Degrees in random uniform minimal factorizations

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We are interested in random uniform minimal factorizations of the $n$-cycle which are factorizations of $(1 \ldots n)$ into a product of $n-1$ transpositions. Our main result is an explicit formula for the joint probability that 1 and 2 appear a given number of times in a uniform minimal factorization. For this purpose, we combine bijections with Cayley trees together with explicit computations of multivariate generating functions.

1 Introduction

Consider the cycle permutation $(1 \ldots n)$ for $n \geq 2$. A minimal factorization is a $(n-1)$-tuple of transpositions $(\tau_1, \ldots, \tau_{n-1})$ such that $\tau_{n-1} \cdots \tau_1 = (1 \ldots n)$. We denote by $\mathcal{M}_n$ the set of all minimal factorizations of the cycle $(1 \ldots n)$. Dénes [De59] first showed that $\mathcal{M}_n$ has cardinality $n^{n-2}$ and several bijective proofs followed afterwards (see [MOS89], [GP93], [GY02] and [Bia04]). Minimal factorizations are linked to other combinatorial objects such as non-crossing partitions [Bia97] and parking functions [Bia02] and more general factorizations have deep connections with enumerative geometry (see e.g. [ACEH18]).

Let $(\tau_1^{(n)}, \ldots, \tau_{n-1}^{(n)})$ be a random minimal factorization chosen uniformly at random in $\mathcal{M}_n$. The study of the behaviour of such a randomly picked minimal factorization is recent (see [FK18], [FK19] and [The20]) and has a rich probabilistic structure: for instance, it is shown in [The20] that such minimal factorizations have connections with Aldous-Pitman fragmentation of the Brownian continuum random tree. Here we are interested in the law of the number of times 1 and 2 appear in $(\tau_1^{(n)}, \ldots, \tau_{n-1}^{(n)})$. In [FK19 Corollary 1.2 (iv)] it was obtained that:

$$\mathbb{P}\left( \Upsilon_1^{(n)} = i, \Upsilon_2^{(n)} = j \right) \xrightarrow{n \to \infty} e^{-2} \frac{n^{i+j-2}}{(i+j-1)!} i^{j-1},$$

where $\Upsilon_1^{(n)} = \#\{1 \leq \ell \leq n-1 : \tau_\ell^{(n)}(k) \neq k\}$ is the number of time $k$ appears in a transposition. We refine this result by finding explicitly the joint distribution for fixed $n$:

**Theorem 1.** For $i, j \geq 1$ and $n \geq i + j$:

$$\mathbb{P}\left( \Upsilon_1^{(n)} = i, \Upsilon_2^{(n)} = j \right) = \frac{n!(n-1)^{n-i-j-1}}{(n-i-j)!(n+1)^{i+j-1}} \frac{(i+j-2)(n-1)}{(i+j-1)(n-i-j+1)} + \frac{i+j-1}{i+j} \frac{(i+j-1)}{(i+j)!}.$$

To show Theorem 1 we explicitly compute the exponential generating function of the (normalized) trivariate generating function $G_n$ defined by

$$G_n(x, y, z) = n^{n-2}E\left[ x^{\Upsilon_1^{(n)}} y^{\Upsilon_2^{(n)}} z^{\Upsilon_3^{(n)}} \right]$$

where $\Upsilon_3^{(n)} = \#\{1 \leq \ell \leq n-1 : \tau_1^{(n)}(k) \cdots \tau_{\ell-1}^{(n)}(k) \neq \tau_\ell^{(n)}(k) \cdots \tau_{n-1}^{(n)}(k)\}$ is the number of transpositions that affect the trajectory of $k$, and then extract the coefficient $[x^iy^jz^k]G_n(x, y, z)$. To this end, there are 4 main steps. First, using a known bijection between minimal factorizations and Cayley trees, we reformulate the problem in terms of a generating function of a trivariate statistic on Cayley trees (Section 2). To compute this generating function, we actually start by computing another generating function $F_n$ obtained by changing one of the three statistics (Section 3). This also yields a result of independent interest by confirming a conjecture [Car19] involving distributional symmetries in uniform Cayley trees (Corollary 10). Finally, we show bijectively that $G_n(x, y, z) = G_n(y, x, z)$ (Section 11), and by combining this with the explicit formula of $F_n$ we get the exponential generating function of $G_n$ (Section 12) and Theorem 1 follows (Section 13).

