Numerical Schemes for Multivalued Backward Stochastic Differential Systems

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Abstract

We define some approximation schemes for different kinds of generalized backward stochastic differential systems, considered in the Markovian framework. We propose a mixed approximation scheme for the following backward stochastic variational inequality

$$dY_t + F(t, X_t, Y_t, Z_t)dt + G(t, X_t, Y_t)dA_t \in \partial \varphi(Y_t)dt + Z_t dW_t,$$

where $(X_t, A_t)_{t \in [0,T]}$ is the unique solution of a reflected forward stochastic differential equation. More precisely, we use an Euler scheme type for the system of decoupled forward-backward variational inequality, combined with Yosida approximation techniques.

Key words and phrases: Euler scheme, Yosida approximation, error estimate, multivalued backward SDEs, reflected SDEs

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1 Introduction

The stochastic differential equations (SDE) with reflecting boundary conditions, also called reflected stochastic differential equations (RSDE) appears from the modeling of different kinds of constrained phenomenon. The elliptic and parabolic PDEs with Neumann type and mixed boundary conditions lead us to probabilistic interpretations, via the Feynman–Kac formula, in terms of reflected diffusion processes, which are solutions of RSDEs. This type of equations were studied for the first time by Skorohod in [22] and after this, considered in general domains (see [11], [14], [21], [23]...).

Since 1990s a lot of researchers focused their attention to numerical schemes, methods and algorithms for the study of the behavior of the solution for RSDEs. In the recent years, some new techniques consist in splitting-step algorithms and mixed penalization methods. The Euler approximation was considered for the first time by Chitashvili and Lazrieva in [5], followed by the Euler-Peano approximation, which was introduced by Saisho in [21]. Lepingle in [10] and

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Slominski in [23] analyzed the corresponding numerical schemes and their rates of convergence. In order to approximate the solution of RSDEs, the penalization method was also very useful (see Menaldi, [14]). Approximation methods for a diffusion reflected and stopped at the boundary appear in the literature in 1998, in the paper of Constantini, Pacchiarotti and Sartoretto [6]. They defined a standard Euler projected approach to stopped reflected diffusions, approach which yields a method with weak order of convergence (1/2 in particular) and they give an easy example where this convergence rate is precise. Regarding adaptive approximations of one-dimensional reflected Brownian motion, it can be used a simple method of two fixed step sizes chosen according to the distance at the boundary.

In the paper [1], Asiminoaei and Răşcanu used a mixed method consisting in penalization and splitting-up for the study of multivalued SDE with reflection at the boundary of the domain. The penalization method was also used by Răşcanu in [19] for the study of the generalized Skorohod problem and of its link to multivalued SDE governed by a general maximal monotone operator (of subdifferential type). Recently, in [7], Ding and Zhang combined the penalization technique with the splitting-step idea to propose some new schemes for the RSDE in the upper half space.

In 1990, Pardoux and Peng introduced in [15] the notion of nonlinear backward stochastic differential equation (for short, BSDE), and they obtained the existence and uniqueness result for this kind of equation. Since then, the interest in BSDEs has kept growing and there have been a lot of works on that subject, both in the direction of the generalization of the equations that appear and in constructing schemes of approximation for them. The backward stochastic variational inequalities (for short, BSVI) were analyzed by Pardoux and Răşcanu in [17] and [18] (the extension for Hilbert spaces case) by a method that consists in a penalizing scheme, followed by its convergence.

Starting with the paper of Pardoux and Peng [16], have been given a stochastic approach for the existence problem of a solution for many type of deterministic partial differential equations (PDE for short). In [17] it is proved, using a probabilistic interpretation, the existence for the viscosity solution for a multivalued PDE (with subdifferential operator) of parabolic and elliptic type. More recently, Maticiuc and Răşcanu in [12], prove an extended result concerning generalized type of BSDE (including an integral with respect to an adapted continuous increasing function and two subdifferential operators). These type of BSVI allows to prove Feynman-Kac type formula for the representation of the solution of PVI with mixed nonlinear multivalued Neumann-Dirichlet boundary conditions.

Even this type of the penalization approach is very useful when we deal with multivalued backward stochastic dynamical systems governed by a subdifferential operator, it fails for the case of a general maximal monotone operator. This motivated a new approach, via convex analysis, for the study of both forward and backward multivalued differential systems. In [20], Răşcanu and Rotenstein identified the solutions of those type of equations with the minimum points of some proper, convex, lower semicontinuous functions, defined on well chosen Banach spaces.

Euler-type approximation schemes for BSDE, and for BSDE with exit time for the forward part of the system, were introduced by Bouchard and Touzi in [4] and Bouchard and Menozzi in [3]. They considered the Markovian framework of a coupled forward-backward stochastic differential system and they defined an adapted backward Euler scheme for the strong approximation of the backward SDE with finite stopping time horizon, namely the first exit time of the forward SDE from a cylindrical domain. In [2], Bouchard and Chassagneux study the discrete-time approximation of the solution of a BSDE with a reflecting barrier.

The paper is organized as follows. Section 2 presents some basic notations, hypothesis and
results that are used throughout this paper. Section 3 is dedicated to the analysis of the behavior of an approximation scheme defined for a backward stochastic variational inequality. In Section 4 we present an existence and uniqueness result for a generalized BSVI and we propose a mixed Euler type approximation scheme for its solution.

2 Notations. Hypothesis. Preliminaries

In all that follows we shall consider a finite horizon \( T > 0 \) and a complete probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) on which is defined a standard \( d \)-dimensional Brownian motion \( W = (W_t)_{t \leq T} \) whose natural filtration is denoted \( \mathbb{F} = \{ \mathcal{F}_t, 0 \leq t \leq T \} \). More precisely, \( \mathbb{F} \) is the filtration generated by \( \mathcal{N}_\mathbb{P} \), the set of all \( \mathbb{P} \)-null sets, i.e. \( \mathcal{F}_t = \sigma \{ W_s, s \leq t \} \lor \mathcal{N}_\mathbb{P} \).

