BERGMAN-TOEPLITZ OPERATORS ON FAT HARTOGS TRIANGLES

TRAN VU KHANH, JIAKUN LIU, PHUNG TRONG THUC

Abstract. In this paper, we obtain some $L^p$ mapping properties of the Bergman-Toeplitz operator

$$f \mapsto T_{K^{-\alpha}} (f) := \int_{\Omega} K_{\Omega}(\cdot, w) K^{-\alpha}(w, w) f(w) \, dV(w)$$

on fat Hartogs triangles $\Omega_k := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^k < |z_2| < 1\}$, where $\alpha \in \mathbb{R}$ and $k \in \mathbb{Z}^+$. 

1. Introduction

In this paper, we continue our study of the “gain” $L^p$-estimate properties of Bergman-Toeplitz operators on pseudoconvex domains following [15]. Let $\Omega \subset \mathbb{C}^n$ be a bounded domain, and let $A^p(\Omega)$ be the closed subspace of holomorphic functions in $L^p(\Omega)$. Given a measurable function $\psi$ on $\Omega$, the Bergman-Toeplitz operator with symbol $\psi$ is defined by

$$(1.1) \quad f \mapsto T_\psi (f)(z) := \int_{\Omega} K(z, w) \psi(w) f(w) \, dV(w),$$

where $K(\cdot, \cdot)$ is the Bergman kernel associated to $\Omega$. Recently, we proved in [15] that for a large class of weakly pseudoconvex smooth domains in $\mathbb{C}^n$, the Bergman-Toeplitz operator $T_\psi$ with $\psi(z) = K^{-\alpha}(z, z)$ maps from $L^p(\Omega)$ to $A^q(\Omega)$ continuously if and only if $\alpha \geq \frac{1}{k} - \frac{1}{q}$, for any $1 < p \leq q < \infty$. As a corollary, we have the associated Bergman projection is self $L^p$ bounded for any $p \in (1, +\infty)$, and moreover, that is sharp in the sense that the Bergman projection is not bounded from $L^p$ to $A^q$ for any $q > p$, (see also [18, 20, 19, 7]).

As a non-smooth case, the fat Hartogs triangle $\Omega_k \subset \mathbb{C}^2$ is defined by

$$(1.2) \quad \Omega_k := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^k < |z_2| < 1\}, \quad \text{for } k \in \mathbb{Z}^+.$$

The model of Hartogs triangles and their variants has recently attracted particular attention through the study of several problems in complex analysis; see e.g. [2, 11, 6, 3, 13]. The fat Hartogs triangle $\Omega_k$ is a pseudoconvex domain but not a hyperconvex domain (see e.g. [21, 4] for the characterisation of pseudoconvexity and hyperconvexity of Reinhardt domains). Nevertheless, the Bergman kernel of $\Omega_k$ can be computed explicitly thanks to the work of Edholm [10]. This important fact provides useful estimates on the Bergman kernel and then the self $L^p$ boundedness of the Bergman projection. In particular, Edholm and McNeal [12] proved that the Bergman projection associated to $\Omega_k$ is self $L^p$-bounded if and only if $p \in \left(\frac{2k+2}{k+2}, \frac{2k+2}{k}\right)$. This generalises the result in the case $k = 1$ by Chakrabarti and Zeytuncu [6]. It should be interesting to add that the Bergman projections associated to the Hartogs triangle domains

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Let $\Omega := \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}$ with $\gamma > 0, \gamma \notin \mathbb{Q}$ are $L^p$ bounded if and only if $p = 2$, see [12].

It is reasonable to expect that for Hartogs triangles $\Omega_k$, the Bergman projections cannot gain the $L^p$ regularity. Moreover, it is also of particular interest to obtain a holomorphic function with higher regularity from an input function in the $L^p$ space. Motivated by this, in this paper we study the Bergman-Toeplitz operators $T_{K^{-\alpha}}$, where $K$ is the Bergman kernel on the diagonal and $\alpha \in \mathbb{R}$. We shall prove that for $1 < p < q < \infty$ such that $\frac{k+2}{2k} - \frac{1}{kp} > \frac{1}{q} > \frac{k}{2k+2}$, the Bergman-Toeplitz operator $T_{K^{-\alpha}}$ is $L^p$-$L^q$ bounded if and only if $\alpha \geq \frac{1}{p} - \frac{1}{q}$. It is natural to show that if $q \geq \frac{2k+2}{k}$, then $T_{K^{-\alpha}}$ cannot be bounded from $L^p(\Omega_k)$ to $A^q(\Omega_k)$ for any $\alpha \in \mathbb{R}$. Surprisingly, for sufficiently small $p, q$, indeed, $\frac{1}{p} + \frac{1}{q} \geq \frac{2k+2}{k}$, we shall point out that there exists $\alpha > 0$ such that $T_{K^{-\alpha}}$ is still bounded from $L^p(\Omega_k)$ to $A^q(\Omega_k)$. The precise statement of our main result is as follows:

**Theorem 1.1.** For $k \in \mathbb{Z}^+$, let $\Omega_k$ be the Hartogs triangle domain defined by [12] and let $1 < p \leq q < \infty$. Then we have the following conclusions:

(i) If $\frac{k+2}{2k} \geq \frac{1}{q}$, then there is no $\alpha \in \mathbb{R}$ such that $T_{K^{-\alpha}} : L^p(\Omega_k) \rightarrow A^q(\Omega_k)$ continuously.

