Relations between Information and Estimation in Scalar Lévy Channels

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Abstract

Fundamental relations between information and estimation have been established in the literature for the scalar Gaussian and Poisson channels. In this work, we demonstrate that such relations hold for a much larger class of observation models. We introduce the natural family of scalar Lévy channels where the distribution of the output conditioned on the input is infinitely divisible. For Lévy channels, we establish new representations relating the mutual information between the channel input and output to an optimal expected estimation loss, thereby unifying and considerably extending results from the Gaussian and Poissonian settings. We demonstrate the richness of our results by working out two examples of Lévy channels, namely the Gamma channel and the Negative Binomial channel, with corresponding relations between information and estimation. Extensions to the setting of mismatched estimation are also presented.

Index Terms

Mutual Information, Relative Entropy, estimation error, SNR (Signal-to Noise Ratio), generalized linear models, Lévy process, exponential family, infinite divisibility, Gaussian channel, Poisson channel, Bregman divergence

I. INTRODUCTION

Deep and elegant relations between fundamental measures of information and fundamental measures of estimation have been discovered for several interesting probabilistic models. Over time, such relations have been subject to interest in communities ranging from information theory to probability and statistical decision theory. For a recent comprehensive treatment of this topic and its implications, we refer to [1]. While both scalar and continuous-time observation models have been extensively discussed in the literature, and intriguing interconnections between both regimes drawn, in this work we focus exclusively on the scalar case.

Our story can be traced back to the early work by Stam [2] in 1959, where “de-Bruijn’s identity” relating the differential entropy of a Gaussian noise corrupted random variable to its Fisher information was presented. However, a more concrete starting point is the recent work by Guo, Shamai and Verdú [3] in 2005. In [3], the authors propose the I-MMSE formula, which presents the derivative with respect to SNR (signal-to-noise ratio) of the mutual information between the input and output of a Gaussian channel, as half the minimum mean squared error in estimating the channel input based on the output. Formally, if $X$, a random variable with finite variance, denotes the channel input, and $Y_\gamma = \gamma X + W$ indicates the channel output at SNR level $\gamma > 0$, where $W \sim N(0, \gamma)$ is an independent Gaussian random variable, then the I-MMSE relationship can be stated as,

$$\frac{\partial}{\partial \gamma} I(X; Y_\gamma) = \mathbb{E}[\ell_G(X, \mathbb{E}[X|Y_\gamma])],$$

(1)

where the Gaussian loss function $\ell_G : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is defined as,

$$\ell_G(x, \hat{x}) = \frac{1}{2} (x - \hat{x})^2.$$

(2)

In other words, for any choice of input distribution, the derivative of the mutual information is equal to half the minimum mean squared error in estimation. It turns out that such a relationship between mutual information and optimal estimation loss is not unique to the Gaussian channel. Similar relations were found for the scalar and continuous-time Poisson Channel in [4] and [5]. Remarkably, the exact same relationship holds in the Poissonian context as well, when the squared error loss is replaced by a natural loss function for the Poisson channel.

Indeed, consider a non-negative random variable $X$, satisfying $\mathbb{E}[X \ln X] < \infty$, and conditioned on $X$, $Y_\gamma \sim \text{Poisson}(\gamma X)$, now denote the Poisson channel input and output at SNR level $\gamma$, respectively. Invoking results from [4] and [5], we can express the relationship corresponding to (1) for the Poisson channel as,

$$\frac{\partial}{\partial \gamma} I(X; Y_\gamma) = \mathbb{E}[\ell_P(X, \mathbb{E}[X|Y_\gamma])],$$

(3)
where the Poisson loss function $\ell_P : [0, \infty) \times [0, \infty) \to [0, \infty]$ is defined as,

$$\ell_P(x, \hat{x}) = x \ln \left( \frac{x}{\hat{x}} \right) - x + \hat{x}.$$  \hspace{1cm} (4)

The similarity between (1) and (3) is quite striking. Indeed, the kinship between these two channel models does not end here. In (6), Verdú extended the I-MMSE result to incorporate mismatch at the decoder. In this setting, the underlying clean signal $X$ is distributed according to $P$, while the decoder believes the true law to be $Q$. For the scalar Gaussian channel model with SNR level $\gamma$, which could be infinity, (6) presents the following relationship between the relative entropy of the true and mismatched output laws, and the difference between the mismatched and matched estimation losses:

$$D(P_{Y_{\gamma}} || Q_{Y_{\gamma}}) = \int_0^\infty E_P[\ell_P(X, E_Q[X|Y_{\alpha}]) - \ell_Q(X, E_P[X|Y_{\alpha}])] d\alpha.$$  \hspace{1cm} (5)

An essentially identical result was established by Atar and Weissman in [5] for the Poisson channel:

$$D(P_{Y_{\gamma}} || Q_{Y_{\gamma}}) = \int_0^\gamma E_P[\ell_P(X, E_Q[X|Y_{\alpha}]) - \ell_P(X, E_P[X|Y_{\alpha}])] d\alpha,$$  \hspace{1cm} (6)

where, as in (3), the overloaded symbol $Y_{\gamma}$ now denotes the output of the Poisson channel with input $X$ at SNR level $\gamma$. Note that the first terms in the right hand sides of the integrands in (5) and (6) denote the average loss incurred when the decoder employs the estimator optimized for law $Q$. The right hand sides therefore indicate the cost incurred due to mismatch in estimation, integrated over a range of SNR values.

Thus, we observe the uncanny connection between the Gaussian and Poissonian observation models, whereby a direct relationship between mutual information, relative entropy and average estimation loss holds verbatim in both models, under the appropriate loss function. Further, the I-MMLE (Mutual Information-Minimum Mean Loss in Estimation) formulae stated in (1), (3), and their mismatched D-MMLE (Relative Entropy-Mean Loss in Estimation) counterparts in (5), (6), hold for any choice of input distributions, as long as they satisfy benign regularity conditions.

In this work, we aim to understand this special connection, and present a clear, unified picture assimilating both classical as well as hitherto unknown results in the world of information and estimation for a wide class of scalar observation models. While our present exposition is restricted to scalar channel models, our results can be extended to more general continuous-time observation models. We consign the full treatment of continuous-time channels to a parallel work in preparation [7] (c.f. also [8] for a related but distinct treatment of pure jump Lévy processes in continuous-time). The scalar case, however, to which we dedicate this paper, is of fundamental importance in its own right. Our main contributions here are fivefold:

1) The introduction (to our knowledge, for the first time in the literature) of scalar Lévy channels, which are a sub-family of the well-known generalized linear models [9] in statistics. Lévy channels satisfy the property that conditioned on the inputs, the outputs are scalar random variables with infinitely divisible distributions. Additionally, they have a natural SNR parameter, which captures the channel quality.

2) For scalar Lévy channels, we present a simple relationship between mutual information and an optimal estimation loss. We also present the generalization of this result to incorporate mismatch at the decoder. Additionally, we provide completely new formulae for expressing the entropy of a random variable, and for the relative entropy between two scalar distributions.

3) We recover results for both the Gaussian and Poissonian settings, for matched and mismatched estimation scenarios, as special cases of our general result. To our knowledge, this is the first unified presentation of information and estimation relationships for these two canonical scalar channels.

4) We present two natural channels, namely the Gamma channel and the Negative Binomial channel, both of which are instances of Lévy channels. For these channels, we use our general result to explicitly derive the information and estimation relationship.

5) We investigate the loss function that emerges in the characterization of mutual information in Lévy channels. In particular, we find a simple general representation for this loss function involving an associated Bregman divergence, in turn uniquely specified by the distribution of the channel output in the absence of any input.

The remainder of this paper is organized as follows. In Section III we introduce scalar Lévy channels as a natural scalar observation model, and discuss some properties underlying this family. In Section IV we present our main theorems on relations between information and estimation for Lévy channels. In Section V we recover, as corollaries of our main result, the fundamental relationships already known for the Gaussian and Poisson channels. In Section VI we introduce and study two special Lévy channels, namely the Gamma channel and the Negative-Binomial channel, from an information and optimal estimation viewpoint. In Section VII we discuss the natural loss function associated with Lévy channels. We present the proofs of our results in Section VII and conclude in Section VIII.
II. Scalar Lévy Channels

Throughout this paper, we only use the natural logarithm $\ln(\cdot)$.

A. A short introduction to Lévy Processes

Lévy processes constitute a fundamental class of stochastic processes. These processes have càdlàg sample paths, and are characterized by stationary and independent increments in time. Important examples include Brownian motion and Poisson processes. We refer the reader to Sato [10] for a comprehensive treatment of Lévy processes. Interestingly, there is a one-to-one correspondence between Lévy processes in law, and infinitely divisible distributions on $\mathbb{R}$. For example, the Gaussian, Poisson, Negative-Binomial, Gamma and Cauchy distributions are all infinitely divisible distributions on $\mathbb{R}$.

One may obtain a Lévy process $\{Y_t\}_{t \geq 0}$ from an infinitely divisible distribution $Z \sim F$ via the following method. We assume that the following cumulant transform of $F$ exists:

$$\kappa(\theta) = \ln \int e^{\theta Z}dF. \quad (7)$$

Then, for any $t > s \geq 0$, we define the increment $Y_t - Y_s$ to have the cumulant transform $(t - s)\kappa(\theta)$, and to be independent of $\{Y_u : u \leq s\}$. By the uniqueness property of cumulant transforms we know that it fully specifies the distribution of $Y_t - Y_s$.

