Some theorems in gravitational and electromagnetic fields

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The paper discusses the influences of velocity curl and field strength on some theorems in the gravitational field and electromagnetic field. With the characteristics of the algebra of quaternions, the theorem of linear momentum, conservation of linear momentum, and conservation of angular momentum etc. can be deduced from the quaternionic definitions of physical quantities. And the strength of gravitational field and electromagnetic field have an influence on some theorems directly. While the velocity curl has an effect on some theorems also.

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I. INTRODUCTION

The quaternions can be used to describe conservation laws and theorems in the electromagnetic field [1] and the gravitational field [2], including the theorem of linear momentum, the conservation of linear momentum, and the conservation of angular momentum, etc. [2]

The concept of the linear momentum was originated by a number of great scientists. The linear momentum of the particle and the conservation of linear momentum were introduced by R. Descartes in 1644. Later, the concept of linear momentum extended from the particle to the electromagnetic field etc. Further, this concept covered the quantum mechanics [1, 2].

The angular momentum is an important concept in the physics, with numerous applications. The concept of the angular momentum covers that in the gravitational field and electromagnetic field.

With the features of the algebra of quaternions [3], we can obtain theorem of linear momentum, conservation of linear momentum, conservation of angular momentum, and conservation of electric current etc. in the gravitational field and electromagnetic field. And we find that the velocity curl and strength have a few influences on the theorems and conservation laws in the gravitational field and electromagnetic field.

II. GRAVITATIONAL FIELD

The theorems in gravitational field can be described by quaternions. In the quaternion space, the coordinates are \( r_0, r_1, r_2, \) and \( r_3 \), with the basis vector \( \mathbf{e}_0 = (1, \mathbf{i}, \mathbf{j}, \mathbf{k}) \). Where, \( r_0 = v_0 t \); \( v_0 \) is the speed of light beam, and \( t \) is the time. The radius vector \( \mathbf{R} = r_0 + \Sigma(r_i \mathbf{i}_j) \), and the velocity \( \mathbf{V} = v_0 + \Sigma(v_j \mathbf{i}_j) \), \( j = 1, 2, 3 ; \) \( i = 0, 1, 2, 3 \).

The gravitational potential is,
\[
\mathbf{A} = a_0 + \Sigma(a_j \mathbf{i}_j)
\]
and the strength \( \mathbf{B} \) of gravitational field,
\[
\mathbf{B} = \hat{\nabla} \times \mathbf{A} = b_0 + \Sigma(b_j \mathbf{i}_j)
\]
where, the \( \circ \) denotes the quaternion multiplication. \( \hat{\nabla} = \partial_0 + \Sigma(\partial_i \mathbf{i}_j) ; \partial_t = \partial / \partial r ; \mathbf{a} = \Sigma(a_j \mathbf{i}_j) ; \nabla = \Sigma(\partial_j \mathbf{i}_j) \).

The gravitational strength \( \mathbf{B} \) includes two components, \( \mathbf{g} / v_0 = \partial_0 \mathbf{a} + \nabla a_0 \) and \( \mathbf{b} = \nabla \times \mathbf{a} \), with the gauge \( b_0 = 0 \).

\[
\begin{align*}
\mathbf{g} / v_0 &= \mathbf{i}_1 (\partial_0 a_1 + \partial_1 a_0) + \mathbf{i}_2 (\partial_0 a_2 + \partial_2 a_0) + \mathbf{i}_3 (\partial_0 a_3 + \partial_3 a_0) \\
\mathbf{b} &= \mathbf{i}_1 (\partial_2 a_3 - \partial_3 a_2) + \mathbf{i}_2 (\partial_3 a_1 - \partial_1 a_3) + \mathbf{i}_3 (\partial_1 a_2 - \partial_2 a_1)
\end{align*}
\]

where, \( \mathbf{a} = 0 \) in the Newtonian gravity.

