Rotation Numbers for $S^2$ diffeomorphisms

John Franks

In these largely expository notes we describe the properties of, and generalize, the function $R$ which assigns a number to a 4-tuple of distinct fixed points of an orientation preserving homeomorphism or diffeomorphism of $S^2$. I am indebted to Patrice LeCalvez for telling me many of the basic properties described here.

1 The function $R_f$ for fixed points.

We let $f$ be an orientation preserving homeomorphism and denote by $X$ the set of fixed points of $f$. We will assume for the moment that $X$ contains at least four distinct points. Given four distinct points $x_1, x_2, x_3, x_4 \in \text{Fix}(f)$ we consider the sets $A = \{x_1, x_2\}$ and $B = \{x_3, x_4\}$. One can puncture at $A$ to obtain an annulus and then consider a lift of $f$ to the universal cover of this annulus. The difference of the rotation numbers of the points of $B$ (with respect to this lift) is an element of $\mathbb{R}$ which is independent of the choice of lift. This gives a real valued function $R_f$ of the four fixed points $\{x_1, x_2, x_3, x_4\}$. This function has some remarkable symmetries under permutations of its arguments which form the content of this note.

We will be interested in the intersection of oriented paths $\alpha : [0,1] \to S^2 \setminus B$ and $\beta : [0,1] \to S^2 \setminus A$ and with $\alpha$ running from $x_1$ to $x_2$, and $\beta$ running from $x_3$ to $x_4$. These paths have an algebraic intersection number determined by the orientation of the paths and the orientation of $S^2$ which we now define. The ends of $\beta$, $x_3$ and $x_4$, are disjoint from $\alpha$ and we can close $\beta$ to a loop by concatenating with a path from $x_4$ to $x_3$ which lies in the complement of the image of $\alpha$. If $\delta \ast \beta$ is loop obtained by concatenating $\delta$ and $\beta$, then its homology class $[\delta \ast \beta] \in H_1(S^2 \setminus A) \cong \mathbb{Z}$ is independent of the choice of $\delta$.

**Definition 1.1.** We define the algebraic intersection number $\alpha \cdot \beta \in \mathbb{Z}$ by

$$[\delta \ast \beta] = (\alpha \cdot \beta)[u],$$

where $u$ is an embedded closed loop in $S^2 \setminus A$, positively oriented with respect to $x_1$, and hence $[u]$ is a generator of $H_1(S^2 \setminus A)$.

The number $\alpha \cdot \beta$ number depends only on the homology classes $[\alpha] \in H_1(S^2 \setminus B, A)$ and $[\beta] \in H_1(S^2 \setminus A, B)$. There is a standard skew-symmetric intersection pairing

$$i : H_1(S^2 \setminus B, A) \times H_1(S^2 \setminus A, B) \to H_0(S^2 \setminus A \cup B) \cong \mathbb{Z}.$$
and the algebraic intersection number of $\alpha \cdot \beta$ is equal to $i([\alpha], [\beta])$.

There are three elementary facts we will have occasion to use

- The intersection is skew-symmetric: $\alpha \cdot \beta = -\beta \cdot \alpha$.
- If $\hat{\alpha}(t) = \alpha(1 - t)$, is the path with reverse parametrization, then $\hat{\alpha} \cdot \beta = -\alpha \cdot \beta$.
- If $g : S^2 \to S^2$ is an orientation preserving homeomorphism then $(g \circ \alpha) \cdot (g \circ \beta) = \alpha \cdot \beta$.

### 1.1 Characterization of $R_f$.

In this section we give a definition of the function $R_f$ and provide several equivalent characterizations (Propositions (1.3), (1.5) and (1.6)). Throughout this section we will assume that $f$ is an orientation preserving homeomorphism and $X = \{x_1, x_2, x_3, x_4\}$ is a subset of $\text{Fix}(f)$. The final result of this section shows that for a fixed $X$ the function $R_f$ is a homomorphism from the group of orientation preserving homeomorphisms which pointwise fix $X$ to $\mathbb{Z}$.

