Quantizing a solitonic string

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ABSTRACT: Quite often the zero mode dynamics on solitonic vortices are described by a non-conformal effective world-sheet sigma model (WSSM). We address the problem of solitonic string quantization in this case. As well-known, only critical strings with conformal WSSMs are self-consistent in ultra-violet (UV) domain. Thus, we look for the appropriate UV completion of the low-energy non-conformal WSSM. We argue that for the solitonic strings supported in well-defined bulk theories the UV complete WSSM has a UV fixed point which can be used for string quantization. As an example, we consider BPS non-Abelian vortices supported by four-dimensional (4D) $\mathcal{N} = 2$ SQCD with the gauge group $\mathrm{U}(N)$ and $N_f$ quark multiplets where $N_f \geq N$. In addition to translational moduli the non-Abelian vortex under consideration carries orientational and size moduli. Their low-energy dynamics are described by a two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric weighted model, namely, $\mathbb{WC}\mathbb{P}(N, N_f - N)$. Given our UV completion of this WSSM we find its UV fixed point. The latter defines a superconformal WSSM. We observe two cases in which this conformal WSSM, combined with the free theory for four translational moduli, has ten-dimensional target space required for superstrings to be critical.

KEYWORDS: Supersymmetric Gauge Theory, Conformal Field Theory, Sigma Models

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1 Introduction

String theory vacua are associated with conformal two-dimensional (2D) sigma models (SMs) on the string world sheet. This SM defines a vacuum for the critical string theory if its Virasoro central charge equals 26 for the bosonic string or 15 for the superstring, see for example, textbook [1]. If not, the Liouville field does not decouple [2] and its central charge adds up to make the total central charge with ghosts included to be zero.

What about solitonic strings? In particular, what can we say about confining solitonic strings present in certain four-dimensional (4D) gauge theories? Their quantization is a major problem — if resolved the solution would give us a first-principle framework for studying hadronic physics.

This problem was first addressed by Polchinski and Strominger [3] for the Abrikosov-Nielsen-Olesen (ANO) vortex [4, 5] in 4D Abelian-Higgs model. For the ANO vortex, the effective theory on the string world sheet reduces to the Nambu-Goto action for the translational zero modes, which in turn reduces to a free theory in the Polyakov formulation [2] and, therefore, is obviously conformal. However, it is not critical in 4D. The authors...
of [3] argued that higher-derivative corrections improve the ultra-violet (UV) behavior of the theory. In particular, a six-derivative term was suggested in [3] which is in fact the Liouville action expressed in terms of the induced metric.

Many solitonic strings present an even more challenging problem: their effective world sheet theory is not conformal. In this paper we consider (as an example) BPS non-Abelian vortices supported in 4D $\mathcal{N} = 2$ supersymmetric QCD (SQCD) with the gauge group $U(N)$ and $N_f \geq N$, $N_f < 2N$ quark flavors. Besides four translational moduli, the non-Abelian vortex have orientational and size moduli. Their low-energy dynamics is described by 2D $\mathcal{N} = (2, 2)$ weighted $\mathbb{C}P$ model ($\mathcal{WCP}(N, \tilde{N})$) on the string world sheet [6–9], with $\tilde{N} = N_f - N$ (see [10–14] for reviews). This model is not conformal for $N_f < 2N$ and is unsuitable for string quantization. It has no world-sheet reparametrization invariance.

This question is puzzling. Say, $\mathcal{N} = 2$ SQCD is a completely well-defined self-consistent theory at all distances. How come we cannot construct a string theory free of pathology for a solitonic vortex string supported by the above 4D theory?

In this paper we build on the Polchinski-Strominger idea [3] that higher derivative corrections should improve UV behavior of the string world-sheet sigma model (WSSM). We will search for the UV completion of the infra-red (IR) WSSM. For a well-defined 4D theory the UV completion of our IR WSSM should have a conformal fixed point in the UV which defines a quantizable string theory for the solitonic vortex. The string states of the UV complete theory should describe hadrons of the original 4D gauge theory.

In this paper we suggest a desired UV completion for the non-Abelian vortex in $\mathcal{N} = 2$ SQCD. Starting from $\mathcal{WCP}(N, \tilde{N})$ model in the IR we find a UV-complete WSSM which satisfies all symmetry requirements. In particular, $\mathcal{N} = (2, 2)$ supersymmetry is most restrictive.

We find two cases in which the UV fixed point becomes a 2D conformal theory (CFT) with a 10D target space required for a superstring to be critical. The target space is of the form $\mathbb{R}^4 \times Y_6$, where $\mathbb{R}^4$ stands for the flat space of our 4D SQCD and comes from four translational zero modes on the vortex, while $Y_6$ is a non-compact Calabi-Yau manifold.

The first case is $\mathcal{N} = 2$ SQCD with the gauge group $U(N = 2)$ and four quark flavors. In this case $Y_6$ is the six dimensional conifold, see [20] for a review. The infrared WSSM is asymptotically free.

The string theory on the conifold was studied previously in our papers [15–18]. There, we considered the non-Abelian vortex in $\mathcal{N} = 2$ SQCD with the gauge group $U(N = 2)$ and four quark flavors, $N_f = 4$. Then the infrared WSSM is already conformal and critical and defines a string theory for a particular value of the coupling constant where the vortex is conjectured to become infinitely thin. The string theory on the conifold was reduced in a certain limit to a noncritical little string theory (LST) (see [19] for a review). The spectrum of the closed string states with the lowest spins was exactly found in [17, 18].

In this paper we do not assume that the non-Abelian vortex is infinitely thin. Instead, the thickness of the vortex sets a scale which plays a crucial role in our construction of the UV complete WSSM. However, it turns out that the UV completion we suggest for the world sheet theory on the non-Abelian vortex in $\mathcal{N} = 2$ SQCD with $N_f = 3$ flavors leads ex-
actly to the string theory on the conifold mentioned above. Therefore, we use the results obtained in [16–18] to describe the hadron spectrum for $\mathcal{N} = 2$ SQCD with $N = 2$ and $N_f = 3$.

Note that one can obtain the spectrum of closed string states at the tree-level approximation only if the associated string theory is weakly coupled, namely the string coupling constant $g_s$ is small. We show that for the case with $N = 2$ and $N_f = 3$ this regime can be arranged, see section 5.2.

The second case in which our UV completion leads to a critical superstring is $\mathcal{N} = 2$ SQCD with the gauge group $U(N = 3)$ and $N_f = N = 3$ quark flavors. In this case the infrared sigma model on the non-Abelian vortex is $\mathbb{C}^P(N - 1 = 2)$ model. Its UV completion has a UV fixed point with a target space described by a non-compact Calabi-Yau manifold $Y_6$ which is the $O(-3)$ line bundle over $\mathbb{C}^P(2)$ and has local $\mathbb{C}^P(2) \times \mathbb{C}$ geometry, see [20] for a review.

This case is new; the detailed study of the associated string theory is left for future work. However, in much the same way as for the conifold case the string spectrum does not contain massless 4D graviton due to non-compactness of the “extra-dimensional” part of the target manifold $Y_6$, cf. [16, 17]. This is of course a desired result since our starting point is 4D $\mathcal{N} = 2$ SQCD without gravity.

The paper is organized as follows. In section 2 we describe general requirements for constructing the UV completion of the IR WSSMs. In section 3 we review non-Abelian vortices in $\mathcal{N} = 2$ SQCD and in particular, describe the IR $\mathcal{W}CP(N, \tilde{N})$ models on the string world sheet. In section 4 we construct UV completion of the world sheet theory and describe its UV fixed point. In section 5 we consider the critical string for the case $N = 2$ and $N_f = 3$ and review our results for the string spectrum obtained in [16–18] for the conifold case. In section 6 we discuss the critical string for the case $N = 3$ and $N_f = 3$ and briefly comment on expected general properties of the resulting string theory. Section 7 summarizes our conclusions.

2 Quest for the UV completion

Schematically, the world sheet theory for a solitonic vortex in 4D Yang-Mills theory can be written as

$$S = \int d^2 \sigma \sqrt{h} \{ \text{IR sigma model + higher derivative terms} \},$$

(2.1)

where $\sigma^\alpha$ ($\alpha = 1, 2$) are the world-sheet coordinates, $h = \det(h_{\alpha\beta})$, where $h_{\alpha\beta}$ is the world-sheet metric understood as an independent variable in the Polyakov formulation [2]. The IR sigma model has the low-energy sigma model action with no more than two derivatives which includes zero modes of the vortex promoted to 2D fields.

Higher derivative corrections run in powers of the ratio $\partial^2 / m_G^2$, where $m_G$ is the scale of masses of the 4D fields which form the vortex solution. If the 4D theory is in the Higgs regime then $m_G$ is the mass of the gauge and Higgs fields.\(^1\) It determines the inverse

\(^1\)For BPS vortex these masses are the same by supersymmetry.
thickness of the vortex and plays the role of the UV cutoff for IR WSSM. At weak coupling \( m_G \) is given by
\[
m_G \sim g \sqrt{T},
\]
where \( T \) is the string tension and \( g \) is a gauge coupling.

While the IR WSSM is known in most cases and can be derived from the 4D gauge theory under consideration, the infinite series of higher-derivative corrections are generally unknown. Still, as we argue in section 1, they are important for the formulation of a well-defined string theory for a given vortex. As usual, in effective theories we can think that higher-derivative corrections appear as a result of integrating out massive fields residing on the string. Since higher-derivative corrections are determined by the 4D mass \( m_G \) we expect that these world-sheet fields have masses \( \geq m_G \). One example of such massive mode is the transverse size of the string itself promoted to a 2D field depending on world sheet coordinates.

