New Isothermic surfaces

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Abstract

In this paper, we consider a method of constructing isothermic surfaces based on Ribaucour transformations. By applying the theory to the cylinder, we obtain a three-parameter family of complete isothermic surfaces that contains n-bubble surfaces inside and outside of the cylinder. In addition, we also obtain one-parameter family of complete isothermic surface with planar ends. Such family of isothermic surfaces do not have constant mean curvature. As application we obtain explicit solutions of the Calapso equation.

Introduction

Ribaucour transformations for hypersurfaces, parametrized by lines of curvature, were classically studied by Bianchi [1]. They can be applied to obtain surfaces of constant Gaussian curvature and surfaces of constant mean curvature, from a given such surface, respectively, with constant Gaussian curvature and constant mean curvature. The first application of this method to minimal surfaces in $\mathbb{R}^3$ was obtained by Corro, Ferreira, and Tenenblat in [7]-[8]. For more application this method, see [9]-[10], [17], [22], [23] and [24].

Using Ribaucour transformations and applying the theory to the cylinder, [8] obtained a two-parameter family of complete linear Weingarten surfaces, that contains n-bubble surfaces inside of the cylinder.

A regular surface $M$ is isothermic if locally, near each non umbilic point of $M$ there exist curvature line coordinates which are conformal with respect to the first fundamental form of $M$.

The study of isothermic surfaces is a very difficult problem because it depends on the integration of an equation with partial derivatives of fourth order (see [25]). Particular classes of these surfaces can be obtained using some transformations from a given isothermic surface.

The theory of isothermic surfaces was studied by eminent geometries as Christoffel [5], Darboux [15]-[16] and Bianchi [11] among others. Particular classes of isothermic surfaces are, the constant mean curvature surfaces, quadrics surfaces, surfaces whose lines of curvature has constant geodesic curvature, in particular, the cyclides of Dupin. The isothermic surfaces are preserved by isometries, dilations and inversions.

In [2] the authors study surfaces with harmonic inverse mean curvature (HIMC surfaces). They distinguish a subclass of $\theta$ – isothermic surfaces, and if $\theta = 0$ then surfaces are isothermic.

In [4] the author show that theory of soliton surfaces, modified in an appropriate way, can be applied also to isothermic immersions in $\mathbb{R}^3$. In this case the so called Sym’s formula gives an
explicit expression for the isothermic immersion with prescribed fundamental forms. The complete classification of the isothermic surfaces is an open problem.

In [3], the author establishes an equation with fourth order partial derivatives from which the problem of obtaining isothermic surfaces apparently becomes much simpler. Such equation (called Calapso equation) defined in [3] given by

\[
\left(\frac{\omega_{1u_1u_2}}{w}\right)_{u_1u_1} + \left(\frac{\omega_{1u_1u_2}}{w}\right)_{u_2u_2} + (\omega^2)_{u_1u_2} = 0,
\]

describes isothermic surfaces in \( \mathbb{R}^3 \).

In [13] the authors introduced the class of radial inverse mean curvature surface (RIMC-surfaces), that are isothermics surfaces. Moreover, were obtained two solutions of the Calapso equation where one can be obtained using [3] and a different one.

In this paper, motivated by [8] and [13] we use the Ribaucour transformations to get a family of isothermic surfaces from a given such surface. As an application of the theory, we obtain a family of complete isothermic surfaces associated to the cylinder with no constant mean curvature.

The families we obtain depend on one or three parameters. One of them is the parameter \( c \neq 0 \) of the Ribaucour transformation. According to the value of \( c \), we can have \( n \)-bubbles surface or not. Precisely, if \( \sqrt{|1+c|} = n/m \) an irreducible rational number, we get an \( n \)-bubble surface. This is an immersed cylinder into \( \mathbb{R}^3 \), with two ends of geometric index \( m \) and \( n \) isolated points of maximum and of minimum for the Gaussian curvature. Otherwise, if \( \sqrt{|1+c|} \) is not a rational number, then isothermic surface is a complete immersion of \( \mathbb{R}^2 \) into \( \mathbb{R}^3 \), not periodic in any variable. Another parameter is \( b \), and depending on the \( b \) sign, we have \( n \)-bubbles surface inside or outside of the cylinder. The last parameter, appear from integrating the Ribaucour transformation. In the family depending on one-parameter, we have planar ends. The position of the planar ends is affect by the value of the parameter. We show explicit examples that check for planar ends and bubbles surfaces.

Also, in this work motivated by [13], for each isothermic surfaces obtained by the Ribaucour transformations, we associates a solution of the Calapso equation. We give explicit solutions of the Calapso equation that depend on functions of a single variable. Applying isometries, dilations, inversions, we obtain new isothermic surfaces. So we get new solutions of the Calapso equation.

1. Ribaucour transformation for isothermic surfaces

In this section, we first recall the theory of Ribaucour transformation for surfaces (see [1] and [7] for more details).

A isothermic surface of \( \mathbb{R}^3 \) is a surface which locally has an orthogonal coordinate system with same coefficients of the first quadratic form.

Let \( M \) be an orientable surface of \( \mathbb{R}^3 \) without umbilic points, with Gauss map we denote by \( N \). We say that \( \tilde{M} \) is associated to \( M \) by a Ribaucour transformation, if and only if, there exists a differentiable function \( h \) defined on \( M \) and a diffeomorphism \( \psi : M \to \tilde{M} \) such that

(a) for all \( p \in M \), \( p + h(p)N(p) = \psi(p) + h(p)N(\psi(p)) \), where \( N \) is the Gauss map of \( \tilde{M} \).
(b) The subset \( p + h(p)N(p) \), \( p \in M \), is a two-dimensional submanifold.
(c) \( \psi \) preserves lines of curvature.

We say that \( \tilde{M} \) is locally associated to \( M \) by a Ribaucour transformation if, for all \( \tilde{p} \), there exists a neighborhood of \( \tilde{p} \) in \( \tilde{M} \) which is associated by a Ribaucour transformation to an open subset of \( M \).

The following result gives a characterization of Ribaucour transformations. For the proof and more details, see [8].
Theorem 1.1. Let $M$ be an orientable surface of $\mathbb{R}^3$, without umbilic points, whose Gauss map is $N$. Let $e_i$, $1 \leq i \leq 2$ be orthonormal principal directions, $-\lambda_i$ the corresponding principal curvatures, i.e. $dN(e_i) = \lambda_i e_i$. A surface $\tilde{M}$ is associated to $M$ by a Bibaucour transformation, if and only if, $M$ and $\tilde{M}$ are associated by a sphere congruence whose radius function $h : M \to \mathbb{R}$ satisfies $1 + h\lambda_i \neq 0$ and
\[
dZ^i(e_i) + Z^i w_{i,j} - Z^j Z^i \lambda_i = 0, \quad 1 \leq i \neq j \leq 2, \tag{1}\]
where
\[
Z^i = \frac{dh(e_i)}{1 + h\lambda_i},
\]
and $w_{i,j}$ are the connection forms of the frame $e_i$.

