THE $L^p$ CARLEMAN ESTIMATE AND A PARTIAL DATA INVERSE PROBLEM

FRANCIS J. CHUNG AND LEO TZOU

Abstract. We construct an explicit Green’s function for the conjugated Laplacian $e^{-x \cdot \omega / h} \Delta e^{-x \cdot \omega / h}$, which let us control our solutions on roughly half of the boundary. We apply the Green’s function to solve a partial data inverse problem for the Schrödinger equation with potential $q \in L^{n/2}$. We also use this Green’s function to derive $L^p$ Carleman estimates similar to the ones in Kenig-Ruiz-Sogge [19], but for functions with support up to part of the boundary.

1. Introduction

In this article we give an explicit construction of a “Dirichlet Green’s function” for the conjugated Laplacian $e^{-x \cdot \omega / h} h^2 \Delta e^{-x \cdot \omega / h}$ on a bounded smooth domain $\Omega \subset \mathbb{R}^n$ for $n \geq 3$. This Green’s function immediately gives various ($L^2$ and $L^p$) Carleman estimates similar to those in Kenig-Ruiz-Sogge [19] and Kenig-Sjöstrand-Uhlmann [20] (linear weight case), for functions in $C^\infty(\Omega)$ with nontrivial boundary conditions.

We also apply the Green’s function to solve the partial data inverse Schrödinger problem with unbounded potential in $L^{n/2}(\Omega)$ for $n \geq 3$.

The main result is the construction of the Green’s function. Let $\omega \in \mathbb{R}^n$ be a unit vector and let $\Gamma \subset \partial \Omega$ be an open subset which is compactly contained in $\{x \in \partial \Omega \mid \nu(x) \cdot \omega > 0\}$. If $p' = \frac{2n}{n+2} < 2 < p = \frac{2n}{n-2}$, we have the following theorem, proved by an explicit construction via heat flow.

Theorem 1.1. Suppose $h > 0$ is sufficiently small. Then there exists an operator $G_\Gamma : L^{p'}(\Omega) \to L^p(\Omega)$ which satisfies

$$e^{-x \cdot \omega / h} h^2 \Delta e^{-x \cdot \omega / h} G_\Gamma = I$$

and the estimates

$$\|G_\Gamma\|_{L^2 \to H^1} \leq C h^{-1}, \quad \|G_\Gamma\|_{L^{p'} \to L^p} \leq C h^{-2}.$$ 

Furthermore, for all $f \in L^{p'}$, $G_\Gamma f \in H^1(\Omega)$ and $G_\Gamma f \mid_\Gamma = 0$.

2000 Mathematics Subject Classification. Primary 35R30.

Key words and phrases. inverse problems, partial data, Calderón problem, Carleman estimate, Green’s function.
We use the Green’s function to prove the following Carleman estimates. Let $H^1(\Omega)$ denote the semiclassical Sobolev space. Define $H^1_\Gamma(\Omega) \subset H^1(\Omega)$ to be the space of functions with vanishing trace along $\Gamma$ and let $H^{-1}_\Gamma(\Omega)$ be its dual.

**Theorem 1.2.** Let $u \in C^2(\bar{\Omega})$ be a function which vanishes along $\partial \Omega$ and $\partial_\nu u |_{\Gamma^c} = 0$. One then has the Carleman estimates
\[
\|u\|_{L^2(\Omega)} \leq \frac{C}{h}\|h^2\Delta_\phi^* u\|_{H^{-1}_\Gamma(\Omega)}, \quad \|u\|_{L^p(\Omega)} \leq C\|\Delta_\phi^* u\|_{L^{p'}(\Omega)}
\]
for all $h > 0$ sufficiently small.

**Remark 1.3.** A modification of the argument presented here can also yield a boundary term of $h^{-1/p} \|\partial_\nu u\|_{L^{p'}(\Gamma)}$ on the left-side of the $L^p$ inequality.

The second estimate differs from other $L^p$ Carleman estimates for the Laplacian in that it allows for $u$ with nontrivial boundary conditions.

Another application of this Green’s function is the resolution of the partial data Calderón problem with unbounded potentials. Let $\Omega$ be a smooth domain contained in $\mathbb{R}^n$, with $n \geq 3$, and let $\omega_0 \in \mathbb{R}^n$ be a unit vector. Define
\[
\Gamma_0^\pm := \{x \in \partial \Omega \mid \pm \nu(x) \cdot \omega_0 \geq 0\}
\]
and let $F \subset \partial \Omega$ be an open neighbourhood containing $\Gamma_0^+$ and $B \subset \partial \Omega$ be an open neighbourhood containing $\Gamma_0^-$.

If zero is not an eigenvalue of the operator $-\Delta + q$, then $q \in L^{n/2}(\Omega)$ gives rise to a well-defined Dirichlet-to-Neumann map
\[
\Lambda_q : H^{\frac{1}{2}}(\partial \Omega) \to H^{-\frac{1}{2}}(\partial \Omega).
\]
(We refer the reader to the appendix of [12] for the definition of the Dirichlet-to-Neumann map for $q \in L^{n/2}(\Omega)$.) We have the following theorem.

**Theorem 1.4.** Let $q_1, q_2 \in L^{n/2}(\Omega)$ be such that $\Lambda_{q_1} f |_{F} = \Lambda_{q_2} f |_{F}$ for all $f \in C_0^\infty(B)$. Then $q_1 = q_2$.

The regularity assumption that $q_j \in L^{n/2}$ is considered optimal in the context of well-posedness theory for the Dirichlet problem; it is also the optimal assumption for the strong unique continuation principle to hold (see [16] for more).

We will provide some brief historical context for these theorems. The construction of the Green’s function for the conjugated Laplace operator was established by Sylvester-Uhlmann [29] using Fourier multipliers with characteristic sets. The authors proved an $L^2$ estimate for their Green’s function and used it to solve the Calderón problem in dimensions $n \geq 3$ for bounded potentials. Chanillo in [3] showed that the Sylvester-Uhlmann Green’s function also satisfies an $L^p \to L^{p'}$ estimate by applying using the result of Kenig-Ruiz-Sogge [19]. This allowed Chanillo to solve the inverse Schrödinger problem with full data for small potentials in the Fefferman-Phong class.
(which contains $L^{n/2}$). Related results were also proved by Lavine-Nachman [23] and Dos Santos Ferreira-Kenig-Salo [12].

The drawback to the Fourier multiplier construction of the Green’s function is that boundary conditions cannot be imposed. Bukhgeim-Uhlmann [2] and Kenig-Sjöstrand-Uhlmann [20] found a way to use Carleman estimates overcome this problem and prove results for the Calderon problem with partial boundary data. Due to its versatility and robustness, this technique has since become the standard tool for solving partial data elliptic inverse problems. The review article [18] contains an excellent overview of recent work in partial data Calderón-type problems; examples for other elliptic inverse problems can be found in [27], [28], [21], [8], and [7].

The Carleman estimates in these papers are typically proved via an integration-by-parts procedure so that boundary conditions can be kept in check. The limitation of this approach is that only $L^2$-type estimates can be derived; none of the available techniques adapt well to $L^p$ setting for functions with boundary conditions. Thus for $q \not\in L^\infty$, there are very few partial data results for the Calderón problem – see [22] for an example of what can be obtained by previous methods.

The Carleman estimate approach has the additional drawback that the Green’s function one “constructs” is an abstract object arising from general statements in functional analysis, like the Hahn-Banach or Riesz representation theorems. This makes partial data reconstruction procedures like the ones in [24] much more difficult to implement in a concrete setting than equivalent ones like [24] for full data.

The Green’s function we construct in Theorem 1.1 has the explicit representation of the Fourier multiplier Green’s function of Sylvester-Uhlmann while at the same time allowing the boundary control of the Carleman estimate approaches. Furthermore, due to its explicit representation as a parametrix, one can easily deduce $L^p$-type estimates as well as $L^2$-type estimates. In a forthcoming article the authors intend to apply the Green’s function constructed here to the problem of reconstruction. One expects that in the context of computational algorithms this Green’s function would open the door to direct inversion methods for partial data Calderón problems in $n \geq 3$ which is parallel to the full data case examined in [1, 9, 10, 11].

We give a brief exposition of our approach. The key observation is that there is a global $\Psi$DO factorization of the conjugated Laplacian $h^2\Delta_\phi := e^{-\omega \cdot x/h} h^2 \Delta e^{\omega \cdot x/h}$ into an elliptic operator $J$ resembling a heat flow and a first-order operator $Q$ which has the same characteristic set as $h^2\Delta_\phi$. One can then construct an inverse for $J$ (and thus $h^2\Delta_\phi$) with Dirichlet boundary conditions by solving the heat flow with zero initial condition.

This way of factoring $h^2\Delta_\phi$ is in the spirit of [4]. However, in our case the factorization is global and occurs on the level of symbols so there will be error terms and they pose a challenge in the construction of the parametrix. As such this necessitates a modified factorization which differs from that of [4] (see (4.7) and the discussions which follow) to obtain the suitable estimates for the remainders of the parametrix.
This article is organized in the following way. In Section 2 we develop a $\Psi DO$ calculus which is compatible with our symbol class - proofs are given in the appendix. In Section 3 we invert a heat flow in the context of this $\Psi DO$ calculus and solve the Dirichlet problem for this heat flow. In section 4 we restate some facts about the Sylvester-Uhlmann Green’s function in the semiclassical setting and derive a factorization for the operator $h^2 \Delta \phi$ involving the heat operator described in the previous section. In section 5 we use this factorization to construct a parametrix with Dirichlet boundary conditions, and in section 6 we turn the parametrix into a Dirichlet Green’s function $G_\Gamma$ and prove Theorem 1.2. Section 7 is devoted to proving Theorem 1.4 using complex geometric optics solutions constructed with the help of $G_\Gamma$.

Acknowledgements: The authors would like to thank the organizers of the Program on Inverse Problems at the Institut Henri Poincaré, where this project began. We would also like to thank Henrik Shahgholian of KTH and Yishao Zhou of Stockholm University for their hospitality during the summer of 2016. In addition, we would like to thank Boaz Haberman for several helpful discussions, and Sagun Chanillo for helping to explain the proof of Lemma 1.2.

2. Elementary Semiclassical $\Psi DO$ theory

We collect a set of facts about semiclassical pseudodifferential operators and also use this opportunity to establish some notations and conventions which we will use throughout. Proofs are contained in the Appendix.

2.1. Mixed Sobolev Spaces. In this article we define the semiclassical Sobolev spaces with the norm

\begin{equation}
\|u\|_{W^{k,r}_{scl}(\mathbb{R}^n)} := \|\langle hD \rangle^k u\|_{L^r}.
\end{equation}

For $k \in \mathbb{N}$ it turns out that this definition is equivalent to the one involving derivatives:

\begin{equation}
\|u\|_{W^{k,r}_{scl}} = \sum_{|\alpha| \leq k} \|\langle hD \rangle^\alpha u\|_{L^r}.
\end{equation}

(Hereafter we will drop the “scl” subscript: unless otherwise stated, all of our Sobolev spaces will be semiclassical.) Choose coordinates $(x', x_n)$ on $\mathbb{R}^n$, with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$, and let $(\xi', \xi_n)$ be the corresponding coordinates on the cotangent space. An immediate consequence of the norm equivalence stated above is that $\langle \xi' \rangle$ is a multiplier from $W^{1,r}(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$. Indeed,

\begin{equation}
\|\langle hD' \rangle u\|_{L^r(\mathbb{R}^n)} = \int_{-\infty}^{\infty} \|\langle hD' \rangle u(x', x_n)\|_{L^r_{x'}} dx_n
\end{equation}

\begin{equation}
\leq \int_{-\infty}^{\infty} \sum_{|\alpha| \leq 1} \|\langle hD \rangle^\alpha u(x', x_n)\|_{L^r_{x'}} dx_n \leq \sum_{|\alpha| = 1} \|\langle hD \rangle^\alpha u\|_{L^r}.
\end{equation}
Now define the mixed Sobolev norms for $u \in C_0^\infty$ by
\begin{equation}
\|u\|_{W^{k,r}(\mathbb{R}^{n-1})W^{r,r}(\mathbb{R}^n)} := \|\langle hD \rangle^k \langle hD \rangle^\ell u\|_{L^r}
\end{equation}
and use these to define the mixed norm spaces $W^{k,r}(\mathbb{R}^{n-1})W^{r,r}(\mathbb{R}^n)$. For convenience we will drop the $\mathbb{R}^{n-1}$ and $\mathbb{R}^n$ in this notation and use the convention that the first $W^{k,r}$ denotes multiplication by $\langle hD \rangle^k$ and the second $W^{r,r}$ denotes multiplication by $\langle hD \rangle^\ell$.

With this definition we have that for $k \geq 0$,
\begin{equation}
W^{-k,r}W^{\ell,r} \subset W^{1-k,r}(\mathbb{R}^n).
\end{equation}
Indeed, one can write
\[
\begin{align*}
u &= \langle hD \rangle^k \langle hD \rangle^{-k} \langle hD \rangle^{-\ell+k} \langle hD \rangle^{-k} \langle hD \rangle^\ell u \\
\text{and use the fact that } \langle hD \rangle^k \langle hD \rangle^{-k} \text{ is a multiplier on } L^r \text{ by } \|hD\|_{L^\infty} \text{ and that} \\
\langle hD \rangle^{-k} \langle hD \rangle^\ell u \in L^r &\iff u \in W^{-k,r}W^{\ell,r}.
\end{align*}
\]

2.2. Tangential Calculus. We denote the Hörmander symbols by $S^0_1(\mathbb{R}^n)$. We also consider symbols in the class $S^k_0(\mathbb{R}^n)$. In this article we will work with product symbols of the form $ba(x', \xi) \in S^k_0(\mathbb{R}^{n-1})S^j_0(\mathbb{R}^n) := S^k_1S^j_1$ where $b(x', \xi') \in S^k_1(\mathbb{R}^{n-1})$ and $a(x', \xi) \in S^j_1(\mathbb{R}^n)$ for $j = 0, 1$. Observe that if $a(x', \xi) \in S^j_1$, then derivatives with respect to either $x'$ or $\xi$ are a finite sum of symbols in $S^j_1$. For convenience we begin with the following Calderón-Vaillancourt type estimate for (classical) PDO with symbols in $S^0_1(\mathbb{R}^n)$ which can be obtained by following the argument of Theorem 9.7 in [30].

**Proposition 2.1.** Let $a(x, \xi)$ be a symbol in $S^0_1(\mathbb{R}^n)$. Then for all $1 < r < \infty$
\begin{equation}
\|a(x, D)u\|_{L^r} \leq C_{r,n} \sum_{|\alpha| \leq k(n), |\beta| \leq k(n)} p_{\alpha, \beta}(a) \|u\|_{L^r}
\end{equation}
where $p_{\alpha, \beta}$ is the semi-norm defined by $p_{\alpha, \beta}(a) := \sup_{x, \xi} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)\langle \xi \rangle^{|\beta|}$ and $k(n) \in \mathbb{N}$ depends on the dimension only.

We shall henceforth denote by $k(n)$ to be the smallest integer for which Proposition 2.1 holds. Note that in $\mathbb{R}^n$ there is a relation between classical and semiclassical quantization of a symbol $a \in S^\infty$ given by
\[
Op_h(a)u(x) = \sqrt{h}^{-n} A_h u_h(x/\sqrt{h})
\]
where $u_h$ is defined by $(\mathcal{F}u_h)(\xi) = (\mathcal{F}u)(\xi/\sqrt{h})$ and $A_h = a_h(x, D)$ for $a_h(x, \xi) := a(\sqrt{h}x, \sqrt{h}\xi)$ (\mathcal{F} denotes the classical Fourier transform). This identity combined
with estimate \((2.6)\) gives us a semiclassical version of Calderón-Vaillancourt: for all \(1 < r < \infty\) and \(h > 0\) sufficiently small,
\[
(2.7) \quad \|Op_h(a)u\|_{L^r} \leq \sum_{|\alpha|,|\beta|\leq k(n)} p_{\alpha,\beta}(a)\|u\|_{L^r} + C\sqrt{h}\|u\|_{L^r}.
\]

For symbols in \(S^k_1S^{-\ell}_1 \cup S^k_1S^{-k(n)-\ell}_0\), we have the following mapping properties.

**Proposition 2.2.** If \(b(x',\xi') \in S^k_1\) and \(a(x',\xi) \in S^\ell_1 \cup S^{-k(n)+\ell}_0\) then
\[
ba(x',hD) : W^{m,r}W^{\ell,r} \to W^{m-k,r}W^{\ell-\ell,r}
\]
with norm
\[
\|ab(x',hD)\| \leq C \sup_{|\alpha| \leq k(n)} |(1 + \Delta_{x'})^{N/2}\partial_{\xi'}^\alpha a(x',\xi)||\xi||^{|\alpha|-\ell} \sup_{|\alpha| \leq k(n)} |(1 + \Delta_{x'})^{N/2}\partial_{\xi'}^\alpha b(x',\xi')||\xi'||^{|\alpha|-k}.
\]

In addition, we have the following compositional calculus result.

**Proposition 2.3.** If \(a \in S^{k_1}_1S^{\ell_1}_1 \cup S^{k_1}_1S^{-k(n)+\ell_1}_0\) and \(b \in S^{k_2}_1S^{\ell_2}_1 \cup S^{k_2}_1S^{-k(n)+\ell_2}_0\) then
\[
b(x'hD)a(x',hD) = ab(x',hD) + h \sum_{|\alpha|=1} (\partial_{\xi'}^\alpha b \partial_{x'}^\alpha a)(x',hD) + h^2 m(x',hD)
\]
where \(m(x',hD) : W^{k,r}W^{\ell,r} \to W^{k-k_1-k_2,r}W^{\ell-\ell_1-\ell_2,r}\).

For proofs of Proposition 2.2 and Proposition 2.3, see the Appendix.

**Remark 2.4.** We have omitted stating the mapping properties on \(H^k_0\) spaces since \(S^k_0S^{\ell}_0 \subset S^{k+\ell}(\mathbb{R}^n)\) and the calculus for these symbols on weighted \(L^2\)-Sobolev spaces are well documented. See for example [26].

