Supermembrane theory on a curved constant background

M.P. García del Moral\textsuperscript{a,1} C. Las Heras\textsuperscript{a,2} P. Leon\textsuperscript{a,3} J.M. Pena\textsuperscript{b,4} A. Restuccia\textsuperscript{a,5}

\textsuperscript{a}Departamento de Física, Universidad de Antofagasta, Aptdo 02800, Chile
\textsuperscript{b}Departamento de Física, Facultad de Ciencias, Universidad Central de Venezuela, A.P. 47270, Caracas 1041-A, Venezuela

\textsuperscript{1}E-mail: maria.garciadelmoral@uantof.cl, camilo.lasheras@ua.cl, pablo.leon@ua.cl, jpena@fisica.ciens.ucv.ve ; joselen@yahoo.com, alvaro.restuccia@uantof.cl

Abstract: In this work we obtain the Hamiltonian description of the Supermembrane theory formulated in the Light Cone Gauge (L.C.G.) on $M_9 \times T^2$ background with constant bosonic three-forms $C_{\pm ab}$. We analyze three different cases depending on the particular values of the constants $(C_{+ab}, C_{-ab})$. When it is imposed a 2-form flux condition over $C_+$ and vanishing $C_-$, it coincides with the Hamiltonian of a supermembrane theory irreducibly wrapped around the 2-torus with a vanishing three-form $C_{\mu\nu\rho} = 0$, shifted by a constant term. The 2-torus target space flux condition of the first theory induces a pullback worldvolume flux that can be identified with the topological invariant associated to the irreducibility of the wrapping condition that appears in the second theory considered. Both theories exhibit a nonvanishing central charge condition in the algebra. The M2-brane theory with constant $C_+$ exhibits discreteness of the supersymmetric spectrum as the theory of irreducible wrapping does, in distinction with the M2-brane case with $C_+ \neq 0$ and vanishing flux condition that has continuous spectrum. We also obtain the Hamiltonian description and constraints of the Supermembrane theory for the case of $C_- \neq 0$. This case is more subtle due to the role of $X^-$ component in the constraint, however with a proper redefinition of the canonical variables the $X^-$ can be decoupled. One of the backgrounds considered here coincides with the asymptotic limit of the supergravity solution found by the authors [1], which is generated by an M2-brane acting as a source.

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1 Introduction

Supermembranes are 2 + 1 supersymmetric extended objects that evolve in an eleven dimensional target space [2]. They are described by a nonlinear interacting field theory invariant under global supersymmetry, local diffeomorphisms and local kappa symmetry [3]. Supermembranes are part of the building blocks of M-theory and they are sources of 11D supergravity [2, 4], a relation that was emphasized in [5]. The fact that the eleven dimensional supergravity contains a 3-form gauge potential that couples to a membrane, just like the 2-form gauge potential couples to a string, led the authors in [2] to formulate the supermembrane coupled to eleven dimensional supergravity. Moreover, in [1] it is shown that the supermembrane emerges as an exact solution of supergravity field equations.

Another remarkable property of Supermembrane theory is that all the five string theories at least at kinematical level and by double dimensional reduction can be obtained from it [6–9]. Matrix theory [10] on the other hand, was originally formulated as a regularized limit of the non-zero mode sector of the 11D supermembrane formulated in the Light Cone Gauge (L.C.G.), though the interpretation in [10] is very different from the supermembrane point of view [3]. Recently it has been shown that the theory when it is compactified on toroidal backgrounds it reproduces U-duality as a symmetry [11–13].
The Supermembrane theory was originally expected to describe the microscopic degrees of freedom of M-theory, however when formulated on 11D Minkowski background [3], it was rigorously proved in the context of matrix model regularization introduced by [14] that it has continuous spectrum form \([0, \infty)\) [15], and the toroidal compactification by itself does not change this behaviour [16]. This property led to the community to re-interpret this theory as a second quantized theory and to reformulate it in terms of a system of interacting \(D0\)-branes [10]. Then \(D0\)-branes became as candidates for the M-theory d.o.f that under the effect of some (unknown) interaction would render the spectrum discrete. There are however few cases described in the literature -up to our knowledge- in which the formulation of supermembrane theory exhibits discreteness of the spectrum [17, 18]. The so called supermembrane with central charges [19] that is irreducibly wrapped [20] around a 2-torus, -condition that has been extended to other backgrounds, see for example the case \(M_7 \times T^4\), [21], \(M_4 \times T^7\), in [22] and the extension to a G2 orbifold structure in [23] -and the supermembrane theory formulated on a pp-wave background [24, 25] whose matrix model regularization corresponds to the BMN matrix model [26] and whose properties of discreteness were proven on [18].

Since M-theory is a candidate for unification theory, at least a sector of the theory will be described in terms of Supermembrane theory. Consequently it becomes increasingly clear the need to obtain a formulation of the theory on more general backgrounds. In the literature it has not been too much studied due to its complexity. Previous works studied it in the L.C.G. for the bosonic part of the supermembrane on arbitrary curved backgrounds in the formalism of the superspace to second order in grassmann variables [27]. In [28] it was studied the supermembrane on Antide Sitter spaces, in particular on \(AdS_4 \times S^7\) and \(AdS_7 \times S^4\).

In this paper we want to study a simple but interesting case: The supermembrane theory formulated in the LCG with some of the constant bosonic three-forms different from zero. We will perform our study by characterizing its supersymmetric Hamiltonian and studying its constrains. For the case with only \((C_+ \neq 0)\) we will establish its connection with the Supermembrane theory irreducibly wrapped called by the authors as Supermembrane with central charge [19, 30]. We will comment the properties of the obtained Hamiltonians in all cases. The study of matrix models in the Light Cone Gauge in the presence of constant bosonic three forms was performed in [29].

The paper is organized as follows: In section 2 we discuss three different scenarios for the 11D supermembrane formulated in the Light Cone Gauge (L.C.G) on a 11D Minkowski metric background and toroidal ba$k$dground$ss$ in the presence of constant non vanishing three forms \(C_{\pm ab}\) and assuming that \(C_{+-a} = C_{abc} = 0\). We obtain the Hamiltonians associated to each of the scenarios. In section 3 we discuss the consistence of the curved background solutions proposed and we relate one of them with a solution found in [1]. This particular solution has an M2-brane as a source and in the asymptotic limit becomes Minkowski with a constant three-form. In section 4 we review the construction of the supermembrane with irreducible wrapping. In section 5 we see the equivalence of a supermembrane with only \(C_+ \neq 0\) compactified on a torus in which a flux condition has been imposed and the supermembrane irreducibly wrapped around a torus. In section 6 we discuss the physical
implications of the above scenarios when they are toroidally compactified and finally we present our conclusions.

