GENERIC CODING WITH HELP AND AMALGAMATION FAILURE

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Abstract. We show that if \( M \) is a countable transitive model of ZF and if \( a, b \) are reals not in \( M \), then there is a \( G \) generic over \( M \) such that \( b \in L[a, G] \). We then present several applications such as the following: if \( J \) is any countable transitive model of ZFC and \( M \not\subseteq J \) is another countable transitive model of ZFC of the same ordinal height \( \alpha \), then there is a forcing extension \( N \) of \( J \) such that \( M \cup N \) is not included in any transitive model of ZFC of height \( \alpha \). Also, assuming \( 0^\# \) exists, letting \( S \) be the set of reals generic over \( L \), although \( S \) is disjoint from the Turing cone above \( 0^\# \), we have that for any non-constructible real \( a \), \( \{ a \oplus s : s \in S \} \) is cofinal in the Turing degrees.

§1. Introduction. If \( 0^\# \) exists, it is not in any (set) forcing extension of \( L \). On the other hand, Mostowski showed that for any real \( x \), there are reals \( g_1, g_2 \) both Cohen generic over \( L \) such that \( x \) is computable from the Turing join of \( g_1 \) and \( g_2 \), written \( x \leq_T g_1 \oplus g_2 \) (see [4] for a proof).

In this paper we investigate the following question (assuming \( 0^\# \) exists): given an arbitrary \( g_1 \in {}^{\omega}\omega \) not in \( L \), is there a real \( g_2 \) generic over \( L \) such that \( 0^\# \leq_T g_1 \oplus g_2 \)?

We will see that the answer is yes. Although there is a limit to what reals are generic over \( L \), there is no limit to what reals are constructible from a fixed non-constructible real and a real that is generic over \( L \). Here is the general formulation:

**Definition 1.1.** Let \( M \) be a countable transitive model of ZFC. Let \( C \) be Cohen forcing.

1. Fix \( \mathbb{P} \in M \). No real \( \bar{a} \in {}^{\omega}\omega \) is \((\mathbb{P}, M)\)-helpful if for any \( x \in {}^{\omega}\omega \), there is a \( G \) that is \( \mathbb{P}\)-generic over \( M \) such that \( x \in L(\bar{a}, G) \).

Now fix a countable transitive model \( M \) of ZFC. Let \( C \) be Cohen forcing.

1. Fix \( \mathbb{P} \in M \). No real \( \bar{a} \in M \) is \((\mathbb{P}, M)\)-helpful: if \( x \in {}^{\omega}\omega \) codes the ordinal \( \text{Ord} \cap M \), then \( x \not\in L(\bar{a}, G) \) for any \( G \) that is \( \mathbb{P}\)-generic over \( M \).

2. Every real Cohen generic over \( M \) is \((\mathbb{C}, M)\)-helpful (see Corollary 5.5 of [4]).

3. Miha Habić (unpublished) and the first author (see [3] just after “nodes of compatibility”) have independently shown that every real unbounded over \( M \) is \((\mathbb{C}, M)\)-helpful.

4. The first author has shown that every real Sacks generic over \( M \) is \((\mathbb{C}, M)\)-helpful (unpublished).

5. The central result of this paper (Theorem 1.3) is that every real not in \( M \) is \((\mathbb{H}, M)\)-helpful, where \( \mathbb{H} \) is “Tree–Hechler” forcing.

6. The question of whether every real not in \( M \) is \((\mathbb{C}, M)\)-helpful remains open.
**Definition 1.2.** The forcing $\mathbb{H}$, called Tree–Hechler forcing, consists of all trees $T \subseteq \omega_1^{\omega}$ such that for all $t \sqsupseteq \text{Stem}(T)$ in $T$,

\[ \{ z \in \omega : t \upharpoonright z \notin T \} \text{ is finite.} \]

The ordering is by inclusion.

That is, a condition in Tree–Hechler forcing has cofinite splitting beyond its stem. Consider a tree $T \subseteq \omega_1^{\omega}$ and a node $t \in T$. By a successor of $t$ we always mean some $t \upharpoonright z \in T$ for $z \in \omega$. By $T | t$ we mean the set of all $s \in T$ that are comparable to $t$. $\text{Stem}(T)$ is the longest element of $T$ that is comparable with all other elements of $T$.

Let $M$ be a transitive model of ZF and suppose $G$ is $\mathbb{H}^M$-generic over $M$. Let $g = \bigcup \bigcap G$. That is, $g : \omega \to \omega$ is the union of all the stems of the trees $T \in G$. The set $G$ can be recovered from $g$ (and $M$). We will treat $g : \omega \to \omega$ as the object which is encoding information.

The poset $\mathbb{H}$ is $\sigma$-centered, because any two conditions with the same stem are comparable. Thus, $\mathbb{H}$ is c.c.c. Combining this with the fact that $|\mathbb{H}| = 2^\omega$, we have the following: there are only $2^\omega$ maximal antichains in $\mathbb{H}$. So, if $M$ is a transitive model of ZFC and $(2^\omega)^M$ is countable, then there is an $\mathbb{H}^M$-generic over $M$.