**Definition 1.2.** Let $f : S^2 \to S^2$ be an orientation preserving homeomorphism and suppose $\{x_1, x_2, x_3, x_4\}$ is a subset of distinct points of $\text{Fix}(f)$. Let $A = \{x_1, x_2\}$ and $B = \{x_3, x_4\}$ and let $\alpha$ be an oriented path in $S^2 \setminus B$ running from $x_1$ to $x_2$ and let $\beta$ be an oriented path in $S^2 \setminus A$ running from $x_3$ to $x_4$. Then we define

$$R_f(x_1, x_2, x_3, x_4) = \alpha \cdot (f \circ \beta) - \alpha \cdot \beta.$$  

The notation implies that $R_f(x_1, x_2, x_3, x_4)$ depends only on $x_1, x_2, x_3, x_4 \in \text{Fix}(f)$ and not on the choice of $\alpha$ and $\beta$. This is indeed true and follows immediately from the following proposition which gives an alternate description of $R_f$. As in the definition above we let $\{x_1, x_2, x_3, x_4\}$ be a set of distinct points in $\text{Fix}(f)$ and define $A = \{x_1, x_2\}$ and $B = \{x_3, x_4\}$.

**Proposition 1.3.** Let $V = S^2 \setminus A$ and let $\tilde{f} : \tilde{V} \to \tilde{V}$ be the lift of $f$ to the universal covering space of $V$ which fixes $\tilde{x}_3$, a lift of $x_3$. Let $T : \tilde{V} \to \tilde{V}$ be the generator of the group of covering translations corresponding to a loop in $V$ with intersection number $+1$ with some (and hence any) oriented path from $x_1$ to $x_2$. If $\tilde{x}_4 \in \tilde{V}$ is a lift of $x_4$, then $R_f(x_1, x_2, x_3, x_4) = n \in \mathbb{Z}$ where $\tilde{f}(\tilde{x}_4) = T^n(\tilde{x}_4)$.

**Proof.** Let $\tilde{x}_3$ be a lift of $x_3$ and let $\tilde{f}$ be the lift of $f$ which fixes $\tilde{x}_3$. If $\alpha$ and $\beta$ are as in Definition (1.2) choose $\tilde{\alpha}$ and $\tilde{\beta}$ lifts of $\alpha$ and $\beta$ with $\tilde{\beta}$ running from $\tilde{x}_3$ to a point which is a lift of $x_4$. We denote this point by $\tilde{x}_4$ and define $n$ by $\tilde{f}(\tilde{x}_4) = T^n(\tilde{x}_4)$. The value of $n$ would be the same if we used any other lift of $x_4$ in place of $\tilde{x}_4$.

The path $\tilde{f}\tilde{\beta}$ runs from $\tilde{x}_3$ to $\tilde{f}(\tilde{x}_4) = T^n(\tilde{x}_4)$. It is homotopic to the concatenation of the path $\beta$ from $\tilde{x}_3$ to $\tilde{x}_4$ with a path $\tilde{\gamma}$ from $\tilde{x}_4$ to $T^n(\tilde{x}_4)$. If $\gamma$ is the closed loop
in $V$ obtained by projecting $\tilde{\gamma}$ then $[\gamma]$ is $n$ times the generator of $H_1(V)$ which has intersection number +1 with $\alpha$. Hence
\[
\alpha \cdot (f \circ \beta) = \alpha \cdot \gamma + \alpha \cdot \beta
\]
and
\[
R_f(x_1, x_2, x_3, x_4) = \alpha \cdot (f \circ \beta) - \alpha \cdot \beta = \alpha \cdot \gamma = n. \tag{1}
\]

**Remark 1.4.** We note that $R_f(x_1, x_2, x_3, x_4)$ is sometimes well defined even if the points $x_1, x_2, x_3,$ and $x_4$ are not all distinct. In particular if $x_1 = x_2$ but $\{x_2, x_3, x_4\}$ are distinct then $R_f(x_2, x_2, x_3, x_4) = 0$ since we may choose the constant path for $\alpha$. Similarly if $x_3 = x_4$ but $\{x_1, x_2, x_3\}$ are distinct then $R_f(x_1, x_2, x_3, x_3) = 0$ since we may choose the constant path for $\beta$. If $x_2 = x_3$ or $x_1 = x_4$ the value of $R_f$ is undefined for the general homeomorphism $f$. However, if $f$ is a $C^1$ diffeomorphism we can define $R_f$ when $x_2 = x_3$ or $x_1 = x_4$, but the value may no longer be an integer, or even rational. We will do this below in Section (1.4).

Since $\beta$ and $f \circ \beta$ have the same endpoints, $x_3$ and $x_4$, we can concatenate the paths $f \circ \beta$ and $\beta^- = \beta(1-t)$ to form a loop $\gamma$ in $S^2 \setminus A$ which also defines $R_f(x_1, x_2, x_3, x_4)$.

