A New Proof of the Lester’s Perimeter Theorem in Euclidean Space

O. Demirel*, L. Aslan and D. Topal
Afyon Kocatepe University, Department of Mathematics, Ahmet Necdet SEZER Campus, 03200, Afyonkarahisar, Turkey
Email: odemirel@aku.edu.tr*, leylaaslan3356@gmail.com, damla.1tpl@gmail.com

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1. Introduction
It is well known that some geometric transformations can be characterized by the properties of they preserve. For instance, collinearity preserving bijections of Euclidean n-space $E^n (2 \leq n < \infty)$ characterize the affine transformations and this theorem is known as the fundamental theorem of affine geometry. The Möbius transformations of the extended complex plane can be characterized by as transformations preserving quadruples of concyclic points. In Minkowski space the Alexandrov’s theorem which describes Lorentz transformations as the transformations of Minkowski space preserving the speed of light. In Euclidean space $E^n (2 \leq n < \infty)$ the Beckman-Quarles theorem which identifies as motions those functions from $E^n$ to itself preserving pairs of points of a given fixed distance apart. More precisely Beckman-Quarles theorem (Beckman and Quarles 1953) states that a function from $E^n$ to itself which preserves the relation $|x - y| = Q$ for a fixed $Q \in \mathbb{R}^n$ must be an Euclidean motion where $|x - y|$ denotes the distance between $x, y \in E^n$. This theorem plays a major role in our result. G. Martin (unpublished) characterized the equiaffine transformations (affine and area preserving) of $E^2$ via the injections which preserves triangles with area 1 as follows, see (Lester 1985).

**Theorem 1.1:** An injection from Euclidean plane to itself which preserves triangles with area 1 must be equiaffine, see (Lester 1985).

J. Lester generalized this theorem to the Euclidean space $E^n$ as follows:

**Theorem 1.2:** Let $f$ be an injection from Euclidean space $E^n (2 \leq n < \infty)$ to itself which preserves triangles with area 1 must be a Euclidean motion, see (Lester 1986).

J. Lester also obtained the following results using triangles of perimeter 1 instead of triangles of area 1.

**Theorem 1.3:** Let $f$ be an injection from Euclidean plane $E^2$ to itself which preserves triangles of perimeter 1 must be a Euclidean motion, see (Lester 1985).

**Theorem 1.4:** Let $f$ be an injection from Euclidean space $E^n (2 \leq n < \infty)$ to itself which preserves triangles of perimeter 1 must be a Euclidean motion, see (Lester 1986).

2. A New Proof of the Lester’s Perimeter Theorem in Euclidean Space

**Lemma 2.1:** Let $F_1 = (c, 0, \cdots, 0)$ and $F_2 = (-c, 0, \cdots, 0)$ be two distinct points in Euclidean space $E^n$, where $0 < c < \frac{1}{4}$ and $2 < n < \infty$. If $XF_1F_2$ is a triangle of perimeter 1 then must be a point on n-dimensional rotated ellipsoid with equation

$$\frac{x_1^2}{a^2} + \sum_{i=2}^{n} \frac{x_i^2}{b_i^2} = 1$$
where \( a + c = \frac{1}{2} \), \( a^2 = b^2 + c^2 \) and \( x_i \) is the natural coordinate function defined by \( x_i(p_1, p_2, \cdots, p_n) = p_i \).

**Proof:** Let us take a point \( X = (x_1, x_2, \cdots, x_n) \) such that
\[
|X - F_1| + |X - F_2| + |F_1 - F_2| = 1.
\]
Clearly
\[
|X - F_1| + |X - F_2| = 1 - 2c = 2a
\]
which implies \( a + c = \frac{1}{2} \). Hence we get
\[
\sum_{i=2}^{n} x_i^2 + (x_1 - c)^2 + \sum_{i=2}^{n} x_i^2 + (x_1 + c)^2 = 2a
\]
which implies
\[
a^2 (a^2 - c^2) = x_1^2 (a^2 - c^2) + a^2 \sum_{i=2}^{n} x_i^2.
\]
Let’s denote the number \( a^2 - c^2 \) by \( b^2 \). Clearly we get
\[
a^2 b^2 = x_1^2 b^2 + a^2 \sum_{i=2}^{n} x_i^2
\]
and this yields the equation which we desired.

