INvariably GENERATION OF PROSOLUBLE GROUPS

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Abstract. A group $G$ is invariably generated by a subset $S$ of $G$ if $G = \langle g^s \mid s \in S \rangle$ for each choice of $g(s) \in G$, $s \in S$. Answering two questions posed by Kantor, Lubotzky and Shalev in [8], we prove that the free prosoluble group of rank $d \geq 2$ cannot be invariably generated by a finite set of elements, while the free soluble profinite group of rank $d$ and derived length $l$ is invariably generated by precisely $l(d - 1) + 1$ elements.

1. Introduction

Following [2] we say that a subset $S$ of a group $G$ invariably generates $G$ if $G = \langle g^s \mid s \in S \rangle$ for each choice of $g(s) \in G$, $s \in S$. We also say that a group $G$ is invariably generated (IG for short) if $G$ is invariably generated by some subset $S$ of $G$; when $S$ can be chosen to be finite, we say that $G$ is FIG. A group $G$ is IG if and only if it cannot be covered by a union of conjugates of a proper subgroup, which amount to saying that in every transitive permutation representation of $G$ on a set with more than one element there is a fixed-point-free element. Using this characterization, Wiegold [13] proved that the free group on two (or more) letters is not IG. Kantor, Lubotzky and Shalev studied invariable generation in finite and infinite groups. For example in [7] they proved that every finite group $G$ is invariably generated by at most $\log_2 |G|$ elements. In [8] they studied invariable generation of infinite groups, with emphasis on linear groups, proving that a finitely generated linear group is FIG if and only if it is virtually soluble.

Let $G$ be a profinite group. Then generation and invariable generation in $G$ are interpreted topologically. Just as every finite group is IG, every profinite group $G$ is also IG. Indeed every proper subgroup of a profinite group $G$ is contained in a maximal open subgroup $M$, and, since $M$ has finite index, $G$ cannot coincide with the union $\bigcup_{g \in G} M^g$. On the other hand, finitely generated profinite groups are not necessarily FIG. In fact by [2] Proposition 2.5, there exist 2-generated finite groups $H$ with $d_I(H)$ (the minimal number of invariable generators) arbitrarily large. This implies that the free profinite of rank $d \geq 2$ is not FIG. In [8] the following questions are asked: Are finitely generated prosoluble groups FIG? Are finitely generated soluble profinite groups FIG?

We prove that the first question has in general a negative answer:

**Theorem 1.** The free prosoluble group of rank $d \geq 2$ is not FIG.

We will deduce Theorem 1 from the following result (see Theorem 8). Let $G$ be a finite 2-generated soluble group and let $p$ be the smallest prime divisor of $|G|$. Then either $d_I(G) \geq p$ or there exists a prime $q > p$ such that $d_I(G) < d_I(C_q \wr G)$, where

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$C_q \wr G$ is the wreath product with respect to the regular permutation representation of $G$.

In contrast, the second question has a positive answer. More precisely we can adapt the arguments used in the proof of Theorem 1 to show:

**Theorem 2.** Let $F$ be the free soluble profinite group of rank $d$ and derived length $l$. Then $d_I(F) = l(d - 1) + 1$.

Denote by $d(G)$ the smallest cardinality of a generating set of a finitely generated profinite group $G$. Clearly if $G$ is pronilpotent, then $d(G) = d_I(G)$. More precisely, by [2 Proposition 2.4] a finitely generated profinite group $G$ is pronilpotent if and only if every generating set of $G$ invariably generates $G$. But what can we say about the difference $d_I(G) - d(G)$ when $G$ is a prosupersoluble group? In this case $G/\text{Frat}(G)$ is metabelian, so Theorem 2 implies that $d_I(G) - d(G) \leq d(G) - 1$. Although supersolubility is a quite strong property and in particular a metabelian group is not in general supersoluble, the previous estimate is sharp.

**Theorem 3.** Let $F$ be the free prosupersoluble group of rank $d$. Then $d_I(F) = 2d - 1$.

2. Preliminaries

A profinite group is a topological group that is isomorphic to an inverse limit of finite groups. The textbooks [11] and [14] provide a good introduction to the theory of profinite groups. In the context of profinite groups, generation and invariable generation are interpreted topologically. By a standard argument (see e.g. [14 Proposition 4.2.1]) it can be proved that a profinite group $G$ is invariably generated by $d$ elements if and only if $G/N$ is invariably generated by $d$ elements for every open normal subgroup $N$ of $G$. Therefore in the following we will mainly work on finite groups.