Remark 1.2 In a local coordinate system by lines of curvature, the function $h(u_1, u_2)$ is a differentiable function which satisfies a second-order nonlinear partial differential equation corresponding to (1). One can linearize the problem of obtaining the function $h$. This is a consequence of the following result (see [9] for a proof and more details)

Proposition 1.2 If $h$ is a nonvanishing function which satisfies (1), then
\[
\frac{1}{h} \sum_{i=1}^{2} Z^i w_i,
\]
is a closed 1-form and there is a nonvanishing function $\Omega$, defined on a simply connected domain, such that
\[
d\Omega(e_i) = \frac{\Omega}{h} Z^i.
\]

For each nonvanishing function $h$, which is a solution of (1), we consider $\Omega$ as above and we define
\[
\Omega_i = d\Omega(e_i), \quad W = \frac{\Omega}{h}.
\]

With this notation,
\[
dh(e_i) = \frac{\Omega_i}{W} \left( 1 + \frac{\Omega \lambda_i}{W} \right), \quad 1 + h\lambda_i = 1 + \frac{\Omega \lambda_i}{W}, \quad Z^i = \frac{\Omega_i}{W}. \tag{2}
\]

The next result shows that (1) is equivalent to a linear system, for more details, see [8].

Proposition 1.4 A function $h$ is a solution of (1) defined on a simply connected domain, if and only if, $h = \frac{\Omega}{W}$, where $\Omega$ and $W$ are functions which satisfy
\[
d\Omega_i(e_i) = \Omega_j w_{i,j}(e_j), \quad \text{for } i \neq j, \tag{3}
\]
\[
d\Omega = \sum_{i=1}^{n} \Omega_i w_i, \tag{4}
\]
\[
dW = - \sum_{i=1}^{n} \Omega_i \lambda_i w_i. \tag{5}
\]
One can show that the Ribaucour transformation of a surface is given in terms of the solution of the above system (see [7] for a proof and more details)

**Theorem 1.5** Let $M$ be an orientable surface of $\mathbb{R}^3$ parametrized by $X : U \subseteq \mathbb{R}^2 \rightarrow M$, without umbilic points. Assume $e_i, 1 \leq i \leq 2$ are orthogonal principal directions, $-\lambda_i$ the corresponding principal curvatures, and $N$ is a unit vector field normal to $M$. A surface $\tilde{M}$ is locally associated to $M$ by a Ribaucour transformation if and only if there is differentiable functions $W, \Omega, \Omega_i : V \subseteq U \rightarrow \mathbb{R}$ which satisfy (3), (4), (5), $W(W + \lambda_i \Omega) \neq 0$ and $\tilde{X} : V \subseteq U \rightarrow \tilde{M}$, is a parametrization of $\tilde{M}$ given by

$$\tilde{X} = X - \frac{2\Omega}{S} \left( \sum_{i=1}^{2} \Omega_i e_i - WN \right),$$

where

$$S = \sum_{i=1}^{2} (\Omega_i)^2 + W^2.$$  \hspace{1cm} (7)

Moreover, the normal map of $\tilde{X}$ is given by

$$\tilde{N} = N + \frac{2W}{S} \left( \sum_{i=1}^{2} \Omega_i e_i - WN \right),$$

and the principal curvatures and coefficients of the first fundamental form of $\tilde{X}$, are given by

$$\tilde{\lambda}_i = \frac{WT_i + \lambda_i S}{S - \Omega T_i},$$

$$g_{ii} = \left( \frac{S - \Omega T_i}{S} \right)^2 g_{ii}$$

where $\Omega_i$, $\Omega$ and $W$ satisfy (3), (4), (5), $S$ is given by (7), $g_{ii}, 1 \leq i \leq 2$ are coefficients of the first fundamental form of $X$, and

$$T_i = 2 \left( d\Omega_i(e_i) + \sum_k \Omega_k w_{ki}(e_i) - W\lambda_i \right).$$ \hspace{1cm} (11)

**Remark 1.6** If $M$ is parametrized by orthogonal curvature lines, we will assume that $e_i$ are given by $\frac{\nabla_{a_i}}{a_i}$, where $a_i = \sqrt{g_{ii}}$. In this case, system (3), (4) and (5) can be rewritten by

$$\Omega_{i,j} = \Omega_j \frac{a_j}{a_{j,i}}, \text{ for } i \neq j,$$

$$\Omega_{i} = a_i \Omega_i,$$

$$W_{i} = -a_i \Omega_i \lambda_i.$$ \hspace{1cm} (12)

The two result following provides a sufficient condition for a Ribaucour transformation to transform a isothermic surface into another such surface. The first one is

**Theorem 1.7** Let $M$ be a surfaces of $\mathbb{R}^3$ parametrized by $X : U \subseteq \mathbb{R}^2 \rightarrow M$, without umbilic points and let $\tilde{M}$ parametrized by (6) be associated to $M$ by a Ribaucour transformation, such that
the normal lines intersect at a distance function $h$. Assume that $h = \frac{\Omega}{W}$ is not constant along the lines of curvature and the function $\Omega$, $\Omega_i$ and $W$ satisfy one of the additional relation

$$T_1 + T_2 = \frac{2S}{\Omega} \quad \text{or} \quad T_1 - T_2 = 0 \quad (13)$$

where $S$ is given by $\frac{\Omega}{2}$ and $T_i$, $1 \leq i \leq 2$ are defined by $\frac{\Omega}{14}$. Then $\widetilde{M}$ parameterized by $\frac{6}{1}$ is a isothermic surface, if and only if $M$ is isothermic surface.

**Proof:** Suppose that $\widetilde{M}$ is a isothermic surface, then the coefficients of the first fundamental form of $\widetilde{X}$ satisfy, $g_{11} = g_{22}$. So, using $\frac{10}{14}$, we have

$$\left(\frac{S - \Omega T_1}{S}\right)^2 g_{11} = \left(\frac{S - \Omega T_2}{S}\right)^2 g_{22}, \quad (14)$$

where $g_{ii}$, $1 \leq i \leq 2$ are the coefficients of the first fundamental form of $X$.

If $T_1 + T_2 = \frac{2S}{\Omega}$, then isolating $T_1$ and substituting in $\frac{14}{14}$, we get $g_{11} = g_{22}$. On the other hand, if $T_1 - T_2 = 0$, then we have from $\frac{14}{14}$ that $g_{11} = g_{22}$. Therefore, $M$ is a isothermic surface.

Conversely, suppose that $M$ is a isothermic surface, then using $\frac{13}{13}$, immediately from $\frac{10}{10}$, we obtain that $\widetilde{M}$ is a isothermic surface.

