Neutrinos With $Z_3$ Symmetry and
New Charged-Lepton Interactions

Ernest Ma

Physics Department, University of California, Riverside, California 92521

Abstract

An approximate $Z_3$ family symmetry is proposed for leptons which results in a neutrino mass matrix with $\sin^2 2\theta_{atm} = 1$ and $\tan^2 \theta_{sol} = 0.5$, but the latter value could easily be smaller. A generic requirement of this approach is the appearance of three Higgs doublets at the electroweak scale, resulting in possibly observable flavor violating leptonic decays, such as $\mu \rightarrow eee$ and $\mu \rightarrow e\gamma$.

Talk at Neutrino Oscillations in Venice, December 2003.
1 Introduction

With the recent experimental progress in measuring atmospheric \[^{1}\] and solar \[^{2}\] neutrino oscillations, the mass-squared differences of the 3 active neutrinos and their mixing angles are now known with some precision. In the previous talk \[^{3}\], the mixing pattern resulting in $\sin^2 2\theta_{\text{atm}} = 1$ and $\tan^2 \theta_{\text{sol}} = 0.5$ was advocated. Here I show how this can be implemented in a complete theory of leptons, using the discrete family symmetry $Z_3 \times Z_2$, which is only broken spontaneously and by explicit soft terms in the Lagrangian.

Motivated by the idea that $\mathcal{M}_\nu$ should satisfy \[^{4, 5}\]

$$ U \mathcal{M}_\nu U^T = \mathcal{M}_\nu, \quad (1) $$

where $U$ is a specific unitary matrix, a very simple form of $\mathcal{M}_\nu$ is here proposed:

$$ \mathcal{M}_\nu = \mathcal{M}_A + \mathcal{M}_B + \mathcal{M}_C, \quad (2) $$

where

$$ \mathcal{M}_A = A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{M}_B = B \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathcal{M}_C = C \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (3) $$

This results in

$$ \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} \sqrt{2}/3 & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}, \quad (4) $$

with

$$ m_1 = A - B, \quad (5) $$

$$ m_2 = A - B + 3C, \quad (6) $$

$$ m_3 = A + B. \quad (7) $$

This explains atmospheric neutrino oscillations with $\sin^2 2\theta_{\text{atm}} = 1$ and solar neutrino oscillations with $\tan^2 \theta_{\text{sol}} = 1/2$. Whereas the mixing angles are fixed, the proposed $\mathcal{M}_\nu$ has the flexibility to accommodate the three patterns of neutrino masses often mentioned, i.e.
(I) the hierarchical solution, e.g. $B = A$ and $C << A$;

(II) the inverted hierarchical solution, e.g. $B = -A$ and $C << A$;

(III) the degenerate solution, e.g. $C << B << A$.

In all cases, $C$ must be small.

2 Relevance of $Z_3 \times Z_2$ Symmetry

In the above, the mixing matrix has $U_{e3} = 0$. This is the consequence of a symmetry, i.e. $\mathcal{M}_\nu$ of Eq. (2) is invariant under the $Z_2$ transformation [6]

$$U_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad U_2^2 = 1,$$ \hfill (8)

However, since $C$ is always small, the possible symmetry of $\mathcal{M}_A + \mathcal{M}_B$ should be considered as the dominant one. This turns out to be $Z_3$ [7], i.e.

$$U_B = \begin{pmatrix} -1/2 & -\sqrt{3}/8 & -\sqrt{3}/8 \\ \sqrt{3}/8 & 1/4 & -3/4 \\ \sqrt{3}/8 & -3/4 & 1/4 \end{pmatrix}, \quad U_B^3 = 1.$$ \hfill (9)

Note that $U_B$ commutes with $U_2$ and $\mathcal{M}_\nu = \mathcal{M}_A + \mathcal{M}_B$ is the most general solution of

$$U_B \mathcal{M}_\nu U_B^T = \mathcal{M}_\nu.$$ \hfill (10)

