Nonplanar Periodic Solutions for Spatial Restricted N+1-Body Problems

Fengying Li and Shiqing Zhang and Xiaoxiao Zhao
Yangtze Center of Mathematics and College of Mathematics, Sichuan University,
Chengdu 610064, People’s Republic of China

Abstract: We use variational minimizing methods to study spatial restricted N+1-body problems with a zero mass moving on the vertical axis of the moving plane for N equal masses. We prove that the minimizer of the Lagrangian action on the anti-T/2 or odd symmetric loop space must be a non-planar periodic solution for any \( N \geq 2 \).

Keywords: Restricted N+1-body problems; nonplanar periodic solutions; variational minimizers; Jacobi’s necessary conditions.

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1 Introduction and Main Result

Spatial restricted 3-body model was studied by Sitnikov [5]. Mathlouthi [3] etc. studied the periodic solutions for the spatial circular restricted 3-body problems by minimax variational methods.

In this paper, we study spatial circular restricted N+1-body problems with a zero mass moving on the vertical axis of the moving plane for N equal masses. Suppose point masses \( m_1 = \cdots = m_N = 1 \) move centered at the center of masses on a circular orbit. The motion for the zero mass is governed by the gravitational forces of \( m_1, \cdots, m_N \). Let \( \rho_j = e^{\sqrt{-1}2\pi j} \) and

\[
q_1(t) = re^{\sqrt{-1}2\pi t} \rho_1, \cdots, q_j(t) = \rho_j q_1(t), \cdots, q_N(t) = re^{\sqrt{-1}2\pi t}
\]

satisfy the Newtonian equations:

\[
m_i \ddot{q}_i = \frac{\partial U}{\partial q_i}, \quad i = 1, \cdots, N,
\]

where

\[
U = \sum_{1 \leq i < j \leq N} \frac{m_im_j}{|q_i - q_j|}.
\]

The orbit \( q(t) = (0, 0, z(t)) \in \mathbb{R}^3 \) for zero mass satisfies the following equation

\[
\ddot{q} = \sum_{i=1}^{N} \frac{m_i (q_i - q)}{|q_i - q|^3}.
\]

Define

\[
f(q) = \int_0^1 \left[ \frac{1}{2} |\dot{q}|^2 + \sum_{i=1}^{N} \frac{1}{|q_i - q|^3} \right] dt, \quad q \in \Lambda_i,
\]

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then
\[ f(q) = \int_{0}^{1} \left[ \frac{1}{2}|z'|^2 + \frac{N}{\sqrt{r^2 + z^2}} \right] dt \triangleq f(z), \quad q \in \Lambda_i, \] (1.6)

where
\[ \Lambda_1 = \left\{ \begin{array}{l}
q(t) = (0,0,z(t))|z(t) \in W^{1,2}(R/Z, R) \\
z(t + \frac{T}{2}) = -z(t), \quad q(t) \neq q_i(t), \quad \forall t \in R, i = 1,2, \ldots, N \end{array} \right\}, \]
\[ \Lambda_2 = \left\{ \begin{array}{l}
q(t) = (0,0,z(t))|z(t) \in W^{1,2}(R/Z, R) \\
q(-t) = -q(t)
\end{array} \right\}, \]
\[ W^{1,2}(R/Z, R) = \left\{ x(t) \left| x(t), \dot{x}(t) \in L^2(R, R) \right. \left. \quad x(t + 1) = x(t) \right\}. \]

Notice that the symmetry in \( \Lambda_1 \) is related with Italian symmetry [1].

In this paper, our main result is the following:

**Theorem 1.1** The minimizer of \( f(q) \) on the closure \( \overline{\Lambda}_i \) of \( \Lambda_i \) (i=1,2) is a nonplanar and noncollision periodic solution.

## 2 Proof of Theorem 1.1

We define the inner product and equivalent norm of \( W^{1,2}(R/Z, R) \):
\[ \langle u, v \rangle = \int_{0}^{1} (uv + u' \cdot v') dt, \]
\[ ||u|| = \left[ \int_{0}^{1} |u|^2 dt \right]^{\frac{1}{2}} + \left[ \int_{0}^{1} |u'|^2 dt \right]^{\frac{1}{2}} \]
(2.1)
\[ \cong \left[ \int_{0}^{1} |u'|^2 dt \right]^{\frac{1}{2}} + |u(0)|. \]
(2.2)

**Lemma 2.1** (Palais’s Symmetry Principle[4]) Let \( \sigma \) be an orthogonal representation of a finite or compact group \( G \) in the real Hilbert space \( H \) such that for \( \forall \sigma \in G, f(\sigma \cdot x) = f(x) \), where \( f : H \to R. \)

Let \( S = \{ x \in H| \sigma \cdot x = x, \forall \sigma \in G \} \). Then the critical point of \( f \) in \( S \) is also a critical point of \( f \) in \( H. \)

By Palais’s Symmetry Principle, we know that the critical point of \( f(q) \) in \( \overline{\Lambda}_i \) is a noncollision periodic solution of Newtonian equation [1,4].

In order to prove Theorem 1.1, we need

**Lemma 2.2** [6] Let \( X \) be a reflexive Banach space, \( S \) be a weakly closed subset of \( X \), \( f : S \to R \cup +\infty, \quad f \neq +\infty \) is weakly lower semi-continuous and coercive \( f(x) \to +\infty \) as \( ||x|| \to +\infty \), then \( f \) attains its infimum on \( S \).