If $G$ is a finite soluble group, the minimal number of generators for $G$ can be computed in terms of the structure of $G$-modules of the chief factors of $G$ with the following formula due to Gaschütz [4].

**Proposition 4.** Let $G$ be a finite soluble group. For every irreducible $G$-module $V$ define $r_G(V) = \dim_{\text{End}_G(V)} V$, set $\theta_G(V) = 0$ if $V$ is a trivial $G$-module, and $\theta_G(V) = 1$ otherwise, and let $\delta_G(V)$ be the number of chief factors $G$-isomorphic to $V$ and complemented in an arbitrary chief series of $G$. Then

$$d(G) = \max_V \left( \theta_G(V) + \left\lceil \frac{\delta_G(V)}{r_G(V)} \right\rceil \right)$$

where $V$ ranges over the set of non $G$-isomorphic complemented chief factors of $G$ and $\lceil x \rceil$ denotes the smallest integer greater or equal to $x$.

There is no similar formula for the minimal size of the invariable generating sets. The best result in this direction is a criterion we gave in [1] to decide whether an invariable generating set of a group $G$ can be lifted to an extension over an abelian normal subgroup. To formulate this result, we need to recall some notation from [1].

Let $G$ be a finite group acting irreducibly on an elementary abelian finite $p$-group $V$. For a positive integer $u$ we consider the semidirect product $V^u \rtimes G$:
unless otherwise stated, we assume that the action of \( G \) is diagonal on \( V^u \), that is, \( G \) acts in the same way on each of the \( u \) direct factors. In [1 Proposition 8] we proved the following.

**Proposition 5.** Suppose \( G \) acts faithfully and irreducibly on \( V \) and \( H^1(G, V) = 0 \). Assume that \( g_1, \ldots, g_d \) invariably generate \( G \). There exist some elements \( w_1, \ldots, w_d \in V^u \) such that \( g_1w_1, g_2w_2, \ldots, g_dw_d \) invariably generate \( V^u \rtimes G \) if and only if

\[
u \leq \sum_{i=1}^d \dim_{\text{End}_G(V)} C_V(g_i).
\]

The assumption \( H^1(G, V) = 0 \) in the case of soluble groups is assured by the following unpublished result by Gaschütz (see [12, Lemma 1]).

**Lemma 6.** Let \( G \neq 1 \) be a finite soluble group and let \( V \) be an irreducible \( G \)-module. Then \( H^1(G, V) = 0 \).

In the following we will use this straightforward consequence of Proposition 5.

**Corollary 7.** Let \( G \neq 1 \) be a finite soluble group and let \( V \) be an irreducible \( G \)-module. Assume that \( x_1, \ldots, x_d \) invariably generate \( V^u \rtimes G \), where \( x_i = v_ig_i \) with \( v_i \in V^u \) and \( g_i \in G \). Then \( g_1, \ldots, g_d \) invariably generate \( G \) and

\[
u \leq \sum_{i=1}^d \dim_{\text{End}_{G/C_G}(V)} C_V(g_i).
\]

**Proof.** Clearly, \( g_1, \ldots, g_d \) invariably generate \( G \). Denote by \( \overline{g_i} \) the image of \( g_i \) in the quotient group \( G/C_G(V) \). By Lemma 6 and Proposition 5 we have

\[
u \leq \sum_{i=1}^d \dim_{\text{End}_{G/C_G}(V)} C_V(\overline{g_i}).
\]

Since \( \dim C_V(\overline{g_i}) = \dim C_V(g_i) \), the result follows. \( \square \)

3. **Proof of Theorem 4**

If \( G \) is a finite group, \( \pi(G) \) is the set of primes dividing the order of \( G \).

**Theorem 8.** Let \( G \) be a 2-generated finite soluble group. Either \( d_I(G) \geq \min \pi(G) \) or there exists a finite soluble group \( H \) having \( G \) as an epimorphic image and such that

- \( d(H) = 2 \);
- \( d_I(H) > d_I(G) \);
- \( \min \pi(H) = \min \pi(G) \).

