On invariants of link maps in dimension four

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Abstract

We affirmatively address the question of whether the proposed link homotopy invariant $\omega$ of Li is well-defined. It is also shown that if one wishes to adapt the homotopy invariant $\tau$ of Schneiderman-Teichner to a link homotopy invariant of link maps, the result coincides with $\omega$.

1 Introduction

A link map $S^2 \cup S^2 \to S^4$ is a map from a union of 2-spheres with pairwise disjoint images, and a link homotopy is a homotopy through link maps. To a link map $f$, Kirk ([4], [5]) assigned a pair of integer polynomials $\sigma(f) = (\sigma_+(f), \sigma_-(f))$ which is invariant under link homotopy and vanishes if $f$ is link homotopic to a link map that embeds either component. He posed the still-open problem of whether $\sigma(f) = (0, 0)$ is sufficient to link nullhomotope $f$. In [6], Li sought to define a link homotopy invariant $\omega(f) = (\omega_+(f), \omega_-(f))$ to detect link maps in the kernel of $\sigma$. When $\sigma_{\pm}(f) = 0$, the mod 2 integer $\omega_{\pm}(f)$ obstructs embedding by counting (after a link homotopy) weighted intersections between $f(S^2) \pm$ and its Whitney disks in the complement of $f(S^2 \mp)$. While the examples with $\sigma(f) = (0, 0)$ but $\omega(f) \neq (0, 0)$ in that paper were found to be in error by Pilz ([8]), the latter did not address another issue. Namely, the proof that $\omega$ is invariant under link homotopy relies implicitly on the assumption that a pair of link homotopic abelian link maps are link homotopic through abelian link maps. The first purpose of this note is to prove this assumption correct and that $\omega$ is a link homotopy invariant.

Theorem 1. If $f$ and $g$ are link homotopic link maps such that $\sigma(f) = (0, 0) = \sigma(g)$, then $\omega(f) = \omega(g)$.

The invariants $\sigma$ and $\omega$ may then be viewed, respectively, as primary and secondary obstructions to link homotoping to an embedding. For the problem of homotoping a map $S^2 \to Y^4$ to an embedding, the homotopy invariants $\mu$ of Wall ([10]) and $\tau$ of Schneiderman and Teichner ([9]) form an analogous pair of obstructions. Our second purpose is to show that if one adapts $\tau$ to the setting of link homotopy in the natural way, one obtains $\omega$. For a link map $f : S^2_+ \cup S^2_- \to S^4$, where the signs are used to distinguish each component, write $X_{\pm} = S^4 \setminus f(S^2_{\mp})$ and let $f_{\pm}$ denote the restricted map $f|S^2_{\pm} : S^2_{\pm} \to X_{\mp}$.

Theorem 2. Let $f$ be a link map with $\sigma_+(f) = 0$. Then $f$ is link homotopic to a link map $g$ such that $\tau(g_+)$ is defined, and one has that $\tau(g_+) = 0$ if and only if $\omega_+(f) = 0$.

Assume all manifolds are equipped with basepoints and orientations arbitrarily unless otherwise specified.
2 Proof of Theorem 1

A link map $f$ is said to be abelian if $\pi_1(X_+) \cong \mathbb{Z}$ and $\pi_1(X_-) \cong \mathbb{Z}$, and an abelian link homotopy is a link homotopy through abelian link maps. We will say $f$ is good if it is abelian and each restricted map $f_\pm$ is a self-transverse immersion with vanishing signed self-intersection number. Theorem 1 will be shown using the following lemma.

Lemma 1. If $f$ and $g$ are regularly homotopic good link maps such that $\sigma(f) = (0, 0) = \sigma(g)$, then $\omega(f) = \omega(g)$.

Proof of Theorem 1. A link map $f$ may be first be perturbed so that it restricts to a self-transverse immersion on each component 2-sphere; local cusp homotopies may then be performed so that these immersions each have vanishing signed self-intersection number. Finger moves of $f(S^2_+)$ in the complement of $f(S^2_-)$, followed by finger moves of $f(S^2_-)$ in the complement of $f(S^2_+)$, may then serve to abelianize $\pi_1(X_+)$ and $\pi_1(X_-)$ (see [1, p. 205]; also [4], [6]). Denote the resulting good link map by $f'$. In [6], the author gives an algorithm for computing the pair of mod 2 integers $\omega(f') = (\omega_+(f'), \omega_-(f'))$ and defines $\omega(f) = \omega(f')$. Suppose $f''$ is another good link homotopy representative of $f$, then $f'$ and $f''$ are regularly homotopic by [5, Theorem 2.4], so $\omega(f'') = \omega(f')$ by Lemma 1. Thus $\omega$ does not depend on the choice of good representative.

It remains to prove Lemma 1. We make use of ideas from [3], to which the reader is referred for more details on finger and Whitney moves along chords. Throughout the rest of this paper, $Y$ will denote a 4-manifold. Let $k : S^2 \to Y$ be an immersion. A chord $\gamma$ attached to $k(S^2)$ is a continuous arc in $Y$ whose endpoints are distinct points of $k(S^2)$ (minus its double points) and whose interior is disjoint from $k(S^2)$. A chord is simple if this arc is an embedding. If two simple chords $\gamma$ and $\gamma'$ for $k(S^2)$ are ambient isotopic in $Y$ by an isotopy fixing $k(S^2)$, then a finger move along either chord yields an ambient isotopic immersion. The following result is a ready consequence of transversality and the isotopy extension theorem.

