The Duals of the 2-Modular Irreducible Modules of the Alternating Groups

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Abstract. We determine the dual modules of all irreducible modules of alternating groups over fields of characteristic 2.

Key words: symmetric group; alternating group; dual module; irreducible module; characteristic 2

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1 Introduction and statement of the result

Let $S_n$ be the symmetric group of degree $n \geq 1$ and let $k$ be a field of characteristic $p > 0$. In [7, Theorem 11.5] G. James constructed all irreducible $kS_n$-modules $D^\lambda$ where $\lambda$ ranges over the $p$-regular partitions of $n$. Here a partition is $p$-regular if each of its parts occurs with multiplicity less than $p$.

As the alternating group $A_n$ has index 2 in $S_n$, the restriction $D^\lambda\downarrow_{A_n}$ is either irreducible or splits as a direct sum of two non-isomorphic irreducible $kA_n$-modules. Moreover, every irreducible $kA_n$-module is a direct summand of some $D^\lambda\downarrow_{A_n}$.

Henceforth we will assume, unless stated otherwise, that $k$ is a field of characteristic 2 which is a splitting field for the alternating group $A_n$. For this, it suffices that $k$ contains the finite field $\mathbb{F}_4$. D. Benson [1] has classified all irreducible $kA_n$-modules:

Proposition 1.1. Let $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_{2s-1} > \lambda_{2s} \geq 0)$ be a strict partition of $n$. Then $D^\lambda\downarrow_{A_n}$ is reducible if and only if

(i) $\lambda_{2j-1} - \lambda_{2j} = 1$ or 2, for $j = 1, \ldots, s$,

(ii) $\lambda_{2j-1} + \lambda_{2j} \not\equiv 2 \pmod{4}$, for $j = 1, \ldots, s$.

In this note we determine the dual of each irreducible $kA_n$-module. Now $D^\lambda\downarrow_{A_n}$ is a self-dual $kA_n$-module, as $D^\lambda$ is a self-dual $kS_n$-module. So we only need to determine the dual of an irreducible $kA_n$-module which is a direct summand of $D^\lambda\downarrow_{A_n}$, when this module is reducible.

Theorem 1.2. Let $\lambda$ be a strict partition of $n$ such that $D^\lambda\downarrow_{A_n}$ is reducible. Then the two irreducible direct summands of $D^\lambda\downarrow_{A_n}$ are self-dual if $\sum_{j=1}^{s} \lambda_{2j}$ is even and are dual to each other if $\sum_{j=1}^{s} \lambda_{2j}$ is odd.
For example $D^{(7,5,1)}_{A_{13}} \cong S \oplus S^*$, for a non self-dual irreducible $kA_{13}$-module $S$, and $D^{(5,4,3,1)}_{A_{13}}$ decomposes similarly. On the other hand $D^{(7,6)}_{A_{13}} \cong S_1 \oplus S_2$ where $S_1$ and $S_2$ are irreducible and self-dual.

In order to prove Theorem 1.2, we use the following elementary result, which requires the assumption that $k$ has characteristic 2:

**Lemma 1.3.** Let $G$ be a finite group and let $M$ be a semisimple $kG$-module which affords a non-degenerate $G$-invariant symmetric bilinear form $B$. Suppose that $B(tm, m) \neq 0$, for some involution $t \in G$ and some $m \in M$. Then $M$ has a self-dual irreducible direct summand.

**Proof.** We have $M = \bigoplus_{i=1}^{n} M_i$, for some $n \geq 1$ and irreducible $kG$-modules $M_1, \ldots, M_n$. Write $m = \sum m_i$, with $m_i \in M_i$, for all $i$. Then

\[
B(tm, m) = \sum_{1 \leq i \leq n} B(tm_i, m_i) + \sum_{1 \leq i < j \leq n} (B(tm_i, m_j) + B(tm_j, m_i))
\]

\[
= \sum_{1 \leq i \leq n} B(tm_i, m_i).
\]

The last equality follows from the fact that $\text{char}(k) = 2$ and

\[
B(tm_i, m_j) = B(m_i, t^{-1}m_j) = B(m_i, tm_j) = B(tm_j, m_i).
\]

Without loss of generality $B(tm_1, m_1) \neq 0$. Then $B$ restricts to a non-zero $G$-invariant symmetric bilinear form $B_1$ on $M_1$. As $M_1$ is irreducible, $B_1$ is non-degenerate. So $M_1$ is isomorphic to its $kG$-dual $M_1^*$. $lacksquare$

## 2 Known results on the symmetric and alternating groups

### 2.1 The irreducible modules of the symmetric groups

We use the ideas and notation of [7]. In particular for each partition $\lambda$ of $n$, James defines the Young diagram $[\lambda]$ of $\lambda$, and the notions of a $\lambda$-tableau and a $\lambda$-tabloid.