The source \( S \) of gravitational field includes the linear momentum density \( \mathbf{S}_g = m \mathbf{v} \) and an extra part \( v_0 \Delta m \).
\[
\mu S = -(S / v_0 + \hat{\nabla}) \ast \circ \mathbf{B} = \mu g S_g - \mathbf{B} \ast \circ \mathbf{B} / v_0
\]
where, \( m \) is the mass density; \( \ast \) denotes the quaternion conjugate; \( \mu \) and \( \mu_g \) are the constants; \( \mathbf{B} \ast \circ \mathbf{B} / (2 \mu_g^2) \) is the field energy density; \( \Delta m = - \mathbf{B} \ast \circ \mathbf{B} / (\mu_g^2) \).

The applied force density \( \mathbf{F} \) is defined from the linear momentum density \( \mathbf{F} = \mu S / \mu_g \). And the latter is the extension of the \( \mathbf{S}_g \).
\[
\mathbf{F} = v_0 (S / v_0 + \hat{\nabla}) \ast \circ \mathbf{P}
\]
where, the applied force density includes the inertial force density and the gravitational force density, etc.

The angular momentum density \( \mathbf{L} \) is defined from the linear momentum density \( \mathbf{P} = \mu S / \mu_g \). The latter is the extension of the \( \mathbf{S}_g = m \mathbf{V} \).
\[
\mathbf{L} = (\mathbf{R} + \kappa_x \mathbf{X}) \circ \mathbf{P}
\]
where, \( \mathbf{X} = \Sigma(x \mathbf{i}_0) \), with \( \kappa_x \) being the coefficient.

And the total energy density \( \mathbf{W} \) is defined from the angular momentum density \( \mathbf{L} \).
\[
\mathbf{W} = v_0 (S / v_0 + \hat{\nabla}) \circ \mathbf{L}
\]

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where, the total energy includes the potential energy, the kinetic energy, the torque, and the work, etc. in the gravitational field.

By the total energy density $W$, we obtain the external power density $N$ as follows.

$$N = v_0(\mathcal{B}/v_0 + \mathcal{L}) \circ W$$

(9)

where, the external power density $N$ includes the power density etc. in the gravitational field.

A. Theorem of linear momentum

In the quaternion space, the inertial mass density is $m$, and the gravitational mass density $\hat{m} = m + \Delta m$. The linear momentum density $\mathcal{F} = p_0 + \Sigma(p_j i_j)$, with $p_0 = \hat{m}v_0$ and $p_j = mv_j$.

By Eq.(6), the applied force density $F$ is

$$\mathcal{F} = f_0 + \Sigma(f_j i_j)$$

(10)

where, $f_0 = \partial p_0/\partial t + v_0 \Sigma(\partial p_j/\partial r_j) + \Sigma(b_j p_j)$.

In the quaternion space, the vectorial part $f$ of applied force density $\mathcal{F}$ can be decomposed from Eq.(10).

$$f = \Sigma(f_j i_j)$$

$$= v_0 \partial_0 j + p_0 h^*$$

$$+ h^* \times p + v_0 \nabla^* p_0 + v_0 \nabla^* \times p$$

(11)

where, $h = \Sigma(b_j i_j)$, $p = \Sigma(p_j i_j)$, $\nabla = \Sigma(i_j \partial_j)$.

The above can be rewritten as follows,

$$f = \partial p/\partial t + F$$

(12)

where, $F = p_0 h^* + v_0 \nabla^* p_0 + (h + v_0 \nabla)^* \times p$. In some cases, there exist the $f = 0$.

In case of the time $t$ is only the independent variable, the $\partial p/\partial t$ will become the $dp/\partial t$. And then we obtain the theorem of linear momentum.

$$(f - F)dt = d(mw)$$

(13)

where, $v = \Sigma(v_j i_j)$.

Further, if the $(f - F) = 0$, the conservation of linear momentum can be derived from the above.

$$d(mw) = 0$$

(14)

The above means the force density $(f - F)$ covers the gravity density etc., but does not include the inertial force density from Eq.(11). And then the theorem of linear momentum is only one of simple cases of Eq.(10). So is the conservation of linear momentum.

