Time Operators and Time Crystals

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We investigate time operators in the context of quantum time crystals in ring systems. We demonstrate that a self-adjoint time operator with a periodic time evolution can be derived for a free particle on a ring system: the conventional Aharonov-Bohm time operator is obtained by taking the infinite-radius limit. We also reveal the relationship between our time operator and a \( PT \)-symmetric time operator. We find that both time operators indeed describe the periodic time evolution of a quantum time crystal.

**Introduction**—In the framework of the standard quantum mechanics, time is not an observable but just a parameter. One of the reasons for this is the difficulty to define a self-adjoint time operator \( \hat{T} \) conjugate to a Hamiltonian operator \( \hat{H} \) which satisfies the canonical commutation relation

\[
[\hat{H}, \hat{T}] = i\hbar.
\]

(1)

This difficulty lies in the difference between self-adjoint operators and symmetric operators even though these are both Hermitian operators. The existence of orthogonal eigenstates and real eigenvalues is ensured for self-adjoint operators but not ensured for symmetric operators. Hence, although observables have to be represented by self-adjoint operators, most of the time operators which satisfy Eq. (1) are symmetric operators. For example, the Aharonov-Bohm time operator

\[
\hat{T}_R = -\frac{m}{2}(\hat{x}\hat{p}^{-1} + \hat{p}^{-1}\hat{x})
\]

(2)

which describes the arrival time of a free particle on a line \( \mathbb{R} \) is a symmetric operator. How to define a self-adjoint time operator is still an open problem.

We consider the above problem in the context of a quantum time crystal (QTC). A QTC is a quantum mechanical state which spontaneously breaks time translation symmetry \( \mathbb{R} \). The idea of a QTC ground state was extended to our previous work of decoherence-induced QTC. A QTC promotes time from a parameter to a physical quantity. So, it is desirable to consider a time operator for a QTC. However, if we regard Eq. (1) as the equation of motion of the Heisenberg operator \( \hat{T}(t) = e^{i\hat{H}t/\hbar}e^{-i\hat{T}t/\hbar} \), then it is clear from the equation that the time described by Eq. (1) is linear and not periodic. Therefore, one needs a self-adjoint time operator with the periodicity of a QTC in order to solve these problems in quantum physics.

**Quantum Mechanics on \( S^1 \)—** Let \( \mathcal{H} \) be the Hilbert space of square-integrable functions with the periodic boundary condition \( \langle \theta | \psi \rangle = \psi(\theta + 2\pi), |\psi\rangle \in \mathcal{H}, \theta \in S^1 \). The time operator in Eq. (1) was defined such that it satisfies a commutation relation similar to the position-momentum commutation relation \( [\hat{x}, \hat{p}] = i\hbar \). However, the canonical commutation relation \( [\hat{\theta}, \hat{\pi}_\theta] = i\hbar \) with the angular position operator \( \hat{\theta} \) and the canonical
angular momentum operator $\hat{\pi}_\theta$ does not hold because $\theta$ is a multivalued operator and its eigenvalues are ill-defined [20]. In order to solve this problem, we use the cosine operator $\hat{\mathcal{C}}$, the sine operator $\hat{\mathcal{S}}$ [21] and the unitary operator $\hat{W} = \hat{e}^{i\hat{\theta}} = \hat{C} + i\hat{S}$ [21] which satisfy the commutation relations on $\mathcal{H}$

$$[\hat{\pi}_\theta, \hat{\mathcal{S}}] = -i\hbar \hat{\mathcal{C}}, \quad [\hat{\pi}_\theta, \hat{\mathcal{C}}] = i\hbar \hat{\mathcal{S}}, \quad [\hat{\pi}_\theta, \hat{W}] = \hbar \hat{W},$$
and

$$\hat{\pi}_\theta |\psi\rangle = i\hbar |\psi\rangle \quad \text{and} \quad \hat{\mathcal{S}} |\psi\rangle = |\psi\rangle \quad \text{for} \quad |\psi\rangle \in D(\hat{\pi}_\theta), \quad \hat{W} |\psi\rangle = |\psi\rangle \quad \text{for} \quad |\psi\rangle \in D(\hat{W}).$$

