CRYSTAL STRUCTURE OF LEVEL ZERO
EXTREMAL WEIGHT MODULES

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ABSTRACT. We consider the crystal structure of the level zero extremal weight modules $V(\lambda)$, using the crystal base of the quantum affine algebra constructed in [K4]. This approach yields an explicit form for extremal weight vectors in the $U^-$ part of each connected component of the crystal, which are given as Schur functions in the imaginary root vectors. We show the map $\Phi_\lambda$ (K4, §13) induces a correspondence between the global crystal base of $V(\lambda)$ and elements $s_{c_0}(z^{-1})G(b), b \in B_0(U_q[z^\pm 1]W')$.

0. INTRODUCTION.

This paper arises from the study of the modified quantum algebra $\tilde{U}_q(\mathfrak{g}) = \oplus_{\lambda \in P} U_q(\mathfrak{g})a_\lambda$ and its associated crystal structure. For $\mathfrak{g}$ a simple Lie algebra the crystal of $\tilde{U}_q(\mathfrak{g})$ has a Peter-Weyl type decomposition and is isomorphic to the crystal of the quantum coordinate ring $\oplus_{\lambda \in P^+} V(\lambda) \otimes V(-\lambda)$, where $P^+$ is the set of dominant weights of $\mathfrak{g}$, and $V(\lambda)$ denotes the irreducible highest weight module of weight $\lambda$.

This decomposition fails when $\tilde{\mathfrak{g}}$ is the affine Lie algebra associated to $\mathfrak{g}$. In fact, the crystal of $\tilde{U}_q(\tilde{\mathfrak{g}})$ naturally decomposes into pieces according to the level of $\lambda \in P, B(\tilde{U}_q(\tilde{\mathfrak{g}})) = B(\tilde{U}_q(\tilde{\mathfrak{g}}))_0 \oplus B(\tilde{U}_q(\tilde{\mathfrak{g}}))_\pm$, and $B(\tilde{U}_q(\tilde{\mathfrak{g}}))_\pm$ again have Peter–Weyl type decompositions. However, as level zero weights are not Weyl group conjugates of dominant weights, a similar analysis is impossible for $B(\tilde{U}_q(\tilde{\mathfrak{g}}))_0$.

In studying the level zero part of the crystal, Kashiwara introduced (K4) the extremal weight modules $V(\lambda)$. These are a natural variation on highest weight modules, since when $\lambda \in P^+, V(\lambda)$ is the usual irreducible highest weight module generated by $v_\lambda$. For each $\lambda = \sum_i \lambda_i A_i \in P$, $V(\lambda)$ has a basis given by a subset of “*-extremal” elements of the global crystal base of $U_q(\mathfrak{g})a_\lambda$. The term extremal refers to the fact that the usual highest weight relations, $E_i v_\lambda = 0$, $F_i^{(\lambda_i + 1)} v_\lambda = 0$, are replaced by the more general condition that $E_i v_\lambda = 0$, $F_i^{(\lambda_i + 1)} v_\lambda = 0$, if $\lambda_i \geq 0$, and $F_i v_\lambda = 0$, $E_i^{(-\lambda_i + 1)} v_\lambda = 0$, if $\lambda_i \leq 0$, as well as new relations determined by the Weyl group.

We study the crystal structure of $V(\lambda)$ for $\lambda = \sum_i \lambda_i \varpi_i$, $\lambda_i \geq 0$, where $\varpi_i \in P_0^+$ are the fundamental level zero weights. Kashiwara proves (K4) that the $V(\varpi_i)$ are affinizations of certain finite dimensional $U_q^{\ell}$–modules which have global crystal bases. In [K4, §13], Kashiwara conjectures a description of the crystal structure of $V(\lambda)$ for arbitrary $\lambda \in P_0^+$ in terms of an $m$–fold tensor product of the crystal
of $V(\varpi_i)$ and Schur functions. These conjectures also imply a Peter–Weyl type decomposition of $B(\hat{U}_q(\mathfrak{g}))$, which will appear in forthcoming work.

The purpose of this note is to verify these conjectures in the symmetric untwisted case using the crystal base of $B(\hat{U}_q(\mathfrak{g}))$ constructed in [BCP]. The key property of this basis is that it too contains Schur functions naturally, and each component of the crystal of $V(\lambda)$ contains an extremal element corresponding to one of these Schur functions.

