STABILITY OF FIBRATIONS OVER ONE-DIMENSIONAL BASES

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ABSTRACT. We introduce and study a new notion of stability for varieties fibered over curves, motivated by Kollár’s stability for homogeneous polynomials with integral coefficients [14]. We develop tools to study geometric properties of stable birational models of fibrations whose fibers are complete intersections in weighted projective spaces. As an application, we prove the existence of standard models of threefold degree 1 del Pezzo fibrations, settling a conjecture of Corti [9].

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1. INTRODUCTION

Finding good models of varieties fibered over one-dimensional schemes is a central problem in geometry and arithmetic, with some of the key examples being: Tate’s minimal models of elliptic curves [28], Néron models of abelian varieties [6, 21], semistable reduction for curves [5], Sarkisov’s standard models of conic bundles [26], Kollár’s theory of semistable hypersurfaces over PIDs [14].

In this paper, we solve a concrete problem in birational geometry of threefolds (Corti’s conjecture on the existence of standard models of degree 1 del Pezzo fibrations) by developing a new theory of semistability for fibrations over a one-dimensional base, which we call Kollár semistability. For the main application—Kollár semistability of weighted hypersurfaces (or their intersections) in weighted projective spaces—our theory is a common generalization of both Tate’s and Kollár’s theories. We give an overview of Corti’s conjecture in §1.1 and of the general theory in §1.2.

1.1. Corti’s conjecture on standard models of degree 1 del Pezzo fibrations. Finding standard models for Mori fiber spaces (MFS) $f : X \to Z$ over a positive-dimensional base $Z$ is crucially useful for birational geometry, and is already an interesting problem when $X$ is a threefold. It could be thought of as a post-MMP step to further simplifying birational models of a given uniruled variety $X$. Standard models behave nicely under predictions in terms of birational rigidity, see Section 6 for further explanation and speculations.

When $\dim X = 3$, the two types of MFS are conic bundles and del Pezzo fibrations (of degree $d \in \{1, \ldots, 9\}$). Sarkisov developed a satisfying theory of standard models of conic
bundles over a surface [26, Theorem 1.13]. Motivated by this, Corti proposed the following notion of a standard model for a threefold del Pezzo fibration:

**Definition 1.1** ([9, Definitions 1.8 and 1.13]). Suppose $Z$ is an (essentially) smooth irreducible one-dimensional scheme over an algebraically closed field $k$. Let $K = k(Z)$ be the function field of $Z$ and $\pi: X \to Z$ be a flat projective morphism with the generic fiber a smooth del Pezzo surface $X_K$ over $K$ of degree $d := K_{X_K}^2 \in \{1, \ldots, 9\}$. We say that $\pi: X \to Z$ is a standard model of $X_K$ over $Z$, or a standard del Pezzo fibration of degree $d$, if:

1. $X$ has terminal singularities.
2. $\pi$ has integral fibers.
3. $-aK_X$ is a $\pi$-ample line bundle, where

$$a = \begin{cases} 
1 & \text{if } d \geq 3, \\
2 & \text{if } d = 2, \\
6 & \text{if } d = 1.
\end{cases}$$

In this paper, we address the following:

**Problem.** Given a del Pezzo surface $X_K$ over $K$, can we find a standard model of $X_K$ over $Z$?

**Remark 1.2.** A standard model $\pi: X \to Z$ is a Mori fiber space if and only if $\text{Pic}(X_K) = \mathbb{Z}$.

**Remark 1.3** (Reduction to the local case). By descent, the problem of finding a standard model immediately reduces to the local case $Z = \text{Spec} \mathcal{O}_{C,p}$, where $\mathcal{O}_{C,p}$ is a local ring of a closed point on a smooth curve $C$ over $k$. Indeed, suppose $\pi: X \to C$ is any model of $X_K$ over a curve $C$. Let $T \subset C$ be the finitely many points where the fiber of $\pi$ is singular. The disjoint union of $C \setminus T$ and $\{\text{Spec} \mathcal{O}_{C,p}\}_{p \in T}$ is an fpqc covering of $C$. Since fpqc descent is effective for projective schemes endowed with a choice of a very ample line bundle (see e.g., [29, Theorem 4.38]), in our case, the $a$-Gorenstein del Pezzo surfaces, we can glue a smooth family over $C \setminus T$ and standard models over spectra of the local rings $\{\text{Spec} \mathcal{O}_{C,p}\}_{p \in T}$ into a global standard model over $C$.

In the complex case, Corti established the existence of standard models for $d \geq 2$:

**Theorem 1.4** (Corti [9, Theorems 1.10 and 1.15]). Suppose $d \geq 2$ and $k = \mathbb{C}$. Let $X_K$ be a smooth del Pezzo surface of degree $d$ over $K = \mathbb{C}(Z)$, the function field of a smooth complex curve $Z$. Then there exists a standard model of $X_K$ over $Z$.

In this paper, we prove Corti’s conjecture [9, Conjecture 1.16], establishing the existence of standard models in the remaining degree 1 case:

**Theorem 1.5.** Suppose $Z$ is an (essentially) smooth irreducible one-dimensional scheme over an algebraically closed field $k$ with char($k$) $\neq 2,3$. Let $K = k(Z)$ be the function field of $Z$. Suppose $X_K$ is a smooth del Pezzo surface of degree 1 over $K$. Then there exists a standard model of $X_K$ over $Z$.

A more precise result will be given in Theorem 1.9 below. Our methods also extend Corti’s Theorem 1.4 for $d = 2$ to every algebraically closed field $k$ with char($k$) $\neq 2$ (see Theorem 4.5).

1.2. **Kollár stability: First examples.** Let $R$ be a DVR with a uniformizer $t$ and the fraction field $K = \text{Frac}(R)$. Set $\Delta = \text{Spec} R$. Suppose we are given a flat projective morphism $\pi: X \to \Delta$, where $X$ is normal and $\pi_* \mathcal{O}_X = \mathcal{O}_\Delta$. We denote this by $X/\Delta$, and refer to it as a fibration. We say that $\pi': X' \to \Delta$ is a (birational) model of $X/\Delta$ if there is a birational map $\chi: X \dashrightarrow X'$
such that $\pi' \circ \chi = \pi$ and such that $\chi$ induces an isomorphism between the generic fibers of $X/\Delta$ and $X'/\Delta$:

$$
\begin{array}{ccc}
X & \xrightarrow{\chi} & X' \\
\pi & & \pi' \\
\downarrow & & \downarrow \\
\Delta & & \\
\end{array}
$$

A priori there are many models for a given fibration $X/\Delta$. Hence it is natural to ask: What is the best model for a given fibration $X/\Delta$? Or, equivalently, given a projective scheme $X_K$ over $K$, what is its best model over $R$? It is crucial to note that we do not allow base change, motivated by questions in birational geometry.

We will answer these questions, in a great generality, by defining a notion of Kollár stability of $X/\Delta$ in Section 2. Here is a brief overview of our theory: As to be expected, stability depends on several choices. Suppose that there exists a proper parameter space $M$ over $R$ such that the generic fiber of $X/\Delta$ is a $K$-valued point of $M$. By properness of $M$, after possibly passing to a birational model, we can assume that $\pi: X \to \Delta$ is a pullback of the universal family over $M$ via a morphism $f_\pi: \Delta \to M$. Assume that $M$ is endowed with an additional structure given by an action of a group scheme $G$, a choice $L \in \text{Pic}^G(M)$ of a $G$-linearized line bundle, and a choice of its $G$-invariant section $\mathcal{D} \neq 0 \in \text{H}^0(M, L)^G$. Given all of this, we can make the following:

**Definition 1.6.** We say that $\pi: X \to \Delta$ is a $\mathcal{D}$-semistable model of $X_K$ if $f_\pi(K) \notin \text{Supp}(\mathcal{D})$ and the $t$-valuation of the Cartier divisor $f_\pi^*(\mathcal{D})$ on $\Delta$ is minimal among all maps $f: \Delta \to M$ with $f_\pi(K) \in G(K) \cdot f(K)$.

Before we delve into the general theory, we give two examples.

1.2.1. **Kollár stability of cubics in $\mathbb{P}^3$, after [14].** Suppose $X_K$ is a smooth degree 3 del Pezzo surface over $K$. Then $X_K$ is isomorphic to a cubic hypersurface in $\mathbb{P}^3_K$, given by some form $\tilde{F} \in \text{H}^0(\mathbb{P}^3_K, \mathcal{O}(3))$. A cubic form $F \in \text{H}^0(\mathbb{P}^3_R, \mathcal{O}(3))$ is called a semistable (cubic) model of $X_K$ over $R$ if $\tilde{F} \in \text{GL}_4(K) \cdot F$ and for every weight system $\rho = (w_1, w_2, w_3, w_4) \in \mathbb{Z}^4$ and every choice of homogeneous coordinates $x, y, z, w$ on $\mathbb{P}^3_R$, we have

$$
(1.1) \quad \text{mult}_\rho(F) \leq \frac{3}{4}(w_1 + w_2 + w_3 + w_4), \quad \text{where}
$$

$$
\text{mult}_\rho(F) := \text{the minimum of the } t \text{-valuations of all the coefficients of } F(t^{w_1}x, t^{w_2}y, t^{w_3}z, t^{w_3}w).
$$

This definition is obtained from Definition 1.6 by taking $M = \text{H}^0(\mathbb{P}^3_R, \mathcal{O}(3))$, $G = \text{GL}_4 = \text{Aut}_R(\text{Cox}(\mathbb{P}^3))$, and $\mathcal{D}$ to be the discriminant divisor on $M$.

**Theorem 1.7** (Kollár, cf. [14, Proposition 6.4.1]).

1. A semistable cubic model over $R$ exists for every smooth degree 3 del Pezzo over $K$.
2. Suppose $R = \mathcal{O}_{C, p}$ is the local ring of a closed point of a smooth algebraic curve over an algebraically closed field $k$. Then a semistable cubic model is a standard del Pezzo fibration of degree 3 over $\Delta = \text{Spec } R$.

A remarkable feature of this result is that semistability has a numerical criterion, given by (1.1), phrased in terms of the equation $F$. Once the reader accepts the existence of a semistable model, its geometric properties, such as terminality of the total space, can be derived from this numerical criterion.
1.2.2. *T-semistable degree 1 del Pezzo fibrations.* The novelty in our approach to Theorem 1.5 (Corti’s conjecture) is to treat it as a stability problem, and not as a problem in the birational geometry of threefolds. Namely, we find a correct analogue Theorem 1.7 for degree 1 del Pezzos, which we now explain.

A smooth degree 1 del Pezzo surface $X_K$ over $K$ can be written as a sextic hypersurface in a weighted projective space $\mathbb{P}_K(1,1,2,3)$. Unfortunately, no notion of semistability for such sextics leads to standard models (see Remark 5.6 for a technical explanation), so instead we write $X_K$ as a complete intersection

\[(1.2) \quad \tilde{F}(x,y,z,w,s) = \tilde{H}(x,y,z,w,s) = 0,
\]
in $\mathbb{P}_K(1,x,1,y,2,z,3_w,3_s)$, where $\deg \tilde{F} = 6$ and $\deg \tilde{H} = 3$. For example, we can write $X_K$ as a sextic in the variables $x,y,z,w$ in $\mathbb{P}_K(1,1,2,3)$ and take $\tilde{H} = s$. We call the resulting ideal $I = (\tilde{F}, \tilde{H}) \subset K[x,y,z,w,s]$ a $(6,3)$-intersection over $K$.

We say that an ideal $I = (F,H) \subset R[x,y,z,w] = \text{Cox}(\mathbb{P}_R(1,1,2,3,3))$ is a $(6,3)$-intersection over $R$ if $\deg F = 6$, $\deg H = 3$, and $H \notin (t)$ and $F \notin (H,t)$. By properness, every $(6,3)$-intersection $(\tilde{F}, \tilde{H})$ over $K$ can be completed uniquely to a $(6,3)$-intersection $I = (F,H)$ over $R$ with $(\tilde{F}, \tilde{H}) = (F,H) \otimes_R K$.

Given a choice of quasihomogeneous coordinates $x,y,z,w,s$ — namely a sequence of degree 1,1,2,3,3 elements of $\text{Cox}(\mathbb{P}_R(1,1,2,3,3))$ that generate this graded $R$-algebra, a weight system $\rho = (w_1, w_2, w_3, w_4, w_5) \in \mathbb{Z}^5$, and an element $A \in R[x,y,z,w,s]$, we define $\text{mult}_\rho(A)$ to be the minimum of the valuations of all the coefficients of $A(t^{w_1}x, t^{w_2}y, t^{w_3}z, t^{w_4}w, t^{w_5}s) \in K[x,y,z,w,s]$.

We can now make the following:

**Definition 1.8.** Fix $0 < \epsilon \ll 1$. A $(6,3)$-intersection $I = (F,H)$ over $R$ is called a *T-semistable* $(6,3)$-intersection model of $X_K$ over $R$ if:

1. The ideal $(F,H) \otimes_R K$ is a $(6,3)$-intersection over $K$ defining a smooth del Pezzo surface in $\mathbb{P}_K(1,1,2,3,3)$ isomorphic to $X_K$ over $K$.
2. For every choice of generators $I = (F,H)$, for every choice of quasihomogeneous coordinates in $R[x,y,z,w]$, and for every weight system $\rho = (w_1, w_2, w_3, w_4, w_5)$, we have:

\[(1.3) \quad \text{mult}_\rho(F) + \frac{6}{7} \left( \text{mult}_\rho(H) - \sum_{i=1}^{5} w_i \right) + \epsilon(2 \text{mult}_\rho(H) + 3w_3 - 2(w_4 + w_5)) \leq 0.
\]

If for some system of coordinates and a weight system $\rho$, the inequality (1.3) is satisfied (resp., violated), we will say that $(F,H)$ is $\rho$-semistable (resp., $\rho$-unstable).

The following result establishes Theorem 1.5 (Corti’s conjecture):

**Theorem 1.9 (\(\Rightarrow\) Theorem 1.5).**

1. Every smooth degree 1 del Pezzo $X_K$ over $K$ has a T-semistable $(6,3)$-intersection model over $R$.
2. Suppose $R = \mathcal{O}_{C,p}$ is the local ring of a closed point of a smooth algebraic curve over an algebraically closed field $k$ with $\text{char}(k) \neq 2,3$. Suppose $I = (F,H)$ is a T-semistable $(6,3)$-intersection model. Let $X := \{F = H = 0\} \subset \mathbb{P}_R(1,1,2,3,3)$. Then $X \to \Delta$ is a standard del Pezzo fibration of degree 1 over $\Delta$.

Part (1) of this theorem uses the machinery of Kollár semistability (see Definition 1.6) and necessitates a careful choice of a parameter space $M$ with a group action and of an invariant Cartier divisor $\mathcal{D}$ that governs stability. Specifically, it follows immediately from the following result that we establish in §4.5:
**Theorem 1.10** (=Proposition 4.8). Let $M$ be the parameter space of intersections of weighted hypersurfaces of degree 6 and 3 in $\mathbb{P}(1,1,2,3,3)$. Let $G = \text{Aut}_R(\mathbb{P}(1,1,2,3,3))$. Then there exists a $G$-invariant divisor $\mathcal{D}^{ter}$ on $M$, supported on the locus of singular intersections, such that every $\mathcal{D}^{ter}$-semistable model in $M$ is a $T$-semistable $(6,3)$-intersection.

The reader who wishes to bypass entirely Sections 2, 3 and 4, can take Part (1) of Theorem 1.9 as a blackbox. Part (2) of Theorem 1.9 is proved as Theorem 5.1 in Section 5, where we use the numerical criterion of Definition 1.8 to verify that a $T$-semistable model is a standard del Pezzo fibration.

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We are now in position to formulate our definition of stability:

**Definition 2.1** (\(\mathcal{D}\)-(semi)stability). An \(R\)-point \(f: \Delta \to M\) is \(\mathcal{D}\)-semistable if

1. \(f(\text{Spec } K) \notin \text{Supp}(\mathcal{D})\) (equivalently, \(f^*(\mathcal{D}) \neq 0\) as a Cartier divisor on \(\Delta\)).
2. \[(2.2) \quad \text{deg } f^*\mathcal{D} := \sum_{(t) \in \text{m-Spec}(R)} \text{val}_t(f^*\mathcal{D}) \leq \sum_{(t) \in \text{m-Spec}(R)} \text{val}_t((f')^*\mathcal{D}) = \text{deg}(f')^*\mathcal{D}\]

for every model \(f'\) of \(f\).

We say that \(f: \Delta \to M\) is \(\mathcal{D}\)-stable if it is \(\mathcal{D}\)-semistable and in addition every model \(f': \Delta \to M\) for which the equality holds in (2.2) satisfies \(f'(\Delta) \in G(R) \cdot f(\Delta)\).

In view of (2.2), we will also call a \(\mathcal{D}\)-semistable model, a \(\mathcal{D}\)-minimizer. Stability then means that the \(\mathcal{D}\)-minimizer is unique, up to an action of \(G(R)\). If \(\rho \in G(K)\) is such that

\[
\sum_{(t) \in \text{m-Spec}(R)} \text{val}_t(f^*\mathcal{D}) > \sum_{(t) \in \text{m-Spec}(R)} \text{val}_t((\rho \cdot f)^*\mathcal{D}),
\]

then we say that \(\rho\) destabilizes \(f\), and that \(f\) is unstable with respect to \(\rho\), or simply \(\rho\)-unstable.

For the applications we have in mind, the distinction between semistability and stability will play no role and so we rarely invoke the concept of \(\mathcal{D}\)-stability.

When \(M\) is a parameter space representing a functor of flat families of schemes, such as for example the Hilbert scheme of a fixed projective space, the \(R\)-points of \(M\) are simply fibrations over \(\Delta\) fibered in the objects parameterized by \(M\). Our applications will be essentially of this form.

2.1.3. **Reduction to DVRs.**

**Lemma 2.2.** Suppose a group scheme \(G\) satisfies the following condition:

\((\dagger)\) For every \(\rho \in G(K)\), there exist finally many irreducible elements \(\{t_i\}_{i=1}^n \in R\) and elements \(\rho_i \in G(R[1/t_i])\) such that

\[\rho = \rho_1 \cdots \rho_n.\]

Then an \(R\)-point \(f \in M(R)\) is \(\mathcal{D}\)-semistable if and only if the induced point \(f \in M(R(t))\) is \(\mathcal{D}\)-semistable for every \((t) \in \text{m-Spec}(R)\).

**Proof.** If \(\rho_i \in G(R[1/t_i])\), then \(\rho = \rho_1 \cdots \rho_n \in G(K)\) destabilizes \(f \in M(R)\) if and only if some \(\rho_i\) destabilizes \(f \in M(R(t_i)).\)

Condition \((\dagger)\) of Lemma 2.2 is satisfied for \(GL_n\) and \(SL_n\) by the elementary divisor theorem, and for \(G = \text{Aut}_{\mathcal{O}}(\text{Cox}(\mathbb{P}(c_1, \ldots, c_n)))\), the group of graded automorphisms of the Cox ring of a weighted projective space by a similar argument. These are the only groups considered in our applications.

