THE CENTRAL LIMIT THEOREM FOR FUNCTION SYSTEMS ON THE CIRCLE

TOMASZ SZAREK AND ANNA ZDUNIK

Abstract. The Central Limit Theorem for Iterated Functions Systems on the circle is proved. We study also ergodicity of such systems.

1. Introduction

In this paper we deal with iterated function systems generated by finite families of homeomorphisms of the circle.

Our main goals are the following: first, to prove the Central Limit Theorem (CLT) for Lipschitz continuous observables, and the Markov process generated by the IFS. This is done under natural mild assumptions, i.e, minimality of the action of the corresponding semigroup on the circle. No additional regularity of the maps is required. In this way, we answer the question which was left open in our previous paper [12]. The proof is based on the result due to Maxwell and Woodroofe (see [9]) which provides a sufficient condition for the Central Limit Theorem for an arbitrary stationary Markov chain. It is worth mentioning here that our considerations allow us to show the CLT for IFS’s starting at an arbitrary initial distribution. Similar result has been obtained recently by Komorowski and Walczuk (see [7]) but developed techniques allow them to consider only Markov chains satisfying spectral gap property in the Wasserstein metric.

Our second purpose is to provide some insights into Markov operators with the e-property. The e–property is a very useful tool in studying ergodic properties of Markov operators and semigroups of Markov operators. It was introduced to deal with Stochastic Partial Differential Equations in infinite dimensional Hilbert spaces (see for instance [6]) but it is also very helpful while studying IFS’s.

In Sections 3 and 4 we show how this property can be easily verified and then used to provide alternative proofs of some known results: ergodicity and asymptotic stability of the iterated function systems, again, under natural mild assumptions.

2000 Mathematics Subject Classification. Primary 60F05, 60J25; Secondary 37A25, 76N10.

Key words and phrases. Iterated Function Systems, Markov operators, invariant measures, central limit theorem.

The research partially supported by the Polish NCN grants 2016/21/B/ST1/00033 (Tomasz Szarek) and 2014/13/B/ST1/04551 (Anna Zdunik).
2. Notation and basic information about Markov operators.

Since we shall deal with systems on the circle, we restrict this short presentation to the caes of compact metric spaces. The general theory is developed for Polish spaces.

Let \((S, d)\) be a compact metric space. By \(\mathcal{M}_1(S)\) we denote the set of all probability measures on the \(\sigma\)-algebra of Borel sets \(\mathcal{B}(S)\). By \(C(S)\) we denote the family of all continuous functions equipped with the supremum norm \(\| \cdot \|\) and by \(\text{Lip}(S)\) we denote the family of all Lipschitz functions. For \(f \in \text{Lip}(S)\) by \(\text{Lip} f\) we denote its Lipschitz constant.

For brevity we shall use the notion of scalar product:

\[
\langle f, \mu \rangle := \int_S f(x) \mu(dx)
\]

for any bounded Borel measurable function \(f : S \to \mathbb{R}\) and \(\mu \in \mathcal{M}_1(S)\).

For any \(\mu, \nu \in \mathcal{M}_1(S)\) we define the Wasserstein distance by the formula

\[
W_1(\mu, \nu) = \sup \{|\langle f, \mu \rangle - \langle f, \nu \rangle| : \text{Lip} f \leq 1\}.
\]

An operator \(P : \mathcal{M}(S) \to \mathcal{M}(S)\) is called a Markov operator if it satisfies the following two conditions:

1) positive linearity: \(P(\lambda_1 \mu_1 + \lambda_2 \mu_2) = \lambda_1 P \mu_1 + \lambda_2 P \mu_2\) for \(\lambda_1, \lambda_2 \geq 0\); \(\mu_1, \mu_2 \in \mathcal{M}(S)\);
2) preservation of the norm: \(P \mu(S) = \mu(S)\) for \(\mu \in \mathcal{M}(S)\).

A Markov operator \(P\) is called a Feller operator if there is a linear operator \(U : C(S) \to C(S)\) such that

\[
\int_S U f(x) \mu(dx) = \int_S f(x) P \mu(dx) \quad \text{for } f \in C(S), \mu \in \mathcal{M}.
\]

A Markov operator \(P : \mathcal{M}(S) \to \mathcal{M}(S)\) is called nonexpansive (with respect to the Wasserstein metric) if

\[
W_1(P \mu, P \nu) \leq W_1(\mu, \nu) \quad \text{for any } \mu, \nu \in \mathcal{M}_1(S).
\]

A measure \(\mu_\ast\) is called invariant if \(P \mu_\ast = \mu_\ast\). An operator \(P\) is called asymptotically stable if it has a unique invariant measure \(\mu_\ast \in \mathcal{M}_1(S)\) such that the sequence \((P^n \mu)_{n \in \mathbb{N}}\) converges in the \(\ast\)-weak topology to \(\mu_\ast\) for any \(\mu \in \mathcal{M}_1(S)\), i.e.,

\[
\lim_{n \to \infty} \int_S f(x) P^n \mu(dx) = \int_S f(x) \mu_\ast(dx)
\]

for any \(f \in C(S)\).

