A SPACE OF GENERALIZED BROWNIAN MOTION
PATH-VALUED CONTINUOUS FUNCTIONS WITH
APPLICATION

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Abstract. In this paper, we introduce the paths space $C_{0}^{gBm}$ which is consists of generalized Brownian motion path-valued continuous functions on $[0,T]$. We next present several relevant examples of the paths space integral. We then establish the existence of the analytic Feynman integral over the paths space for certain bounded cylinder functionals on $C_{0}^{gBm}$.

1. Introduction

Let $(\mathcal{B},\nu)$ denote an abstract Wiener space and let $C_{0}(\mathcal{B})$ be the space of $\mathcal{B}$-valued continuous functions $\mathbf{r}$ which are defined on $[0,T]$ with $\mathbf{r}(0) = 0$, see [39]. In [Ryu], Ryu improved several theories on $C_{0}(\mathcal{B})$ which are developed in classical and abstract Wiener spaces. Since then the concepts of the analytic Feynman integral and the analytic Fourier–Feynman transform, and related topics have been developed on the Wiener paths space $C_{0}(\mathcal{B})$ extensively; references include [7, 11, 12, 23, 37, 38, 43]. In [Ryu], Ryu suggested a cylinder measure $m_{\mathcal{B}}$ on the space $C_{0}(\mathcal{B})$ and constructed the general Wiener integration theorem: given a multi-dimensional tuple $(t_{1},t_{2},\ldots,t_{n}) \in \mathbb{R}^{n}$ with $0 = t_{0} < t_{1} < t_{2} < \cdots < t_{n} \leq T$, and a Borel measurable function $f: \mathbb{R}^{n} \to \mathbb{C}$,

\[
\int_{C_{0}(\mathcal{B})} f(\mathbf{r}(t_{1}),\mathbf{r}(t_{2}),\ldots,\mathbf{r}(t_{n}))dm_{\mathcal{B}}(\mathbf{r}) = \int_{\mathbb{R}^{n}} f\left(\sqrt{t_{1} - t_{0}x_{1}}, \sqrt{t_{1} - t_{0}x_{1}} + \sqrt{t_{2} - t_{1}x_{2}}, \ldots, \sum_{j=1}^{n} \sqrt{t_{j} - t_{j-1}x_{j}}\right) \times d\nu(x_{1},\ldots,x_{n})
\]

in the sense that if either side exists, both sides exist and the equality holds. The concrete formulation of the cylinder measure $m_{\mathcal{B}}$ and the applications to the theory of analytic Feynman integral, see [7, 11, 12, 23, 37, 38, 43] and the references cited therein.

On the other hand, in [13, 14, 15, 16, 18, 22, 24], the authors defined the generalized analytic Feynman integral and the generalized analytic Fourier–Feynman transform on the function space $C_{a,b}[0,T]$, and studied their properties with related topics. The function space $C_{a,b}[0,T]$, induced by a generalized Brownian motion process (GBMP), was introduced by Yeh in [44], and was used extensively in [17, 19, 20, 21, 25].

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A GBMP on a probability space \((\Omega, \Sigma, P)\) and a time interval \([0, T]\) is a Gaussian process \(Y \equiv \{Y_t\}_{t \in [0, T]}\) such that \(Y_0 = c\) almost surely for some constant \(c \in \mathbb{R}\), and for any set of time moments \(0 = t_0 < t_1 < \cdots < t_n \leq T\) and any Borel set \(B \subset \mathbb{R}^n\), the measure \(P(I_{t_1}, \ldots, t_n, B)\) of the cylinder set \(I_{t_1}, \ldots, t_n, B\) of the form \(I_{t_1}, \ldots, t_n, B = \{\omega : (Y_{t_1}(\omega), \ldots, Y_{t_n}(\omega)) \in B\}\) is given by

\[
P(I_{t_1}, \ldots, t_n, B) = \int_B K_n(t, \eta)\,d\eta_1 \cdots d\eta_n
\]

where

\[
K_n(t, \eta) = \left(\frac{2\pi}{n!}\right)^{n/2} \int_{\mathbb{R}^n} \prod_{j=1}^n (b(t_j) - b(t_{j-1}))^{-1/2}
\]

\[
x \exp \left\{ - \frac{1}{2} \sum_{j=1}^n \frac{(\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1}))^2}{b(t_j) - b(t_{j-1})} \right\}
\]

and where \(\eta_0 = c\), \(a(t)\) is a continuous real-valued function on \([0, T]\), and \(b(t)\) is an increasing continuous real-valued function on \([0, T]\). Thus, the GBMP \(Y\) is determined by the continuous functions \(a(\cdot)\) and \(b(\cdot)\). For more details, see [14, 15]. Note that when \(c = 0\), \(a(t) \equiv 0\) and \(b(t) = t\) on \([0, T]\), the GBMP reduces a standard Brownian motion (Wiener process).

We set \(c = a(0) = b(0) = 0\). Then the function space \(C_{a,b}[0, T]\) induced by the GBMP \(Y\) determined by \(a(\cdot)\) and \(b(\cdot)\) can be considered as the space of continuous sample paths of \(Y\), see [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25], and one can see that for each \(t \in [0, T]\),

\[
e_t(x) \sim N(a(t), b(t)),
\]

where \(e_t : C_{a,b}[0, T] \times [0, T] \to \mathbb{R}\) is the coordinate evaluation map given by \(e_t(x) = x(t)\) and \(N(m, \sigma^2)\) denotes the normal distribution with mean \(m\) and variance \(\sigma^2\).

We are obliged to point out that a standard Brownian motion is stationary in time and is free of drift, whereas a GBMP is generally not stationary in time and is subject to a drift \(a(t)\).

In this paper, we thus first attempt to construct the paths space \(C_{a,b}^{\text{GBm}} \equiv C_{a,b}^{\text{GBm}}(C_{a,b}[0, T])\) which is consisted of generalized Brownian motion path-valued continuous functions on \([0, T]\). We next present several relevant examples of the paths space integral. As an application, we then establish the existence of the analytic paths space Feynman integral of bounded cylinder functionals \(F\) of the form

\[
F(\bar{x}) = \int_{\mathbb{R}^m} \exp \left\{ i \sum_{j=1}^m \sum_{k=1}^n (g_j, \bar{x}(s_k)) - v_{j,k} \right\} d\nu(\bar{v}), \quad \bar{x} \in C_0
\]

where \(\nu\) is a complex Borel measure on \(\mathbb{R}^m\) and \((g, \bar{x}(s))\) denotes the Paley–Wiener–Zygmund (PWZ) stochastic integral. In Section 2 below we present a more detailed survey of paths space and a motivation of the topic in this paper.

2. Motivation I

Survey on the classical Wiener space \(C_0[0, T]\)

Given a positive real \(T > 0\), let \(C_0[0, T]\) denote one-parameter Wiener space, that is, the space of all real-valued continuous functions \(x\) on the interval \([0, T]\) with \(x(0) = 0\). Let \(\mathcal{M}\) denote the class of all Wiener measurable subsets of \(C_0[0, T]\).

and let $m_w$ denote Wiener measure. Then, as is well-known, $(C_0[0,T], \mathcal{M}, m_w)$ is a complete probability measure space. The coordinate process $\tilde{W} \equiv \{\tilde{W}_t\}_{t \in [0,T]}$ given by $\tilde{W}_t(x) = x(t)$ on $C_0[0,T] \times [0,T]$ is a Brownian motion. Thus Wiener measure $m_w$ is a Gaussian measure on $C_0[0,T]$ with mean zero and covariance function $r(s,t) = \min\{s,t\}$ in view of following illustration.

The Brownian motion (equivalently, Wiener process) on a probability space $(\Omega, \Sigma, P)$ and a time interval $[0,T]$ is a Gaussian process $W \equiv \{W_t\}_{t \in [0,T]}$ such that $W_0 = 0$ almost surely, and for any set of time moments $0 = t_0 < t_1 < \cdots < t_n \leq T$ and any Borel set $B \subset \mathbb{R}^n$, the measure $P(I_{t_1,\ldots,t_n,B})$ of the cylinder set $I_{t_1,\ldots,t_n,B}$ of the form

$$I_{t_1,\ldots,t_n,B} = \{\omega \in \Omega : (W_{t_1}(\omega), \ldots, W_{t_n}(\omega)) \in B\}$$

is given by

$$\left((2\pi)^n \prod_{j=1}^n (t_j - t_{j-1})\right)^{-1/2} \int_B \exp \left\{- \frac{1}{2} \sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}}\right\} du_1 \cdots du_n$$

where $u_0 = 0$. The coordinate process $\tilde{W} : C_0[0,T] \times [0,T] \to \mathbb{R}$ defined by $\tilde{W}(x,t) = W_t(x) = x(t)$ is also a Brownian motion. Thus the Wiener space $C_0[0,T]$ can be considered as the space of all sample paths of a Brownian motion. We observe that for any $t_1, t_2 \in [0,T]$ with $t_1 < t_2$,

$$\tilde{W}(x,t_2) - \tilde{W}(x,t_1) \sim N(0,t_2 - t_1),$$

where $N(m,\sigma^2)$ denotes the normal distribution with mean $m$ and variance $\sigma^2$. Given the time moments $0 = t_0 < t_1 < \cdots < t_n \leq T$, define a function $P_{(t_1,\ldots,t_n)} : C_0[0,T] \to \mathbb{R}^n$ by $P_{(t_1,\ldots,t_n)}(x) = (x(t_1), \ldots, x(t_n))$. Then the Wiener measure $m_w(I_{t_1,\ldots,t_n,B})$ of the cylinder set $I_{t_1,\ldots,t_n,B} = \{x \in C_0[0,T] : P_{(t_1,\ldots,t_n)}(x) \in B\}$ with a Borel set $B$ in $\mathbb{R}^n$ is given by (2.1). Furthermore, the probability distribution $m_w \circ P_{(t_1,\ldots,t_n)}^{-1}$ and the Lebesgue measure $m^*_L$ on $\mathbb{R}^n$ are mutually absolutely continuous. Thus the Radon–Nikodym derivative of $m_w \circ P_{(t_1,\ldots,t_n)}^{-1}$ with respect to $m^*_L$ is given by

$$\frac{m_w \circ P_{(t_1,\ldots,t_n)}^{-1}}{m^*_L}(u_1,\ldots,u_n) = \left((2\pi)^n \prod_{j=1}^n (t_j - t_{j-1})\right)^{-1/2} \exp \left\{- \frac{1}{2} \sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{t_j - t_{j-1}}\right\}$$

with $u_0 = 0$. In fact, for any subset $E$ of $\mathbb{R}^n$, $E$ is Lebesgue measurable if and only if $P_{(t_1,\ldots,t_n)}(E)$ is Wiener measurable. For more details, see [10] [33] and references cited therein.

For each $v \in L^2[0,T]$ and $x \in C_0[0,T]$, we let $\langle v, x \rangle$ denote the Paley–Wiener–Zygmund (PWZ) stochastic integral [34] [10] [41]. It is known that for each $v \in L^2[0,T]$, the PWZ stochastic integral $\langle v, x \rangle$ exists for $m_w$-a.s. $x \in C_0[0,T]$ and it is a Gaussian random variable with mean $0$ and variance $\|v\|_2^2$. It is also known that for $v_1, v_2 \in L^2[0,T]$,

$$\int_{C_0[0,T]} \langle v_1, x \rangle \langle v_2, x \rangle dm_w(x) = \langle v_1, v_2 \rangle_2$$
where \((\cdot, \cdot)_2\) denotes the \(L_2\)-inner product. Furthermore, if \(v \in L_2[0, T]\) is of bounded variation on \([0, T]\), then the PWZ stochastic integral \(\langle v, x \rangle\) equals the Riemann–Stieltjes integral \(\int_0^T v(t)dx(t)\).

