A perturbative approach for the dynamics of the quantum Zeno subspaces

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Abstract

In this paper we investigate the dynamics of the quantum Zeno subspaces which are the eigenspaces of the interaction Hamiltonian, belonging to different eigenvalues. Using the perturbation theory and the adiabatic approximation, we get a general expression of the jump probability between different Zeno subspaces. We applied this result in some examples. In these examples, as the coupling constant of the interactions increases, the measurement keeps the system remaining in its initial subspace and the quantum Zeno effect takes place.

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1 Introduction

The quantum Zeno effect [1][2] has attracted great attentions. This phenomenon is caused by the influence of the measurement on the evolution of a quantum system. Frequent measurements can inhibit the decay of any unstable system [3], and the short time behavior of the survival probability is not exponential but quadratic. The deviation from the exponential decay has been confirmed in a tunnelling experiment by Wilkinson et al [4]. Moreover, it was also predicted that frequent measurements (but not too frequent) could accelerate the decay process. This is so-called quantum anti-Zeno effect. The quantum Zeno effect and anti-Zeno effect had been discussed in ref.[5][6]. Both effects were first observed recently in an atomic tunnelling system [7].

Misra and Sudarshan’s theorem [3] proved that a system was forced to evolve inside a subspace, related to a projection operator, by frequently observations, but not remaining in its initial state which belonged to the subspace. This idea was developed by Facchi et al to frame the Zeno dynamics of a whole system including a detector apparatus [8]. The system can just evolve in a set of orthogonal subspaces of the total Hilbert space which belong to different eigenvalues of the interaction Hamiltonian in the infinitely strong coupling limit. These subspaces, which the measurement process is able to distinguish, are called quantum Zeno subspaces.

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Quantum zeno dynamics is not absolutely developed yet. Up to now, the dynamics of quantum
Zeno subspaces in ref.[8] has discussed the “infinitely strong measurement” limit. But the finitely
strong measurement is untouched. In this paper, we combine perturbation theory and adiabatic
approximation to describe such process. We obtain an expression for the jump probability between
two different Zeno subspaces of the interaction Hamiltonian. Therefore we have a general method to
deal with the dynamics of quantum Zeno dynamics.

The organization of this paper is as follows. In Section 2 we briefly review the quantum Zeno
subspace theorem. In section 3, we apply the perturbative method to get a general express of the
jump probability between different Zeno subspaces of the interaction Hamiltonian. This method is
unlike in ref.[6]. we regard the free Hamiltonian of the measured system and the detector apparatus
as a perturbation of the Hamiltonian describing the interaction of them. We use this expression
to analyze two time-independent measurements in Section 4. We also use it to show the QZE by
performing the measurement on Heisenberg spin chain in Section 5. The Section 6 concludes the
summary of our results and the discussions for them.

2 Quantum Zeno subspaces

We briefly introduce the quantum Zeno subspace theorem. This theorem is developed by Facchi
et al [8].

Consider a quantum system described by the Hamiltonian $H$:

$$H = H_0 + H_{\text{meas}}(K)$$

(1)

Here $H_0$ is the free Hamiltonian of the measured system and the detector apparatus, $H_{\text{meas}}(K)$ denotes
the interaction of them and $K$ is a set of coupling parameters. If $K$ is simply a coupling constant, we
can simplify the above Hamiltonian into the form

$$H = H_0 + KH_{\text{meas}}$$

(2)

The evolution of the system is described by the unitary operator $U(t)$, which is completely determined
by the total Hamiltonian $H$. In the $K \to \infty$ limit, benefit from the adiabatic theorem [8][9][10], the
evolution operator

$$u(t) = \lim_{K \to \infty} U(t)$$

(3)

has the property:

$$[u(t), P_n] = 0,$$

(4)

where

$$H_{\text{meas}} P_n = \varepsilon_n P_n, \quad P_n P_m = \delta_{nm}.$$  

(5)

