STABLE TYPE OF THE MAPPING CLASS GROUP

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ABSTRACT. We use dynamics of the Teichmüller geodesic flow to show that the action of the mapping class group on the space of projective measured foliations has stable type $III_\lambda$ for some $\lambda > 0$. We do this by generalizing a criterion due to Bowen for a number to be in the stable ratio set, and proving some Patterson-Sullivan type results for the Thurston measure on $PMF$.

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1. INTRODUCTION AND STATEMENT OF RESULTS

Let $(X, d, \nu)$ be a compact metric space endowed with a probability measure and $G$ a countable group acting quasi-invariantly on $(X, \nu)$.

The ratio set of the action, denoted by $RS(G \rtimes (X, \nu))$ is the essential range of the Radon-Nikodym cocycle.

Definition 1.1 (Ratio Set). A number $r \in \mathbb{R}$ is said to be in $RS(G \rtimes (X, \nu))$ if for every positive measure set $A \subset X$ and $\epsilon > 0$ there is a subset $A' \subset A$ of positive measure and a nonidentity element $g \in G$ such that

- $gA' \subset A$
- $\left| \frac{\text{deg}}{\text{deg}}(b) - r \right| \leq \epsilon$ for all $b \in A'$.

The extended real number $+\infty$ is said to be in $RS(G \rtimes (X, \nu))$ if and only if for every positive measure set $A \subset X$ and $n > 0$ there exists a positive measure subset $A' \subset A$ and an element $g \in G$ such that

- $gA' \subset A$
- $\frac{\text{deg}}{\text{deg}}(b) > n$ for all $b \in A'$.

In [11], Bowen and Nevo defined the stable ratio set $SRS(G \rtimes (X, \nu))$ to be intersection over all probability measure preserving actions $G \rtimes (Y, \kappa)$ of the ratio sets of the product actions $G \rtimes (X \times Y, \nu \times \kappa)$.  

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Definition 1.2 (Stable Ratio Set). A number $r \in \mathbb{R} \cup \{\infty\}$ is in the stable ratio set $\text{SRS}(G \actson (X, \nu))$ if $r \in \text{RS}(G \actson (X \times Y, \nu \times \kappa))$ for every probability measure preserving action $G \actson (Y, \kappa)$.

By [13] if the action $G \actson (X, \nu)$ is ergodic and nonatomic, $\text{RS}(G \actson (X, \nu)) \setminus \{0, \infty\}$ is a closed multiplicative subgroup of $\mathbb{R}$ and can thus be classified as one of the following types:

- $II$ if $\text{RS}(G \actson (X, \nu)) = \{1\}$
- $III_0$ if $\text{RS}(G \actson (X, \nu)) = \{0, 1, \infty\}$
- $III_\lambda$, $\lambda > 1$ if $\text{RS}(G \actson (X, \nu)) = \{0, \lambda^n, \infty : n \in \mathbb{Z}\}$
- $III_1$ if $\text{RS}(G \actson (X, \nu)) = [0, \infty]$.

The action of $G$ on $(X, \nu)$ is called weak mixing if for every probability measure preserving ergodic action of $G$ on a space $(K, \mu)$ the induced action of $G$ on $(X \times K, \nu \times \mu)$ is ergodic.

It follows that if $G \actson (X, \nu)$ is weak mixing, its stable ratio set is one of the four types just described. This is called the stable type of the action.

In [11], Bowen and Nevo used this notion to prove pointwise ergodic theorems for a large class of (nonamenable) groups, with the principal condition being that they admit a nonsingular action of stable type $III_\lambda$ for some $\lambda > 0$.

In [12] Bowen proves that for $G$ a Gromov hyperbolic group, $X$ its Gromov boundary, and $\nu$ the Patterson-Sullivan measure on $X$, if $G \actson (X, \nu)$ is weak mixing then it has stable type $III_\lambda$ for some $\lambda \in (0, 1]$. In this paper, we prove an analogous result for the mapping class group $\text{Mod}(S)$ of a surface $S$ of genus at least 2 acting on the space $\text{PMF}$ of projective measured foliations with the Thurston measure.

Theorem 1.3. The action $\text{Mod}(S) \actson (\text{PMF}, \nu)$ has stable type $III_\lambda$ for some $\lambda > 0$.

We prove Theorem 1.3 by introducing the notion of a family of functions $\Upsilon_n : G \times X \times X, n \in \mathbb{N}$ admissible relative to a collection of subsets $\Omega(n, m), n, m \in \mathbb{N}$ of $\text{PMF}$ for $G \actson (X, \nu)$. This generalizes Bowen’s notion of admissible family from [12]. We show in Section 2 that the existence of a relatively admissible family for a weakly mixing action $G \actson (X, \nu)$ implies that action has stable type $III_\lambda$ for some $\lambda > 0$. We then show in Sections 4 and 5 that there exists a relatively admissible family for the action of the mapping class group $\text{Mod}(S)$ on $\text{PMF}$ with the Thurston measure.

While the Teichmüller space $\text{Teich}(S)$ is not globally hyperbolic in any reasonable sense (e.g. it is not Gromov hyperbolic and not $\text{CAT}(0)$), some parts of it exhibit many aspects of hyperbolicity. In particular, Teichmüller geodesic segments spending a uniform proportion of the time over compact parts of moduli space resemble those in Gromov hyperbolic spaces. The Thurston measure can be considered as a conformal density for the Teichmüller metric, and in Section 4 we use this conformal property to prove a relative analogue of Sullivan’s shadow lemma estimating shadows from a fixed origin of balls in $\text{Teich}(S)$ where the connecting segment spends a uniform proportion in the thick part. The general strategy of the proof is to use recurrence estimates of the Teichmüller geodesic flow to show that various quantities are asymptotically dominated by the contribution of the thick part. This allows us to construct “relative” versions of Bowen’s admissible families.
Roughly, the subsets $\Omega(n, m)$ consist of elements of $PMF$ corresponding to geodesic rays from the basepoint $o$ that look hyperbolic near distance $n$ from $o$, with the hyperbolicity weakening as $m$ grows. The functions $\Upsilon_n$ are roughly defined as follows.

$$\Upsilon_n(g, b, b') = \frac{1_{Y_n(b)}(g) 1_{Z_n(g)}(b')}{|Y_n(b)| \nu(Z_n(g))}.$$ 

Here, $|Y_n(b)|$ denotes the cardinality of $Y_n(b)$. A mapping class element $g$ is in $Y_n(b)$ if it moves $o$ a distance of approximately $2n$, $[o, go]$ fellow travels $[o, b]$ for time slightly more than halfway and $[o, go]$ keeps exhibiting hyperbolic behavior after separating from $[o, b]$; $Z_n(g)$ is the subset of $PMF$ consisting of those $b'$ such that $[o, go]$ follows $[o, b']$ slightly less than half way and $[o, b']$ keeps exhibiting hyperbolic behavior after separating from $[o, go]$.

The connection with stable type is made by the following:

**Theorem 1.4.** Suppose $\Upsilon_n, n \in \mathbb{N}$ is admissible relative to $\Omega(m, n), m, n \in \mathbb{N}$ for $G \acts (X, \nu)$. For each $m$ let $\zeta_m$ be any weak-* limit of the $\zeta_{n,m}$ as $n \to \infty$. Let $\zeta$ be any weak-* limit of the $\zeta_m$. Then $e^{T}$ is contained in the stable ratio set of $G \acts (X, \nu)$ for every $T$ in the support of $\zeta$.

It seems that a simplified version of our argument in Section 5 can be used to construct pseudo-admissible families for the actions of nonuniform lattices in manifolds of pinched variable negative curvature on their boundary spheres, proving an analogue of Theorem 1.3 for these actions.

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2. **Relatively-Admissible Families**

Let $(X, d, \nu)$ be a compact metric space endowed with a probability measure and $G$ act ergodically and quasi-invariantly on $X$. The action of $G$ on $(X, \nu)$ is called weak mixing if for every probability measure preserving ergodic action of $G$ on a space $(K, \mu)$ the induced action of $G$ on $(X \times K, \nu \times \mu)$ is ergodic.

