Universal Relations for Gravitational-Wave Asteroseismology of Protoneutron Stars

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State-of-the-art numerical simulations of core-collapse supernovae reveal that the main source of gravitational waves is the excitation of protoneutron star modes during postbounce evolution. In this work we derive universal relations that relate the frequencies of the most common oscillation modes observed, i.e., $g$ modes, $p$ modes, and the $f$ mode, with fundamental properties of the system, such as the surface gravity of the protoneutron star or the mean density in the region enclosed by the shock. These relations are independent of the equation of state, the neutrino treatment, and the progenitor mass and, hence, can be used to build methods to infer protoneutron star properties from gravitational-wave observations alone. We outline how these measurements could be done and the constraints that could be placed on the protoneutron star properties.

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Core-collapse supernova (CCSN) explosions are a promising source of gravitational waves (GW) and might be one of the next discoveries of current or future ground-based GW observatories. The most common CCSN type, a neutrino-driven explosion, is expected to be observable with Advanced LIGO and Virgo within our Galaxy [1] at a rate of about three per century [2]. CCSN events mark the end of the life of massive stars ($8\ M_\odot$–$100\ M_\odot$) following the formation of a heavy iron core that collapses under its own gravity. The end result is the formation of a protoneutron star (PNS) at densities above nuclear-matter density with a radius of $\sim 30\ km$ and a stalled accretion shock at $\sim 100\ km$. This situation is maintained for 0.1–2 s, while accretion proceeds onto the PNS as it cools down through neutrino emission. The shock-PNS system may suffer instabilities, in particular, convection and the standing accretion shock instability (SASI). Those instabilities are crucial for shock revival, as they allow for an enhanced energy deposition in the postshock region by the neutrinos streaming out of the PNS. At the same time these instabilities break spherical symmetry and induce perturbations of the PNS that produce a rich, stochastic GW signal that lasts until the onset of explosion, if successful, or up to the formation of a black hole. Detailed descriptions of the CCSN explosion mechanism and GW emission can be found, e.g., in Refs. [3,4], respectively.

Numerical simulations have shown that the GW signal, albeit highly stochastic, displays some clear trends in the time-frequency plane (spectrograms), as in the example shown in the left-hand panel of Fig. 1. This is the result of
the excitation of PNS $g$ modes and, in some cases, of SASI modes [5–12]. Recent work [13–15] has established that
the features observed in the GW spectrums can be very accurately matched to the $l = 2$ PNS eigenmodes, $l$ being
the order of the spherical-harmonic decomposition. Those are obtained from the solution of an eigenvalue problem at
each time step of the simulation, using angular-averaged profiles of the simulation data as a background. Particularly
important for an accurate matching has been the inclusion of general relativity and space-time perturbations [15].
This analysis enables a detailed study of the behavior of CCSN GW signals in spectrums without the need of
performing computationally challenging multidimensional simulations. Instead, affordable spherically symmetric
simulations serve the purpose, supplemented with the computation of the eigenfrequencies for $l = 2$ perturbations,
responsible for the GW emission. Those simulations, however, do not allow us to compute the amplitude of the
waveform but only its frequency evolution. In this Letter we show that at each instant during the PNS evolution, the
characteristic frequency for the different modes does not depend on the exact structure of the PNS but can be
estimated from the general properties of the remnant. Furthermore, the relations we derive are universal, as they
do not depend on the equation of state (EOS), progenitor star, or neutrino transport. Therefore, they provide the
potential to be used as a basis for parameter inference algorithms once GW observations from CCSN become
available.

