Replacing Pfaffians and applications

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Abstract

We present some Pfaffian identities, which are completely different from the Plücker relations. As consequences we obtain a quadratic identity for the number of perfect matchings of plane graphs, which has a simpler form than the formula by Yan et al (Graphical condensation of plane graphs: a combinatorial approach, Theoret. Comput. Sci., to appear), and we also obtain some new determinant identities.

Keywords: Pfaffian; Perfect matching; Skew adjacency matrix; Plücker relation.

1 Introduction

Let $A = (a_{ij})_{n \times n}$ be a skew symmetric matrix of order $n$ and $n$ is even. Suppose that
\[\pi = \{(s_1, t_1), (s_2, t_2), \ldots, (s_{\frac{n}{2}}, t_{\frac{n}{2}})\}\]
is a partition of $[n]$, that is, $[n] = \{s_1, t_1\} \cup \{s_2, t_2\} \cup \ldots \cup \{s_{\frac{n}{2}}, t_{\frac{n}{2}}\}$, where $[n] = \{1, 2, \ldots, n\}$. Define:
\[b_\pi = \text{sgn}(s_1t_1s_2t_2\ldots s_{\frac{n}{2}}t_{\frac{n}{2}}) \prod_{l=1}^{\frac{n}{2}} a_{s_lt_l},\]
where $\text{sgn}(s_1t_1s_2t_2\ldots s_{\frac{n}{2}}t_{\frac{n}{2}})$ denotes the sign of the permutation $s_1t_1s_2t_2\ldots s_{\frac{n}{2}}t_{\frac{n}{2}}$. Note that $b_\pi$ depends neither on the order in which the classes of the partition are listed nor on the order of the two elements of a class. So $b_\pi$ indeed depends only on the choice of the partition $\pi$. The Pfaffian of $A$, denoted by $Pf(A)$, is defined as
\[Pf(A) = \sum_\pi b_\pi,\]

\textsuperscript{1}This work is supported by FMSTF(2004J024) and NSFF(E0540007)
\textsuperscript{2}Partially supported by NSC94-2115-M001-017

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where the summation is over all partitions of \([n]\), which are of the form of \(\pi\). For the sake of convenience, we define the Pfaffian of \(A\) to be zero if \(A\) is a skew symmetric matrix of odd order. The following result is well known:

**Proposition 1.1 (Cayley Theorem, [1])** For any skew symmetric matrix \(A = (a_{ij})_{n \times n}\) of order \(n\), we have

\[
\det(A) = [Pf(A)]^2.
\]

Suppose that \(G = (V(G), E(G))\) is a weighted graph with the vertex set \(V(G) = \{1, 2, \ldots, n\}\), the edge set \(E(G) = \{e_1, e_2, \ldots, e_m\}\) and the edge-weight function \(\omega : E(G) \to \mathbb{R}\), where \(\omega(e) = \omega_e = a_{ij} (\neq 0)\) if \(e = (i, j)\) is an edge of \(G\) and \(\omega_e = a_{ij} = 0\) otherwise, and \(\mathbb{R}\) is the set of real numbers. Suppose \(G^e\) is an orientation of \(G\). Let \(A(G^e) = (b_{ij})_{n \times n}\) be the matrix of order \(n\) defined as follows:

\[
b_{ij} = \begin{cases} 
a_{ij} & \text{if } (i, j) \text{ is an arc in } G^e, \\
-a_{ij} & \text{if } (j, i) \text{ is an arc in } G^e, \\
0 & \text{otherwise}.
\end{cases}
\]

\(A(G^e)\) is called the skew adjacency matrix of \(G^e\) (see [17]). Obviously, \(A(G^e)\) is a skew symmetric matrix, that is, \((A(G^e))^T = -A(G^e)\).

Given a skew symmetric matrix \(A = (a_{ij})_{n \times n}\) with \(n\) even, let \(G = (V(G), E(G))\) be a weighted graph with the vertex set \(V(G) = \{1, 2, \ldots, n\}\), where \(e = (i, j)\) is an edge of \(G\) if and only if \(a_{ij} \neq 0\), and the edge-weight function is defined as \(\omega_e = |a_{ij}|\) if \(e = (i, j)\) is an edge of \(G\) and \(\omega_e = 0\) otherwise. Define \(G^e\) as the orientation of \(G\) in which the direction of every edge \(e = (i, j)\) of \(G\) is from vertices \(i\) to \(j\) if \(a_{ij} > 0\) and from vertices \(j\) to \(i\) otherwise. We call \(G^e\) to be the corresponding directed graph of \(A\). Obviously, \(A = A(G^e)\). It is not difficult to see that the Pfaffian \(Pf(A)\) of \(A\) can be defined as

\[
Pf(A) = \sum_{\pi \in \mathcal{M}(G)} b_\pi,
\]

where the summation is over all perfect matchings \(\pi = \{(s_1, t_1), (s_2, t_2), \ldots, (s_{\frac{n}{2}}, t_{\frac{n}{2}})\}\) of \(G\), and \(b_\pi\) is the product of all \(\omega(s_i, t_i)\) for \(1 \leq i \leq \frac{n}{2}\).

Pfaffians have been studied for almost two hundred years (see [13, 28] for a history), and continue to find numerous applications, for example in matching theory [17] and in the enumeration of plane partitions [28]. It is interesting to extend Leclerc’s combinatorics
of relations for determinants [15] to the analogous rules for Pfaffians. By tools from multilinear algebra Dress and Wenzel [31] gave an elegant proof of an identity concerning pfaffians of skew symmetric matrices, which yields the Grassmann-Plücker identities (for more details see [31], Sect. 7). Okada [22] presented a Pfaffian identity involving elliptic functions, whose rational limit gives a generalization of Schur’s Pfaffian identity. Knuth [13] used a combinatorial method to give an elegant proof of a classical Pfaffian identity found in [20]. Hamel [6] followed Knuth’s approach and introduced other combinatorial methods to prove a host of Pfaffian identities from physics in [7, 21, 30]. Hamel also provided a combinatorial proof of a result in [27] and a new vector-based Pfaffian identity and gave an application to the theory of symmetric functions by proving an identity for Schur Q-functions. For some related recent results see also [8, 9, 18, 23].

This paper is inspired by two results, one of which is that we can use the Pfaffian method to enumerate perfect matchings of plane graphs (see [11, 12]). Inspired by the Dodgson’s Determinant-Evaluation Rule in [4] and the Plücker relations for Pfaffians, Propp [24], Kuo [14] and Yan et al [33] obtained a method of graphical vertex-condensation for enumerating perfect matchings of plane bipartite graphs. The second is that by using the Matching Factorization Theorem in [2] Yan et al [32] found a method of graphical edge-condensation for counting perfect matchings of plane graphs. It is natural to ask whether there exist some Pfaffian identities completely different from the Plücker relations, which can result in some formulas for the method of graphical edge-condensation for enumerating perfect matchings of plane graphs. The results in Section 3 answer this question in the affirmative. As applications, we obtain two new determinant identities in Section 4.1 and we prove a quadratic relation for the number of perfect matchings of plane graphs in Section 4.2, which has a simpler form than the formula in [32].

