Kalb-Ramond field and Dirac-Kähler equation

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Matrix and tensor formulations of the relativistic wave equation providing description both an electromagnetic field (photon) and a massless Kalb-Ramond field with the zero helicity (notoph) are given. It is shown that this equation is a particular case of the Dirac-Kähler system.

I. INTRODUCTION

It was demonstrated by Ogievetski and Polubarinov [1] in 1966 that if we use the tensor-potential \( \Phi_{\mu\nu} \) (\( \Phi_{\mu\nu} = -\Phi_{\nu\mu} \)) to describe massless field then corresponding massless particle has a single polarization state associated with zero helicity. This particle was called by the authors [1] as "notoph".

When in the theory of the photon the potential and the field intensity are represented by the four-vector and the second-rank antisymmetric tensor, respectively, whereas in the theory of the notoph the opposite statement is true, namely the antisymmetric tensor corresponds to the potential, the four-vector corresponds to the field intensity. The last statement needs some explanation. In fact, the notoph field intensity is given by a completely antisymmetric third-rank tensor, however, its components are like the components of the axial four-vector under the Lorentz group transformations.

In the authors’[1] opinion, the notoph is not a scalar particle with zero mass and helicity. Really, if the notoph is converted to a pair, the total angular momentum of the pair equals unity. The authors [1] came to the conclusion also that the notoph could not be associated with well-known interactions.

Kalb and Ramond [2] practically rediscovered in 1974 the notoph on the basis of another approach. The definition “Kalb-Ramond field” is now became established. To describe interactions of a charged particle represented in 4-dimensional space-time as a closed string, they suggested using a tensor-potential \( \Phi_{\mu\nu} \). Unlike Ogievetski and Polubarinov, these authors have considered a massless particle with one degree of freedom described by the tensor-potential \( \Phi_{\mu\nu} \) as a spin-0 massless particle (massless scalar meson). The theory of Kalb – Ramond field (notoph) is discussed in different aspects in many publications (see for example [3-6] and references cited there). In the present paper we propose a new approach to this subject based on the theory of relativistic wave equations (RWE) [7-9].

II. PHOTON AND NOTOPH

It is well known that the photon possessing two polarization states (helicity \( \pm 1 \)) is described by the vector-potential \( A_\mu \) obeying the second-order equation

\[
□A_\mu = \partial_\mu \partial_\nu A_\nu = -j_\mu ,
\]

where \( j_\mu \) is electric current.

Equation (1) is invariant with respect to the gauge transformations

\[
A_\mu \to A_\mu + \partial_\nu \Lambda (x) .
\]

The field intensity, invariant with respect to transformations (2), corresponds to the second-rank antisymmetric tensor

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.
\]

A massless vector field can be also described by the tensor-potential \( \Phi_{\mu\nu} \) (\( \Phi_{\mu\nu} = -\Phi_{\nu\mu} \)) that satisfies equation (1)

\[
□\Phi_{\mu\nu} = \partial_\mu \partial_\lambda \Phi_{\lambda\nu} + \partial_\nu \partial_\lambda \Phi_{\lambda\mu} = -j_{\mu\nu} ,
\]

where \( j_{\mu\nu} \) is the tensor current (\( j_{\mu\nu} = -j_{\nu\mu} , \partial_\mu j_{\mu\nu} = 0 \)). Equation (4) is invariant with respect to the gauge transformations

\[
\Phi_{\mu\nu} \to \Phi_{\mu\nu} + \partial_\mu \Lambda (x) - \partial_\nu \Lambda (x) .
\]
One can choose a gauge under which an interaction-free equation corresponding to equation (11) takes the following form

\[ \Box \Phi_{\mu v} = 0, \]
\[ \partial_{\mu} \Phi_{\mu v} = 0. \] (6)

Equations (6), (7) still remain invariant with respect to the gauge transformations of (5) on the condition of the following restriction of the gauge functions

\[ \Box \Lambda_{\mu} - \partial_{\mu} \partial_{\nu} \Lambda_{\nu} = 0. \] (8)

The third-rank antisymmetric tensor \( F_{\mu v \lambda} \) connected with the tensor-potential \( \Phi_{\mu v} \) by the relation

\[ F_{\mu v \alpha} = \partial_{\mu} \Phi_{\nu \alpha} + \partial_{\alpha} \Phi_{\mu v} + \partial_{v} \Phi_{\alpha \mu}, \] (9)

is considered as a Kalb-Ramond field intensity.

