A $q$-deformed Uncertainty Relation

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Abstract

Within the formulation of a $q$-deformed Quantum Mechanics a qualitative undercut of the $q$-deformed uncertainty relation from the Heisenberg uncertainty relation is revealed. When $q$ is some fixed value not equal to one, recovering of ordinary quantum mechanics and the corresponding recovering condition are discussed.

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The Heisenberg uncertainty relation is a direct result of the Heisenberg commutation relation (Heisenberg algebra). According to the present tests of quantum electrodynamics quantum theories based on the Heisenberg algebra are correct at least down to $10^{-17}$ cm. A question arises whether there is a possible modification of the Heisenberg algebra at short distances much smaller than $10^{-17}$ cm. In search for such possibilities at short distances (or high energy scales) consideration of the space structure is a useful guide. Recently as a possible candidate of short distance new physics a $q$-deformed quantum mechanics is proposed in the framework quantum group. Quantum groups are a generalization of symmetry groups which have been successfully used in physics. A general feature of spaces carrying a quantum group structure is that they are noncommutative and inherit a well-defined mathematical structure from the quantum group symmetries. In applications in physics questions arise whether the structure can be used for physics at short distances and what phenomena could be linked to it. Starting from such a noncommutative space as configuration space a generalisation to a phase space is obtained. Such noncommutative phase space is a $q$-deformation of the quantum mechanical phase space and thus all the machinery used in quantum mechanics can be applied in $q$-deformed quantum mechanics. A $q$-deformed Heisenberg algebra, as a generalization of Heisenberg’s algebra, is established in $q$-deformation phase space.

Starting from the $q$-deformed Heisenberg algebra Ref. obtained a $q$-deformed uncertainty relation and found that the Heisenberg uncertainty relation is undercut. This is a qualitative deviation from the Heisenberg uncertainty relation. It therefore raises the question when $q$ is some fixed value not equal to one where or not the Heisenberg uncertainty relation is recovered and what is the corresponding condition. In this letter we investigate the above important open question in the framework of the $q$-deformed harmonic oscillator proposed in Ref. We find a further qualitative deviation of the $q$-deformed uncertainty relation from the Heisenberg uncertainty relation.

The $q$-deformed harmonic oscillators were first studied by Macfarlane and Biedenharn. Ref. find a general ansatz of the creation and annihilation operators in terms of the $q$-deformed phase space variables, the position operator $X$, momentum operator $P$,
and scaling operator $U$ which satisfy the $q$-deformed Heisenberg algebra

$$q^{1/2}XP - q^{-1/2}PX = i\hbar U, \quad UX = q^{-1}XU, \quad UP = qPU,$$

where $X$ and $P$ are hermitian and $U$ is unitary:

$$X^\dagger = X, \quad P^\dagger = P, \quad U^\dagger = U^{-1}.$$ (2)

In (1) the parameter $q$ is real and $q > 1$. The operator $U$ closely relates to properties of dynamics and plays an essential role in $q$-deformed quantum mechanics. The definition of the algebra (1) is based on the definition of the hermitian momentum operator $P$. However, if $X$ is assumed to be a hermitian operator in a Hilbert space the usual quantization rule $P \rightarrow -i\partial_X$ does not yield a hermitian momentum operator. Ref. [1] showed that a hermitian momentum operator $P$ is related to $\partial_X$ and $X$ in a nonlinear way by introducing a scaling operator $U$

$$U^{-1} \equiv q^{1/2}[1 + (q - 1)X\partial_X], \quad \bar{\partial}_X \equiv -q^{-1/2}U\partial_X, \quad P \equiv -\frac{i}{2}(\partial_X - \bar{\partial}_X),$$

Where $\bar{\partial}_X$ is the conjugate of $\partial_X$. From (1) and (2) it follows that the $q$-deformed commutation relation is

$$XP - PX = i\hbar \frac{U + U^{-1}}{q^{1/2} + q^{-1/2}},$$ (3)

which yields a $q$-deformed uncertainty relation and shows an undercut of Heisenberg’s minimal uncertainty relation. Because of the complicated relations among $X$, $P$ and $U$, from the above equation it is not clear when $q$ is some fixed value not equal to one whether Heisenberg’s uncertainty relation can be recovered. We now investigate this question in an equivalent framework of algebra (1), the $q$-deformed harmonic oscillator proposed in Ref. [4].

