On the existence of representations of finitely presented groups in compact Lie groups

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Abstract

Given a finite, connected 2-complex $X$ such that $b_2(X) \leq 1$ we establish two existence results for representations of the fundamental group of $X$ into compact connected Lie groups $G$, with prescribed values on certain loops. If $b_2(X) = 1$ we assume $G = \text{SO}(3)$ and that the cup product on $H^1(X; \mathbb{Q})$ is non-zero.

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1 Introduction

In this paper we will prove two existence results for representations of finitely presented groups into compact, connected Lie groups. The results are most elegantly stated when the group is realized as the fundamental group of a finite, connected 2-dimensional CW complex $X$, as the conditions depend only on the cohomology of $X$. This viewpoint also reveals the relations to gauge theory on 3-manifolds, in which this work has its origin.

We now state our main results. Throughout the paper we use integral coefficients for (co)homology unless otherwise specified. The $j$’th Betti number will be denoted $b_j$.

Theorem 1.1 Let $b_2(X) = 0$, and suppose $\gamma_1, \ldots, \gamma_r \in \pi_1(X)$ map to linearly independent elements of $H_1(X; \mathbb{Q})$, where $r \geq 1$. Then for any compact, connected Lie group $G$ and any $t_1, \ldots, t_r \in G$ there exists a homomorphism $\phi: \pi_1(X) \to G$ such that $\phi(\gamma_i) = t_i$ for $i = 1, \ldots, r$.

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A reformulation of the theorem in terms of finitely presented groups is given in Theorem 2.1.

Since the group homology \( H_2(\pi_1(X)) = 0 \), a theorem of Stallings [19, Theorem 7.4] says that the elements \( \gamma_1, \ldots, \gamma_r \) form a basis for a free subgroup of \( \pi_1(X) \). This also follows from Theorem 1.1, because \( SO(3) \) contains a free subgroup \( F_2 \) of rank 2 [21, Theorem 2.1] and \( F_2 \) contains free subgroups of any finite rank.

We now describe our second existence theorem. Suppose \( b_2(X) = 1 \) and let \( \sigma \) be a generator of \( H^2(X) \approx \mathbb{Z} \). Then \( \mu : H^1(X) \times H^1(X) \to \mathbb{Z}, \ (\alpha, \beta) \mapsto \langle \alpha \cup \beta, \sigma \rangle \) is bilinear and skew-symmetric, so it defines an element

\[
\mu \in \Lambda^2(H_1(X)/T),
\]

where \( T \subset H_1(X) \) is the torsion subgroup.

The image of an element \( \gamma \in \pi_1(X) \) in \( H_1(X)/T \) will be denoted \( \bar{\gamma} \).

**Theorem 1.2** Let \( b_2(X) = 1 \), and let \( \varpi \in H^2(X;\mathbb{Z}/2) \) with \( \langle \varpi, \sigma \rangle \neq 0 \). Suppose \( \gamma_1, \ldots, \gamma_{r-2} \in \pi_1(X) \) satisfy

\[
\mu \wedge \bar{\gamma}_1 \wedge \cdots \wedge \bar{\gamma}_{r-2} \neq 0
\]

in the exterior product \( \Lambda^r(H_1(X)/T) \), where \( r \geq 2 \). Then there exists a homomorphism \( \phi : \pi_1(X) \to SO(3) \) such that \( \phi(\gamma_i) = 1 \) for \( i = 1, \ldots, r-2 \) and \( \omega_2(\phi) = \varpi \). The image of such a homomorphism \( \phi \) is not contained in a maximal torus in \( SO(3) \).

The Stiefel-Whitney class \( \omega_2(\phi) \) is defined in Section 5. Note that the Hurewicz homomorphism \( \pi_2(X) \to H_2(X) \) is zero, because \( \mu \neq 0 \). Therefore, \( H_2(\pi_1(X)) \approx \mathbb{Z} \).

Theorem 1.2 will be deduced from the more explicit Theorem 4.1, which concerns representations of free groups in \( SU(2) \).

We now describe some relations of this paper to 3-manifold topology. Let \( Y \) be a compact, connected, oriented 3-manifold with non-empty boundary. By [6, Prop. 4.2.7 and 4.2.13] \( Y \) has a handle decomposition with no 3-handles (and only one 0-handle), and with handles attached in order of increasing index. Hence, \( Y \) is homotopy equivalent to a 2-complex \( X \), so Theorems 1.1 and 1.2 apply with \( Y \) in place of \( X \). (In fact, the author came across these theorems while studying moduli spaces of Bogomolny monopoles over such manifolds \( Y \).)
Now suppose $b_2(Y) = 1$. Then the boundary $\partial Y$ can have at most two components. The form $\mu_0$ can be thought of as a triple cup product, because if $\sigma' \in H^1(Y, \partial Y)$ is the Poincaré dual of $\sigma$ (identifying the (co)homology of $X$ and $Y$) and $[Y] \in H_3(Y, \partial Y)$ is the fundamental class then

$$\langle \alpha \cup \beta, \sigma \rangle = \langle \alpha \cup \beta \cup \sigma', [Y] \rangle.$$  

It is not hard to see that if $\alpha$ lies in the image of $H^1(Y, \partial Y) \to H^1(Y)$ then $\iota_\alpha(\mu) = 0$, where $\iota_\alpha$ denotes contraction with $\alpha$. This implies that $\mu = 0$ if $\partial Y$ is connected and $b_1(Y) < 3$.

As an example, let $V$ be an oriented integral homology 3-sphere and $L$ an oriented link in $V$ with components $L_1, L_2$. Let $Y \subset V$ be the complement of an open tubular neighbourhood of $L$. Then $b_2(Y) = 1$, and $\mu$ is easily computed using the formula (7) in Section 6. Namely, if $a, b$ is any basis for $H_1(Y) = \mathbb{Z}^2$ then

$$\mu = \pm \text{lk}(L_1, L_2) a \wedge b,$$

where $\text{lk}$ denotes the linking number. Thus, if the linking number is non-zero then Theorem 1.2 asserts the existence of a flat connection in the non-trivial $\text{SO}(3)$ bundle over $Y$. This instance of the theorem can also be deduced from a result of Harper–Saveliev [10].

It clearly suffices to prove Theorems 1.1 and 1.2 for $r = b_1(X)$. Whereas the proof of Theorem 1.1 consists in computing the degree of a certain diffeomorphism of $G^n$, where $n$ is the first Betti number of the 1-skeleton of $X$, the main point in the proof of Theorem 1.2 is to express the quantity

$$|T| \det(\mu \wedge \bar{\gamma}_1 \wedge \cdots \wedge \bar{\gamma}_{r-2})$$

as an intersection number in the space $Q^*_n$ of conjugacy classes of irreducible representations $F_n \to \text{SU}(2)$, where $F_n$ denotes the free group on $n$ generators. This is reminiscent of the definition of Casson’s invariant as an intersection number [1]. There are also several examples of invariants of knots or links that can be expressed as intersection numbers, see for instance [15, 4, 9].

