Effects of impurities in noncentrosymmetric superconductors

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Abstract Effects of disorder on superconducting properties of noncentrosymmetric compounds are discussed. Elastic impurity scattering, even for scalar impurities, leads to a strongly anisotropic mixing of the electron states in the bands split by the spin-orbit coupling. We focus on the calculation of the critical temperature $T_c$, the upper critical field $H_{c2}$, and the spin susceptibility $\chi_{ij}$. It is shown that the impurity effects on the critical temperature are similar to those in multi-band centrosymmetric superconductors, in particular, Anderson’s theorem holds for isotropic singlet pairing. In contrast, scalar impurities affect the spin susceptibility in the same way as spin-orbit impurities do in centrosymmetric superconductors. Another peculiar feature is that in the absence of inversion symmetry scalar disorder can mix singlet and triplet pairing channels. This leads to significant deviations of the upper critical field from the predictions of the Werthamer-Helfand-Hohenberg theory in the conventional centrosymmetric case.

1 Introduction

Discovery of superconductivity in a heavy-fermion compound CePt$_3$Si (Ref. [1]) has stimulated considerable interest, both experimental and theoretical, in the properties of superconductors whose crystal lattice lacks a center of inversion. The list of noncentrosymmetric superconductors has been steadily growing and now includes dozens of materials, such as Ulr (Ref. [2]), CeRhSi$_3$ (Ref. [3]), CeIrSi$_3$ (Ref. [4]), Y$_2$C$_3$ (Ref. [5]), Li$_2$(Pd$_{1-x}$Pt$_x$)$_3$B (Ref. [6]), and many others.

A peculiar property of noncentrosymmetric crystals is that the spin-orbit (SO) coupling of electrons with the crystal lattice qualitatively changes the nature of the Bloch states, lifting the spin degeneracy of the electron bands almost every-
where in the Brillouin zone. The resulting nondegenerate bands are characterized by a complex spin texture and a nontrivial wavefunction topology in the momentum space.[7] This has profound consequences for superconductivity, including unusual nonuniform superconducting phases, both with and without magnetic field,[8][9][10][11][12][13] magnetoelastic effect,[14][15][16][17][18] and a strongly anisotropic spin susceptibility with a large residual component at zero temperature.[15][19][20][21] These and other properties are discussed in other chapters of this volume.

In this chapter we present a theoretical review of the effects of nonmagnetic impurities in superconductors without inversion symmetry. In Sect. 2, the disorder-averaged Green’s functions in the normal and superconducting states are calculated. In Sect. 3, the equations for the superconducting gap functions renormalized by impurities are used to find the critical temperature $T_c$. In Sect. 4, the upper critical field $H_{c2}$ is calculated at arbitrary temperatures. In Sect. 5, we calculate the spin susceptibility, focusing, in particular, on the effects of impurities on the residual susceptibility at $T = 0$. Sect. 6 contains a discussion of our results. Throughout this chapter we use the units in which $k_B = \hbar = 1$.

2 Impurity scattering in normal and superconducting state

Let us consider one spin-degenerate band with the dispersion given by $\varepsilon_0(k)$, and turn on the SO coupling. The Hamiltonian of noninteracting electrons in the presence of scalar impurities can be written in the form $H = H_0 + H_{imp}$, where

$$H_0 = \sum_{k, \alpha \beta} [\varepsilon_0(k) \delta_{\alpha \beta} + \gamma(k) \sigma_{\alpha \beta}] a_{k\alpha}^\dagger a_{k\beta}^\dagger,$$

(1)

$\alpha, \beta = \uparrow, \downarrow$ is the spin projection on the $z$-axis, $\sum_k$ stands for the summation over the first Brillouin zone, $\sigma$ are the Pauli matrices, and the chemical potential is included in $\varepsilon_0(k)$. The “bare” band dispersion satisfies $\varepsilon_0(-k) = \varepsilon_0(k)$, $\varepsilon_0(g^{-1}k) = \varepsilon_0(k)$, where $g$ is any operation from the point group $G$ of the crystal. The electron-lattice SO coupling is described by the pseudovector $\gamma(k)$, which has the following symmetry properties: $\gamma(k) = -\gamma(-k)$, $g \gamma(g^{-1}k) = \gamma(k)$. Its momentum dependence crucially depends on $G$, see Ref. [7]. For example, in the case of a tetragonal point group $G = C_{4v}$, which is realized, e.g., in $\text{CePt}_3\text{Si}$, $\text{CeRhSi}_3$, and $\text{CeIrSi}_3$, the simplest expression for the SO coupling is $\gamma(k) = \gamma_0(k_y \hat{x} - k_x \hat{y})$, which is also known as the Rashba model.[22] In contrast, in a cubic crystal with $G = O$, which describes $\text{Li}_2(\text{Pd}_{1-x}\text{Pt}_x)_3\text{B}$, we have $\gamma(k) = \gamma_0 k$.

The impurity scattering is described by the following Hamiltonian:

$$H_{imp} = \int d^3r \sum_\alpha U(r) \psi_\alpha^\dagger(r) \psi_\alpha(r).$$

(2)
The impurity potential \( U(r) \) is a random function with zero mean and the correlator 
\( \langle U(r_1)U(r_2) \rangle_{\text{imp}} = n_{\text{imp}} U_0^2 \delta(r_1 - r_2) \), where \( n_{\text{imp}} \) is the impurity concentration, and \( U_0 \) is the strength of an individual point-like impurity.

The Hamiltonian \( H \) can be diagonalized by a unitary transformation \( u_{\alpha\lambda}(k)c_{k\lambda} \), where \( \lambda = \pm \) is the band index (helicity), and

\[
  u_{+\lambda} = \frac{1}{\sqrt{2}} \sqrt{1 + \gamma_k \frac{\gamma_k}{|\gamma_k|}}, \quad u_{-\lambda} = \frac{1}{\sqrt{2}} \sqrt{1 - \gamma_k \frac{\gamma_k}{|\gamma_k|}}.
\]

with the following result:

\[
  H = \sum_k \sum_{\lambda = \pm} \xi_k(k)c_{k\lambda}^\dagger c_{k\lambda}.
\]

The energy of the fermionic quasiparticles in the \( \lambda \)th band is given by \( \xi_k(k) = \epsilon_0(k) + \lambda \gamma_k |\gamma_k| \).

This expression is even in \( k \) despite the antisymmetry of the SO coupling, which is a manifestation of the Kramers degeneracy: the states \( |k\lambda \rangle \) and \( |-k, \lambda \rangle \) are related by time reversal and therefore have the same energy. In real noncentrosymmetric materials, the SO splitting between the helicity bands is strongly anisotropic. Its magnitude can be characterized by \( E_{\text{SO}} = 2 \max_k |\gamma_k| \).

For instance, in CePt\(_3\)Si \( E_{\text{SO}} \) ranges from 50 to 200 meV (Ref. \([23]\)), while in Li\(_2\)Pd\(_3\)B it is 30 meV, reaching 200 meV in Li\(_3\)Pt\(_3\)B (Ref. \([24]\)).

In the band representation the impurity Hamiltonian \( \Sigma \) becomes

\[
  H_{\text{imp}} = \frac{1}{\mathcal{V}} \sum_{k\lambda} \sum_{\lambda'} U(k - k')w_{\lambda\lambda'}(k, k')c_{k\lambda}^\dagger c_{k'\lambda'},
\]

where \( \mathcal{V} \) is the system volume, \( U(q) \) is the Fourier transform of the impurity potential, and \( w_{\lambda\lambda'}(k, k') = \tilde{w}(k)\tilde{u}(k') \). We see that the impurity scattering amplitude in the band representation is momentum-dependent, even for isotropic scalar impurities, and also acquires both intraband and interband components, the latter causing mixing of the helicity bands. In the case of a slowly-varying random potential, keeping only the forward-scattering contribution \( U(q) \sim \delta_{q,0} \), one obtains:

\[
  w_{\lambda\lambda'}(k, k') = \delta_{\lambda\lambda'}\delta_{k,k'}, \text{i.e. the bands are not mixed.}
\]

The electron Green’s function in the helicity band representation is introduced in the standard fashion: \( G_{\lambda\lambda',\tau\tau'}(k, \omega_n) = \frac{\epsilon_{k\lambda}}{[\omega_n - \xi_k(k)]} \). In the absence of impurities, we have \( G_{0,\lambda\lambda',\tau\tau'}(k, \omega_n) = \delta_{\lambda\lambda'}/|\omega_n - \xi_k(k)| \), where \( \omega_n = (2n + 1)\pi T \) is the fermionic Matsubara frequency.

We will now show that the impurity-averaged Green’s function remains band-diagonal. The disorder averaging with the Hamiltonian \( \Sigma \) can be performed using the standard methods,\(^{[25]} \) resulting in the Dyson equation of the form \( \hat{G}^{-1} = \hat{G}_0^{-1} - \Sigma \), where \( \hat{G} \) is the average Green’s function and \( \Sigma \) is the impurity self-energy, see Fig. \([1]\). In the Born approximation, taking the thermodynamic limit \( \mathcal{V} \to \infty \), we have

\[
  \hat{\Sigma}(k, \omega_n) = n_{\text{imp}} U_0^2 \int \frac{d^3k'}{(2\pi)^3} \tilde{w}(k, k') \hat{G}(k', \omega_n) \tilde{w}(k', k).
\]
Fig. 1. The impurity self-energy in the band representation. The dashed line corresponds to \( n_{\text{imp}} U_0^2 \), the vertices include the anisotropy factors \( \hat{w}(k, k') \), and the solid line is the average Green’s function of electrons in the normal state. It is shown in the text that the self-energy is nonzero only if \( \lambda_1 = \lambda_4 \) and \( \lambda_2 = \lambda_3 \).

Seeking solution of the Dyson equation in a band-diagonal form: \( G_{\lambda \lambda'} = G_{\lambda} \delta_{\lambda \lambda'} \), the integrand on the right-hand side of Eq. (6) can be written as follows:

\[
\hat{u}(k') \hat{G}(k', \omega_n) \hat{u}^\dagger(k') = \frac{G_+(k', \omega_n) + G_-(k', \omega_n)}{2} \hat{t}_0 + \frac{G_+(k', \omega_n) - G_-(k', \omega_n)}{2} \hat{\gamma}(k') \hat{\tau},
\]

where \( \hat{\tau} \) are the Pauli matrices, and \( \hat{\gamma} = \gamma / |\gamma| \). The second line in this expression vanishes after the momentum integration, therefore \( \hat{\Sigma}(k, \omega_n) = \hat{\Sigma}(\omega_n) \hat{t}_0 \). The real part of the self-energy renormalizes the chemical potential, while for the imaginary part we obtain: \( \text{Im} \Sigma(\omega_n) = -\Gamma \text{sign} \omega_n \). Here \( \Gamma = \pi n_{\text{imp}} U_0^2 N_F \) is the elastic scattering rate, with \( N_F \) defined as follows: \( N_F = (N_+ + N_-) / 2 \), where \( N_\lambda = \gamma^{-1} \sum_k \delta(\xi_\lambda(k)) \) is the Fermi-level density of states in the \( \lambda \)th band. Thus we arrive at the following expression for the average Green’s function of the band electrons:

\[
G_{\lambda \lambda'}(k, \omega_n) = \frac{\delta_{\lambda \lambda'}}{i\omega_n - \xi_\lambda(k) + iT \text{sign} \omega_n}.
\] (7)

This derivation is valid under the assumption that the elastic scattering rate is small compared with the Fermi energy \( \varepsilon_F \), which justifies neglecting the diagrams with crossed impurity lines in the self-energy in Fig. 1.