**Remark 1.8** Let $X : U \subseteq R^2 \rightarrow M$ a isothermic parametrized for $M$. So, the first fundamental form of $X$ is given by $I = e^{2\varphi}(du_1^2 + du_2^2)$. Thus, the first additional relation of $\frac{13}{13}$ is equivalent to

$$\Delta \Omega - e^{2\varphi}(\lambda_1 + \lambda_2)W = \frac{Se^{2\varphi}}{\Omega} \quad (15)$$

In fact, under these conditions $T_i$, $1 \leq i \leq 2$, given by $\frac{11}{11}$, can be rewritten as

$$T_1 = \frac{2}{e^{\varphi}} \left(\Omega_{1,1} + \Omega_{2,2} + W\lambda_1 e^{2\varphi}\right),$$

$$T_2 = \frac{2}{e^{\varphi}} \left(\Omega_{2,2} + \Omega_{1,1} - W\lambda_2 e^{2\varphi}\right).$$

Using $\frac{12}{12}$ in this last equation, we get

$$T_1 = \frac{2}{e^{2\varphi}} \left(\Omega_{1,1} + \Omega_{2,2} + W\lambda_1 e^{2\varphi}\right),$$

$$T_2 = \frac{2}{e^{2\varphi}} \left(\Omega_{2,2} + \Omega_{1,1} - W\lambda_2 e^{2\varphi}\right).$$

Therefore, $T_1 + T_2 = \frac{2}{e^{2\varphi}} (\Delta \Omega - e^{2\varphi}W(\lambda_1 + \lambda_2))$ and the first additional relation of $\frac{13}{13}$ is equivalent to $\frac{15}{15}$.

**Remark 1.9** Let $X$ as in the previous remark. Then the parameterization $\widetilde{X}$ of $\widetilde{M}$, locally associated to $X$ by a Ribaucour transformation, given by $\frac{6}{6}$, is defined on

$$V = \{(u_1, u_2) \in U; \Omega T_1 - S \neq 0\}.$$
2. Families of isothermic surfaces associated to the cylinder.

In this section, by applying Theorem 1.7, using the Remark 1.8 and 1.9 to the cylinder, we obtain a three-parameter family of complete isothermic surfaces. As obtained in [8], using examples, we check that there are n-bubble surfaces outside which are 1-periodic, have genus zero and two ends of finite geometric index. In addition, we have 1-periodic n-bubble surfaces inside of the cylinder and one-parameter family of complete isothermic surfaces with planar ends.

Theorem 2.1 Consider the cylinder parametrized by

\[ X(u_1, u_2) = (\cos(u_2), \sin(u_2), u_1), \quad (u_1, u_2) \in \mathbb{R}^2 \]  

as isothermic surface where the first fundamental form is \( I = du_1^2 + du_2^2 \). A parametrized surface \( \tilde{X}(u_1, u_2) \) is isothermic surface locally associated to \( X \) by a Ribaucour transformation as in Theorem 1.7, if and only if, up to a rigid motion of \( \mathbb{R}^3 \), it is given by

\[ \tilde{X} = X - \frac{2}{2b + c(f - g)} \left( f'X_{u_1} + g'X_{u_2} - gN \right) \]  

defined on \( V = \{ (u_1, u_2) \in \mathbb{R}^2; f + g \neq 0 \} \) where \( N \) is the inner unit normal vector field of the cylinder, \( c \neq 0, b \) is real constant, and \( f(u_1), g(u_2) \) are solutions of the equations

\[ f'' - cf = b, \quad g'' + (1 + c)g = b \]  

with initial conditions satisfying

\[ [(f')^2 - cf^2 - 2bf + (g')^2 + (1 + c)g^2 - 2bg](u_1^0, u_2^0) = 0. \]  

Moreover, the normal map of \( \tilde{X} \) is given by

\[ \tilde{N} = N - \frac{2g}{(2b + c(f - g))(f + g)} \left( f'X_{u_1} + g'X_{u_2} - gN \right) \]  

Proof: Consider the first fundamental form of the cylinder \( ds^2 = du_1^2 + du_2^2 \) and the principal curvatures \( \lambda_1 = 0, \lambda_2 = -1 \). Using [12], to obtain the Ribaucour transformations, we need to solve the following of equations

\[ \Omega_{i,j} = 0, \quad \Omega_{i,i} = \Omega_i, \quad W_{i,i} = -\Omega_i \lambda_i, \quad 1 \leq i \neq j \leq 2. \]  

Since \( \Omega_{12} = 0 \), it follows that \( \Omega = f_1(u_1) + g_2(u_2) \). Therefore \( \Omega_1 = f_1' \) and \( \Omega_2 = g_2' \). Moreover, \( W = g_2 + a \), where \( a \) is a real constant. Thus, from [1], \( S = (f_1')^2 + (g_2')^2 + (g_2 + a)^2 \).

Using the Remark 1.8, the associated surface will be isothermic when \( \Delta \Omega + W = \frac{S}{\Omega} \). Therefore, we obtain that the functions \( f_1 \) and \( g_2 \) satisfy

\[ f_1'' + g_2'' + g_2 + a = \frac{(f_1')^2 + (g_2')^2 + (g_2 + a)^2}{f_1 + g_2}. \]  

(21)

Differentiate this last equation with respect \( x_1 \) and \( x_2 \), we get

\[ f_1''' = f_1' \left( \frac{f_1'' - g_2' - g_2 - a}{f_1 + g_2} \right), \quad g_2''' + g_2' = -g_2 \left( \frac{f_1'' - g_2' - g_2 - a}{f_1 + g_2} \right). \]  

(22)
Differentiate \( \frac{f''_1 - g''_2 - g_2 - a}{f_1 + g_2} \), with respect \( x_i, 1 \leq i \leq 2 \) and using (22), we get \( \left( \frac{f''_1 - g''_2 - g_2 - a}{f_1 + g_2} \right)_i = 0 \).

Therefore \( \frac{f''_1 - g''_2 - g_2 - a}{f_1 + g_2} = c \), where \( c \) is a real constant. Thus, we have that \( f_1 \) and \( g_2 \) satisfy

\[
\begin{align*}
f''_1 - cf_1 + ca &= b, \\
g''_2 + (1 + c)g_2 + a + ac &= b.
\end{align*}
\]

Now defining \( f = f_1 - a \) and \( g = g_2 + a \), we obtain that \( f \) and \( g \) satisfy (18), with real constant \( c \neq 0 \), because if \( c = 0 \), then \( \tilde{X} \) is degenerate. Moreover, using (21) we get that the initial conditions satisfying (19) and using the Theorem 1.5, \( \tilde{X} \) is given by (17) and from Remark 1.9 is defined in \( V = \{(u_1, u_2) \in \mathbb{R}^2; f + g \neq 0\} \).

**Remark 2.2** Each isothermic surfaces associated to the cylinder as in Theorem 2.1, is parametrized by lines of curvature and from (10), the metric is given by \( ds^2 = \psi^2(du_1^2 + du_2^2) \), where

\[
\psi = \frac{|c(f + g)|}{|2b + c(f - g)|},
\]

Moreover, from (9), the principal curvatures of the \( \tilde{X} \) are given by

\[
\begin{align*}
\tilde{\lambda}_1 &= -\frac{2g(b + cf)}{c(f + g)^2}, \\
\tilde{\lambda}_2 &= -\frac{cf^2 - 2bf - cg^2}{c(f + g)^2}.
\end{align*}
\]

**Proposition 2.3** Consider the isothermic surfaces associated to the cylinder parametrized by (17). Then the mean curvature of the \( \tilde{X} \) is given by

\[
\tilde{H} = -\frac{1}{2} - \frac{b}{c(f + g)}.
\]

**Proof:** In fact, using Remark 2.2 is easy to get

\[
\tilde{\lambda}_1 + \tilde{\lambda}_2 = -\frac{(cf + cg + 2b)}{c(f + g)}.
\]

Therefore, \( \tilde{H} \) is given by (26).

