ASYMPTOTIC STABILITY OF A BOUNDARY LAYER TO THE EULER–POISSON EQUATIONS FOR A MULTICOMPONENT PLASMA

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Abstract. The main concern of this paper is to analyze a boundary layer called a sheath that occurs on the surface of materials when in contact with a multicomponent plasma. For the formation of a sheath, the generalized Bohm criterion demands that ions enter the sheath region with a high velocity. The motion of a multicomponent plasma is governed by the Euler–Poisson equations, and a sheath is understood as a monotone stationary solution to those equations. In this paper, we prove the unique existence of the monotone stationary solution by assuming the generalized Bohm criterion. Moreover, it is shown that the stationary solution is time asymptotically stable provided that an initial perturbation is sufficiently small in weighted Sobolev space. We also obtain the convergence rate, which is subject to the decay rate of the initial perturbation, of the time global solution toward the stationary solution.

1. Introduction. The theory of plasma systems, comprising electrons and several ion species, is important in various fields of plasma technology such as gas discharge, plasma–surface interactions, and nuclear fusion. Consequently, interest in the investigation of multicomponent plasmas has been increasing. In particular, we are concerned with the boundary layer problem for multicomponent plasmas. This problem occurs in plasma devices when the plasma is in contact with any surface. Owing to the difference in the mobility of electrons and ions, the surface has a negative potential with respect to the plasma. The non-neutral potential region between the plasma and the surface is called a sheath (for details, see [4, 8, 11]).

A plasma system for $k$ components of ions can be described by the Euler equations for the ion density $n_i(t,x)$ and ion velocity $u_i(t,x)$ of the $i$-th component:

\begin{equation}
(n_i)_t + (n_i u_i)_x = 0, \tag{1a}
\end{equation}

\begin{equation}
(m_i n_i u_i)_t + (m_i n_i u_i^2 + p_i n_i)_x = e_i n_i v_x, \quad x > 0, \quad t > 0, \quad i = 1, \ldots, k, \tag{1b}
\end{equation}

coupled with the Poisson equation for the electrostatic potential $-v(t,x)$:

\begin{equation}
\varepsilon v_{xx} = \sum_{i=1}^{n} e_i n_i + e n_e, \quad x > 0, \quad t > 0. \tag{1c}
\end{equation}
The constants \( m_i > 0 \) and \( e_i \neq 0 \) denote the mass and charge of the \( i \)-th ion, respectively. In addition, \( \varepsilon > 0 \) is the permittivity. We assume that the pressure \( p_i \) can be described by an adiabatic law, that is,

\[
 p_i(n_i) = \begin{cases} 
 \kappa T_i n_i & \text{if } \gamma_i = 1, \\
 C_i \exp(S_i/c_{vi}) n_i^{\gamma_i} & \text{if } \gamma_i > 1,
\end{cases} \tag{2}
\]

where \( \kappa \) is the Boltzmann constant and \( C_i \) is a positive constant. Moreover, \( T_i, S_i, \gamma_i, \) and \( c_{vi} \) are the temperature, entropy, adiabatic constant, and specific heat at constant volume of the \( i \)-th ion, respectively. The electron density \( n_e \) is assumed to obey the Boltzmann relation

\[
 n_e = n_{e+} \exp \left( \frac{e}{\kappa T_e} \right). \tag{3}
\]

Here, the constants \( n_{e+} > 0, e < 0, \) and \( T_e > 0 \) are the reference density value, charge, and temperature of the electron, respectively.

For the formation of a sheath, Langmuir [7] concluded that the positive ions must enter the sheath region with a high velocity. For the simple case wherein the plasma contains electrons and only one component of mono-valence ions, Bohm [3] derived the original Bohm criterion for the velocity \( u_1 \) as

\[
 \kappa T_e + \kappa \gamma_1 T_1 < m_1 u_1^2, \quad u_1 < 0. \tag{4}
\]

Riemann [12] extended this criterion to multicomponent plasmas by studying the stationary problem of (1) over a half space. The generalized Bohm criterion is

\[
 \frac{e^2 n_e}{\kappa T_e} - \sum_{i=1}^{k} \frac{e_i^2 n_i}{m_i u_i^2 - \kappa \gamma_i T_i} > 0, \tag{5a}
\]

\[
 \kappa \gamma_i T_i - m_i u_i^2 < 0, \quad u_i < 0, \quad i = 1, \ldots, k. \tag{5b}
\]

Note that (5a) does not uniquely define a criterion for the velocity of each ion component. This complexity arises from the interaction effects among multiple ions, which is implicitly provided through the electrostatic potential. We call (5b) the supersonic outflow condition.

Let us mention mathematical results regarding the formation of a sheath and the original Bohm criterion (4). Ha and Slemrod [5] formulated the problem of sheath formation as a free boundary problem of (1) with \( k = 1 \) and \( T_1 = 0 \) assuming (4) on the boundary. They discussed the time global solvability but not the asymptotic behavior of the solution. It is reasonable to expect that the asymptotic state is given by a stationary solution, since a sheath is observed as a stationary boundary layer. The papers [1, 2, 9, 13] studied this expectation for initial–boundary value problems of (1) with \( k = 1 \). Ambroso, Mélats, and Raviart [2] showed the unique existence of a monotone stationary solution over a one-dimensional bounded domain under assumption (4). The paper [1] numerically showed that the solution to (1) approaches the stationary solution constructed in [2] as the time variable becomes large. Suzuki [13] derived a necessary and sufficient condition, including (4), for the unique existence of a monotone stationary solution over a half space. Moreover, the stability of the stationary solution was shown under a condition slightly stronger than (4) in [13]. Recently, Nishibata, Ohnawa, and Suzuki [9] obtained the stability theorem under (4). These results rigorously clarify that a sheath is regarded as a stationary solution. Furthermore, they ensure the mathematical validity of the original Bohm criterion (4).
The main purpose of the present paper is the rigorous justification of the generalized Bohm criterion (5), because the generalized criterion is more important than the original criterion in plasma technology. More precisely, we extend the existence and stability theorems in [9, 13] to (1) for multicomponent plasmas.

Outline of the paper. The remainder of this paper is organized as follows. In Section 2, we formulate an initial–boundary value problem of (1) over a half space and mention our main results on the unique existence and asymptotic stability of the stationary solution. The generalized Bohm criterion provides a sufficient condition for existence. Moreover, we obtain the stability theorem by assuming the additional condition \( u_1 + \ldots + u_k \rightarrow \). In Section 3, we begin detailed discussions on the proof of the unique existence of the stationary solution. Here, the stationary problem is reduced to a boundary value problem for a scalar equation for the potential. The reduced problem can be solved by a basic ordinary differential equation theory. Section 4 is devoted to the proof of the stability theorem. First, we discuss briefly the unique existence of the time local solution to the initial–boundary value problem. Second, we derive an a priori estimate with suitable weight functions to construct the time global solution. Furthermore, the a priori estimate gives the convergence rate, which is subject to the decay rate of the initial perturbation, of the global solution toward the stationary solution.

Notation. For a non-negative integer \( l \), \( H^l(\Omega) \) denotes the \( l \)-th order Sobolev space in the \( L^2 \) sense, equipped with the norm \( \| \cdot \|_l \). We note \( H^0 = L^2 \) and \( \| \cdot \| := \| \cdot \|_0 \). \( C^k([0,T];H^j) \) denotes the space of \( k \)-times continuously differentiable functions on the interval \([0,T]\) with values in \( H^j \). Moreover, the function space \( X_i^j \) is defined as

\[
X_i^j([0,T]) := \bigcap_{k=0}^i C^k([0,T];H^{j+i-k}(\Omega)), \quad X_i([0,T]) := X_i^0([0,T])
\]

for \( i, j = 0, 1, 2, \ldots \). For a non-negative integer \( k \geq 0 \), \( B^k(\Omega) \) denotes the space of functions whose derivatives up to \( k \)-th order are continuous and bounded over \( \Omega \). In addition, \( B^\infty(\Omega) := \cap_{k=0}^\infty B^k(\Omega) \). The inner product for complex vectors \( v = \sum_{i=1}^n v_i \) and \( w = \sum_{i=1}^n w_i \) is defined as

\[
\langle v, w \rangle := \sum_{i=1}^n v_i \bar{w}_i.
\]

Furthermore, \( C \) and \( c \) denote generic positive constants.

