Chern–Simons Vortices in the Gudnason Model

Xiaosen Han
Institute of Contemporary Mathematics
School of Mathematics
Henan University
Kaifeng, Henan 475004, PR China

Chang-Shou Lin
Department of Mathematics
National Taiwan University
Taipei, Taiwan 10617, ROC

Gabriella Tarantello
Dipartimento di Matematica
Universitè di Roma “Tor Vergata”
Via della Ricerca Scientifica
00133 Rome, Italy

Yisong Yang
Department of Mathematics
Polytechnic Institute of New York University
Brooklyn, New York 11201, USA

Abstract

We present a series of existence theorems for multiple vortex solutions in the Gudnason model of the $\mathcal{N} = 2$ supersymmetric field theory where non-Abelian gauge fields are governed by the pure Chern–Simons dynamics at dual levels and realized as the solutions of a system of elliptic equations with exponential nonlinearity over two-dimensional domains. In the full plane situation, our method utilizes a minimization approach, and in the doubly periodic situation, we employ an-inequality constrained minimization approach. In the latter case, we also obtain sufficient conditions under which we show that there exist at least two gauge-distinct solutions for any prescribed distribution of vortices. In other words, there are distinct solutions with identical vortex distribution, energy, and electric and magnetic charges.

1 Introduction

The classical Abelian Higgs model defined by the Lagrangian action density

$$\mathcal{L}_{AH} = -\frac{1}{4} F_\mu F^\mu \nu + \frac{1}{2} D_\mu \phi D^\mu \phi - \frac{\lambda}{8} (|\phi|^2 - 1)^2,$$  \hspace{1cm} (1.1)
is of foundational importance in theoretical physics. Here \( \phi \) is a complex-valued scalar field called the Higgs field, \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) the electromagnetic field strength generated from a real-valued gauge field, \( A_\mu \), \( D_\mu \phi = \partial_\mu \phi - iA_\mu \phi \) the gauge-covariant derivative, \( \lambda > 0 \) a coupling parameter, \( \mu, \nu = 0, 1, 2, 3 \) are the \((3 + 1)\)-dimensional Minkowski spacetime coordinate indices, and the metrics \((g^{\mu \nu}) = (g_{\mu \nu}) = \text{diag}(1, -1, -1, -1)\) are used to raise or lower indices. For example, in quantum field theory, the model provides a mathematically simplest thought-laboratory allowing various fundamental concepts such as spontaneous symmetry-breaking, the onset and annihilation of the Goldstone particles, and the Higgs mechanism to be formulated and explored \([13, 35, 53]\).

Phenomenologically, the model in its static limit and temporal gauge \( A_0 = 0 \) is the celebrated Ginzburg–Landau theory \([26]\) for superconductivity. In two spatial dimensions, topological defects in the form of the Abrikosov vortices \([1, 21]\) can be generated from the model which, when coupled with the Einstein equations, provide indispensable structures, known as cosmic strings, giving rise to centers of curvature concentrations in spacetime and hence forming the seeds for matter accretion in the early universe \([68, 71, 72]\). On the other hand, however, a statement known as the Julia–Zee theorem \([11, 62]\) says that the static solutions of the equations of motion of (1.1) as well as of that of the general non-Abelian Yang–Mills–Higgs model, in two-spatial dimensions and of finite energy, must stay in the temporal gauge, \( A_0 = 0 \). In other words, vortices of the Yang–Mills–Higgs model, Abelian or non-Abelian, do not carry electric charge and can only be purely magnetic. Nevertheless, electrically \textit{and} magnetically charged vortices, called dyonic vortices, are needed in many areas of applications such as high-temperature superconductivity \([43, 47]\), the Bose–Einstein condensates \([36, 72]\), the quantum Hall effect \([59]\), optics \([9]\), and superfluids \([56]\). Therefore, it will be important to modify the classical Yang–Higgs–Higgs theory so that dyonic vortices are accommodated. In a series of pioneering studies, Jackiw and Templeton \([38]\), Schonfeld \([54]\), Deser, Jackiw, and Templeton \([19, 20]\), Paul and Khare \([51]\), de Vega and Schaposnik \([66, 67]\), and Kumar and Khare \([44]\) developed a modified Yang–Mills–Higgs theory in which the Chern–Simons topological terms \([17, 18]\) are implemented into the action density. Although these terms fail to be gauge-invariant locally, they preserve gauge-invariance globally and thus render the theory the same gauge invariance as in the Yang–Mills–Higgs theory. More importantly, the presence of the Chern–Simons terms make the coexistence of electric and magnetic charges a necessity. Mathematically, however, the presence of the Chern–Simons terms and the nontrivial temporal component of the gauge field leads us to face a much more complicated form of the equations of motion and an existence theory for radially symmetric solutions has only been obtained rather recently \([15]\). This difficulty motivated people to explore possible BPS (after the seminal studies of Bogomol’nyi \([10]\) and Prasad and Sommerfield \([52]\)) reductions of the problem and brought into light the works of Hong, Kim, and Pac \([33]\) and Jackiw and Weinberg \([39]\), which sparked a great development of the subject of construction of the Chern–Simons–Higgs vortex solutions up to today. In their approach, it may be understood that the initial Lagrangian action density to be modified is simply that of the Abelian Higgs model, with the addition of a Chern–Simons term controlled by a coupling parameter \( \kappa \), which may be written as

\[
\mathcal{L}_{\text{MCSH}} = -\frac{1}{4e^2} F_{\mu \nu} F^{\mu \nu} - \frac{\kappa}{4} \epsilon^{\mu \nu \rho} A_\mu F_{\nu \rho} + D_\mu \phi D^\mu \phi - V(|\phi|^2),
\]

where \( e > 0 \) denotes a coupling parameter imposed on the Maxwell kinetic density term and \( V \) is the Higgs potential density function, as in \([51, 66]\). Since in (1.2) both the Maxwell and Chern–Simons terms are present, the model is referred to as the Maxwell–Chern–Simons–Higgs model. In
the limit $\epsilon \to \infty$, the Maxwell term is switched off and the model (1.2) becomes

$$L_{\text{ACSH}} = -\frac{\kappa}{4} \epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} + D_\mu \phi D^\mu \phi - V(|\phi|^2),$$

(1.3)

which is known as the Abelian Chern–Simons–Higgs model [33,39] in which the gauge field dynamics is governed solely by the Chern–Simons term. It is shown in [33,39] that, when the Higgs potential function $V$ is chosen to be

$$V(|\phi|^2) = \frac{1}{\kappa^2} |\phi|^2 (1 - |\phi|^2)^2,$$

(1.4)

which is analogous with the critical choice $\lambda = 1$ for (1.1) studied in [40], the equations of motion of (1.3) may be reduced into the BPS system

$$D_1 \phi + i D_2 \phi = 0, \quad F_{12} = \frac{2}{\kappa^2} |\phi|^2 (1 - |\phi|^2).$$

(1.5)

The multiple vortex solutions, realizing a prescribed distribution of vortices located at $p_1, \ldots, p_n$ and carrying the total electric and magnetic charges, $Q_e$ and $Q_m$, given by

$$Q_e = \kappa Q_m, \quad Q_m = \int F_{12} \text{d}x = 2\pi n,$$

(1.6)

may be constructed in terms of the variable $u = |\phi|^2$ via solving the master equation [33,39]

$$\Delta u = \alpha u (e^u - 1) + 4\pi \sum_{s=1}^n \delta_{p_s}(x),$$

(1.7)

where $\alpha = \frac{4}{\kappa^2}$, whose structure has been shown [64,74] to be much richer and more challenging than that of the classical Abelian Higgs model. For example, in contrast to the Abelian Higgs model (1.1) where the solution of finite energy realizing any prescribed distribution of vortices is unique [40,65,70], the solutions in the Abelian Chern–Simons–Higgs model are not unique and further categorized [22,34,37] into topological and non-topological solutions, which have led to a rapid development of analytic methods and the harvest of a rich vista of results [11,12,14,16,60,61] regarding issues such as existence, uniqueness, nonexistence, asymptotic properties, approximation, etc., of the solutions.

The purpose of the present article is to develop an existence theory for non-Abelian Chern–Simon–Higgs vortices in the Gudnason model [27,28]. Historically, after the formulation of the Abelian Chern–Simons–Higgs model in [33,39], a great deal of activities quickly evolved around developing [23] and analyzing [45,48,73] non-Abelian extensions of the model, both non-relativistic and relativistic. What distinguishes the Gudnason model [27,28] from the classical non-Abelian Chern–Simons–Higgs models [23,63,67] are that the former may be derived in a supersymmetric gauge field theory framework as those in the study of monopole confinement mechanism [5,31,32,46,57,58] and that a kind of bi-level Chern–Simons dynamics is present as in the Bagger–Lambert–Gustavsson theory [6–8,29] and the Aharony–Bergman–Jafferis–Maldacena theory [2] to govern gauge-field kinematics. These studies have prompted a great amount of research activities over the past few years and the nonlinear partial differential equation problems unearthed offer truly rich opportunities for analysts in exploring new techniques and ideas.

We shall present two types of results. The first type concerns the existence of solutions over the full plane subject to the boundary behavior corresponding to the asymptotic vacuum state with a completely broken symmetry [27,28]. Our method is based on a variational reformulation
of the problem and a coercive minimization approach. The second type concerns the existence, nonexistence, and multiplicity of solutions over a doubly periodic domain. The main difficulty we encounter here is the constraint problem which makes it hard to develop a general variational method as in the full plane situation where there is no constraint to tackle. In this situation we compromise to solve the system in the special case with two equations. Although this case is limited, the results are rich and structures are challenging. Here we extend the inequality-constrained techniques [11, 48, 63] to resolve the equality-constraint difficulty and obtain existence and multiplicity results for solutions.

A brief outline of the rest of the paper is as follows. In the next section, we first review the Gudnason model [27, 28] and the associated non-Abelian vortex equations, also called the master equations [27, 28], to be studied in this paper. In the following section, we reduce the master equations into a system of nonlinear equations to be analyzed and state our main existence theorems regarding the solutions. The subsequent sections are then devoted to the proofs of these theorems by developing and utilizing variational techniques. In the last section, we briefly summarize and comment on our results.

2 The Gudnason model

Consider the standard Minkowski spacetime \( \mathbb{R}^{2,1} \) of signature \((+ - -)\) and use \( \mu, \nu = 0, 1, 2 \) to denote the temporal and spatial coordinate indices. The Gudnason model [27] is formulated as an \( \mathcal{N} = 2 \) supersymmetric Yang–Mills–Chern–Simons–Higgs theory with the general gauge group \( G = U(1) \times G' \) where \( G' \) is a non-Abelian simple Lie group represented by matrices. As in [27], use the index \( a = 1, \ldots, \dim(G') \) to label the non-Abelian gauge group generators and the index 0 the Abelian one, \( U(1) \). Then the gauge potential \( A_\mu \) taking values in the Lie algebra of the group \( G \) may be written as \( A_\mu = A^a_\mu t^a \) where \( A^a_\mu \) are real-valued vector fields and \( t^a \) (\( a = 0, u \)) the generators of \( G \) which are normalized to satisfy

\[
t^0 = \frac{1}{\sqrt{2N}}, \quad \text{Tr}(t^a t^b) = \frac{1}{2} \delta^{ab},
\]

where \( N \) is the dimension of the fundamental representation space of \( G' \). The gauge field strength tensor \( F_{\mu\nu} \) is then given by

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu].
\]

Use \( \phi \) to denote the Higgs scalar field in the adjoint representation of \( G \) and the \( N \times N_f \) matrix \( H \) to contain \( N_f \)-flavor matter (quark) fields in the fundamental representation of \( G \). Then their gauge-covariant derivatives are

\[
\mathcal{D}_\mu \phi = \partial_\mu \phi + i[A_\mu, \phi], \quad \mathcal{D}_\mu H = \partial_\mu H + iA_\mu H.
\]

With this preparation, the Lagrangian action density of the Yang–Mills–Chern–Simons–Higgs theory of the Gudnason model [27], omitting the Fermion part, is written as

\[
\mathcal{L}_{\text{YMCSH}} = -\frac{1}{4g^2}(F^a_{\mu\nu})^2 - \frac{1}{4e^2}(F^0_{\mu\nu})^2 - \frac{\mu}{8\pi} e^{\mu\nu\rho} A^a_\mu \partial_\nu A^a_\rho + \frac{1}{3} f^{abc} A^a_\mu A^b_\nu A^c_\rho - \frac{\kappa}{8\pi} e^{\mu\nu\rho} A^0_\mu \partial_\nu A^0_\rho
\]

\[
+ \frac{1}{2g^2} (\mathcal{D}_\mu \phi^a)^2 + \frac{1}{2e^2} (\partial_\mu \phi^0)^2 + \text{Tr}(\mathcal{D}_\mu H)(\mathcal{D}^\mu H) + \text{Tr}|\phi H - Hm|^2
\]

\[
- \frac{g^2}{2} \left( \text{Tr}(HH^\dagger t^a) - \frac{\mu}{4\pi} \phi^a \right)^2 - \frac{e^2}{2} \left( \text{Tr}(HH^\dagger t^0) - \frac{\kappa}{4\pi} \phi^0 - \frac{\xi}{\sqrt{2N}} \right)^2,
\]
where $\epsilon^{\mu \nu}$ is the Kronecker skew-symmetric tensor, $f^{abc}$ are the structural constants of the non-Abelian gauge group $G'$, $e, g \in \mathbb{R}$ the Abelian and non-Abelian Yang–Mills coupling constants, $\kappa \in \mathbb{R}$ the Abelian Chern–Simons coupling, $\pi \in \mathbb{R}$ the so-called Fayet–Iliopoulos parameter, and the notation $N$ denotes the non-Abelian Chern–Simons coupling constant which should not be confused with the spacetime coordinate index, $m$ is a mass matrix, $\xi \in \mathbb{R}$ is the concrete situations, $G$ are governed by the Lagrangian action density (2.12) which are rather complicated.

In order to investigate the solutions of these equations, Gudnason [27] applies the Bogomol’nyi method to carry out a completion of square analysis for the string tension of the model, obtained from the Lagrangian action density (2.12) are [27, 28] governed by the Lagrangian action density (2.12) are [27, 28]

$$\mathcal{L}_{\text{CSH}} = -\frac{\mu}{8\pi} \epsilon^{\mu \nu \rho} \left( A^0_\mu \partial_\nu A^a_\rho - \frac{1}{3} f^{abc} A^a_\mu A^b_\nu A^c_\rho \right) - \frac{\kappa}{8\pi} \epsilon^{\mu \nu \rho} A^0_\mu \partial_\nu A^0_\rho$$

$$+ \text{Tr}(\mathcal{D}_\mu H)(\mathcal{D}^\mu H) - 4\pi^2 \text{Tr} \left( \left( \frac{1}{\kappa N} \text{Tr}(HH^\dagger) - \xi \right) \mathbf{1}_N + \frac{2}{\mu} \text{Tr}(HH^\dagger t^a t^a) \right) H^2,$$

where $\mathbf{1}_N$ denotes the $N \times N$ identity matrix. The equations of motion of the Gudnason model [27, 28] are [27, 28]

$$\frac{\mu}{8\pi} \epsilon^{\mu \nu \rho} F^a_{\mu \nu} = -i \text{Tr} \left( H^\dagger t^a \mathcal{D}^\rho H - (\mathcal{D}^\rho H)^\dagger t^a H \right),$$

$$\frac{\kappa}{8\pi} \epsilon^{\mu \nu \rho} F^0_{\mu \nu} = -i \text{Tr} \left( H^\dagger t^0 \mathcal{D}^\rho H - (\mathcal{D}^\rho H)^\dagger t^0 H \right),$$

$$\mathcal{D}_\mu \mathcal{D}^\mu H = -4\pi^2 \left( \frac{1}{\kappa N} \text{Tr}(HH^\dagger) - \xi \right) \mathbf{1}_N + \frac{2}{\mu} \text{Tr}(HH^\dagger t^a t^a) H^2$$

$$- \frac{8\pi^2}{\kappa N} \text{Tr} \left\{ \left( \frac{1}{\kappa N} \text{Tr}(HH^\dagger) - \xi \right) \mathbf{1}_N + \frac{2}{\mu} \text{Tr}(HH^\dagger t^a t^a) \right\} H$$

$$- \frac{16\pi^2}{\mu} \text{Tr} \left\{ \left( \frac{1}{\kappa N} \text{Tr}(HH^\dagger) - \xi \right) \mathbf{1}_N + \frac{2}{\mu} \text{Tr}(HH^\dagger t^b t^b) \right\} H$$

$$H$$

$$\text{Tr}(HH^\dagger t^a t^a),$$

which are rather complicated. To proceed, Gudnason [27] applies the Bogomol’nyi method to carry out a completion of square analysis for the string tension of the model, obtained from the integration of the Hamiltonian component of the stress tensor, in the static limit, and derives the following BPS system of equations:

$$\overline{\mathcal{D}} H = 0,$$

$$F^a_{12} t^a = \frac{16\pi^2}{\mu \kappa N} \text{Tr}(HH^\dagger) - \xi \text{Tr}(HH^\dagger t^a t^a) + \frac{16\pi^2}{\mu^2} \text{Tr}(HH^\dagger t^b) \text{Tr}(HH^\dagger t^a t^b) t^a,$$

$$F^0_{12} t^0 = \frac{8\pi^2}{\kappa^2 N^2} \text{Tr}(HH^\dagger t^a t^a) - \xi \mathbf{1}_N + \frac{16\pi^2}{\mu \kappa N} \text{Tr}(HH^\dagger t^a t^a)^2 \mathbf{1}_N,$$

where $\overline{\mathcal{D}} = \mathcal{D}_1 + i \mathcal{D}_2$. In order to investigate the solutions of these equations, Gudnason [28] specifies the concrete situations, $G' = SO(2M)$ and $G' = USp(2M)$ with $N = 2M$ so that the equations become

$$\overline{\mathcal{D}} H = 0,$$

$$F^a_{12} t^a = \frac{2\pi^2}{\mu \kappa M} \text{Tr}(HH^\dagger) - \xi \text{Tr}(HH^\dagger t^a) t^a + \frac{2\pi^2}{\mu^2} \text{Tr}(HH^\dagger t^b) \text{Tr}(HH^\dagger t^a t^b) t^b,$$

$$F^0_{12} t^0 = \frac{2\pi^2}{\kappa^2 M^2} \text{Tr}(HH^\dagger) \text{Tr}(HH^\dagger t^a t^a) - \xi \mathbf{1}_{2M} + \frac{2\pi^2}{\mu \kappa M} \text{Tr}(HH^\dagger (HH^\dagger t^a) t^a) \mathbf{1}_{2M},$$
in which \( \langle A \rangle_J = A - J^\dagger A' J \) for a \( 2M \times 2M \) matrix with

\[
J = \begin{pmatrix} 0 & 1_M \\ \epsilon 1_M & 0 \end{pmatrix}, \quad \epsilon = \pm 1 \text{ depending on whether } G' = SO(2M) \text{ or } USp(2M).
\] (2.22)

Then, choosing \( H_0 \) as a suitable background matrix realizing a prescribed distribution of vortices and using a moduli matrix ansatz of the form \( H = S^{-1}H_0, \ S = sS' \), which splits the variables into the Abelian one \( \omega = |s|^2 \) and the non-Abelian one \( \Omega' = S'(S')^\dagger \), so that \( \Omega = SS^\dagger = \omega \Omega' \), the BPS equations (2.19)–(2.21) are shown to become the so-called master equations [28]

\[
\overline{\partial}(\Omega' \partial \Omega'^{-1}) = \frac{\pi^2}{\mu \kappa M} (\text{Tr}(\Omega_0 \Omega^{-1}) - \xi)\langle \Omega_0 \Omega^{-1} \rangle_J + \frac{\pi^2}{\mu^2} \langle (\Omega_0 \Omega^{-1})^2 \rangle_J,
\] (2.23)

\[
\overline{\partial} \ln \omega = -\frac{\pi^2}{\kappa^2 M^2} \text{Tr}(\Omega_0 \Omega^{-1})(\text{Tr}(\Omega_0 \Omega^{-1}) - \xi) - \frac{\pi^2}{\mu \kappa M} \text{Tr}(\Omega_0 \Omega^{-1}\langle \Omega_0 \Omega^{-1} \rangle_J),
\] (2.24)

where \( \overline{\partial} = \partial_1 + i \partial_2 \).

