Finite Temperature Strings*

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Abstract

In this talk a review of earlier work on finite temperature strings was presented. Several topics were covered, including the canonical and microcanonical ensemble of strings, the behavior of strings near the Hagedorn temperature as well as speculations on the possible phases of high temperature strings. The connection of the string ensemble and, more generally, statistical systems with an exponentially growing density of states with number theory was also discussed.

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1. Introduction

The theory of fundamental strings has attracted a lot of interest in the past decade as a possible candidate for a unified theory of all the known interactions. Despite heroic efforts, crucial questions still remain unanswered. For example, the basic degrees of freedom of this theory are still not well understood. In particular near the Planck scale, where we expect the stringy behavior to become more evident, it is very poorly understood what is the correct configuration space and which states dominate the dynamics of string theory at that scale.

One way to probe this region would be to study a very hot ensemble of strings. There is, however, a difficulty as one tries to heat the ensemble past a limiting temperature $T_H = \beta_H^{-1}$, the famous Hagedorn temperature. The number of states of string theory at a specific energy level increases exponentially with the energy. As a result, the partition function is infinite for $\beta < \beta_H$ due to the competition between the entropy and the Boltzmann factor $e^{-\beta E}$. What happens when we try to increase the temperature past this point is, despite much effort, merely a speculation.

Since the effective string tension vanishes as one approaches $T_H$ one possibility is that pumping more energy into the ensemble to increase its temperature merely results in the formation of longer and “wigglier” strings by exciting higher oscillation modes [3]. This might be related to a “string uncertainty principle”

$$\Delta x \approx \frac{\hbar c}{E} + \frac{G_N E}{g^2 c^5},$$

where as we try to probe smaller distances by scattering more energetic strings, we obtain, after a point, poorer resolution due to the formation of longer and longer strings.

Another suggestion is that the Hagedorn transition is really a first or second order phase transition, where above $T_H$ we obtain a large genus zero contribution to the free energy. In a string theory with smooth world sheets we expect the genus zero contribution to the partition function to be zero. The reason is that the partition function is calculated by toroidally compactifying the (euclidean) time direction with radius $R = \beta/2\pi$. Since the sphere is simply connected, it cannot wrap around the time direction and it should give a trivial $\beta$ dependence. This suggests that smooth Riemann surfaces are not appropriate for describing the high temperature phase of the string. The following picture is suggested to hold [1–3]: The modes $\varphi$ and $\varphi^*$ that wind once around $S^1$ become tachyonic at $T_H$ and are supposed to acquire nonzero expectation values $\langle \varphi \rangle$ and $\langle \varphi^* \rangle$ surrounding
which we should look for stable solutions (in fact due to the dilaton the transition occurs at $T_{\text{crit}} < T_H$). Then the $\varphi$ vertex operator creates tiny holes on the world sheet that wind around the time direction tearing the world sheet. This suggests that the degrees of freedom in the high-$T$ phase are drastically reduced. In fact the large $T$ behavior of the free energy implies that there are fewer fundamental gauge invariant degrees of freedom than any known field theory that lives in the embedding space (for example for the closed strings one finds that the number of degrees of freedom is the same as that of a two dimensional field theory!).

At this point it is useful to consider the parallel with low energy QCD. It has been claimed long ago that large $N$ QCD [7] or the strong coupling limit of QCD [8] can be formulated as a string theory (for a recent review see [9] and references therein). In favor of this point of view is the early success of the dual resonance models in describing the low energy meson resonances. There are, however, difficulties with this picture. For example the string in the relevant non-critical dimensions is not Lorentz invariant in the light cone quantization and has one too many oscillators because of the Liouville mode in the Polyakov quantization.

It is believed that $SU(N)$ QCD undergoes a first order deconfining transition at some temperature $T_{\text{dec}}$. This can be the analogue of the Hagedorn transition of string theory. We understand the high energy regime because of asymptotic freedom. The basic degrees of freedom are weekly interacting quarks and gluons. In the strong coupling limit perturbative QCD is no longer a good description anymore, since the perturbation series, even as an asymptotic series, is not a good approximation. Hadrons, mesons and glueballs are the basic degrees of freedom and an effective string theory description in terms of thin chromo-electric flux tubes might be possible. The low temperature phase is described by smooth world sheets and it is expected that such a description breaks down in the high temperature regime; it is the underlying field theory that gives the basic degrees of freedom in that case. In string theory, however, a 26 or 10 dimensional field theory would be inappropriate for the high temperature phase, since such a theory would not have a good ultraviolet behavior.

The study of finite temperature strings also suggests some perhaps deep connection between multiplicative number theory and statistical systems with an exponentially growing density of states [10–13]. There exist several examples, like the bosonic, fermionic or parafermionic Riemann gas and the known string and conformal field theories, where the partition function is related to well studied number theoretic multiplicative arithmetic.
functions and modular forms. Understanding such a connection for physically interesting string theories could be of great importance for calculating and studying the behavior of relevant physical quantities. The number theoretic methods developed are quite powerful and can be used to calculate the partition function or give a very good estimate of the asymptotic behavior of the level densities of the above models. We should also not underestimate the possible mathematical importance of these questions.