Our strategy to find a UV completion for the IR WSSM will be as follows. Instead of attempting to find an infinite series of higher derivative-corrections we include in the world sheet-theory massive states with mass \( \geq m_G \). At first sight this task looks hopeless since we have to determine way too many massive modes. Fortunately, this is not the case. In fact, we are interested only in discrete normalizable modes localized near our string. Non-normalizable modes such as the continuous spectrum of modes with the plane-wave asymptotics have nothing to do with the string — they describe perturbative excitations present in the bulk of our 4D theory.

Thus our task is to find a few normalizable massive modes, the mode associated with the string transverse size being the first priority. In principle this can be done by an honest calculation, however, in this paper we will conjecture UV completions for the BPS non-Abelian vortices in \( \mathcal{N} = 2 \) SQCD using the following general requirements.

(i) The UV completion should have a UV fixed point.

(ii) It should be \( \mathcal{N} = (2, 2) \) supersymmetric for the BPS vortex in \( \mathcal{N} = 2 \) supersymmetric 4D theory. This is the most restrictive requirement.

(iii) The UV completion of the world sheet theory cannot have extra global symmetries not present in 4D theory (and in the infrared WSSM). In fact, we found that this requirement is too restrictive and did not allow us to find any reasonable UV completion. Therefore we replace it with a somewhat relaxed version which is still physically reasonable.

(iii\textsuperscript{relaxed}) If the UV completion has an additional global symmetry not present in our 4D theory the string states have all to be singlets with respect to this symmetry. Then it becomes a “phantom” symmetry.

This relaxed version is minimally necessary. Indeed, the hadronic states made of strings cannot be charged with respect to a symmetry absent in the underlying 4D SQCD.
To conclude this section we address the following problem. One may worry that looking for the UV completion of the WSSM we can go to high enough energy at which vortex can emit states living in the bulk of the 4D SQCD. In fact, this is exactly what happen even at low energy in SQCD with $N_f > N$. As we review in the next section this theory has a Higgs branch formed by perturbative massless states (bifundamental quarks, see next section). They can be thought as a “$\pi$-mesons” of our 4D SQCD. Clearly vortex can emit these “$\pi$-mesons”, so one may worry that the problem is essentially four dimensional and WSSM on the vortex cannot describe physics.

We suggest in this paper that the solution of this problem can be divided into two stages. At the first stage we ignore interactions with 4D bulk states and look for a well-defined WSSM. This allows us to quantize the string and find the spectrum of string states which we interpret as hadrons of 4D SQCD.

At the second stage we can describe interactions of hadrons (string states) with “$\pi$-mesons” using effective Lagrangian description with vertices similar to pion-nucleon vertices in QCD. If the “$\pi$-meson” energy is small the “$\pi$-meson” does not probe the internal structure of a hadron. Hadron interacts as a whole and these interactions can be described by an effective Lagrangian. The parameter of this approximation is the product of the “$\pi$-meson” energy and the size of the hadron which we assume to be small.

Moreover, if we include massive states with mass of order of $m_G$ the problem at the second stage becomes even more complicated. Besides interaction with “$\pi$-mesons” we have to take into account interactions with massive 4D states. This is similar to nucleon interactions with $\rho$-mesons in QCD.

To summarize at the second stage we suggest to use the same strategy that is used in effective low energy descriptions in QCD.

In this paper we limit ourselves to the first stage. Studies of interactions of string states with 4D perturbative states are left for a future work.

It is also worth noting here that the notion of UV completion of WSSM we use in this paper should be understood with care. Clearly the string formation and confinement are IR problems rather then UV problems. At very high energies we have quarks and gauge bosons rather then strings and hadrons. UV completion here assumes energies which are still lower than a certain scale $M_{2D}^{UV}$, which can be thought as a deconfinement scale. This scale is the true UV cutoff for the WSSM.

3 Non-Abelian vortices

3.1 Four-dimensional $\mathcal{N} = 2$ SQCD

Non-Abelian vortex-strings were first found in 4D $\mathcal{N} = 2$ SQCD with the gauge group $U(N)$ and $N_f \geq N$ quark flavors supplemented by the FI $D$ term $\xi$ [6–9], see for example [12, 13] for a detailed review of this theory. In particular, the matter sector of the $U(N)$ theory contains $N_f$ quark hypermultiplets each consisting of the complex scalar fields $q_{kA}$ and $\bar{q}_{Ak}$ (squarks) and their fermion superpartners — all in the fundamental representation of the SU($N$) gauge group. Here $k = 1, \ldots, N$ is the color index while $A$ is the flavor index, $A = 1, \ldots, N_f$. In this paper we assume the quark mass parameters to vanish. In addition,
we introduce the FI parameter $\xi$ in the $U(1)$ factor of the gauge group. It does not break $\mathcal{N} = 2$ supersymmetry.

At weak coupling, $g^2 \ll 1$ (here $g^2$ is the $U(N)$ gauge coupling), this theory is in the Higgs regime in which squarks develop vacuum expectation values (VEVs). The squark vacuum expectation values (VEVs) are

$$\langle q^{kA} \rangle = \sqrt{\xi} \begin{pmatrix} 1 & \ldots & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \ldots & 1 & \ldots & 0 \end{pmatrix}, \quad \langle \bar{q}^{kA} \rangle = 0,$$

where we present the squark fields as matrices in the color ($k$) and flavor ($A$) indices.

These VEVs break the $U(N)$ gauge group. As a result, all gauge bosons are Higgsed. The Higgsed gauge bosons combine with the screened quarks to form long $\mathcal{N} = 2$ multiplets with the mass

$$m_G \sim g \sqrt{\xi}. \quad (3.2)$$

In addition to the $U(N)$ gauge symmetry, the squark condensate (3.1) breaks also the flavor $SU(N_f)$ symmetry. A diagonal global $SU(N)$ combining the gauge $SU(N)$ and an $SU(N)$ subgroup of the flavor $SU(N_f)$ group survives, however. This is a well known phenomenon of color-flavor locking.

Thus, the unbroken global symmetry of our 4D SQCD is

$$SU(N)_{C+F} \times SU(\tilde{N}) \times U(1)_B, \quad (3.3)$$

see [12, 13] for more details. Above, $\tilde{N} = N_f - N$.

The unbroken global $U(1)_B$ factor in eq. (3.3) is identified with a baryonic symmetry. Note that what is usually identified as the baryonic $U(1)$ charge is a part of our 4D theory gauge group. “Our” $U(1)_B$ is an unbroken (by squark VEVs) combination of two $U(1)$ symmetries: the first is a subgroup of the flavor $SU(N_f)$ and the second is the global $U(1)$ subgroup of $U(N)$ gauge symmetry.

The 4D theory has a Higgs branch $\mathcal{H}$ formed by massless quarks which are in the bifundamental representation of the global group (3.3) and carry baryonic charge, see [16] for more details. The dimension of this branch is

$$\text{dim } \mathcal{H} = 4N\tilde{N}. \quad (3.4)$$

At large $\xi$ the theory is weakly coupled. Namely, the gauge coupling freezes at the scale $m_G$ (see (3.2)) and at $m_G \gg \Lambda$ we have

$$\frac{8\pi^2}{g^2(m_G)} = (N - \tilde{N}) \ln \frac{m_G}{\Lambda} \gg 1, \quad (3.5)$$

were $\Lambda$ is the dynamical scale of the 4D $SU(N)$ gauge theory.

As was already noted, we consider $\mathcal{N} = 2$ SQCD in the Higgs phase: $N$ squarks condense. Therefore, the non-Abelian vortex strings at hand confine monopoles. In the $\mathcal{N} = 2$
bulk theory the above strings are 1/2 BPS-saturated; hence, their tension is determined exactly by the FI parameter,
\[ T = 2\pi \xi. \] (3.6)

However, the monopoles cannot be attached to the string endpoints. In fact, in the U(N) theories confined monopoles are junctions of two distinct elementary non-Abelian strings [8, 9, 22] (see [12, 13] for a review). As a result, in 4D \( \mathcal{N} = 2 \) SQCD we have monopole-antimonopole mesons in which monopole and antimonopole are connected by two confining strings. In addition, in the U(N) gauge theory we can have baryons appearing as a closed “necklace” configurations of \( N \times \) (integer) monopoles [12, 13]. For the U(2) gauge group the lightest baryon presented by such a “necklace” configuration consists of two monopoles, see figure 1.

Both stringy monopole-antimonopole mesons and monopole baryons with spins \( J \sim 1 \) have masses determined by the string tension, \( \sim \sqrt{\xi} \) and are heavier at weak coupling than perturbative states with masses \( m_G \sim g\sqrt{\xi} \). Thus they can decay into perturbative states\(^2\) and in fact at weak coupling we do not expect them to appear as stable closed string states. Below we will confirm this expectation from the sting theory side.

If we make \( \xi \) small, \( \xi \ll \Lambda \) our 4D theory becomes weakly coupled in the dual description, see [23] for a review. The dual gauge group \( U(\tilde{N}) \times U(1)^{N-\tilde{N}} \) is Higgsed. Vortices are supported in the dual theory too. They still confine monopoles. Quarks and gauge bosons of the original theory are in the instead-of-confinement phase and form monopole mesons and baryons of the type shown in figure 1. For \( N_f > N \) we expect that these states are heavy and unstable.

