QUANTUM STATISTICAL MECHANICS AND CLASS FIELD THEORY

JORGE PLAZAS

ABSTRACT. In this short communication we survey some known results relating noncommutative geometry to the class field theory of number fields. These results appear within the context of quantum statistical mechanics where some arithmetic properties of a given number field can be realized in terms of the structure of equilibrium states of a quantum statistical mechanical system.

1. INTRODUCTION

In the last ten years some very interesting results relating noncommutative geometry to class field theory have emerged. The first instance of this connection was explored by Bost and Connes in [2] where they related the class field theory of the field of rational numbers \( \mathbb{Q} \) to the structure of equilibrium states of a particular quantum statistical mechanical system. Various results generalizing some aspects of this construction, based on quantum statistical mechanical systems related to other number fields, have appeared since then ([1, 4, 12, 13, 14]). In particular the problem of finding quantum statistical mechanical systems that encode the explicit class field theory of quadratic imaginary extensions of \( \mathbb{Q} \) has been solved recently ([7, 8]). The existence of quantum statistical mechanical systems with rich arithmetical properties opens a new approach to the study of explicit class field theory using the tools of quantum statistical mechanics. The purpose of this article is to give a brief introduction to this topic.

I want to thank the organizers of the 2005 summer school “Geometric and topological methods for quantum field theory” and Max Planck Institute for their support.

2. BASICS IN CLASS FIELD THEORY

The main objects of study in algebraic number theory are number fields, by definition a number field \( K \) is a finite degree extension of the field of rational numbers \( \mathbb{Q} \). Once we fix an algebraic closure \( \bar{K} \) of \( K \) we would like to understand the absolute Galois group \( \text{Gal}(\bar{K}|K) \), which turns out to be very difficult even when \( K = \mathbb{Q} \). One step in understanding the group \( \text{Gal}(\bar{K}|K) \) consists in studying its abelianization \( \text{Gal}(\bar{K}|K)^{ab} \), this abelian group is the Galois group of \( K^{ab} \), the maximal abelian extension of \( K \), so we have:

\[
\text{Gal}(K^{ab}|K) = \text{Gal}(\bar{K}|K)^{ab}
\]

2000 Mathematics Subject Classification. Primary 58B34, 11R37; Secondary 82B10, 11R56.

Key words and phrases. quantum statistical mechanics, explicit class field theory, number field, equilibrium state.
The field $K^{ab}$ may be obtained as the limit over all finite normal abelian extensions of $K$ in $\hat{K}$. Abelian class field theory studies these abelian extensions together with the corresponding Galois groups. One of the main results of the theory characterizes the group $Gal(K^{ab}/K)$ as a quotient of the group of units of a topological ring constructed from the field $K$ alone. This topological ring is the idèle class group $\mathcal{C}_K$. Let us briefly recall its construction.

Let $K$ be a number field and let $\mathcal{O}_K$ be its ring of integers, i.e. $\mathcal{O}_K$ consists of the elements in $K$ which are roots of monic polynomials with coefficients in $\mathbb{Z}$. There is a one to one correspondence between the prime ideals of $\mathcal{O}_K$ and the finite places of $K$, which are equivalence classes of nonarchimedean valuations $|\cdot|$ on $K$. For instance every prime number $p \in \mathcal{O}_K = \mathbb{Z}$ gives rise to a finite place corresponding to the $p$-adic absolute value $|\cdot|_p$ in $\mathbb{Q}$. Given a finite place $\nu$ of $K$ we denote by $K_{\nu}$ the completion of $K$ with respect to the metric induced by $\nu$ and denote by $\mathcal{O}_\nu$ the ring of integers of $K_{\nu}$. For each $\nu$ the ring $\mathcal{O}_\nu$ is an open compact subring of $K_{\nu}$ and we may form the restricted topological product

$$\mathbb{A}_{f,K} = \prod_{\nu}(K_{\nu}:\mathcal{O}_{\nu})$$

where $\nu$ runs over the finite places of $\mathcal{O}_K$. This restricted product is by definition the subset of the cartesian product $\prod_{\nu}K_{\nu}$ consisting of elements for which all but a finite number of coordinates lie in the subrings $\mathcal{O}_{\nu}$. It is given the weakest topology for which the sets $\prod_{\nu \in F}K_{\nu} \times \prod_{\nu \notin F}\mathcal{O}_{\nu}$, for $F$ a finite set of places, are open. The topological ring $\mathbb{A}_{f,K}$ is called the ring of finite adèles of $K$. If we add to $\mathbb{A}_{f,K}$ the product of the completions of $K$ with respect to the infinite places we get $\mathbb{A}_K$, the ring of adèles of $K$. Infinite places are given by equivalence classes of archimedean valuations and correspond to the $[K: \mathbb{Q}]$ different embeddings of $K$ in $\mathbb{C}$.

