Dynamical behaviour of travelling wave solutions to the conformable time-fractional modified Liouville and mRLW equations in water wave mechanics

Abdulla - Al - Mamun a,b,c,*, Samsun Nahar Ananna a,b,c, Tianqing An a, Nur Hasan Mahmud Shahen d,e, Md. Asaduzzaman c, Foyjonessa e

a Department of Mathematics, College of Science, Hohai University, Nanjing-210098, PR China
b School of Science and Engineering, AM's Research Academy, Dhaka, Bangladesh
c Department of Mathematics, Islamic University, Kushtia-7005, Bangladesh
d Department of Arts and Sciences, Bangladesh Army University of Science and Technology, Saidpur-5310, Bangladesh
e Department of Mathematics, European University of Bangladesh, Dhaka-1216, Bangladesh

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ABSTRACT

In this current study, we described a modified extended tanh-function (mETF) method to find the new and efficient exact travelling and solitary wave solutions to the modified Liouville equation and modified regularized long wave (mRLW) equation in water wave mechanics. Travelling wave transformation decreases the leading equation to traditional ordinary differential equations (ODEs). The standardized balance technique provides the instruction of the portended polynomial related result stimulated from the mETF method. The substitution of this result follows the preceding step. Balancing the coefficients of the like powers of the portended solution leads to a system of algebraic equations (SAE). The solution of that SAE for coefficients provides the essential connection between the coefficients and the parameters to build the exact solution. Here the acquired solutions are hyperbolic, rational, and trigonometric function solutions. Our mentioned method is straightforward, succinct, efficient, and powerful and can be emphasized to establish the new exact solutions of different types of nonlinear conformable fractional equations in engineering and further nonlinear treatments.

1. Introduction

From the past few decades, fractional calculus is one of the raising issues of nonlinear dynamics. Most of the real problems are now converted into fractional partial differential equations (FPDEs) model in mathematical physics. The investigation of the travelling wave solutions of FPDEs plays a dynamic role in describing the nature of nonlinear problems in the region of applied discipline and engineering. The smooth function is identified by $F(\omega) = ce^{\omega c}$ for $c, d \in \mathbb{R}$. Then we have the equation $\nabla^2 \omega = ce^{\omega c}$, called the Liouville equation. This equation seems not only in fluid dynamics but also in various problem of mathematical physics for example the field theory and plasma physics. Hence, its exact solutions have been gained with many mathematical methods. Although some nonlinear PDEs are integrable, they may not be so relaxed to integrate. Different categories of nonlinear wave structures have been introduced typically to establish various problems in physical science such as the expansion of wave and shallow water waves, the heat flow phenomena, plasma physics, Geophysics, optical fibres, chemical kinematics, Mathematical biology, electricity, computational fluid mechanics, and quantum mechanics [1, 2, 3, 4, 5, 6, 7, 8, 9]. Consequently, lots of apprehension has been tried to find the new and authentic exact solution of time-fractional NPDEs by various research. Several dominant systems have been established for travelling. Solitary wave solutions of FPDEs, for example the expanded tanh-function method [10, 11, 12, 13], the $(G'/G^2)$-expansion method [14], the method of characteristics [15], the residual power series method [16, 17, 18], q-homotopy analysis transform method [19], the Shehu transform method [20], Sine-Gordon expansion method [21], the new extended direct algebraic method [22], the Jacobi elliptic expansion technique [23, 24, 25, 26].

* Corresponding author at: Department of Mathematics, College of Science, Hohai University, Nanjing-210098, PR China.
E-mail addresses: abdullamamun21@gmail.com (A.A. Mamun), ananna0907@gmail.com (S.N. Ananna), antqi@hhu.edu.cn (T. An), asad@math.iu.ac.bd (Md. Asaduzzaman).

