Abstract

It is shown how $W$-algebras emerge from very peculiar canonical transformations with respect to the canonical symplectic structure on a compact Riemann surface. The action of smooth diffeomorphisms of the cotangent bundle on suitable generating functions is written in the BRS framework while a $W$-symmetry is exhibited. Subsequently, the complex structure of the symmetry spaces is studied and the related BRS properties are discussed. The specific example of the so-called $W_3$-algebra is treated in relation to some other different approaches.
1 Introduction

In the last decade, a large body of literature has been devoted to the study of the so-called $W$-algebras. The latter were first introduced as higher spin extension of the Virasoro algebra, through the operator product expansion (OPE) of the stress-energy tensor and primary fields in two dimensional Conformal Field Theory (CFT). Whenever the OPE of some (primary) fields is needed for dynamical analysis, the associated CFT provides the computational artillery and selects the monomials of the expansion. Since a way of extending the bidimensional conformal symmetry yields the notion of $W$-algebras, in this context CFT arises as a perturbative groundstate around which one expands the theory. $W$-algebras have been widely occurred in two dimensional physics, such as the 2d anisotropic oscillator, Coulombian and generalized hartmann potentials, gravitation ($W$-gravity), condensed matter (quantum Hall effect), integrable models (KdV, Toda), phase transitions in two dimensions, intermediate statistics, solitary waves since the intrinsic intertwining between internal and space time symmetries realized in these algebra provide an amazing landscape to discuss dynamics in a physical context. The interested reader is referred to some review papers such as, and many others which have been implicitly quoted for the sake of brevity.

However, it turns out that the OPE mechanism is not the most natural way to get an algebra. Indeed, the discovery of composition laws in products between quantum fields at coincident points, when inserted in Green functions of complete set of states, do not allow to define an algebra. Beyond them, further mathematical properties have to be imposed, so that a lot of mathematical verifications, spread in the literature, have to be performed. Anyhow the role played by CFT in all the basic aspects of $W$-algebras is crucial for these reasons and their mathematical aspects and geometrical origin are important questions.

Furthermore, the distinctive feature of the (local) conformal symmetry in two dimensions is that it provides an infinite number of conservation laws. It turns out that this infinite symmetry has several consequences as was first shown in; a conformal symmetry is a (local) diffeomorphism (i.e. a locally smooth and invertible map) which acts on the Hilbert space of the theory as a symmetry.

For all these above reasons one can ask whether a more canonical approach, such as the ones leading to Noether theorem in Lagrangian Field theory, or canonical transformations, may provide the construction of a certain type of $W$-algebras. Indeed, if we can find some kind of space-time transformations (due to the local character of these algebras) where all the formal
requirements (associativity, existence of inverse and so on) are fulfilled, then the usual calculation artillery of Field Theory allows a realization of these algebras. In doing so, a larger insight in the intrinsic nature of this symmetry could be achieved.

Already Witten \[9\] has suggested that these algebras could have a space-time origin as symplectic diffeomorphisms, and Hull \[10\] gave some examples in terms of diffeomorphisms acting on fields (the so-called $W$-matter).

In this paper we shall construct some of these $W$-algebras from \textit{only coordinate symmetries} on the cotangent bundle. In particular, we shall derive generating functions for a very restrictive class of canonical transformations whose reduction will provide in general an infinite chain of smooth changes of coordinates from a background and to new coordinate frames.

In mathematical terms, we shall study a realisation of the algebra of diffeomorphisms on the cotangent bundle over a world-sheet $\Sigma$, $\text{Diff}_0(T^*\Sigma)$, which extends the $W$-algebra. A particular attention will be devoted to the complex structure mappings induced by these canonical transformations.

In Section 2 we shall describe the geometrical approach which will lead to B.R.S transformations as they will be given in Section 3.

The general reduction to $W$-symmetry will be described in Section 4 and in particular the $w_\infty$ symmetry will be derived. A particular attention will be paid on the role of complex structure of each two dimensional graded space whose diffeomorphism algebra will generate the $W_n$ algebras. In particular the generalization of the Beltrami differential parameters will also be discussed. Our approach will be complementary to those given in \[11, 12, 13\].

In order to illustrate the construction, we shall discuss in Section 5 the $W_3$ case. Two different diffeomorphism symmetries will give rise to two kinds of $W_3$-algebras respectively. Firstly, a geometrical meaning will be given for the $W_3$-algebra already discussed by Sorella et al \[14\] and found through the OPE mechanism by many authors \[15\]. Secondly, it will be shown how the $W_3$ algebra given by Grimm et al \[17\], and Ader and coworkers \[18, 19\] arises from the construction.

## 2 The Geometrical Approach

Given a smooth compact 2d-surface $\Sigma$, we shall right away choose a prescribed local complex analytic coordinates $(z, \bar{z})$. On the smooth cotangent bundle $T^*\Sigma$, we will use throughout the
paper local adapted complex coordinates \((z, \bar{z}; y, \bar{y})\), with fibre coordinates \((y, \bar{y})\) for the natural coframe associated to the holonomic coordinates \((z, \bar{z})\). The former will be shorthandly denoted by \((z, y)\). The canonical 1-form, \(\theta\) on \(T^*\Sigma\) then locally writes in the local chart \(U_{(z,y)}\) of \(T^*\Sigma\),

\[
\theta|_{U_{(z,y)}} = y_z dz + \bar{y}_{\bar{z}} d\bar{z}.
\]  

(2.1)

The fundamental 2-form, \(\Omega \equiv d\theta\), reads in local adapted coordinates on \(T^*\Sigma\),

\[
\Omega|_{U_{(z,y)}} = dy_z \wedge dz + d\bar{y}_{\bar{z}} \wedge d\bar{z},
\]  

(2.2)

and is closed. By Stokes theorem, one also has

\[
\int_{\Sigma} \Omega = \int_{\partial \Sigma} \theta,
\]  

(2.3)

which vanishes if \(\Sigma\) is without boundary.

Let us now consider a smooth change of local coordinates on \(T^*\Sigma\), \((Z, \bar{Z}; Y, \bar{Y})\), or \((Z, Y)\) for short. The canonical form then locally reads,

\[
\theta|_{U_{(Z,Y)}} = Y_Z dZ + \bar{Y}_{\bar{Z}} d\bar{Z},
\]  

(2.4)

and the corresponding fundamental 2-form,

\[
\Omega|_{U_{(Z,Y)}} = d\theta = dY_Z \wedge dZ + d\bar{Y}_{\bar{Z}} \wedge d\bar{Z}.
\]  

(2.5)

This smooth change of local coordinates on \(T^*\Sigma\) turns out to be a canonical transformation if the fundamental 2-form remains invariant, which means that, on \(U_{(z,y)} \cap U_{(Z,Y)}\),

\[
\Omega|_{U_{(z,y)}} = \Omega|_{U_{(Z,Y)}},
\]  

(2.6)

This condition selects particular coordinate transformation laws which are usually called canonical transformations \([21]\). Eq.(2.6) implies that on \(U_{(z,y)} \cap U_{(Z,Y)}\),

\[
\theta|_{U_{(z,y)}} - \theta|_{U_{(Z,Y)}} = dF,
\]  

(2.7)

where \(F\) is a generating function in the base coordinates \((z, \bar{z}; Z, \bar{Z})\) on \(\pi U_{(z,y)} \cap \pi U_{(Z,Y)}\) which is diffeomorphic to \(\mathbb{R}^2 \times \mathbb{R}^2\) and \(\pi\) is the projection on \(\Sigma\). It will be however more convenient
to use the local coordinates \((z, Y)\) by introducing the generating function \(\Phi(z, Y)\) through a Legendre transformation of \(F\),

\[
d\Phi(z, Y) \equiv d\left(F(z, Z) + Y_Z Z + \overline{Y_Z Z}\right) = y_z dz + \overline{y_z} d\overline{z} + dY_Z Z + d\overline{Y_Z Z}. \tag{2.8}
\]

The function \(\Phi(z, Y)\) is the generating function of the canonical transformation \((z, y) \rightarrow (Z, Y)\). It is locally defined (up to a total derivative) in the independent coordinates \((z, Y)\) of the smooth trivial bundle \(\Sigma \times \mathbb{R}^2\) and has a non-singular Hessian, \(|\frac{\partial^2 \Phi}{\partial z \partial Y}\| \neq 0\). On \(\Sigma \times \mathbb{R}^2\) the total differential is

\[
d = dz \frac{\partial}{\partial z} + d\overline{z} \frac{\partial}{\partial \overline{z}} + dY_Z \frac{\partial}{\partial Y_Z} + d\overline{Y_Z} \frac{\partial}{\partial \overline{Y_Z}} \equiv dz + dY_Z, \tag{2.9}
\]

and \(d^2 = 0\) yields \(d^2 = d^2_y Z = dz dY_Z + dY_Z d_z = 0\), or in local coordinates,

\[
\begin{bmatrix} \partial_z, \frac{\partial}{\partial Y_Z} \end{bmatrix} = 0, \quad \begin{bmatrix} \overline{\partial_z}, \frac{\partial}{\partial \overline{Y_Z}} \end{bmatrix} = 0, \tag{2.10}
\]

with the complex conjugate expressions. Since \(d^2 \Phi = 0\) we get the important identities:

\[
\overline{\partial}_y z = \partial \overline{\pi_Z}, \quad \frac{\partial}{\partial \overline{Y_Z}} Z = \frac{\partial}{\partial Y_Z} \overline{Z}, \quad \frac{\partial}{\partial Y_Z} y_z = \partial Z, \quad \frac{\partial}{\partial \overline{Y_Z}} y_z = \partial \overline{Z}, \tag{2.11}
\]