3. Heat Flow

Define coordinates on \(\mathbb{R}^n\) and let \(\mathbb{R}^n_+\) denote the upper half space \(\{x_n > 0\}\). Let \(F(x',\xi') \in S^1_1(\mathbb{R}^{n-1})\), and define the semiclassical pseudodifferential operator
\[
(3.1) \quad j(x',hD) = h\partial_{x_n} + F(x',hD')
\]
on \(\mathbb{R}^n\). It follows by considering the \(\xi'\) and \(\xi_n\) direction separately and applying the semiclassical Calderón-Vaillancourt theorem that \(j(x',hD)\) is a bounded operator \(j(x',hD) : W^{1,r}(\mathbb{R}^n) \to L^r(\mathbb{R}^n)\) for \(1 < r < \infty\). As we will see in the following section, one of the factors of the conjugated Laplacian has this form. In this section we will prove some basic facts about the existence and \(L^p\) mapping properties of the inverse of such an operator. This extends the \(L^2\) theory explained in [5].

To obtain an inverse, we will assume that \(F\) obeys the ellipticity condition
\[
(3.2) \quad c\langle\xi'\rangle \leq \text{Re}F(x',\xi') \leq C\langle\xi'\rangle
\]
uniformly in \( x' \), for some constants \( c, C > 0 \). This ensures that the principal symbol

\[
j(x, \xi) := i\xi_n + F(x', \xi')
\]

is uniformly elliptic. We will also assume a finiteness condition on \( F \): that there exists \( X' > 0 \) such that for \( |x'| > X' \),

\[
(3.3) \quad \nabla_{x'} F(x', \xi') = 0.
\]

We need an extra condition to ensure that the symbol \( j^{-1} \) is in the suitable calculus. We assume that there exists a first order symbol \( i\xi_n + F_- (x', \xi') \) with compact characteristic set, such that \( D_{x'} F_- (x', \xi') \) is supported in \( |x'| < X' \), and

\[
(i\xi_n + F)(i\xi_n + F_-) = p(x', \xi) + a_0
\]

where \( p(x', \xi) \) is a second order polynomial in \( \xi \) with compact characteristic set and \( a_0 \in S^{-\infty} (\mathbb{R}^{n-1}) \).

The reason why we need this extra assumption is that \( (i\xi_n + F)^{-1} \) is not in general in the class \( S^{-1}_1 (\mathbb{R}^n) \). However, if \( \chi \in C^\infty_0 (\mathbb{R}^n) \) is identically 1 on a neighbourhood containing the characteristic sets of \( i\xi_n + F_- \) and \( p \), then we can derive the following expansion:

\[
(1 - \chi(\xi))(i\xi_n + F)^{-1} = (1 - \chi(\xi))(i\xi_n + F_-)
\]

\[
= \left( \frac{1}{p(x', \xi)} - \frac{a_0}{p(p + a_0)} \right).
\]

Since \( \chi \) is identically one on the characteristic set of \( p \), it follows that \((1 - \chi(\xi))/p(x', \xi)\) is a symbol in \( S^{-2}_1 (\mathbb{R}^n) \), and so

\[
(1 - \chi(\xi))(i\xi_n + F)^{-1} = (i\xi_n + F_-)
\]

\[
= \left( S^{-2}_1 - \frac{a_0(1 - \chi)}{p(p + a_0)} \right).
\]

Now observe that

\[
\frac{a_0}{p(p + a_0)} = \frac{a_0}{p^2} - \frac{a_0^2}{p^2(p + a_0)};
\]

and we can repeat this procedure indefinitely to obtain

\[
(1 - \chi(\xi))(i\xi_n + F)^{-1} = (i\xi_n + F_-)
\]

\[
\left( S^{-2}_1 + a_0 S^{-4}_1 + \cdots + a_0^m S^{-k(n)-1}_1 + a_0^{m+1} S^{-k(n)-2}_0 \right)
\]

where we are using \( S^k_j \) to represent a symbol from the class \( S^k_j (\mathbb{R}^n) \). Now \((i\xi_n + F_-) S^{-2}_1 \in S^{-1}_1 + S^{-1}_1 S^{-2}_1 \), and the same holds for \((i\xi_n + F_-)(a_0 S^{-4}_1 + \cdots + a_0^m S^{-k(n)-1}_1) \). Finally, \((i\xi_n + F_-) a_0^{m+1} S^{-k(n)-2}_0 \in \text{span}(S^{-\infty} S^{-1-k(n)}_0) \), so

\[
(3.4) \quad (1 - \chi(\xi))(i\xi_n + F)^{-1} \in \text{span}(S^0_1 S^{-1}_1 + S^{-\infty} S^{-1-k(n)}_0 + S^1_1 S^{-2}_1).
\]

Meanwhile \( \chi(\xi) j^{-1}(x', \xi) \in S^{-\infty} (\mathbb{R}^n) \), so we can use (3.4) in conjunction with Proposition 2.2 to get that

\[
(3.5) j^{-1}(x', hD) : L^p_\delta \to H^1_\delta, \quad \delta \in \mathbb{R}, \quad j^{-1}(x', hD) : L^r \to W^{1,r}, \quad 1 < r < \infty.
\]

The operator \( j^{-1}(x', hD) \) also turns out to have desirable support properties.
Lemma 3.1. If \( u \in L^r(\mathbb{R}^n) \) is supported only in \( \{ x_n \geq 0 \} \) then \( j^{-1}(x', hD)u \in W^{1,r}(\mathbb{R}^n) \) has trace zero along \( \{ x_n = 0 \} \) and vanishes identically on the set \( \{ x_n < 0 \} \).

Proof. For \( u \in C_c^\infty(\mathbb{R}^n) \), we can write
\[
j^{-1}(x', hD)u(x) = h^{-n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \frac{\mathcal{F}^h u(\xi)}{i\xi_n + F(x', \xi')} e^{i \xi \cdot x} d\xi_n d\xi'
\]
where \( \mathcal{F}^h \) is the semiclassical Fourier transform. We can write out the Fourier transform in the \( x_n \) variable to get
\[
j^{-1}(x', hD)u(x) = h^{-n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \frac{\mathcal{F}^h u(\xi', t)}{i\xi_n + F(x', \xi')} e^{i \xi_n (x_n - t)} d\xi_n dt e^{i \xi \cdot x'} d\xi'.
\]
Now we can use the residue theorem to evaluate the \( d\xi_n \) integral explicitly, and we get
\[
j^{-1}(x', hD)u(x) = h^{-n} \int_{\mathbb{R}^{n-1}} \int_{-\infty}^{\xi_n} \mathcal{F}^h_{x'} u(\xi', t) e^{\frac{i}{h}(x_n - t)} F(x', \xi') dt e^{i \xi \cdot x'} d\xi'.
\]
For \( u \in C_c^\infty(\mathbb{R}^n) \), the lemma follows immediately from this representation. Then the lemma holds for general \( u \in L^r(\mathbb{R}^n) \) by using a density argument involving the bounds in (3.3).

Henceforth we will refer to the support property given in Lemma 3.1 as “preserving support in the \( x_n \) direction”.

We can turn \( j^{-1}(x', hD) \) into a proper inverse. We first prove a composition type lemma for the operator \( j^{-1}(x', hD) \).

Lemma 3.2. Let \( a(x', \xi') \in S^1_1(\mathbb{R}^{n-1}) \). Then
\[
a(x', hD')j^{-1}(x', hD) = (a j^{-1})(x', hD) + h \sum_{|\alpha|=1} (j^{-2} \partial _\xi ^\alpha a_\xi ^\alpha F)(x', hD) + h^2 m(x', hD)
\]
where \( m(x', hD) \) and \( \sum_{|\alpha|=1} (j^{-2} \partial _\xi ^\alpha a_\xi ^\alpha F)(x', hD) \) map \( L^r \rightarrow L^r \) with norm bounded by a constant independent of \( h \). Furthermore, the commutator \( [a(x', hD'), j^{-1}(x', hD)] = h m(x, hD) \) with
\[
m(x, hD) : L^r \rightarrow L^r, \quad m(x, hD) : L^2_\delta \rightarrow L^2_\delta.
\]

Proof. The expansion (3.4) allows us to write \( j^{-1}(x', \xi) \) as span of elements in
\[
S^0_1 S^{1-1} + S^{-\infty} S^{1-1-k(n)} + S^1_1 S^{-2}.
\]
We can therefore apply Proposition 2.3 to each term to obtain
\[
a(x', hD')j^{-1}(x', hD) = a j^{-1}(x', hD) + h m_1(x', hD) + h^2 m_2(x', hD)
\]
where
\[ m_1(x', \xi) = \sum_{|\alpha|=1} (j^{-2} \partial^\alpha_\xi a \partial^\alpha_\xi F)(x', \xi), \quad \text{and} \quad m_2(x', hD) : L^r \to L^r. \]

Using expansion (3.4) again we see that \( m_1(x', \xi) \) is a symbol in the span of \( S_1^1 S_1^{-1} + S^{-\infty} S_0^{-k(n)-1} + S_2^2 S_1^{-2}. \)

Therefore, it maps \( L^r \to L^r \) by Proposition 2.2 and the fact that \( W^{-k,r} W_{\ell,r} \subset W^{l-k,r}(\mathbb{R}^n). \) To obtain the commutator statement, repeat the argument for the composition \( j^{-1}(x', hD) a(x', hD'). \)

Now we can use \( j^{-1} \) to build a proper inverse which preserves support in the \( x_n \) direction. Moreover, the inversion can still be carried out even if \( j \) is perturbed by a small tangential operator \( hF_0. \)

**Proposition 3.3.** Suppose \( F_0(x', \xi') \in S_1^0(\mathbb{R}^{n-1}) \) obeys the same finiteness condition (3.3) as \( F, \) and consider the operator
\[
J := j(x', hD) + hF_0(x', hD).
\]

For \( h > 0 \) sufficiently small there exists an inverse \( J^{-1} : L^r \to W^{1,r} \) of the form
\[
J^{-1} = j^{-1}(x', hD)(1 + hm_1(x', hD) + h^2 m_2(x', hD))^{-1}
\]

where \( m_1(x', hD), m_2(x'hD) : L^r \to L^r. \)

Furthermore, \( J^{-1} \) preserves support in the \( x_n \) direction: if the support of \( u \in L^r \) is contained in \( x_n \geq 0 \) then \( J^{-1} u \) has vanishing trace on \( \{x_n = 0\} \) and vanishes identically when \( \{x_n < 0\}. \) The same holds for mapping properties on \( H^k_{\delta} \) spaces.

**Proof.** We write
\[
J j^{-1}(x', hD) = h \partial_n j^{-1}(x', hD) + F(x', hD) j^{-1}(x', hD) + hF_0(x', hD) j^{-1}(x', hD).
\]

We can apply Proposition 2.3 to the first term, using the expansion (3.4) for \( j^{-1} \), and Lemma 3.2 to the second and third terms to obtain
\[
J j^{-1}(x', hD) = 1 + hm_1(x', hD) + h^2 m_2(x', hD)
\]

where \( m_1(x', \xi) = \sum_{|\alpha|=1} (j^{-2} \partial^\alpha_\xi a \partial^\alpha_\xi F)(x', \xi) + (j^{-1} F_0)(x', \xi'), \)

and \( m_2(x', hD) : L^r \to L^r. \) Using expansion (3.4) again we see that \( m_1(x', \xi) \) is a symbol in the span of \( S_1^1 S_1^{-1} + S^{-\infty} S_0^{-k(n)-1} + S_2^2 S_1^{-2}. \)

Therefore, it maps \( L^r \to L^r \) by Proposition 2.2 and the fact that \( W^{-k,r} W_{\ell,r} \subset W^{l-k,r}(\mathbb{R}^n). \)
Observe that in equation (3.6) since \( J \) is a differential operator in the \( x_n \) direction, it preserves support in the \( x_n \) direction when acting on \( W^{1,r} \). The operator \( j^{-1}(x', hD) : L^r \rightarrow W^{1,r} \) preserves support in \( x_n \) by Lemma 3.1 and thus the left side preserves support in the \( x_n \) direction. We may conclude from this that the right side preserves \( x_n \) support as well and in particular \( hm_1(x', hD) + h^2m_2(x', hD) \) preserves \( x_n \) support. This means that inverting the right-side by Neumann series preserves support in the \( x_n \) direction.

One final consequence of the structure of \( J^{-1} \) we obtained in Proposition 3.3 is the following disjoint support property:

**Lemma 3.4.** Let \( 1_{\mathbb{R}^n} \) be the indicator function for \( x_n \leq 0 \) and \( \epsilon > 0 \). Then for all \( f \in L^r(\mathbb{R}^n) \),

\[
\| J^{-1} 1_{\mathbb{R}^n} f \|_{W^{1,r}(\{x_n \geq \epsilon\})} \leq C_\epsilon h^2 \| f \|_{L^r}.
\]

**Proof.** Let \( \zeta_\epsilon(x_n) \) be a smooth cutoff function which is identically one on \( \{x_n \geq \epsilon\} \) and identically zero on an open set containing \( \{x_n \leq 0\} \). Then

\[
\| J^{-1} 1_{\mathbb{R}^n} f \|_{W^{1,r}(\{x_n \geq \epsilon\})} \leq \| \zeta_\epsilon J^{-1} 1_{\mathbb{R}^n} f \|_{W^{1,r}(\mathbb{R}^n)}.
\]

Therefore it suffices to show that

\[
\| \zeta_\epsilon J^{-1} 1_{\mathbb{R}^n} f \|_{W^{1,r}(\mathbb{R}^n)} \leq C_\epsilon h^2 \| f \|_{L^r}.
\]

From Proposition 3.3 we have that

\[
J^{-1} = j^{-1}(x' hD)(1 + hm_1(x', hD) + h^2m_2(x', hD))^{-1}
\]

where \( (1 + hm_1(x', hD) + h^2m_2(x', hD))^{-1} \) is given by the Neumann series

\[
(1 + hm_1(x', hD) + h^2m_2(x', hD))^{-1} = 1 + \sum_{k=1}^{\infty} (hm_1(x', hD) + h^2m_2(x', hD))^k.
\]

Therefore, by (3.5) we can write

\[
J^{-1} = j^{-1}(x', hD)(1 + hm_1(x', hD)) + h^2M(x', hD)
\]

where \( M : L^r \rightarrow W^{1,r} \) is bounded uniformly in \( h \). Using this expression for \( J^{-1} \) it suffices to show that

\[
\zeta_\epsilon j^{-1}(x', hD)(1 + hm_1(x', hD)) 1_{\mathbb{R}^n} : L^r(\mathbb{R}^n) \rightarrow W^{1,r}(\mathbb{R}^n)
\]

with norm bounded by \( O(h^2) \). We will demonstrate this for the principal part \( \zeta_\epsilon j^{-1}(x', hD) 1_{\mathbb{R}^n} \) and leave the lower order term, which can be written out explicitly using (3.7), to the reader. By using (3.4) we see that the symbol

\[
j^{-1} \in \text{span}(S^0_1S^{-1}_0 + S^{-\infty}S_0^{-1-k(n)} + S^1_1S^{-2}_1).
\]
We will only show the estimate for \( \zeta \)Op\(_h\)(\( S_1^1S_1^{-2} \))1\(_{\mathbb{R}^n} \) and the others are treated in the same way. Suppose \( b \in S_1^1(\mathbb{R}^{n-1}) \) and \( a \in S_1^{-2}(\mathbb{R}^n) \), by Proposition 2.3 we see that

\[
\zeta b(x', hD)1_{\mathbb{R}^n} = \zeta b(x', hD')a(x', hD)1_{\mathbb{R}^n} + h\zeta \sum_{|\alpha|=1} (\partial_\xi^\alpha b)(x', hD')(\partial_\xi^\alpha a)(x', hD)1_{\mathbb{R}^n} + h^2m(x, hD)
\]

where \( m(x', hD) : L^r \rightarrow W^{-1,r}W^{2,r} \subset W^{1,r}(\mathbb{R}^n) \) by (2.4).

Since \( \zeta \) is a function of \( x_n \) only, it commutes with operators from \( S_1^k(\mathbb{R}^{n-1}) \), and thus estimating \( \zeta b(x', hD)1_{\mathbb{R}^n} \) with \( b \in S_1^k(\mathbb{R}^{n-1}) \) and \( a \in S_1^{-2}(\mathbb{R}^n) \) amounts to estimating terms of the form \( \zeta Op_h(S_1^{-2}(\mathbb{R}^n))1_{\mathbb{R}^n} \). Standard disjoint support properties of \( \Psi DO \) then give the desired estimates. \( \square \)

4. Green’s Functions on \( \mathbb{R}^n \)

The purpose of this discussion is to find a way to invert

\[
h^2\Delta_\phi := h^2e^{-\phi/h} \Delta e^{\phi/h}, \quad \phi(x) := x_n
\]

with a suitable boundary condition and good \( L^{p'} \rightarrow L^p \) estimates. We begin with the operator on \( \mathbb{R}^n \) given by the Fourier multiplier \( \frac{1}{|\xi|^2 + 2i\xi_n + 2\xi_1} \). We give a semiclassical formulation of an estimate established in Sylvester-Uhlmann [29].

**Lemma 4.1.** The Fourier multiplier \( \frac{1}{|\xi|^2 + 2i\xi_n + 2\xi_1} \) maps \( L^2_\delta \rightarrow H^2_{\delta-1} \) for \( \delta > 0 \) with norm bounded by \( h^{-1} \).

**Proof.** Consider the multiplier given by \( \frac{1}{|\xi|^2 + 2i\xi_n + 2\xi_1} \). By the result of [29],

\[
Op_h \left( \frac{1}{|\xi|^2 + 2i\xi_n + 2\xi_1} \right) : L^2_\delta \rightarrow L^2_{\delta-1}.
\]

Observe that \( |\xi|^2 + 2i\xi_n + 2\xi_1 = |\xi_1 + 1|^2 + \sum_{j=2}^{n} |\xi_j|^2 + 2i\xi_n - 1 \). Since shifting in the Fourier coordinate is equivalent to multiplying by a complex linear phase,

\[
Op_h \left( \frac{1}{|\xi|^2 + 2i\xi_n - 1} \right) = e^{-ix_1/h}Op_h \left( \frac{1}{|\xi|^2 + 2i\xi_n + 2\xi_1} \right) e^{ix_1/h}
\]

and the proof is complete. \( \square \)

It turns out that the Fourier multiplier \( \frac{1}{|\xi|^2 + 2i\xi_n - 1} \) also satisfies \( L^{p'} \rightarrow L^p \) estimates for \( p = \frac{2n}{n-2} \) and \( p' = \frac{2n}{n+2} \). We describe below the semiclassical formulation of a result by Kenig-Ruiz-Sogge [19] and Chanillo [3] – see also Haberman [14].

