MORE $\ell_r$ SATURATED $\mathcal{L}^\infty$ SPACES

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Abstract. Given $r \in (1, \infty)$, we construct a new $\mathcal{L}^\infty$ separable Banach space which is $\ell_r$ saturated.

1. Introduction

The Bourgain-Delbaen spaces [7] are examples of separable $\mathcal{L}^\infty$ spaces containing no isomorphic copy of $c_0$. They have played a key role in the solution of the scalar-plus-compact problem by Argyros and Haydon [3], where a Hereditarily Indecomposable $\mathcal{L}^\infty$ space is presented with the property that every operator on the space is a compact perturbation of a scalar multiple of the identity.

There has recently been an interest in the study $\mathcal{L}^\infty$ spaces of the Bourgain-Delbaen type. Freeman, Odell and Schlumprecht [8] showed that every Banach space with separable dual is isomorphic to a subspace of a $\mathcal{L}^\infty$ space having a separable dual. The aim of this paper is to present a method of constructing, for every $1 < r < \infty$, a new $\mathcal{L}^\infty$ space which is $\ell_r$ saturated.

Our approach shares common features with the Argyros-Haydon work. More precisely we combine, as in [3], the Bourgain-Delbaen method [7] yielding exotic $\mathcal{L}^\infty$ spaces, with the Tsirelson type space $T(A_n, b)$ which will be later proved to be isomorphic to $\ell_r$. In particular, if $b_1 = b_2 = \ldots = b_n = \theta$, $T(A_n, b)$ coincides with $T(A_n, \theta)$ and the latter is known to be isomorphic to $\ell_p$ for some $p \in (1, \infty)$ (see [3]).

This paper is organized as follows. In the second section, for a given $r \in (1, \infty)$, we construct a Banach space $X_r$. To do this, we first choose $n \in \mathbb{N}$, $n > 1$, and a finite sequence $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ of positive real numbers with $b_1 < 1$, $b_2, b_3, \ldots, b_n < \frac{1}{2}$ such that $\sum_{i=1}^n b_i^{r'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$. The definition of $X_r$ combines the Bourgain-Delbaen method with the Tsirelson type space $T(A_n, \mathbf{b})$ which will be later proved to be isomorphic to $\ell_r$. In particular, if $b_1 = b_2 = \ldots = b_n = \theta$, $T(A_n, \mathbf{b})$ coincides with $T(A_n, \theta)$ and the latter is known to be isomorphic to $\ell_p$ for some $p \in (1, \infty)$ (see [3]).

It is worth noticing that for $n = 2$ the spaces $X_r$ essentially coincide with the original Bourgain-Delbaen spaces $X_{a,b}$. Thus, our construction of $\mathcal{L}^\infty$ spaces which are $\ell_r$ saturated spaces, can be considered as a generalization of the Bourgain-Delbaen method. We must point out here that when $n = 2$,
our proof of the fact that $X_r$ is $\ell_r$ saturated, differs from Haydon’s (see [9]) corresponding one for $X_{n,b}$. To be more specific, $X_r$ has a natural FDD $(M_k)$. Given a normalized skipped block basis $(u_k)$ of $(M_k)$ with the supports of the $u_k$’s lying far enough apart, then it is not hard to check that $(u_k)$ dominates $(e_k)$, the natural basis of $T(A_n, \overline{b})$. The same holds for every normalized block basis of $(u_k)$. To obtain a normalized block basis of $(u_k)$ equivalent to $(e_k)$, we select a sequence $I_1 < I_2 < \ldots$ of successive finite subsets of $N$ such that $\lim_k \| \sum_{i \in I_k} u_i \| = \infty$. Such a choice is possible by the domination of $(e_k)$ by $(u_k)$. We set $v_k = \| \sum_{i \in I_k} u_i \|^{-1} \sum_{i \in I_k} u_i$ and show that some subsequence of $(v_k)$ is dominated by $(e_k)$. To accomplish this we adapt the method of the analysis of the members of a finite block basis of $(e_k)$ with respect to a functional in the natural norming set of $T(A_n, \overline{b})$ (see [9]), to the context of the present construction. We believe that this approach yields a more transparent proof than Haydon’s, at least for the upper $\ell_r$ estimate.

The rest of the paper is devoted to the proof of the main property, namely that $X_r$ is $\ell_r$ saturated. In Section 3, we define the tree analysis of the functionals $\{e_\gamma^* : \gamma \in \Gamma\}$ which is a 1-norming subset of the unit ball of $X_r$. The tree analysis is similar to the corresponding one used in the Tsirelson and mixed Tsirelson spaces [1]. In the following two sections we establish the lower and upper norm estimates for certain block sequences in the space $X_r$.

In the final section we show that every block basis of $(M_k)$ admits a further normalized block basis $(x_k)$ such that every normalized block basis of $(x_k)$ is equivalent to the natural basis of the space $T(A_n, \overline{b})$. Zippin’s theorem [12] yields the desired result.

2. Preliminaries

In this section we define the space $X_r$ combining the Bourgain-Delbaen construction [7] and the Tsirelson type constructions [2], [4].

Before proceeding, we recall some notation and terminology from [3]. Let $n \in N$ and $0 < b_1, b_2, \ldots, b_n < 1$ with $\sum_{i=1}^n b_i > 1$ and there exists $r' \in (1, \infty)$ such that $\sum_{i=1}^n b_i^r = 1$. We may also assume without loss of generality that $b_1 > b_2 > \ldots > b_n$. We define $W[(A_n, \overline{b})]$ to be the smallest subset $W$ of $c_{00}(N)$ with the following properties:

1. $\pm e_k^* \in W$ for all $k \in N$,
2. whenever $f_i \in W$ and max supp $f_i < \min$ supp $f_{i+1}$ for all $i$, we have $\sum_{i \leq a} b_i f_i \in W$, provided that $a \leq n$.

We say that an element $f$ of $W[(A_n, \overline{b})]$ is of Type 0 if $f = \pm e_k^*$ for some $k$ and of Type I otherwise; an element of Type I is said to have weight $b_a$ for some $a \leq n$ if $f = \sum_{i=1}^n f_i$ for a suitable sequence $(f_i)$ of successive elements of $W[(A_n, \overline{b})]$. The Tsirelson space $T(A_n, \overline{b})$ is defined to be the completion of $c_{00}$ with
respect to the norm
\[ \|x\| = \sup\{ \langle f, x \rangle : f \in W[A_n, b] \}. \]

We may also characterize the norm of this space implicitly as being the smallest function \( x \mapsto \|x\| \) satisfying
\[ \|x\| = \max \left\{ \|x\|_{\infty}, \sup_{n} \sum_{i=1}^{n} b_i \|E_i x\| \right\}, \]
where the supremum is taken over all sequences of finite subsets \( E_1 < E_2 < \cdots < E_n \).

We shall now present the fundamental aspects related to the Bourgain-Delbaen construction.

For the interested readers we mention that the following method can be characterized as the "dual" construction of the construction presented in [3]. This characterization is based on the fact that in [3] a particular kind of basis is given to \( \ell_1(\Gamma) \) and the Bourgain-Delbaen type space \( X \) is seen as the predual of its dual, which is \( \ell_1(\Gamma) \).

Let \((\Gamma_q)_{q \in \mathbb{N}}\) be a strictly increasing sequence of finite sets and denote their union by \( \Gamma = \bigcup_{q \in \mathbb{N}} \Gamma_q \).

