$A_4$ family symmetry and quark-lepton unification

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Abstract: We present a model of quark and lepton masses and mixings based on $A_4$ family symmetry, a discrete subgroup of an $SO(3)$ flavour symmetry, together with Pati-Salam unification. It accommodates tri-bimaximal neutrino mixing via constrained sequential dominance with a particularly simple vacuum alignment mechanism emerging through the effective D-term contributions to the scalar potential.

Keywords: Beyond Standard Model, Quark Masses and SM Parameters.
1. Introduction

There has recently been considerable interest in the use of the discrete group $A_4$ as a family symmetry [1–25]. A particularly attractive feature of $A_4$ is the possibility of obtaining non-trivial vacuum alignment in a simpler way than for continuous family symmetries [18–20]. Such non-trivial vacuum alignments are of interest since they can lead to tri-bimaximal neutrino mixing [26]. In particular, in the framework of the see-saw mechanism with sequential dominance (SD) [27–30], such non-trivial vacuum alignment can lead to constrained sequential dominance (CSD) [31] in which tri-bimaximal neutrino mixing arises from simple relations between Yukawa couplings involving the dominant and leading sub-dominant right-handed neutrinos.

Despite the great interest in $A_4$ as a family symmetry, there does not yet exist in the literature a model in which quarks and leptons are unified. Part of the reason for this is that the left and right handed chiral components of the quarks and leptons are usually required to transform differently under the $A_4$ family symmetry [1–3, 7, 9, 15–22]. If both helicity components transform in the same way then the $A_4$ family symmetry does not prevent trivial invariant operators which give a mass matrix contribution proportional to the unit matrix [32], rather than the desired hierarchical form. The situation is rather similar to the case of $SO(3)$ family symmetry since $A_4$ may be regarded as a discrete subgroup of $SO(3)$. In the case of $SO(3)$ the solution to this problem is to accept the left-right asymmetry, and to construct partially unified models based on Pati-Salam gauge group [31]. Such models can in principle be embedded directly into string theory, and may be consistent with $SO(10)$ in a 5D framework [33], without the need for an explicit 4D $SO(10)$ GUT, which in any case suffers from the doublet-triplet splitting problem. However, to best of
our knowledge, no such Pati-Salam unified model with $A_4$ family symmetry exists in the literature.

In this paper we present a realistic model of quark and lepton masses and mixings based on $A_4$ family symmetry and Pati-Salam unification. The model goes along the lines of the $SO(3)$ and Pati-Salam model discussed in detail in [33], and shares many of the desirable features of that model, in particular the flavons entered at the lowest possible order, which allowed the messenger sector to be explicitly specified. Also, as in [33], tribimaximal neutrino mixing emerges from the see-saw mechanism with CSD arising from vacuum alignment. However, whereas the vacuum alignment in $SO(3)$ [31], assumed in [33], was rather involved, here, with the discrete subgroup $A_4$, it will become remarkably simple. Here we will use the discrete radiative vacuum alignment mechanism proposed in [34] for the $\Delta(27)$ discrete symmetry model, based on discrete D-terms rather than the F-term mechanism discussed in [20] for discrete subgroups of $SO(3)$ and $SU(3)$. In fact the $A_4$ model presented here as a discrete version of the $SO(3)$ models discussed in [31, 33], mirrors the $\Delta(27)$ model discussed in [34] which is a discrete version of the $SU(3)$ models discussed in [35–37].

2. The model

The model is based on a high-energy Pati-Salam $SU(4)_C \otimes SU(2)_L \otimes SU(2)_R$ supersymmetric model with Yukawa sector driven by a discrete subgroup of $SO(3)$, the $A_4$ flavour symmetry and a pair of extra symmetry factors $U(1) \otimes Z_2$ to forbid some unwanted operators. The construction goes along similar lines as in the case of a fully $SO(3)$ invariant model studied in [33]. However, sticking to a discrete subgroup of a Lie-group brings in several qualitative changes that require a separate treatment. In particular, it provides for a very effective tool to address the vacuum alignment issues that often make the SUSY models based on continuous flavour symmetries rather cumbersome due a proliferation of extra degrees of freedom.

2.1 The field content and symmetry breaking

The full set of the effective theory matter, Higgs and flavon fields and their transformation properties are given in Table II. We embed the left-handed Standard Model matter fields into a triplet of $A_4$ while keeping the right-handed matter transform as the $SO(3)$-like $A_4$ singlet$^1$. Apart of the pair of MSSM light Higgs doublets $h$ (arranged into the traditional Pati-Salam bidoublet) driving the electroweak symmetry breakdown we use a pair of heavy Higgs bosons $H$ and $H'$ to break the Pati-Salam gauge symmetry at a high scale and provide the Majorana mass terms for the right-handed neutrinos.

