Cycles in graphs of fixed girth with large size

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Abstract

Consider a family of graphs having a fixed girth and a large size. We give an optimal lower asymptotic bound on the number of even cycles of any constant length, as the order of the graphs tends to infinity.

1 Introduction

All graphs we consider in this article are simple graphs. We denote the size of $G$ by $e(G)$ and the order of $G$ by $v(G)$. A $j$-path in $G$ is a path of length $j$ in $G$. A $j$-cycle in $G$ is a cycle of length $j$ in $G$, and it is called an even cycle if $j$ is even. The girth of a graph $G$ is the length of the shortest cycles in $G$. For $x \in V(G)$, let $\Gamma^k_G(x) = \Gamma^k(x)$ denote the set of vertices of $G$ having distance exactly $k$ from the vertex $x$.

In the following, the big-$O$ notations $f(n) = O(g(n))$ are understood as $f(n) = O(g(n))$ as $n \to \infty$, where $n$ denotes the order of a graph. The same applies to $\Theta$.

It is easy to see that a graph having large girth cannot have too many edges. The famous Erdős girth conjecture asserts the existence of graphs of any given girth with a size of maximum possible order.

**Conjecture 1** (Erdős girth conjecture). For any positive integer $m$, there exist a constant $c > 0$ depending only on $m$ and a family of graphs $\{G_n\}$ such that $v(G_n) = n$, $e(G_n) \geq cn^{1+1/m}$ and $\text{girth}(G_n) > 2m$.

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Indeed, such size is maximum by the result of Bondy and Simonovits \cite{2}, in which an explicit constant is given. They showed that a graph $G_n$ of order $n$ with $\text{girth}(G_n) > 2m$ has a size less than $100mn^{1+1/m}$.

This conjecture has been proved true for $m = 1, 2, 3, 5$. See \cite{5}, \cite{3}, \cite{1} and \cite{7}. For a general $m$, Sudakov and Verstraëte \cite{6} showed that if such graphs exist, then they contain at least one cycle of any even length between and including $2m + 2$ and $Cn$, for some constant $C > 0$.

For $\ell > m$, by counting the number of $(2\ell - 2m)$-paths, one can show that the number of $2\ell$-cycles in such graphs has an order not greater than $O(n^{2\ell/m})$, provided that $\deg_{G_n}(x) = \Theta(n^{1/m})$ for any vertex $x$ in $G_n$. We will see in Section \ref{section_reduction} that in the asymptotic case, one can assume this without loss of generality. This suggests the definition of \emph{almost regularity} given in Section \ref{section_reduction}.

In this article, we give a lower bound on the number of $2\ell$-cycles when $\ell = O(1)$, and conclude that the number of $2\ell$-cycles is $\Theta(n^{2\ell/m})$. The precise statement is the following.

\textbf{Theorem 2.} For any real number $c > 0$ and integers $M, m$ with $M > m \geq 2$, there exist a constant $\alpha > 0$ and an integer $N$ such that if $\{G_n\}$ is a family of graphs satisfying $v(G_n) = n$, $e(G_n) \geq cn^{1+1/m}$ and $\text{girth}(G_n) > 2m$, then for $n \geq N$ and $m + 1 \leq \ell \leq M$, the number of $2\ell$-cycles in $G_n$ is at least $\alpha n^{2\ell/m}$.

We proceed as follows. In Section 2, we show that by adjusting the threshold $N$ in our theorem, we can further assume that the graphs have some nice properties, namely bipartite and almost regular. In Section 3, we count the number of short even cycles in $G_n$ up to length $4m$. Finally, in Section 4, we extend the argument to longer cycles, completing the proof of the main theorem.

\section{Reduction to a simpler case}

In this section, we show that it suffices to consider only bipartite graphs which are almost regular, defined as follows.

\textbf{Definition 3.} Suppose $\{G_n\}$ is a family of graphs with $v(G_n) = n$ and $\text{girth}(G_n) > 2m$, we say that $\{G_n\}$ is \emph{almost regular} if there exist $c_1, c_2 > 0$ such that $c_1 n^{1/m} \leq \deg(x) \leq c_2 n^{1/m}$ for any vertex $x \in V(G_n)$. 
It is a well-known fact that any graph has a bipartite subgraph with at least half of its edges. It remains to construct subgraphs whose maximum and minimum degree is of order $n^{1/m}$. To achieve this, we repeatedly apply a theorem of Bondy and Simonovits [2], which states that if an $n$-vertex graph $G_n$ has girth larger than $2m$, then $G_n$ has less than $100mn^{1+1/m}$ edges.