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2 Bijection between minimal factorizations and Cayley trees

2.1 The bijection

Here we explain how to associate a labeled tree with a minimal factorization which will be an essential tool for us. We refer to [FK19] for details and proofs. Fix an integer $n \geq 2$.

**Definition 2.** For $(\tau_1, \ldots, \tau_{n-1}) \in \mathcal{M}_n$, we define a labeled tree $F(\tau_1, \ldots, \tau_{n-1})$ with $n$ vertices labeled from 1 to $n$ where an edge labeled $l$ is drawn between the vertices labeled $a$ and $b$ if and only if $\tau_l = (a, b)$ (see figure 1 for an example).

![Figure 1](image1.png)

Figure 1: Representation of $F$ on the left and $E$ on the right when $n = 10$ for the minimal factorization $((9 \ 10), (7 \ 9), (1 \ 5), (2 \ 5), (3 \ 5), (8 \ 9), (4 \ 5), (1 \ 6), (1 \ 7))$ of $(1 \ldots 10)$. The double circle represents the root of the tree $E$.

Clearly $F$ is injective since a minimal factorization can be easily read on its associated tree. Actually the tree $F$ gives too much information, indeed it is still possible to retrieve the associated minimal factorization when we erase the vertex-labels and keep only the edge-labels. More precisely we have:

**Definition 3.** If $(\tau_1, \ldots, \tau_{n-1}) \in \mathcal{M}_n$, then we construct a rooted, edge-labeled tree $E(\tau_1, \ldots, \tau_{n-1})$ by doing the following on the tree $F(\tau_1, \ldots, \tau_{n-1})$:

- We root the tree at the vertex labeled 1.
- We erase all the vertex-labels (and keep only the edge-labels).

**Proposition 4.** The map $E$ gives a bijection between the set $\mathcal{M}_n$ and the set $\mathcal{C}_n'$ of rooted trees with $n-1$ edges labeled from 1 to $n-1$.

The set $\mathcal{C}_n'$ is clearly in bijection with the set $\mathcal{C}_n$ of Cayley trees with $n$ vertices (i.e. trees with $n$ vertices labeled from 1 to $n$). Indeed if $t \in \mathcal{C}_n$ we create $\alpha(t) \in \mathcal{C}_n'$ by rooting the tree $t$ at the vertex labeled 1, then by pulling all the vertex-labels (except 1 which is erased from $\alpha(t)$) towards the root into the nearest edge. We then subtract 1 from all the labels (see figure 2 for an example). The map $\alpha$ is clearly a bijection.

![Figure 2](image2.png)

Figure 2: A tree $t \in \mathcal{C}_6$ on the left transformed into $\alpha(t) \in \mathcal{C}_6'$ on the right by pulling the vertex-labels towards the root and subtracting 1.

In particular $\mathcal{M}_n$ has the same cardinality as the set of Cayley trees with $n$ vertices $\mathcal{C}_n$ which is known to be $n^{n-2}$. This explains the renormalizing term in the definition of $G_n$. In the article [FK19] the authors
For example, applying \( \text{Find}_{10} \) to the tree on the right of figure 1 gives back the vertex-labels on the left of figure 1.

**Proposition 6.** For \( t = E(f) \in \mathcal{C}_n' \):
\[
\text{Find}_n(t) = F(f).
\]

### 2.2 Reformulation in terms of Cayley trees

In order to prove Theorem 1, we start with rewriting the generating function \( G_n \) defined in the Introduction in terms of Cayley trees. To this purpose we introduce some notation. For \( A \) a subset of \( \{1, \ldots, n\} \) we denote by \( \mathcal{C}_A \) the set of trees with \(|A|\) vertices labeled in a one-to-one manner with the elements of \( A \). Notice that if \( A = \{1, \ldots, n\} \) then \( \mathcal{C}_A = \mathcal{C}_n \) is the set of Cayley trees with \( n \) vertices. If \( t \in \mathcal{C}_A \) and \( i \in A \), we denote by \( \text{deg}_i(t) \) the degree of the vertex \( i \) (where "the vertex labeled \( i \)" has to be understood as "the vertex labeled \( i \)" in the tree \( t \)). Similarly, we denote by \( \text{deg}_i(t) \) the set of vertices composing this path but without including the first one which is the vertex \( i \) (see figure 1 for an example).