The proof of Theorem 1.1 is given in Section 2. An essential ingredient here is a result of Gerstenhaber-Rothaus [5] which provides a formula for the degree of the diffeomorphism of $G^n$ defined by an $n$-tuple of elements of $F_n$. The remaining sections of the paper are occupied with the proof of Theorem 1.2. In Section 3 a collection of submanifolds of $Q^*_n$ is exhibited that represents a basis for the homology group $H_{3n-6}(Q^*_n; \mathbb{Q})$. In Section 4 this is used in combination with the degree formula to compute an intersection number, thereby proving the existence of certain representations $F_n \to \text{SU}(2)$. Section 5 describes the second Stiefel-Whitney class.
of a representation $\pi_1(X) \to \text{SO}(3)$ in terms of the cellular homology of $X$. Section 6 provides a formula for the 2-form $\mu$ when the generator $\sigma$ is represented by a map from a closed surface into $X$. In Section 7 these ingredients are brought together to prove Theorem 1.2. In the appendix a linearization map is defined on the commutator subgroup of any group whose second rational homology group vanishes; this may shed some light on the formula for $\mu$.

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2 Representations and degrees

Throughout this section $G$ will denote a non-trivial compact connected Lie group of rank $m$.

Let $F_n$ be the free group generated by the symbols $y_1, \ldots, y_n$. We identify the abelianization of $F_n$ with $\mathbb{Z}^n$. The image of an element $w \in F_n$ in $\mathbb{Z}^n$ will be denoted $\bar{w}$. In particular, $\bar{y}_i$ is the $i$’th element of the standard basis for $\mathbb{Z}^n$. Each element $w$ of $F_n$ defines a map $w_G : G^n \to G$.

Namely, $w_G(g_1, \ldots, g_n)$ is the image of $w$ under the unique group homomorphism $F_n \to G$ that maps the generator $y_i$ to $g_i$.

The following proposition will be essential to the proofs of both Theorem 1.1 and Theorem 1.2.

**Proposition 2.1 (Gerstenhaber-Rothaus [5])** If $w_1, \ldots, w_n \in F_n$ then the induced map

$$f := (w_1)_G \times \cdots \times (w_n)_G : G^n \to G^n$$

has degree $\deg(f) = (\det(\bar{w}_1, \ldots, \bar{w}_n))^m$. □

The case $n = 1$ is a classical theorem of Hopf [12] (see also [7, p.174]). The general case can be proved by combining Hopf’s theorem with the fact (see [7, p. 169]) that the two maps $\mu, \mu' : G \times G \to G$ given by

$$\mu(g, h) = gh, \quad \mu'(g, h) = hg$$

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induce the same map in rational cohomology. Alternatively – and this is the approach taken in [5] – one can compute the effect of $f$ on the top cohomology of $G^n$ by means of another theorem of Hopf [13] which says that the rational cohomology of $G$ is an exterior algebra generated by homogeneous elements of odd degree.

**Theorem 2.1** Let $H$ be a group with generators $y_1, \ldots, y_n$ and relations $w_1, \ldots, w_s$, where $\tilde{w}_1, \ldots, \tilde{w}_s$ are linearly independent. Let $A := H/[H, H]$ be the abelianization of $H$. If $h_1, \ldots, h_r \in H$ map to linearly independent elements of $A \otimes \mathbb{Q}$ then for any $t_1, \ldots, t_r \in G$ there exists a homomorphism $\phi : H \to G$ such that $\phi(h_i) = t_i$ for $i = 1, \ldots, r$.

**Proof.** We may assume $r = \text{rank}(A)$. Choose a lift $z_i \in F_n$ of $h_i$ for each $i$ and let $f : G^n \to G^n$ be the map defined by the $n$ elements $w_1, \ldots, w_s, z_1, \ldots, z_r \in F_n$ as in Proposition 2.1. Then the set of homomorphisms $\phi$ as in the theorem can be identified with $f^{-1}(1, \ldots, 1, t_1, \ldots, t_r)$. Thus, it suffices to show that $f$ has non-zero degree, since this implies that $f$ is surjective.

Let $N$ be the kernel of the projection $\pi : F_n \to H$, and let $K$ be the kernel of the induced homomorphism $\tau : \mathbb{Z}^n \to A$. Then we have a commutative diagram

$$
\begin{array}{cccccc}
1 & \to & N & \to & F_n & \to & H & \to & 1 \\
& \sigma' \downarrow & \sigma \downarrow & \sigma'' \downarrow & \\
0 & \to & K & \to & \mathbb{Z}^n & \tau & \to & 0
\end{array}
$$

where $\sigma, \sigma''$ are the abelianization maps. Because $\pi$ maps the commutator subgroup of $F_n$ onto the commutator subgroup of $H$, the map $\sigma'$ is surjective. But this just means that $K$ is generated by $\tilde{w}_1, \ldots, \tilde{w}_s$, hence these elements form a basis for $K$.

By the elementary divisors theorem [14] there is a basis $b_1, \ldots, b_n$ for $\mathbb{Z}^n$ and integers $p_1, \ldots, p_s$ such that $p_1 b_1, \ldots, p_s b_s$ is a basis for $K$. This implies that $b_{s+1}, \ldots, b_n$ map to a basis for $A/T$, where $T$ is the torsion subgroup of $A$, and

$$
\tilde{w}_1 \wedge \cdots \wedge \tilde{w}_s = \pm p_1 \cdots p_s b_1 \wedge \cdots \wedge b_n = \pm |T| b_1 \wedge \cdots \wedge b_n.
$$
Therefore, by Proposition 2.1 we have
\[
\deg(f) = (\det(\bar{w}_1, \ldots, \bar{w}_s, \bar{z}_1, \ldots, \bar{z}_r))^m \\
= \pm(|T| \det(b_1, \ldots, b_s, \bar{z}_1, \ldots, \bar{z}_r))^m \\
= \pm(|T| \det(\bar{h}_1, \ldots, \bar{h}_r))^m \\
\neq 0,
\]
where \(h\) denotes the image in \(A/T\) of an element \(h \in H\).