**Lemma 4.2.** The Fourier multiplier satisfies the estimate

\[
\left\| Op_h \left( \frac{1}{|\xi|^2 + 2i\xi_n - 1} \right) u \right\|_{L^p} \leq \frac{C}{h^2} \| u \|_{L^{p'}}
\]

for all \( u \in L^{p'}(\mathbb{R}^n) \).
Proof. We begin with a classical estimate for $Op_1 \left( \frac{1}{|\xi|^2 + 2i\xi_n - 1} \right)$ due to [19]. Let $u \in S$ be a Schwartz function satisfying $\hat{u}(\xi', \xi_n) = 0$ for whenever $\xi_n$ is close to zero. For these $u$, we have $Op_1 \left( \frac{1}{|\xi|^2 + 2i\xi_n - 1} \right) u \in W^{2,p'}(\mathbb{R}^n)$ and we can therefore apply Theorem 2.4 of [19] to obtain

\begin{equation}
\left\| Op_1 \left( \frac{1}{|\xi|^2 + 2i\xi_n - 1} \right) u \right\|_{L^p} \leq C\|u\|_{L^{p'}}.
\end{equation}

We would like to use a density argument to show that the above holds for all $u \in L^{p'}$. Indeed, let $u \in S$ be any Schwartz function and define for all $\delta > 0$ the Schwartz function

$\hat{u}_\delta(\xi) := \theta(\xi_n) \hat{u} := \theta(\xi_n/\delta) \hat{u}$

where $\theta : \mathbb{R} \to [0, 1]$ is a smooth bump function which is identically 1 near the origin. By the dominated convergence theorem and Plancherel one sees that $\lim_{\delta \to 0} \|u_\delta\|_{L^2} = 0$. For the $L^1$ norm, observe that

$u_\delta(x) = \int_{\mathbb{R}} e^{i\xi_n x_n} \theta(\xi_n) \int_{\mathbb{R}} e^{-i\xi_n t} u(x', t) dt d\xi_n = \int_{\mathbb{R}} \hat{\theta}(s) u(x', x_n - s) ds$

so one has $\|u_\delta\|_{L^1} \leq \delta \|u\|_{L^1} \int_{\mathbb{R}} |\hat{\theta}(s)| ds \leq C \|u\|_{L^1}$. Riesz-Thorin interpolation then yields that $\|u_\delta\|_{L^{p'}} \to 0$ for $p' = 2n/(n + 2)$. The function $u - u_\delta$ is then an element of $S$ whose Fourier transform vanishes in a neighbourhood of $\xi_n = 0$ which converges to $u$ in $L^{p'}$ and thus (41) is valid for all $u \in L^{p'}$ by density.

Denote by $u_h(x) := u(hx)$ and insert $u_h$ into the estimate (41) in place of $u$. We get

$\left\| \left( Op_h \left( \frac{1}{|\xi|^2 + 2i\xi_n - 1} \right) u \right)_{h} \right\|_{L^p} \leq C\|u_h\|_{L^{p'}}.$

Making the change of variable $y = hx$ and computing the norms on both sides we get the desired semiclassical estimate stated in the Lemma. The $h^{-2}$ factor arises from the fact that $1/p' - 1/p = 2/n$. \hfill \Box

In order to deal with domains with non-flat boundaries, we will actually need to deal with domains “flattened” by a coordinate change of the type

\begin{equation}
\gamma : (y_n, y'_n) \mapsto (x_n, x'_n) = (y_n - f(y'_n), y'_n).
\end{equation}

Under this change of variables, the conjugated Laplacian $-e^{-y_n/h}(\sum_{j=1}^n h^2 \partial_{y_j}^2)e^{y_n/h}$ becomes the operator

$h^2 \tilde{\Delta}_\phi = Op_h((1 + |K|^2)\xi_n^2 - 2 \xi_n (i - \xi' \cdot K) - (1 - |\xi'|^2))$

where $K(x') := \nabla f(x')$. The next proposition concerns the Green’s function for $h^2 \tilde{\Delta}_\phi$. 
Proposition 4.3. The Green’s function defined by $\tilde{G}_\phi := \gamma^* G_\phi$ satisfies $h^2 \tilde{\Delta}_\phi \tilde{G}_\phi = Id$ and has the bounds
\[
\|\tilde{G}_\phi\|_{L^2_h \rightarrow H^1_s} \leq Ch^{-1}, \quad \|\tilde{G}_\phi\|_{L^{p'} \rightarrow L^p} \leq Ch^{-2}.
\]
Furthermore, we can split $\tilde{G}_\phi = \tilde{G}_\phi^c + (\tilde{G}_\phi - \tilde{G}_\phi^c)$ such that $(\tilde{G}_\phi - \tilde{G}_\phi^c)$ is a $\Psi$DO with symbol in $S^{-1}_1(\mathbb{R}^n)$ and $\tilde{G}_\phi^c$ is constant coefficient, so one can write $G_\phi = G_\phi^c + (G_\phi - G_\phi^c)$ where
\[
G_\phi^c = G_\phi \chi_0(hD) = \chi_1(hD)G_\phi \chi_0(hD)
\]
where $\chi_0, \chi_1 \in C^\infty_c(\mathbb{R}^n)$, with $\chi_0$ identically 1 in the ball of radius 2 and $\chi_1$ identically 1 on the support of $\chi_0$.

Since the characteristic set of $G_\phi$ is disjoint from the support of $1 - \chi_0$, the operator $(G_\phi - G_\phi^c) : L^p(\mathbb{R}^n) \rightarrow W^{2,p}(\mathbb{R}^n)$ is a $\Psi$DO with symbol in $S^{-2}_1(\mathbb{R}^n)$.

The mapping properties of $G_\phi^c$ come from the mapping properties of $G_\phi$ and the fact that $\chi_1(hD)$ has compactly supported symbol.

The estimates for the pull-back operator $\tilde{G}_\phi$ follows naturally from the estimates for $G_\phi$ since the Jacobian of $\gamma$ is identity outside of a compact set. \qed

The characteristic set of $G_\phi$ lies in the sphere $|\xi'| = 1$, and so in particular if $G_\phi^c$ is multiplied by a Fourier side cutoff function supported away from that sphere, the resulting operator is well behaved. The following lemma makes this somewhat more precise.

Lemma 4.4. Let $\tilde{\rho}(\xi')$ be a smooth function with support compactly contained in $|\xi'| < 1$. Then $\tilde{\rho}\tilde{G}_\phi^c = Op_h(S^{-\infty}(\mathbb{R}^n)) + hm(x', hD)\tilde{G}_\phi$ for some $m(x', \xi)$ in $S^{-\infty}(\mathbb{R}^n)$.

Proof. By Proposition 4.3 $\tilde{G}_\phi^c = \gamma^*(\chi G_\phi)$ where $\gamma^*$ is the pull-back by the diffeomorphism given by $(x', x_n) \mapsto (x', x_n - f(x'))$. We compute
\[
\tilde{\rho}(hD')\gamma^*(\chi(hD)G_\phi) = \tilde{\rho}(hD')\gamma^*(\chi(hD))\tilde{G}_\phi = \tilde{\rho}(hD')\tilde{\chi}(x', hD) + hOp_h(S^{-\infty}(\mathbb{R}^n))\tilde{G}_\phi
\]
where $\tilde{\chi}(x', \xi) = \chi(D\gamma(x')^T\xi)$ is the pull-back symbol. By the composition formula in Proposition 2.3
\[
\tilde{\rho}(hD')\gamma^*(\chi(hD)G_\phi) = (Op_h(\tilde{\rho}\tilde{\chi}) + hOp_h(S^{-\infty}(\mathbb{R}^n)))\tilde{G}_\phi = \gamma^*(Op_h(\chi\rho)G_\phi) + hOp_h(S^{-\infty}(\mathbb{R}^n))\tilde{G}_\phi
\]
where $\rho(x', \xi) = \tilde{\rho}(\xi' + \xi_n K(x'))$ is the push-forward symbol. Observe that since $G_\phi$ is constant coefficient, $Op_h(\chi\rho)G_\phi = Op_h \left( \frac{\chi(\xi)\rho(x', \xi)}{\xi'^2 + \xi_n^2 + 1 + 2\xi_n} \right)$. Since $\tilde{\rho}(\xi')$ vanishes in an
Introduce a second cutoff \( \tilde{\chi}(x', \xi) \) and factor an open neighbourhood of \( |\xi'| = 1 \), the symbol

\[
\frac{\chi(\xi)\rho(x', \xi)}{|\xi'|^2 + \xi_n^2 - 1 + 2i\xi_n}
\]

belongs to \( S^{-\infty}(\mathbb{R}^n) \).

4.1. Modified Factorization. To add boundary determination to the Green’s function, we want to take advantage of the fact that \( h^2\tilde{\Delta}_\phi \) factors into two parts, one of which is elliptic and resembles the operator described in Section 3.

Indeed, the symbol of \( \frac{1}{1+K^2}h^2\tilde{\Delta}_\phi \) factors formally as

\[
\xi_n^2 - 2\xi_n \frac{(i - \xi' \cdot K)}{1 + |K|^2} - \frac{(1 - |\xi'|^2)}{1 + |K|^2} = \left( \xi_n - i \left( \frac{(1 + iK \cdot \xi') - \sqrt{(1 + iK \cdot \xi')^2 - (1 - |\xi'|^2)(1 + |K|^2)}}{1 + |K|^2} \right) \right) \times \left( \xi_n - i \left( \frac{(1 + iK \cdot \xi') + \sqrt{(1 + iK \cdot \xi')^2 - (1 - |\xi'|^2)(1 + |K|^2)}}{1 + |K|^2} \right) \right)
\]

and the second factor here is elliptic. The problem is that the square root is not smooth at its branch cut, so this does not give a proper factorization at the operator level. The obvious thing to do is to take a smooth approximation to the square root, but for our purposes we will require something more subtle.

We take the branch of the square root that has non-negative real part, and seek to avoid the branch cut, which happens when the argument of the square root lies on the negative real axis. From examination of the square root, we see that this occurs when \( K \cdot \xi' = 0 \) and \( |\xi'|^2 \leq |K|^2(1 + |K|^2)^{-1} \). By ensuring that \( \xi' \) avoids this set, we can guarantee that the argument of the square root stays away from the branch cut.

Thus let \( 0 < c < c' < 1 \) be a constant such that \( \frac{|K|^2}{1+|K|^2} < c \) for all \( x' \) and let \( \tilde{\rho}(\xi') \) be a smooth function in \( \xi' \) such that \( \tilde{\rho}_0 = 1 \) for \( |\xi'|^2 \leq c \) and \( \text{supp}(\tilde{\rho}_0) \subset B_{\sqrt{c'}} \). Introduce a second cutoff \( \rho \) such that it is identically 1 on \( |\xi'|^2 \leq c' \) but \( \text{supp}(\rho) \subset B_1 \). Observe that

\[
(4.3) \inf_{\xi \in \text{supp} \tilde{\rho}, \ x' \in \mathbb{R}^{n-1}} \left| \xi_n^2 - 2\xi_n \frac{(i - \xi' \cdot K)}{1 + |K|^2} - \frac{(1 - |\xi'|^2)}{1 + |K|^2} \right| > 0.
\]

Since the branch cut of the square root occurs when \( |\xi'|^2 \leq |K|^2(1 + |K|^2)^{-1} \), it follows that for \( \xi' \) in the support of \( 1 - \tilde{\rho}_0 \), the function \( (1 + iK \cdot \xi')^2 - (1 - |\xi'|^2)(1 + K^2) \) stays uniformly away from the branch cut of the square root. As such we may define

\[
(4.4) r := (1 - \tilde{\rho}_0) \sqrt{(1 + iK \cdot \xi')^2 - (1 - |\xi'|^2)(1 + |K|^2)}
\]

and factor

\[
(4.5) \xi_n^2 - 2\xi_n \frac{(i - \xi' \cdot K)}{1 + |K|^2} - \frac{(1 - |\xi'|^2)}{1 + |K|^2} = (\xi_n - \tilde{a}_- + hm_0)(\xi_n - \tilde{a}_+ - hm_0) + \tilde{a}_0 + h \sum_{|\alpha| = 1} \partial_{\xi_n}^\alpha \partial_{\xi'}^\alpha \tilde{a}_- + hm_0 \tilde{a}_+ + h^2 m_0^2
\]
Proof. We use the fact that \( \tilde{a}_+ \) for some compactly supported smooth function \( \chi \) for some \( \tilde{X} \) of radius 2. This means that

\[
\begin{align*}
\tilde{a}_+ &= \pm \left( \frac{(1 + iK \cdot \xi')}{{1 + |K|^2}^{1/2}} \right) \\
\tilde{a}_0 &= \pm \left( \frac{(1 + iK \cdot \xi')^2 - (1 - |\xi'|^2)(1 + |K|^2) - \gamma^2}{{1 + |K|^2}^{1/2}} \right).
\end{align*}
\]

Observe that the support of \( a_0 \) is compactly contained in the interior of the set where \( \tilde{\rho} = 1 \).

We now quantize \( \tilde{a}_0 \) to see that

\[
\frac{1}{1 + K^2} h^2 \tilde{\Delta}_\phi = QJ + \tilde{a}_0(x', hD') - \tilde{h}e_1(x', hD') + h^2 \tilde{e}_0(x', hD')
\]

where \( \tilde{e}_1 = m_0 \tilde{e}_- \in S^{-1}_1(\mathbb{R}^{n-1}), \tilde{e}_0 \in S^0(\mathbb{R}^{n-1}) \), and \( Q \) and \( J \) are the operators with symbols \( \xi_n - a_- + hm_0 \) and \( \xi_n - a_+ + hm_0 \) respectively. Observe that the \( O(h) \) term in the composition formula for \( QJ \) is killed by one of the \( O(h) \) terms in \( \tilde{a}_0 \).

Although this decomposition still gives us an \( O(h) \) error, the symbol \( \tilde{e}_1 \) vanishes when \( |\xi'| = 1 \). In particular it vanishes on the characteristic set of \( h^2 \tilde{\Delta}_\phi \), and as the following lemma shows, it means that \( h\tilde{e}_1(x', hD')\tilde{G}_\phi \) behaves one order of \( h \) better than would be otherwise expected. This will help us with estimates later on.

Lemma 4.5. Let \( \tilde{E}_1 \) denote \( \tilde{e}_1(x', hD') \). The operator \( \tilde{E}_1\tilde{G}_\phi \) is of the form

\[
\tilde{E}_1\tilde{G}_\phi = (\tilde{E}_1\tilde{G}_\phi)^c + Op_h(S^{-1}_1S^{-2}_1) + h\tilde{e}_1(x', hD')\tilde{G}_\phi + hOp_h(S^{-\infty}(\mathbb{R}^{n}))\tilde{G}_\phi
\]

with \( \tilde{e}_1 \in S^{-1}_1(\mathbb{R}^{n-1}) \) and

\[
(\tilde{E}_1\tilde{G}_\phi)^c : L^2 \rightarrow_{h^m} H^k, \quad (\tilde{E}_1\tilde{G}_\phi)^c : L^{p'} \rightarrow_{h^{-m}} H^k \quad \forall k \in \mathbb{N}.
\]

Here the notation \( T : X \rightarrow_{h^m} Y \) indicates that the norm of the operator \( T \) from \( X \) to \( Y \) is bounded by \( O(h^m) \).

Proof. We use the fact that \( \tilde{e}_1 \) takes value zero on the characteristic set of \( \tilde{G}_\phi \). First write

\[
\tilde{E}_1 = \tilde{e}_1(x', hD') = Op_h(\tilde{a}_-m_0\tilde{a}_-) = Op_h(\tilde{a}_-m_0)Op_h(\tilde{a}_-) + hOp_h\tilde{e}_1(x', hD')
\]

for some \( \tilde{e}_1 \in S^{-1}_1(\mathbb{R}^{n-1}) \). Note that

\[
Op_h(\tilde{a}_-)\tilde{G}_\phi = Op_h(\tilde{a}_-)\gamma^*(\chi G_\phi) + Op_h(\tilde{a}_-)\gamma^*((1 - \chi)G_\phi)
\]

for some compactly supported smooth function \( \chi(\xi) \) which is identically 1 on the ball of radius 2. This means that

\[
\tilde{E}_1\tilde{G}_\phi = \frac{1}{\xi_n + i|\xi'|}(1 - |\xi'|^2).\]

We compute the \( Op_h(\tilde{a}_-)\gamma^*(\chi G_\phi) \) portion of this operator.

\[
Op_h(\tilde{a}_-)\gamma^*(\chi hD)G_\phi = Op_h(\tilde{a}_-)\gamma^*(\chi hD)\gamma^*(G_\phi)
\]

\[
= Op_h(\tilde{a}_-)\gamma^*(\chi(x, hD) + hOp_h(S^{-\infty}))(\chi G_\phi)
\]

with \( m_0(x', \xi') := -\tilde{a}_-^{1} \sum_{|\alpha|=1} \partial^{\alpha}_{\xi} \tilde{a}_- \partial^{\alpha}_{x} \tilde{a}_+ \). Here the \( \tilde{a}_+ \) and \( \tilde{a}_0 \) are defined by

\[
(\tilde{a}_+)_+ = \pm \left( \frac{(1 + iK \cdot \xi')}{{1 + |K|^2}^{1/2}} \right)
\]

\[
(\tilde{a}_0)_+ = \pm \left( \frac{(1 + iK \cdot \xi')^2 - (1 - |\xi'|^2)(1 + |K|^2) - \gamma^2}{{1 + |K|^2}^{1/2}} \right).
\]
where \( \tilde{\chi}(x, \xi) \in S^{-\infty} \) is the pulled-back symbol of \( \chi(\xi) \). Continuing by composing \( Op_h(\tilde{a}_-\tilde{a}_+)\tilde{\chi}(x, hD) \) using symbol calculus,

\[
Op_h(\tilde{a}_-\tilde{a}_+)^\gamma(\chi G_\phi) = \gamma^* (Op_h(a_-a_+(\chi(hD)G_\phi)) + hOp_h(S^{-\infty})^\gamma^*(G_\phi)
\]

where \( a_\pm(x, \xi) := (\gamma^* a_\pm)(x, \xi) = \tilde{a}_\pm(x', D\gamma^T \xi) \). We claim \( Op_h(a_-a_+(\chi(hD)G_\phi)) \) can be written as the sum of a \( \PsiDO \) with symbol in \( S^{-\infty}(\mathbb{R}^n) \) and an operator

\[
(4.9) \ (Op_h(a_-a_+(\chi(hD)G_\phi))^\epsilon: L^2 \to h^k \ H^k, \ \ (Op_h(a_-a_+(\chi(hD)G_\phi))^\epsilon: L^{p'} \to h^{-1} \ H^k.
\]

Inserting this into (4.8) would give us the Lemma.

