Regularity of solutions to elliptic equations with Grushin’s operator

Xiaohuan Wang\textsuperscript{a,b} and Jihui Zhang\textsuperscript{a}

\textsuperscript{a} Institute of Mathematics, School of Mathematical Science
Nanjing Normal University, Nanjing 210023, China
\textsuperscript{a} School of Mathematics and Statistics, Henan University
Kaifeng, Henan 475001, China
xiaohuanw\texttt{126.com}

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Abstract

In this paper, we consider the regularity of solutions to elliptic equation with Grushin’s operator. By using the Feynman-Kac formula, we first get the expression of heat kernel, and then by using the properties of heat kernel, the optimal regularity of solutions will be obtained. The novelty of this paper is that the Grushin’s operator is a degenerate operator.

Keywords: Grushin’s operator; Schauder estimate; $L^p$-theory.
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1 Introduction

The regularity of solutions to second order elliptic equation has been extensively studied by many authors, see the book \cite{2}. But for degenerate elliptic equation, there is few work about the regularity results until now. In this paper, we focus on a special degenerate elliptic equation–Grushin’s elliptic equation. The main reason why we can deal with it is that we can get the expression of heat kernel by using the probability method.

For Grushin’s operator, many authors studied it. Beckner \cite{3} obtained the Sobolev estimates for the Grushin’s operator in low dimensions by using hyperbolic symmetry and conformal geometry. Riesz transforms and multipliers for the Grushin’s operator was considered by Jotsaroop et al. \cite{4}. Tri \cite{8} studied the generalized Grushin’s equation. $L^p$-estimates for the wave equation associated to the Grushin operator was studied by Jotsaroop-Thanavelu \cite{5}. The fundamental solution for a degenerate parabolic pseudo-differential operator covering the Grushin’s operator was obtained by Tsutsumi \cite{9, 10}. Furthermore, Tsutsumi \cite{11} constructed a left parametrix for a pseudo-differential operator. We remark that Tsutsumi did not give the exact expression of heat kernel.

In the book \cite{1}, they gave the expression of heat kernel (see Page 191), but he expression is hard to use because they used the inverse Fourier transform. Similar degenerate elliptic equation was studied by Robinson-Sikora \cite{7} and the Hardy inequalities for Grushin’s operator was considered by \cite{12}.

In this paper, in view of probability point, we give a new expression and then get the regularity of solution by using the properties of heat kernel.

This paper is arranged as follows. In next section, some preliminaries are given and the main results will be proved in section 3. Throughout this paper, we write $C$ as a general positive constant and $C_i, i = 1, 2, \cdots$ as a concrete positive constant.
2 Main results

Consider the Grushin’s operator
\[ \mathcal{L} = \frac{1}{2}(\partial_x^2 + x^2 \partial_y^2), \]
which is the generator of the diffusion process \((X_t, Y_t)\), where \((X_t, Y_t)\) satisfies
\[
\begin{align*}
\begin{cases}
    dX_t &= dW_t^1, \\
    dY_t &= X_t dW_t^2,
\end{cases}
\end{align*}
\]
\begin{align*}
    X_0 &= \mu_1, \\
    Y_0 &= \mu_2.
\end{align*}

Here \(W_t^i\) \((i = 1, 2)\) are standard i.i.d Brownian motion. It is easy to see that the process \((X_t, Y_t)\) is a Gaussian stochastic process. Direct calculations show that
\[
\begin{align*}
    \mathbb{E}(X_t Y_t) &= (\mu_1 \mu_2), \\
    \text{Cov}(X_t, Y_t) &= (t \mu_1^2 + \frac{1}{2} t^2).
\end{align*}
\]

Therefore, we get the heat kernel of the operator \(\mathcal{L}\) is
\[
K(t, x, y, \mu_1, \mu_2) = \frac{1}{2\pi t^{3/2}} \exp \left\{ -\frac{(x - \mu_1)^2}{t} - \frac{[\mu_1(x - \mu_1) - y + \mu_2]^2}{t^2} \right\},
\]
which yields that
\[
\begin{align*}
    \nabla_x K(t, x, y) &= \frac{-2x}{t} K(t, x, y), \\
    \nabla_y K(t, x, y) &= \frac{-y}{t^2} K(t, x, y).
\end{align*}
\]

We first want to solve the following elliptic equation
\[
\mathcal{L}^\lambda_b := (\mathcal{L} - \lambda) u + b \cdot \nabla u = f. \tag{2.1}
\]

**Theorem 2.1** Assume that \(b \in C^\beta_b(\mathbb{R}^2)\). There exists a \(\lambda_0 > 0\) such that for any \(f \in C^\beta_b(\mathbb{R}^2)\) and \(\lambda > \lambda_0\), there is a unique solution \(u \in C^{1+\beta}_b(\mathbb{R}^2)\) to equation (2.1) such that
\[
\lambda^\delta \|u\|_{C^{1+\beta}_b(\mathbb{R}^2)} \leq \|f\|_{C^\beta_b(\mathbb{R}^2)}, \tag{2.2}
\]
where \(\delta > 0\) is defined in Lemma 3.1.

