On multiplier processes under weak moment assumptions

Shahar Mendelson\textsuperscript{1,2}

January 26, 2016

Abstract

We show that if $V \subset \mathbb{R}^n$ satisfies a certain symmetry condition (closely related to unconditionaity) and if $X$ is an isotropic random vector for which $\|\langle X, t \rangle\|_{L^p} \leq L \sqrt{p}$ for every $t \in S^{n-1}$ and $p \lesssim \log n$, then the corresponding empirical and multiplier processes indexed by $V$ behave as if $X$ were $L$-subgaussian.

1 Introduction

The motivation for this work comes from various problems in Learning Theory, in which one encounters the following random process.

Let $X = (x_1, \ldots, x_n)$ be a random vector on $\mathbb{R}^n$ (whose coordinates $(x_i)_{i=1}^n$ need not be independent) and let $\xi$ be a random variable that need not be independent of $X$. Set $(X_i, \xi)_{i=1}^N$ to be $N$ independent copies of $(X, \xi)$, and for $V \subset \mathbb{R}^n$ define the centred multiplier process

$$
\sup_{v \in V} \frac{1}{\sqrt{N}} \sum_{i=1}^N (\xi_i \langle X_i, v \rangle - \mathbb{E} \xi \langle X, v \rangle).
$$

(1.1)

Multiplier processes are often studied in a more general context, in which the indexing class need not be a class of linear functionals on $\mathbb{R}^n$. Instead, one may consider an arbitrary probability space $(\Omega, \mu)$ and in which case $F$ is a class of functions on $\Omega$. Let $X_1, \ldots, X_N$ be independent, distributed

\textsuperscript{1}Department of Mathematics, Technion, I.I.T., Haifa, Israel and Mathematical Sciences Institute, The Australian National University, Canberra, Australia, Email: shahar@tx.technion.ac.il

\textsuperscript{2}Supported in part by the Israel Science Foundation.
according to $\mu$, and the multiplier process indexed by $F$ is
\[
\sup_{f \in F} \left| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (\xi_i f(X_i) - \mathbb{E} \xi f(X_i)) \right|.
\] (1.2)

Naturally, the simplest multiplier process is when $\xi \equiv 1$ and (1.2) is the standard empirical process.

Controlling a multiplier process is relatively straightforward when $\xi \in L_2$ and is independent of $X$. For example, one may show (see, e.g., [20], Chapter 2.9) that if $\xi$ is a mean-zero random variable that is independent of $X_1, \ldots, X_N$ then
\[
\mathbb{E} \sup_{f \in F} \left| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (\xi_i f(X_i) - \mathbb{E} \xi f(X_i)) \right| \leq C \|\xi\|_{L_2} \mathbb{E} \sup_{f \in F} \left| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \varepsilon_i f(X_i) \right|,
\]

where here and throughout the article, $(\varepsilon_i)_{i=1}^{N}$ are independent, symmetric $\{-1, 1\}$-valued random variables that are independent of $(X_i, \xi_i)_{i=1}^{N}$, and $C$ is an absolute constant.

This estimate and others of its kind show that multiplier processes are as 'complex' as their seemingly simpler empirical counterparts. However, the results we are looking for are of a different nature: estimates on multiplier processes that are based on some natural complexity parameter of the underlying class $F$, and that exhibits the class' geometry.

It turns out that chaining methods lead to such estimates, and the structure of $F$ may be captured using the following parameter, which is a close relative of Talagrand’s $\gamma$-functionals [19].

**Definition 1.1** For a random variable $Z$ and $p \geq 1$, set
\[
\|Z\|_{(p)} = \sup_{1 \leq q \leq p} \frac{\|Z\|_{L_q}}{\sqrt{q}}.
\]

Given a class of functions $F$, $u \geq 1$ and $s_0 \geq 0$, put
\[
\Lambda_{s_0,u}(F) = \inf \sup_{f \in F} \sum_{s \geq s_0} 2^{s/2} \|f - \pi_{sf}\|_{(u^2 2^s)},
\] (1.3)

where the infimum is taken with respect to all sequences $(F_s)_{s \geq 0}$ of subsets of $F$, and of cardinality $|F_s| \leq 2^{2^s}$. $\pi_{sf}$ is the nearest point in $F_s$ to $f$ with respect to the $(u^2 2^s)$ norm.

Let
\[
\bar{\Lambda}_{s_0,u}(F) = \Lambda_{s_0,u}(F) + 2^{s_0/2} \sup_{f \in F} \|\pi_{sf}\|_{(u^2 2^{s_0})}.
\]
To put these definitions in some perspective, $\|Z\|_{(p)}$ measures the local-subgaussian behaviour of $Z$, and the meaning of ‘local’ is that $\| \cdot \|_{(p)}$ takes into account the growth of $Z$’s moments up to a fixed level $p$. In comparison, $\|Z\|_{\psi_2} \sim \sup_{q \geq 2} \|Z\|_{L_q}/\sqrt{q}$, implying that for $2 \leq p < \infty$, $\|Z\|_{(p)} \lesssim \|Z\|_{\psi_2}$; hence, for every $u \geq 1$ and $s \geq s_0$,

$$\Lambda_{s_0,u}(F) \lesssim \inf \sup_{f \in F} \sum_{s \geq s_0} 2^{s/2} \|f - \pi_s f\|_{\psi_2},$$

and $\tilde{\Lambda}_{0,u}(F) \leq c\gamma_2(F, \psi_2)$ (see [19] for a detailed study on generic chaining and the $\gamma$ functionals).

