A Padé-Weierstrass technique for the rigorous enforcement of control limits in power flow studies

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Abstract

A new technique is presented for solving the problem of enforcing control limits in power flow studies. As an added benefit, it greatly increases the achievable precision at nose points. The method is exemplified for the case of Mvar limits in generators regulating voltage on both local and remote buses. Based on the framework of the Holomorphic Embedding Loadflow Method (HELM), it provides a rigorous solution to this fundamental problem by framing it in terms of optimization. A novel Lagrangian formulation of power-flow, which is exact for lossless networks, leads to a natural physics-based minimization criterion that yields the correct solution. For networks with small losses, as is the case in transmission, the AC power flow problem cannot be framed exactly in terms of optimization, but the criterion still retains its ability to select the correct solution. This foundation then provides a way to design a HELM scheme to solve for the minimizing solution. Although the use of barrier functions evokes interior point optimization, this method, like HELM, is based on the analytic continuation of a germ (of a particular branch) of the algebraic curve representing the solutions of the system. In this case, since the constraint equations given by limits result in an unavoidable singularity at $s = 1$, direct analytic continuation by means of standard Padé approximation is fraught with numerical instabilities. This has been overcome by means of a new analytic continuation procedure, denominated Padé-Weierstrass, that exploits the covariant nature of the power flow equations under certain changes of variables. One collateral benefit of this procedure is that it can also be used when limits are not being enforced, in order to increase the achievable numerical precision in highly stressed cases.

Keywords: Load flow analysis, power system analysis computing, power system modeling, power system simulation, power engineering computing

PACS: 88.80.hh, 88.80.hm, 84.30.Bv, 02.60.Cb, 02.60.Gf

2010 MSC: 14H50, 14H81, 30B10, 30B40, 30B70, 30E10, 94C99

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1. Introduction

The so-called power flow (or load flow) problem consists in finding the steady state of a power network where sources and loads are specified in terms of constant power. It is one of the cornerstones of power systems analysis, upon which many other analyses and tools are built. Mathematically, the core problem consists in a system of nonlinear algebraic equations for the bus voltages, given simply by the power balance equations at every bus. There are however additional controls, mostly dealing with voltage regulation, that need to be taken into account. The most important and ubiquitous among these is voltage regulation by synchronous generators, exerted either locally or at a remote bus. Other important examples are under-load tap changers (ULTC) in transformers, automatically switched shunt capacitor/reactor banks, HVDC line controls, and modern FACTS devices. In all these cases the control device can be represented as one or more additional equality constraint equations representing how the regulated magnitude is to be restricted; and correspondingly, new variables (which can be thought of as the regulating resources) are added to the system in order to keep a balance with between the total numbers of equations and unknowns. The problem is that all controls have resource limits: generators have reactive Mvar limits, ULTC have a limited range, etc. When a regulating resource hits a limit, the behavior of the control changes: the regulated magnitude is now allowed to vary from the commanded setpoint, while the resource remains at its saturation value. This switching behavior turns the power flow into a hard computational problem, particularly for large and complex network models.

An early review by Stott [1] on powerflow methods discusses the two main approaches for enforcing limits, which are still used in most iterative power flow engines today (for a more recent review, see [2]). One straightforward method consists in “type-switching”, where the equations change as needed: the controlling parameter (e.g. the Q injection for a generator; or the tap ratio for an ULTC) is initially an independent variable; but if, after solving the power flow, it hits a limit, it becomes a fixed parameter and the corresponding constraint equation is removed. This method requires solving different power flows as the switching takes place. The other strategy is based on using “feedback adjustments” [3, 4]: the controlling resource is considered always a parameter, but it is modified on-the-fly between iterations, according to how the controlled voltage deviates from the setpoint. This has the advantage that the equations remain fixed, so that the iterative scheme is simpler.

However, both approaches face the problem of slow convergence due to “rebounds”: when a given control saturates during an intermediate step of these methods, it may later require to be reverted back to the unsaturated state. This happens because of nonlinear interactions among controls, particularly when several of them are close to their limits. Sometimes these rebounds enter into cycles, preventing convergence [2]. Many methods have been developed
to avoid this sort of interactions [5, 6], but their justification remains heuris-
tic. To this day, power flow practitioners still have to rely on a combina-
tion of personal experience and heuristics in order to deal with these convergence
problems [7, 8], which usually constitute a time-consuming task. This state of
affairs is somewhat surprising since transmission planners always need to deal
with peak cases, which are quite close to feasibility boundaries and contain a
large percentage of generators (more than 30% is not unusual) saturated at their
Mvar limits.

There have been many studies characterizing the effects of limits in the tran-
sition to voltage collapse [9, 10, 11, 12, 13, 14], but none of these have resulted
in a practical power flow method. In order to find further insights with ac-
tual application to the general power flow problem, one needs to look into the
related areas of Optimal Power Flow (OPF). Many authors have approached
the problem of calculating maximum loadability in the presence of limits using
OPF techniques [15], but it was specifically the representation of control
limits by means of complementarity constraints [16, 17, 18] that triggered new
approaches to the general power flow [19, 20], most notably [21]. One key idea
in these papers is the realization that a power flow with control limits can be
tackled as a constrained optimization problem, even though its aim is not com-
puting a loadability limit. Of course this begs the question of what objective
function to use, and in our opinion none of these works justify their choices sat-
isfactorily. Having said that, using at least some minimization criterion seems
a more principled approach than just using traditional adjustment heuristics or
type-switching, since neither one of those techniques allow for a characteriza-
tion of the solution they arrive to. One should be aware that there could be in
general more than one valid power flow solution, differing in how control devices
end up either in their saturated or un-saturated state.

This paper presents a general-purpose power flow method in which the prob-
lem of enforcing control limits is solved rigorously, based on a minimization
principle that stems from the fundamental physics of electrical networks. The
method is developed for the case of Mvar limits in generators regulating voltage
on both local and remote buses. Since it is based on the framework of the Hol-
omorphic Embedding Loadflow Method (HELM) [22, 23, 24], the method also
inherits the properties of being constructive and therefore produces unequivocal
results: it yields the correct operating solution when the case is feasible, and it
detects infeasibility otherwise.

The paper is structured as follows. Section 2 presents the problem definition
and lays down the nomenclature and conventions used in the method. Section 3
develops a Lagrangian formulation of power flow problem, starting from the
nonlinear DC problem (not to be confused with the linear “DC” approximation
to AC problems) and extending it to lossless AC networks. This provides the
minimization criterion underlying the proposed method, since it shows how to
understand the complementarity constraints in the framework of a constrained
minimization procedure. Section 4 then describes how the HELM scheme is im-
plemented for this problem. In contrast to barrier methods used in traditional
nonlinear programming (such as Interior Point Methods), the HELM technique
is purely based on power series and their analytic continuation. This section shows how to accommodate the constraint equations and, more importantly, how to ensure that the embedding guarantees that the HELM reference germ is consistent with reactive limits and voltage setpoints. Having laid down the essentials of the method, Section 5 then deals with a technical obstacle in the analytic continuation: the unavoidable singularity found at \( s = 1 \), which provokes a very slow convergence of the Padé approximants. This non-trivial numerical problem has been overcome thanks to a new technique based on certain changes of variables having the property that they leave the embedded power flow equations formally invariant under the change. This allows to construct the analytic continuation, still based on Padé approximants, incrementally in stages, similarly to the classic Weierstrass idea of analytic continuation along a path. Since this technique is absolutely crucial to the success of practical implementations on a computer, the method has been named after it. Finally, Section 6 shows a sample of numerical results. Appendix Section Appendix A recapitulates all steps of the method in summarized form.

2. The problem

Let us establish the nomenclature by considering a general power system comprised of constant-power injections \( S_i = P_i + jQ_i \) and constant-current injections \( I_i \). The power flow equations are given by the current balance at each bus,

\[
\sum_{j=0}^{n} y_{ij} (V_i - V_j) + Y_{i}^{\text{sh}} V_i = I_i + \frac{S_i^*}{V_i^*} \quad (i = 1, \ldots, n),
\]

where \( V_i \) are bus voltages, \( y_{ij} \) are branch admittances, and \( Y_{i}^{\text{sh}} \) are shunt admittances to ground, which are assumed to group contributions from line charging susceptances, shunts from transformer modeling, shunt reactor/capacitor banks, and any loads modeled as a constant admittance. As it is commonplace in power systems, the active sign convention will be used here, so that generators inject positive current \( I_i \) and power \( P_i \), while loads inject negative values for those same variables. The network is composed of a total of \( n + 1 \) buses where, without loss of generality, a swing bus is assumed at index 0 (and a swing voltage \( V_0 \) should be specified). It is also customary to define the transmission admittance matrix as:

\[
Y_{ij} = \begin{cases} 
\sum_{k=0}^{n} y_{ik} & \text{if } i = j \\
-y_{ij} & \text{if } i \neq j 
\end{cases}
\]

so that the powerflow equations can be expressed in a more compact form:

\[
\sum_{j=0}^{n} Y_{ij} V_j + Y_{i}^{\text{sh}} V_i = I_i + \frac{S_i^*}{V_i^*} \quad (i = 1, \ldots, n).
\]

These are the power flow equations in their most fundamental form, where all bus injections on the r.h.s. are specified. Since all magnitudes are complex-valued, system (2) consists of \( 2n \) equations in \( 2n \) unknowns, the complex voltages
V_i. In order to contemplate voltage regulation or any other type of control, one includes additional constraint equations and variables to (2), always obeying the basic algebraic rule that the number of independent equations and the number of variables has to match—otherwise one ends up with an over/under-determined system.

2.1. Voltage regulation (PV buses) with no Q-limits

In the following, the formulation will focus on the most ubiquitous type of control: the automatic voltage regulation (AVR) performed by generators. This kind of control may be either local or remote. When it is local to the bus, the bus is commonly referred to as “PV-type”. Whether local or remote, for the purposes of steady-state power flow, the control materializes as an additional equation constraining the voltage modulus of the controlled bus i, \(|V_i| = V_i^{sp}\) (the setpoint). Correspondingly, the reactive power injection \(Q_j\) of the controlling generator \(j\) becomes a new variable in the system. If there are several generators regulating the same voltage, some sharing factors need to be used in order to obtain a single effective degree of freedom, to avoid under-determinacy.

Therefore, when the controlling resources (in this case, generator Mvars) are assumed unlimited, the system of power flow equations simply gets augmented with new equations (equality constraints) and new variables. In the holomorphic embedding method, unlimited controls are easily incorporated into the formulation, after taking proper care of embedding the new constraints in a way that preserves both holomorphicity and consistency with the reference state at \(s = 0\). This can be done in a different number of ways, as shown in [25, 23, 26]. For the treatment of standard PV buses, our favored embedding scheme is as follows:

\[
i \in \text{PQ}: \sum_{j=0}^{n} Y_{ij} V_j(s) + s Y_{i}^{sh} V_i(s) = s I_i + \frac{s S_i^*}{V_i^*(s^*)} \tag{3}
\]

\[
k \in \text{PV}: \sum_{j=0}^{n} Y_{kj} V_j(s) + s Y_{k}^{sh} V_k(s) = s I_k + \frac{s P_k - j Q_k(s)}{V_k^*(s^*)} \tag{4}
\]

\[
k \in \text{PV}: V_k(s) V_k^*(s^*) = 1 + s (W_k^{sp} - 1). \tag{5}
\]

Here \(W_k^{sp}\) is defined as \((V_k^{sp})^2\). The swing voltage is assumed to be \(V_0 = 1\); this can be achieved either by performing a global normalization of variables, or more simply by embedding the swing as \(V_0(s) = 1 + s (V_0 - 1)\).