$P_n$ is a orthogonal operator which projects the total Hilbert space onto $\mathcal{H}_{P_n}$, the eigenspace of $H_{\text{meas}}$
belonging to the eigenvalue $\varepsilon_n$. These subspaces are called quantum Zeno subspaces. If the eigenvalue
is degenerate, the corresponding quantum Zeno subspace is the plus of the degenerate eigenspaces.
Therefore they are in general multidimensional. We can see that the operator $u(t)$ is diagonal with
respect to $H_{\text{meas}}$. Moreover, if $H$ is time-independent, the evolution operator $u(t)$ can be explicitly
given by

$$u(t) = \exp\{-i \sum_n (P_n H_0 P_n + K \varepsilon_n P_n) t\}.$$  

(6)
Let the system in the initial density matrix $\rho_0$. In the $K \to \infty$ limit, the density matrix of the system is
\[ \rho(t) = u(t)\rho_0 u(t)^\dagger, \]
and the probability to find the system in $H_{P_n}$ is
\[ p_n(t) = \text{Tr}[\rho(t)P_n] = \text{Tr}[u(t)\rho_0 u(t)^\dagger P_n] = \text{Tr}[u(t)\rho_0 P_n u(t)^\dagger] = \text{Tr}[\rho_0 P_n] = p_n(0). \]
From this result, it is clear to see that the probability in each quantum Zeno subspace does not change during the measurement process. If the initial density matrix belongs to a quantum Zeno subspace $\rho_0 = P_n \rho_0 P_n$, the system will remain there forever and the QZE takes place.

In $K \to \infty$ limit, the interaction Hamiltonian plays the leading role and determines the evolution of the system. Each quantum Zeno subspace evolves individually, so the probability of each subspace does not leak out to another, although the system does not remain in its initial state.

3 An approximate method

In Sec.2, Quantum Zeno subspaces have been investigated in $K \to \infty$ limit. On the other hand, we pay great attention to the finitely strong time-dependent measurement. We want to know the jump probability between different Zeno subspaces and more details about the quantum Zeno effect. We find that if the free Hamiltonian $H_0(t)$ compared with the interaction Hamiltonian $KH_{\text{meas}}(t)$ is a perturbation and $KH_{\text{meas}}(t)$ satisfies the adiabatic approximation condition, the jump probability is mainly from a contribution of the second-order approximation of the density matrix.

3.1 Perturbation theory

We still consider the system whose time-dependent Hamiltonian has the form (2). The Hamiltonian $H_0(t)$ and $KH_{\text{meas}}(t)$ are Hermitian operators respectively. $H_0(t)$ is a perturbation of $KH_{\text{meas}}(t)$. The time evolution operator $U(t, 0)$ is determined by the Schrödinger equation
\[ i\frac{d}{dt}U(t, 0) = (H_0(t) + H_{\text{meas}}(t))U(t, 0), \quad U(0, 0) = 1. \]
Here Plank’s constant $\hbar$ equals 1. Let us review in summary the solution of the general expression of $U(t, 0)$ [9]. We assume $U^{(0)}(t, 0)$ is the unitary time evolution operator corresponding to $KH_{\text{meas}}(t)$:
\[ i\frac{d}{dt}U^{(0)}(t, 0) = KH_{\text{meas}}(t)U^{(0)}(t, 0), \quad U^{(0)}(0, 0) = 1. \]
We change the Schrödinger representation into the intermediate “representation” by the unitary transformation $U^{(0)\dagger}(t, 0)$:
\[ U_I(t, 0) = U^{(0)\dagger}(t, 0)U(t, 0). \]
The Schrödinger equation in this “representation” reads
\[ i\frac{d}{dt}U_I(t, 0) = H_0(t)U_I(t, 0), \]
where
\[ H_{10}(t) = U^{(0)}(t, 0) H_0 U^{(0)}(t, 0). \]  
(14)

The formal solution of Eq.(13) is
\[ U_I(t, 0) = T \exp(-i \int_0^t H_{10}(t') dt'). \]  
(15)

Here T denotes the time-ordering. According to Eq.(12), we get the expansion for \( U(t, 0) \):
\[ U(t, 0) = U^{(0)}(t, 0) + \sum_{n=1}^{\infty} U^{(n)}(t, 0), \]  
(16)

where
\[ U^{(n)}(t, 0) = (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n U^{(0)}(t, t_n) H_0(t_n) \cdot \cdots \cdot U^{(0)}(t_2, t_1) H_0(t_1) U^{(0)}(t_1, 0). \]  
(17)

The expansion is power series in \( H_0(t) \). If the measurement is strong and \( U^{(0)}(t, 0) \) is very close to \( U(t, 0) \), the series converge rapidly. In the first-order approximation, we have
\[ U(t, 0) = U^{(0)}(t, 0) + (-i) \int_0^t dt_1 U^{(0)}(t, t_1) H_0(t_1) U^{(0)}(t_1, 0). \]  
(18)

