Flattened Partitions and Subexceedant Functions

Fufa Beyene* and Roberto Mantaci†

* Department of Mathematics, Addis Ababa University, Addis Ababa, Ethiopia and CoRS
  Email: fufa.beyene@aau.edu.et

† IRIF, Université de Paris, Paris, France and CoRS
  Email: mantaci@irif.fr

Abstract

In this paper we study some properties of flattened partitions and their canonical forms, as well as the distribution of the statistics of runs and right-to-left minima over the set of flattened partitions. We show the later one is given by the shifted distribution of the Stirling number of the second kind. We also enumerate some interesting sub-classes of the class of flattened partitions, in particular one class enumerated by powers of 2. We use one of the constructive proofs given in the paper to implement an algorithm for the exhaustive generation of flattened partitions by number of runs.

Keywords: Set partition; Flattened partition; Canonical form; Algorithm

2020 Mathematics Subject Classification: 05A05; 05A15; 05A19;

1. Introduction

A partition of the set \([n] = \{1, 2, \ldots, n\}\) is a collection of non-empty subsets of \([n]\) such that \([n]\) is the disjoint union of these subsets. It is well known that set partitions of \([n]\) with \(k\) blocks are counted by the Stirling Numbers of the second kind, and that set partitions of \([n]\) are counted by the Bell numbers, as given in [4,16].

In the standard representation of a set partition (defined by T. Mansour in [8]) the elements in a block are arranged increasingly and the blocks are arranged in increasing order of their first elements.

T. Mansour in [8] gave also a way to encode a set partition (in its standard representation) by its canonical form, that is, every integer is encoded by the number of the block it belongs to.

Canonical forms are subexceedant functions, that is, functions \(f\) over \([n]\) such that \(1 \leq f(i) \leq i\) for all \(i \in [n]\). Note that canonical forms of set partitions are in fact the so-called restricted growth functions (RGF), which are a special case of subexceedant functions.

In 2009, D. Callan [5] introduced the “flattening” operation on set partitions, consisting in obtaining a permutation from a set partition by removing the brackets from its standard representation. Since then these objects named flattened partitions are getting the attention of different researchers and a few findings are emerging as a result, for instance, see [9,12]. Flattened partitions are counted by the shifted Bell numbers.

In this paper we study some properties of flattened partitions and their canonical forms, as well as the distribution of some statistics (runs and right-to-left minima) over the set of flattened partitions. We also enumerate some interesting sub-classes of the class of flattened partitions.

In particular in Section 2 we show that canonical forms of flattened partitions can be bijectively related to RGFs of one size smaller.

We present in Section 3 a combinatorial bijective proof of the recurrence relation satisfied by flattened partitions over \([n]\) having \(k\) runs, a recurrence relation that was conjectured by Nabawanda and Rakotondrajao [12]. We also give another proof of the same results by working on the canonical forms of the flattened partitions.

F. Rakotondrajao and the second author in [10] gave a new way to encode permutation with subexceedant functions. In [2] we introduced a variation of their encoding and defined a bijection \(ϕ\) associating to each subexceedant function \(f = f_1 \ldots f_n\) the permutation \(σ = (n f_n) \ldots (1 f_1)\). On the other hand, in [3] we characterised those permutations whose codes are restricted growth functions and hence are in bijection with set partitions. We named these permutations “Bell permutations of the second kind”.

In Section 4 we associate to each flattened partition a Bell permutation of the second kind, and show that the number of Bell permutations of the second kind over \([n]\) with \(k\) weak exceedances is equal to the number of
flattened partitions over \([n]\) with \(k\) runs.

In Section 5 we show that the distribution of right-to-left minima over flattened partitions is the same as the distribution of number of blocks over set partitions of one size smaller (and hence also given by shifted Stirling numbers of the second kind). A. Munagi in \([11]\) proved that the set partitions over \([n]\) with \(k\) blocks with no two consecutive integers in the same block are also counted by the shifted Stirling numbers of the second kind. So in this section we also show that these partitions bijectively correspond to flattened partitions over \([n]\) with \(k\) right-to-left minima.

The notion of non-crossing partitions was introduced by Rodica Simion in \([15]\). In Section 6 we study the class of flattened non-crossing partitions and we show that it is counted by powers of 2, and finally Section 7 presents an exhaustive generation algorithm for flattened partitions.

1.1 Definitions, Notations and Preliminaries

**Definition 1.1.** A function \(f : [n] \rightarrow [n]\) such that \(1 \leq f(i) \leq i\) for all \(i, 1 \leq i \leq n\) is called a subexceedant function.

We denote by \(\mathfrak{f}_n\) the set of all subexceedant functions over \([n]\).

Let \(f = f(1)f(2)\ldots f(n) \in \mathfrak{f}_n\). Then:
1. set of images of \(f\) is denoted by \(Im(f)\) and its cardinality by \(IMA(f)\).
2. the set of left-to-right maxima of \(f\) is given by \(LrMax(f) = \{f_j | f_j > f_i, i < j\}\) and \(lrmax(f) = |LrMax(f)|\).
3. the set of left-to-right weak maxima positions of \(f\) is \(LwMp(f) = \{i | f_i \geq f_j, 1 \leq i \leq n, j < i\}\) and its cardinality is denoted by \(lwmp(f)\). The set of left-to-right weak maxima is given by \(LwM(f) = \{f_i | i \in LwMp(f)\}\).

**Example 1.1.** Take \(f = 121132342 \in \mathfrak{f}_9\). Then \(Im(f) = \{1, 2, 3, 4\}, IMA(f) = 4, LrMax(f) = \{f_1, f_2, f_5, f_9\} = \{1, 2, 3, 4\}, lrmax(f) = 4, LwMp(f) = \{1, 2, 5, 7, 8\}, lwmp(f) = 5, \) and \(LwM(f) = \{1, 2, 3, 4\}\).

**Definition 1.2.** A set partition \(P\) of \([n]\) is defined as a collection \(B_1, \ldots, B_k\) of nonempty disjoint subsets such that \(\bigcup_{i=1}^{k} B_i = [n]\). The subsets \(B_i\) will be referred to as “blocks”.

**Definition 1.3.** The block representation of a set partition \(P = B_1/\ldots/B_k\) is said to be standard representation if the blocks \(B_1, \ldots, B_k\) are sorted in such way that \(\min(B_1) < \min(B_2) < \cdots < \min(B_k)\) and if the elements of every block are arranged in an increasing order.

We will refer to set partitions in their standard representations.

We will denote by \(\mathfrak{P}(n)\) the set of all set partitions over \([n]\) and \(b_n = |\mathfrak{P}(n)|\) the \(n\)-th Bell number.

**Definition 1.4.** The canonical form of a set partition of \([n]\) is a \(n\)-tuple indicating the block of the standard representation in which each integer occurs, that is, \(f = f_1f_2\ldots f_n\) such that \(j \in B_i\) for all \(j\) with \(1 \leq j \leq n\).

**Example 1.2.** If \(P = 138/247/56 \in \mathfrak{P}(n)\), then its canonical form is \(f = 12134431\).

**Definition 1.5.** A restricted growth function (RGF) over \([n]\) is a subexceedant function \(f = f_1\ldots f_n\) such that \(f_1 = 1\) and \(f_i \leq 1 + \max\{f_1, \ldots, f_{i-1}\}\) for \(2 \leq i \leq n\).

The canonical forms of a set partitions are restricted growth functions (RGF). We denote by \(RGF(n)\) the set of all restricted growth functions over \([n]\).

**Remark 1.1.** In \([3]\) it is shown that if \(f = f_1f_2\ldots f_n\) is the canonical form of a set partition, then the set \(\{f_1, f_2, \ldots, f_i\}\) is an integer interval for all \(i \in [n]\).

