Abstract

Motivated by the growing interest in today’s massive parallel computing capabilities we analyze a queueing network with many servers in parallel to which jobs arrive according to a Poisson process. Each job, upon arrival, is split into several pieces which are randomly routed to specific servers in the network, without centralized information about the status of the servers’ individual queues. The main feature of this system is that the different pieces of a job must initiate their service in a synchronized fashion. Moreover, the system operates in a FCFS basis. The synchronization and service discipline create blocking and idleness among the servers, which is compensated by the fast service time attained through the parallelization of the work. We analyze the stationary waiting time distribution of jobs under a many servers limit and provide exact tail asymptotics; these asymptotics generalize the famous Cramér-Lundberg approximation for the single-server queue.

Keywords: Queueing networks with synchronization, many servers queue, Cramér-Lundberg approximation, high-order Lindley equation, cloud computing, MapReduce, weighted branching processes, stochastic fixed-point equations, large deviations.

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1 Introduction

Cloud computing is an emerging paradigm for accessing shared computing and storage resources over the internet. “Clouds” consist of hundreds of thousands of servers that provide scalable, on-demand data storage and processing capacity to end users. Operators of these large facilities achieve economies of scale and can make efficient use of their computing infrastructure by optimizing how tasks are processed across an interconnected and distributed computer network. End users of cloud computing benefit from reduced capital expenditures and from the flexibility that the scalable paradigm offers. Worldwide demand for cloud computing has experienced rapid growth that is expected to continue into the foreseeable future.

Motivated by this rapidly emerging phenomenon, we analyze a queueing model for a large network of parallel servers. Throughout the paper we use the generic term server to represent a computing unit, e.g., a computer or a processor in the network. Jobs arrive to the network at random times and are split at the time of arrival into a number of pieces. These pieces are then randomly routed to specific servers, where they join the corresponding queue. The first reason for assigning these job
fragments to specific servers is that maintaining a single centralized queue is not scalable in systems of this size, and keeping information about the individual status of each queue to make informed routing decisions can be costly. The second, maybe more important reason, is the requirement for data locality. Since data sets processed on these clouds can be in petabyte size, transporting them can be prohibitive, thus, jobs need to be assigned to servers in close proximity of data storage. In this setting, random routing is equivalent to assuming that data storage is uniformly distributed throughout the cloud. The service requirements of each of the pieces of a job are allowed to be random and possibly dependent. The main distinctive features of this model are: 1) all the pieces of a job must begin their service at the same time, i.e., in a synchronized fashion; 2) jobs are processed in a first-come-first-serve (FCFS) basis, i.e., each of the individual queues at the servers follow a FCFS service discipline. We refer to these two characteristics as the synchronization and fairness requirements.

The fairness requirement is common in many queueing systems where jobs originate from different users, while the job synchronization is a distinctive characteristic of this model that allows us to incorporate the need to exchange information among the different pieces of a job during their processing. This is certainly the case for many scientific applications that involve simulations of complex systems, including: wireless networks, neuronal networks, bio-molecular and biological systems. The fairness and synchronization requirements create, nonetheless, blocking and idleness that are not present in other distributed systems, e.g., multi server queues where the different fragments of a job can be processed independently, and can therefore be thought of as batch arrivals. This lack of efficiency is compensated by the service speed attained through the parallel processing, which can be considerable for very large jobs. See Figure 1.

Figure 1: Queueing model for a server cloud where jobs need to be synchronized. In this figure, the green job is being processed and will be followed by the purple one; note that the second server from the bottom will be idle until the green and purple jobs complete their service. The blue and the yellow jobs, nonetheless, have no servers in common, and therefore can begin their service as soon as all their fragments reach the front of their queues, regardless of their time of arrival to the network.
We analyze in this paper the stationary waiting time of jobs (excluding service) in the many server asymptotic regime. In particular, after establishing sufficient conditions for the stability of the finite system, we show that the limiting stationary waiting time $W$ is given by the endogenous solution to the following high-order Lindley equation:

$$W \overset{D}{=} \left( \max_{1 \leq i \leq N} (W_i + \chi_i - \tau_i) \right)^+,$$

where the $\{W_i\}_{i \in \mathbb{N}}$ are i.i.d. copies of $W$, independent of $(N, \chi_1, \tau_1, \chi_2, \tau_2, \ldots)$, $N$ is the number of pieces of a job, $\tau_i$ is the limiting inter arrival time between piece $i$ and the job immediately in front of it at its assigned queue, and $\chi_i$ is the service time of the fragment of the job in front of the $i$th piece; $\overset{D}{=}$ stands for equality in distribution and $x^+ = \max\{0, x\}$. Note that for $N \equiv 1$, (1.1) reduces to the classical Lindley equation, satisfied by the GI/GI/1 queue. Recursion (1.1) was termed “high-order Lindley equation” and studied in the context of queues with synchronization in [16].

Moreover, by applying the main result in [14] for the maximum of the branching random walk, we provide the exact asymptotics for the tail distribution of $W$. These asymptotics naturally generalize the classical Cramér-Lundberg approximation for the simple random walk to the branching random walk. More precisely, we show that under the Cramér root and derivative conditions

$$E \left[ \sum_{i=1}^{N} e^{\theta(\chi_i - \tau_i)} \right] = 1 \quad \text{and} \quad 0 < E \left[ \sum_{i=1}^{N} (\chi_i - \tau_i)e^{\theta(\chi_i - \tau_i)} \right] < \infty,$$

we have that

$$P(W > x) \sim H e^{-\theta x}$$

as $x \to \infty$ for some $0 < H < \infty$. Throughout the paper, $f(x) \sim g(x)$ as $x \to \infty$ stands for $\lim_{x \to \infty} f(x)/g(x) = 1$.

We end this section with an overview of applications and related models. Although the inspiration for this work comes from massive distributed systems in cloud computing, such as MapReduce and its open source implementation Hadoop, our model is not meant to provide an exact description. The main idea of MapReduce/Hadoop is to divide large data sets into smaller units, then process these smaller units on a large number of parallel servers and finally assemble the partial answers into the final solution. The initial phase of this framework, called the mapping, divides a new job into tasks/files of similar size, e.g., 64 (or 128) megabytes (MB) in size. Irrespective of the size of the original job, all smaller jobs are of equal size - 64 (or 128) MBs, but the number of these smaller jobs and the servers to which they are assigned depends on the original job size. After (or in some implementations during) the execution of the mapping phase the system begins a reducing phase, again on a number of parallel servers. The reduce phase merges the partial answers from the processed tasks into one final answer. Often, the completion of a job consists of several repetitions in sequence of the map-reduce process.

To explain the similarities between our proposed model and MapReduce it is worth elaborating some more on its synchronization and blocking characteristics. In the original FCFS implementation, if there are two jobs A and B arriving in that order, all mapping tasks for job A will execute before any mapping tasks from job B can begin. As tasks from job A are finishing their mapping phase
and moving on to the reduce phase, job B mapping tasks are scheduled. This is similar to the synchronous server assignment that we consider in our model. Furthermore, due to the observation that reducers need to have all the outputs from the mapping phase in order to perform their work, job B reducers cannot start until all job A reducers and all job B mappers have finished. Therefore, there is a natural queueing system with blocking of servers. In the Hadoop framework, a job needs both mappers and reducers available in order to start processing, and the reducers can only begin once all (some) mappers have finished, therefore, in some instances, the reducers must start all tasks of a job at the same time. The blocking problem is a well-known weakness of MapReduce, and although a number of new scheduling disciplines have emerged and replaced the original FCFS implementation, the blocking problem remains.

A closely related model to our present work is that of [11], which considers a problem where each job requires a synchronous execution on a random number of parallel servers. The main difference from our setup, besides the restriction to i.i.d. exponentially distributed service times, is that we assign the pieces of a job to specific servers at the time of arrival, while the problem in [11] waits until the required number of servers is free and then assigns the pieces to these servers. We point out that this is equivalent to having perfect information about the workloads at each server and routing the pieces to the servers with the smallest workloads. Therefore, the model in [11] provides a benchmark that can be used to quantify the value of having centralized information, a question that we will address in a sequel to this paper.

Some of the ideas used in the proofs of the main result in this paper are borrowed from [16], where the authors considered a queueing system with $m$ different types of servers and $n$ identical servers of each type (for a total of $m \times n$ servers), and where each arriving job requires service from exactly one server of each type, i.e., each job needs $m$ parallel servers, and is assigned upon arrival to one of the $n$ possible choices for each type. Theorem 2 in [16] shows that the steady-state distribution of the waiting time converges weakly, as $n \to \infty$, to the endogenous solution of the high-order Lindley equation (1.1) with $N \equiv m$. Besides allowing $N$ to be random and the service requirements of the fragments of a job to be dependent, this paper shows not only the weak convergence of the steady-state waiting time, but also of all its moments.

Another related scenario of virtual path allocation in communication networks was studied in [20] and the follow up work. In that problem, an establishment of a virtual path requires synchronous allocation of channels (which can be viewed as servers) on a number of interconnected links. Hence, the problem can be viewed as synchronous job processing on a specific random number of servers. In general, this is a multi-resource allocation problem that appears in service engineering and consulting project management, e.g., managing a project may require assigning specific people of different skills to simultaneously work on the project. The framework in [20] was the blocking-loss model, and the main performance measure was the loss probability, i.e., there was no queueing. In addition to queueing, our model also considers a many servers limit, which brings an additional level of tractability and, as previously stated, is related to the prior work in [16]. In this context, we believe that our methodology will be applicable to resource allocation problems in large-scale networks, where random routing and a large number of links would justify the limiting approximation. Other applications mentioned in [11] are: the deployment of fire engines in firefighting, jury selection, and the staffing of surgeons and medical personnel in emergency surgery.

The remainder of the paper is organized as follows. Section 2 contains the mathematical description
of our queueing model; Section 3 describes the analysis of the stationary waiting time of jobs in the network, with the main result of this paper in Section 3.2 and the tail asymptotics of the limiting waiting time in Section 3.3. Finally, Section 4 contains the proof of the main theorem.

2 Model description

We consider a sequence of queueing networks indexed by their number of servers, \( n \). Each of the \( n \) servers are identical and operate in parallel. Arrivals to the \( n \)th network occur according to a Poisson process with rate \( \lambda n \) for some parameter \( \lambda > 0 \). Each job, upon arrival to the network, is split into a random number of pieces, usually proportional to the total service requirement of the job. The size of a job, i.e., the number of pieces into which it is split, is determined by some distribution \( f_n(k) \), \( k = 1, 2, \ldots, m_n \), where \( m_n \) is a bound on the number of pieces a job can have and is chosen to satisfy \( m_n \leq n \); this condition ensures that each piece can be routed to a different server. Once a job has been split, say into \( k \) pieces, its fragments are routed randomly to \( k \) different servers in the network (i.e., with all \( n!/((n-k)! \) possible assignments equally likely), forming a queue at their assigned servers. An equivalent way of describing the arrival of jobs into the \( n \)th network is to use the thinning property of the Poisson process and think of independent Poisson processes, each generating jobs of size \( k \), \( k = 1, 2, \ldots, m_n \), at rate \( f_n(k)\lambda n \).

The service times of each of the fragments of a job are allowed to be dependent, but are assumed to be independent of the number of pieces. We will denote by \( B \) the marginal distribution of the service requirement of a fragment of a job; note that the random routing and the independence from the size of the job imply that all fragments have the same marginal distribution. The randomness of the fragments’ service requirements can be used to include rounding effects and small variations on the type of processing that they need. The sizes of jobs and of the service requirements of their fragments are independent of the arrival process.

In order to model the synchronization and fairness characteristics of the network, we will assign to each job a tag (not to be confused with the label that will be introduced later). More precisely, a job having \( \lambda \) pieces receives a tag of the form \((s_1, s_2, \ldots, s_k)\), \( s_i \in \{1, 2, \ldots, n\} \) for all \( i \), \( s_i \neq s_j \) for \( i \neq j \), representing the different servers to which its fragments are sent for processing.

