Different forms of Euler’s theorem for homogeneous functions to solve partial differential equation

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Abstract: In this paper we will discuss Euler’s theorem for homogenous functions to solve different order partial differential equations. We will see that how we can predict the solution of partial differential Equation using different approaches of this theorem. In fact we also consider the case when more than two independent variables will be involved in the partial differential equation whenever dependent functions will be homogenous functions. We will throw a light on one method called Ajayous rules to predict the solution of homogenous partial differential equation.

Keywords: Homogenous functions, Partial Differential equations, Ajayous Rules, order of partial differential equation

1. Introduction

Leonard Euler was one of the greatest mathematicians. His contribution is unforgettable. He has given so many powerful results of mathematics and one of those results is Euler’s Theorem for homogenous function. Engineering students used to study this theorem because Partial differential equation is one of the important fields of mathematics that every engineer should know. Euler’s theorem for homogenous function gives easy way to deal with specific type of partial differential equation which further deals with homogenous function. Using this result we can avoid cumbersome calculation and can get the final result in few steps only. In this paper will discuss extension and generalization of this theorem.

2. Literature review

[1] By definition any function z is said to be homogenous function of x, y with degree ‘n’ if \( z(tx, ty) = t^n z(x, y) \) where \( t > 0 \) and if \( z \) is such type of function then Euler’s theorem states that

\[ xz_x + yz_y = nz \]

This is a partial differential equation of first order

For second order partial differential equation

\[ x^2 z_{xx} + y^2 z_{yy} + 2xyz_{xy} = n(n-1)z \]

Second order result has been generated by using first order result of this theorem.

Similarly for third order results becomes

\[ x^3 \frac{\partial^3 z}{\partial x^3} + 3x^2y \frac{\partial^3 z}{\partial^2 x \partial y} + y^3 x \frac{\partial^3 z}{\partial^2 y \partial x} + y^3 \frac{\partial^3 z}{\partial y^3} = n(n-1)(n-2)z \]

We can expand this result for further higher order partial differential equation by using the lower partial differential equation.

[2] Now if \( z \) is homogenous function of \( x \) and \( y \) with degree \( n \) where every partial derivatives of order upto \( m \) exist and are continuous then which we will get is as follow

\[ x^m \frac{\partial^m z}{\partial x^m} + m x^{m-1} y \frac{\partial^m z}{\partial x^{m-1} \partial y} + \frac{1}{2} m(m-1)x^{m-2}y^2 \frac{\partial^m z}{\partial x^{m-2} \partial y^2} + \ldots + m c_k x^{m-k} y^k \frac{\partial^m z}{\partial x^{m-k} \partial y^k} + \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\]

\[ my^{m-1} \frac{\partial^m z}{\partial x \partial y^{m-1}} + y^m \frac{\partial^m z}{\partial y^m} = n(n-1)(n-2) \ldots \ldots \ldots \ldots (n-m+1)z, \text{ for } m \leq n \]

From above result we can easily conclude that if \( z \) is a homogenous function of \( x \) and \( y \) with degree \( n \) whose all partial derivatives exists for order upto ‘n’ are continuous then

\[ x^n \frac{\partial^n z}{\partial x^n} + nx^{n-1} y \frac{\partial^n z}{\partial x^{n-1} \partial y} + n c_k x^{n-k} y^k \frac{\partial^n z}{\partial x^{n-k} \partial y^k} + \ldots + n c_k x^{n-k} y^k \frac{\partial^n z}{\partial x^{n-k} \partial y^k} + \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\]

\[ my^{n-1} \frac{\partial^n z}{\partial x \partial y^{n-1}} + y^n \frac{\partial^n z}{\partial y^n} = n! z, \text{ for } n \in N \]

Next case would be when given function is composite one i.e. if \( z = f(u) \) is homogenous function with
degree $n$ in $x$ and $y$ then Euler’s theorem for homogenous function for first order will give this following result

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n f(u)$$

For second order if second order partial derivative exists and continuous then result is

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)H_1(u) \quad \text{where} \quad g(u) = n f(u) \quad \text{and} \quad H_1(u) = g'(u) - 1$$