Assume further that \(\nabla_y h(x, \cdot) \in C^\beta_b(\mathbb{R})\) for any \(x \in \mathbb{R}\), where \(h = b \) or \(f\). Then there is a unique solution \(u \in C^{1+\beta}_b(\mathbb{R}^2)\) to equation (2.1) such that
\[
\lambda^\delta \|u\|_{C^{2+\beta}_b(\mathbb{R}^2)} \leq \|f\|_{C^\beta_b(\mathbb{R}; C^{1+\beta}(\mathbb{R}))}.
\]

Let
\[
u(x, y) := (\lambda - \mathcal{L})^{-1} f(x) = \int_0^\infty e^{-\lambda t} K(t, \cdot) * f(x, y) dt.
\]

Next, we consider the \(L^p\)-regularity of \(u(x, y)\).
\textbf{Theorem 2.2} Assume that $f(\cdot, y) \in L^p(\mathbb{R})$ for every $y \in \mathbb{R}$ with and $f(x, \cdot) \in W^{s, q}(\mathbb{R}, \mathbb{R})$ for every $y \in \mathbb{R}$, $0 < s < 1$ and $q > 1$. Then we have the following estimates:

\begin{align*}
\|u\|_{L^p(\mathbb{R}^2)} &\leq C\lambda^{-1 - \frac{3}{2p} + \frac{1}{q}} \|f\|_{L^p(\mathbb{R}, L^s(\mathbb{R}))}, \\
\|\nabla_x u\|_{L^p(\mathbb{R}^2)} &\leq C\lambda^{-\frac{3}{2} - \frac{1}{2p} + \frac{1}{q}} \|f\|_{L^p(\mathbb{R}, L^s(\mathbb{R}))}, \\
\|\nabla_y u\|_{L^p(\mathbb{R}^2)} &\leq C\lambda^{-s - \frac{3}{2p} + \frac{1}{q}} \|f\|_{L^p(\mathbb{R}, W^{s, q}(\mathbb{R}))}.
\end{align*}

Moreover, if we take $0 < s < 1$ and $p, q, r > 1$ such that $-\frac{3}{2r} + \frac{1}{r} < 0$ and $-s - \frac{3}{2r} + \frac{1}{r} + 1 < 0$, we have

\[ \|\nabla_x^2 u\|_{L^r(\mathbb{R}^2)} \leq C, \quad \|\nabla_y^2 u\|_{L^r(\mathbb{R}^2)} \leq C. \]

\section{Proof of Main results}

Denote $[\cdot]_\beta$ by the semi-norm of $C^\beta$. Set

\[ \mathcal{A} := \left\{ f(x, y) : \begin{array}{c}
(\cdot, y) \in L^\infty(\mathbb{R}) \text{ for any } y \in \mathbb{R}, f(x, \cdot) \in C^\beta_b(\mathbb{R}) \\
\text{for any } x \in \mathbb{R} \text{ with } 0 < \beta < 1
\end{array} \right\} \]

\textbf{Lemma 3.1} Assume that $f \in \mathcal{A}$. Then

\begin{align*}
\|u\|_{L^\infty(\mathbb{R}^2)} &\leq C\lambda^{-\frac{1}{2}} \|f\|_{L^\infty(\mathbb{R}^2)}, \\
\|\nabla_y u\|_{L^\infty(\mathbb{R}^2)} &\leq C\lambda^{-\beta} \|f\|_{L^\infty(\mathbb{R}, C^\beta_b(\mathbb{R}))}, \\
\|\nabla_x u\|_{L^\infty(\mathbb{R}^2)} &\leq C\lambda^{-\frac{1}{2} + \beta} \|f\|_{L^\infty(\mathbb{R}, C^\beta_b(\mathbb{R}))}, \\
[\nabla_y u]_\beta &\leq C\lambda^{-\delta} \|f\|_{C^\beta(\mathbb{R}^2)},
\end{align*}

where $0 < \delta < (\beta \wedge (1 - \beta))$. That is to say, $\nabla u \in C^\beta_b(\mathbb{R}^2)$.