where $\hbar$ is the reduced Planck constant and $l$ is an integer. Through the position representation $\hat{\pi}_\theta \rightarrow -i\frac{\partial}{\partial \theta}$, it is clear that the commutation relations in Eq. (3) are the operator versions of $\frac{\partial}{\partial \theta} \sin \theta = \cos \theta, \quad \frac{\partial}{\partial \theta} \cos \theta = -\sin \theta,$ and $\frac{\partial}{\partial \theta} \sin \theta = i e^{i\theta}$, respectively. A physical example of $\hat{\mathcal{C}}$ is the charge density amplitude of an incommensurate charge density wave which is what we used to model a QTC [13]. The complete orthonormal set of momentum eigenstates $\{|\psi\rangle\}_{n=-\infty}^{\infty}$ spans $\mathcal{H}$ such that any state $|\psi\rangle \in \mathcal{H}$ can be written as $|\psi\rangle = \sum_{n=-\infty}^{\infty} c_n |\psi\rangle$. Moreover, let $\hat{F}(\hat{\pi}_\theta)$ be any function of $\hat{\pi}_\theta$, then it immediately follows from (4) that

$$[\hat{F}(\hat{\pi}_\theta), \hat{W}] |\psi\rangle = \hat{W} \hat{F}(\hat{\pi}_\theta) |\psi\rangle,$$

$$\delta \hat{F}(\hat{\pi}_\theta) \equiv \hat{F}(\hat{\pi}_\theta + \hbar) - \hat{F}(\hat{\pi}_\theta).$$

**Self-Adjoint Time Operator on $S^1$**— Let $T$ be a symmetric or a self-adjoint operator on a Hilbert space $\mathcal{H}$ and $H$ be a Hamiltonian operator on $\mathcal{H}$. Then, we say that $T$ is a generalized time operator of $\hat{H}$ if $(\hat{T}, \hat{H}, \hat{K}(t))$ satisfies the generalized weak Weyl relation (GWWR) [14]

$$\hat{T} e^{-it\hat{H}/\hbar} |\psi\rangle = e^{-it\hat{H}/\hbar} (\hat{T} + \hat{K}(t)) |\psi\rangle$$

with $|\psi\rangle \in D(\hat{T})$ and $D(\hat{K}(t)) = \mathcal{H}$ (where $D(\cdot)$ denotes operator domain). The bounded self-adjoint operator $\hat{K}(t)$ is called the commutation factor of the GWWR. Moreover, if $\hat{K}(t)$ is differentiable with respect to $t$, then $(\hat{T}, \hat{H}, \hat{K}(0))$ (where the dot denotes time derivative) satisfies the generalized canonical commutation relation

$$[\hat{H}, \hat{T}] |\psi\rangle = -i\hbar \hat{K}(0) |\psi\rangle$$

for $|\psi\rangle \in D(\hat{T}\hat{H}) \cap D(\hat{H}\hat{T})$. In our case, we require $\hat{K}(0) = -\hat{C}$ for a reason that will become clear in a moment. The case $\hat{K}(t) = -t$ in the GWWR gives the weak Weyl relation $\hat{T} e^{-it\hat{H}/\hbar} |\psi\rangle = e^{-it\hat{H}/\hbar} (\hat{T} - t) |\psi\rangle$ which is a stronger version of Eq. (11) [22]. In general, there is a hierarchy of time operators which satisfy stronger and weaker versions of the canonical commutation relation [14] [23].