The expression for annihilation and creation operators $a$ and $a^\dagger$ in terms of $X$, $P$, and $U$ are

$$a = \alpha U^{-2M} + \beta U^{-M}P, \quad a^\dagger = \bar{\alpha} U^{2M} + \bar{\beta} P U^M,$$ (4)

where $M = 0, 1, 2, \ldots$, $\alpha$ and $\beta$ are complex numbers. From the ansatz (1) it follows that $a$ and $a^\dagger$ satisfy the following algebra:

$$aa^\dagger - q^{-2M}a^\dagger a = 1$$ (5)
with the condition (up to a phase of $\alpha$)

$$\alpha = \frac{e^{i\phi}}{(1 - q^{-2M})^{1/2}}.$$  \hfill (6)

The ansatz (4) is determined by the requirement of the equivalence of algebras (1) and (5). The operators $\hat{a}$ and $\hat{a}^\dagger$ are related to the operator $X$ in a complicated way (In (4) $X$ is nonlinearly included in the operator $U$.)

The $q$-deformed phase space variables $X$, $P$ and the scaling operator $U$ can be expressed in terms of the usual canonical variables $\hat{x}$ and $\hat{p}$ as follows [1]:

$$X = \frac{\hat{z} + \frac{1}{2}}{\hat{z} + \frac{1}{2}} \hat{x}, \quad P = \hat{p}, \quad U = q^\hat{z},$$  \hfill (7)

where $\hat{z} = -i(\hat{x}\hat{p} + \hat{p}\hat{x})/2\hbar$, $[A] = (q^A - q^{-A})/(q - q^{-1})$, and $\hat{x}$ and $\hat{p}$ satisfy

$$[\hat{x}, \hat{p}] = i\hbar, \quad \hat{x}^\dagger = \hat{x}, \quad \hat{p}^\dagger = \hat{p}.$$ \hfill (8)

From (7) and (8) it follows that $X$, $P$ and $U$ satisfy (1) and (2). The algebra (8) is realized as follows:

$$\hat{x} = x, \quad \hat{p} = p + \frac{\gamma}{\sqrt{1 - q^{-2M}}}, \quad p = -i\hbar \partial_x,$$ \hfill (9)

where $\gamma$ is a real constant. Let $q = e^f$, ($0 < f \ll 1$). In the limit $f \to 0$, there are two singular terms in the expression of $a$ in (4). The condition of cancellation of two singular terms is

$$\beta \gamma = -e^{i\phi},$$ \hfill (10)

which leads to $\alpha \beta = \bar{\alpha} \bar{\beta}$. In the limit $f \to 0$, from (4), (6), (7), (9) and (10) it follows that

$$a \to e^{i\phi} \left(\frac{1}{2} i \gamma x - \frac{1}{\gamma} p\right) = a_0.$$ \hfill (11)

The usual expression $a_0 = \sqrt{m\omega/2\hbar x} + i \sqrt{1/2m\omega} \hbar p$ is recovered if we choose

$$e^{i\phi} = \mp i, \quad \gamma = \pm \sqrt{2m\omega}, \quad \beta = \frac{i}{\sqrt{2m\omega}}.$$ \hfill (12)

The $q$-deformed Hamiltonian is

$$H_\omega = \hbar \omega a^\dagger a.$$ \hfill (13)
In the limit $f \to 0$, (13) reduces to the undeformed one $H_\omega \to H_0 = \hbar \omega a_0^\dagger a_0$.

