TRACTABILITY FRONTIER FOR DUALLY-CLOSED TEMPORAL QUANTIFIED CONSTRAINT SATISFACTION PROBLEMS

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Abstract. A temporal (constraint) language is a relational structure with a first-order definition in the rational numbers with the order. We study here the complexity of the Quantified Constraint Satisfaction Problem (QCSP) for temporal constraint languages.

Our main contribution is a dichotomy for the restricted class of dually-closed temporal languages. We prove that QCSP for such a language is either solvable in polynomial time or it is hard for NP or coNP. Our result generalizes a similar dichotomy of QCSPs for equality languages [BC10], which are relational structures definable by Boolean combinations of equalities.

1. Introduction

The main goal of computational complexity is to characterize the difficulty of decision problems by placing them into complexity classes according to the time resources or space resources required to solve the problem. Much more convenient and mathematically elegant, however, is to work with formalisms such as Constraint Satisfaction Problems (CSPs) or Quantified Constraint Satisfaction Problems (QCSPs), considered in this paper, that can express infinitely many decision problems.

The formalism of Constraint Satisfaction Problems express Boolean satisfiability problems: 3-SAT, 2-SAT; \(k\)-coloring and solving equations over finite fields as well as many problems in different branches of Artificial Intelligence such as Temporal Reasoning. An instance of the Constraint Satisfaction Problem CSP(\(\Gamma\)) parametrized by a relational structure \(\Gamma\) is a primitive-positive (pp) sentence of the form: \((\exists v_1 \ldots \exists v_n (R(v_{i_1}, \ldots, v_{i_k}) \land \ldots)),\) where the inner quantifier-free part is the conjunction of relational symbols, from the signature of \(\Gamma\), with variables.

This framework on one hand allows to define many important decision problems. On the other hand, complexity classifications of the flavour of Schaefer’s theorem [Sch78] these problems tend to display are of theoretical interest. Directly relevant to this paper are temporal CSPs that are CSP(\(\Gamma\)) for \(\Gamma\) with a first-order definition in (\(\mathbb{Q}; <\)), called temporal (constraint) languages. In [BK09], nine large classes of tractable (solvable in polynomial time) temporal CSPs has been identified and it was proved that all other such problems are NP-complete. These nine tractable classes contain many decision problems studied previously and scattered across the literature, e.g., the Betweenness and the Cyclic Ordering Problem mentioned in [GJ78], and network satisfaction problem for Point Algebra [VKvB89]. Furthermore, temporal CSP(\(\Gamma\)) has been studied in the literature by [BC10].

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for so-called Ord-Horn languages [NB95], and AND/OR precedence constraints in scheduling [MSS04].

A natural generalization of the CSP is the Quantified Constraint Satisfaction Problem (QCSP) for a relational structure Γ, denoted by QCSP(Γ) that next to existential allows also universal quantifiers in the input sentence. Similarly as the CSP, this problem has been studied widely in the literature, see e.g. [BBJK03, Che12]. In this paper, we study QCSP(Γ) for temporal languages. Although a number of partial results has been obtained [BC10, ZM21, CM12, CW08b, CW08a, CW12, CBW14, Wro14b], these efforts did not lead to a full complexity classification of all temporal QCSP(Γ). One of the reasons is that QCSPs are usually harder to classify than CSPs. This also holds in our case. For instance, temporal CSPs are at most NP-complete, whereas temporal QCSPs can be at most PSPACE-complete. In more detail, nine tractable classes of temporal CSPs identified in [BK09] are given by so-called polymorphisms that are in the case of temporal languages Γ operations from \( \mathbb{Q}^k \) for some \( k \in \mathbb{N} \) to \( \mathbb{Q} \) that preserve Γ (homomorphisms from \( \Gamma^k \) to \( \Gamma \)). The first of these classes is the class preserved by constant operations, the other polymorphisms (all binary) that come into play are named: min, max, mx, dual-mx, mi, dual-mi, ll, dual-ll. Although constant polymorphisms make CSP trivial, QCSP for a temporal language preserved by a constant polymorphism may be even PSPACE-complete [CW08a]. When it comes to min, max, mx, dual-mx, these operations provide tractability for both temporal CSPs and QCSPs [CBW14], the complexity of temporal QCSPs preserved by mi and dual-mi is not known. But it is known that ll and dual-ll do not in general provide tractability for temporal QCSPs [BC10, Wro14b]. However, Guarded Ord-Horn Languages identified in [CW12], are all preserved by both ll and dual-ll and provide tractability for temporal QCSPs. Other partial results obtained in the literature present somehow restricted classifications of temporal QCSPs. Such classifications has been provided in the following three cases.

1. For a while it has been known [BC10] (see [CM12] for a different proof) that a QCSP of an equality language is in P, it is NP-complete or coNP-hard. Recently the problem has been solved completely by showing that such a problem is always either in LOGSPACE, it is NP-complete or PSPACE-complete [ZM21].

2. Positive languages that are relational structures with a positive definition (first-order definition using \( \land, \lor, \text{and} \leq \) only, negation is not used) in \((\mathbb{Q}, \leq)\) were classified in [CW08a, CW08b] where it was proved that a corresponding QCSP is in LOGSPACE, is NLOGSPACE-complete, P-complete, NP-complete or PSPACE-complete.

3. A temporal language Γ is dually-closed if in addition to a relation \( R \) in Γ, it also contains a relation \( -R \) obtained from \( R \) by replacing each tuple \( t = (t[1], \ldots, t[n]) \) with \( -t = (-t[1], \ldots, -t[n]) \) where \( - \) is a unary operation that sends a rational number \( q \) to \( -q \). A temporal language is Ord-Horn if it can be defined by a conjunction of so-called Ord-Horn clauses of the form: \((x_1 \neq y_1 \lor \cdots \lor x_k \neq y_k \lor xRy)\) where \( R \in \{<, \leq, =\} \) and both the disjunction of disequalities and a literal \( xRy \) can be omitted. It was proved in [Wro14b] that QCSP for a dually-closed Ord-Horn language is in P if it is Guarded Ord-Horn and it is coNP-hard otherwise.
The main contribution of this paper is the tractability frontier of QCSPs for all dually-closed temporal languages. We prove that for every dually-closed temporal language \( \Gamma \) the problem QCSP(\( \Gamma \)) is either in \( \text{P} \) or it is \( \text{NP-hard} \) or \( \text{coNP-hard} \). The result is built on all three classifications listed above and clearly generalizes the third of the listed classifications. It also generalizes the original result on equality languages \([\text{BC10}]\): see Section 3 for details. Although our proof is based on previously developed classifications, there are many interesting languages that are not captured by any of them. For instance, the language \((Q; \{(x,y,z) \mid (x = y = z) \lor (x < y < z) \lor (x > y > z)\}, <)\), where \(<\) is a shortcut for \(\{(x,y) \in Q^2 \mid x < y\}\), is dually-closed but neither it is an equality language nor positive nor Ord-Horn.

To provide our classification we carry out a careful analysis of unary operations preserving dually closed temporal languages. In particular we analyse operations that are non-injective. In the case of temporal CSPS, such analysis is not necessary since all corresponding languages give rise to trivial and therefore tractable problems. But it is not true not only for temporal QCSPs but also for temporal abduction, whose complexity has been investigated in \([\text{SW13, Wro14 a}]\). Similar analysis will be also helpful in order to complete the project of identifying temporal languages with the so-called local-to-global consistency, provided so far only for Ord-Horn languages \([\text{Wro12}]\).

1.1. Outline. We start with preliminaries in Section 2. Partial results from the literature that we use in this paper are presented in Section 3. In Section 4 we present first some dually-closed temporal languages that give rise to hard QCSPs and then we use these results to analyse dually-closed temporal languages that are preserved by a constant operation. We show for corresponding QCSPs that either one of the hard problems reduces to it or it is captured by one of the three classifications listed above. In Section 5 we complete the classification of QCSPs for dually closed temporal constraint languages.

2. Preliminaries

We write \([n]\) for the set \(\{1, \ldots, n\}\) with \(n \in \mathbb{N}\) and \(t = (t[1], \ldots, t[n])\) for an \(n\)-ary tuple \(t\). The \(i\)-th entry of the tuple \(t\), we denote by \(t[i]\).

2.1. Formulas and definability. We consider two restricted forms of first-order (fo)-formulas. Let \(\tau\) be a signature. A first-order \(\tau\)-formula is a \(\forall \exists \land\)-formula if it has the form \(Q_1v_1 \ldots Q_nv_n(\psi_1 \land \cdots \land \psi_m)\), where each \(Q_i\) is a quantifier from \(\{\forall, \exists\}\), and each \(\psi_i\) is an atomic \(\tau\)-formula of the form \(R(x_1, \ldots, x_k)\) where \(R \in \tau\). A primitive positive (pp)-formula is a \(\forall \land\)-formula where all quantifiers are existential. A \(\forall \land\)-sentence (a pp-sentence) is a \(\forall \land\)-formula (pp-formula) without free variables.