Fix a $\lambda$-tableau $x$. So $x$ is a filling of $[\lambda]$ with the symbols $\{1, \ldots, n\}$. The corresponding $\lambda$-tabloid is $\{x\} := \{\sigma(x) \mid \sigma \in R_x\}$, where $R_x$ is the row stabilizer of $x$. We regard $\{x\}$ as an ordered set partition of $\{1, \ldots, n\}$. The $\mathbb{Z}$-span of the $\lambda$-tabloids forms the $\mathbb{Z}\mathcal{S}_n$-lattice $M^\lambda$, and the set of $\lambda$-tabloids is an $\mathcal{S}_n$-invariant $\mathbb{Z}$-basis of $M^\lambda$.

Recall from [7, Section 4] that corresponding to each tableau $x$ there is a polytabloid $e_x := \sum \text{sgn}(\sigma)x$ in $M^\lambda$. Here $\sigma$ ranges over the permutations in the column stabilizer $C_x$ of the tableau $x$. The Specht lattice $S^\lambda$ is defined to be the $\mathbb{Z}$-span of all $\lambda$-polytabloids. In particular $S^\lambda$ is a $\mathbb{Z}\mathcal{S}_n$-sublattice of $M^\lambda$; it has as $\mathbb{Z}$-basis the polytabloids corresponding to the standard $\lambda$-tableaux (i.e., the numbers increase from left-to-right along rows, and from top-to-bottom along columns).

Now James defines $\langle , \rangle$ to be the symmetric bilinear form on $M^\lambda$ which makes the tabloids into an orthonormal basis. As the tabloids are permuted by the action of $\mathcal{S}_n$, it is clear that $\langle , \rangle$ is $\mathcal{S}_n$-invariant.

Suppose now that $\lambda$ is a strict partition and consider the unique permutation $\tau \in R_x$ which reverses the order of the symbols in each row of the tableau $x$. In [7, Lemma 10.4] James shows that $\langle \tau e_x, e_x \rangle = 1$, as $\{x\}$ is the only tabloid common to $e_x$ and $e_{\tau x}$ (in fact James proves that $\langle \tau e_x, e_x \rangle$ is coprime to $p$, if $\lambda$ is $p$-regular, for some prime $p$). Set $J^\lambda := \{x \in S^\lambda \mid \langle x, y \rangle \in 2\mathbb{Z}, \text{ for all } y \in S^\lambda\}$. Then $2S^\lambda \subseteq J^\lambda$ and it follows from [7, Theorem 4.9] that $D^\lambda := (S^\lambda/J^\lambda) \otimes_{\mathbb{Z}} k$ is an absolutely irreducible $k\mathcal{S}_n$-module, for any field $k$ of characteristic 2.
2.2 The real 2-regular conjugacy classes of the alternating groups

A conjugacy class of a finite group $G$ is said to be 2-regular if its elements have odd order. R. Brauer proved that the number of irreducible $kG$-modules equals the number of 2-regular conjugacy classes of $G$ [4]. Now Brauer’s permutation lemma holds for arbitrary fields [3, footnote 19]. So it is clear that the number of self-dual irreducible $kG$-modules equals the number of real 2-regular conjugacy classes of $G$.

We review some well known facts about the 2-regular conjugacy classes of the alternating group. See for example [8, Section 2.5].

Corresponding to each partition $\mu$ of $n$ there is a conjugacy class $C_\mu$ of $S_n$; its elements consist of all permutations of $n$ whose orbits on $\{1, \ldots, n\}$ have sizes $\{\mu_1, \ldots, \mu_\ell\}$ (as multiset). So $C_\mu$ is 2-regular if and only if each $\mu_i$ is odd.

Let $\mu$ be a partition of $n$ into odd parts. Then $C_\mu \subseteq A_n$. If $\mu$ has repeated parts then $C_\mu$ is a conjugacy class of $A_n$. As $C_\mu$ is closed under taking inverses, $C_\mu$ is a real conjugacy class of $A_n$.

Now assume that $\mu$ has distinct parts. Then $C_\mu$ is a union of two conjugacy classes $C_\mu^\pm$ of $A_n$. Set $m := \frac{n-\ell(\mu)}{2}$ and let $z \in C_\mu$. Then $z$ is inverted by an involution $t \in S_n$ of cycle type $(2^m, 1^{n-2m})$. Since $C_{S_n}(z) \cong \prod \mathbb{Z}/\mu_i \mathbb{Z}$ is odd, $t$ generates a Sylow 2-subgroup of the extended centralizer $C_{S_n}^*(z)$ of $z$ in $S_n$. It follows that $z$ is conjugate to $z^{-1}$ in $A_n$ if and only if $t \in A_n$. This shows that $C_\mu^\pm$ are real classes of $A_n$ if and only if $\frac{n-\ell(\mu)}{2}$ is even. This and the discussion above shows:

**Lemma 2.1.** The number of self-dual irreducible $kA_n$-modules equals the number of non-strict odd partitions of $n$ plus twice the number of strict odd partitions $\mu$ of $n$ for which $\frac{n-\ell(\mu)}{2}$ is even.