**Proof of Theorem 1.1:** Let

\[
C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0
\]
be the cellular chain complex of \(X\), so that \(C_k = H_k(X^k, X^{k-1})\), where \(X^k\) is the \(k\)-skeleton of \(X\). Then \(X^1\) is homotopy equivalent to a wedge sum \(\vee_nS^1\) of \(n\) circles for some \(n \geq 0\). Let \(x_0 \in X^1\) and \(d_0 \in S^1\) be base-points and fix a basis \(y_1, \ldots, y_n\) for the free group \(\pi_1(X^1, x_0)\).

Let the 2-cells in \(X\) be numbered from 1 to \(s\). For \(i = 1, \ldots, s\) choose a base-point preserving map \(\ell_i : S^1 \to X^1\) that is homotopic to the attaching map of the \(i\)'th 2-cell. Let \(w_i \in \pi_1(X^1, x_0) = F_n\) be the class represented by \(\ell_i\). Then \(H := \pi_1(X, x_0)\) has the presentation

\[
H = \langle y_1, \ldots, y_n \mid w_1, \ldots, w_s \rangle.
\]
Moreover, if \(a_i \in C_2\) is the generator corresponding to the \(i\)'th 2-cell then

\[
\partial a_i = \bar{w}_i \in \mathbb{Z}^n = H_1(X^1),
\]
where we identify \(H_1(X^1)\) with its image in \(H_1(X^1, X^0)\). Thus, the assumption \(b_2(X) = 0\) means precisely that \(\bar{w}_1, \ldots, \bar{w}_s\) are linearly independent, so the theorem follows from Theorem 2.1.

**3 Homology and intersection numbers**

This section is concerned with the homology of certain quotient spaces \(Q^*_n\) associated to the Lie group \(\text{Sp}(1)\) of unit quaternions. (Of course, \(\text{Sp}(1) \approx \text{SU}(2)\).)

We begin by explaining our orientation conventions. If \(W\) is a smooth oriented manifold and \(Z \subset W\) an oriented submanifold then the normal bundle \(NZ\) will be oriented so that for any Euclidean metric in the tangent bundle \(TW\) the isomorphism

\[
TW|_Z = NZ \oplus TZ
\]
preserves orientations. If $V$ is another smooth oriented manifold and $f : V \to W$ is transverse to $Z$ then the submanifold $f^{-1}Z$ will be oriented such that the isomorphism $N(f^{-1}Z) \to f^*(NZ)$ preserves orientations. Given a smooth fibre bundle $\pi : P \to V$ where $P$ is oriented as manifold, the fibres $F$ of $\pi$ will be oriented such that for any Euclidean metric in $TP$ the isomorphism

$$TP|_F = TF \oplus \pi^*(TV)|_F$$

preserves orientations. The latter convention prevents some annoying signs.

We shall need an extension of the notion of intersection number defined in [8]. Let $V, W, Z$ be as above. Suppose $Z$ is closed as a subset of $W$ and that

$$\dim V + \dim Z = \dim W.$$ 

For any smooth map $f : V \to W$ such that $f^{-1}Z$ is compact we define the intersection number $I(f, Z)$ as follows. Let $g : V \to W$ be any smooth map transverse to $Z$ such that $g$ is compactly homotopic to $f$, by which we mean that $g$ is homotopic to $f$ relative to the complement of a compact subset of $V$. Then we set

$$I(f, Z) := \#g^{-1}Z,$$

where $\#$ denotes the number of points counted with sign. If $V$ is in fact a submanifold of $W$ and $f$ is the inclusion map then we define the intersection number $V \cdot Z := I(f, Z)$.

Throughout this section we will use the notation

$$G := \text{Sp}(1), \quad G' := \text{Sp}(1)/\pm 1 = \text{SO}(3). \quad (2)$$

For any integer $n \geq 2$ we define a right $G'$–space $P_n$ as follows. Let $P_n := G^n$ as smooth, oriented manifold, and let $g \in G$ act on $P_n$ by conjugation with $g^{-1}$ in each factor. Since $-1 \in G$ acts trivially, this right action of $G$ descends to a right action of $G'$ on $P_n$. Let $P_n^*$ be the open subset of $P_n$ consisting of those points in $P_n$ that have trivial stabilizer in $G'$. Since $P_n$ is Hausdorff and $G'$ is compact, the quotient space

$$Q_n := P_n/G'$$

is Hausdorff. Let $Q_n^*$ be the image of $P_n^*$ in $Q_n$, which is an open subset of $Q_n$. Then $Q_n^*$ has a canonical smooth structure such that $\pi : P_n^* \to Q_n^*$
is a principal $G'$-bundle. It is easy to see that $Q_2^*$ is diffeomorphic to $\mathbb{R}^3$. Each fibre of $\pi$ inherits an orientation from $G'$, and since $G'$ is connected, the base manifold $Q^*_n$ is orientable. Let $Q^*_n$ have the orientation compatible with the orientations of the total space and fibres of $\pi$ as stipulated above.

**Proposition 3.1** $b_{3n-6}(Q^*_n) = n(n-1)/2$.

**Proof.** In this proof we use real coefficients for (co)homology. The proposition clearly holds for $n = 2$, so let $n \geq 3$. Set

$$R_n := P_n \setminus P^*_n,$$

so that $R_n$ consists of those $n$-tuples of elements from $G$ such that all elements are contained in the same maximal torus. We will show that

$$b_{3n-6}(Q^*_n) = b_{3n-3}(P^*_n) = b_2(R_n) = n(n-1)/2.$$

The last equality is a special case of results by Baird [2], but we include a direct proof here for completeness. Identifying $S^2$ with the unit sphere in the space of pure quaternions (ie quaternions with zero real part) we have a surjective map

$$\chi_n : S^2 \times T^n \to R_n, \quad (x, z) \mapsto (\text{Re}(z_j) + \text{Im}(z_j) \cdot x)_{j=1,\ldots,n},$$

where $T^n := U(1)^n$ and $z = (z_1, \ldots, z_n)$. Clearly, $\chi_n$ factors through a map $\tilde{R}_n \to R_n$, where

$$\tilde{R}_n := (S^2 \times T^n)/(x, z) \sim (-x, z^{-1}),$$

which is a smooth manifold. For each $\epsilon \in \{-1, 1\}^n$ let $B_\epsilon \approx \mathbb{R}P^2$ be the image of $S^2 \times \{\epsilon\} \subset S^2 \times T^n$ in $R^n$. Then $R_n$ is obtained from $\tilde{R}_n$ by collapsing each $B_\epsilon$ to a point $x_\epsilon$, where $x_\epsilon \neq x_{\epsilon'}$ if $\epsilon \neq \epsilon'$. Let $N$ be a closed tubular neighbourhood of $B := \cup_\epsilon B_\epsilon$ in $\tilde{R}_n$ and set $Z := \tilde{R}_n \setminus \text{int}(N)$. Because each $B_\epsilon$ is a rational homology ball it follows from the Mayer-Vietoris sequences for $(N, Z)$ and the image of $(N, Z)$ in $R_n$ that the projection $\tilde{R}_n \to R_n$ induces an isomorphism