Considering the real and imaginary parts of voltages, this is a system of \(2n_{\text{PQ}} + 3n_{\text{PV}}\) equations, in \(2n_{\text{PQ}} + 3n_{\text{PV}}\) unknowns. However, these equations can be simplified and the dimensionality reduced to \(2n_{\text{PQ}} + n_{\text{PV}}\) equations and unknowns, as shown in Appendix A.

2.2. Dealing with Q-limits: complementarity equations

The focus is now switched to the problem of resource limits, which is the main subject of the paper. When a generator reaches either limit, \(Q_j^{\text{min}}\) or \(Q_j^{\text{max}}\), its controlling injection saturates at that limit, and the controlled voltage can
no longer be sustained at the setpoint. Moreover, the sign of the control error is important, as it should be consistent with the sign of the control sensitivity: if the generation saturates at $Q_{\text{max}}$, the achieved voltage should be $V \leq V^{\text{sp}}$, while if it saturates at $Q_{\text{min}}$ one should have $V^{\text{sp}} \leq V$. One should be careful with the special case where the PV bus condition has reversed sensibility, which corresponds to a power flow scenario on the “wrong side” of the generator’s Q-V curve. Although this is a mathematically valid power flow solution, it cannot be considered operationally correct, as it makes voltage regulation unstable. Further, it can be shown how, from the point of view of either HELM or a continuation power flow, these anomalous cases graze a point of collapse somewhere along the path. In the following, we will assume that the user has corrected these anomalies beforehand, so that all generator voltage controls have the correct sensitivity, $\partial V/\partial Q > 0$.

Given the complementary nature of the constraints, they may be expressed through the equations:

\[
\left( V_i V_i^* - W^{\text{sp}} \right) \left( Q_{\text{i max}} - Q_i \right) \left( Q_i - Q_{\text{i min}} \right) = 0 \tag{6}
\]

\[
\left( Q_{\text{i max}} - Q_i \right) \geq 0; \quad \left( Q_i - Q_{\text{i min}} \right) \geq 0 \tag{7}
\]

plus the additional consistency requirements regarding voltage control sensitivities:

\[
Q_i = Q_{\text{i max}} \implies |V_i| \leq V^{\text{sp}}
\]

\[
Q_i = Q_{\text{i min}} \implies |V_i| \geq V^{\text{sp}} \tag{8}
\]

The problem now is finding an equation or set of equations which, when appropriately embedded, would yield a solution satisfying all of these conditions [6]–[8] at $s = 1$. In contrast with the unlimited case, where the equality constraint [5] can be embedded in quite a straightforward way, the limited case contains a number of deep subtleties that make it much more non-obvious. Attempting an ad-hoc embedding of [6] with no regard to the other conditions is likely to end up in a faulty HELM scheme. In the following section, it will be demonstrated how, thanks to a suitable Lagrangian formulation of the power flow problem, this problem can be correctly framed as one of minimization, and thus obtain a HELM scheme that properly enforces all of the above equality and inequality constraints.

3. Lagrangian formulation of power flow

This section shows how the power flow equations can be derived from a Lagrangian formulation, much in the spirit of classical mechanics. This is then used to show how the problem of limits can be tackled in terms of minimization, and from there devise a suitable HELM scheme. Other authors have also approached the same problem using OPF-like minimization techniques [21], but here the treatment specifically addresses the holomorphic embedding formulation. In particular, it will be shown to have an intriguing resemblance to
interior point methods, but with the advantage of remaining a method based on holomorphic functions and their analytic continuation.

3.1. (Nonlinear) DC power flow

In order to show the essential points of the Lagrangian formulation, the attention is first turned to the pure DC problem, where equations are simpler. The nonlinear power flow equations of an all-DC network may be written as:

$$\sum_{j=0}^{n} g_{ij} (V_i - V_j) + G_{sh}^i V_i = I_i + \frac{P_i^*}{V_i^*} \quad (i = 1, \ldots, n), \quad (9)$$

This is not the commonly used “DC power flow” approximation to AC; rather, it is the exact DC counterpart to the AC problem [1], where voltages, conductances, and power and current injections are all real magnitudes. Systems of this type are not common in utility grids, but they do appear for instance in spacecraft, shipboard systems, and some modern microgrids [24]. In any case, this system will be shown to be quite useful as it clarifies up many issues that are otherwise obscured in the AC formulation.

It is straightforward to verify that equations (9) can be derived from the minimization of the following Lagrangian:

$$L = \frac{1}{2} \sum_{j < i} g_{ij} (V_i - V_j)^2 + \frac{1}{2} \sum_{i} G_{sh}^i V_i^2 - I_i V_i - P_i \ln V_i, \quad (10)$$

where the double summation of the first term avoids counting links twice, but does include the swing (at index $j = 0$); while the single summation of the rest of the terms runs only over non-swing buses. For now the term Lagrangian is used here to refer to what is essentially a potential energy function (in the spirit of classical physics), but concepts of constrained optimization and Lagrangian duality will be used shortly after.

The fact that the powerflow equations admit a description in terms of a potential function is remarkable and should not be dismissed as a mere curiosity. It follows that the powerflow problem can be seen as equivalent to a problem of finding static equilibria in classical mechanics. It is thus instructive to analyze the physical meaning of each term in this Lagrangian, as it can provide fruitful insights. The first term is the sum of the power losses ($I^2 R$) over all transmission branches of the network (divided by two). The next term accounts for the power consumption due to shunt conductances. The next term, $I_i V_i$, is the power supplied by constant-current injections at each bus. The last term, $P_i \ln V_i$, is also power supplied into the network, although the particular form of its expression will become clearer shortly. Reading (10) as a problem in classical mechanics, one could envision a hypothetical system made up of masses connected with springs and sitting under some peculiar “gravitational” fields. Conductances $g_{ij}$ and $G_{sh}^i$ would play the role of spring constants, while the rest of the terms could be interpreted as some sort of on-site potential producing
a force field acting locally on each bus. It is also interesting to note that the
operational solution corresponds to a stable equilibrium of the system (a mini-
mum), while all other power flow solutions ("black branches" [22]) correspond
to unstable equilibria (saddle points), although this will not be proven here.

The role of constant power injections becomes clearer when (10) is viewed
as resulting from a slightly different, constrained minimization problem:

\[
\mathcal{E}(\{V_i\}) \equiv \frac{1}{2} \sum_{j<i} g_{ij} (V_i - V_j)^2 + \frac{1}{2} \sum_i G_{th}^i V_i^2 - I_i V_i
\]

minimize: \( \mathcal{E}(\{V_i\}) \)  
subject to: \( V_k - V_{k}^{sp} = 0 \quad (k \in \{V\text{-regulated}\}) \)  (11)

Here the voltage is regulated to a setpoint \( V_{k}^{sp} \) on those buses for which there
was a non-zero constant-power injection specified in problem (9). Since negative
voltages are not allowed, these constraint equations can be written equivalently
as \( \ln V_{k}^{sp} - \ln V_k = 0 \). This constrained problem is then solved by the standard
method of Lagrange multipliers, using the following Lagrangian:

\[
\mathcal{L} \equiv \mathcal{E} - \sum_k P_k \ln \left( \frac{V_k}{V_{k}^{sp}} \right)
\]

which is the same as (10), except for additional constant terms \( P_i \ln V_{i}^{sp} \) which
do not depend on the voltages. The stationarity conditions then take the same
form as the original power flow equations (9). Therefore, constant power in-
jections can be seen as the Lagrange multipliers of minimization problem (11).

Additionally, this view has the advantage of having a more direct physical in-
terpretation in terms of a fundamental energy balance, since now the primal
minimization problem is on \( \mathcal{E} \): power losses, plus power consumption, minus
power supply. Under this light, the standard power flow problem (9), in which
constant power injections (both load and generation) are specified, could be
interpreted as minimizing the transmission network losses. This is a key insight
and a powerful guide when approaching the AC case.

Adding now resource constraints to the regulation resources:

minimize: \( \mathcal{E}(\{V_i\}) \)  
subject to: \( V_k - V_{k}^{sp} = 0 \quad (k \in \{V\text{-regulated}\}) \)  
\( P_k^{\min} \leq P_k \leq P_k^{\max} \)  (13)

Note how these inequality constraints are on the dual variables of problem (11).
Let us solve the maximization of the dual problem by means of logarithmic
barrier methods, by defining:

\[
\mathcal{B}_\mu(\{P_k\}) \equiv \mathcal{E} - \sum_k P_k \ln \left( \frac{V_k}{V_{k}^{sp}} \right) + \mu \ln (P_k^{\max} - P_k) + \mu \ln (P_k - P_k^{\min})
\]
where $\mu > 0$ is the barrier parameter that is made to converge to zero. The stationarity conditions for maximizing $B_\mu$ yield:

$$-\ln \left( \frac{V_k}{V_{sp}^k} \right) - \frac{\mu}{P_{k}^{\text{max}} - P_k} + \frac{\mu}{P_k - P_{k}^{\text{min}}} = 0$$ (14)

Rearranging,

$$(\ln V_k - \ln V_{sp}^k) (P_{k}^{\text{max}} - P_k) (P_k - P_{k}^{\text{min}}) = \mu \left( P_k^{\text{max}} + P_{k}^{\text{min}} - 2P_k \right)$$

In the limit $\mu \to 0$, this equation is essentially the same as the complementarity constraint (6) (when translated to the DC problem), since $\ln V_k = \ln V_{sp}^k$ if and only if $V_k = V_{sp}^k$. The rest of the conditions are also met: the solution will satisfy $P_{k}^{\text{min}} \leq P_k \leq P_{k}^{\text{max}}$ because the barrier guarantees that if the starting point is interior, the solution will remain so; and in case of saturation, the correct control sensitivities are satisfied as well, as it is readily verified by inspecting (14). Therefore this suggests a HELM scheme where using $\mu = \mu_0 (1 - s)$ to embed (14) can accomplish all the desired goals. The next subsection will fully develop this idea for the AC case.

### 3.2. AC power flow

This treatment finds an exact parallel in the AC case. However, the authors have found this is only possible in the limit of an ideal, lossless transmission network, where all branches (i.e. lines, transformers, shunts) have zero resistance. This is only a limitation as far as it concerns the characterization of the solution provided by the method, not its precision. For networks in which this ideality limit is strongly violated, one can still take the proposed HELM scheme at face value and compute a feasible power flow satisfying control limits—the only difference is that in this case it is no longer strictly true that the resulting solution is minimizing something. Still, resistances in transmission networks are typically low ($R/X \lesssim 0.1$), so that, simply invoking a continuity argument, one should expect that using this method should lead to a solution that approximately minimizes the Lagrangian shown here, which will be shown to be related to the net reactive power losses.

