OPTIMAL CUSTOMER BEHAVIOR IN OBSERVABLE AND UNOBSERVABLE DISCRETE-TIME QUEUES

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Abstract. This paper studies the effect of information suppression on Naor’s model as well as on Edelson and Hildebrand’s model under geometric distribution. We set the suitable non-cooperative games and search for their Nash equilibria under the observable and unobservable system. In each case, we analyze the effects of information level on the customers’ equilibrium and socially optimal balking strategies as well as on the profit maximization of the system manager. The socially optimal behavior and the inefficiency of the equilibrium strategies are quantified via the price of anarchy measure. We discuss a comparison study of the profit maximization and social welfare under an imposed admission fee. Also, the impact of information on the selfish and social optimal joining rates is examined. Numerical results are presented to exemplify the impact of system parameters on the optimal behavior of customers under different information levels.

1. Introduction. In the last decades, queueing systems with the game-theoretic approach and strategic behavior of customers have been given considerable attention. The economic analysis with strategic customers is an emerging field in the study of queueing theory, complementing initial analysis in queueing systems that involved the performance measure of systems with non-deciding customers. In such a study, a certain reward-cost structure is enforced on a system that measures the customers wish for service and their reluctance for waiting. Giving customers a choice to determine whether to join or balk permits us to deal with the circumstance as a non-cooperative game, with the relevant players taking decisions with the objective of maximizing individual expected profit. Customers act strategically and enter the queue only when the expected waiting cost is less than the reward received upon being served. This behavior is studied as a game among the potential customers, and it helps to determine the corresponding equilibrium strategy. Concerning the information that is available to the customers, broadly two types

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of models observable and unobservable have been studied in the literature. Extensive literature related to continuous-time queueing model with the game theoretic approach reported in the monograph of Hassin and Haviv [14] and a recent book of Hassin [13].

Naor [19] first developed the well known M/M/1 queue under an economic framework using a linear cost-reward structure. In his study, an arriving customer is aware of the number of customers in the system before deciding on joining or balking the system and referred to as the observable model. This study was complemented by Edelson and Hildebrand [6], who analyzed the corresponding unobservable M/M/1 system, where customers make decisions without knowing the state of the system. Consequently, any information available involving the state of the system influences their joining/balking decision. In both the models, a toll is imposed to induce the socially desired behavior among customers, otherwise their selfish behavior will result in longer queues than are socially desired. Hassin [12] did the first fundamental work on information in Naor’s model. He studied the effect of suppressing queue length information on the social optimization and revenue maximization. The advantages of information suppression over price regulation when it can not be controlled socially is shown. Chen and Frank [4] studied the effect of information suppression on the system throughput under a fixed admission fee. They observed that the decision on information suppression depends strongly on the service value relative to waiting cost. Shone et al. [23] compared the observable and unobservable models concerning the individually and socially optimal arrival rates. They showed that the decision of suppressing queue length information may or may not affect the effective arrival rates. A detailed survey on such works on information suppression can be found in Ibrahim [16] and chapter 3 of Hassin [13]. From the viewpoint of the customers as well as of the service provider, the impact of the information is of concern, since their decisions may vary when studying such models. The literature related to equilibrium strategies in Markovian queues with various levels of information available to customers has several variants, such as vacation models in [3, 11, 24, 9], impatient customers in [21, 22], the price of anarchy in [8, 5], an unreliable server in [2, 17] and many others.

The analysis of discrete-time queueing systems has received extensive attention due to the wide applicability in various fields of computer networks and communications systems, including time-division multiple access, asynchronous transfer mode multiplexers in the broadband integrated services digital network, and slotted carrier-sense multiple access protocols. The benefit of studying the discrete-time queues is that one may find the continuous-time results as a limiting case and also model the events of arrivals and departures simultaneously. In the case of simultaneity, their order may be taken care of as either by late arrival systems with delayed access (LAS-DA) or by early arrival systems (EAS). The details on the LAS-DA and EAS systems are available in [15]. Although significant progress has been made in the continuous-time queues with strategic customers, discrete-time counterparts have got less attention because of the complexities involved in their steady-state analysis that arises due to the occurrences of simultaneous arrivals and departures at slot boundaries. It appears that the strategic customer behavior in basic discrete-time Geo/Geo/1 queues has received only limited attention, see [18, 26, 25, 7, 1] and the references therein. These works emphasize the equilibrium and social balking strategies as individual entities under different service situations. In most of the works, non-optimal behavior of selfish individuals is intrinsic under
equilibrium. This inefficiency in the individual behavior under equilibrium can be dealt with the help of the price of anarchy (PoA). Recently, Goswami and Panda [10] studied partial information disclosure policy in a discrete time set up and made a comparison of the policies with respect to revenue maximization. Panda and Goswami [20] studied the equilibrium and social behavior of customers in a general bulk service discrete-time queue. The present work is different from [20] in several ways. First, we measure the inefficiency of the social behavior under Nash equilibrium and quantify it with the help of price of anarchy. Second, we study the profit maximization problem from the system manager’s point of view and advise to impose a suitable entry fee on all joining customers for optimal profit. Third, we compare the results between observable and unobservable models and suggest to utilize the level of information for social optimization of the resources as well as revenue maximization of the manager, depending on the system congestion. Although there are some works on the discrete time queues with strategic customers, but the authors did not find any work on the effect of suppressing information on the social optimization and revenue maximization in a discrete time queueing system.

In the present study, the effect of information on Naor’s model as well as Edelson and Hildebrand’s model under a discrete-time set up is analyzed. An attempt has been made with the help of PoA to measure the extent to which the cost of the Nash equilibrium might exceed that of the optimal solution. To the best of authors’ knowledge, this work is the first to study information effect on Naor’s model under geometric distribution.