While this paper was in preparation, Nakajima [N] released a preprint where the same results are obtained. The primary difference in the proofs is that here we avoid using an explicit description of Lusztig’s braid group action on extremal weight modules. We obtain the key property of connected components of the crystal of $V(\lambda)$ using the basis [BCP] directly, and this requires less calculation.

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1. The algebra $U_q(\mathfrak{g})$ and background.

In this section, we review very briefly the relevant material on quantum affine algebras and crystal bases.

Let $\hat{\mathfrak{g}}$ be a symmetric untwisted affine Lie algebra over $\mathbb{Q}$ with Cartan datum $(\tilde{I}, \cdot)$. Choose a Cartan subalgebra $\mathfrak{t} \subset \hat{\mathfrak{g}}$ such that simple roots $\{\alpha_i\}_{i \in \tilde{I}}$ and the simple coroots $\{h_i\}_{i \in \tilde{I}}$ are linearly independent and $\dim \mathfrak{t} = |\tilde{I}| + 1 = \text{rank } \hat{\mathfrak{g}} + 1$. Denote by $\langle \cdot, \cdot \rangle: \mathfrak{t}^* \times \mathfrak{t} \to \mathbb{Q}$ the canonical pairing.

Fix the root lattice and coroot lattice by

$$Q = \oplus_i \mathbb{Z} \alpha_i \subset \mathfrak{t}^*$$

and

$$Q^\vee = \oplus_i \mathbb{Z} h_i \subset \mathfrak{t}.$$ 

We assume that $\tilde{I} = \{0, 1, \ldots, n\}$ and that the index $0 \in \tilde{I}$ is such that $(I = \tilde{I} \setminus \{0\}, \cdot)$ is the Cartan datum of the underlying finite type algebra $\mathfrak{g}$.

Set $Q_{\pm} = \pm \sum_i \mathbb{Z}_{\geq 0} \alpha_i$ and $Q^\vee_\pm = \pm \sum_i \mathbb{Z}_{\geq 0} h_i$. Let $\delta \in Q_+$ be the unique element satisfying $\{\lambda \in Q; (h_i, \lambda) = 0 \text{ for every } i\} = \mathbb{Z} \delta$. Similarly we define $c \in Q^\vee_+$ by $\{h \in Q^\vee; (h, \alpha_i) = 0 \text{ for every } i\} = \mathbb{Z} c$. We write

$$\delta = \sum_i a_i \alpha_i \quad \text{and} \quad c = \sum_i a_i^\vee h_i.$$ 

We choose a weight lattice $P \subset \mathfrak{t}^*$, and $\Lambda_i \in P, i \in \tilde{I}$, satisfying

$$\alpha_i \in P \quad \text{and} \quad h_i \in P^* \quad \text{for any } i \in \tilde{I}.$$ 

(1.2)

$$\text{For every } i \in \tilde{I}, \langle h_j, \Lambda_i \rangle = \delta_{ji}.$$ 

(1.3)

We set $P^0 = \{\lambda \in P; \langle c, \lambda \rangle = 0\}$. For $i = 1, \ldots, n$, let $\varpi_i = \Lambda_i - a_i^\vee \Lambda_0$. We denote by $P^+_\uparrow$ the dominant level zero weights of the form $\lambda = \sum_{i=1}^n m_i \varpi_i, m_i \geq 0$.

Denote by $\mathcal{R}^+$ the set of positive roots for $\hat{\mathfrak{g}}$. Let $\mathcal{R}_{>}$ (resp. $\mathcal{R}_{<}$) denote the set of $\alpha \in \mathcal{R}^+$ for which the real part of $\alpha$ is positive (resp. negative). We identify the set of positive imaginary roots (with multiplicity) by defining $\mathcal{R}_0 = \{k \delta \mid k > 0\} \times \{1, \ldots, n\} = \{k \delta^{(i)} \mid k > 0, i = 1, \ldots, n\}$. Then define the set of positive roots with multiplicity as

