MATHEMATICAL PROPERTIES OF THE REGULAR *
*-REPRESENTATION OF MATRIX *-ALGEBRAS WITH APPLICATIONS TO SEMIDEFINITE PROGRAMMING

CRISTIAN DOBRE
Johann Bernoulli Institute for Mathematics and Computer Science
University of Groningen
Groningen, The Netherlands

(Communicated by Yanqin Bai)

ABSTRACT. In this paper we give a proof for the special structure of the Wedderburn decomposition of the regular *-representation of a given matrix *-algebra. This result was stated without proof in: de Klerk, E., Dobre, C. and Pasechnik, D.V.: Numerical block diagonalization of matrix *-algebras with application to semidefinite programming, Mathematical Programming-B, 129 (2011), 91–111; and is used in applications of semidefinite programming (SDP) for structured combinatorial optimization problems. In order to provide the proof for this special structure we derive several other mathematical properties of the regular *-representation.

1. Introduction. Semidefinite programming (SDP) may be described as linear programming (LP) with positive semidefinite matrix variables. For given symmetric \( n \times n \) matrices \( A_0, \ldots, A_m \) and \( b \in \mathbb{R}^m \), the standard SDP problem is defined as:

\[
\begin{align*}
\inf & \quad \text{trace}(A_0X) \\
\text{s.t.} & \quad \text{trace}(A_iX) = b_i \quad (i = 1, \ldots, m) \\
& \quad X \succeq 0,
\end{align*}
\]

where \( X \succeq 0 \) means \( X \) must be symmetric positive semidefinite.

The SDP’s for which one can use the results in this paper have large data matrices \( A_0, \ldots, A_m \), and they are not tractable without exploiting the structure in this data. They formed the motivation to study the Wedderburn decomposition of the regular *-representation of a given matrix *-algebras.

Of particular interest is a structure called algebraic symmetry, where the SDP data matrices are contained in a low-dimensional matrix *-algebra. (Recall that a matrix *-algebra is a linear subspace of \( \mathbb{C}^{n \times n} \) that is closed under multiplication and taking complex conjugate transposes.) Although this structure may seem exotic, it arises in a surprising number of applications, and first appeared in a paper by Schrijver [28] in 1979 on bounds for binary code sizes. (Another early work on algebraic symmetry in SDP is by Kojima et al. [22].)

2010 Mathematics Subject Classification. Primary: 90C22; Secondary: 06B15.

Key words and phrases. Matrix algebras, regular *-representation, algebraic symmetry, semidefinite programming, pre-processing.

The work of C. Dobre was partially supported by Vici grant 639.033.907 from the Netherlands Organization for Scientific Research (NWO).
More recent applications are surveyed in [14, 8, 30] and include bounds on kissing numbers [1], bounds on crossing numbers in graphs [20, 18], bounds on code sizes [29, 7, 23], truss topology design [13, 3], partitioning problem [16], traveling salesman problem [17] etc.

Algebraic symmetry may be exploited since matrix *-algebras have a canonical block diagonal structure after a suitable unitary transform. This result is due to Wedderburn [31] and dates back to 1907. Block diagonal structure may in turn be exploited by interior point algorithms. For some examples of SDP instances with algebraic symmetry, the required unitary transform is known beforehand, e.g. as in [29]. For other examples, like the instances in [20, 18], it is not. When this is the case one may perform numerical pre-processing in order to obtain the required unitary transformation. Murota et al. [27] presented a practical randomized algorithm that may be used for pre-processing of SDP instances with algebraic symmetry; and later this work has been extended by Maehara and Murota [25].

A nice survey on invariant semidefinite programs (finite and infinite dimensional) is given in the chapter Invariant semidefinite programs of the Handbook on Semidefinite, Conic and Polynomial Optimization, chapter written by Bachoc, Gijswijt, Schrijver and Vallentin [2]. They present how to reduce the matrix sizes by regular *-representation and by block diagonalization. Since the latter approach gives the finest decomposition of a matrix *-algebra the authors do not take into consideration combining the two techniques.

However, if we do not have the analytical expression of the block diagonalization, it is not always possible to conduct numerical computations directly to the original matrix *-algebra due to the size of the data matrices. In such situations it makes sense to first use an isomorphic representation (i.e. regular *-representation) of the matrix *-algebra, which will reduce the size of the data matrices to the dimension of the algebra (i.e the cardinality of its basis); and then perform the canonical block decomposition. For example, in [19], the $\vartheta'$ number of the so-called Erdös-Renyi graphs was studied. These graphs, denoted by $ER(q)$ are determined by a single parameter $q > 2$, which is prime. The number of vertices (which will give the size of the matrices in the corresponding SDP relaxation) is $n = q^2 + q + 1$, but the dimension of the algebra is only $2q + 11$. Note that, for example, if $q = 157$, one has $n = 24807$, making it impossible to solve the problem numerically without exploiting the symmetry. Moreover, the Wedderburn decomposition of the algebra is not known in closed form [19]. The $\vartheta'$-number of a graph was introduced in [26] as a strengthening of the Lovász $\vartheta$-number [24] upper bound on the co-clique number of a graph. The $\vartheta'$-number was also studied in detail for Hamming graphs in the seminal paper by Schrijver [28].

Another structured combinatorial optimization problem where the regular *-representation has proven to be useful in reducing the size of the corresponding SDP relaxation is computing the crossing number of complete bipartite graphs. Recall that the crossing number $\text{cr}(G)$ of a graph $G$ is the minimum number of intersections of edges in a drawing of $G$ in the plane.