**Definition 2.1** (Relatively Admissible Families). A family of functions $\Upsilon_n : G \acts X \times X \to \mathbb{R}, n, m \in \mathbb{N}$ will be called admissible relative to a family of closed subsets $\Omega(n, m) \subset X$ if:

- There are $D(m) > 0$ with $\lim_{m \to \infty} D(m) = 0$ such that
  $$\nu(\Omega(n, m)) > 1 - D(m)$$
  for all $n, m \in \mathbb{N}$

- For each $m$ there is a function $f_m : \mathbb{N} \to \mathbb{R}$ with $f_m(n) \to 0$ as $n \to \infty$ such that for all $(g, b, b')$ with $b \in \Omega(n, m)$ and $\Upsilon_n(g, b, b') > 0$ we have $d(b, b') < f_m(n)$ and $d(g^{-1}b, g^{-1}b') < f_m(n)$

- Let
  $$R(g, \eta) = \log \frac{d\nu \circ g}{d\nu}(\eta).$$
For each $m$ there are constants $C(m) > 0$, $N(m) > 0$ such that if $n > N(m)$ then
\[ |R(g^{-1}, b) + |R(g^{-1}, b')| < C(m) \]
and
\[ |R(g^{-1}, b) - R(g^{-1}, b')| \geq 1/C(m) \]
for a.e. $(g, b, b')$ with $b \in \Omega(n, m)$ and $\Upsilon_n(g, b, b') > 0$.

- $\sum_{g \in G} \int \Upsilon_n(g, b, b')d\nu(b') = 1$ for every $n > 0$ and $b \in PMF$.
- There exists constants $C(m) > 0$ such that the following three quantities are bounded above by $C(m)$ for all $n > N(m)$
\[ \int_{b \in \Omega(n, m)} \sum_{g \in G} \Upsilon_n(g, b, b')d\nu(b) \]
for a.e. $b' \in X$
\[ \int_{b \in \Omega(n, m)} \sum_{g \in G} \Upsilon_n(g, b, gb') \frac{d\nu \circ g}{d\nu}(b')d\nu(b) \]
for a.e. $b' \in X$
\[ \int \sum_{g \in G} 1_{\Omega(n, m)}(gb)\Upsilon_n(g, gb, b') \frac{d\nu \circ g}{d\nu}(b')d\nu(b') \]
for a.e. $b \in X$.

Define a measure $\zeta_{n,m}$ on $\mathbb{R}$ by
\[ \zeta_{n,m}(E) = \sum_{g \in G} \int \int 1_E(R(g^{-1}, b') - R(g^{-1}, b))1_{\Omega(n, m)}(b)\Upsilon_n(g, b, b')d\nu(b)d\nu(b') \]

In this section we will prove:

**Theorem 2.2.** Suppose $\Upsilon_n, n \in \mathbb{N}$ is admissible relatively to $\Omega(m, n), m, n \in \mathbb{N}$ for $G \triangleleft X, \nu$. For each $m$ let $\zeta_m$ be any weak-* limit of the $\zeta_{n,m}$ as $n \to \infty$. Let $\zeta$ be any weak-* limit of the $\zeta_m$. Then $e^T$ is contained in the stable ratio set of $G \triangleleft X, \nu$ for every $T$ in the support of $\zeta$.

Note, by the third bullet of Definition 2.1 and the fact that the $\zeta_{n,m}$ are measures of total mass $1 - D(m) < ||\zeta_{n,m}|| < 1$, for each $m$ such a weak-* limit $\zeta_m$ must exist and have support bounded away from 0. Moreover since $\zeta_{n,m}(E) \geq \zeta_{n,m'}(E)$ for $m > m'$ and all measurable $E$ we have $\zeta_m(E) \geq \zeta_m'(E)$ so any weak * limit $\zeta$ of the $\zeta_m$ is a probability measure whose support has a nonzero point.

It follows that the stable ratio set is not contained in $\{0, 1, \infty\}$.

We thus obtain

**Corollary 2.3.** If $G \triangleleft X, \nu$ is weak mixing, and there exists a relatively admissible family for this action, then the action has stable type $III_\lambda$ for some $\lambda \geq 1$.

Define the following operators.
\[ L_{n,m}f(b, t) = \sum_{g \in G} \int f(b, t + R(g^{-1}, b') - R(g^{-1}, b))1_{\Omega(n, m)}(b)\Upsilon_n(g, b, b')d\nu(b') \]
Proof.

Let $L$ exist some $X$ for each Lemma 2.4.

Let $W = \|W_n\|$ be the $L^1$ norm of each $n,m$.

Case $Y_n,m$:

Because $\sum_{g \in G} \int_{\Omega(n,m)} \nu(g, b, b') R(g, b') d\nu(b) \leq C(m)$,

$$||W_n,m f|| = \int \int |W_n,m f| d\nu d\theta = \sum_{g \in G} \int \int f(b', t) 1_{\Omega(n,m)}(b) \nu(g, b, b') d\nu(b') d\nu(b) d\theta(t)$$

$$\leq C(m) \int f(b', t) d\nu(b') d\theta(t) = C(m)||f||$$

Case $X_n,m$:

Because $\sum_{g \in G} \int_{\Omega(n,m)} \nu(g, gb') R(g, b') d\nu(b) \leq C(m)$,

$$||X_n,m f|| = \int \int |X_n,m f| d\nu d\theta = \sum_{g \in G} \int \int f(g^{-1} b', t + R(g^{-1}, b')) 1_{\Omega(n,m)}(b) \nu(g, b, b') d\nu(b') d\nu(b) d\theta(t)$$

$$= \sum_{g \in G} \int \int f(g^{-1} b', t) \frac{d\theta}{d\nu} A_{R(g^{-1}, b')} \nu(g, b, b') d\nu(b') d\nu(b) d\theta(t)$$

$$\leq C'(m) \sum_{g \in G} \int \int f(g^{-1} b', t) 1_{\Omega(n,m)}(b) \nu(g, b, b') d\nu(b') d\nu(b) d\theta(t)$$

$$\leq C'(m) \sum_{g \in G} \int \int f(b', t) \frac{d\nu}{d\nu}(b') 1_{\Omega(n,m)}(gb) \nu(g, b, gb') d\nu(b') d\nu(b) d\theta(t)$$

$$\leq C'(m) C(m) \int f(b', t) d\nu(b') d\theta(t) = C'(m) C(m)||f||$$

Case $Y_n,m$:

Because $\sum_{g \in G} \int_{\Omega(n,m)} \nu(g, gb') R(g, b') d\nu(b) \leq C(m)$,

$$||Y_n,m f|| = \int \int |Y_n,m f| d\nu d\theta = \sum_{g \in G} \int \int f(g^{-1} b, t + R(g^{-1}, b')) 1_{\Omega(n,m)}(b) \nu(g, b, b') d\nu(b') d\nu(b) d\theta(t)$$
Lemma 2.5. For all \( f \in L^1(\nu \times \vartheta) \),

\[
\lim_{n \to \infty} \sup ||W_{n,m}f - f||_1 \leq D(m)||f||_1
\]

\[
\lim_{n \to \infty} ||X_{n,m}f - Y_{n,m}f||_1 = 0
\]

Proof. Without loss of generality let \( f \) be a continuous function with compact support on \( X \times \mathbb{R} \). Let \( \text{Var}_{n,m}(f) = \sup_{d(x,y)<f_m(n)}|f(x,t) - f(y,t)| \). Note \( \text{Var}_{n,m}(f) \to 0 \) as \( n \to \infty \). If \( b \notin \Omega(n,m) \) then clearly \( W_{n,m}f = 0 \). On the other hand, since \( \sum_{g \in G} \text{Var}_{n,m}(g(b,b')d\nu(b')) = 1 \) for every \( n, m > 0 \) and \( b \in PMF \) and for every \( (g,b,b') \) with \( b \in \Omega(n,m) \) and \( \text{Var}_{n,m}(g,b,b') > 0 \) we have \( d(b,b') < f_m(n) \) and \( d(g^{-1}b,g^{-1}b') < f_m(n) \) we have

\[
||W_{n,m}f(b,t) - f(b,t)|| \leq \text{Var}_{n,m}(f)
\]

whenever \( b \in \Omega(n,m) \). Thus

\[
||W_{n,m}f - f||_1 \leq \text{Var}_{n,m}(f) + D(m)||f||_\infty
\]

and hence

\[
\lim_{n \to \infty} \sup ||W_{n,m}f - f||_1 \leq D(m)||f||_\infty \leq D(m)||f||_1
\]

Since compactly supported continuous functions are \( L^1 \) dense and the \( ||W_{n,m}||_1 \leq C(m) \) for all \( n \), the first statement of the lemma follows. The second statement is proved similarly.

\[
\square
\]

Let \( G \) act on \( X \times \mathbb{R} \) by \( g(b,t) = (gb, t + R(g,b)) \) and on \( L^1(\nu \times \vartheta) \) by \( \tilde{g}f = f \circ g^{-1} \)

Proposition 2.6. For every \( G \) invariant function \( f \in L^1(\nu \times \vartheta) \) we have

\[
\lim_{n \to \infty} ||f - L_{n,m}f|| \leq D(m)||f||
\]

Proof. Since \( f \) is \( G \) invariant we have \( X_{n,m} = W_{n,m} \) and \( Y_{n,m} = L_{n,m} \). Now

\[
||X_{n,m}f - Y_{n,m}f|| = ||W_{n,m}f - L_{n,m}f|| \geq ||f - L_{n,m}f|| - ||f - W_{n,m}f||
\]

As \( n \to \infty \) we have

\[
||X_{n,m}f - Y_{n,m}f|| \to 0
\]

and

\[
\lim_{n \to \infty} \sup ||f - W_{n,m}f|| \leq D(m)||f||
\]

proving the proposition.

\[
\square
\]

This immediately implies
Lemma 2.9 \((Lemma 3.8 \text{ in } [12])\) for every \(T \in \mathcal{G}\) and taking limits as \(m \to \infty\) we get

\[
\lim_{m \to \infty} \lim_{n \to \infty} ||f - L_{n,m}|| = 0
\]

Recall the measure \(\zeta_{n,m}\) on \(\mathbb{R}\) defined by

\[
\zeta_{n,m}(E) = \sum_{g \in G} \int \int 1_{E}(R(g^{-1}, b') - R(g^{-1}, b)) 1_{\Omega(n,m)}(b) \Upsilon_{n,m}(g, b, b') \, d\nu(b) \, d\nu(b')
\]

The following is Theorem 2.2 with ”ratio set” in place of ”stable ratio set”.