$$H_k(\tilde{R}_n) \cong H_k(R_n)$$

for each $k$. Now, $H^*(\tilde{R}_n)$ is isomorphic to the 1-eigenspace of the endomorphism of $H^*(S^2 \times T^n)$ induced by the involution $(x, z) \mapsto (-x, z^{-1})$. This yields in particular

$$b_2(R_n) = n(n-1)/2.$$
Using the fact that $\chi_1 : S^2 \times S^1 \to R_1 = S^3$ has degree 2, one finds that the inclusion $R_n \to P_n$ induces a surjection $H_3(R_n) \to H_3(P_n)$. Since $H_2(P_n) = 0$ the homology sequence of the pair $(P_n, R_n)$ shows that the connecting homomorphism

$$H_3(P_n, R_n) \xrightarrow{\sim} H_2(R_n)$$

is an isomorphism.

Now observe that $P_n$ and $R_n$ are real algebraic subsets of $\mathbb{R}^{4n}$. This is obvious for $P_n$. To see that it holds for $R_n$, note that

$$R_n = \{(g_1, \ldots, g_n) \in P_n \mid g_p g_q = g_q g_p \text{ for } 1 \leq p < q \leq n\}.$$ 

Therefore, by [11] both $P_n$ and $R_n$ admit triangulations. (In the case of $P_n$ this is of course elementary.) By [18] Cor. 6.1.11 and Thm. 6.2.17] we have a Poincaré duality isomorphism

$$H^3(P_n, R_n) \approx H_{3n-3}(P_n^*) .$$

We now apply the homology spectral sequence of the fibration $P_n^* \to Q_n^*$.

Since $P_n$ is simply-connected and

$$\dim P_n - \dim(S^2 \times T^n) = 2n - 2 \geq 4,$$

one can show by a transversality argument that $P_n^* = P_n \setminus \chi_n(S^2 \times T^n)$ is simply-connected. Because the fibre $G'$ of the bundle $P_n^* \to Q_n^*$ is path-connected, we conclude that $Q_n^*$ is simply-connected. Hence the fibration is orientable and the $E^2$-page of the spectral sequence is

$$E^2_{p,q} \approx H_{p}(Q^*_n; H_q(G')) .$$

Since $Q_n^*$ is a non-compact $(3n - 3)$-manifold, we have $H_{3n-3}(Q_n^*) = 0$. Therefore,

$$H_{3n-3}(P_n^*) \approx E^\infty_{3n-6,3} \approx E^2_{3n-6,3} \approx H_{3n-6}(Q_n^*)$$

and the proposition is proved. \qed

For each pair $p, q$ of integers satisfying $1 \leq p < q \leq n$ we have a $G'$-equivariant embedding

$$P_2 \to P_n, \quad (g, h) \mapsto (1, \ldots, 1, g, 1, \ldots, 1, h, 1, \ldots, 1),$$

where $g$ and $h$ appear in the $p$'th and $q$'th place, resp. This map induces a topological embedding $I_{p,q} : Q_2 \to Q_n$ that restricts to a smooth embedding
$Q^*_2 \to Q^*_n$ whose image we denote by $W_{p,q}$. Clearly, $W_{p,q} = J_{p,q}(Q_2) \cap Q^*_n$ is a closed subset of $Q^*_n$. Letting $i, j$ denote the usual anti-commuting quaternions for the time being, the composition of the map

$$J_{p,q} : P_{n-2} \to P^*_n,$$

$$(g_1, \ldots, g_{n-2}) \mapsto (g_1, \ldots, g_{p-1}, i, g_p, \ldots, g_{q-2}, j, g_{q-1}, \ldots, g_{n-2}).$$

with the projection $P^*_n \to Q^*_n$ is a smooth embedding $P_{n-2} \to Q^*_n$ whose image we denote by $V^\prime_{p,q}$. Let $W_{p,q}$ and $V^\prime_{p,q}$ have the orientations inherited from $Q^*_2$ and $P^*_{n-2}$, resp.

For each positive integer $m$ let $G$ act on $G^m$ by conjugation in each factor. This (left) action descends to an action of $G'$ on $G^m$ and we have an associated bundle of Lie groups

$$E_{m,n} := P^*_n \times_{G'} G^m \to Q^*_n.$$

Because the action of $G'$ on $G^m$ preserves orientations, each fibre of $E_{m,n}$ inherits an orientation from $G^m$. Since the base space $Q^*_n$ is oriented, we get an induced orientation on the total space $E_{m,n}$ by the above convention. The fixed-point $-1 \in G$ gives rise to a section of $E_{1,n}$ whose image we denote by $-1$.

For $p, q$ as above the commutator map

$$P_n \to G, \quad (g_1, \ldots, g_n) \mapsto [g_p, g_q]$$

is $G'$-equivariant (in the sense that it intertwines the right action of $h^{-1}$ on $P^*_n$ with the left action of $h$ on $G$, for $h \in G'$) and therefore defines a section $\zeta_{p,q}$ of $E_{1,n}$. At this point we recall the following well known facts about $G$:

**Lemma 3.1** (i) Two unit quaternions anti-commute if and only if their real parts vanish and their imaginary parts are orthogonal.

(ii) $1 \in G$ is the only singular value of the commutator map

$$P_2 \to G, \quad (g, h) \mapsto [g, h].$$

It follows from the second part of the lemma that $\zeta_{p,q}$ is transverse to $-1$. Set

$$V_{p,q} := \zeta_{p,q}^{-1}(-1) \subset Q^*_n$$

as oriented manifold.
Lemma 3.2 $V_{1,2} \subset Q^*_2$ consists of a single point, which is positively oriented.

Proof. From Lemma 3.1 (i) we see that $V_{1,2}$ consists of a single point. The sign can be determined by an explicit computation, which we omit. □

Proposition 3.2 In $Q^*_n$ the following hold:

(i) The submanifolds $W_{p,q}$ and $V_{p,q}$ intersect transversely in a single point, which is positively oriented.

(ii) $W_{p,q}$ and $V_{p',q'}$ are disjoint for $(p, q) \neq (p', q')$.

(iii) $V'_{p,q} = (-1)^{p+q+1} V_{p,q}$.