In the other direction, in order to obtain an infinitely divisible random variable from a Lévy process $Y_t$, we just need to take $Y_1$.

By the Lévy–Khintchine formula [10], given any Lévy process $\{Y_t\}$, there exist constants $a \in \mathbb{R}$, $\sigma \geq 0$, a non-negative measure $\nu(\cdot)$ on $\mathcal{B}(\mathbb{R}_0)$ associated with $X_1$ such that $\int_{\mathbb{R}_0} \min(1, z^2) \nu(dz) < \infty$. Here $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$, and $\mathcal{B}(\mathbb{R}_0)$ denotes all Borel-measurable sets on $\mathbb{R}_0$. We call the tuple $(a, \sigma, \nu(\cdot))$ Lévy characteristics of the Lévy process $\{Y_t\}$. Particularly, we call the number $\sigma$ volatility, and the measure $\nu(dz)$ the Lévy measure of the Lévy process $\{Y_t\}$.

In plain terms, the Lévy–Khintchine formula shows that any Lévy process is essentially a mixture of a continuous Brownian motion part and a pure jump part, the latter of which in turn is a mixture of various jump sizes characterized by the measure $\nu(dz)$, where $z$ denotes the jump size. Formally, the Lévy–Itô decomposition [10] gives the following representation of any Lévy process $\{Y_t\}$:

$$Y_t = at + \sigma W_t + \int_0^t \int_{|z| \leq 1} z(d\mu - \nu(dz))ds + \int_0^t \int_{|z| > 1} zd\mu, \quad (8)$$

where $W_t$ is a standard Brownian motion, and is independent of the jump measure $\mu = \mu(dt, dz)$. The jump measure $\mu(dt, dz)$ is defined to satisfy the following relations: $\forall \Gamma \in \mathcal{B}(\mathbb{R}_0)$,

$$\mu([0, t] \times \Gamma) = \sum_{0 < s \leq t} \mathbb{I}(\Delta Y_s \in \Gamma), \quad (9)$$

where $\Delta Y_s = Y_s - Y_{s-}$. The measure $\nu(dz)$ is defined such that

$$\int_0^t \int_{\Gamma} (\mu(ds, dz) - \nu(dz)ds), \quad (10)$$

is a martingale indexed by $t$ for any $\Gamma \in \mathcal{B}(\mathbb{R}_0)$. In other words, $\nu(dz)ds$ is the compensator for $\mu(ds, dz)$.

B. The Scalar Lévy Channel

We now proceed to the characterization of the Lévy scalar channels. Throughout our exposition, we restrict ourselves to one-dimensional scalar random variables. Extensions to random vectors are straightforward.

We say a random variable $Y_\gamma$ is the output of a scalar Lévy channel at SNR $\gamma$ in the absence of input signals, if it satisfies

$$\ln (\mathbb{E}e^{\theta Y_\gamma}) = \gamma \kappa(\theta), \quad (11)$$

where $\kappa(\cdot)$ is the cumulant transform of some infinitely divisible random variable $Z \sim F$. The measure of $Y_\gamma$ in this situation is denoted as $\mathbb{P}_\gamma^\theta$. Note that there is a one-to-one correspondence between $\kappa(\cdot)$ and the Lévy characteristics $(a, \sigma, \nu(\cdot))$.

**Definition 1 (Scalar Lévy Channels):** For an infinitely divisible probability measure $\mathbb{P}_\theta^0$ corresponding to the output of a scalar Lévy channel in the absence of input, and a scalar random variable $X$, we define the output $Y_\gamma$ of the Lévy channel at SNR level $\gamma$ conditioned on input $X$ to be distributed according to law $\mathbb{P}_\theta^\gamma$, which is given by the generalized linear model [9] in (12):

$$\frac{d\mathbb{P}_\theta^\gamma}{d\mathbb{P}_0^\theta}(Y_\gamma) = e^{\theta Y_\gamma - \gamma \kappa(\theta)}, \quad (12)$$

where $\theta = \phi'(X)$ and $\phi(\cdot)$ is the Fenchel-Legendre dual of $\kappa(\cdot)$.
By Banerjee et al. [11], we know the random variable $X$ is in fact the mean of $Y_\gamma$ conditioned on $X$:

$$X = \int Y_\gamma d\mathbb{P}_\theta^? = \kappa'(\theta),$$

and, since $\kappa(\theta)$ is strictly convex, parameters $X, \theta$ are related via a bijection. The case $\theta = 0$ corresponds to no channel input, which is equivalent to $X = \kappa'(0)$.

We now shed some light on the parameter $\gamma$ which is an integral part of our characterization of the Lévy channel above. An important feature of the channel in Definition 1 is that it endows $\gamma$ with a very natural interpretation as the “SNR level” for the Lévy channel. It turns out, that we can place the different $\{Y_\gamma\}$ on the same probability space to construct a Lévy process $\{Y_\gamma\}_{0\leq\gamma\leq T}$, such that the marginal distribution of $Y_\gamma \sim \mathbb{P}_\theta^0$. Kuchler and Sorensen [12, Chap. 2] showed that under the following change of measure (now on processes in $[0, T]$),

$$\frac{d\mathbb{P}_\theta^{[0,T]}}{d\mathbb{P}_0^{[0,T]}}(Y_\gamma, 0 \leq \gamma \leq T) = e^\theta Y_T - T\kappa(\theta),$$

the process $\{Y_\gamma\}_{0\leq\gamma\leq T}$ is in fact another Lévy process under the new law $\mathbb{P}_\theta^{[0,T]}$. It was also shown in [12] that if the Lévy characteristics of process $Y$ are $(a, \sigma, \nu(dz))$ under $\mathbb{P}_0^{[0,T]}$, then the Lévy characteristics of $Y$ are $(a + \sigma^2\theta, \sigma, e^{\theta z}\nu(dz))$ under $\mathbb{P}_\theta^{[0,T]}$.

Of course, for any fixed $\gamma \in [0, T]$, if we restrict the measures $\mathbb{P}_\theta^{[0,T]}$ and $\mathbb{P}_0^{[0,T]}$ to $\sigma\{Y_\gamma\}$, we obtain exactly the same change of measure formula as in [12]. Hence, the change of measure formula in Definition 1 can be viewed as restrictions of the general change of measure formula (14) to $\sigma$-algebras generated by scalar random variables $Y_\gamma$. The change of measure in (14) is usually called the Esscher’s change of measure in the theory of stochastic processes, and has various applications, c.f. [13].

Note that the right hand side of the likelihood ratio (14) depends on the entire history of the process $\{Y_\gamma : 0 \leq \gamma \leq T\}$ only through its final value $Y_T$. This implies that given all the observations $\{Y_\gamma : 0 \leq \gamma \leq T\}$, the last observation $Y_T$ is a sufficient statistic for $\theta$. In other words, we have a natural degradedness in the observations, in terms of the index $\gamma$. This in our opinion gives the parameter $\gamma$ the most suitable interpretation as the signal-to-noise ratio of the channel.

Indeed, we will shortly see that Lévy channels encompass several scalar observation models, namely all models corresponding to infinitely divisible laws. In fact, the family of scalar Lévy channels is a subset of the class of generalized linear models, which have several applications in machine learning and statistics. However, we do trivially note that there are several distributions in the exponential family that are not infinitely divisible, and hence do not fall under the rubric of Lévy channels. Some examples related exponentially. Now, specializing (12) to this case, we obtain

$$\frac{d\mathbb{P}_\theta^\gamma}{d\mathbb{P}_0^\gamma} = e^{\theta Y_T - (e^\theta - 1)\gamma},$$

A simple calculation shows that under $\mathbb{P}_\theta^\gamma$, the random variable $Y_\gamma$ has the following distribution:

$$Y_\gamma \mid X \sim \mathcal{N}(\gamma X, \gamma).$$

This is precisely the definition of the Gaussian channel at SNR level $\gamma$.

In the Poisson channel setting, under $\mathbb{P}_0^\gamma$, the output $Y_\gamma$ is usually taken to be a Poisson random variable with parameter $\gamma \geq 0$. In this case, we have $\kappa(\theta) = e^\theta - 1$, and $X = \kappa'(\theta) = e^\theta$. Thus, the channel input $X$ and the natural parameter $\theta$ are related exponentially. Now, specializing (12) to this case, we obtain

$$\frac{d\mathbb{P}_\theta^\gamma}{d\mathbb{P}_0^\gamma} = e^{\theta Y_T - (e^\theta - 1)\gamma}.$$
III. MAIN RESULTS

“It is even speculated (in [3]) that information and estimation satisfy similar relationships as long as the output has independent increments conditioned on the input.”

– Guo, Shamai and Verdú [4]

To some extent, our work gives a clear affirmative answer to the above suspicion raised in the context of the Gaussian and Poissonian results from Section I. In this section, we will observe that analogous to the Gaussian and Poisson channel, we are able to obtain, for the Lévy channel, a precise formula expressing the mutual information as an optimal estimation loss. As we will shortly demonstrate, the “correct” loss function that presents itself in this formula, is intimately connected with the Gaussian and Poissonian loss functions that we visited in Section I.