Lemma 2. Let $C$ be a compact subset of a 4-manifold $Y$, and let $k : S^2 \to Y$ be an immersion. Suppose $\alpha$ and $\beta$ are simple chords on $k(S^2)$ with common endpoints $p, q$. If $\alpha$ and $\beta$ are path homotopic in $Y\setminus C$ through chords on $k(S^2)$, then $\alpha$ and $\beta$ are ambient isotopic in $Y$ by an isotopy that carries $\alpha$ to $\beta$ and fixes $k(S^2)$ and $C$.

We first show that, roughly speaking, finger moves and Whitney moves of a single component of a link map in the complement of the other component commute.

Lemma 3. Let $f$ be a link map such that the restricted maps $f_\pm$ are self-transverse immersions. Suppose that an immersion $g_+$ is obtained from $f_+$ by
performing a Whitney move, followed by a finger move, in $S^4 \setminus f(S^2_2)$. Then, up to ambient isotopy in $S^4$ fixing $f(S^2_2)$, $g_+$ may be obtained from $f_+$ by performing a finger move, followed by a Whitney move, in $S^4 \setminus f(S^2_2)$.

**Proof.** By the hypotheses, there is an intermediate link map $f'$ such that $f'_- = f_-$. And a Whitney move performed in a 4-ball $B \subset S^4 \setminus f(S^2_2)$ changes $f_+$ to $f'_+$, and a finger move of $f'_+$ along a chord $\gamma \subset S^4 \setminus f(S^2_2)$ changes $f'_+$ to $g_+$.

If $\gamma$ is disjoint from $B$, then the lemma is immediate. Otherwise, we may assume that $\gamma$ intersects $B \setminus f'(S^2_2)$ along the interior of $\gamma$ in a collection of $n \geq 0$ properly embedded arcs. One may then homotop $\gamma$ in a collar of $\partial B \setminus f'(S^2_2)$ to a union $\gamma \cup_{i=1}^n \alpha_i$, of a simple chord $\gamma \subset S^4 \setminus f(S^2_2)$ on $f'(S^2_2)$ that intersects $B$ at precisely one point $p \in \partial B$, and $n$ simple loops $\{\alpha_i\}_{i=1}^n$ in $B \setminus f'(S^2_2)$ based at $p$. But as inclusion induces a surjection $\pi_1(\partial B \setminus f'(S^2_2), p) \to \pi_1(B \setminus f'(S^2_2), p)$, we may further deduce that $\gamma$ is path homotopic in $S^4 \setminus f(S^2_2)$ through chords on $f'(S^2_2)$ to a simple chord $\gamma'$ that misses $B$. Thus, by Lemma 2 there is an ambient isotopy in $S^4$ from $\gamma$ to $\gamma'$ that fixes $f'(S^2_2)$ and $f(S^2_2)$. \qed

We further require that, roughly speaking, a Whitney move of one component of a link map commutes with a finger move of the other. The proof is similar to that of Lemma 3 but we include it for completeness.

**Lemma 4.** Suppose that $f$ and $g$ are good link maps such that $g_+$ is obtained from $f_+$ by performing a Whitney move in $S^4 \setminus f(S^2_2)$ and $g_-$ is obtained from $f_-$ by performing a finger move in $S^4 \setminus g(S^2_2)$. Then (up to ambient isotopy in $S^4$ fixing $f(S^2_2)$) $g_-$ may be obtained from $f_-$ by performing a finger move in $S^4 \setminus f(S^2_2)$ and (up to ambient isotopy in $S^4$ fixing $g(S^2_2)$) $g_+$ may be obtained by performing a Whitney move of $f_+$ in $S^4 \setminus g(S^2_2)$.

**Proof.** Let $B$ be a 4-ball in $S^4 \setminus f(S^2_2)$ such that a Whitney move performed in $B$ changes $f_+$ to $g_+$, and let $\gamma$ be a simple chord in $S^4 \setminus g(S^2_2)$ on $f(S^2_2)$ such that a finger move of $f_-$ along $\gamma$ changes $f_-$ to $g_-$. If $\gamma$ is disjoint from $B$ then the lemma holds without the need for an additional isotopy (note that $f(S^2_2)$ and $g(S^2_2)$ coincide outside $B$). Otherwise, we may assume that $\gamma$ intersects $B \setminus g(S^2_2)$ along the interior of $\gamma$ in a finite collection of properly embedded arcs. Since inclusion induces a surjection $\pi_1(\partial B \setminus g(S^2_2)) \to \pi_1(B \setminus g(S^2_2))$, as in the proof of Lemma 3 one may path homotop $\gamma$ through chords on $g(S^2_2)$ to a simple chord that misses $B$. Now apply Lemma 2. \qed

We can now prove that if two good link maps are regularly link homotopic then they are connected by an abelian link homotopy.

**Lemma 5.** If $f$ and $g$ are regularly homotopic good link maps, then there is a regular homotopy from $f$ to $g$ consisting of a sequence of abelian link homotopies that alternately fix one component.

**Proof.** As in the proof of [5] Theorem 2.4], there is a regular homotopy taking $f$ to $g$ consisting of a sequence of regular homotopies that alternately fix one component. By Lemmas 3 and 4 this sequence can be chosen to first consist of
finger moves (and ambient isotopies) alternately fixing one component, carrying $f$ to a link map $f'$, then a sequence of Whitney moves (and ambient isotopies) alternately fixing one component, carrying $f'$ to $g$.

