Creating Entanglement Using Integrals of Motion

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A quantum Galilean cannon is a 1D sequence of $N$ hard-core particles with special mass ratios, and a hard wall; conservation laws due to the reflection group $A_1N$ prevent both classical stochastization and quantum diffusion. It is realizable through specie-alternating mutually repulsive bosonic soliton trains. We show that an initial disentangled state can evolve into one where the heavy and light particles are entangled, and propose a sensor, containing $N_{\text{total}}$ atoms, with a $\sqrt{N_{\text{total}}}$ times higher sensitivity than in a one-atom sensor with $N_{\text{total}}$ repetitions.

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In a 1D system of hard particles, by tuning the ratios between the particle masses one can choose a variety of distinct regimes of motion [1,2]. While generic values result in thermalization, for some special ones the system maps to known multidimensional kaleidoscopes [3,4]. The outcome velocities then become determined by the initial velocities only, independent of the initial positions if the particle ordering is preserved. In the quantum version, the eigenstates are finite superpositions of plane waves, with no diffraction [5]. In particular a (run in reverse) Galilean cannon—a hard wall followed by a hard-core particles with mass ratios of $1 : \frac{1}{3} : \frac{1}{5} : \frac{1}{10} : \cdots : \frac{2}{N(N+1)}$—will evolve from a state where only the lightest particle is moving (approaching the others from infinity), to a state where all of the particles are moving away from the wall, with the same speeds. The final state of the heavy particles is very different from the initial one, yet highly predictable, presenting an opportunity: if created, a superposition of the final and the initial one, yet highly predictable, presenting an opportunity: if created, a superposition of the final and initial states would be a Schrödinger cat state [1]: just as the $\alpha$-particle controls the well-being of the cat, so the state of the lightest particle controls either (a) the rest of the particles or (b) the heaviest particle, after the others are detected. And yet the whole system is in a pure state. The stored entanglement stored is ready to be used, and we will propose an interferometric application.

Consider a system of $N$ 1D hard-core particles with masses $m_1$, $m_2$, $\ldots$, $m_N$ on the half-line $x > 0$, bounded by a hard wall at $x = 0$. The Hamiltonian is given by the kinetic energy,

$$
\hat{H} = -\sum_{i=1}^{N} \frac{k^2}{2m_i} \frac{\partial^2}{\partial x_i^2} ;
$$

the wavefunction $\Psi(x_1, x_2, \ldots, x_N)$ satisfies the boundary conditions

$$
\Psi\bigg|_{x_1=0} = \Psi\bigg|_{x_1=x_2} = \cdots = \Psi\bigg|_{x_{N-1}=x_N} = 0 .
$$

The coordinate transformation $x_i = \sqrt{\mu/m_i} z_i$ for $i = 1, 2, \ldots, N$, where $\mu$ is an arbitrary mass scale that can be chosen at will, converts the system to a single $N$-dimensional particle of mass $\mu$ moving inside a mirror-walled wedge formed by $N$ mirrors; the outward normalized normals to its mirrors are given by $n_i = -e_1$ and $n_i = \sqrt{m_i/(m_{i-1}+m_i)}e_{i-1} - \sqrt{m_{i-1}/(m_{i-1}+m_i)}e_i$ for $i = 2, 3, \ldots, N$, where $e_i$ is the unit vector along the $z_i$-axis.

For a generic set of masses, sequential reflections about the mirrors generate an infinite set of spatial transformations. However, for every full reflection group [7] of a regular multidimensional polyhedron (i.e. Platonic solid), there is a set of masses whose corresponding system of mirrors generates that group [4]. In these cases, the eigenstates of the system can be found exactly through Bethe ansatz [3,4]: they are given by finite linear combinations of plane waves. In particular, the “Galilean cannon” set of masses $m_1$, $m_2 = m_1/3$, $m_3 = m_1/6$, $\ldots$, $m_N = \frac{2}{N(N+1)} m_1$ corresponds to the symmetry group of a regular $N$-dimensional tetrahedron [5].