### 2.1 Impurity averaging in superconducting state

In the limit of strong SO coupling, i.e. when the band splitting exceeds all superconducting energy scales, the Cooper pairing between the electrons with opposite momenta occurs only if they are from the same nondegenerate band. The pairing interaction in the strong SO coupling case is most naturally introduced using the basis of the exact band states,[19, 23, 26] which already incorporate the effects of the crystal lattice potential and the SO coupling. The total Hamiltonian including the pairing interaction is given by \( H = H_0 + H_{\text{imp}} + H_{\text{int}} \), where the first two terms are given by Eqs. (1) and (5) respectively, and the last term has the following form:
Physically, the pairing interaction is mediated by some bosonic excitations, e.g. phonons, and is effective only at frequencies smaller than a cutoff frequency $\omega_c$, which has to be included in the appropriate Matsubara sums. Alternatively, the cutoff can be imposed on the momenta in Eq. (8), as in the original Bardeen-Cooper-Schrieffer (BCS) model. The diagonal elements of the pairing potential $V_{\lambda, \lambda'}$ describe the intraband Cooper pairing, while the off-diagonal ones correspond to the pair scattering from one band to the other.

The pairing potential can be represented in the following form: $V_{\lambda, \lambda'}(k, k') = t_\lambda(k) t_{\lambda'}(k') \tilde{V}_{\lambda, \lambda'}(k, k')$, see Ref. [27]. Here $t_\lambda(k) = -t_\lambda(-k)$ are non-trivial phase factors originating in the expression for the time reversal operation for electrons in the helicity bands: $K(k, \lambda) = t_\lambda(k) - k, \lambda.$ [19] [26] While the components of $\tilde{V}_{\lambda, \lambda'}$ are even in both $k$ and $k'$ and invariant under the point group operations: $\tilde{V}_{\lambda, \lambda'}(g^{-1} k, g^{-1} k') = \tilde{V}_{\lambda, \lambda'}(k, k')$. The latter can be expressed in terms of the basis functions of the irreducible representations of the point group. [28] In general, the basis functions are different for each matrix element. Neglecting this complication, and also considering only the one-dimensional representation corresponding to the pairing channel with the maximum critical temperature, one can write

$$\tilde{V}_{\lambda, \lambda'}(k, k') = -V_{\lambda, \lambda'} \phi_{\lambda}(k) \phi_{\lambda'}^*(k'),$$

where the coupling constants $V_{\lambda, \lambda'}$ form a symmetric positive-definite $2 \times 2$ matrix, and $\phi_{\lambda}(k)$ are even basis functions. While $\phi_+(k)$ and $\phi_-(k)$ have the same symmetry, their momentum dependence does not have to be the same. The basis functions are assumed to be real and normalized: $\langle \phi_{\lambda}(k) \rangle^2 = 1$, where the angular brackets denote the Fermi-surface averaging in the $\lambda$th band: $\langle \ldots \rangle_\lambda = (1/N_\lambda) \sum_k \langle \ldots \rangle$. 

Treating the pairing interaction (8) in the mean-field approximation, one introduces the superconducting order parameters in the helicity bands, which have the following form: $\Delta_{\lambda}(k, q) = t_{\lambda}(k) \tilde{\Delta}_{\lambda}(k, q)$. The superconducting order parameter is given by a set of complex gap functions, one for each band, which are coupled due to the interband scattering of the Cooper pairs and other mechanisms, e.g. impurity scattering. Thus the overall structure of the theory resembles that of multi-band superconductors. [29] [30] If the pairing corresponds to a one-dimensional representation, see Eq. (9), then we have $\tilde{\Delta}_{\lambda}(k, q) = \eta_{\lambda}(q) \phi_{\lambda}(k)$.

Important particular case is a BCS-like model in which the pairing interaction is local in real space:

$$H_{\text{int}} = -V \int d^3 r \, \psi^\dagger_\lambda(r) \psi^\dagger_\lambda(r) \psi_\lambda(r) \psi_\lambda(r),$$

where $V > 0$ is the coupling constant. One can show [27] that in this model there is no interband pairing for any strength of the SO coupling, the order parameter has only
one component η, the gap symmetry corresponds to the unity representation with \( \phi_{\lambda}(k) = 1 \), and all coupling constants in Eq. (2) take the same value: \( V_{\lambda,\lambda'} = V/2 \).

Let us calculate the impurity-averaged Green’s functions in the superconducting state. To make notations compact, the normal and anomalous Green’s functions\(^{25}\) can be combined into a \( 4 \times 4 \) matrix \( \mathcal{G}(k_1, k_2; \tau) = -\langle T_{\tau} C_{k_1}(\tau) C_{k_2}^+(0) \rangle \), where \( C_k = (c_{k\lambda}, c_{-k\lambda}^T)^T \) are four-component Nambu operators. Averaging with respect to the impurity positions restores translational invariance: \( \langle \mathcal{G}(k_1, k_2; \omega_n) \rangle_{\text{imp}} = \delta_{k_1, k_2} \mathcal{G}(k, \omega_n) \), where

\[
\mathcal{G}(k, \omega_n) = \begin{pmatrix}
\hat{G}(k, \omega_n) & -\hat{F}(k, \omega_n) \\
-\hat{F}^T(k, \omega_n) - \hat{G}^T(-k, -\omega_n)
\end{pmatrix},
\]

and the hats denote \( 2 \times 2 \) matrices in the band space. The average Green’s function satisfies the Gor’kov equations: \( \langle \mathcal{G}_0^{-1} - \Sigma_{\text{imp}} \rangle \mathcal{G} = 1 \), where

\[
\mathcal{G}_0^{-1}(k, \omega_n) = \begin{pmatrix}
i\omega_n - \xi(k) & -\tilde{\Lambda}(k) \\
-\tilde{\Lambda}^+(k) & i\omega_n + \tilde{\xi}(k)
\end{pmatrix},
\]

and the impurity self-energy in the self-consistent Born approximation is

\[
\Sigma_{\text{imp}}(k, \omega_n) = n_{\text{imp}} U_{12}^2 \int \frac{d^3k'}{(2\pi)^3} W(k, k') \mathcal{G}(k', \omega_n) W(k', k),
\]

which is the Nambu-matrix generalization of Eq. (5). The \( 4 \times 4 \) matrix \( W \) is defined as follows: \( W(k, k') = \text{diag} \{ |\tilde{\omega}(k, k')|, -\tilde{\omega}^T(-k', -k) \} \). It is straightforward to show that \( |\tilde{\omega}^T(-k', -k)\rangle_{\lambda,\lambda'} = t_{\lambda'}^T(k) t_{\lambda}^*(-k') w_{\lambda,\lambda'}(k, k') \). We assume the disorder to be sufficiently weak, so that it is legitimate to use the Born approximation. Although there are some interesting qualitative effects in the opposite limit of strong disorder, such as the impurity resonance states,\(^{31}\) these are beyond the scope of our study.

In the absence of impurities, the Green’s functions have the following form: \( G_{0,\lambda,\lambda'}(k, \omega_n) = \delta_{\lambda,\lambda'} G_{0,\lambda}(k, \omega_n) \), and \( F_{0,\lambda,\lambda'}(k, \omega_n) = \delta_{\lambda,\lambda'} t_{\lambda}(k) F_{0,\lambda}(k, \omega_n) \), where

\[
G_{0,\lambda} = -\frac{i\omega_n + \xi_{\lambda}}{\omega_n^2 + \xi_{\lambda}^2 + |\Delta_{\lambda}|^2}, \quad F_{0,\lambda} = \frac{\Delta_{\lambda}}{\omega_n^2 + \xi_{\lambda}^2 + |\Delta_{\lambda}|^2}.
\]

In the presence of impurities, we seek solution of the Gor’kov equations in a band-diagonal form and require, for consistency, that the Nambu-matrix components of the self-energy are also band-diagonal. Then,

\[
\begin{pmatrix}
\Sigma_{11}(k, \omega_n) & \Sigma_{12}(k, \omega_n) \\
\Sigma_{21}(k, \omega_n) & \Sigma_{22}(k, \omega_n)
\end{pmatrix} = \delta_{\lambda,\lambda'} \begin{pmatrix}
\Sigma_1(\omega_n) & t_{\lambda}(k) \Sigma_2(\omega_n) \\
\Sigma_2(\omega_n) & -\Sigma_1(-\omega_n)
\end{pmatrix},
\]

where \( \Sigma_1 \) and \( \Sigma_2 \) satisfy the equations
Absorbing the real part of $\Sigma_1$ into the chemical potential, we have $\Sigma_1(\omega_n) = i\tilde{\Sigma}_1(\omega_n)$, where $\tilde{\Sigma}_1$ is odd in $\omega_n$.

Solving the Gor’kov equations we obtain the following expressions for the disorder-averaged Green’s functions:

$$G_\lambda(k,\omega_n) = -\frac{i\omega_n + \xi_\lambda(k)}{\omega_n^2 + \xi_\lambda^2(k) + |D_\lambda(k,\omega_n)|^2},$$

$$\tilde{F}_\lambda(k,\omega_n) = \frac{D_\lambda(k,\omega_n)}{\omega_n^2 + \xi_\lambda^2(k) + |D_\lambda(k,\omega_n)|^2},$$

where $\tilde{\omega}_n = \omega_n - \tilde{\Sigma}_1(\omega_n)$ and $D_\lambda(k,\omega_n) = \tilde{D}_\lambda(k) + \Sigma_2(\omega_n)$. Substituting these into Eqs. (16), we arrive at the self-consistency equations for the Matsubara frequency and the gap functions renormalized by impurities:

$$\tilde{\omega}_n = \omega_n + \frac{\Gamma}{2} \sum_{\lambda} \rho_\lambda \left\langle \frac{\omega_n}{\sqrt{\omega_n^2 + |D_{\lambda}(k,\omega_n)|^2}} \right\rangle_{\lambda},$$

$$D_\lambda(k,\omega_n) = \eta_\lambda \phi_\lambda(k) + \frac{\Gamma}{2} \sum_{\lambda'} \rho_{\lambda'} \left\langle \frac{D_{\lambda'}(k',\omega_n)}{\sqrt{\omega_n^2 + |D_{\lambda'}(k',\omega_n)|^2}} \right\rangle_{\lambda'},$$

where

$$\rho_\pm = \frac{N_+}{N_F} = 1 \pm \delta$$

are the fractional densities of states in the helicity bands. The parameter $\delta = (N_+ - N_-)/(N_+ + N_-)$ characterizes the strength of the SO coupling.