Clearly, \( X = (t_1, 0, \cdots, 0) \) since \( XF_1F_2 \) is a triangle of perimeter 1. Notice that the points \( A_1 = (a, 0, \cdots, 0) \), \( A_2 = (-a, 0, \cdots, 0) \), \( F_1 = (c, 0, \cdots, 0) \) and \( F_2 = (-c, 0, \cdots, 0) \) are collinear.

**Corollary 2.1.** Let \( F_1 \) and \( F_2 \) be two distinct points in Euclidean space \( E^n (2 < n < \infty) \) such that
\[
|F_1 - F_2| = 2c < \frac{1}{2}
\]
and define the set
\[
\Omega = \{ X \in E^n : XF_1F_2 \text{is a triangle of perimeter 1} \}.
\]
Then the locus of all points \( X \in E^n \) is an \( n \)-dimensional rotated ellipsoid drilled by two points. These two points are clearly the vertices of the ellipsoid.

**Lemma 2.2.** Let \( f \) be an injection from Euclidean space \( E^n (2 < n < \infty) \) to itself which preserves triangles of perimeter 1. Then \( f \) preserves the right angles.

**Proof:** Let \( l_1 \) and \( l_2 \) be two distinct lines in \( E^n \) which meets perpendicularly. Denote the common point of these lines by \( F_1 \). Now take a point on \( l_1 \) say \( F_2 \), such that
\[
0 < |F_1 - F_2| = 2c < \frac{1}{2}.
\]
Now following the same way in the proof of Lemma 2.1, one can easily construct the \( n \)-dimensional rotated ellipsoid \( \Omega \) with focal points \( F_1 \) and \( F_2 \). Clearly \( l_1 \) and \( \Omega \) meets at two points, say \( A \) and \( B \). Now draw the Euclidean line passing through \( F_2 \) and parallel to \( l_2 \). Obviously, this line and \( \Omega \) meets at two points and denote them by \( C \) and \( D \). It is clear that either \( AC \parallel BD \) or \( AD \parallel BC \). Without loss of generality we may assume \( AC \parallel BD \). Clearly one can easily see that \( AF_1F_2C \) is a rectangle which consists of four triangles \( AF_1F_2, F_1F_2C, F_1C, F_2A \). The perimeter of these triangles is 1. Clearly, by hypothesis, the perimeter of the triangles \( A'F_1F_2', F_1F_2'C', F_2C'A' \) and \( C'A'F_2' \) is 1, where
\[
f(A) = A', f(C) = C', f(F_1) = F_1', f(F_2) = F_2'.
\]
This implies that \( A'F_1F_2'C' \) is a rectangle, see (Lester 1985). Finally one can easily see that \( AF_1'F_2' \). This yields us \( f(l_1) = f(l_2) \) which finishes the proof.

**Lemma 2.3:** Let \( f \) be an injection from Euclidean space \( E^n (2 < n < \infty) \) to itself which preserves triangles of perimeter 1 and \( a, c \) be two positive real numbers satisfying \( a + c = \frac{1}{2} \) and \( 0 < c < \frac{1}{2} \). If \( A \) and \( B \) are two points satisfying \( |A - B| = 1 - \frac{1}{2}c < \frac{1}{2} \) then \( f \) preserves the midpoint of \( A \) and \( B \).

