WEAK COMMUTATIVITY BETWEEN TWO ISOMORPHIC POLYCYCLIC GROUPS.

BRUNO CÉSAR R. LIMA AND RICARDO N. OLIVEIRA

Abstract. The operator of weak commutativity between isomorphic groups $H$ and $H^\psi$ was defined by Sidki as
$$\chi(H) = \langle H, H^\psi \mid [h, h^\psi] = 1 \forall h \in H \rangle.$$ It is known that the operator $\chi$ preserves group properties such as finiteness, solubility and also nilpotency for finitely generated groups. We prove in this work that $\chi$ preserves the properties of being polycyclic and polycyclic by finite. As a consequence of this result, we conclude that the non-abelian tensor square $H \otimes H$ of a group $H$, defined by Brown and Loday, preserves the property polycyclic by finite. This last result extends that of Blyth and Morse who proved that $H \otimes H$ is polycyclic if $H$ is polycyclic.

1. Introduction

Given two groups $H$ and $H^\psi$ which are isomorphic via $\psi : h \mapsto h^\psi$, the following group construction was introduced and analysed in [9]
$$\chi(H) = \langle H, H^\psi \mid [h, h^\psi] = 1 \forall h \in H \rangle.$$ The weak commutativity group $\chi(H)$ maps onto $H$ by $h \mapsto h$, $h^\psi \mapsto h$ with kernel $L(H) = \langle h^{-1}h^\psi, h \in H \rangle$ and maps onto $H \times H$ by $h \mapsto (h, 1)$, $h^\psi \mapsto (1, h)$ with kernel $D(H) = [H, H^\psi]$. It is an important fact that $L(H)$ and $D(H)$ commute. Let $T(H)$ be defined as the subgroup of $H \times H \times H$ generated by $\{(h, h), (1, h, 1) \mid h \in H \}$. Then $\chi(H)$ maps onto $T(H)$ by $h \mapsto (h, h, 1)$, $h^\psi \mapsto (1, h, h)$, with kernel $W(H) = L(H) \cap D(H)$, an abelian group. A further normal subgroup of $\chi(H)$ is $R(H) = \langle H, L(H), H^\psi \rangle$ where the quotient $\chi(H)/R(H)$ is isomorphic to the Schur Multiplier $M(H)$. Proving properties of $\chi(H)$ depend crucially upon understanding $R(H)$. Thus, if $H$ is polycyclic by finite then so is $\chi(H)$ and to prove more generally that $\chi(H)$ is polycyclic by finite depends upon showing that $R(H)$ is finitely generated.

We will make use of the following facts from [9, p. 201].

Lemma 1. The groups $D(H)$, $L(H)$, $W(H)$ and $R(H) = [H, L(H), H^\psi]$ satisfy:

(i) $R(H) \triangleleft \chi(H)$, $R(H)$ is $\Psi$-invariant;
(ii) $[W(H), H] \leq R(H) \leq W(H) \leq D(H)$;

Date: September 22, 2014.

1991 Mathematics Subject Classification. 20E25.

Key words and phrases. weak commutativity, polycyclic groups, Schur multiplier.

We are grateful to Professor S. Sidki for suggesting the problem and for his support and encouragement.

The first author acknowledges support from PROCAD-CAPES for Sandwich Doctorate studies.

The second author acknowledges support from PROCAD-CAPES for post-doctoral studies.
(iii) \[ [h_1, h_2^x]h_3 = [h_1, h_2^x]h_3^y \] for any \( h_1, h_2, h_3 \in H \).

(iv) \( D(H) \) centralizes \( L(H) \);

(v) \[ [h_1, h_2^x]h_3 = [h_1 h_3^x, (h_2 h_3^y)^y] \] holds in \( \chi(H) \mod R(H) \), for any elements \( h_1, h_2, h_3 \in H \).

We have the following diagram of subgroups of \( \chi(H) \):

Our main result in this paper is

**Theorem 1.** Let \( H \) be a group which is polycyclic by finite. Then the group \( \chi(H) \) and the non-commutative tensor square \( H \otimes H \) are also polycyclic by finite.