The description of the interaction arising in the case of open strings (the string ends considered as point charged particles of opposite sign) involves, together with the potential \( \Phi_{\mu v} \), an ordinary 4-dimensional vector-potential. In other words, to describe the interaction of open strings in 4-dimensional space-time, one needs two fields: an ordinary electromagnetic field (having two polarization states with the helicity \( \pm 1 \)) and the Kalb-Ramond field (with the helicity 0).

Now let us compare the first-order tensor formulation of the Kalb-Ramond field (notoph) to the formulation of the electromagnetic field (photon). Maxwell’s equations may be written in the following form

\[ \partial_{\nu} F_{\mu \nu} = j_{\mu}, \]
\[ -\partial_{\mu} A_{\nu} + \partial_{\nu} A_{\mu} + F_{\mu \nu} = 0. \] (10)

The equations of the Kalb-Ramond field are of the form [3]

\[ \partial_{\mu} F_{\mu v \alpha} = j_{v \alpha}, \]
\[ \partial_{\mu} \Phi_{v \alpha} + \partial_{\alpha} \Phi_{\mu v} + \partial_{v} \Phi_{\alpha \mu} + F_{\mu v \alpha} = 0. \] (11)

System (10) is invariant with respect to gauge transformations (2) and system (11) is invariant with respect to gauge transformations (5). Here \( F_{\mu \nu} \) and \( F_{\mu v \alpha} \) are field intensities; \( A_{\mu} \) and \( \Phi_{\mu v} \) are potentials of the considered field systems.

In the case of open strings the source \( j_{\mu \nu} \) describes the current created by the string “body”, \( j_{\mu} \) is the current caused by the string ends. They are connected by the following formula [2]:

\[ j_{\alpha} = \partial_{\mu} j_{\mu \alpha}. \] (12)

### III. RELATIVISTIC WAVE EQUATIONS AND MASSLESS VECTOR FIELD

An analysis of systems (10) and (11) may be achieved by writing them in the matrix form of the first-order relativistic wave equation (RWE). In the interaction-free case we have

\[ (\gamma_{\mu} \partial_{\mu} + \gamma_{0}) \psi = 0. \] (13)

Under the Lorentz group a wave function \( \psi \) transforms as some set of irreducible representations of this group forming what is known as linking scheme. The matrix \( \gamma_{0} \) may be proportional to the unity matrix

\[ \gamma_{0} = \kappa I, \] (14)

where \( \kappa \) is parameter with a dimension of mass (\( \hbar = c = 1 \)). In this case equation (13) describes a massive particle. Should the matrix \( \gamma_{0} \) be a singular (projective) one, null matrix including, equation (13) gives description for a
massless particle (see [9]). The matrix $\gamma_4$ plays the main role in determining the spin structure of the field described by equation (13). In the Gel’fand- Yaglom basis [8] it has the form of a direct sum of so called spin blocks

$$\gamma_4 = \oplus \sum_s C^s \times I_{2s+1}. \quad (15)$$

The spin block $C^s$ in (15) is constructed from the elements $C^s_{\tau \tau'}$, where $\tau \sim (l_1, l_2)$ and $\tau' \sim (l'_1, l'_2)$ are linking irreducible Lorentz group representations, i.e. such representations for which $l'_1 = l_1 \pm \frac{1}{2}$, $l'_2 = l_2 \pm \frac{1}{2}$. In such a case, the block $C^s$ includes the elements $C^s_{\tau \tau'}$, corresponding only to the linking representations $\tau$ and $\tau'$ with following restrictions:

$$|l_1 - l_2| \leq S \leq |l_1 + l_2|, \quad |l'_1 - l'_2| \leq S \leq |l'_1 + l'_2|. \quad (16)$$

The representations satisfying the requirements of (16) are regarded as forming the spin block $C^s$. In the massive case, one can state the particle has the spin $S$ when the matrix $\gamma_4$ (15) involves the spin block $C^s$ with nonzero eigenvalues. It is clear that for the description of the spin $S$ the linking scheme must have, at least, two irreducible linking representations forming the block $C^s$. On going to the massless case ($\kappa I \rightarrow 0$, where $|\gamma_0| = 0$), there may be a possibility of the disappearance of the some spin values or of spin projections existing for the massive field. However, the appearance of new spin values, not found for a massive analog, is excluded.