For any Markov operator \(P\) we define the the multifunction \(\mathcal{P} : 2^S \to 2^S\) by the formula

\[
\mathcal{P}(A) = \bigcup_{x \in A} \text{supp} P \delta_x \quad \text{for } A \subset S.
\]
THE CENTRAL LIMIT THEOREM FOR FUNCTION SYSTEMS ON THE CIRCLE

3. E-property

The e–property seems to be a very useful tool in studying ergodic properties of Markov operators and semigroups of Markov operators on Polish spaces. Following [6], we say that a Feller operator $P$ satisfies the e–property if for any $x \in S$ and a Lipschitz function $f \in C(S)$ we have

$$\lim_{y \to x} \sup_{n \in \mathbb{N}} |U^n f(y) - U^n f(x)| = 0,$$

i.e. if the family of iterates $\{U^n f : n \in \mathbb{N}\}$ is equicontinuous.

**Proposition 1.** Let $P$ be a Feller operator. If $P$ satisfies the e–property, then

$$\text{supp } \mu \cap \text{supp } \nu = \emptyset$$

for any different ergodic invariant measures $\mu, \nu \in \mathcal{M}(S)$.

**Proof** The proof may be derived from [3] (see also [5, 6]). Indeed, in Lemma 3.4 we proved that if $x \in \text{supp } \mu$, where $\mu \in \mathcal{M}_1(S)$ is an ergodic invariant measure, then the sequence $(n^{-1} \sum_{k=1}^n P^n \delta_x)_{n \geq 1}$ converges in the $*$–weak topology to $\mu$. Hence our assertion follows immediately.

D. Worm slightly generalized the e–property introducing the Cesàro e–property (see [13]). Namely, a Feller operator $P$ will satisfy the Cesàro e–property at $x \in S$ if for any Lipschitz function $f \in C(S)$ we have

$$\lim_{y \to x} \sup_{n \in \mathbb{N}} \left| \frac{1}{n} \sum_{k=1}^n U^k f(y) - \frac{1}{n} \sum_{k=1}^n U^k f(x) \right| = 0.$$

For Feller operators with the Cesàro e–property the following proposition holds. Its proof is the same as the proof of Proposition 2 in [5].

**Proposition 2.** Let $(S, d)$ be a compact metric space and let $P$ be a Feller operator. Assume that there exists an open subset $S_0 \subset S$ such that $P(S_0) \subset S_0$ and $\mu(S_0) = 1$ for any invariant measure $\mu \in \mathcal{M}_1(S)$. If $P$ satisfies the Cesàro e–property at any point $x \in S_0$, then for any ergodic invariant measure $\mu_* \in \mathcal{M}_1$ and every $x \in S_0 \cap \text{supp } \mu_*$ the sequence $(\frac{1}{n} \sum_{k=1}^n P^n \delta_x)_{n \geq 1}$ converges weakly to $\mu_*$.

4. Ergodicity for iterated function systems on the circle

Iteration of homeomorphisms on the circle has been widely studied recently. For further references see [1, 2, 11, 12] and the references therein. The main purpose of this section is to prove that Markov operators corresponding to iterated function systems on the circle have strong metric properties, i.e. nonexpansiveness, the e–property and Cesàro e–property. These properties imply straightforwardly the ergodic properties of the systems. In this way we may easily derive ergodicity under the most general condition on the system (see [8]).

Let $S^1$ denote the circle with the counterclockwise orientation. We will denote by $[x, y]$ the closed interval form $x$ to $y$ according to this orientation. The distance between $x, y \in S^1$
is the shorter of the lengths of the intervals \([x, y]\) and \([y, x]\). We will denote this distance by \(d(x, y)\).

By \(H^+\) we shall denote the set of all orientation preserving circle homeomorphisms. Let \(\Gamma = \{g_1, \ldots, g_k\} \subset H^+\) be a finite collection of homeomorphisms. Put \(\Sigma_n = \{1, \ldots, k\}^n\), and let \(\Sigma_* = \bigcup_{n=1}^{\infty} \Sigma_n\) be the collection of all finite words with entries from \(\{1, \ldots, k\}\). For a sequence \(i \in \Sigma_*, i = (i_1, \ldots, i_n)\), we denote by \(|i|\) its length (equal to \(n\)).

We consider the action of the semigroup generated by \(\Gamma\), i.e., the action of all compositions \(g_i = g_{i_n} \circ \cdots \circ g_{i_1}\), where \(i = (i_1, \ldots, i_n) \in \Sigma_*\).

**Definition 3.** The orbit of a point \(x \in S^1\) is the set
\[
\mathcal{O}(x) = \{g_i(x) : i \in \Sigma_*\}.
\]

In the case when all the orbits are dense the action of \(\Gamma\) is called minimal. Equivalently, the action of \(\Gamma\) is minimal if for every \(\Gamma\)-invariant closed subset \(A \subset S^1\) either \(A = \emptyset\) or \(A = S^1\).

Let \((p_1, \ldots, p_k)\) be a probability distribution on \(\{1, \ldots, k\}\). Clearly, it defines a probability distribution \(p\) on \(\Gamma\), by putting \(p(g_j) = p_j\). We assume that all \(p_i\)'s are strictly positive. The pair \((\Gamma, p)\) will be called an iterated function system.