The forcing $\mathbb{H}$ is discussed in [9], along with other versions of Hechler forcing, where it is called $\mathbb{D}_{\text{tree}}$. A key ingredient for us is that $\mathbb{H}$ admits a “rank analysis” of its dense sets (see Definition 5.9 and Lemma 5.10). In [2], Jörg Brendle and Benedikt Löwe carry out a rank analysis of $\mathbb{H}$. The original rank analysis of a Hechler-like forcing was done by James Baumgartner and Peter Dordal in [1] for the nondecreasing function version of Hechler forcing (although we discovered “reachability” independently of these works).

Here is our main result:

**Theorem 1.3.** (Generic Coding with Help) *Let $M$ be a transitive model of ZF such that $\mathcal{P}^M(\mathbb{H}^M)$ is countable. Then given any $\bar{a}, x \in \omega$ such that $\bar{a} \notin M$, there is a $G$ that is $\mathbb{H}^M$-generic over $M$ such that $x \leq_T \bar{a} \oplus (\bigcup \bigcap G)$.*

Here, $\bar{a}$ is the “help” which is being used to code $x$. Theorem 1.3 has several interesting applications, which we will present first. Then for completeness we will include a proof of Theorem 1.3.

One striking application is Theorem 2.1, which shows that given two distinct countable transitive models $M, J$ of ZFC of the same height (meaning $\text{Ord} \cap M = \text{Ord} \cap J$), there is a forcing extension of one which does not amalgamate with the other (where two models of the same height $\alpha$ are said to amalgamate iff they are both included in a countable transitive model $W$ of ZFC of height $\alpha$). This answers a question posed by the first author [3] concerning the Hyperuniverse Program: the model $L_\alpha$ is the only node of compatibility of height $\alpha$ (see our discussion after Corollary 2.2).

Theorem 1.3 is a consequence of Lemma 5.11, the “Main Lemma”. However, this was not the Main Lemma’s original purpose in the literature. This lemma originated from the second author’s thesis [5] where it appeared in a game theoretic form that does not explicitly refer to forcing. In that version, Players I and
II play to build a descending sequence through $\mathbb{H}$, where Player I makes $\leq$-extensions but Player II makes $\leq_\Delta$-extensions (to be defined later). The goal of this game was to prove results like Proposition 5.12. In [6] and [7] such results are proved, and the current version of the Main Lemma appears in [6]. We want to emphasize that the Main Lemma may have applications other than Theorem 1.3 and Proposition 5.12.

§2. Amalgamation failure for C.T.M.’s. The Generic Coding with Help theorem implies in a strong way that c.t.m.’s (countable transitive models) of ZFC of the same ordinal height cannot be amalgamated:

THEOREM 2.1. Let $J$ be a c.t.m. of ZFC of ordinal height $\alpha < \omega_1$. Let $M \not\subseteq J$ be another c.t.m. of ZFC of height $\alpha$. Then there is a forcing extension $N$ of $J$ such that $M \cup N$ is not included in any c.t.m. of ZFC of height $\alpha$.

PROOF. Fix $\lambda < \alpha$ and $x \subseteq \lambda$ such that $x \in M - J$. This is possible because $J$ and $M$ are models of ZFC and $M \not\subseteq J$. That is, following the proof of Theorem 13.28 in [8], first fix $X \in M - J$. Now let $x \in M$ be a bounded subset of $\text{Ord} \cap M = \alpha$ such that $X$ in any transitive model of ZFC which contains $x$ can be formed by first bijecting the transitive closure $tc(\{X\})$ of $\{X\}$ with an ordinal $\lambda' < \alpha$, and then encoding the binary relation $\in \mid tc(\{X\})$ as a subset of $\lambda' \times \lambda'$, and then encoding that binary relation by a single set $x \subseteq \lambda$ for some $\lambda < \alpha$. Such an $x$ cannot be in $J$.

Let $g'_0$ and $g''_0$ be mutually Col($\omega, \lambda$)-generic over $J$. Since they are mutually generic, $J[g'_0] - J$ and $J[g''_0] - J$ are disjoint. Let $g_0$ be one of $g'_0$ or $g''_0$ such that $x \notin J[g_0]$.

Now $g_0$ codes a surjection from $\omega$ to $\lambda$. Let $\tilde{x} \subseteq \omega$ be induced from this surjection and $x$. By this we mean if $W$ is any transitive model of ZFC with contains $g_0$, then $x \in W$ iff $\tilde{x} \in W$. Now $\tilde{x} \notin J[g_0]$.

Let $y \in {}^\omega \omega$ be a real that codes a well-ordering of $\omega$ of order type $\alpha$ (so $y$ cannot be in any c.t.m. of ZFC of height $\alpha$). By Theorem 1.3, let $g_1$ be $\mathbb{H}_{[g_0]}$-generic over $J[g_0]$ such that

$$y \leq_T \tilde{x} \oplus \left(\bigcup g_1\right).$$

Let $N = J[g_0][g_1]$. Now suppose, towards a contradiction, that there is some transitive model $W \supseteq M \cup N$ of ZFC of ordinal height $\alpha$. Because $x \in M \subseteq W$ and $g_0 \in N \subseteq W$, we have $\tilde{x} \in W$. But also $g_1 \in N \subseteq W$, so $y \in W$, which is impossible.

