Fractional exclusion statistics: the concept and some applications

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Abstract. I present briefly the concept of fractional exclusion statistics (FES) and I analyze two models of interacting particle systems. After I show that the models are equivalent by transforming one into the other by redefining some terms in the Hamiltonians, I calculate the heat capacity of the system and compare it with the heat capacity of an ideal Bose or Fermi gas. I show that under certain conditions the heat capacities may not be equal. I show that the interacting particle system may always be described as an ideal gas obeying FES and I show the method to construct such a gas.

1. Introduction
Considerable insight may be gained into the microscopic and macroscopic physical properties of a system of interacting particles if this can be described in a certain approximation as a gas of ideal quasiparticles. The difficulty in finding such a description is due to the fact that in general the interaction between the particles cannot be completely removed in the quasiparticle formulation or the physical properties of the quasiparticle gas do not follow closely enough the physical properties of the original system. A method by which a system of interacting particles may be described as an ideal gas of particles with identical physical properties is provided by the fractional exclusion statistics (FES).

The concept of FES was introduced by Haldane [1] and the thermodynamic properties of FES systems were calculated mainly by Wu [2] and Isakov [3]. A FES system consists of a countable number of species of particles, indexed here by capital letters, I, J = 0, 1, ... . Each species consists of a finite number of single-particle states, GI, and particles, NI − GI will also be called the dimension of the species I. The total number of particles in the system is N = ∑I NI.

The key characteristics of FES is that the dimensions of the species depend on the particle populations, namely if NI changes by δNI, for any I, then GI changes by δGI = −αIJδNI for any J. The coefficients αIJ are called the FES parameters. If I denote by GI(0) the dimensions of the species in the absence of any particles in the system (N = 0), then for a given particle distribution, say {NI}, the species dimensions are

\[ G_I = G_I^{(0)} - \sum_J \alpha_{IJ} N_J. \]  

(1)
If the particles are bosons, then the total number of micro-configurations in the system is [2, 4]

\[ W^(-) (\{N_I\}) = \prod_I \frac{(G_I^{(0)} + N_I - 1 - \sum_J \alpha_{IJ} N_J)!}{N_I! (G_I^{(0)} - 1 - \sum_J \alpha_{IJ} N_J)!} = \prod_I \frac{(G_I + N_I - 1)!}{N_I! (G_I - 1)!}, \]

(2a)

whereas if they are fermions, it is

\[ W^+ (\{N_I\}) = \prod_I \frac{(G_I^{(0)} - \sum_J \alpha_{IJ} N_J)!}{N_I! (G_I^{(0)} - N_I - \sum_J \alpha_{IJ} N_J)!} = \prod_I \frac{G_I!}{N_I! (G_I - N_I)!}. \]

(2b)

In general, by the superscripts (−) and (+) I shall denote Bose and Fermi systems, respectively. If the numbers \{G_I\}, \{G_I^{(0)}\} and \{N_I\} are integers whereas \alpha_{IJ}'s are real, then the numbers of the type \(G_I^{(0)} - \sum_J \alpha_{IJ} N_J\) that appear in the Eqs. (2) are defined up to an arbitrary constant of the order 1. This choice has no physical relevance in the thermodynamic limit, when all the number involved are much bigger than 1. Nevertheless, in mesoscopic systems such a constant cannot be neglected in rigorous calculations. Then the physical quantities that remain an integers are \(G_I\) and \(N_I\) [5, 6, 7].

Denoting \(\beta = 1/(k_BT)\), where \(T\) is the temperature of the system, introducing the notation \(n_I \equiv N_I/G_I\), and associating an energy \(\epsilon_I\) and a chemical potential, \(\mu_I\) (\(I = 0, 1, \ldots\)) to each of the species in the system, one can write the grandcanonical partition function,

\[ Z^{(\pm)} = W^{(\pm)} \prod_I \exp [\beta (\epsilon_I - \mu_I) n_I G_I], \]

(3)

which, if it is maximized with respect to the \(n_I\)'s leads to the equations for the equilibrium populations [4],

\[ \beta (\mu_I - \epsilon_I) + \ln \frac{[1 \mp n_I^{(\pm)}]}{n_I^{(\pm)}} = \mp \sum_J \alpha_{IJ} \ln [1 \mp n_J^{(\pm)}]. \]

(4)

In Eq. (4) the upper signs stand for fermions and the lower ones for bosons. For ideal Bose and Fermi gases \(\alpha_{IJ} = 0\) for any \(I\) and \(J\) and Eq. (4) renders the well-known Fermi (for upper signs) and Bose (for lower signs) populations, as expected.