To describe multiple vortices by the master equations (2.23)–(2.24), we take as in [27, 28] the moduli matrix \( H_0(z) \) of the form

\[
H_0(z) = \prod_{i=1}^{M} \prod_{s=1}^{n_i} (z - z_{i,s}) D_0(z),
\]

where

\[
D_0(z) = \text{diag} \left\{ \prod_{s=1}^{n_1} (z - z_{1,s}), \ldots, \prod_{s=1}^{n_M} (z - z_{M,s}), \prod_{s=1}^{n_1} (z - z_{1,s})^{-1}, \ldots, \prod_{s=1}^{n_M} (z - z_{M,s})^{-1} \right\},
\]

and \( z_{i,s} \) are prescribed points on the complex plane, \( s = 1, \ldots, n_i, i = 1, \ldots, M; n_i \) are nonnegative integers, \( i = 1, \ldots, M \). We easily see that

\[
H_0^T(z) J H_0(z) = \prod_{i=1}^{M} \prod_{s=1}^{n_i} (z - z_{i,s}) J,
\]

\[
\det(H_0(z)) = \left( \prod_{i=1}^{M} \prod_{s=1}^{n_i} (z - z_{i,s}) \right)^{2M},
\]

\[
\Omega_0 = H_0(z) H_0(z)^\dagger = \prod_{i=1}^{M} \prod_{s=1}^{n_i} |z - z_{i,s}|^2 D_0(z)^\dagger D_0(z).
\]

With the further ansatz

\[
\Omega' = \text{diag}(e^{\chi_1}, \ldots, e^{\chi_M}, e^{-\chi_1}, \ldots, e^{-\chi_M}), \quad \omega = e^\psi,
\] (2.25)

where \( \chi_1, \ldots, \chi_M, \psi \) are real-valued functions, and a direct computation, the equations (2.23)–(2.24)
Then the equations (2.26)–(2.27) become

\[
\begin{align*}
\bar{\partial} \chi_j &= -\frac{\pi^2}{\mu \kappa M} \left( \prod_{k=1}^M \prod_{s=1}^{n_k} |z - z_{k,s}|^2 \sum_{i=1}^{n_i} \left[ \prod_{s=1}^{n_i} |z - z_{i,s}|^2 e^{-\psi - \chi_i} + \prod_{s=1}^{n_j} |z - z_{i,s}|^{-2} e^{-\psi + \chi_i} \right] - \xi \right) \\
\bar{\partial} \psi &= -\frac{\pi^2}{\mu \kappa M^2} \left( \prod_{k=1}^M \prod_{s=1}^{n_k} |z - z_{k,s}|^2 \sum_{i=1}^{n_i} \left[ \prod_{s=1}^{n_i} |z - z_{i,s}|^2 e^{-\psi - \chi_i} + \prod_{s=1}^{n_j} |z - z_{i,s}|^{-2} e^{-\psi + \chi_i} \right] - \xi \right) \\
&\quad \times \prod_{k=1}^M \prod_{s=1}^{n_k} |z - z_{k,s}|^{4} \left( \prod_{s=1}^{n_j} |z - z_{j,s}|^4 e^{-2\psi - 2\chi_j} - \prod_{s=1}^{n_j} |z - z_{j,s}|^4 e^{-2\psi + 2\chi_j} \right),
\end{align*}
\]

(2.26)

(2.27)

These are the master equations which govern the multiple vortex solutions of the non-Abelian Chern–Simons–Higgs model of Gudnason [27, 28]. Below we aim to establish a series of existence theorems for the solutions of these equations over the full plane and over doubly periodic cell domains.

### 3 Non-Abelian vortex equations and existence theorems

We consider the non-Abelian Chern–Simons–Higgs vortex equations (2.26)–(2.27). With

\[
u = -\psi + \sum_{i=1}^{n_i} \sum_{s=1}^{n_i} \ln |z - z_{i,s}|^2, \quad u_j = -\chi_j + \sum_{s=1}^{n_j} \ln |z - z_{j,s}|^2, \quad j = 1, \ldots, M,
\]

we see that

\[
e^u = \prod_{i=1}^{n_i} \prod_{s=1}^{n_i} |z - z_{i,s}|^2 e^{-\psi}, \quad e^{u_j} = \prod_{s=1}^{n_j} |z - z_{j,s}|^2 e^{-\chi_j}, \quad j = 1, \ldots, M.
\]

Then the equations (2.26)–(2.27) become

\[
\begin{align*}
\Delta u &= \frac{\alpha^2}{M^2} \left( \sum_{i=1}^M \left[ e^{u+u_i} + e^{u-u_i} \right] - \xi \right) \left( \sum_{j=1}^M \left[ e^{u+u_j} + e^{u-u_j} \right] \right) \\
&\quad + \frac{\alpha \beta}{M} \sum_{i=1}^M \left( e^{u+u_i} - e^{u-u_i} \right)^2 + 4\pi \sum_{i=1}^M \sum_{s=1}^{n_i} \delta_{p_{i,s}}(x), \\
\Delta u_j &= \frac{\alpha \beta}{M} \left( \sum_{i=1}^M \left[ e^{u+u_i} + e^{u-u_i} \right] - \xi \right) \left( e^{u+u_j} - e^{u-u_j} \right) \\
&\quad + \beta^2 (e^{2u+2u_j} - e^{2u-2u_j}) + 4\pi \sum_{s=1}^{n_j} \delta_{p_{j,s}}(x), \quad j = 1, \ldots, M.
\end{align*}
\]

(3.1)

(3.2)
where we set $\alpha = \frac{\pi}{K}$, $\beta = \frac{\pi}{\mu}$, $p_{i,s} = z_{i,s}, i = 1, \ldots, M$. When $M = 1$ such equations were first obtained in [27]. It will be convenient to take the rescaled parameters and translated variables

$$\alpha \frac{\xi}{2M} \mapsto \alpha, \quad \beta \frac{\xi}{2M} \mapsto \beta, \quad u \mapsto u + \ln \frac{\xi}{2M}, \quad u_j \mapsto u_j, \quad j = 1, \ldots, M.$$  

Then the equations (3.1) and (3.2) are renormalized into the form

$$\Delta u = \frac{\alpha^2}{M^2} \left( \sum_{i=1}^{M} \left[ e^{u+u_i} + e^{u-u_i} - 2 \right] \right) \left( \sum_{j=1}^{M} \left[ e^{u+u_j} + e^{u-u_j} \right] \right) \quad (3.3)$$

$$\Delta u_j = \frac{\alpha \beta}{M} \sum_{i=1}^{M} \left[ e^{u+u_i} + e^{u-u_i} - 2 \right] \left( e^{u+u_j} - e^{u-u_j} \right)$$

$$+ \beta^2 \left( e^{2u+2u_j} - e^{2u-2u_j} \right) + 4\pi \sum_{s=1}^{n_j} \delta_{p_{j,s}}(x), \quad j = 1, \ldots, M. \quad (3.4)$$

It will be interesting at this spot to compare the classical Abelian Chern–Simons–Higgs vortex equation (1.7) with the above system of non-Abelian vortex equations, (3.3) and (3.4), in the Gudnason model [27, 28], which will be our focus in the present work.

We will consider the equations (3.3)–(3.4) in two cases. In the first case we study the problem (3.3)–(3.4) over the full plane $\mathbb{R}^2$ with the topological boundary conditions

$$u \to 0, \quad u_j \to 0 \text{ as } |x| \to \infty, \quad j = 1, \ldots, M, \quad (3.5)$$

realizing the asymptotic vacuum state with completely broken symmetry [27, 28].

We have the following existence theorem.

**Theorem 3.1** For any sets of points

$$Z_i = \{p_{i,1}, \ldots, p_{i,n_i}\} \subset \mathbb{R}^2, \quad i = 1, \ldots, M, \quad (3.6)$$

and the parameters $\alpha, \beta > 0, \ M \geq 1$, the system of nonlinear elliptic equations (3.3)–(3.4) subject to the boundary condition (3.5) admits a solution over $\mathbb{R}^2$ which possesses the quantized integrals

$$\frac{\alpha^2}{M^2} \int_{\mathbb{R}^2} \left( \sum_{i=1}^{M} \left[ e^{u+u_i} + e^{u-u_i} - 2 \right] \right) \left( \sum_{j=1}^{M} \left[ e^{u+u_j} + e^{u-u_j} \right] \right) \ dx$$

$$+ \frac{\alpha \beta}{M} \sum_{i=1}^{M} \int_{\mathbb{R}^2} \left( e^{u+u_i} - e^{u-u_i} \right)^2 \ dx = -4\pi \sum_{i=1}^{M} n_i, \quad (3.7)$$

$$\frac{\alpha \beta}{M} \int_{\mathbb{R}^2} \left( \sum_{j=1}^{M} \left[ e^{u+u_j} + e^{u-u_j} - 2 \right] \right) \left( e^{u+u_i} - e^{u-u_i} \right) \ dx$$

$$+ \beta^2 \int_{\mathbb{R}^2} \left( e^{2u+2u_i} - e^{2u-2u_i} \right) \ dx = -4\pi n_i, \quad i = 1, \ldots, M. \quad (3.8)$$
Furthermore the boundary condition (3.5) is realized exponentially fast so that there hold the following asymptotic estimates near infinity:

\[ u^2 + \sum_{i=1}^{M} u_i^2 = O(e^{-m(1-\varepsilon)|x|}), \quad |\nabla u|^2 + \sum_{i=1}^{M} |\nabla u_i|^2 = O(e^{-m(1-\varepsilon)|x|}), \quad (3.9) \]

where \( m = 2\sqrt{2} \min\{\alpha, \beta\} \) and \( \varepsilon \in (0, 1) \) is an arbitrarily small parameter.

In the second case we consider the equations (3.3)–(3.4) over a doubly periodic domain \( \Omega \) with \( M = 1 \). That is, in this case we study a \( 2 \times 2 \) version of (3.3)–(3.4). For convenience we rewrite the system (3.3)–(3.4) with \( M = 1 \) as follows

\[ \Delta U = \alpha^2 (e^{U+V} + e^{U-V}) (e^{U+V} + e^{U-V} - 2) + \alpha \beta (e^{U+V} - e^{U-V})^2 + 4\pi \sum_{j=1}^{n} \delta_{p_j}(x), \quad (3.10) \]

\[ \Delta V = \alpha \beta (e^{U+V} - e^{U-V}) (e^{U+V} + e^{U-V} - 2) + \beta^2 (e^{2U+2V} - e^{2U-2V}) + 4\pi \sum_{j=1}^{n} \delta_{p_j}(x). \quad (3.11) \]

We have the following existence results.

**Theorem 3.2** Let \( \Omega \) be a doubly periodic domain in \( \mathbb{R}^2 \) and \( p_1, \ldots, p_n \in \Omega \) which need not to be distinct with repeated \( p \)'s counting for multiplicities. Assume that \( \beta > \alpha > 0 \).

1. If the equations (3.10) and (3.11) have a solution, then there holds the condition

\[ 8\pi n \leq \alpha \beta |\Omega|. \quad (3.12) \]

2. Every solution \((U, V)\) of (3.10) and (3.11) satisfies

\[ e^U < 1, \quad e^{U+V} < 1, \quad e^{U-V} < 1. \quad (3.13) \]

3. For any given constant \( \sigma > 1 \), assume

\[ \frac{\beta}{\alpha} < \sigma. \quad (3.14) \]

Then, there exist a positive constant \( M_\sigma \) such that when \( \alpha > M_\sigma \) the equations (3.10) and (3.11) admit at least two distinct solutions over \( \Omega \), one of which satisfies the behavior

\[ e^{U+V} \to 1, \quad e^{U-V} \to 1, \quad \text{as} \quad \alpha \to +\infty \quad (3.15) \]

pointwise a.e. in \( \Omega \). Furthermore, any solution \((U, V)\) of (3.10) and (3.11) possesses the quantized integrals

\[ \alpha^2 \int_{\Omega} (e^{U+V} + e^{U-V}) (e^{U+V} + e^{U-V} - 2) \, dx + \alpha \beta \int_{\Omega} (e^{U+V} - e^{U-V})^2 \, dx = -4\pi n, \quad (3.16) \]

\[ \alpha \beta \int_{\Omega} (e^{U+V} - e^{U-V}) (e^{U+V} + e^{U-V} - 2) \, dx + \beta^2 \int_{\Omega} (e^{2U+2V} - e^{2U-2V}) \, dx = -4\pi n. \quad (3.17) \]

In Section 4, we establish Theorem 3.1 using a direct minimization method which extends the techniques in [40, 63, 73, 74]. In Section 5, we prove Theorem 3.2 by utilizing and extending an inequality-constrained variational method originally developed in [11] and further developed in [49, 50, 63], and subsequently in [48] for a context that relates more to the situation considered here. In this respect, see also [30].
4 Proof of Theorem 3.1

In this section we prove the existence of topological solutions for the equations (3.3)–(3.4).