2. Problem statement and main results.

2.1. Problem statement. We begin the formulation of our problem by introducing new variables \( (\phi, \varepsilon_0, d, \rho_i, q_i, K_i) \) as

\[
\phi := -\frac{e}{\kappa T_e} v, \quad \varepsilon_0 := \frac{\kappa^2 T_e^2}{e^2} \varepsilon > 0, \quad d := \kappa T_e n_{+} > 0,
\]

\[
\rho_i := m_i n_i, \quad q_i := -\frac{\kappa T_e}{e m_i} \varepsilon_i \neq 0, \quad K_i := \left\{ \begin{array}{ll}
\frac{\kappa T_i}{m_i} > 0 & \text{if } \gamma_i = 1, \\
\gamma_i C_i \exp \frac{S_i}{c_v n} > 0 & \text{if } \gamma_i > 1
\end{array} \right.
\]

for \( i = 1, \ldots, k \) to reduce the number of physical constants. By substituting the above values in (1) and applying assumptions (2) and (3), we have
\[(\rho_1)_t + (\rho_1 u_1)_x = 0, \quad (6a)\]
\[(u_i)_t + \frac{1}{2}(u_i^2)_x + K_i \rho_i^{\gamma_i-2} (\rho_i)_x = q_i \phi_x, \quad i = 1, \ldots, k, \quad (6b)\]
\[\varepsilon \phi_{xx} = \sum_{i=1}^{k} q_i \rho_i - d \exp(-\phi), \quad x > 0, \quad t > 0. \quad (6c)\]

The Bohm criterion (5) is rewritten in a form corresponding to (6) as
\[d \exp(-\phi) - \sum_{i=1}^{k} \frac{q_i^2 \rho_i}{u_i^2 - K_i \rho_i^{\gamma_i-1}} > 0, \quad (7a)\]
\[K_i \rho_i^{\gamma_i-1} - u_i^2 < 0, \quad u_i < 0, \quad i = 1, \ldots, k, \quad (7b)\]

where we have used the ideal gas law \(p_i(n_i) = \kappa n_i T_i\).

We discuss the existence and asymptotic stability of the stationary solution to an initial–boundary value problem of (6) over the one-dimensional half space \(\mathbb{R}_+ := \{x > 0\}\). The initial and boundary data are prescribed as
\[(\rho_i, u_i)(0, x) = (\rho_{i0}, u_{i0})(x), \quad \lim_{x \to \infty} (\rho_{i0}, u_{i0})(x) = (\rho_{i+}, u_{i+}), \quad i = 1, \ldots, k, \quad (8)\]
\[\phi(t, 0) = \phi_b, \quad (9)\]

where \(\rho_{i+}, u_{i+}, \text{ and } \phi_b\) are constants. We take the reference point of the potential \(\phi\) at \(x = \infty\), that is,
\[\lim_{x \to \infty} \phi(t, x) = 0. \]

The quasi-neutrality condition
\[\sum_{i=1}^{k} q_i \rho_{i+} - d = 0 \quad (10)\]
is necessary for the classical solvability of the Poisson equation (6c).

We establish the unique existence of the solution \((\rho_1, u_1, \ldots, \rho_k, u_k, \phi)\) to (6), (8), and (9) in the region where the conditions
\[\inf_{x \in \mathbb{R}_+} \rho_i > 0, \quad i = 1, \ldots, k, \quad (11)\]
\[\sup_{x \in \mathbb{R}_+} (K_i \rho_i^{\gamma_i-1} - u_i^2) < 0, \quad \sup_{x \in \mathbb{R}_+} u_i < 0, \quad i = 1, \ldots, k, \quad (12)\]
\[\inf_{x \in \mathbb{R}_+} \sum_{i=1}^{k} q_i \rho_i > 0 \quad (13)\]
hold under the same initial assumptions
\[\inf_{x \in \mathbb{R}_+} \rho_{i0} > 0, \quad \sup_{x \in \mathbb{R}_+} (K_i \rho_i^{\gamma_i-1} - u_{i0}^2) < 0, \quad \sup_{x \in \mathbb{R}_+} u_{i0} < 0, \quad \inf_{x \in \mathbb{R}_+} \sum_{i=1}^{k} q_i \rho_{i0} > 0, \quad (14)\]
\[\rho_{i+} > 0, \quad K_i \rho_{i+}^{\gamma_i-1} - u_{i+}^2 < 0, \quad u_{i+} < 0, \quad \sum_{i=1}^{k} q_i \rho_{i+} > 0. \quad (15)\]

Condition (12), the supersonic outflow condition, ensures that all characteristic speed of the hyperbolic system of \(2k\) equations in (6a) and (6b) are negative. Therefore, one boundary condition (9) is necessary and sufficient for the well-posedness of the initial–boundary value problem of (6), (8), and (9). We use (13) to obtain
bounds of the solution \( \phi \) to the Poisson equation (6c). Note that (11) ensures (13) for the case wherein the plasma consists of electrons, positive ions, and no negative ions, that is, \( q_i > 0 \) for any \( i = 1, \ldots, k \).

In studying the generalized Bohm criterion (7), we must pay attention to (7a). In fact, (7b) is the supersonic outflow condition contained in the formulation of our initial–boundary value problem.

2.2. Stationary solution. In this subsection we discuss the unique existence of the stationary solution \((\hat{R}, \hat{\phi}) := (\hat{\rho}_1, \hat{u}_1, \ldots, \hat{\rho}_k, \hat{u}_k, \hat{\phi})\) that is time-independent. This solution satisfies

\[
\frac{1}{2} (\hat{u}_i^2)_x + K_i \hat{\rho}_i^{\gamma_i - 2} (\hat{\phi})_x = q_i \hat{\phi}_x, \quad i = 1, \ldots, k,
\]

(16a)

\[
\varepsilon_0 \hat{\phi}_{xx} = \sum_{i=1}^k q_i \hat{\rho}_i - d \exp (-\hat{\phi})
\]

(16c)

together with conditions (8)–(13), that is,

\[
\lim_{x \to \infty} (\hat{\rho}_1, \hat{u}_1, \ldots, \hat{\rho}_k, \hat{u}_k)(x) = (\rho_{1+}, u_{1+}, \ldots, \rho_{k+}, u_{k+}) =: R_+,
\]

(17a)

\[
\hat{\phi}(0) = \phi_0, \quad \lim_{x \to \infty} \hat{\phi}(x) = 0,
\]

(17b)

\[
\inf_{x \in \mathbb{R}_+} \hat{\rho}_i > 0, \quad \sup_{x \in \mathbb{R}_+} (K_i \hat{\rho}_i^{\gamma_i - 1} - \hat{u}_i^2) < 0, \quad \sup_{x \in \mathbb{R}_+} \hat{u}_i < 0, \quad i = 1, \ldots, k,
\]

(17c)

\[
\inf_{x \in \mathbb{R}_+} \sum_{i=1}^k q_i \hat{\rho}_i > 0.
\]

(17d)

Before mentioning our main result on the existence of the stationary solution, let us define the Sagdeev potential \( V \), which plays crucial roles in our analysis, as follows:

\[
V(\hat{\phi}) := \int_0^{\hat{\phi}} \sum_{i=1}^k q_i f_i^{-1} (q_i \eta) - d \exp (-\eta) \, d\eta.
\]

(18)

Here, we define the function \( f_i \) as

\[
f_i(\hat{\rho}_i) := \begin{cases} K_i \log \hat{\rho}_i + \frac{\hat{\rho}_i^2 + u_{i+}^2}{2 \hat{\rho}_i^2} - K_i \log \rho_{i+} - \frac{u_{i+}^2}{2} & \text{if } \gamma_i = 1, \\ \frac{K_i}{\gamma_i - 1} \hat{\rho}_i^{\gamma_i - 1} + \frac{\hat{\rho}_i^2 + u_{i+}^2}{2 \hat{\rho}_i^2} - K_i \frac{\rho_{i+}^{\gamma_i - 1} - u_{i+}^2}{2} & \text{if } \gamma_i > 1 \end{cases}
\]

(19)

and restrict its domain to

\[
I_i := (0, \rho_{i+} M_i^{2/(\gamma_i + 1)}), \quad M_i := \frac{|u_{i+}|}{\sqrt{K_i \rho_{i+}^{\gamma_i - 1}}},
\]

(20)

where \( M_i \) corresponds to the Mach number at \( x = \infty \) for the \( i \)-th ions. Notice that \( f_i \) is invertible on \( I_i \) and hence, \( V \) is well-defined.

The second-order derivative of the Sagdeev potential \( V \) at \( \hat{\phi} = 0 \) is equal to the left-hand side of (7a). In other words,

\[
V''(0) = d - \sum_{i=1}^k \frac{q_i^2 \hat{\rho}_i}{u_{i+}^2 - K_i \hat{\rho}_i^{\gamma_i - 1}}.
\]
The unique existence result of the stationary solution is as follows.