**Proposition 1.5.** If $\beta^-(t) = \beta(1-t)$ and $\gamma = (f \circ \beta) \ast \beta^-$ then
\[
R_f(x_1, x_2, x_3, x_4) = \alpha \cdot \gamma.
\]

An easy consequence of Proposition (1.3) is another useful description of $R_f$. We use that fact that the space of homeomorphisms fixing three distinct points $\{x_1, x_2, x_3\}$ is contractible and so there is an isotopy $f_t(x)$ $f_0 = id$ and $f_1 = f$ such that for all $t \in [0, 1]$, $f_t(x_i) = x_i$ for $1 \leq i \leq 3$. This isotopy is unique up to homotopy as a path in the space of homeomorphisms fixing the points $\{x_1, x_2, x_3\}$.

**Proposition 1.6.** Suppose $\alpha$ is a path from $x_1$ to $x_2$ which is disjoint from the set $\{x_3, x_4\}$. Let $\gamma_0$ be the closed loop in $S^2 \setminus \{x_1, x_2, x_3\}$ defined by $f_t(x_4)$ for $t \in [0, 1]$ then
\[
R_f(x_1, x_2, x_3, x_4) = \alpha \cdot \gamma_0.
\]

**Proof.** This follows immediately from the proof of Proposition (1.3) because $\gamma_0$ lifts to a path $\tilde{\gamma}_0$ in the universal covering space of $S^2 \setminus \{x_1, x_2\}$ with the same endpoints as the path $\tilde{\gamma}$ in the proof of Proposition (1.3). Hence loops $\gamma$ and $\gamma_0$ are homotopic in $S^2 \setminus \{x_1, x_2\}$ so from equation (1) we have
\[
R_f(x_1, x_2, x_3, x_4) = \alpha \cdot \gamma = \alpha \cdot \gamma_0.
\]
An important property of the function $\mathcal{R}_f$ is that if $G$ is a subgroup of Homeo($S^2$) which fixes a set $X$ pointwise then the function which assigns to $f \in G$ the function $\mathcal{R}_f$ is a homomorphism. For $f, g \in$ Homeo($S^2$) we will denote their composition by $fg$.

**Proposition 1.7.** Suppose $\{x_1, x_2, x_3, x_4\}$ is a set of distinct points in $\text{Fix}(f) \cap \text{Fix}(g)$. Then

$$\mathcal{R}_{fg}(x_1, x_2, x_3, x_4) = \mathcal{R}_f(x_1, x_2, x_3, x_4) + \mathcal{R}_g(x_1, x_2, x_3, x_4).$$

**Proof.** We use Proposition (1.6). As an isotopy from $id$ to $fg$ we choose $f_t g_t$, $t \in [0, 1]$ where $f_t$ and $g_t$ are isotopies from $id$ to $f$ and $g$ respectively, each of which fixes the points $x_1, x_2,$ and $x_3$. The loop $\gamma_0$ given by $f_t g_t(x_4)$ is homotopic to the loop which is the concatenation of $\gamma_1$ given by $g_t(x_4) = f_0 g_t(x_4)$ with $\gamma_2$ given by $f_t(x_4) = f_t(g_1(x_4))$.

Hence

$$\mathcal{R}_{fg}(x_1, x_2, x_3, x_4) = \alpha \cdot \gamma_0$$

$$= \alpha \cdot \gamma_1 + \alpha \cdot \gamma_2$$

$$= \mathcal{R}_g(x_1, x_2, x_3, x_4) + \mathcal{R}_f(x_1, x_2, x_3, x_4).$$

\[\square\]

### 1.2 Basic Symmetries of $\mathcal{R}_f$.

We want to understand the effect permuting the variables of $\mathcal{R}_f$. We first establish basic relations – equations (2), (3), (4), and (5), below. In Section (2) we establish relations applicable to any permutation of the variables of $\mathcal{R}_f$.

We will write elements of $S_4$, the symmetric group of order four, in the standard way as a product of cycles. For example $(12)(34)$ is the product the transposition of 1 and 2 and the transposition of 3 and 4. Recall that the group $S_4$ is generated by the three transpositions $\sigma_i = (i, i+1)$, $1 \leq i \leq 3$.

If $x = (x_1, x_2, x_3, x_4)$ is a four-tuple of elements of $\text{Fix}(f)$ we will denote the four-tuple $(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})$ by $x_{\sigma}$. An element of $S_4$ which we will have occasion to use several times is the cyclic permutation $(123)$. We will denote this element by $\tau$.