Formally, let \( X \) denote the space of channel inputs as defined in Definition 1. The reconstruction space for estimating the channel input \( x \in X \), denoted by \( \hat{X} \) is a function space. Each reconstruction \( \hat{x} \in \hat{X} \) is a collection of scalars indexed by \( \mathbb{R} \). In other words, \( \hat{x} = \{ \hat{x}_z : z \in \mathbb{R} \} \). We will now introduce the loss function for the Lévy channel.

**Definition 2 (Loss function for Lévy Channels):** The loss function \( \ell_e : X \times \hat{X} \rightarrow [0, \infty] \), for the Lévy channel with characteristics \((\sigma, \nu(dz))\), is defined as,

\[
\ell_e(x, \hat{x}) = \sigma^2 \ell_\phi(x, \hat{x}_0) + \int_{\mathbb{R}_0} \ell_P(e^{e^{\phi}(x)z}, \hat{x}_z) \nu(dz),
\]

where the loss functions \( \ell_\phi \) and \( \ell_P \) are as defined in Section I. The loss function for Lévy channels has some interesting properties. First, it is always non-negative, and achieves zero if and only if

\[
\hat{x}_0 = x, \hat{x}_z = e^{e^{\phi}(x)z}, z \neq 0.
\]

Second, note that unlike estimation problems in scalar channels encountered thus far, the reconstructed signal, is not a scalar but rather takes values in a function space. It can be viewed as performing a scalar estimate for every jump size for the pure jump part of the Lévy process indexed by SNR level, in addition to a single scalar estimate for the continuous part of the channel output.

We are now in a position to present our main result for information and estimation in the Lévy channel, which presents a formula for the mutual information between the input and output of the Lévy channel. We first define the input regularity property as follows:

**Definition 3 (Input Regularity Property):** We say that the input \( X \) to a scalar Lévy channel satisfies the input regularity property, if the following conditions hold:

1) if \( \sigma \neq 0 \), then \( \mathbb{E}(\phi'(X))^2 < \infty \);

2) \( \mathbb{E} \int_{\mathbb{R}_0} \phi'(X)ze^{e^{\phi}(X)z}\nu(dz) < \infty \)

where, as before, \( \phi(\cdot) \) is the Fenchel-Legendre dual of \( \kappa(\cdot) \) introduced in Section I, and is specified by the channel. Quantities \( \sigma, \nu(dz) \) are Lévy characteristics of the Lévy channels.

Our first main result is the following:

**Theorem 1:** Let \( X \) be a scalar random variable distributed according to law \( P \). If \( X \) satisfies the Input Regularity Property, we have:

\[
\frac{\partial}{\partial \gamma} I(X; Y_\gamma) = \mathbb{E}[\ell_e(X, \hat{X}_{\gamma}^P)],
\]

where \( \hat{X}_{\gamma}^P \), defined as,

\[
\hat{X}_{\gamma,z}^P = \begin{cases} 
\mathbb{E}_P[\phi'(X)|Y_\gamma] & \text{if } z = 0, \\
\mathbb{E}_P[e^{e^{\phi}(X)z}|Y_\gamma] & \text{if } z \neq 0,
\end{cases}
\]

is the optimal (minimum mean loss) reconstruction.

Note that the result in (21) presents the derivative of the mutual information between the input and output with respect to the SNR as the optimal mean loss in estimating the channel input, according to the loss function specified in (19). It is strikingly similar to the I-MMLE results encountered in Section I for the Gaussian and Poissonian loss functions that we visited in Section I.

A particularly interesting case arises when \( \gamma = \infty \), since it gives us a completely new expression for the entropy of a random variable. As a consequence of Theorem 1 we have the following:

**Theorem 2:** If \( X \) is a discrete real-valued random variable satisfying the Input Regularity Property, then

\[
H(X) = \int_0^\infty \mathbb{E}[\ell_e(X, \hat{X}_{\gamma}^P)] d\gamma.
\]

Note that the above representation of entropy in Theorem 2 is quite intriguing. In particular, it holds for any Lévy channel. Further, the left hand side, as is well known, is invariant to one-to-one transformations, a fact that is not at all intuitive for the
right hand side. Indeed, the fact that such a functional representation for the entropy of a random variable holds in general, is surprising.

We will now visit the mismatched estimation setting, and present a result analogous to Theorem 1 in this direction. Recall that in the case of the mismatched decoder, the true law governing the channel input is \( P \) while the decoder incorrectly believes it to be \( Q \). It thus employs the estimator optimized for \( Q \). Using a sub-optimal decoder will incur an additional loss. This loss is also termed “cost of mismatch”. In the following theorem, we demonstrate that two quantities, namely the relative entropy and the cost of mismatch in estimation emerge.

Theorem 1: Let \( X \) be a scalar random variable, and \( P \) and \( Q \) be two laws on \( \mathcal{X} \) such that the following conditions hold:

1) Under \( P \), \( X \) satisfies the Input Regularity Property;
2) At SNR level \( \gamma \),
\[
\mathbb{E}_P \int_0^\gamma \mathbb{E}_Q \left[ \int_{R_0} e^{\phi(X)z} \nu(dz) \right] \, d\alpha = \int_0^\gamma \mathbb{E}_Q \left( \frac{dP_X}{dQ_X} \int_{R_0} e^{\phi(X)z} \nu(dz) \right) \, d\alpha < \infty,
\]
(24)
\[
\int_0^\gamma \mathbb{E}_P \ell_\mathcal{L}(X, \hat{X}_\alpha^P) \, d\alpha < \infty.
\]
(25)
Then, for the Lévy channel, we have,
\[
D(P_\gamma || Q_\gamma) = \int_0^\gamma \mathbb{E}_P [\ell_\mathcal{L}(X, \hat{X}_\alpha^Q) - \ell_\mathcal{L}(X, \hat{X}_\alpha^P)] \, d\alpha,
\]
(26)
where \( P_\gamma \) (\( Q_\gamma \)) denotes the law of the channel output at SNR level \( \gamma \), when the true (mismatched) input law is \( P \) (\( Q \)). When we consider the case \( \gamma = \infty \), we obtain the following representation of relative entropy:

Theorem 4: Let \( X \) be a scalar random variable, and \( P \) and \( Q \) be two laws on \( \mathcal{X} \) such that the following conditions hold:

1) Under \( P \), \( X \) satisfies the Input Regularity Property;
2) \( \forall 0 \leq \gamma < \infty \), we have
\[
\mathbb{E}_P \int_0^\gamma \mathbb{E}_Q \left[ \int_{R_0} e^{\phi(X)z} \nu(dz) \right] \, d\alpha = \int_0^\gamma \mathbb{E}_Q \left( \frac{dP_X}{dQ_X} \int_{R_0} e^{\phi(X)z} \nu(dz) \right) \, d\alpha < \infty,
\]
(27)
\[
\int_0^\gamma \mathbb{E}_P \ell_\mathcal{L}(X, \hat{X}_\alpha^Q) \, d\alpha < \infty.
\]
(28)
Then we have
\[
D(P || Q) = \int_0^\infty \mathbb{E}_P [\ell_\mathcal{L}(X, \hat{X}_\alpha^Q) - \ell_\mathcal{L}(X, \hat{X}_\alpha^P)] \, d\gamma,
\]
(29)
We have managed to show that for the general scalar observation model outlined in Section II there is a fundamental relation between mutual information and an optimal estimation loss. Further, this result can be generalized to incorporate mismatch at the decoder, in which a similar relationship involving the relative entropy and the cost of mismatch in estimation emerges. Both these results are in the same exact form as the classical results for the Gaussian and Poisson channels that were discussed in Section II. In addition, we obtain new formulae for the entropy of a real valued random variable, and the relative entropy between two distributions.

IV. RECOVERING GAUSSIAN AND POISSONIAN RESULTS

In this section, we demonstrate how Theorems 1 and 4 recover the known results for the scalar Gaussian and Poisson channel models.

Recall that under the Gaussian noise channel, the output at SNR level \( \gamma \) is equivalently expressed as,
\[
Y_\gamma = \gamma X + W,
\]
(30)
where \( W \) is a Gaussian random variable \( \mathcal{N}(0, \gamma) \) independent of \( X \). Thus in the absence of an input, the Lévy characteristics of the Gaussian simply satisfy \( a = 0, \sigma = 1, \nu(d\ell) \equiv 0 \). Under this framework, it is easy to see that the loss function in (2) collapses to \( \ell_\mathcal{G} \), i.e., the squared error loss function defined in (2). Also recall, from Section II that for the Gaussian channel we have \( \theta = X, \kappa(\theta) = \frac{1}{2} \theta^2, \phi(X) = \frac{1}{2} X^2 \). Now, invoking the Input Regularity Property, we see that any input to the Gaussian channel must satisfy \( \mathbb{E}[X^2] < \infty \), which is consistent with classical results, c.f. [3] Theorem 1. Further, an application of Theorem 1 directly gives us the I-MMSE formula [3], as stated in (1)-(2),
\[
\frac{\partial}{\partial \gamma} I(X; Y_\gamma) = \mathbb{E}[\ell_\mathcal{G}(X, E[X|Y_\gamma])].
\]
(31)
By an essentially identical argument, we know from Theorem 4.1 that if $E_P X^2 < \infty$, $\int_0^\gamma E_P(X - \tilde{X}_o^Q)^2 d\alpha < \infty$ for all $\gamma > 0$, we have

$$D(P||Q) = \int_0^\infty E_P[\ell_P(X, E_Q[X|Y_\alpha]) - \ell_P(X, E_P[X|Y_\alpha])] d\alpha,$$

which is Verdú’s result in [6].