\textbf{Proof.} Simply calculations show that

\begin{align*}
\|u\|_{L^\infty(\mathbb{R}^2)} &= \left\| \int_0^\infty e^{-\lambda t} K(t, \cdot) * f(x, y) dt \right\|_{L^\infty(\mathbb{R}^2)} \\
&\leq \int_0^\infty e^{-\lambda t} \|K(t, \cdot)\|_{L^1(\mathbb{R})} \|f\|_{L^\infty(\mathbb{R}^2)} dt \\
&\leq C\lambda^{-1} \|f\|_{L^\infty(\mathbb{R}^2)}, \\
\|\nabla_x u\|_{L^\infty(\mathbb{R}^2)} &= \left\| \int_0^\infty e^{-\lambda t} \nabla_x K(t, \cdot) * f(x, y) dt \right\|_{L^\infty(\mathbb{R}^2)} \\
&\leq \int_0^\infty e^{-\lambda t} \|\nabla_x K(t, \cdot)\|_{L^1(\mathbb{R})} \|f\|_{L^\infty(\mathbb{R}^2)} dt \\
&\leq C\lambda^{-\frac{1}{2}} \|f\|_{L^\infty(\mathbb{R}^2)},
\end{align*}

Moreover, if $f(\cdot, y) \in C^\beta_b(\mathbb{R})$ for any $y \in \mathbb{R}$, it holds that

\[ [\nabla_y u]_\beta \leq C\lambda^{-\delta} \|f\|_{C^\beta(\mathbb{R}^2)}, \]

where $0 < \delta < (\beta \wedge (1 - \beta))$. That is to say, $\nabla u \in C^\beta_b(\mathbb{R}^2)$. 

\[ \]
and

\[ \|\nabla_y u\|_{L^\infty(\mathbb{R}^2)} \]

\[ = \| \int_0^\infty e^{-\lambda t} \nabla_y K(t, \cdot) * f(x, y) dt \|_{L^\infty(\mathbb{R}^2)} \]

\[ = \| \int_0^\infty e^{-\lambda t} \int_{\mathbb{R}^2} \nabla_y K(t, x - u, y - v) (f(u, v) - f(u, y)) dudv dt \|_{L^\infty(\mathbb{R}^2)} \]

\[ \leq \| f \|_{L^\infty(\mathbb{R}, C^a_y(\mathbb{R}^2))} \| \int_0^\infty e^{-\lambda t} \int_{\mathbb{R}^2} |\nabla_y K(t, x - u, y - v)| \cdot |y - v|^\beta dudv dt \|_{L^\infty(\mathbb{R}^2)} \]

\[ \leq C \| f \|_{L^\infty(\mathbb{R}, C^a_y(\mathbb{R}^2))} \int_0^\infty e^{-\lambda t} t^{-1+\beta} dt \]

\[ \leq C \lambda^{-\beta} \| f \|_{L^\infty(\mathbb{R}, C^a_y(\mathbb{R}^2))}, \]

where we used the fact that

\[ \int_{\mathbb{R}^2} \nabla_y K(t, x - u, y - v) f(u, y) dudv = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \nabla_y K(t, x - u, y - v) dv \right) f(u, y) du = 0. \]

We also remark that the meaning of \( \| f \|_{L^\infty(\mathbb{R}, C^a_y(\mathbb{R}^2))} \) is that we take infinity norm for the first variable and take Hölder norm for the second variable.

Recall the following interpolation inequality

\[ [u]_{\sigma + (1-\sigma)\gamma} \leq ([u]_\sigma)^\sigma ([u]_\gamma)^{1-\sigma}, \quad 0 \leq \alpha < \gamma \leq 1, \quad \sigma \in (0, 1). \]

Now, if \( 0 < \beta < 1 \), applying the above inequality with \( \alpha = 0, \gamma = 1 \) and \( \beta = \gamma(1-\sigma) \), we have

\[ [\nabla_x u]_\beta = \left[ \int_0^\infty e^{-\lambda t} \nabla_x K(t, \cdot) * f(x, y) dt \right]_\beta \]

\[ \leq \int_0^\infty e^{-\lambda t} \int_{\mathbb{R}^2} |\nabla_x K(t, \cdot, \cdot)|_\beta (x, y) dxdy \| f \|_{L^\infty(\mathbb{R}^2)} dt \]

\[ \leq \| f \|_{L^\infty(\mathbb{R}^2)} \int_0^\infty t^{-\frac{1}{2}} - \beta e^{-\lambda t} dt \]

\[ \leq C \lambda^{-\frac{1}{2} + \beta} \| f \|_{L^\infty(\mathbb{R}^2)}. \]