**Proposition 1.8.** Suppose $\{x_1, x_2, x_3, x_4\}$ is a set of distinct points in $\text{Fix}(f)$. Then

$$\mathcal{R}_f(x) + \mathcal{R}_f(x_\tau) + \mathcal{R}_f(x_\tau^2) = 0, \quad (2)$$

$$\mathcal{R}_f(x_{\sigma_1}) = \mathcal{R}_f(x_{\sigma_2}) = -\mathcal{R}_f(x) \quad (3)$$

**Proof.** Equation (3) follows immediately from the definition of $\mathcal{R}_f$ since reversing the parametrization of $\alpha$ to get $\bar{\alpha}$, a path from $x_2$ to $x_1$ changes the sign of the intersection number. Hence

$$\mathcal{R}_f(x_{\sigma_1}) = \bar{\alpha} \cdot (f \circ \beta) - \bar{\alpha} \cdot \beta$$

$$= -\alpha \cdot (f \circ \beta) + \alpha \cdot \beta$$

$$= -\mathcal{R}_f(x).$$

4
The argument for \( \sigma_3 \) is similar.

To show equation (2) we choose paths \( \alpha_1 \) from \( x_1 \) to \( x_2 \), \( \alpha_2 \) from \( x_2 \) to \( x_3 \), and \( \alpha_3 \) from \( x_3 \) to \( x_1 \). The concatenation of these three paths is a closed loop \( \eta \) in \( S^2 \). Let \( \gamma \) be as defined in Proposition (1.6). Then

\[
R_f(x) + R_f(x_\tau) + R_f(x_{\eta2}) = \alpha_1 \cdot \gamma + \alpha_2 \cdot \gamma + \alpha_3 \cdot \gamma = \eta \cdot \gamma = 0,
\]

since the intersection number of any two closed loops in \( S^2 \) is 0. \( \square \)

**Proposition 1.9.** Suppose \( \{x_1, x_2, x_3, x_4, w\} \) is a set of distinct points in \( \text{Fix}(f) \). Then

\[
R_f(x_1, x_2, x_3, x_4) = R_f(x_3, x_4, x_1, x_2), \quad \text{(4)}
\]

\[
R_f(x_1, x_2, x_3, x_4) = R_f(x_1, w, x_3, x_4) + R_f(w, x_3, x_4), \quad \text{and} \quad \text{(5)}
\]

\[
R_f(x_1, x_2, x_3, x_4) = R_f(x_1, x_2, x_3, w) + R_f(x_1, x_2, w, x_4). \quad \text{(6)}
\]

**Proof.** To show equation (4) we note that Proposition (1.7) implies \( R_f = -R_{f^{-1}} \). Let \( \alpha \) (respectively \( \beta \)) be a path from \( x_1 \) to \( x_2 \) (respectively \( x_3 \) to \( x_4 \)) which we choose so that \( \alpha \cdot \beta = 0 \). Then

\[
R_f(x_1, x_2, x_3, x_4) = \alpha \cdot (f \circ \beta) = -(f \circ \beta) \cdot \alpha \quad \text{since} \quad (f \circ \beta) \text{ is skew-symmetric,}
\]

\[
= -\beta \cdot (f^{-1} \circ \alpha)
\]

\[
= -R_{f^{-1}}(x_3, x_4, x_1, x_2)
\]

\[
= R_f(x_3, x_4, x_1, x_2).
\]

To show equation (5) we choose paths \( \alpha_1 \) from \( x_1 \) to \( w \) and \( \alpha_2 \) from \( w \) to \( x_2 \) and let \( \alpha \) be the path from \( x_1 \) to \( x_2 \) obtained by concatenating them. Then

\[
R_f(x_1, x_2, x_3, x_4) = \alpha \cdot (f \circ \beta) - \alpha \cdot \beta
\]

\[
= \alpha_1 \cdot (f \circ \beta) - \alpha_1 \cdot \beta + \alpha_2 \cdot (f \circ \beta) - \alpha_2 \cdot \beta
\]

\[
= R_f(x_1, w, x_3, x_4) + R_f(w, x_2, x_3, x_4).
\]

Equation (6) follows from equations (4) and (5). \( \square \)

### 1.3 \( R_f \) for diffeomorphisms

For a homeomorphism \( f \) the value of \( R_f(x_1, x_2, x_3, x_4) \) is generally undefined if either of \( x_1, x_2 \) is equal to either of \( x_3, x_4 \). In the case of \( C^1 \) diffeomorphisms we can remove this restriction. This done essentially by incorporating the infinitesimal rotation number at the fixed point, say \( x_1 = x_3 \), in question.

