The existence and regularity theory for abstract semi-linear time-fractional evolution equations

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Abstract

In this paper, we investigate abstract time-fractional evolution equations with nonlinear perturbations. We construct solutions of Lipschitz perturbation problems in arbitrary large time interval independent of the Lipschitz constants. We will extend well-known results for standard evolution equations such as the blow-up alternative, to the time-fractional evolution equations. We also prove the differentiability with respect to time of the solution when the perturbation is sufficiently smooth. The differentiability enables us to use the maximum principle. The theory on general Banach spaces enables us to deduce space regularity result easily.

1 Introduction

In this paper, we study the following abstract time-fractional Cauchy problem

\[
\begin{cases}
D_\alpha^\alpha (u - u_0)(t) + Au(t) + B(t, u(t)) = f(t) \text{ in } (0, T], \\
u(0) = u_0
\end{cases}
\]

(1.1)

in a Banach space \(X\). The operator \(D_\alpha^\alpha\) is the Riemann-Liouville fractional differential operator of order \(\alpha \in (0, 1)\) defined by

\[
D_\alpha^\alpha (u)(t) := \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t (t - \tau)^{-\alpha} u(\tau)d\tau,
\]

(1.2)

and \(A : D(A) \to X\) is a generator of some analytic semigroup.

The linear time-fractional evolution equation of the form

\[
\begin{cases}
D_\alpha^\alpha (u - u_0)(t) + Au(t) = f(t) \text{ in } (0, T], \\
u(0) = u_0
\end{cases}
\]

(1.3)

was originally proposed in order to model the so-called anomalous diffusion phenomenon, which is different from the usual diffusion of materials based on Brownian motion ([3], [20]).

There are many studies on the time-fractional Cauchy problem (1.3). In [14], Mu, Ahmad and Huang investigated the existence and regularity of a solution when the external force term \(f\) has the time regularity \(f \in \mathcal{F}^{\tau,\beta}((0, T]; X)\), where \(\mathcal{F}^{\tau,\beta}((0, T]; X)\) is a weighted Hölder continuous space. Luchko [24] established the maximum principle for (1.3), but only under the condition that the solution is time differentiable. Guidetti [7], [8] investigated the maximal regularity properties of (1.3) in an abstract framework. The mild-solution formula plays an important role in the theory of evolution equations, which is shown by [17], [25] to be given by the following,

\[
u(t) = \int_0^\infty h_\alpha(\theta) T((t - \tau)^\alpha) f(\tau)d\tau + \int_0^t \int_0^\infty (t - \tau)^\alpha u_0(\theta) T((t - \tau)^\alpha \theta) f(\tau)d\tau d\theta \]

(1.4)

where \(h_\alpha\) is a probability density function defined on \((0, \infty)\) such that

\[
\int_0^\infty \theta^\gamma h_\alpha(\theta)d\theta = \frac{\Gamma(1 + \gamma)}{\Gamma(1 + \alpha \gamma)} \text{ for all } \gamma \in (-1, \infty),
\]

and \(T(t)\) denotes the semigroup generated by \(A\).
As for semi-linear and nonlinear problems, Zhou and Jiao [24] proved the existence of a local-in-time mild solution with nonlocal conditions,

$$u(0) + g(u) = u_0$$

where $g : C([0, T]; X) \rightarrow X$ is a given function, while they imposed some restrictions on the growth of the perturbations in order to use the Banach fixed-point theorem. Akagi [11] showed the solvability of (1.1) when $X$ is a Hilbert space and $A$ is replaced by subdifferential $\partial \varphi$ for some functional $\varphi$.

There are also some studies on specific models. Inc, Yusuf, Aliyu and Baleanu [19] investigated the one-dimensional time-fractional Allen-Cahn equation and Klein-Gordon equation. Liu, Cheng, Wangb and Zhao [12] performed numerical analysis of time-fractional Allen-Cahn and Cahn-Hilliard phase-field models. These models describe the phase separation process of iron alloys. Zang, Sun [22] concerned with the blow-up and global existence of a solution to the time-fractional Fujita equation. Zhang, Li and Su [23] investigated time-fractional Ginzburg-Landau equation which is used in the superconductivity theory.

It is also important for some approximation problems that the existence time $T > 0$ can be chosen independently of the Lipschitz constant of the nonlinear term $B$. Regularized approximation equations enable us to consider a problem with some singularities. For example, in [18], the authors studied an equation that describes the heterogeneous chemical catalyst with the absorption term $B(u) = u^{-\beta} \chi_{u>0}$ by considering a regular approximation of $B$. In their discussion, it is essential that the solutions of approximation equations are defined in the same interval. Moreover, in the Stefan problem, the so-called penalty method is often used to construct the solution. This method also introduces the approximation equations to obtain a solution of the original problem. For the detail, we refer to [9], [10]. Nowadays there are some studies on Stefan problems in which the time-fractional heat equations is used as the governing equation, such as [16], [21].

These examples motivate us to establish a unified and general theory for the uniform-in-time solvability and regularity of solutions to (1.1). The novelty of this paper is establishing the long-in-time existence of a solution to (1.1) and a regularity of the solution in a unified approach. To this end, we introduce a weighted metric space of $X$-valued curves in such a way that the Banach fixed point theorem works merely with standard semigroup estimates. Moreover, we prove the regularity with respect to time variable $t$ by considering an auxiliary evolution equation that is obtained by formally differentiating (1.1) with respect to $t$. It is proved that a solution to the auxiliary equation turns out to be the derivative of the solution to (1.1) and this yields the desired regularity properties of the solution.

Before stating the main theorems, we define the following weighted Hölder continuous function space.

**Definition 1.1.** For indices $0 < \beta < r < 1$, let $F^{r, \beta}((0, T]; X)$ denotes the function space of all continuous functions $f$ on $(0, T]$ with the following properties:

1. $\lim_{t \to 0^+} t^{-r} f(t)$ exists.
2. $f$ is $\beta$ Hölder continuous with the weight $s^{1-r+\beta}$, that is,

$$\sup_{0 \leq s < t \leq T} \frac{s^{1-r+\beta}\|f(t) - f(s)\|}{(t-s)^{\beta}} < \infty.$$

3. the following holds,

$$\lim_{t \to 0^+} w_f(t) = \lim_{t \to 0^+} \sup_{0 \leq s < t \leq T} \frac{s^{1-r+\beta}\|f(t) - f(s)\|}{(t-s)^{\beta}} = 0.$$

This function space become Banach space with the norm;

$$\|f\|_{F^{r, \beta}} = \sup_{0 \leq t \leq T} t^{1-r}\|f(t)\| + \sup_{0 \leq s < t \leq T} \frac{s^{1-r+\beta}\|f(t) - f(s)\|}{(t-s)^{\beta}}.$$

If there is no danger of confusion, we shall write $F^{r, \beta}$ instead of $F^{r, \beta}((0, T]; X)$. It is obvious that the embedding $F^{r, \beta'} \subset F^{r, \beta}$ holds for $0 < \beta' < \beta < r \leq 1$. When $g \in C^\alpha([0, T]; X)$ satisfies $g(0) = 0$, $f(t) = t^{-1}g(t)$ belongs to $F^{r, \sigma}((0, T]; X)$. Also, for $0 < \beta < r = 1$, $f \in C^{\beta}$ belongs to $F^{1, \beta}$. We refer to [21] in detail.

First of all, we consider the Lipschitz perturbation problem where $B$ satisfies

$$\|B(t, u) - B(t, v)\| \leq L\|u - v\| \text{ for all } u, v \in X, ~ t \geq 0$$

(1.5)
Theorem 1.1. Suppose that (1.5) is satisfied. Let $T, \beta, r$ satisfy $T > 0, 0 < \beta < r \leq 1, 0 < \max(\beta, 1 - \alpha) < r$ and $\beta < \alpha$. Then, for all $f \in F^{r,\beta}((0,T];X), u_0 \in D(A)$, the semi-linear problem ([1.1]) has a unique solution

$$u \in C([0,T];X) \cap C((0,T];D(A)) \cap F^{r,\beta}((0,T];X).$$

Furthermore, the time regularity of $u$ can be improved as follows.

1. In addition, if $u_0 \in D(A^\alpha)$, $f \in C([0,T];X)$ for $\alpha^{-1} \beta \leq q \leq \beta < \alpha$, then $u \in C^\beta([0,T];X)$.

2. In addition, if $u_0 \in D(A)$ and $f \in F^{1,\beta}((0,T];X)$, then $u \in C^\alpha([0,T];X)$.

We briefly mention the time regularity. At first, the additional assumption of $u_0 \in D(A^\alpha)$ deduces the Hölder continuity of the term

$$t \mapsto \int_0^\infty h_\alpha(\theta) T(t^\alpha \theta) u_0 d\theta$$

in ([1.4]). In general, we can only expect the strong continuity of $T(t)$ at $t = 0$. Therefore $u_0$ must have sufficient regularity. Also, roughly speaking, the assumption of $u_0 \in D(A)$ and $f \in F^{1,\beta}((0,T];X)$ means that $D^\rho_\alpha(u - u_0)$ is in $C([0,T];X)$. That is formally, the $\alpha$-order derivative is continuous on $[0,T]$, hence the original function must be in $C([0,T];X)$. We justify this argument in section 3.

Differentiability of the solution is essential for the use of maximum principle. However, to the author’s best knowledge there has been no research on the differentiability in time of the solution in semi-linear problem ([1.1]). We deduce the regularity in time of the solution when $f$ and $B$ are sufficiently regular to use the maximum principle shown by [24] which is useful in the study of the quantitative properties of the solution. We do not assume time dependence of $B$ in this theorem. The condition for indices $r', \beta'$ is required to obtain the well-posedness of the auxiliary equation in the proof.

Theorem 1.2. We assume the following conditions,

1. Time regularity

$$f \in F^{r,\beta}((0,T];X) \cap C^1([0,T];X), \quad f' \in F^{r',\beta'}((0,T];X)$$

where we take $r, \beta$ as in Theorem [1.1] and $r', \beta'$ such that $0 < \beta' < r' \leq 1, 0 < \max(\beta', 1 - \alpha) < r', \beta' < \alpha$ and $\alpha/2 \leq r'$.

2. The nonlinear part $B : X \to X$ is Fréchet differentiable, its derivative $B' : X \to B(X)$ is uniformly bounded, that is, there exists $L > 0$ such that $\|B(u)\|_{B(X)} \leq L$ for all $u \in X$, and $B'$ is also Lipschitz continuous with Lipschitz constant $L' > 0$.

3. The initial condition $Au_0 + B(u_0) - f(0) \in D(A)$ are satisfied.

Then, the solution of the problem ([1.1]) is differentiable with respect to time in $(0,T]$ and $u' \in F^{\alpha-\gamma,\gamma}((0,T];X)$ where

$$0 < \gamma < \alpha/2, \quad \alpha - \gamma \leq r', \quad \gamma \leq \beta'.$$

Also, we organize the general result for blow-up of the solution with locally Lipschitz perturbation in the sense that

$$\|B(t,u) - B(t,v)\| \leq L(\rho)\|u - v\| \quad \text{for all } u, v \in \{w \in X; \|w\| \leq \rho\}$$

where $L : \mathbb{R} \to \mathbb{R}$ is increasing function, and for each $u \in X$, $t \mapsto B(t,u)$ is bounded, with reference to the idea of [22, 23].