**Theorem 2.1.** (i) Let $R_+$ defined in (17a) satisfy (15) and

$$V''(0) = d - \sum_{i=1}^{k} \frac{q_i^2 \rho_i}{u_i + K_i \rho_i} > 0. \quad (21)$$

There exists a positive constant $\delta$ such that if $|\phi_b| \leq \delta$, then the stationary problem of (16) and (17) has a unique monotone solution $(\tilde{R}, \tilde{\phi})$ satisfying

$$\tilde{R} \in C^1(\mathbb{R}_+), \quad \tilde{\phi} \in C(\mathbb{R}_+) \cap C^2(\mathbb{R}_+). \quad (22)$$

Moreover, the stationary solution belongs to $B^\infty(\mathbb{R}_+)$ and satisfies

$$|\partial_x^j (\tilde{R} - R_+)| + |\partial_x^j \tilde{\phi}| \leq C|\phi_b| e^{-cx} \quad \text{for} \quad j = 0, 1, 2, \ldots, \quad (23)$$

where $c$ and $C$ are positive constants.

(ii) Let $R_+$ satisfy (15) and $V''(0) = 0$. There exists a positive constant $\delta$ such that if $|\phi_b| \leq \delta$ and $V(\phi_b) \geq 0$, then the stationary problem of (16) and (17) has a unique monotone solution $(\tilde{R}, \tilde{\phi})$ satisfying (22).

(iii) Let $R_+$ satisfy (15) and $V''(0) < 0$. If $\phi_b \neq 0$, the stationary problem of (16) and (17) does not admit any solutions $(\tilde{R}, \tilde{\phi})$ satisfying (22). If $\phi_b = 0$, then a constant state $(\tilde{R}, \tilde{\phi}) = (R_+, 0)$ is a unique solution.

In the above theorem, $V(\phi_b) \geq 0$ in assertion (ii) is a necessary condition for existence. We derive this condition in Section 3. Suzuki [13] constructed non-monotone stationary solutions for the case $k = 1$. Thus, monotonicity is required for uniqueness.

2.3. **Asymptotic stability.** This subsection is devoted to the discussion of the asymptotic stability of the stationary solution in Theorem 2.1 (i). To this end, we use perturbations from the stationary solution, such that

$$\psi_i := \rho_i - \tilde{\rho}_i, \quad \eta_i := u_i - \tilde{u}_i, \quad i = 1, \ldots, k,$$

$$\sigma := \phi - \tilde{\phi}.$$

For notational convenience, we use vectors with $2k$ components:

$$\Psi := \psi_1, \eta_1, \ldots, \psi_k, \eta_k, \quad q := q_1, 0, \ldots, q_k, 0.$$

Subtracting (6) from (16), together with the mean-value theorem, yields

$$A^0[\Psi, \tilde{R}] \psi_i + A^1[\Psi, \tilde{R}] \eta_i = \sigma x b[\Psi, \tilde{R}] + h[\Psi, \tilde{R}, \tilde{R}_x], \quad (24a)$$

$$\frac{\varepsilon_0}{d} \sigma_{xx} - \sigma = \frac{1}{d} \langle \psi, q \rangle + g[\sigma, \tilde{\phi}], \quad (24b)$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $\mathbb{C}^{2k}$. Here, the $2k \times 2k$ symmetric matrices $A^0 = (a^0_{lm})$ and $A^1 = (a^1_{lm})$ are defined as

$$a^0_{lm}[\Psi, \tilde{R}] := \begin{cases} K_i(\psi_i + \tilde{\rho}_i)^{\gamma_i - 2} & \text{if } (l, m) = (2i - 1, 2i - 1), \\ K_i(\psi_i + \tilde{\rho}_i) & \text{if } (l, m) = (2i, 2i), \\ 0 & \text{otherwise}, \end{cases}$$

$$a^1_{lm}[\Psi, \tilde{R}] := \begin{cases} K_i(\psi_i + \tilde{\rho}_i)^{\gamma_i - 2}(\eta_i + \tilde{u}_i) & \text{if } (l, m) = (2i - 1, 2i - 1), \\ K_i(\psi_i + \tilde{\rho}_i)^{\gamma_i - 1} & \text{if } (l, m) = (2i - 1, 2i), (2i, 2i - 1), \\ (\psi_i + \tilde{\rho}_i)(\eta_i + \tilde{u}_i) & \text{if } (l, m) = (2i, 2i), \\ 0 & \text{otherwise}, \end{cases}$$

otherwise,
where \( i = 1, \ldots, k \). Note that \( A^0 \) and \( -A^1 \) are positive definite under conditions (11) and (12). Moreover, \( b = \iota(b_1, \ldots, b_{2k}) \) and \( h = \iota(h_1, \ldots, h_{2k}) \) denote vectors with \( 2k \) components:

\[
\begin{align*}
  b_l[\Psi, \tilde{R}] &:= \begin{cases} 
    0 & \text{if } l = 2i - 1, \\
    q_l(\psi_i + \tilde{\rho}_i) & \text{if } l = 2i,
  \end{cases} \\
  h_l[\Psi, \tilde{R}, \tilde{R}_x] &:= \begin{cases} 
    -K_i(\psi_i + \tilde{\rho}_i)^{\gamma_i-2}(\psi_i\tilde{u}_{ix} + \eta_i\tilde{\rho}_{ix}) & \text{if } l = 2i - 1, \\
    (\psi_i + \tilde{\rho}_i)\left\{\eta_i\tilde{u}_{ix} + K_i((\psi_i + \tilde{\rho}_i)^{\gamma_i-2} - \tilde{\rho}_i^{\gamma_i-2})\tilde{\rho}_{ix}\right\} & \text{if } l = 2i,
  \end{cases}
\end{align*}
\]

where \( i = 1, \ldots, k \). The scalar function \( g \) is

\[
g[\Psi, \tilde{R}] := \left\{ \int_0^1 \exp \left(-\theta \sigma - \tilde{\phi}\right) d\theta - 1 \right\} \sigma.
\]

The initial and boundary conditions to (24) are derived from (8) and (9):

\[
\begin{align*}
  \Psi(0, x) &= \Psi_0(x) := \iota(\rho_{10} - \tilde{\rho}_1, u_{10} - \tilde{u}_1, \ldots, \rho_{k0} - \tilde{\rho}_k, u_{k0} - \tilde{u}_k)(x), \\
  \sigma(t, 0) &= 0.
\end{align*}
\]

Let us mention difficulties of our stability analysis. Linearizing system (24) around the asymptotic state at \( x = \infty \) by substituting \( \Psi = 0, \sigma = 0, \tilde{R} = R_+, \) and \( \tilde{R}_x = 0 \) into \( A^0, A^1, b, h, \) and \( g \) in (24), we have

\[
\begin{align*}
  A^0[0, R_+]\Psi_t + A^1[0, R_+]\Psi_x &= \sigma_x b[0, R_+], \\
  \frac{\varepsilon_0}{d} \sigma_{xx} - \sigma &= \frac{1}{d}(\Psi, q).
\end{align*}
\]

\( ^1 \)Notice that the real parts of all spectra of this system are zero under the assumption

\[
u_+ = u_{1+} = \ldots = u_{k+}.
\]

This creates difficulty for our problem, because a standard energy method is not applicable. The author \([9, 13]\) overcomes this issue for the case \( k = 1 \) by using the weighted energy method with a weight function

\[(1 + \beta x)\lambda \quad \text{or} \quad e^{\beta x}.
\]

Furthermore, we see that this method is applicable in our analysis of the multicomponent plasma as follows. Multiply (26) by \( e^{\beta x/2} \) and then introduce new unknown functions \( (\Psi, \Sigma) := (e^{\beta x/2}\Psi, e^{\beta x/2}\sigma) \) to obtain

\[
\begin{align*}
  A^0[0, R_+]\Psi_t + A^1[0, R_+]\Psi_x &= \frac{\beta}{2} A^1[0, R_+]\Psi = \Sigma_x b[0, R_+] - \frac{\beta}{2} \Sigma b[0, R_+], \\
  \frac{\varepsilon_0}{d} \left( \Sigma_{xx} - \beta \Sigma_x + \frac{\beta^2}{4} \Sigma \right) - \Sigma &= \frac{1}{d}(\Psi, q).
\end{align*}
\]

The spectral analysis for system (28) yields Proposition 2.2 which implies an advantage for the weighted energy method. Note that we cannot have explicit formulas for the spectra of (28) for the multicomponent case, even though there are formulas for the one-component case (see [9]). In the Appendix, Proposition 2.2 will be proved using a combination of several general theories, because a direct calculation using the formulas is not available.

