An approximation algorithm for the longest cycle problem in solid grid graphs

Asghar Asgharian Sardroud ∗
Alireza Bagheri †
Department of Computer Engineering & IT,
Amirkabir University of Technology, Tehran, Iran.

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Abstract

Although, the Hamiltonicity of solid grid graphs are polynomial-time decidable, the complexity of the longest cycle problem in these graphs is still open. In this paper, by presenting a linear-time constant-factor approximation algorithm, we show that the longest cycle problem in solid grid graphs is in APX. More precisely, our algorithm finds a cycle of length at least $\frac{2n}{3} + 1$ in 2-connected $n$-node solid grid graphs.

Keywords: Longest cycle, Hamiltonian cycle, Approximation algorithm, Solid grid graph.

1 Introduction

The longest cycle and path problems are well-known NP-hard problems in graph theory. There are various results which show that these problems are hard to approximate in general graphs. For example, assuming that $P \neq NP$, it has been shown that there is no polynomial-time constant-factor approximation algorithm for the longest path problem and also it is not possible to find a path of length $n - n^v$ in polynomial-time in Hamiltonian graphs [14]. The Color coding technique introduced by Alon et al. [1] is one of the first approximation algorithms for these problems which can find paths and cycles of length $\log n$. Later, Björklund et al. introduced another technique with better approximation ratio, i.e. $O(n \log \log n)$, for finding long paths [3, 9]. To our knowledge, the result of Gabow [8], which can find a cycle or path of length $\exp(\Omega(\sqrt{\log l / \log \log l}))$ in graphs with the longest cycle of length $l$, is the best polynomial-time approximation algorithm for finding the longest cycles. The results also show that these problems are hard to approximate

∗Email: asgharian@aut.ac.ir
†Email: al_bagheri@aut.ac.ir
even in bounded-degree and Hamiltonian graphs \[6, 7\]. These problems are even harder to approximate in the case of directed graphs as showed in \[4\]. For more related results on approximation algorithms on general graphs see \[2, 10, 18\].

There are few classes of graphs in which the longest path or the longest cycle problems are polynomial \[5, 11, 12, 15, 16, 17\]. In the case of grid graphs, Itai et al. \[13\] showed that the Hamiltonian path and cycle problems are NP-complete. Grid graphs are vertex-induced subgraphs of the infinite integer grid \(G^\infty\). Later, Umans et al. showed that the Hamiltonicity of solid grid graphs, i.e. the grid graphs in which each internal face has length four, is decidable in polynomial time \[19\]. However, to our knowledge, there is no result on finding or approximating the longest cycle in this class of graphs, but there is only a \(\frac{5}{6}\)-approximation algorithm for finding the longest paths in grid graphs that have square-free cycle covers \[20\]. In this paper, we introduce a linear-time constant-factor approximation algorithm for the longest cycle problem in solid grid graphs. Our algorithm first finds a vertex-disjoint cycle set containing at least \(\frac{2n + 1}{3}\) of the vertices of a given 2-connected, \(n\)-vertex solid grid graph and then merge them into a single cycle.

We organized the paper as follows. In section 2, we present the terminology and some preliminary concepts. Our algorithm for finding the cycle set of the desired length is given in section 3, and in section 4 we show that these cycles can be merged into a single cycle of the same size. Finally, in section 5, we conclude the paper.

