Multiparticle entanglement and ranks of density matrices

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Based on the ranks of reduced density matrices, we derive necessary conditions for the separability of multiparticle arbitrary-dimensional mixed states, which are equivalent to sufficient conditions for entanglement. In a similar way we obtain necessary conditions for the separability of a given mixed state with respect to partitions of all particles of the system into subsets. The special case of pure states is discussed separately.

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Entanglement is not only at the heart of quantum mechanics but also a fundamental physical resource in quantum information theory [1]. On the one hand, entanglement strongly demonstrates quantum nonlocality [2], one of the essential features of quantum mechanics; on the other hand, entanglement also plays a key role in many parts of quantum communication and computation [3], such as quantum cryptography, [4] quantum dense coding [5], quantum teleportation [6], quantum one-way computing [7] and so on. Thus it is of great importance to characterize entanglement of quantum states.

One aspect of characterizing entanglement is to distinguish entangled states from separable ones. There has been much significant progress on this problem in many different directions [8, 9]. Important concepts include Schmidt decomposition [10], Bell inequality [11], partial transpose [12], positive maps [13], and entanglement witnesses [14]. However, it is still an open question how to detect whether a multiparticle arbitrary-dimensional state is entangled. Another aspect of characterizing entanglement is to classify entanglement of quantum states. Based on the separability properties of certain partitions of systems into subsystems, Dür et al. [15] have proposed a complete, hierarchic classification of a family of states, where the states, which have the same number of particles and the corresponding particles have the same Hilbert-space dimensions, are put into different levels of a hierarchy with respect to their entanglement properties. Given a set of many particles and a partition of that set into parts, it is an interesting question to ask whether a given multiparticle state is separable with respect to the given partition. In general, this question is still open though many partial results [16, 17] were obtained during the last few years.

In this work, we propose necessary conditions for the separability of multiparticle arbitrary-dimensional mixed states based on the ranks of reduced density matrices. These are equivalent to sufficient conditions for entangled states. An important advantage of these conditions is that they are completely operational for detecting the separability of quantum states. The word “operational” is used to emphasize, as pointed out by Bruß [18], that an operational criterion can be applied to an explicit density matrix ρ, giving some immediate answer like “ρ is separable,” or “ρ is entangled,” or “this criterion is not strong enough to decide whether ρ is separable or entangled.” In a similar way we propose necessary conditions to determine the separability properties of the partitions of all particles in a given mixed state, that is, to determine whether a given mixed state of several subsystems is entangled. Finally we consider pure states. Two necessary and sufficient conditions for entangled and fully entangled pure states, respectively, are proposed. This allows us to present a simple procedure to determine the type of entanglement of a given pure state. In this procedure, we separate all the particles in a given pure state, without destroying entanglement of the initial state, into the parts of a special partition, where every part contains either X(>1) fully entangled particles or only a single particle.

Let us first consider the definition of entanglement [18]. A pure state ρ of N particles A1, A2, ... , AN is called entangled when it can not be written as

\[ \rho = \otimes_{i=1}^{N} \rho_{Ai}, \]  

where \( \rho_{Ai} \) is the single-particle reduced density matrix given by \( \rho_{Ai} \equiv \text{Tr}_{\{A_j\}}(\rho) \) for \( \{A_j\} \) all \( A_j \neq A_i \). A mixed state ρ of N particles A1, A2, ... , AN, described by M probabilities \( p_j \) and M pure states \( \rho^j \) as \( \rho = \sum_{j=1}^{M} p_j \rho^j \), is called entangled when it can not be written as

\[ \rho = \sum_{j=1}^{M} p_j \otimes_{i=1}^{N} \rho^j_{Ai}, \]  

where \( p_j > 0 \) for \( j = 1, 2, \ldots , M \) with \( \sum_{j=1}^{M} p_j = 1 \).

