Uniform measures and countably additive measures

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Abstract

Uniform measures are defined as the functionals on the space of bounded uniformly continuous functions that are continuous on bounded uniformly equicontinuous sets. If every cardinal has measure zero then every countably additive measure is a uniform measure. The functionals sequentially continuous on bounded uniformly equicontinuous sets are exactly uniform measures on the separable modification of the underlying uniform space.

1 Introduction

The functionals that we now call uniform measures were originally studied by Berezanskiı̆ [1], Csiszár [2], Fedorova [3] and LeCam [17]. The theory was later developed in several directions by a number of other authors; see the references in [19] and [21].

Uniform measures need not be countably additive, but they have a number of properties that have traditionally been formulated and proved for countably additive measures, or countably additive functionals on function spaces. The main result in this paper, in section 3, is that countably additive measures are uniform measures on a large class of uniform spaces (on all uniform spaces, if every cardinal has measure zero).

Section 4 deals with the functionals that behave like uniform measures on sequences of functions; or, equivalently, like countably additive measures on bounded uniformly equicontinuous sets. In the case of a topological group with its right uniformity, these functionals were defined by Ferri and Neufang [6] and used in their study of topological centres in convolution algebras.

2 Notation

In the whole paper, linear spaces are assumed to be over the field \( \mathbb{R} \) of reals. Uniform spaces are assumed to be Hausdorff. Uniform spaces are described by uniformly continuous pseudometrics ([11], Chap. 15), abbreviated u.c.p.

When \( d \) is a pseudometric on a set \( X \), define

\[
\text{Lip}(d) = \{ f : X \to \mathbb{R} \mid |f(x)| \leq 1 \text{ and } |f(x) - f(x')| \leq d(x, x') \text{ for all } x, x' \in X \}.
\]
Then Lip(d) is compact in the topology of pointwise convergence on X, as a topological subspace of the product space \( \mathbb{R}^X \).

When X is a uniform space, denote by \( U_b(X) \) the space of bounded uniformly continuous functions \( f : X \to \mathbb{R} \) with the norm \( \| f \| = \sup \{ |f(x)| : x \in X \} \). Let \( Coz(X) \) be the set of all cozero sets in X; that is, sets of the form \( \{ x \in X : f(x) \neq 0 \} \) where \( f \in U_b(X) \). Let \( \sigma(Coz(X)) \) be the sigma-algebra of subsets of X generated by \( Coz(X) \).

When d is a pseudometric on a set X, denote by \( \mathcal{O}(d) \) the collection of open sets in the (not necessarily Hausdorff) topology defined by d. Note that if d is a u.c.p. on a uniform space X then \( \mathcal{O}(d) \subseteq Coz(X) \).

Denote by \( M(X) \) the norm dual of \( U_b(X) \), and consider three subspaces of \( M(X) \):

1. \( M_{u}(X) \) is the space of those \( \mu \in M(X) \) that are continuous on Lip(d) for every u.c.p. d on X, where Lip(d) is considered with the topology of pointwise convergence on X. The elements of \( M_{u}(X) \) are called uniform measures on X.

2. \( M_{r}(X) \) is the space of \( \mu \in M(X) \) for which there is a bounded (signed) countably additive measure \( m \) on the sigma-algebra \( \sigma(Coz(X)) \) such that
   \[
   \mu(f) = \int f \, dm \quad \text{for} \quad f \in U_b(X).
   \]

3. \( M_{ur}(X) \) is the space of those \( \mu \in M(X) \) that are sequentially continuous on Lip(d) for each u.c.p. d. That is, \( \lim_n \mu(f_n) = 0 \) whenever d is a u.c.p. on X, \( f_n \in \text{Lip}(d) \) for \( n = 1, 2, \ldots \), and \( \lim_n f_n(x) = 0 \) for each \( x \in X \).

When X is a topological group \( G \) with its right uniformity, \( M_{ur}(X) \) is the space Leb\(^s\)(G) in the notation of [6].

Clearly \( M_{u}(X) \subseteq M_{ur}(X) \) for every uniform space X. By Lebesgue’s dominated convergence theorem ([17], 123C), \( M_{r}(X) \subseteq M_{ur}(X) \) for every X.

For any uniform space X, let \( eX \) be the set X with the weak uniformity induced by all uniformly continuous functions from X to \( \mathbb{R} \) ([14], p. 129). Let \( eX \) be the cardinal reflection \( X_{\aleph_1} \) ([13], p. 52 and 129), also known as the separable modification of X. Thus \( eX \) is a uniform space on the same set as X, and a pseudometric on X is a u.c.p. on \( eX \) if and only if it is a separable u.c.p. on X. Note that \( U_b(X) = U_b(eX) = U_b(eX) \) and \( M(X) = M(eX) = M(eX) \).