Our analysis is based on 25 1D simulations with different combinations of numerical codes, gravity approximations,
EOSs, and progenitor stars. The simulations are performed with the AENUS-ALCAR code [16,17] and the CoCoNuT
code [18]. Both are multidimensional Godunov-based Eulerian hydrodynamics codes for spherical polar coordi-
nates and include neutrino transport. Both codes include multigroup treatment for three neutrino species (electron
neutrinos and antineutrinos and heavy flavor neutrinos). AENUS-ALCAR uses an algebraic Eddington-factor method
with an $M_1$ closure [17], whereas CoCoNuT uses a less sophisticated approach [19] based on a stationary transport
solution with only a one-moment closure. In both cases neutrino interactions include charged-current reactions of
electron neutrinos and antineutrinos with nucleons and nuclei, neutrino scattering off nucleons and nuclei, and
neutrino production by nucleon bremsstrahlung. AENUS-
ALCAR also includes electron-positron pair processes and
inelastic scattering off electrons. Regarding the gravity
treatment, AENUS-ALCAR can use either Newtonian gravity
or a pseudorelativistic potential (TOV-A in Ref. [20]) while
CoCoNuT uses general relativity in the XCFC formulation
[21]. We use six different finite-momentum EOSs in our
simulations: LS220 [22], GShen-NL3 [23], HShen [24], SFHo [25], BHB-A [26], and HShen-Λ [24], the latter two
including Λ hyperons. However, hyperons are not included
in the neutrino interaction rates. As initial data we use a
selection of presupernova models from Ref. [27], including
solar-metallicity models between 11 $M_\odot$ and 75 $M_\odot$ and
one 20 $M_\odot$ model with $10^{-4}$ solar metallicity. The com-
plete list of simulations can be found in the Supplemental
Material [28].

For each simulation and at each time after bounce, the
eigenmode frequencies of the region including the PNS and
the shock are determined using the code GREAT [15] which
includes corrections for space-time perturbations of the
lapse and conformal factor in general relativity. This
eigenmode calculation does not provide a classification of
the modes by itself. We use the procedure described in
Ref. [15] to classify modes in $f$ modes ($l^f$), $p$ modes ($l^p$),
and $g$ modes ($l^g$), where $l$ indicates the number of radial
nodes. We restrict ourselves to $l = 2$ modes, which are the
dominant ones for GW emission. Compared to our mode
classification presented in Ref. [15], we here relabel some
previously misclassified modes: one of the $p$ modes is
reclassified as $g_1$, for all other $g$ modes $n$ is increased by
one ($g_n \rightarrow g_{n+1}$), and for all $p$ modes above the reclassified $p$ mode $n$ is decreased by one ($p_{n+1} \rightarrow p_n$).
The new and wider set of simulations used in this work (in Ref. [15]
only two simulations were considered) shows that the
lowest-order $g$ mode had been misclassified as a $p$ mode.
This is supported by three aspects of the simulations: the
energy density distribution of the mode, much more
concentrated in the PNS interior, the time evolution of the
frequency, which often presents crossings with other $p$
modes, and the behavior of the universal relations found in
this work (see below).

We focus on the $f$ mode and on the lowest-order $g$ and $p$
modes, which have been shown to be the only ones
observable in the GW spectrum [13,15]. We obtain the
eigenfrequency of each mode considering all evolution
times and all simulations as a single dataset and try to find
relations between this frequency and the properties of the
system, namely, the PNS mass and radius ($M_{\text{PNS}}$ and
$R_{\text{PNS}}$), the shock radius ($R_{\text{shock}}$), the total mass inside the shock
($M_{\text{shock}}$), the central density and pressure ($\rho_c$ and
$P_c$), as well as different thermodynamical quantities at different radii. To find the best possible relations we systematically
perform fits of the eigenfrequencies with polynomials of the
form $f = a + bx + cx^2 + dx^3$, with $x = A^\alpha B^\beta C^\gamma D^\delta$
and where $A$, ..., $D$ are all different combinations of the quantities defining the system and $\alpha$, ..., $\delta$ are exponents
ranging in the interval $[-3, 3]$ in steps of 0.5. Universal
relations for each of the modes are built using the best-
fitting combinations together with our intuition on the
physical processes that should determine the frequencies.
The results for the fits and the combination used for $x$ are
presented in Table I. For most of the fits we have considered
$\alpha = 0$, i.e., imposed that the frequency goes to zero as $x$
vanesishes. $d \neq 0$ was considered only in one case since in all
others this coefficient did not improve the fit significantly.