Recall that the canonical gaussian process indexed by $F$ consists of centred gaussian random variable $G_f$, and the covariance structure of the process is endowed by the inner product in $L_2(\mu)$. Let

$$\mathbb{E} \sup_{f \in F} G_f = \sup_{f \in F'} \{ \mathbb{E} \sup_{f \in F'} G_f : F' \subset F, F' \text{ is finite} \},$$

and note that if the class $F \subset L_2(\mu)$ is $L$-subgaussian, that is, if for every $f, h \in F \cup 0$,

$$\|f - h\|_{\psi_2(\mu)} \leq L \|f - h\|_{L_2(\mu)},$$

then $\tilde{\Lambda}_{s_0,u}(F)$ may be bounded using the canonical gaussian process indexed by $F$. Indeed, by Talagrand’s Majorizing Measures Theorem [18, 19], for every $s_0 \geq 0$,

$$\tilde{\Lambda}_{s_0,u}(F) \lesssim L \{ \mathbb{E} \sup_{f \in F} G_f + 2^{s_0/2} \sup_{f \in F} \|f\|_{L_2(\mu)} \}.$$  

As an example, let $V \subset \mathbb{R}^n$ and set $F = \{ \langle v, \cdot \rangle : v \in V \}$ to be the class of linear functionals endowed by $V$. If $X$ is an isotropic, $L$-subgaussian vector, it follows that for every $t \in \mathbb{R}^n$,

$$\|\langle X, t \rangle\|_{\psi_2} \leq L \|\langle X, t \rangle\|_{L_2} = L \|t\|_{\ell_2^n}.$$  

Therefore, if $G = (g_1, \ldots, g_n)$ is the standard gaussian vector in $\mathbb{R}^n$, $\ell_s(V) = \mathbb{E} \sup_{v \in V} |\langle G, v \rangle|$ and $d_2(V) = \sup_{v \in V} \|v\|_{\ell_2^n}$, one has

$$\tilde{\Lambda}_{s_0,u}(F) \lesssim L \{ \mathbb{E} \sup_{v \in V} \langle G, v \rangle + 2^{s_0/2} \sup_{v \in V} \|\langle X, v \rangle\|_{L_2} \} \lesssim L (\ell_s(V) + 2^{s_0/2} d_2(V)).$$

As the following estimate from [9] shows, $\tilde{\Lambda}$ can be used to control a multiplier process in a relatively general situation.
Theorem 1.2 For \( q > 2 \), there are constants \( c_0, c_1, c_2, c_3 \) and \( c_4 \) that depend only on \( q \) for which the following holds. Let \( \xi \in L_q \) and set \( \xi_1, \ldots, \xi_N \) to be independent copies of \( \xi \). Fix an integer \( s_0 \geq 0 \) and \( w, u > c_0 \). Then, with probability at least
\[
1 - c_1 w^{-q} N^{-(q/2)-1} \log^q N - 2 \exp(-c_2 u^2 2^{s_0}),
\]
\[
\sup_{f \in F} \left| \frac{1}{\sqrt{N}} \sum_{i=1}^N (\xi_i f(X_i) - \mathbb{E} \xi f) \right| \leq c_3 w u \|\xi\|_{L_q} \tilde{\Lambda}_{s_0, c_4}(F).
\]

It follows from Theorem 1.2 that if \( D(V) = \left( \frac{\ell_2(V)}{d_2(V)} \right)^2 \) then with probability at least
\[
1 - c_2 w^{-q} N^{-(q/2)-1} \log^q N - 2 \exp(-c_3 u^2 D(V)),
\]
\[
\sup_{f \in F} \left| \frac{1}{\sqrt{N}} \sum_{i=1}^N (\xi_i \langle v, X_i \rangle - \mathbb{E} \xi \langle v, X \rangle) \right| \lesssim L w u \|\xi\|_{L_q} \ell_4(V). \tag{1.4}
\]

There are other generic situations in which \( \tilde{\Lambda}_{s_0, u}(F) \) may be controlled using the geometry of \( F \) (for example \[13, 9\] when \( F \) is a class of linear functionals on \( \mathbb{R}^n \) and \( X \) is an unconditional, log-concave random vector). However, there is no satisfactory theory that describes \( \tilde{\Lambda}_{s_0, u}(F) \) for an arbitrary class \( F \); such results are highly nontrivial.

Moreover, because the definition of \( \Lambda_{s_0, u}(F) \) involves \( \| \|_{(p)} \) for every \( p \), class members must have arbitrarily high moments for \( \Lambda_{s_0, u} \) to be well defined.

In the context of classes of linear functionals on \( \mathbb{R}^n \), one expects an analogous result to Theorem 1.2 to be true even if the functionals \( \langle X, t \rangle \) do not have arbitrarily high moments. A realistic conjecture is that if for each \( t \in S^{n-1} \)
\[
\|\langle X, t \rangle\|_{L_q} \leq L \sqrt{q} \|\langle X, t \rangle\|_{L_2}
\]
then a subgaussian-type estimate like (1.4) should still be true.

In what follows we will not focus on such a general result that is likely to hold for every \( V \subset \mathbb{R}^n \). Rather, we will concentrate our attention on situations where a subgaussian estimate like (1.4) is true, but linear functionals only satisfy
\[
\|\langle X, t \rangle\|_{L_q} \leq L \sqrt{q} \|\langle X, t \rangle\|_{L_2} \quad \text{for every } 2 \leq q \lesssim \log n.
\]
The obvious example in which only $\sim \log n$ moments should suffice is $V = B_1^n$ (or similar sets that have $\sim n$ extreme points). Having said that, the applications that motivated this work require a broader spectrum of sets that only need that number of moments to exhibit a subgaussian behaviour as in (1.4).

**Question 1.3** Let $X = (x_1, ..., x_n)$ be an isotropic random vector and assume that $\|x_i\|_{L_q} \leq L\sqrt[q]{q}$ for every $2 \leq q \leq p$. If $\xi \in L_{q_0}$ for some $q_0 > 2$, how small can $p$ be while still having that

$$\mathbb{E} \sup_{v \in V} \left| \frac{1}{N} \sum_{i=1}^{N} \xi \langle X_i, v \rangle - \mathbb{E} \langle X, v \rangle \right| \leq C(L, q_0) \|\xi\|_{L_{q_0} \ell^* (V)}$$

We will show $p \sim \log n$ suffices for a positive answer to Question 1.3 if the norm $\|z\|_{V^\circ} = \sup_{v \in V} |\langle v, z \rangle|$ satisfies the following unconditionality property:

**Definition 1.4** Given a vector $x = (x_i)^n_{i=1}$, let $(x^*_i)^n_{i=1}$ be the non-increasing rearrangement of $(|x_i|)^n_{i=1}$.

The normed space $(\mathbb{R}^n, \|\|)$ is $K$-unconditional with respect to the basis $\{e_1, ..., e_n\}$ if for every $x \in \mathbb{R}^n$ and every permutation of $\{1, ..., n\}$

$$\| \sum_{i=1}^{n} x_i e_i \| \leq K \left( \sum_{i=1}^{n} x_{\pi(i)} e_i \right),$$

and if $y \in \mathbb{R}^n$ and $x^*_i \leq y^*_i$ for $1 \leq i \leq n$ then

$$\| \sum_{i=1}^{n} x_i e_i \| \leq K \left( \sum_{i=1}^{n} y_i e_i \right)$$

**Remark 1.5** This is not the standard definition of an unconditional basis, though every unconditional basis (in the classical sense) on an infinite dimensional space satisfies Definition 1.4 for some constant $K$ (see, e.g., [1]).