(ii) If $\frac{k+2}{2k} - \frac{1}{kp} > \frac{1}{q} > \frac{k}{2k+2}$, then $T_{K^{-\alpha}} : L^p(\Omega_k) \rightarrow A^q(\Omega_k)$ continuously if and only if $\alpha \geq \frac{1}{p} - \frac{1}{q}$.

(iii) If $\frac{1}{p} \geq \frac{k+2}{2k} - \frac{1}{kp}$, then $T_{K^{-\alpha}} : L^p(\Omega_k) \rightarrow A^q(\Omega_k)$ continuously if and only if $\alpha \geq \frac{1}{p} - \frac{k+2}{2k}$.

The proof of this theorem is divided into four small theorems below. In §2, Theorem [2,2], and [2,2], we provide the proof of part (i) and the sufficient conditions of (ii) and (iii) in Theorem [1,1]. In §3, Theorems [3,3] and [3,4], we prove the necessary conditions of (ii) and (iii) in Theorem [1,1]. The following corollary is a direct consequence of Theorem [1,1].

**Corollary 1.2.** Let $P$ be the Bergman projection associated to $\Omega_k$. Then

(i) $P$ maps from $L^p(\Omega_k)$ to $A^q(\Omega_k)$ continuously if and only if $p \in (\frac{2k+2}{k+2}, \frac{2k+2}{k})$; and

(ii) $P$ cannot map from $L^p(\Omega_k)$ to $A^q(\Omega_k)$ continuously if $1 < p < q < \infty$.

We remark that Corollary [1,2] (i) has been proved in [11, Theorem 1.2]. Finally, throughout the paper we use $\preceq$ and $\succeq$ to denote inequalities up to a positive multiplicative constant; and $\approx$ for the combination of $\preceq$ and $\succeq$.

2. **Sufficient conditions**

We first recall the basic properties of the Bergman kernel associated to $\Omega_k$. The Bergman kernel of $\Omega_k$ can be computed explicitly as (see [11])

$$K(z, w) = \frac{2k(s) t^2 + q_k(s) t + s^k p_k(s)}{k\pi^2 (1-t^2)(t-s^k)^2}$$

for $z = (z_1, z_2), w = (w_1, w_2) \in \Omega_k$, where $s := z_1 \bar{w_1}, t := z_2 \bar{w_2}$,

$$p_k(s) := \sum_{j=1}^{k-1} j (k-j) s^{j-1} \text{ and } q_k(s) := \sum_{j=1}^{k} (j^2 + (k-j)^2 s^k) s^{j-1}.$$  

Here, we use the convention $\sum_{j=1}^{k-1} \cdots := 0$ if $k = 1$. Since $|s^k| < |t| < 1$, the Bergman kernel has the upper bound

$$|K(z, w)| \preceq \frac{|z_2 \bar{w}_2|}{|1 - z_2 \bar{w}_2|^2 |z_2 \bar{w}_2 - (z_1 \bar{w}_1)^k|^2}.$$  


Combining the upper bound with the fact that \( p_k(s) \geq 0 \) and \( q_k(s) \geq 1 \) for \( s \in \mathbb{R}^+ \), we obtain the “sharp” estimate of the Bergman kernel on the diagonal

\[
K(z, z) \approx \frac{|z_2|^2}{(1 - |z_2|^2)^2 (|z_2|^2 - |z_1|^{2k})^2}
\]

for \( z = (z_1, z_2) \in \Omega_k \).

Theorem 2.1 (i) is covered in the following theorem.

**Theorem 2.1.** Let \( q > \frac{2k+2}{k} \). Then \( T_{K^{\alpha}} \) cannot map from \( L^p(\Omega_k) \) to \( A^q(\Omega_k) \) continuously, for any \( p > 0 \) and \( \alpha \in \mathbb{R} \).