2 The supermembrane theory on a constant 3-form background

In this preliminary section we will analyse the supermembrane theory formulated in the L.C.G on a target space with a toroidal topology, in particular on $M_9 \times T^2$, with a Minkowski metric but in the presence of a constant bosonic three-form. The analysis of the bosonic membrane on a general background with trivial topology was performed in [27]. The starting point is the action of the 11D Supermembrane theory found in [2],

$$S[Z(\xi)] = T_3 \int d^3 \xi \left[ \frac{1}{2} \sqrt{-g} g^{uv} \pi^u \hat{\pi}^n \eta_{\hat{m} \hat{n}} + \frac{1}{2} \sqrt{-g} + \frac{1}{6} \varepsilon^{uvw} \pi^M \pi^N \pi^L C_{LMN} \right],$$

(2.1)

where, $M, N, L,$ are superspace indices; $\mu, \nu, \lambda$ bosonic target space indices; $\alpha, \beta,$ fermionic target space indices; $\hat{m}, \hat{n},$ bosonic tangent space indices and $\hat{\alpha}, \hat{\beta},$ fermionic tangent space indices. $u, v, w$ are bosonic worldvolume indices. The action is described in terms of the supervielbein $E^A_M$, the induced worldvolume metric $g_{uv}$ and the super three form $C_{LMN}$. The embedding coordinates in the superspace formalism are $Z(\xi) = (X^\mu(\xi), \theta^a(\xi))$ with $\xi^u (u = 0, 1, 2)$ the worldvolume coordinates. Pullback of the supervielbein to the worldvolume is given by $\pi^u = \frac{\partial Z^M}{\partial \xi^u} E^A_M$ and the worldvolume induced metric then is expressed as, $g_{uv} = \pi^u \pi^n \eta_{\hat{m} \hat{n}}$. We take the base manifold to be a product $\Sigma \times \mathbb{R}$, where $\Sigma$ is a compact Riemann surface, we will consider the particular case of a torus.

Under the above assumptions the even embedding maps $X^\mu$ which always appear as one-forms $dX^\mu$ decompose into an exact one-form plus a harmonic one-form. The latter may have nontrivial periods on the basis of homology of the compact base Riemann suerface $\Sigma$. The odd embedding maps $\theta^a$ are always single valued on the base manifold. This action $S[Z(\xi)]$ is invariant under two local transformations given by:

- Reparametrizations of the world volume

$$\delta Z^M = \eta^u(\xi) \partial_u Z^M \tag{2.2}$$

- Local fermionic transformations ($\kappa$ symmetry)

$$\delta Z^M E^\hat{a}_M = 0, \quad \delta Z^M E^\hat{a}_M = (1 - \Gamma) \kappa(\xi), \tag{2.3}$$

where

$$\Gamma = \frac{\varepsilon^{uvw} \pi^\hat{m}_u \pi^\hat{n}_v \pi^\hat{l}_w \Gamma_{\hat{m} \hat{n} \hat{l}}}{6 \sqrt{-g}} \text{ with } \Gamma^2 = 1. \tag{2.4}$$

We notice that the harmonic one forms are invariant under the above local transformation which only involve the exact one-forms. In the following we consider a target space $M_9 \times T^2$ with a non vanishing constant bosonic three form. Let us start with the noncompact $M_{11}$
background with a non-vanishing constant bosonic three form. Trivially the action of the supermembrane takes the following form:

\[
S = \int d^3 \xi \left\{ -\sqrt{-g} - \varepsilon^{uvw} (\bar{\theta} \Gamma^\rho \partial_u \theta C_{\rho uv} + \bar{\theta} \Gamma_{\mu v} \partial_u \theta) \left[ \frac{1}{2} \partial_a X^\mu (\partial_v X^\nu + \bar{\theta} \Gamma^\nu \partial_v \theta) \right] \right.

\[+ \frac{1}{6} \bar{\theta} \Gamma^\mu \partial_u \theta \bar{\theta} \Gamma^\nu \partial_v \theta \right\} - \frac{1}{6} \varepsilon^{uvw} \partial_a \partial_v \partial_v \partial_a X^\rho C_{\rho uv},
\]

(2.5)

where as already mentioned \( dX^\mu \) contains an exact and a harmonic part. The last term associated to the constant 3-form in the noncompact target space is a total derivative since it is expressed in terms of single valued functions over the base and hence it vanishes. Once the target space becomes compactified as \( M_9 \times T^2 \), because of the nonlinear terms containing factors of \( \partial_a X^\mu \) the harmonic sector couples in a nontrivial way with the exact one. Hence, although, (2.5) looks similar to the action of a supermembrane on a flat target space with trivial topology, once it becomes toroidally compactified, it is very different from it. We give its explicit expression in section 5. Moreover in the compactified case, in distinction with the noncompact one, the last term in (2.5) for constant bosonic three form is a total derivative of a multivalued function (due to the harmonic contribution). Therefore its integral is not zero, not even constant.

Now we impose the LCG on the exact sector. This gauge fixing procedure follows exactly as in the flat, trivial topology case because the local gauge transformations only involve the exact part of \( X^\mu \). We will denote by \((\sigma^1, \sigma^2)\) the local spatial coordinates on \( \Sigma \) and \( \tau \) the worldvolume time. We have

\[
X^+(\xi) = X^+(0) + \tau \quad \text{so that} \quad \partial_a X^+ = \delta_{a0}, \quad \text{and} \quad \gamma^+ \theta = 0 \quad (2.6)
\]

where we have decomposed the target space coordinates \( X^\mu \) in terms of \((X^+, X^-, X^a)\) with \( a = 1, \ldots, 9 \) labelling the transverse components. The only residual local symmetry is the invariance under diffeomorphisms preserving the area of the Riemann surface associated to the spatial part of the worldvolume coordinates. This is a first class constraint. The metric \( G_{\mu \nu} \) and the three-form also become decomposed under the L.C.G. Since we will closely follow the notation introduced in [27] let us summarize their results for the bosonic Hamiltonian description of the supermembrane

\[
H = \int d^2 \sigma \left\{ \frac{G_{++}}{P_+ - C_-} \left[ \frac{1}{2} \left( P_a - C_a - \frac{P_+ - C_-}{G_{++}} G_{a+} \right)^2 + \frac{1}{4} (\varepsilon^{rs} \partial_r X^a \partial_s X^b)^2 \right] \right. 

\[- \frac{P_- - C_+}{2 G_{++}} G_{++} - C_+ - C_{+-} + \varepsilon^a \phi_r \right\}
\]

(2.7)

with

\[
C_a = -\varepsilon^{rs} \partial_r X^a \partial_s X^b C_{-ab} + \frac{1}{2} \varepsilon^{rs} \partial_r X^a \partial_s X^c C_{abc},
\]

\[
C_+ = \frac{1}{2} \varepsilon^{rs} \partial_r X^a \partial_s X^b C_{+-ab},
\]

\[
C_{+-} = \varepsilon^{rs} \partial_r X^a \partial_s X^b C_{+-ab}.
\]

(2.8)
subject to the area preserving diffeomorphims (APD) given by,
\[ \phi_r = P_a \partial_r X^a + P_- \partial_r X^- \approx 0 \] (2.9)
where they have fixed \( G_{--} = G_{a--} = 0 \) using the residual symmetry generated by the target space diffeomorphisms.