Returning back to tensor systems (10) and (11), we can write them, with no regard to the sources, in the form (13), where $\gamma_4$ and $\gamma_0$ are dimensional matrices $10 \times 10$ ($\gamma_0$ is singular), $\psi$ is 10-component wave function. In case (10) we have $\psi = \left( A_\mu, F_{\mu\nu} \right)$ and $\psi = \left( F_{\mu\nu}, \Phi_{\mu\nu} \right)$ in case (11). The linking schemes for the Lorentz group irreducible representations associated with systems (10) and (11) are of the form

$$(0, 1) - \left( \frac{1}{2}, \frac{1}{2} \right) \sim (1, 0), \quad (17)$$

$$(0, 1) - \left( \frac{1}{2}, \frac{1}{2} \right)' \sim (1, 0), \quad (18)$$

where $\left( \frac{1}{2}, \frac{1}{2} \right)$ is four-vector representation, $[0, 1] \oplus (1, 0)$ is second-rank antisymmetric tensor representation, $\left( \frac{1}{2}, \frac{1}{2} \right)'$ is representation of the tensor $F_{\mu\nu}$ identical to pseudo-vector representation, sign $\sim$ denotes linking. Let us denote the representations entering into (14): $\left( \frac{1}{2}, \frac{1}{2} \right) \sim 1, (0, 1) \sim 2, (1, 0) \sim 3$, and into (15): $\left( \frac{1}{2}, \frac{1}{2} \right)' \sim 1, (0, 1) \sim 2, (1, 0) \sim 3$. Imposing constraints on the elements $C^s_{\tau \tau'}$, following from the relativistic invariance requirement for RWE (13) and from the possibility of its derivation from the corresponding Lagrangian [8], we get the following form of the matrix $\gamma_4$ in the Gel’fand-Yaglom basis:

$$\gamma_4 = \left( \begin{array}{cc} C^0 & C^1 \\ C^1 \times I_3 & \end{array} \right), \quad C^0 = 0, \ C^1 = \frac{1}{\sqrt{2}} \left( \begin{array}{cccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right). \quad (19)$$

For the Kalb-Ramond field we have

$$\gamma_4 = \left( \begin{array}{cc} C^0 & C^1 \\ C^1 \times I_3 & \end{array} \right), \quad C^0 = 0, \ C^1 = \frac{1}{\sqrt{2}} \left( \begin{array}{cccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{array} \right). \quad (20)$$

The matrix $\gamma_0$ in these cases is, respectively, of the forms

$$\gamma_0 = \left( \begin{array}{c} 0_4 \\ I_6 \end{array} \right) \quad (21)$$

and

$$\gamma_0 = \left( \begin{array}{c} I_4 \\ O_6 \end{array} \right). \quad (22)$$

Comparison between the matrices $\gamma_4$ (19) and $\gamma_4$ (20) demonstrates that they have the same spin structure both the Kalb-Ramond field (notoph) and the electromagnetic field (photon). Differences in the matrix structure of $\gamma_0$ for these
cases lead to the differences of possible physical states (helicity values) for the fields under consideration. The matrices (19), (20) and (21), (22) could not be simultaneously transformed to each other by any unitary transformation. It means, as evident, the physical nonequivalence of electromagnetic and Kalb-Ramond fields.

It is important to note, there exist possibility for their simultaneous description. In [4] this problem has been considered on the basis of the modified Bargmann-Wigner formalism for the symmetric second-rank spinor. We propose another method of describing the Maxwell-Kalb-Ramond field.