2 Some Lemmas

In order to present the following lemmas, we need to introduce some notation and terminology. If \( I \) is a subset of \([n]\), we use \( A_I \) to denote the minor of \( A \) by deleting rows and columns indexed by \( I \). If \( I = \{i_1, i_2, \ldots, i_l\} \subseteq [n] \) and \( i_1 < i_2 < \ldots < i_l \), we use \( Pf_A(i_1i_2\ldots i_l) := Pf_A(I) \) to denote the Pfaffian of \( A_{[n]\setminus I} \). Following Knuth’s notation in [13], for two words \( \alpha \) and \( \beta \) we define \( s(\alpha, \beta) \) to be zero if either \( \alpha \) or \( \beta \) has a repeated
letter, or if $\beta$ contains a letter not in $\alpha$. Or, if these are not the case, $s(\alpha, \beta)$ denotes the sign of the permutation that takes $\alpha$ into the word $\beta(\alpha \backslash \beta)$ (where $\alpha \backslash \beta$ denotes the word that remains when the elements of $\beta$ are removed from $\alpha$). Let $S$ be a subset of $\{1, 2, \ldots, n\}$. We call $S$ an even subset if $|S|$ is even and an odd one otherwise.

Dress et al. [3] used tools from multilinear algebra to prove a Pfaffian identity, which was found by Wenzel [31], as follows:

**Lemma 2.1** (Wenzel [31] and Dress et al. [3]) For any two subsets $I_1, I_2 \subseteq [n]$ of odd cardinality and elements $i_1, i_2, \ldots, i_t \in [n]$ with $i_1 < i_2 \ldots < i_t$ and $\{i_1, i_2, \ldots, i_t\} = I_1 \triangle I_2 := (I_1 \backslash I_2) \cup (I_2 \backslash I_1)$, if $A = (a_{ij})_{n \times n}$ is a skew symmetric matrix with $n$ even, then

$$
\sum_{\tau=1}^{t} (-1)^{\tau} Pf_A(I_1 \triangle \{i_\tau\}) Pf(I_2 \triangle \{i_\tau\}) = 0.
$$

A direct result of Lemma 2.1 is the following lemma, which will play an important role in the proofs of our main results.

**Lemma 2.2** Suppose that $A = (a_{ij})_{n \times n}$ is a skew symmetric matrix with $n$ even and $\alpha$ is an even subset of $[n]$. Let $\beta = \{i_1, i_2, \ldots, i_2p\} \subseteq [n] \backslash \alpha$, where $i_1 < i_2 < \ldots < i_{2p}$. Then, for any fixed $s \in [2p]$, we have

$$
Pf_A(\alpha)Pf_A(\alpha \beta) = \sum_{l=1}^{2p} (-1)^{l+s+1} Pf_A(\alpha i_s i_l) Pf_A(\alpha \beta \backslash i_s i_l),
$$

where $Pf_A(\alpha i_s i_s) = 0$.

The following result is a special case of Lemma 2.2.

**Corollary 2.1** Suppose that $A = (a_{ij})_{n \times n}$ is a skew symmetric matrix and $\{i, j, k, l\} \subseteq [n]$. Then

$$
Pf(A_{\{i,j,k,l\}})Pf(A) = Pf(A_{\{i,j\}})Pf(A_{\{k,l\}}) - Pf(A_{\{i,k\}})Pf(A_{\{j,l\}}) + Pf(A_{\{i,l\}})Pf(A_{\{j,k\}}).
$$

(2.1)

**Remark 2.1** There exists a similar formula on the determinant to Corollary 2.1 as follows, which is called the Dodgson’s Determinant–Evaluation Rule (see [4]):

$$
\det(A_{\{1,n\}}) \det(A) = \det(A_{11}) \det(A_{nn}) - \det(A_{1n}) \det(A_{n1}),
$$

(2.2)

where $A$ is an arbitrary matrix of order $n$ and $A_{ij}$ is the minor of $A$ by deleting the $i$-th row and the $j$-th column.
The following result shows the relation between the Pfaffian and the determinant.

**Lemma 2.3 (Godsil [5])** Let $A$ be a square matrix of order $n$. Then

$$Pf \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} = (-1)^{\frac{n(n-1)}{2}} \det(A).$$

Let $A = (a_{st})_{n \times n}$ be a skew symmetric matrix of order $n$ and $G^e$ the corresponding directed graph. Suppose $(i, j)$ is an arc in $G^e$ and hence $a_{ij} > 0$. Let $G^e$ be a directed graph with vertex set $\{1, 2, \ldots, n+1, n+2\}$ obtained from $G^e$ by deleting the arc $(i, j)$ and adding three arcs $(i, n+1), (n+1, n+2)$ and $(n+2, j)$ with weights $\sqrt{a_{ij}}, 1$ and $\sqrt{a_{ij}}$, respectively (see Figures 1(a) and (b) for an illustration). For convenience, if $a_{ij} = 0$ we also regard $\overline{G}^e$ as a directed graph obtained from $G^e$ by adding three arcs $(i, n+1), (n+1, n+2)$ and $(n+2, j)$ with weights 0, 1 and 0. The following lemma will play a key role in the proofs of our main results.

![Figure 1: (a) The directed graph $G^e$. (b) The directed graph $\overline{G}^e$.](image)

**Lemma 2.4** Suppose that $A = (a_{st})_{n \times n}$ is a skew symmetric matrix and $G^e$ is the corresponding directed graph. Let $\overline{G}^e$ be the directed graph with $n + 2$ vertices defined above and $A(\overline{G}^e)$ the skew adjacency matrix of $\overline{G}^e$. Then

$$Pf(A) = Pf(A(\overline{G}^e)).$$

**Proof** Let $G$ and $\overline{G}$ be the underlying graphs of $G^e$ and $\overline{G}^e$, and let $A(G^e)$ be the skew adjacency matrices of $G^e$. Hence $A(G^e) = (a_{st})_{n \times n}$ and $A(\overline{G}^e) = (b_{st})_{(n+2) \times (n+2)}$, where

$$b_{st} = \begin{cases} 
  a_{st} & \text{if } 1 \leq s, t \leq n \text{ and } (s, t) \neq (i, j), (j, i), \\
  \sqrt{a_{ij}} & \text{if } (s, t) = (i, n+1) \text{ or } (n+2, j), \\
  -\sqrt{a_{ij}} & \text{if } (s, t) = (n+1, i) \text{ or } (j, n+2), \\
  1 & \text{if } (s, t) = (n+1, n+2), \\
  -1 & \text{if } (s, t) = (n+2, n+1), \\
  0 & \text{otherwise.}
\end{cases}$$
By the definitions above, we have

\[ Pf(A) = Pf(A(G^e)). \]

Hence we only need to prove

\[ Pf(A(G^e)) = Pf(A(G^e)). \]

Note that, by the definition of the Pfaffian, we have

\[ Pf(A(G^e)) = \sum_{\pi \in \mathcal{M}(G)} b_{\pi}, \quad Pf(A(G^e)) = \sum_{\pi \in \mathcal{M}(\overline{G})} b_{\pi}, \]

where \( \mathcal{M}(G) \) and \( \mathcal{M}(\overline{G}) \) denote the sets of perfect matchings of \( G \) and \( \overline{G} \).

