On Holographic Entanglement Entropy and Higher Curvature Gravity

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Abstract: We examine holographic entanglement entropy with higher curvature gravity in the bulk. We show that in general Wald’s formula for horizon entropy does not yield the correct entanglement entropy. However, for Lovelock gravity, there is an alternate prescription which involves only the intrinsic curvature of the bulk surface. We verify that this prescription correctly reproduces the universal contribution to the entanglement entropy for CFT’s in four and six dimensions. We also make further comments on gravitational theories with more general higher curvature interactions.
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1. Introduction

In their seminal work [1], Ryu and Takayanagi made a proposal for the calculation of entanglement entropy of the boundary field theory in the context of gauge/gravity duality — see also [2, 3]. Their approach is both simple and elegant. Given a particular spatial region $V$ in the boundary theory, their proposal for the entanglement entropy between this region and its complement is

$$S(V) = \frac{2\pi}{\ell_{d-1}} \, \text{ext}_{m \sim V} [A(m)]$$

(1.1)

where $m \sim V$ indicates that $m$ is a bulk surface that is homologous to the boundary region $V$ [4, 5]. The symbol ‘ext’ indicates that one should extremize the area over all such surfaces $m$.\(^\text{1}\) This result applies where the bulk is described by classical Einstein gravity. Hence we might note the similarity between this expression (1.1) and that for black hole entropy. In a more general holographic framework, one would evaluate the area using the Einstein-frame metric [4].

There is also a fair amount of evidence to support this conjecture [4]:

- As shown in [1], it reproduces precisely the entanglement entropy of a $d = 2$ CFT for an interval of length $\ell$ on a circle of circumference $2\pi R$ [6, 7]

$$S(V) = \frac{c}{3} \log \left( \frac{2R}{\delta} \sin \frac{\ell}{2R} \right),$$

(1.2)

where $c$ is the central charge and $\delta$ is a short-distance cut-off. While this result applies for the vacuum, this holographic expression (1.1) can easily be shown to reproduce the expected entanglement entropy at finite temperature for $d = 2$.

- The leading (divergent) term in $S(V)$ takes precisely the form expected for the ‘area law’ contribution to the entanglement entropy in a $d$-dimensional CFT [2, 3]. That is, the leading contribution is proportional to $A(\partial V)/\delta^{d-2}$.

- As expected, if one considers a pure state in the boundary CFT (which is dual to a fixed bulk geometry without a horizon), one finds that $S(V) = S(\bar{V})$ where $\bar{V}$ denotes the complement of $V$.

\(^\text{1}\)If the calculation is done in a Minkowski signature background, the extremal area is only a saddle point. However, if one first Wick rotates to Euclidean signature, the extremal surface will yield the minimal area. In either case, the area must be suitably regulated to produce a finite answer. Note that for a $d$-dimensional boundary theory, the bulk has $d+1$ dimensions while the surface $m$ has $d-1$ dimensions. We are using ‘area’ to denote the $(d-1)$-dimensional volume of $m$. 
Given two boundary regions, \( V \) and \( U \), one readily shows that this construction (1.1) obeys the necessary inequality known as ‘strong subadditivity’ [8]. That is,
\[
S(V \cup U) + S(V \cap U) \leq S(V) + S(U).
\] (1.3)

In a slightly different context, this approach reproduces the Bekenstein-Hawking entropy of an eternal black hole. Recall that in the context of the AdS/CFT correspondence, the two asymptotic boundaries of an eternal black hole are associated with the original CFT and its thermofield double [9]. The horizon entropy can then be associated with the entanglement entropy between these two sets of degrees of freedom. Applying eq. (1.1) in this context, the region \( V \) becomes the entire boundary (of one asymptotic region) and \( m \) is then the black hole horizon \([4]\).

A standard approach to calculating entanglement entropy (EE) makes use of the replica trick [6, 10]. Unfortunately, this technique makes use of a singular background geometry as an intermediate tool and the natural holographic translation involves a singular bulk manifold [5]. It seems that without a full understanding of string theory or quantum gravity in the bulk, we will not be able to work with this bulk geometry in a controlled way. In particular, it is not possible to properly evaluate the saddle-point action in the gravitational bulk theory. Hence despite various efforts [5], a constructive proof of the Ryu-Takayanagi proposal (1.1) is still unknown.\(^2\) With a complete derivation, one could easily take into account the appearance of higher curvature terms in the bulk gravity theory, \(e.g.,\) to calculate finite \(N_c\) or finite \(\lambda\) corrections to holographic EE. Without such a derivation in hand, we set out here to explore holographic entanglement entropy in higher curvature gravity.

Previous results provide important suggestions as to how we should proceed to extend eq. (1.1) in the presence of higher curvature interactions in the bulk theory. First of all, as long as the prescription is one of minimizing a ‘surface functional’, we expect to have a formalism where the holographic EE satisfies subadditivity (1.3), as in [8]. Hence the question becomes how to define the appropriate surface functional given a particular higher curvature gravity action. As noted above, there is a close connection between holographic entanglement entropy and black hole entropy. In particular, to extend the description of the horizon entropy of an eternal black hole in terms of holographic EE, it must be the case that evaluating the new surface functional on an event horizon yields the correct black hole entropy in the higher curvature gravity theory. A first suggestion then would be that the surface functional simply coincides

\(^2\)However, see [11] for recent progress in this direction.
with the expression for Wald’s formula [12] for black hole entropy in higher curvature gravity. Unfortunately, as we will show below, this proposal fails! However, we must still demand that the new functional should be compatible with Wald’s formula on an event horizon. To make further progress, our discussion will focus on Lovelock gravity [13] in the bulk. We regard the latter as simply a convenient toy model with which we can easily make explicit calculations and one which may provide some useful insights into more general bulk theories.

An overview of the paper is as follows: We begin with a brief review of some useful background material in section 2. We describe Lovelock gravity and Wald’s entropy formula. We also describe another prescription, which we denote $S_{JM}$, for black hole entropy specifically derived for Lovelock gravity [14]. Further we also review a field theory calculation of the universal contribution to entanglement entropy in even dimensional CFT’s [2, 15]. In section 3, we show that the suggestion of replacing eq. (1.1) by an extremization of Wald’s entropy formula fails to provide the correct EE in general. For Lovelock gravity, this leaves us with the $S_{JM}$ prescription and we verify this proposal by comparing the universal contribution to the holographic EE to the analogous CFT results for a variety of geometries in four and six dimensions in sections 4 and 5. We return to considering holographic EE for general gravitational actions in section 6. In particular, our analysis there points out a new ambiguity in the prescription for holographic EE in Lovelock gravity. However, we are able to eliminate this potential ambiguity by considering the details of the variational problem. We conclude with a brief discussion of our results, including some interesting applications, in section 7. There are also four appendices which provide some of the useful technical details.

While proceeding with this project, we learned that the same topic was also being studied by J. de Boer, M. Kulaxizi and A. Parnachev — see talk by M. Kulaxizi [16]. Their results appear in [17]. We also note that the effect on holographic entanglement entropy from a certain higher curvature interaction, the gravitational Chern-Simons term, in three-dimensional AdS space was studied by [18].

2. A few preliminaries

Our primary aim in this paper is to explore the contribution of higher curvature interactions in the bulk gravity theory to holographic entanglement entropy. In the next few sections, we will focus our attention on Lovelock gravity [13]. The latter provides a useful toy model where one can readily perform explicit calculations. We return to more general considerations in sections 6. Hence, we begin below with a brief review of Lovelock gravity to set the context for our discussion in the following sections. Next,
as alluded to above, there is a close connection between black hole entropy and holographic entanglement entropy and so we also review two proposals for the latter in higher curvature gravity. First, there is Wald’s entropy formula \cite{Wald:1993nt}, which can be applied for any covariant gravity action, and then an earlier result derived specifically for Lovelock gravity \cite{Herdeiro:2016tmi}. Finally, setting aside gravity and holography, we also review a calculation of entanglement entropy in even dimensional CFT’s \cite{Klich:2013sma,Kitaev:2015wca}. In these purely field theoretic calculations, the universal contribution to the entanglement entropy is related to the central charges in the trace anomaly. The results for general CFT’s must be reproduced in our calculations of the holographic entanglement entropy and so provides a crucial test in extending the latter to higher curvature gravity.

2.1 Lovelock gravity

Lovelock gravity \cite{Lovelock:1971yv} is the gravitational theory in higher dimensions with higher curvature interactions proportional to the Euler density of higher even dimensional manifolds. The general Lovelock action in $d + 1$ dimensions can be written as

$$ I = \frac{1}{2\ell_p^{d-1}} \int d^{d+1}x \sqrt{-g} \left[ \frac{d(d-1)}{L^2} + R + \sum_{p=2}^{\left\lfloor \frac{d+1}{2} \right\rfloor} c_p L^{2p-2} \mathcal{L}_{2p}(R) \right], \quad (2.1) $$

where $\left\lfloor \frac{d+1}{2} \right\rfloor$ denotes the integer part of $(d + 1)/2$ and $c_p$ are dimensionless coupling constants for the higher curvature terms. These higher order interactions are defined as

$$ \mathcal{L}_{2p}(R) \equiv \frac{1}{2^p} \delta^{\nu_1 \nu_2 \ldots \nu_{2p-1} \nu_{2p}}_{\mu_1 \mu_2 \ldots \mu_{2p-1} \mu_{2p}} R^{\mu_1 \mu_2 \nu_1 \nu_2 \ldots R^{\mu_{2p-1} \mu_{2p}}_{\nu_{2p-1} \nu_{2p}}, \quad (2.2) $$

which is proportional to the Euler density on a $2p$-dimensional manifold. Here, we are using $\delta^{\mu_1 \mu_2 \ldots \mu_{2p-1} \mu_{2p}}_{\nu_1 \nu_2 \ldots}$ to denote the totally antisymmetric product of $2p$ Kronecker delta symbols. Of course, the cosmological constant and Einstein terms could be incorporated into this scheme as $\mathcal{L}_0$ and $\mathcal{L}_1$, respectively. However, we exhibit them explicitly above to establish our normalization for the Planck length, as well as the length scale $L$. By construction, it is clear that in $d + 1$ dimensions, all Lovelock $\mathcal{L}_p$ terms with $p > (d + 1)/2$ must vanish — hence the explicit restriction on the sum in eq. (2.1) is not really required. For $p = (d + 1)/2$, $\mathcal{L}_{2p}$ is topological. While this last term does not contribute to the gravitational equations of motions, it can contribute to black hole entropy \cite{Herdeiro:2016tmi,Kravchuk:2017dol}.

The original motivation to construct this action (2.1) was that the resulting equations of motion are only second order in derivatives \cite{Lovelock:1971yv}. Another interesting feature
of these theories is the equivalence between metric and Palatini formulations [21]. Earlier studies also found exact (asymptotically flat) black hole solutions to the classical equation of motion [22] and the exact form of the Gibbons-Hawking surface term is known [23]. Recently, there has been renewed interest in these theories in the context of the AdS/CFT correspondence. In particular, asymptotically AdS black hole solutions were found for Lovelock gravity [19, 24, 25]. These exact solutions then proved useful in discussions of holographic hydrodynamics and consistency of the boundary CFT [19, 24, 26, 27]. Further these models have also been shown to satisfy a holographic c-theorem [20, 28].

Anticipating our application to the AdS/CFT correspondence, we have explicitly included a negative cosmological constant in the action (2.1). The theory then has AdS$_{d+1}$ vacua with a curvature scale $\tilde{L}^2 = L^2 / f_\infty$ where $f_\infty$ is a root of the following expression:

$$1 = f_\infty - \sum_{p=2}^{[d/2]} \lambda_p (f_\infty)^p .$$

(2.3)

To simplify this expression, we have introduced the following notation

$$\lambda_p = (-)^p \frac{(d - 2)!}{(d - 2p)!} c_p .$$

(2.4)

Note that the topological term (i.e., $p = (d + 1)/2$) does not contribute to determining the AdS scale and so the upper limit on the sum here is not the same as in the action (2.1). In general, this equation yields $[d/2]$ different roots for $f_\infty$. We are only interested in the positive real roots, since these correspond to AdS$_{d+1}$ vacua. However, for many of these roots, the graviton is in fact a ghost-like excitation, i.e., its kinetic term has the wrong sign [29, 30] and further, even if the latter problem is evaded, the vacuum typically does not support nonsingular black hole solutions [30]. In fact, there is at most one root which yields a ghost-free AdS vacuum which supports black hole solutions, as described in detail in [30]. Further, in a regime where the $\lambda_p$ are not large, this will be the smallest positive root and it is continuously connected to the single root ($f_\infty = 1$) that remains in the Einstein gravity limit, i.e., $\lambda_p \to 0$. Implicitly, we will be working in this regime of the coupling space and with this particular root in the following.

Of particular interest in the following, will be the central charges of the boundary CFT for even $d$. For any CFT in an even number of dimensions, the central charges can be defined in terms of the trace anomaly — see eq. (2.13) and the discussion in section 2.3. Now in the context of the AdS/CFT correspondence, general techniques have been developed to holographically evaluate the trace anomaly and determine the
corresponding central charges [31]. When the bulk theory is described by Einstein gravity, one finds that all of the charges are essentially equal, being determined by the ratio $(\tilde{L}/\ell_P)^{d-1}$. However, with the introduction of higher curvature terms in the bulk gravity, the central charges become functions of the new (dimensionless) couplings, as well as the ratio of the AdS scale to the Planck scale, and so the charges can be (at least partially) distinguished in such an extended holographic set up [32, 33, 19, 27].

In general, determining all of the central charges is a fairly involved calculation, however, there is a simple short-cut to calculate $A$ presented in [34]. Given any general covariant action for the bulk gravity theory, $A$ is determined by simply evaluating the value of the Lagrangian in the AdS$_{d+1}$ vacuum. With the conventions of [20, 35], which we have adopted here,

$$A = -\frac{\pi^{d/2} L^{d+1}}{d \Gamma(d/2)} \mathcal{L} |_{AdS} .$$

We emphasize that the right-hand side is evaluated with the theory in Minkowski signature and we refer the interested reader to [20] for further details. In the case of the Lovelock action (2.1), evaluating the above expression is a straightforward exercise, which yields

$$A = \frac{\pi^{d/2}}{\Gamma(d/2)} \left[ 1 - \sum_{p=2}^{[d/2]} \frac{(d-1)p}{d+1-2p} \lambda_p (f_\infty)^{p-1} \right].$$

Here we have used eqs. (2.3) and (2.4) to arrive at this result. Note that in the case of the topological term with $p = (d+1)/2$, one would add an extra term to eq. (2.6) of the form

$$\delta A = \frac{\pi^{d/2}}{\Gamma(d/2)} \left( \frac{d-1}{\ell_P^{d-1}} \right) \times (-)^{d+3 \choose 2} \frac{(d+1)!}{2d} c_{d+1}^{d+1}. $$

2.2 Horizon entropy

As noted in the introduction, there is a close connection between black hole entropy and holographic entanglement entropy. For any (covariant) theory of gravity, the black hole entropy can be calculated using Wald’s entropy formula [12]

$$S = -2\pi \int_{\text{horizon}} d^{d-1}x \sqrt{h} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu}^{\rho\sigma}} \hat{\varepsilon}^{\mu\nu} \hat{\varepsilon}^{\rho\sigma},$$

where $\mathcal{L}$ denotes the gravitational Lagrangian and $\hat{\varepsilon}_{\mu\nu}$ is the binormal to the horizon normalized by $\hat{\varepsilon}_{\mu\nu} \hat{\varepsilon}^{\mu\nu} = -2$ (assuming a Minkowski signature), while $h$ is the determinant of the induced metric $h_{\alpha\beta}$ on the horizon. Now this prescription can easily be
applied to the Lovelock theory (2.1) yielding

\[ S_W = \frac{2\pi}{\ell_P^{-1}} \int_{\text{horizon}} d^{d-1}x \sqrt{h} \left[ 1 + \sum_{p=2}^{[\frac{d+1}{2}]} p c_p L^{2p-2} \mathcal{L}_{2p-2}(R^\parallel) \right]. \]  

(2.9)

Here \( R^\parallel \) denotes the components of the curvature tensor projected onto the horizon, \( i.e., \)

\[ \left[ R^\parallel \right]_{\alpha\beta\gamma\delta} = h^\alpha_{\alpha'} h^\beta_{\beta'} h^\gamma_{\gamma'} h^\delta_{\delta'} R^{\alpha'\beta'\gamma'\delta'}. \]  

(2.10)

We note, however, that this expression for the horizon entropy is not unique. In particular, black hole entropy in the Lovelock theory was studied in [14], which preceded (and in part, motivated) the derivation of Wald’s formula (2.8). Using a Hamiltonian approach, this earlier work [14] derived the following expression

\[ S_{JM} = \frac{2\pi}{\ell_P^{-1}} \int_{\text{horizon}} d^{d-1}x \sqrt{h} \left[ 1 + \sum_{p=2}^{[\frac{d+1}{2}]} p c_p L^{2p-2} \mathcal{L}_{2p-2}(R) \right], \]  

(2.11)

where \( R^{\alpha\beta\gamma\delta} \) are the components of the intrinsic curvature tensor of the slice of the event horizon on which this expression is evaluated.

In fact, there is no disagreement between eqs. (2.9) and (2.11) in the context for which they were derived. Both derivations [12, 14] assumed that the relevant horizon was a Killing horizon, \( i.e., \) the black hole background is stationary with a Killing vector \( \chi^\mu \) which becomes null on the horizon. The geometry is remarkably constrained in this case [36] and it is straightforward to show, in particular on the bifurcation surface, that the extrinsic curvatures vanishes. Now recall that the Gauss-Codazzi equations relate the intrinsic curvature to the projection of the full spacetime curvature with [37]

\[ \left[ R^\parallel \right]_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \sum_{i=1}^{2} \eta_{ij} \left( K_{\alpha\gamma}^i K_{\beta\delta}^j - K_{\alpha\delta}^i K_{\beta\gamma}^j \right). \]  

(2.12)

To describe this result, we need to introduce some formalism, which will also be useful in later discussion.\(^4\) There is a pair of unit vectors \( n^\parallel_i \) (with \( i = 1, 2 \)) which are orthogonal to the surface (on which eq. (2.12) is evaluated) and to each other. Then

\(^4\)The surface of interest in the present discussion is a slice of the black hole horizon, however, we will also apply the same formalism to the bulk surface used in calculating holographic entanglement entropy. Both are codimension two surfaces embedded in the relevant spacetime. Let us also comment on our index conventions throughout the paper. Directions in the full (AdS) geometry are labeled with letters from the second half of the Greek alphabet, \( i.e., \mu, \nu, \rho, \cdots \). Letters from the ‘second’ half of the
\( \eta^{ij} = n^i \eta^{j\mu} \) is the Minkowski\(^5\) metric in the transverse tangent space spanned by these vectors and \( \eta_{ij} \) is the inverse of this metric. We also have tangent vectors \( t^\mu_{\alpha} \) along the surface, which are defined in the usual way with \( t^\mu_{\alpha} = \partial X^\mu / \partial \sigma^\alpha \) where \( X^\mu \) and \( \sigma^\alpha \) are the coordinates in the full embedding space and along the surface, respectively. The induced metric is then given by \( h_{\alpha\beta} = t^\mu_{\alpha} t^\nu_{\beta} g_{\mu\nu} \). We may also define this induced metric as a bulk tensor with \( h_{\mu\nu} = g_{\mu\nu} - \eta_{ij} n^i_{\mu} n^j_{\nu} \). The second fundamental forms are defined for the surface with \( K^i_{\alpha\beta} = -t^\mu_{\alpha} t^\nu_{\beta} \nabla_{\mu} n^i_{\nu} \).

