Symmetric-Asymmetric Structures of Entropy Production

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Abstract

We study entropy productions in a system with a thermal reservoir and a control parameter reservoir. Given dynamics can be logarithmically decomposed into an equilibrium operator and a nonequilibrium factor, they are governed by each reservoir. By the decomposition, total entropy production is systematically decomposed into two parts. Alternatively, given dynamics can be logarithmically decomposed using an arbitrary symmetric operator, and total entropy production is also decomposed into two parts which are given by non-adiabatic and adiabatic entropy productions. However, they differ from ones of an equilibrium operator and nonequilibrium factor. By linear-decomposition using equilibrium operator, we can know that where entropy is generated between two reservoirs. As an example, entropy production rates in steady states are calculated in a system consisting of two-level particles. If the particles can perform work, the system becomes two-level Carnot engine.

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According to the 2'nd law of thermodynamics, average entropy change is always larger than or equal to zero, \( \langle \Delta S \rangle \geq 0 \), in macroscopic level. However, there definitely exist entropy decreasing events in microscopic level. The events are explained very well by fluctuation theorems (FTs). Moreover, the 2'nd law of thermodynamics can be derived from FTs \([1, 2]\). For the reasons, FTs are considered to be generalized versions of the 2'nd law of thermodynamics, and various types of FTs have been developed ever since the first FT was introduced by Evans, Cohen, and Morriss in 1993 \([3]\). Among them, according to three detailed FTs \([1]\), non-adiabatic and adiabatic entropy productions are defined as \( \Delta S_a \equiv \ln(\mathcal{P}/\mathcal{P}^+) = \Delta S_{sys} + \Delta S_{ex} \) and \( \Delta S_{na} \equiv \ln(\mathcal{P}/\mathcal{P}^+) = \Delta S_{res} - \Delta S_{ex} = \Delta S_{hk} \), respectively, where \( \Delta S_{sys} \) is system entropy production, \( \Delta S_{res} \) is reservoir entropy production, \( \Delta S_{ex} \) is excess entropy production, and \( \Delta S_{hk} \) is house-keeping entropy production \([2, 4–8]\).

In the present work, we study entropy productions in a system with a thermal reservoir and another reservoir involved with control parameter introducing symmetric-asymmetric decomposition of dynamics. The decomposition can be performed logarithmically or linearly. By log-decomposition, time evolution operator is decomposed into an equilibrium (or symmetric) and a nonequilibrium (or asymmetric) factor, and total entropy production is also systematically decomposed. By linear-decomposition, we can know that where entropy is generated in nonequilibrium steady states. As an example, entropy production rates in each reservoir is calculated in a system consisting of two-level particles. If the system can perform work, the systems becomes two-level Carnot engine \([9–17]\) in steady states.

Let us consider an ergodic system with two reservoirs. One is thermal (or heat) reservoir \( \mathcal{R}_h \) of inverse temperature \( \beta_h = 1/k_B T_h \), and another one is \( \mathcal{R}_c \) of control parameter \( \lambda \). When the system evolves along a path, \( i_0 \xrightarrow{\lambda_1} i_1 \xrightarrow{\lambda_2} \cdots \xrightarrow{\lambda_T} i_T \), by a schedule of control parameter \( \lambda \), path probability in forward direction is given by

\[
\mathcal{P} = \prod_{t=1}^{T} w_{it, i_{t-1}}(\lambda_t) p_{i_0}(0),
\]

where \( w_{ij}(\lambda) \) is a time evolution operator controlled by \( \lambda \). (If \( i \neq j \), \( w_{ij}(\lambda) \) plays a role of transition probability from \( j \) to \( i \). On the other hand, if \( i = j \), that is given by the probability to be staying in previous state \( j \).) In reversed direction, \( i_0 \xleftarrow{\lambda_1} i_1 \xleftarrow{\lambda_2} \cdots \xleftarrow{\lambda_T} i_T \), reversed path probability is given by