Choosing the background functions

\[ u_i^0(x) = - \sum_{s=1}^{n_i} \ln \left(1 + \lambda |x - p_{i,s}|^{-2}\right), \quad \lambda > 0, \quad i = 1, \ldots, M, \tag{4.1} \]

which satisfy

\[ \Delta u_i^0 = -h_i + 4\pi \sum_{s=1}^{n_i} \delta_{p_{i,s}}, \quad h_i(x) = 4\lambda \sum_{s=1}^{n_i} \frac{1}{(\lambda + |x - p_{i,s}|^2)^2}, \tag{4.2} \]

we see that the new variables

\[ u = \sum_{i=1}^{M} u_i^0 + f, \quad u_j = u_j^0 + f, \quad j = 1, \ldots, M, \tag{4.3} \]

allow us to recast the equations (3.3)–(3.4) into

\[ \Delta f = \frac{\alpha^2}{M^2} \left( \sum_{j=1}^{M} \left[ \frac{\sum_{k=1}^{M} u_k^0 + u_j^0 + f + f_j}{e^{k-1}} - \frac{\sum_{k=1}^{M} u_k^0 - u_j^0 + f - f_j}{e^{k-1}} \right] - 2 \right) \left( \sum_{j=1}^{M} \left[ \frac{\sum_{k=1}^{M} u_k^0 + u_j^0 + f + f_j}{e^{k-1}} + \frac{\sum_{k=1}^{M} u_k^0 - u_j^0 + f - f_j}{e^{k-1}} \right] \right) \]

\[ + \frac{\alpha \beta}{M} \sum_{i=1}^{M} \left( \frac{\sum_{k=1}^{M} u_k^0 + u_i^0 + f + f_i}{e^{k-1}} - \frac{\sum_{k=1}^{M} u_k^0 - u_i^0 + f - f_i}{e^{k-1}} \right)^2 + h_i, \tag{4.4} \]

\[ \Delta f_i = \frac{\alpha \beta}{M} \left( \sum_{j=1}^{M} \left[ \frac{\sum_{k=1}^{M} u_k^0 + u_j^0 + f + f_j}{e^{k-1}} + \frac{\sum_{k=1}^{M} u_k^0 - u_j^0 + f - f_j}{e^{k-1}} \right] - 2 \right) \left( \sum_{j=1}^{M} \left[ \frac{\sum_{k=1}^{M} u_k^0 + u_j^0 + f + f_j}{e^{k-1}} - \frac{\sum_{k=1}^{M} u_k^0 - u_j^0 + f - f_j}{e^{k-1}} \right] \right) \]

\[ + \beta^2 \left( \frac{2 \sum_{k=1}^{M} u_k^0 + 2u_i^0 + 2f + 2f_i}{e^{k-1}} - \frac{2 \sum_{k=1}^{M} u_k^0 - 2u_i^0 + 2f - 2f_i}{e^{k-1}} \right) + h_i, \quad i = 1, \ldots, M. \tag{4.5} \]

The topological boundary condition (3.5) becomes

\[ f \to 0, \quad f_i \to 0 \quad \text{as} \quad |x| \to \infty, \quad i = 1, \ldots, M. \tag{4.6} \]

It can be checked that the equations (4.1) and (4.5) are the Euler–Lagrange equations of the functional

\[ I(f, f_1, \ldots, f_M) \]

\[ = \int_{\mathbb{R}^2} dx \left\{ \frac{M}{\alpha} |\nabla f|^2 + \frac{1}{\beta} \sum_{i=1}^{M} |\nabla f_i|^2 + \sum_{i=1}^{M} \left( \frac{\alpha}{M} \left[ \frac{\sum_{k=1}^{M} u_k^0 + u_i^0 + f + f_i}{e^{k-1}} + \frac{\sum_{k=1}^{M} u_k^0 - u_i^0 + f - f_i}{e^{k-1}} \right] - 2 \right) \right\} \]

\[ + \beta \left[ \frac{\sum_{k=1}^{M} u_k^0 + u_i^0 + f + f_i}{e^{k-1}} - \frac{\sum_{k=1}^{M} u_k^0 - u_i^0 + f - f_i}{e^{k-1}} \right]^2 \right\} + \frac{2M}{\alpha} \sum_{i=1}^{M} f_i h_i + \frac{2}{\beta} \sum_{i=1}^{M} f_i h_i \right\}. \tag{4.7} \]

We consider the functional $I$ over $W^{1,2}(\mathbb{R}^2)$. Here and in what follows we use $W^{1,2}(\mathbb{R}^2)$ to denote the usual Sobolev space of scalar-valued or vector-valued functions. It is not difficult to see that the functional $I$ is continuous, differentiable and lower semi-continuous on $W^{1,2}(\mathbb{R}^2)$. The
important thing is that we can show that the functional $I$ is coercive and bounded from below over $W^{1,2}(\mathbb{R}^2)$, which will be carried out later. Then we can conclude that the functional $I$ admits a critical point $(f, f_1, \ldots, f_M) \in W^{1,2}(\mathbb{R}^2)$, which is a weak solution to the equations (4.4)–(4.5). By the following inequality
\[
\|e^w - 1\|_2^2 \leq C \exp \left( C \|w\|_W^2 \right), \quad \forall w \in W^{1,2}(\mathbb{R}^2),
\]
we see that the right hand side of the equations (4.4)–(4.5) belongs to $L^2(\mathbb{R}^2)$. Then using elliptic $L^2$-estimates and a bootstrap argument, we find that the solution $(f, f_1, \ldots, f_M)$ is smooth. In particular, $(f, f_1, \ldots, f_M)$ lies in $W^{2,2}(\mathbb{R}^2)$ which ensures that $(f, f_1, \ldots, f_M)$ satisfies the boundary condition (4.6). Then, in view of (4.3), we see that the problem consisting of the equations (3.3) and (3.4) admits a solution $(u, u_1, \ldots, u_M)$ satisfying the boundary condition (3.5).

We now establish the exponential decay estimates for the solution. We first note that
\[
e^{u+u_i} + e^{u-u_i} - 2 = (e^{u+u_i} - 1) + (e^{u-u_i} - 1) = e^{\xi_i}(u + u_i) + e^{\xi''}(u - u_i)
\]
\[
e^{u+u_i} - e^{u-u_i} = 2e^{\xi_i}u_i.
\]
where $\xi_i$, $\xi''_i$, and $\xi_i$ are between 0 and $u + u_i$, 0 and $u - u_i$, and $u + u_i$ and $u - u_i$, respectively, $i = 1, \ldots, M$.

By virtue of (4.8) and (4.9), we see that the equations (3.3)–(3.4) may be rewritten as
\[
\Delta u = \frac{\alpha^2}{M^2} \left( \sum_{j=1}^M \left[ e^{u+u_j} + e^{u-u_j} \right] \right) \left( \sum_{i=1}^M \left[ (e^{\xi_i} + e^{\xi''_i})u + (e^{\xi_i'} - e^{\xi''_i})u_i \right] \right)
\]
\[
+ \frac{4\alpha\beta}{M} \sum_{i=1}^M e^{2\xi_i}u_i^2,
\]
\[
\Delta u_i = \frac{2\alpha\beta}{M} e^{\xi_i} \left( \sum_{j=1}^M \left[ (e^{\xi_j} + e^{\xi''_j})u + (e^{\xi_j'} - e^{\xi''_j})u_j \right] \right) u_i
\]
\[
+ 2\beta^2 \left( e^{u+u_i} + e^{u-u_i} \right) e^{\xi_i}u_i, \quad i = 1, \ldots, M,
\]
where $|x| > R$ and $R > 0$ is taken to be sufficiently large so that $R > |p_{i,s}|$ for $s = 1, \ldots, n_i$ and $i = 1, \ldots, M$.

To proceed further, we set
\[
U = u^2 + \sum_{i=1}^M u_i^2.
\]
Then we can compute for $|x| > R$ the result
\[
\Delta U \geq 2u \Delta u + 2 \sum_{i=1}^{M} u_i \Delta u_i \\
\geq \frac{2\alpha^2}{M^2} \left( \sum_{j=1}^{M} [e^{u+u_j} + e^{u-u_j}] \right) \left( \sum_{i=1}^{M} [e^{\xi_i'} + e^{\xi_i^\prime}] \right) u^2 + 4\beta^2 \sum_{i=1}^{M} (e^{u+u_i} + e^{u-u_i}) e^{\xi_i_i} u_i^2 \\
- \frac{2\alpha^2}{M^2} \left( \sum_{j=1}^{M} [e^{u+u_j} + e^{u-u_j}] \right) \left( \sum_{i=1}^{M} |e^{\xi_i'} - e^{\xi_i''}| |u_i| \right) |u| \\
- \frac{8\alpha\beta}{M} \sum_{i=1}^{M} e^{2\xi_i} u_i^2 |u| - \frac{4\alpha\beta}{M} \sum_{i=1}^{n} e^{\xi_i} \left( \sum_{j=1}^{M} (e^{\xi_j'} + e^{\xi_j''}) |u_j| + |e^{\xi_j'} - e^{\xi_j''}||u_j|| \right) u_i^2. \tag{4.13}
\]

Applying (3.5) and the Schwartz inequality in (4.13), we see that for any arbitrarily small $\varepsilon \in (0,1)$ there is $R_\varepsilon > R$ such that $U$ satisfies the elliptic inequality
\[
\Delta U \geq 8(\min \{\alpha, \beta\})^2 \left( 1 - \frac{\varepsilon}{2} \right) U, \quad |x| \geq R_\varepsilon. \tag{4.14}
\]

Applying a comparison function argument to (4.14) and using the boundary property $U = 0$ at infinity, we can find a sufficient large constant $C(\varepsilon) > 0$ such that
\[
U(x) \leq C(\varepsilon)e^{-2\sqrt{T}\min \{\alpha, \beta\}(1-\varepsilon)|x|}, \quad |x| \geq R_\varepsilon. \tag{4.15}
\]

Next, using $\partial$ to denote one of the two partial derivatives, $\partial_1$ and $\partial_2$, we obtain from (3.3) and (3.4) the results
\[
\Delta(\partial u) = \frac{\alpha^2}{M^2} \left( \sum_{j=1}^{M} [e^{u+u_j} + e^{u-u_j}] \right)^2 \partial u + \frac{\alpha^2}{M^2} \left( \sum_{i=1}^{M} [e^{u+u_i} + e^{u-u_i}] \right) \left( \sum_{i=1}^{M} [e^{u+u_i} - e^{u-u_i}] \right) \partial u_i \\
+ \frac{\alpha^2}{M^2} \left( \sum_{j=1}^{M} [e^{u+u_i} + e^{u-u_i} - 2] \right) \left( \sum_{j=1}^{M} [e^{u+u_j} + e^{u-u_j}] \partial u_j + \sum_{j=1}^{M} [e^{u+u_j} - e^{u-u_j}] \partial u_j \right) \\
+ \frac{2\alpha\beta}{M} \sum_{i=1}^{M} (e^{u+u_i} - e^{u-u_i}) \left( [e^{u+u_i} - e^{u-u_i}] \partial u + [e^{u+u_i} + e^{u-u_i}] \partial u_i \right), \tag{4.16}
\]
\[
\Delta(\partial u_i) = 2\beta^2 (e^{2u+2u_i} + e^{2u-2u_i}) (\partial u_i) + 2\beta^2 (e^{2u+2u_i} - e^{2u-2u_i}) (\partial u) \\
+ \frac{\alpha\beta}{M} \sum_{j=1}^{M} [e^{u+u_j} + e^{u-u_j} - 2] \left( [e^{u+u_i} - e^{u-u_i}] \partial u + [e^{u+u_i} + e^{u-u_i}] \partial u_i \right) \\
+ \frac{\alpha\beta}{M} (e^{u+u_i} - e^{u-u_i}) \left( \sum_{j=1}^{M} [e^{u+u_j} + e^{u-u_j}] \partial u + \sum_{j=1}^{M} [e^{u+u_j} - e^{u-u_j}] \partial u_j \right). \tag{4.17}
\]

Applying the $L^2$-estimate and the fact that $u, u_1, \ldots, u_M \in W^{2,2}$ outside $B_R = \{ x \in \mathbb{R}^2 \mid |x| > R \}$, we see in view of the above equations that $\partial u, \partial u_1, \ldots, \partial u_M \in W^{2,2}$ outside $B_R$ as well. Therefore
\[
\partial u, \partial u_1, \ldots, \partial u_M \to 0 \quad \text{as} \quad |x| \to \infty. \tag{4.18}
\]
As before, we set

\[ V = (\partial u)^2 + \sum_{i=1}^{M} (\partial u_i)^2. \quad (4.19) \]

Similar to the case with the function \( U \) defined in (4.12), we may apply the Schwartz inequality and use the equations (4.16) and (4.17) to obtain the elliptic inequality

\[ \Delta V \geq 8(\min\{\alpha, \beta\})^2 \left(1 - \frac{\varepsilon}{2}\right) V, \quad |x| \geq R. \quad (4.20) \]

Thus \( V \) enjoys the same exponential decay estimate as \( U \) as stated in (4.15).

Now we only need to prove the coerciveness and a bound from below for the functional \( I(f, f_1, \ldots, f_M) \) over \( W^{1,2}(\mathbb{R}^2) \).

Using the elementary inequality

\[ \tilde{\alpha}(a + b)^2 + \tilde{\beta}(a - b)^2 \geq 2\min\{\tilde{\alpha}, \tilde{\beta}\}(a^2 + b^2), \quad \forall \tilde{\alpha}, \tilde{\beta} > 0, \forall a, b \in \mathbb{R}, \]

with

\[ a = \sum_{k=1}^{M} u_k^0 + u_0^0 + f + f_i - 1, \quad b = \sum_{k=1}^{M} u_k^0 - u_0^0 + f - f_i - 1, \]

we see that the functional \( I(f, f_1, \ldots, f_M) \) over \( W^{1,2}(\mathbb{R}^2) \) satisfies

\[ I(f, f_1, \ldots, f_M) \]

\[ \geq \int_{\mathbb{R}^2} dx \left\{ \frac{M}{\alpha} |\nabla f|^2 + \frac{1}{\beta} \sum_{i=1}^{M} |\nabla f_i|^2 + 2 \min\left\{ \frac{\alpha}{M}, \beta \right\} \sum_{i=1}^{M} \left( \sum_{k=1}^{M} u_k^0 + u_0^0 + f + f_i - 1 \right)^2 \right. \]

\[ + 2 \min\left\{ \frac{\alpha}{M}, \beta \right\} \sum_{i=1}^{M} \left( \sum_{k=1}^{M} u_k^0 - u_0^0 + f - f_i - 1 \right)^2 \left[ 2M \sum_{i=1}^{M} f_i h_i + 2 \sum_{i=1}^{M} f_i h_i \right]. \quad (4.21) \]

To proceed we need the following lemma in [69].

**Lemma 4.1** The function \( h_i \) belongs to \( L^2(\mathbb{R}^2) \) with

\[ \|h_i\|_2 \leq \frac{C}{\sqrt{\lambda}}, \quad (4.22) \]

for some positive constant \( C \) independent of \( \lambda \) and \( e^{u_i^0} - 1 \in L^p(\mathbb{R}^2) \) for any \( p \geq 2, i = 1, \ldots, M \).

Here and in what follows we use \( C \) to denote a positive constant which may take different values at different places.

Using the Hölder inequality and (4.22), we have

\[ \int_{\mathbb{R}^2} \left( \frac{2M}{\alpha} \sum_{i=1}^{M} f_i h_i + \frac{2M}{\beta} \sum_{i=1}^{M} f_i h_i \right) dx \geq - \frac{2M}{\alpha} \sum_{i=1}^{M} \|f_i\|_2 \|h_i\| - \frac{2M}{\beta} \sum_{i=1}^{M} \|f_i\|_2 \|h_i\| \]

\[ \geq - \frac{C}{\sqrt{\lambda}} \left( \|f\|_2 + \sum_{i=1}^{M} \|f_i\|_2 \right). \quad (4.23) \]

In what follows we need to control the \( L^2 \)-norm of \( f \) and \( f_j \) (\( j = 1, \ldots, M \)) by the positive terms in (4.21). Now we deal with the third term on the right-hand side of (4.21). Since

\[ e^{u_i^0} - 1 \in L^2(\mathbb{R}^2), \quad i = 1, \ldots, M, \]
it is easy to check that
\[ \sum_{k=1}^{M} u_k^0 + u_i^0 - 1 \in L^2(\mathbb{R}^2), \quad \sum_{k=1}^{M} u_k^0 - u_i^0 - 1 \in L^2(\mathbb{R}^2), \quad i = 1, \ldots, M. \] (4.24)

To proceed further, we need the inequality
\[ |e^t - 1| \geq \frac{|t|}{1 + |t|}, \quad \forall t \in \mathbb{R}, \] (4.25)
which follows directly from the elementary inequalities \( e^t - 1 \geq t, \forall t \geq 0, \) and \( 1 - e^{-t} \geq \frac{t}{1 + t}, \forall t \geq 0. \)

With (4.24) and (4.25), we have
\[
\int_{\mathbb{R}^2} \left( \sum_{k=1}^{M} u_k^0 + u_i^0 + f + f_i - 1 \right)^2 dx = \int_{\mathbb{R}^2} \left( \sum_{k=1}^{M} u_k^0 + u_i^0 \left[ e^{f_i} - 1 \right] + \sum_{k=1}^{M} u_k^0 + u_i^0 - 1 \right)^2 dx \\
\geq \frac{1}{2} \int_{\mathbb{R}^2} 2 \sum_{k=1}^{M} u_k^0 + 2u_i^0 \left( e^{f_i} - 1 \right)^2 - \int_{\mathbb{R}^2} \left( \sum_{k=1}^{M} u_k^0 + u_i^0 - 1 \right)^2 dx \\
\geq \frac{1}{2} \int_{\mathbb{R}^2} e^2 \sum_{k=1}^{M} u_k^0 + 2u_i^0 \frac{|f + f_i|^2}{2 + 2|f_i|} dx - C, \quad i = 1, \ldots, M. \tag{4.26}
\]

By the definition of \( u_i^0, \) we see that \( e^2 \sum_{k=1}^{M} u_k^0 + 2u_i^0 \) satisfies \( 0 \leq e^2 \sum_{k=1}^{M} u_k^0 + 2u_i^0 < 1, \) vanishes at the vortex point \( p_i, (s = 1, \ldots, n_i, i = 1, \ldots, M), \) and approaches 1 at infinity. As in [40, 63], we decompose \( \mathbb{R}^2 \) as follows,
\[ \mathbb{R}^2 = \Omega_1^i \cup \Omega_2^i, \quad i = 1, \ldots, M, \tag{4.27} \]
where
\[ \Omega_1^i = \left\{ x \in \mathbb{R}^2 \bigg| e^2 \sum_{k=1}^{M} u_k^0 + 2u_i^0 \leq \frac{1}{2} \right\}, \quad \Omega_2^i = \left\{ x \in \mathbb{R}^2 \bigg| e^2 \sum_{k=1}^{M} u_k^0 + 2u_i^0 \geq \frac{1}{2} \right\}, \quad i = 1, \ldots, M. \]

To deal with the right-hand side of (4.26), we need the inverse Hölder inequality (cf. [69]):

**Lemma 4.2** For any measurable functions \( g_1, g_2 \) on the domain \( \Omega, \) there holds the inequality
\[ \int_{\Omega} |g_1 g_2| dx \geq \left( \int_{\Omega} |g_1|^q dx \right)^{\frac{1}{q}} \left( \int_{\Omega} |g_2|^{q'} dx \right)^{\frac{1}{q'}}, \tag{4.28} \]
where \( q, q' \in \mathbb{R}, 0 < q < 1, q' < 0 \) and \( \frac{1}{q} + \frac{1}{q'} = 1. \)

On \( \Omega_1^i, \) we have \( 0 \leq e^2 \sum_{k=1}^{M} u_k^0 + 2u_i^0 \leq \frac{1}{2} \) and \( e^2 \sum_{k=1}^{M} u_k^0 + 2u_i^0 \) tends to 0 at most at order \( 4 \sum_{k=1}^{M} n_k + 4n_i \) near the vortex points. Then, by taking \( q_i' \) satisfying
\[ -\frac{1}{2 \sum_{k=1}^{M} n_k + 2n_i} < q_i' < 0, \]
we see that the integrals
\[ \int_{\Omega_i} e^{2q_i \sum_{k=1}^{M} u_k^0 + 2q_i u_i^0} \, dx, \quad i = 1, \ldots, M \]
exist.