The following Cameron–Martin space plays an important role in this paper. Let

\[
C'_0[0, T] = \left\{ w \in C_0[0, T] : w(t) = \int_0^t v(\tau)d\tau \text{ for some } v \in L_2[0, T] \right\}.
\]

Then \(C'_0 \equiv C'_0[0, T]\) is a real separable infinite dimensional Hilbert space with inner product

\[
(w_1, w_2)_{C'_0} = \int_0^T Dw_1(\tau)Dw_2(\tau)d\tau
\]

where \(Dw(\tau) = \frac{dm(\tau)}{d\tau}\). Given any \(w \in C'_0[0, T]\), we use the notation \((w, x)^\sim\) to denote the PWZ stochastic integral \(\langle Dw, x \rangle\). Then for \(w, x \in C'_0[0, T]\), \((w, x)^\sim = (w, x)_{C'_0}\) and equation (2.3) above can be rewritten as follows: for \(w_1, w_2 \in C'_0[0, T]\),

\[
\int_{C'_0[0, T]} (w_1, x)^\sim(w_2, x)^\sim dm_w(x) = (w_1, w_2)_{C'_0}.
\]

For each \(t \in [0, T]\), let

\[
\beta_t(s) = \int_0^s \chi_{[0,t]}(\tau)d\tau = \begin{cases} s, & 0 \leq s \leq t \\ t, & t < s \leq T \end{cases}.
\]

Then the family of functions \(\{\beta_t : 0 \leq t \leq T\}\) from \(C'_0[0, T]\) has the reproducing property

\[
(w, \beta_t)_{C'_0} = w(t)
\]

for all \(w \in C'_0[0, T]\). Note that \(\beta_t(s) = \min\{s, t\}\), the covariance function of the Brownian motion \(\bar{W}\) discussed above. We also note that for each \((x, t) \in C_0[0, T] \times [0, T]\),

\[
\int_{C_0[0,T]} \chi_{[0,t]}(\tau)d\tau = (\beta_t, x)^\sim.
\]

We will discuss the Wiener integral of three kinds of tame functions on \(C_0[0, T]\). Given an \(n\)-tuple \((t_1, \ldots, t_n)\) of time moments with \(0 = t_0 < t_1 < \cdots < t_n \leq T\), let \(F : C_0[0, T] \to \mathbb{C}\) be a tame function given by

\[
F(x) = f(x(t_1), x(t_2), \ldots, x(t_n))
\]

where \(f : \mathbb{R}^n \to \mathbb{C}\) is a Lebesgue measurable function. Then applying equation (2.2), it follows that

\[
\int_{C_0[0,T]} F(x)dm_w(x)
\]

\[
= \int_{C_0[0,T]} f(x(t_1), x(t_2), \ldots, x(t_n))dm_w(x)
\]

\[
= \left(\frac{2\pi}{n} \prod_{j=1}^n (t_j - t_{j-1})\right)^{-1/2}
\]

\[
\times \int_{\mathbb{R}^n} f(u_1, u_2, \ldots, u_n) \exp \left\{ -\frac{1}{2} \sum_{j=1}^n \frac{[u_j - u_{j-1}]^2}{t_j - t_{j-1}} \right\} d\bar{u}
\]
where $u_0 = 0$. For each $j \in \{1, 2, \ldots, n\}$, let $\alpha_j(\tau) = \chi_{[0, t_j]}(\tau)$. Then $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a linearly independent set of functions in $L_2[0, T]$, and $\langle \alpha_j, x \rangle = x(t_j)$ for each $j \in \{1, 2, \ldots, n\}$.

Next we consider the second kind of tame function $F$ on $C_0[0, T]$ given by

$$F(x) = f(x(t_1), x(t_2) - x(t_1), \ldots, x(t_n) - x(t_{n-1})).$$

For each $j \in \{1, 2, \ldots, n\}$, let

$$X_j(x) = x(t_j) - x(t_{j-1}).$$

Then $X_j$’s form a set of independent Gaussian random variables such that $X_j \sim N(0, t_j - t_{j-1})$ for each $j \in \{1, 2, \ldots, n\}$. Thus, by the change of variables theorem, it follows that

$$\int_{C_0[0, T]} F(x)dm_w(x) = \int_{C_0[0, T]} f(x(t_1), x(t_2) - x(t_1), \ldots, x(t_n) - x(t_{n-1}))dm_w(x) = \left(2\pi\right)^{n/2} \prod_{j=1}^{n} (t_j - t_{j-1})^{-1/2} \int_{\mathbb{R}^n} f(v_1, \ldots, v_n) \exp \left\{-\frac{1}{2} \sum_{j=1}^{n} \frac{v_j^2}{t_j - t_{j-1}}\right\}d\vec{v}.$$  

For each $j \in \{1, 2, \ldots, n\}$, let $\delta_j(\tau) = \chi_{[t_{j-1}, t_j]}(\tau)$. Then $\{\delta_1, \delta_2, \ldots, \delta_n\}$ is an orthogonal set of functions in $L_2[0, T]$, and $\langle \delta_j, x \rangle = X_j(x)$ for each $j \in \{1, 2, \ldots, n\}$.

Finally the third kind of tame function $F$ we consider is given by

$$F(x) = f\left(\frac{x(t_1)}{\sqrt{t_1}}, \frac{x(t_2) - x(t_1)}{\sqrt{t_2 - t_1}}, \ldots, \frac{x(t_n) - x(t_{n-1})}{\sqrt{t_n - t_{n-1}}}\right).$$

For each $j \in \{1, 2, \ldots, n\}$, let

$$Y_j(x) = \frac{x(t_j) - x(t_{j-1})}{\sqrt{t_j - t_{j-1}}}.$$  

Then $Y_j$’s form a set of i.i.d. Gaussian random variables. We note that for each $j \in \{1, 2, \ldots, n\}$, $Y_j \sim N(0, 1)$. Thus, by the change of variables theorem, it follows that

$$\int_{C_0[0, T]} F(x)dm_w(x) = \int_{C_0[0, T]} f\left(\frac{x(t_1)}{\sqrt{t_1}}, \frac{x(t_2) - x(t_1)}{\sqrt{t_2 - t_1}}, \ldots, \frac{x(t_n) - x(t_{n-1})}{\sqrt{t_n - t_{n-1}}}\right)dm_w(x) = \int_{\mathbb{R}^n} f(w_1, \ldots, w_n)\nu_G(d\vec{w}),$$

where $\nu_G$ is the standard Gaussian measure on $\mathbb{R}^n$ given by

$$\nu_G(d\vec{w}) = (2\pi)^{-n/2} \exp \left\{-\frac{1}{2} \sum_{j=1}^{n} w_j^2\right\}dw_1 \cdots dw_n.$$  

For each $j \in \{1, 2, \ldots, n\}$, let $\gamma_j(\tau) = (t_j - t_{j-1})^{-1/2} \chi_{[t_{j-1}, t_j]}(\tau)$. Then $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ is an orthonormal set of functions in $L_2[0, T]$, and $\langle \gamma_j, x \rangle = Y_j(x)$ for each $j \in \{1, 2, \ldots, n\}$.
In the last expression of (2.7), we consider the following transformation \( S : \mathbb{R}^n \to \mathbb{R}^n \) given by
\[
(2.10) \quad (v_1, v_2, \ldots, v_n) \xrightarrow{S} (u_1, u_2, \ldots, u_n) = (v_1 + v_2, \ldots, v_1 + v_2 + \cdots + v_n).
\]
Then it follows that
\[ v_j = u_j - u_{j-1} \]
for all \( j \in \{1, \ldots, n\} \) and
\[
\mathcal{J} \left( \frac{u_1, \ldots, u_n}{v_1, \ldots, v_n} \right) = 1
\]
where \( \mathcal{J} \) denotes the Jacobi symbol. In these setting, equation (2.7) can be rewritten by
\[
(2.11) \quad \int_{C_0[0, T]} f(x(t_1), x(t_2), \ldots, x(t_n)) dm_w(x)
= \left( \prod_{j=1}^{n} 2\pi (t_j - t_{j-1}) \right)^{-1/2} \times \int_{\mathbb{R}^n} f(v_1, v_1 + v_2, \ldots, v_1 + v_2 + \cdots + v_n) \exp \left\{ -\frac{1}{2} \sum_{j=1}^{n} \frac{v_j^2}{t_j - t_{j-1}} \right\} d\nu.
\]
Next, in the last expression of (2.7), we consider the following transformation \( T : \mathbb{R}^n \to \mathbb{R}^n \) given by
\[
(2.12) \quad (w_1, w_2, \ldots, w_n) \xrightarrow{T} (u_1, u_2, \ldots, u_n)
= \left( \sqrt{t_1 - t_0} w_1, \sqrt{t_1 - t_0} w_1 + \sqrt{t_2 - t_1} w_2, \ldots, \sum_{j=1}^{n} \sqrt{t_j - t_{j-1}} w_j \right).
\]
Then it follows that
\[ w_j = \frac{u_j - u_{j-1}}{\sqrt{t_j - t_{j-1}}} \]
for each \( j \in \{1, \ldots, n\} \) and
\[
\mathcal{J} \left( \frac{u_1, \ldots, u_n}{w_1, \ldots, w_n} \right) = \prod_{j=1}^{n} \sqrt{t_j - t_{j-1}}.
\]
In these setting, equation (2.7) can also be rewritten by
\[
(2.13) \quad \int_{C_0[0, T]} f(x(t_1), \ldots, x(t_n)) dm_w(x)
= \int_{\mathbb{R}^n} f \left( \sqrt{t_1 - t_0} w_1, \sqrt{t_1 - t_0} w_1 + \sqrt{t_2 - t_1} w_2, \ldots, \sum_{j=1}^{n} \sqrt{t_j - t_{j-1}} w_j \right) d\nu_G(\vec{w}),
\]
where \( \nu_G \) is the standard Gaussian measure given by (2.9).

**Remark 2.1.** One can see that the general Wiener integration theorem (1.1) for measurable functionals \( f \) on the Wiener paths space \( C_0(\mathcal{B}) \) is a natural extension of (2.13).
3. Preliminaries

In this section, we present the brief backgrounds which are needed in the following sections.

Let \( a(t) \) be an absolutely continuous real-valued function on \([0, T]\) with \( a(0) = 0 \) and \( a'(t) \in L^2[0, T] \), and let \( b(t) \) be a strictly increasing, continuously differentiable real-valued function with \( b(0) = 0 \) and \( b'(t) > 0 \) for each \( t \in [0, T] \). The GBMP \( Y \) determined by \( a(t) \) and \( b(t) \) is a Gaussian process with mean function \( a(t) \) and covariance function \( r(s, t) = \min\{b(s), b(t)\} \). For more details, see [16, 20, 22, 44, 45]. Applying [45, Theorem 14.2], one can construct a probability measure space \((C_{a,b}[0, T], \mathcal{B}(C_{a,b}[0, T]), \mu)\) where \( C_{a,b}[0, T] \) is the space of continuous sample paths of (a separable version of) the GBMP \( Y \) (it is equivalent to the Banach space of continuous functions \( x \) on \([0, T]\) with \( x(0) = 0 \) under the sup norm) and \( \mathcal{B}(C_{a,b}[0, T]) \) is the Borel \( \sigma \)-field of \( C_{a,b}[0, T] \) induced by the sup norm. We then complete this function space to obtain the complete probability measure space \((C_{a,b}[0, T], \mathcal{W}(C_{a,b}[0, T]), \mu)\) where \( \mathcal{W}(C_{a,b}[0, T]) \) is the set of all Wiener measurable subsets of \( C_{a,b}[0, T] \).

**Remark 3.1.** The coordinate process \( e : C_{a,b}[0, T] \times [0, T] \to \mathbb{R} \) defined by \( e(x, t) \equiv e_t(x) = x(t) \) is also the GBMP determined by \( a(t) \) and \( b(t) \).

**Remark 3.2.** Let \( C_{a,b}^n[0, T] \) be the product of \( n \) copies of \( C_{a,b}[0, T] \). Since the space \( C_{a,b}[0, T] \) endowed with the uniform topology is separable, the Borel \( \sigma \)-field \( \mathcal{B}(C_{a,b}[0, T]) \) is the space of continuous functions \( x \) on \([0, T]\) with \( x(0) = 0 \) under the sup norm) and \( \mathcal{B}(C_{a,b}[0, T]) \) is the Borel \( \sigma \)-field of \( C_{a,b}[0, T] \) induced by the sup norm. We then complete this function space to obtain the complete probability measure space \((C_{a,b}[0, T], \mathcal{W}(C_{a,b}[0, T]), \mu)\) where \( \mathcal{W}(C_{a,b}[0, T]) \) is the set of all Wiener measurable subsets of \( C_{a,b}[0, T] \).

