Four-Spinor Reference Sheets

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Abstract

Some facts about 4-spinors listed and discussed. None, well perhaps some, of the work is original. However, locating formulas in other places has proved a time-consuming process in which one must always worry that the formulas found in any given source assume the other metric (I use \{-1, -1, -1, +1\}) or assume some other unexpected preconditions. Here I list some formulas valid in general representations first, then formulas using a chiral representation are displayed, and finally formulas in a special reference frame (the rest frame of the 'current’ \(j\)) in the chiral representation are listed. Some numerical and algebraic exercises are provided.

1 General Representation

We can use any four complex numbers as the components of a 4-spinor in a given representation, \(\psi = \text{col}\{a + bi, c + di, e + fi, g + hi\}\), where ‘col’ indicates a column matrix and the eight numbers \(a...h\) are real. The 4-spinor generates four real-valued vectors: two light-like, one time-like and one space-like. These may be defined using the gamma matrices of the representation as follows:

\[
\begin{align*}
  j^\mu &\equiv \overline{\psi} \gamma^\mu \psi ; \\
  a^\mu &\equiv \overline{\psi} \gamma^\mu \gamma^5 \psi ; \\
  r^\mu &\equiv \overline{\psi} \gamma^\mu \left(\frac{1 + \gamma^5}{2}\right) \psi ; \\
  s^\mu &\equiv \overline{\psi} \gamma^\mu \left(\frac{1 - \gamma^5}{2}\right) \psi,
\end{align*}
\]

(1)

where \(\overline{\psi} \equiv \psi^\dagger \gamma^4\), \(\mu\) is one of \{1,2,3,4\}, and \(\gamma^5 \equiv -i\gamma^1\gamma^2\gamma^3\gamma^4\). Note that the vectors are representation independent; the substitution \(\gamma^\mu \rightarrow S^{-1}\gamma^\mu S\) and \(\psi \rightarrow S^{-1}\psi\) doesn’t change the vectors. By using a specific representation, perhaps the one displayed below in (3), one can show after some algebra that (i) \(r\) and \(s\) are light-like vectors and that (ii) \(j\) is time-like and that (iii) \(a\) is space-like. An exception occurs (iv) when \(r\) or \(s\) is zero; then \(j\) and \(a\) are light-like.

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Figure 1: The vectors make parallelograms.

Since the gammas in (1) are sandwiched between common factors of $\overline{\psi}$ and $\psi$, we see that the following are true:

$$j^\mu = r^\mu + s^\mu ; \quad a^\mu = r^\mu - s^\mu ; \quad 2r^\mu = j^\mu + a^\mu ; \quad 2s^\mu = j^\mu - a^\mu .$$  \hspace{1cm} (2)

The vectors can be arranged in parallelograms, see Fig. 1.

The scalar product of $j$ with itself, $j^2 \equiv j^\mu j_\mu$, is the same as that for $a$, $a^\mu a_\mu = -j^2$, except for the sign. The two vectors are ‘orthogonal’, $j^\mu a_\mu = 0$. We collect scalar products in Table 1.

| Vector | $j$ | $a$ | $r$ | $s$ |
|--------|-----|-----|-----|-----|
| $j$    | $j^2$ | 0   | $j^2/2$ | $j^2/2$ |
| $a$    | $-j^2$ | $-j^2/2$ | $j^2/2$ |
| $r$    | 0   | $j^2/2$ |
| $s$    | 0   | 0   |
2 Chiral Representation [CR]

To get specific formulas for the vectors in terms of the components of the 4-spinor $\psi$ one must choose a representation for the gammas. I choose a chiral representation [CR]:

$$\gamma^k = \begin{pmatrix} 0 & \sigma^k e^{-i\delta} \\ -\sigma^k e^{i\delta} & 0 \end{pmatrix}; \quad \gamma^4 = \begin{pmatrix} 0 & -e^{i\delta} \\ -e^{-i\delta} & 0 \end{pmatrix}; \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

[CR] (3)

where $\delta$ is an arbitrary phase angle, $k$ is any one of $\{1,2,3\}$, ‘1’ is the unit 2x2 matrix, and the Pauli matrices are the 2x2 matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[CR] (4)

One may check that the gammas (3) satisfy $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2 g^\mu\nu \cdot 1$, where ‘1’ is the unit 4x4 matrix and $g^\mu\nu = \text{diag}\{-1, -1, -1, +1\}$ is the 4x4 metric tensor.