Using the inverse Hölder inequality (4.28), we can get
\[
\int_{\Omega_i} e^{2 \sum_{k=1}^{M} u_k^0 + 2q_i u_i^0} \frac{|f + f_i|^2}{(1 + |f + f_i|)^2} \, dx \geq \left( \int_{\Omega_i} \frac{|f + f_i|^{2q_i}}{(1 + |f + f_i|)^{2q_i}} \, dx \right)^{\frac{1}{q_i}} \left( \int_{\Omega_i} e^{2 \sum_{k=1}^{M} u_k^0 + 2q_i u_i^0} \, dx \right)^{\frac{1}{q_i}} \geq C \left( \int_{\Omega_i} \frac{|f + f_i|^{2q_i}}{(1 + |f + f_i|)^{2q_i}} \, dx \right)^{\frac{1}{q_i}}, \quad i = 1, \ldots, M, \tag{4.29}
\]
where
\[ 0 < q_i < \frac{1}{2 \sum_{k=1}^{M} n_k + 2n_i + 1}, \quad i = 1, \ldots, M. \]

Noting
\[ 0 \leq \frac{|f + f_i|}{1 + |f + f_i|} < 1, \quad i = 1, \ldots, M \]
and applying the Young inequality, we obtain
\[
\left( \int_{\Omega_i} \frac{|f + f_i|^{2q_i}}{(1 + |f + f_i|)^{2q_i}} \, dx \right)^{\frac{1}{q_i}} \geq \left( \int_{\Omega_i} \frac{|f + f_i|^2}{(1 + |f + f_i|)^2} \, dx \right)^{\frac{1}{q_i}} \geq C \int_{\Omega_i} \frac{|f + f_i|^2}{(1 + |f + f_i|)^2} \, dx - C, \quad i = 1, \ldots, M. \tag{4.30}
\]
Combining (4.29) and (4.30), we have
\[
\int_{\Omega_i} e^{2 \sum_{k=1}^{M} u_k^0 + 2q_i u_i^0} \frac{|f + f_i|^2}{(1 + |f + f_i|)^2} \, dx \geq C \int_{\Omega_i} \frac{|f + f_i|^2}{(1 + |f + f_i|)^2} \, dx - C, \quad i = 1, \ldots, M. \tag{4.31}
\]
On the other hand, over \( \Omega_2 \), it is easy to get
\[
\int_{\Omega_2} e^{2 \sum_{k=1}^{M} u_k^0 + 2q_i u_i^0} \frac{|f + f_i|^2}{(1 + |f + f_i|)^2} \, dx \geq \frac{1}{2} \int_{\Omega_2} \frac{|f + f_i|^2}{(1 + |f + f_i|)^2} \, dx, \quad i = 1, \ldots, M. \tag{4.32}
\]
Hence, from (4.26), (4.31), and (4.32), we infer that
\[
\int_{\mathbb{R}^2} \left( e^{\sum_{k=1}^{M} u_k^0 + f + f_i} - 1 \right)^2 \geq C \int_{\mathbb{R}^2} \frac{|f + f_i|^2}{(1 + |f + f_i|)^2} \, dx - C, \quad i = 1, \ldots, M. \tag{4.33}
\]
Now repeating the procedure in getting (4.33), we have
\[
\int_{\mathbb{R}^2} \left( e^{\sum_{k=1}^{M} u_k^0 - u_i^0 + f - f_i} - 1 \right)^2 \geq C \int_{\mathbb{R}^2} \frac{|f - f_i|^2}{(1 + |f - f_i|)^2} \, dx - C, \quad i = 1, \ldots, M. \tag{4.34}
\]
To proceed further, we invoke the following standard interpolation inequality over $\mathbb{R}^2$:

$$\int_{\mathbb{R}^2} w^4 dx \leq 2 \int_{\mathbb{R}^2} w^2 dx \int_{\mathbb{R}^2} |\nabla w|^2 dx, \quad \forall w \in W^{1,2}(\mathbb{R}^2).$$  \hfill (4.35)

Using (4.35), we obtain

$$\left( \int_{\mathbb{R}^2} |f + f_i|^2 dx \right)^2 = \left( \int_{\mathbb{R}^2} \frac{|f + f_i|}{1 + |f + f_i|} \left[ 1 + |f + f_i| |f + f_i| \right] dx \right)^2 \leq \int_{\mathbb{R}^2} \frac{|f + f_i|^2}{(1 + |f + f_i|)^2} dx \int_{\mathbb{R}^2} \left( |f + f_i| + |f + f_i|^2 \right)^2 dx \leq 4 \int_{\mathbb{R}^2} \frac{|f + f_i|^2}{(1 + |f + f_i|)^2} dx \int_{\mathbb{R}^2} |f + f_i|^2 dx \left( \int_{\mathbb{R}^2} |\nabla (f + f_i)|^2 + 1 \right) \leq \frac{1}{2} \left( \int_{\mathbb{R}^2} |f + f_i|^2 dx \right)^2 + C \left( \left[ \int_{\mathbb{R}^2} \frac{|f + f_i|^2}{(1 + |f + f_i|)^2} dx \right]^4 + \int_{\mathbb{R}^2} |\nabla |f + f_i|^2| dx + 1 \right), \hfill (4.36)$$

which implies

$$\|f + f_i\| \leq C \left( \int_{\mathbb{R}^2} \frac{|f + f_i|^2}{(1 + |f + f_i|)^2} dx + \int_{\mathbb{R}^2} |\nabla (f + f_i)|^2 dx + 1 \right) \leq C \left( \int_{\mathbb{R}^2} \frac{|f + f_i|^2}{(1 + |f + f_i|)^2} dx + \int_{\mathbb{R}^2} \left[ |\nabla f|^2 + |\nabla f_i|^2 \right] dx + 1 \right), \quad i = 1, \ldots, M. \hfill (4.37)$$

Similarly, we have

$$\|f - f_i\| \leq C \left( \int_{\mathbb{R}^2} \frac{|f - f_i|^2}{(1 + |f - f_i|)^2} dx + \int_{\mathbb{R}^2} \left[ |\nabla f|^2 + |\nabla f_i|^2 \right] dx + 1 \right), \quad i = 1, \ldots, M. \hfill (4.38)$$

Then, in view of (4.37), (4.38), and the following simple inequality

$$\|f\| + \|f_i\| \leq 2(\|f + f_i\| + \|f - f_i\|), \quad i = 1, \ldots, M,$$

we see that

$$\|f\| + \sum_{i=1}^{M} \|f_i\| \leq C \left( \int_{\mathbb{R}^2} |\nabla f|^2 dx + \sum_{i=1}^{M} \int_{\mathbb{R}^2} \left[ |\nabla f_i|^2 + \frac{|f + f_i|^2}{(1 + |f + f_i|)^2} + \frac{|f - f_i|^2}{(1 + |f - f_i|)^2} \right] dx \right), \hfill (4.39)$$

From (4.21), (4.23), (4.33) and (4.34), we conclude that

$$I(f, f_1, \ldots, f_M) \geq C \left( \int_{\mathbb{R}^2} |\nabla f|^2 dx + \sum_{i=1}^{M} \int_{\mathbb{R}^2} \left[ |\nabla f_i|^2 + \frac{|f + f_i|^2}{(1 + |f + f_i|)^2} + \frac{|f - f_i|^2}{(1 + |f - f_i|)^2} \right] dx \right) - \frac{C}{\sqrt{\lambda}} \left( \|f\| + \sum_{i=1}^{M} \|f_i\| \right) - C. \hfill (4.40)$$
At this point, combining (4.39) and (4.40) and taking $\lambda$ sufficiently large, we can get

$$I(f, f_1, \ldots, f_M) \geq C \left( \int_{\mathbb{R}^2} |\nabla f|^2 \, dx + \sum_{i=1}^{M} \int_{\mathbb{R}^2} \left[ |\nabla f_i|^2 + \frac{|f + f_i|^2}{(1 + |f + f_i|)^2} + \frac{|f - f_i|^2}{(1 + |f - f_i|)^2} \right] \, dx \right) - C. \quad (4.41)$$

Then applying (4.39) in the right hand side of (4.41), we have

$$I(f, f_1, \ldots, f_M) \geq C \left( \|f\|_{W^{1,2}(\mathbb{R}^2)} + \sum_{i=1}^{M} \|f_i\|_{W^{1,2}(\mathbb{R}^2)} \right) - C, \quad (4.42)$$

which says that the functional $I(f, f_1, \ldots, f_M)$ is coercive and bounded from below over $W^{1,2}(\mathbb{R}^2)$. Therefore the existence of a critical point as a global minimizer of $I$ in $W^{1,2}(\mathbb{R}^2)$ follows immediately.

In order to establish the results regarding the quantized integrals (3.7) and (3.8), we note that the background functions $u_0^i (i = 1, \ldots, M)$ defined in (4.1) obey the decay estimates

$$|\nabla u_0^i(x)| = O(|x|^{-3}) \quad \text{as} \quad |x| \to \infty, \quad i = 1, \ldots, M. \quad (4.43)$$

On the other hand, since the solution $(u, u_1, \ldots, u_M)$ of (3.3)–(3.4) obtained decays at infinity according to (3.9), we see that $(f, f_1, \ldots, f_M)$ set forth in (4.3) satisfies

$$|\nabla f(x)| + \sum_{i=1}^{M} |\nabla f_i(x)| = O(|x|^{-3}) \quad \text{as} \quad |x| \to \infty. \quad (4.44)$$

Using (4.44) and the divergence theorem, we arrive at

$$\int_{\mathbb{R}^2} \Delta f \, dx = 0, \quad \int_{\mathbb{R}^2} \Delta f_i \, dx = 0, \quad i = 1, \ldots, M. \quad (4.45)$$

Moreover, integrating directly, we have

$$\int_{\mathbb{R}^2} h_i \, dx = 4\pi n_i, \quad i = 1, \ldots, M. \quad (4.46)$$

Finally, integrating (4.4) and (4.5) over $\mathbb{R}^2$ and applying (4.45) and (4.46), we obtain the quantized integrals (3.7) and (3.8) stated in the theorem.

The proof of Theorem 3.1 is now complete.

5 Proof of Theorem 3.2

In this section we establish the existence of solutions to (3.10)–(3.11) over a doubly periodic domain. We will make a variational formulation of the problem. Then we can carry out a constrained minimization procedure to find the critical points for the associated functional. The key step is to find some inequality-type constraints, from which we can define a suitable admissible set. This procedure was initiated in [11] and refined in [48–50] and [30].

We first give a priori estimates of the solutions to (3.10)–(3.11).

Proposition 5.1 Let $(U, V)$ be a solution of (3.10)–(3.11). Then $U < 0, U + V < 0, U - V < 0$ throughout $\Omega$. 

17
Proof. Let \((U, V)\) be a solution of (3.10) and (3.11). Introduce a transformation \(f = U + V, g = U - V\). From (3.10) and (3.11), we conclude that \(f\) and \(g\) satisfy the equations

\[
\Delta f = (\alpha + \beta)^2 e^f (e^f - 1) + (\alpha - \beta)^2 e^g (e^g - 1) - (\beta^2 - \alpha^2) (e^f + e^g) (e^g - 1) - (\beta^2 - \alpha^2) (e^f + e^g) (e^f - 1), \tag{5.1}
\]

\[
\Delta g = (\alpha + \beta)^2 e^g (e^g - 1) + (\alpha - \beta)^2 e^f (e^f - 1) - (\beta^2 - \alpha^2) (e^f + e^g) (e^f - 1). \tag{5.2}
\]

We first show that \(U < 0\) in \(\Omega\). From (3.10), we see that

\[
\Delta U \geq \alpha^2 e^U (e^V + e^{-V} - 2e^{-U}) e^U (e^V - e^{-V}) + 4\pi \sum_{j=1}^{n} \delta_{p_j} \cdot \tag{5.3}
\]

\[
\geq 2\alpha^2 e^U (e^V + e^{-V}) (e^U - 1) + 4\pi \sum_{j=1}^{n} \delta_{p_j}. \tag{5.4}
\]

Then, by maximum principle, we have \(U < 0\) throughout \(\Omega\).

To prove \(U + V < 0\), we argue by contradiction. Assume that there exists a point \(\tilde{x} \in \Omega\) such that

\[
f(\tilde{x}) = \max_{x \in \Omega} f(x) \geq 0.
\]

From the equation (5.1), we have \(g(\tilde{x}) \geq 0\). Then we obtain \(U(\tilde{x}) = \frac{1}{2} (f(\tilde{x}) + g(\tilde{x})) \geq 0\), which contradicts the conclusion \(U < 0\) in \(\Omega\). Therefore, we have \(f < 0\) in \(\Omega\).

Similarly, if there is a point \(\tilde{x} \in \Omega\) such that

\[
g(\tilde{x}) = \max_{x \in \Omega} g(x) \geq 0,
\]

then by the equation (5.2), we see that \(f(\tilde{x}) \geq 0\), which again leads to a contradiction. Hence the conclusion follows.

By Proposition 5.1, the second part of Theorem 3.2 follows.

Let \(u_0\) be the unique solution of the following problem (see [4])

\[
\Delta u_0 = -\frac{8\pi n}{|\Omega|} + 8\pi \sum_{j=1}^{n} \delta_{p_j} \text{ on } \Omega; \quad \int_{\Omega} u_0 dx = 0.
\]

For convenience, we introduce the following new variables:

\[
U = \frac{u_0}{2} + \frac{u + v}{2}, \quad V = \frac{u_0}{2} + \frac{u - v}{2}, \tag{5.3}
\]

which reduce the equations (3.10)–(3.11) into the form

\[
\Delta \frac{u + v}{2} = \alpha^2 (e^{u_0 + u} + e^v) (e^{u_0 + u} + e^v - 2) + \alpha \beta (e^{u_0 + u} - e^v)^2 + \frac{4\pi n}{|\Omega|}, \tag{5.4}
\]

\[
\Delta \frac{u - v}{2} = \alpha \beta (e^{u_0 + u} - e^v) (e^{u_0 + u} + e^v - 2) + \beta^2 (e^{2u_0 + 2u} - e^{2v}) + \frac{4\pi n}{|\Omega|}. \tag{5.5}
\]
To make a variational reformulation of the problem, we rewrite (5.4) and (5.5) equivalently as

$$\frac{1}{2} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) \Delta u + \frac{1}{2} \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) \Delta v = 2e^{u_o + u} \left( [\alpha + \beta] [e^{u_o + u} - 1] + [\alpha - \beta] [e^v - 1] \right) + \frac{4\pi n}{|\Omega|} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right),$$

(5.6)

$$\frac{1}{2} \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) \Delta u + \frac{1}{2} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) \Delta v = 2e^v \left( [\alpha - \beta] [e^{u_o + u} - 1] + [\alpha + \beta] [e^v - 1] \right) + \frac{4\pi n}{|\Omega|} \left( \frac{1}{\alpha} - \frac{1}{\beta} \right).$$

(5.7)

Therefore, in the sequel we only need to solve (5.6) and (5.7).

We will work on the space $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$, where $W^{1,2}(\Omega)$ denotes the set of $\Omega$-periodic $L^2$-functions whose derivatives are also in $L^2(\Omega)$. We denote the usual norm on $W^{1,2}(\Omega)$ by $\| \cdot \|$ as given by

$$\| w \|^2 = \| w \|_2^2 + \| \nabla w \|_2^2 = \int_\Omega w^2 dx + \int_\Omega |\nabla w|^2 dx.$$

It is easy to see that the solutions of (5.6) and (5.7) are critical points of the functional

$$I_{\alpha\beta}(u, v) = \frac{1}{4} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) \left( \| \nabla u \|^2 + \| \nabla v \|^2 \right) + \frac{1}{2} \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) \int_\Omega \nabla u \cdot \nabla v dx + \alpha \int_\Omega (e^{u_o + u} + e^v - 2)^2 dx + \beta \int_\Omega (e^{u_o + u} - e^v)^2 dx + \frac{4\pi n}{|\Omega|} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) \int_\Omega v dx.$$

(5.8)

In the following subsections we will apply a constrained minimization approach to find a first critical point and the mountain pass theorem to find a second critical point of the above functional, respectively.

### 5.1 Constrained minimization

Let $(u, v)$ be a solution of (5.6) and (5.7), which is also a solution of (5.4) and (5.5). Then integrating these equations over $\Omega$, we obtain the following constraints

$$\alpha \int_\Omega (e^{u_o + u} + e^v - 2) (e^{u_o + u} + e^v) dx + \beta \int_\Omega (e^{u_o + u} - e^v)^2 dx + \frac{4\pi n}{\alpha} = 0,$$

(5.9)

$$\alpha \int_\Omega (e^{u_o + u} + e^v - 2) (e^{u_o + u} - e^v) dx + \beta \int_\Omega (e^{2u_o + 2u} - e^{2v}) dx + \frac{4\pi n}{\beta} = 0,$$

(5.10)

or equivalently,

$$\int_\Omega (e^{u_o + u} - 1) e^{u_o + u} dx - \gamma \int_\Omega (e^v - 1) e^{u_o + v} dx + \frac{2\pi n}{\alpha \beta} = 0,$$

(5.11)

$$\int_\Omega (e^v - 1) e^v dx - \gamma \int_\Omega (e^{u_o + u} - 1) e^v dx + \frac{2\gamma \pi n}{\alpha \beta} = 0,$$

(5.12)

where we define

$$\gamma \equiv \frac{\beta - \alpha}{\beta + \alpha}.$$

(5.13)
throughout the rest of the work. Under our assumption on $\alpha, \beta$, that is, $\beta > \alpha > 0$, we see that $0 < \gamma < 1$.

It can be checked that the constraints (5.9) and (5.10) are the quantized integrals (3.16) and (3.17) stated in Theorem 3.2.