**Proposition 2.8.** For each \(m\) let \(\zeta_{m}\) be any weak-* limit of the \(\zeta_{n,m}\) as \(n \to \infty\). Let \(\zeta\) be any weak-* limit of the \(\zeta_{m}\). Then \(e^{T}\) is contained in the ratio set of \(G \acts (X, \nu)\) for every \(T\) in the support of \(\zeta\).

**Lemma 2.9 (Lemma 3.8 in [12]).** Suppose \(e^{T}\) is not in the ratio set of \(G\) acting on \((B, \nu)\). Then there exists an \(\epsilon > 0\) and a \(G\)-invariant, positive measure set \(A \subset B \times \mathbb{R}\) such that for every \((b, t), (b, t') \in A\) and \((b, t + T + t') \notin A\).

**Proof of Proposition 2.8.** Let \(T\) be an element of the support of \(\zeta\). To obtain a contradiction, suppose that \(e^{T}\) it is not in the ratio set. Let \(A\) and \(\epsilon\) be as in the previous lemma. Let \(f\) be the characteristic function of \(A\). Note,

\[
L_{m,n}f(b, t) = \int 1_{\Omega(n,m)}(b) f(b, t + t') d\zeta_{n,m}(t')
\]

where \(\zeta_{n,m,b}\) is the probability measure given by

\[
\zeta_{n,m,b}(E) = \sum_{g \in G} \int 1_{E}(R(g^{-1}, b') - R(g^{-1}, b)) \Upsilon_{n,m}(g, b, b') \, d\nu(b')
\]

Thus

\[
||f - L_{n,m}f|| = \int 1_{\Omega(n,m)}(b) |f(b, t)| - \int f(b, t + t') d\zeta_{n,m,b}(t') |d\nu(b)| dt
\]

\[
\geq \int_{A} 1_{\Omega(n,m)}(b) f(b, t) - \int f(b, t + t') d\zeta_{n,m,b}(t') |d\nu(b)| dt
\]

\[
\geq \int_{A \setminus \Omega(n,m) \times \mathbb{R}} \zeta_{n,m,b}(T - \epsilon, T + \epsilon) |d\nu(b)| dt
\]

\[
= (\nu \times \theta)(A \cap (\Omega(n,m) \times \mathbb{R})) \zeta_{m,n}(T - \epsilon, T + \epsilon)
\]

\[
\geq ((\nu \times \theta)(A) - D(m)) \zeta_{m,n}(T - \epsilon, T + \epsilon)
\]

The second inequality holds because by the Lemma 2.9, if \((b, t) \in A\) and \(t' \in (T - \epsilon, T + \epsilon)\) then \((b, t + t') \notin A\) so \(f(b, t) - f(b, t') = 1\).

Fixing \(m\) and taking limits as \(n \to \infty\) gives

\[
D(m) \geq ((\nu \times \theta)(A) - D(m)) \zeta_{m}(T - \epsilon, T + \epsilon)
\]

and taking the limit as \(m \to \infty\) we get

\[
(\nu \times \theta)(A) \zeta(T - \epsilon, T + \epsilon) = 0
\]

contradicting that \(T\) is in the support of \(\zeta\). \(\Box\)
Thus we obtain that the action of $G$ on $(X, \nu)$ does not have type $III_0$ proving Theorem 1.4 with “ratio set” in place of “stable ratio set”.

To prove Theorem 2.2, we will show that given any ergodic measure preserving action of $G$ on a probability space $(K, \kappa)$ there exists a topological model for this action and an pseudo-admissible family $\Upsilon_{n,m}$ for this action with limit measure $\zeta'$ such that if $T$ is in the support of $\zeta'$ then $T$ is also in the support of $\zeta'$.

**Lemma 2.10** (Prop 3.10 in [12]). Let $\Gamma \curvearrowright (X, \mu)$ be an ergodic pmp action. Then there exists a compact metric space $(K, d_K)$ with a Borel probability measure $\kappa$ and a continuous action $\Gamma \curvearrowright K$ such that

- $\Gamma \curvearrowright (X, \mu)$ is measurably conjugate to $\Gamma \curvearrowright (K, \kappa)$
- for every $\epsilon > 0$ and $x, y \in K$,

$$1/3 \leq \frac{\kappa(B(x, \epsilon))}{\kappa(B(y, \epsilon))} \leq 3$$

where for example, $B(x, \epsilon) = \{ z \in K : d_K(x, z) \leq \epsilon \}$.

**Proof of Theorem 2.2.** Let $\Gamma \curvearrowright (K, \kappa)$ be an ergodic probability measure preserving action. By Lemma 2.10, we may assume that $(K, d_K)$ is a compact metric space such that for every $\epsilon > 0$ and $x, y \in K$,

$$1/3 \leq \frac{\kappa(B(x, \epsilon))}{\kappa(B(y, \epsilon))} \leq 3.$$ 

Given an integer $n \geq 1$ and $g \in \Gamma$, let $0 < \rho(n, g) < 1/n$ be such that for every $x, y \in K$ with $d_K(x, y) \leq \rho(n, g)$, $d_K(g^{-1}x, g^{-1}y) \leq 1/n$. Define $\Upsilon_n : \Gamma \times X \times K \times X \times K \to \mathbb{R}$ by

$$\Upsilon_n(g, b, k, k') := \frac{1_{B(k, \rho(n, g))}(k') \Upsilon_n(g, b, b')}{\kappa(B(k, \rho(n, g)))}.$$ 

It is an easy exercise using the above estimates to check that $\{ \Upsilon_n \}_{n=1}^\infty$, $\Omega(m, n) \times K$ is an admissible family for $G \curvearrowright (B \times K, \nu \times \kappa)$ with $d_{B \times K}$, a metric on $B \times K$, given by $d_{B \times K}((b, k), (b', k')) = d_B(b, b') + d_K(k, k')$. Since $G \curvearrowright (K, \kappa)$ is measure preserving,

$$R(g, b, k) := \log \frac{d(\nu \times \kappa) \circ g}{d(\nu \times \kappa)}(b, k) = R(g, b).$$

Thus, for any $E \subset \mathbb{R}$

$$\zeta_{n,m}(E) = \sum_{g \in \Gamma} \int \int_{1_{E}} \left( R(g^{-1}, b') - R(g^{-1}, b) \right) 1_{\Omega(n,m)}(b) \Upsilon_n(g, b, b') d\nu(b') d\nu(b)$$

$$= \sum_{g \in \Gamma} \int \int_{1_{E}} \left( R(g^{-1}, b', k') - R(g^{-1}, b) \right) 1_{\Omega(n,m)}(b) \Upsilon_n(g, b, k, b', k') d\nu \times \kappa(b, k') d\nu \times \kappa(b, k).$$

Thus, Prop 3.8 implies the ratio set of the action $\Gamma \curvearrowright (B \times K, \nu \times \kappa)$ contains $\epsilon''$. Since $\Gamma \curvearrowright (K, \kappa)$ is arbitrary, the proof is complete. \(\square\)
3. Background on the Geometry of Teichmüller Space