$$\mathcal{R}^+ = \mathcal{R}_> \cup \mathcal{R}_0 \cup \mathcal{R}_<.$$ 

(1.4)
1.1. Quantum affine algebras and crystals. We denote by $U_q = U_q(\widehat{g})$ the quantum affine algebra on generators $F_i$'s, the $E_i$'s and $K_h$ ($h \in P^*$), see [BCP] for details. Let us denote by $U_q^-(\widehat{g})$ (resp. $U_q^+(\widehat{g})$) the subalgebra of $U_q$ generated by the $F_i$'s (resp. by the $E_i$'s). Introduce the divided powers $F_i^{(n)} = F_i^n/[n]_q!$, $E_i^{(n)} = E_i^n/[n]_q!$, where $[n]_q = (q^n - q^{-n})/(q - q^{-1})$. We denote by $(U_q)_Q$ the subalgebra of $U_q$ generated by the $F_i^{(n)}$'s, the $E_i^{(n)}$'s ($i \in \bar{I}$) and $K_h$ ($h \in P^*$) over $Q[q,q^{-1}]$.

We refer to [K2] for a comprehensive introduction to crystals. We mention the details relevant to this paper. A crystal $B$ is graded by a weight lattice $P_B$ containing simple roots $\alpha_i$, $i \in I'$, for some Cartan datum $(I',\cdot)$. For each $b \in B$ we denote its weight by $wt(b) \in P_B$. $B$ also comes with operators $\hat{\epsilon}_i, \hat{\imath}_i : B \to B \sqcup \{0\}, i \in I'$ such that if $\hat{\epsilon}_i(b) \neq 0$ (resp. $\hat{\imath}_i(b) \neq 0$), $wt(\hat{\epsilon}_i(b)) = wt(b) + \alpha_i$ (resp. $wt(\hat{\imath}_i(b)) = wt(b) - \alpha_i$). Given two crystals $B_1, B_2$, their tensor product, $B_1 \otimes B_2$ denotes the crystal $\{b_1 \otimes b_2 | b_1 \in B_1, b_2 \in B_2\}$ with crystal operators defined in [ip.cit., §7.3].

The algebra $U_q^-(\widehat{g})$ (resp. $U_q^+(\widehat{g})$) has a crystal base (see [K1]) denoted by $(L(\infty), B(\infty))$ (resp. $(L(-\infty), B(-\infty))$). The term crystal base refers to the following additional requirements: $L(\infty)$ is a $Q[q]$ subalgebra of $U_q^-(\widehat{g})$ which is invariant under crystal operators $\hat{\epsilon}_i, \hat{\imath}_i, i \in I$, such that $B(\infty)$ is a basis of $L(\infty)/qL(\infty)$ and forms a crystal under the operators induced from $\hat{\epsilon}_i$ and $\hat{\imath}_i$. A similar description holds for $(L(-\infty), B(-\infty))$. We denote by $u_\infty$ (resp. $u_{-\infty}$) the unique element of $B(\infty)$ (resp. $B(-\infty)$) of weight 0.

Let $-\iota$ be the $Q$-algebra automorphism of $U_q$ sending $q$ to $q^{-1}$, $K_h$ to $K_{-h}$, and fixing $E_i, F_i$. Let $M$ be a $U_q$-module with a crystal base $(L, B)$ (see [K1], §2). Define a bar involution $-\bar{}$ on $M$ to be an involution satisfying $\bar{ab} = \bar{b}\bar{a}$ for $a \in U_q$ and $u \in M$. Let $M_Q$ be a subalgebra of $M$ such that

$$\bar{M_Q} = M_Q, \text{ and } u - \bar{u} \in (q - 1)M_Q \text{ for every } u \in M_Q.$$  

Let $E = L \cap T \cap M_Q$. If the map $f : E \to L/qL$ is an isomorphism of $Q[q]$ modules then we say $M$ has a global base. If $b \in B$, we set $G(b) = f^{-1}(b)$. In this case $\bar{G}(b) = G(b)$ and $G(b)$ is called the globalization of $b$.

Let us denote by $U_q(\widehat{g})$ the modified quantum universal enveloping algebra $\oplus_{a \in P} U_q A_a$ (see [L2, K3]). Then $U_q(\widehat{g})$ has a crystal base $(L, B(U_q(\widehat{g})))$. Let $T_\lambda$ be the crystal consisting of a single element $t_\lambda$ of weight $wt(t_\lambda) = \lambda$, with $\hat{\epsilon}_i(t_\lambda) = \hat{\iota}_i(t_\lambda) = 0$. As a crystal, $B(U_q(\widehat{g}))$ is regular and isomorphic to

$$\bigsqcup_{\lambda \in P} B(\infty) \otimes T_\lambda \otimes B(-\infty).$$

The property of being regular allows one to define a Weyl group $W$ action on the crystal. For each $i \in \bar{I}$, define

$$S_i \cdot b = \begin{cases} \hat{\iota}_i^{(h_i,wt(b))} b & \text{ if } \langle h_i, wt(b) \rangle \geq 0, \\ \hat{\epsilon}_i^{(h_i,wt(b))} b & \text{ if } \langle h_i, wt(b) \rangle \leq 0. \end{cases}$$

Then by [K3] the $S_i$ satisfy the defining relations of the Weyl group.