**Definition 2.1** We say that a job having tag \( \mathbf{r} = (r_1, r_2, \ldots, r_l) \) is a predecessor of a job having tag \( \mathbf{s} = (s_1, s_2, \ldots, s_k) \) if it arrived before the job having tag \( \mathbf{s} \) and they have at least one server in common (i.e., \( r_i = s_j \) for some \( 1 \leq i \leq l \) and \( 1 \leq j \leq k \)). We use the term immediate predecessor if there are no pieces of other jobs in between the two jobs at the server they have in common.

In terms of this definition, the synchronization rule is that the job having tag \( \mathbf{s} = (s_1, s_2, \ldots, s_k) \) cannot begin its service, which is to be done in parallel by servers \( s_1, s_2, \ldots, s_k \), until all its immediate predecessors have completed their service. The fairness rule says that if the job with tag \( \mathbf{r} \) is a predecessor of the job with tag \( \mathbf{s} \), then it will begin its service before the job with tag \( \mathbf{s} \) does.

The first thing we need to establish is the stability of the \( n \)th system described above. Formally, we can think of our model as a superposition of \( \sum_{k=1}^{m_n} \frac{n!}{(n-k)!} \) independent marked point processes. Each of these processes generates jobs of size \( k \) with server assignments \((s_1, \ldots, s_k)\), according to
a Poisson process with rate $f_n(k)\lambda n/(n!/(n-k)!)$, and having i.i.d. marks $\{(\chi^{(1)}_i, \ldots, \chi^{(k)}_i)\}_{i \geq 1}$ corresponding to the service requirements of the fragments. Once we have the appropriate stability conditions, our goal is to describe the distribution of the steady-state waiting time of jobs (excluding service). The approach that we will follow is based on a standard queueing theory technique, where at time $t_0 < 0$ we assume that there are no jobs in the network, and then we look at the waiting time of the first job to arrive after time zero. We will refer to this job as the “tagged” job, and we will describe its waiting time by analyzing a graph containing all the information of which jobs need to complete their service before the tagged job can initiate its own. We will prove that the waiting time of the tagged job, after having taken the limit $t_0 \to -\infty$, is finite almost surely, which is the standard approach for establishing the stability of queueing systems using Loynes’ lemma and Palm distributions. We refer the reader to [8] for more details on this general technique.

After establishing appropriate conditions for the stationarity of the $n$th system, we will take the limit as $n$ goes to infinity (i.e., the many servers limit) and show that the steady-state waiting time in the $n$th system converges in distribution to some finite random variable $W$. Moreover, we will show that we also have convergence of all the moments of order $p > 0$, which can be very useful for analyzing other measures of performance besides the distribution of the stationary waiting time.

3 Analysis of the steady-state waiting time

In order to analyze the waiting time of the tagged job described above we borrow the technique used in [16] and construct a so-called predecessor graph. A predecessor of the tagged job will be any job that arrived before time zero and that needs to complete its service before the tagged job can begin its own, as described by Definition 2.1. The predecessor graph will contain all the information needed to determine the waiting time of the tagged job, in particular, the times of arrival, the number of pieces, and the service requirements of these pieces, of all the predecessors of the tagged job.

3.1 The predecessor graph

To construct the predecessor graph we look at time in reverse, starting from the time the tagged job arrived, say $T_1 \geq 0$, and ending at time $t_0$. The tagged job is split into a random number of pieces, say $\hat{N}_0 = \hat{N}_0(n)$, where $\hat{N}_0$ is distributed according to $f_n$. Each of these pieces will be routed to one of the $n$ servers in the network, where it will either find the server empty or join a queue. Suppose that the tagged job needs to be processed by servers $(s_1, \ldots, s_{\hat{N}_0})$, then any job that is directly in front of the queue at any of the servers $s_i$, $1 \leq i \leq \hat{N}_0$ is an immediate predecessor of the tagged job. To construct the first set of edges in the graph we draw an edge from the tagged job to its immediate predecessors. Moreover, each edge is assigned a vector of the form $(\hat{\tau}_i, \chi_i)$, $1 \leq i \leq \hat{N}_0$, where $\hat{\tau}_i$ is the inter arrival time between the tagged job and its $i$th immediate predecessor, and $\chi_i$ is the service requirement of the piece of the immediate predecessor that is in front of the corresponding piece of the tagged job. Also, if a job is an immediate predecessor of more than one fragment of the tagged job, say it requires service at servers $s_i$ and $s_j$, then $\hat{\tau}_i = \hat{\tau}_j$, although we may still have $\chi_i \neq \chi_j$ with $\chi_i, \chi_j$ possibly dependent. Finally, if a piece of the tagged
job finds its server empty upon arrival, then there is simply no edge to be drawn. Hence, the number of outbound edges of the tagged job is smaller or equal than $\hat{N}_\emptyset$.

Iteratively, once we have identified all the immediate predecessors of the tagged job we repeat the process described above with each one of them. We will call the predecessor graph $G_n(t_0)$, since it will depend on both the number of servers $n$ and the time $t_0$ at which the system starts empty. Since $G_n(t_0)$ will resemble a tree, it will be useful to use tree notation to refer to the predecessors of the tagged job. More precisely, let $\mathbb{N}_+ = \{1, 2, 3, \ldots \}$ be the set of positive integers and let $U = \bigcup_{r=0}^{\infty} (\mathbb{N}_+)^r$ be the set of all finite sequences $i = (i_1, i_2, \ldots, i_r) \in U$, where by convention $\mathbb{N}_+^0 = \{\emptyset\}$ contains the null sequence $\emptyset$. To ease the exposition, for a sequence $i = (i_1, i_2, \ldots, i_k) \in U$ we write $i|_t = (i_1, i_2, \ldots, i_t)$, provided $k \geq t$, and $i|0 = \emptyset$ to denote the index truncation at level $t$, $k \geq 0$. To simplify the notation, for $i \in \mathbb{N}_+$ we simply use $i = i_1$, that is, without the parenthesis. Also, for $i = (i_1, \ldots, i_k)$ we will use $(i, j) = (i_1, \ldots, i_k, j)$ to denote the index concatenation operation, if $i = \emptyset$, then $(i, j) = j$.

We now label the tagged job $\emptyset$, and its immediate predecessors $i$, with $1 \leq i \leq \hat{N}_\emptyset$; the jobs in the next level of predecessors will have labels of the form $(i_1, i_2)$, and in general, any job in the predecessor graph will have a label of the form $i = (i_1, i_2, \ldots, i_k)$, $k \geq 1$. With this notation, $\hat{N}_i$ denotes the number of pieces that the job with label $i$ in the graph is split into, $\hat{T}_{(i,j)}$ will denote the inter arrival time between job $i$ and its $j$th immediate predecessor (a job with label $(i,j)$), and $\chi_{(i,j)}$ denotes the service requirement of the fragment immediately in front of the queue of the $j$th piece of job $i$. Note that the tag of a job, which contains the specific server assignments, allows us to identify the immediate predecessors of a given job, but it plays no role afterwards. Therefore, we will use the labels, not tags, to identify jobs in the predecessor graph. See Figure 2.

![Figure 2: The predecessor graph $G_n(t_0)$. The numbers in each node indicate the size of the job (number of pieces). Some nodes have fewer outbound edges than the size of the job, meaning that the corresponding piece found its server empty upon arrival. Nodes with multiple inbound edges correspond to jobs that are immediate predecessors to more than one job in the graph.](image-url)
We point out that in case a job is an immediate predecessor to more than one job in the graph (or to more than one piece of the same job), the corresponding edges will merge into the common predecessor. Moreover, in this case, the common predecessor is assigned more than one label (e.g., if a job is an immediate predecessor to both jobs $i = (i_1, \ldots, i_k)$ and $j = (j_1, \ldots, j_l)$, then such job can be identified by two different labels, one of the form $(i, s)$ and another of the form $(j, t)$). Furthermore, the merged paths and the subgraph they define from that point onwards will have multiple labels as well. See Figure 3.

![Diagram](image)

**Figure 3**: Multiple labels due to common immediate predecessors.

**Remark 3.1** We allow multiple labels for jobs that are immediate predecessors to more than one job (or more than one piece of the same job), since we need to keep track of the service time of each fragment of a job, as well as of the inter arrival time between the fragment and its immediate predecessor, which can be thought of as “edge” attributes in the graph.

To analyze the waiting time of the tagged job we now derive a high-order Lindley recursion. To this end, let $W_i^{(n,t_0)}$ denote the waiting time of the job having label $i$ in the network with $n$ servers and that starts empty at time $t_0$. Then, if $B_i$ denotes its set of immediate predecessors, we have

$$W_0^{(n,t_0)} = \max \left\{ 0, \max_{i \in B_0} (\chi_i - \hat{\tau}_i + W_i^{(n,t_0)}) \right\},$$

with the boundary condition that the first job to arrive after time $t_0$, and any other job that arrives thereafter and is the first one to use its assigned servers, will have a waiting time of zero (recall that the system is empty at time $t_0$). To analyze (3.1) define $\hat{X}_i \equiv 0$ and $\hat{X}_i = \chi_i - \hat{\tau}_i$ for $i = (i_1, \ldots, i_k)$. Also, let $\mathcal{A}_0 = \{\emptyset\}$, and $\mathcal{A}_r = \{(i_1, \ldots, i_r) \in \mathcal{G}_0(t_0)\}$ for $r \geq 1$, denote the set of labels at graph distance $r$ from the tagged job, i.e., labels whose corresponding job is connected to the tagged job by a directed path of length $r$. We also define for $i \in \mathcal{A}_k$ the sets of labels at graph distance $r$ from $i$ given by

$$B_{i,r} = \left\{ j \in \mathcal{A}_{k+r} : j = (i, i_{k+1}, \ldots, i_{k+r}) \right\}; \quad B_i \equiv B_{i,1}.$$

Next, let

$$\kappa = \max\{r \in \mathbb{N}_+ : |B_{0,r}| > 0\},$$

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where $|A|$ denotes the cardinality of set $A$. Note that $\kappa$ is a random variable and corresponds to the maximum length of any directed path in $\mathcal{G}_n(t_0)$. Furthermore, for any $i \in \hat{A}_\kappa$ we have $W_i^{(n,t_0)} = 0$, and therefore, for any $i \in \hat{A}_{\kappa-1}$,

$$W_i^{(n,t_0)} = \max \left\{ 0, \max_{j \in \mathcal{B}_i} \hat{X}_j \right\}.$$ 

Similarly, iterating (3.1) we obtain for $i \in \hat{A}_{\kappa-2}$,

$$W_i^{(n,t_0)} = \max \left\{ 0, \max_{j \in \mathcal{B}_i} \left( \hat{X}_j + W_j^{(n,t_0)} \right) \right\}$$

$$= \max \left\{ 0, \max_{j \in \mathcal{B}_{i,1}} \hat{X}_j, \max_{j \in \mathcal{B}_{i,2}} \left( \hat{X}_{j|1} + \hat{X}_j \right) \right\}.$$ 

In general, after iterating (3.1) $\kappa$ times we obtain

$$W_0^{(n,t_0)} = \max \left\{ 0, \max_{j \in \mathcal{B}_{0,1}} \hat{X}_j, \ldots, \max_{j \in \mathcal{B}_{\kappa}} \left( \hat{X}_{j|1} + \hat{X}_{j|2} + \cdots + \hat{X}_j \right) \right\}.$$ (3.2)

### 3.2 Main result

Having now introduced a suitable notation and derived a recursive equation for the waiting time, we need to identify conditions under which this queueing system will be stable. Once we have stability we need to identify the stationary distribution of the waiting time. The key idea to solve both problems is that the predecessor graph is very close to being a tree; more precisely, the only thing that prevents it from being a tree is the occasional arrival of a job that is an immediate predecessor to two or more jobs in $\mathcal{G}_n(t_0)$. It turns out that under the scaling we consider in our model (arrival rate equal to $\lambda n$), the probability of this occurring within the timeframe needed for the tagged job to start its service is very small (geometrically small). Once we show that this is the case, taking the limit as $t_0 \to -\infty$ will yield the stability of the network.