Now, finger moves and ambient isotopy preserve abelianess, so $f$ and $f'$ are connected by a sequence of abelian, regular link homotopies alternately fixing one component. On the other hand, $f'$ is obtained from $g$ by a sequence of finger moves (and ambient isotopies) alternatively fixing one component, so these link maps are also connected by abelian, regular link homotopies alternately fixing one component. □

Proof of Lemma 1. By Lemma 5, the link map $f$ is carried to $g$ by a sequence of abelian, regular link homotopies that alternately fix each component. Lemma 1 then follows from the following proposition, which can be deduced from the proof of [6, Proposition 4.2] (and which is expounded upon in [8, Satz 4.14]). □

Proposition 1. Let $h$ be a good link map with $\sigma_+(h) = 0$.

(i) If a good link map $h'$ is obtained from $h$ by performing a regular homotopy of $h_-$ in $S^1 \setminus h(S^2)$, then $\omega_+(h') = \omega_+(h)$. □

(ii) If a good link map $h'$ is obtained from $h$ by performing a regular homotopy of $h_-$ in $S^1 \setminus h(S^2)$, then $\omega_+(h') = \omega_+(h)$.

Remark. Part (i) of this proposition is essentially a special case of the proof in [9] that the $\tau$-invariant is well-defined, while part (ii) is unique in that the ambient manifold $X_-$, into which $h_+$ maps, is allowed to change.

3 Proof of Theorem 2

We begin with some preliminary definitions concerning algebraic intersections of immersed surfaces in 4-manifolds. The reader is referred to [2] for more details on the subject.

3.1 Intersection numbers in 4-manifolds

Suppose $A$ and $B$ are properly immersed, self-transverse 2-spheres or 2-disks in a 4-manifold $Y$. Suppose further that $A$ and $B$ are transverse and that each is equipped with a path (a whisker) connecting it to the basepoint of $Y$.

For an intersection point $x \in A \cap B$, let $\lambda(A, B)_x \in \pi_1(Y)$ denote the homotopy class of a loop that runs from the basepoint of $Y$ to $A$ along its whisker, then along $A$ to $x$, and back to the basepoint along $B$ and its whisker. Define $\text{sign}_{A, B}(x)$ to be 1 or $-1$ depending on whether or not, respectively, the orientations of $A$ and $B$ induce the orientation of $Y$ at $x$. The (algebraic) intersection “number” $\lambda(A, B)$ between $A$ and $B$ is then defined as the sum in the group ring $\mathbb{Z}[\pi_1(Y)]$ of $\text{sign}(x)\lambda(A, B)_x$ over all such intersection points. The value of $\lambda(A, B)$ is invariant under homotopy rel boundary of $A$ or $B$ ([2]), but depends on the choice of basepoint of $Y$ and the choices of whiskers and orientations.
The following two observations will be useful. If \( x, y \in A \cap B \), then the product of \( \pi_1(\Sigma) \)-elements \( \lambda(A, B)_x (\lambda(A, B)_y)^{-1} \) is represented by a loop that runs from the basepoint to \( A \) along its whisker, along \( A \) to \( x \), then along \( B \) to \( y \), and back to the basepoint along \( A \) and its whisker. Secondly, if \( D_A \subset A \) is a 2-disk that is equipped with the same whisker and oriented consistently with \( A \), then \( \lambda(A, B)_x = \lambda(D_A, B)_x \) and \( \text{sign}_{A,B}(x) = \text{sign}_{D_A,B}(x) \) for each \( x \in D_A \cap B \).

### 3.1.1 Surgering tori to 2-spheres

Suppose \( T \) is an embedded torus (or punctured torus, resp.) in \( Y \setminus \text{int} B \) and suppose there is a circle \( \delta_1 \subset T \) that is nullhomotopic in \( Y \). Choose an immersed 2-disk \( D \) in \( Y \) that is bound by \( \delta_1 \) and transverse to \( T \) and \( B \), and choose a normal vector field \( \phi \) to \( \delta_1 \) on \( T \). Let \( \delta'_1 \subset T \) denote a nearby push-off of \( \delta_1 \) along \( \phi \). Extend \( \phi \) over \( D \) and let \( D' \) denote a pushoff of \( D \) along \( \phi \), bound by \( \delta'_1 \) and which we may assume is also transverse to \( B \). If \( D \) is oriented and \( D' \) has the orientation induced as the pushoff, then intersections between \( B \) and \( D \cup D' \) occur as finitely many nearby pairs of points \( \{x_i, x'_i\}_{i=1}^n \), where \( x_i \in \text{int} D \) and \( x'_i \in \text{int} D' \) are of opposite sign. Thus, removing from \( T \) the interior of the annulus bound by \( \delta_1 \cup \delta'_1 \) and attaching \( D \cup D' \) yields an immersed 2-sphere (or 2-disk with boundary \( \partial T \), resp.) \( S \) in \( Y \) such that the intersections between \( B \) and \( S \) are transverse and occur precisely at the pairs of points \( \{x_i, x'_i\}_{i=1}^n \). Furthermore, the algebraic intersections between \( S \) and \( B \) may be calculated using the following lemma. Let \([\alpha]\) denote the class in \( \pi_1(\Sigma) \) of a based loop \( \alpha \) in \( Y \), let \( \overline{\gamma} \) denote the reverse of a path \( \gamma \), and let \(*\) denote composition of paths.