### 3 Gap equations and the critical temperature

The Gor’kov equations must be supplemented by self-consistency equations for the order parameter components, which have the form usual for two-band superconductors. In particular, for a uniform order parameter, $\eta_\lambda(q) = \eta_\lambda \delta(q)$, we have $\sum_{\lambda'} V_{\lambda,\lambda'}^{-1} \eta_{\lambda'} = T \sum_n \int \tilde{F}_\lambda(k,\omega_n) \phi_\lambda(k)$ (recall that the basis functions are assumed to be real). Using Eq. (17), we obtain:

$$\sum_{\lambda'} V_{\lambda,\lambda'}^{-1} \eta_{\lambda'} = \pi \rho_\lambda N_F T \sum_n \left\langle \frac{D_\lambda(k,\omega_n)}{\sqrt{\omega_n^2 + |D_\lambda(k,\omega_n)|^2}} \right\rangle_{\lambda}.$$
These equations are called the gap equations and, together with Eqs. (18) and (19), completely determine the properties of disordered noncentrosymmetric superconductors in the uniform state. The prime in the Matsubara sum means that the summation is limited to \(\omega_n \leq \omega_c\), where \(\omega_c\) is the BCS frequency cutoff.

The superconducting critical temperature can be found from Eq. (21) after linearization with respect to the order parameter components. It follows from Eq. (18) that \(\tilde{\omega}_n = \omega_n + \Gamma \text{sign} \omega_n \) near \(T_c\), and we obtain from Eq. (19) that \(\Sigma(\omega_n) = (\Gamma/2\omega_n) \sum_{\lambda} \rho_{\lambda}(\phi_{\lambda})\eta_{\lambda}\) (here and below we omit, for brevity, the arguments of the basis functions and the subscripts \(\lambda\) in the Fermi-surface averages). Therefore the linearized gap equations take the form

\[
a_{\lambda\lambda'} = \frac{1}{N_F} V^{-1}_{\lambda\lambda'} - \rho_{\lambda} \delta_{\lambda\lambda'} S_{01} - \frac{\Gamma}{2} \rho_{\lambda} \rho_{\lambda'} \langle \phi_{\lambda} \rangle \langle \phi_{\lambda'} \rangle S_{11},
\]

(22)

with \(S_{kl} = \pi T \sum_{\omega_n} \frac{1}{(\omega_n + \Gamma)} \). The Matsubara sums here can be easily calculated:

\[
S_{01} = \pi T \sum_{\omega_n} \frac{1}{\omega_n + \Gamma} = \ln \frac{\pi}{\epsilon C} \omega_c \pi T - \mathcal{F} \left(\frac{T}{\Gamma}\right),
\]

where \(C \approx 0.577\) is Euler’s constant, \(\mathcal{F}(x) = \psi \left(\frac{1}{2} + \frac{1}{2\pi x}\right) - \psi \left(\frac{1}{2}\right)\),

(23)

and \(\psi(x)\) is the digamma function. Note that the expression (23) for the impurity correction to \(S_{01}\) is valid if \(\Gamma \ll \omega_c\), when it is legitimate to extend the summation in \(2\pi T \sum_{\omega_n} [1/(\omega_n + \Gamma) - 1/\omega_n]\) to infinity and express the result in terms of the digamma functions, see Ref. [32]. Similarly, we obtain: \(S_{11} = \mathcal{F}(T/\Gamma)/\Gamma\). It is convenient to introduce the following notation for dimensionless coupling constants:

\[
g_{\lambda\lambda'} = \frac{N_F V_{\lambda\lambda'} \rho_{\lambda'}}{\rho_{\lambda}} = \frac{V_{\lambda\lambda'}}{N_{\lambda'}},
\]

(24)

(note that the matrix \(\hat{g}\) is not symmetric, in general). Then, the superconducting critical temperature \(T_c\) is found from the equation \(\det(\hat{\tau}_0 + \hat{g}\hat{M}) = 0\), where

\[
M_{\lambda\lambda'} = -\delta_{\lambda\lambda'} \ln \frac{2\epsilon C \omega_c}{\pi T_c} + \left(\delta_{\lambda\lambda'} - \frac{\rho_{\lambda} \rho_{\lambda'}}{2} \langle \phi_{\lambda} \rangle \langle \phi_{\lambda'} \rangle\right) \mathcal{F} \left(\frac{T_c}{\Gamma}\right),
\]

(25)

see Ref. [33].

In the absence of impurities, the second term in \(M_{\lambda\lambda'}\) vanishes, and we obtain the critical temperature of a clean superconductor:

\[
T_{c0} = \frac{2\epsilon C \omega_c}{\pi} e^{-1/g},
\]

(26)

where

\[
g = \frac{g_{++} + g_{--}}{2} + \sqrt{\left(\frac{g_{++} - g_{--}}{2}\right)^2 + g_{+-}g_{-+}}
\]

(27)
is the effective coupling constant. In the presence of impurities, the cases of conventional and unconventional pairing have to be considered separately.

**Unconventional pairing.** In this case $\langle \phi_\lambda \rangle = 0$, and we obtain the following equation for $T_c$:

$$\ln \frac{T_c}{T_c} = F \left( \frac{T_c}{\Gamma} \right).$$  \hspace{1cm} (28)

The reduction of the critical temperature is described by a universal function, which has the same form as in isotropic centrosymmetric superconductors with magnetic impurities,[34] or in anisotropically paired centrosymmetric superconductors with nonmagnetic impurities.[35, 28] In particular, at weak disorder, i.e. in the limit $\Gamma \ll T_{c0}$, we have

$$T_c = T_{c0} - \frac{\pi}{4} \frac{\Gamma}{T_{c0}}.$$  \hspace{1cm} (29)

We see that the critical temperature depends on nonmagnetic disorder, but in contrast to the unconventional case, the effect is not described by a universal Abrikosov-Gor’kov function.[36, 33] At weak disorder the suppression is linear in the scattering rate, but with a non-universal slope:

$$T_c = T_{c0} - \frac{1}{g} \left[ c_1 - \frac{\pi}{4} \frac{1}{g} \left( c_3 + \frac{c_5}{2 \sqrt{c_4}} \right) \right] \Gamma.$$  \hspace{1cm} (30)

In the case of strong impurity scattering, $\Gamma \gg T_{c0}$, we use $F(x) = \ln(1/x) + O(1)$ at $x \to 0$, to find that the critical temperature approaches the limiting value given by

$$T_c^* = T_{c0} \exp \left( \frac{1}{g} - \frac{c_1}{2c_3} \right),$$  \hspace{1cm} (31)

i.e. superconductivity is not completely destroyed by impurities. The explanation is the same as in the conventional two-gap superconductors, see e.g. Refs. [37, 38, 39]: Interband impurity scattering tends to reduce the difference between the gap magnitudes in the two bands, which costs energy and thus suppresses $T_c$, but only until both gaps become equal. One can show that both the coefficient in front of $\Gamma$ in Eq. (30) and the exponent in Eq. (31) are negative, i.e. $T_c^* < T_{c0}$. 

**Conventional pairing.** Assuming a completely isotropic pairing with $\phi_\lambda = 1$, we obtain:

$$\ln \frac{T_c}{T_c} = \frac{1}{2} \frac{c_1}{c_2 + c_3 F(x)} + \sqrt{c_4 + c_5 F(x)} + c_6 F^2(x).$$  \hspace{1cm} (29)

where $x = T_c / \Gamma$, and

$$c_1 = \rho_\pm (g_{--} - g_{++}) + \rho_- (g_{++} - g_{--}), \hspace{0.5cm} c_2 = \frac{g_{++} + g_{--}}{2}, \hspace{0.5cm} c_3 = \frac{\det \hat{g}}{2},$$

$$c_4 = \left( \frac{g_{++} - g_{--}}{2} \right)^2 + g_{+-} g_{-+}, \hspace{0.5cm} c_5 = (c_2 - c_1) \det \hat{g}, \hspace{0.5cm} c_6 = c_3^2.$$

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In the case of strong impurity scattering, $\Gamma \gg T_{c0}$, we use $F(x) = \ln(1/x) + O(1)$ at $x \to 0$, to find that the critical temperature approaches the limiting value given by

$$T_c^* = T_{c0} \exp \left( \frac{1}{g} - \frac{c_1}{2c_3} \right),$$  \hspace{1cm} (31)

i.e. superconductivity is not completely destroyed by impurities. The explanation is the same as in the conventional two-gap superconductors, see e.g. Refs. [37, 38, 39]: Interband impurity scattering tends to reduce the difference between the gap magnitudes in the two bands, which costs energy and thus suppresses $T_c$, but only until both gaps become equal. One can show that both the coefficient in front of $\Gamma$ in Eq. (30) and the exponent in Eq. (31) are negative, i.e. $T_c^* < T_{c0}$. 

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**Effects of impurities in noncentrosymmetric superconductors**

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**Unconventional pairing.** In this case $\langle \phi_\lambda \rangle = 0$, and we obtain the following equation for $T_c$:

$$\ln \frac{T_c}{T_c} = F \left( \frac{T_c}{\Gamma} \right).$$  \hspace{1cm} (28)

The reduction of the critical temperature is described by a universal function, which has the same form as in isotropic centrosymmetric superconductors with magnetic impurities,[34] or in anisotropically paired centrosymmetric superconductors with nonmagnetic impurities.[35, 28] In particular, at weak disorder, i.e. in the limit $\Gamma \ll T_{c0}$, we have

$$T_c = T_{c0} - \frac{\pi}{4} \frac{\Gamma}{T_{c0}}.$$  \hspace{1cm} (29)

We see that the critical temperature depends on nonmagnetic disorder, but in contrast to the unconventional case, the effect is not described by a universal Abrikosov-Gor’kov function.[36, 33] At weak disorder the suppression is linear in the scattering rate, but with a non-universal slope:

$$T_c = T_{c0} - \frac{1}{g} \left[ c_1 - \frac{\pi}{4} \frac{1}{g} \left( c_3 + \frac{c_5}{2 \sqrt{c_4}} \right) \right] \Gamma.$$  \hspace{1cm} (30)

In the case of strong impurity scattering, $\Gamma \gg T_{c0}$, we use $F(x) = \ln(1/x) + O(1)$ at $x \to 0$, to find that the critical temperature approaches the limiting value given by

$$T_c^* = T_{c0} \exp \left( \frac{1}{g} - \frac{c_1}{2c_3} \right),$$  \hspace{1cm} (31)

i.e. superconductivity is not completely destroyed by impurities. The explanation is the same as in the conventional two-gap superconductors, see e.g. Refs. [37, 38, 39]: Interband impurity scattering tends to reduce the difference between the gap magnitudes in the two bands, which costs energy and thus suppresses $T_c$, but only until both gaps become equal. One can show that both the coefficient in front of $\Gamma$ in Eq. (30) and the exponent in Eq. (31) are negative, i.e. $T_c^* < T_{c0}$. 