2.1.4. **Stack-theoretic interpretation.** The quadruple \((G, M, \mathcal{L}, \mathcal{D})\) defines quotient stack \(\mathcal{M} := [M/G]\), a line bundle \(\mathcal{L}\) on \(\mathcal{M}\), and a global section \(\mathcal{D}\) of \(\mathcal{L}\). We then say that an \(R\)-point \(f: \Delta \to \mathcal{M}\) is \(\mathcal{D}\)-semistable if \(f^*(\mathcal{D}) \neq 0\) and \(\text{val}_t(f^*\mathcal{D}) \geq \text{val}_t((f')^*\mathcal{D})\) for every other \(R\)-point \(f'\) of \(\mathcal{M}\) such that \(f(\text{Spec } K)\) and \(f'(\text{Spec } K)\) are isomorphic \(K\)-points of \(\mathcal{M}\). There is however a subtle distinction between the resulting \(\mathcal{D}\)-stability on the quotient stack \(\mathcal{M}\) and on the original space \(M\) arising from the possibility that \(R\)-points of \(M\) do not necessarily surject onto the \(R\)-points of \(\mathcal{M}\). Working over a field \(k\), if \(R\) is a complete ring, or \(G\) is a special group in the sense of Serre [27] (see also [24] for a modern exposition) every \(R\)-point of \(\mathcal{M}\) comes from an \(R\)-point of \(M\): In these cases every étale \(G\)-torsor over \(\Delta\) is a Zariski \(G\)-torsor, and so the two notions of \(\mathcal{D}\)-semistability of \(R\)-points of \(M\) and \(\mathcal{M}\) are equivalent.
2.2. Numerical criterion for $\mathcal{D}$-semistability. With Lemma 2.2 in mind, we now assume that $R$ is a DVR.

2.2.1. Hilbert-Mumford-Kollár index. Suppose $f \in M(R)$ and $\rho \in G(K)$. Let $f' = \rho \cdot f \in M(R)$ as defined in (2.1.2). Fix isomorphisms

$$(2.3) \quad i: f^*(\mathcal{L}) \simeq \mathcal{O}_{\Delta} = \tilde{R},$$

$$(2.4) \quad i': (f')^*(\mathcal{L}) \simeq \mathcal{O}_{\Delta} = \tilde{R},$$

where $\tilde{R}$ is simply the coherent sheaf given by the free rank one $R$-module on Spec $R$. (These isomorphisms are defined up to a unit in $R$.)

The $G$-linearization of $\mathcal{L}$ induces a $K$-linear isomorphism

$$\rho^{-1}: \mathcal{L}_{f'(\text{Spec } K)} \rightarrow \mathcal{L}_{f(\text{Spec } K)}$$

and so, by composition, we obtain an isomorphism

$$(2.5) \quad K = \mathcal{O}_{\text{Spec } K} \simeq \mathcal{L}_{f'(\text{Spec } K)} \rightarrow \mathcal{L}_{f(\text{Spec } K)} \simeq \mathcal{O}_{\text{Spec } K} = K.$$ 

By construction, the isomorphism in (2.5) is given by a canonical element in $K/R^*$. We denote the valuation of this element by $\mu_{\mathcal{L}}^\rho(f)$. In plain terms, if $e_{f'}$ is the generator of $\mathcal{L}_{f'(\text{Spec } K)}$ that extends to a generator of $\mathcal{L}|_{f'(\Delta)}$, then

$$\rho^{-1}(e_{f'}) = t_{\mu_{\mathcal{L}}^\rho(f)}(e_f),$$

where $e_f$ is a generator of $\mathcal{L}_{f(\text{Spec } K)}$ that extends to a generator of $\mathcal{L}|_{f(\Delta)}$. We call $\mu_{\mathcal{L}}^\rho(f)$ the Hilbert-Mumford-Kollár index of $f$ with respect to $\rho$ (abbreviated as HMK-index). Our naming convention will be explained below.

It is immediate from the above definition that for every $G$-invariant regular section $\mathfrak{D}$ of $\mathcal{L}$, we have

$$(f^* \mathfrak{D}) = (t_{\mu_{\mathcal{L}}^\rho(f)}(f')^* \mathfrak{D}),$$

and so

$$(2.6) \quad \text{val}_t(f^* \mathfrak{D}) = \text{val}_t((f')^* \mathfrak{D}) + \mu_{\mathcal{L}}^\rho(f).$$

It follows that

**Proposition 2.3.** A map $f: \Delta \rightarrow M$ is $\mathcal{D}$-semistable if and only if $\mu_{\mathcal{L}}^\rho(f) \leq 0$ for all $\rho \in G(K)$.

We record for the future use some functorial properties of the HMK-index that follow immediately from the definition:

**Proposition 2.4.** Suppose $f: \Delta \rightarrow M$ is an $R$-point and $\rho, \phi \in G(K)$. The following holds:

1. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are two $G$-linearized line bundles on $M$, then

$$(2.7) \quad \mu_{\mathcal{L}_1 + \mathcal{L}_2}^\rho(f) = \mu_{\mathcal{L}_1}^\rho(f) + \mu_{\mathcal{L}_2}^\rho(f).$$

2. If $h: M \rightarrow N$ is a $G$-morphism and $\mathcal{L}$ is a $G$-linearized line bundle on $N$, then

$$\mu_{h^* \mathcal{L}}^\rho(f) = \mu_{\mathcal{L}}^\rho(h \circ f).$$

3. $\mu_{\mathcal{L}}^{f \circ \phi}(f) = \mu_{\mathcal{L}}^\rho(\rho \cdot f) + \mu_{\mathcal{L}}^\rho(f)$.

4. If $\rho \in G(R)$, then $\mu_{\mathcal{L}}^\rho(f) = 0$. 


2.2.2. One-parameter subgroups. Fix a morphism $\iota: \text{Spec}(K) \to \mathbb{G}_{m,R} = \text{Spec} R[z, z^{-1}]$ given by $z \mapsto t$. Then for every one-parameter subgroup $\rho: \mathbb{G}_{m,R} \to G$, the morphism $\iota$ defines a corresponding $K$-point of $G$ given by $\rho \circ \iota: \text{Spec} K \to G$. Following Mumford, we denote the resulting $K$-point of $G$ by $(\rho)$. We also write $\mu^\rho_x$ instead of $\mu^\rho_{(\rho)}$. 

For every one-parameter subgroup $\rho: \mathbb{G}_{m,R} \to G$ and every $\rho$-fixed $K$-point $x \in M(K)$, each of the line bundles $L \in \text{Pic}^G(M)$ defines a one-dimensional representation $L_x$ of $\mathbb{G}_{m,K}$. Then the integer obtained by pairing of this character with $\rho$ is precisely $\mu^\rho_x(x)$. Plainly, $\rho$ acts on the fiber $L_x$ by

$$z \cdot w = \mu^\rho_x(x) w, \quad \text{for } w \in L_x. \quad (2.8)$$

2.2.3. Cartan-Iwahori-Matsumoto decomposition. Recall that for $G = \text{GL}(n)$ and $G = \text{SL}(n)$, the elementary divisors theorem says that the double coset of every element of $G(K)$ with respect to the subgroup $G(R)$ contains an element of the form $\langle \rho \rangle$ for some one-parameter subgroup $\rho$ of $G$. (If the DVR $R$ is complete with an algebraically closed residue field, this is true more generally for reductive groups by a theorem of Iwahori [20, p.52]). In this case, we say that $G$ has Cartan-Iwahori-Matsumoto decomposition, see [2].

Combining properties (3) and (4) of Proposition 2.4, we have:

**Proposition 2.5.** Suppose $G$ has Cartan-Iwahori-Matsumoto decomposition. Then a map $f$ is semistable if and only if $\mu^\rho(f) \leq 0$ for all one-parameter subgroups $\rho: \mathbb{G}_{m,R} \to G$.

**Remark 2.6.** If we work over a base field $k$, and $f: \Delta \to M$ is a constant map whose image is the $k$-point $x \in X$, then $\mu_k^\rho(f)$ is the usual Hilbert-Mumford index of $x$ with respect to $L$ and $\rho$. This explains naming $\mu_k^\rho(f)$ Hilbert-Mumford-Kollár index.

2.3. Motivating example après Kollár. In this subsection, we recast Kollár’s notion of stability for homogeneous polynomials, which motivates our whole approach, in the language of $\mathcal{D}$-semistability. We work over a field $k$ in this example.

Let $M = \overline{M} = \mathbb{P}(U)^1$, where $U$ is an algebraic representation of a reductive algebraic group $G$.

We have $\text{Pic}^G(M) = \mathbb{Z} \oplus \text{Ch}(G)$, where $\text{Ch}(G)$ is the character group of $G$. The first summand of $\text{Pic}^G(M)$ is generated by $\mathcal{O}(1)$ with its canonical $G$-linearization induced by the $G$-action on $U$, and the second by the $G$-linearized line bundles

$$\{ \mathcal{O}^\chi \mid \chi \in \text{Ch}(G) \}$$

where $\mathcal{O}^\chi$ is the trivial line bundle linearized by the character $\chi$.

Suppose $L = \mathcal{O}(m)^\chi := \mathcal{O}(m) \otimes \mathcal{O}^\chi$, and $\mathcal{D}$ is a $G$-invariant section of $L$. We now explicate the numerical criterion of Proposition 2.5 for $\mathcal{D}$-semistability of a map $f: \Delta \to M$.

Suppose $\dim U = n$ and $\rho$ is a 1-PS (one-parameter subgroup) $\rho$ of $G$. Choose a basis $u_1, \ldots, u_n$ of $U$ on which $\rho$ acts diagonally as

$$\rho(t) = \text{diag}(t^{u_1}, \ldots, t^{u_n}).$$

Consider now an $R$-point $f: \Delta \to M$ given by an equation $F(t) = \sum_{i=1}^n F_i(t)u_i \in U \otimes_k R$, where $F_i(t) \in R$, and $F(0) \neq 0 \in U$. Then we have the following notion of multiplicity as defined by Kollár:

$$\text{mult}_\rho(F) := \max\{ N \mid \rho \cdot F(t) \in (t^{N}) \} = \min\{ \text{val}_t(F_i(t)t^{u_i}) \mid i = 1, \ldots, n \}. \quad (2.9)$$

\footnote{Our convention is that $\mathbb{P}(U)$ means the space of lines in $U$.}
Setting

$$F^\rho(t) := \frac{\rho \cdot F(t)}{t^{\mu_{\rho}(F)}} = \frac{1}{t^{\mu_{\rho}(F)}} \sum_{i=1}^{n} F_i(t) t^{w_i} u_i,$$

we see that the $R$-point $\rho \cdot f : \Delta \to \mathbb{P}(U)$ is given precisely by the equation $F^\rho(t) \in R \otimes U$.

**Lemma 2.7.** With setup as above, we have

$$\mu_{\rho}^{O(1)}(f) = \mu_{\rho}(F(t)),$$

$$\mu_{\rho}^{O^\times}(f) = -\langle \chi, \rho \rangle,$$

where $\langle \chi, \rho \rangle$ is the integer obtained by pairing $\chi$ and $\rho$. Consequently, for $L = \mathcal{O}(m)^\chi$,

$$\mu_{\rho}^L(f) = m \mu_{\rho}(F(t)) - \langle \chi, \rho \rangle.$$

**Proof.** We first establish (2.11). By Proposition 2.4, we have

$$\mu_{\rho}^{O(1)}(f) = -\mu_{\rho}^{O(-1)}(f).$$

Since $f : \Delta \to \mathbb{P}(U)$ is given by $F(t) \in R \otimes U$ with $F(0) \neq 0$, we have a canonical isomorphism

$$H^0(\Delta, f^*\mathcal{O}(-1)) = RF(t).$$

Similarly, we have a canonical isomorphism

$$H^0(\Delta, (\rho \cdot f)^*\mathcal{O}(-1)) = RF^\rho(t).$$

Since $t^{\mu_{\rho}(F)} = \rho \cdot F(t) / F^\rho(t)$, by definition, we see that $\mu_{\rho}^{O(-1)}(f) = -\mu_{\rho}(F(t))$ and the claim follows.

For (2.12), note that $\rho$ sends the generator 1 of $H^0(\Delta, f^*\mathcal{O}^\chi) = R$ to $t^\langle \chi, \rho \rangle$ times the generator 1 of $H^0(\Delta, (\rho \cdot f)^*\mathcal{O}^\chi) = R$.

□

### 2.3.1. Homogeneous polynomials.

Suppose $V$ is an $n$-dimensional vector space and $G = \text{GL}(V)$. Then $U = \text{Sym}^d V$ is the space of degree $d$ homogeneous polynomials in $n$ variables. The group $G$ has exactly one character, namely the determinant $\chi = \text{det}$. We take $\mathfrak{D}$ to be the discriminant divisor on $\mathbb{P}(U)$. Then $\mathfrak{D}$ is a $G$-invariant section of $L = \mathcal{O}(m)^{-\frac{m}{d}}\chi$, where $m = n(d-1)^{n-1}$ is the degree of the discriminant. We see that for $F \in R \otimes U$, with $F(0) \neq 0$, our definition of multiplicity of $F$ with respect to any one-parameter subgroup $\rho$ of $G$ coincides with Kollár’s definition of multiplicity. Furthermore, the Hilbert-Mumford-Kollár index of $f$ with respect to $\rho$ acting in some basis of $V$ diagonally with weights $w_1, \ldots, w_n$ is

$$\mu_{\rho}^L(f) = m \left( \mu_{\rho}(F) - \frac{d}{n} (w_1 + \cdots + w_n) \right).$$

Summarizing, we conclude that our definition of $\mathfrak{D}$-semistability in $\mathbb{P}(\text{Sym}^d V)$ coincides with Kollár’s definition of semistability for generically smooth families of degree $d$ hypersurfaces in $\mathbb{P}^{n-1}$ as given in [14, Definition (3.3)].

### 3. Kollár stability of weighted hypersurfaces

In this section, we develop Kollár stability of weighted (Cartier) hypersurfaces in arbitrary weighted projective spaces, generalizing Kollár’s theory for ordinary hypersurfaces in [14]. We start with some generalities. While cumbersome at the first glance, they allow us to work over an arbitrary DVR.
3.1. Parameter spaces with group action. Fix $n$ positive integers $\{c_i\}_{i=1}^n$ and let $S := \mathbb{Z}[x_1, \ldots, x_n]$ be the graded ring with grading given by $\text{wt}(x_i) = c_i$. We let $\mathbb{P}(c_1, \ldots, c_n) = \text{Proj}_\mathbb{Z} S$. Given a ring $A$, we let $S_A := S \otimes A = A[x_1, \ldots, x_n]$ and $\mathbb{P}_A(c_1, \ldots, c_n) = \text{Proj}_A S_A$. We set
\[
(3.1) \quad G = \text{Aut}_{gr} \text{Cox}(\mathbb{P}(c_1, \ldots, c_n))
\]
to be the group scheme of graded automorphisms of the Cox ring of $\mathbb{P}(c_1, \ldots, c_n)$. For a ring $A$ we have that $G(A) = \text{Aut}_{gr}(S_A)$ is the group of graded $A$-algebra automorphisms of $S_A$.

We emphasize that the generators $x_i$’s of $A[x_1, \ldots, x_n]$ are not fixed throughout, but are determined only up to an element of $G(A)$. Any such choice of generators with $\text{wt}(x_i) = c_i$ will be called a system of quasihomogeneous coordinates (or simply, coordinates) in $S_A$.

We will be interested in two kinds of parameter spaces associated to $S = \mathbb{Z}[x_1, \ldots, x_n]$. To define them, let $\pi: \mathbb{P}(c_1, \ldots, c_n) \to \text{Spec} \mathbb{Z}$ be the structure morphism. The first is the space of degree $d$ hypersurfaces in $\mathbb{P}(c_1, \ldots, c_n)$:
\[
(3.2) \quad \text{Hyp}(d) := \mathbb{P}(\pi_* \mathcal{O}_{\mathbb{P}(c_1, \ldots, c_n)}(d)).
\]
Note that Hyp$(d)$ is smooth and projective over Spec $\mathbb{Z}$ and Pic(Hyp$(d)$) $\simeq \mathbb{Z}$.

Suppose now $e < d$ are positive integers. Let $q: \mathcal{Q} \to \text{Hyp}(e)$ be the universal degree $e$ hypersurface, where $\mathcal{Q} \hookrightarrow \text{Hyp}(e) \times_{\text{Spec} \mathbb{Z}} \mathbb{P}(c_1, \ldots, c_n)$. We define
\[
(3.3) \quad \text{Int}(d, e) := \mathbb{P}(q_* \mathcal{O}_{\mathcal{Q}}(d)),
\]
to be the space of $(d, e)$-intersections in $\mathbb{P}(c_1, \ldots, c_n)$. Note that Int$(d, e)$ is smooth and projective over Spec $\mathbb{Z}$ and Pic(Int$(d, e)$) $\simeq \mathbb{Z}^2$.

The group scheme $G$ acts on Hyp$(d)$ and Int$(d, e)$ via its natural action on $S$. This will be the only group action we consider for the rest of the paper.

3.2. Stability of DVR-valued points. Suppose now $R$ is a DVR with a uniformizer $t$ and the fraction field $K$. We let $k = R/(t)$ be the residue field. For $F \in S_R = R[x_1, \ldots, x_n]$, we denote by $F_0$ its image in $S_k = k[x_1, \ldots, x_n]$.

Our goal is to understand stability of $R$-points of Hyp$(d)$ and Int$(d, e)$ with respect to various $G$-linearized line bundles on these parameter spaces. We begin with describing $R$-points of Hyp$(d)$ and Int$(d, e)$:

3.2.1. An $R$-point $f: \text{Spec}(R) \to \text{Hyp}(d)$ is simply an element $F \in R[x_1, \ldots, x_n]_d$ such that $F_0 \neq 0 \in k[x_1, \ldots, x_n]$. We call such $F$ the equation of $f$. (It is of course, defined up to a unit in $R$.)

3.2.2. An $R$-point $f: \text{Spec}(R) \to \text{Int}(d, e)$ is an ideal $I = (F, H) \subset R[x_1, \ldots, x_n]$ such that $H \in R[x_1, \ldots, x_n]_e$ and $F \in R[x_1, \ldots, x_n]_d$ and such that $H_0 \neq 0 \in k[x_1, \ldots, x_n]_e$ and $F_0 \notin (H_0)$. We call such ideal $I$ a $(d, e)$-intersection, and $F$ and $H$ the equations of $I$. Note that $H$ is defined up to a unit in $R$, but $F$ is defined only up to addition of a multiple of $H$.

3.2.3. The action of $G(K)$ on $R$-points of Hyp$(d)$. Suppose $\rho \in G(K)$ and $f: \text{Spec}(R) \to \text{Hyp}(d)$ is an $R$-point with the equation $F \in R[x_1, \ldots, x_n]_d$. Just as in the case of homogeneous polynomials (cf. (2.9)), we define the $\rho$-multiplicity of $F$ to be:
\[
(3.4) \quad \text{mult}_\rho(F) := \max \left\{ N \mid \frac{\rho \cdot F}{t^N} \in R[x_1, \ldots, x_n] \right\}.
\]
Set
\[
(3.5) \quad F^\rho := \frac{\rho \cdot F}{t^{\text{mult}_\rho(F)}}.
\]
2.2.2 Remark 3.1. \(\text{Spec}(R[1, \ldots, x_n])\) will often write \(\rho\), the vector \((w_1, \ldots, w_n)\) Pic\(\langle G, 3.8 \rangle\).

Remark 3.1. The name one-parameter subgroup comes from a group scheme morphism \(\mathbb{G}_m, R := \text{Spec}(R[z, z^{-1}]) \to G\) and the \(K\)-point of \(G\) given by \(t \in K\); see 2.2.2.