It will first be shown how the power flow equations of an ideal lossless AC network can be derived from the minimization of a suitable Lagrangian function. In a lossless network all line and transformer conductances vanish, so that all admittances, including shunts, become pure imaginary. Using real and imaginary components, equation (14) becomes:

$$-b_{ij} \text{Im}(V_i - V_j) - b_i^{\text{sh}} \text{Im} V_i = \text{Re} I_i + \frac{P_i \text{Re} V_i + Q_i \text{Im} V_i}{|V_i|^2}$$

$$b_{ij} \text{Re}(V_i - V_j) + b_i^{\text{sh}} \text{Re} V_i = \text{Im} I_i + \frac{P_i \text{Im} V_i - Q_i \text{Re} V_i}{|V_i|^2}$$ (15)
Written in this form, it is straightforward to verify that these equations can be derived from the minimization of the following Lagrangian:

\begin{equation}
\mathcal{L} = -\frac{1}{2} \sum_{ij} b_{ij} \left\{ (\text{Re} V_i - \text{Re} V_j)^2 + (\text{Im} V_i - \text{Im} V_j)^2 \right\} - \frac{1}{2} \sum_i b_{sh}^i \left\{ (\text{Re} V_i)^2 + (\text{Im} V_i)^2 \right\} - \frac{1}{2} \sum_i (\text{Re} I_i \text{Im} V_i - \text{Im} I_i \text{Re} V_i) - \frac{1}{2} \sum_i Q_i \ln \left\{ (\text{Re} V_i)^2 + (\text{Im} V_i)^2 \right\} - \sum_i P_i \tan^{-1} \left( \frac{\text{Im} V_i}{\text{Re} V_i} \right), \tag{16}
\end{equation}

where the index \( j \) in the double summation includes the swing bus. If the complex voltages and complex current injections are now reinterpreted to be vectors in 2-dimensional real space, (16) can be written in the more compact form:

\begin{equation}
\mathcal{L} = -\frac{1}{2} \sum_{ij} b_{ij} \| V_i - V_j \|^2 - \frac{1}{2} \sum_i b_{sh}^i \| V_i \|^2 - \frac{1}{2} \sum_i \| I_i \times V_i \| - \frac{1}{2} \sum_i Q_i \ln \| V_i \|^2 - \sum_i P_i \theta_i, \tag{17}
\end{equation}

Let us analyze the meaning of each term in this expression. The first term is the sum of the reactive power losses \((I^2 X)\) over all network branches (divided by two). Since the susceptance \( b_{ij} \) of transmission lines, except in exceptional cases such as equivalents, is always negative, this first term is always positive. The next term is similar in nature, as it accounts for the reactive power injection/consumption due to shunt admittance terms. These originate mainly from in line charging susceptances, but may also include contributions from shunt banks and constant-impedance load models. The next term, containing the cross product of “vectors” \( I_i \) and \( V_i \) at each bus, shows the effect of local constant-current injections. Its interpretation in a mechanical analogue may not be obvious, but note that it also has dimensions of power. The next term may be interpreted as a sort of potential energy provided by an external on-site “field” produced by a local “charge” \( Q_i \). It is a local logarithmic potential acting on each bus. The last term may also be interpreted as produced by an on-site field, this time mediated by external charges \( P_i \) acting on each node and exerting a sort of transversal force that tries to rotate the voltage vector. Therefore all terms in this Lagrangian consist of magnitudes having dimensions of power, and can be interpreted as being some sort of potential energy, if one forgets about the time dimension. As in the DC case, one can see this as a pseudo-mechanical analogue, where the power flow equations are re-interpreted as the equations for the static equilibria of the system (one could think of particles moving in a 2D space, interconnected with springs and having on-site potentials acting on them). This analogy may be exploited in multiple ways by drawing on the
techniques and intuition from classical mechanics, but here the focus will be on the constrained problem.

Analogously to the DC case, voltage regulation can be seen as the following constrained minimization problem:

$$\begin{align*}
\text{minimize:} & \quad \mathcal{E}(\{V_i\}) \\
\text{subject to:} & \quad |V_k|^2 - W_k^{\text{sp}} = 0 \quad (k \in \{\text{V-regulated}\})
\end{align*}$$

Here $\mathcal{E}$ is defined as $\mathcal{L}$ in [17], but omitting the terms $\frac{1}{2} Q_k \ln \|V_k\|^2$ for voltage-regulated buses $k$. Again, this problem is solved with the standard technique of Lagrange multipliers:

$$\mathcal{L} = \mathcal{E} - \frac{1}{2} \sum_k Q_k \ln \frac{|V_k|^2}{W_k^{\text{sp}}} + \mu \ln (Q_k^{\text{max}} - Q_k) + \mu \ln (Q_k - Q_k^{\text{min}})$$

which is the same as [17], except for additional constant terms $Q_k \ln W_k^{\text{sp}}$ which do not depend on the voltages. The stationarity conditions then take the same form as the original power flow equations [15]. Similarly to the DC case, reactive power injections can be seen as the Lagrange multipliers of minimization problem [18]. The physical interpretation for the minimization of $\mathcal{E}$ in terms of energy is in general not so evident as in DC, but it can still be recovered if all active power injections $P_i$ are made zero. In that limit, it can be shown that all angles are zero, all voltages and currents are real, and the system actually becomes mathematically equivalent to the DC case.

Let us now contemplate resource constraints, i.e., Mvar limits:

$$\begin{align*}
\text{minimize:} & \quad \mathcal{E}(\{V_i\}) \\
\text{subject to:} & \quad |V_k|^2 - W_k^{\text{sp}} = 0 \quad (k \in \{\text{V-regulated}\}) \\
& \quad Q_k^{\text{min}} \leq Q_k \leq Q_k^{\text{max}}
\end{align*}$$

These inequality constraints are on the dual variables of problem [18]. Consider then the maximization of the dual problem by means of logarithmic barrier methods, by defining:

$$B_\mu(\{Q_k\}) = \mathcal{E} - \frac{1}{2} \sum_k Q_k \ln \frac{|V_k|^2}{W_k^{\text{sp}}} + \mu \ln (Q_k^{\text{max}} - Q_k) + \mu \ln (Q_k - Q_k^{\text{min}})$$

where $\mu > 0$ is the barrier parameter. The stationarity conditions for maximizing $B_\mu$ yield:

$$-\frac{1}{2} \ln \frac{|V_k|^2}{W_k^{\text{sp}}} - \frac{\mu}{Q_k^{\text{max}} - Q_k} + \frac{\mu}{Q_k - Q_k^{\text{min}}} = 0$$

Rearranging, one obtains:

$$\left(\ln |V_k|^2 - \ln W_k^{\text{sp}}\right) \left(Q_k^{\text{max}} - Q_k\right) \left(Q_k - Q_k^{\text{min}}\right) = 2\mu (Q_k^{\text{max}} + Q_k^{\text{min}} - 2Q_k)$$
In the limit $\mu \to 0$, this equation is essentially the same as the complementarity constraint (6), since $\ln |V_k|^2 = \ln W_k^{\text{sp}}$ if and only if $|V_k|^2 = W_k^{\text{sp}}$. The rest of the conditions are also met: (7) is satisfied because the barrier terms guarantee that the solution will remain interior if the starting point is so; and the signs of the control errors, given by (8), can be verified to be the correct ones by inspecting (22). When the solution is such that the $Q_k^{\text{max}}$ constraint is active, one obtains $|V_k|^2 \leq W_k^{\text{sp}}$; and vice versa.

Therefore this suggests a HELM scheme where (22) is embedded by using, for instance, $\mu = \mu_0 (1 - s)$. Together with some additional precautions, to be described in the next section, this equation replaces (5) and provides a HELM scheme to solve the problem of Q-limits. This establishes the relationship between the proposed HELM scheme and the above minimization problem. The HELM solution is thus characterized as minimizing $E$ in (20), which, in the limit of the ideal lossless network, is essentially the net amount of reactive losses.

Finally, this result also suggests an apparent resemblance between so-called Interior Point Methods (IPM) and the analytic continuation technique used in HELM. However, just as in the case of homotopy methods, IPM and similar barrier methods are based on continuity and differentiability, not holomorphicity (which is a much stronger requirement).

4. HELM scheme. Consistency requirements

Following the above, a HELM scheme is proposed to solve the problem of voltage regulation while observing Q-limits, using the canonically embedded system (3)–(4) and embedding (22) as follows:

$$V_k(s)V_k^*(s^*) - W_k^{\text{sp}} = \frac{\mu_k (1 - s)}{Q_k(s) - Q_k^{\text{min}}} - \frac{\mu_k (1 - s)}{Q_k^{\text{max}} - Q_k(s)} (24)$$

where $\mu_k$ are suitable constants to be discussed below (following IPM methods, they are chosen to be the same for both terms). As discussed in the previous section, in the limit $s \to 0$ these equations reproduce the complementarity constraints (6)–(8). At this point, it is important to remark that there are possibly many other methods to obtain solutions satisfying the complementarity equations, but this is a method informed by the Lagrangian. In case there are several feasible solutions, the method presented here will select the one minimizing the Lagrangian (17), which has shown to be rooted in the physics of the transmission network.

On the other hand, the following treatment will consider the possibility that the voltage regulation may be remote, i.e. that the controlled bus and its controlling injection are not on the same bus. There is nothing in the constraint equations preventing this, as it would still produce the same, well-defined algebraic problem. The only difference in such case is that, in a strict sense, it would no longer be possible to interpret the resulting solution as the minimizer of the constrained problem shown in the previous section. However, if one restricts this to the most common case, which is regulation at a distance of one bus, the
interpretation is expected to still hold approximately, in the sense that other feasible solutions would yield a higher value of the Lagrangian.

Therefore from now on the notation will label the set of \( n_{PV} \) reactive bus injections with indexes \( a \), and the set of their corresponding \( n_{PV} \) regulated bus voltages with \( b \). Note that the treatment given here does not contemplate concurrent control (i.e. shared responsibility regulation) of the same bus originated from different buses. This problem has no unique solution in general, as it depends on the details of the var-sharing policy, priority rules among generators, etc.