In order to reveal the possible deviation of the $q$-deformed uncertainty relation yielded by the algebra (11), or the equivalent algebra (13), from Heisenberg’s uncertainty relation, we decompose $a$ and $a^\dagger$ into a pair of quadrature operators $Q$ and $K$:

$$a = \frac{Q}{2D_1} + i \frac{K}{2D_2}, \quad a^\dagger = \frac{Q}{2D_1} - i \frac{K}{2D_2}. \quad (14)$$

Where $Q$ and $K$ are hermitian operators, $D_1$ and $D_2$ are positive real parameters and have, respectively, the dimensions of the position and momentum. Introducing (14) into (5), we obtain an equivalent $q$-deformed commutation relation of the algebra (1):

$$QK - KQ = i \bar{\hbar} - i \bar{\hbar} (1 - q^{-2M}) \left( \frac{Q^2}{4D_1^2} + \frac{K^2}{4D_2^2} \right), \quad (15)$$

with a condition

$$\frac{4D_1 D_2}{1 + q^{-2M}} = \hbar. \quad (16)$$

From (15) we now study minimal uncertainties in the position and momentum. We start with (8),

$$|[(Q - \bar{Q}) + i\eta(K - \bar{K})]|^2 \geq 0, \quad (17)$$

where $\bar{F} = \langle i | F | i \rangle$. For any real $\eta$, (15) and (17) yield

$$(\Delta K)^2 \left[ \eta - \frac{\hbar A}{2(\Delta K)^2} \right]^2 - \frac{\hbar^2 A^2}{4(\Delta K)^2} + (\Delta Q)^2 \geq 0, \quad (18)$$

where

$$A = 1 - (1 - q^{-2M}) \left[ \frac{(\Delta Q)^2 + (\bar{Q})^2}{4D_1^2} + \frac{(\Delta K)^2 + (\bar{K})^2}{4D_2^2} \right], \quad (19)$$

Choosing $\eta = \hbar A / 2(\Delta K)^2$, (18) yields the following uncertainty relation:

$$\Delta Q \Delta K \geq \frac{\hbar}{2} - \frac{\hbar}{2} (1 - q^{-2M}) \left[ \frac{(\Delta Q)^2 + (\bar{Q})^2}{4D_1^2} + \frac{(\Delta K)^2 + (\bar{K})^2}{4D_2^2} \right], \quad (20)$$

Because of $1 - q^{-2M} > 0$, (20) shows that the Heisenberg minimal uncertainty relation $\Delta Q \Delta K = \frac{\hbar}{2}$ can be undercut (7).

Defining

$$f(\Delta Q, \Delta K) = \Delta Q \Delta K - \frac{\hbar}{2} \left[ 1 - (1 - q^{-2M}) \left( \frac{(\Delta Q)^2 + (\bar{Q})^2}{4D_1^2} + \frac{(\Delta K)^2 + (\bar{K})^2}{4D_2^2} \right) \right], \quad (21)$$
conditions of \((\Delta K)_{\text{min}}\) (or \((\Delta Q)_{\text{min}}\))
\[ \frac{\partial}{\partial \Delta Q} f(\Delta Q, \Delta K) = 0 \quad \text{or} \quad \frac{\partial}{\partial \Delta K} f(\Delta Q, \Delta K) = 0, \quad f(\Delta Q, \Delta K) = 0 \]
yield
\[ \left[ \frac{\hbar(1 - q^{-2M})}{4D_1^2} \right] \Delta Q + \Delta K = 0 \quad \text{(22)} \]
or
\[ \Delta Q + \left[ \frac{\hbar(1 - q^{-2M})}{4D_2^2} \right] \Delta K = 0. \quad \text{(23)} \]
(22) (or (23)) shows that the only non-negative solution is
\[ (\Delta Q)_{\text{min}} = (\Delta K)_{\text{min}} = 0 \quad \text{(24)} \]
and \(f((\Delta Q)_{\text{min}}, (\Delta K)_{\text{min}}) = 0\) yields
\[ \frac{(\bar{Q})^2}{D_1^2} + \frac{(\bar{K})^2}{D_2^2} = \frac{4}{1 - q^{-2M}}. \quad \text{(25)} \]