3 Bressoud’s bijection

We need a special case of a partition identity of I. Schur [9]. This was already used by Benson in his proof of Proposition 1.1:

**Proposition 3.1** (Schur, 1926). The number of strict partitions of $n$ into odd parts equals the number of strict partitions of $n$ into parts congruent to $0, \pm 1$ (mod 4) where consecutive parts differ by at least 4 and consecutive even parts differ by at least 8.

D. Bressoud [5] has constructed a bijection between the relevant sets of partitions. We describe a simplified version of this bijection.

Let $\mu = (\mu_1 > \mu_2 > \cdots > \mu_\ell)$ be a strict partition of $n$ whose parts are all odd. We subdivide $\mu$ into ‘blocks’ of at most two parts, working recursively from largest to smallest parts. Let $j \geq 1$ and suppose that $\mu_1, \mu_2, \ldots, \mu_{j-1}$ have already been assigned to blocks. We form the block $\{\mu_j, \mu_{j+1}\}$ if $\mu_j = \mu_{j+1} + 2$, and the block $\{\mu_j\}$ otherwise (if $\mu_j \geq \mu_{j+1} + 4$). Let $s$ be the number of resulting blocks of $\mu$.

Next we form the sequence of positive integers $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_s)$, where $\sigma_j$ is the sum of the parts in the $j$-th block of $\mu$. Then the $\sigma_j$ are distinct, as the odd parts form a decreasing sequence, with minimal difference 4, and the even parts form a decreasing sequence, with minimal difference 8. Moreover, each even $\sigma_j$ is the sum of a pair of consecutive odd integers. So $\sigma_j \not\equiv 2 \pmod{4}$, for all $j > 0$.

We get a composition $\zeta$ of $n + 2s(s-1)$ by defining

$$\zeta_1 = \sigma_1, \quad \zeta_2 = \sigma_2 + 4, \quad \ldots, \quad \zeta_s = \sigma_s + 4(s-1).$$

The even $\zeta_j$ form a decreasing sequence, with minimal difference 4, and the odd $\zeta_j$ form a weakly decreasing sequence ($\zeta_j = \zeta_{j+1}$ if and only if $\zeta_j, \zeta_{j+1}$ represent two singleton blocks $\{2k-1\}$ and $\{2k-5\}$ of $\mu$, for some $k \geq 0$).
Choose a permutation $\tau$ such that $\zeta_{r1} \geq \zeta_{r2} \geq \cdots \geq \zeta_{rs}$. Then we get a strict partition $\gamma$ of $n$ by defining

$$\gamma_1 = \zeta_{r1}, \quad \gamma_2 = \zeta_{r2} - 4, \ldots, \gamma_s = \zeta_{rs} - 4(s - 1).$$

By construction, the minimal difference between the parts of $\gamma$ is 4 and the minimal difference between the even parts of $\gamma$ is 8. Moreover, $\gamma_j \equiv \zeta_{rj} \pmod{4}$. So $\gamma_j \not\equiv 2 \pmod{4}$. Then $\mu \to \gamma$ is Bressoud’s bijection.

Finally form a strict partition $\lambda$ of $n$ which has $2s - 1$ or $2s$ parts, by defining

$$(\lambda_{2j-1}, \lambda_{2j}) = \begin{cases} \left( \frac{\gamma_j}{2} + 1, \frac{\gamma_j}{2} - 1 \right), & \text{if } \gamma_j \text{ is even or} \\ \left( \frac{\gamma_j}{2}, \frac{\gamma_j}{2} - 1 \right), & \text{if } \gamma_j \text{ is odd.} \end{cases}$$

Then $\lambda$ satisfies the constraints (i) and (ii) of Proposition 1.1. Conversely, it is easy to see that if $\lambda$ satisfies these constraints, then $\lambda$ is the image of some strict odd partition $\mu$ of $n$ under the above sequence of operations.

**Lemma 3.2.** Let $\mu$ be a strict-odd partition of $n$ and let $\lambda$ be the strict partition of $n$ constructed from $\mu$ as above. Then $\frac{n - \ell(\mu)}{2} = \sum \lambda_{2j}$.