3.3. \(G\)-linearized line bundles. To obtain a numerical criterion for Kollár semistability of an \(R\)-point \(f: \text{Spec}(R) \to M = \text{Hyp}(d)\) (or \(M = \text{Int}(d, c)\)) in terms of its defining equation over \(R\), we must compute the Hilbert-Mumford-Kollár index \(\mu^G(f)\) for every \(G\)-linearized line bundle \(L \in \text{Pic}^G(M)\) and every one-parameter subgroup \(\rho\) of \(G\).

Remark 3.2. The \(\text{Ch}(G)\)-portion of \(\text{Pic}^G(M)\) is the pullback of \(\text{Pic}^G(\text{Spec}(R))\) so is independent of \(M\).

Suppose there are exactly \(s\) distinct integers among \(c_i\)’s, namely
\[v_1 < v_2 < \cdots < v_s.\]

For each \(i = 1, \ldots, s\), we have a character of \(G_K\) given by the determinant of a \(K\)-linear transformation obtained by restricting an element of \(G_K\) to the \(K\)-subspace
\[K\langle x_j \mid x_j \text{ has weight } \leq v_i \text{ in } S \rangle \subset S/(x_1, \ldots, x_n)^2.\]

The resulting \(s\) characters freely generate the character group of \(G_K\). It follows that \(\text{Ch}(G) \simeq \mathbb{Z}^s\) and so
\[\text{Pic}^G(M) \simeq \text{Pic}(M) \oplus \mathbb{Z}^s.\]
3.3.1. Irrelevant one-parameter subgroup. Notice that given a system of quasihomogeneous coordinates \(x_1, \ldots, x_n\), we have the irrelevant one-parameter subgroup \(\rho_{irr}\) acting via
\[
\rho_{irr} \cdot (x_1, \ldots, x_n) = (t^{c_1} x_1, \ldots, t^{c_n} x_n).
\]
Since \(\rho_{irr}\) fixes all the points of \(M\), a necessary condition for a \(G\)-linearized line bundle \(L\) to have a nonzero \(G\)-invariant section is that \(\mu^{\rho_{irr}}_L([X]) = 0\) for some (equivalently, every) \([X] \in M\). This implies that the subspace of \(\text{Pic}^G(M) \otimes \mathbb{Q}\) spanned by \(G\)-linearized line bundles with invariant nonzero sections has dimension at most \(s\).

3.3.2. Tautological line bundles. We treat parameter spaces \(M = \text{Hyp}(d)\) and \(M = \text{Hyp}(d,e)\) at the same time. Let \(\pi: \mathbb{P}(c_1, \ldots, c_n) \to \text{Spec} R\) be the structure morphism.

Suppose \(q: \mathcal{Q} \to M\) is the universal family over \(M\), where \(\mathcal{Q} \to \mathbb{P}(c_1, \ldots, c_n)\). (Note that \(q\) is flat when \(M = \text{Hyp}(d)\) but not when \(M = \text{Hyp}(d,e)\)). Let \(\mathcal{I}\) be the ideal sheaf of \(\mathcal{Q}\) in \(\mathbb{P}(c_1, \ldots, c_n)\) and \(\text{pr}_1: \mathbb{P}(c_1, \ldots, c_n) \to M\) the projection morphism.

We obtain the following \(G\)-linearized line bundles
\[
T_m := \det \pi_*(\mathcal{O}_F(m)) = \det (\text{pr}_1)_*(\text{pr}_2^* \mathcal{O}_F(m)), m \geq 1
\]
\[
J_m := \det (\text{pr}_1)_*(\mathcal{I}(m)), \quad \begin{cases} m \geq d, \text{if } M = \text{Hyp}(d) \\ d \geq m \geq e, \text{if } M = \text{Int}(d,e). \end{cases}
\]
Note that \((\text{pr}_1)_*(\mathcal{I}(m))\) is a vector bundle on \(\text{Int}(d,e)\) for \(m \leq d\).

3.3.3. Hilbert-Mumford-Kollár indices. Let \(\rho\) be a one-parameter subgroup acting by \(\rho \cdot x_i = t^{w_i} x_i\) in some coordinates. Let \(\text{mon}(S_r)\) be the set of all monomials in \(S_r\) in same coordinates. Denote the \(\rho\)-weight of a monomial by \(w_\rho(\prod x_i^{d_i}) = \sum d_i w_i\) and by \(w_\rho(S_r)\) the sum of the \(\rho\)-weight of all monomials in \(S_r\). The following result computes HMK-indices of the tautological line bundles on \(M\):

**Proposition 3.3.** Suppose \(\rho\) is a one-parameter subgroup acting by \(\rho \cdot x_i = t^{w_i} x_i\) in some coordinates.

(A) \(M = \text{Hyp}(d)\). For \(f: \text{Spec}(R) \to M\) given by an equation \(F = F(x_1, \ldots, x_n) \in S_d\), we have
\[
\mu^T_\rho(f) = -w_\rho(S_m),
\]
\[
\mu^J_\rho(f) = -(w_\rho(S_{m-d}) + |\text{mon}(S_{m-d})| \cdot \text{mult}_\rho(F)).
\]

(B) \(M = \text{Int}(d,e)\). For \(f: \text{Spec}(R) \to M\) given by a \((d,e)\)-intersection ideal \((F,H)\), we have
\[
\mu^T_\rho(f) = -w_\rho(S_m),
\]
\[
\mu^I_\rho(f) = -(w_\rho(S_{m-e}) + |\text{mon}(S_{m-e})| \cdot \text{mult}_\rho(H)), \quad e \leq m \leq d - 1,
\]
\[
\mu^f_\rho(f) \leq -(w_\rho(S_{d-e}) + |\text{mon}(S_{d-e})| \cdot \text{mult}_\rho(H) + \text{mult}_\rho(F)).
\]
If \((F,H)\) are adapted to \(\rho\), then equality holds in (3.15).

**Proof.** (A) \(M = \text{Hyp}(d)\). For \(f: \Delta \to M\) with the equation \(F \in S_d\), we have natural identifications of \(\mathcal{O}_\Delta\)-modules:
\[
f^*(J_m) = R \bigwedge_{p \in \text{mon}(S_{m-d})} pF,
\]
\[
f^*(T_m) = R \bigwedge_{p \in \text{mon}(S_m)} p.
\]
Similarly, \( \rho \cdot f : \Delta \to M \) has equation \( F^\rho \) and we also have natural identifications:
\[
(3.18) \quad (\rho \cdot f)^*(J_m) = R \bigwedge_{p \in \text{mon}(S_{m-d})} pF^\rho,
\]
\[
(3.19) \quad (\rho \cdot f)^*(T_m) = R \bigwedge_{p \in \text{mon}(S_m)} p.
\]
Since
\[
\rho \cdot \left( \bigwedge_{p \in \text{mon}(S_{m-d})} pF \right) = \bigwedge_{p \in \text{mon}(S_{m-d})} (\rho \cdot p)(\rho \cdot F) = \bigwedge_{p \in \text{mon}(S_{m-d})} (t^{w_p(p)}(t^{\text{mult}_\rho(F)}F^\rho)),
\]
\[
= t^{w_p(S_{m-d}) + |\text{mon}(S_{m-d})| \text{mult}_\rho(F)} \bigwedge_{p \in \text{mon}(S_{m-d})} pF^\rho.
\]
\[
\rho \cdot \left( \bigwedge_{p \in \text{mon}(S_m)} p \right) = \bigwedge_{p \in \text{mon}(S_m)} (t^{w_p(p)}),
\]
the claims follow from the definition of the HMK-index in §2.2.1.

(B) \( M = \text{Hyp}(d,e) \). The first two equalities are identical to Part (A), so we only need to prove (3.15). Suppose \( f : \Delta \to M \) is given by a \((d,e)\)-intersection \((F,H)\). We have natural identifications of \( O_{\Delta}\)-modules:
\[
(3.20) \quad f^*(J_d) = R \cdot F \wedge \bigwedge_{p \in \text{mon}(S_{d-e})} pH.
\]
The point \( \rho \cdot f : \Delta \to M \) has equation \((A,H^\rho)\), where \( F^\rho = t^q A \text{ (mod } H^\rho)\), where \( q \geq 0 \), see §3.2.4. Since
\[
(3.21) \quad \rho \cdot \left( F \wedge \bigwedge_{p \in \text{mon}(S_{d-e})} pH \right) = t^{\text{mult}_\rho(F)}F^\rho \wedge \bigwedge_{p \in \text{mon}(S_{d-e})} (t^{w_p(p)}(t^{\text{mult}_\rho(H)}H^\rho))
\]
\[
= t^{q + \text{mult}_\rho(F) + w_p(S_{d-e}) + |\text{mon}(S_{d-e})| \text{mult}_\rho(H)} A \wedge \bigwedge_{p \in \text{mon}(S_{d-e})} pH^\rho,
\]
we have
\[
\mu_{\rho,d}^\Delta(f) = -(q + w_p(S_{d-e}) + |\text{mon}(S_{d-e})| \text{mult}_\rho(H) + \text{mult}_\rho(F)),
\]
and the claim follows.

\[ \square \]

**Corollary 3.4.** Let \( M = \text{Hyp}_{(c_1,\ldots,c_n)}(d) \). Let \( v_1 < \cdots < v_s \) be the distinct integers among \( \{c_1,\ldots,c_n\} \). For a one-parameter subgroup \( \rho \) with weight system \((w_1,\ldots,w_n)\), we will write \( W_{v_j}(\rho) = \sum_{i: c_i = v_j} w_i \), and let \( r_j \) denote the number of variables with weight \( v_j \).

1. For every \( G \)-linearized line bundle \( L \in \text{Pic}^G(M) \), there exist unique rational numbers \( \{a_i(L)\}_{i=0}^s \) such that for every one-parameter subgroup \( \rho \) of \( G \) with weight system \((w_1,\ldots,w_n)\) and every \( R \)-point with the equation \( F(x_1,\ldots,x_n) \):
\[
(3.22) \quad \mu_{\rho}^L(F) = a_0(L) \text{mult}_\rho(F) - \sum_{j=1}^s a_j(L)W_{v_j}(\rho)
\]
(2) Moreover, if \( \mathcal{L} \) has a nonzero \( G \)-invariant section, then \( \{ a_i(\mathcal{L}) \}_{i=0}^s \) must satisfy
\[
d_{a_0}(\mathcal{L}) = \sum_{j=1}^s r_j v_j a_j(\mathcal{L}).
\]

(3) Conversely, for every sequence of rational numbers \( \{ a_j \}_{j=0}^s \) there exists a unique line bundle \( \mathcal{L}(a_0, \ldots, a_s) \) whose HMK-index with respect to every one-parameter subgroup \( \rho \) as above is
\[
\mu^\mathcal{L}(a_0, \ldots, a_s)(F) = a_0 \text{ mult}_\rho(F) - \sum_{j=1}^s a_j W_{v_j}(\rho).
\]

Proof. For a fixed \( m \), \( w_\rho(S_m) \) is a \( \mathbb{Z} \)-linear combination of \( s \) linear functionals \( W_{v_j}(\rho) \), where \( j = 1, \ldots, s \). It follows from Proposition 3.3(A) that each \( \mu^T_\rho(F) \) and \( \mu^J_\rho(F) \) is a \( \mathbb{Z} \)-linear combination of \( \text{mult}_\rho(F) \), and \( W_{v_j}(\rho) \)`s, \( j = 1, \ldots, s \). Moreover, the vectors \( \{ T_{v_j} \}_{j=1}^s \) and \( J_\rho \), form an upper-triangular matrix in the basis \( W_{v_1}(\rho), \ldots, W_{v_s}(\rho), \text{mult}_\rho(F) \), and so span the full \( \mathbb{Q} \)-vector space of \( \mathbb{Q} \)-linear combinations of \( W_{v_1}(\rho), \ldots, W_{v_s}(\rho), \text{mult}_\rho(F) \). Since the dimension of this space is \( s + 1 \), the rank of \( \text{Pic}^G(M) \), (1) and (3) follow.

For (2), recall from §3.3.1 that a \( G \)-linearized line bundle with an invariant nonzero global section must satisfy
\[
0 = \mu^\mathcal{L}_{\rho_{irr}}(F) = a_0(\mathcal{L})d - \sum_{i=1}^s r_i c_i a_i(\mathcal{L}) \quad \text{(for every } F \in S_d) .
\]

\[\square\]

Remark 3.5. Given a \( G \)-linearized line bundle \( \mathcal{L} \) with an invariant nonzero global section, it will be convenient to write \( \mathcal{L} = [a_1(\mathcal{L}), \ldots, a_s(\mathcal{L})] \), where the numbers \( a_i(\mathcal{L}) \) are uniquely determined by Corollary 3.4.

3.3.4. Equations of \( G \)-invariant divisors on \( M = \text{Hyp}_{\mathbb{P}(c_1, \ldots, c_n)}(d) \). In some coordinates \( x_1, \ldots, x_n \), consider the universal weighted homogeneous polynomial of degree \( d \) in \( \mathbb{P}(c_1, \ldots, c_n) \):
\[
U = \sum_{\text{mon}(S_d)} b_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n},
\]
where \( b_{i_1, \ldots, i_n} \) are coordinates on \( H^0(\mathbb{P}(c_1, \ldots, c_n), \mathcal{O}(d)) \). A \( G \)-invariant effective divisor \( \mathcal{D} \) on \( M = \text{Hyp}_{\mathbb{P}(c_1, \ldots, c_n)}(d) \) is defined by a \( G \)-semi-invariant homogeneous form in the variables \( b_{i_1, \ldots, i_n} \) (with coefficients in \( R \)), which we denote by the same letter \( \mathcal{D} \). In particular, this form is semi-invariant with respect to every one-parameter subgroup \( \rho \) of \( G \). For \( \rho_\ell := (0, \ldots, 0, 1, 0, \ldots, 0) \)
acting with weight 1 on \( x_\ell \) and weight 0 on all other variables, we have that \( \rho_\ell \) acts on \( b_{i_1, \ldots, i_n} \) by
\[
\rho_\ell(b_{i_1, \ldots, i_n}) = t^i b_{i_1, \ldots, i_n}.
\]
Then the \( \rho_\ell \)-degree of \( \mathcal{D} \) is precisely the integer \( a_\ell(\mathcal{O}(\mathcal{D})) \) given by Corollary 3.4. The homogeneous degree of \( \mathcal{D} \) is
\[
\deg \mathcal{D} = \frac{1}{d} \sum_{\ell=1}^n c_\ell a_\ell(\mathcal{O}(\mathcal{D})) = a_0(\mathcal{O}(\mathcal{D})) .
\]

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Given now an $R$-point $F$ of $M$, such that $F(K) \notin \text{Supp}(\mathcal{D})$, we have by Corollary 3.4 that for every one-parameter subgroup $\rho = (w_1, \ldots, w_n)$,

\begin{equation}
\mu^\mathcal{O}(\mathcal{D})(F) = \deg \mathcal{D} \cdot \text{mult}_\rho(F) - \sum_{\ell=1}^{n} a_\ell(\mathcal{O}(\mathcal{D}))w_\ell.
\end{equation}

Note that this is in complete analogy with the homogeneous case described by (2.14).

3.4. Boundary divisors. We proceed to define geometrically meaningful boundary divisors on $\text{Hyp}(d)$, with the view towards understanding their HMK indices. Throughout, we work under the following assumptions:

- $\text{char}(K) \nmid d$.
- $\text{lcm}(c_1, \ldots, c_n) \mid d$ and $d > c_i$ (equivalently, $d \geq 2c_i$) for all $i$. This ensures that the degree $d$ hypersurface is Cartier, and that the singular points of all such hypersurfaces form a dense subset of $\mathbb{P}(c_1, \ldots, c_n)$.

3.4.1. The discriminant divisor. For some system of coordinates $x_1, \ldots, x_n$, let $T \subset \mathbb{P}(c_1, \ldots, c_n)$ be the torus defined by $x_1x_2 \cdots x_n \neq 0$. Let $\delta = \delta_{\mathbb{P}(c_1,\ldots,c_n)} \subset M = \text{Hyp}(d)$ be the discriminant divisor defined as the closure of the locus parameterizing hypersurfaces that are singular at a point of $T$. The weighted projective space being a toric variety and $\mathcal{O}(d)$ a very ample line bundle, this is an example of a more general construction of an A-discriminant from [11, Chapter 8].

The divisor $\delta$ is defined by an irreducible $G$-semi-invariant polynomial, homogeneous in the variables $b_{i_1,\ldots,i_n}$, and semi-invariant with respect to the $\rho_\ell$-action for every $\ell = 1, \ldots, n$.

By the assumption $\text{char}(K) \nmid d$, we can define the A-discriminant as the A-resultant of the partials $\frac{\partial U}{\partial x_1}, \ldots, \frac{\partial U}{\partial x_n}$ of the universal degree $d$ polynomial $U$ from (3.25).

Remark 3.6. If $n = 1$, we bypass the geometric definition and simply take the discriminant of $U = b_{d/c_1}x_1^{d/c_1}$ to be given by the equation $b_{d/c_1} = 0$.

3.4.2. The boundary divisors from singularities of $\mathbb{P}(c_1, \ldots, c_n)$. For an integer $m \geq 1$, let $J_m \subset S$ be the ideal generated by the variables whose weight is not divisible by $m$. We define $V_m \subset \mathbb{P}(c_1, \ldots, c_n)$ to be closed subscheme defined by $J_m$. If $m_1, \ldots, m_k$ is the subsequence of $c_1, \ldots, c_n$ of the multiples of $m$, then $V_m \simeq \mathbb{P}(m_1, \ldots, m_k)$. We define $\delta[m]$ to be the closure of the locus parameterizing weighted degree $d$ hypersurfaces whose restriction to $V_m$ has a vanishing A-discriminant as a hypersurface in $\mathbb{P}(m_1, \ldots, m_k)$. Clearly, $\delta[1] = \delta_{\mathbb{P}(c_1,\ldots,c_n)}$ is just the discriminant divisor from §3.4.1. In affine terms, we have a surjective linear map

\[ \pi_m : H^0(\mathbb{P}(c_1, \ldots, c_n), \mathcal{O}(d)) \to H^0(V_m, \mathcal{O}_{V_m}(d)) = H^0(\mathbb{P}(m_1, \ldots, m_k), \mathcal{O}(d)). \]

Then

\begin{equation}
\delta[m] = \pi_m^* (\delta_{\mathbb{P}(m_1,\ldots,m_k)}).
\end{equation}

Algebraically, $\delta[m]$ is the discriminant of the weighted homogeneous polynomial obtained from $U$ by setting $x_i = 0$ for every $i$ such that $m \nmid c_i$. It is also homogeneous in the variables $b_{i_1,\ldots,i_n}$ and semi-invariant with respect to the $\rho_\ell$-action for every $\ell = 1, \ldots, n$.

Remark 3.7. In our applications, all the weights $c_i$’s will be coprime. In this case, we have that for $m \geq 2$, $V_m$ is a point if $m = c_i$, for some $i$, and empty otherwise. Then $\delta[m]$ is defined simply as the vanishing locus of the coefficient of $x_i^d$ in $U$; cf. Remark 3.6 above.
3.4.3. A detour on resultants. To understand the numbers \( a_\ell(\mathcal{O}(\delta[m])) \) that are necessary for the computation of the HMK-indices with respect to \( \mathcal{O}(\delta[m]) \), we need to recall some background on A-resultants, as it is applicable in our case.