**Proof.** Note that if \( T_{K^{\alpha}} \) is bounded from \( L^p(\Omega_k) \) to \( A^q(\Omega_k) \) for some \( \alpha_0 < 0 \) then \( T_{K^{\alpha}} \) maps from \( L^p(\Omega_k) \) to \( A^q(\Omega_k) \) continuously for any \( \alpha \geq 0 \). Therefore, we may assume \( \alpha \geq 0 \). In order to prove Theorem 2.1 we shall show that \( T_{K^{\alpha}}(\bar{z}_2) = \frac{\bar{z}_2}{z_2} \) for some non-zero constant \( c \). Then the conclusion follows since \( \bar{z}_2 \in L^p(\Omega_k) \) for any \( p > 0 \), but \( \frac{1}{z_2} \notin L^q(\Omega_k) \) if \( q \geq \frac{2k+2}{k} \) (by a simple calculation).

Now, we recall the fact (see [11])

\[
\{ z^\alpha : \alpha \in \mathcal{B} := \{ (\beta_1, \beta_2) \in \mathbb{Z}^2 : \beta_1 \geq 0, \beta_1 + k (\beta_2 + 1) > -1 \} \}
\]

forms an orthogonal basis for \( A^2(\Omega_k) \). It suffices to prove \( \langle K^{-\alpha} z_2, z_2^\beta \rangle = 0 \) for any \( \beta \in \mathcal{B} \), apart from \( \beta = (0, -1) \). To see this, observe that the function \( \psi(z) := K^{-\alpha}(z, z) \) can be represented as \( \psi(z) = \Phi(|z_1|, |z_2|) \), for a bounded function \( \Phi : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}^+ \). Therefore

\[
\langle K^{-\alpha} z_2, z_2^\beta \rangle = \int_{\Omega_k} \Phi(|z_1|, |z_2|) \frac{z_1^{\beta_1} z_2^{\beta_2}}{z_2} \, dz
\]

\[
= \left( \int_0^{2\pi} e^{-i\theta_1 \beta_1} \, d\theta_1 \right) \left( \int_0^{2\pi} e^{-i\theta_2 (\beta_2 + 1)} \, d\theta_2 \right) \left( \int_U \Phi(r_1, r_2) r_1^{\beta_1+1} r_2^{\beta_2+2} \, dr_1 dr_2 \right)
\]

\[= 0,
\]

unless \( \beta = (0, -1) \). Here \( U := \{(r_1, r_2) : 0 \leq r_1, r_1^k < r_2 < 1 \} \). This completes the proof of Theorem 2.1. 

The next theorem is the main goal of this section, in which we prove the sufficient conditions of (ii) and (iii) in Theorem 2.1.

**Theorem 2.2.** Let \( 1 < p \leq q < 2 + \frac{2}{k} \). If \( p \), \( q \) and \( \alpha \) satisfy either

(i) \( \frac{1}{q} < \frac{k+2}{2k} - \frac{1}{kp} \) and \( \alpha \geq \frac{1}{p} - \frac{1}{q} \), or

(ii) \( \frac{1}{q} \geq \frac{k+2}{2k} - \frac{1}{kp} \) and \( \alpha > \frac{1}{p} - \frac{1}{q} - \left( \frac{k+2}{2k} - \frac{1}{kp} \right) \)

then \( T_{K^{\alpha}} \) maps from \( L^p(\Omega_k) \) to \( A^q(\Omega_k) \) continuously.

One of the fundamental tools to establish the self \( L^p \) boundedness of the Bergman projection is Schur’s test lemma (see [18], [11]). In our recent work [15], Schur’s test lemma has been generalised for studying the \( L^p - L^q \) mapping property of Toeplitz operators.

**Lemma 2.3.** ([13], Theorem 5.1] Let \( (X, \mu) \) and \( (Y, \nu) \) be measure spaces with \( \sigma \)-finite, positive measures; let \( 1 < p \leq q < \infty \) and \( \eta \in \mathbb{R} \). Let \( K : X \times Y \rightarrow \mathbb{C} \) and \( \psi : Y \rightarrow \mathbb{C} \) be measurable functions. Assume that there exist positive measurable functions \( h_1, h_2 \) on \( Y \) and \( g \) on \( X \) such that

\[
h_1^{-1} h_2 \psi \in L^p \left( Y, d\nu \right)
\]
and the inequalities
\[
\int_\Omega |K(x,y)|^{\eta p^*} h_1^{\eta p^*}(y) \, d\nu(y) \leq C_{11} g^{\eta p^*}(x),
\]
(2.3) \[
\int_\Omega |K(x,y)|^{(1-\eta)q} g^q(x) \, d\mu(x) \leq C_{22} h_2^q(y),
\]
hold for almost every \(x \in (X,\mu)\) and \(y \in (Y,\nu)\), where \(\frac{1}{p} + \frac{1}{p^*} = 1\) and \(C_{11}, C_{22}\) are positive constants.