\[
\bar{\mathcal{P}} = \prod_{t=1}^{T} w_{it, i_{t-1}}(\lambda_t) p_{i_T}(T),
\]
where over-bar indicates reversed path. Then total path entropy production is defined as

$$\Delta S_{tot} = \ln \frac{P}{\bar{P}}. \quad (3)$$

The above three equations are systematically decomposed by log-decomposition of dynamics, which means dividing $w_{ij}(\lambda)$ logarithmically into two parts, in a general form,

$$\ln w_{ij}(\lambda) = \ln \epsilon_{ij}(\lambda) + \ln \nu_{ij}(\lambda). \quad (4)$$

Here, $\epsilon_{ij}(\lambda)$ can be defined either as an equilibrium operator $\epsilon_{ij}$ (which can be defined by Hamiltonian in equilibrium states) or as a symmetric operator $\epsilon_{ij}^s(\lambda)$ (which reduces nonequilibrium flux to zero in steady states). Then the factor $\nu_{ij}(\lambda)$ can be defined,

$$\nu_{ij}(\lambda) \equiv \frac{w_{ij}(\lambda)}{\epsilon_{ij}(\lambda)}. \quad (5)$$

Before decomposing dynamics using $\epsilon_{ij}$, it has to be defined neutral (or equilibrium) parameter $\lambda_{eq}$ such that $\nu_{ij}(\lambda_{eq}) = 1$ for all $i$ and $j$. If $\lambda = \lambda_{eq}$, it can be regarded as a case that $R_c$ is removed from the system, when equilibrium operator $\epsilon_{ij}$ is given by

$$\epsilon_{ij} = w_{ij}(\lambda_{eq}), \quad (6)$$

which can be defined by Hamiltonian as an actual equilibrium operator. Hence we can say that $\epsilon_{ij}$ obeys Boltzmann factor, so

$$\frac{\epsilon_{ij}}{\epsilon_{ji}} = \frac{p_{ij}^{\text{eq}}}{p_{ji}^{\text{eq}}} = e^{\beta_s(E_i-E_j)}, \quad (7)$$

Here, $E_i$ is energy level of state $i$, and $p_i^{\text{eq}} = p_i^{\text{st}}(\lambda_{eq})$ is Boltzmann distribution where $p_i^{\text{st}}(\lambda)$ is a steady state distribution for fixed $\lambda$.

Using $\epsilon_{ij}$, given dynamics can be decomposed into two parts, $\ln w_{ij}(\lambda) = \ln \epsilon_{ij} + \ln \nu_{ij}(\lambda)$, by which forward path probability can be decomposed,

$$\ln \mathcal{P} = \ln \mathcal{P}_\epsilon + \ln \mathcal{P}_\nu, \quad (8)$$

where

$$\mathcal{P}_\epsilon \equiv \prod_{t=1}^{T} \epsilon_{i_t;i_{t-1}} p_{i_0}(0), \text{ and } \mathcal{P}_\nu \equiv \prod_{t=1}^{T} \nu_{i_t;i_{t-1}}(\lambda_t).$$
Note that $P_{\nu}$ is not path probability but a *path-probability-like* quantity. In the same manner, reversed path probability can be decomposed,

$$\ln P = \ln \bar{P}_\epsilon + \ln \bar{P}_\nu$$

where

$$\bar{P}_\epsilon = \prod_{t=1}^{T} \epsilon_{t-1|t} p_{tT}(T), \text{ and } \bar{P}_\nu = \prod_{t=1}^{T} \nu_{t-1|t}(\lambda_t).$$