Only in the “true” strong coupling domain \( g^2 \sim 1 \) or \( m_G \sim \Lambda \) we expect that hadrons shown in figure 1 become stable and can be described as closed string states of the soliton string theory.

### 3.2 World-sheet sigma model

The presence of the color-flavor locked group \( SU(N)_{C+F} \) is the reason for the formation of the non-Abelian vortex strings [6–9]. The most important feature of these vortices is the presence of the orientational zero modes. As we already mentioned, in \( \mathcal{N} = 2 \) SQCD these strings are 1/2 BPS-saturated.\(^2\)

\(^2\)Their quantum numbers with respect to the global group (3.3) allow these decays, see [12, 13].
Let us briefly review the model emerging on the world sheet of the non-Abelian string [12, 13].

The translational moduli fields (they decouple from all other moduli) in the Polyakov formulation [2] are given by the action

\[ S_{\text{trans}} = \frac{T}{2} \int d^2 \sigma \sqrt{h} h^{\alpha \beta} \partial_\alpha x^\mu \partial_\beta x_\mu + \text{fermions}, \quad (3.7) \]

where \( x^\mu (\mu = 1, \ldots, 4) \) describe the \( \mathbb{R}^4 \) part of the string target space.

If \( N_f = N \) the dynamics of the orientational zero modes of the non-Abelian vortex, which become orientational moduli fields on the world sheet, are described by two-dimensional \( \mathbb{N} = (2, 2) \) supersymmetric \( \mathbb{CP}(N - 1) \) model.

If one adds additional quark flavors, non-Abelian vortices become semilocal — they acquire size moduli [24]. In particular, for the non-Abelian semilocal vortex at hand, in addition to the complex orientational moduli \( \rho^P \) (here \( P = 1, \ldots, N \)), we must add the size moduli \( \rho^K \) (where \( K = 1, \ldots, \tilde{N} \)), see [6, 9, 24–28]. The size moduli are also complex. The low-energy dynamics of the orientational and size moduli are described by the weighted \( \mathbb{CP} \) model, which we denote \( \mathcal{WCP}(N, \tilde{N}) \).

The gauged formulation of \( \mathcal{WCP}(N, \tilde{N}) \) is as follows [37]. One introduces the \( \mathrm{U}(1) \) charges \( \pm 1 \), namely +1 for \( \rho^P \)'s and −1 for \( \rho^K \)'s. The bosonic part of the action reads

\[ S_{\text{IR}} = \int d^2 \sigma \sqrt{h} \left\{ h^{\alpha \beta} \left( \tilde{\nabla}_\alpha \tilde{n} P \nabla_\beta n^P + \nabla_\alpha \tilde{\rho}_K \tilde{\nabla}_\beta \rho^K \right) 
+ 2|\sigma|^2 |n^P|^2 + 2|\sigma|^2 |\rho^K|^2 + \frac{e^2}{2} \left(|n^P|^2 - |\rho^K|^2 - \beta^2 \right)^2 \right\}, \quad (3.8) \]

where

\[ \nabla_\alpha = \partial_\alpha - i A_\alpha, \quad \tilde{\nabla}_\alpha = \partial_\alpha + i A_\alpha, \quad (3.9) \]

while \( A_\alpha \) and the complex scalar \( \sigma \) form a bosonic part of an auxiliary gauge supermultiplet. In the limit \( e^2 \to \infty \) the gauged linear model (3.8) reduces to \( \mathcal{WCP}(N, \tilde{N}) \).

Classically the coupling constant \( \beta \) in (3.8) is related to the 4D \( \mathrm{SU}(2) \) gauge coupling \( g^2 \) via [12, 13]

\[ \beta = \frac{4\pi}{g^2}. \quad (3.10) \]

In quantum theory the 2D coupling \( \beta \) runs. The relation (3.10) is imposed at the UV cutoff for the effective 2D theory (3.8). This UV cutoff is given by the scale \( m_G \) which determines the inverse thickness of the vortex [12, 13].

Below \( m_G \) the IR WSSM is asymptotically free with the coupling \( \beta \) given by

\[ \beta(\mu) = \frac{(N - \tilde{N})}{2\pi} \log \frac{\mu}{\Lambda}, \quad (3.11) \]

where \( \mu \) is the normalization point below \( m_G \).

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Both the orientational and the size moduli have logarithmically divergent norms, see e.g. [25]. After an appropriate infrared regularization, logarithmically divergent norms can be absorbed into the definition of relevant two-dimensional fields [25]. See also [29] for the discussion of different choices for quark \( \mathrm{U}(1) \) charges which lead to a power non-normalizable orientational and size moduli of the vortex. In these cases orientational and size moduli cannot be promoted to fields living on the string world sheet.
Note that the first (and the only) coefficient of the $\beta$ functions is the same for the 4D SQCD and the IR WSSM. This ensures that the scale of $\mathbb{WCP}(N, \tilde{N})$ coincides with $\Lambda$ of the 4D theory [12, 13].

The global symmetry of the IR WSSM (3.8) is

$$\text{SU}(N) \times \text{SU}(\tilde{N}) \times \text{U}(1)_{B},$$

i.e. exactly the same as the unbroken global group in the 4D theory (3.3). The fields $n$ and $\rho$ transform in the following representations:

$$n : (N, 1, 0), \quad \rho : (1, \tilde{N}, 1).$$

Physically the profile of a semilocal vortex in the plane orthogonal to the string axis has a two-layer structure. It has a hard core of radius $m_{G}^{-1}$ formed by heavy 4D fields and a long-range tail with power fall-off of the profile functions at infinity. The tail is formed by massless quark fields fluctuating along the Higgs branch. Moduli $\rho^{K}$ characterize “sizes” of the massless tail of the vortex [24, 27, 28]. For $N_{f} = N$ size moduli $\rho^{K}$ disappear and the model (3.8) reduces to $\mathbb{CP}(N - 1)$ in the gauge formulation [30].

To conclude this section we note, that one can add small masses to quarks in 4D SQCD. This will result in adding twisted masses (equal to 4D quark masses) to $n$ and $\rho$-fields in (3.8), see [12, 13]. The twisted masses do not break $N = (2, 2)$ supersymmetry. They can be introduced by gauging a global $U(1)$ symmetry associated with each $n^{P}$ or $\rho^{K}$ field and then freezing all components of the gauge multiplet, while the constant values of the $\sigma$ fields will determine the mass [34]. For simplicity we do not introduce quark masses in this paper.

### 3.3 2D-4D correspondence

As was mentioned above confined monopoles of 4D SQCD are junctions of two different elementary non-Abelian strings. In the WSSM they are seen as kinks interpolating between different vacua of $\mathbb{WCP}(N, \tilde{N})$ model. This ensures 2D-4D correspondence: the coincidence of the BPS spectra of monopoles in 4D SQCD in the quark vacuum (given by the exact Seiberg-Witten solution [21]) and kinks in 2D $\mathbb{WCP}(N, \tilde{N})$ model. This coincidence was observed in [31, 32] and explained later in [8, 9] using the picture of confined bulk monopoles which are seen as kinks in the world sheet theory. A crucial point is that both the monopoles and the kinks are BPS-saturated states, and their masses cannot depend on the non-holomorphic parameter $\xi$ [8, 9]. This means that, although the confined monopoles look physically very different from unconfined monopoles on the Coulomb branch of 4D SQCD, their masses are the same. Moreover, these masses coincide with the masses of kinks in the world-sheet theory.

The 2D-4D correspondence imposes another very restrictive requirement on the possible UV completion of the IR $\mathbb{WCP}(N, \tilde{N})$ model on the string world sheet in addition to those discussed in section 2:

(iv) The UV completion should have the same spectrum of the BPS kinks as IR $\mathbb{WCP}(N, \tilde{N})$ model since it is fixed by 4D SQCD.

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4Confined monopoles, being junctions of two distinct 1/2-BPS strings, are 1/4-BPS states in the bulk theory [8].
Below we briefly review the BPS kink spectrum in $\mathbb{WCP}(N, \tilde{N})$ model, see [33] for details. It is fixed by the exact effective twisted superpotential [32, 34–37]. Integrating out the fields $n^P$ and $\rho^K$ we obtain the following exact twisted superpotential:

$$W_{\text{WCP}}(\sigma) = \frac{1}{4\pi} \left\{ (N - \tilde{N}) \sqrt{2} \sigma \ln \frac{\sqrt{2} \sigma}{\Lambda} - (N - \tilde{N}) \sqrt{2} \sigma \right\},$$

(3.14)

where we use one and the same notation $\sigma$ for the twisted superfield [37] and its lowest scalar component. Minimizing this superpotential with respect to $\sigma$ we get the equation for the $\sigma$ VEVs (the so-called twisted chiral ring equation),

$$\left(\sqrt{2} \sigma\right)^N = \Lambda^{(N-\tilde{N})} \left(\sqrt{2} \sigma\right)^{\tilde{N}}.$$

(3.15)

It is seen that $\tilde{N}$ roots of this equation are at $\sigma = 0$ ("zero vacua") while $(N - \tilde{N})$ roots ("$\Lambda$-vacua") are

$$\sqrt{2} \sigma = e^{\frac{2\pi i}{N-\tilde{N}}} \Lambda, \quad k = 1, \ldots (N - \tilde{N}).$$

(3.16)

The masses of the BPS kinks interpolating between two vacua are given by the differences of the superpotential (3.14) calculated at distinct roots [31, 32, 34],

$$M_{\text{BPS}} = 2 |W_{\text{WCP}}(\sigma_1) - W_{\text{WCP}}(\sigma_2)| = \frac{N - \tilde{N}}{2\pi} \Lambda \left| e^{\frac{2\pi i}{N-\tilde{N}}} - 1 \right|,$$

(3.17)

where we present the mass of the kink interpolating between the neighboring $\Lambda$-vacua with $k = 0$ and $k = 1$.