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**Effects of impurities in noncentrosymmetric superconductors**
In the BCS-like model (10) the pairing is isotropic and described by a single coupling constant \( V_{\lambda \lambda'} = V / 2 \), and we have \( g = N_F V \). Although the expression (26) for the critical temperature in the clean case has the usual BCS form, the analogy is not complete, because the order parameter resides in two nondegenerate bands, and \( N_F = (N_+ + N_-) / 2 \). In the presence of impurities, the right-hand side of Eq. (29) vanishes, therefore there is an analog of Anderson’s theorem: The nonmagnetic disorder has no effect on the critical temperature.

Another important particular case, possibly relevant to CePt\(_3\)Si, is the model in which only one, say \( \lambda = + \), band is superconducting, while the other band remains normal.\(^{[23, 33]}\) This can be described by setting \( V_{+} = V_{-} = 0 \). Using Eq. (25), we obtain the following equation for the critical temperature:

\[
\ln \frac{T_c^0}{T_c} = c_0 \phi \left( \frac{T_c}{\Gamma} \right), \tag{32}
\]

where \( c_0 = 1 - \rho_+ (\phi_+)^2 / 2 \). At weak disorder we have \( T_c = T_c^0 - c_0 (\pi \Gamma / 4) \), while at strong disorder \( T_c = T_c^0 (\pi T_c^0 / e C \Gamma)^{1/c_0} \). If the pairing is anisotropic but conventional, then, unlike the unconventional case with \( \langle \phi_+ \rangle = 0 \), the superconductivity is never completely destroyed, even at strong disorder.

### 3.1 Isotropic model

Finding the superconducting gap at arbitrary temperatures and impurity concentrations from the nonlinear gap equations (21) is more difficult than the calculation of \( T_c \). We focus on the case when the pairing is completely isotropic, i.e. \( \phi_+ (k) = \phi_-(k) = 1 \) and \( \Delta_{\lambda} (k) = \eta_{\lambda} \). The order parameter components can be chosen to be real, and the gap equations take the following form:

\[
\sum_{\lambda'} V_{\lambda \lambda'}^{-1} \eta_{\lambda'} = \pi \rho_{\lambda} N_F T \sum_n \frac{D_{\lambda} (\omega_n)}{\sqrt{\tilde{\omega}_n^2 + D^2_{\lambda} (\omega_n)}}. \tag{33}
\]

We further assume that the difference between \( \rho_+ \) and \( \rho_- \) can be neglected and the pairing strength, see Eq. (9), does not vary between the bands: \( V_{++} = V_{--} > 0 \). For the interband coupling constants, we have \( V_{+-} = V_{-+} \). The gap equations have two solutions: \( \eta_+ = \eta_- = \eta \) and \( \eta_+ = - \eta_- = \eta \). In the spin representation, the former corresponds to the singlet state, while the latter – to the “protected” triplet state.\(^{[20]}\)

The impurity responses of these two states turn out to be very different. \( \eta_+ = \eta_- = \eta \). In this case \( D_+ (\omega_n) = D_- (\omega_n) = D (\omega_n) \), and Eqs. (18) and (19) take the following form:

\[
\omega_n = \omega_n + \Gamma \frac{\omega_n}{\sqrt{\omega_n^2 + D^2}}, \quad D = \eta + \Gamma \frac{D}{\sqrt{\omega_n^2 + D^2}}.
\]
Introducing $Z(\omega_n) = 1 + \Gamma / \sqrt{\omega_n^2 + \eta^2}$, the solution of these equations is $D(\omega_n) = Z(\omega_n) \eta$, $\tilde{\omega}_n = Z(\omega_n) \omega_n$. Therefore, the gap equation (33) becomes

$$\eta = \pi g_1 T \sum_n \frac{D}{\sqrt{\tilde{\omega}_n^2 + D^2}} = \pi g_1 T \sum_n \frac{\eta}{\sqrt{\omega_n^2 + \eta^2}},$$

(34)

where $g_1 = (V_{++} + V_{+-}) N_f$. The scattering rate has dropped out, therefore there is an analog of Anderson’s theorem: neither the gap magnitude nor the critical temperature, are affected by impurities. Namely, we have $T_c(\Gamma) = T_{c0}$, see Eq. (26) with $g = g_1$, while the gap magnitude at $T = 0$ is given by the clean BCS expression: $\eta(T = 0) = \eta_0 = (\pi/e^2)T_{c0}$.

$$\eta_+ = -\eta_-. \eta_+$$

In this case $D_+(\omega_n) = -D_-(\omega_n) = \eta$, and we obtain from Eqs. (18), (19) and (33):

$$\tilde{\omega}_n = \omega_n + \Gamma \frac{\tilde{\omega}_n}{\sqrt{\tilde{\omega}_n^2 + \eta^2}}, \quad \eta = \pi g_2 T \sum_n \frac{\eta}{\sqrt{\omega_n^2 + \eta^2}},$$

(35)

where $g_2 = (V_{++} - V_{+-}) N_f$. In the absence of impurities, the critical temperature is given by the BCS expression (26) with $g = g_2$. If $V_{+-} > 0$ (attractive interband interaction), then $g_2 < g_1$ and the phase transition occurs into the state $\eta_+ = \eta_-$. However, if $V_{+-} < 0$ (repulsive interband interaction), then $g_2 > g_1$ and the phase transition occurs into the state $\eta_+ = -\eta_-$. In contrast to the previous case, both the critical temperature and the gap magnitude are now suppressed by disorder. The former is determined by the equation (29), which takes the same universal form as the Abrikosov-Gor’kov equation (28). Superconductivity is completely destroyed if the disorder strength exceeds the critical value $\Gamma_c = \langle \pi/2e^2 \rangle T_{c0}$.

To find the gap magnitude at $T = 0$ as a function of $\Gamma$ we follow the procedure described in Ref. [40]. Replacing the Matsubara sum by a frequency integral in the second of equations (35), we obtain:

$$\ln \frac{\eta_0}{\eta} = \int_0^\infty d\omega \left( \frac{1}{\sqrt{\omega^2 + \eta^2}} - \frac{1}{\sqrt{\tilde{\omega}^2 + \eta^2}} \right),$$

(36)

where $\eta_0 = 2\Gamma_c$ is the BCS gap magnitude in the clean case, and $\tilde{\omega}$ satisfies the equation $\tilde{\omega} = \omega + \Gamma \tilde{\omega} / \sqrt{\tilde{\omega}^2 + \eta^2}$. Transforming the second term on the right-hand side of Eq. (36) into an integral over $\tilde{\omega}$ we arrive at the following equation:

$$\ln \frac{\Gamma_c}{\Gamma} = \frac{\pi x}{4} - \ln(2x) + \theta(x - 1) \left[ \ln(x + \sqrt{x^2 - 1}) - \frac{x}{2} \arctan \sqrt{x^2 - 1 - \frac{x^2 - 1}{2x}} \right],$$

(37)

where $x = \Gamma / \eta$. This equation does not have solutions at $\Gamma > \Gamma_c$, which is consistent with the complete suppression of superconductivity above the critical disorder.
strength. In the weak disorder limit, \( \Gamma \ll \Gamma_c \), the solution is \( x \simeq \Gamma / 2 \Gamma_c \), while at \( \Gamma \rightarrow \Gamma_c \) we have \( x \simeq \sqrt{\Gamma_c / (2 (\Gamma_c - \Gamma))} \).

4 Upper critical field at arbitrary temperature

In this section, we calculate the upper critical field \( H_{c2}(T) \) of a disordered noncentrosymmetric superconductor described by the BCS-like model \((10)\). We assume a uniform external field \( \mathbf{H} \) and neglect the paramagnetic pair breaking. The noninteracting part of the Hamiltonian is given by

\[
\hat{h} = E_0(\mathbf{K}) + \gamma(\mathbf{K}) \hat{\sigma} + U(\mathbf{r}),
\]

where \( \mathbf{K} = -i \nabla + (e/c) \mathbf{A}(\mathbf{r}) \), \( \mathbf{A} \) is the vector potential, and \( e \) is the absolute value of the electron charge. The superconducting order parameter in the model \((10)\) is represented by a single complex function \( \eta(\mathbf{r}) \). According to Sect.\( \ref{sect:order-parameter} \) the zero-field critical temperature is not affected by scalar impurities. The critical temperature at \( \mathbf{H} \neq 0 \), or inversely the upper critical field as a function of temperature, can be found from the condition that the linearized gap equation \( |V^{-1} - T \sum_\nu \langle X(\omega_\nu) \rangle \eta(\mathbf{r})| = 0 \) has a nontrivial solution. Here the operator \( \hat{X}(\omega_\nu) \) is defined by the kernel

\[
X(\mathbf{r}, \mathbf{r}'; \omega_\nu) = \frac{1}{2} \left\langle \text{tr} \hat{g} \hat{G}(\mathbf{r}, \mathbf{r}'; \omega_\nu) \hat{g} \hat{G}^T(\mathbf{r}, \mathbf{r}'; -\omega_\nu) \right\rangle_{\text{imp}},
\]

where \( \hat{g} = i \hat{\sigma}_z \). The angular brackets denote the impurity averaging, and \( \hat{G}(\mathbf{r}, \mathbf{r}'; \omega_\nu) \) is the Matsubara Green’s functions of electrons in the normal state, which satisfies the equation

\[
(i \omega_\nu - \hat{h}) \hat{G}(\mathbf{r}, \mathbf{r}'; \omega_\nu) = \delta(\mathbf{r} - \mathbf{r}'),
\]

with \( \hat{h} \) given by expression \((38)\).