Next, we consider the derivative of the second variable.

\[ [\nabla_y u]_\beta = \left[ \int_0^\infty e^{-\lambda t} \nabla_y K(t, \cdot) * f(x, y) dt \right]_\beta \]

\[ \leq \int_0^\infty e^{-\lambda t} \sup_{x, \hat{x}, y, \hat{y} \in \mathbb{R}, \hat{x} \neq x, \hat{y} \neq y} \left( \frac{1}{|x - \hat{x}|^\beta + |y - \hat{y}|^\beta} \right) \int_{\mathbb{R}^2} \nabla_y K(t, x - u, y - v) (f(u, y - v) - f(u, \hat{y} - v)) dudv \]

\[ + \int_{\mathbb{R}^2} \nabla_y K(t, u, \hat{y} - v) (f(x - u, v) - f(\hat{x} - u, v)) dudv \right) dt \]

\[ = \int_0^\infty e^{-\lambda t} \sup_{x, \hat{x}, y, \hat{y} \in \mathbb{R}, \hat{x} \neq x, \hat{y} \neq y} \left( \frac{1}{|x - \hat{x}|^\beta + |y - \hat{y}|^\beta} [I_1 + I_2] \right) dt. \]
Let us estimate $I_1$. By dividing the real line into two parts, we have

$$I_1 = \int_{\mathbb{R}^2} \nabla_y K(t, x - u, y - v)(f(u, v) - f(u, y))du dv$$

$$+ \int_{\mathbb{R}^2} \nabla_y K(t, x - u, \hat{y} - v)(f(u, \hat{y}) - f(u, v))du dv$$

$$= \int_{\mathbb{R}} \int_{|y-v|\leq 2|y-\hat{y}|} \nabla_y K(t, u, y - v)(f(x - u, v) - f(x - u, y))du dv$$

Notice that $|v| \leq \hat{y}$. Similarly, we can obtain

$$I = I_{11} + \cdots + I_{14}$$

Let us estimate $I_{11}$-$I_{14}$. By using the form of heat kernel, we get

$$|I_{11}| = \left| \int_{\mathbb{R}} \int_{|y-v|\leq 2|y-\hat{y}|} \nabla_y K(t, u, y - v)(f(x - u, v) - f(x - u, y - u^2))du dv \right|$$

$$\leq C t^{-\frac{3}{2}} \int_{\mathbb{R}} e^{-\frac{\gamma^2}{2}} \left( \int_{|y-v|\leq 2|y-\hat{y}|} \frac{|y-v|}{t^2} e^{-\frac{(y-v)^2}{t^2}} |y-v|^\beta dv \right) du$$

$$= C t^{-\frac{3}{2}} \int_{\mathbb{R}} e^{-\frac{\gamma^2}{2}} \left( \int_{|z|\leq 2|y-\hat{y}|} \frac{|z|}{t^2} e^{-\frac{z^2}{t^2}} |z|^\beta dv \right) du$$

$$\leq C t^{-1+\beta} |y-\hat{y}|^\beta.$$
where $0 < \gamma < 1$ and $\theta > 0$ depends on $h$ satisfying

$$
\lim_{h \to 0} \theta^n = \frac{\Gamma^2(1 + \gamma)}{\Gamma(1 + 2\gamma)}.
$$

Denote

$$
\tilde{K}(t, v) = \frac{v}{t^\gamma} e^{-\frac{v^2}{t}}.
$$

By using the above fractional mean value formula with $\gamma > \beta$ and the interpolation inequality in Hölder space

$$
|I_{13}| \leq C t^{-\frac{3}{4}} |y - \hat{y}|^\beta \int_{\mathbb{R}} e^{-\frac{u^2}{4t}} \int_{|y - v > 2|y - \hat{y}|} |\tilde{K}(t, y - v)| |y - \hat{y}|^{-\beta} |y - v|^{\beta} dv du
$$

$$
\leq C t^{-\frac{3}{4}} |y - \hat{y}|^\beta \int_{\mathbb{R}} e^{-\frac{u^2}{4t}} \int_{|y - v > 2|y - \hat{y}|} |\tilde{K}(t, y - v)| |y - v|^{\gamma} dv
$$