Using the previous proposition, we immediately get

**Corollary 2.4** Consider the isothermic surfaces associated to the cylinder parametrized by (17). Then \( \tilde{X} \) is \(-\frac{1}{2} - cmc\), if and only if, \( b = 0 \).

The next result, describes the behavior of \( \tilde{X} \), in the neighborhood of \( p_0 \) where \( (2b + c(f - g))(p_0) = 0 \).
Proposition 2.5 Let \( M = 2b + c(f - g) \) and consider \( p_0 \in \mathbb{R}^2 \) such that \( M(p_0) = 0 \) and \( M(p) \neq 0 \), for all \( p \in V - \{ p_0 \} \). Let \( \tilde{X} : V - \{ p_0 \} \to \mathbb{R}^3 \) be a isothermic surface locally associated by a Ribaucour transformation to cylinder given by Theorem 2.1, where \( V = \{(u_1, u_2) \in \mathbb{R}^2; f + g \neq 0\} \). Then for any divergent curve \( \gamma : [0,1) \to V - \{ p_0 \} \) such that \( \lim_{t \to 1} \gamma(t) = p_0 \) the length of \( \tilde{X}(\gamma) \) is infinite.

Proof: Let \( p_0 \in \mathbb{R}^2 \) such that \( M(p_0) = 0 \), then using (18) and (21), we have \( S(p_0) = 0 \). After a translation, we may assume that \( p_0 = (0,0) \). It follows from Remark 2.3 that the first fundamental form of \( \tilde{X} \) is given by \( \tilde{I} = \psi^2 (du_1^2 + du_2^2) \), where \( \psi = \frac{|c(f + g)|}{|M|} \).

At \( p_0 \) we have \( M(0,0) = 0 \) and \( S(0,0) = 0 \), therefore

\[
f'(0) = 0, \quad g'(0) = 0, \quad g(0) = 0 \quad \text{and} \quad f(0) = \frac{-2b}{c}.
\] (27)

Differentiate \( M \) and evaluating and \( (0,0) \), we get \( M_{,i}(0,0) = M_{,ij}(0,0) = 0 \) for \( 1 \leq i \neq j \leq 2 \) and \( M_{,ii} = -cb \). Hence, by considering the Taylor expansion of \( M \) on neighborhood of \((0,0)\), we obtain

\[
M(p) = \frac{-cb|p|^2}{2} + R, \quad \text{with} \quad \lim_{|p| \to 0} \frac{R}{|p|^2} = 0, \quad p = (u_1, u_2).
\] (28)

Therefore, we have

\[
\lim_{|p| \to 0} \frac{|p|^2 \tilde{\psi}}{M} = \frac{4}{c},
\]

where \( \tilde{\psi} = \frac{c(f + g)}{M} \).

Hence, there exists \( \delta > 0 \) such that, \( 0 < |p| < \delta \) we have \( |p|^2 \tilde{\psi} > \frac{2}{|c|} \).

Let divergent curve \( \gamma : [0,1) \to V - \{(0,0)\} \), where \( V = \{(u_1, u_2) \in \mathbb{R}^2; f + g \neq 0\} \), such that \( \lim_{t \to 1} \gamma(t) = (0,0) \). Then its length is

\[
\int \tilde{X}(\gamma(t)) d|\gamma'(t)| dt > \left| \int \frac{\gamma'(t)}{\gamma(t)} dt \right| \geq \left| \int \frac{2}{|c|} \right| \int \frac{1}{\gamma(t)^2} dt = \infty.
\] (29)

As in [S], we introduce a notation, which will be useful in the following results.

A rotation of angle \( \theta \) in the \( xy \) plane of \( \mathbb{R}^3 \) will be denoted by

\[
R_\theta = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}.
\] (30)

We denote by \( T_\delta \) the translation defined by

\[
T_\delta(x, y, z) = (x, y, z + \delta).
\] (31)

Remark 2.6 Consider the isothermic surfaces associated to the cylinder parametrized by (17).

Then from (18), the functions \( f \) and \( g \) are given by

\[
f = \begin{cases}
- b_1 \cosh(\sqrt{c} u_1) + b_1 \sinh(\sqrt{c} u_1) - \frac{b_1}{2}, & \text{if } c > 0, \\
- \frac{a_1}{2}, & \text{if } c < 0,
\end{cases}
\] (32)
and from (19), the constant satisfy the algebraic relation

\[ \frac{b^2}{c(1 + c)} = c(a_1^2 - b_1^2) - (1 + c)(a_2^2 + b_2^2) \quad \text{if} \quad c > 0, \]

\[ \frac{b^2}{c(1 + c)} = c(a_1^2 + b_1^2) - (1 + c)(a_2^2 + b_2^2) \quad \text{if} \quad -1 < c < 0, \]

\[ a_1^2 + b_1^2 + a_2^2 + b_2^2 - (b + b_2)^2 = 0 \quad \text{if} \quad c = -1, \]

\[ \frac{b^2}{c(1 + c)} = c(a_1^2 - b_1^2) - (1 + c)(a_2^2 - b_2^2) \quad \text{if} \quad c < -1. \]

Moreover, if \( \tilde{X} \) is not \( \frac{1}{2} - cn\), then from Corollary 2.5, using (34) and (37), respectively, we have \( a_1^2 - b_1^2 > 0 \) if \( c > 0 \) and \( a_2^2 - b_2^2 > 0 \) if \( c < -1 \).

Using the Remark 2.5, we get immediately

**Corollary 2.7** Consider the isothermic surfaces associated to the cylinder parametrized by (17). Excluding the \( \frac{1}{2} - cn\), we have that up to rigid motions of \( \mathbb{R}^3 \), the surface \( \tilde{X} \) is determined by the functions

i) If \( c > 0 \), then

\[ f = \sqrt{A_1} \cosh(\sqrt{c} u_1) - \frac{b}{c}, \quad g = \sqrt{B_1} \sin(\sqrt{1 + c} u_2) + \frac{b}{1 + c}, \]

where \( \frac{b^2}{c(1 + c)} = cA_1 - (1 + c)B_1 \), with \( B_1 > 0 \).

ii) If \( -1 < c < 0 \), then

\[ f = \sqrt{A_1} \sin(\sqrt{-c} u_1) - \frac{b}{c}, \quad g = \sqrt{B_1} \sin(\sqrt{1 + c} u_2) + \frac{b}{1 + c}, \]

where \( \frac{b^2}{c(1 + c)} = cA_1 - (1 + c)B_1 \), with \( A_1 > 0 \) and \( B_1 > 0 \).

iii) If \( c < -1 \), then

\[ f = \sqrt{A_1} \sin(\sqrt{-c} u_1) - \frac{b}{c}, \quad g = \sqrt{B_1} \cosh(\sqrt{-1 - c} u_2) + \frac{b}{1 + c}, \]

where \( \frac{b^2}{c(1 + c)} = cA_1 - (1 + c)B_1 \), with \( A_1 > 0 \).