Theorem 1.3. Suppose that $B$ satisfies ([1.6]). Then, ([1.1]) has the unique local solution. Furthermore, suppose that $T^* > 0$ is the maximal existence time. Then, the solution satisfies either one of the following:

1. $T^* = \infty$.

2. $T^* < \infty$ and $\lim_{t \to T^*} \|u(t)\| = \infty$.
The outline of this paper is as follows. In section 2, we introduce some notations, function spaces, and some preliminary results which shall be used later. In section 3, we introduce the abstract theory of fractional power of operator, and characterize the time-fractional operator \( T \) by the abstract framework. In section 4, we prove Theorem 1.1. The outline of the proof is as follows. At first, we show that there exists a unique mild solution no matter how large the \( T \) and \( L \) are. The mild solution essentially becomes the usual one for \( (1.4) \) because of the existence theorem given by [14]. In section 5, we prove the differentiability in time of the solution when \( f \) and \( B \) have appropriate regularities. This result is essential to use maximum principle due to [24]. In section 6, we investigate the locally Lipschitz perturbation problems. In the usual evolution equation theory, the property so-called blow-up must holds. The key idea is the extension of solutions via integral equations due to [22], [23]. In section 7, we apply our results to a specific model. We investigate the combustion model with the perturbation part must holds. The key idea is the extension of solutions via integral equations due to [22], [23]. In section 7, we apply our results to a specific model. We investigate the combustion model with the perturbation part must holds. The key idea is the extension of solutions via integral equations due to [22], [23].

2 preliminaries

2.1 Notations

Let \( X \) be a Banach space equipped with the norm \( \| \cdot \| \). The liner operator \( A : X \supset D(A) \to X \) is called sectorial, if it satisfies the following conditions,

\[
\sigma(A) \subset \Sigma_\omega = \{ \lambda \in \mathbb{C}; \ | \arg \lambda | < \omega \} \text{ for some } 0 < \omega < \pi/2,
\]

\[
\| (\lambda - A)^{-1} \|_{B(X)} \leq \frac{M}{|\lambda|} \quad \forall \lambda \notin \Sigma_\omega \text{ for some } M > 0
\]  

where \( \sigma(A) \) is the spectrum set of \( A \). The condition (2.1) implies that the resolvent set of \( A \) contains 0, that is, \( A^{-1} \in B(X) \). More precisely, for \( \delta < \| A^{-1} \|^{-1} \), \( \{ \lambda \in \mathbb{C}; |\lambda| \leq \delta \} \subset \rho(A) \). We do not assume the density of \( D(A) \) in order to enable us to apply the theory to a wide variety of function spaces to gain some space regularity. We can define the fractional power of the sectorial operator, which we shall organize in Appendix.

As we mentioned in the introduction, according to [17], [25], the solution of (1.3) necessarily satisfies the following representation formula, which is the main tool to deduce some properties of the solution.

\[
u(t) = \int_0^\infty h_\alpha(\theta) T(t^{\alpha} \theta) u_0 d\theta + \alpha \int_0^t \int_0^\infty (t-\tau)^{\alpha-1} \theta h_\alpha(\theta) T((t-\tau)^{\alpha} \theta) f(\tau) d\tau d\theta
\]

where \( h_\alpha \) is a probability density function defined on \((0, \infty)\) such that

\[
\int_0^\infty \theta^\gamma h_\alpha(\theta) d\theta = \frac{\Gamma(1+\gamma)}{\Gamma(1+\alpha\gamma)} \quad \forall \gamma \in (-1, \infty),
\]

and \( T(t) \) denotes the semigroup generated by \( A \) which satisfies the following estimates;

\[
\| T(t) \|_{B(X)} \leq C_1 e^{-C_2 t} \leq C_1 \quad \forall t \geq 0,
\]

\[
\| A^\delta T(t) \|_{B(X)} \leq C\delta t^{-\delta} \quad \forall t > 0, \quad \forall \delta > 0.
\]  

Moreover, for all \( 0 < \delta \leq 1 \),

\[
\| [T(t) - I] A^{-\delta} \| \leq \int_0^t A^{1-\delta} T(\tau) d\tau \leq C\delta \int_0^t \tau^{\delta-1} d\tau \leq C\delta t^\delta, \quad 0 < t < \infty.
\]  

2.2 Results on the existence and regularity

We shall employ the following result when we conclude the mild solution of (1.1) necessarily be the usual one.
Proposition 2.1. (see \cite{[7], [8]}) Suppose that $f \in F_{r, \beta}([0, T]; X)$ and $u_0 \in \overline{D(A)}$. Then, for all $0 < \max(\beta, 1 - \alpha) < r \leq 1$ and $0 < \beta < \alpha < 1$, (1.3) has unique solution $u \in C([0, T]; X) \cap C((0, T]; D(A))$.

It is known that the following maximal regularity results for (1.3) hold.

Proposition 2.2. (see \cite{[6], [8]}) For $0 < \alpha < 1$, the following conditions are necessary and sufficient, in order that there exists a unique solution $u$ of (1.3) such that $D_t^\alpha (u - u_0), Au$ are bounded values in $(X, D(A))_{\theta, \infty}$:

1. $f \in C([0, T]; X) \cap B([0, T]; (X, D(A))_{\theta, \infty})$,
2. $u_0 \in \overline{D(A)}, Au_0 \in (X, D(A))_{\theta, \infty}$.

Proposition 2.3. (see \cite{[6], [8]}) For $\theta \in (0, \alpha)$, the following conditions are necessary and sufficient, in order that there exists a unique solution $u$ of (1.3) such that $D_t^\alpha (u - u_0), Au$ belong to $C^\theta([0, T]; X)$:

1. $f \in C^\theta([0, T]; X)$,
2. $u_0 \in D(A), Au_0 + f(0) \in (X, D(A))_{\theta/\alpha, \infty}$.

When we observe the quantitative property, the next statement is essential.

Proposition 2.4. (see \cite{[24], [16]}) Let $g \in C([0, T]) \cap C^1((0, T])$ satisfies $g' \in L^1(0, T)$ and attains its maximum over the interval $[0, T]$ at $t_0 \in (0, T]$, then, for all $0 < \alpha < 1$,

$$
\int_0^{t_0} (t_0 - \tau)^{-\alpha} g'(\tau) d\tau \geq 0.
$$

Moreover, if $g$ is not constant on $[0, t_0]$, then

$$
\int_0^{t_0} (t_0 - \tau)^{-\alpha} g'(\tau) d\tau > 0.
$$

We note that if $g$ is differentiable, then

$$
D_t^\alpha (g - g(0))(t) = \frac{1}{\Gamma(1 - \alpha)} \frac{d}{dt} \int_0^t (t - \tau)^{-\alpha} (g(\tau) - g(0)) d\tau
$$

$$
= \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - \tau)^{-\alpha} g'(\tau) d\tau
$$

where the last term is the so-called Caputo derivative. Therefore, we have to confirm the time differentiability of the solution when we use Proposition 2.4 to (1.3), or just (1.3).

3 Fractional power of operator and time-fractional derivative

In this section, we introduce a formalization of the time-fractional derivative in an abstract framework due to \cite{[7], [8]}. Let an operator $B$ be as follows:

$$
\begin{cases}
B : \{v \in C^1([0, T]; X) : v(0) = 0\} \rightarrow C([0, T]; X), \\
Bv = -\frac{dv}{dt}.
\end{cases}
$$

(3.1)

For $0 < \alpha < 1$, the power of the operator $B$ is as follow;

$$
B^{-\alpha} = \frac{1}{2\pi i} \int_{\Gamma} \lambda^{-\alpha} (\lambda - B)^{-1} d\lambda.
$$

(3.2)

where we take the path $\Gamma$ as in the Appendix. We can confirm that the formula coincides with,

$$
(B^{-\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau) d\tau.
$$

(3.3)

In the abstract point of view, the inverse of (3.2) $B^\alpha := (B^{-\alpha})^{-1}$ is the $\alpha$ power of $B$. And the domain of $B^\alpha$ is the range of $B^{-\alpha}$, that is, $D(B^\alpha) = R(B^{-\alpha})$. On the other hand, we can check the formula (3.3) is inverse mapping of $g \mapsto D_t^\alpha (g - g(0))$. This implies that $D_t^\alpha$ and $B^\alpha$ is the same operator. These properties are mentioned in \cite{[7], [8]}. 


Proposition 3.1. The operator $B$ defined (3.1) is sectorial in $C([0, T]; X)$, and its fractional power is given by (3.3). In particular,

$$B^\alpha (g - g(0)) = D_t^\alpha (g - g(0)).$$

Proof. For all $\lambda \in \mathbb{C}$ and $f \in C([0, T]; X)$,

$$v = (\lambda - B)^{-1} f \iff \frac{dv}{dt} + \lambda v = f, \; v(0) = 0 \iff v(t) = \int_0^t e^{-\lambda(t-s)} f(s)ds.$$

Therefore, the power of $B$ must be calculated as follows;

$$(B^{-\alpha} f)(t) = \frac{1}{2\pi i} \int_{\Gamma} \lambda^{-\alpha} \left[ \int_0^t e^{-\lambda(t-s)} f(s)ds \right] d\lambda$$

$$= \frac{1}{2\pi i} \int_0^t f(s) \left[ \int_{\Gamma} \lambda^{-\alpha} e^{-\lambda(t-s)} d\lambda \right] ds.$$

So, it is sufficient to show that for all $b > 0$,

$$\frac{1}{2\pi i} \int_{\Gamma} \lambda^{-\alpha} e^{-b\lambda} d\lambda = \frac{1}{\Gamma(\alpha)} b^{\alpha - 1}$$

holds. This formula is obtained by the Hankel’s formula and the transformation of the path $\Gamma$ to the Hankel contour. The resolvent estimate (2.2) shall be obtained easily;

$$\|v(t)\| \leq \int_0^t |e^{-\lambda(t-s)}||f(s)||ds$$

$$\leq \frac{1}{|\lambda|} \left[ 1 - e^{-|\lambda|t} \right] \|f\|_{C([0, T]; X)} \leq \frac{1}{|\lambda|} \|f\|_{C([0, T]; X)}.$$

Proposition 3.2. (see [7], [8]) Let $B$ be as in (3.1). Then,

$$(C([0, T]; X), D(B))_{\theta, \infty} = \{ f \in C^\theta([0, T]; X) : f(0) = 0 \}$$

satisfies with the equivalent norms.

Proposition 3.1 deduces the inclusion for $\beta > \theta$,

$$\{ f \in C^\beta([0, T]; X) : f(0) = 0 \} \subset D(B^\theta) \subset \{ f \in C^\theta([0, T]; X) : f(0) = 0 \}.$$  

In symbolic terms, a Hölder continuous function even slightly stronger than $\alpha$ is necessarily $\alpha$-order differentiable, and its derivative is continuous on $[0, T]$. Also, if a $\alpha$-order derivative is continuous on $[0, T]$, then the original function is necessarily $\alpha$-order Hölder continuous. This is an analogy with the integro-differential.