From (5.9), we see that
\[
\alpha \int_{\Omega} (e^{u_0 + u} + e^v - 2) \, dx + \beta \int_{\Omega} (e^{u_0 + u} - e^v)^2 \, dx = 2 \alpha \left( \int_{\Omega} [1 - e^{u_0 + u}] \, dx + \int_{\Omega} [1 - e^v] \, dx \right) - \frac{4\pi n}{\alpha}.
\] (5.14)

We know that $W^{1,2}(\Omega)$ can be decomposed as follows,
\[
W^{1,2}(\Omega) = \mathbb{R} \oplus \dot{W}^{1,2}(\Omega),
\]
where
\[
\dot{W}^{1,2}(\Omega) = \left\{ w \in W^{1,2}(\Omega) \middle| \int_{\Omega} w \, dx = 0 \right\}
\] is a closed subspace of $W^{1,2}(\Omega)$.

Then, we can decompose $u, v$ into the form
\[
u = u' + c_1, \quad v = v' + c_2,
\]
where
\[
\int_{\Omega} u' \, dx = 0, \quad \int_{\Omega} v' \, dx = 0, \quad c_1 = \frac{1}{|\Omega|} \int_{\Omega} u \, dx, \quad c_2 = \frac{1}{|\Omega|} \int_{\Omega} v \, dx.
\]
Then (5.11) and (5.12) can be rewritten in the form
\[
e^{2c_1} \int_{\Omega} e^{2u_0 + 2u'} \, dx - Q_1(u', v', e^{c_2}) e^{c_1} + \frac{2\pi n}{\alpha \beta} = 0,
\] (5.15)
\[
e^{2c_2} \int_{\Omega} e^{2v'} \, dx - Q_2(u', v', e^{c_1}) e^{c_2} + \frac{2\gamma \pi n}{\alpha \beta} = 0,
\] (5.16)
where
\[
Q_1(u', v', e^{c_2}) \equiv (1 - \gamma) \int_{\Omega} e^{u_0 + u'} \, dx + \gamma e^{c_2} \int_{\Omega} e^{u_0 + u' + v'} \, dx,
\] (5.17)
\[
Q_2(u', v', e^{c_1}) \equiv (1 - \gamma) \int_{\Omega} e^{v'} \, dx + \gamma e^{c_1} \int_{\Omega} e^{u_0 + u' + v'} \, dx.
\] (5.18)

Hence the equations (5.15)–(5.16) are solvable with respect to $c_1$ and $c_2$ if and only if
\[
(Q_1(u', v', e^{c_2}))^2 \geq \frac{8\pi n}{\alpha \beta} \int_{\Omega} e^{2u_0 + 2u'} \, dx,
\] (5.19)
\[
(Q_2(u', v', e^{c_1}))^2 \geq \frac{8\gamma \pi n}{\alpha \beta} \int_{\Omega} e^{2v'} \, dx.
\] (5.20)

From Proposition 5.1, we see that, for a solution $(u, v)$ of (5.6)–(5.8), $u_0 + u < 0, v < 0$, namely, $u_0 + u' + c_1 < 0, v' + c_2 < 0$. Then from (5.19) we obtain
\[
\frac{8\pi n}{\alpha \beta} \int_{\Omega} e^{2u_0 + 2u'} \, dx \leq \left( \int_{\Omega} e^{u_0 + u'} \, dx \right)^2 \leq |\Omega| \int_{\Omega} e^{2u_0 + 2u'} \, dx,
\]
which gives a necessary condition for the existence of solutions to (5.6)–(5.7)

\[ \alpha \beta \geq \frac{8\pi n}{|\Omega|}. \]  

(5.21)

Then we get the first conclusion of Theorem 3.2.

Now we take the following constraints

\[ \left( \int_{\Omega} e^{u_0 + u'} dx \right)^2 \geq \frac{8\pi n}{(1 - \gamma)^2 \alpha \beta} \int_{\Omega} e^{2u_0 + 2u'} dx, \]  

(5.22)

\[ \left( \int_{\Omega} e^{v'} dx \right)^2 \geq \frac{8\gamma \pi n}{(1 - \gamma)^2 \alpha \beta} \int_{\Omega} e^{2v'} dx. \]  

(5.23)

We introduce the following admissible set

\[ A = \left\{ (u', v') \in \dot{W}^{1,2}(\Omega) \times \dot{W}^{1,2}(\Omega) \left| (u', v') \right. \right. \text{ satisfies (5.22) - (5.23)} \right\}. \]  

(5.24)

Thus, for any \((u', v') \in A\), we can find a solution of the equations (5.15)–(5.16) with respect to \(c_1\) and \(c_2\) by solving the following equations

\[ e^{c_1} = \frac{Q_1(u', v', e^{c_2}) + \sqrt{[Q_1(u', v', e^{c_2})]^2 - \frac{8\pi n}{\alpha \beta} \int_{\Omega} e^{2u_0 + 2u'} dx}}{2 \int_{\Omega} e^{2u_0 + 2u'} dx}, \]  

\[ \equiv g_1(e^{c_2}), \]  

(5.25)

\[ e^{c_2} = \frac{Q_2(u', v', e^{c_1}) + \sqrt{[Q_2(u', v', e^{c_1})]^2 - \frac{8\gamma \pi n}{\alpha \beta} \int_{\Omega} e^{2v'} dx}}{2 \int_{\Omega} e^{2v'} dx}, \]  

\[ \equiv g_2(e^{c_1}). \]  

(5.26)

Indeed, letting

\[ F(X) \equiv X - g_1(g_2(X)), \]

we can solve (5.25)–(5.26) by finding the zeros of the function \(F(\cdot)\). Therefore, it is sufficient to prove the following proposition.

\textbf{Proposition 5.2} For any \((u', v') \in A\), the equation

\[ F(X) = X - g_1(g_2(X)) = 0 \]

admits a unique positive solution \(X_0\).

By this proposition, for any \((u', v') \in A\), we can get a solution of (5.15) and (5.16) with respect to \(c_1, c_2\).

\textbf{Proof of the Proposition 5.2.} By (5.25) and (5.26) it is easy to see that

\[ g_i(X) > 0, \quad \forall X \geq 0, \quad i = 1, 2. \]  

(5.27)

Then, we see that \(F(0) = -g_1(g_2(0)) < 0\). We check that

\[ \frac{d g_1(X)}{d X} = \frac{\gamma g_1(X) \int_{\Omega} e^{u_0 + u' + v'} dx}{\sqrt{[Q_1(u', v', X)]^2 - \frac{8\pi n}{\alpha \beta} \int_{\Omega} e^{2u_0 + 2u'} dx}}, \]  

(5.28)

\[ \frac{d g_2(X)}{d X} = \frac{\gamma g_2(X) \int_{\Omega} e^{u_0 + u' + v'} dx}{\sqrt{[Q_2(u', v', X)]^2 - \frac{8\gamma \pi n}{\alpha \beta} \int_{\Omega} e^{2v'} dx}}, \]  

(5.29)
which are all positive. Then, we see that $g_i(X) \ (i = 1, 2)$ is strictly increasing for all $X > 0$.

After a direct computation, we obtain
\[
\lim_{X \to +\infty} \frac{g_1(X)}{X} = \frac{\gamma}{\frac{\Omega}{e^{2u_0+u'+v'}dx}},
\]
\[
\lim_{X \to +\infty} \frac{g_2(X)}{X} = \frac{\gamma}{\frac{\Omega}{e^{2u_0+u'+v'}dx}},
\]
from which it follows that
\[
\lim_{X \to +\infty} \frac{F(X)}{X} = 1 - \frac{\gamma^2}{\frac{\Omega}{e^{2u_0+u'+v'}dx}} \geq 1 - \gamma^2 > 0.
\]
Therefore, we have
\[
\lim_{X \to +\infty} F(X) = +\infty.
\]
Noting that $F(0) < 0$, then we conclude that the equation $F(X) = 0$ has at least one solution $X_0 > 0$.

Next we show that the solution is also unique. From (5.28) and (5.29), and (5.22) and (5.23), we obtain
\[
\frac{dF(X)}{dX} = 1 - \frac{g_1(g_2(X))g_2(X)\gamma^2 (\int_{\Omega} e^{u_0+u'+v'}dx)^2}{\sqrt{[Q_1(u', v', g_2(X))]^2 - \frac{8\pi n}{\alpha\beta} \int_{\Omega} e^{2u_0+2u'}dx \sqrt{[Q_2(u', v', X)]^2 - \frac{8\gamma\pi n}{\alpha\beta} \int_{\Omega} e^{2v'}dx}}}
\]
\[
> 1 - \frac{g_1(g_2(X))}{X} = \frac{F(X)}{X}.
\]
Then we have
\[
\frac{d}{dX} \left( \frac{F(X)}{X} \right) > 0,
\]
which says that $\frac{F(X)}{X}$ is strictly increasing for $X > 0$. As a result, $F(X)$ is strictly increasing for $X > 0$. Then $F(X)$ has a unique zero point. Then the proof of Proposition 5.2 is complete.

By the above discussion we see that, for any $(u', v') \in A$, we can get pair $(c_1(u', v'), c_2(u', v'))$ given by (5.25)–(5.26), which solves (5.15)–(5.16), such that $(u, v)$ defined by
\[
u = u' + c_1(u', v'), \quad v = v' + c_2(u', v')
\]
satisfies (5.9)–(5.10).

In what follows we consider the minimization problem
\[
\min \{ J_{\alpha\beta}(u', v') \mid (u', v') \in A \},
\]
where $J_{\alpha\beta}(u', v')$ is defined by
\[
J_{\alpha\beta}(u', v') = I_{\alpha\beta}(u' + c_1(u', v'), v' + c_2(u', v')),
\]
$(c_1(u', v'), c_2(u', v'))$ is given by (5.25)–(5.26). From (5.8) and (5.14), we see that
\[
J_{\alpha\beta}(u', v') = \frac{1}{4} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) \left( \|\nabla u'\|^2 + \|\nabla v'\|^2 \right) + \frac{1}{2} \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) \int_{\Omega} \nabla u' \cdot \nabla v' dx
\]
\[
+ 2\alpha \left( \int_{\Omega} [1 - e^{u_0+u'}e^{c_1}] dx + \int_{\Omega} [1 - e^{v'}e^{c_2}] dx \right) - \frac{4\pi n}{\alpha}
\]
\[
+ 4\pi n \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) c_1 + 4\pi n \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) c_2.
\]
\[
(5.31)
\]
Remark 5.1 It follows from the Jensen inequality and (5.33) that

\[ e^{c_1} \int_\Omega e^{u_0+u'} dx \leq |\Omega|, \quad e^{c_2} \int_\Omega e^{v'} dx \leq |\Omega|. \]  

(5.32)

Remark 5.1 It follows from the Jensen inequality and (5.33) that

\[ e^{c_1} \leq 1, \quad e^{c_2} \leq 1. \]

Proof. From (5.25) and (5.26), we obtain

\[ e^{c_1} \leq \frac{Q_1(u', v', e^{c_2})}{\int_\Omega e^{2u_0+2u'} dx}, \quad e^{c_2} \leq \frac{Q_2(u', v', e^{c_1})}{\int_\Omega e^{2v'} dx}. \]  

(5.33)

(5.34)

Then it follows from (5.33), (5.34), (5.17), (5.18), and the Hölder inequality that

\[ e^{c_1} \leq \frac{(1 - \gamma) \int_\Omega e^{u_0+u'} dx}{\int_\Omega e^{2u_0+2u'} dx} + \frac{\gamma (1 - \gamma) \int_\Omega e^{u_0+u'+v'} dx \int_\Omega e^{v'} dx}{\int_\Omega e^{2u_0+2u'} dx \int_\Omega e^{2v'} dx} + \frac{\gamma^2 \left( \int_\Omega e^{u_0+u'+v'} dx \right)^2}{\int_\Omega e^{2u_0+2u'} dx \int_\Omega e^{2v'} dx} e^{c_1}, \]

which enables us to conclude that

\[ e^{c_1} \leq \frac{1}{1 + \gamma} \left( \frac{\int_\Omega e^{u_0+u'} dx}{\int_\Omega e^{2u_0+2u'} dx} + \frac{\gamma \int_\Omega e^{u_0+u'+v'} dx \int_\Omega e^{v'} dx}{\int_\Omega e^{2u_0+2u'} dx \int_\Omega e^{2v'} dx} \right). \]  

(5.35)

Similarly, we have

\[ e^{c_2} \leq \frac{1}{1 + \gamma} \left( \frac{\int_\Omega e^{v'} dx}{\int_\Omega e^{2v'} dx} + \frac{\gamma \int_\Omega e^{u_0+u'+v'} dx \int_\Omega e^{u_0+u'} dx}{\int_\Omega e^{2u_0+2u'} dx \int_\Omega e^{2v'} dx} \right). \]  

(5.36)

Using (5.35) and (5.36) and the Hölder inequality, we have

\[ e^{c_1} \int_\Omega e^{u_0+u'} dx \leq \frac{1}{1 + \gamma} \left( \frac{\left( \int_\Omega e^{u_0+u'} dx \right)^2}{\int_\Omega e^{2u_0+2u'} dx} + \frac{\gamma \int_\Omega e^{u_0+u'+v'} dx \int_\Omega e^{u_0+u'} dx \int_\Omega e^{v'} dx}{\int_\Omega e^{2u_0+2u'} dx \int_\Omega e^{2v'} dx} \right) \leq |\Omega|, \]

\[ e^{c_2} \int_\Omega e^{v'} dx \leq \frac{1}{1 + \gamma} \left( \frac{\left( \int_\Omega e^{v'} dx \right)^2}{\int_\Omega e^{2v'} dx} + \frac{\gamma \int_\Omega e^{u_0+u'+v'} dx \int_\Omega e^{u_0+u'} dx \int_\Omega e^{v'} dx}{\int_\Omega e^{2u_0+2u'} dx \int_\Omega e^{2v'} dx} \right) \leq |\Omega|. \]

Thus the lemma follows.

Estimates of the type contained in the following lemma were observed first in [48].
**Lemma 5.2** For any \((u', v') \in A\) and \(s \in (0, 1)\), it holds

\[
\int_{\Omega} e^{u_0 + u'} dx \leq \left( \frac{(1 - \gamma)^2 \alpha \beta}{8 \pi n} \right)^{1-s} \left( \int_{\Omega} e^{s u_0 + su'} dx \right)^{\frac{1}{s}},
\]

(5.37)

\[
\int_{\Omega} e^{v'} dx \leq \left( \frac{(1 - \gamma)^2 \alpha \beta}{8 \gamma \pi n} \right)^{1-s} \left( \int_{\Omega} e^{s v'} dx \right)^{\frac{1}{s}}.
\]

(5.38)

**Proof.** Let \(s \in (0, 1)\), \(a = \frac{1}{2-s}\) such that \(sa + 2(1-a) = 1\). Then using the Hölder inequality and (5.22) we have

\[
\int_{\Omega} e^{u_0 + u'} dx \leq \left( \int_{\Omega} e^{su_0 + su'} dx \right)^a \left( \int_{\Omega} e^{2u_0 + 2u'} dx \right)^{1-a}
\]

\[
\leq \left( \frac{(1 - \gamma)^2 \alpha \beta}{8 \pi n} \right)^{1-a} \left( \int_{\Omega} e^{su_0 + su'} dx \right)^a \left( \int_{\Omega} e^{u_0 + u'} dx \right)^{2(1-a)}
\]

which implies

\[
\int_{\Omega} e^{u_0 + u'} dx \leq \left( \frac{(1 - \gamma)^2 \alpha \beta}{8 \pi n} \right)^{\frac{1-a}{2a-1}} \left( \int_{\Omega} e^{su_0 + su'} dx \right)^{\frac{a}{2a-1}}
\]

\[
= \left( \frac{(1 - \gamma)^2 \alpha \beta}{8 \pi n} \right)^{\frac{1-s}{s}} \left( \int_{\Omega} e^{su_0 + su'} dx \right)^{\frac{s}{s}}.
\]

Analogously, we can obtain (5.38).

Next we show that the functional \(J_{\alpha \beta}\) is coercive and bounded from below on \(A\). To this end, we will use the Trudinger–Moser inequality (see [24])

\[
\int_{\Omega} e^{u} dx \leq C_1 \exp \left( \frac{1}{16 \pi} \left\| \nabla w \right\|_2^2 \right), \quad \forall w \in W^{1,2}(\Omega),
\]

(5.39)

where \(C_1\) is a positive constant.

