Models with varying structure

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Abstract

In this paper the problems of the retrospective analysis of models with time-varying structure are considered. These models include contamination models with randomly switching parameters and multivariate classification models with an arbitrary number of classes. Our main task here is to classify observations with different stochastic generation mechanisms. A new classification method is proposed. We analyze its properties both theoretically and empirically. The asymptotic optimality of the proposed method (by the order of convergence to zero of the estimation error) is also established. At the end of the paper we consider multivariate change-in-mean models and multivariate regression models.

Keywords. Multivariate stochastic model, time-varying structure, dependent observations, $\psi$-mixing conditions, $\psi$-weak dependence, type 1 error, type 2 error, asymptotic optimality, regression model, switching coefficients

1. Introduction

The previous papers of the authors (see, e.g., "Statistical analysis of models with varying structure" (Applied Econometrics, 2015, in Russian), "Multivariate models with varying structure: a binary case" (Review of Applied and Industrial Mathematics, 2016, in Russian) were devoted to the main particular cases of the general problem: how to split univariate mixtures of probabilistical distributions and to perform multivariate classification with only two classes of observations (ordinary observations and outliers). In this paper we consider the general problem of multivariate classification with an arbitrary number of classes of observations.

First, let us mention previous important steps into this field. Models with switching regimes have a long pre-history in statistics (see, e.g., Lindgren (1978)).
A simple switching model with two regimes has the following form:

\[ Y_t = X_t \beta_1 + u_{1t} \text{ for the 1st regime} \]
\[ Y_t = X_t \beta_2 + u_{2t} \text{ for the 2nd regime}. \]

For models with endogenous switchings usual estimation techniques for regressions are not applicable. Goldfeld and Quandt (1973) proposed \textit{regression models with Markov switchings}. In these models probabilities of sequential switchings are supposed to be constant. Usually they are described by the matrix of probabilities of switchings between different states.

Another modification of the regression models with Markov switchings was proposed by Lee, Porter (1984). The following transition matrix was studied:

\[ \Lambda = [p_{ij}]_{i,j=0,1}, \quad p_{ij} = P\{I_t = j|I_{t-1} = i\}. \]

Lee and Porter (1984) consider an example with railway transport in the US in 1880-1886s which were influenced by the cartel agreement. The following regression model was considered:

\[ \log P_t = \beta_0 + \beta_1 X_t + \beta_2 I_t + u_t, \]

where \( I_t = 0 \) or \( I_t = 1 \) in dependence of 'price wars' in the concrete period.

Cosslett and Lee (1985) generalized the model of Lee and Porter to the case of serially correlated errors \( u_t \).

Many economic time series occasionally exhibit dramatic breaks in their behavior, associated with with events such as financial crises (Jeanne and Mason, 2000; Cerra, 2005; Hamilton, 2005) or abrupt changes in government policy (Hamilton, 1988; Sims and Zha, 2004; Davig, 2004). Abrupt changes are also a prevalent feature of financial data and empirics of asset prices (Ang and Bekaert, 2003; Garcia, Luger, and Renault, 2003; Dai, Singleton, and Wei, 2003).

The functional form of the 'hidden Markov model' with switching states can be written as follows:

\[ y_t = c_{s_t} + \phi y_{t-1} + \epsilon_t, \]

where \( s_t \) is a random variable which takes the values \( s_t = 1 \) and \( s_t = 2 \) obeying a two-state Markov chain law:

\[ Pr(s_t = j|s_{t-1} = i, s_{t-2} = k, \ldots, y_{t-1}, y_{t-2}, \ldots) = Pr(s_t = j|s_{t-1} = i) = p_{ij}. \]
A model of this form with no autoregressive elements \((\phi = 0)\) appears to have been first analyzed by Lindgren (1978) and Baum, et al. (1980). Specifications that incorporate autoregressive elements date back in the speech recognition literature to Poritz (1982), Juang and Rabiner (1985), and Rabiner (1989). Markov-switching regressions were first introduced in econometrics by Goldfeld and Quandt (1973), the likelihood function for which was first calculated by Coslett and Lee (1985). General characterizations of moment and stationarity conditions for Markov-switching processes can be found in Tjostheim (1986), Yang (2000), Timmermann (2000), and Francq and Zakoian (2001).

A useful review of modern approaches to estimation in Markov-switching models can be found in Hamilton (2005).

However, the mechanism of Markov chain modeling is far not unique in statistical description of dependent observations. Besides Markov models, we can mention martingale and copula approaches to dealing with dependent data, as well as description of statistical dependence via different coefficients of 'mixing'. All of these approaches are interrelated and we must choose the most appropriate method for the concrete problem. In this paper we choose the mixing paradigm for description of statistical dependence.

Now let us mention some important problems which lead to stochastic models with switching regimes.

**Splitting mixtures of probabilistic distributions**

In the simplest case we suppose that the d.f. of observations has the following form:

\[
F(x) = (1 - \epsilon)F_0(x) + \epsilon F_1(x),
\]

where \(F_0(x)\) is the d.f. of ordinary observations; \(F_1(x)\) is the d.f. of abnormal observations; \(0 \leq \epsilon < 1\) is the probability of obtaining an abnormal observation.

We need to test the hypothesis of statistical homogeneity (no abnormal observations) of an obtained sample \(X^N = \{x_1, x_2, \ldots, x_N\}\). If this hypothesis is rejected then we need to classify this sample into sub-samples of ordinary and abnormal observations.

**Estimation for regression models with abnormal observations**

The natural generalization of the previous model is the regression model with abnormal observations

\[
Y = X\beta + \epsilon,
\]
where \( Y \) is the \( n \times 1 \) vector of dependent observations; \( X \) is the \( n \times k \) matrix of predictors; \( \beta \) is \( k \times 1 \) vector of regression coefficients; \( \epsilon \) is the \( n \times 1 \) vector of random noises with the d.f. of the following type:

\[
 f_\epsilon(x) = (1 - \delta)f_0(x) + \delta f_1(x),
\]

where \( 0 \leq \delta < 1 \) is the probability to obtain an abnormal observation; \( f_0(x) \) is the density function of ordinary observations; \( f_1(x) \) is the density function of abnormal observations. For example, in the model with Huber’s contamination [Huber, 1985]:

\[
 f_0(\cdot) = \mathcal{N}(0, \sigma^2), \quad f_1(\cdot) = \mathcal{N}(0, \Lambda^2), \quad \Lambda >> \lambda > 0.
\]

**Estimation for regression models with changing coefficients**

Regression models with changing coefficients is another generalization of the contamination model. We suppose that regression coefficients of this model can change (switch) from the level \( \beta_0 \) to \( \beta_1 \) and the mechanism of this change is random. We need to test the hypothesis about the absence of switchings for each coefficient (\( \epsilon = 0 \)) and in the case of rejection of this hypothesis to classify observations into different groups.

We need again to test the hypothesis of statistical homogeneity of an obtained sample and to divide this sample into sub-samples of ordinary and abnormal observations if the homogeneity hypothesis is rejected.

The goal of this paper is to propose methods which can solve these problems effectively. Theoretically, we mean estimation of type 1 and type 2 errors in testing the statistical homogeneity hypothesis and with estimation of contaminations parameters in the case of rejectiong this hypothesis. Practically, we propose procedures for implementation of these methods for univariate and multivariate models.

Problems considered in this paper differ substantially from classical change-point problems in which we suppose that distances between various regimes are big enough. In this paper we consider contamination models with coefficients changing in a random way.

The structure of this paper is as follows. In sections 2 and 3 we consider univariate models with switching effects. In section 2 for binary mixtures of probabilistic distributions we prove theorem 1 about exponential convergence to zero of type 1 error in classification (to detect switches for a statistically homogenous sample) as the sample size \( N \) tends to infinity; theorem 2 about exponential convergence to zero of type 2 error (vice versa, to accept stationarity hypothesis for a sample with switches). In section 3.3 we prove theorem 3 which
establishes the lower bound for the error of classification for binary mixtures. From theorems 2 and 3 we conclude that the proposed method is asymptotically optimal by the order of convergence to zero of the classification error.

Different generalizations of the proposed method for the case of univariate models with multiple switching regimes and for multivariate models with switching regimes are considered in sections 3.4 and 3.5. Results of a detailed Monte Carlo study of the proposed method for different stochastic models with switching regimes are presented.

In section 4 we consider multivariate models. Multivariate classification problems are considered in section 4.1. Section 4.2 deals with multivariate regression models.

2. Problems statement

2.1. Change-in-mean problems

Suppose the d.f. of the observations is the binary mixture

\[ f(x) = (1 - \epsilon)f_0(x) + \epsilon f_1(x), \]

where the density functions \( f_0(\cdot), f_1(\cdot) \) and the parameter \( \epsilon \) are unknown. We also suppose that

\[ \mathbb{E}_0(x) = \int x f_0(x) dx = 0, \quad \mathbb{E}_1(x) = \int x f_1(x) dx = h \neq 0, \]

where everywhere in this paper we denote by \( \mathbb{P}_0(\mathbb{E}_0) \) measure (mathematical expectation) of the sequence \( X^N \) under the condition \( \epsilon = 0 \) (no ‘abnormal’ observations.

The problem is to classify all obtained observations into subsamples of ordinary data and outliers.

