Influence of long-range interactions on the critical behavior of the Ising model

T. Blanchard\textsuperscript{1}, M. Picco\textsuperscript{1} and M. A. Rajabpour\textsuperscript{2,3}

\textsuperscript{1} CNRS, LPTHE, Université Pierre et Marie Curie, UMR 7589 - 4 place Jussieu, 75252 Paris cedex 05, France, EU
\textsuperscript{2} SISSA and INFN, Sezione di Trieste - via Bonomea 265, 34136 Trieste, Italy, EU
\textsuperscript{3} Instituto de Física de São Carlos, Universidade de São Paulo - Caixa Postal 369, 13560-590, São Carlos, SP, Brazil

received 14 December 2012; accepted in final form 13 February 2013
published online 12 March 2013

PACS 64.60.F- – Equilibrium properties near critical points, critical exponents
PACS 75.10.Hk – Classical spin models
PACS 05.10.Cc – Renormalization group methods

Abstract – We study the ferromagnetic Ising model with long-range interactions in two dimensions. We first present results of a Monte Carlo study which shows that the long-range interactions dominate over the short-range ones in the intermediate regime of the interaction range. Based on a renormalization group analysis, we propose a way of computing the influence of the long-range interactions as a dimensional change.

Copyright © EPLA, 2013

Introduction. – In recent years there has been a lot of interest in the statistical physics of classical and quantum systems with long-range interactions, for a review see [1]. The role of quasi-stationary states and ergodicity breaking in long-range interacting systems was investigated in [2] and [3]. In [4] the approaching to equilibrium for long-range quantum systems was examined and there has been a lot of enthusiasm in investigating the entanglement entropy in long-range spin chains [5–7]. Very recently an experiment was conducted on a quantum system with tunable long-range interactions [8].

In the present study we focus on the Ising model which is probably the most studied model in statistical mechanics, especially in the context of critical phenomena. Most of the studies about the Ising model are concentrated around the short-range case which is exactly solvable in one and two dimensions [9]. In three dimensions the problem was perturbatively studied using the $\epsilon$-expansion technique [10] of the renormalization group (RG) combined with the Borel resummation of the perturbation series, see [11] and references therein. Most recently the problem was revisited by using conformal bootstrap technique [12]. Although now there are little unknown facts around short-range Ising model the long-range Ising model is still the subject of many contradicting theoretical and numerical studies. We define the long-range Ising model as

$$\mathcal{H} = - \sum_{\langle ij \rangle} \frac{J_{ij}}{r_{ij}} S_i S_j,$$

where the sum is over all pairs of spins of a $d$-dimensional system and $J > 0$. In [13] the $\epsilon$-expansion technique was applied to the above problem shortly after the introduction of the method. Three regimes were discovered: a) the classical regime $0 < \sigma < d/2$ with mean-field critical exponents; b) the intermediate regime $d/2 < \sigma < 2$, where the exponents are functions of $\sigma$ and c) the short-range regime $\sigma > 2$ where the exponents are the same as in the short-range Ising model. The conjectures around the first regime are already proved in [14] and the results of the third regime are widely accepted. The intermediate regime has been the subject of many controversies in the last forty years. In [13] Fisher et al. obtained the expression for $\eta$ and $\gamma$ the susceptibility exponent up to $\epsilon^2$ for $\sigma < 2$ with $\epsilon = 2\sigma - d$. They observed a discrepancy for both exponents at this order at $\sigma = 2$ between their expression for $\sigma < 2$ and their short-range value for $\sigma > 2$. The case of the exponent $\eta$ is special because it gets no corrections at this order so that it sticks to its classical regime’s value; i.e., $\eta = 2 - \sigma$. Shortly after, Sak [15] argued that there is no such discontinuity for $\eta$, $\gamma$ and $\varphi$ the crossover exponent because if one looks at the long-range interaction as a perturbation of the short-range Ising, $\sigma = 2$, one can discover that the short-range Ising exponents should be extended to $\sigma = 2 - \eta_{SR}$ and so there is no discontinuity in the critical exponents.