This presentation is organized as follows. In section two we introduce the main relevant concepts. In section three we present the connection between string theory at finite temperature and number theory. We stress the relation between the level density \( p(N) \) of the bosonic string, the Dedekind eta function \( \eta(\tau) \) and Ramanujan’s \( \tau \)-function \( \tau(n) \). We explain how one can use Hardy and Ramanujan’s results to compute \( p(N) \). In section four we make this connection deeper by discussing the Riemann gas of bosons, fermions or parafermions. In section five we discuss the microcanonical and canonical ensemble of strings emphasizing the limit of validity of the canonical ensemble near the Hagedorn temperature. In section six we present the derivation of the asymptotic density of states for the microcanonical ensemble and the effect of introducing chemical potentials in order to impose conservation laws in the computation of \( \beta_H \). In section seven we describe how we can obtain the partition function with the string path integral on the torus. In section eight we discuss the possible physical behavior of the string ensemble near \( \beta_H \) and in section nine we discuss the results of Kogan and Sathiapalan and Atick and Witten.

2. Strings as a Statistical Mechanical System

Consider a \( d \) dimensional string theory described by the embedding

\[
X : \Sigma \rightarrow \mathbb{R}^{d-1,1}
\]

of the Riemann surface (complex curve) \( \Sigma \) with coordinates \((\sigma, \tau)\) into the \( d \) dimensional Minkowski space \( \mathbb{R}^{d-1,1} \) with cartesian coordinates \( X^\mu, \quad \mu = 1, 2, \ldots, d \). The classical free equations of motion

\[
(\partial^2_\sigma - \partial^2_\tau)X^\mu = 0
\]

and the boundary conditions, which for the open string are \( X'^\mu(\sigma, \tau) = 0 \) for \( \sigma = 0 \) and \( \pi \), determine the mode expansion

\[
x^\mu(\sigma, \tau) = x_0^\mu + p^\mu \tau + i \sum_{n \neq 0} \frac{\alpha_n^\mu}{\sqrt{\tau}} e^{-in\tau} \cos n\sigma .
\]

3
Upon quantization, the constraint $(L_0 - a)|\chi> = 0$ on the physical states determines the spectrum of the theory

$$\frac{1}{2}M^2 = \hat{N} - a.$$  \hspace{1cm} (3)

In the above formulas $L_0$ is the zero mode of the stress energy tensor, a a normal ordering constant and $\hat{N} = \sum_{n=1}^{\infty} n\alpha_{-n} \cdot \alpha_n = \sum_{n=1}^{\infty} n\alpha_n^+ \cdot \alpha_n$ is the level number operator. The number of states at a particular level increases very rapidly with $N$ and as we will soon see, it increases exponentially with $N$ for large $N$. In order to compute the number of states at level $N$, we consider the generating function for the level degeneracies

$$F(z) \equiv \text{Tr } z^{\hat{N}} = \sum_{N=0}^{\infty} d(N) z^N$$ \hspace{1cm} (4)

where $d(N)$ is the number of mass eigenstates at level $N$. We can compute $F(z)$ using

$$F(z) = \text{Tr } z^{\sum_{n=1}^{\infty} n\alpha_n^+ \cdot \alpha_n} = \prod_{n=1}^{\infty} \text{Tr } z^{n\alpha_n^+ \cdot \alpha_n}$$

$$= \left\{ \prod_{n=1}^{\infty} \left( \frac{1}{1 - z^n} \right) \right\}^{d-2}$$

$$= |f(z)|^{d-2},$$

where $\dagger f(z) = \prod_{n=1}^{\infty} \frac{1}{1 - z^n} = \sum_{n=0}^{\infty} p(n) z^n$ is the classical partition function of Euler. It is the generating function for the number of unrestricted partitions of $N$ into positive integers.

Proof: Since

$$f(z) = \prod_{r=1}^{\infty} (1 + z^r + z^{2r} + z^{3r} + \ldots),$$ \hspace{1cm} (6)

a typical term $z^k$ in $f(z)$ is obtained by taking one contribution from each factor $r = 1, \ldots, \infty$ such that $z^k = (z^{k_1})(z^{2k_2})(z^{3k_3})\ldots(z^{nk_n})$ with $k = k_1 + 2k_2 + 3k_3 + \ldots + nk_n$. This yields a partition of $k$ into sets $\{k_i\}$.

3. Singularities of $f(z)$ and Asymptotics of $p(N)$

The function $f(z)$ has an essential singularity at all rational points of the unit circle $S^1$ i.e. the set of points $\{ e^{\frac{2\pi ip}{q}} : p, q \in \mathbb{Z} \} \equiv \{z_{p,q}\}$. The essential singularity arises

\dagger The tachyon may be accounted for by modifying this to $F(z) = \frac{1}{z} |f(z)|^{d-2}$.
because \( z_{p,q}^n = 1 \) for an infinite number of integers \( n = k \cdot q \forall k \in \mathbb{Z} \). In order to estimate the asymptotic behavior of the density of states we have to study the behavior of \( f(z) \) as \( z \to 1^- \). A crude estimate gives

\[
\lim_{z \to 1^-} f(z) = \lim_{z \to 1^-} \prod_{n=1}^{\infty} \frac{1}{1 - z^n} = \lim_{z \to 1^-} \exp \left( - \sum_{n=1}^{\infty} \ln(1 - z^n) \right) \\
= \lim_{z \to 1^-} \exp \left( \sum_{m,n=1}^{\infty} \frac{(z^n)^m}{m} \right) = \lim_{z \to 1^-} \exp \left( \sum_{m=1}^{\infty} \frac{z^m}{m(1 - z^m)} \right). \tag{7}
\]

This is the first appearance of exponential growth in the problem. One can improve the above estimate by using the modular properties of \( f(z) \). Writing \( z \) as \( e^{2\pi i \tau} \), the classical partition function takes the form

\[
f(\tau) = \prod_{n=1}^{\infty} \frac{1}{1 - e^{2\pi i n \tau}}, \tag{8}
\]

which is the partition function of a single boson on the torus with modular parameter \( \tau \).