## 2 Preliminaries

In this section, we present the definitions which is used during the paper and the necessary concepts about solid grid graphs. Grid graphs are vertex-induced subgraphs of the infinite integer grid whose vertices are the integer coordinated points of the plane and there is an edge between any two vertices of unit distance. Let \(G\) be a solid grid graph, i.e. a grid graph that has no inner face of length more than four. We consider solid grid graphs as plane graphs, considering their natural embedding on the integer grid. The vertices of \(G\) adjacent to the outer face are called boundary vertices, and the set of boundary vertices of \(G\) form its boundary. The boundary of connected plane graph \(G\) should be a closed walk, i.e. a cycle in which vertices and edges may be repeated, which is called boundary walk (considering that a single vertex is a walk of length zero). We use \(|W|\) to refer to the number of (not necessarily distinct) edges of a closed walk \(W\). Each cut vertex of \(G\), i.e. a vertex of \(G\) that its removal makes \(G\) disconnected, is a repeated vertex in its boundary walk and vice versa, therefor, \(G\) is 2-connected, if and only if its boundary is a cycle. If \(G\) is not connected, its boundary should be a set of closed walks, i.e. the set of boundary walks of its connected components. Let cycle \(C\) be the boundary of \(G\). We say a vertex of boundary cycle \(C\) is convex vertex, flat vertex or concave vertex respectively, if its degree in \(G\) is two, three or four. The embedding of any cycle of the plane graph \(G\) is a simple rectilinear polygon, as a result, in each cycle of \(G\) the number of convex
vertices should be four more than the number of concave vertices. Also, note that, because solid grid graphs are vertex-induced, their boundary cycles can not contain two consecutive concave vertices. We define two edges of $G$ to be parallel edges if they are not incident to a common vertex, but both of them are adjacent to the same inner face. When $G'$ is a subgraph of $G$, we use the notation $G \setminus G'$ to denote the graph obtained from $G$ after removing all the vertices of $G'$ and their incident edges. It is easy to show that $G \setminus G'$ is also a solid grid graph, when $G'$ is the boundary of $G$, or it is a maximal 2-connected subgraph of $G$.

3 Finding the Cycle Set

Let $G$ be a 2-connected, $n$-node solid grid graph and $C$ be its boundary cycle. Given such a graph $G$, we present an algorithm that finds a set of vertex-disjoint cycles $S$ in $G$ containing at least $\frac{2n}{3} + 1$ of the vertices of $G$. In the next section, by merging these cycles, we construct a cycle of the desired length.

Let $S$ be initially empty. We add $C$ to $S$, and since the $G \setminus C$ may be not 2-connected, we repeat the process recursively on its 2-connected disjoint subgraphs. Let $\{G'_1, \ldots, G'_m\}$ be a maximal set of disjoint maximal 2-connected subgraphs of $G \setminus C$. The pseudocode of the procedure for constructing cycle set $S$ is given in algorithm 3.1

Algorithm 3.1 The algorithm of finding the cycle set $S$

```
procedure FindCycleSet(G)
1: if $|G| < 4$
2: return $\emptyset$
3: let $C$ be the boundary cycle of $G$
4: let $\{G'_1, \ldots, G'_m\}$ be the set of subgraphs of $G$ as defined above
5: $S \leftarrow \{C\}$
6: foreach $G'_i$, $1 \leq i \leq m$ do
7: if not $G'_i$ is a connected component of $G \setminus C$ having $|G'_i| = 4$
8: $S \leftarrow S \cup \text{FindCycleSet}(G'_i)$
9: return $S$
```

The line 7 of the algorithm, excludes some length four cycles from $S$, because these cycles may be unmergable in the next step of our algorithm. The following lemma shows that the sum of the lengths of the cycles in $S$ is at least $\frac{2n}{3} + 1$.