Let \( L^2_{a,b}[0, T] \) be the space of functions on \([0, T]\) which are Lebesgue measurable and square integrable with respect to the Lebesgue–Stieltjes measures on \([0, T]\) induced by \( a(\cdot) \) and \( b(\cdot) \); i.e.,

\[
L^2_{a,b}[0, T] = \left\{ v : \int_0^T v^2(s) db(s) < +\infty \text{ and } \int_0^T v^2(s) db(s) < +\infty \right\}
\]

where \( |a(\cdot)| \) denotes the total variation function of \( a(\cdot) \). Then \( L^2_{a,b}[0, T] \) is a separable Hilbert space with inner product defined by

\[
(u, v)_{a,b} = \int_0^T u(t)v(t) dm_{|a|, b}(t) \equiv \int_0^T u(t)v(t) db(t) + |a|(t),
\]

where \( m_{|a|, b} \) denotes the Lebesgue–Stieltjes measure induced by \( |a|(\cdot) \) and \( b(\cdot) \). In particular, note that \( \|u\|_{a,b} \equiv \sqrt{(u, u)_{a,b}} = 0 \) if and only if \( u(t) = 0 \) a.e. on \([0, T]\). Furthermore, \( (L^2_{a,b}[0, T], \| \cdot \|_{a,b}) \) is a separable Hilbert space.

Next, let

\[
C'_{a,b}[0, T] = \left\{ w \in C_{a,b}[0, T] : w(t) = \int_0^t z(s) db(s) \text{ for some } z \in L^2_{a,b}[0, T] \right\}.
\]
For \( w \in C'_{a,b}[0,T] \), with \( w(t) = \int_0^t z(s)db(s) \) for \( t \in [0,T] \), let \( D : C'_{a,b}[0,T] \to L^2_{a,b}[0,T] \) be defined by the formula

\[
Dw(t) = z(t) = \frac{w'(t)}{\theta(t)}.
\]

Then \( C'_{a,b} \equiv C'_{a,b}[0,T] \) with inner product

\[
(w_1, w_2)_{C'_{a,b}} = \int_0^T Dw_1(t)Dw_2(t)dt
\]

is also a separable Hilbert space.

**Remark 3.3.** Note that the two separable Hilbert spaces \( L^2_{a,b}[0,T] \) and \( C'_{a,b}[0,T] \) are (topologically) homeomorphic under the linear operator \( D \) given by equation (3.1). The inverse operator of \( D \) is given by \((D^{-1}z)(t) = \int_0^t z(s)db(s) \) for \( t \in [0,T]\). But the linear operator \( D \) is not isometric.

In this paper, in addition to the conditions put on \( a(t) \) above, we now add the condition

\[
\int_0^T |a'(t)|^2d|a|(t) < +\infty.
\]

Then, the function \( a : [0,T] \to \mathbb{R} \) satisfies the condition (3.2) if and only if \( a(\cdot) \) is an element of \( C'_{a,b}[0,T] \). Under the condition (3.2), we observe that for each \( w \in C'_{a,b}[0,T] \) with \( Dw = z \),

\[
(w, a)_{C'_{a,b}} = \int_0^T Dw(t)Da(t)dt = \int_0^T z(t)\frac{a'(t)}{\theta(t)}db(t) = \int_0^T z(t)da(t).
\]

For each \( w \in C'_{a,b}[0,T] \) and \( x \in C_{a,b}[0,T] \), we let \( (w,x)^{\sim} \) denote the PWZ stochastic integral [13, 25]. It is known that for each \( w \in C'_{a,b}[0,T] \), the PWZ stochastic integral \( (w,x)^{\sim} \) exists for s-a.e. \( x \in C_{a,b}[0,T] \) and it is a Gaussian random variable with mean \( (w,a)_{C'_{a,b}} \) and variance \( \|w\|_{C'_{a,b}}^2 \). It also follows that for \( w, x \in C'_{a,b}[0,T] \),

\[
(w, x)^{\sim} = (w, x)_{C'_{a,b}}
\]

and that for \( w_1, w_2 \in C'_{a,b}[0,T] \),

\[
\int_{C_{a,b}[0,T]} (w_1, x)^{\sim}(w_2, x)^{\sim}d\mu(x) = (w_1, w_2)_{C'_{a,b}} + (w_1, a)_{C'_{a,b}}(w_2, a)_{C'_{a,b}}.
\]

Thus the random variable \( (w, x)^{\sim} \) is normally distributed with

\[
(w, x)^{\sim} \sim N((w, a)_{C'_{a,b}}, \|w\|_{C'_{a,b}}^2).
\]

Furthermore, if \( Dw = z \in L^2_{a,b}[0,T] \) is of bounded variation on \([0,T]\), the PWZ stochastic integral \( (w, x)^{\sim} \) equals the Riemann–Stieltjes integral \( \int_0^T z(t)dx(t) \).

For each \( t \in [0,T] \), let

\[
\beta_t(s) = \int_0^s \chi_{[0,t]}(\tau)db(\tau) = \begin{cases} b(s), & 0 \leq s \leq t \\ b(t), & t \leq s \leq T \end{cases}.
\]
Then the family of functions $\{\beta_t : 0 \leq t \leq T\}$ from $C^t_{a,b}[0,T]$ has the reproducing property

$$(w, \beta_t)_{C^t_{a,b}} = w(t)$$

for all $w \in C^t_{a,b}[0,T]$. Note that for any $s, t \in [1, 2]$, $\beta_t(s) = \min\{b(s), b(t)\}$, the covariance function associated with the generalized Brownian motion $Y$ used in this paper. We also note that for each $x \in C_{a,b}[0,T]$,

$$(3.6) \quad x(t) = \int_0^T \chi_{[0,t]}(\tau)d\tau = (\beta_t, x)^\sim.$$

Using the change of variable theorem, it follows the function space integration formula:

$$(3.7) \quad \int_{C_{a,b}[0,T]} \exp\{\rho(w, x)^\sim\}d\mu(x) = \exp\left\{\frac{\rho^2}{2}\|w\|_{C_{a,b}}^2 + \rho(a, w)_{C_{a,b}}\right\}$$

for every $\rho > 0$.

4. Motivation II
Change of variables theorem on the function space $C_{a,b}[0,T]$

We shall discuss a change of variables theorem, such as (2.13), on the function space $C_{a,b}[0,T]$.

Given an $n$-tuple $(t_1, \ldots, t_n)$ of time moments with $0 = t_0 < t_1 < \cdots < t_n \leq T$, let $F : C_{a,b}[0,T] \to \mathbb{C}$ be a tame function given by

$$F(x) = f(x(t_1), x(t_2), \ldots, x(t_n))$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a Lebesgue measurable function. We consider the following transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given by

$$(4.1) \quad T_{\vec{t}}(\vec{w}) = \left(L_{\vec{t},1}(\vec{w}), L_{\vec{t},2}(\vec{w}), \ldots, L_{\vec{t},n}(\vec{w})\right)$$

where

$$(4.2) \quad L_{\vec{t},k}(\vec{w}) = \sum_{l=1}^k \sqrt{b(s_l) - b(s_{l-1})}(w_l - a(t_l)) + a(t_k)$$

for each $k = 1, \ldots, n$. Let $\vec{u} = T_{\vec{t}}(\vec{w})$. Then it follows that for each $j \in \{1, 2, \ldots, n\}$,

$$\sqrt{b(t_j) - b(t_{j-1})}(w_j - a(t_j)) = (u_j - a(t_j)) - (u_{j-1} - a(t_{j-1}))$$

or, equivalently,

$$(4.3) \quad (w_j - a(t_{j-1})) = \frac{u_j - a(t_j) - (u_{j-1} - a(t_{j-1}))}{\sqrt{b(t_j) - b(t_{j-1})}}.$$

In this case, we see that

$$(4.4) \quad J\left(\frac{u_1, \ldots, u_n}{w_1, \ldots, w_n}\right) = \prod_{j=1}^n \sqrt{b(t_j) - b(t_{j-1})}.$$
Using (2.1), (4.3) and (4.4), it follows that
\[
\int_{C_{a,b}[0,T]} f(x(t_1), \ldots, x(t_n))d\mu(x)
= \int_{\mathbb{R}^n} f(u_1, \ldots, u_n)K_n(\tilde{t}, \tilde{u})d\tilde{u}
= \int_{\mathbb{R}^n} f(u_1, \ldots, u_n)\left(\prod_{j=1}^{n} 2\pi(b(t_j) - b(t_{j-1}))\right)^{-1/2}
\times \exp\left\{-\frac{1}{2} \sum_{j=1}^{n} \frac{(u_j - a(t_j)) - (u_{j-1} - a(t_{j-1}))^2}{b(t_j) - b(t_{j-1})}\right\}d\tilde{u}
\]
\[
= \int_{\mathbb{R}^n} f(T_{\tilde{t}}(w_1, \ldots, w_n)) \exp\left\{-\frac{1}{2} \sum_{j=1}^{n} (w_j - a(t_{j-1}))^2\right\}
\times \left(\prod_{j=1}^{n} 2\pi(b(t_j) - b(t_{j-1}))\right)^{-1/2} \left|f\left(\frac{u_1, \ldots, u_n}{w_1, \ldots, w_n}\right)\right|d\tilde{w}
\]
\[
= (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(T_{\tilde{t}}(w_1, \ldots, w_n)) \exp\left\{-\frac{1}{2} \sum_{j=1}^{n} (w_j - a(t_{j-1}))^2\right\}d\tilde{w}
\]
\[
= \int_{\mathbb{R}^n} f(T_{\tilde{t}}(w_1, \ldots, w_n))d\nu^{a,\tilde{t}}_G(w),
\]
where \(\nu^{a,\tilde{t}}_G\) is the Gaussian measure on \(\mathbb{R}^n\) (with mean vector \((a(t_1), a(t_2), \ldots, a(t_n))\)) given by
\[
\nu^{a,\tilde{t}}_G(B) = (2\pi)^{-n/2} \int_B \exp\left\{-\frac{1}{2} \sum_{j=1}^{n} (w_j - a(t_{j-1}))^2\right\}d\tilde{w}
\]
for \(B \in \mathbb{R}^n\).

In view of this observation we will construct a paths space associated with the GBMP determined by the continuous function \(a\) and \(b\).