In any event, given eq. (2.12), it is clear that the curvatures in the two expressions for the horizon entropy agree when \( K^i_{\alpha\beta} = 0 \). Hence the two separate proposals will agree in evaluating the horizon entropy for a stationary black hole with a Killing horizon. Now a natural extension of eq. (1.1) to Lovelock gravity would be that the holographic entanglement entropy would be found by extremising the expression which yields black hole entropy. Hence, in fact, eqs. (2.9) and (2.11) provide two natural candidates for the holographic entanglement entropy. Further, as we will find below, in calculating the holographic entanglement entropy, the relevant extrinsic curvatures do not vanish in general and so these two expressions really provide distinct proposals.

### 2.3 Entanglement entropy and the trace anomaly

We turn now to a CFT calculation of entanglement entropy, without reference to holography. The results of these field theory calculations will provide a benchmark against which we can compare our holographic calculations of entanglement entropy. For a conformal field theory in an even number of spacetime dimensions, the coefficient of the universal term in the entanglement entropy can be determined through the trace anomaly. This result relies on a common modification of the usual replica trick [6] which is prevalent in the high energy physics literature and which gives the calculations a geometric character [10]. This ‘geometric approach’ was first used to establish the connection between entanglement entropy and the trace anomaly for two-dimensional CFT’s [7]. Later, similar results were also found for higher dimensions in [2, 15]. In the following, we will not present the details of these calculations, focusing on the results instead, and so we refer the interested reader to [20] for a comprehensive discussion.

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Latin alphabet, \textit{i.e.}, \( i, j, k, \ldots \), correspond to directions in the background geometry of the boundary CFT. Meanwhile, directions along the entangling surface in the boundary are denoted with letters from the beginning of the Latin alphabet, \textit{i.e.}, \( a, b, c, \ldots \), and directions along the corresponding bulk surface are denoted with letters from the beginning of the Greek alphabet, \textit{i.e.}, \( \alpha, \beta, \gamma, \ldots \). Finally, we use hatted letters from the later part of the Latin alphabet to denote the frame or tangent indices in the transverse space to both of these surfaces, \textit{i.e.}, \( \hat{i}, \hat{j} \).

\(^5\) If the embedding geometry had a Euclidean signature, then this transverse metric would simply be a Kronecker delta \( \delta^{ij} \).
However, first let us recall the trace anomaly \[ 38 \],
\[
\langle T^i_i \rangle = \sum_n B_n I_n - 2 (-)^{d/2} A E_d + B' \nabla^i J^i ,
\] (2.13)
which defines the central charges for a CFT in an even number of dimensions, \( d = 2p \). Each term on the right-hand side is a Weyl invariant constructed from the background geometry. In particular, \( E_d \) is the Euler density in \( d \) dimensions. Using the expressions presented earlier in eq. (2.2), we write \( E_{2p} = \mathcal{L}_{2p}(R)/[(4\pi)^p \Gamma(p + 1)] \). This normalization ensures that integrated over a \( d \)-dimensional sphere: \( \oint_{S_d} d^d x \sqrt{g} E_d = 2 \). A general construction of the \( I_n \) can be found in \[ 39 \]. In this approach, the natural building blocks of these invariants are the Weyl tensor \( C_{ijkl} \), the Cotton tensor \( C_{ikj} \) and the Bach tensor \( B_{ij} \) (as well as covariant derivatives of these). A useful observation is that these basis tensors all vanish on a conformally flat background and hence, \( e.g., I_n|_{S^d} = 0 \). Finally, the last term in eq. (2.13) is a conformally invariant but also scheme-dependent total derivative. That is, this last contribution can be eliminated by the addition of a finite and covariant counter-term to the effective action. In any event, we note that these terms play no role in the following simply because they are total derivatives. A final note on our conventions\(^6\) is that the stress tensor is defined by \( T_{ij} \equiv -2/\sqrt{-g} \delta I/\delta g^{ij} \) in Minkowski signature. However, in Euclidean signature, the sign is flipped to \( T_{ij} \equiv 2/\sqrt{g} \delta I / \delta g^{ij} \).

Now consider calculating the entanglement entropy in the CFT using the geometric approach mentioned above. First,\(^7\) a certain entangling surface \( \Sigma \) is chosen which divides the initial time slice into two separate regions, \( V \) and \( \bar{V} \), as illustrated in figure 1. Following [2, 15], we consider the variation of the entanglement entropy under a uniform scale transformation of the system. This technique can only be successfully applied when the geometry for which we are calculating the entanglement entropy contains a single scale \( \ell \). Then the analysis of [20] leads to the following expression:
\[
\ell \frac{\partial S_{EE}}{\partial \ell} = 2\pi \int_{\Sigma} d^{d-2}x \sqrt{h} \varepsilon^{ij} \tilde{\varepsilon}_{kl} \left[ \sum_n B_n \frac{\partial I_n}{\partial R^{ij}_{kl}} - 2 (-)^{d/2} A \frac{\partial E_d}{\partial R^{ij}_{kl}} \right].
\] (2.14)
where \( \tilde{\varepsilon}_{ij} \) is the two-dimensional volume form in the space transverse to \( \Sigma \). Implicitly, for these computations, the background geometry has Euclidean signature and hence \( \tilde{\varepsilon}_{ij} \tilde{\varepsilon}^{ij} = 2 \). The last term in this expression can be further simplified using [20]
\[
2\pi \varepsilon^{ij} \tilde{\varepsilon}_{kl} \frac{\partial E_d}{\partial R^{ij}_{kl}} = E_d - 2(\mathcal{R}) .
\] (2.15)
\(^6\)Our conventions are adopted from [20, 35] and so we refer the reader there for a more detailed discussion.
\(^7\)Actually the first step in applying the replica trick is Wick rotate to Euclidean signature.
Figure 1: Initial time slice divided into two regions $V$ and $\bar{V}$ by the entangling surface $\Sigma$. That is, this contribution is replaced by the Euler density in $d-2$ dimensions but constructed using the intrinsic curvatures on $\Sigma$. This simplification relies on an implicit assumption in this construction, which is that the extrinsic curvatures for $\Sigma$ vanish, and also uses eq. (2.12). Now it is straightforward to verify that the integral on the right-hand side of eq. (2.14) is scale invariant. Hence we can integrate with respect to the scale $\ell$ to arrive at

$$S_{\text{EE}} = \log(\ell/\delta) \int_{\Sigma} d^{d-2}x \sqrt{h} \left[ 2\pi \tilde{\varepsilon}^{ij} \tilde{\varepsilon}_{kl} \sum_n B_n \frac{\partial I_n}{\partial R^{ij} kl} - 2 (-)^{d/2} A E_{d-2} \right],$$  

(2.16)

where $\delta$ is the short-distance cut-off that we use to regulate the calculations. Hence this calculation has identified the universal contribution to the the entanglement entropy, i.e., the term proportional to $\log \delta$ in even $d$. Further, the above result shows that the coefficient of this term is some linear combination of all of the central charges, where the precise linear combination depends on the geometry of the entangling surface $\Sigma$ and of the background geometry [2, 15].

Let us add a few additional remarks about this calculation: As commented above, an implicit assumption in the above calculation is that the extrinsic curvatures of the entangling surface vanish [15]. Otherwise we should expect that additional ‘corrections’ involving the extrinsic curvature must appear in the final result. As we will see below, these corrections were identified for $d = 4$ in [15]. However, there is, in fact, a stronger assumption at play in these calculations. Namely, in applying the geometric approach to calculate entanglement entropy, one should assume that there is a rotational symmetry around the $\Sigma$ [20] and in fact, it is this symmetry that ensures that $K_{ab}^i = 0$. We will see in section 5.2 that a new class of corrections (independent of the extrinsic curvatures) can arise when the rotational symmetry is not present. As a final note, we observe that if this calculation was performed for a CFT in an odd number of spacetime dimensions,
the result would vanish because the trace anomaly is zero for odd $d$. However, this is in keeping with the expectation that there is no logarithmic contribution to the entanglement entropy for odd $d$ and with a smooth entangling surface $\Sigma$.

We now explicitly apply eq. (2.16) for $d = 4$ and 6. These results will then be the central consistency tests for our holographic calculations of entanglement entropy for Lovelock gravity.

2.3.1 Entanglement entropy for $d = 4$

The trace anomaly for four dimensions is well studied and we present it here using the more conventional nomenclature for the central charges:

$$\langle T^i_i \rangle = \frac{c}{16\pi^2} C_{ijkl} C^{ijkl} - \frac{a}{16\pi^2} \left( R_{ijkl} R^{ijkl} - 4 R_{ij} R^{ij} + R^2 \right),$$

(2.17)

where $C_{ijkl} C^{ijkl} = R_{ijkl} R^{ijkl} - 2 R_{ij} R^{ij} + R^2 / 3$ is the square of the four-dimensional Weyl tensor and the second term is proportional to the four-dimensional Euler density.

We have discarded the scheme-dependent total derivative in this expression, as it plays no role in our analysis. Comparing to eq. (2.13), we have $a = A$ and $c = 16\pi^2 B_1$ with $I_1 = C_{ijkl} C^{ijkl}$. Now applying eq. (2.16), the universal contribution to the entanglement entropy becomes

$$S_{EE} = \log(\ell/\delta) \frac{1}{4\pi} \int_{\Sigma} d^2 x \sqrt{h} \left[ c C^{ijkl} \tilde{\epsilon}_{ij} \tilde{\epsilon}_{kl} - 2 a \mathcal{R} \right],$$

(2.18)

where $\mathcal{R}$ is the Ricci scalar for the intrinsic geometry on $\Sigma$.

Similar expressions were previously derived in [2, 15]. Our results can be brought closer to the form presented there using the formalism introduced in section 2.2. We introduce two orthonormal vectors $n^i_\hat{i}$ (with $\hat{i} = 1, 2$) which span the transverse space to the entangling surface $\Sigma$. Then volume form and metric in this space can be written as $\tilde{\epsilon}_{ij} = n^i_\hat{i} n^j_\hat{j} - n^i_\hat{j} n^j_\hat{i}$ and $\tilde{g}^{\hat{i}\hat{j}} = \delta_{ij} n^i_\hat{i} n^j_\hat{j}$, respectively. Now a useful identity which follows from these definitions is

$$\tilde{\epsilon}_{ij} \tilde{\epsilon}_{kl} = \tilde{g}^{\hat{i}\hat{j}} \tilde{g}^{\hat{k}\hat{l}} - \tilde{g}^{\hat{i}\hat{k}} \tilde{g}^{\hat{j}\hat{l}}.$$

(2.19)

Using this identity, we can express our result for the universal contribution to the entanglement entropy as

$$S_{EE} = \log(\ell/\delta) \frac{1}{2\pi} \int_{\Sigma} d^2 x \sqrt{h} \left[ c C^{ijkl} \tilde{g}^{\hat{i}\hat{k}} \tilde{g}^{\hat{j}\hat{l}} - a \mathcal{R} \right],$$

(2.20)

As described above, this result can be reliably applied when there is a rotational symmetry in the transverse space about $\Sigma$ (which ensures that the extrinsic curvatures
on $\Sigma$ vanish). Ref. [15] examined the possible corrections to eq. (2.20) when $K^i_{ab} \neq 0$. The extended result which accounts for this possibility can be written [15]

$$S_{EE} = \log(\ell/\delta) \frac{1}{2\pi} \int_{\Sigma} d^2x \sqrt{h} \left[ c \left( C^{ijkl} \tilde{g}_{ik} \tilde{g}_{jl} - K^i_{a} K^i_{b} + \frac{1}{2} K^i_{a} K^i_{b} \right) - a R \right].$$

(2.21)

It is interesting to note that a holographic calculation was used in [15] to fix the final coefficients of the extrinsic curvature terms in this expression. For comparison purposes in section 4, it is also useful to write this expression as

$$S_{EE} = \log(\ell/\delta) \frac{1}{2\pi} \int_{\Sigma} d^2x \sqrt{h} \left[ c \left( C^{abcd} h_{ac} h_{bd} - K^i_{a} K^i_{b} + \frac{1}{2} K^i_{a} K^i_{b} \right) - a R \right],$$

(2.22)

where $h_{ab}$ is the induced metric on $\Sigma$. The equivalence of the two expressions in eqs. (2.21) and (2.22) follows because the Weyl tensor is traceless, i.e., $C^{ijkl} \tilde{g}_{ik} = 0$ and we can express the induced metric as a bulk tensor with $h_{ij} = g_{ij} - \tilde{g}_{ik} \tilde{g}^{ik}$.

Further in [15], this result was applied to evaluate the entanglement entropy for various surfaces embedded in flat space. In this case, the Weyl curvature vanishes and the entanglement entropy is determined entirely by the contributions coming from the extrinsic curvatures and from the intrinsic Ricci scalar. Considering the case where the entangling surface is a two-sphere of radius $R$, one finds that the two extrinsic curvature terms cancel. The entanglement entropy (2.22) then becomes

$$S_{EE} = -4 a \log(R/\delta),$$

(2.23)

where we have substituted $R$ as the relevant scale $\ell$. Another simple case to consider is when the entangling surface is chosen to be an infinite cylinder, in which case the intrinsic curvature vanishes. If we let the radius of the cylinder be $R$ and we introduce regulator scale $H$ along the length of the cylinder, the entanglement entropy (2.22) then becomes

$$S_{EE} = -c \frac{H}{R} \log(R/\delta).$$

(2.24)

Hence with these two choices for the entangling surface, we are able to isolate the two central charges with a calculation of the entanglement entropy.

In section 4, we will use the above results to test our proposal for holographic entanglement entropy in Lovelock gravity. One comment, perhaps worth making at this point, is that while the derivation of eq. (2.22) did account for the possibility that the extrinsic curvature was nonvanishing [15], no consideration was given to whether the transverse space to $\Sigma$ possessed a rotational symmetry. As we discuss in section 7, the latter does not seem to lead to any difficulties in $d = 4$. 

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2.3.2 Entanglement entropy for \( d = 6 \)

In six dimensions, the trace anomaly (2.13) can be explicitly written as [40]

\[
\langle T^i_i \rangle = \sum_{n=1}^{3} B_n I_n + 2A E_6
\]  

(2.25)

where

\[
\begin{align*}
I_1 &= C_{kij}C^{imnj}C_{m}^{kl}C_{n}^{ij}, & I_2 &= C_{ij}^{kl}C_{mn}^{im}C_{mn}^{ij}, \\
I_3 &= C_{iklm}(\nabla^2 \delta^j_i + 4R^j_i - \frac{6}{5}R \delta^j_i)C^{jklm}, \\
E_6 &= \frac{1}{384\pi^3} \mathcal{L}_6
\end{align*}
\]

(2.26)

with \( \mathcal{L}_6 \) defined in eq. (2.2). We also explicitly write out \( \mathcal{L}_6 \) in eq. (5.2). In eq. (2.25), we have again discarded the scheme-dependent total derivative. The above choice for the basis of the conformal invariants has the virtue that the non-topological terms \( I_n \) all vanish when evaluated on a conformally flat space. Now from eq. (2.16), the universal contribution to the entanglement entropy becomes

\[
S_{EE} = \log(\ell/\delta) \int d^4x \sqrt{h} \left[ 2\pi \sum_{n=1}^{3} B_n \frac{\partial I_n}{\partial R^{ij}_{kl}} \bar{\varepsilon}^{ij} \bar{\varepsilon}_{kl} + 2A E_4 \right] \Sigma
\]  

(2.27)

where

\[
\begin{align*}
\frac{\partial I_1}{\partial R^{ij}_{kl}} \bar{\varepsilon}^{ij} \bar{\varepsilon}_{kl} &= 3 \left( C^{jmnk}C_{mn}^{il} \bar{\varepsilon}^{ij} \bar{\varepsilon}_{kl} - \frac{1}{4} C^{iklm}C^{ij}_{klm} \bar{g}_{ij} + \frac{1}{20} C^{ijkl}C_{ijkl} \right), \\
\frac{\partial I_2}{\partial R^{ij}_{kl}} \bar{\varepsilon}^{ij} \bar{\varepsilon}_{kl} &= 3 \left( C^{iklm}C_{mn}^{ij} \bar{\varepsilon}^{ij} \bar{\varepsilon}_{kl} - C^{iklm}C^{ijkl}_{klm} \bar{g}_{ij} + \frac{1}{5} C^{ijkl}C_{ijkl} \right), \\
\frac{\partial I_3}{\partial R^{ij}_{kl}} \bar{\varepsilon}^{ij} \bar{\varepsilon}_{kl} &= 2 \left( \Box C^{ijkl} + 4R^i_mC^{mjkl} - \frac{6}{5}R C^{ijkl} \right) \bar{\varepsilon}^{ij} \bar{\varepsilon}_{kl} - 4C^{ijkl}R_{ik} \bar{g}_{jl} \\
&\quad + 4C^{iklm}C^{ij}_{klm} \bar{g}_{ij} - \frac{12}{5} C^{ijkl}C_{ijkl}.
\end{align*}
\]

(2.28)

The expressions above have been simplified using the identities: \( \bar{\varepsilon}^{ij} \bar{\varepsilon}^{ij} = 2 \) and \( \bar{g}^{ij}_{ik} = \bar{\varepsilon}^{ij} \bar{\varepsilon}_{kl} g^{jl} \).