Write the 4-spinor $\psi$ as follows

$$\psi = \begin{pmatrix} r \cos(\theta_R/2) \exp(-i\phi_R) \exp(i\frac{\alpha-\beta}{2}) \\ r \sin(\theta_R/2) \exp(+i\phi_R) \exp(i\frac{\alpha-\beta}{2}) \\ l \cos(\theta_L/2) \exp(-i\phi_L) \exp(i\frac{\alpha+\beta}{2}) \\ l \sin(\theta_L/2) \exp(+i\phi_L) \exp(i\frac{\alpha+\beta}{2}) \end{pmatrix}.$$

[CR] (5)

The given four complex numbers making up the components of $\psi$ determine the eight real numbers $r$, $\theta_R$, $\phi_R$, $l$, $\theta_L$, $\phi_L$, $\alpha$, and $\beta$, within the usual additive $n\pi$’s. By (1), (3), and (5) one finds an expression for $j^2$:

$$j^2 = 2r^2l^2(1 + \cos \theta_R \cos \theta_L + \cos \phi_R \cos \phi_L \sin \theta_R \sin \theta_L + \sin \phi_R \sin \phi_L \sin \theta_R \sin \theta_L).$$

[CR] (6)

By (1), with the parameters in (5) and the representation (3), one finds specific formulas for $r$ and $s$,

$$\{r^1, r^2, r^3, r^4\} = \{r^2 \sin \theta_R \cos \phi_R, r^2 \sin \theta_R \sin \phi_R, r^2 \cos \theta_R, r^2 \};$$

[CR] (7)

$$\{s^1, s^2, s^3, s^4\} = \{-l^2 \sin \theta_L \cos \phi_L, -l^2 \sin \theta_L \sin \phi_L, -l^2 \cos \theta_L, l^2 \};$$

[CR] (8)

Clearly the angles $\theta$ and $\phi$ are polar and azimuthal angles of the spatial directions of $r$ and $s$. Specific formulas for $j$ and $a$ follow immediately from (2), (7), and (8).

With the chiral representation the 4-spinor splits into two 2-spinors, $\psi = \text{col}\{\rho, \lambda\}$, where ‘col’ means column matrix. The 2-spinor $\rho$ is right-handed and the other, $\lambda$, is left-handed,
referring to their Lorentz transformation properties. By (5), (7), and (8) one sees that the right 2-spinor $\rho$ determines $r$ and the left 2-spinor $\lambda$ determines $s$. The 2x2 rotation matrix $R(\kappa, \hat{n})$ for a rotation through an angle $\kappa$ about the direction $\hat{n}$ is the same for both right and left 2-spinors, $R(\kappa, \hat{n}) = \exp(-i\hat{n}_k\sigma^k\kappa/2)$. The 2x2 boost matrix $B(u, \hat{n})$ for a boost of speed $\tanh u$ in the direction $\hat{n}$ differs for right and left 2-spinors: $B_R(u, \hat{n}) = \exp(+\hat{n}_k\sigma^k u/2)$ and $B_L(u, \hat{n}) = \exp(-\hat{n}_k\sigma^k u/2)$.

A rotation through an angle $\kappa$ about the direction $\hat{n}$ changes the 4-spinor $\psi$: $\psi \rightarrow [\cos(\kappa/2) \cdot 1 - i \sin(\kappa/2) n_k \gamma^5 \gamma^4 \gamma^k] \psi$, where ‘1’ is the unit 4x4 matrix. The rotation through $\kappa$ about $\hat{n} = \{0,0,1\}$ changes $\{j^1, j^2\}$ to $\{\cos \kappa j^1 - \sin \kappa j^2, \sin \kappa j^1 + \cos \kappa j^2\}$, leaving $j^3$ and $j^4$ unchanged.