Similarly for third order if partial derivative upto third order exists and continuous then result is

$$x^3 \frac{\partial^3 u}{\partial x^3} + 3 \left[ x^2 y \frac{\partial^3 u}{\partial x^2 \partial y} + y^2 x \frac{\partial^3 u}{\partial y^2 \partial x} + y^3 \frac{\partial^3 u}{\partial y^3} \right] = g(u)(g(u)H_1(u) + (g'(u) - 1)(g'(u) - 2))$$

$$= \frac{g(u)H_2(u)}{u}$$

Where $H_2(u) = g(u)H_1(u) + (g'(u) - 1)(g'(u) - 2)$

So in general if partial derivatives of $u$ exists and continuous upto $m$' order where $z = f(u)$ is homogenous function with degree $n$ in $x$ and $y$ then

$$x^m \frac{\partial^m u}{\partial x^m} + m x^{m-1} y \frac{\partial^m u}{\partial x^{m-1} \partial y} + \frac{1}{2} m(m - 1)x^{m-2} \frac{\partial^m u}{\partial x^{m-2} \partial y^2} + \cdots + m^c_k x^{m-k} y^k \frac{\partial^m u}{\partial x^{m-k} \partial y^k}$$

$$+ \cdots + m x^{m-1} y \frac{\partial^m u}{\partial x^{m-1} \partial y} + y^m \frac{\partial^m u}{\partial y^m} = g(u)H_{m-1}(u) \quad \text{for} \quad m \leq n$$

where $H_m = g(u)H_{m-1}(u) + (g'(u) - 1)(g'(u) - 2) \cdots + (g(u) - m), m > 1$

$$g(u) = n f(u) \quad \text{and} \quad H_1(u) = (g(u) - 1)$$

[3] Now next approach is Ajoyous rules
Here one symbol is used i.e. $u_{x^p y^q}$ $u$ is differentiated with $x$ and $y$ partially ‘$p$’ times and ‘$q$’ times respectively.

Now Euler’s theorem for homogenous function for higher ordered is as follows:

For second order it is

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n - 1)u$$

For third order it is

$$x^3 \frac{\partial^3 u}{\partial x^3} + y^3 \frac{\partial^3 u}{\partial y^3} + 2x^2 \frac{\partial^3 u}{\partial x^2 \partial y} + 2y^2 \frac{\partial^3 u}{\partial y^2 \partial x} + 3x^2 y \frac{\partial^3 u}{\partial x^2 \partial y^2} + 3xy^2 \frac{\partial^3 u}{\partial x \partial y^3} + 4x^2 y \frac{\partial^3 u}{\partial x^2 \partial y^2} = n^2(n - 1)u$$

For fourth order

$$x^4 \frac{\partial^4 u}{\partial x^4} + y^4 \frac{\partial^4 u}{\partial y^4} + 5x^3 \frac{\partial^4 u}{\partial x^3 \partial y} + 5y^3 \frac{\partial^4 u}{\partial x^3 \partial y} + 4x^3 \frac{\partial^4 u}{\partial x^3 \partial y} + 4y^3 \frac{\partial^4 u}{\partial x^3 \partial y} + 15x^2 y^2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + 15xy^2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + 4x^2 y^4 \frac{\partial^4 u}{\partial x^2 \partial y^2}$$

$$+ 4xy^3 \frac{\partial^4 u}{\partial x^2 \partial y^2} + 6x^2 y^3 \frac{\partial^4 u}{\partial x^2 \partial y^2} + 8xy \frac{\partial^4 u}{\partial x \partial y^2} = n^3(n - 1)u$$

For fifth order

$$x^5 \frac{\partial^5 u}{\partial x^5} + y^5 \frac{\partial^5 u}{\partial y^5} + 9x^4 \frac{\partial^5 u}{\partial x^4 \partial y} + 9y^4 \frac{\partial^5 u}{\partial x^4 \partial y} + 19x^3 \frac{\partial^5 u}{\partial x^3 \partial y^2} + 19y^3 \frac{\partial^5 u}{\partial x^3 \partial y^2} + 8x^2 y^3 \frac{\partial^5 u}{\partial x^2 \partial y^3} + 8y^2 x^3 \frac{\partial^5 u}{\partial x^2 \partial y^3} + 10x^3 y^2 \frac{\partial^5 u}{\partial x^3 \partial y^2} + 36x^2 y^3 \frac{\partial^5 u}{\partial x^2 \partial y^3} + 36xy^3 \frac{\partial^5 u}{\partial x \partial y^3} + 57x^2 y^3 \frac{\partial^5 u}{\partial x^2 \partial y^3} + 57xy^2 \frac{\partial^5 u}{\partial x^2 \partial y^3}$$

$$+ 5xy^4 \frac{\partial^5 u}{\partial x \partial y^4} + 5x^4 y \frac{\partial^5 u}{\partial x^4 \partial y} + 54x^2 y^2 \frac{\partial^5 u}{\partial x^2 \partial y^2} + 16xy^3 \frac{\partial^5 u}{\partial x \partial y^3} = n^4(n - 1)u$$