We set \( \Delta_0 = \Gamma_0 \) and \( \Delta_q = \Gamma_q \setminus \Gamma_{q-1} \) for \( q = 1, 2, \ldots \)
Assume furthermore that to each \( \gamma \in \Delta_q \), \( q \geq 1 \), we have assigned a linear functional \( c_{\gamma}^*: \ell^\infty(\Gamma_{q-1}) \to \mathbb{R} \). Next, for \( n < m \) in \( \mathbb{N} \), we define by induction, a linear operator \( i_{n,m} : \ell^\infty(\Gamma_n) \to \ell^\infty(\Gamma_m) \) as follows:

For \( m = n + 1 \), we define \( i_{n,n+1} : \ell^\infty(\Gamma_n) \to \ell^\infty(\Gamma_{n+1}) \) by the rule
\[ (i_{n,n+1}(x))(\gamma) = \begin{cases} x(\gamma), & \text{if } \gamma \in \Gamma_n \\ c_{\gamma}^*(x), & \text{if } \gamma \in \Delta_{n+1} \end{cases} \]

for every \( x \in \ell^\infty(\Gamma_n) \).

Then assuming that \( i_{n,m} \) has been defined, we set \( i_{n,m+1} = i_{m,m+1} \circ i_{n,m} \).

A direct consequence of the above definition is that for \( n < l < m \) it holds that \( i_{n,m} = i_{l,m} \circ i_{n,l} \). Finally we denote by \( i_n : \ell^\infty(\Gamma_n) \to \mathbb{R}^\Gamma \) the direct limit \( i_n = \lim_{m \to \infty} i_{n,m} \).

We assume that there exists a \( C > 0 \) such that for every \( n < m \) we have \( \|i_{n,m}\| \leq C \). This implies that \( \|i_n\| \leq C \) and therefore \( i_n : \ell^\infty(\Gamma_n) \to \ell^\infty(\Gamma) \) is a bounded linear map. In particular, setting \( X_n = i_n[\ell^\infty(\Gamma_n)] \), we have that \( X_n \overset{C}{=} \ell^\infty(\Gamma_n) \) and furthermore \((X_n)_{n \in \mathbb{N}}\) is an increasing sequence of subspaces of \( \ell^\infty(\Gamma) \). We also set \( X_{BD} = \bigcup_{n \in \mathbb{N}} X_n \hookrightarrow \ell^\infty(\Gamma) \) equipped with the supremum norm. Evidently, \( X_{BD} \) is an \( \ell^\infty \) space.

Let us denote by \( r_n : \ell^\infty(\Gamma) \to \ell^\infty(\Gamma_n) \) the natural restriction map, i.e. \( r_n(x) = x|_{\Gamma_n} \). We will also abuse notation and denote by \( r_n : \ell^\infty(\Gamma_m) \to \ell^\infty(\Gamma_n) \) the restriction function from \( \ell^\infty(\Gamma_m) \) to \( \ell^\infty(\Gamma_n) \) for \( n < m \).

**Notation 2.1.**
We also extend the functional \( c_\gamma^* : \ell^\infty(\Gamma_n) \to \mathbb{R} \) to a functional \( c_\gamma^* : X_{BD} \to \mathbb{R} \) by the rule \( c_\gamma^*(x) = (c_\gamma^* \circ r_{q-1})(x) \) when \( \gamma \in \Delta_q \).

As it is well known from [3] and [7], instead of the Schauder basis of \( X_{BD} \), it is more convenient to work with a FDD naturally defined as follows:

For each \( q \in \mathbb{N} \) we set \( M_q = i_q[\ell^\infty(\Delta_q)] \).

We briefly establish this fact in the following proposition and then continue with the details of the construction of \( X_r \).

**Proposition 2.2.** The sequence \((M_q)_{q \in \mathbb{N}}\) is a FDD for \( X_{BD} \).

**Proof.** For \( q \geq 0 \) we define the maps \( P_{\{q\}} : X_{BD} \to M_q \) with
\[
P_{\{q\}}(x) = i_q(r_q(x)) - i_{q-1}(r_{q-1}(x))
\]

It is easy to check that each \( P_{\{q\}} \) is a projection onto \( M_q \) and that for \( q_1 \neq q_2 \) and \( x \in M_{q_2} \), we have \( P_{\{q_1\}}(x) = 0 \). Also we have that \( \|P_q\| \leq 2C \).

We point out that in a similar manner one can define projections on intervals of the form \( I = (p, q] \) so that \( P_I(x) = \sum_{i=p+1}^q P_{\{i\}}(x) \) for which we can readily verify the formula
\[
P_I(x) = i_q(r_q(x)) - i_p(r_p(x))
\]

Note that \( \|P_I\| \leq 2C \). This shows that indeed \((M_q)_q \) is a FDD generating \( X_{BD} \). \( \square \)

For \( x \in X_{BD} \) we denote by \( \text{supp} x \) the set \( \text{supp} x = \{ q : P_{\{q\}}(x) \neq 0 \} \) and by \( \text{ran} x \) the minimal interval of \( \mathbb{N} \) containing \( \text{supp} x \).

**Definition 2.3.** A block sequence \((x_i)_{i=1}^\infty \) in \( X_{BD} \) is called skipped (with respect to \((M_q)_{q \in \mathbb{N}}\)), if there is a subsequence \((q_i)_{i=1}^\infty \) of \( \mathbb{N} \) so that for all \( i \in \mathbb{N} \), \( \max \text{supp} x_i < q_i < \min \text{supp} x_{i+1} \).

In the sequel, when we refer to a skipped block sequence, we consider it to be with respect to the FDD \((M_q)_{q \in \mathbb{N}}\).

Let \( q \geq 0 \). For all \( \gamma \in \Delta_q \) we set \( d_\gamma^* = e_\gamma \circ P_{\{q\}} \). Then the family \((d_\gamma^*)_{\gamma \in \Gamma} \) consists of the biorthogonal functionals of the FDD \((M_q)_{q \geq 0} \). Notice that for \( \gamma \in \Delta_q \),
\[
d_\gamma^*(x) = P_q(x)(\gamma) = i_q(r_q(x))(\gamma) - i_{q-1}(r_{q-1}(x))(\gamma) =
\]
\[
= r_q(x)(\gamma) - c_\gamma^*(r_{q-1}(x)) = x(\gamma) - c_\gamma^*(x) =
\]
\[
= c_\gamma^*(x).
\]

The sequences \((\Delta_q)_{q \in \mathbb{N}} \) and \((c_\gamma^*)_{\gamma \in \Gamma} \) are determined as in [3], section 4 and Theorem 3.5.

We give some useful notation. For fixed \( n \in \mathbb{N} \) and \( \overline{b} = (b_1, b_2, \ldots, b_n) \) with \( 0 < b_1, b_2, \ldots, b_n < 1 \), for each \( \gamma \in \Delta_q \) we assign
\[
(a) \ \text{rank} \gamma = q
\]
(b) age of \( \gamma \) denoted by \( a(\gamma) = a \) such that \( 2 \leq a \leq n \)
(c) weight of \( \gamma \) denoted by \( w(\gamma) = b_a \)

In order to proceed to the construction, we first need to fix a positive integer \( n \) and a descending sequence of positive real numbers \( b_1, \ldots , b_n \) such that \( b_1 < 1, b_i < \frac{1}{n}, \) for every \( i = 2, \ldots , n \) and \( \sum_{i=1}^{n} b_i > 1. \) Let \( r \in (1, \infty) \) be such that \( \sum_{i=1}^{n} b_i' = 1 \) and \( \frac{1}{b_1} + \frac{1}{b_n} = 1. \) Now we shall define the space \( X_r \) by using the Bourgain-Delbaen construction that was presented in the preceding paragraphs.