We partition the sets of perfect matchings of \( G \) and \( G^e \) as follows:

\[ \mathcal{M}(G) = \mathcal{M}_1 \cup \mathcal{M}_2, \quad \mathcal{M}(\overline{G}) = \overline{\mathcal{M}}_1 \cup \overline{\mathcal{M}}_2, \]

where \( \mathcal{M}_1 \) is the set of perfect matchings of \( G \) each of which contains edge \( e = (i, j) \), \( \mathcal{M}_2 \) is the set of perfect matchings of \( G \) each of which does not contain edge \( e = (i, j) \), \( \overline{\mathcal{M}}_1 \) is the set of perfect matchings of \( \overline{G} \) each of which contains both of edges \( (i, n+1) \) and \( (n+2, j) \), and \( \overline{\mathcal{M}}_2 \) is the set of perfect matchings of \( \overline{G} \) each of which contains edge \( (n+1, n+2) \).

Suppose \( \pi \) is a perfect matching of \( G \). If \( \pi \in \mathcal{M}_1 \), then there exists uniquely a perfect matching \( \pi' \) of \( G - i - j \) such that \( \pi = \pi' \cup \{(i, j)\} \). It is clear that there is a natural way to regard \( \pi' \) as a matching of \( \overline{G} \). Define: \( \overline{\pi} = \pi' \cup \{(i, n+1), (n+2, j)\} \). Hence \( \overline{\pi} \in \overline{\mathcal{M}}_1 \). Similarly, if \( \pi \in \mathcal{M}_2 \), we can define: \( \overline{\pi} = \pi \cup \{(n+1, n+2)\} \) and hence \( \overline{\pi} \in \overline{\mathcal{M}}_2 \). It is not difficult to see that the mapping \( f : \pi \mapsto \overline{\pi} \) between \( \mathcal{M}(G) \) and \( \mathcal{M}(\overline{G}) \) is bijective.

Hence we only need to prove that for any perfect matching \( \pi \) of \( G \) we have \( b_{\pi} = b_{\overline{\pi}} \). By the definition of \( \overline{\pi} \), if \( \pi = \{(s_1, t_1), (s_2, t_2), \ldots, (s_{l-1}, t_{l-1}), (i, j), (s_l, t_{l+1}), \ldots, (s_n, t_m)\} \in \mathcal{M}_1 \), then \( \overline{\pi} = \{(s_1, t_1), (s_2, t_2), \ldots, (s_{l-1}, t_{l-1}), (i, n+1), (n+2, j), (s_l, t_{l+1}), \ldots, (s_n, t_m)\} \in \overline{\mathcal{M}}_1 \). Note that

\[
sgn(s_1t_1 \ldots s_{l-1}t_{l-1}is_{l+1}t_{l+1} \ldots s_nt_m) = sgn(s_1t_1 \ldots s_{l-1}t_{l-1}i(n+1)(n+2)js_{l+1}t_{l+1} \ldots s_nt_m); \\
b_{s_1t_1} \cdot \ldots \cdot b_{s_{l-1}t_{l-1}}b_{i(n+1)}b_{i(n+2)}b_{s_l,t_{l+1}} \cdot \ldots \cdot b_{s_n,t_m} = \\
\frac{a_{s_1t_1} \cdot \ldots \cdot a_{s_{l-1}t_{l-1}} \sqrt{A_{ij}A_{ij}}a_{s_{l+1}t_{l+1}} \cdot \ldots \cdot a_{s_nt_m}}{a_{s_1t_1} \cdot \ldots \cdot a_{s_{l-1}t_{l-1}}a_{ij}a_{s_{l+1}t_{l+1}} \cdot \ldots \cdot a_{s_nt_m}}.
\]

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Thus we have showed that if $\pi \in \mathcal{M}_1$ then we have $b_\pi = b_{\overline{\pi}}$. Similarly, we can prove that if $\pi \in \mathcal{M}_2$ then we have $b_\pi = b_{\overline{\pi}}$. So we have proved that $Pf(A(G)) = Pf(A(\overline{G}))$, and the lemma follows.

3 New Pfaffian identities

We first need to introduce some notation. In this section, we assume that $A = (a_{ij})_{n \times n}$ is a skew symmetric matrix with $n$ even. Suppose $E = \{(i_l, j_l) | l = 1, 2, \ldots, k\}$ is a subset of $[n] \times [n]$ such that $i_1 \leq i_2 \leq \ldots \leq i_k$ and $i_l < j_l$ for $1 \leq l \leq k$. We define a new skew symmetric matrix $E(A)$ of order $n$ from $A$ and $E$ as follows:

$$E(A) = (b_{ij})_{n \times n}, \quad b_{ij} = \begin{cases} a_{ij} & \text{if } (i, j) \notin E \text{ and } i < j, \\ -a_{ji} & \text{if } (j, i) \notin E \text{ and } i > j, \\ 0 & \text{otherwise.} \end{cases}$$

By the definition of $E(A)$, it is obtained from $A$ by replacing all $(i_l, j_l)$ and $(j_l, i_l)$—entries with zeros and not changing the other entries and hence it is a skew symmetric matrix.

For example, if $A = (a_{ij})_{4 \times 4}$ is a skew symmetric matrix and $E = \{(1, 4), (2, 3), (3, 4)\}$, then

$$E(A) = \begin{pmatrix} 0 & a_{12} & a_{13} & 0 \\ -a_{12} & 0 & 0 & a_{24} \\ -a_{13} & 0 & 0 & 0 \\ 0 & -a_{24} & 0 & 0 \end{pmatrix}.$$

Now, we can state one of our main results as follows.

**Theorem 3.1** Suppose $A = (a_{ij})_{n \times n}$ is a skew symmetric matrix of order $n$ and $E = \{(i_l, j_l) | l = 1, 2, \ldots, k\}$ is a non empty subset of $[n] \times [n]$ such that $i_1 \leq i_2 \leq \ldots \leq i_k$, $i_l < j_l$ for $l \in [k]$. Then, for any fixed $p \in [k]$, we have

$$Pf(E(A))Pf(A) = Pf(E_p(A))Pf(E_{\overline{p}}(A)) + \sum_{1 \leq l \leq k, l \neq p} a_{i_l j_l} \left[ f(p, l) Pf(E(A)_{(i_p, j_p)}) Pf(A_{(j_p, i_l)}) - g(p, l) Pf(E(A)_{(i_l, j_p)}) Pf(A_{(j_l, i_p)}) \right],$$

where $E_p = E \setminus \{(i_p, j_p)\}$, $E_{\overline{p}} = \{(i_p, j_p)\}$, $f(p, l) = s([n], i_p j_l) s([n], j_l i_p)$ and $g(p, l) = s([n], j_l i_p) s([n], j_p i_l).$
Proof Let $G^e$ be the corresponding directed graph of $A$ defined as above, whose vertex set is $[n]$. Let $G^e$ be the directed graph with the vertex set $[n+2k]$ obtained from $G^e$ by replacing each arc between every pair of vertices $i_l$ and $j_l$ with three arcs $(i_l, n+2l−1), (n+2l−1, n+2l)$ and $(n+2l, i_l)$ with weights $\sqrt{a_{il}}, 1$ and $\sqrt{a_{ji}}$ if $(i_l, j_l)$ is an arc of $G^e$ and with three arcs $(j_l, n+2l−1), (n+2l−1, n+2l)$ and $(n+2l, i_l)$ with weights $\sqrt{a_{ji}}, 1$ and $\sqrt{a_{il}}$ if $(j_l, i_l)$ is an arc of $G^e$, respectively. For the case $a_{ilj} > 0$ for $1 \leq l \leq k$, Figure 2 (a) and (b) illustrate the procedure constructing $G^e$ from $G^e$. Suppose $\overline{A} = A(G^e)$ is the skew adjacency matrix of $G^e$.

