New sum rule identities and duality relation for the Potts $n$-point correlation function

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Abstract

It is shown that certain sum rule identities exist which relate correlation functions for $n$ Potts spins on the boundary of a planar lattice for $n \geq 4$. Explicit expressions of the identities are obtained for $n = 4, 5$. It is also shown that the identities provide the missing link needed for a complete determination of the duality relation for the $n$-point correlation function. The $n = 4$ duality relation is obtained explicitly. More generally we deduce the number of correlation identities for any $n$ as well as an inversion relation and a conjecture on the general form of the duality relation.
The Potts model \([1]\), which is a generalization of the two-component Ising model to \(q\) components for arbitrary \(q\), has been a subject matter of intense interest in many fields ranging from condensed matter to high-energy physics. For reviews on the Potts model and its relevance see, for example, \([2,3]\). However, exact results on the Potts model have proven to be extremely elusive. Rigorous results known to this date are limited, and include essentially only a closed-form evaluation of its free energy for \(q = 2\), the Ising model \([4]\), and critical properties for the square, triangular and honeycomb lattices \([5–7]\). Much less is known about its correlation functions.

In this Letter we report on new sum rule identities for the Potts \(n\)-point correlation function. Specifically, we show that, as a consequence of being a many-component system, the correlation functions of Potts spins on the boundary of a planar lattice must necessarily satisfy certain identities when \(n \geq 4\). We further show that these identities lead to the complete determination of a correlation duality relation which, in its simplest form, has proven to be useful in determining the equilibrium crystal shape of the Ising model \([8,9]\). Our results are very general and hold for any planar lattice or graph with arbitrary interactions.

Consider the \(q\)-state Potts model on a planar lattice, or graph, \(L\) of \(N\) sites and \(E\) edges. Let \(i, j, \cdots, m, \ell\) be \(n\) sites on the boundary ordered as shown in Fig. 1, and let \(\sigma_i\) denote the state of the spin at site \(i\). Two spins of \(L\) at sites \(i'\) and \(j'\) interact with an interaction \(K_{ij}\delta(\sigma_{i'}, \sigma_{j'})\), where \(\sigma_{i'}, \sigma_{j'} = 1, 2, \ldots, q\). Define the \(n\)-point correlation function \([10]\)

\[
P_n(\sigma, \sigma', \cdots, \sigma^{(n)}) = \frac{\delta(\sigma_i, \sigma)\delta(\sigma_j, \sigma')\cdots\delta(\sigma_\ell, \sigma^{(n)})}{Z}
\]

as the probability that the \(n\) spins are in respective definite spin states \(\sigma, \sigma', \cdots, \sigma^{(n)}\). In particular, the correlation function

\[
\Gamma_n = q^n P_n(\sigma, \sigma, \cdots, \sigma) - 1
\]

vanishes identically if the \(n\) spins are completely uncorrelated.

It is convenient to write \(P_{ij\cdots\ell} = P_n(i, j, \cdots, \ell) = Z_{ij\cdots\ell}/Z\), where \(i, j, \cdots, \ell = 1, 2, \cdots, q\), \(Z\) is the partition function, and \(Z_{ij\cdots\ell}\) the partial partition function, namely, the sum of
Boltzmann factors with the boundary spin states fixed at $i, j, \ldots, \ell$. Then we have the following theorem.

**Theorem:** (i) The correlation functions $P_n$, $n \geq 4$, are related by certain sum rule identities. Particularly, for $n = 4$ and 5, the identities are

\[
P_{1212} = P_{1213} + P_{2131} - P_{1234}
\]

\[
P_{12112} = P_{12113} + P_{21331} - P_{12334}
\]

\[
P_{12123} = P_{12134} + P_{21314} - P_{12345}
\]

and eight other relations obtained by cyclically permuting the five indices in (4).