Consider now $Gl_1(\mathbb{A}_K)$, the group of units of $\mathbb{A}_K$. $K^*$ can be embedded diagonally as a discrete subgroup of $Gl_1(\mathbb{A}_K)$. The quotient $\mathcal{C}_K = Gl_1(\mathbb{A}_K)/K^*$ is called the idèle class group of $K$. It turns out that the abelian extensions of $K$ correspond to the normal subgroups of $\mathcal{C}_K$. In particular the following important theorem holds (cf. [9], Section II.3.7):

**Theorem 2.1.** Let $D_K$ be the connected component of the identity in $\mathcal{C}_K$. There is a canonical isomorphism:

$$\theta : \mathcal{C}_K/D_K \rightarrow Gal(K^{ab}|K)$$

For the case $K = \mathbb{Q}$ a lot more is known. By the Kronecker Weber theorem (cf. [LS] Section 20) every abelian extension of $\mathbb{Q}$ is contained in a cyclotomic extension, that is, in some extension of the form $\mathbb{Q}(\zeta_n)$ where $\zeta_n$ is a primitive $n$-th root of unity for some $n$. The cyclotomic extension $\mathbb{Q}(\zeta_n)$ is abelian and its Galois group $Gal(\mathbb{Q}(\zeta_n)|\mathbb{Q})$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^*$. The maximal abelian extension of $\mathbb{Q}$ is the field $\mathbb{Q}^{cycl}$ obtained by adjoining to $\mathbb{Q}$ all roots of unity and $\mathbb{C}_\mathbb{Q}/D_\mathbb{Q} = \mathbb{Z}^*$ where $\mathbb{Z}^* = \lim_{\leftarrow n} \mathbb{Z}/n\mathbb{Z}$.

For a general number field $K$, one would also like to know which elements of $\hat{K}$ generate $K^{ab}$ over $K$ and how the Galois group $Gal(K^{ab}|K)$ acts on these generators. This is the content of Hilbert’s 12th problem known also as the explicit class field theory problem. Up to now the only number fields apart from $\mathbb{Q}$ for which a complete solution to the explicit class field theory problem is known are quadratic imaginary fields, that is fields of the form $K = \mathbb{Q}(\sqrt{-d})$, where $d \in \mathbb{N}^+$ is a positive
integer. In this case the answer is given by the theory of complex multiplication which characterizes the field $\mathbb{Q}(\sqrt{-d})^{ab}$ in terms of data coming from an elliptic curve (cf. [17]). Given an embedding $\mathbb{Q}(\sqrt{-d}) \hookrightarrow \mathbb{C}$ the ring of integers of $\mathbb{Q}(\sqrt{-d})$ gives rise to a lattice in $\mathbb{C}$ and we obtain an elliptic curve as the quotient of $\mathbb{C}$ by this lattice. The curve $E$ has a natural abelian group structure and $\mathbb{Q}(\sqrt{-d})^{ab}$ is generated by the values of an analytic function $F$ on $E$ at the points of finite order of $E$. The function $F$ is given by $P^u$ where $P$ is the Weierstrass function of $E$ and $u$ is the order of the group of automorphisms of $E$.

3. Basics in quantum statistical mechanics

Quantum statistical mechanics studies statistical ensembles of quantum mechanical systems. From a mathematical point of view, a quantum statistical mechanical system consists of a set of observables $A$ having the structure of a $C^*$-algebra and a time evolution of the system given by a one-parameter group of automorphisms $\sigma_t$ of the algebra of observables, $\sigma_t \in \text{Aut}(A), \ t \in \mathbb{R}$. Classically the state of a system is specified by a probability measure on the phase space, integration of an observable against this measure gives its mean value corresponding to the particular state. In a quantum mechanical setting, probability measures are replaced by states on the algebra of observables. By definition a state on a unital $C^*$-algebra $A$ is a linear map $\varphi : A \to \mathbb{C}$ satisfying:

- $\varphi(a^*a) \geq 0$ for all $a \in A$.
- $\varphi(1) = 1$.