2.2 The Yukawa sector

In what follows we shall use upper indices for $A_4$ triplet components while the lower indices stand for the various species of structures in the game. The symmetries defined above allow

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$^1$There are in general three inequivalent one-dimensional representations of $A_4$; our choice follows the observation that at the string level the extra singlets (i.e. $SO(3)$ non-invariant ones) usually come from higher representations of the gauge group and thus seem disfavoured, at least in simplest schemes.
for the following contributions to the Yukawa superpotential:

\[
W_Y = \frac{1}{M} y_{23} F.\phi_{23} F_1^c h + \frac{1}{M} y_{123} F.\phi_{123} F_2^c h + \frac{1}{M} y_3 F.\phi_3 F_3^c h + \frac{1}{M^2} y_{G;1} F.\tilde{\phi}_{23} F_3^c \Sigma h + \quad (2.1)
\]

\[
+ \frac{1}{M^2} y_{13} F.(\phi_2 \times \phi_3) F_3^c h + \frac{1}{M^2} y_{13} F.(\phi_2 \times \phi_3) F_3^c h + \frac{1}{M^2} y_{23} I_1(F, \tilde{\phi}_{23}, \tilde{\phi}_{23}, \phi_3) F_3^c h + \ldots
\]

where \( x \times y \) is the standard \( SO(3) \) cross-product, \( (x \times y)^i = s^{ijk} x^j y^k \) (with \( s^{ijk} \) being +1 for each permutation of \( \{i,j,k\} \in \{1,2,3\} \)) corresponds to the extra (symmetric) vector product in \( A_4 \) while \( I_1(x,y,u,v) \) denotes the available independent quartic \( A_4 \) invariants, as discussed in Appendix A. Note that \( M \) is a generic symbol for the mass of the relevant messenger sector fields giving rise to the desired effective vertices in eq. (2.1). For sake of conciseness, we shall not discuss the messenger sector here and defer an interested reader to the study [33] for an example of such analysis.

After the spontaneous breakdown of the flavour symmetry (for details see Section 2.5 and Table 2) the Yukawa matrices generated from this superpotential piece read:

\[
Y_{f;LR}^f = \begin{pmatrix}
0 & y_{123} \varepsilon^f_{123} \\
y_{23} \varepsilon^f_{23} & y_{123} \varepsilon^f_{123} + C^f y_{G;1} \varepsilon^f_{23} \sigma & \bar{y}_{13} \varepsilon^f_{13} \\
-y_{23} \varepsilon^f_{23} & -y_{23} \varepsilon^f_{23} - C^f y_{G;1} \varepsilon^f_{23} \sigma & y_{3} \varepsilon^f_{3}
\end{pmatrix}
\]

where

\[
\varepsilon^f_x \equiv \frac{|\langle \phi_x \rangle|}{M_f}.
\]

parametrize the relevant flavon VEVs normalized to the masses of the corresponding messenger fields, \( C^f = -2, 0, 1, 3 \) (for \( f = u, \nu, d, e \)) are the traditional Clebsch-Gordon coefficients entering the effective Yukawa vertex in (2.2) including the Higgs field \( \Sigma \) (transforming

\[\text{Table 1: The basic Higgs, matter and flavon content of the model.}\]

| field | \( SU(4) \otimes SU(2)_L \otimes SU(2)_R \) | \( A_4 \) | \( U(1) \) | \( Z_2 \) |
|-------|---------------------------------|--------|--------|--------|
| \( F \) | \((4,2,1)\) | 3 | 0 | + |
| \( F_1^c \) | \((\overline{4},1,2)\) | 1 | +2 | - |
| \( F_2^c \) | \((\overline{4},1,2)\) | 1 | +1 | + |
| \( F_3^c \) | \((4,1,2)\) | 1 | -3 | - |
| \( h \) | \((1,2,2)\) | 1 | 0 | + |
| \( H, \overline{H} \) | \((4,1,2), (\overline{4},1,2)\) | 1 | \( \pm 3 \) | + |
| \( H', \overline{H'} \) | \((4,1,2), (\overline{4},1,2)\) | 1 | \( \mp 3 \) | + |
| \( \Sigma \) | \((15,1,3)\) | 1 | -1 | - |
| \( \phi_1 \) | \((1,1,1)\) | 3 | +4 | + |
| \( \phi_2 \) | \((1,1,1)\) | 3 | 0 | + |
| \( \phi_3 \) | \((1,1,1)\) | 3 | +3 | - |
| \( \phi_{23} \) | \((1,1,1)\) | 3 | -2 | - |
| \( \phi_{23} \) | \((1,1,1)\) | 3 | 0 | - |
| \( \phi_{123} \) | \((1,1,1)\) | 3 | -1 | + |
like $(15, 1, 3)$ under the Pati-Salam symmetry) responsible for the distinct charged sector hierarchies à la Georgi and Jarlskog [38] and $\sigma$ denotes the VEV of the Georgi-Jarlskog field $\sigma \equiv \langle \Sigma \rangle / M_F$. The effective couplings $\bar{y}_{23}$ and $\bar{y}_{13}$ stem from the multiple contributions to the 13 and 23 elements of $Y_{LR}^f$ due to the higher number of relevant cubic and quartic $A_4$ invariants.