In last subsection, we discuss different behaviours of \( R(H) \), specially for certain polycyclic groups \( H \).

2. **Connection with the augmentation ideal of \( Z(H) \)**

Given a group \( H \) with identity 1, consider the group ring \( Z(H) \) and its augmentation ideal \( A_Z(H) \) which is generated by \( \{ h - 1 \mid h \in H \} \). Consider also \( I_2(H) \) the ideal of \( Z(H) \) generated by \( \{(h - 1)^2 \mid h \in H \} \). Let \( \tilde{G} = Z(H) \rtimes H \) be semidirect product of \( Z(H) \) by \( H \), where \( Z(H) \) is written additively and conjugation of \( Z(H) \) by \( H \) is described as right multiplication

\[
\left( \sum (x_h h, 1) \right)^{0, h_1} = \sum (x_h h h_1, 1).
\]

In addition, let \( G \) be the subgroup of \( \tilde{G} \) generated by \( A_Z(H) \times \{1\} \) and \( \{0\} \times H \). Then, \( N = I_2(H) \times \{1\} \) is a normal subgroup of \( \tilde{G} \).

**Proposition 1.** Let \( L = L(H) \) and \( L' \) be its derived subgroup. Then the application

\[
\epsilon : H \cup H^\psi \rightarrow \frac{G}{N}
\]
defined by $\epsilon : h \to N(0, h)$ and $h^\phi \to N(0, h)^{(1, 1)}$, $\forall h \in H$, extends to an epimorphism

$\epsilon : \chi(H) \to \frac{G}{N}$

with kernel $L'$. Furthermore, if $H$ is a finitely generated group then so is the quotient group $\frac{L}{L'}$.

Proof. Since, for $h \in H$

$[(0, h), (1, 1)] = (0, h)^{-1}(0, h)^{(1, 1)}$

$= (1, 1)(0, -h)(1, 1)$

$= (-h + 1, 1) \in A_2(H) \times \{1\}$,

we obtain

$A_2(H) \times \{1\} = [(0) \times H, (1, 1)]$

and

$G = \left\langle \{1\} \times H, [(1) \times H]^{(1, 1)} \right\rangle$

$\cong \left\langle H, H^\phi \mid [H, \phi] = 1 \right\rangle$

by [9, p. 189]. Given $h \in H$ we conclude that in $G$,

$[(0, h), (0, h)]^{(1, 1)} = 1 \iff [(1, 1), (0, h), (0, h)] = 1$,

$[(1, 1), (0, h), (0, h)] = ((h - 1)^2, 1)$.

Therefore

$\epsilon : h \to N(0, h), h^\phi \to N(0, h)^{(1, 1)}$

extends to an epimorphism $\epsilon : \chi(H) \to \frac{G}{N}$ and $\text{Ker} = L'$.

As

$\chi(H) = L \cdot H$ and $L' = \frac{A_2(H) \times \{1\}}{N}$

it follows that

$\frac{L}{L'} \cong \frac{A_2(H)}{I_2(H)}$

Let $S = \{a_1, \ldots, a_s\}$ be a generating set for $H$. The following equations hold in $A_2(H)$ modulo $I_2(H)$ for all $a_i, a_j \in S$:

$(a_i - 1)^2 = 0$, $a_i^2 = 2a_i - 1$,

$a_i^k = ka_i - (k - 1) \forall k \in \mathbb{Z}$;

$(a_j a_i)^{-1} = 2 - a_j a_i$,

$a_i^{-1} a_j^{-1} = (2 - a_i)(2 - a_j) = a_i a_j - 2a_i - 2a_j + 4$,

$a_j a_i = -a_i a_j + 2a_j + 2a_i - 2$.

We conclude that $\{I_2(H) + a_{i_1} a_{i_2} \cdots a_{i_s} \mid 1 \leq i_1 < i_2 < \ldots i_s \leq n\}$ generates the abelian group $\frac{Z_n(H)}{I_2(H)}$ and in particular, $\frac{A_2(H)}{I_2(H)}$ is a finitely generated. \hfill \Box

Lemma 2. ([11]) Let $H$ a group finitely presented. Then the Schur Multiplier of $H$ is finitely generated.

