Some remarks on general sum–connectivity coindex

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Abstract: Let $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$ be a simple connected graph with $n$ vertices, $m$ edges and a sequence of vertex degrees $d_1 \geq d_2 \geq \cdots \geq d_n > 0$, $d_i = d(v_i)$. The general sum–connectivity coindex is defined as

$$H_\alpha(G) = \sum_{i \sim j} (d_i + d_j)^\alpha,$$

while multiplicative first Zagreb coindex is defined as

$$P_1(G) = \prod_{i \sim j} (d_i + d_j).$$

Here $\alpha$ is an arbitrary real number, and $i \sim j$ denotes that vertices $i$ and $j$ are not adjacent. Some relations between $H_\alpha(G)$ and $P_1(G)$ are obtained.

Keywords: Topological indices and coindices, sum–connectivity coindex, multiplicative Zagreb coindex.

1 Introduction

Let $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$, $E = \{e_1, e_2, \ldots, e_m\}$, be a simple connected graph with $n = |V|$ vertices and $m = |E|$ edges. With $d_1 \geq d_2 \geq \cdots \geq d_n > 0$, $d_i = d(v_i)$, a sequence of vertex degrees of $G$ is designated. If vertices $v_i$ and $v_j$ are adjacent, we write $i \sim j$, otherwise we write $i \sim j$. We define values $\overline{\Delta}_e$ and $\overline{d}_e$ as

$$\overline{\Delta}_e = \max_{i \sim j} \{d_i + d_j\} \quad \text{and} \quad \overline{d}_e = \min_{i \sim j} \{d_i + d_j\}.$$

A topological index of a graph is a numerical quantity which is invariant under automorphisms of the graph.

Two vertex-degree based topological indices, the first and the second Zagreb index, $M_1$ and $M_2$, are defined as [7, 8]

$$M_1 = M_1(G) = \sum_{i=1}^{n} d_i^2 \quad \text{and} \quad M_2 = M_2(G) = \sum_{i \sim j} d_id_j.$$
As shown in [12], the first Zagreb index can be also expressed as

\[ M_1 = \sum_{i<j} (d_i + d_j). \]

A so-called forgotten topological index, \( F \), is defined as [6]

\[ F = F(G) = \sum_{i=1}^{n} d_i^3. \]

By analogy to \( M_1 \), the invariant \( F \) can be written in the following way

\[ F = \sum_{i<j} (d_i^2 + d_j^2). \]

The general sum–connectivity index was conceived in [17] as

\[ H_\alpha(G) = \sum_{i<j} (d_i + d_j)^\alpha, \]

where \( \alpha \) is an arbitrary real number. Some special cases of this index are the first Zagreb index \( M_1(G) = H_1(G) \), the harmonic index \( H(G) = 2H_{-1}(G) \) [5], the sum–connectivity index \( SC(G) = H_{-1/2}(G) \) [18], and hyper–Zagreb index \( HM(G) = H_2(G) \) [13]. It is not difficult to see that

\[ HM(G) = \sum_{i<j} (d_i + d_j)^2 = F(G) + 2M_2(G). \]

In [4] a concept of coindices was introduced. In this case the sum runs over the edges of the complement of \( G \). Thus, the first and the second Zagreb coindices are defined as [4]

\[ \overline{M}_1(G) = \sum_{i<j} (d_i + d_j) \quad \text{and} \quad \overline{M}_2(G) = \sum_{i<j} d_id_j, \]

and the forgotten Zagreb coindex as [3] (see also [10]) as

\[ \overline{F}(G) = \sum_{i<j} (d_i^2 + d_j^2). \]

The general sum–connectivity coindex was defined in [14] as

\[ \overline{H}_\alpha(G) = \sum_{i<j} (d_i + d_j)^\alpha, \]

where \( \alpha \) is an arbitrary real number. Again, some special cases of \( \overline{H}_\alpha(G) \) are apart from \( \overline{M}_1(G) \), the sum–connectivity coindex \( SC(G) = \overline{H}_{-1/2}(G) \), the harmonic coindex \( \overline{H}(G) = \overline{H}_1(G) \), the harmonic coindex \( \overline{H}_1(G) \), the harmonic.
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2\(H_{-1}(G)\), the hyper Zagreb coindex \(HM(G) = H_2(G)\) [15]. It is not difficult to see that the following identity holds

\[HM(G) = \overline{F}(G) + 2\overline{M}_2(G).\]

