Rectangular Section Model for Analysis of the Resistance of Eccentrically Compressed Masonry Structures

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Abstract. The paper presents section model for analysis of the resistance of masonry structures with rectangle cross-section subjected to the eccentric compression. The nonlinear stress-strain relationship for masonry in compression is assumed taking into account the effect of masonry softening. The formulae for normalized ultimate axial force and bending moment are derived in the analytical form by integrating the cross-sectional equilibrium equations. The calculated results are presented in the form of interaction diagrams and compared with those based on the parabola-rectangle diagram for masonry in compression. It has been shown that the section resistance is strongly influenced by the masonry softening and depends on the limiting value \( \varepsilon_{mu} \) for masonry in compression.

1. Introduction
The resistance of masonry structures subjected to the eccentric compression is considered as a theoretical problem as well as an experimental one. Such structures and members are frequently encountered in the engineering practice, e.g. columns, industrial chimneys, antenna carriers. For determining the resistance of cross-sections of the structures like these a variety of physical models of materials and methods are applied. Most often a simplified approach is used based on the linear or rectangular stress distribution in masonry [4, 6, 12, 13]. For the design of cross-sections parabola-rectangle diagram for masonry may be also assumed which was introduced by Nieser and Engel in German code [9] as well as is recommended in above mentioned codes [12, 13]. For analysis of the resistance of cross-sections deformation models combined with physical nonlinearity of masonry have been proposed by Lechman [6, 7] based on the above mentioned parabola-rectangle diagram. In analytical modelling of masonry structures the stress-strain relationship of masonry \( \sigma_m - \varepsilon_m \) is of primary importance. Therefore, big efforts have been made by many researchers to describe adequately the behaviour of masonry under compression based on the results of the comprehensive experimental study [1, 2, 5, 8, 10]. For this purpose numerical techniques are proposed to attain appropriate material parameters for both ascending and descending parts of \( \sigma_m - \varepsilon_m \) curve. The stress-strain relationships \( \sigma_m - \varepsilon_m \) for masonry recommended in EN 1996-1-1 and PN-EN 1996-1-1 [12, 13] are interpreted in figure 1. Despite the variety of methods and calculation procedures concerning this problem, there are no appropriate analytical solutions based on the nonlinear material law for determining the section resistance with allowance for the masonry softening. The paper presents the derivation of analytical formulae determining the resistance of masonry rectangular cross-sections of structures subjected to the eccentric compression, based on the nonlinear stress-strain relationship for masonry \( \sigma_m - \varepsilon_m \). The stress-strain relation of masonry \( \sigma_m - \varepsilon_m \) in compression for short-term uniaxial loading is adopted from EC2 for nonlinear structural analysis [11], assumed as:
\[ \sigma_m = \frac{k \eta_m - \eta_m^2 \eta_m}{1 + (k - 2) \eta_m^2} f \]  

where: \( \eta_m = \epsilon_m/\epsilon_{m1} \), \( \epsilon_{m1} \) – the strain at peak stress on the \( \sigma_m - \epsilon_m \) diagram, \( k = 1.05 \frac{E_{mm} \mid \epsilon_{m1} \mid}{f} \), \( E_{mm} \) – secant modulus of elasticity of masonry. This stress-strain relation adequately represents the behaviour of masonry units of group 1 by introducing four parameters: \( f, \epsilon_{m1}, \epsilon_{mu} \) and \( E_{mm} \).

Figure 1. Representation of the stress-strain relation for masonry \( \sigma_m - \epsilon_m \): 1) actual diagram 2) ideal parabola-rectangle diagram 3) design parabola-rectangle diagram

2. Derivation of formulae for the resistance
The rectangular cross-section is subjected to the axial force \( N \) and the bending moment \( M \) (Figure 2).

Let us consider the section under combined compression and bending.

In the presented derivation the following assumptions are introduced:
- Plane cross-sections remain plane
- Elasto-plastic stress-strain relationship for masonry is used
- The tensile strength of masonry is ignored
- The ultimate strain for masonry is defined as \( \epsilon_{mu} \).

In further considerations the corresponding dimensionless coordinates are used:

\[ \xi = x / t, \quad \xi' = x' / t, \]  

where: \( \xi \), \( x \) – coordinate describing the location of the neutral axis, \( \xi', x' \) – coordinate of any point of the cross-section, \( t, b \) – the thickness and the width of cross-section, respectively.

