Opaque predicates, veiled sets and their logic

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Abstract. Motivated by considerations in the foundations of quantum mechanics and inspired by the literature on vague predicates, we introduce the concept of an opaque predicate. While in the case of vague predicates there is a kind of indeterminacy with respect to the predicate, in the sense that the vagueness concerns whether a well-determined object satisfies it or not, in the case of opaque predicates the indeterminacy is with regard to the objects which should satisfy them. In other words, their extensions are not well-defined, despite the fact that the conditions for an object to satisfy the predicates are well-known. We suggest that such opaque predicates (and more generally, what we call opaque relations) can be characterized by a logic which encompasses a semantics founded in quasi-set theory, and call their extensions veiled sets.

1. Vagueness and Opacity

"Vagueness is a feature of scientific as of other discourse."

Max Black (1966)

Peirce famously characterised vagueness in the following terms: “A proposition is vague when there are possible states of things concerning which it is intrinsically uncertain whether, had they been contemplated by the speaker, he would have regarded them as excluded or allowed by the proposition. By intrinsically uncertain we mean not uncertain in consequence of any ignorance of the interpreter, but because the speaker’s habits of language were indeterminate”.

In this context, two issues immediately arise: (i) whether vagueness can be dismissed as merely a feature of ‘natural’ language which will effectively evaporate with the introduction of some formal system, and (ii) given a negative answer to (i), whether vagueness requires the use of some form of non-classical logic. These are the issues with which standard discussions of vagueness have been concerned. Thus Wright, for example, emphasises that vague predicates lack ‘sharp boundaries’ and argues forcefully that “... the utility and point of the classifications expressed...”

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by many vague predicates would be frustrated if they were supplied with sharp boundaries” [39, p. 227]. In his own terms, the most ‘profound’ example he gives is that of colour predicates, where the elimination of vagueness would incur the price of jeopardising contact between language and empirical reality. Thus, vagueness is a phenomenon of ‘semantic depth’, in the sense that “[i]t is not usually a matter simply of our lacking an instruction where to ‘draw the line’; rather the instructions we already have determine that the line is not to be drawn” [ibid.]. However, if ‘the line is not to be drawn’, then clearly classical logic is not to be used. Following Putnam [32], Mott has recently suggested that vague predicates can be accommodated within intuitionistic logic if they are taken to be partial, in precisely the above sense: that is, in certain situations, their application simply cannot be decided at all [27]. Thus, the application of a colour predicate can be decided simply by using our sense organs, in those situations where it can be decided at all and it is precisely because of the existence of the complement of such situations that the predicate can be termed ‘vague’. At the end of his paper Mott suggests that intuitionistic mathematics might be the appropriate framework for quantum physics, noting that “[i]f sometimes there really is no fact of the matter whether a certain object is a table, or a certain shade is red, or a certain man bald, then perhaps sometimes there is no fact of the matter exactly where an electron is either” [ibid., p. 147].

However, to talk of there being no fact of the matter as to the location of an electron is to assume that one is talking about a well-defined individual, such that the application of the predicate regarding its location cannot be determined. As we have indicated elsewhere, this is certainly one way of considering the quantum situation (26, 16, 18). Nevertheless, our approach here is different and has not to our knowledge been explored in the literature. Quantum particles may also be regarded as ‘non-individuals’ in a certain sense [31] (see [14], [24]) and now the issues shift from the applicability of the predicates to the determinateness of the objects. It is this which marks the difference between vagueness and what we shall call opacity. With regard to the latter, it is not an issue of the predicate lacking ‘sharp boundaries’ but rather of the objects to which the predicate applies lacking individuality. In the case of opaqueness, then, the grounds for dismissing it as an aspect of natural language are even weaker than the case of vagueness. We would like to emphasise this point: if what quantum mechanics tells us about how the world could be is taken seriously, as we think it should be, then, in the sense delineated here, this ‘world’ is opaque (cf. M. Black who argues for an analysis of vagueness in order to avoid the “wholesale destruction of the formal sciences” [4, p. 27]). Granted this, a completely different formal framework is required to capture this opacity –one which includes both syntactic and semantic elements. It is precisely such a framework that we sketch below.

From a more mathematical point of view, vague predicates are considered to differ from the usual ‘Fregean’ predicates ([37]) in the following sense. A unary predicate letter $P$ of (say) a first-order language is Fregean if it provides a bipartition on a domain $D$ of objects to which the language makes reference. In other words, there exist $D_1$ and $D_2$ such that $D_1 \cap D_2 = \emptyset$, $D_1 \cup D_2 = D$ and $D_1 = \{x \in D : P(x)\} \neq \emptyset$.