There are many natural examples of $K$-unconditional norms, including all the $\ell_p$ norms. Moreover, the norm $\sup_{v \in V} \sum_{i=1}^{n} v^*_i z^*_i$ is 1-unconditional. In fact, if $V \subset \mathbb{R}^n$ is closed under permutations and reflections (sign-changes), then $\| \cdot \|_{V^\circ}$ is 1-unconditional. Finally, since the maximum of two $K$-unconditional norms is $K$-unconditional, it follows that if $\| \cdot \|_{V^\circ}$ is $K$-unconditional, so is the norm $\sup_{v \in V \cap B_2^n} \langle \cdot, v \rangle$.

We will show the following:

5
Theorem 1.6 There exists an absolute constant $c_1$ and for $K \geq 1$, $L \geq 1$ and $q_0 > 2$ there exists a constant $c_2$ that depends only on $K$, $L$ and $q_0$ for which the following holds. Consider

- $V \subset \mathbb{R}^n$ for which the norm $\| \cdot \|_V = \sup_{v \in V} |\langle v, \cdot \rangle|$ is $K$-unconditional with respect to the basis $\{e_1, \ldots, e_n\}$.
- $\xi \in L_{q_0}$ for some $q_0 > 2$.
- An isotropic random vector $X \in \mathbb{R}^n$ which satisfies that
  \[ \max_{1 \leq j \leq n} \| \langle X, e_j \rangle \|_p \leq L \]  for $p = c_1 \log n$.

If $(X_i, \xi_i)_{i=1}^N$ are independent copies of $(X, \xi)$ then

\[ \mathbb{E} \sup_{v \in V} \left| \frac{1}{\sqrt{N}} \sum_{i=1}^N (\xi_i \langle X, v \rangle - \mathbb{E} \xi \langle X, v \rangle) \right| \leq c_2 \| \xi \|_{L_{q_0} \ell_\ast(V)}. \]

The proof of Theorem 1.6 is based on the study of a conditioned Bernoulli process. Indeed, a standard symmetrization argument (see, e.g., [8, 20]) shows that if $(\varepsilon_i)_{i=1}^N$ are independent, symmetric, $\{-1, 1\}$-valued random variables that are independent of $(X_i, \xi_i)_{i=1}^N$ then

\[ \mathbb{E} \sup_{v \in V} \left| \frac{1}{\sqrt{N}} \sum_{i=1}^N \varepsilon_i \xi_i \langle X, v \rangle \right| \leq C \mathbb{E} \sup_{v \in V} \left| \frac{1}{\sqrt{N}} \sum_{i=1}^N \varepsilon_i \xi_i \langle X, v \rangle \right| \]

for an absolute constant $C$; a similar bound hold with high probability, showing that it suffices to study the supremum of the conditioned Bernoulli process

\[ \sup_{v \in V} \left| \frac{1}{\sqrt{N}} \sum_{i=1}^N \varepsilon_i \xi_i \langle X, v \rangle \right| = (\ast). \]

Put $x_i(j) = \langle X_i, e_j \rangle$ and set $Z_j = N^{-1/2} \sum_{i=1}^N \varepsilon_i \xi_i x_i(j)$, which is a sum of iid random variables. Therefore, if $Z = (Z_1, \ldots, Z_n)$ then

\[ (\ast) = \sup_{v \in V} \langle Z, v \rangle. \]

The proof of Theorem 1.6 follows by showing that for a well-chosen constant $C(L, q)$ the event

\[ \{ Z_j^* \leq C \mathbb{E} g_j^* \text{ for every } 1 \leq j \leq n \} \]
is of high probability, and if the norm $\|\cdot\|_{V^*} = \sup_{v \in V} \langle \cdot, v \rangle$ is $K$-unconditional then

$$\sup_{v \in V} \langle Z, v \rangle \leq C_1(K, L, q) \mathbb{E} \sup_{v \in V} \langle G, v \rangle.$$ 

Before presenting the proof of Theorem 1.6, let us turn to one of its outcomes – estimates on the random Gelfand widths of a convex body. We will present another application, motivated by a question in the rapidly developing area of Spare Recovery in Section 3.

Let $V \subset \mathbb{R}^n$ be a convex, centrally symmetric set. A well known question in Asymptotic Geometric Analysis has to do with the diameter of a random $m$-codimensional section of $V$ (see, e.g., [14, 15, 16, 2]). In the past, the focus was on obtaining such estimates for subspaces selected uniformly according to the Haar measure, or alternatively, according to the measure endowed via the kernel of an $m \times n$ gaussian matrix (see, e.g. [17]). More recently, there has been a growing interest in other notions of randomness, most notably, generated by kernels of other random matrix ensembles. For example, the following was established in [12]:

**Theorem 1.7** Let $X_1, \ldots, X_m$ be distributed according to an isotropic, $L$-subgaussian random vector on $\mathbb{R}^n$, set $\Gamma = \sum_{i=1}^m \langle X_i, \cdot \rangle e_i$ and put

$$r_G(V, \gamma) = \inf \{ r > 0 : \ell_*(V \cap rB_2^n) \leq \gamma r \sqrt{m} \}.$$

Then, with probability at least $1 - 2 \exp(-c_1(L)m)$

$$\text{diam}(\ker(\Gamma) \cap V) \leq r_G(V, c_2(L)),$$

for constants $c_1$ and $c_2$ that depends only on $L$.

A version of Theorem 1.7 was obtained under a much weaker assumption: the random vector need not be $L$-subgaussian; rather, it suffices that it satisfies a weak small-ball condition.

**Definition 1.8** The isotropic random vector $X$ satisfies a small-ball condition with constants $\kappa > 0$ and $0 < \varepsilon \leq 1$ if for every $t \in S^{n-1}$,

$$\Pr(\langle X, t \rangle \geq \kappa) \geq \varepsilon.$$

The analog of gaussian parameter $r_G$ for a general random vector $X$ is

$$r_X(V, \gamma) = \inf \{ r > 0 : \mathbb{E} \sup_{v \in V \cap rB_2^n} | \frac{1}{\sqrt{m}} \sum_{i=1}^m \langle X_i, v \rangle | \leq \gamma r \sqrt{m} \}.$$
Clearly, if $X$ is $L$-subgaussian then $r_X(V, \gamma) \leq r_G(V, cL\gamma)$ for a suitable absolute constant $c$.