First we delete vertices of small degree.

**Lemma 4.** For any real number $c > 0$ and integer $m \geq 2$, there exists a constant $\beta > 0$ such that any bipartite graph $G$ of order $n$, size at least $cn^{1+1/m}$ and girth larger than $2m$ has a subgraph $H$ having at least $\beta n$ vertices, at least $\frac{9c}{10}n^{1+1/m}$ edges and minimum degree at least $\frac{c}{10}n^{1/m}$.

**Proof.** Let $G = H_0$. For $i \geq 1$, inductively define $H_i$ to be the subgraph of $H_{i-1}$ induced by all vertices having degree at least $\frac{c}{10}n^{1/m}$ in $H_{i-1}$. Then for all $i$ we have $e(H_i) \geq e(H_{i+1})$ and

$$e(H_i) \geq e(H_0) - (v(H_0) - v(H_i)) \frac{c}{10}n^{1/m}$$

$$\geq cn^{1+1/m} - n \frac{c}{10}n^{1/m}$$

$$= \frac{9c}{10}n^{1+1/m},$$

and so there exists some $j \geq 0$ such that $H_i = H_j$ for all $i \geq j$.

Set $H = H_j$, which has girth larger than $2m$ and size at least $\frac{9c}{10}n^{1+1/m}$. Then,

$$\frac{9c}{10}n^{1+1/m} < 100m \cdot v(H)^{1+1/m},$$

or $v(H) \geq \beta n$, where

$$\beta = \left( \frac{9c}{1000m} \right)^{m/(m+1)}.$$

**Lemma 5.** For every real number $c > 0$ and integer $m \geq 2$, there exists a constant $\gamma > 0$ such that any bipartite graph $G$ of order $n$, size at least $cn^{1+1/m}$ and girth larger than $2m$ has a subgraph $H$ having at least $\frac{c}{2} n$ vertices, at least $\frac{c}{10} n^{1+1/m}$ edges and maximum degree at most $\gamma n^{1/m}$.

**Proof.** For any $\gamma > 0$, let $S_\gamma$ be the set of vertices of $G$ having degree at least $\gamma n^{1/m}$, and let $T_\gamma$ be the remaining vertices of $G$. We want to find a $\gamma$ so
large that $H$ can be chosen as the subgraph $G[T_\gamma]$ of $G$ induced by $T_\gamma$. It suffices to find $\gamma$ large enough so that $e(G[S_\gamma]) < e(G)/4$, $e(T_\gamma, S_\gamma) < e(G)/2$ and $v(G[T_\gamma]) \geq n/2$.

Since both $G$ and $G[S_\gamma]$ have girth larger than $2m$, we can apply the result of [2] twice to obtain

$$e(G[S_\gamma]) < 100m|S_\gamma|^{1+1/m}$$

$$\leq 100m \left( \frac{2e(G)}{\gamma n^{1/m}} \right)^{1+1/m}$$

$$\leq 100m \left( \frac{2 \cdot 100mn^{1+1/m}}{\gamma n^{1/m}} \right)^{1+1/m}$$

$$= \left( \frac{2}{\gamma} \right)^{1+1/m} (100m)^{2+1/m} n^{1+1/m}.$$  

To satisfy the first condition, we choose $\gamma$ large enough so that $e(G[S_\gamma]) < \frac{e(G)}{4} < \frac{100m}{4} n^{1+1/m}$, or

$$\gamma > 2 \cdot 100m \cdot 4^{m/(m+1)} > 400m.$$  

The second condition can be obtained via its contrapositive. Suppose $e(S_\gamma, T_\gamma) \geq \frac{c}{2} n^{1+1/m}$. Let $G_\gamma$ be the subgraph of $G$ induced by $E(S_\gamma, T_\gamma)$. Then apply Lemma 4 to $G_\gamma$, we get a subgraph $H_\gamma$ of $G_\gamma$ deleting the edges in $E(T_\gamma)$. Then, in $G_\gamma$, every vertex in $S_\gamma \cap V(H_\gamma)$ still has degree at least $\gamma n^{1/m}$ and every vertex in $T_\gamma \cap V(H_\gamma)$ has degree at least $\frac{c}{20} n^{1/m}$. Note that any $m$-path in $G_\gamma$ has at most $[m/2]$ internal vertices in $T_\gamma$, therefore the number of $m$-paths in $G_\gamma$ is at least