The main contributions of the present study can be summarized as follows:

• This is the first time that Naor’s [19] (and Edelson and Hildebrand’s [6]) model is solved under geometric distribution.
• PoA is a multi-modal function of the service rate. In the observable case, it is concave downward and bounded below by one, whereas in the unobservable case, it is unity when equilibrium and optimal joining strategies match, otherwise it is infinity.
• The socially optimal behavior can be induced by imposing an admission fee.
• Informed customers’ optimal joining rate is higher as compared to the uninformed customers for smaller values of service valuation and uninformed customers decide to follow the crowd. It helps the social planner to control the arrival of customers in both cases by selecting the level of information to be revealed from time to time.

The rest of this paper is organized as follows. Section 2 presents the mathematical description of the discrete-time model. Sections 3 and 4 analyze the Geo/Geo/1 queue with observable and unobservable policy, respectively. We obtain the equilibrium threshold strategy, social balking strategy, and price of anarchy for both the policies. Section 5 illustrates the numerical results showing the effect of system parameters on the equilibrium and social optimal behavior of customers and profit of the manager. Section 6 concludes the paper.

2. System model. Consider a discrete-time single server queueing system. Let the time axis be divided into fixed size slots of equal intervals as 0, 1, . . . , t, . . . . A potential arrival occurs in the slot \((t^- , t)\) and a potential departure takes place in the slot \((t, t^+)\). Even if a customer arrives in the interval \((t^-, t)\) and finds the server to be idle, his service will not start until the time interval \((t, t^+)\) is reached, i.e., we consider a late arrival system with delayed access (LAS-DA). For details on the
LAS-DA concept, readers may refer to Hunter [15]. The various time epochs at which events take place are described in Fig. 1. The state of the system changes only around the slot boundaries, that is, both arrivals and departures are possible only at slot boundaries. Further, we presume that arrival within \((0^-, 0)\) and departure in \((0, 0^+)\) are not possible. The inter-arrival times are independent and geometrically distributed with probability mass function (p.m.f.)

\[
P(A = n) = \bar{\lambda}^{n-1}\lambda, \quad n \geq 1, \quad 0 < \lambda < 1,
\]

where the random variable \(A\) is the generic of inter-arrival times and \(\bar{x} = 1 - x\) for any real number \(x \in [0, 1]\). Here, \(\lambda (\bar{\lambda})\) is the probability of a potential arrival (no arrival) in every slot. The service times \((S)\) are also independent and geometrically distributed with p.m.f.

\[
P(S = n) = \bar{\mu}^{n-1}\mu, \quad n \geq 1, \quad 0 < \mu < 1,
\]

where \(\mu\) is the probability that only one customer departs in a slot and \(\bar{\mu}\) is the complementary probability with no customer departure in a slot. The waiting customers are served following the first-come, first-served (FCFS) service discipline. The maximum system load \(\rho = \lambda/\mu\) is always positive. For the existence of steady-state distribution, the effective arrival rate (joining rate) must be smaller than the service rate \(\mu\). The inter-arrival and service times are considered to be mutually independent. The above discrete-time queueing system may be represented as a Geo/Geo/1 queue under the LAS-DA policy.

The problem associated in queueing systems with strategic behavior is that an arriving customer decides whether to enter or to balk depending on his expected waiting cost and the reward received after service completion. Individual customers’ joining decisions affect the waiting delay and thus the benefits of all other customers. The behavior of an arriving customer is considerably dependent on the information/knowledge about the system-length at the instant of arrival. We assume that a fixed reward \(R\) is gained by each customer after their service completion and at the same time they are charged a waiting cost of \(C\) units per time unit to remain in the system. Using a linear cost-reward function, we study a customer’s expected net benefit, defined as \(R - CT(n)\) in the observable case and \(R - CW\) in the unobservable case, where \(T(n)\) denotes the mean sojourn time of a tagged customer that encounters \(n\) customers in the system on joining and \(W\) represents the mean sojourn time of an arriving customer in the unobservable case. When an arriving customer encounters an empty system, he will join if his net benefit is positive, that is, \(R > C\mu\). Define \(\nu_s = \frac{R\mu}{C}\), and the above criterion becomes \(\nu_s > 1\). This criterion guarantees that the reward for service exceeds the expected cost for an arriving customer who encounters an empty system. In case of equality, customers decision is to
join the system. Customers desire to maximize their expected net benefit by taking decisions only at their arrival instants. The decisions of customers are irrevocable in the sense that retrials of balking customers and reneging of entering customers are not allowed. We assume that the balking customers will get no reward. The customers are assumed to be identical and will enter the queue if and only if the reward is more than their expected sojourn time.

3. Observable Geo/Geo/1 queue. In this section, we consider an observable discrete-time queueing model wherein customers upon arrival are informed of the present system-length. This information will help in their decision making to join or balk the system. We are interested in the equilibrium solution of the model under individual and social optimization. Here, the (Nash) equilibrium solution is a pure strategy (join or balk) of threshold type, i.e., there exists a positive integer $n_e$ such that an arriving customer will join the system if and only if the system-length upon arrival is smaller than $n_e$. This pure threshold strategy is a dominant one in the sense that it is the best response against any other strategies. Alternatively it maximizes the customer’s net benefit irrespective of the strategies adopted by other customers. The socially optimal solution is also a pure threshold type strategy. Before the equilibrium threshold analysis, we need the steady-state solution of the underlying observable discrete-time queue.