If twisted masses were non-zero then the equation (3.15) and the kink spectrum would become much more complicated [31–34]. In particular, due to the presence of branches in the logarithmic functions in (3.14) each kink comes together with a tower of "dyonic" kinks carrying global U(1) charges (for more details see e.g. [33, 38]).

The masses obtained from (3.17) were shown to coincide with those of the monopoles and dyons in the bulk theory. The latter are given by the period integrals of the Seiberg-Witten curve.

4 The UV completion of WSSM

As we have already mentioned, our WSSM in (3.8) is not conformal for $N_f < 2N$ and cannot serve as a sigma model for the string quantization. In this section we suggest its UV completion using requirements outlined in section 2 and section 3.3. As the simplest choice we can add a massive complex field $\rho_H$ with mass $\sim m_G$ to the $\mathbb{WCP}(N, \tilde{N})$ model (3.8), which physically describes fluctuations of the string hard core. We give the supermultiplet of these fields the charge $(\tilde{N} - N)$ with respect to the auxiliary U(1) gauge field in (3.8). This ensures that the associated coupling does not run at scales above $m_G$.

To preserve $N = (2, 2)$ supersymmetry and the U(1) gauge invariance while making $\rho_H$ heavy we exploit a procedure similar to that of introducing a twisted mass for this field [34].

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5 The reader can keep in mind e.g. the case $N = 2$, $N_f = 3$, and $\tilde{N} = 1$, see below.
Namely, we gauge the global U(1) symmetry associated with $\rho_H$. To this end we introduce an extra gauge multiplet with bosonic components given by gauge field $B_\alpha$ and a complex scalar $\sigma_B$ assuming that $H$ has electric charge +1 with respect to the second gauge field. Also, in order to avoid a second D-flatness condition in the UV we take the second gauge coupling $e_2^2$ finite rather than tending it to infinity (while $e^2 \to \infty$ in (3.8)). Moreover, to get rid of free field $\sigma_B$ in the UV we introduce a twisted superpotential $W_{\text{tree}}(\sigma_B)$ which makes the second gauge multiplet heavy.

Then, the bosonic action takes the following form

$$
S_{\text{complete}} = \int d^2\sqrt{h} \left\{ h^{\alpha\beta} \left( \nabla_{\alpha} \tilde{n}_P \nabla_{\beta} n^P + \nabla_{\alpha} \tilde{\rho}_K \nabla_{\beta} \rho^K + \nabla_{\alpha} \tilde{\rho}_H \nabla_{\beta} \rho_H \right) + \frac{1}{4e_B^2} B_{\alpha\beta} B^{\alpha\beta} + \frac{1}{e_B^2} \partial_\alpha \sigma_B \partial^\alpha \sigma_B - i \sqrt{2} W_{\text{tree}} B_{01} + 2|\sigma|^2 |n|^2 + 2|\sigma|^2 |\rho^K|^2 + 2|\sigma_B - (N - \tilde{N})|\sigma|^2 |\rho_H|^2 + e_2^2 \left( |n|^2 - |\rho^K|^2 - (N - \tilde{N}) |\rho_H|^2 - \beta_1 \right)^2 + e_2^2 \left( |\rho_H|^2 + \sqrt{2} W_{\text{tree}} - \beta_2 \right)^2 \right\},
$$

(4.1)

where $P = 1, \ldots, N$ and $K = 1, \ldots, \tilde{N}$, while

$$
\nabla^H_{\alpha} = \partial_\alpha - i(N - \tilde{N}) A_\alpha + i B_\alpha, \quad \tilde{\nabla}^H_{\alpha} = \partial_\alpha + i(N - \tilde{N}) A_\alpha - i B_\alpha,
$$

(4.2)

$B_{\alpha\beta}$ is the field strength of the gauge field $B_\alpha$ and $W_{\text{tree}}$ is the derivative of the superpotential with respect to $\sigma_B$. We specify the exact form of the superpotential $W_{\text{tree}}(\sigma_B)$ later.

The subscript $H$ means “heavy”. The new field $\rho_H$ has an interpretation of a transverse size of the string core. This field enters in the action (4.1) on the same footing as the “tail sizes” $\rho^K$, namely, all $\rho$ fields have negative electric charge. The difference is that $\rho_H$ is heavy with mass $\sim m_G$ as we will show below.

At the scales $\mu$ above the scale $m_G$ the coupling constant $\beta_1$ does not run since the sum of the electric charges of all fields $n^P$, $\rho^K$ and $\rho_H$ is zero. The coupling constant $\beta_2$ is asymptotically free,

$$
\beta_2 = \frac{1}{2\pi} \log \frac{\mu}{\Lambda_H},
$$

(4.3)

where $\Lambda_H$ is the position of the IR pole of $1/\beta_2$. The absolute value of this scale is identified with the mass $m_G$ in our 4D SQCD,

$$
|\Lambda_H| = m_G,
$$

(4.4)

while the phase is related to the $\theta$-angle of the gauge field $B_\alpha$. For simplicity we assume this phase vanishing.

Below this scale $\rho_H$ can be integrated out and the model (4.1) reduces to (3.8). The running of its coupling is determined by

$$
\beta = \beta_1 + (N - \tilde{N}) \beta_2 = \beta_1 + \frac{N - \tilde{N}}{2\pi} \log \frac{\mu}{\Lambda_H},
$$

(4.5)

where we use the fact that VEV of $|\rho_H|$ is given by $\beta_2$ ($W_{\text{tree}}$ is effectively zero, see below).
Comparing this with eq. (3.11) we see that

$$\beta_1 = \frac{(N - \tilde{N})}{2\pi} \log \frac{\Lambda_H}{\Lambda}. \quad (4.6)$$

4.1 Exact superpotential

The exact twisted superpotential for the model (4.1) is given by

$$W_{\text{eff}}(\sigma, \sigma_B) = \frac{1}{4\pi} \left\{ (N - \tilde{N}) \sqrt{2} \sigma \ln \frac{\sqrt{2} \sigma}{\Lambda} + 4\pi W_{\text{tree}} \right. \\
+ \sqrt{2} \left[ \sigma_B - (N - \tilde{N}) \sigma \right] \ln \frac{\sqrt{2} \left[ \sigma_B - (N - \tilde{N}) \sigma \right]}{\Lambda_H} - \sqrt{2} \sigma_B \right\}. \quad (4.7)$$

Minimizing this superpotential with respect to $\sigma$ and $\sigma_B$ we find two vacuum equations, namely

$$\begin{aligned}
(N - \tilde{N}) \ln \frac{\sqrt{2} \sigma}{\Lambda} - (N - \tilde{N}) \ln \frac{\sqrt{2} \left[ \sigma_B - (N - \tilde{N}) \sigma \right]}{\Lambda_H} &= 0 \quad (4.8) \\
\ln \frac{\sqrt{2} \left[ \sigma_B - (N - \tilde{N}) \sigma \right]}{\Lambda_H} + \frac{4\pi}{\sqrt{2}} W_{\sigma_B}^{\text{tree}} &= 0. \quad (4.9)
\end{aligned}$$

In appendix A we construct the tree superpotentials $W_{\text{tree}}^{\sigma_B}$ for two cases, $\tilde{N} = 0$ and $\tilde{N} > 0$. With this choice the vacuum equations (4.8) and (4.9) reduce to

$$\begin{aligned}
(N - \tilde{N}) \ln \frac{\sqrt{2} \sigma}{\Lambda} &= \Lambda^{(N - \tilde{N})} (\sqrt{2} \sigma)^{\tilde{N}}, \quad (4.10) \\
\sqrt{2} \left[ \sigma_B - (N - \tilde{N}) \sigma \right] &= \Lambda_H. \quad (4.11)
\end{aligned}$$

The superpotentials $W_{\text{tree}}^{\sigma_B}$ constructed in the appendix A satisfy the following conditions

$$W_{\text{tree}}^{\sigma_B} \big|_{\text{vac}} = 0, \quad W_{\sigma_B}^{\text{tree}} \big|_{\text{vac}} = 0, \quad \frac{\partial^2}{\partial \sigma_B^2} W_{\text{tree}}^{\sigma_B} \big|_{\text{vac}} \to \infty, \quad (4.12)$$

where $\big|_{\text{vac}}$ means that $\sigma_B$ is taken to be equal to solutions of vacuum equations (4.10) and (4.11). In particular, the second condition above ensures that the derivative of the superpotential $W_{\sigma_B}^{\text{tree}}$ in eq. (4.9) vanish. Moreover, the third condition makes $\sigma_B$ infinitely heavy.