Although some other forms of RG also appeared [16] in the last forty years, Sak’s argument has been widely accepted [17]. An especially interesting numerical simulation done by Blöte and Luijtjen [18] completely
ruled out any jump in the value of \( \eta \). The motivation of our work comes from a recent numerical work done by one of us [19] in which we improved the algorithm of Blöte and Luijten. This algorithm uses the fact that for a ferromagnetic model, one can use clusters of spins to improve the speed of the simulations as is done in the Wolff cluster algorithm for short-range ferromagnetic models [20]. The improvement in [19] concerns the construction of the clusters by optimizing the search of connected spins over large regions. With this new algorithm, we can simulate systems up to size 5120 \times 5120 with a typical update time of order one second on an ordinary workstation.

By analyzing much bigger sizes than in previous studies, we concluded that neither Fisher et al.’s procedure nor Sak’s machinery fit with the numerics, in particular in the intermediate regime and close to the boundary with the short-range regime. In the present study, we will concentrate on two aspects of this problem. First, we will compare the long-range behavior with the short-range one in two dimensions. We provide numerical evidences that the long-range interactions dominate for \( \sigma < 2 \). Next, we will propose another way to compute the \( \eta \) exponent. The main idea is to make a correspondence between

\[
A_1 = \int d^d x \left( \frac{1}{2} |\nabla^2 s_b(x)|^2 + \frac{\lambda_0}{4!} |s_b(x)|^4 \right),
\]

with \( \epsilon' = 2 \sigma - d \) and

\[
A_2 = \int d^D x \left( \frac{1}{2} |\nabla s_b(x)|^2 + \frac{\lambda_0}{4!} |s_b(x)|^4 \right),
\]

with \( \epsilon = 4 - D \). The first expression \( A_1 \) is a formal way of writing in real space a model with long-range interactions. The second expression \( A_2 \) is the usual way of expressing a short-range \( \phi^4 \) model for a \( D \)-dimensional theory. For \( \sigma \approx 2 \) and with the condition \( \epsilon = \epsilon' \), the \( \epsilon \)-expansion of both models, \( A_1 \) and \( A_2 \) will be the same apart from a term proportional to \( \delta \sigma = 2 - \sigma \). Thus the computation will be done from the model \( A_2 \) with \( D = 4 + d - 2 \sigma \). The deviation of \( \sigma \) from 2 is replaced by a deformation of the dimension from \( d \) to \( D \).

**Long range vs. short range.** – In [19], it was observed that the behavior of the model with long-range interactions and for \( \sigma < 2 \) is different from what is expected for the short-range model. Then we must worry about the relevance of the long-range interactions compared to the short-range ones. If we start from a short-range model and consider the addition of long-range term \( g \sum_{i,j} S_i S_j / r_{ij}^{d+\sigma} \) as a small perturbation, then a simple dimensional argument predicts the relevance of the perturbation as a function of the dimension of \( g \) [17]. Since for large distances we have \( \langle S_i S_j \rangle \approx r_{ij}^{-d-\eta_{SR}} \), with \( \eta_{SR} = \frac{1}{2} \) in two dimensions, we expect that the dimension of \( g \) is \( 2 - \eta_{SR} - \sigma \). Then the result is that the perturbation is relevant for \( \sigma < 2 - \eta_{SR} \) and irrelevant otherwise. We

Fig. 1: (Color online) \( B(g, L) \) vs. \( L \) for \( \sigma = 1.6 \) (top panel) and \( \sigma = 1.8 \) (bottom panel). The short-range (SR) results are depicted in red and have the largest \( B(g, L) \). The dotted line corresponds to the extrapolated value for the long-range (LR) model (see [19]).

We will now test this argument. We consider the case of the perturbation as a function of \( \sigma \) and \( g \). We will use the magnetic cumulant defined as

\[
B(g, L, K) = \frac{\langle m^2 \rangle_{g,L,K}}{\langle m \rangle^2},
\]