Then, the Toeplitz operator \(T_\psi\) associated to the kernel \(K\) and the symbol \(\psi\) defined by
\[
(T_\psi f)(x) := \int_\Omega K(x,y) f(y) \psi(y) \, d\nu(y),
\]
is bounded from \(L^p(Y,\nu)\) onto \(L^q(X,\mu)\). Furthermore,
\[
\|T_\psi\|_{L^p(Y,\nu) \to L^q(X,\mu)} \leq C_{11}^{\frac{1}{p}} C_{22}^{\frac{1}{q}} \|h_1^{1-\eta} h_2^q\|_{L^q(Y,\nu)}.
\]

In order to employ Lemma \(2.3\), we first establish integral estimates on the Bergman kernel of the fat Hartogs triangle \(\Omega_k\).

**Proposition 2.4.** Let \(a, b, c\) be real numbers satisfying
\[
a \geq 1, \quad -1 < b < 0 \quad \text{and} \quad -a + 2b + c + \frac{2}{k} > -2.
\]
Then for any \(z \in \Omega_k\),
\[
\int_{\Omega_k} |K(z,w)|^a |r(w)|^b |w_2|^c dV(w) \leq K^{a-1}(z,z) |r(z)|^b |z_2|^{a-2b-2},
\]
where \(r(z) := \left( |z_2|^2 - |z_1|^{2k} \right) \left( |z_2|^2 - 1 \right)\).

**Proof.** Set \(J(z) := \int_{\Omega_k} |K(z,w)|^a |r(w)|^b |w_2|^c dV(w)\). By (2.1), it follows
\[
J(z) \leq \int_{\Omega_k} \frac{|z_2 w_2|^a \left( |w_2|^2 - |w_1|^{2k} \right)^b \left( 1 - |w_2|^2 \right)^b |w_2|^c}{|1 - z_2 w_2|^{2a} |2z_2 w_2 - z_1 |^{2a}} dV(w)
\]
\[
= \int_{D(z_1 \langle w_2 \rangle)} \frac{|z_2|^a |w_2|^a + c \left( 1 - |w_2|^2 \right)^b}{|1 - z_2 w_2|^{2a} |2z_2 w_2 - z_1 |^{2a}} \int_{D(|w_1| \langle w_2 \rangle)} \left( |w_2|^2 - |w_1|^{2k} \right)^b dV(w_1) dV(w_2),
\]
where \(D\) is the open unit disk in \(C\) and \(D(|w_2| \langle w_1 \rangle) := \left\{ w_1 \in C : |w_1| < |w_2|^\frac{k}{2} \right\}\). By the change of variable \(u := \frac{w_1}{w_2}\), the expression in the bracket \([\cdot]\) can be rewritten as
\[
\left[ \cdots \right] = |w_2|^{-2a + 2b} |z_2|^{-2a} \int_{D(|w_2| \langle w_1 \rangle)} \left| 1 - \frac{|w_1|}{|w_2|} \right|^{-2a} \left( 1 - \frac{|w_1|^k}{|w_2|^k} \right)^b dV(w_1)
\]
\[
= \frac{1}{k^2} |w_2|^{-2a + 2b + \frac{k}{2}} |z_2|^{-2a} \int_{D(|w_2| \langle w_1 \rangle)} \left| 1 - \frac{1}{z_2 u} \right|^{-2a} \left( 1 - |u|^2 \right)^b |u|^{\frac{k}{2} - 2} dV(u).
\]
By Lemma 2.5 below, the integral term in the last line of (2.6) is dominated by \( \left( 1 - \left| \frac{z}{z_2} \right|^2 \right)^{2a+b+2} \).

Thus, (2.6) continues as
\[
[\cdots] \leq |w_2|^{-2a+2b+c+\frac{p}{2}} |z_2|^{-2a} \left( 1 - \left| \frac{z}{z_2} \right|^2 \right)^{-2a+b+2}.
\]

Therefore,
\[
J(z) \leq |z_2|^{2a-2b-4} \left( |z_2|^2 - |z_1|^2 \right)^{-2a+b+2} \int_{D(0)} |1 - w_2 \bar{z_2}|^{-2a} \left( 1 - |w_2|^2 \right)^b |w_2|^{-a+2b+c+\frac{p}{2}} dV(w_2)
\]
\[
\leq K^{a-1} (z, z) |r(z)|^b |z_2|^a \left( |z_2|^2 - |z_1|^2 \right)^{-2a+b+2} \left( 1 - |z_2|^2 \right)^{-2a+b+2},
\]
where the second inequality follows by using Lemma 2.5 again since \(-a + 2b + c + \frac{2}{k} > -2\); and the last one follows by (2.2).

The proof of Proposition 2.4 is complete but we have skipped a crucial technical point that we face now.