5. The paths space \(C_0\)

Let \(C^\text{Bm}_0 \equiv C^\text{Bm}_0(C_{a,b}[0,T])\) be the class of all \(C_{a,b}[0,T]\)-valued continuous functions \(x\) on the compact interval \([0,T]\) with \(x(0) = 0\). From [39] it follows that the class \(C^\text{Bm}_0\) is a real separable Banach space with the norm
\[
\|x\|_{C_0} = \sup_{s \in [0,T]} \|x(s)\|_{C_{a,b}[0,T]}
\]
and the minimal \(\sigma\)-field making the mapping \(x \rightarrow x(s)\) measurable is the Borel \(\sigma\)-field \(\mathcal{B}(C^\text{Bm}_0)\) on \(C^\text{Bm}_0\). Furthermore, the generalized Brownian motion process in \(C_{a,b}[0,T]\) induces a probability measure \(\mu^{\text{Bm}}\) on \((C^\text{Bm}_0, \mathcal{W}(C^\text{Bm}_0))\) where \(\mathcal{W}(C^\text{Bm}_0)\) is the complete \(\sigma\)-field in the sense of Carathéodory extension on the Borel \(\sigma\)-field \(\mathcal{B}(C^\text{Bm}_0)\). We will introduce a concrete form of \(\mu^{\text{Bm}}\). Let \(\tilde{s} = (s_1, \ldots, s_n)\) be given with \(0 = s_0 < s_1 < \cdots < s_n \leq T\), and let \(T_{\tilde{s}} : C^\text{Bm}_0[0,T] \rightarrow C^\text{Bm}_0[0,T]\) be defined by
\[
(T_{\tilde{s}}(\tilde{x}))_i = T_{\tilde{s}}(x_1, \ldots, x_n) = (L_{\rho,1}(\tilde{x}), L_{\rho,2}(\tilde{x}), \ldots, L_{\rho,n}(\tilde{x}))
\]
where
\begin{align}
L_{\tilde{\sigma},k}(\tilde{x}) &= \sum_{l=1}^{k} \sqrt{b(s_l) - b(s_{l-1})}(x_l - a) + a
\end{align}
for each \( k = 1, \ldots, n \). Let \( \mu^n = \times^n \mu = \mu \times \cdots \times \mu \) be the product measure on the product function space \( \mathcal{C}_{a,b}^n[0, T] \). We then define a set function \( \mu_\tilde{\sigma} \) on \( \mathcal{B}(\mathcal{C}_{a,b}^n[0, T]) \)
by
\begin{align}
\mu_\tilde{\sigma}(E) &= \mu^n(T^{-1}_\tilde{\sigma}(E))
\end{align}
for every \( E \in \mathcal{B}(\mathcal{C}_{a,b}^n[0, T]) \). Then \( \mu_\tilde{\sigma} \) is a Borel measure. Next let \( \Psi_\tilde{\sigma} : \mathcal{C}_0^{gBm} \to \mathcal{C}_{a,b}^n[0, T] \) be the function with
\begin{align}
\Psi_\tilde{\sigma}(\tilde{r}) &= (r(s_1), r(s_2), \ldots, r(s_n)).
\end{align}
For Borel subsets \( B_1, B_2, \ldots, B_n \) in \( \mathcal{B}(\mathcal{C}_{a,b}[0, T]) \), \( \Psi^{-1}_\tilde{\sigma}(\prod_{l=1}^{n} B_l) \) is called a cylinder set with respect to \( B_1, B_2, \ldots, B_n \). For each positive integer \( n \) and the cylinder function \( \Psi_\tilde{\sigma} \) given by (2.12) with \( \tilde{s} = (s_1, \ldots, s_n) \), let
\[ I_{\tilde{s}} = \left\{ \Psi^{-1}_\tilde{\sigma}(\prod_{l=1}^{n} B_l) : B_1, B_2, \ldots, B_n \in \mathcal{B}(\mathcal{C}_{a,b}[0, T]) \right\} \]
and let \( I = \cup I_{\tilde{s}} \) where the union is over all ordered multidimensional tuples \( \tilde{s} \). Then for each \( G \in I \), there is a multidimensional tuple \( \tilde{s} = (s_1, \ldots, s_n) \) with \( 0 = s_0 < s_1 < \cdots < s_n \leq T \) such that \( G \in I_{\tilde{s}} \). Given any positive integer \( n \) and Borel subsets \( B_1, B_2, \ldots, B_n \) in \( \mathcal{B}(\mathcal{C}_{a,b}[0, T]) \), we define a set function \( \mu_{c_{0}^{gBm}} \) on \( I \) by
\begin{align}
\mu_{c_{0}^{gBm}}\left( \Psi^{-1}_\tilde{\sigma}(\prod_{l=1}^{n} B_l) \right) &= \mu_\tilde{\sigma}(\prod_{l=1}^{n} B_l).
\end{align}
Then \( \mu_{c_{0}^{gBm}} \) is well-defined and is countably additive on \( I \). Using the Carathéodory extension process, it can be extended on the \( \sigma \)-field \( \mathcal{W}(c_{0}^{gBm}) \), where \( \mathcal{W}(c_{0}^{gBm}) \) is the completion of the Borel \( \sigma \)-field \( \mathcal{B}(c_{0}^{gBm}) \). The extended measure on \( \mathcal{W}(c_{0}^{gBm}) \) will be again denoted by \( \mu_{c_{0}^{gBm}} \). Hence we have the measure space \((c_{0}^{gBm}, \mathcal{W}(c_{0}^{gBm}), \mu_{c_{0}^{gBm}})\). This measure space is called the space of generalized Wiener paths.

**Remark 5.1.** The transform \( T_{\tilde{\sigma}} \) given by (5.1) is formulated based on the transform (2.12) together with the probability law of the cylinder function (5.4).

Applying the techniques similar to those used in the proof of the Kolmogorov extension theorem [45], pp.4–17, we obtain the following two lemmas.

**Lemma 5.2.** For each multidimensional tuple \( \tilde{s} \) with \( 0 = s_0 < s_1 < \cdots < s_n \leq T \), the class \( I_{\tilde{s}} \) is a \( \sigma \)-field. Furthermore, the class \( I = \cup I_{\tilde{s}} \) is a field of subsets of \( c_{0}^{gBm} \).

**Lemma 5.3.** The set function \( \mu_{c_{0}^{gBm}} \) is well-defined and is countably additive on the field \( I \). Furthermore, \( \mu_{c_{0}^{gBm}} \) can be extended uniquely to be a probability measure on the \( \sigma \)-field \( \sigma(I) \) generated by \( I \).

**Remark 5.4.** The \( \sigma \)-field \( \sigma(I) \) generated by \( I \) coincides with the Borel \( \sigma \)-field \( \mathcal{B}(c_{0}^{gBm}) \). Thus \( \mathcal{W}(c_{0}^{gBm}) \) is the completion of \( \sigma(I) \).
Using the Carathéodory extension process, we also obtain the following lemma.

**Lemma 5.5.** The measure $m_{C_0}$ can be extended uniquely on the complete σ-field $\mathcal{W}(C_0^{\text{gBm}})$.

The extended measure on $\mathcal{W}(C_0^{\text{gBm}})$ will be again denoted by $\mu_{C_0^{\text{gBm}}}$. Hence we have the complete measure space $(C_0^{\text{gBm}}, \mathcal{W}(C_0^{\text{gBm}}), \mu_{C_0^{\text{gBm}}})$. This measure space is called the space of GBMP paths (henceforth, GBMP paths space or paths space).

Now, we introduce a paths space integration theorem on the paths space $C_0^{\text{gBm}}$.

**Theorem 5.6** (Paths Space Integration Theorem). Let $\vec{s} = (s_1, \ldots, s_n)$ be given with $0 = s_0 < s_1 < \cdots < s_n \leq T$ and let $F : C_{a,b}^n[0,T] \to \mathbb{C}$ be a $\mathcal{W}(C_{a,b}^n[0,T])$-measurable function. Then

$$\int_{C_0^{\text{gBm}}} F(x_1, \ldots, x_n) d\mu_{C_0^{\text{gBm}}}(x) = \int_{C_{a,b}^n[0,T]} F(T_x(x_1, \ldots, x_n)) d\mu(x_1, \ldots, x_n),$$

where $=*$ means that if either side exists, both sides exist and equality holds.

6. **Proof of the paths space integration theorem**

In order to prove the paths space integration theorem, we need the following lemmas.

**Lemma 6.1** (27). Let $T$ be a measurable transform from a measure space $(X, S, \mu)$ into a measurable space $(Y, T)$, and let $g$ be an extended real valued measurable function on $Y$. Then

$$\int_Y g(y) d\mu \circ T^{-1}(y) = \int_X g(T(x)) d\mu(x)$$

in the sense that if either integral exists, then both sides exist and they are equal.

**Lemma 6.2.** Let $\vec{s} = (s_1, \ldots, s_n)$ be a multidimensional tuple with $0 = s_0 < s_1 < \cdots < s_n \leq T$. Then,

(i) $\Psi^{-1}_\vec{s}(B)$ is in $\mathcal{B}(C_0^{\text{gBm}})$ for every $B \in \mathcal{B}(C_{a,b}^n[0,T])$.

(ii) For a subset $B$ of $C_{a,b}^n[0,T]$ with $\Psi^{-1}_\vec{s}(B) \in \mathcal{B}(C_0^{\text{gBm}})$, $B$ is in $\mathcal{B}(C_{a,b}^n[0,T])$.

**Proof.** (i) The cylinder function $\Psi^{-1}_\vec{s}$ given by (5.24) is continuous, it is $\mathcal{B}(C_0^{\text{gBm}}) - \mathcal{B}(C_{a,b}^n[0,T])$-measurable. Hence $\Psi^{-1}_\vec{s}(B) \in \mathcal{B}(C_0^{\text{gBm}})$ for any $B \in \mathcal{B}(C_{a,b}^n[0,T])$.

(ii) Given a multidimensional tuple $\vec{s} = (s_1, \ldots, s_n)$ with $0 = s_0 < s_1 < \cdots < s_n \leq T$, define a map $H_\vec{s} : C_{a,b}^n[0,T] \to C_0^{\text{gBm}}$ by $H_\vec{s}(\vec{F}) \equiv H_\vec{s}(x_1, \ldots, x_n)$ to be the
polyhedral path in \( C_{0}^{\beta m} \) such as
\[
H_{x}(x_{1},\ldots,x_{n})(s) = \begin{cases} 
  x_{j-1} + \frac{s-s_{j-1}}{s_{j}-s_{j-1}}(x_{j} - x_{j-1}), & s \in [s_{j-1}, s_{j}], \quad j = 1,\ldots,n, \\
  x_{n}, & s \in [s_{n}, T] 
\end{cases}
\]
where \( x_{0} = 0 \) (the zero function on \([0,T]\)).

We note that the Borel \( \sigma \)-field \( B(C_{0}^{\beta m}) \) (resp. \( B(C_{a,b}[0,T]) \)) can be generated by the uniform topology induced by the sup norm \( \| \cdot \|_{C_{0}^{\beta m}} \) (resp. \( \| \cdot \|_{C_{a,b}[0,T]} \)). Given \( \vec{x} = (x_{1},\ldots,x_{n}) \in C_{a,b}^{n}[0,T] \), let \( \langle \vec{x}_{n} \rangle = ((x_{1},\ldots,x_{n})) \) be a sequence in \( C_{a,b}^{n}[0,T] \) which converges to \( \vec{x} \), i.e., for each \( j = 1,\ldots,n \), \( \lim_{n \to \infty} x_{k,j} = x_{j} \) in \( (C_{a,b}[0,T], \| \cdot \|_{C_{a,b}[0,T]}) \). From this and the definition of \( H_{x} \), it follows that \( H_{x}(\vec{x}_{n}) \) converges to \( H_{x}(\vec{x}) \), uniformly on \([0,T] \), as \( k \to \infty \). Hence the map \( H_{x} : (C_{a,b}^{n}[0,T], B(C_{a,b}^{n}[0,T])) \to (C_{0}^{\beta m}, B(C_{0}^{\beta m})) \) is continuous, and so is \( B(C_{a,b}^{n}[0,T]) \). By Remark 6.3, \( B(C_{a,b}^{n}[0,T]) \) is \( B(C_{0}^{\beta m}) \)-measurable. It thus follows that \( H_{x}^{-1}(\mathcal{Q}_{x}^{-1}(B)) \in B(C_{a,b}^{n}[0,T]) \) for each \( B \in B(C_{a,b}^{n}[0,T]) \). To complete the proof of the assertion (ii), it thus suffices to show that for each \( B \in B(C_{a,b}^{n}[0,T]) \), \( H_{x}^{-1}(\mathcal{Q}_{x}^{-1}(B)) = B \). First, take any \( (x_{1},\ldots,x_{n}) \in H_{x}^{-1}(\mathcal{Q}_{x}^{-1}(B)) \). Then, \( (x_{1},\ldots,x_{n}) \in \mathcal{Q}_{x}^{-1}(B) \) and hence \( \mathcal{Q}_{x}(H_{x}(x_{1},\ldots,x_{n})) \in B \). Thus, it follows that
\[
(x_{1},\ldots,x_{n}) = (H_{x}(x_{1},\ldots,x_{n})(s_{1}),\ldots,H_{x}(x_{1},\ldots,x_{n})(s_{n})) = \mathcal{Q}_{x}(H_{x}(x_{1},\ldots,x_{n})) \in B.
\]
Conversely, by the inverse image property of maps, it follows that for each \( B \in B(C_{a,b}^{n}[0,T]) \), \( B \subset (\mathcal{Q}_{x} \circ H_{x})^{-1}(B) = H_{x}^{-1}(\mathcal{Q}_{x}^{-1}(B)) \), as desired. \( \square \)

Remark 6.3. Lemma 6.2 tells us that given any multidimensional tuple \( \vec{s} = (s_{1},\ldots,s_{n}) \) with \( 0 = s_{0} < s_{1} \leq \cdots \leq s_{n} \leq T \) and any subset \( B \) of \( C_{a,b}^{n}[0,T] \), \( \mathcal{Q}_{x}^{-1}(B) \in B(C_{0}^{\beta m}) \) if and only if \( B \in B(C_{a,b}^{n}[0,T]) \).

Our next theorem follows quite readily from the techniques developed in [6, Section 3] and Remark 6.3.