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the associated (gravity) waves depends directly on the for the excitation of these modes and the phase velocity of because buoyancy inside the PNS is responsible (better in AENUS-ALCAR). Although the errors in the fits are (better in COCONUT) or to the different neutrino treatment may be due to the different gravity treatment in either code upper frequency part of the fits. This systematic difference plots), which leads to slightly different relations at the two different codes (indicated with different symbols in the only systematic behavior found is caused by employing the EOS or on the progenitor used for the simulation. The in Table I). This scatter does not depend systematically on the case we observe a scatter of the data around the best fit, which can be characterized by a standard deviation \( \sigma \) (given in Table I). This scatter does not depend systematically on the EOS or on the progenitor used for the simulation. The only systematic behavior found is caused by employing two different codes (indicated with different symbols in the plots), which leads to slightly different relations at the upper frequency part of the fits. This systematic difference may be due to the different gravity treatment in either code (better in CoCoNuT) or to the different neutrino treatment (better in AENUS-ALCAR). Although the errors in the fits are fairly small, the observed systematics indicates that they could be decreased even further by performing more complete numerical simulations, which might be necessary for the purpose of inference (as described below).

The lowest-order \( g \) modes, \( 2g_1 \) and \( 2g_2 \), depend primarily on the surface gravity of the PNS, i.e., \( M_{\text{PNS}}/R_{\text{PNS}}^2 \). The reason is because buoyancy inside the PNS is responsible for the excitation of these modes and the phase velocity of the associated (gravity) waves depends directly on the surface gravity. Measuring the frequency of these modes gives an idea of the compactness of the PNS.

The \( f \) mode and \( p \) modes depend on the square root of the mean density inside the shock, i.e., \( \sqrt{M_{\text{shock}}/R_{\text{shock}}^3} \), as their frequency is primarily determined by the local sound speed and the size of the region containing the modes. The frequency of these modes track primarily the location of the shock. The general behavior matches the dependence seen in previous work [7] and is expected according to general theory of modes in stars [29]. We note that the \( 2g_1 \) mode, improperly labeled as a \( p \) mode in Ref. [15], behaves as a \( g \) mode, since its frequency depends on the PNS surface gravity. Attempts to fit it with a \( p \)-mode dependence result in very poor fits. This is further evidence that, despite its high frequency, this mode is indeed a \( g \) mode.

The \( 2g_3 \) mode presents a significantly different behavior from other \( g \) modes. Our analysis shows that, despite being classified as a \( g \) mode, it cannot be fitted using the PNS surface gravity \( (R^2 = 0.76) \). Instead, in the best fit (see Table I), the dominant behavior is related to the mean

| Mode   | \( x \)               | \( a \)               | \( b \times 10^3 \) | \( c \times 10^6 \) | \( d \times 10^{10} \) | \( R^2 \) | \( \sigma \) |
|--------|-----------------------|-----------------------|---------------------|---------------------|----------------------|---------|-----------|
| \( 2f \) | \( \sqrt{M_{\text{shock}}/R_{\text{shock}}^3} \) | \( \cdots \) | 2.00 ± 0.01 | -8.5 ± 0.1 | \( \cdots \) | 0.967 | 45  |
| \( 2p_1 \) | \( \sqrt{M_{\text{shock}}/R_{\text{shock}}^3} \) | \( \cdots \) | 3.12 ± 0.01 | 9.3 ± 0.2 | \( \cdots \) | 0.991 | 61  |
| \( 2p_2 \) | \( \sqrt{M_{\text{shock}}/R_{\text{shock}}^3} \) | \( \cdots \) | 5.68 ± 0.03 | 14.7 ± 0.7 | \( \cdots \) | 0.983 | 123 |
| \( 2p_3 \) | \( \sqrt{M_{\text{shock}}/R_{\text{shock}}^3} \) | \( \cdots \) | 8.78 ± 0.04 | -4 ± 1 | \( \cdots \) | 0.979 | 142 |
| \( 2g_1 \) | \( M_{\text{PNS}}/R_{\text{PNS}}^2 \) | \( \cdots \) | 18.3 ± 0.05 | -225 ± 2 | \( \cdots \) | 0.982 | 140 |
| \( 2g_2 \) | \( M_{\text{PNS}}/R_{\text{PNS}}^2 \) | \( \cdots \) | 12.4 ± 0.01 | -378 ± 5 | 4.24 ± 0.08 | 0.967 | 76  |
| \( 2g_3 \) | \( \sqrt{M_{\text{shock}}/R_{\text{shock}}^3} p_c/\rho_c^{-5/2} \) | 905 ± 3 | -1.13 ± 0.02 | 2.2 ± 0.5 | \( \cdots \) | 0.925 | 41  |