Observe now that eq. (4.11) shows that the mass of the field $\rho_H$ is equal to $\Lambda_H$,

$$m_{\rho_H} = \Lambda_H, \quad (4.13)$$

see the third line in the eq. (4.1).
Eq. (4.10) is precisely the chiral ring equation (3.15) for the IR \( \mathbb{WCP}(N, \tilde{N}) \) model (3.8). Roots of this equation are given by (3.16). Calculating the mass of the BPS kink interpolating between two neighboring \( \Lambda \)-vacua in the model (4.1) we get

\[
M_{\text{BPS}} = 2 \left| W_{\text{eff}}(\sigma^{(1)}, \sigma^{(1)}_{B}) - W_{\text{eff}}(\sigma^{(2)}, \sigma^{(2)}_{B}) \right| = \frac{1}{2\pi} \sqrt{2}(\sigma^{(1)}_{B} - \sigma^{(2)}_{B})
\]

where we used eq. (4.11). Note, that \( W_{\text{tree}} \) does not contribute to masses of BPS kinks due to the first condition in (4.12).

We see that the BPS kink spectrum in our UV completion of WSSM (4.1) coincides with the one (3.17) in the IR \( \mathbb{WCP}(N, \tilde{N}) \) model.

To summarize we outline the procedure to give a large mass \( \Lambda_{H} \) to the field \( H \). Our procedure is similar to the standard method of introducing a twisted mass [34]. In the standard method the physical degrees of freedom of extra gauge multiplet are frozen by sending \( e_{B} \) to zero, while the VEV of \( \sigma_{B} \) defines the twisted mass. Our procedure assumes that \( e_{B} \) is finite and we freeze the physical degrees of freedom of the second gauge multiplet introducing the superpotential \( W_{\text{tree}} \). The advantage is that it allows us to keep the BPS kink spectrum of the IR WSSM intact. Thus we meet a very restrictive requirement (iv), see section 3.3.

4.2 UV fixed point

Our WSSM (4.1) has a UV fixed point since the coupling \( \beta_{1} \) does not run, while the coupling \( 1/\beta_{2} \) is asymptotically free and goes to zero in the UV. The total bosonic world-sheet action is given by the sum of (4.1) and (3.7),

\[
S = S_{\text{trans}} + S_{\text{complete}} .
\]

The UV fixed point of this \( \mathcal{N} = (2, 2) \) WSSM defines our superstring theory.

Since \( e_{B} \) is finite the \( D_{B} \)-flatness condition does not survive in the UV. However, the first \( D \)-flatness condition in (4.1), namely

\[
|n^{P}|^{2} - |\rho^{K}|^{2} - (N - \tilde{N}) |\rho_{H}|^{2} = \beta_{1},
\]

\[
P = 1, \ldots, N, \quad K = 1, \ldots, \tilde{N},
\]

(supplemented by factorization with respect to the U(1) gauge phase) survives in the UV and determines the “extra-dimensional” target space of our string sigma model.

The above UV conformal sigma model satisfies all requirements of section 2 and section 3.3 except the condition (iii). Clearly, adding the field \( \rho_{H} \) increases the global symmetry of the model in the UV. For example, if \( (N - \tilde{N}) \neq 1 \) we obtain an extra U(1) symmetry. Therefore, below we will use the relaxed version of the condition (iii) and check that it is satisfied, see section 2.

The number of real bosonic degrees of freedom in (4.16) is

\[
2(N + \tilde{N} + 1 - 1) = 2(N + \tilde{N}),
\]
where $2 \times (+1)$ arises from the $\rho_H$ field, while $2 \times (-1)$ is associated with D-flatness condition and one U(1) phase eaten by the Higgs mechanism. Adding four translational moduli from (3.7) we get ten dimensional target space if

$$N + \tilde{N} = 3. \quad (4.18)$$

This is a condition of criticality for our superstring.

Note, that the components (e.g. $\sigma$) of the auxiliary gauge multiplet are "composite" fields and do not represent independent physical degrees of freedom in the UV. In contrast, since $e_B$ is finite the components of $B_\alpha$ gauge multiplet (say $\sigma_B$) are independent degrees of freedom. We introduced superpotential $W^{tree}$ to freeze $\sigma_B$.

The condition of criticality (4.18) has two solutions,\(^6\)

$$N = 2, \quad \tilde{N} = 1 \quad (4.19)$$

and

$$N = 3, \quad \tilde{N} = 0. \quad (4.20)$$

In both cases the target space of the 2D sigma model has the form

$$\mathbb{R}^4 \times Y_6, \quad (4.21)$$

where $Y_6$ is a non-compact Calabi-Yau manifold.

Note that our 2D sigma model preserves $N = (2, 2)$ supersymmetry which is a necessary condition for a superstring to have $N = 2$ space-time supersymmetry in 4D [39, 40].

5 String theory on the conifold

In this section we will consider a critical string theory on $\mathbb{R}^4 \times Y_6$ emerging in 4D SQCD with the U(2) gauge group and $N_f = 3$ flavors. It corresponds to the first solution, eq. (4.19). The electric charge of $\rho_H$ is $-1$ in this case so we have two $n$-fields with charge $+1$ and two $\rho$-fields with charge $-1$. The model (4.1) reduces in the UV to $W_{CP}(2, 2)$ model.

As was mentioned in section 1, the string theory based on this sigma model was studied earlier in our papers [15–18]. In these papers we considered the non-Abelian vortex in $\mathcal{N} = 2$ SQCD with gauge group U($N = 2$) and four quark flavors, $N_f = 4$. In that case the IR WSSM was given by $W_{CP}(2, 2)$. The latter model is conformal and critical and defines a string theory at a particular value of the coupling constant where the vortex was conjectured to become infinitely thin.

Now we do not assume that the non-Abelian vortex is infinitely thin. Now we consider U($N = 2$) SQCD with $N_f = 3$, while the additional $\rho_H$ field describes the size of the core of the non-Abelian vortex. However, in the UV limit our UV completion of the world sheet theory (4.1) reduces to $W_{CP}(2, 2)$. Thus, we can use the results obtained in [15–18] to describe the spectrum of the closed string states in SQCD with three quark flavors. Below we review and reinterpret these results.

\(^6\)The solution with $N = 1, \tilde{N} = 2$ gives the same theory as in (4.19) if we take $\beta$ to be negative. In 4D SQCD it corresponds to a dual description in the regime $m_G \ll \Lambda$. 

- 14 -
Note that the global symmetry of \( \text{WCP}(2, 2) \) model is

\[
\text{SU}(2) \times \text{U}(1)_B \times \text{SU}(2)_{\text{extra}}
\]

so we have an extra \( \text{SU}(2) \) symmetry in our WSSM compared to the symmetry of the 4D theory, see (3.3) for \( N = 2, \bar{N} = 1 \). In this section we will see that the string states are not charged with respect to this “UV symmetry”.

The \( D \)-flatness condition takes the form

\[
|n^P|^2 - |\rho|^2 - |\rho_H|^2 = \beta_1, \quad P = 1, 2,
\]

and a \( \text{U}(1) \) phase is gauged away. This condition defines a non-compact six dimensional Calabi-Yau space, the conifold, see [20] for a review.

The non-compactness is the most crucial feature of our “extra-dimensional” space \( Y_6 \). Most of the modes have non-normalizable wave functions over \( Y_6 \) and therefore do not produce dynamical fields in 4D. Only normalizable over \( Y_6 \) modes localized near the tip of the conifold can be interpreted as hadrons of 4D theory.

It is easy to see that normalizable localized states can arise only at strong coupling in 4D SQCD. To see this we note that at weak coupling in 4D, \( m_G \gg \Lambda \), according to (4.6) we have weak coupling in WSSM too, \( \beta_1 \gg 1 \). In this regime the space defined by (5.2) approaches a flat six dimensional space. It is clear that in this limit there are no localized discrete states on \( Y_6 \). The spectrum of states is continuous, with the plane-wave asymptotics of the wave functions. All these states are non-normalizable. The same is true for \( m_G \ll \Lambda \) when \( \beta_1 \ll -1 \). Only at strong coupling \( m_G \sim \Lambda \) or \( \beta_1 \sim 0 \) do we have a chance to find normalizable states.

### 5.1 Massless baryon

The only 4D massless state found in [16] is the one associated with the deformation of the conifold complex structure. All other modes arising from the massless 10D graviton have non-normalizable wave functions over the conifold. In particular, the 4D graviton associated with a constant wave function over the conifold is absent [16]. This result matches our expectations since from the very beginning we started from \( \mathcal{N} = 2 \) SQCD in the flat four-dimensional space without gravity.

Let us construct the \( \text{U}(1) \) gauge-invariant “mesonic” variables from the fields \( n \) and \( \rho \),

\[
w^{PS} = n^P \rho^S.
\]

Here \( \rho^S = (\rho, \rho_H) \), \( S = 1, 2 \).

These variables are subject to the constraint \( \det w^{PS} = 0 \), or

\[
\sum_{n=1}^{4} w_n^2 = 0,
\]

where

\[
w^{PS} \equiv \sigma_n^{PS} w_n,
\]
and the $\sigma$ matrices above are $\left(1, -i\sigma^a\right)$, $a = 1, 2, 3$. Equation (5.4) defines the conifold $Y_6$. It has the Kähler Ricci-flat metric and represents a non-compact Calabi-Yau manifold [20, 37, 41]. It is a cone which can be parametrized by the non-compact radial coordinate

$$\tilde{r}^2 = \sum_{n=1}^{4} |w_n|^2$$  \hspace{1cm} (5.5)

and five angles, see [41]. Its section at fixed $\tilde{r}$ is $S_2 \times S_3$.

