$k$-Symplectic Pontryagin’s Maximum Principle for some families of PDEs

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Abstract An optimal control problem associated with the dynamics of the orientation of a bipolar molecule in the plane can be understood by means of tools in differential geometry. For first time in the literature $k$-symplectic formalism is used to provide the optimal control problems associated to some families of partial differential equations with a geometric formulation. A parallel between the classic formalism of optimal control theory with ordinary differential equations and the one with particular families of partial differential equations is established. This description allows us to state and prove Pontryagin’s Maximum Principle on $k$-symplectic formalism. We also consider the unified Skinner-Rusk formalism for optimal control problems governed by an implicit partial differential equation.

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1 Introduction

Boscain et al. in [6] study the controllability of the equation

\[ \frac{1}{i} \frac{\partial \Psi(t, \theta)}{\partial t} = -\frac{\partial^2 \Psi(t, \theta)}{\partial \theta^2} + u_1(t) \cos \theta \Psi(t, \theta) + u_2(t) \sin \theta \Psi(t, \theta), \quad (1) \]

which models the rotation motion of a bipolar rigid molecule confined to a plane with two control electric fields. See references in the paper for more details about the problem origin and its interests.

The study of controllability in [6] does not approach the problem of existence and construction of suitable controls for governing the position of the molecule. The controls are obtained depending on the purpose to be achieved. For instance, if the controls are related with the energy needed to take the molecule to a particular position or to track a molecule path in the configuration space, then we might be interested in minimizing the energy consumption. In other words we have associated an optimal control problem to the above control equation. Note that control equation is a particular kind of second-order and control-linear partial differential equation.

In this paper, a geometric approach is considered to deal with optimal control problems for particular families of control partial differential equations, similar to the above mentioned example.

Following the same lines as in our previous paper on Pontryagin’s Maximum Principle [3], PMP, we extend the geometric method to optimal control problems with some control partial differential equations. As in the classical PMP, to succeed in this extension is necessary to extend the control system in a proper way and to optimize a suitable function, the Pontryagin Hamiltonian, instead of the functional.

Before entering into the details, let us provide some historical background on optimal control theory. L. S. Pontryagin talked publicly for the first time about the Maximum Principle in 1958, in the International Congress of Mathematicians that was held in Edinburgh, Scotland. This Principle was developed by a research group on automatic control created by Pontryagin in the fifties. He was engaged in applied mathematics by his friend A. Andronov and because scientists in the Steklov Mathematical Institute were asked to carry out applied research, especially in the field of aircraft dynamics.

At the same time, in the regular seminars on automatic control in the Institute of Automatics and Telemechanics, A. Feldbaum introduced Pontryagin and his colleagues to the time-optimization problem. This allowed them to study how to find the best way of piloting an aircraft in order to defeat a zenith fire point in the shortest time as a time-optimization problem.

Since the equations for modelling the aircraft’s problem are nonlinear and the control of the rear end of the aircraft runs over a bounded subset, it was necessary to reformulate the calculus of variations known at that time. Taking into account ideas suggested by McShane in [14], Pontryagin and his collaborators managed to state and prove the Maximum Principle, which was published in Russian in 1961 and translated into English [16] the following year. See [5] for more historical remarks.

Although geometric control theory has been studied since the sixties, geometric optimal control theory started to be developed in the nineties [11, 19]. However, no geometric description has been made for optimal control problems governed by partial differential equations [10, 18, 20]. In this paper we extend for first time the geometric description of optimal control problems to those governed by some classes of partial differential equations in order to solve optimal control problems that can be found in the physical world, as mentioned above.
With this purpose in mind, the natural geometric background to use is $k$-symplectic formalism (Günther standard polysymplectic), which is a generalization of the symplectic formalism in classical mechanics. The $k$-symplectic formalism makes possible to geometrically interpret some problems such as the vibrating string within field theory [1] and other problems [15]. Locally speaking, these problems correspond with Lagrangian and Hamiltonian functions that do not depend on the base coordinates, usually denoted by $(t^1, \ldots, t^k)$. When a dependence on the base coordinates exists, the $k$-cosymplectic formalism is necessary. In other words the $k$-cosymplectic formalism is the generalization of the cosymplectic formalism used in non-autonomous mechanics to field theories [12,13].

However, the control Eq. (1) under study is of second order. Hence we need to extend the $k$-symplectic formalism for optimal control problems developed in this paper to implicit control differential equations. In this framework we transform the Eq. (1) into a first order one in such a way that we obtain an implicit equation.

The paper is organized as follows: In Sect. 2 we define the setting to describe optimal control problems governed by an explicit first-order partial differential equation using the $k$-symplectic formalism. A parallel between this formulation and the geometric description of optimal control problems governed by ordinary differential equations is considered in order to stress the similarities and differences between both problems.

One of the key points to prove Pontryagin’s Maximum Principle consists of extending in a suitable way the control system by adding new coordinates which contain the information related to the cost function, the so called extended system. In this paper, the $k$-symplectic formalism for the extended optimal control problems only works under some particular assumptions on the cost function, which turn out to include the most typical cost functions in the literature. To prove Pontryagin’s Maximum Principle on $k$-symplectic formalism in Sect. 2.1 we define the elementary perturbation vectors on that formalism.

After this first approach to tackle optimal control problems, we are going to consider in Sect. 3 the unified Skinner-Rusk formalism for $k$-cosymplectic implicit dynamical systems. Following the lines of [2], we adapt the above formalism to describe a novel unified formalism for optimal control problems governed by an implicit partial differential equation in Sect. 4. This generalized unified formalism will allow us to consider interesting problems associated with higher order control partial differential equations, in particular the problem that has motivated our study, see Sect. 5. In this last section, we consider the control partial differential equation that models the orientation of a bipolar molecule in the plane studied in [6], as described at the beginning, when a control-quadratic cost function is considered. Hence, the use of $k$-symplectic formalism and all the generalizations we have made in the previous sections to deal with optimal control problems on partial differential equations are fully justified.

In the sequel, unless otherwise stated, all the manifolds are real, second countable and $C^\infty$. The maps are assumed to be also $C^\infty$. Sum over all repeated indices is understood.

## 2 $k$-Symplectic formalism for optimal control problems governed by an explicit first-order partial differential equation

We first recall briefly the essential definitions and notations in the $k$-symplectic formalism. Let $Q$ be a $n$-dimensional manifold and $\tau_Q : TQ \rightarrow Q$ be the natural tangent bundle projection. The $k$-tangent bundle or the bundle of $k^1$-velocities of $Q$, denoted by $T^k_1 Q$, is the Whitney sum of $k$ copies of the tangent bundle $TQ$, that is,
\[ T^1_k Q = TQ \oplus \cdots \oplus TQ. \]  

(2)

The elements of \( T^1_k Q \) are \( k \)-tuple \((v_{1q}, \ldots, v_{kq})\) of tangent vectors on \( Q \) at the same point \( q \in Q \).

The canonical projection \( \tau^k_Q : T^1_k Q \to Q \) is defined as follows

\[ \tau^k_Q(v_{1q}, \ldots, v_{kq}) = q. \]

If \((V, (q^i))\) is a local chart on \( Q \), then it induces a local chart \((T^1_k V, (q^i, v^i_A))\) on \( T^1_k Q \), where \( T^1_k V = (\tau^k_Q)^{-1}(V) \).

A \( k \)-vector field on \( Q \) is a section \( \mathbf{X} : Q \to T^1_k Q \) of the canonical projection \( \tau^k_Q \). Hence a \( k \)-vector field \( \mathbf{X} \) defines a family of \( k \) ordinary vector fields \([X_1, \ldots, X_k]\) on \( Q \) through the canonical projections \( \tau^{k:A}_Q : T^1_k Q \to TQ \) onto the \( A \)-th component of \( T^1_k Q \), that is,

\[ \tau^{k:A}_Q(v_{1q}, \ldots, v_{kq}) = v_{Aq}, \]

where \( A = 1, \ldots, k \). Note that \( X_A = \tau^{k:A}_Q \circ \mathbf{X} \).

An integral section of \( \mathbf{X} \) is a map \( \sigma : \mathbb{R}^k \to Q \), \( t = (t^1, \ldots, t^k) \to \sigma(t) \) such that

\[ T^1_k \sigma = \left( \frac{\partial \sigma}{\partial t^1}, \ldots, \frac{\partial \sigma}{\partial t^k} \right)_{\sigma(t)} = \mathbf{X} \circ \sigma. \]

We introduce now the notion of control system in the \( k \)-symplectic formalism. Consider a control set \( U \subset \mathbb{R}^l \). We need the notion of a \( k \)-vector field \( \mathbf{X} \) defined along the projection \( \pi_1 : Q \times U \to Q \). Such a \( k \)-vector field is defined by making the following diagram commutative:

\[ \begin{array}{cccccc}
\text{J} \subset \mathbb{R}^k & \xrightarrow{\phi} & \mathbb{R}^k \times U & \xrightarrow{\pi_1} & Q \\
& \searrow & \uparrow \tau^k_Q & \nearrow & \downarrow \\
\xrightarrow{(T^1_k)(\pi_1 \circ \phi)} & & \xrightarrow{\mathbf{X}} & & \xrightarrow{T^1_k Q}
\end{array} \]

where \( J \) is a subset of \( \mathbb{R}^k \).

An integral section of a \( k \)-vector field \( \mathbf{X} \) defined along the projection \( \pi_1 : Q \times U \to Q \) is a map \( \phi = (\sigma, u) : J \subset \mathbb{R}^k \to Q \times U \) such that

\[ (T^1_k)(\pi_1 \circ \phi) = \mathbf{X} \circ \phi, \]

or in other terms,

\[ T^{(t^1, \ldots, t^k)}(\pi_1 \circ \phi) \frac{\partial}{\partial t^A} = X_A \left( \phi \left( t^1, \ldots, t^k \right) \right) = X_A \left( \left( t^1, \ldots, t^k \right), u \left( t^1, \ldots, t^k \right) \right), \]

equivalently \( (\pi_1 \circ \phi)_* \frac{\partial}{\partial t^A} = X_A \circ \phi \) for \( A = 1, \ldots, k \).

Let \( F : Q \times U \to \mathbb{R}^k \) be a regular enough map. Such a function, which is usually called the cost function in the literature, allows us to define the functional

\[ \mathcal{F}[\phi] = \int_{\text{Dom} \phi} (F \circ \phi) dS, \]

(3)

where \( dS = dt^1 \wedge \cdots \wedge dt^k \), i.e. the usual volume form in \( \mathbb{R}^k \).