HELM now requires that the embedding is designed in a way that a unique and meaningful reference powerflow state is obtained at \( s = 0 \): the zero-injection state, in which all voltages are equal to the swing. It is then postulated that the operational solution, in contrast to the many other power flow solutions, is the one that is analytically continued to \( s = 1 \) from this reference state (when it exists). This is based on physical arguments, discussed at length in [22, 23]. An important question then is whether (24) can be consistent with the HELM reference solution, given that limits \( Q_{\min} \), \( Q_{\max} \) come from engineering constraints and could have any value. This consistency can be examined by considering the power series of the variables involved, and checking (24) at order zero. Using the notation \( V_b(s) = \sum_n V_{b[n]} s^n \), \( Q_a(s) = \sum_n Q_{a[n]} s^n \) for power series coefficients, one obtains at zero order:

\[
V_b[0]V_b^*[0] - W_{sp}^b = \frac{\mu_a}{Q_a[0] - Q_{a[\min]}} - \frac{\mu_a}{Q_{a[\max]} - Q_a[0]}
\]

and since the reference solution has \( V_b[0] = 1 \), \( Q_a[0] = 0 \) for all buses:

\[
1 - W_{sp}^b = -\mu_a \frac{Q_{a[\max]} + Q_{a[\min]}}{Q_{a[\max]} Q_{a[\min]}}
\]  

(25)

There are two different consistency requirements to observe here. The first one is that the reference state should not violate limits (or hit exactly a limit), which implies \( Q_{a[\min]} < 0, Q_{a[\max]} > 0 \). In the parlance of interior point methods, this is is the analogue of ensuring that the starting point is “interior”. The second requirement is that (25) should be satisfied with a positive value of the barrier constant \( \mu_a \). Since \( Q_{a[\min]} Q_{a[\max]} < 0 \), this implies \( \text{sgn}(1 - W_{sp}^b) = \text{sgn}(Q_{a[\max]} + Q_{a[\min]}) \). Neither one of these two requirements are guaranteed to be satisfied, in general. Therefore one is forced to embed both the limits and the setpoints, in a way that the above conditions are satisfied at order zero. Note how this was already done in the unlimited PV case, as \( W_{sp}^b(s) = 1 + s(W_{sp} - 1) \) in (5). In this case, the embeddings should be designed to require that:

\[
Q_{a[\min]}[0] < 0, \quad Q_{a[\max]}[0] > 0
\]

\[
\text{sgn} (1 - W_{sp}^b[0]) = \text{sgn} (Q_{a[\max]}[0] + Q_{a[\min]}[0])
\]

(26)

For one-sided limits, the last expression needs to be replaced by \( W_{sp}^b[0] > 1 \) when only \( Q_{a[\max]} \) is present, or by \( W_{sp}^b[0] < 1 \) when only \( Q_{a[\min]} \) is present. All
these conditions can be accomplished by means of many possible embeddings. However, in keeping with the general principle of introducing the minimum amount of changes needed, our preferred method is to embed limits and setpoints linearly, choosing suitable guard constants:

\[ Q_{a}^{\min}(s) = Q_{a}^{\min} + \delta Q_{a}^{\min}(1 - s) \]
\[ Q_{a}^{\max}(s) = Q_{a}^{\max} + \delta Q_{a}^{\max}(1 - s) \]  
\[ W_{b}^{sp}(s) = W_{b}^{sp} + \delta W_{b}^{sp}(1 - s) \]  

(27)

If both limits are present, one reasonable choice for these constants consists in fixing \( \mu_{a} = 1 \) and using the same embedding for \( W_{b}^{sp} \) as in the unlimited case, plus the following symmetrizing embedding for the limits:

\[ Q_{a}^{\min}(s) = -Q_{a}^{\max} - Q_{a}^{\min} + sQ_{a}^{\max} + Q_{a}^{\min} \]
\[ Q_{a}^{\max}(s) = Q_{a}^{\max} - Q_{a}^{\min} + sQ_{a}^{\max} + Q_{a}^{\min} \]  

(28)

which leads to the following embedding constants in (27):

\[ \delta Q_{a}^{\min} = \delta Q_{a}^{\max} = -\frac{Q_{a}^{\max} + Q_{a}^{\min}}{2} \]
\[ \delta W_{b}^{sp} = 1 - W_{b}^{sp} \]  

(29)

For buses with one-sided limits, one needs to choose values for the guard constants in (27) so that the zero order coefficients satisfy the sign requirements and have reasonable values. For instance, in case one has only a lower limit \( Q_{a}^{\min} \):

\[ \delta Q_{a}^{\min} = Q_{a}^{\min}[0] - Q_{a}^{\min} \]
\[ \delta W_{b}^{sp} = W_{b}^{sp}[0] - W_{b}^{sp} \]  
\[ \mu_{a} = Q_{a}^{\min}[0](W_{b}^{sp}[0] - 1) \]  

(30)

so that one reasonable choice could be \( W_{b}^{sp}[0] = 0.9 \text{ pu} \) and \( Q_{a}^{\min}[0] = -10 \text{ pu} \).

For buses where there is only an upper limit \( Q_{a}^{\max} \), one obtains:

\[ \delta Q_{a}^{\max} = Q_{a}^{\max}[0] - Q_{a}^{\max} \]
\[ \delta W_{b}^{sp} = W_{b}^{sp}[0] - W_{b}^{sp} \]  
\[ \mu_{a} = Q_{a}^{\max}[0](W_{b}^{sp}[0] - 1) \]  

(31)

and the corresponding reasonable choice could be \( W_{b}^{sp}[0] = 1.1 \text{ pu} \) and \( Q_{a}^{\min}[0] = +10 \text{ pu} \). However, to keep the following treatment as general as possible, no particular embedding constants will be assumed; it will only be assumed that the embeddings are linear, as given by (27).

The HELM procedure can now be constructed by obtaining the linear system that allows obtaining all unknown coefficients at order \( N \) from the knowledge
of coefficients at orders $N−1$ and lower. This system has been referred to as the $N$-th order representation of the embedded equations [22]. Solving these linear systems in sequence is what constructs the germ of the reference solution, which is later to be analytically continued into the final solution at $s = 1$ (if it exists). For the base power flow equations (3) and (4), Appendix A provides the $N$-th order representation formulas (in the more general Padé-Weierstrass framework). Here the focus is put on the constraint equations, in order to show how their contribution results in an enlarged but well-defined linear system.

The embedding selected in (27) directly provides the power series corresponding to setpoints and limits. They have just two coefficients:

- $Q_a^{\text{min}}(s) = Q_a^{\text{min}}[0] + Q_a^{\text{min}}[1] s$
- $Q_a^{\text{max}}(s) = Q_a^{\text{max}}[0] + Q_a^{\text{max}}[1] s$
- $W_{b}^{\text{sp}}(s) = W_b^{\text{sp}}[0] + W_b^{\text{sp}}[1] s$

If you chose the symmetrizing embedding [28], the values of $\mu_a$ can be chosen all $\mu_a = 1$; otherwise they will be given by the constraint equation [24] at zero order:

$$\mu_a = (W_{b}^{\text{sp}}[0] - 1) \frac{Q_a^{\text{max}}[0] Q_a^{\text{min}}[0]}{Q_a^{\text{max}}[0] + Q_a^{\text{min}}[0]}$$

Now the following auxiliary series will be defined:

$$B_a^{(+/−)}(s) \equiv \frac{\mu_a(1−s)}{Q_a(s)−Q_a^{(\text{min/max})}(s)} = \sum_{m=0}^{\infty} B_a^{(+/−)[m]} s^m$$

Since $Q_a[0] = 0$, its zero-order values are:

$$B_a^{(+/−)[0]} = -\frac{\mu_a}{Q_a^{(\text{min/max})}[0]}$$

For $N ≥ 1$, equating the $N$-th power coefficients on each side of the constraint equation [24] yields:

$$\sum_{m=0}^{N} V_b[m] V_b^*[N−m] = W_b^{\text{sp}}[1] \delta_{1,N} = B_a^- [N] + B_a^+ [N]$$

where $\delta_{1,N}$ is the Kronecker delta. Using the known values $V_b[0] = 1$, the convolution sum on the right hand side decomposes as:

$$2\Re V_b[N] + \sum_{m=1}^{N−1} V_b[m] V_b^*[N−m]$$

Note that, even though this expression involves complex magnitudes, the sum is real, given the symmetry of the convolution.
The power series coefficients of $B^a_{+/−}(s)$ can be obtained in terms of those of $Q_a(s)$ from their definition. For instance, by equating the $N$-th order coefficients of each side of equation $B^a_{+/−}(s) (Q_a(s) − Q^{\text{max}}_a(s)) = \mu_a (1 − s)$, for $N \geq 1$:

$$\sum_{m=0}^{N} B^a_{+}[m] Q_a[N − m] − Q^{\text{max}}_a[0] B^a_{+}[N] = −Q^{\text{max}}_a[1] B^a_{+}[N − 1] = −\mu_a \delta_{1,N}$$

and analogously for $B^a_−$. Since $Q_a[0] = 0$, one obtains:

$$Q^{\text{max}}_a[0] B^a_{+}[N] = B^a_{+}[0] Q_a[N] + \sum_{m=1}^{N−1} B^a_{+}[m] Q_a[N − m] = −B^a_{+}[N − 1] Q^{\text{max}}_a[1] + \mu_a \delta_{1,N}$$

Moving all terms in (32) that include coefficients at order $N$ to the left hand side, one obtains:

$$2 \text{Re} V_b[N] − \left( \frac{B^a_{−}[0]}{Q_{\text{min}}[0]} + \frac{B^a_{+}[0]}{Q^{\text{max}}_a[0]} \right) Q_a[N] = \mathcal{T}_a[N − 1]$$

where the symbol $\mathcal{T}_k[N − 1]$ is used as a convenient shorthand to group all other terms, which are either known quantities or involve coefficients of order $N − 1$ and lower. In the case of one-sided limits, the expressions are very similar and straightforward to obtain.

This set of equations (33) complement the rest, to form a linear system of dimension $2n_{\text{PQ}} + 3n_{\text{PV}}$, involving variables $\text{Re} V_j[N]$, $\text{Im} V_j[N]$, and $Q_a[N]$. Actually, variables $Q_a[N]$ can be easily eliminated from (33) in terms of $\text{Re} V_b[N]$ (it is verified that their pivot cannot be zero), and substituted into the rest of the equations to obtain a linear system of dimension $2(n_{\text{PQ}} + n_{\text{PV}})$. The method then proceeds as usual [22]: one solves these linear systems in sequence, starting from $N = 1$ until enough power series terms are obtained to either obtain convergence of detect oscillation in the Padé approximant sequences.

5. The Padé-Weierstrass method for analytic continuation

The preceding scheme is able to produce a well-defined germ, but its analytic continuation to $s = 1$ by means of Padé approximants suffers from an intrinsic problem: by its very construction, the embedded complementarity constraint (24) creates a singularity of the algebraic curve at $s = 1$, since this is a branching point for many other solutions. Such solutions correspond to the many possible choices in which one can saturate the controls (i.e. different selections of buses switched to PQ-type). All of them satisfy the power flow equations as well as the complementarity constraint, but most of them violate one or more of the other conditions, i.e. (7) and (8). And if there existed any configuration satisfying all the requirements (something that cannot be ruled out, due to the
nonlinearity of the problem), it would be “worse” than the solution produced by the proposed HELM scheme, in the sense of the minimization criterion provided by the Lagrangian formulation on which the method is grounded (the HELM solution approximately minimizes reactive power losses).