where \( K = \beta J \). For each value of \( g \), we consider the quantity \( B_c(g, L, L') \) which corresponds to the crossing of \( B(g, L, K) \) and \( B(g, L', K) \) as a function of \( K \). By choosing a set of increasing values \( L \) and \( L' \) not too far apart, we can determine for each pair the value of \( K \) which corresponds to the crossing and \( B_c(g, L, L') \) is expected to converge towards a finite limit for \( L \to +\infty \). In the following, we will always consider \( L' = 2L \) and then we will just denote the crossing value by \( B(g, L) \). In [19], it was determined that the corresponding quantity for the long-range interaction model, which can be considered as the limit \( g \to \infty \), converges to a value smaller than the one of the short-range model for \( \sigma \leq 2 \). In fig. 1, we present the measured values of \( B(g, L) \) for \( \sigma = 1.6 \) and \( \sigma = 1.8 \). For the first case, we observe a clear tendency for \( B(g, L) \) to converge towards the same limit as the LR model (which is shown as a dotted line) and this for all the values of \( g \) in the range \( g = 0.01 \) up to \( g = 1 \). For the second
case, the situation is less clear. For the small values of \( g \lesssim 0.1 \), it seems first that \( B(g, L) \) converges towards the model with short-range interactions. For larger values of the perturbation, we just observe that \( B(g, L) \) increases with the size. While it can be assumed that this just corresponds to the flow towards the value for the SR model, one can also invoke the effect of strong finite-size corrections.

In fact, since we are considering a case in which there is both a flow towards either the LR model or the SR model and very strong finite-size effects, it is difficult to know which one is the dominating effect. We then adopt another strategy. We will look in the following to the quantity which one is the dominant effect. We then adopt another and very strong finite-size effects, it is difficult to know corrections.

**Renormalization group approach.** – In this section we propose a new way of doing RG analysis for long-range Ising model. Although our analysis shares some similarities with the work of Yamazaki the final results are more general [16]. We implement our RG analysis around the critical point [21, 22], then we can avoid calculating more complicated integrals. Based on the arguments of the last section one can write the Lagrangian of the long-range Ising model, forgetting about the irrelevant short-range part, with respect to the renormalized coupling and field as

\[
\mathcal{L} = \frac{1}{2} \left| \nabla \sigma \right|^2 s(x)^2 + \frac{\lambda \mu \epsilon}{4!} \left| \sigma \right|^4 (Z - 1) \frac{1}{2} \left| \nabla \sigma \right|^2 s(x)^2 + \frac{\lambda \mu \epsilon}{4!} (Z^2 Z_{\Lambda} - 1) \left| \sigma \right|^4 \cdots ,
\]

where \( \epsilon = 2\sigma - d \), \( s = Z^{-1/2} s_b \) and \( \lambda_{0} = \lambda_{\mu} \epsilon \) with \( s_b \) and \( \lambda_{0} \) as bare parameters. The scale \( \mu \) is introduced because we want to do the expansion around the massless theory [21]. The third and fourth terms are designed to remove divergent contributions in vertex functions. The renormalization conditions are

\[
\Gamma^{(2)}(0) = 0, \quad \Gamma^{(2)}(p, -p) = \frac{1}{2} \frac{\lambda}{2d} \mu \epsilon \left| \sigma \right|^4 \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{1}{\left| q_1 \right|^2 \left| q_2 \right|^2 \left| p - q_1 - q_2 \right|^{d}},
\]

\[
\Gamma^{(4)}(p_1, p_2, p_3, p_4) = \lambda \mu \epsilon \left( Z_{\Lambda} Z_{\Lambda}^{2} - \frac{\lambda}{2} \frac{4!}{(2d)^3} \mu \epsilon \left| \sigma \right|^4 \int \frac{d^d q}{(2\pi)^d} \frac{1}{\left| q \right|^4 \left| p_1 + p_2 - q \right|^{d}} \right).
\]
The integrals are infrared divergent for \( \epsilon' > 0 \) and need to be calculated by analytical continuation from the convergent region. This is the same situation as in the usual \( \epsilon \)-expansion in the short-range Ising model. Although the integrals are complicated, they can be calculated using the formulas in [21], and after using the renormalization conditions we will have

\[
Z = 1 - \frac{\lambda^2}{3\sigma(4\pi)^2} \Gamma^3[\frac{d-\sigma}{2}] \Gamma[\frac{3\sigma}{2} - d + 1],
\]

\[
Z_{\lambda} = 1 - \frac{3\lambda}{2(4\pi)^{d/2}} [\Gamma[\epsilon'/2] \Gamma'[\sigma'/2]].
\]