Let us consider the following linking scheme

\[(\frac{1}{2}, \frac{1}{2}) \quad (0, 1) \quad (1, 0), (\frac{1}{2}, \frac{1}{2})'\]

(23)

According to the Gel’fand-Yaglom method [8] we form zero-mass RWE, where \(\psi\) is the 14-component wave function \(\psi = (A_\mu, F_{\mu\nu\alpha}, \Phi_{\mu\nu})\). The matrices \(\gamma_4\) and \(\gamma_0\) are of the form

\[\gamma_4 = \left(\begin{array}{cc} C^0 & I_3 \\ C^1 \times & \end{array}\right), \quad C^0 = (O_2), \quad C^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}, \]

(24)

\[\gamma_0 = \begin{pmatrix} I_8 & \\ O_6 \end{pmatrix} = \begin{pmatrix} I_2 & I_6 \\ & O_6 \end{pmatrix},\]

(25)

and we use the following numbering of the corresponding representations \((\frac{1}{2}, \frac{1}{2}) \sim 1, \quad (\frac{1}{2}, \frac{1}{2})' \sim 2, \quad (0, 1) \sim 3, \quad (1, 0) \sim 4\) involved in (23). Structure (24) of the matrix \(\gamma_4\) and its spin blocks \(C_0, C_1\) demonstrate that a massless field described by this RWE correspond to spin value 1 in the case of particle with a mass. The eigenvalues \(\pm 1\) of the spin block \(C_1\) are two-fold degenerate. This degeneracy is associated with the description of a photon and the notoph on the basis of single indecomposable (irreducible) RWE. The matrix \(\gamma_0\) (25) “cuts out” a half (three of six) of the states with spin equal 1, leaving for the photon and the notoph three degrees of freedom in total.

The tensor form of RWE (13), (23)–(25) is following

\[\partial_\nu \Phi_{\mu\nu} + A_\mu = 0,\]

\[\partial_\mu \Phi_{\nu\alpha} + \partial_\alpha \Phi_{\mu\nu} + \partial_\nu \Phi_{\alpha\mu} + F_{\mu\nu\alpha} = 0,\]

\[\partial_\nu A_\alpha + \partial_\alpha A_\nu + \partial_\mu F_{\mu\nu\alpha} = 0.\]

(26a–26c)

With regard to the sources we have

\[\partial_\nu \Phi_{\mu\nu} + A_\mu = 0,\]

\[\partial_\mu \Phi_{\nu\alpha} + \partial_\alpha \Phi_{\mu\nu} + \partial_\nu \Phi_{\alpha\mu} + F_{\mu\nu\alpha} = 0,\]

\[\partial_\nu A_\alpha + \partial_\alpha A_\nu + \partial_\mu F_{\mu\nu\alpha} = j_{\nu\alpha}.\]

(27a–27c)

System (27) may be reduced to the correct second-order equations for the potentials \(A_\mu\) and \(\Phi_{\mu\nu}\). Based on equation (27a) and acting on equation (27b) with the operator \(\partial_\mu\) and substituting \(\partial_\mu F_{\mu\nu\alpha}\) into equation (27c), we obtain

\[\Box \Phi_{\nu\alpha} = -j_{\nu\alpha}.\]

(28)

We get

\[\Box A_\mu = -j_\mu,\]

(29)

acting on equation (27c) with \(\partial_\nu\), using definition (12) of open strings \((j_\alpha = \partial_\mu j_{\mu\alpha})\) and based on the antisymmetrical properties of tensors \(F_{\mu\nu\alpha}\) and \(\Phi_{\mu\nu}\).

The field intensity tensor connected with the potential \(A_\mu\) is not appear in manifest form in system (27). It is in equation (27) in the following combination

\[\partial_\nu A_\alpha - \partial_\alpha A_\nu \equiv F_{\nu\alpha}.\]

(30)
With regard to (30), one can rewrite equation (27c) in the form

$$-F_{v\alpha} + \partial_\mu F_{\mu v\alpha} = j_{v\alpha}$$

(31)

In the absence of the sources it looks as follows

$$\partial_\mu F_{\mu v\alpha} = F_{v\alpha}. $$

(32)

Equation (32) means that in the case of open string the Maxwell-Kalb-Ramond field is regarded as a single field realizing the interactions. In the case when only the interaction of closed strings or point charges are considered, components of the Maxwell-Kalb-Ramond field can be described separately. Assuming in (27) that $A_\mu = 0$, we obtain system (11) describing the Kalb-Ramond field. We get the equation

$$\partial_\alpha F_{v\alpha} = j_v,$$

(33)

acting on equation (31) with $\partial_\alpha$ and taking into account equation (12). We obtain Maxwell system (10) by considering equation (33) together with definition (30) and omitting all other equations in (27) associated with the string body.