The estimation method is as follows:

1) From the initial sample \( X^N \) compute the estimate of the mean value:

\[ \theta_N = \frac{1}{N} \sum_{i=1}^{N} x_i \]

2) Fix the numbers \( 0 < \kappa < B \) and parameter \( b \in \mathbb{B} \overset{\text{def}}{=} [\kappa, B] \) and classify observations as follows: if an observation falls into the interval \((\theta_N - b, \theta_N + b)\), then we place it into the sub-sample of ordinary observations, otherwise - to the sub-sample of abnormal observations.
3) Then for each \( b \in \mathbb{B} \) we obtain the following decomposition of the sample \( X^N \) into two sub-samples

\[
X_1(b) = \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{N_1}\}, \quad |\tilde{x}_i - \theta_N| < b,
\]

\[
X_2(b) = \{\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{N_2}\}, \quad |\hat{x}_i - \theta_N| \geq b
\]

Denote by \( N_1 = N_1(b), N_2 = N_2(b), N = N_1 + N_2 \) the sizes of the sub-samples \( X_1 \) and \( X_2 \), respectively.

The parameter \( b \) is chosen so that the sub-samples \( X_1 \) and \( X_2 \) are separated in the best way. For this purpose, consider the following statistic:

\[
\Psi_N(b) = \frac{1}{N^2} (N_2 \sum_{i=1}^{N_1} \tilde{x}_i - N_1 \sum_{i=1}^{N_2} \hat{x}_i).
\]

4) Define the boundary \( C > 0 \) and compare it with the value \( J = \max_{b \in \mathbb{B}} |\Psi_N(b)| \) on the set \( b \in \mathbb{B} \). If \( J \leq C \) then we accept the hypothesis \( H_0 \) about the absence of abnormal observations; if, however, \( J > C \) then the hypothesis \( H_0 \) is rejected.

Remark that testing the hypotheses \( H_0, H_1 \) does not require knowledge of the distribution law of observations.

2.2. Regression models with time-varying structure

Here the following model of observations is considered:

\[
Y = X\alpha + u_i = X(\zeta \beta^0 + (1 - \zeta) \beta^1) + U,
\]

where

\[
Y = (y_1, \ldots, y_N)^T \text{ is a } N \times 1 \text{ vector of dependent observations (here and below the sign } ^T \text{ denotes matrix transposition)};
\]

\[
X \text{ - } N \times k \text{ matrix of predictors};
\]

\[
U \text{ - } N \times 1 \text{ vector of centered random noises};
\]

\[
\alpha \text{ - } k \times 1 \text{ vector of model coefficients},
\]

\[
\zeta \text{ - Bernoulli distributed random variable (independent from } U\text{) with two states: 1 with probability } (1 - \epsilon) \text{ and 0 with probability } \epsilon \text{ for a certain unknown parameter } 0 < \epsilon < 1. \text{ Here } \beta^0 \neq \beta^1, k \text{ is the number of model coefficients.}
\]

In words we suppose that coefficients of this model can switch from the level \( \beta^0 \) into the level \( \beta^1 \), and the mechanism of these switchings is random. We need to test the hypothesis of no switches in each coefficient \( (\epsilon = 0) \).

Below we propose the method of solving this problem by means of its reduction to the previous problem.
3. Main results

3.1. Assumptions

The results given below are based upon two main assumptions. The first assumption is formulated in the form of a condition of diminishing dependence between the past and the future of observed processes as the distance between them increases. The second condition takes the form of Cramer’s assumption about the speed of decrease of 'tails' of distributions.

A1.

a) Mixing conditions

On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two $\sigma$-algebras from $\mathcal{F}$.

Consider the following measure of dependence between $\mathcal{H}_1$ and $\mathcal{H}_2$:

$$\psi(\mathcal{H}_1, \mathcal{H}_2) = \sup_{A \in \mathcal{H}_1, B \in \mathcal{H}_2, \mathbb{P}(A) \mathbb{P}(B) \neq 0} \left| \frac{\mathbb{P}(AB)}{\mathbb{P}(A)\mathbb{P}(B)} - 1 \right|$$

Suppose $\{y_n\}, n \geq 1$ is a sequence of random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Denote by $\mathcal{F}_s^t = \sigma\{y_i : s \leq i \leq t\}, 1 \leq s \leq t < \infty$ the minimal $\sigma$-algebra generated by random variables $y_i, s \leq i \leq t$. Define

$$\psi(n) = \sup_{t \geq 1} \psi(\mathcal{F}_1^t, \mathcal{F}_{1+n})$$

We say that a random sequence $\{y_n\}$ satisfies the $\psi$-mixing condition if the function $\psi(n)$ (which is also called the $\psi$-mixing coefficient) tends to zero as $n$ goes to infinity.

The $\psi$-mixing condition is satisfied in most practical cases. In particular, for a Markov chain (not necessarily stationary), if $\psi(n) < 1$ for a certain $n$, then $\psi(k)$ goes to zero at least exponentially as $k \to \infty$ (see Bradley, 2005, theorem 3.3).

b) Nowadays, however, the notion of "weak dependence" of observations is more often used:

**Definition 2** (Doukhan, Louhichi, 1999). The sequence $\{X_i\}$ is called $(\theta, \mathcal{L}, \psi)$-weak dependent (or simply $\psi$-weak dependent), if there exists a sequence $\theta = (\theta_r)$ tending to zero as $r \to \infty$, and the function $\psi$ with the argument $(f, h, n, m) \in \mathcal{L}_n \times \mathcal{L}_m \times \mathbb{N}^2$ such that for any sets of indices $(i_1, \ldots, i_n)$ and $(j_1, \ldots, j_m)$ ($i_1 \leq \cdots \leq i_n < i_n + r < j_1 \leq \cdots \leq j_m)$:

$$|\text{Cov}(f(X_{i_1}, \ldots, X_{i_n}), h(X_{j_1}, \ldots, X_{j_m}))| \leq \psi(f, h, n, m)\theta_r.$$
It is often supposed that

\[ \theta_r = e^{-\beta r}, \quad \beta > 0. \]

The 'weak dependence' condition is true in majority of practical cases. In particular, Ango Nze, Doukhan (2004) showed that \( \psi \)-weak dependence assumption generalizes conditions of mixing, association, etc., for Gaussian sequences and 'Bernoulli shifts'. They proved that all ARMA and bilinear processes are \( \psi \)-weak dependent. We can assume \( \psi \)-weak dependence while considering all practically important cases in statistics.

**A2. Cramer condition**

We say that the sequence \( \{y_n\} \) satisfies the **uniform Cramer condition** if there exists \( T > 0 \) such that for each \( i \), \( \mathbb{E} \exp(ty_i) < \infty \) for \( |t| < T \).

For a centered sequence \( \{y_n\} \) this condition is equivalent to the following (see Petrov, 1987): there exist \( g > 0, H > 0 \) such that

\[ \mathbb{E}e^{ty_n} \leq e^{\frac{1}{2}gt^2}, \quad |t| \leq H, \]

for all \( n = 1, 2, \ldots \).

**We assume that conditions A1 and A2 hold true everywhere in the paper.**

For any \( x > 0 \) let us choose the number \( \gamma(x) \) from the following condition:

\[ \ln(1 + \gamma(x)) = \begin{cases} \frac{x^2}{4g}, & x \leq gH \\ \frac{xH}{4}, & x > gH \end{cases}, \]

where \( g, H \) are taken from the uniform Cramer condition.

For the chosen \( \gamma(x) \), let us find such integer \( \phi_0(x) \geq 1 \) from the \( \psi \)-mixing condition that \( \psi(l) \leq \gamma(x) \) for \( l \geq \phi_0(x) \).

In the following theorem the exponential upper estimate for type 1 error is obtained for the proposed method.

**3.2. Method**

Below we use the statistic \( \Psi_N(b) \) defined in the previous section. We note that it is a variant of the statistic that first appeared in our papers and books (Brodsky, Darkhovsky, 1986, 1993, 2000) devoted to the analysis of change-point problems. Methodologically, it ascends to Kolmogorov’s test for detection differently distributed random samples and to Hurst test in R/S analysis.

**Theorem 1.**
Let \( \epsilon = 0 \). Suppose the d.f. \( f_0(\cdot) \) is symmetric w.r.t. zero and bounded. If \( \psi \)-mixing and Cramer’s conditions are satisfied then for any \( 0 < \kappa < B \) there exists \( C > 0 \) such that the following estimate holds:

\[
P_0\{ \sup_{b \in B} |\Psi_N(b)| > C \} \leq L_1 \exp(-L_2(C)N),
\]

where the constants \( L_1, L_2 > 0 \) do not depend on \( N \).

However, if \( \psi \)-weak dependence and Cramer’s conditions are satisfied then

\[
P_0\{ \sup_{b \in B} |\Psi_N(b)| > C \} \leq L_1 \exp(-L_2(C)\sqrt{N}),
\]

where, again, the constants \( L_1, L_2 > 0 \) do not depend on \( N \).

The proof of Theorem 1 is given in the Appendix.

Now consider characteristics of this method in case \( \epsilon h \neq 0 \). Here we again assume that \( E_0 x_i = 0, i = 1, \ldots, N \).

Put (for some fixed \( \epsilon, h \))

\[
r(b) = \int_{\epsilon h - b}^{\epsilon h + b} f(x)dx, \quad d(b) = \int_{\epsilon h - b}^{\epsilon h + b} f(x)dx
\]

\[
\Psi(b) = r(b) - \epsilon hd(b).
\]

In the following theorem type 2 error is studied.

**Theorem 2.**

1) Suppose \( \psi \)-mixing and Cramer’s conditions are satisfied and there exists \( r^* = \sup_{b \in B} r(b) \). Suppose also that the density function \( f(x) \) is continuous and there exists \( f''(\cdot) \neq 0 \). Then for \( 0 < C < \max_{b \in B} |\Psi(b)| \) we have

\[
P_\epsilon\{ \max_{b \in B} |\Psi_N(b)| \leq C \} \leq L_1 \exp(-L_2(\delta)N)).
\]

where \( \delta = \max_{b \in B} |\Psi(b)| - C > 0 \).