Lemma 6.4 (Converse measurability theorem). Let \( \vec{s} = (s_{1},\ldots,s_{n}) \) be as in Lemma 6.2. For a subset \( B \) of \( C_{a,b}^{n}[0,T] \) with \( \mathcal{Q}_{x}^{-1}(B) \in W(C_{a,b}^{n}[0,T]) \), \( B \) is in \( W(C_{0}^{\beta m}) \).

In view of equation (5.5), it follows the following corollaries.

Corollary 6.5. Given any multidimensional tuple \( \vec{s} = (s_{1},\ldots,s_{n}) \) with \( 0 = s_{0} < s_{1} \leq \cdots \leq s_{n} \leq T \) and any subset \( B \) of \( C_{a,b}^{n}[0,T] \), the following assertions are equivalent.

(i) \( B \) is in \( W(C_{a,b}^{n}[0,T]) \);
(ii) \( B \) is in \( W(C_{a,b}^{n}[0,T]) \);
(iii) \( \mathcal{Q}_{x}^{-1}(B) \) is in \( W(C_{0}^{\beta m}) \).

Corollary 6.6. Given any multidimensional tuple \( \vec{s} = (s_{1},\ldots,s_{n}) \) with \( 0 = s_{0} < s_{1} \leq \cdots \leq s_{n} \leq T \) and any subset \( B \) of \( C_{a,b}^{n}[0,T] \), the following assertions are equivalent.
(i) $B$ is a $\mu^n$-null set in $W(C_{a,b}^n[0,T])$;
(ii) $B$ is a $\mu^T$-null set in $W(C_{a,b}^n[0,T])$;
(iii) $\Psi_{\vec{s}}^{-1}(B)$ is a $\mu_{C_{0}^{BM}}$-null set in $W(C_{0}^{BM})$.

Our next lemma follows quite readily from Corollary 6.6.

**Lemma 6.7.** Let $\vec{s} = (s_1, \ldots, s_n)$ be as in Lemma 6.2. Then, $\Psi_{\vec{s}}^{-1}(B)$ is in $W(C_{0}^{BM})$ for every $B \in \mathcal{M}(C_{a,b}^n[0,T])$. In other words the cylinder function $\Psi_{\vec{s}}$ given by (5.4) is $W(C_{0}^{BM})$-measurable.

We are now ready to present the proof of Theorem 5.6.

**Proof of Theorem 5.6.** We may assume, without loss of generality, that $F$ is a real-valued function. We first note that for any $W(C_{a,b}^n[0,T])$-measurable function $F$ on $C_{a,b}^n[0,T]$,

$$F(x(s_1), \ldots, x(s_n)) = F \circ \Psi_{\vec{s}}(x).$$

Thus, by Lemma 6.7, $F(x(s_1), \ldots, x(s_n))$ is $W(C_{0})$-measurable, as a function of $x$. Next, using (5.4), (6.7), (5.5), (5.3), and (6.7) again, it follows that

$$\int_{C_{0}^{BM}} F(x(s_1), \ldots, x(t_n))d\mu_{C_{0}^{BM}}(x) = \int_{C_{0}^{BM}} F(\Psi_{\vec{s}}(x))d\mu_{C_{0}^{BM}}(x) = \int_{C_{a,b}^n[0,T]} F(w_1, \ldots, w_n)d\mu_{C_{a,b}^n} \circ \Psi_{\vec{s}}^{-1}(w_1, \ldots, w_n) = \int_{C_{a,b}^n[0,T]} F(w_1, \ldots, w_n)d\mu_{\vec{w}} = \int_{C_{a,b}^n[0,T]} F(T_{\vec{s}}(x_1, \ldots, x_n))d\mu_{\vec{w}}(x).$$

This completes the proof of the theorem. \qed

7. Examples

In this section we present interesting examples to which equation (5.6) can be applied. Our examples involves the PWZ stochastic integrals $(w, x(s))$~. Thus, in this section, we have to guarantee the existence of the PWZ stochastic integral $(w, x(s))$~ for $x \in C_{0}^{BM}$. But, in view of Corollary 6.6 we obtain the following lemma.

**Lemma 7.1.** For each $s \in (0, T]$ and $w \in C_{0}^n[0,T]$, $(w, x(s))$~ exists for $\mathcal{M}_{C_{0}}$-a.s. $x \in C_{0}$.

We now ready to present several examples to which equation (5.6) can be applied.

**Example 7.2.** We note that given a function $w$ in $C_{a,b}^n[0,T]$, the PWZ stochastic integral $(w, \cdot)$~ : $C_{a,b}^n[0,T] \to \mathbb{R}$ is a Gaussian random variable with mean $(w, a)_{C_{a,b}^n}$.
and variance $\|w\|_{a,b}^2$. Then using this fact, \((5.6)\) with $n = 1$, and \((3.3)\) with $x$ replaced with $a$, it follows that for each $s \in (0, T]$, 
\[
\begin{align*}
\int_{C_0}[w, s(s)]^2 d\mu_{C_0}([x]) \\
= \int_{C_0}[(w, x - a) - \sqrt{b(s)(x - a)}]^2 d\mu(x) \\
= \int_{C_0}[(w, a) - \sqrt{b(s)(w, a)}]^2 d\mu(x) \\
= \sqrt{b(s)(w, a)} - \sqrt{b(s)(w, a)} + (w, a)_{C_{a,b}} \\
= (w, a)_{C_{a,b}}^2
\end{align*}
\]
and
\[
\begin{align*}
\int_{C_0}[w, s(s)]^2 d\mu_{C_0}([x]) \\
= \int_{C_0}[(w, x)^2 - (w, a)^2 + (w, a)_{C_{a,b}}^2] d\mu(x) \\
= b(s) \int_{C_0}[(w, x)^2 - (w, a)^2 + (w, a)_{C_{a,b}}^2] d\mu(x) \\
+ 2\sqrt{b(s)(w, a)} \int_{C_0}[(w, x)^2 - (w, a)^2 + (w, a)_{C_{a,b}}^2] d\mu(x) \\
= b(s)\|w\|_{C_{a,b}}^2 + (w, a)_{C_{a,b}}^2.
\end{align*}
\]

**Example 7.3.** Let $w_1, w_2 \in C_{a,b}[0, T]$ and let $s_1, s_2 \in (0, T]$ with $s_1 < s_2$. Then using \((5.6)\) with $n = 2$, the Fubini theorem, \((3.4)\), and \((3.3)\), it follows that
\[
\begin{align*}
\int_{C_0}(w_1, s(s_1))\sim (w_2, s(s_2))\sim d\mu_{C_0}(x) \\
= \int_{C_0^2}[(w_1, x_1)^2 - (w_1, a)^2 + (w_1, a)_{C_{a,b}}^2] d\mu(x) \\
\times \left\{ \sqrt{b(s_1)(w_2, x_1)^2 - (w_2, a)^2 + (w_2, a)_{C_{a,b}}^2} \right\} \\
+ \sqrt{b(s_2) - b(s_1)(w_2, x_2)^2 - (w_2, a)^2 + (w_2, a)_{C_{a,b}}^2} d\mu^2(x_1, x_2) \\
= b(s_1)(w_1, w_2)_{C_{a,b}}^2 + (w_1, a)_{C_{a,b}}^2 (w_2, a)_{C_{a,b}}^2.
\end{align*}
\]
In particular, taking $s_1, s_2 \in (0, T]$ and $t, t_1, t_2 \in [0, T]$, and using \((3.6)\) and \((3.5)\), we obtain that
\[
\begin{align*}
\int_{C_0}(s_1(t), s_2(t)) d\mu_{C_0}(x) = \min\{b(s_1), b(s_2)\} b(t) + a^2(t)
\end{align*}
\]
and
\[
\begin{align*}
\int_{C_0}(s_1(t_1), s_2(t_2)) d\mu_{C_0}(x) \\
= \min\{b(s_1), b(s_2)\} \min\{b(t_1), b(t_2)\} + a(t_1)a(t_2).
\end{align*}
\]
Example 7.4. Let $s \in (0, T]$ be fixed. Using equation (5.6) with $n = 1$ and (3.7), it follows that given any nonzero real number $\rho$ and a function $w$ in $C^0_{a,b}[0,T]$, 

$$
\int_{C^0_{a,b}} \exp \{ i\rho (w, f(s)) - \} d\mu_{C^0_{a,b}}(f) 
= \int_{C_{a,b}[0,T]} \exp \{ i\rho (w, \sqrt{b(s)}(x-a) + a) - \} d\mu(x) 
= \int_{C_{a,b}[0,T]} \exp \{ i\rho \sqrt{b(s)}(w,x) - \} d\mu(x) \exp \{ - i\rho \sqrt{b(s)}(w,a)C_{a,b} + i\rho (w,a)C_{a,b} - \} 
= \exp \left\{ - \frac{1}{2} \rho^2 b(s)\|w\|^2_{C_{a,b}} + i\rho (w,a)C_{a,b} - \right\}.
$$

Example 7.5. Let $s_1, s_2 \in (0, T]$ be given with $s_1 < s_2$. Using (5.6) and the Fubini theorem, and applying (3.7), it follows that given any nonzero real number $\rho$ and any functions $w_1$ and $w_2$ in $C^0_{a,b}[0,T]$, 

$$
\int_{C^0_{a,b}} \exp \{ i\rho (w_1, f(s_1)) - \} + i\rho (w_2, f(s_2)) - \} d\mu_{C^0_{a,b}}(f) 
= \int_{C^2_{a,b}[0,T]} \exp \{ i\rho (w_1, \sqrt{b(s_1)}(x_1-a) + a) - \} d\mu(x_1) 
+ \int_{C_{a,b}[0,T]} \exp \{ i\rho \sqrt{b(s_2)}(w_2,x_2) - \} d\mu(x_2) 
\times \exp \{ - i\rho \sqrt{b(s_2)}(w_1 + w_2, a)C_{a,b} - \} + i\rho (w_1, a)C_{a,b} - \} 
= \exp \left\{ - \frac{1}{2} \rho^2 b(s_1)\|w_1 + w_2\|^2_{C_{a,b}} + i\rho (w_1, a)C_{a,b} - \right\}.
$$

By an induction argument, it follows that given any $n$-tuple $s = (s_1, \ldots, s_n)$ with $0 = s_0 < s_1 \leq \cdots \leq s_n \leq T$ and any set $\{w_1, \ldots, w_n\}$ of functions in $C^0_{a,b}[0,T]$, 

$$
\int_{C^0_{a,b}} \exp \left\{ i\rho \sum_{k=1}^n (w_k, f(s_k)) - \right\} d\mu_{C^0_{a,b}}(f) 
= \exp \left\{ - \rho^2 \sum_{k=1}^n (b(s_k) - b(s_{k-1}))\left\| \sum_{i=k}^n w_i \right\|^2_{C_{a,b}} + i\rho \sum_{k=1}^n (w_k, a)C_{a,b} - \right\}.
$$

8. Analytic paths space Feynman integral

As an application of the paths space integral, we suggest an analytic paths space Feynman integral for functionals $F$ on $C^0_{a,b}$. In this section, we give a class
of certain bounded cylinder functionals whose analytic paths space integral and analytic paths space Feynman integral on $C_0^{gBm}$ exist.

A subset $B$ of $C_0^{gBm}$ is said to be scale-invariant measurable provided $\rho B$ is $\mathcal{W}(\mathcal{C})$-measurable for all $\rho > 0$, and a scale-invariant measurable set $N$ is said to be a scale-invariant null set provided $\mu_{C_0^{gBm}}(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). A functional $F$ is said to be scale-invariant measurable provided $F$ is defined on a scale-invariant measurable set and $F(\rho \cdot)$ is $\mathcal{W}(C_{a,b}[0,T])$-measurable for every $\rho > 0$.

Throughout this and next sections, for each $\lambda \in \tilde{\mathbb{C}}_+$, $\lambda^{-1/2}$ is always chosen to have positive real part.