**Proof.** By Dirichlet’s theorem on primes in arithmetic progressions, there exists a prime \( q \) such that the exponent of \( G \) divides \( q - 1 \). Let \( \mathbb{F} \) be the field of order \( q \). By a result of Brauer (see e.g. [3 B 5.21]) \( \mathbb{F} \) is a splitting field for \( G \) so

\[ V := \mathbb{F}G = V_1^{n_1} \oplus \cdots \oplus V_r^{n_r} \]

where the \( V_j \) are absolutely irreducible \( \mathbb{F}G \)-modules no two of which are \( G \)-isomorphic, and \( n_j = \dim \mathbb{F}V_j \). Consider the semidirect product \( H = V \rtimes G \); note that \( H \) is isomorphic to \( C_q \rtimes G \) with respect to the regular permutation representation of \( G \). By [9 Corollary 2.4], as \( C_q \) and \( G \) have coprime orders, \( d(C_q \rtimes G) = \max(d(G), d(C_q) + 1) = 2 \).
Clearly $d_1(G) \leq d_1(H)$. Assume $d_1(G) = d_1(H) = d$. By Corollary 7 applied to each homomorphic image $V_2^{n_j} \times G$, it follows that there exists an invariable generating set $g_1, \ldots, g_d$ of $G$ such that, for any $j$

$$n_j \leq \sum_{i=1}^{d} \dim F C_{V_j}(g_i).$$

Multiplying by $n_j$ we get

$$n_j^2 \leq \sum_{i=1}^{d} n_j \dim F C_{V_j}(g_i).$$

It follows that:

$$|G| = \sum_{j=1, \ldots, r} n_j^2 \leq \sum_{i=1, \ldots, d} \sum_{j=1, \ldots, r} n_j \dim F C_{V_j}(g_i) = \sum_{i=1, \ldots, d} \dim F C_{V G}(g_i).$$

On the other hand, by Lemma 9 below,

$$\dim F C_{V G}(g_i) = \frac{|G|}{|g_i|}$$

and therefore

$$1 \leq \sum_{i=1}^{d} \frac{1}{|g_i|}.$$ 

Since $d = d_1(G)$ we have $g_i \neq 1$ for every $i$, hence $|g_i| \geq p = \min \pi(G)$. Therefore

$$1 \leq \sum_{i=1}^{d} \frac{1}{|g_i|} \leq \frac{d}{p}$$

which implies that $p \leq d$, as required. \hfill \Box

**Lemma 9.** If $g \in G$, then $\dim F C_{V G}(g) = |G : \langle g \rangle|.$

**Proof.** Let $t_1, \ldots, t_r$ be a left transversal of $\langle g \rangle$ in $G$. Assume that $x \in C_{V G}(g)$. As every element of $G$ can be uniquely written in the form $t_ig^j$, we can write $x = \sum_{i,j} a_{i,j}t_ig^j$, where $a_{i,j} \in F$, and, since $xg = x$, we have in particular

$$a_{i,j} = a_{i,j+1}$$

for every $i$ and $j$. Hence $x = \sum_{i} b_it_i(1+g+\cdots+g^{[g]-1})$, for some $b_i \in F$. Conversely, every $F$-linear combination of the elements $t_i(1+g+\cdots+g^{[g]-1})$ is centralized by $g$. In other words the elements $t_i(1+g+\cdots+g^{[g]-1}), \ 1 \leq i \leq r$, are a basis for $C_{V G}(g).$ \hfill \Box

**Corollary 10.** For every $d \in \mathbb{N}$, there exists a finite 2-generated soluble group $G$ with $d_1(G) \geq d$. 