B. Theorem of angular momentum

In the quaternion space, the vectorial part $w$ of total energy density $W$ can be decomposed from Eq.(8).

$$w = \Sigma(w_j i_j)$$

$$= v_0 \partial_0 l + (h + v_0 \nabla) l_0 + (h + v_0 \nabla) \times l$$

(15)

The above can be rewritten as follows,

$$w = \partial l/\partial t + W$$

(16)

where, $l = \Sigma(mw_j) i_j$; $W = (h + v_0 \nabla) l_0 + (h + v_0 \nabla) \times l$.

In some cases, there exist the $w = 0$.

When the time $t$ is only the independent variable, the $\partial l/\partial t$ will become the $dl/\partial t$. And we obtain the theorem of angular momentum.

$$(w - W) dt = dl$$

(17)

In case of the $(w - W) = 0$, the conservation of angular momentum can be derived from the above.

$$dl = 0$$

(18)

The above means the torque density $(w - W)$ covers the term $h \times l$ etc., but does not include the term $\partial l/\partial t$ from Eq.(16). The theorem of angular momentum is only one of simple cases of Eq.(15). And so is the conservation of angular momentum in the gravitational field.

C. Theorem of torque

In the quaternion space, the vectorial part $n$ of external power density $N$ can be decomposed from Eq.(9).

$$n = \Sigma(n_j i_j)$$

$$= v_0 \partial_0 w + (h + v_0 \nabla)^* w_0$$

$$+(h + v_0 \nabla)^* \times w$$

(19)

The above can be rewritten as follows,

$$n = \partial w/\partial t + N$$

(20)

where, $N = (h + v_0 \nabla)^* w_0 + (h + v_0 \nabla)^* \times w$. In some cases, there exist the $n = 0$.

When the time $t$ is only the independent variable, the $\partial w/\partial t$ will become the $dw/\partial t$. And we obtain the theorem of torque in the gravitational field.

$$(n - N) dt = dw$$

(21)

In case of the $(n - N) = 0$, the conservation of torque can be derived from the above.

$$dw = 0$$

(22)

The above means that the vectorial part $(n - N)$ covers the term $h^* \times w$ etc., but does not include the term $\partial w/\partial t$ from Eq.(20). The theorem of torque is only one of simple cases of Eq.(19). And so is the conservation of torque in the gravitational field.

| TABLE I: The quaternion multiplication table. |
|-----------------------------------------------|
| i   | i_1 | i_2 | i_3 |
| 1   | 1   | i_1 | i_2 | i_3   |
| i_1 | i_1 | i_2 | i_3 | i_1   |
| i_2 | i_2 | i_3 | i_1 | i_2   |
| i_3 | i_3 | i_1 | i_2 | i_3   |
The gravitational field and electromagnetic field both can be illustrated by the quaternion, and their quaternion spaces will be combined together to become the octonion space. In other words, the characteristics of gravitational field and electromagnetic field can be described with the octonion space at the same time.

In the quaternion space for the gravitational field, the basis vector \( \mathbb{E}_g = (1, i_1, i_2, i_3) \), and the radius vector \( \mathbb{R}_g = (r_0, r_1, r_2, r_3) \), with the velocity \( V_g = (v_0, v_1, v_2, v_3) \). For the electromagnetic field, the basis vector \( \mathbb{E}_e = (I_0, I_1, I_2, I_3) \), the radius vector \( \mathbb{R}_e = (R_0, R_1, R_2, R_3) \), and the velocity \( V_e = (V_0, V_1, V_2, V_3) \), with \( \mathbb{E}_e = \mathbb{E}_g \circ \mathbb{I}_0 \).

The \( \mathbb{E}_e \) is independent of the \( \mathbb{E}_g \). Both of them can be combined together to become the basis vector \( \mathbb{E} \) of the octonion space.

\[
\mathbb{E} = (1, i_1, i_2, i_3, I_0, I_1, I_2, I_3)
\]

(23)

The radius vector \( \mathbb{R}(r_0, r_1, r_2, r_3, R_0, R_1, R_2, R_3) \) in the octonion space is

\[
\mathbb{R} = r_0 + \Sigma(i_j r_j) + \Sigma(I_i R_i)
\]

(24)

and the velocity \( V(v_0, v_1, v_2, v_3, V_0, V_1, V_2, V_3) \) is

\[
V = v_0 + \Sigma(i_j v_j) + \Sigma(I_i V_i)
\]

(25)

where, \( r_0 = v_0 t \); \( v_0 \) is the speed of light; \( t \) is the time; The symbol \( \circ \) denotes the quaternion multiplication.