iv) If \( c = -1 \), then

\[ f = \sqrt{A_1} \sin(u_1) + b, \quad g = \frac{b}{2} u_2^2 + a_2 u_2 + b_2, \]

where \( A_1 + a_2^2 + b_2^2 - (b + b_2)^2 = 0 \), with \( A_1 > 0 \).
Proof: Consider the isothermic surfaces associated to the cylinder parametrized by (17) that is not $\frac{1}{2} - \text{cmc}$. If $c > 0$, then using (32), we get

\[ f = \sqrt{A_1} \cosh(\sqrt{c} u_1 + A_2) - \frac{b}{c}, \quad g = \sqrt{B_1} \sin(\sqrt{1 + c} u_2 + B_2) + \frac{b}{1 + c}, \]

where $\frac{b^2}{c(1 + c)} = cA_1 - (1 + c)B_1$, with $B_1 > 0$ and the constants $A_2$ and $B_2$, without loss of generality, may be considered to be zero. One can verify that the surfaces with different values of $A_2$, $B_2$ are congruent by rigid motions of $R^3$. In fact, using the notation $\tilde{X}_{bc:A_2B_2}$ for the surface $\tilde{X}$ with fixed constants $A_2$ and $B_2$, we have

\[ \tilde{X}_{bc:A_2B_2} = R_{\frac{b}{\sqrt{1 + c}}} \tilde{X}_{b00} \circ h + T_{\frac{A_2}{\sqrt{1 + c}}}, \]

where $h(u_1, u_2) = \left(u_1 + \frac{A_2}{\sqrt{c}} u_2 + \frac{B_2}{\sqrt{1 + c}}\right)$.

Finally, with analogous argument, we have (39), (40) and (41).

Proposition 2.8 Consider the isothermic surfaces associated to the cylinder parametrized by (17), that is not $\frac{1}{2} - \text{cmc}$. Let $M = 2b + c(f - g)$ and suppose there is $p_0 = (u_1^0, u_2^0)$ such that $M(p_0) = 0$. Then $p_0 = \left(0, \frac{(4k + 1)\pi}{2\sqrt{1 + c}}\right)$, $k \in Z$ if $c > 0$, $p_0 = \left(\frac{(2k - 1)\pi}{2\sqrt{-c}}, \frac{(2k - 1)\pi}{2\sqrt{1 + c}}\right)$, $k \in Z$ if $-1 < c < 0$, $p_0 = \left(\frac{(4k + 1)\pi}{2\sqrt{-c}}, 0\right)$, $k \in Z$ if $c < -1$ and $p_0 = \left(\frac{(2k - 1)\pi}{2}, 0\right)$, $k \in Z$ if $c = -1$. Moreover, the functions $f$ and $g$ given by (38) and (39) become

\[ f = \begin{cases} \frac{1}{c} \left(\cosh(\sqrt{c} u_1) + 1\right), & \text{if } c > 0 \\ \frac{-1}{c} \left(\sin(\sqrt{-c} u_1) + \epsilon_1\right), & \text{if } -1 \leq c < 0, \epsilon_1^2 = 1, \\
\frac{-1}{c} \sin(\sqrt{-c} u_1) + 1, & \text{if } c < -1 \end{cases} \]

\[ g = \begin{cases} \frac{1}{1 + c} \left(\sin(\sqrt{1 + c} u_2) - 1\right), & \text{if } c > 0 \\ \frac{1}{1 + c} \left(\sin(\sqrt{1 + c} u_2) + \epsilon_1\right), & \text{if } -1 < c < 0, \epsilon_1^2 = 1, \\
\epsilon_2 u_2, & \text{if } c = -1, \epsilon_1^2 = 1, \\
\frac{-1}{1 + c} \left(\cosh(\sqrt{-1 - c} u_2) - 1\right), & \text{if } c < -1 \end{cases} \]

Proof: Let $p_0 = (u_1^0, u_2^0)$ such that $M(p_0) = 0$. Using (18) and (21), in $p_0$ we have

\[ f'(u_1^0) = 0, \quad g'(u_2^0) = 0, \quad g(u_2^0) = 0 \quad \text{and} \quad f(u_1^0) = \frac{-2b}{c}. \]

If $c > 0$, then using (38) and (44), we obtain $u_1^0 = 0$, $\sqrt{A_1} = \frac{-b}{c}$, $b < 0$. Moreover we have $\cos(\sqrt{1 + c} u_2^0) = 0$ and $\sqrt{B_1} \sin(\sqrt{1 + c} u_2^0) = \frac{-b}{1 + c}$. Thus $u_2^0 = \frac{(4k + 1)\pi}{2\sqrt{1 + c}}$.  

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If \( b < k (43) \). As before, without loss of generality, we can consider \( b = -1 \). In fact, substituting \( f \) and \( g \) above in (23), (24), and (25), we obtain that the first and second fundamental forms of the \( \bar{X} \), do not depend on \( b \). Thus, \( f \) and \( g \), are given by (42) and (43).

Without loss of generality, we can consider \( b = 0 \). In fact, using (39) and (44), we get \( u_1^0 = (2k - 1)\pi \) and \( u_2^0 = (2k - 1)\pi / 2 \sqrt{1 + c} \), \( k \in Z \).

If \( b > 0 \), then we have \( \sqrt{A_1} = b/c \), and \( \sqrt{B_1} = b/1 + c \). Thus, \( f \) and \( g \) given by (39) can be rewritten by

\[
\frac{f}{c} = \frac{-b}{c} \left( \sin(\sqrt{-c} u_1) + 1 \right), \quad \frac{g}{c} = \frac{-b}{1 + c} \left( \sin(\sqrt{1 + c} u_2) - 1 \right).
\]

Substituting \( f \) and \( g \) above in (23), (24), and (25), we obtain that the first and second fundamental forms of the \( \bar{X} \), do not depend on \( b \). Therefore, without loss of generality, we can consider \( b = -1 \). Thus, \( f \) and \( g \), are given by (42) and (43), with \( \epsilon_1 = -1 \).

On the other hand, if \( b > 0 \), then we have \( \sqrt{A_1} = -b/c \), and \( \sqrt{B_1} = -b/1 + c \). Thus, \( f \) and \( g \) given by (39) can be rewritten by

\[
\frac{f}{c} = \frac{-b}{c} \left( \sin(\sqrt{-c} u_1) + 1 \right), \quad \frac{g}{c} = \frac{-b}{1 + c} \left( \sin(\sqrt{1 + c} u_2) + 1 \right).
\]

As before, without loss of generality, we can consider \( b = 1 \). Thus, \( f \) and \( g \), are given by (42) and (43), with \( \epsilon_1 = 1 \).

If \( c < -1 \), then using (40) and (44), we get \( u_2^0 = 0, \sqrt{B_1} = -b/c, \ b > 0 \).