In the limit \(q \to 1\), from (11), (12), (14) and (16) it follows that
\[ D_1 = d_1(q) \sqrt{\hbar/2m\omega}, \quad D_2 = d_2(q) \sqrt{m\omega \hbar/2}, \quad d_1(q)d_2(q) = \frac{1}{2}(1 + q^{-2M}), \quad \text{(26)} \]
where \(d_1\) and \(d_2\) are dimensionless and satisfy the limiting conditions
\[ d_1(q)_{q \to 1} \to 1, \quad d_2(q)_{q \to 1} \to 1. \quad \text{(27)} \]

When \(q \to 1\), Eq. (20) reduces to the Heisenberg uncertainty relation. But when \(q\) is some fixed value not equal to one, (24) shows that there are states when the condition (25) is met \(\Delta Q\) and \(\Delta K\) can simultaneously equal to zero. This is a qualitative deviation from the Heisenberg uncertainty relation. It therefore raises a question whether or not the Heisenberg uncertainty relation is recovered for fixed \(q\) and what is the recovering condition. In fact, (15) shows that when \(||Q^2||\) and \(||K^2||\) satisfy (||A|| is the norm of \(A\))
\[ 0 \leq ||Q^2|| \leq \frac{4D_1^2}{1 - q^{-2M}}, \quad 0 \leq ||K^2|| \leq \frac{4D_2^2}{1 - q^{-2M}} \quad \text{(28)} \]
the ordinary quantum mechanical behavior is approximately reproduced.

The canonical conjugate pair of operators \(Q\) and \(K\) defined in (14) are not the position and momentum. But from (11), (12), (14), (26) and (28) it follows that in the limit \(q \to 1\),
\( Q \) and \( K \) approach, respectively, the undeformed position operator \( x \) and momentum operator \( p \).

The correct Hamiltonian of the \( q \)-deformed harmonic oscillators is \( H_\omega \) defined in (13). But the Hamiltonian \( H_{Q,K} = \frac{1}{2m}K^2 + \frac{1}{2}m\omega^2Q^2 \) which is extensively considered in literature is not the correct one, though in the limit \( q \rightarrow 1 \) it approaches to \( H_\omega + \frac{1}{2} \). Ref. [11] also noticed the differences between \( H_\omega \) and \( H_{Q,K} \) (Ref. [11] simply took \( d_1(q) = d_2(q) = 1 \)) that \( H_\omega \) does possess conventional physical properties but \( H_{Q,K} \) probably does not permit a consistent physical interpretation.

In order to further confirm the existence of the qualitative deviation from the Heisenberg uncertainty relation, we investigate the coherent states of electromagnetic fields by a different method. For a single mode field of frequency \( \omega \) the electric field operator \( E(t) \) is represented as \( E(t) = E_0[\hat{a}\exp(-i\omega t) + \hat{a}^\dagger\exp(i\omega t)] \) where \( \hat{a} \) and \( \hat{a}^\dagger \) are the photon annihilation and creation operators. \( \hat{a} \) and \( \hat{a}^\dagger \) can be decomposed into a pair of dimensionless conjugate quadrature operators \( X_e \) and \( Y_e \):

\[
\hat{a} = \frac{1}{2}(X_e + iY_e), \quad \hat{a}^\dagger = \frac{1}{2}(X_e - iY_e)
\]

where \( X_e \) and \( Y_e \) are hermitean. In terms of \( X_e \) and \( Y_e \) the operator \( E(t) \) is then expressed as \( E(t) = E_0(X_e \cos \omega t + Y_e \sin \omega t) \).