Let $S$ be a closed surface of genus $g \geq 2$. Let $\text{Mod}(S)$ be the associated mapping class group. The Teichmüller space $\text{Teich}(S)$ is the space of all marked or hyperbolic structures on $S$ up to isotopy. We endow it with the Teichmüller metric. Thurston showed that $\text{Teich}(S) \simeq \mathbb{R}^{6g-6}$ has a natural compactification by the space $\text{PMF} \simeq S^{6g-7}$ of projective classes of measured foliations $\text{MF}$ on $S$, which has many analogies with the compactification of hyperbolic space by its boundary sphere $\mathbb{S}$. The space $Q(S)$ of quadratic differentials can be thought as a cotangent bundle of $\text{Teich}(S)$. A quadratic differential $q$ is determined by its vertical and horizontal measured foliations $q^+$ and $q^-$ respectively. For each $o \in \text{Teich}(S)$ there is a unique Teichmüller geodesic through $o$ in the direction of $q$. Moreover, if $\eta \in \text{PMF}$ is uniquely ergodic, for any $\eta' \in \text{PMF}$ there is a unique Teichmüller geodesic with forward and backward directions $\eta$ and $\eta'$. By Masur’ criterion for unique ergodicity, geodesics in non-uniquely ergodic directions eventually exit forever every thick part $\text{Teich}(S)$. Furthermore, if $q^+$ is uniquely ergodic, the geodesic ray $g_\eta q$ converges to $[q^+]$ \cite{MS}. The Busemann cocycle
\[ \beta_z(x,y) = d(x,z) - d(y,z), \]
for $x, y, z \in \text{Teich}(S)$ extends continuously to uniquely ergodic $z \in \text{PMF}$. There is a unique probability measure $\nu$ of maximal entropy for the Teichmüller geodesic flow on $Q^1(S)/\text{Mod}(S)$, the so called Masur-Veech measure, and it is in the Lebesgue measure class with respect the period coordinates on $Q(S)$. Its entropy is $h = 6g-6$. Let $m$ be the Thurston measure on $\text{PMF}$. The measured foliations which are not uniquely ergodic have measure 0 \cite{R}. For each $x \in \text{Teich}(S)$ define
\[ \nu_x(A) = m(\{\eta \in \text{PMF} : [\eta] \in A, \text{Ext}_x \eta \leq 1\}) \]
We call these normalized Thurston measures on $\text{PMF}$. The measures $\nu_x, x \in \text{Teich}(S)$ form a conformal density for the action of $\text{Mod}(S)$ on $\text{PMF}$ in the sense that
\[ \nu_x \circ g^{-1} = \nu_{gx} \]
and
\[ \frac{d\nu_x}{d\nu_y}(\eta) = e^{h\beta_x(x,y)} \]
for all $g \in \text{Mod}(S), x, y \in \text{Teich}(S)$ and $\eta \in \text{PMF}$ uniquely ergodic. We can write the lift $\tilde{\nu}$ of $\mu$ to $Q^1(S)$ as
\[ d\tilde{\nu}(q) = \exp(h\beta_{[q^+]}(a, \pi(q))) \exp(h\beta_{[q^-]}(a, \pi(q))) d\nu_a([q^+]) d\nu_a([q^-]) \]
for any $o \in \text{Teich}(S)$. The expression makes sense because almost every quadratic differential has uniquely ergodic vertical and horizontal measured foliations. The measures $\mu$ and $\nu_x$ are thus the analogues in the Teichmüller setting of Bowen-Margulis and Patterson-Sullivan measures respectively.

For $\epsilon > 0$ let $\text{Teich}_\epsilon(S)$ be the $\epsilon$-thick part of $\text{Teich}(S)$, which consists of all hyperbolic structures on $S$ with no nontrivial curves of hyperbolic length less than $\epsilon$. By Mumford’s criterion $M_\epsilon(S) = \text{Teich}_\epsilon(S)/\text{Mod}(S)$ is compact for all $\epsilon > 0$. The following is Theorem A of \cite{D} due to Dowdall-Duchin-Masur

**Proposition 3.1.** For each $\epsilon, \theta > \theta' > 0$ there is an $L > 0$ and $\delta > 0$ such that if $I \subset [x, y] \subset \text{Teich}(S)$ is a geodesic subinterval of length at least $L$ and a proportion
of at least \( \theta \) of \( I \) lies in \( \text{Teich}_c(S) \), then for all \( z \in \text{Teich}(S) \) the intersection \( I \cap \text{Nbd}_d([x, z] \cup [y, z]) \) has measure at least \( \theta'((I) \).

The following property of Teichmüller geodesics, also indicative of hyperbolicity in the thick part, is due to Rafi [4].

**Proposition 3.2.** For each \( A > 0 \) and \( \epsilon > 0 \) there exists a constant \( K > 0 \) such that for points \( x, x', y, y' \in \text{Teich}_c(S) \) with \( d_T(x, x') \leq A \) and \( d_T(y, y') \leq A \) the geodesic segments \([x, y] \) and \([x', y'] \) \( K \)-fellow travel in a parametrized fashion, and for \( \eta \in \text{PMF} \) such that \([x, \eta]\) and \([x', \eta]\) are contained in \( \text{Teich}_c(S) \), the geodesic rays \([x, \eta]\) and \([x', \eta]\) \( K \)-fellow travel in a parametrized fashion.

4. **Construction of a Pseudo-Admissible Family for \( \text{Mod}(S) \bowtie \text{PMF} \)**

Let \( \epsilon > 0 \) be such that \( \nu(M_\epsilon(S)) > 0.9999 \).

Let \( L \) and \( \delta \) be the ones provided by Proposition 3.1 for this \( \epsilon \) and \( \theta = 0.9, \theta' = 0.8 \).

Let \( K \) be the one provided by Proposition 3.2 with \( 2\delta \) in place of \( A \).

Let \( 0 < \epsilon' < \epsilon \) be such that \( \text{Nbd}_d\delta(K \text{Teich}_c(S) \subset \text{Teich}_{c'}(S)) \).

Let \( L_1 > L \) and \( \delta_1 > \delta \) be the ones provided by Proposition 3.1 for \( \epsilon' \) in place of \( \epsilon \) and \( \theta = 0.6, \theta' = 0.55 \).

Let \( K_1 > K \) be the one provided by Proposition 3.2 with \( 2\delta_1 \) in place of \( A \).

Let \( 0 < \epsilon'' < \epsilon' \) be such that \( \text{Nbd}_d\delta(K_1 \text{Teich}_{c'}(S) \subset \text{Teich}_{c''}(S)) \).

Let \( L_2 > L_1 \) and \( \delta_2 > \delta_1 \) be the ones provided by Proposition 3.1 for \( \epsilon'' \) in place of \( \epsilon \) and \( \theta = 0.6, \theta' = 0.55 \).

Let \( K_2 > K_1 \) be the one provided by Proposition 3.2 with \( 2\delta_2 \) in place of \( A \). Assume without loss of generality that \( \delta \) is more than twice the diameter of a fundamental domain of \( \text{Teich}_c(S) \) and \( \delta_1 \) is more than twice the diameter of a fundamental domain of \( \text{Teich}_{c'}(S) \).

Define \( \Omega(n, m) \subset \text{PMF} \) to be the set of \( b \in \text{PMF} \) such that for any \( t > m \) at least 0.9999 of each of \( \gamma_{go, o[n - t, n]} \) and \( \gamma_{o,b([n, n + t])} \) lies in \( \text{Teich}_{c'}(S) \).

Note, it follows that if \( n > 2000m \) at least 0.9 of \( \gamma_{o,b([n - 3m, n - (i - 1)m])} \) lies in \( \text{Teich}_c(S) \) for \( i = -1000, \ldots, 1000 \).

For each \( b \in \Omega(n, m) \) such that \( b \notin \Omega(n, k) \) for \( k < m \) define \( Y_n \subset \text{Mod}(S) \) to be the set of \( g \in \text{Mod}(S) \) such that:

- \( d(o, go) \in (2n - 20m, 2n + 20m) \)
- \( -100m \leq \beta(g, o) \leq -50m \)
- At least 99 percent of \( \gamma_{go, o[n - 21m - t, n - 21m]} \) lies in \( \text{Teich}_{c'}(S) \) for all \( n - 21m \geq t \).

For each such \( g \in \text{Mod}(S) \) let \( Z_n(g) \) be the set of \( b' \in \text{PMF} \) such that:

- At least 90 percent of \( b'([n - 9m, n - 8m]) \) lies in \( \text{Teich}_{c'}(S) \).
- For every \( t \leq n - 20m \), \( d(\gamma_{o,go}(t), \gamma_{o,b'}(t)) \leq K_1 \).
- For some \( t \in [n - 10m, n - 9m] \), \( d(\gamma_{o,go}(t), \gamma_{o,b'}(t)) \geq K_1 \).

For each \( b, b' \in \text{PMF} \) and \( g \in \text{Mod}(S) \) let

\[
\Upsilon_n(g, b, b') = \frac{1_{Y_n(b)}(g) 1_{Z_n(g)}(b')}{\nu(Z_n(g))}.
\]

Roughly, the \( \Omega(n, m) \) are elements of \( \text{PMF} \) corresponding to geodesic rays from the basepoint \( o \) that look hyperbolic near distance \( n \) from \( o \), with the hyperbolicity
For the first estimate, note: from the definition of \( \Upsilon_n \) follows by Proposition 4.8 and the definition of \( Z \) for each \( t > n \) becomes equidistributed. The second follows by Proposition 4.7. The third \( \nu(Z_n(g)) \approx_m e^{hn} \).