**Definition 7.** For \( t \in \mathcal{C}_A \) and \( i \in A \) we consider the longest path of vertices, starting from \( i \), such that each vertex of the path has the smallest label among the ones that are both adjacent to the previous vertex on the path and have a greater label than the label of the previous vertex on the path. We denote by \( L_i(t) \) the set of vertices composing this path but without including the first one which is the vertex \( i \) (see figure 1 for an example).

**Definition 8.** For \( t \in \mathcal{C}_n \) we denote by \( \text{deg}_2(t) \) the degree, in \( t \), of the last vertex of the path \( L_i(t) \) (see figure 1 for an example).

Recall from Section 2.1 the bijection \( \mathcal{E} \) between \( \mathcal{M}_n \) and \( \mathcal{C}_n' \) as well as a bijection \( \alpha \) from \( \mathcal{C}_n \) to \( \mathcal{C}_n' \) (illustrated in figure 2). Notice that if \( f \in \mathcal{M}_n \) and \( t = \alpha \circ \mathcal{E}(f) \) then
\[
(\mathcal{M}_1(f), \mathcal{M}_2(f), \mathcal{M}_3(f)) = (\text{deg}_1(t), \text{deg}_2(t), |L_1(t)|).
\]

where \( \mathcal{M}_i(f) \) is the number of times \( i \) appears in \( f \) and \( \mathcal{M}_j(f) \) is the number of transpositions in \( f \) that affect the trajectory of \( j \). The last identity allows us to reformulate the definition of \( G_n \) for \( n \geq 2 \):
\[
G_n(x, y, z) = \sum_{t \in \mathcal{C}_n} x^{\text{deg}_1(t)} y^{\text{deg}_2(t)} z^{|L_1(t)|}.
\]

In the next Section, in order to compute \( G_n \), we introduce another generating function \( F_n \) whose definition is similar to (2) except that \( \text{deg}_2 \) is replaced with \( \text{deg}_2 \).
With this in mind we can decompose the quantity in distinct ways to attach the tree to a real-valued function. It is obviously true for \( n \geq 2 \).

The last equality is a well known result on Cayley trees. Actually, it turns out we have an explicit formula for \( F_n(x, y, z) \).

**Proposition 9.** For \( n \geq 3 \),

\[
F_n(x, y, z) = xyz \left[ x(n - 2 + x)^{n-3} \left( 1 - \frac{yz}{z + y - 1} \right) + (x + y + z + n - 3)^{n-3} \left( n - 2 + y + z + \frac{xyz}{z + y - 1} \right) \right].
\]

Formula (3) still holds in the cases \( y = 1 \) and \( z = 1, n = 1 \). Proposition 9 implies that \( F_n \) is symmetric in \( y \) and \( z \) thus we have the following Corollary which confirms a conjecture made by Caraceni [Car19]:

**Corollary 10.** Let \( T_n \) be a random uniform Cayley tree with \( n \) vertices, then \( (\deg_1(T_n), |L_1(T_n)|, \deg_2(T_n)) \) and \( (\deg_1(T_n), \deg_2(T_n), |L_1(T_n)|) \) have the same law.

It would be very interesting to obtain a direct bijective proof of Corollary 10. Formula (3) with \( y = 1 \) was first conjectured in [FK19] Conjecture 1.4 and was proved by O. Angel & J. Martin [AM19]. Below, we give Angel and Martin’s proof of the case \( y = 1 \) which will be useful to deduce the general case.

*Proof of Proposition 9* for \( y = 1 \). We show by induction on \( n \geq 2 \) that \( F_n(x, 1, z) = xz f_n(x + z) \) where \( f_n \) is a real-valued function. It is obviously true for \( n = 2 \) with \( f_2 = 1 \). Suppose that is true for all \( 2 \leq k \leq n - 1 \) with \( n \geq 3 \).