As it is well-known from Stahl’s theory [27], the near-diagonal sequences of Padé approximants cease to converge at singularities (and on the minimal cut-set joining them). For the HELM scheme developed in the previous section this means that, even if the power flow is feasible, the approximants are no longer guaranteed to converge at \( s = 1 \). Actually, unless the case is outright infeasible (in which case the approximants will clearly oscillate), what happens is that the convergence rate gets very slow as \( s \) approaches 1. In the unlimited HELM method, this would only happen when calculating a case exactly at a point of collapse. In both situations, due to the limits of floating point arithmetic (truncation and round-off), there is a point at which high order Padé approximants can no longer be calculated with enough precision to improve the results. The net result is a ceiling to the achievable precision in the solution as \( s \to 1 \). Numerical experiments show that, in the particular case of the Q-limits HELM scheme, the magnitude of the precision loss is usually too large to obtain acceptable results. On the other hand, in case of powerflow infeasibility, a voltage collapse point will be met earlier at some \( s_{\text{crit}} < 1 \), as shown in [23]; this will be more easily detected as oscillations in the approximants when evaluated at \( s = 1 \).

In any case, given the nature of the HELM reference solution at \( s = 0 \), it is always possible to find a real value \( 0 < s_0 < 1 \) such that the Padé approximants do converge to the required precision. This means it is always possible to obtain a partial solution \( V_i(s_0) \) whose values are exact, within machine precision. What follows is a method that exploits this property and comes up with a method to perform the analytic continuation of the reference germ in a series of stages that approach \( s \to 1 \), greatly enhancing the numerical precision. This is effectively an analytic continuation along a path on the real axis, in the same spirit as Weierstrass, but using a novel technique. A classical Weierstrass procedure would use the initial power series to construct a second power series at a point \( s_0 \) inside its radius of convergence, and then repeat this process. However, Henrici [28] shows that this naive approach yields in practice very poor numerical precision. Here, by contrast, the method uses partial solutions and the power flow equations in order to re-expand the new germs along the path. Since the procedure makes use of Padé approximants as a key element for obtaining partial solutions to very high precision, the method is termed Padé-Weierstrass (to be abbreviated as P-W).

5.1. Two-bus DC system

The essential elements of the Padé-Weierstrass method can be shown in the simplest power flow problem, the two-bus DC system. For further simplicity, the swing voltage will be normalized to 1. The embedded equation reads:

\[
G(V(s) - 1) = s \frac{P}{V(s)}
\]  

(34)
As always, the aim is to obtain the analytic continuation of the HELM germ to \( s = 1 \). Assume now that the convergence of Padé approximants is so slow at \( s = 1 \) that the achievable precision is degraded (though in this simple system, this only happens when the solution is very close to point of voltage collapse). One can always find a value \( 0 < s_0 < 1 \) such that the approximants converge to any required tolerance (within machine precision limits). Let us use these values \( s_0 \) and \( V(s_0) \) to rewrite the equation in terms of the following renormalized parameter and voltage:

\[
\begin{align*}
  s &\equiv s_0 + (1 - s_0)s' \\
  V(s) &\equiv V(s_0)V'(s')
\end{align*}
\]  

The parameter change \( s \rightarrow s' \) is a conformal mapping that takes the range \([s_0, 1]\) in \( s \)-space into the range \([0, 1]\) in \( s' \)-space. Using these, (34) reads:

\[
V(s_0)G\left(V(s_0)V'(s') - 1\right) = \frac{s_0 P}{V(s)} + s' (1 - s_0) \frac{P}{V'(s')}
\]

Using the fact that \( G(V(s_0) - 1) = s_0 \frac{P}{V(s_0)} \), the right hand side of this equation yields:

\[
V(s_0)G\left(V(s_0) - 1\right)V'(s') + V(s_0)G\left(V'(s') - 1\right) = s_0 PV'(s') + V(s_0)G\left(V'(s') - 1\right)
\]

and the final equation becomes:

\[
G'\left(V'(s') - 1\right) = \Gamma'\left(\frac{1}{V'(s')} - V'(s')\right) + s' \frac{P'}{V'(s')}
\]

where the new constants are defined as \( G' \equiv V(s_0)G \), \( \Gamma' \equiv s_0 P \), and \( P' \equiv (1 - s_0)P \). This equation describes a new power flow problem, very similar to (34) except for the appearance of two new terms which will be jointly referred to as a \( \Gamma \)-term. Before analyzing in detail its meaning and role, let us remark that the term itself is completely invariant under this renormalization procedure. That is, assume that instead of (34) one is given the following initial power flow problem:

\[
G\left(V(s) - 1\right) = \Gamma\left(\frac{1}{V(s)} - V(s)\right) + s \frac{P}{V(s)}
\]

Then it can be verified that, under the change (35), this system also transforms exactly into system (36), with these new parameters:

\[
\begin{align*}
  G' &\equiv V(s_0)G \\
  \Gamma' &\equiv \Gamma + s_0 P \\
  P' &\equiv (1 - s_0)P
\end{align*}
\]

This result is quite remarkable. Conceptually, this is similar to the way certain changes of coordinates leave physical laws invariant, i.e. the principle of
general covariance. For instance Newton’s equations of classical mechanics are invariant under Galilean transformations, and Einstein’s equations of Special Relativity are invariant under Lorentz transformations. What we have here is that, when the embedded powerflow problem is written as (37), it is invariant under the “change of reference frame” given by (35).

This property can be exploited to good effect. The whole procedure described here, which will be referred to as a single Padé-Weierstrass stage, can be repeated multiple times, for suitable values $s'_0, s''_0$, etc. At each stage one always obtains a new powerflow problem that it is formally the same, where only the parameters have changed. The great advantage is that, as it will be shown, each successive stage produces a better-conditioned HELM problem from the numerical point of view. Then, if at some stage $k$ the Padé approximants are able to converge with the requested precision at $s^{(k)} = 1$, one can simply undo all the changes (35) and thus obtain the sought solution to the initial problem.

Let us now turn to the $\Gamma$-term. On the first P-W stage it is zero, but it will appear at all subsequent stages. The term can be thought of as a combination of a constant-power injection and a constant-admittance shunt. Their respective currents cancel out when $V'(s') = 1$, which happens at $s' = 0$. This is consistent with the HELM reference state (no flows, all voltages equal to the swing). However, the novelty is that this $\Gamma$-term is not embedded in $s$, so it remains to be seen how the HELM procedure can work. It turns out that given the specific form of this term, whereby their currents cancel out at $s = 0$, the standard HELM method can be carried through without any problem, as shown now.

To kickstart the method on (37), consider first the equation obtained for the power series coefficients at zero order. It is verified to be consistent with the HELM reference state, $V(0) = 1$, since $V^{-1}(0) = V[0] = 1$. Recall that the power series coefficients for the function $1/V(s)$, symbolically represented as $V^{-1}[N]$ (not to be confused with $1/V[N]$), can be obtained from those of $V(s)$ by the relationship:

$$\sum_{m=0}^{N} V[m] V^{-1}[N-m] = \delta_{N,0}$$

(38)

At orders $N \geq 1$, one obtains:

$$GV[N] = \Gamma \left( V^{-1}[N] - V[N] \right) + PV^{-1}[N-1]$$

Using (38), one finally obtains:

$$(G + 2\Gamma) V[N] = PV^{-1}[N-1] - \Gamma \sum_{m=1}^{N-1} V[m] V^{-1}[N-m]$$

(39)

All terms on the right hand side are order $N - 1$ or lower, so they can be computed from coefficients obtained at previous steps. Therefore the HELM scheme can be carried out without any problem.
There is one final aspect that deserves attention. The coefficient of $V[N]$ in (39) could in principle become zero if left unchecked. In a full $n$-bus system, the analogous problem would manifest itself as the matrix of the HELM linear system becoming singular. Let us analyze for instance what would happen when trying to solve the two-bus system (34) for a case right at the voltage collapse point, i.e. $P = P_{crit} = -G/4$. Performing one step of the P-W procedure, one chooses a point $0 < s_0 < 1$ for which the approximants converge to $V(s_0)$ within the required precision. Since this problem can be solved in closed form, one finds $V(s_0) = (1 + \sqrt{1 - s_0})/2$. Therefore, at the next P-W stage the pivot coefficient in (39) becomes:

$$G' + 2\Gamma' = \frac{1}{2} G \left(1 - s_0 + \sqrt{1 - s_0}\right)$$

From this expression one can observe that, when the target state at $s = 1$ sits near or at a singularity, there is a certain trade-off in the choice of $s_0$. Even if the approximants allowed it, getting too close in one step (i.e. $s_0 \approx 1$) would produce a numerically unstable system in the next P-W step; so it may be more advantageous to settle for a smaller value even if this means taking a few more P-W steps, if the overall numerical stability improves.

Next it will be shown how this method can be extended to the full AC power flow, including voltage regulation and Q-limits.

5.2. Full AC system: main equations

The P-W method will now be developed for the full AC power flow problem, as represented by the embedded system (3)–(4) and including the observance of Q-limits by means of the complementarity constraint (24) as shown in Section 4. As in the two-bus DC system, the end result will be a sequence of HELM subproblems all formally identical to the initial one, where only the parameters get re-defined. At each successive stage of the P-W process, the numerical stability of each HELM subproblem improves, so that it is possible to reach much higher levels of precision at $s = 1$, even when a singularity is very close. When dealing with limits, this is a must, since the constraint equation creates a singularity exactly at $s = 1$, as discussed at the beginning of this Section.

In order to simplify the exposition, constant-current injections (either load or generation) will be omitted for now; their treatment is not given here but, it can be shown that the method is easily adapted to contemplate them as well. Let us first consider the equation for PQ buses, (3), which here will be written as:

$$i \in \text{PQ}: \sum_{j=0}^{n} Y_{ij} V_j(s) + sY_{ish} V_i(s) = \frac{sS_i^*}{V_i^*(s^*)} + \Gamma_i \left(\frac{1}{V_i^*(s)} - V_i(s)\right)$$

(40)

Here $\Gamma_i$ is a parameter whose value is zero on the first stage of the method, but in subsequent stages it will be non-null, in general. As in the DC example, assume now that the system is solved with the standard HELM method and a real value $0 < s_0 < 1$ is chosen such that the Padé approximants converge with the required tolerance. The values $s_0$ and $V_i(s_0)$ will now be used to rewrite
the equation in terms of the renormalized parameter \( s' \) and voltage variables \( V'(s') \) defined by (35). Evaluating (40) at \( s = s_0 \) and multiplying both sides by \( V_i^*(s_0) \), one has:

\[
\sum_{j=0}^{n} V_i^*(s_0) Y_{ij} V_j(s_0) + s_0 Y_{i}^{sh} |V_i(s_0)|^2 = s_0 S_i^* + \Gamma_i \left( 1 - |V_i(s_0)|^2 \right)
\]  

(41)

where the fact that \( s_0 \) is chosen real has been used to simplify the expression. A non-real value of \( s_0 \) could in principle be chosen instead, thus leading to a path analytic continuation that would meander around the complex plane in order to get from \( s = 0 \) to \( s = 1 \). Uniqueness of the result would still be guaranteed, since by Stahl’s theorem Padé approximants will keep avoiding the minimal cut set, and therefore there will be no “branch jumps” no matter what path is chosen. However, as shown in [23], physics-based arguments dictate that analytic continuation of the white germ is actually required to exist along all points \( 0 \leq s_0 \leq 0 \) on the real axis, in order to be called the operational solution. Therefore the method requires that \( s_0 \) is chosen real.