$$
\leq C t^{-1+\gamma-\beta} |y - \hat{y}|^\beta \int_{\mathbb{R}} e^{-\frac{u^2}{4t}} \int_{0}^{\infty} |v| e^{-\frac{v^2}{4t}} dv
$$

$$
\leq C t^{-1+\gamma-\beta} |y - \hat{y}|^\beta.
$$

Lastly, by using the properties of heat kernel $K$, it is easy to see that

$$
\int_{|y^2 - (y - v)| > 2|y - \hat{y}|} \nabla_y K(t, u, y - v) dv
$$

$$
= \int_{|y - v| > 2|y - \hat{y}|} \nabla_y K(t, u, y - v) dv
$$

$$
= (2\pi)^{-1} t^{-\frac{3}{2}} e^{-\frac{u^2}{4t}} e^{-\frac{v^2}{4t}} |v = -2|y - \hat{y}| |
$$

$$
= 0.
$$

Using the above equality and similar to the operation of $I_{13}$, we have

$$
|I_{14}| = \left| \int_{\mathbb{R}} \int_{|y - v| > 2|y - \hat{y}|} \nabla_y K(t, u, \hat{y} - v) (f(u, \hat{y} - u^2) - f(u, y - u^2)) dv du \right|
$$

$$
= \left| \int_{\mathbb{R}} \int_{|y - v| > 2|y - \hat{y}|} (\nabla_y K(t, u, \hat{y} - v) - \nabla_y K(t, u, y - v)) \times (f(u, \hat{y} - u^2) - f(u, y - u^2)) dv du \right|
$$

$$
= \left| \int_{\mathbb{R}} (f(x - u, \hat{y}) - f(x - u, y)) dv \int_{|y - v| > 2|y - \hat{y}|} \nabla_y K(t, u, \hat{y} - v) - \nabla_y K(t, u, y - v) dv \right|
$$

$$
\leq C \|f\|_{L^\infty(\mathbb{R}, C^\alpha(\mathbb{R}))} (y - \hat{y})^\beta \int_{\mathbb{R}} \left| \nabla_y K(t, u, \hat{y} - v) - \nabla_y K(t, u, y - v) \right| dv
$$

$$
\leq C \|f\|_{L^\infty(\mathbb{R}, C^\alpha(\mathbb{R}))} t^{-1+\gamma-\beta} |y - \hat{y}|^\beta.
$$

Substituting $I_{11} - I_{14}$ into $I_1$, we get

$$
\int_{0}^{\infty} e^{-\lambda} \left( \sup_{x, \hat{x}, y, \hat{y} \in \mathbb{R}, x \neq \hat{x}, y \neq \hat{y}} \frac{1}{|x - \hat{x}|^\beta + |y - \hat{y}|^\beta} |I_1| \right) dt
$$

$$
\leq C \int_{0}^{\infty} e^{-\lambda} \left( t^{-1+\beta} + t^{-1+\gamma-\beta} \right) dt
$$

$$
\leq C \lambda^{-(\beta \wedge (\gamma - \beta))}.
$$
Similarly, we can prove that if \( f(\cdot, y) \in C^\beta_b(\mathbb{R}) \) for any \( y \in \mathbb{R} \),
\[
\int_0^\infty e^{-\lambda t} \left( \sup_{\hat{x}, \hat{y}, \hat{y}' \in \mathbb{R}, \hat{x} \neq \hat{y}, \hat{y}' \neq \hat{y}} \frac{1}{|x - \hat{x}|^\beta + |y - \hat{y}'|^\beta} |I_2| \right) dt \leq C \lambda^{-(\beta \wedge (\gamma - \beta))}.
\]
Summing the above discussion, we obtain
\[
[\nabla_y u|_\beta \leq C \lambda^{-(\beta \wedge (\gamma - \beta))} ||f||_{C^\beta(\mathbb{R}^2)},
\]
Noting that the above inequality holds for \( 0 < \gamma < 1 \). The proof of this lemma is complete. \( \square \)

**Remark 3.1** It is well known that if \( f \in C^\alpha(\mathbb{R}^n) \), then the solution \( u \) of the following equation
\[
\frac{\partial u}{\partial t} - \Delta u = f, \quad u_0 = 0,
\]
belongs to \( C^{2+\alpha}(\mathbb{R}^n) \), which is the Schauder theory. Noting that the heat kernel of above equation is Gauss heat kernel, that is
\[
K(t, x) = \frac{1}{(2\pi t)^{n/2}} e^{-\frac{x^2}{2t}}.
\]
It is easy to see that \( x \sim \sqrt{t} \). But in our case, different axis has different scaling, that is,
\[
x \sim \sqrt{t}, \quad y \sim t.
\]
Thus when we take derivative for variable \( x \), we can get \( t^{-\frac{1}{2}} \), and take double derivative for variable \( x \), we will get \( t^{-1} \). But if we take derivative for variable \( y \), we shall get \( t^{-1} \), which is different from the classical case. In Schauder theory, we can get \( C^{2+\alpha}(\mathbb{R}^n) \) estimates, but in our case the \( C^{1+\alpha}(\mathbb{R}^n) \) should be optimal.