As a result, total entropy production \((\ref{eq:tot})\) can be decomposed as follows

$$\Delta S_{tot} = \Delta S_\epsilon + \Delta S_\nu$$

where the equilibrium and nonequilibrium parts of path entropy production are

$$\Delta S_\epsilon \equiv \ln \frac{P_\epsilon}{\bar{P}_\epsilon} = \ln \prod_{t=1}^{T} \frac{\epsilon_{t|t-1} p_{oT}(0)}{\epsilon_{t-1|t} p_{tT}(T)}$$

$$\Delta S_\nu \equiv \ln \frac{P_\nu}{\bar{P}_\nu} = \ln \prod_{t=1}^{T} \frac{\nu_{t|t-1}(\lambda_t)}{\nu_{t-1|t}(\lambda_t)}$$

respectively. Since \((\ref{eq:sys})\) and \((\ref{eq:res})\), they can be expressed as

$$\Delta S_\epsilon = \Delta S_{sys} + \Delta S_{\epsilon-ex}$$

$$\Delta S_\nu = \Delta S_{res} - \Delta S_{\epsilon-ex}$$

where

$$\Delta S_{sys} \equiv \ln \frac{p_{oT}(0)}{p_{tT}(T)}$$

$$\Delta S_{res} \equiv \ln \prod_{t=1}^{T} \frac{w_{t|t-1}(\lambda_t)}{w_{t-1|t}(\lambda_t)} \text{ and}$$

$$\Delta S_{\epsilon-ex} \equiv \ln \prod_{t=1}^{T} \frac{p_{t\nu}^{eq}}{p_{t\nu}} = \ln \frac{p_{t\nu}^{eq}}{p_{t\nu}}.$$  \(\text{(17)}\)

Here, we call $\Delta S_{\epsilon-ex}$ as *equilibrium* excess entropy, which is path-independent. If the system is driven out of equilibrium by external driving such that $\lambda_0 = \lambda_{eq}$ and $\lambda_T \neq \lambda_{eq}$, it is
calculated as follows

$$\langle \Delta S_{\epsilon-ex} \rangle = \sum_{\text{over all paths}} \mathcal{P} \Delta S_{\epsilon-ex}$$

$$= \sum_{i_T, \ldots, i_0} w_{i_T i_{T-1}} \cdots w_{i_1 i_0} p_i(0) \ln \frac{p_{eq}^{i_T}}{p_{eq}^{i_0}}$$

$$= \sum_{i_T} p_{i_T}(T) \ln \frac{p_{eq}^{i_T}}{p_{i_0}(0)} \ln \frac{p_{eq}^{i_0}}{p_{eq}^{i_T}}$$

$$= -\beta_h Q_{ex}.$$  

(18)

Here, $Q_{ex} = \langle E \rangle_T - \langle E \rangle_0$ is excess heat [4], where $\langle E \rangle_t \equiv \sum p_i(t) E_i$.

In a different way, symmetric operator $\epsilon^*_ij(\lambda)$ can be defined to reduce nonequilibrium flux to zero, i.e. to satisfy detailed balance (or -like) condition,

$$\epsilon^*_ij(\lambda) p^st_j(\lambda) = \epsilon^*_ji(\lambda) p^st_i(\lambda),$$  

(19)

in steady states. Then asymmetric factor is defined as $\nu^*_ij(\lambda) \equiv w_{ij}(\lambda)/\epsilon^*_ij(\lambda)$. Note that $\epsilon^*_ij(\lambda)$ is an arbitrary symmetric operator which can be defined in nonequilibrium steady states, so dependent on $\lambda$ differently from equilibrium operator $\epsilon_{ij}$. It may be defined as $\epsilon^*_ij(\lambda) \equiv w_{ij}(\lambda)$ for forward direction and $\epsilon^*_ji(\lambda) \equiv w^+_ji(\lambda)$ for reversed direction, but we remain it as an arbitrary one.