\( ^1 \)For the case \( k = 1 \), all spectra are written explicitly in \([9, 13]\). For the other cases, we can check by applying the Fourier and Laplace transformations to (26) and taking the inner product of the resultant eigenvalue problem with \( \tilde{\Psi} \), even though the spectra are not known explicitly.
Proposition 2.2. Let the asymptotic state $R_+$ defined in (17a) satisfy (15) and (27). Then the following three assertions are equivalent:

(i) The real part of all spectra of (28) in the whole space $\mathbb{R}$ is negative for sufficiently small $\beta > 0$.

(ii) The generalized Bohm criterion (21) holds.

(iii) The $2k \times 2k$ symmetric matrix

$$S_\xi := -A^1[0, R_+] - \frac{u_+}{d} q' q - \frac{1}{d} \left( \frac{\varepsilon_0}{d} \xi^2 + 1 \right)^{-1} (q'b[0, R_+] + b[0, R_+]') q$$

is uniformly positive definite for an arbitrary $\xi \in \mathbb{R}$.

Even if we use the weighted energy method, another difficulty in the analysis of the multicomponent plasma still exists. In (24), the interaction effects among multiple ions are provided implicitly through the electrostatic potential. We need delicate estimates of these effects to obtain the stability theorem under the generalized Bohm criterion (21). On the other hand, Proposition 2.2 means that the interaction effects can be handled well in the Fourier space, and we explicitly write these effects in the Fourier space using an algebraic equation. To apply Fourier analysis to our initial–boundary value problem over $\mathbb{R}$, a suitable extension from the half space $\mathbb{R}_+$ to the whole space $\mathbb{R}$ is required. In Section 4, we prove the stability theorem by combining the weighted energy method and Fourier analysis together with a suitable extension. This new technical method is worth noting in the stability analysis of the multicomponent plasma.

We are now in a position to state our stability theorem.

Theorem 2.3. Let the asymptotic state $R_+$ satisfy conditions (15), (21), and (27).

(i) Suppose that $e^{\alpha x/2} \Psi_0 \in H^2(\mathbb{R}_+)$ for some positive constant $\alpha$. There exist positive constants $\beta(\leq \alpha)$ and $\delta$ such that if $|\phi_0| + \|e^{\beta x/2} \Psi_0\|_2 \leq \delta$, then the initial–boundary value problem of (24) and (25) has a unique solution $(\Psi, \sigma)$ satisfying $(e^{\beta x/2} \Psi, e^{\beta x/2} \sigma) \in \mathbb{X}_2([0, \infty)) \times \mathbb{X}_2^2([0, \infty))$. Moreover, the following decay estimate holds:

$$\|e^{\beta x/2} \Psi(t)\|_2^2 + \|e^{\beta x/2} \sigma(t)\|_2^2 \leq C \|e^{\beta x/2} \Psi_0\|_2^2 e^{-\gamma t},$$

(29)

where positive constants $C$ and $\gamma$ are independent of the time $t$.

(ii) Let $\lambda$ and $\nu$ satisfy $\lambda \geq 2$ and $\nu \in (0, \lambda]$. Suppose that $(1 + \alpha x)^{\lambda/2} \Psi_0 \in H^2(\mathbb{R}_+)$ for some positive constant $\alpha$. There exist positive constants $\beta(\leq \alpha)$ and $\delta$ such that if $|\phi_0| + \|(1 + \beta x)^{\lambda/2} \Psi_0\|_2 \leq \delta$, then the initial–boundary value problem of (24) and (25) has a unique solution $(\Psi, \sigma)$ satisfying $(1 + \beta x)^{\lambda/2} \Psi, (1 + \beta x)^{\lambda/2} \sigma) \in \mathbb{X}_2([0, \infty)) \times \mathbb{X}_2^2([0, \infty))$. Moreover, the following decay estimate holds:

$$\|(1 + \beta x)^{\lambda/2} \Psi(t)\|_2^2 + \|(1 + \beta x)^{\lambda/2} \sigma(t)\|_2^2 \leq C \|(1 + \beta x)^{\lambda/2} \Psi_0\|_2^2 (1 + \beta t)^{-\lambda + \zeta}$$

(30)

for an arbitrary $\zeta \in [\nu, \lambda]$, where the positive constant $C$ is independent of the time $t$.

3. Unique existence of the stationary solution. In this section we show the unique existence of a monotone stationary solution. We start from reducing the stationary system (16) to a scalar equation for $\bar{\phi}$.

At the moment, we assume the existence of a monotone solution satisfying (22) to the stationary problem of (16) and (17) and drive a scalar equation which $\bar{\phi}$
Lemma 3.1. \( \phi \) is a desired monotone stationary solution to (16) and (17). The uniqueness of a solution.

where we have used (17a) and \( f \) is defined in (19). Here, \( f_i \) is strictly decreasing over the interval \( I_i \) defined in (20) and strictly increasing over \( (0, \infty) \). We define an inverse function \( f^{-1} \) by restricting the domain of \( f_i \) to \( I_i \) since the asymptotic state \( \rho_i \) belongs to \( I_i \), owing to (17a) and \( \lim_{x \to \infty} \phi(x) = 0 \). Then it holds that

Substitute these equalities into (16c), multiply the resultant equation by \( \phi(x) \), integrate the result over \( (0, \infty) \), and then utilize condition (17a) and \( \lim_{x \to \infty} \phi(x) = 0 \). Then we have a scalar equation for \( \phi \):

where \( V \) is defined in (18). Note that this equation requires the necessary condition \( V(\phi) \leq 0 \).

On the other hand, if the boundary value problem of (33) and (17b) has a solution, then the boundary value problem of (33) and (17b) has a unique monotone solution \( \phi \in C^2(\mathbb{R}_+) \) satisfying

Then it is immediately seen that

\begin{align*}
(\hat{\rho}_1, \hat{u}_1, \ldots, \hat{\rho}_k, \hat{u}_k) := \left( f_1^{-1}(\phi), \frac{\rho_1 + u_1}{f_1^{-1}(\phi)}, \ldots, f_k^{-1}(\phi), \frac{\rho_k + u_k}{f_k^{-1}(\phi)} \right)
\end{align*}

is a desired monotone stationary solution to (16) and (17). The uniqueness of a monotone stationary solution to (16) and (17) also follows from the uniqueness of the solution to (33) and (17b). Hence, it is sufficient to show the unique solvability of the boundary value problem of (33) and (17b) in order to prove Theorem 2.1.

In the remainder of this section, we show the next lemma.

**Lemma 3.1.** (i) Let \( V''(0) > 0 \). There exists a positive constant \( \delta \) such that if \( |\phi_b| \leq \delta \), then the boundary value problem of (33) and (17b) has a unique monotone solution \( \phi \in C(\mathbb{R}_+) \cap C^2(\mathbb{R}_+) \) satisfying (34). Moreover, \( \phi \) is a unique solution to (33) and (17b) in order to prove Theorem 2.1.

(ii) Let \( V''(0) = 0 \). There exists a positive constant \( \delta \) such that if \( |\phi_b| \leq \delta \) and \( V(\phi_b) \geq 0 \), then the boundary value problem of (33) and (17b) has a unique monotone solution \( \phi \in C(\mathbb{R}_+) \cap C^2(\mathbb{R}_+) \) satisfying (34).

(iii) Let \( V''(0) < 0 \). If \( \phi_b \neq 0 \), the boundary value problem of (33) and (17b) does not admit any solutions \( \phi \in C(\mathbb{R}_+) \cap C^2(\mathbb{R}_+) \). If \( \phi_b = 0 \), \( \phi = 0 \) is a unique solution.
Proof. The Sagdeev potential \( V \) is analytic near the point \( \phi = 0 \) and \( V(0) = V'(0) = 0 \). Thus the Taylor expansion gives

\[
V(\phi) = \sum_{n=2}^{\infty} \frac{V^{(n)}(0)}{n!} \phi^n. \tag{36}
\]

The proof is divided into three cases: \( V''(0) > 0, V''(0) = 0, \) and \( V''(0) < 0 \).

(i) Case \( V''(0) > 0 \). Owing to (36), there exists a positive constant \( \delta \) such that if \( |\phi| \leq \delta, V(\phi) \) is non-negative. We prove the existence of a monotone decreasing solution for the case \( \phi_b \in (0, \delta] \), since the other case is shown in a similar manner. When \( \phi_b \in (0, \delta] \), we rewrite the equation (33) into the equivalent equation

\[
\dot{\phi}_x = -\sqrt{\frac{2}{\varepsilon_0}} V(\phi),
\]

where \( \sqrt{V} \) is Lipschitz continuous in \([-\delta, \delta]\). A basic ordinary differential equation theory gives the unique solvability of this equation with (17b).