As described above, this result can be reliably applied for entangling surfaces with rotational symmetry in the transverse space, in which case there is zero extrinsic curvature. In section 5, we will explicitly evaluate eq. (2.27) for various surfaces satisfying these constraints to test our proposal for holographic entanglement entropy in Lovelock gravity. It would, of course, be interesting to extend the above expression (2.27) to
account for the possibility that the extrinsic curvature is nonvanishing, following the approach of [15] in $d = 4$. However, one quickly realizes extending these calculations to $d = 6$ is arduous task and further, we will show in section 5 that there are other corrections unrelated to the extrinsic curvature.

3. Not Wald entropy!

Our goal is to understand how to compute the holographic EE in the presence of higher curvature interactions in the bulk theory. As discussed in the introduction it seems that as long as the prescription is one of minimizing a ‘surface functional’, we can expect that the holographic EE will satisfy subadditivity, as in [8]. Further, the close connection of entanglement entropy and black hole entropy suggests that the new functional must coincide with the expression for Wald entropy (2.8) when evaluated on a black hole horizon. The simple suggestion would then be that we extend eq. (1.1) to higher curvature gravity by extremizing precisely the Wald formula over the bulk surfaces homologous to the boundary region of interest. The recent discussion of [11] on spherical entangling surfaces would seem to lend some credence to this prescription.

Unfortunately, we can easily show that the naïve first guess above for the extension of eq. (1.1) to higher curvature theories simply fails to provide the correct holographic EE for general entangling surfaces. For this purpose, we will use two results that were originally derived in [20, 35]. We only present these results here and refer the interested reader to [20] for further details. We can keep the discussion general and do not need to specify the gravitational theory, beyond that it has a covariant action. Then, in an AdS$_{d+1}$ background, the gravitational equations of motion yield

$$\left. \frac{\delta \mathcal{L}}{\delta R_{\mu \nu \rho \sigma}} \right|_{\text{AdS}} = -\frac{\bar{L}^2}{4d} \mathcal{L}|_{\text{AdS}} \left( \delta_{\mu}^\rho \delta_{\nu}^\sigma - \delta_{\mu}^\sigma \delta_{\nu}^\rho \right). \tag{3.1}$$

Further, motivated by the short-cut to calculating the $A$-type trace anomaly [34], the authors of [20, 35] found

$$a^*_d \equiv -\frac{\pi^{d/2} \bar{L}^{d+1}}{d \Gamma (d/2)} \mathcal{L}|_{\text{AdS}}. \tag{3.2}$$

Here $a^*_d$ is a specific central charge characterizing the $d$-dimensional boundary CFT.\footnote{It was also shown that $a^*_d$ provides a measure of the density of the degrees of freedom in the boundary CFT [20, 41].}

The above expression generalizes eq. (2.5) for odd or even $d$. Recall that for even $d$, we have $a^*_d = A$, the coefficient of the $A$-type trace anomaly. Now our candidate for the holographic EE is to extremize the Wald entropy (2.8) evaluated on $(d-1)$-dimensional
surfaces $m$ homologous to the boundary region of interest. For simplicity, we will test this proposal in pure AdS$_{d+1}$ space, which will suffice to consider the cases where the entangling surface is embedded in flat Minkowski space $R^{1,d-1}$, a cylindrical background $R \times S^{d-1}$ or any number of conformally flat backgrounds — see, e.g., [42]. Hence we consider evaluating the expression for Wald entropy on some bulk surface $m$,

$$
S_w = -2\pi \int_m d^{d-1} x \sqrt{h} \frac{\partial L}{\partial R_{\mu \nu \rho \sigma}} \hat{\varepsilon}^{\mu \nu} \hat{\varepsilon}^{\rho \sigma} = \frac{2\pi}{\pi^{d/2}} \frac{\Gamma(d/2)}{L_{d-1}^{d-1}} \hat{a}_d^* \int m d^{d-1} x \sqrt{h}. \quad (3.3)
$$

We used eqs. (3.1) and (3.2) to produce the second expression on the right above. Hence with this prescription, the calculation of holographic EE would again reduce to extremizing the area of the bulk surface. Further the entire result would always be proportional to the central charge $a_d^*$, independent of the choice of the entangling surface $\Sigma$. In particular, for even $d$, the coefficient of the logarithmic contribution would be proportional to $A$. However, this result is simply incorrect. As we saw in section 2.3, field theoretic calculations indicate that this universal contribution to the entanglement entropy is proportional to a linear combination of all of the central charges appearing in the trace anomaly (2.13). Further the specific linear combination appearing here depends on the geometry of the entangling surface $\Sigma$ and of the background in which $\Sigma$ is embedded.

Hence the proposal that holographic EE would be calculated by extremizing the Wald entropy clearly contradicts the general expectations from purely CFT calculations. In the next two sections, we focus our discussion on Lovelock gravity. While Wald entropy is ruled out, in this case, there remains a second natural candidate in eq. (2.11) for the new functional with which to calculate holographic EE. In the following, we will verify the proposal that extremizing the expression for $S_{JM}$ over the bulk surfaces $m$ will properly determine the EE for the holographic CFT’s dual to Lovelock gravity.

4. EE for $d = 4$ holographic CFT

Here we focus on the case of a four-dimensional boundary theory. In this case with Lovelock gravity in a five-dimensional bulk, only the curvature-squared interaction contributes to the action (2.1) leaving

$$
I = \frac{1}{2\ell_p^3} \int d^5 x \sqrt{-g} \left[ \frac{12}{L^2} + R + \frac{\lambda L^2}{2} \mathcal{L}_4 \right]. \quad (4.1)
$$

Comparing to the notation of section 2.1, we have $\lambda = \lambda_2 = 2c_2$ and explicitly evaluating $\mathcal{L}_4$ using eq. (2.2) yields

$$
\mathcal{L}_4 = R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma} - 4 R_{\mu \nu} R^{\mu \nu} + R^2. \quad (4.2)
$$
In this case, eq. (2.3) reduces to a simple quadratic equation for which the physical (ghost-free) root is

\[ f_\infty = \frac{1 - \sqrt{1 - 4\lambda}}{2\lambda} . \]  

(4.3)

The two central charges appearing in the trace anomaly (2.17) can be calculated using the techniques of [31] yielding [33]

\[ c = \frac{\pi^2 \bar{L}^3}{\ell_p^3} (1 - 2\lambda f_\infty) , \quad a = \frac{\pi^2 \bar{L}^3}{\ell_p^3} (1 - 6\lambda f_\infty) . \]  

(4.4)

Now we would like to test the proposal that the holographic entanglement entropy in this theory is determined by extremizing the expression in eq. (2.11) over the bulk surfaces \( m \) homologous to the appropriate boundary region. Explicitly evaluating this functional for the present case yields

\[ S_{JM} = \frac{2\pi}{\ell_p^3} \int_m d^3x \sqrt{h} \left[ 1 + \lambda L^2 R \right] + \frac{4\pi}{\ell_p^3} \int_{\partial m} d^2x \sqrt{h} \lambda L^2 K , \]  

(4.5)

where \( R \) denotes the Ricci scalar for the intrinsic geometry on \( m \). Similarly, \( K \) denotes the trace of the extrinsic curvature of the boundary \( \partial m \). We have added this ‘Gibbons-Hawking’ boundary term in eq. (4.5) to provide a good variational principle in extremizing this functional. In the rest of this section, we test our hypothesis by evaluating the holographic entanglement entropy and comparing to the general results derived of any CFT, which are presented in section 2.3.1. We will first consider the cases where the entangling surface is a sphere\(^9\) and an infinite cylinder and our holographic results will match precisely with the CFT results in eqs. (2.23) and (2.24). We conclude this section by showing that by applying the techniques developed in [34] to the functional (4.5), in fact, we can recover the general result (2.22) for the EE with any (smooth) entangling surface.

### 4.1 EE of the sphere

In this case, it is convenient to parameterize the \( AdS_5 \) metric as follows

\[ ds^2 = \frac{\bar{L}^2}{z^2} \left( dz^2 - dt^2 + dr^2 + r^2d\Omega_2^2 \right) . \]  

(4.6)

Recall that the AdS scale is given by \( \bar{L}^2 = L^2/f_\infty \). Here, the asymptotic boundary is approached with \( z \to 0 \) and as usual, we regulate the evaluation of eq. (4.5) by

\(^9\)The case of the sphere can be analyzed without restricting \( d \) to a particular value. This analysis is presented in Appendix C
introducing a short distance regulator with \( z = z_{\text{min}} = \delta \). Of course, with this choice of coordinates, the boundary metric is simply flat Minkowski space in spherical polar coordinates. We will calculate the entanglement entropy for the interior of a sphere \( r = R \) on the \( t = 0 \) surface in the AdS boundary.

As shown in Appendix C, the surface that minimizes (4.5) can be parameterized as
\[
  r(\theta) = R \cos \theta , \quad z(\theta) = R \sin \theta , \quad \delta/R \leq \theta \leq \frac{\pi}{2} .
\]
(4.7)
Upon evaluating eq. (4.5) for this surface, the leading term is a non-universal contribution proportional to \( R^2/\delta^2 \). However, if we focus on the universal logarithmic contribution, we find
\[
  S_{\text{JM}} = -4 \pi^2 \frac{\bar{L}^3}{\ell_p^3} \left[ 1 - 6 \lambda f_\infty \right] \log\left( R/\delta \right) + \ldots
\]
(4.8)
Moreover, given the central charges in eq. (4.4), we see that this result is proportional to \( a \) and the result (4.8) can be expressed as
\[
  S_{\text{JM}} = -4 a \log\left( R/\delta \right) + \cdots
\]
(4.9)
With the ellipsis, we are denoting the power-law divergent and finite terms. This result agrees precisely with that given in eq.(2.23), which was derived from purely CFT techniques. At this point, let us also observe that in our calculation of the entanglement entropy, the surface term in eq. (4.5) does not contribute to the above logarithmic term (4.8).

### 4.2 EE of the cylinder.

In the case of the cylinder, we choose the following coordinates to parameterize the AdS\(_5\) space
\[
  ds^2 = \frac{\bar{L}^2}{z^2} \left( dz^2 - dt^2 + dx^2 + dr^2 + r^2 d\phi^2 \right).
\]
(4.10)
With this choice of coordinates, the boundary metric is again flat Minkowski space now in cylindrical polar coordinates. We choose the entangling surface as the cylinder \( r = R \) on the \( t = 0 \) surface in this boundary geometry. In the following, we also introduce a regulator length \( H \) for the \( x \) direction, i.e., along the length of the cylinder. The rest of our notation is the same as in the previous subsection.

In evaluating the functional (4.5), let us parameterize the surface \( m \) with \( r(z) \). We have made some general analysis for a cylindrical entangling surface in a \( d \)-dimensional boundary theory in Appendix C.3. Reducing these results to \( d = 4 \), we arrive at
\[
  S_{\text{JM}} = \frac{4 \pi^2 \bar{L}^3 H}{\ell_p^3} \int_{\delta}^{z_{\text{max}}} dz \frac{r}{z^3} \sqrt{r'^2 + 1} \left( 1 + 2 f_\infty \lambda \frac{1 - zr'/r}{1 + r'^2} \right),
\]
(4.11)
where \( z_{\text{max}} \) denotes the maximal radius which the surface \( m \) reaches. Extremizing the above expression leads to the following ‘equation of motion’

\[
z(1 - 2f_\infty \lambda) + r'[\{(1 + 4f_\infty \lambda)zr' + 3r(1 - 2f_\infty \lambda + r^2)\} \]

\[
= \frac{r''z}{1 + r'^2} \left( 6f_\infty \lambda zr' + r(1 - 2f_\infty \lambda + (1 + 4f_\infty \lambda)r'^2) \right).
\]

(4.12)

To identify the universal contribution in eq. (4.11), it suffices to solve this equation asymptotically by substituting the expansion

\[
r(z) = r_0 + r_1 z + r_2 z^2 + \cdots. \quad (4.13)
\]

which yields

\[
r_1 = 0, \quad r_2 = -\frac{1}{4r_0}. \quad (4.14)
\]

Applying the boundary condition \( r(z = 0) = R \), we find

\[
r(z) = R \left( 1 - \frac{z^2}{4R^2} + \cdots \right). \quad (4.15)
\]

Substituting this asymptotic expansion back into the expression for the EE (4.11) and using the results for the central charges (4.4), we finally obtain

\[
S_{\text{JM}} = -\frac{c}{2} \frac{H}{R} \log(R/\delta) + \cdots, \quad (4.16)
\]

where ellipsis again denotes the finite and nonuniversal contributions. Once again, our computation of the holographic EE using eq. (4.5) is in precise agreement with general result (2.24) for the universal logarithmic contribution.

### 4.3 General case: EE as the Graham-Witten anomaly

Here, we consider general (smooth) entangling surfaces \( \Sigma \) in the boundary CFT and apply the methods developed in [34] to evaluating eq. (4.5). This general formalism allows the holographic evaluation of trace anomalies for submanifolds, \( i.e., \) Graham-Witten anomalies [43]. Further the approach rests on the so-called Penrose-Brown-Henneaux (PBH) transformations, which correspond to the subgroup of bulk diffeomorphisms which generate Weyl transformations of the boundary metric. This approach enables us to perturbatively evaluate the metric and the shape of the minimal surface in the vicinity of the AdS boundary without resorting to either the gravitational equations
of motion or to the short distance cutoff $\delta$. In fact, this feature completely fixes the necessary geometry for $d = 4$, whereas for higher dimensions, one still needs to consider the equations of motion in order to fix various constants which cannot be determined on the basis of PBH transformations.

We start from a brief review of the general method and the interested reader can find the details in the original papers [34]. We denote the dimensions of the AdS boundary and of the submanifold $\Sigma$ embedded in the boundary as $d$ and $k$, respectively. For the initial discussion, we leave $d$ and $k$ as general and however, at the end of the discussion, we will focus on $d = 4$ and $k = 2$, as is relevant for the holographic EE here.

In the Fefferman-Graham (FG) gauge, coordinates are chosen for the bulk metric [39]

$$ds^2 = G_{\mu\nu}dX^\mu dX^\nu = \frac{\tilde{L}^2}{4} \left( \frac{d\rho}{\rho} \right)^2 + \frac{1}{\rho} g_{ij}(x, \rho) dx^i dx^j. \quad (4.17)$$

where $g_{ij}(x, \rho)$ admits a Taylor series expansion in the radial coordinate $\rho$:

$$g_{ij}(x, \rho) = g_{ij}^{(0)}(x) + \frac{1}{\rho} g_{ij}^{(1)}(x) \rho + \frac{2}{\rho^2} g_{ij}^{(2)}(x) \rho^2 + \cdots. \quad (4.18)$$

The asymptotic AdS boundary is approached with $\rho \to 0$. The first term in this expansion, $g_{ij}(x, 0) = g_{ij}^{(0)}(x)$ is identified with the background metric of the dual CFT. Exploring the transformation properties of the $g_{ij}(x, \rho)$ under the PBH diffeomorphisms, which preserve the FG gauge, one can essentially determine the remaining coefficients in the Taylor series for $n < d/2$ — see appendix A for further details. The embedding of the $(k+1)$-dimensional submanifold $m$ into the $(d+1)$-dimensional bulk is described by $X^\mu = X^\mu(y^a, \tau)$, where $X^\mu = \{x^i, \rho\}$ are the bulk coordinates and $\sigma^a = \{y^a, \tau\}$ are the coordinates on surface $m$ (with $a = 1, \ldots, k$). Reparameterizations of $m$ can be fixed by imposing

$$\tau = \rho \quad \text{and} \quad h_{\alpha\beta} = 0, \quad (4.19)$$

where $h_{\alpha\beta}$ denotes the induced metric on $m$.

By definition, the PBH transformations preserve the FG gauge (4.17), however, they do change $\rho$ in general. Thus to stay within the above gauge (4.19), one needs to apply the compensating world-volume diffeomorphism on $m$. The requirement of preserving eq. (4.19) uniquely fixes the induced (by the PBH transformation) world-volume diffeomorphism and the transformation rule of the embedding functions $X^\mu(y^a, \tau)$. Let us make a Taylor expansion or the embedding functions in $\tau$,

$$X^i(\tau, y^a) = X^i_0(y^a) + \frac{1}{\rho} X^i_1(y^a) \tau + \cdots, \quad (4.20)$$

where $X^i_0(y^a)$ describes the position of $\partial m$ on the boundary of AdS. In the case of interest, this matches the position of the entangling surface $\Sigma$ in the boundary metric.
Now studying the above transformation rules order by order, one can determine the higher coefficients, e.g.,

\[ X^i(g^a) = \frac{\tilde{L}^2}{2k} K^i(g^a), \]  

(4.21)

with \( K^i \) being the trace of the second fundamental form of the boundary submanifold \( \Sigma \), i.e., \( K^i = n^j K_{ab}^{(0)} h^{ab} \) where \( h^{ab} \) is the induced metric on \( \Sigma \) — see below. As a result, the induced metric on \( m \) compatible with the gauge choice (4.19) is also determined as

\[ h^\tau\tau = \frac{\tilde{L}^2}{4\tau^2} \left( 1 + \frac{\tilde{L}^2}{k^2} K^i K^j g_{ij} \tau + \cdots \right), \quad h_{ab} = \frac{1}{\tau} \left( h^{(0)}_{ab} + h^{(1)}_{ab} \tau + \cdots \right), \]  

(4.22)

with

\[ h^{(0)}_{ab} = \partial_a X^i \partial_b X^j g_{ij} \quad \text{and} \quad h^{(1)}_{ab} = g_{ab} - \frac{\tilde{L}^2}{k} K^i K^j g_{ij}. \]  

(4.23)

An explicit expression for \( g_{ab}^{(1)} \), which appears in the last formula, can be found in appendix A.