A boost of speed $\tanh u$ in the direction $\hat{n}$ changes the 4-spinor $\psi$: $\psi \rightarrow [\cosh(\alpha/2) \cdot 1 + \sinh(\alpha/2) n_k \gamma^4 \gamma^k] \psi$, where ‘1’ is the unit 4x4 matrix. The boost of speed $\tanh u$ in the direction $\hat{n} = \{0,0,1\}$ changes $\{j^3, j^4\}$ to $\{\cosh u j^3 + \sinh u j^4, \sinh u j^3 + \cosh u j^4\}$, leaving $j^1$ and $j^2$ unchanged.

3 \hspace{1cm} j$/$t$ frame

By applying the appropriate boost (3 parameters: $u, \hat{n}^1, \hat{n}^2$ which determines $\hat{n}^3$) we get a new $j$ which has no spatial components; the new $j$ is in its proper frame. Call this the ‘$j$-time frame.’ In this frame the spinor has equal right and left 2-spinors within a phase, $\rho = e^{-i\beta} \lambda$, and the light-like vectors $r$ and $s$ point in opposite directions. The transformed 4-spinor may be written in the form

$$\psi = \sqrt{\frac{j}{2}} \begin{pmatrix} \cos(\theta/2) \exp(-i\phi/2) \exp(i\alpha - \beta/2) \\ \sin(\theta/2) \exp(+i\phi/2) \exp(i\alpha - \beta/2) \\ \cos(\theta/2) \exp(-i\phi/2) \exp(i\alpha + \beta/2) \\ \sin(\theta/2) \exp(+i\phi/2) \exp(i\alpha + \beta/2) \end{pmatrix}, \quad [CR]$$

(i) $[\{\theta, \phi\}]$ where $\{\theta, \phi\}$ are the $\{\text{polar, azimuthal}\}$ angles indicating the direction of $r$ and $a$ which is opposite to the direction of $s$. The overall phase is $\alpha/2$ and the phase shift from the right 2-spinor to the left 2-spinor is $\beta$. The four angles $\{\theta, \phi, \alpha, \beta\}$, the magnitude of $j$, and the three parameters $u, \hat{n}^1, \hat{n}^2$ of the boost amount to eight real numbers which is the same number needed to specify the four complex numbers making up a 4-spinor in a given representation. Thus we still have a general form for the 4-spinor.

(ii) $[\alpha]$ Rotating $\psi$ in the $j$-time frame, (9), leaves $j$ alone and changes the values of $\{\theta, \phi, \alpha\}$. If the rotation axis is in the direction of $a$, $\hat{n}^k = a^k / \sqrt{j^2 + (a^4)^2}$ with $a^4 = 0$ in this frame, then the effect on $\alpha$ is especially simple: $\alpha$ changes by the negative of the rotation
angle $\kappa$, $\alpha \rightarrow \alpha - \kappa$. Rotating by $\kappa = \alpha$ about $a$ brings $\alpha$ to zero, $\alpha \rightarrow 0$. Therefore we may interpret $\alpha$, twice the overall phase of $\psi$ in this frame, as a rotation angle.

The way this works can be seen as follows. When the direction $a$ is along $\{1,0,0\}$, the angles $\theta$ and $\phi$ in (9) are $\theta = \pi/2$ and $\phi = 0$ or $\pi$. For $\phi = 0$ the right and left 2-spinors are given by $\rho = \lambda = \exp(i\alpha/2) \text{col}\{1,1\}$ if we take $\beta = 0$ and $j = 4$. As noted above, the effect of a rotation is to multiply both $\rho$ and $\lambda$ by the same 2x2 matrix $R(\kappa, \hat{n})$. The rotation matrix $\exp(-i\sigma^1\kappa/2)$ for $\hat{n} = \{1,0,0\}$ is a linear combination of the Pauli matrix $\sigma^1$ and the unit 2x2 matrix. But the 2-spinors are eigenspinors of $\sigma^1$ and the unit 2x2 matrix with eigenvalue 1, so the effect of the rotation matrix $\exp(-i\sigma^1\kappa/2)$ is to change the phase of $\rho$ and $\lambda$ by $-\kappa/2$. In short, the two 2-spinors are eigenspinors of the rotation matrix with the same eigenvalue which is the common phase factor $\exp(-i\kappa/2)$.