To describe the stationary distribution of the waiting time we first observe that, provided the first time that two paths in the predecessor graph merge occurs after the tagged job has initiated its service, we have that the $\hat{\tau}$’s will be i.i.d. exponential random variables with some rate $\lambda_n^*$ and the $\chi$’s will be i.i.d. with distribution $B$. It follows that under the same conditions that guarantee the stability of the network we would have that, after taking the limit as $t_0 \to -\infty$ and the number of servers $n \to \infty$, $W_0^{(n,t_0)}$ would have to converge to a solution to the stochastic fixed-point equation

$$W \overset{D}{=} \max \left\{ 0, \max_{1 \leq i \leq N} \left( \chi_i - \tau_i + W_i \right) \right\},$$ (3.3)

where $\{W_i\}$ are i.i.d. copies of $W$, independent of $(N, \chi_1, \tau_1, \chi_2, \tau_2, \ldots)$, with $N$ having distribution $f \overset{d}{=} \lim_{n \to \infty} f_n$, the $\chi$’s i.i.d. having distribution $B$, and the $\tau$’s i.i.d. exponential random variables with rate $\lambda_n^* \overset{d}{=} \lim_{n \to \infty} \lambda_n^*$, all random variables independent of each other. Note also that we have replaced the set over which the maximum is computed, $i \in \mathcal{B}_0$, with $1 \leq i \leq N$, since in stationarity all the pieces of a job have an immediate predecessor.
It turns out that $[3.3]$ has multiple solutions [9], unlike the standard Lindley equation for $N \equiv 1$. It is the structure of $(3.2)$ that will allow us to identify the correct one. As we will see in the following sections, the appropriate solution is the so-called endogenous one, which is also the minimal one in the usual stochastic order sense.

To identify the value of $\lambda^*$ note that in the time reversed setting, for the system with $n$ servers, we can think of independent Poisson processes each generating jobs with a tag of the form $(s_1, s_2, \ldots, s_k)$, for $1 \leq k \leq m_n$, and $s_i \in \{1, 2, \ldots, n\}$ for all $i$, at rate

$$\lambda_k = \frac{f_n(k)\lambda n}{n!(n-k)!}.$$ 

Moreover, a piece of a job requiring service at server $s_i$ can have as an immediate predecessor any job of any size requiring service at server $s_i$. In particular, there are a total of $\binom{n-1}{k-1}k!$ possible predecessors of size $k$, and therefore, the inter arrival time between the piece of the job requiring service from server $s_i$ and its predecessor is exponentially distributed with rate

$$\lambda^* = \sum_{k=1}^{m_n} \lambda k \left(\frac{n-1}{k-1}\right) k! = \sum_{k=1}^{m_n} \frac{f_n(k)\lambda n}{\binom{n}{k}} \left(\frac{n-1}{k-1}\right) = \lambda \sum_{k=1}^{m_n} k f_n(k). \quad (3.4)$$

It follows that, assuming $f_n$ is uniformly integrable,

$$\lambda^* = \lambda \sum_{k=1}^{\infty} k f(k) = \lambda E[N].$$

Equation $[3.3]$ is known in the literature [16, 9, 14] as a high-order Lindley equation, and the behavior of its endogenous solution is given in Section 3.3.

Before we formulate the main result of this paper it is convenient to specify the conditions we need to impose on $f_n, B$ and $\lambda$; $\Rightarrow$ denotes convergence in distribution.

**Assumption 3.2** Suppose that $f_n$ is a distribution on $\{1, 2, \ldots, m_n\}$, with $m_n \leq n$, $B_k(x)$ is a joint distribution on $\mathbb{R}_+^k$ for each $k \in \mathbb{N}_+$, and $\lambda > 0$.

i) Suppose there exists a distribution $f$ on $\mathbb{N}_+$, having finite mean, such that $f_n \Rightarrow f$ as $n \to \infty$ and $f_n$ is uniformly integrable.

ii) Suppose there exists $\beta > 0$ such that

$$E \left[ \sum_{i=1}^{N} e^{\beta(x_i - \tau_i)} \right] = \frac{\lambda^*}{\lambda^* + \beta} E[N] E \left[ e^{\beta x_1} \right] < 1,$$

where $N$ is distributed according to $f$, $\{x_i\}$ are i.i.d. random variables with distribution $B$, independent of $N$, and $\{\tau_i\}$ are i.i.d. exponentially distributed random variables with rate $\lambda^* = \lambda E[N]$, independent of $(N, x_1, \ldots, x_N)$.

iii) Suppose that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{m_n} k^2 f_n(k) = 0.$$
To give some examples of distributions for which Assumption 3.2 is satisfied, let \( N \) be distributed according to \( f \) and consider
\[
f_n(k) = P\left(\min\{N,m_n\} = k\right) \quad \text{or} \quad f_n(k) = P(N = k|N \leq m_n).
\]
In both cases, provided \( E[N] < \infty \), we can take any \( m_n \to \infty \), including \( m_n = n \), and have \( f_n \) uniformly integrable, since the monotone convergence theorem gives \( E[\min\{N,m_n\}] \to E[N] \) and \( E[N|N \leq m_n] = E[N1(N \leq m_n)]/P(N \leq m_n) \to E[N] \). Finally, Assumption 3.2 (iii) would be satisfied in both examples, with \( m_n = n \), if \( E[\hat{N}^{1+\epsilon}] < \infty \) for some \( \epsilon > 0 \), since then
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} k^2 f_n(k) \leq \lim_{n \to \infty} \frac{n^{1-\epsilon}}{n} \sum_{k=1}^{n} k^{1+\epsilon} f_n(k) = E[\hat{N}^{1+\epsilon}] \lim_{n \to \infty} n^{-\epsilon} = 0.
\]
In case \( E[\hat{N}^{1+\epsilon}] = \infty \) for all \( \epsilon > 0 \), one would need to take \( m_n = o(n) \) to obtain
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} k^2 f_n(k) \leq \lim_{n \to \infty} \frac{m_n^n}{n} \sum_{k=1}^{n} k f_n(k) = E[N] \lim_{n \to \infty} \frac{m_n}{n} = 0.
\]

We are now ready to formulate the main theorem.

**Theorem 3.3** Let \( W_{\emptyset}^{(n,t_0)} \) denote the waiting time, excluding service, of the tagged job (the first job to arrive after time zero) when we start the system empty at time \( t_0 < 0 \) and the network consists of \( n \) servers. Suppose that
\[
E[\hat{N}]E\left[e^{\beta(\chi - \tilde{\tau})}\right] < 1 \quad (3.5)
\]
for some \( \beta > 0 \), where \( \hat{N} \) has distribution \( f_n \), \( \chi \) has distribution \( B \) and \( \tilde{\tau} \) is exponentially distributed with rate \( \lambda_n^* \) and is independent of \( \chi \). Then, for any fixed number of servers \( n \),
\[
\lim_{t_0 \to -\infty} W_{\emptyset}^{(n,t_0)} = W^{(n)} \quad \text{a.s.}
\]
for some finite random variable \( W^{(n)} \). Moreover, provided Assumption 3.2 is satisfied,
\[
W^{(n)} \Rightarrow W,
\]
as \( n \to \infty \), where \( W \) is the endogenous solution to (3.3). Furthermore, for any \( p \geq 1 \),
\[
E \left[\left(W^{(n)}\right)^p\right] \to E \left[W^p\right] < \infty, \quad n \to \infty.
\]

The key idea for the proof of the stability result is to couple the predecessor graph with a weighted branching tree \([17, 12]\) and show that the waiting time of the tagged job is dominated by the maximum of the random walks along all the paths of the tree. The identification of the limit with the endogenous solution to the high-order Lindley equation (3.3) will follow from a similar coupling argument between the predecessor graph and a weighted branching tree, in which we will show that with high enough probability the tagged job will initiate its service before we observe the first merging of paths. Since the number of pieces and the rate of the exponential inter arrival times both depend on \( n \), the continuity properties of the stochastic fixed-point equation will also play a role in the analysis.

For more details on the full characterization of all the solutions to (3.3) we refer the interested reader to the work in \([9]\). We now describe the asymptotic properties of the endogenous solution.
3.3 Analyzing the limit: Generalized Cramér-Lundberg

As stated in Theorem 3.3, the stationary waiting time in the system with \( n \) servers converges to the endogenous solution to the stochastic fixed-point equation (3.3), which receives its name since it can be explicitly constructed on a weighted branching tree. For completeness, we now briefly describe the construction of a weighted branching tree.

Let \((Q,N,C_1,C_2,\ldots)\) be a vector with \( N \in \mathbb{N} \cup \{\infty\} \), and \( Q,\{C_i\} \) real-valued. Given a sequence of i.i.d. vectors \( \{(Q_i,N_i,C_{(i,1)},C_{(i,2)},\ldots)\}_{i \in U} \) having the same distribution as the generic branching vector \((Q,N,C_1,C_2,\ldots)\), we use the random variables \( \{N_i\}_{i \in U} \) to determine the structure of the tree as follows. Let \( A_0 = \{\emptyset\} \) and

\[
A_r = \{(i,i_r) \in U : i \in A_{r-1}, 1 \leq i_r \leq N_i\}, \quad r \geq 1,
\]

be the set of individuals in the \( r \)th generation. Next, assign to each node \( i \) in the tree a weight \( \Pi_i \) according to the recursion

\[
\Pi_0 = 1, \quad \Pi_(i,j) = C_{(i,j)} \Pi_i.
\]

Each weight \( \Pi_i \) is also usually multiplied by its corresponding value \( Q_i \) to construct solutions to non-homogeneous stochastic fixed-point equations.

In the general formulation, the vector \((Q,N,C_1,C_2,\ldots)\) is allowed to be arbitrarily dependent, although for the special case appearing in this paper we will have \( N < \infty \) a.s., \( Q \equiv 1 \), and the \( \{C_i\} \) nonnegative, i.i.d., and independent of \( N \). For more details we refer the reader to [17, 12, 13].

To make the connection between the high-order Lindley equation (3.3) and the main result in [14], let \( R = e^W \), \( R_i = e^{W_i} \), \( Q \equiv 1 \), and \( C_i = e^{\chi_i - \tau} \) to obtain

\[
R \overset{D}= Q \lor \left( \bigvee_{i=1}^{N} C_i R_i \right), \quad (3.7)
\]

where \( x \lor y \) denotes the maximum of \( x \) and \( y \). We refer to (3.7) with a generic branching vector of the form \((Q,N,C_1,C_2,\ldots)\) with the \( \{C_i\} \) nonnegative, and the \( \{R_i\} \) i.i.d. copies of \( R \) independent of \((Q,N,C_1,C_2,\ldots)\), as the branching maximum equation.

It is easy to verify, as was done in [14], that the random variable

\[
R \overset{D}= \bigvee_{r=0}^{\infty} \bigvee_{j \in A_r} \Pi_j Q_j
\]

is a solution to (3.7), known in the literature as the endogenous solution [11, 9]. Moreover, when \( Q \geq 0 \), the endogenous solution is also the minimal one in the usual stochastic order sense (see [9] and also the survey paper [11] for additional references and a wide variety of max-plus equations). Taking logarithms on both sides of (3.7) (with \( Q \equiv 1 \)), we obtain that the endogenous solution to (3.3) is given by

\[
W \overset{D}= \bigvee_{r=0}^{\infty} \bigvee_{j \in A_r} S_j, \quad (3.8)
\]

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where \( S_\emptyset = 0, S_j = \log \Pi_j = X_{j|1} + X_{j|2} + \cdots + X_j \) for \( j \neq \emptyset \), and \( X_1 = \chi_1 - \tau_1 \). Furthermore, it was shown in [14] (see Lemma 3.1) that this endogenous solution is finite a.s. provided

\[
E \left[ \sum_{i=1}^N e^{\beta X_i} \right] < 1
\]

for some \( \beta > 0 \), which we will refer to as the stability condition.