As the initial state for the reverse Galilean cannon (Fig. 1 insert), we take the tensor product of the Gaussian wavepackets for each particle, supplemented by its images needed to satisfy the boundary conditions [2]. The corresponding solution of the time-dependent Schrödinger equation with the Hamiltonian [1], subject to the boundary conditions [2] is

$$
\Psi(z_1, z_2, \ldots, z_N) = \left( \sum_{\vec{g}} (-1)^{P(\vec{g})} \hat{g} \right) \prod_{i=1}^{N} \psi(z_i, t | z_i^{(0)}, v_{z,i}, \sigma_{z,i}) ,
$$

where

$$
\psi(z, t | z^{(0)}, v_z, \sigma_z) = A(t) e^{-\frac{\{z-(z^{(0)})\}^2}{2\mu \sigma_z^2} - \frac{\text{Re} \{v_z (z^{(0)} - z) \}}{2\mu \sigma_z^2} + \text{Im} \{v_z (z^{(0)} - z) \}}},
$$

is the one-body Gaussian wavepacket expanding freely; $A(t) = \left(2\pi \sigma_z^2 [1 + (\text{i}ht)/(2\mu \sigma_z^2)]\right)^{-1/4}$ is the normalization constant; the operators $\hat{g}$ are the transformations.
of space generated by the sequential application of reflections about the $N$ mirrors given by the normals $\mathbf{n}_i$ described above. This solution is obtained in the same way as the Bethe eigenstates in the case of hard-wall kaleidoscopes \cite{1}, which, in turn, is a generalization of a general solution for kaleidoscopes with Robin’s boundary conditions \cite{2,3,4}, which was inspired by the Bethe ansatz solutions for a gas of bosons \cite{5,6,7}. The set of $\hat{g}$’s forms the reflection group $A_N$, containing $(N+1)!$ elements \cite{8}. $\mathcal{P}(\hat{g})$ is the parity of the group element, i.e. the parity of the number of reflections about the $N$ generating mirrors (given in particular by the normals $\mathbf{n}_i$ above) needed to produce this element.

Having obtained the general solution for the problem, let us return to the physical coordinates $x_1, x_2, \ldots, x_N$. For the sequence depicted in the inset of Fig. 1 (“the Galilean cannon run in reverse”), the initial velocities of the particles are $v_{x,N} = -V_y^{(0)} < 0$ and $v_{x,1} = v_{x,2} = \ldots = v_{x,N-1} = 0$. At the final stage of the process, each particle moves away from the wall with the same speed $V_x = \sqrt{m_N/MV_y^{(0)}} = V_y^{(0)}/N$, where $M = \sum_{i=1}^{N} m_i$ is the total mass of the system. The initial distances between the particles, and between the leftmost particle and the wall, are assumed much greater than the widths of their initial packets: $x_i^{(0)} \gg \sigma_{x,1}$ and $x_i^{(0)} - x_{i-1}^{(0)} \gg \max(\sigma_{x,i}, \sigma_{x,i-1})$ for $i = 2, 3, \ldots, N$. In this case, the initial state is close to a product state of individual non-overlapping states of finite support: the images, while formally present in the expression \cite{9}, will be exponentially small at $t = 0$ (but will come to prominence at later times, as the particle wavepackets move around and broaden).