For $\phi = \pi$, the 2-spinor $\rho = \lambda = \exp(i\alpha/2) \text{col}\{-1,1\}$ is an eigenspinor of $\sigma^1$ with eigenvalue $-1$, so the common phase factor is $\exp(+i\kappa/2)$. In Table 2, we collect the change in angles $\{\theta, \phi, \alpha\}$ due to rotations of angle $\kappa$ about the coordinate axes.

(iii) $[\beta]$ The phase $\beta$ is changed, $\beta \rightarrow \beta \pm \kappa$ sign depending on eigenvalue, when the right-handed 2-spinor $\rho$ is rotated by $\kappa$ and $\lambda$ is rotated through $-\kappa$, both rotations taking place about $a$. In this case none of the angles $\{\theta, \phi, \alpha\}$ changes and the magnitude of $j$ doesn’t change.

(iv) $[j]$ An operation that changes only the magnitude of $j$ while leaving $\{\theta, \phi, \alpha, \beta\}$ alone can be found. If the right 2-spinor $\rho$ is boosted along the direction of $a$ by $\tanh u$ and $\lambda$ is boosted by the same speed but in the opposite direction $-a$, then the magnitude of $j$ changes, $j \rightarrow [\cosh u - \sinh u]j$.

Thus the 4-spinor parameters $\{\theta, \phi, \alpha\}$ can each be changed by a suitable rotation applied to $\psi$, $\beta$ alone can be changed by applying a counter-clockwise rotation to the right-handed 2-spinor $\rho$ and the equal clockwise rotation to $\lambda$, and the magnitude of $j$ alone can be changed by boosting $\rho$ forward and boosting $\lambda$ backward.
Table 2: Changes \(\{\Delta \theta, \Delta \phi, \Delta \alpha\}\) due to a rotation of angle \(\kappa\) about each coordinate axis. Values of \(\{\theta, \phi, \alpha\}\) are provided that give the components of the eigenspinors. The \(x^1\) and \(x^2\) eigenspinors are not normalized.

| Eigenspinor → | \(x^1\) | \(x^1\) | \(x^2\) | \(x^2\) |
|--------------|--------|--------|--------|--------|
| Components → | \(\text{col}\{-1, 1\}\) | \(\text{col}\{1, 1\}\) | \(\text{col}\{i, 1\}\) | \(\text{col}\{-i, 1\}\) |
| \(\theta, \phi, \alpha\) → | \(\{\frac{\pi}{2}, \pi, -\pi\}\) | \(\{\frac{\pi}{2}, 0, 0\}\) | \(\{\frac{\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}\}\) | \(\{\frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2}\}\) |
| Rotation Axis ↓ | | | | |
| \(x^1\)-axis | \(\{0, 0, +\kappa\}\) | \(\{0, 0, -\kappa\}\) | \(\{+\kappa, 0, 0\}\) | \(\{-\kappa, 0, 0\}\) |
| \(x^2\)-axis | \(\{-\kappa, 0, 0\}\) | \(\{+\kappa, 0, 0\}\) | \(\{0, 0, +\kappa\}\) | \(\{0, 0, -\kappa\}\) |
| \(x^3\)-axis | \(\{0, +\kappa, 0\}\) | \(\{0, +\kappa, 0\}\) | \(\{0, +\kappa, 0\}\) | \(\{0, +\kappa, 0\}\) |