\footnote{We have chosen this formulation as neutral between the claims of realist and anti-realist; see [38], for example.}
The set $D_1$ is the extension of $P$ and if $a \in D_1$, we say that ‘$a$ has the property $P’$; otherwise, that is, if $a \in D_2$, we say that ‘$a$ does not have the property $P’$. Vague predicates are then characterized as those predicates which do not provide such a bipartition in the domain. In other words, there are objects $a \in D$ such that neither $a \in D_1$ nor $a \in D_2$ holds. For these objects, it is ‘vague’ whether they have or not have the property ascribed by $P$.

The idea of vagueness provided by such an analysis, we emphasize, is concerned with the vagueness of the predicates involved, and not with the objects the language is making reference to. In fact, if we consider if a certain (well-determined) person, who could be classified as a philosopher, is or is not a profound thinker, then it may be vague whether he/she is profound or not, since the criteria for ‘profundity’ is vague. The same occurs with ‘tall’, ‘intelligent’ and so on. Returning to quantum mechanics, there are situations which simply cannot be characterized in terms of ‘vagueness’ in the above sense. We prefer to classify these cases in another fashion and call the predicates involved opaques. The following example, we hope, reinforces what was said above. Suppose we are dealing with a collection of $n$ electrons and that we intend to measure their spin in a certain chosen direction. In other words, we might consider the predicate ‘to have spin up in the (chosen) direction’. It is known that physicists are able to specify precisely what conditions electrons must obey in order to satisfy the predicate, hence the situation is not one concerning vagueness in the sense explained above. However, physicists have no means to determine which are the electrons of the aggregate that have spin up in the given direction. In fact, it turns out that there may be $m$ electrons ($m < n$), say, with spin up in that direction, but if another measurement is made, despite finding the same number $m$ of electrons with spin up in the considered direction, there is no way of assuring that both collections coincide; that is, we have no grounds for asserting that the electrons of this last collection are ‘the same’ as those of the former. There is a strong indeterminacy here concerning the objects of such collections, and this is one of the basic metaphysical interpretations of quantum mechanics.

Another example might be the following: Suppose we are considering the six electrons there are in the level $2p$ of a sodium atom. All the electrons coincide with respect to all their quantum numbers, so there is no way of distinguishing them. Even so, physicists reason as if there are six ‘entities’ (it is difficult to use the word ‘individuals’ in this case —see [36]) in that level. Then consider the predicate ‘to be one of the electrons of the level $2p$ of a sodium atom’. How may we ascribe to this predicate a well-defined extension? This is not possible without ambiguity.

Situations of this sort, in which the indeterminacy resides not with the predicate, but with the individuals instead, motivates our discussion on opaque predicates. The question to be answered now concerns the mathematical treatment of these entities.

2. A logic with opaque predicates

The basic intuitive idea of a logical system encompassing opaque predicates is that for such a predicates, their extensions cannot be defined as standard sets since such a set (Menge) is, according to the well known ‘definition’ coined by Cantor, “any collection into a whole of definite and separate objects of our intuition or of our thought” [3, p. 85]. In accordance with what we have said above, the goal is
to represent the idea that it is not possible to distinguish (that is, to ‘separate’) in a strong sense between the individuals that satisfy the predicate.

Our strategy will be as follows. We intend to characterize as opaque those predicates whose extension is a collection of objects (technically, a quasi-set) whose elements cannot be distinguished from one another. The problem is that, if we consider the predicates of a certain language, say a first-order language, we have no criteria for distinguishing among those predicates which should be considered as opaque. The system described below permits a clear definition of opaque predicates in accordance with the above motivations. The reasons to use such a logic will be explained in the last section.

In what follows we present the main features of the logic $\mathcal{L}_{\text{op}}$, which is a slight modified version of the system $\mathcal{L}_\omega$ presented in [2]. Let us begin by defining the concept of type: we call $\Pi$ the set of types, recursively defined as being the smallest set such that: (a) $e_1, e_2 \in \Pi$, and (b) if $\tau_1, \ldots, \tau_n \in \Pi$, then $\langle \tau_1, \ldots, \tau_n \rangle \in \Pi$.

$e_1$ and $e_2$ are the types of the individuals; the objects of type $e_1$ are called ‘m-objects’ (short for ‘microobjects’) and may be intuitively thought of as elementary particles of modern physics as in the example of opacity we have described in the previous section. The objects of type $e_2$ are such that classical logic applies to them in all its aspects.

In more precise words, $\mathcal{L}_{\text{op}}$ is a higher-order logic whose language has the following categories of primitive symbols: connectives: $\neg$ and $\rightarrow$ ($\land, \lor$ and $\leftrightarrow$ are introduced as usual), the universal quantifier $\forall$ ($\exists$ is defined in the standard way), parentheses and comma. With respect to variables and constants, for each type $\tau \in \Pi$ there exists a denumerably infinite collection of variables $X_1^\tau, X_2^\tau, \ldots$ of type $\tau$ and a (possibly empty) set of constants $A_1^\tau, A_2^\tau, \ldots$ of that type; we use $X^\tau, Y^\tau$ and $C^\tau, D^\tau$ perhaps with subscripts as metavariables for variables and constants of type $\tau$ respectively.