Take $\alpha = [n], \beta = \{n+1, n+2, \ldots, n+2k\} = \{x_i | x_i = n+i, 1 \leq i \leq 2k\}$. Take $q = 2p−1$. Hence $x_q = n+2p−1$ and $(-1)^{l+q+1} = (-1)^l$. By Lemma 2.2, we have

$$P_{f_{\overline{A}}}(\alpha)P_{f_{\overline{A}}}(\alpha\beta) = \sum_{l=1}^{2k} (-1)^l P_{f_{\overline{A}}}(\alpha x_q x_l)P_{f_{\overline{A}}}(\alpha\beta \setminus x_q x_l). \tag{3.1}$$

By the definitions of $E(A)$ and $G^e$ and Lemma 2.4, we have

$$P_{f_{\overline{A}}}(\alpha) = Pf(E(A)), \quad P_{f_{\overline{A}}}(\alpha\beta) = Pf(A). \tag{3.2}$$

We set

$$a_{vl} = -P_{f_{\overline{A}}}(\alpha x_q x_{2l−1})P_{f_{\overline{A}}}(\alpha\beta \setminus x_q x_{2l−1}),$$

$$b_{vl} = P_{f_{\overline{A}}}(\alpha x_q x_{2l'})P_{f_{\overline{A}}}(\alpha\beta \setminus x_q x_{2l'}),$$

that is,

$$a_{vl} = -P_{f_{\overline{A}}}(\alpha(n+2p−1)(n+2l−1))P_{f_{\overline{A}}}(\alpha\beta \setminus (n+2p−1)(n+2l−1)), \tag{3.3}$$

$$b_{vl} = P_{f_{\overline{A}}}(\alpha(n+2p−1)(n+2l'))P_{f_{\overline{A}}}(\alpha\beta \setminus (n+2p−1)(n+2l')). \tag{3.4}$$

By Lemma 2.4, it is not difficult to see that

$$b_p = P_{f_{\overline{A}}}(\alpha x_q x_p)P_{f_{\overline{A}}}(\alpha\beta \setminus \{x_q x_p\}) = Pf(E_p(A))Pf(\overline{E_p}(A)). \tag{3.5}$$
Note that $a_p = 0$. Hence we have

$$Pf_\alpha(A)Pf_\beta(A) = \sum_{l=1}^{2k} (-1)^l Pf_\alpha(x_q x_l)Pf_\beta(x_l)$$

$$= Pf(E(A))Pf(E(A)) + \sum_{1 \leq l' \leq k, l' \neq p} (a_{l'} + b_{l'}). \quad (3.6)$$

Obviously, if $a_{i_p,j_p} = 0$ then theorem is trivial. Hence we may assume that $a_{i_p,j_p} \neq 0$.

First, we prove that if $a_{i_p,j_p} > 0$ then the theorem holds. From (3.2) and (3.6) it suffices to prove the following claim:

**Claim** For any $l' \in [k]$ and $l' \neq p$, if $a_{i_p,j_p} > 0$ we have

$$a_{l'} + b_{l'} = s([n], i_p j_{l'}) s([n], j_{l'} i_p) a_{i_p,j_p} a_{i_{l'},j_{l'}} Pf(E(A)_{i_p,j_p}) Pf(A_{j_{l'},i_{l'}})$$

$$+ s([n], j_{l'} i_p) s([n], i_p j_{l'}) a_{i_{l'},j_{l'}} a_{i_p,j_p} Pf(E(A)_{i_p,j_p}) Pf(A_{j_{l'},i_{l'}}). \quad (3.7)$$

Suppose $a_{i_p,j_p} > 0$. Then $(i_p, n + 2p - 1), (n + 2p - 1, n + 2p)$ and $(n + 2p, j_p)$ are three arcs of $G^e$ with weights $\sqrt{a_{i_p,j_p}}$, 1 and $\sqrt{a_{i_p,j_p}}$. We need to consider two cases:

(a) $a_{i_{l'},j_{l'}} \geq 0$;

(b) $a_{i_{l'},j_{l'}} < 0$.

If $a_{i_{l'},j_{l'}} \geq 0$, then $(i_{l'}, n + 2l' - 1), (n + 2l' - 1, n + 2l')$ and $(n + 2l', j_{l'})$ are three arcs of $G^e$ with weights $\sqrt{a_{i_{l'},j_{l'}}}$, 1 and $\sqrt{a_{i_{l'},j_{l'}}}$. Suppose $X$ is a subset of the vertex set of $G$. Let $\overline{G}[X]^e$ be the directed subgraph of $G^e$ induced by $X$ and $\overline{G}[X]$ the underlying graph of $\overline{G}[X]^e$. Note that $\overline{G}[\alpha(n+2l'-1)(n+2p-1)]$ contains two pendant edges $(i_{l'}, n+2i_{l'})$ and $(i_p, n+2p-1)$. Each perfect matching $\pi$ of $\overline{G}[\alpha(n+2l'-1)(n+2p-1)]$ can be denoted by $\pi = \pi' \cup \{(i_{l'}, n+2i_{l'}-1), (i_{l'}, n+2p-1]\}$, where $\pi'$ is a perfect matching of $\overline{G}[\alpha \backslash \{i_{l'}, i_p]\}$. Set

$$Pf(A(\overline{G}[\alpha(n+2l'-1)(n+2p-1)]^e)) = \sum_{\pi \in \mathcal{M}(\overline{G}[\alpha(n+2l'-1)(n+2p-1)])} b_\pi,$$

$$Pf(A(\overline{G}[\alpha \backslash \{i_{l'}, i_p\}]^e)) = \sum_{\pi' \in \mathcal{M}(\overline{G}[\alpha \backslash \{i_p, i_{l'}\}])} b_{\pi'},$$

where $\mathcal{M}(G)$ is the set of perfect matchings of a graph $G$. By the definitions of $b_\pi$ and $b_{\pi'}$, it is not difficult to see that

$$b_\pi = \text{sgn}(p-l') s([n], i_{l'} i_p) \sqrt{a_{i_{l'},j_{l'}} b_{\pi'}},$$

where $\text{sgn}(x)$ denotes the sign of $x$. By the definition of $E(A)$, we have

$$Pf(A(\overline{G}[\alpha \backslash \{i_{l'}, i_p\}]^e)) = Pf(E(A)_{i_{l'},i_p}).$$
Hence we have proved the following:

\[ Pf_{\overline{X}}(\alpha(n + 2l - 1)(n + 2p - 1)) = \text{sgn}(p - l') s([n], i_p j'v) \sqrt{a_{ij'jp} a_{ij'j'}} Pf(E(A)_{ij'p}) \]  \quad (3.8)

Similarly, we can prove the following:

\[ Pf_{\overline{X}}(\alpha\beta\mid(n + 2l - 1)(n + 2p - 1)) = \text{sgn}(p - l') s([n], j_p j'v) \sqrt{a_{ij'jp} a_{ij'j'}} Pf(A_{ij'p}) \]  \quad (3.9)

\[ Pf_{\overline{X}}(\alpha(n + 2l')(n + 2p - 1)) = \text{sgn}(l' - p) s([n], i_p j'v) \sqrt{a_{ij'jp} a_{ij'j'}} Pf(E(A)_{ij'p}) \]  \quad (3.10)

\[ Pf_{\overline{X}}(\alpha\beta\mid(n + 2l')(n + 2p - 1)) = \text{sgn}(l' - p) s([n], j_p j'v) \sqrt{a_{ij'jp} a_{ij'j'}} Pf(A_{ij'p}) \]  \quad (3.11)

Then (3.7) is immediate from (3.3), (3.4), (3.8) – (3.11). Hence if \( a_{ij'j'} \geq 0 \) then the claim follows.