Moreover we have \( \cos(\sqrt{-c} u_1^0) = 0 \) and \( \sqrt{A_1} \sin(\sqrt{-c} u_1^0) = -b/c \). Thus \( u_1^0 = (4k + 1)\pi / 2 \sqrt{-c} \), \( k \in Z \) and \( \sqrt{B_1} = -b/c \). Therefore \( f \) and \( g \) given by (40) can be rewritten by

\[
\frac{f}{c} = \frac{-b}{c} \left( \sin(\sqrt{-c} u_1) + 1 \right), \quad \frac{g}{c} = \frac{-b}{1 + c} \left( \cosh(\sqrt{-1 + c} u_2) - 1 \right).
\]

As before, without loss of generality, we can consider \( b = 1 \). Thus, \( f \) and \( g \), are given by (42) and (43).

If \( c = -1 \), similarly to the previous cases, using (41) and (44), we have \( p_0 = \left( (2k - 1)\pi, -a_2 / b \right) \) and

\[
f = |b| (\sin(u_1) + \epsilon_1), \quad \epsilon_1^2 = 1, \quad g = \frac{1}{2b} (bu_2 + a_2)^2.
\]

Without loss of generality, we can consider \( a_2 = 0 \). In fact, using (30), we have

\[
\bar{X}_{bcw} = R_{w} \bar{X}_{bcw} \circ h,
\]

where \( h(u_1, u_2) = \left( u_1, u_2 + \frac{a_2}{b} \right) \) and \( \bar{X}_{bcw} \) is the surface \( \bar{X} \) with fixed constant \( a_2 \).

As before, without loss of generality, we can consider \( |b| = 1 \). Thus, \( f \) and \( g \), are given by (42) and (43).

\[
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\]
Proposition 2.9 Any isothermic surfaces associated to the cylinder $\tilde{X}$, given by Theorem 2.1 is complete.

Proof: For any divergent curve $\gamma : [0, 1) \to V - \{p_0\}$ such that $\lim_{t \to 1} \gamma(t) = p_0$ where $p_0$ are given by Proposition 2.8, then from Proposition 2.5 the length of $\tilde{X}(\gamma)$ is infinite.

For divergent curves $\gamma(t) = (u_1(t), u_2(t))$, such that $\lim_{t \to \infty} (u_1^2 + u_2^2) = \infty$, we have $l(\tilde{X} \circ \gamma) = \infty$.

In fact, if $c > 0$, then the functions $f$ and $g$ are given by (38), and from (23) the coefficients of the first fundamental form is $\psi = \frac{|c(f + g)|}{|2b + c(f - g)|}$. Therefore, $\lim_{|u_1| \to \infty} \psi = 1$ uniformly in $u_2$.

Hence, there exist $k > 0$ such that $|\psi(u_1, u_2)| > \frac{1}{2}$ for $(u_1, u_2) \in R^2$ with $|u_1| > k$. Let

$$m = \min \left\{ |\psi(u_1, u_2)|; (u_1, u_2) \in [-k, k] \times \left[ 0, \frac{2\pi}{\sqrt{1 + c}} \right] \right\}.$$  

(50)

Note that, $g(u_2) = g(u_2 + \frac{2\pi}{\sqrt{1 + c}})$, therefore $|\psi(u_1, u_2)| > m$ in $[-k, k] \times R$. Consider $m_0 = \min \{ m, \frac{1}{2} \}$, then $|\psi(u_1, u_2)| > m_0$ in $R^2$. Thus $l(\tilde{X} \circ \gamma) = \infty$. The case $c \leq -1$ is analogous.

Finally if $-1 < c < 0$, then the functions $f$ and $g$ are given by (39), and from (23) the coefficients of the first fundamental form is $\psi = \frac{|c(f + g)|}{|2b + c(f - g)|}$. In this case, let

$$m_0 = \min \left\{ |\psi(u_1, u_2)|; (u_1, u_2) \in \left[ 0, \frac{2\pi}{\sqrt{1 + c}} \right] \times \left[ 0, \frac{2\pi}{\sqrt{1 + c}} \right] \right\}.$$  

(51)

Note that, $g(u_2) = g(u_2 + \frac{2\pi}{\sqrt{1 + c}})$ and $f(u_1) = f(u_1 + \frac{2\pi}{\sqrt{1 + c}})$, therefore $|\psi(u_1, u_2)| > m_0$ in $R^2$.

Thus $l(\tilde{X} \circ \gamma) = \infty$ and we conclude that $\tilde{X}$ is a complete surface.

Remark 2.10 As obtained in [8], the $n$ points of maximum ( respectively minimum ) for the Gaussian curvature of the family of isothermic surfaces given by (17), generate 1-periodic n-bubble surfaces outside of the cylinder. In addition, we can also have 1-periodic n-bubble surfaces inside of the cylinder, since $2b + c(f - g) \neq 0$. We see this with some examples.

i) For $c > 0$, if $\sqrt{1 + c} = \frac{n}{m}$, then generate 1-periodic n-bubble surfaces with two ends of geometric index $m$. And more, if $b > 0$ then the n-bubble surfaces inside ( see Figure 1 ), if $b < 0$ then the n-bubble surfaces outside ( see Figure 2 ).

![Figure 1](image)

Figure 1: In the Figure 1 above, we have $c = 3$, $b = 4\sqrt{6}$ and $\sqrt{1 + c} = \frac{2}{3}$. Thus we have 2-bubble surfaces inside.
Figure 2: In the Figure 2 above we have $c = 3$, $b = -4\sqrt{6}$ and $\sqrt{1 + c} = \frac{2}{1}$. Thus we have 2-bubble surfaces outside.

ii) For $-1 < c < 0$, if $\sqrt{-c}$ and $\sqrt{1 + c}$ are a rational number, then we have 1-periodic bubbles surfaces inside 1-periodic bubbles surfaces vertically (see Figure 3). If $\sqrt{-c}$ and (or) $\sqrt{1 + c}$ are not rational number, then generate bubble surfaces inside bubbles surfaces, but not 1-periodic.

Figure 3: In the Figure 3 above we have $c = \frac{-16}{25}$, $b = \frac{12\sqrt{73}}{125}$ and $\sqrt{-c} = \frac{4}{5}$, $\sqrt{1 + c} = \frac{3}{5}$.

iii) For $c < -1$, if $b > 0$, then we have 1-periodic bubbles surfaces outside (see Figure 4). If $b < 0$, then we have 1-periodic bubbles surfaces inside (see Figure 5).

Figure 4: In the Figure 4 above we have $c = -5$, $b = \frac{4\sqrt{5}}{5} > 0$.
iv) For $c = -1$, in this case we have 1-periodic bubble surfaces outside vertically (see Figure 6)

Remark 2.11 As obtained in [9], the family of isothermic surfaces given by (17), generate planar ends. This occurs whenever there are $p_0 \in \mathbb{R}^2$ such that $(2b + c(f - g))(p_0) = 0$. In this case, $p_0$ are given by Proposition 2.8 and the functions $f$ and $g$ are given by (42) and (43). Besides that, the $\tilde{X}$ depends only on a parameter.
Figure 8: In Figure 8 above, we have $c = \frac{-16}{25}$, in $p_0 = \left(\frac{5(2k-1)\pi}{8}, \frac{5(2k-1)\pi}{6}\right)$ we have planar ends.