2) If \( \psi \)-weak dependence condition is satisfied instead of \( \psi \)-mixing, then

\[
P_\epsilon\{ \max_{b \in B} |\Psi_N(b)| \leq C \} \leq L_1 \exp(-L_2(\delta)\sqrt{N}).
\]

where \( \delta = \max_{b \in B} |\Psi(b)| - C > 0 \).

3) For solving estimation problems, we suppose that the underlying model is

\[
f(x) = (1 - \epsilon)f_0(x) + \epsilon f_0(x - h),
\]

(1)
where \(0 < \epsilon < 1/2, h > 0\) are unknown positive parameters.

Let us consider the equation:

\[
 f(\epsilon h - b^*) = f(\epsilon h + b^*)
\]

Here we suppose that equation (!!) has a unique root \(b^*\) (for fixed \(\epsilon, h\)).

Then \(b^*_N \to b^* \text{ P}_\epsilon\text{-a.s. as } N \to \infty\), where \(b^*_N > 0\) is the estimate of \(b^*\): \(b^*_N \in \arg \max_{b \in \mathbb{R}} |\Psi_N(b)|\). Consider the following estimates of \(\epsilon\) and \(h\):

\[
 \hat{\epsilon}_N \hat{h}_N = \theta_N \\
 1 - \hat{\epsilon}_N = \frac{f_0(\theta_N - b^*_N - \hat{h}_N) - f_0(\theta_N + b^*_N - \hat{h}_N)}{f_0(\theta_N + b^*_N) - f_0(\theta_N - b^*_N)}.
\]

Then the estimates \(\hat{\epsilon}_N, \hat{h}_N\) converge \(\text{ P}_\epsilon\text{-a.s. to the true values of the parameters } \epsilon, h\), respectively, as \(N \to \infty\).

The proof of theorem 2 is given in the Appendix.

**Simulations**

We note that all constants in the above upper estimates of type 1 and type 2 errors are purely qualitative by their nature. Therefore simulations of the proposed method are essential in the analysis of its properties.

In the first series of tests the following mixture model was studied:

\[
 f_\epsilon(x) = (1 - \epsilon)f_0(x) + \epsilon f_0(x - h), \quad f_0(\cdot) = \mathcal{N}(0,1), \quad 0 \leq \epsilon < 1/2.
\]

First, the critical thresholds of the decision statistic \(\max_{b \in \mathbb{R}} |\Psi_N(b)|\) were computed. For homogenous samples of different size (i.e. without switches), p-quantiles of the decision statistic were computed. For this purpose, a Gaussian random sample with determined parameters was generated. After that all steps of the above described method were done. The values of the method’s parameters: \(\kappa = 0.04, B = 50\).

The maximum of the absolute value of the decision statistic was computed. This procedure was iterated 1000 times and the variation series of the maximums of the absolute values of the decision statistic was constructed. Then p-quantiles (with \(p = 0.95\) and \(p = 0.99\)) in this series were computed. The obtained results are given in Table 1.

**Table 1.**

| \(N\)  | 50     | 100    | 300    | 500    | 800    | 1000   | 1200   | 1500   | 2000   |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| \(\alpha = 0.95\) | 0.1681 | 0.1213 | 0.0710 | 0.0534 | 0.044  | 0.0380 | 0.037  | 0.034  | 0.029  |
| \(\alpha = 0.99\) | 0.1833 | 0.1410 | 0.0869 | 0.0666 | 0.050  | 0.0471 | 0.0390 | 0.038  | 0.035  |
In the second series of tests the quantile value for \( p = 0.95 \) was chosen as the critical threshold \( C \) in experiments with non-homogenous samples (for \( \epsilon \neq 0 \)). For different sample sizes in 1000 independent trials of each test, the estimate of type 2 error \( w_2 \) (i.e. the frequency of the event \( \max_{b \in B} |\Psi_N(b)| < C \) for \( \epsilon > 0 \)). The results are presented in table 2.

### Table 2.

| \( \epsilon = 0.1 \) | \( h=2.0 \) | \( C \) | \( w_2 \) | \( h=1.5 \) | \( C \) | \( w_2 \) |
|---|---|---|---|---|---|---|
| \( N \) | 300 | 0.0710 | 0.26 | 800 | 0.0534 | 0.15 |
|  | 500 | 0.044 | 0.044 | 1000 | 0.038 | 0.05 |
|  | 800 | 0.037 | 0.02 | 1200 | 0.029 | 0.02 |
|  | 1200 | 0.022 | 0.03 | 2000 | 0.022 | 0.03 |
|  | 3000 | 0.022 | 0.03 | 2000 | 0.022 | 0.03 |

### 3.3. Asymptotic optimality

Now consider the question about the asymptotic optimality of the proposed method in the class of all estimates of the parameter \( \epsilon \). The a priori theoretical lower bound for the estimation error of the parameter \( \epsilon \) in the model with i.i.d. observations with d.f. \( f_\epsilon(x) = (1 - \epsilon)f_0(x) + \epsilon f_1(x) \) is given in the following theorem.

**Theorem 3.** Let \( \mathcal{M}_N \) be the class of all estimates of the parameter \( \epsilon \). Then for any \( 0 < \delta < \epsilon \),

\[
\liminf_{N \to \infty} \inf_{\hat{\epsilon}_N \in \mathcal{M}_N} \sup_{0 < \epsilon < 1/2} \frac{1}{N} \ln \mathbf{P}_\epsilon \{ |\hat{\epsilon}_N - \epsilon| > \delta \} \geq -\delta^2 J(\epsilon),
\]

where \( J(\epsilon) = \int \frac{(f_0(x) - f_1(x))^2}{f_\epsilon(x)} \, dx \) is the generalized \( \chi^2 \) distance between densities \( f_0(x) \) and \( f_1(x) \) and \( \mathbf{P}_\epsilon \) is the measure corresponding to the density \( f_\epsilon(x) \).

**Proof.**

Remark that it suffices to consider consistent estimates of the parameter \( \epsilon \) (for non-consistent estimates the limit in the left hand of the above inequality is equal to zero). This class is not empty because of the method proposed in the paper.

Suppose \( \hat{\epsilon}_N \) is any consistent estimate of \( \epsilon \) and \( 0 < \delta < \delta' \). Consider the random variable \( \lambda_N = \lambda_N(x_1, \ldots, x_N) = I\{ |\hat{\epsilon}_N - \epsilon| > \delta \} \), where \( I(A) \) is the indicator of the set \( A \).
Thus, or On the other hand, Therefore, choosing \( d > 0 \):

\[
P_\epsilon \{|\hat{\epsilon}_N - \epsilon| > \delta\} = E_\epsilon \lambda_N \geq E_\epsilon (\lambda_N \{f(X^N, \epsilon + \delta')/f(X^N, \epsilon) < e^d\}),
\]

where \( f(X^N, \epsilon) \) is the likelihood function of the sample \( X^N \) of observations with the density function \( f_\epsilon(x) \), i.e.

\[
f(X^N, \epsilon) = \prod_{i=1}^{N} [(1 - \epsilon) f_0(x_i) + \epsilon f_1(x_i)].
\]

Further,

\[
E_\epsilon (\lambda_N \{f(X^N, \epsilon + \delta')/f(X^N, \epsilon) < e^d\}) \geq
\]

\[
\geq e^{-d} E_{\epsilon + \delta'} (\lambda_N \{f(X^N, \epsilon + \delta')/f(X^N, \epsilon) < e^d\}) \geq
\]

\[
\geq e^{-d} (P_{\epsilon + \delta'} \{|\hat{\epsilon}_N - \epsilon| > \delta\} - P_{\epsilon + \delta'} \{f(X^N, \epsilon + \delta')/f(X^N, \epsilon) > e^d\}).
\]

Since \( \hat{\epsilon}_N \) is a consistent estimate, \( P_{\epsilon + \delta'} \{|\hat{\epsilon}_N - \epsilon| > \delta\} \to 1 \) as \( N \to \infty \).

Let us consider the probability \( P_{\epsilon + \delta'} \{f(X^N, \epsilon + \delta')/f(X^N, \epsilon) > e^d\} \). We have

\[
\ln \frac{f(X^N, \epsilon + \delta')}{f(X^N, \epsilon)} = \sum_{i=1}^{N} \ln \left(1 + \delta' \frac{f_1(x_i) - f_0(x_i)}{f_\epsilon(x_i)}\right) =
\]

\[
= \delta' \sum_{i=1}^{N} \frac{f_1(x_i) - f_0(x_i)}{f_\epsilon(x_i)} + o(\delta').
\]

On the other hand,

\[
E_{\epsilon + \delta'} \frac{f_1(x_i) - f_0(x_i)}{f_\epsilon(x_i)} = \delta' \int \frac{(f_1(x_i) - f_0(x_i))^2}{f_\epsilon(x_i)} dx_i = \delta' J(\epsilon).
\]

Therefore, choosing \( d = N((\delta')^2 + \kappa) J(\epsilon), \kappa = o((\delta')^2) \), we obtain

\[
P_{\epsilon + \delta'} \{f(X^N, \epsilon + \delta')/f(X^N, \epsilon) < e^d\} \to 0 \text{ as } N \to \infty.
\]

Thus,

\[
P_\epsilon \{|\hat{\epsilon}_N - \epsilon| > \delta\} \geq (1 - o(1)) e^{-N\delta^2 J(\epsilon)},
\]

or

\[
\liminf_{N \to \infty} \inf_{\epsilon_N \in \mathcal{M}_N} \sup_{0 < \epsilon < 1/2} \frac{1}{N} \ln P_\epsilon \{|\hat{\epsilon}_N - \epsilon| > \delta\} \geq -\delta^2 J(\epsilon),
\]

Theorem 3 is proved.