The impurity average of the product of two Green’s functions in Eq.\( \ref{eq:gap-function} \) can be represented graphically by the ladder diagrams, see Fig.\( \ref{fig:ladder-diagrams} \) (as before, we assume the disorder to be sufficiently weak for the diagrams with crossed impurity lines to be negligible). In order to sum the diagrams, we introduce an impurity-renormalized gap function \( \hat{D}(\mathbf{r}, \omega_\nu) \), which a matrix in the spin space satisfying the following integral equation:

\[
\hat{D}(\mathbf{r}, \omega_\nu) = \eta(\mathbf{r}) \hat{g} + \frac{1}{2} n_{\text{imp}} U_0^2 \hat{g} \int d^3 \mathbf{r}' \text{tr} \hat{g} \langle \hat{G}(\mathbf{r}, \mathbf{r}'; \omega_\nu) \rangle_{\text{imp}} \hat{D}(\mathbf{r}', \omega_\nu) \langle \hat{G}^T(\mathbf{r}, \mathbf{r}'; -\omega_\nu) \rangle_{\text{imp}} + \frac{1}{2} n_{\text{imp}} U_0^2 \hat{g} \int d^3 \mathbf{r}' \text{tr} \hat{g} \langle \hat{G}(\mathbf{r}, \mathbf{r}'; \omega_\nu) \rangle_{\text{imp}} \hat{D}(\mathbf{r}', \omega_\nu) \langle \hat{G}^T(\mathbf{r}, \mathbf{r}'; -\omega_\nu) \rangle_{\text{imp}}.
\]

This can be easily derived from the ladder diagrams in Fig.\( \ref{fig:ladder-diagrams} \) by representing each “rung” of the ladder as a sum of spin-singlet and spin-triplet terms:

\[
n_{\text{imp}} U_0^2 \delta_{\mu \nu} \delta_{\rho \sigma} = \frac{1}{2} n_{\text{imp}} U_0^2 g_{\mu \rho} g^\dagger_{\sigma \nu} + \frac{1}{2} n_{\text{imp}} U_0^2 g_{\mu \rho} g^\dagger_{\sigma \nu},
\]

where \( g_{\mu \rho} \) are the components of the electron charge.
where $\hat{g} = i\hat{\varphi}\delta_2$. 

Seeking solution of Eq. (40) in the form $\hat{D}(r, \omega_n) = d_0(r, \omega_n)\hat{g} + \hat{d}(r, \omega_n)\hat{g}$, we obtain a system of four integral equations for $d_a$, where $a = 0, 1, 2, 3$:

$$
\sum_{b=0}^{3} [\delta_{ab} - \Gamma \hat{G}_{ab}(\omega_n)] d_b(r, \omega_n) = \eta(r)\delta_{a0}.
$$

(42)

Here the operators $\hat{G}_{ab}(\omega_n)$ are defined by the kernels

$$
\hat{G}_{ab}(r, \omega_n) = \frac{1}{2N_F} \text{tr} \hat{g}_a \hat{G}(r, \omega_n) \hat{g}_b \hat{G}^T(r, \omega_n),
$$

(43)

with $\hat{g}_0 = \hat{g}$, and $\hat{g}_a = \hat{g}_a$ for $a = 1, 2, 3$. We see that, in addition to the spin-singlet component $d_0(r, \omega_n)$, impurity scattering also induces a nonzero spin-triplet component $\hat{d}(r, \omega_n)$. The linearized gap equation contains only the former. Indeed, using Eqs. (42) we obtain:

$$
\frac{1}{N_F V} \eta(r) - \pi T \sum_n \frac{d_0(r, \omega_n) - \eta(r)}{\Gamma} = 0.
$$

(44)

It is easy to see that the triplet component does not appear in the centrosymmetric case. Indeed, in the absence of the Zeeman interaction the spin structure of the Green’s function is trivial: $G_{ab}(r, \omega_n) = \delta_{ab} G(r, \omega_n)$. Then it follows from Eq. (43) that $\hat{G}_{ab}(\omega_n) = \delta_{ab} \hat{G}(\omega_n)$, therefore $d_0 = (1 - \Gamma \hat{G})^{-1}\eta$ and $\hat{d} = 0$.

The next step is to find the spectrum of the operators $\hat{G}_{ab}(\omega_n)$. At zero field, the average Green’s function has the following form:

$$
\hat{G}_0(k, \omega_n) = \sum_{\lambda = \pm} \tilde{N}_{\lambda}(k) G_{\lambda}(k, \omega_n),
$$

(45)

where $\tilde{N}_{\lambda} = (1 + \lambda \gamma / 2)$ are the helicity band projection operators, and $G_{\lambda}(k, \omega_n)$ are the impurity-averaged Green’s functions in the band representation, see Eq. (7). At $H \neq 0$, we have $\langle \hat{G}(r, \omega_n) \rangle_{\text{imp}} = \hat{G}_0(r - r', \omega_n) \exp[(ie/c) \int_{r'}^{r} A dr]$, where the integration is performed along a straight line connecting $r$ and $r'$. This approximation is legitimate if the temperature is not very low, so that the Landau level quantization can be neglected. It follows from Eq. (45) that $\hat{G}_{ab}(\omega_n) = Y_{ab}(q, \omega_n)$, where $D = -i\nabla + (2e/c)A$ and

\[
\begin{array}{ccc}
\beta & \gamma \\
\alpha & \delta \\
\end{array}
\]

\[
\begin{array}{ccc}
\beta & \gamma \\
\alpha & \delta \\
\end{array}
\]

Fig. 2 Impurity ladder diagrams in the Cooper channel. Lines with arrows correspond to the average Green’s functions of electrons, $\hat{g} = i\hat{\varphi}$, and the impurity (dashed) line is defined in the text, see Eq. (41).
The Cooper ladder is equal to isotropic bands with singlet-triplet mixing occurs due to the SO coupling and vanishes when between the densities of states and the Fermi velocities in the two bands and setting interband term in

\[ Y_{ab}(\mathbf{q}, \omega_n) = \frac{1}{2\pi N_F} \int \frac{d^3 k}{(2\pi)^3} \text{tr} \hat{g}_a^\dagger \hat{C}_0(\mathbf{k} + \mathbf{q}, \omega_n) \hat{g}_b \hat{C}_0^\dagger (\mathbf{-k}, -\omega_n). \]  

Substituting here expressions (45) and calculating the spin traces, we obtain for the singlet-singlet and singlet-triplet contributions:

\[ Y_{00} = \frac{1}{2} \sum_\lambda \rho_\lambda \left( \frac{1}{|\omega_n| + \Gamma + i\nu_\lambda(\mathbf{k})q \text{ sign } \omega_n/2} \right), \]

\[ Y_{0i} = Y_{i0} = \frac{1}{2} \sum_\lambda \rho_\lambda \left( \frac{\hat{y}_i(\mathbf{k})}{|\omega_n| + \Gamma + i\nu_\lambda(\mathbf{k})q \text{ sign } \omega_n/2} \right), \]

where \( \nu_\lambda = \partial \xi_\lambda / \partial k \) is the quasiparticle velocity in the \( \lambda \)th band. We see that the singlet-triplet mixing occurs due to the SO coupling and vanishes when \( \rho_+ = \rho_- = 1 \) and \( \nu_+ = \nu_- = \nu_F \). The triplet-triplet contributions can be represented as follows:

\[ Y_{ij} = Y_{ij}^{(1)} + Y_{ij}^{(2)}, \]

where

\[ Y_{ij}^{(1)} = \frac{1}{2} \sum_\lambda \rho_\lambda \left( \frac{\hat{y}_i(\mathbf{k}) \hat{y}_i(\mathbf{k})}{|\omega_n| + \Gamma + i\nu_\lambda(\mathbf{k})q \text{ sign } \omega_n/2} \right), \]

and

\[ Y_{ij}^{(2)} = \frac{1}{2\pi N_F} \sum_\lambda \int \frac{d^3 k}{(2\pi)^3} \left( \delta_{ij} - \hat{y}_i \hat{y}_j - i\lambda \epsilon_{i,j} \hat{y}_i \right) G_\lambda(\mathbf{k} + \mathbf{q}, \omega_n) G_{-\lambda}(\mathbf{-k}, -\omega_n). \]

The singlet impurity scattering channel, which is described by the first term in Eq. (41), causes only the scattering of intraband pairs between the bands. In contrast, the triplet impurity scattering can also create interband pairs, which are described by \( Y_{ij}^{(2)} \).

It is easy to show that if the SO band splitting exceeds both \( \omega_c \) and \( \Gamma \), then the interband term in \( Y_{ij} \) is smaller than the intraband one. Let us consider, for example, isotropic bands with \( \xi_{\pm}(\mathbf{k}) = \epsilon_0(\mathbf{k}) \pm \gamma \). Neglecting for simplicity the differences between the densities of states and the Fermi velocities in the two bands and setting \( \mathbf{q} = 0 \), we obtain from Eqs. (49) and (50):

\[ Y_{ij}^{(1)}(0, \omega_n) = \frac{\delta_{ij}}{3(|\omega_n| + \Gamma)} = Y_{\text{intra}}(\omega_n) \delta_{ij}, \]

\[ Y_{ij}^{(2)}(0, \omega_n) = \frac{2\delta_{ij}}{3(|\omega_n| + \Gamma)(1 + r^2)} = Y_{\text{inter}}(\omega_n) \delta_{ij}, \]

where \( r(\omega_n) = \gamma / (|\omega_n| + \Gamma) \). Due to the BCS cutoff, the maximum value of \( \omega_n \) in the Cooper ladder is equal to \( \omega_c \), therefore \( r_{\min} \sim E_{SO} / \max(\omega_c, \Gamma) \). We assume that this ratio is large, which is a good assumption for real materials, therefore

\[ \max_n \frac{Y_{\text{inter}}(\omega_n)}{Y_{\text{intra}}(\omega_n)} = \frac{2}{1 + r_{\min}^2} \sim \left[ \frac{\max(\omega_c, \Gamma)}{E_{SO}} \right]^2 \ll 1, \]
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at all Matsubara frequencies satisfying $|\omega_n| \leq \omega_c$.