Now, let us consider a free particle on $S^1$ with a moment of inertia $I$ and a Hamiltonian $\hat{H} = \hat{\pi}^2_{\theta}/2I$. Then, it immediately follows from Eq. (6) that

$$[\hat{H}, \hat{W}] = \hat{W} \delta \hat{H} \Rightarrow \hat{[H}, \hat{W} (\delta \hat{H})^{-1}] = \hat{W},$$

$$[\hat{H}, \hat{W}^\dagger] = -\delta \hat{H} \hat{W}^\dagger \Rightarrow \hat{[H}, (\delta \hat{H})^{-1} \hat{W}^\dagger] = -\hat{W}^\dagger$$
holds in $\mathcal{H}$. Consequently, we can define the time operator

$$\hat{T}_{S1} = -\text{Im}[\hbar \delta \hat{W} (\delta \hat{H})^{-1}],$$

$$[\hat{H}, \hat{T}_{S1}] = i\hbar \hat{\mathcal{C}}$$

with $D(\hat{T}_{S1}) = \mathcal{H}$. The commutation relation Eq. (9) has the same form as $[\hat{\pi}_\theta, \hat{\mathcal{S}}] = -i\hbar \hat{\mathcal{C}}$ in Eq. (3). Besides, it is a known fact from spectral analysis that the real and imaginary parts of a bounded operator are self-adjoint operators. So, because $\hat{W}$ and $(\delta \hat{H})^{-1}$ are bounded operators, $\hat{T}_{S1} = -\text{Im}[\hbar \delta \hat{W} (\delta \hat{H})^{-1}]$ is a self-adjoint time operator which satisfies Eq. (7).

**Infinite-Radius Limit**— Next, we show that $\hat{T}_{S1}$ reduces to $\hat{T}_R$ in the infinite-radius limit $R \rightarrow \infty$. Using the identity operators $\int_{-\pi}^{\pi} d\theta |\theta\rangle \langle \theta| = 1, \quad \int_{-\infty}^{\infty} dx |x\rangle \langle x| = 1$

$$\sum_{l \in \mathbb{Z}} |\psi_l\rangle \langle \psi_l| = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} dk |k\rangle \langle k| = 1$$

with $x = R\theta$, $k = l/R$, $p = h k$, $\langle \theta |\psi_l\rangle = \langle x |k\rangle$, $I = m R^2$, $\mu_i = \langle \psi_l |\hbar (\delta \hat{H})^{-1} |\psi_l\rangle$, $e^{i\theta} \approx 1 + i\theta$ and $\mu_i/R \rightarrow \infty$ we obtain

$$\hat{T}_{S1} \approx -\text{Im} \left[ \sum_{l \in \mathbb{Z}} \int_{-\pi}^{\pi} d\theta |\theta\rangle \langle \theta| \left( 1 + i\theta \right) \mu_i |l\rangle \langle l| \right]$$

$$= -\text{Re} \left[ \sum_{l \in \mathbb{Z}} \int_{-\pi}^{\pi} d\theta |\theta\rangle \langle \theta| \theta \mu_i |l\rangle \langle l| \right]$$

$$\rightarrow -\text{Re} \left[ \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dx |x\rangle \langle x| \frac{m x^2}{h k} |k\rangle \langle k| \right]$$

$$= -\frac{m}{2} (\hat{x} \hat{p} - \hat{p} \hat{x})$$

Similarly, one can show that $R\hat{\mathcal{S}} \rightarrow \hat{x}, \hat{\mathcal{C}} \rightarrow 1$ as $R \rightarrow \infty$ and $\hat{\pi} = \hat{\pi}/R$, hence

$$\hat{T}_{S1} \rightarrow \hat{T}_R,$$

$$[\hat{\mathcal{S}}, \hat{\pi}] = i\hbar \hat{\mathcal{C}} \rightarrow [\hat{x}, \hat{p}] = i\hbar,$$

$$[\hat{H}, \hat{T}_{S1}] = i\hbar \hat{\mathcal{C}} \rightarrow [\hat{H}, \hat{T}_R] = i\hbar,$$

