I summarise what lattice methods can contribute to our understanding of the phenomenology of QCD at large $N_c$ and describe some recent work on the physics of $SU(N_c)$ gauge theories. These non-perturbative calculations show that there is indeed a smooth $N_c \to \infty$ limit and that it is achieved by keeping $g^2 N_c$ fixed, confirming the usual diagrammatic analysis. The lattice calculations support the crucial assumption that the theory remains linearly confining at large $N_c$. Moreover we see explicitly that $N_c = 3$ is 'close to' $N_c = \infty$ for many physical quantities. We comment on the fate of topology and the deconfining transition at large $N_c$. We find that multiple confining strings are strongly bound. The string tensions of these $k$-strings are close to the M(-theory)QCD-inspired conjecture that $\sigma_k \propto \sin(\pi k/N_c)$ as well as to 'Casimir scaling', $\sigma_k \propto k(N_c - k)$, with the most accurate recent calculations favouring the former. We point out that closed $k$-strings provide a natural way for non-perturbative effects to introduce $O(1/N_c)$ corrections into the pure gauge theory, in contradiction to the conventional diagrammatic expectation.

1 Introduction

As we have seen at this meeting, large-$N_c$ arguments are very useful in illuminating many aspects of QCD. Although we may not know very much about the detailed physics at $N_c = \infty$, we can get a long way by assuming that there is a smooth large $N_c$ limit which is confining and that $N_c = 3$ is 'close to' that limit. Analysis of the colour flow in Feynman diagrams tells us that this limit is achieved by keeping constant the 't Hooft coupling, $\lambda = g^2 N_c$, and that the leading corrections are $O(1/N_c)$ in QCD and $O(1/N_c^2)$ in the gauge theory.

The question I want to address in this talk is: are the above assumptions correct and does the colour flow counting survive if we go beyond diagrams to a fully non-perturbative calculation? The technique I shall use is to discretise the theory onto a space-time lattice, calculate various mass ratios via computer simulation, do this for a large enough range of lattice spacings that one can confidently extrapolate to the continuum limit; and finally, repeat the exercise for a large enough range of $N_c$ that one can confidently control the approach to $N_c = \infty$.

I will do this for $SU(N_c)$ gauge theories with no quarks. These are calculations that can be – and have been – done on workstations. The states of this theory are purely gluonic so one may call them 'glueballs'. If we take...
the $N_c \to \infty$ limit of this glueball spectrum, what we obtain is the glueball spectrum of $QCD_{N_c=\infty}$, since we expect no mixing between glueballs and quarkonia at leading order in $1/N_c$ (at least for $m_q \neq 0$). The next step would be to do $QCD_{N_c}$ in the ‘quenched’ approximation, where all quark vacuum bubbles are neglected. This can be regarded as a relativistic valence quark approximation to the theory. At any fixed $m_q \neq 0$ it has $QCD_{N_c=\infty}$ as its $N_c \to \infty$ limit, since quark vacuum bubbles do not appear at leading order in $1/N_c$, and so it can be used to determine the quarkonium physics of that theory. Such a calculation would be interesting and should be possible using a Teraflop computer of the kind that is becoming available to a number of lattice groups. If however one wants to look at the chiral limit at large $N_c$, then one needs to include quark loops and these are calculations that are not for the near future.

In the next section I briefly remind you how one calculates masses using lattice simulation and I give you an explicit example to demonstrate that such calculations are indeed possible. I then proceed to describe the results of a calculation of the lightest few glueballs which shows that the approach to $N_c = \infty$ is remarkably rapid: even SU(2) is ‘close to’ SU($\infty$) for many quantities. These calculations provide some explicit evidence that linear confinement survives at large $N_c$ and that the limit is indeed achieved by keeping $g^2N_c$ fixed. A much larger calculation of this kind is now in progress. I then summarise the lattice results for the tensions of $k$-strings and what they imply for various model/theoretic expectations. These string tension calculations are also interesting because they provide an explicit example of how non-perturbative physics may violate the usual large $N_c$ diagrammatic colour-counting results. Finally I briefly summarise what we are learning about the deconfining transition and about topology.