Writing (40) in terms of (35), also multiplying both sides by \( V_i^*(s_0) \):

\[
\sum_{j=0}^{n} V_i^*(s_0) Y_{ij} V_j(s_0) V_j'(s')
\]

\[
+ s_0 |V_i(s_0)|^2 Y_i^{sh} V_i'(s') + s'(1 - s_0) |V_i(s_0)|^2 Y_i^{sh} V_i'(s') =
\]

\[
\frac{s_0 S_i^*}{V_i^{sh}(s_0)} + s'(1 - s_0) S_i^* \frac{1}{V_i^{sh}(s_0)} + \Gamma_i \left( \frac{1}{V_i^{sh}(s_0)} - |V_i(s_0)|^2 V_i'(s') \right)
\]  

(42)

In passing, note that if the swing voltage (index \( j = 0 \)) is embedded as \( V_0 = 1 + s(V_{sw} - 1) \), then it is straightforward to obtain its expression in terms of the new parameter:

\[
V_0'(s') = \frac{V_0(s)}{V_0(s_0)} = 1 + s'(1 - s_0)(V_{sw} - 1)
\]

It is useful to introduce now the shorthand notation:

\[
\hat{Y}_{ij} = V_i^*(s_0) Y_{ij} V_j(s_0)
\]

However, unlike the original matrix \( Y_{ij} \), this new admittance matrix would not satisfy the transmission condition (i.e., the sum of all columns does not yield the zero column vector). In fact, the sum of its columns can be readily calculated using (41):

\[
\sum_{j=0}^{n} \hat{Y}_{ij} = -s_0 Y_i^{sh} |V_i(s_0)|^2 + s_0 S_i^* + \Gamma_i \left( 1 - |V_i(s_0)|^2 \right)
\]

This can be fixed by defining the new admittance matrix as follows:

\[
Y_i'_{ij} = \begin{cases} 
\hat{Y}_{ij} & \text{if } i \neq j \\
\hat{Y}_{ii} - \sum_{i=0}^{n} \hat{Y}_{il} & \text{if } i = j
\end{cases}
\]  

(43)
With this definitions and after straightforward algebraic rearrangements and simplifications of (42), one arrives at the following equation for PQ buses:

\[
\sum_{j=0}^{n} Y'_{ij}V'_j(s') + s'Y'^{\text{sh}}_{ij}V'_i(s') = \frac{s'S'^{\text{re}}_{i}}{V'^{\text{re}}_{i}(s'^{\text{re}})} + \Gamma'_{i} \left( \frac{1}{V'^{\text{re}}_{i}(s'^{\text{re}})} - V'_i(s') \right)
\]  

(44)

where the new parameters are given by:

- \( Y'^{\text{sh}}_{i} \equiv (1 - s_0)|V_i(s_0)|^2 Y_{sh} \)
- \( \Gamma'_{i} \equiv \Gamma_i + s_0 S_i' \)
- \( S_i' \equiv (1 - s_0) S_i \)

Equation (44) is exactly the same as the original one, (40). In this sense, it can be said that the system is invariant under the Padé-Weierstrass transformation given by (35).

Analogously, the equations for PV buses can be shown to be invariant as well. Following the notation introduced in Section 4, which allows for one-on-one remote voltage regulation, the equations corresponding to regulating buses are written as:

\[
a \in \text{PV}:
\sum_{j=0}^{n} Y_{aj}V_j(s) + sY^{\text{sh}}_{a}V_a(s) = \frac{sP_a - jQ_a(s)}{V'^{\text{re}}_{a}(s'^{\text{re}})} + \Gamma_a \left( \frac{1}{V'^{\text{re}}_{a}(s'^{\text{re}})} - V_a(s) \right)
\]  

(45)

Note that these buses are not strictly speaking PV-type, as the formulation here is more general. Buses labeled \( a \) control the voltage of buses labeled \( b \); but a bus is said to be PV-type when \( a = b \). However, for the sake of simplicity, the set of buses labeled \( a \) will be referred to as “PV buses” just to avoid longer expressions such as “the buses having voltage-controlling reactive injections”.

In this case, one also needs to specify how the new variables \( Q_a(s) \) should transform. In contrast to voltages, it turns out that the transformation that works is additive:

\[
Q_a(s) \equiv Q_a(s_0) + Q'_a(s')
\]  

(46)

Following now a procedure completely analogous to the PQ case, it is straightforward to verify that the equations for PV buses (45) are also invariant under the transformation given by (35) and (46). In this case the new parameters are:

- \( Y'^{\text{sh}}_{a} \equiv (1 - s_0)|V_a(s_0)|^2 Y^{\text{sh}}_{a} \)
- \( \Gamma'_a \equiv \Gamma_a + s_0 P_a - jQ_a(s_0) \)
- \( P'_a \equiv (1 - s_0) P_a \)

5.3. Full AC system: complementarity constraints

Let us now analyze how the complementarity constraint equations (24) transform under the change given by (35), (46). Consider the most general case,
where limits and setpoints are embedded and voltage regulation may be remote, as described in Section 4:

\[ V_b(s)V_b^*(s^*) - W_b^{sp}(s) = -\frac{\mu_a(1-s)}{Q_{a}^{\text{max}}(s) - Q_{a}(s)} + \frac{\mu_a(1-s)}{Q_{a}(s) - Q_{a}^{\text{min}}(s)} \] (47)

It is readily verified that this equation is also invariant under the change, if one defines the new transformed parameters as follows:

\[ \mu'_a = \frac{(1-s_0)\mu_a}{|V_b(s_0)|^2}, \]
\[ W_b^{sp}(s') = \frac{W_b^{sp}(s_0)}{|V_b(s_0)|^2} + \frac{(1-s_0)\delta W_b^{sp}}{|V_b(s_0)|^2}(1-s') \]
\[ Q_{a}^{\text{min}}(s') \equiv Q_{a}^{\text{min}}(s_0) - Q_{a}(s_0) + \delta Q_{a}^{\text{min}}(1-s_0)(1-s') \]
\[ Q_{a}^{\text{max}}(s') \equiv Q_{a}^{\text{max}}(s_0) - Q_{a}(s_0) + \delta Q_{a}^{\text{max}}(1-s_0)(1-s') \]

For the transformed problem, the consistency requirements at \( s' = 0 \) are automatically satisfied, as the corresponding equation is equivalent to the original one when \( s = s_0 \), and \( s_0 \) is interior. Namely, one has \( Q_{a}^{\text{min}}(s' = 0) < 0 \), \( Q_{a}^{\text{max}}(s' = 0) > 0 \), and:

\[ \text{sgn} \left( 1 - W_b^{sp}(s' = 0) \right) = \text{sgn} \left( Q_{a}^{\text{min}}(s' = 0) + Q_{a}^{\text{max}}(s' = 0) \right) \]

This last condition may be satisfied even with a value of zero, i.e. \( W_b^{sp}(s' = 0) = 1 \); this poses no problem since the transformed coefficients \( \mu'_a \) are already known. Therefore after the first P-W stage there is no need to introduce any additional guards in the embedding of neither the setpoint nor the limits.

This completes then the P-W procedure for the full AC system including Q-limits. At each stage, one obtains (using standard HELM) a partial solution at a value \( 0 < s_0 < 1 \) such that the required precision is satisfied (which is always possible, by Stalh). Then one uses this partial solution in \( s \)-space in order to define a transformed problem in \( s' \)-space, using (35), (46). The transformed problem turns out to be formally the same as the original one, so that the whole procedure can be repeated recursively, until one is able to reach the required precision at \( s^{(k)} = 1 \) at some P-W stage \( k \). The solution to the original problem is then obtained by undoing all the transformations defined by using (35), (46). In effect, this is a path-analytic continuation in the spirit of Weierstrass, but the re-expansion of the power series at intermediate points is obtained using the exact function values (within machine precision, using Padé approximants) instead of the initial power series.

5.4. Full AC system: HELM scheme

It only remains to see how the HELM scheme is slightly modified by the presence of the new terms \( \Gamma \) introduced by the P-W procedure. This only affects the main power flow equations; the complementarity constraint equations

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remain exactly the same as in the first stage, so that the \(N\)-th order representation and the resulting HELM scheme are exactly the same as those shown in Section 3.

From the equation for PQ buses, (40), one obtains the following relationship for the power series coefficients at order \(N\) (the so-called \(N\)-th order representation):

\[
\sum_{j=0}^{n} Y_{ij} V_{j}[N] + \Gamma_i \left( V_{i}[N] - V_{i}^{-1*}[N] \right) =
\]

\[
S_i^* V_i^{-1*}[N-1] - Y_i^{sh} V_i[N-1]
\]

Recall that the coefficients of \(1/V(s)\), \(V_i^{-1}[N]\), can be obtained in terms of those of \(V(s)\) by (38). Initial values are \(V_i[0] = V_i^{-1}[0] = 1\). Therefore:

\[
\sum_{j=0}^{n} Y_{ij} V_{j}[N] + 2\Gamma_i \text{Re} V_{i}[N] = R_i[N-1]
\]

where all terms on the right hand side depend only on coefficients of order \(N-1\) or less:

\[
R_i[N-1] \equiv S_i^* V_i^{-1*}[N-1] - Y_i^{sh} V_i[N-1]
\]

\[
- \Gamma_i \sum_{m=1}^{N-1} V_i^*[m] V_i^{-1*}[N-m]
\]

Analogously, from the equation for PV buses, (45), one obtains:

\[
\sum_{j=0}^{n} Y_{aj} V_{j}[N] + \Gamma_a \left( V_{a}[N] - V_{a}^{-1*}[N] \right) =
\]

\[
P_a V_a^{-1*}[N-1] - Y_a^{sh} V_a[N-1] - j \sum_{m=0}^{N} Q_a[m] V_a^{-1*}[N-m]
\]

Making use of initial values \(V_a^{-1}[0] = 1\), \(Q_a[0] = 0\), and moving all terms of order \(N\) to the left hand side, one obtains:

\[
\sum_{j=0}^{n} Y_{aj} V_{j}[N] + 2\Gamma_a \text{Re} V_{a}[N] + j Q_a[N] = R_a[N-1]
\]

where all terms on the right hand side depend only on coefficients of order \(N-1\) or less:

\[
R_a[N-1] \equiv P_a V_a^{-1*}[N-1] - Y_a^{sh} V_a[N-1]
\]

\[
- \Gamma_i \sum_{m=1}^{N-1} V_a^*[m] V_a^{-1*}[N-m] - j \sum_{m=1}^{N-1} Q_a[m] V_a^{-1*}[N-m]
\]
and this concludes the description of the method. Recall that variables $Q_a[N]$ can be eliminated from the constraint equation (33), thus leading to a linear system of dimension $2(n_{PQ} + n_{PV})$ in the variables $\text{Re}V[N]$, $\text{Im}V[N]$.