$$\frac{1}{2} \nu \left( \frac{c}{20} n^{1/m} \right)^{[m/2]} \left( \gamma n^{1/m} \right)^{[m/2]} \geq \frac{1}{2} \left( \frac{9c}{2000m} \right)^{m/(m+1)} \left( \frac{c\gamma}{20} \right)^{[m/2]} n^2.$$  

But the number of $m$-paths in $G$ cannot be larger than $n^2$, since otherwise there is a pair of vertices being the endpoints of two $m$-paths, contradicting the girth of $G$ is larger than $2m$. Hence,

$$\frac{1}{2} \left( \frac{9c}{2000} \right)^{m/(m+1)} \left( \frac{c\gamma}{20} \right)^{[m/2]} < 1.$$  

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or
\[ \gamma < \frac{20}{c} \left( 2 \left( \frac{2000}{9c} \right)^{m/(m+1)} \right)^{1/[m/2]} < \frac{40}{c} \left( \frac{2000}{9c} \right)^{3/(m+1)}. \]

Therefore, if
\[ \gamma > \frac{40}{c} \left( \frac{2000}{9c} \right)^{3/(m+1)}, \]
then \( e(S_\gamma, T_\gamma) < \frac{c n^{1+1/m}}{2} \leq e(G) \).

Finally, since \( |S_\gamma| \leq \frac{200 m}{\gamma n} \), the third condition is fulfilled by (1). This finishes the proof. \( \square \)

3 Counting short cycles

From now on, we suppose that \( G \) is a bipartite \( n \)-vertex graph having at least \( cn^{1+1/m} \) edges with girth larger than \( 2m \), such that for some constants \( c_1, c_2 > 0 \), there holds \( c_1 n^{1/m} \leq \deg_G(x) \leq c_2 n^{1/m} \) for any vertex \( x \in V(G) \).

In this section, we give a lower bound on the number of the \( 2^{\ell} \)-cycles in \( G \), for each \( m + 1 \leq \ell \leq 2m \).

We first sketch the idea. Let \( x \) be a vertex in \( G \). Suppose we have a path of odd length \( k \) in \( \Gamma_G^m(x) \cup \Gamma_G^{m+1}(x) \) with endpoints \( w_0 \in \Gamma_G^m(x) \) and \( w_k \in \Gamma_G^{m+1}(x) \). For each neighbor \( y \) in \( \Gamma_G^m(x) \) of \( w_k \), the four paths joining \( x \) to \( w_0 \), \( w_0 \) to \( w_k \), \( w_k \) to \( y \), and \( y \) to \( x \) form a closed walk, which contains a cycle, as shown in Figure 1. We show in Section 3.1 that generically these paths are internally disjoint, i.e. the length of the cycle is \( 2m + k + 1 \). Then we count in Section 3.2 the number of such paths and the number of neighbours of \( w_k \). Finally, we obtain the desired lower bound in Section 3.3.

3.1 Internally disjoint closed walk

Note that for distinct \( i, j \leq m + 1 \), we know that \( \Gamma_G^i(x) \cap \Gamma_G^j(x) \) is empty because \( G \) is bipartite and has girth larger than \( 2m \). In particular, the subgraph \( G_x \) of \( G \) induced by the vertices \( \Gamma_G^m(x) \cup \Gamma_G^{m+1}(x) \) is bipartite with bipartition \( \{ \Gamma_G^m(x), \Gamma_G^{m+1}(x) \} \). Hence, any path of odd length in \( G_x \) has one endpoint in \( \Gamma_G^m(x) \) and the other endpoint in \( \Gamma_G^{m+1}(x) \).