Due to the Bernoulli assumption one obtains:

\[ \epsilon = (1 - \frac{\xi'}{\xi}) \epsilon' \]  

where: \( \epsilon' \) - the maximum compressive strain in masonry.
The equilibrium equation of the axial forces in the cross-section takes the following form:

\[ \int_{A_m} \sigma_m dA_m + N = 0 \]  

(4)

where \( dA_m \) – element of the masonry area \( A_m \).

The sectional equilibrium of the bending moments about the symmetry axis of the rectangle can be expressed in the form:

\[ \int_{A_m} \sigma_m (0.5t - x') dA_m - M = 0 \]  

(5)

Figure 2. The rectangular cross-section. Distribution of strain \( \varepsilon \) and stresses in masonry \( \sigma_m \) across the section
On the basis of a combinatorial approach, all possible cases of stress distribution in masonry have been considered. Taking into account the physical relation (1) and geometrical relationship (3) in the equilibrium equations (4) and (5), the problem leads to the purely mathematical task consisted in the searching the antiderivatives of the following functions of variable $\xi$

\[ f_N(\xi) = \frac{k(k - \xi')k_2 - (k - \xi')k_2}{1 + (k - 2)(\xi' - \xi)} \]

\[ f_M(\xi) = \frac{k(k - \xi')k_2 - (k - \xi')k_2}{1 + (k - 2)(\xi' - \xi)} (0.5 - \xi') \]

where: $k_2 = \epsilon' / (\epsilon_m \xi')$.

Using the found antiderivatives of functions (6) – (7), one obtains the formulae for the normalized ultimate cross-sectional forces $n_{mu}$ and $m_{mu}$:

\[ n_{mu} = \frac{(1/(k - 2))\left[ W_2 + 0.5k_2\xi^2 - (1/(k - 2))\left[(W_2 - W_3)\ln W - \xi\right)\right]}{W_1} \]

\[ m_{mu} = \frac{(1/(k - 2))\left[ 0.5(W_1 + (1/(k - 2)))\xi + 0.5\left[ -W_1 + 0.5k_2 - \left[(1/(k - 2))\right]\xi^2 + \left[ -k_2\xi^3 - \left(1/(k - 2)\right)\right]\xi^3 + \left[ 0.5\ln W + \xi - \left(1/k\right)\right] W_1 + \xi - \left(1/k\right)\right] W_2}\right]}{W_1} \]

$N_u, M_u$ - ultimate values of axial force and bending moment,

\[ n_{mu} = \frac{N_u}{bf}, \quad m_{mu} = \frac{M_u}{br^2f}, \]

where: $k_2 = \epsilon' / (\epsilon_m \xi')$, $W_1 = k - k_2 \xi$, $W_2 = k - 2k_2 + 1$, $W_3 = k - (k - 2)k_2$, $W_4 = 1 + (k - 2)k_2 \xi'$; $\delta = 0.5((-1)^k + 1)$.

In a similar way one obtains the formulae for rectangular cross-sections wholly in compression. The geometrical relationship can be expressed in this case as:

\[ \xi = \frac{E_{mu}}{-\xi + E_{mu}}, \quad \epsilon' = \epsilon_{mu}, \quad -\left(1/k\right)\epsilon = \epsilon_i \]

where: $k_2 = \epsilon' / (\epsilon_i \xi')$.

The obtained set of equations (8) – (11) and (13) – (15) describe the cross-section under consideration at any stage. The resistance of masonry cross-section is reached when the ultimate compressive strain...
in masonry $\varepsilon_{mu}$ is reached anywhere in that section. This means that the following conditions shall be satisfied:

$$\varepsilon' = \varepsilon_{mu},$$  \hspace{1cm} (16)

$$-(\frac{1}{\xi} - 1)\varepsilon' = \varepsilon_t, \quad -3.5 \leq \varepsilon_t \leq 10. \hspace{1cm} (17)$$