**Lemma 3.1.** The sum of the length of the cycles in $S$ is at least $\frac{2n}{3} + 1$.

To prove Lemma 3.1 we need two other auxiliary lemmas, so we defer it after the statements of Lemmas 3.2 and 3.3. Let $G_1, \ldots, G_k$ be the connected components of $G \setminus C$ and $W_1, \ldots, W_k$ be respectively their boundary walks. Also, for each connected component $G_i$, let $C_i$ be the cycle of $G$ that immediately encloses $G_i$. To be more precise, we can construct such cycles $C_1, C_2, \ldots$ and $C_k$ by preforming a set of split operations on $C$ as follows. For each pair of connected components $G_i$ and $G_j$, their should
be a pair of vertices \( u \) and \( v \) in \( G \) whose removal disconnects the vertices of \( G_i \) from the vertices of \( G_j \) in \( G \). Based on the fact that \( u \) and \( v \) are adjacent or not, we use the split operation shown respectively in figure (a) or (b) to split the cycle \( C \) into two cycles \( C' \) and \( C'' \). Note that, we need at most four new edges to construct \( C' \) and \( C'' \) from \( C \). We repeat the split operation recursively on \( C' \) and \( C'' \) until we obtain \( k \) cycles \( C_1, \ldots, C_k \) such that each cycle \( C_i \), \( 1 \leq i \leq k \), encloses only a connected component \( G_i \) of \( G \setminus C \). To construct the desired cycle set, only \( k - 1 \) split operations are required, and in each split operation we use at most four new edges. So, we should have the following equation:

\[
|C| + 4(k - 1) \geq \sum_{i=1}^{k} |C_i|
\]  

(1)

Constructing these cycles is not necessary for finding the final long cycle, however, we need these cycles in the proof of our lemmas. In Lemma 3.2 we show that the length of the boundary walk of \( G_i \) is less than the length of its enclosing cycle \( C_i \).

**Lemma 3.2.** Let \( G_i \) be a connected component of \( G \setminus C \) and \( W_i \) and \( C_i \) be respectively its boundary walk and enclosing cycle. Then we have

\[
|C_i| \geq |W_i| + 8.
\]

**Proof.** If \( G_i \) be a single vertex then \( |C_i| \) is at least eight and the lemma holds. Otherwise, let \( C_i \) and \( W_i \) be directed in clockwise order. For some
edges of $W_i$ there is a distinct parallel edge in $C_i$ (as an example see the
edges $e_1$ and $e'_1$ in Figure 2). Moreover, for each group of at most two
consecutive edges of $W_i$ which have no parallel edges in $C_i$, there is a
distinct concave vertex in $C_i$ (for example the edges $e_2$ and $e_3$ and the
vertex $u$ in Figure 2). Instead, for each convex vertex $v$ of $C_i$, the two
edges of $C_i$ incident to $v$ have no parallel edge in $W_i$. Therefore, knowing
that the number of convex vertices in $C_i$ is equal to the number of concave
vertices plus four, we have $|W_i| \leq |C_i| - 8$.

![Figure 2: An example cycle $C_i$ and walk $W_i$](image)

Each 2-connected subgraph $G'_i$, $1 \leq i \leq m$, in Algorithm 3.1 is a
subgraph of a connected component of $G \setminus C$. Thus, without loss of
generality, let $G'_1, \ldots, G'_h_i$ be subgraphs of $G_i$ for some $1 \leq i \leq h_i \leq m$.
Also, we call a vertex of $G$ a free vertex if it is not in any of cycles of $S$.

**Lemma 3.3.** For each connected component $G_i$ of $G \setminus C$, if $|G_i| > 4$ we have $|G_i| - \sum_{j=1}^{h_i} |G'_j| \leq \frac{|W_i|}{2} + 1$.

**Proof.** First, note that, because $G'_i$, $1 \leq j \leq m$, is maximal and 2-
connected, each vertex of $G_i$ which is not in any $G'_j$, $1 \leq j \leq m$, should be
on the boundary of $G_i$, i.e. $W_i$, and they should be free vertices. Therefore,
to prove the lemma, we show that at most $\frac{|W_i|}{2} + 1$ of the vertices of
$W_i$ are free vertices. Let $W_i$ contains $x$ duplicated edges (i.e. the edges
that are repeated in $W_i$ two times). Removing all the $x$ duplicated edges
domains $W_i$, including the resulting isolated vertices, will result a set of closed
walks $Y_i$. The length of $|Y_i|$, i.e. the sum of the lengths of its closed walks,
should be $|W_i| - 2x$. Also, the vertices that are in $W_i$ but not in $Y_i$ should
be free vertices, because they are adjacent only to the outer face. There
should be at most $x + 1$ such distinct vertices. In addition, we will show
that there is at most $\frac{|W_i|}{2}$ free vertices in $Y_i$. Thus, the total number of
free vertices of $W_i$ is not more than $\frac{|W_i|}{2} + x + 1$ which is equal to $\frac{|W_i|}{2} + 1$.