If the initial multidimensional Gaussian wavepacket is sufficiently long, the various parts of the superposition \cite{10} will start overlapping at intermediate stages of the time evolution, forcing the particles to entangle—despite the closeness of the initial state to a product state. The most promising is the superposition between 1. the initial packet and 2. the “outgoing” one, where all the particles are moving with a velocity $+V_x$: here the state of the lightest particle controls the state of each of the heavier ones, including the heaviest. It turns out that for a properly tuned set of the initial conditions, there will be regions of space where these two waves spatially overlap and all other parts of \cite{10} are exponentially small. Indeed, one can show that if a classical trajectory passes through the point $(x_1^{(sc)}, x_2^{(sc)}, \ldots, x_N^{(sc)}) = (\ell, 3\ell, 6\ell, \ldots, N(\ell + 1/2))$, it will do so twice, once during the initial leg of the evolution and once during the final. (Here and below, $\ell$ is an arbitrary length scale, and the subscript ‘sc’ stands for “self-crossing.”) The distances between the particles increase linearly with the index: $x_j^{(sc)} - x_{j-1}^{(sc)} = j\ell$. At the exact middle point of the time evolution (which can be shown to equal the time when the lightest particle would hit the wall if there were no other particles present), the state around the point of self-crossing is close to

$$\Psi(x_1, x_2, \ldots, x_N) \approx e^{-i(m_NV_y^{(0)}(x_N - x_N^{(sc)}) + \phi)} + e^{+i(m_NV_y^{(0)}(x_N - x_N^{(sc)}) + m_NV_x(x_N - x_N^{(sc)}) + \phi)},$$

where the relative phase $\phi$ can be approximated, using the eikonal approximation, as $\hbar\dot{\phi} = 2m_1V_x^{(0)}\ell$. In the
state above, the coordinate of the lightest particle is entangled with the center-of-mass position for the remaining bodies. Below, we will use this entanglement as a way to improve the sensitivity of interferometric measurements. As for the “cat” per se, we have the position of the center of mass of the particles being spread over an \( N^2 \ell \) range. If this seems too abstract, one can also generate entanglement between two localized objects, one light and one heavy: suppose the intermediate particles 2, 3, ..., \( N - 1 \) have been detected at particular positions. The particles 1 and \( N \) remain entangled, in spite of the \( N - 2 \) hard walls between them formed by the detected intermediate particles. For example, when the intermediate particles are detected at their “self-crossing” values, the state of the system becomes

\[
\Psi_{1,N}(x_1, x_N) \propto e^{-i(m_N V_0^i (x_N - x_N(x_N^{(m_N)})) + \Phi)} + e^{i(m_1 V_0 (x_1 - x_1(x_1^{(m_1)})) + m_N V_0(x_N - x_N(x_N^{(m_N)})) + \Phi)}.
\]

This state is a paradigmatic Schrödinger cat state where a light particle \( (x_N) \), the “\( \alpha \)-particle”, is entangled with a heavy one \( (x_1) \), the “\( \alpha \)cat”.

Figure 2 shows the results of time propagation according to the above scheme, for \( N = 4 \). At the classical self-crossing point, the incident and the outgoing waves dominate: the clear interference fringes in the \( x_1 - x_4 \) plane are a signature of that. The absence of diffraction is a sign of integrability. Also note that for a generic set of masses, the most probable outcome of the process is the equipartition of energy. In this case, the velocity of the heaviest particle will be \( \sqrt{N(N + 1)/2} \), \( N \approx 1 \) times lower than in the integrable case, leaving it effectively at rest.

We have numerically computed the Rényi entropy \( S_2[\text{particle } 1] = -\ln[\text{Tr}[\rho_{\text{particle } 1}^2]] \) for the reduced density matrix \( \rho_{\text{particle } 1} \) of the heaviest particle, for the state truncated to the square area in Fig. 1 and \( (x_2, x_3) \) fixed to (3, 6) \( \ell \). To compute the entropy, we discretized the \( (x_1, x_4) \) space into a square grid, and in doing so, reduced the computation to a standard setting where the Hilbert space has a finite number of dimensions. We verified that for small enough grid spacing, the entropy does not depend on the value of the spacing. As expected, \( S_2[\text{particle } 1] \approx \ln(1.991) \), close to \( \ln 2 \), indicating two element-wise-distinct sets of particle momenta.