Figure 9: In Figure 9 above, we have $c = -5$, in $p_0 = \left(\frac{(4k+1)\pi}{2\sqrt{5}}, 0\right)$ we have planar ends.

3. Solution of the Calapso Equation.

In \cite{3}, we have that for each isothermic surface, we have a solution of the Calapso equation. In the section, we obtain solution of the Calapso Equation.

The Calapso equation defined in \cite{Calapso} given by

\[
\left(\frac{\omega, u_1 u_2}{w}\right)_{u_1 u_1} + \left(\frac{\omega, u_1 u_2}{w}\right)_{u_2 u_2} + (\omega^2)_{u_1 u_2} = 0,
\]

describes isoghermic surfaces in $R^3$.

**Remark 3.1** \cite{13} Let $X(u_1, u_2)$ be an isothermic surface with the first fundamental form given by

\[
I = e^{2\phi}(du_1^2 + du_2^2).
\]

Then the functions $\omega = \sqrt{2}e^\phi H$ and $\Omega = \sqrt{2}e^\phi H'$ are solutions of the Calapso equation, where $H$ is the mean curvature of $X$ and $H'$ is the skew curvature of $M$.

**Proposition 3.2** Consider the isothermic surfaces associated to the cylinder parametrized by \cite{17}.
Then the mean and skew curvature of the $\tilde{X}$ are given by

$$\tilde{H} = -\frac{1}{2} - \frac{b}{c(f + g)},$$

(53)

$$\tilde{H}' = \frac{M - 2b}{M\psi^2},$$

(54)

where the functions $f$ and $g$ are given by (38)-(41), $M = 2b + c(f - g)$ and $\psi$ are the coefficients of the first fundamental form of the $\tilde{X}$, given by (23).

**Proof:** In fact, the (53) is given by Proposition 2.3. For provide (54), just replace (24) and (25) in $\tilde{H}' = \tilde{\lambda}_1 - \tilde{\lambda}_2$ and we conclude the prove.

Using Remark 3.1 and Proposition 3.2, we get immediately

**Corollary 3.3** Consider the isothermic surfaces associated to the cylinder parametrized by (17), whose first fundamental form is given by

$$I = \left(\frac{c(f + g)}{M}\right)^2 (du_1^2 + du_2^2),$$

where the functions $f$ and $g$ are given by (38)-(41) and $M = 2b + c(f - g)$.

Then the functions $\omega = \frac{\sqrt{2}(M + 2cg)}{2M}$ and $\Omega = \frac{\sqrt{2}(f - g)}{f + g}$, with $\epsilon = 1$ if $c > 0$ and $\epsilon = -1$ if $c < 0$, are solutions of the Calapso equation.

**Example 3.4** Consider the isothermic surfaces associated to the cylinder given by Figure 1. In this case, we have $b = 4\sqrt{6}$, $c = 3$, $f(u_1) = 2\cosh(\sqrt{3}u_1) - \frac{4\sqrt{6}}{3}$ and $g(u_2) = \sin(2u_2) + \sqrt{6}$. Using the Corollary 3.3, we have

$$\omega = \frac{\sqrt{2}(7\sqrt{6} + 6\cosh(\sqrt{3}u_1) + 3\sin(2u_2))}{2\sqrt{6} + 12\cosh(\sqrt{3}u_1) - 6\sin(2u_2)},$$

$$\Omega = \frac{\sqrt{2}(-7\sqrt{6} + 6\cosh(\sqrt{3}u_1) - 3\sin(2u_2))}{-\sqrt{6} + 6\cosh(\sqrt{3}u_1) + 3\sin(2u_2)}$$

Figure 10: In the figure above we have the graphics of the solutions of the Calapso equation.

**Example 3.5** Consider the isothermic surfaces associated to the cylinder given by Figure 2. In this case, we have $b = -4\sqrt{6}$, $c = 3$, $f(u_1) = 2\cosh(\sqrt{3}u_1) + \frac{4\sqrt{6}}{3}$ and $g(u_2) = \sin(2u_2) - \sqrt{6}$. 16
Using the Corollary 3.3, we have

\[ \omega = \frac{\sqrt{2}(-7\sqrt{6} + 6\cosh(\sqrt{3}u_1) + 3\sin(2u_2))}{-2\sqrt{6} + 12\cosh(\sqrt{3}u_1) - 6\sin(2u_2)} \]
\[ \Omega = \frac{\sqrt{2}(7\sqrt{6} + 6\cosh(\sqrt{3}u_1) - 3\sin(2u_2))}{\sqrt{6} + 6\cosh(\sqrt{3}u_1) + 3\sin(2u_2)} \]

The graphics for these solutions of the Calapso equation are similar to the graphics in Figure 6.

**Example 3.6** Consider the isothermic surfaces associated to the cylinder given by Figure 3.
In this case, we have \( c = -\frac{16}{25}, b = \frac{12\sqrt{73}}{125}, f(u_1) = 2\sin(\frac{2}{5}u_1) + \frac{3\sqrt{73}}{20} \) and \( g(u_2) = \sin(\frac{3}{5}u_2) + \frac{4\sqrt{73}}{15} \).
Using the Corollary 3.3, we have

\[ \omega = \frac{\sqrt{2}(7\sqrt{73} + 120\sin(\frac{2}{5}u_1) + 60\sin(\frac{3}{5}u_2))}{50\sqrt{73} - 240\sin(\frac{4}{5}u_1) + 120\sin(\frac{3}{5}u_2)} \]
\[ \Omega = \frac{\sqrt{2}(7\sqrt{73} - 120\sin(\frac{2}{5}u_1) + 60\sin(\frac{3}{5}u_2))}{25\sqrt{73} + 120\sin(\frac{4}{5}u_1) + 60\sin(\frac{3}{5}u_2)} \]

![Figure 11: In the figure above we have the graphics of the Example 3.6 solutions of the Calapso equation.](image)

**Example 3.7** Consider the isothermic surfaces associated to the cylinder given by Figure 4.
In this case, we have \( c = -5, b = \frac{4\sqrt{5}}{3}, f(u_1) = \frac{\sin(\sqrt{5}u_1)}{3} + \frac{4\sqrt{5}}{15} \) and \( g(u_2) = \frac{\cosh(2u_2)}{2} - \frac{\sqrt{5}}{3} \).
Using the Corollary 3.3, we have

\[ \omega = \frac{-\sqrt{2}(18\sqrt{5} + 10\sin(\sqrt{5}u_1) + 15\cosh(2u_2))}{4\sqrt{5} + 20\sin(\sqrt{5}u_1) - 30\cosh(2u_2)} \]
\[ \Omega = \frac{-\sqrt{2}(18\sqrt{5} + 10\sin(\sqrt{5}u_1) - 15\cosh(2u_2))}{-2\sqrt{5} + 10\sin(\sqrt{5}u_1) + 15\cosh(2u_2)} \]

**Example 3.8** Consider the isothermic surfaces associated to the cylinder given by Figure 6.
In this case, we have \( c = -1, b = 2, f(u_1) = \sin(u_1) + 2 \) and \( g(u_2) = u_2^2 - \frac{\sqrt{3}}{4} \).
Using the Corollary 3.3, we have

\[ \omega = \frac{-\sqrt{2}(-11 + 4u_2^2 + 4\sin(u_1))}{-10 - 8u_2^2 + 8\sin(u_1)} \]

\[ \Omega = \frac{-\sqrt{2}(11 - 4u_2^2 + 4\sin(u_1))}{5 + 4u_2^2 + 4\sin(u_1)} \]

Example 3.9 Consider the isothermic surfaces associated to the cylinder given by Figure 7. In this case, we have \( c = 3 \), \( f(u_1) = \frac{\cosh(u_1) + 1}{3} \) and \( g(u_2) = \frac{\sin(2u_2) - 1}{4} \).

