Triangle-free induced subgraphs of the unitary polarity graph

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Abstract

Let $\perp$ be a unitary polarity of a finite projective plane $\pi$ of order $q^2$. The unitary polarity graph is the graph with vertex set the points of $\pi$ where two vertices $x$ and $y$ are adjacent if $x \in y^\perp$. We show that a triangle-free induced subgraph of the unitary polarity graph of an arbitrary projective plane has at most $(q^4 + q)/2$ vertices. When $\pi$ is the Desarguesian projective plane $\text{PG}(2, q^2)$ and $q$ is even, we show that the upper bound is asymptotically sharp, by providing an example on $q^4/2$ vertices. Finally, the case when $\pi$ is the Figueroa plane is discussed.

Keywords: polarity graph, triangle-free, interlacing, Figueroa plane

1 Introduction

Let $\pi$ be a finite projective plane of order $q$. A polarity $\perp$ of $\pi$ is an involutory bijective map sending points to lines and lines to points which reverses incidence. The polarity graph $G(\pi, \perp)$ is the graph with vertex set the points of $\pi$ where two vertices $x$ and $y$ are adjacent if $x \in y^\perp$. Remark that we could have defined the graph $G(\pi, \perp)$ equivalently with the lines of $\pi$ as vertices. This graph is not simple: for every absolute point $x$ we have $x \in x^\perp$, which means that every absolute point gives rise to a loop. A classical theorem by Baer [2] states that every polarity has at least $q + 1$ points, which implies that there a polarity graph has at least $q + 1$ loops. With a slight abuse of notation we will identify the
vertices of the polarity graph with the points (or lines) of the plane. We will say for example that a point $x$ is adjacent to another point $y$. For all definitions and notions regarding projective planes and polarities not mentioned in Section 2, we refer the reader to [3, 8, 18].

Polarity graphs and their properties have been the subject of study over the last few years. Questions regarding their independence number [14, 22, 23], chromatic number [27] and other properties have been posed and (partially) answered. The motivation behind this line of research lies first of all in the fact that these graphs possess a lot of structure and interesting features. More importantly, polarity graphs are related to some classes of problems in extremal graph theory, among which Ramsey problems and Turán-type problems. For example in the latter, Füredi [12, 13] has shown that the unique graph with the most edges among all graphs on $q^2 + q + 1$ vertices not containing $C_4$ as a subgraph is the polarity graph, where $\perp$ is an orthogonal polarity, i.e., a polarity with $q + 1$ absolute points, which is the least possible as we already mentioned.

Recently, Loucks and Timmons [20] have drawn attention to the following problem.

**Question 1.1.** What is the largest set of non-absolute vertices in $G(\pi, \perp)$ inducing a triangle-free subgraph?

This problem first appeared in [19], where the authors considered the case when $\pi = PG(2, q)$ and $\perp$ an orthogonal polarity. They used a construction due to Parsons [24] to obtain a 3-uniform hypergraph on $n = q^2$ vertices, $q$ odd, of girth five with $\frac{1}{2}n^{3/2} - \frac{1}{6}n^{1/2}$ edges, which is asymptotically the best possible. Parsons’ construction on which they relied is exactly a triangle-free induced subgraph of $G(\pi, \perp)$.

In this article, we investigate the case when $\perp$ is a unitary polarity. Then the order of the projective plane is necessarily a square, say $q^2$, there are $q^3 + 1$ absolute points and the set of absolute points forms a unital $U$. Note that there are unitals which do not arise from a unitary polarity, see [3] for further results on this topic. We denote by UP($q^2$) the unitary polarity graph for an arbitrary projective plane of order $q^2$. In the first part of the paper, by refining the techniques used in [20], we obtain the following upper bound for a triangle-free induced subgraph of UP($q^2$).

**Theorem 1.2.** Let $S$ be a subset of non-absolute vertices of UP($q^2$) inducing a triangle-free subgraph, then

$$|S| \leq \frac{q^4 + q}{2}.$$  

Moreover, if equality holds and $\ell$ is a line of $\pi$, then $|\ell \cap S| \in \{\frac{q^2 - q}{2}, \frac{q^2 + q}{2}\}$. 

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In the second part of the paper we deal with the case when \( \pi \) is the Desarguesian projective plane \( \text{PG}(2, q^2) \). We will denote this graph by \( \text{DUP}(q^2) \). When \( q \) is even, we are able to show that the upper bound is asymptotically sharp.

**Theorem 1.3.** There exists a set of non-absolute vertices of \( \text{DUP}(q^2) \) inducing a triangle-free subgraph of size \( q^4/2 \).

In the last part of the paper we consider the case when \( \pi \) is the Figueroa plane \( F \). The plane \( F \) is obtained by the Desarguesian plane \( \text{PG}(2, q^3) \), by distorting certain lines. Sometimes a polarity \( \perp \) of the Desarguesian projective plane is “inherited” and gives rise to a polarity \( \perp' \) of the Figueroa plane \( F \). We show that, in the case of “inherited” polarities, a triangle-free induced subgraph of \( \mathcal{G}((\text{PG}(2, q^3), \perp)) \) gives rise to a triangle-free induced subgraph of \( \mathcal{G}(F, \perp') \).

## 2 Preliminaries about \( \text{UP}(q^2) \)

Before we can prove these results, we need some structural information about \( \text{UP}(q^2) \), in particular about the neighbourhood structure. If \( x \) is a point of \( \pi \), then \( x^\perp \) denotes its polar line. Suppose first that \( x \) is an absolute point, that is, its polar line contains \( x \) itself. Therefore, \( x \) is adjacent to \( q^2 \) vertices and has a loop. Let \( y \) be a neighbour of \( x \), then \( x^\perp \cap y^\perp = \{x\} \), which implies that \( y \) has no neighbours in \( N(x) \). This means that the subgraph induced by \( x \) and its neighbours looks like a star.

On the other hand, if \( x \) is a non-absolute point, then it is adjacent to \( q^2 + 1 \) other vertices. Among these there are \( q + 1 \) absolute points, while the remaining \( q^2 - q \) are non-absolute points. Let \( y \) be a non-absolute neighbour of \( x \), then \( x^\perp \) and \( y^\perp \) intersect in a third point \( z \). Hence, \( x, y, z \) form a triangle in \( \text{UP}(q^2) \). Moreover, \( z \) is the unique common neighbour of \( x \) and \( y \). This implies that non-absolute neighbours of \( x \) come in adjacent pairs, giving rise to \( (q^2 - q)/2 \) triangles with common vertex \( x \).