2. FES in general interacting particle systems

The FES is the general method by which interacting particle systems can be described as ideal gases of quasiparticles. Let me exemplify it here by two generic models.

2.1. The models

In the first model the total energy of the system is equal to the sum of the energies of the quasiparticles, namely

\[ E(\{n_i\}) \equiv \sum_i \tilde{\epsilon}_i n_i, \]

(5)

where \(i\) are sets of quantum numbers, each set defining a quasiparticle state; \(\tilde{\epsilon}_i\) and \(n_i\) are the quasiparticle energy and the population of the state \(i\), respectively. The entropy of the system has the typical form,

\[ S^{(\pm)}(T, \mu) = \mp k_B \sum_i \left\{ n_i^{(\pm)}(T, \mu) \ln [1 \mp n_i^{(\pm)}(T, \mu)] \pm n_i^{(\pm)}(T, \mu) \ln [n_i^{(\pm)}(T, \mu)] \right\}, \]

(6)

where again the upper signs are for fermions, the lower ones are for bosons; by \(n_i^{(\pm)}(T, \mu)\) I denote the equilibrium population of the state \(i\), at temperature \(T\) and chemical potential
\( U^{(\pm)}(T, \mu) \equiv \sum_i \tilde{\epsilon}_i n_i(T, \mu) \) and \( N^{(\pm)}(T, \mu) \equiv \sum_i n_i(T, \mu) \) are the (equilibrium) internal energy and total particle number of the system, then the equilibrium populations, \( n_i^{(\pm)}(T, \mu) \), are determined by minimizing the grandcanonical potential,

\[
\Omega^{(\pm)}(T, \mu) = U^{(\pm)}(T, \mu) - TS^{(\pm)}(T, \mu) - \mu N^{(\pm)}(T, \mu),
\]

with respect to \( n_i^{(\pm)} \)'s. This procedure leads to the populations

\[
n_i^{(\pm)}(T, \mu) = \left[ e^{\delta \tilde{\epsilon}_i^{(1)} - \mu} \pm 1 \right]^{-1}, \tag{8}
\]

where

\[
\tilde{\epsilon}_i^{(1)} = \tilde{\epsilon}_i + \sum_j \frac{\partial \tilde{\epsilon}_i}{\partial n_j} n_j \tag{9}
\]

are the new quasiparticle energies.

We observe that the populations (8) are not the ideal Bose or Fermi gas populations, with the energies \( \tilde{\epsilon}_i \), but they depend on the more complicated new quasiparticle energies, \( \tilde{\epsilon}_i^{(1)} \). Moreover,

\[
\tilde{E}^{(1)}(\{n_i\}) \equiv \sum_i \tilde{\epsilon}_i^{(1)} n_i = \sum_i \tilde{\epsilon}_i n_i + \sum_i \sum_j \frac{\partial \tilde{\epsilon}_i}{\partial n_j} n_i n_j \equiv E(\{n_i\}) + \frac{1}{2} \sum_i \sum_j \tilde{V}_{ij} n_i n_j \tag{10}
\]

where I have identified

\[
\tilde{V}_{ij} = 2 \partial \tilde{\epsilon}_i / \partial n_i. \tag{11}
\]

In the second model we start with a total energy of the system of the Fermi liquid theory form [8],

\[
E(\{n_i\}) = \sum_i \epsilon_i n_i + \frac{1}{2} \sum_{ij} V_{ij} n_i n_j, \tag{12}
\]

where \( \epsilon_i \) is the free particle energy and \( V_{ij} \) characterizes the particle-particle interaction energy.