Remark 3.1. Let $Y = L^p(0, T), D(B) = W^{1, p}(0, T)$, then we get

$$D(B^\alpha) \subset (L^p(0, T), W^{1, p}(0, T))_{\alpha, \infty} = B^{\alpha, p, \infty}(0, T)$$

where we define Besov space as follow;

$$B^{s, p, q} := (L^p, W^{m, p})_{s/m, q}.$$

For any bounded $\Omega \subset \mathbb{R}^N$ with sufficiently smooth boundary, we employ the following inclusion (see, [4]),

$$sp > N \Rightarrow B^{s, p, q}(\Omega) \subset C(\overline{\Omega}).$$

When $N = 1, m = 1$, this embedding holds if $\alpha > 1/p$. Therefore, in the Lebesgue and Sobolev space perspective, $\alpha > 1/p$ differentiable function naturally become continuous. This fact has already been mentioned in [11] when $p = 2$ and $X$ is Hilbert space.
4 Proof of Theorem 1.1

We first show the existence of time global mild-solution in the framework of continuous functions. Then, the regularity guarantees that the mild-solution is actually a classical one. We solve the semi-linear problem \((1.1)\) by using the Banach-fixed point theorem with reference to [11]. The idea is to define a function space with a small enough norm for given \(T\) and \(L\). We equip the norm
\[
\|u\|_\mu = \sup_{0 \leq t \leq T} e^{-\mu t} \|u(t)\|
\]
to \(C([0,T]; X)\). We define the map \(S : C([0,T]; X) \to C([0,T]; X)\) by
\[
S : v \mapsto u(t) = \int_0^\infty h_\alpha(\theta)T(t^\alpha \theta)\,d\theta u_0 + \alpha \int_0^t \int_0^\infty (t - \tau)^{\alpha-1} \theta h_\alpha(\theta)T((t - \tau)^{\alpha-1} \theta) [f(\tau) - B(\tau v)] \,d\tau d\theta.
\]

**Proposition 4.1.** For all \(L > 0, T > 0\), there exists sufficiently large \(\mu > 0\) such that the \(S\) becomes contraction mapping.

*Proof.* For arbitrary \(v_1, v_2 \in C([0,T]; X)\), let \(u_1 = Sv_1, u_2 = Sv_2\). We can easily see that
\[
\|u_1(t) - u_2(t)\| \leq \frac{LC_1 \alpha}{\Gamma(1 + \alpha)} \int_0^t e^{\mu \tau} e^{-\mu (t - \tau)} (t - \tau)^{\alpha-1} \|v_1(\tau) - v_2(\tau)\| \,d\tau
\]
\[
e^{-\mu t}\|u_1(t) - u_2(t)\| \leq \frac{LC_1 \alpha}{\Gamma(1 + \alpha)} \int_0^t e^{-\mu (t - \tau)} (t - \tau)^{\alpha-1} \,d\tau \|v_1 - v_2\|_\mu.
\]
We note that
\[
\int_0^t e^{-\mu (t - \tau)} (t - \tau)^{\alpha-1} \,d\tau \leq \int_0^T e^{-\mu \tau} \tau^{\alpha-1} \,d\tau
\]
and
\[
e^{-\mu \tau} \tau^{\alpha-1} \leq \tau^{\alpha-1}, e^{-\mu \tau} \tau^{\alpha-1} \to 0 \text{ pointwise in } (0,T) \text{ as } \mu \to \infty.
\]
So, we set
\[
A_\mu = \int_0^T e^{-\mu \tau} \tau^{\alpha-1} \,d\tau \to 0 \text{ as } \mu \to \infty
\]
to conclude
\[
\|u_1 - u_2\|_\mu = \sup_{0 \leq t \leq T} e^{-\mu t} \|u_1(t) - u_2(t)\| \leq A_\mu \frac{LC_1 \alpha}{\Gamma(1 + \alpha)} \|v_1 - v_2\|_\mu.
\]
In particular, we choose so large \(\mu > 0\) that
\[
A_\mu LC_1 \alpha / \Gamma(1 + \alpha) < 1.
\]

**Proposition 4.2.** Let
\[
S_0(t) = \int_0^\infty h_\alpha(\theta)T(t^\alpha \theta)\,d\theta u_0
\]
for \(u_0 \in X\).

1. If \(u_0 \in \overline{D(A)}\), then \(S_0 \in \mathcal{F}^{r,\beta}((0,T]; X)\) for all \(0 < \beta < r < 1\) and \(\beta < \alpha\).
2. If \(u_0 \in D(A^r)\) for \(\alpha^{-1} \beta \leq q \leq 1\) and \(\beta < \alpha\), then \(S_0 \in \mathcal{C}^q((0,T]; X)\).

*Proof.*

1. The strong continuity of the semigroup at \(t = 0\) guarantees the existence of \(\lim_{t \to 0} t^{1-r} \|S_0(t)\|\). Suppose that \(s < t\), then for arbitrary \(x \in X\) and \(0 < \eta < 1\),
\[
\|T(t)x - T(s)x\| \leq \int_s^t \|AT(\tau)\| \|x\| \,d\tau
\]
\[
= \int_s^t \|T(\tau - s)A^\eta T(s)\| \|x\| \,d\tau
\]
\[
\leq C_\eta \int_s^t (\tau - s)^\eta \|x\| \,d\tau
\]
\[
\leq C_\eta (t - s)^\eta \|x\|
\]
Therefore,
\[
\| S_0(t) - S_0(s) \| \leq \int_0^\infty h_\alpha(\theta) \| T(t^\alpha \theta)u_0 - T(s^\alpha \theta)u_0 \| d\theta
\]
\[
\leq C_\eta \int_0^\infty h_\alpha(\theta) (t^\alpha \theta - s^\alpha \theta)^{\eta} s^{\alpha \eta} d\theta \| u_0 \|
\]
\[
\leq C_{\eta, \alpha} \| u_0 \| (t - s)^{\alpha \eta}
\]
Hence we take \( \eta = \alpha^{-1} \beta \) to evaluate
\[
\frac{s^{1 - r + \beta}}{(t - s)^2} \| S_0(t) - S_0(s) \| \leq C_{\eta, \alpha} \| u_0 \| s^{1 - r} \to 0
\]
as \( t \to 0 \).

2. The condition \( u_0 \in D(A^q) \) deduces
\[
\| S_0(t) - u_0 \| = \int_0^\infty h_\alpha(\theta) \| (T(t^\alpha \theta) - I)A^{-q} \| \| A^q u_0 \| d\theta
\]
\[
\leq C_q \int_0^\infty h_\alpha(\theta) \| A^q \| \| A^q u_0 \| d\theta
\]
\[
\leq C_q \Gamma(1 + q) \| A^q u_0 \| t^\beta.
\]
Therefore, we prove the Hölder continuity at \( t = 0 \). The continuity on \( (0, T] \) can be obtained by using the following estimate,
\[
\| T(t^\alpha \theta)u_0 - T(s^\alpha \theta)u_0 \| \leq \| (T(t^\alpha \theta - s^\alpha \theta) - I)A^{-q} \| \| T(s^\alpha \theta)\| \| A^q u_0 \|
\]

\[\square\]

**Proposition 4.3.** Let
\[
S_1(g)(t) := \int_0^t \int_0^\infty (t - \tau)^{\alpha - 1} \theta h_\alpha(\theta) T((t - \tau)^\alpha \theta) g(\tau) d\tau d\theta.
\]

1. If \( g \in \mathcal{F}^{r, \beta}((0, T]; X) \) for \( 0 < \beta < r \leq 1 \) and \( \beta < \alpha \), then \( S_1(g) \in \mathcal{F}^{r, \beta}((0, T]; X) \)

2. If \( g \in C([0, T]; X) \), then \( S_1(g) \in C^{\beta}([0, T]; X) \).

**Proof.** Let \( g \in \mathcal{F}^{r, \beta}((0, T]; X) \). The simple calculation yields
\[
\| S_1(g)(t) - S_1(g)(s) \| \leq \left\| \int_s^t (t's formula) \pm \int_0^s (t's formula) - \int_0^s (s's formula) \right\|
\]
\[
\leq \int_s^t \int_0^\infty \theta h_\alpha(\theta)(t - \tau)^{\alpha - 1} \| T((t - \tau)^\alpha \theta) \|_{B(X)} \tau^{-1 - r} \| g(\tau) \| d\tau d\theta
\]
\[
+ \int_0^s \int_0^\infty \theta h_\alpha(\theta) \| (s - \tau)^{\alpha - 1} T((s - \tau)^\alpha \theta) - (t - \tau)^{\alpha - 1} T((t - \tau)^\alpha \theta) \| g(\tau) \| d\tau d\theta
\]
\[
\leq C_1 \| g \|_{\mathcal{F}^{r, \beta}} \int_s^t (t - \tau)^{\alpha - 1} \tau^{-1 - r} d\tau
\]
\[
+ \| g \|_{\mathcal{F}^{r, \beta}} \int_0^s \theta h_\alpha(\theta)(s - \tau)^{\alpha - 1} \| T((s - \tau)^\alpha \theta) - T((t - \tau)^\alpha \theta) \| \tau^{-1} d\tau d\theta
\]
\[
+ \frac{C_1}{\Gamma(1 + \alpha)} \int_0^s |(s - \tau)^{\alpha - 1} - (t - \tau)^{\alpha - 1}| \| g(\tau) \| d\tau
\]
\[
=: (J_1 + J_2 + J_3)
\]
The estimate for \( J_1, J_2, J_3 \) is as follows,
We select arbitrary $0 < \delta < 1$ to gain the following estimate,

\[
(s - \tau)^{\alpha-1} \|T((t - \tau)^{\alpha} \theta) - T((s - \tau)^{\alpha} \theta)\|_{\tau^{-1}} \\
= (s - \tau)^{\alpha-1} \tau^{-1} \|T((t - \tau)^{\alpha} \theta) - (s - \tau)^{\alpha} \theta\| - \|A^{-\delta} A^{\delta} T((s - \tau)^{\alpha} \theta)\| \\
\leq C_{\delta}(s - \tau)^{\alpha-1} \tau^{-1} |(t - \tau)^{\alpha} \theta - (s - \tau)^{\alpha} \theta|^\delta (s - \tau)^{-\delta \alpha \theta - \delta} \\
= C_{\delta}(s - \tau)^{(1-\delta)\alpha-1} \tau^{-1}(t - s)^{\alpha \delta}.
\]

Therefore

\[
J_2 \leq C_{\delta} \int_0^s (s - \tau)^{(1-\delta)\alpha-1} \tau^{-1}(t - s)^{\alpha \delta} d\tau \\
\leq C_{\delta} B(\alpha(1 - \delta), r) s^{\alpha(1-\delta) + \beta} s^{\beta - 1}(t - s)^{\alpha \delta} \\
J_2 \leq C_{\delta} B(\alpha(1 - \delta), r) s^{\alpha(1-\delta) + \beta}(t - s)^{\alpha \delta}
\]

We note that $\tau \mapsto \tau^{\alpha-1}$ is in $F^{\alpha - \sigma \alpha}$ for all $\sigma < \alpha$. Then, there exists some $C_{\sigma}$ such that

\[
J_3 = \frac{C_1}{\Gamma(1 + \alpha)} \int_0^s \left[(s - \tau)^{\alpha-1} - (t - \tau)^{\alpha-1}\right] \|g(\tau) - g(s)\| d\tau \\
+ \frac{C_1}{\Gamma(1 + \alpha)} \int_0^s \left[(s - \tau)^{\alpha-1} - (t - \tau)^{\alpha-1}\right] \|g(s)\| d\tau \\
\leq \frac{C_1 C_{\sigma} \|g\|_{F^{\alpha-\beta}}}{\Gamma(1 + \alpha)} (t - s)^{\sigma} \int_0^s (s - \tau)^{\alpha-2\sigma} (s - \tau)^{\beta \tau^{-1} - \beta} d\tau \\
+ \frac{C_1 \|g\|_{F^{\alpha-\beta}}}{\Gamma(1 + \alpha)} (t - s)^{\sigma - 1} \int_0^s [(s - \tau)^{\alpha-1} - (t - \tau)^{\alpha-1}] d\tau
\]

We denote the appropriate constants as $C$ to deduce

\[
J_3 \leq C(t - s)^{\sigma} B(\alpha + 2\sigma, r - \beta) s^{\alpha - 2\sigma + \beta} s^{\beta - 1 - \beta} + C s^{\beta - 1 - \beta} s^{\beta}(t - s)^{\alpha} \\
J_2 \leq C(t - s)^{\sigma} + C s^{\beta}(t - s)^{\alpha}.
\]

Especially, we select $\sigma = \beta$.