By rewriting \( W \) as

\[
W = \max \left\{ 0, \max_{j \in A_1} X_j, \max_{j \in A_2} (X_{j|1} + X_j), \max_{j \in A_3} (X_{j|1} + X_{j|2} + X_j), \ldots \right\},
\]

the similarities with (3.2) become apparent. To give some additional intuition as to why (3.8) is the appropriate solution, it is helpful to recall the \( N \equiv 1 \) case, where Lindley’s equation is known to have a unique solution whenever \( E[X_1] < 0 \). Moreover, this solution can be expressed in terms of the supremum of the random walk \( S_k = X_1 + \cdots + X_k, S_0 = 0 \). A standard proof of this relation consists in iterating the recursion

\[
W_{n+1} = \max \left\{ 0, X_n + W_n \right\}, \quad W_0 = 0,
\]

to obtain

\[
W_{n+1} = \max \left\{ 0, X_n, X_n + X_{n-1}, \ldots, X_n + X_{n-1} + \cdots + X_1 \right\} \overset{D}{=} \max_{0 \leq k \leq n} S_k.
\]

It follows by taking the limit as \( n \to \infty \) on both sides that the stationary waiting time in the FCFS GI/GI/1 queue satisfies

\[
W \overset{D}{=} \max_{k \geq 0} S_k.
\]

This last observation is also the key for establishing the celebrated Cramér-Lundberg approximation, which says that if there exists a solution \( \theta > 0 \) such that \( E[e^{\theta X_1}] = 1 \), then

\[
P(W > x) \sim K e^{-\theta x}, \quad x \to \infty,
\]

for some constant \( K > 0 \) (see, e.g., [7], Chapter XIII). The constant \( \theta \) is known in the literature as the Cramér-Lundberg root, and \( K \) can be written in terms of the limiting overshoot in the renewal process defined by the ladder heights of \( \{S_k\} \).

It is then to be expected that a similar story can be told in the branching setting. This is indeed the case, as was recently shown in [14]. There, for the endogenous solution to the more general branching maximum recursion (3.7), it was shown that under the natural root condition \( E \left[ \sum_{i=1}^N C_i^\theta \right] = 1 \) and the derivative condition \( 0 < E \left[ \sum_{i=1}^N C_i^\theta \log C_i \right] < \infty \), we have that

\[
P(R > x) \sim H x^{-\theta}, \quad x \to \infty,
\]

for some constants \( \theta, H > 0 \). Note that for the high-order Lindley’s equation (3.3), this condition translates into the existence of a root \( \theta > 0 \) such that \( E \left[ \sum_{i=1}^N e^{\theta X_i} \right] = 1 \). The power-law
asymptotics of $R$ is a consequence of the Implicit Renewal Theorem on Trees from [12, 13], which constitutes a powerful tool for the analysis of many different types of branching recursions (e.g., the maximum recursion [6, 14], the linear recursion or smoothing transform [2, 4, 5, 12, 13, 3], the discounted tree sum [11], etc). This theorem is in turn a generalization of the Implicit Renewal Theorem of Goldie [10] for non-branching recursions, which can be used to analyze the random coefficient autoregressive process of order one and the reflected random walk, among others. The name “implicit” refers to the fact that the Renewal Theorem is applied to a random variable $R$ (e.g., the solution to a stochastic fixed-point equation) without having knowledge of its distribution, which in turn leads to the resulting constant $H$ in the asymptotics to be implicitly defined in terms of $R$ itself. As useful as the Implicit Renewal Theorem on Trees is, it does not on its own give the strict positivity of the constant $H$, so unlike the $N≡1$ case where a ladder heights argument gives that $H > 0$, the branching case needs a more careful analysis (see [14] for more details).

We conclude this section with the theorem describing the asymptotic behavior of the endogenous solution $W$, which is a direct consequence of Theorem 3.4 in [14] and the observation that the $\{\tau_i\}$ are exponentially distributed and independent of $(N, \chi_1, \ldots, \chi_N)$, giving the required non-arithmetic condition.

**Theorem 3.4** Let $W$ be given by (3.8) and suppose that for some $\theta > 0$, $E \left[ \sum_{i=1}^{N} e^{\theta(\chi_i - \tau_i)} \right] = 1$ and $0 < E \left[ \sum_{i=1}^{N} e^{\theta(\chi_i - \tau_i)}(\chi_i - \tau_i) \right] < \infty$. In addition, assume

1) $E \left[ \left( \sum_{i=1}^{N} e^{\chi_i - \tau_i} \right)^{\theta} \right] < \infty$, if $\theta > 1$; or,

2) $E \left[ \left( \sum_{i=1}^{N} e^{\theta(\chi_i - \tau_i)/(1+\epsilon)} \right)^{1+\epsilon} \right] < \infty$ for some $0 < \epsilon < 1$, if $0 < \theta \leq 1$.

Then,

$$P(W > x) \sim H e^{-\theta x}, \quad x \to \infty,$$

where $0 < H < \infty$ is given by

$$H = \frac{E \left[ 1 \vee \sum_{i=1}^{N} e^{\theta(\chi_i - \tau_i + W_i)} - \sum_{i=1}^{N} e^{\theta(\chi_i - \tau_i + W_i)} \right]}{\theta E \left[ \sum_{i=1}^{N} e^{\theta(\chi_i - \tau_i)}(\chi_i - \tau_i) \right]}.$$

4 **Proofs**

This section of the paper contains the proof of Theorem 3.3. To ease the exposition we separate the main theorem into two parts, the first one concerning the existence of a stationary waiting time for a fixed number of servers and a fixed arrival rate of jobs, the second one establishing the limiting distribution of the stationary waiting time as the number of servers and the arrival rate of jobs grow to infinity.

We start by summarizing some of the notation that will be used throughout this section, starting with all the random variables involved in the predecessor graph. Let

$$\hat{U}_0 = 0, \quad \hat{U}_r = \max_{j \in \hat{A}_r} \hat{S}_j, \quad r \geq 1,$$
where $\hat{S}_j = \hat{X}_{j|1} + \hat{X}_{j|2} + \cdots + \hat{X}_1$, $\hat{X}_1 = \chi_1 - \hat{\tau}_1$, and $\hat{A}_r$ is the set of labels (not jobs) in the predecessor graph at graph distance $r$ from the tagged job. Note that $|B_{\emptyset,r}| \leq |\hat{A}_r|$, since $B_{\emptyset,r}$ refers to the set of labels in $G_n(t_0)$, where there are some jobs/fragments that do not have any predecessors, and as $t_0 \to -\infty$ all jobs/fragments will have have one. We also point out that since every time multiple paths in the graph merge (i.e., every time a job that is an immediate predecessor to multiple fragments arrives) all the jobs from that point onwards will have multiple labels, then some of the $\{\hat{N}_i\}$ will be repeated, and are therefore not independent. Similarly, the $\{\hat{\tau}_i\}$ correspond to the inter arrival times between jobs in the predecessor graph (the length of the edges), and are therefore, in general, neither independent of each other nor of the $\{\hat{N}_i\}$. More precisely, the marginal distribution of each of the $\hat{\tau}_i$ is exponential with rate $\lambda_n^*$, but conditionally on knowing that $i$ shares a predecessor with one or more other jobs, its rate changes and all the inter arrival times corresponding to edges that merge into the same job become dependent. Finally, the $\{\chi_i\}$ are identically distributed with marginal distribution $B$, and their dependance with the $\{\hat{\tau}_i\}$ and $\{\hat{N}_i\}$ is limited to the multiplicity of the labels (i.e., labels referring to edges that lie on merged paths have the same service requirements). Nonetheless, service requirements of the form $\chi$ with $i \neq j$ may also be dependent if they correspond to fragments of the same job (i.e., they correspond to service requirements on paths that merge). We will identify labels belonging to the same job in the predecessor graph through the equivalence relation $i \sim j$.

The analysis of the predecessor graph will become tractable once we identify suitable approximations where the merging of paths due to common predecessors does not occur, in other words, where the predecessor graph is truly a tree. Since these approximations will be used several times in the proofs it is convenient to define them upfront.

In general, we will use $\hat{N}$ to refer to a random variable having distribution $f_n$, $N$ to refer to a random variable having distribution $f$, $\chi$ to refer to a random variable having distribution $B$, $\hat{\tau}$ to denote an exponential random variable with rate $\lambda_n^*$, and $\tau$ to denote an exponential random variable with rate $\lambda^*$.

Let $\{\hat{N}_i\}$, denote a sequence of i.i.d. copies of $\hat{N}$, $\{\chi_i\}$ an i.i.d. sequence of copies of $\chi$, and $\{\hat{\tau}_i\}_{i \in U}$ an i.i.d. sequence of copies of $\hat{\tau}$, all independent of each other. Use the $\{\hat{N}_i\}$ to define a branching process by setting $\hat{A}_0 = \{\emptyset\}$ and $\hat{A}_r = \{(i,i_r) : i \in \hat{A}_{r-1}, 1 \leq i_r \leq \hat{N}_i\}$ for $r \geq 1$. Next, set $\hat{X}_1 = \chi_1 - \hat{\tau}_1$, $\hat{S}_j = \hat{X}_{j|1} + \hat{X}_{j|2} + \cdots + \hat{X}_j$ and define

$$\hat{U}_0 = 0, \quad \hat{U}_r = \max_{j \in \hat{A}_r} \hat{S}_j, \quad r \geq 1.$$ 

Similarly, by repeating the construction given above after removing the $\sim$ from all the random variables we obtain

$$U_0 = 0, \quad U_r = \max_{j \in \hat{A}_r} S_j, \quad r \geq 1,$$

where $S_j = X_{j|1} + X_{j|2} + \cdots + X_j$ and $X_1 = \chi_1 - \tau_1$. We point out that if $f_n = f$ for all $n \geq n_0$, then there is no difference between all the $\sim$ random variables and those without it.

We are now ready to prove the first part of Theorem 3.3.

**Theorem 4.1 (Stability)** Let $W_{t_0}^{(n,t_0)}$ denote the waiting time, excluding service, of the tagged job (the first job to arrive after time zero) when we start the system empty at time $t_0 < 0$. Suppose
that
\[ E[N]E \left[ e^{\beta (\chi - \tau)} \right] < 1 \tag{4.1} \]
for some $\beta > 0$, where $N$ has distribution $f_n$, $\chi$ has distribution $B$ and $\tau$ is exponentially distributed with rate $\lambda_n^*$ and is independent of $\chi$. Then, for any fixed number of servers $n$,
\[ \lim_{t_0 \to -\infty} W_{t_0}^{(n)} = W^{(n)} \quad \text{a.s.} \]
for some finite random variable $W^{(n)}$.

**Proof.** Since
\[ W_{t_0}^{(n)} = \bigvee_{r=0}^{\kappa} \bigvee_{j \in B_{t_0}, r} \left( \hat{X}_{j|1} + \hat{X}_{j|2} + \cdots + \hat{X}_j \right), \tag{4.2} \]
and as $t_0 \to -\infty$ we have that $\kappa \to \infty$ a.s. and $B_{t_0} \uparrow \hat{A}$, it follows by monotone convergence that
\[ \lim_{t_0 \to -\infty} W_{t_0}^{(n)} = \bigvee_{r=0}^{\infty} \hat{U}_r \triangleq W^{(n)} \quad \text{a.s.} \]
Therefore, it only remains to verify that $W^{(n)} < \infty$ a.s.