When the electric charge is combined with the mass to become the electron or the proton etc., we obtain the \( R_1 I_1 = r_1 i_1 \circ I_0 \) and \( V_1 I_1 = v_1 i_1 \circ I_0 \), with \( i_0 = 1 \).

The potential of the gravitational and electromagnetic fields are \( \mathbb{A}_g = (a_0, a_1, a_2, a_3) \) and \( \mathbb{A}_e = (A_0, A_1, A_2, A_3) \) respectively. They are combined together to become the potential \( \mathbb{A} = \mathbb{A}_g + k_{eg} \mathbb{A}_e \), with \( k_{eg} \) being the coefficient.

The strength \( \mathbb{B}(b_0, b_1, b_2, b_3, B_0, B_1, B_2, B_3) \) consists of the gravitational strength \( \mathbb{B}_g \) and the electromagnetic strength \( \mathbb{B}_e \). The gauge, \( b_0 = 0 \), and \( B_0 = 0 \).

\[
\mathbb{B} = \mathbb{B}_g + k_{eg} \mathbb{B}_e
\]

(26)

The gravitational strength \( \mathbb{B}_g \) in Eq.(2) includes two components, \( \mathbb{g} = (g_01, g_02, g_03) \) and \( \mathbb{b} = (g_23, g_31, g_12) \), while the electromagnetic strength \( \mathbb{B}_e \) involves two parts, \( \mathbb{E} = (B_01, B_02, B_03) \) and \( \mathbb{B} = (B_23, B_31, B_12) \).

\[
\mathbb{E}/v_0 = I_1(\partial_0 A_1 + \partial_1 A_0) + I_2(\partial_0 A_2 + \partial_2 A_0) + I_3(\partial_0 A_3 + \partial_3 A_0)
\]

\[
\mathbb{B} = I_1(\partial_1 A_2 - \partial_2 A_1) + I_2(\partial_1 A_3 - \partial_3 A_1) + I_3(\partial_1 A_0 - \partial_0 A_1)
\]

(27)

(28)

In the octonion space, the electric current density \( \mathbb{S}_e = qV_g \circ \mathbb{I}_0 \) is the source for the electromagnetic field, and the linear momentum density \( \mathbb{S}_g \) for the gravitational field. The source \( \mathbb{S} \) satisfies,

\[
\mu \mathbb{S} = - (\mathbb{E}/v_0 + \mathbb{\diamond})^* \circ \mathbb{B}
\]

\[
= \mu_\text{g}^2 \mathbb{S}_g + k_{eg} \mu_\text{e}^2 \mathbb{S}_e - (\mathbb{B}^* \circ \mathbb{B}/v_0)
\]

(29)

where, \( k_{eg}^2 = \mu_\text{g}^2/\mu_\text{e}^2 \); \( q \) is the electric charge density; \( \mu_\text{e}^2 \) is the constant; * denotes the conjugate of octonion.

\[
\mathbb{B}^* \circ \mathbb{E}/\mu_\text{g}^2 = \mathbb{B}_g^* \circ \mathbb{B}_g/\mu_\text{g}^2 + \mathbb{B}_e^* \circ \mathbb{B}_e/\mu_\text{e}^2
\]

(30)

The force density \( \mathbb{F} \) is defined from linear momentum density \( \mathbb{P} = \mu \mathbb{S}/\mu_\text{g}^2 \), which is the extension of the \( \mathbb{S}_g \).

\[
\mathbb{F} = v_0(\mathbb{E}/v_0 + \mathbb{\diamond})^* \circ \mathbb{P}
\]

(31)

where, the force density includes the gravity density, the inertial force density, Lorentz force density \( \mathbb{I}_1 \), and the interacting force density between the electromagnetic strength with magnetic moment, etc.

