The Synchronization of Clock Rate and the Equality of Durations Based on the Poincaré-Einstein-Landau Conventions

Zhao Zheng\textsuperscript{1} Tian Guihua\textsuperscript{2} Liu Liao\textsuperscript{1} and Gao Sijie\textsuperscript{1}

\textsuperscript{1}Department of Physics, Beijing Normal University, Beijing 100875, China
\textsuperscript{2}School of Science, Beijing University of Posts And Telecommunications, Beijing 100876, China.

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Abstract

There are two important basic questions in the measurement of time. The first one is how to define the simultaneity of two events occurring at two different places. The second one is how to define the equality of two durations. The first question has been solved by Einstein, Landau and others on the convention that the velocity of light is isotropic and it is a constant in empty space. But no body has answered the second question until today. In this paper, on the same convention about the velocity of light given by Poincaré, Einstein, Landau and others, we find the solution to the definition of the equality of two durations. Meanwhile, we also find answer to the question about the definition of the synchronization of rate of clocks located at different places.

Key words: equality of two durations, simultaneity, synchronization of clock rate, duration, velocity of light.

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1 Introduction

Both the simultaneity of clocks located at different places and the equality of durations are important problems in research about time. Poincaré thought that the two problems are related to each other, and that they can only acquire meaning by convention. He guessed that one could solve the problems by the convention that the velocity of light might be isotropic and might be a constant in empty space\cite{1, 2, 3, 4}.

*Email: Zhaoz43@hotmail.com
In the theory of the relativity, Einstein defined the simultaneity of clocks located at different places by the convention that the velocity of light is homogeneous and isotropic, and it is a universal constant in empty space\cite{5,6}. Landau pointed out that one can define globally the simultaneity of coordinate time only in a time-orthogonal frame\cite{7}.

However they did not discuss how to define the equality of durations of a clock.

We find that the synchronization of the moment of clocks located at different places and the synchronization of the rate of these clocks are different. The synchronization of the moment of clocks at different places might be difficult, while synchronization of their rate turns out easier. We suggested a new programme to synchronize the rate of clocks, where we do not synchronize the moment of the clocks. Following the convention on the velocity of light given by Poincaré, Einstein and Landau, we got the condition of transitivity of synchronization of clock rate. In the space-time where the condition is valid, we can define the identical “clock rate” in the whole space-time, although there may not exist “simultaneous hypersurfaces”. In those papers, we stressed the synchronization of rates of clocks located at different space points\cite{8,9,10,11}.

In this paper, with the programme to synchronize the rate of coordinate clocks, we define the equality of durations of a clock by the above convention on velocity of light. As a result, we not only answer the first problem on the measurement of time advanced by Poincaré (i.e. the definition on synchronization of clocks in different space points), but also find the solution to his second problem (i.e. the definition of the equality of durations).

The paper is organized as following. In the Sec.2, we introduce Poincaré’s ideas about measurement of time. In the Sec.3 and Sec.4, we present the results of research on simultaneity given by Einstein, Landau and others. Then we give the result of research on synchronization of rate of clocks in Sec.5. In Sec.6, we get the definition on the equality of durations. In the Sec.7, short conclusion and discussion are given.

2 Poincaré’s Notion of the Simultaneity and the Measurement of Time

How to define the simultaneity of two events occurring at two different places? How to define the equality of two durations? These two important basic questions have been discussed by many famous philosophers, physicists and mathematicians. Henri Poincaré gave inspired analysis on them before the birth of A. Einstein’s theory of relativity\cite{1,2,3,4}.

In “The Measure of Time” (1898) and “The Value of Science” (1905) Poincaré pointed out: “we have not a direct intuition of simultaneity nor of the equality of two durations (time intervals)”. “It is difficult to separate
the qualitative problem of simultaneity from the quantitative problem of the measurement of time.”

“The qualitative problem of simultaneity is made of depend upon the quantitative problem of the measurement of time.”

In “Science and Hypothesis” (1902) Poincaré wrote:

“(1). There is no absolute space, 

(2)There is no absolute time. When we say that two periods are equal, the statement has no meaning, and can only acquire a meaning by a convention.