### 2.3 The Majorana sector

The Majorana mass matrix is obtained from the superpotential of the form

$$W_M = \frac{1}{M^3} w_1 F_{12}^2 H H' \phi_{23}^2 + \frac{1}{M^3} w_2 F_{22}^2 H H' \phi_{123}^2 + \frac{1}{M^3} w_3 F_{33}^2 H^2 +$$

$$+ \frac{1}{M^4} F_{11}^2 H^2 \left[ w_4 (\phi_{123} \times \tilde{\phi}_{23}) \phi_3 + w_4' (\phi_{123} \ast \tilde{\phi}_{23}) \phi_3 \right] +$$

$$+ \frac{1}{M^4} F_{12}^2 F_{22}^2 H H' \left[ w_5 (\phi_{23} \times \phi_{123}) \phi_2 + w_5' (\phi_{23} \ast \phi_{123}) \phi_2 \right] +$$

$$+ \frac{w_6^i}{M^5} F_{22}^2 H^2 I_i (\phi_3, \phi_3, \phi_{23}, \tilde{\phi}_{23}) + \frac{w_6^i}{M^5} F_{23}^2 H^2 I_i (\phi_2, \phi_3, \tilde{\phi}_{23}, \phi_{23}) + \ldots$$

where as before $I_i$ stand for the various $A_4$ quartic invariants and the ellipsis denotes the higher order terms. It is easy to verify that the Majorana mass matrix emerging from here reads

$$M_{RR}^{\nu} = \begin{pmatrix}
O(\varepsilon_{23}^2 \delta_H, \varepsilon_{123}^{\nu}, \varepsilon_{23}^{\nu}, \varepsilon_{23}^{\nu}) \\
\ldots \\
O(\varepsilon_{123}^{\nu}, \varepsilon_{23}^2 \delta_H, \varepsilon_{23}^{\nu}, \varepsilon_{23}^{\nu}) \\
O(1)
\end{pmatrix} \frac{\langle H \rangle^2}{M} \quad (2.3)$$

where only the relevant terms are displayed because the mixing in the right-handed neutrino sector due to the off-diagonal terms is negligible.

### 2.4 The generic results

In order to achieve a good fit to all the quark and lepton masses and mixing parameters one has to assume a hierarchy among the flavon VEV parameters $\varepsilon_x^{f}$. Since the relevant VEV scales emerge from a radiative symmetry breaking mechanism, as discussed in Section 2.3, it is completely natural to expect a certain hierarchy among them that in turn propagates to the order of magnitude differences in $\varepsilon_x^{f}$. The only extra assumption concerns the magnitude of the VEV of $H'$ entering the Majorana sector analysis $\langle H' \rangle \equiv \delta_H \langle H \rangle$ with $\delta_H \ll 1$. However, a similar radiative mechanism like in the flavon case can play a role here thus making such an assumption as natural as the previous ones.

As it was shown previously in the context of an $SO(3)$ model [33], the structures under consideration lead to a good fit of all the quark and lepton mass and mixing data provided

$$\delta_H, \frac{\varepsilon_{123}^{\nu}, \varepsilon_{23}^{\nu}}{\varepsilon_{23}^{\nu}} \sim O(10^{-3}) \text{ while } \frac{\varepsilon_{23}^{\nu}}{\varepsilon_{23}^{\nu}} \sim O(1):$$

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3Note that the role of the $\phi_{12}$ flavon of [33] is played here by $\phi_2$ with an advantage of a particular simplicity of the vacuum alignment mechanism, see Section 2.3. Moreover, the difference among the vacuum structure of the flavons associated with the Georgi-Jarlskog mechanism [38] (i.e. $\phi_{23}$) with respect to [33] is essentially harmless for the fit of the quark and lepton masses and also the CKM parameters coming predominantly from the above-diagonal entries are expected to remain stable enough. Thus, there is no need to perform a dedicated numerical analysis and the interested reader is again deferred to the one given in [33].
• The naturalness of the hierarchy among the third and second generation Yukawa couplings as well as a moderate suppression of the $V_{cb}$ CKM mixing parameter are traced back to the higher-order origin of the relevant (Georgi-Jarlskog and 2-3 entry) operators.

• The first generation masses as well as the smallness of the $V_{ub}$ CKM mixing descend from the hierarchy of the relevant flavon VEVs as discussed in the next section.

• The neutrino sector conforms to the CSD conditions [31]. The particular structure of the neutrino Yukawa matrix together with the hierarchy of the charged lepton Yukawa couplings leads to approximate tri-bimaximal mixing in the neutrino sector [26] characterized by the approximately maximal atmospheric mixing $\tan \theta_{23} \approx 1$, large solar mixing angle obeying $\sin \theta_{12} \approx 1/\sqrt{3}$ and a small reactor angle $\theta_{13} \approx 0$, in good agreement with the latest neutrino data, see e.g. [39, 40] and references therein.

• Concerning the light neutrino mass spectrum, the large hierarchy in $Y_{\nu LR}$ is effectively undone in the seesaw formula by the particular form of the Majorana mass matrix (2.3).

Thus, the model provides a very good description of all the known quark and lepton masses and mixing parameters. The only missing ingredient is the mechanism leading to the desired correlations among the VEVs of the various triplet flavon components shown in Table 2.

2.5 The vacuum alignment mechanism

The discrete nature of the flavour symmetry leads to a particularly simple option to achieve all the desired vacuum structures displayed in Table 3. As discussed in [34], in such a class of models the supergravity (SUGRA) induced D-terms can naturally lead to a set of extra quartic terms in the effective scalar potential. Such a set of terms, however, lead to a lift of the would-be degenerate vacua potentially emerging in a continuous case and thus makes the vacuum alignment mechanism straightforward. To force the system to depart from the symmetric state we shall assume a variant of a radiative symmetry breaking mechanism, as we now discuss.