The procedure for finding the physical properties of this system is similar to the procedure employed for the previous model. We write the entropy like in Eq. (6), the grandcanonical potential like in Eq. (7) and determine the equilibrium particle distribution by minimizing \( \Omega^{(\pm)} \) with respect to the populations. The equilibrium population has again the expression (8), but with a quasiparticle energy,

\[
\tilde{\epsilon}_i^{(2)} = \frac{\partial E}{\partial n_i} = \epsilon_i + \sum_j V_{ij} n_j, \tag{13}
\]

The sum of the quasiparticle energies is

\[
\tilde{E}^{(2)}(\{n_i\}) \equiv \sum_i \tilde{\epsilon}_i^{(2)} n_i = \sum_i \epsilon_i n_i + \sum_i \sum_j V_{ij} n_i n_j \equiv E(\{n_i\}) + \frac{1}{2} \sum_i \sum_j V_{ij} n_i n_j. \tag{14}
\]

We observe that the model 1 and model 2 are very similar to each other. They can be viewed as completely equivalent if we redefine the quasiparticle energies of model 2 as

\[
\tilde{\epsilon}_i \equiv \epsilon_i + \frac{1}{2} \sum_j V_{ij} n_j. \tag{15}
\]

With this redefinition, the total energy of the system in the model 2 has the same expression (5) as in model 1, whereas the identification (11) makes the expressions for the total quasiparticle energies, \( \tilde{E}^{(1)} \) and \( \tilde{E}^{(2)} \), similar. Now we can discuss only model 1 and the results may be immediately extended to model 2.
2.2. Transforming the interacting particle systems into ideal particle systems

Let us first check if the thermodynamic properties of these interacting particle systems are those of ideal Bose or Fermi gases. For this let me calculate the heat capacity, \( C_V^{(\pm)} \equiv \left( \frac{\partial U^{(\pm)}}{\partial T} \right)_{N=\text{const.}} = T \equiv \left( \frac{\partial S^{(\pm)}}{\partial T} \right)_{N=\text{const.}} \), by using the formula

\[
\frac{C_V^{(\pm)}}{T} = \left( \frac{\partial S^{(\pm)}}{\partial T} \right)_N = \left( \frac{\partial S^{(\pm)}}{\partial T} \right)_\mu \left( \frac{\partial N^{(\pm)}}{\partial T} \right)_\mu \left( \frac{\partial N^{(\pm)}}{\partial \mu} \right)_T^{-1},
\]

with

\[
\frac{\partial S^{(\pm)}}{\partial T} = k_B \sum_i \ln \left( \frac{1 - n_i^{(\pm)}}{n_i^{(\pm)}} \right) \frac{\partial n_i^{(\pm)}}{\partial T}, \quad \frac{\partial S^{(\pm)}}{\partial \mu} = k_B \sum_i \ln \left( \frac{1 - n_i^{(\pm)}}{n_i^{(\pm)}} \right) \frac{\partial n_i^{(\pm)}}{\partial \mu},
\]

\[
\frac{\partial N^{(\pm)}}{\partial T} = \sum_i \frac{\partial n_i^{(\pm)}}{\partial T}, \quad \frac{\partial N^{(\pm)}}{\partial \mu} = \sum_i \frac{\partial n_i^{(\pm)}}{\partial \mu}.
\]

Since \( n_i^{(\pm)}(T, \mu) \) is given by Eq. (8), taking into account that \( \varepsilon_i^{(2)} \) depends on both, \( T \) and \( \mu \), due to its dependence on \( \{n_i\} \), then the temperature variation of \( n_i^{(\pm)} \) should be calculated from the system

\[
\frac{\partial n_i^{(\pm)}(T, \mu)}{\partial (k_B T)} = \frac{\beta^2 (\tilde{\varepsilon}_i^L - \mu) e^{\beta (\tilde{\varepsilon}_i^L - \mu)}}{\left[ e^{\beta (\tilde{\varepsilon}_i^L - \mu)} + 1 \right]^2} \left[ 1 - \frac{1}{\beta (\tilde{\varepsilon}_i^L - \mu)} \sum_j V_{ij} \frac{\partial n_j^{(\pm)}(T, \mu)}{\partial (k_B T)} \right],
\]