A self-polar triangle of \( \pi \) (with respect to \( \perp \)) is a triangle each of whose vertices has the opposite side as polar line. From the discussion above it follows that triangles in \( \text{UP}(q^2) \) are in one-to-one correspondence with self-polar triangles of \( \pi \) and that there are exactly

\[
\frac{1}{3}(q^4 - q^3 + q^2)\frac{q^2 - q}{2} = \frac{q^3(q^2 - q + 1)(q - 1)}{6}
\]

of such triangles. Here and in the sequel we use the term triangle to refer to a triangle in \( \text{UP}(q^2) \) or to a self-polar triangle of \( \pi \).
3 The upper bound

In [20], an upper bound for the number of vertices of UP(q^2) inducing a triangle-free subgraph was proved. We refine their argument in order to obtain a better upper bound, see Theorem 1.2. To do so, we use techniques from spectral graph theory.

Given two subsets of vertices S, T in a regular graph, let \( e(S, T) \) denote the number of edges having a vertex in \( S \) and a vertex in \( T \). If \( S = T \), then we simply write \( e(S) \) instead of \( e(S, T) \). Note that \( e(S, T) = e(T, S) \). The following result, which first appeared in [1], is known as the expander mixing lemma and it is a useful tool to estimate \( e(S, T) \). Furthermore, it has found several applications in finite geometry over the last years [4, 21, 23, 26].

**Lemma 3.1.** Let \( G = (V, E) \) be a \( d \)-regular graph on \( n \) vertices with eigenvalues \( d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \). Let \( \lambda = \max(\lambda_2, |\lambda_n|) \) be the second largest eigenvalue (in absolute value) and \( S \subseteq V \), then the following inequality holds:

\[
\left| 2e(S) - \frac{d|S|^2}{n} \right| \leq \lambda |S| \left( 1 - \frac{|S|}{n} \right).
\]

We will apply the expander mixing lemma to the graph \( \Gamma \) obtained from UP(q^2) by deleting its \( q^3 + 1 \) absolute points. Hence, \( \Gamma \) is a \( (q^2 - q) \)-regular graph on \( q^4 - q^3 + q^2 \) vertices. To apply the lemma, the second largest eigenvalue of \( \Gamma \) is needed. This can be derived by using a technique due to Haemers, called eigenvalue interlacing. In what follows, we recall some definitions and results from [16].

**Definition 3.2.** Two sequences of real numbers \( \lambda_1 \geq \cdots \geq \lambda_n \) and \( \mu_1 \geq \cdots \geq \mu_m \) with \( m < n \) interlace if

\[
\lambda_i \geq \mu_i \geq \lambda_{n-m+i} \text{ for } 1 \leq i \leq m.
\]

**Theorem 3.3.** Let \( H \) be an induced subgraph of a graph \( G \), then the eigenvalues of \( H \) interlace those of \( G \).

From [14], the eigenvalues of UP(q^2) are \( q^2 + 1, q, -q \) with multiplicities \( 1, (q^4 + 2q^2 - q)/2, (q^4 + q)/2 \), respectively. Therefore, by Theorem 3.3, the eigenvalues of \( \Gamma \), which are \( q^2 - q = \mu_1 \geq \cdots \geq \mu_m \), with \( m = q^4 - q^3 + q^2 \), have to satisfy

\[
q = \lambda_2 \geq \mu_2 \geq \lambda_{q^3+3} = q \quad \text{and} \quad -q = \lambda_m \geq \mu_m \geq \lambda_n = -q.
\]

Either way, the second largest eigenvalue of \( \Gamma \) (in absolute value) equals \( q \).
Proposition 3.4. Let $S$ be a subset of non-absolute vertices of $\text{UP}(q^2)$ inducing a triangle-free subgraph, then
\[ |S| \leq \frac{q^4 + q}{2}. \]

Proof. Applying the expander mixing lemma to $\Gamma$, we find
\[
|2e(S) - \frac{(q^2 - q)|S|^2}{q^4 - q^3 + q^2}| \leq q|S| \left( 1 - \frac{|S|}{q^4 - q^3 + q^2} \right). \tag{1}
\]

Recall that every vertex $x$ is adjacent to $q^2 - q$ vertices in $\Gamma$, which come in pairs to form $(q^2 - q)/2$ triangles with common vertex $x$. Suppose that $x \in S$, then $x$ can be adjacent to at most one vertex of each triangle. This implies that $x$ has at most $(q^2 - q)/2$ neighbours in $S$ and hence
\[
2e(S) = \sum_{x \in S} d_S(x) \leq \frac{q^2 - q}{2} |S|,
\]
where $d_S(x)$ denotes the number of neighbours in $S$ of a vertex $x$. We can assume that $|S| \geq (q^4 - q^3 + q^2)/2$, otherwise the proposition is vacuously true. Then (1) becomes
\[
\frac{(q^2 - q)|S|^2}{q^4 - q^3 + q^2} - q|S| \left( 1 - \frac{|S|}{q^4 - q^3 + q^2} \right) \leq 2e(S) \leq \frac{q^2 - q}{2} |S|.
\]

Now solving the previous inequality for $|S|$ proves the result. \hfill \square

The next step is to find out whether the upper bound can be attained. Assume that equality holds. From the proof of Proposition 3.4, we have that each vertex $x \in S$ has degree $d_S(x) = (q^2 - q)/2$. From a geometrical point of view, this means that if $x \in S$, then $|x^\perp \cap S| = (q^2 - q)/2$, i.e., a certain number of lines intersect the set $S$ in a constant number of points. By again using eigenvalue interlacing, we will show that an even stronger property holds: for any line $\ell$ we have
\[
|\ell \cap S| \in \left\{ \frac{q^2 - q}{2}, \frac{q^2 + q}{2} \right\},
\]
i.e. $S$ is a two-intersection set. These point sets have been intensively studied in the literature, see for example [7, 25] and references therein.

Definition 3.5. The interlacing of two sequences of real numbers $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mu_1 \geq \cdots \geq \mu_m$, $m < n$, is tight if there exists $0 \leq k \leq m$ such that
\[
\lambda_i = \mu_i \text{ for } 1 \leq i \leq k \text{ and } \lambda_{n-m+i} = \mu_i \text{ for } k+1 \leq i \leq m.
\]
Theorem 3.6. Let $A$ be a symmetric $n \times n$ matrix partitioned as

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,m} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,m} \end{pmatrix},$$

such that $A_{i,i}$ is a square matrix for all $1 \leq i \leq m$. The quotient matrix $B$ is the $m \times m$ matrix with entries the average row sums of the blocks of $A$. More precisely,

$$B = (b_{i,j}), \quad b_{i,j} = \frac{1}{n_i} 1^t A_{i,j} 1,$$

where $1$ denotes the all one column vector and $n_i$ is the number of rows of $A_{i,j}$. Then the following holds

1. The eigenvalues of $B$ interlace those of $A$;

2. if the interlacing is tight, then $A_{i,j}$ has constant row and column sums for $1 \leq i, j \leq m$.