For convenience, we will use the following notation. For a state ρ of N particles A1, A2, ... , AN, the reduced density matrix obtained by tracing ρ over particle Ai is written as \( \rho_{R(i)} = \text{Tr}_{\{A_i\}}(\rho) \) where R(i) denotes the set of the remaining \( (N - 1) \) particles other than particle Ai. In the same way, \( \rho_{R(i,j)} = \text{Tr}_{\{A_i\}}(\rho_{R(i)}) = \text{Tr}_{A_j}(\text{Tr}_{\{A_j\}}(\rho)) = \text{Tr}_{A_k}(\text{Tr}_{\{A_k\}}(\rho)) \) denotes the reduced density matrix obtained by tracing ρ over particles Ai and Aj, \( \rho_{R(i,j,k)} = \text{Tr}_{A_l}(\text{Tr}_{\{A_l\}}(\rho_{R(i,j)}))) \), and so on. In view of these relations, ρ can be called 1-level-higher density matrix of \( \rho_{R(i)} \), 2-level-higher density matrix of \( \rho_{R(i,j)} \), and 1-level-higher density matrix of \( \rho_{R(i,j)} \) and 2-level-higher density matrix of \( \rho_{R(i,j,k)} \).
and so on. It is obvious that the number of the 1-level-higher density matrices of a reduced density matrix can be greater than 1. For example, the 1-level-higher density matrices of $\rho_{R(i)}$ are $\rho_{R(i)}$ and $\rho_{R(j)}$.

The rank of a matrix $\rho$, denoted as $\text{rank}(\rho)$, is the maximal number of linearly independent row vectors (also column vectors) in the matrix $\rho$. The rank of the density matrix of a pure state has the following basic property:

**Lemma 1.** A state is pure if and only if the rank of its density matrix $\rho$ is equal to 1, i.e., $\text{rank}(\rho) = 1$.

**Proof.** A state $\rho$ is pure if and only if $\rho^2 = \rho$ holds, that is, $\rho$ is a projection operator onto a one-dimensional subspace so that only one eigenvalue is equal to 1, all the other ones being zero. Thus the number of linearly independent row vectors of $\rho$ is equal to 1. Therefore $\text{rank}(\rho) = 1$ holds for a pure state $\rho$.

Conversely, for a density matrix $\rho$ with $\text{rank}(\rho) = 1$, since there is only one linearly independent row vector of $\rho$, it is possible to rewrite the density matrix in a new form with only one element, whose value is equal to 1, by selecting a suitable basis. In that basis, $\rho^2 = \rho$ is evident and hence $\rho$ is pure.

Now we discuss necessary conditions for separable states.

**Theorem (Separability Condition).** If a state $\rho$ of $N$ particles $A_1, A_2, \ldots, A_N$ is separable, then the rank of any reduced density matrix of $\rho$ must be less than or equal to the ranks of all of its 1-level-higher density matrices, i.e.,

$$\text{rank}(\rho_{R(i)}) \leq \text{rank}(\rho)$$

holds for any $A_i \in \{A_1, A_2, \ldots, A_N\}$; and

$$\begin{cases} 
\text{rank}(\rho_{R(i),j}) \leq \text{rank}(\rho_{R(i)}) \\
\text{rank}(\rho_{R(i),j}) \leq \text{rank}(\rho_{R(j)})
\end{cases}$$

holds for any pair of all particles; and so on.

**Proof.** Here we will only give the proof for mixed states. Pure states will be considered in Lemma 2. For simplicity, we only prove (3). The remaining inequalities can be proved in a similar way.