Suppose \( p \in \text{Fix}(f) \). Let \( \alpha : [0, 1] \to S^2 \) and \( \beta : [0, 1] \to S^2 \) be \( C^1 \) embeddings with \( \alpha(0) = \beta(0) = p \) with distinct tangent vectors at \( p \). We can blow up the point \( p \).
to obtain a map \( \hat{f} \) on the compactification of \( S^2 \setminus \{p\} \). The action of \( \hat{f} \) on the circle which compactifies \( S^2 \setminus \{p\} \) is the projectivization of \( Df_p \). We denote the compactified space (which is topologically a closed disk) by \( M \). The points on the boundary of \( M \) which correspond to the tangents to \( \alpha \) and \( \beta \) at \( p \) are distinct and the ends of paths \( \hat{\alpha} \) and \( \hat{\beta} \) which agree with \( \alpha \) and \( \beta \) on \( (0, 1) \). We define \( \alpha \cdot \beta \) to be \( \hat{\alpha} \cdot \hat{\beta} \).

We can now define \( R_f(p, x_2, p, x_4) \). Choose embeddings \( \alpha \) and \( \beta \) satisfying

(1) \( \alpha(0) = \beta(0) = p \in \text{Fix}(f) \).

(2) \( \alpha(1) = x_2, \beta(1) = x_4 \) with \( x_1, x_2 \in \text{Fix}(f) \).

(3) If \( v \) and \( w \) are the tangents at \( p \) to \( \alpha \) and \( \beta \) respectively then

\[
\frac{d f^n_p(v)}{\|d f^n_p(v)\|} \neq \frac{w}{\|w\|},
\]

for all \( n \in \mathbb{Z} \).

**Lemma 1.10.** Suppose the paths \( \alpha \) and \( \beta \) satisfy (1) - (3) above. Then the limit

\[
\lim_{n \to \infty} \alpha \cdot (f^n \circ \beta) - \alpha \cdot \beta
\]

exists and is independent of the choice of \( \alpha \) and \( \beta \).

**Proof.** Blow up the points \( p \) and \( x_2 \) to form a closed annulus \( \mathbb{A} \). Choose the lift \( \tilde{f} : \tilde{\mathbb{A}} \to \tilde{\mathbb{A}} \) which fixes a lift of \( x_4 \). Let \( C \) be the circle added when \( p \) was blown up; so \( C \) is one component of the boundary of \( \mathbb{A} \). Parametrize \( C \) so that \( \hat{\alpha}(0) \) is 0 and let \( \tilde{C} \) be the universal cover of \( C \) which we may think of as a component of the boundary of \( \tilde{\mathbb{A}} \).

Then \( \tilde{f}|_{\tilde{C}} \) is a lift of \( f|_{C} \) and it has a well defined translation number \( \tau(\tilde{f}) \) defined to be

\[
\tau(\tilde{f}) = \lim_{n \to \infty} \frac{\tilde{f}^n(\tilde{p}_2) - \tilde{p}_2}{n}
\]

where \( \tilde{p}_2 \) is the lift of the endpoint \( \beta(0) \). This limit always exists by the standard theory of rotation numbers for circle homeomorphisms. The annulus \( \mathbb{A} \) depends only on \( p \) and \( x_2 \). The lift \( \tilde{f} \) is determined by \( x_4 \).

Finally we note that

\[
(\alpha \cdot (f^n \circ \beta) - \alpha \cdot \beta) = -(\tilde{f}^n(\tilde{p}_2) - \tilde{p}_2) + K,
\]

where \( K \) satisfies \( |K| \leq 2 \). It follows that

\[
\lim_{n \to \infty} \frac{\alpha \cdot (f^n \circ \beta) - \alpha \cdot \beta}{n} = -\tau(\tilde{f}).
\]

\( \square \)
Definition 1.11. We define
\[ R_f(p, x_2, p, x_4) = \lim_{n \to \infty} \frac{\alpha \cdot (f^n \circ \beta) - \alpha \cdot \beta}{n}. \]

The number \( R_f(x_1, p, x_3, p) \) is defined analogously as are the values of \( R_f \) when one of first two and one of its second two variables are equal to \( p \). We also observe that if \( p_1, p_2 \in \text{Fix}(f) \) then \( R_f(p_1, p_2, p_1, p_2) \) is also defined and equal to the difference of the rotation numbers of \( \hat{f} \) on the two boundary components of the annulus \( \mathbb{A} \) obtained by blowing up both \( p_1 \) and \( p_2 \).

It is easy to check that \( R_f \) as defined still satisfies the symmetries of Propositions (1.9) and (1.9).