Replacing $\epsilon_{ij}(\lambda)$ with $\epsilon^*_ij(\lambda)$ in (14), given dynamics can be decomposed as $\ln w_{ij}(\lambda) = \ln \epsilon^*_ij(\lambda) + \ln \nu^*_ij(\lambda)$ by which path probabilities and entropy production are also systematically decomposed, i.e. $\ln \mathcal{P} = \ln \mathcal{P}_{\epsilon^*} + \ln \mathcal{P}_{\nu^*}$, $\ln \tilde{\mathcal{P}} = \ln \tilde{\mathcal{P}}_{\epsilon^*} + \ln \tilde{\mathcal{P}}_{\nu^*}$, and $\Delta S_{tot} = \Delta S_{\epsilon^*} + \Delta S_{\nu^*}$. Since (19), it is simply shown that

$$\Delta S_{\epsilon^*} = \ln \frac{\mathcal{P}_{\epsilon^*}}{\mathcal{P}_{\epsilon}} = \ln \frac{\mathcal{P}}{\mathcal{P}_{+}} = \Delta S_{na}$$

(20)

$$\Delta S_{\nu^*} = \ln \frac{\mathcal{P}_{\nu^*}}{\mathcal{P}_{\nu}} = \ln \frac{\mathcal{P}}{\mathcal{P}_{-}} = \Delta S_{a}.$$  

(21)

Therefore we claim that entropy productions relevant to $\epsilon^*_ij(\lambda)$ or $\nu^*_ij(\lambda)$ are given by $\Delta S_{na}$ or $\Delta S_{a}$, regardless of how $\epsilon^*_ij(\lambda)$ is defined.

However, $\Delta S_{\epsilon}$ or $\Delta S_{\nu}$ are not same with them. As well known, $\Delta S_{na} = \Delta S_{sys} + \Delta S_{ex}$ and $\Delta S_{a} = \Delta S_{res} - \Delta S_{ex}$, where (ordinary nonequilibrium) excess entropy is given by $\Delta S_{ex} = \sum_{t=1} T \ln \frac{\nu^*_{ij}(\lambda)}{\nu^*_{ij}(\lambda)}[1]$. Comparing it with (17), we can see that $\Delta S_{ex} \neq \Delta S_{\epsilon-ex}$ generally.
If a system in an equilibrium state begins to be driven out of it for fixed \( \lambda \neq \lambda_{eq} \), the system goes to a nonequilibrium steady state in which nonequilibrium entropy production rate is, from (12), given by

\[
\langle \dot{S}_\nu \rangle = \frac{1}{2} \sum_{ij} [w_{ij}(\lambda)p_{j}^{st}(\lambda) - w_{ji}(\lambda)p_{i}^{st}(\lambda)] \ln \frac{\nu_{ij}(\lambda)}{\nu_{ji}(\lambda)}
\]  

(22)

It is \( \langle \dot{S}_\nu \rangle = \langle \dot{S}_{res} \rangle \), since \( \langle \dot{S}_{\epsilon-ex} \rangle = 0 \) in steady states. As well known, \( \Delta S_a = \langle \dot{S}_{hk} \rangle = \langle \dot{S}_{res} \rangle \) in steady states, further we have shown that \( \Delta S_{\nu^*} = \Delta S_a \) in (21), so they are same with each other, \( \langle \dot{S}_\nu \rangle = \langle \dot{S}_{\nu^*} \rangle \), in steady states. However, we will see that linearly decomposed ones of them differ from each other.

In linear-decomposition of dynamics using \( \epsilon_{ij} \), given dynamics is decomposed as

\[
w_{ij}(\lambda) = \epsilon_{ij} + n_{ij}(\lambda),
\]

(23)

where \( n_{ij}(\lambda) \equiv w_{ij}(\lambda) - \epsilon_{ij} \) is a nonequilibrium operator being controlled by \( \lambda \). If given system is in equilibrium, \( n_{ij}(\lambda_{eq}) = 0 \) (or \( \nu_{ij} = 1 \)) for all \( i \) and \( j \), and vice versa. Applying (24) on (22), that is decomposed as

\[
\langle \dot{S}_\nu \rangle = \langle \dot{S}_h \rangle + \langle \dot{S}_c \rangle
\]