**Lemma 6.** Let \( \delta_2 \) be an oriented, simple circle on \( T \) that intersects each of \( \delta_1 \) and \( \delta'_1 \) exactly once, at points \( z \) and \( z' \) (respectively), and is tangent to \( \phi \) at \( z \). Let \( \iota \) be a path in \( Y \) from its basepoint to \( z \). If \( S \) and \( D \) are oriented consistently and both equipped with the whisker \( \iota \), then

\[
\lambda(S, B) = (1 - [\iota \ast \delta_2 \ast \overline{\iota}]) \lambda(D, B).
\]

**Proof.** For each \( i \), let \( \gamma_i \) be a path on \( D \) connecting \( z \) to \( x_i \) (that does not pass through any double points) and let \( \gamma'_i \) be its pushoff along \( \phi \), connecting \( z' \) to \( x'_i \). Let \( \beta_i \) be a path on \( B \) from \( x_i \) to \( x'_i \) (that does not pass through any double points), and let \( \delta_2 \) be the arc \( \delta_2 \cap S \), oriented to run from \( z' \) to \( z \). Then the product \( \lambda(S, B)_{x_i} (\lambda(S, B)_{x'_i})^{-1} \) is represented by the loop

\[
\iota \ast \gamma_i \ast \beta_i \ast \overline{\gamma'_i} \ast \delta_2 \ast \overline{\iota}.
\]

Homotoping \( S \) (rel boundary) by collapsing \( D' \) onto \( D \) except near its intersections with \( B \), one sees that the loop \( 1 \) is homotopic in \( Y \) to the loop \( \iota \ast \delta_2 \ast \overline{\iota} \). Thus, equipping \( D \) with the whisker \( \iota \) and the same orientation as \( S \), we have

\[
\lambda(S, B)_{x'_i} = [\iota \ast \delta_2 \ast \overline{\iota}] \lambda(S, B)_{x_i} = [\iota \ast \delta_2 \ast \overline{\iota}] \lambda(D, B)_{x_i},
\]
and \( \text{sign}_{S,B}(x'_i) = -\text{sign}_{D,B}(x'_i) \). Summing over all such pairs of intersections yields

\[
\lambda(S, B) = \sum_i \text{sign}_{S,B}(x_i)\lambda(S, B)_{x_i} + \text{sign}_{S,B}(x'_i)\lambda(S, B)_{x'_i}
\]

\[= \sum_i (1 - [\ell \ast \delta_2 * r]) \text{sign}_{D,B}(x_i)\lambda(D, B)_{x_i}
\]

\[= (1 - [\ell \ast \delta_2 * r])\lambda(D, B). \]

### 3.2 Unknotted immersions and link maps

Two immersions \( k_0, k_1 : S^2 \to \mathbb{R}^4 \) are said to be equivalent if there are orientation-preserving self-diffeomorphisms \( h \) of \( S^2 \) and \( H \) of \( \mathbb{R}^4 \), respectively, such that \( k_1 \circ h = H \circ k_0 \). Denote the standard embedding \( S^2 \subset \mathbb{R}^4 \) by \( u_0^0 \). By applying local cusp homotopies, \( d \) of positive sign and \( e \) of negative sign, to \( u_0^0 \), one obtains an unknotted immersion, denoted \( u_d^e : S^2 \to \mathbb{R}^4 \). Note that \( u_d^e \) is unique up to equivalence; we say that an immersion \( k : S^2 \to \mathbb{R}^4 \) (or its image) is unknotted if \( k \) is equivalent to \( u_d^e \) for some \( d, e \geq 0 \). See [3] for more details.

Identify \( S^4 = \mathbb{R}^4 \cup \{ \infty \} \).

**Lemma 7.** A link map \( f \) is link homotopic to a good link map \( g \) such that \( g(S^2) \) is unknotted in \( \mathbb{R}^4 \subset S^4 \).

**Proof.** As in the proof of Theorem 1, we may assume after a link homotopy that \( f \) is a good link map. By [3, Lemma 3], there is a family of disjoint chords attached to \( f(S^2) \) such that finger moves along them change \( f(S^2) \) into an unknotted immersion in \( \mathbb{R}^4 = S^4 \setminus \{ \infty \} \). As these chords may be assumed to miss \( f(S^2 + \) ), we have the required result.

For an immersed 2-sphere \( A \) in a 4-manifold \( Y \), let \( \omega_2(A) \in \mathbb{Z}_2 \) denote the second Stiefel Whitney number of the normal bundle of \( A \) in \( Y \). The results in [7] readily generalize to give the following.

**Lemma 8.** Suppose \( k : S^2 \to \mathbb{R}^4 \) is an unknotted, self-transverse immersion with \( d \) double points, and let \( Y \) denote the complement in \( S^4 \) of \( k(S^2) \). Then \( \pi_2(X_\perp) \) is a free \( \mathbb{Z}[\mathbb{Z}] \)-module on \( d \) generators, the Hurewicz map \( \pi_2(Y) \to H_2(Y) \) surjects and \( \omega_2(A) = 0 \) for any immersed 2-sphere \( A \) in \( Y \).

**Remark.** Indeed, the complement of an open tubular neighborhood of \( k(S^2) \) in \( S^4 \) has a handlebody decomposition consisting of one 0-handle, one 1-handle, and \( d \) zero-framed 2-handles attached along unknotted circles in \( S^3 \) which are nullhomotopic in the boundary of the union of the 0- and 1-handle.

Let \( k : S^2 \to Y \) be a self-transverse immersion and suppose \( p \) is a double point of \( k(S^2) \). An accessory circle for \( p \) is an (oriented) simple circle on \( k(S^2) \) that passes through exactly one double point, \( p \), and changes sheets there.
Lemma 9. Let \( f \) be a good link map such that \( f(S^2) \) is unknotted in \( \mathbb{R}^4 \subset S^4 \). Equip \( f(S^2) \) with a whisker in \( X^- \) and fix an identification of \( \pi_1(X^-) \) with \( \mathbb{Z}(s) \) so as to write \( \mathbb{Z}[\pi_1(X^-)] = \mathbb{Z}[s,s^{-1}] \). Label the double points of \( f(S^2) \) by \( \{p_i\}^d_{i=1} \) and choose an accessory circle \( \alpha_i \) for \( p_i \) for each \( 1 \leq i \leq d \).