**Lemma 5.3** For any \((u', v') \in A\), the functional \(J_{\alpha \beta}\) satisfies

\[
J_{\alpha \beta}(u', v') \geq \frac{1}{4 \beta} \left( \left\| \nabla u' \right\|_2^2 + \left\| \nabla v' \right\|_2^2 \right) - C_{\alpha \beta},
\]

(5.40)

where

\[
C_{\alpha \beta} \equiv \frac{8 \pi n^2 (\alpha + \beta)}{\alpha^2} \left( \ln \alpha \beta + \ln C_1 + \frac{1}{8 \pi n} \ln \frac{1 - \gamma}{2} \right) - \frac{8 \pi n}{\alpha} \ln \frac{1 - \gamma}{2}
\]

\[
+ 4 \pi n \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) \max_{x \in \Omega} u_0 - 4 \pi n^2 \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) \left( 1 + \frac{\beta}{\alpha} \right) \ln \gamma.
\]

(5.41)

**Proof.** From (5.25) and (5.26), we see that

\[
e^{c_1} \geq \frac{(1 - \gamma) \int_{\Omega} e^{u_0 + u'} dx}{2 \int_{\Omega} e^{2u_0 + 2u'} dx}, \quad e^{c_2} \geq \frac{(1 - \gamma) \int_{\Omega} e^{v'} dx}{2 \int_{\Omega} e^{2v'} dx}.
\]

Then by (5.22) and (5.23), we obtain

\[
e^{c_1} \geq \frac{4 \pi n}{(1 - \gamma) \alpha \beta \int_{\Omega} e^{u_0 + u'} dx}, \quad e^{c_2} \geq \frac{4 \pi n}{(1 - \gamma) \alpha \beta \int_{\Omega} e^{v'} dx}.
\]
which lead to

\[
c_1 \geq \ln \frac{4\pi n}{1-\gamma} - \ln \alpha \beta - \ln \int_{\Omega} e^{u_0 + u'} dx,
\]

(5.42)

\[
c_2 \geq \ln \frac{4\gamma \pi n}{1-\gamma} - \ln \alpha \beta - \ln \int_{\Omega} e^{u'} dx.
\]

(5.43)

For any \(s \in (0,1)\), using Lemma 5.2 and the Trudinger–Moser inequality (5.39), we have

\[
\ln \int_{\Omega} e^{u_0 + u'} dx \leq \frac{1-s}{s} \left( \ln \left[ \frac{1-\gamma}{8\pi n} \right] + \ln \alpha \beta \right) + \frac{1}{s} \ln \int_{\Omega} e^{su_0 + su'} dx
\]

\[
\leq \frac{s}{16\pi} \| \nabla u' \|^2 + \frac{1-s}{s} \left( \ln \left[ \frac{1-\gamma}{8\gamma \pi n} \right] + \ln \alpha \beta \right) + \frac{1}{s} \ln \int_{\Omega} e^{su'} dx
\]

\[
\ln \int_{\Omega} e^{u'} dx \leq \frac{1-s}{s} \left( \ln \left[ \frac{1-\gamma}{8\gamma \pi n} \right] + \ln \alpha \beta \right) + \frac{1}{s} \ln \int_{\Omega} e^{su'} dx
\]

\[
\leq \frac{s}{16\pi} \| \nabla u' \|^2 + \frac{1-s}{s} \left( \ln \left[ \frac{1-\gamma}{8\gamma \pi n} \right] + \ln \alpha \beta \right) + \frac{1}{s} \ln C_1.
\]

(5.44)

(5.45)

Then, from (5.31), (5.14), (5.42)–(5.45), we see that

\[
J_{\alpha \beta}(u', v') \geq \left( \frac{1}{2\beta} - \frac{s n}{4} \ln \left[ \frac{1}{\alpha} + \frac{1}{\beta} \right] \right) \| \nabla u' \|^2 + \left( \frac{1}{2\beta} - \frac{s n}{4} \ln \left[ \frac{1}{\alpha} + \frac{1}{\beta} \right] \right) \| \nabla v' \|^2
\]

\[
+ \frac{8\pi n}{\alpha} \ln \frac{1-\gamma}{2} - 4\pi n \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) \max_{x \in \Omega} u_0
\]

\[
- \frac{8\pi n}{s\alpha} \left( \ln \alpha \beta + \ln C_1 + \ln \left[ \frac{1-\gamma}{8\gamma \pi n} \right] \right) + \frac{4\pi n}{s} \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) \ln \gamma.
\]

(5.46)

Now by taking

\[
s = \frac{\alpha}{n(\alpha + \beta)};
\]

in (5.46), we get (5.40).

It is easy to see that \(J_{\alpha \beta}(u', v')\) is weakly lower semi-continuous on \(A\). Then by lemma 5.3 we infer that the infimum of \(J_{\alpha \beta}(u', v')\) can be attained on \(A\).

In the sequel we will show that, when \(\alpha, \beta\) satisfy \(\beta > \alpha > 0\), (3.14), and \(\alpha\) is sufficiently large, a minimizer can only be an interior point of \(A\).

**Lemma 5.4** The functional \(J_{\alpha \beta}\) satisfies

\[
\inf_{(u', v') \in \partial A} J_{\alpha \beta}(u', v') \geq 2|\Omega|\alpha - \frac{16\pi n}{(1+\gamma)(1-\gamma)^2\alpha} - \frac{4\gamma \sqrt{2\pi n |\Omega|}}{1-\gamma^2} - C_{\alpha \beta},
\]

(5.47)

where \(C_{\alpha \beta}\) is defined by (5.41).

**Proof.** On the boundary of \(A\), we have

\[
\left( \int_{\Omega} e^{u_0 + u'} dx \right)^2 = \frac{8\pi n}{(1-\gamma)^2\alpha \beta} \int_{\Omega} e^{2u_0 + 2u'} dx
\]

(5.48)

or

\[
\left( \int_{\Omega} e^{u'} dx \right)^2 = \frac{8\gamma \pi n}{(1-\gamma)^2\alpha \beta} \int_{\Omega} e^{2u'} dx.
\]

(5.49)
If (5.48) holds, using (5.35) and the Hölder inequality, we obtain

\[
e^{c_1} \int_{\Omega} e^{u_0 + u'} dx \leq \frac{1}{1 + \gamma} \left( \frac{\int_{\Omega} e^{u_0 + u'} dx}{\int_{\Omega} e^{2u_0 + 2u'} dx} \right)^2 + \frac{\gamma \int_{\Omega} e^{u_0 + u'} dx \int_{\Omega} e^{u_0 + u'} dx}{\int_{\Omega} e^{2u_0 + 2u'} dx \int_{\Omega} e^{2u'} dx}
\]

\[
\leq \frac{8\pi n}{(1 + \gamma)(1 - \gamma)^2 \alpha \beta} + \frac{2\gamma \sqrt{2\pi n|\Omega|}}{(1 - \gamma^2) \sqrt{\alpha \beta}} + \frac{8\pi n}{(1 + \gamma)(1 - \gamma)^2 \alpha^2} + \frac{2\gamma \sqrt{2\pi n|\Omega|}}{(1 - \gamma^2) \alpha},
\]

which leads to

\[
2\alpha \left( \int_{\Omega} [1 - e^{u_0 + u'} e^{c_1}] dx + \int_{\Omega} [1 - e^{u'} e^{c_2}] dx \right) \geq 2|\Omega|\alpha - \frac{16\pi n}{(1 + \gamma)(1 - \gamma)^2 \alpha} - \frac{4\gamma \sqrt{2\pi n|\Omega|}}{1 - \gamma^2} - C_{\alpha \beta},
\]

Therefore, using similar estimates for \( c_1, c_2 \) as in Lemma 5.3 on the boundary of \( A \) we have

\[
J_{\alpha \beta}(u', v') \geq 2|\Omega|\alpha - \frac{16\pi n}{(1 + \gamma)(1 - \gamma)^2 \alpha} - \frac{4\gamma \sqrt{2\pi n|\Omega|}}{1 - \gamma^2} - C_{\alpha \beta},
\]

where \( C_{\alpha \beta} \) is defined by (5.41). Then the proof of Lemma 5.4 is complete.

As a test function, we use, as in [48], the solution characterized by Tarantello [63]. Namely, from [63], we know that, for \( \lambda \) sufficiently large, there exists a solution \( w_\lambda \) of the equation

\[
\Delta w = \lambda e^{u_0 + w} (e^{u_0 + w} - 1) + \frac{8\pi n}{|\Omega|},
\]

satisfying \( w_\lambda = c_\lambda + w_\lambda', c_\lambda = \frac{1}{|\Omega|} \int_{\Omega} w_\lambda dx, \int_{\Omega} w_\lambda' dx = 0 \), such that \( u_0 + w_\lambda < 0 \) in \( \Omega \), \( c_\lambda \to 0 \), and \( w_\lambda' \to -u_0 \) pointwise, as \( \lambda \to +\infty \).

In view of \( e^{u_0} \in L^\infty(\Omega) \) and the dominated convergence theorem, we have

\[
e^{u_0 + w_\lambda} \to 1 \quad \text{strongly in} \quad L^p(\Omega) \quad \text{for any} \ p \geq 1,
\]

as \( \lambda \to +\infty \). In particular,

\[
\int_{\Omega} e^{2u_0 + 2w_\lambda'} dx \to |\Omega|,
\]

as \( \lambda \to +\infty \). Therefore, for \( \alpha_0 \) large and fixed \( \varepsilon \in (0, 1) \), we can find \( \lambda_\varepsilon \) to ensure that \( (w_\lambda', 0) \in A \) for every \( \alpha > \alpha_0 \) and

\[
\frac{(1 - \gamma^2) |\Omega|}{\int_{\Omega} e^{2u_0 + 2w_\lambda'} dx - \gamma^2 |\Omega|} \geq 1 - \varepsilon.
\]

By the Jensen inequality,

\[
\int_{\Omega} e^{u_0 + w_\lambda'} dx \geq |\Omega|.
\]

Then in view of (5.25) and (5.26) we get

\[
e^{c_1(w_\lambda', 0)} \geq \frac{Q_1(w_\lambda', 0, e^{c_2(w_\lambda', 0)})}{2 \int_{\Omega} e^{2u_0 + 2w_\lambda'} dx} \left( 1 + \frac{1}{\alpha \beta Q_1^2(w_\lambda', 0, e^{c_2(w_\lambda', 0)})} \right) \frac{\int_{\Omega} e^{2u_0 + 2w_\lambda'} dx}{\int_{\Omega} e^{2u_0 + 2w_\lambda'} dx} - \frac{4\pi n}{\alpha \beta (1 - \gamma |\Omega|)} - 4\pi n
\]

\[
\geq \frac{Q_1(w_\lambda', 0, e^{c_2(w_\lambda', 0)})}{\int_{\Omega} e^{2u_0 + 2w_\lambda'} dx} - \frac{4\pi n}{\alpha \beta (1 - \gamma |\Omega|)}.
\]
Similarly, we have

\[ e^{c_2(w_{k,0}^{'},0)} \geq 1 - \gamma + \gamma e^{c_1(w_{k,0}^{'},0)} - \frac{4\gamma n}{\alpha \beta (1 - \gamma) |\Omega|} . \]  

(5.53)

Therefore it follows from (5.52) and (5.53) that

\[ e^{c_1(w_{k,0}^{'},0)} \geq \frac{\gamma |\Omega|}{\int_{\Omega} e^{2u_0 + 2w_{k,\alpha}^{'}} \, dx} \left( 1 - \gamma + \gamma e^{c_1(w_{k,0}^{'},0)} - \frac{4\gamma n}{\alpha \beta (1 - \gamma) |\Omega|} \right) + \frac{\gamma^2 |\Omega|}{\int_{\Omega} e^{2u_0 + 2w_{k,\alpha}^{'}} \, dx} - \frac{4\gamma n(1 + \gamma^2)}{\alpha \beta (1 - \gamma) |\Omega|} \]

which implies

\[ e^{c_1(w_{k,0}^{'},0)} \geq \frac{(1 - \gamma^2) |\Omega|}{\int_{\Omega} e^{2u_0 + 2w_{k,\alpha}^{'}} \, dx} - \frac{4\gamma n(1 + \gamma^2)}{\alpha \beta (1 - \gamma) (1 - \gamma)^2 |\Omega|} . \]  

(5.54)

Similarly, we get

\[ e^{c_2(w_{k,0}^{'},0)} \geq \frac{(1 - \gamma^2) |\Omega|}{\int_{\Omega} e^{2u_0 + 2w_{k,\alpha}^{'}} \, dx} - \frac{8\pi n \gamma}{\alpha \beta (1 + \gamma) (1 - \gamma)^2 |\Omega|} . \]  

(5.55)

Then, combining (5.54), (5.55), and (5.51), we conclude that for all \( \alpha > \alpha_0, \)

\[ e^{c_1(w_{k,0}^{'},0)} \geq 1 - \varepsilon - \frac{4\pi n(1 + \gamma^2)}{\alpha \beta (1 + \gamma) (1 - \gamma)^2 |\Omega|} , \]

\[ e^{c_2(w_{k,0}^{'},0)} \geq 1 - \varepsilon - \frac{8\pi n \gamma}{\alpha \beta (1 + \gamma) (1 - \gamma)^2 |\Omega|} . \]

Therefore, we obtain that for all \( \alpha > \alpha_0, \)

\[ \int_{\Omega} \left( 1 - e^{c_1(w_{k,0}^{'},0)} e^{u_0 + w_{k,\alpha}^{'}} \right) \, dx \leq \varepsilon |\Omega| + \frac{4\pi n(1 + \gamma^2)}{\alpha \beta (1 + \gamma) (1 - \gamma)^2 |\Omega|} , \]

(5.56)

\[ \int_{\Omega} \left( 1 - e^{c_2(w_{k,0}^{'},0)} \right) \, dx \leq \varepsilon |\Omega| + \frac{8\pi n \gamma}{\alpha \beta (1 + \gamma) (1 - \gamma)^2 |\Omega|} . \]

(5.57)

**Lemma 5.5** Assume \( \beta > \alpha > 0 \) and that (3.14) holds. There exists a positive constant \( M_\sigma \) such that, when \( \alpha > M_\sigma, \) we have

\[ J_{\alpha,\beta}(w_{\lambda,0}^{'},0) - \inf_{(u',v') \in \partial A} J_{\alpha,\beta}(u',v') < -1 . \]  

(5.58)

**Proof.** Fix \( \varepsilon \in (0, \frac{1}{4}) \), we choose \( w_{\lambda,0}^{'}, \) that satisfies (5.56)–(5.57). Noting that \( e^{c_1} \leq 1, e^{c_2} \leq 1, \) then by (5.31), (5.56) and (5.57) we have

\[ J_{\alpha,\beta}(w_{\lambda,0}^{\prime},0) \leq \frac{1}{4} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) \| \nabla w_{\lambda,0}^{\prime} \|_2^2 + 4\varepsilon |\Omega| + \frac{8\pi n(1 + \gamma)}{(1 - \gamma)^2 \beta} - \frac{4\pi n}{\alpha} , \]

(5.59)
where $C_{\varepsilon}$ is a positive constant depending only on $\varepsilon$.

Then by Lemma 5.4 we have

$$J_{\alpha\beta}(w', 0) - \inf_{(w', v') \in \partial A} J_{\alpha\beta}(w', v') \leq 2\alpha|\Omega|(2\varepsilon - 1) + C_{\varepsilon} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) + \frac{8\pi n(1 + \gamma)}{(1 - \gamma)^2 \beta} + \frac{16\pi n \varepsilon}{(1 + \gamma)(1 - \gamma)^2} + C_{\alpha\beta},$$

(5.60)

where $C_{\alpha\beta}$ is defined by (5.41).

By the assumption $\beta > \alpha > 0$ and (3.14), we easily deduce the following estimate

$$\frac{2}{1 + \sigma} \leq 1 - \gamma < 1,$$

(5.61)

and

$$C_{\alpha\beta} \leq C_{\sigma} \left( \frac{\ln \alpha}{\alpha} + 1 \right),$$

(5.62)

with a suitable constant $C_{\sigma} > 0$ depending on $\sigma > 1$ only. By using the above estimates in (5.60), we obtain the desired conclusion.

Using Lemma 5.3 and 5.5 we infer that under the assumption $\beta > \alpha > 0$ and (3.14), when $\alpha$ is sufficiently large the problem (5.30) admits a minimizer $(u'_\alpha, v'_\alpha)$, which lies in the interior of $A$. Consequently,

$$DJ_{\alpha\beta}(u'_\alpha, v'_\alpha) = 0,$$

and

$$u_\alpha = u'_\alpha + c_1(u'_\alpha, v'_\alpha), \quad v_\alpha = v'_\alpha + c_2(u'_\alpha, v'_\alpha)$$

(5.63)

gives rise to a critical point for the functional $I_{\alpha\beta}$, namely, a weak solution to (5.6) and (5.7).

In what follows we study the behavior of the solution given by (5.63).

**Lemma 5.6** Let $(u_\alpha, v_\alpha)$ be defined by (5.63). Then

$$e^{u_\alpha + u_\alpha} \to 1, \quad e^{v_\alpha} \to 1$$

(5.64)

as $\alpha \to +\infty$ pointwise a.e. in $\Omega$, and in $L^p(\Omega)$ for any $p \geq 1$.

**Proof.** Using (5.59), we obtain that for any $\varepsilon > 0$ (small) there exists $\alpha_\varepsilon > 0$ such that when $\alpha > \alpha_\varepsilon$,

$$\inf_{(u', v') \in A} J_{\alpha\beta}(u', v') \leq 4\varepsilon\alpha|\Omega| + C_{\varepsilon} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) + \frac{8\pi n(1 + \gamma)}{(1 - \gamma)^2 \beta} - \frac{4\pi n \varepsilon}{\alpha}. $$

(5.65)

By a similar argument as in Lemma 5.3 we have

$$\inf_{(u', v') \in A} J_{\alpha\beta}(u', v') = J_{\alpha\beta}(u'_\alpha, v'_\alpha) \geq \alpha \int_{\Omega} (e^{u_0 + u_\alpha} + e^{v_\alpha} - 2)^2 dx + \beta \int_{\Omega} (e^{u_0 + u_\alpha} - e^{v_\alpha})^2 dx - C_{\alpha\beta},$$

$$\geq 2\alpha \left( \int_{\Omega} [e^{u_0 + u_\alpha} - 1]^2 dx + \int_{\Omega} [e^{v_\alpha} - 1]^2 dx \right) - C_{\alpha\beta},$$

(5.66)
where $C_{\alpha\beta}$ is defined by (5.41).

Then by means of (5.61), (5.62), (5.65), and (5.66), we see that

$$\limsup_{\alpha \to +\infty} \left( \int_\Omega [e^{u_0 + u_\alpha} - 1]^2 \, dx + \int_\Omega [e^{v_\alpha} - 1]^2 \, dx \right) \leq 2|\Omega|\varepsilon, \quad \forall \varepsilon > 0,$$

which enables us to conclude that,

$$e^{u_0 + u_\alpha} \to 1, \quad e^{v_\alpha} \to 1,$$

in $L^2(\Omega)$, as $\alpha \to +\infty$. Since by Proposition 5.1, we also know that

$$e^{u_0 + u_\alpha} < 1, \quad e^{v_\alpha} < 1$$

in $\Omega$. Then we have

$$e^{u_0 + u_\alpha} \to 1, \quad e^{v_\alpha} \to 1$$

pointwise a.e. in $\Omega$ as $\alpha \to +\infty$. At this point, we may complete the proof of Lemma 5.6 by dominated convergence theorem.