For \( 1 \leq i \leq m \) and any vertex \( w \in \Gamma_G^i(x) \), there is a unique \( (x, w) \)-path \( P_w^x \) of length \( i \) in \( G \). Note that for any two vertices \( y_1, y_2 \in \Gamma_G^m(x) \), the intersection of the paths \( P_{y_1}^x \) and \( P_{y_2}^x \) must be a path, of which \( x \) is an
The four paths form a closed walk, which contains a cycle endpoint. The following lemma guarantees that if $w_0 \in \Gamma^m_G(x)$ and $w_k \in \Gamma^{m+1}_G(x)$, there is at most one neighbour $u \in \Gamma^1_G(x)$ so that $P^x_{w_0}$ and $P^x_u$ intersect internally, see Figure 2.

Figure 1: The four paths form a closed walk, which contains a cycle

Figure 2: Other neighbours of $w_k$ give internally disjoint paths

Lemma 6. Suppose two vertices $y_1, y_2 \in \Gamma^m_G(x)$ share a common neighbour $w \in \Gamma^{m+1}_G(x)$, then the paths $P_{y_1}$ and $P_{y_2}$ are internally disjoint.

Proof. Suppose the paths $P^x_{y_1}$ and $P^x_{y_2}$ intersects internally, then their intersection must be a path of length $L \geq 1$, with endpoints $x$ and $v$, for some
Figure 3: A cycle of length at most $2m$ formed

$v \in \Gamma^L_G(x)$. Thus, the union of the paths $P_{y_1 \setminus v}$, $P_{y_2 \setminus v}$ and the edges $(y_2, w), (w, y_1)$ is a cycle of length $2(m - L) + 2 \leq 2m$ in $G$, as in Figure 3, contradiction.

Now, given a path $P = (w_0, w_1, \ldots, w_k)$ of odd length $k \leq 2m - 1$ in $G_x$ with $w_0 \in \Gamma^m_G(x)$ and $w_k \in \Gamma^{m+1}_G(x)$. Note that $V(P) \cap \Gamma^1_G(w_k) = \{w_{k-1}\}$ as $\text{girth}(G_n) > 2m$. Let $y \in \Gamma^1_{G_x}(w_k)$. As shown in Figure 4, the four paths $P_{w_0}$, $P$, $(w_k, y)$ and $P_y$ contain a cycle of length $2m + k + 1$, with at most two exceptions, namely $y = w_{k-1}$ and $y = u$.

Figure 4: Most neighbours of $w_k$ give cycles of length $2m + k + 1$
3.2 Number of paths in $G_x$

Note that in $G_x$, the minimum degree can be as small as 1. Instead of counting the number of paths of a given length in $G_x$, we work with a subgraph of $G_x$ having large minimum degree. We adopt the result from Section 2.

It is easy to see that $v(G_x) \leq n$ and $e(G_x) \geq \left| \Gamma^m_{G}(x) \right| c_1 n^{1/m} \geq (c_1 n^{1/m})^{m+1} = c_1^{m+1} n^{1+1/m}$.

Using Lemma 4, we obtain a bipartite subgraph $H_x$ of $G_x$ having order at least $\left( \frac{9c_1^{m+1}}{1000m} \right)^{1/(1+m)} n$, size at least $\frac{9c_1^{m+1}}{10} n^{1+1/m}$, and $\frac{c_1^{m+1}}{10} n^{1/m} \leq \deg_{H_x}(u) \leq c_2 n^{1/m}$, for any vertex $u \in V(H_x)$, with bipartition $\{A_x, B_x\}$, where $A_x = \Gamma^m_{G}(x) \cap H_x$, and $B_x = \Gamma^{m+1}_{G}(x) \cap H_x$.

**Lemma 7.** Let $k$ be an odd number satisfying $1 \leq k \leq 2m - 1$. The number of $k$-paths in $G_x$ is at least

$$\frac{9c_1^{(m+1)(k+1)}}{10^{k+1} c_2} n^{1+k/m}.$$  

**Proof.** The result follows from

$$|B_x| \geq \frac{e(H_x)}{c_2 n^{1/m}} \geq \frac{9c_1^{m+1}}{10c_2} n$$

and that the number of $k$-paths in $G_x$ is at least

$$|B_x| \left( \frac{c_1^{m+1}}{10} n^{1/m} \right)^k.$$

\[\square\]

3.3 Lower bound on the number of short cycles

The work in the preceding sections allows us to find a lot cycles in $G$. It is clear that a $2\ell$-cycle can be counted by at most $2\ell$ times as each vertex of the cycle can play the role of $x$ once.