**Definition 1.6.** "Flattening" a set partition of \([n]\) consists in obtaining a permutation by concatenating the blocks of the set partition \(P = B_1/\ldots/B_k\) of \([n]\) written in standard form. The result of this operation is denoted \(\text{Flatten}(P) = \pi\). We call this permutation a flattened partition. A maximal increasing subsequence of consecutive integers of the flattened partition is called a run.

We will denote by \(F_n\) the set of all flattened partitions over \([n]\) and by \(a_n\) its cardinality.

Let \(\pi = \pi(1)\pi(2)\ldots \pi(n) \in F_n\). Then the set of right-to-left minima of \(\pi\) is given by \(Rlmin(\pi) = \{\pi(i) | \pi(i) < \pi(j), j > i\}\), and its cardinality \(rlmin(\pi) = |Rlmin(\pi)|\).
Example 1.3. The flattened partition of \( P = 126/3/48/57 \) is the permutation \( \pi = 12634857 \). Note that \( P \) has four blocks while \( \pi \) has three runs.

Further \( \text{rlmin}(\pi) = \{1, 2, 3, 4, 5, 7\} \) and \( \text{rlmin}(\pi) = 6 \).

Theorem 1.1. (\[12\]) The number of set partitions over \([n]\) and the number of flattened partitions over \([n+1]\) are equal. That is, \( b_n = a_{n+1} \) for all \( n \geq 1 \).

Let \( P = B_1/\ldots/B_k \) be a set partition over \([n]\). Then the corresponding flattened partition in \( F_{n+1} \) is constructed as follows:

1. arrange each minimum element of the block at the end of its block and then increase every integer by 1.
2. Finally attach the integer 1 at the front. Consequently, we construct the set partition over \([n]\) corresponding to a flattened partition \( \pi \in F_{n+1} \) as follows. Put bars after every right-to-left minimum of \( \pi \). Then delete 1 and decrease every integer by 1, and arrange the elements of a block in increasing order.

Example 1.4. Consider \( P = 14/258/37/6 \in \Psi(8) \). Then we write it as 41/582/73/6. Again by increasing every integer by 1 we have 52/693/84/7. Hence the resulting flattened partition becomes \( \pi = 152693847 \in F_9 \). Conversely, for \( \pi = 152693847 \in F_9 \) we have \( \text{rlmin}(\pi) = \{1, 2, 3, 4, 7\} \). So by putting bars after each right-to-left minimum we have 1/52/693/84/7 \( \rightarrow \) 14/258/37/6 = \( P \).

Remark 1.2. The number of flattened partitions over \([n+1]\) that start with the factor 12 equals the total number of flattened partitions over \([n]\).

Proof. Let \( \pi \in F_{n+1} \) that start with 12. Then delete 1 from the front and decrease every integer in \( \pi \) by 1 to obtain \( \pi' \in F_n \).

Corollary 1.1. The exponential generating function \( A(x) \) of the number of flattened partitions has the closed differential form

\[
A'(x) = e^x - 1
\]

Corollary 1.2. For all positive integer \( n \geq 1 \), the number \( a_n \) of flattened partitions over \([n]\) is given by

\[
a_n = \frac{1}{e} \sum_{m \geq 0} \frac{m^{n-1}}{m!}
\]

2. Canonical Forms of Flattened Partitions

We characterize flattened partition as a set partition \( P = B_1/\ldots/B_k \), where \( \text{max}(B_i) > \text{min}(B_{i+1}) \), \( i = 1, \ldots, k-1 \). In this case we say that \( P \) has \( k \) runs.

In this section we study the canonical forms of flattened partitions. We encode each integer in a flattened partition by the number of run it belongs to.

The canonical form of any flattened partition is a subexceedant function (because, every integer \( i \in [n] \) is in one of the first \( i \) runs). However, every subexceedant function is not necessarily the canonical form of a flattened partition.

Here we characterize subexceedant functions that correspond to flattened partitions.

Note that if \( f \) is the canonical form of a flattened partition \( \pi \) with \( k \) runs, then \( IMA(f) = k \).

Example 2.1. Let \( \pi = 152693847 \). Then its canonical form is \( f = 123412432 \) and \( IMA(f) = 4 \). This means 1 and 5 are in the first run, 2, 6 and 9 are in the second run, and so on.

Proposition 2.1. There is a bijection between the set of flattened partitions over \([n]\) and the set, \( T_n \) of subexceedant functions \( f = f_1f_2\ldots f_n \) over \([n]\) satisfying the following:

1. \( f \) is a RGF, or equivalently \( LrMax(f) = [k] \), where \( IMA(f) = k \), and

2. every left-to-right maximum \( i > 1 \) in \( f \) has at least one occurrence of \( i-1 \) on its right.

Proof. If \( \pi \) is a flattened partition, then \( \pi \) is a set partition and its canonical form \( f \) satisfies condition (1).

If \( \pi = B_1B_2\ldots B_k \) is a flattened partition with \( k \) runs, then \( \text{min}(B_i) < \text{min}(B_{i+1}) \) and \( \text{max}(B_i) > \text{min}(B_{i+1}) \), \( i = 1, 2, \ldots, k-1 \). Note that every leftmost occurrence in \( f \) is a left-to-right maximum. The positions of leftmost and rightmost occurrences of \( i \) in \( f \) correspond to the minimum and the maximum elements of the run \( B_i \), respectively. By definition, the minimum of the run \( B_{i+1} \) is smaller than the maximum of the run \( B_i \). This implies the second condition.
Remark 2.1. Let $f = f_1 \ldots f_n$ be the canonical form of a set partition $P = B_1/\ldots/B_k$ over $[n]$ with $k$ blocks. Then $f_i \in LrMax(f)$ if and only if $i = \min(B_{f_i})$.

Remark 2.2. Let $f = f_1 f_2 \ldots f_n \in RGF(n)$. If $LrMax(f) = \{f_{i_1}, \ldots, f_{i_s}\}$ and $f_j \notin LrMax(f)$, then there is some $r \leq k$ such that $i_r < j$ and $f_j = f_{i_r} + 1$ or $f_j = f_{i_r}$. If $f_j < f_{i_r}$ for all $r$, $1 < r \leq k$ and $i_r < j$, then $f$ would not be an integer interval for all $i \in [n]$ and hence a RGF.

We want to give a combinatorial proof of the result of Theorem 1.1 in terms of canonical forms. Let $f = f_1 f_2 \ldots f_n \in RGF(n)$. For each $f_i$ we define $u_i$, the number of unique left-to-right maxima of $f$ that are less than or equal to $f_i$ on the positions $1, \ldots, i - 1$ with the sequence $u = (u_1, \ldots, u_n)$.

Let $\delta = (\delta_1, \ldots, \delta_n)$, where

$$
\delta_i = \begin{cases} 
1, & \text{if } f_i \text{ is non-unique left-to-right maximum of } f \\
0, & \text{otherwise}
\end{cases}
$$

Define a mapping $\alpha : RGF(n) \mapsto T_{n+1}$, where $T_{n+1}$ is the set of canonical forms of flattened partitions over $[n + 1]$, by $\alpha(f) = f'$, where $f = f'_1 \ldots f'_n$ and $f^* = f'_1 f'_2 \ldots f'_n$ is obtained from $f$ as follows:

$$f^* = f - u + \delta$$

Example 2.2. Take $f = 1213124$. Then $LrMax(f) = \{f_1, f_2, f_3, f_7\} = \{1, 2, 3, 4\}$ and $u = (0, 0, 0, 1, 0, 0, 2), \delta = (1, 1, 0, 0, 0, 0, 0)$. Let $\alpha(f) = f' = f'_1 f'_2 \ldots f'_7 \in T_8$. Then $f' = f'_1 f'_2 \ldots f'_6 = 12313123$.

Remark 2.3. Let $f = f_1 f_2 \ldots f_n \in RGF(n)$ and $\alpha(f) = f' = f'_1 \ldots f'_n$, where $f^* = f'_1 \ldots f'_n$. Then $f_i \in LrMax(f)$ if and only if $i \in LwMp(f^*)$.