The appropriate definition of equilibrium states in this context was given by Haag, Hugenholtz and Winnink in [11]. Given a quantum statistical mechanical system $(A, \sigma_t)$ an equilibrium state will be a state $\varphi$ on the algebra $A$ satisfying certain compatibility condition with respect to the time evolution of the system. This condition, known as the KMS condition (after Kubo, Martin and Schwinger) depends on a thermodynamic parameter $\beta = \frac{1}{T}$, the inverse temperature of the system.

**Definition 3.1.** Let $(A, \sigma_t)$ be a quantum statistical mechanical system. A state $\varphi$ on $A$ satisfies the KMS condition at inverse temperature $0 < \beta < \infty$ if for every $a, b \in A$ there exist a bounded holomorphic function $F_{a,b}$ on $\{z \in \mathbb{C} \mid 0 < \Im(z) < \beta\}$, continuous on the closed strip, such that

$$F_{a,b}(t) = \varphi(a \sigma_t(b)), \quad F_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a), \quad \forall t \in \mathbb{R}$$

We call such state a $\text{KMS}_{\beta}$ state.

A $\text{KMS}_{\infty}$ state is by definition a weak limit of $\text{KMS}_{\beta}$ states as $\beta \to \infty$. This definition is stronger than the more commonly used definition of ground states as states for which the KMS condition holds with $\beta = \infty$. The definition of $\text{KMS}_{\infty}$ states as weak limits of $\text{KMS}_{\beta}$ states is better behaved and more appropriate for the applications we will describe below.

It turns out (cf. [3], Section 5.3) that for each $\beta$ the set of $\text{KMS}_{\beta}$ states associated to the time evolution $\sigma_t$ is a compact convex space. We denote by $E_{\beta}$ the space of extremal points of the space of $\text{KMS}_{\beta}$ states.

**Example 3.2.** Let $\beta > 0$ and let $A = \mathcal{B}(\mathcal{H})$ be the algebra of bounded operators on a Hilbert space $\mathcal{H}$. Given a positive self adjoint operator $H \in A$ we can define
a time evolution on $A$ by

$$\sigma_t(a) = e^{itH}ae^{-itH}, \quad a \in A$$

If the operator $e^{-\beta H}$ is trace class then

$$\varphi(a) = \frac{1}{Z} Tr(a e^{-\beta H}), \quad Z = Tr(e^{-\beta H})$$

is a $KMS_\beta$ state.

More generally, given a quantum statistical mechanical system $(A, \sigma_t)$ we can look for representations of the algebra $A$ as an algebra of bounded operators in a Hilbert space $H$. Then, given a positive self adjoint operator $H$ such that the time evolution $\sigma_t$ has the form above, we call

$$Z(\beta) = Tr(e^{-\beta H})$$

the partition function of the system. The poles of this function correspond to the critical temperatures at which phase transitions of the system may take place.

Consider a quantum statistical mechanical system $(A, \sigma_t)$. A group $G \subset Aut(A)$ such that

$$\sigma_t g = g \sigma_t \quad \forall g \in G, \quad \forall t \in \mathbb{R}$$

is called a symmetry group of the system. Then acts on the space of $KMS_\beta$ states for any $\beta$ and hence on $\mathcal{E}_\beta$. Inner automorphisms coming from unitaries invariant under the time evolution act trivially on equilibrium states.

For some of the applications we will describe below it is also important to consider symmetries of the system induced by endomorphisms of the algebra $A$ (5 [7] [8]). An endomorphism of $A$ is by definition a $*-$morphism $\rho : A \to A$ commuting with $\sigma_t$. Such an endomorphism acts on the set of $KMS_\beta$ states $\varphi$ for which $\varphi(\rho(1)) \neq 0$ by $\rho^* \varphi = \frac{1}{\varphi(\rho(1))} \varphi \circ \rho$. Inner endomorphisms coming from isometries invariant under the time evolution act trivially on equilibrium states. Symmetries induced by endomorphisms were introduced in the context of super selection sectors developed by Doplicher, Haag and Roberts.