We are ready to give a lower bound on the number of short even cycles, up to length $4m$ in $G$.  

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Proposition 8. Let $m$ be a positive integer. Let $G$ be a bipartite $n$-vertex graph having girth larger than $2m$ and $c_1 n^{1/m} \leq \deg_G(v) \leq c_2 n^{1/m}$ for any vertex $v \in V(G)$, for some $c_1, c_2 > 0$. Then for $m+1 \leq \ell \leq 2m$, the number of $2\ell$-cycles in $G$ is at least $\alpha_{\ell} n^{2\ell/m}$, where

$$\alpha_{\ell} = \frac{9}{2\ell c_2} \left( \frac{c_1^{m+1}}{10} \right)^{2\ell-2m+1} > 0.$$ 

Proof. Using Lemma 7 with $k = 2\ell - 2m - 1$ and the observation above, the number of $2\ell$-cycles in $G$ is at least

$$\frac{v(G)}{2\ell} \left( \min_{x \in V(G)} \deg(H_x) \right) \left( \min_{x \in V(G)} \text{number of (}2\ell - 2m - 1\text{)-paths in } H_x \right)$$

$$\geq \frac{n}{2\ell} \left( \frac{c_1^{n+1}}{10} n^{1/m} \right) \left( \frac{9c_1^{(2\ell-2m)(m+1)}}{10^{2\ell-2m}c_2} n^{(2\ell-m-1)/m} \right)$$

$$= \frac{9}{2\ell c_2} \left( \frac{c_1^{m+1}}{10} \right)^{2\ell-2m+1} n^{2\ell/m}.$$

4 Proof of main theorem

To count the number of longer cycles, we observe that $H_x$ has all the nice properties we wanted, namely bipartite, almost regular and large girth, we can apply Proposition 8 to $H_x$ and get many short cycles in $H_x$. From them, we obtain a lot of paths in $H_x$, and each of them corresponds to many longer cycles in $G$ as in Section 3. These longer cycles give many longer paths in $H_x$ and again, each of these paths corresponds to many even longer cycles in $G_x$. Eventually, we have Theorem 2.

For simplicity, we will assume that $m$ is even from now on. For odd $m$, one can proceed similarly.

Changing the parameters in Proposition 8, the number of $2\ell$-cycles in $H_x$ is at least

$$\frac{9}{2\ell c_2} \left( \frac{c_1^{(m+1)^2}}{10^{m+2}} \right)^{2\ell-2m+1} n^{2\ell/m}.$$
for $2\ell \in L_0 := \{3m, 3m + 2, 3m + 4, \ldots, 4m\}$, and so the number of paths of length $2\ell - m \in \{2m, 2m + 2, 2m + 4, \ldots, 3m\}$ in $H_x$ is at least

$$\frac{9}{c_2} \left(\frac{c_1^{(m+1)}2}{10^{m+2}}\right)^{2\ell-2m+1} n^{2\ell/m}.$$ 

Then, for $2\ell \in L_0$, the number of $((2\ell - m) + 2m + 2)$-cycles in $G$ is at least

$$\frac{n}{2\ell + m + 2} \frac{c_1^{2(m+1)}}{100} \frac{9}{c_2} \left(\frac{c_1^{(m+1)2}}{10^{m+2}}\right)^{2\ell-2m+1} n^{2\ell/m}$$

$$= \frac{9}{(2\ell + m + 2)c_2} \frac{c_1^{2(m+1)}}{100} \left(\frac{c_1^{(m+1)2}}{10^{m+2}}\right)^{2\ell-2m+1} n^{(2\ell+m+2)/m},$$

or for $2\ell \in L_1 := \{4m + 2, 4m + 4, \ldots, 5m + 2\}$, the number of $2\ell$-cycles in $G$ is at least $\alpha_\ell n^{2\ell/m}$, where $\alpha_\ell$ is a positive constant depending on $m, c_1, c_2, \ell$ only. Repeating the same argument with the sets $L_j := \{3m + j(m + 2), 3m + j(m + 2) + 2, \ldots, 4m + j(m + 2)\}$, the proof of Theorem 2 is completed.

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