We say two c.t.m.’s $N, M$ of ZFC of height $\alpha$ are compatible iff there is a c.t.m. $W$ of ZFC of height $\alpha$ such that $N \cup M \subseteq W$.

COROLLARY 2.2. Given any two distinct c.t.m.’s of ZFC of the same height, there is a forcing extension of one that is not compatible with the other.

The first author asked (see [3]) if for a given $\alpha < \omega_1$, whether $L_\alpha$ was the only c.t.m. of ZFC of height $\alpha$ that was compatible with every c.t.m. of ZFC of height $\alpha$ (that
is, whether \( L_\alpha \) was the only node of compatibility of height \( \alpha \) in the Hyperuniverse. Now we see the answer is yes: If \( M \neq L_\alpha \) is a c.t.m. of ZFC of height \( \alpha \), then \( M \) is not compatible with a certain forcing extension of \( L_\alpha \).

**Remark 2.3.** Mostowski’s result in the introduction was used by him for a result about amalgamation (see [4]): Let \( J \) be a c.t.m. of ZF of ordinal height \( \alpha < \omega_1 \). Let \( x \) be a real which codes \( \alpha \). Let \( c_1, c_2 \) be two reals Cohen generic over \( J \) such that \( x \leq_T c_1 \oplus c_2 \). Then \( J[c_1] \) and \( J[c_2] \) are not compatible.

§3. A complex set disjoint from a Turing cone. As mentioned before, if \( 0^\# \) exists (or even just \( \omega_1 \) is inaccessible in \( L \)), then given any real \( x \), there are two Cohen generics \( s_1, s_2 \) over \( L \) such that \( x \leq_T s_1 \oplus s_2 \). So, let \( S \) be the complement of the Turing cone above \( 0^\# \) (the Turing cone above \( a \in \omega_\omega \) is the set of all \( b \in \omega_\omega \) such that \( b \geq_T a \)). Every real generic over \( L \) is in \( S \). Now \( S \) is small in one sense, because it is disjoint from a Turing cone. But it is large in another sense, because \( \{ s_1 \oplus s_2 : s_1, s_2 \in S \} \) is cofinal in the Turing degrees. We get a variation of this phenomenon using the Generic Coding with Help Theorem (1.3). Let \([x]\) denote the Turing degree of \( x \in \omega_\omega \).

**Proposition 3.1.** Assume \( 0^\# \) exists. Let \( S \subseteq \omega_\omega \) be the set of all reals of the form \( s = \bigcup \bigcap G \) for some \( G \) that is \( \mathbb{H}^L \)-generic over \( L \). The set \( S \) is disjoint from the Turing cone above \( 0^\# \). On the other hand, for any real \( \bar{a} \notin L \), the set \( S^* := \{ [\bar{a} \oplus s] : s \in S \} \) is cofinal in the Turing degrees.

Also, if \( x \) is any real such that \( x \geq_T \bar{a} \) and \( x \) computes a length \( \omega \) enumeration of \( \mathbb{R} \cap L \), then \([x] \in S^* \) (so \( S^* \) contains a Turing cone).

**Proof.** It is well known that no generic extension of \( L \) contains \( 0^\# \). Hence, \( 0^\# \) is not Turing reducible to any \( s = \bigcup \bigcap G \) for a \( G \) that is \( \mathbb{H}^L \)-generic over \( L \). That is, \( S \) is disjoint from the Turing cone above \( 0^\# \).

Now fix a real \( \bar{a} \) not in \( L \). Pick any \( x \in \omega_\omega \). By Theorem 1.3 there is some \( G \) that is \( \mathbb{H}^L \)-generic over \( L \) such that letting \( s = \bigcup \bigcap G \), we have \( x \leq_T \bar{a} \oplus s \). Hence, \( S^* \) is cofinal in the Turing degrees.

For the last part, again fix a real \( \bar{a} \notin L \) and let \( x \geq_T \bar{a} \) be a real which computes a length \( \omega \) enumeration of \( \mathbb{R} \cap L \). There is some \( G \) that is \( \mathbb{H}^L \)-generic over \( L \) such that letting \( s = \bigcup \bigcap G \), we have \( x \leq_T \bar{a} \oplus s \). However, by the proof of Theorem 1.3, fix an \( s \) like this that can be built using \( \bar{a} \), \( x \), and a length \( \omega \) enumeration of \( \mathbb{R} \cap L \) (using that the dense subsets of \( \mathbb{H}^L \) in \( L \) are coded by reals in \( L \)). So we can have \( s \leq_T \bar{a} \oplus x \). We now have

\[
x \leq_T \bar{a} \oplus s \leq_T \bar{a} \oplus (\bar{a} \oplus x) \leq_T \bar{a} \oplus x = x,
\]

so \( x =_T \bar{a} \oplus s \), and so \([x] \in S^* \).