Consider a slightly more general situation of the parameter space of \((d_1, \ldots, d_n)\)-intersections in \( \mathbb{P}(c_1, \ldots, c_n) \). It parameterizes \( n \)-tuples of weighted homogeneous polynomials

\[
g_i = \sum_{m \in \text{mon}(S_{d_i})} q_{im} m, \quad (i = 1, \ldots, n)
\]

in \( S \) of weighted degrees \( d_1, \ldots, d_n \), respectively. It is an affine space with coordinates \( \{q_{im} : i = 1, \ldots, n, m \in \text{mon}(S_{d_i})\} \).

Let \( L \) be the algebraic closure of the field \( K(q_{im}) \). We have the following result:

**Proposition 3.8.** Assume \( c_1 = 1 \). The A-resultant of \( g_1, \ldots, g_n \) is a homogeneous irreducible polynomial in the variables \( \{q_{im} : i = 1, \ldots, n, m \in \text{mon}(S_{d_i})\} \). For every \( \ell \), it is separately homologous in the variables \( \{q_{im} : m \in \text{mon}(S_{d_i})\} \) of degree \( A_\ell(d_1, \ldots, d_n) \), where \( A_\ell(d_1, \ldots, d_n) \) equals to:

1. The length of the finite \( L \)-algebra
   \[ A_\ell := L[x_1, x_2^{-1}, \ldots, x_n, x_n^{-1}] / (g_j(1, x_2, \ldots, x_n) : j \neq \ell). \]
2. The length of a subscheme of the torus \( T \) given by the ideal \( (g_j : j \neq \ell) \).
3. \( \frac{1}{c_1 \cdots c_n} \) times the length of the subscheme of \( \mathbb{P}^{n-1} \setminus \mathbb{V}(x_1 \cdots x_n) \) given by the ideal
   \[ (g_j(x_1^{c_1}, x_2^{c_2}, \ldots, x_n^{c_n}) : j \neq \ell). \]

**Remark 3.9.** We can apply the above proposition to compute the degrees of homogeneity of the resultant also in the case \( c_1 = \gcd(c_1, \ldots, c_n) \) by passing to \( \mathbb{P}(1, c_2/c_1, \ldots, c_n/c_1) \).

**Proof.** We dehomogenize by setting \( x_1 = 1 \). We can now use the definition of the A-resultant of the \( n \)-tuple \( \{g_i(1, x_2, \ldots, x_n)\}_{i=1}^n \) as in [11, Proposition-Definition 1.1, p.252]. Specifically, using \( c_1 = 1 \), the affine lattices associated to the Newton polytopes of \( g_i(1, x_2, \ldots, x_n) \), \( i = 1, \ldots, n \), naturally embed into and generate \( \mathbb{Z}^n \) via \( x_i^{i_1} \cdots x_n^{i_n} \mapsto (i_2, \ldots, i_n) \). The statement about the degree of homogeneity of the resultant in the coefficients of \( g_\ell \) now follows by [22, Theorem 1.1] and the subsequent discussion on [22, p.379]; see also [11, p.255].

**Corollary 3.10.** (1) Suppose \( c_1 = 1 \). The weight of the semi-invariant \( \delta[1] = \delta_{\mathbb{P}(c_1, \ldots, c_n)} \) with respect to the \( \rho_\ell \)-action is

\[
\frac{d}{c_\ell} A_\ell(d - c_1, \ldots, d - c_n),
\]

where \( A_\ell(d - c_1, \ldots, d - c_n) \) is as defined in Proposition 3.8.

(2) For a positive integer \( m \), let \( m_1, \ldots, m_k \) be the subsequence of \( c_1, \ldots, c_n \) of the multiples of \( m \), i.e., \( \{m_1, \ldots, m_k\} = \{c_i : m \mid c_i\} \). Assume \( m_1 = m \). Then the degree of the semi-invariant \( \delta[m] \) with respect to the \( \rho_\ell \) is

\[
\frac{d}{c_\ell} A_\ell((d - m_1)/m, \ldots, (d - m_k)/m), \quad \text{if } m \mid c_\ell,
\]

and is zero otherwise.

**Proof.** By (3.28), (1) implies (2).

To prove (1), we use a standard specialization trick: Consider the Fermat-like polynomial

\[
F = \sum_{i=1}^n u_i x_i^{c_i}.
\]
By the assumption \( \text{char}(K) \nmid d \), and so the discriminant of \( F \) is the resultant of \( g_i = u_i x_i^{d_i - 1} \), \( i = 1, \ldots, n \). Since this resultant does not vanish when \( u_1 \cdots u_n \neq 0 \), it must be equal (up to a scalar) to the monomial \( \prod_{i=1}^n u_i^{(d - c_i, \ldots, d - c_n)} \), where we used the degree of homogeneity with respect to \( u_i \) established in Proposition 3.8. Since \( u_i \) has \( \rho_i \)-weight \( \frac{d_i}{c_i} \) if \( i = \ell \), and zero otherwise, we conclude. \[\square\]

The following lemma gives a useful estimate on \( A_i(d_1, \ldots, d_n) \) under some further simplifying assumptions:

**Lemma 3.11.** Suppose \( c_1 = 1 \) and \( c_i \mid d_i \) for all \( i = 1, \ldots, n \). Then we have

\[
A_1(d_1, \ldots, d_n) = \prod_{j \neq 1} d_j / c_j.
\]

If \( c_1 > 1 \), we have a bound

\[
A_\ell(d_1, \ldots, d_n) \leq \frac{d_1}{c_\ell} \prod_{j \neq 1, \ell} d_j / c_j.
\]

**Proof.** The polynomial \( g_j \) defines a degree \( d_j \) hypersurface in \( \mathbb{P}(c_1, \ldots, c_n) \). Applying interpretation (2) of Proposition 3.8, Bezout’s theorem gives a bound

\[
A_\ell(d_1, \ldots, d_n) \leq \frac{1}{c_1 \cdots c_n} \prod_{j \neq \ell} d_j,
\]

with equality exactly when we can find \( g_1, \ldots, \hat{g}_\ell, \ldots, g_n \) that have no common root in \( \mathbb{V}(x_1 \ldots x_n) \).

If \( c_1 = 1 \), then using \( c_i \mid d_i \), the specialization \( g_i = x_i^{d_i} - x_1^{d_i} \) shows that this is the case for \( \ell = 1 \).

If \( c_\ell > 1 \), then \( g_1, \ldots, \hat{g}_\ell, \ldots, g_n \) necessarily vanish at the point \( x_1 = \cdots = x_\ell = \cdots = x_n = 0 \) whenever \( c_\ell \) does not divide any of the \( d_i, i \neq \ell \). We have a strict inequality in this case. \(\square\)

**3.4.4. Example:** The resultant of a \((5, 5, 4, 3)\)-intersection in \( \mathbb{P}(1, 1, 2, 3) \).

**Lemma 3.12.** The degrees of homogeneity of the resultant of degree 5, 5, 4, 3 forms \( g_1, g_2, g_3, g_4 \) in \( \mathbb{P}(1, 1, 2, 3) \) are:

1. \( A_1(5, 5, 4, 3) = A_2(5, 5, 4, 3) = 10 \),
2. \( A_3(5, 5, 4, 3) = 12 \),
3. \( A_4(5, 5, 4, 3) = 16 \).

**Proof.** (1) follows directly from Lemma 3.11. For (2), we note that \( g_1(1, x_2, x_3, x_4) \) is linear in the weight 3 variables \( x_4 \), and so \( A_2(5, 5, 4, 3) \) is the number of intersections of two generic degree 5 weighted hypersurfaces in \( \mathbb{P}(1, 1, 2) \) away from the vertex, which is easily seen to be \( 12 = (25 - 1)/2 \). For (3), consider the intersection of three degrees 5, 5, 4 homogeneous forms \( g_1(x_1, x_2, x_3^5, x_4^3), g_2(x_1, x_2, x_3^5, x_4^3), g_3(x_1, x_2, x_3^5, x_4^3) \) in \( \mathbb{P}^3 \). If \( g_1, g_2, g_3 \) are chosen generically, these hypersurfaces have no common roots in \( x_1 x_2 x_3 = 0 \) with the exception of the point \( [0 : 0 : 0 : 1] \), where they all vanish with multiplicities, 2, 2, 1, respectively. Their tangent cones at this point have no common zero, and so passing to the blow-up of \( \mathbb{P}^3 \) at \( [0 : 0 : 0 : 1] \), we conclude that

\[
A_4(5, 5, 4, 3) = \frac{1}{6}(100 - 4) = 16.
\]

\[\square\]
Corollary 3.13. Let $\delta \in \text{Hyp}_{P(1,1,2,3)}(6)$ be the discriminant divisor. Then for an $R$-point $F \in \text{Hyp}_{P(1,1,2,3)}(6)$ with $F(K) \notin \text{Supp}(\delta[1])$, we have

$$
\mu_{\rho}^{\delta[1]}(F) = 48 \text{ mult}_{\rho}(F) - 60(w_1 + w_2) - 36w_3 - 32w_4.
$$

3.4.5. The cone generated by the boundary divisors. It would be interesting to understand the cone of effective $\mathbb{Q}$-linear combinations of the boundary divisors $\delta[m]$ for an arbitrary $(c_1, \ldots, c_n)$, but for our purposes it will be sufficient to understand the simplest case when

$$1 = c_1 = c_2 = \cdots = c_{r-1} < c_r < c_{r+2} < \cdots < c_n,$$

where $c_i$’s are all pairwise coprime. In this case, in the notation of Remark 3.5, we have:

$$
\mathcal{O}(\delta[1]) = \left[\frac{d}{1} A_1(d-c_1, \ldots, d-c_n), \frac{d}{c_r} A_r(d-c_1, \ldots, d-c_n), \ldots, \frac{d}{c_n} A_n(d-c_1, \ldots, d-c_n)\right]
$$

$$
\mathcal{O}(\delta[c_i]) = \left[0, \ldots, 0, \frac{d}{c_i}, 0, \ldots, 0\right], \quad i \geq r.
$$

Proposition 3.14 (The balanced line bundle). The cone of effective linear combinations of the boundary divisors $\mathcal{O}(\delta[m])$ is simplicial with extremal rays $\mathcal{O}(\delta[c_i])$, $i = 1, \ldots, s$. Moreover, this cone contains a line bundle

$$
\mathcal{L}^{\text{bal}} = \left[\frac{d}{\sum c_i}, \ldots, \frac{d}{\sum c_1}\right].
$$

Proof. By Lemma 3.11, for $\ell \geq r$, we have (in)qualities

$$
\frac{d}{\ell} A_{\ell}(d-c_1, \ldots, d-c_n) = d \prod_{j=2}^{n} \frac{d-c_j}{c_j} > \frac{d-1}{c_\ell} \frac{d}{c_\ell} \prod_{j \neq 1, \ell}^{n} \frac{d-j}{c_j} > \frac{d}{c_\ell} A_\ell(d-c_1, \ldots, d-c_n),
$$

where we have used $d-c_\ell > (d-1)/c_\ell$.

\[ \square \]

3.4.6. Weighted hypersurfaces with $\mathbb{G}_m$-action and their HMK-indices. Suppose the $c_1, \ldots, c_n$ is a sequence of weights such that $c_1 = 1$ and all the $c_i$’s greater than 1 are coprime. (This ensures that $\mathbb{P}(c_1, \ldots, c_n)$ has isolated singularities.) We continue working under the assumption $\text{char}(K) \nmid d$.

Definition 3.15 ((Partial) Fermat hypersurfaces). Let $S_R = R[x_1, \ldots, x_n]$ be the graded ring with grading given by the weight vector $\vec{c} = (c_1, \ldots, c_n)$. Define a Fermat hypersurface of weighted degree $d$ to be a smooth hypersurface given by the equation

$$
\frac{d}{x_1^{c_1}} + \cdots + \frac{d}{x_n^{c_n}}.
$$

For $\ell \in \{1, \ldots, n\}$, we define the $\ell$th partial Fermat hypersurface of weighted degree $d$ to be

$$
f_\ell := x_1^{d} + \cdots + x_{\ell-1}^{d} + x_{\ell+1}^{d} + \cdots + x_n^{d}.
$$

Let $\rho_\ell$ be the one-parameter subgroup of $G$ given by $\rho(t) = tx_\ell$, and $\rho_\ell(x_i) = x_i$ for $i \neq \ell$. Evidently, $f_\ell$ is $\rho_\ell$-invariant.

Proposition 3.16. Consider the $\ell$th partial Fermat $f_\ell$ of degree $d$, with its $\mathbb{G}_m$-action $\rho_\ell$.

$$
\mu_{\rho_\ell}^{\mathcal{O}(\delta[1])}(f_\ell) = -\frac{d}{c_\ell} A_\ell(d-c_1, \ldots, d-c_n),
$$

$$
\mu_{\rho_\ell}^{\mathcal{O}(\delta[m])}(f_\ell) = \begin{cases} 
-\frac{d}{c_\ell}, & \text{if } m = c_\ell, \\
0, & \text{if } m \geq 2 \text{ and } m \neq c_\ell.
\end{cases}
$$
Proof. This follows from Corollary 3.10. □

4. Examples

In this section, we give explicit examples of the theory developed so far. Three of the examples will treat stability of weighted hypersurfaces:

(1) Sextics in $\mathbb{P}(1,2,3)$ and Tate’s minimal models of elliptic fibrations from the point of view of Kollár stability; see §4.1.
(2) Quartics in $\mathbb{P}(1,1,1,2)$ and interpretation of [9, Section 4] in terms of Kollár stability; see §4.2.
(3) Sextics in $\mathbb{P}(1,1,2,3)$ and first steps towards establishing the existence of standard models of degree one del Pezzo fibrations.

We also consider two example of $(d,e)$-intersections in weighted projective spaces:

(1) Kollár stability of $(4,2)$-intersections in $\mathbb{P}(1,1,1,2,2)$ and application to the standard models of degree 2 del Pezzo fibrations; see §4.3.
(2) Kollár stability of $(6,3)$-intersections in $\mathbb{P}(1,1,2,3,3)$.

In the last example, we introduce the concept of T-stability of a $(6,3)$-intersection in $\mathbb{P}(1,1,2,3,3)$ which will be the crucial tool in Section 5.

Throughout $R$ is a DVR with a fraction field $K$ and valuation $\text{val}: K \to \mathbb{Z} \cup \{\infty\}$.

4.1. Sextics in $\mathbb{P}(1,2,3)$ and Tate’s minimal Weierstrass models of elliptic curves.

Every elliptic curve over $K$ has a model over $R$ given by a sextic equation in $\mathbb{P}^R(1,x,2,z,3,w)$:

\[(4.1) \quad F(x,z,w) = ow^2 + (p_1xz + p_2x^4)w + q_1z^3 + q_2x^2z^2 + q_3x^4z + q_4x^6 = 0.\]

Moreover, one can always arrange for
\[(4.2) \quad o = q_1 = 1,\]
in which case (4.1) is known as the Weierstrass form.

In [28], Tate famously defined the minimal Weierstrass model to be the model over $R$ satisfying (4.2) and minimizing the valuation of the discriminant of $F(x,z,w)$. Tate used the minimal model in his algorithm for determining the Kodaira-Néron classification of the central fiber in the associated elliptic fibration. Here, we demonstrate that Tate’s minimal model is an example of Kollár $\mathcal{O}$-semistable model for an appropriate choice of $\mathcal{O}$ on the space of sextics in $\mathbb{P}(1,2,3)$.

By §3.4, the three boundary divisors on $\text{Hyp}_{\mathbb{P}(1,2,3)}(6)$ are:

(1) $\delta = \delta[1]$, the closure of the locus of sextics singular at a non-singular point of $\mathbb{P}(1,2,3)$.
(2) $\delta[2]$, sextics through $\frac{1}{2}(1,1)$-point.
(3) $\delta[3]$, sextics through $\frac{1}{3}(1,2)$-point.

By Corollary 3.10, with respect to a one-parameter subgroup $\rho = (w_1, w_2, w_3)$, acting diagonally in some coordinates $x, z, w$, the HMK-indices of these boundary divisors are:

\[
\begin{align*}
\mu^\delta[1]_\rho(F) &= 7 \text{mult}_\rho(F) - 12w_1 - 6w_2 - 6w_3, \quad (\text{char}(K) \neq 2,3) \\
\mu^\delta[2]_\rho(F) &= \text{mult}_\rho(F) - 3w_2. \\
\mu^\delta[3]_\rho(F) &= \text{mult}_\rho(F) - 2w_3.
\end{align*}
\]
Remark 4.1. Explicitly, for the reader who might be surprised to learn that the degree of the discriminant of an elliptic curve can be \(7\) (just as we were), the equation of \(\delta[1]\) is
\[
\begin{align*}
3.10 & \quad p_1^3 p_2^3 q_1 + 27 op_2^4 q_2 - p_1^2 p_2 q_2 - 36 op_1 p_2^2 q_2 + 8 op_1^2 p_2^2 q_2 - 16 o^2 p_2^2 q_2 + p_1^5 p_2 q_3 \\
&+ 30 o p_1^2 p_2 q_3 - 8 o p_2^2 q_2 q_3 + 16 o^2 p_1 p_2^2 q_3 - 96 o^2 p_1 p_2 q_3 + q_3 \\
&+ 8 o^2 p_2 q_3 - 16 o^2 q_2 q_3 + q_3 - p_1^4 q_3 - 36 op_1 p_2 q_1 q_4 - 12 o p_1^2 q_4 \\
&+ 144 o^2 p_1 p_2 q_1 q_4 - 48 o p_1^2 q_1^2 + 64 o^3 q_1 q_3 + 126 o^2 p_2 q_1 q_4 + 12 o p_1^2 q_1 \\
&- 144o^2 p_1 p_2 q_1 q_4 - 48 o p_1^2 q_1^2 q_3 + 28 o^2 p_2 q_1 q_3 q_4 + 432 o^3 q_1 q_3^2 q_4.
\end{align*}
\]

Proposition 4.2. Let \(D^{Gor} := \epsilon \delta[1] + \delta[2] + \delta[3]\), where \(0 < \epsilon \ll 1\). Then \(D^{Gor}\)-semistable model is the Tate’s minimal Weierstrass model.

Proof. Consider \(F\) as in (4.1). If \(val(o) \geq 1\), then for \(\rho = (1, 1, 1)\), we have \(\text{mult}_\rho(F) \geq 3\). Then
\[
\mu^\delta[1]_\rho(F) \geq -3, \quad \mu^\delta[2]_\rho(F) \geq 0, \quad \mu^\delta[3]_\rho(F) \geq 1,
\]
and, consequently, \(\mu^\delta\rho(D) > -3\epsilon + 1 > 0\).

If \(val(q_1) \geq 1\), then for \(\rho = (1, 1, 2)\), we have \(\text{mult}_\rho(F) \geq 4\) and
\[
\begin{align*}
\mu^\delta[1]_\rho(F) &= 7 \text{mult}_\rho(F) - 12 w_1 - 6 w_2 - 6 w_3 \geq 28 - 30 = -2, \\
\mu^\delta[2]_\rho(F) &= \text{mult}_\rho(F) - 3 w_2 \geq 1, \\
\mu^\delta[3]_\rho(F) &= \text{mult}_\rho(F) - 2 w_3 \geq 0,
\end{align*}
\]
and, consequently, \(\mu^\delta\rho(D) > -2\epsilon + 1 > 0\).