We set \( \Delta_0 = \emptyset, \Delta_1 = \{ 0 \} \) and recursively define for each \( q > 1 \) the set \( \Delta_q. \)

Assume that \( \Delta_p \) have been defined for all \( p \leq q. \) We set

\[
\Delta_{q+1} = \{ (q+1, a, p, \eta, \varepsilon e^*_\xi) : 2 \leq a \leq n, p \leq q, \varepsilon = \pm 1, \ e^*_\xi \in S_{\ell^1(\Gamma_q)}, \ \xi \in \Gamma_q \setminus \Gamma_p, \eta \in \Gamma_p, \ b_{a-1} = w(\eta) \}.
\]

For \( \gamma \in \Delta_{q+1} \) it is clear that the first coordinate is the rank of \( \gamma, \) while the second is the age \( a(\gamma) \) of \( \gamma. \) The functionals \( (c^*_\gamma)_{\gamma \in \Delta_{q+1}} \) are defined in a way that depends on \( \gamma \in \Delta_{q+1}. \) Namely, let \( x \in \ell^\infty(\Gamma_q). \)

(i) For \( \gamma = (q+1, 2, p, \eta, \varepsilon e^*_\xi) \) we set

\[
c^*_\gamma(x) = b_1 x(\eta) + b_2 \varepsilon e^*_\xi(x - i_{p,q}(r_p(x))).
\]

(ii) For \( \gamma = (q+1, a, p, \eta, \varepsilon e^*_\xi) \) with \( a > 2 \) we set

\[
c^*_\gamma(x) = x(\eta) + b_a \varepsilon e^*_\xi(x - i_{p,q}(r_p(x))).
\]

We may now define sequences \( (i_q), (\Gamma_q), (X_q) \) in a similar manner as before and set \( X_r = \bigcup_{q \in \mathbb{N}} X_q. \) Assuming that \( (i_q) \) is uniformly bounded by a constant \( C, \) we conclude that the space \( X_r \) is a subspace of \( \ell^\infty(\Gamma). \) The constant \( C \) is determined as in \cite{[3]} Theorem 3.4, by taking \( C = \frac{1}{2} \). Thus, for every \( m \in \mathbb{N}, \|i_m\| \leq C. \) This implies that \( \|P_I\| \leq 2C \) for every \( I \) interval.

**Remark 2.4.** In the case of \( n = 2, \) i.e. \( \overline{b} = (b_1, b_2), \) the space \( X_r \) essentially coincides with the Bourgain-Delbaen space \( X_{b_1, b_2}, \) since every \( \gamma \in \Gamma \) is of age 2.

**Remark 2.5.** As it is shown in Proposition 6.2, the choice of \( r, \) based on the fixed \( n \) and \( \overline{b}, \) yields that \( \mathcal{T}(A_n, \overline{b}) \cong \ell_r. \) Moreover, the ingredients of the "Tsirelson type spaces" theory that are used throughout this paper are essentially the same with the corresponding ones in \cite{[3]}. The basic difference in our approach is that we use only one family \( \mathcal{T}(A_n, \overline{b}) \) for some appropriate \( n \) and \( \overline{b}. \)
3. The Tree Analysis of $e^*_\gamma$ for $\gamma \in \Gamma$

We begin by recalling the analysis of $e^*_\gamma$ in [3] section 4. The only difference is that in our case all the functionals $e^*_\gamma$ have weight depending on their age which is greater or equal to 2.

3.1. The evaluation Analysis of $e^*_\gamma$ for $\gamma \in \Gamma$. First we point out that for $q \in \mathbb{N}$ every $\gamma \in \Delta_{q+1}$ admits a unique analysis as follows:

Let $a(\gamma) = a \leq n$. Then using backwards induction we determine a sequence of sets $\{p_i, q_i, \varepsilon_i e^*_\xi_i\}_{i=1}^a \cup \{\eta_i\}_{i=2}$ with the following properties.

(i) $p_1 < q_1 < \cdots < p_a < q_a = q$.
(ii) $\varepsilon_i = \pm 1$, $\operatorname{rank} \xi_i \in (p_i, q_i]$ for $1 \leq i \leq a$ and $\operatorname{rank} \eta_i = q_i + 1$ for $2 \leq i \leq a$.
(iii) $\eta_1 = \gamma$, $\eta_i = (\operatorname{rank} \eta_i, i, p_i, \eta_{i-1}, \varepsilon_i e^*_\xi_i)$ for every $i > 2$.

Moreover, following similar arguments as in [3] Proposition 4.6 it holds that,

$$e^*_\gamma = \sum_{i=2}^a d^*_{\eta_i} + \sum_{i=1}^a b_i \varepsilon_i e^*_\xi_i \circ P_{[p_i, q_i]} = \sum_{i=2}^a e^*_{\eta_i} \circ P_{[q_i, q_i+1]} + \sum_{i=1}^a b_i \varepsilon_i e^*_\xi_i \circ P_{[p_i, q_i]}.$$

We set $g_\gamma = \sum_{i=2}^a d^*_{\eta_i}$ and $f_\gamma = \sum_{i=1}^a b_i \varepsilon_i e^*_\xi_i \circ P_{[p_i, q_i]}$.

3.2. The r-Analysis of the functional $e^*_\gamma$. Let $r \in \mathbb{N}$ and $\gamma \in \Delta_{q+1}$. Let $a(\gamma) = a \leq n$ and $\{p_i, q_i, \varepsilon_i e^*_\xi_i\}_{i=1}^a \cup \{\eta_i\}_{i=2}$ the evaluation analysis of $\gamma$. We define the r-analysis of $e^*_\gamma$ as follows:

(a) If $r \leq p_1$, then the r-analysis of $e^*_\gamma$ coincides with the evaluation analysis of $e^*_\gamma$.
(b) If $r \geq q_a$, then we assign no r-analysis to $e^*_\gamma$ and we say that $e^*_\gamma$ is r-indecomposable.
(c) If $p_1 < r < q_a$, we define $i_r = \min\{i : r < q_i\}$. Note that this is well-defined. The r-analysis of $e^*_\gamma$ is the following triplet

$$\{(p_i, q_i)_{i \geq i_r}, \{\varepsilon_i \xi_i\}_{i \geq i_r}, \{\eta_i\}_{i \geq \max(2, i_r)}\}.$$

where $p_i$ is either the same or $r$ in the case that $r > p_i$.

Next we introduce the tree analysis of $e^*_\gamma$ which is similar to the tree analysis of a functional in a Mixed Tsirelson space (see [3] Chapter II.1). Notice that the evaluation analysis and the r-analysis of $e^*_\gamma$ form the first level of the tree analysis that we are about to present.

We start with some notation. We denote by $(\mathcal{T}, \leq \mathcal{T})$ a finite partially ordered set which is a tree. Its elements are finite sequences of natural
numbers ordered by the initial segment partial order. For every \( t \in \mathcal{T} \), we denote by \( S_t \) the immediate successors of \( t \).

Assume now that \( (p_t, q_t)_{t \in \mathcal{T}} \) is a tree of intervals of \( \mathbb{N} \) such that \( t \preceq s \) \iff \( (p_t, q_t) \supset (p_s, q_s) \) and \( t, s \) are incomparable \iff \( (p_t, q_t) \cap (p_s, q_s) = \emptyset \). For such a family \( (p_t, q_t)_{t \in \mathcal{T}} \) and \( t, s \) incomparable we shall denote by \( t < s \) \iff \( (p_t, q_t) < (p_s, q_s) \) (i.e. \( q_t < p_s \)).