If \( a_{ij'j'} < 0 \), then \( (j'v, n + 2l' - 1), (n + 2l' - 1, n + 2l') \) and \( (n + 2l', i_v) \) are three arcs of \( \overline{G'} \) with weights \( \sqrt{-a_{ij'j'}} \) and \( \sqrt{-a_{ij'j'}} \). Similarly, we can prove the following:

\[ Pf_{\overline{X}}(\alpha(n + 2l' - 1)(n + 2p - 1)) = \text{sgn}(p - l') s([n], i_p j'v) \sqrt{a_{ij'jp} a_{ij'j'}} Pf(A(E(A)_{ij'p})) \]  \quad (3.12)

\[ Pf_{\overline{X}}(\alpha\beta\mid(n + 2l' - 1)(n + 2p - 1)) = \text{sgn}(p - l') s([n], j_p j'v) \sqrt{a_{ij'jp} a_{ij'j'}} Pf(A_{ij'p}) \]  \quad (3.13)

\[ Pf_{\overline{X}}(\alpha(n + 2l')(n + 2p - 1)) = \text{sgn}(l' - p) s([n], i_p j'v) \sqrt{a_{ij'jp} a_{ij'j'}} Pf(E(A)_{ij'p}) \]  \quad (3.14)

\[ Pf_{\overline{X}}(\alpha\beta\mid(n + 2l')(n + 2p - 1)) = \text{sgn}(l' - p) s([n], j_p j'v) \sqrt{a_{ij'jp} a_{ij'j'}} Pf(A_{ij'p}) \]  \quad (3.15)

Then (3.7) is immediate from (3.3), (3.4), (3.12) – (3.15). Hence if \( a_{ij'j'} < 0 \) then the claim follows.

Hence we have proved that if \( a_{ijp} > 0 \) then the theorem holds.

If \( a_{ijp} < 0 \), we consider \( Pf(-A) \) and \( Pf(-E(A)) \). Note that \( (-A)_{ijp} > 0 \). The result proved above implies that

\[ Pf(-E(A)) Pf(-A) = Pf(E_p(-A)) Pf(E_p(-A)) - a_{ijp} \sum_{1 \leq l \leq k, l \neq p} (-a_{ijl}) \times \]

\[ [f(p, l) \times Pf(E(-A)_{ijpl}) Pf((-A)_{ijpl})] - g(p, l) Pf(E(-A)_{ijpl}) Pf((-A)_{ijpl}) \]  \quad (3.16)

Note that by the definition of the Pfaffian we have \( Pf(-A) = (-1)^{\frac{p}{2}} Pf(A) \). By (3.16), we can show that we have

\[ Pf(E(A)) Pf(A) = Pf(E_p(A)) Pf(E_p(A)) + \]

\[ \sum_{1 \leq l \leq k, l \neq p} Pf(E(-A)_{ijpl}) Pf((-A)_{ijpl}) \]
\[ a_{i_p j_p} \sum_{1 \leq l \leq k, l \neq p} a_{i_l j_l} \left[ f(p, l) P f(E(A)_{i_p j_l}) P f(A_{i_l j_l}) - g(p, l) P f(E(A)_{i_p j_l}) P f(A_{i_l j_l}) \right], \]

which implies that if \( a_{i_p j_p} < 0 \) then the theorem also holds.

Hence we have proved the theorem. \( \blacksquare \)

**Corollary 3.2** With the same notation as Theorem 3.1, for any fixed \( p \in [k], \)

\[ P f(E(A)) P f(A) = P f(E_p(A)) P f(E_p(A)) + \]

\[ a_{i_p j_p} \sum_{1 \leq l \leq k, l \neq p} a_{i_l j_l} \left[ f(p, l) P f(E(A)_{i_p j_l}) P f(A_{i_l j_l}) - g(p, l) P f(E(A)_{i_p j_l}) P f(A_{i_l j_l}) \right]. \]

**Proof** Let \( A^T \) be the transpose of \( A \). Note that \( P f(A^T) = (-1)^{\frac{k}{2}} P f(A) \). The corollary follows immediately from Theorem 3.1 by considering the transpose of \( A \). \( \blacksquare \)

The following result is a special case of Theorem 3.1 and Corollary 3.2.

**Corollary 3.3** Suppose \( A = (a_{ij})_{n \times n} \) is a skew symmetric matrix of order \( n \) and \( E = \{(i_l, j_l)| l = 1, 2, \ldots, k\} \) is a non empty subset of \([n] \times [n]\) such that \( i_1 < j_1 < i_2 < j_2 < \ldots i_l < j_l < \ldots < i_k < j_k \). Then

\[ P f(E(A)) P f(A) - P f(E_1(A)) P f(E_1(A)) = a_{i_l j_l} \sum_{l=2}^k a_{i_l j_l} \left[ P f(E(A)_{i_l j_l}) P f(A_{i_l j_l}) - P f(E(A)_{i_l j_l}) P f(A_{i_l j_l}) \right] \]

\[ = a_{i_l j_l} \sum_{l=2}^k a_{i_l j_l} \left[ P f(E(A)_{i_l j_l}) P f(A_{i_l j_l}) - P f(E(A)_{j_l j_l}) P f(A_{i_l j_l}) \right]. \]

**Remark 3.2** The Pfaffian identities in Theorem 3.1 and Corollaries 3.2 and 3.3 express the product of Pfaffians of two skew symmetric matrices \( E(A) \) and \( A \) in terms of the Pfaffians of the minors of \( E(A) \) and \( A \), where \( E(A) \) is a skew symmetric matrix obtained from \( A \) by replacing some non zero entries \( a_{i_l j_l} \) and \( a_{j_l j_l} \) of \( A \) with zeros. On the other hand, an obvious observation in the Pfaffian identities known before, which belong to the Plücker relations, is that the related matrices are either a skew symmetric matric \( A \) or some minors of \( A \). Hence the Pfaffian identities in Theorem 3.1 and Corollaries 3.2 and 3.3 are completely new and different from the Plücker relations.

**Example 3.1** Let \( A = (a_{ij})_{4 \times 4} \) and \( E = \{(1,2), (3,4)\} \). Then, by Corollary 3.3, we have
4 Applications

As applications of some results in Section 3, we obtain some determinant identities different from the Plücker relations in Section 4.1 and we prove a quadratic relation for the number of perfect matchings of plane graphs in Section 4.2, which has a simpler form than the formula in [32].