We conclude that a \(D^{Gor}\)-semistable model has \(val(o) = val(q_1) = 0\) and so avoids \(\delta[2]\) and \(\delta[3]\) entirely. Then a \(D^{Gor}\)-semistable model minimizes the degree of the discriminant \(\delta[1]\) subject to the condition \(val(o) = val(q_1) = 0\), which is precisely the definition of the Tate’s model.

\[\Box\]

Remark 4.3. The formula for \(\mu^\delta[1]_\rho(F)\) is derived under the assumption \(\text{char}(K) \neq 2, 3\), in which case it suffices to take \(\epsilon < 1/6\) in the statement of the proposition. Note however that when \(0 < \epsilon \ll 1\), the exact formula for \(\mu^\delta[1]_\rho\) is irrelevant, and so the proposition remains true even in characteristics 2 and 3.

4.2. Quartics in \(\mathbb{P}(1, 1, 1, 2)\) and degree 2 del Pezzo fibrations. Assume \(\text{char}(K) \neq 2\). Consider the space \(\text{Hyp}_{\mathbb{P}(1, 1, 1, 2)}(4)\) of degree 4 weighted hypersurfaces in \(\mathbb{P}(1, 1, 1, 2)\). By §3.4, we have two boundary divisors:

(1) \(\delta[1]\), the closure of the locus of quartics singular at a non-singular point of \(\mathbb{P}(1, 1, 1, 2)\).

(2) \(\delta[2]\), quartics through the \(\frac{1}{2}(1, 1, 1)\) point of \(\mathbb{P}(1, 1, 1, 2)\).

The \(K\)-valued points of \(M \setminus (\delta_0 \cup \delta_1)\) parameterize smooth degree 2 del Pezzo surfaces over \(K\) (this is true for any field \(K\)).

By Corollary 3.10, with respect to a one-parameter subgroup \(\rho = (w_1, w_2, w_3, w_4)\) acting diagonally in some coordinates \(x_1, x_2, x_3, x_4, \) and an \(R\)-point of \(\text{Hyp}_{\mathbb{P}(1, 1, 1, 2)}(4)\) with the equation \(F \in R[x_1, x_2, x_3, x_4]\), the HMK-indices of these boundary divisors are:
\[
\begin{align*}
\mu^\delta[1]_\rho(F) &= 40 \text{mult}_\rho(F) - 36(w_1 + w_2 + w_3) - 26 w_4, \\
\mu^\delta[2]_\rho(F) &= \text{mult}_\rho(F) - 2 w_4.
\end{align*}
\]

It follows that for
\[
\mathcal{D}^{val} := \frac{1}{45}(\delta[1] + 5 \delta[2]),
\]

\[20\]
given by Proposition 3.14, we have

$$\mu^{\mathcal{D}_{bal}}_\rho(f) = \text{mult}_\rho(F) - \frac{4}{5}(w_1 + w_2 + w_3 + w_4).$$

We have the following result recasting a bulk of [9, Section 4] in the language of Kollár stability (hence we omit the proof):

**Proposition 4.4.** Suppose $R = \mathcal{O}_{C,p}$ is the local ring of a smooth curve over an algebraically closed field of characteristic not 2. Suppose $F \in R[x_1, x_2, x_3, x_4]$ is $\mathcal{D}_{bal}$-semistable. Then $F = 0$ defines a threefold del Pezzo fibration $X \to \text{Spec} R$ of degree 2 such that $X$ has terminal singularities and the central fiber $X_0$ is reduced, and either irreducible or a union of two hypersurfaces in $\mathbb{P}(1,1,1,2)$ of degree 2 each passing through the $\frac{1}{2}(1,1,1)$-point.

For a general choice of $A, B \in R[x_1, x_2, x_3, x_4][I]$, the family

$$t(x_1^2 + G(x_1, x_2, x_3)) + A(x_1, x_2, x_3)B(x_1, x_2, x_3) = 0,$$

has a central fiber that is a union of two degree 2 hypersurfaces each passing through the $\frac{1}{2}(1,1,1)$-point and is $\mathcal{D}$-semistable for any choice of a $G$-invariant effective boundary divisor $\mathcal{D}$. In the next subsection, we illustrate how to improve Proposition 4.4 by working with $(4,2)$-intersections in $\mathbb{P}(1,1,1,2)$.

### 4.3. (4,2)-intersections in $\mathbb{P}(1,1,1,2)$.

Assume $\text{char}(K) \neq 2$. Let $M = \text{Int}_{\mathbb{P}(1,1,1,2)}((4,2)$ be the parameter space of $(4,2)$ intersections in $\mathbb{P}(1,1,1,2)$, and $G = \text{Aut}_{gr}(\text{Cox}(\mathbb{P}(1,1,1,2)))$.

#### 4.3.1. Boundary divisors.

Let $M^0$ be the locus in $M$ of $(4,2)$-intersection $(F, H)$ where $H = x_5$. Then the translate $G \cdot M^0$ is an open $G$-invariant subscheme of $M$ whose complement has codimension 2 (it is defined by the vanishing of the coefficients of $x_4$ and $x_5$ in $H$). As $x_5 = 0$ defines a closed subscheme in $\mathbb{P}(1,1,1,2)$ isomorphic to $\mathbb{P}(1,1,1,2)$, we have an isomorphism $M^0 \simeq \text{Hyp}_{\mathbb{P}(1,1,1,2)}((4)$ given by $(F, x_5) \mapsto (F(x_1, \ldots, x_4, 0)$. This isomorphism is $\text{Aut}(\text{Cox}(\mathbb{P}(1,1,1,2)))$ acts on $\mathbb{Z}[x_1, \ldots, x_4]$ and $\mathbb{G}_m$ acts by $\lambda \cdot x_5 = \lambda x_5$. (In particular, this $\mathbb{G}_m$ acts trivially on $\text{Hyp}_{\mathbb{P}(1,1,1,2)}((4)$.)

By smoothness of $\text{Hyp}_{\mathbb{P}(1,1,1,2)}((4)$ and $M$, the $G$-translates of the boundary divisors $\delta[1]$ and $\delta[2]$ inside $\text{Hyp}_{\mathbb{P}(1,1,1,2)}((4)$ that we defined in 3.4 extend uniquely to divisors on $M$, which we continue to denote $\delta[1]$ and $\delta[2]$. Moreover, if in some coordinates $x_1, \ldots, x_5$, we have $I = (F(x_1, \ldots, x_4), x_5)$, then for every one-parameter subgroup $\rho$ with weight system $(w_1, \ldots, w_4, w_5)$ in this coordinates, we have

$$\mu^\delta[\rho](F, H) = \mu^{\delta[\rho]}_\rho(F),$$

where $\rho' = (w_1, \ldots, w_4)$.

By Proposition 3.3 (B), we compute the Hilbert-Mumford-Kollár indices with respect to the line bundles $T_1, T_2, J_2, J_4$, and $J_1' := J_4 - 5J_2 - T_2$ of an $R$-point $f$: $\text{Spec}(R) \to M$ given by a $(4,2)$-intersection $(F, H)$:

$$\mu_{\rho}^{T_1}(f) = -w_\rho\{x_1, x_2, x_3\}$$
$$= -(w_1 + w_2 + w_3),$$

$$\mu_{\rho}^{T_2}(f) = -w_\rho(S_2) = w_\rho\{x_1^2, x_1 x_2, x_2^2, x_1 x_3, x_2 x_3, x_3^2, x_4, x_5\}$$
$$= -(4(w_1 + w_2 + w_3) + (w_4 + w_5)),$$

$$\mu_{\rho}^{J_2}(f) = -\text{mult}_\rho(H),$$

$$\mu_{\rho}^{J_1'}(f) \leq -5\text{mult}_\rho(H_1) + \text{mult}_\rho(F_1) + 4(w_1 + w_2 + w_3) + (w_5 + w_5)).$$

$$\mu_{\rho}^{J_1}(f) \leq -\text{mult}_\rho(F).$$
Clearly, \( T_1, T_2, J_2, \) and \( J'_4 \) form a basis of \( \text{Pic}^G(M) \otimes \mathbb{Q} \).

We proceed to express the boundary divisors \( \delta[1] \) and \( \delta[2] \) in terms of the above basis with the goal of computing their HMK-indices. To do so, let \( \left\{ f_\ell \right\}_{\ell=1,4} \) be the partial Fermats as in Definition 3.15. Then we have the following two distinguished points of \( M, I_1 := (f_1, x_5) = (x_2^4 + x_3^3 + x_4^4, x_5) \) and \( I_4 := (f_4, x_5) = (x_1^4 + x_2^3 + x_3^4, x_5) \). The point \( I_\ell \) is fixed by the one-parameter subgroup \( \rho_\ell \) acting via \( \rho \cdot x_\ell = t x_\ell \), by \( \rho_{\text{ter}} = (1, 1, 2, 2) \), and by \( \rho_5 = (0, 0, 0, 1) \).

We now collect the computations that we have obtained so far:

| \( \mathcal{L} \) | \( T_1 \) | \( T_2 \) | \( J_2 \) | \( J'_4 \) | \( \delta[1] \) | \( \delta[2] \) |
|------------------|---------|---------|-------|-------|--------|--------|
| \( \mu_{\rho_1}^{\mathcal{L}}(I_1) \) | \(-1\) | \(-4\) | \(0\) | \(0\) | \(-36\) | \(0\) |
| \( \mu_{\rho_4}^{\mathcal{L}}(I_4) \) | \(0\) | \(-1\) | \(0\) | \(0\) | \(-26\) | \(-2\) |
| \( \mu_{\rho_{\text{ter}}}^{\mathcal{L}}(I_\ell) \) | \(-3\) | \(-16\) | \(-2\) | \(-4\) | \(0\) | \(0\) |
| \( \mu_{\rho_5}^{\mathcal{L}}(I_\ell) \) | \(0\) | \(-1\) | \(-1\) | \(0\) | \(0\) | \(0\) |

Using the method of indeterminate coefficients, we obtain:

\[
\delta[1] = -68T_1 + 26T_2 - 26J_2 - 40J'_4,
\]

\[
\delta[2] = -8T_1 + 2T_2 - 2J_2 - J'_4.
\]

In particular, from (4.6), we conclude that for \( f \) with equations \( (F, H) \), we have

\[
\mu_\rho^{[1]}(f) \geq 40 \text{ mult}_\rho(F_1) + 26 \text{ mult}_\rho(H_1) - 36(w_1 + w_2 + w_3) - 26(w_4 + w_5),
\]

\[
\mu_\rho^{[2]}(f) \geq \text{ mult}_\rho(F_1) + 2 \text{ mult}_\rho(H_1) - 2(w_4 + w_5).
\]

For \( \mathcal{D}^{\text{bol}} := \frac{1}{35}\left[\delta[1] + 5\delta[2]\right] \), we then have

\[
\mu_\rho^{\mathcal{D}^{\text{bol}}}(f) \geq \text{ mult}_\rho(F_1) + \frac{4}{5}\text{ mult}_\rho(H_1) - \frac{4}{5}(w_1 + w_2 + w_3 + w_4 + w_5).
\]

4.3.2. Corti’s standard models in degree 2. Let \( \mathcal{D}^{\text{ter}} = \mathcal{D}^{\text{bol}} + \epsilon\delta_1 \), where \( 0 < \epsilon \ll 1 \). Then we have the following result:

**Theorem 4.5.** Suppose \( R = \mathcal{O}_{C, \beta} \) is the local ring of a smooth curve over an algebraically closed field of characteristic not 2. Suppose \( (F, H) \in M \) is \( \mathcal{D}^{\text{ter}} \)-semistable. Then \( F = H = 0 \) defines a threefold del Pezzo fibration \( X \to \Delta \) of degree 2 such that \( X \) has terminal singularities and the central fiber \( X_0 \) is integral.

The proof is similar, but easier, compared to Theorem 5.1, with a part of the proof following directly from [9, Section 4]. Therefore, we omit details.

4.4. Sextics in \( \mathbb{P}(1,1,2,3) \): Towards standard models of degree 1 del Pezzo fibrations. Let \( M := \text{Hyp}(1,1,2,3)(6) \) be the space of sextics in \( \mathbb{P}(1,1,2,3) \), and \( G = \text{Aut}(\text{Cox}(\mathbb{P}(1,1,2,3))) \).

We have three boundary divisors in \( M \):

1. \( \delta[1] \), the closure of the locus of quartics singular at a non-singular point of \( \mathbb{P}(1,1,2) \).
2. \( \delta[2] \), sextics through the \( \frac{1}{3}(1,1,1) \)-point.
3. \( \delta[3] \), sextics through the \( \frac{1}{4}(1,1,2) \)-point.

The \( K \)-valued points of \( M \setminus (\delta[1] \cup \delta[2] \cup \delta[3]) \) parameterize smooth degree 1 del Pezzo surfaces over \( K \) (for any field \( K \)).

Assume now \( \text{char}(K) \neq 2,3 \). Given \( F \in R[x_1, x_2, x_3, x_4]_6 \) and a one-parameter subgroup \( \rho \) of \( G \) acting by \( \rho \cdot x_i = t^w x_i \), we have by Proposition 3.10 and Lemma 3.13 that
\[
\mu_\rho^{[1]}(F) = 48 \text{mult}_\rho(F) - 60(w_1 + w_2) - 36w_3 - 32w_4,
\]
\[
\mu_\rho^{[2]}(F) = \text{mult}_\rho(F) - 3w_3,
\]
\[
\mu_\rho^{[3]}(F) = \text{mult}_\rho(F) - 2w_4.
\]

It follows that for the divisor
\[
D^{bal} := \frac{1}{70}(6\delta[1] + 8\epsilon[1] + 14\epsilon[3]),
\]
given by Proposition 3.14, we have
\[
\mu_\rho^{bal}(F) = \text{mult}_\rho(F) - \frac{6}{7}(w_1 + w_2 + w_3 + w_4).
\]

In analogy with Theorem 4.4, we can prove

**Theorem 4.6.** Suppose \( R = \mathcal{O}_{C, P} \) is the local ring of a smooth curve over an algebraically closed field of characteristic not 2 or 3. Suppose \( F \in R[x_1, x_2, x_3, x_4][6] \) is \( D^{bal} \)-semistable. Then \( F = 0 \)
defines a threefold del Pezzo fibration \( X \to \Delta \) of degree 1 such that \( X \) has terminal singularities
and the central fiber \( X_0 \) is reduced, and either irreducible or a union of two hypersurfaces in \( \mathbb{P}(1, 1, 2, 3) \) of degree 3 each passing through the \( \frac{1}{3}(1, 1, 2) \)-point.

Since this result, and its proof, is subsumed by Theorem 5.1, we omit the details.

**4.4.1. Gorenstein canonical models.** We use Kollár stability to give a quick self-contained proof of the fact that every degree 1 del Pezzo fibration over a smooth curve has a model with Gorenstein canonical singularities. This fact has been shown by Corti [9, Corollary-Definition 3.4 and Proposition 3.13] and Loginov [18, Corollary B] in characteristic 0 using MMP techniques.

**Theorem 4.7.** Suppose \( R = \mathcal{O}_{C, P} \) is the local ring of a smooth curve over an algebraically closed field of characteristic not 2 or 3. Let \( \mathcal{D}^{Gor} := \epsilon \delta_0 + \delta_1 + \delta_2 \), where \( 0 < \epsilon \ll 1 \). Suppose \( F \in R[x_1, x_2, x_3, x_4][6] \) is \( \mathcal{D}^{Gor} \)-semistable. Then \( F = 0 \) defines a threefold del Pezzo fibration \( X \to \Delta \) of degree 1 such that \( X \) has Gorenstein canonical singularities and the central fiber \( X_0 \) is integral.

**Proof.** By (4.10), \( \mathcal{D}^{Gor} \)-semistability implies the following condition for any 1-PS \( \rho \) acting with weights \( (w_1, w_2, w_3, w_4) \) in some coordinates on \( \mathbb{P}(1, 1, 2, 3) \):
\[
(2 + 48\epsilon) \text{mult}_\rho(F) \leq 3w_3 + 2w_4 + \epsilon(60w_1 + 60w_2) + 36w_3 + 32w_4).
\]

Take \( \epsilon = \frac{1}{N^2} \), where \( N > 0 \). Write \( F = a(t)x_1^2 + b(t)x_2^3 + G \), where \( G \in (x_1, x_2) \). Then \( \mathcal{D}^{Gor} \)-semistability with respect to \( \rho = (N, N, 1, 2) \) implies that \( (2 + \frac{48}{N^2}) \text{mult}_\rho(F) \leq 7 + \frac{2}{N^2}(100 + 120N) \). It follows that \( \text{mult}_\rho(F) \leq 3 \) and so \( b(0) \neq 0 \). Similarly, \( \mathcal{D}^{Gor} \)-semistability with respect to \( \rho = (N, N, 1, 1) \) implies that \( \text{mult}_\rho(F) \leq 2 \), and so \( a(0) \neq 0 \). We conclude that every \( \mathcal{D}^{Gor} \)-semistable family \( X \to \Delta \) has a central fiber of the form \( x_1^2 + x_2^3 + \cdots = 0 \), a sextic that avoids the singularities of \( \mathbb{P}(1, 1, 2, 3) \). Such sextics are Gorenstein and necessarily integral, by degree considerations.

Consider now a Gorenstein singular point \( P \in X_0 \), which after a change of coordinates, we can take to be \( P = [1 : 0 : 0 : 0] \). We have \( F = x_1^2 + x_2^3 + G(x_1, x_2, x_3) \), where \( G \in (x_1, x_2)^2 \). Then for \( \rho = (0, 1, 2, 3) \), (4.12) gives
\[
(2 + 48\epsilon) \text{mult}_\rho(F) \leq 12 + \epsilon(96 + 72 + 60) = 12 + 228\epsilon.
\]
Thus \( \text{mult}_\rho(F) \leq 5 \). By the recognition criterion of Du Val singularities given in Lemma 5.3 below, the hyperplane section \( x_2 = c \), where \( c \in k \) is general, has a Du Val singularity at \( P \). It follows that \( P \) is a cDV Gorenstein singularity, hence canonical.
An example of $x_1^2 + x_3^3 + t^2(x_1^6 + x_2^6) = 0$ shows that $X$ can have non-isolated singularities.

4.5. Stability of $(6,3)$-intersections in $\mathbb{P}(1,1,2,3,3)$. We let $M := \text{Int}_{\mathbb{P}(1,1,2,3,3)}(6,3)$ be the parameter space of $(6,3)$-intersections in $\mathbb{P}(1,1,2,3,3)$.

Then we have
\begin{align*}
\mu^{T_1}_\rho(f) &= -w_\rho(x_1, x_2) = -(w_1 + w_2), \\
\mu^{T_2}_\rho(f) &= -w_\rho(x_1^2, x_1 x_2, x_2^3, x_3) = -(3(w_1 + w_2) + w_3), \\
\mu^{T_3}_\rho(f) &= -w_\rho(x_1^3, x_1^2 x_2, x_1 x_2^2, x_3, x_2 x_3, x_4, x_5) \\
&= -(7(w_1 + w_2) + 2w_3 + (w_4 + w_5)).
\end{align*}

(4.13)
\[\mu^{J_3}_\rho(f) = -\text{mult}_\rho(H)\]
\[\mu^{J_5}_{\rho}(f) \leq -\text{mult}_\rho(F) - 8\text{mult}_\rho(H) - 7(w_1 + w_2) - 2w_3 - (w_4 + w_5).\]
\[\mu^{J_5-7J_3}_\rho(f) \leq -\text{mult}_\rho(F),\]
where the equality holds in the last two lines whenever $(F, H)$ are equations of $f$ adapted to $\rho$.

Consider now the three partial Fermats $\{f_\ell\}_{\ell=1,3,4}$ as defined in Definition 3.15. Then we have the following three distinguished points of $M$: $I_\ell := (f_\ell, x_5)$, $\ell = 1, 3, 4$. The point $I_\ell$ is fixed by the one-parameter subgroup $\rho_\ell$ acting via $\rho \cdot x_\ell = tx_\ell$, by $\rho_{\text{irr}} = (1,1,2,3,3)$, and by $\rho_5 = (0,0,0,0,1)$.