We say that a relation \(R\) is (fo)-definable (\(\forall \land\)-, or pp-definable) in a relational structure \(\Gamma\) if \(R\) has the same domain as \(\Gamma\), and there is a fo-formula (\(\forall \land\)-, or pp-formula) \(\phi\) in the signature of \(\Gamma\) such that \(\phi\) holds exactly on those tuples that are contained in \(R\). A relational structure \(\Delta\) is fo-definable in \(\Gamma\) if every relation in \(\Delta\) is fo-definable in \(\Gamma\).

2.2. Temporal languages and formulas. In this paper a temporal formula is a fo-formula built from quantifiers, logical connectivities and relational symbols: \(<, \leq, \neq, =\). A temporal relation is a relation with a fo-definition in \((Q; <, \leq, \neq, =)\) and a temporal (constraint) language is a relational structure over a finite signature consisting of the domain \(Q\) and a finite number of temporal relations.
2.3. QCSPs. Let $\Gamma$ be a relational structure over a finite signature. The Quantified Constraint Satisfaction Problem for $\Gamma$, denoted by QCSP($\Gamma$), is the problem to decide if a given $\forall \exists$-sentence over a signature of $\Gamma$ is true in $\Gamma$.

Lemma 1. ([BC10] [BBJ08]) Let $\Gamma_1, \Gamma_2$ be constraint languages. If $\Gamma_1$ has a $\forall \exists$-definition in $\Gamma_2$, then QCSP($\Gamma_1$) is logarithmic space reducible to QCSP($\Gamma_2$).

The above lemma states that we can reduce between QCSPs problems by providing appropriate $\forall \exists$-definitions. In this paper we restrict ourselves to pp-definitions but rather work with their characterization by polymorphisms than with them directly.

2.4. Polymorphisms. Let $t_1, \ldots, t_k$ be a $m$-tuples over $Q$ and let $f$ be a function (called also an operation) $f : D^k \to D$. Then we write $f(t_1, \ldots, t_k)$ for the tuple obtained from the tuples $t_1, \ldots, t_k$ by applying $f$ componentwise, i.e., for the $m$-tuple $t = (f(t_1[1], \ldots, t_k[1]), \ldots, f(t_1[m], \ldots, t_k[m]))$. If for all $t_1, \ldots, t_k \in R$ the tuple $t$ is also in $R$, then we say that $f$ is a polymorphism (of preserves) $R$. If it happens for all relations in $\Gamma$, then $f$ is a polymorphism of (preserves) $\Gamma$. If $f$ does not preserve $R$ or $\Gamma$, then we say that $f$ violates $R$ or $\Gamma$, respectively.

Observe that an automorphism is a unary polymorphism that preserves all relations and their complements. The set of automorphisms of $\Gamma$ is denoted by $\text{Aut}(\Gamma)$. An orbit of a $k$-tuple $t$ wrt. $\text{Aut}(\Gamma)$ is the set $\{ \alpha(t) \mid \alpha \in \text{Aut}(\Gamma) \}$.

The set of all polymorphisms of a temporal language $\Gamma$, denoted by $\text{Pol}(\Gamma)$, forms an algebraic object called a clone [Sze86], which is a set of operations defined on a set $Q$ that is closed under composition and that contains all projections. Moreover, $\text{Pol}(\Gamma)$ is also closed under interpolation (see Proposition 1.6 in [Sze86]): we say that a $k$-ary operation $f$ is interpolated by a set of $k$-ary operations $F$ if for every finite subset $A$ of $Q$ there is some operation $g \in F$ such that $f(a) = g(a)$ for every $a \in A^k$. We say that $F$ locally generates an operation $g$ if $g$ is in the smallest clone that is closed under interpolation and contains all operations in $F$.

Since all temporal languages are preserved by all automorphisms of $(\mathbb{Q}; <)$, the following definition makes sense. We say that a set of operations $F$ generates an operation $g$ if $F$ together with all automorphisms of $(\mathbb{Q}; <)$ locally generates $g$. In case that $F$ contains just one operation $f$, we also say that $f$ generates $g$. We have the following easy observation.

Observation 2. Let $\Gamma$ be a temporal language. If $\Gamma$ is preserved by all operations in $F$ and $F$ generates $g$, then $g$ also preserves $\Gamma$.

To prove that operations in $F$ generate $g$, we use the following result from [BK09].

Lemma 3. An operation $f$ generates $g$ if and only if every temporal relation that is preserved by $f$ is also preserved by $g$.

For temporal languages we have the following characterization of pp-definability in terms of polymorphisms [BN06].

Theorem 4. Let $\Gamma_1, \Gamma_2$ be temporal languages, then $\Gamma_2$ has a pp-definition in $\Gamma_1$ if and only if $\text{Pol}(\Gamma_1) \subseteq \text{Pol}(\Gamma_2)$.

A direct consequence of Theorem 4 is that if a temporal language $\Gamma_1$ does not have a pp-definition in $\Gamma_2$, then there is an operation over $Q$ which is a polymorphism of $\Gamma_2$ but is not a polymorphism of $\Gamma_1$. The next lemma [BK09] allows us to bound the arity of such polymorphism.
Lemma 5. Let $\Gamma$ be a relational structure and let $R$ be a $k$-ary relation that is a union of $l$ orbits of $k$-tuples of $\text{Aut}(\Gamma)$. If $R$ is violated by a polymorphism $g$ of $\Gamma$ of arity $m \geq l$, then $R$ is also violated by an $l$-ary polymorphism of $\Gamma$.

Provided we have some more knowledge about $\text{Pol}(\Gamma)$, the following observation allows us to reduce the arity of a considered polymorphism even more.

Observation 6. Let $t_1, \ldots, t_n, t \in \mathbb{Q}^m$. If $\Gamma$ is preserved by $f : \mathbb{Q}^n \to \mathbb{Q}$ and $g : \mathbb{Q}^{n-1} \to \mathbb{Q}$ such that $f(t_1, \ldots, t_n) = t$ and $g(t_1, \ldots, t_n-1) = t$ then $\Gamma$ is preserved by $h(x_1, \ldots, x_{n-1}) := f(x_1, \ldots, x_{n-1}, g(x_1, \ldots, x_{n-1}))$ such that $h(t_1, \ldots, t_{n-1}) = t$.

2.5. Operations of Special Importance for this Paper. Let $- : \mathbb{Q} \to \mathbb{Q}$ be the unary operation such that for every $q \in \mathbb{Q}$ we have $-(q) = -q$. Let $f : \mathbb{Q}^n \to \mathbb{Q}$. The dual of $f$ is the operation $-f(-x_1, \ldots, -x_n)$ denoted by $\overline{f}$.

Let $e$ be any order-preserving bijection between $(-\infty, \pi)$ and $(\pi, \infty)$. Then the operation $\text{cyc} : \mathbb{Q} \to \mathbb{Q}$ is defined by $e(x)$ for $x < \pi$ and by $e^{-1}(x)$ for $x > \pi$.

We say that $f : \mathbb{Q}^k \to \mathbb{Q}$ for some $k \in \mathbb{N}$ is a constant operation if $f$ sends every $k$-tuple of rational numbers to the same rational number.

We define the operation $\text{wave} : \mathbb{Q} \to \mathbb{Q}$ to be equal to $-x$ for $x < 0$, to 0 for $0 \leq x \leq 1$ and to $(x-1)$ for $x > 1$. We define the operation $\text{peak}$ to be equal to $-1$ for $x \neq 0$ and equal to 1 for $x = 0$. Further, we define $\text{su}_i : \mathbb{Q} \to \mathbb{Q}$ where $i \in \mathbb{N}$ to be an operation that satisfies $\text{su}_i(x) = 0$ for $x < 0$; $\text{su}_i(x) = j$ for $[j-1,j)$ if $0 < j < i$; and $\text{su}_i(x) = i$ for $[i-1, +\infty)$. We define $\text{ic}$ to be an operation such that $\text{ic}(x) = x$ for all $(x < 0)$ and $\text{ic}(x) = 0$ for $(x \geq 0)$; and $\text{ci}$ to be an operation such that $\text{ci}(x) = 0$ for $x < 0$ and $\text{ci}(x) = x$ for $(x \geq 0)$. Observe that $\text{ic}$ and $\text{ci}$ are the duals of each other.

Let $\text{pp}$ be an arbitrary binary operation on $\mathbb{Q}$ such that $\text{pp}(a_1, b_1) \leq \text{pp}(a_2, b_2)$ if and only if one of the following cases applies:

- $a_1 \leq 0$ and $a_1 \leq a_2$
- $0 < a_1$, $0 < a_2$, and $b_1 \leq b_2$.

All operation that satisfy the above conditions generate each other and therefore the same clone. The dual of $\text{pp}$ is known as dual-$\text{pp}$.

Let $\text{ll}$ be a binary operation on $\mathbb{Q}$ such that $\text{ll}(a_1, b_1) < \text{ll}(a_2, b_2)$ if

- $a_1 \leq 0$ and $a_1 < a_2$, or
- $a_1 \leq 0$ and $a_1 = a_2$ and $b_1 < b_2$, or
- $a_1, a_2 > 0$ and $b_1 < b_2$, or
- $a_1 > 0$ and $b_1 = b_2$ and $a_1 < a_2$.