If $LrMax(f) = \{f_{i_1}, \ldots, f_{i_s}\}$ and $i_r < j < i_{r+1}$, $r = 1, \ldots, k - 1$, then $f_j \leq f_{i_r}$. Thus $\alpha(f(j)) = f'_j < \alpha(f(i_r)) = f'_{i_r}$. That is, $j \notin LwMp(f^*)$.

We note that $\alpha$ is well defined and it is easy to see that for all $f \in RGF(n)$ we have $\alpha(f) = f' \in T_{n+1}$. That is, every left-to-right maximum of $f'$ is left-to-right weak maximum of $f'$ and hence $f'$ is a RGF. By Remark 2.2, every non-left-to-right maximum $s$ has left-to-right maximum $s + 1$ from its left.

Note that under $\alpha$ each unique left-to-right maximum of $f$ corresponds to left-to-right weak maximum but not left-to-right maximum of $f'$. Thus the number of left-to-right maxima of $f'$,

$$lrmax(f') = lrmax(f) + 1 - l$$

where $l$ is the number of unique left-to-right maxima of $f$.

Further, if $f_i \in LrMax(f)$ is not unique, then there is $f_j = f_{i_r} + 1$ and $u_i = u_j$. Since $\alpha(f(i)) = f'_i \in LrMax(f')$ and $\alpha(f(j)) = f'_j$, we have $f'_i = f'_j + 1$. Therefore, $f' \in T_{n+1}$.

Again let $f = f'_1 \ldots f'_n \in T_{n+1}$. Then we define a map $\beta : T_{n+1} \mapsto RGF(n)$ such that $\beta(f') = f = f_1 f_2 \ldots f_n$, where $f$ is obtained from $f'$ as follows.

Let $f^* = f'_1 \ldots f'_n$ and take $LwMp(f^*)$. We define for each $i \in [n]$, the number $v_i$ of non-strict left-to-right weak maxima of $f^*$ that are less than or equal to $f'_i$ on the positions $1, \ldots, i - 1$. Let $v = (v_1, \ldots, v_n)$.

Further, let $\delta' = (\delta'_1, \ldots, \delta'_n)$, where

$$
\delta'_i = \begin{cases} 
1, & \text{if } f'_i \text{ is strict left-to-right weak maximum of } f^* \\
0, & \text{otherwise}
\end{cases}
$$

Then $f = f_1 f_2 \ldots f_n$ is obtained from $f^*$ as follows:

$$f = f' + v - \delta'$$

Note that $\beta = \alpha^{-1}$.

Example 2.3. Take $f' = 122134321 \in T_9$. Then $f^* = 22134321$ and $v = (0, 0, 0, 1, 1, 1, 1, 0), \delta' = (1, 0, 0, 1, 1, 0, 0, 0)$. Thus, $f = 12134431 \in RGF(8)$.

As a result we have the following proposition.

Proposition 2.2. The mapping $\alpha$ from the set $RGF(n)$ to the set $T_{n+1}$ is a bijection.
Because each run except the last has length at least 2.

The following result was conjectured by Nabawanda and Rakotondrajao [12] who also were able to justify the first term of the right hand side. We provided the combinatorial proof for the second.

**Theorem 3.1.** The number \( a_{n,k} \) of flattened partitions of \([n]\) having \( k \) runs satisfies the following recurrence relation:
\[
a_{n,k} = ka_{n-1,k} + (n-2)a_{n-2,k-1}, \quad n \geq 2, \quad k \geq 1,
\]
where \( a_{0,0} = 1, \ a_{1,0} = 0, \ a_{1,1} = 1 \).

In order to prove this result, we partition the set \( F_{n,k} \) of flattened partitions over \([n]\) with \( k \) runs into two subsets:
\[
F_{n,k}^{(1)} := \text{the set of elements of } F_{n,k} \text{ in which the removal of the integer } n \text{ does not decrease the number of runs},
\]
\[
F_{n,k}^{(2)} := \text{the set of elements of } F_{n,k} \text{ in which the removal of the integer } n \text{ decreases the number of runs. That is,}
\]
in the factor \( xny \) of \( \pi \in F_{n,k}^{(2)} \) we have \( x < y \).

We will denote the cardinality of these subsets by \( a_{n,k}^{(1)} \) and \( a_{n,k}^{(2)} \), respectively.

**Example 3.1.** 12435 \( \in F_{5,2}^{(1)} \) and 15234 \( \in F_{5,2}^{(2)} \).

For each element \((i, \pi) \in [n-2] \times F_{n-2,k-1}\) define the mapping
\[
\psi : [n-2] \times F_{n-2,k-1} \rightarrow F_{n,k}^{(2)} \text{ such that } \psi(i, \pi) = \pi', \text{ where } \pi' \text{ is a partition obtained from } \pi \text{ by:}
\]
a. increasing by 1 all integers greater than \( i \), and
b. inserting the factor \( n \) \( i + 1 \) immediately after the rightmost integer of the set \( \{1, 2, \ldots, i\} \).

**Lemma 3.1.** For all \((i, \pi) \in [n-2] \times F_{n-2,k-1} \), we have indeed \( \psi(i, \pi) \in F_{n,k}^{(2)} \).

**Proof.** Since \( \pi \in F_{n-2,k-1} \) and the procedure inserts the factor \( n \) \( i + 1 \) immediately after the rightmost integer of the set \( \{1, \ldots, i\} \), all integers to the right of \( i + 1 \) are greater than \( i + 1 \) and \( i + 1 \) begins a new run. Thus the resulting partition is flattened with the number of runs increased by 1. Further, \( n \) follows some integer in the set \( \{1, \ldots, i\} \). Hence its removal decreases the number of runs. So \( \pi' \in F_{n,k}^{(2)} \).

**Proposition 3.1.** The map \( \psi \) defined above is a bijection.

**Proof.** We prove that \( \psi \) is both injective and surjective.

First let us assume that \((i_1, \pi_1) \neq (i_2, \pi_2)\) for \( i_1, i_2 \in [n-2] \) and \( \pi_1, \pi_2 \in F_{n-2,k-1} \). Let \( \psi(i_1, \pi_1) = \pi_1' \) and \( \psi(i_2, \pi_2) = \pi_2' \). Then \( \pi_1' \) and \( \pi_2' \) are flattened partitions in \( F_{n,k}^{(2)} \) by the previous lemma. We consider two cases.

1. If \( i_1 \neq i_2 \), then in one of the resulting partitions \( n \) is followed by \( i_1 + 1 \) while in the other \( n \) is followed by \( i_2 + 1 \).
2. If \( i_1 = i_2 \) and \( \pi_1 \neq \pi_2 \), then the two flattened partitions \( \pi_1 \) and \( \pi_2 \) have at least two entries in which they differ. Thus inserting \( n \) \( i_1 + 1 = n \) \( i_2 + 1 \) at the rightmost of the set \( \{1, 2, \ldots, i_1 = i_2\} \), results in different flattened partitions \( \pi_1' \) and \( \pi_2' \).

Thus, in both cases, \( \pi_1' = \psi(i_1, \pi_1) \neq \psi(i_2, \pi_2) = \pi_2' \). Hence \( \psi \) is injective.

Next, consider any \( \pi' \in F_{n,k}^{(2)} \) then \( n \) does not appear in the last position. Let \( j > 1 \) be the integer following \( n \) in \( \pi' \). We exhibit a pair \( (i, \pi) \in [n-2] \times F_{n-2,k-1} \) such that \( \psi(i, \pi) = \pi' \). Define \( \pi \) to be the flattened partition obtained from \( \pi' \) by deleting \( n \) and decreasing by 1 every integer greater than or equal to \( j+1 \) in the resulting word. Note that if \( n \) follows the integer \( i \) in \( \pi' \), then \( i < j \) and deleting the factor \( n \) from \( \pi' \) reduces the number of runs by 1 and the size of the partition by 2. Then \( \pi \in F_{n-2,k-1} \) and \( \psi(j+1, \pi) = \pi' \). Thus, \( \psi \) is surjective.