(ii) The number of correlation identities for a given $n$ is $a_n = b_n - c_n$, where $b_n$ and $c_n$ are generated respectively from the generating functions

\[
\exp(e^t - 1) = \sum_{n=0}^{\infty} b_n t^n / n!
\]

\[
(1 - \sqrt{1 - 4t})/2t = \sum_{n=0}^{\infty} c_n t^n.
\]

**Proof:** The identity (3) is equivalent to

\[
Z_{1212} = Z_{1213} + Z_{2131} - Z_{1234},
\]

which we represent graphically in Fig. 2. Consider the high-temperature expansion of $Z_{ijk\ell}$ in the form [11] of

\[
Z_{ijk\ell} = \sum_G q^{n(G)} \prod_{i',j'} \left( e^{K_{i'j'}} - 1 \right).
\]

Here, as a consequence of the fact that the four boundary sites are fixed in definite spin states, the summation is taken over all graphs $G \subseteq \mathcal{L}$ in which there are $n(G)$ clusters excluding those connected to the four boundary sites.

Apply the expansion (8) to the four $Z$’s in (7). It is clear that, as a consequence of $\mathcal{L}$ being planar, we have $Z_{1212} = T_1 + T_2 + T_3$ where $T_1$ is the sum of graphs where sites $i$ and $k$ belong to the same cluster, $T_2$ those graphs where sites $j$ and $\ell$ belong to the
same cluster, and $T_3$ graphs $i, j, k, \ell$ all belong to different clusters. It is also clear that we have $Z_{1213} = T_1 + T_3$, $Z_{2131} = T_2 + T_3$ and $Z_{1234} = T_3$. The identity (7), and hence (3), now follows as a sum rule condition. Clearly, the existence of (3) is a consequence of the planar connectivity topology. It can also be shown that all $n = 5$ identities are generated by inserting one boundary site to the diagrams in Fig. 2, resulting in identities (4) shown graphically in Fig. 3. One can proceed in a similar fashion to derive sum rules for $n \geq 6$, and thus we have established (i). We remark that the sum rules manifest themselves only for $q \geq 4$, and therefore do not apply to the Ising model.

To enumerate $a_n$, the number of correlation identities for a given $n$, it is instructive to consider the case $n = 4$. First, by enumeration we find that there are 15 distinct $Z_{ijkl}$. For each $Z_{ijkl}$ we construct its graph as in Fig. 2 and connect sites in the same state by drawing connecting lines exterior to $L$, resulting in a “connectivity” of the four points. (There is no distinction in connectivity topology between drawing connecting lines within or exterior to $L$). A well-nested connectivity, or well-nested $Z$ for brevity, is one in which the connecting lines do not intersect [12]. For $n = 4$, 14 of the 15 $Z$’s, which are shown in Fig. 4, are well-nested. Only $Z_{1212}$ which, for precisely the same planar topology reason noted in the above, is not well-nested.

More generally for a given $n$-point correlation function $Z_{ij...m\ell}$, or $Z$ for brevity, one connects in its graph sites in the same state to arrive at an $n$-point connectivity. Let there be altogether $b_n$ distinct connectivities of which $c_n$ are well-nested. To each $Z$ which is not well-nested, we follow the procedure describe in the above, namely, expanding graphically in a high-temperature series. Since all graphs in the expansion do not contain intersecting lines, by applying the principle of inclusion-exclusion [13] we eventually arrive at a sum rule expressing the particular correlation function in question in terms of well-nested ones. This gives rise to an identity for this particular $Z$. Furthermore, since each $Z$ has a unique graphical expansion, all identities are distinct. It follows that the number of sum rule identities, $a_n$, is equal to the number of $Z$’s which are not well-nested, namely, $b_n - c_n$.