From now on, we assume that \( \text{Dom} \phi = I_1 \times \cdots \times I_k = [t^1_0, t^1_i] \times \cdots \times [t^k_0, t^k_i] =: I \).
Before stating the optimal control problem on $k$-symplectic formalism, we remind here the classical optimal control problem with cost function $G: Q \times U \to \mathbb{R}$.

**Statement 21** (*Optimal control problem, OCP*) Given $(Q, U, X, G, I)$. Find a curve $(\gamma, u): I \subset \mathbb{R} \to Q \times U$ joining the points $x_0$ and $x_f$ in $Q$ such that

(i) it is an integral curve of the vector field $X$ defined along the projection $\pi_1: Q \times U \to Q$, i.e. $\dot{\gamma}(t) = X(\gamma(t), u(t))$;

(ii) it minimizes the functional $\int_I G(\tilde{\gamma}(t), \tilde{u}(t))dt$ among all the integral curves $(\tilde{\gamma}, \tilde{u})$ of $X$ on $Q \times U$ joining $x_0$ and $x_f$.

**Statement 22** (*$k$-symplectic optimal control problem, kOCP*) Given $(Q, U, X, F, I)$. Find a map $\phi = (\sigma, u): I = I_1 \times \cdots \times I_k \subset \mathbb{R}^k \to Q \times U$ passing through the points $q_0$ and $q_f$ in $Q$ such that

(i) it is an integral section of the $k$-vector field $X = (X_1, \ldots, X_k)$ defined along the projection $\pi_1: Q \times U \to Q$, i.e.

$$T_k^1(\pi_1 \circ \phi) = X \circ \phi,$$

where $t = (t^1, \ldots, t^k) \in I_1 \times \cdots \times I_k$;

(ii) it minimizes the functional $\int_{I_1 \times \cdots \times I_k} F(\tilde{\phi}(t))d^k t$ among all the integral sections $\tilde{\phi}$ of $X$ on $Q \times U$ passing through $q_0$ and $q_f$, where $d^k t = dt^1 \wedge \cdots \wedge dt^k$.

Let us compare the frameworks in the traditional optimal control problems and in the $k$-symplectic optimal control problems.

**Classical OCP for ODE**

$$\begin{array}{ccc}
Q \times U & \xrightarrow{T} & Q \\
\gamma & \circ \pi_1 & \downarrow \tau_Q \\
(\gamma, u) & & \phi = (\sigma, u)
\end{array}$$

**$k$-symplectic OCP**

$$\begin{array}{ccc}
Q \times U & \xrightarrow{T} & Q \\
\phi = (\sigma, u) & \circ \pi_1 & \downarrow \tau_Q \\
I = I_1 \times \cdots \times I_k \subset \mathbb{R}^k & & \phi = (\sigma, u)
\end{array}$$

Observe that the classical OCP has associated a problem of explicit ordinary differential equations, whereas the equations in the $k$-symplectic optimal control problem are explicit partial differential equations.

Since late fifties the most efficient tool to solve optimal control problem is Pontryagin’s Maximum Principle, which provides us with necessary conditions for optimality [16]. One of the key points to prove that Principle for classical optimal control theory consists of extending the control system in a suitable way. To be more precise, $Q$ is extended to the manifold $\tilde{Q} = \mathbb{R} \times Q$ with local coordinates $\tilde{x} = (x^0, x^i)$ and the corresponding extended vector field is given by

$$\tilde{X}(\tilde{x}, u) = G(x, u) \frac{\partial}{\partial x^0(\tilde{x}, u)} + X(x, u).$$

Note that the system of ordinary differential equations which determines the integral curves of $\tilde{X}$ can be decoupled in the following sense: we first integrate $\frac{dx^i}{dt} = X^i(x, u)$ and then
we have

$$x^0(t) = \int_{t_0}^{t} G(\gamma(s), u(s))ds,$$

for any $t \in I = [t_0, t_1]$.

Unfortunately, in order to extend coherently the optimal control problem on $k$-symplectic formalism we need some extra assumptions on the cost function.

**Assumption 1** The Lie derivative of the cost function with respect to each $X_A$ is zero, that is, $L_{X_A} F = 0$.

**Assumption 2** The control functions $u : I \to U$ are locally constants.

We can justify these assumptions as follows. In a first try to extend the control system we will add $k$ new variables $(q^{01}, \ldots, q^{0k})$ such that for every $A = 1, \ldots, k$

$$\frac{\partial q^{0A}}{\partial t^A} = F,$$  

$$\frac{\partial q^{0A}}{\partial t^B} = 0, \text{ for } B \neq A. \quad (5)$$

Once we have an integral section of $X$, we integrate (5) and obtain

$$q^{0A}[t^1, \ldots, \widehat{t^A}, \ldots, t^k](t^A) = \int_{t^A}^{t^A} F(q^1, \ldots, q^n, u^1, \ldots, u^l)(t^1, \ldots, s, \ldots, t^k) ds.$$  

The Eq. (6) is satisfied by $q^{0A}$ for $B \neq A$ if

$$\frac{\partial q^{0A}}{\partial t^B} = \int_{t^A}^{t^A} \left( \frac{\partial F}{\partial q^i} \frac{\partial q^i}{\partial t^A} + \frac{\partial F}{\partial u^a} \frac{\partial u^a}{\partial t^A} \right) (t^1, \ldots, s, \ldots, t^k) ds = 0.$$  

Note that if both assumptions are satisfied the equations will be immediately satisfied. Thus these assumptions are necessary to guarantee the compatibility of the system of partial differential equations when we extend the control system.

Moreover, having in mind [9] Assumption 2 is reasonable when dealing with control systems. These two assumptions include the most typical cost functions considered in optimal control problems such as control-quadratic, constant function 1 (that is, time optimal), etc. Hence these assumptions do not impose great restrictions according to the literature.

Under the above assumptions, let us consider now the extended $k$-symplectic optimal control problem. In order to preserve the same philosophy as in classical control theory, we will have to add $k$ new coordinates $(q^{0B})_{B=1,\ldots,k}$. Then the extended manifold in $k$-symplectic formalism is given by $\tilde{Q} = \mathbb{R}^k \times Q$. If $X = (X_1, \ldots, X_k)$ is the $k$-vector field on $Q$, then the *extended $k$-vector field* $\tilde{X}$ on $\tilde{Q}$ is given by $(\tilde{X}_1, \ldots, \tilde{X}_k)$ where

$$\tilde{X}_A = F \delta^B_A \frac{\partial}{\partial q^B} + X_A = F \frac{\partial}{\partial q^0A} + X_A, \text{ for every } A = 1, \ldots, k, \quad (7)$$

where $\delta^B_A$ is the Kronecker’s delta and $F$ is the cost function.
Remark 21 Note that if the projection of an integral section since the order of integration does not matter. 

Because of (6) \( q^{0_A} \) is constant when we fix \( t^A \). On the other hand, due to Assumption 1 and (5) \( q^{0_A} \) is constant along integral curves of \( X_A \) for every \( A = 1, \ldots, k \).

Statement 23 (Extended k-symplectic optimal control problem) Given \((\hat{Q}, U, \hat{X}, F, I)\). Find a map \( \hat{\phi} = (\hat{\sigma}, u): I \subset \mathbb{R}^k \rightarrow \hat{Q} \times U \) passing through the points \((0, q_0)\) in \( \hat{Q} \) and \( q_f \) in \( \hat{Q} \) such that

(i) it is an integral section of the \( k \)-vector field \( \hat{X} = (\hat{X}_1, \ldots, \hat{X}_k) \) defined along the projection \( \hat{T}_1: \hat{Q} \times U \rightarrow \hat{Q} \), i.e. locally

\[
T_k(\hat{T}_1 \circ \hat{\phi}) = \hat{X} \circ \hat{\phi}, \quad \text{i.e.} \quad \frac{\partial \sigma^{0_B}}{\partial t^A}(t) = F(\phi(t))\delta^B_A, \quad \frac{\partial \sigma^i}{\partial t^A}(t) = X^i_A(\phi(t)),
\]

where \( t = (t^1, \ldots, t^k) \in I_1 \times \cdots \times I_k, \delta^B_A \) is the Kronecker’s delta, for every \( A, B = 1, \ldots, k; i = 1, \ldots, n \);

(ii) it minimizes each functional

\[
\mathcal{F}_A[\phi](t) = \int_{t_0^A}^{t^A} F(\phi(t)) \left( t^1, \ldots, A^i, \ldots, t^k \right) \, ds,
\]

for \( A = 1, \ldots, k \), among all the integral sections \( \hat{\phi} \) of \( \hat{X} \) on \( \hat{Q} \times U \) passing through \( q_0 \) and \( q_f \) such that \( \phi = \pi_{Q\times U} \circ \hat{\phi} \) for \( \pi_{Q\times U}: \hat{Q} \times U \rightarrow Q \times U \).

Remark 21 Note that if the projection of an integral section \( \hat{\phi}: I_1 \times \cdots \times I_k \rightarrow \hat{Q} \times U \) of \( \hat{X} \) to \( \phi \) on \( Q \times U \) minimizes each functional in (8), then the projection of the integral section \( \hat{\phi} \) to \( \phi \) on \( Q \times U \) minimizes the functional

\[
\mathcal{F}[\phi] = \int_{I_1 \times \cdots \times I_k} (F \circ \phi) \, d^k t
\]

since the order of integration does not matter.

Hence, in contrast with classical optimal control theory, in \( k \)-symplectic formalism the extended optimal control problem and the optimal control problem are not equivalent. However, solutions to the extended problem in Statement 23 are also solutions to the original \( k \)-symplectic optimal control problem in Statement 22. As we will see later on, the adapted version of Pontryagin’s Maximum Principle in \( k \)-symplectic formalism provides us with necessary conditions for optimality of the \( k \) functionals in (8) for those cost functions satisfying assumptions 1 and 2.