All operations satisfying these conditions generate each other and the same clone. The dual of $\text{ll}$ is known as dual-$\text{ll}$.

3. Partial Classifications

In this section we present some results from the literature that we use for our classification.

3.1. Equality Languages. We say that a temporal language $\Gamma$ is an equality language if it is definable by Boolean combinations of equalities ($=$). A temporal language is an equality language if and only if $\text{Pol}(\Gamma)$ contains all permutations of $\mathbb{Q}$, i.e., all bijunctive functions from $\mathbb{Q}$ to $\mathbb{Q}$.
3.2. Dually-Closed Temporal Languages. A temporal language $\Gamma$ is dually-closed if $f \in \text{Pol}(\Gamma)$ always whenever $\overline{f} \in \text{Pol}(\Gamma)$. Since $\overline{-}$ is a permutation of $\mathbb{Q}$, we have that it preserves every equality language and hence every equality language is also dually-closed.

Let $R \subseteq \mathbb{Q}^n$. The dual of $R$ is the relation $\{(-t[1], \ldots, -t[n]) \mid t \in R\}$ denoted by $\overline{R}$. It is easy to prove $Wro14b$ that $\Gamma$ is a dually-closed language if and only if the following holds: $\Gamma$ pp-defines $R$ if and only if it pp-defines $\overline{R}$.

3.3. Ord-Horn and Guarded Ord-Horn Languages. Recall from the introduction that a temporal language is Ord-Horn if it can be defined by a conjunction of Ord-Horn clauses of the form: $(x_1 \neq y_1 \lor \cdots \lor x_p \neq y_p)$ where $R \in \{<, \leq, =\}$ and both the disjunction of disequalities and a literal $xRy$ can be omitted. They have the following algebraic characterization $Wro12$.

**Proposition 7.** Let $\Gamma$ be a temporal language. Then $\Gamma$ is Ord-Horn if and only if it is preserved by both $ll$ and dual-$ll$.

The only tractable case for dually-closed QCSP($\Gamma$) is where $\Gamma$ is a guarded Ord-Horn language.

**Definition 8.** We say that a temporal language $\Gamma$ is Guarded Ord-Horn (GOH) if every relation in $\Gamma$ is definable by a GOH formula defined as follows.

1. A Basic OH formula which is in one of the following forms is a GOH formula:
   - $x = y$, $x \leq y$,
   - $(x_1 \neq y_1 \lor \cdots \lor x_p \neq y_p)$, or
   - $(x_1 \neq x_2 \lor \cdots \lor x_1 \neq x_q) \lor (x_1 < y_1) \lor (y_1 \neq y_2 \lor \cdots \lor y_1 \neq y_q')$.
2. If $\psi_1$ and $\psi_2$ are GOH formulas, then $\psi_1 \land \psi_2$ is a GOH formula.
3. If $\psi$ is a GOH formula, then
   
   
   \[
   (x_1 \leq y_1) \land \cdots \land (x_m \leq y_m) \land \\
   (x_1 \neq y_1 \lor \cdots \lor x_m \neq y_m \lor \psi)
   \]

   is a GOH formula.

What might be interesting is that the algorithm for the following tractability result uses local consistency methods.

**Theorem 9.** ($[CW12]$) Let $\Gamma$ be a GOH structure. Then QCSP($\Gamma$) is solvable in polynomial time.

3.4. Dually-Closed Ord-Horn Languages. For temporal languages that are both dually-closed and Ord-Horn we have the following dichotomy.

**Theorem 10.** Let $\Gamma$ be a dually-closed Ord-Horn temporal language. Then QCSP($\Gamma$) is in $P$ if $\Gamma$ is Guarded Ord-Horn. Otherwise, it is coNP-hard.

3.5. Temporal Languages. We say that a temporal language is positive if every relation in $\Gamma$ has a positive definition in $(\mathbb{Q}; \leq)$, a first-order definition with use of $\land, \lor,$ and $\leq$ only. No negation is in use. The following algebraic characterization of positive temporal languages is easy to prove.

**Proposition 11.** Let $\Gamma$ be a temporal language, then $\Gamma$ is positive if and only if it is preserved by wave.

The following dichotomy is a consequence of the classification in $[CW08a, CW08b]$. 

Theorem 12. Let $\Gamma$ be a positive temporal language. Then $\text{QCSP}(\Gamma)$ is in $P$ if it is preserved either by pp or dual-pp. Otherwise it is NP-hard.

4. Languages Preserved by a Constant Operation

First we present some temporal languages that give rise to coNP-hard and NP-hard QCSPs.

Theorem 13. Consider the following relations:

- $\text{BetwC} = \{(x, y, z) \in \mathbb{Q}^3 \mid (x < y < z \vee x > y > z \vee x = y = z)\}$;
- $\text{CyclC} = \{(x, y, z) \in \mathbb{Q}^3 \mid (x < y < z \vee y < z < x \vee z < x < y \vee x = y = z)\}$;
- $\text{EqXor} = \{(x, y, z) \in \mathbb{Q}^3 \mid (x = y \vee x = z)\}$;
- $\text{EqOr} = \{(x_1, \ldots, x_n) \in \mathbb{Q}^n \mid (\vee_{i,j \in [n], i \neq j} x_i = x_j)\}$;
- $S = \{(x, y, z) \in \mathbb{Q}^3 \mid (x = y = z) \vee (x \neq y \land x \neq z \land z \neq y)\}$.

It holds that $\text{QCSP}(\mathbb{Q}; I)$, $\text{QCSP}(\mathbb{Q}; S)$, $\text{QCSP}(\mathbb{Q}; \text{BetwC})$ and $\text{QCSP}(\mathbb{Q}; \text{CyclC})$ are coNP-hard; and $\text{QCSP}(\mathbb{Q}; \text{EqXor})$ as well as $\text{QCSP}(\mathbb{Q}; \text{EqOr}_n)$ for all $n \geq 3$ are NP-hard.

Proof. To prove the theorem, we need one auxiliary relation: $I = \{(x, y, z) \mid (x = y \rightarrow y = z)\}$. By Theorem 5.5 in [BC10], it follows that $\text{QCSP}(I)$ and $\text{QCSP}(S)$ are coNP-hard as well as that $\text{EqXor}$ and $\text{EqOr}_n$ for all $n \geq 3$ are NP-hard. To prove that $\text{QCSP}(\text{BetwC})$ and $\text{QCSP}(\text{CyclC})$ are coNP-hard, we will show that they pp-define $I$. Indeed, let $R \in \{\text{BetwC}, \text{CyclC}\}$. We claim that $I(x, y, z) = \exists u \exists v \ (R(x, y, u) \land R(x, y, v) \land R(u, v, z))$. In both cases, we have that if $x$ and $y$ have the same value, then also $u$ and $v$ and in consequence also $z$ has the same value. We have now to show that if $x$ has a different value than $y$, then $z$ can have an arbitrary value.

We first consider the case where $x$ has the value less than $y$. If $R$ is BetwC, then $u$ and $v$ have to be greater than $y$ and $u$ can be greater than $v$, and hence $z$ can be greater than $y$, equal to $y$, between $x$ and $y$, equal to $x$ as well as less than $x$. If $R$ is CyclC, then we can have $x$ to be less than $y$ less than $u$ less than $v$ but still $z$ can be less than $x$, equal to $x$, between $x$ and $y$, equal to $y$ as well as greater than $y$.

The second case we consider is when $y$ has the value less than $x$. If $R$ is BetwC then the analysis is symmetrical to that in the previous paragraph since this relation is preserved by $\rightarrow$. If $R$ is CyclC, then we can place $y$, $u$, and $v$ so that $y < u < v < x$. Now, it is straightforward to check that $z$ can be between $v$ and $x$, equal to $x$, greater than $x$, less than $y$, equal to $y$ but also between $y$ and $u$. 

We will now show that if a dually-closed temporal constraint language is preserved by a constant operation, then $\text{QCSP}(\Gamma)$ pp-defines one of relations in Theorem 13 and $\text{QCSP}(\Gamma)$ is hard, or $\Gamma$ is an equality language, a positive language or a dually-closed Ord-Horn language. We start with an auxiliary lemma.

Lemma 14. Let $f : \mathbb{Q} \rightarrow \mathbb{Q}$ be an operation that is neither constant nor it preserves $<$, then one of the following holds.

1. The operation $f$ is injective and then:
   a. $f$ preserves $\text{Betw} = \{(x, y, z) \in \mathbb{Q}^3 \mid (x < y < z \lor x > y > z)\}$ and generates $\rightarrow$; or
   b. $f$ preserves $\text{Cycl} = \{(x, y, z) \in \mathbb{Q}^3 \mid (x < y < z \lor y < z < x \lor z < x < y)\}$ and generates $\cyc$; or
(c) $f$ preserves $\text{Sep} = \{(x_1, y_1, x_2, y_2) \in \mathbb{Q}^4 \mid (x_1 < x_2 < y_1 < y_2) \lor (x_1 < y_1 < x_2 < y_2) \lor (y_1 < y_2 < x_1 < x_2) \lor (x_1 < y_1 < y_2 < x_1) \lor (y_2 < x_1 < x_2 < y_1) \lor (y_2 < y_1 < x_2 < x_1)\}$ and generates both $-$ and $\text{cyc}$; or

(d) $f$ generates all permutations.