We denote by \( L^r_{ad}(\Omega; C([0, T]; \mathbb{R}^k)) \), \( r \in [1, \infty) \), the closed linear subspace of adapted stochastic processes \( f \in L^r(\Omega, \mathbb{F}, \mathbb{P}; C([0, T]; \mathbb{R}^k)) \), i.e. \( f(\cdot, t) : \Omega \to \mathbb{R}^k \) is \( \mathcal{F}_t \)-measurable for all \( t \in [0, T] \) and \( \mathbb{E} \left( \sup_{t \in [0, T]} |f(t)|^r \right) < \infty \). Also, we shall use the notation \( L^r_{ad}(\Omega; L^q([0, T]; \mathbb{R}^k)) \), \( r, q \in [1, \infty) \) the Banach space of \( \mathcal{F}_t \)-measurable stochastic processes \( f : \Omega \times [0, T] \to \mathbb{R}^k \) such that \( \mathbb{E} \left( \int_0^T |f(t)|^q \, dt \right)^{r/q} < \infty \).

Consider the following data:

- the continuous coefficient functions \( b : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m \), \( \sigma : [0, T] \times \mathbb{R}^m \to \mathbb{R}^{m \times d} \), \( g : \mathbb{R}^m \to \mathbb{R}^n \) and \( F : [0, T] \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n \times d} \to \mathbb{R} \), which satisfies the following standard assumptions:
  - for some constants \( \alpha \in \mathbb{R} \), \( \beta, \gamma \geq 0 \) and for all \( t \in [0, T] \), \( x, \tilde{x} \in \mathbb{R}^m \), \( y, \tilde{y} \in \mathbb{R}^n \) and \( z, \tilde{z} \in \mathbb{R}^{n \times d} \):
    - \( (i) \ |b(t, x) - b(t, \tilde{x})| + \|\sigma(t, x) - \sigma(t, \tilde{x})\| \leq L |x - \tilde{x}| \),
    - \( (ii) \ (y - \tilde{y})^t F(t, x, y, z) - F(t, x, \tilde{y}, z) \leq \alpha |y - \tilde{y}|^2 \),
    - \( (iii) \ |F(t, x, y, z) - F(t, x, \tilde{y}, \tilde{z})| \leq \beta |z - \tilde{z}| \),
  - and there exist some constants \( M > 0 \) and \( p, q \in \mathbb{N} \) such that, for all \( t \in [0, T] \), \( x \in \mathbb{R}^m \) and \( y \in \mathbb{R}^n \):
    - \( (2) \ |g(x)| \leq M(1 + |x|^q) \) and \( |F(t, x, y, 0)| \leq M(1 + |x|^p + |y|) \).
- the function \( \varphi : \mathbb{R}^n \to (-\infty, +\infty] \) which is a proper convex lower semicontinuous function and satisfies that there exist \( M > 0 \) and \( r \in \mathbb{N} \) such that, for all \( x \in \mathbb{R}^m \):
    - \( (3) \ |\varphi(g(x))| \leq M(1 + |x|^r) \).

The following theorem summarizes some already well known results concerning forward and backward SDE, considered in the Markovian framework (for the proof see Karatzas & Shreve [8], for forward case, and Pardoux & Răşcanu [17] for the backward system).

**Theorem 1** Let \((t, x) \in [0, T] \times \mathbb{R}^m \) be fixed. Under the assumptions \( (1), (2) \) and \( (3) \), the forward-backward coupled system

\[
\begin{align*}
\frac{dX_s^{t,x}}{dt} &= b(s, X_s^{t,x}) \, ds + \sigma(s, X_s^{t,x}) \, dW_s, \quad s \in [0, T], \\
\frac{dY_s^{t,x}}{dt} + F(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) \, ds &\in \partial \varphi(Y_s^{t,x}) \, ds + Z_s^{t,x} \, dW_s, \quad s \in [0, T], \\
X_t^{t,x} &= x, \quad Y_0^{t,x} = g(X_0^{t,x}),
\end{align*}
\]

is a unique solution to the above backward SDE, considered in the Markovian framework (for the proof see Karatzas & Shreve [8].
has a unique solution, i.e. there exist a unique process \( X^{t,x} \in L^2_{ad}(\Omega; C([0,T]; \mathbb{R}^m)) \) such that

\[
X^{t,x}_s = x + \int_t^s b(r, X^{t,x}_r)dr + \int_t^s \sigma(r, X^{t,x}_r)dW_r, \quad s \in [0,T],
\]

and respectively

\[
(Y^{t,x}, Z^{t,x}, U^{t,x}) \in L^2_{ad}(\Omega; C([0,T]; \mathbb{R}^n)) \times L^2_{ad}(\Omega; L^2([0,T]; \mathbb{R}^{n\times d})) \times L^2_{ad}(\Omega; L^2([0,T]; \mathbb{R}^n)),
\]

such that

\[
\begin{cases}
Y^{t,x}_s + \int_s^T U^{t,x}_r dr = g(X^{t,x}_T) + \int_s^T 1_{[t,T]}(r)F(r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r)dr - \int_s^T Z^{t,x}_r dW_r, \quad s \in [0,T] \\
U^{t,x}_s \in \partial \varphi \left( Y^{t,x}_s \right), \quad d\mathbb{P} \times ds \text{ on } \Omega \times [0,T].
\end{cases}
\]