Theorem 2. If $H$ be a polycyclic (polycyclic by finite) group then so is $\chi(H)$. 

Proof. We will prove the assertion for $H$ polycyclic by finite; the proof for $H$ polycyclic is similar. Denote $L(H) = L, D(H) = D$. It follows directly from the above diagram of subgroups that

$$\frac{\chi(H)}{D} \cong H \times H, \quad \frac{H}{H'} \cong \frac{\chi(H)}{DL} \cong \frac{\chi(H)}{D}$$

and

$$\frac{DL}{D} \cong \frac{L}{W} \cong H' \cdot H$$

are all polycyclic by finite. Since $\frac{L}{L'} \frac{L'W}{W} \left(= \frac{L'}{L'\cap W}\right)$ are finitely generated groups, we conclude that $\frac{L}{L'\cap W}$ is polycyclic by finite. Since $W = D \cap L$ and $[D, L] = 1$, it follows that $L' \cap W \leq L' \cap Z(L)$. Therefore, by a theorem of Schur [10, p. 19], the group $L' \cap W$ is isomorphic to a subgroup of the Schur Multiplier of $\frac{L}{L'\cap W}$. Since $\frac{L}{L'\cap W}$ is polycyclic by finite, by the above lemma, it follows that $M(\frac{L}{L'\cap W})$ is finitely generated. Therefore, $L' \cap W$ is finitely generated and consequently, $L$ is polycyclic by finite and finally so is $\chi(H)$. □
3. Connection with the non-commutative tensor square

We recall the non-commutative tensor square introduced by Brown-Loday [2]

\[ H \otimes H = \langle h_1 \otimes h_2 \mid h_1 h_2 \otimes h_3 = (h_1 h_2 \otimes h_3, h_2) \rangle, \]

and the group defined by N. Rocco [7]

\[ \nu(H) = \langle H, H^\psi \mid [h_1, h_2^\psi h_3^\psi] = [h_1 h_2^\psi, (h_2^\psi)^\psi], \forall h_1, h_2, h_3 \in H \rangle. \]

Rocco showed in [7] that there exist an isomorphism between the subgroup \( \tau(H) = [H, H^\psi] \) of \( \nu(H) \) and \( H \otimes H \). Moreover Brown-Loday [2] showed that for \( J(H) \), is the kernel of the epimorphism \( \tau(H) \to H' \) defined by \( [h_1, h_2^\psi] \mapsto [h_1, h_2] \), and for \( \Delta(H) = \langle [h, h^\psi] \mid h \in H \rangle \), then the Schur Multiplier \( M(H) \) is isomorphic the quotient \( J(H)/\Delta(H) \).

It was shown in [1] that if \( H \) is polycyclic then so is \( \nu(H) \). Thus, the following corollary generalizes this result.

**Corollary 1.** Let \( H \) be a polycyclic by finite group. Then \( \nu(H) \) and \( H \otimes H \) are polycyclic by finite groups.

**Proof.** By [7, p. 68-69] we have that \( \frac{\nu(H)}{\Delta(H)} \cong \frac{\nu(H)}{\Delta(H)} \), where the diagonal group \( \Delta(H) = \langle [h, h^\psi] \mid h \in H \rangle \) is a normal finitely generated abelian subgroup of \( \nu(H) \) and \( R(H) = [H, L, H^\psi] \) is a subgroup of \( \chi(H) \).

4. The subgroup \( R(H) \) for a polycyclic group \( H \)

**Proposition 2.** Let \( H \) be a group and \( T \) a transversal for \( H' \) in \( H \). Then,

\[ R(H) = \left\langle [h_1, h_2^\psi h_3^\psi] [h_1 h_3, (h_2 h_3)^\psi]^{-1} \mid h_1, h_2, h_3 \in H \right\rangle^T. \]

**Proof.** First let’s show that

\[ R(H) = \left\langle [h_1, h_2^\psi h_3^\psi] [h_1 h_3, (h_2 h_3)^\psi]^{-1} \mid h_1, h_2, h_3 \in H \right\rangle^{\chi(H)}. \]

The subgroup of \( \chi(H) \),

\[ J(H) = \left\langle [h_1, h_2^\psi h_3^\psi] [h_1 h_3, (h_2 h_3)^\psi]^{-1} \mid h_1, h_2, h_3 \in H \right\rangle^{\chi(H)} \]

is contained in \( R(H) \), by Lemma 1 (v).