The crossing number of the complete bipartite graph $K_{r,s}$ is only known in a few special cases (like $\min\{r, s\} \leq 6$), and it is therefore interesting to obtain lower bounds on $\text{cr}(K_{r,s})$. (There is a well known upper bound on $\text{cr}(K_{r,s})$ via a drawing which is conjectured to be tight.)
De Klerk et al. [20] showed that one may obtain a lower bound on $cr(K_{r,s})$ via the optimal value of a suitable SDP problem, namely:

$$
cr(K_{r,s}) \geq \frac{s}{2} \left( s \min_{X \geq 0, X \geq 0} \{ \text{trace}(QX) \mid \text{trace}(JX) = 1 \} - \left\lfloor \frac{r}{2} \right\rfloor \left\lfloor \frac{r-1}{2} \right\rfloor \right),$$

where $Q$ is a certain (given) matrix of order $n = (r-1)!$, and $J$ is the all-ones matrix of the same size. In [20] it was proven that one can restrict the optimization to the centralizer algebra of $\text{aut}(Q)$, say $A_{\text{SDP}}$. For this SDP problem the algebra $A_{\text{SDP}}$ is a coherent algebra and an orthogonal basis $B_1, \ldots, B_d$ of zero-one matrices of $A_{\text{SDP}}$ is available. For $r = 9$ for example, using the regular $*$-representation of $A_{\text{SDP}}$, the dimension of the SDP constraint was reduced from $n = 40320$ to $d = 2438$. Further, the Wedderburn decomposition of the regular $*$-representation of $A_{\text{SDP}}$ yields linear matrix inequalities involving matrices with maximum size 12. This improved significantly the computational time of the underlying SDP problem, see [15].

Outline. The paper is structured as follows. In Section 2 we review some basic properties of matrix $*$-algebras. In particular, the canonical block decomposition is described. Section 3 introduces the regular $*$-representation of a given matrix $*$-algebra and includes an extension of a theorem due to de Klerk et al [18]. Section 4 proves that the regular $*$-representation is invariant to changing the basis of the matrix $*$-algebra, up to an orthogonal transformation. Finally, using this result, in Section 5 we prove a special block structure of the Wedderburn decomposition of the regular $*$-representation.

2. Basic properties of matrix $*$-algebras. In what follows we give a review of decompositions of matrix $*$-algebras over $\mathbb{C}$, with an emphasis on the constructive (algorithmic) aspects.

Definition 2.1. A set $\mathcal{A} \subseteq \mathbb{C}^{n \times n}$ is called a matrix $*$-algebra over $\mathbb{C}$ (or a matrix $\mathbb{C}$-$*$-algebra) if, for all $X, Y \in \mathcal{A}$:

- $\alpha X + \beta Y \in \mathcal{A}$ for all $\alpha, \beta \in \mathbb{C}$;
- $X^* \in \mathcal{A}$;
- $XY \in \mathcal{A}$.

A matrix $\mathbb{C}$-$*$-subalgebra of $\mathcal{A}$ is said to be maximal if it is not contained in any proper $\mathbb{C}$-$*$-subalgebra of $\mathcal{A}$. (Recall that a subset of a set is proper if it is not the empty set or the set itself.)

In applications one often encounters matrix $\mathbb{C}$-$*$-algebras with the following additional structure.

Definition 2.2. Assume that a given set of zero-one $n \times n$ matrices $\{A_1, \ldots, A_d\}$ has the following properties:

1. $\sum_{i \in I} A_i = I$ for some index set $I \subseteq \{1, \ldots, d\}$ and $\sum_{i=1}^d A_i = J$;
2. $A_i^T \in \mathcal{A}$ for each $i$;
3. $A_i A_j \in \text{span}\{A_1, \ldots, A_d\}$ for all $i, j$.

Then $\{A_1, \ldots, A_d\}$ is called a coherent configuration.

Thus, a coherent configuration is a basis of zero-one matrices of a (possibly non-commutative) matrix $*$-algebra. Such an algebra is called a coherent algebra. Moreover, when the elements of the set $\{A_1, \ldots, A_d\}$ commute and $I \subseteq \{A_1, \ldots, A_d\}$, the basis of zero-one matrices is called an association scheme.
Proposition 1 (see e.g., Section 1.5 in [9]). The elements of a commutative matrix $\mathbb{C}^*$-algebra have a common set of orthonormal eigenvectors. These may be viewed as the columns of a unitary matrix $Q$, i.e., $Q^*Q = I$.

More information on coherent configurations and related structures may be found in [11] and [4].

As a consequence of Proposition 1, any element of a commutative matrix $\mathbb{C}^*$-algebra can be diagonalized using the same unitary matrix $Q$. If the commutativity does not appear then one could block diagonalize the elements of the algebra as we will see further in this section.

For matrices $A_1, A_2$, the direct sum is defined as
\[
A_1 \oplus A_2 := \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},
\]
and we will denote the iterated direct sum of $A_1, ..., A_n$ by $\bigoplus_{i=1}^n A_i$. If all $A_i$ are equal we define:
\[
t \circ A := \bigoplus_{i=1}^t A.
\]

Let $\mathcal{A}$ and $\mathcal{B}$ be two matrix $\mathbb{C}^*$-algebras. Then the direct sum of $\mathcal{A}$ and $\mathcal{B}$ is:
\[
\mathcal{A} \oplus \mathcal{B} := \{ M \oplus M' \mid M \in \mathcal{A}, M' \in \mathcal{B} \}.
\]

We say that $\mathcal{A}$ is a zero algebra if $\mathcal{A}$ consists only of the zero matrix.