Then \( \pi_2(X^-) \cong (\oplus_i \mathbb{Z})[s,s^{-1}] \) and there is a \( \mathbb{Z}[s,s^{-1}] \)-basis represented by self-transverse, immersed, whiskered 2-spheres \( \{A_i\}^d_{i=1} \) in \( X^- \) with the following properties. For each \( 1 \leq i \leq d \), there is an integer Laurent polynomial \( q_i \in \mathbb{Z}[s,s^{-1}] \) such that

\[
\lambda(f(S^2), A_i) = (1-s)^2 q_i(s)
\]

and \( q_i(1) = \text{lk}(f(S^2), \alpha_i) \). Moreover, if for any \( 1 \leq j \leq d \) the loop \( \alpha_j \) bounds a 2-disk in \( S^4 \) that intersects \( f(S^2) \) exactly once, then we may choose \( A_j \) so that

\[
\lambda(f(S^2), A_j) = (1-s)^2.
\]

Proof. For \( t_0, t'_0 \in \mathbb{R} \), \( t'_0 > t_0 \), let \( \mathbb{R}^3[t_0] \) denote the hyperplane of \( \mathbb{R}^4 \) whose fourth coordinate \( t \) is \( t_0 \), and let \( \mathbb{R}^3[t_0, t'_0] = \{(x,t) \in \mathbb{R}^4 : x \in \mathbb{R}^3, t_0 \leq t \leq t'_0 \} \).

Figure 1 gives a “moving picture” description of an immersed 2-disk \( U \) (appearing as an arc in each slice \( \mathbb{R}^3[t_0, t_0] \), \( t_0 \in [-1,1] \)) in a 4-ball \( N \subset \mathbb{R}^3[-1,1] \), with a single self-transverse double point \( p \subset \mathbb{R}^3[0] \). In this figure we have labeled a loop \( \alpha \subset \mathbb{R}^3[0] \) on \( U \) that changes sheets at \( p \) and bounds a 2-disk \( D \). For each \( 1 \leq i \leq d \), let \( \mathring{U}_i \) be a 2-disk on \( S^2 \) that contains the two preimages of the double point \( p_i \), and no other double point preimages. There is a diffeomorphism \( \Gamma_i \) of \( N \) onto a 4-ball neighborhood of \( p_i \) in \( S^4 \) that takes \( U \) to \( f(\mathring{U}_i) \), \( \alpha \) to \( \alpha_i \) and \( p \) to \( p_i \). Choose a 4-ball neighborhood \( N^+ \subset N \) of \( p \) so that the (smaller 4-ball) \( \Gamma_i(N^+) \) is disjoint from \( f(S^2) \). There is a torus \( T \) in \( N^+ \setminus U \) that

intersects \( D \) exactly once; see Figure 2. The torus appears as a cylinder in each of \( \mathbb{R}^3[-1] \) and \( \mathbb{R}^3[1] \), and appears as a pair of circles in \( \mathbb{R}^3[t_0] \) for \( t_0 \in (-1,1) \).

By Alexander duality the linking pairing

\[
H_2(X^-) \times H_1(f(S^2)) \to \mathbb{Z}
\]

defined by \( (R,v) \mapsto R \cdot \Upsilon \), where \( v = \partial \Upsilon \subset S^4 \), is nondegenerate. Thus, as the loops \( \{\alpha_i\} \) represent a basis for \( H_1(f(S^2)) \cong \mathbb{Z}^d \), we have that \( H_2(X^-) \cong \mathbb{Z}^d \) and (after orienting) the so-called linking tori \( \{T_i\} \), defined by \( T_i = \Gamma_i(T) \),
Figure 2

Figure 3

represent a basis. We proceed to apply the construction of (3.1.1) (twice, successively) to turn these tori into 2-spheres.

In Figure 2 we have illustrated an oriented circle $\delta$ on $T$ which intersects $\mathbb{R}^3[-1]$ and $\mathbb{R}^3[1]$ each in an arc, and appears as a pair of points in $\mathbb{R}^3[t_0]$ for $t_0 \in (-1, 1)$. Notice that $\delta$ is isotopic in $N^+ \setminus U$ to the circle $\hat{\delta} \subset \mathbb{R}^3[1]$ that is also illustrated in Figure 2. By attaching the trace of such an isotopy to the 2-disk $\hat{\Delta} \subset \mathbb{R}^3[1]$ illustrated, bound by $\hat{\delta}$, one may obtain an embedded 2-disk $\Delta \subset N^+$ that is bound by $\delta$ and intersects $U$ precisely where $\hat{\Delta}$ does. These intersection points are the endpoints of an arc $\gamma \subset \mathbb{R}^3[1]$ on $U$, shown in Figure 2. Let $\gamma_i = \Gamma_i(\gamma) \subset f(S^2)$. In Figure 3(a) we have illustrated in $\mathbb{R}^3[1]$ the restriction of a tubular neighborhood of $U$ to $\gamma$. Identifying this tubular neighborhood with $\gamma \times D^2$, we may assume the embedding $\Gamma_i$ carries $\gamma \times D^2$ onto the restriction over $\gamma_i$ of a tubular neighborhood of $f(S^2)$ that is disjoint from $f(S^2)$. Let $\Theta$ be the punctured torus in $N \setminus U$ given by