§4. Larger sets are generically generic. The Generic Coding with Help Theorem shows that reals not in \( M \) are “helpful”. The following theorem shows that *any set of ordinals not in \( M \) is helpful*, provided \( M \) contains the supremum of the set of
ordinals and that we pass to an outer model of \( V \) in which a large enough cardinal has become countable.

**Theorem 4.1.** Let \( M \) be a transitive model of ZF. Let \( \lambda \) be a cardinal such that \( \lambda \in M \). Let \( P = (\text{Col}(\omega, \lambda) \otimes \mathbb{H})^M \). Let \( V \) be an outer model of \( V \) in which \( P^M(P) \) is countable. Let \( X \in \mathcal{P}^V(\lambda) \). Let \( A \in \mathcal{P}^V(\lambda) - M \). Then there is a \( G \) in \( V \) such that

1. \( G \) is \( P \)-generic over \( M \),
2. \( X \in L(A, G) \).

**Proof.** Using the same mutual generic technique as in the second paragraph of the proof of Theorem 2.1, let \( g_0 \in V \) be \( \text{Col}(\omega, \lambda) \)-generic over \( M \) so that \( A \not\in M[g_0] \). Let \( \tilde{a} \in \omega \omega \) be such that for every transitive model \( N \) of ZF such that \( g_0 \in N \), we have \( A \in N \) iff \( \tilde{a} \in N \). Now \( \tilde{a} \not\in M[g_0] \). Let \( \tilde{x} \in \omega \omega \) be such that for every transitive model \( N \) of ZF such that \( g_0 \in N \), we have \( X \in N \) iff \( \tilde{x} \in N \).

Force over \( M[g_0] \) by \( \mathbb{H}^{M[\tilde{a}]} \) to get \( g_1 \) so that \( \tilde{x} \leq_T \tilde{a} \oplus (\bigcup g_1) \). Let \( G := g_0 \ast g_1 \), so \( G \in V \) is \( P \)-generic over \( M \). \( L(A, G) \) is a model of ZF and it contains \( g_0 \) and \( A \), so it contains \( \tilde{a} \). It also contains \( g_1 \), therefore it contains \( \tilde{x} \). Since it contains \( g_0 \) and \( \tilde{x} \), it contains \( X \).

Note that if \( \lambda = \omega \) in the theorem above, then we can simply take \( P \) to be \( \mathbb{H}^M \).

**§5. Proof of Generic Coding with Help Theorem.** Theorem 1.3 follows from the Main Lemma of [6]. For completeness we give a full proof here.

**5.1. Evasiveness and the Sticking Out Lemma.** We will now start to prove the theorem. This subsection helps to clarify how we use the hypothesis \( \tilde{a} \not\in M \).

**Definition 5.1.** Let \( M \) be a transitive model of ZF. A set \( A \subseteq \omega \) is evasive with respect to \( M \) iff it is infinite and it has no infinite subsets in \( M \).

**Fact 5.2.** Given any \( \tilde{a} \in \omega \omega \), there is a set \( A \subseteq \omega \) such that \( \tilde{a} =_T A \) and \( A \) is computable from every infinite subset of itself.

Thus if \( M \) is a transitive model of ZF and \( \tilde{a} \in \omega \omega - M \), then if \( A \) comes from the fact above, then \( A \) is evasive with respect to \( M \).

**Lemma 5.3. (Sticking out lemma)** Let \( M \) be a transitive model of ZF. Let \( A \subseteq \omega \) be evasive with respect to \( M \). Then if \( B \subseteq \omega \) is infinite and in \( M \), then \( B - A \) is infinite.

**Proof.** Assume towards a contradiction that \( B - A \) is finite. Then \( B - A \in M \). Since both \( B \) and \( B - A \) are in \( M \), we have \( B \cap A \in M \) as well. At the same time, since \( B \) is infinite and \( B - A \) is finite, \( B \cap A \) must be finite. So now we have shown that \( B \cap A \) is an infinite subset of \( A \) that is in \( M \), which contradicts \( A \) being evasive with respect to \( M \). \( \square \)

**5.2. Decoding an \( x \in \omega \omega \) from an \( \mathbb{H} \) generic and an \( A \subseteq \omega \).** Suppose \( G \) is generic for \( \mathbb{H} \). Recall that \( g := \bigcup \bigcap G \) is a function from \( \omega \) to \( \omega \). The idea is to look at each \( n \in \omega \) such that \( g(n) \in A \). Which element of \( A \) this \( g(n) \) actually is will give us a piece of encoded information. For each \( n \) such that \( g(n) \not\in A \), no information is being encoded. Here is what we mean precisely:
Definition 5.4. Fix a computable function \( \theta : \omega \to \omega \) such that

\[
(\forall m \in \omega) \theta^{-1}(m) \text{ is infinite.}
\]

Given an infinite \( A \subseteq \omega \), let \( e_A : \omega \to A \) be the strictly increasing enumeration of \( A \). Let \( \eta_A : A \to \omega \) be the function \( \theta \circ e_A^{-1} \).

Note that for each \( m \in \omega \), \( \eta_A^{-1}(m) \subseteq A \) is infinite.