Proof. Part (i) is a consequence of Lemma 3.2 whereas (ii) is trivial. Lemma 3.1 (i) implies that $V_{p,q}' = V_{p,q}$ as smooth manifolds. On the other hand, one easily checks that $W_{p,q} \cdot V_{p',q} = (-1)^{p+q+1}$, proving (iii). □

It follows from the proposition that the $V_{p,q}$’s represent linearly independent classes in $H_{3n-6}(Q^*_n; \mathbb{Q})$. Combining this with Proposition 3.1 we obtain

Corollary 3.1 The oriented submanifolds $\{V_{p,q}\}_{1 \leq p < q \leq n}$ of $Q^*_n$ represent a basis for $H_{3n-6}(Q^*_n; \mathbb{Q})$.

Now let $n = 2\rho$, where $\rho$ is a positive integer. The $G'$–equivariant map

$$c_\rho : P_{2\rho} \to G, \quad (g_1, \ldots, g_{2\rho}) \mapsto \prod_{\ell=1}^{\rho} [g_{\ell}, g_{\ell+\rho}]$$

defines a section $s_\rho$ of $E_{1,2\rho}$, which is transverse to $-1$ by Lemma 3.1 (ii). Set

$$M_\rho := s_\rho^{-1}(-1) \subset Q^*_{2\rho}$$
as oriented manifold. Because $c_\rho^{-1}(-1)$ is a compact subset of $P^*_n$, the space $M_\rho$ is compact.

The spaces $M_\rho$ have been the subject of much research, see for instance [20] and the references therein. The following basic result might be known, but we have been unable to find a reference.

Proposition 3.3 The class in $H_{6\rho-6}(Q^*_{2\rho}; \mathbb{Q})$ represented by $M_\rho$ is given by

$$[M_\rho] = \sum_{\ell=1}^{\rho} [V_{\ell,\ell+\rho}].$$
Proof. It follows from Lemma 3.2 that

\[ W_{\ell,\ell+\rho} \cdot M_\rho = 1, \]

and one clearly has \( W_{p,q} \cap M_\rho = \emptyset \) unless \((p,q) = (\ell,\ell+\rho)\) for some \(\ell\). The proposition now follows from Corollary 3.1 and Proposition 3.2. \( \square \)

4 Representations and commutators

We will now use the results of Section 3 to prove an existence theorem for representations of free groups into \( \text{Sp}(1) \), from which we will deduce Theorem 1.2.

Suppose \( v_0, \ldots, v_k \) are elements of the free group \( F_n \) such that \( v_0 \) is a product of commutators,

\[ v_0 = \prod_{\ell=1}^{\rho} [u_\ell, u_{\ell+\rho}], \tag{5} \]

where \( u_1, \ldots, u_{2\rho} \in F_n \). Set \( L := \mathbb{Z}^n \) and

\[ \lambda := \sum_{\ell=1}^{\rho} \bar{u}_\ell \wedge \bar{u}_{\ell+\rho} \in \Lambda^2 L. \]

In the appendix it is shown that \( \lambda \) is in fact determined by \( v_0 \), but we will not need this fact. Let \( \epsilon_0, \ldots, \epsilon_k \in \{\pm 1\} \) with \( k \geq 0 \) and \( \epsilon_0 = -1 \).

Theorem 4.1 Suppose

\[ \lambda \wedge \bar{v}_1 \wedge \cdots \wedge \bar{v}_k \neq 0 \quad \text{in} \quad \Lambda^{k+2}L. \]

Then there exists a homomorphism \( \psi : F_n \to \text{Sp}(1) \) satisfying

\[ \psi(v_i) = \epsilon_i, \quad i = 0, \ldots, k. \]

As an application, let \( H \) be the group with generators \( y_1, \ldots, y_n \) and relations \( v_0, \ldots, v_k \). Then \( \psi \) induces a homomorphism \( \phi : H \to \text{SO}(3) \) such that

- the image of \( \phi \) is not contained in a maximal torus of \( \text{SO}(3) \),
- \( \phi \) does not lift to a homomorphism \( H \to \text{Sp}(1) \).
Proof of Theorem 4.1: We may assume \( k = n - 2 \). The set of equivalence classes of such representations \( \psi \) form a subset \( \mathcal{R} \subset Q_n^* \). In fact, because \( v_0 \) is a product of commutators we have \( \mathcal{R} \subset Q_n^* \). We will express \( \mathcal{R} \) in a different way, making use of (5). Let \( f_0 : Q_n \to Q_{2\rho} \) be the map defined by \( u_1, \ldots, u_{2\rho} \). Then \( U := f_0^{-1}Q_{2\rho}^* \) is an open subset of \( Q_n^* \). Then

\[
 f := f_0|_U : U \to Q_{2\rho}^*
\]
is a proper map. The elements \((v_1, \ldots, v_{n-2})\) define a \( G' \)-equivariant map

\[
 h : P_n \to G^{n-2}
\]
which in turn determines a section \( \xi_0 \) of \( E_{n-2,n} \). Set \( \xi := \xi_0|_U \). Let \( \mathcal{E} \) denote the image of the section of \( E_{n-2,n} \) corresponding to the fixed-point \((\epsilon_1, \ldots, \epsilon_{n-2})\) of \( G^{n-2} \). Then

\[
 \mathcal{R} = (f \times \xi)^{-1}(M_\rho \times \mathcal{E}).
\]

We will show that the intersection number

\[
 \kappa := I(f \times \xi, M_\rho \times \mathcal{E})
\]
is non-zero, which will imply that \( \mathcal{R} \) is non-empty. Note that the intersection number is indeed well-defined, since \( M_\rho \) and \( \mathcal{E} \) are closed subsets of \( Q_{2\rho}^* \) and \( E_{n-2,n} \) resp. and \( f^{-1}M_\rho \) is a compact subset of \( U \).

**Proposition 4.1** \( \kappa = \det(\lambda \wedge \bar{v}_1 \wedge \cdots \wedge \bar{v}_{n-2}) \neq 0 \).

Here, ‘det’ denotes the standard isomorphism \( \Lambda^n L \to \mathbb{Z} \).