2 Calculating Masses

To calculate a mass we construct some operator $\phi(t)$ with the quantum numbers of the state and then use the standard decomposition of the Euclidean correlator in terms of energy eigenstates

$$C(t) \equiv \langle \phi(t)\phi(0) \rangle = \sum_n |\langle n|\phi|\Omega \rangle|^2 \exp\{-E_nt\} \quad (1)$$

where $|n\rangle$ are the energy eigenstates, with $E_n$ the corresponding energies, and $|\Omega\rangle$ is the vacuum state. To be able to evaluate the corresponding Euclidean Feynman Path Integrals we discretise continuous space-time to a lattice and truncate the infinite volume to a finite hypertorus. We now have a finite
number of degrees of freedom and can calculate Feynman Path Integrals using standard Monte Carlo techniques.

The lattice degrees of freedom are SU($N_c$) matrices that reside on the links of the lattice. In our partition function the fields are weighted with $\exp\{S\}$ where $S$ is the standard plaquette action

$$S = -\beta \sum_p \left( 1 - \frac{1}{N_c} \text{ReTr} U_p \right),$$

and $U_p$ is the ordered product of the matrices on the boundary of the plaquette $p$. For smooth fields this action reduces to the usual continuum action with $\beta = 2N_c/g^2$. However the fields that dominate the Feynman Path Integral are rough, all the way to the scale of the lattice spacing $a$. For these fields we can define a running lattice coupling $g_L(a)$ which reduces in the continuum limit to a coupling $g(a)$ in our favourite scheme:

$$\beta \equiv \frac{2N_c}{g_L^2(a)} \xrightarrow{a \to 0} \frac{2N_c}{g^2(a)} \quad (3)$$

So by varying the inverse lattice coupling $\beta$ we vary the lattice spacing $a$.

If we use a lattice action with reflection positivity, such as the simple plaquette action in eqn(2), then the decomposition in eqn(1) remains valid, except that now $t = am_t$, so that we obtain the energies from eqn(1) as $aE_n$ i.e. in units of the lattice spacing.

Having calculated some masses $am_i$ at a fixed value of $a$ we can remove lattice units by taking ratios: $am_i/am_j = m_i/m_j$. This ratio differs from the desired continuum value by lattice corrections. For our action the functional form of the leading correction is known to be $O(a^2)$. Thus for small enough $a$ we can extrapolate our calculated mass values

$$\frac{m_i(a)}{m_j(a)} = \frac{m_i(0)}{m_j(0)} + ca^2 m_k^2(a) \quad (4)$$

where $c$ depends on $i,j$ and $k$ and the $a$-dependence of $m_k(a)$ will make differences at $O(a^4)$. At this point we have obtained the mass ratios of the continuum theory which is the ultimate goal of our lattice calculations.

If we want to calculate the lightest mass using eqn(4) then it is clear that we have to go to large enough $t$ that the contribution of the excited states has died away and the correlation function has acquired a simple exponential fall-off with $t$. At large $t$, however, the value of the correlation function becomes very small and it is not obvious that a numerical approach, with finite errors, will be accurate enough. To demonstrate that it can be, I show in Fig.1 the correlation function used to extract the lightest SU(4) $J^{PC} = 0^{++}$ glueball.
mass in an ongoing calculation. On this plot a simple exponential is a straight line and it is clear that the corresponding mass can be determined very accurately. It is also clear that the simple exponential decay already starts at small $t$. This means that the operator we are using must be a good approximation to the lightest glueball wavefunctional. This is no accident; it has been obtained by a variational procedure which is an important ingredient in the successful lattice calculation of glueball masses, but one which I have no time to describe further here.

We have just seen that lattice calculations of masses are indeed possible. But are they accurate enough to permit a controlled extrapolation to the continuum limit? To demonstrate that the answer to this question is yes, I plot in Fig. 2 the (preliminary) $0^{++}$ masses obtained from the same calculation at four different values of $a$. The masses have been expressed in terms of the confining string tension, $a^2 \sigma$, which has also been also calculated, and the ratio is plotted against $a^2 \sigma$. As we note from eqn (4), the leading lattice correction is $O(a^2)$ which means that the continuum extrapolation at sufficiently small $a$ will be a straight line – as shown in the plot. This provides an example of a typical continuum extrapolation.
Figure 2. The lightest SU(4) scalar glueball mass, \( m_{0^{++}} \), expressed in units of the string tension, \( \sigma \), plotted against the latter in lattice units. The \( a \to 0 \) continuum extrapolation, using a leading lattice correction, is shown.