At this point, we set the dimensions to \( d = 4 \) and \( k = 2 \). Now applying the above results, we find the following expansion for the intrinsic Ricci scalar on \( m \),

\[ \mathcal{R} = \frac{6}{L^2} + \left( \mathcal{R}_\Sigma + \frac{2}{L^2} h^{(0)}_{ab} g_{ab} - \frac{1}{2} K^i K^j g_{ij} \right) \tau + \cdots \]  

(4.24)

where \( \mathcal{R}_\Sigma \) is the intrinsic curvature scalar on the boundary surface \( \Sigma \), i.e., the entangling surface. Next using a variant of Gauss-Codazzi relation [37], we re-express the term in parentheses above as

\[ \mathcal{R}_\Sigma + \frac{2}{L^2} h^{(0)}_{ab} g_{ab} - \frac{1}{2} K^i K^j g_{ij} = h^{ac} h^{bd} C_{abcd} - \left( \text{tr}(K^i K^j) - \frac{1}{2} K^i K^j \right) g_{ij}. \]  

(4.25)

Now we combine all of these results together in evaluating \( S_{JM} \), our surface functional for the holographic EE in eq. (4.5). Note that the asymptotic expansions above suffice in identifying the logarithmic contribution and we find using eq. (4.4),

\[ S_{EE} = \frac{\log(\ell/\delta)}{2\pi} \int_{\Sigma} d^2 x \left( h^{(0)}_{ab} \right)^{1/2} \left[ c \left( h^{ac} h^{bd} C_{abcd} - \text{tr}(K^i K^j) + \frac{1}{2} K^i K^j \right) - a \mathcal{R}_\Sigma \right] + \cdots, \]  

(4.26)

where \( \ell \) is some macroscopic scale that emerges from the CFT geometry. Now if we account for the slightly different notation here and in section 2.3.1, we see that this

\[ ^{10} \text{Note that we are adopting the notation of [34] here by contracting the extrinsic curvatures with a normal vector, i.e., } K^i_{ab} = n_j^{(0)} K_{ab}^{(0)}. \]
holographic result for an arbitrary (smooth) entangling surface \( \Sigma \) precisely matches the universal entropy term (2.22) derived from purely field theoretic considerations. Hence this final test seems a strong indication that \( S_{\text{JM}} \) is the correct surface functional to replace the area in eq. (1.1) when defining holographic EE in Lovelock gravity.

As a final comment, we note that the boundary term which we added to eq. (4.5) only contributes to power-law divergent and finite terms in the holographic EE and does not contribute to the universal term (4.26).

5. EE for \( d = 6 \) holographic CFT

We now turn to the case of a six-dimensional boundary theory. In this case with a seven-dimensional bulk, the curvature-squared and -cubed interactions contribute to the Lovelock action (2.1) yielding

\[
I = \frac{1}{2\ell_p^6} \int d^7 x \sqrt{-g} \left[ \frac{30}{L^2} + R + \frac{L^2}{12} \lambda \mathcal{L}_4(R) - \frac{L^4}{24} \mu \mathcal{L}_6(R) \right]. \tag{5.1}
\]

Comparing to the notation of section 2.1, we have \( \lambda = \lambda_2 = 12c_2 \) and \( \mu = \lambda_3 = -24c_3 \). Further, \( \mathcal{L}_4 \) is given in eq. (4.2) while explicitly evaluating \( \mathcal{L}_6 \) using eq. (2.2) yields

\[
\mathcal{L}_6 = 4 R_{\mu\nu \rho\sigma} R_{\rho\sigma}^\tau R_{\tau\chi}^{\mu\nu} - 8 R_{\mu \nu \rho \sigma} R_{\rho \sigma}^\tau R_{\tau \chi}^{\mu \nu} - 24 R_{\mu \nu \rho \sigma} R_{\rho \sigma}^{\tau \chi} R_{\tau \chi}^{\mu \nu} - 24 R_{\mu \nu \rho \sigma} R_{\rho \sigma}^{\mu \nu} R_{\rho \sigma}^{\tau \chi} R_{\tau \chi}^{\mu \nu} + 3 R_{\mu \nu \rho \sigma} R_{\rho \sigma}^{\mu \nu} R_{\rho \sigma}^{\tau \chi} R_{\tau \chi}^{\mu \nu} - 24 R_{\mu \nu \rho \sigma} R_{\rho \sigma}^{\mu \nu} R_{\rho \sigma}^{\tau \chi} R_{\tau \chi}^{\mu \nu} + 16 R_{\mu \nu} R_{\rho \sigma} R_{\rho \sigma}^{\mu \nu} - 12 R_{\mu \nu} R_{\rho \sigma} R_{\rho \sigma}^{\mu \nu} R - R^3, \tag{5.2}
\]

Substituting \( d = 6 \) into eq. (2.3) yields the cubic equation

\[
1 = f_\infty - f_\infty^2 \lambda - f_\infty^3 \mu. \tag{5.3}
\]

In principle, we can again solve for \( f_\infty \) analytically, however, the precise expression will not be needed in the following. Note that implicitly we choose the particular root (the smallest positive root) which gives the physical vacuum, as discussed in detail in [30]. With \( d = 6 \), there are four central charges appearing in the trace anomaly, as discussed in section 2.3. The holographic expressions for the central charges were calculated in [19]:

\[
B_1 = \frac{\tilde{L}^5 - 9 + 26 f_\infty \lambda + 51 f_\infty^2 \mu}{288}, \quad B_2 = \frac{\tilde{L}^5 - 9 + 34 f_\infty \lambda + 75 f_\infty^2 \mu}{1152}, \quad B_3 = \frac{\tilde{L}^5 - 9 + 2 f_\infty \lambda - 3 f_\infty^2 \mu}{384}, \quad A = \frac{\pi^3 \tilde{L}^5 3 - 10 f_\infty \lambda - 45 f_\infty^2 \mu}{6}. \tag{5.4}
\]
Of course, this expression for $A$ agrees with the general expression given in eq. (2.6).

Now we would like to further test the proposal that the holographic EE in Lovelock gravity is given by extremizing the expression in eq. (2.11) over the bulk surfaces $m$ homologous to the appropriate boundary region. For seven-dimensional Lovelock gravity, eq. (2.11) becomes

$$S_{JM} = \frac{2\pi}{\ell_p^5} \int_m d^5x \sqrt{h} \left[ 1 + \frac{\lambda}{6} L^2 R - \frac{\mu}{8} L^4 \left( \mathcal{R}_{\mu\nu\rho\sigma} \mathcal{R}^{\mu\nu\rho\sigma} - 4 \mathcal{R}_{\mu\nu} \mathcal{R}^{\mu\nu} + \mathcal{R}^2 \right) \right].$$  \hspace{1cm} (5.5)

While we could add an appropriate set of surface terms [23], we will not consider their contributions in the following. The focus of our analysis will be the universal logarithmic contributions but, as found in the previous section with $d = 4$, the surface terms will only make power-law contributions in the short distance cut-off $\delta$. That is, they only contribute power-law divergent or finite terms in the entanglement entropy.

Following the strategy of the previous section 4, we extract the universal log-term in the holographic calculation and compare with the corresponding CFT result (2.27). As previously noted, the latter result can only be reliably applied with the entangling surfaces where there is a rotational symmetry in the transverse space [20]. Hence to test the proposal for holographic EE, we start by applying it to various surfaces which possess the desired rotational symmetry. We will find that in these cases there is full agreement between (5.5) and the general CFT results (2.27). However, we note that the restriction for the CFT analysis in section 2.3 is commonly stated as demanding that the extrinsic curvature of the entangling surface should vanish, e.g., [15]. Hence we also examine the holographic EE for surfaces with zero extrinsic curvature but without a rotational symmetry in the transverse space. In this case we find a discrepancy between eqs. (5.5) and (2.27). We argue that eq. (2.27) is incomplete, i.e., unable to properly determine the universal contribution, for these cases. However, we are able to use holography to construct the additional curvature terms which must be added to eq. (2.27) to correctly determine the universal EE.

### 5.1 Entangling surfaces with rotational symmetry

As noted above, the CFT results of section 2.3 are only reliable when the entangling surface has a rotational symmetry in the transverse space [20]. This symmetry is not generally present when time is singled out in geometries of the form $R_t \times \mathcal{M}_{d-1}$. Rather the Euclidean$^{11}$ background must have a high degree of symmetry. As simple example, we could take the six-dimensional boundary geometry to be $S^6$ and the desired rotational symmetry would result by choosing $\Sigma$ as a maximal $S^4$ within this geometry.

$^{11}$Recall that these CFT calculations are performed after Wick rotating to Euclidean signature.
A natural question would be: what is the interpretation of the resulting entanglement entropy? A simple Wick rotation of the $S^6$ back to Minkowski signature would yield six-dimensional de Sitter space. In this case, $\Sigma$ would become the equator of a constant time slice which has an $S^5$ geometry. The EE would then be interpreted as measuring the entanglement of the CFT between the two halves of this time slice. In fact, this will be the entanglement entropy across the cosmological horizon of the de Sitter geometry.\footnote{See \cite{20, 35} for a different interpretation of this particular calculation.}

In fact, in the preceding example, one can readily see that the universal contribution to the EE is simply proportional to the central charge $A$. The reason being that the boundary geometry is conformally flat and examining eqs. (2.27) and (2.28), we see that the expressions multiplying the $B_n$ are all proportional to the Weyl tensor. Hence the latter contributions must all vanish in this particular case. However, this example is instructive, as we learn that to probe these terms in eq. (2.27), we must choose the boundary geometry to not be conformally flat. As a result, the bulk geometry will not be simply pure AdS space. However, we will only need to understand the details of the asymptotic geometry, as described in appendix A. In fact to determine the universal contribution, \textit{i.e.}, the logarithmic term, in the holographic EE with $d = 6$, we will have to carry this asymptotic expansion to second order. For the boundary geometries chosen below, these expansions are explicitly constructed in appendix B.

We consider four different six-dimensional boundary geometries in the following: a) $R \times S^2 \times S^3$, b) $R^3 \times S^3$, c) $R^2 \times S^4$ and d) $S^3 \times S^3$. Recall that the entangling surface is a four-dimensional submanifold which we wish to choose in a symmetric way so that there is a rotational symmetry in the transverse two dimensions. For example, the backgrounds (a), (b) and (d) contain an $S^3$ and we can choose $\Sigma$ to wrap a maximal $S^1$ in this component of the geometry, as well as filling the other three dimensions of the background. Similarly, in the geometry (c), $\Sigma$ can wrap a maximal $S^2$ inside the $S^4$ and also the $R^2$ component of the boundary geometry. Alternatively, $\Sigma$ can wrap the entire $S^4$ and sit at a point in the $R^2$. In the latter case, there is a rotational symmetry in the plane $R^2$. Similarly, in the geometry (b), we can also choose $\Sigma$ to wrap $R \times S^3$, which leaves a rotational symmetry about the line $R$ in the $R^3$.

With the bulk metrics given in appendix B, one must solve for the asymptotic expansion (4.20) of the bulk surface $m$ which minimizes the entropy functional (5.5). Because of the rotational symmetry about $\Sigma$, the extrinsic curvatures vanish on this surface. Then as can be seen from eq. (4.21), this vanishing implies that the expansion (4.20) could only begin at second order in $\tau$. However, we have also verified that in fact for all of our examples below the second order term also vanishes leaving

\[ X^i(y^a, \tau) = X^i(y^a) + O(\tau^3). \] (5.6)
The universal contribution in the holographic EE can be evaluated by plugging these expressions back into (5.5) and extracting the logarithmic divergence. We now present the results of our holographic calculations and of the CFT analysis (2.27) for these universal contributions for the various geometries:

**a)** \( R \times S^2 \times S^3 \) with \( \Sigma = R \times S^2 \times S^1 \):

\[
S_{\text{JM}} = \frac{3\pi V_\Sigma}{100 R_1^4 R_2^4} \frac{\tilde{L}^5}{36 \ell_p^5} \left( 8 R_1^2 R_2^2 (3 - 13 f_\infty \lambda - 30 f_\infty^2 \mu) - 3 R_1^4 (7 - 17 f_\infty \lambda - 30 f_\infty^2 \mu) + 12 R_2^4 (3 - 13 f_\infty \lambda - 30 f_\infty^2 \mu) \right) \log(\ell/\delta), \tag{5.7}
\]

\[
S_{\text{EE}} = -\frac{3\pi V_\Sigma}{100 R_1^4 R_2^4} \left( B_1 (17 R_1^4 + 72 R_1^2 R_2^2 + 108 R_2^4) - 4 B_2 (13 R_1^4 + 8 R_1^2 R_2^2 + 12 R_2^4) + 16 B_3 (17 R_1^4 + 32 R_1^2 R_2^2 + 48 R_2^4) \right) \log(\ell/\delta). \tag{5.8}
\]

where \( R_1 \) and \( R_2 \) are the radii of curvature for the \( S^3 \) and \( S^2 \), respectively, and \( \ell \) is some macroscopic scale of the CFT geometry. Further \( V_\Sigma \) is the volume of the entangling surface, i.e., for \( \Sigma = R \times S^2 \times S^1 \), \( V_\Sigma = 8\pi^2 R_1 R_2^2 H \) where \( H \) is a regulator length along the \( R \) factor. In fact, we do not explicitly need to evaluate \( V_\Sigma \) to compare the two expressions above. Using the holographic expression (5.4) for the four central charges, we find \( S_{\text{JM}} = S_{\text{EE}} \). We present the remaining results more briefly.

**b)** \( R^3 \times S^3 \) with \( \Sigma = R^3 \times S^1 \):

\[
S_{\text{JM}} = \frac{\pi V_\Sigma}{25 R_1^4} \frac{\tilde{L}^5}{4 \ell_p^5} (3 - 13 f_\infty \lambda - 30 f_\infty^2 \mu) \log(\ell/\delta), \tag{5.9}
\]

\[
S_{\text{EE}} = -\frac{9\pi V_\Sigma}{25 R_1} \left( 9 B_1 - 4 B_2 + 64 B_3 \right) \log(\ell/\delta). \tag{5.10}
\]

where \( R_1 \) is the radius of the \( S^3 \).

**b')** \( R^3 \times S^3 \) with \( \Sigma = R \times S^3 \):

\[
S_{\text{JM}} = \frac{\pi V_\Sigma}{25 R_1^3} \frac{\tilde{L}^5}{4 \ell_p^5} (3 - 13 f_\infty \lambda - 30 f_\infty^2 \mu) \log(\ell/\delta), \tag{5.11}
\]

\[
S_{\text{EE}} = -\frac{9\pi V_\Sigma}{25 R_1} \left( 9 B_1 - 4 B_2 + 64 B_3 \right) \log(\ell/\delta). \tag{5.12}
\]

**c)** \( R^2 \times S^4 \) with \( \Sigma = R^2 \times S^2 \):

\[
S_{\text{JM}} = -\frac{3\pi V_\Sigma}{400 R_1^4} \frac{\tilde{L}^5}{\ell_p^5} (9 - 19 f_\infty \lambda - 30 f_\infty^2 \mu) \log(\ell/\delta), \tag{5.13}
\]

\[
S_{\text{EE}} = -\frac{3\pi V_\Sigma}{100 R_1^4} \left( 17 B_1 - 52 B_2 + 912 B_3 \right) \log(\ell/\delta). \tag{5.14}
\]
c’) $R^2 \times S^4$ with $\Sigma = \text{pt.} \times S^4$:

$$S_{\text{JM}} = \frac{\pi V_\Sigma}{50R_1^4 \frac{L^5}{4\ell_\text{P}^5}} \left(99 - 389f_\infty \lambda - 2070f_\infty^2 \mu\right) \log(\ell/\delta), \quad (5.15)$$

$$S_{\text{EE}} = \frac{9\pi V_\Sigma}{50R_1^4} \left(\frac{25}{3\pi^3} A - 17B_1 + 52B_2 - 592B_3\right) \log(\ell/\delta). \quad (5.16)$$

where $R_1$ is the radius of the $S^4$.

d) $S^3 \times S^3$ with $\Sigma = S^1 \times S^3$:

$$S_{\text{JM}} = \frac{\pi V_\Sigma}{25R_1^4 R_2^4} \frac{L^5}{4\ell_\text{P}^5} \left(3 - 13f_\infty \lambda - 30f_\infty^2 \mu\right) \log(\ell/\delta), \quad (5.17)$$

$$S_{\text{EE}} = -\frac{\pi V_\Sigma}{25R_1^4 R_2^4} \left(\frac{R_1^2 + R_2^2}{4}\right)^2 9(9B_1 - 4B_2 + 64B_3) \log(\ell/\delta), \quad (5.18)$$

where $R_1$ is the radius $S^3$ wrapped by the $S^1$ and $R_2$ is the radius of the other $S^3$.

Again we do not need to explicitly specify $V_\Sigma$ in these expressions to make the comparison of $S_{\text{JM}}$ and $S_{\text{EE}}$. In every case, the holographic result $S_{\text{JM}}$ is in complete agreement with $S_{\text{EE}}$ given by the CFT formula (2.27), when we substitute in the holographic expressions for the central charges (5.4). Further note that with these tests, we have probed all of the individual terms appearing in the CFT result, i.e., all three $B_n$, as well as $A$, appear in the expressions above. Hence once again, we have found strong indications that $S_{\text{JM}}$ is the correct surface functional to extend the standard definition (1.1) of holographic EE to Lovelock gravity in the bulk.

5.2 Entangling surfaces without rotational symmetry

As we noted above, the restriction for the CFT analysis in section 2.3 is commonly stated as demanding that the extrinsic curvature of $\Sigma$ should vanish, e.g., [15], rather than requiring a rotational symmetry around this symmetry. Hence here we examine the holographic EE for such surfaces, namely with zero extrinsic curvature but without a rotational symmetry in the transverse space. In particular, we focus on the first three backgrounds above, where we can think of the geometry as $R_t \times M_{d-1}$. Then, because of the simple product form of the geometry, if the entangling surface $\Sigma$ lies in the ‘spatial’ geometry $M_{d-1}$, the extrinsic curvature associated with the normal vector in the time direction vanishes, i.e., $K^I_{ab} = 0$. At the same time, in our examples, the ‘spatial’ geometry contains various sphere components $S^n$. If the entangling surface is chosen to wrap a maximal $S^{n-1}$ within one of these, we have ensured the vanishing of the remaining extrinsic curvatures associated with a normal vector within $M_{d-1}$. Of course, as is clear in this construction, there will also be no rotational symmetry
around \( \Sigma \). Below, we present the results for the universal contribution to the EE from our holographic calculations using eq. (5.5) and from the CFT analysis (2.27) in these geometries:

**a) \( R \times S^2 \times S^3 \) with \( \Sigma = S^1 \times S^3 \):**

\[
S_{\text{JM}} = -\frac{\pi V_\Sigma}{1600 R_1 R_2} \left[ \frac{\tilde{L}^5}{\ell_p^5} \right] \left[ 3(1 + 4 f_\infty \lambda + 15 f_\infty^2 \mu) R_1^4 - 16(3 - 13 f_\infty \lambda - 30 f_\infty^2 \mu) R_2^1, \
-4(33 - 103 f_\infty \lambda - 210 f_\infty^2 \mu) R_1^2 R_2^2 \right] \log(\ell/\delta),
\]

(5.19)

\[
S_{\text{EE}} = -\frac{\pi V_\Sigma}{200 R_1^4 R_2^2} \left[ \frac{\tilde{L}^5}{\ell_p^5} \right] \left[ 3(51 B_1 - 156 B_2 - 304 B_3) R_1^4 - 24(9 B_1 - 4 B_2 + 64 B_3) R_2^4 \
-6(19 B_1 + 36 B_2 - 16 B_3) R_1^2 R_2^2 \right] \log(\ell/\delta),
\]

(5.20)

\[
\Delta S = S_{\text{JM}} - S_{\text{EE}} = \frac{\pi V_\Sigma}{64 R_1^4 R_2^2} \left[ \frac{\tilde{L}^5}{\ell_p^5} \right] \left[ 1 - 2 f_\infty \lambda - 3 f_\infty^2 \mu \right] \log(\ell/\delta),
\]

(5.21)

where \( R_1 \) and \( R_2 \) are the radii of the \( S^3 \) and \( S^2 \), respectively. Note that \( \Sigma \) fills the entire \( S^3 \) and wraps a maximal \( S^1 \) within the \( S^2 \) component. As before, the volume \( V_\Sigma \) is not needed to compare the two results. In the last line above, we have substituted the holographic expression for the central charges (5.4) into \( S_{\text{EE}} \) and we can see that there is a discrepancy between the two results. Note, however, that the difference \( \Delta S \) is proportional to \( B_3 \) in eq. (5.4). We present the remaining results more briefly.