1.4 \( R_f \) for periodic points

From Proposition (1.7) it is clear that for \( q \in \mathbb{Z} \) we have \( R_{fq} = qR_f \). It follows that if \( \{x_1, x_2, x_3, x_4\} \) is a subset of distinct points in \( \text{Fix}(f) \), then
\[ R_f(x_1, x_2, x_3, x_4) = \frac{1}{q} R_{fq}(x_1, x_2, x_3, x_4). \]

Definition 1.12. Suppose \( \{x_1, x_2, x_3, x_4\} \) is a subset of distinct points in \( \text{Fix}(f^q) \). We define
\[ R_f(x_1, x_2, x_3, x_4) = \frac{1}{q} R_{fq}(x_1, x_2, x_3, x_4). \]

Because of the linearity of this definition it is clear that Propositions (1.7), (1.8), and (1.9) all remain valid when the argument of \( R_f \) is \( x = (x_1, x_2, x_3, x_4) \), a 4-tuple of distinct points in \( \text{Per}(f) \), the set of periodic points of \( f \).

2 The complete symmetries of \( R_f \).

If \( X \) is a set we want to investigate the class of functions of four variables \( F : X^4 \to \mathbb{R} \) which possess certain symmetries under permutation of the variables. The simplest example with the symmetries which interest us occurs when \( X = \mathbb{R} \) and we define the quadratic function \( q : \mathbb{R}^4 \to \mathbb{R} \) by
\[ q(x_1, x_2, x_3, x_4) = (x_1 - x_2)(x_3 - x_4). \]

As before we will write elements of \( S_4 \) in the standard way as a product of cycles and we recall that \( \sigma_i \) denotes the transposition \((i, i + 1), 1 \leq i \leq 3 \). The group \( S_4 \) is generated by these three transpositions.

If \( x = (x_1, x_2, x_3, x_4) \) is a four-tuple of elements of some set we will denote the four-tuple \((x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)})\) by \( x_\sigma \). As before an element of \( S_4 \) which we will denote the cyclic permutation \((123)\) by \( \tau \).
We are interested in the vector space of all functions \( F : X^4 \to \mathbb{R} \) which satisfy the following three equations.

\[
F(x) + F(x_\tau) + F(x_{\tau^2}) = 0, \tag{7}
\]

\[
F(x_{\sigma_i}) = F(x_{\sigma_3}) = -F(x) \tag{8}
\]

\[
F(x_1, x_2, x_3, x_4) = F(x_1, w, x_3, x_4) + F(w, x_2, x_3, x_4), \quad \text{for all } w \in X. \tag{9}
\]

Equation (7) says that if we cyclically permute the first three arguments of \( F \) and sum the values, the result is 0. This is a kind of Jacobi identity for \( F \). Equation (8) says that \( F \) is skew-symmetric in its first two variables and in its last two. Equation (9) says that as a function of its first two variables \( F \) is a coboundary. We remark that it will be shown below (see Remark (2.2)) that a consequence of these relations is that \( F(x_1, x_2, x_3, x_4) = F(x_3, x_4, x_1, x_2) \) so it follows that \( F \) is also a coboundary in its last two variables, i.e., for all \( w \in X \),

\[
F(x_1, x_2, x_3, x_4) = F(x_1, x_2, x_3, w) + F(x_1, x_2, w, x_4).
\]

To understand the symmetries of \( F \) is useful to introduce a function to \( \mathbb{R}^3 \) containing the three values obtained by cyclically permuting the first three arguments of \( F \). Hence we define the function \( \mathbb{F} : X^4 \to \mathbb{R}^3 \) by

\[
\mathbb{F}(x) = (F(x), F(x_\tau), F(x_{\tau^2})) \tag{10}
\]

An immediate consequence of equation (7) is the fact that the image of \( \mathbb{F} \) lies in the subspace of \( \mathbb{R}^3 \) given by \( y_1 + y_2 + y_3 = 0 \).

The symmetries of \( F \) under the action of \( S_4 \) permuting its arguments are most easily described by means of a representation of \( S_4 \) in \( GL(3, \mathbb{Z}) \). Since the group \( S_4 \) is generated by the three transpositions \( \sigma_i = (i, i + 1) \), \( 1 \leq i \leq 3 \), we can define a representation \( \Theta \) by specifying its value on these three elements of \( S_4 \). We define

\[
\Theta(\sigma_1) = \Theta(\sigma_3) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix},
\]

\[
\Theta(\sigma_2) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.
\]

Proposition 2.1. The function \( \Theta \) defined for \( \sigma_i, 1 \leq i \leq 3 \), determines a homomorphism from \( \Theta : S_4 \to GL(3, \mathbb{Z}) \).

