FUNCTORIAL DESINGULARIZATION OF QUASI-EXCELLENT SCHEMES IN CHARACTERISTIC ZERO: THE NON-EMBEDDED CASE

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Abstract. We prove that any noetherian quasi-excellent scheme of characteristic zero admits a strong desingularization which is functorial with respect to all regular morphisms. We show that as an easy formal consequence of this result one obtains strong functorial desingularization for many other spaces of characteristic zero including quasi-excellent stacks and formal schemes, and complex and non-archimedean analytic spaces. Moreover, these functors easily generalize to non-compact setting by use of converging blow up hypersequences with regular centers.

1. Introduction

1.1. Motivation.

1.1.1. Historical overview. Encouraged by Hironaka’s work [Hir] on resolution of singularities, Grothendieck introduced quasi-excellent (or qe) schemes in [EGA, IV, §7.9] in order to provide a natural general framework for desingularization. Grothendieck observed that the schemes studied by Hironaka were schemes of finite type over a local qe scheme \( k \), and proved that if any integral scheme of finite type over a base scheme \( k \) admits a desingularization in the weakest possible sense then \( k \) is quasi-excellent. Grothendieck conjectured that the converse is probably true, and thus any qe scheme admits a desingularization, and claimed without proof that the conjecture holds true for noetherian qe schemes over \( \mathbb{Q} \) as can be proved by Hironaka’s method. The latter claim was never checked, so desingularization of qe schemes of characteristic zero remained a conjecture until the author’s work [Tem] in 2008. Although Grothendieck’s claim that Hironaka’s highly complicated method applies to all qe schemes over \( \mathbb{Q} \) was not checked in [Tem], and so the stature of his claim is still unclear, it was shown by much easier methods that desingularization of qe schemes over \( \mathbb{Q} \) can be easily obtained from Hironaka’s results and can be deduced by a harder work from desingularization of varieties. The latter is important, since many relatively simple proofs for varieties are now available, and they establish a canonical resolution which is compatible with all smooth morphisms between varieties, as opposed to the non-constructive original Hironaka’s method.

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1.1.2. New goals. The main disadvantage of the results from [Tem] was that the desingularization results for qe schemes were weaker than their analogs for varieties in two aspects: (a) the centers of resolving blow ups were not regular, (b) no functoriality/canonicity was achieved. The aim of this paper is to strengthen the methods of [Tem] in order to construct a desingularization by blowing up only regular centers and so that the whole blow up sequence is functorial in all regular morphisms (in particular, we cannot rely on the method of [Hir] anymore since its resolution is not functorial). Our main results are Theorems 1.2.1 and 5.3.2 providing non-embedded desingularization of noetherian qe schemes over $\mathbb{Q}$ and non-compact objects including all qe (formal) schemes over $\mathbb{Q}$ and analytic spaces in characteristic zero. In comparison with the known non-embedded desingularization of varieties our results give the strongest known desingularization with the exception mentioned in Remark 1.2.2(ii): we do not treat non-embedded desingularization of generically non-reduced schemes.

Although our method follows the method of [Tem] closely, we tried to make the paper as self-contained as possible. So, the familiarity with [Tem] can be useful but is not necessary. In particular, we give/recall all needed definitions. As opposed to [Tem], this time we split the exposition to two separate papers concerning non-embedded and embedded desingularization, and this paper deals only with the non-embedded case without boundary. Note that unlike the case of varieties we have to deal with both cases because one cannot deduce the non-embedded case from the embedded one – there are qe schemes that cannot be locally embedded into regular ones.

1.2. Main result.

1.2.1. Strong non-embedded desingularization. We refer to §§2.1,2.2 for the terminology, which includes the notions of quasi-excellent schemes, regular loci and morphisms, and blow up sequences. Here is the strong non-embedded desingularization theorem for generically reduced noetherian quasi-excellent schemes in characteristic zero. Recall that "strong" means that the desingularization blows up only regular subschemes.

**Theorem 1.2.1.** For any noetherian quasi-excellent generically reduced scheme $X = X_0$ over $\text{Spec}(\mathbb{Q})$ there exists a blow up sequence $\mathcal{F}(X) : X_n \to X_0$ such that the following conditions are satisfied:

(i) the centers of the blow ups are disjoint from the preimages of the regular locus $X_{\text{reg}}$;
(ii) the centers of the blow ups are regular;
(iii) $X_n$ is regular;
(iv) the blow up sequence $\mathcal{F}(X)$ is functorial with respect to all regular morphisms $X' \to X$, in the sense that $\mathcal{F}(X')$ is obtained from $\mathcal{F}(X) \times_X X'$ by omitting all empty blow ups.

1.2.2. On the non-reduced case.

**Remark 1.2.2.** (i) Theorem 1.2.1 implies that the same claim holds for all noetherian qe schemes over $\mathbb{Q}$. Indeed, given an arbitrary such $X$ with reduction $\tilde{X}$ we can consider the blow up sequence $i_* \mathcal{F}(\tilde{X}) : X' \to X$ which is the push-forward of $\mathcal{F}(\tilde{X}) : \tilde{X}' \to \tilde{X}$ with respect to the closed immersion $i : \tilde{X} \to X$ (see §4.2.1). Then the reduction of $X'$ is $\tilde{X}'$, so it is regular. Next we kill all
generically non-reduced components by blowing them up along their reductions, obtaining a blow up $X'' \rightarrow X'$ with generically reduced $X''$. Finally, it remains to apply $\mathcal{F}(X'') : X''' \rightarrow X''$ to construct a functorial desingularization $X''' \rightarrow X$.

(ii) We ignore on purpose the case when $X$ is not reduced along an irreducible component because in this case the assertion of the Theorem is not the "right" (or interesting) version of desingularization, and it does not assert anything new as we saw in (i). A much more subtle claim is that $X$ can be made equisingular or normally flat along its reduction. Loosely speaking, such desingularization does not kill non-reduced components, but makes their nilpotent structure "as smooth as possible". Such strong desingularization is known for varieties (already due to Hironaka), but it is not currently achieved by our method because rig-equisingular formal varieties do not have to be algebraizable, as opposed to rig-regular ones. So the method of §3.1 does not apply straightforwardly to the generically non-reduced case. I expect that an approximate algebraization from [Elk, Th. 5] should suffice to deal with rig-equisingular formal varieties, but this is a subject for future research.

1.3. Overview.

1.3.1. The black box strategy. Now let us discuss our method and the structure of the paper. In order to construct a desingularization $\mathcal{F}$ of all qe schemes over $\mathbb{Q}$ we use as a black box any algorithm $\mathcal{F}_{\text{Var}}$ which desingularizes varieties of characteristic zero and is functorial with respect to all regular morphisms (i.e. satisfies the conditions (i), (iii) and (iv) from the Theorem). Moreover, if $\mathcal{F}_{\text{Var}}$ is strong then we can also achieve that $\mathcal{F}$ is strong, i.e. satisfies the condition (ii) as well. (We consider both strong and non-strong cases mainly because it does not cost us any extra-work.)

1.3.2. Extending $\mathcal{F}_{\text{Var}}$. It was checked in a very recent work [BMT] that the algorithm of Bierstone-Milman is functorial with respect to all regular morphisms (not necessarily of finite type). Thus, for the sake of concreteness, we can start with the algorithm $\mathcal{F}_{\text{Var}}$ from [BM2]. (Alternatively, it is shown in [BMT] that one can start with any algorithm $\mathcal{F}_{\mathbb{Q}}$ for varieties over $\mathbb{Q}$ and extend it to all varieties using approximation results of [EGA, IV, §8.8]. In such case, one can also build on many other available algorithms.) In §2 we fix our terminology and recall many basic facts about blow ups, desingularizations and formal schemes. We warn the reader that the terminology differs in part from that of [Tem], including blow ups from §2.2.1, desingularization from §2.3.1 and qe formal schemes from §2.4.3.

We extend in §3 the functor $\mathcal{F}_{\text{Var}}$ to pairs $(X, Z)$, where $X$ is a qe scheme over $\mathbb{Q}$ and $Z \hookrightarrow X$ is a Cartier divisor containing the singular locus of $X$ and isomorphic to disjoint union of varieties. Each $\mathcal{F}_{\text{Var}}(X, Z)$ is a desingularization of $X$, and it remains unclear if it really depends on $Z$. Similarly to the method of [Tem] this construction goes by passing to the formal completion of $X$ along $Z$, thus obtaining a rig-regular formal variety $\mathfrak{X} = \bar{X}_Z$, and algebraizing $\mathfrak{X}$ by a variety $X'$. Then $\mathcal{F}_{\text{Var}}(X')$ induces desingularizations on $\mathfrak{X}$ and $X$, and the main problem we have to solve is that the choice of $X'$ is absolutely non-canonical. Moreover, and this causes the main trouble, even the ground field of $X'$ can be chosen in many ways. This complication is by-passed by proving that all information about the singularities of $\mathfrak{X}$ can be extracted already from a sufficiently thick nilpotent neighborhood $X_n \hookrightarrow \mathfrak{X}$ of its closed fiber $\mathfrak{X}_n$. Moreover, everything is determined by the scheme $X_n$, and is independent of a variety structure of $X_n$, which is not unique. In particular, we
prove in §3.2 that $\mathcal{F}_{\text{Var}}(X) := \widehat{\mathcal{F}_{\text{Var}}(X')}$ is canonically defined already by $X_n$, and is therefore independent of the choice of the algebraization $X'$. This section is the heart of the paper, and it is the main novelty since [Tem]. As in loc.cit., Elkik’s results from [Elk] are the main tool we are using for algebraization.

1.3.3. Constructing $\mathcal{F}$. In §4 we use the desingularizations $\mathcal{F}_{\text{Var}}(X, Z)$ to construct another desingularization functor $\mathcal{F}$ which applies to all generically reduced noetherian qe schemes, and this is done by induction on codimension similarly to the argument from [Tem, 2.3.4]. This time we must work more carefully in order not to lose functoriality of the algorithm and regularity of the centers, but the basic idea is the same. Actually, we construct a sequence of functors $\mathcal{F}^{\leq d}$ which desingularize a qe scheme $X$ over the set $X^{\leq d}$ of points of codimension at most $d$. The construction is inductive: given the blow up sequence $\mathcal{F}^{\leq d-1}(X) : X' \to X$ we insert few new blow ups which resolve its centers and the source over few ”bad” points of codimension $d$. Actually, after localization at the bad points we are dealing with schemes whose singular loci are disjoint unions of varieties, so the functor $\mathcal{F}_{\text{Var}}$ suffices to construct $\mathcal{F}^{\leq d}(X)$ by patching in $\mathcal{F}^{\leq d-1}(X)$ over the bad points.

Remark 1.3.1. (i) Because of functoriality, we must build $\mathcal{F}$ from scratch, so most probably it differs from $\mathcal{F}_{\text{Var}}$ even for simple varieties.

(ii) It remains to be an interesting open question how far one can push the standard desingularization functors $\mathcal{F}_{\text{Var}}$ defined in terms of derivative ideals, including the question if one can extend $\mathcal{F}_{\text{Var}}$ to all qe schemes of characteristic zero.

(iii) The author conjectures that the existing algorithms can be extended (up to minor modifications) to all schemes over $\mathbb{Q}$ which locally admit closed immersions into regular affine schemes $U = \text{Spec}(A)$ with enough derivatives, and that the extended algorithm is compatible with all regular morphisms. The condition on derivatives means that the sheaf $\text{Der}_{U/\mathbb{Z}}$, which does not have to be quasi-coherent and can even have zero stalks, admits global sections $\partial_1, \ldots, \partial_n$ such that for any point $u \in U$ the images of $\partial_i$’s generate the tangent space $(m_u/m_u^2)^*$. Since existence of the above derivatives is equivalent to existence of an (analog of) Taylor homomorphism $A \to A[[T]]$ due to Bierstone-Milman, this conjecture agrees with Bierstone-Milman philosophy. It is a future project by Bierstone, Milman and the author to try to generalize the algorithm from [BM2] accordingly to the conjecture.

(iv) The above conjecture implies functorial desingularization of formal varieties of characteristic zero. In particular, it would drastically simplify our work in this paper by proving a much more general result on formal desingularization than we prove in §3.1.5, and such approach would not use Elkik’s results (we desingularize all qe formal schemes over $\mathbb{Q}$ in §5, but this is based on the intermediate result of §3.1.5).

1.3.4. Applications to other categories. Finally, in §5 we use the functorial desingularization of qe schemes over $\mathbb{Q}$ to establish desingularization in other categories. We show that our result implies desingularization of stacks over $\mathbb{Q}$ with regular atlas (not necessarily of finite type), of complex/rigid/Berkovich analytic spaces of characteristic zero, and of qe formal schemes over $\mathbb{Q}$. The results about stacks (not covered by varieties) and formal schemes are new. Desingularization of blow ups of affine formal schemes over $\mathbb{Q}$ was proved in [Tem, 4.3.2], but it was impossible to globalize this result without canonicity of the construction, see [Tem, 4.3.3] and the remark after it. The analytic desingularization is well known – it is
done by absolutely the same method as its algebraic analog, but strictly speaking it required a parallel proof until now. We show that it is a formal consequence of desingularization of qe schemes. Since the latter is obtained in this paper from a desingularization $\mathcal{F}_Q$ of $Q$-varieties, we see that all desingularization theories can be built in a very algebraic way using a single algorithm $\mathcal{F}_Q$. At first glance, this might look as a sharp surprise in view of existence of non-algebraizable formal and analytic singularities. Finally, using canonical desingularization we also desingularize non-compact objects in §5.3, including locally noetherian qe schemes over $Q$. In particular, this settles completely Grothendieck’s conjecture in characteristic zero (not quasi-compact qe schemes were not treated in [Tem]).

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## 2. Setup

### 2.1. Schemes and morphisms.

#### 2.1.1. Varieties. Variety or algebraic variety in this paper always means a scheme $X$ which admits a finite type morphism $X \to \text{Spec}(k)$ to the spectrum of a field. If such a morphism is fixed then we say that $X$ is a $k$-variety and $k$ is the ground field of $X$. The reader should be aware that an abstract scheme $X$ may admit many different structures of an algebraic variety (especially, when $X$ is not reduced).

#### 2.1.2. Ideals and closed subschemes. Given a scheme $X$, we will freely pass between closed subsets and reduced closed subschemes of $X$. Also, we will freely pass between ideals $I \subset \mathcal{O}_X$ and closed subschemes $Z \hookrightarrow X$. 
2.1.3. **Pro-open subsets and subschemes.** A pro-open subset of a scheme $X$ is a subset $S \subset |X|$ which coincides with the intersection of all its neighborhoods. An equivalent condition is that $X$ is closed under generalizations. If $(S, \mathcal{O}_X|_S)$ is a scheme then we call it the pro-open subscheme corresponding to $S$. For example, if $x \in X$ then $X_x = \text{Spec}(\mathcal{O}_{X,x})$ is a pro-open subscheme of $X$.