Definition 2.3. A matrix $\mathbb{C}^*$-algebra is called simple if it has no nontrivial ideal. (An ideal of $\mathcal{A}$ is a $\ast$-subalgebra that is closed under both left and right multiplication by elements of $\mathcal{A}$.)

Definition 2.4. A matrix $\mathbb{C}^*$-algebra is called basic if
\[
\mathcal{A} = t \circ \mathbb{C}^{s \times s} := \{ t \circ M \mid M \in \mathbb{C}^{s \times s} \}
\]
for some integers $s, t$.

Definition 2.5. Two matrix $\mathbb{C}^*$-algebras $\mathcal{A}, \mathcal{B} \subset \mathbb{C}^{n \times n}$ are called equivalent if there exists a unitary matrix $Q \in \mathbb{C}^{n \times n}$ such that
\[
\mathcal{B} = \{ Q^* M Q \mid M \in \mathcal{A} \} =: Q^* \mathcal{A} Q.
\]

Proposition 2 (see e.g., Section 2.2 in [6]). Every matrix $\mathbb{C}^*$-algebra $\mathcal{A}$ containing the identity is equivalent to a direct sum of simple matrix $\mathbb{C}^*$-algebras.

Proposition 3 (see e.g., Section 2.2 in [6]). Every simple matrix $\mathbb{C}^*$-algebra $\mathcal{A}$ containing the identity is equivalent to a basic matrix $\mathbb{C}^*$-algebra.

Propositions 2 and 3 imply the so-called fundamental structure theorem for matrix $\mathbb{C}^*$-algebras, which is as follows:

Theorem 2.6 (see [31]). If $\mathcal{A} \subseteq \mathbb{C}^{n \times n}$ is a matrix $\ast$-algebra that contains the identity, then there exist a unitary matrix $Q$ and positive integers $p$ and $n_i, t_i$ ($i = 1, \ldots, p$) such that
\[
Q^* \mathcal{A} Q = \bigoplus_{i=1}^p t_i \circ \mathbb{C}^{n_i \times n_i}.
\]

Thus, \( \dim(\mathcal{A}) = \sum_{i=1}^p n_i^2 \) and \( n = \sum_{i=1}^p t_i n_i \).

If the identity does not belong to $\mathcal{A}$, then in view of Definition 2.5, each matrix $\ast$-algebra over $\mathbb{C}$ is equivalent to a direct sum of basic algebras and possibly a zero algebra. A detailed proof of this result is given e.g., in [6] (Theorem 1 there).
3. Regular *-representation of a matrix *-algebra.

**Definition 3.1** (see e.g., Section 1 in [5]). A representation of an algebra $\mathcal{A}$ is a vector space $V$ together with a homomorphism of algebras $\varphi : \mathcal{A} \mapsto \text{End}(V)$, where $\text{End}(V)$ denotes the set of endomorphisms from $V$ to $V$.

**Definition 3.2.** When $V = \mathcal{A}$ and $\varphi : \mathcal{A} \mapsto \text{End}(\mathcal{A})$ is given by $\varphi(A)Y = Ay$ $\forall Y \in \mathcal{A}$, one obtains the regular representation of $\mathcal{A}$. Moreover, when $\mathcal{A}$ has an involution operation, say $*$, and $\varphi(A^*) = \varphi(A)^*$ $\forall A \in \mathcal{A}$, one obtains the regular *-representation.

Note that in the definition above $A^*$ is the involution of $A$, and $\varphi(A)^*$ is the adjoint of the linear operator $\varphi(A)$. We will use the notation $\varphi_A := \varphi(A)$, so $\varphi_A(Y) = Ay$ $\forall Y \in \mathcal{A}$.

Assume now that $\mathcal{A}$ has an orthogonal basis of real matrices $B_1, \ldots, B_d \in \mathbb{R}^{n \times n}$, with $B_i^* \in \{B_1, \ldots, B_d\}$ for any $i = 1, \ldots, d$. This situation is not general, but it is usual for the applications in semidefinite programming that we have considered in the introduction.

We normalize this basis with respect to the Frobenius norm:

$$D_i := \frac{1}{\sqrt{\text{trace}(B_i^* B_i)}} B_i \quad (i = 1, \ldots, d),$$

and define multiplication parameters $\gamma_{i,j}^k$ via:

$$D_i D_j = \sum_{k=1}^{d} \gamma_{i,j}^k D_k,$$

and subsequently define the $d \times d$ matrices $L_k$ $(k = 1, \ldots, d)$ via

$$(L_k)_{ij} = \gamma_{i,j}^k, \quad (i, j = 1, \ldots, d).$$

**Lemma 3.3.** For any $k = 1, \ldots, d$, $L_k$ is the matrix representation of the linear operator $\varphi_{D_k}$ with respect to the basis $\{D_1, \ldots, D_d\}$.

**Proof.** Since $D_k \in \mathcal{A}$, for any $k = 1, \ldots, d$ we have

$$\varphi_{D_k}(D_j) = D_k D_j = \sum_{i=1}^{d} (L_k)_{ij} D_i, \quad (j = 1, \ldots, d),$$

which completes the proof. \hfill $\square$

Therefore, we will work with the matrix representation of the linear operator $\varphi_{D_k}$. The matrices $L_k$ form the basis of a matrix *-algebra, say $\mathcal{A}^{rcg}$. We will abuse terminology slightly by calling $\mathcal{A}^{rcg}$ the regular *-representation of $\mathcal{A}$ (with respect to the basis $\{D_1, \ldots, D_d\}$).