Let $L_t$ be the number of customers in the system at time $t^+$. According to the assumptions in the LAS-DA model, a customer who finishes service and leaves at $t^+$ is not counted in $L_t$ while the one arriving at $t^-$ is counted in $L_t$. The derivation of the system-length process $\{L_t, t \geq 0\}$ builds a discrete-time Markov chain with state space $\{0, 1, 2, \ldots, n_e\}$. This model can be represented by the finite buffer discrete-time Geo/Geo/1/$n_e$ queue and its state transition diagram is depicted in Fig. 2. Let us define the steady-state probability that there are $k$ customers in the system by $P_0 = \frac{\lambda}{\lambda + \mu} P_0 + \frac{\lambda \mu}{\lambda + \mu} P_1$, $P_1 = \lambda P_0 + (\lambda \bar{\mu} + \lambda \mu) P_1 + \bar{\lambda} \bar{\mu} P_2$, $P_n = \lambda \bar{\mu} P_{n-1} + (\lambda \bar{\mu} + \lambda \mu) P_n + \lambda \mu P_{n+1}$, $2 \leq n \leq n_e - 2$, $P_{n_e-1} = \lambda \bar{\mu} P_{n_e-2} + (\lambda \bar{\mu} + \lambda \mu) P_{n_e-1} + \mu P_{n_e}$, $P_{n_e} = \lambda \bar{\mu} P_{n_e-1} + \bar{\mu} P_{n_e}$.

Using (1a) - (1d) recursively, the steady-state probabilities are obtained as $P_i = \left\{ \begin{array}{ll} P_0, & \text{for } 1 \leq i \leq n_e - 1, \\ \frac{r^i}{1-\mu} P_0, & \text{for } i = n_e, \end{array} \right.$
where \( r = \frac{\lambda(1-\mu)}{(1-\mu)\mu} \). Using the normalization condition \( \sum_{n=0}^{\bar{n}_e} \bar{P}_n = 1 \), we get the probability of an empty (or idle) system, as

\[
P_0 = \frac{1 - \rho}{1 - \rho^2 r^{\bar{n}_e}} = \frac{\bar{\mu}(\mu - \bar{\lambda})}{\mu \bar{\mu} - \lambda \bar{\lambda} r^{\bar{n}_e}}.
\]

The expected number of customers entering the system in unit time is

\[
\lambda(1 - P_{n_e}) = \lambda \left[ 1 - \frac{\bar{\lambda}(\mu - \lambda) r^{n_e}}{\mu \bar{\mu} - \lambda \bar{\lambda} r^{n_e}} \right] = \frac{\lambda \mu (\bar{\mu} - \bar{\lambda} r^{n_e})}{\mu \bar{\mu} - \lambda \bar{\lambda} r^{n_e}}.
\] (2)

The mean system-length when customers follow the joining threshold \( n_e \), is

\[
L(n_e) = \sum_{i=1}^{n_e} i \bar{P}_i = \lambda \bar{\lambda} \left[ \frac{1}{\mu - \lambda} - \frac{r^{n_e}(\mu \bar{\mu} - \lambda \bar{\lambda})}{(\mu - \lambda)(\mu \bar{\mu} - \lambda \bar{\lambda} r^{n_e})} - \frac{n_e r^{n_e}}{\mu \bar{\mu} - \lambda \bar{\lambda} r^{n_e}} \right].
\] (3)

**Remark 1.** The steady-state probabilities are derived with the assumption that \( \lambda \neq \mu \). The steady-state probabilities can be similarly derived under the equality condition \( \lambda = \mu \). For the sake of completeness, we present the probabilities only and corresponding performance measures can be computed in a similar manner to the inequality case,

\[
P_i = \begin{cases} \bar{\mu} P_0, & \text{for } i = 1, 2, \ldots, n_e - 1, \\ P_0, & \text{for } i = n_e, \end{cases}
\]

where

\[
P_0 = \frac{\bar{\mu}}{2\bar{\mu} + n_e - 1}.
\]

3.1. **Equilibrium behavior.** If the tagged customer encounters \( n \) customers in the system just before his arrival epoch, then his mean sojourn time will be the sum of \( n + 1 \) regular mean service times with rate \( \mu \).

\[
T(n) = \frac{n + 1}{\mu}, \quad n = 0, 1, 2, \ldots.
\]

Now, the expected net benefit of the tagged customer after his service completion can be expressed as \( \Delta(n) = R - CT(n) \). Thus, the tagged customer will join the queue if and only if \( \Delta(n) \geq 0 \). In the observable queues, customers who wish to maximize their net benefit will follow a pure threshold strategy, that is, there exists a positive integer \( n_e \) such that the future arriving customers will enter the system if the number of customers present in the system is smaller than \( n_e \). When arriving customers observe \( n_e \) or more, they prefer to balk as their net benefit becomes negative. Thus, the pure threshold \( n_e \) can be obtained by solving the inequalities \( \Delta(n_e) \geq 0 \), and \( \Delta(n_e + 1) < 0 \). Combining the two inequalities, we get

\[
n_e \leq \frac{R \mu}{C} < n_e + 1,
\]

or equivalently,

\[
n_e = \left\lfloor \frac{R \mu}{C} \right\rfloor,
\]

where \( \lfloor x \rfloor \) denotes the greatest integer less than or equal to \( x \). Here, the equilibrium balking threshold depends on the service rate, waiting cost and the service completion reward, but not on the actual arrival rate.
3.2. Socially optimal behavior. In this section, we are interested to find a
threshold which maximizes the overall social benefit of all customers in the system.
Consider an arbitrary threshold strategy \( n \). Customers upon arrival will balk if they find \( n \) customers in the system, otherwise will join. The probability of observing \( n \) customers in the system is \( P_n \), which is same as the balking probability. The probability that an arrival joins will be \( 1 - P_n \). Due to the BASTA property, the probability that an arrival encounters the system on state \( n \) and balks is \( P_n \). Hence, the social benefit per time unit when all present customers follow the equilibrium threshold policy “while arriving at time \( t \), enter if \( L_t < n \) and balk otherwise” is given by

\[
\Delta_s(n) = \lambda(1 - P_n) R - C L(n),
\]

where \( L(n) \) can be obtained from (3). On simplification, the expected total profit under threshold strategy \( n \) is given by

\[
\Delta_s(n) = \lambda R \left( 1 - \frac{\bar{\lambda}(\mu - \lambda) n}{\mu \bar{\mu} - \lambda \lambda r^n} \right) - C \bar{\lambda} \left[ \frac{1}{\mu - \lambda} - \frac{r^n(\mu \bar{\mu} - \lambda \lambda) n}{\mu \bar{\mu} - \lambda \lambda r^n} \right].
\]