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24, 25, 26), sub equation method [27], Homotopy Analysis Method [28], the Fractional Laplace Differential Transform Method [29], the Hirota’s bilinear scheme [30], extended simple equation technique [31, 32, 33], (G′/G)-expansion technique [34, 35, 36, 37], (G′/G)1/G-expansion method [38], the advanced exp (−ϕ(z))-expansion method [39, 40], the variational iteration method [41, 42, 43], the general projective Riccati equation scheme [44], the exponential function method [45], extended direct algebraic system [46, 47], modified simple equation approach [48, 49, 50] and auxiliary method [51]. Inside, an expected result with parameters is supposed to be a result of leading equations and the relationships between the parameters are examined. The sense is humbly on the likeness with exponential-type results to the ODEs with constant coefficients. These expected results are of several forms, casing exponential, trigonometric, hyperbolic, or rational functions, and more. Furthermore, the interaction of several categories of waves was also determined by spending analytical performances [52, 53].

Modified Liouville and the modified regularized long wave scheme are used in physics to leverage the nonlinear wave scheme. A symmetric explanation of the mRLW equation is exposed to describe softly nonlinear ion acoustic and space charge wave. The equation holds hyperbolic secular squared solitary waves and has four known invariants. Here in our existing work we have employed fruitfully the mETF method to catch the exact travelling wave solution of the modified Liouville equation and the mRLW equation as results we have achieved more naively and power solutions as compared to previous literature results. The obtained our new solutions have numerous applications to solve nonlinear problems arising in nonlinear sciences.

This research focuses on the dynamical analysis of time-fractional modified Liouville and mRLW equations with the direct application of the mETF method. Our mentioned method has some benefits in contrast to the tanh-function method. It only follows a simpler algorithm to yield an algebraic system and yields kink and singular soliton solutions with no extra effort [54, 55]. The main important methodology of this method is too explicit the exact solutions of FNLEs that satisfying the Nonlinear ODE of the form, \( u' + a_1 u + \sum_{i=1}^{m} a_i u^i = 0 \). Our mentioned method over the other existing scheme [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50] gives some direct and brief form of exact travelling wave solution. Several authors have been established this mETF method to find out the exact solution of various NLLEs in deference sense of derivative such as (Jumarie’s modified Riemann-Liouville derivatives, Conformable derivative, Kerr law nonlinearity) [56, 57, 58, 59]. Still, there are no fruitful studies not found yet on our preferred WBBM equation with this method. Here our obtained new exact solution of newly introduced WBBM equations is more accurate and efficient and very friendly applicable in various fields of mathematical physics, engineering, and wave analysis-related treatments. So we can claim that our proposed study, dynamical analysis of newly introduced WBBM equations with the mETF method’s aid, is novel in the sense of conformable derivative. With the virtue of some computational software (Maple, Mathematica), we showed the obtained solutions by taking fruitful values of the included parameters by depicting sketches to simplify the physical explanation appropriately. Improved Liouville and mRLW equations are used in physics to handle the nonlinear wave scheme. By considering this fact, we employed the mETF technique successfully to discover the exact travelling wave solutions of improved Liouville and mRLW equations in water wave mechanics. The mRLW equation appeared in Gardner’s study [56]. Our new solutions have numerous applications to resolve nonlinear problems rising in nonlinear disciplines and mathematical physics. We firmly believe that B-spline finite elements established on estimated solutions describing solitary waves’ motion were examined in this condition with the mentioned scheme precisely.

The rest of the paper is decorated as follows: In section 2, the narration of the conformable time-fractional differential equation. In section 3, the mETF method has been described. In segment 4, we apply this proposed scheme to the conformable time-fractional modified Liouville and mRLW equations. In section 5, results and discussion, in section 6, conclusions are gathered.