The terms of type $\tau$ are the variables and the constants of that type; so, we have individual terms of type $e_1$ and of type $e_2$. We use $U^\tau, V^\tau$, perhaps with subscripts, as syntactical variables for terms of type $\tau$. The atomic formulas are defined in the usual way: if $U^\tau$ is a term of type $\tau = \langle \tau_1, \ldots, \tau_n \rangle$ and $U_1^\tau, \ldots, U_n^\tau$ are terms of types $\tau_1, \ldots, \tau_n$ respectively, then $U^\tau(U_1^\tau, \ldots, U_n^\tau)$ is an atomic formula. Other formulas are defined as in a standard way.

The postulates of $\mathcal{L}_{\text{op}}$ (axiom schemata and inference rules) are the following:

(A1): $A$, where $A$ comes from a tautology in $\neg$ and $\rightarrow$ by uniform substitution of formulas of $\mathcal{L}_{\text{op}}$ for the variables.

(A2): $\forall X^\tau(A \rightarrow B) \rightarrow (A \rightarrow \forall X^\tau B)$, where $X^\tau$ does not occur free in $A$.

(A3): $\forall X^\tau A(X^\tau) \rightarrow A(U^\tau)$ where $U^\tau$ is a term free for $X^\tau$ in $A(X^\tau)$ and of the same type of $X^\tau$.

(R1): From $A$ and $A \rightarrow B$ to infer $B$.

(R2): From $A$ to infer $\forall X^\tau A$.

The syntactical concepts of free and bound occurrences of a variable in a term or in a formula, such as those of sentence (closed formula), theorem, consistent set of formulas, etc. can be defined without difficulty. The logic $\mathcal{L}_{\text{op}}$ still encompasses a comprehension axiom, which can be stated as follows:

If $U^\tau(X_1^\tau, \ldots, X_n^\tau)$ is a formula in which the variables $X_1^\tau, \ldots, X_n^\tau$ are free and if $X_i^\tau$ is a predicate of type $\tau = \langle \tau_1, \ldots, \tau_n \rangle$, then
The concept of identity is introduced for all objects except those denoted by terms of type $e_1$. This justifies the distinction we considered between the types of the individuals. The idea conforms itself with Schrödinger’s (see also [2] pp. 27–29), who said that the concept of identity does not make sense for the elementary particles of modern physics [34] pp. 17–18, [6]. The definition is briefly stated as follows:

**Definition 2.1.** If $\tau \neq e_1$, then:

$$U^\tau = V^\tau \iff \forall X^{(\tau)}(X^{(\tau)}(U^\tau) \leftrightarrow X^{(\tau)}(V^\tau))$$

Then our logic characterizes also in a syntactical way a certain category of objects (namely, those denoted by the terms of type $e_1$) about which we cannot say either that they are identical or that they are distinct. These objects play the role of those objects whose aggregates are intended to constitute the extensions of the opaque predicates. In this sense, we can introduce in a more precise way the following definition of what is to be understood by an opaque relation:

**Definition 2.2.** An opaque relation is any term $U^\tau$ of type $\tau$ = $\langle \tau_1, \ldots, \tau_n \rangle$ where every $\tau_i$ is obtained recursively from $e_1$ (that is, for every $i$, $\tau_i$ is $e_1$ itself, or $\langle e_1 \rangle$, or $\langle e_1, e_1 \rangle$ and so on).

It must be realized that we do not intend to enter here into the familiar discussion on what is to be considered as a ‘predicate’. We simply use a logical terminology and call a ‘predicate’ an unary relation of type $\langle i \rangle$, where $i$ is $e_1$ or $e_2$. Then, opaque predicates are particular cases of the opaque relations expressed by the above definition.

The problem, as we will recall in the beginning of the next section, is to find an adequate way of interpreting these predicates, since it would make no sense to ascribe a set, that is, a collection of distinguishable objects, as the extension of such predicates.