Comparing results of theorems 2 and 3 we conclude that the proposed method is asymptotically optimal by the order of convergence of the estimates of a mixture parameters to their true values.
### 3.4. Generalizations: non-symmetric distribution functions

Results obtained in theorems 1 and 2 can be generalized to the case of non-symmetric distribution functions. Suppose the d.f. \( f_0(\cdot) \) is asymmetric w.r.t. zero. Then we can modify the proposed method as follows.

1. From the initial sample \( X^N = \{x_1, \ldots, x_N\} \) compute the mean value \( \theta_N = \frac{1}{N} \sum_{i=1}^{N} x_i \) and the sample \( Y^N = \{y_1, \ldots, y_N\} \); \( y_i = x_i - \theta_N \). Then we divide the sample \( Y^N \) into two sub-samples \( I_1(b), I_2(b) \) as follows:

\[
y_i \in \begin{cases} 
I_1(b) = \{\tilde{y}_1, \ldots, \tilde{y}_{N_1(b)}\}, & -\phi(b) \leq y_i \leq b \\
I_2(b) = \{\hat{y}_1, \ldots, \hat{y}_{N_2(b)}\}, & y_i > b \text{ or } y_i < -\phi(b),
\end{cases}
\]

where the function \( \phi(b) \) is defined from the following condition: \( 0 = \int_{-\phi(b)}^{b} y f_0(y) \, dy \). \( f_0(y) = f_0(x - \theta_N) \). \( N = N_1(b) + N_2(b) \) and \( N_1(b), N_2(b) \) are sample sizes of \( I_1(b), I_2(b) \), respectively.

2. As before we compute the statistic

\[
\Psi_N(b) = \frac{1}{N_2} \frac{N_1(b)}{N_2(b)} \sum_{i=1}^{N_1(b)} \tilde{y}_i - \frac{N_1(b)}{N_2(b)} \sum_{i=1}^{N_2(b)} \hat{y}_i.
\]

3. Then the value \( J = \max_{b \in B} |\Psi_N(b)| \) is compared with the threshold \( C \). If \( J \leq C \) then the hypothesis \( H_0 \) (no abnormal observations) is accepted; if, however, \( J > C \) then the hypothesis \( H_0 \) is rejected and the estimate of the parameter \( \epsilon \) is constructed.

4. For this purpose, define the value \( b_N^* \):

\[
b_N^* \in \arg \max_{b \in \mathbb{B}} |\Psi_N(b)|.
\]

Then

\[
\epsilon_N^* = \frac{N_2(b_N^*)}{N}.
\]

Consider application of this method for the study of the classic \( \epsilon \)-contamination model:

\[
f_\epsilon(\cdot) = (1 - \epsilon)N(\mu, \sigma^2) + \epsilon N(\mu, \Lambda^2), \quad \Lambda^2 >> \sigma^2, \quad 0 \leq \epsilon < 1/2.
\]

For this model, the method described above has the form:

1. From the sample of observations \( X^N = \{x_1, \ldots, x_N\} \) the mean value estimate \( \hat{\mu} = \frac{1}{N} \sum_{i=1}^{N} x_i \) was computed.
2. The sequence \( y_i = (x_i - \hat{\mu})^2, \ i = 1, \ldots, N \) and its empirical mean \( \theta_N = \sum_{i=1}^{N} y_i/N \) are computed.

3. Then for each \( b \in \mathbb{B} \), the sample \( Y^N = \{y_1, \ldots, y_N\} \) is divided into two sub-samples in the following way: for \( \theta_N(1 - \phi(b)) \leq y_i \leq \theta_N(1 + b) \) put \( \tilde{y}_i = y_i \) (the size of the sub-sample \( N_1 = N_1(b) \)), otherwise put \( \hat{y}_i = y_i \) (the size of the sub-sample \( N_2 = N_2(b) \)). Here we choose the function \( \phi(b) \) from the following condition:

\[
\theta_N(1+b) \int_{\theta_N(1-\phi(b))}^{\theta_N(1+b)} y f_0(y) dy = 0,
\]

where \( f_0(\cdot) = N(0, (1 - \epsilon)^2 \sigma^2) \).

From here we obtain:

\[
\phi(b) = 1 - \frac{b}{e^b - 1}.
\]

4. For any \( b \in \mathbb{B} \), the following statistic is computed:

\[
\Psi_N(b) = \frac{1}{N^2} (N_2 \sum_{i=1}^{N_2} \tilde{y}_i - N_1 \sum_{i=1}^{N_1} \hat{y}_i).
\]

where \( N = N_1 + N_2 \), \( N_1 = N_1(b) \), \( N_2 = N_2(b) \) are sizes of sub-samples of ordinary and abnormal observations, respectively.

5. Then, as above, the threshold \( C > 0 \) is chosen and compared with the value \( J = \max_b |\Psi_N(b)| \). If \( J \leq C \) then the hypothesis \( H_0 \) (no abnormal observations) is accepted; if, however, \( J > C \) then the hypothesis \( H_0 \) is rejected and the estimate of the parameter \( \epsilon \) is constructed as follows.

Define the value \( b^*_N \):

\[
b^*_N \in \arg \max_{b > 0} |\Psi_N(b)|.
\]

Then

\[
\epsilon^*_N = N_2(b^*_N)/N.
\]

Remark. For estimation of the threshold, we use the approach described in 2.1.3.

In experiments the critical values of the statistic \( \max_b |\Psi_N(b)| \) were computed. For this purpose, as above, for homogenous samples (for \( \epsilon = 0 \)), \( \alpha \)-quantiles of the decision statistic \( \max_b |\Psi_N(b)| \) were computed (\( \alpha = 0.95, 0.99 \)). The results obtained in 5000 trials of each test are presented in table 3.

Table 3.
The quantile value for $\alpha = 0.95$ was chosen as the critical threshold $C$ in experiments with non-homogenous samples (for $\epsilon \neq 0$). For different sample sizes in 5000 independent trials of each test, the estimate of type 2 error $w_2$ (i.e. the frequency of the event $\max_{b} |\Psi_N(b)| < C$ for $\epsilon > 0$) and the estimate $\hat{\epsilon}$ of the parameter $\epsilon$ were computed. The results are presented in tables 4 and 5.

Table 4.

| $\Lambda = 3.0$ | $\epsilon = 0.05$ |
|----------------|-----------------|
| $N$ | 300 | 500 | 800 | 1000 |
| $C$ | 0.1570 | 0.1419 | 0.1252 | 0.1244 |
| $w_2$ | 0.27 | 0.15 | 0.06 | 0.04 |
| $\hat{\epsilon}$ | 0.064 | 0.056 | 0.052 | 0.05 |

Table 5.

| $\Lambda = 5.0$ | $\epsilon = 0.01$ |
|----------------|-----------------|
| $N$ | 1000 | 1200 | 1500 | 2000 | 3000 |
| $C$ | 0.1244 | 0.1146 | 0.1107 | 0.1075 | 0.1019 |
| $w_2$ | 0.25 | 0.20 | 0.15 | 0.10 | 0.04 |
| $\hat{\epsilon}$ | 0.0135 | 0.013 | 0.012 | 0.011 | 0.010 |

3.5. Generalizations: multiple switchings

Suppose we obtain the data $X^N = \{x_1, \ldots, x_N\}$, where the d.f. of an observation $x_i$ can be written as follows:

$$f(x_i) = (1 - \epsilon_1 - \cdots - \epsilon_k) f_0(x_i) + \epsilon_1 f_1(x_i) + \cdots + \epsilon_k f_k(x_i),$$

where $\epsilon_1 \geq \epsilon_2 \geq \cdots \geq \epsilon_k \geq 0$, $0 \leq \epsilon_1 + \cdots + \epsilon_k < 1$, $|E_1 f_1| < |E_2 f_2| < \cdots < |E_k f_k|$.

In particular, we suppose the d.f. $f_0(x)$ is symmetric and unimodal w.r.t. $E_0 f_0$

$$f(x_i) = (1 - \epsilon_1 - \cdots - \epsilon_k) f_0(x_i) + \epsilon_1 f_0(x_i - h_1) + \cdots + \epsilon_k f_0(x_i - h_k),$$
and \( H = \epsilon_1 h_1 + \cdots + \epsilon_k h_k \neq 0 \).

Our goal is to test the hypothesis \( \epsilon_s = 0, s = 1, \ldots, k \) (no switches). In this section we denote by \( E_i, i = 0, 1, \ldots, k \), the mathematical expectation of random variables corresponding to the d.f. with shift \( E_i f_i(E_0 f_0 \overset{\text{def}}{=} 0) \).

This model has the following sense. In the case of a binary switching we have ordinary and abnormal observations. In the case of multiple switchings abnormal observations are from different classes. We do in analogy with the general form of this method.

1. From the initial sample \( X^N \) compute the estimate of the mean value:

\[
\theta_N = \frac{1}{N} \sum_{i=1}^{N} x_i
\]

1.2 Fix the numbers \( 0 < \kappa < B \) and parameter \( b \in B \overset{\text{def}}{=} [\kappa, B] \) and classify observations as follows: if an observation falls into the interval \( (\theta_N - b, \theta_N + b) \), then we place it into the sub-sample of ordinary observations, otherwise - to the sub-sample of abnormal observations.

1.3. Then for each \( b \in B \) we obtain the following decomposition of the sample \( X^N \) into two sub-samples

\[
X_1(b) = \{ \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{N_1} \}, \quad |\tilde{x}_i - \theta_N| < b,
\]

\[
X_2(b) = \{ \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{N_2} \}, \quad |\hat{x}_i - \theta_N| \geq b
\]

Denote by \( N_1 = N_1(b), N_2 = N_2(b), N = N_1 + N_2 \) the sizes of the sub-samples \( X_1 \) and \( X_2 \), respectively.