The following result is proven by de Klerk, Pasechnik and Schrijver [18] in the case when $\mathcal{A}$ is the centralizer algebra of a group. However, their arguments go through for any matrix *-algebra; and we present here the extended proof for the completeness of this section.

**Theorem 3.4.** The bijective linear mapping $\Phi : \mathcal{A} \mapsto \mathcal{A}^{rcg}$ such that $\Phi(D_k) = L_k$ $(k = 1, \ldots, d)$ defines a *-isomorphism from $\mathcal{A}$ to $\mathcal{A}^{rcg}$. Thus, $\Phi$ is an algebra isomorphism with the additional property

$$\Phi(A^*) = \Phi(A)^* \quad \forall A \in \mathcal{A}.$$
Proof. For any \( Y \in \mathcal{A} \) we define as before the linear operator \( \varphi_Y : \mathcal{A} \mapsto \mathcal{A} \) by
\[
\varphi_Y(X) = YX \quad \forall X \in \mathcal{A}.
\] (5)
Using Lemma 3.3 we have that \( L_k := \Phi(D_k) \) is the matrix corresponding to the linear operator \( \varphi_{D_k} \) in the basis \( D_1, ..., D_d \). Thus, for any \( Y = \sum_k y_k D_k \in \mathcal{A} \), \( \Phi(Y) \) is the matrix corresponding to the linear operator
\[
\varphi_Y = \varphi_{\sum_k y_k D_k} = \sum_k y_k \varphi_{D_k}
\]
in the basis \( D_1, ..., D_d \).
Using (5) we have for any \( Y, Z \in \mathcal{A} \):
\[
\varphi_{YZ}(X) = YZX = \varphi_Y(\varphi_Z(X)) = (\varphi_Y \circ \varphi_Z)(X) \quad \forall X \in \mathcal{A}.
\]
Therefore, for any \( Y, Z \in \mathcal{A} \) we have \( \Phi(YZ) = \Phi(Y)\Phi(Z) \). Thus, \( \Phi \) is an algebra homomorphism.
\( \Phi(Y) = 0 \) implies that \( YX = 0 \ \forall X \in \mathcal{A} \), and in particular we obtain \( YY^* = 0 \), which implies that \( Y = 0 \). Therefore, \( \Phi \) is injective and by construction we conclude that it is a bijection.
We still need to show that \( \Phi \) is a *-isomorphism (i.e., it preserves symmetry). To do so, we need to show that \( \Phi(Y^*) = \Phi(Y)^* \).
On the one hand, by definition of \( \varphi_Y \) we have \( \varphi_Y(D_j) = YD_j \); on the other hand, using the fact that \( \Phi(Y) \) is the matrix of operator \( \varphi_Y \) in the basis \( D_1, ..., D_d \) we obtain:
\[
YD_j = \sum_{i=1}^d \Phi(Y)_{ij} D_i.
\]
Using the orthonormality of the basis \( D_1, ..., D_d \), in the above relation, we take the inner product with the matrices \( D_i \) and use the linearity of the operator. Hence,
\[
\text{trace}(D_i^T YD_j) = \sum_{i=1}^d \Phi(Y)_{ij} \text{trace}(D_i^T D_i) = \Phi(Y)_{ij}.
\]
In the same way:
\[
Y^* D_i = \sum_{i=1}^d \Phi(Y^*)_{ti} D_t
\]
and we take the inner product with the matrices \( D_j \). Notice that if \( A \in \mathbb{C}^{n \times n} \) then \( \text{trace}(A^*) = \overline{\text{trace}(A)} \). From the orthonormality of the basis, the right-hand side becomes \( \Phi(Y^*)_{jj} \). Hence,
\[
\Phi(Y^*)_{ji} = \text{trace}(D_j^T Y^* D_i) = \text{trace}(D_j^T Y^* D_i)^* = \text{trace}(D_j^T Y D_j)^* = \Phi(Y)_{ij},
\]
therefore the preservation of the symmetry is proved. \( \square \)
Since \( \Phi \) is a homomorphism, \( A \) and \( \Phi(A) \) have the same eigenvalues (up to multiplicities) for all \( A \in \mathcal{A} \). As a consequence, we have the following theorem.

**Theorem 3.5.** Let \( \{D_1, ..., D_d\} \) be an orthonormal basis of a matrix *-algebra \( \mathcal{A} \), \( \{L_1, ..., L_d\} \) the basis of the regular *-representation of \( \mathcal{A} \) (i.e., \( \mathcal{A}^{reg} \)) as defined in (4), and \( x \in \mathbb{R}^d \). We have
\[
\sum_{i=1}^d x_i D_i \succeq 0 \iff \sum_{i=1}^d x_i L_i \succeq 0.
\]
Example 3.1. Consider the 5-cycle (pentagon), denoted $C_5$. The automorphism group of $C_5$ is the so-called dihedral group on 5 elements and has order $|\text{aut}(C_5)| = 10$. The centralizer algebra of this group has the following basis:

$$B_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$  

The purpose of this example is to illustrate the regular $^*$-representation of a matrix $^*$-algebra. Hence, for details about centralizer algebras the reader is referred to Section 4 in [21]. Further, we normalize the basis $B_1, B_2, B_3$ to get

$$D_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad D_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$D_3 = \frac{1}{\sqrt{10}} \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$  