It can be established that the function \( \Delta_s \)'s dependence on \( n \) is discretely unimodal with strategy \( n_o \), that is, it is increasing in \( \{0, 1, \ldots, n_o\} \) and decreasing in \( \{n_o, n_o + 1, n_o + 2, \ldots\} \). It is related with two inequalities

\[
\lambda R \left[ \frac{\bar{\lambda}(\mu - \lambda) r^{n_o}}{\mu \bar{\mu} - \lambda \lambda r^{n_o}} - \frac{\bar{\lambda}(\mu - \lambda) r^{n_o+1}}{\mu \bar{\mu} - \lambda \lambda r^{n_o+1}} \right] - C \bar{\lambda} \left[ \frac{r^{n_o}(\mu \bar{\mu} - \lambda \lambda) n}{\mu \bar{\mu} - \lambda \lambda r^{n_o}} + \frac{n_o r^{n_o}}{\mu \bar{\mu} - \lambda \lambda r^{n_o+1}} \right] < 0,
\]

and

\[
\lambda R \left[ \frac{\bar{\lambda}(\mu - \lambda) r^{n_o-1}}{\mu \bar{\mu} - \lambda \lambda r^{n_o-1}} - \frac{\bar{\lambda}(\mu - \lambda) r^{n_o}}{\mu \bar{\mu} - \lambda \lambda r^{n_o}} \right] - C \bar{\lambda} \left[ \frac{r^{n_o-1}(\mu \bar{\mu} - \lambda \lambda) n}{\mu \bar{\mu} - \lambda \lambda r^{n_o-1}} + \frac{(n_o - 1) r^{n_o-1}}{\mu \bar{\mu} - \lambda \lambda r^{n_o-1}} \right] > 0.
\]

Simplifying the above inequalities, we get equivalent inequalities as

\[
R(1 - \rho)^2 < \frac{\bar{C}}{\mu} \left[ (n_o + 1)(1 - \rho) - \mu(1 - \rho) - \bar{\lambda}(1 - \rho r^{n_o}) \right]
\]

and

\[
R(1 - \rho)^2 > \frac{\bar{C}}{\mu} \left[ n_o(1 - \rho) - \mu(1 - \rho) - \bar{\lambda}(1 - \rho r^{n_o-1}) \right]
\]

The inequalities (5) and (6) may be put into the following form

\[
\frac{n_o(1 - \rho) - \mu(1 - \rho) - \bar{\lambda}(1 - \rho r^{n_o-1})}{(1 - \rho)^2} \leq \frac{R \mu}{\bar{C}} \leq \frac{(n_o + 1)(1 - \rho) - \mu(1 - \rho) - \bar{\lambda}(1 - \rho r^{n_o})}{(1 - \rho)^2}.
\]

To deal with (7), it is suitable to consider a function \( g(\nu) \) of two independent variables \( \rho > 0 \) and \( \nu \geq 1 \) as

\[
g(\nu) = \frac{\nu(1 - \rho) - \mu(1 - \rho) - \bar{\lambda}(1 - \rho \nu^{-1})}{(1 - \rho)^2},
\]

then \( n_o = \lfloor \nu^* \rfloor \), where \( \nu^* \) is the unique solution to

\[
g(\nu) = R \mu / \bar{C}.
\]
Putting the value of $\rho$ arbitrary but fixed, it is noted that $g(\nu)$ is an increasing function of $\nu$. Therefore, the integers between which $\nu^*$ lies will follow the inequalities related with (7). Further, simplification shows that $n_o \leq n_e$.

3.3. Price of anarchy. Price of anarchy (PoA) is the ratio of optimal and equilibrium social benefit. It measures the degree to which non-cooperation estimates cooperation. It is also, often used to measure the efficiency of a system degradation due to selfish behaviour of its customers. It is given as

$$
PoA(\rho, \nu_s) = \frac{\mu(\bar{\lambda}r^{n_o}) - \frac{\bar{\lambda}}{\nu_s}}{\frac{\mu(\bar{\lambda}r^{(\nu_s)})}{\mu(\mu-\lambda r^{(\nu_s)})} - \frac{\bar{\lambda}}{\nu_s}} \left\{ \frac{1}{\mu-\lambda} - \frac{r^{n_o}(\mu-\nu_s)}{(\mu-\lambda)(\mu-\lambda r^{(n_o)})} - \frac{n_o r^{n_o}}{\mu-\lambda r^{(n_o)}} \right\}.$$

$$
= \frac{P_{\text{join}}(n_o) - \frac{\bar{\lambda}}{\nu_s} L(n_o)}{P_{\text{join}}(n_e) - \frac{\bar{\lambda}}{\nu_s} L(n_e)} \quad (\nu_s = R\mu/C) \quad (8)
$$

where $P_{\text{join}}(n)$ and $L(n)$ are the probability that an arriving customer joins the system and the expected system-length under a threshold policy $n$, respectively.