IV. DIRAC-KÄHLER AND MAXWELL-KALB-RAMOND FIELDS

Let us consider now the Dirac-Kähler field. The name "Dirac-Kähler equation" (DKE) was introduced in [10]. Its vector form was discovered by Darwin [11]. Fundamental mathematical properties of the DKE were established by Kähler [12]. The DKE has been rediscovered independently in different mathematical approaches (see, e.g., ref. [13-14] and references therein). A tensor formulation of DKE is based on the following system

$$\partial_v \varphi_{\mu\nu} + \partial_\mu \varphi + \varphi_\mu = 0,$$

$$\partial_\mu \varphi_{v\lambda} + \partial_\lambda \varphi_{\mu v} + \partial_v \varphi_{\lambda\mu} + \partial_\rho \varphi_{\mu v\lambda} + \varphi_{\mu v\lambda} = 0,$$

$$-\partial_\mu \varphi_v + \partial_v \varphi_\mu + \partial_\lambda \varphi_{\lambda\mu v} + \varphi_{\mu v} = 0,$$

$$\partial_\lambda \varphi_{\mu v} + \varphi = 0,$$

$$\partial_\mu \varphi_{v\lambda\rho} - \partial_v \varphi_{\mu\lambda\rho} + \partial_\lambda \varphi_{\mu v\rho} - \partial_\rho \varphi_{\mu v\lambda} + \varphi_{\mu v\lambda\rho} = 0.$$ (34)

The symbol “$\varphi$” is used here for all Lorentz covariants: $\varphi$ is scalar, $\varphi_\mu$ is four-vector, $\varphi_{\mu\nu\lambda}, \varphi_{\mu\nu\lambda\rho}$ are antisymmetric tensors of the 2, 3 and 4 ranks, respectively. System (34) has three massless analogs. One of them is derived from (34) if one put in the first two equations the terms $\varphi_\mu$ and $\varphi_{\mu\nu\lambda}$ are equal to zero (see for example [15]). The second system is obtained if one omit the terms $\varphi_{\mu\nu}, \varphi$ and $\varphi_{\mu\nu\lambda\rho}$ in third, fourth, and fifth equations. The third possibility unifies both above mentioned massless systems.

One has in the second case

$$\partial_v \varphi_{\mu\nu} + \partial_\mu \varphi + \varphi_\mu = 0,$$ (35a)

$$\partial_\mu \varphi_{v\lambda} + \partial_\lambda \varphi_{\mu v} + \partial_v \varphi_{\lambda\mu} + \partial_\rho \varphi_{\mu v\lambda} + \varphi_{\mu v\lambda} = 0,$$ (35b)

$$-\partial_\mu \varphi_v + \partial_v \varphi_\mu + \partial_\lambda \varphi_{\lambda\mu v} + \varphi_{\mu v} = 0,$$ (35c)

$$\partial_\lambda \varphi_{\mu v} + \varphi = 0,$$ (35d)

$$\partial_\mu \varphi_{v\lambda\rho} - \partial_v \varphi_{\mu\lambda\rho} + \partial_\lambda \varphi_{\mu v\rho} - \partial_\rho \varphi_{\mu v\lambda} + \varphi_{\mu v\lambda\rho} = 0.$$ (35e)

System (35) is invariant with respect to the gauge transformations

$$\varphi_\mu \rightarrow \varphi_\mu + \partial_\mu \Lambda (x),$$

$$\varphi \rightarrow \varphi - \Lambda (x),$$ (36)

(including the case when the sources are present) where the gauge function $\Lambda (x)$ satisfies the equation

$$\Box \Lambda (x) = 0.$$ (37)