2.1.4. **Filtration by codimension.** For a locally noetherian scheme $X$ by $X < d$, $X \geq d$, etc., we denote the subsets of $X$ consisting of all points of codimension strictly less than $d$, large than $d$, etc. We note that $S = X < d$ is a pro-open subset, but usually it does not underly a pro-open subscheme. We will use in the sequel filtration by codimension $\emptyset = X < 0 \subset X < 1 \subset X < 2 \subset \ldots$, which however may be infinite.

**Remark 2.1.1.** Even if $X$ is noetherian its dimension can be infinite, due to some pathological examples by Nagata. For this reason, one should use noetherian induction instead of a more naive induction by dimension. In all cases of practical interest, however, noetherian schemes are finite dimensional.

2.1.5. **Schematical closure.** If $U$ is a pro-open subscheme in a locally noetherian scheme $S$ and $Z_U \hookrightarrow U$ is a closed subscheme then by the schematical closure of $Z_U$ in $S$ we mean the schematical image $Z$ of the morphism $i : Z_U \to S$ (i.e. the minimal closed subscheme of $S$ such that $i$ factors through it). The following Lemma indicates that this construction works as fine as in the case when $U$ is open in $S$, and $Z$ is actually the minimal extension of $Z_U$ to a closed subscheme in $S$.

**Lemma 2.1.2.** Keep the above notation, then $Z_U$ coincides with the restriction of $Z$ on $U$.

**Proof.** It suffices to prove that $Z_U$ admits any extension to a closed subscheme $Z \hookrightarrow S$ since the minimal extension then exists by local noetherianity of $S$. An arbitrary extension was constructed in [Tem, 2.1.1] as follows: first one uses [EGA, IV$_4$, 8.6.3] to extend $Z_U$ to a closed subscheme $Z_V$ of a sufficiently small open neighborhood $V$ of $U$, and then one extends $Z_V$ to $S$ using [EGA, I$_{\text{new}}$, 6.9.7].

2.1.6. **Schematical density.** Assume that $S$ is a scheme with a pro-open subset $U$. We say that $U$ is schematically dense in $S$ if for any proper closed subscheme $S' \hookrightarrow S$ there exists $u \in U$ with $S' \times_S \text{Spec}(\mathcal{O}_{S,u})$ a proper subscheme of $\text{Spec}(\mathcal{O}_{S,u})$ (possibly empty). For a locally noetherian $S$ this happens if and only if $U$ contains the generic points of all irreducible and embedded components of $S$.

2.1.7. **U-admissibility.** An $S$-scheme $X$ is called $U$-admissible if the preimage of $U$ in $X$ is schematically dense. This follows the terminology of [RG] and we will not use the notion ”admissible” in other meanings.

2.1.8. **Regular morphisms.** A morphisms $f : Y \to X$ of schemes is called regular if it is flat and has geometrically regular fibers. For morphisms of finite type regularity is equivalent to smoothness, so it can be viewed as a generalization of smoothness to morphism not necessarily of finite type. A homomorphism of algebras $f : A \to B$ is regular if $\text{Spec}(f)$ is regular.

**Remark 2.1.3.** (i) We remark that Hironaka uses the notion of universally regular morphisms instead of regular morphisms reserving the notion of regularity to what we call reg morphisms below. However, our definition of regularity is the standard definition, see for example [Mat].
(ii) Let us temporarily say that a morphism \( f : Y \to X \) is \( \text{reg} \) if it is flat and has regular fibers. Then a morphism is regular in our sense if and only if it is universally reg.

(iii) It seems that reg morphisms are not worth a study (and a special name). For example, one can easily construct an example of a flat family of integral curves over \( \mathbf{A}_k^1 \), where the general fiber is regular but all closed fibers are singular.

2.1.9. **Singular locus.** We define the regular locus \( X_{\text{reg}} \) of a scheme \( X \) as a set of points at which \( X \) is regular, and the singular locus \( X_{\text{sing}} \) is defined as the compliment of \( X_{\text{reg}} \).

2.1.10. **Compatibility with regular morphisms.** It is well known that regular/singular locus is compatible with regular morphisms, i.e. for a regular morphisms \( f : Y \to X \) we have that \( Y_{\text{sing}} = f^{-1}(X_{\text{sing}}) \) and \( Y_{\text{reg}} = f^{-1}(X_{\text{reg}}) \). See, for example, [Mat, 23.7].

2.1.11. **Singular locus of a morphism.** By the singular locus \( g_{\text{sing}} \) of a morphism \( g : Y \to X \) we mean the set of points \( y \in Y \) at which \( g \) is not regular (i.e. \( g_y : \text{Spec}(\mathcal{O}_{Y,y}) \to \text{Spec}(\mathcal{O}_{X,g(y)}) \) is not regular). We will not use this notion beyond the following Lemma, where compatibility with regular morphisms is established.

**Lemma 2.1.4.** (i) If \( f : Z \to Y \) and \( g : Y \to X \) are morphisms of schemes and \( f \) is regular, then \( (g \circ f)_{\text{sing}} = f^{-1}(g_{\text{sing}}) \).

(ii) If \( g : A \to B \) and \( f : B \to C \) are local homomorphisms between local rings and \( f \) is regular, then \( g \) is regular if and only if \( g \circ f \) is regular.

**Proof.** Obviously, (ii) is a particular case of (i). On the other hand, the claim that \( z \in (g \circ f)_{\text{sing}} \) if and only if \( z \in f^{-1}(g_{\text{sing}}) \) reduces to checking (ii) for the local homomorphisms \( g_y : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y} \) and \( f_z : \mathcal{O}_{Y,y} \to \mathcal{O}_{Z,z} \), where \( y = f(z) \) and \( x = g(y) \). Since \( f_z \) is faithfully flat, \( g_y \) is flat if and only if \( f_z \circ g_y \) is flat. The homomorphism \( f_z \otimes \mathcal{O}_{X,x}, b(z) : \mathcal{O}_{Y,y}/m_x \mathcal{O}_{Y,y} \to \mathcal{O}_{Z,z}/m_z \mathcal{O}_{Z,z} \) is the base change of \( f_z \), hence it is regular, and by §2.1.10 we obtain that \( \mathcal{O}_{Y,y}/m_x \mathcal{O}_{Y,y} \) is regular if and only if \( \mathcal{O}_{Z,z}/m_z \mathcal{O}_{Z,z} \) is regular. This concludes the proof.

2.1.12. **Quasi-excellent schemes.** For shortness, we will abbreviate the word quasi-excellent as \( \text{qe} \). Since \( \text{qe} \) schemes are defined by two conditions which are of an interest of their own, we introduce corresponding classes of schemes. We say that \( X \) is an \( N \)-scheme if it is locally noetherian and for any \( Y \) of finite type over \( X \) the regular locus \( Y_{\text{reg}} \) is open. We say that \( X \) is a \( G \)-scheme if for any point \( x \in X \) the completion homomorphism \( \mathcal{O}_{X,x} \to \hat{\mathcal{O}}_{X,x} \) is regular. It was proved by Grothendieck that it suffices to check this only at the closed points. A scheme is \( \text{qe} \) if it is both \( G \) and \( N \) scheme. If in addition \( X \) is universally catenary then it is called excellent. We say that a ring is \( G \), \( N \), \( \text{qe} \) or excellent if its spectrum is so. We list few basic well known properties of \( \text{qe} \) schemes.

**Lemma 2.1.5.** (i) If \( X \) is \( N \), \( G \), \( \text{qe} \) or excellent then any \( X \)-scheme of finite type is so.

(ii) \( X \) is a \( G \)-scheme if and only if the local rings of all closed points \( x \in X \) are \( G \)-rings.

(iii) If \( X \) is a \( G \)-scheme then any completion homomorphism \( X \to \hat{X} \) is regular.
2.1.13. **Quasi-excellence and completions.** Unfortunately, the G-property can be lost when passing to formal completions. The question whether completion of a qe scheme is qe was open until very recently, when it was solved affirmatively by Ofer Gabber.

**Theorem 2.1.6** (Gabber). *Let $A$ be a noetherian ring with an ideal $I$ such that $A$ is complete in the $I$-adic topology. Then $A$ is qe if and only if $A/I$ is qe. In particular, formal completion preserves quasi-excellence.*

**Remark 2.1.7.** (i) This Theorem will be used in establishing basic properties of qe formal schemes and their morphisms since it allows to prove sharpest results. All our applications to usual schemes will be limited to the characteristic zero case. Since these applications only use some results about formal varieties (which are qe by results of Valabrega), one could avoid any usage of Theorem 2.1.6 similarly to [Tem], at cost of working with cumbersome intermediate formulations.

(ii) The main intermediate progress towards Gabber’s Theorem was done in the paper [NN] by Nishimura-Nishimura, where the same result was proved conditionally assuming the weak resolution of singularities for local qe schemes. In particular, this settled the case of characteristic zero by using Hironaka’s theorem (which covers local qe schemes). Alternatively, one can use the results of [Tem] as the desingularization input.

(iii) Gabber strengthened the proof of [NN] so that desingularization of local qe schemes is replaced with a regular cover in the topology generated by alterations and flat quasi-finite covers. This argument is outlined in Gabber’s letter to Laszlo. The existence of such a regular cover for any qe scheme is a subtle and important result by Gabber whose written version will (hopefully) be available soon. Actually, it is the only desingularization result established for all qe schemes.

2.1.14. **Categories.** In this paper, we denote by $\mathcal{QE}$ the category of generically reduced noetherian qe schemes and by $\mathcal{QE}_{\text{reg}}$ its subcategory containing only regular morphisms. Of main interest for our needs are the full subcategories $\operatorname{Var}_{p=0,\text{reg}}$ and $\mathcal{QE}_{p=0,\text{reg}}$ of $\mathcal{QE}_{\text{reg}}$. The objects in $\operatorname{Var}_{p=0,\text{reg}}$ are disjoint unions of generically reduced varieties of characteristic zero and the objects of $\mathcal{QE}_{p=0,\text{reg}}$ are generically reduced noetherian qe schemes over $\mathbb{Q}$. The reason to consider disjoint unions of varieties defined over different fields will become clear in §2.3.5.

2.2. **Blow up sequences.**

2.2.1. **Blow ups.** Basic facts about blow ups can be found in [Tem, §2.1] or in the literature cited there. Recall that the blow up $f : \text{Bl}_V(X) \to X$ along a closed subscheme $V$ is the universal morphism such that $f^*(V)$ is a Cartier divisor. We will use the terminology of [Tem] with one important exception: when we say that $Y \to X$ is a blow up (or $Y$ is a blow up of $X$), we always mean that the center $V$ of a blow up is fixed. So, strictly speaking, we always mean (usually implicitly) that an isomorphism $Y \simeq \text{Bl}_V(X)$ is fixed. The reason for this change is that we will study functoriality, and the choice of $V$ increases canonicity of constructions, see, for example, §2.2.5. Following the convention of [Kol] we call a blow up $\text{Bl}_V(X) \to X$ the empty or trivial blow up. This is the only blow up we will sometimes ignore. However, even empty blow ups play important synchronizing role when patching local desingularizations, see §§2.3.4–2.3.6. Note that though any blow up along a Cartier divisor induces an isomorphism on the level of schemes, it may play
a non-trivial role for functorial desingularization. In our case this is mainly the synchronization (similarly to empty blow up), but in the embedded case such blow ups induce non-trivial operations of strict and controlled transforms, so they cannot be ignored by no means.

2.2.2. Blow up sequences. A price one has to pay for using a finer notion of blow ups is that the composition \( X'' \to X' \to X \) of blow ups cannot be considered as a blow up in a natural way: though \( X'' \to X \) is isomorphic to a blow up \( \text{Bl}_W(X) \to X \), it is not clear how to choose \( W \) canonically. Therefore we define a blow up sequence of length \( n \) to be a composition of \( n \) blow ups \( X_n \xrightarrow{f_n} \cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0 \) with centers \( V_i \hookrightarrow X_i \). It will be convenient to denote a blow up sequence as \( f : X_n \to X_0 \) and say that \( V_i \)'s are its centers. We stress that a blow up sequence is not defined by the morphism \( X_n \to X \) even when we blow up regular schemes along regular centers, see [Kol, 3.33]. Using the dashed arrows we will sometimes split blow up sequences, e.g. \( f : X_n \to X_{i+1} \to X_i \to X_0 \).

2.2.3. Equality/isomorphism of blow up sequences. An isomorphism between two blow up sequences \( X'_n \to X_0 \) and \( X_n \to X_0 \) is a set of isomorphisms \( X'_i \cong X_i \) which identify \( X_0 \)'s and the centers. Obviously, if two blow up sequences of \( X \) are isomorphic then the isomorphism is unique. For this reason, we will often say by slight abuse of language that two such blow up sequences are equal, and this cannot cause any confusion. For example, this agreement allows us to say "unique" in Lemma 2.2.1 below instead of "unique up to unique isomorphism".

2.2.4. Trivial extension. We say that a blow up sequence \( \tilde{f} : \tilde{X} \to X \) is a trivial extension of a blow up sequence \( f : X' \to X \) if \( f \) is obtained from \( \tilde{f} \) by removing few empty blow ups (here we invoke our agreement about equality of blow up sequences).

2.2.5. Pushing forward with respect to pro-open immersions. One advantage of having the center \( V \) of a blow up \( f_U : U' \to U \) fixed is that for any pro-open immersion \( U \to X \) with a locally noetherian \( X \) we can canonically extend \( f_U \) to a blow up \( f : X' \to X \). Simply take the schematic closure of \( V \) in \( X \) to be the center of \( f \), then \( f_U = f \times_X U \) by Lemma 2.1.2. Iterating the same construction we obtain the following easy result.

**Lemma 2.2.1.** Assume that \( f_U : U' \to U \) is a blow up sequence and \( i : U \hookrightarrow X \) is a pro-open immersion with a locally noetherian \( X \). Then there exists a unique blow up sequence \( f : X' \to X \) such that \( f_U = f \times_X U \) and the centers of \( f_U \) are schematically dense in the centers of \( f \).

The blow up sequence \( f \) will be called the pushforward of \( f_U \) with respect to the pro-open immersion \( i \).

2.2.6. \( T \)-supported blow up sequences. Assume that \( X \) is an \( S \)-scheme, \( U \to S \) is an open subscheme, \( T = S \setminus U \) and \( V = X \times_S U \). Then we say that a blow up sequence \( X' \to X \) is \( T \)-supported if its centers lie over \( T \). In order to avoid any confusion with \( V \)-admissibility of the centers, we will not say that \( f \) is "\( V \)-admissible" in this situation (as opposed to [RG] and [Tem]). Actually, we will often be interested in the following two extreme cases: (a) the centers of \( f \) are \( V \)-admissible, and (b) the centers of \( f \) are \( T \)-supported. Note that (a) takes place if and only if \( f \) is the
pushforward of its restriction $f \times_X V$ with respect to $i : V \hookrightarrow X$, and (b) takes place if and only if $f \times_X V$ is trivial.