Thus the interband contributions to the Cooper ladder can be neglected, and we obtain:

$$ Y_{ab}(\mathbf{q}, \omega_n) = \frac{1}{2} \sum_\lambda \rho_\lambda \left( \frac{\Lambda_{\lambda,a}(\mathbf{k}) \Lambda_{\lambda,b}(\mathbf{k})}{|\omega_n| + \Gamma + i\nu_{\lambda}(\mathbf{k}) |\mathbf{q}| \text{sign} \omega_n/2} \right), \quad (52) $$

where $\Lambda_{\lambda,0}(\mathbf{k}) = 1$ and $\Lambda_{\lambda,a}(\mathbf{k}) = \lambda \hat{\Lambda}_{a}(\mathbf{k})$ for $a = 1, 2, 3$. Making the substitution $\mathbf{q} \to \mathbf{D}$, we represent $\hat{\Lambda}_{ab}$ as a differential operator of infinite order:

$$ \hat{\Lambda}_{ab}(\omega_n) = \frac{1}{2} \int_0^\infty du \ e^{-u(|\omega_n|+\Gamma)} \sum_\lambda \rho_\lambda \left( \Lambda_{\lambda,a}(\mathbf{k}) \Lambda_{\lambda,b}(\mathbf{k}) e^{-i\nu_{\lambda}(\mathbf{k}) |\mathbf{D}| \text{sign} \omega_n/2} \right). \quad (53) $$

In order to solve Eqs. (42), with the operators $\hat{\Lambda}_{ab}(\omega_n)$ given by expressions (53), we follow the procedure described in Ref. [42]. Choosing the $z$-axis along the external field: $\mathbf{H} = H_{z}\hat{z}$, we introduce the operators $a_{\pm} = \ell_H(D_{x} \pm iD_{y})/2$, and $a_{3} = \ell_H D_{z}$, where $\ell_H = \sqrt{c/eH}$ is the magnetic length. It is easy to check that $a_{+} = a_{+}^{\dagger}$ and $[a_{-}, a_{+}] = 1$, therefore $a_{\pm}$ have the meaning of the raising and lowering operators, while $a_{3} = a_{3}^{\dagger}$ commutes with both of them: $[a_{3}, a_{\pm}] = 0$. It is convenient to expand both the order parameter $\eta$ and the impurity-renormalized gap functions $d_{\alpha}$ in the basis of Landau levels $|N,p\rangle$, which satisfy $a_{-}|N,p\rangle = \sqrt{N+1}|N+1,p\rangle$, $a_{-}|N,p\rangle = \sqrt{N}|N-1,p\rangle$, and $a_{3}|N,p\rangle = p|N,p\rangle$, where $N = 0, 1, \ldots$, and $p$ is a real number characterizing the variation of the order parameter along the field. We have

$$ \eta(\mathbf{r}) = \sum_{N,p} \eta_{N,p} |N,p\rangle, \quad d_{\alpha}(\mathbf{r}, \omega_n) = \sum_{N,p} d_{N,p}^{\alpha}(\omega_n) \langle N,p | \mathbf{r} \rangle. \quad (54) $$

According to Eqs. (42), the expansion coefficients can be found from the following algebraic equations:

$$ \sum_{N',p',b} \left[ \delta_{ab} \delta_{N'N} \delta_{pp'} - \Gamma \langle N,p | \hat{\Lambda}_{ab}(\omega_n) | N',p' \rangle \right] d_{N',p'}^{\alpha}(\omega_n) = \delta_{\alpha 0} \eta_{N,p}. \quad (55) $$

Substituting the solutions of these equations into

$$ \frac{1}{N_{p}V} \eta_{N,p} - \pi T \sum_{\pi} \frac{d_{N,p}^{\alpha}(\omega_n)}{\Gamma} = 0, \quad (56) $$

see Eq. (43), and setting the determinant of the resulting linear equations for $\eta_{N,p}$ to zero, one arrives at an equation for the upper critical field.

### 4.1 $H_{c2}(T)$ in a cubic crystal

In the general case, i.e. for arbitrary crystal symmetry and electronic band structure, the procedure outlined above does not yield an equation for $H_{c2}(T)$ in a closed
form, since all the Landau levels are coupled. In order to make progress, we focus on the cubic case, \( G = 0 \), with a parabolic band and the SO coupling given by \( \gamma(\mathbf{k}) = \gamma_0 \hat{\mathbf{k}} \). As for the parameter \( \delta \), which characterizes the difference between the band densities of states, see Eq. (20), we assume that

\[
\delta_c \ll |\delta| \leq 1,
\]

where \( \delta_c = \max(\omega_c, \Gamma)/\epsilon_F \ll 1 \). While the first inequality follows from the condition [31], which ensures the smallness of the interband contribution to the Cooper impurity ladder, the second one is always satisfied, with \(|\delta| \to 1\) corresponding to the rather unrealistic limit of extremely strong SO coupling.

In order to solve the gap equations, we make a change of variables in the triplet component: \( d_\pm = (d_1 \pm id_2)/\sqrt{2} \). Then, Eqs. (42) take the following form:

\[
\begin{pmatrix}
1 - \Gamma \hat{\mathcal{H}}_{00} & -\Gamma \hat{\mathcal{H}}_{03} & -\Gamma \hat{\mathcal{H}}_{02} - \Gamma \hat{\mathcal{H}}_{0+} \\
-\Gamma \hat{\mathcal{H}}_{03} & 1 - \Gamma \hat{\mathcal{H}}_{33} & -\Gamma \hat{\mathcal{H}}_{32} - \Gamma \hat{\mathcal{H}}_{3+} \\
-\Gamma \hat{\mathcal{H}}_{02} - \Gamma \hat{\mathcal{H}}_{0+} & -\Gamma \hat{\mathcal{H}}_{32} - \Gamma \hat{\mathcal{H}}_{3+} & 1 - \Gamma \hat{\mathcal{H}}_{22}
\end{pmatrix}
\begin{pmatrix}
d_0 \\
d_3 \\
d_+ \\
d_-
\end{pmatrix}
= \begin{pmatrix}
\eta \\
0 \\
0 \\
0
\end{pmatrix},
\]

(58)

where

\[
\hat{\mathcal{H}}_{0\pm} = \hat{\mathcal{H}}_{03} + i\hat{\mathcal{H}}_{02}, \quad \hat{\mathcal{H}}_{3\pm} = \hat{\mathcal{H}}_{33} \pm \Gamma \sqrt{2}, \quad \hat{\mathcal{H}}_{2\pm} = \hat{\mathcal{H}}_{11} \pm \Gamma \sqrt{2},
\]

with \( \hat{\mathcal{H}}_{ab} = \hat{\mathcal{H}}_{ba} \) given by Eqs. (53).

According to Eq. (55), one has to know the matrix elements of the operators \( \hat{\mathcal{H}}_{ab}(\omega_n) \) in the basis of the Landau levels \( |N,p\rangle \). After some straightforward algebra, see Ref. [43] for details, we find that \( \hat{\mathcal{H}}_{00}, \hat{\mathcal{H}}_{03}, \hat{\mathcal{H}}_{33}, \) and \( \hat{\mathcal{H}}_2 \) are diagonal in the Landau levels, while for the rest of the operators (59) the only nonzero matrix elements are as follows: \( \langle N, p | \hat{\mathcal{H}}_2 | N+1, p \rangle = \langle N+1, p | \hat{\mathcal{H}}_3 | N, p \rangle = \langle N, p | \hat{\mathcal{H}}_3 | N, p \rangle = \langle N+1, p | \hat{\mathcal{H}}_2 | N+2, p \rangle = \langle N+2, p | \hat{\mathcal{H}}_2 | N, p \rangle \). Therefore, the Landau levels are decoupled, and for \( \eta(r) = \hat{\eta}(r) |N,p\rangle \) (\( \hat{\eta} \) is a constant) the solution of Eqs. (58) has the following form:

\[
\begin{pmatrix}
d_0(r, \omega_n) \\
d_3(r, \omega_n) \\
d_+(r, \omega_n) \\
d_-(r, \omega_n)
\end{pmatrix}
= \begin{pmatrix}
d_0^0(\omega_n)(r) |N,p\rangle \\
d_0^3(\omega_n)(r) |N,p\rangle \\
d_+^0(\omega_n)(r) |N+1,p\rangle \\
d_+^0(\omega_n)(r) |N-1,p\rangle
\end{pmatrix},
\]

(60)

At arbitrary magnitude of the SO band splitting, the singlet-triplet mixing makes the equation for \( H_{22}(T) \) in noncentrosymmetric superconductors considerably more cumbersome than in the Helfand-Werthamer problem [32], even in our “minimal” isotropic model. It is even possible that, at some values of the parameters, the maximum critical field is achieved for \( N > 0 \) and \( p \neq 0 \), the latter corresponding to a disorder-induced modulation of the order parameter along the applied field. Leaving
these exotic possibilities aside, here we just assume that \( N = p = 0 \). Then the only nonzero components of the impurity-renormalized gap function, see Eq. (54), are \( d_{0,0}^0 \) and \( d_{0,0}^+ \).

It is convenient to introduce the reduced temperature, magnetic field, and disorder strength:

\[
t = \frac{T}{T_c}, \quad h = \frac{2H}{H_0}, \quad \zeta = \frac{\Gamma}{\pi T_c},
\]

where \( H_0 = \Phi_0/\pi \xi_0^2 \), \( \Phi_0 = \pi c/e \) is the magnetic flux quantum, and \( \xi_0 = v_F/2\pi T_c \) is the superconducting coherence length (\( v_F \) is the Fermi velocity). Using the expression for the critical temperature to eliminate both the frequency cutoff and the coupling constant from Eq. (56), we arrive at the following equation for the upper critical field \( h_{c2}(t) \):

\[
\ln \frac{1}{t} = 2 \sum_{n \geq 0} \left[ \frac{1}{2n+1} - t \frac{w_n(1 - \zeta p_n) - \zeta \delta^2 q_n^2}{(1 - \xi w_n)(1 - \zeta p_n) + \zeta^2 \delta^2 q_n^2} \right], \quad (61)
\]

where

\[
w_n = \int_0^\infty d\rho e^{-\theta_n \rho} \int_0^1 ds e^{-h \rho^2 (1 - s^2)/4},
\]

\[
p_n = \int_0^\infty d\rho e^{-\theta_n \rho} \int_0^1 ds \left[ 1 - \frac{h}{2} \rho^2 (1 - s^2) \right] e^{-h \rho^2 (1 - s^2)/4}, \quad (62)
\]

\[
q_n = \int_0^\infty d\rho e^{-\theta_n \rho} \int_0^1 ds \sqrt{\frac{h}{4} \rho (1 - s^2)} e^{-h \rho^2 (1 - s^2)/4},
\]

where \( \theta_n = (2n+1)t + \zeta \).

In the clean limit, i.e. at \( \zeta \to 0 \), or if the SO band splitting is negligibly small, i.e. at \( \delta \to 0 \), the Helfand-Werthamer equation for \( H_{c2} \) in a centrosymmetric superconductor[42] is recovered. Therefore, the absence of inversion symmetry affects the upper critical field only if disorder is present. One can expect that the effect will be most pronounced in the “dirty” limit, in which Eq. (61) takes the form of a universal equation describing the magnetic pair breaking in superconductors:[44]

\[
\ln \frac{1}{t} = \Psi \left( 1 + \frac{\sigma}{t} \right) - \Psi \left( \frac{1}{2} \right), \quad (63)
\]

Here the parameter

\[
\sigma = \frac{2 + \delta^2}{12 \zeta} h
\]

characterizes the pair-breaker strength. Note that the corresponding expression in the centrosymmetric case is different: \( \sigma_{CS} = h/6 \zeta \). Analytical expressions for the upper critical field can be obtained in the weak-field limit near the critical temperature:

\[
h_{c2}|_{t\to1} = \frac{24 \zeta}{(2 + \delta^2) \pi^2 (1 - t)}, \quad (65)
\]
and also at low temperatures:

\[ h_{c2}=\frac{\sqrt{2}\zeta}{2+\delta^2}\zeta. \]  

We see that nonmagnetic disorder suppresses the orbital pair breaking and thus enhances the upper critical field.