Lemma 3.7. Let $S$ be a subset of non-absolute vertices of $\text{UP}(q^2)$ inducing a triangle-free subgraph, with $|S| = (q^4 + q)/2$. Then, if $x$ is a point of $\pi$, we have that

$$|x^\perp \cap S| = \begin{cases} q^2 - q \quad & \text{if } x \in S, \\ q^2 + q \quad & \text{if } x \notin S. \end{cases}$$

Proof. Let $A$ be the adjacency matrix of $\text{UP}(q^2)$ (with loops). We can partition the points of $\pi$ into three sets: the set of absolute points $U$, the set of interest $S$ and their complement $R$. Hence $|R| = q(q - 1)(q^2 - q + 1)/2$. Using this partition, we get a partition of $A$ as follows

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{pmatrix}.$$

Consider the quotient matrix

$$B = \begin{pmatrix} |U|^{-1}e(U) & |U|^{-1}e(U,S) & |U|^{-1}e(U,R) \\ |S|^{-1}e(S,U) & |S|^{-1}e(S) & |S|^{-1}e(S,R) \\ |R|^{-1}e(R,U) & |R|^{-1}e(R,S) & |R|^{-1}e(R) \end{pmatrix}.$$

We already know a few entries. Indeed, by hypothesis, $|S|^{-1}2e(S) = (q^2 - q)/2$. Moreover, every absolute point is incident with exactly one absolute point (itself),
so \(|U|^{-1}e(U) = 1\) and every non-absolute point is incident with \(q + 1\) absolute points, hence \(|S|^{-1}e(S, U) = |R|^{-1}e(R, U) = q + 1\). Analogously, since every point of \(S\) is incident with exactly \((q^2 - q)/2\) points of \(R\) we get \(|S|^{-1}e(S, R) = (q^2 - q)/2\). As \(e(U, S) = e(S, U)\), it follows that \(|U|^{-1}e(U, S) = (q^2 + q)/2\), and similarly \(|U|^{-1}e(U, R) = (q^2 - q)/2\) and \(|R|^{-1}e(R, S) = (q^2 + q)/2\). Lastly, since the sum of the elements of a row of \(A\) equals \(q^2 + 1\), we obtain \(|R|^{-1}2e(R) = (q^2 - 3q)/2\). Collecting these values gives

\[
B = \begin{pmatrix}
1 & q^2 + q & q^2 - q \\
q + 1 & q^2 - q & q^2 - q \\
q + 1 & q^2 + q & q^2 - 3q
\end{pmatrix}.
\]

The eigenvalues of \(B\) are \(q^2 + 1, -q, -q\), which shows that the interlacing is tight. By Theorem 3.6, every block \(A_{i,j}\) has constant row sum and constant column sum. This means that every vertex in \(U\) or \(R\) is adjacent to precisely \((q^2 + q)/2\) vertices in \(S\).

**Remark 3.8.** Let \(S\) be a subset of non-absolute vertices of \(UP(q^2)\) inducing a triangle-free subgraph, with \(|S| = (q^4 + q)/2\). Then, in the language of [9], we have that \(S\) is an intriguing set of \(\Gamma\), which could also be shown using Proposition 3.8 of that article.

**Remark 3.9.** Let \(S\) be a subset of non-absolute vertices of \(UP(q^2)\) inducing a triangle-free subgraph such that \(|S| = (q^4 + q)/2\). Then the set \(S \cup U\) is a two-intersection set. Indeed, if \(x \in S\), then \(|x^\perp \cap S| = (q^2 - q)/2\) and hence \(|x^\perp \cap (S \cup U)| = (q^2 + q)/2 + 1\). On the other hand, if \(x \notin S\), then \(|x^\perp \cap S| = (q^2 + q)/2\) and hence \(|x^\perp \cap (S \cup U)|\) equals either \((q^2 + q)/2 + 1\) or \((q^2 + q)/2 + q + 1\), according as \(x \in U\) or \(x \notin U\). It follows that \(S \cup U\) is a set of \((q + 2)(q^3 + 1)/2\) points such that every line meets \(S \cup U\) is either \((q^2 + q + 2)/2\) or \((q^2 + 3q + 2)/2\) points. Since no such a set exists in \(PG(2, 4)\) or in \(PG(2, 9)\), see [25], it follows that in these cases the upper bound of Proposition 3.4 cannot be attained.

### 4 The Desarguesian plane

Let \(\pi\) be the Desarguesian projective plane \(PG(2, q^2)\), with \(q = p^h\), \(p\) a prime, \(h\) a positive integer. The set of absolute points of a unitary polarity of \(PG(2, q^2)\) is called a *Hermitian curve*. In this case, if \(q\) is even, by means of constructive arguments, we are able to show a lower bound close to the upper bound of Theorem 1.2. In particular we will prove the existence of a triangle-free subgraph of \(DUP(q^2)\) having \(q^4/2\) vertices, see Theorem 1.3. The strategy is the following:
we will fix a unitary polarity $\perp$ and hence a Hermitian curve $\mathcal{U}$; we will consider a pencil $\mathcal{P}$ consisting of $q$ Hermitian curves such that $\mathcal{U}$ is contained in $\mathcal{P}$ and elements in $\mathcal{P}$ pairwise intersect at a common point. Then we will select $q/2$ Hermitian curves in $\mathcal{P} \setminus \{\mathcal{U}\}$ and show that the set of points covered by these Hermitian curves distinct from their common point possesses the required properties.

4.1 A lower bound

The projective plane $\text{PG}(2, q^2)$ will be represented via homogeneous coordinates over the Galois field $\mathbb{F}_q$, i.e., represent the points of $\text{PG}(2, q^2)$ by $\langle (x, y, z) \rangle$, $x, y, z \in \mathbb{F}_q$, $(x, y, z) \neq (0, 0, 0)$, and similarly lines by $\langle [a, b, c] \rangle$, $a, b, c \in \mathbb{F}_q$, $[a, b, c] \neq [0, 0, 0]$. Incidence is given by $ax + by + cz = 0$. To avoid awkward notation the angle brackets will be dropped in what follows. The group consisting of all projectivities of $\text{PG}(2, q^2)$ is denoted by $\text{PGL}(3, q^2)$. The point $U_i$ is the point with 1 in the $i$-th position and 0 elsewhere. As any two Hermitian curves are projectively equivalent [18, Chapter 5], we may assume that $\mathcal{U}$ has equation

$$X_1^qX_2 + X_1X_2^q + X_3^{q+1} = 0.$$  

In other words, the matrix defining the polarity is the matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

two vertices $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ in $\text{DUP}(q^2)$ are adjacent if and only if

$$(x_1 \ y_1 \ z_1) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2^q \\ y_2^q \\ z_2^q \end{pmatrix} = 0,$$

and

$$(x, y, z)^\perp = [y^q, x^q, z^q].$$

Let $G \cong \text{PGU}(3, q^2)$ be the subgroup of $\text{PGL}(3, q^2)$ leaving $\mathcal{U}$ invariant. We shall find it helpful to work with the elements of $\text{PGL}(3, q^2)$ as matrices in $\text{GL}(3, q^2)$ and the points of $\text{PG}(2, q^2)$ as column vectors, with matrices acting on the left.