4. The Bost-Connes system

In [2] Bost and Connes constructed a remarkable quantum statistical mechanical system $(A, \sigma_t)$ in which the structure of equilibrium states is related in a deep way with the class field theory of $\mathbb{Q}$. The group $\mathcal{C}_Q/D_Q = \hat{\mathbb{Z}}^*$ acts as symmetries of this system and the algebraic numbers generating the maximal abelian extension of $\mathbb{Q}$ can be recovered as values of the equilibrium states at zero temperature on the observables corresponding to an arithmetic subalgebra of $A$. What is more remarkable about this system is the fact that the action of the Galois group on this values commutes with the action of $\mathcal{C}_Q/D_Q = \hat{\mathbb{Z}}^*$ on the equilibrium states. In this section we describe this system and its main features.

Let $A$ be the $C^*$-algebra generated by two sets of elements $\{e(r) \mid r \in \mathbb{Q}/\mathbb{Z}\}$ and $\{\mu_n \mid n \in \mathbb{N}^+\}$ with relations:

1. $\mu_{n}^* \mu_n = 1 \quad \forall n \in \mathbb{N}^+$
2. $\mu_n \mu_k = \mu_{nk} \quad \forall n, k \in \mathbb{N}^+$
3. $e(0) = 1, \quad e(r)^* = e(-r), \quad e(r)e(s) = e(r + s) \quad \forall r, s \in \mathbb{Q}/\mathbb{Z}$
4. $\mu_n e(r) \mu_n^* = \frac{1}{n} \sum_{n s = r} e(s) \quad \forall n \in \mathbb{N}^+, \forall r \in \mathbb{Q}/\mathbb{Z}$
Define a time evolution on the $C^*$-algebra $A$ by taking
\[ \sigma_t(\mu_n) = n^t \mu_n, \quad n \in \mathbb{N}^+, \quad t \in \mathbb{R} \]
\[ \sigma_t(e(r)) = e(r), \quad r \in \mathbb{Q}/\mathbb{Z}, \quad t \in \mathbb{R} \]

We will refer to the quantum statistical mechanical system $(A, \sigma_t)$ as the Bost-Connes system. We summarize the main results of [2] about the structure of this system. As above we denote by $\mathcal{E}_\infty$ the set of extremal $KMS_\infty$ states.

**Theorem 4.1** (Bost, Connes).

- The group $\mathbb{Z}_Q/D_Q = \mathbb{Z}^*$ acts as a group of symmetries of the system $(A, \sigma_t)$.
- Let $A_Q$ be the $\mathbb{Q}$-subalgebra of $A$ generated over $\mathbb{Q}$ by the sets of elements \{ $e(r)$ | $r \in \mathbb{Q}/\mathbb{Z}$ \} and \{ $\mu_n, \mu_n^+$ | $n \in \mathbb{N}^+$ \}. Then for every $\varphi \in \mathcal{E}_\infty$ and every $a \in A_Q$ the value $\varphi(a)$ is algebraic over $\mathbb{Q}$. Moreover for any $\varphi \in \mathcal{E}_\infty$ one has $\varphi(A_Q) \subset \mathbb{Q}^{ab}$ and $\mathbb{Q}^{ab}$ is generated by numbers of the form $\varphi(a)$ with $\varphi \in \mathcal{E}_\infty$ and $a \in A_Q$.
- For all $\varphi \in \mathcal{E}_\infty$, $\gamma \in Gal(\mathbb{Q}^{ab}|\mathbb{Q})$ and $a \in A_Q$ one has
  \[ \gamma \varphi(a) = \varphi(\theta^{-1}(\gamma)a) \]
  where $\theta : \mathbb{Z}^* \rightarrow Gal(\mathbb{Q}^{ab}|\mathbb{Q})$ is the class field theory isomorphism.

The system $(A, \sigma_t)$ was originally introduced by Bost and Connes in the context of Hecke algebras. The inclusion of rings $\mathbb{Z} \subset \mathbb{Q}$ induces an inclusion matrix groups $P_\mathbb{Z}^+ \subset P_\mathbb{Q}^+$ where $P_\mathbb{Q}^+ = \{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} | b \in \mathbb{Q}, a \in \mathbb{Q}_+^* \}$ and $P_\mathbb{Z}^+ = \{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} | c \in \mathbb{Z} \}$. This inclusion of matrix groups is almost normal in the sense that the orbits of $P_\mathbb{Z}^+$ acting on the left on the set of right cosets $P_\mathbb{Q}^+/P_\mathbb{Z}^+$ are finite and vice versa.