(2) The operation $f$ is of infinite image but is not injective and then generates $\text{cl}$ or $\text{ic}$.

(3) The operation $f$ is of finite image and then it generates $\text{su}_1$ or peak.

**Proof.** The proof consists of three parts which correspond to three items from the formulation of the lemma.

**(Part One)** If $f$ is injective, then by the arguments in the proofs of Propositions 17 and 19 in [BK09], it is generated by automorphisms of $(\mathbb{Q};<)$. Since $f$ does not preserve $<$, by Cameron’s theorem [Cam76] (see also Theorems 4 and 13 in [BK09]), we have that one of the items in Case 1 holds.

**(Part Two)** We now turn to the case where $f$ is of infinite image but is not injective. Before we continue, we make an observation.

**Observation 15.** Let $f : \mathbb{Q} \to \mathbb{Q}$ be an operation that takes infinitely many values in $(a, b)$ where $a, b \in \mathbb{Q} \cup \{-\infty, +\infty\}$, then there is an infinite sequence of rational numbers contained in $(a, b)$:

1. Either of the form $q_1 > q_2 > \cdots$ such that:
   (a) $f(q_1) > f(q_2) > \cdots$,
   (b) $f(q_1) < f(q_2) < \cdots$;
2. Or of the form $q_1 < q_2 < \cdots$ such that:
   (a) $f(q_1) > f(q_2) > \cdots$,
   (b) $f(q_1) < f(q_2) < \cdots$;

**Proof.** Let $S$ be the set of elements in $(a, b)$ such that for all $x, y \in (a, b)$ we have $f(x) \neq f(y)$ and let $c \in S$. Now, either $S_1 := S \cap (a, c)$ or $S_2 := S \cap (c, b)$. We consider only the second case in which we have that either for every $q \in S_2$ there is $p > q$ and we are in Case 2, or there is $p \in S_2$ such that for every $\delta > 0$, there is $q \in S_2$ such that $p < q$ and $|p - q| \leq \delta$ and we are in Case 1. Now an easy application of Infinite Ramsey Theorem gives us subsequence which is either strictly increasing or strictly decreasing.

We now show that in Cases 1b and 2a the operation $f$ generates $\text{--}$. We convey the proof only in the second case. We use Lemma 3. Let $t$ be an $n$-ary tuple (of some $n$-ary relation) such that $p_1 < \cdots < p_k$ are all pairwise different values in $t$. It is enough to observe that $-(t) = \beta(f(\alpha(t)))$ where $\alpha, \beta \in \text{Aut}(\mathbb{Q};<)$ are such that $\alpha$ sends $p_1, \ldots, p_k$ to $q_1, \ldots, q_k$ and $\beta$ sends $f(q_1), \ldots, f(q_k)$ to $-(p_1), \ldots, -(p_k)$. \hfill $\square$

Let now $f$ be a non injective function that takes infinitely many values for arguments in the interval $(a_1, a_2)$ with $a_1, a_2 \in \mathbb{Q} \cup \{-\infty, +\infty\}$. Then we are in one of the four cases of Observation 15. To simplify the proof, we will always assume that we are either in Case 2b or in Case 2a. The proofs for two other cases are always similar.

Since $f$ is non-injective, there are $c_1 < c_2$ such that $f(c_1) = f(c_2)$. Furthermore, $f$ takes infinitely many values either for arguments less than $a_1$, or arguments greater than $a_1$. In the first case there are infinitely many values taken above $f(a_1)$ or below $f(a_1)$. By the previous assumptions, we are either in Case 2b or Case 2a.
In the second case $f$ generates $-$. In any case, either by considering $g := f$ or $g := -f$, we have one of the two following situations. In the first situation, there is a sequence of rational numbers $q_1 < q_2 < \cdots < c_1 < c_2$ such that $g(q_1) < g(q_2) < \cdots < g(c_1) = g(c_2)$. In the second situation there is a sequence of rational numbers $q_1 < q_2 < \cdots < c_1 < c_2$ such that $g(c_1) = g(c_2) < g(q_1) < g(q_2) < \cdots$. In both situations the operation $g$ generates $ic$. We provide the proof only for the first one. We use Lemma $6$. Let $t$ be a tuple of length $n$ in some relation $R$ with pairwise different values $p_1 < \cdots < p_k < 0 \leq p_{k+1} < \cdots < p_{k+l}$. We prove that $ic(t)$ is in $R$ by the induction on $l$. If $l = 0$, then $ic(t) = \alpha(t)$ where $\alpha \in Aut(Q; <)$ is the identity on $Q$. Assume now that we are done for $l = m$. We will prove the claim for $l = (m + 1)$. Let $t$ be a tuple in $R$ with pairwise different values $p_1 < \cdots < p_k < 0 \leq p_{k+1} < \cdots < p_{k+m+1}$ and $\alpha \in Aut(Q; <)$ such that $\alpha(p_1) < \cdots < \alpha(p_k) < \alpha(p_{k+1}) < 0 \leq \alpha(p_{k+2}) < \cdots < \alpha(p_{k+m+1})$. By the induction hypothesis, the relation $R$ contains a tuple $t_1 = ic(\alpha(t))$ with values $r_1 < \cdots < r_k < r_{k+1} < \cdots < r_{k+2} = 0$, where for all $i \in [k+1]$ we have $r_i = ic(\alpha(p_i))$ and $r_{k+2} = ic(\alpha(p_{k+2})) = \cdots = ic(\alpha(p_{k+m+1})) = 0$. Let now $\beta \in Aut(Q; <)$ be such that for $i \in [k]$ it holds $\beta(r_i) = q_i$ and $\beta(r_{k+1}) = c_1$ and $\beta(r_{k+2}) = c_2$, and $\gamma \in Aut(Q; <)$ be such that for all $i \in [k]$ we have $\gamma(q_i) = p_i$ as well as $\gamma(g(q_1)) = \gamma(g(c_2)) = 0$. Observe that $\gamma(g(t_1))$ is equal to $ic(t)$. It follows that $R$ contains $ic(t)$ and we are done.

From now on, we assume that $f$ takes finitely many values for arguments in the interval $(-\infty, c_1)$ and infinitely in the interval $(c_1, \infty)$. Again, we consider only Cases 2A and 2B from Observation 15. In the first situation we look at $g := f$ and in the second situation we look at $g := -f$. In any case we have either a sequence of rational numbers $d_1 < d_2 < c_1 < q_1 < q_2 < \cdots$ such that either $g(d_1) = g(d_2) < g(q_1) < g(q_2) < \cdots$ or $g(q_1) < g(q_2) < \cdots < g(d_1) = g(d_2)$. In both cases we have that $g$ generates $ci$. The proof is similar to the proof that $g$ generates $ic$ from the previous paragraph. This completes the second part of the proof of the lemma.

(Part Three) If $f$ is of finite image, then either there are two intervals $[a_1, a_2]$ and $[a_3, a_4]$ with $a_1 < a_2 < a_3 < a_4$ as well as $b_1 \neq b_2$ such that for all $x \in [a_1, a_2]$ we have $f(x) = b_1$ and for all $x \in [a_3, a_4]$ it holds $f(x) = f(y) = b_2$ or almost all rational numbers are sent by $f$ to the same value. We will show that in the first case, the operation $f$ generates $su_1$, whereas in the second case it generates $peak$. We start with first case. Observe that without loss of generality we can assume that $b_1 < b_2$. Indeed, if $b_1 > b_2$, then instead of $f$ we consider $g = f(\alpha(f))$ where $\alpha \in Aut(Q; <)$ satisfies $\alpha(b_1) = a_3$ and $\alpha(b_2) = a_1$. The operation $g$ satisfies $g(x) = a_1$ for all $x \in [a_1, a_2]$ and $g(x) = a_3$ for all $x \in [a_3, a_4]$. By Lemma 6 it is enough to show that every relation preserved by $f$ is also preserved by $su$. Let $R$ be any relation preserved by $f$ and $t$ a tuple in $R$. We have to show that $t_1 = su(t)$ is also a tuple in $R$. Let $q_1, \ldots, q_k < 0 \leq q_{k+1} < \cdots < q_l$ be pairwise different values occurring in $t$; $\alpha \in Aut(Q; <)$ such that $\alpha$ sends $q_1, \ldots, q_k$ to an interval $[a_1, a_2]$ and $q_{k+1}, \ldots, q_l$ to $[a_3, a_4]$ and $\beta \in Aut(Q; <)$ such that $\beta(b_1) = 0$ and $\beta(b_2) = 1$. Observe that $\beta(su(\alpha(t))) = t_1$. It follows that $f$ generates $su_1$.