We verify our claim. Observe that

\[
(4.10) \ a_+a_- = (1 - \rho_0)^2(1 - |\xi' + \xi_n K|^2) + (1 + iK \cdot (\xi' + \xi_n K))^2 - (1 - \rho_0)^2(1 + iK \cdot (\xi' + \xi_n K))^2
\]

where \( \rho_0(x', \xi) = \tilde{\rho}_0(\xi + \xi_n K) \). Now \( \tilde{\rho}_0(x', \xi') = 0 \) if \( |\xi'| \geq c' \) for some \( c' < 1 \) and \( K(x') \) is uniformly bounded. Therefore \( \rho_0(x', \xi) = 0 \) if

\[
\frac{1 + c'}{2} \leq |\xi'| \leq 2 - c' \quad \text{and} \quad |\xi_n| \leq \frac{1 - c'}{2 \sup_{x'} |K(x')| + 1}.
\]

Since the characteristic set of the the Fourier multiplier \( \frac{1}{\xi_n^2 + i2\xi_n(1 - |\xi'|^2)} \) is compactly contained in this set, let \( \chi_2(\xi) \) be a cutoff which is supported in this set and 1 in a neighbourhood of the characteristic set and define

\[
(Op_h(a_-a_+(\chi(hD)G_\phi))^\epsilon := Op_h(a_-a_+(\chi(hD)\chi_2(hD)G_\phi).
\]

Write

\[
(Op_h(a_-a_+(\chi(hD)G_\phi)) = (Op_h(a_-a_+(\chi(hD)G_\phi))^\epsilon + (Op_h(a_-a_+(\chi(hD)(1 - \chi_2(hD)))G_\phi).
\]

The second expression is \( \PsiDO \) of order \( -\infty \) since it vanishes identically near the characteristic set and is therefore a compactly supported smooth multiplier.

It remains to establish (4.9) for the part containing the characteristic set. Since \( \rho_0 \) vanishes identically on the support of \( \chi_2 \), it follows from (4.10) that

\[
Op_h(a_+a_-)(\chi(hD))\chi_2(hD)G_\phi = Op_h(\frac{(1 - |\xi'| + \xi_n K|^2)}{(1 + |K|^2)^2}) \chi(hD)\chi_2(hD)G_\phi.
\]

Note since \( Op_h(\frac{(1 - |\xi'| + \xi_n K|^2)}{(1 + |K|^2)^2}) \) is a differential operator, proving (4.9) amounts to proving estimates for the operators \( Op_h(\frac{(1 - |\xi'|^2)\chi(\xi)}{\xi_n^2 + i2\xi_n(1 - |\xi'|^2)}) \) and \( Op_h(\frac{\xi_n(\xi)\xi'}{\xi_n^2 + i2\xi_n(1 - |\xi'|^2)}) \).

Crucially, these are both bounded Fourier multipliers with compact support and therefore map \( L^2 \to H^k \) for all \( k \in \mathbb{N} \) with norm \( O(1) \). Therefore

\[
Op_h(a_+a_-)\chi_2(hD)\chi(hD) : L^2 \to H^k
\]

with norm \( O(1) \).

Moving on to the \( L^{p'} \to H^k \) estimate we write \( \chi(hD)G_\phi = \chi(hD)G_\phi\chi_{100}(hD) \) where \( \chi_{100}(\xi) \) is identically 1 on the support of \( \chi \). The estimate is then a result of
the $L^2$ estimate and the fact that $\chi_{100}(hD) : L^{p'} \to_{h^0} W^{1,p'} \hookrightarrow_{h^{-1}} L^2$ by Sobolev embedding.

\[ \square \]

5. Parametrices on the Half-Space

In this section we construct parametrices for $h^2\tilde{\Delta}_\phi$ on the upper half space which give vanishing trace on the boundary. By a change of variables, we will later use these to build the Green’s function of Theorem \[\text{(1.1)}\]. Because the factoring in \[\text{(4.7)}\] contains a large error term $A_0$ at small frequencies, we will perform two separate constructions – one for the large frequency case and one for the small frequency case. We split the two frequency cases by using the cutoff function $\tilde{\rho}$ from Proposition \[\text{4.3}\], and is a suitable parametrix for the operator $hD$.

\[ \square \]

A contains a large error term these to build the Green’s function of Theorem \[\text{1.1}\]. Because the factoring in \[\text{(4.7)}\] give vanishing trace on the boundary. By a change of variables, we will later use Proposition \[\text{5.1}\].

The map $P_1$ has mapping properties like those of $\tilde{G}_\phi$.

**Proposition 5.1.** The map $P_1$ satisfies, for $\delta > 0$,

\[ P_1 : L^p_\delta(R^n) \to_{h^{-1}} H^1_{\delta-1}(R^n), \quad P_1 : L^p(R^n) \to_{h^{-2}} L^p(R^n). \]

Furthermore, $P_1 v \in H^1_{\text{loc}}(R^n)$ with $P_1 v \mid_{x_n=0} = 0$ for all $v \in L^p(R^n)$.

**Proof.** The weighted $L^2$ Sobolev norms come as a direct consequence of the mapping properties of $\tilde{G}_\phi$ and the fact that $J$, $J^{-1}$ arise from $S^h_0(R^n)$.

For the mapping property from $L^p(R^n) \to L^p(R^n)$, we split $\tilde{G}_\phi = \tilde{G}_\phi^c + (\tilde{G}_\phi - \tilde{G}_\phi^c)$ and observe

\[ J^+ J\tilde{G}_\phi^c : L^{p'}(R^n) \to_{\delta-h^{-2}} W^{k,p'}_{\delta} \to_{h^{-1}} W^{k-1,p'} \to_{h^{-1}} W^{1,p}. \]

\[ J^+ (\tilde{G}_\phi - \tilde{G}_\phi^c) : L^{p'}(R^n) \to_{\delta-h^{-2}} W^{2,p'} \to_{h^{-1}} W^{1,p'} \to_{h^{-1}} L^2 \to_{h^{-1}} H^1 \to_{h^{-1}} L^p. \]

The above diagram also shows that $P_1 v \in H^1_{\text{loc}}$ for all $v \in L^p(R^n)$ by omitting the last Sobolev embedding. The trace property then comes from the definition of $P_1$ and Proposition \[\text{3.3}\].

\[ \square \]

We now have the following proposition for $P_1$. In the following statement we denote $1_{\tilde{\Omega}}$ to be the indicator function of $\tilde{\Omega}$. If $v \in L^p(\tilde{\Omega})$ we use the notation $1_{\tilde{\Omega}} v$ to denote its trivial extension to a function in $L^p(R^n)$.
Proposition 5.2. Let $\tilde{\Omega} \subset \mathbb{R}^n_+$ be a bounded domain with $\partial \tilde{\Omega} \cap \{x_n = 0\} \neq \emptyset$. Denote by $1_{\tilde{\Omega}}$ the indicator function of $\tilde{\Omega}$. Then $P_l$ is a parametrix at large frequencies with vanishing trace on the boundary of the upper half space, in the sense that for all $v \in L^p(\tilde{\Omega})$,

$$1_{\tilde{\Omega}}h^2\tilde{\Delta}_\phi P_l 1_{\tilde{\Omega}}v = (1 - \tilde{\rho}(hD')) + R_l + hR'_l v,$$

with

$$P_l 1_{\tilde{\Omega}}v \in H^1_{loc}(\mathbb{R}^n), \quad P_l 1_{\tilde{\Omega}}v |_{\partial \tilde{\Omega} \cap \{x_n = 0\}} = 0,$$

where $R_l = 1_{\tilde{\Omega}}R_l 1_{\tilde{\Omega}}$ and $R'_l = 1_{\tilde{\Omega}}R'_l 1_{\tilde{\Omega}}$ have the estimates

$$R_l : L^2 \rightarrow_h L^2, \quad R_l : L^p \rightarrow_{h^0} L^2, \quad R'_l : L^r \rightarrow_{h^0} L^r, \quad 1 < r < \infty.$$

To prove this, we compute in the sense of distributions on $\mathbb{R}^n_+$ acting on $C^\infty_0(\mathbb{R}^n_+)$. Using (4.7)

$$h^2\tilde{\Delta}_\phi P_l = (1 - \tilde{\rho})(1 + K^2)(QJ + \tilde{\Delta}_\phi + \tilde{\rho}J \tilde{\Delta}_\phi + [h^2\tilde{\Delta}_\phi, \tilde{\rho}J \tilde{\Delta}_\phi] + h^2\tilde{\Delta}_\phi, \tilde{\rho})J \tilde{\Delta}_\phi + h(1 - \tilde{\rho})(1 + K^2)\tilde{\Delta}_\phi + h(1 - \tilde{\rho})(1 + K^2)\tilde{\Delta}_\phi + [h^2\tilde{\Delta}_\phi, \tilde{\rho}]J \tilde{\Delta}_\phi$$

The first term requires some care. Testing this operator against $v \in C^\infty_0(\mathbb{R}^n)$ and $u \in C^\infty_0(\mathbb{R}^n)$ yields $\langle Q^*(1 + K^2)(1 - \tilde{\rho})^*u, (1_{\mathbb{R}^n_+}J \tilde{\Delta}_\phi)v \rangle_{L^2(\mathbb{R}^n)}$. The operator $Q^*$ is a PDO in the $\xi'$ direction but it is only a differential operator in the $\xi_n$ direction. Therefore the support does not spread in the $x_n$ direction. The operator $\tilde{\rho}(hD')$ is an operator only in the $\xi'$ direction and therefore does not spread support in the $x_n$ direction. As such $Q^*(1 + K^2)(1 - \tilde{\rho})^*u$ vanishes in an open neighbourhood containing the closure of the lower half space and therefore for all $u \in C^\infty_0(\mathbb{R}^n_+)$ and $v \in C^\infty_0(\mathbb{R}^n)$,

$$\langle Q^*(1 + K^2)(1 - \tilde{\rho})^*u, (1_{\mathbb{R}^n_+}J \tilde{\Delta}_\phi)v \rangle_{L^2(\mathbb{R}^n)} = \langle Q^*(1 + K^2)(1 - \tilde{\rho})^*u, (J \tilde{\Delta}_\phi)v \rangle_{L^2(\mathbb{R}^n)}.$$

Therefore we may continue our computation:

$$h^2\tilde{\Delta}_\phi P_l = (1 - \tilde{\rho})(1 + K^2)(QJ \tilde{\Delta}_\phi) + (1 - \tilde{\rho})(1 + K^2)\tilde{\Delta}_\phi + h(1 - \tilde{\rho})(1 + K^2)\tilde{\Delta}_\phi + h(1 - \tilde{\rho})(1 + K^2)\tilde{\Delta}_\phi + [h^2\tilde{\Delta}_\phi, \tilde{\rho}]J \tilde{\Delta}_\phi$$

At this juncture we invoke the factorization (4.7) again and plug the relation

$$h^2\tilde{\Delta}_\phi - (1 + K^2)(\tilde{\Delta}_\phi - h\tilde{\Delta}_1 + h^2\tilde{\Delta}_0) = (1 + K^2)QJ$$

into the first term. Since $h^2\tilde{\Delta}_\phi G_\phi = I$, we get for all $v \in C^\infty_0(\mathbb{R}^n)$,

$$h^2\tilde{\Delta}_\phi P_l v = (1 - \tilde{\rho})v + R_1 v + R_2 v + [h^2\tilde{\Delta}_\phi, \tilde{\rho}]J \tilde{\Delta}_\phi v$$

as a distribution on $\mathbb{R}^n_+$ (ie integrating against functions in $C^\infty_0(\mathbb{R}^n_+)$) where

$$R_1 = h(1 - \tilde{\rho})(1 + K^2)\tilde{\Delta}_1 (1 - J^*J)G_\phi$$
(5.3) \[ R_2 = (1 - \tilde{\rho})(1 + K^2)(\tilde{A}_0 - \tilde{A}_0J^+J + h^2 \tilde{E}_0 - h^2 \tilde{E}_0J^+J)\tilde{G}_\phi. \]

In the following three lemmas, we claim that the remainder terms in (5.1) have the form of the remainders in Proposition 5.2. The estimates for the terms in \( R_2 \) do not use the finer structures of \( \tilde{G}_\phi \) while the estimates for terms in \( R_1 \) takes advantage of smallness of operators whose symbol is zero on the characteristic set of \( \tilde{G}_\phi \).

**Lemma 5.3.**
\[
[h^2 \tilde{\Delta}_\phi, \tilde{\rho}]J^+J \tilde{G}_\phi = h^2 R'_0 + h R'_0
\]
where \( R'_0 : L^r \to L^r, R''_0 : L^2_\delta \to h^{-1} L^2_{\delta^{-1}}, \) and \( R''_0 : L^{p'} \to h^{-2} L^p. \)

**Lemma 5.4.** The operator \( R_1 \) from (5.2) can be written as \( R_1 = R'_1 + R''_1 \) where
\[
\|R'_1\|_{L^2 \to L^2} + h \|R'_1\|_{L^{p'} \to L^2} \leq C h, \quad \|R''_1\|_{L^2_\delta \to L^2_{\delta^{-1}}} + h \|R''_1\|_{L^{p'} \to L^p} \leq C h
\]

**Lemma 5.5.** The operator \( R_2 \) from equation (5.3) maps \( R_2 : L^2_\delta \to L^2_{\delta^{-1}} \) with norm \( O(h) \) while \( R_2 : L^{p'} \to L^p \) with norm \( O(1) \).

The basic idea is that Lemma 5.5 follows from the smallness of \( h^2 \tilde{E}_0 \) and the fact that \( A_0 \) is supported only where \( (1 - \tilde{\rho}) \) is zero. Lemma 5.4 follows from the good behaviour of \( \tilde{E}_1 \) given by Lemma 4.3 and Lemma 5.3 follows from the good behaviour of \( \tilde{G}_\phi \) off of the support of \( \tilde{\rho} \).

**Proof of Lemma 5.5** The terms involving \( h^2 \tilde{E}_0 \) can be estimated directly using the estimates for \( \tilde{G}_\phi \) and \( \tilde{P}_1 \) in Propositions 4.3 and 4.1. The terms involving \( \tilde{A}_0 \) can be estimated by observing that since \( \tilde{\rho}(\xi') \) is chosen to be identically 1 in a neighbourhood of the support of \( \tilde{a}_0(x', \xi') \), the operator
\[
(1 - \tilde{\rho}(hD'))(1 + K^2)\tilde{A}_0 \in h^\infty Op_h(S^\infty(\mathbb{R}^{n-1})).
\]

**Proof of Lemma 5.4** We begin with the \( h \tilde{E}_1 \tilde{G}_\phi \) term in (5.2). By Lemma 4.5
\[
(5.4) \quad h \tilde{E}_1 \tilde{G}_\phi = h(\tilde{E}_1 \tilde{G}_\phi^c) \ast + h Op_h(S^1_1 S^{-2}) + h^2 \vec{c}'(x', hD') \phi \tilde{G}_\phi + h^2 Op_h(S^{-\infty}(\mathbb{R}^n)) \tilde{G}_\phi
\]
with \( \vec{c}' \in S^0_1(\mathbb{R}^{n-1}) \) and
\[
(\tilde{E}_1 \tilde{G}_\phi^c)^c : L^2 \to h^0 H^k, \quad (\tilde{E}_1 \tilde{G}_\phi^c)^c : L^{p'} \to h^{-1} H^k \quad \forall k \in \mathbb{N}.
\]

Our task is to sort the terms in this operator into the \( R'_1 \) bin and the \( R''_1 \) bin. The first term of (5.4) fits the mapping properties of objects in the \( R'_1 \) bin. The second term of (5.4) is a \( \Psi DO \) in the \( S^1_1 S^{-2} \) class and therefore belongs to the \( R''_1 \) bin by Proposition 2.2. By Proposition 4.3 the third term of (5.4) can be written as
\[
h^2 \vec{c}'(x', hD') \tilde{G}_\phi = h^2 \vec{c}'(x', hD') \tilde{G}_\phi^c + h^2 \vec{c}'(x', hD')(\tilde{G}_\phi - \tilde{G}_\phi^c)
\]
where \( (\tilde{G}_\phi - \tilde{G}_\phi^c) \) is a \( \Psi DO \) with symbol in \( S^{-2}_{1,2}(\mathbb{R}^n) \). The \( \Psi DO \) part is in the \( R'_1 \) bin since it behaves well on \( W^{r,k} \) spaces and the estimate for \( L^{p'} \to L^2 \) can be obtained.
by doing semiclassical Sobolev embedding. For the characteristic part, \( \tilde{G}_\phi \) takes \( L^2_{\delta} \rightarrow h^{-1} H^p_{k-1} \) and \( L^{p'} \rightarrow h^{-2} W^{k,p} \). Therefore the characteristic part belongs to the \( R''_1 \) bin.

The reasoning for the third term of (5.4) also applies to the last term of (5.4) and shows that it can also be sorted into the \( R'_1 \) and \( R''_1 \) bin.

We proceed next with the \( h\tilde{E}_1 J^+ J\tilde{G}_\phi \) term of (5.2):

\[
(5.5) \quad h\tilde{E}_1 J^+ J\tilde{G}_\phi = h J^+ J\tilde{E}_1 \tilde{G}_\phi + h[\tilde{E}_1, J^{-1}] 1_{R^+_n} J\tilde{G}_\phi + h J^+[J, \tilde{E}_1]\tilde{G}_\phi.
\]

In the above calculation we commuted \( \tilde{E}_1 \) and \( 1_{R^+_n} \) since \( \tilde{E}_1 \) only acts in the \( x' \) direction.