This is the first connection we see with additive number theory (combinatorics) and conformal field theory. The function \( f(z) \) is closely related to an important modular form, the Dedekind eta function \( \eta(\tau) \) via

\[
\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) = z^{\frac{1}{24}} / f(z). \tag{9}
\]

This function has many fascinating number theoretic properties. For example

\[
\eta^{24}(\tau) = \frac{z}{f(z)^{24}} = z \left\{ \prod_{n=1}^{\infty} (1 - z^n) \right\}^{24} \equiv \sum_{n=1}^{\infty} \tau(n) z^n \tag{10}
\]

defines Ramanujan’s \( \tau \)-function \( \tau(n) \).

We list below some interesting properties of \( \tau(n) \)

1) Ramanujan conjectured that \( \tau(n) \) is a multiplicative function, i.e.

\[
\tau(mn) = \tau(m) \tau(n) \quad \text{if} \quad (m, n) = 1,
\]

where \((m, n)\) is the greatest common divisor of \( m \) and \( n \). This was proved by Mordell [14].
2) $\tau(m)\tau(n) = \sum_{d|m,n} d^{11} \tau(\frac{mn}{d^2})$.

3) If $\tau(p) \equiv 0 \pmod{p}$ then $\tau(pm) \equiv 0 \pmod{p} \quad \forall n$.

4) Dyson [15] proved that

$$\tau(n) = \sum_{a,b,c,d,e} \frac{(a-b)(a-c)(a-d)(a-e)(b-c)(b-d)(b-e)(c-d)(c-e)(d-e)}{1!2!3!4!}$$

where $a, b, c, d, e$ satisfy

$$a, b, c, d, e \equiv 1, 2, 3, 4, 5 \pmod{5},$$

$$a + b + c + d + e = 0,$$

and

$$a^2 + b^2 + c^2 + d^2 + e^2 = 10n.$$ 

The function $\eta(\tau)$ is a modular form of weight $+\frac{1}{2}$ since

$$\eta(-1/\tau) = (-i\tau)^{1/2} \eta(\tau). \quad (11)$$

Using (11) we can map the region $z \to 1^-$, where $f(z)$ has singularities, to the region $z' \to 0^+$, where $f(z)$ is regular and close to 1. Under $\tau \to -1/\tau, z = e^{2\pi i \tau} \to z' = e^{-2\pi i / \tau}$ and since $z' = \exp\left(\frac{4\pi^2}{\ln z}\right)$, we obtain

$$f(z') = \frac{(z')^{1/24}}{\eta(-1/\tau)} = \left(-\frac{\ln z}{2\pi}\right)^{-1/2} z^{-1/24} f(z) (z')^{1/24}.$$ 

Thus

$$\lim_{z \to 1^-} f(z) = \lim_{z \to 1^-} \left(-\frac{\ln z}{2\pi}\right)^{1/2} z^{\frac{23}{24}} \exp\left(\frac{-\pi^2}{6 \ln z}\right) \quad (12)$$

Note that (12) is a slightly improved estimate of the behavior of $f(z)$ as $z \to 1^-$ as compared to (7).

To extract $p(N)$ from the above results we follow Hardy and Ramanujan [16] who first applied the techniques of complex analysis to the combinatoric problem of determining the asymptotic behavior of the number of unrestricted partitions of an integer $N$.

$$p(N) = \frac{1}{2\pi i} \oint dz \frac{f(z)}{z^{N+1}}, \quad (13)$$
where the contour is taken around the origin and within and close to the unit circle. Then

\[ p(N) = \frac{1}{2\pi i} \oint dz \left( \frac{-\ln z}{2\pi} \right)^{1/2} \exp \left[ \frac{-\pi^2}{6\ln z} - (N + \frac{23}{24}) \ln z \right] . \]  

(14)

A saddle point evaluation of \( p(N) \) (see appendix) leads to the result

\[ p(N) \approx \frac{1}{4\sqrt{3N}} \exp \left( \frac{\pi}{\sqrt{3}} \sqrt{N} \right) . \]  

(15)

A better result for \( p(N) \) was derived by Hardy and Ramanujan. They showed that

\[ p(N) = \frac{1}{2\pi \sqrt{2}} \frac{d}{dN} \left( e^{c\lambda_N} \right) \frac{1}{\lambda_N} + \frac{(-1)^N}{2\pi} \frac{d}{dN} \left( \frac{e^{\frac{1}{3}c\lambda_N}}{\lambda_N} \right) + \ldots \]  

(16)

where \( \lambda_N = \sqrt{N - \frac{1}{24}} \) and \( c = \pi \sqrt{\frac{2}{3}} \). For \( N = 100 \) and \( 200 \), summing 6 terms of this series gives \( p(100) = 190,569,291.996 \) and \( p(200) = 3,972,999,029,338.004 \), whereas the exact results are \( p(100) = 190,569,292 \) and \( p(200) = 3,972,999,029,388 \) respectively \[16\].

Later an exact analytical result was found by Rademacher \[17\] using a slight variant of the Hardy-Ramanujan analysis

\[ p(N) = \frac{1}{\pi \sqrt{2}} \sum_{q=1}^{\infty} A_q(N) q^{1/2} \frac{d}{dN} \left( \sinh \left( \frac{c\lambda_N}{q} \right) \right) , \]  

(17)

where \( A_q(N) \) is a certain sum of roots of unity.