### 3.3. The Tree Analysis of the Functional \( e^*_\gamma \)

Let \( \gamma \in \Delta_{q+1} \) with \( a(\gamma) = a \leq n \). A family of the form \( \mathcal{F}_\gamma = \{ \xi_t, (p_t, q_t) \}_{t \in \mathcal{T}} \) is called the tree analysis of \( e^*_\gamma \) if the following are satisfied:

1. \( \mathcal{T} \) is a finite tree with a unique root denoted as \( \emptyset \).
2. We set \( \xi_\emptyset = \gamma, (p_\emptyset, q_\emptyset) = (1, q) \) and let \( \{p_t, q_t, e^*_\xi_t\}_{t \in \mathcal{T}} = \emptyset \cup \{ \eta_t \}_{t \in \mathcal{T}} \) the evaluation analysis of \( \xi_\emptyset \). Set \( S_\emptyset = \{ (1), (2), \ldots, (a) \} \) and for every \( s = (i) \in S_\emptyset \), \( \{ \xi_s, (p_s, q_s) \} = \{ \xi_i, (p_t, q_t) \} \).
3. Assume that for a \( t \in \mathcal{T} \) \( \{ \xi_t, (p_t, q_t) \} \) has been defined. There are two cases:

   a. If \( e^*_\xi_t \) is \( p_t \)-decomposable, let
      \[
      \{ (p_i, q_i) \}_{i \geq i_{p_t}}, \{ \xi, \ddot{s} \}_{i \geq i_{p_t}}, \{ \eta_i \}_{i \geq \max \{ 2, i_{p_t} \}}
      \]
      the \( p_t \) analysis of \( e^*_\xi_t \). We set \( S_t = \{ (t \ddott{i}) : i \geq i_{p_t} \} \) and
      \[
      S_t = S_t^a, \text{ if } \eta_{i_{p_t}} \text{ exists }\]
      \[
      S_t = S_t \setminus \{ (t \ddott{i_{p_t}}) \}, \text{ otherwise}
      \]
      Then, for every \( s = (t \ddott{i}) \in S_t \), we set \( \{ \xi_s, (p_s, q_s) \} = \{ \xi_i, (p_t, q_t) \} \) where \( \{ \xi, \ddot{s} \}_{i \geq i_{p_t}} \) is a member of the \( p_t \) analysis of \( e^*_\xi_t \).

   b. \( e^*_\xi_t \) is \( p_t \)-indecomposable, then \( \xi_t \) consists a maximal node of \( \mathcal{F}_\gamma \).

**Notation 3.2.** For later use we need the following:

For every \( t \in \mathcal{T} \), \( e^*_\xi_t = f_t + g_t \), where \( f_t = \sum_{s \in S_t} b_s \xi_s e^*_s \circ P_{(p_s, q_s)} \) and \( g_t = \sum_{s \in S_t} d^*_s \eta_{s_{t-1}} \) and for \( s = (t \ddott{i}) \in S_t^a \), \( \eta_{(t \ddott{i})} = (\text{rank } \eta_{(t \ddott{i-1})}, i, p_{(t \ddott{i-1})}, \eta_{(t \ddott{i-1})}, e^*_s(\xi_{(t \ddott{i-1})})) \).

In the rest of the paper, we set \( f_t = f_t^\gamma \) and \( g_t = g_t^\gamma \).

**Lemma 3.3.** Let \( x \in \mathbb{X} \) and \( \gamma \in \Gamma \). Then,

\[
\prod_{0 \leq s \leq t_x} (\xi_s b_s) (f_s + g_s) (x)
\]

where \( t_x = \max \{ t : \text{ran } x \subseteq (p_t, q_t) \} \).

**Proof.** Let \( \mathcal{F}_\gamma = \{ \xi_t, (p_t, q_t) \}_{t \in \mathcal{T}} \) a tree analysis of \( \gamma \).

If \( \{ t : \text{ran } x \subseteq (p_t, q_t) \} = \emptyset \), then \( e^*_\gamma (x) = f_0 (x) + g_0 (x) \) and the equality holds.

If \( \{ t : \text{ran } x \subseteq (p_t, q_t) \} \neq \emptyset \), we can find \( \{ t_1 < t_2 < \ldots < t_m \} \in \mathcal{T} \) such that \( t_1 \in S_\emptyset \) and \( t_m = t_x \).
For every \( t \in \mathcal{T} \) with \( t < t_x \), \( g_t(x) = 0 \). Indeed, for every \( s \in S^\text{pr}_t \), \( d^*_m(x) = e^*_n \circ P_{g_{t_x+1}}(x) = 0 \) because \( \text{ran } x \subseteq (p_{t_x}, q_{t_x}) \subseteq (p_x, q_x) \).

So, we have that

\[
e^*_t(x) = f_0(x) = \sum_{s \in S_0} b_s \varepsilon_x e^*_s \circ P_{(p_x, q_x)}(x) = b_t \varepsilon_x \varepsilon_t^* e^*_t(1)
\]

\[
= b_t \varepsilon_t^* f_t(x) = b_t \varepsilon_t^* b_t \varepsilon_t^* \circ P_{(p_t, q_t)}(x) = b_t b_t \varepsilon_t^* f_t(x) = \ldots = \prod_{\varepsilon_t^* b_t(x)} (p_t + q_t)(x)
\]

setting \( e_0 = b_0 = 1 \).

\[\Box\]

**Corollary 3.4.** If \((f_{t_x}, (p_{t_x}, q_{t_x}))\) is a maximal node, then \( e^*_t(x) = 0 \).

**Proof.** Let \((f_{t_x}, (p_{t_x}, q_{t_x}))\) be a maximal node. Then \( f_{t_x}(x) = 0 \) and \( g_{t_x}(x) = 0 \) and from Lemma 3.3 we deduce that \( e^*_t(x) = 0 \). \[\Box\]

### 4. The lower estimate

**Definition 4.1.** An \( \phi \in W(A_n, \mathcal{B}) \) is said to be a proper functional if it admits a tree analysis \( (\phi_t)_{t \in \mathcal{T}} \) such that for every non-maximal node \( t \in \mathcal{T} \) the set \( \{\phi_s : s \in S_t\} \) has at least two non-zero elements.

We denote by \( W_{pr}(A_n, \mathcal{B}) \) to be the subset of \( W(A_n, \mathcal{B}) \) consisting of all proper functionals. For every \( t \in \mathcal{T} \) it holds that \( \phi_t = \sum_{s \in S_t} b_s \phi_s \) with \( \{b_s\}_{s \in S_t} \subseteq \{b_1, b_2, \ldots, b_n\} \) and \( b_0 = 1 \).

**Lemma 4.2.** The set \( W_{pr}(A_n, \mathcal{B}) \) 1-norms the space \( \mathcal{T}(A_n, \mathcal{B}) \).

**Proof.** We shall show that for every \( \phi \in W(A_n, \mathcal{B}) \) there exists \( g \in W_{pr}(A_n, \mathcal{B}) \) such that \( |\phi(m)| \leq g(m) \) \( \forall m \in \mathbb{N} \). Since the basis is 1-unconditional the previous statement yields the result.

To this end, let \( \phi \in W(A_n, \mathcal{B}) \). Then using a tree analysis \( (\phi_t)_{t \in \mathcal{T}} \) of \( \phi \) we easily see that for every \( m \in \text{supp } f \), there exists a maximal node \( t_m \in \mathcal{T} \) with \( \phi_{t_m} = \varepsilon_m e^*_m \) and \( \phi(m) = \varepsilon_m \prod_{t \leq t_m} b_t \).

For every \( m \in \text{supp } \phi \) we set \( K_m = \{t \in \mathcal{T} : t < t_m \text{ and } \#S_t > 1\} \). Then it is easy to see that the functional \( g = \sum_{m \in \text{supp } \phi} (\prod_{t \in K_m} b_t) e^*_m \) is a functional belonging to \( W_{pr}(A_n, \mathcal{B}) \). Moreover, since \( b_t < 1 \) for every \( t \in \mathcal{T} \) we get that \( |\phi(m)| \leq g(m) \) \( \forall m \in \mathbb{N} \). \[\Box\]

**Lemma 4.3.** Let \( \phi \in W_{pr}(A_n, \mathcal{B}) \) and \( l \in \mathbb{N} \). If \( \text{maxsupp } \phi = l \), then \( h(\mathcal{T}_\phi) \leq l \).

**Proof.** Let \( \theta_n \) be the amount of nodes at the \( n \) level of \( \mathcal{T}_\phi \). Since \( \phi \) is proper, it holds that \( \theta_{n+1} > \theta_n \) for every \( n \in \mathbb{N} \). Assume to the contrary that \( h(\mathcal{T}_\phi) > l \), i.e. \( h(\mathcal{T}_\phi) = l + k \) for some \( k \in \mathbb{N} \). Then,

\[\theta_1 = 1, \quad \theta_2 = 2, \quad \ldots, \theta_{l+k} \geq l + k\]
Since, the $l + k$ level of $T_{\phi}$ consists of functionals of the form $e_i^\ast$, we deduce that $\maxsupp(\phi) \geq l + k > l$, which leads to a contradiction.