To establish the finiteness of $W^{(n)}$ we note that it suffices to show that
\[ P(\hat{U}_r > 0 \text{ i.o.}) = 0. \]
This in turn will follow from the Borel-Cantelli Lemma once we show that
\[ P(\hat{U}_r > 0) \leq c^r \tag{4.3} \]
for some constant $0 < c < 1$. Therefore, we focus on showing (4.3).

Recall from the observations made at the beginning of this section that the $\{\hat{\tau}_i\}_{i \in G_n(t_0)}$ are neither i.i.d. nor independent of the $\{N_i\}_{i \in G_n(t_0)}$. More precisely, recall that for each piece of a job requiring service at server $s_i$ there are $\binom{n-1}{k-1}k!$ possible immediate predecessors of size $k$ (i.e., jobs that also require service from server $s_i$). Since the arrival of jobs is assumed to follow a Poisson process, this lead in (3.4) to the inter arrival time between a fixed piece of a job and its unique immediate predecessor to be exponentially distributed with rate $\lambda_n^*$. The problem arises when the piece of a job has two or more immediate predecessors, in which case the rate for the exponential changes.

Consider an arrival that is predecessor to two jobs (or two pieces of the same job), $j_1$ and $j_2$, and note that there must be two different servers, say $s_{i_1}$ and $s_{i_2}$, that are required by the arriving job and that are also assigned to jobs $j_1$ and $j_2$, respectively. There are only $\binom{n-2}{k-2}k!$ possible jobs of size $k$ requiring service by servers $s_{i_1}$ and $s_{i_2}$, and therefore, the rate at which such a predecessor arrives is given by
\[ \lambda_n^{(2)} = \sum_{k=2}^{m_n} \lambda_k \binom{n-2}{k-2} k!. \]
In general, a job that is predecessor to jobs \( j_1, j_2, \ldots, j_r \) in the graph arrives at a rate
\[
\lambda_n^{(r)} = \sum_{k=r}^{m_n} \lambda_k \binom{n-r}{k-r} k! \leq \sum_{k=1}^{m_n} \lambda_k \binom{n-1}{k-1} k! = \lambda_n^*.
\]

As for the lack of independence between the \( \{\hat{\tau}_i\} \), note that the inter arrival times between pieces of jobs that have a common immediate predecessor are dependent. The sequence \( \{\hat{\tau}_i\} \) is also dependent on the \( \{\hat{N}_i\} \), since a large number of jobs awaiting for a predecessor to arrive increases the probability of an arriving job being predecessor to two or more pieces at a time. Hence, the analysis of \( P(\hat{U}_r > 0) \) needs some care.

We start by using Markov’s inequality to obtain
\[
P(\hat{U}_r > 0) = P(\max_{j \in \hat{A}_r} \hat{S}_j > 0) \leq E \left[ \bigvee_{j \in \hat{A}_r} e^{\beta \hat{S}_j} \right] \leq E \left[ \sum_{j \in \hat{A}_r} e^{\beta \hat{S}_j} \right].
\]

Now rewrite the last expectation as follows
\[
E \left[ \sum_{j \in \hat{A}_r} e^{\beta \hat{S}_j} \right] = \sum_{j \in \mathbb{N}_r^+} E \left[ e^{\beta \hat{S}_j} 1(j \in \hat{A}_r) \right],
\]
and notice that
\[
1(j \in \hat{A}_r) = \prod_{k=0}^{r-1} 1(j_{k+1} \leq \hat{N}_{j_{k+1}}),
\]
and is therefore independent of the \( \{\hat{\tau}_{j,k}\}_{k=1}^r \). Since all labels along a path correspond to different jobs, then the vectors \( \{(\hat{N}_{j,k}, \chi_{(j,k,1)}, \ldots, \chi_{(j,k,\hat{N}_{j,k})})\}_{k=0}^{r-1} \) are i.i.d. copies of \( (\hat{N}, \chi_1, \ldots, \chi_N) \), where the \( \{\chi_i\} \) are i.i.d. copies of \( \chi \), independent of \( \hat{N} \). To eliminate the dependence between this last sequence and the inter arrival times note that we can replace the \( \{\hat{\tau}_{j,k}\}_{k=1}^r \) with i.i.d. copies of \( \hat{\tau} \), independent of \( \{(\hat{N}_{j,k}, \chi_{(j,k,1)}, \ldots, \chi_{(j,k,\hat{N}_{j,k})})\}_{k=0}^{r-1} \) to obtain
\[
e^{\beta \hat{S}_j} 1(j \in \hat{A}_r) \leq_{s.t.} e^{\beta \hat{S}_j} 1(j \in \hat{A}_r),
\]
where \( \leq_{s.t.} \) denotes the usual stochastic order, and the \( \{\hat{S}_i\} \) were defined at the beginning of this section. It follows that
\[
P(\hat{U}_r > 0) \leq E \left[ \sum_{j \in \hat{A}_r} e^{\beta \hat{S}_j} \right] \leq E \left[ \sum_{j \in \hat{A}_r} e^{\beta \hat{S}_j} \right], \quad (4.4)
\]
where the last expectation can be computed using standard weighted branching processes arguments (see, e.g., [12]) and is given by
\[
E \left[ \sum_{j \in \hat{A}_r} e^{\beta \hat{S}_j} \right] = \left( E \left[ \sum_{i=1}^{N} e^{\beta (\chi_i - \hat{\tau}_i)} \right] \right)^r = \left( E[\hat{N}] E \left[ e^{\beta (\chi - \hat{\tau})} \right] \right)^r. \quad (4.5)
\]
Setting $c = E[\hat{N}]E\left[e^{\beta(X-Y)}\right] < 1$ completes the proof. ■

For the second part of the main theorem we will prove that

$$W^{(n)} \xrightarrow{W_p} W \quad \text{as } n \to \infty$$

for any $p \geq 1$, where $W_p$ denotes the Wasserstein distance of order $p$ (see, e.g., [19], Chapter 6). This is equivalent to convergence in distribution plus convergence of all the moments of order up to $p$ (see Theorem 6.8 in [19]).

To this end, we will consider three different sets of processes that will yield intermediate approximations between $W^{(n)}$ and $W$. In particular, we will show that if $\mu_n$ is the probability measure of $W^{(n)}$, $\hat{\nu}_k$ is the probability measure of $\bigvee_{r=0}^k \hat{U}_r$, $\tilde{\nu}_k$ is that of $\bigvee_{r=0}^k \tilde{U}_r$, $\nu_k$ of $\bigvee_{r=0}^k U_r$, and, finally, $\mu$ is the probability measure of

$$W = \bigvee_{r=0}^\infty U_r,$$

then

$$W_p(\mu_n, \mu) \leq W_p(\mu_n, \hat{\nu}_{r_n}) + W_p(\hat{\nu}_{r_n}, \tilde{\nu}_{r_n}) + W_p(\tilde{\nu}_{r_n}, \nu_{r_n}) + W_p(\nu_{r_n}, \mu) \to 0 \quad (4.6)$$

as $n \to \infty$, for some $r_n \to \infty$.

The technical difficulty in the proofs lies in the need to construct explicit couplings of the pairs of probability measures involved for which we can show that their difference converges to zero in $L_p$ norm. We point out that although $W^{(n)}$ and $\bigvee_{r=0}^{r_n} \hat{U}_r$, as well as $W$ and $\bigvee_{r=0}^{r_n} U_r$, are naturally defined on the same probability space, all other pairs are not.

**Theorem 4.2 (Many servers limit)** Let $W^{(n)}$ denote the waiting time, excluding service, of a job in stationarity. Then, provided Assumption 3.2 is satisfied,

$$W^{(n)} \xrightarrow{W_p} W, \quad n \to \infty,$$

for any $p \geq 1$, where $W$ is the endogenous solution to the high-order Lindley equation (3.3).

The proof of Theorem 4.2 is based on series of results.

**Lemma 4.3** Suppose Assumption 3.2 (i)-(ii) is satisfied. Then, for any $r_n \to \infty$ as $n \to \infty$, and any $p \geq 1$, we have that

$$\lim_{n \to \infty} E\left[\left|W^{(n)} - \bigvee_{r=0}^{r_n} \hat{U}_r\right|^p\right] = 0 \quad \text{and} \quad \lim_{n \to \infty} E\left[\left|W - \bigvee_{r=0}^{r_n} U_r\right|^p\right] = 0.$$ 

In particular, this implies that, as $n \to \infty$,

$$W_p(\mu_n, \hat{\nu}_{r_n}) \to 0 \quad \text{and} \quad W_p(\nu_{r_n}, \mu) \to 0.$$
Proof. Let $\beta > 0$ be the one from Assumption 3.2 (ii) and let $\rho_\beta = E[N]E\left[e^{\beta (\chi - \bar{\tau})}\right] < 1$. Fix $0 < \epsilon < 1 - \rho_\beta$ and note that

$$E[\tilde{N}]E\left[e^{\beta (\chi - \bar{\tau})}\right] = E[\tilde{N}]E\left[e^{\beta \chi}\right] \left(\frac{\lambda_n^*}{\lambda_n^* + \beta}\right).$$

By Assumption 3.2 (i) we have that $E[\tilde{N}] \to E[N]$, and therefore $\lambda_n^* \to \lambda^*$ as $n \to \infty$. Hence,

$$\lim_{n \to \infty} E[\tilde{N}]E\left[e^{\beta (\chi - \bar{\tau})}\right] = \rho_\beta. \quad (4.7)$$

It follows that for large enough $n$,

$$E[\tilde{N}]E\left[e^{\beta (\chi - \bar{\tau})}\right] \leq \rho_\beta + \epsilon < 1.$$

Next, note that

$$E\left[\left|W^{(n)} - \sqrt{r_n} \hat{U}_r\right|^p\right] = E\left[\left(\left(\sum_{r=r_n+1}^{\infty} \hat{U}_r - \sqrt{r_n} \hat{U}_r\right)^+\right)^p\right] \leq E\left[\left(\sum_{r=r_n+1}^{\infty} \hat{U}_r\right)^+\right]^p.$$

To analyze the last expectation note that by Markov’s inequality,

$$E\left[\left(\hat{U}_r\right)^+\right] = \int_0^\infty P\left(\left(\hat{U}_r\right)^+ > x\right) dx = \int_0^\infty P\left(e^{\beta \hat{U}_r} > e^{\beta x^{1/p}}\right) dx \leq E\left[e^{\beta \hat{U}_r}\right]\int_0^\infty e^{-\beta x^{1/p}} dx = E\left[\bigvee_{j \in \tilde{A}_r} e^{\beta \hat{S}_j}\right] \frac{p}{\beta^p} \int_0^\infty u^{p-1}e^{-u} du,$$

where $\int_0^\infty u^{p-1}e^{-u} du = E[\xi^{p-1}] < \infty$ with $\xi$ exponentially distributed with rate one. Letting $C_{\beta,p} = pE[\xi^{p-1}] / \beta^p$ gives

$$E\left[\left(\hat{U}_r\right)^+\right] \leq C_{\beta,p} E\left[\bigvee_{j \in \tilde{A}_r} e^{\beta \hat{S}_j}\right] \leq C_{\beta,p} E\left[\sum_{j \in \tilde{A}_r} e^{\beta \hat{S}_j}\right].$$

Moreover, as shown in the proof of Theorem 4.1 (see (4.4) and (4.5)), we have

$$E\left[\sum_{j \in \tilde{A}_r} e^{\beta \hat{S}_j}\right] \leq E\left[\sum_{j \in \tilde{A}_r} e^{\beta \hat{S}_j}\right] = \left(E[\tilde{N}]E\left[e^{\beta (\chi - \bar{\tau})}\right]\right)^r.$$

It follows that, for sufficiently large $n$,

$$\sum_{r=r_n+1}^{\infty} E\left[\left(\hat{U}_r\right)^+\right] \leq C_{\beta,p} \sum_{r=r_n+1}^{\infty} \left(E[\tilde{N}]E\left[e^{\beta (\chi - \bar{\tau})}\right]\right)^r \leq C_{\beta,p} \sum_{r=r_n+1}^{\infty} (\rho_\beta + \epsilon)^r = O((\rho_\beta + \epsilon)^n)$$
as \( n \to \infty \), since \( \rho_\beta + \epsilon < 1 \).