The scalar Poisson channel at SNR level $\gamma$, specifies the output distribution as a Poisson distribution with parameter $\gamma$ in the absence of input signals. Under this model, the Lévy characteristics of the scalar Poisson channel are $\kappa = 0$, $\sigma = 0$, $\nu(dz) = \delta_1$, where the measure $\delta_x$ denotes a point mass at $x$. Again, it is straightforward to see that the loss function in (19) collapses to the natural Poisson loss function $\ell_P$ defined in [4]. Note that for the Poisson channel, its input $X$ and the natural parameter $\theta$ satisfy the relation $X = e^{\theta}$, or $\theta = \ln X$. Thus, the Input Regularity Property tells us that the input should satisfy $E[X \ln X] < \infty$, a condition identical to the treatment in the literature, c.f. [5, Section V.A]. An application of Theorem 1 to this specialized setting, gives us the relationship between mutual information and minimum mean loss in estimation for the Poisson channel,

$$\frac{\partial}{\partial \gamma} I(X; Y_\gamma) = E[\ell_P(X, E[X|Y_\gamma])].$$

Applying Theorem 2 to the Poisson channel, we obtain the following corollary:

**Corollary 5**: Let non-negative random variable $X$ be the input to the Poisson channel, $P, Q$ are two probability measures on $\mathcal{X}$. If $E_P X \ln X < \infty$ and $0 \leq \gamma < \infty$,

$$\int_0^\gamma E_P(\ell_P(X, E_Q[X|Y_\alpha])) d\alpha < \infty, \quad E_P \int_0^\gamma \ell_P(X, E_Q[X|Y_\alpha]) d\alpha < \infty,$$

then

$$D(P||Q) = \int_0^\infty E_P[\ell_P(X, E_Q[X|Y_\alpha]) - \ell_P(X, E_P[X|Y_\alpha])] d\alpha.$$  

It is worth mentioning that Corollary 5 is a strengthened version of Atar and Weissman [5 Thm 4.1]. Note that in [5 Thm 4.1] and [11], the mismatched estimation results are stated with the very strong condition that $P$ and $Q$ are probability measures supported on interval $[a, b], 0 < a < b < \infty$. This condition appears to be restrictive to the authors, since it eliminates the possibility of using zero input in the Poisson channels. Indeed, it was conjectured in [5] that their Theorem 4.1 holds under weaker assumptions, which Corollary 5 presents.

The above discussion demonstrates that Theorems 1 and 3 indeed present a unified picture of relations between information and estimation in an important class of channels. For instance, they recover the classical results for both the Gaussian and Poisson channels as special cases of a general result. We will now argue that the family of channels encompassed by our results is much richer. We will illustrate this via two interesting and natural examples of Lévy channels, and for these channels derive new estimation theoretic formulae for the mutual information and relative entropy.

V. TWO NEW EXAMPLES: THE GAMMA AND NEGATIVE BINOMIAL CHANNELS

The Gamma distribution is an infinitely divisible law on $\mathbb{R}$. In this section we introduce the scalar Lévy channel that is induced via the Gamma distribution on $\mathbb{R}$. We call this the scalar Gamma channel, or simply the Gamma channel.

Recall that the Gamma distribution is a continuous-valued two-parameter probability distribution. We say that a random variable $Z$ follows the Gamma distribution with “shape” parameter $k > 0$ and “scale” parameter $s > 0$, or equivalently $Z \sim \Gamma(k, s)$, if it has the following probability density function:

$$f(z; k, s) = \frac{z^{k-1}e^{-z/s}}{s^k \Gamma(k)}, z > 0.$$  

The Gamma distribution satisfies infinite divisibility with respect to the shape parameter, for a fixed scale. We now consider the Lévy channel corresponding to the Gamma distribution. In the absence of an input, i.e. $\theta = 0$, the law $P_\theta^0$ imposes the following distribution on the channel output:

$$Y_\gamma \sim \Gamma(\gamma, 1),$$

where, as before, $\gamma > 0$ is the channel SNR level. Note that the scale parameter is chosen to be 1 for convenience, and is not a requirement. Also, for the Gamma channel, the cumulant $\kappa_1^\gamma(\theta) = \ln \left( \frac{1}{1-\theta} \right)$. We are now in a position to define the scalar Gamma channel, according to (12).

**Definition 4 (Gamma Channel)**: Let $X$ be the input non-negative random variable, which is related to the natural parameter $\theta$, via (13), i.e. $X = \kappa_1^\gamma(\theta) = \frac{1}{1-\theta}$. The output of the Gamma channel at SNR level $\gamma$ is governed by $P_\theta^\gamma$ and satisfies

$$Y_\gamma|X \sim \Gamma(\gamma, X).$$
An equivalent characterization of the Gamma channel emerges by specifying the Lévy characteristics \((a, \sigma, \nu(dz))\) of the channel. For the Gamma channel, \(a = 1 - e^{-1}, \sigma = 0\) and the jump-size distribution is given by \(\nu(dz) = z^{-1}e^{-z}dz, z > 0\). The interested readers are referred to Protter [14, Chap. 1.4] to how these Lévy characteristics are derived.

The channel defined in (38) is a very simple scalar observation model. The input simply modulates the scale parameter of a Gamma distributed random variable. It is therefore striking that this probabilistic model joins the elite group of channels which enjoy a unique relationship between mutual information and optimal estimation. We summarize this in the following result, a direct consequence of Theorem 1.

**Theorem 6:** Let non-negative random variable \(X \sim P\) be the input to the Gamma channel, which satisfies the Input Regularity Property, i.e., \(E X < \infty\). Let \(Y_\gamma\) denote the channel output at SNR \(\gamma\). Then, we have,

\[
\frac{\partial}{\partial \gamma} I(X; Y_\gamma) = E[\ell_\gamma(X, \hat{X}_\gamma)] ,
\]

where the loss function \(\ell_\gamma\) is defined as,

\[
\ell_\gamma(x, \hat{x}) = \int_{\mathbb{R}^+} \ell_P(e^{(1-1/z)}z, \hat{x}_z)(z^{-1}e^{-z})dz .
\]

Recall that \(\ell_P\) is simply the Poisson loss function introduced in (4).

**Theorem 7:** Let \(X\) be the input to the Gamma channel, and conditions in Theorem 3 are satisfied. Let \(P_{Y_\gamma}\) and \(Q_{Y_\gamma}\) denote the output laws when the input is distributed according to \(P\) and \(Q\) respectively. Then, we have

\[
D(P_{Y_\gamma}||Q_{Y_\gamma}) = \int_0^\gamma E_P[\ell_\gamma(X, \hat{X}_\gamma)] - \ell_\gamma(X, \hat{X}_\alpha)] d\alpha,
\]

Recall that the loss function \(\ell_\gamma\) is as defined in (40).

Thus, we present a new relationship for the Gamma channel, in which we express the derivative of the relative entropy between the true and mismatched output distributions with respect to the SNR, as a difference between the average estimation losses incurred when the decoder uses the mismatched and optimal estimators respectively.

One point we wish to focus on concerns the specific form of the loss functions that emerge in our characterization of Lévy channels, such as in (40) for the Gamma channel. A natural question to ask in this context is, how does one visualize loss calculations, it follows from (12) that under \(P_\gamma\), the conditional distribution \(Y_\gamma|\theta\) becomes,

\[
Y_\gamma|\theta \sim NB(\gamma, 1/2).
\]

Again, we mention that the second parameter is chosen to be \(1/2\) for convenience, and is not a requirement. After some simple calculations, it follows from (12) that under \(P_\gamma\), the conditional distribution \(Y_\gamma|\theta\) becomes,

\[
Y_\gamma|\theta \sim NB(\gamma, 1/2) e^{\theta}.
\]

The cumulant transform of \(P_\gamma\) for the Negative Binomial distribution is

\[
\kappa_{NB}(\theta) = \ln \left( \frac{1}{1 - \theta} \right).
\]

Let \(X\) be the channel input related to the natural parameter \(\theta\) via \(X = \kappa_{NB}'(\theta) = \frac{\theta}{1 - \theta} \). We now define the scalar Negative Binomial channel.

**Definition 5 (Negative Binomial Channel):** The Negative Binomial channel with non-negative input \(X\), at SNR level \(\gamma\) is defined via the following conditional distribution,

\[
Y_\gamma|X \sim NB(\gamma, \frac{X}{1+X}).
\]
Note that for the Negative Binomial distribution, we have $\sigma = 0$, and $\nu(k) = \frac{1}{k+\nu}$, $k \in \mathbb{N}_+$, in the corresponding Lévy characteristics. The interested readers are referred to [10] Chap.1 Example 4.6] for derivations of these Lévy characteristics.