In this paper we are going to extend those previous results by finding the supersymmetric Hamiltonian of the theory when it is formulated on the following background:

\[ G_{++} = 1, \quad G_{ab} = \delta_{ab}, \quad G_{++} = G_{--} = 0 \] (2.10)

They correspond to solutions of the field equations of \( D = 11 \) Supergravity.

### 2.1 The supermembrane with only \( C_{+ab} \neq 0 \)

The supersymmetric Lagrangian in this background takes the form

\[ L = -\sqrt{-g} \Delta - \varepsilon^{rs} \partial_r X^a \bar{\partial} \Gamma^a \partial_s \theta + C_+, \] (2.11)

where

\[ C_+ = \frac{1}{2} \varepsilon^{rs} \partial_r X^a \partial_s X^b \bar{g}_{ab}, \] (2.12)

where we are denoting as in [3], \( \Delta = -g_{00} + u_r \bar{g}_{rs} u_s \) being \( u_r \equiv g_{0r}, \bar{g}_{rs} g_{st} = \delta^{rt} \) and \( g \equiv \text{det} g = -\Delta \bar{g} \) (with \( \varepsilon^{0rs} = \varepsilon^{rs} \)). With, as found in [3]:

\[ \bar{g}_{rs} \equiv g_{rs} = \partial_r X^a \partial_s X^b \delta_{ab}, \] (2.13)
\[ \bar{g}_{r0} \equiv u_r = \partial_r X^- + \partial_0 X^a \partial_s X^b \delta_{ab} + \bar{\theta} \Gamma^a \partial_s \theta, \] (2.14)
\[ \bar{g}_{00} = 2 \partial_0 X^- + \partial_0 X^a \partial_0 X^b \delta_{ab} + 2 \bar{\theta} \Gamma^a \partial_0 \theta. \] (2.15)

Now it is easy to see that the canonical momenta are not different from the ones obtained in [3] for the flat Minkowski target space with \( C_{\mu
u} = 0 \)

\[ P_a = \sqrt{\frac{\bar{g}}{\Delta}} (\partial_0 X_a - u_r \bar{g}^{rs} \partial_s X_a), \] (2.16)
\[ P_- = \sqrt{\frac{\bar{g}}{\Delta}}, \] (2.17)
\[ S = \sqrt{\frac{\bar{g}}{\Delta}} \Gamma^- \theta. \] (2.18)

Since the harmonic sector does not depend on \( \tau \), there are no conjugate momenta associated to them. The Hamiltonian takes the form

\[ H = \frac{1}{P_-} \left[ \frac{1}{2} P_a P^a + \frac{1}{4} \left( \varepsilon^{rs} \partial_r X^a \partial_s X^b \right)^2 \right] + \varepsilon^{rs} \bar{\partial} \Gamma^a \partial_r \theta \partial_r X^a - C_+, \] (2.19)

subject to the primary constraints

\[ P_a \partial_r X^a + P_- \partial_r X^- + S \partial_\tau \theta \approx 0, \] (2.20)
\[ S + P_- \Gamma^- \theta \approx 0. \] (2.21)
The Hamiltonian density (2.19) has the same form as the Hamiltonian density of the supermembrane on a flat, topologically trivial, target space in [3], shifted by $C_+$. However in our formulation the bosonic and fermionic potential couple in a nontrivial way the harmonic and exact modes providing mass terms to the Hamiltonian density. Therefore, the structure of the densities are very different, even their spectrum may become qualitatively different.

### 2.2 The supermembrane with only $C_{-ab} \neq 0$

In this case the Lagrangian (2.1) can be rewritten in the LCG by

$$\mathcal{L} = -\sqrt{-\bar{g}} \Delta - \varepsilon^{rs} \partial_r X^a \bar{\theta} \Gamma^- \partial_s \theta - \varepsilon^{rs} \left[ \partial_0 X^b \partial_r X^a \bar{\theta} \Gamma^- \partial_s \theta - \frac{1}{2} \partial_r X^b \partial_s X^a \partial_0 \bar{\theta} \Gamma^- \theta \right] C_{-ab} + \partial_0 X^+ C_- + \partial_0 X^a C_a,$$

(2.22)

where

$$C_a = -\varepsilon^{r\bar{s}} \partial_r X^- \partial_{\bar{s}} X^b C_{-ab},$$

(2.23)

$$C_- = \frac{1}{2} \varepsilon^{r\bar{s}} \partial_r X^a \partial_{\bar{s}} X^b C_{-ab}.$$  

(2.24)

Then the canonical momenta now is modified by the presence of the nonvanishing three-form contribution

$$P_a = \sqrt{\frac{\bar{g}}{\Delta}} (\partial_0 X_a - u_r \varepsilon^{r\bar{s}} \partial_{\bar{s}} X_a) + \varepsilon^{rs} \bar{\theta} \Gamma^- \partial_s \theta \partial_r X^b C_{-ab} + C_a,$$

(2.25)

$$P_- = \sqrt{\frac{\bar{g}}{\Delta}} + C_-,$$

(2.26)

$$S = -\left( \sqrt{\frac{\bar{g}}{\Delta}} + C_- \right) \Gamma^- \theta,$$

(2.27)

The Hamiltonian density takes the following form

$$\mathcal{H} = \frac{1}{P_- - C_-} \left[ \frac{1}{2} \left( P_a - C_a - \varepsilon^{rs} \bar{\theta} \Gamma^- \partial_s \theta \partial_r X^b C_{-ab} \right)^2 + \frac{1}{4} \left( \varepsilon^{rs} \partial_r X^a \partial_s X^b \right)^2 \right]$$