Proof. Choose a smooth map \( f_1 : U \to Q_{2\rho}^* \) that is transverse to \( M_\rho \) and compactly homotopic to \( f \). (See the beginning of Section 3 for the definition of ‘compactly homotopic’.) Then \( N := f_1^{-1}M_\rho \) is a compact, oriented \((3n - 6)\)-dimensional submanifold of \( U \). Choose a smooth section \( \xi_1 \) of \( E_{n-2,n}|_U \) compactly homotopic to \( \xi \) such that \( \xi_1|_N \) is transverse to \( \mathcal{E} \). Then \( f_1 \times \xi_1 \) is transverse to \( M_\rho \times \mathcal{E} \), so

\[
 \kappa = #(f_1 \times \xi_1)^{-1}(M_\rho \times \mathcal{E})
\]

\[
 = #(\xi_1|_N)^{-1}\mathcal{E}
\]

\[
 = I(\xi|_N, \mathcal{E}).
\]

For any \( v \in F_n \) and \( 1 \leq p \leq n \) let \( \deg_p(v) \) denote the \( p \)'th component of \( \bar{v} \in \mathbb{Z}^n \).
Lemma 4.1 The class in $H_{3n-6}(Q^*_n;\mathbb{Q})$ represented by $N$ is given by

$$[N] = \sum_{p<q} c_{p,q}[V_{p,q}],$$

(6)

where

$$c_{p,q} = \sum_{\ell=1}^{\rho} [\text{deg}_p(u_\ell)\text{deg}_q(u_{\ell+\rho}) - \text{deg}_q(u_\ell)\text{deg}_p(u_{\ell+\rho})].$$

Proof. That $[N]$ can be expressed in the form (6) follows from Corollary 3.1. Set $W_{p,q} := W_{p,q} \cap U$. Then $f_{p,q} := f|_{W_{p,q}} : W_{p,q} \to Q^*_{2\rho}$ is a proper map and

$$c_{p,q} = W_{p,q} \cdot N = W_{p,q} \cdot I(f_{p,q}, M_\rho) = \sum_{\ell=1}^{\rho} I(f_{p,q}, V_{\ell,\ell+\rho}).$$

Let $\psi_{p,q,\ell} : P_2 \to P_2$ be the composition of the embedding $\tilde{P}_2 \to P_n$ in (3) with the map $P_n \to P_2$ defined by $(u_\ell, u_{\ell+\rho})$. From Lemma 3.2 and Proposition 2.1 we obtain

$$I(f_{p,q}, V_{\ell,\ell+\rho}) = \text{deg}(\psi_{p,q,\ell}) = \text{deg}_p(u_\ell)\text{deg}_q(u_{\ell+\rho}) - \text{deg}_q(u_\ell)\text{deg}_p(u_{\ell+\rho}).$$

To prove the first of the above two equalities we choose a smooth map $b : W_{p,q} \to Q^*_{2\rho}$ transverse to $V_{\ell,\ell+\rho}$ and compactly homotopic to $f_{p,q}$. We then observe that any such homotopy can be lifted to a $G'$-equivariant smooth map $B : P^*_n|_{W_{p,q}} \times [0,1] \to P^*_{2\rho}$ such that $B(\cdot,0)$ is the map induced by $u_1, \ldots, u_{2\rho}$.

Returning to the proof of Proposition 4.1 we have

$$\kappa = I(\xi|_N, \mathcal{E}) = \sum_{p<q} c_{p,q}I(\xi|_{V_{p,q}}, \mathcal{E})$$

$$= \sum_{p<q} (-1)^{p+q+1} c_{p,q} I(\xi|_{V'_{p,q}}, \mathcal{E}),$$

where the second equality comes from Proposition 3.2(iii). Set

$$V := \tilde{v}_1 \wedge \cdots \wedge \tilde{v}_{n-2}.$$ 

Because $E_{n-2,n}|_{V_{p,q}}$ is trivial we have

$$I(\xi|_{V'_{p,q}}, \mathcal{E}) = \text{deg}(h \circ J_{p,q}) = (-1)^{p+q+1} \det(e_p \wedge e_q \wedge V),$$

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where $J_{p,q} : P_{n-2} \to P_n^*$ is the map in (3) and $e_1, \ldots, e_n$ is the standard basis for $L = \mathbb{Z}^n$. Thus,

$$\kappa = \sum_{p < q} c_{p,q} \det(e_p \wedge e_q \wedge V) = \det(\lambda \wedge V).$$

\square

5 The second Stiefel–Whitney class of an $SO(3)$ representation

Let $X$ be a finite, connected CW-complex with base-point $x_0$, and let

$$\phi : H := \pi_1(X, x_0) \to SO(3)$$

be any homomorphism. Let $\tilde{X} \to X$ be the universal covering. A choice of base-point in $\tilde{X}$ lying above $x_0$ makes $\tilde{X}$ a principal $H$-bundle, so we can associate to $\phi$ an $SO(3)$-bundle $E_\phi \to X$, whose second Stiefel-Whitney class we denote by $\omega_2(\phi) \in H^2(X; \mathbb{Z}/2)$. The aim of this section is to describe $\omega_2(\phi)$ in terms of its value on any element of $H_2(X; \mathbb{Z}/2)$. (A description of $\omega_2(\phi)$ in terms of group cohomology can be found in [17, Section 3.1].)

We will use the same notation as in the proof of Theorem 1.1 in Section 2, so in particular, $C_s$ denotes the cellular chain complex of $X$. Let $c = \sum_i a_i \otimes c_i$ be any cycle in $C_2 \otimes \mathbb{Z}/2$, where $c_i \in \mathbb{Z}/2$. For each $j$ choose a lift $q_j \in Sp(1)$ of $\phi(y_j) \in SO(3)$, and let $\psi : F_n \to Sp(1)$ be the unique homomorphism such that $\psi(y_j) = q_j$. Then $\psi(w_i) = \pm 1$ for each $i$.

Proposition 5.1 Let $\delta$ denote the group isomorphism $\{\pm 1\} \to \mathbb{Z}/2$. Then

$$\langle \omega_2(\phi), [c] \rangle = \sum_{i=1}^8 c_i \delta(\psi(w_i)).$$

Here $[c] \in H_2(X; \mathbb{Z}/2)$ denotes the homology class of $c$. The proposition holds more generally for $SO(N)$ representations, $N \geq 3$, if one replaces $Sp(1)$ by $Spin(N)$.

Proof. Let $J$ be the set of indices $i \in \{1, \ldots, s\}$ such that $c_i = 1$, and let $d_0 \in S^1 = \partial D^2$ be a base-point as before. Choose an embedding $h : D^2 \times J \to int(D^2)$ and a map $g : D^2 \to X$ such that $g(d_0) = x_0$ and the following properties hold:

- $g$ maps the complement of the image of $h$ into $X^1$,
- for each $i \in J$ the map $g(h(\cdot, i)) : D^2 \to X$ is the characteristic map of the $i$'th 2-cell of $X$. 

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• $g|_{S^1}$ represents the element $\prod_{i \in J} w_i$ of $\pi_1(X^1, x_0)$, where the product is taken according to some ordering of $J$.