**Proposition 4.4.** Let $(x_k)_{k \in \mathbb{N}}$ be a normalized skipped block sequence in $X_r$ and $(q_k)_{k \in \mathbb{N}}$ a strictly increasing sequence of integers such that $\text{supp} x_k \subset (q_k + k, q_{k+1})$. Then, for every sequence of positive scalars $(a_k)_{k \in \mathbb{N}}$ and for every $l \in \mathbb{N}$, it holds that

$$
\| \sum_{k=1}^l a_k e_k \|_{T(A_n, b)} \leq C \| \sum_{k=1}^l a_k x_k \|_{\infty}
$$

where $(e_k)_{k \in \mathbb{N}} \subseteq T(A_n, b)$ and $C$ is an upper bound for the norms of the operators $i_m$ in $X_r$.

**Proof.** Let $\phi \in W(A_n, b)$. From Lemma 4.2, we may assume that $\phi$ is proper. We will use induction on the height of the tree $T_{\phi}$.

If $h(T_{\phi}) = 0$ (i.e., $f$ is maximal), then $\phi$ is of the form $\phi = \varepsilon_k e_k^\ast$ with $\varepsilon_k = \pm 1$. We observe that $|\phi(\sum_{k=1}^l a_k e_k)| = |a_k| = a_k$. From Proposition 4.8, we can choose $\gamma \in \Gamma_{q_{l+1} - 1 - 1} \setminus \Gamma_{q_k + k}$ such that $|x_k(\gamma)| \geq \frac{1}{C} \|x_k\| = \frac{1}{C}$. Then, $|\phi(\sum_{k=1}^l a_k e_k)| = a_k \leq C|a_k||x_k(\gamma)| = C|e_k^\ast(a_k x_k)| \leq C|e_k^\ast(\sum_{k=1}^l a_k x_k)|$. We assume that for every $\phi \in W(A_n, b)$ with $h(T_{\phi}) = h > 0$ and $\maxsupp(\phi) = l_0$, there exists $\gamma \in \Gamma$, such that:

1. $\gamma \in \Gamma_{q_{l_0 + 1} - 1} \setminus \Gamma_{q_0 + 1}
2. h(T_{\phi}) = h(F_{\gamma}) \leq l_0
3. |\phi(\sum_{k=1}^l a_k e_k)| \leq C|\sum_{k=1}^l a_k x_k(\gamma)|$ for every $l \geq l_0$

Observe that assumption (1) yields $x_{l_0} < \text{rank} \gamma < x_{l_0 + 1}$, while assumption (2) gives us that $\minsupp x_{l_0 + 1} - \maxsupp x_{l_0} > h(T_{\phi})$. Indeed,

$$
x_{l_0} < q_{l_0 + 1} < \text{rank} \gamma \leq q_{l_0 + 1} + h \leq q_{l_0 + 1} + l_0 < q_{l_0 + 1} + (l_0 + 1) < x_{l_0 + 1}
$$

and $\minsupp x_{l_0 + 1} - \maxsupp x_{l_0} > l_0 + 1 > l_0 \geq h(F_{\gamma})$.

Let $\phi \in W(A_n, b)$ with $h(T_{\phi}) = h + 1$, $l_0 = \maxsupp(\phi)$ and let $(\phi_t)_{t \in \mathcal{T}}$ be the tree analysis of $\phi$. Then, $\psi$ is of the form $\psi = \sum_{s \in S} b_s \phi_s$, $\#S \leq n$. We observe that for every $s \in S \setminus \{1\}$, $h(T_{\phi_s}) = h$. We set $p_s = 1$, for every $s \in S \setminus \{1\}$, $p_s = \min\{q_k + k : k \in \text{supp} \phi_s\}$ and for every $s \in S \setminus \{1\}$, $r_s = q_{l_0 + 1} + h$ where $l_0 = \maxsupp(\phi_s)$.

We next apply the inductive hypothesis to obtain $\xi_s \in \Gamma_{r_s} \setminus q_{l_s + 1}$ with $h(T_{\phi_s}) = h(F_{\xi_s})$ such that

$$
|\phi_s(\sum_{k=1}^l a_k e_k)| = |\phi_s(\sum_{k \in \text{supp} \phi_s} a_k e_k)| \leq C\varepsilon_s \sum_{k \in \text{supp} \phi_s} a_k x_k(\xi_s)
$$

$$
= C\varepsilon_s e_{\xi_s}^\ast(\sum_{k \in \text{supp} \phi_s} a_k x_k) = C\varepsilon_s e_{\xi_s}^\ast \circ P_{\{p_s, r_s\}}(\sum_{k=1}^l a_k x_k),
$$

with $\varepsilon_s$ such that $e_{\xi_s}^\ast(\sum_{k \in \text{supp} \phi_s} a_k x_k) = |\sum_{k \in \text{supp} \phi_s} a_k x_k(\xi_s)|$. 


Let $\gamma \in \Gamma$ have analysis $\{p_s, r_s, \varepsilon_s e^{\ast}_{\xi_s}\}_{s \in S_0} \cup \{\eta_s\}_{s \in S_0 \setminus \{1\}}$ where $\eta_s \in \Delta_{r_{s+1}}$. Observe that rank $\xi_s \in (q_{l_s+1}, r_s] \subset (p_s, r_s]$. It is clear that for every $s \in S_0 \setminus \{1\}$, $d_{\eta_s}(\sum_{k=1}^l a_k x_k) = 0$. Indeed, support $x_{l_s} < q_{l_s+1} < q_{s+1} + (h + 1) = r_s + 1 \leq q_{l_s+1} + (l_s + 1) <$ support $x_{l_s+1}$.

Therefore,

$$|\phi(\sum_{k=1}^l a_k e_k)| \leq \sum_{s \in S_0} |b_s \phi_s(\sum_{k \in \text{supp} \phi_s} a_k e_k)| \leq C \sum_{s \in S_0} b_s e_s e^{\ast}_{\xi_s} \circ P_{\{p_s, r_s\}}(\sum_{k=1}^l a_k x_k)
\leq C |\sum_{k=1}^l a_k x_k(\gamma)|$$

It is clear that $h(T_{\phi}) = h(F_{\gamma}) \leq l_0$ and $x_{l_0} < \text{rank} \gamma < x_{l_0+1}$. \hfill $\square$

**Corollary 4.5.** For every block sequence in $\mathfrak{X}_r$ there exists a further block sequence satisfying inequality (11).

5. The upper estimate

Let $(y_i)_{i \in \mathbb{N}}$ be a normalized skipped block sequence in $\mathfrak{X}_r$. From Corollary 4.5, we can find a further block sequence of $(y_i)_j$, still denoted by $(y_i)_j$, satisfying inequality (11).