**Lemma 2.3** (Poincare-Wirtinger Inequality) Let \( q \in W^{1,2}(R/Z, R^N) \) and \( \int_{0}^{T} q(t) dt = 0 \), then
\[ \int_{0}^{T} |\dot{q}(t)|^2 dt \geq \left( \frac{2\pi}{T} \right)^2 \int_{0}^{T} |q(t)|^2 dt. \]
(2.3)

**Lemma 2.4** \( f(q) \) in (1.6) attains its infimum on \( \Lambda_1 = \Lambda_1 \) or \( \Lambda_2 = \Lambda_2. \)

**Proof.** By Lemma 2.2 and Lemma 2.3, it is easy to prove Lemma 2.4.
Lemma 2.5 (Jacobi’s Necessary Condition[2]) If the critical point \( u = \tilde{u}(t) \) corresponds to a minimum of the functional \( \int_a^b F(t, u(t), u'(t))\,dt \) and if \( F_{u'u'} > 0 \) along this critical point, then the open interval \((a, b)\) contains no points conjugate to \( a \), that is, for \( \forall c \in (a, b) \), the following boundary value problem:

\[
\left\{ \begin{array}{l}
-\frac{d}{dt}(Ph') + Qh = 0, \\
h(a) = 0, \quad h(c) = 0,
\end{array} \right.
\]

has only the trivial solution \( h(t) \equiv 0 \), \( \forall t \in (a, c) \), where

\[
P = \frac{1}{2} F_{u'u'}|_{u=\tilde{u}},
\]

\[
Q = \frac{1}{2} (F_{uu} - \frac{d}{dt}F_{uu'})|_{u=\tilde{u}}.
\]

Lemma 2.6 The radius \( r \) for the moving orbit of \( N \) equal masses is

\[
r = \left( \frac{1}{4\pi} \right)^{\frac{2}{3}} \left[ \sum_{1 \leq j \leq N-1} \csc\left( \frac{\pi j}{N} \right) \right]^{\frac{1}{3}}.
\]

Proof. By (1.1)-(1.3), we have

\[
\ddot{q}_N = \sum_{j \neq N} \frac{q_j - q_N}{|q_j - q_N|^3}.
\]

Substituting (1.1) into (2.7), we have

\[
-4\pi^2 = \sum_{j \neq N} \frac{\rho_j - \rho_N}{r^3|\rho_j - \rho_N|^3}
\]

\[
4\pi^2 r^3 = \sum_{j \neq N} \frac{1 - \rho_j}{|1 - \rho_j|^3}
\]

\[
= \frac{1}{4} \sum_{1 \leq j \leq N-1} \csc\left( \frac{\pi j}{N} \right)
\]

Then

\[
r^3 = \frac{1}{16\pi^2} \sum_{1 \leq j \leq N-1} \csc\left( \frac{\pi j}{N} \right).
\]

Therefore

\[
r = \left( \frac{1}{4\pi} \right)^{\frac{2}{3}} \left[ \sum_{1 \leq j \leq N-1} \csc\left( \frac{\pi j}{N} \right) \right]^{\frac{1}{3}}.
\]

Lemma 2.7 ([8]) \( \sum_{j=1}^{N-1} \csc\left( \frac{\pi j}{N} \right) = \frac{4}{\pi} \).

For the functional (1.6), let

\[
F(z, z') = \frac{1}{2} |z'|^2 + \frac{N}{\sqrt{r^2 + z^2}}.
\]

Then the second variation of (1.6) in the neighborhood of \( z = 0 \) is given by

\[
\int_0^1 (Ph'^2 + Qh^2)\,dt,
\]
where
\[
P = \frac{1}{2} F_{zz'}|z=0 = \frac{1}{2},
\]
\[
Q = \frac{1}{2} (F_{zz} - \frac{d}{dt} F_{zz'})|z=0 = -\frac{N}{2r^3}.
\]

The Euler equation of (2.12) is called the Jacobi equation of the original functional (1.6), which is
\[
-\frac{d}{dt} (P h'^2) + Q h = 0,
\]
That is,
\[
h'' + \frac{N}{r^3} h = 0.
\]

Next, we study the solution of (2.16) with initial values \(h(0) = 0, \ h'(0) = 1\). It is easy to get
\[
h(t) = \sqrt{\frac{r^3}{N}} \cdot \sin \sqrt{\frac{N}{r^3}} t,
\]
which is not identically zero on \([0, \frac{1}{2}]\), but we will prove \(h(\frac{1}{2}) = 0\), and \(h(c) = 0\) for some \(c \in (0, \frac{1}{2})\).

Notice that
\[
\sqrt{\frac{N}{r^3}} = \sqrt{\frac{4\pi}{N}} \left( \sum_{j \neq N} \text{csc} \frac{\pi}{N} j \right)^{-\frac{1}{2}}
\]

Hence
\[
\left. \frac{1}{2} \right. \sqrt{\frac{N}{r^3}} = \sqrt{\frac{4\pi}{N}} \left( \sum_{j \neq N} \text{csc} \frac{\pi}{N} j \right)^{-\frac{1}{2}} \cdot 2\pi
\]
\[
= \sqrt{\frac{4\pi}{N}} \cdot 2\pi = N\pi.
\]

So
\[
h(\frac{1}{2}) = 0.
\]

Given \(N \geq 2\), choose \(0 < c = \frac{1}{2N} < \frac{1}{2}\) such that \(2Nc = 1\), then
\[
\sqrt{\frac{N}{r^3}} c = 2N\pi c = \pi
\]
Therefore
\[
\sin \sqrt{\frac{N}{r^3}} c = \sin \pi = 0.
\]

Hence \(q(t) = (0,0,0)\) is not a local minimum for \(f(q)\) on \(\tilde{\Lambda}_i = \Lambda_i (i = 1,2)\). So the minimizers of \(f(q)\) on \(\Lambda_i\) are not always at the center of masses, they must oscillate periodically on the vertical axis, that is, the minimizers are not always co-planar, hence we get the non-planar periodic solutions.

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