In the general case, \( H_{c2}(T) \) can be calculated analytically only in the vicinity of the critical temperature using the Ginzburg-Landau free energy expansion. The results for the impurity response turn out to be nonuniversal, i.e. dependent on the pairing symmetry, the values of the intra- and interband coupling constants, and the densities of states in the helicity bands.\[33\]

5 Spin susceptibility

In this section we calculate the magnetic response of a noncentrosymmetric superconductor, neglecting the orbital magnetic interaction and taking into account only the Zeeman coupling of the electron spins with a uniform external field \( H \):

\[ H_{Zeeman} = -\mu_B \sum_{k,\alpha\beta} \sigma_{\alpha\beta} d^\dagger_{k\alpha} a_{k\beta} = -H \sum_{k,\lambda\lambda'} m_{\lambda\lambda'}(k)c^\dagger_{k\lambda} c_{k\lambda'}, \]  

where \( \mu_B \) is the Bohr magneton. The components of the spin magnetic moment operator in the band representation have the following form:

\[ \hat{m}_x = \mu_B \left( \frac{\hat{\gamma}_x}{\gamma_{\perp}} - \frac{(\gamma_{\perp} \hat{\gamma}_x + i\gamma_{\perp})}{\gamma_{\perp}} \right), \]

\[ \hat{m}_y = \mu_B \left( \frac{\hat{\gamma}_y}{\gamma_{\perp}} - \frac{(\gamma_{\perp} \hat{\gamma}_y - i\gamma_{\perp})}{\gamma_{\perp}} \right), \]

\[ \hat{m}_z = \mu_B \left( \frac{\hat{\gamma}_z}{\gamma_{\perp}} - \frac{(\gamma_{\perp} \hat{\gamma}_z - i\gamma_{\perp})}{\gamma_{\perp}} \right), \]  

where \( \gamma_{\perp} = \sqrt{\gamma_x^2 + \gamma_y^2} \).

The magnetization of the system is expressed in terms of the Green’s functions as follows:

\[ m_i = \left( \frac{1}{\gamma} \right) \Gamma \sum_{\lambda\lambda'} \text{tr} m_{\lambda\lambda'}(k) G_{\lambda\lambda'}^{11}(k,\omega_n), \]

where \( G_{\lambda\lambda'}^{11}(k,\omega_n) \) is the impurity-averaged 4 \( \times \) 4 matrix Green’s function in the presence of magnetic field (recall that the upper indices label the Nambu matrix components, see Sect. 2.1). In a weak field, we have:

\[ \hat{m}_x = \mu_B \gamma_{\perp}/\gamma \]  

Treating the Zeeman interaction, Eq. (67), as a small perturbation and expanding \( \mathcal{G} \) in powers of \( H \), we obtain:

\[ \chi_{ij} = -T \sum_n \frac{1}{\gamma} \sum_{kk'} \langle \text{Tr} M_i(k) \mathcal{G}(k,\omega_n) M_j(k') \mathcal{G}(k',\omega_n) \rangle_{\text{imp}}, \]  

where \( \mathcal{G}(k,\omega_n) \) is the impurity-averaged Green’s function.
where \( M_i(k) = \text{diag} [\hat{m}_i(k), -\hat{m}_i^T(-k)] \). The Green’s functions here are unaveraged 4 \times 4 matrix Green’s functions at zero field, and the trace is taken in both the electron-hole and helicity indices.

In the clean case, one can evaluate expression (69) by summing over the Matsubara frequencies first, followed by the momentum integration. The susceptibility tensor can be represented as \( \chi_{ij} = \chi_{ij}^\perp + \chi_{ij}^\parallel + \tilde{\chi}_{ij} \) (Ref. [21]), where

\[
\chi_{ij}^\perp = -\mu_B^2 T \sum_n \int \frac{d^3 k}{(2\pi)^3} \hat{\gamma}_j \hat{\gamma}_i (G_{\lambda}^2 + |F_{\lambda}|^2) = \mu_B^2 N_{\lambda} \langle \hat{\gamma}_j \hat{\gamma}_i Y_\lambda \rangle \tag{70}
\]

are the intraband contributions, determined by the thermally-excited quasiparticles near the Fermi surfaces. Here \( Y_\lambda(k, T) = 2 \int_0^\infty d\xi (-\partial f/\partial E_\lambda) \) is the angle-resolved Yosida function, \( f(\epsilon) = (e^{\epsilon/T} + 1)^{-1} \) is the Fermi-Dirac distribution function, \( E_\lambda(k) = \sqrt{\xi^2 + |\Delta_\lambda(k)|^2} \) is the energy of quasiparticle excitations in the \( \lambda \)th band, and, as in the previous sections, the subscripts \( \lambda \) in the Fermi-surface averages are omitted for brevity. The interband contribution is given by

\[
\tilde{\chi}_{ij} = -2\mu_B^2 T \sum_n \int \frac{d^3 k}{(2\pi)^3} (\delta_{ij} - \hat{\gamma}_j \hat{\gamma}_i) \left( G_+ G_- + \text{Re} \tilde{F}_+ \tilde{F}_- \right)
\]

\[
\simeq -\mu_B^2 \int \frac{d^3 k}{(2\pi)^3} \frac{\delta_{ij} - \hat{\gamma}_j \hat{\gamma}_i}{|\gamma|} \left[ f(\xi_+) - f(\xi_-) \right]. \tag{71}
\]

Since \( \tilde{\chi}_{ij} \) is determined by all quasiparticles in the momentum-space shell “sandwiched” between the Fermi surfaces, it is almost unchanged when the system undergoes the superconducting transition, in which only the electrons near the Fermi surface are affected.

Collecting together the inter- and intraband contributions, we arrive at the following expression for the spin susceptibility of a clean superconductor:

\[
\chi_{ij} = \tilde{\chi}_{ij} + \mu_B^2 N_F \sum_\lambda \rho_\lambda \langle \hat{\gamma}_j \hat{\gamma}_i Y_\lambda \rangle. \tag{72}
\]

At zero temperature there are no excitations \( (Y_\lambda = 0) \) and the intraband terms are absent, but the susceptibility still has a nonzero value given by \( \tilde{\chi}_{ij} \). The temperature dependence of the susceptibility in the superconducting state at \( 0 < T < T_c \) is determined by the intraband terms, with the low-temperature behavior depending crucially on the magnitude of the SO coupling at the gap nodes. While in the fully gapped case the intraband susceptibility is exponentially small in all directions, in the presence of the lines of nodes it is proportional to either \( T \) or \( T^3 \), depending on whether or not the zeros of \( \Delta_\lambda(k) \) coincide with those of \( \gamma(k) \), see Ref. [45]. In Fig. 3 the temperature dependence of \( \chi_{ij} \) is plotted for a Rashba superconductor with \( \gamma(k) = \gamma_y (k \times \hat{z}) \) and a cylindrical Fermi surface (referred to as the 2D model), and also for a cubic superconductor with \( \gamma(k) = \gamma_0 k \) and a spherical Fermi surface (the 3D model). In both cases, the gaps in the two helicity bands are assumed to be isotropic and have the same magnitude.
Fig. 3. The clean-case temperature dependence of the transverse components of the susceptibility for the 2D model (the solid line), and of all three components for the 3D model (the dashed line); $\chi_P = 2\mu_B^2N_F$ is the Pauli susceptibility.

Now let us include the scalar disorder described by Eq. (2), or, equivalently, Eq. (5). After the impurity averaging, Eq. (69) is represented by a sum of the ladder diagrams containing the average Green’s functions (17). In contrast to the clean case, it is not possible to calculate the Matsubara sums before the momentum integrals. To make progress, one should add to and subtract from Eq. (69) the normal-state susceptibility. It is easy to show that the latter is not affected by impurities and is therefore given by $\chi_{N,ij} = \tilde{\chi}_{ij} + \mu_B^2N_F\sum_\lambda \rho_\lambda \langle \hat{\gamma}_i \hat{\gamma}_j \rangle$, see Eq. (72). Then,

$$\chi_{ij} - \chi_{N,ij} = -T \sum_n \frac{1}{\gamma} \sum_{kk'} \left( \langle \text{Tr} M_i G M_j \rangle_{\text{imp}} - \langle \text{Tr} M_i G N M_j \rangle_{\text{imp}} \right),$$

(73)

where $G_N$ is the unaveraged $4 \times 4$ matrix Green’s function in the normal state with impurities, and the Matsubara summation is limited to the frequencies $|\omega_n| \leq \omega_c$, at which the gap function is nonzero.

Due to convergence of the expression on the right-hand side of Eq. (73), one can now do the momentum integrals first. The second term vanishes, while the calculation of the ladder diagrams in the first term is facilitated by the observation that the integrals of the products of the Green’s functions from different bands are small compared with their counterparts containing the Green’s functions from the same band. The argument is similar to the one leading to Eq. (51). For example, assuming that both the SO band splitting and the pairing are isotropic, i.e. $\xi_{\pm}(k) = \epsilon_0(k) \pm \gamma$ and $D_\pm = D$, we have

$$\max_n \left\{ \int d\epsilon_n G_{\lambda\lambda} G_{-\lambda\lambda} \right\} = \max_n \frac{1}{1 + \gamma^2/\Omega_n^2} \sim \left[ \max (\omega_c, \Gamma) \right]^2 \ll 1,$$

where $\Omega_n = \sqrt{\omega_n^2 + |D|^2}$. In the same way, one can also obtain estimates for the momentum integrals containing anomalous Green’s functions:
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\[
\text{max}_n \int \frac{d\omega_n}{\pi} G_{\lambda} \bar{F}_{\lambda} = \text{max}_n \frac{1 - i\gamma / \omega_n}{1 + \gamma^2 / \Omega_n^2} \sim \frac{\text{max}(\omega_n, \Gamma)}{E_{SO}} \ll 1,
\]

\[
\text{max}_n \int \frac{d\omega_n}{\pi} \bar{F}_{\lambda} \bar{F}_{\lambda} = \text{max}_n \frac{1}{1 + \gamma^2 / \Omega_n^2} \sim \left[ \frac{\text{max}(\omega_n, \Gamma)}{E_{SO}} \right]^2 \ll 1.
\]

Thus we see that it is legitimate to keep only the same-band contributions to the impurity ladder on the right-hand side of Eq. (73).