**Definition 8.1.** Let $F : C_0 \to \mathbb{C}$ be a scale-invariant measurable functional such that the paths space integral

$$J(\lambda) = \int_{C_0^{gBm}} F(\lambda^{-1/2}x) d\mu_{C_0^{gBm}}(x)$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic paths space integral of $F$ over $C_0^{gBm}$ with parameter $\lambda$, and for $\lambda \in \mathbb{C}_+$ we write

$$\int_{C_0^{gBm}}^{\mathrm{an}_{\lambda}} F(x) d\mu_{C_0^{gBm}}(x) = J^*(\lambda).$$

Let $q \neq 0$ be a real number and let $F$ be a functional such that $\int_{C_0^{gBm}}^{\mathrm{an}_{\lambda}} F(x) d\mu_{C_0^{gBm}}(x)$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the analytic paths space \textit{Feynman integral} of $F$ with parameter $q$ and we write

$$\int_{C_0^{gBm}}^{\mathrm{an}_{\lambda}} F(x) d\mu_{C_0^{gBm}}(x) = \lim_{\lambda \to -iq} \int_{C_0^{gBm}}^{\mathrm{an}_{\lambda}} F(x) d\mu_{C_0^{gBm}}(x)$$

where $\lambda$ approaches $-iq$ through values in $\mathbb{C}_+$.

### 8.1. Cylinder functionals in $\tilde{\mathcal{G}}_{m,s}$.

A functional $F$ is called a cylinder functional on $C_0^{gBm}$ if there exists a finite subset $\{h_1, \ldots, h_k\}$ of $C_{a,b}'[0,T]$ such that

$$F(x) = \psi((h_1, x(s))^\sim, \ldots, (h_k, x(s))^\sim), \quad x \in C_0^{gBm}$$

where $\psi$ is a complex-valued Borel measurable function on $\mathbb{R}^k$. It is easy to show that for given cylinder functional $F$ of the form (8.2) there exists an orthonormal subset $\{g_1, \ldots, g_m\}$ of $C_{a,b}'[0,T]$ such that $F$ is expressed as

$$F(x) = f((g_1, x(s))^\sim, \ldots, (g_m, x(s))^\sim), \quad x \in C_0^{gBm}$$

where $f$ is a complex Borel measurable function on $\mathbb{R}^m$. Thus we lose no generality in assuming that every cylinder functional on $C_{a,b}[0,T]$ is of the form (8.3).

**Definition 8.2.** Let $\mathcal{M}(\mathbb{R}^m)$ denote the space of complex-valued Borel measures on $\mathcal{B}(\mathbb{R}^m)$. It is well known that a complex-valued Borel measure $\nu$ necessarily has a finite total variation $||\nu||$, and $\mathcal{M}(\mathbb{R}^m)$ is a Banach algebra under the norm $|| \cdot ||$ and with convolution as multiplication.
For \( \nu \in M(\mathbb{R}^m) \), the Fourier transform \( \hat{\nu} \) of \( \nu \) is a complex-valued function defined on \( \mathbb{R}^m \) by the formula

\[
(8.4) \quad \hat{\nu}(\vec{u}) = \int_{\mathbb{R}^m} \exp \left\{ i \sum_{j=1}^m u_j v_j \right\} d\nu(\vec{v})
\]

where \( \vec{u} = (u_1, \ldots, u_m) \) and \( \vec{v} = (v_1, \ldots, v_m) \) are in \( \mathbb{R}^m \).

Let \( G_m = \{g_1, \ldots, g_m\} \) be an orthonormal subset of \( C_{a,b}[0,T] \). Given \( s \in (0,T] \), define a functional \( F : \mathbb{C}_0 \rightarrow \mathbb{C} \) by

\[
(8.5) \quad F(\vec{x}) = \hat{\nu}((g_1, \vec{x}(s))^\sim, \ldots, (g_m, \vec{x}(s))^\sim), \quad \vec{x} \in C^{\text{an}}_0
\]

where \( \hat{\nu} \) is the Fourier transform of \( \nu \) in \( M(\mathbb{R}^m) \). Then \( F \) is a bounded cylinder functional since \( |\hat{\nu}(\vec{u})| \leq \|\nu\| < +\infty \). Let \( \mathbb{F}_{G_m,s} \) be the space of all functionals \( F \) on \( C^{\text{an}}_0 \) having the form \( (8.5) \). Note that \( F \in \mathbb{F}_{G_m,s} \) implies that \( F \) is scale-invariant measurable on \( C^{\text{an}}_0 \).

We first show that the analytic paths space integral of the functional \( F \) given by equation \( (8.5) \) exists.

**Theorem 8.3.** Let \( F \in \mathbb{F}_{G_m,s} \) be given by equation \( (8.5) \). Then for each \( \lambda \in \mathbb{C}_+ \), the analytic paths space integral \( \int_{C^{\text{an}}_0} F(\vec{x}) d\mu_{C^{\text{an}}_0}(\vec{x}) \) exists and is given by the formula

\[
(8.6) \quad \int_{C^{\text{an}}_0} F(\vec{x}) d\mu_{C^{\text{an}}_0}(\vec{x}) = \int_{\mathbb{R}^m} \exp \left\{ -\frac{b(s)}{2\lambda} \sum_{j=1}^m v_j^2 + i\lambda^{-1/2} \sum_{j=1}^m (g_j, a) c_{a,b} v_j \right\} d\nu(\vec{v}).
\]

**Proof.** By \( (8.1) \), \( (8.4) \), the Fubini theorem, \( (5.6) \) with \( n = 1 \), \( (3.6) \), and the fact that the set \( \{g_1, \ldots, g_m\} \) is orthonormal in \( C_{a,b}[0,T] \), we have that for all \( \lambda > 0 \),

\[
J(\lambda) = \int_{C^{\text{an}}_0} F(\lambda^{-1/2} \vec{x}) d\mu_{C^{\text{an}}_0}(\vec{x})
\]

\[
= \int_{\mathbb{R}^m} \int_{C^{\text{an}}_0} \exp \left\{ i\lambda^{-1/2} \sum_{j=1}^m (g_j, \vec{x}(s))^\sim v_j \right\} d\mu_{C^{\text{an}}_0}(\vec{x}) d\nu(\vec{v})
\]

\[
= \int_{\mathbb{R}^m} \int_{C_{a,b}[0,T]} \exp \left\{ i\lambda^{-1/2} \sum_{j=1}^m (g_j, \sqrt{b(s)}(x-a) + a)^\sim v_j \right\} d\mu(x) d\nu(\vec{v})
\]

\[
= \int_{\mathbb{R}^m} \int_{C_{a,b}[0,T]} \exp \left\{ i\lambda^{-1/2} \left( \sqrt{b(s)} \sum_{j=1}^m g_j v_j, x \right)^\sim \right\} d\mu(x)
\]

\[
\times \exp \left\{ -i\lambda^{-1/2} \sqrt{b(s)} \sum_{j=1}^m (g_j, a) c_{a,b} v_j + i\lambda^{-1/2} \sum_{j=1}^m (g_j, a) c_{a,b} v_j \right\} d\nu(\vec{v})
\]

\[
= \int_{\mathbb{R}^m} \exp \left\{ -\frac{1}{2\lambda} \left\| \sqrt{b(s)} \sum_{j=1}^m g_j v_j \right\|^2 + i\lambda^{-1/2} \left( \sqrt{b(s)} \sum_{j=1}^m g_j v_j, a \right) \right\}
\]

\[
\times \exp \left\{ -i\lambda^{-1/2} \sqrt{b(s)} \sum_{j=1}^m (g_j, a) c_{a,b} v_j + i\lambda^{-1/2} \sum_{j=1}^m (g_j, a) c_{a,b} v_j \right\} d\nu(\vec{v})
\]
\[ J^*(\lambda) = \int_{\mathbb{R}^m} \exp \left\{ -\frac{b(s)}{2}\sum_{j=1}^{m} \frac{m}{v_j^2} + i\lambda^{-1/2} \sum_{j=1}^{m} (g_j, a)_{C_{a,b}^\prime} v_j \right\} d\nu(\bar{\nu}) \]

Now let

\[ J^*(\lambda) = \int_{\mathbb{R}^m} \exp \left\{ -\frac{b(s)}{2}\sum_{j=1}^{m} v_j^2 + i\lambda^{-1/2} \sum_{j=1}^{m} (g_j, a)_{C_{a,b}^\prime} v_j \right\} d\nu(\bar{\nu}). \]

for \( \lambda \in \mathbb{C}_+. \) Then \( J^*(\lambda) = J(\lambda) \) for all \( \lambda > 0. \)

We will use the Morera theorem to show that \( J^*(\lambda) \) is analytic on \( \mathbb{C}_+ \). First, let \( \{\lambda_l\}_{l=1}^{\infty} \) be a sequence in \( \mathbb{C}_+ \) such that \( \lambda_l \to \lambda \) through \( \mathbb{C}_+ \). Then \( \lambda_l^{-1/2} \to \lambda^{-1/2} \) and Re(\( \lambda_l \)) \( \neq 0 \) for all \( l \in \mathbb{N} \). Thus we have that for each \( l \in \mathbb{N} \),

\[ \left| \exp \left\{ -\frac{b(s)}{2|\lambda_l|} \sum_{j=1}^{m} v_j^2 + i\lambda_l^{-1/2} \sum_{j=1}^{m} (g_j, a)_{C_{a,b}^\prime} v_j \right\} \right| \]

\[ = \exp \left\{ -\frac{b(s) \text{Re}(\lambda_l)}{2|\lambda_l|^2} \sum_{j=1}^{m} v_j^2 - i|\lambda_l|^{-1/2} \sum_{j=1}^{m} (g_j, a)_{C_{a,b}^\prime} v_j \right\} \]

\[ = \exp \left\{ -\frac{1}{2} \sum_{j=1}^{m} \left( \frac{\sqrt{b(s) \text{Re}(\lambda_l)}}{|\lambda_l|} v_j + |\lambda_l|^{-1/2} \left( \frac{\sqrt{b(s) \text{Re}(\lambda_l)}}{|\lambda_l|} \right)^2 \sum_{j=1}^{m} (g_j, a)_{C_{a,b}^\prime} v_j \right) \right\} \]

\[ \leq \exp \left\{ \frac{|\lambda_l|^2 (\text{Im}(\lambda_l^{-1/2}))^2}{2b(s) \text{Re}(\lambda_l)} \sum_{j=1}^{m} (g_j, a)_{C_{a,b}^\prime}^2 \right\}. \]

Since \( \nu \in \mathcal{M}(\mathbb{R}^m) \), we see that

\[ \left| \int_{\mathbb{R}^m} \exp \left\{ \frac{|\lambda_l|^2 (\text{Im}(\lambda_l^{-1/2}))^2}{2b(s) \text{Re}(\lambda_l)} \sum_{j=1}^{m} (g_j, a)_{C_{a,b}^\prime}^2 \right\} d\nu(\bar{\nu}) \right| \]

\[ \leq \exp \left\{ \frac{|\lambda_l|^2 (\text{Im}(\lambda_l^{-1/2}))^2}{2b(s) \text{Re}(\lambda_l)} \sum_{j=1}^{m} (g_j, a)_{C_{a,b}^\prime}^2 \right\} ||\nu|| < +\infty \]

for each \( l \in \mathbb{N} \). Furthermore, we have that

\[ \lim_{l \to \infty} \int_{\mathbb{R}^m} \exp \left\{ \frac{|\lambda_l|^2 (\text{Im}(\lambda_l^{-1/2}))^2}{2b(s) \text{Re}(\lambda_l)} \sum_{j=1}^{m} (g_j, a)_{C_{a,b}^\prime}^2 \right\} d\nu(\bar{\nu}) \]

\[ = \lim_{l \to \infty} \exp \left\{ \frac{|\lambda_l|^2 (\text{Im}(\lambda_l^{-1/2}))^2}{2b(s) \text{Re}(\lambda_l)} \sum_{j=1}^{m} (g_j, a)_{C_{a,b}^\prime}^2 \right\} \nu(\mathbb{R}^m) \]

\[ = \exp \left\{ \frac{|\lambda|^2 (\text{Im}(\lambda^{-1/2}))^2}{2b(s) \text{Re}(\lambda)} \sum_{j=1}^{m} (g_j, a)_{C_{a,b}^\prime}^2 \right\} \nu(\mathbb{R}^m) \]

\[ = \int_{\mathbb{R}^m} \exp \left\{ \frac{|\lambda|^2 (\text{Im}(\lambda^{-1/2}))^2}{2b(s) \text{Re}(\lambda)} \sum_{j=1}^{m} (g_j, a)_{C_{a,b}^\prime}^2 \right\} d\nu(\bar{\nu}). \]
Thus, by Theorem 4.17 in [42, p.92], $J^*(\lambda)$ is continuous on $\mathbb{C}_+$. Next, using the fact that
\[ k(\lambda) \equiv \exp \left\{ - \frac{b(s)}{2\lambda} \sum_{j=1}^{m} v_j^2 + i\lambda^{-1/2} \sum_{j=1}^{m} (g_j, \alpha)_{C_{\alpha}^{\prime}} v_j \right\} \]

is analytic on $\mathbb{C}_+$, and applying the Fubini theorem, it follows that
\[ \int_{\triangle} J^*(\lambda) d\lambda = \int_{\mathbb{R}^m} \int_{\triangle} k(\lambda)d\lambda d\nu(\vec{v}) = 0 \]

for all rectifiable simple closed curve $\triangle$ lying $\mathbb{C}_+$. Thus by the Morera theorem, $J^*(\lambda)$ is analytic on $\mathbb{C}_+$. Therefore the analytic paths space integral $J^*(\lambda) = \int_{\mathbb{C}_0^{\text{an}}} F(\vec{x}) d\mu_{\mathbb{C}_0^{\text{an}}} (\vec{x})$ exists and is given by equation (8.6).