The first term above can be handled exactly the same as the \( h\tilde{E}_1 \tilde{G}_\phi \) term – note that the argument for the terms in (5.4) shows that each of the constituent terms of \( h\tilde{E}_1 \tilde{G}_\phi \) in (5.4) maps to \( W^{1,r} \), and so applying \( 1_{R^+_n} J \) presents no difficulty. For the first commutator term of (5.5), Lemma 3.2 and Proposition 3.3 show that \([\tilde{E}_1, J^{-1}]=hm(x, hD)\) for some

\[
m(x, hD): L^r \rightarrow L^r, \quad m(x, hD): L^2_{\delta} \rightarrow L^2_{\delta}.
\]

Therefore, splitting \( \tilde{G}_\phi \) into its characteristic part \( \tilde{G}_\phi^c \) and its \( \Psi DO \) part \( \tilde{G}_\phi - \tilde{G}_\phi^c \) as in Proposition 4.3, we have

\[
L^{p'} \tilde{G}_\phi - \tilde{G}_\phi^c \rightarrow W^{2,p'} \tilde{E}_1 J^+ \rightarrow h^{-1} L^2 \rightarrow \frac{1}{h} \rightarrow L^2 \rightarrow [J^{-1}, \tilde{E}_1] \rightarrow L^2, \quad L^2 \tilde{G}_\phi - \tilde{G}_\phi^c \rightarrow H^2 J^+ \rightarrow h^{-1} \rightarrow [E_1, J^{-1}] \rightarrow L^2
\]

and so \( h[\tilde{E}_1, J^{-1}] 1_{R^+_n} J(\tilde{G}_\phi - \tilde{G}_\phi^c) \) belongs to \( R'_1 \) bin. For the characteristic part

\[
L^{p'} \tilde{G}_\phi^c \rightarrow W^{k,p} \tilde{E}_1 J^+ \rightarrow k^{-1,p} \rightarrow L^p \rightarrow [J^{-1}, \tilde{E}_1] \rightarrow L^p, \quad L^2 \tilde{G}_\phi^c \rightarrow H^k J^+ \rightarrow h^{-1} \rightarrow [E_1, J^{-1}] \rightarrow L^k
\]

and therefore \( h[\tilde{E}_1, J^{-1}] 1_{R^+_n} J\tilde{G}_\phi^c \) belongs to the \( R''_1 \) bin.

For the \([J, \tilde{E}_1]\tilde{G}_\phi \) term, splitting \( \tilde{G}_\phi \) into its characteristic part \( \tilde{G}_\phi^c \) and its \( \Psi DO \) part \( \tilde{G}_\phi - \tilde{G}_\phi^c \) we have

\[
L^{p'} \tilde{G}_\phi - \tilde{G}_\phi^c \rightarrow W^{2,p'} \frac{[J, \tilde{E}_1]}{h} \rightarrow W^{1,p'} h^{-1} J^+ \rightarrow H^1, \quad L^2 \tilde{G}_\phi - \tilde{G}_\phi^c \rightarrow H^2 \frac{[J, \tilde{E}_1]}{h} \rightarrow H^1 J^+ \rightarrow H^1
\]

Therefore \( h J^+[J, \tilde{E}_1](\tilde{G}_\phi - \tilde{G}_\phi^c) \) belongs to the \( R'_1 \) bin. For the part with characteristic set, \( J^+[J, \tilde{E}_1]\tilde{G}_\phi \) behaves like

\[
L^{p'} \tilde{G}_\phi \rightarrow W^{k,p} \frac{[J, \tilde{E}_1]}{h} \rightarrow W^{k-1,p} \rightarrow W^{1,p}, \quad L^2 \tilde{G}_\phi \rightarrow H^k \frac{[J, \tilde{E}_1]}{h} \rightarrow H^k J^+ \rightarrow H^k
\]

and therefore \( h J^+[J, \tilde{E}_1]\tilde{G}_\phi \) belongs to \( R''_1 \) bin.

\[\square\]

**Proof of Lemma 5.3** We have

\[
[h^2 \Delta_\phi, \tilde{p} J^+ J\tilde{G}_\phi = [K^2, \tilde{p}] h^2 D_n^2 J^+ J\tilde{G}_\phi - 2[K \cdot hD_{x'}, \tilde{p}] hD_n J^+ J\tilde{G}_\phi.
\]
Some care will be needed in treating the term involving $h^2 D_n^2$ hitting $J^+ = J^{-1} 1_{R_0''}$. We are only considering the expressions as maps to distributions on $\mathbb{R}^n_+$, so for all $u \in C_0^\infty(\mathbb{R}^n_+)$ and $v \in C_0^\infty(\mathbb{R}^n)$,

$$\langle hD_n u, hD_n J^{-1} 1_{\mathbb{R}^n_+} v \rangle = \langle hD_n u, (1 - FJ^+)v \rangle = \langle u, hD_n v - Fv - FJ^+ v \rangle.$$

Here we used the fact that $J = hD_n + F(x', hD')$ for some $F(x', \xi') \in S^1_1(\mathbb{R}^{n-1})$ and the tangential operator $F(x', hD')$ commutes with the indicator function of the upper half-space.

Combining the two expressions we obtain

$$(5.6) \quad [h^2 \tilde{\Delta}_\phi, \tilde{\rho}]J^+ J\tilde{G}_\phi = [K^2, \tilde{\rho}](hD_n - F - FJ^+) J\tilde{G}_\phi - 2 [K \cdot hD_{x'}, \tilde{\rho}](1 - FJ^+) J\tilde{G}_\phi.

We decompose $\tilde{G}_\phi$ in (5.6) into its $\Psi DO$ part and its characteristic part as stated in Proposition 4.3. The part of (5.6) containing the $\Psi DO$ is a bounded map from $L^2_{\omega R_0}$ to $L^2_{\omega R_0}$.

For the part containing the characteristic set, we expand $[h^2 \tilde{\Delta}_\phi, \tilde{\rho}]J^+ J\tilde{G}_\phi^c$ as

$$[K^2, \tilde{\rho}](hD_n - F - FJ^+) J\tilde{G}_\phi^c - 2 [K \cdot hD_{x'}, \tilde{\rho}](1 - FJ^+) J\tilde{G}_\phi^c

+ [K^2, \tilde{\rho}](1 - \tilde{\rho}_1)(hD_n - F - FJ^+) J\tilde{G}_\phi^c - 2 [K \cdot hD_{x'}, \tilde{\rho}](1 - \tilde{\rho}_1)(1 - FJ^+) J\tilde{G}_\phi^c$$

where $\tilde{\rho}_1(\xi')$ is chosen to be identically 1 in a neighbourhood compactly containing the support of $\tilde{\rho}$ but vanishes identically in a neighbourhood of $|\xi'| = 1$. By disjoint support, $[K^2, \tilde{\rho}](1 - \tilde{\rho}_1)$ and $[K \cdot hD_{x'}, \tilde{\rho}](1 - \tilde{\rho}_1)$ both belong to $h^\infty S^{-\infty}(\mathbb{R}^{n-1})$. Since $\tilde{G}_\phi^c : L^2_2 \to h^{-1} H^k_{L^2}$ and $L' \to h^{-2} W^{k, p}$, the second line in the above expression for $[h^2 \tilde{\Delta}_\phi, \tilde{\rho}]J^+ J\tilde{G}_\phi^c$ can be sorted into the $h^2 R''_0$ bin.

The only thing remaining is to treat the terms on the support of $\tilde{\rho}_1$. We will treat the first term and the second term is dealt with in the same manner. We commute $\tilde{\rho}_1(hD')$ so that it appears next to $\tilde{G}_\phi^c$:

$$[K^2, \tilde{\rho}][hD_n - F - FJ^+] J\tilde{G}_\phi^c = [K^2, \tilde{\rho}](hD_n - F - FJ^+) J\tilde{\rho}_1 \tilde{G}_\phi^c + h^2 R''_0.$$

We are able to throw all the commutator terms with $\tilde{\rho}_1$ into the $h^2 R''_0$ bin by using Proposition 3.3, Lemma 3.2, and Proposition 2.3 in conjunction with the mapping properties of $\tilde{G}_\phi^c$ given by Proposition 4.3. Since $\tilde{\rho}_1(\xi')$ vanishes identically near $|\xi'| = 1$, Lemma 4.4 asserts that,

$$\tilde{\rho}_1 \tilde{G}_\phi^c = Op_h(S^{-\infty}(\mathbb{R}^n)) + hm(x', hD) \tilde{G}_\phi^c$$

for some $m(x, \xi) \in S^{-\infty}(\mathbb{R}^n)$ and therefore

$$[K^2, \tilde{\rho}][hD_n - F - FJ^+] J\tilde{G}_\phi^c = hR'_0 + h^2 R''_0.$$
Proof of Proposition 5.2. The estimates for $R_l$ and $R'_l$ come from Lemmas 5.3 and 5.5 in conjunction with (5.1). The trace property of the operator $R_l \mathbf{1}_\Omega$ on $\partial \tilde{\Omega} \cap \{x_n = 0\}$ is a result of Proposition 5.1. Note that the $L^2$ bounds in Proposition 5.2 are unweighted because of the conjugation with indicator functions of $\tilde{\Omega}$. □

5.2. Parametrix for $h^2 \tilde{\Delta}_\phi$ at Small Frequency. Here we want to look for a parametrix for $h^2 \tilde{\Delta}_\phi$ at low frequencies. We begin by defining $p(x', \xi)$ to be the symbol of $h^2 \tilde{\Delta}_\phi$:

$$p(x', \xi) := (1 + K^2)\xi_n^2 - 2\xi_n (i - \xi' \cdot K) - (1 - |\xi'|^2).$$

Now define

$$P_s := \tilde{\rho}(x', hD).$$

The following proposition says that $P_s$ inverts $h^2 \tilde{\Delta}_\phi$ at small frequencies, up to an $O(h)$ error.

Proposition 5.6. $P_s$ is a bounded operator $P_s : L^r \to W^{2,r}$ for all $r \in (1, \infty)$. Moreover for all $r \in (1, \infty)$,

$$h^2 \tilde{\Delta}_\phi P_s = \tilde{\rho} + hR_s$$

for some $R_s : L^r \to L^r$ bounded uniformly in $h$.

Proof. We want to use the symbol calculus developed in Section 2. However, we have the complication that $1/p(x', \xi)$ is not a proper symbol, because of the zeros of $p(x', \xi)$. Therefore it is not immediately evident that $\tilde{\rho}/p(x', \xi)$ lies in the symbol class $S^{-\infty} S^{-1}_1$, as we would want.

We can remedy this by writing

$$\tilde{\rho}(\xi')/p(x', \xi) = (1 - \chi_{100}(\xi))\tilde{\rho}(\xi')/p(x', \xi) + \chi_{100}(\xi)\tilde{\rho}(\xi')/p(x', \xi)$$

where $\chi_{100}(\xi) \in S^{-\infty}(\mathbb{R}^n)$ is a smooth cutoff function supported only for $|\xi| < 100$, and identically one in the ball $|\xi| \leq 50$.

Now note that by (4.3), $p(x', \xi)$ is properly elliptic on the support of $\tilde{\rho}(\xi')$, so $\chi_{100}(\xi)\tilde{\rho}(\xi')/p(x', \xi) \in S^{-\infty}(\mathbb{R}^n)$. Moreover, since the characteristic set of $p(x', \xi)$ lies well inside the set where $\chi_{100} \equiv 1$, we have that $(1 - \chi_{100}(\xi))/p(x', \xi) \in S^{-2}_1(\mathbb{R}^n)$.

Therefore $P_s$ can be understood as the sum of two operators, one of which is in the symbol class $S^{-\infty}(\mathbb{R}^n)$ and the other of which is in the symbol class $S^{-\infty} S^{-1}_1(\mathbb{R}^n)$. Then Proposition 2.2 asserts that $P_s : L^r \to W^{2,r}$ is a bounded operator and Proposition 2.3 asserts that

$$h^2 \tilde{\Delta}_\phi \text{Op}_h \left( \frac{\tilde{\rho}}{p} \right) = \text{Op}_h((1 - \chi_{100})\tilde{\rho}) + \text{Op}_h(\chi_{100}\tilde{\rho}) + hR_{-1} = \text{Op}_h(\tilde{\rho}) + hR_s$$

as we wanted. □
It turns out that our small frequency parametrix preserves support in the $x_n$ direction.

**Proposition 5.7.** Suppose $v \in L^r(\mathbb{R}^n)$, with $1 < r < \infty$, and $\text{supp}(v)$ is contained in the closure of $\mathbb{R}^n_+$. Then both $\text{supp}(P_s v)$ and $\text{supp}(R_s v)$ are contained in $\mathbb{R}^n_+$, where $R_s$ is the operator from Proposition 5.6. In particular, $P_s v |_{x_n=0} = 0$ if $\text{supp}(v) \subset \mathbb{R}^n_+$.

**Proof.** Let $v \in C_c^\infty(\mathbb{R}^n)$. Then

\begin{equation}
(5.7) \quad \text{Op}_h \left( \frac{\hat{p}}{p} \right) v(x) = h^{-n} \int_{\mathbb{R}^n} \frac{\hat{p}(\xi')}{p(x', \xi)} \hat{v}(\xi) e^{i\xi \cdot x/h} \, d\xi.
\end{equation}

We split the integral on the right into $x'$ and $x_n$ variables and get

\[ h^{-n} \int_{\mathbb{R}^{n-1}} e^{i\xi' \cdot x'/h} \int_{-\infty}^{\infty} \frac{\hat{p}(\xi')}{p(x', \xi)} \hat{v}(\xi) e^{i\xi_n x_n/h} \, d\xi_n \, d\xi'. \]

Consider the inner integral

\[ \int_{-\infty}^{\infty} \frac{\hat{p}(\xi')}{p(x', \xi)} \hat{v}(\xi) e^{i\xi_n x_n/h} \, d\xi_n. \]

For fixed $\xi'$ and $x'$, we can write the Fourier transform of $v$ in the $\xi_n$ variable explicitly to get

\begin{equation}
(5.8) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\hat{p}(\xi')}{p(x', \xi)} \mathcal{F}_{x'} v(\xi', s) e^{i\xi_n(x_n-s)/h} \, d\xi_n \, ds.
\end{equation}

We want to evaluate the inner integral using the residue calculus. Since $e^{i\xi_n(x_n-s)/h}$ is analytic, we need to understand the zeros of $p(x', \xi)$ as a polynomial in $\xi_n$. Factoring, we have

\[ p(x', \xi) = -(1 + |K|^2)(\xi_n - a_+)(\xi_n - a_-) \]

where

\[ a_+ = \frac{1 + iK \cdot \xi' \pm \sqrt{(1 + iK \cdot \xi')^2 - (1 - |\xi'|^2)(1 + |K|^2)}}{1 + |K|^2}. \]

Therefore $p(x', \xi)$, viewed as a polynomial in $\xi_n$, has two roots: $a_+$ and $a_-$. Since we are taking the standard branch of the square root, it follows that $a_+$ has positive imaginary part. Meanwhile, if the imaginary part of $a_-$ vanishes, then by proper choice of $\xi_n$, the factor $(\xi_n - a_-)$ can be made to vanish. On the other hand $\hat{p}(\xi')$ is defined to have support only where $p$ is elliptic, and so the imaginary part of $a_-$ does not approach zero on the support of $\hat{p}(\xi')$. Moreover $a_-$ has positive imaginary part when $\xi' = 0$, and it is continuous in $x'$ and $\xi'$ except when $i\sqrt{(1 + iK \cdot \xi')^2 - (1 - |\xi'|^2)(1 + |K|^2)}$ is entirely real, and so $a_-$ also lies in the upper half of the complex plane for all $x'$ and $\xi'$. 
Therefore evaluating the inner integral of (5.8) using the residue calculus over the appropriate contours, we get
\[
2\pi i \int_{-\infty}^{\infty} \frac{\tilde{\rho}(\lambda') F(x', v, \xi, s)}{(1 + |K|^2) (a_+ - a_-)} ds
\]
at least for \( a_+ \neq a_- \). Note that since \( a_\pm \) both have positive imaginary part on the support of \( \tilde{\rho}(\lambda') \), this integral converges. Now (5.9)
\[
\text{Op}_h \left( \frac{\tilde{\rho}}{p} \right) f = 2\pi i h^{-n} \int_{\mathbb{R}^{n-1}} e^{i\lambda' \cdot x'/h} \int_{-\infty}^{\infty} \tilde{\rho}(\lambda') F(x', v, \xi, s) (e^{ia_-(x_n-s)/h} - e^{ia_+(x_n-s)/h}) \frac{ds d\lambda'}{(1 + |K|^2) (a_+ - a_-)}
\]
At first glance this integral may have issues with convergence when \( a_+ - a_- \to 0 \). However, on the set where \( a_+ = a_- \), the residue calculus tells us that the integral vanishes, and near this set we have
\[
\lim_{a_+ - a_- \to 0} \frac{e^{ia_-(x_n-s)/h} - e^{ia_+(x_n-s)/h}}{(1 + |K|^2) (a_+ - a_-)} = \frac{i(x_n - s)}{h(1 + |K|^2)} e^{ia_-(x_n-s)/h}.
\]
Therefore the integral on the right side of (5.9) converges, and so this provides an honest representation of \( \text{Op}_h (\tilde{\rho}/p) \), at least when \( v \in C_c^\infty (\mathbb{R}^n) \). Note that we are not claiming that this integral proves \( L^r \) boundedness: the non-smoothness of \( a_\pm \) makes this non-obvious. Rather, we want to use this representation of the operator to prove the support property. If \( v \in C_0^\infty (\mathbb{R}^n) \) is supported only in the upper half space \( x_n > 0 \), it is clear from (5.9) that
\[
(5.10) \quad P_s v(x', x_n) = 0 \text{ for } x_n \leq 0.
\]
Now from Proposition 5.6 we have
\[
\| P_s v \|_{W^{2,r}(\mathbb{R}^n)} \leq \| v \|_{L^r(\mathbb{R}^n)},
\]
and it follows from the trace theorem that for any fixed \( x_n \),
\[
(5.11) \quad h \| P_s v(\cdot, x_n) \|_{W^{1,r}(\mathbb{R}^{n-1})} \leq \| v \|_{L^r(\mathbb{R}^n)}.
\]
Therefore if \( v \in L^r(\mathbb{R}^n) \) is supported only in the upper half space, we can approximate it with \( C_0^\infty \) functions supported in the upper half space and use the support property for those functions, together with (5.11), to conclude that
\[
\text{Op}_h \left( \frac{\tilde{\rho}}{p} \right) v(x', x_n) = 0
\]
for \( x_n \leq 0 \). This shows that \( P_s \) has the desired support property. The support property for \( R_s \) then follows from writing
\[
\hbar^2 \Delta_d P_s - \tilde{\rho}(hD') = hR_s
\]
and noting that every operator on the left hand side of this equation has the desired support property. 
\[\square\]
6. Dirichlet Green’s function and Carleman estimates

6.1. Green’s Function For Single Graph Domains. By combining Sections 5.1 and 5.2 we see that $1_{\tilde{\Omega}}(P_s + P_l)1_{\tilde{\Omega}}$ is a parametrix for the operator $h^2\tilde{\Delta}_\phi$ in the domain $\tilde{\Omega}$. As one expects, this parametrix can be modified into a Green’s function.