Proof. The set \( \sigma_1, \sigma_2, \sigma_3 \in S_4 \) is a set of generators. It is easily checked that \( \Theta \) respects the relations \( \sigma_i^2 = id, \sigma_1\sigma_3 = \sigma_3\sigma_1 \) and \( \sigma_i\sigma_i+1\sigma_i = \sigma_i+1\sigma_i\sigma_i+1 \), which are a complete set of relations defining \( S_4 \) so \( \Theta \) is a homomorphism. \( \square \)
Remark 2.2. Since the transposition \((13) = \sigma_1 \sigma_2 \sigma_1\) and \((24) = \sigma_2 \sigma_3 \sigma_2\) it is easy to calculate
\[
\Theta((13)) = \Theta((24)) = \begin{bmatrix}
0 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix},
\]
and hence that \(\Theta((13)(24)) = I\). It is easy to check that the kernel of \(\Theta\) is a group isomorphic to \(\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}\) which is generated by the elements \((12)(34)\) and \((13)(24)\) of \(S_4\). One can also check that the image of \(\Theta\) is isomorphic to \(S_3\).

Theorem 2.3. Suppose for all \(x \in X^4\) the function \(F : X^4 \to \mathbb{R}\) satisfies equations (7) and (8) and we define \(F : X^4 \to \mathbb{R}^3\) by
\[
F(x) = (F(x), F(x_\tau), F(x_\tau^2)).
\]
Then for every \(\sigma \in S_4\) and every \(x \in X^4\)
\[
F(x_\sigma) = \Theta(\sigma)F(x).
\] (11)

We postpone the proof of this result to the next section. Note that this result about the symmetries of \(F\) and \(F\) requires only that \(F\) satisfy equations (7) and (8). For the next result we will additionally use the coboundary condition, equation (9). We note that given any function of two variables \(g : X^2 \to \mathbb{R}\) if we define
\[
F(x) = g(x_1, x_3) - g(x_1, x_4) - g(x_2, x_3) + g(x_2, x_4),
\]
then it is straightforward to check that \(F\) satisfies equations (7) and (8). The next theorem asserts that the converse of this is also true.

Theorem 2.4. Suppose for all \(x \in X^4\) the function \(F : X^4 \to \mathbb{R}\) satisfies equations (8), and (9) Then there exists a function \(g : X^2 \to \mathbb{R}\) such that
\[
F(x_1, x_2, x_3, x_4) = g(x_1, x_3) - g(x_1, x_4) - g(x_2, x_3) + g(x_2, x_4).
\] (12)
If we normalize by picking two distinguished elements \(a, b \in X\) and specifying that \(g(a, w) = g(w, b) = 0\) for all \(w \in X\) then \(g\) is unique.

Proof. Given \(F\) satisfying equations equations (8) and (9), and two distinguished elements \(a, b \in X\) we define
\[
g(u, v) = F(u, a, v, b).
\]
Skew-symmetry in the first two variables and last two variables implies \(F(a, a, v, b) = 0\) and \(F(u, a, b, b) = 0\), so \(g(a, w) = g(w, b) = 0\) for all \(w \in X\).

To show that equation (12) is satisfied we make repeated use of the skew-symmetry in the first two or last two variables (equation (8)) and of the coboundary relation in the first two and last two variables (equation (9)).
\[ F(x_1, x_2, x_3, x_4) = F(x_1, a, x_3, x_4) + F(a, x_2, x_3, x_4) \]
\[ = F(x_1, a, x_3, x_4) - F(x_2, a, x_3, x_4) \]
\[ = (F(x_1, a, x_3, b) + F(x_1, a, b, x_4)) - (F(x_2, a, x_3, b) + F(x_2, a, b, x_4)) \]
\[ = F(x_1, a, x_3, b) - F(x_1, a, x_4, b) - F(x_2, a, x_3, b) + F(x_2, a, x_4, b) \]
\[ = g(x_1, x_3) - g(x_1, x_4) - g(x_2, x_3) + g(x_2, x_4). \]

To show uniqueness, suppose \( g_1 : X^2 \to \mathbb{R} \) is another function satisfying the conclusion of the theorem. Then for any \( u, v \in X \)
\[ g(u, v) = F(u, a, v, b) \]
\[ = g_1(u, v) - g_1(u, b) - g_1(a, v) + g_1(a, b) \]
\[ = g_1(u, v). \]

\[ \square \]

3 Proof of Theorem (2.3)

**Theorem (2.3)** Suppose for all \( x \in X^4 \) the function \( F : X^4 \to \mathbb{R} \) satisfies equations (7) and (8) and we define \( F : X^4 \to \mathbb{R}^3 \) by
\[ F(x) = (F(x), F(x_\tau), F(x_\tau^2)). \]

Then for every \( \sigma \in S_4 \) and every \( x \in X^4 \)
\[ F(x_\sigma) = \Theta(\sigma)F(x). \quad (13) \]

**Proof.** Since \( \Theta \) is a homomorphism it suffices to prove that \( F(x_\sigma) = \Theta(\sigma)F(x) \) for each \( \sigma \) in the set of generators of \( \{\sigma_1, \sigma_2, \sigma_3\} \).