4.1 New determinant identities

We first need to introduce some notation and terminology. Throughout this subsection, we will assume $A = (a_{ij})_{n \times n}$ is an arbitrary matrix of order $n$ and $E = \{(i_l, j_l)|1 \leq l \leq k\} \subseteq [n] \times [n]$, where $a_{i_l j_l} \neq 0$. Define a new matrix of order $n$ from $A$ and $E$, denoted by $E[A] = (b_{st})_{n \times n}$, where $b_{st} = \begin{cases} a_{st} & \text{if } (s, t) \notin E, \\ 0 & \text{otherwise.}\end{cases}$ In other words, $E[A]$ is an $n \times n$ matrix obtained from $A$ by replacing all entries $a_{i_l j_l}$ for $1 \leq l \leq k$ with zeros and not changing the other entries. For example, if $A = (a_{ij})_{4 \times 4}$, $E = \{(1, 2), (2, 2), (3, 1)\}$, by the definition of $E[A]$ we have

$$E[A] = \begin{pmatrix} a_{11} & 0 & a_{13} & a_{14} \\ a_{21} & 0 & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}, \{(3, 4)\}[A] = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$
Lemma 4.5 If $A = (a_{ij})_{n \times n}$ is a matrix of order $n$ and $A^* = \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix}$, then, for any $i, j \in [n]$, $i \neq j$, we have

(i) $Pf(A^*_{i,j}) = 0$,

(ii) $Pf(A^*_{i,n+i,j}) = (-1)^{\frac{1}{2}(n-1)(n-2)} det(A_{ij})$,

(iii) $Pf(A^*_{i,n+j,i}) = (-1)^{\frac{1}{2}(n-1)(n-2)} det(A_{ji})$,

where $A_{ij}$ denotes the minor of $A$ obtained by deleting the $i$–th row and $j$–th column from $A$.

Proof Note that $A^*_{i,j} = \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix}$, where $B$ is an $(n - 2) \times n$ matrix obtained from $A$ by deleting two rows indexed by $i$ and $j$. Obviously, $\det \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} = 0$. Hence by Cayley Theorem we have $\left[ Pf(A^*_{i,j}) \right]^2 = \det \begin{pmatrix} 0 & B \\ -B^T & 0 \end{pmatrix} = 0$, which implies that $Pf(A^*_{i,j}) = 0$. Similarly, by Lemma 2.3 we can prove (ii) and (iii). Hence the lemma follows.

Theorem 4.2 Let $A = (a_{ij})_{n \times n}$ be a matrix of order $n$ and $E = \{(i_1, j_1)|1 \leq l \leq k\}$ a non empty subset of $[n] \times [n]$, where $i_1 \leq i_2 \leq \ldots \leq i_k$. Then for a fixed $p \in [k]$ we have

$$\det(E[A]) \det(A) = \det(E_p[A]) \det(E_p^T[A]) - \sum_{1 \leq l \leq k, l \neq p} (-1)^{i_1 + j_1 + j_p + j_p a_{i_p j_p} a_{i_j j_i}} \det(E[A]) \det(A_{i_j j_i})$$

where $E_p = E\setminus \{(i_p, j_p)\}$ and $E_p^T = \{(i_p, j_p)\}$.

Proof Define: $A^* = \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix} = (a^*_{ij})_{2n \times 2n}$ and $E^* = \{(i_1, n + j_1)|1 \leq l \leq k\}$. By Theorem 3.1, we have

$$Pf(E^*(A^*)) Pf(A^*) = Pf(E^*_p(A^*)) Pf(E^*_p(A^*)) +$$

$$a^*_{i_p(n+j_p)} \sum_{1 \leq l \leq k, l \neq p} a^*_{i_l(n+j_l)} \left[ f(p, l) Pf(E^*(A^*)_{i_p, n+j_l}) Pf(A^*_{i_l, n+j_l}) - g(p, l)Pf(E^*(A^*)_{i_p, n+j_l}) Pf(A^*_{i_l, n+j_l}) \right]$$

$$g(p, l)Pf(E^*(A^*)_{i_p, n+j_l}) Pf(A^*_{i_l, n+j_l})]$$

(4.1)

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where \( f(p, l) = s([2n], i_p(n + j_l))s([2n], (n + j_p)i_l) \) and \( g(p, l) = s([2n], i_p i_l)s([2n], (n + j_p)(n + j_l)) \). It is not difficult to see that we have the following:

\[
a^*_{i_p(n+j_p)} = a_{i_p, j_p}, \quad a^*_{i(n+j_l)} = a_{i, j_l}, \quad f(p, l) = -(-1)^{i_p+j_p+i_l+j_l}.
\] (4.2)

By Lemma 2.3 and the definitions of \( A^* \) and \( E^*(A^*) \), we have

\[
P f(E^*(A^*)) = (-1)^{\frac{1}{2}n(n-1)} \det(E[A]), \quad P f(A^*) = (-1)^{\frac{1}{2}n(n-1)} \det(A),
\] (4.3)

\[
P f(E^*_p(A^*)) = (-1)^{\frac{1}{2}n(n-1)} \det(E_p[A]), \quad P f(E^*_p(A^*)) = (-1)^{\frac{3}{2}n(n-1)} \det(E^*_p[A]).
\] (4.4)

By (i) in Lemma 4.5, we have

\[
P f(E^*(A^*))_{\{i_p, i_l\}} = 0,
\] (4.5)

and by (ii) and (iii) in Lemma 4.5, we have

\[
P f(E^*(A^*))_{\{i_p, n+j_l\}} = (-1)^{\frac{1}{2}(n-1)(n-2)} \det(E[A]_{i_p, j_l}),
\] (4.6)

\[
P f(A^*_p(n+j_p, i_l)) = (-1)^{\frac{1}{2}(n-1)(n-2)} \det(A_{i_p, j_l}).
\] (4.7)

The theorem is immediate from (4.1) – (4.7) and hence we have completed the proof of the theorem.

In the proof of Theorem 4.2, (4.1) is obtained from Theorem 3.1. Obviously, a corresponding identity to (4.1) can be obtained from Corollary 3.2. Similarly, by this identity we can prove the following:

**Theorem 4.3** Let \( A = (a_{ij})_{n \times n} \) be a matrix of order \( n \) and \( E = \{(i_l, j_l)|1 \leq l \leq k\} \) a non empty subset of \([n] \times [n]\), where \( i_1 \leq i_2 \leq \ldots \leq i_k \). Then for a fixed \( p \in [k] \) we have

\[
\det(E[A]) \det(A) = \det(E_p[A]) \det(E^*_p[A]) - \sum_{1 \leq l \leq k, l \neq p} (-1)^{i_p+j+p+i_l+j_l} a_{i_p, j_p} a_{i_l, j_l} \det(E[A]_{i_l, j_l}) \det(A_{i_p, j_l}),
\]

where \( E_p = E \setminus \{(i_p, j_p)\} \) and \( E^*_p = \{(i_p, j_p)\} \).

The following result is immediate from Theorems 4.2 and 4.3.