2.2.7. **Strict transform.** The following definitions and facts about blow up sequences follow from the well known particular case of usual blow ups (i.e. the sequences of length one), which can be found in [Con1, §1]. Given a closed subscheme $Z_0 \hookrightarrow X_0$ we define the *strict transform* $Z_n = f_i^!(Z_0)$ of $Z_0$ under $f : X_n \rightarrow X_0$ as the iterative strict transform of $Z_0$ with respect to $f_i$'s. Note that $Z_n \rightarrow Z_0$ is the blow up sequence whose centers are the scheme-theoretic preimages of the centers $V_i$ of $f$. More concretely, $f$ induces a blow up sequence $Z_n \rightarrow Z_0$, where $Z_i = f_i^!(Z_{i-1})$ is isomorphic to the blow up of $Z_{i-1}$ along $V_{i-1} \times_{X_{i-1}} Z_{i-1}$.

2.2.8. **Flat base changes.** Any blow up sequence $f : X_n \rightarrow X_0$ is compatible with any flat base change $Y_0 \rightarrow X_0$ in the sense that $Y_1 := Y_0 \times_{X_0} X_1$ is a flat $X_1$-scheme isomorphic to the blow up of $Y_0$ along $Y_0 \times_{X_0} V_0$, and similarly for all further blow ups in the sequence.

2.3. **Desingularization.**

2.3.1. **Desingularization of schemes.** Desingularization of a scheme $X$ is an $X_{\text{sing}}$-supported blow up sequence $f : X' \rightarrow X$ with regular $X'$. Though the definition makes sense for any locally noetherian scheme $X$, we will study only the case when $X$ is generically reduced. The reason for this was explained in Remark 1.2.2.

2.3.2. **Strong desingularization.** A desingularization of a scheme is called strong if the centers of its blow ups are regular schemes. We remark that most of the recent approaches based on order reduction of marked ideals lead to a desingularization which is not strong. In particular, this is the case for the algorithms from [Wi] and [Kol], see [Kol, 3.106] or [BM2, 8.2]. A strong desingularization for varieties of characteristic zero can be found in [Hir], [BM1], or [Vil].

2.3.3. **Compatibility with morphisms.** We say that desingularizations $g' : \overline{X} \rightarrow X'$ and $g : \overline{X} \rightarrow X$ of $X'$ and $X$, respectively, are compatible with respect to a flat morphism $f : X' \rightarrow X$ if $g'$ is a trivial extension of $g \times_X X'$, and they even coincide in the case when $f : X' \rightarrow X$ is surjective. We will see in Lemma 2.3.1 that the latter condition has simple but subtle and important consequences. Actually, we will be only interested in the case when $f$ is regular.

2.3.4. **Functorial desingularization.** If $\mathcal{C}$ is a subcategory of $\text{QCE}_{\text{reg}}$, then by a functorial (strong) desingularization on $\mathcal{C}$ we mean a rule (or a blow up sequence functor) $\mathcal{F}$ which to each object $X$ from $\mathcal{C}$ assigns a (strong) desingularization $\mathcal{F}(X) : \overline{X} \rightarrow X$ in a way compatible with the morphisms from $\mathcal{C}$, i.e. for any morphism $f : X' \rightarrow X$ from $\mathcal{C}$ the desingularizations $\mathcal{F}(X)$ and $\mathcal{F}(X')$ are compatible with respect to $f$. Clearly, one can view functorial desingularization as a functor to an appropriate category of blow up sequences, but we do not need to develop such formalized approach. A non-functorial desingularization corresponds to the case when $\text{Mor}(\mathcal{C})$ consists of identities only.

**Lemma 2.3.1.** If $\mathcal{F}$ is a functorial desingularization on $\mathcal{C}$ and $f, g : X' \rightarrow X$ are two morphisms in $\mathcal{C}$, then $f^* \mathcal{F}(X) = g^* \mathcal{F}(X)$ as blow up sequences (taking into account the empty blow ups).
Proof. Since \( \overline{f} = f \sqcup \text{Id}_X \) and \( \overline{g} = g \sqcup \text{Id}_X \) are surjective regular morphisms from \( \overline{X} = X' \sqcup X \) to \( X \), we have that \( \overline{f}^* \mathcal{F}(X) = \overline{g}^* \mathcal{F}(X) \). Restricting this equality over \( X' \hookrightarrow \overline{X} \) gives the required equality.

Remark 2.3.2. (i) Up to empty blow ups both \( f^* \mathcal{F}(X) \) and \( g^* \mathcal{F}(X) \) are equal to \( \mathcal{F}(X') \), so the Lemma actually asserts that the empty blow ups are inserted in the same places.

(ii) The proof might look as casuistics, but it has a real meaning. Desingularizing \( X \) and \( X' \) simultaneously we have to compare their singularities and decide which one should be blown up earlier, and the trace of this information on \( X' \) hides in the empty blow ups.

2.3.5. Restriction to affine subcategory.

Lemma 2.3.3. Let \( \mathcal{C} \) be any subcategory in \( \mathcal{QE} \) such that

(i) \( \mathcal{C} \) is closed under taking disjoint unions;

(ii) if \( f : Y \hookrightarrow X \) is an open immersion and \( X \) is in \( \mathcal{C} \) then \( f \) is in \( \mathcal{C} \).

Then any desingularization functor \( \mathcal{F} \) on \( \mathcal{C} \) is uniquely determined by its restriction \( \mathcal{F}_{aff} \) onto the full subcategory \( \mathcal{C}_{aff} \) formed by the affine schemes from \( \mathcal{C} \). Moreover, any (strong) desingularization functor \( \mathcal{F}_{aff} \) on \( \mathcal{C}_{aff} \) extends uniquely to a (strong) desingularization functor on \( \mathcal{C} \).

We only outline the proof of the Lemma, since the argument is known. If an object \( X \) of \( \mathcal{C} \) is covered by open affine subschemes \( X_1, \ldots, X_n \) then \( X' = \sqcup_{i=1}^n X_i \) is in \( \mathcal{C} \) and we can consider the desingularization \( \mathcal{F}_{aff}(X') : \sqcup_{i=1}^n Y_i \to X' \). The blow up sequences \( Y_i \to X_i \) agree over the intersections \( X_i \cap X_j \) by Lemma 2.3.1, and hence glue to a global desingularization \( \mathcal{F}(X) : Y \to X \). We refer to [Kol, 3.37] or [BM2, §7.1] for a detailed proof along this line.

2.3.6. Abstract invariant.

Remark 2.3.4. (i) It is critical for Lemma 2.3.3 to have disjoint unions in \( \mathcal{C} \) since \( \mathcal{F}_{aff}(X') \) contains more information than all blow up sequences \( \mathcal{F}_{aff}(X_i) \). This is exactly the information about the order of the blow ups in \( \mathcal{F}_{aff}(X') \) (in particular, sometimes we may simultaneously perform few blow ups from different \( \mathcal{F}_{aff}(X_i) \)'s).

Alternatively, one can notice that we glue trivial extensions of \( \mathcal{F}_{aff}(X_i) \)'s rather than these sequences themselves, and the list of inserted empty blow ups is the additional information. Obviously, we cannot combine a single blow up sequence \( \mathcal{F}_{aff}(X') \) without this information.

(ii) If an algorithm is controlled by an invariant, as in [W] or [BM2], then the condition (i) in the lemma is redundant. In this case, the invariant dictates the order of the blow ups, so the gluing is obvious. Thus, we use disjoint unions to implicitly encode nearly the same information as contained in the invariant.

(iii) To the best of my knowledge (which might be very incomplete), the idea to use disjoint unions instead of invariants is due to Kollar, see [Kol, 3.38]. However, as we will immediately see both approaches are rigorously equivalent.

(iv) To make sense of the above claims we associate to any desingularization functor \( \mathcal{F} \) an ordered set as follows. If \( X, Y \) are two schemes from \( \mathcal{C} \) with desingularizations \( \mathcal{F}(X) : X_0 \to X_0 = X \) and \( \mathcal{F}(Y) : Y_0 \to Y_0 = Y \) and points \( x \in X_i \) and \( y \in Y_j \), then we say that \( x \) and \( y \) are \( \mathcal{F} \)-equivalent if \( \mathcal{F}(X \sqcup Y) \) simultaneously blows them up for the first time. We warn the reader that the equivalence class of \( x \) depends on the whole tower \( X_i \to X \) rather then only on the local situation on
\(X_i\) (i.e. it depends on the history, at least to some extent). We denote the above equivalence class as \(\text{inv}(x)\), and the set \(\text{inv}(\mathcal{F})\) of all such equivalence classes is, obviously, a totally ordered set. Theoretically, for each \(x\) as above we have associated an invariant \(\text{inv}(x) \in \text{inv}(\mathcal{F})\) controlling \(\mathcal{F}\), though for practical applications one might often wish to have a more constructive description of the invariant.

2.3.7. Functorial desingularization of varieties. The strong desingularization from \([Hir]\) is not functorial (and algorithmic). The desingularization from \([BM1]\) is given in an explicit algorithmic way, and the functoriality of the latter algorithm with respect to smooth morphism was checked later in \([BM2]\). Finally, it was observed in \([BMT]\) that a general regular morphism between varieties can be reduced to smooth morphisms using certain limit procedures. Using this, compatibility with all regular morphisms was established in \([BMT]\), and the result was also extended to disjoint unions of varieties. We formulate the latter theorem for reader’s convenience, and it will be used essentially when we will generalize its assertion to \(\mathcal{Q}^p = 0, \text{reg}\).

**Theorem 2.3.5.** There exists functorial strong desingularization \(\mathcal{F}_{\text{Var}}\) on the category \(\text{Var}_{p=0, \text{reg}}\) such that the set \(\text{inv}(\mathcal{F}_{\text{Var}})\) of invariants is countable.

**Remark 2.3.6.** It follows from \([BMT]\) that any desingularization functor \(\mathcal{F}_{\text{Var}}\) on \(\text{Var}_{p=0, \text{reg}}\) is induced from its restriction \(\mathcal{F}_{\mathcal{Q}}\) to the varieties over \(\mathcal{Q}\) because any object of \(\text{Var}_{p=0, \text{reg}}\) admits a regular morphism to a variety over \(\mathcal{Q}\). It follows immediately that \(\text{inv}(\mathcal{F}_{\text{Var}}) = \text{inv}(\mathcal{F}_{\mathcal{Q}})\), but the latter set is countable because the geometry over \(\mathcal{Q}\) is countable (there are countably many varieties, points and blow up sequences over \(\mathcal{Q}\)). So, countability in Theorem 2.3.5 is automatic.

2.4. Formal analogs. In this section we recall very briefly basic notions from the theory of formal desingularization, and we refer to \([Tem, \S 3]\) for details. All formal schemes are assumed to be locally noetherian. Formal schemes and their ideals will be denoted as \(\mathfrak{X}, \mathfrak{Y}, \mathfrak{J} \subset \mathcal{O}\mathfrak{X}\), etc.

2.4.1. Closed fiber. The maximal ideal of definition defines a closed subscheme \(\mathfrak{X}_s\) called the closed fiber of \(\mathfrak{X}\). Topologically, \(|\mathfrak{X}| = |\mathfrak{X}_s|\).

2.4.2. Support of ideals. We say that an ideal \(\mathfrak{I} \subset \mathcal{O}\mathfrak{X}\) is supported on a closed subscheme \(\mathfrak{Z} = \text{Spf}(\mathcal{O}\mathfrak{Y}/\mathfrak{J})\) if \(\mathfrak{J}^n \subset \mathfrak{I}\) for large \(n\). So, an ideal is open if and only if it is supported on \(\mathfrak{X}_s\).

**Remark 2.4.1.** (i) For an open ideal \(\mathfrak{J}\) one can also define its support set-theoretically as \(|\text{Spf}(\mathcal{O}\mathfrak{X}/\mathfrak{J})|\) or as the associated reduced closed subscheme of \(\mathfrak{X}_s\) which is the reduction of \(\text{Spec}(\mathcal{O}\mathfrak{X}/\mathfrak{J})\).

(ii) In general, one can define support set-theoretically using a generic fiber of \(\mathfrak{X}\) (there are different definitions of the latter in rigid, analytic or adic geometries).

2.4.3. Quasi-excellent formal schemes. We give the following definition of quasi-excellence, which is a priori more restrictive than its analog in \([Tem, \S 3.1]\). A locally noetherian formal scheme \(\mathfrak{X}\) is quasi-excellent or qe if for any morphism \(\text{Spf}(A) \rightarrow \mathfrak{X}\) of finite type the ring \(A\) is qe. It follows from the Gabber’s Theorem 2.1.6 that \(\mathfrak{X}\) is qe if and only if it admits a covering by open subschemes \(\text{Spf}(A_i)\) with qe rings \(A_i\). In particular, Gabber’s theorem implies that all reasonable definitions of quasi-excellence coincide (including this definition and the definition in \([Tem]\)) and that quasi-excellence is preserved under taking formal completion along a closed subscheme.
2.4.4. Formal blow ups. In the affine case, a formal blow up $\hat{\text{Bl}}_I(\text{Spec}(A)) \rightarrow \text{Spec}(A)$ is the formal completion of the blow up $\text{Bl}_I(\text{Spec}(A)) \rightarrow \text{Spec}(A)$. This definition is compatible with formal localizations on the base and hence generalizes to the case of general formal blow up $\hat{\text{Bl}}_I(\mathfrak{X}) \rightarrow \mathfrak{X}$ along an ideal $\mathfrak{I} \subset O_{\mathfrak{X}}$.

2.4.5. Charts. Thus, the formal blow ups are glued from the charts $\text{Spf}(A/I,g)$ with $g \in I$, where we set $A/I,g = \hat{A}/I,g$. (We warn the reader that $A/I,g$ is not a subring of $A_{(g)}$ though $A/I,g \subset A_{(g)}$.)

2.4.6. Compatibility with usual blow ups. Formal completion is compatible with (formal) blow ups, i.e. it takes blow ups of schemes to formal blow ups of formal schemes.

2.4.7. Support. If $\mathfrak{X}$ is a formal $\mathfrak{S}$-scheme and $\mathfrak{I} \hookrightarrow \mathfrak{S}$ is a closed formal subscheme then we say that a formal blow up $\hat{\text{Bl}}_I(\mathfrak{X}) \rightarrow \mathfrak{X}$ is $\mathfrak{I}$-supported if $\mathfrak{I}$ is $\mathfrak{I} \times_{\mathfrak{X}} \mathfrak{X}$-supported, i.e. $\mathfrak{I}^nO_{\mathfrak{X}} \subset \mathfrak{I}$ where $\mathfrak{I} = \text{Spf}(O_{\mathfrak{S}}/\mathfrak{I})$.

2.4.8. Formal blow up sequences. A formal blow up sequence $f : \mathfrak{X}_n \rightarrow \mathfrak{X}$ is defined in an obvious way. Such a sequence is $\mathfrak{I}$-supported for a formal subscheme $\mathfrak{I} \hookrightarrow \mathfrak{X}$ if all centers of $f$ are $\mathfrak{I}$-supported.