We can add axioms of extensionality and infinity in the standard way to our logic by adapting those of [20 Chap. 4]. Let us mention here only the case of the axiom of choice, which might be thought as problematic due to the senselessness of the concept of identity regarding the objects of type $e_1$. We may use a weaker form which covers only those situations in which there are no objects of type $e_1$ involved, which can be adapted from that presented in [20 p. 156]. More precisely, if $X_{1}^{\tau_1}$ and $X_{2}^{\tau_1}$ are variables such that $\tau_1 \neq e_1$, $Y_{1}^{\tau_2}$ and $Y_{2}^{\tau_2}$ are variables such that $\tau_2 \neq e_1$, $Z_{1}^{(\tau_1, \tau_2)}$ and $Z_{2}^{(\tau_1, \tau_2)}$ are also variables, then the axiom is:

$$\forall Z_{1}^{(\tau_1, \tau_2)} \exists Z_{2}^{(\tau_1, \tau_2)} (\forall X_{1}^{\tau_1} (\exists Y_{1}^{\tau_2} (Z_{1}^{(\tau_1, \tau_2)} (X_{1}^{\tau_1}, Y_{1}^{\tau_2}) \rightarrow 
\exists Y_{1}^{\tau_2} (Z_{2}^{(\tau_1, \tau_2)} (X_{1}^{\tau_1}, Y_{1}^{\tau_2}) \land Z_{1}^{(\tau_1, \tau_2)} (X_{1}^{\tau_1}, Y_{1}^{\tau_2})))) \rightarrow 
\forall X_{1}^{\tau_1} \forall X_{2}^{\tau_1} \forall Y_{1}^{\tau_1} \forall Y_{2}^{\tau_1} (Z_{2}^{(\tau_1, \tau_2)} (X_{1}^{\tau_1}, Y_{1}^{\tau_2}) \land Z_{2}^{(\tau_1, \tau_2)} (X_{1}^{\tau_1}, Y_{2}^{\tau_2}) \rightarrow Y_{1}^{\tau_2} = Y_{2}^{\tau_2}))$$

Other usual syntactical notions are defined in the standard way, such as for instance the concept of $\vdash A$, $\Gamma \vdash A$ for a set $\Gamma$ of sentences, and so on.

Now let us consider the semantical counterpart of our logic.
3. Semantics

3.1. The mathematical framework. Let us recall once more that within the scope of the logic \( L_{op} \), we can consider objects for which there is no meaning in talking about either their identity or their diversity. But, when we consider the semantical aspects of such a logic, in the sense of an association of certain objects of a mathematical structure to the terms of the language, it is convenient that such a procedure should reflect the intuitive aspects of the logic. Hence, it is convenient that the terms of type \( e_1 \) (those to which the usual concept of identity cannot be applied) should not have a well-defined interpretation. In fact, if we ascribe to the constants of the type \( e_1 \) well-determined individuals of the domain (elements of a set, say, as in the usual semantics), since to the elements of a set the relation of equality makes sense, we are leaving aside the basic idea we intend to capture, namely, that of the ‘non-individuality’ of certain elements. So, a ‘natural’ semantics for our logic should be presented by using a mathematical device in which we can talk about objects that cannot be individualized, that is, strongly indistinguishable objects.

In other words, we are not acting in conformity with the intuitive aspects we intend to capture if we use standard set theory to build the mathematical structure in which the language of \( L_{op} \) is to be interpreted. Thus, instead of using a set theory like Zermelo-Fraenkel in the metamathematics, we use the quasi-set theory \( \Omega \) (for details, see [23]).

Roughly speaking, the theory \( \Omega \) is a mathematical device for treating collections of indistinguishable objects. The theory allows the presence of a certain kind of \textit{Urelemente} (the so called \( m \)-atoms) to which the usual concept of identity does not apply. The underlying logic of \( \Omega \) is classical quantificational logic without identity; the specific symbols are three unary predicate letters \( m(x) \) (read ‘\( x \) is an \( m \)-atom’), \( M(x) \) (read ‘\( x \) is an \( M \)-atom’—that is, a standard \textit{Urelement}) and \( Z(x) \) (read ‘\( x \) is a standard set’). Such ‘sets’ are characterized in \( \Omega \) as quasi-sets whose transitive closure do not contain \( m \)-atoms). Furthermore, the language still encompasses two binary predicate symbols \( \in \) (membership) and \( \equiv \) (indistinguishability) and a unary functional symbol \( qc \) (quasi-cardinality). A quasi-set (qset for short) is defined as an entity which is not an \textit{Urelement} (that is, it is anything that is neither an \( m \)-atom nor a classical atom). We write \( Q(x) \) for saying that \( x \) is a quasi-set. The concept of quasi-cardinal is introduced in such a way that it extends the concept of cardinal for arbitrary qsets; some additional remarks on this concept shall be mentioned below.