One of the main results of [BCP] is an explicit construction of a crystal base for $U_q^\pm(\widehat{g})$, and by [L6] one of $B(U_q(\widehat{g}))$. We denote by $T_i$ ($= T_{i,1}$ in [L2, Chapter 37]) the automorphism of $U_q$ corresponding to the simple reflection $s_i$, $i = 0, \ldots, n$. 





For each \( w \in W \), the \( T_i \)'s define an automorphism \( T_w \) of \( U_q \). Using these automorphisms, for each \( \alpha \in \mathcal{R}_+ \cup \mathcal{R}_- \), \( i = 1 \ldots n, k > 0 \), we define root vectors \( E_\alpha \), \( \hat{P}_{\alpha,k} \in U_q^+ (\hat{\mathfrak{g}}) \) as in \( \text{BCP} \).

The \( \hat{P}_{\alpha,k} (1 \leq i \leq n, k > 0) \) are used to construct a basis of the imaginary parts of \( (U_q)^+_Q \) as follows. Let \( c_0 \) be an \( n \)-tuple of partitions \((\rho^{(1)}, \rho^{(2)}, \ldots, \rho^{(n)}) \) where each \( \rho^{(i)} = (\rho^{(i)}_1 \geq \rho^{(i)}_2 \geq \ldots) \). For a partition \( \rho \), denote by \( \rho' \) its transpose. For each \( i \), define Schur functions in the \( \hat{P}_{i,k} \) by

\[
S_{\rho^{(i)}} = \det(\hat{P}_{i,\rho'^{(i)}-k+m})_{1 \leq k,m \leq t},
\]

where \( t \geq l(\rho'^{(i)}) \). This puts the \( \hat{P}_{i,k} \) in the role of elementary symmetric functions.

Denote the product over \( i = 1, \ldots, n \) of \( S_{\rho^{(i)}} \) by

\[
S_{c_0}^+ = \prod_{i=1}^n S_{\rho^{(i)}}.
\]

**Definition 1.1.** Let \( c_+ \in \mathbb{N}^{R_>} \) (resp. \( c_- \in \mathbb{N}^{R_-} \)) and \( c_0 \) as above. Denote by \( c = (c_+, c_0, c_-) \). Each \( c \) indexes a basis element of \( (U_q)^+_Q \).

\[
B^+_{c} = (E_{c_+}) \cdot S_{c_0}^+ \cdot (E_{c_-}),
\]

which when specialized at \( q = 1 \) becomes an element of the Kostant \( \mathbb{Z} \)-form of \( U_{q=1}^+ (\hat{\mathfrak{g}}) \). When we refer to an element of this type, we will call \( B^+_{c} \) purely imaginary if \( c_+ = c_- = 0 \).

**Proposition 1.1.** \( \text{BCP} \) \( (L(-\infty), \overline{B^+_c} u_{-\infty} \mod qL(-\infty)) \) forms a crystal base of \( U^+_q \).

There are two more involutions of \( U_q \) which we refer to: Let \( \psi \) be the automorphism of \( U_q \) which sends \( E_i \) to \( F_i \), \( F_i \) to \( E_i \), and \( K_h \) to \( K_{-h} \). It gives a bijection \( B(\infty) \simeq B(-\infty) \). Let \( * \) be the anti-automorphism of \( U_q \) which fixes \( E_i \) and \( F_i \), and sends \( K_h \) to \( K_{-h} \). Restricted to \( U^+_q \), \( * \) gives an bijection \( B(\infty) \simeq B(\infty) \). For the calculations in this paper we use the crystal bases \( (L(-\infty), \overline{(B^+_c)}^* \mod qL(-\infty)) \) of \( U_q^+ \) and \( (L(\infty), \psi(\overline{B^+_c}) \mod qL(\infty)) \) of \( U^-_q \). In what follows, we replace the definitions of root vectors in \( \text{BCP} \) by those obtained by applying the involutions \( *, - \), and \( \psi \) as described. So \( E_{\alpha} \) actually refers to \( \overline{(E_{\alpha})}^* \), \( F_{\alpha} \) actually refers to \( \psi(E_{\alpha}) \), \( \hat{P}_{i,-k} \) refers to \( \psi(\hat{P}_{i,k}) \), \( B^+_{c} = \psi(\overline{B^+_c}) \), etc. The purpose of applying these involutions is to arrange the root ordering of the crystal bases to aid the calculations.