Let $g \in C([0, T]; X)$. For $s < t$, a simple calculation yields

\[
\|S_1(g)(t) - S_1(g)(s)\| \leq \left\| \int_0^t (t's & formula) \pm \int_0^s (t's & formula) - \int_0^s (s's & formula) \right\| \\
\leq \int_0^t \int_0^\infty \theta h_{\alpha}(\theta)(t - \tau)^{\alpha-1} \|T((t - \tau)^{\alpha} \theta)\|_{B(X)} \|g(\tau)\| d\tau d\theta \\
+ \int_0^s \int_0^\infty \theta h_{\alpha}(\theta)(s - \tau)^{\alpha-1} T((s - \tau)^{\alpha} \theta) - (t - \tau)^{\alpha-1} T((t - \tau)^{\alpha} \theta)\|g(\tau)\| d\tau d\theta \\
\leq \frac{C_1 \|g\|_{F^{\alpha-\beta}}}{\Gamma(1 + \alpha)} \int_0^t (t - \tau)^{\alpha-1} d\tau + \frac{C_1 \|g\|_{F^{\alpha-\beta}}}{\Gamma(1 + \alpha)} \int_0^s (s - \tau)^{\alpha-1} - (t - \tau)^{\alpha-1} d\tau \\
+ \|g\| \int_0^\infty \theta h_{\alpha}(\theta) \left| T((t - \tau)^{\alpha} \theta) - T((s - \tau)^{\alpha} \theta) \right| (s - \tau)^{\alpha-1} d\tau d\theta \\
= (J_1 + J_2 + J_3)
\]
Now, the evaluation of $J_1, J_2$ is as follows,

$$J_1 = \frac{C_1 \| g \|}{\Gamma(1 + \alpha)} \int_s^t (t - \tau)^{\alpha - 1} d\tau \leq \frac{C_1 \| g \|}{\alpha \Gamma(1 + \alpha)} (t - s)^\alpha$$

$$J_2 = \frac{C_1 \| g \|}{\Gamma(1 + \alpha)} \int_0^s |(s - \tau)^{\alpha - 1} - (t - \tau)^{\alpha - 1}| d\tau$$

$$= \frac{C_1 \| g \|}{\alpha \Gamma(1 + \alpha)} (s^\alpha - t^\alpha + (t - s)^\alpha).$$

On the other hand, for arbitrary $0 < \delta < 1$,

$$(s - \tau)^{\alpha - 1} \| T((t - \tau)^{\alpha} \theta) - T((s - \tau)^{\alpha} \theta)\|$$

$$= (s - \tau)^{\alpha - 1} \| T((t - \tau)^{\alpha} \theta - (s - \tau)^{\alpha} \theta) - I \| A^{-\delta} A^\delta T((s - \tau)^{\alpha} \theta)\|$$

$$\leq C_\delta (s - \tau)^{\alpha - 1} \| (t - \tau)^{\alpha} \theta - (s - \tau)^{\alpha} \theta \| \delta (s - \tau)^{-\delta_1 \alpha - \delta}$$

$$\leq C_{\delta, \alpha} (s - \tau)^{(1 - \delta) - 1} (t - s)^{\alpha \delta}$$

where $C_{\delta, \alpha}$ depends on $\delta, \alpha$. Therefore,

$$J_3 \leq \frac{C_{\delta, \alpha}}{\Gamma(1 + \alpha)} \int_0^s (s - \tau)^{\alpha(1 - \delta) - 1} d\tau (t - s)^{\alpha \delta}$$

$$\leq \frac{C_{\delta, \alpha} \tau^{\alpha(1 - \delta)}}{\alpha(1 - \delta) \Gamma(1 + \alpha)} (t - s)^{\alpha \delta}.$$

\[ \square \]

Proof of Theorem 1.1 Let $u$ denotes the fixed-point of $S$. For any $f \in F^{r, \beta}$, the existence and uniqueness for the equation with external force term $f - B(u) \in F^{r, \beta}$;

$$D_0^\alpha (w - w(0)) + Aw = f - B(u), \ w(0) = u_0$$

is assured by Proposition 2.1. This solution $w$ coincides with $u$ because the representation formula of (1.4) necessarily be satisfied. Proposition 4.2, 4.3 provides the improvement of time regularity, $u \in C^{\beta}([0, T]; X)$ when $u \in D(A^\alpha)$ and $f \in C([0, T]; X)$. Suppose that $u \in D(A)$ and $f \in F^{1, \beta}([0, T]; X)$. It is sufficient to prove $Au \in C([0, T]; X)$ because of (3.3). The continuity of

$$AS_0(t) = \int_0^\infty h_\alpha(\theta) T(t^\alpha \theta) A u_0 d\theta$$

at $t = 0$ is obvious. It is sufficient to prove that for any $g \in F^{1, \beta}([0, T]; X),$

$$AS_1(g)(t) = \int_0^t \int_0^\infty (t - \tau)^{\alpha - 1} \theta h_\alpha(\theta) AT((t - \tau)^{\alpha} \theta) g(\tau) d\tau d\theta \to 0$$

as $t \to 0$. Indeed,

$$AS_1(g)(t) = \int_0^t \int_0^\infty (t - \tau)^{\alpha - 1} \theta h_\alpha(\theta) AT((t - \tau)^{\alpha} \theta) [g(\tau) - g(t)] d\tau d\theta$$

$$+ \int_0^t \int_0^\infty (t - \tau)^{\alpha - 1} \theta h_\alpha(\theta) AT((t - \tau)^{\alpha} \theta) g(t) d\tau d\theta$$

$$= \int_0^t \int_0^\infty (t - \tau)^{\alpha - 1} \theta h_\alpha(\theta) AT((t - \tau)^{\alpha} \theta) [g(\tau) - g(t)] d\tau d\theta + [I - \Phi(t)] g(t)$$

See (5.3) for the definition of $\Phi(t)$. On the other hand, we estimate

$$\int_0^t \int_0^\infty (t - \tau)^{\alpha - 1} \theta h_\alpha(\theta) AT((t - \tau)^{\alpha} \theta) || g(\tau) - g(t) || d\tau d\theta$$

$$\leq C \int_0^t (t - \tau)^{\beta - 1} \tau^{-\beta} d\tau w_\alpha(t) \to 0 \text{ as } t \to 0$$
and
\[ \|g(t) - \Phi(t)g(t)\| \leq \|I - \Phi(t)\| \cdot \|g(t) - g(0)\| + \|g(0) - \Phi(t)g(0)\| \leq C\|g(t) - g(0)\| + \|g(0) - \Phi(t)g(0)\| \to 0 \text{ as } t \to 0 \]
because of the strong continuity of \( \Phi(t) \) at \( t = 0 \).

Remark 4.1. Our results shall be extended to the non-local perturbation
\[ B : C([0, T]; X) \to C([0, T]; X) \]
such that
\[ \|B(u) - B(v)\|_{C([0, T]; X)} \leq L\|u - v\|_{C([0, T]; X)} \]
for some \( L > 0 \) and maps the (weighted) Hölder space to itself.

5 Proof of Theorem 1.2
In this section, we will prove Theorem 1.2. If we differentiate both sides of (1.1) formally by \( t \), we should be able to derive the equation which \( u' \) should satisfy. For preparation, we at first investigate evolution equations with integral initial conditions.

5.1 Integral initial condition equation
We investigate the following equation
\[ \begin{cases} D^\alpha_t u + Au = f, & \text{in } (0, T], \\ (I^\alpha u)(0) = x_0 \end{cases} \tag{5.1} \]
where
\[ I^\alpha u(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - \tau)^{-\alpha} u(\tau) d\tau. \tag{5.2} \]
We assume that \( f \in F^{\alpha, \beta}((0, T]; X) \) for some \( 0 < \beta < r \leq 1 \). The representation formula for mild solution shall be changed from (1.4).

Proposition 5.1. The solution of (5.1) is given by the following;
\[ u(t) = \alpha \int_0^\infty h_\alpha(\theta) T(t_\theta x_0) d\theta + \alpha \int_0^t \int_0^\infty \theta h_\alpha(\theta)(t - \tau)^{\alpha - 1} T((t - \tau)^\theta f(\tau)) d\tau d\theta \tag{5.3} \]
Proof. The formula (5.3) can be deduced from (5.2) by the same way as in [25]. In fact, we apply the Laplace transform to (5.2) and use the fact (see, for instance [13]),
\[ \mathcal{L}[D^\alpha_t g](s) = s^\alpha \mathcal{L}[g](s) - I^\alpha g(0) \]
where \( \mathcal{L} \) is Laplace transform
\[ \mathcal{L}[g](s) := \int_0^\infty e^{-st} g(t) dt. \]
Therefore, we denote \( U = \mathcal{L}[u], F = \mathcal{L}[f] \), then,
\[ s^\alpha U(s) - x_0 + AU(s) = F(s) \Rightarrow U(s) = -(-s^\alpha - A)^{-1}(x_0 + F(s)). \]
We use the fact that for each \( \lambda \leq 0 \),
\[ (\lambda - A)^{-1} = -\int_0^\infty e^{\lambda T} dt, \]
Then,
\[ U(s) = \int_0^\infty e^{s^\alpha t}(x_0 + F(s)) dt. \]
The rest of the calculations are the same as in [25]. The second term of (5.3) is obviously the solution for
\[ D_\alpha^2 w + Aw = f, \quad I^{\alpha-1}(w)(0) = 0. \]
We denote the first term of (5.3) by \( w_0(t) \) and let
\[
\Phi(t) = \int_0^\infty h(\theta)T(t^\alpha \theta)d\theta, \tag{5.4}
\]
We observe
\[
w_0(t) = \int_0^\infty h(\theta)A^{-1}(\alpha t^{\alpha-1}\theta)AT(t^\alpha \theta)x_0d\theta
= -\int_0^\infty h(\theta)A^{-1} \frac{d}{dt} \left[ T(t^\alpha \theta)x_0 \right] d\theta
= -A^{-1} \frac{d}{dt} \Phi(t)x_0,
\]
that is,
\[ Aw_0(t) = -\Phi'(t)x_0. \]
On the other hand, (1.4) means that \( \Phi(t)x_0 \) is the solution for
\[ \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \Phi'(t)x_0d\tau = -A \Phi(t)x_0. \]
Therefore,
\[ \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} w_0(\tau)d\tau = -A^{-1} \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \Phi'(t)x_0d\tau
= \Phi(t)x_0
\]
which means,
\[ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} w_0(\tau)d\tau = \Phi'(t)x_0 = -Aw_0(t). \]
If \( x_0 \in \overline{D(A)} \), the Lebesgue convergence theorem yields when \( t \to 0 \),
\[
\frac{\alpha}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} w_0(\tau)d\tau
\]
\[ \xrightarrow{t \to 0} \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 (1-\tau)^{-\alpha} \int_0^\infty \theta h(\theta)T(\tau^{\alpha} \theta)x_0d\tau d\theta
\]
\[ \xrightarrow{t \to 0} \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 (1-\tau)^{-\alpha} \int_0^\infty \theta h(\theta)T(\tau^{\alpha} \theta x_0)d\tau d\theta
\]
\[ \xrightarrow{t \to 0} \frac{\alpha}{\Gamma(1-\alpha)} \int_0^1 (1-\tau)^{-\alpha} \int_0^\infty \theta h(\theta)x_0d\tau d\theta
\]
\[ = \frac{\alpha}{\Gamma(1-\alpha)} B(1-\alpha, \alpha) \frac{\Gamma(2)}{\Gamma(1+\alpha)} x_0
\]
\[ = x_0 \]

Next we investigate the following Lipschitz perturbation problem
\[
\begin{aligned}
\begin{cases}
D_\alpha^2 u + Au + B(u) = f, & \text{in } (0, T],
(I^\alpha u)(0) = x_0.
\end{cases}
\end{aligned} \tag{5.5}
\]
Proposition 4.2 and the fact that \( t \mapsto t^{\alpha-1} \) is in \( F^{\alpha-\gamma, \gamma} \) deduce the next lemma.