Moreover, for all \( p \geq 2 \), there exists some constant \( C_p > 0 \), \( q \in \mathbb{N}^* \), such that, for all \( t, \tilde{t} \in [0,T] \), \( x, \tilde{x} \in \mathbb{R}^n \):

\[
\begin{align*}
(j) & \quad \mathbb{E} \left( \sup_{s \in [0,T]} |X^{t,x}_s|^p \right) \leq C_p (1 + |x|^p), \\
(jj) & \quad \mathbb{E} \left( \sup_{s \in [0,T]} |X^{t,x}_s - X^{\tilde{t},\tilde{x}}_s|^p \right) \leq C_p (1 + |x|^p + |\tilde{x}|^p) (|t - \tilde{t}|^{p/2} + |x - \tilde{x}|^p), \\
(jjj) & \quad \mathbb{E} \left( \sup_{s \in [0,T]} |Y^{t,x}_s|^2 \right) \leq C (1 + |x|^2), \\
(jv) & \quad \mathbb{E} \left( \sup_{s \in [0,T]} |Y^{t,x}_s - Y^{\tilde{t},\tilde{x}}_s|^2 \right) \leq C_2 \left[ \mathbb{E} \left| g(X^{t,x}_T) - g(X^{\tilde{t},\tilde{x}}_T) \right|^2 + \mathbb{E} \int_0^T 1_{[t,T]}(r)F(r, X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r) - 1_{[\tilde{t},T]}(r)F(r, X^{\tilde{t},\tilde{x}}_r, Y^{\tilde{t},\tilde{x}}_r, Z^{\tilde{t},\tilde{x}}_r)^2 dr \right].
\end{align*}
\]

## 3 Approximations schemes for BSVI

We will consider a partition of \([0,T]\),

\[
\pi = \{ t_i = ih : 0 \leq i \leq n \}, \quad \text{with } h := T/n, \quad n \in \mathbb{N}^*,
\]

on which we approximate the solution of the backward stochastic variational inequality [9]. For the numerical simulations of the forward part, the most standard approach consists in approximating the SDE in a proper way on each interval \([t_i, t_{i+1}]\) by the classical Euler scheme (see, e.g. Kloeden & Platen [9]):

\[
\begin{align*}
X^{t_i+1}_h &= X^{t_i}_h + b \left( X^{t_i}_h \right) h + \sigma \left( X^{t_i}_h \right) \left( W_{t_{i+1}} - W_{t_i} \right), \quad i = 0, \ldots, n-1 \\
X^{0}_h &= X_0.
\end{align*}
\]

We remark that the above numerical scheme is easy to implement since it requires only the simulation of \( d \)-independent Gaussian variables for the Brownian increments, providing a weak error of \( h \) order.

For \( t \in [t_i, t_{i+1}] \) let

\[
X^h_t = X^{t_i}_h + b \left( X^{t_i}_h \right) (t - t_i) + \sigma(X^{t_i}_h) (W_t - W_{t_i}).
\]

We have the following estimation of the error given by the Euler scheme (see [9]).
Proposition 2 Under the assumptions (I) on the coefficients $b$ and $\sigma$, for all $p \geq 1$, there exists $C_p > 0$ such that
\[
\max_{0, n - 1} \mathbb{E} \left( \sup_{t \in [0, T]} |X_t - X_t^h|^p + \sup_{t \in [t_i, t_{i+1}]} |X_t - X_t^i|^p \right)^{1/p} \leq C_p \sqrt{h}.
\]

Here and subsequently we will consider the one-dimensional BSDE case.

Using the Yosida approximation $\nabla \varphi$ of the multivalued operator $\partial \varphi$, with $\varepsilon = h^a$ and $a \in (0, 1/2)$ (the way of choosing this constant will be detailed later), we deduce that the following approximate equation
\[
Y_t^h + \int_t^T \nabla \varphi_h \left( Y_r^h \right) dr = g(X_T) + \int_t^T F(r, X_r, Y_r^h, Z_r^h) dr - \int_t^T Z_r^h dW_r, \quad \forall t \in [0, T], \quad \mathbb{P} - a.s.,
\]

admits a unique solution $(Y_t^h, Z_t^h) \in L^2_{ad}(\Omega; C([0, T]; \mathbb{R})) \times L^2_{ad}(\Omega; L^2([0, T]; \mathbb{R}^d))$.

Further, inspired by the paper of Bouchard and Touzi, [4], let us define an Euler type approximation for the Yosida approximation process $Y^\varepsilon$. For an intuitive introduction, let $Y_T^h := g(X_T^h)$ be the initial condition, and, for $i = n - 1, 0$, remark that
\[
Y_t^h \sim Y_{t+i}^h + h \left[ F(t_i, X_{t_i}, Y_{t_i}^h, Z_{t_i}^h) - \nabla \varphi_h(\nabla \varphi_h) \right] - Z_{t_i}^h(W_{t+i} - W_{t_i});
\]

taking the conditional expectation $\mathbb{E}^i (\cdot) := \mathbb{E} (\cdot \mid \mathcal{F}_{t_i})$, we obtain
\[
Y_{t+i}^h \sim \mathbb{E}^i (Y_{t+i}^h) + h \left[ F(t_i, X_{t_i}, Y_{t_i}^h, Z_{t_i}^h) - \nabla \varphi_h(\nabla \varphi_h) \right].
\]

If we multiply (8) by $W_{t+i} - W_{t_i}$ it follows
\[
Z_{t_i}^h \sim \frac{1}{h} \mathbb{E}^i (Y_{t+i}^h (W_{t+i} - W_{t_i})).
\]

Therefore, we propose the following implicit discretization procedure, which define the pair $(Y^h, Z^h)$ inductively, for $i = n - 1, 0$ :
\[
\begin{align*}
\tilde{Y}_T^h & := g(X_T^h), \quad \tilde{Z}_T^h = 0, \\
\tilde{Y}_{t_i}^h & := \mathbb{E}^i, h(\tilde{Y}_{t+i}^h) + h \left[ F(t_i, X_{t_i}, Y_{t_i}^h, Z_{t_i}^h) - \nabla \varphi_h(\nabla \varphi_h) \right], \\
\tilde{Z}_{t_i}^h & := \frac{1}{h} \mathbb{E}^i, h(\tilde{Y}_{t+i}^h (W_{t+i} - W_{t_i})), \\
\tilde{U}_{t_i}^h & := \nabla \varphi_h(\mathbb{E}^i, h(\tilde{Y}_{t+i}^h)),
\end{align*}
\]

where $\mathbb{E}^i (\cdot) := \mathbb{E} (\cdot \mid \mathcal{F}_{t_i})$ and $F_{t_i}^h := \sigma(X_{t_i}^h : 0 \leq j \leq i)$.