Proposition 4.4. For all \( b' \in \text{PMF} \)
\[
\nu\{b \in \Omega(n,m) : b' \in \bigcup_{g \in Y_n(b)} Z_n(g)\} \lesssim_m e^{-hn}
\]

Proposition 4.5. For all \( b' \in \text{PMF} \) the number of \( g \in \text{Mod}(S) \) with \( gb' \in Z_n(g) \) and \( g \in Y_n(b) \) for some \( b \in \Omega(n,m) \) has cardinality \( \lesssim_m e^{hn} \).

Proposition 4.6. For each \( b \in \Omega(n,m) \)
\[
|\{g \in \text{Mod}(S) : gb \in \Omega(n,m), g \in Y_n(gb)\}| \lesssim_m e^{hn}
\]

Proposition 4.7. For each \( b \in \Omega(n,m) \), \( g \in Y_n(b) \), \( b' \in Z_n(g) \) we have
\[
b, b', g^{-1}b, g^{-1}b' \in \text{pr}_{o,g^{-1}o}B_{\delta_1}(\gamma_{o,g^{-1}o}(t))
\]
for some \( t > n - 122m \) with \( \gamma_{o,g^{-1}o}(t) \in \text{Teich}(S) \)

Proposition 4.8. For each \( b \in \Omega(n,m) \), \( g \in Y_n(b) \) and \( b' \in Z_n(g) \) we have
\[-6m \leq \beta_o(go,o) \leq 21m\]

We are now ready to verify the conditions of Definition 1.1. The first bullet point follows since by ergodicity of the Teichmüller geodesic flow, almost all geodesic rays from \( o \) become equidistributed. The second follows by Proposition 4.7. The third follows by Proposition 4.8 and the definition of \( Z_n(g) \). The fourth is immediate from the definition of \( \Upsilon_n \). We now verify the estimates of the fifth bullet point. For the first estimate, note:
\[
\int_{b \in \Omega(n,m)} \sum_{g \in \text{Mod}(S)} \Upsilon_n(g, b, b') d\nu(b) = \int_{b \in \Omega(n,m)} 1_{Y_n(b)} \sum_{g \in Y_n(b)} \frac{1}{\nu(Z_n(g))} d\nu(b) \lesssim_m \int_{b \in \Omega(n,m)} \frac{e^{hn}}{Y_n(b)} \sum_{g \in Y_n(b)} 1_{Z_n(g)}(b') d\nu(b)
\]
Furthermore by the triangle inequality if \( \eta \) Note, of \( \{b \in \Omega(n, m) : b' \in \cup_{g \in Y_n(b)} Z_n(g)\} \leq_m 1 \)

For the second estimate of the fifth bullet point note that if \( \Upsilon_n(g, b, gb') \neq 0 \) then
\[
\frac{d\nu \circ g}{d\nu}(b') = e^{-h\beta_{\nu}(g^{-1}o, o)} = e^{h\beta_{\nu}(go, o)} \leq e^{15hm}
\]

thus
\[
\int_{\Omega(n, m)} \sum_{g \in \text{Mod}(S)} \Upsilon_n(g, b, gb') \frac{d\nu \circ g}{d\nu}(b') d\nu(b) \leq_m \int_{\Omega(n, m)} \sum_{g \in \text{Mod}(S)} \Upsilon_n(g, b, gb') = \int_{b \in \Omega(n, m)} \frac{1}{|Y_n(b)|} \sum_{g \in Y_n(b)} \frac{1}{\nu(Z_n(g))} \sum_{b \in \Omega(n, m)} \Upsilon_n(g, b, gb') d\nu(b) \leq_m 1
\]

To see the last inequality note
\[
\nu\{b \in \Omega(n, m) : g \in Y_n(b), gb' \in Z_n(g)\} \leq \nu\{b \in \Omega(n, m) : gb' \in \bigcup_{k \in Y_n(b)} Z_n(k)\} \leq_m e^{-hn}
\]

for each \( b' \) by Proposition 4.4 and the number of nonzero terms in the sum is at most \( \leq_m e^{hn} \) by Proposition 4.5.

For the final estimate, note that if \( \int_{\Omega(n, m)} (gb) \Upsilon_n(g, gb, b') \neq 0 \) then
\[
\frac{d\nu \circ g}{d\nu}(b) = e^{-h\beta_{\nu}(g^{-1}o, o)} = e^{h\beta_{\nu}(go, o)} \leq e^{100hm}
\]

so
\[
\int_{\Omega(n, m)} \sum_{g \in \text{Mod}(S)} \Upsilon_n(g, gb, b') \frac{d\nu \circ g}{d\nu}(b') d\nu(b') \leq_m \int_{g \in \text{Mod}(S)} \Upsilon_n(g, gb, b') d\nu(b') = \int_{g \in \text{Mod}(S)} \frac{1}{|Y_n(g)|} \nu(Z_n(g)) \int_{b \in \Omega(n, m)} \Upsilon_n(g, gb, b') d\nu(b') = \int_{g \in \text{Mod}(S)} \frac{1}{|Y_n(g)|} \nu(Z_n(g)) \leq_m e^{-hn} \nu(g \in \text{Mod}(S) : gb' \in \Omega(n, m), g \in Y_n(gb)) \leq_m 1
\]

This completes the proof.

5. Proofs of Propositions in Section 4

We begin by proving the following analogue of Sullivan’s shadow Lemma:

**Lemma 5.1.** For each \( r > 0, \theta, \epsilon > 0 \) and \( R > 0 \) there exists a \( C > 0 \) with the following property: for every \( g \in \text{Mod}(S) \) such that any initial length \( \geq R \) segment of \( [o, g^{-1}o] \) spends a proportion at least \( \theta \) in \( \text{Teich}_{e}(S) \) we have
\[
C^{-1} e^{-hd(go, o)} \leq \nu_o(pr_oB_r(go)) \leq Ce^{-hd(go, o)}
\]

**Proof.** Note,
\[
\nu_o(pr_oB_r(\gamma o)) = \int_{\eta \in pr_oB_r(\gamma o)} e^{-h\beta_{\nu}(n, \gamma o)} d\nu_{\gamma o}
\]

Furthermore by the triangle inequality if \( \eta \in pr_oB_r(\gamma o) \) we have
\[
d(o, \gamma o) - 2r \leq \beta_{\eta}(o, \gamma o) \leq d(o, \gamma o)
\]
Thus
\[ \nu_o(pr_{\gamma^{-1}\circ B_r}(o))e^{-\hd(o,\gamma_o)} = \nu_{\gamma_o}(pr_oB_r(\gamma_o))e^{-\hd(o,\gamma_o)} \leq \nu_o(pr_oB_r(\gamma_o)) \]
\[ \leq e^{2hr-\hd(o,\gamma_o)}\nu_{\gamma_o}(pr_oB_r(\gamma_o)) \leq e^{2hr-\hd(o,\gamma_o)}\|\nu_o\|. \]
So
\[ \nu_o(pr_{\gamma^{-1}\circ B_r}(o))e^{-\hd(o,\gamma_o)} \leq \nu_o(pr_oB_r(\gamma_o)) \leq e^{2hr-\hd(o,\gamma_o)}\|\nu_o\|. \]
This gives an upper bound.

For the lower bound, we need to show that \( \nu_o(pr_{\gamma^{-1}\circ B_r}(o)) \) is bounded away from 0 independent of \( \gamma \) as long as any initial length \( \geq R \) segment of \( [o, \gamma^{-1}\circ o] \) spends a proportion at least \( \theta \) in \( \text{Teich}_\epsilon(S) \) for which it would suffice to show that there is a \( E > 0 \) such that for all \( y \in \text{Teich}(S) \) such that any initial length \( \geq R \) segment of \( [o, y] \) spends a proportion at least \( \theta \) in \( \text{Teich}_\epsilon(S) \) \( \nu_o(pr_{\gamma^{-1}\circ B_r}(o)) \) > \( E \).

Suppose not. Then there is a sequence of such \( y_n \in \text{Teich}(S) \) converging to \( \zeta \in \text{PMF} \) with \( \nu_o(pr_{\gamma^{-1}\circ B_r}(o)) \rightarrow 0 \). By Masur’s criterion, \( \zeta \) is uniquely ergodic. Thus, \( \nu_o(pr_{\gamma^{-1}\circ B_r}(o)) = 0 \) which is impossible since \( \nu_o \) has full support on \( \text{PMF} \) and \( pr_{\gamma^{-1}\circ B_r}(o) \) contains an open set.