We now consider the case where $f$ sends almost all values to $b$. Since $f$ is not a constant operation, there is $a_1 \in Q$ such that $f(a_1) = b_1 \neq b$. Again, without loss of generality we can assume that $b < b_1$. If it is not the case, then instead of $f$ we consider $f(\alpha(f))$ where $\alpha \in Aut(Q; <)$ satisfies $\alpha(b_1) = a$ and $\alpha(b) = a_1$ for
some $a < a_1$ with $f(a) = b$. Let now $R$ be a relation preserved by $f$ and $t \in R$, by Lemma 5 we have to show that $\text{peak}(t)$ is in $R$. Let $q_1, \ldots, q_k, q_{k+1}, \ldots, q_l$ be pairwise different values in $t$ different from 0 and such that $q_1 < \cdots < q_k < 0 < q_{k+1} < \cdots < q_l$. Let $\alpha \in \text{Aut}(Q; \leq)$ be such that it sends $q_1, \ldots, q_k$ to rational numbers less than $a_1$ such that $f(\alpha(q_1)) = \cdots = f(\alpha(q_k)) = b$, it sends 0 to $a_1$ and $q_{k+1}, \ldots, q_l$ to rational numbers greater than $a_1$ such that $f(\alpha(q_{k+1})) = \cdots = f(\alpha(q_l)) = b$. Further, let $\beta \in \text{Aut}(Q; \leq)$ satisfy $\beta(b) = -1$ and $\beta(b_1) = 1$. Observe that $\beta(f(\alpha(t))) = \text{peak}(t)$. It follows that $f$ generates peak.

Now, we make a first serious step. We show that either $\Gamma$ pp-defines BetwC, which by Theorem 13 gives rise to the hard QCSP or $\Gamma$ is preserved by one of few polymorphisms.

**Lemma 16.** Let $\Gamma$ be a temporal language preserved by a constant operation. If $\Gamma$ does not pp-define BetwC, then $\Gamma$ is preserved by ll, dual-ll, pp or dual-pp, s1, peak, ic, ci, −, cyc or all permutations.

**Proof.** If $\Gamma$ does not pp-define BetwC, then by Theorem 11 it follows that there is an operation $f$ that preserves $\Gamma$ and violates BetwC. By Lemma 5 we can assume that $f$ is a ternary operation such that for tuples $t_1, t_2, t_3$ satisfying $t_1[1] < t_1[2] < t_1[3]$, $t_2[1] > t_2[2] > t_2[3]$ and $t_3[1] = t_3[2] = t_3[3]$ we have $f(t_1, t_2, t_3) = t$ and $t \notin$ BetwC. Since $\Gamma$ is preserved by a constant operation, it is preserved by all constant operations, in particular some constant operation $g$ satisfying $g(t_1, t_2) = t_3$. By Observation 6 we have that $\Gamma$ is preserved by an operation $h : \mathbb{Q}^2 \to \mathbb{Q}$ such that $h(t_1, t_2) = t$.

Observe that the operation $h$ violates Betw defined in the formulation of Lemma 14. If $h$ preserves $<$, then by Lemma 49 in [BK09], we have that $g$ generates pp, dual-pp, ll, or dual-ll. From now on, we assume that $h$ violates $<$. We now consider the situation where $h$ violates $\leq$. If it is the case, then there are tuples $s_1, s_2$ such that $s_1[1] < s_1[2]$, $s_2[1] = s_2[2]$, and such that for $s = h(s_1, s_2)$ we have $s[1] > s[2]$. Since $\Gamma$ is preserved by a constant operation we can again apply Observation 6 and obtain that there is a unary operation $h_s$ preserving $\Gamma$ such that $h_s(s_1) = s$. Since $h_s$ is neither a constant operation nor it preserves $<$, it follows by Lemma 14 that $h_s$ generates s1, peak, ic, ci, −, cyc or all permutations. From now on we assume that $h$ preserves $\leq$. Let $h_1 := h(x, x)$. If $h_1$ is neither a constant operation nor it preserves $<$, then we are again done by Lemma 14. Thus, we have two cases to consider to complete the proof.

First we look at the situation where $h_1$ is a constant operation that sends all the rational numbers to $a$. Since $t \notin$ BetwC, there are $i \neq j$ in $[3]$ such that $t[i] \neq t[j]$. It follows that either $t[i]$ or $t[j]$ is different than $a$. Assume without loss of generality that $t[i] = b \neq a$. Let $d_1, d_2 \in \mathbb{Q}$ be such that $d_1 < \min(t[i], t[j]) < d_2$ and $\alpha \in \text{Aut}(\mathbb{Q}; <)$ be such that it sends all rational numbers $q \in (-\infty, d_1) \cup (d_2, \infty)$ to $q$; the interval $[d_1, t[i])$ to $[d_1, t[i])$, and $[t[j], d_2]$ to $[t[j], d_2]$. Due to Cantor’s theorem such $\alpha$ clearly exists. Consider $h_a(x) = h(x, \alpha(x))$. Observe that there is an infinite sequence $q_1 < q_2 < \cdots < t[i][1] < p_1 < p_2 < \cdots$ of rational numbers such that $h_a(q_1) = h_a(q_2) = \cdots = h_a(p_1) = h_a(p_2) = \cdots = a$ and $h(t[i]) = b$. Now as in the second paragraph of the third part of the proof of Lemma 14 we can show that $h_a$ and in consequence $f$ generates peak.

The last case to consider is where $h_1$ preserves $<$. Since $h$ violates $<$ and preserves $\leq$, there are $c_1, c_2 \in \mathbb{Q}^2$ such that $c_1[i] < c_2[i]$ for $i \in [2]$ and $f(c_1) = f(c_2)$. 

Let \( d_1, d_2 \in \mathbb{Q}^2 \) be such that \( d_1 < \min(c_1[1], c_1[2]) < \max(c_2[1], c_2[2]) < d_2 \) and \( \alpha \in \text{Aut}(\mathbb{Q}; <) \) be such that it sends all rational numbers \( q \in (-\infty, d_1) \cup (d_2, \infty) \) to \( q \); the interval \([d_1, c_2[1]) \) to \([d_1, c_1[1]) \); the interval \([c_2[1], c_2[2]) \) to \([c_1[1], c_1[2]) \); the interval \([c_2[2], d_2) \) to \([c_2[1], d_2) \). Such \( \alpha \) clearly exists. Consider \( h_\alpha(x) = h(x, \alpha(x)) \).

Since \( h_\alpha \) is not injective and not constant, it follows by Lemma 14 that \( h_\alpha \) and in consequence \( f \) generates peak, su_1, ic, ci, \(-\), cyc, or all permutations. \( \square \)

From now on, we provide lemmas which takes care of polymorphisms listed in Lemma 16. We show that in each of these cases, we can reduce our classification to the existing ones. We first take a look at the situation where \( \Gamma \) is preserved by ll, dual-ll, pp or dual-pp.

**Lemma 17.** Let \( \Gamma \) be a dually-closed temporal language such that \( \Gamma \) is preserved by pp, dual-pp, ll, or dual-ll. Then \( \Gamma \) is a dually-closed Ord-Horn constraint language.

**Proof.** Since \( \Gamma \) is dually-closed, we have two cases to consider. If \( \Gamma \) is preserve by ll or dual-ll, then it is preserved by the both operations. It follows by Proposition 4 then \( \Gamma \) is an Ord-Horn language.

It remains to consider the case where \( \Gamma \) is preserved by both pp and dual-pp. By [CBW14], if \( \text{Pol}(\Gamma) \) contains pp then every \( R \) in \( \Gamma \) can be defined as a conjunction of clauses of the form:

\[
x \neq y_1 \lor \cdots \lor x \neq y_k \lor x \geq z_1 \lor \cdots \lor x \geq z_l.
\]

The language \( \Gamma \) is also preserved by dual-pp. In that case as we show every clause \( \{ \} \) in the definition of every relation in \( \Gamma \) satisfies \( l = 1 \). Suppose not. Then there is \( R \) in \( \Gamma \) that does not have an Ord-Horn definition. Let \( \phi \) be a definition of \( R \) in terms of clauses of the form \( \{ \} \) with a minimal number of literals and let \( \psi \) be clause in \( \phi \) for which \( l \geq 2 \). By the minimality of \( \phi \), it is satisfied by assignments \( t_1, t_2 : \text{Var}(\phi) \to \mathbb{Q} \) such that \( t_i \) violates all literals in \( \psi \) except for \( x \geq z_i \). Let \( \alpha_1 \in \text{Aut}(\mathbb{Q}; <) \) be such that \( \alpha_1(t_1(z_1))) \leq \alpha_1(t_1(x)) \leq 0 < \alpha_1(t_1(z_2)) \). To reach the contradiction, we will show that \( f := \text{dual-pp}(\alpha_1(t_1), t_2) \) does not satisfy any disjunct of \( \psi \). By the definition of dual-pp, it follows that \( f(x) < f(z_1) \) and \( f(x) < f(z_2) \). All other literals are violated since dual-pp preserves \( < \) and \( = \). It completes the proof of the lemma. \( \square \)

The operations ic and ci are the duals to each other and as we show in the proof of Theorem 25 \( \{ \text{ic, ci} \} \) generates su_1. Since we restrict ourselves to dually-closed languages, the next lemma applies also to the case where \( \Gamma \) is preserved by ic or ci.