(and their c.c.). The generating function \(\Phi\) induces the following canonical transformation defined by,

\[
(z, \overline{z}) = (z, \overline{z}), \quad y_z(z, Y) = \partial \Phi(z, Y), \tag{2.12}
\]
\[
Z(z, Y) = \frac{\partial}{\partial Y_Z} \Phi(z, Y), \quad (Y_Z, \overline{Y_Z}) = (Y_Z, \overline{Y_Z}), \tag{2.13}
\]

with also the complex conjugate coordinates. In account of this choice of generating function, \(((z, \overline{z}); y_z, \overline{y_z})\) becomes a family of local smooth sections of the cotangent bundle \(T^*\Sigma\) parametrized by \(Y\) and over \(\pi U_{(z,y)} \cap \pi U_{(Z,Y)}\).

As is well known, by the implicit function theorem, eq.(2.12) is locally solved in \(Y(z, y)\) and plugging into Eq.(2.13) one ends with

\[
Z(z, y) = \left(\frac{\partial}{\partial Y_Z} \Phi(z, Y)\right)_{|Y = Y(z, y)}. \tag{2.14}
\]

* From now on, we shall reserve \(\partial\)'s for \(\partial \equiv \partial_z, \overline{\partial} \equiv \overline{\partial}_z\).
One also checks that
\[ \theta|_{U(z,y)} = d_z \Phi(z, Y)|_{Y = Y(z,y)}, \] (2.15)
where the 1-form, \( d_z \Phi(z, Y) \), on \( \Sigma \times \mathbb{R}^2 \), is evaluated on a solution \( Y(z,y) \) of eq.(2.12). Accordingly, the fundamental 2-form simply reads,
\[ \Omega|_{U(z,y)} = d\theta|_{U(z,y)} = d_y \left( d_z \Phi(z, Y)|_{Y = Y(z,y)} \right) = \left( d_y d_z \Phi(z, Y) \right)|_{Y = Y(z,y)}, \] (2.16)
where the r.h.s. may be viewed as the restriction onto solutions \( Y(z,y) \) of a 2-form exact on each factor of the product \( \Sigma \times \mathbb{R}^2 \). An important remark is in order. Due to \( d^2 \Phi = 0 \), the 2-forms \( dY_z Z \wedge dY_Z + dY_z \overline{Z} \wedge d\overline{Y}_Z \) or \( d_y d_z \Phi(z, Y) \) identically vanish in \( \Omega \). This yields two very particular classes of canonical transformations, in particular, those which allow reparametrizations of \( Z(z,Y) \) in the \( Y_Z \) fibre coordinate only. One has the following theorem.

**Theorem 1** On the smooth trivial bundle \( \Sigma \times \mathbb{R}^2 \), the vertical holomorphic change of local coordinates,
\[ Z((z,\overline{z}), (Y_Z, \overline{Y}_Z)) \longrightarrow Z((z,\overline{z}), \mathcal{F}(Y_Z), \overline{Y}_Z), \] (2.17)
where \( \mathcal{F} \) is a holomorphic function in \( Y_Z \), while the horizontal holomorphic change of local coordinates,
\[ y_z(z,\overline{z}, (Y_Z, \overline{Y}_Z)) \longrightarrow y_z(f(z), \overline{z}, (Y_Z, \overline{Y}_Z)), \] (2.18)
where \( f \) is a holomorphic function in \( z \), are both canonical transformations.

The latter type of canonical transformations states that the fundamental 2-form remains unchanged under local holomorphic changes of the local \( z \)-coordinate on the basis, namely holomorphic changes of charts. In the former, one can restrict oneself to a very particular situation. Since any smooth change of local complex coordinates on the base Riemann surface \( \Sigma \),
\[ (z,\overline{z}) \longrightarrow (Z(z,\overline{z}), \overline{Z}(z,\overline{z})) \],
can be obtained by the generating function,
\[ \Phi(z, Y) = Z(z,\overline{z}) Y_Z + \overline{Z}(z,\overline{z}) \overline{Y}_Z, \] (2.19)
one may consider more generally on \( \Sigma \times \mathbb{R}^2 \) the following holomorphically split generating function in the vertical direction, namely,
\[ \Phi(z, Y) = \Phi_1((z,\overline{z}), Y_Z) + \overline{\Phi}_1((z,\overline{z}), \overline{Y}_Z), \] (2.20)
Considering the expansion around \( Y_Z = \overline{Y}_Z = 0 \), the generating function \( \Phi \) will be written in formal power series,

\[
\Phi(z, Y) = \sum_{n \geq 1} \left[ \frac{1}{n!} Y^n_Z \left( \frac{\partial}{\partial Y Z} \right)^n \Phi_1(z, Y) \big|_{Y_Z = 0} \right] + \sum_{n \geq 1} \left[ \frac{1}{n!} \overline{Y}_Z^n \left( \frac{\partial}{\partial \overline{Y}_Z} \right)^n \Phi_1(z, Y) \big|_{\overline{Y}_Z = 0} \right]
\]

\[\equiv \sum_{n \geq 1} \left[ Y^n_Z Z^{(n)}(z, \overline{z}) \right] + \sum_{n \geq 1} \left[ \overline{Y}_Z^n \overline{Z}^{(n)}(z, \overline{z}) \right] \tag{2.21}\]

where we have set \( \Phi(z, Y) \big|_{Y_Z, \overline{Y}_Z = 0} = \Phi_0(z, \overline{z}) \) and introduced together with its complex conjugate counterpart,

\[Z^{(n)}(z, \overline{z}) \equiv \left[ \frac{1}{n!} \left( \frac{\partial}{\partial Y Z} \right)^n \Phi_1(z, Y) \right] \big|_{Y_Z = 0} \tag{2.22}\]

as the jet coordinates of \( \Phi \) at \( Y = 0 \) with respect to \( Y \) parametrized by \((z, \overline{z})\). Making use of these jet coordinates will provide a new scenario for generating \( W \)-algebras. Note also that taking \( Y = 0 \) implies \( y = 0 \).

We shall now introduce some quantities which will be useful in the sequel. Let us denote,

\[
\lambda(z, Y) = \partial_y \frac{\partial}{\partial Y Z} \Phi(z, Y), \quad \lambda(z, Y)\mu(z, Y) = \overline{\lambda}_y \frac{\partial}{\partial \overline{Y}_Z} \Phi(z, Y),
\]

\[
\lambda(y, z) = \overline{\lambda}_y \frac{\partial}{\partial \overline{Y}_Z} \Phi(z, Y), \quad \lambda(y, z)\overline{\mu}(z, Y) = \partial_{\overline{Y}_Z} \Phi(z, Y). \tag{2.23}\]

Before being restricted to the solutions \( Y = Y(z, y) \), the globally defined fundamental 2-form \( (\ref{2.16}) \) rewrites in the \((z, Y)\) independent local coordinates,

\[
\Omega(z, Y) = \left( \lambda dY_Z + \lambda_d dY_Z \right) \wedge dz + \left( \overline{\lambda}_d Y_Z + \lambda y dY_Z \right) \wedge d\overline{z}
\]

\[= dY_Z \wedge \lambda \left( dz + \mu d\overline{z} \right) + dY_{\overline{z}} \wedge \overline{\lambda} \left( d\overline{z} + \overline{\mu} dz \right) \tag{2.24}\]

from which one can read off by construction,

\[
d_y z(z, Y) = \lambda(z, Y) \left( dz + \mu(z, Y) d\overline{z} \right) \tag{2.25}\]

\[
d_y y_z(z, Y) = \lambda(z, Y) \left( dY_Z + \frac{\overline{\lambda}(z, Y) \overline{\mu}(z, Y)}{\lambda(z, Y)} dY_{\overline{z}} \right)
\]

