A non-perturbative method for time-dependent problems in quantum mechanics

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Abstract. A powerful method for calculating the eigenvalues of a Hamiltonian operator consists of converting the energy eigenvalue equation into a matrix equation by means of an appropriate basis set of functions. The convergence of the method can be greatly improved by means of a variational parameter in the basis functions determined by the principle of minimal sensitivity. In the case of the quartic anharmonic oscillator and of a symmetrical double-well potential we choose an effective oscillator frequency. In the case of nonsymmetrical potential we add a coordinate shift in a two-parameter variational calculation. The method not only gives the spectrum, but also an approximation to the energy eigenfunctions. Consequently it can be used to solve the time-dependent Schrödinger equation using the method of stationary states. We apply it to the time development of two different initial wave functions in the double-well slow roll potential.

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1. Introduction

We present a method for obtaining arbitrarily precise approximations to the solution of the time-dependent Schrödinger equation with a potential which fulfills the condition $\lim_{|x| \to \infty} V(x) = +\infty$ (i.e. a potential which only admits bound states). Although there are many examples of potentials of this kind, only a limited number of them can be solved exactly, the best known example being the simple harmonic oscillator (SHO), which is a standard topic in virtually any quantum mechanics textbook and which can be used to model many physical systems.

In this paper we consider problems which cannot be solved analytically and where an alternative strategy must be found. Perturbation theory is the standard tool which is used to deal with such problems; unfortunately, the straightforward application of perturbation theory to some problems is not practical because the perturbation series is divergent. There are various methods to overcome this apparent limitation; for example the linear delta expansion (LDE) \[1, 2\] and other variants of variational perturbation theory (VPT) \[3, 4, 5, 6, 7\]. Loosely speaking, these techniques, although differing in the details, are based on the powerful idea that one can obtain a new expansion in some “unnatural” parameter (i.e. one not appearing in the original problem) and that the sequence of approximants resulting from this expansion can be made to converge very fast by suitably choosing a variational parameter. For example, the linear delta expansion works by interpolating the full Hamiltonian with the Hamiltonian corresponding to a soluble model, which depends on an arbitrary parameter, and by applying perturbation theory to it. The parameter is then determined by means of the principle of minimal sensitivity (PMS) \[8\]. Since the optimum value of the adjustable parameter given by the PMS depends upon the natural parameters in the Hamiltonian, the result corresponds to a non-perturbative result, i.e. to a non-polynomial expression in the natural parameters.

Among other applications of the LDE and VPT we mention an improved Lindstedt-Poincaré method \[9, 10, 11\], the calculation of the period of classical oscillators \[12, 13, 14\], the spectrum of a quantum potential with the WKB method \[15\] and the acceleration of the convergence of mathematical series \[16\]. However, it was found that the LDE fails to give the correct long-time behavior of the wave-function in the quantum mechanical version of the slow-roll scenario of inflation\[17\]. In successive orders it is able to approximate the exact time development more and more accurately, but only up to the time where the wave function has spread out and is beginning to contract again. The Hartree-Fock method does give a general qualitative picture of the time-development at later times, but is very far from being accurate.

Here we propose an alternative method that consists of converting the eigenvalue equation into a matrix equation by taking matrix elements with respect to harmonic oscillator wave functions of arbitrary frequency $\Omega$, and then determining $\Omega$ by some version of the PMS criterion. The particular criterion used here was first proposed and utilized in \[18\]. Having thus obtained the approximate energy eigenvalues and
eigenfunctions, one can then use the method of stationary states to calculate the time-dependence of the state for a given initial configuration. It turns out that this method is extremely accurate at even quite small orders, and has no problem with the long-time behaviour.

The article is organized as follows: in Section 2 we describe the method in general terms and apply it to calculating the spectrum of various anharmonic oscillators; in Section 3 we use the method of stationary states to follow the time development of two initial wave-functions for the double-well potential that has been used in slow roll inflation and compare our results with those found in the literature; finally, in Section 4 we draw our conclusions.

2. Energy Spectrum

2.1. The Method

We tackle the problem of solving the energy eigenvalue equation

$$\hat{H}\psi_n = E_n \psi_n$$

(1)

by converting it to a matrix equation, using an orthonormal basis of wave functions of the quantum harmonic oscillator, depending upon an arbitrary frequency $\Omega \equiv \alpha^2$:

$$\phi_n(x) = N_n e^{-\alpha^2 x^2/2} H_n(\alpha x),$$

(2)

where the normalization constant is $N_n = (\alpha/(2^n n! \sqrt{\pi}))^{1/2}$. $\hat{H}$ is the Hamiltonian for a particle in a one-dimensional potential that supports only bound states.