The axioms of indistinguishability state that \( \equiv \) has the properties of an equivalence relation. The \textit{extensional equality} \( =_{E} \) is defined in the following way: \( x =_{E} y \) iff \( (Q(x) \land Q(y) \land \forall z (z \in x \leftrightarrow z \in y)) \lor (M(x) \land M(y) \land x \equiv y) \). That is, in

\begin{footnote}{In classical logic and mathematics, the concept of ‘indistinguishability’ cannot be separated from that of ‘identity’ by force of Leibniz’ Law. This ‘identification’ of identity and indistinguishability (agreement with respect to properties), let us recall, is in the core of Ramsey’s criticism of the treatment of identity presented in \textit{Principia Mathematica} \textit{(vis., Leibniz’ Law)}. In a certain sense, our logics vindicate Ramsey’s claim that “There is nothing self-contradictory (…) in \( a \) and \( b \) [where \( a \neq b \)] having all their elementary properties in common. Hence, since this is logically possible, it is essential to have a symbolism which allows us to consider this possibility and does not exclude it by definition” [33], p. 182].\end{footnote}

\begin{footnote}{As explained in that work, there have been presented several versions of quasi-set theory (in reality, they constitute distinct theories; see our references).\end{footnote}
extensional entities are indistinguishable standard \textit{Urelemente} or qsets which have ‘exactly the same’ elements. It can be proven that the extensional equality has all the formal properties of classical identity.\footnote{Hence, by ‘indistinguishable standard \textit{Urelemente}, as in the last phrase, we understand ‘identical \textit{Urelemente}’. From now on, we will use ‘\(=\)’ instead of ‘\(\equiv\)’.} The substitutivity principle is valid only with respect to indistinguishable objects which are not \(m\)-atoms. In symbols, \(\forall x\forall y (\neg m(x) \land \neg m(y) \rightarrow (x \equiv y \rightarrow (A(x, x) \rightarrow A(x, y))))\) with the usual syntactical restrictions. Furthermore, the axioms for the concept of quasi-cardinal generalize the concept of cardinal for arbitrary qsets. Since the identity relation cannot be applied to (a pair of) \(m\)-atoms and since it is postulated that every qset has a quasi-cardinal, then there is a precise sense in saying that the objects of a qset whose elements are indistinguishable \(m\)-atoms can only be aggregated in certain quantities, but that they cannot be ordered or counted.\footnote{Then the \(m\)-atoms have some of the characteristics generally attributed to quanta. See \cite{24}.}

One of the most peculiar axioms of \(\Omega\) is the ‘weak’ axiom of extensionality, which states that qsets having ‘the same quantity of elements of the same sort’ are indistinguishable qsets (this idea can be stated conveniently by means of the concept of quasi-cardinal and by passing the quotient by the relation of indistinguishability). Among other things, this axiom permits us to prove an interesting result, which we call ‘the theorem of the unobservability of permutations’\footnote{The concept of ‘one’ \(m\)-atom can be made precise by using the ‘strong singleton’ of an \(m\)-atom. More precisely, the strong singleton of \(x\) is a qset which is a subqset (in the usual sense) of the qset of all the objects indistinguishable from \(x\) (this qset is provided by an axiom like the pair axiom of ZFC, but using the indistinguishability relation instead of equality) which has quasi-cardinal 1. In \(\Omega\), we can prove that such a qset exists.}: if in a qset we exchange one \(m\)-atom by an indistinguishable one, then the resulting qset is indistinguishable from the original qset. This can be viewed as the formal counterpart of the unobservability of particle permutations in quantum mechanics. An important qsets for our purposes here are the ‘pure’ qsets, that is, those qsets whose elements are \(m\)-atoms only. Certain particular pure qsets will be taken to be the extensions of the opaque predicates of the logic \(\mathsf{Log}_{op}\), as we will see below.

Furthermore, it is convenient to recall that \(\Omega\) involves standard mathematics, in the sense that all axioms of Zermelo-Fraenkel set theory may be suitably translated into the language of quasi-sets and (these translations) proven as theorems of \(\Omega\).\footnote{For further information on this particular topic, see for instance \cite{22, 8, 23}.} all the set-theoretical operations (union, cartesian products, difference between quasi-sets, etc.) can be performed in \(\Omega\) similarly as in standard set theory; consequently, we don’t need to pay attention here to the terminology, which is used as in the standard set theories.

Let us still mention in brief about an axiom which states that \(\text{qc}(P(x)) = 2^{\text{qc}(x)}\), where \(P(x)\) is the quasi-set of all the subquasi-sets of \(x\) (defined in the usual way). Here, \(2^{\text{qc}(x)}\) may be understood as follows. If \(\alpha\) is the cardinal which is the quasi-cardinal of \(x\), then \(2^{\text{qc}(x)}\) is \(\text{card}(\alpha^{\alpha}2)\), that is, the cardinal of the collection (which is a ‘set’) of all functions from \(\alpha\) to \(2 = \{0, 1\}\).\footnote{Another axiom says that every quasi-set has an unique quasi-cardinal which is a cardinal defined in the ‘classical’ part of \(\Omega\). So, the quasi-cardinals are ‘sets’ in \(\Omega\), obeying the standard axioms of \(\mathsf{ZF}\) set theory.} It should also be realised that since
both \( \alpha \) and 2 are ‘classical’ objects (in the sense that they obey the axioms of \( \text{ZF} \)), then \( ^2 \) is also a ‘set’ and hence the definition of \( \text{card}(^2) \) makes sense.