The point $U_2$ clearly belongs to $\mathcal{U}$ and its polar line $U_2^\perp$ is the line $\ell : X_1 = 0$. We can consider the pencil $\mathcal{P}$ consisting of the $q$ Hermitian curves given by

$$\mathcal{U}_\lambda : \lambda X_1^{q+1} + X_1^qX_2 + X_1X_2^q + X_3^{q+1} = 0,$$
where $\lambda \in \mathbb{F}_q$. When $\lambda = 0$ we retrieve $U$, that is $U_0 = U$. Note that each of the Hermitian curves of $\mathcal{P}$ is contained in the pencil generated by $U : X_1^q X_2 + X_1 X_2^q + X_3^{q+1} = 0$ and the degenerate Hermitian curve $X_1^{q+1} = 0$. Every point not on the line $\ell$ belongs to exactly one Hermitian curve of $\mathcal{P}$, while the point $U_2$ is common to all Hermitian curves in $\mathcal{P}$.

**Lemma 4.1.** There exists a subgroup $K$ of $G$ of order $q^3$ acting regularly on points of $U_\lambda \setminus \{U_2\}, \lambda \in \mathbb{F}_q$.

**Proof.** Let $K$ be the subgroup of $G$ whose elements are associated with the following matrices

$$
\begin{pmatrix}
1 & 0 & 0 \\
a & 1 & -b^q \\
b & 0 & 1
\end{pmatrix},
$$

where $(1, a, b) \in \mathcal{U}$. Then $K$ is a group of order $q^3$. Straightforward calculations show that if $P \in U_\lambda$ and $g \in K$, then $P^g \in U_\lambda$ and that the stabilizer in $K$ of a point $P \in U_\lambda \setminus \{U_2\}$ is trivial. $\square$

**Proposition 4.2.** Let $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}_q \setminus \{0\}$, not necessarily distinct. If $U_{\lambda_1} \cup U_{\lambda_2} \cup U_{\lambda_3}$ contains a triangle, not containing the common point $U_2$, then

$$
\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3 = 0.
$$

(2)

**Proof.** Suppose that we do have three points $P, Q, R$ forming a triangle and contain in $U_{\lambda_1} \cup U_{\lambda_2} \cup U_{\lambda_3}$. Taking into account Lemma 4.1, we may assume that $P$ is the point $(1, x_1, 0) \in U_{\lambda_1}$, where

$$
\lambda_1 + x_1 + x_1^q = 0.
$$

(3)

The second point $Q = (1, x_2, y_2) \in U_{\lambda_2}$ has to be on the line $P^\perp : x_1^q X_1 + X_2 = 0$, which implies that $x_2 = -x_1^q$. Here we find by (3) that

$$
\lambda_2 - x_1^q - x_1 + y_2^{q+1} = \lambda_1 + \lambda_2 + y_2^{q+1} = 0.
$$

(4)

For the third point $R$, we find in the same way that it should be of the form $(1, -x_1^q, y_3) \in U_{\lambda_3}$, where

$$
\lambda_1 + \lambda_3 + y_3^{q+1} = 0.
$$

(5)

Moreover, $R \in Q^\perp$, where $Q^\perp : -x_1 X_1 + X_2 + y_2^q X_3 = 0$. This implies that

$$
-x_1 - x_1^q + y_2^q y_3 = \lambda_1 + y_2^q y_3 = 0.
$$

(6)
Remark that $y_2$ nor $y_3$ can equal zero, for otherwise one of $Q$ or $R$ would have to be $U_2$. Multiplying this last equation by $y_2 y_3^q$ and using (4), (5), (6), we obtain that

$$\lambda_1 y_2 y_3^q + y_2^{q+1} y_3^{q+1} = \lambda_1 (-\lambda_1 y_2^q) + (-\lambda_1 - \lambda_2)(-\lambda_1 - \lambda_3) =$$

$$= -\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_1 \lambda_3 = 0,$$

as $\lambda_1 \in \mathbb{F}_q$. □

Therefore, if we can find a set $\Lambda \subseteq \mathbb{F}_q \setminus \{0\}$ such that for every $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$ equation (2) is never satisfied, then the set $\bigcup_{\lambda \in \Lambda} U_\lambda \setminus \{U_2\}$ induces a triangle-free subgraph on $|\Lambda|q^3$ vertices.

**Remark 4.3.** Note that, although Proposition 4.2 is useless in the case when $p = 3$, in all the other case it provides the (weak) lower bound $q^3$ for the number of vertices of a triangle-free induced subgraph of $\text{DUP}(q^2)$.

The last step is to construct a set $\Lambda$ so that (2) is never satisfied for any three elements in $\Lambda$.

**Lemma 4.4.** If $q$ is even, there exist $q/2$ additive subgroups $H$ such that $1 \notin H$ and $|H| = q/2$.

**Proof.** Consider $\mathbb{F}_q$ as a vector space $V$ over $\mathbb{F}_2$, then the additive subgroups of $\mathbb{F}_q$ are in one-to-one correspondence with subspaces of this vector space $V$. In particular, a subgroup of size $q/2$ corresponds to a hyperplane of $V$. It is immediate that any of the $q/2$ hyperplanes not through the vector corresponding to 1, gives rise to a subgroup $H$ satisfying all conditions. □

**Lemma 4.5.** Let $q$ be even and let

$$\Lambda = \{1/(1 + a) \mid a \in H\},$$

where $H$ is an additive subgroup of $\mathbb{F}_q$ of order $q/2$ not containing 1. Then for any three elements in $\Lambda$, (2) is never satisfied.