It is necessary to truncate the infinite-dimensional matrix $H_{n\ell}$ to some finite dimension, say $N \times N$, and then its eigenvalues can be calculated by simple matrix diagonalization. It is known that as $N$ increases the approximation for the energy levels should steadily improve. This is indeed the case, but for an arbitrary $\Omega$ the convergence may be quite slow. This leads one to seek for some criterion to choose an optimum value of $\Omega$. The criterion we shall adopt here, which is essentially that adopted in [18], is the principle of minimal sensitivity applied to the trace of the truncated matrix.

The rationale behind this principle is that the eigenvalues and other exact quantities of the problem are independent of $\Omega$ but any approximate result coming from the diagonalization method for finite $N$ exhibits a spurious dependence on the oscillator frequency. This also applies to the trace of the matrix, the sum of the eigenvalues. However, for finite $N$ a spurious $\Omega$ dependence will emerge in the sum. A reasonable criterion for choosing $\Omega$ is therefore to take it at a stationary point of $T_N \equiv \sum_{n=0}^{N-1} H_{nn}$, so that this invariance is respected, at least locally. Thus we impose the PMS condition

$$\frac{\partial}{\partial \Omega} T_N = 0.$$  

(3)

The reason for applying this condition to the trace is that $T_N$ is a simple quantity to evaluate, and moreover it is invariant under the unitary transformation associated with a change of basis. Once $\Omega$ is so determined, one obtains an approximation to the first $N$
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eigenvalues and eigenvectors of \( \hat{H} \) by a numerical diagonalization of the truncated \( N \times N \) matrix. One could also contemplate applying the PMS to the determinant, which of course shares the property of invariance under unitary transformations, but this would be a much more cumbersome calculation, and could well introduce many spurious PMS values.

In the following sections we will need the harmonic oscillator matrix elements of \( x^p \). Closed formulas have been given in Ref. [19], which we adapt here for completeness.

\[
(x^{2r})_{n\ell} = \frac{\sqrt{n!}!}{(2\sqrt{\alpha})^{2r}} \sum_{k=0}^{\min(n,r-\lambda)} \frac{(2r)!}{2^{2r-k-\lambda}(r-\lambda-k)!(n-k)!(2\lambda+k)!k!}
\]

(4)

for \( \ell - n = 2\lambda \), and

\[
(x^{2r+1})_{n\ell} = \frac{\sqrt{n!}!}{(2\sqrt{\alpha})^{2r+1}} \sum_{k=0}^{\min(n,r-\lambda)} \frac{(2r+1)!}{2^{2r-k-\lambda+\frac{3}{2}}(r-\lambda-k)!(n-k)!(2\lambda+1+k)!k!}
\]

(5)

for \( \ell - n = 2\lambda + 1 \). These formulas assume \( \ell \geq n \), but the matrix is symmetric. In both cases \( r \) must be greater than \( \lambda \). For all other values of \( p, n \) and \( \ell \) the matrix elements vanish.

In addition to these we will need the matrix elements of \( p^2 \), which are given by

\[
(p^2)_{n\ell} = -\frac{1}{2} \alpha^2 \left[ \sqrt{\ell(\ell-1)} \delta_{\ell,n+2} - (2\ell+1) \delta_{\ell,n} + \sqrt{(\ell+1)(\ell+2)} \delta_{\ell,n-2} \right]
\]

(6)

2.2. The quartic anharmonic oscillator

We take the Hamiltonian as

\[
\hat{H} = \frac{1}{2}(p^2 + m^2x^2) + gx^4,
\]

(7)

whose matrix elements in the basis of the wave functions of Eq. (2) are

\[
H_{n\ell} = \frac{1}{2}(p^2)_{n\ell} + \frac{1}{2}m^2(x^2)_{n\ell} + g(x^4)_{n\ell}.
\]

(8)

The trace to order \( N \) is given by

\[
\frac{4}{N} T_N = N(\Omega - \frac{m^2}{\Omega}) + \frac{g}{\Omega^2}(1 + 2N^2).
\]

(9)

It turns out that \( T_N \) has a single minimum, located at

\[
\Omega_{PMS} = \frac{m^2}{X^{1/3}} - \frac{1}{3} X^{1/3},
\]