The number $c_n$ has been evaluated by Blöte and Nightingale [12] in a consideration of
the transfer matrix formulation of the Potts model, and is found to be generated by (6). To enumerate \( b_n \) we note that it is precisely the number of ways that \( n \) objects can be partitioned into indistinguishable parts. Let there be \( m_\nu \) parts of \( \nu \) objects each, \( \nu = 1, 2, \cdots \). Then we have \( b_n = \sum_{m_\nu=0}^\infty \prod_{\nu=1}^\infty \left[ \frac{(n!)^{m_\nu}}{(\nu!)^{m_\nu}} \right] \) where the prime over the summation indicates the condition \( \sum_{\nu=1}^\infty \nu m_\nu = n \). This leads to the generating function (5) and thus establishes (ii). Particularly, we find \( a_4 = 15 - 14 = 1 \), \( a_5 = 52 - 42 = 10 \), \( a_6 = 203 - 132 = 71 \), \( a_7 = 877 - 429 = 448 \). We have verified these numbers by explicitly enumerating all connectivities for \( n \leq 6 \).

**Duality relation for \( P_n \):**

It has been known for some time that the two-point correlation function of an Ising model is related to its counterpart in the dual space. The usual derivation of this relation involves embedding expansions of the correlation functions on the lattice followed by an explicit term-by-term identification [14,15]. In a recent paper one of us [10] introduced a new approach to this problem which invokes only a repeated use of an elementary duality consideration [16]. The new approach, which is very general, also permits the extension of the duality analysis to the Potts model for \( n = 2, 3 \) [10]. However, an extension of the analysis of [10] to \( n \geq 4 \) ran into an apparent snag of inadequacy of conditions [17]. Here we show that the correlation identities derived in the above provide the missing link, and with the help of these identities we determine the duality relation for any \( n \).

The consideration of [10] is based on the fundamental duality relation [16]

\[
Z = qCZ^* \tag{9}
\]

relating the partition function \( Z \) of any planar lattice, or graph, to the partition function \( Z^* \) on the dual. Here, \( C = q^{-N^*} \prod_{\text{edges}} (e^{K_{ij}} - 1) \), with \( N^* \) being the number of sites of the dual and the product taken over all edges. The interaction \( K_{ij}^* \) dual to \( K_{ij} \) is given by

\[
(e^{K_{ij}} - 1)(e^{K_{ij}} - 1) = q. \tag{10}
\]

Starting from \( \mathcal{L} \) we consider a lattice \( \mathcal{L}^* \) formed by introducing \( n \) spins \( \alpha, \beta, \gamma \cdots, \delta \) to
the boundary of the dual of $\mathcal{L}$ (Cf. Fig. 1), each interacting with neighboring dual spins within $\mathcal{L}$. (Note that $\mathcal{L}^*$ now has $N^* + n - 1$ sites and is not the dual of $\mathcal{L}$.) Let $Z_{\alpha \beta \gamma \cdots \delta}^*$ be the partial dual partition function of $\mathcal{L}^*$ with the $n$ boundary spins fixed in the respective definite states. Our goal is to obtain a duality relation in the form of a linear transformation relating the $Z_{ij \cdots m\ell}$ to $Z_{\alpha \beta \gamma \cdots \delta}^*$.

Regard the $b_n$ well-nested connectivities (such as those shown in Fig. 4 for $n = 4$) as auxiliary lattices, and apply the fundamental duality relation to each one of them [17]. Applying the duality on $\mathcal{L}$ itself, for example, we obtain (9) which can be written as an equation relating linear combinations of the $Z$ and $Z^*$ [10]. Applying the duality to the well-nested connectivity $\mathcal{L}_n$ in which all $n$ points are connected to a common point with interactions $K$ as in $\mathcal{L}_4$ shown in Fig. 4(b), we obtain

$$Z_{\text{aux}(n)} = \left( \frac{qC}{q^{n-1}} \right) (u - 1)^n Z^*_{\text{aux}(n)},$$