At $\beta_1 = 0$ the conifold develops a conical singularity, so both $S_2$ and $S_3$ can shrink to zero. The conifold singularity can be smoothed out in two distinct ways: by deforming the Kähler form or by deforming the complex structure. The first option is called the resolved conifold and amounts to introducing a non-zero $\beta_1$ in (5.2). This resolution preserves the Kähler structure and Ricci-flatness of the metric. If we put $\rho^K = 0$ in (5.2) we get the $\mathbb{CP}(1)$ model with the $S_2$ target space (with radius $\sqrt{\beta_1}$). The resolved conifold has no normalizable zero modes. In particular, the modulus $\beta_1$ which becomes a scalar field in four dimensions has non-normalizable wave function over the $Y_6$ manifold [16].

As explained in [16, 42], non-normalizable 4D modes can be interpreted as (frozen) parameters of the 4D theory. The $\beta_1$ field is the most straightforward example of this, since the 2D coupling $\beta_1$ is related to the ratio $m_G/\Lambda$ in 4D SQCD, see (4.6).

If $\beta_1 = 0$ another option exists, namely a deformation of the complex structure [20]. It preserves the Kähler structure and Ricci-flatness of the conifold and is usually referred to as the deformed conifold. It is defined by deformation of eq. (5.4), namely,

$$\sum_{n=1}^{4} w_n^2 = b, \hspace{1cm} (5.6)$$

where $b$ is a complex number. Now the $S_3$ can not shrink to zero, its minimal size is determined by $b$.

The modulus $b$ becomes a 4D complex scalar field. The effective action for this field was calculated in [16] using the explicit metric on the deformed conifold [41, 43, 44],

$$S(b) = T \int d^4x |\partial_\mu b|^2 \log \frac{T^2 L^4}{|b|}, \hspace{1cm} (5.7)$$

where $L$ is the size of $\mathbb{R}^4$ introduced as an infrared regularization of logarithmically divergent $b$ field norm.\(^7\)

We see that the norm of the $b$ modulus turns out to be logarithmically divergent in the infrared. The modes with the logarithmically divergent norm are at the borderline between normalizable and non-normalizable modes. Usually such states are considered as “localized” on the string. We follow this convention.

The field $b$, being massless, can develop a VEV. Thus, we have a new Higgs branch in 4D $\mathcal{N} = 2$ SQCD which opens up only for the critical value of the coupling constant $\beta_1 = 0$ ($m_G = \Lambda$).

\(^7\)The infrared regularization on the conifold $\tilde{r}_{\text{max}}$ translates into the size $L$ of the 4D space because the variables $\rho$ in (5.5) have an interpretation of the vortex string sizes, $\tilde{r}_{\text{max}} \sim TL^2$. 

- 16 -
In [16] the massless state $b$ was interpreted as a baryon of 4D $\mathcal{N} = 2$ SQCD. Let us explain this. From eq. (5.6) we see that the complex parameter $b$ (which is promoted to a 4D scalar field) is a singlet with respect to both SU(2) factors in (5.1). What about its baryonic charge?

Since

$$w_n = \frac{1}{2} \text{Tr} [(\bar{\sigma} n)_{KP} n^P \rho^K]$$

we see that the $b$ state transforms as

$$(1, 2, 1),$$

where we used (3.13) and (5.6). Three numbers above refer to the representations of (5.1). In particular, it has the baryon charge $Q_B(b) = 2$.

As shown in [16] our string on the conifold is of type IIA. For type IIA superstring the complex scalar associated with deformations of the complex structure of the Calabi-Yau space enters as a component of a massless 4D $\mathcal{N} = 2$ hypermultiplet, see [45] for a review. Instead, for type IIB superstring it would be a component of a vector BPS multiplet. Non-vanishing baryonic charge of the $b$ state confirms our conclusion that the string under consideration is of type IIA. The associated hypermultiplet is explicitly constructed in [18].

5.2 Massive states

In fact the critical string theory on the conifold is hard to use for calculating the spectrum of massive (non-BPS) string modes because the supergravity approximation does not work at $\beta_1 = 0$. In this section we review the results obtained in [17] based on the little string theory (LST) approach, see [17] for details. Namely, we used the equivalent formulation of our string theory on the conifold as a non-critical $c = 1$ string theory with the Liouville field $\phi$ and a compact scalar $Y$ at the self-dual radius formulated on the target space [46, 47]

$$\mathbb{R}^4 \times \mathbb{R}_\phi \times S^1.$$  \hfill (5.10)

This theory has a linear in $\phi$ dilaton, such that string coupling is given by

$$g_s = e^{-\frac{1}{\sqrt{2}} \phi}.$$  \hfill (5.11)

The value of the background charge of the Liouville field ($= \sqrt{2}$) ensures that the central charge of the supersymmetrized $c = 1$ theory is equal to 9, exactly what is needed for criticality.

Generically the above equivalence is formulated in a certain limit between the critical string on the non-compact Calabi-Yau spaces with an isolated singularity on the one hand, and non-critical $c = 1$ string with the additional Ginzburg-Landau $\mathcal{N} = 2$ superconformal system [46], on the other hand. In the conifold case the extra Ginzburg-Landau factor in (5.10) is absent [48].

The Ginzburg-Landau superconformal system, if present, would have a superpotential defined by the left-hand side of eq. (5.4). In this case the vertex operators would contain dependence on powers of fields $w_n$ charged with respect to SU(2)$\times$ SU(2) factor in (5.1),

\[\text{JHEP12(2019)050}\]
cf. [46]. However, since the Ginzburg-Landau system is absent for the conifold case the string states are not charged with respect to SU(2) factors of the global group. As we will see below, they all have baryonic charge.

In fact the $c = 1$ non-critical string theory on (5.10) can also be described in terms of two-dimensional black hole [49], which is the SL$(2, \mathbb{R})/U(1)$ coset Wess-Zumino-Novikov-Witten theory [46, 47, 50, 51] at level

$$k = 1,$$

(5.12)

where $k$ is the total level of the Kač-Moody algebra in the supersymmetric version (the level of the bosonic part of the algebra is then $k_b = k + 2 = 3$).

In [52] it was shown that $\mathcal{N} = (2, 2)$ SL$(2, \mathbb{R})/U(1)$ coset which can be exactly solved by algebraic methods is a mirror description of the $c = 1$ Liouville theory. The target space of this theory has the form of a semi-infinite cigar; the field $\phi$ associated with the motion along the cigar cannot take large negative values due to semi-infinite geometry. In this description the string coupling constant at the tip of the cigar is $g_s \sim 1/b$. If we following [46] take $b$ large the string coupling at the tip of the cigar will be small and the string perturbation theory becomes reliable, cf. [46, 59]. In particular, we can use the tree-level approximation to obtain the string spectrum.

The vertex operators for the string theory on the manifold (5.10) are constructed in [46], see also [48, 50]. Primaries of the $c = 1$ part for large positive $\phi$ (where the target space becomes a cylinder $\mathbb{R}_\phi \times S^1$) take the form

$$V_{j,m} \sim \exp \left( \sqrt{2}j\phi + i\sqrt{2}mY \right),$$

(5.13)

where $2m$ is integer. Scaling dimension of the primary operator (5.13) is

$$\Delta_{j,m} = m^2 - j(j + 1).$$

(5.14)

The spectrum of the allowed values of $j$ and $m$ in (5.13) was exactly determined using the Kač-Moody algebra for the coset SL$(2, \mathbb{R})/U(1)$ in [50, 53–56], see [57] for a review. We will look for string states with normalizable wave functions over the “extra dimensions” which we will interpret as hadrons in 4D $\mathcal{N} = 2$ SQCD. These states come from the discrete spectrum. For $k = 1$ we are left with only two allowed values of $j$ [17],

$$j = -\frac{1}{2}, \quad m = \pm \left\{ \frac{1}{2}, \frac{3}{2}, \ldots \right\}$$

(5.15)

and

$$j = -1, \quad m = \pm \{ 1, 2, \ldots \},$$

(5.16)

where $j = -1/2$ case corresponds to the logarithmically normalizable modes like in eq. (5.7).

For scalar states in 4D the GSO projection restricts the integer $2m$ for the operator in (5.13) to be odd [46, 58], and we have only one possibility $j = -\frac{1}{2}$, see (5.15). This determines the masses of the 4D scalars [17],

$$\frac{(M^S_m)^2}{8\pi T} = m^2 - \frac{1}{4}. $$

(5.17)
Figure 2. Spectrum of spin-0 and spin-2 states as a function of the baryonic charge. Closed and open circles denote spin-0 and spin-2 states, respectively.

In particular, the state with $m = \pm 1/2$ is the massless baryon $b$, associated with deformations of the conifold complex structure [17], while states with $m = \pm (3/2, 5/2, \ldots)$ are massive 4D scalars.

At the next level we consider 4D spin-2 states. The GSO projection selects now $2m$ to be even, $|m| = 0, 1, 2, \ldots$ [46], thus we are left with only one allowed value of $j$, $j = -1$ in (5.16). Moreover, the value $m = 0$ is excluded. This leads to the following expression for the masses of spin-2 states [17]:

$$\frac{M_{\text{spin-2}}^2}{8\pi T} = m^2, \quad |m| = 1, 2, \ldots$$

(5.18)

We see that all spin-2 states are massive. This confirms the result in [16] that no massless 4D graviton appears in our theory. It also matches the fact that our “boundary” theory, 4D $\mathcal{N} = 2$ QCD, is defined in flat space without gravity.

The momentum $m$ in the compact $Y$ direction of the vertex operator (5.13) is related to the baryon charge of a string state [17],

$$Q_B = 4m.$$  

(5.19)

All states reviewed above are baryons. Their masses as a function of the baryon charge are shown in figure 2.