3 Origin of $\mathcal{M}_C$

Since $\mathcal{M}_C$ is small and breaks the symmetry of $\mathcal{M}_A + \mathcal{M}_B$, it is natural to think of its origin in terms of the well-known dimension-five operator [8]

$$\mathcal{L}_{\text{eff}} = \frac{f_{ij}}{2\Lambda}(\nu_i\phi^0 - l_i\phi^+)(\nu_j\phi^0 - l_j\phi^+) + H.c.,$$ \hfill (11)
where \((\phi^+, \phi^0)\) is the usual Higgs doublet of the Standard Model and \(\Lambda\) is a very high scale. As \(\phi^0\) picks up a nonzero vacuum expectation value \(v\), neutrino masses are generated, and if \(f_{ij}v^2/\Lambda = C\) for all \(i, j\), \(\mathcal{M}_C\) is obtained. Since \(\Lambda\) is presumably of order \(10^{16}\) to \(10^{18}\) GeV, \(C\) is of order \(10^{-3}\) to \(10^{-5}\) eV. This range of values is just right to encompass all three solutions mentioned above.

As for the form of \(\mathcal{M}_C\), it may be understood as coming from effective universal interactions among the leptons at the scale \(\Lambda\). For example, if Eq. (11) has an \(S_3\) symmetry as generated by \(U_2\) and \(U_3\)

\[
U_C = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}, \quad U_C^3 = 1,
\]

the most general form of \(\mathcal{M}_C\) would be

\[
\mathcal{M}_C = C \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + C' \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

However, the \(C'\) term can be absorbed into \(\mathcal{M}_A\), so again \(\mathcal{M}_\nu\) of Eq. (2) is obtained. This form of the neutrino mass matrix has in fact been discussed as an ansatz in a number of recent papers [9, 10, 11, 12, 13]. In particular, let it be rewritten as

\[
\mathcal{M}_\nu = (A + C) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - B \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + C \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + C' \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]

Note that each of the above four matrices is a group element of \(S_3\). This is the recent proposal of Harrison and Scott [12]. The difference here is that the underlying symmetry of \(\mathcal{M}_\nu\) has been identified, thus allowing a complete theory of leptons to be constructed.

4 Origin of \(\mathcal{M}_A + \mathcal{M}_B\)

To accommodate the proposed \(Z_3\) symmetry in a complete theory, the Standard Model of particle interactions is now extended [5] to include three scalar doublets \((\phi^0_i, \phi^-_i)\) and one
very heavy triplet $(\xi^+, \xi^+, \xi^0)$. The leptonic Yukawa Lagrangian is given by
\[
\mathcal{L}_Y = h_{ij} [\xi^0 \nu_i \nu_j - \xi^+(\nu_i l_j + l_i \nu_j)]/\sqrt{2} + \xi^{++} l_i l_j] + f^k_{ij} (l_i \phi^0_j - \nu_i \phi^+_j) l^c_k + H.c.,
\]
where, under the $Z_3$ transformation,
\[
(\nu, l)_i \rightarrow (U_B)_{ij} (\nu, l)_j, \quad l^c_k \rightarrow l^c_k,
\]
\[
(\phi^0, \phi^-)_i \rightarrow (U_B)_{ij} (\phi^0, \phi^-)_j, \quad (\xi^{++}, \xi^+, \xi^0) \rightarrow (\xi^{++}, \xi^+, \xi^0).
\]

This means
\[
U_B^T h U_B = h, \quad U_B^T f^k U_B = f^k,
\]
resulting in
\[
h = \begin{pmatrix}
a - b & 0 & 0 \\
0 & a & -b \\
0 & -b & a
\end{pmatrix}, \quad f^k = \begin{pmatrix}
a_k - b_k & d_k & d_k \\
-d_k & a_k & -b_k \\
-d_k & -b_k & a_k
\end{pmatrix}.
\]

Note that $h$ has no $d$ terms because it has to be symmetric. Note also that both $h$ and $f$ are invariant under $U_2$ of Eq. (8). The neutrino mass matrix $\mathcal{M}_A + \mathcal{M}_B$ is obtained with $A = 2a\langle \xi^0 \rangle$ and $B = 2b\langle \xi^0 \rangle$. The fact that it is proportional to a single vacuum expectation value is important for the preservation of the $Z_3$ symmetry. As for the smallness of $\langle \xi^0 \rangle$, it is fully explained \[14\] [15] as the analog of the canonical seesaw mechanism in the case of very large and positive $m^2_{\xi}$.