**Proof.** Let $p$ be a prime number with $d \leq p$ and consider the set $\Omega_p$ of the finite 2-generated soluble groups whose order is divisible by no prime smaller than $p$. Assume by contradiction, that $d_1(G) < d$ for every $G \in \Omega_p$ and let $G^*$ be a group in $\Omega_p$ such that $d_1(G^*) = \max_{G \in \Omega_p} d_1(G)$. Since $d_1(G^*) \leq d$ and $d \leq p$, by the Theorem 8 there exists $H$ in $\Omega_p$ with $d_1(G^*) < d_1(H)$, and this contradicts the maximality of $d_1(G^*)$. \hfill \Box
Proof of Theorem 1. Let $F$ be the $d$-generated free prosoluble group, with $d \geq 2$. Assume that $F$ is FIG. In particular $d_I(H) \leq d_I(F)$ for every 2-generated finite soluble group $H$, but this contradicts Corollary 10. □

4. Proof of Theorem 2

We need, as a preliminary result, a formula for the minimal number of generators of a $G$-module.

Lemma 11. Let $G$ be a finite group. Assume that $A$ is a direct product

$$A = A_1^{n_1} \times \cdots \times A_r^{n_r},$$

where, for each $i$, $A_i$ is a finite elementary abelian $p_i$-group for a prime number $p_i$, $A_i$ is an irreducible $\mathbb{F}_{p_i}$-$G$-module and $A_i$ is not $G$-isomorphic to $A_j$ for $i \neq j$. Then the minimal number of elements needed to generate $A$ as $G$-module is

$$d_G(A) = \max_{i \in \{1, \ldots, r\}} \left( \left\lfloor \frac{n_i}{r_G(A_i)} \right\rfloor \right),$$

where $\lfloor x \rfloor$ denotes the smallest integer greater or equal to $x$.

Proof. If $J_i$ is the Jacobson radical of $\mathbb{F}_{p_i}G$, then $\mathbb{F}_{p_i}G/J_i$ is semisimple and Artinian, hence we can apply the Wedderburn-Artin theorem (see e.g. [6, Lemma 1.11, Theorems 1.14 and 3.3]) and we conclude that $A_i$ occurs precisely $\dim_{\text{End}(A_i)}(A_i) = r_G(A_i)$ times in $\mathbb{F}_{p_i}G/J_i$. Then, by [5, Lemma 7.12], $A$ can be generated, as $G$-module, by

$$d_G(A) = \max_{i \in \{1, \ldots, r\}} \left( \left\lfloor \frac{n_i}{r_G(A_i)} \right\rfloor \right)$$

elements. □

Proposition 12. Let $G$ be a finite soluble $d$-generated group of derived length $l$. Then $d_I(G) \leq l(d - 1) + 1$.

Proof. The proof is by induction on $l$. If $l = 1$, then $G$ is abelian and $d_I(G) = d(G) \leq d = (d - 1) + 1$.

Assume $l > 1$ and let $A$ be the last non-trivial term of the derived series of $G$. Then $dl(G/A) = l - 1$. Since $d_I(G) = d_I(G/\text{Frat}(G))$, without loss of generality we can assume $\text{Frat}(G) = 1$. Then $A$ is a direct product of complemented minimal normal subgroups of $G$ and we can write

$$A = A_1^{n_1} \times \cdots \times A_r^{n_r}$$

where each $A_i$ is an elementary abelian $p_i$-group, for a prime number $p_i$, $A_i$ is an irreducible $\mathbb{F}_{p_i}$-$G$-module and $A_i$ is not $G$-isomorphic to $A_j$ for $i \neq j$. Therefore by Lemma 11

$$d_G(A) = \max_{i \in \{1, \ldots, r\}} \left( \left\lfloor \frac{n_i}{r_G(A_i)} \right\rfloor \right).$$

On the other hand, by Proposition 3

$$d \geq d(G) = \max_V \left( \theta_G(V) + \left\lfloor \frac{\delta_G(V)}{r_G(V)} \right\rfloor \right)$$

where $V$ ranges over the set of non $G$-isomorphic complemented chief factors of $G$. Note that $\theta_G(A_i) = 1$ for every $i$. Indeed, if we assume that $A_i$ is a trivial $G$-module, then, as $\text{Frat}(G) = 1$, we have $G = A_i \times H$ for a complement $H$ of $A_i$. 

in $G$. Hence $G' = H'$ and $G'$ does not contain $A_i$, contradicting the fact that $A_i$ is a subgroup of the last term of the derived series of $G$.