Therefore, we have that

$$\|\sum_{i=1}^m y_i\|_{\infty} \geq \frac{1}{C} \|\sum_{i=1}^m e_i\|_{\mathcal{T}(A_n, \mathfrak{b})}$$

For every $j \in \mathbb{N}$, set $M_j = \{1, 2, \ldots, n^j\}$. It is easily checked, after identifying $M_j$ with $\{1, 2, \ldots, n\}$, that the functional $f_j = \sum_{s \in M_j} (\prod_{i=1}^{n_j} b_{s_i}) e^{\ast}_s$ belongs to $\mathcal{W}(A_n, \mathfrak{b})$ where $s_i$ is the $i$-th coordinate of $s$, for each $i = 1, 2, \ldots, n$ and $\sum_{s \in M_j} (\prod_{i=1}^{n_j} b_{s_i}) e^{\ast}_s$. Using the fact that $\#M_j = n^j$, we obtain that

$$\|\sum_{i=1}^{n^j} e_i\|_{\mathcal{T}(A_n, \mathfrak{b})} = \|\sum_{s \in M_j} e_s\|_{\mathcal{T}(A_n, \mathfrak{b})} \geq f_j(\sum_{i=1}^{n^j} e_i) = (\sum_{i=1}^{n^j} b_i)^j.$$

Also, for every $m \in \mathbb{N}$ large enough we may find $j \in \mathbb{N}$ such that $n^{j+1} > m > n^j$. From the above and the unconditionality of the basis of the space $\mathcal{T}(A_n, \mathfrak{b})$, it follows that

$$\|\sum_{i=1}^m y_i\|_{\infty} \geq \frac{1}{C} \|\sum_{i=1}^m e_i\|_{\mathcal{T}(A_n, \mathfrak{b})} \geq \frac{1}{C} \|\sum_{i=1}^{n^j} e_i\|_{\mathcal{T}(A_n, \mathfrak{b})} = (\sum_{i=1}^{n^j} b_i)^j.$$

We conclude that $\|\sum_{i=1}^m y_i\|_{\infty} \to \infty$ as $\sum_{i=1}^m b_i > 1$.

We next choose a further block sequence $(x_k)_{k \in \mathbb{N}}$ of $(y_i)_{i \in \mathbb{N}}$ with some additional properties. Let $\varepsilon > 0$ and choose a descending sequence $(\varepsilon_k)_{k}$ of positive reals such that $\sum_{k=1}^{\infty} \varepsilon_k < \varepsilon$. We can also find an increasing
sequence \((n_k)_k\) of positive integers and a sequence \((F_k)_k\) of successive subsets of \(\mathbb{N}\) such that the following are satisfied:

1. For every \(k \in \mathbb{N}\), \(\frac{1}{n_k} < \varepsilon_k\).
2. For every \(k \in \mathbb{N}\), \(\|\sum_{l \in F_k} y_l\| > n_k\). This is possible, due to the above notation.

We have thus constructed a normalized skipped block sequence \((x_k)_{k \in \mathbb{N}}\) of the form \(x_k = \sum_{l \in F_k} \lambda_l y_l\), where \(\lambda_l = \frac{1}{\|\sum_{i \in F_k} y_l\|}\). Notice that \(|\lambda_l| < \varepsilon_k\) for every \(l \in F_k\).

Let \(\gamma \in \Gamma\) with tree analysis \(F_\gamma = \{\xi_t, (p_t, q_t)\}_{t \in T}\). For every \(k \in \mathbb{N}\), we set \(t_0 = \max\{t : \text{ran } x_k \subseteq (p_t, q_t)\}\). Notice that if for a given \(x_k\), \(t_0\) is non-maximal, then there exist at least two immediate successors of \(t_0\), say \(s_1, s_2\) such that the corresponding intervals \((p_{s_1}, q_{s_1})\), \((p_{s_2}, q_{s_2})\) intersect \(\text{ran } x_k\). For later use we shall denote by \((p_{s_0}, q_{s_0})\) the first interval in the natural order of disjoint segments of the natural numbers that intersects \(x_0\). Notice that \(s_0\) is not necessarily the first element of \(S_t\).

For the pair \(\gamma, (x_k)_{k \in \mathbb{N}}\) and for every \(t \in T\) we define the following sets:
\[D_t = \bigcup_{s \not\approx t}\{k : s = t_0\},\]
\[K_t = D_t \setminus \bigcup_{s \not\subseteq S_t} D_s = \{k : t = t_0\}\] and \(E_t = \{s \in S_t : D_s \neq \emptyset\}\).

We now set \(x_k = x'_k + x''_k + x'''_k\) where,
\[x'_k = x_k \mid (p_{s_0}, q_{s_0})\], \(x''_k = x_k \mid \bigcup_{s \in S_{t_0} \not\subseteq (p_{s_0}, q_{s_0})} (p_s, q_s)\) and \(x'''_k = x_k - x'_k - x''_k\).

Remark 5.1. (1) The sets \(D_t, K_t, E_t\) are determined by the chosen pair \(\gamma, (x_k)_k\). For a different pair, these sets may differ as well. For example, let \(k \in K_t\), for the pair \(\gamma, (x_k)_k\). Then \(t = t_0\) for \(x_k\). By the construction of \(x'_k\), there exists \(s_k \in S_t\) such that \(x'_k = x_k \mid (p_{s_k}, q_{s_k})\).

Thus, taking the pair \(\gamma, (x'_k)_k\) the same \(k\) belongs to \(K_{s_k}\).

(2) For every \(k \in \mathbb{N}\), \(|g_{t_k}(x_k)| \leq 2Cn \varepsilon_k\).

Indeed, from the definition of \((x_k)_{k \in \mathbb{N}}\) we have that
\[
|g_{t_k}(x_k)| \leq \sum_{s \in S_{t_k}^{p_k}} |d_{t_k}^*(x_k)| \leq \sum_{s \in S_{t_k}^{p_k}} |e_{t_k}^* \circ P_{\{q_{s+1}\}}(\sum_{l \in F_k} \lambda_l y_l)| \leq \sum_{s \in S_{t_k}^{p_k}} \|e_{t_k}^*\| \|P_{\{q_{s+1}\}}\| \|\lambda_l y_l\| \leq \sum_{s \in S_{t_k}^{p_k}} 2C \varepsilon_k \leq 2C \varepsilon_k (\sharp S_{t_k}) \leq 2Cn \varepsilon_k.
\]

(3) It is obvious that \(g_{t_k}(x_k) = g_{t_k}(x''_k), f_{t_k}(x'''_k) = 0\) and for every \(t < t_k\), \(g_t(x''_k) = 0\).

Lemma 5.2. For the pairs \(\gamma, (x'_k)_k\) and \(\gamma, (x''_k)_k\) it holds that \(#(K_t \cup E_t) \leq n\).

Proof. Let \(t \in T\) and let \(k \in K_t\).

We set \(s_k = \max\{s \in S_t : (p_s, q_s) \cap \text{ran } x'_k \neq \emptyset\}\). From the definition of \(t_k\),
notice that $\#S_t \geq 2$. It holds that $s_k \not\in E_t$.
Indeed, from the definition of $t_k$, $s_k$ we have that $(p_{s_k}, q_{s_k}) \cap \gamma_k = \gamma_k$. Since $s_k \in S_{t_k}$, $(p_{s_k}, q_{s_k}) \subseteq (p_{t_k}, q_{t_k})$. It follows that $(p_{s_k}, q_{s_k}) \subseteq \gamma_k$.
Therefore, we can define a one-to-one map $G : \gamma \to S_{t} \setminus E_{t}$, hence $\#\gamma + \#E_{t} \leq \#S_{t} \leq n$.
The proof for the pair $\gamma, (x''_k)_{k \in \mathbb{N}}$ is similar. \hfill $\Box$

**Proposition 5.3.** Let $(x_k)_{k \in \mathbb{N}}$ be as above. Then for every $\gamma \in \Gamma$ there exist $\phi_1, \phi_2 \in W(\mathcal{A}_n, \overline{b})$ such that for every sequence $(a_k)_{k \in \mathbb{N}}$ of positive scalars, for every $l \in \mathbb{N}$ it holds that,

\[
\left| \sum_{k=1}^{l} a_k x_k(\gamma) \right| \leq \frac{1}{b_n} \left( \phi_1 + \phi_2 \right) \left( \sum_{k=1}^{l} a_k e_k \right) + 2Cn \varepsilon \left( \sum_{k=1}^{l} a_k^r \right)\]

**Proof.** Let $\gamma \in \Delta_{q+1}$ with $a(\gamma) = a \leq n$. Let $\mathcal{F}_\gamma = \{ \xi_t, (p_t, q_t) \}_{t \in \mathcal{T}}$, where $\xi_0 = \gamma$, be the tree analysis of $\gamma$. We may assume that $\bigcup_{k=1}^{l} \gamma_k \subseteq (p_0, q_0)$.