A separable mixed state $\rho$ of $N$ particles $A_1, A_2, \ldots, A_N$ and its reduced density matrix $\rho_{R(i)}$ can be written as

$$\rho = \sum_{j=1}^{M} p_j \rho_j = \sum_{j=1}^{M} p_j \bigotimes_{i=1}^{N} \rho_{A_i}^{j}$$

$$\rho_{R(i)} = \sum_{j=1}^{M} p_j \rho_{R(i),j} = \sum_{j=1}^{M} p_j \bigotimes_{k,z=1}^{N} \rho_{A_k}^{j}.$$  

According to Lemma 1, any pure state can be considered a basis vector in its vector space. Thus $M$ pure states $\rho_j$, where $\rho_j = \bigotimes_{i=1}^{N} \rho_{A_i}^{j}$, $\rho_{A_i}^{j} \in \mathcal{H}_{A_i}$, for $j = 1, 2, \ldots, M$, are $M$ basis vectors that span a vector space $U \subset \bigotimes_{k,z=1}^{N} \mathcal{H}_{A_k}$. Here $\mathcal{H}_{A_i}$ denotes the Hilbert space of particle $A_i$. The maximal number of linearly independent vectors among these $M$ basis vectors is the rank of $\rho$, $\text{rank}(\rho)$, and at the same time, it is the dimension of vector space $U$.

In a similar way, $M$ basis vectors $\rho_{R(i),j}$, where $\rho_{R(i),j} = \bigotimes_{k,z=1}^{N} \rho_{A_k}^{j}$ for $j = 1, 2, \ldots, M$, span a vector space $V \subset \bigotimes_{k,z=1}^{N} \mathcal{H}_{A_k}$ with the dimension $\text{rank}(\rho_{R(i)})$.

From the construction of the vector spaces $U$ and $V$ it is clear that $V$ is a linear subspace of $U$, and hence its dimension is not greater than that of $U$. This proves (4) since the dimensions of the vector spaces are equal to the ranks of the density matrices.

The separability conditions (3,4) for mixed states are not sufficient. For example, an important family of the biqubit mixed states are the so called Werner states [18], which are mixtures of a maximally entangled biqubit pure state and the separable biqubit maximally mixed state. These states are fully characterized by the fidelity $F$, which measures the overlap of the maximally entangled biqubit pure state with the Werner states. Though the Werner states do satisfy the separability conditions (3,4), they are entangled for $F > 1/2$.

The necessary (but not sufficient) conditions (3,4) for a mixed state to be separable are logically equivalent to the following sufficient (but not necessary) conditions for a mixed state to be entangled:

**Corollary 1.** Given a mixed state $\rho$, if the rank of at least one of the reduced density matrices of $\rho$ is greater than the rank of one of its 1-level-higher density matrices, then the state $\rho$ is entangled.

For a given mixed state, there are hierarchical relations among all possible partitions of the particles (e.g. in Ref. [15]). For example, consider a partition of all particles into $i$ parts. If we allow some of the parts to act together as a new composite part, then we obtain a new partition into $j$ parts with $j < i$. In a way similar to the proof of the separability conditions (3,4), we obtain the following interesting separability properties of the partitions of the particles in a given mixed state:

**Corollary 2.** Consider a mixed state $\rho = \sum_{j=1}^{M} p_j \rho_j$ and a partition of the particles. If any two parts $U$ and $V$ in the partition are separable, that is, the state of the composition $(U + V)$ of parts $U$ and $V$ can be written as

$$\rho(U + V) = \sum_{j=1}^{M} p_j \rho_j^{(U + V)} = \sum_{j=1}^{M} p_j (\rho_U^j \otimes \rho_V^j)$$

where $\rho_U^j \in \mathcal{H}_U$, $\rho_V^j \in \mathcal{H}_V$ and $\rho_j^{(U + V)} \in \mathcal{H}_{(U + V)}$, then the ranks of the two reduced density matrices $\rho_U^j$ and $\rho_V^j$...
both are less than or equal to the rank of \( \rho_{(U + V)} \), i.e.,

\[
\begin{align*}
\text{rank}(\rho_U) & \leq \text{rank}(\rho_{(U + V)}) \\
\text{rank}(\rho_V) & \leq \text{rank}(\rho_{(U + V)})
\end{align*}
\] (7)

The \( M \) basis vectors \( \rho^0_U \) and the \( M \) basis vectors \( \rho^i_U \) span two linear subspaces of the composite vector space spanned by the \( M \) basis vectors \( \rho^0_{(U + V)} \). Thus as the dimensions of the two linear subspaces, \( \text{rank}(\rho_U) \) and \( \text{rank}(\rho_V) \) both are not greater than \( \text{rank}(\rho_{(U + V)}) \), the dimension of the composite vector space, Corollary 2 is proved. The Werner states again show that the separability conditions for mixed states in Corollary 2 are not sufficient.