Therefore \( \psi \) is a bijection. \( \Box \)

**Example 3.2.** Let \( i = 3 \) and \( \pi = 13524 \in F_{5,2} \). Then \( \psi(i, \pi) \) is obtained as follows

Increase each integer greater than 3 in \( \pi = 13245 \) by one. This gives us 13625. Then insert the factor 7(3+1) = 74 into the position at the rightmost of the integers 1, 2, 3. Thus the unique position to insert the factor is between 2 and 5. Thus, \( \psi(3, 13524) = \pi' = 1362745 \in F_{7,3}^{(2)} \).

Now we are ready to present the proof of Theorem 3.1

**Proof.** The left-hand side counts the number of flattened partitions in the set \( F_{n,k} \).

**Case-1:** We show that the first part of the right hand side counts the number of elements in \( F_{n,k}^{(1)} \).

Consider any flattened partition \( \pi \) in the set \( F_{n-1,k} \). We insert \( n \) at the end of each of the runs of the partition \( \pi \) to obtain a flattened partition having the same number of runs and length increased by one. This operation gives us the elements of \( F_{n,k}^{(1)} \). Since we have \( k \) different possibilities where to insert \( n \) for each \( \pi \in F_{n-1,k} \), this contributes \( a_n^{(1)} = k a_{n-1,k} \) to the number \( a_{n,k} \).

**Case-2:** We show that the second part of the right hand side counts the number of elements in \( F_{n,k}^{(2)} \). By Proposition 3.1 the sets \([n-2] \times F_{n-2,k-1}^{(1)} \) and \( F_{n,k}^{(2)} \) have the same cardinality. That is, \( (n-2)a_{n-2,k-1} = a_n^{(2)} \).

This implies that the second part of the recurrence relation contributes \( (n-2)a_{n-2,k-1} \) to the number \( a_{n,k} \). Hence the recurrence relation \( a_{n,k} = k a_{n-1,k} + (n-2)a_{n-2,k-1} \) holds true for all \( (n,k) \geq (0,0) \). \( \Box \)

Let \( T_{n,k} = \{ f \in F_n | \text{Im}(f) = [k] \} \). So \( \text{card} T_{n,k} = a_{n,k} \). The set \( T_{n,k} \) contains the canonical forms of the flattened partitions in \( F_{n,k} \). Note that every element \( f = f_1 \ldots f_n \in T_{n,k} \) has the property that each \( f' \) obtained from \( f \) by deleting \( f_n \) is either in \( T_{n-1,k} \) or not. If \( f' \notin T_{n-1,k} \), then the left-to-right maximum \( m = f_n \) in \( f' = f_1 f_2 \ldots f_{n-1} \) has no occurrence of \( m-1 \) on its right.

Therefore we can provide a bijective proof of Theorem 3.1 in terms of canonical forms.

**Proof.**

**Case-i:** Let \( f = f_1 f_2 \ldots f_{n-1} \in T_{n-1,k} \). Then by concatenating an integer \( i \in [k] \) at the end of \( f \) we obtain a unique \( f' \in T_{n,k} \). This is because \( f' \) satisfies the conditions 1 and 2 of Proposition 2.1 if and only if \( f \) does. This contributes \( k a_{n,k} \) to the number \( a_{n,k} \).

**Case-ii:** Let \( f = f_1 f_2 \ldots f_{n-2} \in T_{n-2,k-1} \) and \( i \in [n-2] \). Let \( m = \max_{1 \leq j \leq i} \{ f_j \} \).

Then we construct each \( f' = f'_1 f'_2 \ldots f'_{n-1} \in T_{n,k} \) associated to \( (i, f) \) as follows:

Increase by 1 all \( f_j \)'s such that \( f_j \geq m \) and \( j > i \), insert \( m+1 \) at the position \( i+1 \) and concatenate \( m \) at the end.

For instance take \( i = 12132 \) and \( i = 3 \). Then the above construction gives us \( (i, f) \mapsto f' \) as follows:

We have \( m = \max_{1 \leq j \leq 3} \{ f_j \} = \max_{1 \leq j \leq 3} \{ 1, 2, 3 \} = 2 \). Then \( f'_1 = f_1 = 1, f'_2 = f_2 = 2, f'_3 = f_3 = 3, f'_4 = m+1 = 3, f'_5 = f_4 + 1 = 4, f'_6 = f_5 = 3, f'_7 = m = 2 \). Thus, \( f' = 1213432 \).

Note that if \( f^* = f'_1 \ldots f'_{i-1} \) is obtained from \( f' \) by deleting \( f'_i \), then \( f^* \notin T_{n-1,k} \). That is, \( f'_{i+1} = m+1 > f'_i, 1 \leq j \leq i \). So \( m+1 \in \text{Im} f^* \) and it has no occurrence of \( m \) from its right since \( f'_n = m \) is deleted and every \( f_j \) with \( f_j \geq m \) and \( j > i \) is increased by 1.

Conversely, let \( f' = f'_1 f'_2 \ldots f'_{n-1} \in T_{n,k} \) where the left-to-right maximum value \( m = f'_n \) in \( f^* = f'_1 f'_2 \ldots f'_{n-1} \) has no occurrence of \( m-1 \) on its right.

Then we construct \( f \in T_{n-2,k-1} \) from \( f' \) as follows:

First set \( m = f'_n \) and find \( i \) such that \( f'_{i+1} = m+1 \), the left-to-right maximum in \( f' \). Since \( f' \) satisfies condition (2) of Proposition 2.1 such left-to-right maximum \( m+1 \) indeed exists.

Then delete \( f'_{i+1} \) and \( f'_n \), and reduce by one every integer greater than or equal to \( m \) from those to the right of \( f'_{i+1} \).
From this we see that $f \in T_{n-2,k-1}$. Therefore, $(i, f) \mapsto f'$, where $i \in [n - 2]$, $f \in T_{n-2,k-1}$ and $f' = f'_1 f'_2 \ldots f'_{n-1} \in T_{n,k}$ with the left-to-right maximum $m = f'_{n-1}$ in $f' = f'_1 f'_2 \ldots f'_{n-1}$ has no occurrence of $m - 1$ on its right is a bijection. So, this contributes $(n - 2)a_{n-2,k-1}$ to the number $a_{n,k}$. 

\[ \square \]

4. Bell permutations of the second kind corresponding to Flattened partitions

In [3] we introduced a family of permutations of $\mathfrak{S}_n$ counted by Bell numbers (and therefore in bijection with the set partitions over $[n]$). We called this permutations “Bell permutations of the second kind” and noted the set $\mathfrak{B}(n)$. In this section we want to study those of Bell-permutations of the second kind that are in bijection with flattened partitions over $[n]$. We start by recalling the definition of $\mathfrak{B}(n)$.

F. Rakotondrajao and the second author introduced in [10] a permutation code associating with every permutation a subexceedant function. The permutation associated to a subexceedant function $f = f_1 \ldots f_n$ is

$$\sigma = (1 f_1) \cdots (n f_n)$$

This is bijection $F_n \mapsto \mathfrak{S}_n$ is noted $\phi$. In [2] we introduced a slightly different variant of $\phi$ to a subexceedant function

$$\sigma = (n f_n) \cdots (1 f_1)$$

this bijection $F_n \mapsto \mathfrak{S}_n$ is noted $\tilde{\phi}$.

In [3] the family $\mathfrak{B}(n)$ is defined as $\mathfrak{B}(n) = \tilde{\phi}(\text{RGF}(n))$. That is, the permutation whose code by $(\tilde{\phi})^{-1}$ is the canonical form of a partition.