4. Quantum Statistical Mechanics and Multiplicative Number Theory

Multiplicative number theory in the context of statistical systems with exponentially growing number of states has been considered first by Julia and Spector\[10–11\]. Consider, instead of the usual Fock space of a quantum field theory, a different labelling of the quantum states. Place the creation operators \( \{ a_i^\dagger \} \) in cardinal order and associate the \( i^{th} \) prime number with the \( i^{th} \) creation operator. Then label the state \( \prod_i (a_i^\dagger)^{r_i} |0 \rangle \) by the integer \( N = \prod_i p_i^{r_i} \). Since the factorization of \( N \) into prime numbers is unique, this provides a basis for the Hilbert space of states. This is known as the Gödel numbering \[18\]. We are arithmetizing this way a free quantum field theory the same way as the formal
calculus of propositions is arithmetized by associating the \(i\)th symbol in a proposition with the \(i\)th prime number and determining the corresponding multiplicity by the symbol itself.

A simple example of a system with exponential growth of the number of states is the Riemann gas. The partition function of such a system is defined as

\[
Z(\beta) = \text{Tr} e^{-\beta H},
\]

with energies \(E_i = \ln p_i\). Then

\[
Z(\beta) = \sum_{\{r_i\}} \exp(-\beta \sum_i r_i \ln p_i) = \sum_{N=1}^{\infty} \frac{1}{N^\beta} = \zeta(\beta)
\]

is the (ordinary) Riemann zeta function. The partition function therefore, has a simple pole at \(\beta = 1\). This can alternatively be seen by computing the density of states

\[
\rho(E) = \frac{1}{\Delta E} \rho(N) \Delta(N) = \frac{1}{\ln \left(\frac{N+1}{N}\right)} = N \left(1 + \frac{1}{2N} + \ldots\right) = \exp(E) + \frac{1}{2} + \mathcal{O}(e^{-E}).
\]

We see that \(\beta_H = 1\) corresponds to the simple pole of \(\zeta(\beta)\) at \(\beta = 1\).

Generally, multiplicative generating functions of the form (Dirichlet series)

\[
F(\beta) = \sum_{n=1}^{\infty} \frac{f(n)}{n^\beta}
\]

are of interest, where \(f(n)\) is a general multiplicative arithmetic function on the natural numbers valued in some field (usually the field of real or complex numbers). As in the case of the ordinary Riemann zeta function, \(F(\beta)\) generally has a simple set of singularities but a rich structure of complex zeroes.

The case \(f(n) = \mu(n)\), where \(\mu(n)\) is the Möbius function, gives the partition function of the fermionic analog of (18). This gives \(Z_F(\beta) = \frac{\zeta(\beta)}{\zeta(2\beta)}\). Similarly the system of \(k\)-parafermions is associated with \(\mu_k\)

\[
f(n) \equiv \mu_k(n) = \begin{cases} 
1 & \text{if } n = \prod_i (p_i)^{r_i} \text{ with } 0 \leq r_i \leq k-1; \\
0 & \text{otherwise.}
\end{cases}
\]

8
It gives

\[ Z_k(\beta) = \frac{\zeta(\beta)}{\zeta(k\beta)}. \]  

(23)

Parafermions of order 2 and \( \infty \) correspond to fermions and bosons respectively. Note that a representation of the ordinary Riemann zeta function can be obtained by tensoring an infinite set of parafermionic gases of order \( k \) at successively lower temperatures, such that

\[ \zeta(\beta) \equiv Z_\infty(\beta) = \prod_{n=0}^{\infty} Z_k(k^n\beta). \]  

(24)

We hope that the above exposition has given a flavor of the important connection between arithmetic gases and multiplicative number theory. Much remains to be understood for the case of statistical systems that arise from string theories and conformal field theories.

5. The String Microcanonical and Canonical Ensemble

We are now ready to compute the microcanonical density of states for the string ensemble. Since the mass spectrum is given by

\[ \frac{1}{4} M^2 = N - \text{const}, \]  

(25)

the asymptotic form for \( p(N) \) translates into the string density of states

\[ \Omega(M) \sim M^{-\alpha} \exp(\beta M). \]  

(26)

The microcanonical density of states \( \Omega(E) \) is defined by

\[
\Omega(E) = \text{Tr} \, \delta(E - \tilde{H}) = \int \frac{d\tilde{\beta}}{2\pi} \text{Tr} \, e^{i\tilde{\beta}(E - \tilde{H})} = \int_{\beta_0 - i\infty}^{\beta_0 + i\infty} \frac{d\beta}{2\pi i} e^{\beta E} \text{Tr} \, e^{-\beta \tilde{H}} = \int_{\beta_0 - i\infty}^{\beta_0 + i\infty} \frac{d\beta}{2\pi i} e^{\beta E} Z(\beta),
\]  

(27)

where \( \beta = i\tilde{\beta} \) and \( \beta_0 \) is chosen for the convergence of \( Z(\beta) \equiv \text{Tr} \, e^{-\beta \tilde{H}} \), the canonical partition function. Inverting the Laplace transform we obtain

\[
Z(\beta) = \int_0^{\infty} dE \Omega(E) e^{-\beta E}
\]  

(28)
In a typical statistical mechanical system with \(N\) degrees of freedom \(\Omega(E) \sim E^N\) for large \(E\). Thus the entropy \(S = \ln \Omega(E) \sim N \ln E\) is extensive. For large \(N\), the integrand is sharply peaked and the canonical partition function can be evaluated by a saddle point approximation. Since

\[
Z(\beta) = \int_0^\infty dE \exp (\ln \Omega(E) - \beta E)
\]

the saddle point occurs at \(E_0\) such that

\[
\frac{d \ln \Omega(E)}{dE} \bigg|_{E=E_0} = \beta = \frac{N}{E_0},
\]

i.e. \(E_0 = \frac{N}{\beta} = NT\). Thus fixing the temperature \(T\) (canonical ensemble) is equivalent in the thermodynamic limit to fixing \(E\) in the microcanonical ensemble at \(NT\) and \(T\) is really the average energy per degree of freedom. The fluctuations are given by

\[
\begin{align*}
Z(\beta) &= Z_0 \int_0^\infty dE \exp \left\{ \frac{1}{2} (E - E_0)^2 \frac{\partial^2 S}{\partial E^2} (S(E) - \beta E) \right\} \\
&= Z_0 \int_0^\infty dE \exp \left\{ \frac{1}{2} (E - E_0)^2 \frac{\partial^2 S(E)}{\partial E^2} \right\}.
\end{align*}
\]