In the octonion space, the angular momentum density

\[
\mathbb{L} = (\mathbb{R} + k_{eg} \mathbb{X}) \circ \mathbb{P}
\]

(32)

where, \( \mathbb{P} = \mu \mathbb{S}/\mu_\text{g}^2 \); \( \mathbb{X} = \Sigma(x_i I_i) + \Sigma(X_i I_i) \).

And the total energy density \( \mathbb{W} \) is defined from the angular momentum density \( \mathbb{L} \).

\[
\mathbb{W} = v_0(\mathbb{E}/v_0 + \mathbb{\diamond}) \circ \mathbb{L}
\]

(33)

where, the total energy includes the potential energy, the kinetic energy, torque, and work, etc. in the gravitational field and the electromagnetic field.

By means of the total energy density \( \mathbb{W} \), we obtain the external power density

\[
\mathbb{N} = v_0(\mathbb{E}/v_0 + \mathbb{\diamond})^* \circ \mathbb{W}
\]

(34)

where, the external power density includes the power density in the gravitational and electromagnetic fields.

### A. Theorem of linear momentum

In the octonion space, the gravitational mass density \( \mathbb{m} = m + \Delta m \), with \( \Delta m = -\mathbb{B}^* \circ \mathbb{B}/(\mu_\text{g}^2 v_0^2) \). The linear momentum density \( \mathbb{P} = \mathbb{p}_0 + \Sigma(p_j I_j) \) and \( \Sigma(\mathbb{F}_i I_i) \). And, \( m \) is the inertial mass density; \( \mathbb{P}_i = MV_i \); \( M = k_{eg} \mu_\text{e}^2 q/\mu_\text{g}^2 \); \( p_0 = \mathbb{m} v_0 \); \( p_j = mv_j \).

By Eq.(31), the applied force density \( \mathbb{F} \) is

\[
\mathbb{F} = f_0 + \Sigma(f_j I_j) + \Sigma(F_i I_i)
\]

(35)
In the octonion space, the vectorial part \( f \) of applied force density \( \mathbb{F} \) can be decomposed from Eq.(35).
\[
f = \Sigma(f_j i_j) + \Sigma(F_i I_i)
\] (36)

The above can be rewritten as follows,
\[
f = \partial z / \partial t + \mathbb{F}
\] (37)

where, \( \mathbb{F} = \{ \Sigma(f_j i_j) - \partial z / \partial t \} + \Sigma(F_i I_i) \), which is the extension of Eq.(12). \( z = \Sigma(m v_j) i_j \).

When the time \( t \) is only the independent variable, the \( \partial z / \partial t \) will become \( dz / dt \). And we obtain the theorem of linear momentum in the octonion space.
\[
(f - \mathbb{F}) dt = dz
\] (38)

Further, if the \( f - \mathbb{F} = 0 \), the conservation of linear momentum can be derived from the above.
\[
dz = 0
\] (39)

The above means the \( f - \mathbb{F} \) covers the gravity density and Lorentz force density etc., but does not include the inertial force density from Eq.(37). And the theorem of the linear momentum is only one of simple cases of Eq.(36). So is the conservation of linear momentum in the gravitational field and electromagnetic field.

### B. Theorem of angular momentum

In the octonion space, from the angular momentum density,
\[
\mathbb{L} = l_0 + \Sigma (l_j i_j) + \Sigma (L_i I_i)
\] (40)
we have the total energy density
\[
\mathbb{W} = w_0 + \Sigma (w_j i_j) + \Sigma (W_i I_i)
\] (41)

In the octonion space, the vectorial part \( w \) of total energy density \( \mathbb{W} \) can be decomposed from Eq.(41).
\[
w = \Sigma (w_j i_j) + \Sigma (W_i I_i)
\] (42)

The above can be rewritten as follows.
\[
w = \partial j / \partial t + \mathbb{W}
\] (43)

where, \( \mathbb{W} = \{ \Sigma (w_j i_j) - \partial j / \partial t \} + \Sigma (W_i I_i) \), which is the extension in Eq.(16). \( h = \Sigma (b_j i_j); j = \Sigma (l_j i_j) \).