as $R \rightarrow \infty$. Therefore, we conclude that $\hat{T}_{S1}$ is indeed a self-adjoint analogue of the Aharonov-Bohm time operator $\hat{T}_R$ on $S^1$. 
Time Evolution of the Time Operator—For a free particle on a ring we have $1/\mu = (1 + 2l)/\mu_0$. So, using the identity operator $1 = \sum_i \langle \psi_i | \psi_i \rangle$, one can show that the Heisenberg operator $\hat{T}_{S^1}(t) = e^{i\hat{H}t/\hbar}\hat{T}_{S^1}e^{-i\hat{H}t/\hbar}$ becomes

$$\hat{T}_{S^1}(t) = \text{Im} \left[ \sum_i \mu_i e^{i(2\nu + 1)t/\mu_0} |\psi_{i+1}\rangle \langle \psi_i| \right],$$

(11)

hence $\hat{T}_{S^1}(t)$, $\hat{K}(t) = \hat{T}_{S^1}(t) - \hat{T}_{S^1}$, and their expectation values with a general state $|\psi\rangle \in \mathcal{H}$ are periodic with the period $2\pi/\mu_0$ because $(2l + 1)$ is an integer. The periodic time evolution of $\hat{T}_{S^1}(t)$ is also present in the Heisenberg equation of motion

$$\frac{d}{dt}\hat{T}_{S^1}(t) = -\text{Re} \left[ \sum_i e^{i(2\nu + 1)t/\mu_0} |\psi_{i+1}\rangle \langle \psi_i| \right].$$

We note that the period is proportional to the particle’s moment of inertia which diverges as $R \to \infty$. Therefore, this periodicity, which was already discovered in our previous work on QTC [12], is intrinsic to ring systems.

Extension to PT-Symmetric Time Operators—The commutation relation $[\hat{\pi}_\theta, W] = \hbar \hat{W}$ in Eq. (3) also reduces to $[\hat{x}, \hat{p}] = i\hbar$ in the infinite-radius limit $R \to \infty$. So, we also consider a time operator which satisfies a similar commutation relation. In particular, we consider a time operator $\hat{T}_{S^1}^{PT}$ with $\mathcal{P}T$ (space-time inversion) symmetry: Operators with $\mathcal{P}T$-symmetry have real eigenvalues if their eigenstates are $\mathcal{P}T$-symmetric as well [17, 18]. The parity operator $\mathcal{P}$ in $S^1$ satisfies the following properties [24]: $\mathcal{P}^2 = 1$, $\mathcal{P}W\mathcal{P}^{-1} = W^t$, $\mathcal{P}\hat{\pi}_\theta \mathcal{P}^{-1} = -\hat{\pi}_\theta$, and $\mathcal{P}|\psi\rangle = |\psi\rangle$. We show that the time reversal operator $\mathcal{T}$ shares the same properties. The time reversal operator for a bosonic particle is an antiunitary operator which satisfies $\mathcal{T}^\dagger \mathcal{T} = 1$. Time reversal changes the direction of motion of a particle on $S^1$, i.e. $\hat{\pi}_\theta \mathcal{T}|\psi\rangle = -i\hbar \mathcal{T}|\psi\rangle$ should be satisfied. Therefore, we have $\mathcal{T}|\psi\rangle = a_\ell |\psi\rangle$ with a coefficient $a_\ell$. Ant-unitarity of $\mathcal{T}$ implies $a_\ell = 1$. Then, using Eq. (4) and Eq. (3) we readily obtain $\mathcal{T}\hat{\pi}_\theta \mathcal{T}^{-1} = -\hat{\pi}_\theta$ and $\mathcal{T}W\mathcal{T}^{-1} = W^t$. Therefore, $\mathcal{T}[\mathcal{P}T, \hat{T}_{S^1}] = 0$, $\mathcal{T}|\psi\rangle = |\psi\rangle$. (12)