As mentioned above the trajectories that minimize (8) also minimize (9), but not necessarily in the other way around. Remember that to minimize a multiple integral does not imply that every simple integral involved is minimized. Thus, the necessary conditions for optimality described in Sect. 2.1 in the \( k \)-symplectic version of Pontryagin’s Maximum Principle are more restrictive than the traditional necessary conditions for optimality in [3,16].
The elements of extended optimal control problems in classical formalism and k-symplectic formalism are summarized in the following diagrams:

Classical extended OCP for ODE  

\[
\text{HA} \quad \text{HA} \\
\end{array} 
\]

\[
\begin{array}{c}
\text{formalism neither in the}
\end{array} 
\]

\[
\text{this Hamiltonian problem is not equivalent to the optimal control problems in the classical}
\]

\[
\text{ciated with each of the extended optimal control problems. It is important to remark here that}
\]

\[
\text{the following family of equations}
\]

\[
\begin{array}{c}
\text{123}
\end{array} 
\]

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\]

\[
\omega
\]

\[
\text{where}
\]

\[
\text{for}
\]

\[
\text{The Hamiltonian for the extended optimal control problem in classical theory is given by}
\]

\[
H(\hat{p}, u) = \langle \hat{p}, \hat{X}(\hat{x}, u) \rangle = p_0 G(x, u) + p_i X^i(x, u).
\]

\[
\text{For each control}\ u, \text{the Hamiltonian vector field}\ \hat{X}^{[u]}_H = \hat{X}_H(\cdot, u) \text{ satisfies the following Hamilton’s equation}
\]

\[
i_{\hat{X}^{[u]}_H} \omega = dH^{[u]},
\]

\[
\text{where}\ \omega \text{ is the canonical symplectic structure on}\ T^* \hat{Q} \text{. Locally}\ \omega = dx^0 \wedge dp_0 + dx^i \wedge dp_i
\]

\[
in\text{natural local coordinates}\ (x^0, x^i, p_0, p_i)\ \text{in}\ T^* \hat{Q}.
\]

\[
\text{For the extended k-symplectic optimal control problem we consider k Hamiltonian functions}\ H_A : (T^*_k)^* \hat{Q} \times U \rightarrow \mathbb{R}\ \text{defined as follows}
\]

\[
H_A(\hat{p}, u) = \langle \hat{p}^A, \hat{X}_A(\hat{q}, u) \rangle = \sum_{B=1}^k p_{0_B}^A F(q, u) \delta_B^A + \sum_{j=1}^n p_j^A X_j^A(q, u)
\]

\[
= p_{0_A}^A F(q, u) + \sum_{j=1}^n p_j^A X_j^A(q, u), \quad (10)
\]

\[
in\text{natural local coordinates}\ (q^{0_1}, \ldots, q^{0_k}, q^1, \ldots, q^n, (p_{0_1}^A, \ldots, p_{0_k}^A, p_1^A, \ldots, p_n^A)_{A=1,\ldots,k})\ \text{for}\ (T^*_k)^* \hat{Q}.
\]

\[
\text{For each control}\ u, \text{the Hamiltonian k-vector field}\ \hat{X}^{*[u]} = (\hat{X}^{*[u]}_1, \ldots, \hat{X}^{*[u]}_k) \text{ must satisfy the following family of equations}
\]

\[
i_{\hat{X}_A^{*[u]}} \omega^A = dH_A^{[u]} \quad \text{for every} \quad A = 1, \ldots, k. \quad (11)
\]

The canonical k-symplectic structure on\ (T^*_k)^* \hat{Q} \text{ is given by}\ (\omega_1, \ldots, \omega_k)\ \text{ where}\ \omega_A = (\pi^A)^* \omega, \pi^A : (T^*_k)^* \hat{Q} \rightarrow T^* \hat{Q} \text{ is the projection onto the}\ A\text{-copy and}\ \omega\ \text{is the canonical symplectic structure on}\ T^* \hat{Q} \text{. Locally}\ \omega_A = dq^{0_i} \wedge dp_{0_j}^A + dq^i \wedge dp_i^A.
If for each control the Hamiltonian \( k \)-vector field \( \hat{X}^{(u)} \) is solution to (11), then it is solution to the following Hamilton-De Donder-Weyl equations

\[
\sum_{A=1}^{k} i_{\hat{X}^{(u)}_A} \omega^A = \sum_{A=1}^{k} dH^{(u)}_A = d \left( \sum_{A=1}^{k} H^{(u)}_A \right) = dH, \tag{12}
\]

associated with the Hamiltonian \( H \): \( (T^1_k)^* \hat{Q} \times U \rightarrow \mathbb{R} \) given by

\[
H(\hat{p}, u) = \sum_{A=1}^{k} H_A(\hat{p}, u) = \sum_{A=1}^{k} \langle \hat{p}^A, \hat{X}_A(\hat{q}, u) \rangle,
\]

where \( \hat{p} \in (T^1_k)^* \hat{Q} \). By the superposition principle all the solutions of (11) are solutions to (12) because both systems are linear in the momenta. However, (12) has more solutions apart from the ones coming from (11). In fact, for every \( A \in \{1, \ldots, k\} \) the \( k \)-vector field \( \hat{X}^* \) associated with the Hamiltonian \( H \) satisfies

\[
\hat{X}^*_A = (Y_A)^0_b \frac{\partial}{\partial q^0_b} + (Y_A)^i \frac{\partial}{\partial q^i} + (Y_A)^C_0_B \frac{\partial}{\partial p^B_0} + (Y_A)^C_j \frac{\partial}{\partial p^C_j}.
\]

From (11) we obtain

\[
(Y_A)^0_b = X^0_A = F \delta_A^B, \quad (Y_A)^A_0_b = 0, \quad (Y_A)^i_j = X^i_A, \quad (Y_A)^A_i = -p^A_0 \frac{\partial F}{\partial q^i} - p^j_A \frac{\partial X^j_A}{\partial q^i}, \tag{13}
\]

for every \( A = 1, \ldots, k \). Note that the Hamiltonian \( k \)-vector field \( \hat{X}^* = (\hat{X}^*_1, \ldots, \hat{X}^*_k) \) is not completely determined because the following functions

\[
(Y_A)^C_0_b, \quad (Y_A)^C_j
\]

remain undetermined for \( C \neq A \) and for every \( A = 1, \ldots, k \).

On the other hand, from (12) the Hamiltonian \( k \)-vector field \( \hat{X}^* = (\hat{X}^*_1, \ldots, \hat{X}^*_k) \) must satisfy

\[
(Y_A)^0_b = X^0_A = F \delta_A^B, \quad (Y_A)^A_0_b = 0, \quad (Y_A)^i_j = X^i_A, \quad \sum_{A=1}^{k} (Y_A)^i_j = \sum_{A=1}^{k} \left( -p^A_0 \frac{\partial F}{\partial q^i} - p^j_A \frac{\partial X^j_A}{\partial q^i} \right). \tag{15}
\]

By comparing (13) and (15) it is clear that all the solutions to (11) are also solutions of (12), not in the other way around. Neither the Hamiltonian \( k \)-vector field \( \hat{X}^{(u)} \) solution to (11), nor the Hamiltonian \( k \)-vector field \( \hat{X}^{(u)} \) solution to the Hamilton-De Donder-Weyl equations are fully determined. For the first one, the functions in (14) remain undetermined. For the second one, the functions in (14) remain undetermined and maybe some of the \( (Y_A)^i_j \) involved in (15).

However, we can reduce in an intrinsic way the number of functions that remain undetermined in the above mentioned Hamiltonian \( k \)-vector fields \( \hat{X}^{(u)} \) in such a way that the Hamiltonian \( k \)-vector field \( \hat{X}^{(u)} \) solution to (11) is fully determined. Note that \( (T^1_k)^* \hat{Q} = (T^1_k)^* (\mathbb{R}^k \times \hat{Q}) \simeq (T^1_k)^* \mathbb{R}^k \times (T^1_k)^* \hat{Q} \), which has two natural projections \( pr_1 \) and \( pr_2 \) from \( (T^1_k)^* \hat{Q} \) to \( (T^1_k)^* \mathbb{R}^k \) and \( (T^1_k)^* \hat{Q} \), respectively. Consider now the canonical
projections \( \pi^k_C : (T^1_k)^*Q \to T^*Q \) and \( \pi^k_{\mathbb{R}^k} : (T^1_k)^*\mathbb{R}^k \to T^*\mathbb{R}^k \) to the \( C \)th component of \((T^1_k)^*Q\) and \((T^1_k)^*\mathbb{R}^k\), respectively.

The conditions

\[
\begin{align*}
T \left( \pi^k_C \circ \text{pr}_1 \right) (\hat{X}_A^*) &= 0, \\
T \left( \pi^k_{\mathbb{R}^k} \circ \text{pr}_2 \right) (\hat{X}_A^*) &= 0,
\end{align*}
\]

for every \( C \neq A \) imply locally that \((Y_A)^C_{0B} = 0\) and \((Y_A)^C_j = 0\) for \( C \neq A \) and for every \( C \neq A \).

Under conditions \((16), (17)\), given an initial condition \( \hat{\beta}_0 \) in \((T^1_k)^*Q\) there exists a unique integral section \( \hat{\beta} : I_1 \times \cdots \times I_k \to (T^1_k)^*Q \) of the Hamiltonian \( k \)-vector field solution to \((11)\). It is clear from the local Eq. \((13)\) that once \( A \) is fixed, \( \hat{p}^B \) does not appear in the set of equations in \((11)\) associated with \( A \) and only \( \hat{p}^A \)'s appear.

2.1 Elementary perturbation vectors and Pontryagin’s Maximum Principle on \( k \)-symplectic formalism

Now let us introduce the notion of elementary perturbation in \( k \)-symplectic formalism that allows us to define later the \( k \)-symplectic tangent perturbation cones. These elements are essential to prove the \( k \)-symplectic Pontryagin’s Maximum Principle, Theorem 3.

First fix a surface \((\hat{\sigma}, u) : I_1 \times \cdots \times I_k \to \hat{Q} \times U\). Let \( \pi_A \) be a 3-tuple \( \{r_A, l_A, u_A\} \) where \( r_A, l_A \in \mathbb{R} \) and \( u_A \in U \subset \mathbb{R}^l \). The \( A \)th-\( \text{elementary perturbation of the control} \ u \) is defined as follows

\[
u[\pi_A^s](t^1, \ldots, t^k) = \begin{cases} u_A, & t^A \in [r_A - l_As, r_A], \\ u(t^1, \ldots, t^k), & \text{elsewhere}. \end{cases}
\]

Associated to this control \( \nu[\pi_A^s] \), the mapping \( \varphi[\pi_A^s] : I_1 \times \cdots \times I_k \to \hat{Q} \) is the integral section of the \( k \)-vector field \( \hat{\Sigma}^{\nu[\pi_A^s]} \) with initial condition \( \varphi(t_0^1, \ldots, t_0^k) \) at \( (t_0^1, \ldots, t_0^k) \).