**Lemma 2.5.** Let \( a, b \) and \( c \) be real numbers such that \( a \geq 1, \ -1 < b < 0 \) and \( c > -2 \).

Then, for any \( v \in D \),
\[
I_{a,b,c}(v) := \int_D |1 - u \bar{v}|^{-2a} \left( 1 - |u|^2 \right)^b |u|^c dV(u) \leq \left( 1 - |v|^2 \right)^{-2a+b+2}.
\]

**Proof.** This lemma is a slight extension of [11, Lemma 3.2], in which the estimate
\[
I_{1,b,c}(v) \leq (1 - |v|^2)^b
\]
has been proved for \(-1 < b < 0\) and \(-2 < c < 0\). The estimate (2.8) can be automatically extended to the case \( c > -2 \). Now, if \( a \geq 1 \) then
\[
|1 - u \bar{v}|^{-2a} \leq (1 - |v|)^{2-2a} \leq (1 - |v|^2)^{2-2a};
\]
and hence by (2.8)
\[
I_{a,b,c}(v) \leq (1 - |v|^2)^{2-2a} I_{1,b,c}(v) \leq (1 - |v|^2)^{-2a+b+2},
\]
provided \(-1 < b < 0\) and \( c > -2 \). \( \square \)

Now we are ready to prove Theorem 2.2.

**Proof of Theorem 2.2.** Let \( p' = \frac{p}{p^*} \), \( 0 < \beta < \min\left( \frac{2}{p}, \frac{1}{p} \right) \) and \( \gamma < (1 + \beta)(1 - \frac{1}{p}) \). Combining the choice of \( p', \beta, \gamma \) and the hypothesis \( 1 < p \leq q < 2 + \frac{2}{k} \), it is clear that the relations
\[
a \geq 1, \ -1 < b < 0 \text{ and } -a + 2b + c + \frac{2}{k} > -2
\]
satisfy for both choices
\[
(a, b, c) = (1, -\beta p', (2\beta - \gamma)p') \text{ and } (a, b, c) = \left( \frac{q}{p'}, -\beta q, (2\beta - \frac{1}{p'})q \right).
\]
Thus, by Proposition 2.3, the following integral estimates hold
\begin{align}
(2.9) \int_{\Omega_k} |K(z, w)| |r(w)|^{-\beta p'} |w_2|^{(2\beta - \gamma) p'} dV(w) & \leq |r(z)|^{-\beta p'} |z_2|^{2\beta p' - 1}, \\
(2.10) \int_{\Omega_k} |K(z, w)|^q |r(z)|^{-\beta q} |z_2|^{(2\beta p' - 1) q} dV(z) & \leq K^{\frac{q}{p} - 1} (w, w) |r(w)|^{-\beta q} |w_2|^\frac{q}{p} + 2\beta q - 2.
\end{align}

This can be read that the integral estimates in Lemma 2.3 hold with \( \eta = \frac{1}{p'} \),
\[ h_1(w) := |r(w)|^{-\beta} |w_2|^{2\beta - \gamma}, \quad g(z) := |r(z)|^{-\beta} |z_2|^{2\beta - \gamma} \]
and
\[ h_2(w) := K^{\frac{q}{p} - \frac{1}{p} q} (w, w) |r(w)|^{-\beta} |w_2|^\frac{q}{p} + 2\beta q - 2. \]

In order to conclude that the Bergman-Toeplitz operator \( T_{K^{-\alpha}} : L^p(\Omega_k) \to A^q(\Omega_k) \) continuously, we shall choose \( \gamma < (1 + \frac{2}{p})(1 - \frac{1}{p}) \) such that
\[ A(w) := h_1^{-1}(w) h_2(w) K^{-\alpha}(w, w) = K^{\frac{q}{p} - \frac{1}{p} q} (w, w) |w_2|^{\gamma + \frac{q}{p} - \frac{2}{p}} \]
is uniformly bounded for all \( w \in \Omega_k \).

If \( \frac{1}{q} < \frac{k + 2}{2k} - \frac{1}{kp} \) and \( \alpha \geq \frac{1}{p} - \frac{1}{q} \), by choosing \( \gamma = \frac{q}{p} - \frac{1}{p} q \), we have \( A(w) \leq 1 \) for all \( w \in \Omega_k \). Moreover, with this choice, the requirement \( \gamma < (1 + \frac{2}{p})(1 - \frac{1}{p}) \) is equivalent to the given condition \( \frac{1}{q} < \frac{k + 2}{2k} - \frac{1}{kp} \). This proves the first part of Theorem 2.3.