The function \( \sum_{i=1}^{k} q_i f_i^{-1}(q, \phi) \) is positive on \( \phi = 0 \) owing to (10). Thus, by retaking \( \delta \) sufficiently small if necessary, we see that the solution \( \phi \) constructed above satisfies (34). One can show (35) by straightforward computations together with (36) and \( V''(0) > 0 \). The proof of assertion (i) is complete.

(ii) Case \( V''(0) = 0 \). It is seen from the analyticity of \( V \) and \( V \neq 0 \) that there exists an integer \( l_0 \in \mathbb{N} \) such that \( V^{(l_0)}(0) \neq 0 \) and \( V^{(l)}(0) = 0 \) for \( l < l_0 \). This means that \( V(\phi) \) is approximated by \( V^{(l_0)}(0) \phi^{l_0}/l_0! \) for suitably small \( \phi \). \footnote{We cannot take \( \phi_b \) satisfying \( V(\phi_b) \geq 0 \) if \( l_0 \) is even and \( V^{(l_0)}(0) < 0 \); we can take \( \phi_b \) satisfying \( V(\phi_b) \geq 0 \) for the other case.} Provided that \( \phi_b \) satisfies \( V(\phi_b) \geq 0, V(\phi) \) is non-negative between \( \phi = 0 \) and \( \phi = \phi_b \). Hence, by the same argument as in the case \( V''(0) > 0 \), we establish the unique existence of a monotone solution to (33) and (17b).

(iii) Case \( V''(0) < 0 \). We see from (36) and \( V''(0) < 0 \) that \( V(\phi) \) is negative near the point \( \phi = 0 \). However, \( V(\phi) \) must be non-negative owing to (33). Hence, no solution exists for \( \phi_b \neq 0 \). \( \square \)

4. Asymptotic stability of stationary solution. In this section we discuss the asymptotic stability of the stationary solution of which the asymptotic state \( R_+ \) satisfies (21). The time local solvability of the initial–boundary value problem of (24) and (25) is summarized as follows.

Lemma 4.1. (i) Suppose that \( e^{\alpha x^2/2} \Psi_0 \in H^2(\mathbb{R}_+) \) for some constant \( \alpha > 0 \) and \( \Psi_0 + R \) satisfies (14). Let \( \beta \) be a constant satisfying \( \beta \in (0, \alpha] \) and

\[
\beta^2 < \frac{4d}{\varepsilon_0} e^{m_*}, \quad m_* := \min \left\{ \inf_x (-\tilde{\phi}), \inf_x \log \frac{1}{d} \langle \tilde{R}, \mathbf{q} \rangle, \inf_x \log \frac{1}{d} \langle \Psi_0 + \tilde{R}, \mathbf{q} \rangle \right\}.
\]

Then there exists a constant \( T > 0 \) such that the initial–boundary value problem of (24) and (25) has a unique solution \( (\Psi, \sigma) \) satisfying \( e^{\beta x^2/2} \Psi, e^{\beta x^2/2} \sigma \in X_2([0, T]) \times X_2^*([0, T]) \) and (11)–(13).
(ii) Suppose that \((1 + \alpha x)^{\beta/2} \Psi_0 \in H^2(\mathbb{R}_+)\) for some constants \(\alpha, \lambda > 0\) and \(\Psi_0 + \tilde{R}\) satisfies (14). Let \(\beta\) be a constant satisfying \(\beta \in (0, \alpha]\) and
\[
\beta^2 < \frac{4d}{\varepsilon_0 \lambda} e^{m_*}.
\]
Then there exists a constant \(T > 0\) such that the initial-boundary value problem of (24) and (25) has a unique solution \((\Psi, \sigma)\) satisfying \(((1 + \beta x)^{\lambda/2} \Psi, (1 + \beta x)^{\lambda/2} \sigma) \in X_2([0, T]) \times X_2^2([0, T])\) and (11)–(13).

Because Lemma 4.1 is shown in a similar manner as in [13], we omit its proof and focus on the proof of Proposition 4.2, which is an a priori estimate. For notational convenience, we introduce
\[
N_w(T) := \sup_{0 \leq \tau \leq t} \|( \sqrt{w}(\Psi)(t) \|_2 \text{ for } w(x) = e^{\beta x} \text{ or } w(x) = (1 + \beta x)^{\lambda}.
\]

**Proposition 4.2.** Let the asymptotic state \(R_+\) satisfy conditions (15), (21), and (27).

(i) Suppose that \((\Psi, \sigma)\) with \((e^{\alpha x/2} \Psi, e^{\alpha x/2} \sigma) \in X_2([0, T]) \times X_2^2([0, T])\) is a solution to (24) and (25), and \(\Psi + \tilde{R}\) satisfies (11)–(13). There exist positive constants \(\beta(\leq \alpha)\) and \(\delta\) such that if \(|\phi_0| + N_w(T) \leq \delta\), then it holds that
\[
e^{\gamma t} \|( e^{\beta/2} \Psi(t) \|_2^2 + \beta \int_0^t e^{\gamma \tau} \| e^{\beta x/2} \Psi(\tau) \|_2^2 d\tau \leq C \| e^{\beta x/2} \Psi_0 \|_2^2
\]
for \(t \in [0, T]\), where the positive constants \(C\) and \(\gamma\) are independent of \(T, \phi_0, \) and \(\beta\).

(ii) Let \(\lambda\) and \(\nu\) satisfy \(\lambda \geq 2\) and \(\nu \in (0, \lambda]\). Suppose that \((\Psi, \sigma)\) with \(((1 + \alpha x)^{\lambda/2} \Psi, (1 + \alpha x)^{\lambda/2} \sigma) \in X_2([0, \infty)) \times X_2^2([0, \infty))\) is a solution to (24) and (25), and \(\Psi + \tilde{R}\) satisfies (11)–(13). There exist positive constants \(\beta(\leq \alpha)\) and \(\delta\) such that if \(|\phi_0| + N_{(1 + \beta x)^\nu}(T) \leq \delta\), then it holds that
\[
(1 + \beta t)^\gamma \|(1 + \beta x)^{\lambda/2} \Psi(t) \|_2^2 + \beta \int_0^t (1 + \beta \tau)^\gamma \|(1 + \beta x)^{(\lambda - 1)/2} \Psi(\tau) \|^2_2 d\tau
\]
\[
\leq C \|(1 + \beta x)^{\lambda/2} \Psi_0 \|_2^2 + C \beta \gamma \int_0^t (1 + \beta \tau)^{\gamma - 1} \|(1 + \beta x)^{\lambda/2} \Psi(\tau) \|^2_2 d\tau
\]
for \(t \in [0, T]\), \(\zeta \in [\nu, \lambda]\), and \(\gamma \in [0, \infty)\), where the positive constant \(C\) is independent of \(T, \phi_0, \beta, \zeta, \) and \(\gamma\).

Notice that the time global solution \((\Psi, \sigma)\) can be constructed by the standard continuation argument using the time local solvability developed in Lemma 4.1 and the a priori estimate in Proposition 4.2. Once the global solution is constructed, it is obvious that the resultant \((\Psi, \sigma)\) satisfies (37) and (38) for \(t \in [0, \infty)\). The decay estimate (29) immediately follows from (37) together with (40) in Lemma 4.3. In addition, similarly as in [6, 10], applying an induction argument to (38) gives the decay estimate (30). Hence, it suffices to show Proposition 4.2 in order to prove Theorem 2.3. The rest of this section is devoted to the derivation of the a priori estimate.