The necessary separability conditions for the partitions in Corollary 2 can again be reformulated as sufficient entanglement conditions of the partitions: given a mixed state and a partition of the particles, consider any two parts in the partition. If the rank of at least one of the reduced density matrices of the two parts is greater than the rank of the density matrix of the composition of these two parts, then these two parts are entangled.

Now we discuss pure states.

**Lemma 2.** A pure state is entangled if and only if the rank of at least one of its reduced density matrices is greater than 1.

**Proof.** — If a pure state is entangled, according to Schrödinger’s definition of entanglement [10]: “The whole is in a definite state, the parts taken individually are not”, then at least one of the states obtained by tracing the original state over some particles is mixed. By Lemma 1 the rank of this reduced state is greater than 1. Conversely, if the rank of one reduced density matrix of a pure state is greater than 1, then the reduced state is mixed, and according to Schrödinger’s definition, the original state is entangled.

An important subclass of the multiparticle entangled states are the so-called fully entangled states [17], which cannot be reduced to mixtures of states where a smaller number of particles are entangled. For example, triqubit states that are not of the forms \( \rho_1 \otimes \rho_{23}, \rho_2 \otimes \rho_{13}, \) and \( \rho_3 \otimes \rho_{12} \), or mixtures of these states are fully entangled, such as the Greenberger-Horne-Zeilinger (GHZ) state [21]. In terms of the ranks of reduced density matrices, we obtain the following necessary and sufficient condition for a pure state to be fully entangled:

**Corollary 3.** A pure state is fully entangled if and only if the ranks of its all reduced density matrices are greater than 1.

**Proof.** — A pure state is fully entangled if and only if every particle and every multi-particle combination in the system are entangled with the remaining particles. That is, the states of every individual particle and every individual multi-particle combination are mixed, i.e., the ranks of all reduced density matrices are greater than 1, and vice versa.

For a given pure state \( \rho \), if its particles are separated into two parts \( U \) and \( V \), then the Schmidt decomposition of state \( \rho \) is written as

\[
\rho = \sum_{i=1}^{k} \lambda_i |u_i \rangle \langle v_i| 
\] (8)

where \( |u_i \rangle \in \mathcal{H}_U, \langle v_i| \in \mathcal{H}_V \) and \( \sum_{i=1}^{k} \lambda_i = 1 \) with \( \lambda_i > 0 \). Here the number \( k \) is called the Schmidt rank of \( \rho \), which is the rank of the reduced density matrix \( \rho_U \) (and \( \rho_V \)):

\[
\text{rank}(\rho_U) = \text{rank}(\rho_V). \quad (9)
\]

Then we obtain the following useful Lemma:

**Lemma 3.** Given a pure state \( \rho \), if its particles are separated into two parts \( U \) and \( V \), then \( \text{rank}(\rho_U) = 1 \) holds if and only if these two parts are separable, i.e., \( \rho = \rho_U \otimes \rho_V \).