The observation below will be very useful in the development of our results for the analytic paths space Feynman integral of functionals given by equation (8.6).

If $a(t) \equiv 0$ on $[0, T]$, then for all $F \in \hat{\mathcal{F}}_{G_m,s}$ given by equation (8.5), the analytic paths space Feynman integral $\int_{\mathbb{C}_0^{\text{an}}} F(\vec{x}) d\mu_{\mathbb{C}_0^{\text{an}}} (\vec{x})$ will always exist for all real $q \neq 0$ and be given by the formula
\[ \int_{\mathbb{C}_0^{\text{an}}} F(\vec{x}) d\mu_{\mathbb{C}_0^{\text{an}}} (\vec{x}) = \int_{\mathbb{R}^m} \exp \left\{ - \frac{ib(s)}{2q} \sum_{j=1}^{m} v_j^2 \right\} d\nu(\vec{v}). \]

However for $a(t)$ as in Section 3 and proceeding formally using (8.6) and (8.11), we observe that $\int_{\mathbb{C}_0^{\text{an}}} F(\vec{x}) d\mu_{\mathbb{C}_0^{\text{an}}} (\vec{x})$ will be given by equation (8.9) below if it exists. But the integral on the right-hand side of (8.9) might not exist if the real part of
\[ \left\{ - \frac{ib(s)}{2q} \sum_{j=1}^{m} v_j^2 + i(-iq)^{-1/2} \sum_{j=1}^{m} (g_j, \alpha)_{C_{\alpha}^{\prime}} v_j \right\} \]

is positive.

We emphasize that any functional $F \in \hat{\mathcal{F}}_{G_m,s}$ is bounded on $\mathbb{C}_0^{\text{an}}$, because
\[ |F(\vec{x})| = \left| \int_{\mathbb{R}^m} \exp \left\{ i \sum_{j=1}^{m} (g_j, \vec{x}(s)) v_j \right\} d\nu(\vec{v}) \right| \leq \int_{\mathbb{R}^m} \left| \exp \left\{ i \sum_{j=1}^{m} (g_j, \vec{x}(s)) v_j \right\} \right| d\nu(\vec{v}) \leq \int_{\mathbb{R}^m} d\nu(\vec{v}) = \|\nu\| < +\infty. \]

However, there is a functional $F$ in $\hat{\mathcal{F}}_{G_m,s}$ which is not analytic paths space Feynman integrable on $\mathbb{C}_0^{\text{an}}$. In order to present an example of the functional $F$ which is not analytic paths space Feynman integrable, we consider the class
\[ \hat{\mathcal{F}}_{G_1,s} = \left\{ F : F(x) = \tilde{\nu}(x) = \int_{\mathbb{R}} \exp \left\{ i(g, \vec{x}(s))^{-1} v \right\} d\nu(v) \right\} \]

for some $\nu \in \mathcal{M}(\mathbb{R})$. Next we consider a measure $\alpha$ on $\mathbb{R}$ which is concentrated on the set of natural numbers $\mathbb{N}$ and is given by
\( \alpha(\{m\}) = 1/m^2 \) for each \( m \in \mathbb{N} \). Then \( \alpha \) is an element of \( \mathcal{M}(\mathbb{R}) \). Consider the functional \( F \in \mathbb{F}_{G_m,s} \) given by

\[
F(x) = \int_{\mathbb{R}} \exp \left\{ i (g, \hat{y}(s))^\ast v \right\} d\alpha(v).
\]

In this case, by equation (8.9) below, we have that for a positive real number \( q > 0 \),

\[
\int_{\mathbb{C}_0^{G_m}} F(y) d\mu_{\mathbb{C}_0^{G_m}}(y)
\]

(8.7)

\[
= \int_{\mathbb{R}} \exp \left\{ -\frac{ib(s)}{2q} v^2 + i(-iq)^{-1/2} (g, a)_{C_{a,b}} v \right\} d\alpha(v)
\]

\[
= \sum_{m=1}^{\infty} \exp \left\{ -\frac{ib(s)m^2}{2q} + \left( 1 - \frac{1}{\sqrt{2q}} + \frac{i}{\sqrt{2q}} \right) (g, a)_{C_{a,b}} m \right\} \frac{1}{m^2}.
\]

Then, we have

\[
L = \lim_{m \to \infty} \exp \left\{ -\frac{ib(s)(m+1)^2}{2q} + \left( 1 - \frac{1}{\sqrt{2q}} + \frac{i}{\sqrt{2q}} \right) (g, a)_{C_{a,b}} (m + 1) \right\} \frac{1}{(m+1)^2}.
\]

(8.8)

\[
= \lim_{m \to \infty} \exp \left\{ -\frac{ib(s)m^2}{2q} + \left( 1 - \frac{1}{\sqrt{2q}} + \frac{i}{\sqrt{2q}} \right) (g, a)_{C_{a,b}} m \right\} \frac{1}{m^2}.
\]

If \( (g, a)_{C_{a,b}} < 0 \), then \( L > 1 \) and so, by the d’Alembert ratio test, we see that the series in the last expression of (8.7) diverges.

Consequently, in view of this example, we clearly need to impose additional restrictions on the functionals in \( \mathbb{F}_{G_m,s} \) to establish the existence of the analytic Feynman integral on \( \mathbb{C}_0^{G_m} \).

For a positive real number \( q_0 \), we define a subclass \( \mathbb{F}_{G_m,s}^{q_0} \) of \( \mathbb{F}_{G_m,s} \) by \( F \in \mathbb{F}_{G_m,s}^{q_0} \) if and only if

\[
\int_{\mathbb{R}^m} \exp \left\{ \left\| a \right\|_{C_{a,b}} \sqrt{2q_0} \sum_{j=1}^m |v_j| \right\} d\nu(|\vec{v}|) < +\infty.
\]

(8.8)

where \( \nu \) and \( F \) are related by (8.5).

Note that in case \( a(t) \equiv 0 \) and \( b(t) = t \) on \([0, T]\), the function space \( C_{a,b}[0, T] \) reduces to the classical Wiener space \( C_0[0, T] \) and \( (g_j, a)_{C_{a,b}} = 0 \) for all \( j = 1, \ldots, n \).

In this case, it follows that for all \( q_0 > 0 \), \( \mathbb{F}_{G_m,s}^{q_0} = \mathbb{F}_{G_m,s} \).

**Theorem 8.4.** Given a positive real number \( q_0 \), let \( F \in \mathbb{F}_{G_m,s}^{q_0} \) be given by (8.5). Then, for all real \( q \) with \( |q| > |q_0| \), the analytic paths space Feynman integral

\[
\int_{\mathbb{C}_0^{G_m}} F(y) d\mu_{\mathbb{C}_0^{G_m}}(y)
\]

(8.9)

\[
= \int_{\mathbb{R}^m} \exp \left\{ -\frac{ib(s)}{2q} \sum_{j=1}^m v_j^2 + i(-iq)^{-1/2} \sum_{j=1}^m (g_j, a)_{C_{a,b}} v_j \right\} d\nu(|\vec{v}|).
\]
Thus by the dominated convergence theorem, it follows equation (8.9).

and so, by condition (8.8),

\[ k \exists a sufficiently large l \]

Proof. Let \( \{ \lambda_i \}_{i=1}^\infty \) be a sequence of complex numbers such that \( \lambda_i \to -iq \) through \( \mathbb{C}_+ \) and for each \( l \in \mathbb{N} \), let

\[
f_l(\vec{v}) = \exp \left\{ -\frac{b(s)}{2\lambda_l} \sum_{j=1}^m v_j^2 + i\lambda_l^{-1/2} \sum_{j=1}^n (g_j, a) c_{a,b} v_j \right\}.
\]

Then \( f_l(\vec{v}) \) converges to

\[
f(\vec{v}) = \exp \left\{ -\frac{ib(s)}{2q} \sum_{j=1}^m v_j^2 + i(-iq)^{-1/2} \sum_{j=1}^n (g_j, a) c_{a,b} v_j \right\}.
\]

By Theorem 8.3 for all \( l \in \mathbb{N} \), \( \int_{\mathbb{R}^m} f_l(\vec{v}) d\nu(\vec{v}) \) exists. Since \( |\arg(\lambda_l^{-1/2})| < \pi/4 \) for every \( l \in \mathbb{N} \) and \( \lambda_l^{-1/2} = \text{Re}(\lambda_l^{-1/2}) + i\text{Im}(\lambda_l^{-1/2}) \to (i(-iq)^{-1/2} = 1/\sqrt{2q}) \) + i\text{sign}(q)/\sqrt{2q} \), we see that \( \text{Re}(\lambda_l^{-1/2}) > |\text{Im}(\lambda_l^{-1/2})| \) for every \( l \in \mathbb{N} \) and there exists a sufficiently large \( k \in \mathbb{N} \) such that \( |\text{Im}(\lambda_l^{-1/2})| < 1/\sqrt{2q} \) for every \( l \geq k \). Thus, using the Cauchy–Schwartz inequality, it follows that for each \( l \geq k \),

\[
|f_l(\vec{v})| = \left| \exp \left\{ -\frac{b(s)}{2\lambda_l} \left( |\text{Re}(\lambda_l^{-1/2})|^2 - |\text{Im}(\lambda_l^{-1/2})|^2 \right) \right. \right. \\
\left. \left. \quad + i\text{Re}(\lambda_l^{-1/2})\text{Im}(\lambda_l^{-1/2}) \right) \sum_{j=1}^m v_j^2 \right. \\
\left. \quad + i \left( \text{Re}(\lambda_l^{-1/2}) + i\text{Im}(\lambda_l^{-1/2}) \right) \sum_{j=1}^m (g_j, a) c_{a,b} v_j \right| \\
\leq \exp \left\{ -\text{Im}(\lambda_l^{-1/2}) \sum_{j=1}^m (g_j, a) c_{a,b} v_j \right\} \\
\leq \exp \left\{ |\text{Im}(\lambda_l^{-1/2})| |a| c_{a,b} \sum_{j=1}^m |v_j| \right\} \\
< \exp \left\{ \frac{|a| c_{a,b}}{\sqrt{2q}} \sum_{j=1}^m |v_j| \right\}
\]

and so, by condition (8.8),

\[
\left| \int_{\mathbb{R}^m} f_l(\vec{v}) d\nu(\vec{v}) \right| \leq \int_{\mathbb{R}^m} |f_l(\vec{v})| d\nu(\vec{v}) \\
< \int_{\mathbb{R}^m} \exp \left\{ \frac{|a| c_{a,b}}{\sqrt{2q}} \sum_{j=1}^m |v_j| \right\} d\nu(\vec{v}) < +\infty.
\]

Also, by condition (8.8), we have

\[
\left| \int_{\mathbb{R}^m} f(\vec{v}) d\nu(\vec{v}) \right| \leq \int_{\mathbb{R}^m} \exp \left\{ \frac{|a| c_{a,b}}{\sqrt{2q}} \sum_{j=1}^m |v_j| \right\} d\nu(\vec{v}) \\
< \int_{\mathbb{R}^m} \exp \left\{ \frac{|a| c_{a,b}}{\sqrt{2q}} \sum_{j=1}^m |v_j| \right\} d\nu(\vec{v}) < +\infty.
\]