(24)

where

\[
\langle \dot{S}_h \rangle = \frac{1}{2} \sum_{ij} [\epsilon_{ij}p_{j}^{st}(\lambda) - \epsilon_{ji}p_{i}^{st}(\lambda)] \ln \frac{\nu_{ij}(\lambda)}{\nu_{ji}(\lambda)}
\]

(25)

\[
\langle \dot{S}_c \rangle = \frac{1}{2} \sum_{ij} [n_{ij}(\lambda)p_{j}^{st}(\lambda) - n_{ji}(\lambda)p_{i}^{st}(\lambda)] \ln \frac{\nu_{ij}(\lambda)}{\nu_{ji}(\lambda)}
\]

(26)

Therefore, we can know where entropy is generated by linear-decomposition using \( \epsilon_{ij} \) in steady states: \( \langle \dot{S}_h \rangle \) is generated in \( \mathcal{R}_h \), and the second one \( \langle \dot{S}_c \rangle \) is generated in \( \mathcal{R}_c \). If entropy production rate is, however, linearly decomposed into \( \epsilon_{ij}^*(\lambda) \) and \( n_{ij}^*(\lambda) \equiv w_{ij}(\lambda) - \epsilon_{ij}^*(\lambda) \), we can not know where entropy is generated between them, because both of \( \epsilon_{ij}^*(\lambda) \) and \( n_{ij}^*(\lambda) \) are dependent on \( \lambda \).

As an example, let us consider a system consisting of two-level particles. Each particle can be in either a state of low energy \( E_1 \) or another state of high energy \( E_2 \). After the system has been driven out of equilibrium by \( \nu_{12} = \lambda > 1 \) and \( \nu_{21} = 1 \), it will be in a nonequilibrium steady state. However, the steady state is quietly different from general cases of nonequilibrium steady states. In steady states of the system,

\[
\langle \dot{S}_\nu \rangle = 0 \quad \text{because} \quad w_{12}(\lambda)p_{2}^{st}(\lambda) = w_{21}(\lambda)p_{1}^{st}(\lambda)
\]

(27)
for any \( \lambda \). Though, the system is clearly in a nonequilibrium steady state if \( \lambda \neq \lambda_{eq} \), because there exists non-zero heat flux from a reservoir passing the system to another reservoir. The detailed balance (like) equation in (27) can be rewritten as \( \lambda \epsilon_{12} p_{2}^{st} = \epsilon_{21} p_{1}^{st} \), so (25) and (26) have a relation such that

\[
\langle \dot{S}_c \rangle = -\langle \dot{S}_h \rangle = (\lambda - 1)\epsilon_{12} p_{2}^{st} \ln \lambda \geq 0.
\]  

(28)

The equality happens only if \( \lambda = \lambda_{eq} \).

If the particles in the system can perform work, and if \( \mathcal{R}_c \) is another cold thermal reservoir of \( \lambda = \beta_c > \beta_h \) (or \( T_c < T_h \)), then the system becomes a two-level Carnot engine [9–17]. Roughly, entropy production rate in \( \mathcal{R}_h \) can be written as \( \langle \dot{S}_h \rangle = -\beta_h \dot{Q}_h \), where \( \dot{Q}_h \) is a rate of heat flowing out of \( \mathcal{R}_h \) into the system; \( \langle \dot{S}_c \rangle = \beta_c \dot{Q}_c \) is entropy production rate in \( \mathcal{R}_c \), where \( \dot{Q}_c \) is a rate of heat flowing out of the system into \( \mathcal{R}_c \). (Beware of the sign of entropy productions.) Hence, from (28), we can see that \( \beta_h \dot{Q}_h = \beta_c \dot{Q}_c \). Reflecting on the first law of thermodynamics, it is simply derived that

\[
\eta = 1 - \frac{T_c}{T_h}.
\]  

(29)

where \( \eta \) is efficiency. That is exactly same with efficiency of (two-level) Carnot engine.