The Markov operator \(P : \mathcal{M}(S^1) \to \mathcal{M}(S^1)\) of the form
\[
P\mu = \sum_{g \in \Gamma} p(g)\mu \circ g^{-1},
\]
where \(\mu \circ g^{-1}(A) = \mu(g^{-1}(A))\) for \(A \in \mathcal{B}(S^1)\), describes the evolution of distribution due to action of randomly chosen homeomorphisms from the collection \(\Gamma\). It is a Feller operator, i.e., the operator \(U : C(S^1) \to C(S^1)\) given by the formula
\[
Uf(x) = \sum_{g \in \Gamma} p(g)f(g(x)) \quad \text{for } f \in C(S^1) \text{ and } x \in S^1
\]
is its dual. We shall illustrate usefulness of the notion of e-property, providing a very simple proof of the following:

**Proposition 4.** Let \(\Gamma^{-1} = \{g_1^{-1}, \ldots, g_k^{-1}\}\) act minimally and let \((p_1, \ldots, p_k)\) be a probability distribution on \(\{1, \ldots, k\}\). Then the operator \(P\) corresponding to the iterated function system \((\Gamma, p)\) satisfies the e-property. Moreover, \(P\) admits a unique invariant measure.

**Proof.** Let \(\tilde{\mu} \in \mathcal{M}_1\) be an arbitrary invariant measure for the iterated function system \((\Gamma^{-1}, p)\). Since \(\Gamma^{-1}\) acts minimally, the support of \(\tilde{\mu}\) equals \(S^1\). We easily check that \(\tilde{\mu}\) has no atoms. To do this take the atom \(u\) with maximal measure. From the fact that \(\tilde{\mu}\) is invariant for \(P\) we obtain that \(F = \{v \in S^1 : \tilde{\mu}(\{v\}) = \tilde{\mu}(\{u\})\}\) is invariant for \(\Gamma\) and consequently it is also invariant for \(\Gamma^{-1}\), i.e. \(g_i(F) = F\) for \(i = 1, \ldots, k\) and \(g_i^{-1}(F) = F\). This contradicts the assumption that \(\Gamma^{-1}\) acts minimally. Indeed, from the fact that
\[
\tilde{\mu}(\{v\}) = \sum_{i=1}^{k} p_i\tilde{\mu}(\{g_i(v)\}),
\]
we obtain that \( \tilde{\mu}(\{g_i(v)\}) = \tilde{\mu}(\{v\}) \) for all \( i = 1, \ldots, k \). Since \( g_i \) are homeomorphisms and the set \( F \) is finite we obtain that \( g_i(F) = F \) for \( i = 1, \ldots, k \) and \( g_i^{-1}(F) = F \), which is impossible, since the action of \( \Gamma^{-1} \) is minimal.

Define the function \( \chi : S^1 \times S^1 \to \mathbb{R}_+ \) by the formula

\[
\chi(x, y) = \min(\tilde{\mu}([x, y]), \tilde{\mu}([y, x])) \quad \text{for} \ x, y \in S^1.
\]

It is straightforward to check that \( \chi \) is a metric and convergence in \( \chi \) is equivalent to the convergence in \( d \).

Further, we may check that for any function \( f : S^1 \to \mathbb{R} \) satisfying \( |f(x) - f(y)| \leq \chi(x, y) \) for \( x, y \in S^1 \) we have

\[
|Uf(x) - Uf(y)| \leq \chi(x, y) \quad \text{for} \ x, y \in S^1.
\]

This follows from the definition of the operator \( U \) and the fact that \( |f(x) - f(y)| \leq \tilde{\mu}([x, y]) \) and \( |f(x) - f(y)| \leq \tilde{\mu}([y, x]) \)

\[
|Uf(x) - Uf(y)| \leq \sum_{g \in \Gamma} p(g)|f(g(x)) - f(g(y))| \leq \sum_{g \in \Gamma} p(g)\tilde{\mu}([g(x), g(y)]) = \tilde{\mu}([x, y])
\]

and analogously

\[
|Uf(x) - Uf(y)| \leq \sum_{g \in \Gamma} p(g)|f(g(x)) - f(g(y))| \leq \sum_{g \in \Gamma} p(g)\mu([g(y), g(x)]) = \mu([y, x])
\]

and hence

\[
|Uf(x) - Uf(y)| \leq \chi(x, y)
\]

for any \( x, y \in S^1 \). This finishes the proof of the e-property of the operator \( P \).

To complete the proof of our theorem we would like to apply Proposition \[2\]. Therefore we have to check that \( \text{supp} \mu \cap \text{supp} \nu \neq \emptyset \) for any \( \mu, \nu \in M_1(S^1) \) invariant for \( P \). Assume, contrary to our claim, that \( \text{supp} \mu \cap \text{supp} \nu = \emptyset \). Take the set \( \Lambda \) of all intervals \( I \subset S^1 \setminus (\text{supp} \mu \cup \text{supp} \nu) \) such that one of its ends belongs to \( \text{supp} \mu \) but the second to \( \text{supp} \nu \). Observe that \( \tilde{\mu}(I) > 0 \) for all \( I \in \Lambda \) and that there exists \( I_0 \) such that \( \tilde{\mu}(I_0) = \inf_{I \in \Lambda} \tilde{\mu}(I) \). We easily see that \( g(I_0) \in \Lambda \). Indeed, we have

\[
\tilde{\mu}(I_0) = \sum_{g \in \Gamma} p(g)\tilde{\mu}(g(I_0))
\]