\(^1\)Setting rather \( \Phi(z, Y) \big|_{Y_Z, \overline{Y}_Z = 0} = \Phi_0(z, \overline{z}) \), induces that 1-form \( y \) is defined up to the coboundary \( d\Phi_0 \). Also \( \Phi \) is connected to the identity map \( zY_Z + \overline{Y}_Z \).
(and the c.c. expressions). The latter reveal a parametrization of complex structures by a Beltrami differential depending on the vertical coordinate \( Y, \mu(z, Y) \), on the base surface \( \Sigma \) according to the complex coordinates \((z, \bar{z})\), and by \( \frac{\bar{\lambda}(z, Y)\bar{\mu}(z, Y)}{\lambda(z, Y)} \) on the vertical fiber when \((Y_Z, \bar{Y}_Z)\) complex coordinates are used. The restriction condition \( d\Omega|_{Y=\bar{Y}(z,y)} = 0 \) gives,

\[
d\Omega|_{U(z,y)} = \overline{\lambda}(z, Y)dz \wedge dz \wedge dY_Z + \overline{\lambda}(z, Y)\overline{\mu}(z, Y))d\bar{z} \wedge dz \wedge d\bar{Y}_Z + \delta(\lambda(z, Y)\mu(z, Y))dz \wedge d\bar{z} \wedge dY_Z + \delta\lambda(z, Y)dz \wedge d\bar{z} \wedge d\bar{Y}_Z \\
+ d\bar{Y}_Z \wedge \frac{\partial}{\partial Y_Z}\left(\lambda(z, Y)\left(dz + \mu(z, Y)d\bar{z}\right)\right) \wedge dY_Z \\
+ dY_Z \wedge \frac{\partial}{\partial Y_Z}\left(\overline{\lambda}(z, Y)\left(d\bar{z} + \overline{\mu}(z, Y)d\bar{z}\right)\right) \wedge d\bar{Y}_Z = 0,
\]

and yields the Beltrami identities,

\[
\overline{\lambda}(z, Y) = \partial(\mu(z, Y)\lambda(z, Y)), \quad \frac{\partial}{\partial \bar{Y}_Z}\lambda(z, Y) = \frac{\partial}{\partial Y_Z}\left(\overline{\lambda}(z, Y)\overline{\mu}(z, Y)\right)
\]

(2.27)

together with their complex conjugate expressions. The Liouville theorem follows from,

\[
\det\left|\frac{\partial Z(z, Y)}{\partial z}\right| = \lambda(z, Y)\overline{\lambda}(z, Y)(1 - \mu(z, Y)\overline{\mu}(z, Y)) = \det\left|\frac{\partial y(z, Y)}{\partial Y}\right|.
\]

(2.28)

Therefore, one can consider two distinct smooth changes of local coordinates which are indeed strictly related through the given generating function \( \Phi \). The former on the base, \((z, \bar{z}) \rightarrow (Z, \bar{Z})\) for fixed \((Y_Z, \bar{Y}_Z)\), and the latter on the fiber over \((z, \bar{z})\), \((y, \bar{y}) \rightarrow (Y_Z, \bar{Y}_Z)\).

Accordingly, the transformation laws of the derivative operators on each factor of the base or on the cotangent fiber respectively read,

\[
\frac{\partial}{\partial z} = \frac{\partial_z - \overline{\mu}(z, Y)\overline{\partial}\bar{z}}{\lambda(z, Y)(1 - \mu(z, Y)\overline{\mu}(z, Y))},
\]

(2.29)

\[
\frac{\partial}{\partial Y_Z}|_{Y = Y(z, y)} = \lambda(z, Y)\left(\frac{\partial}{\partial y_z} + \mu(z, Y)\frac{\partial}{\partial \bar{y}_z}\right)|_{Y = Y(z, y)} \equiv \lambda(z, Y)|_{Y = Y(z, y)}\mathcal{D}^z,
\]

(2.30)

(with their c.c. expressions) and where the combination \( \mathcal{D}^z \) of the vertical derivatives with respect to the \((y, \bar{y})\) fiber coordinates has been introduced. First, it has to be noted that Eqs.(2.27) infer the following important identity,

\[
\mathcal{D}^z \ln \overline{\lambda}(z, Y)|_{Y = Y(z, y)} = \frac{\overline{\mathcal{D}^z \mu}(z, Y)|_{Y = Y(z, y)} + \mu(z, Y)|_{Y = Y(z, y)}\mathcal{D}^z\overline{\mu}(z, Y)|_{Y = Y(z, y)}(z, Y)}{1 - \mu(z, Y)|_{Y = Y(z, y)}\overline{\mu}(z, Y)|_{Y = Y(z, y)}},
\]

(2.31)
with of course the c.c. formula, and secondly, from Eq.(2.10) one gets,

\[
\begin{align*}
\left[ \partial_z, D^z \right] &= -\partial_z \log \lambda(z, Y)|_{Y=Y(z,y)} D^z, \\
\left[ \overline{\partial}_z, D^z \right] &= -\overline{\partial}_z \log \lambda(z, Y)|_{Y=Y(z,y)} D^z \\
&= -\left( \partial \mu(z, Y)|_{Y=Y(z,y)} D^z - \mu(z, Y)|_{Y=Y(z,y)} \left[ \partial_z, D^z \right] \right),
\end{align*}
\]

(2.32)

(and the c.c. expressions), and,

\[
\left[ D^z, \overline{\partial}^z \right] = (D^z \overline{\partial}(z, Y)|_{Y=Y(z,y)}) \frac{\partial}{\partial y_z} - (\overline{\partial}^z \mu(z, Y)|_{Y=Y(z,y)}) \frac{\partial}{\partial \overline{\partial} z},
\]

(2.33)

from which one verifies that \( y_z \) and \( \overline{\partial}_z \) turn out to be independent fiber coordinates,

\[
\left[ \frac{\partial}{\partial y_z}, \frac{\partial}{\partial \overline{\partial} z} \right] = 0.
\]

(2.34)

3 The smooth diffeomorphism action

So far we have considered canonical transformations as changes of local coordinates on the cotangent bundle \( T^*\Sigma \) while those generated by holomorphically split generating functions in the vertical coordinates have been preferred. Since our strategy amounts to choosing \((z, Y)\) as independent local coordinates with generating function \( \Phi(z, Y) \), it is recalled that \( y_z = \partial \Phi(z, Y) \) defines a section of the cotangent bundle \( T^*\Sigma \xrightarrow{\pi} \Sigma \). One may wonder how \( \Phi(z, Y) \) is affected by the action of smooth diffeomorphisms of the trivial bundle \( \Sigma \times \mathbb{R}^2 \) as a manifold with coordinates \((z, Y)\). In order to implement the diffeomorphism action, the BRS differential algebra setting will be used, see e.g. [22]. We consider the action of smooth diffeomorphisms homotopic to the identity map, \( \varphi_t = id_{\Sigma} + t c + o(t) \), where \( c = \xi^z(z, Y)\partial + \xi^{z^*}(z, Y)\overline{\partial} + \eta^z(z, Y) \frac{\partial}{\partial y_z} + \eta^{z^*}(z, Y) \frac{\partial}{\partial \overline{\partial} z} \equiv \xi \cdot \partial + \eta \cdot \overline{\partial} \), is the smooth Faddeev-Popov ghost associated to vector fields with respect to the background complex coordinates \((z, Y)\), \( s z = s Y = 0 \). The corresponding infinitesimal action on fields over the cotangent bundle is obtained by a (graded) Lie derivative encoded in a nilpotent BRS operation \( S \equiv L_c = i_c d - d i_c \), \( S^2 = 0 \).

\(^{4}\)Mathematically speaking, \( c \) is the generator of the Grassmann algebra of the dual of the Lie algebra of smooth diffeomorphisms.
Then at the infinitesimal level, the BRS variation of $\Phi(z,Y)$ locally writes,

$$S\Phi(z,Y) = L_c \Phi(z,Y) = \xi(z,Y) \cdot y(z,Y) + \eta(z,Y) \cdot Z(z,Y) \equiv \Lambda(z,Y),$$  \hspace{1cm} (3.1)

where $\Lambda(z,Y)$ is a Grassmann function subject to $S\Lambda(z,Y) = 0$. The variation of the fundamental 2-form (2.16) in $U(z,y) \cap U(Z,Y)$ writes,

$$S\Omega(z,Y) = dY_Z d_z \Lambda(z,Y),$$  \hspace{1cm} (3.2)

and for the canonical 1-form one has, in virtue of Eq.(2.15),

$$S\theta|_{U(z,y)} = d_z \Lambda(z,Y)|_{Y=Y(z,y)}$$

$$S\theta|_{U(Z,Y)} = \left[d \left( Y_Z \frac{\partial}{\partial Y_Z} \Lambda(z,Y) + Y_Z \frac{\partial}{\partial Z} \Lambda(z,Y) \right) - dY_Z \Lambda(z,Y) \right]|_{z=z(Z,Y)}$$  \hspace{1cm} (3.3)

in such a way that the invariance of $\Omega$ and $\theta$ writes,

$$S \int_{\Sigma} \Omega = S \int_{\partial \Sigma} \theta = 0.$$  \hspace{1cm} (3.4)