If the temperature of the system \( T \) can be measured in the steady state (though it is not easy in general), probability distribution can be given by Boltzmann distribution, \( p_i^{st} \sim e^{-\beta E_i} \) for \( i = 1 \) or 2 where \( \beta = 1/k_B T \). From \( \lambda = \nu_{12}/\nu_{21} > 1 \), \( w_{12} p_{2}^{st} = w_{21} p_{1}^{st} \) and \( \epsilon_{12} p_{2}^{eq} = \epsilon_{21} p_{1}^{eq} \), we can see that \( \ln \lambda = \ln \frac{\nu_{12}}{\nu_{21}} = \ln \frac{w_{12}/\epsilon_{12}}{w_{21}/\epsilon_{21}} = \ln \frac{w_{12} p_{2}^{eq}}{w_{21} p_{1}^{eq}} = \ln \frac{p_{1}^{eq}}{p_{2}^{eq}} = (\beta - \beta_h)(E_2 - E_1) > 0 \). Hence, \( \beta > \beta_h \) or \( T < T_h \). That is because \( \mathcal{R}_c \) of low temperature is attached to the system.

In the case that the system was in an equilibrium at low temperature \( T_c \) is driven out of it by attaching hot reservoir \( \mathcal{R}_h \) on the system, it is obtained that \( \beta < \beta_c \) or \( T > T_c \). Therefore, we can say that heat in \( \mathcal{R}_h \) is flowing into the system, and some a mount of which performs works, then the rest is flowing out of it into \( \mathcal{R}_c \). During the process, \( \langle \dot{S} \rangle = 0 \). Therefore the system performs work with same efficiency of Carnot engine.

Alternatively, \( w_{ij}(\lambda) = \epsilon^{*}_{ij}(\lambda) + n^{*}_{ij}(\lambda) \) might be tried, but \( \langle \dot{S}_{\nu} \rangle = \langle \dot{S}_{h^\nu} \rangle = \langle \dot{S}_{c^\nu} \rangle = 0 \) since \( \ln \frac{\nu_{12}}{\nu_{21}} = 0 \) for all \( \lambda \) in the system.

In the present work, we have proposed symmetric-asymmetric decomposition in a system with two reservoirs. In the system, given dynamics can be logarithmically decomposed as
by which path probabilities and entropy production are systematically decomposed,

\[ w_{ij} = \epsilon_{ij} + \nu_{ij} \]

\[ \ln P = \ln P_\epsilon + \ln P_\nu \]

\[ \ln \bar{P} = \ln \bar{P}_\epsilon + \ln \bar{P}_\nu \]

\[ \Delta S_{\text{tot}} = \Delta S_\epsilon + \Delta S_\nu \]

as seen in (4), (8), (9) and (10). That is intuitive indeed, since entropy is defined by log of probability.

We have distinguished \( \epsilon_{ij} \) or \( \nu_{ij} \) from \( \epsilon^{*}_{ij}(\lambda) \) or \( \nu^{*}_{ij}(\lambda) \). Among them, \( \epsilon_{ij} \) is an operator governed by a thermal reservoir \( R_h \), and \( \nu_{ij}(\lambda) \) is a factor governed by another reservoir \( R_c \) of control parameter \( \lambda \). So \( \epsilon_{ij} \) can be defined by Hamiltonian when \( \lambda = \lambda_{eq} \) (or when \( R_c \) is removed), so which obeys Boltzmann factor. Then under the influence of \( R_c \), the system is driven out of original equilibrium by \( \nu_{ij}(\lambda) = \frac{w_{ij}(\lambda)}{\epsilon_{ij}} \) for \( \lambda \neq \lambda_{eq} \). Decomposed entropy productions, \( \Delta S_\epsilon \) and \( \Delta S_\nu \), are given by

\[ \Delta S_\epsilon = \Delta S_{\text{sys}} + \Delta S_{\epsilon - \text{ex}} \]

\[ \Delta S_\nu = \Delta S_{\text{res}} - \Delta S_{\epsilon - \text{ex}} \]

as seen in (13) and (14).