Let us first consider the case of a single triplet $\phi$. Apart from the obvious $SO(3)$ invariant $(\phi^\dagger \phi)^2$ the discrete $A_4$ symmetry admits for instance a contraction like

$$I_0(\phi^\dagger, \phi, \phi^\dagger, \phi) \equiv \sum_{i=1}^{3} \phi^i \phi^i \phi^i \phi^i \phi^i$$

(2.4)

that breaks the rotational degeneracy of the would-be $SO(3)$ symmetric vacua. Assuming that the scalar potential is governed by the terms

$$V \equiv -M^2_\phi (\phi^\dagger \phi) + \lambda I_0 (\phi^\dagger, \phi, \phi^\dagger, \phi) + \Lambda (\phi^\dagger \phi)^2 + \ldots$$

(2.5)

4Here we choose to write the standard $SO(3)$-invariant term $\Lambda (\phi^\dagger \phi)^2$ in the basis that exhibits the convexity of the potential rather than in terms of the “$I_1, i_7$” independent invariant advocated in Appendix 4. It is indeed trivial to see that $(\phi^\dagger \phi)^2 = (I_0 + 2I_1)(\phi, \phi^\dagger, \phi)$. 

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it is easy to verify that the only vacuum structures that can arise in such a case (i.e. when all the mixing terms are negligible) are
\[
\langle |\phi| \rangle \propto (1,1,1) \quad \text{and/or} \quad \langle |\phi| \rangle \propto (1,0,0), (0,1,0), (0,0,1)
\]
(2.6)
where only the magnitudes of the components are so far specified. What matters is the sign can be exploited.

The various linearly independent components but the equality of their magnitudes, and their complex orthogonality, which matters in achieving tri-bimaximal mixing via CSD is not the absolute phases of the flavon components but the equality of their magnitudes, and their complex orthogonality, which we shall shortly discuss.

All this leads us to the following realization of the vacuum alignment mechanism: suppose each of the fields \(\phi_{123}, \phi_1\) and \(\phi_3\) has a potential of the form in Eq.(2.5), simply repeated for each field. Suppose each of the fields develop negative mass-squares through radiative effects around scales \(M_{123}, M_1\) and \(M_3\) and let us arrange the \(\lambda\)-terms in the leading piece of the scalar potential in Eq.(2.3) so that they pick up VEVs in the directions allowed by eq. (2.6), in particular:
\[
\langle |\phi_{123}| \rangle \propto (1,1,1) \quad \text{and} \quad \langle |\phi_1| \rangle \propto (1,0,0), \quad \langle |\phi_3| \rangle \propto (0,0,1)
\]
(2.7)
The stability of such a setup requires that the mixing arising from the \(\text{“inhomogeneous”}\) terms like\(^6\) \(I_i(\phi_{123}^\dagger, \phi_{123}, \phi_{123}^\dagger, \phi_{123})\), \(i \in \{0,1,2\}\) should be suppressed with respect to the \(\text{“pure”}\) ones \(I_i(\phi_{123}^\dagger, \phi_{123}, \phi_{123}^\dagger, \phi_{123})\) and \(I_i(\phi_{123}^\dagger, \phi_{123}, \phi_{123}^\dagger, \phi_{123})\).

Subsequently, \(\langle \tilde{\phi}_{23} \rangle\) and \(\langle \tilde{\phi}_{23} \rangle\) can be generated if the interactions with the first stage fields \(\phi_{123}\) and \(\phi_{123}\) are dominated by the terms
\[
V \ni -M_{23}^2|\phi_{23}|^2 + \lambda_{123}|\phi_{123}^\dagger\phi_{23}|^2 + \lambda_1|\phi_{123}^\dagger\phi_{23}|^2 - \tilde{M}_{23}^2|\tilde{\phi}_{23}|^2 + \tilde{\lambda}_{123}|\tilde{\phi}_{123}^\dagger\tilde{\phi}_{23}|^2 + \tilde{\lambda}_1|\tilde{\phi}_{123}^\dagger\tilde{\phi}_{23}|^2 + \ldots
\]
(2.8)
where the ellipsis stands for \(SO(3)\) (and thus also \(A_4\)) invariant terms of the form \(A_\phi(\phi, \phi)^2\) necessary to lift the flat directions. If \(\lambda_{123}\) and \(\tilde{\lambda}_{123}\) are positive, the VEVs of \(\phi_{23}\) and \(\tilde{\phi}_{23}\) driven to the directions orthogonal to \(\langle \phi_{123} \rangle\) while \(\lambda_1\), \(\tilde{\lambda}_1\) make their first component vanish and thus \(\langle |\phi_{23}| \rangle, (|\tilde{\phi}_{23}|) \propto (0,1,1)\). Concerning the above mentioned ambiguity in fixing the phases of the vacuum alignment emerging from the simple potential (2.3),

\(^5\)The alignment of \(\langle |\phi| \rangle \propto (1,0,0)\) and \(\langle |\phi| \rangle \propto (0,0,1)\) is a just a choice of basis that we are free to make as long as there are no interactions binding the VEVs of \(\phi_1\) and \(\phi_3\) together. On the other hand, to make sure \(\phi_1\) does not coincide with \(\phi_3\) spontaneously a mixing term like \(|\phi_1^\dagger, \phi_3|^2\) with a positive coefficient can be exploited.