Now, we can define the $\mathcal{P}T$-symmetric time operator by $\hat{T}_{S^1}^{PT} = \hat{f} - \hbar W (\delta \hat{H})^{-1}$, where $f$ is a real $\mathcal{P}T$-symmetric operator which commutes with $\hat{H}$. From a calculation similar to Eq. (10) we see that $\hat{T}_{S^1}^{PT}$ diverges as $R \to \infty$ unless we choose $\hat{f} = \hbar (\delta \hat{H})^{-1}$. So, we obtain the $\mathcal{P}T$-symmetric Aharonov-Bohm time operator on $S^1$

$$\hat{T}_{S^1}^{PT} = \hbar (1 - \hat{W}) (\delta \hat{H})^{-1}$$

$$= \hat{T}_{S^1}^{Re} + i\hat{T}_{S^1}^{Im}$$

(13)

which satisfies the commutation relation

$$[\hat{H}, \hat{T}_{S^1}^{PT}] |\psi\rangle = -\hbar \hat{W} |\psi\rangle.$$ (14)

$\hat{T}_{S^1}$ is defined in Eq. (8). The real part $\hat{T}_{S^1}^{Re}$ is given by

$$\hat{T}_{S^1}^{Re} = \text{Re}(\hat{T}_{S^1}^{PT}) = \hbar (\delta \hat{H})^{-1} - \frac{\hbar}{2} i \hat{W} (\delta \hat{H})^{-1} + (\delta \hat{H})^{-1}\hat{W}^{t}$$

(15)

and satisfies the commutation relation

$$[\hat{H}, \hat{T}_{S^1}^{Re}] = -i\hbar \hat{S}.$$ (16)

The eigenstates and eigenvalues of $\hat{T}_{S^1}^{PT}$ are calculated using biorthogonal quantum mechanics [25]: Suppose that the time operator $\hat{T}_{S^1}^{PT}$ and its Hermitian conjugate $(\hat{T}_{S^1}^{PT})^\dagger$ satisfy the eigenvalue equations

$$\hat{T}_{S^1}^{PT} |\phi_l\rangle = \tau_l |\phi_l\rangle,$$

$$\hat{T}_{S^1}^{PT}^\dagger |\chi_l\rangle = \tau_l^* |\chi_l\rangle.$$ (17)

Let us adopt the position representation $\hat{\pi}_\theta \rightarrow -i\hbar \frac{\partial}{\partial \theta}$, which implies $\delta \hat{H} \rightarrow \frac{\hbar^2}{T} (-i\frac{\partial}{\partial \theta} + \frac{1}{2})$. Then, Eq. (17) and Eq. (18) are equivalent to the following differential equations:

$$\phi_l(\theta) = \frac{\hbar}{T} \left( -i \frac{\partial}{\partial \theta} + \frac{1}{2} \right) \phi_l(\theta),$$

$$\left( 1 - e^{i\theta} \right) \phi_l(\theta) = \frac{\tau_l \hbar}{T} \left( -i \frac{\partial}{\partial \theta} + \frac{1}{2} \right) \phi_l(\theta),$$

$$\left( 1 - e^{-i\theta} \right) \chi_l(\theta) = \frac{\tau_l^* \hbar}{T} \left( -i \frac{\partial}{\partial \theta} + \frac{1}{2} \right) \chi_l(\theta),$$

which have the orthonormal solutions

$$\phi_l(\theta) = \phi_{l0} \left( 1 - e^{i\theta} \right) e^{i\nu_l\frac{\theta}{2}} e^{-i\frac{\nu_l}{2}},$$

$$\chi_l(\theta) = \chi_{l0} e^{i\nu_l\frac{\theta}{2}} e^{i\frac{\nu_l}{2}},$$

$$\int_{-\pi}^{\pi} d\theta \phi_{l*}(\theta) \phi_{m}(\theta) = 2\pi \chi_{l0} \phi_{m0} \delta_{l,m} = \delta_{l,m}$$

where $\nu_l = \frac{I}{2\hbar} - \frac{1}{2}$ was introduced for brevity. The periodic boundary condition requires $\nu_l$ to be an integer. Therefore, we obtain the eigenvalues

$$\tau_l = \tau_l^* = \frac{2I}{(2m + 1)\hbar} = \mu_{\nu_l}.$$ (19)

$I = m R^2$ implies that these eigenvalues are interpreted as the time required for a free particle with velocity $v = (\nu_l + \frac{1}{2}) \frac{\hbar}{mR}$ to move a distance $R$ on the ring; i.e. the time required to make a full rotation is $2\pi / v = 2\pi R / v$ (Fig. 2). This period is expected to be proportional to the angular momentum $\hbar$, so we set $\nu_l \sim I$.