To the general canonical transformations given by Eqs.(2.12) and (2.13) preserving the symplectic form $\Omega$ as stated in Eq.(2.6), there will correspond the infinitesimal variations respectively,

$$Sy_z(z,Y) = \partial_z \Lambda(z,Y), \hspace{1cm} SZ(z,Y) = \frac{\partial}{\partial Y_Z} \Lambda(z,Y) \equiv \Upsilon^Z(z,Y),$$  \hspace{1cm} (3.5)

and their complex conjugates. The latter is the infinitesimal transformation of the new local holomorphic coordinate $Z$ in terms of the ghost vector $c = \Upsilon^Z \partial_Z + \Upsilon^\overline{Z} \partial_{\overline{Z}}$ expressed in the new system $(Z, \overline{Z})$. We also have,

$$S\lambda(z,Y) = \frac{\partial}{\partial Y_Z} \partial_z \Lambda(z,Y), \hspace{1cm} S \left( \lambda(z,Y) \overline{\mu}(z,Y) \right) = \frac{\partial}{\partial Y_Z} \partial_z \Lambda(z,Y).$$  \hspace{1cm} (3.6)

In order to restore the explicit dependence in the $Y$ coordinates, from Eqs.(3.3) and (2.31), one defines,

$$SZ(z,Y) \equiv \Upsilon^Z(z,Y) = \frac{\partial}{\partial Y_Z} \Lambda(z,Y) \equiv \lambda(z,Y)C(z,Y),$$  \hspace{1cm} (3.7)

so that,

$$C(z,Y) = \frac{1}{\lambda(z,Y)} \Upsilon^Z(z,Y)$$  \hspace{1cm} (3.8)
We now restrict ourselves to the solutions $Y = Y(z, y)$ according to the strategy defined in Eq. (2.30),

$$
C(z, Y)|_{Y = Y(z, y)} = \mathcal{D}^\varpi \left( \Lambda(z, Y)|_{Y = Y(z, y)} \right) \\
= c(z, y) + \mu(z, Y)|_{Y = Y(z, y)} \overline{c}(z, y),
$$

where we have set,

$$
c(z, y) = \frac{\partial}{\partial y_z} \left( \Lambda(z, Y)|_{Y = Y(z, y)} \right), \quad \overline{c}(z, y) = \frac{\partial}{\partial \overline{y}_{\overline{z}}} \left( \Lambda(z, Y)|_{Y = Y(z, y)} \right). \quad (3.10)
$$

Thus Eq. (3.8) when restricted to the solutions $Y = Y(z, y)$ is given by (3.9) in terms of a derivative of $\Lambda$ with respect the background $(y_z, y_{\overline{z}})$ local fiber coordinates. This ansatz is of course strongly supported by the implicit function theorem which is supposed to be understood from now. Moreover, one has the following identity,

$$
\left[ \mathcal{S}, \mathcal{D}^\varpi \right] = -\mathcal{C}(z, Y) \partial_z \log \lambda(z, Y) \mathcal{D}^\varpi - \partial \mathcal{C}(z, Y) \mathcal{D}^\varpi \\
= \mathcal{C}(z, Y) \left[ \partial, \mathcal{D}^\varpi \right] - \partial \mathcal{C}(z, Y) \mathcal{D}^\varpi \quad (3.11)
$$

which yields the variations,

$$
\mathcal{S} \lambda(z, Y) = \partial \left( \lambda(z, Y) \mathcal{C}(z, Y) \right) \quad (3.12)
$$

$$
\mathcal{S} \mu(z, Y) = \mathcal{C}(z, Y) \partial \mu(z, Y) - \mu(z, Y) \partial \mathcal{C}(z, Y) + \overline{\mathcal{C}}(z, Y) \quad (3.13)
$$

$$
\mathcal{S} \mathcal{C}(z, Y) = \mathcal{C}(z, Y) \partial \mathcal{C}(z, Y) \quad (3.14)
$$

$$
\mathcal{S} c(z, y) = (c(z, y) \partial + \overline{c}(z, y) \overline{\partial}) c(z, y) \quad (3.15)
$$

(and their c.c. expressions).

4 Towards a $W$-algebra presentation

We have previously chosen canonical transformations described by the change of local complex coordinates parametrized by $Y$,

$$
Z(z, \overline{z}) \rightarrow Z((z, \overline{z}), Y_{\overline{z}}) = Z(z, \overline{z}) + \sum_{n \geq 2} n Z^{(n)}(z, \overline{z}) Y_{\overline{z}}^{n-1} \quad (4.1)
$$
where \( Z^{(n)}(z, \bar{z}) \) has been defined in Eq. (2.22). The choice of \( Y_{Z} \) as an independent variable, accordingly gives that the mappings,

\[
(z, \bar{z}) \rightarrow (Z^{(n)}, \bar{Z}^{(n)}), \quad n \geq 1,
\]

generate a tower of smooth changes of local complex coordinates on the base Riemann surface, each of them will be shown to be local solutions of a Beltrami like equation.

4.1 The complex structures underlying the \( Z^{(n)} \) local complex coordinates

It is now shown how each \((Z^{(n)}, \bar{Z}^{(n)})\) coordinates define new complex coordinates pertaining to a complex structure. Indeed, by construction one can write,

\[
d_z Z^{(n)}(z, \bar{z}) = \left[ \frac{1}{n!} \left( \frac{\partial}{\partial Y_{Z}} \right)^n \partial \Phi(z, Y) \right] dz + \left[ \frac{1}{n!} \left( \frac{\partial}{\partial Y_{Z}} \right)^n \bar{\partial} \Phi(z, Y) \right] \bigg|_{Y_{Z}=0, \bar{Y}_{Z}=0}
\]

\[
\equiv \lambda^{Z(n)}_z(z, \bar{z}) [dz + \mu^{\bar{z}}((z, \bar{z}), n) d\bar{z}]
\]

(4.3)

where we have introduced

\[
\lambda^{Z(n)}_z(z, \bar{z}) \equiv \left[ \frac{1}{n!} \left( \frac{\partial}{\partial Y_{Z}} \right)^n \partial \Phi(z, Y) \right] \bigg|_{Y_{Z}=0, \bar{Y}_{Z}=0} \equiv \partial Z^{(n)}(z, \bar{z})
\]

(4.4)

\[
\lambda^{Z(n)}_{\bar{z}}(z, \bar{z}) \equiv \left[ \frac{1}{n!} \left( \frac{\partial}{\partial Y_{Z}} \right)^n \bar{\partial} \Phi(z, Y) \right] \bigg|_{Y_{Z}=0, \bar{Y}_{Z}=0} \equiv \bar{\partial} Z^{(n)}(z, \bar{z}).
\]

(4.5)

Since the generating function is supposed to be complex analytic in \( Y \), both the convergence of the series (2.21) and the requirement that the mappings, \((z, \bar{z}) \rightarrow (Z^{(n)}, \bar{Z}^{(n)})\), preserve the orientation lead to the condition \(|\mu^{\bar{z}}((z, \bar{z}), n)| \leq 1\).

A Beltrami identity is immediately recovered for each level \( n \),

\[
\left[ \frac{1}{n!} \left( \frac{\partial}{\partial Y_{Z}} \right)^n \bar{\partial} \Phi(z, Y) \right] \bigg|_{Y_{Z}=0, \bar{Y}_{Z}=0} = \bar{\partial} \lambda^{Z(n)}_{z}(z, \bar{z}) = \partial (\lambda^{Z(n)}_{\bar{z}}(z, \bar{z}) \mu^{\bar{z}}((z, \bar{z}), n))
\]

(4.6)

The expression for \( \lambda^{Z(n)}_{z} \) is non local in \( \mu^{\bar{z}}((z, \bar{z}), n) \). It is now clear that the quantity \( \mu^{\bar{z}}((z, \bar{z}), n) \) encodes the complex structure of the space \( Z^{(n)} \). From the very definitions, a direct computation yields the following expansion

\[
\mu^{\bar{z}}((n, (z, \bar{z}))) = \sum_{k=1}^{n} \omega_{(k-1)}(n, (z, \bar{z})) \mu^{(k)}_{\bar{z}}(z, \bar{z}),
\]

(4.7)
in terms of local \((-n, 1)\)-conformal fields, \((\mu^{(1)}_z(n, (z, \overline{z}))) = \mu(n, \overline{z}) \equiv \mu(z, \overline{z}) \) for \( n = 1 \),

\[
\mu^{(n)}_z(z, \overline{z}) = \left[ \frac{1}{n!} \left( D^n_z \right) \right] \Phi(z, Y) \bigg|_{Y_z, Y_{\overline{z}}=0} \equiv \left[ \frac{1}{n!} \left( D^n_z \right) \right] \mu(z, Y) \bigg|_{Y_z, Y_{\overline{z}}=0} \quad (4.8)
\]

and with coefficients \((k-1, 0)\)-conformal fields, non local in the \( \mu^{(n)}_z((z, \overline{z}), m) \) of orders \( m \leq n \),

\[
\omega_{(k-1)}(n, (z, \overline{z})) \equiv k! \left( \prod_{j=1}^{k} \lambda^{(p_j)}_z a_j \right) \left( \sum_{i=1}^{k} a_i = k, \text{ and } \sum_{i=1}^{k} p_i a_i = n, \text{ with } n \geq p_1 > \cdots > p_n \geq 0 \right) \quad (4.9)
\]

with \( \omega_0(z, \overline{z}) = 1 \). From Eq. (4.7) we can state the Theorem,

**Theorem 2** The complex structure of the \((Z^{(n)}, \overline{Z}^{(n)})\) can be described by parameters \( \mu^{(n)}_z((z, \overline{z}), n) \) which extend to these spaces the Beltrami multipliers. The parameters \( \mu^{(n)}_z((z, \overline{z}), n) \) (for a given \( n \)), depend in a non local way on their partners \( \mu^{(n)}_z((z, \overline{z}), j) \) for \( j \leq n \).