The proof involving \( W \) and the \( \{U_r\}_{r \geq 0} \) is essentially the same as is therefore omitted. □

The following result regarding the contribution of all the paths with multiple labels in the predecessor graph is the most technical one in the paper, since it is where the subtle dependence introduced by the merging of paths plays a role.

**Lemma 4.4** For \( r \geq 1 \) define \( M_r = \{ i \in \hat{A}_r : i \sim j \text{ for some } j \neq i \} \) to be the set of labels in the predecessor graph at graph distance \( r \) from the tagged job belonging to jobs with multiple labels. Then, for any \( \beta > 0 \) and \( \bar{\rho}_\beta = E[\bar{N}]E[e^{\beta(x-\hat{\tau})}] \),

\[
E \left[ \sum_{i \in M_r} e^{\beta \hat{S}_i} \right] \leq \frac{r}{n} E \left[ \bar{N}^2 \right] \left( \bar{\rho}_\beta \right)^r.
\]

**Proof.** We first write for \( r \geq 1 \),

\[
E \left[ \sum_{i \in M_r} e^{\beta \hat{S}_i} \right] = \sum_{i \in [\hat{N}]_r} E \left[ e^{\beta \hat{S}_i} 1(i \in M_r) \right].
\]

Now note that along a path all jobs are different and therefore the service requirements \( \{\chi_{i|k}\}_{k=1}^{r-1} \) are i.i.d. with distribution \( B \), and are independent of the \( \{\hat{N}_{i|k}\}_{k=0}^{r-1} \). The inter arrival times \( \{\hat{\tau}_{i|k}\}_{k=1}^{r-1} \) do depend on the \( \{\hat{N}_{i|k}\}_{k=0}^{r-1} \) in the sense that a large number of jobs in the predecessor graph at the time a job arrives increases its probability of being an immediate predecessor to more than one job, and therefore influences the rate of the corresponding \( \hat{\tau} \). It follows that if we replace them by i.i.d. copies of \( \hat{\tau} \) independent of everything else, we obtain

\[
E \left[ e^{\beta \hat{S}_i} 1(i \in M_r) \right] \leq \left( E \left[ e^{\beta(x-\hat{\tau})} \right] \right)^r P(i \in M_r).
\]

To compute the last probability let \( C_i \) denote the event that \( i \) is a common immediate predecessor to two or more jobs/fragments in the predecessor graph, and note that

\[
1(i \in M_r) = \sum_{s=1}^{r} 1((i|s-1) \in M_{s-1}, (i|s) \in M_s) \leq \sum_{s=1}^{r} 1(i \in A_r, C_{i|s}),
\]

and therefore,

\[
\sum_{i \in [\hat{N}]_r} P(i \in M_r) \leq \sum_{s=1}^{r} \sum_{i \in [\hat{N}]_r} E \left[ \prod_{k=0}^{r-1} 1(i_{k+1} \leq \hat{N}_{i|k}) 1(C_{i|s}) \right].
\]

Next, let \( \sigma_1 = T_1 - \hat{\tau}_{i1} - \cdots - \hat{\tau}_1 \) denote the time at which job \( i \) arrived to the predecessor graph. Define \( \mathcal{F}_t = \sigma(\{\hat{N}_j, s_j, \chi_j, \hat{\tau}_j\) : \( \sigma_j > t \) \) to be the sigma algebra containing the “history” of the
predecessor graph over the interval \((t, T_1]\), and note that \(F_{\sigma_1}\) does not reveal whether \(C_i\) occurred nor the value of \(\hat{N}_i\). Now note that for \(1 \leq s \leq r - 1\),

\[
\sum_{i \in N^s_+} E \left[ \prod_{k=0}^{r-1} 1(i_{k+1} \leq \hat{N}_{i[k]} 1(C_{i[s]}) \right] = \sum_{i \in N^s_+} E \left[ \prod_{k=0}^{s-1} 1(i_{k+1} \leq \hat{N}_{i[k]} 1 \left( C_{i[s]} \right) \bigg| F_{\sigma_i[s]} \right) E \left[ \prod_{k=s}^{r-1} 1(i_{k+1} \leq \hat{N}_{i[k]} 1(C_{i[s]}) \bigg| F_{\sigma_i[s]} \right) \right].
\]

Moreover, since \(\{\hat{N}_{i[k]}\}_{k=s}^{r-1}\) are independent of \(F_{\sigma_i[s]}\), then

\[
E \left[ \prod_{k=s}^{r-1} 1(i_{k+1} \leq \hat{N}_{i[k]} 1(C_{i[s]}) \bigg| F_{\sigma_i[s]} \right] = E \left[ 1(i_{s+1} \leq \hat{N}_{i[s]} 1(C_{i[s]}) \bigg| F_{\sigma_i[s]} \right] \prod_{k=s+1}^{r-1} P(i_{k+1} \leq \hat{N}_{i[k]}),
\]

with the convention that \(\prod_{a=\alpha}^{\beta} x_i \equiv 1\) if \(a > b\).

To analyze the last conditional expectation let \(K_i\) be the number of pieces of jobs that are available at time \(t\), where by available we mean that they they do not have an immediate predecessor in \((t, T_1]\). Note that the event \(C_j\) is a function of \(K_{\sigma_j}\) and \(\hat{N}_j\) only; more precisely, for any \(j \in N^s_+\) we have

\[
P(C_j|\hat{N}_j, K_{\sigma_j}) = \frac{(K_{\sigma_j} - 1)}{(K_{\sigma_j})} \leq \frac{\hat{N}_j}{n}.
\]

It follows that

\[
E \left[ 1(j_{s+1} \leq \hat{N}_j) C_j \bigg| F_{\sigma_j} \right] = E \left[ 1(j_{s+1} \leq \hat{N}_j) P(C_j|\hat{N}_j, K_{\sigma_j}) \bigg| F_{\sigma_j} \right] \leq E \left[ 1(j_{s+1} \leq \hat{N}_j) \frac{\hat{N}_j}{n} \bigg| F_{\sigma_j} \right] = \frac{1}{n} E[\hat{N}_j 1(j_{s+1} \leq \hat{N}_j)].
\]

It follows that, for \(1 \leq s \leq r - 1\),

\[
\sum_{i \in N^s_+} E \left[ \prod_{k=0}^{r-1} 1(i_{k+1} \leq \hat{N}_{i[k]} 1(C_{i[s]}) \right] \leq \sum_{i \in N^s_+} E \left[ \prod_{k=0}^{s-1} 1(i_{k+1} \leq \hat{N}_{i[k]} \right] \frac{1}{n} E[\hat{N}_{i[s]} 1(i_{s+1} \leq \hat{N}_{i[s]})] \prod_{k=s+1}^{r-1} P(i_{k+1} \leq \hat{N}_{i[k]} \right) = \frac{1}{n} \left( E[\hat{N}] \right)^{r-1} E[\hat{N}^2].
\]

For \(s = r\) note that the same arguments used above give

\[
\sum_{i \in N^s_+} E \left[ \prod_{k=0}^{r-1} 1(i_{k+1} \leq \hat{N}_{i[k]} 1(C_i) \right] = \sum_{i \in N^s_+} \prod_{k=0}^{r-1} P(i_{k+1} \leq \hat{N}_{i[k]} P(C_i) \leq \sum_{i \in N^s_+} \prod_{k=0}^{r-1} P(i_{k+1} \leq \hat{N}_{i[k]} \frac{E[\hat{N}_i]}{n} = \frac{1}{n} \left( E[\hat{N}] \right)^{r+1}.
\]
We conclude that

\[
E \left[ \sum_{i \in M_r} e^{\beta S_i} \right] \leq \left( E \left[ e^{\beta (X - \bar{\tau})} \right] \right)^r \left( \sum_{s=1}^{r-1} \frac{1}{n} \left( E[\bar{N}] \right)^{r-1} E[\bar{N}^2] + \frac{1}{n} \left( E[\bar{N}] \right)^{r+1} \right)
\]

\[
= \frac{1}{n} (\tilde{\rho}_\beta)^r \left( (r-1) \frac{E[\bar{N}^2]}{E[\bar{N}]} + E[\bar{N}] \right).
\]

Noting that \( 1 \leq (E[\bar{N}])^2 \leq E[\bar{N}^2] \) completes the proof. 

\[\blacksquare\]

**Lemma 4.5** Let \( X_1, X_2, Y_1, Y_2 \) be nonnegative random variables and let \( p \geq 1 \). Then

\[
(E[|X_1 \lor X_2 - Y_1 \lor Y_2|^p])^{1/p} \leq (E[X_2^p])^{1/p} + (E[Y_2^p])^{1/p} + (E[|X_1 - Y_1|^p])^{1/p}.
\]

**Proof.** First note that for any real numbers \( x_1, x_2 \) we have that

\[
x_1 \lor x_2 = (x_2 - x_1)^+ + x_1,
\]

from where we obtain that for \( y_1, y_2 \) also real,

\[
|x_1 \lor x_2 - y_1 \lor y_2| \leq (x_2 - x_1)^+ + |x_1 - y_1| + (y_2 - y_1)^+.
\]

Moreover, provided \( x_1, y_1, x_2, y_2 \geq 0 \), we have that

\[
|x_1 \lor x_2 - y_1 \lor y_2| \leq x_2 + |x_1 - y_1| + y_2.
\]

Substituting in the random variables and using Minkowski’s inequality gives the result. 

\[\blacksquare\]

The following proposition contains the main coupling between the predecessor graph and its weighted branching tree approximation. Its proof relies on the bound provided by Lemma 4.4.

**Proposition 4.6** Suppose that Assumption 3.2 is satisfied. Then, for \( \nu_k \), the probability measure of \( \bigvee_{r=0}^k \tilde{U}_r \), and \( \tilde{\nu}_k \), the probability measure of \( \bigvee_{r=0}^k \tilde{U}_r \), we have that for any \( n \to \infty \), and any \( p \geq 1 \),

\[
W_p(\nu_{\tilde{r}_n}, \tilde{\nu}_{\tilde{r}_n}) \to 0 \quad n \to \infty.
\]

**Proof.** From the definition of the Wasserstein metric, we need to construct a coupling of \( \nu_{\tilde{r}_n} \) and \( \tilde{\nu}_{\tilde{r}_n} \) for which we can show that their \( L_p \) distance converges to zero. We will do this by defining a weighted branching tree that will be very close to the predecessor graph restricted to predecessors at graph distance at most \( r_n \) of the tagged job (i.e., whose labels are of the form \( \mathbf{i} = (i_1, \ldots, i_r) \) with \( r \leq r_n \)). To start, define \( M_r = \{ \mathbf{i} \in A_r : \mathbf{i} \sim \mathbf{j} \text{ for some } \mathbf{j} \prec \mathbf{i} \} \) as in Lemma 4.4. To construct the weighted branching tree we proceed inductively starting from the tagged job.