4.1. **Estimate and formula for elliptic equation.** In this section we discuss the estimate and formula of the solution \(\sigma\) to the elliptic equation (24b) in order to show Proposition 4.2. It is straightforward to show the elliptic estimate in Lemma 4.3 by a standard energy method.
Lemma 4.3. Let \( w(x) \) be either \( e^{\beta x} \) or \((1 + \beta x)^c\). Under the same assumption as in Proposition 4.2,
\[
\|\sqrt{w} \partial_l^i \sigma\|_2 \leq C\|\sqrt{w} \partial_l^i \Psi\|, \quad \|\sqrt{w} \partial_l^i \sigma\|_2 \leq C\|\sqrt{w} \partial_l^i \Psi\| \quad \text{for } l = 0, 1, \tag{39}
\]
\[
\|\sqrt{w}\sigma\|_{2+j} \leq C\|\sqrt{w}\Psi\|_j, \quad \|\sqrt{w}\sigma\|_{2+j} \leq C\|\sqrt{w}\Psi\|_j \quad \text{for } j = 0, 1, 2. \tag{40}
\]

We drive the formula for \( \sigma \) and \( \sigma_t \) by combination of the odd extension and
Fourier transformation. Multiply \((24b)\) by the weight function \( \sqrt{w} \).
Apply the operator \( \partial_l^i \) for \( l = 0, 1 \) to the resultant equation. The result is
\[
\frac{\varepsilon_0}{d} \big(\sqrt{w} \partial_Q^i \sigma\big)_x - \big(\sqrt{w} \partial_Q^i \Psi\big) = \frac{1}{d} \big(\sqrt{w} \partial_Q^i \Psi, q\big) + \partial_Q^i \big(\mathcal{G}[w, \sigma, \bar{\phi}]\big), \tag{41}
\]
where
\[
\mathcal{G}[w, \sigma, \bar{\phi}] := \frac{\varepsilon_0}{d} \left(2 \left\{\big(\sqrt{w}\big)_x \sigma\right\}_x - \big(\sqrt{w}_x \sigma\big)_x\right) + \sqrt{w} g[\sigma, \bar{\phi}].
\]
Applying the odd extension to the right-hand side of \((41)\), we consider the problem
\[
\frac{\varepsilon_0}{d} Q_{txx} - Q_t = \left[\frac{1}{d} (q, \sqrt{w}_x \partial^i \Psi) + \partial^i \mathcal{G}[w, \sigma, \bar{\phi}]\right]_{\text{odd}} \quad \text{in } \mathbb{R},
\]
where \([f]_{\text{odd}}(x) := f(x) \text{ if } x > 0, -f(-x) \text{ if } x < 0. \) A unique solution \( Q_t \in H^2(\mathbb{R}) \)
to this problem is explicitly given by utilizing Fourier analysis. The odd function \( Q_t \) also satisfies \( Q_t = \left[\sqrt{w}_x \partial^i \sigma\right]_{\text{odd}}, \) and therefore the next lemma holds.

Lemma 4.4. Let \( w(x) \) be either \( e^{\beta x} \) or \((1 + \beta x)^c\). Under the same assumption as in Proposition 4.2,
\[
\left[\sqrt{w}_x \partial^i \sigma\right]_{\text{odd}} = \mathcal{F}^{-1} \left[\frac{-1}{\varepsilon_0 d^{-1} \xi^2} + \mathcal{F} \left[\frac{1}{d} (\sqrt{w}_x \partial^i \Psi, q) + \partial^i \mathcal{G}[w, \sigma, \bar{\phi}]\right]_{\text{odd}}\right] \quad \text{for } l = 0, 1, \tag{42}
\]
where \( \mathcal{F} \) is the Fourier operator.

4.2. A priori estimate. This section is devoted to the completion of the proof of
Proposition 4.2. Applying the operator \( \partial_l^i \partial_j^i \) for \( l = 0, 1 \) and \( j = 0, 1 \) to \((24a)\) gives
\[
A^0[\Psi, \bar{R}] \partial_l^i \partial_j^i \Psi_t + A^1[\Psi, \bar{R}] \partial_l^i \partial_j^i \Psi_x = (\partial_l^i \partial_j^i \sigma_x) b[\Psi, \bar{R}] + \mathcal{H}_{l,j}, \quad \tag{43}
\]
where
\[
\mathcal{H}_{l,j} := - \partial_l^i \partial_j^i (A^0[\Psi, \bar{R}] \Psi_t) + A^0[\Psi, \bar{R}] \partial_l^i \partial_j^i \Psi_t - \partial_l^i \partial_j^i (A^1[\Psi, \bar{R}] \Psi_x)
+ A^1[\Psi, \bar{R}] \partial_l^i \partial_j^i \Psi_x + \partial_l^i \partial_j^i (\sigma_x b[\Psi, \bar{R}]) - (\partial_l^i \partial_j^i \sigma_x) b[\Psi, \bar{R}]
+ \partial_l^i \partial_j^i (b[\Psi, \bar{R}, \bar{R}])].
\]
We first derive the estimates of \( \partial_l^i \Psi \) and \( \partial_l^i \Psi_x \) from these equations since it is
convenient to handle the boundary terms that arise during the course of our energy estimates when we integrate by parts. After establishing the estimates for \( \partial_l^i \Psi \) and \( \partial_l^i \Psi_x \), we rewrite these estimates in terms of the spatial derivatives of the solutions
by the relations
\[
c\|\sqrt{w}_x \Psi\|_{1+l} \leq \sum_{m=0}^l \|\sqrt{w}_x \partial_l^i \psi\| \leq C\|\sqrt{w}_x \psi\|_{1+l}, \tag{44a}
\]
\[
c\|\sqrt{w}_x \Psi\|_{1+l} \leq \sum_{m=0}^l \|\sqrt{w}_x \partial_l^i \psi\| \leq C\|\sqrt{w}_x \psi\|_{1+l}. \tag{44b}
\]
we have $w(x) = e^{\beta x}$ and $(1 + \beta x)^{\ell}$. These relations follow from (42) since $A^0$ and $-A^1$
are positive definite under (11) and (12).

Taking the inner product of (42) with the vector $2\partial_t^l \tilde{\Psi}$ and substituting $j = 0$, we have

\[
\left(\langle \partial_t^l \Psi, A^0(x) \tilde{\Psi}, \tilde{R} \partial_t^l \Psi \rangle \right) + \left(\langle \partial_t^l \Psi, A^1(x) \tilde{R} \partial_t^l \Psi \rangle - 2\langle \partial_t^l \sigma, \partial_t^l \Psi, b \tilde{\Psi} \rangle \right) + 2\langle \partial_t^l \sigma, \partial_t^l \Psi, b \tilde{\Psi} \rangle = \mathcal{I}_l, \tag{44}
\]

where

\[
\mathcal{I}_l := \langle \partial_t^l \Psi, (A^0(x) \tilde{R}) \partial_t^l \Psi \rangle + \langle \partial_t^l \Psi, (A^1(x) \tilde{R}) \partial_t^l \Psi \rangle - 2\langle \partial_t^l \sigma, \partial_t^l \Psi, b \tilde{\Psi} \rangle.
\]

Similarly as above, for the case $j = 1$, it holds that

\[
\left(\langle \partial_t^l \Psi_x, A^0(x) \tilde{R} \partial_t^l \Psi_x \rangle \right) + \left(\langle \partial_t^l \Psi_x, A^1(x) \tilde{R} \partial_t^l \Psi_x \rangle \right) - 2\langle \partial_t^l \sigma, \partial_t^l \Psi_x, b \tilde{\Psi} \rangle = \mathcal{J}_l, \tag{45}
\]

where

\[
\mathcal{J}_l := \langle \partial_t^l \Psi_x, (A^0(x) \tilde{R}) \partial_t^l \Psi_x \rangle + \langle \partial_t^l \Psi_x, (A^1(x) \tilde{R}) \partial_t^l \Psi_x \rangle + \langle \partial_t^l \Psi_x, b \tilde{\Psi} \rangle + 2\langle \partial_t^l \Psi_x, b \tilde{\Psi} \rangle.
\]

Here, owing to (23), (40), and the Sobolev and Schwartz inequalities, $\mathcal{I}_l$ and $\mathcal{J}_l$ are estimated as

\[
||\mathcal{I}_l, \mathcal{J}_l|| \leq C \langle |\phi|, N_{w}(T) \rangle \frac{w}{\beta w} \langle |\langle \partial_t^l \Psi, \partial_t^l \Psi_x, \partial_t^l \Psi_x, b \tilde{\Psi} xx, \partial_t^l \sigma, \partial_t^l \sigma \rangle|^2, \tag{46}
\]

where $C$ is independent of $|\phi|$, $N_{w}(T)$, and $\beta$.