Let \( \aleph \) be a cardinal number, and let \( A \) be a set of cardinality \( \aleph \). As in [12], say that \( \aleph \) has measure zero if \( m(A) = 0 \) for every non-negative countably additive measure \( m \) defined on the sigma-algebra of all subsets of \( A \) and such that \( m(\{ a \}) = 0 \) for all \( a \in A \). A related notion, not used in this paper, is that of a nonmeasurable cardinal as defined by Isbell [13], using two-valued measures \( m \) in the preceding definition.

It is not known whether every cardinal has measure zero. The statement that every cardinal has measure zero is consistent with the usual axioms of set theory. A detailed discussion of this and related properties of cardinal numbers can be found in [9] and [14].

Let d be a pseudometric on a set X. A collection \( W \) of nonempty subsets of X is uniformly \( d \)-discrete if there exists \( \varepsilon > 0 \) such that \( d(x, x') \geq \varepsilon \) whenever \( x \in V, x' \in V', V, V' \in W, V \neq V' \).
A set $Y \subseteq X$ is **uniformly $d$-discrete** if the collection of singletons $\{\{y\} \mid y \in Y\}$ is uniformly $d$-discrete.

Let $X$ be a uniform space. A set $Y \subseteq X$ is **uniformly discrete** if there exists a u.c.p. $d$ on $X$ such that $Y$ is uniformly $d$-discrete. Say that $X$ is a (uniform) D-space [18] if the cardinality of every uniformly discrete subset of $X$ has measure zero.

This generalizes the notion of a topological D-space as defined by Granirer [12] and further discussed by Kirk [16] in the context of topological measure theory. A topological space $T$ is a D-space in the sense of [12] if and only if $T$ with its fine uniformity ([13], I.20) is a uniform D-space. If $X$ is a uniform space and $Y \subseteq X$ is uniformly discrete in $X$ then $Y$ is also uniformly discrete in $X$ with its fine uniformity. Therefore, if $X$ is a topological D-space in the sense of [12] then it is also a uniform D-space.

Since the countable infinite cardinal $\aleph_0$ has measure zero, every uniform space $X$ such that $X = eX$ is a D-space. Thus every uniform subspace of a product of separable metric spaces is a D-space. Moreover, the statement that every uniform space is a D-space is consistent with the usual axioms of set theory.

### 3 Measures on uniform D-spaces

The uniform spaces $X$ for which $M_\sigma(X) \subseteq M_d(X)$ were investigated by several authors [11] [3] [4] [10] [17]. The opposite inclusion $M_d(X) \subseteq M_\sigma(X)$ has not attracted as much attention. Theorem 2 in this section characterizes the uniform spaces $X$ for which $M_d(X) \subseteq M_\sigma(X)$.

**Lemma 1** Let $d$ be a pseudometric on a set $X$, and $\varepsilon > 0$. Then there exist sets $W_n$ of nonempty subsets of $X$, $n = 1, 2, \ldots$, such that

1. $\bigcup_{n=1}^{\infty} W_n$ is a cover of $X$;
2. for each $n$, $W_n \subseteq O(d)$;
3. for each $n$, the $d$-diameter of each $V \in W_n$ is at most $\varepsilon$;
4. each $W_n$ is uniformly $d$-discrete.

The lemma is essentially the theorem of A.H. Stone about $\sigma$-discrete covers in metric spaces. For the proof, see the proof of 4.21 in [15].

The next theorem is the main result of this paper. It generalizes a known result about separable measures on completely regular topological spaces — Proposition 3.4 in [16].

**Theorem 2** For any uniform space $X$, the following statements are equivalent:

(i) $X$ is a uniform D-space.

(ii) $M_\sigma(X) \subseteq M_d(X)$. 

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In view of Theorem 2 and the remarks in section 2, the statement that \( M_\sigma(X) \subseteq M_u(X) \)
for every uniform space \( X \) is consistent with the usual axioms of set theory.