2. Conformable fractional derivative

The conformable derivative of order \( \gamma \) with regards to the independent variable \( t \) is described as

\[
D^\gamma_t (y(t)) = \lim_{\sigma \to 0} \frac{y(t + \sigma t^\gamma) - y(t)}{\sigma}, \quad t > 0, \quad \gamma \in (0, 1]
\]

for a purpose \( y = y(t) : [0, \infty) \to \mathbb{R} \). This well-defined fractional derivative is accomplished by satisfying some well-known essential conditions. If \( y \) is \( \gamma \)-differential in some \( (0, \sigma) \), \( \sigma > 0 \), and then \( \lim_{\sigma \to 0^+} D^\gamma_t (y(t)) \) exists, then

\[
D^\gamma_t (y(0)) = \lim_{\sigma \to 0^+} D^\gamma_t (y(t))
\]

Later, Abdeljawad has also explored chain rule, exponential functions, Gronwalls inequality, definite and indefinite integration by parts, Laplace transform and Taylor power series expansions for conformable derivative in the process of fractional order. The definition of a conformable fractional order derivative can effortlessly overwhemled the difficulty of current modified Riemann Liouville derivative definition.

**Theorem 1.** Consider that the order of derivative \( \gamma \in (0, 1) \), and suppose that \( h = h(t) \) and \( k = k(t) \) are \( \gamma \)-differentiable for all positive value of \( t \). Then,

- \( D^\gamma_t (a_1 h + a_2 k) = a_1 D^\gamma_t (h) + a_2 D^\gamma_t (k) \).
- \( D^\gamma_t (\text{t}^m) = m \text{t}^{m-\gamma}, \quad \forall m \in \mathbb{R} \).
- \( D^\gamma_t (\mu) = 0, \quad \forall \mu(t) = \mu \).
- \( D^\gamma_t (h k) = h D^\gamma_t (k) + k D^\gamma_t (h) \).
- \( D^\gamma_t \left( \frac{1}{f} \right) = \frac{f^\gamma(t) - f^\gamma(0)}{f(t)} \).
- \( D^\gamma_t (h(t) k(t)) = \text{t}^{1-\gamma} \frac{d}{dt} \left( k(t) \right) \).

for all \( a_1, a_2 \in \mathbb{R} \). Conformable fractional differential operator obeys some important essential stuff like the chain rule, Taylor series expansion, and Laplace transforms.

**Theorem 2.** Assume \( h = h(t) \) be a \( \gamma \) conformable differentiable function and assume that \( k \) is differentiable and well-defined in the range of \( h \). Then,

\[
D^\gamma_t (h(t) k(t)) = \text{t}^{1-\gamma} k(t) h^\gamma(t) \left( k(t) \right)
\]
The conformable fractional derivative has two benefits over the conventional fractional derivatives. First, the conformable fractional derivative definition is likely and it fulfills most of the belongings which the classical integral derivative has such as linearity, product rule, quotient rule, power rule, chain rule, vanishing derivatives for constant functions, Rolle’s theorem, and mean value theorem. Second, the conformable derivative brings us a lot of expediency when it is applied for modelling many physical problems, because the differential equations with conformable fractional derivative are easier to solve mathematically than those linked with the Riemann-Liouville or Caputo fractional derivative. In fact, various researchers have already applied conformable fractional derivative to numerous fields and a lot of matching procedures were established [59, 60, 61].

3. Description of the mETF method

In this part, we give a brief explanation of the mETF method. Considering a given nonlinear PDE

$$F(u, u_t, u_x, u_{xx}, \ldots) = 0$$

We first consider its travelling solutions $u(x, t) = u(\psi)$, $\psi = x \pm ct$, then equation (3) converts an ODE. The next essential step is that the solution we are observing is expressed in the form.

$$u(\psi) = A_0 + \sum_{j=1}^{n} a^j (A_j + B_j a^{-2})$$

and

$$a^j = b + a^2$$

where $b$ is a parameter to be determined, $a = a(\psi) = \frac{dx}{d\psi}$. The parameter $n$ can be found by matching the uppermost order linear term with nonlinear terms [62]. Inserting (4) and (5) into the ODE will produce an SAE concerning $A_j$, $B_j$, $b$ and $c$ (where $j = 1, 2, \ldots, n$) because all the coefficients of $a^j$ have to vanish. With the support of Mathematica, one can govern $A_j$, $B_j$, $b$ and $c$. Equation (5) has general solutions:

Case-I. Hyperbolic function solution, when $(b < 0)$:

$$a = -\sqrt{-b} \tanh(\sqrt{-b} \psi)$$

$$a = -\sqrt{-b} \coth(\sqrt{-b} \psi)$$

Case-II. Rational function solution, when $(b = 0)$:

$$a = -\frac{1}{\psi}$$

Case-III. Trigonometric function solution, when $(b > 0)$:

$$a = \sqrt{b} \tan(\sqrt{b} \psi)$$

$$a = -\sqrt{b} \cot(\sqrt{b} \psi)$$

In the next part, we study some nonlinear equations of unusual interest in physics and mathematics to demonstrate this method.

4. Application of mETF method

4.1. Modified Liouville equation

Considering the time-fractional form of MLE is as follows [2]:

$$a^2 u_{xx} - D^\alpha_t u + pe^{\theta u} = 0$$

(9)

$D^\alpha_t$ is the fractional derivative in a conformable sense. Where $\alpha > 0$ and $a$, $p$ and $q$ are real arbitrary constants. Consider the travelling waves transformations

$$u(x, t) = U(\psi), \text{ where } \psi = mx + c \frac{t}{\gamma} \text{ and } U = e^{\theta u}$$

(10)

By replacing the above transformations of equation (10) into equation (9) and integrating with respect to $\psi$, we have the resulting ODE as:

$$\left(\frac{m^2 a^2}{q} - \frac{c^2}{q}\right)(U''U - U'^2) + pU^3 = 0$$

(11)

With the virtue of homogeneous balancing of the uppermost order derivative term $U'''U$ and the nonlinear term $U^3$ in equation (11), we find that $N = 2$. Therefore, our suggested method allows us to use the auxiliary solution of the form:

$$U(\psi) = A_0 + A_1 a + A_2 a^2 + B_1 a^{-1} + B_2 a^{-2}$$

(12)

Now putting the value of $U$, $U'''$ and $U^3$ in equation (11), as well as equating the coefficients of like power of $a$ from the above equation, we get the resulting SAE:
\[
\frac{1}{q} \left( pqA_3^2 + b^2c^2A_1^2 - a^2b^2m^2A_1^2 - 2b^2c^2A_2A_3 + 2a^2b^2m^2A_0A_2 - 8bc^2A_1B_1 + 8a^2b^2m^2A_1B_1 + 6pqA_0A_1B_1 + c^2B_1^2 - a^2m^2B_1^2 + 3pqA_2B_1^2 \\
- 2c^2A_0B_2 + 2a^2m^2A_0B_2 + 5pqA_2^2B_1^2 - 32bc^2A_2B_2 + 32a^2b^2m^2A_2B_2 + 6pqA_0A_2B_2 \right) = 0
\]

\[
\frac{1}{q} \left( -2bc^2A_0A_1 + 2a^2b^2m^2A_0A_1 + 3pqA_0A_1 + 2b^2c^2A_1A_2 - 2a^2b^2m^2A_0A_2 + 3pqA_2A_1 - 18bc^2A_2B_1 + 18a^2b^2m^2A_2B_1 + 6pqA_0A_2B_1 \\
- 8c^2A_0B_2 + 8a^2m^2A_0B_2 + 6pqA_2B_2 \right) = 0
\]

\[
\frac{1}{q} \left( -2bc^2A_0B_1 + 2a^2b^2m^2A_0B_1 + 3pqA_2B_1 - 8bc^2A_2B_1 + 8a^2b^2m^2A_2B_1 + 3pqA_2B_1^2 - 18bc^2A_2B_2 + 18a^2b^2m^2A_2B_2 + 6pqA_0A_2B_2 + 2c^2B_2B_2 \\
- 2a^2m^2B_1B_2 + 6pqA_2B_2 \right) = 0
\]