We denote by \( \mathcal{P}_n \) the set of couples \((A, B)\) with \( A \) and \( B \) two subsets of \( \{1, \ldots, n\} \) such that \( A \cup B = \{1, \ldots, n\}, A \cap B = \varnothing, 1 \in A \) and \( n \in B \). Let \( t \in \mathcal{C}_n \), set the vertex 1 to be the root of \( t \). Consider cutting the tree \( t \) by removing the edge between the vertex \( n \) and its parent to end up with two trees \( t_1 \in \mathcal{C}_A \) and \( t_2 \in \mathcal{C}_B \) for some \((A, B) \in \mathcal{P}_n \). Actually given two trees \( t_1 \in \mathcal{C}_A \) and \( t_2 \in \mathcal{C}_B \) with \((A, B) \in \mathcal{P}_n \) there are \( |A| \) distinct ways to attach \( t_2 \) to \( t_1 \) by joining the vertex \( n + 1 \) of \( t_2 \) to one of \( t_1 \)’s vertices to obtain a tree \( t \in \mathcal{C}_n \).

With this in mind we can decompose the quantity \( F_n(x, 1, z) \) depending on where \( t_2 \) is attached to \( t_1 \):

\[
F_n(x, 1, z) = xz + \sum_{(A,B) \in \mathcal{P}_n} \sum_{\substack{|A| \geq 1 \\atop t_1 \in \mathcal{C}_A \\atop t_2 \in \mathcal{C}_B}} (|A| - 2)x^{\deg_1(t_1)}z^{\deg_2(t_1)} + x^{\deg_2(t_2)+1}z^{\deg_1(t_2)} + x^{\deg_1(t_1)}z^{\deg_2(t_1)+1}.
\]
The first term corresponds to the case where \( t_1 \) has only 1 vertex (i.e. \( |A| = 1 \)). If \( |A| > 1 \) then the vertex 1 and the last vertex of the path \( L_1(t_1) \) are distinct in \( t_1 \) and we have to consider three cases: 1) we attach \( t_2 \) to a vertex of \( t_1 \) which is neither 1, nor the last vertex of \( L_1(t_1) \). 2) We attach \( t_2 \) to the vertex 1. 3) We attach \( t_2 \) to the last vertex of \( L_1(t_1) \). We then have:

\[
F_n(x, 1, z) = xz + \sum_{a=2}^{n-1} \binom{n-2}{a-1} F_a(x, 1, z)(a-2 + x + z).
\]

By induction we conclude that:

\[
F_n(x, 1, z) = xz + xz \sum_{a=2}^{n-1} \binom{n-2}{a-1} f_a(x+z)(a-2 + x + z).
\]

So \( F_n(x, 1, z)/(xz) \) depends only on \( x + z \), thus induction is shown. We then just need to take \( z = 1 \) and use the case \( y = z = 1 \) to conclude.

To prove the general case of Proposition \( \text{[9]} \) we will use the particular cases \( y = 1 \) and \( z = 1 \). We will also use the following Abel’s binomial Theorem \( \text{[Rio79, p. 18]} \).

**Proposition 11.** For every integer \( n \geq 0 \) the following identity holds:

\[
\sum_{k=0}^{n} \binom{n}{k} x(x - k z)^{k-1} (y + k z)^{n-k} = (x + y)^n.
\]

Three useful variants can be deduced from this identity.

**Corollary 12.** For every integer \( n \geq 0 \),

\[
\begin{align*}
\text{Variant 1} & \quad \sum_{k=0}^{n} \binom{n}{k} (x + k)^{k-1} (y - k)^{n-k} = \frac{(x + y)^n}{x}; \\
\text{Variant 2} & \quad \sum_{k=0}^{n} \binom{n}{k} (x + k)^{k} (n - k + y)^{n-k-1} = \frac{(x + y + n)^n}{y}; \\
\text{Variant 3} & \quad \sum_{k=0}^{n} \binom{n}{k} (x + k)^{k-1} (n - k + y)^{n-k-1} = \frac{x + y}{xy} (x + y + n)^{n-1}.
\end{align*}
\]