**Proof.** Assume on the contrary that there are three elements in $\Lambda$ satisfying (2). Then we would have

$$\frac{1}{(1 + a)(1 + b)} + \frac{1}{(1 + b)(1 + c)} + \frac{1}{(1 + a)(1 + c)} = 0,$$

for elements $a, b, c \in H$. After clearing the denominators, we find that

$$a + b + c = 1,$$

which is a contradiction as the left hand size is an element of $H$, while 1 is not in $H$. □
Remark 4.6. If $q$ is odd, let $P$ be a point of $PG(2, q^2)$ not in $U$ and let $T_P$ be the set of non-absolute points distinct from $P$ lying on the $q+1$ lines containing $P$ and tangent to $U$. Then $|T_P| = (q^2 - 1)(q + 1)$. We claim that $T_P$ contains no triangle. Otherwise, if $Q_1, Q_2, Q_3$ were a triangle contained in $U$, we would obtain a configuration consisting of seven points: $P, Q_1, Q_2, Q_3, PQ_1 \cap U, PQ_2 \cap U, PQ_3 \cap U$ and seven lines $P_1, Q_1, Q_2, Q_3, PQ_1, PQ_2, PQ_3$ such that through each point there pass three lines and each line contains three points, i.e., a Fano plane $PG(2, 2)$. On the other hand, if $q$ is odd, $PG(2, 2)$ cannot be embedded in $PG(2, q^2)$. Note that a larger set containing $T_P$ and not containing triangles can be obtained by adding to $T_P$ the $q^2 - q$ non-absolute points on the line $P \perp$, this gives the (weak) lower bound $q^3 + 2q^2 - 2q - 1$ for the number of vertices of a triangle-free induced subgraph of $DUP(q^2)$, in the case when $q$ is odd.

4.2 Properties of the graph

Assume that $q$ is even and let $\Sigma$ be a triangle-free induced subgraph of $DUP(q^2)$ on $q^4/2$ vertices constructed in subsection 4.1. Here, we investigate further properties of the graph $\Sigma$. To start off, we prove that $\Sigma$ is regular.

Proposition 4.7. The graph $\Sigma$ is $q(q - 1)/2$-regular.

Proof. Let $P$ be a point of $PG(2, q^2)$, $P \notin \ell$. As we have partitioned all points of $PG(2, q^2)$ into the union of the $q$ sets $U_\lambda \setminus \{U_2\}$ and the line $\ell$, it is easy to see that the line $P_{-1}$ contains a point of $\ell$, is secant to $q - 1$ Hermitian curves of $P$ and is tangent to exactly one Hermitian curve of $P$. Consider a vertex $v \in U_\lambda \setminus \{U_2\}$, $\lambda \in \Lambda$ and let $U_\mu$ be the unique Hermitian curve of $P$ such that $|v_{-1} \cap U_\mu| = 1$. Taking into account Lemma 4.1, we can assume that the point $v$ has coordinates $(1, x, 0)$, where $x + x^q = \lambda$. Then $v_{-1}$ has dual coordinates $[x^q, 1, 0]$ and we have to find $\mu$ such that $v_{-1}$ is tangent to $U_\mu$. This means finding $\mu$ such that

$$\mu X_{1}^{q+1} + (x + x^q)X_{2}^{q+1} + X_{3}^{q+1} = 0$$

has only one solution. It follows immediately that $\lambda = \mu$. Hence, every vertex $v \in U_\lambda$ has exactly one neighbour in $U_\lambda$. Moreover, it has $q + 1$ neighbours in $\Sigma$ on the $q/2 - 1$ other Hermitian curves $U_\lambda, \lambda \in \Lambda \setminus \{\mu\}$, which implies that the degree in $\Sigma$ of every vertex of $\Sigma$ is $q(q - 1)/2$. □

In fact, with a similar proof, one can show the exact intersection numbers for any line with $S$. 

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Corollary 4.8. Let $x$ be a point of $\text{PG}(2, q^2)$, then

$$|x^\perp \cap \Sigma| = \begin{cases} 
q^2 - q^2 & \text{if } x \in \Sigma, \\
q^2 + q^2 & \text{if } x \in \text{PG}(2, q^2) \setminus (S \cup \ell), \\
q^2 & \text{if } x \in \ell \setminus \{U_2\}, \\
0 & \text{if } x = U_2.
\end{cases}$$

Remark that $q(q-1)/2$-regularity is the best we can achieve. Indeed, as we have already seen, every non-absolute point $v$ is adjacent to $q(q-1)$ other non-absolute points and these neighbours come in pairs to form $q(q-1)/2$ triangles with common vertex $v$, so $v$ can be adjacent to at most one vertex in each of these triangles. The fact that $\Sigma$ is $q(q-1)/2$-regular implies that $v$ is adjacent to exactly one vertex in each of the triangles. In other words, if a triangle of $\text{DUP}(q^2)$ contains a vertex of $\Sigma$, it contains another vertex of $\Sigma$. Thus we have shown the following result.

Corollary 4.9. Every triangle of $\text{DUP}(q^2)$ has either 0 or 2 vertices in common with $\Sigma$.

This property allows us to show that the subgraph $\Sigma$ is maximal with the triangle-free property, i.e., we cannot add any vertex not in $\Sigma$ without creating a triangle.

Proposition 4.10. The graph $\Sigma$ is maximal with respect to the triangle-free property.

Proof. Suppose we could add another vertex $v$. This vertex $v$ has at least one neighbour in $\Sigma$ as $v^\perp$ intersects any $U_\lambda$ in at least one point. Therefore, consider a triangle $T$ containing $v$ and a vertex of $v^\perp \cap \Sigma$. From the previous Corollary, we know that the triangle $T$ actually has its third vertex in $\Sigma$ and hence we cannot add $v$ to $\Sigma$ without creating a triangle. \hfill $\Box$

The next result shows that $\Sigma$ can be chosen in such a way that it has girth 5.

Proposition 4.11. For $q$ an even prime power, there exists a $q(q-1)/2$-regular graph on $q^4/2$ vertices of girth 5.

Proof. We can show the result for $q \leq 16$ using Magma [5], so suppose $q \geq 32$ for the remainder of the proof.

Taking into account Lemma 4.4 and Lemma 4.5, let $\Lambda = \{1/(1 + a) \mid a \in H\}$, where $H$ is an additive subgroup of $\mathbb{F}_q$ of order $q/2$ not containing 1. Let us
consider a non-zero element \( a \in H \) and let \( \lambda_1 = 1/(1 + a) \in \Lambda \). Let \( b \in H \), with \( a \neq b \) such that \( b \) is not a solution of none of the following equations:

\[
X^2 + (a + 1)X + a^3 = 0, \quad X^3 + aX + a(a + 1) = 0, \quad (7)
\]

\[
X^2 + (a + 1)X + a^2 + a + 1 = 0. \quad (8)
\]