The parameter \( b \) is chosen so that the sub-samples \( X_1 \) and \( X_2 \) are separated in the best way. For this purpose, consider the following statistic:

\[
\Psi_N(b) = \frac{1}{N^2} (N_2 \sum_{i=1}^{N_1} \tilde{x}_i - N_1 \sum_{i=1}^{N_2} \hat{x}_i).
\]

1.4. Define the boundary \( C > 0 \) and compare it with the value \( J = \max |\Psi_N(b)| \) on the set \( b \in B \). If \( J \leq C \) then we accept the hypothesis \( H_0 \) about the absence of abnormal observations; if, however, \( J > C \) then the hypothesis \( H_0 \) is rejected.

2. As a result, we obtain two sub-samples: ordinary observations and outliers at the first step of the algorithm.

3. Then we remove all found 'ordinary' observations from the sample and repeat steps 1 and 2.
4. So we proceed further until a sub-sample without switches is obtained (i.e. the decision threshold $C$ is not exceeded).

Remark that the 1st type error for multiple switchings can be estimated like in the binary case (we do not formulate this result). As to the 2nd type error (i.e. the probability that we stop at the 1st step of the method because the decision threshold is not exceeded) just observe that a binary switch is a particular case of the general multiple switching situation (when all $\epsilon_i$ beginning from $i = 2$ are equal to zero).

Therefore

$$P_\epsilon\{2\text{nd type error, multiple switches}\} \leq P_\epsilon\{2\text{nd type error, binary case}\} \leq L_1 \exp(-\beta(\delta, N)),$$

for $0 \leq \delta \leq \max_{b \in B} |\Psi(b)| - C$.

**Theorem 4.**

Suppose $0 < C < \max_{b \in B} |\Psi(b)|$. Then the 2nd type error probability is estimated from above as follows:

$$P_\epsilon\{2\text{nd type error}\} \leq L_1 \exp(-\beta(\delta, N)),$$

where $0 \leq \delta = \max_{b \in B} |\Psi(b)| - C$

**Example**

Let us consider the following example. Suppose we have the model with three classes of observations:

$$f(x_i) = (1 - \epsilon_1 - \epsilon_2) f_0(x_i - h_1) + \epsilon_1 f_0(x_i - h_2) + \epsilon_2 f_0(x_i - h_3), \quad i = 1, \ldots, N,$$

where $f_0(\cdot) = \mathcal{N}(0, 1)$; $x_i$ are i.r.v.’s.

The problem is to test the stationarity hypothesis: $H_0 : \epsilon_1 = \epsilon_2 = 0$.

Concretely, in this model the following parameters were chosen:

$$\epsilon_1 = 0.3; \quad \epsilon_2 = 0.15$$

$$h_1 = 1, \quad h_2 = 3, \quad h_3 = 7.$$

In experiments we estimated the type 2 error probability $\hat{\omega}_2$.

The following results were obtained (each cell of this table is the average in 1000 replications):

**Table 6.**
4. Multivariate models

4.1. Multivariate classification

Binary mixtures

Now let us consider the multivariate classification problem with binary mixtures. Suppose multivariate observations are of the following type:

\[
Y^N = \{Y^n\}_{n=1}^N, \quad Y^n = (y^n_1, \ldots, y^n_k).
\]

The multivariate density function of the vector \(Y^n\) is

\[
f(Y^n) = (1 - \epsilon)f_0(Y^n) + \epsilon f_1(Y^n),
\]

where \(f_0(\cdot)\), \(f_1(\cdot)\) are the d.f.’s of ordinary and abnormal observations, respectively; the d.f. \(f_0(\cdot)\) is supposed to be symmetric w.r.t. its mean vector.

First, let us consider the case \(E_1(Y^n) = a \neq 0\), i.e. changes in mean of abnormal observations. Remark that the baseline "change-in-mean" problem is usually considered in many methods of 'cluster analysis' in which different distances between multivariate 'points' of characteristics (even without references to density functions and mathematical expectations of observations) are considered.

The method can be formulated in analogy with the univariate case:

1) From the initial sample \(Y^N\) compute the estimate of the mean value:

\[
\theta_N = \frac{1}{N} \sum_{i=1}^{N} Y^i.
\]

2) Fix the parameter \(b > 0\) and classify observations as follows:

- if \(\|Y^i - \theta_N\| \leq b\), then we place \(Y^i\) into the sub-sample of ordinary observations \(\{\tilde{Y}^i\}\);
- if \(\|Y^i - \theta_N\| > b\), then we place \(Y^i\) into the sub-sample of abnormal observations \(\{\hat{Y}^i\}\).

As a result, for each \(b > 0\) we obtain the decomposition of the sample \(Y^N\) into sub-samples of ordinary and abnormal observations. Suppose the size of ordinary sub-sample is \(N_1(b)\) and the size of abnormal sub-sample is \(N_2(b)\).
3) The parameter \( b \) can be chosen in order to separate the sub-samples of ordinary and abnormal observations (\( \{ \tilde{Y}^i \} \) and \( \{ \hat{Y}^i \} \), respectively) in the best way. For this purpose, consider the following statistic:

\[
\Psi_N(b) = \frac{1}{N^2}(N_2 \sum_{i=1}^{N_1} \tilde{Y}^i - N_1 \sum_{i=1}^{N_2} \hat{Y}^i).
\]

4) Define the boundary \( C > 0 \) and compare it with the value \( J = \max_{b \in \mathbb{B}} \| \Psi_N(b) \| \). If \( J \leq C \) then we accept the hypothesis \( H_0 \) about the absence of abnormal observations; if, however, \( J > C \) then the hypothesis \( H_0 \) is rejected.

For this method, in analogy with the univariate case we can formulate results about type 1 and type 2 error probabilities. For example, the exponential upper estimate for type 1 error probability is formulated as follows:

Let \( \epsilon = 0 \). Suppose the d.f \( f_0(\cdot) \) is symmetric w.r.t. zero and bounded. Then for all \( \kappa, B : 0 < \kappa < B \) there exists \( C > 9 \) such that

\[
P_0\{ \sup_{b \in \mathbb{B}} \| \Psi_N(b) \| \leq C \} \leq L_1 \exp(-\beta(C,N)),
\]

where \( \mathbb{B} = [\kappa, B] \).

Imitation modeling

In this example the following multivariate Gaussian model was considered:

\[
f(X) = (1 - \epsilon)f_0(X) + \epsilon f_1(X).
\]

where \( f_0(X) \) is the two-dimensional Gaussian d.f. with the vector of means \( \mu_1 = (0 \ 0)' \) and the covariance matrix \( \text{Cov}(x_i) = \begin{pmatrix} 0.745 & -0.07 \\ -0.07 & 0.01 \end{pmatrix} \), and \( f_1(X) \) is the two-dimensional Gaussian d.f. with the vector of means \( \mu_2 = (0 \ 0.25)' \) and the same correlation matrix. Here \( \epsilon = 0.2 \).

In this model it is a priori known that switchings occur in the second coordinate of observations. Therefore from the beginning we consider this second coordinate (which is connected with the first coordinate in virtue of our two-dimensional model).

First, the critical thresholds of the decision statistic \( \max_{b \in \mathbb{B}} |\Psi_N(b)| \) were computed. For homogenous samples of different size (i.e. without switches), \( p \)-quantiles of the decision statistic were computed. For this purpose, a Gaussian random sample with determined parameters was generated. After that all steps of the above described method were done. The values of the method’s parameters: \( \kappa = 0.04, B = 50 \).
The maximum of the absolute value of the decision statistic was computed. This procedure was iterated 1000 times and the variation series of the maximums of the absolute values of the decision statistic was constructed. Then \( p \)-quantiles (with \( p = 0.95 \) and \( p = 0.99 \)) in this series were computed. The obtained results are given in Table 7.

**Table 7.**

| \( N \) | 50 | 100 | 200 | 300 | 500 | 700 | 1000 | 1500 |
|--------|----|-----|-----|-----|-----|-----|------|------|
| \( \alpha = 0.95 \) | 0.0066 | 0.0059 | 0.0041 | 0.0037 | 0.0027 | 0.0024 | 0.0019 | 0.0016 |
| \( \alpha = 0.99 \) | 0.014 | 0.0083 | 0.0057 | 0.0045 | 0.0037 | 0.0036 | 0.0024 | 0.0020 |

In the second series of tests the quantile value for \( p = 0.95 \) was chosen as the critical threshold \( C \) in experiments with non-homogenous samples (for \( \epsilon \neq 0 \)). For different sample sizes in 1000 independent trials of each test, the estimate of type 2 error \( w_2 \) (i.e. the frequency of the event \( \max_{b \in \mathbb{B}} |\Psi_N(b)| < C \) for \( \epsilon > 0 \)). The results are presented in table 8.

**Table 8.**

| \( N \) | 100 | 200 | 300 | 500 | 700 | 1000 | 1500 |
|--------|-----|-----|-----|-----|-----|------|------|
| \( C \) | 0.0059 | 0.0041 | 0.0037 | 0.0027 | 0.0024 | 0.0019 | 0.0016 |
| \( w_2 \) | 0.110 | 0.019 | 0.002 | 0 | 0 | 0 | 0 |

The results obtained witness about the fact that the quality of this method increases with the growing sample size. Here: \( w_2 \) is the frequency of type 2 error, \( C \) is the decision threshold.

In analogy with the univariate case we can generalize this method to the case of multiple switchings.

### 4.2. Switching regressions

Let us first remind the considered model of observations:

\[
Y = X\beta + U = X(\zeta\beta_0 + (1 - \zeta)\beta_1) + U,
\]

where

- \( Y \) is a \( N \times 1 \) vector of dependent observations \( y_1, y_2, \ldots, y_N \);
- \( X \) is a \( N \times k \) matrix of predictors;
- \( U \) is a \( N \times 1 \) vector of centered random noises \( u_1, u_2, \ldots, u_N \);
ζ is a $k \times 1$ vector of r.v.’s $ζ_1, ζ_2, \ldots, ζ_k$ independent of $u_1, \ldots, u_N$ and identically distributed according to Bernoulli law:

$$
P(ζ_j = 1) = 1 - P(ζ_j = 0) = ε_j, \quad j = 1, 2, \ldots, k,
$$

for certain unknown parameters $0 < ε_j < 1$, $j = 1, \ldots, k$.