This important geometrical statement it is the basis for the physical discussion of the problem.

The previous arguments show that the quantities \( \mu^{(n)}_z((z, \overline{z})) \) parametrize the change of the complex structure of the \((Z^{(n)}, \overline{Z}^{(n)})\) in terms of the index \( n \).

The role of the parameters \( \mu^{(j)}_z((z, \overline{z})) \) can be understood from Eq (4.7) since it is easy to realize that the coordinate system \((Z^{(n)}, \overline{Z}^{(n)})\) will have a local complex structure \( \mu \) as \((Z, \overline{Z})\) does, if and only if \( \mu^{(j)}_z((z, \overline{z})) = 0 \) for \( 2 \leq j < n \).

Obviously few examples can better clarify this point; this will be performed in the most simple cases in the next Section.

We do not pretend to exhaust and to classify all the \( W \) algebras in our context (more examples and a deeper insight in critical situations will be needed), but we hope to reduce to an unique geometrical setting the more common \( W \) algebras studied in the literature.

### 4.2 The \( W \)-symmetry

We derive now a \( W \)-symmetry from the previous construction combining both the diffeomorphism action and the canonical transformations via the B.R.S machinery. For each \( n \) we define the diffeomorphism action on the local complex coordinate \( Z^{(n)} \) by,

\[
SZ^{(n)}(z, \overline{z}) \equiv Y^{(n)}(z, \overline{z}) \equiv \left[ \frac{1}{n!} \left( \frac{\partial}{\partial Y_Z} \right)_Z^n \Lambda(z, Y) \right] |_{Y_z, Y_{\overline{z}}=0} \quad (4.10)
\]
which are invariant ghost functions since the $S$ nilpotency gives:

$$S\Upsilon^{(n)}(z, \overline{z}) = 0. \quad (4.11)$$

In more details, one has

$$\Upsilon^{(n)}(z, \overline{z}) = \left[ \frac{1}{n!} \left( \frac{\partial}{\partial Y_{\overline{z}}} \right)^n \lambda(z, Y) C(z, Y) \right]_{Y_{\overline{z}}, Y_{z} = 0}$$

$$= \lambda_z^{(n)}(z, \overline{z}) \sum_{k=1}^{n} \omega_{(k-1)}(n, (z, \overline{z})) C^{(k)}(z, \overline{z}) \quad (4.12)$$

where we have introduced the following $(-n, 0)$-conformal ghost fields,

$$C^{(n)}(z, \overline{z}) = \left[ \frac{1}{n!} \left( \frac{\partial}{\partial Y_{\overline{z}}} \right)^n \Lambda(z, Y) \right]_{Y_{\overline{z}}, Y_{z} = 0}, \quad n = 1, 2, \cdots, \quad C^{(1)}(z, \overline{z}) \equiv C(z, \overline{z}), \quad (4.13)$$

and where the non local coefficients $\omega_{(k-1)}(n, (z, \overline{z}))$ have been previously introduced in Eq.(4.9).

From the very definitions one obtains the obvious law of transformation under diffeomorphisms,

$$SC^{(n)} = \sum_{r=1}^{n} r C^{(r)} \partial_z C^{(n-r+1)}. \quad (4.14)$$

Moreover, one can see that the $(-n, 1)$-conformal fields (4.8) are also given by,

$$\mu^{(n)}_{z} = \frac{\partial}{\partial \overline{z}} C^{(n)}, \quad (4.15)$$

so that by a trick related to diffeomorphisms [23], namely, $\{S, \frac{\partial}{\partial \overline{z}}\} = \mathcal{J}$, one easily obtains their BRS variations,

$$S\mu^{(n)}_{z}(z, \overline{z}) = \mathcal{J} C^{(n)}(z, \overline{z}) + \sum_{r=1}^{n} r \left( C^{(r)}(z, \overline{z}) \partial \mu^{(n-r+1)}_{z}(z, \overline{z}) - \mu^{(r)}_{z}(z, \overline{z}) \partial C^{(n-r+1)}(z, \overline{z}) \right). \quad (4.16)$$

For the non local fields one gets,

$$S\lambda^{(n)}_{z}(z, \overline{z}) = \left[ \frac{1}{n!} \left( \frac{\partial}{\partial Y_{\overline{z}}} \right)^n \partial \lambda(z, Y) \right]_{Y_{\overline{z}}, Y_{z} = 0}$$

$$= \left[ \frac{1}{n!} \left( \frac{\partial}{\partial Y_{\overline{z}}} \right)^{(n-1)} \lambda(z, Y) C(z, Y) \right]_{Y_{\overline{z}}, Y_{z} = 0}$$

$$= \partial \left( \lambda^{(n)}_{z}(z, \overline{z}) \mathcal{K}^{(n)}((z, \overline{z}), n) \right) \quad (4.17)$$
where the new ghost vector field,
\[ K^z((z, \overline{z}), n) = \sum_{k=1}^{n} \omega_{(k-1)}(n, (z, \overline{z})) C^{(k)}(z, \overline{z}) \]  \hspace{1cm} (4.18)
has as variation,
\[ SK^z((z, \overline{z}), n) = K^z((z, \overline{z}), n) \partial K^z((z, \overline{z}), n), \]  \hspace{1cm} (4.19)
and since by construction,
\[ S\lambda^Z_{n}(z, \overline{z}) = S\partial Z_{n} = \partial \Upsilon_{n} = \partial (\lambda^Z_{n}(z, \overline{z}) K^z((z, \overline{z}), n)), \]  \hspace{1cm} (4.20)
one finally gets,
\[ S\mu^z_{n}(z, \overline{z}) = K^z((z, \overline{z}), n) \partial \mu^z_{n}(z, \overline{z}) - \mu^z_{n}(z, \overline{z}) \partial K^z((z, \overline{z}), n) + \overline{\partial} K^z((z, \overline{z}), n). \]  \hspace{1cm} (4.21)
For each \( n \) a diffeomorphism BRS-structure with a ghost \( K^z((z, \overline{z}), n) \) (non local in the complex structure parameters) can be put into evidence.

While the case \( n = 1 \) identifies the diffeomorphism symmetry, we show now that for each sector of the grading \( n = 1 \cdots \) we shall individuate a \( W \) symmetry.

The new ghost fields defined by:
\[ c^{(p,q)}(z, \overline{z}) = \left[ \frac{1}{p!q!} \left( \frac{\partial}{\partial y_{z}} \right)^{p} \left( \frac{\partial}{\partial y_{\overline{z}}} \right)^{q} \Lambda(z, Y) \right]_{Y_{z}, Y_{\overline{z}}=0} \]  \hspace{1cm} (4.22)
with \( c^{(1,0)}(z, y) = c^z(z, y) \mid_{y=0} \) transform as,
\[ SC^{(p,q)}(z, \overline{z}) = \sum_{r=0, \ldots, p \atop s=0, \ldots, q \atop r+s > 0} \left( r c^{(r,s)}(z, \overline{z}) \partial_{z} c^{(p-r+1,q-s)}(z, \overline{z}) + s c^{(r,s)}(z, \overline{z}) \partial_{z} c^{(p-r,q-s+1)}(z, \overline{z}) \right). \]  \hspace{1cm} (4.23)
Notice that the variations of \( c^{(p,q)} \) contain the fields \( c^{(r,s)} \) with degrees \( r \leq p \) and \( s \leq q \). The previous transformations (4.23) induce the following commutation relations, for currents defined through the nilpotent functional BRS operator,
\[ \delta \equiv \int_{\Sigma} d^{2} z \wedge d\overline{z} \left( c^{(p,q)}(z, \overline{z}) T_{(p,q)}(z, \overline{z}) + SC^{(p,q)}(z, \overline{z}) \frac{\delta}{\delta c^{(p,q)}} \right), \]  \hspace{1cm} (4.24)
namely, \( \{ \delta, \delta \} = 0 \), leads to,
\[
\left[ T_{(p,q)}(z,\bar{z}), T_{(r,s)}(z',\bar{z}') \right] = p \partial_{z'} \delta^{(2)}(z' - z) T_{(p+r-1,q+s)}(z,\bar{z}) - r \partial_z \delta^{(2)}(z - z') T_{(p+r-1,q+s)}(z',\bar{z}')
\]
\[
+ q \partial_{z'} \delta^{(2)}(z' - z) T_{(p+r,q+s-1)}(z,\bar{z}) - s \partial_z \delta^{(2)}(z - z') T_{(p+r,q+s-1)}(z',\bar{z}').
\]
which turn out to be a realization of the so-called \( W_\infty \)-algebra according to [10], if no limit is put on the grading indices. Otherwise, if a truncation criterium is given (by fixing a suitable upper limit on the series in Eq.(2.21)) a \( W_n \)-algebra (with \( n \) obviously finite) can be constructed as will be seen in the next section for the most simple examples.