Since \( H_{\text{meas}}(t) \) is time-dependent, its eigenspaces can shift during the measurement process, as well as the eigenvalues \( \varepsilon_n(t) \). We have
\[ H_{\text{meas}}(t) P_n(t) = \varepsilon_n(t) P_n(t), \]  
(19)
\[ P_n(t) P_m(t) = \delta_{nm}. \]  
(20)

The Hilbert space corresponding to the projection \( P_n(t) \) is in general multidimensional. We suppose at the initial time the quantum system is in \( \rho_0 \). \( \rho_0 \) belongs to a quantum Zeno subspace. It means
\[ \rho_0 = P_n(0) \rho_0 P_n(0). \]  
(21)

Under the continuous measurement, the density matrix at time \( t \) becomes
\[ \rho(t) = U(t, 0) \rho_0 U^\dagger(t, 0). \]  
(22)

Using Eq.(18), the density matrix can be obtained up to second-order approximation:
\[ \rho^{(0)}(t) = U^{(0)}(t, 0) \rho_0 U^{(0)}\dagger(t, 0), \]  
(23)
\[ \rho^{(1)}(t) = U^{(0)}(t, 0) \rho_0 U^{(1)}\dagger(t, 0) + U^{(1)}(t, 0) \rho_0 U^{(0)}\dagger(t, 0), \]  
(24)
\[ \rho^{(2)}(t) = U^{(0)}(t, 0) \rho_0 U^{(2)}\dagger(t, 0) + U^{(2)}(t, 0) \rho_0 U^{(0)}\dagger(t, 0) + U^{(1)}(t, 0) \rho_0 U^{(1)}\dagger(t, 0). \]  
(25)
3.2 Adiabatic approximation

The property Eq.(4) \[8\] of the Quantum Zeno subspaces is derived from the adiabatic theorem \[9\][10]. Similarly we apply the adiabatic approximation to solve the time-dependent measurement problem.

Throughout the measurement process, we suppose the eigenvalues and the eigenspaces of the interaction Hamiltonian $KH_{\text{meas}}(t)$ satisfy \[9\]:

(i) the eigenvalues remain distinct:

$$\varepsilon_n(t) \neq \varepsilon_m(t), \quad m \neq n; \quad (26)$$

(ii) the derivatives $dP_n(t)/dt, d^2P_n(t)/dt^2$ are well-defined and piece-wise continuous.

We define a unitary operator $A(t)$ having the property

$$P_n(t) = A(t)P_n(0)A^\dagger(t), \quad A(0) = 1. \quad (27)$$

The physical significance of the unitary transformation $A(t)$ is that: it takes any set of basis vectors of $H_{\text{meas}}(0)$ over into a set of basis vectors of $H_{\text{meas}}(t)$, each eigenvectors of $H_{\text{meas}}(0)$ being carried over into one of the eigenvectors of $H_{\text{meas}}(t)$ that derive from it by continuity. It is determined by the following equation

$$\frac{d}{dt}A(t) = M(t)A(t), \quad (28)$$

where $M(t)$ is a Hermitian operator

$$M(t) = i \sum_n (dP_n(t)/dt)P_n(t). \quad (29)$$

We assume that $KH_{\text{meas}}(t)$ satisfy the adiabatic approximation condition

$$|\frac{\alpha_{mn}^{\text{max}}}{\varepsilon_m^{\text{min}}}| \ll K^2, \quad (30)$$

where

$$\varepsilon_m^{\text{min}} = \min |\varepsilon_m(t) - \varepsilon_n(t)|, \quad m \neq n, \quad (31)$$

and

$$\alpha_m^{\text{max}} = \max \left( \sum_{m \neq n} |\alpha_{mn}(t)|^2 \right) \quad (32)$$

$$\alpha_{mn}(t) = -\frac{\langle m|\frac{dH_{\text{meas}}(t)}{dt}|n\rangle_t}{\varepsilon_{mn}(t)} \quad (33)$$

Here $|n\rangle_t$ is the initial eigenvector of $H_{\text{meas}}(t)$ belonging to the eigenvalue $\varepsilon_n(t)$, $|m\rangle_t$ belonging to $\varepsilon_m(t)$. $\varepsilon_{mn}(t)$ is the “Bohr frequency” of the transition $n \rightarrow m$. Therefore the zero-approximation $U^{(0)}(t,0)$ of the time evolution operator determined by $KH_{\text{meas}}(t)$ has the asymptotic property

$$U^{(0)}(t,0)P_n(0) = P_n(t)U^{(0)}(t,0), \quad (34)$$

and $U^{(0)}(t,0)$ can be expressed approximately in the form

$$U^{(0)}(t,0) \simeq A(t)\Phi(t), \quad (35)$$
where

\[ \Phi(t) = \sum_n \exp(-i\varphi_n(t))P_n(0), \quad (36) \]
\[ \varphi_n(t) = \int_0^t K\varepsilon(t')dt'. \quad (37) \]