We now present a characterization of the derivative of the mutual information between the channel input and output, with respect to the SNR level $\gamma$ in the Negative Binomial Channel.

**Theorem 8:** Let non-negative random variable $X$ satisfy the Input Regularity Property, i.e. $\mathbb{E}X \ln \left(\frac{2X}{1+X}\right) < \infty$. Let $Y_\gamma$ denote the output of the Negative Binomial channel with input $X$ at SNR level $\gamma$. Then,

$$\frac{\partial}{\partial \gamma} I(X;Y_\gamma) = \mathbb{E}[\ell_{NB}(X, \hat{X}_\gamma^P)],$$

where the loss function $\ell_{NB}$ is defined as,

$$\ell_{NB}(x, \hat{x}) = \sum_{z \in \mathbb{N}_+} \ell_P(e^z \ln(2x/(1+x)), \hat{x}_z) \frac{1}{z^{2z}}.$$  

Recall that $\ell_P$ is simply the Poisson loss function introduced in (4).

Analogous to our discussion so far, we now present the corresponding result for mismatched estimation.

**Theorem 9:** Let non-negative random variable $X$ be the input to the Negative Binomial channel, and conditions in Theorem 5 are satisfied. Let $P_{Y_\gamma}$ and $Q_{Y_\gamma}$ denote the output laws when the input is distributed according to $P$ and $Q$ respectively. Then, we have,

$$D(P_{Y_\gamma} || Q_{Y_\gamma}) = \int_0^\infty \mathbb{E}_P[\ell_{NB}(X, \hat{X}_\gamma^Q) - \ell_{NB}(X, \hat{X}_\gamma^P)] d\alpha,$$

Recall that the loss function $\ell_{NB}$ is as defined in (48).

The so called Negative Binomial channel has appeared in the literature before, in the context of relations between information and estimation, c.f. [15], [16]. However, our approach is quite different from existing approaches, and in our point of view, much more natural. We will illustrate the key differences between our approach and existing approaches in Negative Binomial Channels in the next section.

We conclude the section, with a short remark. The theorems that we have presented for both the Gamma and Negative Binomial channel express fundamental measures of information in the form of an optimal estimation loss. The answers that we obtain turn out to be non-trivial and hard to obtain via elementary methods. Underlying these results is the categorization of a large and interesting family of channels. While we have illustrated the applicability and novelty of our main results via these two examples, we note in passing that a similar analysis can be performed for every infinitely divisible distribution via considering the corresponding Lévy channel that spawns from it.

**VI. CHANNELS AND LOSS FUNCTIONS: A DISCUSSION**

We begin this section with a result that relates the relative entropy between two scalar distributions of the same exponential family, and the underlying distribution itself. It turns out, that under benign conditions, the relative entropy is exactly given by the Bregman divergence corresponding to the Legendre dual of the cumulant of said exponential family. This does not appear to be a new result, for example, a mention of this fact has been made in [17]. First, we review this result.

An arbitrary exponential family is defined as

$$\frac{dP_\theta}{dP_0}(Y) = e^{\theta Y - \kappa(\theta)},$$

where $\kappa(\theta)$ is the cumulant transform of distribution $P_0$. It was proved in Banerjee et.al [11] that if we define the mean parameter $x \in \mathbb{R}$ to be

$$x(\theta) = \int Y dP_\theta,$$

we have

$$\frac{dP_\theta}{dP_0}(Y) = e^{-d_\phi(Y,x(\theta)) + d_\phi(Y,x(0))},$$

where $d_\phi(x_1, x_2) = \phi(x_1) - \phi(x_2) - \phi'(x_2)(x_1 - x_2)$ is the Bregman divergence generated by convex function $\phi(\cdot)$, and $\phi(\cdot)$ is the Fenchel–Legendre dual of $\kappa(\cdot)$. It was also proved in [11] that

$$x = \kappa'(\theta), \quad \theta = \phi'(x),$$

which implies that there is a bijection between $x$ and $\theta$ since $\kappa(\cdot)$ is strictly convex.

The following lemma represents the relative entropy between exponential family distributions using Bregman divergences.

**Lemma 10:** Let $P_1$ denote $P_{\theta_1}, P_2$ denote $P_{\theta_2}$ in [50], $x_i = \kappa'(\theta_i), i = 1, 2$, we have

$$D(P_1 || P_2) = \phi(x_1) - \phi(x_2) - \phi'(x_2)(x_1 - x_2).$$
Applying Lemma 10 to Lévy channels with SNR $\gamma$, we have the following corollary since the Fenchel–Legendre transform of $\gamma \kappa(\theta)$ is $\gamma \phi(x/\gamma)$:

**Corollary 11:** Let $P_1^\gamma$ denote the output distribution of a Lévy channel with SNR $\gamma$ with deterministic input $x_1$, $P_2^\gamma$ denote the output distribution of the same Lévy channel with SNR $\gamma$ with deterministic input $x_2$, we have

$$\frac{\partial}{\partial \gamma} D(P_1^\gamma \| P_2^\gamma) = \phi(x_1) - \phi(x_2) - \phi'(x_2)(x_1 - x_2).$$

(55)

Now we are in the position to discuss the key differences between our approach and existing approaches to establishing information-estimation results in channels beyond Gaussian and Poisson.

Nearly all the literature attempting to generalize the information-estimation relationships restrict themselves to exponential families. At the very least, the generalized results should apply to the cases where the inputs are point masses, i.e., non-random.

In other words, suppose $\gamma \kappa$ of $\phi$ being a convex function does not imply $f(x) = x \phi'(x)$ is also a convex function, hence $d_f(x, y)$ may not be a Bregman divergence, and is not necessarily non-negative. Coincidentally, for Gaussian and Poisson models, the Bregman divergence generated by $x \phi'(x)$ and $\phi(x)$ are equal up to a multiplicative factor. Indeed, we have

$$D(P_1 \| P_2) = \phi(ax_1) - \phi(ax_2) - \phi'(ax_2)(ax_1 - ax_2),$$

(60)

and the parameter $a$ is the parameter with respect to which [15, 16] analyze the derivatives of mutual information and relative entropy.

Essentially, [15, 16] worked on generalizing (60) to random inputs $X$. However, it seems to the authors that even in the degenerate scalar case (60), the parameter $a$ may not display consistent properties for different channel models. Indeed, if we take derivatives on both sides of (60), we will obtain

$$a \frac{\partial}{\partial a} D(P_1 \| P_2) = d_f(ax_1, ax_2),$$

(61)

where $f(x) = x \phi'(x)$, $d_f(x, y) = f(x) - f(y) - f'(y)(x - y)$. However, $\phi(x)$ being a convex function does not imply $f(x) = x \phi'(x)$ is also a convex function, hence $d_f(x, y)$ may not be a Bregman divergence, and is not necessarily non-negative. Coincidentally, for Gaussian and Poisson models, the Bregman divergence generated by $x \phi'(x)$ and $\phi(x)$ are equal up to a multiplicative factor. Indeed, we have

$$\phi(x) = \frac{1}{2} x^2, \quad d_\phi(x, y) = \frac{1}{2}(x - y)^2,$$

(62)

$$x \phi'(x) = x^2, \quad d_{x \phi'(x)}(x, y) = (x - y)^2,$$

(63)

for the Gaussian model, and

$$\phi(x) = x \ln x - x + 1, \quad d_\phi(x, y) = x \ln \left( \frac{x}{y} \right) - x + y,$$

(64)

$$x \phi'(x) = x \ln x, \quad d_{x \phi'(x)}(x, y) = x \ln \left( \frac{x}{y} \right) - x + y,$$

(65)

for the Poisson model.

The Bregman divergences listed in [15, 16] for Binomial and Negative Binomial models are all Bregman divergences generated by $x \phi'(x)$, and it so happens that they are both strictly convex. However, if we consider the Gamma distribution, we have

$$\phi_T(x) = x - 1 - \ln(x),$$

(66)
which implies
\[ f(x) = x\phi'_\gamma(x) = x - 1, \quad df(x, y) \equiv 0. \] (67)

As a consequence of this parametrization, results in [15], [16] do not take the forms of (1) and (5).

Further, we can show that even if we consider random inputs in the Gamma distribution, if we follow the definition of parameter \( a \) in [15], [16], we would obtain that the mutual information between input and output is invariant with respect to the parameter \( a \). Note that in our definition of the Gamma channel, we have
\[ Y_\gamma|X \sim \Gamma(\gamma, X), \] (68)
however, if we follow [15], [16], then we have
\[ Y_a|X \sim \Gamma(k, aX/k), \] (69)
where \( k \) is some fixed positive constant.

**Lemma 12:** Suppose \( P, Q \) are two probability measures of non-negative random variable \( X \). If we have \( Y_a|X \sim \Gamma(k, aX/k), k > 0 \), then
\[ \frac{\partial}{\partial a} f(X; Y_a) \equiv 0 \] (70)
\[ \frac{\partial}{\partial a} D(P_{Y_a} || Q_{Y_a}) \equiv 0. \] (71)

Lemma 12 shows the parameterization in [15], [16] may lead to some strange results that do not capture the infinite-divisibility of the Gamma distribution.