**Proof.** This proof is adapted from the author’s unpublished manuscript [18].

To prove that (i) implies (ii), let \( X \) be a D-space. To show that \( M_\sigma(X) \subseteq M_u(X) \), it is enough to show that \( \mu \in M_u(X) \) for every non-negative \( \mu \in M_\sigma(X) \), in view of the Jordan decomposition of countably additive measures ([8], 231F). Take any \( \mu \in M_\sigma(X) \), \( \mu \geq 0 \) and any \( \varepsilon > 0 \). Let \( m \) be the non-negative countably additive measure on \( \sigma(\text{Coz}(X)) \) such that
\[
\mu(f) = \int f \, dm \quad \text{for} \quad f \in U_b(X).
\]

Let \( d \) be a u.c.p. on \( X \), and \( \{f_\alpha\}_{\alpha} \) a net of functions \( f_\alpha \in \text{Lip}(d) \) such that \( \lim_\alpha f_\alpha(x) = 0 \)
for every \( x \in X \). Our goal is to prove that \( \lim_\alpha \mu(f_\alpha) = 0 \).

For the given \( X \), \( d \) and \( \varepsilon \), let \( \mathcal{W}_n \) be as in Lemma 1. If \( V \in \mathcal{W}_n \) for some \( n \) then choose a point \( x_V \in V \). Let \( T_n = \{x_V \mid V \in \mathcal{W}_n\} \) for \( n = 1, 2, \ldots \).

Fix \( n \) for a moment. For each subset \( \mathcal{W}' \subseteq \mathcal{W}_n \) we have \( \bigcup \mathcal{W}' \in \mathcal{O}(d) \subseteq \text{Coz}(X) \). Thus for each \( S \subseteq T_n \) we may define \( \tilde{m}(S) = m(\bigcup \{V \in \mathcal{W}_n \mid x_V \in S\}) \), and \( \tilde{m} \) is a countably additive measure defined on all subsets of \( T_n \). Since the set \( T_n \) is uniformly discrete and \( X \) is a D-space, it follows that the cardinality of \( T_n \) is of measure zero, and there exists a countable set \( S_n \subseteq T_n \) such that
\[
m(\bigcup \{V \in \mathcal{W}_n \mid x_V \in T_n \setminus S_n\}) = \tilde{m}(T_n \setminus S_n) = 0.
\]

Denote \( P = \bigcup_{n=1}^\infty S_n \) and \( Y = \{x \in X \mid d(x,P) \leq \varepsilon\} \). If \( V \in \mathcal{W}_n \) for some \( n \) and \( x_V \in P \) then \( V \subseteq Y \), by property 3 in Lemma 1. Therefore
\[
X \setminus Y \subseteq \bigcup_{n=1}^\infty \{V \in \mathcal{W}_n \mid x_V \in T_n \setminus S_n\}
\]
and \( m(X \setminus Y) = 0 \).

Define \( g_\alpha(x) = \sup_{\beta \geq \alpha} |f_\beta(x)| \) for \( x \in X \). Then \( g_\alpha \in \text{Lip}(d) \), \( g_\alpha \geq g_\beta \) for \( \alpha \leq \beta \), and \( \lim_\alpha g_\alpha(x) = 0 \) for every \( x \in X \).

Since the set \( P \) is countable, there is an increasing sequence of indices \( \alpha(n) \), \( n = 1, 2, \ldots \),
such that \( \lim_n g_{\alpha(n)}(x) = 0 \) for every \( x \in P \), hence \( \lim_n g_{\alpha(n)}(x) \leq \varepsilon \) for every \( x \in Y \). Thus
\[
\lim_\alpha |\mu(f_\alpha)| \leq \lim_\alpha \mu(g_\alpha) \leq \lim_n \mu(g_{\alpha(n)}) = \lim_n \left( \int_Y g_{\alpha(n)} \, dm + \int_{X \setminus Y} g_{\alpha(n)} \, dm \right) \leq \varepsilon \, m(X)
\]
which proves that \( \lim_\alpha \mu(f_\alpha) = 0 \).

To prove that (ii) implies (i), assume that \( X \) is not a D-space. Thus there is a u.c.p. \( d \) on \( X \), a subset \( P \subseteq X \) and a non-negative countably additive measure \( m \) defined on all subsets of \( P \) such that
\begin{itemize}
  \item \( d(x,y) \geq 1 \) for \( x,y \in P \), \( x \neq y \);
  \item \( m(x) = 0 \) for each \( x \in P \);
  \item \( m(P) = 1 \).
\end{itemize}
Define $\mu(f) = \int f \, dm$ for $f \in U_b(X)$. Clearly $\mu \in M_\sigma(X)$.