(11)

where $u = e^K$, and $Z_{\text{aux}(n)}$ and $Z^*_{\text{aux}(n)}$ are respectively the partition functions of $\mathcal{L}_n$ and its dual. Now, both sides of (11) are polynomials of degree $n$ in $u$. Since (11) holds for arbitrary $u$, the coefficients of all powers of $u$ must be equal. However, it suffices to equate only the coefficients of the highest power of $u$ (equating other coefficients leads simply to linear combinations of equations to be obtained from other connectivities). On the LHS we have $Z_{\text{aux}(n)} = q(u + q - 1)^n Z_{11 \cdots 1} + O(u^{n-1})$, and find the coefficient of $u^n$ to be $qZ_{11 \cdots 1}$. On the RHS we have $(u - 1)^n Z^*_{\text{aux}(n)} = q(u + q - 1)^n Z^*_{11 \cdots 1} + \text{other terms}$, after using (10), and the coefficient of $u^n$ is a linear combination of the $b_n Z^*$. This leads immediately to an expression for $Z_{11 \cdots 1}$. For $n = 4$, for example, we obtain

$$Z_{1111} = \frac{C}{q^2} \left[ Z^*_{1111} + q_1(Z^*_{2111} + Z^*_{1211} + Z^*_{1121} + Z^*_{1112}) + q_1(Z^*_{1122} + Z^*_{1221}) 
+ q_1 q_2(Z^*_{123} + Z^*_{213} + Z^*_{231} + Z^*_{132}) + q_1(Z^*_{1212} + q_2 Z^*_{2112} + q_2 Z^*_{2131} + q_2 q_3 Z^*_{1234}) \right]$$

$$= \{1 + q_1(1,1,1,1) + q_1(1,1) + q_1 q_2(1,1,1,1) + q_1(1,q_2,q_2,q_2 q_3)\},$$

(12)

where $q_m = q - m$, $m = 1, 2, \cdots$, and in the last line we have introduced a short-handed notation. An immediate consequence of (12) is the result
\[
\Gamma_4 = q_1(p_{2111} + p_{1211} + p_{1121} + p_{1112} + p_{1212} + p_{1122} + p_{1221}) \\
+ q_1 q_2 (p_{1123} + p_{2113} + p_{2311} + p_{1231} + p_{2131}) + q_1 q_2 q_3 p_{1234},
\]

where we have introduced \(\mathcal{L}_n\), \(Z^* = q Z_{1111}'\), as well as \(p_{\alpha\beta\gamma\delta} = Z_{\alpha\beta\gamma\delta}' / Z_{1111}'\). For general \(n\) the consideration of \(\mathcal{L}_n\) leads to

\[
\Gamma_n = q_1 \sum p_{211\ldots 1} + q_1 q_2 \sum p_{231\ldots 1} + \cdots + q_1 q_2 \cdots q_{n-1} p_{123\ldots n}
\]

where the meaning of the summations is obvious.

Applying (10) to all \(c_n\) auxiliary lattices of well-nested connectivities in this fashion and equating the coefficients of the highest power of \(u\) in each case, we obtain \(c_n\) equations for the \(b_n\) unknown \(Z\)'s. Since \(c_n < b_n\) for \(n \geq 4\), it appears that there are more unknowns than equations and that the equations are inadequate [17]. However, after combining the \(c_n\) equations with the \(b_n - c_n\) sum rule identities, we have precisely \(b_n\) equations, and the duality relation can now be completely determined!

In the case of \(n = 4\), the solution of the 15 equations leads to, in addition to (12),

\[
Z_{2111} = \{1 + (-1, -1, q_1, q_1) + (q_1, -1) + q_2(q_1, -1, -1, -1) - (1, q_2, q_2, q_2 q_3)\}
\]

\[
Z_{1122} = \{1 + (-1, q_1, -1, q_1) - (1, 1) - q_2(1, 1, 1, 1) + (q_1, q_1 q_2, -q_2, -q_2 q_3)\}
\]

\[
Z_{1123} = \{1 - (1, -q_1, 1, 1) - (1, 1) + (2, 2, -q_2, -q_2) + (-1, -q_2, 2, 2 q_3)\}
\]

\[
Z_{1212} = \{1 - (1, 1, 1, 1) + q_1(1, 1) - q_2(1, 1, 1, 1) + Q(-1, q_1 q_2, q_1 q_2, q_2 q_3 r)\}
\]