We hope to have convinced the reader that the parametrization in our framework is natural and captures the core properties of the Gaussian and Poisson distributions. In fact, we conjecture that scalar Lévy channels are the largest family of channels for which one can establish information-estimation results paralleling all existing results in the Gaussian and Poisson observation models.

Now we turn to another intriguing application of Lemma 10. It turns out Lemma 10 allows us to extract a simple and elegant description for the seemingly convoluted loss functions that we have encountered in the previous section. For the rest of this section, we will discuss the interpretation of the general Lévy channel loss function \( \ell_L \), and its specializations to channels of interest.

Recall that the reconstruction space for the loss function in (19) is an entire continuum of scalar values. We want to reduce the representation of this loss function to a single reconstruction parameter. To this effect, we define the “representative loss” function as follows.

**Definition 6 (Representative Loss Function for the Lévy Channel):** We define the representative loss function for the scalar Lévy channel as,
\[ \tilde{\ell}_L(x, y) = \ell_\Gamma(x, \hat{X}_{\delta_y}), \] (72)
where \( \hat{X}_{\delta_y} \) is the optimal reconstruction when the input to the channel is a fixed value \( y \), i.e.,
\[ \hat{X}_{\delta_y}(z) = \begin{cases} y & \text{if } z = 0, \\ e^{\phi'(y)z} & \text{if } z \neq 0. \end{cases} \] (73)

We will now demonstrate that this is a very natural representation of the Lévy channel loss function. From the definition above, and the discussion in section [11] it is easy to see that upon specializing to the Gaussian case, we obtain,
\[ \tilde{\ell}_G(x, y) = \ell_G(x, y) = \frac{1}{2} (x - y)^2. \] (74)

Similarly, specializing to the Poisson channel, one also recovers the Poisson loss function,
\[ \tilde{\ell}_P(x, y) = \ell_P(x, y) = x \ln \left( \frac{x}{y} \right) - x + y. \] (75)

We now demonstrate our main result in this section.

**Theorem 13:** Let \( P_1 \) denote the output distribution of a Lévy channel under SNR \( \gamma \) with deterministic input \( x_1 \), \( P_2 \) denote the output distribution of the same Lévy channel under SNR \( \gamma \) with deterministic input \( x_2 \), we have
\[ \frac{\partial}{\partial \gamma} D(P_1 || P_2) = \tilde{\ell}_L(x_1, x_2) \] (76)
\[ = \sigma^2 \ell_G(x_1, x_2) + \int_{\mathbb{R}_0} \ell_P(e^{\phi'(x_1)z}, e^{\phi'(x_2)z}) \nu(dz) \] (77)
\[ = \phi(x_1) - \phi(x_2) - \phi'(x_2)(x_1 - x_2) \] (78)
Theorem 13 shows that under deterministic inputs, the seemingly convoluted loss function for Lévy channels collapses to a crisp closed form formula, which in turn is simply the Bregman divergence generated by the convex function $\phi(x)$.

Finally, we specialize Theorem 13 to the Gamma and Negative Binomial channels for a simple representation of their respective loss functions, which were defined in Section V.

For the Gamma channel, we have $\kappa_\Gamma(\theta) = -\ln(1-\theta)$, whose Fenchel–Legendre transform is given by,

$$\phi_\Gamma(x) = x - 1 - \ln x, x > 0.$$  \hfill (79)

Thus, we have,

$$\tilde{\ell}_\Gamma(x_1, x_2) = d_\Gamma(x_1, x_2) = \frac{x_1}{x_2} - \ln \left( \frac{x_1}{x_2} \right) - 1.$$  \hfill (80)

The loss function $\tilde{\ell}_\Gamma(x_1, x_2)$ is also called Itakura-Saito distance [11], and has proved to play an important role in linear inverse problems as investigated by Csiszár [18]. To visualize this loss function, we fix $x_1 = 1$ and vary $x_2$, to obtain Figure 1.

For Negative Binomial channels, we have $\kappa_{NB}(\theta) = \ln \left( \frac{1}{1-\theta} \right)$. The Fenchel–Legendre dual of $\kappa_{NB}(\theta)$ is,

$$\phi_{NB}(x) = x \ln x - (1+x) \ln(1+x) + x \ln 2 + \ln 2, x \geq 0.$$  \hfill (81)

Hence, we know,

$$\tilde{\ell}_{NB}(x_1, x_2) = d_{NB}(x_1, x_2) = x_1 \ln \left( \frac{x_1}{x_2} \right) + (1+x_1) \ln \left( \frac{1+x_2}{1+x_1} \right).$$  \hfill (82)

To visualize $\tilde{\ell}_{NB}(x_1, x_2)$, we fix $x_1 = 1$ and vary $x_2$, to obtain Figure 2.
VII. Proofs

A. Proof of Theorem 1

Theorem 1 can be obtained via direct application of Theorem 3. Indeed, mutual information $I(X; Y_\gamma)$ is expressible as

$$I(X; Y_\gamma) = \mathbb{E}_X D(P_{Y_\gamma|X} \| P_{Y_\gamma}),$$

(83)

where $P_{Y_\gamma|X}$ is the marginal distribution of output of the Lévy channel under point mass input $\delta_X$, and $P_{Y_\gamma}$ is the marginal distribution of $Y_\gamma$ under input $P_X$.

It is easy to verify that conditions in Theorem 3 are satisfied if we assume $X$ satisfies the Input Regularity Property. Thus,

$$I(X; Y_\gamma) = \int_0^\gamma \mathbb{E}_P \ell_{\mathcal{L}}(X, \hat{X}_\alpha) d\alpha.$$

(84)

The claim follows from taking derivatives on both sides with respect to $\gamma$.

B. Proof of Theorem 2

According to Theorem 1 for any $\gamma$, we have

$$I(X; Y_\gamma) = \int_0^\gamma \mathbb{E}_P \ell_{\mathcal{L}}(X, \hat{X}_\alpha) d\alpha.$$

(85)

Taking $\gamma \to \infty$ on both sides, we have

$$\lim_{\gamma \to \infty} I(X; Y_\gamma) = \int_0^\infty \mathbb{E}_P \ell_{\mathcal{L}}(X, \hat{X}_\alpha) d\alpha,$$

(86)

For discrete random variables $X$,

$$\lim_{\gamma \to \infty} I(X; Y_\gamma) = H(X).$$

(87)

C. Proof of Theorem 3

By the bijection between infinitely divisible distributions and Lévy processes, given any Lévy channel, we construct a Lévy process $\{Y_s, 0 \leq s \leq t\}$ such that $Y_s \sim P_0^a$. Kuchler and Sorensen [12, Chap. 2] showed that under the following change of measure

$$\frac{d\mathbb{P}^a_{[0,t]}}{d\mathbb{P}^a_0}(Y_s, 0 \leq s \leq t) = e^{Y_s - t\kappa(\theta)},$$

(88)

the process $Y_s$ is another Lévy process under $d\mathbb{P}^a_{[0,t]}$. If we restrict the Radon–Nikodym derivative to $\sigma\{Y_s\}$, we obtain the definition of Lévy channel at SNR $s$. Note that in this settings, the time $t$ parameter replaces the SNR parameter $\gamma$.

We define the non-negative martingale process

$$L_t = \frac{d\mathbb{P}^a_{[0,t]}}{d\mathbb{P}^a_0}(Y_s, 0 \leq s \leq t) = e^{Y_s - t\kappa(\theta)}.$$

(89)

Using the Lévy–Itô decomposition in (3), we have

$$L_t = e^{\theta (\sigma t + \sigma W_t + \int_0^t \int_{|z| \leq 1} z (d\mu - \nu(dz)ds) + \int_0^t \int_{|z| > 1} z d\mu)} - t\kappa(\theta)$$

$$= e^{\sigma^2 \theta^2 t + \int_0^t \int_0^\theta (\theta z d\mu - (\theta^2 - 1) \nu(dz)ds)},$$

(90)

(91)

where we used the cumulant transform of Gaussian and Poisson models in the second step. Define the martingale

$$M_t = \sigma \theta W_t + \int_0^t \int_{\mathbb{R}_0} (e^{\theta z} - 1) (\mu(ds, dz) - \nu(dz)ds),$$

(92)

we can verify that $L_t$ satisfies the following stochastic differential equation:

$$L_t = 1 + \int_0^t L_s \cdot dM_s.$$

(93)

Indeed, according to [19], the solution of (93) is unique and is given by

$$L_t = e^{\varphi_t},$$

(94)

where

$$\varphi_t = M_t - \frac{1}{2} (M^2)_t + \sum_{s \leq t} (\ln(1 + \Delta M_s) - \Delta M_s).$$

(95)
Plugging \( M_t \) into the general solution verifies that \( L_t \) is indeed the unique solution.