3 SU\((N_c)\) Glueball Masses

In \( \text{Fig. 2} \) we calculated the lightest and first excited \( 0^{++} \) glueball masses and the lightest tensor \( 2^{++} \) glueball mass. We took the ratio to the string tension and extrapolated to the continuum limit as described in Section \( \text{Fig. 3} \). We did this for SU(2), SU(3), SU(4) and SU(5) gauge theories. In Fig. 3 I plot these continuum mass ratios against \( 1/N_c^2 \). We expect the leading correction at large \( N_c \) to be \( O(1/N_c^2) \),

\[
\frac{m_i}{m_j}_{|_{N_c}} = \frac{m_i}{m_j}_{|_{\infty}} + \frac{c_{ij}}{N_c^2}, \tag{5}
\]

which is a simple straight line on our plot. Remarkably, as we see in Fig. 3, the mass ratios for all values of \( N_c \) can be described by just the leading correction and the corresponding coefficients are modest in magnitude.

This shows us that there is indeed a smooth large \( N_c \) limit with a finite confining string tension. Moreover for these quantities SU(3) is clearly close
to SU(∞). Indeed, so is SU(2). That is to say, SU(N_c) gauge theories are close to SU(∞) for all values of N_c.

4 't Hooft Coupling

We have seen that there is a smooth large-N_c limit. Is it achieved by keeping constant the 't Hooft coupling, λ ≡ g^2N_c, as suggested by the standard analysis of diagrams? In D=2+1 the coupling g^2 has dimensions of mass and the question is simply whether g^2N_c/√σ goes to a non-zero finite constant as N_c → ∞. The answer is found to be yes. Here in D=3+1 the coupling runs and is dimensionless. The question therefore becomes: is the smooth large N_c limit achieved by keeping fixed the running 't Hooft coupling, as defined on some scale l that is fixed in units of some quantity that partakes of the smooth large-N_c limit, such as the string tension? To test this we use eqn(3) which tells is that a suitable definition of a running 't Hooft coupling is

\[ \lambda_I(a) = g_I^2(a)N_c = \frac{2N_c^2}{\beta(\text{ReTr} U_p/N_c)} \]  

(6)

The extra factor involving the plaquette is a standard mean-field (or tadpole) improved version of β and the naïve λ(a) we would derive from it. Such improvements are customary because the naïve lattice coupling is known to
be very poor in the sense of having very large higher order corrections.

We extract $\lambda_I(a)$ and $a\sqrt{\sigma}$ for various values of $a$. The latter expresses $a$ in physical units so that a plot of $\lambda_I(a)$ against $a\sqrt{\sigma}$ is a plot of how the coupling runs. If the large $N_c$ limit requires a fixed 't Hooft coupling then we would expect that such plots tend to a fixed curve as $N_c \to \infty$. As we see in Fig. 4 not only does this seem to be the case, but the limit is already achieved at the smallest non-trivial values of $N_c$.

5 k-Strings

We can consider confining strings between static colour charges in arbitrary representations. However gluon screening means that the effective representation can be changed dynamically. It is therefore useful to label charges by their transformation properties under the centre of the group since gluons transform trivially under the centre. Suppose the source transforms by a factor of $z^k$ under a global gauge transformation $z$ belonging to the centre $Z_{N_c}$ of the SU($N_c$) group. Call the lightest confining string joining sources in this class the $k$-string. The usual string between quarks is the $k = 1$ string. What is the tension $\sigma_k$ of such a string as a function of $k$ and $N_c$? There are some
conjectures (see for details). For example, a form

$$\frac{\sigma_k}{\sigma_1} = \frac{\sin \frac{k \pi}{N_c}}{\sin \frac{\pi}{N_c}}. \quad (7)$$

has been conjectured in an M(-theory)QCD approach to QCD. Another relevant example is the old Casimir scaling hypothesis

$$\frac{\sigma_k}{\sigma} = \frac{k(N_c - k)}{N_c - 1} \quad (8)$$

as well. (Note we use $\sigma \equiv \sigma_1$ from now on.)