Each free vertex of $Y_i$ should be adjacent to an inner face $f$ of $G_i$
which is not in any $G'_j$, $1 \leq j \leq h_i$ (see Figure 3 as an example). Because
of the maximality of $G'_j$, $1 \leq j \leq m$, $f$ can not share any edge with
another inner face of $G_i$, so it should be adjacent to a cut vertex $v_c$ of
$G_i$. Also, $v_c$ should be adjacent to another face $f'$ of $G_i$, and because
of maximality of $G'_j$, $1 \leq j \leq m$, $f'$ can not contain free vertices. Hence, $v_c$
is a free vertex, and this ensures that $Y_i$ can not contain more than
three consecutive free vertices. Moreover, the fact that $f'$ can not contain
Figure 3: An example in which three consecutive vertices of $Y_i$ are free vertices. Dotted edges are edges of $Y_i$ and dark edges are edges of the cycles in $S$.

Proof of lemma 3.1. To prove the lemma, we first show that in each iteration of construction of $S$ in algorithm 3.1 the number of vertices left unused, i.e. the free vertices, is not more than $\frac{|C_i|-4}{2}$. Consider a connected component $G_i$ of $G \setminus C$. If $|G_i| \leq 4$, all the vertices of $G_i$ is free vertices, and one can easily check that $|G_i| < \frac{|C_i|-4}{2}$ holds. Otherwise, using lemma 3.3 the number of vertices left free on $G_i$ is less than $\frac{W_i}{2}$ which is less than $\frac{|C_i|-8}{2}$ by lemma 3.2. Next, summing the maximum number of vertices left free in each connected component of $G \setminus C$, the total number of vertices left free in a single step of the algorithm should be at most $\sum_{i=1}^{k} \frac{|C_i|-4}{2}$, which is not more than $\frac{|C|}{2}$ by Equation 1. Considering all the steps of the algorithm, the total number of vertices left free can not be more than $\frac{|S|}{2}$. Therefore, we have $n \leq |S| + \frac{|S|-4}{2}$ which shows that $|S| \geq \frac{2n}{3} + 1$.

4 Merging the Cycles

After finding the cycle set $S$, the next step of the algorithm is to merge all the cycles of $S$ into a single cycle. Except the boundary cycle of $G$, each cycle $C_{in} \in S$ is nested immediately inside a cycle $C_{out} \in S$ which is called its outer cycle. Also, $C_{in}$ is called an inner cycle of $C_{out}$. Our algorithm starts from the outermost cycle of $S$, and merge each cycle with its inner cycles using one of the merge operations which are shown in Figure 4. But, $S$ may contains some cycles that are not mergeable with their outer cycles using our merge operations. These cycles are diamond-shaped cycles. More precisely, a diamond-shaped cycle is a boundary cycle if it contains no flat vertex, and a solid grid graph which its boundary is a diamond-shaped cycle is called a diamond-shaped grid graph.

Lemma 4.1. Let $C_{in}$ be a cycle in $S$ and $C_{out}$ be its outer cycle. For each flat vertex $v$ in $C_{in}$, there is at least one distinct flat vertex in $C_{out}$.