(10)

where

\[
X \equiv \frac{3}{N} \left[-9g(1 + 2N^2) + \sqrt{3} \left(-N^2m^6 + 27g^2(1 + 2N^2)^2\right)^{\frac{1}{2}} \right]
\]

(11)

The graph of this trace (divided by \( N \)) against \( \Omega \) is shown in Fig. 1, exhibiting a global minimum. Here we have taken the parameters as \( m = 1 \), \( g = 1000 \), and later multiplied the eigenvalues by a factor of 2 for comparison with the results of [20] corresponding to \( \beta = 2000 \). By taking \( \Omega \) at the minimum, in accordance with PMS, we obtain a precise approximation to the spectrum.
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Figure 1. Trace of the Hamiltonian matrix ($\mathcal{N}$) normalized by the number of states as a function of the variational parameter $\Omega$. We use the parameters $m = 1$, $g = 4000$, corresponding to the case studied in [20]. Different curves represent different subspaces.

The error in the energy of the $n$th excited state, i.e. $\Delta = |E_n^{(N)} - E_n^{(100)}|$, decreases exponentially with $N$, as is shown in Fig. 2 for the ground state and for the 49th excited state. $E_n^{(N)}$ is the energy of the $n$th state obtained by considering the $N \times N$ subspace, whereas $E_n^{(100)}$ is the energy of the $n$th state corresponding to the largest subspace considered in this paper, which is used as reference.

For $N = 100$ we obtain the energy of the ground state of the Hamiltonian $p^2 + x^2 + 2gx^4$ as

$$E_0^{(100)} = 13.38844170100806193900617690280728652296098988517435666039$$

which agrees in the first 58 (underlined) digits with the corresponding result of Table 1 of [20], obtained with a different method. Much higher precision can be obtained by enlarging the $N \times N$ subspace: this costs little additional effort because, once the PMS has been applied, the Hamiltonian matrix is fully numerical and its eigenvalues/eigenvectors can be calculated numerically with accuracy and speed.

It is well known that the accuracy of the eigenvalues given by the method of Rayleigh-Ritz decreases as the quantum number increases: this happens because the influence of the states which lie outside the subspace $N \times N$ is felt more strongly by the “border states”, i.e. those states which fall on the border of the selected subspace. In order to calculate highly excited states without enlarging the subspace too much one can center the subspace around that particular state and apply the procedure mentioned above. This has been done in Fig. 3, where we have plotted the ratio $\Delta_n = (E_n^{(N)} - E_n^{(WKB)})/E_n^{(WKB)}$, considering a square subspace of elements $H_{ij}$ centred around the $n = 100$ state. $E_n^{(WKB)}$ is obtained using the analytical formula for the energy of the quantum anharmonic oscillator obtained in [15]. The flattening of the error around $n$ for the largest subspaces considered ($81 \times 81$ and $161 \times 161$) signals that the numerical results obtained with the present method have reached the precision of
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Figure 2. Error in the energy of the ground and 49th excited states as a function of \( n \) and for subspaces of different dimensions. We use the parameters \( m = 1 \), \( g = 4000 \), corresponding to the case studied in [20].

Figure 3. \( \Delta_n = (E^{(N)}_n - E^{(WKB)}_n)/E^{(WKB)}_n \) as a function of the dimension of the subspace.

the WKB formula. We conclude that the present method can be used to obtain the energies and wave functions of arbitrarily high excited states with the desired accuracy. As expected, the error \( \Delta_n \) is maximal for the “border states”.

When the centred subspace is restricted to one element \( N = 1 \) the method reduces to a well-known simple variational calculation.

2.3. The double-well potential

The same method can be applied with very little change to the Hamiltonian which has been used in the consideration of slow roll inflation in the early universe, namely

\[
H = \frac{1}{2}p^2 + \lambda(x^2 - a^2)^2/24 + \text{const}.
\]
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\[ \frac{1}{2}p^2 - \frac{1}{2}m^2x^2 + gx^4, \]  \hspace{1cm} (12)

with \( m^2 = \lambda a^2/6 \) and \( g = \lambda/24 \).

The parameters will be taken as \( a = 5 \) and \( \lambda = 0.01 \), as these have been used in previous work on the subject (see, for example, [17, 22, 23]).

All that is needed in this case is to reverse the sign of \( m^2 \) in the formulas given in the previous subsection. We obtain similar accuracy for the eigenvalues with very little effort. These will be used in Section 3 to give the time development of a given initial configuration.