**Lemma 18.** Let \( \Gamma \) be an operation preserved by su_1, then \( \Gamma \) is positive or QCSP(\( \Gamma \)) is NP-hard.

**Proof.** We first consider the case where \( \Gamma \) is preserved by all \( \text{su}_i \) with \( i \in \mathbb{N} \). In this case, as we show, \( \Gamma \) is a positive language. To this end, we have to show that \{su_1 \ | \ i \in \mathbb{N} \} generate wave. Let \( t \) be an \( n \)-tuple with pairwise different values \( q_1 < \cdots < q_a < 0 \leq q_{a+1} < \cdots < q_{a+b} \leq 1 < q_{a+b+1} < \cdots < q_{a+b+c} \). Let \( \alpha \in \text{Aut}(\mathbb{Q}; <) \) be such that it sends \( q_i \) for \( i \in [a] \) to \( i - 1 \), \( q_i \) for \( i \in [a+1, \ldots, b] \) to the interval \([a, a+1] \) and \( q_i \) for \( i \in [b+1, \ldots, c] \) to \( a + i \) and \( \beta \) such that it sends \( \text{su}_a(\alpha(q_i)) \) for \( i \in [a] \) to \( q_i ; \text{su}_b(\alpha(q_{a+1})) \) to 0 and \( \text{su}_c(\alpha(q_{a+b+i})) \) for \( i \in [c] \) to \( (q_{a+b+i} - 1) \). Observe that \( \beta(\text{su}_a(\alpha(t))) \) = wave(t).

The second case holds if there exists \( k \in \mathbb{N} \) such that \( \Gamma \) is preserved by \( \text{su}_k \) but is not preserved by \( \text{su}_{k+1} \). Let \( R \in \Gamma \) and \( t \in R \) such that \( t_s = \text{su}_{k+1}(t) \) is not in \( R \).
For the sake of simplicity assume that $t$ is an injective tuple, that is, all its entries are pairwise different. Observe that $t$ has to be of length $n > (k + 1)$. Let $\Pi_1, \ldots, \Pi_{k+1}$ be a partition of $[n]$ such that $i, j \in \Pi_a$ for $a \in [l]$ if and only if $t_s[i] = t_s[j]$. Consider the relation $R_s$ pp-defined by $R(x_1, \ldots, x_n) \land \bigwedge_{a \in [k+1]} \bigwedge_{i, j \in \Pi_a} x_i = x_j$. Intuitively, $R_s$ is just $R$ where coordinates from the same $\Pi_i$ are identified. Since $R$ and also $R_s$ are preserved by $s_u$ and $R_s$ is of arity $k + 1$, it is easy to see that $R_s$ is preserved by wave and hence it is a positive relation. By Theorem 12 it follows that a positive relation $R$ gives rise to NP-hard QCSP unless it is preserved by pp or dual-pp. We now show that $R_s$ is preserved by none of these operations, which imply that QCSP($\mathbb{Q}; R_s$) and hence QCSP($\Gamma$) is NP-hard and completes the proof of the lemma. Since $t$ is in $R$ and $R$ is preserved by $s_u$, it follows that $R$ contains also both $t_1$ such that $t_1[\Pi_1] < t_1[\Pi_2 \cup \Pi_3] < \cdots < t[\Pi_{k+1}]$ and $t_2$ such that $t_2[\Pi_1 \cup \Pi_2] < t[\Pi_3] < \cdots < t[\Pi_{k+1}]$. [NOTATION: We write $t[S]$ where $S \subseteq [n]$ for the value $t[i]$ that is common for all $i \in S$.] By the pp-definition, we have that $R_s$ contains tuples $t'_1$ and $t'_2$ such that $t'_1[\Pi_1] < t'_1[\Pi_3] < \cdots < t'_1[\Pi_{k+1}]$ and $t'_2[\Pi_1] = t'_2[\Pi_3] < \cdots < t'_2[\Pi_{k+1}]$. Let now $\alpha \in Aut(\mathbb{Q}; \angle)$ be such that $\alpha(t'_1[\Pi_1]) < 0 < \alpha(t'_1[\Pi_3])$. Observe now that $t'_3 = pp(\alpha(t'_2), t'_1)$ satisfies $t'_3[\Pi_1] < t'_3[\Pi_3] < \cdots < t'_3[\Pi_{k+1}]$. Hence there is $\beta \in Aut(\mathbb{Q}; \angle)$ such that $\beta(t'_3) = t^*$. It contradicts the fact that $t^*$ is not in $R$. It follows that $R_s$ is not preserved by pp. To show that the relation is not preserved by dual-pp, we proceed in the similar way with the difference that we take $\alpha$ such that $\alpha(t'_2[\Pi_1]) < 0 < \alpha(t'_2[\Pi_3])$ and $t'_3 = pp(\alpha(t'_2), t'_1)$.\[\square\]

The next case to consider is where $\Gamma$ is preserved by peak.

Lemma 19. Let $\Gamma$ be a temporal language preserved by peak. If $\Gamma$ defines neither $EqXor$ nor $EqOr$, then $\Gamma$ is preserved by all permutations.

Proof. We need some definitions. Let $n$ be a natural number, $t$ an $n$-ary tuple and $S \subseteq [n]$. If for all $i, j \in S$, we have $t[i] = t[j]$, then we write $t[S]$ to indicate the value which is common for all $t[i]$ with $i \in S$. We say that an $n$-ary tuple $t$ is an ordered $k$-partition of $[n]$ if there is an underlying partition $\{\Pi_1, \ldots, \Pi_k\}$ of $[n]$ such that for all $i, j \in [n]$ we have $t[i] = t[j]$ if and only if $i, j \in \Pi_l$ for some $l \in [k]$ and $t[\Pi_l] < t[\Pi_{l+1}]$ for all $i \in [k-1]$. Let $t_1, t_2$ be two ordered $k$-partitions of $[n]$ with the same underlying partition $\{\Pi_1, \ldots, \Pi_k\}$ of $[n]$. Then there is an automorphism $\alpha \in Aut(\mathbb{Q}; \angle)$ such that $t_1 = \alpha(t_2)$. Hence for an $n$-ary relation $R$, we have $t_1 \in R$ if and only if $t_2 \in R$. We will write $t[\Pi_1] < \cdots < t[\Pi_k]$ for an ordered $k$-partition of $[n]$ with an underlying partition $\{\Pi_1, \ldots, \Pi_k\}$ of $[n]$. We also need a special treatment of ordered 2-partitions. We say that a tuple $t$ is an $[a, b]$ 2-partition if $|\Pi_1| = a$ and $|\Pi_2| = b$.

If $\Gamma$ pp-defines neither $EqXor$ nor $EqOr$ for any $n \geq 3$, then there are operations $f_x$ and $f_1, f_2, \ldots$ preserving $R$ such that $f_x$ violates $EqXor$ while $f_n$ for $n \geq 3$ violates $EqOr$. Let $R$ be an $n$-ary relation and $t$ a tuple in $R$. For the sake of simplicity we assume that the values in $t$ are pairwise different. To show that all permutation of $t_o$ are in $R$ we will prove that for all $k \leq n$, the relation $R$ contains all ordered $k$-partitions of $n$.

First Part of the Proof. In the first part of the proof, we prove that $R$ contains all 2-partitions of $n$. By induction on $m = \min(a, b) - 1$ we show that every $[a, b]$ 2-partition of $[n]$ is in $R$. If $m = 0$, then $a = 1$ or $b = 1$. In this case we just use the operation peak and automorphisms of $Aut(\mathbb{Q}; \angle)$. Indeed, let $t'[\Pi_1 \setminus \{i\}] < t[\{i\}]$.
be an ordered $[n-1,1]$ 2-partition of $[n]$. Observe that $t$ is equal to $\beta(\text{peak}(\alpha(s)))$ where $\alpha, \beta$ are automorphisms of $(Q;\prec)$ such that $\alpha$ sends $s[i]$ to 0; and $\beta$ sends $-1$ and 1 to $t([n] \setminus \{i\})$ and $t(\{i\})$, respectively. Now from $t([n] \setminus \{i\}) < t(\{i\})$ we can obtain any $[1,n-1]$ 2-partition of $[n]$ by first sending $t([n] \setminus \{i\})$ and $t(\{i\})$ to 0 and 1, respectively, it flips the values; and then by using an appropriate automorphism of $(Q;\prec)$. Assume now that we are done for $[a,b]$ 2-partitions with $m$, as defined above, equal to $l$. We will now prove that the claim holds for $l+1$. By the observation above, we have that $R$ is preserved by $f_x$. Since $f_x$ violates EqOr, by Theorem 4 and Lemma 5, there are tuples $s, s_1, s_2, s_3, s_4, s_5$ such that

- $s_1[1] = s_1[2] = s_1[3]$,
- $s_2[1] = s_2[2] < s_2[3]$,
- $s_3[1] = s_3[2] > s_3[3]$,
- $s_4[1] = s_4[3] < s_4[2]$, and
- $s_5[1] = s_5[3] > s_5[2]$.

and we have $f(s_1, s_2, s_3, s_4, s_5) = s$ and $s$ such that $s[1] \neq s[2]$ and $s[1] \neq s[3]$.