One can then form a convolution algebra $\mathcal{H}(P_\mathbb{Z}^+, P_\mathbb{Q}^+)$ given by finitely supported $\mathbb{C}$-valued functions on $P_\mathbb{Z}^+ \backslash P_\mathbb{Q}^+/P_\mathbb{Z}^+$, this algebra is called the Hecke algebra of the pair $(P_\mathbb{Z}^+, P_\mathbb{Q}^+)$. $\mathcal{H}(P_\mathbb{Q}^+; P_\mathbb{Z}^+)$ can be completed to a $C^*$-algebra with a natural time evolution. It is shown in [2] that in this way one obtains the system $(A, \sigma_t)$ described above.

Let $(\epsilon_n)_{n \in \mathbb{N}^+}$ be the canonical basis of the Hilbert space $l^2(\mathbb{N}^+)$. For any element $\gamma \in Gal(\mathbb{Q}^{ab}|\mathbb{Q})$ one can define a representation $\pi_\gamma$ of the algebra $A$ in $l^2(\mathbb{N}^+)$ by
\[ \pi_\gamma(\mu_n) \epsilon_k = \epsilon_{nk}, \quad n, k \in \mathbb{N}^+ \]
\[ \pi_\gamma(e(r)) \epsilon_k = \gamma(e^{2\pi i r}) \epsilon_k, \quad k \in \mathbb{N}^+, r \in \mathbb{Q}/\mathbb{Z} \]

For any of these representations the time evolution of the system is implemented by the operator $H \epsilon_k = (\log k) \epsilon_k$, $k \in \mathbb{N}^+$ and so the partition function of the system is the Riemann zeta function
\[ Z(\beta) = \text{Tr}(e^{-\beta H}) = \sum_k \frac{1}{k^\beta} = \zeta(\beta) \]

The Bost-Connes system exhibits a phenomenon called spontaneous symmetry breaking. For $0 < \beta \leq 1$ the system $(A, \sigma_t)$ admits only one equilibrium state and the symmetry group of the system acts trivially on this state. For $\beta > 1$ the space $\mathcal{E}_\beta$ is parameterized by $\hat{\mathbb{Z}}^*$ and the symmetry group acts transitively on this space.
5. SOME GENERALIZATIONS TO OTHER NUMBER FIELDS

The work of Bost and Connes has inspired several authors to construct quantum statistical mechanical systems associated to other number fields. These constructions generalize in different directions some of the results in [2].

In [13], M. Laca and I. Raeburn realized the algebra of observables of the Bost-Connes system as a semigroup crossed product algebra. Given a discrete group $\Gamma$ its group algebra $\mathbb{C} \Gamma$ can be completed to a $\mathbb{C}^*$-algebra $\mathbb{C}^*(\Gamma)$ by considering all unitary irreducible representations of $\mathbb{C} \Gamma$ as an algebra of operators on some Hilbert space. If a semigroup $S$ acts by endomorphisms on the algebra $\mathbb{C}^*(\Gamma)$ one can twist the product and the convolution on $\mathbb{C}^*(\Gamma)$ by the action of $S$ getting a crossed product $\mathbb{C}^*(\Gamma) \rtimes S$. The $\mathbb{C}^*$-algebra of the Bost-Connes system is then given by a semigroup crossed product $\mathbb{C}^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}^+$ where the action of $\mathbb{N}^+$ on $\mathbb{C}^*(\mathbb{Q}/\mathbb{Z})$ is a right inverse to the action coming from the natural multiplication $(n, r) \mapsto nr$ where $n \in \mathbb{N}^+$ and $r \in \mathbb{Q}/\mathbb{Z}$.

Arledge, Laca and Raeburn considered in [1] crossed products of the form $\mathbb{C}^*(K/O) \rtimes O^x$ for $K$ an arbitrary number field and $O$ its ring of integers. They characterize the faithful representations of these algebras and realized them as Hecke algebras.