Specially, if the interaction Hamiltonian is time-independent, \( A(t) \) equals 1 at any time, and \( \Phi(t) \) is

\[ \Phi(t) = \sum_n \exp(-iK\varepsilon_n(t))P_n. \quad (38) \]

### 3.3 Jump probability

We investigate the jump probability from Zeno subspace \( \mathcal{H}_{P_n(0)} \) to \( \mathcal{H}_{P_m(t)} \) under the action of the perturbation \( H_0(t) \). The jump probability is

\[ W(P_n(0) \rightarrow P_m(t)) = Tr\{P_m(t)\rho(t)\}. \quad (39) \]

Since the initial density matrix belongs to the quantum Zeno subspace \( \mathcal{H}_{P_n} \), we have

\[ P_m(0)\rho_0 = \rho_0 P_m(0) = 0, \quad m \neq n. \quad (40) \]

Using Eq.(23), (24), (25) and (34), we get the expansion of the jump probability up to second-order. From Eq.(40), we find the zero-order term and first-order term is 0 and the jump probability is mainly from the contribution of the second-order term

\[
W^{(2)}(P_n(0) \rightarrow P_m(t)) = \begin{align*}
&= Tr\{P_m(t)\rho^{(2)}(t)\} \\
&= Tr\{U^{(0)}(t,0)P_m(0)\rho_0 U^{(2)\dagger}(t,0)\} + Tr\{U^{(2)}(t,0)\rho_0 P_m(0)U^{(0)\dagger}(t,0)\} \\
&\quad + Tr\{P_m(t)U^{(1)}(t,0)\rho_0 U^{(1)\dagger}(t,0)\} \\
&= Tr\{P_m(t)U^{(1)}(t,0)\rho_0 U^{(1)\dagger}(t,0)\}. \quad (41)
\end{align*}
\]

Using Eq.(17), we get the probability defined by the integral equation

\[ W(P_n(0) \rightarrow P_m(t)) = \int_0^t dt_1 \int_0^t dt_2 Tr\{P_m(t)U^{(0)}(t,t_1)H_0(t_1)U^{(0)\dagger}(t_1,0)\rho_0 U^{(0)\dagger}(t_2,0)H_0(t_2)U^{(0)\dagger}(t,t_2)\}. \quad (42) \]

From Eq.(35) and the composition law

\[ U^{(0)}(t,0) = U^{(0)}(t,t')U^{(0)}(t',0), \quad (43) \]

we can replace \( U^{(0)}(t,t') \) by the asymptotic form

\[ U^{(0)}(t,t') \simeq A(t)\Phi(t)\Phi^\dagger(t')A^\dagger(t'). \quad (44) \]

Eq.(42) can be simplified to

\[ W(P_n(0) \rightarrow P_m(t)) = \int_0^t dt_1 \int_0^t dt_2 Tr\{A^\dagger(t_1)H_0(t_1)A(t_1)\rho_0 A^\dagger(t_2)H_0(t_2)A(t_2)P_m(0)\} \]
\[ \times \exp\{i \int_{t_2}^{t_1} K(\varepsilon_m(t') - \varepsilon_n(t'))dt'\}. \quad (45) \]
There are two assumptions for the validity of Eq.(45): the free Hamiltonian $H_0(t)$ can be regarded as a perturbation of the interaction Hamiltonian $KH_{\text{meas}}(t)$ and $KH_{\text{meas}}(t)$ changes sufficiently slowly to satisfy the adiabatic approximation condition. With the enhancement of the coupling constant $K$, the phase factor vibrates rapidly and the integration tends to decline. The decay of the system is inhibited by the measurement. Eq.(45) is the main result of this paper. It can describe the problem of the finitely strong measurement. We will discuss the quantum Zeno effect in the following two Sections. However, in the “infinitely strong measurement” limit $K \rightarrow \infty$, Eq.(45) tends to zero. Therefore the system remains in its initial Zeno subspace $\mathcal{H}_{P_n(0)}$ forever. This is the result in ref.[8].