Since the union of the solutions of the equations (7) and (8) consists of at most 7 distinct elements of \( \mathbb{F}_q \), we can always find such an element \( b \) if \( q \geq 32 \). Let \( P_1 := (1, x_1, 0) \in U_{\lambda_1} \) and \( P_2 := (1, x_1^q, 0) = P_1^1 \cap U_{\lambda_1} \), its unique neighbour on \( U_{\lambda_1} \). Take another point \( Q_1 := (1, x_1^q, z) \in P_1^1 \cap U_{\lambda_2} \), with \( \lambda_2 = 1/(1 + b) \in \Lambda \). \( \lambda_1 \). Hence \( x_1 + x_1^q = \lambda_1 \) and \( z^{q+1} = \lambda_1 + \lambda_2 \). Its unique neighbour on \( U_{\lambda_2} \) is \( Q_2 := (1, x_1 + \lambda_1 + \lambda_2, z) = Q_1^1 \cap U_{\lambda_2} \). Then \( R := P_2^1 \cap Q_2^1 = (1, x_1, \lambda_2/z^q) \) belongs to \( U_{\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2} \). Note that, since \( b \) is not a solution of (8), we have that \( \lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2 \neq 0 \). Hence \( P_1 P_2 R Q_2 Q_1 \) is a cycle of length 5 in \( \Sigma \) if and only if

\[
\frac{\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2}{\lambda_1 + \lambda_2} \in \Lambda. \quad (9)
\]

On the other hand, a straightforward calculation shows that (9) holds true if and only if

\[
x := \frac{a^2 b + a b^2 + a b + 1}{a^2 + b^2 + a b + a + b + 1} \in H.
\]

Note that \( x \neq 1 \), otherwise at least one among \( a \) and \( b \) should be 1. Analogously to the proof of Lemma 4.4, view \( \mathbb{F}_q \) as a vector space \( V \) over \( \mathbb{F}_2 \). Since \( b \) is not a solution of the equations (7), we have that the vector subspace of \( V \) generated by \( a, b, x \) does not contain 1. Therefore, we can find a hyperplane \( H \) such that \( a, b, x, 1 \in H \) and \( 1 \notin H \). This hyperplane corresponds to an additive subgroup of size \( q/2 \), which concludes the proof.

\[\square\]

5 The Figueroa plane

The finite Figueroa planes are non-Desarguesian projective planes of order \( q^3 \) for all prime powers \( q \geq 2 \). These planes were constructed algebraically in 1982 by Figueroa [11], and Hering and Schaeffer [17], and synthetically in 1986 by Grundhöfer [15]. All Figueroa planes of finite square order possess a unitary polarity and hence admit unitals [10]. In general it can be seen that under certain assumptions, a polarity \( \rho \) of the Desarguesian projective plane is “inherited” and gives rise to a polarity \( \rho' \) of the Figueroa plane \( F \). In this section we show that, in the case of “inherited” polarities, a triangle-free induced subgraph of \( G(PG(2, q^3), \rho) \) gives rise to a triangle-free induced subgraph of \( G(F, \rho') \).
5.1 Construction of Figueroa planes

Let $\alpha$ be an order 3 collineation of the classical projective plane $PG(2, q^3)$ of order $q^3$ over the finite field $\mathbb{F}_{q^3}$, where the fixed points of $\alpha$ constitute a subplane isomorphic to $PG(2, q)$. The points and lines of $PG(2, q^3)$ are partitioned into distinct types, as follows. A point $x$ of $PG(2, q^3)$ belongs to $O_1$ if $x^{\alpha} = x$, or to $O_2$ if $x, x^{\alpha}, x^{\alpha^2}$ are distinct and on a line, or to $O_3$ if $x, x^{\alpha}, x^{\alpha^2}$ are distinct and not on a line. Types of lines $L_1, L_2, L_3$ of $PG(2, q^3)$ are defined dually. Points of $O_1$ and lines of $L_1$ thus constitute a subplane $D$ isomorphic to $PG(2, q)$. If $x$ is a point of $O_2$, then it is on the unique line of $L_1$ containing $x, x^{\alpha}, x^{\alpha^2}$. Conversely, if a point is on a unique line of $L_1$, then it belongs to $O_2$ since a point of $O_1$ is on $q + 1$ lines of $L_1$ and a point of $O_3$ is on no line of $L_1$. It follows that if (and only if) a point is on no line of $L_1$, then it belongs to $O_3$.

Let $\mu$ be an involutory bijection between the points of $O_3$ and the lines of $L_3$ given as follows: if $x \in O_3$ and $\ell \in L_3$, then $x^{\mu} = x^{\alpha}x^{\alpha^2}$, and $\ell^{\mu} = \ell^{\alpha} \cap \ell^{\alpha^2}$. The Figueroa plane $\mathcal{F}$ is obtained by the introduction of a new incidence between the set of points $P$ and the set of lines $L$ of $PG(2, q^3)$, so that (viewing a line as a point set) the $q^3 - q^2 - q - 2$ points of $O_3$ on the line $\ell = x^{\alpha}x^{\alpha^2} \in L_3$ distinct from $x^{\alpha}$ and $x^{\alpha^2}$ are replaced by other points of $O_3$ to form a new line. More precisely, as a set of points, a line of $PG(2, q^3)$ belonging to $L_1$ or to $L_2$ remains unchanged as a line in $\mathcal{F}$. As for a line $\ell = x^{\alpha}x^{\alpha^2} \in L_3$ in $PG(2, q^3)$, where $x \in O_3$, let $y_i, 1 \leq i \leq q^3 - q^2 - q - 2$, be the remaining points of $O_3$ on $\ell$. Consider the pencil of lines of $L_3$ on the point $x \in O_3$. Other than $xx^{\alpha}$ and $xx^{\alpha^2}$, the remaining lines of $L_3$ in the pencil are given by $z_iz_i^{\alpha}, 1 \leq i \leq q^3 - q^2 - q - 2$, where each $z_i$ is a point of $O_3$. Let $\ell_{\mathcal{F}}$ be the set of points obtained from $\ell$ by replacing each $y_i$ with $z_i^{\alpha^2}$. Then, $\ell_{\mathcal{F}}$ is the Figueroa line corresponding to the line $\ell$ of $L_3$, see also [6]. Note that

$$P \in \ell_{\mathcal{F}} \cap O_3 \text{ if and only if } \ell^{\mu} \in P^{\mu}.$$

We observe the following property.

**Lemma 5.1.** Let $\ell_1, \ell_2 \in L_3$ such that $\ell_1 \cap \ell_2 \in O_2$. Then the line $\ell_1^{\mu} \ell_2^{\mu}$ belongs to $L_2$.