Thus by the dominated convergence theorem, it follows equation (8.9). \( \square \)
8.2. Functionals in \( \hat{F}_{G_{m,s}} \). Let \( n \) and \( m \) be positive integers. Given an \( n \)-tuple \( \vec{s} = (s_1, \ldots, s_n) \) with \( 0 = s_0 < s_1 < \cdots < s_n \leq T \) and an orthonormal set \( G_m = \{g_1, \ldots, g_m\} \) of functions in \( C'_a[0,T] \), let \( \hat{F}_{G_{m,s}} \) be the space of all functionals \( F \) on \( C_0^{g_{0m}} \) of the form

\[
F(\vec{f}) = \hat{\nu}((g_1, \vec{f}(s_1))^{\sim}, \ldots, (g_s, \vec{f}(s_n))^{\sim}) = \int_{\mathbb{R}^m} \exp \left\{ i \sum_{j=1}^{m} \sum_{k=1}^{n} u_{j,k} v_{j,k} \right\} d\nu(\vec{v})
\]

for \( \vec{x} \in C_0^{g_{0m}} \), where \( \nu \) is an element of \( \mathcal{M}(\mathbb{R}^{mn}) \), the class of all complex-valued Borel measures on \( \mathbb{R}^{mn} \) with finite total variation, and \( \hat{\nu} \) denotes the Fourier–Stieltjes transform of \( \nu \) given by

\[
\hat{\nu}(\vec{u}) = \int_{\mathbb{R}^m} \exp \left\{ i \sum_{j=1}^{m} \sum_{k=1}^{n} u_{j,k} v_{j,k} \right\} d\nu(\vec{v})
\]

where \( \vec{u} = (u_1, 1, \ldots, u_n, u_2, 1, \ldots, u_{m,1}, \ldots, u_{m,n}) \in \mathbb{R}^{mn} \) and \( \vec{v} = (v_1, 1, \ldots, v_n, v_2, 1, \ldots, v_{m,1}, \ldots, v_{m,n}) \in \mathbb{R}^{mn} \).

Also, for a positive real number \( q_0 \), we define a subclass \( \tilde{F}_{G_{m,s}}^{\gamma_{0m}} \) of \( \hat{F}_{G_{m,s}} \) by \( F \in \tilde{F}_{G_{m,s}}^{\gamma_{0m}} \) if and only if

\[
\int_{\mathbb{R}^m} \exp \left\{ \frac{\|a\|_{C'_a}}{\sqrt{2q_0}} \sum_{j=1}^{m} \sum_{k=1}^{n} |v_{j,k}| \right\} d|\nu|(\vec{v}) < +\infty.
\]

where \( \nu \) and \( F \) are related by \((8.10)\).

Our next theorem shows the analytic paths space integral exists for all \( F \in \hat{F}_{G_{m,s}} \). The following summation formula

\[
\sum_{k=1}^{n} \sum_{l=1}^{k} A_l B_k = \sum_{l=1}^{n} \sum_{k=l}^{n} A_l B_k
\]

will be helpful to prove the theorem.

**Theorem 8.5.** Let \( F \in \hat{F}_{G_{m,s}} \) be given by equation \((8.11)\). Then for each \( \lambda \in \mathbb{C}_+ \), the analytic paths space integral \( \int_{C_0^{g_{0m}}} F(\vec{x}) d\mu_{C_0^{g_{0m}}}(\vec{x}) \) exists and is given by the formula

\[
\int_{C_0^{g_{0m}}} F(\vec{x}) d\mu_{C_0^{g_{0m}}}(\vec{x}) = \int_{\mathbb{R}^m} \exp \left\{ -\frac{1}{2\lambda} \sum_{j=1}^{m} \sum_{l=1}^{n} \left[ b(s_l) - b(s_{l-1}) \right] \sum_{k=1}^{n} v_{j,k} \right\}^2 + i\lambda^{-1/2} \sum_{j=1}^{m} \sum_{k=1}^{n} (g_j, a)_{C'_a, k} v_{j,k} \nu(\vec{v}).
\]

**Proof:** By \((8.10)\), \((8.11)\), the Fubini theorem, \((5.6)\) together with \((5.1)\) and \((5.2)\), we first obtain that for all \( \lambda > 0 \),

\[
J(\lambda) = \int_{C_0^{g_{0m}}} F(\lambda^{-1/2} \vec{x}) d\mu_{C_0^{g_{0m}}}(\vec{x})
\]

\[
= \int_{\mathbb{R}^m} \int_{C_0^{g_{0m}}} \exp \left\{ i\lambda^{-1/2} \sum_{j=1}^{m} \sum_{k=1}^{n} (g_j, \vec{f}(s_k))^{\sim} v_{j,k} \right\} d\mu_{C_0^{g_{0m}}}(\vec{x}) d\nu(\vec{v})
\]
respectively. Using (8.12), the first and the second triple summations in the last expression of (8.13) can be rewritten by

\begin{equation}
\sum_{j=1}^{m} \sum_{k=1}^{n} \left[ \sum_{l=1}^{k} \sqrt{b(s_l) - b(s_{l-1})} (g_j, x_l)^\sim \right] v_{j,k}
\end{equation}

(8.14)

and

\begin{equation}
\sum_{j=1}^{m} \sum_{k=1}^{n} \left[ \sum_{l=1}^{k} \sqrt{b(s_l) - b(s_{l-1})} (g_j, a) c_{a,b}^\prime \right] v_{j,k}
\end{equation}

(8.15)

respectively. Using (8.14), (8.15), (3.7), and the fact that the set \( \{g_1, \ldots, g_n\} \) is orthonormal in \( C_{a,b}^0[0,T] \), it follows that for all \( \lambda > 0 \),

\begin{equation}
J(\lambda) = \int_{\mathbb{R}^m} \int_{C_{a,b}^0[0,T]} \exp \left\{ i\lambda^{-1/2} \sum_{j=1}^{m} \sum_{k=1}^{n} \left[ \sum_{l=1}^{k} \sqrt{b(s_l) - b(s_{l-1})} (g_j, x_l)^\sim \right] v_{j,k} \right\} d\mu^n(\vec{x}) d\nu(\vec{v})
\end{equation}

But, using (8.12), the first and the second triple summations in the last expression of (8.13) can be rewritten by

\begin{equation}
\int_{\mathbb{R}^m} \int_{C_{a,b}^0[0,T]} \exp \left\{ i\lambda^{-1/2} \sum_{j=1}^{m} \sum_{k=1}^{n} \left[ \sum_{l=1}^{k} \sqrt{b(s_l) - b(s_{l-1})} (g_j, x_l)^\sim \right] v_{j,k} \right\} d\mu^n(\vec{x}) \times \exp \left\{ -i\lambda^{-1/2} \sum_{j=1}^{m} \sum_{k=1}^{n} \left[ \sum_{l=1}^{k} \sqrt{b(s_l) - b(s_{l-1})} (g_j, a) c_{a,b}^\prime \right] v_{j,k} \right\} d\nu(\vec{v})
\end{equation}

endequation

\begin{align*}
J(\lambda) &= \int_{\mathbb{R}^m} \int_{C_{a,b}^0[0,T]} \exp \left\{ i\lambda^{-1/2} \sum_{j=1}^{m} \sum_{k=1}^{n} \left[ \sum_{l=1}^{k} \sqrt{b(s_l) - b(s_{l-1})} (g_j, x_l)^\sim \right] v_{j,k} \right\} d\mu^n(\vec{x}) d\nu(\vec{v}) \\
&= \int_{\mathbb{R}^m} \int_{C_{a,b}^0[0,T]} \exp \left\{ i\lambda^{-1/2} \sum_{j=1}^{m} \sum_{k=1}^{n} \left[ \sum_{l=1}^{k} \sqrt{b(s_l) - b(s_{l-1})} (g_j, x_l)^\sim \right] v_{j,k} \right\} d\mu^n(\vec{x}) \times \exp \left\{ -i\lambda^{-1/2} \sum_{j=1}^{m} \sum_{k=1}^{n} \left[ \sum_{l=1}^{k} \sqrt{b(s_l) - b(s_{l-1})} (g_j, a) c_{a,b}^\prime \right] v_{j,k} \right\} d\nu(\vec{v}).
\end{align*}
\[
\times \exp \left\{ -i\lambda^{-1/2} \sum_{l=1}^{n} \sum_{j=1}^{m} \left( \sum_{k=l}^{n} \sqrt{b(s_l) - b(s_{l-1})}(g_j, a) c_{a,b}^{*} \right) v_{j,k} \right. \\
\left. + i\lambda^{-1/2} \sum_{l=1}^{m} \sum_{k=1}^{n} (g_j, a) c_{a,b}^{*} v_{j,k} \right\} d\nu(\vec{v}) \\
= \int_{\mathbb{R}^{mn}} \left( \prod_{l=1}^{n} \int_{C_{a,b}[0,T]} \times \exp \left\{ -\frac{1}{2\lambda} \left( \sum_{l=1}^{m} \sum_{j=1}^{n} \sqrt{b(s_l) - b(s_{l-1})} v_{j,k} \right)^2 \right. \right. \\
\left. \left. \left. + i\lambda^{-1/2} \left( \sum_{l=1}^{m} \sum_{j=1}^{n} \sqrt{b(s_l) - b(s_{l-1})}(g_j, a) c_{a,b}^{*} \right) v_{j,k} \right) \right\} d\nu(\vec{v}) \\
= \int_{\mathbb{R}^{mn}} \exp \left\{ -\frac{1}{2\lambda} \sum_{l=1}^{n} \sum_{j=1}^{m} \left( \sum_{k=l}^{n} \sqrt{b(s_l) - b(s_{l-1})} v_{j,k} \right)^2 \right. \\
\left. \left. + i\lambda^{-1/2} \sum_{j=1}^{m} \sum_{k=1}^{n} (g_j, a) c_{a,b}^{*} v_{j,k} \right\} d\nu(\vec{v}) \right. \\
= \int_{\mathbb{R}^{mn}} \exp \left\{ -\frac{1}{2\lambda} \sum_{j=1}^{m} \sum_{l=1}^{n} \left[ b(s_l) - b(s_{l-1}) \right] \left( \sum_{k=l}^{n} v_{j,k} \right)^2 \right. \\
\left. \left. + i\lambda^{-1/2} \sum_{j=1}^{m} \sum_{k=1}^{n} (g_j, a) c_{a,b}^{*} v_{j,k} \right\} d\nu(\vec{v}). \right.
\]

Next, let \( J^*(\lambda) \) be given by the last expression of \([8.16]\) for each \( \lambda \in \mathbb{C}_+ \). Then using the techniques similar to those used in the proof of Theorem \([8.3]\) we can show that \( J^*(\lambda) = J(\lambda) \) for all \( \lambda > 0 \) and \( J^*(\lambda) \) is analytic on \( \mathbb{C}_+ \). This completes the proof. \( \square \)

Our next theorem follows quite readily from the techniques developed in the proof of Theorem \([8.4]\).
Theorem 8.6. Given a positive real number $q_0$, let $F \in \hat{T}_{q_0}^{G_m, \vec{s}}$ be given by (8.10). Then, for all real $q$ with $|q| > |q_0|$, the generalized analytic Feynman integral $\int_{C_{q_0}^{G_m}} \text{anf}_q C_{G_m} \mu_{C_{q_0}^{G_m}}(x) d\mu_{C_{q_0}^{G_m}}(x)$ exists and is given by the formula

$$
\int_{C_{q_0}^{G_m}} \text{anf}_q C_{G_m} \mu_{C_{q_0}^{G_m}}(x) d\mu_{C_{q_0}^{G_m}}(x) = \int_{\mathbb{R}^m} \exp \left\{ -\frac{i}{2q} \sum_{j=1}^{m} \sum_{l=1}^{n} \left[ b(s_l) - b(s_{l-1}) \right] \left( \sum_{k=l}^{n} v_{j,k} \right)^2 + i(-iq)^{-1/2} \sum_{j=1}^{m} \sum_{k=1}^{n} (g_j, a) c_{a,b}^{g_j} v_{j,k} \right\} d\nu(\vec{v}).
$$

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