This may be rewritten in terms of the microcanonical (formal) specific heat \(C_V\). Since \(S_M = \ln \Omega(E)\) and \(T_M = (\frac{\partial E}{\partial S})^{-1}\), we have

\[
\frac{\partial^2 S}{\partial E^2} = -\frac{1}{T^2} \frac{dT}{dE} \bigg|_{V} = -\frac{1}{T^2} \frac{1}{(C_V)_M},
\]

where \(C_V = \frac{dE}{dT}\). Then

\[
(C_V)_M = -\frac{1}{T^2} \left( \frac{\partial^2 S}{\partial E^2} \right)^{-1}.
\]

Thus (31) becomes

\[
Z(\beta) = Z_0 \exp \left\{ -\frac{1}{2} (E - E_0)^2 \frac{1}{T^2} \frac{1}{(C_V)_M^{-1}} \right\}.
\]

Convergence requires that \(C_M > 0\) in order that saddle point approximation is valid.

For strings \(\Omega(E) \sim E^{-\alpha} \exp \beta_M E\) and as one approaches \(T_H\) from below the integrand \(\Omega(E)E^{-\beta_M E}\) flattens out. Typically \(\alpha > 1\) and \(Z(\beta)\) is well defined for \(\beta = \beta_H\), but it diverges for \(\beta > \beta_H\). Fixing \(T\) as \(\beta \to \beta_H^+\) no longer corresponds to fixing a precise \(E\) and
the canonical and microcanonical ensemble need not agree. The entropy \( S = \ln \Omega(E) \sim \beta H E - \alpha \ln E \) is no longer extensive as well. For further discussion see section 7.

For free particles

\[
Z(\beta) = \prod_{k,b} (1 - e^{-\beta E_{k,b}})^{-1} \prod_{k,f} (1 + e^{-\beta E_{k,f}}).
\]

Therefore

\[
\ln Z(\beta) = \sum_{r=1}^{\infty} \left( \frac{f_B(\beta r)}{r} - (-1)^r \frac{f_F(\beta r)}{r} \right),
\]

where the single-string partition functions \( f_B \) and \( f_F \) are

\[
f_B = \sum_{k,b} e^{-\beta E_{k,b}} = \int_0^{\infty} dE e^{-\beta E} \omega_B(E),
\]

\[
f_F = \sum_{k,f} e^{-\beta E_{k,f}} = \int_0^{\infty} dE e^{-\beta E} \omega_F(E).
\]

Substituting (33) into (27) we obtain the multi-string density of states from the single string density of states

\[
\Omega(E) = \int_{\beta_0-i\infty}^{\beta_0+i\infty} \frac{d\beta}{2\pi i} e^{\beta E} \exp \left\{ \sum_{r=1}^{\infty} \left( \frac{f_B(\beta r)}{r} - (-1)^r \frac{f_F(\beta r)}{r} \right) \right\}.
\]

The \( r = 1 \) term gives the familiar Maxwell-Boltzmann expression

\[
\Omega(E) = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{E_1}^{E} dE_1 \omega(E_1) \delta(E - \sum_i E_i)
\]

6. From the One-String Density of States to the Multi-String Density of States for Toroidal Compactification

Consider \( d \)-dimensional strings compactified on an internal manifold of dimension \( c \) that is for simplicity taken to be a torus with radii \( R_i \) with \( i = 1, 2, \ldots, c \). In computing the full density of states, it is important to incorporate the effects of conservation laws [19] such as conservation of momentum, winding number and, in the case of the heterotic string, of
charges or windings/momenta in the internal directions. For every conserved charge $Q_A$ we introduce a chemical potential $\mu_A$ and compute

$$\Omega(E, \mu_A) = tr \left[ e^{2\pi i \mu_A Q_A} \delta(E - H) \right]$$

$$Z(\beta, \mu_A) = tr \left[ e^{2\pi i \mu_A Q_A} e^{-\beta H} \right].$$

(39)

The condition that the total conserved charge be some fixed value $q_A$ (typically $q_A = 0$) is then enforced by multiplying $\Omega(E, \mu_A)$ or $Z(\beta, \mu_A)$ by $\exp\left\{ -2\pi i \mu_A q_A \right\}$ and integrating $\mu_A$ over the range $(-\frac{1}{2}, \frac{1}{2})$

$$\Omega(E, q_A) = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\mu_A \Omega(E, \mu_A) e^{-2\pi i \mu_A q_A}.$$  

(40)

The full density of states will be derived from the single-string density of states as discussed in the previous section. In order to compute the single string density of states, consider the energy of a general string state

$$E^2 = k^2 + \sum_{i=1}^{c} \left( \frac{n_i^2}{R_i^2} + (2R_i)^2 m_i^2 \right) + 4n_L + 4n_R.$$  

(41)

Here $k$ is the momentum in the non-compact directions and $n_i$ and $m_i$ are the momentum and winding quantum numbers, respectively. The integers $n_L$ and $n_R$ label the internal level numbers describing oscillatory mode excitations of the string.