\[
\frac{1}{q} \left( 3pqA_0A_2^2 - 8bc^2A_0A_2 + 8a^2b^2m^2A_0A_2 + 3pqA_0A_2 + 2b^2c^2A_2^2 - 2a^2b^2m^2A_2^2 - 4c^2A_1B_1 + 4a^2m^2A_1B_1 + 6pqA_1A_2B_1 - 16c^2A_2B_2 \\
+ 16a^2m^2A_2B_2 + 3pqA_2B_2 \right) = 0
\]

\[
\frac{1}{q} \left( -4b^2c^2A_1B_1 + 4a^2b^2m^2A_1B_1 + 3pqA_0B_1^2 - 8bc^2A_0B_2 + 8a^2b^2m^2A_0B_2 + 3pqA_0B_2 - 16b^2c^2A_2B_2 + 16a^2b^2m^2A_2B_2 + 6pqA_1B_1B_2 \\
+ 2c^2B_2^2 - 2a^2m^2B_2^2 + 3pqA_2B_2 \right) = 0
\]

\[
\frac{1}{q} \left( -2c^2A_0A_1 + 2a^2b^2m^2A_0A_1 + pqA_1 - 2bc^2A_1A_2 + 2a^2b^2m^2A_1A_2 + 6pqA_0A_1A_2 - 10c^2A_1B_1 + 10a^2b^2m^2A_1B_1 + 3pqA_2A_2 \right) = 0
\]

\[
\frac{1}{q} \left( -2b^2c^2A_0B_1 + 2a^2b^2m^2A_0B_1 + pqB_1^2 - 10b^2c^2A_1B_1 + 10a^2b^2m^2A_1B_1 - 2bc^2B_1B_2 + 2a^2b^2m^2B_1B_2 + 6pqA_0B_1B_2 + 3pqA_1B_2 \right) = 0
\]

\[
\frac{1}{q} \left( -c^2A_1^2 + 2a^2b^2m^2A_0A_2 + 6a^2m^2A_0A_2 + 3pqA_0A_2 + 3pqA_2 \right) = 0
\]

\[
\frac{1}{q} \left( -b^2c^2B_1^2 + a^2b^2m^2B_1^2 - 6b^2c^2A_0B_2 + 6a^2b^2m^2A_0B_2 + 3pqB_2B_2 + 3pqA_2B_2 \right) = 0
\]

\[
\frac{1}{q} \left( -4c^2A_1A_2 + 4a^2b^2m^2A_1A_2 + 3pqA_1A_2 \right) = 0
\]

\[
\frac{1}{q} \left( -4b^2c^2B_1B_2 + 4a^2b^2m^2B_1B_2 + 3pqB_1B_2 \right) = 0
\]

\[
\frac{1}{q} \left( -2c^2A_2^2 + 2a^2b^2m^2A_2^2 + pqA_2 \right) = 0
\]

\[
\frac{1}{q} \left( -2b^2c^2B_2^2 + 2a^2b^2m^2B_2^2 + pqB_2 \right) = 0
\]

The solution of the above system for \( b, A_0, A_1, A_2, b, B_1, B_2 \) provides several solution sets

**Set-1:**

\[
b = 0, \quad A_0 = 0, \quad A_1 = 0, \quad A_2 = \frac{2(c^2 - a^2m^2)}{pq}, \quad B_1 = 0, \quad B_2 = 0
\]

**Set-2:**

\[
b = b, \quad A_0 = \frac{2(bc^2 - a^2b^2m^2)}{pq}, \quad A_1 = 0, \quad A_2 = 0, \quad B_1 = 0, \quad B_2 = \frac{2(b^2c^2 - a^2b^2m^2)}{pq}
\]