**Corollary 4.4** Let \( A = (a_{ij})_{n \times n} \) be a matrix of order \( n \) and \( E = \{(i_l, j_l)|1 \leq l \leq k\} \) a non empty subset of \([n] \times [n]\), where \( i_1 \leq i_2 \leq \ldots \leq i_k \). Then for a fixed \( p \in [k] \) we have

\[
\sum_{l=1}^{k} (-1)^{i_i+j_l} a_{i_l, j_l} \{ \det(A[E]_{i_l, j_l}) \det(A_{i_l, j_l}) - \det(E[A]_{i_l, j_l}) \det(A_{i_l, j_l}) \} = 0.
\]
Example 4.2 Let \( A = (a_{ij})_{3 \times 3}, E = \{(1,1), (2,2), (3,3)\} \) and \( p = 2 \). Then, by Theorems 4.2 and 4.3, we have

\[
\begin{align*}
\begin{vmatrix}
   a_{11} & a_{12} & a_{13} \\
   a_{21} & a_{22} & a_{23} \\
   a_{31} & a_{32} & a_{33}
\end{vmatrix}
\end{align*}
\]

\[
\begin{align*}
\begin{vmatrix}
   a_{11} & a_{12} & a_{13} \\
   a_{21} & a_{22} & a_{23} \\
   a_{31} & a_{32} & a_{33}
\end{vmatrix}
\end{align*}
\]

\[
\begin{align*}
= -a_{11}a_{22}
\begin{vmatrix}
   a_{21} & a_{23} \\
   a_{31} & a_{33}
\end{vmatrix}
\begin{vmatrix}
   a_{12} & a_{13} \\
   a_{32} & a_{33}
\end{vmatrix}
\begin{vmatrix}
   -a_{22}a_{33} \\
   a_{21} & a_{23}
\end{vmatrix}
\begin{vmatrix}
   a_{11} & a_{13} \\
   a_{31} & a_{32}
\end{vmatrix}
\begin{vmatrix}
   0 & a_{12} \\
   a_{21} & a_{23}
\end{vmatrix}
\end{align*}
\]

4.2 Graphical edge–condensation for enumerating perfect matchings

Let \( M(G) \) denote the sum of weights of perfect matchings of a weighted graph \( G \), where the weight of a perfect matching \( M \) of \( G \) is defined as the product of weights of edges in \( M \). It is well known that computing \( M(G) \) of a graph \( G \) is an \( NP \)-complete problem (see [10]). Inspired by (2.2)-Dodgson’s Determinant–Evaluation Rule, Propp [24] first found the method of graphical vertex-condensation for enumerating perfect matchings of plane bipartite graphs as follows:

**Proposition 4.2 (Propp [24])** Let \( G = (U,V) \) be a plane bipartite graph in which \( |U| = |V| \). Let vertices \( a, b, c \) and \( d \) form a 4–cycle face in \( G \), \( a, c \in U \), and \( b, d \in V \). Then

\[
M(G)M(G - \{a,b,c,d\}) = M(G - \{a,b\})M(G - \{c,d\}) + M(G - \{a,d\})M(G - \{b,c\}).
\]

By a combinatorial method, Kuo [14] generalized Propp’s result above as follows.

**Proposition 4.3 (Kuo [14])** Let \( G = (U,V) \) be a plane bipartite graph in which \( |U| = |V| \). Let vertices \( a, b, c \), and \( d \) appear in a cyclic order on a face of \( G \).

(1) If \( a, c \in U \), and \( b, d \in V \), then

\[
M(G)M(G - \{a,b,c,d\}) = M(G - \{a,b\})M(G - \{c,d\}) + M(G - \{a,d\})M(G - \{b,c\}).
\]

(2) If \( a, b \in U \), and \( c, d \in V \), then

\[
M(G - \{a,d\})M(G - \{b,c\}) = M(G)M(G - \{a,b,c,d\}) + M(G - \{a,c\})M(G - \{b,d\}).
\]
By Ciucu’s Matching Factorization Theorem in [2], Yan and Zhang [33] obtained a more general result than Kuo’s for the method of graphical vertex-condensation for enumerating perfect matchings of plane bipartite graphs. Furthermore, Yan et al [32] proved the following results:

**Proposition 4.4 (Yan, Yeh and Zhang [32])** Let $G$ be a plane weighted graph with $2n$ vertices. Let vertices $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ ($2 \leq k \leq n$) appear in a cyclic order on a face of $G$, and let $A = \{a_1, a_2, \ldots, a_k\}$, $B = \{b_1, b_2, \ldots, b_k\}$. Then, for any $j = 1, 2, \ldots, k$, we have

$$
\sum_{Y \subseteq B, |Y| \text{ is odd}} M(G-a_j-Y)M(G-A\{a_j\}\overline{Y}) = \sum_{W \subseteq B, |W| \text{ is even}} M(G-W)M(G-A-W),
$$

where the first sum ranges over all odd subsets $Y$ of $B$ and the second sum ranges over all even subsets $W$ of $B$, $\overline{Y} = B \setminus Y$ and $\overline{W} = B \setminus W$.

The following result, which is a special case of the above theorem, was first found by Kenyon and was sent to “Domino Forum” in an Email (for details, see [32]).

**Corollary 4.5** Let $G$ be a plane graph with four vertices $a, b, c$ and $d$ (in the cyclic order) adjacent to a single face. Then

$$
M(G)M(G-a-b-c-d) + M(G-a-c)M(G-b-d) = M(G-a-b)M(G-c-d) + M(G-a-d)M(G-b-c). \quad (4.8)
$$

By Ciucu’s Matching Factorization Theorem in [2], Yan et al [32] also obtained some results for the method of graphical edge-condensation for enumerating perfect matchings of plane graphs. In this subsection, by using the new Pfaffian identity in Corollary 3.3 we will prove a quadratic relation, which has a simpler form than the formula in [32], for the method of graphical edge-condensation for computing perfect matchings of plane graphs.

We first need to introduce the Pfaffian method for enumerating perfect matchings [11, 12]. If $G^e$ is an orientation of a simple graph $G$ and $C$ is a cycle of even length, we say that $C$ is oddly oriented in $G^e$ if $C$ contains odd number of edges that are directed in $G^e$ in the direction of each orientation of $C$. We say that $G^e$ is a Pfaffian orientation of $G$ if every nice cycle of even length of $G$ is oddly oriented in $G^e$ (a cycle $C$ in $G$ is nice if $G-C$ has perfect matchings). It is well known that if a graph $G$ contains no subdivision
of $K_{3,3}$ then $G$ has a Pfaffian orientation (see [16]). McCuaig [19], McCuaig et al [20], and Robertson et al. [25] found a polynomial-time algorithm to show whether a bipartite graph has a Pfaffian orientation.

**Proposition 4.5** ([12, 17]) Let $G^e$ be a Pfaffian orientation of a graph $G$. Then

$$[M(G)]^2 = \det(A(G^e)),$$

where $A(G^e)$ is the skew adjacency matrix of $G^e$.

**Remark 4.3** Let $G^e$ be a Pfaffian orientation of a graph $G$ and $A(G^e)$ the skew adjacency matrix of $G^e$. By Cayley Theorem and Proposition 4.5, we have

$$M(G) = \pm Pf(A(G^e)),$$

which implies that, for two arbitrary perfect matchings $\pi_1$ and $\pi_2$ of $G$, both $b_{\pi_1}$ and $b_{\pi_2}$ have the same sign.

**Proposition 4.6** (Kasteleyn’s theorem, [11, 12, 17]) Every plane graph $G$ has an orientation $G^e$ such that every boundary face—except possibly the unbounded face—has an odd number of edges oriented clockwise. Furthermore, such an orientation is a Pfaffian orientation.

Now we can prove the following result:

**Lemma 4.6** Let $G$ be a plane graph with four vertices $a, b, c$ and $d$ (in the cyclic order) adjacent to the unbounded face. Let $G^e$ be an arbitrary Pfaffian orientation satisfying the condition in Proposition 4.6 and $A = A(G^e)$ the skew adjacency matrix of $G^e$. Then all $Pf(A_{\{a,b,c,d\}})Pf(A), Pf(A_{\{a,b\}})Pf(A_{\{c,d\}}), Pf(A_{\{a,c\}})Pf(A_{\{b,d\}})$ and $Pf(A_{\{a,d\}})Pf(A_{\{b,c\}})$ have the same sign.