**Theorem 1.9** \cite{11, 10} Let $X$ be an isotropic random vector that satisfies the small-ball condition with constants $\kappa$ and $\epsilon$. If $X_1, \ldots, X_m$ are independent copies of $X$ and $\Gamma = \sum_{i=1}^{m} \langle X_i, \cdot \rangle e_i$, then with probability at least 

$$1 - 2 \exp(-c_0(\epsilon)m)$$

$$\text{diam} (\ker(\Gamma) \cap V) \leq r_X(V, c_1(\kappa, \epsilon)).$$

Theorem \cite{11} implies that if the norm $\|z\|_{V^\circ}$ is $K$-unconditional, and the growth of moments of the coordinate linear functionals $\langle X, e_i \rangle$ for $1 \leq i \leq n$ is $L$-subgaussian’ up to the level $\sim \log n$, then the small-ball condition depends only on $L$ and $r_X(V, c_1(L)) \leq r_G(V, c_2(L, K))$. Therefore, with probability at least $1 - 2 \exp(-c_0(L)m)$ one has the gaussian estimate:

$$\text{diam} (\ker(\Gamma) \cap V) \leq r_G(V, c_2(L, K)),$$

even though the choice of a subspace has been made according to an ensemble that could be very far from a subgaussian one.

We end this introduction with a word about notation. Throughout, absolute constants are denoted by $c, c_1, \ldots$, etc. Their value may change from line to line or even within the same line. When a constant depends on a parameter $\alpha$ it will be denoted by $c(\alpha)$. $A \lesssim B$ means that $A \leq cB$ for an absolute constant $c$, and the analogous two-sided inequality is denoted by $A \sim B$. In a similar fashion, $A \lesssim_{\alpha} B$ implies that $A \leq c(\alpha)B$, etc.

## 2 Proof of Theorem \cite{1.6}

There are two substantial difficulties in the proof of Theorem \cite{1.6}. First, $Z_1, \ldots, Z_n$ are not independent random variables, not only because of the Bernoulli random variables $(\epsilon_i)_{i=1}^N$ that appear in all the $Z_i$’s, but also because the coordinates of $X = (x_1, \ldots, x_n)$ need not be independent. Second, while there is some flexibility in the moment assumptions on the coordinates of $X$, there is no flexibility in the moment assumption on $\xi$, which is only ‘slightly better’ than square-integrable.

As a starting point, let us address the fact that the coordinates of $Z$ need not be independent.
Lemma 2.1 There exist absolute constants \( c_1 \) and \( c_2 \) for which the following holds. Let \( \beta \geq 1 \) and set \( p = 2\beta \log(en) \). If \( (W_j)_{j=1}^n \) are random variables and satisfy that \( \|W_j\|_p \leq L \), then for every \( t \geq 1 \), with probability at least \( 1 - c_1 t^{-2\beta} \),

\[
W_j^* \leq c_2 t L \sqrt{\beta \log(en/j)} \quad \text{for every } 1 \leq j \leq n.
\]

Proof. Let \( a_1, \ldots, a_k \in \mathbb{R} \) and by the convexity of \( t \to t^q \),

\[
\left( \frac{1}{k} \sum_{j=1}^k a_j^2 \right)^{\frac{1}{q}} \leq \frac{1}{k} \sum_{j=1}^k a_j^2.
\]

Thus, given \( (a_i)_{i=1}^n \), and taking the maximum over subsets of \( \{1, \ldots, n\} \) of cardinality \( k \),

\[
\max_{|J_1| = k} \left( \frac{1}{k} \sum_{j \in J_1} a_j^2 \right)^{\frac{1}{q}} \leq \max_{|J_1| = k} \frac{1}{k} \sum_{j \in J_1} a_j^2 \leq \frac{1}{k} \sum_{j=1}^n a_j^2.
\]

When applied to \( a_j = W_j \), it follows that point-wise,

\[
\left( \frac{1}{k} \sum_{j=1}^k (W_j^*)^2 \right)^{\frac{1}{q}} \leq \frac{1}{k} \sum_{i=1}^n W_j^2. \tag{2.1}
\]

Since \( \|W_j\|_p \leq L \) it is evident that \( E W_j^{2q} \leq L^{2q} q^{q} \) for \( 2q \leq p \). Hence, taking the expectation in (2.1),

\[
\left( E \left( \frac{1}{k} \sum_{j=1}^k (W_j^*)^2 \right)^{\frac{1}{q}} \right)^{1/q} \leq q L^2 \cdot \left( \frac{n}{k} \right)^{1/q} \leq c_1 q L^2
\]

for \( q = \beta \log(en/k) \) (which does satisfy \( 2q \leq p \)). Hence, by Chebyshev’s inequality, for \( t \geq 1 \),

\[
Pr \left( \frac{1}{k} \sum_{j=1}^k (W_j^*)^2 \geq (et)^2 c_1^2 L^2 q \right) \leq \frac{1}{t^{2q}} \cdot e^{-2q} = \left( \frac{1}{en} \right)^2 \cdot \frac{1}{t^{-2q}}. \tag{2.2}
\]

Using (2.2) for \( k = 2^j \) and applying the union bound, it is evident that with probability at least \( 1 - 2t^{-2\beta} \), for every \( 1 \leq k \leq n \),

\[
(W_k^*)^2 \leq \frac{1}{k} \sum_{j=k}^n (W_j^*)^2 \leq t^2 L^2 \beta \log(en/k).
\]
Recall that \( q_0 > 2 \) and set \( \eta = (q_0 - 2)/4 \). Let \( u \geq 2 \) and consider the event
\[
A_u = \{ \xi^*_i \leq u\|\xi\|_{L_{q_0}} (eN/i)^{1/q_0} \text{ for every } 1 \leq i \leq N \}.
\]
A standard binomial estimate combined with Chebyshev’s inequality for \(|\xi|_{q_0}\) shows that \( A_u \) is a nontrivial event. Indeed,
\[
Pr \left( \xi^*_i \geq u\|\xi\|_{L_{q_0}} (eN/i)^{1/q_0} \right) \leq \left( \frac{N}{i} \right) Pr^i \left( \xi \geq u\|\xi\|_{L_{q_0}} (eN/i)^{1/q_0} \right) \leq \frac{1}{u^{q_0}},
\]
and by the union bound for \( 1 \leq i \leq n \), \( Pr(A_u) \leq 2/u^{q_0} \).