Definition 5.5. Let \( M \) be a transitive model of ZF. Let \( G \) be \( \mathbb{H}^M \)-generic over \( M \). Let \( A \subseteq \omega \).

Then the real that is \( A \)-encoded by \( G \) is

\[
\langle \eta_A(g(n_i)) : i < \omega \rangle,
\]

where \( g := \bigcup \bigcap G \) and

\[ n_0 < n_1 < \ldots \]

is the increasing enumeration of the set of \( n \in \omega \) such that \( g(n) \in A \). However, if there are only finitely many such \( n \)'s, then the real \( A \)-encoded by \( G \) is the zero sequence.

Observation 5.6. Let \( x \in ^\omega \omega \) be the real \( A \)-encoded by \( G \). Then

\[ x \leq_T A \oplus \left( \bigcup \bigcap G \right). \]

5.3. The stronger \( \leq_A \) ordering and the Main Lemma. Given \( A \subseteq \omega \), there is an ordering \( \leq_A \) defined on \( \mathbb{H} \) which is stronger than \( \leq \). Intuitively, \( T' \leq_A T \) iff \( T' \leq T \) and the stem of \( T' \) does not “hit” \( A \) any more than the stem of \( T \) already does:

Definition 5.7. Let \( A \subseteq \omega \). Then given \( t, t' \in ^{<\omega} \omega \), we write \( t' \sqsupseteq_A t \) iff \( t' \supseteq t \) and

\[
(\forall n \in \text{Dom}(t') - \text{Dom}(t)) t'(n) \notin A.
\]

Definition 5.8. Let \( A \subseteq \omega \). Given \( T, T' \in \mathbb{H} \), we write \( T' \leq_A T \) iff \( T' \leq T \) and \( \text{Stem}(T') \sqsupseteq_A \text{Stem}(T) \).

The content of the Main Lemma soon to come is that as long as \( A \) is evasive with respect to \( M \), we can hit dense subsets of \( \mathbb{H} \) (that are in \( M \)) by making \( \leq_A \) extensions. So, we can construct a generic without being forced to encode unwanted information. Hence, we can alternate between 1) making \( \leq_A \) extensions in order to build an \( \mathbb{H} \) generic but not encoding any information and 2) making non- \( \leq_A \) extensions to encode information. We use a rank analysis to prove the Main Lemma:

Definition 5.9. Given \( S \subseteq ^{<\omega} \omega \) and \( t \in ^{<\omega} \omega \),

\[ \bullet t \text{ is 0-S-reachable iff } t \in S; \]

\[ \bullet t \text{ is } \alpha\text{-S-reachable for some } \alpha > 0 \text{ iff } \{ z \in \omega : t \wedge z \text{ is } \beta - S \text{ - reachable for some } \beta < \alpha \} \]

is infinite;

\[ \bullet t \text{ is S-reachable iff } t \text{ is } \alpha\text{-S-reachable for some } \alpha. \]
Notice that if $t$ is not $S$-reachable, then only a finite set of successors of $t$ can be $S$-reachable.

**Lemma 5.10.** Let $\mathcal{D} \subseteq \mathbb{H}$ be dense. Let

$$S = \{ s \in <\omega \omega : (\exists T \in \mathcal{D}) \text{ Stem}(T) = s \}.$$ 

Fix $t \in <\omega \omega$. Then $t$ is $S$-reachable.

**Proof.** Assume that some fixed $t$ is not $S$-reachable. We will construct a tree $T \in \mathbb{H}$ with stem $t$ such that no $s \supseteq t$ in $T$ is in $S$. Hence, no $T' \leq T$ can be in $\mathcal{D}$.

There is only a finite set of $z \in \omega$ such that $t \sim z$ is $S$-reachable. Let the successors of $t$ in $T$ be those $t \sim z$ that are not $S$-reachable. Now for each $t \sim z_0$ in $T$, there is only a finite set of $z \in \omega$ such that $t \sim z_0 \sim z$ is $S$-reachable. Let the successors of each $t \sim z_0$ in $T$ be those $t \sim z_0 \sim z$ that are not $S$-reachable. Continuing this procedure $\omega$ times yields a tree $T$ such that all $s \supseteq t$ in $T$ are not $S$-reachable. In particular no $s \supseteq t$ in $T$ is in $S$.

**Lemma 5.11.** (Main Lemma) Let $M$ be a transitive model of ZF. Let $A \subseteq \omega$ be evasive with respect to $M$. Let $\mathbb{P} = \mathbb{H}^M$. Let $\mathcal{D} \in \mathcal{P}^M(\mathbb{P})$ be open dense (in $M$). Let $T \in \mathbb{P}$. Then there exists some $T' \leq_A T$ in $\mathcal{D}$.

**Proof.** Let $t = \text{Stem}(T)$. Let

$$S = \{ s <\omega \omega : (\exists T' \in \mathcal{D}) \text{ Stem}(T') = s \}.$$ 

If we can find a $s \supseteq_A t$ in $T \cap S$, then letting $T' \in \mathcal{D}$ be such that $\text{Stem}(T') = s$ and letting $T'' \leq T$ be $T'' = T \upharpoonright s$, then $T' \cap T''$ is in $\mathcal{D}$ (because $\mathcal{D}$ is open), and $\text{Stem}(T' \cap T'') = s$ so $T' \cap T'' \leq_A T$. Hence, we will be done.