As in the case of $\mathbb{Q}$ one has that for any number field $K$ the ring inclusion $O \subset K$ induces an almost normal inclusion of matrix groups $P_0^+ \subset P_K^+$. Harari and Leichtnam studied in [12] a quantum statistical mechanical system corresponding to the Hecke algebra $\mathcal{H}(P_K^+; P_0^+)$. The equilibrium states of this system share many of the properties of the Bost-Connes system. The group $\hat{O}^* = \prod \nu O_\nu^*$, where $\nu$ runs over the finite places of $K$, acts as a symmetry group of the system and the partition function of the system is the Dedekind zeta function of the number field $K$

$$\zeta_K(s) = \sum_a \frac{1}{N(a)^s} = \prod_p \frac{1}{1 - N(p)^{-s}}$$

with a finite number of factors removed from the product. In this expression $a$ runs over the integral ideals of $O$, $p$ runs over the prime ideals of $O$ and $N(a) = [O : a]$ is the absolute norm of the ideal $a$. This system exhibits spontaneous symmetry breaking at the pole of this function; for $0 < \beta \leq 1$ the system admits only one equilibrium state and the symmetry group of the system acts trivially on this state. For $\beta > 1$ the space $\mathcal{E}_\beta$ is parameterized by $\hat{O}^*$ and the symmetry group acts freely and transitively on this space. The results of [12] hold for arbitrary global fields. These include number fields and finite separable extensions of the field of rational functions over a finite field.

A quantum statistical mechanical system similar to the one of [12] was introduced in [4] by Cohen. Her system has the advantage of recovering the full Dedekind zeta function of the number field $K$ as partition function of the system.

Laca and van Frankenhuijsen studied in [14] the Hecke algebra corresponding to the almost normal inclusion of matrix groups

$$\begin{pmatrix} 1 & O \\ 0 & O^* \end{pmatrix} \subset \begin{pmatrix} 1 & K \\ 0 & K^* \end{pmatrix}$$

for an arbitrary number field $K$ with ring of integers $O$. The equilibrium states of the corresponding system are analyzed in the case of fields with class number one. For quadratic imaginary fields of class number one the symmetry group of the
system is isomorphic to the Galois group of the maximal abelian extension of the field and the action of this group on extremal equilibrium states is transitive.

6. Fabulous states for number fields

The work of Bost and Connes together with its various generalizations opens the possibility of approaching the problem of explicit class field theory within the framework of noncommutative geometry. Following these lines one would like to have for an arbitrary number field $K$ a quantum statistical mechanical system which fully incorporates its class field theory, the notion of fabulous states for number fields, introduced in [5] by Connes and Marcolli, encodes this aim.

Given a number field $K$ together with an embedding $K \hookrightarrow \mathbb{C}$ the “problem of fabulous states” asks for the construction of a quantum statistical mechanical system $(\mathcal{A}, \sigma_t)$ such that:

- The group $C_K/D_K$ acts as a group of symmetries of the system $(\mathcal{A}, \sigma_t)$.
- There exists a $K$-subalgebra $\mathcal{A}_K$ of $\mathcal{A}$ such that for every $\varphi \in \mathcal{E}_\infty$ and every $a \in \mathcal{A}_K$ the value $\varphi(a)$ is algebraic over $K$. Moreover $K^{ab}$ is generated over $K$ by numbers of this form.
- For all $\varphi \in \mathcal{E}_\infty$, $\gamma \in Gal(K^{ab}|K)$ and $a \in \mathcal{A}_K$ one has $\gamma \varphi(a) = \varphi(\theta^{-1}(\gamma)a)$ where $\theta : C_K/D_K \to Gal(K^{ab}|K)$ is the class field theory isomorphism.

The ground states of such a system are refereed as “fabulous” in view of its rich arithmetical properties.

7. Q-lattices, quadratic extensions and complex multiplication

For the case of quadratic imaginary fields $K = \mathbb{Q}(\sqrt{-d})$, $d \in \mathbb{N}^+$, the problem of finding a quantum statistical mechanical system having all the properties discussed in the last section has been recently solved by Connes, Marcolli and Ramachandran [7, 8]. Their results rely deeply on the arithmetic properties of Q-lattices which were studied in [5].