4 Time-independent measurement

In the preceding section, we get the jump probability Eq.(45) between different quantum Zeno subspaces. Now we use it to look at time-independent measurement. Furthermore, we assume the free Hamiltonian is time-independent. We consider the repeated measurements separated by the free evolution of the system. The duration of the free evolution is $\tau_F$ and the duration of the measurement is $(\tau - \tau_F)$:

$$H_{\text{meas}}(t) = \theta(\tau - t)\theta(t - \tau_F)P_n.$$  

(46)

Here $\theta(t - \tau_F)$ is Heaviside unit step function. There are two Zeno subspaces $\mathcal{H}_{P_n}$ and $\mathcal{H}_{P_m}$ of $H_{\text{meas}}$ respectively belonging to the eigenvalues 1 and 0. The initial density matrix $\rho_0$ of the system belongs to Hilbert space $\mathcal{H}_{P_n}$. After a measurement, the jump probability from $\mathcal{H}_{P_n}$ to $\mathcal{H}_{P_m}$ is

$$W(P_n \rightarrow P_m, \tau) = Tr\{P_mH_0(0)\rho_0\}$$

$$+ \frac{4\tau_F}{K} \sin \frac{1}{2}K(\tau - \tau_F) \cos \frac{1}{2}K(\tau + \tau_F)$$

$$+ \frac{4}{K^2} \sin^2 \frac{1}{2}K(\tau - \tau_F).$$

(47)

In $\tau_F \rightarrow \tau$ limit(instantaneous measurement [11]), the survival probability exhibit a quadratic behavior at short time:

$$W(P_n, \tau) = 1 - \tau^2/\tau_z^2,$$

where

$$\tau_z^2 = Tr\{P_mH_0(0)\rho_0\}.$$  

(49)

$\tau_z$ is called Zeno time. We perform N measurements at time intervals $\tau$ for a time $t$. With $N$ increasing($\tau \rightarrow 0$), the system will be freezed in its initial subspace(QZE) [8]. This result is correct for the case of the finite coupling constant $K$.

Let us now consider another time-independent continuous measurement described by the following:

$$H_{\text{meas}} = \sum_n \varepsilon_n P_n.$$  

(50)

Similarly we have the initial density of the system belonging to Hilbert space $\mathcal{H}_{P_n}$. The duration of the measurement is $\tau$. The jump probability from Zeno subspace $\mathcal{H}_{P_n}$ to $\mathcal{H}_{P_m}$ is

$$W(P_n \rightarrow P_m, \tau) = Tr\{P_mH_0(0)\rho_0\}$$

$$\times \frac{4\sin^2 \frac{1}{2}K(\varepsilon_m - \varepsilon_n)\tau}{K^2(\varepsilon_m - \varepsilon_n)^2}.$$  

(51)
And the survival probability is
\[ W(P_n, \tau) = 1 - \sum_{m \neq n} W(P_n \rightarrow P_m, \tau) \]
\[ = 1 - \sum_{m \neq n} \text{Tr}\{P_m H_0 \rho_0 H_0\} \frac{4 \sin^2 \frac{1}{2} K (\varepsilon_m - \varepsilon_n)\tau}{K^2 (\varepsilon_m - \varepsilon_n)^2}. \tag{52} \]

When the system evolves under the continuous measurement for a shot time \( \tau \), we perform an ideal measurement (projection) to confirm whether the system survives inside \( \mathcal{H}_{P_n} \). Repeating the above procedure, we have the survival probability in \( \mathcal{H}_{P_n} \) at time \( t = N\tau \)
\[ W(P_n, t) = (W(P_n, \tau))^N \]
\[ \simeq 1 - t \sum_{m \neq n} \text{Tr}\{P_m H_0 \rho_0 H_0\} \frac{4 \sin^2 \frac{1}{2} K (\varepsilon_m - \varepsilon_n)\tau}{K^2 (\varepsilon_m - \varepsilon_n)^2}. \tag{53} \]

where the decay rate is
\[ R = \sum_{m \neq n} \text{Tr}\{P_m H_0 \rho_0 H_0\} \frac{4 \sin^2 \frac{1}{2} K (\varepsilon_m - \varepsilon_n)\tau}{K^2 (\varepsilon_m - \varepsilon_n)^2}. \tag{54} \]

Introducing the functions
\[ G(\varepsilon) = \sum_{m \neq n} \text{Tr}\{P_m H_0 \rho_0 H_0\} \delta(\varepsilon_m - \varepsilon), \tag{55} \]
\[ F(\varepsilon) = \frac{4 \sin^2 \frac{1}{2} K (\varepsilon - \varepsilon_n)\tau}{2\pi K^2 (\varepsilon - \varepsilon_n)^2}, \tag{56} \]
we can recast Eq.(54) as
\[ R = 2\pi \int_{-\infty}^{+\infty} G(\varepsilon) F(\varepsilon) d\varepsilon. \tag{57} \]