\(^6\)Here we again suppress all the triplet indices so that the generic symbols \(I_i(\phi_i^\dagger, \phi_i, \phi_i^\dagger, \phi_i)\) account for the various linearly independent \(A_4\) contractions. There are only 4 such independent structures for \(A = C\) and \(B = D\), three if \(A = B = C = D\) and 2 if on top of that of \(\phi = \phi^\dagger\) (i.e. only if \(\phi\) is strictly neutral), for more details see Appendix A.
| flavon VEV | VEV direction | VEV normalization (scale) |
|-----------|--------------|--------------------------|
| ⟨φ₁⟩     | (1, 0, 0)    | M₁/√2(λ₁ + Λ₁)          |
| ⟨φ₂⟩     | (0, 1, 0)    | M₂/√2(λ₂ + Λ₂)          |
| ⟨φ₃⟩     | (0, 0, 1)    | M₃/√2(λ₃ + Λ₃)          |
| ⟨φ₂₃⟩    | (0, 1, −1)   | M₂₃/2√Λ₂₃               |
| ⟨φ₁₂₃⟩   | (1, 1, 1)    | M₁₂₃/√2(λ₁₂₃ + 3Λ₁₂₃)  |
| ⟨˜φ₂₃⟩   | (0, 1, −1)   | ˜M₂₃/2√˜Λ₂₃             |

Table 2: The vacuum alignment pattern generated by the mechanism specified in the text. The mass scales Mᵢ and the relevant quartic couplings λᵢ and Λᵢ are defined in Section 2.5, Eqs. (2.5) and (2.8). In the “VEV direction” column only the magnitudes of the relevant (in general complex) flavon VEVs are displayed. The minus sign in the case of φ₂₃ and ˜φ₂₃ illustrates the important π-difference of the 2nd and 3rd component VEV phases of φ₂₃ and φ₁₂₃.

in particular φ₁₂₃, the orthogonality condition ⟨φ₁₂₃⟩†⟨φ₂₃⟩ = 0 together with ⟨φ₂₃⟩ = 0 following from the minimisation of (2.5) is just enough to generate θ₁₃ close to zero [30] and tan θ₁₂₃ ∼ 1/√2 regardless any particular arrangement of the ⟨φ₁₂₃⟩ phases. The minus signs in Table 2 are for illustrative purposes and simply denote the π-shift in the relative phases of the components of ⟨φ₁₂₃⟩ and ⟨φ₂₃⟩ (or ⟨˜φ₂₃⟩) arising from the relevant orthogonality conditions.

At this point it is perhaps worth mentioning that the positivity of the λ-couplings above also ensures a better control over the magnitudes of the corresponding VEVs unlike the case of having an interaction with a negative coupling constant when a potentially large negative correction must be compensated by the explicit mass entry from the F-terms. This means that one can handle easily all the relevant scales without a need of an extra tuning of parameters in the would-be “effective wrong-sign masses” that might otherwise arise.

Concerning the alignment of φ₂, a particular shape of its VEV is immaterial as long as it admits a nonzero projection to the second SO(3) coordinate. A particularly elegant setup can be obtained if for instance ⟨φ₂⟩ ∝ (0, 1, 0) is generated via the same mechanism like φ₁,3, repeating just as before the form of potential (2.5) with λ₂ > 0. To make sure the (0,1,0) option is picked up one can employ the orthogonality of all the φ₁, φ₂ and φ₃ VEVs via the mixing terms of the form |φᵢ†φⱼ|² with positive coefficients so that the complete basis of the triplet space is spanned.

The results of our vacuum alignment mechanism are summarized in Table 2. It is easy to see that all the mass scales in Table 2 are essentially free: since the potential (2.8) is fully SO(3) invariant, the anisotropy enters only through the A₄ terms driving the VEVs of φ₁₂₃ and φ₁,2,3 while ⟨φ₂₃⟩, ⟨φ₂₃⟩ can rotate freely to follow the constraints imposed through the interactions with φ₁₂₃₁. Thus, even a small push in any particular direction is enough to imprint the desired alignment to all the relevant VEVs and we are free to choose M₂₃ and M₁₂₃ so that the desired VEV hierarchy is achieved.
3. Conclusions

We have constructed the first complete model of flavour based on $A_4$ family symmetry together with the $SU(4)_C \otimes SU(2)_L \otimes SU(2)_R$ Pati-Salam gauge symmetry. $A_4$ corresponds to the symmetry of the tetrahedron, and is a discrete subgroup of $SO(3)$. Assuming the simple extra symmetry factors $U(1) \otimes Z_2$, we have performed an operator analysis of the model, and shown that the resulting effective Yukawa and Majorana couplings have a similar form to those discussed in [33], and when the messenger sector is completed, the resulting structures provide a good description of the fermion mass and mixing spectrum. In particular, the constrained sequential dominance is realized and tri-bimaximal neutrino mixing results, with calculable deviations expressed in terms of neutrino sum rules [31, 41].