**Set-3:**

\[
b = b, \quad A_0 = \frac{2(bc^2 - a^2b^2m^2)}{pq}, \quad A_1 = 0, \quad A_2 = \frac{2(c^2 - a^2m^2)}{pq}, \quad B_1 = 0, \quad B_2 = 0
\]

**Set-4:**

\[
b = b, \quad A_0 = \frac{4(bc^2 - a^2b^2m^2)}{pq}, \quad A_1 = 0, \quad A_2 = \frac{2(c^2 - a^2m^2)}{pq}, \quad B_1 = 0, \quad B_2 = \frac{2(b^2c^2 - a^2b^2m^2)}{pq}
\]

Using these solution sets, we build solutions to equation (9) as follows:

**When \( b < 0 \), we obtain the following hyperbolic function solutions:**

\[
U_1(x,t) = \frac{2b(c^2 - a^2m^2)Csch[\sqrt{-b(mx + ct)}])^2}{pq}
\]

\[
U_2(x,t) = \frac{2b(c^2 - a^2m^2)Sech[\sqrt{-b(mx + ct)}])^2}{pq}
\]

\[
U_3(x,t) = \frac{8b(c^2 - a^2m^2)Csch[2\sqrt{-b(mx + ct)}])^2}{pq}
\]

**When \( b = 0 \), we obtain the following rational function solution**

\[
U_4(x,t) = \frac{2(c^2 - a^2m^2)}{pq(mx + ct)^2}
\]

**When \( b > 0 \), we obtain the following trigonometric function solution**


\[
U_5(x,t) = \frac{2b(c^2 - a^2m^2) \csc \left( \sqrt{b(mx + \frac{ct}{q})^2} \right)^2}{pq}
\]
\[
U_6(x,t) = \frac{2b(c^2 - a^2m^2) \sec \left( \sqrt{b(mx + \frac{ct}{q})^2} \right)^2}{pq}
\]
\[
U_7(x,t) = \frac{8b(c^2 - a^2m^2) \csc \left( 2\sqrt{b(mx + \frac{ct}{q})^2} \right)^2}{pq}
\]

### 4.2. Modified regularized long-wave (mRLW) equation

The leading form of the mRLW equation is as follows:

\[
D_q^t u + au_x + cu u_x + rD_q^t u_{xx} = 0, \quad t \geq 0 \tag{13}
\]

where \(D_q^t\) conformable fractional differential operator, \(a\), \(c\) and \(q\) are real arbitrary constants that can be determined later. Consider the travelling waves transformations

\[
u(x,t) = u(\psi), \quad \psi = p \left( x - \frac{vt}{q} \right) \tag{14}
\]

By replacing the above transformations of equation (14) into equation (13), we have the following ODE:

\[
(-pv + ap)U + cp U^3 - rvp^3 U'' = 0 \tag{15}
\]

Balance the uppermost order derivative term \(U''\) and nonlinear term \(U^3\) in equation (15), we have \(n = 1\). We assume that the solution of the equation (12) has the form as:

\[
U(\psi) = A_0 + A_1 \alpha + B_1 \alpha^{-1} \tag{16}
\]

Substituting equation (16) along with the equation (5), equation (15) becomes,

\[
\frac{1}{3} p \left( 3aA_0 - 3vA_0 + cA_1^3 + 6cA_0 A_1 B_1 \right) = 0
\]

\[
\frac{1}{3} p \left( 3aA_1 - 3vA_1 - 6b^2p^2vA_1 + 3cA_1^2 A_1 + 3cA_1^2 B_1 \right) = 0
\]

\[
\frac{1}{3} p \left( 3B_1 - 3B_1 - 6b^2p^2vB_1 + 3cA_1^2 B_1 + 3cA_1 B_1^2 \right) = 0
\]

\[
cp A_0 A_1^2 = 0
\]

\[
cp A_0 B_1^2 = 0
\]

\[
\frac{1}{3} p \left( -6b^2p^2vA_1 + cA_1^3 \right) = 0
\]

\[
\frac{1}{3} p \left( -6b^2p^2vB_1 + cB_1^3 \right) = 0
\]