$1 - k \times 1$ vector composed of 1’s.

Here $β_0 \neq β_1$, $k$ - dimensionality of the vector of coefficients $α$ of the model.

For solving this problem, consider the OLS estimate of the vector $β$ (here and below ’ is the symbol of transposition):

$$
\hat{β} = (X'X)^{-1}X'Y = ζβ_0 + (1 - ζ)β_1 + (X'X)^{-1}X'U.
$$

Since the sequence of noises $U$ is centered, the problem is reduced to the above considered problem of detection of switches in the mean of an observed random vector. The matrix of predictors $X$ influences only the random component.

Formally, we need to introduce the following vector $I = (1, 1, \ldots, 1)$ ($N$ units) and consider

$$
\tilde{β} = [ζβ_0 + (1 - ζ)β_1]I + (X'X)^{-1}X'U I.
$$

Then the $(k \times N)$ matrix $β$ consists of $N$ columns of $k \times 1$ vectors with means $β_0$ and $β_1$ changing in a random manner. Each component $j = 1, \ldots, k$ of these vectors $\tilde{β}_i^j$, $i = 1, \ldots, N$ is therefore a univariate random sequence

$$
\tilde{β}_i^j = [ζβ_0^j + (1 - ζ)β_1^j] + ξ_i^j, \quad i = 1, \ldots, N,
$$

where

$$
ξ_i^j = ((X'X)^{-1}X'U I)_i^j.
$$

So the problem of detection of changes in regression coefficients is reduced to the above considered problem of detection switches in the mean value of a univariate random sequence. Remark that the uniform Cramer and the $ψ$-mixing conditions are still satisfied for the process $ξ_i^j$, $i = 1, \ldots, N$. As $Eu_i \equiv 0$ we get that there exist constants $g_1 > 0$, $H_1 > 0$ such that

$$
Ee^tξ_i^j \leq \frac{1}{2^2g_1^2t^2}, \quad |t| \leq H_1,
$$

for all $i = 1, \ldots, N$, $j = 1, \ldots, k$. Moreover, we choose the number $m_0(·)$ from the $ψ$-mixing condition for $ξ_i^j$, $i = 1, \ldots, N$: for any chosen number $γ(·) > 0$: $ψ(l) \leq γ(x)$ for $l \geq m_0(x)$.
For testing the hypothesis of no switches we again consider the decision statistic $\Psi_N(b)$ and compare the maximum of its module with the decision threshold $C > 0$. Then the following theorem holds:

Formally, we can write

$$\tilde{\alpha} = [\zeta \beta_0 + (1 - \zeta) \beta_1] + (X'X)^{-1}X'U.$$  

Each component $j = 1, \ldots, k$ of these vectors $\tilde{\alpha}_j$, $i = 1, \ldots, N$ is therefore a univariate random sequence

$$\tilde{\alpha}_j^i = [\zeta_j^i \beta_0 + (1 - \zeta_j^i) \beta_1] + \xi_j^i, \quad i = 1, \ldots, N,$$

where

$$\xi_j^i = ((X'X)^{-1}X'U I)^j_i.$$

So the problem of detection of changes in regression coefficients is reduced to the above considered problem of detection switches in the mean value of a univariate random sequence. Remark that the uniform Cramer and the $\psi$-mixing conditions are still satisfied for the process $\xi_j^i$, $i = 1, \ldots, N$. As $EU \equiv 0$ we get that there exist constants $g_1 > 0$, $H_1 > 0$ such that

$$E e^{t \xi_j^i} \leq e^{\frac{1}{2} g_1 t^2}, \quad |t| \leq H_1,$$

for all $i = 1, \ldots, N, j = 1, \ldots, k$. Moreover, we choose the number $m_0(\cdot)$ from the $\psi$-mixing condition for $\xi_j^i$, $i = 1, \ldots, N$: for any chosen number $\gamma(x) > 0$: $\psi(l) \leq \gamma(x)$ for $l \geq m_0(x)$.

For testing the hypothesis of no switches we again consider the decision statistic $\Psi_N(b)$ and compare the maximum of its module with the decision threshold $C > 0$. Then the following theorem holds:

**Theorem 5.**

Suppose $\epsilon = 0$, the d.f. of each component of the vector $U$ is symmetric w.r.t. zero and the $\psi$-mixing and the uniform Cramer conditions for $\xi_j^i$, $i = 1, \ldots, N$ are satisfied. Then for any threshold $C > 0$ the following upper estimate for the 1st type error probability holds:

$$P_0 \{\max_{b \in \mathbb{S}} |\Psi_N(b)| > C \} \leq L_1 \exp(-\beta(C, N)),$$

where the function $\beta(C, N)$ is defined in the proof of Theorem 1.

The proof of theorem 5 is based upon the same ideas as the proof of theorem 1. Therefore it is omitted here.
5. Simulations

To 4.2

In the following example the regression model with one deterministic predictor was considered:

\[ y_i = c_1 + c_2 \cdot i + u_i, \quad u_i \sim N(0; 1), \quad i = 1, \ldots, n. \]

\[ \xi \sim U[0; 1] \]

\[ \gamma = [c_1; c_2] = \begin{cases} 
\beta_1, & \epsilon < \xi \leq 1 \\
\beta_2, & 0 \leq \xi \leq \epsilon 
\end{cases} \]

Table 9.

| \( \epsilon = 0.05 \) | \( \beta_1 = [1; 1], \beta_2 = [1; 2] \) |
|---|---|---|---|
| \( N \) | 300 | 500 | 800 | 1000 |
| \( C \) | 0.07 | 0.05 | 0.04 | 0.03 |
| \( w_2 \) | 0.87 | 0.59 | 0.14 | 0.004 |
| \( \hat{\epsilon} \) | 0.08 | 0.059 | 0.052 | 0.05 |

Table 10.

| \( \epsilon = 0.1 \) | \( \beta_1 = [1; 1], \beta_2 = [1; 1.5] \) |
|---|---|---|---|
| \( N \) | 300 | 500 | 800 | 1000 |
| \( C \) | 0.07 | 0.05 | 0.04 | 0.03 |
| \( w_2 \) | 0.83 | 0.65 | 0.13 | 0.0 |
| \( \hat{\epsilon} \) | 0.15 | 0.12 | 0.102 | 0.10 |

Conclusion

In this paper we considered the problems of the retrospective analysis of models with time-varying structure. These models include contamination models with randomly switching parameters and multivariate classification models with an arbitrary number of classes. Our main task here is to classify observations with different stochastic generation mechanisms. We propose a new classification method and analyze its properties both theoretically and empirically. It was proved that type 1 and type 2 errors of the proposed method converge to zero exponentially as the sample size \( N \) tends to infinity. The asymptotic optimality of the proposed
method follows from theorem 3. In this theorem the theoretical lower bound for
the error of estimation of the model’s parameters was established. This bound
is attained for the proposed method (by the order of convergence to zero of the
estimation error). Then we consider generalizations of the proposed method to
the case of non-symmetric d.f.’s of ordinary observations and to the case of an
arbitrary number of classes of observations with different stochastic generation
mechanisms. The multivariate models with time-varying structure are considered
at the end of this paper. Here we consider multivariate change-in-mean models
and multivariate models with time-varying regression coefficients.

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Appendix. Proofs of theorems

Theorem 1. Proof.

In subsequent considerations we use many times the following inequality which will be proved first: let $S_n = \sum_{k=1}^{n} \xi_k$, where $\{\xi_k\}_{k=1}^{\infty}$ is the sequence of r.v.’s, satisfying conditions A1 and A2 (whether $\psi$-mixing or $\psi$-weak dependence), and $E\xi_k \equiv 0$.

Then under A1(a) and A2, for sufficiently large $N$, the following inequality holds true:

$$P \{|S_n|/N > x\} \leq A(x) \exp \left(-B(x)\gamma N\right) \quad (*)$$

where positive functions $A(\cdot)$, $B(\cdot)$ can be computed explicitly.

Under A1(b) and A2, however,

$$P \{|S_n|/N > x\} \leq A(x) \exp \left(-B(x)\gamma \sqrt{N}\right), \quad (**)$$

where, again, positive functions $A(\cdot)$, $B(\cdot)$ can be computed explicitly.
In the sequel we introduce the following notation:

\[
\beta(x, N) = \begin{cases} 
B(x)\gamma N, & \text{in case of } \psi-\text{mixing} \\
B(x)\gamma \sqrt{N}, & \text{in case of } \psi-\text{weak dependence} 
\end{cases}
\]

Then

\[
P\{|S_N|/N > x\} \leq A(x) \exp \left( -\beta(x, N) \right)
\]

Under \(\psi\)-mixing and Cramer's conditions this inequality was proved in Brodsky, Darkhovsky (2000). Here we prove it under assumptions A1(b) and A2.

Under "weak dependence" assumption we proceed from Roussas-Ionnides inequality (see, e.g., Hwang, Shin (2014)) in the following form. Let \(\gamma\) be a certain large number: \(\gamma \geq 2\). Suppose \((p_1, \ldots, p_\gamma)\) and \(q_\gamma\) are positive numbers such that

\[
\frac{1}{p_1} + \cdots + \frac{1}{p_\gamma} = \frac{1}{q_\gamma} < 1,
\]

We assume that \(\xi_1, \ldots, \xi_\gamma\) is a weak dependent sequence with the function \(\theta_r\) in Definition 2. Suppose that \(E\|\xi_i\|^{p_i} < \infty, \quad p_i > 1,\) for \(i = 1, \ldots, \gamma - 1, \gamma \geq 2.\) The analogous boundedness conditions are imposed on the functions \(h\) and \(f\) and their first derivatives.