Next, we calculate the large radius limit of $\hat{T}_{S^1}^{PT}$. The eigenvalues $\tau_l$ diverge as $R \to \infty$ because it takes an infinite amount of time to move an infinite distance. Instead, if $\hat{T}_{S^1}^{PT}$ has maximally broken $\mathcal{P}T$ symmetry; that
is, if all of its eigenvalues $\gamma_l$ are pure-imaginary complex-conjugate pairs, then $-i\hat{T}^{PT}_{S_1}$ (which is not defined in $\mathcal{H}$) has real eigenvalues. In this case, a calculation similar to Eq. (10) gives
\begin{equation}
-\hat{T}^{PT}_{S_1} \rightarrow (\hat{T}^{NH}_R)^\dagger = \hat{T}_R + i\frac{m\hbar}{2}\hat{p}^{-2}.
\end{equation}
$\hat{T}^{NH}_R$ and $(\hat{T}^{NH}_R)^\dagger$ satisfy the canonical commutation relation
\begin{equation}
[H, \hat{T}^{NH}_R] = [\hat{H}, (\hat{T}^{NH}_R)^\dagger] = [\hat{H}, \hat{T}_R] = i\hbar.
\end{equation}
The non-Hermitian time operator $\hat{T}^{NH}_R$ (which is not $\mathcal{PT}$-symmetric) was studied in reference [14]. (Here, the fact that $-i\hat{T}^{PT}_{S_1}$ reduces to $(\hat{T}^{NH}_R)^\dagger$ is merely a problem of definition. Since the eigenvalues of $T^{NH}_{S_1}$ are real in $\mathcal{H}$, we can say that $(\hat{T}^{PT}_{S_1})^\dagger$ is the actual $\mathcal{PT}$-symmetric operator and $\hat{T}^{PT}_{S_1}$ is its complex conjugate. Then, $-i(\hat{T}^{PT}_{S_1})^\dagger$ reduces to $\hat{T}^{NH}_R$ in the large radius limit.)

Discussion and Conclusion—First, we summarize our results. We defined a self-adjoint time operator $\hat{T}_{S_1}$ and a $\mathcal{PT}$-symmetric time operator $\hat{T}^{PT}_{S_1}$ for a free particle on a ring system. $\hat{T}_{S_1}$ is a generalized time operator which satisfies Eq. (14) and reflects the periodic time evolution of a QTC. The eigenstates and eigenvalues of $\hat{T}^{PT}_{S_1}$ are calculated using biorthogonal quantum mechanics. A summary of the time operators is given in Fig. 4. Second, we discuss the significance of considering a ring system. $\hat{K}(t)$ of a ring system is periodic with a radius-dependent period. On the other hand, for a one-dimensional system ($\mathbb{R}$) we have $\hat{K}(t) = t$ which implies

\begin{equation}
\text{Eq. 14 or } \hat{K}(t) = t\hat{C} \text{ with a bounded self-adjoint operator } \hat{C} \text{ which implies } [\hat{T}, \hat{H}] = i\hbar \frac{\partial}{\partial t}.
\end{equation}
However, our results imply that time operators should be defined based on the real-space topology of a quantum system. So, Eq. (14) may not be the only possibility to define self-adjoint time operators, especially if we want time operators for time crystals. Our work may also be generalized to relativistic particles, but quantization of constrained systems is still a subject with active research and the commutation relations Eq. (3) are not guaranteed to hold for relativistic systems as well.