\[
Z_{1213} = \{1 - (1, 1, 1, 1) + (-1, q_1) + (2, -q_2, 2, -q_2) + Q(q_1, s, s, q_3 t)\}
\]

\[
Z_{2131} = \{1 - (1, 1, 1, 1) + (q_1, -1) + (-q_2, 2, -q_2, 2) + Q(q_1, s, s, q_3 t)\}
\]

\[
Z_{1234} = \{1 - (1, 1, 1, 1) - (1, 1) + 2(1, 1, 1, 1) + Q[r, t, t, q_3(2 - 5q)]\}
\]

where \(Q = 1/(q^2 - 3q + 1), r = 2q - 1, s = q^2 - 4q + 2, t = q^2 - 5q + 2\). Expressions for \{\(Z_{1211}, Z_{1121}, Z_{1112}\}\} \{\(Z_{1221}\}\} and \{\(Z_{2113}, Z_{2311}, Z_{1231}\}\} are given by cyclic permutations. We remark that a closer examination shows that all \(Z_{ijkl}\) except the last four in (13) can be determined without the use of the identity (4).
The solution (12) and (13) can be written more compactly by using the fact that $Z^*$, as partial partition functions of $L^*$, satisfy the same sum rules as the $Z$. Particularly, we have $Z^*_{1212} = Z^*_{1213} + Z^*_3 - Z^*_{1234}$. Using this relation and rewriting (1) as

$$P_4(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = A_{1234} + A_{1123}\delta_{12} + A_{2113}\delta_{23} + A_{2311}\delta_{34} + A_{1231}\delta_{14} + A_{1213}\delta_{13} + A_{2131}\delta_{24} + A_{1122}\delta_{12}\delta_{13} + A_{1212}\delta_{13}\delta_{24} + A_{2111}\delta_{23} + A_{1211}\delta_{13} + A_{1121}\delta_{12} + A_{1112}\delta_{13} + A_{1111}\delta_{1234}$$

(16)

where $\delta_{12} = \delta(\sigma_1, \sigma_2), \delta_{13} = \delta_{12}\delta_{23}$, etc. we obtain after some algebra

$$A_{1234} = q^{-4}[1 - (p_{2111} + p_{1211} + p_{1121} + p_{1112} + p_{1122} + p_{1221} + p_{1212}) + 2(p_{1123} + 2p_{1231} + p_{2311} + p_{1231} + 2p_{1213} + 2p_{1234}) - 6p_{1234}]$$

$$A_{1123} = q^{-3}(p_{1211} - p_{1231} - p_{2311} - p_{1213} + 2p_{1234})$$

$$A_{2113} = q^{-3}(p_{1121} - p_{1123} - p_{2131} + 2p_{1234})$$

$$A_{2311} = q^{-3}(p_{1112} - p_{2113} - p_{1123} - p_{1213} + 2p_{1234})$$

$$A_{1231} = q^{-3}(p_{2111} - p_{2311} - p_{2113} - p_{2131} + 2p_{1234})$$

$$A_{1213} = q^{-3}(p_{2121} - p_{1231} - p_{2131} + p_{1234}), \quad A_{2131} = q^{-3}(p_{1222} - p_{1123} - p_{2311} + p_{1234})$$

$$A_{1122} = q^{-2}(p_{1213} - p_{2134}), \quad A_{1221} = q^{-2}(p_{2131} - p_{1234}), \quad A_{1212} = 0$$

$$A_{2111} = q^{-2}(p_{2123} - p_{1234}), \quad A_{1211} = q^{-2}(p_{2113} - p_{1234})$$

$$A_{1121} = q^{-2}(p_{2311} - p_{1234}), \quad A_{1112} = q^{-2}(p_{1231} - p_{1234}), \quad A_{1111} = q^{-1}p_{1234}.$$

(17)

For general $n$ we write in analogous to (16)

$$P_n(\sigma_1, \sigma_2, \ldots, \sigma_n) = A_{12\cdots n} + A_{1123\cdots(n-1)}\delta_{12} + \cdots + A_{11\cdots1}\delta_{12\cdots n}$$