Remark 3 Observe that $\tilde{Y}_{t_i}^h$ is defined implicitly as the solution of a fixed point problem. Since the involved functions are Lipschitz, it is well defined. Moreover, for small values of $h > 0$ it can be estimated numerically in an accurate way.
Remark 4 We can also use an explicit scheme to define
\[ \hat{Y}^h_{t_i} := E^i(h\hat{Y}^h_{t_{i+1}}) + hE^i(h\left[F(t_i, X^h_{t_i}, \hat{Y}^h_{t_i}, \hat{Z}^h_{t_i}) - \nabla \varphi_{h^2}(\hat{Y}^h_{t_i})\right]). \]

The advantage of this scheme is that it does not require a fixed point procedure but, from a numerical point of view, adding a term in the conditional expectation makes it more difficult to estimate. Therefore the implicit scheme can be more tractable in practice.

Remark 5 We have that the filtration \( \mathcal{F}_t \) generated by the Brownian motion coincides with the filtration generated by the diffusion process \( X \), i.e. \( \mathcal{F}_t = \mathcal{F}_t^X \), and, from the Markov property of the process \( X \), it follows that
\[
E^i(\hat{Y}^h_{t_{i+1}}) = E^i(h\hat{Y}^h_{t_{i+1}}) = E(\hat{Y}^h_{t_{i+1}} | X^h_{t_i}),
\]
\[
E^i(h\hat{Y}^h_{t_{i+1}}(W_{t_{i+1}} - W_{t_i})) = E^i(h(\hat{Y}^h_{t_{i+1}}(W_{t_{i+1}} - W_{t_i})) = E(\hat{Y}^h_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) | X^h_{t_i}).
\]

Consider now a continuous version of (9). From the martingale representation theorem there exists a square integrable process \( \tilde{Z}^h \) such that
\[
\hat{Y}^h_{t_{i+1}} = \hat{Y}^h_{t_i} + \int_{t_i}^{t_{i+1}} \tilde{Z}^h_s dW_s,
\]
and, therefore, we define, for \( t \in (t_i, t_{i+1}] \),
\[
\hat{Y}^h_{t_i} := \hat{Y}^h_{t_i} + (t - t_i) \left[f(t_i, X^h_{t_i}, \hat{Y}^h_{t_i}, \hat{Z}^h_{t_i}) - \nabla \varphi_{h^2}(\hat{Y}^h_{t_{i}})\right] + \int_{t_i}^{t} \tilde{Z}^h_s dW_s.
\]

Obviously, we obtain that \( \hat{Y}^h_{t_{i+1}} = \hat{Y}^h_{t_i} \), and, for the simplicity of the notation, we will continue to write \( \hat{Y}^h_{t_i} \) for \( \hat{Y}^h_{t_{i+1}} \).

Remark 6 From (9), (10) and the isometry property, we notice that, for \( i = 0, n - 1 \),
\[
h \tilde{Z}^h_{t_i} = E^i(h\tilde{Y}^h_{t_{i+1}}(W_{t_{i+1}} - W_{t_i})) = E^i(h\int_{t_i}^{t_{i+1}} \tilde{Z}^h_s dW_s) = E^i(\int_{t_i}^{t_{i+1}} \tilde{Z}^h_s ds).
\]

To approximate \( Z^h_{t_i} \) we use
\[
\tilde{Z}^h_{t_i} := \frac{1}{h} E^i(\int_{t_i}^{t_{i+1}} Z^h_s ds), \quad t \in (t_i, t_{i+1})
\]
rather than \( Z^h_{t_i} \), which is the best approximation in \( L^2(\Omega \times [0, T]) \) of \( Z^h \) by adapted processes which are constant on each interval \( [t_i, t_{i+1}) \) (see Lemma 3.4.2 from Zhang [24]):
\[
E\left[\int_{t_i}^{t_{i+1}} |Z^h_s - \tilde{Z}^h_{t_i}|^2 ds\right] \leq E\left[\int_{t_i}^{t_{i+1}} |Z^h_s - \eta|^2 ds\right],
\]
for all \( \mathcal{F}_{t_i} \)-measurable stochastic process \( \eta \).
Proposition 9 Let the assumptions (15) and (16) be satisfied. We have the following estimate,

\[ E|Z_{t_i}^{h} - \bar{Z}_{t_i}^{h}|^2 = \frac{1}{h^2}E\left[ \left| \int_{t_i}^{t_{i+1}} \Delta^h Z_s ds \right|^2 \right] \leq \frac{1}{h} \int_{t_i}^{t_{i+1}} E|\Delta^h Z_s|^2 ds \leq \frac{1}{h} \int_{t_i}^{t_{i+1}} E|\Delta^h Z_s|^2 ds. \]  

Remark 7 From (12), the definition of \( \bar{Z}_{t_i}^{h} \) and Jensen inequality we obtain

\[
E|Z_{t_i}^{h} - \bar{Z}_{t_i}^{h}|^2 = \frac{1}{h^2} E\left[ \left| \int_{t_i}^{t_{i+1}} \Delta^h Z_s ds \right|^2 \right] \leq \frac{1}{h} \int_{t_i}^{t_{i+1}} E|\Delta^h Z_s|^2 ds \leq \frac{1}{h} \int_{t_i}^{t_{i+1}} E|\Delta^h Z_s|^2 ds.
\]

In order to prove an error estimate of the scheme first we use the solution \( (Y_t^h, Z_t^h)_{t \in [0,T]} \) of the approximating equation \( (17) \). The next result is a straightforward consequence of Proposition 2.3 from Pardoux & Răşcanu [17].