Figure 2 shows the fits for the universal relations. In each case we observe a scatter of the data around the best fit, which can be characterized by a standard deviation \( \sigma \) (given in Table I). This scatter does not depend systematically on the EOS or on the progenitor used for the simulation. The only systematic behavior found is caused by employing two different codes (indicated with different symbols in the plots), which leads to slightly different relations at the upper frequency part of the fits. This systematic difference may be due to the different gravity treatment in either code (better in CoCoNuT) or to the different neutrino treatment (better in AENUS-ALCAR). Although the errors in the fits are fairly small, the observed systematics indicates that they could be decreased even further by performing more complete numerical simulations, which might be necessary for the purpose of inference (as described below).

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![Figure 2](https://example.com/figure2.png)

**FIG. 2.** Fits of the different modes. The left-hand panel shows the \( f \) mode and the first three \( p \) modes while the right-hand panel shows the first two \( g \) modes. The results from AENUS-ALCAR and CoCoNuT are represented with solid circles and crosses, respectively. Shaded areas indicate \( 2\sigma \) error intervals.
The right-hand panel of Fig. 1 compares those frequencies with the GW spectrogram of the simulation and shows that both agree within the errors of the universal relations. Note that data from the s20 simulation were not included in the models used to build the universal relations, which are all 1D. This is an indication that our relations are valid even when applied to multidimensional models and therefore are truly universal.

The relations presented in this work may eventually allow us to measure the physical properties of a PNS during its first second of life, out of purely GW information. The measurement of the PNS surface gravity gives information about the EOS of nuclear matter at finite temperature, which has not been probed yet in other astrophysical scenarios (let alone at the laboratory). The only similar scenario is the postmerger evolution of binary neutron stars, but the typical frequencies are somewhat higher and thus less accessible with current GW detectors [31]. Measuring \( \frac{M_{\text{PNS}}}{R_{\text{PNS}}} \) also allows us to constrain the PNS radius. The minimum mass of the iron core at the onset of collapse is set by the Chandrasekhar mass [32] for a gas of degenerate electrons, \( M_{\text{Ch}} = 5.83 Y_e^2 \), where \( Y_e \approx 0.46 \) is the electron fraction for iron, which results in a minimum mass of \( 1.2 M_\odot \). The maximum mass depends on finite temperature effects [27]. Stellar evolution models show that it can be as high as \(~2.5 M_\odot\) although for solar-metallicity models it does not exceed \( 2 M_\odot \) [27]. The iron core collapses into the PNS in a typical timescale of a few 100 ms. After this the mass accretion rate drops significantly due to the lower density of the outer layers. Therefore, at about 0.5 s after bounce the mass of the PNS is very close to the mass of the iron core and is in the range \( 1.2 M_\odot – 2.5 M_\odot \). Similar arguments hold for \( M_{\text{shock}} \), which is approximately equal to \( M_{\text{PNS}} \), because of the small amount of mass between the PNS and the shock. This allows us to transform the
universal relations into constraints for the shock and PNS radii. Figure 3 shows an example of how to constraint the PNS radius by measuring the frequency of the dominant $f_2$ mode. For the s20 model the frequency at 0.5 s postbounce is about 1 kHz, which would correspond to a measurement of the PNS radius in the 2σ confidence interval 28–52 km. The PNS radius computed from the actual simulation is 30 km, which falls within this interval. Similarly, a measurement of the $f$-mode frequency of about 300 Hz places a constraint on the shock radius in the $2\sigma$ interval 64–127 km, at 0.5 s postbounce, to be compared with the value of 70 km obtained from the simulation. In an actual CCSN event there could be additional constraints for the mass of the iron core, which would reduce further the errors in the estimation, e.g., constraints from the neutrino luminosity and spectrum, from the observation of the progenitor star, and from the direct observation of the compact remanent, possibly as a pulsar, decades after the explosion.

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