Let \( \{\tilde{N}_{\mathbf{i}}\}_{\mathbf{i} \in U} \) be a sequence of i.i.d. copies of \( \tilde{N} \), let \( \{\chi'_{\mathbf{i}}\}_{\mathbf{i} \in U} \) be a sequence of i.i.d. copies of \( \chi \), and let \( \{\tilde{\tau}'_{\mathbf{i}}\}_{\mathbf{i} \in U} \) be a sequence of i.i.d. exponential random variables with rate \( \lambda^*_n \), all sequences independent
of each other and of all other random variables used up to now. Next, let \( \tilde{A}_0 = \{ \emptyset \} = \hat{A}_0 \) and \( \hat{N}_0 \equiv \tilde{N}_0 \), which defines \( \hat{A}_1 = \{ i : 1 \leq i \leq \hat{N}_0 \} \). In general, for \( r \geq 1 \) and each \( i \in \hat{A}_r \), set

\[
\hat{N}_i = \begin{cases} \hat{N}_i, & \text{if } i \in M^c_r, \\ \hat{N}_i', & \text{otherwise,} \end{cases} \quad \text{and} \quad \hat{X}_i = \begin{cases} \hat{X}_i, & \text{if } i \in M^c_r, \\ \hat{X}_i', & \text{otherwise.} \end{cases}
\]

Then, use the newly defined \( \{ \hat{N}_i \}_{i \in \hat{A}_r} \) to construct \( \tilde{A}_{r+1} = \{ (i, i_{r+1}) : i \in \hat{A}_r, 1 \leq i_{r+1} \leq \hat{N}_i \} \). We point out that, by construction, \( M^c_r \subseteq \hat{A}_r \cap \hat{A}_r \). Also, the \( \{ (\hat{N}_i, \hat{X}_i(1), \ldots, \hat{X}_i(i_{\hat{N}_i})) \} \) are now i.i.d. with the same distribution as \( (\tilde{N}, \tilde{X}_1, \ldots, \tilde{X}_{\tilde{N}}) \), and therefore define a weighted branching tree.

Recall from the beginning of the section that

\[
\tilde{S}_0 = 0, \quad \tilde{S}_i = \tilde{X}_{i|1} + \tilde{X}_{i|2} + \cdots + \tilde{X}_i, \quad i \neq \emptyset,
\]

and

\[
\tilde{U}_r = \bigvee_{i \in \hat{A}_r} \tilde{S}_i.
\]

We will show that \( E \left[ \left| \bigvee_{r=0}^{\infty} \tilde{U}_r - \bigvee_{r=0}^{\infty} \tilde{U}_r \right|^p \right] \to 0 \) as \( n \to \infty \).

To this end, note that we can split the paths in \( \hat{U}_r \) and \( \tilde{U}_r \) as follows:

\[
\hat{U}_r = \max \left\{ \bigvee_{i \in M^c_r} \hat{S}_i, \bigvee_{i \in M^c_r} \hat{S}^+_i \right\} \equiv \max \left\{ \hat{U}_r^{(1)}, \hat{U}_r^{(2)} \right\},
\]

and

\[
\tilde{U}_r = \max \left\{ \bigvee_{i \in \hat{A}_r \cap M^c_r} \tilde{S}_i, \bigvee_{i \in \hat{A}_r \cap M^c_r} \tilde{S}^+_i \right\} \equiv \max \left\{ \tilde{U}_r^{(1)}, \tilde{U}_r^{(2)} \right\}.
\]

Note that \( \emptyset \in \hat{A}_r \cap M^c_r \), and therefore \( \hat{U}_r^{(1)} \) and \( \tilde{U}_r^{(1)} \) are nonnegative without having to add the positive parts to the corresponding \( \hat{S}_i \) and \( \tilde{S}_i \). Moreover, \( \hat{S}_i \equiv \tilde{S}_i \) for \( i \in M^c_r \) and therefore, \( \hat{U}_r^{(1)} \equiv \tilde{U}_r^{(1)} \). It follows from Lemma 4.5 that

\[
E \left[ \left| \bigvee_{r=0}^{\infty} \hat{U}_r - \bigvee_{r=0}^{\infty} \tilde{U}_r \right|^p \right] = E \left[ \max \left\{ \bigvee_{r=0}^{\infty} \hat{U}_r^{(1)}, \bigvee_{r=0}^{\infty} \hat{U}_r^{(2)} \right\} - \max \left\{ \bigvee_{r=0}^{\infty} \tilde{U}_r^{(1)}, \bigvee_{r=0}^{\infty} \tilde{U}_r^{(2)} \right\} \right]^p \\
\leq \left( E \left[ \left( \bigvee_{r=0}^{\infty} \hat{U}_r^{(2)} \right)^p \right] \right)^{1/p} + \left( E \left[ \left( \bigvee_{r=0}^{\infty} \tilde{U}_r^{(2)} \right)^p \right] \right)^{1/p}.
\]

To analyze the last two expectations we follow the same approach used in the proof of Lemma 4.3 to obtain

\[
E \left[ \left( \bigvee_{r=0}^{\infty} \hat{U}_r^{(2)} \right)^p \right] \leq C_{\beta, p} \sum_{r=1}^{\infty} E \left[ \sum_{i \in M_r} e^{\beta \hat{S}_i} \right],
\]

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where $C_{\beta,p} = pE[\xi^{p-1}] / \beta^p$ with $\xi$ exponentially distributed with rate one, and $\beta > 0$ is the one from Assumption 3.2 (ii). By Lemma 4.4 we have that

$$E \left[ \sum_{i \in M_r} e^{\beta S_i} \right] \leq \frac{r}{n} E[\tilde{N}^2] \left( E[\tilde{N}] E \left[ e^{\beta (\chi - \tilde{r})} \right] \right)^r.$$

By (4.7) we have that $E[\tilde{N}] E \left[ e^{\beta (\chi - \tilde{r})} \right] \rightarrow \rho_\beta$ as $n \rightarrow \infty$. We conclude that, for any $0 < \epsilon < 1 - \rho_\beta$ and sufficiently large $n$,

$$E \left[ \left( \bigvee_{r=0}^{r_n} \tilde{U}_r \right)^p \right] \leq C_{\beta,p} \frac{r}{n} E \left[ \tilde{N}^2 \right] \left( E[\tilde{N}] E \left[ e^{\beta (\chi - \tilde{r})} \right] \right)^r \leq C_{\beta,p} \frac{r}{n} E \left[ \tilde{N}^2 \right] \sum_{r=1}^{r_n} r(\rho_\beta + \epsilon)^r = O \left( \frac{1}{n} E \left[ \tilde{N}^2 \right] \right).$$

The proof that $E \left[ \left( \bigvee_{r=0}^{r_n} \tilde{U}_r \right)^p \right] = O \left( E \left[ \tilde{N}^2 \right] n^{-1} \right)$ follows the same steps and is therefore omitted. We have thus shown that

$$E \left[ \left( \bigvee_{r=0}^{r_n} \tilde{U}_r \right)^p \right] = O \left( \frac{1}{n} E \left[ \tilde{N}^2 \right] \right)$$

as $n \rightarrow \infty$, which in turn implies that $W_p(\hat{\nu}_{r_n}, \nu_{r_n}) \rightarrow 0$ by Assumption 3.2 (iii).

**Lemma 4.7** Let $\{x_i\}_{i \geq 1}$ and $\{y_i\}_{i \geq 1}$ be two sequences of real numbers. Then, for any $k \geq 1$,

$$\max_{1 \leq i \leq k} (x_i + y_i) - \max_{1 \leq i \leq k} x_i \leq \max_{1 \leq i \leq k} |y_i|.$$

**Proof.** Let $i^*$ be such that $x_{i^*} + y_{i^*} \geq x_i + y_i$ for all $1 \leq i \leq k$, and let $j^*$ be such that $x_{j^*} \geq x_i$ for all $1 \leq i \leq k$. Next suppose that $y_{i^*} \geq 0$ and note that

$$\max_{1 \leq i \leq k} (x_i + y_i) - \max_{1 \leq i \leq k} x_i = x_{i^*} + y_{i^*} - \max_{1 \leq i \leq k} x_i \leq \max_{1 \leq i \leq k} |y_i|.$$

If $y_{i^*} \leq 0$ then write

$$\max_{1 \leq i \leq k} (x_i + y_i) - \max_{1 \leq i \leq k} x_i = x_{j^*} - \max_{1 \leq i \leq k} (x_i + y_i) \leq -y_{j^*} \leq \max_{1 \leq i \leq k} |y_i|.$$

The following, and last, preliminary result provides a coupling for two weighted branching trees. As pointed out earlier, this step is unnecessary if $f_n \equiv f$ for all $n$ sufficiently large.

**Proposition 4.8** Suppose that Assumption 3.2 (i)-(ii) is satisfied. Then, for $\tilde{\nu}_k$, the probability measure of $\bigvee_{r=0}^{k} \tilde{U}_r$, $\nu_k$, the probability measure of $\bigvee_{r=0}^{k} U_r$, any $p \geq 1$ and any $r_n \rightarrow \infty$ such that

$$\lim_{n \rightarrow \infty} E[N] - E[\tilde{N}] \left( E[N] \right)^{r_n/p} r_n = 0,$$

we have that

$$W_p(\tilde{\nu}_{r_n}, \nu_{r_n}) \rightarrow 0 \quad n \rightarrow \infty.$$
Proposition 4.6, so we will skip many of the intermediate steps. First, we obtain that the analysis of the two expectations in (4.8) is very similar to the approach used in the proof of Proposition 4.6.

By using Lemma 4.5 we obtain

\[
\tilde{\eta}_1 = -\frac{1}{\lambda^*_n} \log \xi_i \quad \text{and} \quad \eta_1 = -\frac{1}{\lambda^*_n} \log \xi_i,
\]

\[
\tilde{N}_1 = F^{-1}(\xi_i) \quad \text{and} \quad N_1 = F^{-1}(\xi_i),
\]

where \( g^{-1}(t) = \inf\{x \in \mathbb{R} : g(x) \geq t\} \) (this is the standard inverse transform construction).

Now use the \( \{\tilde{N}_1\} \) and the \( \{N_1\} \) to construct two branching trees according to \( \tilde{A}_0 = \emptyset = A_0 \) and \( \tilde{A}_r = \{(i, i_r) : i \in \tilde{A}_{r-1}, 1 \leq i_r \leq \tilde{N}_1\}, A_r = \{(i, i_r) : i \in A_{r-1}, 1 \leq i_r \leq N_1\} \) for \( r \geq 1 \). It remains to construct the sequences of service requirements. For the \( \sim \) weighted branching tree we sample independent of all other random variables, and set \( \tilde{X}_i = \chi_i - \tau_i \) for each \( i \in U \). Then let \( X_i = \chi_i - \tau_i \).

To analyze the difference between the corresponding processes \( \tilde{U}_r \) and \( U_r \), we start by defining the notion of a miscoupling. We say that there has been a miscoupling at node \( i \in A_r \) if \( N_i \neq \tilde{N}_i \) and \( N_{ijk} = \tilde{N}_{ijk} \) for all \( 1 \leq k < r \). Next, define \( C_r = \{i \in A_r : N_{ijk} = \tilde{N}_{ijk} \} \) for all \( 1 \leq k < r \), which corresponds to the set of individuals in both trees that have no miscouplings along their paths.