and since the interval \( g(I_0) \) has the ends belonging to \( \text{supp} \mu \) and to \( \text{supp} \nu \), for \( I_0 \) had and \( g(\text{supp} \mu) \subset \text{supp} \mu \) and \( g(\text{supp} \nu) \subset \text{supp} \nu \) for all \( g \in \Gamma \), we get that there is \( J \in \Lambda \) such that \( J \subset g(I_0) \). Hence \( \tilde{\mu}(g(I_0)) \geq \tilde{\mu}(J) \geq \tilde{\mu}(I_0) \) and from equation \([1]\) it follows that \( \tilde{\mu}(g(I_0)) = \tilde{\mu}(I_0) \) and consequently \( g(I_0) \in \Lambda \). Otherwise there would be \( h \in \Gamma \) and \( J \subset h(I_0) \), \( J \in \Lambda \) and \( \tilde{\mu}(J) < \tilde{\mu}(h(I_0)) \leq \tilde{\mu}(I_0) \), by the fact that \( \text{supp} \tilde{\mu} = S^1 \), contrary to the definition of \( I_0 \). Finally, observe that the set

\[
\mathcal{H} = \{ J \in \Lambda : \tilde{\mu}(J) = \tilde{\mu}(I_0) \}
\]

is finite and all its elements are disjoint open intervals. Further \( g(\mathcal{H}) \subset \mathcal{H} \) and consequently \( g(\mathcal{H}) = \mathcal{H} \) for any \( g \in \Gamma \), by the fact that \( g \) is a homeomorphism. Consequently, for any \( g \) and the finite set \( F \) of all ends of the intervals \( J \) from \( \mathcal{H} \) we have \( g(F) \subset F \) and therefore
$g(F) = F$. Hence $g^{-1}(F) = F$ for any $g \in \Gamma$ and consequently $\Gamma^{-1}$ is not minimal, contrary to our assumption.

The following theorem was proved in [8], with the proof involving a generalization of Lyapunov exponents. We want to provide a very simple argument, based only on the (independently proved) e-property.

**Theorem 5.** If $\Gamma = \{g_1, \ldots, g_k\}$ acts minimally, then for any probability vector $(p_1, \ldots, p_k)$ the iterated function system $(\Gamma, p)$ admits a unique invariant measure.

**Proof.** The iterated function system $(\Gamma^{-1}, p)$ satisfies the e-property. Denote by $\tilde{P}$ and $\tilde{U}$ the Markov operator and the dual operator corresponding to $(\Gamma^{-1}, p)$, respectively. From the proof of the previous proposition it follows that the hypothesis holds provided the unique invariant measure $\tilde{\mu}$ for $(\Gamma^{-1}, p)$ satisfies the condition $\text{supp } \tilde{\mu} = S^1$.

Now assume that $S^1 \setminus \text{supp } \tilde{\mu} \neq \emptyset$. Let $(a, b) \subset S^1 \setminus \text{supp } \tilde{\mu}$. Set

$$S_0 = \bigcup_{n=1}^{\infty} \bigcup_{\{i_1, \ldots, i_n\} \in \Sigma_n} g_{i_1 \cdots i_n}((a, b))$$

and observe that $S_0$ is open and dense in $S^1$, by the minimal action of $\Gamma$. Let $\mu_s \in \mathcal{M}_1$ be an ergodic invariant measure for $(\Gamma, p)$. Since $\text{supp } \mu_s = S^1$ and $g_i(S_0) \subset S_0$ for any $i = 1, \ldots, k$ we have $\mu_s(S_0) > 0$ and $U1_{S_0} = 1_{S_0}$. Thus $\mu_s(S_0) = 1$, by ergodicity of $\mu_s$. We are going to apply Proposition 2 therefore we have to check that the Cesàro e-property holds at any $x \in S_0$. To do this fix $x \in S_0$ and $\varepsilon > 0$. Let $I \subset S_0$ be an open neighbourhood of $x$. Let $f : S^1 \rightarrow \mathbb{R}$ be a Lipschitz function with the Lipschitz constant $L$. Choose a finite set $\{x_0, \ldots, x_N\} \subset S^1$ such that $|x_{i-1}, x_i| < \varepsilon/L$. Since $\frac{1}{n} \sum_{k=1}^{n} \tilde{P}^k \delta_x$ converges weakly to $\tilde{\mu}$ for any $x \in S^1$ and $\tilde{\mu}(I) = 0$ we have

$$\frac{1}{n} \sum_{k=1}^{n} \tilde{U}^k 1_I(x) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

On the other hand, we have

$$\frac{1}{n} \sum_{k=1}^{n} \tilde{U}^k 1_I(x) = \frac{1}{n} \sum_{k=1}^{n} \sum_{\{i_1, \ldots, i_k\} \in \Sigma_k} p_{i_1} \cdots p_{i_k} 1_I(g_{i_1}^{-1} \circ \cdots \circ g_{i_k}^{-1}(x))$$

$$= \frac{1}{n} \sum_{k=1}^{n} \sum_{\{i_1, \ldots, i_k\} \in \Sigma_k} p_{i_1} \cdots p_{i_k} 1_{g_{i_1} \cdots i_k}(I)(x)$$

$$= \frac{1}{n} \sum_{k=1}^{n} \sum_{\{i_1, \ldots, i_k\} \in \Sigma_k} p_{i_1} \cdots p_{i_k} 1_{\{x_{i_1}, \ldots, x_I\}}(g_{i_1} \cdots i_k(I)).$$