The $Z_3$ symmetry is broken by the soft terms of the Higgs sector, thus $v_{1,2} << v_3$ may be assumed. If $d_k, b_k << a_k$ is also assumed (which by itself does not break $Z_3$), then the hierarchy of $m_e, m_\mu, m_\tau$ is understood. The mixing matrix $V_L$ in the $l_i$ basis (such that $V_L \mathcal{M}_i V_L^\dagger$ is diagonal) will be very close to the identity matrix with off-diagonal terms of order $m_e/m_\mu$, $m_e/m_\tau$, and $m_\mu/m_\tau$. This construction allows $\mathcal{M}_\nu$ of Eq. (2) to be in the $(\nu_e, \nu_\mu, \nu_\tau)$ basis as a very good approximation.
5 Flavor Violating Leptonic Decays

The Yukawa couplings of the three Higgs doublets are such that the dominant coupling of \( \phi_0^1 \) is \( (m_\tau/v_3)e\tau^c \) and that of \( \phi_0^2 \) is \( (m_\tau/v_3)\mu\tau^c \). Other couplings are at most of order \( m_\mu/v_3 \) in this model, and some are only of order \( m_e/v_3 \). The smallness of flavor changing decays in the leptonic sector is thus guaranteed, even though they should be present and may be observable in the future. The decays \( \tau^- \to e^-e^+e^- \) and \( \tau^- \to e^-e^+\mu^- \) may proceed through \( \phi_0^1 \) exchange with coupling strengths of order \( m_\mu m_\tau/v_3^2 \approx (g^2/2)(m_\mu m_\tau/M_W^2) \), whereas the decays \( \tau^- \to \mu^-\mu^+\mu^- \) and \( \tau^- \to \mu^-\mu^+e^- \) may proceed through \( \phi_0^2 \) exchange also with coupling strengths of the same order. We estimate the order of magnitude of these branching fractions to be

\[
B \sim \left( \frac{m_\mu^2 m_\tau^2}{m_{1,2}^4} \right) B(\tau \to \mu\nu\nu) \simeq 6.1 \times 10^{-11} \left( \frac{100 \text{ GeV}}{m_{1,2}} \right)^4, \tag{20}
\]

which is well below the present experimental upper bound of the order \( 10^{-6} \) for all such rare decays \[16\].

The decay \( \mu^- \to e^-e^+e^- \) may also proceed through \( \phi_0^1 \) with a coupling strength of order \( m_\mu^2/v_3^2 \). Thus

\[
B(\mu \to eee) \sim \frac{m_\mu^4}{m_1^4} \simeq 1.2 \times 10^{-12} \left( \frac{100 \text{ GeV}}{m_1} \right)^4, \tag{21}
\]

which is at the level of the present experimental upper bound of \( 1.0 \times 10^{-12} \). The decay \( \mu \to e\gamma \) may also proceed through \( \phi_0^2 \) exchange (provided that \( \text{Re}\phi_0^2 \) and \( \text{Im}\phi_0^2 \) have different masses) with a coupling of order \( m_\mu m_\tau/v_3^2 \). Its branching fraction is given by \[17\]

\[
B(\mu \to e\gamma) \sim \frac{3\alpha}{8\pi} \frac{m_\tau^4}{m_{\text{eff}}^4}, \tag{22}
\]

where

\[
\frac{1}{m_{\text{eff}}^2} = \frac{1}{m_{2R}^2} \left( \ln \frac{m_{2R}^2}{m_\tau^2} - \frac{3}{2} \right) - \frac{1}{m_{2L}^2} \left( \ln \frac{m_{2L}^2}{m_\tau^2} - \frac{3}{2} \right). \tag{23}
\]

Using the experimental upper bound \[18\] of \( 1.2 \times 10^{-11} \), we find \( m_{\text{eff}} > 164 \text{ GeV} \).
Once $\phi_1^0$ or $\phi_2^0$ is produced, its dominant decay will be to $\tau^\pm e^\mp$ or $\tau^\pm \mu^\mp$ if each couples only to leptons. If they also couple to quarks (and are sufficiently heavy), then the dominant decay products will be $t \bar{u}$ or $t \bar{c}$ together with their conjugates. As for $\phi_3^0$, it will behave very much as the single Higgs doublet of the Standard Model, with mostly diagonal couplings to fermions. It should also be identified with the $\phi$ of Eq. (15).