2.4. General anharmonic potentials

We now consider general anharmonic potentials of the form:

\[ V(x) = \sum_{j=0}^{N} \kappa_j x^j \]  \hspace{1cm} (13)

where the coefficients \( \kappa_j \) define a polynomial of order \( N \) (even). We require that \( \kappa_N > 0 \) to ensure that only bound states are permitted.

The shifted potential

\[ V(x + \sigma) = \sum_{j=0}^{N} \kappa_j (x + \sigma)^j = \sum_{j=0}^{N} \kappa_j \sum_{k=0}^{j} \binom{j}{k} x^{j-k} \sigma^k, \]  \hspace{1cm} (14)

and the original one have the same spectrum. Therefore we can choose \( \sigma \) to be a variational parameter and apply the PMS as in the case of the adjustable oscillator frequency.

As an example, let us consider the potential:

\[ V(x) = 11 - 118x - 44x^2 + 80x^3 + 16x^4. \]  \hspace{1cm} (15)

Since \( V(x) \) is strongly asymmetric we expect that decomposing it with respect to a basis of SHO wave functions centred at the origin will not be the best choice. We therefore translate the potential [21] by an arbitrary quantity \( -\sigma \) and impose the PMS on \( \Omega \) and \( \sigma \) simultaneously. That is, we impose the two PMS conditions

\[ \frac{\partial T_N}{\partial \Omega} = 0, \quad \frac{\partial T_N}{\partial \sigma} = 0. \]  \hspace{1cm} (16)

Using a 10 × 10 subspace we obtain the optimal values \( \sigma = -3.889 \) and \( \Omega = 31.179 \). Correspondingly we find the energy of the ground state to be:

\[ E_0^{(10)} = -1229.116051045 \]  \hspace{1cm} (17)

where the underlined digits are correct. Using a 40 × 40 subspace we find \( \sigma = -3.583 \) and \( \Omega = 27.431 \), obtaining

\[ E_0^{(40)} = -1229.116051046004597058992 \]  \hspace{1cm} (18)

where the underlined digits are correct. Similar results hold for the excited states.
We notice that $\sigma_{PMS}$ does not correspond to the value for which $V(x)$ has a minimum, i.e. $x \approx -3.979$. In fact $\sigma_{PMS}$ tends to increase as the dimension of the subspace is increased, as a result of the influence of the highly excited states, which are less localized.

It is clear that such a scheme can be applied with limited effort to a general anharmonic potential of arbitrary order: in fact, a suitable choice of $\sigma$ allows one to reach high precision with a limited number of terms. Clearly, since a general potential can be always expanded in a Taylor series around a point, this implies that our method can be easily applied to calculate portions of the spectrum of a potential with arbitrary precision.

In Fig. 5 we have plotted the first 200 states of the spectrum of Eq. (15), using our method with subspaces of $100 \times 100$ and $200 \times 200$ and $500 \times 500$, together with the first-order WKB estimate for comparison.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig4.png}
\caption{Potential $V(x) = 11 - 118x - 44x^2 + 80x^3 + 16x^4$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig5.png}
\caption{Spectrum of the potential of Eq. (15), as a function of the quantum number $n$. The pluses, triangles and crosses correspond to the eigenvalues obtained using our method with subspaces of $100 \times 100$, $200 \times 200$ and $500 \times 500$ respectively. The first-order WKB estimate is also shown for the higher states.}
\end{figure}
3. Time development

3.1. The Method of Stationary States

If solutions of the energy eigenvalue equation (11) are known for a given Hamiltonian \( \hat{H} \) then the time-dependent Schrödinger equation (\( \hbar = 1 \))

\[
i \frac{d}{dt} \Psi = \hat{H} \Psi .
\]

(19)

can be solved by the method of stationary states. Namely, if the initial wave-function at \( t = 0 \) can be expanded as

\[
\Psi(x, t = 0) = \sum_{n=0}^{\infty} a_n \psi_n(x),
\]

(20)

its value at a later time is given by

\[
\Psi(x, t) = \sum_{n=0}^{\infty} a_n e^{-iE_n t} \psi_n(x)
\]

(21)

The method of Section 2 gives us an approximation not only to the spectrum, but also to the energy eigenfunctions, namely

\[
\psi_n(x) = \sum_{k=0}^{N-1} d_{nk} \phi_k(x).
\]

(22)

where \( d_{nk} \) denotes the \( k \)th component of the \( n \)th eigenvector of the truncated Hamiltonian matrix.