With the above remarks, these imaginary root vectors satisfy an important property following from \( \text{BCP} \) Proposition 2.2 and eq. (4.9) :

**Proposition 1.2.**

(i) \( \hat{P}_{i,-k} = F^{(k)}_{\delta_{-\alpha_i}} E^{(k)}_{\alpha_i} + qx \),

(ii) \( \hat{P}_{i,k} = E^{(k)}_{\alpha_i} F^{(k)}_{\delta_{-\alpha_i}} + qx \),

where in (i), \( x \) is a sum of terms \( B^-_{c} \) with coefficients in \( \mathbb{Z}[q] \) where for each term \( c_- \neq 0 \). In (ii), \( x \) is a sum of terms \( B^+_c \) with coefficients in \( \mathbb{Z}[q] \) where for each term \( c_+ \neq 0 \).
2. Extremal weight modules

2.1. Extremal vectors. Let $M$ be an integrable $U_q$-module. A vector $u \in M$ of weight $\lambda \in P$ is called extremal (see [3]) if we can find vectors $\{u_w\}_{w \in W}$ satisfying the following properties:

(2.1) $u_w = u$ for $w = e$,

(2.2) if $\langle h_i, w\lambda \rangle \geq 0$ then $\delta_i u_w = 0$ and $\delta_i'(h_i, w\lambda) u_w = u_{s_i w}$,

(2.3) if $\langle h_i, w\lambda \rangle \leq 0$, then $\delta_i u_w = 0$ and $\delta_i'(h_i, w\lambda) u_w = u_{s_i w}$.

If such $\{u_w\}$ exists, it is unique and $u_w$ has weight $w\lambda$. We denote $u_w$ by $S_w u$.

Similarly, for a vector $b$ of a regular crystal $B$ with weight $\lambda$, we say that $b$ is an extremal vector if it satisfies the following similar conditions: we can find vectors $\{b_w\}_{w \in W}$ such that

(2.4) $b_w = b$ for $w = e$,

(2.5) if $\langle h_i, w\lambda \rangle \geq 0$ then $\delta_i b_w = 0$ and $\delta_i'(h_i, w\lambda) b_w = b_{s_i w}$,

(2.6) if $\langle h_i, w\lambda \rangle \leq 0$ then $\delta_i b_w = 0$ and $\delta_i'(h_i, w\lambda) b_w = b_{s_i w}$.

Then $b_w$ must be $S_w b$.

For $\lambda \in P$, we denote by $V(\lambda)$ the $U_q$-module generated by $u_\lambda$ with the defining relation that $u_\lambda$ is an extremal vector of weight $\lambda$ (see [3] for details). It is proved in [3] that $V(\lambda)$ has a global crystal base $(L(\lambda), B(\lambda))$. Moreover, if $M$ is any integrable $U_q$ module with extremal weight vector $u$ of weight $\lambda$, there is a unique $U_q$ homomorphism $\Phi : V(\lambda) \to M$, such that $\Phi(u_\lambda) = u$. On an integral $\hat{U}_q (\mathfrak{g})$ module $M$, we use the regularized crystal operators $\delta_i, \delta_i' : M \to M$ as defined in ([4], §6). In this context, $\Phi_\lambda$ commutes with the crystal operators $\delta_i, \delta_i'$.

2.2. Extremal weight modules $V(\lambda)$ for $\lambda \in P_+^0$. Denote by $c_0(\lambda)$ the set of $c_0 = (\rho^{(1)}, \rho^{(2)}, \ldots, \rho^{(n)})$ such that for each $i$, $i(\rho^{(i)}) \leq \lambda_i = \langle h_i, \lambda \rangle$. The following is an important corollary to [4], Theorem 5.1].