This axiom has an important consequence. As suggested above, in \( \Omega \) there may exist qsets whose elements are \( m \)-atoms only, called ‘pure’ qsets. Furthermore, it may be the case that the \( m \)-atoms of a pure qset \( x \) are indistinguishable from one another, in the sense of sharing the indistinguishability relation \( \equiv \). In this case, the axioms provide the grounds for saying that nothing in the theory can distinguish among the elements of \( x \). But, then one could ask what is it that sustains the idea that there is more than one entity in \( x \). The answer is obtained through the above mentioned axioms (among others, of course). Since the quasi-cardinal of the power qset of \( x \) has quasi-cardinal \( 2^{\text{qc}(x)} \), then if \( \text{qc}(x) = \alpha \), it results that for every quasi-cardinal \( \beta \leq \alpha \) there exists a subquasi-set \( y \subseteq x \) such that \( \text{qc}(y) = \beta \), according to the remaining axioms about the quasi-cardinality of the subquasi-sets. Thus, if \( \text{qc}(x) = \alpha \neq 0 \), the axioms do not forbid the existence of \( \alpha \) subquasi-sets of \( x \) which can be regarded as ‘singletons’.

Of course the theory cannot prove that these ‘unitary’ subquasi-sets (supposing now that \( \text{qc}(x) \geq 2 \)) are distinct, since we have no way of ‘identifying’ their elements. These ‘singletons’ are indistinguishable in the sense of the axiom of weak extensionality. But what is important is that quasi-set theory is compatible with the existence of distinct and absolutely indistinguishable \( m \)-atoms. This is important, for example, for obtaining a description of quantum statistics in the framework of \( \Omega \) (see \[25\] for details on this point). In other words, it is consistent with \( \Omega \) to maintain that \( x \) has \( \alpha \) elements, which may be regarded as absolutely indistinguishable objects. Since the elements of \( x \) may share the relation \( \equiv \), they may be further understood as belonging to a same ‘equivalence class’ (for instance, being indistinguishable electrons) but in such a way that we cannot assert either that they are identical or that they are distinct from one another (i.e., they act as ‘identical electrons’ in the physicist’s jargon).

3.2. The Generalized Semantics of \( \mathcal{L}_{\text{op}} \). All the developments of this section are performed in the quasi-set theory \( \Omega \). We will not provide here all the details but only the main definitions and results, which are similar to those of classical logic (see for instance \[4\]). The proofs can be adapted without difficulty from the most general case of Schrödinger logics presented in \[7\].

We call \( \mathcal{L} \) the language of \( \mathcal{L}_{\text{op}} \). Let \( D \) be a quasi-set such that \( D = m \cup M \) where \( m \) is a pure qset whose elements are indistinguishable from one another and \( M \) is a non-empty set (in \( \Omega \)).

By a frame for \( \mathcal{L} \) based on \( D \) we mean a quasi-function \( \mathfrak{M} \) whose domain in the set \( \Pi \) of types such that:

1. \( \mathfrak{M}(e_1) = m \)
2. \( \mathfrak{M}(e_2) = M \)
3. For each type \( \tau = \langle \tau_1, \ldots, \tau_n \rangle \in \Pi \), \( \mathfrak{M}(k) \subseteq \mathfrak{P}(\mathfrak{M}_{\tau_1} \times \ldots \times \mathfrak{M}_{\tau_n}) \). If the inclusion in this last expression can be replaced by (extensional) equality, then the frame is standard.

It turns out to be ‘functions’ in the standard sense. As remarked, all the details may be found in \[23\].

\[10\] The application of this formalism to the concept of non-individual quantum particles has been proposed in \[24\].
If we write $\mathfrak{M}_\tau$ instead of $\mathfrak{M}(\tau)$, then the frame can be viewed as a family $(\mathfrak{M}_\tau)_{\tau \in \Pi}$ of qsets satisfying the above conditions. In what follows, we will refer indifferently to both this family and $\mathfrak{F} = \{ X : \exists \tau \in \Pi \land X = \mathfrak{M}(\tau) \}$ as the frame for $\mathfrak{L}$.

A denotation for $\mathfrak{L}$ based on $D$ is a quasi-function $\phi$ whose domain is the set of constants of $\mathfrak{L}$, defined such that $\phi(A^\tau) \in \mathfrak{M}_\tau$ for every $\tau \in \Pi$. So, in particular $\phi(A^{e_1}) \in m$ and $\phi(A^{e_2}) \in M$.