Similarly, the initial wave function can also be expressed as a truncated expansion in terms of the \( \phi_n(x) \):

\[
\Psi_n(x, t = 0) = \sum_{k=0}^{N-1} c_n \phi_n(x).
\]

(23)

By comparison with Eq. (22) we see that

\[
a_n = \sum_{\ell=0}^{N-1} c_{\ell} (d^{-1})_{\ell n}.
\]

(24)

These coefficients can now be used in Eq. (21) to obtain an approximation to the wave function \( \Psi(x, t) \) at any later time \( t \).

3.2. Slow roll inflation

Here we use the Hamiltonian of Eq. (12) with two different initial configurations.

3.2.1. Centred Gaussian

The initial wave function, used in previous studies of slow-roll inflation, is given by

\[
\Psi(x, t = 0) = \left( \frac{m}{2\pi} \right)^{1/4} e^{-mx^2/4}.
\]
Our task is to find the coefficients \( c_n \) in Eq. (23).

By orthonormality,
\[
c_n = \left( \frac{m}{2\pi} \right)^{1/4} \int_{-\infty}^{+\infty} e^{-\beta^2 x^2 / 2} H_n(\alpha x) dx \tag{25}
\]
where \( \beta^2 \equiv m/2 + \alpha^2 \). By means of the change of variable \( y = \alpha x \)
\[
c_n = \frac{N_n}{\alpha} \left( \frac{m}{2\pi} \right)^{1/4} \int_{-\infty}^{+\infty} e^{-\frac{\beta^2}{2\alpha^2} y^2} H_n(y) dy \tag{26}
\]
and the expansion of \( H_n(y) \), namely
\[
H_n(y) = \sum_{k=0}^{[n/2]} (-1)^k \frac{n!}{(n-2k)!k!} 2^{n-2k} y^{n-2k} \tag{27}
\]
we obtain
\[
c_n = \frac{N_n}{\alpha} \left( \frac{m}{2\pi} \right)^{1/4} \sum_{k=0}^{[n/2]} (-1)^k \frac{n!}{(n-2k)!k!} 2^{n-2k} J_k \tag{28}
\]
where
\[
J_k = \int_{-\infty}^{+\infty} y^{n-2k} e^{-\frac{\beta^2}{2\alpha^2} y^2} dy = \left( \frac{\sqrt{2\alpha}}{\beta} \right)^{n-2k+1} \int_{-\infty}^{+\infty} z^{n-2k} e^{-z^2} dz \tag{29}
\]
In fact \( c_n = 0 \) unless \( n \) is even \( n = 2\ell \), and then
\[
c_{2\ell} = \frac{N_{2\ell}}{\alpha} \left( \frac{m}{2\pi} \right)^{1/4} \sum_{k=0}^{\ell} (-1)^k \frac{(2\ell)!}{(2\ell - 2k)!k!} 2^{2(\ell-k)} \left( \frac{\sqrt{2\alpha}}{\beta} \right)^{2(\ell-k)+1} \Gamma \left( \ell - k + \frac{1}{2} \right) \tag{29}
\]
The coefficients \( a_n \) are now given by Eq. (24)

For comparison with previous work, we use our time-dependent wave function to calculate \( \langle x^2 \rangle \), given by
\[
\langle x^2 \rangle = \int \Psi^*(x, t) x^2 \Psi(x, t) \, dx \tag{30}
\]
In terms of the \( \psi_n(x) \) this is
\[
\langle x^2 \rangle = \sum_{n, \ell} a_n^* a_{\ell} e^{-i\omega_{n\ell} t} \int \psi_n(x) x^2 \psi_{\ell}(x) \, dx \tag{31}
\]
where \( \omega_{n\ell} \equiv E_n - E_{\ell} \). Using Eq. (22) this becomes
\[
\langle x^2 \rangle = \sum_{n, \ell} a_n^* a_{\ell} e^{-i\omega_{n\ell} t} \sum_{k,j} d_{nk} d_{\ell j} (x^2)_{kj} \tag{32}
\]
The result for \( \langle x^2 \rangle^{1/2} \) is plotted in Fig. 6. As can be seen, the result is vastly superior to Hartree-Fock, and on this scale cannot be distinguished from that obtained using Fourier transform methods. The ratio of different orders is shown in Fig. 7.

3.2.2. Shifted Gaussian

In this case we consider an initial wave function
\[
\Psi(x, t = 0) = \left( \frac{\mu}{2\pi} \right)^{1/4} e^{-\mu(x-x_0)^2/4},
\]
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Figure 6. \( \langle x^2 \rangle^{1/2} \) versus \( t \) for the slow roll potential of Eq. (12). Various orders of our method compared with the Hartree-Fock method used in [23] and the Fourier transform method of [24].