The random variables we shall use in Lemma 2.1 are
\[
W_j = \frac{Z_j}{BD} 1_{A_u},
\]
for \( u \geq 2 \) and \( 1 \leq j \leq n \).

The following lemma is the crucial step in the proof of Theorem 1.6.

**Lemma 2.2** There exists an absolute constant \( c \) for which the following holds. Let \( X \) be a random variable that satisfies \( \|X\|_{(p)} \leq L \) for some \( p > 2 \) and set \( X_1, \ldots, X_N \) to be independent copies of \( X \). If
\[
W = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \xi_i X_i 1_{A_u},
\]
then \( \|W\|_{(p)} \leq cuL \).

The proof of Lemma 2.2 requires two preliminary estimates on the ‘gaussian’ behaviour of a monotone rearrangements of \( N \) copies of a random variable.

**Lemma 2.3** There exists an absolute constant \( c \) for which the following holds. Assume that \( \|X\|_{(2p)} \leq L \). If \( X_1, \ldots, X_N \) are independent copies of \( X \), then for every \( 1 \leq k \leq N \) and \( 2 \leq q \leq p \),
\[
\| (\sum_{i \leq k} (X^*_i)^2 )^{1/2} \|_{L_q} \leq cL(\sqrt{k \log(eN/k)} + \sqrt{q}).
\]

**Proof.** The proof follows from a comparison argument, showing that up to the \( p \)-th moment, the ‘worst case’ is when \( X \) is a gaussian variable.
Let $V_1, \ldots, V_k$ be independent, nonnegative random variables and set $V'_1, \ldots, V'_k$ to be independent and nonnegative as well. Observe that if $\|V_i\|_{L_q} \leq L\|V'_i\|_{L_q}$ for every $1 \leq q \leq p$ and $1 \leq i \leq N$, then

$$\|\sum_{i=1}^{k} V_i\|_{L_p} \leq L\|\sum_{i=1}^{k} V'_i\|_{L_p}.$$  \hspace{1cm} (2.3)

Indeed, consider all the integer-valued vectors $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k)$, where $\alpha_i \geq 0$ and $\sum_{i=1}^{k} \alpha_i = p$. There are constants $c_{\vec{\alpha}}$ for which

$$\|\sum_{i=1}^{k} V_i\|_{L_p}^p = \mathbb{E} \left( \sum_{i=1}^{k} V_i \right)^p = \mathbb{E} \sum_{\vec{\alpha}} c_{\vec{\alpha}} \prod_{i=1}^{k} V_i^{\alpha_i} = \sum_{\vec{\alpha}} c_{\vec{\alpha}} \prod_{i=1}^{k} \mathbb{E} V_i^{\alpha_i},$$

and an identical type of estimate holds for $(V'_i)$. (2.3) follows if

$$\prod_{i=1}^{k} \mathbb{E} V_i^{\alpha_i} \leq L^p \prod_{i=1}^{k} \mathbb{E} (V'_i)^{\alpha_i},$$

and the latter may be verified because $\|V_i\|_{L_q} \leq L\|V'_i\|_{L_q}$ for $1 \leq q \leq p$.

Let $G = (g_i)_{i=1}^{k}$ be a vector whose coordinates are independent standard gaussian random variables. If $V_i = X_i^2$ and $V'_i = c^2 L^2 g_i^2$, then by (2.3), for every $1 \leq q \leq p$,

$$\|\sum_{i=1}^{k} X_i^2\|_{L_q} \leq c^2 L^2 \|\sum_{i=1}^{k} g_i^2\|_{L_q} = c^2 L^2 \left( \mathbb{E} \|G\|_{L_2}^{2q} \right)^{1/q}.$$  \hspace{1cm} (2.4)

It is standard to verify that

$$\mathbb{E} \|G\|_{L_2}^{2q} \leq c^{2q} (\sqrt{k} + \sqrt{q})^{2q},$$

and therefore,

$$\|\sum_{i=1}^{k} X_i^2\|_{L_q} \lesssim L^2 \max\{k, q\}.$$  \hspace{1cm} (2.5)

By a binomial estimate,

$$\Pr \left( \sum_{i \leq k} (X_i^2) \geq t^2 \right) \leq \binom{N}{k} \Pr \left( \sum_{i \leq k} X_i^2 \geq t^2 \right) \leq \binom{N}{k} t^{-2q} \|\sum_{i \leq k} X_i^2\|_{L_q}^q \lesssim \left( \frac{eN}{k} \right)^k t^{-2q} \cdot L^{2q} \max\{k, q\}^q,$$
and if \( q \geq k \log(eN/k) \) and \( t = euL\sqrt{q} \) for \( u \geq 1 \) then
\[
Pr\left( \left( \sum_{i \leq k} (X^*_i)^2 \right)^{1/2} \geq euL\sqrt{q} \right) \leq u^{-2q}.
\] (2.4)

Hence, setting \( q = k \log(eN/k) \), tail integration implies that
\[
\| (\sum_{i \leq k} (X^*_i)^2)^{1/2} \|_{L_q} \lesssim L\sqrt{k \log(eN/k)},
\]
and if \( q \geq k \log(eN/k) \), one has
\[
\| (\sum_{i \leq k} (X^*_i)^2)^{1/2} \|_{L_q} \lesssim L\sqrt{q},
\]
as claimed.

The second preliminary result we require also follows from a straightforward binomial estimate:

**Lemma 2.4** Assume that \( \|X\|_p \leq L \) and let \( X_1, \ldots, X_N \) be independent copies of \( X \). Consider \( s \geq 1 \), \( 1 \leq q \leq p \) and \( 1 \leq k \leq N \) that satisfies that \( k \log(eN/k) \geq q \). Then
\[
\| (\sum_{i > k} (X^*_i)^s)^{1/s} \|_{L_q} \leq c(s)LN^{1/s},
\]
for a constant \( c(s) \) that depends only on \( s \).