Proposition 8 Under the assumptions \( (1)-(3) \), there exists \( C > 0 \) such that

\[
\sup_{t \in [0,T]} E|Y_t - Y_t^h|^2 + \int_0^T |Z_t - Z_t^h|^2 dt \leq CT(T) h^a,
\]

where \( \Gamma(T) := E\left[ 1 + |g(X_T)|^2 + |X_T|^r + \int_0^T F(0, X_s^h, 0, 0) ds \right] \)

We recall Theorem 3.4.3 from [24], applied for the solution \( (Y_t^h, Z_t^h) \) of (17). To obtain a similar conclusion we have to impose more restrictive assumptions than (13):

- there exists some constant \( K > 0 \), such that

\[
(i) \quad |b(x) - b(\bar{x})| + \|\sigma(x) - \sigma(\bar{x})\| \leq K|\bar{x} - \bar{x}|, \forall x, \bar{x} \in \mathbb{R}^m,
\]

\[
(ii) \quad |F(\xi) - F(\bar{\xi})| \leq K|\xi - \bar{\xi}|, \forall \xi, \bar{\xi} \in [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d,
\]

\[
(iii) \quad |g(y) - g(\bar{y})| \leq K|y - \bar{y}|, \forall y, \bar{y} \in \mathbb{R};
\]

- the function \( \varphi : \mathbb{R}^m \to (-\infty, +\infty) \) is a proper convex lower semicontinuous function and there exist \( M > 0 \) and \( r \in \mathbb{N} \) such that

\[
|\varphi(g(x))| \leq M(1 + |x|^r), \forall x \in \mathbb{R}^m.
\]

Proposition 9 Let the assumptions (15) and (16) be satisfied. We have the following estimate, for some \( C > 0 \),

\[
\max_{i=0, n-1} \sup_{t \in [t_i, t_{i+1}]} E|Y_t^h - Y_{t_i}^h|^2 + \sum_{i=1}^n E\left[ \int_{t_i}^{t_{i+1}} |Z_s^h - \bar{Z}_s^h|^2 ds \right] \leq Ch,
\]

where \( \bar{Z}_s^h := \frac{1}{h} \int_{t_i}^{t_{i+1}} E\left[ \int_{t_i}^{t_{i+1}} Z_s^h ds \right]. \)

Proof. The inequality \( \max_{i=0, n-1} \sup_{t \in [t_i, t_{i+1}]} E|Y_t^h - Y_{t_i}^h|^2 \leq Ch \)

can be obtained by classical calculus, using Itô's formula, Lipschitz property of the coefficient functions and the bounds of the approximate solution \( (Y_t^h, Z_t^h)_{t \in [0,T]} \) of (17) (see Proposition 2.1 and 2.2 from [17]).
For the proof of the inequality
\[ \sum_{i=1}^{n} \mathbb{E} \int_{t_i}^{t_{i+1}} |Z_s^h - \hat{Z}_s^h|^2 ds \leq C h \]
is sufficient to recall the proof of Theorem 3.4.3 from [24].

Using the estimates from the above Propositions we can prove the following:

**Proposition 10** Let the assumptions (15) and (16) be satisfied. Then there exists \( C > 0 \) such that
\[
\sup_{t \in [0,T]} \mathbb{E} |Y_t^h - \hat{Y}_t^h|^2 + \mathbb{E} \int_0^T |Z_t^h - \hat{Z}_t^h|^2 dt \leq C h^{1-2a}.
\]

**Proof.** From (7) and (11) we deduce that, for \( \alpha, \beta > 0 \),
\[
Y_t^h = Y_{t_{i+1}}^h + \int_t^{t_{i+1}} \left[ F(s, X_s^h, Z_s^h) - \nabla \varphi_{h^a}(Y_s^h) \right] ds - \int_t^{t_{i+1}} Z_s^hdW_s,
\]
\[
\hat{Y}_t^h = Y_{t_{i+1}}^h + \int_t^{t_{i+1}} \left[ F(t_i, X_{t_i}^h, \hat{Y}_{t_i}^h) - \nabla \varphi_{h^a}(Y_{t_i}^h) \right] ds - \int_t^{t_{i+1}} \hat{Z}_s^h dW_s.
\]
Throughout the proof let \( \Delta_h F_t := F(t, X_t, Y_t^h, Z_t^h) - F(t, X_{t_i}^h, \hat{Y}_{t_i}^h, \hat{Z}_{t_i}^h) \), \( \Delta_h Y_t := Y_t^h - \hat{Y}_t^h \) and \( \Delta_h Z_t := Z_t^h - \hat{Z}_t^h, t \in [t_i, t_{i+1}] \).