Like the classical case, we can get the \( C^{2+\alpha} \)-estimate for the \( x \)-axis if \( f(\cdot, y) \in C^\beta_b(\mathbb{R}) \) for any \( y \in \mathbb{R} \), but we can not get the same estimate for \( y \)-axis. If we want to get the \( C^{2+\alpha} \)-estimate for the \( y \)-axis, we need more regularity about the second variable. In other words, we have the following results.

**Corollary 3.1** Assume that \( f(\cdot, y) \in C^\beta_b(\mathbb{R}) \) for any \( y \in \mathbb{R} \) and \( \nabla_y f(x, \cdot) \in C^\beta_b(\mathbb{R}) \) for any \( x \in \mathbb{R} \). Then
\[
[\nabla^2 u|_\beta \leq C \lambda^{-\delta} ||f||_{C^\beta(\mathbb{R}, C^{1+\beta}(\mathbb{R}))},
\]
where \( 0 < \delta < (\beta \wedge (1 - \beta)) \). That is to say, \( \nabla^2 u \in C^\beta(\mathbb{R}^2) \).

**Proof of Theorem 2.1** We use Picard’s iteration to solve (2.1). Let \( u_0 = 0 \) and define for \( n \in \mathbb{N} \),
\[
u_n := (\lambda - \mathcal{L})^{-1}(f - b \cdot \nabla u_{n-1}).
\](3.1)
It follows from Lemma 3.1 that
\[
\lambda^n \|u_n\|_{C^{1+\beta}(\mathbb{R}^2)} \leq \|f - b \cdot \nabla u_{n-1}\|_{C^\beta(\mathbb{R})} \leq \|f\|_{C^\beta(\mathbb{R})} + \|b\|_{L^\infty(\mathbb{R}^2)} \cdot \|\nabla u_{n-1}\|_{C^\beta(\mathbb{R})} + \|b\|_{C^\beta(\mathbb{R})} \cdot \|\nabla u_{n-1}\|_{L^\infty(\mathbb{R}^2)},
\]
(3.2)
and
\[
\lambda^\delta \|u_n - u_m\|_{C_b^{1+\beta}(\mathbb{R}^2)} \leq \|b\|_{L^\infty(\mathbb{R}^2)} \cdot \|\nabla u_{n-1} - \nabla u_{m-1}\|_{C_b^\beta(\mathbb{R}^2)} + \|b\|_{C_b^\beta(\mathbb{R}^2)} \cdot \|\nabla u_{n-1} - \nabla u_{m-1}\|_{L^\infty(\mathbb{R}^2)}.
\]

Choosing $\lambda_0$ be large enough so that $C\lambda^{-\delta}\|b\|_{C_b^\beta(\mathbb{R}^2)} < 1/4$ for all $\lambda \geq \lambda_0$, we get
\[
\|u_n\|_{C_b^{1+\beta}(\mathbb{R}^2)} \leq \lambda^{-\delta}(\|f\|_{C_b^\beta(\mathbb{R}^2)} + 2\|b\|_{C_b^\beta(\mathbb{R}^2)}) + \frac{1}{2}\|u_n - 1\|_{C_b^{1+\beta}(\mathbb{R}^2)}
\]
and for all $n \geq m$,
\[
\|u_n - u_m\|_{C_b^{1+\beta}(\mathbb{R}^2)} \leq \frac{1}{2}\|u_n - 1 - u_m - 1\|_{C_b^{1+\beta}(\mathbb{R}^2)}.
\]
Substituting them into (3.2) and (3.3), we obtain
\[
\lambda^{-\delta}\|u_n\|_{C_b^{1+\beta}(\mathbb{R}^2)} \leq C\|f\|_{C_b^\beta(\mathbb{R}^2)}
\]
and for all $n \geq m$,
\[
\lambda^{-\delta}\|u_n - u_m\|_{C_b^{1+\beta}(\mathbb{R}^2)} \leq \frac{C}{2m}.
\]
Hence there is a $u \in C_b^{1+\beta}(\mathbb{R}^2)$ such that (2.2) holds and
\[
\lambda^{-\delta}\|u - u_m\|_{C_b^{1+\beta}(\mathbb{R}^2)} \leq \frac{C}{2m},
\]
and $u$ solves equation (2.1) by taking limits for (3.1). The second result can be obtained similarly.