Notice that, if plotted as a function of the coordinate \( X \equiv X_{\text{COM}} + \frac{1}{2} x_N \), the \( N \)-body density corresponding to the state \( \Psi(x_1, x_2, \ldots, x_N) \) in the vicinity of the self-crossing point shows interference fringes with crest-to-crest distance of \( \Delta X = 2 \pi / (M V_B) \), as if produced by a wave for a single massive particle of mass \( M \) split by a \( \pm M V_B/2 \) beamsplitter. Here, \( X_{\text{COM}} \equiv \sum_{i=1}^N m_i x_i / M \) is the center of mass coordinate. Figure 2 shows a sample interferometric scheme exploiting this effect. In this scheme, the beamsplitter only acts on the lightest particle, while due to the entanglement buildup, the more massive particles also affect the position of the fringes.

Below we will suggest a possible experimental realization in which the role of the particles is played by polymers of atoms—bosonic solitons. Assume that the lightest particle is a polymer consisting of \( N_{\text{light}} \) atoms. The heaviest one will contain \( N_{\text{heavy}} = N_{\text{light}} \times (N + 1)/2 \approx N_{\text{total}} \) atoms, where \( N_{\text{total}} \) is the total number of atoms in the system. Imagine that a phase object, a potential barrier of height \( U \), per atom, acting during a limited time \( \tau \), is introduced between the wall and the default position of the heavy polymer. The best sensitivity to the strength \( U \) of the phase object can be easily estimated as \( \Delta U_{\text{Galilean cannon}} \sim \hbar / (\tau N_{\text{heavy}}) \sim \hbar / (\tau N_{\text{total}}) \), i.e. by the energy-time uncertainty relation for an interferometer formed by the heaviest polymer attempting to measure a barrier of height \( N_{\text{total}} U \) within a time \( \tau \). However, if \( N_{\text{total}} \) individual atoms are used, the maximal sensitivity is only \( \Delta U_{\text{individual atoms}} \sim \hbar / (\tau \sqrt{N_{\text{total}}}) \), which is the sensitivity of a single-atom interferometer further improved by the signal-to-noise reduction using \( N_{\text{total}} \) repetitive measurements. The net relative sensitivity gain produced by the entanglement, for a given number of atoms \( N_{\text{total}} \) available, becomes a gain \( \sim \sqrt{N_{\text{total}}} \).

In conclusion, we showed that for a particular one-dimensional mass sequence, it is possible to realize a protocol in which the system evolves, on its own, from a product state to a state where a heavy particle becomes entangled with a light one, thus realizing Schrödinger’s \( \alpha \)-a-cat-and-an-\( \alpha \)-particle paradigm. We show numerically that the Rényi entropy of the heavy particle can rise to almost \( \ln 2 \). The robustness of the protocol is due to the integrability of the model that protects it from both classical stochasticization and quantum diffraction. We suggest a concrete way to exploit the heavy-light entanglement by proposing an atomic interferometric sensor scheme that shows an \( \sqrt{N_{\text{total}}} \) increase in sensitivity, where \( N_{\text{total}} \) is the total number of atoms employed.