Applying Energy equality we obtain that
\[
\mathbb{E} |\Delta_h Y_t|^2 + \mathbb{E} |\Delta_h Z_t|^2 ds = \mathbb{E} |\Delta_h Y_{t_{i+1}}|^2 + 2 \mathbb{E} \int_t^{t_{i+1}} \Delta_h Y_s \Delta_h F_s ds
\]
\[ - 2 \mathbb{E} \int_t^{t_{i+1}} \Delta_h Y_s \left( \nabla \varphi_{h^a}(Y_s^h) - \nabla \varphi_{h^a}(\hat{Y}_{t_i}^h) \right) ds,
\]
We first compute \( \Delta_h Y_s \cdot \left[ \Delta_h F_s - (\nabla \varphi_{h^a}(Y_s^h) - \nabla \varphi_{h^a}(\hat{Y}_{t_i}^h)) \right] 
\]
for which we use Lipschitz property of \( F \) and \( \nabla \varphi_{h^a} \) :
\[
2 \mathbb{E} \int_t^{t_{i+1}} \Delta_h Y_s \cdot \left[ \Delta_h F_s - (\nabla \varphi_{h^a}(Y_s^h) - \nabla \varphi_{h^a}(\hat{Y}_{t_i}^h)) \right] ds \leq
\]
\[ \leq 2 K \mathbb{E} \int_t^{t_{i+1}} |\Delta_h Y_s|^2 ds + \frac{4}{\alpha} \mathbb{E} \int_t^{t_{i+1}} |X_s - X_{t_i}^h|^2 ds + \frac{4}{\alpha} \mathbb{E} \int_t^{t_{i+1}} |Z_s^h - \hat{Z}_{t_i}^h|^2 ds
\]
\[ + \frac{4}{\alpha} \mathbb{E} \int_t^{t_{i+1}} |Z_s^h - \hat{Z}_{t_i}^h|^2 ds + \frac{1}{\beta h^{2a}} \mathbb{E} \int_t^{t_{i+1}} |Y_s^h - \hat{Y}_{t_i}^h|^2 ds,
\]
where \( \alpha, \beta > 0 \) will be chosen later.

From now on, let \( C > 0 \) be a constant independent of \( h \), constant which can take different values from one line to another.

From Proposition [2] we have that there exists \( C > 0 \) such that
\[
\mathbb{E} |X_s - X_{t_i}^h|^2 \leq 2 \mathbb{E} |X_s - X_{t_i}^h|^2 + 2 \mathbb{E} |X_{t_i} - X_{t_i}^h|^2 \leq C h,
\]
\[
8
\]
and, from Proposition 9,
\[ \mathbb{E}|Y_s^h - \tilde{Y}_{t_i}^h|^2 \leq 2 \mathbb{E}|Y_s^h - Y_{t_i}^h|^2 + 2 \mathbb{E}|Y_{t_i}^h - \tilde{Y}_{t_i}^h|^2 \leq Ch + 2 \mathbb{E}|\Delta h Y_{t_i}|^2. \]

Using (13)
\[ \mathbb{E}|Z_s^h - \tilde{Z}_{t_i}^h|^2 \leq 2 \mathbb{E}|Z_s^h - \tilde{Z}_{t_i}^h|^2 + 2 \mathbb{E}|\tilde{Z}_{t_i}^h - \tilde{Z}_{t_i}^h|^2 = 2 \mathbb{E}|Z_s^h - \tilde{Z}_{t_i}^h|^2 + \frac{2}{h} \int_{t_i}^{t_{i+1}} \mathbb{E}|\Delta h Z_s|^2 ds. \]

Then (18) yields
\[ A_i(t) := \mathbb{E}|\Delta h Y_t|^2 + \int_t^{t_{i+1}} \mathbb{E}|\Delta h Z_s|^2 ds \leq (K^2 \alpha + \beta) \mathbb{E} \int_t^{t_{i+1}} |\Delta h Y_s|^2 ds + B_i, \]
where
\[ B_i := \mathbb{E}|\Delta h Y_{t_{i+1}}|^2 + \frac{8}{\alpha} \int_{t_i}^{t_{i+1}} \mathbb{E}|Z_s^h - \tilde{Z}_{t_i}^h|^2 ds + \frac{8}{\alpha} \int_{t_i}^{t_{i+1}} \mathbb{E}|\Delta h Z_s|^2 ds + \left( \frac{4}{\alpha} + \frac{1}{\beta h^{2a}} \right) Ch^2 + \]
\[ + 2h \left( \frac{4}{\alpha} + \frac{1}{\beta h^{2a}} \right) \mathbb{E}|\Delta h Y_{t_i}|^2. \]

Using a backward Gronwall type inequality we deduce
\[ \mathbb{E}|\Delta h Y_t|^2 \leq B_i e^{(K^2 \alpha + \beta) h} \leq B_i e^{(K^2 \alpha + \beta) h} \leq CB_i, \]
and, therefore,
\[ A_i(t) \leq B_i + (K^2 \alpha + \beta) \int_t^{t_{i+1}} CB_i ds = B_i \left[ 1 + C (K^2 \alpha + \beta) h \right] \leq B_i \left[ 1 + Ch \right], \ h \in (0, 1). \]

The above inequality and the definition of \( B_i \) implies
\[ \left[ 1 - (1 + Ch) \left( \frac{4}{\alpha} + \frac{1}{\beta h^{2a}} \right) 2h \right] \mathbb{E}|\Delta h Y_t|^2 + \left[ 1 - (1 + Ch) \frac{8}{\alpha} \right] \int_{t_i}^{t_{i+1}} \mathbb{E}|\Delta h Z_s|^2 ds \leq \]
\[ \leq (1 + Ch) \left[ \mathbb{E}|\Delta h Y_{t_{i+1}}|^2 + \frac{8}{\alpha} \int_{t_i}^{t_{i+1}} \mathbb{E}|Z_s^h - \tilde{Z}_{t_i}^h|^2 ds + Ch^{2-2a} \right]. \]