(3)Not only have we no direct intuition of the equality of two periods, but we have not even direct intuition of the simultaneity of two events occurring at two different places.”

Poincaré thought that not only the equality of two durations but also the simultaneity of two events occurring at two different places can only acquire meaning by convention.

“This convention, however, is not absolutely arbitrary; it is not the child of our caprice. We admit it because certain experiments have shown us that it will be convenient, 

“The simultaneity of two events, or the order of their succession, the equality of two durations, are to be so defined that the enunciation of the natural laws may be as simple as possible, 

3 The Definition on Simultaneity Given by Einstein

A. Einstein agrees with Poincaré’s idea that the simultaneity of two events occurring at two different places can only acquire meaning by convention. Because “physics” is a science depending on experiment and measurement, so any definitions based on convention about the simultaneity must be operable in experiment and measurement. In the theory of “relativity”, Einstein gave a definition of the simultaneity based on the convention that the space is homogeneous and isotropic for light propagation and that the velocity of light is a universal constant in empty space.

In “on the electrodynamics of moving bodies” (1905) he wrote:

“We have so far defined only an “A time” and a “B time”. We have not defined a common “time” for A and B, for the latter cannot be defined at all unless we establish by definition that the “time” required by light to travel from A to B equals the “time” it requires to travel from B to A. Let a ray of light start at the “A time” \( t_A \) from A towards B. Let it at the “B time” \( t_B \) be reflected at B in the direction of A, and arrive again at A at the “A time” \( t'_A \).

In accordance with definition the two clocks synchronize if
\[ t_B - t_A = t'_A - t_B \] (1)

We assume that this definition of synchronism is free from contradictions, and possible for any number of points; and that the following relations are universally valid:—

1. If the clock at B synchronizes with the clock at A, the clock at A synchronizes with the clock at B.

2. If the clock at A synchronizes with the clock at B and also with the clock at C, the clocks at B and C also synchronize with each other.

Thus with the help of certain imaginary physical experiments we have settled what is to be understood by synchronous stationary clocks located at different places, and have evidently obtained a definition of “simultaneous,” or “synchronous,” of “time.” The “time” of an event is that which is given simultaneously with the event by a stationary clock located at the place of the event, this clock being synchronous, and indeed synchronous for all time determinations, with a specified stationary clock.

In agreement with experience we further assume the quantity

\[ \frac{2AB}{t'_A - t_A} = c \] (2)

to be a universal constant—the velocity of light in empty space.

In Einstein’s above theory, Eq. (1) can be re-written as

\[ \frac{t'_A + t_A}{2} = t_B \] (3)

the “A time” \( \tilde{t}_A \)

\[ \tilde{t}_A = \frac{t'_A + t_A}{2} \] (4)

is defined as the “simultaneous” moment with the “B time” \( t_B \).

With this way, Einstein defined “simultaneity” or “synchronism” of any number of stationary clocks located at different places in an inertial frame.

It can be proved that the above definition and hypotheses are free from contradictions in inertial frames.

However, the “hypothesis 2” may be invalid in an arbitrary reference system in a curved space-time, or even in a non-inertial frame in a flat space-time. It has been proved that the “hypothesis 2” is valid only in time-orthogonal frames[7]. It should be emphasized that even in a time-orthogonal frame, in general, one can synchronize only the “coordinate clocks” over all space, not the “standard clocks”. In other words, one only can globally define “the simultaneity of the coordinate time” but not “the simultaneity of the proper time” in time-orthogonal frames in general.
4 The Condition of Transitivity of Simultaneity Given by Landau

Let us discuss the possibility of synchronizing clocks at different points over all space, i.e. the possibility of “hypothesis 2” being valid[7, 8].