When the time \( t \) is only the independent variable, the \( \partial j / \partial t \) will become \( dj / dt \). And we obtain the theorem of angular momentum in the octonion space.
\[
(w - \mathbb{W}) dt = dj
\] (44)

In case of the \( w - \mathbb{W} = 0 \), the conservation of angular momentum can be derived from the above.
\[
dj = 0
\] (45)

The above means that the torque density \( w - \mathbb{W} \) covers the terms \( h \times j \) etc., but does not include the term \( \partial j / \partial t \) from Eq.(43). And the theorem of angular momentum is only one of simple cases of Eq.(42). So is the conservation of angular momentum in the case for the gravitational and electromagnetic fields.

### C. Theorem of torque

In the octonion space, from the total energy density \( \mathbb{W} \), we obtain the external power density
\[
\mathbb{N} = n_0 + \Sigma (n_j i_j) + \Sigma (N_i I_i)
\] (46)

In the octonion space, the vectorial part \( n \) of external power density \( \mathbb{N} \) can be decomposed from Eq.(46).
\[
n = \Sigma (n_j i_j) + \Sigma (N_i I_i)
\] (47)

The above can be rewritten as follows,
\[
n = \partial y / \partial t + \mathbb{N}
\] (48)

where, \( \mathbb{N} = \{ \Sigma (n_j i_j) - \partial y / \partial t \} + \Sigma (N_i I_i) \), which is the extension in Eq.(20). \( y = \Sigma (w_j i_j) \).

When the time \( t \) is only the independent variable, the \( \partial y / \partial t \) will become \( dy / dt \). And then, we obtain the theorem of torque in the octonion space.
\[
(n - \mathbb{N}) dt = dy
\] (49)

In case of the \( n - \mathbb{N} = 0 \), the conservation of torque can be derived from the above.
\[
dy = 0
\] (50)

The above means that the vectorial part \( n - \mathbb{N} \) covers the terms \( h^* \times y \) etc., but does not include the term \( \partial y / \partial t \) from Eq.(47). And the theorem of torque is only one of simple cases of Eq.(46). So is the conservation of torque in the case for coexistence of gravitational field and electromagnetic field.

### D. Theorem of electric current

In the octonion space, a new physical quantity \( \mathbb{F}_q \) can be defined from Eq.(35).
\[
\mathbb{F}_q = \mathbb{F} \circ I_0 = F_0 + \Sigma (F_j i_j) - \Sigma (f_i I_i)
\] (51)

In the octonion space, the vectorial part \( f_q \) of applied force density \( \mathbb{F}_q \) can be decomposed from Eq.(51).
\[
f_q = \Sigma (F_j i_j) - \Sigma (f_i I_i)
\] (52)

The above can be rewritten as follows.
\[
f_q = \partial Z / \partial t + \mathbb{F}_q
\] (53)

where, \( \mathbb{Z} = \Sigma \{ (MV_j) i_j \}; \mathbb{F}_q = \{ (F_j i_j) - \partial Z / \partial t \} - \Sigma (f_i I_i) \); In some cases, \( f_q = 0 \).

When the time \( t \) is only the independent variable, the \( \partial Z / \partial t \) will become \( dZ / dt \). And we obtain the theorem of electric current in the octonion space.
\[
(f_q - \mathbb{F}_q) dt = dZ
\] (54)

Further, if the \( f_q - \mathbb{F}_q = 0 \), the conservation of electric current can be derived from the above.
\[
dZ = 0
\] (55)
The above means the \((f_\mu - F_\mu)\) covers the gravity density and Lorentz force density etc., but does not include the inertial force density from Eq.(53). And the theorem of the electric current is only one of simple cases of Eq.(52). So is the conservation of electric current in the gravitational field and electromagnetic field.

\[ E. \quad \text{Theorem of magnetic momentum} \]

In the octonion space, a new physical quantity \(W_q\) can be defined from Eq.(41).

\[ W_q = W \circ I_0 = W_0 + \Sigma(W_j i_j) - \Sigma(w_i I_i) \quad (56) \]

In the octonion space, the vectorial part \(w_q\) of total energy density \(W_q\) can be decomposed from Eq.(56).

\[ w_q = \Sigma(W_j i_j) - \Sigma(w_i I_i) \quad (57) \]

The above can be rewritten as follows.

\[ w_q = \partial J/\partial t + W_q \quad (58) \]

where, \( W_q = \{\Sigma(W_j i_j) - \partial W/\partial t\} - \Sigma(w_i I_i)\); \( J = \Sigma(L_j i_j)\); \( H = \Sigma(B_j i_j)\); In some cases, \( w_q = 0 \).