The above formulation is similar to the one obtained in ref.[5] which has analyzed the conditions to obtain the QZE and AZE. But the measurement process is completely different. In that case, the free evolution of the system is interrupted by instantaneous ideal measurements (projections) at time intervals \( \tau \). Furthermore, the measurements force the measured system remaining in its initial state. In our case, the whole system including the detector apparatus is not necessary to do that, but remains in its initial Zeno subspace which is in general multidimensional. The decay rate (57) is the overlap of the factors \( G(\varepsilon) \) and \( F(\varepsilon) \). If the frequency \( \nu \sim 1/\tau \) satisfies
\[ \nu \gg \Gamma_R |\varepsilon_n - \varepsilon_M|, \tag{58} \]
the QZE can be obtained. Here \( \Gamma_R \) is the width of \( G(\varepsilon) \) and \( \varepsilon_M \) is the centre of gravity of \( G(\varepsilon) \). Moreover, the decay rate also reduces as the coupling constant \( K \) increases. In fact, the interaction Hamiltonian which denotes the continuous measurement as well as the free Hamiltonian govern the evolution of the system, the bigger the more influence on it. We see that the evolution of the system under the action of a continuous measurement process is similar to that obtained with pulsed measurements [8].
5 Measurement on Heisenberg spin chain

Let us now investigate another example of a time-dependent measurement on an XYZ Heisenberg spin(1/2) chain at zero temperature. The spin-systems have been discussed in the subject of Adiabatic Quantum Computation [12]. The interesting problem about the Adiabatic Quantum Computation is the investigation of the ground state of spin-systems. In this paper, we will investigate the quantum Zeno effect of the spin-systems which initially is in the ground state. We have the spin chain interacting with a magnetic field which is rotated sufficiently slowly from Z-axis to X-axis without changing its magnitude $h$ during the time $T$ [13]. The total Hamiltonian of the system is

$$H = H_0 - \sum_{j=1}^{n} h((1 - s)\sigma_j^z + s\sigma_j^x), \quad s = \frac{t}{T},$$

(59)

Here $H_0$ represents the free Hamiltonian of the XYZ Heisenberg spin chain

$$H_0 = \sum_{j=1}^{n} (\lambda_1 \sigma_j^x \sigma_{j+1}^x + \lambda_2 \sigma_j^y \sigma_{j+1}^y + \lambda_3 \sigma_j^z \sigma_{j+1}^z),$$

(60)

and the second term denotes the interaction of the field and the spin chain. We use the similar method in ref [13] by Korepin to get the adiabatic approximation condition for the interaction Hamiltonian $I(s)$:

$$hT \gg \sqrt{n/2}.$$  

(61)

This condition, unlike in ref [13], is only for the interaction Hamiltonian. The free Hamiltonian $H_0$ acts as a perturbation.

We define

$$I(s) = -\sum_{j=1}^{n} I_j(s) = -\sum_{j=1}^{n} h((1 - s)\sigma_j^z + s\sigma_j^x),$$

(62)

where

$$I_j(s) = h((1 - s)\sigma_j^z + s\sigma_j^x).$$

(63)

Introducing the matrices

$$A(s) = \begin{pmatrix} \frac{\sqrt{2K^2(s)-2K(s)(1-s)}}{K(s)-(1-s)} & \frac{\sqrt{2K^2(s)+2K(s)(1-s)}}{K(s)-(1-s)} \\ \frac{\sqrt{2K^2(s)-2K(s)(1-s)}}{K(s)-(1-s)} & \frac{-K(s)-(1-s)}{\sqrt{2K^2(s)+2K(s)(1-s)}} \end{pmatrix},$$

(64)

$$A^\dagger(s) = \begin{pmatrix} \frac{\sqrt{2K^2(s)-2K(s)(1-s)}}{K(s)-(1-s)} & \frac{\sqrt{2K^2(s)+2K(s)(1-s)}}{K(s)-(1-s)} \\ \frac{\sqrt{2K^2(s)+2K(s)(1-s)}}{K(s)-(1-s)} & \frac{-K(s)-(1-s)}{\sqrt{2K^2(s)+2K(s)(1-s)}} \end{pmatrix},$$