Then, Table 1 will give us the coefficients $\gamma_{i,j}^k$, $i, j, k = 1, \ldots, 3$.

$$\begin{array}{|c|c|c|c|}
\hline
 & D_1 & D_2 & D_3 \\
\hline
D_1 & \frac{1}{\sqrt{5}}D_1 & \frac{1}{\sqrt{5}}D_2 & \frac{1}{\sqrt{5}}D_3 \\
\hline
D_2 & \frac{1}{\sqrt{5}}D_2 & \frac{1}{\sqrt{5}}D_1 + \frac{1}{\sqrt{10}}D_3 & \frac{1}{\sqrt{10}}(D_2 + D_3) \\
\hline
D_3 & \frac{1}{\sqrt{5}}D_3 & \frac{1}{\sqrt{10}}(D_2 + D_3) & \frac{1}{\sqrt{5}}D_1 + \frac{1}{\sqrt{10}}D_2 \\
\hline
\end{array}$$

**Table 1.** Multiplication table of the normalized matrices.

Further, using (3) and (4), we can easily compute by hand the matrices $L_1, L_2, L_3$, that form the basis of the regular $^*$-representation:

$$L_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & \frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & 0 & \frac{1}{\sqrt{10}} \\ 0 & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix},$$

$$L_3 = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{5}} \\ 0 & \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{10}} & 0 \end{pmatrix}.$$  

Notice that in this toy example we have reduced the size of the basis matrices from $n = 5$ to $d = 3$. 


4. A change of basis. By Wedderburn’s theorem, any matrix $C^*\text{-algebra } A$ that contains the identity takes the form
\[ Q^*AQ = \oplus_{i=1}^{t} t_i \odot \mathbb{C}^{n_i \times n_i}, \quad (6) \]
for some integers $t$, $t_i$, and $n_i$ ($i = 1, \ldots, t$), and some unitary $Q$.

Our further goal is to show that the Wedderburn decomposition of $A^{reg}$ has a special structure that does not depend on the values $t_i$ ($i = 1, \ldots, t$). To this end, the lemmas in this section show how the regular $^*$-representation behaves when the orthonormal basis of the matrix $^*$-algebra $A$ is changed.

**Lemma 4.1.** The regular $^*$-representations of $A$ and $Q^*AQ$ are the same.

**Proof.** Denote by $A_Q$ the algebra after block diagonalization. We have that \( \{Q^*D_1Q, \ldots, Q^*D_dQ\} \) is a basis for $A_Q$. We will prove that applying the regular $^*$-representation to both $A$ and $A_Q$ yields the same matrices denoted earlier in this section by $L_1, \ldots, L_d$.

If we denote $D'_i := Q^*D_iQ$, then from (3), by multiplying with $Q^*$ and $Q$ to the left and right respectively we obtain:
\[ Q^*D_iD_jQ = \sum_{k=1}^{d} \gamma_{i,j}^k Q^*D_kQ, \]

Further, since $Q$ is unitary, we have
\[ Q^*D_iQQ^*D_jQ = \sum_{k=1}^{d} \gamma_{i,j}^k Q^*D_kQ, \]
and using the earlier notation \( D'_iD'_j = \sum_{k=1}^{d} \gamma_{i,j}^k D'_k \), which proves that we have the same values $\gamma_{i,j}^k$ so we obtain the same regular $^*$-representation for both $A$ and $A_Q$. \[ \square \]

This implies that, when studying $A^{reg}$, we may assume without loss of generality that $A$ takes the form
\[ A = \oplus_{i=1}^{t} t_i \odot \mathbb{C}^{n_i \times n_i}. \]

**Lemma 4.2** (see e.g., Sections 0.1.0 and 1.0.1 in [12]). Let $V$ be a vector space of dimension $d:=\text{dim}(V)$, let $L : V \to V$ be a linear operator, and $B = \{B_1, \ldots, B_d\}$, $B' := \{B'_1, \ldots, B'_d\}$ two bases of $V$. Then there exists a matrix $S \in \mathbb{R}^{d \times d}$, independent of $L$, such that
\[ M^L_B = S^{-1} M^L_{B'}, \]
where $M^L_B$ is the matrix representation of $L$ with respect to basis $B$ and $S$ is the transition matrix from $B$ to $B'$.

**Corollary 1.** Let $V$ be a vector space of dimension $d:=\text{dim}(V)$, let $L : V \to V$ be a linear operator, and $B = \{D_1, \ldots, D_d\}$, $B' := \{D'_1, \ldots, D'_d\}$ two orthonormal bases of $V$. Then there exists a unitary matrix $Q \in \mathbb{R}^{d \times d}$, independent of $L$, such that
\[ M^L_B = Q^* M^L_{B'} Q, \]
where $M^L_B$ is the matrix representation of $L$ with respect to basis $B$, and $Q$ is the unitary transition matrix from $B$ to $B'$. \[ \square \]
Proof. Let $B$ denote the unitary matrix containing the orthonormal vectors of $B$, and $B'$ denote the unitary matrix containing the orthonormal vectors of $B'$. If $Q$ is the transition matrix, then $B = Q^* B'$. Since both $B$ and $B'$ are unitary matrices, it follows that $Q$ is also unitary. Using Lemma 4.2 we conclude the proof.