Proof. Let $G_{in}$ and $G_{out}$ be respectively the grid graphs that $C_{in}$ and $C_{out}$ be their boundaries and $u$ be the vertex of $G$ outside of the cycle $C_{in}$ adjacent to $v$. See the upper parts of Figures 4(a), (b) and (c). If $u$ is not a free vertex, then it should be in $C_{out}$, and at least one of its two incident
edges in $C_{\text{out}}$ should be parallel to one of the edges of $C_{\text{in}}$ incident to $v$ (upper part of Figure 4 (a)). But if $u$ is a free vertex, considering this fact that $C_{\text{in}}$ and $C_{\text{out}}$ are boundary cycles of some solid grid graphs and $G_{\text{in}}$ is a maximal 2-connected subgraph of $G_{\text{out}} \setminus C_{\text{out}}$, the configuration of $C_{\text{in}}$ and $C_{\text{out}}$ around the vertices $u$ and $v$ must be isomorphic to one of the configurations which is depicted in the upper parts of Figures 4(b) and (c). Hence, in Figure 4(a) either $u$ or one of its two adjacent vertices in $C_{\text{out}}$, in Figure 4(b) one of the vertices $x$ and $y$, and in Figure 4(c) both of the vertices $x$ and $y$ are flat vertices of $C_{\text{out}}$. Thus, for each flat vertex of $C_{\text{in}}$ there is a flat vertex in $C_{\text{out}}$.

Let $C_i$ be a diamond-shaped cycle and $G_i$ be a solid grid graph which its boundary is $C_i$. If $G_i \setminus C_i$ be 2-connected, it should be diamond-shaped. Otherwise, by Lemma 4.1, $C_i$ should has a flat vertex. So, the diamond-shaped cycles in $S$ can be grouped into some groups of nested diamond-shaped cycles (for an example see figure 6). We have the following lemma about the innermost diamond-shaped cycles. Note that, the length-four cycle is the smallest diamond-shaped cycle.

**Lemma 4.2.** Let $C_d$ be an innermost diamond-shaped cycle in $S$ and $G_d$ be the solid grid graph whose boundary is $C_d$. Then, either $|C_d| = 4$ and the outer cycle of $C_d$ is not diamond-shaped or there is at least one free vertex in the boundary of $G_d \setminus C_d$.

**Proof.** First let $|C_d| = 4$, and the outer cycle $C_{\text{out}}$ of $C_d$ is diamond shaped, as depicted in 6(c), and $G_{\text{out}}$ be the solid grid graph which its boundary is $C_{\text{out}}$. In this case, $G_d$ is a connected component of $G_{\text{out}}$ which contradicts the line 7 of Algorithm 3.1. Therefore, $C_{\text{out}}$ is not diamond-shaped when $|C_d| = 4$. For the case that $|C_d| > 4$, because $C_d$ is an innermost cycle in $S$, either $G_d \setminus C_d$ is a set of isolated vertices, as shown.
Figure 5: Examples of diamond-shaped cycles. Only (a) and (c) can be innermost cycles in $S$.

Figure 6: Converting a group of nested diamond-shaped cycles into a group of non-diamond-shaped cycles using a free vertex in Figure 5(a), or it is a length-four cycle not in $S$, as depicted in Figure 5(b). Clearly, in both cases there should be a free vertex in the boundary of $G_d \setminus C_d$.

Using the free vertices that their existence proved in Lemma 4.2, we replace each group of nested diamond-shaped cycles in $S$, except the length-four diamond-shaped cycles, by a set of non-diamond-shaped cycles as depicted in Figure 6 and we name the resulting cycle set $S'$. So, $S'$ does not contain any diamond-shaped cycles, except the diamond-shaped cycles of length four. Lemma 4.3 insures that we can merge all the cycles of $S'$ starting from the innermost cycles.

**Lemma 4.3.** Using the merge operations of Figure 4, all the cycles of $S'$ can be merged into a single cycle $C$ containing a given boundary edge $e$ of $G$.

**Proof.** If $S'$ contains only one cycle, the lemma holds easily. Otherwise, let $C_{in}$ be a cycle of $S'$ and $C_{out}$ be its outer cycle, and let $G_{in}$ and $G_{out}$ be respectively the grid graphs that $C_{in}$ and $C_{out}$ are their boundary cycles.