2.4.9. Singular locus. Singular locus of a qc formal scheme is a reduced closed subscheme $\mathfrak{X}_{\text{sing}}$ or the corresponding ideal $\mathfrak{I} \subset O_{\mathfrak{X}}$. For an affine formal scheme $\text{Spf}(A)$ this is the ideal that defines $\text{Spec}(A)$, and this local definition globalizes because formal localizations are regular morphisms on qc formal schemes. Singular loci are compatible with formal completions: if a scheme $X$ and its formal completion $\hat{X} = \hat{X}_{\mathfrak{I}}$ are qc then $\mathfrak{X}_{\text{sing}}$ is the completion of $\mathfrak{X}_{\text{sing}}$ along $I\mathfrak{O}_{\mathfrak{X}_{\text{sing}}}$ by [Tem, 3.1.4]. One defines the non-reduced locus of a qc $\mathfrak{X}$ similarly, and says that $\mathfrak{X}$ is regular or reduced if the corresponding locus is empty.

Remark 2.4.2. Though I do not know such examples, it seems probable that regularity (and even reducedness) can be destroyed by formal localization of a noetherian adic ring. If this is the case then these notions do not make any sense for general noetherian formal schemes. At the very least, some examples show that reducedness can be destroyed by formal localizations in the non-noetherian case.

2.4.10. Desingularization. A (strong) desingularization of $\mathfrak{X}$ is defined similarly to the scheme case: it is an $\mathfrak{X}_{\text{sing}}$-supported formal blow up sequence with regular source (and regular centers).

2.4.11. Rig-regularity. We say that $\mathfrak{X}$ is rig-regular if its singular locus is given by an open ideal (i.e. is $\mathfrak{X}$-supported). Note that a desingularization in the rig-regular case is given by blowing up open ideals, i.e. blowing up formal subschemes which are usual schemes.

Lemma 2.4.3. Let $X$ be a qc scheme with a closed subscheme $Z$ such that the formal completion $\mathfrak{X} = \hat{X}_Z$ is qc, then

(i) any $Z$-supported blow up sequence $f : \mathfrak{X}' \rightarrow \mathfrak{X}$ is the completion of a uniquely defined $Z$-supported blow up sequence $f : X' \rightarrow X$;

(ii) if $\mathfrak{X}_{\text{sing}} \subset Z$ then any (strong) desingularization $f : \mathfrak{X}' \rightarrow \mathfrak{X}$ is the completion of a uniquely defined (strong) desingularization $f : X' \rightarrow X$. 

Proof. Any $Z$-supported closed subscheme of $X$ is given by an open ideal, so it is defined already as a closed subscheme in $X$. So, the center $V_0$ of the first blow up of $f$ is a closed subscheme of $X$ and we can set $X_1 = \text{Bl}_{V_0}(X)$. Then $X_1$ is the completion of $X_1$, hence we can algebraize the center $V_1 \hookrightarrow X_1$ by $V_1 \hookrightarrow X_1$, and proceed by induction on the length of $f$. This proves (i), and (ii) follows from (i) and the compatibility of formal completions with singular loci. \hfill \Box

2.4.12. Regular morphisms. A morphism $f : \mathcal{Y} \to X$ between qe formal schemes is regular if there exist affine coverings $X_i = \text{Spf}(A_i)$ and $\mathcal{Y}_i = \text{Spf}(B_i)$ of $X$ and $\mathcal{Y}$ such that $f(\mathcal{Y}_i) \subseteq X_i$ and the induced homomorphism $A_i \to B_i$ is regular.

2.4.13. Completions of regular morphisms.

Lemma 2.4.4. Let $A$ be a qe ring with an ideal $I$, $\hat{A}$ be its $I$-adic completion and $B$ be a noetherian $I$-adic $\hat{A}$-ring. Then the homomorphism $\hat{A} \to B$ is regular if and only if the homomorphism $A \to B$ is regular.

Proof. The direct implication is obvious since the completion homomorphism $A \to \hat{A} = C$ is regular by quasi-excellence of $A$. Conversely, suppose that $A \to B$ is regular. Since any prime ideal in $B$ is contained in an open prime ideal, in order to prove that $C \to B$ is regular we should show that for any open prime ideal $q \subseteq B$ with the preimage $p \subseteq C$ the homomorphism $f : C_p \to B_q$ is regular. Moreover, in view of Andre’s Theorem on localization of formal smoothness, see [And], it suffices to check that $f$ is formally smooth because $C$ is qe by Gabber’s Theorem 2.1.6. Recall that by [EGA, 0IV, 19.7.1 and 22.5.8] formal smoothness of the local homomorphism $f$ is equivalent to its flatness and geometric regularity of its closed fiber. The ideals $p$ and $r = p \cap A$ are open, hence $\hat{A}_r \to \hat{C}_p$, and then $rC_p = pC_p$ by quasi-excellence of $A$. Thus the closed fiber $B_q/pB_q \to \hat{B}_q$ of $f$ is geometrically regular over the residue field $C_p/pC_p \to \hat{A}_r/r\hat{A}_r$ by regularity of $A \to B$. Finally, the completion homomorphism $C_p \to \hat{C}_p$ is flat because $C_p$ is noetherian and the homomorphism $\hat{C}_p \to \hat{A}_r \to \hat{B}_q$ is flat because $A \to B$ is flat, hence $g : C_p \to \hat{B}_q$ is flat and we deduce that $f$ is flat because $g$ is its composition with the faithfully flat completion homomorphism $B_q \to \hat{B}_q$. \hfill \Box

Corollary 2.4.5. Let $f : Y \to X$ be a regular morphism between qe schemes, and $\mathcal{I} \subseteq \mathcal{O}_X$ and $\mathcal{J} \subseteq \mathcal{O}_Y$ be ideals with the completions $\hat{X} = \hat{X}_\mathcal{I}$ and $\hat{Y} = \hat{Y}_\mathcal{J}$. Then the completion $f : \hat{\mathcal{Y}} \to \hat{X}$ of $f$ is regular.

Proof. We can assume that $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$, and let $\hat{A}$ and $\hat{B}$ be the $\mathcal{I}$-adic and the $\mathcal{J}$-adic completions, respectively. Then the homomorphism $A \to B \to \hat{B}$ is regular, and therefore $\hat{A} \to \hat{B}$ is regular by Lemma 2.4.4. \hfill \Box

2.4.14. Regularity and affine subschemes.

Lemma 2.4.6. If a morphism $f : \mathcal{Y} \to X$ between qe formal schemes is regular and $X' = \text{Spf}(A')$ and $\mathcal{Y}' = \text{Spf}(B')$ are open formal subschemes of $X$ and $\mathcal{Y}$ such that $f(\mathcal{Y}') \subseteq X'$, then the induced homomorphism $A' \to B'$ is regular.

Proof. First we prove that regularity survives formal localizations. Namely, assume that $\mathcal{Y} = \text{Spf}(B)$, $X = \text{Spec}(A)$, the homomorphism $A \to B$ is regular, $A' = A_{\{f\}}$ and $B' = B_{\{g\}}$. Obviously, $A' \to B'$ is the completion of a regular homomorphism $A_f \to B_g$, which is a localization of $A \to B$. Hence $A' \to B'$ is a regular homomorphism by Corollary 2.4.5.
To complete the proof it now suffices to prove the following claim. Assume that \( \mathcal{Y} = \text{Spf}(B) \), \( \mathfrak{X} = \text{Spf}(A) \), \( \mathfrak{X} = \text{Spf}(A_j) \) and \( \mathcal{Y} = \mathcal{Y}_i = \text{Spf}(B_i) \) such that \( A_i = A_i(f)_i \), \( B_i = B_i(g)_i \), \( f(\mathcal{Y}_i) \subset \mathfrak{X}_i \) and the homomorphisms \( A_i \to B_i \) are regular. In particular, the compositions \( A \to A_i \to B_i \) are regular. Then we claim that the homomorphism \( A \to B \) is regular. Assume to the contrary that \( f : \text{Spec}(B) \to \text{Spec}(A) \) is not regular. Since any point of \( \text{Spec}(B) \) specializes to a point of \( \text{Spf}(B) \), there exists a point \( x \in \mathcal{Y} \subset \text{Spec}(B) \) such that the homomorphism \( f \) is not regular at \( x \). Since the morphisms \( \text{Spec}(B_i) \to \text{Spec}(B) \) are regular, the composed morphisms \( \text{Spec}(B_i) \to \text{Spec}(B) \to \text{Spec}(A) \) are not regular at the preimage of \( x \) by Lemma 2.1.4(i). However, \( x \) has a non-empty preimage in some \( \text{Spec}(B_i) \) because \( \mathcal{Y}_i \)'s cover \( \mathcal{Y} \). This contradicts the regularity of \( A \to B_i \), and therefore \( f \) is regular. \( \square \)

3. Extending \( \mathcal{F}_{\text{Var}} \) to schemes with small singular locus

Loosely speaking, the aim of §3 is to extend the functor \( \mathcal{F}_{\text{Var}} \) to generically reduced qe schemes over \( \mathbb{Q} \) whose singular locus is sufficiently small. More precisely, we will extend \( \mathcal{F}_{\text{Var}} \) to pairs \((X, Z)\) where \( X \) is a generically reduced noetherian qe scheme over \( \mathbb{Q} \) and \( Z \to X \) is a Cartier divisor isomorphic to a disjoint union of varieties and containing \( X_{\text{sing}} \). This is an intermediate result towards our proof of the main Theorem 1.2.1, so we do not pursue any generality in §3. The question of extending \( \mathcal{F}_{\text{Var}} \) to wider classes of schemes was discussed in Remark 1.3.1. The construction of \( \mathcal{F}_{\text{Var}}(X, Z) \) goes by completing \( X \) along \( Z \) and algebraizing the obtained formal variety, and the main difficulty is to prove that this construction is independent of the algebraization.

3.1. Extending \( \mathcal{F}_{\text{Var}} \) to formal varieties.

3.1.1. Formal varieties. A noetherian formal scheme \( \mathfrak{X} \) is called a formal variety if its closed fiber \( \mathfrak{X}_s \) is a variety.

Remark 3.1.1. (i) Formal varieties are called special formal schemes in [Tem, §3.2].

(ii) It is easy to prove that an equicharacteristic \( \mathfrak{X} \) is a formal variety if and only if locally it is of the form \( \text{Spf}(k[[T_1, \ldots, T_n]][[S_1, \ldots, S_m]]/I) \) where \( k \) is any field of definition of \( \mathfrak{X}_s \); see [Tem, 3.2.1] for a proof. Note that the latter formal scheme is of finite type over \( \text{Spf}(k[[S_1, \ldots, S_m]]) \) because of the equality

\[
 k[[T_1, \ldots, T_n]][S_1, \ldots, S_m] = k[[S_1, \ldots, S_m]][T_1, \ldots, T_n]
\]

3.1.2. Rig-smoothness. Let \( A \) be an adic ring and \( B = A\{T_1, \ldots, T_n\}/I \) be topologically finitely generated over \( A \). Following [Elk], one defines a Jacobian ideal \( H_{B/A} \) (see also [Tem, §3.3]). The construction of \( H_{B/A} \) is compatible with formal localizations, hence one obtains a Jacobian ideal \( H_{\mathcal{Y}/\mathfrak{X}} \) for any finite type morphism \( f : \mathcal{Y} \to \mathfrak{X} \). One can view the corresponding closed subscheme of \( \mathcal{Y} \) as the non-smoothness locus of \( f \). In particular, \( f \) is smooth if and only if \( H_{\mathcal{Y}/\mathfrak{X}} = 0 \).

An arbitrary morphism \( f : \mathcal{Y} \to \mathfrak{X} \) is rig-smooth if it is of finite type and \( H_{\mathcal{Y}/\mathfrak{X}} \) is open.

Remark 3.1.2. (i) Intuitively, rig-smoothness means that the "generic fiber" of \( f \) is smooth.
(ii) If $X = \text{Spf}(k[[\pi]])$ then the generic fiber $f_{\eta} : \mathfrak{X}_\eta \to \mathfrak{X}_\eta$ can be defined in the categories of rigid, analytic or adic spaces. In this case, rig-smoothness of $f$ is equivalent to smoothness of $f_{\eta}$ by [Tem, 3.3.2].

### 3.1.3. Algebraization of formal varieties

A formal variety is called (locally) algebraizable if (locally) it is isomorphic to a formal completion of a variety.

**Remark 3.1.3.** (i) It is well known that formal singularities can be non-algebraizable. So, a general formal variety does not have to be locally algebraizable.

(ii) The main algebraization tool is [Elk, Th. 7] by Renee Elkik. This Theorem implies that if $A$ possesses a principal ideal of definition and $\text{Spf}(B) \to \text{Spf}(A)$ is a rig-smooth morphism then $\text{Spf}(B)$ is $A$-algebraizable.

(iii) It is an interesting open question if the assumption about the principal ideal of definition is redundant. Because of this assumption we have to introduce the class of principal formal varieties below.

### 3.1.4. Locally principal formal varieties

By a locally principal formal variety we mean a pair $(\mathfrak{X}, \mathfrak{I})$ consisting of a formal variety $\mathfrak{X}$ and a locally invertible and open ideal $\mathfrak{I} \subset \mathcal{O}_{\mathfrak{X}}$, and we say that $(\mathfrak{X}, \mathfrak{I})$ is principal if $\mathfrak{I}$ is isomorphic to $\mathcal{O}_{\mathfrak{X}}$.

**Remark 3.1.4.** (i) It is easy to see that an affine equicharacteristic formal such scheme $\mathfrak{X}$ with an ideal $\mathfrak{I}$ is a principal formal variety if and only if it is of finite type over $(\mathfrak{S} = \text{Spf}(k[[\pi]]), (\pi))$, where $k$ is a field of definition of $\mathfrak{X}$ and $\pi$ is a generator of the ideal of definition (we refer to [Tem, 3.2.3] for details).

(ii) In the zero characteristic case, a principal formal variety $\mathfrak{X}$ is rig-regular if and only if it is rig-smooth over an $\mathfrak{S}$ as in (i).

(iii) It follows from (ii) and the theorem of Elkik that any affine rig-regular principal formal variety of characteristic zero is algebraizable; see [Tem, 3.3.1] for details.

### 3.1.5. Desingularization of rig-regular locally principal formal varieties

Let $\hat{\text{Var}}_{p=0}$ denote the category of all rig-regular locally principal formal varieties of characteristic zero with regular morphisms.

**Theorem 3.1.5.** There exists unique up to unique isomorphism desingularization functor $\hat{\mathcal{F}}_{\text{Var}}$ on $\hat{\text{Var}}_{p=0}$ such that $\hat{\mathcal{F}}_{\text{Var}}$ is compatible with $\mathcal{F}_{\text{Var}}$ under formal completions. Moreover, $\hat{\mathcal{F}}_{\text{Var}}$ is strong if and only if $\mathcal{F}_{\text{Var}}$ is strong.