Given \( \epsilon > 0 \), define the map

\[
\varphi[\pi_A] : I_1 \times \cdots \times I_k \times [0, \epsilon] \to \hat{Q}, \\
(t, s) \mapsto \varphi[\pi_A](t, s) = \varphi[\pi_A^s](t).
\]

For every \( t \in I_1 \times \cdots \times I_k, \varphi[\pi_A]^t : [0, \epsilon] \to \hat{Q} \) is given by \( \varphi[\pi_A]^t(s) = \varphi[\pi_A](t, s) \). The curve \( \varphi[\pi_A]^t \) depends continuously on \( s \) and on \( \pi_A = \{r_A, l_A, u_A\} \).

From \( \varphi[\pi_A^s] \) we can define a curve as follows

\[
\varphi[\pi_A^s](t^1, \ldots, t^k) : I_A \to \hat{Q}, \\
t^A \mapsto \varphi[\pi_A^s](t^1, \ldots, t^k) = \varphi[\pi_A^s](t^1, \ldots, t^A, \ldots, t^k).
\]

This curve is an integral curve of \( \hat{X}_A^{\nu[\pi_A^s]} \) with initial condition \( (t_0^A, \varphi(t_0^1, \ldots, t_0^k)) \).

**Proposition 1** Let \( r_A \in I_A \). If \( \nu[\pi_A^s] \) is an elementary perturbation of \( u \) specified by the data \( \pi_A = \{r_A, l_A, u_A\} \), then the curve \( \varphi[\pi_A^s] \) is differentiable at \( s = 0 \) and its tangent vector \( \hat{v}[\pi_A] \) is
\[
[\tilde{X}_A(\tilde{\sigma}(t^1, \ldots, r_A, \ldots, t^k), u_A) - \tilde{X}_A(\tilde{\sigma}(t^1, \ldots, r_A, \ldots, t^k), u(t^1, \ldots, r_A, \ldots, t^k))]l_A
\]

(19)

for fixed \( (t^1, \ldots, r_A, \ldots, t^k) \in I_1 \times \cdots \times I_k \).

**Proof** In local coordinates \((q_0^1, \ldots, q_0^k, q^1, \ldots, q^n)\) for \( \tilde{Q} \), note that

\[
q^i \circ \tilde{\sigma} \left[ \pi^x_A \right] \left( t^1, \ldots, \hat{t}^A, \ldots, t^k \right) (r_A) - q^i \circ \tilde{\sigma} \left[ \pi^x_A \right] \left( t^1, \ldots, \hat{t}^A, \ldots, t^k \right) (t_0^A)
\]

\[= \int_{t_0^A}^{r_A} X^i_A \left( \tilde{\sigma} \left[ \pi^x_A \right] \left( t^1, \ldots, \hat{t}^A, \ldots, t^k \right) (t), u \left[ \pi^x_A \right] \left( t^1, \ldots, t, \ldots t^k \right) \right) dt \]

for every \( i \in \{0, \ldots, 0_k, 1, \ldots n\} \).

To compute the derivative of \( \varphi^t_{\pi_A} \) at \( s = 0 \) with \( \mathbf{t} = (t^1, \ldots, r_A, \ldots, t^k) \) we use the definition of the derivative:

\[
\frac{d}{ds} \bigg|_{s=0} (q^i \circ \varphi^t_{\pi_A}) (s) = \lim_{s \to 0} \left( \frac{(q^i \circ \varphi^t_{\pi_A}) (s) - (q^i \circ \varphi^t_{\pi_A}) (0)}{s} \right)
\]

\[= \lim_{s \to 0} \left( \frac{q^i \circ \tilde{\sigma} \left[ \pi^x_A \right] \left( t^1, \ldots, r_A, \ldots, t^k \right) - q^i \circ \tilde{\sigma} \left( t^1, \ldots, r_A, \ldots, t^k \right)}{s} \right)
\]

\[= \lim_{s \to 0} \left( \frac{\int_{r_A}^{r_A - l_A s} X^i_A \left( \tilde{\sigma} \left[ \pi^x_A \right] \left( t^1, \ldots, \hat{t}^A, \ldots, t^k \right) \right) (t), u \left[ \pi^x_A \right] \left( t^1, \ldots, t, \ldots t^k \right) \right) dt}{s} \]

\[= \lim_{s \to 0} \left( \frac{\int_{r_A}^{r_A - l_A s} X^i_A \left( \tilde{\sigma} \left[ \pi^x_A \right] \left( t^1, \ldots, \hat{t}^A, \ldots, t^k \right) \right) (t), u \left( t^1, \ldots, t, \ldots t^k \right) \right) dt}{s} \]

\[= C. \]

Let us use now the following equation

\[
\int_{t-s}^{t} X(\gamma(h), u(h))dh = s X(\gamma(t), u(t)) + o(s),
\]

(20)

in the above formula having in mind that \( o(s) \) tends to 0 when \( s \) tends to 0. Then,

\[
C = \lim_{s \to 0} \left( \frac{X^i_A \left( \tilde{\sigma} \left[ \pi^x_A \right] \left( t^1, \ldots, \hat{t}^A, \ldots, t^k \right) \right) (r_A), u \left( t^1, \ldots, r_A, \ldots t^k \right) l_A s}{s} \]

\[= \frac{X^i_A \left( \tilde{\sigma} \left[ \pi^x_A \right] \left( t^1, \ldots, \hat{t}^A, \ldots, t^k \right) \right) (r_A), u \left( t^1, \ldots, r_A, \ldots t^k \right) l_A s + o(s)}{s} \]

\( \square \)
\[
= \lim_{s \to 0} \left( X^i_A \left( \hat{\sigma} \left[ \pi^{\mathcal{A}}_A \right] (t^1, \ldots, \hat{t}^A, \ldots, t^k) (r_A), u_A \right) (r_A), u_A \right) l_A
\]

for each \( i \in \{0, 1, 0_k, 1, \ldots, n \} \).

Note that the tangent vector in Proposition 2.1 is in \((\tau^{k,A}_Q (T^1_k \hat{Q}))_{(t^j_0, \ldots, t^j_l)} = T_{(t^j_0, \ldots, t^j_l)} \hat{Q} \) .

The vector \( \hat{\pi}[\pi_A] \) is called the \textbf{Ath-elementary perturbation vector associated to the perturbation data} \( \pi_A = \{r_A, l_A, u_A\} \). It is also called an \textbf{Ath-perturbation vector of class I}.

Following the same lines as in [3] we can define the associated Ath-perturbation vector obtained from \( c \) different Ath-perturbation data \( \pi_{A_1}, \ldots, \pi_{A_c} \) with different and/or same perturbation time \( r_{A_1}, \ldots, r_{A_c} \).

At each copy of the tangent bundle in the \( k \)-tangent bundle \( T^1_k \hat{Q} \), we construct an \textbf{Ath-tangent perturbation cone}

\[
K^A_t = \text{conv} \left( \bigcup_{a < \tau \leq t} \left( \Phi^{\mathcal{A}}_{(t, \tau)} \right)_* \mathcal{V}^A_t \right)
\]

(21)

where \( \mathcal{V}^A_t \) denotes the set of Ath-elementary perturbation vectors at \( \tau \), \( \left( \Phi^{\mathcal{A}}_{(t, \tau)} \right)_* \) is the pushforward of the flow of \( \hat{X}^A_{\pi}[\mathcal{A}] \) with \( \hat{\sigma}(\tau) \) as initial condition at time \( \tau \), \( \text{conv} \mathcal{W} \) denotes the closure of the convex hull of the set \( \mathcal{W} \).

\textbf{Remark 22} If the controls are only measurable and bounded, as usually assumed in control theory, all the perturbations and geometric elements such as vectors, cones, etc. that appear in the paper are only defined at Lebesgue times where the equality (20) is satisfied.

The definition of \( k \) different perturbation cones in (21) implies that the perturbation data associated with different Ath copies are not mixed. As proved in [3, Proposition 3.12], the following result is true for the cones \( K^A_t \) for every \( A = 1, \ldots, k \).

\textbf{Proposition 2} Let \( t^A \in [t^A_0, t^A] \). If \( v \) is a nonzero vector in the interior of \( K^A_t \), then there exists \( \varepsilon > 0 \) such that for every \( s \in (0, \varepsilon) \) there are \( s' > 0 \) and a perturbation of the control \( u[\pi_A] \) such that

\[
\hat{\sigma}[\pi^A_X](t^1, \ldots, \hat{t}^A, \ldots, t^k)(t^A) = \hat{\sigma}(t^1, \ldots, \hat{t}^A, \ldots, t^k)(t^A) + s'v.
\]

This proposition is essential to prove Pontryagin’s Maximum Principle in \( k \)-symplectic formalism.