Then the following inequality holds:

\[
\left| E \left[ \prod_{i=1}^{\gamma} \xi_i \right] - \prod_{i=1}^{\gamma} E[\xi_i] \right| \leq B(\gamma - 1) \theta_r^{1 - \frac{1}{q_\gamma}} \prod_{i=1}^{\gamma} \|\xi_i\|_{p_i},
\]

where the constant \(B\) does not depend on \(\gamma\) and \(r\).

For the proof of theorem 1 we split the sum \(S_n\) into \(\phi\) terms of the following types (choice of \(\phi\) is explained below):

\[
S_n = S_n^1 + \cdots + S_n^\phi,
\]

where

\[
S_n^i = \xi(i) + \xi(i + \phi(x)) + \cdots + \xi(i + \phi(x)(\frac{n-i}{\phi(x)})),
\]

\(n = i, 1, 2, \ldots, \phi(x)\).

Then

\[
P\{|S_n|/n \geq x\} \leq \phi(x) \max_{1 \leq i \leq \phi} P\{|S_n^i \geq (k(i) - 1)x\}.
\]

We need to obtain the exponential upper estimate for the probability \(P\{Z_k > x\}\), where

\[
Z_k = \sum_{j=1}^{k} \xi(i + \phi j).
\]
From Chebyshev’s inequality we obtain
\[ P\{Z_k > x\} \leq e^{-tx} E e^{tZ_k}. \]

From the Roussas-Ionnides inequality we have:
\[
P\{|S_n| \geq C \frac{Cn}{m_N(C)}\} \leq \begin{cases} 
\exp\left(-\frac{nC^2}{2gm_N^2(C)}\right), & C \leq gT \\
\exp\left(-\frac{nCT}{2m_N(C)}\right), & C > gT \\
\exp\left(-\frac{C^2n^2}{2g \sum_{j=1}^{n} p_j}\right), & C \leq gT \frac{\sum_{j=1}^{n} p_j}{n} \\
\exp\left(-\frac{nTC}{2}\right), & C > gT \frac{\sum_{j=1}^{n} p_j}{n}.
\end{cases}
\]

Consider the second term in the right hand:
\[
B(n-1) \theta \frac{1}{q_n(m_N(C))} \exp\left(-\frac{C^2n^2}{2g \sum_{j=1}^{n} p_j}\right), \quad C \leq gT \frac{\sum_{j=1}^{n} p_j}{n}
\]
\[
\exp\left(-\frac{nTC}{2}\right), \quad C > gT \frac{\sum_{j=1}^{n} p_j}{n}.
\]

The direct calculation of the minimum of the function
\[
1 - \frac{1}{q_n} \exp\left(-\frac{C^2n^2}{2g \sum_{j=1}^{n} p_j}\right),
\]
dependent on the arguments \(p_1, \ldots, p_{n-1}, q_n\), on condition that
\[
\frac{1}{p_1} + \ldots + \frac{1}{p_{n-1}} + \frac{1}{q_n} = 1
\]
gives
\[
q_n^* \sim n^2, \quad p_i \sim n, \quad i = 1, \ldots, n-1.
\]

Therefore \(1 - \frac{1}{q_n} > 1/2\) for large enough \(n\). This fact yields the estimate
\[
P\{|S_n|/n \geq C\} \leq m_N(C) \exp\left(-\frac{n}{m_N(C)}C\right) + B(n-1) \exp\left(-(m_N(C))\beta/2\right), \quad \beta > 0.
\]

Then we choose \(m_N(C) \sim \sqrt{N}\) and obtain
\[
P\{|S_n|/n \geq C\} \leq L_1 \exp\left(-L_2(C)\sqrt{N}\right).
\]

So in the case of \(\psi\)-weakly dependent variables we need to choose \(m_N(C) \sim \sqrt{N}\).

The function \(\theta_\phi\) exponentially converges to zero with the increase of \(\phi\) (this fact holds true in most cases):
\[
\theta_\phi \leq e^{-\beta\phi}, \quad \beta > 0,
\]
Then we can take $\phi \sim k$ and conclude that

$$P\{|Z_k|/k > x\} \leq L_1 \exp(-L_2(x)\sqrt{k}),$$

where $L_1, L_2$ are constants not dependent on $k$.

Therefore,

$$P\{|S_n/n\} \leq L_1 \exp(-L_2(x)\sqrt{n}),$$

and

$$\alpha_N = P_0\{\max_{|\alpha N| \leq l \leq N} \|Y_N(l)\| > C\} \leq L_1 \exp(-L_2(C)\sqrt{N})\].$$

where as before $L_1, L_2$ are constants not dependent on $N$.

For $\Psi_N(b)$ we can write:

$$\Psi_N(b) = \left( N \sum_{i=1}^{N_1(b)} \tilde{x}_i - N_1(b) \sum_{i \in N} x_i \right)/N^2. $$

Then

$$P_0\{\sup_{b \in B} |\Psi_N(b)| > C\} \leq P_0\{\sup_{b \in B} \left| \sum_{i=1}^{N_1(b)} \tilde{x}_i \right| > \frac{C N}{2}\}$$

$$+ P_0\{\left| \sum_{i \in N} x_i \right| > \frac{C N}{2}\}. $$

Further,

$$P_0\left\{\sup_{b \in B} \left| \sum_{i=1}^{N_1(b)} \tilde{x}_i \right| > \frac{C N}{2}\right\} \leq \sum_{n=1}^{N} P_0\left\{\sup_{b \in B} \left| \sum_{i=1}^{n} \tilde{x}_i \right| > \frac{C n}{2}\right\} \cap \{N_1(b) = n\}. $$

Consider the function

$$\Delta(b) = \int_{|x| \leq b} f_0(x)dx.$$

the function $\Delta(b)$ is continuous and $\min_{b \in B} \Delta(b) \geq \int_{|x| \leq \kappa} f_0(x)dx \overset{\text{def}}{=} u$.

Now let us split the segment $B = [\kappa, B]$ into equal parts with the interval such that $|\Delta(b_i) - \Delta(b_{i+1})| \leq u/2$. In virtue of uniform continuity of $\Delta(b)$ such split is possible (here $\{b_i\}$ are bounds of this split).

Denote the number of subsegments by $R$, and subsegments themselves by $B_s, s = 1, \ldots, R$. Then

$$P_0\left\{\sup_{b \in B} \left| \sum_{i=1}^{n} \tilde{x}_i \right| > \frac{C n}{2}\right\} \cap \{N_1(b) = n\} \leq R \max_{s} P_0\left\{\sup_{b \in B_s} \left| \sum_{i=1}^{n} \tilde{x}_i \right| > \frac{C n}{2}\right\} \cap \{N_1(b) = n\}. $$

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Consider the fixed subsegment $B_i \defeq [b_i, b_{i+1}]$. From definition of the numbers $N_1(b)$ we obtain for each $b \in B_i$:

$$N_1(b_i)/N \leq N_1(b)/N \leq N_1(b_{i+1})/N$$

Now let us construct estimates for probabilities of the following events:

$$|N_1(b_i)/N - \Delta(b_i)| \leq u/4, \quad |N_1(b_{i+1})/N - \Delta(b_{i+1})| \leq u/4$$

First, let us estimate the probability of deviation of the r.v. $\theta_N$ from its mathematical expectation $E_m \theta_N \equiv 0$.

For each $\gamma > 0$ and sufficiently large $N$, from inequality (*** it follows that

$$P_0\{|\theta_N| > \gamma\} \leq A(\gamma) \exp (-B(\gamma)N)$$

(5)

in case $A1(a)$ (mixing) and

$$P_0\{|\theta_N| > \gamma\} \leq A(\gamma) \exp \left(-B(\gamma)\sqrt{N}\right)$$

(5)

in case $a1(b)(\psi$-weak dependence).

From definition, $N_1(b) = \sum_{k \in N} I(|x_k - \theta_N| \leq b)$.

Then for every fixed $r > 0$ we obtain

$$P_0\{|x_k - \theta_N| \leq b\} \leq P_0\{|x^k| \leq b + r\} + P_0\{|\theta_N| > r\}$$

(6)

Moreover,

$$P_0\{|x_k - \theta_N| \leq b\} \geq P_0\{|x^k| \leq b - r\} - P_0\{|\theta_N| > r\}$$

(7)

For each point $b_i$ from the split of the segment $\mathbb{B}$ we obtain for sufficiently large $N$ (see Brodsky, Darkhovsky (2000)):

$$P_0\left\{ \frac{1}{N} \sum_{k \in N} \left( I(|x_k - \theta_N| \leq b_i) - E_0(I(|x_k - \theta_N| \leq b_i)) \right) \right\} > u/2 \right\} \leq A(u) \exp (-B(u)N)$$

(8)

Denote by $\phi_N(r) = A \exp (-B(r)N)$, where $A$ is a certain constant not depending on $N$. Then it follows from (6) and (7) that

$$\Delta(b_i + r) + \phi_N(r) \geq E_0(I(|x_k - \theta_N| \leq b_i)) \geq \Delta(b_i - r) + \phi_N(r)$$

Since the function $\Delta(\cdot)$ satisfies Lipshitz condition (in virtue of boundedness of the density function), from these inequalities for some $r$ (e.g., $0 < r < u/4$) it follows that for large enough $N > N_0$

$$P_0\left\{ \frac{1}{N} \sum_{k \in N} I(|x_k - \theta_N| \leq b_i) - \Delta(b_i) \right\} > u/4 \right\} \leq A(u) \exp (-B(u)N) \defeq \gamma(u, N)$$

(9)
Estimate (9) is satisfied for each point $b_i$.