Let $Y$ be the result of attaching a 2-cell to $X^1$ with $g|_{S^1}$ as attaching map. Let $f : Y \to X$ be the cellular map obtained by combining $g$ with the inclusion $X^1 \to X$. Then $f_* : H_2(Y; \mathbb{Z}/2) \to H_2(X; \mathbb{Z}/2)$ maps the non-zero element of $H_2(Y; \mathbb{Z}/2)$ to $[c]$. Set $K := \pi_1(Y, x_0)$ and let $\gamma$ denote the composite homomorphism

$$K \xrightarrow{f_*} H \xrightarrow{\psi} SO(3).$$

Let $\tilde{Y} \to Y$ be the universal covering, and choose a base-point in $\tilde{Y}$ lying above $x_0$. The base-point preserving map $\tilde{Y} \to \tilde{X}$ covering $f$ is equivariant with respect to $f_* : K \to H$ and induces an isomorphism $E_\gamma \to f^* E_\phi$. It is now easy to verify the following equivalences:

$$\langle \omega_2(\phi), [c] \rangle = 0 \iff \omega_2(\gamma) = 0 \iff E_\gamma \text{ lifts to an Sp}(1) \text{ bundle} \iff \gamma \text{ lifts to a homomorphism } K \to \text{Sp}(1) \iff \psi(\prod_{i \in J} w_i) = 1 \iff \sum_{i=1}^s c_i \delta(\psi(w_i)) = 0,$$

where in the fourth equivalence we used the fact that $\psi(\prod_{i \in J} w_i)$ is independent of the choice of the lifts $g_j$ since $c$ is a cycle. □

**Proposition 5.2** If the image of $\phi$ is contained in a maximal torus of $SO(3)$, then $\langle \omega_2(\phi), z \rangle = 0$ for every $z \in H_2(X; \mathbb{Z})$.

**Proof.** Let $z$ be represented by a cycle $c = \sum_{i=1}^s c_i a_i$ in $C_2$, where $c_i \in \mathbb{Z}$. Then $w := \prod_i (w_i)^{c_i} \in F_n$ is a product of commutators. Since $\psi$ takes values in a maximal torus of $\text{Sp}(1)$, we have $\psi(w) = 1$, so

$$0 = \delta(\psi(w)) = \sum_i c_i \delta(\psi(w_i)) = \langle \omega_2(\phi), z \rangle.$$

□

### 6 A formula for $\mu$

We now give a formula for the 2-form $\mu$ in Theorem 1.2 that is useful for computations. Suppose the generator $\sigma \in H_2(X)$ is represented by a map $f$ :
\( \Sigma \to X \), where \( \Sigma \) is a closed, connected surface of genus \( \rho \). Let \( b_1, \ldots, b_{2\rho} \) be a symplectic basis for \( H_1(\Sigma) \), so for \( 1 \leq i < j \leq 2\rho \) one has the intersection numbers

\[
 b_i \cdot b_j = \begin{cases} 
 1 & \text{if } j = i + \rho, \\
 0 & \text{else}. 
\end{cases}
\]

For \( \alpha, \beta \in H^1(\Sigma) \) one easily checks that

\[
 \langle \alpha \cup \beta, [\Sigma] \rangle = \sum_{\ell=1}^{\rho} \left[ \langle \alpha, b_{\ell} \rangle \langle \beta, b_{\ell+\rho} \rangle - \langle \beta, b_{\ell} \rangle \langle \alpha, b_{\ell+\rho} \rangle \right].
\]

Letting \( \tilde{b}_j \) denote the image of \( f_* b_j \) in \( H_1(X)/T \) we obtain

\[
 \langle \alpha \cup \beta, \sigma \rangle = \sum_{\ell=1}^{\rho} \left[ \langle \alpha, \tilde{b}_{\ell} \rangle \langle \beta, \tilde{b}_{\ell+\rho} \rangle - \langle \beta, \tilde{b}_{\ell} \rangle \langle \alpha, \tilde{b}_{\ell+\rho} \rangle \right],
\]

so that

\[
 \mu = \sum_{\ell=1}^{\rho} \tilde{b}_{\ell} \wedge \tilde{b}_{\ell+\rho}. \tag{7}
\]

## 7 Proof of Theorem [1.2](#)

We may assume \( r = b_1(X) \). If \( \phi \) is a homomorphism as in the theorem then by Proposition [5.2](#) the image of \( \phi \) cannot be contained in a maximal torus. We now prove the existence of \( \phi \). We continue using the notation introduced in the proof of Theorem [1.1](#) except that we now assume the 2-cells of \( X \) are numbered from 0 to \( s \), with corresponding generators \( a_0, \ldots, a_s \) of \( C_2 \), so that \( H \) has the presentation

\[
 H = \langle y_1, \ldots, y_n \mid w_0, \ldots, w_s \rangle.
\]

We begin by modifying the relations in this presentation. Choose an invertible integral matrix \( M = (m_{ij})_{0 \leq i, j \leq s} \in \GL_{s+1}(\Z) \) with

\[
 \sum_{i=0}^{s} m_{i0} a_i = \sigma.
\]

Choose an automorphism \( \alpha \in \Aut(F_{s+1}) \) that maps to \( M \) under the canonical surjective homomorphism \( \Aut(F_{s+1}) \to \GL_{s+1}(\Z) \). Let \( p : F_{s+1} \to F_n \) be the unique homomorphism with \( p(y_i) = w_i \) for \( i = 0, \ldots, s \), and set

\[
 w'_i := p(\alpha(y_i+1)), \quad i = 0, \ldots, s. \tag{8}
\]
Thus, each $w'_i$ can be expressed as a word in $w_0, \ldots, w_s$, and conversely, each $w_i$ is a word in $w'_0, \ldots, w'_s$. Hence, we have the new presentation

$$H = \langle y_1, \ldots, y_n \mid w'_0, \ldots, w'_s \rangle.$$ 

Choose $\eta_0, \ldots, \eta_s \in \mathbb{Z}/2$ such that for any cycle $c = \sum_{i=0}^s a_i \otimes c_i$ in $C_2 \otimes \mathbb{Z}/2$ one has

$$\langle \varpi, c \rangle = \sum_i c_i \eta_i.$$ 

For $j = 0, \ldots, s$ set

$$\epsilon_j := \delta^{-1}(\sum_i m_{ij} \eta_i) \in \{\pm 1\},$$

where $\delta$ is as in Proposition 5.1. Then

$$\epsilon_0 = \delta^{-1}(\varpi, \sigma) = -1.$$ 

**Lemma 7.1** If $\psi : F_n \to Sp(1)$ is any homomorphism such that $\psi(w'_j) = \epsilon_j$ for each $j$ then the induced homomorphism $\phi : H \to SO(3)$ satisfies $\omega_2(\phi) = \varpi$.