Suppose \( \hat{a} \) and \( \hat{a}^\dagger \) satisfy the algebra (5). From (28) and (5) it follows that

\[
X_eY_e - Y_eX_e = iC
\]

with

\[
C = 2 - 2(1 - q^{-2M})\hat{a}^\dagger \hat{a}.
\]

We now prove that for any state

\[
0 \leq \langle C \rangle \leq 2.
\]

Suppose the existence of a ground \(|0\rangle \) satisfying \( \hat{a}|0\rangle = 0 \) and \( \langle 0|0 \rangle = 1 \). Using the algebra (5) we obtain [4]: \(|n\rangle = \epsilon_n^{1/2}(\hat{a}^\dagger)^n|0\rangle \), and \( \hat{a}^\dagger \hat{a}|n\rangle = \epsilon_n|n\rangle \), \((n = 0, 1, 2, \cdots)\) with \( \epsilon_n = (1 - q^{-2nM})/(1 - q^{-2M}) \). We notice that in the limit \( q \rightarrow 1 \), we have \( \epsilon_n = n \). Because
\( \hat{a}^\dagger \hat{a} \) is hermitean, we may suppose that its eigen states are complete, and any states \(| \rangle \) may be expanded as \(| \rangle = \sum_{n=0}^{\infty} C_n |n\rangle \) with \( \sum_{n=0}^{\infty} |C_n|^2 = 1 \). Thus in any states we have

\[
\langle \hat{a}^\dagger \hat{a} \rangle = \sum_{n=0}^{\infty} |C_n|^2 \epsilon_n = (1 - q^{-2M})^{-1} \left( 1 - \sum_{n=0}^{\infty} |C_n|^2 q^{-2nM} \right)
\]

and \( \langle C \rangle = 2 \sum_{n=0}^{\infty} |C_n|^2 q^{-2nM} \). Thus \( 0 \leq \langle C \rangle \leq 2 \sum_{n=0}^{\infty} |C_n|^2 = 2 \). From (30) to (32) we conclude that Heisenberg’s minimal uncertainty relation \( \Delta X_e \Delta Y_e = 1 \) is undercut. We now investigate the \( q \)-deformed coherent states which were investigated in Refs. [12, 13]. A coherent state \( |\beta\rangle \) satisfies \( \hat{a} |\beta\rangle = |\beta\rangle \). From (5) and (29) it follows that

\[
(\Delta X_e)^2 = (\Delta Y_e)^2 = 1 - (1 - q^{-2M})|\beta|^2 \geq 0.
\]  

(33)

A similar result was obtained in Ref. [14]. Specially, we notice that when

\[
|\beta|^2 \to \frac{1}{1 - q^{-2M}}.
\]  

(34)

yield

\[
(\Delta X_e)_{\text{min}} = (\Delta Y_e)_{\text{min}} \to 0.
\]  

(35)

For the coherent state \( |\beta\rangle \), from (29) it yields \( (\bar{X}_e)^2 + (\bar{Y}_e)^2 = 4|\beta|^2 \). Thus the condition (34) is just the condition (25).

(20) and (30)- (32) show that within the formulation of the algebra (11) or equalent algebra (5) the Heisenberg uncertainty relation is undercut. Specially, (24) and (35) show a qualitative deviation from the Heisenberg uncertainty relation, that is, there are some special states, which permit simultaneous zero minimal uncertainties in a pair of conjugate variables. In a sense (20) may be called 'semi-uncertainty relation'. An example is \( q \)-deformed coherent states of electromagnetic fields. In (33) \( (\Delta X_e)^2, (\Delta Y_e)^2 \) and \( |\beta|^2 \) are quantities which can be measured by experiments. In principle, (33) could test the deviation from the Heisenberg uncertainty relation, and when the condition (34) is met it should yield simultaneous zero uncertainties in two quadratures of electric fields. Of course, any attempt to test the possible \( q \)-deformed effects is challenge, because if \( q \)-deformed quantum mechanics is a correct theory at short distances, its corrections to the present-day physics must be extremely small.
We conclude this paper by clarifying the following interesting questions.

(I) Why the 'undercutting' occurs? In order to track down the origin of this undercutting phenomena a deep understanding of the $q$-deformed commutation relation (3) which is the manifestation of the non-commutativity structure is necessary. We demonstrate that the expectation value of the operator $C_q = (U + U^{-1})(q^{1/2} + q^{-1/2})^2$ in (3) satisfies $\langle |C_q| \rangle \leq 1$.