As in §4.3.1, the boundary divisors $\delta[1]$, $\delta[2]$, and $\delta[3]$ in Hyp$_{\mathbb{P}(1,1,2,3,3)}(6)$ that we defined in §3.4 extend uniquely to divisors on $M$. We now collect the HMK-indices computations that we have obtained so far:

| $\mathcal{L} = \mathcal{L}$ | $T_1$ | $T_2$ | $T_3$ | $J_3$ | $J'_3$ | $\delta[1]$ | $\delta[2]$ | $\delta[3]$ |
|----------------------------|-------|-------|-------|-------|-------|-------------|-------------|-------------|
| $\mu^{\mathcal{L}}_{\rho_1}(I_1)$ | $-1$  | $-3$  | $-7$  | $0$   | $0$   | $-60$       | $0$         | $0$         |
| $\mu^{\mathcal{L}}_{\rho_3}(I_3)$ | $0$   | $-1$  | $-2$  | $0$   | $0$   | $-36$       | $-3$        | $0$         |
| $\mu^{\mathcal{L}}_{\rho_4}(I_4)$ | $0$   | $0$   | $-1$  | $0$   | $0$   | $-32$       | $0$         | $-2$        |
| $\mu^{\mathcal{L}}_{\rho_{\text{irr}}}(I_\ell)$ | $-2$  | $-8$  | $-24$ | $-3$  | $-6$  | $0$         | $0$         | $0$         |
| $\mu^{\mathcal{L}}_{\rho_5}(I_\ell)$ | $0$   | $0$   | $-1$  | $-1$  | $0$   | $0$         | $0$         | $0$         |

The HMK-indices of boundary line bundles are computed using (4.10), and the remaining indices are obtained from (4.13), noting that the equations $(f_\ell, x_5)$ are adapted to each $\rho$.

Solving a simple system of linear equations, we obtain the expression for the boundary divisors in terms of tautological line bundles (we use additive notation for line bundles here):
\begin{align*}
\delta[1] &= -80T_1 - 28T_2 + 32T_3 - 32J_3 - 48J'_3, \\
\delta[2] &= -9T_1 + 3T_2 - J'_3, \\
\delta[3] &= -2T_1 - 4T_2 + 2T_3 - 2J_3 - J'_3.
\end{align*}

In particular,
\begin{align*}
\mu^{\delta[1]}_{\rho}(f) &= 48 \text{mult}_\rho(F_1) + 32 \text{mult}_\rho(H_1) - 60(w_1 + w_2) - 36w_3 - 60(w_4 + w_5), \\
\mu^{\delta[2]}_{\rho}(f) &= \text{mult}_\rho(F_1) - 3w_2, \\
\mu^{\delta[3]}_{\rho}(f) &= \text{mult}_\rho(F_1) + 2 \text{mult}_\rho(H_1) - 2(w_4 + w_5).
\end{align*}

(4.14)

For $\mathfrak{G}^{\text{bol}} := \frac{1}{70}(\delta[1] + 8\delta[2] + 14\delta[3])$, we have
\begin{align*}
\mu^{\mathfrak{G}^{\text{bol}}}_{\rho}(f) &\geq \text{mult}_\rho(F_1) + \frac{6}{7} \text{mult}_\rho(H_1) - \frac{6}{7}(w_1 + w_2 + w_3 + w_4 + w_5).
\end{align*}

(4.15)
Collecting these results, we arrive at a semistability notion that will be central in our proof of Corti’s conjecture in Section 5.

**Proposition 4.8 (T-semistability).** Fix $0 < \epsilon < 1$. Then for an effective boundary divisor
\begin{equation}
\mathcal{D}^{\text{ter}} := \mathcal{D}^{\text{bal}} + \epsilon(\delta[3] - \delta[2]) = \frac{1}{70} (\delta[1] + 8\delta[2] + 14\delta[3]) + \epsilon(\delta[3] - \delta[2])
\end{equation}
a $(6,3)$-intersection $(F, H)$ in $\mathbb{P}_R(1,1,2,3,3)$ is $\mathcal{D}^{\text{ter}}$-semistable only if for every choice of quasi-homogeneous coordinates on $\mathbb{P}_R(1,1,2,3,3)$, and for every weight system $\rho = (w_1, \ldots, w_5)$, we have
\[\text{mult}_\rho(F) + \frac{6}{7}\left(\text{mult}_\rho(H) - \sum_{i=1}^5 w_i\right) + \epsilon(2\text{mult}_\rho(H) - 2(w_4 + w_5) + 3w_3) \leq 0.\]

We will call $\mathcal{D}^{\text{ter}}$-semistable $(6,3)$-intersections $T$-semistable.

Consequently, every $R$-point of $\text{Int}_{\mathbb{P}_R(1,1,2,3,3)}(6,3)$ whose $K$-point defines a smooth degree 1 del Pezzo in $\mathbb{P}_K(1,1,2,3,3)$, or, equivalently, does not lie in $\cup_{i=1}^3 \text{Supp}(\delta[i])$, has a $T$-semistable model over $R$.\footnote{As the next section will illustrate, T stands for “terminality.”}

5. **T-SEMISTABLE AND STANDARD MODELS OF DEGREE 1 DEL PEZZO FIBRATIONS**

In this section, we prove Theorem 1.9 Part (2). We work over an algebraically closed field $k$ with $\text{char}(k) \neq 2,3$. Let $R$ be a DVR (and a $k$-algebra) with the residue field $k$. Let $K$ be the fraction field of $R$, $t$ be the uniformizer, and $\text{val}: K \to \mathbb{Z} \cup \infty$ the $t$-valuation on $K$. We work in $\mathbb{P}_R(1,1,2,3,3)$. Let $S = \text{Cox}(\mathbb{P}_R(1,1,2,3)) = R[x, y, z, w, s]$, where $x, y, z, w, s$ is some choice of quasi-homogeneous coordinates on $\mathbb{P}_R(1,1,2,3,3)$. For $F \in S$, we denote by $F_0$ its residue class in $k[x, y, z, w, s]$. For $f = f(x, y, z, w, s) \in K[x, y, z, w, s]$ (not necessarily homogeneous), we define $\text{val} f$ to be the minimum of the valuations of all the coefficients of $f$.

Recall the definition of a $T$-semistable $(6,3)$-intersection in $\mathbb{P}_R(1,1,2,3,3)$ from Definition 1.8 or Proposition 4.8. The goal of this section is to prove:

**Theorem 5.1 (Theorem 1.9 Part (2)).** For every $T$-semistable $(6,3)$-intersection $(F, H)$, the scheme $X = \{F = H = 0\} \subset \mathbb{P}_R(1,1,2,3,3)$ satisfies:

1. The morphism $X \to \text{Spec} R$ is flat with integral fibers.
2. Every point of $X$ is either 2 or 3-Gorenstein, and $-6K_X$ is relatively ample over $\text{Spec} R$.
3. $X$ has isolated singularities. For every $k$-point $P \in X$, the general elephant through $P$ has an isolated Du Val singularity at $P$. Consequently, $X$ has terminal singularities.

In particular, $X \to \text{Spec} R$ is a standard degree 1 del Pezzo fibration.

The plan of the proof is as follows. Part (1) is proved in Lemma 5.4, which also introduces a key dichotomy of the remainder of the proof. We prove that $X$ has isolated singularities in Lemmas 5.9, 5.15, and 5.18. Once we establish that $X$ is a $(6,3)$-complete intersection in $\mathbb{P}_R(1,1,2,3,3)$ with isolated singularities, then adjunction is applicable and gives $K_X = \mathcal{O}_{\mathbb{P}_R(1,1,2,3)}(-1)|_X$. This proves Part (2) of Theorem 5.1. Finally, we will show that the general element of $| - K_X | = |\mathcal{O}_X(1)|$ through every $k$-point of $X$ has at worst Du Val singularities. This is the most involved part of the analysis. The terminality of the total space then follows by the structure theory of threefold terminal singularities; see Proposition 5.2 below.
5.1. Threefold terminal singularities and their elephants. We collect relevant results from the classification theory of terminal threefold singularities, that are well-known over \( \mathbb{C} \), but perhaps less familiar in our setting, over an algebraically closed field \( k \) of \( \text{char}(k) \neq 2, 3 \). To state them, first recall that for a threefold \( X \), an elephant (resp., a general elephant) is an element (resp., a general element) of the anticanonical linear system \( | - K_X | \). An overarching principle, first formulated by Reid [25, p.393], is that threefold terminal singularities are precisely those with canonical (i.e., Du Val) general elephants. This is known over \( \mathbb{C} \) thanks to the explicit Kollár-Shepherd-Barron-Mori classification ([19],[15, Theorem 6.5]) of terminal threefold singularities, but we are not aware of an analogous statement\(^3\) in positive characteristic. However, for our purposes, only one direction is needed, as given in a recent paper of Kollár, and valid in arbitrary characteristic:

**Proposition 5.2** ([16, Corollary 11]). Suppose \( P \in X \) is an isolated normal threefold singularity and \( E \subset X \) an elephant containing \( P \). If \( E \) has an isolated Du Val singularity at \( P \), then \( P \in X \) is a terminal singularity.

We use the following recognition criterion for Du Val surface singularities:

**Lemma 5.3.** Suppose a hypersurface in \( \text{Spec}\, k[[x,y,z]] \) given by an equation \( f(x,y,z) = 0 \) has an isolated multiplicity 2 singularity at \( P = (0,0,0) \). Then:

\begin{enumerate}
  \item If the tangent cone of \( f \) is reduced, then \( f \) is formally equivalent to \( xy + z^{n+1} = 0 \), a type \( A_n \) singularity, for some \( n \geq 1 \).
  \item If \( f = x^2 + g(y,z) \) and the tangent cone of \( g(y,z) \) is a cubic with at least 2 different linear factors, then \( f \) is formally equivalent to \( x^2 + z(y^2 + z^{n-2}) = 0 \), a type \( D_n \) singularity, for some \( n \geq 4 \).
  \item If \( f = x^2 + y^3 + h(y,z) \), where \( \text{deg} h \geq 4 \), and \( \text{val } h(t^2 y, tz) \leq 5 \), or, equivalently, \( h(y,z) \) has a \( z^4 \), \( yz^3 \), or \( z^5 \) term, then \( P \) is a type \( E_6, E_7, \) or \( E_8 \) singularity.
\end{enumerate}

**Proof.** This follows by the standard argument as in [25, p.375] (see also [17, 3]), with the exception that when \( \text{char}(k) = 5 \), there are two distinct formal isomorphism classes of \( E_8 \) singularities, namely \( x^2 + y^3 + z^5 \) and \( x^2 + y^3 + z^5 + yz^4 \) [4]. \( \square \)

5.2. Integrality of the central fiber and the key dichotomy. We begin by proving that a \( T \)-semistable model is flat over \( \text{Spec}\, R \) and has an integral central fiber. In the process, we demonstrate the necessity of working in \( \mathbb{P}_R(1,1,2,3,3) \). Our analysis of singularities and the proof of terminality then fall into two distinct cases depending on the behavior of the central fiber. The following result describes this dichotomy:

**Lemma 5.4.** Suppose \( (F,H) \) is a \( T \)-semistable \((6,3)\)-intersection. Then \( X := \{ F = H = 0 \} \subset \mathbb{P}_R(1,1,2,3,3) \) is, relatively over \( \text{Spec}\, R \), a complete \((6,3)\)-intersection with an integral central fiber. Moreover, in some system of coordinates, one of the following holds:

\begin{enumerate}
  \item \( H = s \), its vanishing defines \( \mathbb{P}(1,1,2,3) \hookrightarrow \mathbb{P}(1,1,2,3,3) \) in every fiber, and \( X_0 \) is an integral degree 6 hypersurface in \( \mathbb{P}_k(1,1,2,3) \).
  \item \( H_0 = y^3 - xz \), its vanishing defines \( \mathbb{P}_k(1,2,9,9) \hookrightarrow \mathbb{P}_k(1,1,2,3,3) \) in the central fiber, and \( X_0 \) is an integral degree 18 hypersurface in \( \mathbb{P}_k(1,1,2,9,9) \) not containing the line \( \alpha = \beta = 0 \).
\end{enumerate}

The first case allows us to find a standard model of \( X_K \) which is a weighted degree 6 hypersurface in \( \mathbb{P}_R(1,1,2,3) \). The second case is more involved because the ambient weighted

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\(^3\)The point being that, in positive characteristic, there could be other families of terminal singularities not covered by Mori’s list.
projective space also degenerates. That the second case necessarily appears is illustrated in the proof of this lemma.

**Proof of Lemma 5.4.** We claim $H_0$ is irreducible. If not, then in some coordinates, $x | H_0$. For $\rho = (1, 0, 0, 0, 0)$, we have $\text{mult}_\rho(F) \geq 0$ and $\text{mult}_\rho(H) = 1$, making $(F, H) \rho$-unstable by (1.3).

**Remark 5.5.** This is the only place in the proof of terminality where we use the perturbed stability condition (i.e., need $\epsilon > 0$ in Definition 1.8). Note also that we use in an essential way the closedness of the residue field $k$ here.

Up to a change of coordinates, there are exactly two possibilities for an irreducible $H_0$: $H_0 = s$ and $H_0 = y^3 - xz$.

(1) $H_0 = s$. We can change coordinates so that $H = s$. Then $H = 0$ defines $\mathbb{P}(1, 1, 2, 3)$ in every fiber. The central fiber $X_0$ is non-integral if and only if $F_0(x, y, z, w, 0)$ factors non-trivially in $k[x, y, z, w]_6$. Let $A$ be its smallest positive degree factor.

(i) If $\deg A = 1$, then, up to a change of coordinates, $A = y$ and $(F, H)$ is destabilized by $\rho = (0, 1, 0, 0, 0): \text{mult}_\rho(F) = 1$ and $\text{mult}_\rho(H) \geq 0$.

(ii) If $\deg A = 2$, then, up to a change of coordinates, $A = z$ and $(F, H)$ is destabilized by $\rho = (0, 0, 1, 0, 0)$.

(iii) Suppose now $\deg A = 3$. If $A = w$ in some coordinates, then $(F, H)$ is destabilized by $\rho = (0, 0, 0, 1, 0)$. Otherwise, $A$ does not depend on $w$ and so we may write

$$F(x, y, z, w, 0) = A(x, y, z)B(x, y, z) + tG(x, y, z, w).$$

Make a change of coordinates $s' = s - A(x, y, z)$. Then in the new coordinates $(F, H)$ becomes

$$(F, H) = (s'B(x, y, z) + tG(x, y, z, w), s' + A(x, y, z))$$

For $\rho = (0, 0, 0, 0, 1)$, we now have $\text{mult}_\rho(F) = 1$ and $\text{mult}_\rho(H) = 0$, making $(F, H)$ $\rho$-unstable by (1.3).

(2) $H_0 = y^3 - xz$. This cubic is precisely the image of a closed immersion

$$\mathbb{P}_k(1, 2, \beta, 9, s) \hookrightarrow \mathbb{P}_k(1, 1, 2, 3) \quad \text{via} \quad x = \alpha^3, \quad y = \alpha\beta, \quad z = \beta^3.$$  

Semistability with respect to $\rho = (1, 1, 1, 1, 1)$ implies that $F_0 \notin (x, y, z)$, and so the central fiber $X_0$ is a degree 18 hypersurface $F_0(\alpha^3, \alpha\beta, \beta^3, w, s) = 0$ in $\mathbb{P}_k(1, 2, \beta, 9, s)$ with a non-zero quadratic term in $w, s$. If $F_0(\alpha^3, \alpha\beta, \beta^3, w, s)$ factors non-trivially in $k[\alpha, \beta, w, s]$, then necessarily it does into two degree 9 factors with a non-zero linear term in $w, s$ each. After a change of variables, we can assume that $s | F$. Then $(F, H)$ is destabilized by $\rho = (0, 0, 0, 0, 1)$.

□

**Remark 5.6.** The case (1.iii) in the proof of this lemma is the only place in the proof of Theorem 5.1 where starting with a family whose central fiber lives in $\mathbb{P}(1, 1, 2, 3)$, we must use a one-parameter subgroup which is not a $K$-point of $\text{Cox}(\mathbb{P}(1, 1, 2, 3))$ but a $K$-point of $\text{Cox}(\mathbb{P}(1, 1, 2, 3, 3))$. In other words, the reducibility of the central fiber in the family

$$t(w^2 + G(x, y, z)) + A(x, y, z)B(x, y, z) = 0,$$

is the sole reason we consider models in $\mathbb{P}(1, 1, 2, 3, 3)$ instead of $\mathbb{P}(1, 1, 2, 3)$.

This concludes the proof of Part (1) of Theorem 5.1, which shows that $X$ is a threefold fibered in integral surfaces over $\text{Spec} R$. Since the generic fiber of $X$ smooth, we conclude that $X$ is nonsingular in codimension 1. By adjunction, $-K_X = \mathcal{O}_{\mathbb{P}(1, 1, 2, 3, 3)}(1)|_X$ and so $-6K_X$ is a relatively ample line bundle. This establishes Part (2) of Theorem 5.1.
The proof of Part (3) now proceeds according to one of the two possibilities given in Lemma 5.4, and is done in §5.3, 5.4, respectively. In each case, we first show that \( X \) has isolated singularities, and then for every singularity \( P \in X \), find an elephant (= an anticanonical divisor) \( E \) through \( P \) such that \( P \in E \) is Du Val.

Since \(-K_X = \mathcal{O}_{\mathbb{P}(1,1,2,3,3)}(1)|_X\), every hyperplane \( ax + by \), with either \( \text{val} a = 0 \) or \( \text{val} b = 0 \), restricts to an elephant on \( X \) fibered in elliptic curves over the base. We will need the following Bertini-like result:

**Lemma 5.7.** For every choice of coordinates, let \( E \) be the elephant given by the equation \( y = cx \) (resp., \( y = ctx \)), where \( c \in k \) is general. Then \( E_K \) is smooth over \( K \).

**Proof.** After a change of variables affecting only \( z, w, s \), the generic fiber \( X_K \) can be written as a sextic in \( \mathbb{P}(1,1,2,3,3) \) in the Weierstrass form:

\[
w^2 + z^3 + p(x,y)z + q(x,y) = 0,
\]

with \( p(x,y) \in K[x,y]_4 \) and \( q(x,y) \in K[x,y]_6 \). The discriminant \( D(x,y) := 4p^3 + 27q^2 \) is a non-zero binary sextic by the smoothness of \( X_K \). The discriminant of the elliptic curve \( E_K \) is \( D(1,c) \) (resp., \( D(1,ct) \)), which will be nonzero for a general \( c \in k \). The claim follows. \( \square \)

### 5.3. Central fiber in \( \mathbb{P}(1,1,2,3) \)

Suppose we are in case (1) of Lemma 5.4. We can write \((F,H) = (F(x,y,z,w),s)\) and treat \( F \) as a sextic in \( R[x,y,z,w] \). In particular, all of our coordinate changes in this subsection will fix \( s \) and come from \( \text{Aut}_{gr}(R[x,y,z,w]) \). The threefold \( X \) is then a sextic \( F = 0 \) in \( \mathbb{P}(1,1,2,3,3) \).