When the time \( t \) is only the independent variable, the \( \partial J/\partial t \) will become the \( dJ/dt \). And we obtain the theorem of magnetic momentum in the octonion space.

\[ (w_q - W_q)dt = dJ \quad (59) \]

In case of the \((w_q - W_q) = 0\), the conservation of magnetic momentum can be derived from the above.

\[ dJ = 0 \quad (60) \]

The above means that the torque density \((w_q - W_q)\) covers the terms \( H \times J \) etc., but does not include the term \( \partial J/\partial t \) from Eq.(57). And the theorem of magnetic momentum is only one of simple cases of Eq.(56). So is the conservation of magnetic momentum in the case for the gravitational and electromagnetic fields.

\[ F. \quad \text{Theorem of torque-like} \]

In the octonion space, a new physical quantity \(N_q\) can be defined from Eq.(46).

\[ N_q = N \circ I_0 = N_0 + \Sigma(N_j i_j) - \Sigma(n_i I_i) \quad (61) \]

In the octonion space, the vectorial part \(n_q\) of external power density \(N_q\) can be decomposed from Eq.(61).

\[ n_q = \Sigma(N_j i_j) - \Sigma(n_i I_i) \quad (62) \]

The above can be rewritten as follows.

\[ n_q = \partial Y/\partial t + N_q \quad (63) \]

where, \( Y = \Sigma(W_j i_j); \quad N_q = \{\Sigma(N_j i_j) - \partial Y/\partial t\} - \Sigma(n_i I_i)\); In some cases, \( n_q = 0 \).

When the time \( t \) is only the independent variable, the \( \partial W_q/\partial t \) will become the \( dW_q/dt \). And then, we obtain the theorem of torque-like in the octonion space.

\[ (n_q - N_q)dt = dY \quad (64) \]

In case of the \((n_q - N_q) = 0\), the conservation of torque-like can be derived from the above.

\[ dY = 0 \quad (65) \]

The above means that the vectorial part \((n_q - N_q)\) covers the terms \( H' \times Y \) etc., but does not include the term \( \partial Y/\partial t \) from Eq.(62). And the theorem of torque-like is only one of simple cases of Eq.(61). So is the conservation of torque-like in the case for coexistence of gravitational field and electromagnetic field.

\[ IV. \quad \text{INFLUENCES OF VELOCITY CURL} \]

In the octonion compounding space for coexistence of gravitational field and electromagnetic field, the radius vector \( \mathbb{R} \) will be extended to \( \mathbb{R} = \mathbb{R} + k_{rx}\mathbb{X} \), although their basis vector \( \mathbb{E} \) keeps unchanged.

\[ A. \quad \text{Compounding space} \]

In the octonion compounding space, the basis vector remains the same as that in Eq.(23), while the radius vector \( \mathbb{R} \) will be extended to \( \mathbb{R} \), with the octonion quantity \( \mathbb{X} = \Sigma(x_i i_i) + \Sigma(X_i I_i) \).

The radius vector \( \mathbb{R} \) in Eq.(24) and the velocity \( \mathbb{V} \) in Eq.(25) will be extended to \( \mathbb{R} \) and \( \mathbb{V} = \mathbb{V} + k_{rx}\mathbb{A} \) respectively. Their components can be written as follows.

\[ r_i \to \bar{r}_i = r_i + k_{rx}x_i \quad (66) \]
\[ R_i \to \bar{R}_i = R_i + k_{rx}X_i \quad (67) \]
\[ v_i \to \bar{v}_i = v_i + k_{rx}a_i \quad (68) \]
\[ V_i \to \bar{V}_i = V_i + k_{rx}A_i \quad (69) \]