For sixth order

$$x^6 \frac{\partial^6 u}{\partial x^6} + y^6 \frac{\partial^6 u}{\partial y^6} + 14x^5 \frac{\partial^6 u}{\partial x^5 \partial y} + 14y^5 \frac{\partial^6 u}{\partial x^5 \partial y} + 55x^4 \frac{\partial^6 u}{\partial x^4 \partial y^2} + 55y^4 \frac{\partial^6 u}{\partial x^4 \partial y^2} + 65x^3 \frac{\partial^6 u}{\partial x^3 \partial y^3} + 65y^3 \frac{\partial^6 u}{\partial x^3 \partial y^3} + 16x^2 \frac{\partial^6 u}{\partial x^2 \partial y^4}$$

$$+ 16xy^3 \frac{\partial^6 u}{\partial x \partial y^4} + 20y^3 \frac{\partial^6 u}{\partial x \partial y^4} + 140x^2 y^3 \frac{\partial^6 u}{\partial x^2 \partial y^3} + 140x^2 y^3 \frac{\partial^6 u}{\partial x^2 \partial y^3} + 239xy^3 \frac{\partial^6 u}{\partial x \partial y^3} + 239xy^3 \frac{\partial^6 u}{\partial x \partial y^3}$$

$$+ 15x^2 y^2 \frac{\partial^6 u}{\partial x^2 \partial y^2} + 15x^2 y^2 \frac{\partial^6 u}{\partial x^2 \partial y^2} + 70xy^4 \frac{\partial^6 u}{\partial x \partial y^4} + 70xy^4 \frac{\partial^6 u}{\partial x \partial y^4} + 70xy^4 \frac{\partial^6 u}{\partial x \partial y^4}$$

$$+ 6xy^5 \frac{\partial^6 u}{\partial x \partial y^5} + 6x^2 y^5 \frac{\partial^6 u}{\partial x^2 \partial y^5} + 195xy^2 \frac{\partial^6 u}{\partial x \partial y^2} + 195xy^2 \frac{\partial^6 u}{\partial x \partial y^2} + 330x^2 y^2 \frac{\partial^6 u}{\partial x^2 \partial y^2}$$

$$+ 32xy \frac{\partial^6 u}{\partial x \partial y^2} = n^5(n - 1)u$$
Now question is how to generalize this. For that we will use Ajayous rules. These rules are applicable only for order \( N \geq 2 \) and total number of rules are 5

**Rule 1:**

The \( N \)th order partial equation of Euler’s theorem is the form

\[
\sum a_i x^p y^q u_{x^p y^q} = n^{N-1}(n-1)u
\]

Such that, \( 2 \leq p + q \leq N \), \( 0 \leq p \leq N \), \( 0 \leq q \leq N \) and number \( j \) varies from 1 to the total number of terms in the equation.

Also, \( N \geq 2 \), where \( p, q \in W \)

Now here is a table:

| Rule number | Corresponding to the terms | Value of \( a_j \) constant coefficients |
|-------------|----------------------------|------------------------------------------|
| 2           | \( x^2 u_{x^2} + y^2 u_{y^2} \) | \( 2^{N-2} \)                           |
| 3           | \( xy u_{xy} \)               | \( 2^{N-1} \)                           |
| 4           | \( x^3 u_{x^3} + y^3 u_{y^3} \) | \( 1 - \frac{31}{4}(N-3) + \frac{181}{12}(N-3)^2 - \frac{47}{6}(N-3)^3 + \frac{23}{12}(N-3)^4 \) |

**Rule no. 5**

For other remaining terms in order to find the value of \( a_j \) we need to compare expansion of \((x + y)^N\) with the left hand side of partial differential equations and then equate those terms which are similar.