(2.28)

$$+ \varepsilon^{rs} \bar{\theta} \Gamma^- \partial_s \theta \partial_r X^a,$$

It is subject to the primary constraints

$$P_a \partial_r X^a + P_- \partial_r X^- + \bar{S} \partial_r \theta \approx 0,$$

(2.29)

$$S + P_- \Gamma^- \theta \approx 0.$$

(2.30)

The difficulty reported in [27] due to the coupling of $X^-$ in the expression of $C_a$ appear also here. In fact to solve for $X^-$ in (2.29) leads to nonlocal expressions. In order to remove this difficulty we notice that the transformation

$$X^a \rightarrow \hat{X}^a \equiv X^a, \quad X^- \rightarrow \hat{X}^- \equiv X^-,$$

$$P_a \rightarrow \hat{P}_a \equiv P_a - C_a, \quad P_- \rightarrow \hat{P}_- \equiv P_- - C_-,$$

$$\theta^i \rightarrow \hat{\theta}^i \equiv \theta^i, \quad S \rightarrow \hat{S} \equiv S,$$

(2.31)
preserves the Poisson brackets, therefore it is a canonical transformation on phase space. Moreover, the kinetic terms

\[
\int_{\Sigma} \left( P_a \dot{X}^a + P_- \dot{X}^- + S \dot{\theta} \right) = \int_{\Sigma} \left( \hat{P}_a \dot{\hat{X}}^a + \hat{P}_- \dot{\hat{X}}^- + \hat{S} \dot{\hat{\theta}} \right),
\]

(2.32)

remain invariant under (2.31). In addition we find that the Hamiltonian density and the

\[
\hat{H} = \frac{1}{\hat{P}_-} \left[ \frac{1}{2} \left( \hat{P}_a - \varepsilon^{rs} \hat{\theta} \Gamma^- \partial_r \hat{\theta} \partial_s \hat{X}^b C_{-ab} \right)^2 + \frac{1}{4} \left( \varepsilon^{rs} \partial_r \hat{X}^a \partial_s \hat{X}^b \right)^2 \right] + \varepsilon^{rs} \hat{\theta} \Gamma^- \Gamma_a \partial_r \hat{\theta} \partial_s \hat{X}^a.
\]

(2.33)

subject to the new constrains

\[
\phi_r \equiv \hat{P}_a \partial_r \hat{X}^a + \hat{P}_- \partial_r \hat{X}^- + \hat{S} \partial_r \hat{\theta} \approx 0 \quad (2.34)
\]

\[
\chi \equiv \hat{S} + \left( \hat{P}_- + \hat{C}_- \right) \Gamma^- \hat{\theta} \approx 0. \quad (2.35)
\]

As we can observe, the first constraint is invariant when expressed in the new canonical variables. The second one gets modified by the \( C_- \) fermionic contribution, however the algebra of constrains is preserved. Having performed that transformation we may use the residual gauge symmetry generated by the constraints to impose the gauge fixing condition

\[
\hat{P}_- = \sqrt{w},
\]

(2.36)

where \( \sqrt{w} \) is a time independent scalar density. We may then eliminate \( \hat{X}^-, \hat{P}_- \) as canonical variables and obtain a formulation solely in terms of \( \hat{X}^a, \hat{P}_a \). It is also a canonical formulation. The remaining constraint after the partial gauge fixing is, as usual, the area preserving ones:

\[
d \left( \frac{1}{\sqrt{w}} \hat{P}_a d\hat{X}^a + \hat{\theta} \Gamma^- d\hat{\theta} \right) = 0, \quad (2.37)
\]

\[
\int_{\Sigma} d \left( \frac{1}{\sqrt{w}} \hat{P}_a d\hat{X}^a + \hat{\theta} \Gamma^- d\hat{\theta} \right) = 0, \quad (2.38)
\]

where the first constraint is the local integrability condition which must be satisfied in order to have a local solution for \( \hat{X}^- \). The second integral constraint, is the condition that the periods \( d\hat{X}^- \) are trivial and hence \( d\hat{X}^- \) is an exact one form. The formulation is then independent of \( X^- \).

The Hamiltonian density can by written as

\[
\hat{\mathcal{H}} = \frac{1}{\sqrt{w}} \left[ \frac{1}{2} \left( \hat{P}_a - \varepsilon^{rs} \hat{\theta} \Gamma^- \partial_r \hat{\theta} \partial_s \hat{X}^b C_{-ab} \right)^2 + \frac{1}{4} \left( \varepsilon^{rs} \partial_r \hat{X}^a \partial_s \hat{X}^b \right)^2 \right] + \varepsilon^{rs} \hat{\theta} \Gamma^- \Gamma_a \partial_r \hat{\theta} \partial_s \hat{X}^a
\]

(2.39)

subject to (2.37) and (2.38). We can observe the important aspect that the \( X^- \) can be totally decoupled expressing only the theory in terms of the physical (transverse) degrees of freedom. A peculiarity of this model is the shift in the canonical momenta by a fermionic coupling with the three-form \( C_{-ab} \). The role of this shift needs to be studied carefully in
order to determine the spectral properties of the theory in this background. We will not discuss point any further in this paper.

This case received a lot of attention in the past due to its potential connection [27] with the noncommutative matrix models in the case when the target space gets compactified on a torus [31]. We will comment about it in section 6.

2.3 The supermembrane with $C_{-ab} \neq 0$ and $C_{+ab} \neq 0$

This corresponds to the more general case. For this case the Lagrangian (2.1) in the LCG takes the form

$$
\mathcal{L} = -\sqrt{-\bar{g}} \Delta - \varepsilon^{rs} \partial_r X^a \bar{\partial} \Gamma_a \Gamma_b \partial_b \theta - \varepsilon^{rs} \left[ \partial_b X^b \partial_r X^a \bar{\partial} \Gamma_a \partial_b \theta \right. \\
\left. - \frac{1}{2} \partial_r X^b \partial_s X^a \partial_b \partial_s \theta \right] C_{-ab} + C_+ + \partial_0 X^- C_+ + \partial_0 X^a C_a,
$$

(2.40)

where $C_a$ and $C_-$ has the same form as in the previous case and $C_+$ is defined by (2.12).