$$\Theta = \Delta \setminus (\partial \gamma \times \text{int} D^2) \cup_{\partial \gamma \times S^1} (\gamma \times S^1),$$

which has boundary $\delta$. Note that $\Gamma_i(\Theta)$ is disjoint from $f(S^2)$. Form a loop $\beta$ on $\mathbb{R}^3[1] \cap \Theta$ by connecting the endpoints of $\gamma \times \{1\}$ by an arc on $\hat{\Delta} \setminus (\partial \gamma \times \text{int} D^2)$. 
Since $\pi_1(X_-) \cong \mathbb{Z}$ and a loop of the form $\Gamma_i(\{b\} \times S^1)$ ($b \in \text{int } \gamma$) is meridinal to $f(S^2_+)$, by replacing $\gamma \times \{1\} \subset \beta$ by its band sum with oriented copies of $\{b\} \times S^1$ if necessary (see Figure 3(b)) we may assume that $\beta_i = \Gamma_i(\beta)$ is a simple circle bounding an immersed, self-transverse 2-disk $D_i$ in $X_-$ that is transverse to $f(S^2_+)$. Now, as $f(S^2_+)$ misses $\Gamma_i(N^+)$, the loop $\beta_i$ is freely homotopic in $S^4 \setminus f(S^2_+)$ to $\alpha_i$. Consequently,

$$|f(S^2_+ \cdot D_i| = |\text{lk}(f(S^2_+), \alpha_i)|$$

(2)

as non-negative integers. Let $\hat{A}_i$ be the immersed, self-transverse 2-disk in $X_-$ obtained by performing the construction of (3.1.1) with the (embedded) punctured torus $\Gamma_i(\Theta) \subset X_- \setminus f(S^2_+)$, $\beta_i \subset \Gamma_i(\Theta)$, $D_i$ and some choice of normal vector field to $\beta_i$ on $\Gamma_i(\Theta)$. Then, since a loop of the form $\Gamma_i(\{b\} \times S^1)$ ($b \in \text{int } \gamma$) is dual to $\beta_i$ on $\Gamma_i(\Theta)$ and hence represents a generator ($s$ or $s^{-1}$) of $\pi_1(X_-)$, by Equation (2) and Lemma 6 we have (after orienting $\hat{A}_i$ and connecting it to the basepoint of $X_-$)

$$\lambda(f(S^2_+, \hat{A}_i) = (1 - s)\hat{q}_i(s)$$

(3)

for some integer Laurent polynomial $\hat{q}_i \in \mathbb{Z}[s, s^{-1}]$ such that $\hat{q}_i(1) = \text{lk}(f(S^2_+), \alpha_i)$. Moreover, if $|f(S^2_+ \cap D_j| = 1$ for some $j$ then (since we are free to choose the orientation and whisker of $\hat{A}_j$) we may take $\hat{q}_j = 1$.

Now, for each $i$, $\hat{A}_i$ is bound by the circle $\Gamma_i(\delta)$ on the (embedded) linking torus $T_i \subset X_- \setminus f(S^2_+)$. Perform the construction of (3.1.1) with $T_i$, $\Gamma_i(\delta)$, $\hat{A}_i$ and some choice of normal vector field to $\Gamma_i(\delta)$ on $T_i$. Then, since $\delta$ has a dual curve on $T$ that is meridinal to $U$, by Equation (3) and Lemma 6 we have (after orienting $\hat{A}_i$ and connecting it to the basepoint of $X_-$)

$$\lambda(f(S^2_+, \hat{A}_i) = (1 - s)^2q_i(s)$$

for some $q_i \in \mathbb{Z}[s, s^{-1}]$ such that $q_i(1) = \text{lk}(f(S^2_+), \alpha_i)$. As above, if $|f(S^2_+ \cap D_j| = 1$ for some $j$ then we may take $q_j = 1$.

By construction, $\hat{A}_i$ is homologous to $T_i$ for each $i$, so by Lemma 8 the immersed 2-spheres $\{A_i\}_{i=1}^d$ represent a $\mathbb{Z}[s, s^{-1}]$-basis for $\pi_2(X_-)$. 

\[ \square \]

### 3.3 The invariant $\tau$ applied to link maps

In [9], the authors define a homotopy invariant $\tau$ which takes at input a map $k : S^2 \to Y^4$ with vanishing Wall self-intersection $\mu(k)$ and gives output in a quotient $\Pi(Y, k)$ of the group ring $\mathbb{Z}[\pi_1(Y) \times \pi_1(Y)]$ modulo certain relations. The relations are additively generated by the equations

$$(a, b) = -(b, a) \quad (R_1)$$

$$(a, b) = -(a^{-1}, ba^{-1}) \quad (R_2)$$

$$(a, 1) = (a, a) \quad (R_3)$$

$$(a, \lambda(k(S^2), A)) = (a, \omega_2(A) \cdot 1) \quad (R_4)$$

\[ \]
where \(a, b \in \pi_1(Y)\), \(A\) represents an immersed \(S^2\) or \(\mathbb{RP}^2\) in \(Y\) (in the latter case, the group element \(a\) is the image of the nontrivial element in \(\pi_1(\mathbb{RP}^2)\)).