Let $t_g\Pi_1 < t_g\Pi_2$ be any ordered $[n-l-1,l+1]$ ordered 2-partition of $[n]$. Let $i \in \Pi_i$. Since $R$ is preserved by a constant operation we have that it contains a tuple $t_1$ such that all its entries are equal to $s_1[1]$. Moreover, by the induction assumption, we have that $R$ contains all of the following:

- an ordered $[n-l,l]$ 2-partition $t_2[\Pi_1 \cup \{i\}] = s_1[1] < t_2[\Pi_2 \setminus \{i\}] = s_1[3]$;
- an ordered $[l,n-l]$ 2-partition $t_3[\Pi_2 \setminus \{i\}] = s_2[3] < t_3[\Pi_1 \cup \{i\}] = s_2[1]$;
- an ordered $[n-1,1]$ 2-partition $t_4[\Pi_1 \setminus \{i\}] = s_4[1] < t_4[i] = s_4[2]$; and
- an ordered $[1,n-1]$ 2-partition $t_5[i] = s_5[2] < t_5[n] \setminus \{i\} = s_5[1]$.

It is now straightforward to check that $t = f(t_1, t_2, t_3, t_4, t_5)$ satisfies $t[\Pi_1] \neq t[\Pi_1 \setminus \{i\}]$ and $t[\Pi_1] \neq t(\{i\})$. It is now easy to see that by applying peak and appropriate automorphisms of $(Q;\prec)$ we can obtain $t_g\Pi_i < t_g\Pi_{i+1}$. This proves that we can obtain any $[n-l-1,l+1]$ ordered 2-partition of $[n]$. Any $[l+1,n-l-1]$ ordered partition of $[n]$ can be obtained from a corresponding $[n-l-1,l+1]$ ordered partition just by flipping the values. This can be obtained with the use of peak. This proves that $R$ contains all ordered $[a,b]$ 2-partitions where $\min(a, b) = l+1$ and completes the induction step. By mathematical induction we obtain that $R$ contains all ordered 2-partitions.

Second Part of The Proof. Here we show that $R$ contains all ordered $k$-partitions of $[n]$ for $k \leq n$. By the previous part of the proof we have that $R$ contains all 2-partition of $n$, which we will use as a base case in our induction. Assume now that we are done for $k = l$. We will now prove the claim for $k = l+1$. Recall that $f_{l+1}$ is an operation preserving $\Gamma$ and violating EqOr$_{(l+1)}$. Let $s_1, \ldots, s_p$ be a list of all, up to isomorphisms, different, ordered $a$-partition of $[l+1]$ with $a \leq l$. Observe that up to a permutation of $a_1, \ldots, a_p$, we can assume that $f(s_1, \ldots, s_p) = s$ and $s$ satisfies $s[1] < \cdots < s[l+1]$. Let now $t_g[\Pi_1] < \cdots < t_g[\Pi_{l+1}]$ be any ordered $(l+1)$-partition of $[n]$. We will now show that $t_g$ is in $R$. Let $t_1, \ldots, t_p$ be a list of all, up to isomorphisms, ordered $a$-partitions of $[n]$ with $a \leq l$ such that for all $b \in [p]$ and $i \in [l+1]$ we have $t_b[i] = s_b[\Pi_i]$. By the induction hypothesis we have that all tuples $t_1, \ldots, t_p$ are in $R$. Observe that there is an automorphism $\alpha \in Aut(Q;\prec)$ such that $\alpha(f(t_1, \ldots, t_p)) = t_g$. This completes the induction step. By mathematical induction we have that every ordered $k$-partition with $k \leq n$ is in $R$. This completes the proof of the lemma. \hfill \qed
The only remaining polymorphisms from Lemma 16 are $-$ and cyc. We take care of them in three steps. First we look at the case where $\Gamma$ has only $-$ out of these two.

**Lemma 20.** Let $\Gamma$ be a temporal language preserved by a constant operation and by $-$. If $\Gamma$ does not pp-define BetwC, then it is preserved by su, peak, ci, ic, cyc or all permutations.

*Proof.* If $\Gamma$ does not pp-define BetwC, then by Theorem 4 there is an operation $f_b$ that preserves $\Gamma$ and violates BetwC. By Lemma 5 we can assume that $f_b$ is a ternary operation such that for some tuples $t_1, t_2, t_3$ satisfying $t_1[1] < t_1[2] < t_1[3]$, $t_2[3] < t_2[2] < t_2[1]$, and $t_4[1] = t_4[2] = t_4[3]$ we have $f_b(t_1, t_2, t_3) = t$ and $t \notin$ BetwC. Since $\Gamma$ is preserved by a constant operation, it is preserved by all constant operations, in particular some constant operation $f_c$ satisfying $f_c(t_1, t_2, t_3) = t$. By Observation 6 we have that $\Gamma$ is preserved by an operation $f_d : \mathbb{Q}^3 \rightarrow \mathbb{Q}$ such that $f_d(t_1, t_2) = t$. There are also an automorphism $\alpha \in \text{Aut}(\mathbb{Q}; <)$ such that cyc$(\alpha(t_1)) = t_2$ and cyc$(\alpha(t_1)) = t_3$. Thus, by applying Observation 6 again, we conclude that $\Gamma$ is preserved by a unary operation $f$ such that $f(t_1) = t$. Observe that the operation $f$ violates Betw and $<$ and is not a constant operation. We use Lemma 14. First we consider the case where $f$ is injective. Since it violates both Betw and $<$, we have by Item 1 that $f$ generates cyc or all permutations. On the other hand, if $f$ is not injective, then it generates ic, $ci, su_1$, or peak. It completes the proof of the lemma. \hfill $\square$

We now take care of the situation where $\Gamma$ is preserved by cyc.

**Lemma 21.** Let $\Gamma$ be a temporal language preserved by a constant operation and by cyc. If $\Gamma$ does not pp-define CyclC, then it is preserved by $su_1, peak, ci, ic, -$ or all permutations.

*Proof.* If $\Gamma$ does not pp-define CyclC, then by Theorem 4 it follows that there is an operation $f_b$ that preserves $\Gamma$ and violates CyclC. By Lemma 5 we can assume that $f_b$ is an operation of arity four such that some for tuples $t_1, t_2, t_3, t_4$ satisfying $t_1[1] < t_1[2] < t_1[3]$, $t_2[3] < t_2[2] < t_2[1]$, $t_3[3] < t_3[2] < t_3[1] < t_3[2]$, and $t_4[1] = t_4[2] = t_4[3]$ we have $f_b(t_1, t_2, t_3, t_4) = t$ and $t \notin$ BetwC. Since $\Gamma$ is preserved by a constant operation, it is preserved by all constant operations, in particular some constant operation $f_c$ satisfying $f_c(t_1, t_2, t_3) = t_4$. By Observation 6 we have that $\Gamma$ is preserved by an operation $f_d : \mathbb{Q}^5 \rightarrow \mathbb{Q}$ such that $f_d(t_1, t_2, t_3) = t$. There are also automorphisms $\alpha, \beta \in \text{Aut}(\mathbb{Q}; <)$ such that cyc$(\alpha(t_1)) = t_2$ and cyc$(\alpha(t_1)) = t_3$. Thus, by applying Observation 6 twice, we conclude that $\Gamma$ is preserved by a unary operation $f$ such that $f(t_1) = t$. Observe that $f$ violates $<$ and Cycl, defined in the formulation of Lemma 14. If $f$ is injective, then by Item 1 of Lemma 14 we have that $f$ generates $-$ or all permutations. The operation $f$ is also not constant. Thus, if it is not injective, then by the same lemma, it follows that it generates ic, $ci, su_1$ or peak. This completes the proof of the lemma. \hfill $\square$

Finally, we consider the situation where $\Gamma$ is preserved by both $-$ and cyc.

**Lemma 22.** Let $\Gamma$ be a temporal language preserved by a constant operation by $-$, and cyc. If $\Gamma$ does not pp-define $S$, then it is preserved by $su, peak, ci, ic$ or all permutations.
Proof: The proof goes along the lines of the proof of Lemmas 20 and 21. Again, if \( \Gamma \) does not pp-define \( S \), then there is an operation \( f \) that preserves \( \Gamma \) and violates \( S \). Again, we can assume that the arity of \( f \) is the number of orbits of \( 3 \)-tuples with respect to \( Aut(Q; <) \) contained in \( S \). Thus there are tuples \( t_1, t_2, t_3, t_4, t_5, t_6, t_7 \) such that \( t_1[1] < t_4[2] < t_1[3], t_2[1] < t_2[3] < t_1[2], t_5[2] < t_3[1] < t_3[3], t_4[2] < t_4[3] < t_4[1], t_5[3] < t_3[1] < t_5[2], t_6[3] < t_6[2] < t_6[1] \), and \( t_7[1] = t_7[2] = t_7[3] \). Observe that tuples \( t_2, \ldots, t_7 \) can be obtained from \( t_1 \) by applying to \( t_1 \) the operations: \(-, cyc, constant operations and automorphisms of \((Q; <)\). Thus by multiple application of Observation 6, we conclude that there is a unary operation \( f \) that preserves \( \Gamma \) and such that \( f(t_1) = t \) for some \( t \notin S \). Observe that \( f \) violates \(<, Betw, Cycl \) and is not a constant operation. If it is injective, then by Item 1 of Lemma 14 it follows that \( f \) generates all permutations. If \( f \) is not injective, then by the same lemma, we have that \( f \) generates \( ic, ci, su_1 \) or peak. \qed

We can use the above lemmas to prove the following.