As an empirical realization of the scheme presented above we suggest using chains of cold bosonic solitons 20–22. For our scheme, it is necessary to have two internal states available (or, alternatively, two kinds of atoms). We assume that like species attract each other, while the scattering length between the opposite species is tuned to a positive value. For Li\(^7 \) atoms, the desired window in Feshbach magnetic field strength does exist: in particular, at 855 G, the scattering lengths governing a \( (m_F = -1) - (m_F = 0) \) mixture are \( a_{-1,-1} \approx -0.5 a_B \), \( a_{0,0} \approx -10 a_B \), and \( a_{1,-1} \approx +1.0 a_B \), where \( a_B \) is the Bohr radius. 24. One would have to ensure that the kinetic energy of the relative motion of the solitons must be lower than both the intra- and inter-specie interaction energy per particle, to ensure both a suppression of the inelastic effects and an absence of inter-specie transmission. Finally, the soliton sizes must be adjusted to fit the desired mass sequence. Nontrivial in-
of the mass of the original soliton, with a hypercube of dimensions 0 \times 0.1 \times 0.1 \times 0.1 \ell centered at the self-crossing point (1, 3, 6, 10) \ell were kept. The distance between the crests is consistent with the predicted value of $\Delta \tilde{X} = 2\pi/(Mv_x) = 0.0165 \ell$. The velocity kick induced by the beam-splitter is the same as the initial velocity $V_x(0)$ of Fig. 1. The initial wavepacket width of the lightest particle is $\sigma_{x,1} = 0.408 \ell$. The period of time between the beamslitting and detection is $0.0210 Mv_x^2/\hbar$. The remaining parameters are the same as in Fig. 1. Note that thanks to the presence of the beamslitter and an additional mirror, the initial wavepacket is no longer required to be long enough to self-cross. Accordingly, if viewed in the $z$-coordinates, the initial many-body wavepacket is perfectly round.

Integrals of motion present, at the mean-field level, in cold one-dimensional Bose gases may provide a way to accurately divide the gas onto desired fractions, exact in the mean-field limit. For instance, the 1, 1/3, 1/6, 1/10 mass spectrum considered above can be created using three types of quench of the coupling constant: sudden increase by a factor of 4, 25/8, and 9. Starting from a single soliton of a mass $M$, a sequence of quenches $9 \rightarrow 25/8 \rightarrow 8 \rightarrow 4 \rightarrow 4$ would lead to an ensemble of eight solitons of masses $\{3/15, 7/15, 13/15, 19/15\} \times M$ of the mass of the original soliton, with $225/8 \times M$ lost to the thermal atoms. If the third, fifth, sixth, and seventh members of this sequence are further removed, the resulting group $\{2/15, 7/45, 13/45, 19/45\} \times M$ constitutes the desired sequence.

FIG. 2. A Galilean-cannon-based interferometer: an example with $N = 4$ particles. Initially all particles are at the point of self-crossing, $\{x_1, x_2, x_3, x_4\} = (1, 3, 6, \ldots, \left\lfloor \frac{N(N+1)}{2} \right\rfloor = 4, 10) \ell$. At time B, a beam-splitter is applied to the lightest particle. After a series of collisions, each particle returns to its initial position, at the detection time D, where the particle positions are measured. A phase object, both spatially and temporarily localized, is shown as a grey rectangle. While no individual particle position distribution possesses any structure, the distribution of the $X = X_{\text{COM}} + \frac{\pi}{2} x_N$ variable shows interference fringes, whose position is controlled by the phase introduced by the object. The inset shows a sample histogram derived from 79900 simulated realizations of the detection cycle: those were selected from a longer sequence; only the realizations with particle positions within a hypercube of dimensions $0.1 \times 0.1 \times 0.1 \ell$ were kept. The distance between the crests is consistent with the predicted value of $\Delta \tilde{X} = 2\pi/(Mv_x) = 0.0165 \ell$. The velocity kick induced by the beam-splitter is the same as the initial velocity $V_x(0)$ of Fig. 1. The initial wavepacket width of the lightest particle is $\sigma_{x,1} = 0.408 \ell$. The period of time between the beamslitting and detection is $0.0210 Mv_x^2/\hbar$. The remaining parameters are the same as in Fig. 1. Note that thanks to the presence of the beamslitter and an additional mirror, the initial wavepacket is no longer required to be long enough to self-cross. Accordingly, if viewed in the $z$-coordinates, the initial many-body wavepacket is perfectly round.

FIG. 3. Relative size of the mass fluctuation of the lightest particle at which a transition from an integrable to stochastic behavior occurs, as a function of the number of particles.
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