Let \( \mathcal{F}_t^Y \) denote the filtration \( \sigma \{ Y_s : 0 \leq s \leq t \} \). In order to characterize the relative entropy, we need to compute the marginal distribution of \( Y_t \). Let \( P_Y \) denote the marginal distribution of \( Y_t \) when the input \( X \) has distribution \( P \), \( Q_Y \) denote the marginal distribution of \( Y_t \) when the input has distribution \( Q \), and \( R_Y \) denote the marginal distribution of \( Y_t \) when there is no input. According to the Radon–Nikodym theorem, we know that the Radon–Nikodym derivative restricted to \( \sigma \{ Y_t \} \) is equal to the conditional expectation with respect to \( \mathcal{F}_t^Y \) under the whole probability measure corresponding to no input:

\[
\frac{dP_Y}{dR_Y} = \mathbb{E}_R \left[ L_t | \mathcal{F}_t^Y \right], \quad R_Y \text{ -- a.s.}
\] (96)

Denote \( \frac{dP_Y}{dR_Y} \) by \( \bar{L}_t^P \). We have

\[
L_t = 1 + \int_0^t \sigma L_{s-} dW_s + \int_0^t L_{s-} \left( \int_{\mathbb{R}_0} (e^{\theta z} - 1)(\mu(ds, dz) - \nu(dz)ds) \right).
\] (97)

Taking conditional expectations with respect to \( \mathcal{F}_t^Y \) on both sides under \( R_Y \), according to the Fubini-type theorem in [20 Thm. 2], we have

\[
\bar{L}_t^P = 1 + \int_0^t \sigma \mathbb{E}_R[L_{s-} \theta | \mathcal{F}_s^Y] dW_s + \int_0^t \left( \int_{\mathbb{R}_0} (\mathbb{E}_R[L_{s-}(e^{\theta z} - 1) | \mathcal{F}_s^Y])(\mu(ds, dz) - \nu(dz)ds) \right).
\] (98)

By the general formula of conditional expectations, we have

\[
\mathbb{E}_R[L_{s-} \theta | \mathcal{F}_s^Y] = \bar{L}_s^P \mathbb{E}_P[\theta | \mathcal{F}_s^Y]
\]

\[
\mathbb{E}_R[L_{s-}(e^{\theta z} - 1) | \mathcal{F}_s^Y] = \bar{L}_s^P (\mathbb{E}_P[e^{\theta z} | \mathcal{F}_s^Y] - 1),
\] (99) (100)

which implies

\[
\bar{L}_t^P = 1 + \int_0^t \sigma \bar{L}_s^P \mathbb{E}_P[\theta | \mathcal{F}_s^Y] dW_s + \int_0^t \bar{L}_s^P \left( \int_{\mathbb{R}_0} (\mathbb{E}_P[e^{\theta z} | \mathcal{F}_s^Y] - 1)(\mu(ds, dz) - \nu(dz)ds) \right).
\] (101)

Solving this stochastic differential equation using the general solutions given by (94) and (95), we have

\[
\bar{L}_t^P = e^{\theta t},
\] (102)

where

\[
\rho_t^P = \sigma \int_0^t \mathbb{E}_P[\theta | \mathcal{F}_s^Y] dW_s - \frac{1}{2} \sigma^2 \int_0^t (\mathbb{E}_P[\theta | \mathcal{F}_s^Y])^2 ds + \int_0^t \int_{\mathbb{R}_0} \left( \ln (\mathbb{E}_P[e^{\theta z} | \mathcal{F}_s^Y]) \right) d\mu - (\mathbb{E}_P[e^{\theta z} | \mathcal{F}_s^Y] - 1)\nu(dz)ds.
\] (103)

Note that we have similar expressions for \( \bar{L}_t^Q = \frac{dQ_Y}{dR_Y} \). In order to calculate the relative entropy, we have

\[
\ln \frac{dP_Y}{dQ_Y} = \ln \left( \frac{dP_Y}{dR_Y} \frac{dR_Y}{dQ_Y} \right)
\]

\[
= \ln \frac{dP_Y}{dR_Y} - \ln \frac{dQ_Y}{dR_Y}
\]

\[
= \rho_t^P - \rho_t^Q
\]

\[
= \sigma \int_0^t (\mathbb{E}_P[\theta | \mathcal{F}_s^Y] - \mathbb{E}_Q[\theta | \mathcal{F}_s^Y]) dW_s - \frac{1}{2} \sigma^2 \int_0^t (\mathbb{E}_P[\theta | \mathcal{F}_s^Y] - \mathbb{E}_Q[\theta | \mathcal{F}_s^Y])^2 ds
\]

\[
+ \int_0^t \int_{\mathbb{R}_0} \left( \ln \left( \frac{\mathbb{E}_P[e^{\theta z} | \mathcal{F}_s^Y]}{\mathbb{E}_Q[e^{\theta z} | \mathcal{F}_s^Y]} \right) \right) d\mu - (\mathbb{E}_P[e^{\theta z} | \mathcal{F}_s^Y] - \mathbb{E}_Q[e^{\theta z} | \mathcal{F}_s^Y])\nu(dz)ds
\] (104) (105) (106) (107) (108)

By [21 Thm 7.12] and the martingale translation theorem in [22], we know that under probability measure \( P_Y \), the process \( W_t \) is still a Brownian motion. In fact,

\[
\tilde{W}_t = W_t - \int_0^t \mathbb{E}_P[\theta | \mathcal{F}_s^Y] ds
\] (109)

is a Brownian motion under \( P_Y \). Also, under \( P_Y \), the compensator of \( \mu(ds, dz) \) is no longer \( \nu(dz)ds \), but \( \mathbb{E}_P[e^{\theta z} | \mathcal{F}_s^Y]\nu(dz)ds \).
Bearing these in mind, we represent \( \ln \frac{dP_Y}{dQ_Y} \) as

\[
\ln \frac{dP_Y}{dQ_Y} = \sigma \int_0^t \left( \mathbb{E}_P[\theta] - \mathbb{E}_Q[\theta] \right) d\tilde{W}_s + \frac{1}{2} \sigma^2 \int_0^t \left( \mathbb{E}_P[\mathbb{E}_P[\theta] - \mathbb{E}_Q[\theta]] \right)^2 ds
\]

Then we come back to the conditions under which we guarantee the stochastic integrals are martingales. Note that Theorem 3 assumes that

\[
\int_0^t \mathbb{E}_P[\theta] - \mathbb{E}_Q[\theta] ds = \mathbb{E}_P[\theta] - \mathbb{E}_Q[\theta]\]

in general it does not further imply

\[
\mathbb{E}_P[\theta] = \mathbb{E}_Q[\theta]
\]

hence we have \( \tilde{W}_s \) is a zero mean martingale.

Now, temporarily assuming these two quantities are martingales, we obtain the desired result:

\[
\mathbb{E}_P \int_0^t \left( \mathbb{E}_P[\theta] - \mathbb{E}_Q[\theta] \right)^2 ds < \infty,
\]

then

\[
\sigma \int_0^t \left( \mathbb{E}_P[\theta] - \mathbb{E}_Q[\theta] \right) d\tilde{W}_s
\]

is a martingale, hence has mean zero.

Meanwhile, according to [23 Thm 18.7], if

\[
\int_0^t \int_0^t \mathbb{E}_P[\theta|F^Y_s] \ln \left( \frac{\mathbb{E}_P[\theta|F^Y_s]}{\mathbb{E}_Q[\theta|F^Y_s]} \right) \nu(dz)ds < \infty,
\]

then

\[
\int_0^t \int_0^t \mathbb{E}_P[\theta|F^Y_s] \ln \left( \frac{\mathbb{E}_P[\theta|F^Y_s]}{\mathbb{E}_Q[\theta|F^Y_s]} \right) \nu(dz)ds
\]

is a zero mean martingale.

Now, temporarily assuming these two quantities are martingales, we obtain the desired result:

\[
D(P_Y||Q_Y) = \frac{1}{2} \sigma^2 \int_0^t \left( \mathbb{E}_P[\theta] - \mathbb{E}_Q[\theta] \right)^2 ds
\]

\[
+ \int_0^t \int_0^t \mathbb{E}_P[\theta|F^Y_s] \ln \left( \frac{\mathbb{E}_P[\theta|F^Y_s]}{\mathbb{E}_Q[\theta|F^Y_s]} \right) \nu(dz)ds.
\]

Here, in the last step we have used the following facts:

\[
\mathbb{E}_P(X - \mathbb{E}_Q[X|Y])^2 - \mathbb{E}_P(X - \mathbb{E}_P[X|Y])^2 = \mathbb{E}_P(X|Y) - \mathbb{E}_Q[X|Y])^2,
\]

\[
\mathbb{E}_P\mathbb{E}_P(X, \mathbb{E}_P[X|Y]) - \mathbb{E}_P\mathbb{E}_P(X, \mathbb{E}_Q[X|Y]) = \mathbb{E}_P\mathbb{E}_P(X, \mathbb{E}_Q[X|Y]).
\]

Now we come back to the conditions under which we guarantee the stochastic integrals are martingales. Note that Theorem 3 assumes that

\[
\mathbb{E}_P \int_0^t \ell\mathbb{E}_P(X, \hat{X}^\Phi_s)ds < \infty,
\]

hence we have \( \int_0^t \left( \mathbb{E}_P[\theta|F^Y_s] - \mathbb{E}_Q[\theta|F^Y_s] \right)^2 ds < \infty \), which implies that the stochastic integral with respect to \( \tilde{W}_s \) has mean zero.