(18)

and similarly for boundary spins $\sigma_\alpha, \sigma_\beta, \cdots, \sigma_\delta$ of $L^*$.

$$P_n^*(\sigma_\alpha, \sigma_\beta, \cdots, \sigma_\delta) = A_{12\cdots n}^* + A_{1123\cdots(n-1)}^*\delta_{\alpha\beta} + \cdots + A_{11\cdots1}^*\delta_{\alpha\beta\cdots\delta}.$$

(19)

Regard the diagram in Fig. 1 as representing $A_{ij\cdots\ell}$ and construct for each $A$ the associated connectivity as previously described. Then we are led to the following working conjecture:
Conjecture:

\[ A_{ij \ldots \ell} = q^{1-m} A^*_{\alpha \beta \ldots \delta} \]

\[ = 0, \quad \text{otherwise}, \]

(20)

where \( m \) is the number of distinct indices in \( \{i, j, \ldots, \ell\} \). The conjecture is readily verified for \( n = 2, 3, 4 \). In practice, for any given \( n \), one can solve \( A^*_{\alpha \beta \ldots \delta} \) from (19) in terms of \( P^*_{\alpha \beta \ldots \delta} = q^{-1}p^*_{\alpha \beta \ldots \delta} \) by applying the principle of inclusion-exclusion [13]. Details will not be given.

An inversion relation: Since \( \alpha \beta \gamma \cdots \delta \) are boundary sites of \( \mathcal{L}^* \), the transformation relating \( Z^* \) to \( Z \), an inversion process, is given precisely by the same transformation relating \( Z \) to \( Z^* \). Now \( \mathcal{L}^* \) has \( N^* + n - 1 \) sites and its dual has \( N - n + 1 \) sites. Also \( \mathcal{L} \) and \( \mathcal{L}^* \) have the same number of edges. Therefore, we have

\[ Z_{ijk \ldots m\ell} = \left( \prod \left( e^{K_{ij}} - 1 \right) \right) \sum_{\alpha \beta \gamma \cdots \delta} M(ij \cdots m\ell | \alpha \beta \gamma \cdots \delta) Z^*_{\alpha \beta \gamma \cdots \delta}, \]

(21)

\[ Z^*_{\alpha \beta \gamma \cdots \delta} = \left( \prod \left( e^{K^*_{ij}} - 1 \right) \right) \sum_{ij \cdots m\ell} M(\alpha \beta \gamma \cdots \delta | ij \cdots m) Z_{ij \cdots m}, \]

(22)

where \( M \) is a \( b_n \times b_n \) matrix. From Fig. 1 we observe that the resulting spin indices on \( \mathcal{L} \) after the inversion is a counter-clockwise cyclic permutation of the original ordering. Substituting (22) into (21) and making use of (10) and the Euler relation \( E = N + N^* - 2 \), we are led to the identity

\[ M^2(ijk \cdots \ell | j'k' \cdots \ell') = q^{n-1} \delta_{ijj'} \delta_{kk'} \cdots \delta_{ell'}, \]

(23)

which we refer to as an inversion relation. We have explicitly verified this inversion relation for \( n = 2, 3, 4 \). It can also be shown that the \( n = 4 \) inversion relation can be used to deduce the sum rule identity (6).

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Figure captions

Fig. 1. A planar lattice $\mathcal{L}$ and $n$ sites $i, j, \ldots m, \ell$ on the boundary.
Fig. 2. Graphical representation of the sum rule identity (3).

Fig. 3. Graphical representation of the sum rule identities (4).

Fig. 4. The 14 well-nested connectivities for \( n = 4 \) corresponding to (a) \( Z_{1234} \), (b) \( Z_{1111} \), (c) \( Z_{1112} \) occurring 4 times, (d) \( Z_{1123} \) occurring 4 times, (e) \( Z_{1213} \) occurring 2 times, and (f) \( Z_{1122} \) occurring 2 times.
Fig. 2
Fig. 3
Fig. 4