If \( \frac{1}{q} > \frac{k + 2}{2k} - \frac{1}{kp} \) and \( \alpha > \frac{1}{p} - \left( \frac{k + 2}{2k} - \frac{1}{kp} \right) \) then
\[ A(w) \leq |w_2|^{\frac{q}{p} + 2 + \frac{2\alpha + \gamma}{p} - \frac{2}{p}} = |w_2|^{2\alpha + \gamma + \frac{q}{p}}. \]

Here, we have used the fact that \( \alpha > \frac{1}{p} - \left( \frac{k + 2}{2k} - \frac{1}{kp} \right) \geq \frac{1}{p} - \frac{1}{q} \) and \( K^{-1}(w, w) \leq |w_2|^2 \) by (2.2).

Now by choosing \( \gamma = 2\alpha + \frac{1}{p} \), we have \( A \) is uniformly bounded. The proof is complete since the requirement \( \gamma < (1 + \frac{2}{p})(1 - \frac{1}{p}) \) is equivalent to \( \alpha > \frac{1}{p} - \left( \frac{k + 2}{2k} - \frac{1}{kp} \right) \).

\[ \square \]

3. Necessary conditions

To complete the proof of Theorem 1.1, the remaining task is to show the respective lower bounds for \( \alpha \). That is, if \( T_{K^{-\alpha}} \) maps from \( L^p(\Omega_k) \) to \( A^q(\Omega_k) \) continuously, then \( \alpha \) must be greater than or equal to the desired values. We shall prove parts (ii) and (iii) in Theorem 1.1 by using two different approaches, respectively. We remark that the underlying idea of both arguments is to derive the desirable property from singular points.

In the first approach, we illustrate a technique involving the use of the pluricomplex Green function. This method may be applied to a more general context. We first recall that for a bounded pseudoconvex domain \( \Omega \subset \mathbb{C}^n \), the pluricomplex Green function with a pole \( w \in \Omega \) is defined by
\[ G(\cdot, w) := \sup \left\{ u(\cdot) : u \in PSH^{-}(\Omega), \limsup_{z \to w} (u(z) - \log |z - w|) < \infty \right\}. \]

Here \( PSH^{-}(\Omega) \) denotes the set of all negative plurisubharmonic functions in \( \Omega \). The relation between the pluricomplex Green function and the Bergman kernel has been studied by several authors, see e.g. [14, 1, 8, 2]. One of the most important facts from these results is the very
weak assumption on the regularity of the domain. We refer the reader to [16, Chapter 6] for an introduction of the pluricomplex Green function.

The following proposition is first proved by Herbort [14] with the constant on the right hand side depending on the diameter of the domain, and Blocki [3] improved it to the sharp one as follows.

**Proposition 3.1** (Herbort-Blocki). Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ and let $t$ be any positive number. Then for any holomorphic function $f$ on $\Omega$ and any $w \in \Omega$,$$
abla (z, w) < t \\
\int_{\{G(\cdot, w) < t\}} |f(z)|^2 \, dz \geq e^{-2nt} \frac{|f(w)|^2}{K(w, w)}.$$The next lemma provides a relation between the Bergman kernel and the pole $w$ on the sublevel set $\{G(\cdot, w) < -1\}$ for the Hartogs triangles $\Omega_k$.

**Lemma 3.2.** Let $w \in \Omega_k$, $$K(z, z) |z_2|^2 \approx K(w, w) |w_2|^2$$ for any $z \in \{G(\cdot, w) < -1\}$.

**Proof.** We first prove the following elementary fact.

**Claim:** If $a, b \in D$ and $\left| \frac{a - b}{1 - ab} \right| < \frac{1}{e}$ then $1 - |a|^2 \approx 1 - |b|^2$.

**Proof of the claim.** Set $\xi = \frac{b-a}{1-ab}$. Then $a = \frac{b-\xi}{1-\xi b}$ and hence $\frac{1-|a|^2}{1-|b|^2} = \frac{1-|\xi|^2}{1-|b|^2}$. This implies the stated claim by the fact that $$\frac{e - 1}{e + 1} < \frac{1 - |\xi|^2}{1 + |\xi|^2} \leq \frac{1 - |\xi|^2}{1 - |\xi|^2} \leq \frac{e + 1}{e - 1},$$ since $b \in D$. \hfill \Box

We now proceed the proof of Lemma 3.2. Recall that, see e.g. [17], for $z = (z_1, z_2)$ and $w = (w_1, w_2)$,$$G_{D \times D}(z, w) = \max \left\{ \log \left| \frac{z_1 - w_1}{1 - z_1 \bar{w}_1} \right|, \log \left| \frac{z_2 - w_2}{1 - z_2 \bar{w}_2} \right| \right\}.$$Since the map $$F : \Omega_k \longrightarrow D \times D \\\(z_1, z_2) \longrightarrow \left( \frac{z_1}{z_2}, \frac{z_2}{z_2} \right)$$is holomorphic, we obtain$$G_{D \times D}(F(z_1, z_2), F(w_1, w_2)) \leq G_{\Omega_k}((z_1, z_2), (w_1, w_2))$$for any $z, w \in \Omega_k$. Thus for any $z \in \{z \in \Omega_k : G(z, w) < -1\}$, we have$$\left| \frac{z_1}{z_2} - \frac{w_1}{w_2} \right| < \frac{1}{e} \text{ and } \left| \frac{z_2 - w_2}{1 - z_2 \bar{w}_2} \right| < \frac{1}{e}.$$Using the above claim, it follows