**Proof** By (2.1) in Corollary 2.1, we have

$$Pf(A_{\{a,b,c,d\}})Pf(A) = Pf(A_{\{a,b\}})Pf(A_{\{c,d\}}) - Pf(A_{\{a,c\}})Pf(A_{\{b,d\}}) + Pf(A_{\{a,d\}})Pf(A_{\{b,c\}}).$$

(4.9)

Obviously, $A_{\{a,b,c,d\}}, A_{\{a,b\}}, A_{\{c,d\}}, A_{\{a,c\}}, A_{\{b,d\}}, A_{\{a,d\}}$ and $A_{\{b,c\}}$ are the skew adjacency matrices of $G^e - a - b - c - d, G^e - a - b, G^e - c - d, G^e - a - c, G^e - b - d, G^e - a - d$
and $G^e - b - c$, respectively. Note that all the orientations $G^e - a - b - c - d, G^e - a - b, G^e - c - d, G^e - a - c, G^e - b - d, G^e - a - d$ and $G^e - b - c$ of $G - a - b - c - d, G - a - b, G - c - d, G - a - c, G - b - d, G - a - d$ and $G - b - c$ satisfy the condition in Proposition 4.6 and hence are Pfaffian orientations. By Remark 4.3, we have

$$M(G) = \pm Pf(A), \quad M(G - a - b - c - d) = \pm Pf(A_{(a,b,c,d)}).$$

Hence we have proved the following:

$$M(G)M(G - a - b - c - d) = \pm Pf(A)Pf(A_{(a,b,c,d)}). \quad (4.10)$$

Similarly, we can prove the following:

$$M(G - a - b)M(G - c - d) = \pm Pf(A_{(a,b)})Pf(A_{(c,d)}); \quad (4.11)$$

$$M(G - a - c)M(G - b - d) = \pm Pf(A_{(a,c)})Pf(A_{(b,d)}); \quad (4.12)$$

$$M(G - a - d)M(G - b - c) = \pm Pf(A_{(a,d)})Pf(A_{(b,c)}). \quad (4.13)$$

The lemma is immediate from (4.8) – (4.13).

Now we can start to state the main result in this subsection as follows.

**Theorem 4.4** Suppose $G$ is a plane weighted graph with even number of vertices and the weight of every edge $e$ in $G$ is denoted by $\omega_e$. Let $e_1 = a_1b_1, e_2 = a_2b_2, \ldots, e_k = a_kb_k$ ($k \geq 2$) be $k$ independent edges in the boundary of a face $f$ of $G$, and let vertices $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ appear in a cyclic order on $f$ and let $X = \{e_i\mid i = 1, 2, \ldots, k\}$. Then, for any $j = 1, 2, \ldots, k$,

$$M(G)M(G - X) = M(G - e_j)M(G - X\{e_j\}) + \sum_{1 \leq i \leq k, i \neq j} \omega_{e_i}[M(G - b_j - a_i)M(G - X - a_j - b_i) - M(G - b_j - b_i)M(G - X - a_j - a_i)].$$

**Proof** Note that $e_1 = a_1b_1, e_2 = a_2b_2, \ldots, e_k = a_kb_k$ ($k \geq 2$) are $k$ independent edges in the boundary of a face $f$ of $G$. It suffices to prove the following:

$$M(G)M(G - X) = M(G - e_1)M(G - X\{e_1\}) + \sum_{i=2}^{k} \omega_{e_i}[M(G - b_1 - a_i)M(G - X - a_1 - b_i) - M(G - b_1 - b_i)M(G - X - a_1 - a_i)], \quad (4.14)$$
Similarly, we have

\[ P_f(E(A)) = \pm M(G - X)M(G); \quad (4.16) \]

\[ P_f(E_1(A))P_f(E_1(A)) = \pm M(G - X\{e_1\})M(G - e_1); \quad (4.17) \]

\[ P_f(E(A)_{1,2i})P_f(A_{2,2i-1}) = \pm M(G - X - a_i - b_i)M(G - b_i - a_i); \quad (4.18) \]

\[ P_f(E(A)_{1,2i-1})P_f(A_{2,2i}) = \pm M(G - X - a_i - a_i)M(G - b_i - b_i). \quad (4.19) \]

Since every perfect matching of \( G - X \) is also a perfect matching of \( G \), by the definition of the Pfaffian, both \( P_f(A) \) and \( P_f(E(A)) \) have the same sign. Hence by (4.16) we have

\[ P_f(E(A))P_f(A) = M(G - X)M(G). \quad (4.16') \]

Similarly, we have

\[ P_f(E_1(A))P_f(E_1(A)) = M(G - X\{e_1\})M(G - e_1). \quad (4.17') \]

Note that if \( \pi' \) is a perfect matching of \( G - a_i - b_i - a_i - b_i \) \( (i \neq 1) \) then \( \pi = \pi' \cup \{(a_1, b_1), (a_i, b_i)\} \) is a perfect matching of \( G \). By the definition of the Pfaffian, it is not
difficult to see that both $b_\pi$ and $b_{\pi'}$ have the same sign, which implies that both $P f(A)$ and $P f(A_{\{a_1,b_1,a_i,b_i\}})$ have the same sign. Hence $P f(A) P f(A_{\{a_1,b_1,a_i,b_i\}}) \geq 0$. By Lemma 4.6, we have

$$P f(A_{\{a_1,b_i\}}) P f(A_{\{b_1,a_i\}}) \geq 0, P f(A_{\{a_1,a_i\}}) P f(A_{\{b_1,b_i\}}) \geq 0. \quad (4.20)$$

Since every perfect matching of $G - X - a_1 - b_i$ is also a perfect matching of $G - a_1 - b_i$, both $P f(E(A)_{\{a_1,b_i\}})$ and $P f(A_{\{a_1,b_i\}})$ have the same sign. Similarly, both $P f(E(A)_{\{a_1,a_i\}})$ and $P f(A_{\{a_1,a_i\}})$ have the same sign. Hence by (4.20) we have

$$P f(E(A)_{\{a_1,b_i\}}) P f(A_{\{b_1,a_i\}}) \geq 0, P f(E(A)_{\{a_1,a_i\}}) P f(A_{\{b_1,b_i\}}) \geq 0. \quad (4.21)$$

From (4.18), (4.19) and (4.21), we have

$$P f(E(A)_{\{1,2i\}}) P f(A_{\{2,2i-1\}}) = M(G - X - a_1 - b_i) M(G - b_1 - a_i); \quad (4.18')$$

$$P f(E(A)_{\{1,2i-1\}}) P f(A_{\{2,2i\}}) = M(G - X - a_1 - a_i) M(G - b_1 - b_i). \quad (4.19')$$

Note that $a_{12} = \omega_{e_1}$ and $a_{2i-1,2i} = \omega_{e_i}$. It is not difficult to see that (4.14) follows from (4.15) and (4.16') – (4.19'). Hence we have complete the proof of the theorem. \[ \square \]

**Remark 4.4** The formula in Theorem 4.4 for the method of graphical edge-condensation for enumerating perfect matchings of plane graphs has a simpler form than that in Theorem 3.2 in [32].

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