(65)

where

$$K(s) = \sqrt{s^2 + (1-s)^2},$$

(66)

we can rewrite the Eq.(63) in the form:

$$I_j(s) = hK(s)A_j(s)\sigma_j^z A_j^\dagger(s).$$

(67)
We define the instantaneous eigenspaces of $I(s)$ is $\mathcal{H}_{PLm}(s)$ corresponding to the projections $P(Lm, s)$. The denotations of the number $L$ and $m$ are pointed out in the following. At time $t = 0$, the projection $P(Lm, 0)$ is

$$P(Lm, 0) = \bigotimes_{j=1}^{n} P_j(0). \quad (68)$$

Here $P_j(0)$ is the projection onto the eigenspace $\mathcal{H}_{Pj}(0)$ of $I_j(0)$. From Eq.(67), we can easily write the projection at time $s$ in the form:

$$P(Lm, s) = U(s) \bigotimes_{j=1}^{n} P_j(0) U^\dagger(s), \quad (69)$$

where

$$U(s) = \bigotimes_{j=1}^{n} A_j(s). \quad (70)$$

We suppose that the eigenvalue of the above projection is $hLK(s)$ and $m$ is an additional quantum number to distinguish the degenerate eigenspaces belonging to the eigenvalue. The Zeno subspace and the corresponding projection belonging to the eigenvalue $hLK(s)$ are respectively

$$\mathcal{H}_{PL}(s) = \bigoplus_m \mathcal{H}_{PLm}(s), \quad (71)$$

$$P(L, s) = \sum_m P(Lm, s). \quad (72)$$

We assume that the magnitude $h$ of the field is larger than the critical point $h_c = 4$ [14] and the initial state of the measured system is the ground state(ferromagnetic):

$$|n, 0 \rangle = \bigotimes_{j=1}^{n} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (73)$$

Since the ground state is non-degenerate, we denote the corresponding projection by $P(n,0)$. Using Eq.(45), the jump probability from Zeno subspace $\mathcal{H}_{Pn}(0)$ to $\mathcal{H}_{PL}(s)$ is

$$W(P(n,0) \rightarrow P(L,1)) = \int_{0}^{1} ds_1 \int_{0}^{s_1} ds_2 Tr\{U^\dagger(s_1)H_0(s_1)U(s_1)P(n,0)U^\dagger(s_2)H_0(s_2)U(s_2)P(L,0)\} \times \exp\{i \int_{s_2}^{s_1} hT(n-L)K(s') ds' \}. \quad (74)$$

Now for simplicity, let us consider two qubits described by free Hamiltonian $H_0$

$$H_0 = \sigma_1^x \sigma_2^x + 2\sigma_1^y \sigma_2^y + \sigma_1^z \sigma_2^z. \quad (75)$$

$H_0$ can be regard as a perturbation in comparison with $I(s)$ as long as we tune the magnitude $h$ of the field. We denote the projections onto the eigenspaces of $I(s)$ by

$$P_1(s) = P(\uparrow\uparrow, s) \quad P_2(s) = P(\uparrow\downarrow, s) \quad P_3(s) = P(\downarrow\uparrow, s) \quad P_4(s) = P(\downarrow\downarrow, s), \quad (76)$$
which belong to the eigenvalues \((-2hK(s)), 0, 0, (2hK(s))\), respectively. Therefore the Zeno subspaces belonging to the eigenvalues \((-2hK(s)), 0, (2hK(s))\) are respectively

\[ \mathcal{H}_{P_1}(s), \quad \mathcal{H}_{P_2}(s) \bigoplus \mathcal{H}_{P_3}(s), \quad \mathcal{H}_{P_4}(s). \]  

(77)

At \(t=0\), we have

\[
P_1(0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_3(0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_4(0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(78)

From Eq.(64), (65), (70), (74), (75) and (78), We find the jump probability has the simple form

\[
W(P_1(0) \rightarrow P_2(1)) = 0, \quad W(P_1(0) \rightarrow P_3(1)) = 0, \quad W(P_1(0) \rightarrow P_4(1)) = \int_0^1 ds_1 \int_0^1 ds_2 \exp(i \int_{s_2}^{s_1} 4hTK(s')ds')
\]

\[
= \int_0^1 ds_1 \int_0^1 ds_2 \cos(4hT \int_0^{s_1} K(s')ds') - 4hT \int_0^{s_2} K(s')ds')
\]

\[
= (\int_0^1 ds \cos(4hT \int_0^{s} K(s')ds'))^2 + (\int_0^1 ds \sin(4hT \int_0^{s} K(s')ds'))^2.
\]