Consequently, we have

$$\frac{1}{n} \sum_{k=1}^{n} \sum_{\{i_1, \ldots, i_k\} \in \Sigma_k} p_{i_1} \cdots p_{i_k} 1_{\{x_0, \ldots, x_M\}}(g_{i_1} \cdots i_k(I)) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$
Thus, for any $x, y \in I$, the interval $g_{i_1 \cdots i_k}([x, y])$ typically will be located between some points $x_{i-1}$ and $x_i$, so that its length will be less than $\varepsilon/L$. Hence

$$\limsup_{n \to \infty} \left| \frac{1}{n} \sum_{k=1}^{n} U^k f(x) - \frac{1}{n} \sum_{k=1}^{n} U^k f(y) \right|$$

$$\leq \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \sum_{(i_1, \ldots, i_k) \in \Sigma_k} p_{i_1} \cdots p_{i_k} |f(g_{i_1} \circ \cdots \circ g_{i_k}(x)) - f(g_{i_1} \circ \cdots \circ g_{i_k}(y))|$$

$$\leq \limsup_{n \to \infty} \frac{L}{n} \sum_{k=1}^{n} \sum_{(i_1, \ldots, i_k) \in \Sigma_k} p_{i_1} \cdots p_{i_k} |g_{i_1} \cdots g_{i_k}([x, y])| \leq L\varepsilon/L = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the operator $P$ satisfies the Cesáro e-property. This completes the proof. 

5. **Central Limit Theorem**

Let $\Gamma = \{g_1, \ldots, g_k\}$ be a family of homeomorphisms on $S^1$ and let $p = (p_1, \ldots, p_k)$ be a probability vector. Let $\mu \in M_1$ be an invariant measure for the iterated function system $(\Gamma, p)$. By $(X_n)_{n \geq 0}$ we shall denote the stationary Markov chain corresponding to the iterated function system $(\Gamma, p)$. Let $\varphi : S^1 \to \mathbb{R}$ be a Lipschitz function satisfying $\int_{S^1} \varphi \, d\mu = 0$. Set

$$S_n := S_n(\varphi) = \varphi(X_0) + \cdots + \varphi(X_n)$$

and

$$S_n^* = \frac{1}{\sqrt{n}} S_n \quad \text{for } n \geq 1.$$

Our main purpose in this section is to prove that $S_n^*$ is asymptotically normal (the CLT theorem). Maxwell and Woodroofe in $[9]$ studied general Markov chains and formulated a simple sufficient condition for the CLT which in our case takes the form

$$\sum_{n=1}^{\infty} n^{-3/2} \left\| \sum_{k=1}^{n} U^k \varphi \right\| < \infty,$$

where $\| \cdot \|$ denotes the $L^2(\mu)$ norm. More precisely, the result proved in $[9]$ says that if (2) holds, then the limit $\sigma^2 = \lim E(S_n^*)$ exists and is finite, and then the distribution of $S_n^*$ tends to $\mathcal{N}(0, \sigma)$.

We start with recalling some properties of iterated function systems obtained by D. Malicet (see Theorem A and Corollary 2.6 in [8]):

**Proposition 6.** Let $\Gamma = \{g_1, \ldots, g_k\}$ be a family of homeomorphisms on $S^1$ such that there is no measure invariant by $\Gamma$. Let $p = (p_1, \ldots, p_k)$ be a probability vector. If $\Gamma$ acts minimally, then there exists $q \in (0, 1)$ such that:
for every $x \in S^1$ there exists an open neighbourhood $I$ of $x$ and $\Omega \subset \Sigma$ with $P(\Omega) > 0$

such that for $i = (i_1, i_2, \ldots) \in \Omega$ we have

$$|g_{i_1, \ldots, i_1}(I)| \leq q^n;$$

• (asymptotic stability) for any $x \in S^1$ the sequence $(P^n\delta_x)_{n \geq 1}$, where $P$ is the Markov operator corresponding to $(\Gamma, p)$, converges weakly to the unique invariant measure $\mu_*$.

First, let us note that Proposition 6 implies the e-property:

Proposition 7. Under the hypothesis of Proposition 6 the operator $P$ corresponding to $(\Gamma, p)$ satisfies the e–property.

Proof From Proposition 6 it follows that $P$ is asymptotically stable. Since $\text{supp} \mu_* = S^1$, in particular $\text{Int} S^1 \supset \text{supp} \mu_*$, from Theorem 4.8 in [4] we obtain that $P$ satisfies the e–property.

We may formulate the main result of our paper saying that the iterated function system under quite general assumptions fulfills the Central Limit Theorem.

Theorem 8. Let $\varphi : S^1 \to \mathbb{R}$ be an arbitrary Lipschitz function. If $\Gamma = \{g_1, \ldots, g_k\}$ acts minimally and there is no measure invariant by $\Gamma$, then for any probability vector $p = (p_1, \ldots, p_k)$ the iterated function system $(\Gamma, p)$ satisfies the Central Limit Theorem.

Proof We can assume that the Lipschitz constant of the function $\varphi$ is equal to 1. Denote by $|\varphi|$ the supremum norm of $\varphi$.

By Proposition 6 there exist $I \subset S^1$, $q \in (0, 1)$ and $\Omega \subset \Sigma$ such that $\alpha := P(\Omega) > 0$ and

$$|g_{i_1, i_2, \ldots, i_1}(x) - g_{i_1, i_2, \ldots, i_1}(y)| \leq q^n$$

for any $(i_1, i_2, \ldots) \in \Omega$.