Figure 7. Ratio of the results of our method for the slow roll potential: orders 10 and 20 versus 40.

representing a particle localized around \( x = x_o \) at \( t = 0 \). As before, we obtain the coefficients \( c_n \) by orthonormality:

\[
c_n = \int \phi^*_n(x) \Psi(x, t=0) dx = N_n \left( \frac{\mu}{2\pi} \right)^{1/4} \int_{-\infty}^{+\infty} e^{-\frac{\mu^2}{2\beta^2} \left(x - \frac{\mu x_o}{2\beta^2}\right)^2} e^{-\mu x_o^2 \alpha^2 / 4\beta^2} H_n(\alpha x) dx,
\]

where now \( \beta^2 \equiv \mu/2 + \alpha^2 \). With a change of variable to \( y = \beta (x - \mu x_o / 2\beta^2) \),

\[
c_n = \frac{N_n}{\beta} \left( \frac{\mu}{2\pi} \right)^{1/4} e^{-\mu x_o^2 \alpha^2 / 4\beta^2} \int_{-\infty}^{+\infty} e^{-y^2 / 2} H_n \left( \frac{\alpha}{\beta} \left( y + \frac{\mu x_o}{2\beta^2} \right) \right) dy.
\]
We expand the Hermite polynomial, to obtain

\[ H_n(\alpha x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{n!}{(n-2k)!k!} 2^{n-2k} \left( \frac{\alpha}{\beta} \right)^{(n-2k)} \left( y + \frac{\mu x_0}{2\beta} \right)^{n-2k} \]

Therefore

\[ c_n = \frac{N_n}{\beta} \left( \frac{\mu}{2\pi} \right)^{1/4} e^{-\mu x_0^2 \alpha^2 / 4\beta^2} \]

\[ \times \sum_{k=0}^{[n/2]} \sum_{j=0}^{n-2k} (-1)^k \frac{n!}{k! j! (n-2k-j)!} \left( \frac{\alpha}{\beta} \right)^{(n-2k)} \left( \frac{\mu x_0}{2\beta} \right)^{n-2k-j} K_j, \] (35)

where

\[ K_j \equiv \int_{-\infty}^{+\infty} y^j e^{-y^2/2} dy = 2^{(j+1)/2} \left[ 1 + (-1)^j \right] \Gamma(j + 1/2). \] (36)

Finally we obtain

\[ c_n = \frac{N_n}{\beta} \left( \frac{\mu}{2\pi} \right)^{1/4} e^{-\mu x_0^2 \alpha^2 / 4\beta^2} \sum_{k=0}^{[n/2]} \sum_{j=0}^{[n/2]-k} (-1)^k \frac{(n)! 2^{n-2k} \alpha^{-2k-2j}}{k! (2j)! (n-2k-2j)!} \]

\[ \times \left( \frac{\alpha}{\beta} \right)^{(n-2k)} \left( \frac{\mu x_0}{2\beta} \right)^{n-2k-2j} \left( \sqrt{2} \right)^{2j+1} \Gamma(j + 1/2). \] (37)

Figure 8 shows the expectation value \( \langle x \rangle \) as a function of time for four different values of \( \mu \). As expected, as \( \mu \) increases, the frequency of oscillation between the two wells increases as well. The plots in Fig. 8 correspond to the values \( g = 1/2400 \), \( m = 1/(2\sqrt{6}) \) and \( x_0 = 5 \).

**Figure 8.** \( \langle x \rangle \) with four different choices of \( \mu \) and with \( x_0 = 5 \) for the slow roll potential with a shifted Gaussian.
4. Conclusions

We have shown that the matrix method, combined with the principle of minimal sensitivity, is a powerful tool for finding the spectrum of arbitrary polynomial potentials having only bound states. The original method proposed in [18] has been generalized in two ways: for the calculation of higher levels, the matrix can be centred around a higher levels of the SHO, and for noneven potentials the introduction of a shift parameter improves the accuracy of the method for a given dimension $N$. This aspect of the method could be used for a general, non-polynomial potential, in conjunction with its Taylor expansion about a given point.

As a by-product of the calculation of the spectrum, the method also provides approximations to the energy eigenfunctions. Knowing both the energies and their eigenfunctions we are then able to implement the method of stationary states to track the time development of a given initial wave function with good accuracy and for long time scales, as we have demonstrated for the slow-roll potential.

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