String states shown in (5.17) and (5.18) are particular representatives of $\mathcal{N} = 2$ supermultiplets in 4D. Other components can be restored by 4D supersymmetry. This was done in [18] for low-lying states. The massless baryon in (5.17) with $m = \pm 1/2$ is a hypermultiplet, while the first excited state with $m = \pm 3/2$ is a long $\mathcal{N} = 2$ massive vector
supermultiplet. The lowest state with $m = \pm 1$ in (5.18) contains massive spin-2 and vector $\mathcal{N} = 2$ multiplets.

Now we can check that the condition (iii)$_{\text{relaxed}}$ in section 2 is fulfilled. We see that all states found in [17] have baryonic charge and, as was explained above, none of them are charged with respect to SU(2) factors in (5.1).

6 String in U(3) SQCD

In this section we consider the string theory for the non-Abelian vortex in U(3) $\mathcal{N} = 2$ SQCD with $N_f = 3$ quark flavors, see (4.20). In this case $\tilde{N} = 0$ and the IR WSSM does not contain $\rho$ fields at all. The UV completion (4.1) includes $\rho_H$ field with electric charge $-3$. In the UV limit the D-flatness condition reads

$$|n^P|^2 - 3|\rho_H|^2 = \beta_1, \quad P = 1, 2, 3,$$

and one U(1) phase is gauged away. The coupling $\beta_1$ does not run and is determined by eq. (4.6) with $N = 3$ and $\tilde{N} = 0$. The sigma model target space $Y_6$ defined by (6.1) is a non-compact Calabi-Yau manifold which is the $\mathcal{O}(-3)$ line bundle over $\mathbb{C}\mathbb{P}(2)$ and locally has $\mathbb{C}\mathbb{P}(2) \times \mathbb{C}$ structure, see [20] for a review. The string theory on this space is new and here we restrict ourselves to a few general comments leaving the detailed study of this theory for future work.

In much the same way as in the conifold case the string theory on the manifold (6.1) does not contain 4D massless graviton. The reason is that the manifold (6.1) is not compact and 4D graviton which has a constant wave function over $Y_6$ is a non-normalizable state. Of course, this conclusion match our expectations because we started with U(3) $\mathcal{N} = 2$ SQCD without gravity.

Moreover, in much the same way as in the conifold case we have a chance to find normalizable string states only at strong coupling $m_G \sim \Lambda$ or $\beta_1 \sim 0$. To see that this is the case we note that at $|\beta_1| \to \infty$ the manifold $Y_6$ in (6.1) tends to flat space.

The global group of $Y_6$ in (6.1) is

$$\text{SU}(3) \times \text{U}(1)_{\text{extra}},$$

where U(1)$_{\text{extra}}$ is an extra U(1) associated with the global rotation of the $\rho_H$ field. This U(1) is absent in 4D SQCD. Below we will argue that closed string states are not charged with respect to this extra U(1).

Without the analysis of the string theory on the manifold (6.1), to be carried out later, for the time being we formulate pure field theoretical arguments. Note, that global charges of string states come from confined monopoles seen as kinks in the world sheet theory, see [12, 13] and section 3 above. We have shown in section 4.1 that BPS kinks in the model (4.1) coincide with kinks in the IR WSSM. Clearly they are not charged with respect to the extra U(1) symmetry. This implies that closed string states of the theory on the manifold (6.1) are not charged with respect to U(1)$_{\text{extra}}$. The requirement (iii)$_{\text{extra}}$ of section 2 is fulfilled.
7 Conclusions

In this paper we presented a UV completion for a conventional non-Abelian string with \( \mathbb{CP}(N) \)-like models on the world sheet. In our construction the above string flows to a conformal superstring above a certain scale \( m_G \). With the judicious choice of parameters this solitonic string becomes critical. The dependence of the string spectrum on the quark masses is not yet explored. Also, the second of two solutions presented — the \( U(N = 3) \) gauge group and \( N_f = N = 3 \) quark flavors — with the target space described by a non-compact Calabi-Yau manifold \( Y_6 \) has to be further investigated. We plan to address both issues in a forthcoming publication.

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A Twisted tree superpotential

In this appendix we construct the tree twisted superpotential for our UV completion of WSSM (4.1), which satisfy conditions (4.12). Consider first the theory with \( \bar{N} = 0 \). We take
\[
W_{\text{tree}} = C \Lambda \left\{ \left( \frac{\sqrt{2} \sigma_B - \Lambda_H}{N \Lambda} \right)^N - 1 \right\}^2,
\]
where \( C \) is a constant which we take to be large, \( C \to \infty \). This will make \( \sigma_B \) infinitely heavy. The derivative of this superpotential with respect to \( \sigma_B \) reads
\[
W_{\sigma_B} = 2\sqrt{2}C \left\{ \left( \frac{\sqrt{2} \sigma_B - \Lambda_H}{N \Lambda} \right)^N - 1 \right\} \left[ \frac{\sqrt{2} \sigma_B - \Lambda_H}{N \Lambda} \right]^{N-1}.
\]

To check two first conditions in (4.12) we use eq. (4.11) to express \((\sqrt{2} \sigma_B - \Lambda_H)\) in terms of \( \sigma \) and then eq. (3.16) for VEVs of \( \sigma \). It is easy to see that two first conditions in (4.12) are satisfied.

The leading contribution to the mass of \( \sigma_B \) (in the limit \( C \to \infty \)) is proportional to the second derivative of the tree superpotential,
\[
m_{\sigma_B} \sim e_B^2 \left| \frac{\partial^2}{\partial \sigma_B^2} W_{\text{tree}} \right|_{\text{vac}} = 4C \frac{e_B^2}{\Lambda} \left| \frac{\sqrt{2} \sigma_B - \Lambda_H}{N \Lambda} \right|^{2(N-1)}|_{\text{vac}} = 4C \frac{e_B^2}{\Lambda}.
\]
We see that \( \sigma_B \) becomes infinitely heavy in the limit \( C \to \infty \).

Now let us consider theories with \( \bar{N} > 0 \). We take the tree superpotential in the form
\[
W_{\text{tree}} = C \Lambda \left\{ \frac{\sqrt{2} \sigma_B - \Lambda_H}{(N - \bar{N}) \Lambda} \right\}^2 \left\{ \left( \frac{\sqrt{2} \sigma_B - \Lambda_H}{(N - \bar{N}) \Lambda} \right)^{N-\bar{N}} - 1 \right\}^2.
\]
while its derivative reads

\[
\mathcal{W}_{\sigma_B}^{\text{tree}} = \frac{2\sqrt{2}C}{N - \tilde{N}} \left[ \sqrt{2}\sigma_B - \Lambda_H \right] \left\{ \left[ \frac{\sqrt{2}\sigma_B - \Lambda_H}{(N - \tilde{N})\Lambda} \right]^{N - \tilde{N}} - 1 \right\} \\
\times \left\{ (N - \tilde{N} + 1) \left[ \frac{\sqrt{2}\sigma_B - \Lambda_H}{(N - \tilde{N})\Lambda} \right]^{N - \tilde{N}} - 1 \right\}. \tag{A.5}
\]

It is easy to see that two first conditions in (A.12) are satisfied for both \( \Lambda \) and zero-vacua. In particular, the combination \( (\sqrt{2}\sigma_B - \Lambda_H) \) is zero for zero-vacua.

Calculating the second derivative of the tree superpotential we get

\[
m_{\sigma_B} \sim 4C \frac{\epsilon_B^2}{(N - \tilde{N})^2\Lambda} \left( N - \tilde{N} + 1 \right) \left[ \frac{\sqrt{2}\sigma_B - \Lambda_H}{(N - \tilde{N})\Lambda} \right]^{N - \tilde{N}} - 1 \bigg|_{\text{vac}}. \tag{A.6}
\]

The mass is infinite since the absolute value above is nonzero for both \( \Lambda \) and zero-vacua.

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References

[1] M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory*, Cambridge University Press (1987) [insPIRE].

[2] A.M. Polyakov, *Quantum Geometry of Bosonic Strings*, Phys. Lett. 103B (1981) 207 [insPIRE].

[3] J. Polchinski and A. Strominger, *Effective string theory*, Phys. Rev. Lett. 67 (1991) 1681 [insPIRE].

[4] A.A. Abrikosov, *On the Magnetic properties of superconductors of the second group*, Sov. Phys. JETP 5 (1957) 1174 [insPIRE].

[5] H.B. Nielsen and P. Olesen, *Vortex Line Models for Dual Strings*, Nucl. Phys. B 61 (1973) 45 [insPIRE].

[6] A. Hanany and D. Tong, *Vortices, instantons and branes*, JHEP 07 (2003) 037 [hep-th/0306150] [insPIRE].

[7] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, *NonAbelian superconductors: Vortices and confinement in N = 2 SQCD*, Nucl. Phys. B 673 (2003) 187 [hep-th/0307287] [insPIRE].

[8] M. Shifman and A. Yung, *NonAbelian string junctions as confined monopoles*, Phys. Rev. D 70 (2004) 045004 [hep-th/0403149] [insPIRE].