Suppose a light signal travel from some point A in space to another point B infinitely near to it and then back along the same path. L and M are world lines of point A and point B respectively (Fig. 1). Let the light signal start at the “A time” $t_A$ from A towards B, then it at the “B time” $t_B$ is reflected at B in the direction of A and arrives again at A at the “A time” $t_A$. Let

$$dt_{(1)} = t_{A1} - t_B$$

$$dt_{(2)} = t_{A2} - t_B$$

$$t_A = \frac{t_{A1} + t_{A2}}{2}$$

Following Einstein (Sec.2), one can define “A time” $t_A$ being the simultaneous moment with “B time” $t_B$. Substituting Eqs. (5) and (6) into Eq. (7), one has

$$t_A = t_B + \frac{1}{2} [dt_{(2)} + dt_{(1)}]$$

Let us write the interval, separating the space and time coordinates, of two events, as

$$ds^2 = g_{00}dt^2 + 2g_{0i}dx^0dx^i + g_{ij}dx^idx^j, \quad (i, j = 1, 2, 3)$$
As the interval between the two events corresponds to the departure and arrival of the light signal from one point to the other, it is equal to zero. Solving the equation \( ds^2 = 0 \) with respect to \( dx^0 \), one finds two roots:

\[
dt = -g_{0i}dx^i \pm \sqrt{(g_{0i}g_{0j} - g_{00}g_{ij})dx^idx^j} / g_{00}
\]

(10)
corresponding to the propagation of the light signal in the two directions between A and B. So, we have

\[
dt_{(2)} + dt_{(1)} = -\frac{2g_{0i}dx^i}{g_{00}}
\]

(11)

Substituting Eq. (11) into Eq. (8), one gets

\[
\Delta t \equiv t_A - t_B = -\frac{g_{0i}dx^i}{g_{00}}
\]

(12)

One finds that there exists a difference in the values of the coordinate time \( t \) for two simultaneous moments at infinitely near points. Eq. (12) enables people to synchronize clocks located at infinitely near points of space. Carrying out a similar synchronization, one can synchronize clocks located at different points alone any open curves of space, i.e. one can define simultaneity of moments along any open curve. However synchronization of clocks along a closed contour turns out to be impossible in general, because \( \Delta t \) is different from zero when one synchronizes clocks starting out along the contour and returning to the initial point. In fact, one has

\[
\oint \Delta t \neq 0
\]

(13)

Hence it is impossible to synchronize clocks over all space unless the reference system is time-orthogonal

\[
g_{0i} = 0
\]

(14)
or

\[
\oint \Delta t = \oint \left( -\frac{g_{0i}}{g_{00}} \right) dx^i = 0
\]

(15)

5 Transitivity of Synchronization of Clock Rate

In Ref. [8, 9, 10, 11], we gave another kind of clock synchronization, where we do not demand to synchronize “simultaneous moments” of coordinate clocks, only demand to synchronize “rates” of coordinate clocks.

Now let us give the condition on the transitivity of synchronization of clock rate. At the first simultaneous moment of the space points A and B, the time difference of their coordinate clocks is

\[
\Delta t_1 = t_{A1} - t_{B1} = -\left( g_{0i}/g_{00} \right) dx^i, \quad (i = 1, 2, 3)
\]

(16)
At the second simultaneous moment, the time difference is
\[ \Delta t_2 = t_{A2} - t_{B2} = -\left(\frac{g_{0i}}{g_{00}}\right)_2 \, dx^i, \quad (i = 1, 2, 3) \] (17)

The difference between the rates of the two clocks can be given:
\[
\delta (\Delta t) = (\Delta t)_A - (\Delta t)_B \equiv (t_{A2} - t_{A1}) - (t_{B2} - t_{B1})
\]
\[= (t_{A2} - t_{B2}) - (t_{A1} - t_{B1}) = - \left[\left(\frac{g_{0i}}{g_{00}}\right)_2 - \left(\frac{g_{0i}}{g_{00}}\right)_1\right] \, dx^i \] (18)

Therefore the rates of coordinate clocks are the same everywhere, or the synchronization of clock rate is transitive, if and only if
\[
\oint \delta (\Delta t) = 0 \tag{19}
\]
or
\[
\oint \left(\frac{g_{0i}}{g_{00}}\right)_1 \, dx^i = \oint \left(\frac{g_{0i}}{g_{00}}\right)_2 \, dx^i \tag{20}
\]
That means
\[
\frac{\partial}{\partial t} \oint \Delta t = \frac{\partial}{\partial t} \oint \left(\frac{g_{0i}}{g_{00}}\right) \, dx^i = 0 \tag{21}
\]
or
\[
\frac{\partial}{\partial t} \left(\frac{g_{0i}}{g_{00}}\right) = 0 \tag{22}
\]
Eq. \((21)\) requires that the integral along a closed path is a constant independent of time:
\[
\oint \Delta t = \oint \left(\frac{g_{0i}}{g_{00}}\right) \, dx^i = \text{const} \tag{23}
\]