Given a number field $K$ with $[K, \mathbb{Q}] = n$ there is an embedding $K^* \hookrightarrow \text{Gl}_n(\mathbb{Q})$, this induces an embedding $\text{Gl}_1(\mathbb{A}_K) \hookrightarrow \text{Gl}_n(\mathbb{A}_\mathbb{Q})$. It is therefore important to study $\text{Gl}_n$ analogs of the Bost-Connes system. In [5] Connes and Marcolli constructed a quantum statistical mechanical system with group of symmetries $\text{Gl}_2(\mathbb{A}_\mathbb{Q},f)$ and analyzed in detail the arithmetical properties of its equilibrium states. The central notions introduced Connes and Marcolli in [5] are those of Q-lattices and commensurability.

A $n$ dimensional Q-lattice is by definition given by a pair $(\Lambda, \phi)$ where $\Lambda$ is a lattice in $\mathbb{R}^n$ and $\phi : \mathbb{Q}^n/\mathbb{Z}^n \to \mathbb{Q}\Lambda/\Lambda$ is a homomorphism of abelian groups. Two $n$ dimensional Q-lattices $(\Lambda_1, \phi_1)$ and $(\Lambda_2, \phi_2)$ are commensurable if $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$ and $\phi_1 = \phi_2$ modulo $\Lambda_1 + \Lambda_2$. The relation of commensurability is an equivalence relation on the set of $n$ dimensional Q-lattices. The space of commensurability classes of $n$-dimensional Q-lattices is described in terms of a noncommutative $C^*$-algebra $C^*(\mathcal{L}_n)$.

A one dimensional Q-lattice can be rescaled by multiplying it by a positive scale factor so the multiplicative group $\mathbb{R}^*$ acts on the set of one dimensional Q-lattices. The $C^*$-algebra $C^*(\mathcal{L}_n/\mathbb{R}^*_n)$ corresponding to the space of commensurability classes of one dimensional Q-lattices up to scaling has a natural time evolution and a
natural choice of an arithmetic structure given by a $\mathbb{Q}$-subalgebra of $C^*(\mathcal{L}_n/\mathbb{R}_+^*)$. In this way one recovers the Bost-Connes system $[5]$. The group $C^*$ acts by rescaling on the set of two dimensional $\mathbb{Q}$-lattices. As in the one dimensional case the $C^*$-algebra $C^*(\mathcal{L}_n/\mathbb{C}^*)$ corresponding to the space of commensurability classes of two dimensional $\mathbb{Q}$-lattices up to scaling has a natural time evolution and a natural choice of an arithmetic structure. The structure of equilibrium states of the corresponding quantum statistical mechanical system is related to the Galois theory of the field of modular functions. We refer to $[6]$ and $[16]$ for a survey of these results.

Consider now a quadratic imaginary field $K = \mathbb{Q}(\sqrt{-d})$, $d \in \mathbb{N}^+$, with ring of integers $\mathcal{O}$. A one dimensional $K$-lattice is given by a pair $(\Lambda, \phi)$ where $\Lambda$ is a finitely generated $\mathcal{O}$-submodule of $\mathbb{C}$ with $\Lambda \otimes \mathcal{O} K \cong K$ and $\phi : K/\mathcal{O} \to K\Lambda/\Lambda$ is a homomorphism of abelian groups. Two one dimensional $K$-lattices $(\Lambda_1, \phi_1)$ and $(\Lambda_2, \phi_2)$ are commensurable if $K\Lambda_1 = K\Lambda_2$ and $\phi_1 = \phi_2 \mod \Lambda_1 + \Lambda_2$, the relation of commensurability is an equivalence relation on the set of one dimensional $K$-lattices. A one dimensional $K$-lattice is in particular a two dimensional $\mathbb{Q}$-lattice and two one dimensional $K$-lattices are commensurable if and only if the underlying $\mathbb{Q}$-lattices are commensurable. The space of commensurability classes of one dimensional $K$-lattices gives rise to a quantum statistical mechanical system whose ground states are fabulous states for the field $K$ $[7, 8]$. The explicit class field theory of the quadratic imaginary fields can thus be fully recovered in the form discussed above.