Taking \( a \in (0, 1/2) \), we can chose \( h > 0 \) sufficiently small and \( \alpha, \beta > 0 \) large enough such that
\[ C_1 := 1 - (1 + Ch) \left( \frac{4}{\alpha} + \frac{1}{\beta h^{2a}} \right) 2h > 0 \quad \text{and} \quad C_2 := 1 - (1 + Ch) \frac{8}{\alpha} > 0 \]
and, therefore,
\[ C_1 \mathbb{E}|\Delta h Y_t|^2 + C_2 \int_{t_i}^{t_{i+1}} \mathbb{E}|\Delta h Z_s|^2 ds \leq \]
\[ \leq (1 + Ch) \left[ \mathbb{E}|\Delta h Y_{t_{i+1}}|^2 + \frac{8}{\alpha} \int_{t_i}^{t_{i+1}} \mathbb{E}|Z_s^h - \tilde{Z}_{t_i}^h|^2 ds + Ch^{2-2a} \right]. \]
Writing the above inequality for each \( i = 0, \ldots, n - 1 \), we can deduce
\[
\mathbb{E}|\Delta^h Y_{t_i}|^2 \leq (1 + Ch)^n \left[ h^{2-2a} + \mathbb{E}|\Delta^h Y_T|^2 + \sum_{i=1}^n \int_{t_i}^{t_{i+1}} \mathbb{E}|Z_s^h - \bar{Z}_{t_i}^h|^2 \, ds \right], \quad i = 0, n - 1.
\]

From the Lipschitz property of \( g \) and Proposition [9] we obtain, for each \( i = 0, n - 1 \), since \( a \in (0, 1/2) \), that
\[
(22) \quad \mathbb{E}|\Delta^h Y_{t_i}|^2 \leq Ch, \ \forall h \in (0, 1) \text{ small enough.}
\]

For the proof of the inequality concerning \( \|\Delta^h Z_s\|_{L^2(\Omega \times [0,T])} \) we act in the following manner (see, e.g. Bouchard & Touzi [4]). From (21) it follows that
\[
\int_0^T \mathbb{E}|\Delta^h Z_s|^2 \, ds = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}|\Delta^h Z_s|^2 \, ds \leq
\]
\[
\leq (1 + Ch) \left[ \sum_{i=0}^{n-1} \mathbb{E}|\Delta^h Y_{t_{i+1}}|^2 + Ch^{2-2a} n + 8 \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}|Z_s^h - \bar{Z}_{t_i}^h|^2 \, ds \right] - C_1 \sum_{i=0}^{n-1} \mathbb{E}|\Delta^h Y_{t_i}|^2 =
\]
\[
= (1 + Ch) \left[ \mathbb{E}|\Delta^h Y_T|^2 + Ch^{2-2a} + 8 \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}|Z_s^h - \bar{Z}_{t_i}^h|^2 \, ds \right] +
\]
\[
+ \left( 1 + Ch \right) + \left( 1 + Ch \right) \left( \frac{4}{\alpha} + \frac{1}{\beta} \right) 2h - 1 \right) \sum_{i=1}^{n-1} \mathbb{E}|\Delta^h Y_{t_i}|^2 - C_1 \mathbb{E}|\Delta^h Y_0|^2.
\]

and, therefore, from (22),
\[
\int_0^T \mathbb{E}|\Delta^h Z_s|^2 \, ds \leq C \left[ \mathbb{E}|\Delta^h Y_T|^2 + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}|Z_s^h - \bar{Z}_{t_i}^h|^2 \, ds + h^{1-2a} \right] +
\]
\[
+ \left( Ch + \frac{8}{\alpha} h + \frac{2}{\beta} h^{1-2a} \right) \sum_{i=0}^{n-1} \mathbb{E}|\Delta^h Y_{t_i}|^2 \leq
\]
\[
\leq C \left[ \mathbb{E}|\Delta^h Y_T|^2 + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}|Z_s^h - \bar{Z}_{t_i}^h|^2 \, ds + h^{1-2a} + h^{1-2a} Ch n \right] =
\]
\[
= C \left[ Ch + Ch + Ch^{1-2a} \right] \leq Ch^{1-2a}.
\]

Using the definition (20) of \( B_i \) we deduce that \( B_i \leq Ch \) and, respectively, \( \max \mathbb{E}|\Delta^h Y_{t_i}|^2 \leq Ch \), which completes the proof.

Consequently we have proved our main result:

**Theorem 11** There exists the constant \( C > 0 \) which depends only on the Lipschitz constants of the coefficients, such that:

\[
(23) \quad \sup_{t \in [0,T]} \mathbb{E}|Y_t - \hat{Y}_t|^2 + \mathbb{E} \int_0^T \left[ |Y_t - \hat{Y}_t|^2 + |Z_t - \bar{Z}_t|^2 \right] \, dt \leq Ch^{a^\wedge(1-2a)}.
\]
4 Generalized BSVI. A proposed scheme for numerical approximation

Let $\mathcal{D}$ be a open bounded subset of $\mathbb{R}^d$ of the form

$$\mathcal{D} = \{ x \in \mathbb{R}^d : \ell(x) < 0 \}, \quad \text{Bd} \mathcal{D} = \{ x \in \mathbb{R}^d : \ell(x) = 0 \},$$

where $\ell \in C_0^3 (\mathbb{R}^d)$, $|\nabla \ell(x)| = 1$, for all $x \in \text{Bd} \mathcal{D}$. We know that (see, e.g., Lions & Szniitman [11]), for every $(t, x) \in \mathbb{R}_+ \times \mathcal{D}$, there exists a unique pair of $\mathcal{D} \times \mathbb{R}_+-$valued progressively measurable continuous processes $(X_t^{t,x}, A_s^{t,x})_{s \geq 0}$, solution of the reflected SDE:

$$
\begin{align*}
X_t^{t,x} &= x + \int_t^{s\vee t} b(r, X_r^{t,x})dr + \int_t^{s\vee t} \sigma(r, X_r^{t,x})dW_r - \int_t^{s\vee t} \nabla \ell(X_r^{t,x})dA_r^{t,x}, \\
A_s^{t,x} &= \int_t^{s\vee t} 1_{\{X_r^{t,x} \in \text{Bd} \mathcal{D}\}}dA_r^{t,x}.
\end{align*}
$$

Moreover, it can be proved that

$$
\mathbb{E} \sup_{s \in [0,T]} \left( |X_t^{t,x} - X_s^{t,x'}|^p \right) \leq C(|x - x'|^p + |t - t'|^{p/2})
$$

and, for all $\mu > 0$, $\mathbb{E}(e^{\mu A_T^{t,x}}) < \infty$.