\[
\frac{\partial n_i^{(\pm)}(T, \mu)}{\partial \mu} = \frac{\beta e^{\beta (\tilde{\varepsilon}_i^L - \mu)}}{\left[ e^{\beta (\tilde{\varepsilon}_i^L - \mu)} + 1 \right]^2} \left[ 1 - \sum_j V_{ij} \frac{\partial n_j^{(\pm)}(T, \mu)}{\partial \mu} \right].
\]

From Eqs. (16), (17) and (18) we conclude that the heat capacity of the system has the temperature dependence of an ideal Bose or Fermi gas if the expressions in the square brackets in Eqs. (18) are equal (or close enough) to 1.

If the expressions in the parentheses in Eqs. (18) differ significantly from 1, then the system is not thermodynamically equivalent (i.e. does not have the same heat capacity and entropy) [9] with an ideal Bose or Fermi system.

To show how we can transform the gas of model 1 or model 2 into an ideal FES gas, let us use the continuous limit and define the density of states (DOS), \( \hat{\sigma}(\tilde{\varepsilon}) \). We follow the procedure of Ref. [10] and coarse-grain the \( \tilde{\varepsilon} \) axis into elementary, fixed intervals, say \( \delta \tilde{\varepsilon}_I \equiv \tilde{\varepsilon}_{I+1} - \tilde{\varepsilon}_I \). Each interval represents a species of particles, like the ones described in Section 1. Each species contains \( G_I = \hat{\sigma}(\tilde{\varepsilon}) \delta \tilde{\varepsilon}_I \) states and \( N_I = n(\tilde{\varepsilon}_I) \hat{\sigma}(\tilde{\varepsilon}) \delta \tilde{\varepsilon}_I \) particles, where \( n(\tilde{\varepsilon}_I) \equiv N_I/G_I \). Since the energies \( \tilde{\varepsilon}_I \) depend on the populations, this implies that \( \hat{\sigma}(\tilde{\varepsilon}) \) depends also on the populations, so by including a number, \( \delta N_I \) of particles into the species \( I \), the DOS along the whole \( \tilde{\varepsilon} \) axis may change. But since the intervals \( \delta \tilde{\varepsilon}_I \) are fixed and if we fix also the number of particles in each interval (except for the variation \( \delta N_I \)), then the variation of DOS changes the number of states in the species and this gives rise to FES. The FES parameters are [10]

\[
\alpha_{\tilde{\varepsilon}_I} = -\frac{\delta \tilde{\varepsilon}_I}{d \tilde{\varepsilon}_I} \left[ \frac{\partial \tilde{\varepsilon}_I}{\partial \hat{\sigma}(\tilde{\varepsilon}_I)} \hat{\sigma}(\tilde{\varepsilon}_I) \right],
\]

where \( \tilde{\varepsilon}_I \) is the species where it is observed the change in the number of states, after the inclusion of the particles in the species \( I \).
3. Conclusions

After a brief introduction of the fractional exclusion statistics (FES) formalism, I analyzed the thermodynamic properties of two types of interacting particle systems. I showed that the systems are equivalent from the thermodynamics perspective and they can be transformed one into the other by a redefinition of some terms in the Hamiltonian. I also showed that such interacting particle systems may not always be thermodynamically equivalent [9] with a Bose or a Fermi ideal gas, but they can always be described as ideal gases of FES quasiparticles. I also showed the procedure for finding such an ideal FES gas description.

Acknowledgments

Discussions with dr. Gerhard Müller and dr. Alexandru Nemneş are gratefully acknowledged. The work was supported by the Romanian National Authority for Scientific Research CNCS-UEFISCDI projects PN-II-ID-PCE-2011-3-0960 and PN09370102/2009. The travel support from the Romania-JINR Dubna collaboration project Titeica-Markov and project N4063 are gratefully acknowledged.

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