**Lemma 5.1.** Suppose that \( x_0 \in \overline{D(A)} \). Then, \( w_0 \in F^{\alpha-\gamma, \gamma} \) for any \( 0 < \gamma < \alpha/2 \).
We define the map on $F^{\alpha-\gamma,\gamma}$
\[
\tilde{S}: F^{\alpha-\gamma,\gamma} \ni v \mapsto \tilde{S}(v)(t) = \alpha \int_0^\infty \theta h_\alpha(\theta)T(t^\alpha \theta)xd\theta t^{\alpha-1} + \alpha \int_0^t \int_0^\infty \theta h_\alpha(\theta)(t-\tau)^{\alpha-1}T((t-\tau)^\alpha \theta)(f(\tau) - B(v(\tau)))d\tau d\theta.
\]
(5.6)

**Theorem 5.1.** Let $f \in F^{r,\beta}((0,T]; X)$ such that $0 < \beta < r \leq 1$, $\beta < \alpha$, $0 < \max(\beta, 1-\alpha) < r$ and $\alpha/2 \leq r$. Then, (5.6) has the unique solution $u \in F^{\alpha-\gamma,\gamma}((0,T]; X) \cap C((0,T]; D(A))$ where $0 < \gamma < \alpha/2$, $\alpha - \gamma \leq r$ and $\gamma \leq \beta$.

**Proof.** We just need to show that (5.6) is contraction mapping on $F^{\alpha-\gamma,\gamma}$. The condition for $\gamma$ and Proposition 4.3 guarantee that $\tilde{S}$ maps $F^{\alpha-\gamma,\gamma}$ to itself. Now, for each $v_1, v_2 \in F^{\alpha-\gamma,\gamma}$, we denote $\tilde{S}(v_1) = u_1, \tilde{S}(v_2) = u_2$.

The difficulty of the proof is that we have to evaluate the seminorm part of $\| \cdot \|_{F^{r,\beta}}$, not just uniform norm. For simplicity, we denote $\alpha - \gamma = \kappa$, also, we use $\lesssim$ as an evaluation by appropriate constant multiplication to avoid a complicated notation. We equip the norm $\|f\|_\mu = \sup_{0 \leq t \leq T} e^{-\mu t} l^{1-\kappa} \|f(t)\| + \sup_{0 \leq s < t \leq T} e^{-\mu s} \|f(t) - f(s)\|/\|v\|_{\mu}$ for $F^{\kappa,\gamma}((0,T]; X)$ where $\mu > 0$.

**Estimate for uniform norm**

We can easily see that
\[
\|u_1(t) - u_2(t)\| \lesssim \int_0^t e^{\mu(t-\tau)} e^{-\mu t} (t-\tau)^{\alpha-1} (t-\tau)^{\alpha-1} \|v_1(\tau) - v_2(\tau)\| d\tau
\]
\[
e^{-\mu t} \|u_1(t) - u_2(t)\| \lesssim \int_0^t e^{\mu(t-\tau)} e^{-\mu t} (t-\tau)^{\alpha-1} (t-\tau)^{\alpha-1} \|v_1(\tau) - v_2(\tau)\| d\tau
\]
\[
\lesssim \mu \cdot \|v_1 - v_2\|_{\mu}.
\]

For sufficiently small $\epsilon > 0$, Let $p = 1 + \epsilon$ and $q$ be the conjugate of $p$. Because of the Hölder’s inequality
\[
e^{-\mu t} \|u_1(t) - u_2(t)\| \lesssim \left( \int_0^t e^{-\mu q(t-\tau)} d\tau \right)^{1/q} \left( \int_0^t (t-\tau)^{p(\alpha-1)} (t-\tau)^{p(\kappa-1)} d\tau \right)^{1/p} \cdot L \cdot \|v_1 - v_2\|_{\mu}
\]
\[
= B_\mu \left( \int_0^t (t-\tau)^{p(\alpha-1)} r^{p(\kappa-1)} d\tau \right)^{1/p} \cdot \|v_1 - v_2\|_{\mu}
\]
where
\[
\left( \int_0^t e^{-\mu q(t-\tau)} d\tau \right)^{1/q} = \left( \int_0^t e^{-\mu q^\tau} d\tau \right)^{1/q}
\]
\[
\leq \left( \int_0^t e^{-\mu q^\tau} d\tau \right)^{1/q} = B_\mu \rightarrow 0, \mu \rightarrow \infty
\]
The underlined part shall be estimated as follows;
\[
(\text{underline part})^p = \int_0^t t^{p(\alpha-1)} \left( \frac{1-\tau}{t} \right)^{p(\alpha-1)} t^{p(\kappa-1)} \left( \frac{\tau}{t} \right)^{p(\kappa-1)} d\tau
\]
\[
= t^{p(\alpha+\kappa-2)} \int_0^t \left( \frac{1-\tau}{t} \right)^{p(\alpha-1)} t^{p(\kappa-1)} \left( \frac{\tau}{t} \right)^{p(\kappa-1)} d\tau, \ (\tau \leftrightarrow \tau/t, \ d\tau \leftrightarrow td\tau)
\]
\[
= t^{p(\alpha+\kappa-2)+1} \int_0^1 \left( 1-\frac{\tau}{t} \right)^{p(\alpha-1)} t^{p(\kappa-1)} \left( \frac{\tau}{t} \right)^{p(\kappa-1)} d\tau
\]
\[
= t^{p(\alpha+\kappa-2)+1} B(p(\alpha-1) + 1, p(\kappa-1) + 1)
\]
\[
(\text{underline part}) = B(p(\alpha-1) + 1, p(\kappa-1) + 1)^{1/p} t^{p(\alpha+\kappa-2)+1/p} \mu^{\kappa-1}.
\]
That is,
\[ e^{-\mu t^{1-\kappa}}\|u_1(t) - u_2(t)\| \lesssim B_\mu t^{\alpha-1+1/p}\|v_1 - v_2\|_\mu \]

We can choose the \( \epsilon > 0 \) such that the indices meet
\[
\begin{align*}
\alpha - 1 + \frac{1}{p} &= \alpha - \frac{\epsilon}{1 + \epsilon} > 0, \\
p(\alpha - 1) + 1 &= (1 + \epsilon)\alpha - \epsilon > 0, \\
p(\kappa - 1) + 1 &= (1 + \epsilon)\kappa - \epsilon > 0.
\end{align*}
\]

Hence, we conclude
\[
\sup_{0 \leq t \leq T} e^{-\mu t^{1-\kappa}}\|u_1(t) - u_2(t)\| \leq B_\mu C(L,T)\|v_1 - v_2\|_\mu
\]

**Estimate for seminorm**

We set \( V = B(v_1) - B(v_2) \) and \( U = \tilde{S}v_1 - \tilde{S}v_2 \). We shall estimate \( \| U(t) - U(s) \| \). The simple calculation yields

\[
\| U(t) - U(s) \|
\leq \left\| \int_0^t (t\text{'s formula}) \pm \int_0^s (t\text{'s formula}) - \int_0^s (s\text{'s formula}) \right\|
\leq \int_s^t \int_0^\infty \theta h_\alpha(\theta)(t - \tau)^{\alpha-1}\|T((t - \tau)\alpha\theta)\|_{B(X)}\tau^{\kappa-1}T^{1-\kappa}e^{\mu(t - \tau)}e^{-\mu\tau}\|V(\tau)\|d\tau d\theta
\]

\[+ \int_s^t \int_0^\infty \theta h_\alpha(\theta)((s - \tau)^{\alpha-1}T((s - \tau)^{\alpha}\theta) - (t - \tau)^{\alpha-1}T((t - \tau)^{\alpha}\theta))\|\tau^{\kappa-1}T^{1-\kappa}e^{\mu(t - \tau)}e^{-\mu\tau}\|V(\tau)\|d\tau d\theta
\]

\[e^{-\mu t}\|U(t) - U(s)\|
\\lesssim \int_s^t (t - \tau)^{\alpha-1}\tau^{\kappa-1}e^{-\mu(t - \tau)}d\tau\|v_1 - v_2\|_\mu
\]

\[+ \int_s^t \int_0^\infty \theta h_\alpha(\theta)(s - \tau)^{\alpha-1}\tau^{\kappa-1}e^{-\mu(s - \tau)}\|T((s - \tau)^{\alpha}\theta) - T((t - \tau)^{\alpha}\theta))\|_{B(X)}d\tau d\theta\|v_1 - v_2\|_\mu
\]

\[+ \int_s^t \|((s - \tau)^{\alpha-1} - (t - \tau)^{\alpha-1})\|\tau^{\kappa-1}e^{-\mu(s - \tau)}d\tau d\theta\|v_1 - v_2\|_\mu
\]

\[=: (K_1 + K_2 + K_3)\|v_1 - v_2\|_\mu.
\]

We shall estimate \( s^{1-\kappa+\gamma}K_1 \) from above by the product of the term which converges to zero and the power of \( (t - s) \).