3.4. Profit maximization. In the above subsections, it is observed that individual optimization of selfish customers leads to a system with higher joining threshold in compare with the joining threshold in case of social optimization. To reduce the gap between the thresholds $n_e$ and $n_o$, the system manager can use the following policies: the waiting cost can be increased by imposing a toll, or the service reward can be decreased by imposing an appropriate admission fee. Let us assume that the system manager levies an entry fee $q$ on each joining customer. The manager’s objective is to find a suitable threshold that maximizes his profit. The model presumes that the entry fee $q$ is declared and customers take their decision, whether to join or not on the basis of this fee. Therefore, a customer who sees $n$ customers in the system joins only if the reward $R$ is greater than or equal to his expected full price $q + C\frac{n+1}{n}$. Alternatively this behavior is similar to that of customers calculating their service execution by $R - q$ instead of $R$. Considering the fee $q$, the maximal acceptable number of customers in the queue is

$$
n = \left\lfloor \frac{(R - q)\mu}{C} \right\rfloor.
$$

Here, $n$ is the threshold policy followed by the customers and $P_n$ is the probability of finding $n$ customers in the system under this threshold policy. The effective arrival rate is $\lambda(1 - P_n)$. Therefore, the system manager’s profit per time unit is

$$
Z(n) = \lambda(1 - P_n)q.
$$

The manager’s objective is to maximize his profit by choosing a maximum possible entry fee under the threshold policy $n$. Under this threshold policy, the manager may select the maximal entry fee that obeys with the threshold $n$. Thus,

$$
q(n) = R - \frac{Cn}{\mu}. \quad (9)
$$

The price $q$ given in (9) is such that an arrival who sees $n - 1$ customers in the system is indifferent between balking or joining. This price is optimal simply under the presumption that customers in favor of joining break ties. Otherwise, it is preferable to ask a price little less than $q$, and precisely no optimal price exists. So, instead of finding the optimal entry fee $q$, we are interested in the threshold
that maximizes the manager’s profit. Using (9) and (2) for a given threshold \( n \), manager’s profit per time unit is

\[
Z_o(n) = \frac{\lambda (\bar{\mu} - \lambda r^n)}{\mu \bar{\mu} - \lambda \lambda r^n} \left( R - \frac{Cn}{\mu} \right) = R \frac{\lambda (\bar{\mu} - \lambda r^n)}{\mu \bar{\mu} - \lambda \lambda r^n} \left( \nu_s - n \right),
\]

where \( \nu_s = \frac{n \mu}{C} \). A profit-maximizing threshold will satisfy the following condition:

\[
Z_o(n - 1) < Z_o(n) \leq Z_o(n + 1).
\]

Putting the value from (10) in the first condition, we have

\[
\frac{\bar{\mu} - \lambda r^n}{\mu \bar{\mu} - \lambda \lambda r^n} (\nu_s - n) > \frac{\bar{\mu} - \lambda r^{n+1}}{\mu \bar{\mu} - \lambda \lambda r^{n+1}} (\nu_s - n + 1),
\]

or

\[
(\nu_s - n) > \frac{\mu (\bar{\mu} - \lambda r^{n+1})(\mu \bar{\mu} - \lambda \lambda r^n)}{\bar{\mu} r^{n+1}(\mu - \lambda)^2}. \]

Similarly simplifying the second condition, we get

\[
(\nu_s - n - 1) \leq \frac{\mu (\bar{\mu} - \lambda r^n)(\mu \bar{\mu} - \lambda \lambda r^{n+1})}{\bar{\mu} r^n(\mu - \lambda)^2}.
\]

These two cases may be summed up to

\[
n + \frac{\mu (\bar{\mu} - \lambda r^{n-1})(\mu \bar{\mu} - \lambda \lambda r^n)}{\bar{\mu} r^{n-1}(\mu - \lambda)^2} \leq \nu_s < n + 1 + \frac{\mu (\bar{\mu} - \lambda r^n)(\mu \bar{\mu} - \lambda \lambda r^{n+1})}{\bar{\mu} r^n(\mu - \lambda)^2}. \tag{11}
\]

Define \( h(\nu) = \nu + \frac{\mu (\bar{\mu} - \lambda r^{n-1})(\mu \bar{\mu} - \lambda \lambda r^n)}{\bar{\mu} r^{n-1}(\mu - \lambda)^2} \). Clearly, \( h(\nu) \) is an increasing function of \( \nu \). Therefore, \( \nu_s = h(\nu) \) has a unique solution, say \( \nu_m \). Now, \( n_m = \lfloor \nu_m \rfloor \) satisfies the optimal conditions of (11) uniquely. Thus, the manager’s profit will be maximum if all arriving customers follow the threshold strategy \( n_m \) and the profit-maximizing entry fee is \( q_m = R - n_m \frac{C}{\mu} \). The manager’s profit per time unit under this entry fee is

\[
Z_o = \lambda R \frac{\mu (\bar{\mu} - \lambda r^{n_m})}{\mu \bar{\mu} - \lambda \lambda r^{n_m}} \left( 1 - \frac{C n_m}{\mu} \right).
\]

It is instantaneous that \( \nu_s \geq \nu_m \), so that \( n_c \geq n_m \). Thus, individual optimization of selfish customers leads to longer queues than that of the profit maximization of the system manager.