On the other hand, symmetric operator \( \epsilon^{*}_{ij}(\lambda) \) is not directly involved with Hamiltonian. As we have mentioned above, \( \epsilon^{*}_{ij}(\lambda) \) is an arbitrary symmetric operator which reduce nonequilibrium flux in (nonequilibrium) steady states. Using \( \epsilon^{*}_{ij}(\lambda) \), given dynamics can be decomposed into two parts, \( w_{ij}(\lambda) = \epsilon^{*}_{ij}(\lambda) + \nu^{*}_{ij}(\lambda) \), and the relevant productions are

\[ \Delta S_{\epsilon^*} = \Delta S_{\text{na}} = \Delta S_{\text{sys}} + \Delta S_{\text{ex}} \]

\[ \Delta S_{\nu^*} = \Delta S_{\text{a}} = \Delta S_{\text{res}} - \Delta S_{\text{ex}} \]

as seen in (20) and (21), regardless of how \( \epsilon^{*}_{ij}(\lambda) \) is defined. That is because the rate \( \frac{\epsilon^{*}_{ij}(\lambda)}{\nu^{*}_{ij}(\lambda)} \) is restored to \( \frac{w_{ij}(\lambda)}{w^{*}_{ij}(\lambda)} \). However, \( \Delta S_\epsilon \neq \Delta S_{\text{na}} \) and \( \Delta S_\nu \neq \Delta S_{\text{a}} \), because \( \Delta S_{\epsilon - \text{ex}} \neq \Delta S_{\text{ex}} \). Between them, equilibrium excess entropy production is given by

\[ \Delta S_{\epsilon - \text{ex}} = -\beta_h Q_{\text{ex}} \]

when given system is driven out of equilibrium to another steady state, as shown in (18). More detailed about it, we will study in our future works.
By linear-decomposition of dynamics in steady states, nonequilibrium entropy production rate is decomposed into two parts,

\[ \langle \dot{S}_\nu \rangle = \langle \dot{S}_h \rangle + \langle \dot{S}_c \rangle. \]

Since \( \langle \dot{S}_\nu \rangle = \langle \dot{S}_{res} \rangle \) in steady states, \( \langle \dot{S}_h \rangle \) and \( \langle \dot{S}_c \rangle \) are entropy production rates in \( R_h \) and \( R_c \), respectively.

As an example, we have calculated \( \langle \dot{S}_h \rangle \) and \( \langle \dot{S}_c \rangle \) in the system consisting of two-level particles. In steady states, \( w_{12}(\lambda)p_2^{st}(\lambda) = w_{21}(\lambda)p_1^{st}(\lambda) \) for any \( \lambda \), accordingly \( \langle \dot{S}_\nu \rangle = 0 \) always. However, that does not mean entropy is not generated at all in each reservoir. In the steady state, heat is flowing out of \( R_h \) into the system, so entropy in \( R_h \) decreases; the heat in the system is flowing into \( R_c \), so entropy in \( R_c \) increases. Therefore,

\[ \langle \dot{S}_\nu \rangle = \langle \dot{S}_{res} \rangle = 0 \quad \text{but} \quad \langle \dot{S}_c \rangle = -\langle \dot{S}_h \rangle \geq 0 \]

in steady states of the system of two-level particles.

The equality happens only if \( \lambda = \lambda_{eq} \). If the particles in the system can perform work, the system becomes two-level Carnot engine [9–17] of efficiency

\[ \eta = 1 - \frac{T_c}{T_h} \]

in steady states, as seen in (29).

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