The proof is complete. $\square$

**Proof of Theorem 2.2.** For simplicity, we only consider a special case, that is, $f(x, y) = f_1(x)f_2(y)$. Assume that $f_1 \in L^p(\mathbb{R})$ and $f_2 \in L^q(\mathbb{R})$. Denote
\[
K_1(t, y) = \int_\mathbb{R} K(t, x - u, y)f_1(u)du.
\]
By using the above inequality, Minkowski’s inequality and the properties of the heat kernel, we
have
\[
\|u\|_{L^r(\mathbb{R}^2)} = \|\int_0^\infty e^{-\lambda t} K(t, \cdot) * f(x, y) dt\|_{L^r(\mathbb{R}^2)}
\leq \int_0^\infty e^{-\lambda t} \left( \int_{\mathbb{R}^2} |K_1(t, x, y - v)| f_2(v) \, dv \right)^{\frac{1}{r'}} \, dt
= \int_0^\infty e^{-\lambda t} \left( \int_{\mathbb{R}} \|K_1(t, x, \cdot)\|_{L^{r'}(\mathbb{R})} dx \right)^{\frac{1}{r'}} \, dt
\leq \|f_2\|_{L^{r'}(\mathbb{R})} \int_0^\infty e^{-\lambda t} \left( \int_{\mathbb{R}} \|K_1(t, x, \cdot)\|_{L^m(\mathbb{R})} \, dx \right)^{\frac{1}{r'}} \, dt
= \|f_2\|_{L^{r'}(\mathbb{R})} \int_0^\infty e^{-\lambda t} \left( \int_{\mathbb{R}} \int_{\mathbb{R}} |K(t, x - u, y) f_1(u) du|^r \, dx \right)^{\frac{1}{m'}} \, dt
\leq \|f_2\|_{L^{r'}(\mathbb{R})} \|f_1\|_{L^p(\mathbb{R})} \int_0^\infty e^{-\lambda t} \|K(t, \cdot, y)\|_{L^n(\mathbb{R}, L^m(\mathbb{R}))} \, dt
\leq C_1 \int_0^\infty e^{-\lambda t} t^{-\frac{1}{2} + \frac{1}{p'} + \frac{1}{m'}} \, dt
\leq C \lambda^{-1 - \frac{1}{2} + \frac{1}{p'} + \frac{1}{m'}}.
\]

where
\[
1 + \frac{1}{r} = \frac{1}{n} + \frac{1}{p'}, \quad 1 + \frac{1}{r} = \frac{1}{m} + \frac{1}{q}, \quad (3.4)
\]
\[
C_1 = \|f_1\|_{L^p(\mathbb{R})} \|f_2\|_{L^{q'}(\mathbb{R})} \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{-p' x^2 - p' (y-x)^2} \, dx \right)^{\frac{q'}{p'}} \, dy \right)^{\frac{1}{q'}}.
\]

We remark that
\[
\left( \int_{\mathbb{R}} |\nabla_x K(t, x, y)|^r \, dx \right)^{\frac{1}{r'}} \leq C t^{-\frac{1}{2} + \frac{1}{p'}},
\]
\[
\left( \int_{\mathbb{R}} |\nabla_y K(t, x, y)|^r \, dy \right)^{\frac{1}{r'}} \leq C t^{-1 + \frac{1}{p'}}.
\]

Similarly, we obtain
\[
\|\nabla_x u\|_{L^r(\mathbb{R}^2)} \leq C \lambda^{-\frac{1}{2} - \frac{1}{2} + \frac{1}{p'} + \frac{1}{q'}}.
\]

Furthermore, we can get
\[
\|\nabla^2_x u\|_{L^r(\mathbb{R}^2)} \leq C \lambda^{-\frac{1}{2} + \frac{1}{2} + \frac{1}{q'}}.
\]