6. Numerical results

6.1. Improved accuracy and numerical stability near collapse

Before numerical results are presented on power flow cases with Q-limits, let us analyze the effects of the P-W technique on the base HELM method, when no Mvar limits are enforced. In order to do so, let us consider the kind of situations under which the method may suffer from numerical precision problems. There are two possible sources: very close proximity to the feasibility boundary (voltage collapse), and ill-conditioning of the transmission admittance matrix. The last case may occur when some anomalous branch admittances are much larger or much smaller than the rest, ending up in a HELM linear system whose matrix has a bad condition number. This will produce errors in the power series coefficients, and eventually slower convergence. Just as it is done in iterative methods, this is simply prevented by requiring the admittances to be within certain minimum and maximum thresholds.

The other case has a more fundamental cause. A case that is feasible but very near the critical point is a case for which the embedded HELM problem has a singularity at $s_{\text{crit}} = 1 + \varepsilon$, where $0 < \varepsilon << 1$. The point $s_{\text{crit}}$ marks the transition, as the Padé approximants will start to oscillate for $s > s_{\text{crit}}$. The problem is that, even though Stahl’s theorem still guarantees that the Padé approximants converge at $s = 1$ for this case, the convergence rate slows down so much that too many orders of the power series are needed, and the approximants can no longer be computed with precision due to the inherent limits of floating point arithmetic. In simpler terms: as a case approaches the feasibility boundary, the method hits a ceiling of maximum attainable numerical precision.

In order to quantify this effect, and show how the new P-W technique helps in mitigating it, some numerical results are presented below. The important point here is not so much the ability of P-W to obtain "nose points" with much greater precision (after all, it can be shown that the attainable precision of the base HELM method is sufficient for most practical purposes). Rather, the aim here is to provide insight as to why P-W is needed when incorporating limits, since in that case the complementarity constraints always produce a singularity right at $s_{\text{crit}} = 1$.

Figure 1 shows the concept of maximum attainable precision in an actual example, case9241pegase available in MATPOWER [29]. All calculations are performed using standard IEEE double precision throughout, i.e. about 15 digits of precision. The figure plots the overall convergence rate by looking at the maximum update error in voltages across all buses versus $N$, the number of power series terms. This update error is calculated as the difference between two successive Padé approximants down the staircase sequence of the Padé table,
Figure 1: Convergence rates for case9241 (unmodified, no Mvar limits enforced). Maximum attainable precision improves from about $\pm 10^{-7}$ to $\pm 10^{-11}$ pu by performing just one P-W step.

e.g. $[L/M]$ and $[L+1/M]$ (in the notation of [30]), where one normally uses $L = M = N/2$. The results show that in this case the standard HELM algorithm is not able to obtain the voltages with more precision than about $\pm 10^{-7}$ pu, no matter how many series terms are obtained. This happens because at around 25 terms the evaluation of Padé approximants starts suffering from the limitations of floating point arithmetic (catastrophic cancellations and round-off). Performing just one step of the P-W procedure, the new maximum achievable precision in the transformed problem rises to $\pm 10^{-11}$ pu. A second P-W step is sufficient to achieve about $\pm 10^{-15}$, thus exhausting the machine precision.

Figure 2 demonstrates the effect of successive applications of the P-W transformation in a more extreme case. The calculation was performed using case9 from MATPOWER, in which a uniform scaling was applied to all loads and generators in order to bring the system as close as possible to collapse. The location of the critical point was narrowed down to a scaling factor of $\lambda_{\text{crit}} = 2.48539267 \pm 1.0e^{-8}$, which was confirmed using both HELM and MATPOWER’s CPF method. Since the total load at the critical point is around 315 MW, this error margin in the determination of the exact value of $\lambda_{\text{crit}}$ corresponds to about 3 Watts. Therefore, since Mvar limits are not enforced here, the limiting singularity $s_{\text{crit}}$ is very close to, but slightly above 1. At stage zero (base HELM method), it can be observed that the convergence rate is quite slow and the maximum achievable precision seems to be around $\pm 10^{-2}$ pu. After 8 P-W
steps, the convergence rate is so fast that just 9 terms are enough to achieve maximum machine precision.

This behavior can be qualitatively understood in terms of the geometry of the P-W transformation in s-space. Let us recall the parameter transformation, rewriting (35) in this form:

$$s' = \frac{s - s_0}{1 - s_0}$$

It is easy to see that this conformal mapping is a translation and a dilation of s-space, which has the effect of advancing and “zooming in” towards the point $s = 1$. Using conformal maps in order to push unwanted singularities away and thus improve the convergence properties of Padé approximants is a well-known technique that has been applied to general cases [31]. Here it will be shown how this particular transformation works in the HELM problem.

Figure 3 shows graphically the effect that the successive P-W transformations have on the singularities of the problem: all non-relevant singularities get pushed away, eventually leaving the only one that matters: the first one to be encountered on the positive real axis [23]. When this critical singularity is $s_{\text{crit}} > 1$, the successive P-W transformations will keep pushing it away until it is so far that the power series is sufficiently well-conditioned to allow its Padé approximants to converge quickly (one could even obtain a series whose radius of convergence included 1, although this is not strictly necessary). If on the other hand $s_{\text{crit}} < 1$, the P-W procedure would “pull in” the singularity towards...
0 and quickly get stuck (the successive values $s_0$ would approach zero), thus signaling infeasibility. As a by-product, this would also provide a precise value for $s_{\text{crit}}$. Finally, if $s_{\text{crit}} = 1$ then all other singularities are pushed away but $s_{\text{crit}}$ will always remain at a distance of 1. The Padé approximants of every HELM subproblem will still not converge exactly at 1, but all that is needed is that they converge near 1. Then, thanks to the zooming property, one can get extremely close to 1 in the original $s$-space. For instance, assume a case where one always obtains sufficient convergence at $s_0 = 0.9$ in every P-W stage. At the second stage, the starting point $s''_0 = 0$ represents $s = 0.99$. After 10 P-W stages, the partial solution will be just $10^{-10}$ away from 1 in the original $s$-space. This qualitatively explains the good numerical results that are obtained in the next section.

6.2. Examples with Q-limits

Figure 4 shows the convergence rates for case9241pegase when Mvar limits are enforced using the P-W method. Contrast this with Figure 1, where limits are not enforced. The presence of the complementarity constraints introduces an unavoidable singularity at $s = 1$, whose effects can be clearly seen in the convergence rates. However, the application of the P-W technique achieves the solution with good precision.

Table 1 shows the results of applying the HELM-PW method to the top 25 largest cases available in MATPOWER. The numerical tolerances required for
all cases were $10^{-11}$ pu maximum update error in voltages (at each P-W stage), and $10^{-8}$ pu maximum mismatch error in the original power flow equations (nodal currents). The maximum number of power series terms used at each P-W stage was $N_{\text{max}} = 32$. Note that in all these examples regulation is always local to the bus, and that $Q_{\text{min}}, Q_{\text{max}}$ represent the aggregate limits of all generators connected to the bus, in case there is more than one.

The first column shows the number of P-W stages needed in order to meet the required tolerance; this provides a rough idea of the numerical difficulty involved in arriving to the solution. The next three columns show the number of PV buses that end up saturated at either $Q_{\text{min}}$ or $Q_{\text{max}}$, comparing the results of HELM vs. the two limit-enforcing strategies available in MATPOWER’s Newton-Raphson method (strategy 1 consists in type-switching all violated buses at once, on every NR solution; while strategy 2 consists in type-switching one bus at a time). One can observe how there are many cases where the solutions obtained with MATPOWER are different, and a few cases where NR does not converge. Additionally, HELM’s solutions seem to consistently yield a lower total number of saturated buses.

The table also shows the percentage of saturated buses over the total number of PV buses. Interestingly, there seems to be no clear correlation between this number and the number of P-W steps required, which probably indicates that the numerical difficulty of the case may be influenced by other factors, such as proximity to voltage collapse. This issue has been left for further research.
6.3. Some preliminary results on performance

A performance evaluation of the original HELM method, which did not have Q-limit enforcement, was presented in [24], using a MATLAB-based implementation and comparing results to several methods available in MATPOWER. Since our current implementation of the P-W method has not yet undergone a proper optimization, its performance will not be reviewed here. Instead, some scaling arguments will be given in order to roughly characterize the expected performance.

As seen in table 1, it is observed that the method requires a number of P-W steps that varies between 6 and 60, for this sample of cases and a choice of $N_{\text{max}} = 32$. Since each P-W step involves roughly one full HELM computation, this provides an approximate idea of the kind of performance one may expect. There is a certain trade-off between $N_{\text{max}}$ and the resulting number of P-W steps needed to achieve the required tolerance. If one uses a large value for $N_{\text{max}}$ (e.g., more than 60 power series terms), the advancing values $s_0$ at each P-W step will be larger, and therefore the total amount of P-W steps needed will be lower. However, the HELM problem being solved at each of those steps will be more costly, because there will be more series terms to solve. This would involve solving more linear systems, with their corresponding right-hand sides (which involve convolutions); and also computing Padé approximants at larger orders. The authors have not yet tried to find rigorously the optimum point for this trade-off, but initial numerical results indicate that $N_{\text{max}} \approx 30$ gives good results.

7. Conclusions and further research

A new HELM-based method has been presented that deals with control limits in a rigorous way, based on a novel Lagrangian formulation of the power flow equations. The method has been developed in full detail for the case of Mvar limits of voltage-controlling generators in AC networks, which is the most important problem of this type. Like HELM, the method is direct, constructive, and deterministic.

The Lagrangian approach allows framing the problem in terms of constrained minimization, leading to a HELM scheme for calculating that minimum. Two additional key ingredients are also necessary: on the one hand, a proper embedding for setpoints and Q-limit values, in order to ensure consistency of the HELM reference state at $s = 0$. On the other, the application of the P-W technique, which enables the possibility of calculating the analytic continuation of the reference state to $s = 1$ with good numerical precision. P-W exploits a remarkable property of the power flow equations, whereby certain changes of variables and the embedding parameter leave the equations formally invariant. This allows one to calculate the analytic continuation of the power series in several steps, leveraging the power flow equations at each step to re-expand new auxiliary series. The net result is an analytic continuation along a path, with far greater numerical stability.
In contrast to traditional iterative methods, this one directly avoids the convergence problems derived from mutual interactions among controls, which can produce oscillatory behavior, particularly when several nearby controls are close to their saturation limits. The method presented here is closer in spirit to some OPF-like approaches to power flow, although the differences are fundamental, since HELM is based on algebraic curves and complex analysis.

The authors have also applied this method to other types of regulating devices with limits, in the context of DC microgrids [24]. Examples are the sequential shunt unit (SSU) that regulates the output voltage of solar panels aboard spacecraft, and the voltage regulation of DC-DC converters. In those cases the regulation can also be expressed in terms of algebraic constraints involving new regulating injections, and their limits can be accommodated under the same P-W technique even though they are sometimes dependent on voltage.

Future areas to explore include the extensions to other controls, such as automatic tap changers (ULTC transformers), automatic phase-shifting transformers, or inter-area flow constraints. Shared control of a given point originated from different buses is also a challenging goal. On the other hand, the resemblance between the method presented here and certain OPF methods certainly calls for a more detailed study on the possible applications to optimization problems.