**Claim.** For the pairs $\gamma, (x'_k)_{k \in \mathbb{N}}$ and $\gamma, (x''_k)_{k \in \mathbb{N}}$ there exist $\phi_1, \phi_2 \in W(\mathcal{A}_n, \overline{b})$ such that for every sequence of positive scalars $(a_k)_{k \in \mathbb{N}}$ and for every $l \in \mathbb{N}$, it holds that

\[
\left| f_\overline{b} \left( \sum_{k=1}^{l} a_k x'_k \right) \right| \leq \frac{2C}{b_n} \phi_1 \left( \sum_{k=1}^{l} a_k e_k \right)
\]

\[
\left| f_\overline{b} \left( \sum_{k=1}^{l} a_k x''_k \right) \right| \leq \frac{2C}{b_n} \phi_2 \left( \sum_{k=1}^{l} a_k e_k \right)
\]

**Proof of the Claim.** We only prove inequality [3]. The proof of inequality [4] requires the same arguments. We recall that $f_t = \sum_{s \in S_t} b_s e_s (f_s + g_s) \circ P_{(p_s, q_s)}$ for every $t \in \mathcal{T}$ non maximal. From the definition of $(x'_k)_{k \in \mathbb{N}}$, we have that $g_s \circ P_{(p_s, q_s)}(x'_k) = 0$ for every $s \in S_t$. Therefore, $f_t(\sum_{k \in D_t} a_k x'_k) = (\sum_{s \in S_t} b_s e_s f_s \circ P_{(p_s, q_s)}) (\sum_{k \in D_t} a_k x'_k)$. We will use backwards induction on the levels of the tree $\mathcal{T}$, i.e. we shall show that for every $t \in \mathcal{T}$ there exists $\phi^t_1 \in W(\mathcal{A}_n, \overline{b})$ with supp $\phi^t_1 \subseteq D_t$ such that

\[
\left| f_t \left( \sum_{k \in D_t} a_k x'_k \right) \right| \leq \frac{2C}{b_n} \phi^t_1 \left( \sum_{k \in D_t} a_k e_k \right)
\]

Let $0 < h \leq \max\{ |t| : t \in \mathcal{T} \}$
We assume that the proposition has been proved for all $t$ with $|t| = h$.
Let $t \in \mathcal{T}$ with $|t| = h - 1$. Then we have the following cases:

1. If $f_t$ is a maximal node, $f_t(\sum_{k \in D_t} a_k x'_k) = 0$, so there is nothing to prove. Indeed, $K = D_t$, therefore for every $k \in D_t$, from Corollary 3.3 $f_t(x'_k) = 0$ since $t = t_k$. 

(2) If $f_t$ is a non-maximal node, then
\[
 f_t((\sum_{k \in D_t} a_k x_k')) = (\sum_{s \in S_t} b_s \varepsilon_s f_s \circ P_{(\rho, g_0)})(\sum_{k \in D_t} a_k x_k') = \sum_{s \in S_t} b_s \varepsilon_s f_s(\sum_{k \in D_t} a_k x_k') + \sum_{k \in K}(\sum_{s \in S_t} b_s \varepsilon_s f_s)(a_k x_k').
\]
From the fact that, for every $k \in K_t$, $g_t(x_k') = 0$ we get that
\[
 |f_t(x_k')| = |x_k'(\xi_t)| \leq \|x_k'\| \leq 2C = 2C \varepsilon_k^*(e_k).
\]
Moreover, for $s \in E_t$ it holds that $|s| = h - 1$. For every $k \in D_s$, from the inductive hypothesis we obtain
\[
 |\sum_{s \in S_t} b_s f_s(x_k')| = |b_s f_s(x_k')| \leq b_s \frac{2C}{\beta_n} \phi_1^s(e_k).
\]
with $\phi_1^s \in W(A_n, b)$ and supp $\phi_1^s \subset D_s$.
We set $\phi_1^s = (\sum_{s \in E_t} b_s \phi_1^s + \sum_{k \in K_t} b_k e_k')$.
From Lemma 5.2 it is easily checked that $\phi_1^s \in W(A_n, b)$ and it holds that, $|f_t((\sum_{k \in D_t} a_k x_k'))| \leq 2C \phi_1^s(\sum_{k \in D_t} a_k e_k).
\]
\]
Recall that $e_\gamma^* (\sum_{k=1}^l a_k x_k) = g_\theta(\sum_{k=1}^l a_k x_k) + f_\theta(\sum_{k=1}^l a_k x_k)$. The fact that $g_\theta(\sum_{k=1}^l a_k x_k') = g_\theta(\sum_{k=1}^l a_k x_k'') = g_\theta(\sum_{k \in \{m: \tau_m \neq 0\}} a_k x_k''') = f_\theta(\sum_{k \in \{m: \tau_m = 0\}} a_k x_k''') = 0$ implies the following:
\[
 |e_\gamma^* \sum_{k=1}^l a_k x_k| \leq |g_\theta(\sum_{k \in \{m: \tau_m = 0\}} a_k x_k')| + |f_\theta(\sum_{k=1}^l a_k x_k')| + |f_\theta(\sum_{k \in \{m: \tau_m = 0\}} a_k x_k'')|.
\]
\]
\]
From Remark 5.1 we get that,
\[
 |g_\theta(\sum_{k \in \{m: \tau_m = 0\}} a_k x_k')| \leq \sum_{k \in \{m: \tau_m = 0\}} a_k |g_\theta(x_k')| \leq 2C |n| \sum_{k \in \{m: \tau_m = 0\}} a_k \varepsilon_k.
\]
From Lemma 3.3 and Remark 5.1 we have that,
\[
 |f_\theta(\sum_{k \in \{m: \tau_m \neq 0\}} a_k x_k''')| \leq \sum_{k \in \{m: \tau_m \neq 0\}} a_k (\prod_{t \leq t_k} b_t) |g_{t_k}(x_k''')| \leq 2C \frac{1}{2} \sum_{k \in \{m: \tau_m \neq 0\}} a_k \varepsilon_k \leq 2C |n| \sum_{k \in \{m: \tau_m \neq 0\}} a_k \varepsilon_k.
\]
Finally, we conclude that
\[
\left| \sum_{k=1}^{l} a_k x_k(\gamma) \right| \leq 2Cn \sum_{k \in \{m : t_m = \emptyset\}} a_k \varepsilon_k + \frac{2C}{b_n} \phi_1 \left( \sum_{k=1}^{l} a_k e_k \right)
\]
\[+ \frac{2C}{b_n} \phi_2 \left( \sum_{k=1}^{l} a_k e_k \right) + 2Cn \sum_{k \in \{m : t_m \neq \emptyset\}} a_k \varepsilon_k \]
\[\leq \frac{2C}{b_n} \left( \phi_1 + \phi_2 \right) \left( \sum_{k=1}^{l} a_k e_k \right) + 2Cn \max \{ a_k : k \in \mathbb{N} \} \left( \sum_{k=1}^{l} \varepsilon_k \right) \]
\[\leq \frac{2C}{b_n} \left( \phi_1 + \phi_2 \right) \left( \sum_{k=1}^{l} a_k e_k \right) + 2Cn \varepsilon \left( \sum_{k=1}^{l} a_k^r \right)^{\frac{1}{r}}.
\]
where in the last inequality we used the fact that the \( \ell_r \) norm dominates the \( c_0 \) norm. \( \square \)

**Remark 5.4.** From \([4]\) Theorem I.4, we know that \( \| \sum a_k e_k \|_{T(A_n, \overline{b})} \geq M \left( \sum a_k^r \right)^{\frac{1}{r}}. \) This result and the previous Proposition, yield that
\[
\left| \sum_{k=1}^{l} a_k x_k(\gamma) \right| \leq \frac{2C}{b_n} \left( \phi_1 + \phi_2 \right) \left( \sum_{k=1}^{l} a_k e_k \right) + \frac{2Cn \varepsilon}{M} \sum_{k=1}^{l} a_k e_k \|_{T(A_n, \overline{b})}.
\]
For \( \varepsilon = \frac{M}{nb_n}, \)
\[
\left| \sum_{k=1}^{l} a_k x_k(\gamma) \right| \leq \frac{6C}{b_n} \| \sum_{k=1}^{l} a_k e_k \|_{T(A_n, \overline{b})}.
\]
Therefore,
\[
(5) \quad \| \sum_{k=1}^{l} a_k x_k \|_{\infty} \leq \frac{6C}{b_n} \| \sum_{k=1}^{l} a_k e_k \|_{T(A_n, \overline{b})}.
\]

**Corollary 5.5.** For every block sequence in \( X_r \) there exists a further block sequence satisfying inequality (5).