As described in section 1, the number of left-moving (right-moving) excitations at a given level is given by the generating function $F(z) = \sum_{\alpha} z^{n_L} \bar{F}(\bar{z}) = \sum_{\alpha} \bar{z}^{n_R}$ where the sum runs over all possible states $\alpha$. For a bosonic sector $f(z) = \eta^{-24}(z) = \frac{1}{z} f(z)^{24}$ (the extra factor $\frac{1}{2}$ corresponds to the tachyon), while for a superstring sector $f(z) = \frac{\varphi^2(z)}{\eta^{10}(z)}$. As before we compute the asymptotic level densities using a saddle point evaluation

$$d(n) = \frac{1}{2\pi i} \oint \frac{dz}{z^{n+1}} f(z).$$  

(42)

The resulting level densities for left and right sectors are

$$d(n_L) \sim n_L^{-(d+1)/4} e^{4\pi \sqrt{a_L n_L}}, \quad d(n_R) \sim n_R^{-(d+1)/4} e^{4\pi \sqrt{a_R n_R}}$$  

(43)

where $a_{L,R} = \frac{1}{2}$ for a superstring sector and $a_{L,R} = 1$ for a bosonic string sector (so for the heterotic string $a_L = 1$ and $a_R = \frac{1}{2}$). Physical states must also obey the level-matching condition $L_0 - \bar{L}_0 = 0$, or $n_L - n_R = -\sum_i m_i n_i$. This and (41) give

$$8n_L = E^2 - k^2 - \left( \frac{n_i}{R_i} + 2R_i m_i \right)^2,$$

$$8n_R = E^2 - k^2 - \left( \frac{n_i}{R_i} - 2R_i m_i \right)^2.$$
Therefore the single-string density of states is (including the chemical potentials \( \kappa, \mu \) and \( \nu \))

\[
\omega(E, \kappa, \mu, \nu) dE \sim V_D \int \frac{d^Dk}{(2\pi)^D} \sum_{m_i, n_i} \frac{E dE}{4} e^{2\pi i (\kappa \cdot k + \mu \cdot m + \nu \cdot n)} e^{4\pi \sqrt{a_L n_L(E, k, n_i, m_i) + 4\pi \sqrt{a_R n_R(E, k, n_i, m_i)}}} \frac{e^{\left\{4\pi \sqrt{a_L n_L(E, k, n_i, m_i) + 4\pi \sqrt{a_R n_R(E, k, n_i, m_i)}}\right\}}}{n_L(E, k, n_i, m_i)^{(d+1)/4} n_R(E, k, n_i, m_i)^{(d+1)/4}}.
\]

(44)

Here \( D = d - c - 1 \) is the number of non-compact space directions, and \( V_D \) is the volume of the non-compact directions.

Note that for each \( R_i \) there are regions with different behaviour. If \( R_i \gg E \gg 1/R_i \) we ignore the winding quantum numbers \( m_i \) and the sum over the closely spaced momentum levels is well approximated by an integral over a continuous spectrum of momentum; thus the string behaves as if the direction \( i \) were noncompact. Likewise, if \( 1/R_i \gg E \gg R_i \) the sum over momenta drops out and the sum over winding becomes continuous; this situation is dual to the previous one under \( R \rightarrow 1/2R \). Finally we have the truly high-energy regime, \( E \gg R_i, 1/R_i \). Here we can closely approximate the sum over \( m_i, n_i \) by an integral \( dm_i dn_i \).

Furthermore, for large \( E \) we can evaluate the integral (44) by saddle point methods. If we expand the arguments of the square roots to first order in \( k^2, (n_i/R_i \pm 2Rm_i)^2 \) (valid for large \( E \) in the saddle point approximation since the region \( (n_i/R_i \pm 2Rm_i)^2/E^2 \ll 1 \), \( k^2/E^2 \ll 1 \) dominates), the integral becomes quadratic and yields

\[
\omega(E, \kappa, \mu, \nu) \sim \frac{1}{E^{D/2+1}} e^{\beta_H(\kappa, \mu, \nu) E}
\]

(45)

where

\[
\beta_H(\kappa, \mu, \nu) = \beta_H - \sqrt{2\pi} \left\{ \frac{\kappa^2}{\sqrt{a_L} + \sqrt{a_R}} + \left( \frac{\mu}{4R} + \frac{\nu R}{2} \right)^2 + \left( \frac{\mu R - \nu R}{\sqrt{a_R}} \right)^2 \right\}
\]

(46)

and

\[
\beta_H = \sqrt{2\pi} (\sqrt{a_L} + \sqrt{a_R})
\]

is the usual Hagedorn temperature \( \beta \). Thus in effect the Hagedorn temperature depends on the chemical potentials.

This result can now be used to derive the multi-string density of states as sketched in the previous section. First we will compute the Maxwell-Boltzmann contribution to the total density of states, and then we will argue that the corrections due to Bose-Einstein
or Fermi-Dirac statistics do not substantially alter the asymptotic form of the density of states.

The MB expression (38) becomes a sum of terms of the form

\[
\frac{1}{n!} \int \prod_{i=1}^{n} \frac{dE_i}{E_i^{D/2+1}} e^{\beta H(\kappa,\mu,\nu)E_i} \delta \left( E - \sum_{i} E_i \right).
\]

(47)

The integrals over \( E_i \) are divergent at the lower end and must be cut off at some energy \( m_0 \gg M_s \) where the asymptotic expression (45) is no longer valid. It is clear that the dominant contribution to (47) is that in which most of the energy is in one string, \( E_i \approx E \).