**Proof.** Taking \( z = -1 \) in Abel’s binomial formula gives the first variant. Doing the change of index \( k \to n - k \) and the change of variables \( y \to x + n \) and \( x \to y \) in variant 1 gives variant 2. To get variant 3 we begin by differentiating variant 1 with respect to \( y \), so we have:

\[
\sum_{k=0}^{n} \binom{n}{k} (n - k)(x + k)^{k-1} (n - k + y)^{n-k-1} = n \frac{(x + y + n)^{n-1}}{x}.
\]

Denote by \( A \) the left side of variant 3 which we want to compute, then:

\[
nA - n \sum_{k=1}^{n} \binom{n-1}{k-1} (x + k)^{k-1} (n - k + y)^{n-k-1} = n \frac{(x + y + n)^{n-1}}{x}.
\]

By doing the change of index \( k \to k + 1 \) in the last sum and using variant 2 for the resulting sum we get:

\[
nA - n \frac{(x + y + n)^{n-1}}{y} = n \frac{(x + y + n)^{n-1}}{x}.
\]

The expression of \( A \) can be deduced from the last display and thus variant 3 is shown.

**Proof of Proposition [9] in the general case.** Assume \( n \geq 3 \). Once again we will use a "tree-cutting" argument but instead of cutting at vertex \( n \), we cut at vertex 2. More precisely, we denote by \( Q_n \) the set of all couples \((A, B)\) with \( A \) and \( B \) two subsets of \{1, . . . , n\} such that \( A \cup B = \{1, . . . , n\}, A \cap B = \emptyset, 1 \in A \) and \( 2 \in B \). Once again, we decompose the quantity \( F_n(x, y, z) \) depending on where \( t_2 \) is attached to \( t_1 \):

\[
F_n(x, y, z) = \sum_{(A, B) \in Q_n} \sum_{t_2 \in E_A} \left[ (|A| - 1)x^{\deg_1(t_1)}y^{\deg_2(t_2)+1}z^{|L_1(t_1)|} + x^{\deg_1(t_1)+1}y^{\deg_2(t_2)+1}z^{|L_2(t_2)|+1} \right].
\]
The first term appearing after the sums corresponds to attaching $t_2$ to a vertex which is not 1 in $t_1$ and the second term corresponds to attaching $t_2$ to the vertex 1. We then have:

$$F_n(x, y, z) = \sum_{a=2}^{n-1} \binom{n-2}{a-1} (a-1)yF_a(x, 1, z)F_{n-a}(y) + \sum_{a=1}^{n-1} \binom{n-2}{a-1} xyzF_a(x)F_{n-a}(y, 1, z).$$

Now we can use the cases $y = 1$ and $y = z = 1$ to replace the occurrences of $F_a$ in the last display. Let’s compute the first sum which we call $A_n(x, y, z)$, afterwards we will compute the second one, $B_n(x, y, z)$.

$$A_n(x, y, z) = \sum_{a=2}^{n-1} \binom{n-2}{a-1} (a-1)xy^2z(a-2+x+z)^{a-2}(n-a-1+y)^{n-a-2}$$

$$= (n-2)\sum_{a=0}^{n-3} \binom{n-3}{a} xy^2z(a+x+z)^a(n-a-3+y)^{n-a-4}$$

$$= (n-2)xyz(x+y+z+n-3)^{n-3}. $$

The second equality comes from the fact that $(a-1)\binom{n-2}{a-1} = (n-2)\binom{n-3}{a}$. The last equality comes from variant 2 of Abel’s binomial formula. Now for $B_n(x, y, z)$ we need to be careful and isolate the case $a = n-1$ because formula 3 doesn’t apply in the case $(y = 1, n = 1)$.

$$B_n(x, y, z) - x^2yz(x+n-2)^{n-3} = \sum_{a=0}^{n-3} \binom{n-2}{a} x^2y^2z^2(a+x)^{a-1}(n-a-3+y+z)^{n-a-3}$$

$$= \sum_{a=0}^{n-4} \binom{n-3}{a} x^2y^2z^2(a+1+x)^a(n-a-4+y+z)^{n-a-4}$$

$$+ \sum_{a=0}^{n-3} \binom{n-3}{a} x^2y^2z^2(a+x)^{a-1}(n-a-3+y+z)^{n-a-3}.$$  

The second equality comes from the fact that $\binom{n-2}{a} = \binom{n-3}{a-1} + \binom{n-3}{a}$. The last one comes from variants 1 and 2 of Abel’s binomial Theorem. The desired formula follows easily.

$$\square$$

4 A second generating function on Cayley trees and Proof of Theorem 1

The goal now is to compute the exponential generating function of $G_n$:

$$\sum_{n\geq 1} \frac{G_{n+1}(x, y, z)}{n!}.$$ 

Then by identifying coefficients in formula 4, we will be able to prove Theorem 1. The first step is to establish a symmetry property of $G_n$ with a bijective approach.