Taking into account that the function $\varphi$ in (35) satisfies the analogical equation

$$\Box \varphi = 0.$$ (38)
we can assume \( \varphi = 0 \) without loss of the generality. In this way we get the following system
\[
\begin{align*}
\partial_v \varphi_{\mu v} + \varphi_\mu &= 0, \\
\partial_\mu \varphi_{v \lambda} + \partial_\lambda \varphi_{\mu v} + \partial_v \varphi_{\mu \lambda} + \partial_\rho \varphi_{\mu \nu \rho} + \varphi_{\mu \nu \lambda} &= 0, \\
-\partial_\mu \varphi_v + \partial_\nu \varphi_\mu + \partial_\lambda \varphi_{\lambda \mu v} &= 0, \\
\partial_\mu \varphi_{v \lambda \rho} - \partial_\nu \varphi_{\mu \lambda \rho} + \partial_\lambda \varphi_{\mu v \rho} - \partial_\rho \varphi_{\mu \nu \lambda} &= 0.
\end{align*}
\]
We keep also in mind that for \( \varphi = 0 \) equation (35d) follows from (35a).

According to [3], a massless pseudo-scalar field described by the intensity tensor \( \varphi_{\mu v \lambda \rho} \) represents interaction between the second-order membranes in the space with the dimensionality \( d = 4 \). Based on this treatment, we can consider system (39) (or (35)) as a model for the simultaneous description of the fields realizing interactions of all the string and membrane types in 4-dimensional space.

If we eliminate from (39) the tensor \( \varphi_{\mu v \lambda \rho} \) (equation (39d) in this case follows from (39b)), we get system (26), i.e. the description of the Maxwell-Kalb-Ramond massless field. With regard to the sources, we have system (27).

System (35) is formed on the basis of the following linking scheme
\[
(0, 0) \\
(\frac{1}{2}, \frac{1}{2})
\]
\[
(0, 1) \quad (1, 0),
\]
\[
(\frac{1}{2}, \frac{1}{2})' \\
(0, 0)',
\]
where \( (0, 0)' \) is pseudoscalar representation identical to the tensor \( \varphi_{\mu v \lambda \rho} \). Let us number the representations entering into (40) as follows: \((0, 0) \sim 1, (0, 0)' \sim 2, (\frac{1}{2}, \frac{1}{2})' \sim 3, (\frac{1}{2}, \frac{1}{2}) \sim 4, (0, 1) \sim 5, (1, 0) \sim 6, \) Then in the Gel'fand-Yaglom basis, we obtain using all the standard requirements of a RWE theory the following expressions for matrices \( \gamma_4 \) and \( \gamma_0 \)
\[
\gamma_4 = \begin{pmatrix} C^0 \\ C^1 \times I_3 \end{pmatrix}, \quad C^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad C^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix},
\]
\[
\gamma_0 = \begin{pmatrix} O_2 & I_8 \\ I_8 & O_6 \end{pmatrix} = \begin{pmatrix} O_2 & I_2 & I_6 & O_6 \end{pmatrix}.
\]

Omitting the representations \((0, 0)\) and \((0, 0)'\) in the linking scheme of (40), we get at RWE (13) with linking scheme (23) and matrices \( \gamma_4 \) (24) and \( \gamma_0 \) (25), i.e. the theory describing, as has been noted previously, the Maxwell-Kalb-Ramond (photon + notoph) field realizing interactions of the open strings (first-order membranes) in the space with the dimensionality \( d = 4 \).

Therefore, this theory is a particular case of the Dirac-Kähler system.

V. CONCLUSION

So, we see that both cases of massless vector field (with helicity values \( \pm 1 \) and with zero value) must be considered on equal grounds from the point of view of RWE in the Gel'fand-Yaglom approach. In fact, the description of the photon-notoph system is the direct consequence of its consideration in the above mentioned theory. It is also evident that the Kalb-Ramond field equations and the photon-notoph equations are particular massless cases of the Dirac-Kähler equation. It opens new possibilities for applications of the Dirac-Kahler field in the string theory, because the
Kalb-Ramond field appears in the effective low-energy field theory derived from relativistic strings (see for example [16]).

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