The solution of this system for \(A_0, A_1, A_2, B_1, B_2\) provides several solution sets

Set-1:

\[
b = \frac{a - v}{8p^2rv}, \quad A_0 = 0, \quad A_1 = \pm \frac{\sqrt{6p} \sqrt{r} \sqrt{v}}{\sqrt{c}}, \quad B_1 = \pm \frac{\sqrt{6a} \pm \sqrt{6v}}{8 \sqrt{cp} \sqrt{r} \sqrt{v}}
\]

Set-2:

\[
b = \frac{a - v}{2p^2rv}, \quad A_0 = 0, \quad A_1 = 0, \quad B_1 = \pm \frac{\sqrt{2} \sqrt{a^2 - 2av + v^2}}{8 \sqrt{cp} \sqrt{r} \sqrt{v}}
\]

Set-3:

\[
b = \frac{-a + v}{4p^2rv}, \quad A_0 = 0, \quad A_1 = \pm \frac{\sqrt{6p} \sqrt{r} \sqrt{v}}{\sqrt{c}}, \quad B_1 = \pm \frac{\sqrt{6a} \pm \sqrt{6v}}{4 \sqrt{cp} \sqrt{r} \sqrt{v}}
\]

Set-4:

\[
b = \frac{a - v}{2p^2rv}, \quad A_0 = 0, \quad A_1 = \pm \frac{\sqrt{6p} \sqrt{r} \sqrt{v}}{\sqrt{c}}, \quad B_1 = 0
\]

Using these solution sets, we build solutions to equation (12) as follows:
Fig. 1. The above set of figures are representing the function solution of $U_2(x,t)$ for the parameter $a=1$, $c=-0.5$, $m=1$, $p=-2$, $q=1$, $b=-1$ and $\gamma=1$, 0.25, 0.5 ($1^\text{st}$, $2^\text{nd}$, $3^\text{rd}$ column respectively) within the interval $-3 \leq x, t \leq 3$ for both MATLAB and Mathematica plot views.

When $a = v \Rightarrow b = 0$, we obtain the following rational function solution:

$$U_{10,11}(x,t) = \pm \sqrt{\frac{6}{c}} \frac{v}{(\gamma x - t v)}$$

When $b < 0$, we obtain the following hyperbolic function solutions:

$$U_{10,11}(x,t) = \pm \frac{2}{\sqrt{c}} (v-a) \text{Cosh} \left[ \sqrt{1+\frac{2}{\gamma} (x - \frac{t v}{\gamma})} \right]$$

$$U_{12}(x,t) = \frac{2}{\sqrt{c}} (a-v) \text{Cosh} \left[ \frac{\sqrt{1+\frac{2}{\gamma} (x - \frac{t v}{\gamma})}}{\sqrt{2} \sqrt{c}} \right]$$

$$U_{13,14}(x,t) = \pm \frac{2}{\sqrt{c}} (a-v) \text{Tanh} \left[ \frac{\sqrt{1+\frac{2}{\gamma} (x - \frac{t v}{\gamma})}}{\sqrt{2} \sqrt{c}} \right]$$

When $b > 0$, we obtain the following trigonometric function solutions:

$$U_{15,16}(x,t) = \pm \frac{2}{\sqrt{c}} (a-v) \text{Cot} \left[ \frac{\sqrt{1+\frac{2}{\gamma} (x - \frac{t v}{\gamma})}}{\sqrt{2} \sqrt{c}} \right]$$

$$U_{17,18}(x,t) = \pm \frac{2}{\sqrt{c}} (v-a) \text{Csc} \left[ \frac{\sqrt{1+\frac{2}{\gamma} (x - \frac{t v}{\gamma})}}{\sqrt{2} \sqrt{c}} \right]$$

$$U_{19,20}(x,t) = \pm \frac{2}{\sqrt{c}} (a-v) \text{Tan} \left[ \frac{\sqrt{1+\frac{2}{\gamma} (x - \frac{t v}{\gamma})}}{\sqrt{2} \sqrt{c}} \right]$$