**Estimate for \( K_1 \)**

\[ K_1 = \int_s^t (t - \tau)^{\alpha-1}\tau^{\kappa-1}e^{-\mu(t - \tau)}d\tau.
\]

Let \( p = 1 + \epsilon, q \) be its conjugate. Then, by the Hölder’s inequality

\[ K_1 \leq \int_s^t (t - \tau)^{\alpha-1}e^{-\mu(t - \tau)}d\tau \leq \int_s^t e^{-\mu(t - \tau)}d\tau \| v_1 - v_2 \|_\mu \]

\[s^{1-\kappa+\gamma}K_1 \lesssim \int_s^t (t - \tau)^{\alpha-1}e^{-\mu(t - \tau)}d\tau \tau^\gamma
\]

\[ \lesssim \left( \int_s^t e^{-\mu(t - \tau)}d\tau \right)^{1/q} \left( \int_s^t (t - \tau)^{(\alpha-1)p}d\tau \right)^{1/p} \tau^\gamma
\]

\[ \leq C_\mu (t - s)^{\alpha-1+1/p}\tau^\gamma
\]
where
\[
\left( \int_{0}^{t-s} e^{-\mu \tau q \theta} d\tau \right)^{1/q} \leq \left( \int_{0}^{T} e^{-\mu \tau q \theta} d\tau \right)^{1/q} = C_\mu \to 0, \mu \to \infty.
\]
We choose \(0 < \epsilon < \alpha/(1-\alpha)\). Hence,
\[
s^{1-\kappa+\gamma}K_1 \lesssim C_\mu (t-s)^{\alpha-1+1/p} t^\gamma.
\]

**Estimate for \(K_2\)**
\[
K_2 = \int_{0}^{s} \int_{0}^{\infty} \theta h_\alpha(\theta)(s-\tau)^{\alpha-1-\kappa-1}e^{-\mu(s-\tau)}\Vert T((t-\tau)^{\alpha}\theta) - T((s-\tau)^{\alpha}\theta)\Vert d\tau d\theta.
\]
We select arbitrary \(0 < \delta < 1\). The underlined part shall be estimated;
\[
\begin{align*}
&\quad (\text{underline part}) = (s-\tau)^{\alpha-1-\kappa-1}e^{-\mu(s-\tau)}\Vert T((t-\tau)^{\alpha}\theta) - T((s-\tau)^{\alpha}\theta)\Vert \\
&\lesssim (s-\tau)^{\alpha-1-\kappa-1}e^{-\mu(s-\tau)}[(t-\tau)^{\alpha}\theta - (s-\tau)^{\alpha}\theta] \theta \delta (s-\tau)^{-\delta\alpha-\theta} \\
&= e^{-\mu(s-\tau)}(s-\tau)^{\alpha(1-\delta)-1-\kappa}(t-s)^{\alpha\delta}
\end{align*}
\]
We select \(p = 1 + \epsilon\) for some \(\epsilon > 0\), and \(q\) be the conjugate of \(p\) as usual. Therefore
\[
K_2 \lesssim \int_{0}^{s} e^{-\mu(s-\tau)}(s-\tau)^{\alpha(1-\delta)-1-\kappa}(t-s)^{\alpha\delta} d\tau \\
\leq \left( \int_{0}^{s} e^{-\mu(s-\tau)q \theta} e^{-\mu(s-\tau)q \theta} d\tau \right)^{1/q} \left( \int_{0}^{s} \left( (s-\tau)^{\alpha(1-\delta)-1}p \tau^{(\kappa-1)p} \right) d\tau \right)^{1/p} (t-s)^{\alpha\delta} \\
\leq D_\mu \left( \int_{0}^{s} \left( s-\tau \right)^{\alpha(1-\delta)-1} \tau^{\kappa-1} \right)^{1/p} (t-s)^{\alpha\delta}
\]
where
\[
\left( \int_{0}^{s} e^{-\mu(s-\tau)q \theta} e^{-\mu(s-\tau)q \theta} d\tau \right)^{1/q} \leq \left( \int_{0}^{T} e^{-\mu \tau q \theta} d\tau \right)^{1/q} = D_\mu \to 0, \mu \to \infty.
\]
Moreover,
\[
\begin{align*}
&\quad (\text{underline part}) = \int_{0}^{s} s^{(\alpha(1-\delta)-1)p} \left( 1 - \frac{\tau}{s} \right)^{\{\alpha(1-\delta)-1\}p} s^{\alpha(1-\delta)-1}p \left( T \frac{\tau}{s} \right)^{(\kappa-1)p} d\tau \\
&\quad = s\int_{0}^{1} \left( 1 - \tau \right)^{\{\alpha(1-\delta)-1\}p} \tau^{(\kappa-1)p} d\tau \\
&\quad = s\ast B(\{\alpha(1-\delta)-1\}p + 1, (\kappa-1)p + 1)
\end{align*}
\]
where \(\ast = \{\alpha(1-\delta)-1\}p + (\kappa-1)p + 1\). Thus,
\[
K_2 \lesssim D_\mu (t-s)^{\alpha\delta} B(\{\alpha(1-\delta)-1\}p + 1, (r-1)p + 1)^{1/p} s^{\alpha(1-\delta)-1+\gamma+1/p} s^{\kappa-1-\gamma} \\
s^{1-\kappa+\gamma}K_2 \lesssim D_\mu (t-s)^{\alpha\delta} B(\{\alpha(1-\delta)-1\}p + 1, (\kappa-1)p + 1)^{1/p} s^{\alpha(1-\delta)-1+\gamma+1/p}.
\]
We choose appropriate \(\epsilon > 0\) such that
\[
\begin{align*}
&\quad \{\alpha(1-\delta)-1\}p + 1 = (1 + \epsilon)\alpha(1-\delta) - \epsilon > 0, \\
&\quad (\kappa-1)p + 1 = \kappa(1 + \epsilon) - \epsilon > 0.
\end{align*}
\]
Hence,
\[
s^{1-\kappa+\gamma}K_2 \lesssim D_\mu (t-s)^{\alpha\delta} s^{\alpha(1-\delta)-1+\gamma+1/p}.
\]
Estimate for $K_3$

$$K_3 = \int_0^s |(s-\tau)^{\alpha-1} - (t-\tau)^{\alpha-1}| \tau^{\kappa-1} e^{-\mu(s-\tau)} d\tau d\theta$$

We note that $\tau \mapsto \tau^{\alpha-1}$ is in $\mathcal{F}^{\alpha-\sigma, \sigma}$ for all $\sigma < \alpha/2$, then,

$$K_3 \lesssim (t-s)^{\alpha} \int_0^s (s-\tau)^{\alpha-2\gamma} \tau^{\kappa-1} e^{-\mu(s-\tau)} d\tau.$$ 

As before, Let $p = 1 + \epsilon$, and $q$ be its conjugate,

$$(\underline{\text{underline part}}) \leq \left( \int_0^s e^{-\mu(s-\tau)^q} d\tau \right)^{1/q} \left( \int_0^s (s-\tau)^{(\alpha-2\gamma)p} \tau^{(\kappa-1)p} d\tau \right)^{1/p} \leq E_\mu s^{\alpha-2\gamma+1/p} s^{\kappa-1} B((\alpha-1-2\sigma)p+1, (\kappa-1)p+1)^{1/p}$$

where

$$E_\mu = \left( \int_0^T e^{-\mu \tau^p} d\tau \right)^{1/p}.$$ 

That is,

$$s^{1-\kappa+\gamma} K_3 \lesssim E_\mu (t-s)^{\alpha} s^{\alpha-2\gamma+1/p}.$$ 

We select $\sigma < \alpha/2$ and $\epsilon > 0$ such that

$$\begin{cases}
\alpha - 1 - 2\sigma + \gamma + \frac{1}{p} = \alpha - \gamma - 2\sigma - \frac{\epsilon}{1+\epsilon} > 0, \\
(\alpha - 1 - 2\sigma)p + 1 > 0, \\
(\kappa - 1)p + 1 > 0.
\end{cases}$$

Finally, we select again $\epsilon, \delta$ and $\sigma$ such that

$$\frac{\alpha}{2} \leq \min \left( \alpha - \frac{\epsilon}{1+\epsilon}, \alpha \delta \right), \quad \gamma \leq \sigma < \frac{\alpha}{2}$$
for some constant $C$ depends on the semigroup, existence time $T > 0$, $f'$ and $L$. Therefore, Corollary [B.1] yields
\[ \|w(t)\| \lesssim t^{\alpha-1}(1 + bt)^{2-\alpha} e^{bt+1} \lesssim t^{\alpha-1}. \]
where $b' = (CT(\alpha))^{1/\alpha}$. For given $u_1, u_2$,
\[ w_1(t) - w_2(t) = \int_0^t \int_0^\infty \theta h_o(\theta)(t-\tau)^{\alpha-1} T((t-\tau)^\alpha) (B'(u_1(\tau))w_1(\tau) - B'(u_2(\tau))w_2(\tau)) d\tau d\theta \]
\[ \|w_1(t) - w_2(t)\| \lesssim \int_0^t (t-\tau)^{\alpha-1} \|B'(u_1(\tau)) - B'(u_2(\tau))\| \|w_2(\tau)\| d\tau \]
\[ + \int_0^t (t-\tau)^{\alpha-1} \|B'(u_1(\tau))\| \|w_1(\tau) - w_2(\tau)\| d\tau \]
\[ \lesssim \int_0^t (t-\tau)^{\alpha-1} \tau^{\alpha-1} e^{\mu(t-\tau)} e^{-\mu\tau} \|w_1(\tau) - w_2(\tau)\| d\tau \]
\[ + \int_0^t (t-\tau)^{\alpha-1} e^{\mu(t-\tau)} e^{-\mu\tau} \|w_1(\tau) - w_2(\tau)\| d\tau \]
\[ e^{-\mu t} \|w_1(t) - w_2(t)\| \lesssim t^{2\alpha-1} \|u_1 - u_2\| + \int_0^t (t-\tau)^{\alpha-1} e^{-\mu\tau} \|w_1(\tau) - w_2(\tau)\| d\tau \]
Therefore, we use again Corollary [B.1] for $e^{-\mu t} \|w_1(t) - w_2(t)\|$ to obtain
\[ \|w_1(t) - w_2(t)\| \lesssim e^{\mu t} t^{2\alpha-1}(1 + bt)^{2-2\alpha} \|u_1 - u_2\| \]
for appropriate $b > 0$. Hence,
\[ \|U_1(t) - U_2(t)\| \lesssim \int_0^t e^{\mu\tau} t^{2\alpha-1}(1 + b\tau)^{2-2\alpha} d\tau \|u_1 - u_2\| \]
\[ e^{-\mu t} \|U_1(t) - U_2(t)\| \lesssim \int_0^t e^{-\mu(t-\tau)} t^{2\alpha-1}(1 + b\tau)^{2-2\alpha} d\tau \|u_1 - u_2\| \]
For small $\epsilon > 0$, we set $p = 1 + \epsilon$ and its conjugate $q$. Since $(1 + bt)^{2-2\alpha} \leq (1 + bT)^{2-2\alpha}$,
\[ e^{-\mu t} \|U_1(t) - U_2(t)\| \lesssim \int_0^t e^{-\mu(t-\tau)} t^{2\alpha-1} d\tau \|u_1 - u_2\| \]
\[ \lesssim \left( \int_0^t e^{-\mu(t-\tau)} q d\tau \right)^{1/q} \left( \int_0^t t^{2\alpha-1} p d\tau \right)^{1/p} \|u_1 - u_2\|. \]
We choose $\epsilon > 0$ such that
\[ (2\alpha - 1)p + 1 = (2\alpha - 1)(1 + \epsilon) + 1 = 2\alpha + (2\alpha - 1)\epsilon > 0. \]
So we can take sufficiently large $\mu$ so that $u \mapsto U$ becomes contraction mapping. Then, the fixed-point $u$ meets the following properties;
\[ \left\{ \begin{array}{l}
\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\tau)^{-\alpha} u'(\tau) d\tau + Au'(t) + B'(u(t))u'(t) = f'(t), \\
\frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} u'(\tau) d\tau \rightarrow x_0.
\end{array} \right. \]
Integrating both sides, we deduce
\[ \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} u'(\tau) d\tau - x_0 + A(u(t) - u_0) + B(u(t)) - B(u_0) = f(t) - f(0). \]
If we choose
\[ x_0 = -Au_0 - B(u_0) + f(0), \]
the fixed-point \( u \) coincides with the unique solution of
\[
\begin{cases}
D_t^\alpha (u - u_0) + Au + B(u) = f, \\
u(0) = u_0.
\end{cases}
\]
This completes the proof. \( \square \)

6 Proof of Theorem 1.3

In this section, we investigate the locally Lipschitz perturbation problem. For the discussion on the extension of the solution, we refer to [22, 23].