**Proof.** Assume by contradiction that $\ell_1^{\mu} \ell_2^{\mu} \in L_3$. Then $(\ell_1^{\mu} \ell_2^{\mu})^{\mu} \in O_3$. Since $\ell_1^{\mu} \in (\ell_1^{\mu} \ell_2^{\mu})$ and $\ell_2^{\mu} \in (\ell_1^{\mu} \ell_2^{\mu})$, we have that $(\ell_1^{\mu} \ell_2^{\mu})^{\mu} \in (\ell_1)_F$ and $(\ell_1^{\mu} \ell_2^{\mu})^{\mu} \in (\ell_2)_F$. On the other hand $\ell_1 \cap \ell_2 = (\ell_1)_F \cap (\ell_2)_F$, since $\ell_1 \cap \ell_2 \in O_2$. It follows that $(\ell_1^{\mu} \ell_2^{\mu})^{\mu} = \ell_1 \cap \ell_2$, a contradiction. \qed
5.2 Inherited polarities of $\mathcal{F}$

Let $\rho$ be a polarity of $\PG(2, q^3)$ such that $\rho$ and $\alpha$ commute, i.e., $\rho \alpha = \alpha \rho$. Let $\mathcal{X}$ denote the set of $\rho$-absolute points.

**Lemma 5.2.** The following properties hold true:

1) $\mathcal{X}$ is preserved by $\alpha$,

2) the point $P \in \mathcal{O}_i$ if and only if $P^\rho \in \mathcal{L}_i$,

3) the line $\ell \in \mathcal{L}_i$ if and only if $\ell^\rho \in \mathcal{O}_i$,

4) if $P \in \mathcal{O}_3$, then $P^\mu \rho = P^{\rho \mu}$,

5) if $\ell \in \mathcal{L}_3$, then $\ell^\mu \rho = \ell^{\rho \mu}$.

**Proof.** Properties 1), 2) and 3) follow directly from the fact that the collineation $\alpha$ and the polarity $\rho$ commute. To prove 4), let $P$ be a point of $\mathcal{O}_3$, then

$$P^\mu \rho = (P^\rho P^{\alpha^2})^\rho = P^\rho \cap P^{\alpha^2} = P^\rho \cap P^{\rho \alpha^2} = P^{\rho \mu}.$$

Property 5) follows similarly.

Consider the following map $\rho_F$: for points and lines of $\mathcal{O}_1$ or $\mathcal{O}_2$, $\rho_F = \rho$. For a point $x \in \mathcal{O}_3$, $x^{\rho F} = (x^\rho)_F$, where $(x^\rho)_F$ is the line of $\mathcal{F}$ corresponding to the line $x^\rho \in \mathcal{L}_3$ as described in the previous subsection. For a line $\ell \in \mathcal{L}_3$, let $(\ell^\rho)_F = \ell^\rho$. Since $\rho$ commutes with $\mu$, $\rho_F$ is indeed a polarity of $\mathcal{F}$. Furthermore, if $x$ is a point of $\mathcal{O}_3$, then $x$ is $\rho_F$-absolute if and only if $x^{\rho \mu} \in \mathcal{X}$. Hence, if we denote by $\mathcal{X}_F$ the $\rho_F$-absolute points, we have that

$$\mathcal{X}_F = (\mathcal{X} \cap \mathcal{O}_1) \cup (\mathcal{X} \cap \mathcal{O}_2) \cup \{x^{\rho \mu} \mid x \in \mathcal{X} \cap \mathcal{O}_3\}.$$

Since $\mu \rho$ is a bijection, the number of points of $\mathcal{O}_3$ which are $\rho_F$-absolute equals the number of points of $\mathcal{O}_3$ which are $\rho$-absolute. Thus, the number of absolute points of $\rho_F$ is the same as that of $\rho$.

We end this section by considering the self-polar triangles with respect to inherited polarities of $\mathcal{F}$.

**Lemma 5.3.** $T$ is a self-polar triangle with respect to $\rho$, containing at least two points in $\mathcal{O}_1 \cup \mathcal{O}_2$ if and only if $T$ is a self-polar triangle with respect to $\rho_F$, containing at least two points in $\mathcal{O}_1 \cup \mathcal{O}_2$.

**Proof.** Let $T = \{P_1, P_2, P_3\}$, where $P_1, P_2 \in T \cap (\mathcal{O}_1 \cup \mathcal{O}_2)$. Then $P_1^\rho = P_1^{\rho F} = P_2 P_3$ and $P_2^\rho = P_2^{\rho F} = P_1 P_3$. On the other hand, $P_3^\rho = P_1 P_2$ if and only if $P_3 = P_1^\rho \cap P_2^\rho = P_1^{\rho F} \cap P_2^{\rho F}$ if and only if $P_3^{\rho F} = P_1 P_2$, as required.
Remark 5.4. Note that if a triangle $T$ has at least one of its points in $O_1$, then $T$ is contained in $O_1 \cup O_2$, while if two of its points are in $O_1$, then its third point will belong to $O_1$ as well.

Lemma 5.5. There is a bijection between the self-polar triangles of $\text{PG}(2, q^3)$ with respect to $\rho$ and the self-polar triangles of $\mathcal{F}$ with respect to $\rho_{\mathcal{F}}$.

Proof. Taking into account Lemma 5.3 and Remark 5.4, we can consider the self-polar triangles containing no point of $O_1$ and at most one point of $O_2$. Let $T = \{P_1, P_2, P_3\}$ be a self-polar triangle of $\text{PG}(2, q^3)$ with respect to $\rho$. Let $P_1^\rho = \ell_i$, that is, $\ell_1 = P_2 P_3, \ell_2 = P_1 P_3, \ell_3 = P_1 P_2$. Assume first that $T \subseteq O_3$. We show that $T$ is a self-polar with respect to $\rho$ if and only if $T_{\mathcal{F}} = \{\ell_i^\rho, \ell_i^\mu, \ell_i^\mu\}$ is a self polar triangle with respect to $\rho_{\mathcal{F}}$. Indeed,

$$\ell_1^\mu = \ell_2^\mu \cap \ell_3^\mu \iff \ell_1^\mu, \ell_2^\mu, \ell_3^\mu \in (\ell_1^\mu)_{\mathcal{F}}$$

and similarly for any permutation of the indices.

On the other hand, if $P_1 \in O_2$ and $P_2, P_3 \in O_3$, let $P = \ell_2^\mu \cap \ell_3^\mu = (\ell_2^\mu \ell_3^\mu)^\rho = P_2^\rho \cap P_3^\rho$. Then, taking into account Lemma 5.1 and Lemma 5.2, we have that $P \in O_2$. Moreover, $T$ is a self-polar triangle with respect to $\rho$ if and only $T_{\mathcal{F}} = \{\ell_2^\rho, \ell_3^\rho, P\}$ is a self polar triangle with respect to $\rho_{\mathcal{F}}$. Indeed, a similar argument as used above gives $P_2 = P_1^\rho \cap P_3^\rho$ if and only if $\ell_2^\mu = \ell_3^\mu \cap \ell_2^\rho \cap \ell_3^\rho$ and $P_3 = P_1^\rho \cap P_2^\rho$ if and only if $\ell_3^\mu = \ell_2^\mu \cap \ell_2^\rho \cap \ell_3^\rho$. Moreover, since $P \in \ell_i^\mu \cap O_2 \subset (\ell_i^\mu \cap (\ell_i^\mu))_{\mathcal{F}}, i = 2, 3$, it follows that $P \in (\ell_i^\mu)_{\mathcal{F}} = \ell_i^\mu_{\mathcal{F}}$.