5. Result and discussion

The physical explanation of the founded exact travelling wave solutions to the modified Liouville equation and the mRLW equation has been explained in this part. We expose the graphical representation of these solutions and accomplish the different solutions using some suitable parameters. As an outcome, we successfully got the equation including kink solution, dark kink shape solution, periodic wave solution, singular Kink, single soliton, and other types of soliton founded by choosing different free parameters which have the proper physical explanation. To represent the graphs, we have used two computational computer software MATLAB and Mathematica. We have shown and focused on some crucial geometrically interpretable graphs and physical explanation in mathematical physics and water wave mechanics among each exact travelling-wave solution. It’s
essential to notice that every suitable solution has been sketched three times for conformable fractional parameter \( \gamma \). 1st three figures of \( a, b, c \) are from the 3D surface plot with contour using MATLAB views and 4th (d) figure are from 2D combined line chart of \( a, b, c \), respectively in every set of Figs. 1–4. Here Fig. 1(a) represents the anti-bell shape soliton for \( \gamma = 1 \). With the decreasing of fractional parameters \( \gamma = 1 \) to 0.5 and 0.25 the function graph takes the form as singular soliton. The Fig. 1-(d) shows the line chart of combined 2D plot which indicate us to the highest amplitude...
and frequency of the wave. Here Fig. 2-(a) represents the double soliton shape for \( \gamma = 1 \). With the decreasing of fractional parameters \( \gamma = 1 \) to 0.5 and 0.25 the function graph does not have any dynamical changes. The Fig. 2-(d) shows the line chart of combined 2D plot which clear us to show the highest amplitude and frequency of the wave. Here Fig. 3-(a) represents the periodic rogue wave shape for \( \gamma = 1 \). With the decreasing of fractional parameters \( \gamma = 1 \) to 0.5 and 0.25 the function graph does not have any dynamical changes. The Fig. 3-(d) shows the line chart of combined 2D plot which clear us to show the highest amplitude and frequency of the wave. Here Fig. 4-(a) represents the kink shape solutions for \( \gamma = 1 \). With the decreasing of fractional parameters \( \gamma = 1 \) to 0.5 and 0.25 the function graph changes to one soliton shape. The Fig. 1-(d) shows the line chart of combined 2D plot which clear us to show the highest amplitude and frequency of the wave.

6. Conclusion

In this study, exact travelling-wave solutions of modified Liouville and conformable time-fractional mRLW equations are discovered using the mETF method. Expending companionable wave transform, the equations are condensed to some ODEs. Then, the expected solutions are exchanged into the resultant form of ODE. Comparing the coefficients of like power of \( a \) to zero represents some SAE. Solving this system provides the relations between the parameters. Some physical and composite solutions are arrangements of management of a hyperbolic tangent, cotangent, tangent, cotangent, and secant, cosecant functions are unwavering explicitly. Graphical representation of some solutions is represented in some finite fields to understand the effects of \( b \) and \( \gamma \) by using MATLAB and Mathematica. So we demand that the obtained solutions are unique in this current study and thus could be more effective in the study of time-fractional nonlinear water wave mechanics and nonlinear physical phenomena.

Declarations

Author contribution statement

Abdulla - AI - Mamun: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

Samsun Nahar Ananna: Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

Tianqing An, Md. Asaduzzaman: Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data.

Nur Hasan Mahmud Shahen: Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

Foyjonnesa: Contributed reagents, materials, analysis tools or data; Wrote the paper.

Declaration of interests statement

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Data included in article/supplementary material/referenced in article.

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Additional information

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