6 Consequences of an Arbitrary $\mathcal{M}_C$

It should also be noted that as long as $\mathcal{M}_C$ is small, its exact form is not that important. Let

$$\mathcal{M}_C = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix},$$

then it is easily shown that

$$\tan^2 \theta_{sol} = \left( \frac{1 - \sqrt{1 + z^2}}{z} \right)^2,$$

where

$$z = \frac{2\sqrt{2}(d + e)}{2(f - a) + b + c}.$$  \hspace{1cm} (26)

For $z = 2\sqrt{2}$, $\tan^2 \theta_{sol} = 0.5$ is recovered. If $z = 2.2$ instead, $\tan^2 \theta_{sol} = 0.42$ is obtained, which is the central value of this parameter in a recent global fit \cite{19} of all the data. Also, $U_{e3}$ becomes nonzero in general and is given by

$$U_{e3} \simeq \frac{d - e}{2\sqrt{2}B}.$$ \hspace{1cm} (27)

7 Conclusion

If a symmetry is indeed responsible for the observed pattern of neutrino masses and mixing, then $Z_3 \times Z_2$ is a very good candidate. It explains the dominant structure of $\mathcal{M}_\nu$, i.e.
\( \mathcal{M}_A \) and \( \mathcal{M}_B \) of Eq. (3), which are proportional to the \( vev \) of a single Higgs triplet. The remainder, i.e., \( \mathcal{M}_C \) of Eq. (3) or (24), should be considered as a perturbation from another source of neutrino mass, such as the effective dimension-five operator of Eq. (11). Arbitrary charged-lepton mass may be accommodated and flavor violating leptonic decays such as \( \mu \rightarrow eee \) and \( \nu \rightarrow e\gamma \) are predicted to be in the observable range.

8 Acknowledgements

I thank Milla Baldo Ceolin for her great hospitality in Venice. This work was supported in part by the U. S. Department of Energy under Grant No. DE-FG03-94ER40837.

References

[1] C. K. Jung, C. McGrew, T. Kajita, and T. Mann, Ann. Rev. Nucl. Part. Sci. 51, 451 (2001).

[2] Q. R. Ahmad et al., SNO Collaboration, Phys. Rev. Lett. 89, 011301, 011302 (2002); \texttt{nucl-ex/0309004} K. Eguchi et al., KamLAND Collaboration, Phys. Rev. Lett. 90, 021802 (2003).

[3] W. G. Scott, these proceedings.

[4] E. Ma, Phys. Rev. Lett. 90, 221802 (2003).

[5] E. Ma and G. Rajasekaran, Phys. Rev. D68, 071302(R) (2003).

[6] E. Ma, Phys. Rev. D66, 117301 (2002).

[7] E. Ma, \texttt{hep-ph/0308282}

[8] S. Weinberg, Phys. Rev. Lett. 43, 1566 (1979).
[9] P. F. Harrison, D. H. Perkins, and W. G. Scott, Phys. Lett. B530, 167 (2002).

[10] Z.-Z. Xing, Phys. Lett. B533, 85 (2002).

[11] P. F. Harrison and W. G. Scott, Phys. Lett. B535, 163 (2002).

[12] P. F. Harrison and W. G. Scott, Phys. Lett. B557, 76 (2003).

[13] X.-G. He and A. Zee, Phys. Lett. B560, 87 (2003).

[14] E. Ma and U. Sarkar, Phys. Rev. Lett. 80, 5716 (1998).

[15] E. Ma, Phys. Rev. Lett. 81, 1171 (1998).

[16] K. Hagiwara et al., Particle Data Group, Phys. Rev. D66, 011501 (2002).

[17] E. Ma and G. Rajasekaran, Phys. Rev. D64, 113012 (2001).

[18] M. L. Brooks et al., Phys. Rev. Lett. 83, 1521 (1999).

[19] P. Aliani et al., hep-ph/0309156