By Mumford’s compactness criterion and Proposition 3.2 we obtain the following corollary.

**Corollary 5.2.** For every \( \theta, \epsilon > 0 \) and \( R > 0 \) and each \( r > 0 \) larger than twice the diameter of a fundamental domain of \( \text{Teich}_\epsilon(S) \) there exists a \( C > 0 \) with the following property: for every \( x \in \text{Teich}_\epsilon(S) \) such that any initial length \( \geq R \) segment of \( [x, o] \) spends a proportion at least \( \theta \) in \( \text{Teich}_\epsilon(S) \) we have
\[ C^{-1}e^{-\hd(x,o)} \leq \nu_o(pr_oB_r(x)) \leq Ce^{-\hd(x,o)}. \]

The next lemma says that at least a uniform proportion of shadows of balls consists of directions which recur uniformly to the thick part.

**Lemma 5.3.** For every \( \theta > 0, R > 0, \rho > 0, \epsilon > 0 \) with \( \mu(M_\epsilon(S)) \leq \rho \), and \( r > 0 \) there is a \( K > 0 \) such that for each \( \epsilon' > 0 \) with \( \text{Nbd}_K \text{Teich}_\epsilon(S) \subset \text{Teich}_{\epsilon'}(S) \) there are \( M > 0 \) and \( C > 0 \) such that for every \( g \in \text{Mod}(S) \) such that any initial length \( \geq R \) segment of \( [o, \gamma^{-1}\circ o] \) spends a proportion at least \( \theta \) in \( \text{Teich}_{\epsilon'}(S) \) the set of \( \eta \in pr_oB_r(go) \) such that \( \gamma_{\eta, o}[d(o, go), d(o, go) + t] \) spends at a proportion of at least \( \rho \) in \( \text{Teich}_{\epsilon'}(S) \) for every \( t > M \) has measure at least \( Ce^{-\hd(o,go)} \).

**Proof.** Let \( A(o, go, r, M, \epsilon') \) be the set of \( \eta \in pr_oB_r(go) \) such that
\[ \gamma_{\eta, o}[d(o, go), d(o, go) + t] \]
spends at a proportion of at least \( \rho \) in \( \text{Teich}_{\epsilon'}(S) \) for every \( t > M \). By conformality of the Thurston measure,
\[ \frac{\nu_o(A(o, go, r, M, \epsilon'))}{\nu_o(pr_oB_r(go))} \geq e^{-4hr} \frac{\nu_o(g^{-1}A(o, go, r, M, \epsilon'))}{\nu_o(pr_{\gamma^{-1}\circ B_r}(o))}. \]

Note, \( \nu_o(pr_{\gamma^{-1}\circ B_r}(o)) > e > 0 \) for a positive number \( c > 0 \) depending only on \( \theta, \epsilon' \) so
\[ \frac{\nu_o(A(o, go, r, M, \epsilon'))}{\nu_o(pr_oB_r(go))} \geq D\nu_o(g^{-1}A(o, go, r, M, \epsilon')) \]
where \( D \) depends only on \( \epsilon', r, \theta, \rho \).
Moreover, if $\eta \in \text{pr}_o B_r(go)$ then by Proposition 3.2
\[
d(\gamma_{o,\eta}(t), \gamma_{g^{-1},o,\eta}(d(go, o) + t)) \leq K
\]
for all $t \geq 0$ where $K$ depends only on $r$. Hence, if $\gamma_{o,\eta}[0,t]$ spends a proportion of at least $\rho$ in $\text{Teich}_e(S)$ then $\gamma_{o,\eta}[d(o, go), d(o, go) + t]$ spends at a proportion of at least $\rho$ in $\text{Teich}_e(S)$. Let $E(r, M, \epsilon)$ be the set of $\eta \in \text{PMF}$ such that $\gamma_{o,\eta}[0,t]$ spends a proportion of at least $\rho$ in $\text{Teich}_e(S)$ for all $t > M$. Note for large enough $M$ we have $\nu_\varphi(E(r, M, \epsilon)) > 1 - \frac{\epsilon}{2}$ so
\[
\nu(g^{-1}A(o, go, r, M, \epsilon')) \geq \nu(E(r, M, \epsilon) \cap \text{pr}_{g^{-1},o} B_r(o)) \geq \frac{c}{2}
\]
completing the proof.

Again, by Mumford’s compactness criterion and Proposition 3.2 we obtain the following corollary.

**Corollary 5.4.** For each $\theta > 0$, $\rho > 0$, $\epsilon > 0$ with $\mu(\text{Teich}_e(S)/\text{Mod}(S)) \leq \rho$, $R > 0$ and $r > 0$ larger than twice the diameter of a fundamental domain of $\text{Teich}_e(S)$ there is a $K > 0$ such that for every $\epsilon' > 0$ with $\text{Nbd}_K \text{Teich}_e(S) \subset \text{Teich}_e(S)$ there are $M > 0$ and $C > 0$ such that for every $x \in \text{Teich}_e(S)$ such that any initial length $\geq R$ segment of $[x, o]$ spends a proportion at least $\theta$ in $\text{Teich}_e(S)$ the set of $\eta \in \text{pr}_o B_r(x)$ such that $\gamma_{o,\eta}[d(o, x), d(o, x) + t]$ spends at a proportion of at least $\rho$ in $\text{Teich}_e(S)$ for every $t > M$ has measure at least $Ce^{-\delta d(o, x)}$.

From now on, we will be able to restrict our attention to $m, n$ such that $m > L_2, K_2, \delta_2$ and $n > 1000m$ and we will do so without further notice.

**Proposition 5.5 (Proposition 4.2).** For each $b \in \Omega(m, n)$
\[
e^{hn - 100hm} \leq |Y_n(b)| \leq e^{hn + 100hm}
\]

**Proof.** Assume without loss of generality that $b \in \Omega(m, n) \setminus \bigcup_{k < m} \Omega(n, k)$ Note, for each $x \in \text{Teich}(S)$ as $T \to \infty$ the ball $B_T(\gamma_{x,b}(T))$ converges to $H(x, b, (-\infty, 0)]$

Furthermore, if
\[
q \in B_{n-35m}(\gamma_{o,b}(n + 15m))
\]
then by the triangle inequality,
\[
q \in B_{2n-20}(o) \cap B_T(\gamma_{o,b}(T + 50m)).
\]

On the other hand, suppose
\[
q \in B_{2n-20m}(o) \cap B_T(\gamma_{o,b}(T + 50m)).
\]

Since $\gamma_{o,b}[n + 15m, n + 16m]$ spends at least half the time in $\text{Teich}_e(S)$, it follows that $\gamma_{o,b}(n + 15m)$ is within $m$ of either $[o, q]$ or $[\gamma_{o,b}(T + 50m), q]$.

Thus,
\[
B_{n-35m}(\gamma_{o,b}(n + 15m)) \subset B_{2n-20}(o) \cap B_T(\gamma_{o,b}(T + 50m)) \subset B_{n-33m}(\gamma_{o,b}(n + 15m))
\]
and letting $T \to \infty$ we get
\[
B_{n-35m}(\gamma_{o,b}(n + 15m)) \subset B_{2n-20}(o) \cap H(o, b, (-\infty, -50m] \subset B_{n-33m}(\gamma_{o,b}(n + 15m)).
\]
Similarly we have:
\[
B_{n-35m}(\gamma_{o,b}(n + 15m)) \subset B_{2n-20}(o) \cap H(o, b, (-\infty, -50m] \subset B_{n-33m}(\gamma_{o,b}(n + 15m))
\]
\[ B_{n-15m}(\gamma_{o,b}(n+35m)) \subset B_{2n+20}(o) \cap H(o,b,(-\infty,-50m]) \subset B_{n-13m}(\gamma_{o,b}(n+35m)) \]

\[ B_{n-60m}(\gamma_{o,b}(n+40m)) \subset B_{2n-20}(o) \cap H(o,b,(-\infty,-100m]) \subset B_{n-58m}(\gamma_{o,b}(n+40m)) \]

\[ B_{n-40m}(\gamma_{o,b}(n+60m)) \subset B_{2n+20}(o) \cap H(o,b,(-\infty,-100m]) \subset B_{n-38m}(\gamma_{o,b}(n+60m)) \]

Since \( \gamma_{o,b}(n+35m) \) is within \( m \) of \( \text{Teich}_t(S) \) it is within \( m + \rho \leq 2m \) of some point of \( \text{Mod}(S)o \). Thus,

\[ Y_n(b) \subset \text{Mod}(S)o \cap B_{2n+20}(o) \cap H(o,b,(-\infty,-50m]] \subset \text{Mod}(S)o \cap B_{n-13m}(\gamma_{o,b}(n+35m)) \subset \text{Mod}(S)o \cap B_{n-11m}(g_1o) \]