In this section we consider domains with a component of the boundary which coincides with the graph of a function. In particular, let $\Omega$ be a bounded domain in $\mathbb{R}^n$, and suppose $f \in C_0^\infty(\mathbb{R}^{n-1})$ such that $\Omega$ lies in the set $\{x_n > f(x')\}$, with a portion of the boundary $\Gamma \subset \partial \Omega$ lying on the graph $\{x_n = f(x')\}$. Denote by $\gamma$ the change of variable $(x', x_n) \mapsto (x', x_n - f(x'))$.

**Proposition 6.1.** There exists a Green’s function $G_\Gamma$ which satisfies the relation $\langle h^2\Delta_\phi^* u, G_\Gamma f \rangle = \langle u, f \rangle$ for all $u \in C_0^\infty(\Omega)$ and is of the form $\gamma^* G_\Gamma = 1_{\tilde{\Omega}}(P_s + P_l)1_{\tilde{\Omega}}(I + R)$ with $R$ obeying the estimates

$$R : L^{p'}(\tilde{\Omega}) \to_{h^0} L^2(\tilde{\Omega}), \quad R : L^2(\tilde{\Omega}) \to_h L^2(\tilde{\Omega}).$$

The Green’s function $G_\Gamma$ satisfies the estimates

$$G_\Gamma : L^2(\Omega) \to_{h^{-1}} L^2(\Omega), \quad G_\Gamma : L^{p'}(\Omega) \to_{h^{-2}} L^p(\Omega).$$

**Proof.** Change coordinates $(x', x_n) \mapsto (x', x_n - f(x'))$ so that $\tilde{\Gamma} \subset \{x_n = 0\}$ and let $\tilde{\Delta}_\phi$ be the pulled-back conjugated Laplacian. All equalities below are in the sense of distributions in $\tilde{\Omega}$. By Proposition 5.2 and Proposition 5.6 for any $v \in L^{p'}(\tilde{\Omega})$,

$$\langle h^2\tilde{\Delta}_\phi^* u, 1_{\tilde{\Omega}}(P_s + P_l)1_{\tilde{\Omega}}v \rangle = \langle u, v \rangle \quad \forall u \in C_0^\infty(\tilde{\Omega})$$

with $R_s$ and $R_l$ mapping $L^r \to L^r$ with no loss in $h$ while

$$R_l : L^2 \to_h L^2, \quad R_l : L^{p'} \to_{h^0} L^2.$$

Let $S : L^r \to L^r$ denote the inverse of $(1 + hR'_l + hR_s)$ by Neumann series. Then in $\tilde{\Omega}$ we have

$$h^2\tilde{\Delta}_\phi 1_{\tilde{\Omega}}(P_s + P_l)1_{\tilde{\Omega}}S = I + R_lS,$$

with $R_lS : L^2 \to_h L^2$ while $R_lS : L^{p'} \to_{h^0} L^2$. Therefore, for all $v \in L^{p'}(\tilde{\Omega})$ the Neumann series

$$(1 + R_lS)^{-1} v := v - \sum_{k=0}^\infty (-R_lS)^k (R_lS)v \in L^{p'}$$

is well-defined and the series converge in $L^2(\tilde{\Omega})$. The operator $1_{\tilde{\Omega}}(P_s + P_l)1_{\tilde{\Omega}}S(1 + R_lS)^{-1}$ is then a right inverse of $h^2\tilde{\Delta}_\phi$ in $\tilde{\Omega}$. By defining

$$G_\Gamma := \gamma^* 1_{\tilde{\Omega}}(P_s + P_l)1_{\tilde{\Omega}}S(1 + R_lS)^{-1}$$
one obtains the Green’s function in the original coordinates.

For the estimates on $G_T$ and for verifying the trace it is more convenient to work with the operator $1_{\tilde{\Omega}}(P_s + P_l)1_{\tilde{\Omega}}(1 + R_l S)^{-1}$ and deduce the analogous properties for $G_T$. We first check that $1_{\tilde{\Omega}}(P_s + P_l)1_{\tilde{\Omega}}(1 + R_l S)^{-1} v \in H^1_{\text{loc}}(\tilde{\Omega})$ for all $v \in L^{p'}$ and that the trace vanishes on $\tilde{\Gamma} \subset \{x_n = 0\}$. By Proposition 5.1 the operator $P_l$ maps $L^{p'}$ into $H^1_{\text{loc}}$ having vanishing trace on $\{x_n = 0\}$. By Proposition 5.6 $P_l v$ is an element of $W^{2,p'}(\mathbb{R}^n) \hookrightarrow H^1(\mathbb{R}^n)$ which vanishes in $\{x_n < 0\}$ if $v \in L^{p'}(\mathbb{R}^n)$ vanishes in $\{x_n < 0\}$. Therefore we conclude that $1_{\tilde{\Omega}}(P_s + P_l)1_{\tilde{\Omega}}(1 + R_l S)^{-1} v \in H^1(\tilde{\Omega})$ has trace zero on $\tilde{\Gamma}$ for all $v \in L^{p'}(\tilde{\Omega})$ and thus $G_T$ has vanishing trace on $\Gamma$.

To verify the mapping properties of $1_{\tilde{\Omega}}(P_s + P_l)1_{\tilde{\Omega}}(1 + R_l S)^{-1}$ write

$$1_{\tilde{\Omega}}(P_s + P_l)1_{\tilde{\Omega}}(1 + R_l S)^{-1} = 1_{\tilde{\Omega}}(P_s + P_l)1_{\tilde{\Omega}}(I - \sum_{k=0}^{\infty} (R_l S)^k (R_l S))$$

$$= 1_{\tilde{\Omega}}(P_s + P_l)1_{\tilde{\Omega}} - 1_{\tilde{\Omega}}(P_s + P_l)1_{\tilde{\Omega}}(I - \sum_{k=0}^{\infty} (R_l S)^k (R_l S))$$

Since $S : L^r \rightarrow L^r$, inserting an $L^2(\tilde{\Omega})$ function would yield, by Propositions 5.1 and 5.6 an $H^1$ function with a loss of $h^{-1}$ in the first term and no loss in the second. For mappings from $L^{p'}$ we only need to concern ourselves with the first term since the Neumann sum maps $L^{p'} \rightarrow L^2$ with no loss in $h$ and we can refer to the $L^2$ estimate for $1_{\tilde{\Omega}}(P_s + P_l)1_{\tilde{\Omega}}S$.

The mapping properties are then verified by observing, due to Proposition 5.1

$$1_{\tilde{\Omega}}P_l 1_{\tilde{\Omega}}S : L^{p'} \overset{S}{\rightarrow} L^{p'} \overset{1_{\tilde{\Omega}}}{\rightarrow} L^{p'} \overset{P_l}{\rightarrow} L^p \overset{1_{\tilde{\Omega}}}{\rightarrow} L^p.$$ 

And due to Proposition 5.6

$$1_{\tilde{\Omega}}P_s 1_{\tilde{\Omega}}S : L^{p'} \overset{S}{\rightarrow} L^{p'} \overset{1_{\tilde{\Omega}}}{\rightarrow} L^{p'} \overset{P_s}{\rightarrow} W^{2,p'} \overset{h^{-2}}{\hookrightarrow} L^p \overset{1_{\tilde{\Omega}}}{\rightarrow} L^p.$$ 

This finishes the proof of Theorem 1.1 in the case when $\Gamma$ lies in a single graph. In the next section we move on to the general case.

6.2. Proof of Theorem 1.1 - Dirichlet Green’s Function. To prove Theorem 1.1 in the general case, we first develop the necessary tools for gluing together Green’s functions. Let $\Omega$ be a bounded domain and $\Gamma$ be a subset of $\partial \Omega$ which coincides with the graph $\{x_n = f(x')\}$ of a smooth compactly supported function $f$. Without loss of generality we may assume that there is an open neighbourhood $\Omega_\Gamma \subset \mathbb{R}^n$ of $\Gamma$ for which $\Omega_\Gamma \cap \Omega$ lies in the set $\{x_n > f(x')\}$, and that

$$\Omega_\Gamma \cap \partial \Omega \cap \{x_n = f(x')\} = \tilde{\Gamma}.$$ 

Then $\Gamma' := \Omega_\Gamma \cap \partial \Omega$ is an open subset of the boundary such that $\Gamma \subset \subset \Gamma'$ and compact subsets of $\Gamma' \setminus \tilde{\Gamma}$ lies strictly above the graph $x_n = f(x')$. 

Let \( \chi \in C^\infty_0(\mathbb{R}^n) \) be supported inside \( \Omega \) with \( \chi = 1 \) near \( \Gamma \). Then we can arrange that \( \text{supp}(\chi) \cap \partial \Omega \subset \Gamma' \), and for the derivatives of \( \chi \) to have the following support property.

\[
(6.1) \quad \exists \epsilon > 0 \mid \text{supp}(1_\Omega \Delta \chi) \subset \{(x', x_n) \mid x_n \geq f(x') + \epsilon\}.
\]

In this setting choose an open subset \( \mathcal{O} \subset \Omega \cap \{(x', x_n) \mid x_n > f(x')\} \) which contains \( \Gamma' \) as a part of its boundary and whose closure contains the support of \( \chi 1_\Omega \). Set \( G_{\Gamma} \) to be the Green’s function constructed in Proposition 6.1 for the domain \( \mathcal{O} \) with vanishing trace on \( \Gamma \). We may then define

\[
(6.2) \quad \Pi_{\Gamma} : L^{p'}(\Omega) \to_{h^{-2}} L^p(\Omega), \quad \Pi_{\Gamma} : L^2(\Omega) \to_{h^{-1}} H^1(\Omega)
\]

by

\[
\Pi_{\Gamma} : \chi \chi_\Omega(G_\phi - G_{\Gamma}) 1_\mathcal{O}.
\]

Note that \( G_{\Gamma} \) is not defined on the portion of \( \Omega \) that lies below the graph of \( f \), but this point is rendered moot by the multiplication by \( \chi \). Observe that by Proposition 6.1 one has the trace identity

\[
(6.3) \quad \Pi_{\Gamma} v \in H^1(\Omega), \quad (\Pi_{\Gamma} v) |_{\Gamma} = (G_\phi v) |_{\Gamma}, \quad \forall v \in L^{p'}(\Omega).
\]

**Lemma 6.2.** One has the estimates

\[
h^2 \Delta_\phi 1_\Omega \Pi_{\Gamma} 1_\Omega : L^{p'}(\Omega) \to_{h^0} L^2(\Omega), \quad h^2 \Delta_\phi 1_\Omega \Pi_{\Gamma} 1_\Omega : L^2(\Omega) \to_{h^1} L^2(\Omega).
\]

With this lemma we are in a position to construct a general Green’s function for the \( h^2 \Delta_\phi \) on a general domain \( \Omega \). Let \( \omega \in \mathbb{R}^n \) be a unit vector and \( \Gamma \subset \partial \Omega \) be compactly contained in \( \{x \in \partial \Omega \mid \omega \cdot \nu(x) > 0\} \) and write \( \Gamma \) as a union of its connected components \( \Gamma_j \). Without loss of generality we may assume as before that \( \omega = (0', 1) \). For each \( \Gamma_j \) construct \( \chi_j \) and \( \Pi_{\Gamma_j} \) as earlier. One then, by (6.3), has that

\[
\left( G_\phi v - \sum_{j=1}^k \Pi_{\Gamma_j} v \right) |_{\Gamma} = 0, \quad \forall v \in L^{p'}(\Omega).
\]

Furthermore by Lemma 6.2

\[
h^2 \Delta_\phi 1_\Omega \left( G_\phi - \sum_{j=1}^k \Pi_{\Gamma_j} \right) 1_\Omega = I + R'
\]

with

\[
R' : L^2(\Omega) \to_{h} L^2(\Omega), \quad R' : L^{p'}(\Omega) \to_{h^0} L^2(\Omega).
\]

Note that as before we can as before invert by Neumann series since \( L^{p'} \) gets mapped by \( R' \) to \( L^2 \) with no loss and the Neumann series converge in \( L^2 \). Theorem 1.1 is now complete by the estimates of (6.2), Lemma 4.1, and Lemma 4.2. All that remains is to give a proof of Lemma 6.2.

**Proof of Lemma 6.2** By Proposition 6.1, \( G_{\Gamma} \) is by construction a right inverse for \( h^2 \Delta_\phi \) in \( \Omega \), and \( \chi 1_\Omega \) is supported only on \( \Omega \), so \( \chi h^2 \Delta_\phi 1_\Omega G_{\Gamma} v(x) = \chi v(x) \) as distributions on \( \Omega \). Meanwhile \( G_\phi \) is an honest right inverse for \( h^2 \Delta_\phi \) on \( \mathbb{R}^n \), so \( h^2 \Delta_\phi 1_\Omega G_\phi = I \) as distributions on \( \Omega \). Therefore as distributions on \( \Omega \), the only term in \( h^2 \Delta_\phi \Pi_{\Gamma} v(x) \)
is \([h^2\Delta_\phi, \chi_j]1_\Omega(G_\phi - G_{\Gamma^*})1_\Omega v(x)\). To analyze this term we will change coordinates by \((x', x_n) \mapsto (x', x_n - f(x'))\) and mark the pushed forward domains, functions and operators with a tilde. Then by the push-forward expression for the operator \(G_{\Gamma^*}\) stated in Proposition 6.1, the operator in our term becomes
\[
[h^2\tilde{\Delta}_\phi, \tilde{x}]1_{\tilde{\Omega}}(\tilde{G}_\phi - (P_s + P_t)1_{\tilde{\Omega}}(I + R))1_{\tilde{\Omega}}
\]
where
\[
R : L^{p'}(\tilde{\Omega}) \to h^p L^2(\tilde{\Omega}), \quad R : L^2(\tilde{\Omega}) \to h L^2(\tilde{\Omega}).
\]
Computing the commutator \([h^2\tilde{\Delta}_\phi, \tilde{x}]\) explicitly in conjunction with the operator estimates in Proposition 5.6 and Proposition 5.1 we have that
\[
(6.4) \quad [h^2\tilde{\Delta}_\phi, \tilde{x}]1_{\tilde{\Omega}}(\tilde{G}_\phi - (P_s + P_t)1_{\tilde{\Omega}}(1 + R_\delta)^{-1})1_{\tilde{\Omega}} = [h^2\tilde{\Delta}_\phi, \tilde{x}]1_{\tilde{\Omega}}(\tilde{G}_\phi - (P_s + P_t))1_{\tilde{\Omega}} + E
\]
where
\[
E : L^{p'}(\tilde{\Omega}) \to h^p L^2(\tilde{\Omega}), \quad E : L^2(\tilde{\Omega}) \to h L^2(\tilde{\Omega}).
\]
Returning to (6.4), we see that \(E\) has the correct boundedness properties, so it remains only to analyze the first term
\[
[h^2\tilde{\Delta}_\phi, \tilde{x}]1_{\tilde{\Omega}}(\tilde{G}_\phi - (P_s + P_t))1_{\tilde{\Omega}}.
\]
Since we are only doing the computation in \(\tilde{\Omega}\), the first order differential operator \([h^2\tilde{\Delta}_\phi, \tilde{x}]\) commutes with the indicator function \(1_{\tilde{\Omega}}\), and we have
\[
[h^2\tilde{\Delta}_\phi, \tilde{x}]1_{\tilde{\Omega}}(\tilde{G}_\phi - (P_s + P_t))1_{\tilde{\Omega}} = 1_{\tilde{\Omega}}[h^2\tilde{\Delta}_\phi, \tilde{x}](\tilde{G}_\phi - (P_s + P_t))1_{\tilde{\Omega}}.
\]
Now \(P_s\) maps \(L^2\) to \(L^2\) with no loss of \(h\)'s, and \(L^{p'}\) to \(W^{2,p'} \hookrightarrow h^{-1} H^1\). Meanwhile the commutator \([h^2\tilde{\Delta}_\phi, \tilde{x}]\) maps \(H^1\) to \(L^2\) with the gain of \(h\), so the term involving \(P_s\) has the desired behaviour. Therefore the only term of difficulty is
\[
[h^2\tilde{\Delta}_\phi, \tilde{x}]1_{\tilde{\Omega}}(\tilde{G}_\phi - P_t)1_{\tilde{\Omega}} = [h^2\tilde{\Delta}_\phi, \tilde{x}]1_{\tilde{\Omega}}(I - J^* J)\tilde{G}_\phi 1_{\tilde{\Omega}} = 1_{\tilde{\Omega}}[h^2\tilde{\Delta}_\phi, \tilde{x}] J^{-1} 1_{\mathbb{R}^n} J\tilde{G}_\phi 1_{\tilde{\Omega}}.
\]
By (6.1) the term \(1_{\tilde{\Omega}}[h^2\tilde{\Delta}_\phi, \tilde{x}]\) is a first order differential operator whose coefficients are supported in \(\{x_n \geq \epsilon > 0\}\). The proof then follows from Lemma 3.4.

6.3. **Carleman Estimates.** The Carleman estimates in Theorem 1.2 now follow from the existence of the Green’s function \(G_{\Gamma}\).

**Proof of Theorem 1.2.** Let \(u \in C^2(\tilde{\Omega})\) be a function which vanishes along \(\partial \Omega\) and \(\partial_\nu u \mid r = 0\), and let \(v \in C^\infty_0(\tilde{\Omega})\). Integrating by parts, we have
\[
(6.5) \quad \langle h^2\Delta_\phi^* u, G_{\Gamma^*} v \rangle_{\tilde{\Omega}} = \langle u, v \rangle_{\tilde{\Omega}}
\]
with the boundary terms vanishing because of the boundary conditions on \(u\) and the boundary behaviour of \(G_{\Gamma^*} v\). Equation (6.5) implies that
\[
\|h^2\Delta_\phi u\|_{H^{-1}(\tilde{\Omega})} \|G_{\Gamma^*} v\|_{H^1(\tilde{\Omega})} \geq |\langle u, v \rangle_{\tilde{\Omega}}|
\]
and
\[
\|h^2\Delta_\phi u\|_{L^{p'}(\tilde{\Omega})} \|G_{\Gamma^*} v\|_{L^p(\tilde{\Omega})} \geq |\langle u, v \rangle_{\tilde{\Omega}}|.
\]
Applying the boundedness results for $G_\Gamma$ and taking the supremum over $v \in C_0^\infty(\Omega)$ completes the proof.