Since $n_i \leq \delta_G(A_i)$, by equations 4.1 and 4.2 we deduce that
\[
d \geq \max_{i \in \{1, \ldots, r\}} \left( 1 + \left\lceil \frac{n_i}{r_G(A)} \right\rceil \right) = 1 + d_G(A)
\]
hence $d_G(A) \leq d - 1$. Let $a_1, \ldots, a_d - 1$ be a set of generators for $A$ as $G$-module and let $g_1, \ldots, g_t$ be invariable generators for $G$ modulo $A$ with $t = d_I(G/A)$. Then it is straightforward to check that the the elements
\[g_1, \ldots, g_t, a_1, \ldots, a_{d - 1}\]
invariably generate $G$, hence
\[d_I(G) \leq t + (d - 1) = d_I(G/A) + (d - 1).
\]
Since $dI(G/A) = l - 1$, by inductive hypothesis we have that
\[d_I(G/A) \leq (l - 1)(d - 1) + 1,
\]
and we conclude that
\[d_I(G) \leq (l - 1)(d - 1) + 1 + (d - 1) = l(d - 1) + 1,
\]
as required. □

Denote by $dI(G)$ the derived length of a soluble group $G$. It follows from the previous proposition, that if $G$ is a finitely generated soluble profinite group, then $d_I(G) \leq dI(G)(d(G) - 1) + 1$. In order to complete the proof of Theorem 2 it suffices to prove the following result:

**Theorem 13.** Let $d$ be a positive integer and let $p$ be a prime number. For every positive integer $l < \frac{p}{d - 1} + 1$ there exists a finite soluble group $G_I$ such that

- $p = \min\pi(G_I)$,
- $dI(G_I) = l$,
- $d(G_I) = d$,
- $dI(G_I) = l(d - 1) + 1$.

**Proof.** We prove the theorem by induction on $l$. If $l = 1$, then we can take $G_I = C_p^d$. So suppose that a group $G_I$, with the desired properties, has been constructed for $l < \frac{p}{d - 1}$. As in the proof of Theorem 8 if we take a prime $q$ such that the exponent of $G_I$ divides $q - 1$ and we consider the field $F$ be the field of order $q$, then
\[V := FG_I = V_1^{n_1} \oplus \cdots \oplus V_r^{n_r}\]
where the $V_j$ are absolutely irreducible $FG$-modules no two of which are $G$-isomorphic, and $n_j = \dim_F V_j$. Consider the semidirect product $G_{l+1} = V^{d-1} \rtimes G_I$. It can be easily seen that $dI(G_{l+1}) = dI(G_I) + 1 = l + 1$ and that $G_{l+1}$ is isomorphic to the wreath product $C_q^{d-1} \wr G_I$ with respect to the regular permutation representation of $G_I$. In particular, by [9 Corollary 2.4], as $C_q^{d-1}$ and $G_I$ have coprime orders,
\[d(G_{l+1}) = d(C_q^{d-1}) = \max(d(G_I), d(C_q^{d-1}) + 1) = d.
\]
Now let $t = d_I(G_{l+1})$ and suppose that $w_1g_1, \ldots, w_tg_t$, with $w_i \in V^{d-1}$ and $g_i \in G_I$, invariably generate $G_{l+1}$. By Corollary 4 for any $j \in \{1, \ldots, t\}$
\[(d - 1)n_j \leq \sum_{i=1}^{t} \dim_F C_{V_j}(g_i).
\]
As in the proof of Theorem 8, this implies

\[(4.3) \quad d - 1 \leq \sum_{i=1}^{t} \frac{\dim \mathcal{F}_{G_{i}}(g_{i})}{|G_{i}|}.
\]

Notice that \(g_{1}, \ldots, g_{t}\) must invariably generate \(G_{i}\) so \(t \geq d_{i}(G_{i}) = l(d - 1) + 1\) and in particular we may assume \(g_{i} \neq 1\) for every \(i \leq l(d - 1) + 1\). Therefore, by Lemma 9

\[
\frac{\dim \mathcal{F}_{G_{i}}(g_{i})}{|G_{i}|} \leq \frac{1}{p} \quad \text{if} \quad i \leq l(d - 1) + 1.
\]

Since the trivial bound \(\dim \mathcal{F}_{G_{i}}(g_{i})/|G_{i}| \leq 1\) holds for all \(i = l(d - 1) + 2, \ldots, t\), it follows from (4.3) that

\[d - 1 \leq \frac{l(d - 1) + 1}{p} + t - l(d - 1) - 1\]

i.e.