For parabola-rectangle diagram for masonry, the corresponding formulae for $n_{mu}$, $m_{mu}$ take the forms [7]:

$$n_{mu} = \frac{\xi_m - \frac{2}{\xi_{m1}}(\varepsilon' - \xi_m)}{\xi_m - \xi_m} - \frac{1}{2} \frac{\varepsilon'}{\xi_m} (\xi_m - \xi_m) - \frac{1}{2} \frac{\xi_m}{\xi_m} \frac{\varepsilon_m - \xi_m}{\xi_m} + \frac{1}{3} \frac{1}{\xi_m} (\xi_m - \xi_m)^3 \right), \hspace{1cm} (18)$$

$$m_{mu} = \frac{1}{4} \left[ \frac{2}{\xi_m} (\xi_m - \xi_m^2) \right] + \frac{1}{2} \frac{\varepsilon'}{\xi_m} (\xi_m - \xi_m) - \frac{1}{2} \frac{1}{\xi_m} (\xi_m - \xi_m^2) + \frac{1}{3} \frac{1}{\xi_m} (\xi_m - \xi_m^3) + \frac{1}{3} \frac{1}{\xi_m} (\xi_m - \xi_m^4) \right], \hspace{1cm} (19)$$

where: $\xi_m$ - coordinate describing the depth of the plastifying zone of masonry.

3. Results and discussions

The obtained formulae (8) – (15) describe the resistance of rectangular cross-section under consideration as a function of the limiting value of the masonry compressive strain $\varepsilon_{mu}$. If $|\varepsilon_{mu}| > |\varepsilon_m|$, certain masonry instability may occur for the section wholly in compression. It means that Drucker’s stability postulates [3] may not be satisfied for this case. Using the above derived formulae and restrictions resulting from Drucker’s stability postulates, the interaction diagrams with the normalized resistances $n_{mu} - m_{mu}$ have been obtained for the following data (Figure 3): the mean compressive strength of masonry $f = 20$ MPa, $E_m = 20$ GPa, the limiting value $\varepsilon_{mu} = -3.5\%$. The points on interaction curves correspond to the limiting compressive strain $\varepsilon_{mu} = -3.5\%$ and tensile (or less compressive) strain in masonry $\varepsilon_t$, $-3.5\% \leq \varepsilon_t \leq 10\%$. The curve obtained on the basis of the proposed solution (8) – (15) (solid line) in Figure 3 was compared with that resulting from the parabola-rectangle diagram (dashed line) for masonry in compression ($\varepsilon_m = -2.0\%$, $\varepsilon_{mu} = -3.5\%$). It indicates that the solutions based on the parabola-rectangle stress distribution in masonry may lead to the overestimation of the section resistance. It is also apparent that the section resistance is strongly influenced by the masonry softening interpreted as the descending part of the $\sigma_m - \varepsilon_m$ curve.

For $\varepsilon_{mu} = \varepsilon_m = -2\%$, the stability condition by Drucker is completely fulfilled and the obtained curves based on formulae (8) – (15) can be regarded as the carrying capacity curves in the meaning accepted in design codes (Figure 4). It is worth noting that this interaction curve (proposed solution, solid line) is very close to that based on the parabola-rectangle (parabola, dashed line). In a similar way, the corresponding formulae may be obtained for other section shapes.
Figure 3. Interaction curve \( n_{\mu u} - m_{\mu u} \) based on the proposed solution versus that based on the parabola-rectangle diagram (parabola), for masonry rectangular cross-section and the limiting value \( \varepsilon_{\mu u} = -3.5\% \) (at failure)

Figure 4. Comparison of the proposed solution based on the nonlinear relation \( \sigma_{\mu} - \varepsilon_{\mu} \) (1) with that based on the parabola-rectangle diagram (parabola), for masonry rectangular cross-section and limiting value \( \varepsilon_{\mu u} = -2.0\% \)
4. Conclusions
The analytical formulae have been derived for analysing the resistance of masonry rectangular cross-section subjected to the eccentric compression, based on the nonlinear stress-strain relationship for masonry that closely to reality represents the behaviour of masonry in compression. In a similar way the corresponding formulae for other section shapes may be obtained. The formulation like this enables to analyze the behaviour of the cross-section of masonry members and structures in the post-critical phase and in this respect it can be regarded as a valuable solution in the theory of masonry structures. Thus, it can be very useful as far as the prediction and verification of test results are concerned. The resistance of cross-sections is strongly influenced by the masonrysoftening represented by the descending part of the curve $\sigma_m - \varepsilon_m$ for masonry in compression. The proposed approach results in more realistic evaluation of the resistance of cross-sections compared to those based on the parabolic-rectangular diagram for masonry in compression. The presented method can be easily implemented and effectively used not only in the structural design and maintenance of unreinforced masonry structures, but in reinforced ones as well. Further experimental work is needed concerning the post-critical behaviour of masonry eccentrically compressed members and structures.

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