7. Complex Geometrical Optics and the Inverse Problem

Let $\Omega \subset \mathbb{R}^n$, $\omega \in S^{n-1}$ and $\Gamma \subset \partial \Omega$ be an open subset of the boundary compactly contained in $\{x \in \partial \Omega \mid \nu(x) \cdot \omega > 0\}$ where $\nu$ denotes the normal vector. By Theorem 7.1 there exists a Green’s function $G_\Gamma$ for $h^2\Delta\phi$ with vanishing trace on $\Gamma$ and

$$G_\Gamma : L^2(\Omega) \rightarrow_{h^{-1}} L^2(\Omega), \quad G_\Gamma : L^{p'}(\Omega) \rightarrow_{h^{-2}} L^p(\Omega).$$

7.1. Semiclassical solvability. Let $\omega$ be a unit vector and $\Gamma \subset \partial \Omega$ be an open subset which is compactly contained in $\{x \in \partial \Omega \mid \nu(x) \cdot \omega > 0\}$ we have the following solvability result, resembling the one in [23] (see the explanation of this method in [12]), but with an additional term.

Proposition 7.1. Let $L \in L^2(\Omega)$ with $\|L\|_{L^2} \leq Ch^2$, and let $q \in L^{n/2}(\Omega)$. For all $a = a_h \in L^\infty$ with $\|a_h\|_{L^\infty} \leq C$, there exists a solution of

$$h^2(\Delta\phi + q)r = h^2qa + L \quad \text{with estimates } \|r\|_{L^2} \leq o(1) \text{ and } \|r\|_{L^p} \leq O(1).$$

Proof. We try solutions of the form $r = G_\Gamma(\sqrt{|q|}v + L)$ for $v \in L^2$ with $\|v\|_{L^2} \leq Ch^2$. Supposing this can be accomplished, then using $\|L\|_{L^2} \leq Ch^2$,

$$\|r\|_{L^2} \leq \|G_\Gamma(\sqrt{|q|}v)\|_{L^2} + \|G_\Gamma(L)\|_{L^2} \leq \|G_\Gamma(\sqrt{|q|}v)\|_{L^2} + \|G_\Gamma(L)\|_{L^2} \leq \frac{C_\epsilon}{h}\|v\|_{L^2} + \frac{C}{h^2}\|\sqrt{|q|}v\|_{L^{p'}} + Ch$$

where for any $\epsilon > 0$ we decompose $\sqrt{|q|} = \sqrt{|q|^+} + \sqrt{|q|^+}$ with $\sqrt{|q|^+} \in L^\infty$ and $\|\sqrt{|q|^+}\|_{L^n} \leq \epsilon$. Therefore,

$$\|r\|_{L^2} \leq \left(\frac{C_\epsilon}{h} + \frac{C\epsilon}{h^2}\right)\|v\|_{L^2} + Ch = o(1)$$

by taking $h \to 0$ and using that $\|v\|_2 \leq Ch^2$.

For the $L^p$ norm, observe that

$$\|L\|_{L^{p'}} \leq \|L\|_{L^2} \leq Ch^2 \quad \text{and} \quad \|\sqrt{|q|}v\|_{L^{p'}} \leq \|q\|_{L^{n/2}}\|v\|_{L^2} \leq Ch^2.$$

The mapping property of $G_\Gamma$ from $L^{p'} \rightarrow_{h^{-2}} L^p$ then gives the result.

We now show that we can indeed construct such a $v$. Inserting the ansatz into (7.1) and writing $q = e^{i\theta}|q|$ for some $\theta(\cdot) : \Omega \rightarrow \mathbb{R}$ we see that it suffices to construct $v \in L^2$ solving the integral equation

$$(1 + h^2e^{i\theta}\sqrt{|q|}G_\Gamma\sqrt{|q|})v = h^2(e^{i\theta}\sqrt{|q|a} - e^{i\theta}\sqrt{|q|G_\Gamma(L)}).$$
with \( \|v\|_{L^2} \leq C h^2 \). Observe that the right side is \( O(h^2) \) in \( L^2 \) norm due to the fact that \( \|L\|_{L^2} \leq C h^2 \) so it suffices to show that \( h^2 e^{i\theta} \sqrt{|q|} G_{\Gamma} \sqrt{|q|} : L^2 \to L^2 \) is bounded by \( o(1) \) as \( h \to 0 \) and invert by Neumann series. Indeed, writing \( \sqrt{|q|} = \sqrt{|q|^2 + |q'|} \) we have

\[
\sqrt{|q|} G_{\Gamma} \sqrt{|q|} = \sqrt{|q|^2} G_{\Gamma} \sqrt{|q'|} + \sqrt{|q'|} G_{\Gamma} \sqrt{|q'|} + \sqrt{|q|^2} G_{\Gamma} \sqrt{|q'|}.
\]

Each of the three pieces have the following mapping properties:

\[
\begin{align*}
\sqrt{|q|^2} G_{\Gamma} \sqrt{|q'|} & : L^2 \xrightarrow{h^{-1}} L^2 \xrightarrow{h^{-1}} L^2 \\
\sqrt{|q'|} G_{\Gamma} \sqrt{|q'|} & : L^2 \xrightarrow{h^{-2}} L^2 \xrightarrow{h^{-2}} L^2 \\
\sqrt{|q|^2} G_{\Gamma} \sqrt{|q'|} & : L^2 \xrightarrow{h^{-1}} L^2 \xrightarrow{h^{-2}} L^2 \xrightarrow{h^{-2}} L^2
\end{align*}
\]

Therefore we have that \( h^2 e^{i\theta} \sqrt{|q|} G_{\Gamma} \sqrt{|q|} : L^2 \to o(1) \ L^2 \) as \( h \to 0 \). \( \square \)

### 7.2. Ansatz for the Schrödinger equation

We briefly summarize the ansatz construction procedure given in [20]; see also the explanation in [4]. Let \( \phi(x) \) and \( \psi(x) \) be linear functions satisfying \( D(\phi + i\psi) \cdot D(\phi + i\psi) = 0 \). If \( \Gamma \subset \partial \Omega \) is an open subset of the boundary satisfying \( D\phi \cdot \nu(x) \geq \epsilon_0 > 0 \) for all \( x \in \Gamma \), we first look to construct a solution to

\[
h^2 \Delta \phi (e^{i\psi/h} + a_h) = L, \quad (e^{i\psi/h} + a_h) |_{\Gamma} = 0
\]

with \( \|L\|_{L^2} \leq C h^2 \) and \( a_h \in L^\infty \). By the fact that \( \nabla \phi \cdot \nu(x) \geq \epsilon_0 > 0 \) for all \( x \in \Gamma \), we can apply Borel’s lemma to construct \( \ell \in C^\infty \) such that

\[
D\ell \cdot D\ell (x) = d(x, \Gamma)^\infty \quad \ell \bigg|_\Gamma = (\phi + i\psi) \bigg|_\Gamma \quad \partial_x \ell \bigg|_\Gamma = -\partial_x (\phi + i\psi) \bigg|_\Gamma.
\]

Since we are working with linear weights we will need a slightly more general \( h \)-dependent phase function than \( \phi + i\psi \). Let \( \xi \in \mathbb{R}^n \) be a fixed vector which is orthogonal to both \( D\phi \) and \( D\psi \), and \( \psi_h(x) \) be a linear function defined by \( \psi_h(x) = (\xi - \omega) \cdot x \)

\[
(7.2) \quad \omega_h = \frac{1 - \sqrt{1 - h^2|\xi|^2}}{h} D\psi
\]

is a vector of length \( O(h) \). Observe that in this setting the linear function \( \phi + i(\psi + h\psi_h) \) still solves the eikonal equation

\[
D(\phi + i(\psi + h\psi_h)) \cdot D(\phi + i(\psi + h\psi_h)) = 0.
\]

We now construct \( b \in C^\infty (\Omega) \) supported close to \( \Gamma \) such that

\[
(7.3) \quad e^{-i/h} h^2 \Delta (e^{i/h} e^{i\psi_h} b) = d(x, \Gamma)^\infty + O_L(1) (h^2), \quad b \bigg|_\Gamma = -1
\]
Using the fact that $D\ell \cdot D\ell = d(x, \Gamma)^\infty$ and $D\psi_h = \xi - \omega_h$ with $|\omega_h| \leq Ch$ we see that this amounts to solving the transport equation

$$bD\ell \cdot \xi + b\Delta \ell + 2D\ell \cdot Db = d(x, \Gamma)^\infty, \quad b |\Gamma| = -1.$$ 

Taking advantage of the fact that $-\partial_\nu Re(\ell) |\Gamma| = \partial_\nu |\Gamma| \geq \epsilon_0 > 0$ we can again solve the iterative equation and use Borel’s Lemma to construct $b \in C^\infty(\Omega)$ supported in an arbitrarily small neighbourhood of $\Gamma$ satisfying this approximate equation. We have therefore constructed $b \in C^\infty$ solving (7.3).

By the fact that $\nabla \phi \cdot \nu(x) \geq \epsilon_0 > 0$ we have, by choosing the support of $b$ sufficiently small, that $Re(\phi(x) - \ell(x)) \sim d(x, \Gamma)$ on $\text{supp}(b)$. By analyzing separately the case when $d(x, \Gamma) \leq \sqrt{h}$ and $d(x, \Gamma) \geq \sqrt{h}$ we have that (7.3) becomes

$$h^2 \Delta \phi(e^{\frac{\ell - \phi}{h}}) = O_L(h^2), \quad b |\Gamma| = -1$$

By the fact that $h^2 \Delta e^{\frac{\ell + i\psi}{h}} = 0$ and $\ell |\Gamma| = (\phi + i\psi) |\Gamma|$ we have

$$h^2 \Delta \phi(e^{\frac{\ell + i\psi}{h}}) + e^{\frac{i\psi}{h}} = L, \quad \|L\|_{L^\infty} \leq Ch^2, \quad (1 + a_h) |\Gamma| = 0.$$ 

where $a_h := e^{\frac{\ell - \phi + \psi}{h}}$ with $\|a_h\|_{L^\infty} \leq C$ and $a_h(x) \to 0$ for all $x \in \Omega$ as $h \to 0$.

This discussion allows us to construct the suitable CGO for solving our inverse problem. Indeed, let $\omega$ and $\omega'$ be two unit vectors which are mutually orthogonal. Define $\phi(x) = \omega \cdot x$ and $\psi(x) = \omega' \cdot x$. Let $\xi \in \mathbb{R}^n$ be another vector satisfying $\omega \cdot \xi = \omega' \cdot \xi = 0$ and define $\psi_h(x) := (\xi - \omega_h) \cdot x$ where $\omega_h$ is as in (7.2). Construct $\ell, b \in C^\infty(\Omega)$ so that (7.4) is satisfied. Applying Proposition 7.1 to (7.4) proves the following

**Proposition 7.2.** Let $\omega$ and $\omega'$ be two unit vectors which are mutually orthogonal. Let $\Gamma \subset \partial \Omega$ be an open subset compactly contained in $\{x \in \partial \Omega \mid \omega \cdot \nu(x) > 0\}$. For all $q \in L^n/2$ there exists solutions to

$$(\Delta + q)u = 0, \quad u \in H^1(\Omega), \quad u |\Gamma| = 0$$

of the form

$$u = e^{\frac{\omega + i\omega'}{h}x + h\psi_h}(1 + a_h + r)$$

with $\|a_h\|_{L^\infty} \leq C$, $a_h \to 0$ pointwise in $\Omega$ as $h \to 0$. The remainder $r \in L^p$ satisfies the estimates $\|r\|_{L^2} = o(1)$ and $\|r\|_p \leq C$ as $h \to 0$.

### 7.3. Recovering the Coefficients

In this section we prove Theorem 1.4. Let $\omega$ be a unit vector sufficiently close to $\omega_h$ such that there exists an open set $\Gamma_+$ such that $\partial \Omega \setminus B \subset \subset \Gamma_+ \subset \subset \{x \in \partial \Omega \mid \omega \cdot \nu(x) > 0\}$, $\partial \Omega \setminus F \subset \subset \Gamma_+ \subset \subset \{x \in \partial \Omega \mid \omega \cdot \nu(x) < 0\}$

Let $\xi \in \mathbb{R}^n$ be any vector orthogonal to $\omega$ and choose a third vector $\omega'$ of unit length which is perpendicular to both $\xi$ and $\omega$.

By Theorem 7.2 there exists solutions $u_\pm \in H^1(\Omega)$ solving

$$(\Delta + q_1)u_+ = 0, \quad u_+ |\Gamma_+ = 0, \quad (\Delta + q_2)u_- = 0, \quad u_- |\Gamma_- = 0$$

THE $L^n$ CARLEMAN ESTIMATE AND A PARTIAL DATA INVERSE PROBLEM 31
of the form

$$u_\pm = e^{\pm \omega x + ihx + h|x|^2} (1 + a_h^\pm + r_\pm), \quad \|r_\pm\|_{L^2} = o(1), \quad \|r_\pm\|_{L^p} = O(1)$$

where $\psi_h^\pm(x) := (\pm \xi - \omega') \cdot x$.

Since $u_\pm$ are solutions belonging to $H^1(\Omega)$ and vanish on $\partial\Omega\setminus \mathcal{B}$ and $\partial\Omega\setminus \mathcal{F}$ respectively, we have the following boundary integral identity (see Lemma A.1 of [12])

$$\int_{\Omega} \bar{u}_- (q_1 - q_2) u_+ = 0.$$

Inserting the expressions for $u_\pm$ gives

$$0 = \int_{\Omega} e^{2i\xi x} q (1 + a_h^+ a_h^- + a_h^- a_h^+ + a_h^- r_+ + a_h^+ r_- + r_+ + r_+ r_-)$$

where $q = q_1 - q_2$. The function $q \in L^{n/2} \subset L^1$ and

$$\|a_h^\pm\|_{L^\infty} \leq C, \quad \lim_{h \to 0} a_h^\pm(x) = 0 \quad \forall x \in \Omega$$

by (7.4). Therefore, terms $\lim_{h \to 0} \int_{\Omega} e^{2i\xi x} q (a_h^+ a_h^- + a_h^- a_h^+ + a_h^- r_+ + a_h^+ r_- + r_+ + r_+ r_-) = 0$. For the terms involving $\int_{\Omega} e^{2i\xi x} a_h^\pm r_\pm$, we note that for all $\epsilon > 0$ we may split $q = q^\epsilon + q^\epsilon$ where $q^\epsilon \in L^\infty$ while $\|q^\epsilon\|_{L^{n/2}} \leq \epsilon$. Then, using the fact that $\|a_h^\pm\|_{L^\infty} \leq C$,

$$\left| \int_{\Omega} e^{2i\xi x} qa_h^\pm r_\pm \right| \leq C (\|q^\epsilon\|_{L^\infty}\|r_\pm\|_{L^2} + \|q^\epsilon\|_{L^{n/2}}\|r_\pm\|_{L^p})$$

where $p = \frac{2n}{n-2}$. By the estimates on $r_\pm$ given in Proposition 7.2 we have that $\lim_{h \to 0} \|r_\pm\|_{L^2} = 0$ and $\|r_\pm\|_{L^p} \leq C$. Therefore, the limit

$$\lim_{h \to 0} \int_{\Omega} e^{2i\xi x} qa_h^\pm r_\pm \leq C \epsilon$$

for all $\epsilon > 0$ and therefore the limit vanishes. The terms $\int_{\Omega} e^{2i\xi x} (r_- + r_+) + \int_{\Omega} e^{2i\xi x} q r_- r_+$ can be estimated the same way. For the last term, we again decompose, for all $\epsilon > 0$, $q = q^\epsilon + q^\epsilon$. The integral $\int_{\Omega} e^{2i\xi x} qr_- r_+$ is then estimated by

$$\int_{\Omega} |q^\epsilon r_- r_+| + \int_{\Omega} |q^\epsilon r_- r_+| \leq \|q^\epsilon\|_{L^\infty}\|r_-\|_{L^2}\|r_+\|_{L^2} + \|q^\epsilon\|_{L^{n/2}}\|r_-\|_{L^p}\|r_+\|_{L^p}$$

The $L^p$ norms of $r_\pm$ stay uniformly bounded while the $L^2$ norms vanish when $h \to 0$. Therefore the limit

$$\lim_{h \to 0} \int_{\Omega} e^{2i\xi x} qr_- r_+ \leq C \|q^\epsilon\|_{L^{n/2}} \leq C \epsilon$$

for all $\epsilon > 0$ and therefore vanishes.

This means that $\mathcal{F}(q)(\xi) = 0$ for all $\xi$ which are orthogonal to $\omega$. Note that varying $\omega$ in a small neighbourhood does not change the fact that $\Gamma$ lies in the set.
\{x \in \partial \Omega \mid \omega \cdot \nu(x) > 0\}, and so the construction in Proposition 7.2 still applies. Then varying \(\omega\) in a small neighbourhood and using the analyticity of the Fourier transform for \(q\) compactly supported we have that \(q = q_1 - q_2 = 0\). \(\Box\)

8. Appendix

Here we will provide proofs for Proposition 2.2 and Proposition 2.3 from Section 2.

To begin, suppose \(a \in S^k_0(\mathbb{R}^n)\) be a symbol whose spatial dependence is in \(x'\) only and compactly supported. We then have the following expression for the quantization of their product:

\[ a(x', hD)f = \int e^{i\lambda \cdot x'} \int e^{i\xi \cdot x'} \int e^{i\xi \cdot z'} \frac{(1 + \Delta z')^N a(z', h\xi)}{(1 + |\lambda|^2)^N} \mathcal{F}(u)(\xi) d\xi d\lambda \]

Proposition 8.1. Let \(a(x', \xi)\) be in \(S^k_0(\mathbb{R}^n)\) or \(S^{k(n)}_0(\mathbb{R}^n)\) for some \(k(n)\) large depending only on the dimension. Suppose \(a(x', \xi)\) depends only on \(x' \in \mathbb{R}^{n-1}\) in the spatial variable and \(D_{x'} a(x', \xi) = 0\) if \(x'\) is outside of a fixed compact set. If \(b(x', \xi') \in S^l_0(\mathbb{R}^{n-1})\) with \(D_{x'} b(x', \xi') = 0\) outside of a compactly supported set, then

\[ ab(x', hD) : L^r \to L^r \]

with norm

\[ \|ab(x', hD)\|_{L^r \to L^r} \leq C \sup_{x', \xi} |(1 + \Delta_{x'})^N \partial_\xi^\alpha a(x', \xi)||(|\xi|)|^\alpha \sup_{x', \xi'} |(1 + \Delta_{x'})^N \partial_{\xi'}^\alpha b(x', \xi')||(|\xi'|)|^\alpha \]

where the constant \(C\) depends linearly on the volume of the support of \(D_{x'} ba(x', \xi)\) in \(x'\) and \(N\) depends only on the dimension.