In order to get a secondary solution of (5.6) and (5.7), we first show that the solution $(u_\alpha, v_\alpha)$ given by (5.63) is a local minimizer of the functional $I_{\alpha\beta}$.

**Lemma 5.7** Let $(u_\alpha, v_\alpha)$ be defined by (5.63). Then $(u_\alpha, v_\alpha)$ is a local minimizer of the functional $I_{\alpha\beta}$ in $\mathcal{A}$.

**Proof.** It is easy to check that for any $(u', v') \in \mathcal{A}$,

\[
\partial_{c_1} I_{\alpha\beta}(u' + c_1(u', v'), v' + c_2(u', v')) \\
= 2(\alpha + \beta) \left[ e^{2c_1} \int_\Omega e^{2u_0 + 2u'} \, dx - e^{c_1} Q_1(u', v', e^{c_2}) + \frac{2\pi n}{\alpha \beta} \right] = 0,
\]

\[
\partial_{c_2} I_{\alpha\beta}(u' + c_1(u', v'), v' + c_2(u', v')) \\
= 2(\alpha + \beta) \left[ e^{2c_2} \int_\Omega e^{2u_0 + 2u'} \, dx - e^{c_2} Q_2(u', v', e^{c_1}) + \frac{2\gamma \pi n}{\alpha \beta} \right] = 0,
\]

and

\[
\partial^2_{c_1^2} I_{\alpha\beta}(u' + c_1(u', v'), v' + c_2(u', v')) \\
= 2(\alpha + \beta) \left[ 2e^{2c_1} \int_\Omega e^{2u_0 + 2u'} \, dx - e^{c_1} Q_1(u', v', e^{c_2}) \right],
\]

\[
\partial^2_{c_2^2} I_{\alpha\beta}(u' + c_1(u', v'), v' + c_2(u', v')) \\
= 2(\alpha + \beta) \left[ e^{2c_2} \int_\Omega e^{2u_0 + 2u'} \, dx - e^{c_2} Q_2(u', v', e^{c_1}) \right],
\]

\[
\partial^2_{c_1 c_2} I_{\alpha\beta}(u' + c_1(u', v'), v' + c_2(u', v')) \\
= -2(\alpha + \beta) \gamma e^{c_1} e^{c_2} \int_\Omega e^{u_0 + u' + v'} \, dx.
\]
Then, in view of (5.25) and (5.26), we obtain
\[
\partial^2_{c_1} I_{\alpha\beta}(u_\alpha, v_\alpha) \\
= 2(\alpha + \beta) \left\{ \left[ (1 - \gamma) \int_{\Omega} e^{u_0 + u_\alpha} dx + \gamma \int_{\Omega} e^{u_0 + u_\alpha + v_\alpha} dx \right]^2 - \frac{8\pi n}{\alpha \beta} \int_{\Omega} e^{2u_0 + 2u_\alpha} dx \right\}^{\frac{1}{2}}, \\
\partial^2_{c_2} I_{\alpha\beta}(u_\alpha, v_\alpha) \\
= 2(\alpha + \beta) \left\{ \left[ (1 - \gamma) \int_{\Omega} e^{v_\alpha} dx + \gamma \int_{\Omega} e^{u_0 + u_\alpha + v_\alpha} dx \right]^2 - \frac{8\gamma \pi n}{\alpha \beta} \int_{\Omega} e^{2v_\alpha} dx \right\}^{\frac{1}{2}}.
\]

Since \((u'_\alpha, v'_\alpha)\) lies in the interior of \(A\), we obtain
\[
\partial^2_{c_1} I_{\alpha\beta}(u_\alpha, v_\alpha) > 2(\alpha + \beta) \gamma \int_{\Omega} e^{u_0 + u_\alpha + v_\alpha} dx, \\
\partial^2_{c_2} I_{\alpha\beta}(u_\alpha, v_\alpha) > 2(\alpha + \beta) \gamma \int_{\Omega} e^{u_0 + u_\alpha + v_\alpha} dx.
\]

Hence, at the point \((u_\alpha, v_\alpha)\) the Hessian matrix of \(I_{\alpha\beta}(u'_1 + c_1, v'_1 + c_2)\) with respect to \((c_1, c_2)\) is strictly positive definite. Let \(u = u' + c_1, v = v' + c_2\). By continuity, there exists \(\delta > 0\) such that for
\[
\|u - u_\alpha\| + \|v - v_\alpha\| \leq \delta,
\]
we have \((u', v')\) lies in the interior of \(A\) and
\[
I_{\alpha\beta}(u, v) \geq I_{\alpha\beta}(u' + c_1(u', v'), v' + c_2(u', v')) = J_{\alpha\beta}(u', v').
\]

Hence we have
\[
I_{\alpha\beta}(u, v) \geq J_{\alpha\beta}(u', v') \geq \inf_{(u', v') \in A} J_{\alpha\beta}(u', v') = I_{\alpha\beta}(u_\alpha, v_\alpha),
\]
which says that \((u_\alpha, v_\alpha)\) is a local minimizer of \(I_{\alpha\beta}(u, v)\).

### 5.2 The second solution

To find a secondary solution which is actually a mountain-pass critical point, we show that the functional \(I_{\alpha\beta}\) satisfies the Palais–Smale condition.

**Lemma 5.8** Let \(\{(u_m, v_m)\}\) be a sequence in \(W^{1,2}(\Omega) \times W^{1,2}(\Omega)\) satisfying
\[
I_{\alpha\beta}(u_m, v_m) \rightarrow \nu \quad \text{as} \quad m \rightarrow +\infty, \quad (5.67)
\]
\[
\|DI_{\alpha\beta}(u_m, v_m)\|_* \rightarrow 0 \quad \text{as} \quad m \rightarrow +\infty, \quad (5.68)
\]
where \(\nu\) is a constant, \(\| \cdot \|_*\) denotes the norm of the dual space of \(W^{1,2}(\Omega) \times W^{1,2}(\Omega)\). Then \(\{(u_m, v_m)\}\) admits a convergent subsequence in \(W^{1,2}(\Omega) \times W^{1,2}(\Omega)\).

**Proof.** Let \(u_m = c_{1,m} + u'_m, v_m = c_{2,m} + v'_m, \) where
\[
\int_{\Omega} u'_m dx = 0, \quad \int_{\Omega} v'_m dx = 0, \quad c_{1,m} = \frac{1}{|\Omega|} \int_{\Omega} u_m dx, \quad c_{2,m} = \frac{1}{|\Omega|} \int_{\Omega} v_m dx.
\]
A simple computation gives

$$(DI_{\alpha\beta}(u_m, v_m)) (\varphi, \psi)$$

\[
= \frac{1}{2} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) \int \nabla u_m \cdot \nabla \varphi + \nabla v_m \cdot \nabla \psi \, dx + \frac{1}{2} \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) \int (\nabla v_m \cdot \nabla \varphi + \nabla u_m \cdot \nabla \psi) \, dx
\]

\[+ 2\alpha \int_\Omega (e^{u_0+u_m} + e^{v_m} - 2)e^{u_0+u_m} \varphi \, dx + 2\alpha \int_\Omega (e^{u_0+u_m} + e^{v_m} - 2)e^{v_m} \psi \, dx
\]

\[+ 2\beta \int_\Omega (e^{u_0+u_m} - e^{v_m})e^{u_0+u_m} \varphi \, dx - 2\beta \int_\Omega (e^{u_0+u_m} - e^{v_m})e^{v_m} \psi \, dx
\]

\[+ \frac{4\pi n}{|\Omega|} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) \int_\Omega \varphi \, dx + \frac{4\pi n}{|\Omega|} \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) \int_\Omega \psi \, dx.
\]

(5.69)

Taking $(\varphi, \psi) = (1, 0)$ and $(\varphi, \psi) = (0, 1)$ in (5.69), from (5.68) we obtain as $m \to +\infty$,

$$\int_\Omega e^{2u_0+2u_m} \, dx = \int_\Omega e^{u_0+u_m} \, dx = \int_\Omega e^{v_m} \, dx$$

(5.70)

Combining (5.70) and (5.71) gives

$$\alpha \int_\Omega e^{u_0+u_m} + e^{v_m} \, dx$$

as $m \to +\infty$.

Noting that

$$\alpha (e^{u_0+u_m} + e^{v_m} - 2)^2 + \beta (e^{u_0+u_m} - e^{v_m})^2$$

\[= (\alpha + \beta) (e^{2u_0+2u_m} + e^{2v_m} - 2(\alpha - \beta) e^{u_0+u_m} + v_m) - 4\alpha (e^{u_0+u_m} + e^{v_m}) + 4\alpha
\]

and

$$\alpha (e^{u_0+u_m} + e^{v_m} - 2)^2 + \beta (e^{u_0+u_m} - e^{v_m})^2 \geq \alpha \left( [e^{u_0+u_m} + e^{v_m} - 2]^2 + [e^{u_0+u_m} - e^{v_m}]^2 \right)$$

\[= 2\alpha \left( [e^{u_0+u_m} - 1]^2 + [e^{v_m} - 1]^2 \right),
\]

by (5.72) we have

$$2\alpha \int_\Omega \left( [e^{u_0+u_m} - 1]^2 + [e^{v_m} - 1]^2 \right) \, dx + 2\alpha \int_\Omega (e^{u_0+u_m} + e^{v_m}) \, dx + \frac{4\pi n}{\alpha} < 4|\Omega| \alpha + o(1)$$

as $m \to +\infty$.

Then it follows from (5.73) that

$$\int_\Omega (e^{u_0+u_m} + e^{v_m} - 2)^2 \, dx + \int_\Omega (e^{u_0+u_m} - e^{v_m})^2 \, dx \leq 4|\Omega| + o(1),$$

(5.74)

$$\int_\Omega (e^{u_0+u_m} - 1)^2 \, dx + \int_\Omega (e^{v_m} - 1)^2 \, dx \leq 2|\Omega| + o(1),$$

(5.75)

$$\int_\Omega e^{u_0+u_m} \, dx \leq 2|\Omega| + o(1), \quad \int_\Omega e^{v_m} \, dx \leq 2|\Omega| + o(1),$$

(5.76)
Using (5.76) and the Jensen inequality we obtain
\[ e^{c_1,m} \leq 2 + o(1), \quad e^{c_2,m} \leq 2 + o(1), \] (5.77)
as \( m \to +\infty \), which says \( c_{1,m}, c_{2,m} \) are bounded from above. From (5.75)–(5.76) we see that
\[ \int_\Omega e^{2u_0 + 2u_m} \, dx \leq 6|\Omega| + o(1), \quad \int_\Omega e^{2v_m} \, dx \leq 6|\Omega| + o(1), \] (5.78)
as \( m \to +\infty \).

Denote \((u'_m + v'_m)^+ \equiv \max\{u'_m + v'_m, 0\}\). Setting \( \varphi = (u'_m + v'_m)^+ \) in (5.69), we have
\[
(DI_{\alpha \beta}(u_m, v_m))[(u'_m + v'_m)^+, (u'_m + v'_m)^+] = \frac{1}{\alpha} \|\nabla (u'_m + v'_m)^+\|^2_2 + 2(\alpha + \beta) \int_\Omega (e^{u_0 + u_m} - e^{v_m}) (u'_m + v'_m)^+ \, dx
\]
\[ + 8\alpha \int_\Omega e^{u_0 + u_m + v_m} (u'_m + v'_m)^+ \, dx - 4\alpha \int_\Omega (e^{u_0 + u_m + v_m}) (u'_m + v'_m)^+ \, dx
\]
\[ + \frac{8\pi n}{\|\Omega\|} \int (u'_m + v'_m)^+ \, dx
\]
\[ \geq 8\alpha \int_\Omega e^{u_0 + u_m + v_m} (u'_m + v'_m)^+ \, dx - 4\alpha \int_\Omega (e^{u_0 + u_m} + e^{v_m}) (u'_m + v'_m)^+ \, dx
\]
\[ \geq 8\alpha \int_\Omega e^{u_0 + u_m + v_m} (u'_m + v'_m)^+ \, dx
\]
\[ - 4\alpha \left( \left[ \int_\Omega e^{2u_0 + 2u_m} \, dx \right]^{\frac{1}{2}} + \left[ \int_\Omega e^{2v_m} \, dx \right]^{\frac{1}{2}} \right) \|u'_m + v'_m\|^2_2.
\] (5.79)

Then it follows from (5.68), (5.78) and (5.79) that
\[
\int_\Omega e^{u_0 + u_m + v_m} (u'_m + v'_m)^+ \, dx \leq C \left( \|u'_m + v'_m\|^2_2 + \varepsilon_n \|u'_m + v'_m\|^2 \right)
\]
\[ \leq C \left( \|\nabla u'_m\|^2_2 + \|\nabla v'_m\|^2_2 \right),
\] (5.80)
where \( C \) is a suitable positive constant independent of \( m \) and \( \varepsilon_n \to 0 \) as \( m \to \infty \).

Now let \( (\varphi, \psi) = (u'_m, v'_m) \) in (5.69), we see that
\[
(DI_{\alpha \beta}(u_m, v_m))(u'_m, v'_m)
\]
\[ \geq \frac{1}{\beta} \left( \|\nabla u'_m\|^2_2 + \|\nabla v'_m\|^2_2 \right) + 2(\alpha + \beta) \left[ \int e^{2u_0} e^{2v_m} (e^{2u'_m} - 1) u'_m \, dx + \int e^{2u_0} e^{2c_{1,m}} u'_m \, dx \right]
\]
\[ + 2(\alpha + \beta) \left[ \int e^{2c_{2,m}} (e^{2v'_m} - 1) v'_m \, dx + \int e^{2c_{2,m}} v'_m \, dx \right]
\]
\[ - 4\alpha \left( \left[ \int_\Omega e^{2u_0 + 2u_m} \, dx \right]^{\frac{1}{2}} \|\nabla u'_m\|^2_2 + \left[ \int_\Omega e^{2v_m} \, dx \right]^{\frac{1}{2}} \|\nabla v'_m\|^2_2 \right) - C \left( \|\nabla u'_m\|^2_2 + \|\nabla v'_m\|^2_2 \right)
\]
\[ \geq \frac{1}{\beta} \left( \|\nabla u'_m\|^2_2 + \|\nabla v'_m\|^2_2 \right) - C \left( \|\nabla u'_m\|^2_2 + \|\nabla v'_m\|^2_2 \right),
\] (5.81)
where we have used (5.77)–(5.80) and the Hölder inequality, $C$ is a positive constant independent of $m$.

Then using (5.68)–(5.81), we conclude that there exists a positive constant $C$ independent of $m$ such that

$$
\|\nabla u'_m\|_2 + \|\nabla v'_m\|_2 \leq C.
$$

(5.82)

On the other hand, from (5.67) we see that $I_{\alpha\beta}(u_m, v_m)$ is bounded from below. Therefore we have

$$
4\pi n \left[ \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) c_{1,m} + \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) c_{2,m} \right] 
\geq -C - \alpha \int_\Omega \left( e^{u_0 + u_m} + e^{v_m} - 2 \right)^2 \, dx - \beta \int_\Omega \left( e^{v_0 + u_m} - e^{v_m} \right)^2 \, dx 
- \frac{1}{2\alpha} \left( \|\nabla u'_m\|_2^2 + \|\nabla v'_m\|_2^2 \right) 
\geq -C,
$$

(5.83)

where we have used (5.74) and (5.82), $C$ is a positive constant independent of $m$. Then we infer from (5.77) and (5.83) that $c_{1,m}, c_{2,m}$ are bounded from below uniformly with respect to $m$. As a result $c_{1,m}, c_{2,m}$ are bounded for all $m$. Therefore we obtain that $u_m = u'_m + c_{1,m}, v_m = v'_m + c_{2,m}$ are bounded sequence in $W^{1,2}(\Omega)$. Hence there exists a subsequence of $u_m$ and $v_m$, still denoted by $u_m$ and $v_m$, such that

$$
u_m \to \tilde{u}, \quad v_m \to \tilde{v}
$$

weakly in $W^{1,2}(\Omega)$, strongly in $L^p(\Omega)$ for any $p \geq 1$ and pointwise a.e. in $\Omega$ as $m \to \infty$, where $\tilde{u}, \tilde{v} \in W^{1,2}(\Omega)$. Moreover, we see that

$$e^{u_m} \to e^{\tilde{u}}, \quad e^{v_m} \to e^{\tilde{v}} \quad \text{strongly in} \quad L^p(\Omega), \forall p \geq 1
$$

and

$$c_{1,m} = \frac{1}{|\Omega|} \int_\Omega u_m \, dx \to \tilde{c}_1 = \frac{1}{|\Omega|} \int_\Omega \tilde{u} \, dx, \quad c_{2,m} = \frac{1}{|\Omega|} \int_\Omega v_m \, dx \to \tilde{c}_2 = \frac{1}{|\Omega|} \int_\Omega \tilde{v} \, dx
$$

as $m \to \infty$.

Hence, for any $(\varphi, \psi) \in W^{1,2}(\Omega) \times W^{1,2}(\Omega)$, as $m \to \infty$ we have

$$I'_{\alpha\beta}(u_m, v_m)(\varphi, \psi) \to I'_{\alpha\beta}(\tilde{u}, \tilde{v})(\varphi, \psi) = 0,
$$

(5.84)

which says that $(\tilde{u}, \tilde{v})$ is a critical point for the functional $I_{\alpha\beta}$.

In the sequel we show the strong convergence of $(u_m, v_m)$. Let $(\varphi, \psi) = (u_m - \tilde{u}, v_m - \tilde{v})$. Then, from (5.68) and (5.84), we see that

$$
[DI_{\alpha\beta}(u_m, v_m) - DI_{\alpha\beta}(\tilde{u}, \tilde{v})][u_m - \tilde{u}, v_m - \tilde{v}] \leq \varepsilon_n (\|u_m - \tilde{u}\| + \|v_m - \tilde{v}\|) = o(1),
$$

(5.85)

as $m \to \infty$.  