**Proof.** Each pair of consecutive parts $\lambda_{2j-1}$, $\lambda_{2j}$ of $\lambda$ corresponds to a block $B$ of $\mu$. Moreover by our description of Bressoud’s bijection, there are integers $q_1, \ldots, q_s$, with $\sum q_j = 0$ such that

$$(\lambda_{2j-1} + 2q_j, \lambda_{2j} + 2q_j) = \begin{cases} \left( \frac{\mu_i + 1}{2}, \frac{\mu_i - 1}{2} \right), & \text{if } B = \{\mu_i\}, \\ (\mu_i, \mu_{i+1}), & \text{if } B = \{\mu_i, \mu_{i+1}\}. \end{cases}$$

In case $B = \{\mu_i, \mu_{i+1}\}$, we have $\mu_i = \mu_{i+1} + 2$ and thus $\mu_{i+1} - \frac{1}{2} + \frac{\mu_{i+1} - 1}{2} = \lambda_{2j} + 2q_j$. We conclude that

$$\frac{n - \ell(\mu)}{2} = \sum_{i=1}^{\ell(\mu)} \frac{\mu_i - 1}{2} = \sum_{j=1}^{s} (\lambda_{2j} + 2q_j) = \sum_{j=1}^{s} \lambda_{2j}. \quad \blacksquare$$

### 4 Proof of Theorem 1.2

Let $D(n)$ be the set of strict partitions of $n$ and let $S(n)$ be the set of strict partitions of $n$ which satisfy conditions (i) and (ii) in Proposition 1.1. So there are $2|S(n)| + |D(n) \setminus S(n)|$ irreducible $kA_n$-modules.

Next set $S(n)^+ := \{\lambda \in S(n) \mid \sum \lambda_{2j} \text{ is even}\}$. Then it follows from Lemmas 2.1 and 3.2 that the number of self-dual irreducible $kA_n$-modules equals $2|S(n)^+| + |D(n) \setminus S(n)|$. Now $D^\lambda \downarrow_{A_n}$ is an irreducible self-dual $kA_n$-module, for $\lambda \in D(n) \setminus S(n)$. So we can prove Theorem 1.2 by showing that the irreducible direct summands of $D^\lambda \downarrow_{A_n}$ are self-dual for all $\lambda \in S(n)^+$.

Suppose then that $\lambda \in S(n)^+$. Let $\tau \in A_n$ be the permutation which reverses each row of a $\lambda$-tableau, as discussed in Section 2.1. We claim that $\tau \in A_n$. For $\tau$ is a product of $\sum_{i=1}^{2s} \left\lfloor \frac{\lambda_{2j}}{2} \right\rfloor$ commuting transpositions. Now $\left\lfloor \frac{\lambda_{2j-1}}{2} \right\rfloor + \left\lfloor \frac{\lambda_{2j}}{2} \right\rfloor = \lambda_{2j}$, as $\lambda_{2j-1} - \lambda_{2j} = 1$, or $\lambda_{2j-1} - \lambda_{2j} = 2$ and both $\lambda_{2j-1}$ and $\lambda_{2j}$ are odd. So $\sum_{i=1}^{2s} \frac{\lambda_{2j}}{2} = \sum_{j=1}^{s} \lambda_{2j}$ is even. This proves the claim.

Since $D^\lambda$ is irreducible and the form $\langle , \rangle$ is non-zero, $\langle , \rangle$ is non-degenerate on $D^\lambda$. Write $D^\lambda \downarrow_{A_n} = S_1 \oplus S_2$, where $S_1$ and $S_2$ are non-isomorphic irreducible modules. As $\tau \in A_n$, it follows from Lemma 1.3 that we may assume that $S_1$ is self-dual. Now $S_2^* \not\cong S_1^* \cong S_1$ and $S_2^*$ is isomorphic to a direct summand of $D^\lambda \downarrow_{A_n}$. So $S_2$ is also self-dual. This completes the proof of the theorem.
5 Irreducible modules of alternating groups
over fields of odd characteristic

We now comment briefly on what happens when $k$ is a splitting field for $A_n$ which has odd characteristic $p$. Let $sgn$ be the sign representation of $kS_n$. So $sgn$ is 1-dimensional but non-trivial. G. Mullineux defined a bijection $\lambda \rightarrow \lambda^M$ on the $p$-regular partitions of $n$ and conjectured that $D^\lambda \otimes sgn = D^{\lambda^M}$ for all $p$-regular partitions $\lambda$ of $n$. This was only proved in the 1990’s by Kleshchev and Ford–Kleshchev. See [6] for details.

Now $D^\lambda|_{A_n} \cong D^{\lambda^M}|_{A_n}$, and $D^\lambda|_{A_n}$ is irreducible if and only if $\lambda \neq \lambda^M$ See [2] for details. Moreover $D^\lambda$ and $D^{\lambda^M}$ are duals of each other, by [7, Theorem 6.6]. So $D^\lambda|_{A_n}$ is self-dual, if $\lambda \neq \lambda^M$. However when $\lambda = \lambda^M$, we do not know how to determine when the two irreducible direct summands of $D^\lambda|_{A_n}$ are self-dual.

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