### 6. The Main Result

**Proposition 6.1.** Let \((x_k)_{k \in \mathbb{N}}\) be a skipped block sequence in \( X_r \) satisfying \( \text{minsupp} x_{k+1} > \text{maxsupp} x_k + k \) and the conditions of Proposition 5.3. Then \((x_k)_{k \in \mathbb{N}}\) is equivalent to the basis of the Tsirelson space \( T(A_n, \overline{b}) \) for \( n \) and \( \overline{b} \) determined as before.

**Proof.** It is an immediate consequence of Propositions 4.4, 5.3 and Remark 5.4. \( \square \)
**Proposition 6.2.** The space $\mathcal{T}(\mathcal{A}_n, \overline{b})$ is isomorphic to $\ell_p$ for some $p \in (1, \infty)$.

**Proof.** In a similar manner as in [1] Theorem I.4, one can see that for every normalized block sequence $(x_k)_k$ of the basis $(e_j)_j$ and for every scalar sequence $(a_k)$ it holds that, $\| \sum a_k x_k \| \leq \frac{2}{\mu} \| \sum a_k e_k \|$. Zippin’s Theorem [12] yields that $\mathcal{T}(\mathcal{A}_n, \overline{b})$ is isomorphic to some $\ell_p$, for some $p \in (1, \infty)$.

**Remark 6.3.** An alternative proof could also be derived using the Results in Sections 4 and 5. Indeed, let $(y_l)_{l \in \mathbb{N}}$ be a skipped block sequence in $\mathcal{X}_r$. Then, there exists a further block sequence $(x_k)_{k \in \mathbb{N}}$ satisfying simultaneously the assumptions of Corollaries 4.5 and 5.5 Therefore, $(x_k)_{k \in \mathbb{N}}$ satisfies the assumptions of Proposition 6.1.

Let’s observe that every further block sequence $(z_k)_k$ of $(x_k)_k$ is also skipped block and satisfies Proposition 6.1, thus it is equivalent to the basis of the space $\mathcal{T}(\mathcal{A}_n, \overline{b})$. Hence, every block sequence $(z_n)_n$ of $(x_k)_k$ is equivalent to $(x_k)_k$. Zippin’s theorem [12] yields that the space $(x_k)_k >$ is isomorphic to some $\ell_p$. Therefore, $\mathcal{T}(\mathcal{A}_n, \overline{b}) \cong \ell_p$ for some $p \in (1, \infty)$.

In order to determine the exact value of $p$, we need the following Proposition.

**Proposition 6.4.** The space $\mathcal{T}(\mathcal{A}_n, \overline{b})$ is isomorphic to $\ell_r$ with $\frac{1}{r} + \frac{1}{r'} = 1$ and $\sum_{i=1}^n b_i^{r'} = 1$.

**Proof.** First, let observe that for every $x \in c_{00}$, $\|x\| \leq \|x\|_r$. We shall use induction on the cardinality of supp $x$. If $|\text{supp } x| = 1$, it is trivial. Assume that it holds for every $y \in c_{00}$ with $|\text{supp } y| \leq n$ and let $x \in c_{00}$ with $|\text{supp } x| = n + 1$. Then either $\|x\| = \|x\|_{\infty}$ or $\|x\| = \sum_{i=1}^n b_i \|E_i x\|$. for some appropriate subsets $E_1 < E_2 < \ldots < E_n$. In the first case, there is nothing to prove as for every $p \in [r, \infty)$ $\|x\|_{\infty} \leq \|x\|_p$. Therefore we only need to deal with the second case.

It suffices to observe that for every $i = 1, 2, \ldots, n$, the cardinality of supp $E_i x$ is less than supp $x$ and thus, using the inductive hypothesis along with Hölder’s inequality, we get that

$$\|x\| \leq \sum_{i=1}^n b_i \|E_i x\|_r \leq \left( \sum_{i=1}^n b_i^{r'} \right)^{\frac{1}{r'}} \left( \sum_{i=1}^n \|E_i x\|_r^{r'} \right)^{\frac{1}{r'}} = \|x\|_r.$$

By combining the preceding argument with Proposition 6.2, we conclude that $\mathcal{T}(\mathcal{A}_n, \overline{b})$ is isomorphic to $\ell_p$ for some $p \in [r, \infty)$.

For every $l \in \mathbb{N}$ set $M_l = \{1, 2, \ldots, n\}^l$. We have already mentioned that for every $l \in \mathbb{N}$ the functional $f_l = \sum_{s \in M_l} (\prod_{i=1}^l b_{s_i}) e_{s_i}$ belongs to $W(\mathcal{A}_n, \overline{b})$ where $s_i$ is the $i$-th coordinate of $s$, for each $i = 1, 2, \ldots, n$ and $\sum_{s \in M_l} \prod_{i=1}^l b_{s_i} = (\sum_{i=1}^n b_i)^l$. We set $a_s = \prod_{i=1}^l b_{s_i}$ and $x_l = \sum_{s \in M_l} a_s^l e_{s_i}.$
It is easily seen that for every \( l \in \mathbb{N} \), \( \| x_l \| = 1 \). Indeed,
\[
\| x_l \| \leq \| x_l \|_r = \left( \sum_{s \in M_l} a_r'(s)^{\frac{1}{r}} \right)^{\frac{1}{r}} = 1 = f_l(x_l) \leq \| x_l \|.
\]

We claim that for \( p' > r \) and every \( \varepsilon > 0 \) there exists \( l \in \mathbb{N} \) such that \( \| x_l \|_{p'} < \varepsilon \). If the claim holds we are done as \( p \) coincides with \( r \).

**Proof of the Claim:** Notice that for \( p' > r \),
\[
\sum_{i=1}^{n} b_i^{p'(1+\delta)} < \sum_{i=1}^{n} b_i^{p'} = 1.
\]
Thus, there exists \( l \in \mathbb{N} \) such that \( \left( \sum_{i=1}^{n} b_i^{p'(1+\delta)} \right)^{\frac{1}{p'}} < \varepsilon \).

**Theorem 6.5.** For every \( r \in (1, \infty) \) the space \( X_r \) is \( \ell_r \) saturated.

**Proof.** As it was mentioned in the above Remark, for every skipped block sequence in \( X_r \) we can find a further block sequence \( (x_k)_k \) such that the space \( \langle (x_k)_k \rangle \) is isomorphic to \( \ell_r \).

**Remark 6.6.** From the previous Theorem, we deduce that the space \( X_r \) is a separable \( L_\infty \) space which does not contain \( \ell_1 \). Therefore, the results of D.Lewis-C.Stegall \[10\] and A. Pelczyński \[11\] yields that \( X_r^* \) is isomorphic to \( \ell_1 \). Alternatively, one can use the corresponding argument of D. Alspach \[1\] and show directly that \( (M_q) \) is a shrinking FDD for \( X_r \). It then follows that \( (e^*_\gamma)_{\gamma \in \Gamma} \) is a basis for \( X_r^* \), equivalent to the usual \( \ell_1 \)-basis.

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