Appendix A. Method summary

The HELM P-W method is summarized here for the general case corresponding to a set of \( n_{\text{PQ}} \) buses of type PQ (labeled with indexes \( i \) when the distinction is needed) and \( n_{\text{PV}} \) buses (labeled with indexes \( a \)) having a reactive power injection that controls the voltage at a corresponding set of buses (labeled with indexes \( b \)). A bus where \( a = b \) is typically referred to as being PV-type. It is assumed that there are no concurrent controls, i.e. each controlled bus \( b \) has only one corresponding controlling injection \( a \). The injections are assumed to have both upper and lower limits: \( Q_{a}^{\text{min}}, Q_{a}^{\text{max}} \). In case of one-sided limits, the changes in the formulation are all straightforward, except for the special care that must be taken in embedding the limits and the setpoint in the complementarity equation (see examples (30) and (31) in Section 4).

Embedding and reference state

The embedded power flow equations (see Section 2 for notation conventions):

\[
\sum_{j=0}^{n} Y_{ij} V_{j}(s) + s Y_{i}^{\text{sh}} V_{i}(s) = \frac{s S_{i}^{s}}{V_{i}^{s}(s) - V_{i}(s)} + \Gamma_{i} \left( \frac{1}{V_{i}^{s}(s)} - V_{i}(s) \right)
\]

\[
\sum_{j=0}^{n} Y_{aj} V_{j}(s) + s Y_{a}^{\text{sh}} V_{a}(s) = \frac{s P_{a} - j Q_{a}(s)}{V_{a}^{s}(s)} + \Gamma_{a} \left( \frac{1}{V_{a}^{s}(s)} - V_{a}(s) \right)
\]

where all \( \Gamma \) parameters are zero at the first P-W stage. The swing is embedded as \( V_{0}(s) = 1 + s(V_{0} - 1) \), so as to allow a reference state having \( V_{j}[0] = 1 \) for all \( j \). Reactive injection variables are \( Q_{a}[0] = 0 \) for all \( a \).
Reactive limits are contemplated by means of the embedded complementarity equation:

\[ V_b(s)V_b^*(s^*) - W_b^{sp}(s) = -\frac{\mu_a(1-s)}{Q_{a\text{max}}(s) - Q_a(s)} + \frac{\mu_a(1-s)}{Q_a(s) - Q_{a\text{min}}(s)} \]

where \( W_b^{sp} \equiv V_b^{sp}V_b^{sp*} \). Limits and setpoints need to be adequately embedded in order to guarantee certain consistency requirements as described in Section 4.

The following choice is recommended:

\[ \mu_a = 1 \]

\[ Q_{a\text{min}}(s) = Q_{a\text{min}} - \frac{Q_{a\text{max}} + Q_{a\text{min}}}{2}(1-s) \]

\[ Q_{a\text{max}}(s) = Q_{a\text{max}} - \frac{Q_{a\text{max}} + Q_{a\text{min}}}{2}(1-s) \]

\[ W_b^{sp}(s) = W_b^{sp} + (1-W_b^{sp})(1-s) \]

so that the coefficients of their corresponding power series are:

\[ W_b^{sp}[0] = 1 \]

\[ Q_{a\text{max}}[0] = -Q_{a\text{min}}[0] = \frac{Q_{a\text{max}} - Q_{a\text{min}}}{2} \]

\[ W_b^{sp}[1] = (W_b^{sp} - 1) \]

\[ Q_{a\text{max}}[1] = Q_{a\text{min}}[1] = \frac{Q_{a\text{max}} + Q_{a\text{min}}}{2} \]

However, the formulas in the next section will not assume any particular choice for these embedding constants; only the linearity of the embedding is assumed.

**HELM scheme**

At each stage of the P-W procedure, solve the following HELM equations sequentially order by order, to obtain the power series of voltages and reactive injections:

\[ \sum_{j=0}^{n} Y_{ij} V_j[N] + 2\Gamma_i \text{Re} V_i[N] = R_i[N-1] \]

\[ \sum_{j=0}^{n} Y_{aj} V_j[N] + 2\Gamma_a \text{Re} V_a[N] + jQ_a[N] = R_a[N-1] \]

The terms on the right hand side depend only on coefficients of order \( N - 1 \) or less, and are calculated as:

\[ R_i[N-1] \equiv S_i V_i^{-1*}[N-1] - Y_i^{sh} V_i[N-1] \]

\[ -\Gamma_i \sum_{m=1}^{N-1} V_i^*[m] V_i^{-1*}[N-m] \]

\[ R_a[N-1] \equiv P_i V_a^{-1*}[N-1] - Y_i^{sh} V_a[N-1] \]

\[ -\Gamma_i \sum_{m=1}^{N-1} V_a^*[m] V_a^{-1*}[N-m] - j \sum_{m=1}^{N-1} Q_a[m] V_a^{-1*}[N-m] \]
and the coefficients of $1/V(s)$ can be obtained from those of $V(s)$ by:

$$V_j^{-1}[N] = -V_j[N] - \sum_{m=1}^{N-1} V_j[m] V_j^{-1}[N-m]$$

Variables $Q_a[N]$ are to be solved below in terms of Re$V_b[N]$ from the constraint equation, and substituted into the above linear system. To do so, first construct the auxiliary power series $B_a^{(+/)}$, using:

$$B_a^+[0] = -\frac{\mu_a}{Q_a^{max}[0]} Q_a[0]$$

$$B_a^+[N] = \frac{1}{Q_a^{max}[0]} \sum_{m=0}^{N-1} B_a^+[m] Q_a[N-m]$$

$$-B_a^+[N-1] Q_a^{max}[1] + \mu_a \delta_{1,N}$$

and the same expressions for $B_a^-$, using $Q_a^{min}[0]$ and $Q_a^{min}[1]$ instead. Having calculated the coefficients of $B_a^{(+/)}$ up to order $N-1$, the coefficients of $Q_a[N]$ are solved from:

$$2\text{Re}V_b[N] - \left( \frac{B_a^-[0]}{Q_a^{min}[0]} + \frac{B_a^+[0]}{Q_a^{max}[0]} \right) Q_a[N] = T_a[N-1]$$

and substituted into the main equations in order to form a linear system of dimension $2(n_{PQ} + n_{PV})$ in the variables Re$V[N]$, Im$V[N]$. The expression for the right hand side in the above expression is:

$$T_a[N-1] \equiv W_{b}^{s p}[1] \delta_{1,N} - \sum_{m=1}^{N-1} V_b[m] V_b^*[N-m]$$

$$+ \frac{1}{Q_a^{min}[0]} \sum_{m=1}^{N-1} B_a^-[m] Q_a[N-m] - B_a^-[N-1] Q_a^{min}[1] + \mu_a \delta_{1,N}$$

$$+ \frac{1}{Q_a^{max}[0]} \sum_{m=1}^{N-1} B_a^+[m] Q_a[N-m] - B_a^+[N-1] Q_a^{max}[1] + \mu_a \delta_{1,N}$$

Note how the matrix of the linear system stays constant across all orders $N$, so the system needs to be factorized only once (at the beginning of each P-W stage), and the factors are reused to solve the same linear system with a different right hand side at each order. Of course, it is strongly recommended to use a modern sparse LU solver [32], for efficiency. For best results, we recommend using either CXSparse [32] or KLU [33], together with the Approximate Minimum Degree reordering algorithm [34] for reducing factor fill-in.

Set a maximum number of power series terms $N^{max}$ somewhere between 20 to 40. At each order (or maybe after the first ten terms, to save some
work in most cases), evaluate the corresponding Padé approximants at $s = 1$. If convergence within the desired tolerance is obtained, the solution has been found. If $N_{\text{max}}$ is reached without convergence, find an intermediate value $0 < s_0 < 1$ such that convergence of the partial solution $V_j(s_0)$ is obtained (this is guaranteed by Stahl). The maximum achievable $s_0$ can be found by a simple bisection procedure or any other similar method. Also, one can select a value of $s_0$ slightly smaller than the maximum possible, thus trading smaller P-W steps for slightly better overall numerical stability.

**Padé-Weierstrass updates**

The partial solution $V_j(s_0), Q_a(s_0)$ is now used to construct the transformed HELM problem of the next P-W stage. The embedding parameter and the variables transform as:

$$s \equiv s_0 + (1 - s_0)s'$$
$$V_j(s) \equiv V_j(s_0)V'_j(s')$$
$$Q_a(s) \equiv Q_a(s_0) + Q'_a(s')$$

This change leaves all the embedded equations, both the power flow and the complementarity constraints, invariant. One should keep track of all these transformations in order to undo them once the point $s^{(k)} = 1$ can be reached at the $k$-th P-W stage, when numerical tolerances are met.

For the power flow system, the new parameters are as follows. The new transmission admittance matrix is given by:

$$Y'_{ij} = \begin{cases} 
\tilde{Y}_{ij} & \text{if } i \neq j \\
\tilde{Y}_{ii} - \sum_{l=0}^{n} \tilde{Y}_{il} & \text{if } i = j
\end{cases}$$

where: $\tilde{Y}_{ij} \equiv V^*_i(s_0)Y_{ij}V_j(s_0)$

and where both indices $i$ and $j$ run over all buses, including the swing. The new shunt admittances are, also for all buses:

$$Y'^{sh}_{j} \equiv (1 - s_0)|V_j(s_0)|^2 Y^{sh}_{j}$$

Now the power and gamma terms are as follows, depending on the type of bus:

$$\Gamma'_i \equiv \Gamma_i + s_0 S'_i$$
$$\Gamma'_a \equiv \Gamma_a + s_0 P_a - jQ_a(s_0)$$
$$S'_i \equiv (1 - s_0)S_i$$
$$P'_a \equiv (1 - s_0)P_a$$

Finally, the new parameters in the complementarity constraint equation are as follows:

$$\mu'_a \equiv \frac{(1 - s_0)\mu_a}{|V_b(s_0)|^2}$$
$$W'^{\text{sp}}_b[0] \equiv \frac{W^{\text{sp}}_b[0] + s_0W^{\text{sp}}_b[1]}{|V_b(s_0)|^2}$$
$$W'^{\text{sp}}_b[1] \equiv \frac{(1 - s_0)W^{\text{sp}}_b[1]}{|V_b(s_0)|^2}$$
$$Q'^{\text{lim}}_a[0] \equiv Q^{\text{lim}}_a[0] + s_0Q^{\text{lim}}_a[1] - Q_a(s_0)$$
$$Q'^{\text{lim}}_a[1] \equiv (1 - s_0)Q^{\text{lim}}_a[1]$$
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Table 1: Numerical results of the P-W method applied to the top 25 largest cases available in MATPOWER. The MP-1 and MP-2 columns report the MATPOWER results when using NR with the two alternative limit-enforcing strategies \((\text{enforce}\_\text{q\_lims} 1\text{ and } 2,\text{ respectively})\). Results equal to HELM’s are marked with \(=\), differences are highlighted in bold, and non-convergence shown as \(\text{NC}\).