33
Indeed, noting that \( u_\delta \) positive constant \( \alpha \) as the behavior \( m \) as such that Of course, we can find a secondary solution. Then by using Ekeland’s lemma (see [25]), we get a local minim um (\( \to \infty \). Then the lemma follows.

To find a secondary solution, we note that the functional admits a mountain-pass structure. We have shown that, for \( \alpha \) large enough, there exists a solution \((u_\alpha, v_\alpha)\) of (5.6)–(5.7) satisfying the behavior 
\[
e^{u_0+u_\alpha} \to 1, \quad e^{v_\alpha} \to 1,
\]
as \( \alpha \to +\infty \) and this solution is a local minimum for \( I_{\alpha\beta} \) by Lemma [5.7]. Then there exists a positive constant \( \delta_0 > 0 \) such that
\[
I_{\alpha\beta}(u, v) \geq I_{\alpha\beta}(u_\alpha, v_\alpha) \quad \text{if} \quad \|u - u_\alpha\| + \|v - v_\alpha\| \leq \delta_0.
\]

To find a secondary solution, we note that the functional admits a mountain-pass structure. Indeed, noting that \( u_0 + u_\alpha < 0, v_\alpha < 0 \), we have for all \( c < 0 \),
\[
I_{\alpha\beta}(u_\alpha + c, v_\alpha) - I_{\alpha\beta}(u_\alpha, v_\alpha) \leq (4\alpha + \beta)|\Omega| + 4\pi n \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) c.
\]

We consider two situations.

(1) If \((u_\alpha, v_\alpha)\) is not a strictly local minimum for the functional \( I_{\alpha\beta} \), i.e., for all \( 0 < \delta < \delta_0 \),
\[
\inf_{\|u-u_\alpha\|+\|v-v_\alpha\|=\delta} I_{\alpha\beta}(u, v) = I_{\alpha\beta}(u_\alpha, v_\alpha).
\]
Then by using Ekeland’s lemma (see [25]), we get a local minimum \((u_\alpha^\delta, v_\alpha^\delta)\) for \( I_{\alpha\beta} \) such that
\[
\|u_\alpha^\delta - u_\alpha\| + \|v_\alpha^\delta - v_\alpha\| = \delta \quad \text{for any} \quad \delta \in (0, \delta_0).
\]

Of course, we can find a secondary solution.

(2) If \((u_\alpha, v_\alpha)\) is a strictly local minimum for the functional \( I_{\alpha\beta} \), then there exits \( \delta_1 \in (0, \delta_0) \) such that
\[
\inf_{\|u-u_\alpha\|+\|v-v_\alpha\|=\delta_1} I_{\alpha\beta}(u, v) > I_{\alpha\beta}(u_\alpha, v_\alpha).
\]

Moreover, in view of (5.86) we can choose \( |\tilde{c}| > \delta_0 \) such that
\[
I_{\alpha\beta}(u_\alpha + \tilde{c}, v_\alpha) < I_{\alpha\beta}(u_\alpha, v_\alpha) - 1
\]

Now we introduce the paths
\[
\mathcal{P} = \{ \Gamma(t) | \Gamma \in C([0,1]; W^{1,2}(\Omega) \times W^{1,2}(\Omega)) : \Gamma(0) = (u_\alpha, v_\alpha), \ \Gamma(1) = (u_\alpha + \tilde{c}, v_\alpha) \}\n\]

34
and define

\[ m_0 = \inf_{\Gamma \in \mathcal{P}} \sup_{t \in [0,1]} \{ I_{\alpha\beta}(\Gamma(t)) \}. \]

Hence we have

\[ m_0 > I_{\alpha\beta}(u, v). \] (5.87)

Then by Lemma 5.8 we see that the functional \( I_{\alpha\beta} \) satisfies all the hypotheses of the mountain-pass theorem of Ambrosetti–Rabinowitz [3]. Therefore we conclude that \( m_0 \) is a critical value of the functional \( I_{\alpha\beta} \). Noting (5.87), we have an additional solution of the equations (5.6) and (5.7).

Then the proof of Theorem 3.2 is complete.

6 Summary and comments

In this paper, we developed an existence theory for the multiple vortex solutions of the non-Abelian Chern–Simons–Higgs equations in the Gudnason model [27,28] of \( \mathcal{N} = 2 \) supersymmetric field theory where gauge field dynamics is governed by two Chern–Simons terms. In the full-plane situation, we proved the existence of solutions for the general system of nonlinear partial differential equations involving an arbitrary number of unknowns and established the exponential decay estimates for the solutions which realize a family of quantized integrals expressed in terms of vortex numbers. These solutions give rise to multiple vortex field configurations which approach at spatial infinity the vacuum state with completely broken symmetry and known as topological solutions. There are no restrictions to the values of various coupling parameters and vortex numbers or vortex distributions for the existence results to hold. In the doubly periodic domain setting, we confined our study to the case when the vortex equations contain only two unknowns. We derived a necessary condition for the existence of a multiple vortex solution and obtained some sufficient conditions for the existence of two distinct solutions for a given prescribed distribution of vortices. We also established the asymptotic behavior of the solutions as a coupling parameter goes to infinity.

The problems which remain untouched and are of considerable future interest are the existence of nontopological solutions realizing the asymptotic vacuum state with unbroken symmetry and the existence of solutions of the multiple vortex equations involving more than two unknowns. Our methods developed so far are still not sufficiently effective in dealing with these rather challenging problems and new ideas and techniques are called upon in order to make further progress in this area.

References

[1] A. A. Abrikosov, On the magnetic properties of superconductors of the second group, Sov. Phys. JETP 5 (1957) 1174–1182.

[2] O. Aharony, O. Bergman, D. L. Jaferis and J. Maldacena, \( \mathcal{N} = 6 \) superconformal Chern-Simons-matter theories, M2-branes and their gravity duals, J. High Energy Phys. 0810 (2008) 091.

[3] A. Ambrosetti and P. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349–381.

[4] T. Aubin, Nonlinear Analysis on Manifolds: Monge–Ampère Equations, Springer, Berlin and New York, 1982.
[5] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi, and A. Yung, Nonabelian superconductors: vortices and confinement in $\mathcal{N} = 2$ SQCD, Nucl. Phys. B 673 (2003) 187–216.

[6] J. Bagger and N. Lambert, Modeling multiple M2’s, Phys. Rev. D 75 (2007) 045020.

[7] J. Bagger and N. Lambert, Gauge symmetry and supersymmetry of multiple M2-branes, Phys. Rev. D 77 (2008) 065008.

[8] J. Bagger and N. Lambert, Comments on multiple M2-branes, J. High Energy Phys. 0802 (2008) 105.

[9] A. Bezryadina, E. Eugenieva, and Z. Chen, Self-trapping and flipping of double-charged vortices in optically induced photonic lattices, Optics Lett. 31 (2006) 2456–2458.

[10] E. B. Bogomol’nyi, The stability of classical solitons, Sov. J. Nucl. Phys. 24 (1976) 449–454.

[11] L. Caffarelli and Y. Yang, Vortex condensation in the Chern–Simons Higgs model: an existence theorem, Commun. Math. Phys. 168 (1995) 321–336.

[12] D. Chae and O. Yu. Imanuvilov, The existence of nontopological multivortex solutions in the relativistic self-dual Chern–Simons theory, Commun. Math. Phys. 215, 119–142 (2000).

[13] M. Chaichian and N. F. Nelipa, Introduction to Gauge Field Theory, Springer, Berlin and New York, 1984.

[14] H. Chan, C.-C. Fu, and C.-S. Lin, Non-topological multi-vortex solutions to the self-dual Chern–Simons–Higgs equations, Commun. Math. Phys. 231 (2002) 189–221.

[15] R. M. Chen, Y. Guo, D. Spirn, and Y. Yang, Electrically and magnetically charged vortices in the Chern–Simons–Higgs theory, Proc. Roy. Soc. A 465 (2009) 3489–3516.

[16] X. Chen, S. Hastings, J. B. McLeod, and Y. Yang, A nonlinear elliptic equation arising from gauge field theory and cosmology, Proc. Roy. Soc. A 446, 453–478 (1994).

[17] S. S. Chern and J. Simons, Some cohomology classes in principal fiber bundles and their application to Riemannian geometry, Proc. Nat. Acad. Sci. USA 68 (1971) 791–794.

[18] S. S. Chern and J. Simons, Characteristic forms and geometric invariants, Ann. Math. 99 (1974) 48–69.

[19] S. Deser, R. Jackiw, and S. Templeton, Three-dimensional massive gauge theories, Phys. Rev. Lett. 48 (1982) 975–978.

[20] S. Deser, R. Jackiw, and S. Templeton, Topologically massive gauge theories, Ann. Phys. 140 (1982) 372–411.

[21] Q. Du, M. Gunzburger, and J. Peterson, Analysis and approximation of the Ginzburg–Landau model of superconductivity, SIAM Rev. 34 (1992) 54–81.

[22] G. Dunne, Self-Dual Chern–Simons Theories, Lecture Notes in Physics, vol. m 36, Springer, Berlin, 1995.

[23] G. Dunne, R. Jackiw, S.-Y. Pi, and C. Trugenberger, Self-dual Chern–Simons solitons and two-dimensional nonlinear equations, Phys. Rev. D 43 (1991) 1332–1345.

[24] L. Fontana, Sharp borderline Sobolev inequalities on compact Riemannian manifolds, Comment. Math. Helv. 68 (1993) 415–454.

[25] N. Ghousoub, Duality and Perturbation Methods in Critical Point Theory, Cambridge University Press, Cambridge, U. K., 1993.

[26] V. L. Ginzburg and L. D. Landau, On the theory of superconductivity, in Collected Papers of L. D. Landau (edited by D. Ter Haar), pp. 546–568, Pergamon, New York, 1965.
[27] S. B. Gudnason, Non-abelian Chern–Simons vortices with generic gauge groups, *Nucl. Phys. B* **821** (2009) 151–169.

[28] S. B. Gudnason, Fractional and semi-local non-Abelian Chern–Simons vortices, *Nucl. Physics B* **840** (2010) 160–185.

[29] A. Gustavsson, Algebraic structures on parallel M2-branes, *Nucl. Phys. B* **811** (2009) 66–76.

[30] X. Han and G. Tarantello, Doubly periodic self-dual vortices in a relativistic non-Abelian Chern–Simons model, Calc. Var. and PDE, DOI 10.1007/s00526-013-0615-7.

[31] A. Hanany, M. J. Strassler, and A. Zaffaroni, Confinement and strings in MQCD, *Nucl. Phys. B* **513** (1998) 87–118.

[32] A. Hanany and D. Tong, Vortices, instantons and branes, *J. High Energy Phys.* **0307** (2003) 037.

[33] J. Hong, Y. Kim and P.-Y. Pac, Multivortex solutions of the Abelian Chern–Simons–Higgs theory, *Phys. Rev. Lett.* **64** (1990) 2330–2333.

[34] P. A. Horvathy and P. Zhang, Vortices in (Abelian) Chern–Simons gauge theory, *Phys. Rep.* **481** (2009) 83–142.

[35] K. Huang, *Quarks, Leptons, and Gauge Fields*, 2nd ed., World Scientific, Singapore, 1992.

[36] S. Inouye, S. Gupta, T. Rosenband, A. P. Chikkatur, A. Görlitz, T. L. Gustavson, A. E. Leanhardt, D. E. Pritchard and W. Ketterle, Observation of vortex phase singularities in Bose-Einstein condensates, *Phys. Rev. Lett.* **87** (2001), 080402.

[37] R. Jackiw, S.-Y. Pi, and E. J. Weinberg, Topological and non-topological solitons in relativistic and non-relativistic Chern–Simons theory, *Particles, Strings and Cosmology* (Boston, 1990), pp. 573–588, World Sci. Pub., River Edge, NJ, 1991.

[38] R. Jackiw and S. Templeton, How super-renormalizable interactions cure their infrared divergences, *Phys. Rev. D* **23** (1981) 2291–2304.

[39] R. Jackiw and E. J. Weinberg, Self-dual Chern–Simons vortices, *Phys. Rev. Lett.* **64** (1990) 2334–2337.

[40] A. Jaffe and C. H. Taubes, *Vortices and Monopoles*, Birkhäuser, Boston, 1980.

[41] B. Julia and A. Zee, Poles with both magnetic and electric charges in non-Abelian gauge theory, *Phys. Rev. D* **11** (1975) 2227–2232.

[42] Y. Kawaguchi and T. Ohmi, Splitting instability of a multiply charged vortex in a Bose–Einstein condensate, *Phys. Rev. A* **70** (2004) 043610.

[43] D. I. Khomskii and A. Freimuth, Charged vortices in high temperature superconductors, *Phys. Rev. Lett.* **75** (1995) 1384–1386.

[44] C. N. Kumar and A. Khare, Charged vortex of finite energy in nonabelian gauge theories with Chern–Simons term, *Phys. Lett. B* **178** (1986) 395–399.

[45] C.-S. Lin, J. Wei, and D. Ye, Classification and nondegeneracy of SU(n + 1) Toda system with singular sources, *Invent. Math.* **190** (2012) 169–207.

[46] A. Marshakov and A. Yung, Non-Abelian confinement via Abelian flux tubes in softly broken N = 2 SUSY QCD, *Nucl. Phys. B* **647** (2002) 3–48.

[47] Y. Matsuda, K. Nozakib, and K. Kumagaib, Charged vortices in high temperature superconductors probed by nuclear magnetic resonance, *J. Phys. Chem. Solids* **63** (2002) 1061–1063.

[48] M. Nolasco and G. Tarantello, Vortex condensates for the SU(3) Chern–Simons theory, *Commun. Math. Phys.* **213** (2000) 599–639.
[49] M. Nolasco and G. Tarantello, On a Sharp Sobolev-type Inequality on Two-Dimensional Compact Manifolds, *Arch. Rational Mech. Anal.* **145** (1998) 161–195.

[50] M. Nolasco and G. Tarantello, Double vortex condensates in the Chern-Simons-Higgs theory, *Calc. Var. and PDE* **9** (1999), 31–94.

[51] S. Paul and A. Khare, Charged vortices in an Abelian Higgs model with Chern–Simons term, *Phys. Lett. B* **17** (1986) 420–422.

[52] M. K. Prasad and C. M. Sommerfield, Exact classical solutions for the ’t Hooft monopole and the Julia–Zee dyon, *Phys. Rev. Lett.* **35** (1975) 760–762.

[53] L. H. Ryder, *Quantum Field Theory*, 2nd edition, Cambridge U. Press, London, 1996.

[54] J. S. Schonfeld, A massive term for three-dimensional gauge fields, *Nucl. Phys. B* **185** (1981) 157–171.

[55] N. Seiberg and E. Witten, Monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory, *Nucl. Phys. B* **426** (1994) 19–52. Erratum – *ibid.* **B** **430** (1994) 485–486.

[56] S. I. Shevchenko, Charged vortices in superfluid systems with pairing of spatially separated carriers, *Phys. Rev. B* **67** (2003) 214515.

[57] M. Shifman, and A. Yung, Supersymmetric solitons and how they help us understand non-Abelian gauge theories *Rev. Mod. Phys.* **79** (2007) 1139.

[58] M. Shifman and A. Yung, *Supersymmetric Solitons*, Cambridge U. Press, Cambridge, U. K., 2009.

[59] J. B. Sokoloff, Charged vortex excitations in quantum Hall systems, *Phys. Rev. B* **31** (1985) 1924–1928.

[60] J. Spruck and Y. Yang, Topological solutions in the self-dual Chern–Simons theory: existence and approximation, *Ann. Inst. H. Poincaré – Anal. non linéaire*, **12**, (1995) 75–97.

[61] J. Spruck and Y. Yang, The existence of non-topological solitons in the self-dual Chern–Simons theory, *Commun. Math. Phys.* **149**, 361–376 (1992).

[62] J. Spruck and Y. Yang, Proof of the Julia–Zee theorem, *Commun. Math. Phys.* **291** (2009) 347–356.

[63] G. Tarantello, Multiple condensate solutions for the Chern–Simons–Higgs theory, *J. Math. Phys.* **37** (1996) 3769–3796.

[64] G. Tarantello, *Self-Dual Gauge Field Vortices, an Analytic Approach*, Progress in Nonlinear Differential Equations and Their Applications **72**, Birkhäuser, Boston, Basel, Berlin, 2008.

[65] C. H. Taubes, Arbitrary N-vortex solutions to the first order Ginzburg–Landau equations, *Commun. Math. Phys.* **72** (1980) 277–292.

[66] H.J. de Vega and F. Schaposnik, Electrically charged vortices in non-Abelian gauge theories with Chern–Simons term, *Phys. Rev. Lett.* **56** (1986) 2564–2566.

[67] H.J. de Vega and F. Schaposnik, Vortices and electrically charged vortices in non-Abelian gauge theories, *Phys. Rev. D*, **34** (1986) 3206–3213.

[68] A. Vilenkin and E. P. S. Shellard, Cosmic Strings and Other Topological Defects, Cambridge U. Press, Cambridge, 1994.

[69] R. Wang, The existence of Chern–Simons vortices, *Commun. Math. Phys.* **137** (1991) 587–597.

[70] S. Wang and Y. Yang, Abrikosov’s vortices in the critical coupling, *SIAM J. Math. Anal.* **23** (1992) 1125–1140.

[71] Y. Yang, Obstructions to the existence of static cosmic strings in an Abelian Higgs model, *Phys. Rev. Lett.* **73**, 10–13 (1994).
[72] Y. Yang, Prescribing topological defects for the coupled Einstein and Abelian Higgs equations, *Commun. Math. Phys.* **170**, 541–582 (1995).

[73] Y. Yang, The relativistic non-Abelian Chern–Simons equations, *Commun. Math. Phys.* **186** (1997) 199–218.

[74] Y. Yang, *Solitons in Field Theory and Nonlinear Analysis*, Springer-Verlag, New York, 2001.