\textbf{Theorem 3} (\( k \)-symplectic Pontryagin’s Maximum Principle) If \( \hat{\phi}^*(\hat{\sigma}^*, u^*) : I_1 \times \cdots \times I_k \to \hat{Q} \times U \) is a solution of the extended \( k \)-symplectic optimal control problem \( (\hat{Q}, U, \hat{X}, F, I) \), Statement 23, such that \( F \) satisfies assumptions 1 and 2, then there exists \( (\hat{\beta}, u) : I_1 \times \cdots \times I_k \to (T^1_k)^* \hat{Q} \times U \) along \( \hat{\sigma}^* \) such that

\( \mathcal{C} \) Springer
(i) \((\pi^A \circ \hat{\beta}, u)\) along \(\hat{\sigma}^*\) is a solution of (11) for each \(A = 1, \ldots, k\);
(ii) the Hamiltonian \(H^A: (T^1_k)^* \hat{Q} \times U \rightarrow \mathbb{R}\) in (10) along the optimal integral section is equal to the supremum of \(H^A\) over the controls almost everywhere;
(iii) the supremum of the Hamiltonian \(H^A: (T^1_k)^* \hat{Q} \times U \rightarrow \mathbb{R}\) in (10) along the optimal integral section is constant almost everywhere;
(iv) \(\hat{\beta}^A(t) \neq 0 \in T^*_{\hat{\sigma}^*(t)} \hat{Q}\) for each \(t \in I_1 \times \cdots \times I_k\) and for every \(A = 1, \ldots, k\);
(v) \(\beta^A_0(t), \ldots, \beta^A_k(t)\) are constant and \(\beta^A_0\) is non-positive for every \(A = 1, \ldots, k\).

\[\begin{align*}
\textbf{Proof} & \quad \text{As } (\hat{\sigma}^*, u^*) \text{ is a solution of the extended } k\text{-symplectic optimal control problem, if } \\
& \quad \tau \in I_1 \times \cdots \times I_k, \text{ for every initial condition } \beta_\tau \in (T^1_k)^* \hat{Q} \text{ there exists a unique curve } \hat{\beta} \text{ in } \\
& \quad (T^1_k)^* \hat{Q} \text{ satisfying the } k \text{ equations in (11) and the initial condition.} \\
& \quad \text{As in the classical Pontryagin’s Maximum Principle the initial condition must be conveniently chosen so that the rest of conditions in the theorem are fulfilled.} \\
& \quad \text{For each } A \in \{1, \ldots, k\}, \text{ consider the } A\text{-tangent perturbation cone } K^A_{t_f} \subseteq T^*_{\hat{\sigma}^*(t_f)} \hat{Q} \text{ and the vector } (0, \ldots, -1, 0, 0, \ldots, 0) \text{ in } T^*_{\hat{\sigma}^*(t_f)} \hat{Q} \text{ that indicates the decreasing direction of the coordinate} \\
& \quad A \qquad q^{0A}(t) = \int_{t_f}^{t_A} F\left(\hat{\sigma}^* \left(t^1, \ldots, h, \ldots, t^k\right), u^* \left(t^1, \ldots, h, \ldots, t^k\right)\right) \, dh. \\
& \quad \text{Observe that if } (0, \ldots, -1, 0, 0, \ldots, 0) \text{ was in the interior of } K^A_{t_f}, \text{ then there would exist an } A\text{-perturbation data } \pi_A = \{r_A, l_A, u_A\} \text{ such that } (\hat{\sigma}[\pi_A], u[\pi_A]) \text{ passes through the same points on } Q \text{ as } \sigma^* = \pi_2 \circ \hat{\sigma}^*, \text{ but } q^{0A}[\pi_A](t_f) < q^{0A}[\hat{\sigma}^*](t_f). \text{ This is a contradiction with the fact that } (\hat{\sigma}^*, u^*) \text{ is a solution of the extended } k\text{-symplectic optimal control problem. Hence } (0, \ldots, -1, 0, 0, \ldots, 0) \text{ cannot be in the interior of } K^A_{t_f}. \\
& \quad \text{Thus, there exists } \hat{\beta}^A_{t_f} \in T^*_{\hat{\sigma}^*(t_f)} \hat{Q} \text{ such that} \\
& \quad \langle \hat{\beta}^A_{t_f}, \left(0, \ldots, -1, 0, 0, \ldots, 0\right) \rangle \geq 0, \quad (22) \\
& \quad \langle \hat{\beta}^A_{t_f}, \hat{\nu}[\pi_A] \rangle \leq 0 \quad \forall \hat{\nu}[\pi_A] \in K^A_{t_f}. \quad (23) \\
& \quad \text{Condition (22) implies that } \beta^A_0 \leq 0. \text{ Let us explicitly write for each } A \in \{1, \ldots, k\} \text{ the equations for the integral curves of } \hat{X}^A \text{ that satisfy Eq. (11):} \\
& \quad \frac{\partial q^{0B}}{\partial t^A} = F_{\delta^B A}, \quad \frac{\partial p^{0B}_A}{\partial t^A} = 0, \\
& \quad \frac{\partial q^i}{\partial t^A} = X^A_i, \quad \frac{\partial p^j_A}{\partial t^A} = -\frac{\partial H_A}{\partial q^i} = -p^{0A}_A \frac{\partial F}{\partial q^i} - p^j_A \frac{\partial X^j}{\partial q^i}, \\
& \quad \text{for } B = 1, \ldots, k \text{ and } i, j = 1, \ldots, n. \text{ Note that there are no equations for } p^B_i \text{ with } B \neq A. \\
& \quad \text{Hence the momenta whose coordinates are } p^B_i \text{ remain undetermined. They will be determined by solving (11) with } A = B. \text{ Given an initial condition in the } A\text{-copy of } T^*_{\hat{\sigma}^*(t_f)} \hat{Q}, \text{ we just solve the equations in the fiber for } p^A. \\
& \quad \text{If } \hat{\beta}^A_{t_f} = 0 \in T^*_{\hat{\sigma}^*(t_f)} \hat{Q}, \text{ then the solution to (11) in the fiber will be zero along } \hat{\sigma}^* \text{ because of the linearity of the differential equation in the momenta. If the momenta is zero, it does}
\]
not provide us with any information related to the separation condition. Hence, \( \hat{\beta}^A(t) \neq 0 \in T_{\hat{\sigma}^A(t)}^* \hat{Q} \) for every \( t \in I \).

As in the classical Pontryagin’s Maximum Principle, condition (23) and the definition of the \( \mathcal{A} \)-elementary perturbation vector in (19) prove the condition about the supremum of the Hamiltonian \( \mathcal{H}_A : (T_k^1)^* \hat{Q} \times U \rightarrow \mathbb{R} \) in (10) over the controls for each \( A \in \{1, \ldots, k\} \). The constancy of the supremum of the Hamiltonian over the controls is proved analytically, analogously to the classical Pontryagin’s Maximum Principle, see [3, 16, 19] for more details.

From Eq. (11) we deduce that for each \( A \in \{1, \ldots, k\} \), \( \beta^A_B \) is constant for every \( B \in \{1, \ldots, k\} \) along the optimal integral section \( \hat{\phi}^* = (\hat{\sigma}^*, u^*) \).

### 3 Application of unified formalism for \( k \)-cosymplectic to implicit PDEs

We are going to apply the unified Skinner-Rusk formalism for \( k \)-cosymplectic field theories developed in [17, Section 4] to the dynamics description for systems given by implicit partial differential equations. This will be very useful to develop Sects. 4 and 5 so that physical examples associated with higher order control partial differential equations fit in the approach considered in this paper.

The Whitney sum \( T_k^1 \hat{Q} \oplus (T_k^1)^* \hat{Q} \) has natural bundle structures over \( T_k^1 \hat{Q} \) and \( (T_k^1)^* \hat{Q} \). The suitable bundle to describe non-autonomous dynamical systems governed by partial differential equations is \( \mathcal{W} := \mathbb{R}^k \times (T_k^1 \hat{Q} \oplus (T_k^1)^* \hat{Q}) \). Local coordinates for \( \mathcal{W} \) are \((t^B, q^i, v^i_A, p^A_i)\). Let us denote by \( \text{pr}_1 : \mathbb{R}^k \times (T_k^1 \hat{Q} \oplus (T_k^1)^* \hat{Q}) \rightarrow \mathbb{R}^k, \text{pr}_2 : \mathbb{R}^k \times (T_k^1 \hat{Q} \oplus (T_k^1)^* \hat{Q}) \rightarrow T_k^1 \hat{Q} \) and \( \text{pr}_3 : \mathbb{R}^k \times (T_k^1 \hat{Q} \oplus (T_k^1)^* \hat{Q}) \rightarrow (T_k^1)^* \hat{Q} \) the local projections into the first, second and third factor of \( \mathcal{W} \), respectively. Locally,

\[
\begin{align*}
\text{pr}_1 \left( t^B, q^i, v^i_A, p^A_i \right) &= t^B, \\
\text{pr}_2 \left( t^B, q^i, v^i_A, p^A_i \right) &= (q^i, v^i_A) = (q, v), \\
\text{pr}_3 \left( t^B, q^i, v^i_A, p^A_i \right) &= (q^i, p^A_i) = (q, p).
\end{align*}
\]

Let \((d_t, \ldots, d_t) \) and \((\omega_1, \ldots, \omega_k)\) be the canonical forms on \( \mathbb{R}^k \times (T_k^1)^* \hat{Q} \). We denote by \((\vartheta^A, \ldots, \vartheta^A)\) and \((\Omega_1, \ldots, \Omega_k)\) the pullback by \( \text{pr}_1 \) and \( \text{pr}_3 \) of these forms to \( \mathbb{R}^k \times (T_k^1 \hat{Q} \oplus (T_k^1)^* \hat{Q}) \), that is, \( \vartheta^A = (\text{pr}_1)^*(d^A_t) \) and \( \Omega_A = (\text{pr}_3)^*(\omega_A) \) for \( 1 \leq A \leq k \). Locally,

\[
\vartheta^A = d^A_t, \quad \Omega_A = dq^i \wedge dp^A_i. \tag{24}
\]

The coupling function \( \mathcal{C} \) on \( \mathbb{R}^k \times (T_k^1 \hat{Q} \oplus (T_k^1)^* \hat{Q}) \) is defined as follows

\[
\mathcal{C} : \mathbb{R}^k \times T_k^1 \hat{Q} \oplus (T_k^1)^* \hat{Q} \rightarrow \mathbb{R}
\]

\[
(t, v_q, p_q) \mapsto \sum_{A=1}^{k} p^A_q(v_{Aq}) = \sum_{A=1}^{k} (p^A_q v^i_A).
\]

Given a Lagrangian function \( \mathbb{L} \) on \( \mathbb{R}^k \times T_k^1 \hat{Q} \), the Hamiltonian function \( \mathcal{H} \) on \( \mathbb{R}^k \times (T_k^1 \hat{Q} \oplus (T_k^1)^* \hat{Q}) \) is defined as follows

\[
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\]
The problem in the Skinner-Rusk formalism for $k$-cosymplectic field theories consists of finding integral sections $\phi: \mathbb{R}^k \rightarrow \mathbb{R}^k \times (T_1^k Q \oplus (T_1^k)^* Q)$ of an integrable $k$-vector field $Z = (Z_1, \ldots, Z_k)$ on $\mathbb{R}^k \times (T_1^k Q \oplus (T_1^k)^* Q)$ such that:

$$
\sum_{A=1}^{k} i_{Z_A} \omega_A = dH - \sum_{A=1}^{k} \frac{\partial H}{\partial t^A} \partial^A, \quad i_{Z_A} \partial^B = \delta^B_A.
$$

(26)

See [17] for more details.