**a’) \( R \times S^2 \times S^3 \) with \( \Sigma = S^2 \times S^2 \):**

\[
S_{\text{JM}} = -\frac{\pi V_\Sigma}{1200 R_1^2 R_2^2} \left[ \frac{\tilde{L}^5}{\ell_p^5} \right] \left[ 3(7 - 17 f_\infty \lambda - 30 f_\infty^2 \mu) R_1^4 + (39 - 134 f_\infty \lambda - 285 f_\infty^2 \mu) R_2^4 \
-2(87 - 33 f_\infty \lambda - 1950 f_\infty^2 \mu) R_1^2 R_2^2 \right] \log(\ell/\delta),
\]

(5.22)

\[
S_{\text{EE}} = -\frac{\pi V_\Sigma}{100 R_1^2 R_2^2} \left[ \frac{\tilde{L}^5}{\ell_p^5} \right] \left[ 3(17 B_1 - 52 B_2 + 272 B_3) R_1^4 - 8(27 B_1 - 12 B_2 + 32 B_3) R_2^4 \
-6 \left( \frac{25}{3 \pi^3} A + 19 B_1 + 36 B_2 - 336 B_3 \right) R_1^2 R_2^2 \right] \log(\ell/\delta),
\]

(5.23)

\[
\Delta S = S_{\text{JM}} - S_{\text{EE}} = \frac{\pi V_\Sigma}{48 R_1^2 R_2^2} \left[ \frac{\tilde{L}^5}{\ell_p^5} \right] \left[ 1 - 2 f_\infty \lambda - 3 f_\infty^2 \mu \right] \log(\ell/\delta),
\]

(5.24)

where again \( R_1 \) and \( R_2 \) are the radii of the \( S^3 \) and \( S^2 \), respectively.
b) $R^3 \times S^3$ with $\Sigma = R^2 \times S^2$:

$$S_{JM} = -\frac{\pi V_\Sigma}{1200 R_1^4 \ell_5^5} (39 - 134 f_\infty \lambda - 285 f_\infty^2 \mu) \log(\ell/\delta),$$  \hspace{1cm} (5.25)

$$S_{EE} = 2\pi V_\Sigma \frac{25}{25 R_1^4} (27 B_1 - 12 B_2 + 32 B_3) \log(\ell/\delta),$$  \hspace{1cm} (5.26)

$$\Delta S = S_{JM} - S_{EE} = \frac{\pi V_\Sigma}{48 R_1^4 \ell_5^5} (1 - 2 f_\infty \lambda - 3 f_\infty^2 \mu) \log(\ell/\delta),$$  \hspace{1cm} (5.27)

where $R_1$ is the radius of the $S^3$.

c) $R^2 \times S^4$ with $\Sigma = R^1 \times S^3$:

$$S_{JM} = \frac{9\pi V_\Sigma}{1600 R_1^4 \ell_5^5} (13 - 28 f_\infty \lambda - 45 f_\infty^2 \mu) \log(\ell/\delta),$$  \hspace{1cm} (5.28)

$$S_{EE} = \frac{3\pi V_\Sigma}{200 R_1^4} (51 B_1 - 156 B_2 + 1616 B_3) \log(\ell/\delta),$$  \hspace{1cm} (5.29)

$$\Delta S = S_{JM} - S_{EE} = \frac{\pi V_\Sigma}{64 R_1^4 \ell_5^5} (1 - 2 f_\infty \lambda - 3 f_\infty^2 \mu) \log(\ell/\delta),$$  \hspace{1cm} (5.30)

where $R_1$ is the radius of the $S^4$.

Hence in all these examples, there is a mismatch between the holographic result found using eq. (5.5) and the CFT result evaluated with eq. (2.27). A clue as to the nature of this mismatch comes from observing that in each of the above cases, the difference $\Delta S$ is proportional to the holographic expression for $B_3$, given in eq. (5.4). Further, we note that the mismatch persists even in the limit of Einstein gravity, where we set $\lambda = \mu = 0$ which sends $f_\infty \to 1$.

We are specifically probing the EE in geometries where we know the derivation sketched in section 2.3 is not valid and hence the results for $S_{EE}$ are suspect. Hence, in proceeding to examine this mismatch, our working hypothesis will be that in fact the holographic results above are actually the correct ones.

To understand the nature of the mismatch between the two calculations even at vanishing extrinsic curvatures, we resort once more to the powerful techniques developed in [34] to extract analytically the holographic entanglement entropy for general boundary geometries and submanifolds, as shown in section 4.3. Above, we noted that mismatch occurs even when the bulk theory is Einstein gravity where the holographic EE is simply given by eq. (1.1). To simplify the calculations here then, we will restrict our attention to Einstein gravity. However, we will be able to extend our results to remove the discrepancies above for the cubic Lovelock theory.

In case of Einstein gravity, the holographic entanglement entropy is simply the area of the minimal surface, whose expansion near the boundary is readily obtained, analo-
gous to the computations detailed in section 4.3. The only difference in six dimensions, however, is that the measure, i.e., $\sqrt{h}$ in the area integral, begins at order $\tau^{-d/2} = \tau^{-3}$. Hence we extend our expansions to order $\tau^2$ in order to extract the log-divergent term. This is a tedious task, even without the complication of higher derivative corrections in $S_{JM}$. Hence to simplify our calculations further but yet keep enough generality to encompass all of the geometries that are commonly considered, e.g., entangling surfaces as proposed above, we make the following assumptions about the geometry of the entangling surface $\Sigma$ in the boundary metric:

$$K_{ab}^i = 0, \quad R_{mnpr} n^i n^j n^k n^l t_a^r = 0,$$

(5.31)

In terms of the holographic construction, the tangent vectors are given by $t_a^m = \partial_a X^m$ and $R_{mnpr}$ is the boundary curvature tensors constructed from $\tilde{g}_{ij}$. Together the above assumptions imply that on $\Sigma$,

$$R_{mn} n^m t_n^a = 0.$$

(5.32)

With these assumptions, the log term in the holographic entanglement entropy with Einstein gravity is given by

$$S = 2\pi \log\delta \frac{\bar{L}_5}{e^5} \int_{\Sigma} d^4x \sqrt{h} \left[ \frac{1}{2} h^{ij} \tilde{g}_{ij} + \frac{1}{8} (h^{ij} \tilde{g}_{ij})^2 - \frac{1}{4} \tilde{g}_{ij} h^{jk} \tilde{g}_{kl} h^{li} \right],$$

(5.33)

where

$$h^{ij} = h^{ab}_{\partial_a X^i \partial_b X^j}$$

(5.34)

is the tangential projector with respect to $\Sigma$. Alternatively, we can express this tensor as $h^{ij} = \tilde{g}^{ij} - n^i n^j$. Explicit expressions for $\tilde{g}$ and $\tilde{g}$ are given in eq. (A.3). One can check that, subject to our assumptions, the above expression (5.33) is conformally invariant if we restrict to transformations to be independent of the transverse coordinates. Note that each of the terms in CFT result (2.27) is also conformally invariant in the same sense.

Now one can compute the difference between the holographic result (5.33) and the expected CFT result (2.27) for Einstein gravity. Clearly this difference has to be some conformal invariant that vanishes when evaluated on defects that preserve the rotational symmetry in the transverse space. It is interesting that in the difference, terms with explicit covariant derivatives of the curvatures exactly cancel. In light of conformal invariance, the result can be arranged into the following compact form:

$$\Delta S = 4\pi B_3 \log\delta \int_{\Sigma} d^4x \sqrt{h} \left( C_m^r s C_{mnsk} \tilde{g}_{d} \tilde{g}_{rk} - C_m^r s C_{mnlk} \tilde{g}_{dl} \tilde{g}_{rk} \right) + 2C_m^r s C_{mnsk} \tilde{g}_{dl} \tilde{g}_{ks} - 2C_m^r s C_{mnsk} \tilde{g}_{dl} \tilde{g}_{ks},$$

(5.35)
where $\tilde{g}_{ij} = n_i^j n_j^i$ is the metric in the transverse space to $\Sigma$. We have written the coefficient as $B_3 = \frac{L^5}{384\ell^5}$, even though in Einstein gravity the central charges can not really be distinguished since they are all proportional to $\tilde{L}^5/\ell^5_P$. However, our previous results suggest that $\Delta S$ will be proportional to $B_3$ in a more general context.

In the presence of a rotational symmetry in the transverse space, any tensors with an odd number of projectors into the transverse space must vanish. Further, the rotational symmetry guarantees that the extrinsic curvatures all vanish. It is fortuitous then that these observations are consistent with our assumptions above in eq. (5.31). For a tensor that carries only two transverse indices, its symmetric part, should be proportional to the transverse metric $\tilde{g}^\perp$, and the anti-symmetric part, the volume form $\tilde{\epsilon}$. Hence we can write

$$ C_{an,bn_j} = H^{S}_{ab} \tilde{g}^\perp_{n,n_j} + H^{A}_{ab} \tilde{\epsilon}_{n,n_j}, \quad (5.36) $$

where $H^{(A)S}_{ab}$ is some (anti)symmetric tensor, as required by the symmetry of the Weyl tensor, with tangential indices along $\Sigma$. Eq. (5.36) implies, from the cyclic permutation property of the Weyl tensor, that

$$ C_{anb_n,} = 2H^{A}_{ab} \epsilon^\perp_{n,n_j}. \quad (5.37) $$

Substituting these expressions into eq. (5.35), and using the fact that the transverse space is two-dimensional, one can verify that $\Delta S$ vanishes identically in the presence of rotational symmetry.

We have derived $\Delta S$ from a holographic analysis with Einstein gravity, however, our conjecture is that this correction term should be included in general. That is, we propose that the correct result for the universal EE in a six-dimensional CFT is given by the sum of $S_{ee}$ in eq. (2.27) and $\Delta S$ in eq. (5.35) — of course, this should only be the correct result when the extrinsic curvature of $\Sigma$ vanishes. We can test this conjecture with the holographic results of the cubic Lovelock theory above. In this comparison, we have verified that in fact $\Delta S$ in eq. (5.35) precisely matches the difference $\Delta S$ in each of these four examples, which then seems to be strong evidence in favour of our proposal. Again, this proposal only applies when the extrinsic curvature of $\Sigma$ vanishes. When the extrinsic curvature is also nonvanishing, there should be many more correction terms. One could try to determine these terms following the analysis of [15]. First, one constructs all possible conformal invariants involving four derivatives, constructed with the extrinsic curvatures (and the Weyl tensor). Next, one assembles these invariants with arbitrary coefficients in a trial expression which would be added

\[\text{The latter ensures that the components containing only transverse indices, i.e., } C_{n_1n_2n_3n_4}, \text{ cancel altogether.}\]
to $S_{EE} + \Delta S$. Then, one tries to fix the coefficients by testing this CFT result for a variety of entangling surfaces and background geometries against the holographic result for the cubic Lovelock theory.

6. General gravity actions

The two previous sections show quite convincingly that holographic EE in Lovelock gravity is again calculated by minimizing a surface functional and that the appropriate functional is given by the expression $S_{JM}$ in eq. (2.11), originally derived to evaluate black hole entropy in this theory. Now a natural question is whether this success teaches us any lessons for a more general gravity action in the bulk. Unfortunately, the lessons may be limited. For example, the derivation [14] of black hole entropy leading to eq. (2.11) relies on a Hamiltonian formulation which is not readily extended beyond Lovelock gravity, i.e., a generic higher curvature theory will not have second order equations of motion. However, working with Lovelock gravity has certainly provided a great deal of experience with regards to holographic EE and so here we will try to apply this experience to a more general gravity theory.

As an interesting test case, we focus on a general curvature-squared action with $d = 4$

$$I = \frac{1}{2\ell_3^3} \int d^5x \sqrt{-g} \left[ \frac{12}{L^2} + R + L^2 \left( \lambda_1 R_{\mu\nu\rho\sigma} R^\mu\nu\rho\sigma + \lambda_2 R_{\mu\nu} R^{\mu\nu} + \lambda_3 R^2 \right) \right]. \quad (6.1)$$

In examining this theory, a typical approach would be that the curvature-squared terms appear as the first few corrections in a perturbative string expansion, e.g., [44, 45]. In this context, we would have small couplings, i.e., $\lambda_{1,2,3} \ll 1$, and we would only calculate to leading order in any of these parameters. If one attempted to work with the full nonlinear theory (and finite couplings), one encounters the typical problems. For example, the generic action leads to fourth order equations of motion which produces ghost-like excitations in the gravitational theory and from a holographic perspective, this corresponds to producing nonunitary operators to the boundary CFT [20]. Of course, if we tune the couplings as

$$\lambda_1 = \lambda_3 = \lambda/2 \quad \text{and} \quad \lambda_2 = -2\lambda, \quad (6.2)$$

this action (6.1) coincides with Gauss-Bonnet (GB) gravity (4.1) and this problem of higher derivatives is evaded. In the following, we will examine the theory primarily from the perturbative perspective but we will not set the couplings to be small in our analysis as this will allow us consider the case of GB gravity further, as well.
The AdS$_5$ vacua now have curvature $\tilde{L}^2 = L^2/f_\infty$ where \[ 1 = f_\infty - \frac{2}{3} f_\infty^2 (\lambda_1 + 2\lambda_2 + 10\lambda_3). \] (6.3)

If we are treating the higher curvature terms perturbatively, \textit{i.e.,} $\lambda_{1,2,3} \ll 1$, this then would yield

$$f_\infty = 1 + \frac{2}{3} (\lambda_1 + 2\lambda_2 + 10\lambda_3) + \cdots .$$ (6.4)

Of course, with the GB couplings (6.2), the expression (6.3) above agrees with eq. (2.3) for GB gravity. Further, ref. [32] evaluated the holographic trace anomaly for this general action (6.1) and one finds

$$c = \pi^2 \frac{\tilde{L}^3}{\ell_p^3} (1 + 4 f_\infty (\lambda_1 - 2\lambda_2 - 10\lambda_3)) , \quad a = \pi^2 \frac{\tilde{L}^3}{\ell_p^3} (1 - 4 f_\infty (\lambda_1 + 2\lambda_2 + 10\lambda_3)) .$$ (6.5)

Again, with eq. (6.2), this agrees with the result (4.4) determined for GB gravity.

Now we would like to consider holographic EE in the presence of these general curvature-squared interactions. As a first approximation, we take our surface functional to be the Wald formula.\footnote{To simplify the notation slightly, we will assume that we have Wick rotated to Euclidean signature in the following discussion.} Upon evaluating eq. (2.8) for the above action (6.1), the result can be written as

$$S_W = \frac{2\pi}{\ell_p^3} \int_m d^3x \sqrt{h} \left[ 1 + L^2 \left( (2\lambda_1 + \lambda_2 + 2\lambda_3) R^\mu_\nu^\rho_\sigma \tilde{g}_\mu^\perp \tilde{g}_\nu^\perp + (\lambda_2 + 4\lambda_3) R^\alpha_\mu_\beta_\nu h^\perp_{\alpha_\beta} + 2\lambda_3 R A^\alpha_\beta_\gamma_\delta h_{\alpha_\gamma} h_{\beta_\delta} \right) \right] .$$ (6.6)

where $\tilde{g}^\perp_{\mu_\nu}$ and $h_{\alpha_\beta}$ are the transverse and induced metrics for the surface $m$, respectively — here, we are applying the notation introduced in section 2.3.1 to the bulk surface $m$. Note that with the GB couplings (6.2), the coefficients of the first two curvature terms above vanish. In general, if we apply the expressions for the central charges (6.5), then this Wald expression (6.6) will produce results for the EE which agree with those coming from the CFT analysis, \textit{i.e.,} eq. (2.22), but only for entangling surfaces on which the extrinsic curvature vanishes. Applying the techniques of section 4.3, we find the holographic EE contains a logarithmic term of the form

$$S_W = \frac{\log(\ell/\delta)}{2\pi} \int_\Sigma d^2x \sqrt{h} \left[ c C_{abcd} h^{ac} h^{bd} - a \left( R + (K_i^a K_i^a - \frac{1}{2} K_a^a) \right) \right] .$$ (6.7)

Here, the Weyl tensor corresponds to that evaluated for the boundary metric $^{(0)}g_{ij}$ while intrinsic curvature $\mathcal{R}$ and the extrinsic curvatures are evaluated on the entangling
surface $\Sigma$, again embedded in the boundary geometry $g_{ij}$. Note that this expression (6.7) is composed of three independent conformal invariants. Now to fix eq. (6.7) to agree with eq. (2.22) from the pure CFT analysis, we should presumably add extra terms to the surface functional. It is reasonable to assume that these new terms should be covariant and contain only two derivatives, but be independent of the terms already appearing in eq. (6.6). It seems then that the only natural geometric terms will be constructed from the extrinsic curvature of the bulk surface $m$, which we will denote $K^i_{\alpha\beta}$. There are two independent terms and so we write

$$\delta S = \frac{2\pi L^2}{\ell_\text{P}^3} \int_{m} d^3x \sqrt{h} \left( s_1 K^i_{\alpha\beta} K^i_{\beta\alpha} + s_2 K^i_{\alpha\alpha} K^i_{\beta\beta} \right).$$

(6.8)

Now we want to fix the value for constants $s_1$ and $s_2$ so that there is an additional contribution to the logarithmic term with precisely the form

$$\delta S = (a - c) \frac{\log(\ell/\delta)}{2\pi} \int_{\Sigma} d^2x \sqrt{h} \left( K^i_{a b} K^i_{a b} - \frac{1}{2} K^i_{\alpha\alpha} K^i_{\beta\beta} \right).$$

(6.9)

With some further analysis, we find the desired result requires

$$s_1 = -2\lambda_1,$$

(6.10)

while $s_2$ remains undetermined. That is, the term $(K^i_{\alpha\alpha})^2$ in eq. (6.8) only contributes to regular terms in the entanglement entropy and it does not contribute to the universal logarithmic term (or to the nonuniversal divergent terms). In a perturbative framework, this ambiguity cannot be resolved. At zeroth order in the $\lambda_i$, the entropy is simply given by an extremal surface in the AdS bulk, which then necessarily satisfies $K^i_{\alpha\alpha} = 0$, e.g., see [49]. Assuming that $s_2 = O(\lambda_i)$, then this term would only begin to contribute at order $\lambda_i^3$. Hence it simply does not effect the holographic EE at the linear order, which is the order of validity of the present calculations in the perturbative expansion.