The multiplicative first Zagreb coindex was defined in [16] as

\[\overline{\Pi}_1(G) = \prod_{i \neq j}(d_i + d_j).\]

In this paper we determine the bound for the difference

\[\overline{H}_a(G) - \overline{m}(\overline{\Pi}_1(G))^{\alpha/\overline{m}},\]

where \(\overline{m} = \frac{n(n-1)}{2} - m\).

### 2 Preliminaries

In this section we recall some analytical inequalities for the real number sequences that will be used in the subsequent considerations.

Let \(a = (a_i)\) and \(b = (b_i), i = 1, 2, \ldots, n,\) be positive real number sequences with the properties

\[0 < r_1 \leq a_i \leq R_1 < +\infty \quad \text{and} \quad 0 < r_2 \leq b_i \leq R_2 < +\infty.\]

In [1] (see also [11]) the following inequality was proven

\[n \sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i \sum_{i=1}^{n} b_i \leq n^2 \gamma(n)(R_1 - r_1)(R_2 - r_2),\]  

(1)

where

\[\gamma(n) = \frac{1}{n^2} \left(1 - \frac{n}{2} \left(1 - \frac{n}{2}\right)\right) = \frac{1}{4} \left(1 - \frac{(-1)^{n+1} + 1}{2n^2}\right).\]

Equality holds if and only if \(R_1 = a_1 = \cdots = a_n = r_1\) or \(R_2 = b_1 = \cdots = b_n = r_2.\)

For the positive real number sequence \(a = (a_i), i = 1, 2, \ldots, n,\) the following inequality was proven in [9]

\[\left(\sum_{i=1}^{n} \sqrt{a_i}\right)^2 \leq (n-1) \sum_{i=1}^{n} a_i + n \left(\prod_{i=1}^{n} a_i\right)^{1/n},\]

(2)

with equality if and only if \(a_1 = a_2 = \cdots = a_n.\)

For the positive real number sequence \(a = (a_i), i = 1, 2, \ldots, n,\) with the property \(a_1 \geq a_2 \geq \cdots \geq a_n > 0,\) in [2] the following inequality was proven

\[\sum_{i=1}^{n} a_i - n \left(\prod_{i=1}^{n} a_i\right)^{1/n} \geq (\sqrt[n]{a_1} - \sqrt[n]{a_n})^2.\]

(3)

Equality holds if \(a_2 = a_3 = \cdots = a_{n-1} = \sqrt[n]{a_1 a_n}.\)
3 Main results

In the next theorem we establish lower and upper bounds for the difference \( H_\alpha(G) - m(\Pi_1(G))^{\alpha/m} \) depending on the parameters \( \alpha, m, \bar{\Delta}_e \) and \( \bar{\delta}_e \).

Theorem 1. Let \( G \) be a simple graph with \( m \geq 2 \) edges. If \( \alpha \geq 0 \), then

\[
\left( \bar{\Delta}_e^\alpha - \bar{\delta}_e^\alpha \right)^2 \leq H_\alpha(G) - m(\Pi_1(G))^{\alpha/m} \leq m^2 \gamma(m) \left( \bar{\Delta}_e^\alpha - \bar{\delta}_e^\alpha \right)^2 .
\] (4)

If \( \alpha \leq 0 \), \( G \not\cong K_n \), then

\[
\left( \bar{\delta}_e^\alpha - \bar{\Delta}_e^\alpha \right)^2 \leq H_\alpha(G) - m(\Pi_1(G))^{\alpha/m} \leq m^2 \gamma(m) \left( \bar{\delta}_e^\alpha - \bar{\Delta}_e^\alpha \right)^2 .
\]

Equality on the left–hand side holds if \( \alpha = 0 \), or \( d_i + d_j = \sqrt{\bar{\Delta}_e \bar{\delta}_e} \), for any pair of nonadjacent vertices of \( G \). Equality on the right–hand side holds if and only if \( \alpha = 0 \) or \( d_i + d_j \) is a constant for any pair of non adjacent vertices of \( G \).