5 The \( W_3 \) example

In the literature there exist two types of the named \( W_3 \)-algebra, namely the former is obtained by a field realization [10, 14] which is on-shell, while the latter is given by reduction [6, 15, 16, 17]. It will be shown down below how those two realizations of \( W_3 \)-algebra, will be re-obtained from our construction only grounded on the combination of canonical transformations and diffeomorphisms.

5.1 The chiral \( W_3 \)-gravity

We have seen in the preceding section how the sequences of smooth changes of local complex coordinates,
\[
(z, \bar{z}) \longrightarrow (Z^{(n)}, \bar{Z}^{(n)}), \quad n = 1, 2 \cdots \infty
\]
generate the \( W_\infty \) algebra. By the way we can arbitrarily truncate the series to a non empty subset (or at least only one) of these invariance laws and see the physical consequences it will imply. The simplest example is obtained by considering only the smooth change of local complex coordinates,
\[
(z, \bar{z}) \longrightarrow (Z^{(2)}, \bar{Z}^{(2)}),
\]
while neglecting the previous one of the lower order,
\[
(z, \bar{z}) \longrightarrow (Z, \bar{Z}).
\]
This will show, upon imposing the diffeomorphism invariance, the coordinate origin of the chiral $w_3$ algebra previously obtained by Sorella et al \cite{14}.

Following the notation of the previous Section we have successively,

$$\lambda_z^{Z(2)} = \partial Z^{(2)}, \quad \lambda = \partial Z,$$

$$SZ^{(2)} = \Upsilon^{(2)} = \lambda_z^{Z(2)} \mathcal{C} + \lambda^2 \mathcal{C} \equiv \lambda_z^{Z(2)} \mathcal{K}^{z(2)},$$

$$S\lambda_z^{Z^{(2)}} = \partial \Upsilon^{(2)} = \partial \left( \lambda_z^{Z(2)} \mathcal{K}^{z(2)} \right)$$

$$\mathcal{K}^{z(2)} = \mathcal{C} + \frac{\left( \lambda_z^{Z} \right)^2}{\lambda_z^{Z(2)}} \mathcal{C} \equiv \mathcal{C} + \frac{\beta_z}{2} \mathcal{C},$$

where we have set

$$\beta_z = \frac{2(\lambda_z^{Z})^2}{\lambda_z^{Z(2)}}.$$  

The nilpotency condition on $SZ^{(2)}$ gives,

$$SK^{z(2)} = \mathcal{K}^{z(2)}(2) \partial \mathcal{K}^{z(2)}.$$

The decomposition of the ghost $\mathcal{K}^{z(2)}$ turn out to be so useful to see how the underlying coordinate $Z$ behaves under the cotangent diffeomorphisms.

In fact while the $\mathcal{C}$ term is the usual ghost of the $(z, \bar{z}) \rightarrow (Z, \bar{Z})$ mapping, the $\frac{\beta_z}{2} \mathcal{C}$ term upgrades the diffeomorphism to the $(z, \bar{z}) \rightarrow (Z^{(2)}, \bar{Z}^{(2)})$ level. The parameter $\beta_z$ does contain the relative change of the $Z$ coordinate with respect to the background $(z, \bar{z})$ used to describe the $(Z^{(2)}, \bar{Z}^{(2)})$ invariance. So the independent study of the behaviour of $\mathcal{K}^{z(2)}$ and $\frac{\beta_z}{2} \mathcal{C}$ under Eq (5.8) gives more details on the realization of the diffeomorphism. In order to have a more precise geometrical information we shall fix the $\mathcal{C}^{(2)}$ BRS transformation to be,

$$SC^{(2)} = \mathcal{C} \partial \mathcal{C}^{(2)} + 2 \mathcal{C}^{(2)} \partial \mathcal{C},$$

in order to deduce the consistent transformation of $\beta_z$ and to derive the consistent breaking of the symmetry $(z, \bar{z}) \rightarrow (Z, \bar{Z})$.

Now, in term of the $\mathcal{K}(2)$ ghost the previous equation (5.9) rewrites,

$$SC^{(2)} = \mathcal{K}^{z(2)} \partial \mathcal{C}^{(2)} + 2 \mathcal{C}^{(2)} \partial \mathcal{K}^{z(2)} - \frac{3}{2} \beta \mathcal{C}^{(2)} \partial \mathcal{C}^{(2)}$$

$$17$$
where Eq. (5.6) has been used. In particular we get,

\[ S\left( C^{(2)} \partial C^{(2)} \right) = K^z(2) \partial(\partial C^{(2)} \partial C^{(2)}) - 3C^{(2)} \partial C^{(2)} \partial K^z(2). \]  

(5.11)

The nilpotency condition on \( C^{(2)} \) reads,

\[ 0 = S^2 C^{(2)} = \left( SK^z(2) - K^z(2) \partial K^z(2) \right) \partial C^{(2)} - 2C^{(2)} \partial \left( SK^z(2) - K^z(2) \partial K^z(2) \right) - \frac{3}{2} \left( S \beta - \partial(\beta K^z(2)) \right) C^{(2)} \partial C^{(2)}. \]  

(5.12)

Modifying in a consistent way the BRS transformations of both \( C \) and \( \lambda \) while preserving Eq. (5.6), allows one to set,

\[ SC = C \partial C + \mathcal{X}, \]  

(5.13)

where, from the nilpotency condition,

\[ SX = C \partial X - \partial CX. \]  

(5.14)

Now from \( S^2 C^{(2)} = 0 \) we obtain,

\[ \mathcal{X} \partial C^{(2)} = 2C^{(2)} \partial \mathcal{X} \]  

(5.15)

which can be solved by setting,

\[ \mathcal{X} = C^{(2)} \partial C^{(2)} \frac{16}{3} T + \alpha \left( \partial C^{(2)} \partial^2 C^{(2)} - \frac{2}{3} C^{(2)} \partial^3 C^{(2)} \right) \]  

(5.16)

where we have been forced to introduce a spin (2, 0)-conformal field \( T \). Since \( \mathcal{X} \) has to satisfy Eq. (5.14), \( T \) is subject to the consistency condition,

\[ ST = C \partial T + 2 \partial C T + \partial C^{(2)} \mathcal{W} + \frac{2}{3} C^{(2)} \partial \mathcal{W} + \alpha \partial^3 C \]  

(5.17)

which allows us to introduce a spin (3, 0)-conformal field \( \mathcal{W} \) whose BRS behaviour can be calculated from the nilpotency condition applied on (5.17). To sum up, we have computed the most general deformation of the \( C \) ghost field compatible with the fixed variation (5.9) of \( C^{(2)} \).

From the very definition (5.6) of \( K^z(2) \), and after using (5.10), the variation \( SK^z(2) \) can be expressed in terms of both the \( K^z(2) \) and \( C^{(2)} \) ghosts only. One gets,

\[ \left( SK^z(2) - K^z(2) \partial K^z(2) \right) - \left( S \left( \frac{\beta^z}{2} \right) - \partial \left( \frac{\beta^z}{2} K^z(2) \right) \right) C^{(2)} = \mathcal{X} - C^{(2)} \partial C^{(2)} \left( \frac{\beta^z}{2} \right). \]  

(5.18)
Requiring the diffeomorphism symmetry implemented by (5.8), on the one hand, Eq. (5.12) reduces to,

$$\left( C^{(2)} \partial C^{(2)} \right) \left[ S \left( \beta_z \right) - \partial \left( \beta_z \mathcal{K}_z^{(2)} \right) \right] = 0. \quad (5.19)$$

and on the other hand, by plugging (5.16) into (5.18), one gets,

$$C^{(2)} \left( S \left( \frac{\beta_z}{2} \right) - \partial \left( \frac{\beta_z}{2} \mathcal{K}_z^{(2)} \right) \right) = C^{(2)} \partial C^{(2)} \left( \frac{16}{3} T - \frac{\beta_z^2}{2} \right) + \alpha \left( \partial C^{(2)} \partial^2 C^{(2)} - \frac{2}{3} \partial C^{(2)} \partial^3 C^{(2)} \right). \quad (5.20)$$