The very important point is that if we expand \( Z \) with respect to \( \epsilon' \) there will not be any pole and one cannot get sensible contribution to the critical exponent of the field \( s(x) \). However, since the integrals are infrared divergent, the right way to get a sensible perturbation theory is to expand \( Z \) first around \( \sigma = 2 \) and then around \( \epsilon' = 0 \). The situation is very similar to the short-range case; we have an integral which is divergent and would like to control its divergence. If we expand the above equations first around \( \epsilon' \) we actually get a finite term which is apparently wrong. Our choice of order of expansion is not arbitrary and it was actually forced by the divergent integrals. Since we have two parameters dimension \( d \) and \( \sigma \); and they can be changed independently, one can first consider having \( \delta\sigma = 2 - \sigma \) small and then do the perturbation theory with respect to \( \epsilon' \). After expanding \( Z \) and \( Z_{\lambda} \) with respect to \( \delta\sigma \) and then \( \epsilon' \) we get

\[
Z = 1 - \frac{\lambda^2}{12(4\pi)^d} \left[ 1 - \left( \frac{1}{2\epsilon'} + \frac{12\gamma - 13}{8} \right) \delta\sigma + \cdots \right] + O(\lambda^3),
\]

\[
Z_{\lambda} = 1 + \frac{3\lambda}{(4\pi)^{d/2}} \left[ 1 + (1 - \gamma)\delta\sigma + \cdots \right] + O(\lambda^3).
\]

with \( \gamma = 0.5772 \ldots \) as the Euler-Mascheroni constant. Using the Callan-Symanzik equation which states that the derivative of the bare quantities with respect to \( \mu \) is zero, one can get the beta functions as

\[
\beta(\lambda) = \mu \frac{\partial}{\partial\mu} \lambda = -\epsilon' \lambda + \frac{3\lambda^2}{(4\pi)^{d/2}} \left[ 1 + (1 - \gamma) \delta\sigma \right],
\]

\[
\gamma(\lambda) = \mu \frac{\partial}{\partial\mu} \ln Z = \frac{\lambda^2}{6(4\pi)^d} \left[ 1 - \left( \frac{1}{2\epsilon'} + \frac{12\gamma - 13}{8} \right) \delta\sigma \right].
\]

To derive the above formula we first use eq. (15) to get a relation between \( \lambda, \lambda_0 \) and \( \mu \). Using the above beta functions at the critical point where \( \beta(\lambda^*) = 0 \) one can easily get the correction to the mean-field value of the critical exponent \( \delta\eta = \eta - (2 - \sigma) \) as

\[
\delta\eta = \gamma(\lambda^*) = \frac{1}{54} \epsilon'^2 - \frac{1}{108} \epsilon' \delta\sigma - \frac{1}{432} \epsilon'^2 (3 - 4\gamma) \delta\sigma + \cdots.
\]

Based on our prescription it is obvious that in the \( \epsilon' \)-expansion of the \( \eta \) exponent the zeroth-order terms of \( \delta\sigma \) expansion will be the same as the \( \epsilon \)-expansion of the short-range Ising model but with \( \epsilon' = 2\sigma - d \) instead of \( \epsilon = 4 - D \). So in principle close to the \( \sigma = 2 \) we will have

\[
\eta = 2 - \sigma + \frac{1}{54} \epsilon'^2 + \cdots + O(\delta\sigma),
\]

where dots represent the higher-order terms of the \( \epsilon' \)-expansion. Since in our proposal of doing the RG we first expand all the contributions around \( \sigma = 2 \) and then around \( \epsilon' = 0 \) we expect that the dots in the formula (19) are exactly the same as in the short-range Ising model with \( \epsilon' \) instead of \( \epsilon \). The above expansion suggest that for \( \delta\sigma \) small one can argue that the critical exponent of the long-range Ising model in \( d \) dimensions is approximately the same as the critical exponent of \( D = 4 + d - 2\sigma \) short-range Ising model. For the short-range Ising model \( \delta\eta \) is known up to \( \epsilon^n \) for various dimensions [23,24]. The first correction to this value comes from the second and third terms of eq. (18) which are both negative. If the higher-order terms, \( (\delta\sigma)^n \) with \( n \geq 2 \), do not change the sign of the contribution to \( \eta \) one can conclude that the
critical exponent $\eta$ of the short-range Ising model in $D=4+d-2\sigma$ gives an upper bound for the $\delta\eta$ of the long-range Ising model in $d$ dimension. Of course this conjecture needs to be checked by calculating higher loop corrections to the critical exponents. Based on the above arguments we compared in fig. 4 the $\eta$ coming from the numerical calculations for the long-range Ising model in two dimensions with the results coming from the five loop calculation of $D=6-2\sigma$ dimensional short-range Ising model. The results are well comparable in the region 1.75 < $\sigma$ < 2 and as we argued for the smaller values of $\sigma$ the actual values lie below our approximation.