4. Unobservable Geo/Geo/1 queue. In the unobservable queues, customers are not provided with any information about the number of customers present in the system upon their arrival. As the optimal decision of a customer depends on the strategies followed by other customers and the customers are homogeneous, the situation is like a symmetric game. The strategies available to the customers are, either to join or to balk, which are pure strategies and a mixed strategy is a probability distribution one uses to randomly choose among the pure strategies. Since probabilities are continuous, there are infinitely many mixed strategies available to a player. We are interested to find the optimal mixed strategy in the individual and social optimization cases. Here we consider the mixed strategy as the probability of joining \( f \), \( 0 \leq f \leq 1 \). If all the customers follow the same equilibrium strategy \( f \), the evolution of the system-length process forms a discrete-time Markov chain with effective arrival rate \( \lambda f \) and state space \( \{0, 1, 2, \ldots\} \). The stability condition for the underlying queueing model is \( \lambda f < \mu \). The state transition diagram is illustrated.
in Fig. 3. The stationary system-length distributions can be obtained by solving the following set of balance equations.

\[ \begin{align*}
    0 & \quad \lambda_f P_0 = \lambda_f \mu P_1, \\
    & \quad (\lambda_f \mu + \lambda_f \bar{\mu}) P_1 = \lambda_f P_0 + \lambda_f \mu P_2, \\
    & \quad (\lambda_f \mu + \lambda_f \bar{\mu}) P_n = \lambda_f P_{n-1} + \lambda_f \mu P_{n+1}, \quad n \geq 2,
\end{align*} \]

Solving (12a) - (12c), recursively, we get

\[ P_n = r^n P_0, \quad n \geq 1, \quad r = \frac{\lambda_f (1 - \mu)}{(1 - \lambda_f) \mu}, \]

Finally, \( P_0 \) can be obtained using the normalization condition \( \sum_{n=0}^{\infty} P_n = 1 \),

\[ P_0 = 1 - \frac{\lambda_f}{\mu}. \]

The average number of customers in the system is \( L(f) = \sum_{n=1}^{\infty} n P_n = \frac{\lambda_f \overline{X}}{\mu - \lambda_f} \).

Using Little’s rule, the mean sojourn time can be computed to

\[ W(f) = \frac{L(f)}{\lambda_f} = \frac{\overline{X}}{\mu - \lambda_f}. \]

Differentiating \( W(f) \) with respect to \( f \), we have \( W'(f) = \frac{\lambda_f}{(\mu - \lambda_f)^2} > 0 \). Thus, \( W(f) \) is a strictly increasing function of \( f \) for \( f \in [0, 1] \).

4.1. **Equilibrium behavior.** We are interested in the equilibrium joining behavior of an arbitrary tagged customer when all other customers follow the same strategy. Let \( \Delta(f) \) denote the expected net benefit of the tagged customer when all other customers follow the mixed joining strategy \( f \). When \( \lambda_f < \mu \) and all customers follow the mixed joining strategy \( f \), the tagged customer’s net benefit from joining is \( \Delta(f) = R - C \frac{\overline{X}}{\mu - \lambda_f} \). If \( \Delta(f) \) is positive (negative), the tagged customer decides to join (balk) the system. If \( \Delta(f) \) is zero, the tagged customer is indifferent between joining and balking. Now, consider the following three cases to study the behavior of the tagged customer and equilibrium joining strategy \( (f_e) \) under each case.

Case 1. \( \lambda_f < \mu \) and \( R \geq C \frac{\overline{X}}{\mu - \lambda_f} \). If all customers join the system, then the tagged customer receives a positive benefit by joining the system. Thus, joining is an equilibrium strategy, and moreover, it is a dominant strategy.

Case 2. \( \lambda_f < \mu \) and \( R \leq C \frac{\overline{X}}{\mu} \). If all customers balk the system, the tagged customer receives a negative benefit by joining the system. Thus, balking is an equilibrium strategy, and moreover, it is also a dominant strategy.
Case 3. $\lambda f < \mu$ and $C \frac{\lambda}{\mu - \lambda} \geq R \geq C \frac{1}{\mu}$. If all customers balk the system, the tagged customer receives a positive benefit by joining the system, which is more than the benefit of balking. Thus, balking is not an equilibrium strategy. If all customers join the system, then the tagged customer receives a positive benefit by not joining the system, which is more than the benefit of joining. Thus, joining is not an equilibrium strategy. Therefore, there exists a unique equilibrium strategy $f_e$ that satisfies $\Delta(f_e) = 0$ and is given by $f_e = \frac{R\mu - C}{\lambda(R - C)}$.

Specifically, for an individual the best response is joining (not joining) with probability $f_e (1 - f_e)$.

Let $\lambda_e$ be the equilibrium joining rate and $W_e$ be mean sojourn time under equilibrium, when all customers follow the strategy $f_e$. Table 1 depicts the equilibrium joining strategies for the Geo/Geo/1 unobservable queue.

### Table 1. Equilibrium joining strategy

| Case | $f_e$ | $\lambda_e$ | $W_e$ |
|------|-------|-------------|-------|
| $\lambda - \mu - \frac{C\bar{\mu}}{R - C}$ | 1 | $\lambda$ | $\frac{\lambda}{\mu - \lambda}$ |
| $0 \leq \mu - \frac{C\bar{\mu}}{R - C} < \lambda$ | $\frac{1}{\lambda}(\mu - \frac{C\bar{\mu}}{R - C})$ | $\mu - \frac{C\bar{\mu}}{R - C}$ | $\frac{R}{C}$ |
| $\mu - \frac{C\bar{\mu}}{R - C} < 0$ | 0 | 0 | $\frac{1}{\mu}$ |

4.2. Socially optimal behavior. The social benefit is the overall benefit of all potential customers in the system. We are interested to find the socially optimal joining behavior of customers, that is, the socially optimal joining probability $f_s$, $(0 \leq f_s \leq 1)$ which maximizes the social benefit of the system. Thus, the socially optimal joining strategy is of the form “while arriving at time $t$, join with probability $f_s$ and balk with probability $1 - f_s$”. A tagged customer’s net benefit who joins with probability $f$ when all others follow the same joining strategy is $R - C \frac{\lambda f}{\mu - \lambda f}$ and the social benefit is

$$\Delta_s(f) = \lambda f \left( R - C \frac{\lambda f}{\mu - \lambda f} \right).$$