After summarizing briefly the Skinner-Rusk formalism for $k$-cosymplectic field theories, here we are interested in adapting it to find the dynamics of systems described by implicit partial differential equations. An implicit dynamical system $(\mathbb{L}, M)$ is described by the submanifold

$$M = \left\{ \left( t^B, q^i, v^i_A \right) \in \mathbb{R}^k \times (T_1^k Q) \mid \Psi^\alpha \left( t^B, q^i, v^i_A \right) = 0, \ 1 \leq \alpha \leq s \right\},$$

of $\mathbb{R}^k \times T_1^k Q$, where $d\Psi^1 \wedge \cdots \wedge d\Psi^s \neq 0$, and a Lagrangian function $\mathbb{L} \in C^\infty(M)$. This submanifold $M$ of $\mathbb{R}^k \times T_1^k Q$ can be naturally embedded by $t^M: M \hookrightarrow \mathbb{R}^k \times T_1^k Q$.

In order to adapt the above formalism to this kind of dynamical systems, we must define the $k$-symplectic implicit bundle $\mathcal{W}^M = M \times_Q (T_1^k)^* Q$ and the corresponding canonical immersion

$$i^M: \mathcal{W}^M \hookrightarrow \mathcal{W} = \mathbb{R}^k \times (T_1^k Q \oplus (T_1^k)^* Q).$$

(27)

Now we can consider the pullback of the coupling function and the canonical forms on $\mathcal{W}$ to $\mathcal{W}^M$:

$$C^{\mathcal{W}^M} = \left( i^M \right)^* (C), \quad \partial^A_{\mathcal{W}^M} = \left( i^M \right)^* (\partial^A), \quad \omega^M_{\mathcal{W}^M} = \left( i^M \right)^* (\omega_A).$$

Let $\rho^M_1: \mathcal{W}^M \rightarrow M$ be the natural projection, we define the Hamiltonian function $H_{\mathcal{W}^M}: \mathcal{W}^M \rightarrow \mathbb{R}$ as follows

$$H_{\mathcal{W}^M} = C^{\mathcal{W}^M} - \left( \rho^M_1 \right)^* \mathbb{L}. $$

Analogously to (26), the problem of describing the dynamics of $(\mathbb{L}, M)$ consists of finding the integral sections $\phi: \mathbb{R}^k \rightarrow \mathcal{W}^M$ of an integrable $k$-vector field $Z = (Z_1, \ldots, Z_k)$ on $\mathcal{W}^M$ such that

$$\sum_{A=1}^{k} i_{Z_A} \omega^M_{\mathcal{W}^M} = dH_{\mathcal{W}^M} - \sum_{A=1}^{k} \frac{\partial H_{\mathcal{W}^M}}{\partial t^A} \partial^A_{\mathcal{W}^M}, \quad i_{Z_A} \partial^B = \delta^B_A.$$

(28)

Or equivalently, the problems consists of finding the integral sections $\phi: \mathbb{R}^k \rightarrow \mathcal{W}$ of an integrable $k$-vector field $Z = (Z_1, \ldots, Z_k)$ on $\mathcal{W} = \mathbb{R}^k \times (T_1^k Q \oplus (T_1^k)^* Q)$ such that

$$\sum_{A=1}^{k} i_{Z_A} \omega_A = dH_{\mathcal{W}^M} - \sum_{A=1}^{k} \frac{\partial H_{\mathcal{W}^M}}{\partial t^A} \partial^A_{\mathcal{W}^M}, \quad \omega^\alpha \left( t^B, q^i, v^i_A \right) = 0, \quad 1 \leq \alpha \leq s.$$

(29)

This equation is obtained from (28) by rewriting the equations on $\mathcal{W}$ so that the constraints $\psi^\alpha = 0$ in (27) must be added to the equation in a suitable way.
The $A$th vector field $Z_A$ on $\mathcal{W}$ is locally given by

$$Z_A = (Z_A)_B^i \frac{\partial}{\partial t^B} + (Z_A)_B^i \frac{\partial}{\partial q^i} + (Z_A)_B^i \frac{\partial}{\partial v^i_B} + (Y_A)_B^i \frac{\partial}{\partial p_A^B}.$$  

From (29) we first have $(Z_A)_B^i = 1$, $(Z_A)_B^i = 0$ for $B \neq A$. Moreover,

$$\sum_{A=1}^k i_{Z_A} \Omega_A - dH_{W^M} + \sum_{A=1}^k \frac{\partial H_{W^M}}{\partial t^A} \vartheta^A - \lambda_A d\Psi^\alpha + \lambda_A \sum_{A=1}^k \frac{\partial \Psi^\alpha}{\partial t^A} \vartheta^A = (Z_A)^i A d\delta^i + (Z_A)^i A - p_A^i A d\psi^i_A + \frac{\partial \lambda_A}{\partial v^i_A} d\psi^i_A = 0.$$  

Thus,

$$(Z_A)^i = v_A^i, \quad \sum_{A=1}^k (Y_A)^i A \vartheta^A = \frac{\partial \lambda_A}{\partial q^i} - \lambda_A \frac{\partial \psi^\alpha}{\partial q^i}, \quad (Y_A)^i A \vartheta^A = \frac{\partial \lambda_A}{\partial v^i_A} - \lambda_A \frac{\partial \psi^\alpha}{\partial v^i_A}.$$  

By also imposing the conditions (16) and (17) in the $k$-vector field on $\mathcal{W}$, we have

$$Z_A = \frac{\partial}{\partial t^A} + v_A^i \frac{\partial}{\partial q^i} + (Z_A)_B^i \frac{\partial}{\partial q^i} + (Y_A)_i A \frac{\partial}{\partial p_A^i},$$  

with

$$p_A^i = \frac{\partial \lambda_A}{\partial v^i_A} - \lambda_A \frac{\partial \psi^\alpha}{\partial v^i_A}, \quad \sum_{A=1}^k (Y_A)_i A \vartheta^A = \frac{\partial \lambda_A}{\partial q^i} - \lambda_A \frac{\partial \psi^\alpha}{\partial q^i}.$$  

If $Z$ is a solution of (29), then we must start a constraint algorithm in the sense of [7]. To be more precise, each $Z_A$ must be tangent to the submanifold $M_L$ contained in $W^M$ and defined by (32). That is, the following tangency conditions must be satisfied

$$0 = Z_A (\psi^\alpha), \quad 0 = Z_A p_A^B - \frac{\partial \lambda_A}{\partial v^i_B} + \lambda_A \frac{\partial \psi^\alpha}{\partial v^i_B}.$$  

on $M_L$ for every $A = 1, \ldots, k$. Depending on the particular examples, some components will be determined and the constraint algorithm must proceed until stabilization.

**Remark 31** For non-autonomous $k$-symplectic explicit dynamical systems the manifold $M$ is defined by constraints locally given by $\Psi^\alpha (t, q^i, v_A^i) = v_A^i - X_A (t, q^i) = 0$. Hence the above process can be used for this kind of dynamical systems.
4 Unified formalism for optimal control problems governed by an implicit partial differential equation

We extend now the unified formalism for implicit control systems developed in [2, Section 4] to optimal control problems whose dynamics is given by implicit control partial differential equations, instead of just explicit control partial differential equations as developed in Sect. 2. The problem consists of finding the solutions to optimal control problems governed by an implicit partial differential equation by taking advantage of the unified formalism developed in this section. See Sect. 5 for a particular problem where this unified formalism is used.

Let $C$ be the control bundle with natural coordinates $(t^A, q^i, u^a)$. In contrast to the explicit description of control partial differential equations in Sect. 2, let us consider now the case where the control partial differential equations are given implicitly by the following submanifold

$$MC = \left\{(t^B, u^a, q^i, v^j_A) \in C \times Q | \Psi^\alpha \left(t^B, u^a, q^i, v^j_A\right) = 0, \ 1 \leq \alpha \leq s\right\}$$

of $C \times Q$, where $d\Psi^1 \wedge \cdots \wedge d\Psi^s \neq 0$. There exists a natural embedding $ι_{MC}: MC \hookrightarrow C \times Q$. Then the implicit optimal control problem under consideration is determined by $(L, MC)$, where $L \in C^\infty(MC)$ is a Lagrangian function.

Let us define now the $k$-symplectic implicit control bundle $W^{MC} = MC \times (T^1_k)^*Q$ which is a submanifold of $C \times (\mathbb{R}^k \times Q)^* \cong (\mathbb{R}^{k+1} \times (T^1_k Q \oplus (T^1_k)^*Q))$. Then we have, respectively, the canonical immersion and the natural projection:

$$ι_{MC}: W^{MC} \hookrightarrow C \times (\mathbb{R}^k \times Q), \ \sigma_{W} : C \times (\mathbb{R}^k \times Q) \rightarrow W.$$

Now we can consider the pullback of the coupling function in Sect. 3 and the canonical forms on $W$ to $W^{MC}$:

$$C^{W^{MC}} = (σ_W \circ ι_{MC})^*(C), \ \Omega_A^{W^{MC}} = (σ_W \circ ι_{MC})^*(Ω_A), \ \bar{ϕ}^A_{W^{MC}} = (σ_W \circ ι_{MC})^*(ϕ^A)$$

Let $ρ_{MC}^1 : W^{MC} \rightarrow MC$ be the natural projection, the Hamiltonian function $H_{W^{MC}} : W^{MC} \rightarrow \mathbb{R}$ is defined as follows

$$H_{W^{MC}} = C^{W^{MC}} - (ρ_{MC}^1)^*L.$$

The dynamics of the optimal control problem $(L, MC)$ is determined by the solutions of the equations

$$\sum_{A=1}^k i_{Z_A} (Ω_A^{W^{MC}}) = 0, \ i_{Z_A} \bar{ϕ}^B_{W^{MC}} = δ^B_A,$$

for a $k$-vector field $Z = (Z_1, \ldots, Z_k)$ on $W^{MC}$.

In order to work in local coordinates we need the following proposition whose proof is straightforward.

**Proposition 4** For a given $w \in W^{MC}$, the following conditions are equivalent:

1. There exists a $k$-vector field $Z_w \in (T^1_k)^wW^{MC}$ verifying that

$$\sum_{A=1}^k Ω_A^{W^{MC}} ((Z_A)_w, (Y_A)_w) = 0, \ \text{for every } Y_w \in (T^1_k)^wW^{MC}.$$
(2) There exists a $k$-vector field $Z_w \in (T^1_k)_{w}(C \times \mathbb{R}^k \times \mathcal{Q})$ verifying that

(i) $Z_w \in (T^1_k)_w \mathcal{W}^M_C$,

(ii) $\sum_{A=1}^{k} i(Z_A)(\sigma^*_W(\Omega_A))_w \in ((T^1_k)_w \mathcal{W}^M_C)^0$.