**Proposition 2.1.** (i) For any $\lambda \in P_+^0$, any vector in $B(\lambda)$ is connected to an extremal weight vector of the form $b_1 \otimes t_\lambda \otimes u_{-\infty}$, where $b_1$ is purely imaginary with respect to the crystal base.
(ii) Furthermore, all such possible $b_1 \in B(\infty)$ are given by $S_{c_0}^{-1} u_{-\infty} \mod qL(\infty)$ where $c_0 \in c_0(\lambda)$.

**Proof.** (i) By [4], Theorem 5.1] any vector is connected to an extremal weight vector of the form $b_1 \otimes t_\lambda \otimes u_{-\infty}$, where $wt(b_1) = -k \delta$. Using the crystal base to express $b_1$, we take $B_\delta = b_1 \mod qL(\infty)$, for some $c$. Assume that $B_\delta$ isn’t purely imaginary. Since $wt(B_\delta) = -k \delta$, and $B_\delta$ (resp. $B_\delta^-$) consists only of terms in root vectors with positive real part (resp. negative real part), it follows $c_\delta \neq 0$. By [4], Theorem 5.3 (iii)] we have $B_\delta u_\lambda = 0$. However, by assumption $B_\delta u_\lambda \in L(\lambda)$ such that $B_\delta u_\lambda \neq 0 \mod qL(\lambda)$. This is a contradiction. (ii) From Proposition 4.2 we have $\tilde{P}_{k} u_\lambda = F_{k}^{(b)} \delta_{k}^{(b)} u_\lambda$. Since the weights of $V(\lambda)$ are in the convex hull of $W \lambda$ ([4], Corollary 5.2]), this implies that $\tilde{P}_{k} u_\lambda = 0$ for $k > \lambda_i$. Note that for any $i$, $i(\rho^{(i)}) \leq \lambda_i \iff \rho^{(i)}_{-1} \leq \lambda_i$. Since the $\tilde{P}_{k}$ all commute, considering the top row of the determinant $S_{c_0}$, we have $S_{c_0}^{-1} u_{-\infty} = 0$ for $c_0 \notin c_0(\lambda)$. □

Let $z_1$ be the $U_q'$ automorphism of $V(\varpi_i)$ defined in [4], §5.2].
Lemma 2.1. Let $i = 1, \ldots, n$. Then on $V(\varpi_i)$:

\[ \tilde{P}_{i,-1} u_{\varpi_i} = \frac{\tilde{P}_{i,-1} u_{\varpi_i} - z_i^{-1} u_{\varpi_i}}{z_i}, \quad \tilde{P}_{i,-k} u_{\varpi_i} = 0, \quad k > 1. \]

Proof. For $k > 1$ the statement follows from Proposition 1.2. Let $k = 1$. $V(\varpi_i)$ has a unique global basis element of weight $\varpi_i - \delta$, which by definition equals $S_w u_{\varpi_i} = z_i^{-1} u_{\varpi_i}$, where $w = t(\alpha_i)$. Since $\varpi_i$ is regularly $t(\alpha_i)$–dominant (see [K4, §3.1]), it follows from the identity (K3, Appendix B)):

\[ S_j (b_1 \otimes t_\mu \otimes u_{-\infty}) = \tilde{f}_j^n b_1 \otimes t_\mu \otimes u_{-\infty} \text{ if } a = (h_j, wt(b_1) + \mu) \geq 0, \]

that $z_i^{-1} u_{\varpi_i} \mod qL(\varpi_i) \subset B(\infty) \otimes t_{\varpi_i} \otimes u_{-\infty}$.

Since $B_{c^*} u_{\varpi_i} = 0$ for $c_\cdot \neq 0$, the unique element of the crystal of $B(\varpi_i)$ of weight $\varpi_i - \delta$ must be $\tilde{P}_{i,-1} u_{\varpi_i} \mod qL(\varpi_i)$. Note that the globalization of an element in $B(\infty) \otimes t_{\varpi_i} \otimes u_{-\infty}$ remains in $U_q u_{\varpi_i}$. Since for $1 \leq j \neq i \leq n$, $\tilde{P}_{j,-1} u_{\varpi_i} = 0$, we have immediately that $G(\tilde{P}_{i,-1} u_{\varpi_i} \mod qL(\varpi_i)) = \tilde{P}_{i,-1} u_{\varpi_i}$, which completes the proof. \[ \square \]

2.3. The map $\Phi_\lambda$. Let $\lambda = \sum_{i \in I} m_i \varpi_i \in P^0_\varpi$. The module $V' = \bigotimes_{i \in I} V(\varpi_i)_{\otimes m_i}$ has a crystal base $(L(V'), B(V')) = (L(\varpi_i)_{\otimes m_i}, B(\varpi_i)_{\otimes m_i})$. Let $u' = \bigotimes_{i \in I} u_{\varpi_i}^{\otimes m_i}$.