[9] A. Hanany and D. Tong, *Vortex strings and four-dimensional gauge dynamics*, JHEP 04 (2004) 066 [hep-th/0403158] [insPIRE].
10. D. Tong, *TASI lectures on solitons: Instantons, monopoles, vortices and kinks*, in *Theoretical Advanced Study Institute in Elementary Particle Physics: Many Dimensions of String Theory (TASI 2005)*, Boulder, Colorado, 5 June–1 July 2005 (2005) [hep-th/0509216] [inSPIRE].
11. M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, *Solitons in the Higgs phase: The Moduli matrix approach*, *J. Phys. A* **39** (2006) R315 [hep-th/0602170] [inSPIRE].
12. M. Shifman and A. Yung, *Supersymmetric Solitons and How They Help Us Understand Non-Abelian Gauge Theories*, *Rev. Mod. Phys.* **79** (2007) 1139 [hep-th/0703267] [inSPIRE].
13. M. Shifman and A. Yung, *Supersymmetric Solitons*, Cambridge University Press (2009) [inSPIRE].
14. D. Tong, *Quantum Vortex Strings: A Review*, *Annals Phys.* **324** (2009) 30 [arXiv:0809.5060] [inSPIRE].
15. M. Shifman and A. Yung, *Critical String from Non-Abelian Vortex in Four Dimensions*, *Phys. Lett. B* **750** (2015) 416 [arXiv:1502.00683] [inSPIRE].
16. P. Koroteev, M. Shifman and A. Yung, *Non-Abelian Vortex in Four Dimensions as a Critical String on a Conifold*, *Phys. Rev. D* **94** (2016) 065002 [arXiv:1605.08433] [inSPIRE].
17. M. Shifman and A. Yung, *Critical Non-Abelian Vortex in Four Dimensions and Little String Theory*, *Phys. Rev. D* **96** (2017) 046009 [arXiv:1704.00825] [inSPIRE].
18. M. Shifman and A. Yung, *Hadrons of N = 2 Supersymmetric QCD in Four Dimensions from Little String Theory*, *Phys. Rev. D* **98** (2018) 085013 [arXiv:1805.10989] [inSPIRE].
19. D. Kutasov, *Introduction to Little String Theory*, in *Superstrings and Related Matters 2001*, Proc. of the ICTP Spring School of Physics, C. Bachas, K.S. Narain and S. Randjbar-Daemi eds., ICTP Lect. Notes Ser. **7** (2002) 165 [inSPIRE].
20. A. Neitzke and C. Vafa, *Topological strings and their physical applications*, hep-th/0410178 [inSPIRE].
21. N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in N = 2 supersymmetric QCD*, *Nucl. Phys. B* **431** (1994) 484 [hep-th/9408099] [inSPIRE].
22. D. Tong, *Monopoles in the Higgs phase*, *Phys. Rev. D* **69** (2004) 065003 [hep-th/0307302] [inSPIRE].
23. M. Shifman and A. Yung, *Lessons from supersymmetry: “Instead-of-Confinement” Mechanism*, *Int. J. Mod. Phys. A* **29** (2014) 1430064 [arXiv:1410.2900] [inSPIRE].
24. A. Achucarro and T. Vachaspati, *Semilocal and electroweak strings*, *Phys. Rept. 327* (2000) 347 [hep-ph/9904229] [inSPIRE].
25. M. Shifman and A. Yung, *Non-Abelian semilocal strings in N = 2 supersymmetric QCD*, *Phys. Rev. D* **73** (2006) 125012 [hep-th/0603134] [inSPIRE].
26. M. Eto et al., *On the moduli space of semilocal strings and lumps*, *Phys. Rev. D* **76** (2007) 105002 [arXiv:0704.2218] [inSPIRE].
27. M. Shifman, W. Vinci and A. Yung, *Effective World-Sheet Theory for Non-Abelian Semilocal Strings in N = 2 Supersymmetric QCD*, *Phys. Rev. D* **83** (2011) 125017 [arXiv:1104.2077] [inSPIRE].
28. P. Koroteev, M. Shifman, W. Vinci and A. Yung, *Quantum Dynamics of Low-Energy Theory on Semilocal Non-Abelian Strings*, *Phys. Rev. D* **84** (2011) 065018 [arXiv:1107.3779] [inSPIRE].
[29] E. Gerchkovitz and A. Karasik, Vortex-strings in $\mathcal{N} = 2$ SQCD and bulk-string decoupling, JHEP 02 (2018) 091 [arXiv:1710.02203] [inSPIRE].

[30] E. Witten, Instantons, the Quark Model and the $1/n$ Expansion, Nucl. Phys. B 149 (1979) 285 [inSPIRE].

[31] N. Dorey, The BPS spectra of two-dimensional supersymmetric gauge theories with twisted mass terms, JHEP 11 (1998) 005 [hep-th/9806056] [inSPIRE].

[32] N. Dorey, T.J. Hollowood and D. Tong, The BPS spectra of gauge theories in two-dimensions and four-dimensions, JHEP 05 (1999) 006 [hep-th/9902134] [inSPIRE].

[33] M. Shifman and A. Yung, Non-Abelian Confinement in $\mathcal{N} = 2$ Supersymmetric QCD: Duality and Kinks on Confining Strings, Phys. Rev. D 81 (2010) 085009 [arXiv:1002.0322] [inSPIRE].

[34] A. Hanany and K. Hori, Branes and $\mathcal{N} = 2$ theories in two-dimensions, Nucl. Phys. B 513 (1998) 119 [hep-th/9707192] [inSPIRE].

[35] A. D’Adda, A.C. Davis, P. Di Vecchia and P. Salomonson, An Effective Action for the Supersymmetric CP($N^1$) Model, Nucl. Phys. B 222 (1983) 45 [inSPIRE].

[36] E. Witten, Phases of $\mathcal{N} = 2$ theories in two-dimensions, Nucl. Phys. B 403 (1993) 159 [hep-th/9301042] [inSPIRE].

[37] K. Hori and C. Vafa, Mirror symmetry, hep-th/0002222 [inSPIRE].

[38] D. Gepner, Space-Time Supersymmetry in Compactified String Theory and Superconformal Models, Nucl. Phys. B 296 (1988) 757 [inSPIRE].

[39] T. Banks, L.J. Dixon, D. Friedan and E.J. Martinec, Phenomenology and Conformal Field Theory Or Can String Theory Predict the Weak Mixing Angle?, Nucl. Phys. B 299 (1988) 613 [inSPIRE].

[40] P. Candelas and X.C. de la Ossa, Comments on Conifolds, Nucl. Phys. B 342 (1990) 246 [inSPIRE].

[41] S. Gukov, C. Vafa and E. Witten, CFT’s from Calabi-Yau four folds, Nucl. Phys. B 584 (2000) 69 [Erratum ibid. B 608 (2001) 477] [hep-th/9906070] [inSPIRE].

[42] J. Louis, Generalized Calabi-Yau compactifications with $D$-branes and fluxes, Fortsch. Phys. 53 (2005) 770 [inSPIRE].

[43] A. Giveon and D. Kutasov, Little string theory in a double scaling limit, JHEP 10 (1999) 034 [hep-th/9908110] [inSPIRE].

[44] D. Ghoshal and C. Vafa, $c = 1$ string as the topological theory of the conifold, Nucl. Phys. B 453 (1995) 121 [hep-th/9506122] [inSPIRE].
[48] A. Giveon, D. Kutasov and O. Pelc, *Holography for noncritical superstrings*, JHEP 10 (1999) 035 [hep-th/9907178] [INSPIRE].

[49] E. Witten, *On string theory and black holes*, Phys. Rev. D 44 (1991) 314 [INSPIRE].

[50] S. Mukhi and C. Vafa, *Two-dimensional black hole as a topological coset model of c = 1 string theory*, Nucl. Phys. B 407 (1993) 667 [hep-th/9301083] [INSPIRE].

[51] H. Ooguri and C. Vafa, *Two-dimensional black hole and singularities of CY manifolds*, Nucl. Phys. B 463 (1996) 55 [hep-th/9511164] [INSPIRE].

[52] K. Hori and A. Kapustin, *Duality of the fermionic 2-D black hole and N = 2 Liouville theory as mirror symmetry*, JHEP 08 (2001) 045 [hep-th/0104202] [INSPIRE].

[53] L.J. Dixon, M.E. Peskin and J.D. Lykken, *N = 2 Superconformal Symmetry and SO(2,1) Current Algebra*, Nucl. Phys. B 325 (1989) 329 [INSPIRE].

[54] P.M.S. Petropoulos, *Comments on SU(1,1) string theory*, Phys. Lett. B 236 (1990) 151 [INSPIRE].

[55] S. Hwang, *Cosets as gauge slices in SU(1,1) strings*, Phys. Lett. B 276 (1992) 451 [hep-th/9110039] [INSPIRE].

[56] J.M. Evans, M.R. Gaberdiel and M.J. Perry, *The no ghost theorem for AdS_3 and the stringy exclusion principle*, Nucl. Phys. B 535 (1998) 152 [hep-th/9806024] [INSPIRE].

[57] J.M. Evans, M.R. Gaberdiel and M.J. Perry, *The no-ghost theorem and strings on AdS_3*, in *Nonperturbative aspects of strings, branes and supersymmetry*, Proceedings, Spring School on nonperturbative aspects of string theory and supersymmetric gauge theories and Conference on super-five-branes and physics in 5 + 1 dimensions, Trieste, Italy, 23 March–3 April 1998, pp. 435–444 (1998) [hep-th/9812252] [INSPIRE].

[58] D. Kutasov and N. Seiberg, *Noncritical superstrings*, Phys. Lett. B 251 (1990) 67 [INSPIRE].

[59] N. Dorey, *A New deconstruction of little string theory*, JHEP 07 (2004) 016 [hep-th/0406104] [INSPIRE].