Since the operator $P$ is asymptotically stable and it satisfies the e–property there exists $m \in \mathbb{N}$ such that

$$P^m\delta_x(I) > \mu_*(I)/2 > 0 \quad (3)$$

for any $x \in S^1$. Indeed, for any $x \in S^1$ there is $n \in \mathbb{N}$ such that (3) holds for all $m \geq n$, by stability. On the other hand, from the e–property it follows that we may choose some neighbourhood $U$ of $x$ such that the above property will be satisfied for all points from $U$ with the same $m$. By compactness of $S^1$ we may choose $m$ such that (3) holds for any $x \in S^1$.

Further, if the iterated function system $(\Gamma, p)$, where $\Gamma = \{g_1, \ldots, g_k\}$ and $p = (p_1, \ldots, p_k)$, is given it may be rewritten as $\{\tilde{g}_1, \ldots, \tilde{g}_{2k}\}$ and $p = (\tilde{p}_1, \ldots, \tilde{p}_{2k})$ setting

$$\tilde{g}_i = g_i \quad \text{for } i = 1, \ldots, k \quad \text{and} \quad \tilde{g}_{i+k} = g_i \quad \text{for } i = 1, \ldots, k,$$

and

$$\tilde{p}_i = \min_{1 \leq i \leq k} p_i := p \quad \text{for } i = 1, \ldots, k \quad \text{and} \quad \tilde{p}_{i+k} = p_i - p \quad \text{for } i = 1, \ldots, k.$$
By virtue of condition (3) we may assume then that for any \( x \in S^1 \) there exists \((i_1, \ldots, i_m)\Sigma_m\) such that \( g_{i_m,i_{m-1},\ldots,i_1}(x) \in I \) and \( \mathbb{P}(i_1, \ldots, i_m) = p^m \).

Now we are going to evaluate
\[
\sum_{k=1}^{n} \mathbb{E}[g_{i_k,i_{k-1},\ldots,i_1}(x) - g_{i_k,i_{k-1},\ldots,i_1}(y)]
\]
for any \( x, y \in S^1 \) for \( i = 1, \ldots, M \).

For any \( x, y \in S^1 \) by \( C_{x,y} \) we shall denote the subset of \( \Sigma \times \Sigma \) given by the formula
\[
C_{x,y} = [(i_1, \ldots, i_m) \times \Omega] \times [(\tilde{i}_1, \ldots, \tilde{i}_m) \times \Omega],
\]
where \((i_1, \ldots, i_m), (\tilde{i}_1, \ldots, \tilde{i}_m) \in \Sigma_m \times \Sigma_m\) are such that \( g_{i_m,i_{m-1},\ldots,i_1}(x), g_{i_m,i_{m-1},\ldots,i_1}(y) \in I \).

For any \( l \geq 1 \) by \( C_{x,y}^l \) we denote the projection of \( C_{x,y} \) on the first \( l \) coordinates, i.e.
\[
C_{x,y}^l = \{(i_1, \ldots, i_l) \times (\tilde{i}_1, \ldots, \tilde{i}_l) : [(i_1, \ldots, i_l) \times \Sigma] \times [(\tilde{i}_1, \ldots, \tilde{i}_l) \times \Sigma] \cap C_{x,y} \neq \emptyset \}.
\]

Obviously,
\[
\sum_{k=1}^{\infty} \left| g_{i_k,i_{k-1},\ldots,i_1}(x) - g_{i_k,i_{k-1},\ldots,i_1}(y) \right| \leq 2m + (1 - q)^{-1}
\]
for any \((i_1, i_2, \ldots) \times (j_1, j_2, \ldots) \in C_{x,y} \).

Fix \( n \geq 1 \) and \( x, y \in S^1 \). Set \( \mathbb{Q} = \mathbb{P} \times \mathbb{P} \). Let \( k = [n^{1/4}] \) and set \( \beta := p^m \alpha \). We are going to define by induction a sequence of sets \( A_1 \subset \cdots \subset A_k \subset \Sigma_n \times \Sigma_n \) such that for any \( l \in \{1, \ldots, k\} \) we have
\[
\mathbb{Q}(\Sigma_n \times \Sigma_n \setminus A_l) \leq (1 - \beta)^l,
\]
the complement \( \Sigma_n \times \Sigma_n \setminus A_l \) is a union of some product sets of the form
\[
[(i_1, \ldots, i_l) \times \Sigma_{n-l}] \times [(j_1, \ldots, j_l) \times \Sigma_{n-l}],
\]
and
\[
\sum_{l=1}^{n} \left| g_{i_l,i_{l-1},\ldots,i_1}(x) - g_{j_l,i_{l-1},\ldots,i_1}(y) \right| \leq l(2m + (1 - q)^{-1})
\]
for any \((i_1, \ldots, i_n) \times (j_1, \ldots, j_n) \in A_l \). If \( l = 1 \) we define \( A_1 = C_{x,y}^n \). Obviously \( \mathbb{Q}(\Sigma_n \times \Sigma_n \setminus A_1) \leq 1 - \beta \).