The main simplification afforded by the discrete symmetry is in the vacuum alignment sector. Due to the discrete nature of the flavour symmetry a particularly simple vacuum mechanism emerges through the SUGRA induced D-term contributions to the effective scalar potential that lift the $SO(3)$ vacuum degeneracy. We have shown that this discrete version of the radiative symmetry breaking mechanism may be achieved with a minimal number of fields that do not participate directly in the Yukawa sector, and that a realistic model can be constructed which incorporates all these features simultaneously. The $A_4$ model presented here may be regarded as being on the same footing as the $\Delta(27)$ model presented in [34].

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A. Basic properties of the quartic triplet $A_4$ invariants

In this appendix we give a short compendium on the main features of the symmetry group of tetrahedron called $A_4$ and the properties of the basic triplet invariants used in the main text. Where suitable we use the notation of He et al. [23].

As a discrete subgroup of $SO(3)$, $A_4$ admits a triplet representation $3$ and three inequivalent singlets often denoted by $1$, $1'$ and $1''$. Concerning namely the triplets $x = (x^1, x^2, x^3)$ it is convenient to express the action of the group elements by means of its correspondence to the semidirect product $Z_3 \rtimes Z_2$ as:

$$Z_3(x^1, x^2, x^3) \rightarrow (x^2, x^3, x^1)$$
$$Z_2(x^1, x^2, x^3) \rightarrow (x^1, -x^2, -x^3)$$

With this information at hand one can see that apart of the “standard” $SO(3)$-like dot product $(x \cdot y) \equiv x^1 y^1 + x^2 y^2 + x^3 y^3$ and the cross product $(x \times y).z = \varepsilon^{ijk} x^i y^j z^k$ there is an extra symmetric cubic invariant like $(x \ast y).z = s^{ijk} x^i y^j z^k$ where $s^{ijk}$ is $+1$ on all permutations $\{i, j, k\} \in \{1, 2, 3\}$ and zero otherwise.
At the quartic level one can easily check that the basic structures

\begin{align}
I_0(x, y; u, v) &\equiv x_1y_1u_1v_1 + x_2y_2u_2v_2 + x_3y_3u_3v_3 \\
I_1(x, y; u, v) &\equiv x_1y_1u_2v_2 + x_2y_2u_3v_3 + x_3y_3u_1v_1 \\
I_2(x, y; u, v) &\equiv x_1y_1u_3v_3 + x_2y_2u_1v_1 + x_3y_3u_2v_2
\end{align}

(A.2)

are invariant with respect to the action (A.1). However, this is not the end of the story yet as one must consider also the other permutations of the set of parameters \(\{x, y, u, v\}\). The symmetries of \(I_{0,1,2}\) are such that all these expressions are actually invariant with respect to permutations of the first and second pair of arguments (and in case of \(I_0\) even all of them), and thus what matters is just the pairings of \(\{x, y, u, v\}\). In short, the independent structures emerging from eq. (A.2) correspond to \(I_0, I_1\) and \(I_2\) with arguments \((x, y; u, v), (x, u; y, v)\) and \((x, v; y, u)\) only. Due to the maximal symmetry of \(I_0\) one gets only 4 additional relevant structures from \(I_1\) and \(I_2\), namely

\begin{align}
I_3(x, y; u, v) &\equiv I_1(x, u; y, v), I_4(x, y; u, v) \equiv I_2(x, u; y, v) \\
I_5(x, y; u, v) &\equiv I_1(x, v; y, u), I_6(x, y; u, v) \equiv I_2(x, v; y, u)
\end{align}

To demonstrate the completeness of such a “naively” constructed set of invariants it is sufficient to find a mapping of \(I_{0,...,6}\) onto the set of “group-theoretical” purely triplet quartic invariants of the form \((x,y)(u,v) = (3 \otimes 3)_1 \otimes (3 \otimes 3)_1, (x,y)_{\nu}(u,v)_{\nu} = (3 \otimes 3)_{\nu} \otimes (3 \otimes 3)_{\nu}\) (provided \((x,y)_{\nu} \equiv x_1y_1 + \omega x_2y_2 + \omega^2 x_3y_3\) and \((x,y)_{\nu} \equiv x_1y_1 + \omega^2 x_2y_2 + \omega x_3y_3\) stand for the distinct extra \(A(4)\) singlets, \(\omega\) is the cubic root of unity and \(1^\nu \otimes 1^\nu = 1\))  , \((x \times y)(u \times v) = [(3 \otimes 3)_{3a} \otimes (3 \otimes 3)_{3a}]_1, (x \times y)(u \times v) = [(3 \otimes 3)_{3s} \otimes (3 \otimes 3)_{3s}]_1\) and \((x \times y)(u \times v) = [(3 \otimes 3)_{3s} \otimes (3 \otimes 3)_{3s}]_1\), leading to a large number of options upon taking into account the various permutations of 4 objects \(\{x, y, u, v\}\). However, it is obvious that they are by far not all linearly independent, as one can see for instance from the identity \((x \times y)(u \times v) = (x.u)(y.v) - (x.v)(y.u)\). Indeed, these structures can be mapped onto \(I_{0,...,6}\) as