4.1 A symmetry result

Before computing the exponential generating function of $G_n$ we first state a useful symmetry result.

**Proposition 13.** For all $n \geq 2$:

$$G_n(x, y, z) = G_n(y, x, z).$$

**Proof.** We will prove it by finding a bijection $\phi$ in $\mathfrak{M}_n$ which exchanges $\mathbb{I}_1$ and $\mathbb{I}_2$ and keeps $\mathbb{I}_1$ unchanged. For $1 \leq k \leq n$ we set $\gamma(k) = 3 - k \mod n$ so $\gamma$ is a permutation and $\gamma^{-1} = \gamma$. For $(\tau_1, \ldots, \tau_{n-1}) \in \mathfrak{M}_n$ we define $\phi(\tau_1, \ldots, \tau_{n-1}) = \gamma \circ \tau_1 \circ \cdots \circ \tau_{n-1} \circ \gamma$. Notice that for any transposition $\tau = (a, b)$, $\gamma \circ \tau \circ \gamma = (\gamma(a) \gamma(b))$
hence \( \phi(\tau_1, \ldots, \tau_{n-1}) \) is a product of \( n-1 \) transpositions. To see that \( \phi \) has the expected property, we interpret the action of \( \phi \) on the tree \( \mathcal{F} \). The tree \( \mathcal{F} \circ \phi \) is obtained from \( \mathcal{F} \) by relabelling the vertex-labels according to the permutation \( \gamma \) and the edge-labels according to the permutation \( i \mapsto n-i \). Such an edge-relabelling implies that \( \text{Find}_n \) reads through \( \mathcal{E} \circ \phi \) in the exact opposite order than in \( \mathcal{E} \). Thus we easily check that \( \phi(\tau_1, \ldots, \tau_{n-1}) \) sends 2 on 3, 3 on 4, \ldots, \( n \) on 1 and 1 on 2 so \( \phi(\tau_1, \ldots, \tau_{n-1}) \) is a minimal factorization of \( (1 \ldots n) \).

4.2 Computation of the exponential generating function of the second generating function

Before computing the exponential generating function of \( G_n \) let’s introduce the Lambert \( W \) function (see e.g. [JK97]). It is by definition the solution (in the sense of formal series) of \( W(z)e^{W(z)} = z \). Using Lagrange inversion, one can show that

\[
e^{-rW(-z)} = \left[ \frac{-W(-z)}{z} \right]_r = \sum_{n \geq 0} \frac{r(n+r)^{n-1}}{n!} z^n
\]

with the convention that \( 0 \times 0^{-1} = 1 \). These properties of \( W \) will be useful when proving Proposition 14 and Theorem 11.

**Proposition 14.** The following identity on formal series holds:

\[
e^n(y-x) \sum_{n \geq 1} \frac{G_{n+1}(x, y, z)}{n!} r^n = \frac{xyz(x-1)}{x+1-1} e^{-yW(-t)} - \frac{xyz(y-1)}{y+1-1} e^{-zW(-t)} + \frac{xyz^2(y-x)}{(x+z-1)(y+z-1)} e^{-(x+y+z-1)W(-t)}.
\]