$$1 - |z_2|^2 \approx 1 - |w_2|^2 \text{ and } 1 - \left| \frac{z_1}{z_2} \right|^2 \approx 1 - \left| \frac{w_1}{w_2} \right|^2.$$
Now, the desired result can be obtained by (3.1) and the fact
\[
K(z, z)|z_2|^2 \approx \left[ (1 - |z_2|^2) \left( 1 - \frac{1}{|z_2|^2} \right) \right]^{-2}.
\]

We now turn to the proof of the necessary condition in Theorem (ii).

**Theorem 3.3.** Let $1 < p \leq q < 2 + \frac{2}{m}$ and let $\alpha \in \mathbb{R}$. Assume that $T_{K^{-\alpha}} : L^p(\Omega_k) \to A^q(\Omega_k)$ continuously. Then $\alpha \geq \frac{1}{p} - \frac{1}{q}$.

**Proof.** First, we may assume that $\alpha < 1$, otherwise it implies the conclusion. Since $K(w, z)w_2$ is holomorphic in $w$, one has
\[
\int_{\Omega_k} |K(z, w)|^2 K^{-\alpha}(w, w) |w_2|^2 \, dw = \int_{\Omega_k} K(z, w) K^{-\alpha}(w, w) \overline{w_2} K(w, z) w_2 \, dw
\]
\[
= \int_{\Omega_k} K(z, w) K^{-\alpha}(w, w) \overline{w_2} \left( \int_{\Omega_k} K(w, \xi) K(\xi, z) \xi_2 \, d\xi \right) \, dw
\]
\[
= \int_{\Omega_k} \left( \int_{\Omega_k} K(w, \xi) K^{-\alpha}(w, w) K(z, w) \overline{w_2} \, dw \right) K(\xi, z) \xi_2 \, d\xi.
\]

By Hölder’s inequality and the boundedness of $T_{K^{-\alpha}}$, it continues as
\[
\int_{\Omega_k} |K(z, w)|^2 K^{-\alpha}(w, w) |w_2|^2 \, dw \leq \left\| \int_{\Omega_k} K(w, \cdot) K^{-\alpha}(w, w) K(z, w) \overline{w_2} \, dw \right\|_{L^q(\Omega_k)}
\]
\[
\leq \left\| K(z, \cdot) \right\|_{L^p(\Omega_k)} \left\| K(z, \cdot) \right\|_{L^p(\Omega_k)},
\]
where $\frac{1}{q} + \frac{1}{q} = 1$. By Lemma 2.4
\[
\int_{\Omega_k} |K(z, w)|^2 K^{-\alpha}(w, w) |w_2|^2 \, dw \leq |z_2|^{2 - \frac{2}{p} + \frac{2}{q}} K^{2 - \frac{1}{p} + \frac{1}{q}}(z, z)
\]
\[
= |z_2|^{\frac{2}{q} - \frac{2}{p} + \frac{1}{q}} K^{-1} + \frac{1}{q}(z, z).
\]

Note that the appearance of the term $w_2$ above allows us to apply Lemma 2.4. On the other hand, the LHS of (3.2) satisfies
\[
\int_{\Omega_k} |K(z, w)|^2 K^{-\alpha}(w, w) |w_2|^2 \, dw \geq \int_{\{G(z, z) < -1\}} |K(z, w)|^2 K^{-\alpha}(w, w) |w_2|^2 \, dw
\]
\[
= \int_{\{G(z, z) < -1\}} |K(z, w)|^2 |w_2|^{2+2\alpha} \left( K(w, w) |w_2|^2 \right)^{-\alpha} \, dw
\]
\[
\geq \left( K(z, z) |z_2|^2 \right)^{-\alpha} \int_{\{G(z, z) < -1\}} |K(z, w) w_2|^2 \, dw
\]
\[
\geq \left( K(z, z) |z_2|^2 \right)^{-\alpha} K(z, z) |z_2|^4.
\]

Here we have used Lemma 3.2 and Proposition 5.1. From this, (3.2) and (3.3), we get $\alpha \geq \frac{1}{p} - \frac{1}{q}$ by letting $z_2 \to 1$. This completes the proof of Theorem 3.3.
The second approach is to construct an appropriate sequence \( \{f_j\} \) and establish the lower bound from the hypothesis

\[
\text{(3.4) } \sup_j \frac{\|T_{K^{-\alpha}}(f_j)\|_{L^p(\Omega_k)}}{\|f_j\|_{L^p(\Omega_k)}} < \infty.
\]

This approach is standard and depends quite heavily on the intrinsic information of our domains \( \Omega_k \). Let us use this approach to prove the necessary condition in Theorem 3.4.