**Theorem 23.** Let \( \Gamma \) be a dually-closed temporal constraint language preserved by a constant operation, then one of the following holds.

1. The problem QCSP(\( \Gamma \)) is coNP-hard or NP-hard.
2. \( \Gamma \) is a dually-closed Ord-Horn constraint language.
3. \( \Gamma \) is a positive constraint language.
4. \( \Gamma \) is an equality constraint language.

Proof: By Lemma 14, we have that either \( \Gamma \) pp-defines BetwC and then by Theorem 13 the problem QCSP(\( \Gamma \)) is coNP-hard; or \( \Gamma \) is preserved by one of the following operations: pp, dual-pp, ll, dual-ll, su_1, ic, ci, peak, \(-, cyc \), or all permutations. If it is one of the first four operations, then by Lemma 17 we have that \( \Gamma \) is a dually-closed Ord-Horn language and we are in case 2.

Observe now that a dually-closed language preserved by ic or ci is preserved by both ic and ci. These operations are the dual of each other. Moreover, \{ic, ci\} generate su_1. Indeed, for every tuple \( t \) with pairwise different values \( q_1 < \cdots < q_a \leq 0 < q_{a+1} < \cdots < q_{a+b} \) we have that \( ci(\alpha(\text{ic}(t))) = su_1(t) \) where \( \alpha \) is an automorphism of \((Q; <)\) such that \( \alpha(q_1) < \cdots < \alpha(q_a) < 0 \) and \( \alpha(O) = 1 \). It follows by Lemma 18 that if a dually-closed temporal constraint language \( \Gamma \) preserved by su_1, ic, or ci, then \( \Gamma \) is either hard and we are in Case 4 or positive and we are in Case 3. Further, by Lemma 19 and Theorem 13 if \( \Gamma \) is preserved by peak, then \( \Gamma \) is either hard and we are in Case 4 or \( \Gamma \) is an equality language and we are in case 3. What remained to consider is the situation where \( \Gamma \) is preserved by \(-\) or cyc. In the former case we use Lemma 20 and Lemma 21 in the latter. We have that either \( \Gamma \) pp-defines a relation that give rise to hard QCSP or it is preserved by both \(-\) and cyc. In the first case we are in Case 4 and we are done, whereas in the second we use Lemma 22. Here, again, either \( \Gamma \) pp-defines \( S \) and by Theorem 13 we are in Case 4 or one of the previously considered cases holds and we are also done. This completes the proof of the theorem. \qed

5. **Classification**

Here, we prove that QCSP(\( \Gamma \)) for a dually-closed temporal language \( \Gamma \) is either hard or it is in \( P \).
Theorem 24. Let $\Gamma$ be a dually-closed temporal language. Then $\Gamma$ is a Guarded Ord-Horn language and QCSP($\Gamma$) is in $P$. Otherwise QCSP($\Gamma$) is NP-hard or coNP-hard.

Proof. By Theorem 50 in [BK09], it follows that CSP($\Gamma$), and hence also QCSP($\Gamma$) is hard, or $\Gamma$ is preserved by: pp, dual-pp, ll, dual-ll or a constant operation. In first four cases we use Lemma [17] which reduces the problem to the classification of dually-closed Ord-Horn temporal constraint satisfaction problems. In the case where $\Gamma$ is preserved by a constant operation, by Theorem [23] QCSP($\Gamma$) is hard or $\Gamma$ is either a dually-closed Ord-Horn language or an equality language, or a positive language. In the first two cases, it is in fact dually-closed Ord-Horn and hence by Theorem [10] the language $\Gamma$ is Guarded Ord Horn and QCSP($\Gamma$) is in $P$ or QCSP($\Gamma$) is coNP-hard. We now consider the case where $\Gamma$ is positive. By Theorem [12] we have that either QCSP($\Gamma$) is NP-hard, or $\Gamma$ is preserved by pp or dual-pp. In the former case we are done. In the latter case, by Lemma [17] we have that $\Gamma$ is dually-closed Ord-Horn and we are done by Theorem [10].

References

[BBJK03] Ferdinand Börner, Andrei A. Bulatov, Peter Jeavons, and Andrei A. Krokhin. Quantified constraints: Algorithms and complexity. In Computer Science Logic, 17th International Workshop, CSL 2003, 12th Annual Conference of the EACSL, and 8th Kurt Gödel Colloquium, KGC 2003, Vienna, Austria, August 25-30, 2003, Proceedings, pages 58–70, 2003.

[BC10] Manuel Bodirsky and Hubie Chen. Quantified equality constraints. SIAM Journal on Computing, 39(8):3682–3699, 2010. A preliminary version of the paper appeared in the proceedings of LICS’07.

[BK09] Manuel Bodirsky and Jan Kára. The complexity of temporal constraint satisfaction problems. Journal of the ACM, 57(2):1–41, 2009. An extended abstract appeared in the Proceedings of the Symposium on Theory of Computing (STOC’08).

[BN06] Manuel Bodirsky and Jaroslav Nešetřil. Constraint satisfaction with countable homogeneous templates. Journal of Logic and Computation, 16(3):359–373, 2006.

[Cam76] Peter J. Cameron. Transitivity of permutation groups on unordered sets. Mathematische Zeitschrift, 148:127–139, 1976.

[CBW14] Hubie Chen, Manuel Bodirsky, and Michal Wrona. tractability of quantified temporal constraints to the max. Int. J. Algebra Comput., 24(11-14), 2014.

[Che12] Hubie Chen. Meditations on quantified constraint satisfaction. In Logic and Program Semantics - Essays Dedicated to Dexter Kozen on the Occasion of His 60th Birthday, Lecture Notes in Computer Science 7230. Springer, 2012.

[CM12] Hubie Chen and Moritz Müller. An algebraic preservation theorem for aleph-zero categorical quantified constraint satisfaction. Logical Methods in Computer Science, 9(1), 2012.

[CW08a] Witold Charatonik and Michal Wrona. Quantified positive temporal constraints. In Proceedings of CSL, pages 94–108, 2008.

[CW08b] Witold Charatonik and Michal Wrona. Tractable quantified constraint satisfaction problems over positive temporal templates. In LPAR, pages 543–557, 2008.

[CW12] Hubie Chen and Michal Wrona. Guarded ord-horn: A tractable fragment of quantified constraint satisfaction. In 19th International Symposium on Temporal Representation and Reasoning, TIME 2012, Leicester, United Kingdom, September 12-14, 2012, pages 99–106, 2012.

[GJ78] Michael Garey and David Johnson. A guide to NP-completeness. CSLI Press, Stanford, 1978.

[MSS04] Rolf H. Mühring, Martin Skutella, and Frederik Stork. Scheduling with and/or precedence constraints. SIAM Journal on Computing, 33(2):393–415, 2004.
[NB95] Bernhard Nebel and Hans-Jürgen Bürckert. Reasoning about temporal relations: A maximal tractable subclass of Allen’s interval algebra. *Journal of the ACM*, 42(1):43–66, 1995.

[Sch78] Thomas J. Schaefer. The complexity of satisfiability problems. In *Proceedings of the Symposium on Theory of Computing (STOC)*, pages 216–226, 1978.

[SW13] Johannes Schmidt and Michal Wrona. The complexity of abduction for equality constraint languages. In *CSL*, pages 615–633, 2013.

[Sze86] Ágnes Szendrei. *Clones in universal algebra*. Séminaire de Mathématiques Supérieures. Les Presses de l’Université de Montréal, 1986.

[VKvB89] Marc Vilain, Henry Kautz, and Peter van Beek. Constraint propagation algorithms for temporal reasoning: A revised report. *Reading in Qualitative Reasoning about Physical Systems*, pages 373–381, 1989.

[Wro12] Michal Wrona. Syntactically characterizing local-to-global consistency in ord-horn. In *Principles and Practice of Constraint Programming - 18th International Conference, CP 2012, Québec City, QC, Canada, October 8-12, 2012. Proceedings*, pages 704–719, 2012.

[Wro14a] Michal Wrona. Local-to-global consistency implies tractability of abduction. In *Proceedings of the Twenty-Eighth AAAI Conference on Artificial Intelligence, July 27 -31, 2014, Québec City, Québec, Canada.*, pages 1128–1134, 2014.

[Wro14b] Michal Wrona. Tractability frontier for dually-closed ord-horn quantified constraint satisfaction problems. In *Mathematical Foundations of Computer Science 2014 - 39th International Symposium, MFCS 2014, Budapest, Hungary, August 25-29, 2014. Proceedings, Part I*, pages 535–546, 2014.

[ZM21] Dmitriy Zhuk and Barnaby Martin. The complete classification for quantified equality constraints. *CoRR*, abs/2104.00406, 2021.

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