\begin{align}
2I_0(x, y; u, v) &\equiv 2(x.y)(u.v) - [(x \times v)(y \times u) + (x \times u)(y \times u)] \\
4I_1(x, y; u, v) &\equiv [(x \times v)(y \times u) + (x \times u)(y \times u) + (x \times v)(y \times u) + (x \times v)(y \times u)] \\
4I_2(x, y; u, v) &\equiv [(x \times v)(y \times u) + (x \times u)(y \times u) - (x \times v)(y \times u) - (x \times v)(y \times u)] \\
4I_3(x, y; u, v) &\equiv [(x \times v)(y \times u) - (x \times v)(y \times u) + (x \times v)(y \times u) - (x \times v)(y \times u)] \\
4I_4(x, y; u, v) &\equiv [(x \times v)(y \times u) - (x \times v)(y \times u) - (x \times v)(y \times u) + (x \times v)(y \times u)] \\
4I_5(x, y; u, v) &\equiv [(x \times v)(y \times u) - (x \times v)(y \times u) + (x \times v)(y \times u) - (x \times v)(y \times u)] \\
4I_6(x, y; u, v) &\equiv [(x \times v)(y \times u) - (x \times v)(y \times u) - (x \times v)(y \times u) + (x \times v)(y \times u)]
\end{align}

(A.3)

It is then easy to see that whenever \(x = u\) and \(y = v\) only 4 of these structures remain independent (for instance \(I_0, I_3, I_4\) and one from the equal \(I_{1,2,5,6}\)). If on top of that \(y = x^d\) then \(I_3 = I_4^d\) that allows for only three independent terms in a hermitean scalar potential and, finally, if all the arguments coincide there is only 2 such terms like for instance \(I_0\) and one from \(I_{1,...,6}\) left.
References

[1] E. Ma and G. Rajasekaran, *Softly broken a(4) symmetry for nearly degenerate neutrino masses*, Phys. Rev. **D64** (2001) 113012, [hep-ph/0106291](https://arxiv.org/abs/hep-ph/0106291).

[2] E. Ma, *Quark mass matrices in the a(4) model*, Mod. Phys. Lett. **A17** (2002) 627–630, [hep-ph/0203238](https://arxiv.org/abs/hep-ph/0203238).

[3] K. S. Babu, E. Ma, and J. W. F. Valle, *Underlying a(4) symmetry for the neutrino mass matrix and the quark mixing matrix*, Phys. Lett. **B552** (2003) 207–213, [hep-ph/0206292](https://arxiv.org/abs/hep-ph/0206292).

[4] M. Hirsch, J. C. Romao, S. Skadhauge, J. W. F. Valle, and A. Villanova del Moral, *Degenerate neutrinos from a supersymmetric a(4) model*, [hep-ph/0312244](https://arxiv.org/abs/hep-ph/0312244).

[5] M. Hirsch, J. C. Romao, S. Skadhauge, J. W. F. Valle, and A. Villanova del Moral, *Phenomenological tests of supersymmetric a(4) family symmetry model of neutrino mass*, Phys. Rev. **D69** (2004) 093006, [hep-ph/0312265](https://arxiv.org/abs/hep-ph/0312265).

[6] E. Ma, *Non-abelian discrete family symmetries of leptons and quarks*, [hep-ph/0409075](https://arxiv.org/abs/hep-ph/0409075).

[7] E. Ma, *A(4) origin of the neutrino mass matrix*, Phys. Rev. **D70** (2004) 031901, [hep-ph/0404199](https://arxiv.org/abs/hep-ph/0404199).

[8] E. Ma, *Symmetries and neutrino masses*, New J. Phys. **6** (2004) 104, [hep-ph/0405152](https://arxiv.org/abs/hep-ph/0405152).

[9] S.-L. Chen, M. Frigerio, and E. Ma, *Hybrid seesaw neutrino masses with a(4) family symmetry*, Nucl. Phys. **B724** (2005) 423–431, [hep-ph/0504181](https://arxiv.org/abs/hep-ph/0504181).

[10] E. Ma, *Aspects of the tetrahedral neutrino mass matrix*, Phys. Rev. **D72** (2005) 037301, [hep-ph/0505209](https://arxiv.org/abs/hep-ph/0505209).

[11] K. S. Babu and X.-G. He, *Model of geometric neutrino mixing*, [hep-ph/0507217](https://arxiv.org/abs/hep-ph/0507217).

[12] E. Ma, *Tetrahedral family symmetry and the neutrino mixing matrix*, Mod. Phys. Lett. **A20** (2005) 2601–2606, [hep-ph/0508099](https://arxiv.org/abs/hep-ph/0508099).

[13] E. Ma, *Autogenous neutrino mixing*, [hep-ph/0511133](https://arxiv.org/abs/hep-ph/0511133).

[14] A. Zee, *Obtaining the neutrino mixing matrix with the tetrahedral group*, Phys. Lett. **B630** (2005) 58–67, [hep-ph/0508278](https://arxiv.org/abs/hep-ph/0508278).