However, although \( \mathbb{E}_P \int_0^t \ell\mathbb{E}_P(X, \hat{X}^\Phi_s)ds < \infty \) implies inequality

\[
\int_0^t \int_0^t \left( \mathbb{E}_P[\theta|F^Y_s] \ln \left( \frac{\mathbb{E}_P[\theta|F^Y_s]}{\mathbb{E}_Q[\theta|F^Y_s]} \right) \nu(dz)ds < \infty,
\]

in general it does not further imply

\[
\int_0^t \int_0^t \mathbb{E}_P[\theta|F^Y_s] \ln \left( \frac{\mathbb{E}_P[\theta|F^Y_s]}{\mathbb{E}_Q[\theta|F^Y_s]} \right) \nu(dz)ds < \infty.
\]
We take the following approach. Note that for all \( x, y > 0 \), we have
\[
\begin{align*}
|x \ln x - \ln y| &= x \ln(x/y)I(x > y) + x \ln(y/x)I(x < y) \\
&= x \ln(x/y) - x \ln(x/y)I(x < y) + x \ln\left(1 + \frac{y-x}{x}\right)I(x < y) \\
&\leq x \ln(x/y) - x \ln(x/y)I(x < y) + (y-x)I(x < y) \\
&= x \ln(x/y) + (y-x) - (y-x)I(x < y) - (y-x)I(x > y) + x \ln(y/x)I(x < y) + (y-x)I(x < y) \\
&= \ell_{P}(x,y) - (y-x)I(x > y) + x \ln\left(1 + \frac{y-x}{x}\right)I(x < y) \\
&\leq \ell_{P}(x,y) - (y-x)I(x > y) + (y-x)I(x < y) \\
&= \ell_{P}(x,y) + |x-y| \\
&\leq \ell_{P}(x,y) + x + y.
\end{align*}
\]

Hence, it suffices to show that conditions in Theorem 3 guarantee that
\[
\mathbb{E}_{P} \int_{0}^{t} \int_{\mathbb{R}^{0}} \mathbb{E}[e^{\theta z} | F_{s}^{Y}] \nu(dz) ds < \infty, \quad \mathbb{E}_{P} \int_{0}^{t} \int_{\mathbb{R}^{0}} \mathbb{E}[e^{\theta z} | F_{s}^{X}] \nu(dz) ds < \infty.
\]

It follows from the Input Regularity Property under \( P \) that \( \forall t < \infty \)
\[
\mathbb{E}_{P} \int_{0}^{t} \int_{\mathbb{R}^{0}} \mathbb{E}[e^{\theta z} | F_{s}^{Y}] \nu(dz) ds = t \int_{\mathbb{R}^{0}} \mathbb{E}[e^{\theta z}] \nu(dz) < \infty.
\]

We also have
\[
\mathbb{E}_{P} \int_{0}^{t} \int_{\mathbb{R}^{0}} \mathbb{E}[e^{\theta z} | F_{s}^{X}] \nu(dz) ds = \int_{X \times Y} \int_{0}^{t} \int_{\mathbb{R}^{0}} e^{\phi(X)z} \nu(dz) ds dP_{Y_{t}} dQ_{X|Y_{t}} = \mathbb{E}_{Q} \int_{0}^{t} \int_{\mathbb{R}^{0}} \left( \frac{dP_{Y_{t}}}{dQ_{Y_{t}}} \right) e^{\phi(X)z} \nu(dz) ds < \infty,
\]

where we used the assumptions in Theorem 3 in the last step.

The proof is complete.

D. Proof of Theorem 4

According to Theorem 3 we know for all \( \gamma < \infty \),
\[
D(P_{X_{\gamma}} || Q_{Y_{\gamma}}) = \int_{0}^{\gamma} \mathbb{E}_{P}[\ell_{\mathcal{L}}(X, \hat{X}_{\alpha}^{Q_{\gamma}}) - \ell_{\mathcal{L}}(X, \hat{X}_{\alpha}^{P_{\gamma}})] d\alpha.
\]

We will first show that
\[
\lim_{\gamma \to \infty} D(P_{X_{\gamma}} || Q_{Y_{\gamma}}) = D(P || Q).
\]

Note that
\[
D(P_{X_{\gamma}} || Q_{Y_{\gamma}}) \leq D(P_{X_{\gamma}} || Q_{Y_{\gamma}} | P_{X_{\gamma}}) = D(P || Q) + D(P_{Y_{\gamma}} | X || Q_{Y_{\gamma}} | P_{X_{\gamma}}) = D(P || Q),
\]

where the last equality is due to the fact that \( P_{X_{\gamma}} \)-a.s. \( P_{Y_{\gamma}} | X = Q_{Y_{\gamma}} | X \) provided that \( P \ll Q \). The monotonicity of the left hand side implies that the limit exists when \( \gamma \to \infty \), and
\[
\lim_{\gamma \to \infty} D(P_{X_{\gamma}} || Q_{Y_{\gamma}}) \leq D(P || Q).
\]

On the other hand, by the Law of Large Numbers, \( Y_{\gamma} / \gamma \) converges weakly to \( X \) when \( \gamma \to \infty \). Since relative entropy is lower semi-continuous under weak convergence, we have
\[
\liminf_{\gamma \to \infty} D(P_{Y_{\gamma}} || Q_{Y_{\gamma}}) = \liminf_{\gamma \to \infty} D(P_{Y_{\gamma}} / \gamma || Q_{Y_{\gamma}} / \gamma) \geq D(P || Q).
\]
We conclude that \( \lim_{\gamma \to \infty} D(P_{\gamma} \| Q_{\gamma}) = D(P \| Q) \).

Since
\[
\lim_{\gamma \to \infty} \int_0^\gamma \mathbb{E}_P[\ell_{\mathcal{L}}(X, \hat{X}_a^Q) - \ell_{\mathcal{L}}(X, \hat{X}_a^P)] \, da = \int_0^\infty \mathbb{E}_P[\ell_{\mathcal{L}}(X, \hat{X}_a^Q) - \ell_{\mathcal{L}}(X, \hat{X}_a^P)] \, da,
\]
we have
\[
D(P \| Q) = \int_0^\infty \mathbb{E}_P[\ell_{\mathcal{L}}(X, \hat{X}_a^Q) - \ell_{\mathcal{L}}(X, \hat{X}_a^P)] \, da.
\]

**E. Proof of Lemma 10**

According to Banerjee et al. [11], we have
\[
\frac{dP_\theta}{dP_0}(Y) = e^{-d_\phi(Y,x(\theta))+d_\phi(Y,x(0))},
\]
where \( d_\phi(x_1, x_2) = \phi(x_1) - \phi(x_2) - \phi'(x_2)(x_1 - x_2) \) is the Bregman divergence generated by convex function \( \phi(\cdot) \), and \( \phi(\cdot) \) is the Fenchel–Legendre dual of \( \kappa(\cdot) \).

We have
\[
\ln \frac{dP_{\theta_1}}{dP_{\theta_2}}(Y) = \ln \frac{dP_{\theta_1}}{dP_0} - \ln \frac{dP_{\theta_2}}{dP_0} = -d_\phi(Y, x(\theta_1)) + d_\phi(Y, x(\theta_2)) - d_\phi(Y, x(0))
\]
which is the Bregman divergence generated by convex function \( \phi(\cdot) \), and \( \phi(\cdot) \) is the Fenchel–Legendre dual of \( \kappa(\cdot) \).

Taking expectations on both sides with respect to \( P_{\theta} \), we have
\[
D(P_\theta \| P_0) = \phi(x(\theta_1)) - \phi(x(\theta_2)) - \phi'(x(\theta_2))(x(\theta_1) - x(\theta_2)) = \phi(x_1) - \phi(x_2) - \phi'(x_2)(x_1 - x_2).
\]

**F. Proof of Lemma 12**

We first prove that, in the Gamma model indexed by \( \Gamma(k, aX/k) \), the following relationship holds:
\[
a \frac{\partial p(y)}{\partial a} = -\frac{d(yp(y))}{dy}.
\]

Indeed, we have
\[
p(y) = \int_X \frac{k^k y^{k-1} e^{-\frac{ky}{ax}}}{(ax)^k \Gamma(k)} \, dP_X,
\]
and
\[
a \frac{\partial p(y)}{\partial a} = \int_X \frac{k^k y^{k-1} e^{-\frac{ky}{ax}}}{\Gamma(k)(ax)^k} \left( k \Gamma(k) - k \Gamma(k) \right) \, dP_X,
\]
as well as
\[
-\frac{d(yp(y))}{dy} = \int_X \frac{k^k y^{k-1} e^{-\frac{ky}{ax}}}{\Gamma(k)(ax)^k} \, dP_X.
\]
The result for mutual information follows from expressing $I(X; Y_a)$ via

$$I(X; Y_a) = E_X D(P_{Y_a}|X) \| P_{Y_a}|X).$$ (170)

**VIII. Conclusions**

We have introduced the family of scalar Lévy channels, where the output conditioned on the input is a random variable having an infinitely divisible law. We establish new and general relations between fundamental information measures and optimal estimation loss for this class of channels, under natural and explicitly identified loss functions. We conjecture that the scalar Lévy channels are the largest family of channels admitting information-estimation relations that fully parallel the known results for the Gaussian and Poisson models. As corollaries of our results, we unify the known results for the Gaussian and Poisson models, and present novel representations for the entropy and relative entropy. We illustrate our results via two examples: the Gamma and Negative Binomial channels.

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