Following the same steps used in the proof of Proposition 4.6, and with some abuse of notation, split the paths in \( \tilde{U}_r \) and \( U_r \) as follows:

\[
\tilde{U}_r = \max \left\{ \bigvee_{i \in \tilde{A}_r \cap C_r} \tilde{S}_i^+, \bigvee_{i \in \tilde{A}_r \cap \overline{C}_r} \tilde{S}_i^+ \right\},
\]

and

\[
U_r = \max \left\{ \bigvee_{i \in A_r \cap C_r} S_i^+, \bigvee_{i \in A_r \cap \overline{C}_r} S_i^+ \right\} \triangleq \max \left\{ U_r^{(1)}, U_r^{(2)} \right\}.
\]

By using Lemma 4.5 we obtain

\[
E \left[ \left( \bigvee_{r=0}^{n} \tilde{U}_r - \bigvee_{r=0}^{n} U_r \right)^p \right] = E \left[ \max \left\{ \bigvee_{r=0}^{n} \tilde{U}_r^{(1)}, \bigvee_{r=0}^{n} \tilde{U}_r^{(2)} \right\} - \max \left\{ \bigvee_{r=0}^{n} U_r^{(1)}, \bigvee_{r=0}^{n} U_r^{(2)} \right\} \right]^p
\]

\[
\leq \left\{ E \left[ \left( \bigvee_{r=0}^{n} \tilde{U}_r^{(2)} \right)^p \right] \right\}^{1/p} + \left\{ E \left[ \left( \bigvee_{r=0}^{n} U_r^{(2)} \right)^p \right] \right\}^{1/p} \quad (4.8)
\]

\[
+ \left\{ E \left[ \left( \bigvee_{r=0}^{n} \tilde{U}_r^{(1)} - \bigvee_{r=0}^{n} U_r^{(1)} \right)^p \right] \right\}^{1/p}
\]

(4.9)

The analysis of the two expectations in (4.8) is very similar to the approach used in the proof of Proposition 4.6, so we will skip many of the intermediate steps. First, we obtain that

\[
E \left[ \left( \bigvee_{r=0}^{n} \tilde{U}_r^{(2)} \right)^p \right] \leq C_{\beta,p} \sum_{r=1}^{n} E \left[ \sum_{i \in \tilde{A}_r \cap C_r} e^{\beta \tilde{S}_i} \right],
\]
where $C_{\beta,p}$ is a finite constant and $\beta > 0$ is the same one from Assumption 3.2(ii). To compute the expectation on the right hand side let $H_k = \sigma(\{\tilde{N}_i, \tilde{X}_1(\tilde{N}_i), \ldots, \tilde{X}_n(\tilde{N}_i)\} : i \in \tilde{A}_s, 0 \leq s < k)$ for $k \geq 1$ and note that for $r \geq 2$,

$$a_r \triangleq E \left[ \sum_{i \in \tilde{A}_r \cap C_{r}} e^{\beta \tilde{S}_i} \right] = E \left[ \sum_{i \in \tilde{A}_{r-1} \cap C_{r-1}} \sum_{j=1}^{\tilde{N}_i} e^{\beta \tilde{S}_{i,j}} + \sum_{i \in \tilde{A}_{r-1} \cap C_{r-1}} \sum_{j=1}^{\tilde{N}_i} e^{\beta \tilde{S}_{i,j}} 1(\tilde{N}_i \neq N_i) \right] = E \left[ \sum_{i \in \tilde{A}_{r-1} \cap C_{r-1}} \sum_{j=1}^{\tilde{N}_i} e^{\beta \tilde{X}_{i,j}} \left| H_{r-1} \right\right] = \tilde{\rho}_\beta a_{r-1} + E \left[ \tilde{N}_1(\tilde{N} \neq N) \right] \left[ e^{\beta (\chi-\tilde{\chi})} \right] E \left[ \sum_{i \in \tilde{A}_{r-1} \cap C_{r-1}} e^{\beta \tilde{S}_i} \right],$$

where $\tilde{\rho}_\beta = E[\tilde{N}E \left[ e^{\beta (\chi-\tilde{\chi})} \right]$ and $(\tilde{N}, N) = (F_n^{-1}(\zeta), F^{-1}(\zeta))$ in the first of the last three expectations, with $\zeta \sim \text{Uniform}(0,1)$. Letting $E_n = E \left[ \tilde{N}1(\tilde{N} \neq N) \right] E \left[ e^{\beta (\chi-\tilde{\chi})} \right]$ and noting that

$$E \left[ \sum_{i \in \tilde{A}_{r-1} \cap C_{r-1}} e^{\beta \tilde{S}_i} \right] \leq E \left[ \sum_{i \in \tilde{A}_{r-1}} e^{\beta \tilde{S}_i} \right] = (\tilde{\rho}_\beta)^{r-1},$$

gives

$$a_r \leq \tilde{\rho}_\beta a_{r-1} + E_n(\tilde{\rho}_\beta)^{r-1}.$$ 

Iterating this recursion $r - 1$ times,

$$a_r \leq (\tilde{\rho}_\beta)^{r-1} a_1 + (r - 1)E_n(\tilde{\rho}_\beta)^{r-1} = E_n r (\tilde{\rho}_\beta)^{r-1}.$$ 

Since by (4.7) we have that $\tilde{\rho}_\beta \rightarrow \rho_\beta = E[N]E \left[ e^{\beta (\chi-\tilde{\chi})} \right]$ as $n \rightarrow \infty$, then for $0 < \epsilon < 1 - \rho_\beta$ and $n$ sufficiently large,

$$E \left[ \left( \sqrt[n]{\sum_{r=0}^{r_n} \tilde{U}_r^{(2)}} \right)^p \right] \leq C_{\beta,p} E_n \sum_{r=1}^{r_n} r \rho_\beta)^{r-1} \leq C_{\beta,p} E_n \sum_{r=1}^{\infty} r (\rho_\beta + \epsilon)^{r-1} = O(E_n)$$

as $n \rightarrow \infty$. The proof for the expectation involving the $\{U_r^{(2)}\}$ is symmetric with respect to the $\sim$ notation, so we obtain

$$E \left[ \left( \sqrt[n]{\sum_{r=0}^{r_n} \tilde{U}_r^{(2)}} \right)^p \right] + E \left[ \left( \sqrt[n]{\sum_{r=0}^{r_n} U_r^{(2)}} \right)^p \right] = O(E_n). \quad (4.10)$$
It remains to analyze the expectation in (4.9).

The key idea to do this is to note that

$$\bar{U}_r^{(1)} = \max_{j \in C_r} (S_j + E_j),$$

where

$$E_j = \left( 1 - \frac{\lambda^*}{\lambda_n^*} \right) (\tau_1 + \tau_2 + \cdots + \tau_j).$$

By Lemma 4.7

$$E \left[ \bigg| \bigg| \bigg| \frac{r_n}{r_n} \bar{U}_r^{(1)} - \frac{r_n}{r_n} U_r^{(1)} \bigg| \bigg| \bigg| \right]^p \leq E \left[ \bigg| \bigg| \bigg| \frac{r_n}{r_n} \bigg| E_i \bigg| \bigg| \bigg| \right]^p \leq E \left[ \bigg| \bigg| \bigg| \bigg| i \in A_{r_n} \bigg| E_i \bigg| \bigg| \bigg| \right]^p,$$

where for the last identity we used the observation that if $i \in C_r$ for some $r \leq r_n$, then there is at least one $j \in A_{r_n}$ such that $(j|r) = i$ (recall that $f_n(0) = f(0) = 0$), and since all the $\{\tau_i\}$ are nonnegative, $|E_i| \leq |E_j|$. To estimate the last expectation note that since the $\{\tau_i\}_{i \in U}$ are independent of the $\{N_i\}_{i \in U}$, we have

$$E \left[ \bigg| \bigg| \bigg| \bigg| i \in A_{r_n} \bigg| E_i \bigg| \bigg| \bigg| \right]^p = E \left[ \sum_{i \in A_{r_n}} |E_i|^p \right] = E[|A_{r_n}|] \left| \frac{1}{\lambda^*} - \frac{1}{\lambda_n^*} \right| E[Y_{r_n}],$$

where $Y_{r_n}$ is an Erlang random variable with parameters $(r_n, 1)$. Since $E[|A_{r_n}|] = (E[N])^{r_n}$,

$$\left| \frac{1}{\lambda^*} - \frac{1}{\lambda_n^*} \right| = \left| \frac{E[N] - E[\tilde{N}]}{\lambda E[N] E[\tilde{N}]} \right| \leq \frac{|E[N] - E[\tilde{N}]}{\lambda},$$

and

$$E[Y_{r_n}] = \int_0^{\infty} x^{r_n + p - 1} e^{-x} \frac{\Gamma(r_n + p)}{(r_n - 1)!} dx = \frac{\Gamma(r_n + p)}{\Gamma(r_n)},$$

where $\Gamma(t)$ is the gamma function, we have that

$$E \left[ \bigg| \bigg| \bigg| \bigg| \frac{r_n}{r_n} \bar{U}_r^{(1)} - \frac{r_n}{r_n} U_r^{(1)} \bigg| \bigg| \bigg| \right|^p \right] \leq \frac{|E[N] - E[\tilde{N}]|^p}{\lambda^p} (E[N])^{r_n} \frac{\Gamma(r_n + p)}{\Gamma(r_n)}$$

$$= O \left( \left( \frac{|E[N] - E[\tilde{N}]|^p}{(E[N])^{r_n/p} r_n} \right)^{1/p} \right) \tag{4.11}$$

as $n \to \infty$, where in the last step we used that $\lim_{k \to \infty} \Gamma(k) k^\alpha / \Gamma(k + 1) = 1$ for any $\alpha \in \mathbb{R}$.

Combining (4.10) and (4.11) with (4.8) and (4.9), gives

$$\left( E \left[ \bigg| \bigg| \bigg| \bigg| \frac{r_n}{r_n} \bar{U}_r - \frac{r_n}{r_n} U_r \bigg| \bigg| \bigg| \right|^p \right] \right)^{1/p} = O \left( \frac{\epsilon_n^{1/p}}{r_n/r_n} \right),$$

where

$$\left| E[N] - E[\tilde{N}] \right| (E[N])^{r_n/p} r_n \to 0$$

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as $n \to \infty$ by assumption. To see that $\mathcal{E}_n \to 0$ as well, note that

$$
\mathcal{E}_n = \frac{\tilde{\rho}_\beta}{E[N]} \sum_{k=1}^{m_n} kP(N \neq k | \tilde{N} = k) f_n(k) \leq \tilde{\rho}_\beta \max_{1 \leq k \leq m_n} P(N \neq k | \tilde{N} = k).
$$

Since $\tilde{\rho}_\beta \to \rho_\beta < 1$, it only remains to show that the maximum on the right hand side converges to zero. To see this is the case let $R_k = \{ u \in (0,1) : F^{-1}_n(u) = k \}$ and note that for any $1 \leq k \leq m_n$,

$$
P(N \neq k | \tilde{N} = k) = \int_{R_k} \{ 1(F^{-1}(u) \leq k - 1) + 1(F^{-1}(u) \geq k + 1) \} \, du
$$

$$
= \int_{R_k} \{ 1(F^{-1}(u) \leq F^{-1}_n(u) - 1) + 1(F^{-1}(u) \geq F^{-1}_n(u) + 1) \} \, du
$$

$$
= \int_{R_k} 1(|F^{-1}(u) - F^{-1}_n(u)| \geq 1) \, du
$$

$$
\leq \int_0^1 |F^{-1}(u) - F^{-1}_n(u)| \, du,
$$

where the last integral is the Wasserstein distance of order one ($W_1$) between distributions $f_n$ and $f$ (see, e.g., [15, 18]), which converges to zero since $f_n \Rightarrow f$ and $E[\tilde{N}] \to E[N]$ (by Assumption 3.2 (i)), which is equivalent to convergence in $W_1$ (see Theorem 6.8 in [19]). This completes the proof.

Now that we have all the convergence results for each of the pairs of probability measures involved in (4.6), we can give the proof of Theorem 4.2.

**Proof of Theorem 4.2.** Let $\varphi(n) = |E[N] - E[\tilde{N}]|$, which converges to zero as $n \to \infty$ since $f_n$ is uniformly integrable by Assumption 3.2. Now let

$$
r_n = \begin{cases} 
\frac{p}{2 \log E[N]} |\log \varphi(n)|, & \text{if } E[N] > 1, \\
\varphi(n)^{-1/2}, & \text{if } E[N] = 1,
\end{cases}
$$

and note that

$$
|E[N] - E[\tilde{N}]| (E[N])^{r_n/p r_n} = \begin{cases} 
\frac{p}{2 \log E[N]} \varphi(n)^{1/2} |\log \varphi(n)|, & \text{if } E[N] > 1, \\
\varphi(n)^{1/2}, & \text{if } E[N] = 1,
\end{cases}
$$

which converges to zero as $n \to \infty$ in both cases.

The rest is an immediate consequence of (4.6) combined with Lemma 4.3, Proposition 4.6, and Proposition 4.8.

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