**Proof.** Clearly, for every \( 1 \leq i \leq N \) and \( 2 \leq r \leq p \),
\[
Pr (X^*_i \geq t) \leq \binom{N}{i} Pr^i (X \geq t) \leq \binom{N}{i} \left( \frac{\|X\|_{L_r}}{t^r} \right)^i \leq \left( \frac{eN}{i} \cdot \frac{L^r r^{r/2}}{t^r} \right)^i.
\]

Hence, if \( t = L\sqrt{r} \cdot eu \) for \( u \geq 2 \) and \( r = 3\log(eN/i) \), then
\[
Pr \left( X^*_i \geq u \cdot eL\sqrt{3\log(eN/i)} \right) \leq u^{-3i\log(eN/i)}. \tag{2.5}
\]
Applying the union bound for every \( i \geq k \), it follows that for \( u \geq 4 \), with probability at least \( 1 - (u/2)^{-3k\log(eN/k)} \),
\[
X^*_i \leq u \cdot eL\sqrt{3\log(eN/i)}, \text{ for every } k \leq i \leq N. \tag{2.6}
\]
On that event
\[ \left( \sum_{i \geq k} (X_i^*)^s \right)^{1/s} \leq c(s) u LN^{1/s}, \]
and since \( k \log(eN/k) \geq q \), tail integration shows that
\[ \| \left( \sum_{i \geq k} (X_i^*)^s \right)^{1/s} \|_{L_q} \leq c_1 LN^{1/s}. \]

Proof of Lemma 2.2. Recall that \( q_0 = 2 + 4 \eta \), that \( \xi \in L^{q_0} \) and that
\[ W = \frac{1}{\sqrt{N}} \sum_{i=1}^N \varepsilon_i \xi_i X_i I_{A_u}. \]
Note that for every \( (a_i)_{i=1}^N \in \mathbb{R}^N \) and any integer \( 0 \leq k \leq N \),
\[ \left\| \sum_{i=1}^N \varepsilon_i a_i \right\|_{L_q} \lesssim \sum_{i \leq k} a_i^* + \sqrt{q} \left( \sum_{i > k} (a_i^*)^2 \right)^{1/2} \tag{2.7} \]
where the two extreme cases of \( k = 0 \) and \( k = N \) mean that one of the terms in (2.7) is 0.

Set \( r = 1 + \frac{\eta}{q_0} \) and put \( \theta = 1/q_0 \). Since \( (\varepsilon_i)_{i=1}^N \) are independent of \( (X_i, \xi)_{i=1}^N \) and using the definition of the event \( A_u \),
\[ N^{q/2} \mathbb{E} W^q = N^{q/2} \mathbb{E} I_{A_u} \mathbb{E} W^q \leq c^q \mathbb{E} I_{A_u} \left( \left( \sum_{i \leq k} \xi_i^* X_i^* \right)^q + q^{q/2} \left( \sum_{i > k} (\xi_i^*)^2 (X_i^*)^2 \right)^{q/2} \right) \]
\[ \leq c^q \mathbb{E} X \left( \left( \sum_{i \leq k} (N/i)^\theta X_i^* \right)^q + q^{q/2} \left( \sum_{i > k} (N/i)^{2\theta} (X_i^*)^2 \right)^{q/2} \right). \]

By the Cauchy-Schwarz inequality,
\[ \left( \sum_{i \leq k} (N/i)^\theta X_i^* \right)^q \leq \left( \sum_{i \leq k} (N/i)^{2\theta} \right)^{q/2} \left( \sum_{i \leq k} (X_i^*)^2 \right)^{q/2}, \]
and
\[ \sum_{i \leq k} (N/i)^{2\theta} = \sum_{i \leq k} (N/i)^{1/1+2\eta} \leq \frac{c_1}{\eta} N^{1/(1+2\eta)} k^{2\eta/(1+2\eta)} \leq \frac{c_1}{\eta} N. \]
Therefore,
\[ \mathbb{E} \left( \sum_{i \leq k} (N/i)^\theta X_i^* \right)^q \lesssim \eta^{-q/2} N^{q/2} \mathbb{E} \left( \sum_{i \leq k} (X_i^*)^2 \right)^{q/2} = (*). \]
Also, by Hölder’s inequality for \( r = 1 + \eta \) and its conjugate index \( r' \),
\[
\left( \sum_{i > k} (N/i)^{2\theta} (X_i^*)^2 \right)^{q/2} \leq \left( \sum_{i > k} (N/i)^{2\theta r} \right)^{q/2r} \cdot \left( \sum_{i > k} (X_i^*)^{2r'} \right)^{q/2r'}
\]
and
\[
\sum_{i \geq k} (N/i)^{2\theta r} = \sum_{i \geq k} (N/i)^{(1+\eta)/(1+2\eta)} \leq \frac{C_1}{\eta} N.
\]

Hence,
\[
\mathbb{E} \left( \sum_{i > k} (N/i)^{2\theta} (X_i^*)^2 \right)^{q/2} \leq \eta^{-q/2r} N^{q/2r} \mathbb{E} \left( \sum_{i > k} (X_i^*)^{2r'} \right)^{q/2r'} = (**).
\]

Let \( k \in \{0, \ldots, N\} \) be the smallest that satisfies \( k \log(eN/k) \geq q \) (and without loss of generality we will assume that such a \( k \) exists; if it does not, the modifications to the proof are straightforward and are omitted).

Applying Lemma 2.3 for that choice of \( k \),
\[
(*) \leq c^q \eta^{-q/2} N^{q/2} \cdot L^q (\sqrt{k \log(eN/k)} + \sqrt{q})^q \leq c_1^q \eta^{-q/2} L^q N^{q/2} q^{q/2}.
\]

Turning to (**), set \( s = 2r' \sim \max\{\eta^{-1}, 2\} \) and one has to control
\[
\mathbb{E} \left( \sum_{i > k} (X_i^*)^s \right)^{q/s}
\]
for the choice of \( k \) as above. By Lemma 2.4
\[
\mathbb{E} \left( \sum_{i > k} (X_i^*)^s \right)^{q/s} \leq c^q(s) L^q N^{q/2r} = c_1^q(\eta) L^q N^{q/2r'}.
\]
Therefore,
\[
(**) \leq c^q(\eta) L^q N^{q/2r} \cdot N^{q/2r'} = c^q(\eta) L^q N^{q/2}.
\]

Combining the two estimates,
\[
N^{q/2} \mathbb{E} W^q \leq N^{q/2} \cdot c^q(\eta) L^q q^{q/2},
\]
implying that \( \|W\|_{L_q} \leq c(\eta) L \).