Now assume that we have defined \( A_l \) for some \( l \leq n \). Denote by \( B_k \) the set of all pairs of sequences \((i_k \times j_k) := (i_1, \ldots, i_k) \times (j_1, \ldots, j_k)\) which appear in the union (5). Set
\[
A_{l+1} := A_l \cup \bigcup_{k=1}^{n-m-l} (i_k \times j_k) \times C_{g_{i_k,i_{k-1},\ldots,i_1}(x),g_{j_k,j_{k-1},\ldots,j_1}(y)}^{n-k} \bigcup_{k=n-m}^{n} B_k \times \Sigma_{n-k}
\]
To evaluate the probability \( \mathbb{Q}(\Sigma_n \times \Sigma_n \setminus A_{l+1}) \) it is enough to observe that
\[
\mathbb{Q}(\Sigma_{n-k} \times \Sigma_{n-k} \setminus C_{g_{i_k,i_{k-1},\ldots,i_1}(x),g_{j_k,j_{k-1},\ldots,j_1}(y)}^{n-k}) \leq 1 - \beta
\]
for any \((i_1, \ldots, i_k) \times (j_1, \ldots, j_k) \in B_k, k \leq n - m - 1\) and
\[
\mathbb{Q}(\Sigma_n \times \Sigma_n \setminus A_l) \leq (1 - \beta)^l.
\]
On the other hand, note that for every \((i_1, \ldots, i_k) \times (j_1, \ldots, j_k) \in B_k\) the cylinder set \((i_1, \ldots, i_{k-1}) \times (j_1, \ldots, j_{k-1}) \times \Sigma_{n-k} \times \Sigma_{n-k}\) has a nonempty intersection with \(A_i\), so the induction assumption (i) applies, giving, in particular the bound for

\[
\sum_{i=1}^{k-1} |g_{i, i-1, \ldots, i_1}(x) - g_{j, i-1, \ldots, j_1}(y)|
\]

and we obtain

\[
\sum_{i=1}^{n} |g_{i, i-1, \ldots, i_1}(x) - g_{j, i-1, \ldots, j_1}(y)| \leq (l + 1)(2m + (1 - q)^{-1})
\]

for any \((i_1, \ldots, i_n) \times (j_1, \ldots, j_n) \in A_{l+1}^n\).

Let \(\varphi : \mathbb{S}^1 \to \mathbb{R}\) be a Lipschitz function with the Lipschitz constant \(L\). Assume that \(\int_{\mathbb{S}^1} \varphi(x) \mu_\ast(dx) = 0\). We are in a position to evaluate the \(L_2\) norm \(\|\sum_{k=1}^{n} U_k \varphi\|\). We have

\[
\left| \sum_{k=1}^{n} U_k \varphi(x) \right| = \left| \sum_{k=1}^{n} U_k \varphi(x) - \int_{\mathbb{S}^1} \varphi(y) \mu_\ast(dy) \right| = \left| \int_{\mathbb{S}^1} \left[ \sum_{k=1}^{n} U_k \varphi(x) - U_k \varphi(y) \right] \mu_\ast(dy) \right|
\]

\[
\leq \int_{\mathbb{S}^1} \sum_{k=1}^{n} |U_k \varphi(x) - U_k \varphi(y)| \mu_\ast(dy).
\]

On the other hand, we have

\[
\sum_{k=1}^{n} |U_k \varphi(x) - U_k \varphi(y)| = \sum_{k=1}^{n} \mathbb{E} |\varphi(g_{i, k, i-1, \ldots, i_1})(x) - \varphi(g_{j, k, i-1, \ldots, i_1})(y)|
\]

\[
\leq \int_{\Sigma_n} \sum_{k=1}^{n} \left| \varphi(g_{i, k, i-1, \ldots, i_1})(x) - \varphi(g_{j, k, i-1, \ldots, i_1})(y) \right| dQ((i_1, \ldots, i_n), (j_1, \ldots, j_n))
\]

\[
\leq \int_{\Sigma_n} \sum_{k=1}^{n} \left| \varphi(g_{i, k, i-1, \ldots, i_1})(x) - \varphi(g_{j, k, i-1, \ldots, i_1})(y) \right| dQ((i_1, \ldots, i_n), (j_1, \ldots, j_n))
\]

\[
+ \int_{A_{[n/4]}} \sum_{k=1}^{n} \left| \varphi(g_{i, k, i-1, \ldots, i_1})(x) - \varphi(g_{j, k, i-1, \ldots, i_1})(y) \right| dQ((i_1, \ldots, i_n), (j_1, \ldots, j_n))
\]

\[
\leq \int_{\Sigma_n} \sum_{k=1}^{n} \left| \varphi(g_{i, k, i-1, \ldots, i_1})(x) - \varphi(g_{j, k, i-1, \ldots, i_1})(y) \right| dQ((i_1, \ldots, i_n), (j_1, \ldots, j_n))
\]

\[
+ \int_{A_{[n/4]}} \sum_{k=1}^{n} \left| \varphi(g_{i, k, i-1, \ldots, i_1})(x) - g_{j, k, i-1, \ldots, i_1}(y) \right| dQ((i_1, \ldots, i_n), (j_1, \ldots, j_n))
\]

\[
\leq 2n |\varphi| Q(\Sigma_n \times \Sigma_n \setminus A_{[n/4]}) + (n^{1/4} + 1)(2m + (1 - q)^{-1})Q(A_{[n/4]})
\]

\[
\leq 2n |\varphi| (1 - \beta)^{n^{1/4}} + (n^{1/4} + 1)(2m + (1 - q)^{-1}) := a_n.
\]
Consequently, from the above estimates we obtain
\[ \| \sum_{k=1}^{n} U^k \varphi \| \leq \left( \int_{S^1} \left( \int_{S^1} a_n \mu_\ast(dy) \right)^2 \mu_\ast(dx) \right)^{1/2} = a_n. \]

Since the series \( \sum_{n=1}^{\infty} \frac{a_n}{n^{3/2}} \) is convergent, condition (I) holds and the stationary sequence \((\varphi(X_n))_{n \geq 1}\) satisfies the CLT.