Recall that if $\sigma = (1)\sigma(2) \ldots \sigma(n) \in \mathfrak{S}_n$ a permutation over $[n]$, then a weak exceedance of $\sigma$ is a position $i$ such that $\sigma(i) \geq i$; The values of weak exceedances are said to be weak exceedance letters and the subword of $\sigma = (1)\sigma(2) \ldots \sigma(n)$ including all weak exceedance letters is denoted by $\text{w-ExcL}(\sigma)$.

In [3] we also introduced the following definition :

Let $\sigma \in \mathfrak{S}_n$. We define the inverse nearest orbital minorant (inom) of $i \in [n]$ as the integer $j \leq i$ such that $j = \sigma^{-t}(i)$ with $t \geq 1$ chosen as small as possible and we showed that if $\tilde{\phi}(f) = \sigma$, then for all $i \in [n]$ we have $f(i) = \text{inom}(i)$.

Finally, the increasing integer sequence Seq($\sigma$) associated to $\sigma$ is constructed as follows: For $x = 1, \ldots, n$ include $x$ to the sequence Seq($\sigma$) if $x \in \text{Seq}(x)$ is not the inom of some integer smaller than $x$.

**Example 4.1.** Let $\sigma = 435129678 = (1 4)(2 3 5)(6 9 8 7)$. Then $\text{w-Exc}(\sigma) = \{1, 2, 3, 6\}$, $\text{w-ExcL}(\sigma) = \{\sigma(i) : i \in \text{w-Exc}(\sigma)\} = \{4, 3, 5, 9\}$ in an increasing order of weak exceedances. Further we have Seq($\sigma$) = \{1, 2, 5, 6\}.

In [3] we proved that for every Bell-permutation of the second kind $\sigma \in \mathfrak{B}(n)$ with the set of weak exceedance letters $\text{w-ExcL}(\sigma) = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, where $\alpha_i = \sigma(i)$ for $i = 1, 2, \ldots, k$ and Seq($\sigma$) = $\{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ we have

$$\gamma_i \leq \alpha_i, \text{ for all } i = 1, 2, \ldots, k. \quad (2)$$

If we apply $\tilde{\phi}$ to the set $T_n$ of canonical forms of flattened partitions over $[n]$, then we obtain a subset of Bell-permutations of the second kind that we want to characterize here.

In [3] we presented a bijection $\lambda : \mathfrak{B}(n) \mapsto \mathfrak{B}(n)$ such that $\lambda(\sigma) = P$, where $P$ is obtained as follows :

1. If $i \in \text{w-Exc}(\sigma)$, then we insert $\sigma(i)$ in the $i$-th block, 

2. For non-weak exceedance letters $i$, in decreasing order, we put $i$ in the inom($i$)-th block of $P$.

Recall that the elements of Seq($\sigma$) correspond to the minimum elements of the blocks of the set partition $P$, where as the elements of $\text{w-ExcL}(\sigma)$ in increasing order of weak exceedances correspond to the maximum elements of the blocks of $P$, where $\sigma \in \mathfrak{B}(n)$ and $\lambda(\sigma) = P$.

From this remark we deduce the following.
Corollary 4.1. There is a bijection between the set of flattened partitions over \([n]\) and the set of Bell-permutations of the second kind over \([n]\) with the following additional condition:
for every \(\sigma \in \mathcal{B}(n)\) with \(w\)-\text{ExcL}(\sigma)\) and \(\text{Seq}(\sigma)\) as above we have

\[
\alpha_i > \gamma_{i+1}, \quad \text{for all } i \in [k-1]
\]  

(3)

Proof. The restriction of \(\lambda\) to the set of Bell-permutations of the second kind over \([n]\) with the condition in the proposition, and the set of flattened partitions over \([n]\) is a bijection. 

Let us denote by \(\mathcal{B}_F(n)\) the set of Bell permutations of the second kind satisfying condition \(3\).

Example 4.2. 1. Take \(\sigma = 379861254 = (1 \ 3 \ 9 \ 4 \ 8 \ 5 \ 6)(2 \ 7) \in \mathcal{B}(9)\). Then \(w\)-\text{ExcL}(\sigma) = \langle 3, 7, 9, 8, 6 \rangle\) and \(\text{Seq}(\sigma) = (1, 2, 4, 5, 6)\) satisfy condition \(3\) and hence \(\sigma \in \mathcal{B}_F(9)\). Indeed \(\lambda(\sigma) = P\), where \(P = 13/27/49/58/6\). So \(\pi = 132749586\) is a flattened partition.

2. Take \(\sigma = 957834126 = (1 \ 9 \ 6 \ 4 \ 8 \ 2 \ 5 \ 3 \ 7) \in \mathcal{B}(9)\). Then see that \(w\)-\text{ExcL}(\sigma) = \langle 9, 5, 7, 8 \rangle\) and \(\text{Seq}(\sigma) = \langle 1, 3, 7, 8 \rangle\) and \(\alpha_2 = 5 < \gamma_3, \alpha_3 = 7 < \gamma_4 = 8\). So condition \(3\) is failed and \(\sigma \notin \mathcal{B}_F(9)\).

We are going to show that the distribution of weak exceedances in \(\mathcal{B}_F(n)\) is equal to the distribution of runs in the set of flattened partitions over \([n]\).

Remark 4.1. Let \(f(\sigma) = \sigma = \sigma(1)\sigma(2)\ldots\sigma(n-1) \in \mathfrak{S}_{n-1}\). If \(f'\) is obtained from \(f\) by concatenating some \(j \in [n]\) at its end, then \(\sigma' \in \mathfrak{S}_n\) with \(f(\sigma') = \sigma'\) is obtained from \(\sigma\) by \(\sigma' = (jn) \cdot \sigma\). 

Corollary 4.1. The number of Bell-permutations of the second kind in \(\mathcal{B}_F(n, k)\) having \(k\) weak exceedances satisfies the recurrence relation in Theorem \([3, 4]\).

Proof. The proof immediately follows from the proof of the recurrence relation in terms of canonical forms given at the end of the previous section.

Case-i: Let \(\sigma \in \mathcal{B}_F(n-1, k)\) with \(w\)-\text{ExcL}(\sigma) = \langle \alpha_1, \alpha_2, \ldots, \alpha_k \rangle\) and \(\text{Seq}(\sigma) = \langle \gamma_1, \gamma_2, \ldots, \gamma_k \rangle\). Then \(\sigma' = (jn) \cdot \sigma\) for \(j \in [k]\) is the unique Bell-permutation of the second kind with \(w\)-\text{Exc}(\sigma) = \langle j, \gamma_1, \ldots, \gamma_k \rangle\) corresponding to a flattened partition over \([n]\). Observe that \(\text{Seq}(\sigma) = \text{Seq}(\sigma')\) and \(w\)-\text{ExcL}(\sigma') = \langle \alpha'_1, \alpha'_2, \ldots, \alpha'_k \rangle\), where \(\alpha'_i = \alpha_i, i \neq j\) and \(\alpha'_j = n\). Thus condition \(3\) is satisfied for \(\sigma'\). So we have a term \(kt_{n-1, k}\) since \(j\) can be chosen in \(k\) different ways.

Case-ii: Let \(\sigma \in \mathcal{B}_F(n-2, k-1)\) with \(w\)-\text{Exc}(\sigma) = \langle k-1 \rangle\). Then construct \((i, \sigma) \mapsto \sigma'\), where \(\sigma'\) is a Bell-permutation of the second kind with \(w\)-\text{Exc}(\sigma) = \langle j, k-1 \rangle\) corresponding to a flattened partition over \([n]\) as follows:

Take \(\sigma = (n-2) \cdot f_{n-3}) \cdot \ldots \cdot f_i \cdot (1f_i)\),

Then \(\sigma' = (n f'_n) \cdot (n-1) f'_{n-1}) \cdot (j f'_j) \cdot (1f'_i)\),

where \(f' = f'_1 f'_2 \ldots f'_{n-2}\) obtained from \(f = f_1 f_2 \ldots f_{n-2}\) according to the procedure described in \([1]\). Since \(f \in T_{n-2, k-1}\) and \(f' \in T_{n,k}\), the resulting permutation \(\sigma'\) is indeed the required Bell-permutation of the second kind.