On the other hand, applying the operator $\partial_t^l$ for $l = 0, 1$ to (24b) and multiplying the resultant equation by $\langle \partial_t^l \Psi_x, b \tilde{\Psi} \rangle$, we obtain

\[
\frac{\partial_t^l \sigma}{\partial_t^l \sigma} \langle \partial_t^l \Psi_x, b \tilde{\Psi} \rangle = \frac{1}{d} \langle \partial_t^l \Psi, q \rangle \langle \partial_t^l \Psi_x, b \tilde{\Psi} \rangle + \partial_t^l (\langle \sigma, \tilde{\phi} \rangle \langle \partial_t^l \Psi_x, b \tilde{\Psi} \rangle), \tag{47}
\]

Notice that $\langle \partial_t^l \Psi_x, b \tilde{\Psi} \rangle$ is rewritten as

\[
\langle \partial_t^l \Psi_x, b \tilde{\Psi} \rangle = \sum_{i=1}^{k} q_i (\psi_i + \tilde{\psi}_i) \partial_t^l q_i x
\]

\[
= - \langle \partial_t^l \Psi_t, x \rangle - \langle \partial_t^l \Psi_x, q \rangle - \sum_{i=1}^{k} q_i (\eta_i + \tilde{\eta}_i) \partial_t^l \psi_i x - \frac{q_i \hat{H}^{(1-i)}_{l,0}}{K_i (\psi_i + \tilde{\psi}_i)^{\gamma_i-2}} , \tag{48}
\]

where we have solved the odd components of (42) for $j = 0$ with respect to $(\psi_i + \tilde{\psi}_i) \partial_t^l q_i x$ and then substituted them in deriving the second equality. Here, $\hat{H}^{(1-i)}_{l,0}$
denotes the \((2i - 1)\)-th component of \(\mathcal{H}_{t, 0}\) and \(u_+\) is defined in (27). Substituting (48) into the first term of the right-hand side of (47) gives
\[
\left(\frac{\varepsilon_0}{d}\partial_t^2 \sigma_{xx} - \partial_t^2 \sigma\right)\langle \partial_t^2 \Psi, b[\Psi, \tilde{R}] \rangle = -\frac{1}{2d} \langle \partial_t^2 \Psi, q \rangle_t \frac{u_+}{2d} \langle \partial_t^2 \Psi, q \rangle_t + \mathcal{K}_t, \quad (49)
\]
where
\[
\mathcal{K}_t := \partial_t^2 (g[\sigma, \tilde{\phi}])\langle \partial_t^2 \Psi, b[\Psi, \tilde{R}] \rangle
\]
and
\[\frac{1}{d}\langle \partial_t^2 \Psi, q \rangle \sum_{i=0}^{k} (q_i (\eta_i + \tilde{u}_i - u_+) \partial_t^2 \psi_{ix} - \frac{q_i \mathcal{H}_{t, 0}^{2i - 1}}{K_i (\psi_i + \tilde{\rho})^{1 - 2}}).
\]

The Sobolev and Schwarz inequalities together with (23) and (40) yield
\[
|\mathcal{K}_t| \leq C (|\phi_b| + N_u(T)) \frac{w_x}{\beta_w} \frac{w_x}{\beta_w} \left| \langle \partial_t^2 \Psi, \partial_t^4 \Psi, \partial_t^2 \Psi, \partial_t^2 \Psi \rangle \right|^2, \quad (50)
\]
where \(C\) is independent of \(|\phi_b|, N_u(T),\) and \(\beta\). We are now in a position to establish the a priori estimate.

**Proof of Proposition 4.2.** Multiply (44) by \(w\), (45) by \(\varepsilon_0 w / d\), and (49) by \(2w\) respectively. Here, \(w(x) = e^{\beta x}\) or \(w(x) = (1 + \beta x)^\zeta\) with \(\zeta \in [\nu, \lambda]\). Then summing up these three results, integrating it over \(\mathbb{R}_+\), and using the boundary conditions \(\partial_x^2 \sigma(t, 0) = 0\), we have
\[
\frac{d}{dt} \int_{\mathbb{R}_+} w \left( \langle \partial_t^2 \Psi, A^0[\Psi, \tilde{R}] \partial_t^2 \Psi \rangle + \frac{1}{d} \langle \partial_t^2 \Psi, q \rangle^2 + \frac{\varepsilon_0}{d} \langle \partial_t^2 \Psi, A^0[\Psi, \tilde{R}] \partial_t^2 \Psi \rangle \right) dx \\
- \left( \langle \partial_t^2 \Psi, A^1[\Psi, \tilde{R}] \partial_t^2 \Psi \rangle + \frac{u_+}{d} \langle \partial_t^2 \Psi, q \rangle^2 + \frac{\varepsilon_0}{d} \langle \partial_t^2 \Psi, A^1[\Psi, \tilde{R}] \partial_t^2 \Psi \rangle \right) (0, t) = : I_1
\]
\[
- \int_{\mathbb{R}_+} w_x \left( \langle \partial_t^2 \Psi, A^1[0, R_+] \partial_t^2 \Psi \rangle + \frac{u_+}{d} \langle \partial_t^2 \Psi, q \rangle^2 - 2 (\partial_t^2 \sigma) \langle \partial_t^2 \Psi, b[0, R_+] \rangle \right) dx = : I_2
\]
\[
- \int_{\mathbb{R}_+} w_x \frac{\varepsilon_0}{d} \langle \partial_t^2 \Psi, A^1[\Psi, \tilde{R}] \partial_t^2 \Psi \rangle dx = : I_3
\]
\[
= \int_{\mathbb{R}_+} w \left( \langle \partial_t^2 \Psi, (A^1[\Psi, \tilde{R}] - A^1[0, R_+]) \partial_t^2 \Psi \rangle - 2 (\partial_t^2 \sigma) \langle \partial_t^2 \Psi, b[\Psi, \tilde{R}] - b[0, R_+] \rangle \right)
\]
\[
+ w \left( I_1 + \frac{\varepsilon_0}{d} J_1 + 2 \mathcal{K}_t \right) dx.
\]
\[
\leq C (|\phi_b| + N_u(T)) \int_{\mathbb{R}_+} (1 + \beta^{-1}) w_x \left\{ \Psi^2 + \Psi_x^2 + (\partial_t^2 \Psi)^2 + (\partial_t \sigma)^2 \right\} dx, \quad (51)
\]
where we have used (39), (43), (46), and (50) to get the last inequality in (51).

Let us estimate the terms \(I_1, I_2,\) and \(I_3\) in the left-hand side of (51) separately. Notice that \(I_1\) is non-negative since \(-A^1\) is positive definite under conditions (11) and (12). Thus \(I_1\) can be ignored. Similarly, \(I_3\) is estimated from below as
\[
I_3 \geq c \int_{\mathbb{R}_+} w_x (\partial_t^2 \Psi_x)^2 dx. \quad (52)
\]

The term \(I_2\) is rewritten by the integration over \(\mathbb{R}\) together with the odd extension functions \(\sqrt{w_x} \partial_t^2 \Psi_{\text{odd}}\) and \(\sqrt{w_x} \partial_t^2 \sigma_{\text{odd}}\). Use the Parseval theorem and then
substitute the formula, stated in Lemma 4.2, into \( [\sqrt{w_x} \partial_x']_{\text{odd}} \). The result is

\[
I_2 = -\frac{1}{2} \int_{\mathbb{R}} \left\langle [\sqrt{w_x} \partial_x']_{\text{odd}}, \left( A^1[0, R_+] + \frac{u_+}{d} q' q \right) [\sqrt{w_x} \partial_x']_{\text{odd}} \right\rangle \\
- 2[\sqrt{w_x} (\partial_x')_{\text{odd}}][[\sqrt{w_x} \partial_x']_{\text{odd}}, b[0, R_+]] \, dx \\
= \frac{1}{2} \int_{\mathbb{R}} \left( \mathcal{F}[[\sqrt{w_x} \partial_x']_{\text{odd}}, S_{\xi} \mathcal{F}[[\sqrt{w_x} \partial_x']_{\text{odd}})] \right) \, d\xi \\
- \frac{1}{2} \int_{\mathbb{R}} \frac{1}{\varepsilon_0 d - 1\xi^2 + 1} \mathcal{F} \left[ \partial_t' \{ \mathcal{G}[w_x, \sigma, \hat{\phi}] \}_{\text{odd}} \right] \, d\xi,
\]

where \( S_{\xi} \) and \( \mathcal{G} \) are defined in Proposition 2.3 and Lemma 4.2, respectively. Proposition 2.3 and the generalized Bohm criterion (21) ensure that \( S_{\xi} \) is positive definite uniformly in \( \xi \in \mathbb{R} \). Thus, from (39), the Plancherel theorem, and the Sobolev and Schwartz inequalities, we estimate \( I_2 \) from below as

\[
I_2 \geq c \int_{\mathbb{R}_+} w_x (\partial_x')^2 \, dx - C(\beta + |\phi_0| + N_w(T)) \int_{\mathbb{R}_+} w_x (\partial_x')^2 \, dx. \tag{53}
\]

Substitute (52) and (53) into (51) and sum up the results for \( l = 0, 1 \). Then,\(^3\) letting \( \beta, |\phi_0|, \) and \( N_w(T) \) be sufficiently small, we obtain

\[
\frac{d}{dt} \sum_{l=0}^{1} \int_{\mathbb{R}_+} w \left( (\partial_x') \{ A^0[\Psi, R] \partial_x' \Psi \} + \frac{1}{d} (\partial_x') \{ \Psi, A^0[\Psi, R] \partial_x' \Psi \} \right) \, dx \\
+ c \sum_{l=0}^{1} \int_{\mathbb{R}_+} w \left\{ (\partial_x')^2 + (\partial_x')^2 \right\} \, dx \leq 0. \tag{54}
\]

Moreover, substitute \( w(x) = e^{\beta x} \) in (54), multiply it by \( e^{\gamma t} \), and integrate the resultant inequality by parts over \([0, t] \). Letting \( \gamma \) be sufficiently small and utilizing \( A_0 > 0 \) and (43) give (37). Similarly, substituting \( w(x) = (1 + \beta x)^c \) in (54), multiplying it by \((1 + \beta t)^\gamma \), integrating the result by parts over \([0, t] \), and using (43), we have (38). Hence, the proof have been completed. \( \square \)

5. Appendix. This section is devoted to the proof of Proposition 2.2. Notice that assertion (iii) in Proposition 2.2 holds if and only if the \( 2k \times 2k \) symmetric matrix \( S_0 \) is positive definite since the matrix \( A^1[0, R_+] + d^{-1} q' q \) is positive definite owing to (15). We show that assertion (i) is equivalent to \( S_0 > 0 \) in subsection 5.1 and that assertion (ii) is equivalent to \( S_0 > 0 \) in subsection 5.2, respectively.