**Proof of Theorem 1.6.** By Lemma 2.2 for every \( 1 \leq j \leq n \), \( \|W_j\|_p \leq c(\eta) L \), and thus, by Lemma 2.1 with probability at least \( 1 - c_1 t^{-2\beta} \),
\[
W_j^* \leq c(\eta) t L \sqrt{\beta \log(en/j)} \quad \text{for every } 1 \leq j \leq n.
\]
Moreover, \( \Pr(A_n) \geq 1 - 2/u_q \); therefore, with probability at least \( 1 - c_1 t^{-2\beta} - 2u_q \), for every \( 1 \leq j \leq n \),

\[
Z_j^* \leq c(\eta)t u L\|\xi\|_{Lq_0} \sqrt{\beta \log(eN/j)}.
\]

Hence, on that event and because the norm \( \sup_{v \in V} |\langle v, \cdot \rangle| \) is \( K \) unconditional,

\[
\sup_{v \in V} |\langle Z, v \rangle| \leq K c(\eta) \sqrt{\beta tu L\|\xi\|_{Lq_0}} \sup_{v \in V} |\langle Z_0, v \rangle|,
\]

for a fixed vector \( Z_0 \) whose coordinates are \( (\sqrt{\log(en/j)})_{j=1}^n \). Observe that \( |\langle Z_0, e_j \rangle| \lesssim E g_j^* \), and thus

\[
\sup_{v \in V} |\langle Z_0, v \rangle| \leq K \sup_{v \in V} \left\| \sum_{i=1}^n v_i E g_i^* \right\|.
\]

Therefore, by Jensen’s inequality, with probability at least \( 1 - t^{-2\beta} - 2u_q \),

\[
\sup_{v \in V} \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^N \varepsilon_i x_i(j) \right\| = \sup_{v \in V} |\langle Z, v \rangle| \lesssim K c(\eta) \sqrt{\beta tu L\|\xi\|_{Lq_0}} \sup_{v \in V} |\langle G, v \rangle|.
\]

And, fixing \( \beta \) and integrating the tails,

\[
E \sup_{v \in V} \left\| \frac{1}{\sqrt{N}} \sum_{i=1}^N \varepsilon_i x_i(j) \right\| \lesssim K_{\eta, L} \|\xi\|_{Lq_0} \ell_*(V),
\]

as claimed.

\section{Applications in Sparse Recovery}

Sparse recovery is a central topic in modern statistics and signal processing, though the problem we describe below is far from its most general form. Because a detailed description of the subtleties of sparse recovery would be unreasonably lengthy, some statements may appear a little vague. For more information on sparse recovery we refer the reader to the books [3, 5, 4], which are devoted to this topic.

The question in sparse recovery is to identify, or at least approximate, an unknown vector \( v_0 \in \mathbb{R}^n \), and to do so using relatively few linear measurements. The measurements one is given are ‘noisy’, of the form

\[
Y_i = \langle v_0, X_i \rangle - \xi_i \quad \text{for } 1 \leq i \leq N;
\]
$X_1, \ldots, X_N$ are independent copies of a random, isotropic vector $X$ and 
$\xi_1, \ldots, \xi_N$ are independent copies of a random variable $\xi$ that belongs to $L_q$ for some $q > 2$.

The reason for the name “sparse recovery” is that one assumes that $v_0$ is sparse: it is supported on at most $s$ coordinates, though the identity of the support itself is not known. Thus, one would like to use the given random data $(X_i, Y_i)_{i=1}^N$ and select $\hat{v}$ in a wise way, leading to a high probability estimate on the error rate $\|\hat{v} - v_0\|_{\ell_2}$ as a function of the number of measurements $N$ and of the ‘degree of sparsity’ $s$.

In the simplest recovery problem, $\xi = 0$ and the data is noise-free. Alternatively, one may assume that the $\xi_i$’s are independent of $X_1, \ldots, X_N$, or, in a more general formulation, very little is assumed on the $\xi_i$’s.

The standard method of producing $\hat{v}$ in a noise-free problem and when $v_0$ is assumed to be sparse is the basis pursuit algorithm. The algorithm produces $\hat{v}$, which is the point with the smallest $\ell_1$ norm that satisfies 
$$
\langle X_i, v_0 \rangle = \langle X_i, v \rangle \quad \text{for every } 1 \leq i \leq N.
$$

It is well known [12] that if $X$ is isotropic and $L$-subgaussian, $v_0$ is supported on at most $s$ coordinates and one is given

$$
N = c(L) s \log \left( \frac{en}{s} \right) \quad (3.1)
$$

random measurements $(\langle X_i, v_0 \rangle)_{i=1}^N$, then with high probability, the basis pursuit algorithm has a unique solution and that solution is $v_0$.

Recently, it has been observed in [6] that the subgaussian assumption can be relaxed: the same number of measurements as in (3.1) suffice for a unique solution if

$$
\max_{1 \leq j \leq n} \| \langle X, e_j \rangle \|_{(p)} \leq L \quad \text{for } p \sim \log n.
$$

And, the estimate of $p \sim \log n$ happens to be almost optimal. There is an example of an isotropic vector $X$ with iid coordinates for which

$$
\max_{1 \leq j \leq n} \| \langle X, e_j \rangle \|_{(p)} \leq L \quad \text{for } p \sim (\log n)/\log \log n \quad (3.2)
$$

but still, with probability $1/2$ the basis pursuit algorithm does not recover even a 1-sparse vector $v_0$ given the same number of random measurements as in (3.1).

Since ‘real world’ data is not noise-free, some effort has been invested in producing analogs of the basis pursuit algorithm in a ‘noisy’ setup. The
most well known among these procedures is the LASSO (see, e.g., the books \[3, 5\] for more details) in which \( \hat{v} \) is selected to be the minimizer in \( \mathbb{R}^n \) of the functional

\[
v \to \frac{1}{N} \sum_{i=1}^{N} (\langle v, X_i \rangle - Y_i)^2 + \lambda \|v\|_{\ell_1^n},
\]

for a well-chosen of \( \lambda \).

Following the introduction of the LASSO, there have been many variations on the same theme – by changing the penalty \( \|\|_{\ell_1^n} \) and replacing it with other norms. Until very recently, the behaviour of most of these procedures has been studied under very strong assumptions on \( X \) and \( \xi \) – usually, that \( X \) and \( \xi \) are independent and gaussian, or at best, subgaussian.

One may show that Theorem 1.6 can be used to extend the estimates on \( \|\hat{v} - v_0\|_{\ell_2^n} \) beyond the gaussian case thanks to two significant facts:

- The norms used in the LASSO and in many of its modifications happen to have a 1-unconditional dual: for example, among these norms are weighted \( \ell_1^n \) norms, mixtures of the \( \ell_1^n \) and the \( \ell_2^n \) norms, norms that are invariant under permutations, etc.