\[t \geq \left( (l + 1)(d - 1) + 1 - \frac{l(d - 1) + 1}{p} \right).\]

Since we are assuming \(l < \frac{p-1}{d-1}\), we have \(\frac{l(d-1)+1}{p} < 1\) and consequently \(d_{i}(G_{i+1}) = t \geq (l + 1)(d - 1) + 1\). On the other hand, since \(d_{i}(G_{l}) = l + 1\), by Proposition 12 we have \(d_{i}(G_{i+1}) \leq (l + 1)(d - 1) + 1\) and therefore the equality \(d_{l}(G_{i+1}) = (l + 1)(d - 1) + 1\) has been proved. \(\square\)

5. Proof of Theorem 8

**Proposition 14.** For every \(d \in \mathbb{N}\) there exists a finite supersoluble group \(G\) such that \(d(G) = d\) and \(d_{l}(G) \geq 2d - 1\).

**Proof.** Let \(K = C_{2}^{d}\). There are \(\alpha := 2d - 1\) different epimorphisms \(\sigma_{1}, \ldots, \sigma_{\alpha}\) from \(K\) to \(C_{2}\) (\(\sigma_{i} : K \to C_{2}\) is uniquely determined by \(M_{i} = \ker \sigma_{i}\), a \((d - 1)\)-dimensional subspace of \(K\)). To any \(i\), there corresponds a \(K\)-module \(V_{i}\) defined as follows: \(V_{i} \cong C_{3}\) and \(v_{i}^{k} = v_{i}\) if \(k \in M_{i}\), \(v_{i}^{k} = v_{i}^{2}\) otherwise. Let \(W_{i} = V_{i}^{d-1}\) and consider \(G = \left( \prod_{1 \leq i \leq \alpha} W_{i} \right) \rtimes K\). The group \(G\) is supersoluble and, by Proposition 8, it is easy to see that \(d(G) = d\). Now assume that \(g_{1}, \ldots, g_{r}\) invariably generate \(G\). We write \(g_{i} = (w_{i1}, \ldots, w_{i\alpha})k_{i}\) with \(k_{i} \in K\) and \(w_{ij} \in W_{j}\). In particular \(k_{1}, \ldots, k_{r}\) generate \(K\) and, up to reordering the elements \(g_{1}, \ldots, g_{r}\), we can assume that the first \(d\)-elements \(k_{1}, \ldots, k_{d}\) are a basis for \(K\). Let \(M = \langle k_{1}^{-1}k_{2}, \ldots, k_{d-1}k_{d} \rangle\). It can be easily checked that \(M\) is a maximal subgroup of \(K\), so \(M = M_{j}\) for some \(j \in \{1, \ldots, \alpha\}\). Moreover \(k_{i} \notin M_{j}\) for every \(i \in \{1, \ldots, d\}\), in particular \(C_{V_{j}}(k_{i}) = 0\) for every \(i \in \{1, \ldots, d\}\). On the other hand \(w_{1}, w_{1}k_{1}, \ldots, w_{r}k_{r}\) invariably generate \(G\), so, by Corollary 7

\[d - 1 \leq \sum_{1 \leq i \leq r} \dim \mathcal{F}_{G}(k_{i}) = \sum_{d+1 \leq i \leq r} \dim \mathcal{F}_{G}(k_{i}) \leq r - d.
\]

Hence \(r \geq 2d - 1\). \(\square\)

**Proof of Theorem 8.** Let \(F\) be the free prosupersoluble group of rank \(d \geq 2\). By Proposition 14, there exists a finite supersoluble \(d\)-generated group \(G\) such that \(d_{l}(G) \geq 2d - 1\). Hence \(d_{l}(F) \geq 2d - 1\).
To prove the converse, since $d_I(F) = d_I(F/\text{Frat}(F))$, it suffices to consider $G = F/\text{Frat} F$. By [10, Proposition 3.3], $G'$ is abelian hence $d I(G) \leq 2$ and it follows from Proposition 12 that $d_I(G) \leq 2d - 1$. Therefore $d_I(F) = 2d - 1$. □

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