Since formal varieties from the Theorem are locally algebraizable, the uniqueness is obvious. To prove existence we, at the very least, should show that if such an $\mathfrak{X}$ admits two algebraizations, say $\mathfrak{X} \xrightarrow{\sim} \hat{\mathfrak{X}}$ and $\mathfrak{X} \xrightarrow{\sim} \hat{\mathfrak{Y}}$, then the completions of the blow up sequences $\mathcal{F}_{\text{Var}}(X)$ and $\mathcal{F}_{\text{Var}}(Y)$ give rise to the same formal blow up sequence of $\mathfrak{X}$. We will solve this problem in §3.2 by showing that the desingularization of $X$ and $Y$ is canonically determined already by a sufficiently thick infinitesimal neighborhood of $X_s$.

### 3.2. Intrinsic dependence of the desingularization on the singular locus

Intuitively, it is natural to expect that for a scheme $X$ all information about its singularities, including the desingularization information, is contained in a "sufficiently thick" closed subscheme $Y$ with $|Y| = X_{\text{sing}}$. It is a very interesting and
difficult question how to define such property of $Y$ rigorously (i.e. the fact that $Y$ has enough nilpotents to keep all information about the singularities of $X$), but we will not study it here in general. We will only consider a very special case of what we call Elkik fibrations, which suffices to prove Theorem 3.1.5.

3.2.1. Principal affine pairs. By a principal affine pair over a field $k$ we mean a pair $(X, \mathcal{I})$ consisting of an affine $k$-variety $X = \text{Spec}(A)$ with a principal ideal $\mathcal{I} \subset O_X$ corresponding to $I \subset A$. The support of $\mathcal{I}$ will be called the closed fiber and we will denote it $X_s$. The closed fiber underlies schemes $X_n = \text{Spec}(A/I^n)$, and the pair $(X_n, \mathcal{I}_n)$, where $\mathcal{I}_n = \mathcal{I}O_{X_n}$ will be called the $n$-th fiber of $(X, \mathcal{I})$. Henselization or formal completion of the pair will be denoted $(X^h, \mathcal{I}^h)$ and $(\hat{X}, \hat{\mathcal{I}})$. We have natural regular morphisms $\hat{X} \to X^h \to X$ and closed immersions of the above schemes which are compatible with the ideals and induce isomorphisms on the $m$-th fibers for $m \leq n$.

3.2.2. Morphisms of pairs. A morphism between principal pairs $f : (X', \mathcal{I}') \to (X, \mathcal{I})$ is a morphism $h : X \to X'$ such that $\mathcal{I}' = \mathcal{I}O_X$. If $X$, $X'$ and $h$ are defined over $k$ then we say that $f$ is a $k$-morphism. Also, we define a morphism $\overline{f} : (\overline{X'}, \overline{\mathcal{I}}') \to (\overline{X}, \overline{\mathcal{I}})$ between $n$-th fibers, henselizations or completions as a morphism $\overline{h} : X \to X'$ that respects the ideals. Note that we do not impose any condition on the original pairs in this definition. A morphism $\overline{f}$ as above is said to be regular, isomorphism, etc. if $\overline{h}$ is so. Finally, to any morphism $f$ we obviously associate $n$-th fibers, henselization and completion which will be denoted $f_n$, $f^h$ and $\overline{f}$, and similarly to any morphism $\overline{f}$ between completions, henselizations or $n$-th fibers we can associate a morphism $\overline{f}_m$ between the $m$-th fibers, where $m \leq n$.

3.2.3. Elkik pairs. Until the end of §3.2 we assume that the characteristic is zero. By Elkik pair over $k$ we mean a principal affine pair $(X, \mathcal{I})$ over $k$ such that $X_{\text{sing}}$ is contained in the closed fiber $X_s$. Note that for any generator $\pi$ of $I$ the generic fiber of the induced morphism $f : X \to \mathbb{A}^1_k$ is a regular scheme, and hence $f$ is generically smooth by the characteristic zero assumption.

3.2.4. Recovery of a henselian Elkik pair from a sufficiently thick fiber. The following fundamental result about Elkik pairs is an easy corollary of Elkik’s results.

**Proposition 3.2.1.** Let $(X, \mathcal{I})$ and $(X', \mathcal{I}')$ be Elkik pairs over $k$. Then there exist numbers $n_0$ and $r$ depending only on $(\hat{X}, \hat{\mathcal{I}})$ and such that for any $n \geq n_0$ and a smooth $k$-morphism $f_n : (X'_n, \mathcal{I}'_n) \to (X_n, \mathcal{I}_n)$ there exists a smooth $k$-morphism $g : (X^h, \mathcal{I}^h) \to (X^h, \mathcal{I}^h)$ which lifts $f_{n-r}$. Moreover, if $f_n$ is an isomorphism then $g$ can be taken to be an isomorphism, and so the henselizations are $k$-isomorphic if and only if the $n$-th fibers are $k$-isomorphic for a single $n \geq n_0$.

**Proof.** We will assume that $n_0 \gg 3r \gg 1$ are large enough numbers depending on $(\hat{X}, \hat{\mathcal{I}})$ and $(\hat{X}', \hat{\mathcal{I}}')$, and we will eliminate dependency on $(\hat{X}', \hat{\mathcal{I}}')$ later. For simplicity, we will encode the dependency as $n_0(\hat{X}')$, $r(\hat{X})$, etc. It would be possible using the proof and the references to [Elk] to get precise bounds, but we will not care about them. Fix any generator $\pi$ of $\mathcal{I}$ and denote its images in $\mathcal{I}^h$ and $\mathcal{I}_n$ by $\pi$, then $X$ and all its derived schemes are provided with the compatible morphisms to $S = \text{Spec}(k[\pi])$. Let $\pi' \in \mathcal{I}_n$ be the image of $\pi$ under the isomorphism induced by $f_n$, and denote by $\pi'$ any lifting of $\pi'$ to $\hat{\mathcal{I}}'$. We view $X^h$ as an $S$-scheme via the morphism taking $\pi'$ to $\pi$. 


Let us first consider the case when $f_n$ is an isomorphism. By [Elk, Th. 2], if $n_0(\hat{X})$ and $r(\hat{X})$ are large enough then there exists a section of $X \times_S X^h \rightarrow X^h$ which lifts the $(n-r)$-th fiber’s section $f_{n-r} \times_S \text{Id}_{X^h_{n-r}} : X^h_{n-r} \rightarrow X_{n-r} \times_S X^h_{n-r}$. Projecting this section onto $X$ gives a morphism $X^h \rightarrow X$ lifting the isomorphism $f_{n-r}$. By the universal property of henselizations, the latter morphisms factors through a morphism $g : X^h \rightarrow X^h$ lifting $f_{n-r}$. Using [Elk, Th. 2] again, we find a section $g' : X^h \rightarrow X^h$ of $g$ which lifts $f_{n-2r}$, and a section of $g'$ which lifts $f_{n-3r}$. Since $g'$ has left and right inverses, it is an isomorphism lifting $f_{n-2r}$. It remains to replace the above $r$ with $2r$ and to note that since the henselizations are isomorphic, so are the completions, and hence $n_0, r$ depend only on $(\hat{X}, \mathcal{I})$. Strictly speaking, $n_0$ and $r$ also depend on the choice of $\pi$, but this can be easily by-passed by choosing $\pi$ so that $n_0, r$ are minimal possible.

Let now $f_n$ be an arbitrary smooth morphism. By [Elk, Th. 6] the $X_n$-smooth scheme $X'_n$ can be lifted to an $X^h$-smooth scheme $X''$. It follows from the description of the numbers $n_0, r$ in terms of the Jacobian ideals in [Elk] that they can only drop when passing to an $\hat{X}$-smooth formal scheme $\hat{X}''$ (and they are preserved under smooth covers). Thus, the numbers $n_0(\hat{X})$ and $r(\hat{X})$ fit $\hat{X}''$ as well. Then, by the first part, the henselization of $\hat{X}''$ along the ideal generated by $\pi$ is isomorphic to $X^h$ and the isomorphism can be chosen so that it induces identity on the $(n-r)$-th fibers. In particular, the induced morphism $X^h \rightarrow X^h \rightarrow X^h$ is as required.

**Corollary 3.2.2.** Let $(X, \mathcal{I})$, $(X', \mathcal{I}')$, $n$ and $r$ be as in Proposition 3.2.1. Then there exist a smooth morphism $g : X'' \rightarrow X$ and an etale morphism $h : X'' \rightarrow X'$ inducing an isomorphism $X'' \cong X^h$ such that the induced morphism $g_{n-r} \circ h^{-1}_{n-r} : X^h_{n-r} \rightarrow X^h_{n-r}$ equals to $f_{n-r}$.

**Proof.** By Proposition 3.2.1 we can lift $f_{n-r}$ to a smooth morphism $f^h$ between the henselizations. Since $X' = \text{proj lim} X'_n$ where $X'_n \rightarrow X'$ are etale morphisms inducing isomorphisms on henselizations, we can apply [EGA, IV, 8.13.1] to approximate the composed morphism $X^h \rightarrow X^h \rightarrow X$ by an etale morphism $X'_n \rightarrow X$. Then we can take $X'' = X'_n$. \qed

**3.2.5. Restriction of a desingularization on an Elkik fiber.** In §3.2.5, all schemes and morphisms are defined over a field $k$. Let $(X, \mathcal{I})$ be an Elkik pair over $k$, and assume that $f : X^{(p)} \rightarrow X^{(0)}$ is a (strong) desingularization of $X = X^{(0)}$. In particular all centers $V^{(i)} \hookrightarrow X^{(i)}$ of $f$ sit over $X$, and hence we can choose $l$ such that each $V^{(i)}$ is contained in the $l$-th fiber $X^{(i)}_l$. To simplify the notation we set $X' = X^{(1)}$, $V' = V^{(1)}$ and $V = V^{(0)}$. Choose numbers $n_0 = n_0(\hat{X}^{(1)})$ and $r_1 = r(\hat{X}^{(1)})$ as in Proposition 3.2.1, set $n_0 = \max_{1 \leq i \leq p} n_0_i$ and $r = \max_{1 \leq i \leq p} n_0_i$, and choose any number $n \geq \max(l, n_0)$. Then we define the naive restriction of $f$ onto the $n$-th fiber as the sequence $f_n : X^{(p)}_n \rightarrow X^{(0)}_n$ provided with the closed subschemes $V^{(1)} \hookrightarrow X^{(1)}_n$. A serious disadvantage of the naive restriction is that the situation is not fully controlled by $X_n$ because there might be non-trivial $X_n$-automorphisms $\sigma$ of $X'_n$, even though each $X$-automorphism of $X'$ is trivial. So, the subscheme $V \hookrightarrow X$ completely defines the morphism $X' \rightarrow X$, but given only $X_n$ and $V \hookrightarrow X_n$ we cannot reconstruct the morphism $X'_n \rightarrow X_n$. Fortunately, it turns out that any such $\sigma$ is the identity modulo large enough power of $\mathcal{I}$, so sufficiently thick fibers $X'_m$ are determined uniquely.
Lemma 3.2.3. Keep the above notation, and assume that \( n > l + r \). Then the \( X_n \)-scheme \( X'_{n-l-r} \) is determined by \( X_n \) and \( V \) up to a unique \( X_n \)-isomorphism.

Proof. Let us explain first how one can non-canonically construct \( X'_{n-l-r} \) from \( X_n \) and \( V \). Find any realization of \( (X_n, I_n) \) as the \( n \)-th fiber of an Elkik pair \( (Y, \mathcal{I}) \). The original \( (X, \mathcal{I}) \) is a possible choice, but since we are proving intrinsic dependency on \( X_n \) there are other equally good alternatives. Set \( Y' = \text{Bl}_V(Y) \) and consider the fiber \( Y'_{n-l-r} \). To prove the Lemma we should show that the above a priori non-canonical construction is actually canonical, and we will do that by establishing a canonical \( X_n \)-isomorphism \( f'_{n-l-r} : Y'_{n-l-r} \rightarrow X'_{n-l-r} \). By Proposition 3.2.1 there exists an isomorphism \( f^h : Y^h \cong X^h \) which identifies the fiber of an Elkik pair in a way which is canonical up to the choice of \( f \). Since \( f^h \) is determined by \( f \), we can assume \( f = f^h \). Then the \( n \)-th fibers. In the sequel it will be more convenient to work with the formal completion \( \hat{f} : \hat{Y} \rightarrow \hat{X} \) and the completed blow up \( \hat{Y}' \rightarrow \hat{Y} \) (the reason for switching to formal schemes is that the theory of henselian schemes, their blow ups, etc., was not developed in the paper). Since \( V \hookrightarrow X_n \), \( \hat{f} \) induces an isomorphism \( \hat{f}' : \hat{Y}' \rightarrow \hat{X}' \) whose \((n-l-r)\)-th fiber is an isomorphism \( f'_{n-l-r} \). It remains to prove uniqueness of \( f'_{n-l-r} \), and this follows from the following claim by taking \( m = n - r \). For any automorphism \( \phi \) of \( \hat{Y} \) and the induced automorphism \( \phi' \) of \( \hat{Y}' \), if \( \phi \) induces an identity on some \( Y_m \) then \( \phi' \) induces an identity on \( Y_{m-l} \).