**Lemma 4.3.** Let $A^{reg}$ be the regular $*$-representation of $A$ with respect to the orthonormal basis $\{D_1, \ldots, D_d\}$, and let $A'^{reg}$ be the regular $*$-representation of $A$ with respect to the orthonormal basis $\{D'_1, \ldots, D'_d\}$. Then there exists a unitary matrix $Q$ such that $
abla A^{reg} = Q^* A'^{reg} Q$.

**Proof.** Define as before the linear mappings $\Phi : A \mapsto A^{reg}$ such that $\Phi(D_k) = L_k (k = 1, \ldots, d)$, and $\Phi' : A \mapsto A'^{reg}$ such that $\Phi'(D'_k) = L'_k (k = 1, \ldots, d)$. Then we have

\[
A^{reg} = \left\{ \sum_{k=1}^{d} \alpha_k L_k \mid \alpha_k \in \mathbb{C} \right\} \quad \text{and} \\
A'^{reg} = \left\{ \sum_{k=1}^{d} \alpha'_k L'_k \mid \alpha'_k \in \mathbb{C} \right\},
\]

where, by Lemma 3.3, $L_k = \Phi(D_k)$ is the matrix corresponding to the linear operator $\varphi_{D_k}$ in the basis $\{D_1, \ldots, D_d\}$, and $L'_k = \Phi'(D'_k)$ is the matrix corresponding to the linear operator $\varphi_{D'_k}$ in the basis $\{D'_1, \ldots, D'_d\}$.

Thus, for any $A = \sum_k \alpha_k D_k \in A$, $\Phi(A)$ is the matrix corresponding to the linear operator

\[
\varphi_A = \sum_k \alpha_k \varphi_{D_k}
\]

in the basis $D_1, \ldots, D_d$. Moreover, if we write $A = \sum_k \alpha'_k D'_k \in A$, then $\Phi'(A)$ is the matrix corresponding to the linear operator

\[
\varphi_A' = \sum_k \alpha'_k \varphi_{D'_k}
\]

in the basis $D'_1, \ldots, D'_d$.

By Corollary 1, if $A \in A$, the matrix representations of $\varphi_A$ with respect to the two orthonormal bases $\{D_1, \ldots, D_d\}$ and $\{D'_1, \ldots, D'_d\}$ are related via

\[
\Phi(A) = Q^* \Phi'(A) Q,
\]

where $Q$ is some orthonormal matrix that does not depend on $A$. This concludes the proof.

5. **Wedderburn decomposition of regular $*$-representation.** In this section we will prove the main result of this paper, namely: when constructing the Wedderburn decomposition of the regular $*$-representation one obtains in (6) the number of identical blocks (i.e. $t_i$) equal to the size of the identical blocks (i.e. $n_i$). To this end we will prove three lemmas and for the first lemma we need a basic property of the Kronecker product of two matrices. We do not go into details, for further information on this topic the reader is referred to [10].

If $n$ denotes the size of two given matrices $A$ and $B$, then $A \otimes B$ denotes a block matrix with block $ij$ given by $A_{ij} B$ (Kronecker product). One has:
Let \( t \) and \( n \) be given integers. The regular \(*\)-representation of \( t \odot \mathbb{C}^{n \times n} \) is equivalent to \( n \odot \mathbb{C}^{n \times n} \), for the standard basis.

**Proof.** The standard basis of \( t \odot \mathbb{C}^{n \times n} \) clearly has \( n^2 \) elements since we have \( t \) repeated blocks. Let

\[
D_{i_1,i_2} := \frac{1}{\sqrt{t}} I_t \otimes e_{i_1} e_{i_2}^T, \quad (i_1, i_2 = 1, \ldots, n)
\]

denote the normalized basis matrices. Its regular \(*\)-representation will consist of \( n^2 \) dimensional matrices, say \( L_{i_1,i_2}, (i_1, i_2 = 1, \ldots, n) \).

We will show that for all \( i_1, i_2 \) we have

\[
L_{i_1,i_2} = \frac{1}{\sqrt{t}} P^T (I_n \otimes (e_{i_2} e_{i_1}^T)) P,
\]

for some permutation matrix \( P \), and the lemma will therefore be proved.

To this end, for \( i_1, i_2 \in \{1, \ldots, n\} \) let us define \( E^{(i_1,i_2)} := e_{i_1} e_{i_2}^T \). Then, using (3), we have for \( i_1, i_2, j_1, j_2 = 1, \ldots, n \)

\[
\frac{1}{\sqrt{t}} (I_t \otimes E^{(i_1,i_2)})(I_t \otimes E^{(j_1,j_2)}) = \sum_{k_1,k_2=1}^{n} \gamma^{(k_1,k_2)}_{(i_1,i_2),(j_1,j_2)} \frac{1}{\sqrt{t}} I_t \otimes E^{(k_1,k_2)},
\]

for some scalars \( \gamma^{(k_1,k_2)}_{(i_1,i_2),(j_1,j_2)} \). This is equivalent to

\[
\frac{1}{\sqrt{t}} I_t \otimes (E^{(i_1,i_2)} E^{(j_1,j_2)}) = I_t \otimes \sum_{k_1,k_2=1}^{n} \gamma^{(k_1,k_2)}_{(i_1,i_2),(j_1,j_2)} E^{(k_1,k_2)},
\]

which yields:

\[
\frac{1}{\sqrt{t}} E^{(i_1,i_2)} E^{(j_1,j_2)} = \sum_{k_1,k_2=1}^{n} \gamma^{(k_1,k_2)}_{(i_1,i_2),(j_1,j_2)} E^{(k_1,k_2)}.
\]