Following Ref. [21], we obtain the following expression for the spin susceptibility:

\[
\chi_{ij} = \chi_{N,ij} + \frac{2\mu_B N_F}{\Gamma} \sum_n \frac{\partial X_i(\omega_n)}{\partial H_j}, \tag{74}
\]

where \(X_i\) are found from the equations

\[
X_i - \sum_j \left( A_{1,ij} X_j + A_{2,ij} Y_j + A_{2,ij}^* Y_j^* \right) = X_{0,i},
\]

\[
Y_i - \sum_j \left( 2A_{2,ij}^* X_j + A_{3,ij} Y_j + A_{4,ij} Y_j^* \right) = Y_{0,i}. \tag{75}
\]

The notations here are as follows:

\[
A_{1,ij} = \frac{\Gamma}{2} \sum_\lambda \rho_\lambda \left< \frac{\gamma \gamma_j |D_\lambda|^2}{\Omega_n^3} \right>, \quad A_{2,ij} = \frac{\Gamma}{4} \sum_\lambda \rho_\lambda \left< \frac{i\gamma \gamma_j^* \omega_n D_\lambda^*}{\Omega_n^3} \right>,
\]

\[
A_{3,ij} = \frac{\Gamma}{4} \sum_\lambda \rho_\lambda \left< \frac{\gamma \gamma_j (2\omega_n^2 + |D_\lambda|^2)}{\Omega_n^3} \right>, \quad A_{4,ij} = \frac{\Gamma}{4} \sum_\lambda \rho_\lambda \left< \frac{\gamma \gamma_j D_\lambda^2}{\Omega_n^3} \right>,
\]

\[
X_{0,i} = -\mu_B \sum_j A_{1,ij} H_j, \quad Y_{0,i} = -2\mu_B \sum_j A_{2,ij}^* H_j,
\]

and \(\Omega_n = \sqrt{\omega_n^2 + |D_\lambda|^2}\). Due to fast convergence, the Matsubara summation in Eq. (74) can be extended to all frequencies.

5.1 Residual susceptibility for isotropic pairing

The general expression for the spin susceptibility, Eq. (74), is rather cumbersome. On the other hand, application of our results to real noncentrosymmetric materials is complicated by the lack of information about the superconducting gap symmetry and the distribution of the pairing strength between the bands. As an illustration, we focus on the isotropic pairing model introduced in Sect. 3.1. In this model, the order parameter magnitudes in the two bands are the same, but the relative phase can be either 0 or \(\pi\). While in the clean limit the spin susceptibility for both states is given by Eq. (72), the effects of impurities have to be analyzed separately.

\(\eta_+ = \eta_- = \eta\). Solving equations (75) we obtain, in the coordinate system in which \(\langle \gamma \gamma_j \rangle\) is diagonal, the following expression for the nonzero components of
the susceptibility tensor:

\[
\frac{\chi_{ii}(T)}{\chi_P} = 1 - \langle \hat{\gamma}_{i}^{2} \rangle \pi T \sum_{n} \frac{\eta^{2}}{\omega_{n}^{2} + \eta^{2}} \frac{1}{\sqrt{\omega_{n}^{2} + \eta^{2} + \Gamma_{i}}},
\]  

(76)

where \(\Gamma_{i} = (1 - \langle \hat{\gamma}_{i}^{2} \rangle) \Gamma\).

We are particularly interested in the effect of disorder on the residual susceptibility at zero temperature. In this limit, the Matsubara sum in Eq. (76) can be replaced by a frequency integral, which gives

\[
\frac{\chi_{ii}(T = 0)}{\chi_P} = 1 - \langle \hat{\gamma}_{i}^{2} \rangle + \langle \hat{\gamma}_{i}^{2} \rangle \Phi_{1} \left( \frac{\Gamma_{i}}{\eta_{0}} \right),
\]

(77)

where \(\eta_{0} = (\pi / eC) T_{c}\) is the gap magnitude at \(T = 0\), and

\[
\Phi_{1}(x) = 1 - \frac{\pi}{2x} \left( 1 - \frac{4}{\pi \sqrt{1 - x^{2}}} \arctan \sqrt{\frac{1 - x}{1 + x}} \right).
\]

While the first two terms on the right-hand side of Eq. (77) represent the residual susceptibility in the clean case, the last term describes the impurity effect. In a weakly-disordered superconductor, using the asymptotics \(\Phi_{1}(x) \approx \pi x / 4\), we find that the residual susceptibility increases linearly with disorder. In the dirty limit, \(\Gamma \gg \eta_{0}\), we have \(\Phi_{1}(x) \to 1\), therefore \(\chi_{ii}(T = 0)\) approaches the normal-state value \(\chi_{P}\). For the two simple band-structure models (2D and 3D) discussed earlier in this section, the Fermi-surface averages can be calculated analytically, and we obtain the results plotted in Fig. 4.

Thus we see that, similarly to spin-orbit impurities in a usual centrosymmetric superconductor, scalar impurities in a noncentrosymmetric superconductor lead to an enhancement of the spin susceptibility at \(T = 0\). Since the interband contribution is not sensitive to disorder, this effect can be attributed to an increase in the intraband susceptibilities.

\(\eta_{\pm} = -\eta_{-} = \eta\). From Eqs. (74) and (75) we obtain the nonzero components of the susceptibility tensor:

\[
\frac{\chi_{ii}(T)}{\chi_P} = 1 - \langle \hat{\gamma}_{i}^{2} \rangle \pi T \sum_{n} \frac{\eta^{2}}{(\omega_{n}^{2} + \eta^{2})^{3/2} - \Gamma \langle \hat{\gamma}_{i}^{2} \rangle \eta^{2}}.
\]

(78)

We note that for a spherical 3D model with \(\langle \hat{\gamma}_{i}^{2} \rangle = 1/3\) this expression has exactly the same form as the susceptibility of the superfluid \(^{3}\)He-B in aerogel, see Refs. [40] and [47].

At \(T = 0\), the expression (78) takes the following form:

\[
\frac{\chi_{ii}(T = 0)}{\chi_P} = 1 - \langle \hat{\gamma}_{i}^{2} \rangle + \langle \hat{\gamma}_{i}^{2} \rangle \Phi_{2} \left( \frac{\Gamma}{\eta} \right),
\]

(79)

where
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\[ \Phi_2(x) = 1 - \int_{y_{\min}}^{\infty} dy \left[ 1 - \frac{x}{(y^2 + 1)^{3/2}} \right] \frac{1}{(y^2 + 1)^{3/2} - x\langle \hat{\gamma}_i^2 \rangle} \]

and \( y_{\min} = \theta(x - 1)\sqrt{x^2 - 1} \). The last term on the right-hand side of Eq. (79) describes the effect of impurities. According to Sect. 3.1, superconductivity is suppressed above the critical disorder strength \( \Gamma_c = (\pi/2e^F)T_0 \). For a given \( \Gamma \), one should first obtain the gap magnitude from Eq. (37) and then calculate \( \Phi_2(x) \). In the weak disorder limit, we have \( \Phi_2(x) \simeq (3\pi x/16)(1 - \langle \hat{\gamma}_i^2 \rangle) \), i.e. the residual susceptibility increases linearly with disorder. At \( \Gamma \to \Gamma_c \), we have \( \Phi_2(x) \to 1 \) and \( \chi_{ii}(T = 0) \to \chi_P \). The dependence of \( \chi_{ii}(T = 0) \) on the disorder strength for the 2D

Fig. 4 The residual susceptibility at \( T = 0 \) vs disorder strength for \( \eta_+ = \eta_- \). The solid line corresponds to the transverse components in the 2D case (\( \chi_{zz} = \chi_P \) and is disorder-independent), the dashed line – to the diagonal components in the 3D case.

Fig. 5 The residual susceptibility at \( T = 0 \) vs disorder strength for \( \eta_+ = -\eta_- \). The solid line corresponds to the transverse components in the 2D case (\( \chi_{zz} = \chi_P \) and is disorder-independent), the dashed line – to the diagonal components in the 3D case.
and 3D models is plotted in Fig. 5. As in the case $\eta_+ = \eta_-$, the residual susceptibility is enhanced by impurities.

6 Conclusions

Scalar disorder in noncentrosymmetric superconductors causes anisotropic mixing of the electron states in the bands split by the SO coupling. The critical temperature is generally suppressed by impurities, but this happens differently for conventional and unconventional pairing. For all types of unconventional pairing (which is defined as corresponding to a non-unity representation of the crystal point group, with vanishing Fermi-surface averages of the gap functions), the impurity effect on $T_c$ is described by the universal Abrikosov-Gor’kov equation. The same is also true for certain types of conventional pairing, in particular the “protected” isotropic triplet state with $\eta_+ = -\eta_-$. Any deviation from the Abrikosov-Gor’kov curve, in particular, an incomplete suppression of superconductivity by strong disorder, is a signature of conventional pairing symmetry.

The impurity-induced mixing of singlet and triplet pairing channels makes the magnetic response of noncentrosymmetric superconductors with the SO coupling different from the centrosymmetric case. In an isotropic BCS-like model, the upper critical field $H_{c2}$ is enhanced by disorder at all temperatures, the magnitude of the effect depending on the SO coupling strength. In general, the effect of impurities on the slope of $H_{c2}$ is sensitive to the pairing symmetry and the band structure.

Concerning the spin susceptibility, we found that scalar impurities in noncentrosymmetric superconductors act similarly to spin-orbit impurities in centrosymmetric superconductors, in the sense that they enhance the residual susceptibility at $T = 0$. The quantitative details again depend on the band structure, the anisotropy of the SO coupling, and the symmetry of the order parameter.

Most of the experimental work on noncentrosymmetric superconductors has been done on CePt$_3$Si. In this compound the Fermi surface is quite complicated and consists of multiple sheets. It is not known which of them are superconducting. The order parameter symmetry is not known either, although there is experimental evidence that there are lines of nodes in the gap. The data on the impurity effects are controversial. The experimental samples seem to be rather clean, with the ratio of the elastic mean free path $l$ to the coherence length $\xi_0$ ranging from 4 (Ref. [1]) to 10 $\rightarrow$ 27 (Ref. [51]. There are indications that $T_c$ is indeed suppressed by structural defects and/or impurities in some samples. On the other hand, the values of both the critical temperature and the upper critical field in polycrystalline samples are higher than in single crystals. This is opposite to what has been observed in other unconventional superconductors and also disagrees with the theoretical predictions, assuming that the polycrystals are intrinsically more disordered than the single crystals. In addition, the low-temperature behaviour of the penetration depth in disordered samples is unusual and cannot be explained by existing
theoretical models. In order to resolve these issues, more systematic studies of the disorder effects in a wide range of impurity concentrations are needed.

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