Consider now the following generalized backward stochastic variational inequality:

$$
\begin{align*}
dY_t + F(t, X_t, Y_t, Z_t)dt + G(t, X_t, Y_t)dA_t \in \partial \varphi(Y_t)dt + Z_tdW_t, \quad 0 \leq t \leq T, \\
Y_T = g(X_T).
\end{align*}
$$

We will suppose that the functions $F$ and $G$ satisfy the same assumption as the generator function $F$ from the previous section. It is known (see Maticiuc & Răşcanu [13]) that the above equation admits a unique solution, i.e., for all $t \in [0,T]$, $\mathbb{P}$-a.s.,

$$
Y_t + \int_t^T U_s ds = g(X_T) + \int_t^T F(s, X_s, Y_s, Z_s)ds + \int_t^T G(s, X_s, Y_s)dA_s - \int_t^T Z_s dW_s,
$$

where $U_t \in \partial \varphi(Y_t)$, a.e. on $\Omega \times [0,T]$.

**Theorem 12** Under the considered assumptions, the generalized BSVI [22] admits a unique solution $(Y_t, Z_t, U_t)$ of $\mathcal{F}_t$-progressively measurable processes. Moreover, for any $0 \leq s \leq t \leq T$, we have, for some positive constant $C$:

$$
\begin{align*}
(a) \quad & \mathbb{E} \left[ \int_s^t (|Y_r|^2 + ||Z_r||^2) dr + \int_s^t |Y_r|^2 dA_r \right] + \mathbb{E} \sup_{s \leq r \leq t} |Y_r|^2 \leq CM_1, \\
(b) \quad & \mathbb{E}(\varphi(Y_t)) \leq CM_2 \quad \text{and} \quad \mathbb{E} \left[ \int_s^t |U_r|^2 dr \right] \leq CM_2,
\end{align*}
$$

where

$$
M_1 = \mathbb{E} \left[ |\xi|^2 + \int_0^T \left( |F(s, 0, 0)|^2 ds + |G(s, 0)|^2 dA_s \right) \right] \quad \text{and} \quad M_2 = \mathbb{E}(|\xi|^2 + \varphi(\xi)).
$$
For the generalized system considered above, we propose a mixed approximation scheme, considering, for the simplicity of the presentation, only the case \( \varphi \equiv 0 \). Consider the grid of \( [0, T] : \pi = \{ t_i = ih, i \leq n \} \), with \( h := T/n, n \in \mathbb{N}^* \) and we will define \( X^\pi \), the approximating Euler scheme for the reflected process \( X \). We follow the paper [6], where the authors present the standard projected Euler approach to stopped reflected diffusions.

\[
\begin{align*}
X_0^\pi &= x, \quad A_0^\pi = 0, \\
\dot{X}_{t+1}^\pi &= X_t^\pi + b(t_i, X_t^\pi)(t_{i+1} - t_i) + \sigma(t_i, X_t^\pi)(W_t - W_{t_i}), \\
\end{align*}
\]

Taking the projection on the domain, we define

\[
\begin{align*}
X_{t+1}^\pi &= \begin{cases} 
\dot{X}_{t+1}^\pi, & X_{t+1}^\pi \in \mathcal{D}, \\
\text{Pr}_{\mathcal{D}}(\dot{X}_{t+1}^\pi), & X_{t+1}^\pi \notin \mathcal{D},
\end{cases} \\
A_{t+1}^\pi &= \begin{cases} 
A_t^\pi, & X_{t+1}^\pi \in \mathcal{D}, \\
A_t^\pi + \| \text{Pr}_{\mathcal{D}}(\dot{X}_{t+1}^\pi) - \dot{X}_{t+1}^\pi \|, & X_{t+1}^\pi \notin \mathcal{D}.
\end{cases}
\end{align*}
\]

To define an approximation scheme for the generalized BSVI (25) consider \( Y_t^\pi := g(X_t^\pi) \) and, for \( i = \frac{n-1}{h}, 0 \), in an intuitive manner, using the notation \( \Delta A_{t_i}^\pi := A_{t_{i+1}}^\pi - A_{t_i}^\pi \) and \( \Delta W_{t_i} := W_{t_{i+1}} - W_{t_i} \):

\[
Y_{t_i} \sim Y_{t_{i+1}} - G(X_{t_{i+1}}^\pi, Y_{t_{i+1}})\Delta A_{t_i}^\pi - Z_{t_i} \Delta W_{t_i}.
\]

We take the conditional expectation \( \mathbb{E}^{F_i} \) and it follows

\[
Y_{t_i} \sim \mathbb{E}^{F_i}(Y_{t_{i+1}}) - \mathbb{E}^{F_i}|G(X_{t_{i+1}}^\pi, Y_{t_{i+1}})\Delta A_{t_i}^\pi|.
\]

This suggest to define the following approximation scheme:

\[
(27) \quad \begin{cases} 
Y_{t_i}^\pi := \mathbb{E}^{F_i}|Y_{t_{i+1}}^\pi - G(X_{t_{i+1}}^\pi, Y_{t_{i+1}})\Delta A_{t_i}^\pi|, & Y_T^\pi := g(X_T^\pi), \\
Z_{t_i}^\pi := \frac{1}{h} \mathbb{E}^{F_i}|Y_{t_{i+1}}^\pi \Delta W_{t_i} - G(X_{t_{i+1}}^\pi, Y_{t_{i+1}})\Delta A_{t_i}^\pi \Delta W_{t_i}|.
\end{cases}
\]

**Problem 13** The proof of the convergence for the scheme defined by (27) can provide a useful tool for the approximation of the solution for the Generalized BSVI (25). For the moment this is still an open problem, which the interested reader is kindly invited to approach it.

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