Now by the previous lemma, fix some ordinal $\alpha$ such that $t$ is $\alpha$-$S$-reachable. If $\alpha = 0$ we are done, so assume $\alpha > 0$. The set

$$B = \{ z \in \omega : t \sim z \text{ is } \beta - S \text{ - reachable for some } \beta < \alpha \}$$

is infinite and in $M$. Since $A$ is evasive with respect to $M$, $B - A$ must be infinite by the Sticking Out Lemma (Lemma 5.3). Thus, we may fix some $z_0 \in (B - A)$ such that $t \sim z_0 \in T$.

Now $t \sim z_0$ is $\beta$-$S$-reachable for some fixed $\beta < \alpha$. If $\beta = 0$ we are done, and otherwise we may find some $z_1 \in (B - A)$, for an appropriately redefined $B$, such that $t \sim z_0 \sim z_1 \in T$ and $t \sim z_0 \sim z_1$ is $\gamma$-$S$-reachable for some $\gamma < \beta$. We may continue like this but eventually we will have some $t \sim z_0 \sim \ldots \sim z_n$ that is in $S$. 

**5.4. Proof of Generic Coding with Help Theorem.**

**Proof of Theorem 1.3.** Let $M$ be a transitive model of ZF. Let $\mathbb{P} = \mathbb{H}^M$ and assume $\mathcal{P}^M(\mathbb{P})$ is countable. Let $x \in ^\omega \omega$. Let $\tilde{a} \in ^\omega \omega - M$. By Fact 5.2, fix $A \subseteq \omega$ such that $A =_T \tilde{a}$ and $A$ is computable from every infinite subset of itself. Then $A$ is evasive with respect to $M$. It suffices to find a $\mathbb{P}$-generic $G$ over $M$ such that $x \leq_T A \oplus (\bigcup \bigcap G)$. By Observation 5.6, it suffices to find a $\mathbb{P}$-generic $G$ over $M$ such that $x$ is the real $A$-encoded by $G$. 

Since $\mathcal{P}^M(\mathbb{P})$ is countable, let $\langle D_i : i < \omega \rangle$ be an enumeration of the open dense subsets of $\mathbb{P}$ in $M$. We will construct a decreasing $\omega$-sequence

$$T_0 \geq T_1 \geq \cdots$$

def of $\mathbb{P}$-conditions such that each $T_i \in D_i$. Hence

$$G := \{ T \in \mathbb{P} : (\exists i) T \geq T_i \}$$

will be $\mathbb{P}$-generic over $M$. On the other hand, we will construct the sequence of conditions so that $x$ is the real $A$-encoded by $G$.

Since $A$ is evasive with respect to $M$, by Lemma 5.11 (the Main Lemma), let $T_0 \leq_A 1_\mathbb{P}$ be such that $T_0 \in D_0$. Now we will encode $x(0)$; let $T_0' \leq T_0$ be a non-$\leq_A$ extension of $T_0$, extending the stem of $T_0$ by one, such that $\text{Stem}(T_0') = \text{Stem}(T_0) \upharpoonright z$ for a $z \in A$ such that $\eta_A(z) = x(0)$. This is possible because

$$\{ z \in A : \eta_A(z) = x(0) \}$$

is infinite, and so must intersect

$$\{ z \in \omega : \text{Stem}(T_0) \upharpoonright z \in T_0 \}.$$

Next, let $T_1 \leq_A T_0'$ be such that $T_1 \in D_1$. Then, let $T_1' \leq T_1$ be such that

$$\text{Stem}(T_1') = \text{Stem}(T_1) \upharpoonright z$$

for a $z \in A$ such that $\eta_A(z) = x(1)$. Continuing this $\omega$ times, we see that $x$ is the real $A$-encoded by $G$. That is, let $g := \bigcup \bigcap G$. The only $n$'s such that $g(n) \in A$ come from when we made non-$\leq_A$ extensions. And, if $n_0 < n_1 < \cdots$ is the strictly increasing enumeration of these $n$'s, then we see that $\eta_A(g(n_i)) = x(i)$ for each $i$.

\section*{5.5. Another application of the Main Lemma.} As described in the introduction, here is the original kind of result for which the Main Lemma was created. A proof can be found in [7].

\textbf{Proposition 5.12.} Assume $AD^+$. Fix $a \in {}^\omega \omega$. Then there is a Borel (in fact, Baire class one) function $f_a : {}^\omega \omega \rightarrow {}^\omega \omega$ such that whenever $g : {}^\omega \omega \rightarrow {}^\omega \omega$ is a function whose graph is disjoint from $f_a$, then

$$a \in L[C]$$

where $C \subseteq \text{Ord}$ is any $\omega$-Borel code for $g$.

The function $(a, x) \mapsto f_a(x)$ is Borel as well.