We can verify these rules and compare with the above equation which has already been given for \( N = 1 \) to 6

**For 'm' Independent variables:**

Now if \( u \) is a homogenous function of \( x_1, x_2, x_3, \ldots, \ldots, x_m \) variables with degree \( n \) then Euler’s theorem gives first order partial differential equation as follows

\[
x_1 \frac{du}{dx_1} + x_2 \frac{du}{dx_2} + x_3 \frac{du}{dx_3} + \ldots + x_m \frac{du}{dx_m} = nu
\]

Where \( u = u(x_1, x_2, x_3, \ldots \ldots \ldots \ldots x_m) \)

**For higher order \( N=2 \) results are given as**

\[
x_1^2 \frac{\partial^2 u}{\partial x_1^2} + x_2^2 \frac{\partial^2 u}{\partial x_2^2} + x_3^2 \frac{\partial^2 u}{\partial x_3^2} + \ldots + x_m^2 \frac{\partial^2 u}{\partial x_m^2} + 2x_1x_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} + 2x_1x_3 \frac{\partial^2 u}{\partial x_1 \partial x_3} + 2x_1x_4 \frac{\partial^2 u}{\partial x_1 \partial x_4}  \\
+ \ldots + 2x_1x_m \frac{\partial^2 u}{\partial x_1 \partial x_m} + x_2x_3 \frac{\partial^2 u}{\partial x_2 \partial x_3} + 2x_2x_4 \frac{\partial^2 u}{\partial x_2 \partial x_4} + 2x_2x_5 \frac{\partial^2 u}{\partial x_2 \partial x_5}  \\
+ \ldots + 2x_2x_m \frac{\partial^2 u}{\partial x_2 \partial x_m} + x_3x_4 \frac{\partial^2 u}{\partial x_3 \partial x_4} + 2x_3x_5 \frac{\partial^2 u}{\partial x_3 \partial x_5} + 2x_3x_6 \frac{\partial^2 u}{\partial x_3 \partial x_6}  \\
+ \ldots + 2x_3x_m \frac{\partial^2 u}{\partial x_3 \partial x_m} + \ldots + 2x_m-1x_m \frac{\partial^2 u}{\partial x_{m-1} \partial x_m} = (n-1)u
\]

Or

\[
\sum_{i=1}^{m} x_i^2 \frac{\partial^2 u}{\partial x_i^2} + 2x_1 \sum_{i=2}^{m} x_i \frac{\partial^2 u}{\partial x_1 \partial x_i} + 2x_2 \sum_{i=3}^{m} x_i \frac{\partial^2 u}{\partial x_2 \partial x_i} + \ldots + 2x_m-1x_m \frac{\partial^2 u}{\partial x_{m-1} \partial x_m} = (n-1)u
\]

**For \( N=2 \) \( m=2 \) it becomes**

\[
\sum_{i=1}^{2} x_i^2 \frac{\partial^2 u}{\partial x_i^2} + 2x_1 \sum_{i=2}^{2} x_i \frac{\partial^2 u}{\partial x_1 \partial x_i} = n(n-1)u
\]

\[
x_1^2 \frac{\partial^2 u}{\partial x_1^2} + x_2^2 \frac{\partial^2 u}{\partial x_2^2} + 2x_1x_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} = n(n-1)u
\]

**For \( N=2, m=3 \)**
Euler’s theorem for homogenous function is a very important result which makes the way easier to solve the partial differential equations which deal with specific type of functions called homogenous function. We have discussed two approaches here first one was direct and second one was Ajayous rules. But one limitation of Ajayous rule is that it is applicable on partial differential equation of order two or more. We have studied such type of partial differential equation makes the way easier to solve the

Here summation is taken in such a manner that no term is of the form \( x^p \) such that \( p = q \)

We can verify this result easily

3. Conclusion

Euler’s theorem for homogenous function is a very important result which makes the way easier to solve the partial differential equations which deal with specific type of functions called homogenous function. We have discussed two approaches here first one was direct and second one was Ajayous rules. But one limitation of Ajayous rule is that it is applicable on partial differential equation of order two or more. We have studied generalization of this result for higher order and with \( m \) independent variables. This is a very useful tool to solve so many engineering problem involving such type of partial differential equation.

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