Fourth, an operator being $\mathcal{PT}$-symmetric does not necessarily mean that it has real eigenvalues, but it can have real eigenvalues. Typically, in $\mathcal{PT}$-symmetric systems with gain and loss (denoted by $\pm\gamma$) all eigenvalues are real below a critical value $\gamma < \gamma_c$ and $\mathcal{PT}$-symmetry is spontaneously broken otherwise. Therefore, it would be an interesting problem to consider spontaneous breaking of $\mathcal{PT}$-symmetry (of the Hamiltonian or the time operator) in periodically driven Floquet time crystals [8, 12].

Moreover, one of the motivations to define time operators is to derive time-energy uncertainty relations. For the self-adjoint operator $\hat{T}_{S_1}$, the conventional Robertson uncertainty relation $\Delta H \Delta T_{S_1} \geq \frac{1}{4} |\langle H, \hat{T}_{S_1} \rangle| = \frac{1}{2} |\langle \hat{C} \rangle|$ is satisfied [27], and for $\hat{T}^{PT}_{S_1}$ non-Hermitian analogues [24, 28] can be obtained. In all cases, the phase-angular momentum uncertainty relations and the time-energy uncertainty relations for the self-adjoint parts and for the $\mathcal{PT}$-symmetric parts are completely equivalent. This fact is a direct consequence of the similarities between the

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**FIG. 2.** The eigenvalues $\gamma_l$ of $\hat{T}^{PT}_{S_1}$ describes the periodicity in time of a free particle moving on a ring with a constant velocity.

**FIG. 3.** The expectation values $\langle \phi_l | \hat{T}_{S_1} | \phi_l \rangle / \langle \phi_l | \phi_l \rangle$ is shown for $l = 0, 1, 5$ and $100$ for $\mu_0 = 2 \times 10^{-6}$ sec. The amplitude is proportional to $\mu_l$ and the period is $P = 2\pi \mu_0$.

**FIG. 4.** $\hat{T}_{S_1}$ (Eq. (3)) is a self-adjoint time operators and $\hat{T}^{PT}_{S_1}$ (Eq. (13)) is a $\mathcal{PT}$-symmetric time operator. In the large radius limit (from $S^1$ to $\mathbb{R}$), $\hat{T}_{S_1}$ reduces to the Aharonov-Bohm time operator $\hat{T}_R$ (Eq. (2)) and $\hat{T}^{PT}_{S_1}$ reduces to the non-Hermitian operator $\hat{T}^{NH}_R$ (Eq. (14)) [27].
commutation relations.

Finally, we discuss the results in this paper in the context of experiments. The order parameter of a charge density wave (CDW) or a superconductor is a complex scalar $\Delta = |\Delta| e^{i\theta}$. Suppose that we can quantize the phase $\theta$ by $\hat{W} = e^{i\theta}$. Then the commutation relations Eq. (9), Eq. (16), and Eq. (14) give the Heisenberg equation of motions of the time operators:

\[
\frac{d}{dt} \hat{T}_S(t) = -\hat{C}(t),
\]

\[
\frac{d}{dt} \hat{T}_{\text{Re}}S_1(t) = \hat{S}(t),
\]

\[
\frac{d}{dt} \hat{T}_{\text{PT}}S_1(t) = -i\hat{W}(t),
\]

respectively. Actually, the periodic oscillation of $\hat{W}(t)$ was already used in our previous work of decoherence-induced QTC which consists of a ring-shaped incommensurate CDW coupled to a fluctuating magnetic flux [13]. Therefore, our results can be tested in various CDW or superconducting systems, in superfluid systems with spatial periodicity [7, 29] and in other systems which can be described by a single particle on $S^1$.

We expect that our results have many important applications and insights, such as advancement in the physics of constrained systems (ring systems, Möbius systems, etc.) [30, 31] or understanding the space-time structure of the early (topologically non-trivial) universe [32].