The latter claim reduces to a simple computation on charts of \( \hat{Y}' \). Since \( \phi \) acts identically on \( |\hat{Y}| = |Y_m| \) we can assume \( \hat{Y} = \text{Spf}(A) \) is affine, and then \( \hat{V} = V(J) \) for an ideal \( J \subset A \) containing \( \pi^l \), where \( \pi \) is a generator of \( \mathcal{I} \). By \$2.4.5 \$, \( \hat{Y}' \) is covered by charts \( Z_g = \text{Spf}(A(J/g)) \) with \( g \in J \). Note that \( \phi' \) moves \( Z_g \) to \( Z_{\phi(g)} \) and each time \( \text{Spf}(A(J/g)) \) to \( \phi(J/g) \in A(J/\phi(g)) \). In the intersection \( Z_{\phi(g)} = \text{Spf}(A(J\phi(g))) \) we have that

\[
\frac{f}{g} - \phi\left(\frac{f}{g}\right) = \frac{f}{g} - \frac{f + \pi^m a}{g + \pi^m b} = \pi^m l \cdot \frac{b - a g}{\phi(g)} \in \pi^m l A(J^2/\phi(g))
\]

This proves that \( \phi'_{m-l} \) acts trivially on \( (Z_{\phi(g)})_{m-l} \). Also, the same argument shows that the open immersions \( (Z_{\phi(g)})_{m-l} \hookrightarrow (Z_g)_{m-l} \) and \( (Z_{\phi(g)})_{m-l} \hookrightarrow (Z_{\phi(g)})_{m-l} \) are actually isomorphisms. So, \( \phi'_{m-l} \) preserves the charts and acts trivially on them, and hence is an identity. \( \square \)

The lemma implies that for large enough \( n \) the subscheme \( V(0) \hookrightarrow X_n^{(0)} \) defines \( X_n^{(1)} \) up to a unique isomorphism, the subscheme \( V(1) \hookrightarrow X_n^{(1)} \) defines \( X_n^{(2)} \) up to a unique isomorphism, etc. The tower \( X_n^{(p)} \rightarrow \cdots \rightarrow X_n^{(0)} \) induced by the desingularization \( f : X^{(p)} \rightarrow X^{(0)} \) and the centers \( V(i) \hookrightarrow X_n^{(i)} \) for \( 0 \leq i \leq p - 1 \) will be called the \( n \)-th restriction of \( f \) onto the Elkik fiber and will be denoted \( f|_{X_n^{(i)}} \). The restriction allows to encode the desingularization in Elkik fibers in a way which is canonical up to the choice of \( n \). The latter is not a real trouble, since any choice of large enough \( n \) does the job equally well, and the subschemes \( V(i) \hookrightarrow X_n^{(i)} \) and \( V(i) \hookrightarrow X_n^{(i)} \) with \( N \geq n \) are identified by the closed immersion \( X_n^{(i)} \hookrightarrow X_n^{(i)} \). For this reason, we will not worry about \( n \) in the sequel, and each time \( n \) appears in the notation of a restriction we will assume that it is large enough.
3.2.6. Compatibility with smooth $k$-morphisms. Let $(X, \mathcal{I})$ and $(Y, \mathcal{I}')$ be Elkik pairs, $f : X' \to X$ and $g : Y' \to Y$ be some desingularizations, and consider the restriction $\mathcal{J} = f|_{X'}_*$ and $\mathcal{G} = g|_{Y'}_*$ for large enough $n$ (depending on $X$ and $Y$). Given a flat morphism $h : (Y_n^2, \mathcal{I}'_n) \to (X_n^2, \mathcal{I}_n^2)$ between the Elkik fibers we denote by $h^*(\mathcal{J})$ the base change of $\mathcal{J}$ with respect to $h$, i.e. the tower $h^*(X_{n(n-1)}) \to \cdots \to h^*(X_{n(0)})$ with the subschemes $h^*(V^{(i)})$ where by definition $h^*(Z) = Z \times_{X_n^2} Y_n^2$ for an $X_n^2$-scheme $Z$. We say that $\mathcal{J}$ and $\mathcal{G}$ are compatible with $h$ if $\mathcal{G}$ is obtained from $h^*(\mathcal{J})$ by eliminating empty "blow ups", i.e. isomorphisms $h^*(X_{n(i-1)}) \cong h^*(X_{n(n-i-1)})$ with empty centers $h^*(V^{(i-1)})$.

Lemma 3.2.4. Let $\mathcal{F}$ be a desingularization algorithm for $k$-varieties which is compatible with $k$-smooth morphisms. Then the restriction of $\mathcal{F}$ on Elkik fibers is compatible with smooth $k$-morphisms between the fibers. In particular, for large enough $n$ the $n$-th restriction of $\mathcal{F}(X)$ depends up to a canonical isomorphism only on the $k$-variety $X_n^2$ with a principal ideal $\mathcal{I}_n^2$.

Proof. Let $(X, \mathcal{I})$ and $(Y, \mathcal{I}')$ be Elkik pairs over $k$, and $h : (Y_n^2, \mathcal{I}'_n) \to (X_n^2, \mathcal{I}_n^2)$ be a smooth $k$-morphism. By Corollary 3.2.2 we can lift $h_{n^2-r}$ (with $r$ depending only on $X$) to a smooth morphism of Elkik pairs $f : (Z, \mathcal{J}') \to (X, \mathcal{I})$ and an etale morphism of Elkik pairs $g : (Z, \mathcal{J}') \to (Y, \mathcal{I}')$ whose henselization $g'$ is an isomorphism and such that $h_{n^2-r} = f_{n^2-r} \circ g^{-1}_{n^2-r}$. By our assumption, $\mathcal{F}$ is compatible with $f$ and $g$, so $f^*(\mathcal{F}(X))$ is a trivial extension of $\mathcal{F}(Z)$ and similarly for $g$. Moreover, $g^*(\mathcal{F}(Y))$ actually coincides with $\mathcal{F}(Z)$ because $Y_{\text{sing}} \subset Y_s$ and hence $Y_{\text{sing}}$ is contained in the image of $Z$. So, passing to the $n$-th restrictions we obtain that $h^*(\mathcal{F}(X)|_{n^2})$ is a trivial extension of $\mathcal{F}(Z)|_{n^2} \cong \mathcal{F}(Y)|_{n^2}$.

3.2.7. Compatibility with regular morphisms. Now, we are going to generalize Lemma 3.2.4 to the case of regular morphisms $h : Y_n^2 \to X_n^2$. Note that even if $h$ is smooth and $X, Y$ are $k$-varieties, $h$ does not have to be a $k$-morphism. For this reason we do not have an analog of Corollary 3.2.2 anymore. Our strategy will be to reduce to $\mathbb{Q}$-varieties using approximation and to use that Lemma 3.2.4 covers all regular morphisms between $\mathbb{Q}$-varieties.

Proposition 3.2.5. Let $\mathcal{F}$ be a desingularization functor for varieties of characteristic zero that is compatible with all regular morphisms. Then the restriction of $\mathcal{F}$ on Elkik fibers is compatible with all regular morphisms between the fibers. In particular, for large enough $n$ the $n$-th restriction of $\mathcal{F}(X)$ depends only on the scheme $X_n^2$ with a principal ideal $\mathcal{I}_n^2$.

Proof. Let $(X, \mathcal{I})$ and $(Y, \mathcal{I}')$ be Elkik pairs, and $h : (Y_n^2, \mathcal{I}'_n) \to (X_n^2, \mathcal{I}_n^2)$ be a regular morphism (with $n$ large enough). By [BMT] $X$ is the filtered projective limit of a family $X_\alpha$ of affine $\mathbb{Q}$-varieties with smooth affine transition morphisms, and then the projections $X \to X_\alpha$ are regular. Moreover, taking only $\alpha \geq \alpha_0$ we can easily achieve that $\mathcal{I}$ is already defined on each $X_\alpha$, and each pair $(X_\alpha, \mathcal{I}_\alpha)$ is an Elkik fiber over $\mathbb{Q}$. Define in a similar way a family $Y_{\alpha'}$ with limit $Y$. Since $\mathcal{F}$ is compatible with regular morphisms, $\mathcal{F}(X)$ is induced from $\mathcal{F}(X_\alpha)$, and hence the restriction of $\mathcal{F}$ to $X_n^2$ (resp. $Y_n^2$) is induced from its restriction to each $X_{\alpha,n^2}$ (resp. $Y_{\alpha',n^2}$). Fix any $Y_{\alpha',n^2}$, then by [EGA, IV.1, 8.13.1] the morphism $h_{\alpha'} : X_{\alpha,n^2} \to Y_{\alpha',n^2}$ is induced from a morphism $h_{\alpha,\alpha'} : X_{\alpha,n^2} \to Y_{\alpha',n^2}$ with large enough $\alpha$. The latter morphism does not have to be smooth but its restriction
onto a neighborhood $U$ of the image of $X_{n^2}$ is smooth because the composition of $h_{\alpha,\alpha'}$ with the regular morphism $X_{n^2} \to X_{n,n^2}$ is the regular morphism $h_{\alpha'}$. Replacing $X_{n,n^2}$ with $U$ we can assume that $h_{\alpha,\alpha'}$ is a smooth morphism of $\mathbb{Q}$-varieties. Then $h_{\alpha,\alpha'}$ is compatible with the restriction of $F$ by Lemma 3.2.4, and therefore $h$ is compatible with the restriction as well. \hfill \Box

3.3. Functionarial desingularization via algebraization.

3.3.1. Proof of Theorem 3.1.5. Let $(\mathfrak{X}, \mathfrak{J})$ be an affine principal rig-regular formal variety of characteristic zero. Choose a generator $\pi \in \mathfrak{J}$, then a rig-smooth morphism $\mathfrak{X} \to \mathfrak{G} = \text{Spf}(k[[\pi]])$ arises and therefore there exists an algebraization $(\mathfrak{X}, \mathfrak{J}) \to \tilde{(\mathfrak{X}, \mathfrak{I})}$, where $(\mathfrak{X}, \mathfrak{I})$ with $\mathfrak{I} = (\pi)$ is an Elkik pair over $k$. The completion of the desingularization $F_{\text{Var}}(X)$ of $X$ is a desingularization $\hat{f} : \mathfrak{X} \to \mathfrak{X}$, and we proved in Proposition 3.2.5 that $\hat{f}$ intrinsically depends on a fiber $(X_n, \mathcal{I}_n)$. In particular, $\hat{f}$ intrinsically depends on $(\mathfrak{X}, \mathfrak{J})$, so we can denote it as $\hat{F}_{\text{Var}}(\mathfrak{X})$. Furthermore, Proposition 3.2.5 implies that the functor $\hat{F}_{\text{Var}}$ (which is defined so far in the affine case) is compatible with any regular morphism $h : (\mathfrak{Y}, \mathfrak{J}') \to (\mathfrak{X}, \mathfrak{J})$ because the fibers $h_n : (\mathfrak{Y}_n, \mathcal{I}'_n) \to (X_n, \mathcal{I}_n)$ are regular. It follows by the standard gluing argument from §§2.3.5–2.3.6 that the definition of $\hat{F}_{\text{Var}}$ extends to all rig-regular locally principal formal varieties of characteristic zero, and the obtained desingularization is compatible with all regular morphisms.

3.3.2. Desingularization of schemes with small singular locus. Consider the category $\mathcal{C}_{\text{small}}$ as follows. Objects of $\mathcal{C}_{\text{small}}$ are pairs $(X, Z)$, where $X$ is a generically reduced noetherian qe scheme of characteristic zero and $Z$ is a Cartier divisor in $X$ which contains $X_{\text{sing}}$ and is a disjoint union of varieties. Morphisms in $\mathcal{C}_{\text{small}}$ are morphisms $(X', Z') \to (X, Z)$ such that $X' \to X$ is regular and $Z' \sim_X X' \times_X Z$.

**Theorem 3.3.1.** The functor $F_{\text{Var}}$ extends to a desingularization functor $F_{\mathfrak{C}}$ which assigns to a pair $(X, Z)$ a desingularization of $X$ in a way functorial with all morphisms from $\mathcal{C}_{\text{small}}$. If $F_{\text{Var}}$ is strong then $F_{\mathfrak{C}}$ is strong.

**Proof.** To define $F_{\mathfrak{C}}$ we note that $\mathfrak{X} := \tilde{X}_Z$ is a rig-regular locally principal formal variety by [Tem, 3.1.5(ii)], hence it admits a desingularization $\hat{f} = \hat{F}_{\text{Var}}(\mathfrak{X})$ by Theorem 3.1.5. Since $\mathfrak{X}$ is rig-regular, the centers of its desingularization are $\mathfrak{X}_{\mathfrak{X}}$-supported. By Lemma 2.4.3 $\hat{f}$ algebraizes to a desingularization $f : X' \to X$, and $f$ is strong if and only if $\hat{f}$ is strong. It remains to observe that given a morphism $f : (X', Z') \to (X, Z)$ we obtain a morphism of their completions $\tilde{f} : \mathfrak{X}' \to \mathfrak{X}$ because $Z' = X' \times_X Z$. Moreover, if $f$ is regular then $\tilde{f}$ is regular by Corollary 2.4.5. Since $\hat{F}_{\text{Var}}$ is compatible with any regular morphism $\tilde{f}$, we obtain that $F_{\mathfrak{C}}$ is functorial with respect to all morphisms from $\mathcal{C}_{\text{small}}$. \hfill \Box

**Remark 3.3.2.** (i) Though it is reasonable to expect that $F_{\mathfrak{C}}$ is functorial with respect to all regular morphisms $X' \to X$, we did not prove that. Moreover, it is even unclear if the desingularization of $(X, Z)$ is actually independent of the choice of $Z$ (in case, there are few possibilities).

(ii) Since it cannot cause any confusion, we will denote $F_{\mathfrak{C}}$ as $F_{\text{Var}}$ in the sequel.

4. Construction of $\mathcal{F}$

4.1. Induction on codimension.
4.1.1. Unresolved locus. When working on (strong) desingularization of a scheme $X$ we will use the following terminology: given an $X_{\text{sing}}$-supported blow up sequence $f : X' \to X$ by the unresolved locus of $f$ we mean the smallest closed set $f_{\text{sing}} = T \subset X$ such that $f$ induces a (strong) desingularization on $X \setminus T$. Note that $T = f(X'_{\text{sing}})$ (resp. $T$ is the union of $f(X'_{\text{sing}})$ and the images of all singular loci of the centers of $f$).

4.1.2. Desingularization up to codimension $d$. We say that a blow up sequence $f : X' \to X$ is a (strong) desingularization up to codimension $d$ if $f_{\text{sing}} \subset X^{\geq d}$. This happens if and only if $f$ induces a (strong) desingularization on a neighborhood on $X^{\leq d}$.

**Remark 4.1.1.** (i) If $d = \dim(X)$ then desingularization is the same as desingularization up to codimension $d$. On the other side, if $X' \to X$ is a desingularization up to codimension $d - 1$ then $X'$ can have singularities in any codimension and their local structure can be even worse than that of the original $X$. However, the global structure of $X'_{\text{sing}}$ is very simple because it is contractible to a finite set of closed points of $X$. In particular, $X'_{\text{sing}}$ is a disjoint union of proper varieties, while $X_{\text{sing}}$ can be even an infinite dimensional scheme.

(ii) The desingularization of integral schemes in [Tem] is constructed as a blow up sequence $X_n \to X_0 = X$, where each $X_d$ desingularizes $X$ up to codimension $d$ and the blow up $X_{d+1} \to X_d$ is $T$-supported for some closed $T \subset X$ disjoint from $X^{\leq d}$. Thus, the desingularization is built successively by improving the situation over $X^1$, $X^2$, etc. To construct the blow up $X_{i+1} \to X_i$ we should care only for the preimages of finitely many "bad" points from $X^{i+1}$, and, similarly to (i), this reduces the problem to the case when $X_{\text{sing}}$ is a variety. The latter case is reduced to desingularization of varieties by completing and algebraizing, similarly to our desingularization of $C_{\text{small}}$ in §3.3.2.

(iii) We will adopt a similar strategy here with two modifications as follows: (a) we will also insert new $T$-supported blow ups in the middle of the sequence in order to correct the old centers over $X^{d+1}$ (in case of strong desingularization) and in order to make the preimage of each bad point to a Cartier divisor, and (b) we will work with functors $\mathcal{F}^{\leq d}$ on $\mathcal{Q} \mathcal{E}^p_0 \text{-reg}$ rather than with desingularizations of single schemes. In a sense, we will construct $\mathcal{F}$ by establishing an exhausting filtration $\mathcal{F}^{\leq d}$ by its blow up subsequences that desingularize each $X$ up to codimension $d$. This plan will be precisely formulated in 4.1.5.