**Proof.** Fix \( n \geq 1 \). As in the proof of Proposition 9 in the case \( y = 1 \), we denote by \( \mathcal{P}_{n+1} \) the set of couples \((A, B)\) with \( A \) and \( B \) two subsets of \( \{1, \ldots, n+1\} \) such that \( A \cup B = \{1, \ldots, n+1\}, A \cap B = \emptyset, 1 \in A \) and \( n+1 \in B \). Then with a similar “tree-cutting” argument that led to (4) we have:

\[
G_{n+1}(x, y, z) = xyz \sum_{t \in \mathcal{E}_n} y^\deg S_1(t) + \sum_{\substack{(A, B) \in \mathcal{P}_{n+1} \\mid |A| > 1}} \sum_{\substack{t_1 \in \mathcal{E}_A \\mid \deg L_1(t_1) \neq 0 \\mid \deg L_1(t_1) = 0 \\mid \deg L_1(t_1) = 1 \\mid \deg L_1(t_1) = 2}} \frac{[|A| - 2] x^{\deg S_1(t_1)} y^{\deg S_2(t_1)} z^{\deg L_1(t_1)}}{x^{\deg S_1(t_1)} + y^{\deg S_2(t_1)} + z^{\deg L_1(t_1)}} + x^{\deg S_1(t_1)} y^{\deg S_2(t_1)} + z^{\deg L_1(t_1)}
\]

The first term corresponds to the case where \( t_1 \) has only 1 vertex (i.e. \( |A| = 1 \)). If \( |A| > 1 \) then the vertex 1 and the last vertex of the path \( L_1(t_1) \) are distinct in \( t_1 \) and we have to consider three cases: 1) we attach \( t_2 \) to a vertex of \( t_1 \) which is neither 1, nor the last vertex of \( L_1(t_1) \). 2) We attach \( t_2 \) to the vertex 1. 3) We attach \( t_2 \) to the last vertex of \( L_1(t_1) \). Then we have:

\[
G_{n+1}(x, y, z) = xyzF_n(y) + \sum_{a=2}^n \binom{n-1}{a-1} ((a-2)G_a(x, y, z) + xG_a(x, y, z) + yzF_a(x, 1, z)F_{n+1-a}(y)).
\]

By Proposition 13 we get:

\[
G_{n+1}(x, y, z) = xyzF_n(x) + \sum_{a=2}^n \binom{n-1}{a-1} [(a-2)G_a(x, y, z) + yG_a(x, y, z) + xzF_a(y, 1, z)F_{n+1-a}(x)]
\]

Now if we make the difference between the last two equations, we obtain:

\[
0 = xyz(F_n(y) - F_n(x)) + (x - y) \sum_{a=2}^n \binom{n-1}{a-1} G_a(x, y, z) + z \sum_{a=2}^n \binom{n-1}{a-1} [yF_a(x, 1, z)F_{n+1-a}(y) - xF_a(y, 1, z)F_{n+1-a}(x)]
\]
If \( n \geq 2 \), by using the third variant of Abel’s binomial formula and Proposition \([9]\) we get:

\[
\sum_{a=2}^{n} \binom{n-1}{a-1} F_a(x, 1, z) F_{n+1-a}(y) = xz \frac{\frac{x+y+z-1}{x+z-1} (x+y+z+n-2)^{n-2}}{y+z-1} - xyz (x+n-1)^{n-2} y+z-1
\]

and

\[
\sum_{a=2}^{n} \binom{n-1}{a-1} F_a(y, 1, z) F_{n+1-a}(x) = yz \frac{\frac{x+y+z-1}{y+z-1} (x+y+z+n-2)^{n-2}}{x+z-1} - xyz (y+n-1)^{n-2} x+z-1.
\]

So, finally,

\[
0 = (x-y) \sum_{a=2}^{n} \binom{n-1}{a-1} G_a(x, y, z) + u_{n-1}(x, y, z),
\]

where

\[
u_{n-1}(x, y, z) = \frac{xy^2z(x-1)}{x+z-1} (y+n-1)^{n-2} - \frac{x^2yz(y-1)}{y+z-1} (x+n-1)^{n-2} + \frac{xyz^2(y-x)(x+y+z-1)}{(x+z-1)(y+z-1)} (x+y+z+n-2)^{n-2}.
\]

Thus, by Pascal’s inversion formula,

\[
(y-x) G_{n+1}(x, y, z) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} u_k(x, y, z). \tag{7}
\]

Now define for \( n \geq 0 \) the polynomials \( P_n \) by

\[
P_n(u) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (k+u)^{k-1}. \tag{8}
\]