By Rocco [7, p. 68-69] the epimorphism \( \varepsilon : \chi(H) \to \frac{\nu(H)}{\Delta(H)} \) given by \( h \mapsto \Delta(H)h \), \( h^\psi \mapsto \Delta(H)h^\psi \), \( \forall h \in H \), has kernel \( R(H) \). So \( \varepsilon \) induces \( \bar{\varepsilon} : \frac{\chi(H)}{\Delta(H)} \to \frac{\nu(H)}{\Delta(H)} \). On the other hand, the application \( \zeta : \frac{\nu(H)}{\Delta(H)} \to \frac{\chi(H)}{\Delta(H)} \) such that \( \Delta(H)h \mapsto J(H)h \), \( \Delta(H)h^\psi \mapsto J(H)h^\psi \) extend to a epimorphism \( \bar{\zeta} : \frac{\nu(H)}{\Delta(H)} \to \frac{\chi(H)}{\Delta(H)} \), because
Lemma 1 (iii), we have 

$$\frac{\nu(H)}{\Delta(H)} = \left\langle H, H^\psi \mid [h, h^\psi] = 1, [h_1, h_2]^{\psi_{h_3}} = [h_1^{h_3}, (h_2^{h_3})^\psi] = [h_1, h_2]^{\psi_{h_3}}, \forall h, h_1, h_2, h_3 \in H \right\rangle$$

and the relations 

$$[h_1, h_2]^{\psi_{h_3}} = [h_1, h_2]^{\psi_{h_3}} = [h_1^{h_3}, (h_2^{h_3})^\psi], \ [h, h^\psi] = 1, \forall h, h_1, h_2, h_3 \in H$$

holds in $\frac{\chi(H)}{J(H)}$. How $\zeta$ is the identity in $\frac{\chi(H)}{J(H)}$, follows that $R(H) = J(H)$.

Now we have that $R(H) \leq D(H)$, so by Lemma 1 (iii), it follows that

$$R(H) = \left\langle [h_1, h_2]^{\psi_{h_3}}[h_1^{h_3}, (h_2^{h_3})^\psi]^{-1} \mid h_1, h_2, h_3 \in H \right\rangle^H.$$ 

Again, by Lemma 1 (iii) and (iv), we have

$$[R(H), H^\psi] = 1.$$

Then

$$R(H) = \left\langle [h_1, h_2]^{\psi_{h_3}}[h_1^{h_3}, (h_2^{h_3})^\psi]^{-1} \mid h_1, h_2, h_3 \in H \right\rangle^T.$$ 

\[ \square \]

**Proposition 3.** Let $H$ be a polycyclic group with a polycyclic generators $S = \{a_1, a_2, \ldots, a_n\}$ and let $T$ be a transversal for $H'$ in $H$. Then 

$$R(H) = \left\langle [a_i, a_j]^{a_k} [a_i^{a_k} (a_j^{a_k})^\psi]^{-1} \mid a_i, a_j, a_k \in S \right\rangle^T.$$ 

Proof. Let $S = \{a_1, \ldots, a_n\}$ be a generating set for $H$ then by [5, p. 37], the subgroup 

$$K = \left\langle [h_1, h_2]^{\psi_{h_3}}[h_1^{h_3}, (h_2^{h_3})^\psi]^{-1} \mid h_1, h_2, h_3 \in H \right\rangle^{H \ast H^\psi}$$

of the free product $H \ast H^\psi$ has the presentation 

$$J = \left\langle [a_i, a_j]^{a_k} [a_i^{a_k} (a_j^{a_k})^\psi]^{-1} \mid a_i, a_j, a_k \in S \right\rangle^{H \ast H^\psi}.$$ 