Thus if we take \(p = q = r\), then we have \(\|\nabla^2_x u\|_{L^r(\mathbb{R}^2)} \leq C\). However, if we deal with the second variable, it is difficult to get the decay estimate. More precisely, we have for \(\frac{3}{2r} < \frac{1}{p'} + \frac{1}{q'}\),
\[
\|\nabla_y u\|_{L^r(\mathbb{R}^2)} \leq C \lambda^{-\frac{1}{2} + \frac{1}{p'} + \frac{1}{q'}} \rightarrow \infty, \quad \text{as} \quad \lambda \rightarrow \infty.
\]
Hence we must add more regularity on the second variable. Meanwhile, we recall that if \( h \in W^{s,p}(\mathbb{R}^n) \) with \( 0 < s < 1 \), then

\[
\|h\|_{W^{s,p}(\mathbb{R}^n)} = \left( \|h\|_{L^p(\mathbb{R}^n)}^p + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|h(x) - h(y)|^p}{|x - y|^{n+sp}} dxdy \right)^{\frac{1}{p}}.
\]

If we assume that \( f_2 \in W^{s,q}(\mathbb{R}) \) and let

\[
K_2(t, y) = \int_{\mathbb{R}} \nabla_y K(t, x, y - v) f_2(v) dv.
\]

we get

\[
\|\nabla_y u\|_{L^r(\mathbb{R}^2)} = \left\| \int_0^\infty e^{-\lambda t} \nabla_y K(t, \cdot) * f(x, y) dt \right\|_{L^r(\mathbb{R}^2)}
\]

\[
\leq \int_0^\infty e^{-\lambda t} \left( \int_{\mathbb{R}^2} |K_2(t, x - u, y) f_1(u) du|^{\frac{p}{q}} dxdy \right)^{\frac{q}{p}} dt
\]

\[
\leq \|f_1\|_{L^p(\mathbb{R})} \int_0^\infty e^{-\lambda t} \left( \int_{\mathbb{R}} \int_{\mathbb{R}} K_2(t, \cdot, y) dy \right)^{\frac{q}{p}} dt
\]

\[
= \|f_1\|_{L^p(\mathbb{R})} \int_0^\infty e^{-\lambda t} \times \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |\nabla_y K(t, x, v)|^{s+\frac{1}{q}} \frac{f_2(y - v) - f_2(y)}{|v|^{s+\frac{1}{q}}} dv \right)^{\frac{q}{r}} dy \right)^{\frac{1}{r}} dt
\]

\[
\leq \|f_1\|_{L^p(\mathbb{R})} \|f_2\|_{W^{s,q}(\mathbb{R})} \int_0^\infty e^{-\lambda t} \left( \int_{\mathbb{R}} \left( |\nabla_y K(t, x, y)|^m \right)^{m+\frac{m}{q}} dy \right)^{\frac{1}{m}} dt
\]

\[
\leq C_2 \int_0^\infty e^{-\lambda t} t^{-\frac{s}{q} + \frac{1}{2p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{m}} dt
\]

\[
\leq C \lambda^{-s - \frac{3}{2p} + \frac{1}{2p} + \frac{1}{r} + \frac{1}{q} + \frac{1}{m}}
\]

where \( m, n \) satisfy (3.4) and

\[
C_2 = 2\|f_1\|_{L^p(\mathbb{R})} \|f_2\|_{W^{s,q}(\mathbb{R})} \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} \left| y - x^2 \right|^p m \right)^{m+\frac{m}{q}} e^{-p|x^2 - y|^2} dx \right)^{\frac{1}{p}} dy
\]

Moreover, under the condition that \( f_2 \in W^{s,p}(\mathbb{R}) \), we can similarly get

\[
\|\nabla_y^2 u\|_{L^r(\mathbb{R}^2)} \leq C \lambda^{-s - \frac{3}{2p} + \frac{1}{2p} + 1}.
\]

Hence it is easy to see that we can take suitable \( s \in (0, 1) \), \( p > 1 \) and \( r > 1 \) such that \( -s - \frac{3}{2p} + \frac{1}{2p} + 1 \leq 0 \). That is to say, we have

\[
\|\nabla_y^2 u\|_{L^r(\mathbb{R}^2)} \leq C
\]

under the condition that \( f_2 \in W^{s,p}(\mathbb{R}) \). The proof is complete. \( \square \)

**Remark 3.2** It is well known that if \( f \in L^p(\mathbb{R}^n) \), then the solution \( u \) of the following equation

\[
u_t - \Delta u = f, \quad u_0 = 0,
\]

belongs to \( W^{2,p}(\mathbb{R}^n) \), which is the \( L^p \)-theory. Noting that the heat kernel of above equation is Gauss heat kernel, and similar to Remark 3.1, it is easy to find the difference from the classical Laplacian operator. Due to the singularity of the variable \( y \), we must give two different assumptions. Comparing the classical \( L^p \)-theory, in our case the regularity of Theorem 2.2 should be optimal.
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