Now following the same steps of the previous sections, we obtain that the conjugate momenta are not different from (2.25)-(2.27) since the role of $C_+$ does not affect them and it is just a constant shift in the Hamiltonian. The corresponding Hamiltonian density is then,

$$
\mathcal{H} = \frac{1}{P_- - C_-} \left[ \frac{1}{2} \left( \check{P}_a - C_a - \varepsilon^{rs} \partial_r \check{\partial}_s X^b C_{-ab} \right)^2 + \frac{1}{4} \left( \varepsilon^{rs} \partial_r X^a \partial_s X^b \right)^2 \right] \\
+ \varepsilon^{rs} \partial_r \check{\partial}_s \check{\partial}_t X^a - C_+,
$$

(2.41)

subject to the primary constraints

$$
P_a \partial_r X^a + P_- \partial_r X^- + \check{S} \partial_r \theta \approx 0,
$$

(2.42)

$$
S + P_- \Gamma^{-} \theta \approx 0.
$$

(2.43)

Using now the canonical transformation (2.31) the Hamiltonian can be written as

$$
\hat{\mathcal{H}} = \frac{1}{\sqrt{w}} \left[ \frac{1}{2} \left( \check{P}_a - \varepsilon^{rs} \partial_r \check{\partial}_s X^b C_{-ab} \right)^2 + \frac{1}{4} \left( \varepsilon^{rs} \partial_r \check{X}^a \partial_s \check{X}^b \right)^2 \right] + \varepsilon^{rs} \partial^r \check{\partial}^s \check{X}^a - C_+,
$$

(2.44)

subject to the constraints (2.34),(2.35). We can observe that the effect introducing both types of constant three-forms is just is a superposition of the effects generated by each one: a shift $C_+$ in the Hamiltonian density and a shift in the kinetic term given by the coupling of the $C_{-ab}$ to the fermionic sector.

3 Consistence of the curved background

In order to check that $G_{\mu\nu} = \eta_{\mu\nu}$ with $C_{\mu\nu\rho} = \text{const}$ is consistent with supergravity in 11 dimensions, let us consider the bosonic action of supergravity [32]. The equation of motion for the vielbein is [33]

$$
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{3} g_{\mu\nu} F_{\rho\sigma\lambda\tau} F^{\rho\sigma\lambda\tau} - \frac{1}{24} F_{\mu\rho\sigma\lambda} F^{\rho\sigma\lambda},
$$

(3.1)
and, is easy to see that the scalar curvature is given by
\[ R = \frac{1}{36} F_{\rho\sigma\lambda\tau} F^{\rho\sigma\lambda\tau}, \] (3.2)

But, in the chosen background the field strength \( F = dC \) is zero. We have then, that the equation of motion can be written as
\[ R_{\mu\nu} = 0, \] (3.3)
this implies that the vacuum solution must be Minkowski in 11 dimensions. It can be verified that this is not only a maximally symmetric space, but a maximally supersymmetric space [34], because of the vanishing condition for the field strength.

In consequence, the background considered is a solution of supergravity in \( d = 11 \). Moreover, this background contains the case where the background coincides with the asymptotic limit of a supergravity solution generated by an M2-brane acting as a source [1]. The metric is given by [1, 35]
\[ ds^2 = (1 + \frac{k}{r^6})^{-\frac{2}{3}} dx^\mu dx^\nu \eta_{\mu\nu} + (1 + \frac{k}{r^6})^{-\frac{1}{3}} dy^\bar{m} dy^\bar{n} \delta_{\bar{m}\bar{n}}, \] (3.4)
where \( x^\mu = 0, 1, 2, y^\bar{m} = 3, ..., 10 \) and \( r = \sqrt{y^m y^m} \) is the radial isotropic coordinate in the transverse space. On the other hand, the ansatz for the 3-form produces:
\[ C_{\bar{\mu}\bar{\nu}\bar{\sigma}} = \epsilon_{\bar{\mu}\bar{\nu}\bar{\sigma}}(1 + \frac{k}{r^6})^{-1}, \] (3.5)
with the other components set to zero. When \( r \to \infty \), the metric (3.4) goes to Minkowski metric and (3.5) is constant, corresponding in the LCG to \( C_{\pm 12} = \pm \frac{C_{012}}{\sqrt{2}} \). This is the asymptotic limit of (3.4). In fact, the membrane solution (3.4) interpolates between this flat space with \( C_{\mu\nu\rho} = \text{const} \) and \( AdS_4 \times S^7 \) at the horizon when \( r \to 0 \). A flat superspace is characterized by having a Minkowski metric and the gravitino, spin connection and field strength equal to zero [27].

The study of the supermembrane compactified on a torus \( M_9 \times T^2 \), formulated in any of the backgrounds discussed in the last section, requires its consistency with supergravity in \( d = 9 \). There is only one maximal and undeformed 9-dimensional supergravity [36, 37]. This theory can be obtained by Kaluza Klein reduction from Type IIA or IIB supergravity in 10 dimensions. This is in fact, a consequence of the T-duality relation between type IIA and IIB string theory compactified on a circle. If we consider that the target space is Minkowski with the only non trivial components of the 3-form \( C_{\mu\nu\rho} \neq 0 \) set as constant, the supergravity bosonic fields are [36, 37]
\[ e^a_\mu, \ \varphi, \ \tau \equiv \chi + ie^{-\phi}, \ A^0_\mu, \] (3.6)
The discussion for the equation of motion for the vielbein \( e^a_\mu \) in nine dimensions without fluxes will be equivalent to \( d = 11 \) analysis, because the field strenght related to the 1-form
and 2-form are identically zero in the background we are considering. The equations of motion for the scalar field $\varphi$ and the axion-dilaton $\tau$ are

$$d(\star d\varphi) = 0, \quad (3.7)$$
$$d\left(\frac{d\tau}{(Im\tau)^2}\right) = i\frac{d\tau \wedge \star d\bar{\tau}}{(Im\tau)^3}, \quad (3.8)$$

which is a consistent system of two uncoupled equations for $\varphi$ and $\tau$. If we do not restrict ourselves to the condition that $C_{\mu\nu\rho}$ is set constant, the field strength $F^0 = dA^0$ will appear on the right hand side of (3.7) in a term of the form $\frac{2}{\sqrt{7}}e^{\sqrt{7}\varphi}F^0 \wedge \star F^0$ and on a third equation that can be interpreted as a generalized Maxwell equation $d(e^{\sqrt{7}\varphi} \star F^0) = 0$.

The set of equations (3.7) and (3.8) can be written in a coordinate system as

$$\partial_\mu \partial^\mu \varphi = 0, \quad (3.9)$$
$$\partial_\mu \partial^\mu \bar{\tau} = \partial_\mu \bar{\tau} \partial^\mu \bar{\tau}. \quad (3.10)$$

As the choice for $C_{\mu\nu\rho} \neq 0$ to be a constant contains the compactification in $M_9 \times T^2$ of the three cases discussed in section 2, we have verified that all of them are consistent with supergravity in $d = 9$.