If $\phi : H \ast H^\psi \rightarrow \chi(H)$ is the natural epimorphism, we conclude that 

$$\phi(J) = R(H) = \left\langle [a_i, a_j]^{a_k} [a_i^{a_k} (a_j^{a_k})^\psi]^{-1} \mid a_i, a_j, a_k \in S \right\rangle^{\chi(H)}.$$ 

Prop 2 

$$= \left\langle [a_i, a_j]^{a_k} [a_i^{a_k} (a_j^{a_k})^\psi]^{-1} \mid a_i, a_j, a_k \in S \right\rangle^T.$$ 

\[ \square \]

4.1. Different behaviours of $R(H)$. It is difficult in general to obtain information about $R(H)$ and there are a few cases for which it is described. The following remark is helpful in establishing the non-triviality of $R(H)$.

**Remark 1.** Let $H, K$ be groups and $\varphi : H \rightarrow K$ be an epimorphism. Then $\varphi$ extends to an epimorphism $\hat{\varphi} : \chi(H) \rightarrow \chi(K)$ by $\hat{\varphi} : h \rightarrow h^\varphi, h^\psi \rightarrow (h^\varphi)^\psi$ (that is, by having $\hat{\varphi}$ commute with $\psi$). Therefore, $L(H)^{\hat{\varphi}} = L(K)$ and $R(H)^{\hat{\varphi}} = R(K)$. In particular, $R(H)$ is non-trivial provided $R(K)$ is non-trivial.
Information about $R(H)$ is known for finitely generated abelian groups $H$ (see Section 4.2 of [9]).

**Theorem 3.** (Theorem 4.2.1 [9, p. 204]) Let $H$ be an abelian group. Then,

1. $D = W = [L, H], R = [D, H] = [L, 2H]$.
2. $L$ is nilpotent of class $\leq 2$, $L' \leq D \leq Z(L), L' \leq Z(\gamma(H))$.
3. $L' = D^2 = [H^2, H^\psi], R^2 = 1$.

In case $H$ is a finite elementary abelian $2$-group, we have

**Proposition 4.** Let $H$ be an elementary abelian $2$-groups of rank $k$ and order $n (= 2^k)$. Then,

1. $\chi(H)$ is isomorphic to the natural extension of $A_2(H)$ by $H$ of order $2^{n-1}n$, where $A_2(H)$ corresponds to $L(H)$.
2. the derived subgroup $\gamma_2(\chi(H)) = D(H)$.
3. $\gamma_3(\chi(H)) = R(H)$ and has order $2^{n-k-(\frac{k}{2})}$.

**Proof.** The first two items follow directly from the material in Section 4.2 of [9].

The third item follows from

\[
\chi(H)^V = [L(H), H],
\]

\[
[L(H), H, \chi(H)] = [L(H), H, H] = [L(H), H, H^\psi] = R(H)
\]

and from

\[
\frac{|\chi(H)|}{|R(H)|} = |H^2| |M(H)|, |M(H)| = 2^{\frac{k}{2}}.
\]

In view of the above remark, we conclude

**Proposition 5.** Let $H$ be a group which has as homomorphic image an elementary abelian $2$-group of rank at least $3$. Then $R(H)$ is non-trivial.

On the other hand, $R(H)$ can be trivial, as is the case of $H$ a perfect group (see Section 4.4 of [9]). For polycyclic groups, we have

**Proposition 6.** The group $R(H)$ is trivial for the following polycyclic groups:

1. $H$ a finite abelian $p$-group for $p$ odd;
2. $H$ a $2$-generated free nilpotent group of class $2$;
3. $H$ a metacyclic group.

**Proof.**

(i) This case was shown in Theorem 4.2.4 of [9].

(ii) Let $S = \{a_1, a_2\}$ be a generating set for $H$. Then $H/H' = \langle H' a_1, H' a_2 \rangle$ and $H' = \langle [a_1, a_2] \rangle$.