For each $i$, and each of the $\nu = 1, \ldots, m_i$ factors of $V(\varpi_i)_{\otimes m_i}$, we let $z_{i,\nu}$ be the commuting automorphisms defined in [K4, §4.2]. By [K4, Theorem 8.5], the submodule

\[ W' = U_q(\mathfrak{g}) q[z_i^{\pm 1}]_{1 \leq i \leq n, 1 \leq \nu \leq m_i} u' \subset V' \]

has a global crystal base $(L(W'), B(W'))$ such that $L(W') \subset \bigotimes_{i \in I} L(\varpi_i)_{\otimes m_i}$, $B(W') = \bigotimes_{i \in I} B(\varpi_i)_{\otimes m_i}$. Since $W'$ contains the extremal vector $u'$ of weight $\lambda$ we have a unique $U_q$-linear morphism:

\[ \Phi_\lambda : V(\lambda) \to W', \]

sending $u_\lambda$ to $u'$, and which commutes with the crystal operators $e_i, f_i$.

For each $n$–tuple of partition $c_0 = (\rho(1), \rho(2), \ldots, \rho(n))$ we consider the product of Schur functions in the variables $z_i^{-\nu}$ (see [M]):

\[ s_{c_0}(z_i^{-\nu}) = \prod_{i=1}^n s_{\rho(i)}(z_i^{-1}, \ldots, z_i^{-n}). \]

Note that for each $i$, $s_{\rho(i)}(z_i^{-1})$ acts as the 0 map if $m_i < \ell(\rho(i))$. We will omit the indices $i, \nu$ and write $s_{c_0}(z_i^{-1})$.

Using Lemma 2.1 we have:

\[ \Phi_\lambda(S_{c_0} u_\lambda) = s_{c_0}(z_i^{-1}) u'. \]

Proposition 2.2. Let $c_0 = (\rho(1), \rho(2), \ldots, \rho(n))$ be an $n$–tuple of partitions:

\[ \Phi_\lambda(S_{c_0} u_\lambda) = s_{c_0}(z_i^{-1}) u'. \]

Proof. Note that $\sigma \circ (\psi \times \psi) \circ \Delta(a) = \Delta(\psi(a))$ for $a \in U_q$. Since our $\tilde{P}_{i,-k}$ are those given in [BCF] after applying $- \circ \psi$ we have by [BCF, Proposition 3.4] and [3]

\[ \Delta(\tilde{P}_{i,-k}) = \sum_{s=0}^k \tilde{P}_{i,-s} \otimes \tilde{P}_{i,-s-k} + \text{terms acting as 0 on } v_{\varpi_j} \otimes v_{\varpi_j} \text{ for all } j_1, j_2 \in I. \]

This implies that $\Delta^{m_i}(\tilde{P}_{i,-k})$ acts as $e_k(z_i, \ldots, z_i, m_i)$ on $V'$ where $e_k$ is the $k$–th elementary symmetric function. Since polynomials in the $\tilde{P}_{i,-k}$ (resp. elementary
symmetric functions) generate the Schur functions $\overline{S_{c_0}}$ (resp. $s_{c_0}(z^{-1})$) we have $\Phi_\lambda(\overline{S_{c_0}} u_\lambda) = s_{c_0}(z^{-1}) u'$. Since $\Phi_\lambda$ is uniquely defined as a $U_q$ homomorphism, it commutes with the respective $\pi$ actions, where $-\pi$ on $L(W')$ is $c_{n_{\text{norm}}}$ as defined in [K4, §8]. Now since the $z_{i,\nu}$ commute with the bar action on $W'$ the proposition follows. □