Note that in the perturbative framework where $\lambda_i \ll 1$, the couplings $\lambda_2$ and $\lambda_3$ can be varied (and even be set to zero) by field redefinitions (e.g., $\delta g_{\mu\nu} = \alpha_1 R_{\mu\nu} + \alpha_2 R g_{\mu\nu}$) but these changes should not change any physical quantities, e.g., see [44]. The effect of such field redefinitions on the holographic EE may seem a bit mysterious given eq. (6.6). However, it is perhaps clearer when we note that the $\lambda_2$ and $\lambda_3$ contributions there can be rewritten in terms of just $R_{\mu\nu}$ and $R$ — see eq. (7.1) below. In any event, it is reassuring that our results for the universal contribution to the holographic EE in eqs. (6.7) and (6.9) can be written entirely in terms of the central charges (and the

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15Following our notation in footnote 4, the extrinsic curvatures on $\Sigma$ and $m$ are distinguished by the type of indices, i.e., Latin and Greek indices for $\Sigma$ and $m$, respectively.
geometry of the entangling surface $\Sigma$) in the CFT. From this perspective, it is also interesting that the coefficient $s_1$ is fixed in eq. (6.10) in terms of only $\lambda_1$, the single coupling whose value is not subject to field redefinitions.

With such field redefinitions, we could always tune the curvature-squared couplings to

$$\lambda_1 = \lambda, \quad \lambda_2 = -\frac{4}{3}\lambda, \quad \lambda_3 = \frac{1}{6}\lambda.$$  \hspace{1cm} (6.11)

In this case, the higher curvature terms in eq. (6.1) combine as $L^2 \lambda C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$, i.e., the Weyl tensor squared. Further the curvature terms in the Wald contribution (6.6) to the holographic EE would be proportional to $C^{\mu\nu\rho\sigma} \tilde{\varepsilon}_{\mu\nu} \tilde{\varepsilon}_{\rho\sigma}$. In this case, this term in Wald entropy will simply not contribute to any calculation of holographic EE in pure AdS$_5$ and the entire $\lambda$ contribution will come from the correction term (6.8). However, this term proportional to the Weyl tensor can still contribute here in more general backgrounds, such as considered in section 4.3. In fact, following the analysis there, one finds that in a general background, the components of the bulk Weyl tensor scale as $C_{\mu\nu\rho\sigma} \sim \rho$ near the boundary and hence this term will typically contribute to the universal term in the holographic EE.

There is, of course, a well-known higher curvature term in string theory which is quartic in the Weyl tensor [46]. In this case, the interaction would produce a Wald contribution to the holographic EE which is proportional to the Weyl tensor cubed and so again this contribution would vanish in pure AdS$_5$ — implicitly, we will focus on $d = 4$ here. Further following the analysis of section 4.3, this $C^3$ term would generically vanish at least as fast as $\rho^3$ near the asymptotic boundary and hence it would never be able to contribute to the universal EE term. This result is, in fact, essentially required by the consistency of the holographic framework. From the perspective of the boundary theory, these $C^4$ terms introduce corrections of order $1/\lambda^{3/2}$ and $\lambda^{1/2}/N_c^2$ [47] and in particular then, these corrections depend on the 't Hooft coupling $\lambda$. However, the analysis of section 2.3 indicates that the universal contribution to the EE should be proportional to the central charges $a$ and $c$ and in a four-dimensional superconformal gauge theory, it is known that the central charges are independent of the gauge coupling [48]. Therefore this universal term can not receive any $\lambda$-dependent corrections.\footnote{We might add that for $N = 4$ super-Yang-Mills theory in the free field limit, numerical calculations [50] of the EE for a sphere embedded in flat space explicitly confirm that the universal contribution matches the strong coupling result and so also confirms this independence of the gauge coupling.} This is certainly in accord with our conclusion above that the Wald contribution coming from the $C^4$ interaction does not contribute to the universal term. However, just as in our analysis of the curvature-squared interactions above, we expect that the correct functional for the holographic EE will receive corrections beyond this Wald contribution. Hence one
restriction on these corrections is that they can not contribute to the universal EE term for any background or for any entangling surface. In fact, it seems that this constraint is easily satisfied. A preliminary analysis suggests that covariant terms of the form $C^2K^2$, $CK^4$ or $K^6$ all vanish as fast as $C^3$ near the boundary, \(i.e.,\) at least as fast as $\rho^3$. Hence none of these potential contributions to the surface functional would affect the universal EE term. We emphasize that all of these terms, as well as the original $C^4$ interaction, would still make finite contributions to the EE, \(e.g.,\) in a thermal state, the temperature dependence of the EE would receive finite $\lambda$ corrections.

6.1 An ambiguity in holographic EE?

Let us return to considering the curvature squared theory (6.1) with the coefficients tuned as in eq. (6.2) to produce GB gravity. Above we identified an ambiguity in the correction term (6.8), in that we did not fix the coefficient $s_2$. In the perturbative framework, we showed this ambiguity would not affect the results for the holographic EE since the corresponding contribution was always higher order in $\lambda_i$. However, for GB gravity where the couplings are kept finite, the story is more interesting.

Naïvely, our expectation would be that the coefficients in the correction term (6.8) should be fixed so that the Wald expression (6.6) is converted to the expression (4.5) which was successfully tested in section 4. This would, in fact, require that $s_1$ takes the value given in eq. (6.10) but it would also require that $s_2 = -s_1 = 2\lambda_1$. Of course, the analysis above showed that this coefficient is simply not fixed if we only demand that the holographic entanglement entropy reproduce the correct logarithmic term. We note that the latter was precisely the criterion against which we tested eq. (2.11) in sections 4 and 5. Hence our analysis there is actually incomplete, since we have shown here that this leaves certain ambiguities in the definition of the surface functional used to calculate the holographic EE.

Hence we must find another approach to fix this ambiguity. To produce a well-defined variation problem, it is reasonable to require that the equations of motion fixing the extremal surface in the bulk should remain second order. Since the Wald part of the surface functional (6.6) contains only projectors of the bulk curvatures into the surface world-volume, they contribute only terms which are second order in derivatives to the equations of motion. The only source of higher derivative terms comes from the correction term (6.8). Thus we would like to find a suitable ratio of the coefficients, $s_1$, $s_2$, such that any higher derivative terms in the equations of motion cancel.

Since we are varying only the embedding of the surface $m$, we can safely choose a convenient gauge for the background metric. We opt for Riemannian normal coordinates so that the Christoffel symbols are set to zero locally in the vicinity of any point on $m$. Of course, derivatives of the connection will not vanish in general, but one
can show that these terms do not contain cubic or higher derivatives of the embedding function. Therefore we can effectively consider a flat Minkowski background, in which case, eq. (6.8) simply reduces to

\[
\delta S = \frac{2\pi L^2}{\ell_p^3} \int d^3 x \sqrt{h} \left[ \tilde{g}^\perp_{\mu\nu} \partial_\alpha \partial_\beta X^\mu \partial_\gamma \partial_\delta X^\nu \left( s_1 h_1^\alpha \gamma h_1^\beta \delta + s_2 h_2^\alpha \beta h_2^\gamma \delta \right) \right].
\] (6.12)

The four-derivative terms in the equations of motion are then given by

\[
\partial_\alpha \partial_\beta \left( \frac{\delta (\delta S)}{\partial_\alpha \partial_\beta X^\mu} \right) \bigg|_{4\text{-derivative}} = \frac{4\pi L^2}{\ell_p^3} (s_2 - s_1) \sqrt{h} \tilde{g}^\perp_{\mu\nu} h_2^\alpha \beta h_2^\gamma \delta \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta X^\nu.
\] (6.13)

This immediately singles out the special value

\[
s_2 = -s_1.
\] (6.14)

Further we have checked that with this choice of the coefficients, the three-derivative terms also vanish.

Hence, eq. (6.14) guarantees that the variational problem produces only two-derivative equations. However, as noted above, this constraint, together with eq. (6.10), give precisely the necessary coefficients to convert the Wald entropy functional (6.6) to \( S_{2m} \), given in eq. (4.5). Hence we have uniquely determined \( S_{2m} \) as the correct surface functional in calculating holographic EE for GB gravity with two criteria. First, the holographic entanglement entropy must reproduce the correct logarithmic term and second, the variational problem must be second order in derivatives. While we have not investigated the latter criterion in detail for higher Lovelock theories, we note that eq. (2.11) is constructed with extended Euler densities for the intrinsic surface geometry. Of course, they have the same structure as the Lovelock action (2.1) itself and so one expects that an analysis similar to that showing the Lovelock equations are second order would show the variational problem here is also second order. Hence we expect that the same two criteria above will also uniquely select \( S_{2m} \) as the appropriate surface functional to calculate holographic EE for the general Lovelock theories, as well.

7. Discussion

The present paper was an exploration of holographic entanglement entropy for higher curvature gravity theories. We were naturally led to consider a procedure of extremizing some surface functional, similar to the original definition (1.1) for Einstein gravity, in order that the holographic EE satisfies subadditivity (1.3). The close connection with black hole entropy suggests that the new functional might simply be Wald’s formula
(2.8). However, one of our results, in section 3, was that this prescription would fail to provide the correct EE in general. This is unfortunate as it would have given a simple prescription that could be applied quite generally to any higher curvature theory of gravity.

Turning to the special case of Lovelock gravity (2.1), we considered an alternative expression (2.11), which still coincides with Wald’s formula on the Killing horizon of a stationary black hole. In sections 4 and 5, we showed that extremizing $S_{JM}$ yields the correct universal EE contribution for CFT’s in $d = 4$ and 6 with a variety of geometries. In fact, in $d = 4$, we showed that the holographic approach precisely reproduced the general expression (2.22) for the universal contribution for any smooth entangling surface. In $d = 6$, we found a precise match for various geometries where the background geometry was not conformally flat and the entangling surface was chosen so that there was a rotational symmetry around the surface.\(^\text{17}\) While our approach of testing $S_{JM}$ focussed on even dimensions and on the vacuum of the boundary CFT, we expect that the result is quite general. That is, for any Lovelock theory in any dimension and in any asymptotically AdS geometry, the holographic EE can be calculated by extremizing the $S_{JM}$ functional (2.11) for surfaces homologous to the boundary region of interest.

In section 6.1, we found a potential ambiguity in our prescription for Gauss-Bonnet gravity. In particular, a term proportional to the square of the trace of the extrinsic curvature could be added to $S_{JM}$ with an arbitrary coefficient and still leave unchanged the universal EE contribution. We emphasize that this additional term would still modify the finite contributions to the entanglement entropy. However, we argued that the coefficient of this extra term must be set to zero in order to preserve the fact that the variational problem in calculating the holographic EE is still second order in derivatives. While our analysis here focused on Gauss-Bonnet gravity in five bulk dimensions, it extends trivially to any spacetime dimension. We also expect that similar ambiguities will arise for higher Lovelock theories but that again requiring a second order variational problem will uniquely select $S_{JM}$ as the appropriate surface functional.

The goal remains to determine a comprehensive prescription for holographic EE which can be applied to general higher curvature theories. So it is natural to ask whether our success in understanding holographic EE in Lovelock gravity can teach us any lessons for a more general gravity actions in the bulk. Unfortunately, it seems that the lessons may be limited. It is reasonable to expect that the special form of\(^\text{17}\)Note that for the geometries chosen for $d = 6$, the extremal bulk surface has vanishing extrinsic curvature and so on this surface $S_W = S_{JM}$. We also note that the latter observation is not in contradiction with the result in section 3 that $S_W \propto A$ for any entangling surface because this only applies for empty AdS space.
which only involves the intrinsic curvature of the surface on which it is evaluated must be related to the topological origin of the Lovelock theories. However, consider the following analysis: In section 6, we considered a general curvature squared action (6.1) and it is clear that the final surface functional $S_w + \delta S$ depends on more than just the intrinsic geometry, if we examine eq. (6.6). However, we observe that we can rewrite the expression as

$$S_{HEE} = \frac{2\pi}{\ell_p^3} \int_m d^3x \sqrt{h} \left[ 1 + L^2 \left( 2\lambda_1 \mathcal{R} - (4\lambda_1 + \lambda_2) R^{\alpha\beta} h_{\alpha\beta} + (2\lambda_1 + \lambda_2 + 2\lambda_3) R \right) \right],$$

(7.1)

where $\mathcal{R}$ denotes the intrinsic curvature scalar $m$. Producing this final expression relied on a number of geometric identities, e.g., the fact that the Weyl tensor is traceless, but also fixing $s_2$ as in section 6.1. Of course, if we choose the couplings $\lambda_i$ as in eq. (6.2), corresponding to GB gravity, the coefficients of the last two terms vanish and we recover $S_{JM}$ again. On the other hand, one might also consider this expression in a perturbative framework (with $\lambda_i \ll 1$) in which case we can substitute the leading order gravitational equations into the above expression. That is, with $R_{\mu\nu} = -4g_{\mu\nu}/L^2 + O(\lambda_i)$, eq. (7.1) reduces to

$$S_{HEE} = \frac{2\pi}{\ell_p^3} \int_m d^3x \sqrt{h} \left[ 1 + 8(\lambda_1 - \lambda_2 - 5\lambda_3) + 2L^2 \lambda_1 \mathcal{R} \right] + O(\lambda_i^2)$$

(7.2)

Hence there is also a sense that, within the perturbative framework, the functional determining the holographic EE only depends on the intrinsic geometry of the surface. Of course, the final expression would be slightly more complicated if the bulk theory coupled the gravitational theory (6.1) to various matter fields. Then it appears that simplifying with the gravitational equations of motion would introduce matter field terms into eq. (7.2). In any event, it seems that further explorations will be needed before a comprehensive prescription for holographic entanglement entropy is uncovered. It may be interesting to explore these issues with the physically reasonable theories constructed in [20] with cubic curvature interactions.

As discussed in section 2.3, the universal term in the EE of a CFT can be determined in terms of the central charges using an analysis involving the trace anomaly [2, 15] — see also discussion in [20]. The results of this analysis formed the basis of our consistency tests for various prescriptions for holographic EE. However, the CFT analysis can only be applied in situations where there is a rotational symmetry in the transverse space about the entangling surface $\Sigma$. Hence it is known that this analysis does not capture any of the contributions involving the extrinsic curvature [15].
However, in section 5.2, we found however that there can also be various terms just involving the bulk curvature which are also missed in this analysis. The new terms (5.35) which we found there correct the standard result (2.27) for $d = 6$. However, we expect that there will be similar corrections involving only bulk curvatures for any $d \geq 6$. It seems that this was not a problem in $d = 4$ simply because the low dimension limits the number of conformal invariants [15]. Of course, there will also be a variety of further corrections involving extrinsic curvatures to our results in $d = 6$ or for higher dimensions. In any event, our results highlight the necessity of a rotational symmetry about the entangling surface to apply the analysis of [2, 15]. It is incorrect to describe the necessary requirement as saying the extrinsic curvatures must vanish, as is commonly done.

In the holographic framework, when the entangling surface $\Sigma$ has a rotational symmetry boundary, this typically extends to a symmetry about a bulk surface $m_\Sigma$. The latter then naturally becomes the extremal surface in calculating the holographic EE. In such a situation, it also appears that upon analytically continuing back to Minkowski signature, the rotational symmetry will become a Killing symmetry, but further that $m_\Sigma$ becomes the bifurcation of a Killing horizon in the new Minkowski signature spacetime. That is, the resulting bulk geometry has the structure of a black hole. One obstruction to the latter may be if somehow a naked singularity appears along $m_\Sigma$. Another interesting situation, which appears in [20, 11], is when the rotational symmetry only appears after a conformal transformation of the boundary geometry. Of course, the rotational symmetry is only a requirement of a specific CFT analysis [2, 15] and one can not expect that such symmetry arises for a generic entangling surface. Hence, more generally, it would be useful to develop a better understanding of the geometry of the extremal bulk surface appearing in the calculation of holographic EE, perhaps along the lines of [51].

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A. Fefferman-Graham expansion

In a holographic framework, as long as the boundary field theory is conformal in the UV, the dual geometry will approach AdS asymptotically. In this context, the bulk spacetime will admit a Fefferman-Graham expansion as follows

\[ ds^2 = \frac{\tilde{L}^2}{4} d\rho^2 + \frac{1}{\rho} g_{ij}(x, \rho) \, dx^i \, dx^j, \quad (A.1) \]

where \( g_{ij} \) admits a Taylor series expansion in the radial coordinate \( \rho \)

\[ g_{ij}(x, \rho) = g_{ij}^{(0)}(x^i) + \rho \, g_{ij}^{(1)}(x^i) + \rho^2 \, g_{ij}^{(2)}(x^i) + \cdots. \quad (A.2) \]

The leading term \( g_{ij}^{(0)} \) is identified with the boundary CFT metric. The subsequent coefficients \( g_{ij}^{(n)} \) can be determined in terms of \( g_{ij}^{(0)} \) order by order by expanding the gravitational equations of motion — although, for even \( d \), this expansion breaks down at order \( n = d/2 \) where an additional logarithmic term appears.