Of course, the constant in the Eq. \((23)\) could be non-zero. This is a weaker condition than that of time-orthogonality. Obviously, Eq. \((21)\) is only a necessary condition, but not a sufficient condition for constructing simultaneity surfaces.

In the whole space–time which satisfies Eq. \((23)\) but not Eq. \((15)\), one can synchronize all clocks such that rates of the coordinate clocks are the same everywhere, or the synchronization of clock rate is transitive. But the simultaneity is not transitive, or one can not construct simultaneity surfaces globally in the whole space-time.

### 6 Measurement of Duration

The simultaneity of two events occurring at two different places and the transitivity of simultaneity have been defined, by Einstein, Poincaré, Landau and others on the foundation of the convention that the velocity of light is homogeneous and isotropic, and is a universal constant in empty space. Obversely, these important convention and definitions are operable in experiment and measurement.
Now, let us discuss the issue about “the equality of two durations”, i.e. “the equality of two periods or two time intervals”. Poincaré thought that this problem of the measurement of time is related to the problem of simultaneity.

We want to point out that we can define “the equality of two durations” based on the same convention about the velocity of light in empty space, and that this definition is also operable in experiment and measurement.

6.1 Stationary Space-times

Let an observer with a light source and a mirror rest at a fixed space point A in a stationary space-time, and another observer with a mirror rest at the point B infinitely near to A. L and M are respectively the world-lines of the observers resting at the points A and B (FIG.2). Let the observer A emit a light signal at the “A time” $t_{A1}$ towards B. The signal is reflected by the mirror of the observer B at the “B time” $t_{B1}$ in the direction of A, and the light signal arrives again at A at the “A time” $t_{A2}$. When the light signal goes to B and back to A, the difference of the coordinate time is

$$\Delta t_1 = t_{A2} - t_{A1} \quad (24)$$

At the same moment ($t_{A2}$) receiving the light signal reflected by B, the mirror A reflects again the signal in the direction of B. The light signal arrives again at B at the “B time”$t_{B2}$, then it is reflected and comes back to A at the “A time”$t_{A3}$, . . . . . . . The travel of the light signal to and fro between A and B can be regarded as a period of time. The duration of the
The first period is shown in Eq. (24). The duration of the second period is:
\[ \Delta t_2 = t_{A3} - t_{A2} \quad (25) \]

The duration of the nth period is:
\[ \Delta t_n = t_{A(n+1)} - t_{An} \quad (26) \]

There is no reason to think that the “durations” of the above periods are different, because the space-time is stationary. So, one has
\[ \Delta t_1 = \Delta t_2 = \cdots = \Delta t_n \quad (27) \]
i.e.
\[ t_{A2} - t_{A1} = t_{A3} - t_{A2} = \cdots = t_{A(n+1)} - t_{An} \quad (28) \]

Consequently, we give a definition on the equality of durations in a stationary space-time.

Discussion:
(i) Because the distance between the point B and the point A is arbitrary selected, we can define the equality of durations of any length.
(ii) The durations discussed above are the durations of coordination time. The relation between the duration of coordination time \( dt \) and the duration of proper time \( d\tau \) is
\[ d\tau = \sqrt{-g_{00}}dt \quad (29) \]

We have the conclusion that the durations of the proper time of a resting observer, shown by Eq. (29), are equal, that is
\[ d\tau_1 = d\tau_2 = \cdots = d\tau_n, \quad (30) \]

because \( g_{00} \) is independent of time \( t \) in stationary space times.