As a consequence of this last proposition, we can obtain the implicit optimal control equations using condition (2) in Proposition 4 as follows: there exists a $k$-vector field $Z$ on $C \times \mathbb{R}^k \times \mathcal{Q}$ $\mathcal{W}$ such that

(i) $Z$ is tangent to $\mathcal{W}^M_C$;

(ii) the 1-form $\sum^{k}_{A=1} i(Z_A)(\sigma^*_W(\Omega_A))$ is null on the $k$-vector fields tangent to $\mathcal{W}^M_C$.

As usual, the undetermined functions $\lambda_{\alpha}$’s are called Lagrange multipliers.

Now using coordinates $(t^B, u^a, q^i, v^i_A, p^A_i)$ in $C \times \mathbb{R}^k \times \mathcal{Q}$ $\mathcal{W}$, we look for $k$ vector fields

$$Z_A = (Z_A)^B \frac{\partial}{\partial t^B} + (Z_A)^a \frac{\partial}{\partial u^a} + (Z_A)^i \frac{\partial}{\partial q^i} + (Z_A)^i_B \frac{\partial}{\partial v^i_B} + (Y_A)^i_B \frac{\partial}{\partial p^i_B},$$

where $(Z_A)^B, (Z_A)^a, (Z_A)^i, (Z_A)^i_B, (Y_A)^i_B$ are unknown functions on $\mathcal{W}^M_C$ verifying the equation

$$0 = \sum_{A=1}^{k} i_{Z_A} \left( dq^i \wedge dp^A_i - d \left( \sum_{A=1}^{k} (p^A_i v^i_A) - \mathbb{L}(t, u, q, v) \right) \right)$$

$$+ \frac{\partial H_{\mathcal{W}^M_C}}{\partial t^A} dt^A - \lambda_{\alpha} d\Psi^\alpha + \lambda_{\alpha} \frac{\partial \Psi^\alpha}{\partial t^A} dt^A$$

$$= \left( - \sum_{A=1}^{k} (Y_A)^a_i - \lambda_{\alpha} \frac{\partial \Psi^\alpha}{\partial q^i} + \frac{\partial \mathbb{L}}{\partial q^i} \right) dq^i + \left( \frac{\partial \mathbb{L}}{\partial u^a} - \lambda_{\alpha} \frac{\partial \Psi^\alpha}{\partial u^a} \right) du^a$$

$$+ \left( - p^A_i + \frac{\partial \mathbb{L}}{\partial v^i_A} - \lambda_{\alpha} \frac{\partial \Psi^\alpha}{\partial v^i_A} \right) dv^i_A + ((Z_A)^i - v^i_A) dp^A_i.$$

Note that from (35) we have $(Z_A)^A_i = 1$ and $(Z_A)^B_i = 0$ for $A \neq B$. Moreover,

$$\sum_{A=1}^{k} (Y_A)^A_i = \frac{\partial \mathbb{L}}{\partial q^i} - \lambda_{\alpha} \frac{\partial \Psi^\alpha}{\partial q^i},$$

$(Z_A)^i = v^i_A,$

$p^A_i = \frac{\partial \mathbb{L}}{\partial v^i_A} - \lambda_{\alpha} \frac{\partial \Psi^\alpha}{\partial v^i_A},$

$$0 = \frac{\partial \mathbb{L}}{\partial u^a} - \lambda_{\alpha} \frac{\partial \Psi^\alpha}{\partial u^a},$$

together with the tangency conditions.

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for every $A = 1, \ldots, k$. Imposing the conditions (16), (17), we know that $(Y_A)^B = 0$ for $A \neq B$. From here we can start a constraint algorithm in the sense of [7] as follows: the tangency conditions with respect to the constraints (39) and (40) obtained from Eq. (36) give the following equations on $\mathcal{W}^{MC}$:

$$0 = Z_A \left( \frac{\partial L}{\partial u^a} - \lambda_\alpha \frac{\partial \psi^\alpha}{\partial u^a} \right)$$

$$= (Y_A)^A \delta_B^A - \frac{\partial^2 L}{\partial t^A \partial v^j_B} v^j_A - \frac{\partial^2 L}{\partial q^j \partial u^a} (Z_A)^j_C - \frac{\partial^2 L}{\partial v^B_C \partial v^j_B} (Z_A)^j_C - \frac{\partial^2 L}{\partial u^a \partial v^j_B} (Z_A)^j_C + \lambda_\alpha \frac{\partial^2 \psi^\alpha}{\partial t^A \partial u^a} - \frac{\partial^2 \psi^\alpha}{\partial t^A \partial u^a} - \frac{\partial^2 \psi^\alpha}{\partial v^B_C \partial u^a} (Z_A)^j_C - \lambda_\alpha \frac{\partial^2 \psi^\alpha}{\partial t^A \partial u^a} - \frac{\partial^2 \psi^\alpha}{\partial t^A \partial u^a} - \frac{\partial^2 \psi^\alpha}{\partial v^B_C \partial u^a} (Z_A)^j_C - \lambda_\alpha \frac{\partial^2 \psi^\alpha}{\partial t^A \partial u^a} + \lambda_\alpha \frac{\partial^2 \psi^\alpha}{\partial t^A \partial u^a}$$

$$= \frac{\partial^2 \psi^\alpha}{\partial t^A \partial u^a} - \frac{\partial^2 \psi^\alpha}{\partial t^A \partial u^a} - \frac{\partial^2 \psi^\alpha}{\partial t^A \partial u^a} - \frac{\partial^2 \psi^\alpha}{\partial t^A \partial u^a}$$

$$= \frac{\partial^2 \psi^\alpha}{\partial t^A \partial u^a} - \frac{\partial^2 \psi^\alpha}{\partial t^A \partial u^a} - \frac{\partial^2 \psi^\alpha}{\partial t^A \partial u^a} - \frac{\partial^2 \psi^\alpha}{\partial t^A \partial u^a}$$

If the square matrix of size $k + nk$

$$\begin{pmatrix}
\frac{\partial^2 L}{\partial u^a \partial u^a} - \lambda_\alpha \frac{\partial^2 \psi^\alpha}{\partial u^a \partial u^a} & \frac{\partial^2 L}{\partial u^a \partial u^a} - \lambda_\alpha \frac{\partial^2 \psi^\alpha}{\partial u^a \partial u^a} \\
- \frac{\partial^2 \psi^\alpha}{\partial u^a \partial u^a} + \lambda_\alpha \frac{\partial^2 \psi^\alpha}{\partial u^a \partial u^a} & - \frac{\partial^2 \psi^\alpha}{\partial u^a \partial u^a} + \lambda_\alpha \frac{\partial^2 \psi^\alpha}{\partial u^a \partial u^a}
\end{pmatrix}$$

has maximum rank, $(Z_A)^b$ and $(Z_A)^j_B$ are determined in terms of $(Y_A)^A$, which must satisfy the condition (37) coming from (36). The algorithm continues until stabilization.

5 Example: orientation of a bipolar molecule in the plane by means of two external fields

Let us consider now the control partial differential equation studied in [6, Section 8]:

$$i \frac{\partial \Psi(t, \theta)}{\partial t} = \left( -\frac{\partial^2 \Psi(t, \theta)}{\partial \theta^2} + u_1(t) \cos \theta \Psi(t, \theta) + u_2(t) \sin \theta \Psi(t, \theta) \right), \quad (42)$$

where $\Psi$ is an element in a Hilbert space taking values on the complex and $u_1, u_2$ take values in $\mathbb{R}$.

Let us rewrite the problem according to Sect. 4. This equation fits in 2-symplectic formalism where $t^1 = t$ and $t^2 = \theta$. Note that (42) is a partial differential equation on the complex numbers. Hence let us rename

$$i \frac{\partial \Psi(t, \theta)}{\partial t} = \left( -\frac{\partial^2 \Psi(t, \theta)}{\partial \theta^2} + u_1(t) \cos \theta \Psi(t, \theta) + u_2(t) \sin \theta \Psi(t, \theta) \right), \quad (42)$$

where $\Psi$ is an element in a Hilbert space taking values on the complex and $u_1, u_2$ take values in $\mathbb{R}$.

Let us rewrite the problem according to Sect. 4. This equation fits in 2-symplectic formalism where $t^1 = t$ and $t^2 = \theta$. Note that (42) is a partial differential equation on the complex numbers. Hence let us rename
\[ q^1 = \text{Re}\Psi, \quad q^2 = \text{Im}\Psi. \]

In order to rewrite (42) as an implicit partial differential equation we work on a 6-dimensional manifold \( Q \) with local coordinates

\[
\left( q^1, q^2, q^3 = \frac{\partial q^1}{\partial t}, q^4 = \frac{\partial q^2}{\partial t}, q^5 = \frac{\partial q^1}{\partial \theta}, q^6 = \frac{\partial q^2}{\partial \theta} \right)
\]
to transform the second order partial differential equation into first order partial differential equations.