Thus the integral is approximately

\[
\Omega_n(E,\kappa,\mu,\nu) \sim \frac{1}{(n-1)!} e^{\beta H(\kappa,\mu,\nu)E} \left( \frac{2}{Dm_0^{D/2}} \right)^{n-1} \text{ for } D > 0; \\
\sim \frac{1}{(n-1)!} e^{\beta H(\kappa,\mu,\nu)E} \left[ \ln(E/m_0) \right]^{n-1} \text{ for } D = 0.
\]

(48)

Summing over \( n \) then gives

\[
\Omega(E,\kappa,\mu,\nu) \sim \frac{e^{\beta H(\kappa,\mu,\nu)E}}{E^{D/2+1}} e^{2\beta H(\kappa,\mu,\nu)Dm_0^{D/2}} \text{ for } D > 0; \\
\sim \frac{1}{m_0} e^{\beta H(\kappa,\mu,\nu)E} \text{ for } D = 0.
\]

(49)

The density of states without the conservation constraints is given by setting \( \kappa = \mu = \nu = 0 \); this expression has been found in the literature. To obtain the density of states including the constraint of zero total momentum and winding, we use (46) and integrate over \( \kappa,\mu,\nu \) to find

\[
\Omega(E) \sim \frac{e^{\beta H E}}{E^{d-\delta_D}}
\]

(50)

where we define \( \delta_D = 1 \) if \( D = 0 \) and \( \delta_D = 0 \) otherwise. Imposing the constraints changes the power of \( E \); this is in contrast to other thermodynamic situations where conservation laws are not so important essentially because one string carries most of the energy and bears the burden of the conservation laws.

We can also compute corrections to (50) due to Bose-Einstein and Fermi-Dirac statistics. One finds that the effect of these corrections is to multiply (50) by a function that depends on low-energy physics but to leading order not on \( E \).
Thus for the type II string the asymptotic density of states is
\[
\Omega(E) \sim e^{\beta \mu E} E^{10-\delta_D}. \quad (51)
\]
If we consider the heterotic \(E_8 \times E_8\) or \(\text{Spin}(32)/Z_2\) string we see that conservation of the 16 U(1) charges should also be imposed. In the bosonic formulation this is equivalent to momentum/winding conservation for the internal degrees of freedom, and, as one can easily convince oneself by a slight modification of the above argument, gives an extra factor of \(1/E^8\). Thus the asymptotic density of states for the heterotic string is
\[
\Omega(E) \sim e^{\beta \mu E} E^{18-\delta_D}. \quad (52)
\]

It will be useful for us to deduce which string configurations contribute most significantly to the total string density of states; this can be done by inspection of (47) or the corresponding expression including quantum statistics. As we have already noted, the dominant configuration for the integral for a fixed number \(n\) of strings is when \(E_i \approx E\), i.e. when most of the energy is in a single string. For \(D > 0\) it also appears that string configurations with a small total number of strings dominate the sum, but for \(D = 0\) configurations with a large total number of strings dominate the sum. (These qualitative features also hold true when corrections due to quantum statistics are included.) We will return to a discussion of this single string dominance of the energy in section 7.

7. Path-Integral Derivation

In the meantime, we turn to the second derivation of the string density of states \[20\text{-}23\] and \[3\]. Once again (27) is used to relate \(\Omega(E)\) to the complex function \(Z(\beta)\). Next recall that the free energy \(F = -\frac{1}{\beta} \ln Z\) for a single free particle of mass \(M\) can be computed from a first-quantized one-loop path integral on a space with compactified time direction of circumference \(\beta\). The action is \(S_p = \frac{1}{2} \int_0^1 d\tau [e^{-1}(dX/d\tau)^2 + eM^2]\), and in the path integral we integrate over all maps \(X^\mu(\tau)\) of the circle into the target spacetime \(S^1 \times \mathbb{R}^{d-1}\) and over all one-metrics \(g = e^{2\hat{F}}\) on the circle. Explicitly,
\[
2 \ln Z = \sum_{n=-\infty}^{\infty} \int \frac{Dg}{\text{Vol(Diff)}} \int_n DX e^{-S_p(-1)^nF} \quad (53)
\]
where in the path integral over metrics we must eliminate the volume of the diffeomorphism group, and where the boundary condition on the \(X\) path integral is \(X^0(1) = X^0(0) + \)
\( n \beta, \quad X^i(1) = X^i(0) \); \( n \) is the number of times that the circle winds around the compact time direction. We also define \( \hat{F} \) to be zero for bosons and one for fermions; the extra factor \((-1)^n \hat{F}\) corresponds to anti-periodic boundary conditions for fermions in the second-quantized formalism. One easily shows (taking into account the Killing vector on the circle) that

\[
\int \frac{Dg}{\text{Vol(Diff)}} = \int_0^\infty \frac{ds}{s}
\]

where \( s \) is the proper length of the circle. One can also show

\[
\int_n Dxe^{-Sp} = (V\beta)(2\pi s)^{-d/2} e^{-\frac{n^2\beta^2}{2s} - \frac{M^2 s}{2}}
\]

where \( V \) is the spatial volume. These give the free energy

\[
F = -\frac{V}{2} \int_0^\infty \frac{ds}{s} (2\pi s)^{-d/2} \sum_{n=-\infty}^\infty e^{-\frac{n^2\beta^2}{2s} - \frac{M^2 s}{2}} (-1)^n \hat{F}.
\]

(54)

It is fairly simple to demonstrate that this is equivalent to the standard expressions for the free energy of a single free particle.