Proof. For \( \alpha \geq 0 \), \( n := m, a_i = b_i := (d_i + d_j)^{\alpha/2} \), \( R_1 = R_2 = \bar{\Delta}_e^\alpha, r_1 = r_2 = \bar{\delta}_e^\alpha \), with summation performed over all non adjacent vertices of \( G \), the inequality (1) becomes

\[
m \sum_{i \neq j} (d_i + d_j)^{\alpha} - \left( \sum_{i \neq j} (d_i + d_j)^{\alpha/2} \right)^2 \leq m^2 \gamma(m) \left( \bar{\Delta}_e^\alpha - \bar{\delta}_e^\alpha \right)^2 ,
\]

that is

\[
m H_\alpha(G) - \left( \sum_{i \neq j} (d_i + d_j)^{\alpha/2} \right)^2 \leq m^2 \gamma(m) \left( \bar{\Delta}_e^\alpha - \bar{\delta}_e^\alpha \right)^2 . \tag{5}
\]

For \( \alpha \geq 0 \), \( n := m, a_i := (d_i + d_j)^{\alpha} \), where summation is performed over all pairs of non adjacent vertices of \( G \), the inequality (2) transforms into

\[
\left( \sum_{i \neq j} (d_i + d_j)^{\alpha/2} \right)^2 \leq (m - 1) \sum_{i \neq j} (d_i + d_j)^{\alpha} + m \left( \prod_{i \neq j} (d_i + d_j)^{\alpha} \right)^{1/m} ,
\]

that is

\[
\left( \sum_{i \neq j} (d_i + d_j)^{\alpha/2} \right)^2 \leq (m - 1) H_\alpha(G) + m \left( \Pi_1(G) \right)^{\alpha/m} . \tag{6}
\]

Now from (5) and (6) we obtain right-hand side of (4). Equalities in (5) and (6), and consequently in the right-hand side of (4), hold if and only if \( \alpha = 0 \) or \( d_i + d_j \) is a constant for any pair of non adjacent vertices of \( G \).
For $\alpha \geq 0$, $n := m$, $a_i := (d_i + d_j)^\alpha$, $a_1 := \Delta^\alpha$, $a_n := \delta^\alpha$, with summation performed over all pairs of non-adjacent vertices, the inequality (3) becomes

$$\sum_{i \neq j} (d_i + d_j)^\alpha - m \left( \prod_{i \neq j} (d_i + d_j)^\alpha \right)^{1/m} \geq \left( \frac{\frac{\Delta^\alpha}{e}}{\frac{\delta^\alpha}{e}} \right)^2,$$

from which left-hand part of (4) is obtained. Equality in (7), and consequently in (4), holds if $\alpha = 0$ or $d_i + d_j = \sqrt{\Delta \delta}$ for any pair of non-adjacent vertices of $G$.

The case $\alpha < 0$ is proved analogously, thus omitted.

Since for any $m$ holds $\gamma(m) \leq \frac{1}{4}$, we have the next corollary of Theorem 1.

**Corollary 1.** Let $G$ be a simple graph with $m \geq 2$ edges. If $\alpha \geq 0$, then

$$H_\alpha(G) - m (\Pi_1(G))^{\alpha/m} \leq \frac{m^2}{4} \left( \frac{\frac{\Delta^\alpha}{e} - \frac{\delta^\alpha}{e}}{\frac{\Delta^\alpha}{e} - \frac{\delta^\alpha}{e}} \right)^2.$$

If $\alpha \leq 0$ and $G \not\cong K_n$, then

$$H_\alpha(G) - m (\Pi_1(G))^{\alpha/m} \leq \frac{m^2}{4} \left( \frac{\frac{\Delta^\alpha}{e} - \frac{\delta^\alpha}{e}}{\frac{\Delta^\alpha}{e} - \frac{\delta^\alpha}{e}} \right)^2.$$

Equalities hold if and only if $\alpha = 0$, or $d_i + d_j$ is a constant for any pair of non-adjacent vertices of $G$.