Since

\[
E^{(i_1,i_2)} E^{(j_1,j_2)} = e_{i_1} e_{i_2}^T e_{j_1} e_{j_2}^T = \delta_{i_2,j_1} \delta_{i_1,j_2} e_{i_1} e_{i_2}^T,
\]

we have

\[
\gamma^{(k_1,k_2)}_{(i_1,i_2),(j_1,j_2)} = \frac{1}{\sqrt{t}} \delta_{i_2,j_1} \delta_{i_1,k_1} \delta_{j_2,k_2}
\]

\[
= \begin{cases} \frac{1}{\sqrt{t}} & \text{if } k_1 = i_1, \, i_2 = j_1, \, k_2 = j_2 \\ 0 & \text{else} \end{cases}
\]

Using

\[
(L_{i_1,i_2})_{(j_1,j_2),(k_1,k_2)} = \gamma^{(i_1,i_2)}_{(k_1,k_2),(j_1,j_2)}
\]

we obtain

\[
L_{i_1,i_2} = \frac{1}{\sqrt{t}} (e_{i_1} e_{i_2}^T) \otimes I_n.
\]

Following (7) we obtain, for a suitable permutation matrix \( P \),

\[
L_{i_1,i_2} = \frac{1}{\sqrt{t}} P^T (I_n \otimes (e_{i_2} e_{i_1}^T)) P,
\]

and this concludes the proof.

**Lemma 5.2.** Let \( t \) and \( n \) be given integers. The regular \(*\)-representation of \( t \odot \mathbb{C}^{n \times n} \) is equivalent to \( n \odot \mathbb{C}^{n \times n} \), for any choice of orthonormal basis.

**Proof.** By Lemma 5.1 the regular \(*\)-representation of \( t \odot \mathbb{C}^{n \times n} \) is equivalent to \( n \odot \mathbb{C}^{n \times n} \) when using the standard basis \( I_t \otimes (e_{i_1} e_{i_2}^T) \). Lemma 4.3 completes the proof.
Lemma 5.3. Let $A_\alpha$ be matrix $*$-algebras and let $A_\alpha^{reg}$ denote their regular $*$-representations, for $\alpha = 1, \ldots, t$. The regular $*$-representation of $\bigoplus_{\alpha=1}^{t} A_\alpha$ is equivalent to $\bigoplus_{\alpha=1}^{t} A_\alpha^{reg}$.

Proof. Let $\{D_1^\alpha, \ldots, D_{d_\alpha}^\alpha\}$ be a given orthonormal basis of $A_\alpha$ for each $\alpha = 1, \ldots, t$. Denote the regular $*$-representation of each $A_\alpha$ by $A_\alpha^{reg}$, with basis $\{L_1^\alpha, \ldots, L_{d_\alpha}^\alpha\}$. Let $d := \sum_{\alpha=1}^{t} d_\alpha$ be the dimension of $A := \bigoplus_{\alpha=1}^{t} A_\alpha$. We now construct an orthonormal basis, say $\{D_1, \ldots, D_d\}$ of $A$. Each matrix $D_i$ will be block diagonal with exactly one nonzero block given by $D_{i}^\alpha$ for some $\alpha \in \{1, \ldots, t\}$ and $j \in \{1, \ldots, d_\alpha\}$. The position of this nonzero block will correspond to the “position” of $A_\alpha$ in the direct sum $\bigoplus_{\alpha=1}^{t} A_\alpha$.

The exact construction is as follows: matrices $D_1, \ldots, D_d$ are formed from $D_1^1, \ldots, D_{d_1}^1$, respectively; matrices $D_{d_1+1}, \ldots, D_{d_1+d_2}$ are formed from $D_1^2, \ldots, D_{d_2}^2$, respectively, etc. If we denote the regular $*$-representation of $A$ by $A^{reg}$, with basis $\{L_1, \ldots, L_d\} \subset \mathbb{C}^{d \times d}$, then the matrix $L_i$ has exactly the same block structure as $D_i$ ($i = 1, \ldots, d$), by construction. In particular, matrices $L_1, \ldots, L_{d_1}$ are formed from $L_1^1, \ldots, L_{d_1}^1$, respectively, etc. We now have $A^{reg} = \bigoplus_{\alpha=1}^{t} A_\alpha^{reg}$. This completes the proof. \qed

Using the last two lemmas, we can readily prove the following theorem.

Theorem 5.4. The regular $*$-representation of $A := \bigoplus_{i=1}^{t} t_i \otimes \mathbb{C}^{n_i \times n_i}$ is equivalent to $\bigoplus_{i=1}^{t} n_i \otimes \mathbb{C}^{n_i \times n_i}$.

The Wedderburn decomposition of $A^{reg}$ therefore takes the form

$$Q^* A^{reg} Q = \bigoplus_{i=1}^{t} n_i \otimes \mathbb{C}^{n_i \times n_i},$$

for some suitable unitary matrix $Q$.

To end, comparing (6) and (8), we may informally say that the $t_i$ and $n_i$ values are equal for all $i$ in the Wedderburn decomposition of a regular $*$-representation.