4.1.3. Equicodimensional blow up sequences. We say that a blow up sequence or a desingularization $f : X_n \to X_0$ of a locally noetherian scheme $X = X_0$ is equicodimensional if for each center $V_i$ of $f$ there exists a number $d$ such that $V_i$ is disjoint from the preimage of $X^{<d}$ and $V_i$ is $X^{\leq d}$-admissible. Let $g_i : X_i \to X$ denote the natural projection, then the above condition can be re-stated as follows: $g_i(V_i)$ is of pure codimension $d$ and for the discrete set $T_i = g_i(V_i) \cap X^d$ of its maximal points the set $g_i^{-1}(T_i) \cap V_i$ is schematically dense in $V_i$. In the above situation we say that $V_i$ is of pure $X$-codimension $d$.

4.1.4. Filtration by codimension.

**Lemma 4.1.2.** Let $f : X_n \to X_0 = X$ be an equicodimensional blow up sequence. Then there exists a unique blow up sequence $f^{\leq d-1} : X_m \to X$ such that all centers of $f^{\leq d-1}$ are of $X$-codimension strictly smaller than $d$ and if $U$ is obtained from $X$...
by removing the images of all centers of $f$ of $X$-codimension $\geq d$, then $f^{\leq d-1} \times_X U$ is obtained from $f \times_X U$ by removing empty blow ups.

Proof. The lemma dictates what the restriction of $f^{\leq d-1}$ over $U$ is. Since the centers of $f^{\leq d-1}$ are of $X$-codimension strictly smaller than $d$, the whole $f^{\leq d-1}$ must be the pushforward of $f^{\leq d-1}|_U$ with respect to the open immersion $U \hookrightarrow X$ (i.e. we simply take the centers of $f^{\leq d-1}$ to be the schematical closures of the centers of $f^{\leq d-1}|_U$, as was observed in §2.2.6(a)).

4.1.5. The strategy of constructing $F$ and $F^{\leq d}$. Let $F$ be an equicodimensional desingularization functor. Then we can define functors $F^{\leq d}$ by setting $F^{\leq d}(X) = (F(X))^{\leq d}$. The procedure of removing centers of large $X$-codimension is compatible with regular morphisms, so we indeed get a functor. Moreover, each $F^{\leq d}$ is an up to codimension $d$ desingularization functor because $F(X)|_U = F^{\leq d}(X)|_U$ for a neighborhood $U$ of $X$ of $X^{\leq d}$. The sequence of functors $F^{\leq d}$ is compatible in the sense that $(F^{\leq d})^{\leq e} = F^{\leq e}$ for any pair $e \leq d$.

Lemma 4.1.3. (i) Each equicodimensional desingularization functor $F$ defines a compatible sequence $\{F^{\leq d}\}_{d \in \mathbb{N}}$ of equicodimensional desingularizations up to codimension $d$.

(ii) Conversely, assume that $\{F^{\leq d}\}_{d \in \mathbb{N}}$ is a compatible sequence of equicodimensional desingularizations up to codimension $d$, and for any noetherian $X$ and $e \geq d$ the centers of $F^{\leq e}(X)$ of $X$-codimension $> d$ are $T_d$-supported, where $T_d = F^{\leq d}(X)$ is the unresolved locus of $F^{\leq d}(X)$. Then the sequence $\{F^{\leq d}(X)\}_{d \in \mathbb{N}}$ stabilizes for large $d$’s, and hence the sequence $\{F^{\leq d}\}_{d \in \mathbb{N}}$ gives rise to an equicodimensional desingularization functor $F$ on noetherian schemes.

Proof. The assertion of (i) was observed earlier. Assume given a sequence $F^{\leq d}$ and a noetherian scheme $X$ as in (ii). The condition on the centers implies that $F^{\leq d}(X)|_{X \setminus T_d} = F^{\leq d}(X)|_{X \setminus T_d}$. In particular, $T_1 \supseteq T_2 \supseteq T_3 \ldots$ and hence this sequence stabilizes by noetherian induction. However, $T_n$ is of codimension $n$, hence the only possibility for stabilization is that $T_n = \emptyset$ for large enough $n$, and then the sequence $F^{\leq d}(X)$ stabilizes for $d \geq n$.

We will construct successively a compatible family $F^{\leq d}$ as in Lemma 4.1.3(ii). Each $F^{\leq d+1}$ will be obtained by inserting few blow ups into $F^{\leq d}$ in order to correct it over the general points of $F^{\leq d}_{\text{sing}}$. First we have to prove few easy claims about inserting blow ups into a blow up sequence, and this will be done in §4.2.

4.2. Operations with blow up sequences.

4.2.1. Pushing forward with respect to closed immersions.

Lemma 4.2.1. Let $X_0$ be a scheme with a closed subscheme $V_0$ and $g : V_n \dashrightarrow V_0$ be a blow up sequence of length $n$. Then there exists a unique blow up sequence $f : X_n \dashrightarrow X_0$ of length $n$ such that for each $1 \leq i < n$ the $i$-th strict transform $V'_i \hookrightarrow X_i$ of $V_0$ is $V_0$-isomorphic to $V_i$ and for each $0 \leq i < n$ the $i$-th center of $f$ is contained in $V'_i$ and is mapped isomorphically onto the $i$-th center of $g$ by the $V_0$-isomorphism $V'_i \cong V_i$.

Proof. The center of the first blow up of $f$ must coincide with that of $g$. Then the strict transform of $V_0$ is $V_0$-isomorphic to $V_1$ (and the isomorphism is unique
because both are modifications of $V_0$). The isomorphism $V'_1 \sim V_1$ dictates the choice of the second center of $f$, etc.

In the situation of the Lemma, we will say that $f$ is the pushforward of $g$ with respect to the closed immersion $V_0 \hookrightarrow X_0$.

4.2.2. Extending.

**Definition 4.2.2.** Assume that $f : X_n \rightarrow X_0$ is a blow up sequence of length $n$ and with centers $V_i$, $0 \leq m \leq n$ is a number and $U \hookrightarrow X_0$ is an open subscheme such that $V_i$ is $U$-admissible for each $m < i < n$. Let, furthermore, $g : X'_m \rightarrow X_m$ be any $(X_0 \setminus U)$-supported blow up sequence of length $n'$, then by an extension of $f$ with $g$ we mean a blow up sequence of the form

$$f' : X'_n \xrightarrow{f'_{n-1} \circ \cdots \circ f'_m} X'_m \xrightarrow{g} X_m \xrightarrow{f_{m-1} \circ \cdots \circ f_0} X_0$$

of length $n + n'$ obtained from $f$ by inserting $g$ before $f_m$ and such that the following conditions are satisfied: (a) after the base change with respect to the open immersion $U \hookrightarrow X_0$, $f'$ becomes a trivial extension of $f$; (b) for each $i \geq m$ the center $V'_i$ of $f'_i : X'_{i+1} \rightarrow X'_i$ is $U$-admissible. By successive extending of $f$ we mean applying the above extension operation few times.

**Lemma 4.2.3.** Given $f : X_n \rightarrow X_0$ and $g : X'_m \rightarrow X_m$ as in definition 4.2.2, there exists a unique extension $f'$ of $f$ by $g$. The center $V'_m$ of $f'_m$ is naturally isomorphic to the strict transform of the center $V_m$ of $f_m$ under $g$, i.e. $V'_m = g(V_m)$.

**Proof.** The conditions (a) and (b) leave no choice in the definition of the centers $V'_i$ for $m \leq i < n$: we must have that $V_i \times_X U \sim V'_i \times_X U$ and $V'_i$ is the schematic closure of this scheme in $X'_{i-1}$. These rules dictate an inductive construction of the canonical sequence of blow ups $f'_{i+1} : X'_{i+1} = Bl_{V'_i}(X'_i) \rightarrow X'_i$ starting with $f'_m$. Then it is clear that the sequence satisfies all properties an extension should satisfy. Regarding the second claim, we just use the definition of the strict transform and the fact that both $V'_m$ and $V_m$ have a common schematically dense open subscheme which is the preimage of $U$. □

4.2.3. Merging.

**Lemma 4.2.4.** Let $X$ be a scheme with pairwise disjoint closed subschemes $T_1, \ldots, T_n$, $U_i = X \setminus \bigcup_{j \neq i} T_j$ and $U = \bigcup_{i=1}^n U_i$. Assume that $g : U' \rightarrow U$ is a blow up sequence, then there exists a unique $(\bigcup_{i=1}^n T_i)$-supported blow up sequence $f : X' \rightarrow X$ such that $f|_{U_i} = g|_{U_i}$ for each $1 \leq i \leq n$.

In the situation described in this obvious Lemma we will say that $f$ is merged from the the blow up sequence $g$ or from its components $g \times_U U_i$.

4.2.4. Compatibility with flat base changes. Since blow up sequences are compatible with flat base changes in the sense of §2.2.8, one can check straightforwardly that all constructions from §4.2 are also compatible. So, we obtain the following Lemma.

**Lemma 4.2.5.** The operations of pushing forward, merging and extending blow up sequences are compatible with flat base changes.
4.3. The main theorem. Now we have all necessary tools to construct a strong desingularization functor $F$ on $\mathcal{Q}^{p=0,\text{reg}}$ from the functor $F_{\text{Var}}$ which was extended to a functor on $\mathcal{C}_{\text{small}}$ by Theorem 3.3.1. Note, however, that the functor $F$ will not coincide with $F_{\text{Var}}$ even on varieties, since we must build the new desingularization functor from scratch for the sake of compatibility.

Proof of Theorem 1.2.1. By Lemma 4.1.3 it suffices to build a compatible sequence of functors $F^{\leq d}$ which provide an equicodimensional desingularization up to codimension $d$ and such that the centers of $F^{\leq d}$ of $X$-codimension $d$ sit over $T_{d-1} = F^{\leq d-1}_{\text{sing}}$. The construction will be done inductively, and we start with empty $F^{\leq 0}$ since generically reduced schemes are regular in codimension 0. Thus, we can assume that the sequence $F^{\leq 0}, \ldots, F^{\leq d-1}$ is already constructed, and our aim is to construct $F^{\leq d}$. First we will construct $F^{\leq d}(X)$ for a single scheme $X$, and then we will check that the construction is functorial. The required sequence will be obtained by extending the blow up sequence $f = F^{\leq d-1}(X) : X_m \to X_0 = X$ few times. To simplify notation, after each extension we will denote the obtained blow up sequence as $f : X_m \to X_0$, but this should not cause any confusion. By our assumption, $T_{d-1}$ is a closed subset of $X^{\geq d}$, hence it has finitely many points of codimension $d$, which are the generic points of the irreducible components of $T_{d-1}$. Let $T$ denote the set of these points.

Extension 0. We denote by $\overline{T}$ the Zariski closure of $T$ with the reduced scheme structure and extend $f$ by inserting $\text{Bl}_{\overline{T}}(X) \to X$ as the first blow up. As an output we obtain a blow up sequence $F^{\leq d}_{\overline{T}}(X) : X'_m \to X'_0 \to X_0 = X$ of length $m + 1$ where the first center (the inserted one) is regular over a neighborhood of $T$. As agreed above, we set $f = F^{\leq d}_{\overline{T}}(X)$ and increase $m$ by one after this step. As an output we achieve that the scheme-theoretic preimage of $\overline{T}$ in each $X_i$ is a Cartier divisor $D_i$.

Extensions 1, \ldots, $n$. The last $n$ centers of $F^{\leq d}_{\overline{T}}(X)$ are regular over $X^{\leq d} \setminus T$ but do not have to be so over $T$. We remedy this problems by $n$ successive extensions. Let us describe the $i$-th one. It obtains as an input a blow up sequence $f = F^{\leq d}_{i-1}(X)$ in which only the last $n-i$ centers can be non-regular over $T$ and outputs a blow up sequence $F^{\leq d}_i(X)$ with only $n-i-1$ bad blow ups in the end. By our assumption, $X_{i+1} \to X_i$ is the first blow up of $f$ whose center $W$ can be non-regular over $T$. The intersection of $W_{\text{sing}}$ with the preimage of $T$ can be non-empty, but it is definitely contained in $W \cap X_i^{\leq d} \subseteq X^{\leq d-1}$. Consider the blow up sequence $F^{\leq d-1}(W) : W' \to W$, which exists by the induction assumption. Its centers $V_j$ can have singularities only over $W^{\geq d}$, in particular, the image of $(V_j)_{\text{sing}}$ in $X$ is contained in $X^{>d}$. By Lemma 4.2.1, the pushforward $X_i' \to X_i$ of $F^{\leq d-1}(W)$ with respect to the closed immersion $W \to X_i$ is a blow up sequence with centers $V_j$. In particular, its centers are regular over $X^{\leq d}$.

Let now $f' : X'_m \to X'_i \to X_i \to X_0$ be obtained from $f$ by inserting $X'_i \to X_i$ instead of $X_{i+1} \to X_i$ as in Definition 4.2.2. By Lemma 4.2.3 the center of $X_{i+1} \to X_i'$ is the strict transform of $W$, hence it is $W'$. Since $W'$ is regular over $X^{\leq d}$ by the construction, only the last $i-1$ blow ups of $f'$ can be non-regular over $T$ – the blow ups from the sequence $X'_m \to X'_{i+1}$. So, we can set $F^{\leq d}_i(X) = f'$. 
Remark 4.3.1. The scheme $W$ does not have to be integral. So, we essentially exploited here that the resolution functor $F$ is defined at least for all reduced ambient schemes.

Extension $n + 1$. At this stage we already have a blow up sequence $f = F^≤d(X)$ such that all its centers are regular over $X^{≤d}$. The only problem is that though the singular locus of $X_m$ is disjoint from the preimage of $X^{≤d} \setminus T$ it can hit the preimage of $T$. For any $x ∈ T$ consider the pro-open subscheme $X_x := X_m ×_X \text{Spec}(\mathcal{O}_{X,x})$ of $X_m$ and let $D_x$ be the scheme-theoretic preimage of $x$, then $D_x = (D_m)|_{X_x}$ is a Cartier divisor containing $(X_x)_{\text{sing}}$. So, the pair $(X_x, D_x)$ is an object of $\mathcal{C}_{\text{small}}$, and hence so is $(\sqcup_{x ∈ T} X_x, \sqcup_{x ∈ T} D_x)$. Applying $F_{\text{Var}}$ to the latter pair we obtain a list of strong desingularizations $f_x : X'_x → X_x$ for all $x ∈ T$. (Actually, each $f_x$ is $F_{\text{Var}}(X_x, D_x)$ saturated with few synchronizing empty blow ups.) Let $U → X$ be an open neighborhood of $X^{≤d}$ such that the closures $\overline{T}$ of distinct point of $T$ in $U$ are pairwise disjoint, and set $U_m = X_m ×_X U$. Define $g_x : U_x → U_m$ as the pushforward of $f_x$ with respect to the pro-open immersion $X_x → U_m$. Since each $g_x$ is $\overline{T}$-supported, Lemma 4.2.4 implies that we can merge all $g_x$’s into a single blow up sequence $g : U'_m → U_m$. Finally, we define $f' : U'_m → X_m$ to be the pushforward of $g$ with respect to the open immersion $U_m → X_m$, and set $F^{≤d}(X) = f ∘ f'$. Then $F^{≤d}(X)$ coincides with $F^{≤d-1}(X)$ over $X^{≤d} \setminus T$ and coincides with $f ∘ f_x$ over each $\text{Spec}(\mathcal{O}_{X,x})$ for $x ∈ T$. By our construction the latter is a strong desingularization of $\text{Spec}(\mathcal{O}_{X,x})$, hence the constructed $F^{≤d}(X)$ is a strong desingularization of $X$ up to codimension $d$.