### 4 Supermembrane theory irreducibly wrapped around a 2-torus

Let us consider the Hamiltonian of the 11D supermembrane theory formulated on the Light Cone Gauge on a Minkowski target space in [3]. The three-form in this background has the fermionic components different from zero and all bosonic components set to zero, i.e. $C_{\mu\nu\rho} = 0$. The Hamiltonian is

$$H = \int_\Sigma \sqrt{w} \left[ \frac{1}{2} \left( \frac{P_a}{\sqrt{w}} \right)^2 + \frac{1}{4} \left( \frac{\epsilon_{rs}}{\sqrt{w}} \partial_r X^a \partial_s X^b \right)^2 + \bar{\theta} \Gamma_{-}\{X^a, \theta\} \right], \quad (4.1)$$

subject to the first class constraint associated to the invariance under area preserving diffeomorphisms connected to the identity.

$$d(P_a dX^a + \bar{\theta} \Gamma_{-} d\theta) = 0, \quad (4.2)$$

When the space has compact directions a new first class constraint associated to the area preserving diffeomorphisms must be imposed

$$\int_\Sigma d(P_a dX^a + \bar{\theta} \Gamma_{-} d\theta) = 0, \quad (4.3)$$

In the following we are considering as base manifold $\Sigma$ a $g = 1$ compact Riemann surface. We will denote $\sigma^r$, $r = 1, 2$ the local coordinates over $\Sigma$ and $\tau$ the time coordinate with real values.

The above construction can be extended to the $M_9 \times T^2$. The bosonic degrees of freedom are the transverse embedding maps from $\Sigma$ onto $M_9 \times T^2$ and the associated to a Majorana
spinor \( \theta \alpha \) on the target space and scalars on the worldvolume. We denote by \( X^i, i = 1, 2 \) the embedding maps from \( \Sigma \) onto \( T^2 \) torus of the target, and by \( X^m, m = 1, \ldots, 7 \) are maps from \( \Sigma \) onto \( M_9 \).

The maps satisfy a winding condition,

\[
\oint_{C_j} dX^i = m^i_j, \tag{4.4}
\]

where we denote \( C_j \) the homological basis of the 2-torus and by \( m^i_j \) the winding numbers associated to the non-trivial wrapping of the embedding maps \( X^i(\sigma^1, \sigma^2, \tau) \) around the basis of 1-cycles.

In [20] the authors imposed an extra condition to guarantee the irreducibility of the wrapping,

\[
\int_{\Sigma} dX^i \wedge dX^j = \epsilon^{ij} n A, \quad n \in \mathbb{Z}/\{0\} \tag{4.5}
\]

with \( A \) denoting the area of the 2-torus and \( n \) an integer that is chosen to be different from zero. This condition is a quantization condition and ensures that the harmonic modes appear in a nontrivial way in the expression of \( X^i \). This condition is related with the existence of a central charge in the SUSY algebra and for this reason this sector of the Supermembrane has been denoted as Supermembrane with central charges. Indeed, it implies that the supermembrane is a calibrated submanifold [21]. Previous studies of the M2-brane on holomorphic curves was considered in [38] and [39].

From a geometrical point of view, irreducibility condition ensures, the existence of a nontrivial \( U(1) \) principal bundle over the worldvolume of the supermembrane, characterized by the integer \( n \) associated to its first Chern class.

A particular \( n \) fixes and restricts the allowed class of principal fiber bundle where it can be formulated. The existence of this topological invariant avoids that the supermembrane could get unwrapped, as we will see. The canonical connections are \( U(1) \) monopoles expressed in terms of the embedding maps (which are minimal immersions) of the supermembrane in the compactified space. Indeed, this corresponds to have a nontrivial 2-form flux over the supermembrane world-volume,

\[
\int_{\Sigma} F_2 = n \in \mathbb{Z}/\{0\}. \tag{4.6}
\]

The realization of these maps at the level of the supermembrane are [40]

\[
X^+_0 = \frac{1}{2} (\tanh(\phi) + 1), \tag{4.7}
\]

\[
X^-_0 = \frac{1}{2} (\tanh(\phi) - 1), \tag{4.8}
\]

defined over two open sets \( U^+ \) and \( U^- \) on the membrane respectively, and

\[
dY_0 = d\phi, \tag{4.9}
\]
where $\phi$ and $\varphi$ are two harmonic functions defined on $\Sigma$. The minimal connection is given by
\begin{equation}
\hat{A} = X_0dY_0, \tag{4.10}
\end{equation}
whose associated field strength is $F_2 = d\hat{A}$. The irreducible wrapping condition, also called central charge condition is a flux condition over the worldvolume that generalizes the Dirac monopole construction to Riemann surfaces of arbitrary genus $\geq 1$ [40]. The theory defined in this way is a restriction of the supermembrane theory. All configurations must satisfy, in addition, the global constraint (4.5). The constraint (4.5) does not change the local symmetries of the supermembrane theory since it is topological condition. In particular, the invariance under area preserving diffeomorphisms is preserved.

Since $dX^i$ are closed but not exact one-forms they admit a Hodge decomposition. On $\Sigma$ the linear space of harmonic one-forms is two dimensional. We will denote $dX^i_h(\sigma^1, \sigma^2), i = 1, 2$, a normalized basis of harmonic one-forms:
\begin{equation}
\oint_{C_j} dX^i_h = \delta^i_j, \tag{4.11}
\end{equation}
then the one-forms are decomposed as,
\begin{equation}
\begin{aligned}
dX^i &= 2\pi R^i m^j dX^j_h + \delta^{ij} dA_j, \tag{4.12}
\end{aligned}
\end{equation}
where $dA$ is an exact one-form. In order to be consistent with the Hodge decomposition the winding matrix $W$ must have the determinant $n$ fixed and different from zero,
\begin{equation}
\det W = \det \left( \begin{array}{cc} m^1_1 & m^1_2 \\ m^2_1 & m^2_2 \end{array} \right) = n. \tag{4.13}
\end{equation}

On $\Sigma$ there exists a natural metric defined in terms of the harmonic one-forms that is regular,
\begin{equation}
w_{rs} = \partial_r X^i_h \partial_s X^j_h \epsilon_{ij}. \tag{4.14}
\end{equation}
The square root of its determinant is given by
\begin{equation}
\sqrt{w} = \frac{1}{2} \epsilon_{ij} \partial_r X^i_h \partial_s X^j_h \epsilon^{rs}, \tag{4.15}
\end{equation}
$w_{rs}$ is invariant under a canonical change on the basis of homology. It is intrinsically defined on $\Sigma$, it is independent on the embedding of $\Sigma$ onto $T^2$.