To show that the CLT theorem holds for a sequence \((\varphi(X^x_n))_{n \geq 1}\) starting at arbitrary \(x \in S^1\) it is enough to prove that
\[ \left| \mathbb{E} \exp \left( i t \frac{\varphi(X^x_1) + \ldots + \varphi(X^x_n)}{\sqrt{n}} \right) - \mathbb{E} \exp \left( i t \frac{\varphi(X_1) + \ldots + \varphi(X_n)}{\sqrt{n}} \right) \right| \to 0 \quad \text{as} \quad n \to \infty. \]

In the same way as in the above estimates we obtain that
\[
\begin{align*}
&\left| \mathbb{E} \exp \left( i t \frac{\varphi(X^x_1) + \ldots + \varphi(X^x_n)}{\sqrt{n}} \right) - \mathbb{E} \exp \left( i t \frac{\varphi(X_1) + \ldots + \varphi(X_n)}{\sqrt{n}} \right) \right| \\
&\leq \frac{1}{\sqrt{n}} \int_{\mathbb{S}^n \times \mathbb{S}^n} \left| \varphi(g_{i_k,i_{k-1},\ldots,i_1})(x) - \varphi(g_{j_k,j_{k-1},\ldots,j_1})(y) \right| dQ((i_1, \ldots, i_n), (j_1, \ldots, j_n)) \\
&\leq \frac{n|\varphi|(1 - \beta)(n^{1/4}) + (n^{1/4} + 1)(m + (1 - q)^{-1})}{\sqrt{n}} \to 0 \quad \text{as} \quad n \to \infty.
\end{align*}
\]
Since
\[ \mathbb{E} \exp \left( i t \frac{\varphi(X_1) + \ldots + \varphi(X_n)}{\sqrt{n}} \right) \to \exp(-t^2 \sigma^2/2) \quad \text{as} \quad n \to \infty, \]
we obtain
\[ \mathbb{E} \exp \left( i t \frac{\varphi(X^x_1) + \ldots + \varphi(X^x_n)}{\sqrt{n}} \right) \to \exp(-t^2 \sigma^2/2) \quad \text{as} \quad n \to \infty \]
and we are done. 

\begin{remark}
The same theorem holds for Hölder continuous observables \(\varphi\). The above proof goes through with obvious modifications.
\end{remark}

\section*{References}

[1] B. Deroin, V. Kleptsyn, and A. Navas, \textit{Sur la dynamique unidimensionnelle en régularité intermédiaire}, Acta Math. \textbf{199} no. 2, 199-262 (2007).

[2] É. Ghys, \textit{Groups acting on the circle}, L’Enseignement Mathématique \textbf{47}, 329-407 (2001).

[3] S. Hille, K. Horbacz and T. Szarek, \textit{Existence of a unique invariant measure for a class of equicontinuous Markov operators with application to a stochastic model for an autoregulated gene}, Annales mathématiques Blaise Pascal, \textbf{23} no. 2, 171-217 (2016); doi: 10.5802/ambp.360

[4] S. Hille, T. Szarek and M. Ziemlańska, \textit{Equicontinuous families of Markov operators in view of tightness and asymptotic stability}, to be published.

[5] R. Kapica, T. Szarek and M. Ślęczka, \textit{On a unique ergodicity of some Markov processes}, Potential Anal. \textbf{36}, 589-606 (2012).
[6] T. Komorowski, S. Peszat and T. Szarek, *On ergodicity of some Markov processes*, Ann. Probab. 38, 1401-1443 (2010).

[7] T. Komorowski and A. Walczuk, *Central limit theorem for Markov processes with spectral gap in the Wasserstein metric*, Stochastic Processes and Appl. 122, 2155–2184 (2012).

[8] D. Malicet, *Random walks on Homeo(S1)*, preprint: [arXiv:1412.8618](https://arxiv.org/abs/1412.8618).

[9] M. Maxwell and M. Woodroofe, *Central Limit Theorems for additive functionals of Markov chains*, Ann. Probab. 28, 713–724 (2000).

[10] S.P. Meyn and R.L. Tweedie, *Markov chains and stochastic stability*. Communications and Control Engineering Series. Springer-Verlag London, Ltd., London, 1993.

[11] A. Navas, *Groups of Circle Diffeomorphisms*, Chicago Lectures in Mathematics. University of Chicago Press, 2010.

[12] T. Szarek and A. Zdunik, *Stability of iterated function systems on the circle*, Bull. Lond. Math. Soc. 48, no. 2, 365–378 (2016).

[13] Worm, D.T.H. (2010), *Semigroups on spaces of measures*, PhD. thesis, Leiden University, The Netherlands. Available at: [www.math.leidenuniv.nl/nl/theses/PhD/](http://www.math.leidenuniv.nl/nl/theses/PhD/)

TOMASZ SZAREK, INSTITUTE OF MATHEMATICS, UNIVERSITY OF GDANSK, WITA STWOSZA 57, 80-952 GDANSK, POLAND

E-mail address: szarek@intertele.pl

ANNA ZDUNIK, INSTITUTE OF MATHEMATICS, UNIVERSITY OF WARSAW, UL. BANACHA 2, 02-097 WARSZAWA, POLAND

E-mail address: A.Zdunik@mimuw.edu.pl