**Proof of Theorem 1.3** We begin with the construction of a local solution. Let
\[
\tilde{B}(t, u) = \begin{cases}
B(t, u) & \text{if } ||u|| \leq M, \\
B \left( t, \frac{M}{||u||} \right) & \text{if } ||u|| > M.
\end{cases}
\]
We can easily see that \( \tilde{B} \) is globally Lipschitz continuous. Therefore, there exists a unique solution \( u \) for
\[
D_t^\alpha (u - u_0) + Au + \tilde{B}(u) = f, \quad u(0) = u_0.
\]
We take sufficiently large \( M \) such that \( ||u_0|| < M/2 \), then, the continuity of \( u \) guarantees the existence of \( T > 0 \) such that
\[
||u(t)|| < M \quad \forall t \in [0, T].
\]
It means that \( u \) satisfies \( \Box \) on \( [0, T] \). We assume \( T^* < \infty \), and there exists \( C > 0 \) such that for any \( \epsilon > 0 \), we can take \( t_\epsilon \in (T^* - \epsilon, T^*) \) such that \( ||u(t_\epsilon)|| \leq C \). We set
\[
\mathcal{E}_{h, \delta} := \{ v \in C^0([t_\epsilon, t_\epsilon + h]; X); \|v(t_\epsilon)\| = ||u(t_\epsilon)||, \|v - u(t_\epsilon)\|_{C([t_\epsilon, t_\epsilon + h]; X)} \leq \delta \},
\]
\[
\| \cdot \|_{E_{h, \delta}} = \| \cdot \|_{C^0([t_\epsilon, t_\epsilon + h]; X)}
\]
and for each \( v \in E_{h, \delta}, \ t \in [t_\epsilon, t_\epsilon + h], \)
\[
\mathcal{T}(v)(t) := \int_0^\infty h_\alpha(\theta)T(t^\alpha \theta)u_0 d\theta + \alpha \int_0^t \int_0^\infty (t - \tau)^{\alpha - 1} \theta h_\alpha(\theta) T((t - \tau)^\alpha \theta) [f(\tau) - B(u(\tau))] d\tau d\theta
\]
\[
+ \alpha \int_t^\infty \int_0^\infty (t - \tau)^{\alpha - 1} \theta h_\alpha(\theta) T((t - \tau)^\alpha \theta) [f(\tau) - B(v(\tau))] d\tau d\theta.
\]
For any \( v \in E_{h, \delta} \), the representation of mild solution provides \( \mathcal{T}(v)(t_\epsilon) = u(t_\epsilon) \) and
\[
\| \mathcal{T}(v)(t) - u(t_\epsilon) \| = \| \mathcal{T}(v)(t) - \mathcal{T}(v)(t_\epsilon) \|
\]
\[
= \int_0^\infty h_\alpha(\theta) [T(t^\alpha \theta) - T(t_\epsilon^\alpha \theta)] u_0 d\theta
\]
\[
+ \alpha \int_0^t \int_0^\infty (t - \tau)^{\alpha - 1} \theta h_\alpha(\theta) [(t - \tau)^\alpha T((t - \tau)^\alpha \theta) - (t_\epsilon - \tau)^{\alpha - 1} T((t_\epsilon - \tau)^\alpha \theta)] [f(\tau) - B(u(\tau))] d\tau d\theta
\]
\[
+ \alpha \int_t^\infty \int_0^\infty (t - \tau)^{\alpha - 1} \theta h_\alpha(\theta) T((t - \tau)^\alpha \theta) [f(\tau) - B(v(\tau))] d\tau d\theta.
\]
As in the proof of Theorem 1.1, we estimate \( \| \mathcal{T}(v)(t) - u(t_\epsilon) \| \leq (t - t_\epsilon)^2 \). This inequality yields the existence of \( \bar{h} > 0 \) such that if \( h < \bar{h} \), then \( \| \mathcal{T}(v)(t) - u(t_\epsilon) \| \leq \epsilon \), i.e., \( \mathcal{T} \) maps \( E_{h, \delta} \) to itself. We note that we can choose \( \bar{h} \)}.
independently of $\varepsilon$. For arbitrary $v_1, v_2 \in E_{h, \delta}$, $\|B(v_1(t)) - B(v_2(t))\| \leq L(u(t) + \delta)\|v_1(t) - v_2(t)\|$. We write in abbreviated form of $L(u(t) + \delta) \leq L(C + \delta) =: L$.

$$\|T(v_1(t)) - T(v_2(t))\| \leq \frac{LC_1}{\Gamma(\alpha)} \int_{t_s}^{t} (t - \tau)^{\alpha-1}\|v_1(\tau) - v_2(\tau)\|d\tau$$

$$\leq \frac{LC_1 h^\alpha}{\alpha \Gamma(\alpha)}\|v_1 - v_2\|_{E_{h, \delta}}.$$ 

Also, let $U = T(v_1) - T(v_2)$, then, for each $t_s \leq s \leq t \leq t_s + h$,

$$\alpha^{-1}(U(t) - U(s)) = \int_{t_s}^{t} (t's formula) \pm \int_{t_s}^{s} (t's formula) - \int_{t_s}^{s} (s's formula)$$

$$= \int_{t_s}^{t} (t's formula) + \int_{t_s}^{s} [(t's formula) - (s's formula)]$$

$$\|U(t) - U(s)\| \leq \frac{C_1 L}{\Gamma(\alpha)} \int_{s}^{t} (t - \tau)^{\alpha-1}\|v_1(\tau) - v_2(\tau)\|d\tau$$

$$+ \frac{C_1 L}{\Gamma(\alpha)} \int_{t_s}^{s} |(s - \tau)^{\alpha-1} - (t - \tau)^{\alpha-1}|\|v_1(\tau) - v_2(\tau)\|d\tau$$

$$+ \alpha L \int_{t_s}^{s} \int_{0}^{\infty} \theta h_{\alpha}(\theta) (s - \tau)^{\alpha-1} |T((t - \tau)^\alpha \theta) - T((s - \tau)^\alpha \theta)|\|v_1(\tau) - v_2(\tau)\|d\theta$$

$$\leq (J_1 + J_2 + J_3)\|v_1 - v_2\|_{E_{h, \delta}}$$

where

$$J_1 = \frac{C_1 L}{\Gamma(\alpha)} \int_{s}^{t} (t - \tau)^{\alpha-1}d\tau,$$

$$J_2 = \frac{C_1 L}{\Gamma(\alpha)} \int_{t_s}^{s} |(s - \tau)^{\alpha-1} - (t - \tau)^{\alpha-1}| d\tau,$$

$$J_3 = \alpha L \int_{t_s}^{s} \int_{0}^{\infty} \theta h_{\alpha}(\theta) (s - \tau)^{\alpha-1} |T((t - \tau)^\alpha \theta) - T((s - \tau)^\alpha \theta)|d\tau.$$ 

We estimate

$$\|J_1\| \leq \frac{C_1 L}{\alpha \Gamma(\alpha)}(s^\alpha - s^\alpha) \leq \frac{C_1 C_1 L}{\alpha \Gamma(\alpha)}h^{\alpha - \beta}(t - s)^\beta.$$ 

where $C_\alpha$ is the Hölder norm of $\tau \mapsto \tau^\alpha$. Furthermore,

$$\|J_2\| = \frac{C_1 L}{\alpha \Gamma(\alpha)}[(s - t_s)^{\alpha} - (t - t_s)^{\alpha} + (t - s)^{\alpha}]$$

$$\leq \frac{2C_\alpha C_1 L}{\alpha \Gamma(\alpha)}h^{\alpha - \beta}(t - s)^\beta$$

Also, we take auxiliary number $0 < \eta < 1$ to obtain

$$\|J_3\| \leq \frac{C_\alpha \eta \Gamma(2 - \eta)}{\Gamma(1 + \alpha(1 - \eta))} \int_{t_s}^{s} (s - \tau)^{\alpha(1 - \eta) - 1}(t - s)^{\alpha \eta}$$

$$\leq \frac{C_\alpha \eta \Gamma(2 - \eta)}{\Gamma(1 + \alpha(1 - \eta))}h^{\alpha(1 - \eta)}(t - s)^{\alpha \eta}$$

where $C_{\alpha, \eta}$ is appropriate constant depends only on $\alpha, \eta$. We take $\eta$ such that $\beta = \alpha \eta$. Therefore, we choose sufficiently small $h$ such that

$$h^{\alpha - \beta} \leq \frac{1}{2} \min \left( \frac{\alpha \Gamma(\alpha)}{C_1 L}, \frac{\alpha \Gamma(\alpha)}{C_\alpha C_1 L}, \frac{\alpha \Gamma(\alpha)}{2C_\alpha C_1 L}, \frac{\alpha \Gamma(\alpha)}{C_{\alpha, \eta} \Gamma(2 - \eta)} \right) \quad \text{and} \quad h < \tilde{h}$$
in order to make $\mathcal{T}$ be a contraction. Then, its fixed point $v = \mathcal{T}v$ satisfies

\[
v(t) = \int_0^\infty h_\alpha(\theta)[T(t^\alpha\theta)u_0]d\theta + \alpha \int_0^t \int_0^\infty (t - \tau)^{\alpha-1} h_\alpha(\theta)[T((t - \tau)^\alpha\theta)\varepsilon f(\tau) - B(u(\tau))]d\tau d\theta \\
+ \alpha \int_t^\infty \int_0^\infty (t - \tau)^{\alpha-1} h_\alpha(\theta)[T((t - \tau)^\alpha\theta)\varepsilon f(\tau) - B(u(\tau))]d\tau d\theta.
\]

Let

\[
w(t) = \begin{cases} u(t) & \text{for } t \in [0, t_e], \\
v(t) & \text{for } t \in [t_e, t_e + h],
\end{cases}
\]

then $w$ satisfies for all $t \in [0, t_e + h]$

\[
w(t) = \int_0^\infty h_\alpha(\theta)[T(t^\alpha\theta)u_0]d\theta + \alpha \int_0^t \int_0^\infty (t - \tau)^{\alpha-1} h_\alpha(\theta)[T((t - \tau)^\alpha\theta)\varepsilon f(\tau) - B(w(\tau))]d\tau d\theta,
\]

this means that $w$ satisfies

\[
D^\alpha(w - u_0) + Aw + B(w) = f(t, x) \quad \text{in } (0, t_e + h], \ w(0) = u_0
\]

because of the Hölder continuity of $\tau \mapsto B(w(\tau))$. Unrelation between $\epsilon > 0$ and $h > 0$ enables us to take $\epsilon < h$ and deduce contradiction.