Let \( x = (a, b, c, d) \). We define the numbers \( r \) and \( s \) by \( F(a, b, c, d) = r \) and \( F(b, c, a, d) = s \). Then
\[ \begin{bmatrix} F(x) \\ F(x_\tau) \\ F(x_\tau^2) \end{bmatrix} = \begin{bmatrix} F(a, b, c, d) \\ F(b, c, a, d) \\ F(c, a, b, d) \end{bmatrix} = \begin{bmatrix} r \\ s \\ -r - s \end{bmatrix} \quad (14) \]

where the equality of the third component follows from equation (7).

We next conclude
\[ \begin{bmatrix} F(x_{\sigma_1}) \\ F(x_{\sigma_2}) \\ F(x_{\sigma_3}) \end{bmatrix} = \begin{bmatrix} F(b, a, c, d) \\ F(a, c, b, d) \\ F(c, b, a, d) \end{bmatrix} = \begin{bmatrix} -r \\ r + s \\ -s \end{bmatrix}. \quad (15) \]
where the first component comes from equation (8), the third component comes from equation (8) applied to the second component of equation (14) and the second component follows from equation (7).

Since
\[
\begin{bmatrix}
-1 & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
r \\
s \\
-r - s
\end{bmatrix}
= \begin{bmatrix}
-r \\
r + s \\
-r - s
\end{bmatrix}
\]
we have shown \( F(x_{\sigma_1}) = \Theta(\sigma_1)F(x) \).

Likewise from equation (15) we conclude
\[
\begin{bmatrix}
F(x_{\sigma_2}) \\
F(x_{\tau \sigma_2}) \\
F(x_{\tau^2 \sigma_2})
\end{bmatrix}
= \begin{bmatrix}
F(a, c, b, d) \\
F(c, b, a, d) \\
F(b, a, c, d)
\end{bmatrix} = \begin{bmatrix}
r + s \\
s \\
-r 
\end{bmatrix}.
\]

(16)

Hence \( F(x_{\sigma_2}) = \Theta(\sigma_2)F(x) \).

To handle the case of \( \sigma_3 \) we define \( t = -F(x_{\tau^2 \sigma_3}) = -F(d, a, b, c) \) so
\[
\begin{bmatrix}
F(x_{\sigma_3}) \\
F(x_{\tau \sigma_3}) \\
F(x_{\tau^2 \sigma_3})
\end{bmatrix}
= \begin{bmatrix}
F(a, b, d, c) \\
F(b, d, a, c) \\
F(d, a, b, c)
\end{bmatrix} = \begin{bmatrix}
-r \\
r + t \\
-t
\end{bmatrix}.
\]

(17)

where equality of the first components follows by equation (8) and the value of the second component again comes from the fact that the sum of the components is 0 (equation (7)).

From the last component of equation (15) we get \( F(c, b, d, a) = s \) and the second component of (17) shows \( F(b, d, c, a) = -r - t \). By equation (7),
\[
F(c, b, d, a) + F(b, d, c, a) + F(d, c, b, a) = 0.
\]

and hence
\[
s + (-r - t) + F(d, c, b, a) = 0.
\]

So \( F(d, c, b, a) = r + t - s \) and \( F(d, c, a, b) = -r - t + s \). Also \( F(c, a, d, b) = r + s \) by the third component of equation (14). Using equation (7) once again, we see
\[
F(c, a, d, b) + F(a, d, c, b) + F(d, c, a, b) = 0,
\]
and hence
\[
(r + s) + F(a, d, c, b) + (-r - t + s) = 0.
\]

So \( F(a, d, c, b) = -2s + t \). But the third component of equation (17) implies \( F(a, d, c, b) \) is also equal to \(-t\), so we conclude that \(-t = -2s + t\) and hence \( s = t \). Substituting \( s \) for \( t \) in equation (17) gives \( F(x_{\sigma_3}) = \Theta(\sigma_3)F(x) \).