The spaces associated to commensurability classes of $\mathbb{Q}$-lattices can be interpreted as noncommutative versions of Shimura varieties. Classical Shimura varieties have a rich structure coming from a natural action of ad`elic groups on them, they appear in the context of $[5, 7, 8]$ as spaces of extremal $KMS$ states at low temperature. In $[10]$ Ha and Paugam developed further this point of view constructing quantum statistical mechanical systems associated to arbitrary Shimura varieties. Given a general number field $K$ the results in $[10]$ lead in particular to the existence of a quantum statistical mechanical system whose group of symmetries is $\mathcal{C}_K/D_K$ and whose partition function is the Dedekind zeta function $\zeta_K(s)$.

The first case for which there is not yet a complete solution to the explicit class field theory problem is the case of real quadratic fields, $K = \mathbb{Q}(\sqrt{d})$, where $d \in \mathbb{N}^+$ is a square free positive integer. In $[15]$ Manin proposed the use of noncommutative geometry as a geometric framework for the study of abelian class field theory of real quadratic fields. This is the so called “real multiplication program”. The main idea is that noncommutative tori may play a role in the study of real quadratic fields analogous to the role played by elliptic curves in the study of imaginary quadratic fields. Some possible relations between Manin’s program and $\mathbb{Q}$-lattices are discussed in $[16]$.

References

[1] J. Arledge, M. Laca, I. Raeburn, *Semigroup crossed products and Hecke algebras arising from number fields*. Doc. Math. **2** (1997) 115-138.
[2] J.B. Bost, A. Connes, *Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory*. Selecta Math. (N.S.) **1** (1995) 411-457.
[3] O. Bratteli, D.W. Robinson, *Operator algebras and quantum statistical mechanics*. Springer, New York (1981).
[4] P. Cohen, *A C*-dynamical system with Dedekind zeta partition function and spontaneous symmetry breaking*. J. Théor. Nombres Bordeaux 11 (1999) 15-30.

[5] A. Connes, M. Marcolli, *From physics to number theory via noncommutative geometry. Part I. Quantum statistical mechanics of Q-lattices*. In “Frontiers in Number Theory, Physics, and Geometry I.” Springer Verlag, (2006).

[6] A. Connes, M. Marcolli, *Q-lattices: quantum statistical mechanics and Galois theory*. J. Geom. Phys. 56 (2006) 2-23.

[7] A. Connes, M. Marcolli, N. Ramachandran, *KMS states and complex multiplication*. Selecta Math. (N.S.) 11 (2005) 325-347.

[8] A. Connes, M. Marcolli, N. Ramachandran, *KMS states and complex multiplication. Part II*, to appear in the Abel Symposium Volume.

[9] G. Gras, *Class field theory*. Springer Verlag, Berlin, (2003).

[10] E. Ha, F. Paugam, *A Bost-Connes-Marcolli system for Shimura varieties*, IMRP 5 (2005) 237-286.

[11] R. Haag, N.M. Hugenholtz, M. Winnink, *On the equilibrium states in quantum statistical mechanics*. Comm. Math. Phys. 5 (1967) 215-236.

[12] D. Harari, E. Leichtnam, *Extension du phénomène de brisure spontanée de symétrie de Bost-Connes au cas des corps globaux quelconques*. Selecta Math. (N.S.) 3 (1997) 205-243.

[13] M. Laca, I. Raeburn, *A semigroup crossed product arising in number theory*. J. London Math. Soc. (2) 59 (1999), 330-344.

[14] M. Laca, M. van Frankenhuijsen, *Phase transitions on Hecke C*-algebras and class-field theory over Q*. preprint [math.OA/0410305]

[15] Y. Manin *Real multiplication and noncommutative geometry*. In: “The legacy of Niels Henrik Abel”, Springer Verlag, Berlin (2004).

[16] M. Marcolli, *Arithmetic noncommutative geometry*. University Lecture Series, 36. American Mathematical Society, Providence (2005).

[17] J.P. Serre *Complex multiplication*. In: “Algebraic Number Theory.” Proc. Instructional Conf., Brighton (1965).

[18] H. P. F. Swinnerton-Dyer *A brief guide to algebraic number theory*. London Mathematical Society Student Texts, 50. Cambridge University Press, Cambridge (2001).

---

Max Planck Institute for Mathematics, Vivatsgasse 7, Bonn 53111, Germany

E-mail address: plazas@mpim-bonn.mpg.de