Theorem 5.6. Let $\mathcal{Z}$ be a triangle-free set consisting of non-absolute points with respect to $\rho$, then

$$\mathcal{Z}_{\mathcal{F}} = (\mathcal{Z} \cap O_1) \cup \{x^\mu \mid x \in \mathcal{Z} \cap O_3\}$$

is a triangle-free set consisting of non-absolute points with respect to $\rho_{\mathcal{F}}$.

Proof. Assume by contradiction that there exists a self-polar triangle with respect to $\rho_{\mathcal{F}}$, say $T_{\mathcal{F}}$, contained in $\mathcal{Z}_{\mathcal{F}}$, then necessarily $T_{\mathcal{F}}$ is contained in $O_3$. If $T_{\mathcal{F}} = \{P_1^\mu, P_2^\mu, P_3^\mu\}$, then it follows that $T = \{P_1, P_2, P_3\}$ is a self-polar triangle with respect to $\rho$ contained in $\mathcal{Z}$, a contradiction. □

Finally, taking into account Theorem 1.3 and Theorem 5.6, we have the following.
Corollary 5.7. Let $\rho$ be a unitary polarity of $\operatorname{PG}(2, q^6)$, $q$ even, and let $\alpha$ be an order 3 collineation of $\operatorname{PG}(2, q^6)$ fixing a subplane $D \cong \operatorname{PG}(2, q^2)$ pointwise, such that $\rho$ and $\alpha$ commute. Then, there exists a set of non-absolute vertices of $G(F, \rho F)$ inducing a triangle-free subgraph of size $\frac{q^{12} - q^3(q^4 - 1)(q^2 + 1)}{2}$.

Proof. From Theorem 1.3, there exists a set $Z$ of non-absolute vertices of $G(\operatorname{PG}(2, q^6), \rho)$ inducing a triangle-free subgraph of size $\frac{q^{12}}{2}$. From Theorem 5.6, the set $Z_F$ is a triangle-free set consisting of non-absolute points with respect to $\rho_F$, where $|Z_F| = |Z \setminus O_2|$. We will count the number of points of $Z \cap O_2$, which we have to remove, by inspecting the $q^4 + q^2 + 1$ lines of $L_1$. Recall that these lines only contain points of $O_1$ and $O_2$. As every point of $O_2$ lies on exactly one line of $L_1$, we will find every point of $Z \cap O_2$ once.

By construction, $Z$ is the union of $q^3/2$ Hermitian curves of $\operatorname{PG}(2, q^6)$ pairwise meeting in a point $U_2$ of $D$ and having the same tangent line $\ell$ at $U_2$, with their common point $U_2$ deleted. Among these Hermitian curves there are $q/2$ meeting $D$ in a Hermitian curve of $\operatorname{PG}(2, q^2)$. One can see this by using the vector space representation of $F_{q^6}$ over $F_2$: the $q^3/2$ Hermitian curves are parametrized by elements of $F_{q^3}$, which form a hyperplane. As $F_q$ is a subspace of $F_{q^3}$ not properly contained in the hyperplane, as it does not contain the element $1 \in F_q$, this means that it intersects $F_q$ in $q/2$ points, which parametrize the $q/2$ Hermitian curves in $D$. It follows that for a line $r \in L_1$, we can compute $|r \cap (Z \cap O_2)| = |r \cap Z| - |r \cap Z \cap O_1|$ using the intersection properties as stated in Corollary 4.8 in both $\operatorname{PG}(2, q^6)$ and $D = \operatorname{PG}(2, q^2)$ respectively. Therefore, if $r$ is a line of $O_1$, then

$$|r \cap Z| - |r \cap Z \cap O_1| = \begin{cases} \frac{q^6 - q^3}{2} - \frac{q^2 - q}{2} & \text{if } U_2 \notin r \text{ and } r^\rho \notin Z, \\ \frac{q^6 + q^3}{2} - \frac{q^2 + q}{2} & \text{if } U_2 \notin r \text{ and } r^\rho \notin Z, \\ \frac{q^6 - q^2}{2} & \text{if } U_2 \in r \neq \ell, \\ 0 & \text{if } r = \ell. \end{cases}$$

The number of lines corresponding to each case is respectively $q^4/2$, $q^4/2$, $q^2$ and 1. Summing up over all these lines, we obtain the number $|Z \cap O_2|$ which we had to subtract from $q^{12}/2$ to obtain the result. \hfill \Box

6 Conclusion and open problems

In [20] the following question was posed.

Question 6.1. Given a finite projective plane $\pi$ of order $q$ and a polarity $\perp$, is it possible to find a triangle-free subgraph of the polarity graph of size $\frac{1}{2}q^2 + o(q^2)$?
When \( \pi = \text{PG}(2, q) \), this question has been almost completely resolved. Depending on the parity of \( q \) and the type of \( \perp \), there are four possibilities, shown in the table below.

| prime power \( q \) | type       | answer  |
|----------------------|------------|---------|
| even                 | pseudo     | yes [22]|
| odd                  | orthogonal | yes [24]|
| even square          | unitary    | yes     |
| odd square           | unitary    | ?       |

Starting from Question 6.1, we can state three open problems, ranked in what we believe to be increasing difficulty.

**Open problem 1.** Show that there exists a triangle-free induced subgraph of \( \text{DUP}(q^2) \), \( q \) odd, of size \( \frac{1}{2} q^4 + o(q^4) \).

**Open problem 2** (Conjecture 1 in [23]). Prove or disprove that in case 2, i.e. \( \pi = \text{PG}(2, q) \) and \( \perp \) is an orthogonal polarity, Parsons’ examples are the largest. If true, is it possible to show that they are the unique triangle-free induced subgraphs of this size?

As shown by Loucks and Timmons, this can only be true when \( q \) is large enough. Using a computer search, they found larger examples for \( q = 5, 7, 9, 13 \).

**Open problem 3.** What if \( \pi \) is not the Desarguesian projective plane \( \text{PG}(2, q) \)? Can we still answer Question 6.1 in the affirmative?

In the case when \( \pi \) is the Figueroa plane and the unitary polarity is inherited, we showed that the answer is indeed yes. In fact, one can do this for any inherited polarity by Theorem 5.6, but for this article, we restrict ourselves to the unitary case.

Lastly, remark that there exist projective planes of order \( q \) and polarities where the size of the set of absolute points does not belong to \( \{ q + 1, q\sqrt{q} + 1 \} \), see [8].

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