Consequently the sector of Supermembrane theory in $D = 11$ restricted by the previous topological condition (4.5) has a non-trivial and irreducible wrapping over the $T^2$ compactified target-space. The hamiltonian is given by
\begin{equation}
H^{Irred} = \int_{\Sigma} \sqrt{w} d\sigma^1 \wedge d\sigma^2 \left[ \frac{1}{2} \left( \frac{P_m}{\sqrt{w}} \right)^2 + \frac{1}{2} \left( \frac{P_i}{\sqrt{w}} \right)^2 + \frac{1}{4} \{ X^m, X^m \}^2 \right.
\end{equation}
\begin{equation}
+ \frac{1}{2} (D_i X^m)^2 + \frac{1}{4} (F_{ij})^2 \right] + (n^2 \text{Area}_{T^2})
\end{equation}
\begin{equation}
+ \int_{\Sigma} \sqrt{w} d\sigma^1 \wedge d\sigma^2 \left[ \Lambda \left( \frac{P_i}{\sqrt{w}} \right) + \left\{ X^m, \frac{P_m}{\sqrt{w}} \right\} \right]
\end{equation}
\begin{equation}
+ \int_{\Sigma} \sqrt{w} d\sigma^1 \wedge d\sigma^2 \left[ - \bar{\theta} \Gamma_{-\Gamma} D_i \theta - \bar{\theta} \Gamma_{-\Gamma} m \{ X^m, \theta \} - \Lambda \{ \bar{\theta} \Gamma_{-\Gamma}, \theta \} \right], \tag{4.16}
\end{equation}

\begin{center}
- 12 -
\end{center}
where there is a symplectic covariant derivative and symplectic curvature defined

\[ D_i X^m = D_i X^m + \{ A_i, X^m \}, \]
\[ F_{ij} = D_i A_j - D_j A_i + \{ A_i, A_j \}, \tag{4.17} \]

with

\[ D_i \mathbf{\cdot} = 2\pi m^k _i k_i \theta^{kr} \partial_r \hat{X}^j \partial_j \mathbf{\cdot} \]

being \( R_i \) the radii of the two torus, \( k = 1, 2 \) and \( \theta_{kj} \) a matrix that has relation with the monodromy of the theory when formulated on a torus bundle \([11]\), and the symplectic connection transforming under area preserving diffeomorphisms given by

\[ \delta_{\epsilon} A = D\epsilon. \tag{4.18} \]

In \([41]\), the authors show that this hamiltonian classically does not contain string-like configurations. At a quantum level it has the remarkable property of having a supersymmetric discrete spectrum with finite multiplicity, \([17, 18]\) in distinction with the supermembrane compactified on a torus without this restriction (4.5) that is continuous spectrum from \([0, \infty)\) \([15, 16]\). The topological condition implies that the harmonic modes couple in a nontrivial way with the local modes and generate a "mass" term in the Hamiltonian which combined with the contribution of the other terms in the potential render the spectrum discrete. Moreover since the harmonic forms that are used to express the metric \( \sqrt{\omega} \) are BPS solutions of the semiclassical Hamiltonian they also break spontaneously half of the supersymmetry.

5 Equivalence between the Supermembrane theory with non trivial \( C_+ \) and the Supermembrane irreducibly wrapped

Let us consider now the supermembrane on a target space \( M_9 \times T^2 \) with a constant \( C_{+ij} \) three-form, as in section 2. We use the same decomposition of the bosonic embedding maps as in section 4. We then have the following Hamiltonian for the supermembrane,

\[ \mathcal{H} = \sqrt{\omega} \left[ \frac{1}{2} \left( \frac{P_m}{\sqrt{\omega}} \right)^2 + \left( \frac{P_i}{\sqrt{\omega}} \right)^2 + \frac{1}{4} \{ X^i, X^j \}^2 + \frac{1}{2} \{ X^i, X^m \}^2 + \frac{1}{4} \{ X^m, X^n \}^2 - \bar{\theta} \Gamma^{-m} \Gamma_m \{ X^m, \theta \} - \bar{\theta} \Gamma^{-i} \Gamma_i \{ X^i, \theta \} + C_+ \right]. \tag{5.1} \]

subject to the first class constraints

\[ d( P_i dX^i + P_m dX^m + \bar{\theta} \Gamma^- d\theta) = 0, \tag{5.2} \]
\[ \oint_{C_o} d( P_i dX^i + P_m dX^m + \bar{\theta} \Gamma^- d\theta) = 0. \tag{5.3} \]

Now let us impose the following restriction
\[
\int_{T^2} C_+ = \int_{T^2} C_{+ij} dX^i \wedge dX^j = k. \tag{5.4}
\]

on all embedding maps from the base torus to the target space. For \( C_{+ij} = k \in \mathbb{Z}/0 \) this is a flux condition over the 2-torus of the target space.

\[
\int_{T^2} \tilde{F}_2 = k. \tag{5.5}
\]

Although \( C_{+ij} \) is constant, there is a nontrivial contribution from the angular coordinates of \( T^2 \). The flux condition is a restriction on the harmonic sector of the embedding maps, there is no condition on the exact sector.

The pullback of the target space two-form flux induces a flux condition \( F_2 \) on the worldvolume of the supermembrane

\[
\int_{T^2} \tilde{F}_2 = \int_{\Sigma} F_2 = k. \tag{5.6}
\]

Consequently the target space 2-form flux condition induces a central charge condition over the worldvolume and \( k = n \). Therefore the Hamiltonians (5.1) and (4.16) differ only by a shift generated by the integral of the form \( C_+ \)

\[
H^{C_+} = H^{Irred} + T \int_{\Sigma} C_+, \tag{5.7}
\]

where the last term is proportional to \( n \) units of flux and hence it is constant for a given \( n \). One may also consider the theory formulated on the space of all \( U(1) \) bundles and in that case this term is discrete but not constant.

Whenever this equivalence holds, the worldvolume flux induces a target space flux condition on the 2-torus and vice versa due to the minimal immersion map implied by the irreducibility condition. A highly nontrivial property.

Summarizing, a supermembrane in the presence of constant \( C_{+ij} \) on a 2-torus with a flux condition on it, is equivalent to an irreducibly wrapped supermembrane on the same 2-torus with zero bosonic three-form contribution. Both supermembrane theories exhibit a central charge in the supersymmetric algebra.

An immediate consequence of this equivalence between both actions is the fact that a supermembrane formulated on a Minkowski background in the presence of a constant three form \( C_{+ij} \) and compactified on a 2-torus when a flux condition is imposed on it, the theory has a purely discrete spectrum with eigenvalues of finite multiplicity at quantum level.