5.1. Equivalence of assertion (i) to \( S_0 > 0 \). We begin with applying the Fourier and Laplace transformations to (28) to obtain the eigenvalue problem

\[
\mu(\sqrt{-1} \xi) A^0[0, R_+] \hat{\Psi} + \left( \sqrt{-1} \xi - \frac{\beta}{2} \right) A^1[0, R_+] \hat{\Psi} = \left( \sqrt{-1} \xi - \frac{\beta}{2} \right) \hat{\Sigma} b[0, R_+], \tag{55a}
\]

\[
\hat{\Sigma} = -\frac{1}{d} \left\{ -\frac{\varepsilon_0}{d} \left( \sqrt{-1} \xi - \frac{\beta}{2} \right)^2 + 1 \right\}^{-1} \langle \hat{\Psi}, q \rangle \quad \text{for} \quad \xi \in \mathbb{R}. \tag{55b}
\]

Moreover, the next equality follows from the odd components of (55a).

\[
\mu(\sqrt{-1} \xi) q' q \hat{\Psi} + \left( \sqrt{-1} \xi - \frac{\beta}{2} \right) u_+ q' q \hat{\Psi} + \left( \sqrt{-1} \xi - \frac{\beta}{2} \right) q' b[0, R_+] \hat{\Psi} = 0. \tag{56}
\]

\(^3\) We let \(|\phi_0|\) and \(N_w(T)\) be appropriately small compare to \(\beta\).
Assuming \( S_0 > 0 \) hold, we show assertion (i). Take the inner product of (55a) with the vector \((1 + \varepsilon_0 \xi^2 / d)\hat{\psi}\) and of (56) with the vector \(d^{-1}\hat{\psi}\), respectively. Then summing up these results and using (55b) give

\[
2(\text{Re} \mu(\sqrt{-1} \xi)) \left\{ \langle \hat{\psi}, A^0[0, R_+]\hat{\psi} \rangle + \frac{1}{d} \langle \hat{\psi}, q^0 q\hat{\psi} \rangle + \frac{\varepsilon_0}{d} \xi^2 \langle \hat{\psi}, A^0[0, R_+]\hat{\psi} \rangle \right\} \\
+ \beta \left\{ \langle \hat{\psi}, S\xi\hat{\psi} \rangle - \frac{\varepsilon_0}{d} \xi^2 \langle \hat{\psi}, A^1[0, R_+]\hat{\psi} \rangle \right\} = 2\text{Re} \left( h(\sqrt{-1} \xi) \langle \hat{\psi}, q \rangle \langle b, \hat{\psi} \rangle \right).
\]

(57)

Here, \( h(\sqrt{-1} \xi) \) is defined as a function of \( \sqrt{-1} \xi \) and estimated as

\[
|h(\sqrt{-1} \xi)| \leq C \beta^2 (1 + \xi^2)
\]

(58)

for suitably small \( \beta > 0 \), where \( C \) is a positive constant independent of \( \beta \) and \( \xi \). Owing to the positive definiteness of \( A^0, -A^1, \) and \( S\xi \), we see from (57) and (58) that \( \text{Re} \mu(\sqrt{-1} \xi) \leq -C \beta < 0 \) holds for an arbitrary \( \xi \in \mathbb{R} \) by taking \( \beta \) sufficiently small.

Let us prove \( S_0 > 0 \) by assuming assertion (i). Multiply (55a) by \((1 - \varepsilon_0 \beta^2 / 4d)\) and (56) by \(d^{-1}\), respectively. Then add these two results and substitute \( \xi = 0 \) to obtain

\[
\mu(0) M^0 \hat{\psi} + \frac{\beta}{2} M^1 \hat{\psi} = 0,
\]

(59)

\[
M^0 := \left( 1 - \frac{\varepsilon_0 \beta^2}{4d} \right) A^0[0, R_+] + \frac{1}{d} q^0 q, \quad M^1 := S_0 + \frac{\varepsilon_0 \beta^2}{4d} A^1[0, R_+].
\]

Assertion (ii) and (59) mean that the eigenvalues of \((M^0)^{-1} M^1\) are positive. On the other hand, the symmetric matrix \(M^0\) is positive definite for an arbitrary \( \beta < \sqrt{4d/\varepsilon_0} \). Then it is straightforward to show \( M^1 > 0 \) which gives

\[
\langle U, S_0 U \rangle \geq \frac{\varepsilon_0 \beta^2}{4d} \langle U, -A^1[0, R_+] U \rangle \geq c \beta^2 |U|^2 \quad \text{for} \ U \in \mathbb{C}^{2k}.
\]

Consequently, the equivalence of assertion (ii) and \( S_0 > 0 \) have been proved.

5.2. Equivalence of assertion (ii) to \( S_0 > 0 \). We show the equivalence of the generalized Bohm criterion (21) and the positive definiteness of the \(2k \times 2k\) symmetric matrix \( S_0 \). For notational convenience, let us introduce positive constants

\[
d_i := \frac{K_i \rho^i_{-1} t_i}{q^i_{1} \rho^i_{-1}}, \quad l_i := \begin{cases} 1 & \text{if } i = 1, \\ l_{i-1} - l_{i-1}^2 (d_{i-1} + l_{i-1})^{-1} & \text{if } i \geq 2. \end{cases}
\]

It is seen by induction that (21) holds if and only if \( P_k(u^2_+) > 0 \) holds, where \( P_k \) is defined as

\[
P_k(u^2_+) := \begin{cases} -1 + \left( \frac{u^2_+ d}{q^2_{k} \rho^k} - d_1 \right) & \text{if } k = 1, \\ P_{k-1}(u^2_+) \left( \frac{u^2_+ d}{q^2_{k} \rho^k} - d_k \right) - \prod_{i=1}^{k-1} \left( \frac{u^2_+ d}{q^2_{i} \rho^i} - d_i \right) & \text{if } k \geq 2. \end{cases}
\]

Therefore it is sufficient to prove that \( S_0 > 0 \) is equivalent to \( P_k(u^2_+) > 0 \).
The necessary and sufficient condition for $S_0 > 0$ is that the determinants $D^{(i)}$ of all leading principal minors of $S_0$ are positive. Straightforward computations give
\[
D^{(2i-1)} = \begin{cases} 
-u_+ d^{-1} q_1^2(d_1 + l_1) & \text{if } i = 1, \\
-u_+ d^{-2i+1} q_1^2 \left( \prod_{j=1}^{i-1} q_j^4 \rho_j^2 \right) \left( \prod_{j=1}^{i} (d_j + l_j) \right) P_{i-1}(u_+^2) & \text{if } i \geq 2,
\end{cases}
\]
and
\[
D^{(2i)} = d^{-2i} \left( \prod_{j=1}^{i} q_j^4 \rho_j^2 \right) \left( \prod_{j=1}^{i} (d_j + l_j) \right) P_i(u_+^2)
\]
for $i = 1, \ldots, k$. This means that $S_0$ is positive definite if and only if $P_i(u_+^2)$ is positive for $i = 1, \ldots, k$. Then we see from $S_0 > 0$ that $P_k(u_+^2) > 0$ holds. On the other hand, the positivity $P_i(u_+^2) > 0$ for $i = 1, \ldots, k-1$ follows from (15) and $P_k(u_+^2) > 0$. Hence, the positivity $P_k(u_+^2) > 0$ also gives $S_0 > 0$.

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