A direct comparison shows that $\alpha = 0$, and thus

$$S \frac{\beta_z}{2} = \partial \left( \frac{\beta_z}{2} \mathcal{K}_z^{(2)} \right) + \partial C^{(2)} \left( \frac{16}{3} T - \frac{\beta_z^2}{2} \right) + \Sigma \mathcal{C}^{(2)}, \quad (5.21)$$

where $\Sigma$ is a ghost graded zero quadratic differential which labels a $\Phi$-$\Pi$ ambiguity. The nilpotency allows to compute the BRS variation of $\Sigma$. However as above, the latter will be exhibited up to a $\Phi$-$\Pi$ ambiguity and so on. The closure of a minimal BRS algebra will be achieved by imposing,

$$\frac{16}{3} T = \frac{\beta_z^2}{2}, \quad \Sigma = 0, \quad (5.22)$$

so that we are left with

$$S \frac{\beta_z}{2} = \partial \left( \frac{\beta_z}{2} \mathcal{K}_z^{(2)} \right). \quad (5.23)$$

From Eq. (5.17) we derive

$$\mathcal{W} = \left( \frac{3}{2} \right)^{\frac{3}{2}} \left( \frac{\beta_z}{4} \right)^3 \quad (5.24)$$

showing that both the fields $T$ and $\mathcal{W}$ are non local fields. Finally for the sake of consistency, one checks,

$$S \left( \beta \mathcal{K}_z^{(2)} \right) = 0. \quad (5.25)$$

In the case (5.19) the smooth change of complex coordinates $(z, \bar{z}) \rightarrow (Z^{(2)}, \bar{Z}^{(2)})$ will be preserved under diffeomorphisms and the ghost $\mathcal{K}_z^{(2)}$ transforms as a factorized ghost vector field, while

$$S \lambda_{\bar{z}} = \partial \left( \lambda_{\bar{z}} \mathcal{K}_{\bar{z}}^{(2)} \right). \quad (5.26)$$
The parameter \( \lambda^Z_z \) is compatible with the complex structure defined by the \( Z^{(2)} \) coordinates, and the geometrical quantities \( \lambda^Z_z, \lambda^{Z^{(2)}}_z \) are covariant; in particular the object:

\[
J \equiv \frac{\lambda^{Z^{(2)}}_z}{\lambda^Z_z}
\]

transforms as:

\[
S J = K^z(2) \partial J.
\]

The latter implies,

\[
(\bar{\partial} - \mu((z, \bar{z}), 2) \partial) J \equiv \frac{\partial}{\partial Z^{(2)}} J = 0
\]

so that \( J \) is holomorphic in \( Z^{(2)} \). So we can parametrize:

\[
\lambda = \frac{\beta}{2 \mathcal{J}}, \quad \lambda^{Z^{(2)}}_z = \frac{\beta}{2 \mathcal{J}^2},
\]

In conclusion the well-defined (chiral) BRS algebra (already treated in [14] in a BRS framework), defined by

\[
\mathcal{S} \mathcal{C} = \mathcal{C} \partial \mathcal{C} + \frac{1}{2} \beta^2 \mathcal{C}^{(2)} \partial \mathcal{C}^{(2)}, \quad \mathcal{S} \mathcal{C}^{(2)} = \mathcal{C} \partial \mathcal{C}^{(2)} + 2 \mathcal{C}^{(2)} \partial \mathcal{C}, \quad S \beta = \partial \left( \beta \left( \mathcal{C} + \frac{1}{2} \mathcal{C}^{(2)} \beta \right) \right)
\]

describes a diffeomorphism symmetry hidden in the choice of the parameters which leads to a diffeomorphism ghost \( K^z(2) = \mathcal{C} + \frac{\beta}{2} \mathcal{C}^{(2)} \).

The \( \beta_z \) parameter describes the relative tuning between the change of complex coordinates \((Z, \bar{Z}) \rightarrow (Z^{(2)}, \bar{Z}^{(2)})\) with respect to the same \((z, \bar{z})\) background, namely

\[
\beta_z = \frac{\partial K^z(2)}{\partial \mathcal{C}^{(2)}}.
\]

### 5.2 The induced \( W_3 \)-gravity

According to Theorem [1], one considers as a canonical transformation the following vertical smooth change of coordinates \( Z(z, Y) \rightarrow Z'(z, Y) \), where \( Z(z, Y) = \partial_{Y_2} \Phi(z, Y) \) and where \( Z'(z, Y) \) is defined by the following replacement,

\[
Z^{(2)}(z, \bar{z}) \rightarrow Z^{(2)}((z, \bar{z}), Y_Z) \equiv \frac{1}{2} \left( \frac{\partial}{\partial Y_Z} \right)^2 \Phi(z, Y) = \sum_{n \geq 0} \frac{(n + 2)(n + 1)}{2} Y^n Z^{(2+n)}(z, \bar{z})
\]
in the expansion (4.1) of the $Z(z, Y)$ coordinate. One has

$$Z'(z, Y) = Z(z, \xi) + 2 Y Z^{(2)}(z, \xi) + \sum_{n \geq 2} (n + 1)^2 Y^n Z^{(n+1)}(z, \xi),$$  

(5.34)

showing that the corresponding generating function $\Phi'$ coincides with $\Phi$ up to the second order in $Y_Z$. Furthermore, $Z^{(2)}((z, \xi), Y_Z)$ has as BRS variation,

$$S Z^{(2)}((z, \xi), Y_Z) = \frac{1}{2} \left( \frac{\partial}{\partial Y_Z} \right)^2 \Lambda(z, Y) = \frac{1}{2} \sum_{n \geq 0} (n + 2)(n + 1) Y_Z^n \Sigma^{(n+2)}(z, \xi),$$

(5.35)

where in complete analogy with the construction given in section 3 we have,

$$S K^{(2)}_2((z, \xi), Y_Z) = K^{(2)}_2((z, \xi), Y_Z) \partial K^{(2)}_2((z, \xi), Y_Z)$$

(5.36)

Now it is easy to realize that

$$\lambda Z^{(2)}((z, \xi), Y_Z) \equiv \partial Z^{(2)}((z, \xi), Y_Z) = \lambda Z^{(2)}((z, \xi), Y_Z) + \sum_{n \geq 1} (n + 2)(n + 1) Y_Z^n \lambda Z^{(2+n)}(z, \xi)$$

(5.37)

from which, similarly to the previous example, it follows that,

$$K^{(2)}_2((z, \xi), Y_Z) = C(z, \xi) + \frac{\lambda Z^{(2)}((z, \xi), Y_Z)}{\lambda Z^{(2)}((z, \xi), Y_Z)} C^{(2)}(z, \xi) + \omega^2((z, \xi), Y_Z)$$

$$\equiv C(z, \xi) + \frac{\beta_2((z, \xi), Y_Z)}{2} C^{(2)}(z, \xi) + \omega^2((z, \xi), Y_Z)$$

(5.38)

Therefore if we impose the condition Eq(5.36) together with Eqs.(5.13)(5.9), the previous condition $\alpha = 0$ can be avoided since the B.R.S variations of $\beta_2((z, \xi), Y_Z)$ and $\omega^2((z, \xi), Y_Z)$ are at our disposal for this purpose. We do not write these variations since they are unimportant in the treatment. But it has to be noted that in the $(z, \xi)$ plane (with no $Y_Z$ dependence) only the couple of ghost fields $C(z, \xi), C^{(2)}(z, \xi)$ will survive and whose BRS variations are given by,

$$SC = C \partial C - C^{(2)} \partial C^{(2)} \frac{16}{3} \tau + \alpha \left( \partial C^{(2)} \partial^2 C^{(2)} - \frac{2}{3} C^{(2)} \partial^3 C^{(2)} \right)$$

$$SC^{(2)} = C \partial C^{(2)} + 2 C^{(2)} \partial C,$$

(5.39)
where for the sake of definiteness $T$ turns out to be a projective connection, in contrast to the quadratic differential constructed in the previous example. The nilpotency condition infers altogether that $T$ is assigned a well defined transformation law

$$ST = C \partial T + 2 \partial C T - \partial C^{(2)} W - \frac{2}{3} C^{(2)} \partial W + \alpha \partial^3 C,$$

(5.40) in terms of a cubic differential $W$, whose variation is,

$$SW = C \partial W + 3 \partial C W + \frac{2}{3} T \partial \left( C^{(2)} T \right) + \frac{\alpha}{24} \left( \alpha \partial^5 C^{(2)} + 2C^{(2)} \partial^3 T \right.$$

$$+ 10T \partial^3 C^{(2)} + 15 \partial T \partial^2 C^{(2)} + 9 \partial^2 T \partial C^{(2)} \right)\left( C^{(2)} W \right) + \frac{1}{24} \left( \partial^5 C^{(2)} \right.$$

$$+ 2C^{(2)} \partial^3 T + 10T \partial^3 C^{(2)} + 15 \partial T \partial^2 C^{(2)} + 9 \partial^2 T \partial C^{(2)} \right)\right);$$(5.41)