For instance, take \(\sigma = 35412 = (52)(413) = (5 2)(4 3)(3 1)(2 1)\). Then \(\tilde{\phi}^{-1}(\sigma) = f = 12132\) and for \(i = 3, f' = 1213432\) by \([1]\). Hence \(\sigma' = (7 2)(6 3)(5 4)(4 3)(3 1)(2 1) = (72)(64513) = 3765142\).

5. The Distribution of Right-to-Left Minima Over the Set of Flattened Partitions

In this section we consider the distribution of the statistics of right-to-left minima over the set of flattened partitions.

Let us denote by \(F_{n,k}\) the set of flattened partitions over \([n]\) having \(k\) right-to-left minima, and \(t_{n,k} = \text{card} F_{n,k}\).

Proposition 5.1. For all positive integers \(n\) and \(k\) with \(2 \leq k \leq n\):

\[
t_{n,k} = t_{n-1,k-1} + (k-1)t_{n-1,k}, \quad t_{1,1} = 1
\]

Proof. Every element \(\pi \in F_{n,k}\) is obtained either from every \(\pi' \in F_{n-1,k-1}\) by appending \(n\) at its end, or from every \(\pi'' \in F_{n-1,k}\) by inserting \(n\) before each right-to-left minimum that is different from \(1\). If \(\pi \in F_{n-1,k-1}\) and \(\text{Rmin}(\pi) = \{\pi(i_1), \pi(i_2), \ldots, \pi(i_{k-1})\}\), then \(1 = \pi(i_1) < \pi(i_2) < \cdots < \pi(i_{k-1})\). So inserting \(n\) before each \(\pi(i)\) for \(j \neq 1\) allows \(\pi(i_j)\) to be the minimum element of its run. Thus the partition \(\pi''\) is flattened with the same number of right-to-left minima as \(\pi\). The former case contributes \(t_{n-1,k-1}\) to the number \(t_{n,k}\), and the later contributes \((k-1)t_{n-1,k}\) as there are \(k-1\) right-to-left minima except \(1\).
Recall that the recurrence relation satisfied by the Stirling numbers of the second kind is given by \( S(n, k) = S(n-1, k-1) + kS(n, k) \).
From this it is easy to see that the values of \( t_{n,k} \) given in the Proposition 5.1 are shifted values of the Stirling numbers of the second kind. That is, \( t_{n,k} = S(n-1, k-1) \), for all \( n \geq k \geq 1 \).

The following proposition gives us the relation between the statistics of right-to-left minima of flattened partitions and the left-to-right weak maxima positions of the corresponding canonical form.

**Proposition 5.2.** The set of right-to-left minima of a flattened partition over \([n]\) and the set of left-to-right weak maxima positions of its corresponding canonical form are the same.

**Proof.** Let \( \pi = \pi(1) \ldots \pi(n) \in F_n \) having \( k \) right-to-left minima and \( f = f_1 \ldots f_n \) be its canonical form.
If \( \pi(i) \in RLMin(\pi) \), then \( \pi(i) < \pi(j), j > i \). Let \( RLMin(\pi) = \{1 = \pi(i_1) < \pi(i_2) < \ldots < \pi(i_k) = \pi(n)\} \). Then we have \( f_{\pi(i_1)} \leq f_{\pi(i_2)} \leq \ldots \leq f_{\pi(i_k)} \). Assume that \( \pi(j) \notin RLMin(\pi) \). Then there is some integer \( s \) such that \( \pi(j) > \pi(s), s > i \). Hence \( f_{\pi(s)} > f_{\pi(j)} \) and \( \pi(j) \notin LWMP(f) \). Thus \( \{\pi(i_1), \ldots, \pi(i_k)\} \subseteq LWMP(f) \).
Conversely, let \( LWMP(f) = \{i_1, \ldots, i_k\} \). Then \( s \geq k \) and \( f_{i_1} \leq \ldots \leq f_{i_k} \). If \( j \in LWMP(f) \), then \( f_j \geq f_i, i < j \).
That is, there is no integer less than \( j \) in \( \pi \) to its right. Thus \( j \) is less than all integers in \( \pi \) that are to its right. Hence \( j \in RLMin(\pi) \) and \( s = k \). Therefore, \( RLMin(\pi) = LWMP(f) \).

**Example 5.1.** Let \( \pi = 149238576 \). Its corresponding canonical form is \( f = 122134321 \). Thus \( RLmin(\pi) = \{1, 2, 3, 5, 6\} = LWMP(f) \).

Using Proposition 5.2, we present another bijective proof for the recursion formula in Proposition 5.1 for the corresponding set of canonical forms of flattened partitions.

Now we interpret \( t_{n,k} \), the number of canonical forms in \( T_{n,k} \), as the number of set partitions over \([n]\) with \( k \) blocks.

Now we present a bijection between these two equisized classes of partitions. Let us denote by \( P_n \) the set of all set partitions over \([n]\) with blocks of non-consecutive elements.

Note that the number of set partitions over \([n]\) with \( k \) blocks such that no two consecutive integers are in the same block satisfies the recurrence relation in Proposition 5.1.

Now we present a bijection between \( P_n \) and \( F_n \) as follows:
Let \( P = B_1 / B_2 / \ldots / B_k \in P_n \) with \( k \) blocks.
Define a map \( \theta : P_n \rightarrow F_n \) given by \( \pi = \theta(P) \), where \( \pi' \) is obtained as follows:

- For \( i = 2, \ldots, k \):
  - If \( b \in B_i, b \neq min(B_i) \) and \( b-1 \in B_j \) such that \( j < i \), then move \( b \) in to \( B_{i-1} \), and rearrange the elements of \( B_{i-1} \) in increasing order. An integer can only be moved at most once.
  - Concatenate the resulting partition and set it \( \pi \).
Example 5.3. Let \( P = 1358/26/47 \). Then we have \( 1358/26/47 \rightarrow 13568/2/47 \rightarrow 13568/27/4 \rightarrow 13568274 \). Therefore \( \pi = 13568274 \).

Theorem 5.1. The map \( \theta \) is a bijection.

Proof. First we prove that for every \( P \in \mathcal{P}_n \), \( \pi = \theta(P) \in F_n \). The operation moves an element \( b \) of a block \( B_i \) in to the block proceeding it whenever there is \( b - 1 \in B_j, j < i \), and \( \min(B_i) \) never moves for all \( i \). Thus the minimum elements are in increasing order and hence \( \pi \) is a flattened partition. Note that if \( P \) has \( k \) blocks, then \( \pi \) has \( k \) right-to-left minima. That is, let \( b \in B_i \).

- If \( b = \min(B_i) \), then \( b \) is less than all integers to its right in \( \pi \) and \( b \notin RMin(\pi) \).
- If \( b \neq \min(B_i) \), then \( b - 1 \notin B_i \). Let \( b - 1 \notin B_j \). If \( j < i \), then the procedure moves \( b \) in to \( B_{i-1} \) and \( \min(B_i) \) is to the right of \( b \) in \( \pi \) and \( b \notin RMin(\pi) \).
- If \( j > i \), then \( b - 1 \geq \min(B_j) \) and hence \( b > \min(B_j) \) and the procedure moves neither \( b \) nor \( \min(B_j) \). So \( b \notin RMin(\pi) \).

Therefore, \( b \in RMin(\pi) \) if and only if \( b = \min(B_i), i = 1, \ldots, k \).