For the case that $C_{in}$ is a non-diamond-shaped cycle, let $v$ be a flat vertex of $C_{in}$. As described in the proof of Lemma 4.1, there is only
three possible configurations for \(C_{in}\) and \(C_{out}\) around \(v\), as depicted in the upper parts of the Figures 4 (a), (b) and (c). In these cases, we can use respectively the merge operations depicted in Figures 4 (a), (b) and (c) to merge \(C_{in}\) and \(C_{out}\). Moreover, \(C_{in}\) should contain at least two flat vertices, because any cycle in a grid graph has even length and the number of convex vertices in \(C_{in}\) is four more than concave vertices. Therefore, starting from the outermost cycle and each time merging the cycle by one of its inner cycles one can merge all the non-diamond shaped cycles of \(S'\) into a single cycle \(C_{res}\). Note that, the outermost cycle of \(S'\) contains the edge \(e\), and it can be merged by each of its inner cycles using at least two different flat vertices. Therefor, we can chose the merge operations such that the cycle \(C_{res}\) contains the edge \(e\).

Considering Lemma 4.2, the only case that \(C_{in} \in S'\) is diamond-shaped, is when \(|C_{in}| = 4\). Let \(C_{in}\) be such a cycle and let it has no parallel edge with \(C_{res}\), otherwise they can be merged by the merge operation of Figure 4(a). In this case, by line 7 of Algorithm 3.1 and maximality of subgraphs \(G'_i\), there should be a free vertex \(v\) adjacent to one of the four vertices of \(C_{in}\). Because there is no parallel edges between \(C_{in}\) and \(C_{res}\), there are two possible configurations for \(C_{in}\) and \(C_{res}\) around the vertex \(v\). The two possible configurations are depicted in the upper parts of the Figure 4(d) and (e). Thus, based on the fact that vertex \(w\), in these figures, is a free vertex or not, \(C_{in}\) and \(C_{res}\) can be merged respectively using the merge operations of the Figures 4(d) and (e).

We conclude this section summarizing our result in the following theorem.

**Theorem 4.1.** There is a linear-time \(\frac{2}{3}\)-approximation algorithm for finding a longest cycle in solid grid graphs.

**Proof.** The desired approximation algorithm is as follows. Let \(G\) be a largest 2-connected subgraph of a given solid grid graph. First, construct the cycles set \(S\) using Algorithm 3.1, then convert its diamond-shaped cycles to non-diamond shape cycles and make \(S'\) as described before. Constructive proof of Lemma 4.3 gives a method for merging all the cycles of \(S'\), and Lemma 3.4 ensure that the constructed long cycle contains at least two third of the vertices of \(G\). We complete our proof arguing that the introduced approximation algorithm can be implemented in linear time. The boundary cycle \(C\) of \(G\) can be found in time \(|C|\), and by only checking the boundary vertices of \(G\), one can construct a maximal set of disjoint 2-connected components of \(G \setminus C\). Thus, the lines 3 and 4 of Algorithm 3.1 can be implemented in time \(O(|C|)\). Moreover, except the recursive calls, the other lines of the algorithm can be implemented in constant time. Therefore, Algorithms 3.1 can be implemented such that \(\text{FindCycleSet}(G)\) runs in time \(O(|S|)\). The other steps of our algorithm, i.e. constructing cycle set \(S'\) from \(S\), finding the flat vertices of cycles of \(S'\) and merging the cycles of \(S'\) can be implemented in linear time. Thus, the total running time of the algorithm is \(O(|S|)\) which is linear with respect to the size of \(G\).
5 Conclusions

We introduced a linear-time approximation algorithm that, given a 2-connected, $n$-node solid grid graph, can find a cycle containing at least two third of its vertices. Since, cycles are 2-connected, our algorithm is a constant-factor approximation for the longest cycle problem in solid grid graphs. In other words, if the given solid grid graph $G$ is not 2-connected, one can apply our algorithm to the largest 2-connected subgraph of $G$ to find a cycle of the length at least two third of the length of the longest cycle of $G$. A better approximation ratio for the longest cycle problem in solid grid graphs or the longest path problem in this class of graphs can be the subject of future work.

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