Then in virtue of (4) with the probability no less than $(1 - \gamma(u, N))$, for $N > N_0$ we obtain for all $b \in B_i$:

$$(\Delta(b_i) - u/4)N \leq N_1(b) \leq (\Delta(b_i) + u/2)N \quad (10)$$

Now split the set of all values $N_1(b), b \in B_i$ into two subsets: $A_i \overset{\text{def}}{=} \{1 \leq n \leq N : [(\Delta(b_i) - u/4)N] \leq n \leq [(\Delta(b_i) + u/2)N]\}$ and its complement. We obtain $P_0(A_i) \geq (1 - \gamma(u, N))$ при $N > N_0$.

Then

$$P_0 \left\{ \sup_{b \in B_i} \{ \sum_{i=1}^{n} \bar{x}_i > \frac{C}{2} n \} \cap \{ N_1(b) = n \} \right\} \leq \gamma(u, N)$$

$$+ P_0 \left\{ \max_{n \in A_i} \{ \sum_{i=1}^{n} \bar{x}_i > \frac{C}{2} n \} \right\} \quad (11)$$

For the probability in the right hand of (11), we note that $\Delta(b) \geq u$. Hence

$$P_0 \left\{ \max_{n \in A_i} \{ \sum_{i=1}^{n} \bar{x}_i > \frac{C}{2} n \} \right\} \leq A(C) \exp \left( -N(\Delta(b_i) - u/4)B(C) \right) \leq \quad (12)$$

$$\leq A(C) \exp(-NB(C)3/4u)$$

Since all these considerations are valid for every sub-segment, from (11) and (12) we obtain for each $s = 1 \ldots, R$

$$P_0 \left\{ \sup_{b \in B_s} \{ \sum_{i=1}^{n} \bar{x}_i > \frac{C}{2} n \} \cap \{ N_1(b) = n \} \right\} \leq A(C) \exp(-NB(C)u/4) \quad (13)$$

The analogous estimate is valid for the second term in (2).

Taking into account (2), (3), (4), (9), (13), we obtain the exponential estimate from theorem 1.

**Theorem 2. Proof.**

Consider the main decision statistic:

$$\Psi_N(b) = \left( N \sum_{i=1}^{N_1(b)} \bar{x}_i - N_1(b) \sum_{i=1}^{N} x_i \right) / N^2.$$  

Write

$$\frac{1}{N} \sum_{i=1}^{N} \bar{x}_i = \frac{1}{N} \sum_{n=1}^{N} \sum_{i=1}^{n} \bar{x}_i |N_1 = n\} P_e \{ N_1 = n \} =$$

$$= \frac{1}{N} \sum_{n=1}^{N} \sum_{i=1}^{n} E_e(\bar{x}_i - \theta_N | < b | N_1 = n) P_e \{ N_1 = n \}$$

$$= \frac{1}{N} (E_e N_1) \int_{|eh-x|<b} f(x)xdx / \int_{|eh-x|<b} f(x)dx \rightarrow \int_{|eh-x|<b} f(x)xdx, \quad \text{при } N \to \infty$$

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Here we used the relationship
\[ \frac{1}{N} E_N N_1 = \frac{1}{N} E_N \sum_{k=1}^{N} k I(|x^k - \theta_N| \leq b) \rightarrow \int f(x)dx \quad \text{при} \quad N \rightarrow \infty \]

Therefore form this relationship from the law of large numbers and the equality
\[ E_N x^i = \epsilon h \]

we obtain
\[ E_N \Psi_N(b) \rightarrow \Psi(b) \quad \text{при} \quad N \rightarrow \infty, \]
where \( \Psi(b) = r(b) - \epsilon h d(b), \quad r(b) = \int_{|\epsilon h - x| < b} f(x)dx, \quad d(b) = \int_{|\epsilon h - x| < b} f(x)dx. \)

For each \( C > 0 \) write:
\[
\begin{align*}
&\text{P}_\epsilon\{|\Psi_N(b) - \Psi(b)| > C\} \leq \text{P}_\epsilon\{| \sum_{i=1}^{N_1(b)} \tilde{x}^i - N r(b) | > \frac{C}{2}N\} \\
&+ \text{P}_\epsilon\{| \frac{N_1(b)}{N} \sum_{i=1}^{N} x^i - N \epsilon h d(b) | > \frac{C}{2}N\}. \tag{14}
\end{align*}
\]

Consider the first term in the right hand:
\[
\text{P}_\epsilon\{ \sum_{i=1}^{N_1(b)} \tilde{x}^i - N r(b) | > \frac{C}{2}N\} \leq \sum_{n=1}^{N_1(b)} \text{P}_\epsilon\{| \tilde{x}^i | > \frac{C}{2}N + nr(b) \}
\]

Consider this estimate for the first probability in the right hand. Write
\[
\begin{align*}
&\text{P}_\epsilon\{| \sum_{i=1}^{N_1(b)} \tilde{x}^i - N r(b) | > \frac{C}{2}N\} = \sum_{n=1}^{N_1(b)} \text{P}_\epsilon\{| \sum_{i=1}^{n} \tilde{x}^i | < \frac{C}{2}N + nr(b) \} + \sum_{n=1}^{N_1(b)} \text{P}_\epsilon\{| \sum_{i=1}^{n} \tilde{x}^i | < \frac{C}{2}N + nr(b) \}.
\end{align*}
\]

The random value \( \tilde{x}^i - r(b) \) is centered. Therefore we obtain the following exponential upper estimate:
\[
\sum_{n=1}^{N} \text{P}_\epsilon\{| \sum_{i=1}^{n} \tilde{x}^i | > \frac{C}{2}N + nr(b) \} \leq L_1 \exp(-L_2(C)N),
\]

where the constants \( L_1 \) and \( L_2 \) do not depend on \( N \).
The second probability in the right hand of (14) is estimated in analogous way. Therefore for any fixed $b$ we have the following estimate:

$$P_\epsilon \{ |\Psi_N(b) - \Psi(b) | > C \} \leq L_1 \exp(-L_2(C)N).$$

The type 2 error probability:

$$P_\epsilon \{ \max_{b \in B} |\Psi_N(b) - \Psi(b) | < C \} \leq P_\epsilon \{ \max_{b \in B} |\Psi_N(b) - \Psi(b) | - |\Psi(b) | - C \}.$$

Let $\delta = \max_{b \in B} |\Psi(b) | - C$. Then

$$P_\epsilon \{ \max_{b \in B} |\Psi_N(b) | < C \} \leq L_1 \exp(-L_2(\delta)N).$$

If the sequence of observations satisfies $\psi$-weak dependence condition, we use theorem 1 in order to obtain the exponential estimate

$$P_\epsilon \{ \max_{b \in B} |\Psi_N(b) | \leq C \} \leq L_1 \exp(-L_2(\delta)\sqrt{N}).$$

where $\delta = \max_{b \in B} |\Psi(b) | - C > 0$.

As to the proof of 3), remark that the function $\Psi(b) = E_\epsilon \Psi_N(b)$ satisfies the reversed Lipschitz condition in a neighborhood of $b^*$.

In fact, we have $\Psi(b^*) = 0$, $\Psi'(b^*) = 0$ and $\Psi''(b^*) = (f(\epsilon h + b^*) - f(\epsilon h - b^*) + b^*(f'(\epsilon h + b^*) - f'(\epsilon h - b^*)) = 2(b^*)^2 f''(u) \neq 0$, where $0 \leq u = u(b^*) \leq b^*$. Therefore in a small neighborhood of $b^*$ we obtain:

$$|\Psi(b) - \Psi(b^*)| = (b^*)^2 |f''(u(b^*))|(b - b^*)^2 \geq C(b - b^*)^2,$$

for a certain $C = C(b^*) > 0$.

Now for any $0 < \kappa < 1$ consider the event $|b_N - b^*| > \kappa$. Then

$$P_\epsilon \{ |b_N - b^*| > \kappa \} \leq P_\epsilon \{ \max_b |\Psi_N(b_N) - \Psi(b^*) | > \frac{1}{2} C\kappa^2 \} \leq 4 \phi_0(\cdot) \exp(-L(C)N),$$

where $L(C)$ is a certain constant not depending on $N$.

From this inequality it follows that $b_N \to b^*$ $P_\epsilon$-a.s. as $N \to \infty$.

Then

$$\epsilon_N = N_2(b_N)/N, \quad h_N = \theta_N/\epsilon_N$$

are the nonparametric estimates for $\epsilon$ and $h$, respectively.

In general these estimates are asymptotically biased and non-consistent. For construction of consistent estimates of $\epsilon$ and $h$, we need information about the
d.f. \( f_0(\cdot) \). These consistent estimates can be obtained from the following system of equations:

\[
\hat{\epsilon}_N \hat{h}_N = \theta_N \\
\frac{1 - \hat{\epsilon}_N}{\hat{\epsilon}_N} = \frac{f_0(\theta_N - b_N - \hat{h}_N) - f_0(\theta_N + b_N - \hat{h}_N)}{f_0(\theta_N + b_N) - f_0(\theta_N - b_N)}.
\]

The estimates \( \hat{\epsilon}_N \) and \( \hat{h}_N \) are connected with the estimate \( b_N \) of the parameter \( b^* \) via this system of deterministic algebraic equations. Therefore the rate of convergence \( \hat{\epsilon}_N \to \epsilon \) and \( \hat{h}_N \to h \) is determined by the rate of convergence of \( b_N \) to \( b^* \) (which is exponential w.r.t. \( N \)). So we conclude that \( \hat{\epsilon}_N \to \epsilon \) and \( \hat{h}_N \to h \) \( P_\epsilon \)-a.s. as \( N \to \infty \).

Theorem 2 is proved.