Hence, a social optimizer searches for the value of $f$, which maximizes (13). The first and second derivatives of the social benefit with respect to $f$ are

$$\Delta_1(f) = \lambda(R - C) - \frac{\lambda C \bar{\mu}}{(\mu - \lambda f)^2},$$

$$\Delta_2(f) = -2 \frac{\lambda^2 C \bar{\mu}}{(\mu - \lambda f)^3} < 0.$$

Since, $\Delta_2(f) < 0$, $\forall f \in [0, 1]$, $\Delta_s(f)$ is a strictly concave function of $f$ and has a unique maximum, say $f_s$. If all customers follow the joining strategy $f_s$, then all have maximum net benefit, which in turn maximizes the social benefit. Thus, this joining strategy is socially optimal. Presuming that customer uses the socially optimal joining strategy, let $\lambda_s$ and $W_s$ are the socially optimal joining rate and socially optimal mean sojourn time in the system, respectively. Table 2 presents the socially optimal joining strategies for the Geo/Geo/1 unobservable queue.
Table 2. Socially optimal joining strategy

| Case       | $f_s$          | $\lambda_s$ | $W_s$       |
|------------|---------------|-------------|-------------|
| $\lambda \leq \mu - \sqrt{\frac{C\mu}{R-C}}$ | 1             | $\lambda$  | $\frac{\lambda}{\mu-\lambda}$ |
| $\lambda > \mu - \sqrt{\frac{C\mu}{R-C}}$ | $\frac{\mu - \sqrt{\frac{C\mu}{R-C}}}{\lambda}$ | $\mu - \sqrt{\frac{C\mu}{R-C}}$ | $1 + \frac{\mu}{\mu-\lambda} \sqrt{\frac{R-C}{C\mu}}$ |

Now, we compare the equilibrium and socially optimal strategies. After some algebraic simplifications, we get

$$f_e - f_s = \frac{1}{\lambda} \sqrt{\frac{C\mu}{R-C}} \left( 1 - \sqrt{\frac{C\mu}{(R-C)\mu}} \right),$$

which is positive, as $(R - C)\mu > C\bar{\mu}$ implies $R > \frac{C}{\mu}$. Thus, $f_s < f_e$ unless $f_s = 1$. Customers always prefer to join with rate $\lambda_e$ for higher individual benefits. However, the social optimizer should suggest the customers to join with rate $\lambda_s$, which is socially desirable.

4.3. Price of anarchy. Price of anarchy is the ratio of optimal and equilibrium social benefit. It measures the degree to which non-cooperation estimates cooperation. It is also, often used to measure the efficiency of a system degradation due to selfish behavior of its customers. The PoA in the unobservable case is computed using the formula,

$$PoA = \frac{\Delta_s(f_s)}{\Delta_s(f_e)}.$$  

The social benefit under equilibrium is $\Delta_s(f_e) = \lambda_e(R - \frac{C\lambda_e}{\mu - \lambda_e}) = 0$ when $f_e \neq 1$, and $\Delta_s(1) = \lambda R - C \frac{\lambda}{\mu - \lambda}$. The optimal social benefit $\Delta_s(f_s) = \lambda_s(R - \frac{C\lambda_s}{\mu - \lambda_s})$ is $\lambda(R - C \frac{\lambda}{\mu - \lambda})$ when $f_s = 1$ and $(\sqrt{(R - C)\mu} - \sqrt{C\bar{\mu}})^2$, elsewhere. Clearly, $(\sqrt{(R - C)\mu} - \sqrt{C\bar{\mu}})^2$ is positive as $R\mu > C$. Thus, $\Delta_s(f_s) > 0$ for $0 < f_s < 1$. Since, the social benefit under equilibrium is zero for $0 \leq f_e < 1$, the PoA is infinity. PoA is unity under the joining strategy $f_e = 1 = f_s$, as equilibrium and optimal social benefits are equal.

4.4. Profit maximization. Let us assume that the system manager fixes a profit-maximizing entry fee $q$ on each joining customer. A monopoly does not allow for a positive customer surplus, since in such a circumstances the entry fee may be increased without decreasing the arrival rate. Thus, $R = q + CW$ and monopoly’s issue is to maximize $q = R - CW$. As the social optimal target is to maximize the total benefit of the customers and the servers, which is $Z_u = \lambda q + \lambda(R - CW - q)$. The term $\lambda q$ cancels, thus, the entry fees have no impact of the social benefit. Hence, the social job is to maximize $\lambda(R - CW)$, that is, $Z_u = \Delta_u$.

5. Numerical results. In this section, we present some numerical experiments to demonstrate the efficiency of the discrete-time queueing model discussed so far. The four-fold aim of the numerical study is (i) determination of the equilibrium and socially optimal strategies for each level of information; (ii) computation of corresponding social welfare optimization and profit maximization discussed above;
(iii) illustration of the impact of the system parameters on the price of anarchy; and
(iv) a comparison study of the effect of information levels on the selfish and social
optimal joining behavior.

We first consider the observable model (wherein the arriving customers are in-
formed about the number of customers in the system) and explore the sensitivity
of the equilibrium pure threshold $n_e$ and socially optimal threshold $n_o$ concerning
different system parameters $R, C, \mu$ and $\lambda$. The impact of service completion reward
to waiting cost ratio on the threshold policies under individual optimization and
social optimization is illustrated in Fig. 4. A mixed dominance is observed between
the equilibrium thresholds. When the reward cost ratio is higher, the equilibrium
balking threshold dominates the socially optimal threshold, and are equal in $[4, 6)$
and $[6, 8)$. The optimal threshold strategy is smaller than that of the Nash equi-
lbrium strategy is the general observation in Naor’s work and to get the desired
social behavior, Naor suggested imposing an appropriate admission fee. In case of
heavy congestion, the imposed toll may help in system management by deviating
customers from the service station. The discrete-time model shows that the socially
optimal threshold can be induced without imposing any admission fee, rather by
choosing appropriate reward and cost values.