\section*{6. HOD.} By Vopěnka’s Theorem, every real is generic over HOD. But one can ask if there is a single $\mathbb{P} \in \text{HOD}$ such that $(|\mathbb{P}| \leq 2^\omega)_{\text{HOD}}$ and every real is $\mathbb{P}$-generic over HOD. This is relevant to our paper because by Theorem 4.1, if $\tilde{V}$ is an outer model of $V$ in which $\mathcal{P}^\text{HOD}^\prime$ is countable, and $\tilde{a} \in ({}^\omega \omega)^{\tilde{V}}$, then for any $x \in ({}^\omega \omega)^{\tilde{V}}$, there is a $G$ that is $\tilde{a}$-generic over HOD$^\prime$ such that $x \in L(\tilde{a}, G)$. So the question is whether the $\tilde{a}$ can be removed. The answer is no.
**Proposition 6.1.** It is consistent with ZFC that there is a real \( R \) that is not \( \mathbb{P} \)-generic over \( \text{HOD} \) for any \( \mathbb{P} \in \text{HOD} \) such that \( (|\mathbb{P}| \leq 2^\omega)^{\text{HOD}} \). Moreover, this persists to any outer model of \( V \). That is, if \( \hat{V} \) is an outer model of \( V \), then \( R \) is not \( \mathbb{P} \)-generic over \( \text{HOD}^{\hat{V}} \) for any \( \mathbb{P} \in \text{HOD}^{\hat{V}} \) such that \( (|\mathbb{P}| \leq 2^\omega)^{\text{HOD}^{\hat{V}}} \).

**Proof.** Start with \( L \). Let \( C_{\omega_3} \subseteq L \) be the forcing to add a Cohen subset of \( \omega_2 \). Let \( A \subseteq \omega_2 \) be \( C_{\omega_3} \)-generic over \( L \). Let \( X \subseteq \omega_1 \) be generic over \( L[A] \) by almost disjoint coding such that \( A \in L[X] \). Let \( R \subseteq \omega \) be generic over \( L[X] \) by almost disjoint coding such that \( X \in L[R] \). So now

\[
L \subseteq L[A] \subseteq L[X] \subseteq L[R]
\]

and \( \mathcal{P}(\omega_1)^L = \mathcal{P}(\omega_1)^{L[A]} \) (because \( C_{\omega_3} \) is \( <\omega_2 \)-closed). Let \( H = \text{HOD}^{L[R]} \). We will show that \( L[R] \) is not generic over \( H \) by any forcing of size \( (2^\omega)^H \). Moreover, fix any outer model \( N \) of \( L[R] \). We will show that \( N \) satisfies that \( R \) is not generic over \( H \) by any forcing of size \( (2^\omega)^H \).

The forcing \( Q \) to go from \( L[A] \) to \( L[R] \) is weakly homogeneous [11]. This is subtle, because a three step iteration of almost disjoint coding, to code a subset of \( \omega_3 \) into a subset of \( \omega \) may not be weakly homogeneous [11]. Now because \( Q \in L[A] \) is weakly homogeneous, \( H \subseteq L[A] \).

Since \( L \subseteq H \subseteq L[A] \) and \( \mathcal{P}(\omega_1)^L = \mathcal{P}(\omega_1)^{L[A]} \), we have \( \omega^L_1 = \omega^H_1 = \omega^L[A] \) and \( H \) satisfies CH. Suppose towards a contradiction that there is some \( \mathbb{P} \in H \) such that \( R \) is in a generic extension of \( H \) by \( \mathbb{P} \) (meaning there is some \( G \subseteq N \) that is \( \mathbb{P} \)-generic over \( H \) and \( R \in H[G] \)) and \( \mathbb{P} \) has size \( (2^\omega)^H = \omega^H_1 \). Then because \( \mathcal{P}(\omega_1)^L = \mathcal{P}(\omega_1)^H \), a forcing \( \tilde{\mathbb{P}} \) isomorphic to \( \mathbb{P} \) is in \( L \). Also because \( \mathcal{P}(\omega_1)^L = \mathcal{P}(\omega_1)^H \), all dense subsets of \( \tilde{\mathbb{P}} \) in \( H \) are already in \( L \). Let \( G \subseteq \tilde{\mathbb{P}} \) in \( N \) be \( \tilde{\mathbb{P}} \)-generic over \( L \) such that \( R \in L[G] \). Note that this implies \( A \in L[G] \).

By a density argument for \( C_{\omega_3} \), the set \( A \subseteq \omega^L_2 \) has no subset of size \( \omega^L_2 \) in \( L \). On the other hand, let \( \tilde{A} \) be a \( \tilde{\mathbb{P}} \) name for \( A \). For each \( \alpha \in A \), let \( p_\alpha \in G \) be a condition such that \( p_\alpha \Vdash \tilde{\alpha} \in \tilde{A} \). Since \( (|\tilde{\mathbb{P}}| < \omega_2)^L \), fix a \( p \in G \) such that \( p = p_\alpha \) for a size \( \omega^L_2 \) set of \( \alpha \in A \). Now the set

\[
\{ \alpha < \omega^L_2 : p \Vdash \tilde{\alpha} \in \tilde{A} \}
\]

is a size \( \omega^L_2 \) subset of \( A \) in \( L \), which is a contradiction. \( \dashv \)

We mentioned in the proof above that the three step iteration of almost disjoint coding to code a subset of \( \omega_3 \) into a subset of \( \omega \) may not be weakly homogeneous. The argument in the proof above also shows us why: start with \( V = L \) and let \( A \subseteq \omega_3 \) be a Cohen subset of \( \omega_3 \). Let \( R \subseteq \omega \) arise from the three step iteration \( Q \in L[A] \) of almost disjoint coding to code \( A \subseteq \omega_3 \) into a subset of \( \omega \). Suppose towards a contradiction that \( Q \) is weakly homogeneous. Then \( \text{HOD}^{L[R]} \subseteq L[A] \).