(81)

We can see that the jump probability from \(\mathcal{H}_{P_1}(s)\) to \(\mathcal{H}_{P_2}(s) \bigoplus \mathcal{H}_{P_3}(s)\) is zero, which can be explained by the following matrix element:

\[
\langle \uparrow \downarrow, s | H_0 | \uparrow \uparrow, s \rangle = 0. \quad \langle \downarrow \uparrow, s | H_0 | \uparrow \uparrow, s \rangle = 0. \quad \langle \downarrow \downarrow, s | H_0 | \uparrow \uparrow, s \rangle = -1.
\]

(82) (83) (84)

From Eq.(82) and (83), we see that jump process from \(\mathcal{H}_{P_1}(s)\) to \(\mathcal{H}_{P_2}(s) \bigoplus \mathcal{H}_{P_3}(s)\) is forbidden under the action of \(H_0\) at any time. Therefore the jump probability running out of \(\mathcal{H}_{P_1}(s)\) is the contribution of the probability \(W(P_1(0) \rightarrow P_4(1))\). Comparing the matrix elements of \(H_0\) with the minimal energy difference \(2hK(s)\) of \(I(s)\), we also find that \(H_0\) is a perturbation with the condition \(h \geq 4\). On the other hand, from the result of Eq.(61), we get the adiabatic approximation condition \(\hbar T \gg 1\). The above two conditions are for the validity of Eq.(81).

Fig.1 shows the jump probability which varies with the time \(T\) in the form \(\sin^2 T\) without the interaction \(I(s)\). Fig.2 and Fig.3 show two cases of the jump probability determined by Eq.(81). It is given in Fig.2 that the probability of the system jumping out of \(\mathcal{H}_{P_1}(s)\) varies with the magnitude \(h\) of the field, where the duration \(T\) of the measurement is 1 and the magnitude is larger than 9 for the adiabatic approximation condition. We see that the amplitude of the probability declines with the enhancement of the magnitude. In Fig.3, we change the duration of the measurement from 1 to 10 with \(h = 9\). The amplitude tends to zero rapidly with the duration \(T\) increasing. Therefore we find that the measurement does slow down the decay of \(\mathcal{H}_{P_1}(s)\) by enhancing the magnitude of the field or the duration of the measurement. The quantum Zeno effect takes place.
Figure 1: the jump probability running out of $\mathcal{H}_{P_1}(s)$ varies with the time $T$ without the action of the field. The amplitude does not change with $T$ increasing.

Figure 2: The jump probability running out of $\mathcal{H}_{P_1}(s)$ varies with the magnitude $h$ of the magnetic field, where the duration of the measurement $T = 1$. The amplitude declines with $h$ increasing.

Figure 3: The jump probability running out of $\mathcal{H}_{P_1}(s)$ varies with the duration of the measurement, where the magnitude of the magnetic field $h = 9$. The more slowly we rotate the magnetic field the smaller the amplitude is.
6 Conclusions

We have analyzed the time-dependent measurement and get a general expression (45) of jump probability between different Zeno subspaces of the interaction Hamiltonian $K\mathcal{H}_{\text{meas}}(s)$. The validity of the expression has two conditions: the free Hamiltonian can be regarded as a perturbation of $K\mathcal{H}_{\text{meas}}(s)$ and $K\mathcal{H}_{\text{meas}}(s)$ changes efficiently slowly to satisfy the adiabatic approximation condition. Therefore the result is a general perturbative method which describes the dynamics of quantum Zeno subspaces. It can be applied to not only the time-independent measurement, but also the time-dependent one. We use this expression in two time-independent measurement’s examples to explain the quantum Zeno effect. We also use it in a time-dependent measurement on an XYZ Heisenberg spin chain. In this measurement the Zeno subspace we adopt is one-dimensional. We can see that the jump probability out of its initial Zeno subspace reduces as the coupling constant $K$. In fact, the analysis of the multidimensional Zeno subspace is just similar with that of the one-dimensional’s. Now we are able to apprehend the dynamics of the quantum Zeno subspaces more.

It is well known that it is important that one can prepare and/or control the state of the system under consideration at one’s will in quantum information and computation. Recently, a novel mechanism to purify quantum states, based on the Zeno-like measurements, has been proposed [15]. The purification process of states characterized by the specific interactions of the systems was shown [16] to be controlled through the continuous measurements, i.e., the quantum Zeno dynamics. We believe that the perturbative approach for the quantum Zeno dynamics given by us here is helpful to discuss the quantum state purification.

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