The T-semistability of a complete intersection \((F(x,y,z,w),s)\) can then be interpreted in terms of \( F \) alone. Namely, if \((F,H)\) is T-semistable, then \( F(x,y,z,w) \in R[x,y,z,w]_6 \) satisfies:

1. For every weight system \( \rho = (a,b,c,d) \), we have that
   \[
   \text{mult}_{\rho}(F) \leq \frac{6}{t}(a + b + c + d).
   \]
2. \( F_0(x,y,z,w) \) is irreducible in \( k[x,y,z,w]_6 \).

A sextic \( F(x,y,z,w) \) satisfying these conditions will be called T-semistable. A sextic \( F \) satisfying (resp., violating) (5.2) will be called \( \rho \)-semistable (resp., \( \rho \)-unstable).

We begin with a preparatory result:

**Lemma 5.8.** Suppose that, in some coordinates, a T-semistable sextic has equation

\[
(5.3) \quad o(t)w^2 + (p_1(x,y,t)z + p_3(x,y,t))w + p_0(t)z^3 + p_2(x,y,t)z^2 + p_4(x,y,t)z + p_6(x,y,t).
\]

Then it must satisfy one of the following (up to scaling by a unit in \( R \)):

1. \( o(t) = 1 \).
2. \( o(t) = t \), and either
   1. \( p_1(x,y,0) \neq 0 \), or
   4. \( \text{val}(p_0(t)) \leq 1 \), or
   3. \( p_2(x,y,0) \neq 0 \).
3. \( o(t) \geq 2 \), and \( p_1(x,y,0) \neq 0 \).
4. \( o(t) = 2 \), \( p_1(x,y,0) = 0 \), \( p_3(x,y,0) \neq 0 \), and \( p_0(t) = 1 \).

and must have either \( p_3(x,y,0) \neq 0 \), or \( p_4(x,y,0) \neq 0 \), or \( t^2 \mid p_6(x,y,t) \).

In particular, either \( a_3(x,y,z,0) \neq 0 \) or \( \text{val}(a_0(t)) \leq 1 \).

**Proof.** If \( \text{val}(o(t)) \geq 1 \), then stability with respect to \( \rho = (1,1,1,2) \) ensures that either \( p_1(x,y,0) \neq 0 \), or \( \text{val}(p_0(t)) \leq 1 \), or \( p_2(x,y,0) \neq 0 \).
Assume that \( \text{val}(o(t)) \geq 2 \) and \( p_1(x, y, 0) = 0 \). Taking \( \rho = (1, 1, 2, 2) \), we see that \( p_3(x, y, 0) \neq 0 \). Taking \( \rho = (1, 1, 1, 1) \), we see that \( p_0(0) \neq 0 \). Taking \( \rho = (1, 1, 1, 0) \) shows that \( \text{val}(o(t)) = 2 \).

Finally, semistability with respect to \( \rho = (0, 0, 1, 1) \) ensures that either \( p_3(x, y, 0) \neq 0 \), or \( p_4(x, y, 0) \neq 0 \), or \( t^2 \mid p_6(x, y, t) \).

Lemma 5.9. Suppose \( F \) is \( T \)-semistable, then \( X \) has only isolated singularities.

Proof. Suppose \( X \), and hence \( X_0 \), is singular along a curve. We will show that in some coordinates (a component of) this curve is given by \( L_1 = L_2 = 0 \), where \( \{L_1, L_2\} \subset \{x, y, z, w\} \) (Cases I and II below). Since \( X \) is singular along the locus \( L_1 = L_2 = t = 0 \), we must have \( F \in (L_1, L_2, t)^2 \). Then \( F \) is destabilized by the one-parameter subgroup \( \rho \) acting with weight 1 on \( L_i \)'s and weight 0 on the remaining coordinates, leading to a contradiction.

Case I: \( o(t) = 1 \). After a change of variables, we can take \( a_3(x, y, z, t) = 0 \). Then \( a_6(x, y, z, 0) \) is non-reduced. It is not a square since \( F_0 \) is integral, so after a further change of coordinates we have \( L_1^2 \mid a_6(x, y, z, 0) \), where either \( L_1 = x \) or \( L_1 = z \). Then \( L_1 = w = t = 0 \) is in the singular locus of \( X \).

Case II: \( a_3(x, y, z, 0) \neq 0 \) and \( \text{val}(t) \geq 1 \). We have \( F_0 = a_3(x, y, z, 0)w + a_6(x, y, z, 0) \).

Since \( a_3 \) and \( a_6 \) are coprime by integrality, we see that the singular locus of \( X_0 \) must consist of the fibers of the projection map over the 0-dimensional locus \( a_3(x, y, z, 0) = a_6(x, y, z, 0) = 0 \) in \( \mathbb{P}_k(1, 1, 2) \). Up to a change of coordinates, we may take either \( x = y = t = 0 \) or \( y = z = t = 0 \) to be in the singular locus of \( X_0 \).

Case III: \( a_3(x, y, z, 0) = 0 \) and \( o(t) = t \). For \( F = tw^2 + a_6(x, y, z, t) \) to have non-isolated singularities along \( t = 0 \), we must have \( a_6(x, y, z, 0) \) to have non-isolated singularities as a sextic in \( \mathbb{P}_k(1, 1, 2) \). Then \( F_0 = a_6(x, y, z, 0) \) is non-reduced, contradicting the \( T \)-semistability.

We proceed to prove that the isolated singularities of \( X \) are terminal by proving that the general elephant through every singular point has Du Val singularities. Let \( P \in X \) be a singular point. Then up to a change of coordinates, we have:

1. \( P = [1 : 0 : 0 : 0] \), a Gorenstein point.
2. \( P = [0 : 0 : 1 : 0] \), a 2-Gorenstein point.
3. \( P = [0 : 0 : 1 : 1] \), a Gorenstein point.
4. \( P = [0 : 0 : 0 : 1] \), a 3-Gorenstein point.

We begin with:

Lemma 5.10. For every choice of coordinates, let \( E \) be the elephant given by the equation \( y = cx \) (resp., \( y = ctz \)), where \( c \in k \) is general. Then \( E \) has isolated singularities (resp., isolated singularities away from \( x = y = t = 0 \)).

Proof. The generic fiber of \( E \) is smooth by Lemma 5.7. It remains to show that \( E \), given by the sextic equation \( F(x, cx, z, w) \) (resp., \( F(x, ctz, z, w) \)) in \( \mathbb{P}_R(1_x, 2_z, 3_w) \) has isolated singularities in the central fiber (respectively, isolated singularities away from \( x = t = 0 \)). Suppose not, then every component of the one-dimensional singular locus of \( E \) is a non-reduced component of \( E_0 \), and, by the degree considerations, has equations \( L = t = 0 \), where \( L \in \{x, z, w\} \).

Consider first the case of \( y = cx \). Then \( F(x, cx, z, w) \in (L, t)^2 \). If \( L = x \), then \( F(x, y, z, w) \in (x, y, t)^2 \), and is unstable with respect to \( \rho = (1, 1, 0, 0) \) by (5.2). If \( L \in \{z, w\} \), then \( F(x, y, z, w) \in (L, t)^2 \) and is unstable with respect to \( \rho = (0, 0, 1, 0) \) or \( \rho = (0, 0, 0, 1) \).

Consider now \( y = ctz \). Then \( F(x, ctz, z, w) \in (L, t)^2 \), where \( L \in \{z, w\} \) because we consider singularities away from \( x = y = t = 0 \) in this case. Then \( F \in (y, L, t)^2 \) and so is unstable with respect to \( \rho = (0, 1, 1, 0) \) or \( \rho = (0, 1, 0, 1) \) by (5.2). \( \square \)
Proposition 5.11. Let \( P = [1 : 0 : 0 : 0] \in X \) be a singular point. Let \( E \) be the elephant through \( P \) given by the equation \( y = cx \), where \( c \in k \) is general. Then \( P \in E \) is Du Val. Consequently, \( P \in X \) is cDV and hence terminal.

Proof. In the affine chart \( x = 1 \), and local coordinates \( y, z, w, t \), the equation of \( X \) is \( f(y, z, w, t) := F(1, y, z, w) \) and the equation of \( E \) is \( e(z, w, t) := f(ct, z, w, t) \).

By Lemma 5.10, \( P \) is an isolated singularity of \( E \). Since \( F \) is semistable for \( \rho = (0, 1, 1, 1) \), we have

\[
\text{val} e(tz, tw, t) = \text{val} f(ct, tz, tw, t) = \text{val} f(ty, tz, tw, t) = \text{mult}_\rho(F) \leq 2.
\]

Thus \( P \) is a double point of \( E \), and we can use Lemma 5.3 to prove that \( P \in E \) is Du Val. Let \( o(t) \) be the coefficient of \( w^2 \) in \( F \).

Case I: \( o(t) = 1 \). After a change of coordinates, we can write \( f = w^2 + a(y, z, t) \). If \( a(y, z, t) \) has a quadratic term, then so does \( a(ct, z, t) \) and so \( P \in E \) is an \( A \)-singularity by Lemma 5.3(a).

Hence we can assume that

\[
a = p_0(y, t)z^3 + p_1(y, t)z^2 + p_2(y, t)z + p_3(y, t),
\]

where \( \text{val} p_i(ct, t) \geq i \). By \( \rho = (0, 1, 1, 2) \)-semistability, we must have \( \text{val} f(ty, tz, t^2w, t) = \text{mult}_\rho(F) \leq 3 \). It follows that \( \text{val} p_i(ct, t) = i \) for at least one \( i \). Consider the following three subcases:

i) If \( p_0 = 1 \), we can change coordinate \( z \) to arrive at \( f = w^2 + z^3 + p_2(y, t)z + p_3(y, t) \). If \( p_2(ct, t)z + p_3(ct, t) \) has a cubic term, then \( P \in E \) is a \( D \) singularity by Lemma 5.3(b).

Suppose not. Then stability with respect to \( \rho = (0, 1, 2, 3) \) ensures that either \( \text{val} p_3(ct, t) \leq 5 \) or \( \text{val} p_2(ct, t) \leq 3 \), giving a type \( E \) singularity by Lemma 5.3(c).

ii) If \( t \mid p_0 \), and either \( \text{val} p_1(ct, t) = 1 \) or \( \text{val} p_2(ct, t) = 2 \), then we have a type \( D \) singularity by Lemma 5.3(b).

iii) If \( t \mid p_0 \), and \( \text{val} p_1(ct, t) \geq 2 \) and \( \text{val} p_2(ct, t) \geq 3 \), then necessarily \( \text{val} p_3(ct, t) = 3 \). In this case, semistability of \( F \) with respect to \( \rho = (0, 1, 0, 1) \) ensures that \( p_0 = t \). It follows that

\[
e = w^2 + t^3 + tz^3 + z^2p_1(ct, t) + zp_2(ct, t),
\]

an \( E_7 \) singularity by Lemma 5.3(c).

Case II: \( \text{val} o(t) \geq 1 \). If the tangent cone of \( f \) is a reduced quadric, then so is the tangent cone of \( e \) and we have an \( A \) singularity. Assume from now on that the tangent cone of \( f \) is a non-reduced quadric. In particular, it cannot depend on \( w \) and so lies in \( (y, z, t)^2 \). In particular \( xzw, x^2yw \notin F \). Semistability of \( F \) with respect to \( \rho = (0, 1, 1, 0) \), ensures \( tw^2 \in F \). Semistability of \( F \) with respect to \( \rho = (0, 1, 0, 0) \), ensures that either \( z^2x^2 \in F \) or \( z^3 \in F \). If \( z^2x^2 \in F \), then \( e(z, w, t) = z^2 + tw^2 + \cdots \) is a \( D \) singularity. If \( z^3x^2 \notin F \) and \( z^3 \in F \), then the tangent cone of \( e \) must be \( t^2 \), and so \( e(z, w, t) = t^2 + tw^2 + z^3 + \cdots \) is an \( E_6 \) singularity. \( \square \)

5.3.1. Singularities along \( x = y = 0 \). We now analyze the singularities of \( X \) along the line \( x = y = 0 \) passing through both cyclic quotient singularities of \( \mathbb{P}(1, 1, 2, 3) \). We consider an elephant \( E \) given by \( y = cx \), where \( c \in k \) is general. By Lemma 5.10, the singularities of \( E \) are isolated. We prove that they are Du Val along \( x = y = 0 \), thus establishing the terminality of \( X \) along this line.

Lemma 5.12. Suppose \( P = [0 : 0 : 0 : 1] \in X \) is a 3-Gorenstein singularity. Let \( E \) be the elephant through \( P \) given by \( y = cx \), where \( c \in k \) is general. Then \( P \in E \) is Du Val. Hence \( P \in X \) is a terminal singularity.

---

4We write \( m \in F \) (resp., \( m \notin F \)) to indicate that a monomial \( m \) has a nonzero (resp., zero) coefficient in \( F \).
Proof. In the coordinates $\alpha := x^3, \beta := z^3, \gamma := xz,$ and $t$ on the affine chart $w = 1,$ the equations of $E$ are
\begin{align}
\alpha \beta &= \gamma^3, \\
0 &= o(t) + p_1(t)\gamma + p_2(t)\alpha + q_1(t)\beta + q_2(t)\gamma^2 + q_3(t)\alpha \gamma + q_4(t)\alpha^2.
\end{align}

We proceed to consider the possibilities enumerated in Lemma 5.8, noting that only cases (B–D) are possible.

Case (B): $o(t) = t.$ Then (5.4) is a surface $A_2$-singularity. (It is easy to see that $P \in X$ is a $\frac{1}{3}(1, 1, 2)$-point.)

Case (C): $val_t(o(t)) \geq 2$ and $p_1(t) = 1.$ Then (5.4) is a surface $A_n$-singularity.

Case (D): $val_t(o(t)) = 2$ and $p_2(t) = q_1(t) = 1.$ Plugging in $\beta = -t^2 - \alpha - p_1(t)\gamma - q_2(t)\gamma^2 - q_3(t)\alpha\gamma - q_4(t)\beta^2$ into $\alpha\beta = \gamma^3,$ we get a singularity
\[\alpha t^2 + \alpha^2 + \gamma^3 + \cdots = 0,\]
which is formally equivalent to $\alpha^2 + \gamma^3 + t^4 = 0,$ an $E_6$-singularity. (One can see that $P \in X$ is $\frac{1}{3}cD_4$ singularity according to the Mori classification scheme [19],[15, Theorem 6.5].)

\[\Box\]

\textbf{Lemma 5.13.} Suppose $P = [0 : 0 : 1 : 0] \in X$ is a 2-Gorenstein singularity. Let $E$ be the elephant given by $y = cx,$ where $c \in k$ is general. Then $P \in E$ is Du Val. Hence $P \in X$ is a terminal singularity.

\textbf{Proof.} Keep the notation of Lemma 5.8, and set $p_i(t) = p_i(c, cx, t)/x^i.$ In the coordinates $\alpha := x^2, \beta := w^2, \gamma := xw, \text{ and } t,$ in the affine chart $z = 1,$ the equations of $E$ are
\begin{align}
\alpha \beta &= \gamma^2, \\
0 &= o(t)\beta + p_1(t)\gamma + p_3(t)\alpha \gamma + p_0(t) + p_2(t)\alpha + p_4(t)\alpha^2 + p_6(t)\alpha^3 = 0.
\end{align}

Only cases (A–C) of Lemma 5.8 are possible. We can also assume that $p_0(t) = t^k,$ where $k \geq 2,$ since $val_t(p_0(t)) = 1$ implies that $P \in E$ is an $A_1$-singularity.

Case (A): $o(t) = 1.$ Eliminating $\beta,$ we get a surface singularity:
\[\gamma^2 + t^k\alpha + p_2(t)\alpha^2 + p_4(t)\alpha^3 + p_6(t)\alpha^4 = 0.\]

If $val_t(p_2(t)) = 0,$ then we have an $A$-singularity. If $k = 2$ or $val_t(p_2(t)) = 1,$ then we have a $D$-singularity. Suppose $k \geq 3$ and $val_t(p_2(t)) \geq 2.$ Semistability of $F$ with respect to $\rho = (1, 1, 0, 2),$ implies that $val_t(p_4(t)) = 0.$ Now if $k = 3,$ we get an $E_7$-singularity. If $k \geq 4,$ we destabilize $F$ using $\rho = (1, 1, 1, 3).

Case (B): $o(t) = t.$ It $p_1(t) = 1,$ eliminating $\gamma,$ we get a surface singularity:
\[\alpha \beta = (t\beta + t^k + p_3(t)\alpha + p_4(t)\alpha^2 + p_6(t)\alpha^3)^2 + \cdots,\]
which is of type $A.$ If $p_1(t) = 0,$ then $p_3(t) = 1,$ and, eliminating $\alpha,$ we get
\[(t\beta + t^k + \cdots)\beta = \gamma^2\]
which is a $D$-singularity.

Case (C): $p_1(t) = 1.$ This is identical to the case $p_1(t) = 1$ in Case (B) above.

\[\Box\]

\textbf{Lemma 5.14.} Let $P = [0, 0, 1, 1] \in X$ be a Gorenstein singularity. Then $P \in E$ is Du Val. Hence $P \in X$ is a terminal singularity.
Proof. In the coordinates $\alpha := x^3, \beta := z^3 - 1, \gamma := xz$ on the affine chart $w = 1$, the equations of $E$ are
\begin{align}
\gamma^3 &= \alpha(1 + \beta), \\
0 &= (o(t) + q_1(t)) + p_1(t)\gamma + p_2(t)\alpha + q_1(t)\beta + q_2(t)\gamma^2 + q_3(t)\alpha\gamma + q_4(t)\alpha^2.
\end{align}
(5.6)

For $P \in E$ to be singular, we must have $val_t(p_1(t)), val_t(q_1(t)) \geq 1$ and $val_t(o(t) + q_1(t)) \geq 2$. However, if $val_t(q_1(t)) \geq 2$ and $val_t(o(t)) \geq 2$, then $F$ is unstable for $\rho = (1,1,0,0)$. We conclude that $val_t(q_1(t)) = 1$ and $o(t) = -q_1(t)$. Then $E$ is a hypersurface singularity in $\text{Spec} \ k[\beta, \gamma, t]$ with the tangent cone $t\beta = 0$, hence of Type A.

This concludes the proof of Theorem 5.1 in Case (1) of Lemma 5.4.

5.4. Central fiber in $\mathbb{P}_k(1,2,9,9)$. Suppose we are in Case (2) of Lemma 5.4, so that $H_0 = y^3 - xz$. Then in some coordinates we have:
\begin{align*}
H &= y^3 - xz + t^n L(w, s), \ n \geq 1, \ L(w, s) \in R[w, s],
F &= q(w, s) + wb_3(x, y, z) + sa_3(x, y, z) + f_6(x, y, z) + tP(x, y, z, w, s), \ q(w, s) \neq 0.
\end{align*}
Here, and throughout the rest of this subsection, we use lowercase letters to denote elements of $k[x, y, z, w, s]$ and uppercase letters to denote elements of $R[x, y, z, w, s]$.

We first treat the easier case of $rk q(w, s) = 2$ in the next lemma, and then consider the case $rk q(w, s) = 1$ in §5.4.1.

Lemma 5.15. If $rk q(w, s) = 2$, then $X$ has isolated singularities, and for every singular point $P \in X$, the general elephant through $P$ has Du Val Type A singularity at $P$. Thus $X$ is terminal.

Proof. We can change coordinates so that
\begin{align}
H &= y^3 - xz + t^n L(w, s) \\
F &= ws + f_6(x, y, z) + tP(x, y, z).
\end{align}
(5.7)

By Lemma 5.4, the central fiber $X_0$ is an irreducible degree 18 hypersurface $\text{ws} + f_6(\alpha^3, \alpha\beta, \beta^3) = 0$ in $\mathbb{P}_k(1,3,9w,9s)$. Evidently, it has isolated singularities. Hence $X$ has isolated singularities.