In a similar way, the potential \( \mathbb{A} \) and the strength \( \mathbb{B} \) in Eq.(26) will be extended to \( \bar{A} = \mathbb{A} + k_{rx}\mathbb{V} \) and \( \bar{B} = \mathbb{B} + k_{rx}\mathbb{U} \) respectively.

\[ a_i \to \bar{a}_i = a_i + K_{rx}v_i \quad (70) \]
\[ A_i \to \bar{A}_i = A_i + K_{rx}V_i \quad (71) \]
\[ b_i \to \bar{b}_i = b_i + K_{rx}u_i \quad (72) \]
\[ B_i \to \bar{B}_i = B_i + K_{rx}U_i \quad (73) \]

where, \( K_{rx} = 1/k_{rx}, k_{rx} = 1 \).

The velocity curl is

\[ \mathbb{U} = \hat{\phi} \circ \mathbb{V} = \Sigma(u_i i_i) + \Sigma(U_i I_i) \quad . \]

where, \( \hat{\phi} = \Sigma(i_i \partial_i); \quad \partial_i = \partial/\partial \bar{r}_i \); in general, \( \bar{r}_i \approx r_i \).
B. Field equations

In the octonion compounding space, the gauge equations extend into,
\[ \bar{b}_0 = b_0 + K_{rx}u_0 = 0 \quad \bar{B}_0 = B_0 + K_{rx}U_0 = 0 \]
and then the source \( S \) in Eq. (29) will become \( \bar{S} \).
\[ \mu \bar{S} = \mu_0^g \bar{S}_g + k_{eg} \mu_0^s \bar{S}_e - \bar{B}^* \circ \bar{B} / v_0 \]  \hspace{1cm} (74)
where, \( v_0 = v_0 + k_{rx}a_0 \), \( \bar{S}_g = m(\bar{V}_g + k_{rx}A_g) \), \( \bar{S}_e = q(\bar{V}_g \circ J_0 + k_{rx}A_e) \).

The applied force density \( \bar{F} \) is defined from the linear momentum density \( \bar{P} = \mu \bar{S} / \mu_0^g \).
\[ \bar{F} = \bar{v}_0(\bar{B} / v_0 + \diamond) \circ \bar{P} \]  \hspace{1cm} (75)

In the octonion space, the total energy density \( \bar{W} \) is defined from the angular momentum density \( \bar{L} = \bar{R} \circ \bar{P} \),
\[ \bar{W} = \bar{v}_0(\bar{S} / v_0 + \diamond) \circ \bar{L} \]  \hspace{1cm} (76)
and we obtain the external power density
\[ \bar{N} = \bar{v}_0(\bar{S} / v_0 + \diamond) \circ \bar{W} \]  \hspace{1cm} (77)

In the octonion compounding space, the above means that the velocity curl has an influence on field equations. Similarly, the curl has an impact on several theorems, including the theorem of linear momentum, conservation of linear momentum, and theorem of angular momentum etc. in the case for the gravitational field and electromagnetic field.

V. CONCLUSIONS

In the quaternion spaces, from the definition of applied force, we obtain the theorem of linear momentum and the conservation of linear momentum. Similarly, we have the theorem of angular momentum and the conservation of angular momentum etc. And we find the strength in the gravitational field has an influence on the theorems and conservation laws about the linear momentum.

In the octonion spaces, beside the above conservation laws and theorems in gravitational field, we can obtain the conservation of electric current and the conservation of magnetic moment etc. in the electromagnetic field. And we find the strength in the electromagnetic field has an influence on the theorems and conservation laws about the electric current.

In the octonion compounding spaces, we can obtain the conservation laws and theorems similarly, and find the velocity curl has an influence on the theorems and conservation laws in the electromagnetic field and the gravitational field.

It should be noted the study for the conservation laws and the theorems of physical quantities examined only some simple cases under the Galilean transformation in the electromagnetic field and gravitational field. Despite its preliminary character, this study can clearly indicate the conservation of linear momentum etc. are only some of simple inferences due to low velocity curl and weak strength of electromagnetic and gravitational fields. For the future studies, the research will concentrate on only some predictions about the conservation laws and the theorems with high velocity curl and strong strength in the electromagnetic and gravitational fields.

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