- As noted in [7], if \( \Psi \) is a norm, \( B_\Psi \) is its unit ball and \( \hat{v} \) is the minimizer in \( \mathbb{R}^n \) of the functional

\[
v \to \frac{1}{N} \sum_{i=1}^{N} (\langle v, X_i \rangle - Y_i)^2 + \lambda \Psi(v),
\]

then the key to controlling \( \|\hat{v} - v\|_{\ell_2^n} \) is the behaviour of

\[
\sup_{v \in B_\Psi \cap \epsilon B_2^n} \left| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \xi_i \langle X_i, v \rangle - \mathbb{E} \xi \langle X, v \rangle \right|,
\]

which is precisely the type of process that Theorem 1.6 deals with.

It follows from Theorem 1.6 that if \( \xi \in L_q \) for some \( q > 2 \), the expectation of (3.5) is the same as if \( \xi \) and \( X \) were independent and gaussian. Thus, under those conditions, one can expect the ‘gaussian’ error estimate in procedures like (3.4). Moreover, because of (3.2), the condition that linear forms exhibit a subgaussian growth of moments up to \( p \sim \log n \) is necessary, making the outcome of Theorem 1.6 optimal in this context.
The following is a simplified version of an application of Theorem 1.6. We refer the reader to [7] for its general formulation, as well as for other examples of a similar nature.

Let $X$ be an isotropic measure on $\mathbb{R}^n$ that satisfies $\max_{1 \leq j \leq n} \|\langle X, e_j \rangle\|_p \leq L$ for $p \leq c_0 \log(n)$. Set $\xi \in L_q$ for $q > 2$ that is mean-zero and independent of $X$ and put $Y = \langle X, v_0 \rangle - \xi$.

Given an independent sample $(X_i, Y_i)_{i=1}^N$ selected according to $(X, Y)$, let $\hat{v}$ be the minimizer of the functional (3.3).

**Theorem 3.1** Assume that $v_0$ is supported on at most $s$ coordinates and let $0 < \delta < 1$. If $\lambda = c_1(L, \delta)\|\xi\|_{L_q} \sqrt{\log(en)/N}$, then with probability at least $1 - \delta$, for every $1 \leq p \leq 2$

$$\|\hat{v} - v_0\|_p \leq c_2(L, \delta)\|\xi\|_{L_q} s^{1/p} \sqrt{\log(ed) N}.$$ 

The proof of Theorem 3.1 follows by combining Theorem 3.2 from [7] with Theorem 1.6.

**References**

[1] Fernando Albiac and Nigel J. Kalton. *Topics in Banach space theory*, volume 233 of Graduate Texts in Mathematics. Springer, New York, 2006.

[2] Shiri Artstein-Avidan, Apostolos Giannopoulos, and Vitali D. Milman. *Asymptotic geometric analysis. Part I*, volume 202 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2015.

[3] Peter Bühlmann and Sara van de Geer. *Statistics for high-dimensional data*. Springer Series in Statistics. Springer, Heidelberg, 2011. Methods, theory and applications.

[4] Simon Foucart and Holger Rauhut. *A mathematical introduction to compressive sensing*. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, New York, 2013.

[5] Vladimir Koltchinskii. *Oracle inequalities in empirical risk minimization and sparse recovery problems*, volume 2033 of Lecture Notes in Mathematics. Springer, Heidelberg, 2011. Lectures from the 38th Probability Summer School held in Saint-Flour, 2008, École d’Été de Probabilités de Saint-Flour. [Saint-Flour Probability Summer School].

[6] Guillaume Leceré and Shahar Mendelson. Sparse recovery under weak moment assumptions. Technical report, CNRS, ENSAE and Technion, 2014. To appear in Journal of the European Mathematical Society.

[7] Guillaume Leceré and Shahar Mendelson. Regularization and the small-ball method i: sparse recoevry. Technical report, CNRS, ENSAE and Technion, I.I.T., 2015.

[8] Michel Ledoux and Michel Talagrand. *Probability in Banach spaces*, volume 23 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1991. Isoperimetry and processes.
[9] Shahar Mendelson. Upper bounds on product and multiplier empirical processes. Technical report. To appear in Stochastic Processes and their Applications.

[10] Shahar Mendelson. Learning without concentration for general loss function. Technical report, Technion, I.I.T., 2013. arXiv:1410.3192.

[11] Shahar Mendelson. A remark on the diameter of random sections of convex bodies. In Geometric aspects of functional analysis, volume 2116 of Lecture Notes in Math., pages 395–404. Springer, Cham, 2014.

[12] Shahar Mendelson, Alain Pajor, and Nicole Tomczak-Jaegermann. Reconstruction and subgaussian operators in asymptotic geometric analysis. Geom. Funct. Anal., 17(4):1248–1282, 2007.

[13] Shahar Mendelson and Grigoris Paouris. On generic chaining and the smallest singular value of random matrices with heavy tails. J. Funct. Anal., 262(9):3775–3811, 2012.

[14] V. D. Milman. Random subspaces of proportional dimension of finite-dimensional normed spaces: approach through the isoperimetric inequality. In Banach spaces (Columbia, Mo., 1984), volume 1166 of Lecture Notes in Math., pages 106–115. Springer, Berlin, 1985.

[15] A. Pajor and N. Tomczak-Jaegermann. Nombres de Gel’fand et sections euclidiennes de grande dimension. In Séminaire d’Analyse Fonctionnelle 1984/1985, volume 26 of Publ. Math. Univ. Paris VII, pages 37–47. Univ. Paris VII, Paris, 1986.

[16] Alain Pajor and Nicole Tomczak-Jaegermann. Subspaces of small codimension of finite-dimensional Banach spaces. Proc. Amer. Math. Soc., 97(4):637–642, 1986.

[17] Gilles Pisier. The volume of convex bodies and Banach space geometry, volume 94 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1989.

[18] Michel Talagrand. Regularity of Gaussian processes. Acta Math., 159(1-2):99–149, 1987.

[19] Michel Talagrand. Upper and lower bounds for stochastic processes, volume 60 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer, Heidelberg, 2014. Modern methods and classical problems.

[20] Aad W. van der Vaart and Jon A. Wellner. Weak convergence and empirical processes. Springer Series in Statistics. Springer-Verlag, New York, 1996. With applications to statistics.