Let \(f\) be a good link map with \(\sigma_+(f) = 0\) (from which it follows that \(\mu(f_+) = 0\)). For an integer \(k\), let \(k\) denote its image in \(\mathbb{Z}_2\). Letting \(\rho\) denote the mod 2 Hurewicz map \(\pi_1(X-) \to H_1(X-, \mathbb{Z}_2) = \mathbb{Z}_2\), define a ring homomorphism \(\varphi_f : \Pi(X-, f_+) \to \mathbb{Z}_2[t : t^2 = 1]\) by

\[
(a, b) \mapsto \rho(a) + \rho(a)\rho(b) + \rho(b),
\]

and extending linearly mod 2. We now prove a stronger form of Theorem 2.

**Lemma 10.** Let \(f\) be a link map with \(\sigma_+(f) = 0\). After a certain link homotopy of \(f\) we have that \(\varphi_f\) is an isomorphism and takes \(\tau(f_+)\) to \((1 + t)\omega_+(f)\).

**Proof.** By Lemma 7 we may assume \(f\) is a good link map (and so \(\mu(f_+) = \sigma_+(f) = 0\)) such that \(f(S^2)\) is unknotted. We may perform a finger move of \(f(S^2)\) along a chord attached in the complement of and meridional to \(f(S^2)\) so that a slice in \(\mathbb{R}^3[t_0]\) (for some \(t_0\)) of the result is illustrated in Figure 4. This produces a pair of oppositely-signed double points \(\{p^+, p^-\}\) on \(f(S^2)\) such that (in particular) \(p^+\) has an accessory circle bounding an obvious embedded 2-disk in \(\mathbb{R}^3[t_0]\) that intersects \(f(S^2)\) exactly once. Note that \(f(S^2)\) is still unknotted (by [3, Lemma 1]) and, in particular, its complement in \(S^4\) still has abelian fundamental group.

Now, fixing \(f_-\) and \(X_-\), by Lemma 9 we may thus identify \(\pi_1(X_-)\) with \(\mathbb{Z}(s)\) and \(\pi_2(X_-)\) with \(\bigoplus_{i=1}^d \mathbb{Z}[s, s^{-1}]\), for some \(d \geq 0\), such that there is an immersed, whiskered 2-sphere \(A_0\) in \(X_-\) with the property that \(\lambda(f(S^2), A_0) = (1 - s)^2\). Moreover, for any whiskered, immersed 2-sphere \(A\) in \(X_-\) we have \(\lambda(f(S^2), A) = (1 - s)^2q_A(s)\) for some integer Laurent polynomial \(q_A \in \mathbb{Z}[s, s^{-1}]\).

![Figure 4](image-url)

Therefore, if we identify \(\mathbb{Z}[\pi_1(X_-) \times \pi_1(X_-)]\) with \(\mathbb{Z}[s^{\pm 1}, t^{\pm 1}]\) via \((s^n, s^m) = s^nt^m\) (for \(n, m \in \mathbb{Z}\)), by Lemma 8 the ring \(\Pi(X_, f_+)\) is the quotient of the group
ring $\mathbb{Z}[s^\pm 1, t^\pm 1]$ modulo the relations generated additively by the equations:

\[
\begin{align*}
& s^n t^n - s^n = 0 \quad (T_1) \\
& s^n t^m + s^{-n} t^{m-n} = 0 \quad (T_2) \\
& s^n t^m + s^m t^n = 0 \quad (T_3) \\
& s^n t^m (1 - t)^2 = 0 \quad (T_4)
\end{align*}
\]

where $n, m \in \mathbb{Z}$. Note that in reformulating Relation ($R_4$) to obtain Relation ($T_4$) we have used the action of $\pi_1(X_-) = \mathbb{Z}\langle s \rangle$ on $\pi_2(X_-)$.

Let $\equiv$ denote equivalence in $\Pi(X_-, f_+)$, Clearly $\varphi_f$ is surjective; to show injectivity we first show that for any integers $n, m$, one has

\[s^n t^m \equiv t^n + nm + m. \quad (4)\]

By Relations ($T_2$) and ($T_3$) we have $2t^n \equiv 0$ and hence $2s^n \equiv -2t^n \equiv 0$ for each $n \in \mathbb{Z}$. Then Relation ($T_4$) implies that $t^{m+2} \equiv t^m$ for any integer $m$, and it follows by an induction argument that

\[t^m \equiv t^m. \quad (5)\]

Now, $s \equiv -t \equiv t$ and $st \equiv t$ by Relations ($T_1$)-($T_3$). Combining these equivalences with the consequence of Relation ($T_4$) that $st^{m+2} \equiv 2st^{m+1} - st^m$, an induction gives

\[st^m \equiv t \quad (6)\]

for any integer $m$.

Finally, fix $n_0 \in \mathbb{Z}$. By Relation ($T_3$) and Equivalences (5) and (6), we have $s^{n_0} = -t^{n_0} \equiv t^{n_0}$ and $s^{n_0} t = -s^{n_0} \equiv -t \equiv t$. Suppose now that for some $k \geq 1$ Equivalence (4) holds for $n = n_0$ and any $m \in \{0, 1, \ldots, k\}$. Then Relation ($T_4$) implies that

\[
s^{n_0} t^{k+1} = 2s^{n_0} t^k - s^{n_0} t^{k-1} = 2t^{n_0+n_0k+k} - t^{n_0+n_0(k-1)+(k-1)} \equiv t^{n_0+n_0(k+1)+(k+1)},
\]

On the other hand, suppose that for some $k \leq 0$ Equivalence (4) holds for $n = n_0$ and any $m \in \{k, k+1, \ldots, 0, 1\}$; then

\[
s^{n_0} t^{k-1} = 2s^{n_0} t^k - s^{n_0} t^{k+1} = 2t^{n_0+n_0k+k} - t^{n_0+n_0(k+1)+(k+1)} \equiv t^{n_0+n_0(k-1)+(k-1)}.
\]

Thus, by induction Equivalence (4) holds for $n = n_0$ and any integer $m$. But $n_0 \in \mathbb{Z}$ was arbitrary, so the equivalence holds for all integers $n, m$. As $2 \equiv
0 \equiv 2t$, we deduce that $\Pi(X_-, f_+)$ is the group ring $\mathbb{Z}_2(t : t^2 = 1)$ and $\varphi_f$ is injective.