Conclusions. – In this letter we provided further numerical evidences that the long-range interaction have an influence on the critical behavior of the Ising model for $\sigma \leq 2$, in contrast with previous RG studies [13,15]. We proposed a way to compute the influence on $\eta$ of a deviation from $\sigma=2$ based on renormalization group ideas. The main idea is the double expansion with respect to $δ\sigma = 2 - \sigma$ and then $\epsilon' = 2\sigma - d$ in a way that we get a non-trivial contribution to the wave function renormalization. Our analysis shows that close to $\sigma = 2$ one can approximate the $\eta$ exponent of the $d$-dimensional long-range Ising model with the same exponent of $(d = 4 + D - 2\sigma)$-dimensional short-range Ising model. Our results are in excellent agreement with the numerical results [19].

***

We thank A. Gambassi, G. Gori, A. Trombettoni and A. Codello for useful discussions. The work of MAR was supported in part by FAPESP.

REFERENCES

[1] Campa A., Dauxois T. and Ruffo S., Phys. Rep., 480 (2009) 57.
[2] Gabrielli A., Joyce M. and Marcos B., Phys. Rev. Lett., 105 (2010) 210602.
[3] da C. Benetti F. P., Teles T. N., Pakter R. and Levin Y., Phys. Rev. Lett., 108 (2012) 140601.
[4] Kastner M., Phys. Rev. Lett., 106 (2011) 130601.
[5] Barthel T., Dusuel S. and Vidal J., Phys. Rev. Lett., 97 (2006) 220402.
[6] Koffel T., Lewenstein M. and Tagliacozzo L., Phys. Rev. Lett., 109 (2012) 267203.
[7] Cadarso A., Sanz M., Wolf M. M., Cirac J. I. and Perez-Garcia D., arXiv:1209.3898.
[8] Britton J. W. et al., Nature, 484 (2012) 489492.
[9] Baxter R. J., Exactly Solved Models in Statistical Mechanics (Academic Press Inc., London; Harcourt Brace Jovanovich Publishers) 1982.
[10] Fisher M. and Wilson K. G., Phys. Rev. Lett., 28 (1972) 240; Le Guillou J. C. and Zinn-Justin J., Phys. Rev. Lett., 39 (1977) 95.
[11] Guida R. and Zinn-Justin J., J. Phys. A: Math. Gen., 31 (1998) 8103.
[12] El-Showk S., Paulos M. F., Poland D., Rychkov S., Simmons-Duffin D. and Vichi A., Phys. Rev. D, 86 (2012) 025022.
[13] Fisher M. E., Ma S. K. and Nickel B. G., Phys. Rev. Lett., 29 (1972) 917.
[14] Aizenman M. and Fernandez R., Lett. Math. Phys., 16 (1988) 39.
[15] Sak J., Phys. Rev. B, 8 (1973) 281.
[16] Yamazaki Y., Phys. Lett. A, 61 (1977) 207; Physica A, 92 (1978) 446; Nuovo Cimento A, 55 (1980) 59.
[17] Cardy J., Scaling and Renormalization in Statistical Physics, in Cambridge Lect. Notes Phys., 5 (1996) 71.
[18] Luitjen E. and Blöte H. W. J., Phys. Rev. Lett., 89 (2002) 025703.
[19] Picco M., arXiv:1207.1018.
[20] Wolff U., Phys. Rev. Lett., 60 (1988) 1461.
[21] Itzykson C. and Drouffe J.-M., Statistical Field Theory (Cambridge University Press) 1991.
[22] Kleiner H. and Schulte-Frohlinde V., Critical Properties of $\phi^4$-Theories (World Scientific, Singapore) 2001.
[23] Le Guillou J.C. and Zinn-Justin J., J. Phys. (Paris), 48 (1987) 19.
[24] Holovatch Yu., Theor. Math. Phys., 96 (1993) 1099.