**Proof.** — If \( \text{rank}(\rho_U) = 1 \) holds, then \( \text{rank}(\rho_V) = 1 \) holds by Eq. (9), thus states \( \rho_U \) and \( \rho_V \) are pure by Lemma 1. According to the proposition in Ref. [21]: “For two systems \( U \) and \( V \), whenever \( U \) is in a pure state, no correlation exists between \( U \) and \( V \)”, states \( \rho_U \) and \( \rho_V \) are separable. Therefore the whole pure state \( \rho \) can be written as \( \rho = \rho_U \otimes \rho_V \). Conversely, if \( \rho \) is pure and separable with respect to the two parts \( U \) and \( V \), that is, \( \rho = \rho_U \otimes \rho_V \), then the ranks obey

\[
\text{rank}(\rho) = \text{rank}(\rho_U) \ast \text{rank}(\rho_V) = 1, \quad \text{and hence} \quad \text{rank}(\rho_U) = \text{rank}(\rho_V) = 1.
\]

Using the results obtained above, we construct the following procedure to find a special partition of a given pure state \( \rho \) of \( N \) particles \( A_1, A_2, \ldots, A_N \), where each part is the minimal set of particles which cannot be separated any more without destroying entanglement of the initial state, so that the particles are separable when they are in different parts but entangled when they are in one and the same part. Our procedure consists in successively searching for all subsets of growing size which are separable from the rest of the system in the sense of Lemma 3. The maximal set size which has to be checked for separability is \([N/2]\) (the maximal integer less than or equal to \( N/2 \)), since along with every separable set of size \( M \), its complement of size \((N - M)\) also is of course separable from all other particles. In more detail the procedure works as follows:

Step 1. Calculate the rank of \( \rho_{RI} \) for all particles. By Lemma 8 if \( \text{rank}(\rho_{RI}) = 1 \) holds, then \( \rho \) factorizes as \( \rho = \rho_{A_i} \otimes \rho_{RI} \). Suppose there exist \( M_1 \), \( 0 \leq M_1 \leq N \), particles that satisfy \( \text{rank}(\rho_{RI}) = 1 \), then \( \rho \) is the tensor product of \( M_1 \) single-particle parts and a part of \((N - M_1)\) particles. After this step, it is impossible that there exists a separable single particle in the \((N - M_1)\)-particle part. If \((N - M_1) > 3 \), holds, we perform the next step, otherwise the procedure ends.

Step 2. For the part of the remaining \((N_2 = N - M_1)\) particles, calculate the rank of \( \rho_{RI} \) for all two-particle
combinations. If there exist $M_2$, $0 \leq M_2 \leq \lfloor N_2/2 \rfloor$, two-particle combinations that satisfy $\text{rank}(\rho_{R(i,j)}) = 1$, then the part of $N_2$ particles is the tensor product of $M_2$ two-particle parts and a part of $(N_2-2M_2)$ particles. If $(N_2-2M_2) > 5$ holds, we perform the next step, otherwise the procedure ends.

The following steps are similar to steps 1 and 2. In the end, if we obtain separable parts in the procedure, then state $\rho$ can be written as the tensor product of those parts. If we do not obtain any separable part in the procedure, then state $\rho$ is fully entangled.

As an example to explain the procedure in detail, we use the 6-qubit pure state $|\Psi\rangle = (1/2)(|0000000\rangle + |0001111\rangle + |0110000\rangle + |0111111\rangle)$. In step 1, after calculating $\text{rank}(\rho_{R(i)})$ for all qubits, we obtain only $\text{rank}(\rho_{R(1)}) = 1$ so that $\rho = \rho_{A_1} \otimes \rho_{R(1)}$. Since $(6-1) > 3$, we continue. In step 2, for the part of the remaining 5 qubits, after calculating $\text{rank}(\rho_{R(i,j)})$ for all 2-qubit combinations, we obtain only $\text{rank}(\rho_{R(1,2,3)}) = 1$ so that $\rho_{R(1)} = \rho_{(A_2,A_3)} \otimes \rho_{R(1,2,3)}$. Since $(5-2) < 5$, we end the procedure. In the end, state $\rho$ can be written as $\rho = \rho_{A_1} \otimes \rho_{(A_2,A_3)} \otimes \rho_{(A_4,A_5,A_6)}$.