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[1] More precisely, if the eigenvalues of a Hamiltonian are continuous and bounded below, then there is no self-adjoint time operators which satisfy the canonical commutation relation (1).
[2] M. Reed and B. Simon, *Fourier Analysis, Self-Adjointness (Methods of Modern Mathematical Physics, Vol. 2)* (Academic Press, 1975).
[3] J. Muga and C. Leavens, *Phys. Rep.* **338**, 353 (2000).
[4] Y. Aharonov and D. Bohm, *Phys. Rev.* **122**, 1649 (1961).
[5] F. Wilczek, *Phys. Rev. Lett.* **109**, 160401 (2012).
[6] T. Li, Z.-X. Gong, Z.-Q. Yin, H. T. Quan, X. Yin, P. Zhang, L.-M. Duan, and X. Zhang, *Phys. Rev. Lett.* **109**, 163001 (2012).
[7] F. Wilczek, *Phys. Rev. Lett.* **111**, 250402 (2013).
[8] C. W. von Keyserlingk and S. L. Sondhi, *Phys. Rev. B* **93**, 245146 (2016).
[9] N. Y. Yao, A. C. Potter, I.-D. Potirniche, and A. Vishwanath, *Phys. Rev. Lett.* **118**, 030401 (2017).
[10] D. V. Else, B. Bauer, and C. Nayak, *Phys. Rev. X* **7**, 011026 (2017).
[11] S. Choi and others., *Nature* **543**, 9 (2017).
[12] J. Zhang et al., *Nature* **543**, 217 (2017).
[13] K. Nakatsugawa, T. Fujii, and S. Tanda, *Phys. Rev. B* **96**, 094308 (2017).
[14] A. Arai, *Rev. Math. Phys.* **17**, 1071 (2005).
[15] E. Galapon, *Proc. Royal Soc. Lond. A: Math. Phys. Eng. Sci.* **458**, 451 (2002).
[16] E. A. Galapon, R. F. Caballar, and R. T. Bahague Jr, *Phys. Rev. Lett.* **93**, 180406 (2004).
[17] C. M. Bender, S. Boettcher, and P. N. Meisinger, *J. Math. Phys.* **40**, 2201 (1999).
[18] C. M. Bender, *Rep. Prog. Phys.* **70**, 947 (2007).
[19] E. Recami, V. S. Olkhovsky, and S. P. Maydanyuk, *International Journal of Modern Physics A* **25**, 1785 (2010).
[20] P. Carruthers and M. M. Nieto, *Rev. Mod. Phys.* **40**, 411 (1968).
[21] Y. Ohnuki and S. Kitakado, *J. Math. Phys.* **34**, 2827 (1993).
[22] K. Schm"udgen, *J. Funct. Anal.* **50**, 8 (1983).
[23] A. Arai and F. Hiroshima, *Ann. Henri Poincaré* **18**, 2995 (2017).
[24] S. Tanimura, *Progr. Theor. Exp. Phys.* **90**, 271 (1993).
[25] D. C. Brody, *J. Phys. A: Math. Theor.* **47**, 035305 (2014).
[26] We note that $PPT$-symmetric operators are not necessarily non-Hermitian: For instance, $\hat{p}^2$ is a $PPT$-symmetric Hermitian operator.
[27] P. Pfeifer and J. Fröhlich, *Rev. Mod. Phys.* **67**, 759 (1995).
[28] Y.-N. Dou and H.-K. Du, *J. Math. Phys.* **54**, 103508 (2013).
[29] P. Nozières, *Europhys. Lett.* **103**, 57008 (2013).
[30] S. Tanda, T. Tsumeta, Y. Okajima, K. Inagaki, K. Yamaya, and N. Hatakenaka, *Nature* **417**, 397 EP (2002).
[31] G. Povie, Y. Segawa, T. Nishihara, Y. Miyauchi, and K. Itami, *Science* **356**, 172 (2017).
[32] J.-P. Luminet, J. R. Weeks, A. Riazuelo, R. Lehoucq, and J.-P. Uzan, *Nature* **425**, 593 (2003).