[15] B. Adhikary, B. Brahmachari, A. Ghosal, E. Ma, and M. K. Parida, *A(4) symmetry and prediction of u(e3) in a modified altarelli-feruglio model*, Phys. Lett. **B638** (2006) 345–349, [hep-ph/0603059](https://arxiv.org/abs/hep-ph/0603059).

[16] E. Ma, H. Sawanaka, and M. Tanimoto, *Quark masses and mixing with a(4) family symmetry*, [hep-ph/0606103](https://arxiv.org/abs/hep-ph/0606103).
[17] E. Ma, Tribimaximal neutrino mixing from a supersymmetric model with a4 family symmetry, Phys. Rev. D73 (2006) 057304.

[18] G. Altarelli and F. Feruglio, Tri-bimaximal neutrino mixing from discrete symmetry in extra dimensions, Nucl. Phys. B720 (2005) 64–88, hep-ph/0504165.

[19] G. Altarelli and F. Feruglio, Tri-bimaximal neutrino mixing, a(4) and the modular symmetry, Nucl. Phys. B741 (2006) 215–235, hep-ph/0512103.

[20] I. de Medeiros Varzielas, S. F. King, and G. G. Ross, Tri-bimaximal neutrino mixing from discrete subgroups of su(3) and so(3) family symmetry, hep-ph/0512313.

[21] L.avoura and H. Kuhbock, Predictions of an a(4) model with a five-parameter neutrino mass matrix, hep-ph/0610050.

[22] B. Adhikary and A. Ghosal, Constraining cp violation in a softly broken a(4) symmetric model, hep-ph/0609193.

[23] X.-G. He, Y.-Y. Keum, and R. R. Volkas, A(4) flavour symmetry breaking scheme for understanding quark and neutrino mixing angles, JHEP 04 (2006) 039, hep-ph/0601001.

[24] G. Altarelli, An update on models of neutrino masses and mixings, hep-ph/0610164.

[25] G. Altarelli, F. Feruglio, and Y. Lin, Tri-bimaximal neutrino mixing from orbifolding, hep-ph/0610165.

[26] P. F. Harrison, D. H. Perkins, and W. G. Scott, Tri-bimaximal mixing and the neutrino oscillation data, Phys. Lett. B530 (2002) 167, hep-ph/0202074.

[27] S. F. King, Atmospheric and solar neutrinos with a heavy singlet, Phys. Lett. B439 (1998) 350–356, hep-ph/9806440.

[28] S. F. King, Atmospheric and solar neutrinos from single right-handed neutrino dominance and u(1) family symmetry, Nucl. Phys. B562 (1999) 57–77, hep-ph/9904210.

[29] S. F. King, Large mixing angle msw and atmospheric neutrinos from single right-handed neutrino dominance and u(1) family symmetry, Nucl. Phys. B576 (2000) 85–105, hep-ph/9912492.

[30] S. F. King, Constructing the large mixing angle mns matrix in see-saw models with right-handed neutrino dominance, JHEP 09 (2002) 011, hep-ph/0204369.

[31] S. F. King, Predicting neutrino parameters from so(3) family symmetry and quark-lepton unification, JHEP 08 (2005) 105, hep-ph/0506297.

[32] E. Ma, Suitability of a(4) as a family symmetry in grand unification, hep-ph/0607190.
[33] S. F. King and M. Malinsky, *Towards a complete theory of fermion masses and mixings with so(3) family symmetry and 5d so(10) unification*, [hep-ph/0608021](http://arxiv.org/abs/hep-ph/0608021).

[34] I. de Medeiros Varzielas, S. F. King, and G. G. Ross, *Neutrino tri-bi-maximal mixing from a non-abelian discrete family symmetry*, [hep-ph/0607045](http://arxiv.org/abs/hep-ph/0607045).

[35] S. F. King and G. G. Ross, *Fermion masses and mixing angles from su(3) family symmetry*, *Phys. Lett.* **B520** (2001) 243–253, [hep-ph/0108112](http://arxiv.org/abs/hep-ph/0108112).

[36] S. F. King and G. G. Ross, *Fermion masses and mixing angles from su(3) family symmetry and unification*, *Phys. Lett.* **B574** (2003) 239–252, [hep-ph/0307190](http://arxiv.org/abs/hep-ph/0307190).

[37] I. de Medeiros Varzielas and G. G. Ross, *Su(3) family symmetry and neutrino bi-tri-maximal mixing*, *Nucl. Phys.* **B733** (2006) 31–47, [hep-ph/0507176](http://arxiv.org/abs/hep-ph/0507176).

[38] H. Georgi and C. Jarlskog, *A new lepton - quark mass relation in a unified theory*, *Phys. Lett.* **B86** (1979) 297–300.

[39] A. Strumia and F. Vissani, *Implications of neutrino data circa 2005*, [hep-ph/0503246](http://arxiv.org/abs/hep-ph/0503246).

[40] A. Strumia and F. Vissani, *Neutrino masses and mixings and*, [hep-ph/0606054](http://arxiv.org/abs/hep-ph/0606054).

[41] S. Antusch and S. F. King, *Charged lepton corrections to neutrino mixing angles and cp phases revisited*, *Phys. Lett.* **B631** (2005) 42–47, [hep-ph/0508044](http://arxiv.org/abs/hep-ph/0508044).