In Fig. I show the lattice values of $\sigma_{k=2}/\sigma$ as a function of $a^2 \sigma$ obtained in our recent SU(4) and SU(5) calculations. (With continuum extrapolations.) One sees that $\sigma_{k=2}/\sigma \ll 2$ i.e. non-trivial strongly bound $k$-strings do indeed exist. Moreover the continuum value lies between the Casimir scaling and MQCD conjectures, which are numerically very similar.

A more recent and more accurate calculation in SU(4) and SU(6) favours the MQCD conjecture. On the other hand our preliminary anisotropic lattice calculations seem to favour an intermediate value.
6 Corrections to the $N_c = \infty$ Limit?

Suppose that the $k$-string tension is eventually found to satisfy Casimir scaling, as in eqn(8). Consider a $k$-string wrapped around a spatial hypertorus of length $l$. This represents an energy eigenstate of the finite volume Hamiltonian with mass $m_k = l\sigma_k$ with corrections of $O(1/l^2)$ that we can neglect for large enough $l$. Assuming eqn(8) we obtain

$$m_k = l\sigma_k = l\sigma \frac{k(N_c - k)}{N_c - 1} = l\sigma \{ k - \frac{k(k-1)}{N_c} + \ldots \}. \quad (9)$$

Consider the $N_c$-dependence of this, keeping $l$ fixed in units of, say, $\sigma$. We see that the leading correction is $O(1/N_c)$. This contradicts the usual diagrammatic result that the leading correction in SU($N_c$) gauge theories should be $O(1/N_c^2)$.

Perhaps we should not regard a string of fixed $k$ as being the ‘same’ state as $N_c$ is varied. Then the above would not worry us. But suppose that (part of) the glueball spectrum arises from closed strings of flux, as for example in the Isgur-Paton flux tube model. As pointed out in one can form such loops out of $k$-strings as well as out of fundamental strings, leading to sectors of states that are scaled in mass by $\sigma_k/\sigma$. Eqn(8) tells us that such glueball states will then have $O(1/N_c)$ rather than $O(1/N_c^2)$ corrections as $N_c \to \infty$.

Of course we do not yet know whether $k$-strings satisfy Casimir scaling or not. (Note that the same issue arises in D=2+1.) But the general point is that we have an explicit example of how non-perturbative effects – string formation – might lead to a violation of the usual colour counting rules. This is interesting whether or not reality chooses to make use of this possibility.

7 Conclusions

I have not had time to discuss topology. Here one finds that the SU($\infty$) topological susceptibility is non-zero and not very different from the SU(3) one. This is important for our understanding of how the $\eta'$ gets its large mass. Moreover fluctuations that are unambiguously instanton-like disappear from the vacuum as $N_c$ grows. I have also not discussed the deconfining transition: the nature of this transition is being actively investigated.

What I have shown in this talk, using fully non-perturbative calculations, is that the large-$N_c$ limit is smooth, confining and is achieved precociously for many physical quantities. Not only is $N_c = 3$ close to $N_c = \infty$ but so is $N_c = 2$. There are new stable strings at larger $N_c$ and their string tensions are intriguingly close to both the MQCD and Casimir scaling conjectures.
One obtains the $N_c \to \infty$ limit by keeping fixed the ’t Hooft coupling. This is as expected. Not expected was the observation that the $k$-strings provide, in principle, an explicit avenue by which states can acquire anomalous $O(1/N_c)$ corrections.

Lattice calculations which will make our knowledge of $SU(N_c)$ gauge theories much more extensive are under way.

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I have learned a great deal from the talks and various discussions in this ‘interdisciplinary’ meeting; for example the observations made in Section 2 came to me while discussing my lattice calculations with Simon Dalley, Matt Strassler and Aneesh Manohar. My thanks to the organizers for inviting me.

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