It was shown in \([34]\) that these coefficients \( g_{ij}^{(n)} \) are largely fixed by conformal symmetries at the boundary, up to some small ambiguity that must be fixed by the equations of motion. This procedure applies for all \( n < d/2 \) for either odd or even \( d \). Specifically, \( g_{ij}^{(1)} \) and \( g_{ij}^{(2)} \) for arbitrary \( g_{ij}^{(0)} \) are given by \([34]\)

\[ g_{ij}^{(1)} = -\frac{\tilde{L}^2}{d-2} \left( R_{ij} - \frac{g_{ij}^{(0)}}{2(d-2)} R \right) \]

\[ g_{ij}^{(2)} = \tilde{L}^4 \left( k_1 C_{mnkl} C^{mnkl} g_{ij}^{(0)} + k_2 C_{iklm} C_{j}^{k} \right) + \frac{1}{d-4} \left[ \frac{1}{8(d-1)} \nabla_i \nabla_j R - \frac{1}{4(d-2)} \Box R_{ij} + \frac{1}{8(d-1)(d-2)} \Box R g_{ij}^{(0)} \right] \]

\[ - \frac{1}{2(d-2)} R^{kl} R_{ikjl} + \frac{d-4}{2(d-2)^2} R_i^k R_{jk} + \frac{1}{(d-1)(d-2)^2} R R_{ij} \]

\[ + \frac{1}{4(d-2)^2} R^{kl} R_{kl} g_{ij}^{(0)} - \frac{3d}{16(d-1)^2(d-2)^2} R^2 g_{ij}^{(0)} \], \quad (A.3) \]

where the curvature tensors are evaluated for the boundary metric \( g_{ij}^{(0)} \). The constant coefficients \( k_1 \) and \( k_2 \) are precisely the remaining ambiguities (at this order) which

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18 The power series expansion in \( \rho \) can be altered in the back-reaction from other nontrivial fields. A simple example would be when the boundary CFT is deformed by a relevant operator — e.g., see \([52, 53]\).
cannot be determined from symmetries alone. For the cubic Lovelock gravity theory (5.1) in seven dimensions, they are determined to be

\[ k_1 = \frac{f_\infty \lambda + 3 f_\infty^2 \mu}{160(1 - 2 f_\infty \lambda - 3 f_\infty^2 \mu)}, \quad k_2 = -\frac{f_\infty \lambda + 3 f_\infty^2 \mu}{24(1 - 2 f_\infty \lambda - 3 f_\infty^2 \mu)}. \]  

(A.4)

These results (A.3) are all that is needed to extract the logarithmic divergent term in the holographic entanglement entropy for the Lovelock theory in section 5.

**B. Curved boundaries**

In the following, we consider the cubic Lovelock theory in seven (bulk) dimensions, with the action in eq. (5.1). The (vacuum) equations of motion are given by

\[
R_{\mu\nu} + \frac{L^2}{6} \lambda (R_{\mu\rho\sigma\tau} R_{\nu}^{\rho\sigma\tau} - 2 R_{\mu\rho} R_{\nu}^{\rho} - 2 R_{\mu\rho\sigma} R^{\rho\sigma} + R R_{\mu\nu}) \quad (B.1)
\]

\[-\frac{L^4}{8} \mu \left( R_{\mu\nu} R^2 - 4 R_{\mu \nu} R^{\rho\sigma} R_{\rho \sigma} + R_{\mu \nu} R_{\rho \sigma \tau \chi} R^{\rho \sigma \tau \chi} - 4 R_{\mu \rho \sigma} R^{\rho \sigma} R + 8 R_{\mu \rho \sigma} R^{\rho \chi \sigma \omega} R_{\chi \omega} \right.
\]

\[+ 8 R_{\mu \rho \sigma \tau} R^{\rho \sigma \tau} R_{\chi \omega} - 4 R_{\mu \rho \sigma} R^{\rho \tau} R_{\chi \omega} - 4 R_{\mu \rho \sigma} R^{\rho \tau \omega} \chi \omega - 4 R_{\mu \rho \sigma} R^{\rho \chi \tau \omega} R + 8 R_{\mu \rho \sigma \chi} R^{\rho \tau \omega} \chi \omega
\]

\[+ 4 R_{\mu \sigma \tau \omega} R^{\tau \omega \rho} R_{\rho} + 2 R_{\mu \sigma \tau \omega} R^{\tau \omega \rho} R_{\rho} - 4 R_{\mu \sigma \tau \omega} R^{\tau \omega \rho} R_{\rho} + 4 R_{\nu \sigma \tau \omega} R^{\tau \omega \rho} R_{\rho} + 4 R_{\nu \sigma \tau \omega} R^{\tau \omega \rho} R_{\rho}
\]

\[+ 2 R_{\mu \tau \rho \sigma} R^{\tau \omega \rho} R_{\chi \omega} + 8 R_{\mu \rho \sigma} R^{\rho \sigma} - 8 R_{\mu \sigma \tau \omega} R^{\tau \omega \rho} R_{\rho} - 8 R_{\mu \sigma \tau \omega} R_{\nu}^{\tau \omega \rho} R^{\rho} + 8 R_{\nu \sigma \tau \omega} R^{\tau \omega \rho} R_{\rho}
\]

\[-8 R_{\mu \sigma \tau \omega} R^{\tau \omega \rho} R_{\nu}^{\rho} R^{\omega \rho} \chi) - \frac{1}{2} g_{\mu \nu} \left( \frac{30}{L^2} + R + \frac{L^2}{12} \lambda \mathcal{L}_4 - \frac{L^4}{24} \mu \mathcal{L}_6 \right) = 0 .
\]

The above equations can be found in many places in the literature, e.g., see [54].

In sections 5.1 and 5.2, we wish to study asymptotically AdS$_7$ solutions where the boundary metric is not conformally flat. The simplest approach is to construct these solutions using the Fefferman-Graham expansion near the AdS boundary, as in appendix A. In eq. (A.3), we provide explicit formulae for $^{(1)\ ij}_j$ and $^{(2)\ ij}_j$, constructed for a given boundary metric $^{(0)\ ij}_j$. Instead we produced our results here by explicitly solving the equations of motion (B.1), order by order in the asymptotic expansion. For the examples considered in sections 5.1 and 5.2, we found:

**a) $R \times S^2 \times S^3$:**

Consider the following metric ansatz:

\[
ds^2 = \frac{\tilde{L}^2}{z^2} \left( dz^2 + f_1(z) dt^2 + f_2(z) R_z^2 d\Omega_2^2 + f_3(z) R_i^2 d\Omega_3^2 \right),
\]

(B.2)
where $d\Omega_2^2$ and $d\Omega_3^2$ are standard round metrics on a unit two-sphere and three-sphere, respectively. We expand around the asymptotic boundary with

$$f_i(z) = 1 + \sum_{j=1}^{\infty} k_{i,j} z^{2j}.$$  \hfill (B.3)

Now aided by the appropriate computer software, we solve the equations of motion (B.1) order by order in our expansion in powers of $z^2$. To leading order, we find the familiar expression

$$1 - f_\infty + \lambda f_\infty^2 + \mu f_\infty^3 = 0.$$  \hfill (B.4)

At second order, we find:

$$k_{1,1} = \frac{3 R_1^2 + R_3^2}{20 R_1^2 R_3^2}, \quad k_{2,1} = \frac{4 R_1^2 - 3 R_3^2}{20 R_1^2 R_3^2}, \quad k_{3,1} = \frac{R_1^2 - 7 R_3^2}{20 R_1^2 R_3^2}.$$  \hfill (B.5)

At the next order, the coefficients can be expressed as:

$$k_{1,2} = \frac{2 R_1^4 (8 - 9 f_\infty \lambda - 3 f_\infty^2 \mu) - 2 R_1^2 R_3^2 (27 - 86 f_\infty \lambda - 177 f_\infty^2 \mu) + R_3^4 (69 - 142 f_\infty \lambda - 219 f_\infty^2 \mu)}{1600 (1 - 2 f_\infty \lambda - 3 f_\infty^2 \mu) R_1^4 R_3^4},$$

$$k_{2,2} = \frac{R_3^4 (69 - 142 f_\infty \lambda - 219 f_\infty^2 \mu) + 6 R_1^2 R_3^2 (1 - 8 f_\infty \lambda - 21 f_\infty^2 \mu) - 2 R_3^4 (7 + 4 f_\infty \lambda + 33 f_\infty^2 \mu)}{1600 (1 - 2 f_\infty \lambda - 3 f_\infty^2 \mu) R_1^4 R_3^4},$$

$$k_{3,2} = \frac{6 R_1^4 (8 - 9 f_\infty \lambda - 3 f_\infty^2 \mu) - 2 R_1^2 R_3^2 (21 - 38 f_\infty \lambda - 51 f_\infty^2 \mu) - R_3^4 (33 - 54 f_\infty \lambda - 63 f_\infty^2 \mu)}{4800 (1 - 2 f_\infty \lambda - 3 f_\infty^2 \mu) R_1^4 R_3^4}. \hfill (B.6)$$

b) $R^3 \times S^3$:

Consider the following metric ansatz:

$$ds^2 = \frac{L^2}{z^2} \left( dz^2 + f_1(z) \left( dt^2 + dx^2 + dy^2 \right) + f_3(z) R_1^2 d\Omega_3^2 \right),$$  \hfill (B.7)

where $d\Omega_3^2$ is the standard round metric on a unit three-sphere. We expand around the asymptotic boundary with the same expressions as in eq. (B.3) and solve the equations (B.1) order by order in our expansion in powers of $z^2$. As expected, to leading order, we again recover eq. (B.4). At second order, we find:

$$k_{1,1} = \frac{3}{20 R_1^2}, \quad k_{3,1} = -\frac{7}{20 R_1^2}. \hfill (B.8)$$

At the next order, the coefficients can be expressed as:

$$k_{1,2} = \frac{-69 + 142 f_\infty \lambda + 219 f_\infty^2 \mu}{1600 (-1 + 2 f_\infty \lambda + 3 f_\infty^2 \mu) R_1^4}, \quad k_{3,2} = \frac{11 - 18 f_\infty \lambda - 21 f_\infty^2 \mu}{1600 (-1 + 2 f_\infty \lambda + 3 f_\infty^2 \mu) R_1^4}. \hfill (B.9)$$
e) $R^2 \times S^4$:

Consider the following metric ansatz:

$$ds^2 = \frac{L^2}{z^2} \left( dz^2 + f_1(z) \left( dt^2 + dx^2 \right) + f_2(z) R_1^2 \, d\Omega_4^2 \right),$$

(B.10)

where $d\Omega_4^2$ is the standard round metric on a unit four-sphere. We proceed as above solving the equations (B.1) order by order. To leading order, we again recover eq. (B.4). At second order, we find:

$$k_{1,1} = \frac{3}{10} R_1^2, \quad k_{2,1} = -\frac{9}{20} R_1^2.$$  

(B.11)

At the next order, the coefficients can be expressed as:

$$k_{1,2} = \frac{9(-7 + 16 f_\infty \lambda + 27 f_\infty^2 \mu)}{800(-1 + 2 f_\infty \lambda + 3 f_\infty^2 \mu) R_1^4}, \quad k_{2,2} = \frac{-18 + 29 f_\infty \lambda + 33 f_\infty^2 \mu}{800(-1 + 2 f_\infty \lambda + 3 f_\infty^2 \mu) R_1^4}.$$  

(B.12)

d) $S^3 \times S^3$:

Consider the following metric ansatz:

$$ds^2 = \frac{L^2}{z^2} \left( dz^2 + f_1(z) R_1^2 \, d\Omega_3^2 + f_2(z) R_2^2 \, d\Omega_3^2 \right),$$

(B.13)

where $d\Omega_3^2$ is the standard round metric on a unit three-sphere. Proceeding as above yields eq. (B.4) to leading order, whereas at second order, we obtain

$$k_{1,1} = \frac{3 R_1^2 - 7 R_2^2}{20 R_1^2 R_2^2}, \quad k_{2,1} = \frac{3 R_2^2 - 7 R_1^2}{20 R_1^2 R_2^2}.$$  

(B.14)

At third order, we have

$$k_{1,2} = \frac{R_1^4 (11 - 18 f_\infty \lambda - 21 f_\infty^2 \mu) + 2 R_2^2 R_1^2 (21 - 38 f_\infty \lambda - 51 f_\infty^2 \mu) - R_1^4 (69 - 142 f_\infty \lambda - 219 f_\infty^2 \mu)}{1600 (-1 + 2 f_\infty \lambda + 3 f_\infty^2 \mu) R_1^4 R_2^4},$$

$$k_{2,2} = \frac{R_1^4 (11 - 18 f_\infty \lambda - 21 f_\infty^2 \mu) + 2 R_2^2 R_1^2 (21 - 38 f_\infty \lambda - 51 f_\infty^2 \mu) - R_2^4 (69 - 142 f_\infty \lambda - 219 f_\infty^2 \mu)}{1600 (-1 + 2 f_\infty \lambda + 3 f_\infty^2 \mu) R_1^4 R_2^4}.$$

Note that if one trades, e.g., $S^2 \times S^3$ for $H^2 \times H^3$, the signs of all the curvatures are reversed. Hence the contributions to the six-dimensional trace anomaly in the boundary theory are simply flipped and so we do not expect to get a distinct test of our holographic entanglement entropy for Lovelock gravity. Other simple boundary manifolds which should give distinct results for the trace anomaly include: $R^4 \times S^2$, $R^2 \times (S^2)^2$, $R^2 \times S^2 \times H^2$. 

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C. EE in the GB gravity

In this appendix we consider a $d$-dimensional boundary CFT dual to a GB gravity. The EE is investigated in the case when the entangling surface is either a sphere or a cylinder. Thus the number of terms in eq. (2.1) is restricted to $p_{\text{max}} = 2$

$$I = \frac{1}{2\ell_{p-1}^d} \int d^{d+1}x \sqrt{-g} \left[ \frac{d(d-1)}{L^2} + R + \frac{L^2 \lambda}{(d-2)(d-3)} L_4 \right], \quad (C.1)$$

and hence eq. (2.11) reduces to

$$S_{JM} = \frac{2\pi}{\ell_{p-1}^d} \int m d^{d-1}\sqrt{h} \left[ 1 + \frac{2 L^2 \lambda}{(d-2)(d-3)} R \right] + \frac{8\pi}{\ell_{p-1}^{d-1}} \frac{L^2 \lambda}{(d-2)(d-3)} \int_{\partial M} K \quad (C.2)$$

The $AdS_{d+1}$ metric

$$ds^2 = \frac{\bar{L}^2}{z^2} (dz^2 - dt^2 + \sum_{i=1}^{d-1} dx_i^2) \quad (C.3)$$

is an exact solution of the equations of motion in the GB gravity. We introduce a short distance cut-off in the boundary CFT here by setting a minimum value of $z$: $z = z_{\text{UV}} = \delta$. In what follows, we choose either $\sum_i dx_i^2 = dv^2 + r^2 d\Omega_{d-2}^2$ or $\sum_i dx_i^2 = dv^2 + dr^2 + r^2 d\Omega_{d-3}^2$ when $V$ corresponds to a ball, $A_D := \{x_i| r \leq R\}$ or a solid cylinder, $A_C = \{x_i| r \leq R\}$, respectively. Hence, the induced metric on the static minimal surface in the $AdS_{d+1}$ bounded by either $\partial A_D$ or $\partial A_C$, is given respectively by

$$h_D^{ab} dx^a dx^b = \frac{\bar{L}^2}{z^2} [(\dot{r}^2 + \dot{z}^2) du^2 + r^2 d\Omega_{d-2}^2] \quad (C.4)$$

and

$$h_C^{ab} dx^a dx^b = \frac{\bar{L}^2}{z^2} [(\dot{r}^2 + \dot{z}^2) du^2 + dv^2 + r^2 d\Omega_{d-3}^2] \quad (C.5)$$

where $u$ parametrizes the minimal surface in the $(z,r)$ plane and dot denotes the derivative with respect to $u$. Both expressions are of the form

$$ds^2 = ds_{X}^2 + \sum_i e^{2F_i} ds_{Y_i}^2 \quad (C.6)$$

where the conformal factors depend on the $x$ coordinates only, $F_i = F_i(x)$. In this case, one can conveniently decompose the curvature tensor and the associated scalars in terms of $F_i$ fields and the curvature tensor of the $X$ space. The related useful formulae are summarized in appendix D.
C.1 EE for a sphere with general \( d \)

In this case (comparing eqs. (C.4) and (C.6)) we identify a one-dimensional space along the \( u \)-direction with \( X \) of (C.6) and the \((d - 2)\)-dimensional sphere with radius \( \tilde{L} \) is identified with \( Y \), whereas

\[
e^{2F} = \frac{r^2}{z^2} \quad \Rightarrow \quad F = \ln(r/z) . \tag{C.7}
\]

In particular,

\[
R_{abcd}^Y = \frac{1}{\tilde{L}^2}(g_{ab}g_{cd} - g_{ad}g_{bc}) \quad \Rightarrow \quad R_{ab}^Y = \frac{d - 3}{\tilde{L}^2}g_{ab} , \quad R^Y = \frac{(d - 2)(d - 3)}{\tilde{L}^2} , \tag{C.8}
\]

where \( g_{ab}^Y \) is the metric of the unit \((d - 2)\)-dimensional sphere. Using (D.6) yields

\[
R^D = e^{-2F} \left( \frac{(d - 2)(d - 3)}{\tilde{L}^2} - 2(d - 2)\Delta_X(F) - (d - 2)(d - 1)h_{uu}^u \right)^2 , \tag{C.9}
\]

where as before ‘dot’denotes the derivative with respect to \( u \), and

\[
h_{uu} = (h_{uu})^{-1} = \frac{\tilde{L}^2}{z^2}(r^2 + z^2) \quad \Rightarrow \quad \Delta_X(F) = \frac{1}{\sqrt{h_{uu}}} \partial_u \left( \sqrt{h_{uu}} h_{uu}^u \right) . \tag{C.10}
\]

Let us evaluate now the extrinsic curvature \( \mathcal{K} \). The normal outward unit vector to the boundary surface defined by \( u = u_i \), or equivalently by \( r(u_i) = R, z(u_i) = \delta \), is given by

\[
n_a = -\sqrt{h_{uu}} \delta_{uu} , \tag{C.11}
\]

where \( a \) runs over \( u \) and a \((d - 2)\)-dimensional sphere, thus \((i, k \) below run over the \((d - 2)\)-dimensional sphere only)

\[
\mathcal{K} = h_{ab}^u \partial_a n_b |_{u = u_i} = \left[ h_{uu}^u \partial_u n_u + g_{ik} \partial_i n_k \right] |_{u = u_i} = -\left( \frac{(d - 2)}{\sqrt{h_{uu}}} e^{(d - 2)F} \right) |_{u = u_i} , \tag{C.12}
\]

where we used \( 2(\det h_{ab})^{-1/2}\partial_u \sqrt{\det h_{ab}} = h_{ab}^u \partial_u h_{ab} \). As a result, we get

\[
\int_{\partial m} \mathcal{K} = -\tilde{L}^{d-2} S_{d-2} h_{uu}^{-1/2} \partial_u e^{(d - 2)F} |_{u = u_i} . \tag{C.13}
\]