We notice that $U + U^{-1}$ is hermitean, but $U - U^{-1}$ is anti-hermitean, for any state $\langle (U + U^{-1})^2 \rangle \geq 0, \langle (U - U^{-1})^2 \rangle \leq 0$. Thus $\langle (U + U^{-1})^2 \rangle - 4 = \langle (U - U^{-1})^2 \rangle \leq 0$, and $(\langle U + U^{-1} \rangle)^2 \leq \langle (U + U^{-1})^2 \rangle \leq 4$, i.e. $\langle |U + U^{-1}| \rangle \leq 2$. Because $q^{1/2} + q^{-1/2} \geq 2$ for any $q > 0$, we obtain $|\langle C_q \rangle| = |\langle U + U^{-1} \rangle|/(q^{1/2} + q^{-1/2}) \leq 1$. Eq. (3) gives a $q$-deformed uncertainty relation $\Delta X \Delta P \geq \hbar/2 |\langle C_q \rangle|$ which shows that the Heisenberg minimal uncertainty relation $\Delta X \Delta P = \hbar/2$ can be undercut.

(II) Regarding (20) and (35), what are the number states $|n\rangle$ which in the expansion of the coherent states $|\beta\rangle$ or the latter, or the states $|i\rangle$ for the former, give the undercutting? In order to answer this question, for the number states $|n\rangle$ we calculate $(\Delta X_n)^2 = (\Delta Y_n)^2 = 1/4 + (1 + q^{-2M})\epsilon_n/4$ which shows that the number states do not give the undercutting. When $q \rightarrow 1$, we have $\epsilon_n \rightarrow n$, and $(\Delta X_n)^2$ and $(\Delta Y_n)^2$ reduce to the undeformed ones.

(III) What representation of $q$-oscillator algebras lead to the undercutting of Heisenberg’s uncertainty relation or to the opposite case? In order to clarify this matter we investigate $q$-deformed algebras considered in Ref. [11] $aa^\dagger - q^{\pm 1/2}a^\dagger a = q^{\mp N/2}$, where $N = a^\dagger a$ is the number operator. These algebras involve $N$ explicitly, and thus are not natural realization of them. Defining $b_1 = aq^{-(N-1)/4}, b_1^\dagger = q^{-(N-1)/4}a^\dagger$ and $b_2 = aq^{(N-1)/4}, b_2^\dagger = q^{(N-1)/4}a^\dagger$, respectively, for the algebra with $q^{-N/2}$ and $q^{N/2}$, the above algebras with $q^{-N/2}$ and $q^{N/2}$ reduce, respectively, to $b_1b_1^\dagger - q^{-1}b_1^\dagger b_1 = 1$ and $b_2b_2^\dagger - q^{2}b_2^\dagger b_2 = 1$. The former is just the algebra (3) with $M = 1$ which leads to undercutting of Heisenberg’s uncertainty relation. The later, introducing operators $X_2$ and $Y_2$ by $b_2 = 1/2(X_2 + iY_2), b_2^\dagger = 1/2(X_2 - iY_2)$, reduces to $X_2Y_2 - Y_2X_2 = iC_2$ with $C_2 = 2 + 2(q - 1)b_2^\dagger b_2$. By a similar procedure of proving (32) we can prove that for any states $\langle C_2 \rangle \geq 2$ (The equal sign holds only for the case $q = 1$.) Thus the later leads to the opposite case.
Acknowledgement The author would like to thank Prof. J. Wess very much for his many stimulating helpful discussions, to thank Prof. H. A. Kastrup for many helpful comments. He would also like to thank Institut für Theoretische Physik, RWTH-Aachen, for warm hospitality. His work has been supported by the Deutsche Forschungsgemeinschaft (Germany), the National Natural Science Foundation of China under the Grant No. 19674014, and the Shanghai Education Development Foundation.

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