Next, we prove that \( \theta \) is one-to-one. Suppose that \( P \neq P' \), where \( P, P' \in \mathcal{P}_n \). Then there are two possibilities:

1. If \( P \) and \( P' \) have different number of blocks, then we are done since \( \theta(P) \) and \( \theta(P') \) have different number of right-to-left minima.
2. Let \( P = B_1/\ldots/B_k \) and \( P' = B'_1/\ldots/B'_k \), and at least one element \( b \in B_i \) where \( b \notin B'_i \). Then \( b \in B'_i, i \neq j \).

(a) If \( b = \min(B_i) \), then \( b \) is the \( i \)-th right-to-left minimum of \( \theta(P) \) and it would not be the case for \( \theta(P') \).

(b) Let \( b \neq \min(B_i) \) and let \( b - 1 \notin B_m \). Two cases are possible.

- If \( m < i < j \) or \( m < j < i \), then in this case \( \theta \) moves \( b \) in to \( B_{i-1} \) for \( P \) and \( b \) in to \( B'_{j-1} \) for \( P' \). Thus, \( \theta(P) \neq \theta(P') \).

- If \( i < m < j \) or \( j < m < i \), then \( \theta \) moves \( b \) in to \( B'_{j-1} \) only for \( P' \) in the former case or to \( B_{i-1} \) only for \( P \) in the later case.

In either case we have \( \theta(P) \neq \theta(P') \).

Therefore, \( \theta \) is a bijection.

Now we present the inverse of \( \theta \). Let \( \pi \in F_n \) with \( k \) right-to-left minima. Then \( P = \theta^{-1}(\pi) \) is obtained as follows.

- Insert bars before each right-to-left minimum and let \( B_1/\ldots/B_k \) be the resulting partition.

- For \( i = k, \ldots, 2 \) consider the elements in the \( i \)-th block in increasing order:

  - If \( b \in B_i, b \neq \min(B_i) \) and \( b - 1 \notin B_j \) for some \( j \leq i \), then put \( b \) in \( B_{i+1} \) and arrange the elements in each block in increasing order.

It can be seen that \( \theta^{-1} \) constructs the required set partition for the given flattened partition \( \pi \). This is because the right-to-left minima of \( \pi \) are increasing and become the minimum elements of the blocks of the corresponding set partition \( P \), and there will be no consecutive integers in the same block.

For instance, let \( \pi = 13625784 \). Then we have \( 136/2578/4 \). Since \( 6 \in B_1 \) we insert 7 in \( B_3 \). So \( P = 136/258/7/47 \).

There is another class of flattened partitions over \([n]\) counted by \( t_{n-1} \), where \( t_n \) is the number of flattened partitions over \([n]\).

\( \pi = \pi(1)\pi(2)\ldots\pi(n) \in F_n \). Let \( RMin(\pi) = \{ \pi(i_1), \pi(i_2), \ldots, \pi(i_k) \} \). Then define the set \( F_n \) of flattened partitions \( \pi \) such that for all \( j = 1, 2, \ldots, k - 1 \) we have a set \( \{ \pi(i_j + 1), \pi(i_j + 2), \ldots, \pi(i_{j+1}) \} \) with no consecutive elements.

Proposition 5.3. For all \( n \geq 2 \), \( cardF_n = t_{n-1} \), where \( t_n \) is the number of flattened partitions over \([n]\).

Proof. The restriction of a bijection in Theorem 1.1 is a bijection from \( \mathcal{P}_{n-1} \) to \( F_n \).

6. Non-Crossing Flattened Partitions

A non-crossing partition of \( S \) is a partition in which no two blocks “cross” each other, that is, if \( a \) and \( b \) belong to one block and \( x \) and \( y \) to another, they are not arranged in the order \( axby \). If one draws an arch based at \( a \) and \( b \), and another arch based at \( x \) and \( y \), then the two arches cross each other if the order is \( axby \) but not if it is \( axyb \) or \( abxy \). In the latter two orders the partition \( \{ \{a, b\}, \{x, y\}\} \) is non-crossing. For instance see [15].
Example 6.1. In the following diagram $\pi = 125739104678$ is not non-crossing while $\pi' = 12391046785$ is non-crossing flattened partition.

We are interested in non-crossing flattened partitions over $[n]$. We denote by $NF_n$ the set of all non-crossing flattened partitions over $[n]$.

Theorem 6.1. The number $a_n$ of non-crossing flattened partitions over $[n]$ is equal to $2^{n-2}, n \geq 2$, where $a_1 = a_2 = 1$.

Proof. We use induction on $n$, and provide a recursive construction for the flattened partitions of $NF_n$.

For $n = 1$ the assertion is trivially true (initial condition). Assume that the assertion is true for all values less than $n$.

We distinguish two cases, depending on if $n$ is in the same run as $n-1$ or not.

Case-1: If $n$ is in the same run as $n-1$, we delete $n$ and obtain a non-crossing flattened partition in $NF_{n-1}$. Thus, this contributes $a_{n-1} = 2^{n-3}$ by the induction hypothesis.

Case-2: Suppose that $n$ is not in the same run as $n-1$ and that $n$ is the successor of $i$, with $i < n-1$. In this case, all the integers $1, 2, \ldots, i$ are in the first run. Assume that there is $j \in \{1, 2, \ldots, i-1\}$ such that $j$ is not in the same run as $i$. We can choose $j$ such that $j+1$ is in the same run as $i$. If $j$ is not the last element of its run, then the arc relating it to its successor in the partition creates a crossing.

If instead $j$ is the last element of its run, then there is an integer $k > i$ such that $k$ is in the same run of an integer $j' < i$ (and hence creates a crossing with the arc $(i,n)$), because otherwise, the run containing $j$ should be merged with one of the runs containing integers of $[i+1, n-1]$ or the partition would not be flattened.

Thus the first run is uniquely determined by the integer $i$ and the remaining $n-i-1$ integers must form a non-crossing flattened partition. Thus for each $i$ we have $a_{n-i-1}$ such partitions. Again by induction hypothesis we have $a_{n-i-1} = 2^{n-i-3}$ non-crossing flattened partitions for each $i$. Then taking the sum over all possible $i$ we have $\sum_{i=1}^{n-2} a_{n-i-1} = a_{n-2} + \ldots + a_2 + a_1 = \sum_{i=1}^{n-2} 2^{n-i-3}$.

Putting together the two cases we have

$$a_n = 2^{n-3} + (2^{n-4} + \ldots + 2^1 + 1 + 1) = 2^{n-2}$$

7. The Exhaustive Generation

We used the results presented here – and in particular the construction shown in the proof of the recurrence relation for the sets $F_{n,k}$ – to implement an algorithm to generate these objects, that is, an algorithm which, for any fixed integer $n$, solves the following problem:
**Problem:** Flattened-Partitions-Generation

**Input:** an integer \( n \)

**Output:** the set of all flattened partitions in \( F_n \), partitioned into the subsets \( F_{n,k} \).

Rather than implementing recursive algorithms we made use of dynamic programming and obtained iterative algorithms. All algorithms represent a partition as a list of integers and a set of partitions as a list of partitions and hence as a list of lists of integers.

Algorithm 1 is used to generate flattened partitions in \( F^{(1)}_{n,k} \subseteq F_{n,k} \) from \( F_{n-1,k} \) based on the idea discussed in section 3.