Remark 4.3.2. We could use the blow up sequences $F_{\text{Var}}(X_x, D_x)$ instead of $f_x$’s in the construction of $F^{≤d}(X)$ (in other words, we could omit all empty blow ups in $f_x$’s). Although such choice would work as well, our construction seems to be more natural.

It remains to check that $F^{≤d}$ is functorial. By Lemma 4.2.5 the operations of pushing forward, extending and merging of blow ups are compatible with a regular (and even flat) base change $h : (\overline{X}, \overline{S}) → (X, S)$, and we claim that it follows easily that all intermediate constructions in our proof are functorial. Indeed, the set of the maximal points of $h^{-1}(T)$ is exactly the set $\overline{T}$ of points of $\overline{X}^{≤d}$ over which $F^{≤d-1}(\overline{X})$ is not a desingularization. Hence the Zariski closure of $\overline{T}$ coincides with $\overline{T} ×_X \overline{X}$ in Extension 0, and therefore the blow up sequence $F^{≤d}_0$ is functorial. In the $i$-th Extension we have that $\overline{W} = W ×_X \overline{X}$ by inductive functoriality of $F_{i-1}^{≤d}$. Hence $F^{≤d-1}(\overline{W})$ is a trivial extension of $F^{≤d-1}(W) ×_X \overline{X}$ by functoriality of $F^{≤d-1}$, and so $F_i^{≤d}$ is functorial. The last Extension is dealt with similarly. □

Remark 4.3.3. The same proof actually applies to a more general situation, which might include, for example, schemes of arbitrary characteristic. Let $\mathcal{C}$ be a subcategory of $\mathcal{Q}_{\text{reg}}$ which is closed under blow ups and taking subschemes. Define a category $\mathcal{C}_{\text{small}}$ as usual, i.e. the objects are pairs $(X, Z)$ with $X$ in $\mathcal{C}$ and $Z$ a Cartier divisor containing $X_{\text{sing}}$ and isomorphic to a variety, and the morphisms are the regular ones. Then the same argument as above shows that starting with a (strong) desingularization functor $F_{\text{Var}}$ on $\mathcal{C}_{\text{small}}$ one can construct a (strong) desingularization functor $F$ on $\mathcal{C}$. Obviously, in the case of non-strong desingularization one can skip extensions $1, \ldots, n$. 
5. Desingularization in other categories

5.1. Desingularization of stacks.

5.1.1. Stacks. If not said to the contrary, by a stack we always mean an Artin stack \( \mathcal{X} \). In particular, \( \mathcal{X} \) can be represented by a smooth groupoid \( s, t : R \rightrightarrows U \) (which is called an atlas of \( \mathcal{X} \)). Equivalently, \( \mathcal{X} \) admits a smooth cover \( U \to \mathcal{X} \) by a scheme and then \( R = U \times_{\mathcal{X}} U \) with the projections \( s \) and \( t \) onto \( U \) and the multiplication \( m : R \times_{s, U, t} R \to U \) corresponding to the projection \( p_{13} \) of \( R \times_{s, U, t} R \rightrightarrows U \times U \times U \) onto \( R \). We will consider only smooth atlases, and we say that a stack is qe if it admits a qe smooth atlas, i.e. an atlas in which \( U \) is qe (and so \( R \) is also qe).

5.1.2. Blow ups of stacks. We say that a morphism \( f : \mathcal{Y} \to \mathcal{X} \) is a blow up along a closed substack \( \mathcal{Z} \) if it admits an atlas \( (f_1, f_0) : (Y_1 \rightrightarrows Y_0) \to (X_1 \rightrightarrows X_0) \) with \( f_0 \) the blow up along \( Z_0 = \mathcal{Z} \times_{\mathcal{X}} X_0 \) and \( f_1 \) the blow up along \( Z_1 = \mathcal{Z} \times_{\mathcal{X}} X_1 \). For the sake of completeness we remark that using descent of closed subschemes one can show that in this case one can choose \( X_1 \rightrightarrows X_0 \) to be any atlas of \( \mathcal{X} \) and then take \( Y_1 = \mathcal{Y} \times_{\mathcal{X}} X_1 \).

5.1.3. Desingularization.

Theorem 5.1.1. The blow up sequence functor \( \mathcal{F} \) extends uniquely to the 2-category of generically reduced noetherian qe stacks over \( \mathbb{Q} \).

Proof. Given a generically reduced qe stack \( \mathcal{X} \), find a qe atlas \( s, t : R \rightrightarrows U \) of \( \mathcal{X} \). Note that \( U \) and \( R \) are automatically generically reduced. By functoriality of \( \mathcal{F} \) the smooth morphisms \( s \) and \( t \) extend to smooth morphisms \( t_0, \ldots, t_0 = t \) and \( s_0, \ldots, s_0 = s \) between the whole blow up sequences \( \mathcal{F}(R) : R_n \to R \) and \( \mathcal{F}(U) : U_n \to U \). Moreover, the groupoid composition map \( m : R \times_{s, U, t} R \to R \) is smooth, hence it extends to a tower of groupoid maps \( m_i : R_i \times_{s_i, U_i, t_i} R_i \to R_i \). Thus, we have actually constructed a tower of groupoid blow ups \( (R_n \rightrightarrows U_n) \to (R \rightrightarrows U) \), which gives rise to a blow up sequence \( \mathcal{F}(\mathcal{X}) : \mathcal{X}_n \to \mathcal{X}_0 = \mathcal{X} \). Since \( U_n \) is regular the stack \( \mathcal{X}_n \) is regular.

Now, we have to check that the construction is independent of the chart. This reduces to applying the functoriality of \( \mathcal{F} \) few more times by dominating any two charts with a third one and comparing the corresponding blow up sequences. Finally, the compatibility of the construction with regular 1-morphisms and 2-isomorphisms between them is checked similarly, so we skip the details. \( \square \)

5.1.4. Stacks in regular topology. Let us indicate how the above result can be generalized to a wider class of stacks. By regular topology on the category of schemes we mean the natural intermediate topology between the smooth and the fpqc ones. Its objects are regular fpqc morphisms, and its covers are surjective such morphisms. We say that a stack \( \mathcal{X} \) in regular topology is qe if it admits a chart \( s, t : R \rightrightarrows U \) with qe \( R \) and \( U \) and regular morphisms \( s \) and \( t \) (note that this time noetherianity of \( U \) does not guarantee that of \( R \) because even the scheme \( \text{Spec}(\mathbb{C}) \times_{\text{Spec}(\mathbb{Q})} \text{Spec}(\mathbb{C}) \) is not noetherian). Then the same argument as earlier provides a canonical desingularization of such stacks. Moreover, this desingularization is functorial at least with respect to morphisms \( \mathcal{X}' \to \mathcal{X} \) which admit regular qe atlases \( (R' \rightrightarrows U') \to (R \rightrightarrows U) \) (in the sense that the morphisms \( R' \to R \) and \( U' \to U \) are regular).

5.2. Desingularization of formal schemes and analytic spaces.
5.2.1. Categories. Let $\mathcal{C}'$ be any of the following categories: noetherian qe formal schemes over $\mathbb{Q}$, quasi-compact complex analytic spaces (maybe non-separated), quasi-compact analytic $k$-spaces of Berkovich or quasi-compact rigid $k$-analytic spaces for a complete non-Archimedean field $k$ of characteristic zero, which is non-trivially valued in the rigid case. We will be interested in the full subcategory $\mathcal{C}$ of $\mathcal{C}'$ whose objects have nowhere dense non-reduced locus.

Remark 5.2.1. Recall that an analytic $k$-space is an analytic space over a non-Archimedean $k$-field $K$. Therefore, in the case of Berkovich spaces it suffices to take $k = \mathbb{Q}$ or $k = \mathbb{Q}_p$ since other categories are contained in one of these.

5.2.2. Regularity. There is a natural notion of regular morphisms in all these categories: see §2.4.12 for the formal case; regularity is smoothness in the complex analytic case; in the case of Berkovich and rigid spaces we say that $\mathcal{M}(\mathcal{B}) \rightarrow \mathcal{M}(\mathcal{A})$ is regular if $\mathcal{A} \rightarrow \mathcal{B}$ is regular, and globalize this similarly to §2.4.12. Note that Berkovich analytic spaces are excellent by [Duc]. Let $\mathcal{C}_{\text{reg}}$ be obtained from $\mathcal{C}$ by removing all non-regular morphisms.

5.2.3. Desingularization.

Theorem 5.2.2. Let $\mathcal{C}_{\text{reg}}$ be as above. Then $\mathcal{F}$ induces a strong desingularization on $\mathcal{C}_{\text{reg}}$ by formal/analytic blow up sequence functor $\mathcal{F}_\mathcal{C}$.

Proof. Each compact space $X$ from $\mathcal{C}$ can be covered by finitely many charts $X_i$ which are affine formal schemes, affinoid spaces or Stein compacts. Similarly we cover each intersection $X_i \cap X_j$ by spaces $X_{ijk}$ which are affine formal schemes, affinoid spaces or Stein compacts, though the latter coverings may be infinite. It is known that the rings of functions $A_i = \mathcal{O}_X(X_i)$ and $A_{ijk} = \mathcal{O}_X(X_{ijk})$ are excellent noetherian rings in each of these cases, and the homomorphisms $\phi_{ijk}: A_i \rightarrow A_{ijk}$ are regular. Also, the non-reduced locus on $X_i$ is compatible with the non-reduced locus on $X'_i := \text{Spec}(A_i)$ under the morphism $X_i \rightarrow X'_i$ of locally ringed spaces, and so $X'_i$ is generically reduced. The desingularization blow up sequence $\mathcal{F}(X'_i)$ induces a formal/analytic blow up sequence $\mathcal{F}_\mathcal{C}(X_i) : X'_i \rightarrow X_i$ by completing/analytifying the centers. These sequences agree on the intersections because $\mathcal{F}$ is compatible with the regular homomorphisms $\phi_{ijk}$. So, $\mathcal{F}(X'_i)$ glue to a single blow up sequence $\mathcal{F}_\mathcal{C}(X') : X' \rightarrow X$ which desingularizes $X$. Compatibility of $\mathcal{F}_\mathcal{C}$ with regular morphisms follows from compatibility of the original $\mathcal{F}$. □

5.3. Desingularization in the non-compact setting.

5.3.1. Categories. Let $\mathcal{C}'$ be any of the following categories: locally noetherian qe schemes over $\mathbb{Q}$, locally noetherian qe stacks over $\mathbb{Q}$, locally noetherian qe formal schemes over $\mathbb{Q}$, complex analytic spaces, Berkovich analytic $k$-spaces or rigid spaces over $k$ for a complete non-Archimedean field $k$ of characteristic zero. Given $\mathcal{C}'$ we define its subcategories $\mathcal{C}$ and $\mathcal{C}_{\text{reg}}$ as earlier.

5.3.2. Blow up hypersequences. By a hypersequence we mean a totally ordered set $I$ with an initial element $0$ and such that any element $i \in I$ possesses a successor which will be denoted $i+1$. By a hypersequence $\{X_i\}_{i \in I}$ in $\mathcal{C}$ we mean a set of objects of $\mathcal{C}$ ordered by a hypersequence $I$ and provided with a transitive family of morphisms $f_{ij} : X_j \rightarrow X_i$ for each $j \geq i$ (so $f_{ii} = \text{Id}_{X_i}$). If $I$ is a hypersequence with a finite subsequence $i_0 < i_1 < \cdots < i_n$, and $X_{i_0} \rightarrow X_{i_{n-1}} \rightarrow \cdots \rightarrow X_{i_0}$ is a blow up sequence, then we can define its trivial extension $\{X_i\}_{i \in I}$ by taking $f_{ij}$ to
be an empty blow up for any any pair \( j, k \in I \) with \( i_m < j \leq k \leq i_m+1 \) for some \( 0 \leq m < n \). By a blow up hypersequence we mean a hypersequence \( \{X_i\}_{i \in I} \) such that each morphism \( f_i := f_{i+1,i} \) is a blow up, and each point \( x \in X_0 \) possesses a neighborhood \( U \) for which the hypersequence \( \{X_i \times_X U\}_{i \in I} \) is a trivial extension of its finite subsequence.

**Remark 5.3.1.** (i) Any blow up hypersequence converges to an object \( X_\infty \) (if \( I \) has a maximal element \( i \) then \( X_\infty = X_i \)). So, if \( I \) has no maximal element we can extend \( \{X_i\}_{i \in I} \) to a blow up hypersequence \( \{X_i\}_{i \in I \cup \{\infty\}} \).

(ii) Clearly, an infinite blow up sequence which stabilizes over compact subobjects of \( X_0 \) is a blow up hypersequence. Often one can perform few blow ups simultaneously thus ”shrinking” the initial hypersequence to a hypersequence ordered by \( \mathbb{N} \cup \infty \), i.e. to a usual sequence \( \cdots \to X_1 \to X_0 \) augmented by its limit. For example, this is obviously the case when \( X_0 \) is a disjoint union of compact components. However, we saw in Remark 2.3.4 that functorial properties are destroyed by such operation, since one cannot ignore the order of the blow ups even when \( X_0 \) is a disjoint union of compact pieces. The existing desingularization algorithms can be extended to non-compact case functorially only when one allows blow up hypersequences because the resolving invariant takes values in complicated ordered sets rather than in \( \mathbb{N} \) (in particular, the algorithms involve more than one induction loop). Most probably, there are no desingularization algorithms whose set of invariants is \( \mathbb{N} \).

5.3.3. Desingularization.

**Theorem 5.3.2.** Let \( \mathcal{C}_{reg} \) be as above. Then the functor \( F_\mathcal{C} \) from Theorem 5.2.2 induces a strong desingularization functor hyper-\( F_\mathcal{E} \) which assigns to objects of \( \mathcal{C} \) countable algebraic/formal/analytic blow up hypersequences with regular maximal element \( X_\infty \). In particular, the morphism \( X_\infty \to X \) is a functorial desingularization of \( X \) by a single proper morphism.

**Proof.** We act as in the proof of Theorem 5.2.2, though this time a chart \( \{X_i\} \) can be infinite. We have defined resolutions \( F_\mathcal{C}(X_i) \) in the proof of Theorem 5.2.2. Obviously, we can saturate each blow up sequence \( F_\mathcal{C}(X_i) \) with trivial blow ups so that one obtains a blow up hypersequence hyper-\( F_\mathcal{E}(X_i) \) whose objects are parametrized by the invariants of \( F_\mathcal{E} \) (and so there are countably many of them). Then the hypersequences hyper-\( F_\mathcal{E}(X_i) \) agree on the intersections because they agree over each compact subspace in the intersections, and hence glue to a single countable hypersequence hyper-\( F_\mathcal{E}(X) \). \( \square \)

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