Now, we see the contrast between threshold variations under different arrival
and service completion probabilities in Figs. 5 and 6. The threshold strategies are
monotonically increasing with the service completion probability and non-increasing
concerning arrival probability. Equilibrium threshold $n_e$ remains unchanged for
any variation in arrivals. Because the customers arriving at the system after the
tagged customer do not affect the individual optimization, which is intuitive from
the analytical expression. But, the socially optimal threshold $n_o$ decreases as the
system gets congested. Higher waiting time for customers in the queue results in a
larger cost for them to join the system. In this situation, customers prefer to balk.

Next, we study the effect of model parameters on the PoA under the observable
case and unobservable case. Fig. 7 illustrates the monotonic increasing behavior
of the PoA against the arrival rate in the observable case. The rate of increase is
lower for less congested systems and higher for more congested systems. The PoA
becomes unbounded as the system utilization increases beyond 1, that is, $\lambda > 0.5$.
In Fig. 8, the PoA is multimodal with respect to service rate in the observable
case and are prominently observed when the system utilization is more than unity
($\rho > 1$). From the two figures, a distinct characteristic of the PoA concerning
information level is observed. In the observable case, PoA is monotone decreasing
and bounded below by unity for $\rho < 1$, displays multimodal behavior in a heavily
congested system and becomes unbounded for $\rho > 1$, whereas, in the unobservable
model, PoA is 1 when $\nu_s = 1$ and becomes infinity for $\nu_s > 1$. Revealing information
to arriving customers has a positive effect on the social optimization whereas not
revealing has a negative impact. One can control the PoA for optimal social benefit
by selecting suitable model parameters.

In the unobservable model, there exist mixed strategies under equilibrium which
is given by the joining probabilities. The equilibrium and socially optimal joining
strategies are illustrated for different model parameters in Figs. 9-11. In Fig. 9,
the equilibrium mixed strategy always dominates the socially optimal strategy for
smaller values of $R/C$ and are equal for higher values (after it attains 3.8) of $R/C$.
The difference in their value increases with an increase in the reward cost ratio
up to 3.8. Thus, customers behave selfishly when uninformed and are interested in
individual optimization. Similar behavior is observed against service rate in Fig. 10. Under no information situation, socially optimal behavior can be induced without imposing any additional admission fee for joining customers. The selfish and social joining strategies match to unity for less congested systems, and customers’ selfish behavior is dominated for higher arrival rates in Fig. 11. In Fig. 12, the social equilibrium benefit is a concave downward function of $\lambda$. This indicates that the social benefit increases with more customers joining the system, but after a certain value of $\lambda$ (here 0.4) the social benefit starts decreasing. This is because of the heavy congestion that increases the customer waiting times which in turn reduces the social benefit. Further, the profit maximization and social welfare under the optimal profit is compared to the customer arrival rate in Fig. 13. Although both the functions are monotonically increasing, the social welfare function is dominated by the profit maximization function. From the numerical computation, we found that the profit maximizing threshold $n_m$ has the value from $\{2, 3\}$. The lower the value of $n_m$, the smaller is the functional value difference between social welfare and
profit maximization. But for higher threshold value, their difference grows, and the profit maximization dominates the social welfare under profit maximization. These results will help the system manager to suitably choosing the threshold value ($n_m$) under congestion systems to get the maximum benefit under both the policies.

Finally, the optimal joining rates are computed in the observable and unobservable cases with the parameters $R = 20$ and $C = 5$. In the observable case, the individual and socially optimal joining rates are calculated from the relation $\lambda \mu (\bar{\mu} - \lambda r^\nu)/(\mu \bar{\mu} - \lambda \bar{\mu} r^\nu)$ and $\lambda \mu (\bar{\mu} - \lambda r^\nu)/(\mu \bar{\mu} - \lambda \bar{\mu} r^\nu)$, respectively. The corresponding optimal joining rates are calculated from the expressions presented in the table 1 and table 2, respectively. In Fig. 14, the informed customers optimal joining rate is higher as compared to the uninformed customers for smaller values of $\nu$. But, with an increase in $\nu$, the reverse trend is seen. When customers are not informed, it is observed that they decide to follow the crowd. Two intersections are observed between the optimal joining rates of both informed and uninformed customers. These intersection points provide crucial information to the social planner.
in controlling the system congestion. The socially optimal joining rates increase under both information levels, but the informed customers are more interested to join in the less congested case and follow the crowd in the congested systems. In Fig. 15, there is only one intersection between the optimal joining rates of observable and unobservable cases in contrast to the two intersection points observed in the first case (Fig. 14). Also, this result (the unique intersection point) is a deviation from the continuous-time counterpart wherein multiple intersection points occur in the case of socially optimal joining rates [23]. In both the cases, it is confirmed that the social planner can control the arrival of customers by selecting the level of information to be revealed from time to time.

6. Conclusion. In this paper, we discussed the effect of information on Naor’s model in the context of a discrete-time Geo/Geo/1 queue. We studied the equilibrium and social optimal behavior of the strategic customers under two different information situations. The price of anarchy in the observable system is a multimodal function of service rate and is bounded below by unity. We incorporated the profit-maximizing framework, where the manager levies a profit maximizing
entry fee to get maximum profit out of the system. Several new insights in the discrete-time system is observed in the behavior of the price of anarchy and social welfare function. This study helps the service provider to choose proper strategies for maximizing his profit from the system. On the other hand, this helps arriving customers to select a strategy such that their expected waiting time will minimize, which successively increases their expected net benefit. Using the numerical results, system managers can control the optimal arrival rate to get the maximum benefit under both observable and unobservable cases. The methods used in this paper may be used to study an optimal balking strategy in discrete-time batch arrival and batch service queues, which are left for future examinations.

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