For any set $S \subseteq P$, define the function $f_S \in \text{Lip}(d)$ by $f_S(x) = \min(1, d(x, S))$ for $x \in X$. Then $f_S(x) = 0$ for $x \in S$ and $f_S(x) = 1$ for $x \in P \setminus S$. Let $F$ be the directed set of all finite subsets of $P$ ordered by inclusion. We have $\lim_{S \in F} f_S(x) = f_P(x)$ for each $x \in X$, $\mu(f_S) = 1$ for every $S \in F$, and $\mu(f_P) = 0$. Thus $\mu \notin M_u(X)$. \hfill $\Box$

The inclusion $M_\sigma(X) \subseteq M_u(eX)$ in the following corollary is Theorem 2.1 in [5].

**Corollary 3** If $X$ is any uniform space then $M_\sigma(X) \subseteq M_u(eX) \subseteq M_u(cX)$.

**Proof.** As is noted above, $eX$ is a D-space for any $X$. Thus $M_\sigma(eX) \subseteq M_u(eX)$ by Theorem 2. From the definitions of $M_\sigma(X)$, $eX$ and $cX$ we get $M_\sigma(X) = M_\sigma(eX)$ and $M_u(eX) \subseteq M_u(cX)$. \hfill $\Box$

Corollary 3 follows also from Theorem 4 in the next section: $M_\sigma(X) \subseteq M_{u\sigma}(X) = M_u(eX)$.

4 Countably uniform measures

In this section we compare the spaces $M_{u\sigma}(X)$ and $M_u(X)$.

**Theorem 4** If $X$ is any uniform space then $M_{u\sigma}(X) = M_u(eX)$.

**Proof.** To prove that $M_{u\sigma}(X) \subseteq M_u(eX)$, note that if a pseudometric $d$ is separable then $\text{Lip}(d)$ with the topology of pointwise convergence is metrizable, and therefore sequential continuity on $\text{Lip}(d)$ implies continuity.

To prove that $M_u(eX) \subseteq M_{u\sigma}(X)$, take any $\mu \in M_u(eX)$. Let $d$ be a u.c.p. on $X$, $f_n \in \text{Lip}(d)$ for $n = 1, 2, \ldots$, and $\lim_n f_n(x) = 0$ for each $x \in X$. Define a pseudometric $\tilde{d}$ on $X$ by

$$\tilde{d}(x, y) = \sup_n |f_n(x) - f_n(y)| \quad \text{for} \quad x, y \in X.$$  

Then $\tilde{d}$ is a separable u.c.p. on $X$, hence a u.c.p. on $eX$, and $f_n \in \text{Lip}(\tilde{d})$ for $n = 1, 2, \ldots$. Therefore $\lim_n \mu(f_n) = 0$.

In view of Theorem 4, spaces $M_{u\sigma}(X)$ have all the properties of general $M_u(X)$ spaces. For example, every $M_u(X)$ is weak* sequentially complete [12], and the positive part $\mu^+$ of every $\mu \in M_u(X)$ is in $M_u(X)$ [11] [3] [17]. Therefore the same is true for $M_{u\sigma}(X)$.

By Theorem 4 if $X = eX$ then $M_u(X) = M_{u\sigma}(X)$ (cf. [6], 2.5(iii)). To see that the equality $M_u(X) = M_{u\sigma}(X)$ does not hold in general, first consider a uniform space $X$ that is not a uniform D-space. Since $M_\sigma(X) \subseteq M_{u\sigma}(X)$, from Theorem 2 we get $M_u(X) \neq M_{u\sigma}(X)$. However, that furnishes an actual counterexample only if there exists a cardinal that is not of measure zero. Next we shall see that, even without assuming the existence of such a cardinal, there is a space $X$ such that $M_u(X) \neq M_{u\sigma}(X)$.

Let $\hat{X}$ denote the completion of a uniform space $X$. Pelant [20] constructed a complete uniform space $X$ for which $eX$ is not complete. For such $X$, there exists an element $x \in e\hat{X} \setminus X$.  

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Every \( f \in U_b(X) = U_b(eX) \) uniquely extends to \( \widehat{f} \in U_b(e\widehat{X}) \). Let \( \delta_x \in M(X) \) be the Dirac measure at \( x \); that is, \( \delta_x(f) = \widehat{f}(x) \) for \( f \in U_b(X) \). Then \( \delta_x \in M_u(eX) \), therefore \( \delta_x \in M_{u\sigma}(X) \) by Theorem 4. On the other hand, \( \delta_x \not\in M_u(X) \), since \( \delta_x \) is a multiplicative functional on \( U_b(X) \) and \( x \not\in \widehat{X} \) (\cite{ref10}, section 6). Thus \( M_u(X) \neq M_{u\sigma}(X) \).

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