However the nilpotency condition on $W$ holds whatever $\alpha$ is. Requiring that all the variations are globally defined infers that $\alpha = 1$. Thus we get the well-defined exact differential system,

$$SC = C \partial C - \frac{16}{3} C^{(2)} \partial C^{(2)} T + \left( \partial C^{(2)} \partial^2 C^{(2)} - \frac{2}{3} C^{(2)} \partial^3 C^{(2)} \right)$$

$$SC^{(2)} = C \partial C^{(2)} + 2C^{(2)} \partial C$$

$$ST = C \partial T + 2 \partial C T - 24 \partial C^{(2)} W - 16C^{(2)} \partial W + \partial^3 C$$

$$SW = C \partial W + 3 \partial C W + \frac{2}{3} T \partial \left( C^{(2)} T \right) + \frac{1}{24} \left( \partial^5 C^{(2)} \right.$$

$$+ 2C^{(2)} \partial^3 T + 10T \partial^3 C^{(2)} + 15 \partial T \partial^2 C^{(2)} + 9 \partial^2 T \partial C^{(2)} \right)\right);$$(5.42)

This BRS algebra has already been described in a different context by [17], [18].

6 Conclusions

In this paper we have introduced a systematic approach to $W$-algebras based on canonical transformations by means of an abstract symplectic structure. It is however as claimed by Witten and Hull, the action of diffeomorphisms of the cotangent bundle which generates these $W$-symmetries. Our construction provides a BRS formulation of a local symmetry which is recognized to be that of $W$-algebras (see Bilal et al. in [7], and [14, 15, 16, 17, 18, 19]) but without any care about the OPE’s of primary fields. It may also shed some new light in the study of their geometrical structure [16].

As shown for the $W_3$ specific examples, various $W$-algebras may emerge from the construction. Moreover, one may ask oneself about the link between the truncation procedure in the $Y_Z$
variable (which fixes the “rank” of the algebra) and the relative conformal weights of the primary fields whose OPE’s give rise to the algebras. One answer to this question might originate from the locality principle of QFT and the intrinsic symplectic structure introduced through the canonical quantization scheme. Indeed, by fixing the maximum order of the coframe in the $Y_Z$ coordinate, the relative “field momentum representation” in the associated Field theory carries this operation in its tensorial content, but the mechanism is not completely understood by the authors yet. In the paper, we have also presented a possible space-time origin of these algebras which ought to be useful in the study of induced $W$-gravity. Although most of our discussion have been rather complete and general, the absence of a physical-phenomenological framework is the more important limit of the present treatment.

More examples are needed for further investigations in the role played by such symmetries in physics, as recalled in the introduction. This requires either a Lagrangian or an Hamiltonian approach in which the quantization procedure can be performed according to the physical context. The former will be studied elsewhere.

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**References**

1. A.B.Zamolodchikov, Theor.Math.Phys. 65 (1985) 1205

2. V.A.Fateev and A.B Zamolodchikov, Nucl.Phys. B280 [FS18](1987) 644

3. T.Tjin, ”Finite and infinite W algebras”, Doctoral Thesis, hep-th/9308146
   J. de Boer, F. Harmsze, T. Tjin, ”Nonlinear Finite W Symmetries And Applications In Elementary Systems”, Phys. Rept. 272:139-214 (1996), e-Print Archive: hep-th/9503161

4. A.Cappelli, C.A.Trugenberger,G.R.Zemba, ”$W_{1+\infty}$ minimal models and the hierarchy in the Quantum Hall Effect. Int.J.Mod.Phys.A12:1101-1111,(1997), e-Print Archive: hep-th/9610019.

   F. Barbarin, E. Ragoucy, P. Sorba, ”Finite W Algebras And Intermediate Statistics”, Nucl. Phys. B442:425-443, (1995), e-Print Archive: hep-th/9410114.
[5] X. Shen, “W infinity and string theory”, Int. J. Mod. Phys. A7 (1992) 6953-6993.

[6] P.Bouwknegt, K.Shoutens, W Symmetry in conformal field theory Phys. Rep. 223 (1993) 183

[7] A.Bilal, V.V.Fock and I.I.Kogan, Nucl.Phys. B359 (1991) 635.
Y.Matsuo, Comm. Math.Phys 152,317(1993) and Phys. Lett B274,309 (1992)
G.Sotkov and M.Stanishkov, Nucl.Phys. B356 439 (1991)
S.Govindarajan, "Higher dimensions uniformisation of w geometry", [hep-th 9412078]

[8] A.A. Belavin, A.M. Polyakov, A.B. Zamolodchikov, "Infinite conformal symmetry in two dimensional quantum field theory", Nucl.Phys B241(1984) 333.

[9] E. Witten, "Surprises with topological field theories" in "String 90", Proceedings, Superstring Workshop, College Station USA, March 12-17, 1990, by R. Arnowitt and al. eds, World Scientific Publishing, Singapore 1991.

[10] C.M.Hull, “Geometry and W-gravity”, Talk at "Pathways to Fundamental Interactions", 16th John Hopkins Workshop on Current Problems in Particle Theory, Goteborg 1992, [hep-th/9301074].
C.M. Hull, “Classical and quantum W-gravity”, [hep-th/9201057].
C.M. Hull, “W-Geometry”, Comm. Math. Phys. 156, 1973, 245-275, [hep-th/9211113].

[11] R.Zucchini, "Light cone $W_n$ geometry and its symmetries and projective field theory", Class. Quant. Grav. 10, 253-278 (1993).

[12] F. Gieres, "Conformally covariant operators on Riemann Surfaces (with applications to conformal and integrable models)", Int. J. Mod. Phys. A8, 1-58, 1993.

[13] S. Lazzarini, "Flat complex vector bundles, the Beltrami differential, and W algebras", Lett. Math. Phys 41, 207-225, 1997.
S.Lazzarini, "Some remarks on the geometry of $W$-algebras", in “$W$-algebras: Extended Conformal Symetries”, Marseille-Luminy, 3-7 July 1995, R. Grimm et V. Ovsienko Eds, Preprint CPT-95/P.3268.
[14] M. Carvalho, L.C Queiroz Vilar, S. Sorella, "Algebraic Characterization of anomalies in chiral W(3) gravity", Int. J. Mod. Phys. A10 3877-3900 (1995).

[15] C.M. Hull Phys. Lett. 240B (1990) 110, Nucl. Phys. 353B (1991) 707, Phys. Lett. 259B (1991) 621, Phys. Lett. 364B (1991) 621, Phys. Lett. 269B (1991) 267, Nucl. Phys. 367B (1991) 731, Comm. Math. Phys. 156 (1993) 245, Int. J. Mod. Phys. 8 (1993) 507

K. Shoutens, A. Sevrin, P. van Nieuwenhuizen, Comm. Math. Phys. 124 (1989) 87, E. Bergshoeff, C.N. Pope, L.J. Romans, E. Sezgin, X. Shen, K. S. Stelle, Phys. Lett. B243 (1990) 245, Nucl. Phys. B363 (1991) 163.

A.T. Ceresole, M. Frau, J. McCarty, A. Lerda, Phys. Lett. B256 (1991) 72.

[16] H. Ooguri, K. Schoutens and A. Sevrin, “The induced action of W_3 gravity”, Commun. Math. Phys. 145 (1992) 515-539.

[17] D. Garajeu, S. Lazzarini, R. Grimm, "W gauge structure and their anomalies: an algebraic approach", J. Math. Phys. 36, 7043-7072 (1995)

[18] A. Abud, J.-P. Ader, L. Cappiello, "Consistent anomalies of the induced W gravities", Phys. Lett. B396 108-116 (1996)

[19] J.-P. Ader, F. Biet, Y. Noirot, "A geometrical approach to super W-induced gravities in two dimensions", Nucl. Phys. B466, 285-314 (1996)

[20] K. Kodaira, “Complex Manifolds and Deformation of Complex Structures”, Comprehensive Studies in Mathematics, Spinger-Verlag, New-York, 1986.

[21] Y. Choquet-Bruhat, “Géométrie Différentielle & Systèmes extérieurs”, Monographies Universitaires de Mathématiques, Dunod 1968.

Y. Choquet-Bruhat, C. DeWitt-Morette, M. Dillard-Bleick, “Analysis, Manifolds and Physics”, revised Edition 1982, North-Holland 1977.

[22] G. Bandelloni, “Diffeomorphism cohomology in Quantum Field Theory models”, Phys. Rev. D38 (1988) 1156.

[23] G. Bandelloni and S. Lazzarini ; “Diffeomorphism Cohomology In Beltrami Parametrization”, Jour. Math. Phys. 34 (1993) 5413.