Let $K = \langle [h_1, h_2^k] h_3, (h_1 h_2^k)^\psi^{-1} h_1, h_2, h_3 \in H \rangle^{H*H^\psi}$ in the group $H*H^\psi$. By [5, p. 37] we can simplify the generating set of $K$ to

\[
K = \langle [a_1, a_2^b]^k [a_1^b, (a_2^b)^\psi]^{-1} | b \in S \cup \{a_1, a_2\} \rangle^{H*H^\psi}.
\]

Taking the natural epimorphism $\phi : H*H^\psi \to \chi(H)$, it is easy see that $R(H) = \phi(K) = \langle [a_1, a_2^b] [a_1^b, (a_2^b)^\psi]^{-1} | b \in \{a_1, a_2, [a_1, a_2]\} \rangle^{\chi(H)}$. 

Now, the following relations hold in $\chi(H)$,
\[
\begin{align*}
[a_1^a, a_2^\psi]^a_1 &= [a_1^\psi, a_2^a_1] = [a_1^\psi, a_2^a_1], \\
[a_1, a_2^a_2] &= [a_1^a_2, a_2^\psi].
\end{align*}
\]
Since $\gamma_3(H) = 1$, it follows that
\[
\begin{align*}
[a_1, a_2^\psi]^a_1 &= [a_1, a_2^\psi]^a_1 = [a_1, a_2^\psi].
\end{align*}
\]
Thus, $R(H)$ is trivial.

(iii) Let $S = \{a_1, a_2\}$ be a polycyclic generators for $H$. By Proposition 3 we can take
\[
R(H) = \left\langle [a_i, a_j^\psi]^{a_k} [a_i, (a_j^a_k)^\psi]^{-1} \mid a_i, a_j, a_k \in S \right\rangle^T,
\]
were $T$ is any transversal for $H/H'$. In the group $\chi(H)$ the following relations hold,
\[
\begin{align*}
[a_1, a_2^\psi]^a_1 &= [a_1^\psi, a_2]^a_1 = [a_1^\psi, a_2^a_1] \\
&= [a_1, (a_2^a_1)^\psi], \\
[a_1, a_2^\psi]^a_2 &= [a_1^a_2, (a_2)^\psi].
\end{align*}
\]
Thus, $R(H)$ is trivial. \(\square\)

REFERENCES

[1] Blyth, Russell D.; Morse, Robert F.; Computing the nonabelian tensor squares of polycyclic groups. J. Algebra 321 (2009), no. 8, 2139-2148.
[2] R. Brown, J.L. Loday, Van Kampen theorems for diagrams of spaces, Topology 26 (1987) 311-335.
[3] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.7.4; 2014, http://www.gap-system.org.
[4] K. W. Gruenberg, Cohomological Topics in Group Theory, Lecture Notes in Mathematics No 143, Springer-Verlag, New York and Berlin, 1970.
[5] Gupta, Narain D., Rocco, Norai R., Sidki, Said N. Diagonal Embeddings of Nilpotent Groups, Illinois J. of Math., 30 (1986) 274-283.
[6] McDermott, Aidan; The nonabelian tensor product of groups: Computations and structural results, Ph.D. thesis, National Univ. of Ireland, Galway, February 1998.
[7] Rocco, Norai R. On a construction related to the nonabelian tensor square of a group. Bol. Soc. Brasil. Mat. (N.S.) 22 (1991), no. 1, 63-79.
[8] N.R. Rocco, A Presentation for a Crossed Embedding of Finite Solvable Groups, Comm. in Algebra 22 (1994) 1975-1998.
[9] Sidki, Said N., On Weak Permutability between Groups. Journal of Algebra., v.63, p.186 - 225, 1980.
[10] Karpilovsky, G., The Schur Multiplier (London Mathematical Society monographs; new ser. 2) Oxford University Press, 1987.
[11] Stammbach, U., Über die ganzzahlige homologie von gruppen, Expo. Math. 3 4 (1985) 359-372.
SECRETARIA DE ESTADO EDUCAÇÃO DO DISTRITO FEDERAL, BRASÍLIA, DF, BRAZIL
E-mail address: bruno_crlima@hotmail.com

UNIVERSIDADE FEDERAL DE GOIÁS, GOIÂNIA, GOIÁS, BRAZIL
E-mail address: ricardo@ufg.br