Proof. In the constant coefficient case this is a direct consequence of Mihlin’s multiplier theorem applied first to all variable then to \(\xi'\) variables. We can therefore assume without loss of generality that either \(b(x', \xi) = 0\) or \(a(x', \xi) = 0\) for \(x'\) outside of a fixed compact set.

Then apply Minkowski to expression (8.1) for \(N\) chosen to be large enough and for each \(z' \in \text{supp}(ba(\cdot, \xi))\) apply the constant coefficient estimate for Fourier multipliers on \(L^r\). \(\Box\)

An immediate Corollary is the mapping property from Sobolev spaces:

Corollary 8.2. If \(a(x', \xi) \in S^k_1S^{\ell}_1 \cup S^k_0 S^{k(n)+\ell}\) then

\[ a(x', hD) : W^{k,r}W^{\ell,r} \to L^r \]

with norm uniformly bounded in \(h\).

Proof. Pre-composition yields that \(a(x', hD)(hD')^{-k}(hD)^{-\ell}\) is a quantization of a symbol in \(S^k_0S^{\ell}_0 \cup S^k_0 S^{k(n)}\) and therefore takes \(L^r \to L^r\). This shows that

\[ a(x', hD) : W^{k,r}W^{\ell,r} \to L^r \quad \forall 1 < r < \infty. \]
Composition of two ΨDO operators in this class can be described by the composition calculus $b(x, hD)a(x, hD) = ab(x, hD) + h \sum_{|\alpha|=1} (\partial_x^\alpha b \partial_x^\alpha a)(x, hD) + h^2 m(x, hD)$ with the remainder explicitly computed as

\begin{equation}
  m(x, \xi) = \sum_{|\alpha|=2} \int_{\mathbb{R}^2n} \frac{e^{iy\eta}}{\langle \eta \rangle^N} (1 + \Delta_\eta)^N \partial_x^\alpha b(x, \eta + \xi)(1 + \Delta_\eta)^N \int_0^1 \partial_x^\alpha a(x + \theta hy, \xi) d\theta dy d\eta \quad \forall N \in \mathbb{N}
\end{equation}

This leads to the following statement about the remainder term of the composition:

**Lemma 8.3.** Let $a \in S^k_1 S^l_1 \cup S^k_0 S^{-l}(n) + \ell_1$ and $b \in S^{-k_1} S^{-l_1} \cup S^{-k_1} S_0^{-\ell_1-k(n)}$ then one has $b(x', hD)a(x', hD) = ab(x', hD) + h \sum_{|\alpha|=1} (\partial_x^\alpha b \partial_x^\alpha a)(x', hD) + h^2 m(x', hD)$ with $m(x', hD) : L^r \rightarrow L^r$ norm independent of $h > 0$.

**Proof.** We have that

$$b(x'hD)a(x', hD) = ab(x', hD) + h \sum_{|\alpha|=1} \partial_x^\alpha b \partial_x^\alpha a(x', hD) + h^2 m(x', hD)$$

where $m(x, \xi)$ is given by (8.2). By taking $N$ large enough in (8.2) we see that

\begin{equation}
  m(x', hD)u = \sum_{|\alpha|=2} \int_{\mathbb{R}^2n} \frac{e^{iy\eta}}{\langle \eta \rangle^N} \int_0^1 m_{\theta, y, h, \eta}(x', hD) d\theta dy d\eta
\end{equation}

where for each $(\alpha, \theta, y, h, \eta)$, $m_{\theta, y, h, \eta}(x', \xi) \in S^0_S^0 S^{-l}(n)$ is a symbol of the form $m_{\theta, y, h, \eta}(x', \xi) = \langle \eta \rangle^{-N} \langle y \rangle^{-N}(1 + \Delta_\eta)^N \partial_x^\alpha b(x', \eta + \xi)(1 + \Delta_\eta)^N \partial_x^\alpha a(x + \theta hy, \xi)$. Since $a \in S^k_1 S^l_1 \cup S^k_0 S^{-l}(n) + \ell_1$ and $b \in S^{-k_1} S^{-l_1} \cup S^{-k_1} S_0^{-\ell_1-k(n)}$ we may write $a = a^v a^v$ and $b = b^v b^v$ where

$a^v(x', \xi') \in S^k_1, b^v(x', \xi') \in S^{-k_1}, a^v(x', \xi) \in S^l_1 \cup S^{-l}(n) + \ell_1, b^v(x', \xi) \in S^{-l_1} \cup S^{-l_1-k(n)} - \ell_1$.

We see then that for each $(\alpha, \theta, y, h, \eta)$ the symbol $m_{\theta, y, h, \eta}(x', \xi)$ consists of finitely many (depending on the choice of $N$) terms of the from

$$\langle \eta \rangle^{-N} \langle y \rangle^{-N} \partial_x^\alpha b \partial_x^\alpha a^v \partial_x^\alpha \partial_y^\alpha \partial_\eta^\alpha (x', \xi) (\theta h) |^{\beta_1} (\partial_x^\alpha a^v \partial_y^\alpha \partial_\eta^\alpha (x', \xi) (\theta h) |^{\beta_2} (\partial_x^\alpha \partial_y^\alpha \partial_\eta^\alpha (x', \xi) (\theta h) $$

which is a symbol in $S^0_S^0 S^{-l}(n)$. Here $b^v(x', \xi) := b(x', \xi + \eta) + \theta hy, \xi) := a(x' + \theta hy, \xi)$ Applying Proposition 8.4 to each of these terms and choosing $N \geq k(n)$ we have that

$$\sup_{(\alpha, \theta, y, h, \eta)} \| m_{\theta, y, h, \eta}(x', hD) \|_{L^r \rightarrow L^r} \leq C_N$$

Here we used Peeter’s inequality $\frac{\langle \xi \rangle}{\langle \xi + \eta \rangle} \leq C$. Choosing $N \geq n + 2$ in (8.3) we get that

$$\| m(x', hD) \|_{L^r \rightarrow L^r} \leq C \sup_{(\alpha, \theta, y, h, \eta)} \| m_{\theta, y, h, \eta}(x', hD) \|_{L^r \rightarrow L^r} \int_{\mathbb{R}^2n} \langle \eta \rangle^{-N} \langle y \rangle^{-N} d\eta dy.$$

□

The composition formula given by Lemma 8.3 in conjunction with the mapping property asserted in Proposition 8.1 also allows us to deduce, Proposition 2.2 by composition with suitable powers of \( \langle hD \rangle \).

**Proposition 8.4 (Proposition 2.2)**. If \( b(x', \xi') \in S^k_{1} \) and \( a(x', \xi) \in S^\ell_{1} \cup S^{-k(\ell)+\ell}_{0} \) then
\[
ba(x', hD) : W^{m, r} \rightarrow W^{m-k, r}
\]
with norm
\[
\|ba(x', hD)\| \leq C \sup_{x', \xi, |\alpha| \leq k(\alpha)} |(1 + \Delta_{x'})^N \partial_{\xi}^\alpha a(x', \xi)| |(\xi)|^{\alpha-k(\alpha)} \sup_{x', \xi, |\alpha| \leq k(\alpha)} |(1 + \Delta_{x'})^N \partial_{\xi}^\alpha b(x', \xi')| |(\xi')|^{\alpha-k(\alpha)}
\]

**Proof.** Since pre-composition by \( \langle hD \rangle^{-k} \langle hD \rangle^{-\ell} \) amounts to multiplication of symbols without remainders, it suffices to show that symbols \( a(x', \xi) \in S^\ell_{1} \cup S^{-k(\ell)+\ell}_{0} \) take \( L^r \rightarrow W^{-k, r} \). Indeed, by Lemma 8.3 we have that
\[
\langle hD \rangle^{-k} \langle hD \rangle^{-\ell} a(x', hD) = c(x', hD) + h^2 m(x', hD)
\]
where \( c(x', hD) = \langle \xi' \rangle^{-k} \langle \xi \rangle^{-\ell} a(x', \xi) + h \sum_{|\alpha| = 1} \partial_{\xi'}^\alpha ((\xi')^{-k} (\xi)^{-\ell}) \partial_{\xi}^\alpha a(x', \xi) \) and \( m(x', hD) : W^{k, r} \rightarrow W^{-k-1, r} \).

To estimate the operator norm by the size of the symbol, using (8.2) and estimate the remainder as in the proof of Proposition 8.1. \( \Box \)

Now we turn to the proof of Proposition 2.3.

**Proposition 8.5 (Proposition 2.3)**. If \( a \in S^k_{1} \cup S^{-k(\ell)+\ell}_{1} \) and \( b \in S^{k_2}_{1} \cup S^{-k_2(\ell)+\ell_2}_{1} \) then
\[
b(x'hD) a(x', hD) = ab(x', hD) + h \sum_{|\alpha| = 1} (\partial_{\xi'}^\alpha b \partial_{\xi}^\alpha a)(x', hD) + h^2 m(x', hD)
\]
where \( m(x', hD) : W^{k, r} \rightarrow W^{-k-1-2, r} \).

**Proof.** The proof is exactly the same as Proposition 8.1 except that to show the boundedness of the remainder in the mixed Sobolev norms one uses Proposition 2.2.

We have that
\[
b(x'hD) a(x', hD) = ab(x', hD) + h \sum_{|\alpha| = 1} \partial_{\xi'}^\alpha b \partial_{\xi}^\alpha a(x', hD) + h^2 m(x', hD)
\]
where \( m(x, \xi) \) is given by (8.2). By taking \( N \) large enough in (8.2) we see that
\[
m(x', hD) u = \sum_{|\alpha| = 2} \int_{R^{2n}} \frac{e^{i y \cdot \eta}}{(\eta)^N (y)^N} \int_{0}^{1} m_{\theta, y, h, \eta}^{\alpha, j}(x', hD) u \theta d\theta dy d\eta
\]
where for each \( (\alpha, \theta, y, h, \eta) \), \( m_{\theta, y, h, \eta}^{\alpha, j} = S^{\ell}_{0} \cup S^{-k(\ell)}_{0} \) is a symbol of the form \( m_{\theta, y, h, \eta}^{\alpha, j}(x', \xi) = \langle \eta \rangle^{-N} (y)^{-N} (1 + \Delta_{y})^{N} \partial_{\eta}^\alpha b(x', \xi) \). Since
\( a \in S_1^{k_1} S_1^{\ell_1} \cup S_1^{k_1} S_0^{S^{-k(n)+\ell_1}} \) and \( b \in S_1^{-k_1} S_1^{-\ell_1} \cup S_1^{-k_1} S_0^{-k(n)-\ell_1} \) we may write \( a = a^t a^v \) and \( b = b^t b^v \) where
\[
a^t(x', \xi') \in S_1^{k_1}, \quad b^v(x', \xi') \in S_1^{-k_1}, \quad a^v(x', \xi) \in S_1^{\ell_1} \cup S_0^{-k(n)+\ell_1}, \quad b^v(x', \xi) \in S_1^{-\ell_1} \cup S_0^{-k(n)-\ell_1}.
\]

We see then that for each \((\alpha, \theta, y, h, \eta)\) the symbol \( m_{\theta, y, h, \eta}^\alpha(x', \xi) \) consists of finitely many (depending on the choice of \( N \)) terms of the form
\[
\langle \eta \rangle^{-N} \langle y \rangle^{-N} \partial_\xi^{\beta_1} \partial_\xi^{\beta_2} a_{\theta, h, \eta}^\alpha(x', \xi) (\theta h) \partial_\xi^{\beta_1} \partial_\xi^{\beta_2} b_{\theta, h, \eta}^v(x', \xi) (\theta h) \partial_\xi^{\beta_1} \partial_\xi^{\beta_2} a_{\theta, h, \eta}^v(x', \xi)
\]
which is a symbol in \( S_1^0 S_1^0 \cup S_1^{S^{-k(n)}} \). Here \( b_{\eta}(x', \xi) := b(x', \xi + \eta) \) and \( a_{\theta, h, \eta}(x', \xi) := a(x' + \theta h y', \xi) \).

Applying Proposition 8.1 to each of these terms and choosing \( N \geq k(n) \) we have that
\[
\sup_{(\alpha, \theta, y, h, \eta)} \| m_{\theta, y, h, \eta}^\alpha(x', h D) \|_{L^r \rightarrow L^r} \leq C_n
\]
Here we used Peeter's inequality \( \frac{\langle \xi \rangle}{|\xi + \eta|^{n+1}} \leq C \). Choosing \( N \geq n + 2 \) in (8.3) we get that
\[
\| m(x', h D) \|_{L^r \rightarrow L^r} \leq C \sup_{(\alpha, \theta, y, h, \eta)} \| m_{\theta, y, h, \eta}^\alpha(x', h D) \|_{L^r \rightarrow L^r} \int_{\mathbb{R}^{2n}} \langle y \rangle^{-N} \langle \eta \rangle^{-N} dy dy
\]

\[\square\]

**References**

[1] J. Bikowski, K. Knudsen, and J. Mueller. Direct numerical reconstruction of conductivities in three dimensions using scattering transforms, *Inv. Prob.*, 27 (2011), 015002.

[2] A.L. Bukhgeim and G. Uhlmann. Recovering a potential from partial Cauchy data, *Comm. PDE*, 27 (2002), 653–668.

[3] S. Chanillo. A problem in electrical prospection and a n-dimensional Borg-Levinson Theorem. *Proc. Am. Math. Soc.*, 108 (1990), 761-767.

[4] F.J. Chung. A partial data result for the magnetic Schrödinger inverse problem. *Anal. and PDE*, 7 (2014), 117-157.

[5] F.J. Chung. Partial data for the Neumann-to-Dirichlet map. *J. Fourier Ana. and App.*, 21 (2015), 628-665.

[6] F.J. Chung. Determining a magnetic potential from partial Neumann-to-Dirichlet data. *Inv. Prob. and Imag.*, 8 (2014), 959-989.

[7] F.J. Chung, P. Ola, M. Salo, and L. Tzou. Partial data inverse problems for the Maxwell equations. Preprint (2015), [arXiv:1502.01618](http://arxiv.org/abs/1502.01618)

[8] F.J. Chung, M. Salo, and L. Tzou. Partial data inverse problems for the Hodge Laplacian. Preprint (2013), [arXiv:1310.4616](http://arxiv.org/abs/1310.4616)

[9] H. Cornean, K. Knudsen, and S. Siltanen. Towards a d-bar reconstruction method for three-dimensional EIT. *J. Inv. Ill-Posed Problems*, 14, No. 2 (2006), 111-134.

[10] F. Delbary and K. Knudsen. Numerical nonlinear complex geometrical optics algorithm for the 3D Calderón problem, *Inv. Prob. and Imag.*, 8 (2014), 991 - 1012.

[11] F. Delbarya, P.C. Hansen, and K. Knudsen. Electrical impedance tomography: 3D reconstructions using scattering transforms, *Appl. Anal.*, (2011), 1-19.
[12] D. Dos Santos Ferreira, C.E. Kenig, and M. Salo. Determining an unbounded potential from Cauchy data in admissible geometries, Comm. PDE 38 (2013), no. 1, 50-68.
[13] B. Haberman and D. Tataru. Uniqueness in Calderón’s problem with Lipschitz conductivities, Duke Math. J. 162, no. 3 (2013), 497-516.
[14] B. Haberman. Uniqueness in Calderón’s problem for conductivities with unbounded gradient. Comm. Math. Phys. 340, no. 2 (2015), 639-659.
[15] D. Jerison, Carleman inequalities for the Dirac and Laplace operators and unique continuation, Adv. Math. 63 (1986), 118-134.
[16] D. Jerison and C. E. Kenig. Unique continuation and absence of positive eigenvalues for Schrödinger operators, Ann. of Math. 121 (1985), 463-494.
[17] C.E. Kenig and M. Salo. The Calderón problem with partial data on manifolds and applications, Anal. & PDE 6 (2013), 2003–2048.
[18] C.E. Kenig and M. Salo. Recent progress in the Calderón problem with partial data. Contemp. Math., 615 (2014), 193-222.
[19] C.E. Kenig, A. Ruiz, and C.D. Sogge. Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators. Duke Math. J. 55 (1987), 329-347.
[20] C.E. Kenig, J. Sjöstrand, and G. Uhlmann. The Calderón problem with partial data. Ann. of Math. 165 (2007), 567-591.
[21] K. Krupchyk, M. Lassas, and G. Uhlmann, Determining a first order perturbation of the biharmonic operator by partial boundary measurements. J. Funct. Anal. 262 (2012), 1781-1801.
[22] K. Krupchyk and G. Uhlmann. The Calderón problem with partial data for conductivities with 3/2 derivatives. Preprint, arXiv 1508.07102.
[23] R. Lavine and A. Nachman. Unpublished result. Announced in A. Nachman, Inverse scattering at fixed energy. Proceedings of the Xth Congress on Mathematical Physics. L. Schmdgen, ed., Springer-Verlag, Leipzig, Germany 1991, pp 434-441.
[24] A. Nachman. Reconstructions from boundary measurements, Ann. of Math. 128 (1988), 531–576.
[25] A. Nachman and B. Street. Reconstruction in the Calderon problem with partial data, Comm. PDE 35 (2010), 375–390.
[26] M. Salo. Semiclassical pseudodifferential calculus and the reconstruction of a magnetic field M Salo. Comm. PDE 31 (11), 1639 - 1666.
[27] M. Salo and L. Tzou. Carleman estimates and inverse problems for Dirac operators. Math. Ann. 344 (2009), no. 1, p. 161-184.
[28] M. Salo and L. Tzou. Inverse problems with partial data for a Dirac system: a Carleman estimate approach. Adv. Math. 225 (2010), no. 1, p. 487-513.
[29] J. Sylvester and G. Uhlmann. A global uniqueness theorem for an inverse boundary problem. Ann. of Math. 43 (1990), 201-232.
[30] M. Wong Introduction to pseudo-differential operators. World Scientific Publishing, Singapore, 2004.
[31] M. Zworski Introduction to semiclassical analysis. American Mathematical Society, Providence, RI, 2012.