Their exponential generating function is given by

\[
\sum_{n \geq 0} \frac{P_n(u)}{n!} t^n = \sum_{k \geq 0} u^k (k+u)^{k-1} \sum_{n \geq k} \binom{n}{k} (-1)^{n-k} \frac{1}{n!} t^{n-k} = uc^{-t} \sum_{k \geq 0} \frac{(k+u)^{k-1} t^k}{k!} = e^{-t-uW(-t)}.
\]

This, combined with \((7)\), readily gives the desired result. 

\[\square\]

### 4.3 Proof of Theorem \([1]\)

To simplify notation, for \( n \geq 2 \), set

\[
p_{i,j}^n = \delta \left( \deg_1(T_n) = i, \deg_2(T_n) = j \right)
\]

so that

\[
p_{i,j}^n = \frac{1}{n^{n-2}} [x^i y^j] G_n(x, y, 1).
\]

**Proof of Theorem \([7]\)** Fix \( i, j \geq 1 \). We take the formula of Proposition \([14]\) with \( z = 1 \) and divide it by \((y-x)\) to get:

\[
e^t \sum_{n\geq1} \frac{G_{n+1}(x, y, 1)}{n!} t^n = \frac{xy}{y-x} \left[ e^{-yW(-t)} - e^{-xW(-t)} \right] + \frac{1}{y-x} \left[ xe^{-xW(-t)} - ye^{-yW(-t)} \right] + e^{-(x+y)W(-t)}.
\]
We shall identify the coefficient in front of $x^iy^j$ (which is a polynomial in $t$) in this formula. On the left of
this equality, the coefficient is:

$$e^t \sum_{n \geq 1} p_{i,j}^{n+1} \frac{(n+1)^{n-1}}{n!} t^n.$$  

Using the identity $y^k - x^k = (y-x)(y^{k-1} + y^{k-2}x + \cdots + x^{k-1})$ we deduce the coefficient on the right:

$$\frac{(-W(-t))^{i+j}}{i!j!} + \frac{(-W(-t))^{i+j-1}}{(i+j-1)!} + \frac{(-W(-t))^{i+j}}{(i+j)!}.$$  

Therefore

$$\sum_{n \geq 1} p_{i,j}^{n+1} \frac{(n+1)^{n-1}}{n!} t^n = e^{-t} \frac{(-W(-t))^{i+j}}{i!j!} + e^{-t} \frac{(-W(-t))^{i+j-1}}{(i+j-1)!} + e^{-t} \frac{(-W(-t))^{i+j}}{(i+j)!}.$$  

We now fix $n \geq i+j$ and identify the coefficient associated with $t^n$ in the above formula. Fix $1 \leq \ell \leq n$. By
using formula (8) we obtain:

$$[t^n]e^{-t}(-W(-t))^\ell = \ell \sum_{k=\ell}^n \frac{(-1)^{n-k} k^{k-1-\ell}}{(n-k)! (k-\ell)!}.$$  

Recall the definition of $P_n$ in (8). Taking $y = z = 1$ in equation (7) gives:

$$G_{n+1}(x,1,1) = F_{n+1}(x) = P_n(x+1) - P_n(1).$$  

In particular,

$$P_n^{(\ell)}(x+1) = F_n^{(\ell)}(x) = x \frac{(n-1)!}{(n-1-\ell)!} (n+x)^{n-1-\ell} + \ell \frac{(n-1)!}{(n-\ell)!} (n+x)^{n-\ell},$$  

with the convention that $1/(-1)! = 0$. On the other hand, by definition of $P_n$,

$$P_n^{(\ell)}(x) = x \sum_{k=\ell+1}^n (-1)^{n-k} \binom{n}{k} \frac{(k-1)!}{(k-1-\ell)!} (k+x)^{k-1-\ell} + \ell \sum_{k=\ell}^n (-1)^{n-k} \binom{n}{k} \frac{(k-1)!}{(k-\ell)!} (k+x)^{k-\ell}. $$  

We finally obtain:

$$\ell \sum_{k=\ell}^n \frac{(-1)^{n-k} k^{k-1-\ell}}{(n-k)! (k-\ell)!} = \frac{1}{n!} P_n^{(\ell)}(0) = (n-1)^{n-1-\ell} \frac{\ell - 1}{(n-\ell)!}.$$  

Formula (11) then follows.

\[ \square \]

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