Turning to the second part of the lemma, we refer the reader to [6] and [9] for detailed descriptions of the $\omega$ and $\tau$ invariants, respectively, and to [2] for background on framed Whitney disks. We make only a few summarizing remarks.

Since $\sigma_+(f) = 0$, $\pi_1(X_-) = \mathbb{Z}(s)$, and $f_+$ is self-transverse with vanishing signed sum of its double points, the double points of $f(S^2_+)$ may be decomposed into canceling pairs $\{p_i^+, p_i^\prime\}_{i=1}^k$ in the following sense. For each $1 \leq i \leq k$, one has $\text{sign}(p_i^+) = -\text{sign}(p_i^\prime)$ and the preimages of $p_i^\pm$ in $S^2_+$ may be labeled $\{x_i^\pm, y_i^\pm\}$ so that if $\gamma_i$ is an arc on $S^2_+$ connecting $x_i^+$ to $x_i^-$ (and missing all other double point preimages) and $\gamma_i'$ is an arc on $S^2_+$ connecting $y_i^+$ to $y_i^-$ (and missing $\gamma_i$ and all other double point preimages), then the loop $f(\gamma_i) \cup f(\gamma_i') \subset f(S^2_+)$ is nullhomotopic in $X_-$. The arcs $\{\gamma_i, \gamma_i'\}_{i=1}^k$ may be chosen so that the resulting Whitney circles $\{f(\gamma_i \cup \gamma_i')\}_{i=1}^k$ are mutually disjoint, simple circles in $X_-$ such that each bounds an immersed, framed Whitney disk $W_i$ in $X_-$ whose interior is transverse to $f(S^2_+)$. Let $\alpha_i^\pm$ be an arc on $S^2_+$ connecting $x_i^\pm$ to $y_i^\pm$, and let $n_i^\pm$ denote the integer $\text{lk}(f(S^2_+), f(\alpha_i^\pm))$; then $n_i^- = -n_i^+$. (In [6] the non-negative integer $|n_i^+|$ is called the $n$-multiplicity for the pair $\{p_i^+, p_i^\prime\}$, and in [9] the $\pi_1(X_-)$-element $s^{n_i^+}$ is called the primary group element for $W_i$.) Note that $\rho(s^{n_i^+})$ is the mod 2 image of $n_i^+$. Let $i \in \{1, 2, \ldots, k\}$ and suppose $x \in f(S^2_+) \cap \text{int} W_i$. A loop that first goes along $f(S^2_+)$ from its basepoint to $x$, then along $W_i$ to $f(\gamma_i') \subset \partial W_i$, then back along $f(S^2_+)$ to the basepoint of $f(S^2_+)$, determines a $\pi_1(X_-)$-element $s^{m_x}$ (called the secondary group element associated to $x$ in [9]; the non-negative integer $|m_x|$ is called the multiplicity of $x$ in [6]). Associate to $x$ a sign by orienting $W_i$ using the following convention: orient $\partial W_i$ from $p_i^-$ to $p_i^+$ along the $f(\gamma_i)$, then back to $p_i^-$ along $f(\gamma_i)$, the positive tangent to $\partial W_i$ together with an outward-pointing second vector then orient $W_i$. Let

$$J_x^i = n_i^+ + n_i^+ m_x + m_x \in \mathbb{Z}_2$$

and

$$I_x^i = \text{sign}(x) s^{n_i^+} t^{m_x} \in \mathbb{Z}[s^{\pm 1}, t^{\pm 1}].$$

Then Li’s $\mathbb{Z}_2$-valued $\omega_+$-invariant applied to $f$ is defined by

$$\omega_+(f) = \sum_{i=1}^k \sum_{x \in f(S^2_+) \cap \text{int} W_i} J_x^i \mod 2;$$

while, in this special case, the Schneiderman-Teichner invariant $\tau$ applied to $f_+$ is given by the $\mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$-sum

$$\tau(f_+) = \sum_{i=1}^k \sum_{x \in f(S^2_+) \cap \text{int} W_i} I_x^i$$

evaluated in the quotient $\Pi(X_-, f_+)$.
Now, \( \varphi_f(I_x) = t^{J_x} \), and consequently
\[
\varphi_f(\tau(f_+)) = \sum_{i=1}^{k} \sum_{x \in f(S_{2}^i) \cap \text{int } W_i} t^{J_x}.
\]

But \( \varphi(\tau(f_+)) \in \{0, 1, t, 1 + t\} \) must map forward to 0 under the homomorphism \( \Pi(X_-, f_+) \to \Pi(S^4, f_+) = \mathbb{Z}_2 \) induced by the inclusion \( X_- \subset S^4 \) and given by sending \( s, t \mapsto 1 \). Thus
\[
\varphi_f(\tau(f_+)) = \sum_{i=1}^{k} \sum_{x \in f(S_{2}^i) \cap \text{int } W_i} J_x^x \cdot (1 + t) \mod 2
\]
\[
= (1 + t) \omega_+(f).
\]

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