Having these results at hand, one can show that the minimal surface of (C.2), can be conveniently parameterized as follows

\[
r(u) = R \cos(u/R) , \quad z(u) = R \sin(u/R) , \quad \text{where} \quad \delta \leq u \leq \frac{\pi}{2} R . \tag{C.14}
\]
Here we need to introduce the ratio of \( u \) to some scale in the argument of the trigonometric functions above in order to maintain the correct dimensions. We chose \( R \) as the natural scale, however, any other scale can be used instead. Let us proceed by substituting (C.9) into (C.2) and integrating by parts

\[
S_{JM} = -\frac{8\pi}{\ell_p^{d-1}} \frac{L^{d-2} S_{d-2} \lambda}{(d-2)(d-3)} \tilde{L}^2 \left[ h_{uu}^{-1/2} \partial_u e^{(d-2)F} \right]_{u_i}^{u_f} + \frac{2\pi}{\ell_p^{d-1}} L^{d-2} S_{d-2} \int du \sqrt{h_{uu} e^{(d-2)F}} \left( 1 + 2 (\tilde{L}/L) \lambda \left[ e^{-2F} + L^2 h_{uu} \right] \right)
\]

According to (C.13) the Gibbons-Hawking term precisely cancels the boundary contribution which corresponds to the lower bound of the first term in the above expression (the upper bound vanishes due to symmetry). Substituting now the general parametrization

\[
r(u) = f(u/R) \cos(u/R) , \quad z(u) = f(u/R) \sin(u/R) , \quad \delta \leq u \leq \frac{\pi}{2} R ,
\]

yields (with \( x = u/R \))

\[
S_{JM} = \frac{2\pi}{\ell_p^{d-1}} \frac{L^{d-2} S_{d-2}}{(d-2)(d-3)} \int_{\delta/R}^{\pi/2} dx \frac{\cos(x)^{d-2}}{\sin(x)^{d-1}} \sqrt{1 + \left( \frac{d \ln f}{dx} \right)^2}
\]

\[
\times \left( 1 + 2 (\tilde{L}/L)^2 \lambda \left[ \tan^2(x) + \frac{\cos^{-2}(x)}{1 + \left( \frac{d \ln f}{dx} \right)^2} \right] \right)
\]

Since the integrand depends on ‘time’ \( x \) and on the square of the ‘velocity’ \( v(x) := d \ln f/dx \), the corresponding Euler-Lagrange equation is

\[
\frac{d}{dx} \left( \frac{\partial L(x, v(x))}{\partial v(x)} \right) = 0 ,
\]

and it admits the solution \( v = 0 \Leftrightarrow f = \text{const} \). Plugging (C.14) into (C.9), yields

\[
R_D = -\frac{(d-1)(d-2)}{\tilde{L}^2} .
\]

Substituting this result back into (C.2), we finally obtain

\[
S_{JM} = \frac{2\pi}{\ell_p^{d-1}} \left[ 1 - 2 \frac{d-1}{d-3} \right] \int \sqrt{\det h^D_{ab}} + \frac{8\pi}{\ell_p^{d-1}} \frac{\tilde{L}^2 \lambda}{(d-2)(d-3)} \int \partial_m \mathcal{K} ,
\]

\[
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\]
where the area of the surface is given by

$$\int_m d^{d-1} x \sqrt{\det h^D_{ab}} = \tilde{L}^{d-1} S_{d-2} \int_{\delta/R}^1 dy \frac{(1 - y^2)^{d-3}/y^{d-1}}{\Gamma(d/2)} \log(\delta/R) + \cdots$$

(C.21)

where $S_{d-2}$ is the area of the unit $(d-2)$-dimensional sphere and in the last equality we assumed that $d$ is even, since only in that case does the integral contains a subleading logarithmic divergence, which can be evaluated by expanding the integrand in powers of $y$. The rest of the terms are encoded in ellipsis. As a result, the universal term (for even $d$) in the EE is given by

$$S_{JM} = (-)^{d/2-1} \frac{4\pi^{d/2}}{\Gamma(d/2)} \tilde{L}^{d-1} \left[ 1 - 2 \frac{d - 1}{d - 3} \int_0^{\infty} \lambda \right] \log(l/\delta) + \cdots.$$  \hspace{1cm} (C.22)

Now comparing this result with eq. (2.6), we recognize that the pre-factor is proportional to $A$. In fact, our result here matches that in [11].

**C.2 Spherical entangling surfaces beyond GB gravity.**

It was shown in [11] that when the entangling surface corresponds to a sphere, the universal term in the EE will be always proportional to the $A$-type anomaly for even $d$. Therefore inclusion of the higher order interactions (2.2) in the Lovelock gravity will reconstruct (2.6) term by term. To illustrate this idea, let us do one step beyond the GB interaction by taking $p_{\text{max}} = 3$ in (2.1)

$$I = \frac{1}{2\ell^d_{p-1}} \int d^{d+1} x \sqrt{-g} \left[ \frac{d(d - 1)}{L^2} + R + \frac{L^2 \lambda}{(d - 2)(d - 3)} L_4 \right. \\
+ \frac{L^4 \mu}{(d - 2)(d - 3)(d - 4)(d - 5)} L_6 \left. + \cdots \right],$$  \hspace{1cm} (C.23)

then (2.11) becomes

$$S_{JM} = \frac{2\pi}{\ell^d_{p-1}} \left[ 1 + \frac{2}{(d - 2)(d - 3)} R + \frac{3L^4 \mu}{(d - 2)(d - 3)(d - 4)(d - 5)} \mathcal{L}_4(\mathcal{R}) + \cdots \right].$$  \hspace{1cm} (C.24)

The ellipsis denotes the surface terms [23], which are suppressed since they do not contribute to the universal divergence explored in what follows.
Using (D.7) one finds

$$R_{D \mu \nu \rho \sigma} R_{D}^{\mu \nu \rho \sigma} = \frac{2}{(d-2)(d-3)} \left[ e^{-2F} R^Y - (d-2)(d-3)h^{uu} \dot{F}^2 \right]^2 + 4(d-2) \left[ \Delta_X F + h^{uu} \dot{F}^2 \right]^2,$$

$$R_{D \mu \nu} R_{D}^{\mu \nu} = \frac{1}{d-2} \left[ e^{-2F} R^Y - (d-2)(\Delta_X F + (d-2)h^{uu} \dot{F}^2) \right]^2 + (d-2)^2 \left[ \Delta_X F + h^{uu} \dot{F}^2 \right]^2,$$

(C.25)

Substituting (C.14) into these expressions, yields\(^{19}\)

$$R_{D \mu \nu \rho \sigma} R_{D}^{\mu \nu \rho \sigma} = \frac{2(d-1)(d-2)}{L^4},$$

$$R_{D \mu \nu} R_{D}^{\mu \nu} = \frac{(d-1)(d-2)^2}{L^4}. \quad \text{(C.26)}$$

Now combining the latter with (C.19), we get

$$\mathcal{L}_4 = R_{D \mu \nu \rho \sigma} R_{D}^{\mu \nu \rho \sigma} - 4R_{D \mu \nu} R_{D}^{\mu \nu} + R_{D}^2 = \frac{(d-1)(d-2)(d-3)(d-4)}{L^4}. \quad \text{(C.27)}$$

Plugging all the above into (C.24), we finally obtain

$$S_{JM} = \frac{2\pi}{L^{d-1}} \left[ 1 - 2 \frac{d-1}{d-3} f_\infty \lambda + 3 \frac{d-1}{d-5} f_\infty^2 \mu \right] \int_m d^{d-1}x \sqrt{\det h_{ab}^D + \ldots} \quad \text{(C.28)}$$

Substituting (C.21) we recover (2.14) where \(A\) is given by (2.6) with \(p_{\text{max}} = 3\). In fact again, this result for the cubic Lovelock theory matches precisely with the expression derived in [11] for an arbitrary higher curvature theory.

C.3 EE for a cylinder with general \(d\)

In the case of (C.5), we identify a one dimensional space along \(u\)-direction with \(X\) of (C.6), whereas a one dimensional space along \(v\)-direction and a \((d-3)\)-dimensional sphere with radius \(L\) are identified with \(Y_1\) and \(Y_2\) respectively. Hence,

$$e^{2F_1} = \frac{L^2}{z^2} \quad \Rightarrow \quad F_1 = \ln(\frac{L}{z}),$$

$$e^{2F_2} = \frac{r^2}{z^2} \quad \Rightarrow \quad F_2 = \ln(\frac{r}{z}). \quad \text{(C.29)}$$

\(^{19}\)One can extend the argument presented in the case of GB gravity to prove that (C.14) minimizes (C.24).
Substituting into (C.2), yields

\[ R_C = e^{-2F_2} \frac{(d-3)(d-4)}{L^2} - 2\Delta_X (F_1) - 2(d-3)\Delta_X (F_2) - 2h^{uu} \hat{F}_1^2 \\
- (d-3)(d-2)h^{uu} \hat{F}_2^2 - 2(d-3)h^{uu} \hat{F}_1 \hat{F}_2 \quad \text{(C.30)} \]

where dot denotes derivative with respect to \( u \) and

\[ h_{uu} = (h^{uu})^{-1} = \frac{\tilde{L}^2}{z^2} (\tilde{r}^2 + \tilde{z}^2) \Rightarrow \Delta_X (F_i) = \frac{1}{\sqrt{h_{uu}}} \partial_u (\sqrt{h_{uu}} h^{uu} \hat{F}_i ) \quad \text{(C.31)} \]

Next we evaluate the extrinsic curvature \( K \). The normal outward unit vector to the boundary surface defined by \( u = u_i \), or equivalently by \( r(u_i) = R, z(u_i) = \delta \), is given by

\[ n_a = -\sqrt{h_{uu}} \delta_{ua} \quad \text{(C.32)} \]

where \( a \) runs over \( u, v \) and a \( (d-3) \)-dimensional sphere, thus \( (i, k \below run over the \( (d-3) \)-dimensional sphere only)

\[ K = g^{ab} \nabla_a n_b |_{u=u_i} = \left[ h^{uu} \nabla_u n_u + g^{uv} \nabla_v n_u + g^{ik} \nabla_i n_k \right]_{u=u_i} = -\frac{\partial_u e^{F_1 + (d-3)F_2}}{\sqrt{h_{uu}}} \bigg|_{u=u_i}, \quad \text{(C.33)} \]

where we used \( 2(\det h_{ab})^{-1/2} \partial_u \sqrt{\det h_{ab}} = h^{ab} \partial_u h_{ab} \). Thus

\[ \int_{\partial m} K = -\tilde{L}^{d-3} S_{d-3} H \ h_{uu}^{-1/2} \partial_u e^{F_1 + (d-3)F_2} \bigg|_{u_i} \quad \text{(C.34)} \]

where \( H = \int dv \) is the length of the cylinder.

Substituting (C.30) into (C.2) and integrating by parts, yields

\[ S = -\frac{8\pi \tilde{L}^{d-3} S_{d-3} H \lambda}{\ell_p^{d-1}} \left\frac{L^2}{(d-2)(d-3)} \right\ h_{uu}^{-1/2} \partial_u e^{F_1 + (d-3)F_2} \bigg|_{u_i} + \frac{2\pi \tilde{L}^{d-3} S_{d-3} H}{\ell_p^{d-1}} \int du \sqrt{h_{uu}} \]

\[ \times e^{F_1 + (d-3)F_2} \left( 1 + \frac{2(L/\tilde{L})^2 \lambda}{(d-2)} \left[ (d-4)e^{-2F_2} + 2h^{uu} \tilde{L}^2 \hat{F}_1 \hat{F}_2 + (d-4) \tilde{L}^2 h^{uu} \hat{F}_2^2 \right] \right) \]

\[ + \frac{8\pi}{\ell_p^{d-1}} \left( \frac{\tilde{L}^2 \lambda}{(d-2)(d-3)} \int_{\partial m} K \right). \quad \text{(C.35)} \]

According to (C.34), the Gibbons-Hawking term cancels the lower bound of the first term in the above expression. In contrast to the case of the ball, we did not succeed to find a closed analytic expression for the surface which minimizes (C.35) in general \( d \). However, to evaluate the universal divergence, one needs to know the asymptotic expansion of such a surface to order which depends on \( d \). Therefore to proceed further, we must pick a particular value, \( e.g., d = 4 \), for the dimension of the boundary CFT. We illustrate such computation in section 4.2.
D. Curvature tensor for a warped geometry

In this appendix, we derive some results which are useful to evaluate $S_{βi}$ in appendix C. In particular, we determine the general expression for the Riemann curvature tensor $R_{αβγδ}$, Ricci tensor $R_{βδ}$ and Ricci scalar $R$ for a warped geometry of the form:

$$ds^2 = ds_X^2 + \sum_i e^{2F_i} ds_{Y_i}^2$$  \hspace{1cm} (D.1)

where the conformal factors depend on the $x$ coordinates in the base $X$, i.e., $F_i = F_i(x)$. Our convention for the curvature (which matches [37]) is

$$R_{αβγδ} = \frac{1}{2}(g_{αδ,γβ} + g_{βγ,αδ} - g_{αγ,βδ} - g_{βδ,αγ}) + g_{μν}(Γ_μ^{γ},γ,αδ - Γ_μ^{γ,βδ}Γ_μ^{αγ}) \cdot \hspace{1cm} (D.2)$$

In what follows Greek letters $α$, $β$, $γ$, ... run over the base space $X$, whereas Greek letters with a subscript $α_i$, $β_i$, $γ_i$, ... run over directions in the fibre spaces $Y_i$. The nonvanishing components of the Christoffel symbol are given by

$$Γ_{α,βγ} = Γ^X_{α,βγ}, \hspace{0.5cm} Γ_{α_i,β_γ} = e^{2F_i}Γ_{α_i,β_γ}^Y, \hspace{0.5cm} Γ_{α_i,β_δ} = \partial_γ F_i e^{2F_i}g_{α_i,δ}, \hspace{0.5cm} Γ_{α,β_γ} = -∂_α F_i e^{2F_i}g_{δ,γ},$$  \hspace{1cm} (D.3)

where $Γ_{α,βγ} = g_{αδ}Γ^δ_{βγ}$. Further we introduced a notation where superscript $X$ or $Y_i$ indicates the corresponding quantity, above the Christoffel symbol, is calculated for the metric on the corresponding space. It follows that the non-vanishing components of the curvature tensor are

$$R_{αβγδ} = R^X_{αβγδ},$$

$$R_{αβγδ_i} = -e^{2F_i}(\nabla_α \nabla_γ F_i + \partial_α F_i \partial_γ F_i) g^Y_{βδ_i},$$

$$R_{αβγδ_i} = e^{2F_i}R^{Y_i}_{βγδ_i} + (∂F_i)^2 e^{4F_i}(g_{βδ_i}g^Y_{α,δ} - g^Y_{β,δ}g^Y_{α,δ}),$$

$$R_{αβγδ_i} = - (∂F_i \cdot ∇ F_j) e^{2(F_i + F_j)} g^Y_{βδ_i} g^Y_{α,δ} \text{ with } i \neq j \hspace{1cm} (D.4)$$

where all derivatives are evaluated in the $X$ space and $∇$ denotes the covariant derivative compatible with the metric on $X$. Thus the non-vanishing components of the Ricci tensor are

$$R_{βδ} = R^X_{βδ} - \sum_i d_i(\nabla_β \nabla_δ F_i + \partial_β F_i \partial_δ F_i),$$

$$R_{βδ_i} = R^{Y_i}_{βδ_i} - (∇^2 F_i) e^{2F_i}g^Y_{βδ_i} - e^{2F_i}g^Y_{βδ_i} \sum_j d_j(∂F_i \cdot ∇ F_j) \hspace{1cm} (D.5)$$

where $d_i$ corresponds to the dimension of the space $Y_i$ and the Laplacian $∇^2$ is again evaluated on $X$. Finally, the Ricci scalar is

$$R = R^X + \sum_i [e^{-2F_i}R^Y_i - 2d_i(∇^2 F_i) - d_i(∂F_i)^2] - \sum_{ij} d_id_j(∂F_i \cdot ∇ F_j),$$  \hspace{1cm} (D.6)
where $R_X$, $R_Y^i$ are the Ricci scalars of the spaces $X$ and $Y_i$, respectively. In particular, using these expressions, one finds the following following expressions for various invariants

\[
R_{abcd}R^{abcd} = R_X^{\alpha\beta\gamma\delta} + \sum_i \left[ e^{-4F_i} R_Y^{i\alpha\beta\gamma\delta_i} - 4e^{-2F_i} R_Y^{i}(\partial F_i)^2 \right. \\
\left. + 4 d_i \nabla_\alpha \nabla_\gamma F_i \nabla^\alpha \nabla^\gamma F_i + 8 d_i \nabla_\alpha \nabla_\gamma F_i \partial^\alpha F_i \partial^\gamma F_i - 2d_i(d_i-1)(\partial F_i \cdot \partial F_i)^2 \right] \\
+ 4 \sum_{ij} d_i d_j (\partial F_i \cdot \partial F_j)^2 ,
\]

\[
R_{ab}R^{ab} = R_X^{\beta\delta} + \sum_i \left[ e^{-4F_i} R_Y^{i\beta\delta_i} - 2d_i R_X^{\alpha\beta\gamma\delta} (\nabla^\alpha \nabla^\gamma F_i + \partial^\alpha F_i \partial^\gamma F_i) \right. \\
\left. + d_i (\nabla^2 F_i)^2 - 2e^{-2F_i} R_Y^{i}(\nabla^2 F_i) \right. \\
+ \sum_{ij} \left[ d_i d_j (\nabla_\beta \nabla_\delta F_i + \partial_\beta F_i \partial_\delta F_i) (\nabla^\beta \nabla^\delta F_j + \partial^\beta F_j \partial^\delta F_j) \right] \\
\left. - 2d_j (e^{-2F_j} R_Y^{i} - d_i \nabla^2 F_i)(\partial F_i \cdot \partial F_j) \right] + \sum_{i,j,k} d_i d_j d_k (\partial F_i \cdot \partial F_j)(\partial F_i \cdot \partial F_k)
\]

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