By the orbit growth estimate of Theorem 1.1 in [1] this implies that

\[ |Y_n(b)| \leq C e^{h(n-11m)} \]

where \( C \) is a uniform constant. This proves the upper bound. Now we consider the lower bound; let \( W_n(b) \) be the set of \( g \in \text{Mod}(S) \) with \( d(o,go) \in (2n-20m,2n+20m) \) and \( \beta_b(go,a) \in [-100m,-50m]. \) (So \( W_n(b) \) is the same as \( Y_n(b) \) but without the thickness assumptions). Note, \( W_n(b) \) is contained in the intersection of \( \text{Mod}(S)o \) with

\[ (B_{2n+20}(o) \cap H(o,b,(-\infty,-50m]] \setminus (B_{2n+20}(o) \cap H(o,b,(-\infty,-100m]])) \cup (B_{2n-20}(o) \cap H(o,b,(-\infty,-50m)])) \]

\[ \supset B_{n-16m}(g_{20}) \setminus (B_{n-37m}(g_{30}) \cup B_{n-32m}(g_{40})) \]

for some \( g_2,g_3,g_4 \in \text{Mod}(S) \). By Theorem 1.1 in [1] this implies that

\[ |W_n(b)| \geq C e^{h(n-16m)} - C e^{h(n-37m)} - C e^{h(n-32m)} \geq D e^{h(n-16m)} \]

for a uniform constant \( D \). We claim that if \( g \in W_n(b) \) is such that \( \gamma_{g_{20},go}[o,t] \) spends a proportion of at least 0.9999 in \( \text{Teich}_t(S) \) for all \( t > m \) then \( g \in W_n(b) \). By Theorem 2.10 of [1] at least half of \( W_n(b) \) satisfy the property, so the proposition follows if the claim is true. Now, we prove the claim. Note, \( d(g_{20},\gamma_{o,b}(n+35m)) \leq m \).

Note, \( d(go,\gamma_{o,b}(n+40m)) \geq n - 60m \) so \( d(go,\gamma_{o,b}(n+35m)) \geq n - 65m \) and hence \( d(go,g_{20}) \geq n - 66m \).

So, we have

\[ n - 66m \leq d(go,g_{20}) \leq n - 12m \]

\[ n + 34m \leq d(o,g_{20}) \leq n + 36m \]

and

\[ 2n - 20m \leq d(o,go) \leq 2n + 20m. \]

If \( \gamma_{g_{20},go}(t) \) is within a \( \delta \) neighborhood of \( [o,g_{20}] \) we must therefore have

\[ 2n - 20m \leq d(o,go) \leq d(go,g_{20}) + d(o,g_{20}) + \delta - t \leq 2n + 24m + \delta - t \]

so

\[ t \leq 44m + \delta \leq 45m \]
Hence if \( \gamma_{go,go}(t) \) is within a \( \delta \) neighborhood of \([o,go]\) we must have \( t > d(go,go) - 45m > n - 120m \). Note, \( \gamma_{go,go}[n - 121m, n - 120m] \) spends at least 90 percent in \( Teich_e(S) \) so there is a \( t \in [n - 121m, n - 120m] \) with \( \gamma_{go,go}(t) \in Teich_e(S) \) and \( \gamma_{go,go}(t) \) within \( \delta \) of \([o,go]\). Since \( t < n - 120m \), \( \gamma_{go,go}(t) \) cannot be within \( \delta \) of \([o,go]\) so there is an \( s \in [−\delta, \delta] \) with \( d(\gamma_{go,go}(t), \gamma_{go,go}(t+s)) \leq \delta \). By Proposition 3.2 we have \( d(\gamma_{go,go}(t), \gamma_{go,go}(t+s)) \leq K \) for all \( t < n - 121m \) so at least 99 percent of \( \gamma_{go,go}[n - 121m - t, n - 121m] \) lies in \( Teich_e(S) \) for all \( n - 121m \geq t \geq m \).

**Proposition 5.6.** If \( b \in \Omega(n, m) \) and \( g \in Y_n(b) \) then for some \( t \in [n - 9m, n - 8m] \) with \( b(t) \in Teich_e(S) \) we have \( d(b(t), \gamma_{go,go}(t)) \leq 2\delta \)

**Proof.** Assume without loss of generality that \( b \in \Omega(n, m) \setminus \bigcup_{k<m} \Omega(n, k) \) Note, \( go \in B_{2n+20}(o) \cap H(o, b, (−\infty, −50)) \subset B_{n-13m}(b(n + 35m)) \).

Since \( \gamma_{o,b}[n - 10m, n - 9m] \) spends more than 0.9 of the time in \( Teich_e(S) \) there is a \( t \in [n - 10m, n - 9m] \) with \( b(t) \in Teich_e(S) \) such that \( b(t) \) is within \( \delta \) of \([o,go] \cup [b(n), go] \).

However, any point of \([b(n + 35m), go] \) is within \( n - 13m \) of \( go \) while \( d(b(t), go) \geq d(o, go) - d(b(t), o) \geq (2n - 20m) - (n - 9m) = n - 11m > n - 13m + \delta \).

Thus, \( b(t) \) is within \( \delta \) of \([o, go] \) completing the proof.

We therefore obtain:

**Corollary 5.7.** If \( b \in \Omega(n, m) \), \( g \in Y_n(b) \), then for all \( t \leq n - 9m \) we have \( d(b(t), \gamma_{go,go}(t)) \leq K \)

**Corollary 5.8.** If \( b \in \Omega(n, m) \), \( g \in Y_n(b) \), \( b' \in Z_n(g) \) then for all \( t \leq n - 20m \) we have \( d(b'(t), b(t)) \leq K + K_1 \leq 2K_1 \)

Using this and the shadow estimate from Lemma 5.1 we obtain

**Corollary 5.9 (Proposition 4.4).** For almost every \( b' \in \text{PMF} \)

\[
\nu\{b \in \Omega(n, m) : b' \in \bigcup_{g \in Y_n(b)} Z_n(g)\} \leq Ce^{10hm−hn}
\]

where \( C \) does not depend on \( m, n \).

**Proposition 5.10.** If \( b \in \Omega(n, m) \) and \( g \in Y_n(b) \) then for every \( t > n + 61m \) we have \( d(b(t), \gamma_{go,go}(t)) \geq 2K \)

**Proof.** Note, for large enough \( T \) we have

\[
d(go, b(T)) - T = d(go, b(T)) - d(o, b(T)) \geq -101m
\]

If \( d(b(t), \gamma_{go,go}(t)) \leq 2K \) then

\[
T - 101m \leq d(go, b(T)) \leq d(b(T), b(t)) + 2K + d(go, \gamma_{go,go}(t)) \leq (T - t) + 2K + (2n + 20m - t)
\]

so

\[
2t \leq 2n + 121m + 2K \leq 2n + 121m
\]

so \( t \leq n + 61m \).

**Corollary 5.11.** If \( b \in \Omega(n, m) \) and \( g \in Y_n(b) \) then there exists an \( s \in [−\delta, \delta] \) such that \( d(b(t), \gamma_{go,b}(t+s)) \leq K \) for all \( t > 62m \).
Proof. Since $\gamma_{b,b}^n + 61m, n + 62m]$ spends at a proportion of at least 0.9 in $\text{Teich}^e(S)$ there is a $t_0 \in [n + 61m, n + 62m]$ with $b(t_0) \in \text{Teich}^e(S)$ such that $d(b(t_0), \gamma_{b,b}(t_0 + s)) \leq \delta$ for some $|s| < \delta$. By Proposition 3.2, $d(b(t), \gamma_{b,b}(t + s)) \leq K$ for all $t > t_0$. \hfill \Box

Proposition 5.12 (Proposition 4.6). For each uniquely ergodic $b \in \text{PMF}$

$$\{|g \in \text{Mod}(S) : gb \in \Omega(n, m), g \in Y_n(gb)\} \lesssim_m e^{hn}$$

Proof. If $g \in Y_n(gb)$ then

$$d(g^{-1}o, o) \leq 2n + 20m$$

and

$$\beta_b(g^{-1}o, o) = -\beta_b(o, g^{-1}o) = -\beta_{gb}(o, o) \leq 100m$$

Moreover, if $gb \in \Omega(n, m)$ then there exists an $s \in [-\delta, \delta]$ such that

$$d(\gamma_{o,gb}(t), \gamma_{go,gb}(t + s)) \leq K$$

for all $t > 62m$. Hence, at least 60 percent of $\gamma_{go,gb}[n + 65m, n + 66m]$ lies in $\text{Teich}^e(S)$. Note, as $T \to \infty$ we have $B(\gamma_{o,b}(T - 100m)) \to H(o, b, (iny, 100m))$. Suppose $q \in B_{2n+20m}(o) \cap B(\gamma_{o,b}(T - 100m))$. Since $\gamma_{o,b}[n + 61m, n + 62m]$ spends at least 60 percent in $\text{Teich}^e(S)$, it follows that $\gamma_{b,b}(n - 40m)$ is within $105m$ of either $[o, q]$ or $[\gamma_{o,b}(T - 100m), q]$. In the first case,

$$d(\gamma_{o,b}(n - 40m), q) \leq d(o, q) - d(\gamma_{o,b}(n - 40m), o) + 50m \leq (2n + 20m) - (n - 40m) + 210m \leq n + 270m$$