Table 3: A continuation of Table 2

| Eigenspinor → | \(x^3\) | \(x^3\) |
|--------------|--------|--------|
| Components → | \(\text{col}\{0, 1\}\) | \(\text{col}\{1, 0\}\) |
| \(\theta, \phi, \alpha\) → | \(\{\pi, \phi_0, -\phi_0\}\) | \(\{0, \phi_0, \phi_0\}\) |
| Rotation Axis ↓ | | |
| \(x^1\)-axis | \(\{-\kappa, -\phi_0 + \frac{\pi}{2}, +\phi_0 - \frac{\pi}{2}\}\) | \(\{+\kappa, -\phi_0 - \frac{\pi}{2}, -\phi_0 + \frac{\pi}{2}\}\) |
| \(x^2\)-axis | \(\{-\kappa, -\phi_0 + \pi, +\phi_0 - \pi\}\) | \(\{+\kappa, -\phi_0, -\phi_0\}\) |
| \(x^3\)-axis | \(\{0, 0, +\kappa\}\) | \(\{0, 0, -\kappa\}\) |
A    Problems

1. Find $j$, $a$, $r$, and $s$ when
   (i) the 4-spinor $\psi$ has four equal real-valued components: $A = a = c = e = g$ and $0 = b = d = f = h$;
   (ii) as in (i) but with $c$ negative: $A = a = -c = e = g$ and $0 = b = d = f = h$;
   (iii) try $A = a = d = e = f$, $0 = b = c$, and $2A = g$.

2. Use the gammas (3) to find $j$ as a function of $a$ ...

3. Show that $\gamma^1 \cdot \gamma^2 + \gamma^2 \cdot \gamma^1 = 0$ and that $\gamma^2 \cdot \gamma^2 + \gamma^2 \cdot \gamma^2 = -2 \cdot 1$, where ‘1’ is the unit 4x4 matrix.

4. By definition, $\exp[-i\sigma^1 \kappa/2] = \Sigma(-i\sigma^1 \kappa/2)^n/n!$.
   (i) Calculate $(\sigma^1)^2 = \sigma^1 \cdot \sigma^1$.
   (ii) Show $\exp[-i\sigma^1 \kappa/2] = \cos(\kappa/2) \cdot 1 - i \sin(\kappa/2) \sigma^1$, where ‘1’ is the unit 2x2 matrix.

5. Find $r$, $\theta_R$, $\phi_R$, $\alpha$, $\beta$, $\mu$, $\theta_L$, and $\phi_L$ for the 4-spinor of problem 1(iii).

6. The parity operator $P$ has the following effect on a 4-spinor in the chiral representation:
   $P \left( \begin{array} \rho \\ \lambda \end{array} \right) = \left( \begin{array} -\lambda \\ -\rho \end{array} \right)$, where $\rho$ and $\lambda$ are the right- and left-handed 2-spinors. The charge
   conjugation operator $C$ has the following effect: $C\psi = i\gamma^2\psi$.
   Apply $P$, $C$ and $CP$ to the 4-spinor of problem 1(iii) and find the $j$’s and $a$’s.

7. (i) Find a 64 component quantity $\Gamma_{\mu \tau}$ so that $j^\mu = -\Gamma_{\mu \tau} r^\mu s^\tau$ and $\Gamma_{\mu \tau} = -\Gamma_{\tau \mu}$.
   (ii) Show that $0 = r^\mu + s^\mu + \Gamma_{\nu \tau} r^\nu s^\tau$. Interpret that equation using parallel transfer
   and the parallelograms of Figure 1.

References

[1] Among Quantum Mechanics books see, for example: Messiah, A., Quantum Mechanics (North Holland 1966), Volume 2, Chapter XX; Sakurai, J.J., Advanced Quantum Mechanics (Addison-Wesley 1967), Appendices B and C.

[2] Among Quantum Field Theory books see, for example: Itzykson, C. and Zuber, J., Quantum Field Theory (McGraw-Hill 1980), Appendix A-2; Berestetsky, V. B., Lifshitz, E. M., and Pitaevskii, L. P., Quantum Electrodynamics (Pergamon 1980), pp. 76-84; Weinberg, S., The Quantum Theory of Fields (Cambridge University Press, Cambridge, 1995), Volume I, Section 5.4.