In summary, we have proposed separability criteria for multiparticle arbitrary-dimensional mixed states in terms of the ranks of reduced density matrices. Furthermore, we discussed detection and classification of entanglement in multiparticle pure states. As compared to the important necessary condition given by the positivity of the partial transpose \cite{12}, our results are quite convenient to apply but not quite as strong. Combinations of the rank and positive partial transpose criteria have been used to study the separability properties of some special composite systems \cite{24}. It is an interesting problem for the further research to investigate the relation between these two approaches in more detail.

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[1] B. M. Terhal, M. M. Wolf, and A. C. Doherty, Phys. Today 56, No. 4, 46 (2003).
[2] H. M. Wiseman, quant-ph/0509061 and references therein.
[3] M. A. Nielsen and I. L. Chuang, Quantum Information and Computation, (Cambridge Univ. Press, 2000); C. H. Bennett and D. P. DiVincenzo, Nature (London) 404, 247 (2000).
[4] A. K. Ekert, Phys. Rev. Lett. 67, 661 (1991).
[5] C. H. Bennett and S. J. Wiesner, Phys. Rev. Lett. 69, 2881 (1992).
[6] C. H. Bennett et al., Phys. Rev. Lett. 70, 1895 (1993).
[7] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. 86, 5188 (2001).
[8] A. Sen(De), U. Sen, M. Lewenstein, and A. Sanpera, quant-ph/0508032; M. Horodecki, P. Horodecki, and R. Horodecki, in Quantum Information: An Introduction to Basic Theoretical Concepts and Experiments, edited by G. Alber et al., Springer Tracts in Modern Physics, 173 (Springer-Verlag, Berlin, 2001), p. 151; and references therein.
[9] D. Bruß, J. Math. Phys. 43, 4237 (2002).
[10] A. Peres, Quantum Theory: Concepts and Methods, (Kluwer Academic Publishers, Dordrecht, 1993).
[11] J. S. Bell, Physics 2, 195 (1964).
[12] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
[13] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A 223, 1 (1996).
[14] B. M. Terhal, Phys. Lett. A 271, 319 (2000).
[15] W. Dür, J. I. Cirac, and R. Tarrach, Phys. Rev. Lett. 83, 3562 (1999); W. Dür and J. I. Cirac, Phys. Rev. A 61, 042314 (2000).
[16] A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri, Phys. Rev. Lett. 88, 187904 (2002); Phys. Rev. A 71, 032333 (2005); J. Eisert, P. Hyllus, O. Gühne, and M. Curty, Phys. Rev. A 70, 062317 (2004).
[17] J. Uffink, Phys. Rev. Lett. 88, 230406 (2002); M. Seevinck and G. Svetlichny, Phys. Rev. Lett. 89, 060401 (2002).
[18] R. F. Werner, Phys. Rev. A 40, 4277 (1989).
[19] E. Schrödinger, Naturwissenschaften 23, 807 (1935); English translation in Proc. Am. Philos. Soc. 124, 323 (1980).
[20] D. M. Greenberger, M. Horne, and A. Zeilinger, in Bell’s Theorem, Quantum Theory, and Conceptions of the Universe, edited by M. Kafatos, (Kluwer, Dordrecht, 1989), p. 69.
[21] B. d’Espagnat, Conceptual Foundations of Quantum Mechanics, Second Edition, (W. A. Benjamin, Inc., Reading, Massachusetts), p. 52.
[22] H. Lütkepohl, Handbook of Matrices, (John Wiley & Sons, Chichester, 1996).
[23] For $(N-M_1) \leq 3$, the set of all remaining particles cannot contain further separable subsets since all separable single particles have already been found by assumption. Similar considerations hold for the following steps.
[24] B. Kraus, J. I. Cirac, S. Karnas, and M. Lewenstein, Phys. Rev. A 61, 062302 (2000); S. Karnas and M. Lewenstein, Phys. Rev. A 64, 042313 (2001); S. Fei et al., Phys. Rev. A 68, 022315 (2003).