Based on these definitions, we may introduce the concept of an interpretation for $\mathfrak{L}$ based on $D$ as an ordered pair $\mathfrak{A} = ((\mathfrak{M}_\tau)_{\tau \in \Pi}, \phi)$, where $(\mathfrak{M}_\tau)_{\tau \in \Pi}$ is a frame for $\mathfrak{L}$ (based on $D$) and $\phi$ a denotation as above. The interpretation is principal if the frame is standard. It can be shown that the defined predicate of equality (Definition 1.1) is interpreted in the quasi-set $\Delta_{\equiv}(\tau)$, the ‘pseudo-diagonal’ of $\mathfrak{M}_\tau$, namely, the qset whose elements are pairs of indistinguishable objects (see $\phi$).

A valuation for $\mathfrak{L}$ (based on $D$) is a quasi-function $\psi$ whose domain is the collection of terms of $\mathfrak{L}$ and in such a way that $\psi$ is the extension of the denotation quasi-function $\phi$ to the whole set of terms of $\mathfrak{L}$. In other words, $\psi$ is defined as follows:

1. $\psi(A^\tau) \equiv \phi(A^\tau)$ for every constant $A^\tau$. That is, the images of $A^\tau$ by $\phi$ and by $\psi$ are indistinguishable.
2. $\psi(x^{e_1}) \in m$
3. $\psi(X^{e_2}) \in M$
4. $\psi(X^\tau) \in \mathfrak{M}_\tau$ for $\tau \neq e_1, e_2$.

We introduce the concept of a formula $A$ being satisfiable with respect to the interpretation $\mathfrak{A}$ (in symbols: $\mathfrak{A}, \psi \models A$) in accordance with the following clauses:

1. $\mathfrak{A}, \psi \models U^\tau(X_1^\tau, \ldots, X_n^\tau)$ iff $(\psi(X_1^\tau), \ldots, \psi(X_n^\tau)) \in \psi(U^\tau)$, where $U^\tau$ is a term of type $\langle \tau_1, \ldots, \tau_n \rangle$ and $X_i^\tau$ are terms of type $\tau_i$ $(i = 1, \ldots, n)$.
2. The satisfaction clauses for $\neg$, $\rightarrow$ and $\forall$ are introduced as usual.

A formula $A$ is true with respect to the interpretation $\mathfrak{A}$ iff $\mathfrak{A}, \psi \models A$ for every valuation $\psi$ with respect to $\mathfrak{A}$. An interpretation $\mathfrak{A}$ is normal iff every instance of the axioms of $\mathfrak{L}_{op}$ is true in $\mathfrak{A}$, as are all instances of extensionality, separation, infinite and choice. In what follows we will consider only appropriate interpretations.

A normal interpretation which is not principal is a secondary interpretation. A formula $A$ is valid iff it is true with respect to all principal interpretations, and it is satisfiable if there exists a principal interpretation $\mathfrak{A}$ and a valuation $\psi$ such that $\mathfrak{A}, \psi \models A$. The formula is secondarily valid if it is true with respect to all normal interpretations, and it is secondarily satisfiable if is is true with respect to some normal interpretation.

Then, by adapting the proofs presented in $\mathfrak{L}$, we can state without difficulty the following results: (1) $A$ is valid iff $\neg A$ is not satisfiable; (2) $A$ is secondarily valid iff $\neg A$ is not secondarily satisfiable; (3) $A$ is satisfiable iff $\neg A$ is not valid; (4) $A$ is secondarily satisfiable iff $\neg A$ is not secondarily valid and (5) $A$ is valid (respect. secondarily valid) with respect to a normal interpretation iff its universal closure is valid (respect. secondarily valid) with respect to this interpretation (see also $\mathfrak{L}$).

By a model of a set $\Gamma$ of formulas of $\mathfrak{L}$ we understand a normal interpretation $\mathfrak{A}$ such that $\mathfrak{A}, \psi \models A$ for every formula $A \in \Gamma$. If $\mathfrak{A}$ is a principal interpretation, we talk of principal models, or of secondary models if $\mathfrak{A}$ is a secondary interpretation.
The following terminology will be used below: $\Gamma \models A$ means that $A$ holds in every model of $\Gamma$, and $\models A$ means that $A$ is secondarily valid.

The proofs of the theorems below are simple adaptations from those of usual higher-order logic [2], and we don’t think they must be repeated here. A particular case of similar results involving quasi-set semantics was presented with more details in [7].

**Theorem 3.1 (Soundness).** All theorems of $L_{op}$ are secondarily valid.

In other words, $\vdash A$ implies $\models A$; it is not difficult to generalize this result: $\Gamma \vdash A$ implies $\Gamma \models A$.

**Theorem 3.2 (Lindenbaum).** Every consistent set $\Gamma$ of closed formulas of $L$ can be extended to a maximal consistent class $\bar{\Gamma}$ of closed formulas of $L$.