This result suggests an interesting question about the deep nature of certain kind of fluxes, its relation with minimal embeddings and the possibility to extend this result to other backgrounds.

6 Discussion and Conclusions

In this paper we discussed the Hamiltonian formulation of the Supermembrane theory on target space with toroidal topology, flat metric and constant bosonic 3-form. Let us discuss the three different cases:
• **Bosonic Analysis.** If we restrict our analysis to the bosonic piece of the Hamiltonian formulated in the canonical variables there is no substantial difference between all the three cases previously analyzed. All of them can be expressed in terms of the bosonic sector of the supermembrane irreducibly wrapped around the 2-torus. The case of a membrane in the presence of \( C_{+ij} = k \) compactified on a torus and with a flux condition on it, is equivalent to an irreducibly wrapped membrane plus a shift given by \( \int_S C_+ \). The membrane in the presence of a constant \( C_{-ij} \) it exactly corresponds to the bosonic part of a supermembrane on a flat background [3] since - as we have shown- the \( X^- \) contribution decouples completely from the Hamiltonian and from the constraints. Consequently when the theory is compactified on a 2-torus and an extra flux condition is imposed

\[
\int_{T^2} C_- = k. \tag{6.1}
\]

it exactly coincides with the bosonic part of a supermembrane irreducibly wrapped on the 2-torus.

The case of a bosonic membrane theory on a flat metric compactified on a torus in the presence of \( C_{\pm ij} \) is a superposition of the two cases previously considered. For example taking \( C_{+ab} = -C_{-ab} = \text{const.} \) as in ([1]) when a flux condition (6.1) imposed, it is equivalent to the irreducibly wrapped membrane with a shift given by the \( \int C_+ \) contribution.

The flux condition can be also interpreted as wrapping condition on a noncommutative target 2-torus [31].

All of these bosonic theories with nonvanishing constant \( C_{\pm ij} \) have discrete spectrum as expected, since this property also holds to the bosonic sector of the Supermembrane theory when it is formulated on an eleven dimensional Minkowski spacetime.

• **Case** \( C_{+ij} = k \in \mathbb{Z}/\{0\} \). We first consider \( C_{+ab} \) constant on a on a 11D noncompact target space with vanishing \( C_{-ab} \). The Hamiltonian reduces to the one found by De Wit, Hoppe and Nicolai, shifted by a constant. The spectrum is continuous.

In the case when the supermembrane is formulated on a \( T^2 \times M^9 \) target space there are two possibilities: One in which the \( \int_{T^2} C_{+ij} dX^i \wedge dX^j \) vanishes. It corresponds to the Hamiltonian compactification on a torus that extends trivially the result found in [16] and it has a continuous spectrum. It can be understood from the fact that classically it contains-as in the uncompactified case-, string-like spikes that can be attached to the spectrum without any cost of energy. There is a second possibility: \( \int_{T^2} C_+ = k \in \mathbb{Z}/\{0\} \) it corresponds to a flux condition on the 2 torus target space. This flux backreacts on the worldvolume generating a flux on the worldvolume associated to the presence a fixed nontrivial \( U(1) \) fiber bundle whose first chern class is \( k \). It acts as a new constraint on the Hamiltonian and it is associated to the existence of a nontrivial central charge condition. The Hamiltonian corresponds to the Hamiltonian of the Supermembrane theory irreducible wrapped on a flat torus without any 3-form background, shifted by a constant term proportional to \( k \). This means that the super-
membrane wrapped on a 2-torus target space with a 2-form flux induced by a constant \( C_+ \) has the highly nontrivial property of having a discrete spectrum. A relevant comment is that the Supermembrane description of BMN matrix model corresponding to the propagating of supermembrane on a pp-wave [25] has also a nonvanishing \( C_+ \). Semiclassically as mentioned by the authors it has discrete spectrum. In [18] it was proven that the complete regularized model have also a full-fledged supersymmetric discrete spectrum (semi classical discreteness does not imply fullfledged discreteness). The 3-form \( C_+ \) cannot generate a 3-form flux in the case of compactifications on \( T^d \) for \( d \geq 3 \) since \( X^+ \) is proportional to time, but as we have just seen it may produce a 2-form flux on a 2-cycle of the compact manifold. It becomes quite clear that it may play a very relevant role in the formulation of a well-behaved quantum supermembrane action.

- **Case** \( C_{-ij} \). An interesting case occurs when \( C_- \neq 0 \). This case corresponds to a compactification of the supermembrane on a flat metric with nontrivial \( C_- \). The noncommutative Super Yang Mills description of the matrix model on a noncommutative torus \( T^2_\theta \), proposed by [31] can be obtained in terms of a SYM matrix model description in the presence of a B-field by Morita equivalence [42–44] whenever the noncommutative parameter given by,

\[
[X^i, X^j] = \theta^{ij},
\]

is rational.

In [27], it was originally conjectured that the uplift of these models to M-theory could be related to a Supermembrane theory formulated on a flat metric with a nontrivial constant \( C_{-rs} = \theta^{rs} \) based in deformation quantization [45]. Indeed the matrix model for constant three form fields developed by [29] it would be interesting to see in a future work the relation (if any) with the model developed in [41].

The formulation of the theory on a noncommutative torus emerges naturally for the bosonic sector of \( C_{\pm ij} \). For \( C_+ \) it can be also extended directly to the supersymmetric case that is directly connected with the usual BFSS matrix model. The analysis for the \( C_- \) case is more subtle since the three form appears not only in the Hamiltonian but also in the constraints. However as we have shown the constraint can also be resolved and it produces, for the bosonic variables a shift in the canonical momentum variable \( \hat{P}_a \) as already was signalled by the authors of [29]. In the final Hamiltonian one has to realize that there is a nontrivial mixing in the supersymmetric contribution.

\[
(\hat{P}_a - \varepsilon^{rs} \hat{\theta} \Gamma^- \partial_\theta \partial_\tau X^b C_{-ab})^2.
\]

The role of this term in the characterization of the spectrum deserves a deeper study. We leave it for a future work.

Let us consider now the more general case when both types of three forms are switched on \( C_{\pm} \neq 0 \). This background corresponds to the asymptotic solution described by the authors [1, 35] generated by an M2-brane when \( C_{+rs} = C_{-rs} \). The spectral properties
of the supersymmetric Hamiltonian, as in the previous case, deserves a deeper study. A
generalization of these type of backgrounds to more general curved ones arising
from supergravity solutions was considered in [46]. It would be very interesting to
understand their coupling to supermembrane theories.

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