(iii) We can generalize the definition on “the equality of durations” to the whole space-time, and get the conclusion that the durations of coordinate time are equal in the whole stationary space–time, because the condition (19) or (20) of “transitivity of synchronization of clock rate” is valid in any stationary space-times. From Eq. (29), we know that the durations of proper time of the different observers located at the different space points are not equal in general, owing to \( g_{00} \) being a function depending on space points.
6.2 A General Case

Let $\mathcal{L}$ be a congruence of the world-lines of a family of static observers A, B, C, . . . in a curved space-time (FIG.3). It means that a coordinate system covering $\mathcal{L}$ is selected, where $\mathcal{L}$ is just the congruence of the time coordinate curves. We assume the synchronization of clock rates is transitive, but the coordinate system is not time-orthogonal.

Let us synchronize clocks along a closed space contour with a light signal following Einstein and Landau. We know from Eq. (12) that the “B time” $t_B$ of the point B near A is defined as the simultaneous moment of the “A time” $t_A$ when the time difference of the two simultaneous moments is given as

$$\Delta t = \frac{-g_{0i}}{g_{00}} dx^i$$

Similarly, we can define the “C time” $t_C$ of the point C near B as the simultaneous moment of the “B time” $t_B$, . . . . . . When we synchronize clocks along the closed space contour returning to the initial point A, we find that the simultaneous moment, shown by the clock A, is $t_{A2}$, which is different from $t_{A1}$, the difference of the two simultaneous moments is

$$\Delta t_1 = t_{A2} - t_{A1} = \int_1 \left( -\frac{g_{0i}}{g_{00}} \right) dx^i = a_1$$

Using the same method, we continue to synchronize clocks along the same closed contour, we have

$$\Delta t_2 = t_{A3} - t_{A2} = \int_2 \left( -\frac{g_{0i}}{g_{00}} \right) dx^i = a_2$$
\[ \Delta t_n \equiv t_{A(n+1)} - t_{An} = \oint_n \left( -\frac{g_{ij}}{g_{00}} \right) dx^i = a_n \]  

(34)

Owing to the coordinate system being not time-orthogonal, we have

\[ a_i \neq 0, \quad (i = 1, 2, \cdots, n) \]  

(35)

But we know that, from Eqs. (19)-(23),

\[ a_1 = a_2 = \cdots = a_n = a = \text{const} \]  

(36)

Because the synchronization of clock rate is transitive in the coordinate systems, we get the equal time periods from \( t_{A1} \) to \( t_{A2} \), from \( t_{A2} \) to \( t_{A3} \), etc. We can define the equal periods as the equal durations. Thus we obtain the equal "time intervals" of the observer \( A \).

It is easy to know that the constant \( a \) in Eq. (36) will be different when we synchronize clocks along different closed space contours. Because we can select arbitrary contour to synchronize clocks, so we can get arbitrary value of \( a \). It means that we can define arbitrary length of "equality of duration", i.e. arbitrary length of "equality of time intervals". Certainly, that is the duration of coordinate time.

The relation between the coordinate time \( t \) and the proper time \( \tau \) of the observer \( A \) is

\[ d\tau = \sqrt{-g_{00}} dt \]  

(37)

because \( A \) rests in the coordinate system. Therefore, we can define "length" of the durations of proper time by means of

\[ \Delta \tau_k = \int_{k\alpha}^{(k+1)\alpha} \sqrt{-g_{00}} dt. \]  

(38)

In other words, we can parameterize the world line with the duration of proper time \( \Delta \tau_k \) given by Eq. (38).

### 7 Conclusion and discussion

It should be noticed that Eq. (36) is based upon the condition on the transitivity of synchronization of clock rate. Our original purpose suggesting the notion of “the transitivity of synchronization of clock rate” is to find the coordinate clocks whose rates are the same in the whole space. It means that the “time periods” of the same kind of periodic motion are the same through the whole space time. Here we want to emphasize that the condition on the transitivity of synchronization of clock rate is not only the condition of the equality of the rates of clocks located at different space points, but also the condition of the equality of durations of a clock. Therefore, the condition...
not only solves the question about the definition of the synchronization of rate of clocks located at different space points, but also solves the question about the definition of the equality of two durations. It means that the condition on transitivity of synchronization of clock rate is the condition to define identical, equal time intervals through the whole space time.

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