REFERENCES

[1] C. Bachoc and F. Vallentin, New upper bounds for kissing numbers from semidefinite programming, J. Amer. Math. Soc., 21 (2008), 909–924.
[2] C. Bachoc, D. Gijswijt, A. Schrijver and F. Vallentin, Invariant semidefinite programs, in “Handbook on Semidefinite, Conic and Polynomial Optimization” (eds. M. F. Anjos and J. B. Lasserre), Springer, (2012), 219–270.
[3] Y.-Q. Bai, E. de Klerk, D. V. Pasechnik and R. Sotirov, Exploiting group symmetry in truss topology optimization, Optimization and Engineering, 10 (2009), 331–349.
[4] P. J. Cameron, Coherent configurations, association schemes and permutation groups, in “Groups, Combinatorics and Geometry” (eds. A.A. Ivanov, M.W. Liebeck and J. Saxl), World Scientific, Singapore, (2003), 55–71.
[5] P. Etingof, O. Golberg, S. Hensel, T. Liu, A. Schwendner, E. Uvdovina and D. Vaintrob, Introduction to representation theory, preprint. arXiv:0901.0827v3.
[6] D. Gijswijt, “Matrix Algebras and Semidefinite Programming Techniques for Codes,” Ph. D. Thesis, University of Amsterdam, The Netherlands, 2005.
[7] D. Gijswijt, A. Schrijver and H. Tanaka, New upper bounds for nonbinary codes based on the Terwilliger algebra and semidefinite programming, Journal of Combinatorial Theory, 113 (2006), 1719–1731.
[8] K. Gatermann and P. A. Parrilo, Symmetry groups, semidefinite programs, and sums of squares, J. Pure and Applied Algebra, 192 (2004), 95–128.
[9] C. Godsil, “Association Schemes,” Lecture notes, University of Waterloo, 2010. Available from: http://quoll.uwaterloo.ca/mine/Notes/assoc2.pdf.
[10] A. Graham, “Kroneker Products and Matrix Calculus with Applications,” John Wiley and Sons, Chichester, 1981.
11] D. G. Higman, *Coherent algebras*, Linear Algebra Applications, 93 (1987), 209–239.
12] R. A. Horn and C. R. Johnson, “Matrix Analysis,” Cambridge University Press, 1990.
13] Y. Kanno, M. Ohsaki, K. Murota and N. Katoh, *Group symmetry in interior-point methods for semidefinite program*, Optimization and Engineering, 2 (2001), 293–320.
14] E. de Klerk, *Exploiting special structure in semidefinite programming: a survey, of theory and applications*, European Journal of Operational Research, 201 (2010), 1–10.
15] E. de Klerk, C. Dobre and D. V. Pasechnik, *Numerical block diagonalization of matrix *-algebras with application to semidefinite programming*, Mathematical Programming-B, 129 (2011), 91–111.
16] E. de Klerk, C. Dobre, D. V. Pasechnik and R. Sotirov, *On semidefinite programming relaxations of maximum k-section*, Mathematical Programming-B, Online: http://link.springer.com/article/10.1007%2Fs10107-012-0603-2.
17] E. de Klerk and C. Dobre, *A comparison of lower bounds for the Symmetric Circulant Traveling Salesman Problem*, Discrete Applied Mathematics, 159 (2011), 1815–1826.
18] E. de Klerk, D. V. Pasechnik and A. Schrijver, *Reduction of symmetric semidefinite programs using the regular *-representation*, Mathematical Programming-B, 109 (2007), 613–624.
19] E. de Klerk, M. W. Newman, D. V. Pasechnik and R. Sotirov, *On the Lovász θ-number of almost regular graphs with application to Erdős-Rényi graphs*, European Journal of Combinatorics, 31 (2009), 879–888.
20] E. de Klerk, J. Maharry, D. V. Pasechnik, B. Richter and G. Salazar, *Improved bounds for the crossing numbers of K_{m,n} and K_n*, SIAM Journal on Discrete Mathematics, 20 (2006), 189–202.
21] E. de Klerk and R. Sotirov, *Exploiting group symmetry in semidefinite programming relaxations of the quadratic assignment problem*, Mathematical Programming, 122 (2010), 225–246.
22] M. Kojima, S. Kojima and S. Hara, *Linear algebra for semidefinite programming*, in “Research Report B-290,” Tokyo Institute of Technology, (1997), 1–23.
23] M. Laurent, *Strengthened semidefinite bounds for codes*, Mathematical Programming, 109 (2007), 239–261.
24] L. Lovász, *On the Shannon capacity of a graph*, IEEE Transactions on Information theory, 25 (1979), 1–7.
25] T. Maehara and K. Murota, *A numerical algorithm for block-diagonal decomposition of matrix *-algebras with general irreducible components*, Japan Journal of Industrial and Applied Mathematics, 27 (2010), 263–293.
26] R. J. McEliece, E. R. Rodemich and H. C. Rumsey, *The Lovász bound and some generalizations*, Journal of Combinatorics, Information & System Sciences, 3 (1978), 134–152.
27] K. Murota, Y. Kanno, M. Kojima and S. Kojima, *A numerical algorithm for block-diagonal decomposition of matrix *-algebras with application to semidefinite programming*, Japanese Journal of Industrial and Applied Mathematics, 27 (2010), 125–160.
28] A. Schrijver, *A comparison of the Delsarte and Lovász bounds*, IEEE Transactions on Information Theory, 25 (1979), 425–429.
29] A. Schrijver, *New code upper bounds from the Terwilliger algebra*, IEEE Transactions on Information Theory, 51 (2005), 2859–2866.
30] F. Vallentin, *Symmetry in semidefinite programs*, Linear Algebra and Applications, 430 (2009), 360–369.
31] J. H. M. Wedderburn, *On hypercomplex numbers*, Proceedings of the London Mathematical Society, 6 (1907), 77–118.

Received June 2011; 1st revision November 2012; final revision November 2012.

E-mail address: c.dobre@rug.nl