Furthermore, there is the following important result:

**Theorem 3.3.** If $A$ is a closed formula of $L$ which is not a theorem, then there exists a normal interpretation whose domains $M_\tau$ are denumerably infinite, with respect to which $\neg A$ is valid.

Then, based on these theorems, we can state the (weak) completeness theorem for our logic, whose proof can be adapted either from [3] or [7].

**Theorem 3.4 (Completeness).** Every formula of $L_{op}$ which is secondarily valid is a theorem.

That is, $\models A$ implies $\vdash A$. In general, if $\Gamma$ is a set of closed formulas of $L$ which is not inconsistent, then $\Gamma \models A$ implies $\Gamma \vdash A$, that is, if $A$ holds in every model of $\Gamma$, then $A$ is derivable from the formulas of $\Gamma$.

4. Veiled Sets

Having sketched the main features of the generalized quasi-set semantics for our logic, let us turn to a consideration of the opaque predicates and see more carefully how they were interpreted in the framework of the previous section.

An opaque relation, according to our previous definiton, is a term of type $\langle \tau \rangle$ where $\tau \in \Pi$ obtained recursively from the basic type $e_1$. Intuitively, an opaque predicate is an unary opaque relation of type $\tau = \langle e_1 \rangle$. Semantically, to an opaque predicate is associated a subquasi-set of the pure quasi-set $m$. In other words, the extension of such a predicate is a collection of objects for which there is no sense in saying that they are equal or distinct, and this is in conformity with the above mentioned examples from quantum mechanics. So, the semantics of our logic agrees with its syntatic aspects.

Such pure quasi-sets, let us remark, seem to be concealed by a kind of veil, since although they have a well-determined neighbourhood (the membership relation has a standard behaviour), we definitely cannot distinguish between their elements. Furthermore, the ‘unobservability permutation theorem’ mentioned above can be understood as saying that if an element of one of these quasi-sets is exchanged

\footnote{The concepts mentioned here are like the standard ones.}

\footnote{That is, their characteristic function is a (quasi)function from the quasi-set in $\{0, 1\}$, as in usual set-theory, what this intuitively means that, for every $x$, $x$ belongs or does not belong to the quasi-set. So, the present case is distinct from that involving fuzzy sets or quasi-sets (concerning the latter, see [10]).}
by an indistinguishable one, all happens (to us, who are behind the veil) as if nothing had occurred at all. Such quasi-sets could be called *veiled sets*, and are the ‘natural’ extensions of opaque predicates. Furthermore, in accordance with quantum mechanics, indistinguishable $m$-atoms are not like objects (individuals) which merely cannot be identified; there is a strong ‘ontic’ indeterminacy among them.

Before ending the paper, let us comment briefly about a peculiarity of our logic $L_{op}$. We have chosen such a logic as ‘the’ logic of opaque predicates on the grounds of formalizing the intuitive idea that opaque predicates should be characterized as describing properties for which it is not possible to distinguish between the elements that have the property ascribed by the predicate. Of course it would be incorrect to suggest that $L_{op}$ is the only logic of such predicates. Notwithstanding this, someone could ask us why we didn’t use classical logic instead of $L_{op}$ but in such a way that its semantics is represented in quasi-set theory. Then, he/she could say, it should be sufficient to interpret some predicates of the language as veiled sets and the idea of opaque predicates could be achieved. This of course is an interesting idea which could simplify much of the above discussion. But, although we agree with the convenience of using classical logic when possible, in this case we would have no syntactical means of distinguishing opaque predicates from other predicates: only a ‘semantical’ distinction would be provided by using veiled sets as the extensions of some predicates, while to the remaining ones standard sets should be used instead. But we think that we could try to obtain a logical system which provides also a syntactical distinction between the predicates, and our logic may be viewed as an attempt in this direction. In fact, our system provides not only a distinction among predicates of its language, but by postulating that the concept of identity is meaningless for certain entities, it is still in accordance with the intuitive idea of characterizing opaque predicates. So, returning to Peirce’s characterisation, perhaps we can say the following: A proposition is opaque when there are possible states of things concerning which it is intrinsically uncertain whether, had they been contemplated by the speaker, he would have regarded them as excluded or allowed by the proposition. By intrinsically uncertain we mean not uncertain in consequence of any ignorance of the interpreter, or any indeterminacy in the speaker’s habits of language but because of a fundamental ontological indeterminacy with regard to the objects denoted.

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13 The case of the unobservability of permutations in quantum mechanics is ‘didactically’ mentioned by Roger Penrose in his *The emperor’s new mind*: “according to the modern theory [quantum mechanics], if a particle of a person’s body were exchanged with a similar particle in one of the bricks of his house then nothing would have happened at all” [20, p. 360].
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