Similarly, in the second case

$$d(\gamma_{o,b}(n - 40m), q) \leq n + 270m$$

So, letting $T \to \infty$ we get

$$B_{2n+20m}(o) \cap H(o, b, (iny, 100m)) \subset B_{n+270m}\gamma_{o,b}(n - 40m) \subset B_{n+400m}(g_o)$$

for some $g_o \in \text{Mod}(S)$. Thus by Theorem 1.1 of [1]

$$|\text{Mod}(S)o \cap B_{2n+20m}(o) \cap H(o, b, (\infty, 100m))| \leq Ce^{hn+400hm}$$

for some uniform constant $C$. \hfill \Box

Proposition 5.13 (Proposition 4.5). For all $b' \in \text{PMF}$ the number of $g \in \text{Mod}(S)$ with $gb' \in Z_n(g)$ and $g \in Y_n(b)$ for some $b \in \Omega(n, m)$ has cardinality $\lesssim_m e^{hn}$.

Proof. If $gb' \in Z_n(g)$ then

$$\beta_{b'}(g^{-1}o, o) = -\beta_{gb'}(o, o) \leq 100m$$

and $d(g^{-1}o, o) \leq 2n + 20m$. Moreover, at least 60 percent of $\gamma_{b,b}[n - 10m, n - 9m]$ lies in $\text{Teich}^e(S)$ so the result follows by the same argument as Proposition 4.6. \hfill \Box

Proposition 5.14 (Proposition 4.3). For each $b \in \Omega(n, m)$ and $g \in Y_n(b)$ we have

$$Ce^{hn} \leq \nu(Z_n(g)) \leq De^{21hm-hn}$$

for some uniform constants $C$ and $D$.
Proof. Assume without loss of generality that $b \in \Omega(n, m) \setminus \bigcup_{k \leq m} \Omega(n, k)$.

Let $t \in [n - 20m, n - 9m]$ with $b(t) \in \text{Teich}_v(S)$. We have $d(b(t), b'(t)) \leq 2K_1$ and hence $b' \in pr_2B_{2K_1}(b(t))$. By definition of $\Omega(n, m)$ for all $s > 2m$ the segment $b([t - s, t])$ spends at least half the time in $\text{Teich}_v(S)$. By Proposition 3.1 this implies

$$
\nu_o(Z_n(g)) \leq \nu_o(pr_2B_{2K}(b(t))) \leq Ce^{21hm-hn}
$$

where $C$ is independent of $m, n$. For the lower bound, consider $t_1 \in [n - 20m, n - 9m], t_2 \in [n - 10m, n - 9m]$ with $b(t_1) \in \text{Teich}_v(S)$. Note, we have $d(b(t_1), \gamma_{o,go}(t_1)) \leq K$ so $\gamma_{o,go}(t_1) \in \text{Teich}_v(S)$. Moreover, $b([t_1 - t, t_1])$ spends at least half the time in $\text{Teich}_v(S)$ for each $t \in [2m, t_1]$ so $\gamma_{o,go}([t_1 - t, t_1])$ spends at least half the time in $\text{Teich}_v(S)$. Thus, $Z_n(g)$ contains all the $b' \in pr_2B_{2K_1}(\gamma_{o,go}(t_1)) \setminus pr_2B_{K_2}(\gamma_{o,go}(t_1))$ such that $b'([n - 9m, n - 8m])$ spends more than 90 percent of the time in $\text{Teich}_v(S)$. By Proposition 4.4, the $\nu$ measure of the $b' \in pr_2B_{2K_1}(\gamma_{o,go}(t_1))$ such that $b'([n - 9m, n - 8m])$ spends more than 90 percent of the time in $\text{Teich}_v(S)$ is $\geq Ce^{21hm-hn}$ and $\nu_o(pr_2B_{K_2}(\gamma_{o,go}(t_1))) \geq De^{9hm-hn}$ for $C, D$ independent of $n, m$. Thus, for $e^{12hm} > 2D/C$ we have

$$
\nu(Z_n(g)) \geq \frac{D}{2} e^{9hm-hn}
$$

and so obtain the desired result.

□

Proposition 5.15 (Proposition 4.8). For each $b \in \Omega(n, m)$, $g \in Y_n(b)$ and $b' \in Z_n(g)$ we have $-6m \leq \beta_{b'}(g, o) \leq 21m$

Proof. Assume without loss of generality that $b \notin \Omega(n, k)$ for any $k < m$. Assume $T > n > 1000m$. Note, $d(b'^{(n - 20m)}, \gamma_{o,go}(n - 20m)) \leq K_1$ so

$$
d(go, b'(T)) \leq d(b'(n - 20m), \gamma_{o,go}(n - 20m)) + d(b'(n - 20m), b(T)) + d(\gamma_{o,go}(n - 20m), go)
$$

$$
\leq K_1 + (T - n + 20m) + [(2n - 20m) - (n - 20m)] = K_1 + T + 20m
$$

So

$$
d(b'(T), go) - d(b'(T), o) \leq K_1 + 20m \leq 21m
$$

for all $T$ so

$$
\beta_{b'}(g, o) \leq 21m
$$

On the other hand, for some $t \in [n - 10m, n - 9m]$ we have $d(\gamma_{o,go}(t), b'(t)) \geq K_1$ and so for all $t \geq n - 9m$ we have $d(\gamma_{o,go}(t), b'(t)) \geq 2\delta_1$. Since at least 60 percent of $\gamma_{o,go}[n - 9m, n - 8m]$ and $b'([n - 9m, n - 8m])$ is $\text{Teich}_v(S)$, it follows that $\gamma_{o,go}(n - 9m)$ and $b'(n - 9m)$ are both within $m/2 + \delta_2 < m$ of points on $[go, b'(T)]$. Thus,

$$
d(go, b'(T)) + 4m \geq d(go, \gamma_{o,go}(n - 9m)) + d(b'(n - 9m), \gamma_{o,go}(n - 9m)) + d(b'(n - 9m), b(T))
$$

$$
\geq [(2n - 20m) - (n - 9m)] + (T - n + 9m) = T - 2m.
$$

Hence,

$$
d(go, b'(T)) - d(o, b'(T)) \geq -6m
$$

and letting $T \to \infty$ we get

$$
\beta_{b'}(g, o) \geq -6m
$$

□
Proposition 5.16. If \( b \in \Omega(n, m) \) and \( g \in Y_n(b) \) then for every \( t > n + 61m \) we have \( d(b'(t), \gamma_{o, go}(t)) \geq 2K_1 \)

Proof. This is proved in the same way as Proposition 4.10. \( \square \)

Proposition 5.17 (Proposition 4.7). For each \( b \in \Omega(n, m) \), \( g \in Y_n(b) \), \( b' \in Z_n(g) \) we have

\[
b, b', g^{-1}b, g^{-1}b' \in pr_{o, g^{-1}o}B_{\delta_1}(\gamma_{o, g^{-1}o}(t))
\]

for some \( t > n - 122m \) with \( \gamma_{o, g^{-1}o}(t) \in \text{Teich}_c(S) \)

Proof. It is enough to prove that \( b, b' \in pr_{go, o}B_{\delta_1}(\gamma_{go, o}(t)) \)

Note, \( \gamma_{go, o}([n - 122m, n - 121m]) \) spends at least 90 percent in \( \text{Teich}_c(S) \), so there is a \( t \in [n - 122m, n - 121m] \) with \( \gamma_{go, o}(t) \in \text{Teich}_c(S) \) so that \( \gamma_{go, o}(t) \) is within \( \delta_1 \) of \( [go, b] \cup [o, b] \) and also of \( [go, b'] \cup [o, b'] \) Note, \( \gamma_{go, o}(s) = \gamma_{o, go}(s) \) for \( s = d(go, o) - t \geq 2n - 20m - (n - 121m) > n + 100 \) so we must have points \( \gamma_{go, b}(t_1) \) and \( \gamma_{go, b'}(t_2) \) within \( \delta_1 \) of \( \gamma_{go, o}(t) \). By Proposition 3.2 we obtain the desired result. \( \square \)

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