For some specific values of parameter $\alpha$ the following inequalities are obtained.

**Corollary 2.** Let $G$, $G \not\cong K_n$, be a simple graph with $m \geq 2$ edges. Then we have

$$\left( \frac{\sqrt{\Delta} - \sqrt{\delta}}{\sqrt{\Delta} \delta} \right)^2 \leq \frac{1}{2} H(G) - m (\Pi_1(G))^{-1/m} \leq m^2 \gamma(m) \left( \frac{\sqrt{\Delta} - \sqrt{\delta}}{\sqrt{\Delta} \delta} \right)^2 \leq \frac{m^2}{4} \left( \frac{\sqrt{\Delta} - \sqrt{\delta}}{\sqrt{\Delta} \delta} \right)^2,$$

$$\left( \frac{\sqrt{\Delta} - \sqrt{\delta}}{\sqrt{\Delta} \delta} \right)^2 \leq \frac{m^2}{4} \left( \frac{\sqrt{\Delta} - \sqrt{\delta}}{\sqrt{\Delta} \delta} \right)^2,$$

$$\left( \frac{\sqrt{\Delta} - \sqrt{\delta}}{\sqrt{\Delta} \delta} \right)^2 \leq \frac{m^2}{4} \left( \frac{\sqrt{\Delta} - \sqrt{\delta}}{\sqrt{\Delta} \delta} \right)^2,$$

$$\left( \sqrt{\Delta} - \sqrt{\delta} \right)^2 \leq \frac{m^2}{4} \left( \frac{\sqrt{\Delta} - \sqrt{\delta}}{\sqrt{\Delta} \delta} \right)^2.$$

Equalities hold if and only if $\alpha = 0$, or $d_i + d_j$ is a constant for any pair of non-adjacent vertices of $G$. 

The case $\alpha \leq 0$ is proved analogously, thus omitted.
\[
(\overline{\Delta_e} - \overline{\delta_e})^2 \leq H\overline{M}(G) - \overline{m}(\overline{\Pi}_1(G))^{2/m} \leq \overline{m}^2 \gamma(\overline{m}) (\overline{\Delta_e} - \overline{\delta_e})^2 \leq \frac{\overline{m}^2}{4} (\overline{\Delta_e} - \overline{\delta_e})^2.
\]

Equalities in the left-hand sides of the above inequalities hold if \( d_i + d_j = \sqrt{\overline{\Delta_e} \overline{\delta_e}} \) for any pair of non-adjacent vertices \( v_i \) and \( v_j \) of \( G \). Equalities in the right-hand sides of the above inequalities hold if and only if \( d_i + d_j \) is constant for any pair of non-adjacent vertices \( v_i \) and \( v_j \) of \( G \).

Since \( 2\overline{F}(G) \geq H\overline{M}(G) = \overline{F}(G) + 2\overline{M}_2(G) \geq 4\overline{M}_2(G) \), the following is valid.

**Corollary 3.** Let \( G \) be a simple graph with \( m \geq 2 \) edges. Then
\[
4\overline{M}_2(G) - \overline{m}(\Pi_1(G))^{2/m} \leq \overline{m}^2 \gamma(\overline{m}) (\overline{\Delta_e} - \overline{\delta_e})^2 \leq \frac{\overline{m}^2}{4} (\overline{\Delta_e} - \overline{\delta_e})^2,
\]
\[
2\overline{F}(G) - \overline{m}(\Pi_1(G))^{2/m} \geq (\overline{\Delta_e} - \overline{\delta_e})^2.
\]

Equalities hold if and only if \( d_i = d_j \) for any pair of non-adjacent vertices of \( G \).

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