The local coordinates for \( C \times_Q T_2^1 Q \) are \((t^1, t^2, u^1, u^2, q^i, v_1^i, v_2^i)\). Note that apart from (42) we also know that

\[
v_1 = \left( \frac{\partial q^1}{\partial t}, \frac{\partial q^2}{\partial t}, \frac{\partial^2 q^1}{\partial t \partial t}, \frac{\partial^2 q^2}{\partial t \partial t}, \frac{\partial^2 q^1}{\partial \theta \partial t}, \frac{\partial^2 q^2}{\partial \theta \partial t} \right),
\]

\[
v_2 = \left( \frac{\partial q^1}{\partial \theta}, \frac{\partial q^2}{\partial \theta}, \frac{\partial^2 q^1}{\partial t \partial \theta}, \frac{\partial^2 q^2}{\partial t \partial \theta}, \frac{\partial^2 q^1}{\partial \theta \partial \theta}, \frac{\partial^2 q^2}{\partial \theta \partial \theta} \right).
\]

Hence (43) determines some relationships between some coordinates of \( v_1 \) and \( v_2 \). Equations (42) and (43), determine a submanifold \( M_C \) of \( C \times_Q T_2^1 Q \) implicitly defined by the following constraints:

\[
\begin{align*}
\Psi_1 & = v_1 - q^3, \quad \Psi_4 = v_2^2 - q^6, \quad \Psi_7 = -q^3 - v_2^2 + u_1 q^2 \cos \theta + u_2 q^2 \sin \theta, \\
\Psi_2 & = v_1^2 - q^4, \quad \Psi_5 = v_2^3 - v_2, \quad \Psi_8 = q^4 - v_2^2 + u_1 q^1 \cos \theta + u_2 q^1 \sin \theta.
\end{align*}
\]

A general 2-vector field \( Z \) on \( C \times_{\mathbb{R}^2 \times Q} (\mathbb{R}^2 \times (T_2^1 Q \oplus (T_2^1 Q))) \) is locally given by

\[
Z_A = (C_A)^i \frac{\partial}{\partial t} + (C_A)^i \frac{\partial}{\partial \theta} + (D_A)_a \frac{\partial}{\partial u_a} + (E_A)^i \frac{\partial}{\partial q^i} + (F_A)_B \frac{\partial}{\partial v^B} + (G_A)_B \frac{\partial}{\partial p^B}.
\]

Assume that the cost function is control-quadratic in the following way \( L = \frac{1}{2}(u_1^2 + u_2^2) \).

From (35), (36) we have

\[
\begin{align*}
i_{Z_A} dB_A & \rightarrow (C_1)^1 = 1, \quad (C_1)^2 = 0, \quad (C_2)^1 = 0, \quad (C_2)^2 = 1, \\
dp^i_A & \rightarrow (E_A)^i = v^i_A, \\
dq^1 & \rightarrow (G_1)^1 + (G_2)^2 = -\lambda_8 (u_1 \cos \theta + u_2 \sin \theta), \\
dq^2 & \rightarrow (G_1)^2 + (G_2)^1 = -\lambda_7 (u_1 \cos \theta + u_2 \sin \theta), \\
dq^3 & \rightarrow (G_1)^3 + (G_2)^3 = \lambda_1 + \lambda_7, \\
dq^4 & \rightarrow (G_1)^4 + (G_2)^4 = \lambda_2 - \lambda_8, \\
dq^5 & \rightarrow (G_1)^5 + (G_2)^5 = \lambda_3, \\
dq^6 & \rightarrow (G_1)^6 + (G_2)^6 = \lambda_4, \\
dv^i_A & \rightarrow \lambda_1 + p^1 = 0, \quad \lambda_2 + p^2 = 0, \quad \lambda_3 + p^2 = 0, \quad \lambda_4 + p^1 = 0, \\
\lambda_5 + p^2 = 0, \quad \lambda_5 + p^1 = 0, \quad \lambda_6 + p^2 = 0, \quad \lambda_6 + p^1 = 0, \\
\lambda_7 + p^2 = 0, \quad \lambda_8 + p^2 = 0, \quad p^1 = 0, \quad p^1 = 0, \\
du_1 & \rightarrow \lambda_7 q^2 \cos \theta + \lambda_8 q^1 \cos \theta - u_1 = 0, \\
du_2 & \rightarrow \lambda_7 q^2 \sin \theta + \lambda_8 q^1 \sin \theta - u_2 = 0.
\end{align*}
\]
Hence all the controls and Lagrange multipliers are determined:

\[ \begin{align*}
\lambda_1 &= -p_1^1, \quad \lambda_2 = -p_2^1, \quad \lambda_3 = -p_1^2, \quad \lambda_4 = -p_2^2, \quad \lambda_5 = -p_3^2 = p_5^1, \\
\lambda_6 &= -p_2^1 = p_6^1, \quad \lambda_7 = p_6^2, \quad \lambda_8 = p_5^2, \\
u_1 &= p_6^2 q^2 \cos \theta + p_5^2 q^1 \cos \theta, \quad u_2 = p_6^2 q^2 \sin \theta + p_5^2 q^1 \sin \theta.
\end{align*} \]

The cost function can be written as follows:

\[ L = \frac{1}{2} (u_1^2 + u_2^2) = (p_6^2 q^2 + p_5^2 q^1)^2. \]

Since \( p_3^1 = 0 \) and \( p_4^1 = 0 \), we have \( Z_A(p_3^1) = Z_A(p_4^1) = 0 \) for \( A = 1, 2 \). Then, \( (G_1)_3^1 = (G_2)_3^1 = (G_2)_4^1 = (G_1)_4^1 = 0 \). Having this in mind, we have

\[ \begin{align*}
(G_1)_1^1 &= -(G_2)_1^2 - p_5^1 \left( p_6^2 q^2 + p_5^2 q^1 \right), \\
(G_1)_2^2 &= -(G_2)_2^2 - p_6^1 \left( p_6^2 q^2 + p_5^2 q^1 \right), \\
(G_2)_3^2 &= -p_1^1 + p_6^2, \\
(G_2)_4^2 &= -p_1^1 - p_2^2, \\
(G_1)_5^3 &= -(G_2)_5^2 - p_2^1, \\
(G_1)_6^6 &= -(G_2)_6^2 - p_2^2.
\end{align*} \] (44)

Note that the controls satisfy the following relationship \( u_1 \sin \theta = u_2 \cos \theta \). If we impose the tangency condition, we have

\[ \begin{align*}
Z_1(u_1 \sin \theta - u_2 \cos \theta) &= (D_1)_1 \sin \theta - (D_1)_2 \cos \theta = 0, \\
Z_2(u_1 \sin \theta - u_2 \cos \theta) &= u_1 \cos \theta + u_2 \sin \theta + (D_2)_1 \sin \theta - (D_2)_2 \cos \theta = 0.
\end{align*} \]

Thus, \((D_1)_1 = \cos \theta, \quad (D_1)_2 = \sin \theta \) and \((D_2)_1 \sin \theta - (D_2)_2 \cos \theta = -p_6^2 q^2 - p_5^2 q^1 \).

By conditions (16), (17) we have that \((G_A)_i^B = 0 \) for \( A \neq B \).

After imposing the tangency conditions in (41) we obtain

\[ \begin{align*}
Z_A(\Psi^1) &= (F_A)_1^1 - v_A^3 = 0, \quad Z_A(\Psi^2) = (F_A)_1^2 - v_A^4 = 0, \\
Z_A(\Psi^3) &= (F_A)_2^1 - v_A^5 = 0, \quad Z_A(\Psi^4) = (F_A)_2^2 - v_A^6 = 0, \\
Z_A(\Psi^5) &= (F_A)_3^3 - (F_A)_4^4 = 0, \quad Z_A(\Psi^6) = (F_A)_4^2 - (F_A)_5^6 = 0, \\
Z_A(\Psi^7) &= -v_A^3 - (F_A)_5^6 + (D_A)_1 q^2 \cos \theta - \delta^A u_1 q^2 \sin \theta + v_A^2 u_1 \cos \theta \\
&\quad + (D_A)_2 q^2 \sin \theta + \delta^A u_2 q^2 \cos \theta + v_A^2 u_2 \sin \theta = 0, \\
Z_A(\Psi^8) &= v_A^3 - (F_A)_5^5 + (D_A)_1 q^1 \cos \theta - \delta^A u_1 q^1 \sin \theta + v_A^1 u_1 \cos \theta \\
&\quad + (D_A)_2 q^1 \sin \theta + \delta^A u_2 q^1 \cos \theta + v_A^1 u_2 \sin \theta = 0,
\end{align*} \]

Thus,

\[ \begin{align*}
Z_A &= \frac{\partial}{\partial t^A} + v_A^i \frac{\partial}{\partial q^i} + (D_A)_a \frac{\partial}{\partial u_a} + v_A^3 \frac{\partial}{\partial v_A^1} + v_A^4 \frac{\partial}{\partial v_A^2} + v_A^5 \frac{\partial}{\partial v_A^3} + v_A^6 \frac{\partial}{\partial v_A^4} \\
&\quad + (F_A)_1^5 \left( \frac{\partial}{\partial v_A^2} + \frac{\partial}{\partial v_A^1} \right) + (F_A)_1^6 \left( \frac{\partial}{\partial v_A^4} + \frac{\partial}{\partial v_A^5} \right) \\
&\quad + \left( v_A^4 + (D_A)_1 q^1 \cos \theta - \delta^A u_1 q^1 \sin \theta + v_A^1 u_1 \cos \theta + (D_A)_2 q^1 \sin \theta \right)
\end{align*} \]
\[
+ \delta_2^4 u_2 q^1 \cos \theta + v_A^1 u_2 \sin \theta \frac{\partial}{\partial v_2^3} \\
+ \left( -v_A^3 + (D_A)_1 q^2 \cos \theta - \delta_2^4 u_1 q^2 \sin \theta + v_A^2 u_1 \cos \theta + (D_A)_2 q^2 \sin \theta \\
+ \delta_2^4 u_2 q^2 \cos \theta + v_A^2 u_2 \sin \theta \right) \frac{\partial}{\partial v_2^6} \\
+(F_A)_3 \frac{\partial}{\partial v_1^3} + (F_A)_4 \frac{\partial}{\partial v_1^4} + (G_A)_i \frac{\partial}{\partial p_i^A},
\]

where \( t^1 = t, t^2 = \theta, (D_1)_1 = \cos \theta, (D_1)_2 = \sin \theta \) and \( (D_2)_1 \sin \theta - (D_2)_2 \cos \theta = -p_6^2 q^2 - p_5^2 q^1 \) and also Eqs. (44) are satisfied. The optimal sections are integral sections of \( Z = (Z_1, \ldots, Z_k) \).

### 6 Future work

After this first geometric approach to optimal control problems governed by partial differential equations, it remains open to find the way to successfully extend any control system regardless of the nature of the cost function. The main difficulty is to obtain a compatible system of partial differential equations after extending the original control system.

In this paper we have not mentioned the different kind of extremals for optimal control problems. There exist the so-called abnormal extremals which are characterized at first without considering the cost function. As shown in [4], the constraint algorithm in the sense of Gotay–Nester–Hinds is useful to characterize the different kind of extremals in optimal control theory. Now, that the optimal control problems governed by partial differential equations have been understood in the \( k \)-symplectic framework, it seems that the application of the constraint algorithm for \( k \)-presymplectic Hamiltonian systems [8] will characterize the extremals of those problems.

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