**Theorem 3.4.** Let \( 1 < p < q < 2 + \frac{2}{\alpha} \) and let \( \alpha > 0 \). Assume that \( T_{K^{-\alpha}} : L^p(\Omega_k) \to A^q(\Omega_k) \) continuously. Then \( \alpha > \frac{k+1}{kp} - \frac{k+2}{2k} \).

**Proof.** We employ a similar computation as in \[9\. Define the sequence \( \{f_j\}_{j=1}^\infty \) by

\[
f_j(z) := \begin{cases} h(|z_2|) \frac{|z_2|^p}{|z|^p} ; & a_{j+1} < |z_2| < 1, \\ 0 ; & \text{elsewhere,} \end{cases}
\]

where \( a_j := \frac{1}{j} \) and the function \( h : [0, 1] \to (0, \infty) \) is defined by

\[
h(x) := x^{\frac{1}{p} - 1 - \left(2 + \frac{2}{\alpha}\right)\frac{1}{p}} \text{ for } x \in (a_{l+1}, a_l] ; \quad l = 1, 2, \ldots.
\]

We now can easily check that

\[
\|f_j\|_{L^p(\Omega_k)}^p \leq \int_{a_{j+1}}^1 h^p(r_2)r_2^{p+\frac{1}{p}+1}dr_2 \leq \sum_{l=1}^\infty \left( \int_{a_{l+1}}^{a_l} h^p(r_2)r_2^{p+\frac{1}{p}+1}dr_2 \right) 
\]

\[
\leq \sum_{l=1}^\infty l \left( l^{-p} - (l+1)^{-(1+\frac{1}{p})} \right) 
\]

\[
\leq \sum_{l=1}^\infty \frac{1}{l^p}.
\]

Therefore

\[
\text{(3.5) } \sup \left\{ \|f_j\|_{L^p(\Omega_k)}^p : j \in \mathbb{Z}^+ \right\} < c_0 := \sum_{l=1}^\infty \frac{1}{l^p}.
\]

By construction, \( f_j \in L^2(\Omega_k) \), for any \( j \in \mathbb{Z}^+ \). Thus we can make use of the above orthogonal basis \( \mathcal{B} \) of \( A^2(\Omega_k) \) to obtain

\[
T_{K^{-\alpha}}(f_j) = C \left( \int_{a_{j+1}}^1 \int_0^{r_2} (1-r_2)^{2\alpha} \left( r_2 - r_1^k \right)^{2\alpha} r_1 r_2 h(r_2) dr_1 dr_2 \right) \frac{1}{z_2}
\]

for a constant \( C \) independent of \( j \). Therefore

\[
\text{(3.6) } \|T_{K^{-\alpha}}(f_j)\|_{L^q(\Omega_k)}^q \geq \sum_{l=1}^j \left( \int_{a_{l+1}}^{a_l} \left( \int_0^{r_2} (1-r_2)^{2\alpha} \left( r_2 - r_1^k \right)^{2\alpha} r_1 r_2 h(r_2) dr_1 dr_2 \right) \frac{1}{z_2} \right)
\]
We now show that \( \alpha > \frac{k_1}{4p} - \frac{k_2}{2p} \) by contradiction. Assume that \( \alpha \leq \frac{k_1}{4p} - \frac{k_2}{2p} \). Since \((1 - r_2)^{2\alpha} \geq \max \{1 - 2\alpha r_2, 2\alpha - 2\alpha r_2\}\) for any \( \alpha \geq 0 \) and \( r_2 \in (0, 1) \), (3.6) continues as

\[
(3.7) \quad \|T_{K \to \alpha} (f_j)\|_{L^q(\Omega_k)} \geq \sum_{i=1}^{j} \left( \int_{a_{i+1}}^{a_i} r_2^{\frac{1}{d} - 1} \, dr_2 \right) - \sum_{i=1}^{j} \left( \int_{a_{i-1}}^{a_i} r_2^{\frac{1}{d}} \, dr_2 \right).
\]

Note that the first sum of the RHS of (3.7) goes to infinity while the second sum converges as \( j \to \infty \). The contradiction now follows from (3.4), (3.5) and (3.7) by letting \( j \to \infty \).

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*E-mail address*: tkhanh@uow.edu.au  jiakun1@uow.edu.au  ttp754@uowmail.edu.au

Institute for Mathematics and its Applications, School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, NSW 2522, AUSTRALIA.