**Algorithm 1** Exhaustive Generation of Flattened Partitions in \( F^{(1)}_{n,k} \) from the partitions of \( F_{n-1,k} \)

**Procedure:** FUNCTION ONE \( F,n \)

**Ensure:** \( F \) is a list of lists whose elements represent flattened partitions of size \( n-1 \), and \( n \) is the integer to be inserted so that the number of blocks remains the same.

\[
L \leftarrow [\ ]
\]

for \( \pi \) in \( F \) do

for \( t \) in \( \text{Range}(\text{Length}(\pi) - 1) \) do

if \( \pi[t] > \pi[t+1] \) then

\( \pi' \leftarrow \pi.\text{Insert}(t+1,n) \)

\( \text{L.Append}(\pi') \)

end if

end for

\( \pi' \leftarrow \pi.\text{Append}(n) \)

\( \text{L.Append}(\pi') \)

end for

return \( L \)

The exhaustive generation of flattened partitions in \( F_{n,k}^{(2)} \) starts from the set \( [n-2] \times F_{n-2,k-1} \). That is, from a pair \((i,\pi)\) \( \in [n-2] \times F_{n-2,k-1} \) we obtain a flattened partition \( \pi' \in F_{n,k}^{(2)} \) using Algorithm 2. The idea is based on the operation given in section 3.

**Algorithm 2** Generation of a Flattened Partition of \( F_{n,k}^{(2)} \) from an element of \( [n-2] \times F_{n-2,k-1} \)

**Procedure:** FLATTENED_PARTITION_SIZE_INC_BY_TWO\( ((\pi,p)) \)

**Ensure:** \( \pi \) is a flattened partition in \( F_{n-2,k-1} \) and \( p \) is an integer in \([n-2] \).

for \( t \) in \( \text{Range(Length}(\pi)) \) do

if \( \pi[t] > p \) then

\( \pi[t] \leftarrow \pi[t] + 1 \)

end if

end for

\( \text{pos} = \text{Length}(\pi) - 1 \)

while \( \pi[\text{pos}] > p \) do

\( \text{pos} \leftarrow \text{pos} - 1 \)

end while

\( \pi.\text{Insert}(\text{pos} + 1,\text{Length}(\pi) + 2) \)

\( \pi.\text{Insert}(\text{pos} + 2,p + 1) \)

return \( (\pi) \)

Algorithm 3 calls Algorithm 2 and gives us the exhaustive generation algorithm for the set of flattened partitions in \( F_{n,k}^{(2)} \).
Algorithm 3 Exhaustive Generation of Flattened Partitions in $F_{n,k}^{(2)}$

Procedure: FUNCTION_TWO($F$)

Ensure: $F$ is a list of lists whose elements represent a flattened partition of $F_{n-2,k-1}$.

$L ← []$

for $\pi$ in $F$ do

for $p$ in Range(1, Length($\pi$) + 1) do

$\pi' ←-$ FLATTENED_PARTITION_SIZE_INC_BY_TWO($\pi$, $p$)

$L$.Append($\pi'$)

end for

end for

return $L$

Now we present the main exhaustive generation algorithm that generates all and only those flattened partitions in $F_n$ for all possible $n$. The algorithm returns a list of lists of lists of integers, namely the list $F_n = [F_{n,1}, F_{n,2}, \ldots, F_{n,n}]$ where each element $F_{n,k}$ is the list of all flattened partitions of $[n]$ having $k$ runs. Since $F_{n,k} = \emptyset$ if $k > \lceil \frac{n}{2} \rceil$, the algorithm can be optimised by computing only those sets $F_{n,k}$ that are not empty.

As we said, the algorithm is based on dynamic programming. We stock the values of the lists $F_{n-1}$ and $F_{n-2}$ and use them to compute the list $F_n$.

In order to save memory, only the last two lists are kept at any time: the list $F_{n-1}$ will be stock in a variable called LastRow and the list $F_{n-2}$ will be stock in a variable called RowBeforeLast, while list $F_n$ will be affected to the variable CurrentRow.

At the end of each iteration, the three variables are shifted.

Algorithm 4 Exhaustive Generation of Flattened Partitions

Procedure: FLATTENED_PARTITIONS($n$)

Ensure: $F$ is a list of lists whose elements represent a flattened partition of $F_{n-2,k-1}$.

RowBeforeLast ← $[[[1]]]$

LastRow ← $[[[1, 2]], [[]]]$

for $i$ in Range(3, $n + 1$) do

CurrentRow ← $[]$

CurrentRow.Append(FUNCTION_ONE(LastRow[0], $i$))

for $j$ in Range(1, $\lceil \frac{i}{2} \rceil$) do

CurrentRow.Append(FUNCTION_ONE(LastRow[j], $i$))

CurrentRow.Append(FUNCTION_TWO(RowBeforeLast[j - 1]))

end for

RowBeforeLast ← LastRow

LastRow ← CurrentRow

end for

All these algorithms have been implemented in Python.

Example 7.1. When FLATTENED_PARTITIONS($n$) is executed for $n = 5$ we get the list $F_5$:

$[[12345], [13452, 13425, 13524, 13245, 14523, 14235, 12453, 12435, 15234, 12534, 12354], [15243, 14253, 13254], [[]]]$.

Observe that $a_{5,1} = 1$, $a_{5,2} = 11$, $a_{5,3} = 3$, $a_{5,4} = 0$, and $a_5 = 1 + 11 + 3 + 0 = 15$.

Acknowledgements

Both authors are members of the project CoRS (Combinatorial Research Studio), supported by the Swedish government agency SIDA. The most significant advances of this research work have been made during two visits of the first author to IRIF. The first visit was entirely supported by IRIF, the second visit was substantially supported by ISP (International Science Programme) of Uppsala University (Sweden) and partially supported by IRIF. The authors are deeply grateful to these two institutions. We also thank our colleagues from CoRS for valuable discussions and comments.
References

[1] J. Baril and V. Vajnovszki, *A permutation code preserving a double Eulerian bistatistic*, Discrete Applied Mathematics, 224, 9-15 (2017).
[2] F. Beyene and R. Mantaci, *Investigations on a Permutation Code*, Electronic Journal of Combinatorics (Submitted), (2020).
[3] F. Beyene and R. Mantaci, *Another Family of Permutations Counted by the Bell Numbers*, Discrete Mathematics (Submitted), (2020).
[4] M. Bona, *Introduction to Enumerative Combinatorics*, The McGraw Hill Companies, (2007).
[5] D. Callan. *Pattern avoidance in “flattened” partitions*. Discrete Math., 309(12):4187–4191, (2009).
[6] D. Dumont and G. Viennot, *A combinatorial interpretation of the Seidel generation of Genocchi numbers*, Ann. Discrete Math. 6, 77-87, (1980).
[7] D. Foata and D. Zeilberger, *Denert’s Permutation Statistic is indeed Euler-Mahonian*, Studies in Applied Mathematics, 31-59, (1990).
[8] T. Mansour, *Combinatorics of set partitions*. Taylor & Francis Group, LLC, (2013).
[9] T. Mansour, M. Shattuck and S. Wagner, *Counting subwords in flattened partitions of sets*, Discrete Mathematics 338, 1989–2005, (2015).
[10] R. Mantaci and F. Rakotondrajao, *A permutation representation that knows what “Eulerian” means*. Discrete Mathematics and Theoretical Computer Science 4, 101-108, (2001).
[11] A. O. Munagi, *Set Partitions and Separations*, International Journal of Mathematics and Mathematical Sciences 2005:3, 451-643, (2005).
[12] O. Nabawanda and F. Rakotondrajao, *Run Distribution Over Flattened Partitions*, Journal of Integer Sequences, Vol. 23 (2020).
[13] M. Orlov, *Efficient Generation of Set Partitions*, (2002).
[14] G. Rota, *The number of partitions of a set*, Amer. Math. Monthly 71, 498–504, (1964).
[15] R. Simion, *Non-crossing partitions*, Discrete Mathematics, volume 217, numbers 1–3, 367–409, April (2000).
[16] R. P. Stanley, *Enumerative Combinatorics*, Vol. 1, 2nd ed, Cambridge Studies of Advanced Mathematics, Cambridge University Press, (2011).
[17] R. P. Stanley, *Enumerative combinatorics*. Vol. 2. Cambridge Studies of Advanced Mathematics, Cambridge University Press, (1999).