Using the Corollary 3.3, we have

\[ \omega = \frac{\sqrt{2}(7 + 4\cosh(u_1) - 3\sin(2u_2))}{-2 + 8\cosh(u_1) - 6\sin(2u_2)} \]

\[ \Omega = \frac{\sqrt{2}(-7 - 4\cosh(u_1) + 3\sin(2u_2))}{1 + 4\cosh(u_1) + 3\sin(2u_2)} \]
Example 3.10 Consider the isothermic surfaces associated to the cylinder given by Figure 8. In this case, we have $c = -\frac{16}{25}$, $f(u_1) = \frac{25\sin(\frac{2u_1}{16})+25}{16}$ and $g(u_2) = \frac{25\sin(\frac{2u_2}{16})+25}{16}$. Using the Corollary 3.3, we have the solution of the Calapso Equation for this case. The graphic are similar to the graphs in the previous examples.

Example 3.11 Consider the isothermic surfaces associated to the cylinder given by Figure 9. In this case, we have $c = -5$, $f(u_1) = \sin(\sqrt{5}u_1)+\frac{1}{5}$ and $g(u_2) = \cosh(2u_2)-\frac{1}{4}$. Using the Corollary 3.3, we have the solution of the Calapso Equation for this case. The graphic are similar to the graphs in the previous examples.

Remark 3.12 For each isothermic surface obtained in this work, we can apply isometries, dilations, inversions, obtaining new isothermic surfaces. So we can get new solutions of the Calapso equation.

References

[1] Bianchi, L.: Lezioni di geometria Differenziale. Terza Edicione, Nicola Zanichelli Editore (1927).

[2] Bobenko, A.; Eitner, U.; Kitaev, A.: Surfaces with harmonic inverse mean curvature and Painlevé equations. Geom. Dedicata 68, 2, 187-227, (1997).

[3] Calapso, P.: Sulle superficie a linee di curvatura isoterme. Palermo Rend. 17, 275-286, (1903).

[4] Cieslinski, J.; Goldstein, P.; Sym, A.: Isothermic surfaces in $E^3$ as solitons surfaces. Phys. Lett. A. 205, 1 37-43, (1995).

[4] Cieslinski, J.: The Darboux-Bianchi Transformation for isothermic surfaces. Classical results the solitons approach. Differential Geom. Appl. 7, 1, 1-28, (1997).

[5] Chistoffel, E., B.: Ueber einige allgemeine Eigenschaften der Minimumsflächen. J. Reine Angew. Math. 67, 218-228, (1897).

[6] Clarkson, P. A.; Painlevé analysis and the complete integrability of a generalized variable-coefficient Kadomtsev-Petviashvili equation. IMA J. Appl. Math. 67, 1, 27-53, (1990).

[7] Corro, A. M. V.; Ferreira, W. P.; Tenenblat, K.: On Ribaucour transformations for hypersurfaces, Mat. Contemp. 17, 137-160, (1999).

[8] Corro, A. M. V.; Ferreira, W. P.; Tenenblat, K.: Ribaucour transformations for Constant mean curvature and linear weingarten surfaces. Pacific Journal of Mathematics. 212, 2, 265-296, (2003).

[9] Corro, A. M. V.; Ferreira, W. P.; Tenenblat, K.: Minimal surfaces obtained by Ribaucour transformations. Geometriae Dedicata, Nettherlands, 96, 1, 117-150, (2003).

[10] Corro, A. M. V.; Tenenblat, K.: Ribaucour transformation revisited, Comm. Geom. 12, 5, 1055-1082, (2004).

[11] Corro, A. M. V.: Generalized Weingarten surfaces of Bryant type en Hyperbolic 3-space, Mat. Contemp. 30, 71-89, (2006).
[12] Corro, A. M. V.; Martinez A., Milán F.: Complete flat surfaces with two isolated singularities in hyperbolic 3-space, J. Math. Anal. Appl. 366, 582-592 (2010).

[13] Corro, A. M. V.; Fernandes, K. V ; Riveros, C.M.C.: Isothermic surfaces and solutions of the Calapso equation, Serdica Math. J. 44, 341-364, (2018).

[14] Corro, A. V. ; Ferro, M. L. ; Rodrigues, L. A.: K-isothermic Hypersurfaces. NEXUS Mathematicæ, 3, 1-20, (2020).

[15] Darboux,G. Sur les isothermiques, C. R. Acad. Sci. Paris, 128, 1299-1305, (1899).

[16] Darboux,G. Leons sur la thorie des surfaces, Chelsea, Pub. Co, (1972).

[17] Ferreira, W. P.; Tenenblat, K.: Hypersurfaces with flat r-mean curvature and Ribaucour transformations. International Journal of Applied Mathematics and Statistics, 11, 38-51, (2007).

[18] Hertrich-Jeromin U. Introduction to Mobius differential geometry, London Mathematical Society Lecture Noete series, Vol. 300, cambridge University Press, Cambridge, (2003).

[19] Schief W. K.: Isothermic surfaces and the Calapso equation: the full monty. Bakhund and Darboux transformations. The geometry of solitons (Halifax, NS, 1999), CRM Proc. Lecture Notes vol. 29. Providence, RI, Amer. Math. Soc., 393-403, (2001).

[20] Song, D. A.: Laguerre isothermic in $\mathbb{R}^3$ and their Darboux transformation. Sci. China Math. 56, 1, 67-78, (2013).

[21] Sym, A.: Soliton Surfaces. Lett. Nuovo Cimento, 33, 12, 394-400, (1982).

[22] Tenenblat, K.: On Ribaucour transformations and applications to linear Weingarten surfaces. Anais da Academia Brasileira de Ciências (Impresso), ABC, 74, 559-575, (2002).

[23] Tenenblat, K.; Lemes, M. V.: On Ribaucour transformations and minimal surfaces. Matemática Contemporânea, 29, 13-40, (2005).

[24] Tenenblat, K.; Wang: Ribaucour Transformations for Hypersurfaces in Space Forms. Annals of Global Analysis and Geometry, 29, 157-185, (2006).

[25] Weingarten, J.: Ueber die Differentialgleichungen der Oberflächen, welche durch Krümmungslinien in unendlich Kleine Quadrate getheilt werden Können. Berl. Ber. 1883, 1163-1166, (1883).