Next we consider the image of $B(\lambda)$ under $\Phi_\lambda$. By [K4, Theorem 5.1] every element of $B(\lambda)$ is connected to an extremal vector of the form $b_1 \otimes t_\lambda \otimes u_{-\infty}$, which by Proposition 2.3 equals $S^{-}_{c_0} u_{\infty} \otimes t_\lambda \otimes u_{-\infty}$ mod $qL(\lambda)$. Therefore we have,

\begin{equation}
B(\lambda) = \{x_1 x_2 \ldots x_n | x_i \in \{\bar{e}_i, 0, \bar{f}_i\}, c_0 \in c_0(\lambda)\} \setminus \{0\}.
\end{equation}

Since $\Phi_\lambda$ commutes with crystal operators, and the $z_{i,\nu}$ induce automorphisms of the $U_q$-crystal of $V(\varpi_i)$, we have that $\Phi_\lambda(L(\lambda)) \subset L(W')$. Denote by $\Phi_{\lambda|q=0}$ the induced map $L(\lambda)/qL(\lambda) \rightarrow L(W')/qL(W')$.

**Proposition 2.3.** Let $B_0(W')$ be the connected component of $B(W')$ containing $u'$. Then

$$\Phi_{\lambda|q=0} : \{b \mid b \in B(\lambda)\} \rightarrow \{s_{c_0}(z^{-1})b' \mid c_0 \in c_0(\lambda), b' \in B_0(W')\}$$

is a bijection.

**Proof.** We have

$$\Phi_{\lambda|q=0}(B(\lambda)) \setminus \{0\} = \{s_{c_0}(z^{-1})B_0(W')\}.$$ 

Arguing using (2.11), we check that $\Phi_{\lambda|q=0}(B(\lambda))$ is injective. Let $b \in B(\lambda)$ such that $\Phi_{\lambda|q=0}(b) = 0$. Since $b$ is connected by crystal operators to $b_1 \otimes t_\lambda \otimes u_{-\infty}$, where $b_1 = S^{-}_{c_0} u_{\infty}$ mod $qL(\infty)$, $c_0 \in c_0(\lambda)$, this implies $\Phi_{\lambda|q=0}(S^{-}_{c_0} u_\lambda$ mod $qL(\lambda)) = 0$. This contradicts Proposition 2.2. □

**Corollary 2.1.** The map $\Phi_\lambda$ is injective.

**Proof.** Since $\Phi_{\lambda|q=0} : L(\lambda)/qL(\lambda) \rightarrow L(W')/qL(W')$ maps the crystal base $B(\lambda)$ bijectively, it follows $\{s_{c_0}(z^{-1})B_0(W')\}$ is linearly independent in $L(W')/qL(W')$. Write an element $v \in \ker \Phi_\lambda, v \neq 0$, in terms of the global base $\{G(b) \mid b \in B(\lambda)\}$ as $v = \sum b c_0(q) G(b)$. Multiplying by a power of $q$ we may assume that each $c_0(q)$ is regular at $q = 0$, so that $v \mod qL(\lambda) \neq 0$. This implies $\Phi_{\lambda|q=0}(v \mod qL(\lambda)) \neq 0$, which is a contradiction. □

By Proposition 2.3 for each $b \in B(\lambda)$ there exist $b' \in B_0(W')$ and $s_{c_0}(z^{-1})$ such that $\Phi_\lambda(b) = s_{c_0}(z^{-1})b' \mod qL(W')$. Let $G(b), G(b')$ be the respective globalizations of $b$ and $b'$. Then $\Phi_\lambda(G(b)) = s_{c_0}(z^{-1})G(b') \mod qL(W')$. Since $\Phi_\lambda$ commutes with the $-\pi$ involutions, $\Phi_\lambda(G(b)) = s_{c_0}(z^{-1})G(b') \mod q^{-1}L(W')$. We conclude:

**Theorem 1.** $\Phi_\lambda$ induces a bijection between the sets

$$\Phi_\lambda : \{G(b) \mid b \in B(\lambda)\} \rightarrow \{s_{c_0}(z^{-1})G(b) \mid c_0 \in c_0(\lambda), b \in B_0(W')\}.$$

**Remark.** Taken together the results of this section give the conjectures [K4, 13.1, 13.2]. To obtain 13.1 (iii) consider that the crystal $\bigotimes_{i \in I} B(m_i \varpi_i)$ is by Proposition 2.3 in bijective correspondence with $\{s_{c_0}(z^{-1})B_0(W')\}$, and note that $\Phi_\lambda$ factors through $\bigotimes_{i \in I} V(m_i \varpi_i)$. 

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