**Proof:** Let \( F_1 \) and \( F_2 \) be two distinct points in \( E^n \) satisfying \( 0 < |F_1 - F_2| < \frac{1}{2} \). Now construct the \( n \)-dimensional rotated ellipsoid \( \Omega \) with focal points \( F_1, F_2 \) in the same way as in the proof of Lemma 2.1 and Corollary 2.1. Let \( K \) and \( L \) be the vertices of \( \Omega \). Then for each point \( X \) of \( \Omega \) such that \( K = X \neq L \), the perimeter of the triangle \( XF_1F_2 \) is 1. Clearly, there exists appropriate points of \( \Omega \) to get an orthogonal basis of \( E^n \). More precisely, there exists some points on \( \Omega \), say \( X_1, X_2, \cdots, X_{n-1} \) such that the set
\[
\Theta_1 = \{ F_1F_2X_1, F_1X_2, \cdots, F_1X_{n-1} \}
\]
is an orthogonal basis of \( E^n \). Clearly \( K = X_1 \neq L \) for each \( j \in \{1, 2, \cdots, n-1\} \). In addition to this the set
\[
\Theta_2 = \{ F_1F_2X_1, F_1X_2, \cdots, F_1X_{n-1} \}
\]
is also an orthogonal basis of \( E^n \) where \( F_1X_1 = -F_1X_1 \). Since \( f \) preserves the right angles by Lemma 2.2, it is clear that the sets
\[
\Theta_1' = \{ F_1'F_2'X_1', F_1'X_2', \cdots, F_1'X_{n-1}' \}
\]
and
\[
\Theta_2' = \{ F_1'F_2'X_1, F_1X_2, \cdots, F_1X_{n-1} \}
\]
are also orthogonal bases of $E^n$ where $f(Y_i) = Y'_i$, $f(F_i) = F'_i$ and $f(X_j) = X'_j$ for $i \in \{1, 2\}$ and for $j \in \{1, 2, \ldots, n-1\}$. Since $f$ is injective and it preserves the triangles of perimeter 1, one can easily see that $F'_i Y'_i = -F'_i X'_i$ and this implies that $F'_i$ is the midpoint of $X'_i$ and $Y'_i$.

**Corollary 2.2.** Let $f$ be an injection from Euclidean space $E^n$ ($2 < n < \infty$) to itself which preserves triangles of perimeter 1. Then $f$ preserves the Euclidean lines, i.e. $f$ is affine.

Now we are ready to give the general proof of Theorem 1.4 for $2 \leq n < \infty$.

**Proof of Theorem 1.4:** Let $ABC$ be an equilateral triangle with perimeter 1 and denote the midpoints of the sides of $ABC$ by $M'_{ab}, M'_{bc}$ and $M'_{ac}$. Clearly $AM'_{ab} \perp BC, BM'_{bc} \perp AC$ and $CM'_{ac} \perp AB$. Since $f$ preserves the right angles by Lemma 2.2, we get $A'M'_{ac} \perp B'C', B'M'_{bc} \perp A'C'$ and $C'M'_{ab} \perp A'B'$ where $A' = f(A), B' = f(B), C' = f(C)$ and $M'_{ab} = f(M_{ab})$. Moreover by Lemma 2.3, $M'_{ab}, M'_{bc},$ and $M'_{ac}$ are the midpoints of the sides of $A'B'C'$. More precisely

\[
M'_{ab} = \frac{A' + B'}{2}, M'_{bc} = \frac{B' + C'}{2}, M'_{ac} = \frac{A' + C'}{2}
\]

holds. This yields us that the triangle $A'B'C'$ must be an equilateral triangle of perimeter 1. Hence we get $|A' - B'| = |A' - C'| = |B' - C'| = \frac{1}{3}$. Finally we see that $f$ preserves the distance $\frac{1}{3}$. From the Beckman-Quarles theorem (Beckman and Quarles 1953) $f$ must be a motion of $E^n$. It is easy to see that the method we used in the proof of Theorem 1.4 is valid for the proof of Theorem 1.3.

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Chitkara University, Saraswati Kendra, SCO 160-161, Sector 9-C, Chandigarh, 160009, India

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