We proceed to show that the general elephant through a singularity $P \in X$ is Du Val.

Up to a change of coordinates, preserving the form (5.7), we have
\begin{align*}
P &\in \{[1:0:0:0:0],[0:0:1:0:0],[0:0:1:0:1],[0:0:0:0:1]\}.
\end{align*}
In each case we take the elephant $E$ given by $y = ct x$, or $y = cx$, with $c \in k$ general. The generic fiber of such $E$ is smooth by Lemma 5.7 and has reduced central fiber $E_0$ because already the central fiber of the elephant $y = 0$ is reduced.

We provide details only for two cases, with the rest being analogous (and easier than corresponding cases of $rk q(w, s) = 1$).

(1) $P = [1:0:0:0:0]$. We take $E : y = ct x$. Then in the affine chart $x = 1$, we have $z = c^3 t^3 + t^n L(w, s)$ and $E$ has an isolated singularity
\begin{align*}
\text{ws} + f_6(1,ct,c^3t^3 + t^n L(w, s)) + tP(1,ct,c^3t^3 + t^n L(w, s)) = 0,
\end{align*}
with a reduced quadric tangent cone $\text{ws} = 0$, hence of Type A.
(2) $P = [0 : 0 : 0 : 0 : 1]$. We take $E : y = cx$. In the affine chart $s = 1$, and in local coordinates $\alpha = x^3$, $\beta = z^3$, $\gamma = xz$, and $w, t$, the equations of $E$ are
\[
\begin{align*}
\alpha\beta &= \gamma^3, \\
\gamma &= t^6 L(w, 1) + c^3 \alpha, \\
w &= G(\alpha, \beta, \gamma, t).
\end{align*}
\]

This is again a Type A singularity.

$\square$

5.4.1. The case $\text{rk} q(w, s) = 1$.

Lemma 5.16. Suppose $(F, H)$ is $T$-semistable with $H_0 = xz - y^3$ and $F_0(0, 0, 0, w, s)$ a rank 1 quadric. Then after a change of coordinates we have:
\[
\begin{align*}
F &= w^2 + s(ta_3(x, y, z)) + f_6(x, y, z) + tB_6(x, y, z), \\
H &= y^3 - xz + t^6 L(w, s), \quad n \geq 1,
\end{align*}
\]
where
\[
\begin{align*}
(1) & \text{ Either } a_3 \notin (y^3 - xz) \text{ or } f_6 \notin (y^3 - xz). \\
(2) & \text{ Either } ts + a_3(x, y, z) \notin (H). \\
(3) & \text{ Either } ts + a_3(x, y, z) \notin (x, y)^3 + (H), \text{ or } f_6 \text{ has } z^3 \text{ term.} \\
(4) & \text{ Either } ts + a_3(x, y, z) \notin (y, z)^2 + (H), \text{ or } f_6 \notin (y, z)^2, \text{ or } B_6 \text{ has } x^6 \text{ term.}
\end{align*}
\]

Remark 5.17. If $(t^6 L(w, s)) \neq (ts)$, Conditions (2–4) are vacuous. In this case, the standard form (5.8) uniquely determines $F$. If $t^6 L(w, s) = ts$, $F$ is determined only up to a multiple of $H = y^3 - xz + ts$, which explains the phrasing of Conditions (2–4).

Proof. Write the equations of $X$ in the form
\[
\begin{align*}
F &= w^2 + sa_3(x, y, z) + f_6(x, y, z) + tB_6(x, y, z, s) = 0, \\
H &= y^3 - xz + tA_3(x, y, z, w, s) = 0.
\end{align*}
\]

Semistability with respect to $\rho = (1, 1, 1, 1, 0)$ implies that either mult$_\rho(F) \leq 1$ or mult$_\rho(H) \leq 1$. The only possible terms of $\rho$-weight 1 are $ts$ in $H$ or $ts^2$ in $F$. In either case, taking $F = F + s\lambda H$, for an appropriate $\lambda \in k$, ensures that $F$ has $ts^2$ term. We can then eliminate $s$ from $B_6$ to arrive at $B_6 \in \mathbb{R}[x, y, z]$. By changing $x, y, z$, we can eliminate $x, y, z$ from $A_3$, so that $tA_3(x, y, z, w, s) = t^6 L(w, s)$ for $L(w, s) \in \mathbb{R}[w, s]$.

Condition (1) follows from the integrality of $X_0$. Condition (2) follows from semistability with respect to $(1, 1, 2, 3, 2)$, (3) follows from semistability with respect to $\rho = (1, 1, 1, 2, 1)$, and (4) follows from semistability with respect to $\rho = (0, 1, 1, 1, 0)$.

$\square$

Lemma 5.18. Suppose $(F, H)$ is $T$-semistable, then $X$ has only isolated singularities.

Proof. Suppose $X$ is singular along a curve $C \subset X_0$. By the Jacobian criterion, we have
\[
C \subset (w = y^3 - xz = a_3(x, y, z) = f_6(x, y, z) = t = 0),
\]
a cone over a finite set of points in $\mathbb{P}(1_x, 1_y, 2_z)$. After a change of coordinates $x, y, z$ preserving $y^3 - xz$, we can assume $C$ is either a line $t = w = z = y = 0$ or a line $t = w = x = y = 0$.

Suppose $X$ is singular along $t = w = z = y = 0$. Then we can change $F$ by a multiple of $H$ to ensure $F \in (t, w, z, y)^2$. For $\rho = (0, 1, 1, 1, 0)$, we have mult$_\rho(H) = 1$ and mult$_\rho(F) = 2$, making $(F, H)$ $\rho$-unstable.

Suppose $X$ is singular along $t = w = x = y = 0$. Then we can change $F$ by a multiple of $H$ to ensure $F \in (t, w, x, y)^2$. For $\rho = (1, 1, 0, 1, 0)$, we have mult$_\rho(H) = 1$ and mult$_\rho(F) = 2$, making $(F, H)$ $\rho$-unstable. $\square$
We proceed to prove that the isolated singularities of $X$ are terminal. Let $P \in X$ be a singular point. Up to a change of variables preserving the standard form (5.8)

1. $P = [1 : 0 : 0 : 0 : 0]$, a Gorenstein point.
2. $P = [0 : 0 : 1 : 0 : 0]$, a 2-Gorenstein point.
3. $P = [0 : 0 : 1 : 0 : 1]$, a Gorenstein point.
4. $P = [0 : 0 : 0 : 0 : 1]$, a 3-Gorenstein point.

Lemma 5.19. For every choice of coordinates preserving the standard form (5.8), let $E$ be the elephant given by the equation $y = cx$ (resp., $y = ct x$), where $c \in k$ is general. Then $E$ has isolated singularities (resp., isolated singularities away from $x = y = t = 0$).

Proof. The generic fiber of $E$ is smooth by Lemma 5.7. It remains to show that $E$ has isolated singularities along the central fiber (respectively, isolated singularities away from $x = y = t = 0$). Suppose not, and let $C$ be a one-dimensional component of the singular locus of $E$ in $t = 0$. By the Jacobian criterion, we have $C \subset (w = c^3 x^3 - xz = a_3(x, cx, z) = f_6(x, cx, z) = t = 0)$, or $C \subset (w = c^3 x^3 - xz = a_3(x, c t x, z) = f_6(x, c t x, z) = t = 0)$.

Since $y^3 - xz = a_3(x, y, z) = f_6(x, y, z) = 0$ is a finite set in $\mathbb{P}(1_x, 1_y, 2_z)$, a general section $y = cx$ (resp., $y = ct x$) will avoid it unless this finite set contains a point with $y = 0$. The possibilities are then $[x : y : z] = [0 : 1 : 0]$ if $y = cx$ (resp., $[x : y : z] = [0 : 0 : 1], [1 : 0 : 0]$ if $y = ct x$).

Consider first the case of $E : y = cx$. Its singularities lie along the line $x = y = w = 0$. If $f_6$ has $z^3$ term, then $w = x = y = 0$ implies $z = 0$, hence an isolated singularity. If $f_6$ has no $z^3$ term, then $s(st + a_3(x, cx, z)) + \lambda sH$ has either $st$ or $sxz$ term for any $\lambda \in k$ (Condition 3). In this case, $(\partial F/\partial x) (\partial H/\partial t) - (\partial F/\partial t) (\partial H/\partial x)$ is generically non-zero along $w = x = y = t = 0$, and so the claim follows.

Consider now the case of $E : y = ct x$. We need to show that its singularities are isolated along the line $w = y = z = 0$. This follows from Condition 4.

Lemma 5.20. Let $P = [0 : 0 : 0 : 0 : 1] \in X$ be a 3-Gorenstein singularity. Then the generic elephant $E$ through $P$ has Du Val singularities.

Proof. Take $E$ to be given by $y = cx$ where $c \in k$ is general. In the affine chart $s = 1$, we work in the local coordinates $\alpha := x^3, \beta := x(z - c^3 x^2), \gamma := (z - c^3 x^2)^3$, and $w, t$. The equations of $E$ in Spec $k[\alpha, \beta, \gamma, w, t]$ are:

\[
\begin{align*}
\alpha \gamma &= \beta^3, \\
\beta &= t^n L(w, 1), \\
0 &= w^2 + (t + a(\alpha, \beta)) + f(\alpha, \beta, \gamma) + tB(\alpha, \beta, \gamma).
\end{align*}
\]

If $t + a(\alpha, t^n L(w, 1))$ has nonzero $t$ term, we proceed working formally locally and eliminate $\beta$ and $t$ to arrive at a hypersurface

\[
\alpha \gamma = (w^{2n} L(w, 1) + \cdots)^3
\]

in $k[[\alpha, \gamma, w]]$. Hence $E$ has Type A singularity at $P$.

Otherwise, we necessarily have $t^n L(w, 1) = t$ (up to a nonzero scalar) by Condition 3, $t + a(\alpha, t^n L(w, 1))$ is a non-zero multiple of $\alpha$, and $f$ must have a $\gamma$ term. Eliminating $t$ and $\alpha$, we arrive at

\[
(w^2 + \gamma + (\text{higher degree terms})) \gamma = \beta^3,
\]

an $E_6$ singularity. \qed
**Lemma 5.21.** Let $P = [0 : 0 : 1 : 0 : 0] \in X$ be a 2-Gorenstein singularity. Then $P \in E$ is Du Val for a general elephant $E$.

**Proof.** We take $E$ to be given by $y = cx$, where $c \in k$ is general, and change coordinate $z = z + c^3 x^2$. Then the equation of $E$ are

$$
y = cx,

xz = t^6 L(w, s),

0 = w^2 + s(ts + a_3(x, y, z)) + f_6(x, y, z) + tB_6(x, y, z, s).
$$

Since $f_6$ has no $z^3$ term, we must have $ts + a_3(x, y, z) \notin k[x, y]$ (mod $xz - t^6 L(w, s)$).

We work in the affine chart $z = 1$, where the local variables will be $\alpha = w^2$, $\beta = s^2$, $\gamma = ws$, and $t$. Eliminating $x$ using $x = t^6 L(w, s)$, we arrive at

$$
\gamma^2 = \alpha \beta,

0 = \alpha + (c't^3 + c''t^2 + (h.o.t.)) + f(\alpha, \beta, \gamma) + tB(\alpha, \beta, \gamma),
$$

where $c' \neq 0$.

If $B_6$ has a $z^3$ term (resp., $B$ has a constant term), then we get $A_1$-singularity. Otherwise, $f(\alpha, \beta, \gamma) + tB(\alpha, \beta, \gamma)$ is at least quadratic. Eliminating $\alpha$, we arrive at

$$
\gamma^2 - (c't^3 - c''t^2) + (\text{higher order terms}) = 0
$$
in Spec $k[[\beta, \gamma, t]]$. Since $c' \neq 0$, this is a Type D singularity. □

**Lemma 5.22.** Suppose $P = [1, 0, 0, 0, 0] \in X$ is a singular point. A general elephant $E$ of the form $y = ct x$, $c \in k$, has Type $A$ or $D$ singularities at $P$.

**Proof.** We change variables $z = z + c^3 t^3 x^2$, and work in the affine chart $x = 1$, so that $z = t^6 L(w, s)$. The equation of $E$ in Spec $k[[s, w, t]]$ then becomes

$$
w^2 + s(ts + a_3(1, ct, t^6 L(w, s)) + f_6(1, ct, t^6 L(w, s)) + tB(1, ct, t^6 L(w, s)) = 0.
$$

where either $ts + a_3(1, ct, t^6 L(w, s))$ has $ts$ term, or $t^6 L(w, s) = ts$ and $f_6$ has $x^4 z = ts$ term (Condition 4). Hence $P$ is either Type D or Type A singularity. □

**Lemma 5.23.** Suppose $P = [0, 0, 1, 0, 1]$ is a singular point of $X$. Then $P \in E$ is an $A_1$-singularity.

**Proof.** Let $q(t)$ be the coefficient of $F$ in Lemma 5.16. Note that $val_1(q(t)) \geq 1$. In the affine chart $s = 1$, in local coordinates $\alpha := x^3, \beta := z^3 - 1, \gamma := xz$, and $w, t$, the equations of $E$ are then

$$
\gamma^3 = \alpha(\beta + 1),

(5.9)

\gamma = c^3 \alpha + t^6 L(w, 1),

0 = w^2 + (t + a(\alpha, \gamma)) + q(t)(\beta + 1) + Q(\alpha, \gamma),
$$

where $a(\alpha, \gamma)$ is linear and $Q(\alpha, \gamma) \in R[\alpha, \gamma]$ quadratic in $\alpha, \gamma$. If $val(q(t)) \geq 2$, then the fact that $P \in E$ is singular implies that $E$ is singular along the whole line $\alpha = \gamma = w = t = 0$, which is a contradiction. If $val(q(t)) = 1$, then $E$ has equation

$$
w^2 + t\beta + \cdots = 0,
$$

which is $A_1$-singularity. (When $(t^6 L(w, 1)) \neq (ts)$, one can see that $val(q(t)) = 1$ directly from the condition that $P \in E$ is singular.) □

This concludes the proof of Theorem 5.1 in Case (2) of Lemma 5.4.
6. Questions and conjectures

In this section, we state some questions and conjectures arising from our definition of stability for degree 1 and 2 del Pezzo fibrations, and concerning the specific parameter spaces of such del Pezzos appearing in this article.

6.1. Optimality of the models. In defining stability for a given fibration, even for a fixed $G$ and $M$, there is a choice of a $G$-invariant divisor (or, equivalently, a $G$-linearized line bundle $L$ with an invariant section). For a given del Pezzo over $K$, many choices of $L$ may result in stable models over $R$ with a terminal total space and integral fibers. It is possible that not all of these models are isomorphic, in which case it is natural to ask “What is the best choice?” In particular, we can ask the following question.

**Question 1.** Do the line bundle $L_{\text{ter}} := \mathcal{O}(D_{\text{ter}})$ defined in Proposition 4.8 and the CM line bundle $L^\text{CM}$ give the same notion of stability?

From the birational point of view, we can consider optimality in the following manner.

**Definition 6.1.** Let $\pi: X \to Z$ be a morphism from a terminal $\mathbb{Q}$-factorial variety to a curve. Suppose $-K_X$ is $\pi$-ample. Let $X_0$ be the fiber over $0 \in Z$ and suppose that $X_0$ is irreducible. We define the movable canonical threshold of $X$ along $X_0$ to be

$$mct_X(X_0) = \sup \left\{ \lambda \mid \left( X, \frac{\lambda}{n}\mathcal{M} \right) \text{ is canonical along } X_0, \text{ where } \mathcal{M} \subset |-nK_X + lF|, n, l \in \mathbb{Z} \right\}.$$ 

**Remark 6.2.** The movable canonical threshold helps in measuring the singularities of $X$ along $X_0$ and, indirectly, of $X_0$ itself. Indeed, by inversion of adjunction we have $mct_X(X_0) \geq \text{lct}(X_0)$.

**Theorem 6.3** ([7, Theorem 1.5]). Let $Z$ be a smooth curve. Suppose that there is a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\rho} & Y \\
\downarrow \pi & & \downarrow \pi_Y \\
Z & \xrightarrow{=} & Z
\end{array}
$$

such that $\pi$ and $\pi_Y$ are flat morphisms, and $\rho$ is a birational map that induces an isomorphism

$$\rho|_{X \setminus X_0}: X \setminus X_0 \to Y \setminus Y_0$$

where $X_0$ and $Y_0$ are scheme fibers of $\pi$ and $\pi_Y$ over a point $0 \in Z$, respectively. Suppose that the varieties $X$ and $Y$ have terminal and $\mathbb{Q}$-factorial singularities, the divisors $-K_X$ and $-K_Y$ are $\pi$-ample and $\pi_Y$-ample respectively, the fibers $X_0$ and $Y_0$ are irreducible, and $mct_X(X_0) + mct_Y(Y_0) > 1$. Then $\rho$ is an isomorphism.

This theorem suggests a relation between (semi)stability and movable canonical threshold:

**Conjecture 6.4.** For $1 \leq d \leq 3$, there exists a notion of $\mathcal{D}$-stability for degree $d$ del Pezzo fibrations such that every semistable model $\pi: X \to Z$ satisfies $mct_X(X_t) \geq \frac{1}{2}$ for all $t \in Z$.

6.2. Birational rigidity of del Pezzo fibrations.

**Definition 6.5** ([10, Definition 1.2]). A Mori fiber space $\pi: X \to S$ is said to be birationally rigid if the existence of a birational map $\chi: X \dashrightarrow Y$ to a Mori fiber space $\sigma: Y \to T$ implies that there exist a birational selfmap $\alpha: X \dashrightarrow X$ and a birational map $g: S \dashrightarrow T$ such that the following diagram commutes.
and that the induced map on the generic fibers \( X_\eta \) and \( Y_\eta \) is an isomorphism.

It is well known that a birational map between two Mori fiber spaces can be decomposed into so-called elementary Sarkisov links. There are four types of elementary links numbered by I, II, III, IV, which are explicitly described in [8]. For a del Pezzo fibration, links of type III and IV could be initiated only if the \( K \)-condition is not satisfied, see [1] for an analysis in degree 2.

**Definition 6.6.** We say that \( \pi : X \to S \) satisfies the \( K \)-condition if 

\[
-K_X \notin \overline{\text{Mob}(X)},
\]

where \( \overline{\text{Mob}(X)} \) is the interior of the cone of mobile divisors.

One can think of this as the condition which prevents the existence of elementary Sarkisov links of type III and IV. Type I or II links are initiated by an extremal extraction (MMP blow up) of a curve or a point in \( X \). The expectation is that Type I links do not exist on (semistable) del Pezzo fibrations in degree 1, 2 or 3. Type II links however can exist, and these are typically birational maps where the central fiber is changed. We expect that for \( 1 \leq d \leq 3 \), there exists a notion of \( \mathcal{D} \)-stability for degree \( d \) del Pezzo fibrations such that the modified version of Grinenko's conjecture on birational rigidity of del Pezzo fibrations (see [23], [12], and [13]) holds:

**Conjecture 6.7.** Let \( 1 \leq d \leq 3 \) and suppose \( X \to \mathbb{P}^1 \) is a \( \mathcal{D} \)-semistable del Pezzo fibration of degree \( d \) satisfying the \( K \)-condition. Then \( X \) is birationally rigid.

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