**Proof.** For each $j$ one has

$$\sum_i m_{ij} \eta_i = \delta(\psi(w'_j)) = \sum_i m_{ij} \delta(\psi(w_i)),$$

hence $\eta_i = \delta(\psi(w_i))$ and the lemma follows from Proposition 5.1. □

Abelianization of the equation (8) yields

$$w'_j = \sum_i m_{ij} \bar{w}_i.$$ 

In particular,

$$\overline{w'_0} = \partial \sigma = 0,$$

so $w'_0$ is a product of commutators, ie

$$w'_0 = \prod_{\ell=1}^{\rho} [u_{\ell}, u_{\ell+\rho}]$$

for some $u_1, \ldots, u_{2\rho} \in F_n$.

We now express $\mu$ in terms of the $u_\ell$'s. First recall that the oriented model surface $\Sigma_\rho$ of genus $\rho$ is obtained from $D^2$ by pairwise identification
of certain segments of \( S^1 \). These segments parametrize \( 2\rho \) loops in \( \Sigma_{\rho} \) that in turn represent a symplectic basis \( b_1, \ldots, b_{2\rho} \) for \( H_1(\Sigma_{\rho}) \). Choose a map \( f: \Sigma_{\rho} \to X \) representing the generator \( \sigma \in H_2(X) \) such that, in the notation introduced in Section 3, the class \( \bar{b}_\ell \in H_1(X)/T \) is the one represented by \( u_\ell \), for \( \ell = 1, \ldots, 2\rho \). (Compare the map \( g \) in the proof of Proposition 5.1.)

Then \( \mu \) is given by the formula (7).

In the following we use the notation of Section 3. For \( i = 1, \ldots, r-2 \) let \( z_i \in F_n \) be a lift of \( \gamma_i \in H \). To prove the theorem it suffices, by Lemma 7.1, to find a representation \( \psi: F_n \to G = Sp(1) \) such that, in the notation introduced in Section 6, the class \( \tilde{b}_\ell \in H_1(X)/T \) is the one represented by \( u_\ell \), for \( \ell = 1, \ldots, 2\rho \). (Compare the map \( g \) in the proof of Proposition 5.1.)

Then \( \psi(u_\ell) = \epsilon_\ell \), \( \psi(z_i) = 1 \) for \( i = 1, \ldots, r-2 \) and \( j = 0, \ldots, s \). To this end we will apply Theorem 4.1 with \( k = n-2 \) and

\[
(v_0, \ldots, v_{n-2}) = (w'_0, \ldots, w'_s, z_1, \ldots, z_{r-2})
\]

and \( \epsilon_j = 1 \) for \( j > s \). Recall that we have fixed a basis for the free group \( \pi_1(X^1, x_0) \), so we can identify \( H_1(X^1) = L = \mathbb{Z}^n \). Let \( K \) denote the subgroup of \( L \) spanned by the linearly independent elements \( \bar{w}_1, \ldots, \bar{w}_s \). By the elementary divisors theorem we can find a basis \( d_1, \ldots, d_n \) for \( L \) and integers \( m_1, \ldots, m_s \) such that \( \{m_i d_i\}_{1 \leq i \leq s} \) is a basis for \( K \). This implies that \( \{d_i\}_{s+1 \leq i \leq n} \) maps to a basis for \( A_0 := H_1(X)/T \). Therefore,

\[
\det(\lambda \wedge \bar{v}_1 \wedge \cdots \wedge \bar{v}_{n-2}) = \pm |T| \det(\lambda \wedge d_1 \wedge \cdots \wedge d_s \wedge \bar{z}_1 \wedge \cdots \wedge \bar{z}_{r-2})
= \pm |T| \det(\mu \wedge \bar{\gamma}_1 \wedge \cdots \wedge \bar{\gamma}_{r-2})
\neq 0,
\]

since the natural map \( L \to A_0 \) takes \( \bar{z}_i \) to \( \bar{\gamma}_i \) and the induced map \( \Lambda^2 L \to \Lambda^2 A_0 \) takes \( \lambda \) to \( \mu \). Theorem 4.1 now guarantees the existence of the desired representation \( \psi \). This proves Theorem 1.2.

A Commutators and group homology

The purpose of this appendix is to shed some light on the term \( \lambda \) in Theorem 4.1. Let \( G \) be any group and \( H_\ast(G) \) its group homology with integer coefficients. Let \( H_\ast(G; \mathbb{Q}) = H_\ast(G) \otimes \mathbb{Q} \) be the homology with rational coefficients. We identify \( H_1(G) \) with the abelianization of \( G \). The image of an element \( x \in G \) in \( H_1(G; \mathbb{Q}) \) will be denoted \( \bar{x} \). Let \([G, G]\) be the commutator subgroup of \( G \).
Proposition A.1 If $H_2(G)$ is a torsion group (or equivalently, if $H_2(G;\mathbb{Q}) = 0$) then there exists a group homomorphism
\[ \alpha : [G, G] \to \Lambda^2 H_1(G;\mathbb{Q}) \] (9)
that sends any commutator $[x, y]$ to $\tilde{x} \wedge \tilde{y}$.

If $H_2(G) = 0$ then the proposition also holds if $H_1(G;\mathbb{Q})$ is replaced by $H_1(G)$ in (9).

Proof. We use Miller’s description [16] (see also [3]) of $H_2(G)$ in terms of commutators. Let $\langle G, G \rangle$ be the free group on all pairs $\langle x, y \rangle$ with $x, y \in G$. Let $Z(G)$ be the kernel of the homomorphism $\langle G, G \rangle \to [G, G]$ that sends $\langle x, y \rangle$ to $[x, y]$. Miller shows that
\[ H_2(G) \approx Z(G)/B(G), \]
where $B(G)$ is the normal subgroup of $\langle G, G \rangle$ generated by a certain subset $E$. Now let
\[ \beta : \langle G, G \rangle \to \Lambda^2 H_1(G;\mathbb{Q}) \]
be the homomorphism that sends $\langle x, y \rangle$ to $\tilde{x} \wedge \tilde{y}$. It is easily verified that $\beta$ vanishes on $E$, hence on $B(G)$. Since $Z(G)/B(G)$ is a torsion group while $\Lambda^2 H_1(G;\mathbb{Q})$ is torsion free we conclude that $\beta$ vanishes on $Z(G)$. Hence $\beta$ descends to a homomorphism as in the proposition.

Now let $G$ be the free group $F_n$. Then $H_2(G) = 0$, so in the notation of Theorem we have
\[ \lambda = \alpha(v_0). \]

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