By Vopěnka’s Theorem, \( R \) is generic over \( \text{HOD}^{L[R]} \) by a forcing of size \( \omega_2 \). Since \( \mathcal{P}((\omega_2)^L = \mathcal{P}((\omega_2)^{L[A]} \) and \( \text{HOD}^{L[R]} \) is intermediate between \( L \) and \( L[A] \), there must be some \( \tilde{\mathbb{P}} \in L \) of size \( \omega_2 \) such that \( R \) is in a \( \tilde{\mathbb{P}} \)-generic extension \( L[G] \) of \( L \). But now since \( A \in L[G] \) and \( |\tilde{\mathbb{P}}| \leq \omega_2 \), \( A \) has a size \( \omega_3 \) subset in \( L \). This contradicts \( A \) being Cohen generic over \( L \).
§7. Questions

7.1. What can replace $\mathbb{H}$?

**Question 7.1.** Let $M$ be a c.t.m. of ZFC. What are the forcings $P \in M$ such that every real $a \in {}^\omega \omega - M$ is $(P, M)$-helpful? Does Cohen forcing work? What about a forcing which is $\omega$-bounding?

7.2. Generically coding subsets of $\omega_1$ with help. Given a transitive model $M$, it is natural to ask whether subsets of $\omega_1$ can be coded by generics over $M$ with help. By Theorem 4.1, this is possible as long as we pass to a sufficiently larger outer model $V$. We suspect that passing to $\hat{V}$ is not necessary provided that $M$ is large enough.

In terms of being large enough, note that given a forcing $P \in L(\mathbb{R})$,

- there is a surjection of $\omega$ onto $P$ in $L(\mathbb{R})$,
- $P$ is countably closed,
- there is a proper class of Woodin cardinals, and
- CH holds.

then there is a $P$-generic over $L(\mathbb{R})$ in $V$. Here is a proof of this fact (pointed out by Paul Larson): every set of reals in $L(\mathbb{R})$ is the continuous preimage of $\mathbb{R}^\#$, so there are at most $2^{\omega}$ sets of reals in $L(\mathbb{R})$. But, because CH holds, there are $\omega_1$ sets of reals in $L(\mathbb{R})$. So there are $\omega_1$ dense subsets of $P$ in $L(\mathbb{R})$. Let $\langle D_\alpha : \alpha < \omega_1 \rangle$ be an enumeration of all these dense sets. By the fact that $P$ is countably closed, we can hit all $\omega_1$ dense sets by forming a length $\omega_1$ decreasing sequence through $P$ (here we also use that $\subset^{\omega_1} P \subseteq L(\mathbb{R})$).

So, we ask the following (where we have weakened arbitrary help $\bar{a} \in P(\omega_1) - L(\mathbb{R})$ to some fixed help $\bar{a} \in P(\mathbb{R})$):

**Question 7.2.** Assume CH and a proper class of Woodin cardinals. Is there some $\bar{a} \subseteq \mathbb{R}$ and some forcing $P \in L(\mathbb{R})$ that is countably closed such that given any $X \subseteq \omega_1$, there is a $G$ that is $P$-generic over $L(\mathbb{R})$ such that $X \in L(\bar{a}, G, \mathbb{R})$?

Along similar lines, Woodin has conjectured (Section 10.6 of [10]) that assuming CH and a measurable Woodin cardinal, then for any $X \subseteq \omega_1$, there is some $B \subseteq \mathbb{R}$ such that $L(B, \mathbb{R}) \models AD^+$ and $X \in L(B, \mathbb{R})[G]$ for some $G$ that is $\text{Col}(\omega_1, \mathbb{R})$-generic over $L(\mathbb{R}, B)$.

Assume Woodin’s conjecture is true and assume $V$ satisfies CH and has a measurable Woodin cardinal. Let $\mathcal{C}$ be the collection of all inner models of AD$^+$ containing all the reals. Then every subset of $\omega_1$ (and therefore every subset of $\mathbb{R}$ because we are assuming CH) is generic over some model in $\mathcal{C}$. Our question above asks whether the smallest model in $\mathcal{C}$, namely $L(\mathbb{R})$, is still large enough so that $(\exists \bar{a} \subseteq \mathbb{R})(\exists P \in L(\mathbb{R}))(\forall X \subseteq \omega_1)(\exists G$ that is $P$-generic over $L(\mathbb{R})$) $X \in L(\bar{a}, G, \mathbb{R})$.

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