7 Application

Let us consider the following combustion equation

\[
D^\alpha(u - u_0) - \Delta u - e^{-1/u}\chi_{\{u > 0\}} = f(t).
\]  

(7.1)

This model describes some reaction-diffusion processes with the generation of heats, based on the Arrhenius law which states the reaction rates is proportional to $e^{-C/u}$ for some $C > 0$, where $u$ represents the absolute temperature [15]. The perturbation part

\[
r \mapsto \begin{cases} e^{-1/r} & \text{if } r > 0, \\
0 & \text{if } r \leq 0
\end{cases}
\]

is obviously Lipschitz, so we apply Theorem 1.1 and Theorem 1.2 with the following framework;

\[
X = u \in C(\overline{\Omega}), \\
D(A) = \left\{ u \in \bigcap_{p \geq 1} W^{2,p}(\Omega); u, \Delta u \in C(\overline{\Omega}), \ u|_{\partial\Omega} = 0 \right\}.
\]  

(7.2)

where $\Omega$ is bounded and its boundary is sufficiently smooth. It is known that the interpolation property

\[
(X, D(A))_{\theta, \infty} = \left\{ u \in C^2(\overline{\Omega}); u|_{\partial\Omega} = 0 \right\}
\]

is satisfied [2].

**Theorem 7.1.** Let $T > 0$ be arbitrary and $f \in \mathcal{F}^\beta([0, T]; C(\overline{\Omega}))$ for some $0 < \beta < r \leq 1$ and $0 < \max(\beta, 1 - \alpha) < r$. Then,

\[
\left\{ D^\alpha(u', x) - u_0(x))(t) - \Delta u(t, x) - e^{-1/u}\chi_{\{u(x, t) > 0\}} = f(t, x), \quad (t, x) \in (0, T) \times \Omega, \\
u(t, x) = 0 \quad (t, x) \in [0, T] \times \partial\Omega,
\right.

\]  

(7.4)

has the unique solution

\[
u \in C([0, T]; C(\overline{\Omega})) \cap C((0, T]; D(A)) \cap \mathcal{F}^r, \beta((0, T]; X)
\]

If $f$ is differentiable with respect to $t$ and its derivative is of class $\mathcal{F}^{r', \beta'}((0, T]; C(\overline{\Omega}))$ for $0 < \beta' < r' \leq 1$, $A u_0 + B(u_0) - f(0) \in D(A)$, then $u$ is time differentiable and $u' \in \mathcal{F}^{r', \beta'}((0, T]; X)$.  

20
We provide the overview of the fractional power of operator. See [1], [2], [5] for details.

Proposition 2.4 provides further properties to (7.1).

**Proposition 7.1.** Assume \( f \geq 0, u_0 \geq 0 \) and \( f, u_0 \) be sufficiently regular. Then, \( u \geq 0 \).

**Proof.** The selection of Banach space (7.2) and Proposition 2.2 yield the twice space differentiability. If the solution \( u \) attain its minimum at \((t_0, x_0) \in (0, T) \times \Omega \) and \( u(t_0, x_0) < 0 \), then Proposition 2.4 provides

\[
0 > D_t^\alpha (u(\cdot, x_0) - u_0(x_0)) - \Delta u(t_0, x_0) = f(t_0, x_0),
\]

it is nothing but a contradiction.

\[ \square \]

### 8 Acknowledgement

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### A Power of operator

We provide the overview of the fractional power of operator. See [1], [2], [5] for details. Let \( A \) be sectorial, then we define its power for \( \Re z > 0 \) by

\[
A^{-z} := \frac{1}{2\pi i} \int_G \lambda^{-z}(\lambda - A)^{-1}d\lambda,
\]

where we take the branch

\[
\lambda^{-z} = \exp[-z(\log|\lambda| + i \arg \lambda)],
\]

that is, single-valued function on \( \mathbb{C} \setminus (-\infty, 0] \). Also, we take the integral path \( \Gamma = \Gamma_- \cup \Gamma_+ \cup \Gamma_0 \), where

\[
\Gamma_\pm : \lambda = \rho e^{\pm i\omega}, \ \delta \leq \rho < \infty,
\]

\[
\Gamma_0 : \lambda = \delta e^{i\varphi}, \ -\omega \leq \varphi \leq \omega,
\]

for \( \delta < \|A^{-1}\|^{-1} \). The evaluation

\[
|\lambda^{-z}| = |\exp[-z(\log \rho \pm i\omega)]| = e^{\pm (2\pi z)\rho - \Re z}, \ \lambda \in \Gamma_\pm
\]

guarantees that the definition of \( A^{-z} \) makes sense. When \( z = n \in \mathbb{N} \), we transform the path into \( |\lambda| = \delta \) so that we confirm

\[
\frac{1}{2\pi i} \int_{C_\delta} \lambda^{-n}(\lambda - A)^{-1}d\lambda
\]

\[
= -\frac{1}{(n-1)!} \left[ \frac{d^{n-1}}{d\lambda^{n-1}}(\lambda - A)^{-1} \right]_{\lambda=0}
\]

\[
= (-1)^n[(\lambda - A)^{-n}]_{\lambda=0}
\]

\[
= A^{-n}.
\]

where we retake the path \( C_\delta = \{ \lambda \in \mathbb{C}; |\lambda| = \delta \} \). For all \( \Re z, \Re z' > 0 \), we can show the semigroup porperty

\[
A^{-z}A^{-z'} = A^{-(z+z')}
\]

by the standard way. We define \( A^n \) as the inverse of \( A^{-n} \). Let \( z \in \mathbb{R} \) is not an integer. If \( A^{-z}u = 0 \), then for any \( n > z \),

\[
A^{-n}u = A^{-(n-z)}A^{-z}u = 0
\]

Therefore \( u = 0 \). This means that \( A^{-z} \) has inverse. So we define \( A^z = (A^{-z})^{-1}, D(A^z) = R(A^{-z}) \). We can show

\[
R(A^{-z}) \subset R(A^{1-z}), \ i.e., D(A^{z}) \subset D(A^{1-z}) \text{ since } 0 < \Re z_1 < \Re z_2 \text{ implies } A^{z_2} = A^{z_1}A^{(z_2-z_1)}.
\]

We list the important properties for the power of operator theory. We refer to [5], [1], [2] for the detailed proof.
Proposition A.1. Let $A$ be a sectorial operator on Banach space $X$. For $0 < \theta < \theta' < 1$, the following inclusion holds.

$$(X, D(A))_{\theta', \infty} \subset (X, D(A))_{\theta, 1} \subset D(A^{\theta}) \subset (X, D(A))_{\theta, \infty}$$

Proposition A.2. Let $\omega < \pi/2$. Then, for any $0 < t < \infty$ and $0 < \theta < \infty$,

$$\|A^{\theta}T(t)\|_{B(X)} \leq \frac{M\Gamma(\theta)}{\cos\omega} \frac{1}{t^\theta}$$

holds.

Proof. Let

$$E^{\theta}(t) := \frac{1}{2\pi i} \int_{C} \lambda^{\theta} e^{-t\lambda} (\lambda - A)^{-1} d\lambda$$

where

$$\Gamma_{\pm} : \lambda = \rho e^{\pm i\omega}, \ 0 \leq \rho < \infty$$
$$\Gamma_{0} : \lambda = \delta e^{i\varphi}, -\omega \leq \varphi \leq \omega.$$

We can easily check $A^{-\theta}E^{\theta}(t) = E^{\theta}(t)A^{-\theta} = T(t)$, i.e.,

$$A^{\theta}T(t) = \frac{1}{2\pi i} \int_{C} \lambda^{\theta} e^{-t\lambda} (\lambda - A)^{-1} d\lambda,$$

which yields

$$\|A^{\theta}T(t)\|_{B(X)} \leq \frac{M}{\pi} \int_{\delta}^{\infty} \rho^{\theta-1} e^{-t\rho \cos\omega} d\rho + \frac{M}{2\pi} \int_{-\omega}^{\omega} \delta^{\theta} e^{-t\delta \cos\varphi} d\varphi.$$

Passing $\delta \to 0$ completes the proof. \hfill \square

B Special functions

We summarize some properties of special functions such as the Gamma function denoted by

$$\Gamma(z) := \int_{0}^{\infty} t^{z-1} e^{-t} dt \ \Re z > 0,$$

and Beta function denoted by

$$B(x, y) := \int_{0}^{1} (1 - t)^{x-1} t^{y-1} dt \ \Re x, \ Re y > 0.$$

These functions satisfy the following properties (see for instance [13]):

(i) $\Gamma(1 + x) = x\Gamma(x) \ \forall x > 0,$

(ii) $B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)},$

(iii) $\frac{1}{\Gamma(x)} = \frac{1}{2\pi i} \int_{C_H} e^{\lambda} \lambda^{-x} d\lambda$ (Hankel’s formula)

where $C_H$ is an Hankel contour;

$$C_H = C_+ \cup C_- \cup C_0,$$
$$C_+ : \Im \lambda = \epsilon, \ \Re \lambda : -\infty \to 0,$$
$$C_- : \Im \lambda = -\epsilon, \ \Re \lambda : 0 \to -\infty,$$
$$C_0 : e^{i\varphi}, -\pi/2 \leq \varphi \leq \pi/2.$$

for some $\epsilon > 0$. We also introduce the Mittag-Leffler function for $0 \leq t < \infty$

$$E_{\mu, \nu}(t) := \sum_{n=0}^{\infty} \frac{t^{\nu}}{\Gamma(\mu + n\nu)}$$

which is evaluated as follows.
**Lemma B.1.** (see [5]) We estimate

$$E_{\mu,\nu}(t) \leq \frac{2}{\Gamma(\mu)}(1+t)^{2-\mu}e^{t+1}$$

where $\Gamma_0 = \min_{0<\xi<\infty} \Gamma(\xi)$.

Suppose that $\varphi(t,s)$ is real valued function defined for $0 \leq s < t \leq T$ satisfying

$$0 \leq \varphi(t,s) \leq C(t-s)^{\epsilon-1}$$

for some $\epsilon > 0$ and $C > 0$. We introduce the following integral inequality.

**Proposition B.1.** (see [5]) Let $a, b, \mu, \nu > 0$ be constant. If $\varphi$ satisfies

$$\varphi(t,s) \leq a(t-s)^{\mu-1} + b \int_s^t (t-\tau)^{\nu-1}\varphi(\tau,s)d\tau, \quad 0 \leq s < t \leq T,$$

then, the following evaluation holds,

$$\varphi(t,s) \leq a\Gamma(\mu)(t-s)^{\mu-1}E_{\mu,\nu}\left((b\Gamma(\nu))^{1/\nu}(t-s)\right), \quad 0 \leq s < t \leq T.$$

Suppose that $w(t)$ is real valued function for $0 < t \leq T$ satisfying

$$0 \leq w(t) \leq Ct^{\epsilon-1}$$

for some $\epsilon, C > 0$. Replacement of $\varphi(t,s) = w(t-s)$ yields the next corollary.

**Corollary B.1.** We make the same assumptions as in Proposition B.1. If $w$ meets

$$w(t) \leq a\mu^{\mu-1} + b \int_0^t (t-\tau)^{\nu-1}w(\tau)d\tau,$$

then, $w$ satisfies

$$w(t) \leq a\Gamma(\mu)t^{\mu-1}E_{\mu,\nu}\left((b\Gamma(\nu))^{1/\nu}t\right).$$

Especially,

$$w(t) \leq \frac{2a\Gamma(\mu)}{\Gamma(\mu)}t^{\mu-1}(1+b't)^{2-\mu}e^{b't+1}$$

where $b' = (b\Gamma(\nu))^{1/\nu}$.

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