Likewise, the free energy for string is obtained by doing the functional integral over maps of the torus into \( S^1 \times R^{d-1} \), with the (conformal gauge-fixed) action which we may take to be that of the heterotic string (our units are such that \( \alpha' = \frac{1}{2} \))

\[
S = \frac{1}{2\pi} \int d^2\sigma \left( \partial_\tau X^\mu \partial_\tau X_\mu + \bar{\psi}_\mu \partial_\tau \psi_\mu + \psi^\mu \partial_\tau \bar{\psi}_\mu + \lambda^i \partial_\tau \lambda^i \right).
\]

(The \( \lambda_i \) provide a fermionic representation of the gauge degrees of freedom.) The maps are allowed to wind in the time direction on the torus, but not in the space direction, as in the case of the particle. The result is

\[
F = -\frac{V}{4\pi^5} \int_0^\infty \frac{d\tau_2}{\tau_2} \int_{-\frac{\tau_2}{2}}^{\frac{\tau_2}{2}} \frac{1}{\eta(q)\eta(\bar{q})} \text{ch}[^\hat{\cal G}](q)
\]

\[
\sum_{\pi_1,\pi_2=0,1} \bar{C}_{\pi_1\pi_2}(\bar{q}) \sum_n \exp \left\{ -\frac{\beta^2 n^2}{2\pi \tau_2} \right\} (-1)^n \pi_1.
\]

(55)

Here \( q = \exp{2\pi i(\tau_1 + i\tau_2)} \), \( \eta \) is the Dedekind function; \( \text{ch}[^\hat{\cal G}] \) is the character function of the gauge group \( G = E_8 \times E_8 \) or \( \text{SO}(32) \), \( \text{ch}[\hat{E}_8 \times \hat{E}_8] = (\vartheta^8_2 + \vartheta^8_3 + \vartheta^8_4)^2 / 4\eta^{16}, \text{ch}[\hat{\text{SO}}(32)] = (\vartheta^4_2 + \vartheta^4_3 + \vartheta^4_4) / 2\eta^{16} \); and \( C_{00} = \bar{\vartheta}^4_3, C_{01} = -\bar{\vartheta}^4_1, C_{10} = -\bar{\vartheta}^4_1 \), and \( C_{11} = 0 \). This expression is analogous to (54); the integral over \( \pi \tau_2 \) corresponds to that over \( s \), the sum over \( n \) is the
sum over sectors where the time direction on the torus wraps $n$ times around the target time dimension, and the rest of the expression integrated over $\tau_1$ gives the sum over all single string states of $(-1)^n \exp\{-M^2 \pi \tau_2/2\}$. We could therefore alternatively derive this expression directly from (54) by thinking of the string as a collection of particles corresponding to its various modes.

To make this expression look more familiar from the string point of view, we recall that the region $S$ of the $\tau = \tau_1 + i\tau_2$ plane $|\tau_1| < 1/2$, $0 < \tau_2 < \infty$ corresponds to an infinite number of copies of the fundamental domain $F$ defined by $|\tau_1| < 1/2$, $|\tau| > 1$; $S$ is tiled by images of $F$ under maps $\tau' = (p\tau + q)/r\tau + s$, $p, q, r, s \in \mathbb{Z}$, $ps - qr = 1$.

The free energy for the heterotic string can therefore be rewritten as an integral over the fundamental domain

$$F = -\frac{V}{4} \pi^{-5} \int_{F} \frac{d^2 \tau}{2\pi \tau_2} (2\pi \tau_2)^{-5} \frac{1}{\eta(q)^8 \tilde{\eta}(\bar{q})^{12}} \text{ch}[\tilde{G}](q) \sum_{\pi_1, \pi_2 = 0, 1} \bar{C}_{\pi_1 \pi_2}(\bar{q})$$

$$\sum_{m, n} \exp\left\{-\frac{\beta^2}{2\pi \tau_2} \left(m^2 \tau_2^2 + (n - m\tau_1)^2\right)\right\} (-1)^{m\pi_2 + n\pi_1 + mn} \tag{56}$$

and likewise for the type II string

$$F = -\frac{V}{8} \pi^{-5} \int_{F} \frac{d^2 \tau}{2\pi \tau_2} (2\pi \tau_2)^{-5} \frac{1}{\eta(q)^8 \tilde{\eta}(\bar{q})^{8}} \sum_{\pi_1, \pi_2 = 0, 1} \sum_{\bar{\pi}_1, \bar{\pi}_2 = 0, 1} C_{\pi_1 \pi_2}(q) \bar{C}_{\bar{\pi}_1 \bar{\pi}_2}(\bar{q})$$

$$\sum_{m, n} \exp\left\{-\frac{\beta^2}{2\pi \tau_2} \left(m^2 \tau_2^2 + (n - m\tau_1)^2\right)\right\} (-1)^{m(\pi_2 + \bar{\pi}_2) + n(\pi_1 + \bar{\pi}_1)} \tag{57}$$

From (56) one can proceed to calculate the asymptotic density of states.

In these expressions the sum over $m$ is the sum over maps of the torus in which the space direction of the torus is wound $m$ times about the target $S^1$. While one might think of these as corresponding to strange states of the string in which it wraps around the time direction, this interpretation is not necessarily correct if the fundamental expression for the free energy is (55) which directly corresponds to $Z(\beta) = \text{tr} e^{-\beta H}$. This trace contains only ordinary string states and not the strange ‘winding states;’ the latter appear to arise only as mathematical artifacts when we transform the expression for $\ln Z$ as an integral over $S$ into an integral over $F$.

Note the following interesting point. (55) is equivalent to the modular invariant (56) the cosmological constant $\Lambda = 0$. However, if $\Lambda \neq 0$, (55) gives $\infty \cdot \Lambda + F'$ and (56) gives $\Lambda + F'$ where $\Lambda$ is cosmological constant as it is usually defined in string theory (as an
integral over $\mathcal{F}$) and $F'$ is the temperature dependent part of $F$. This is also a difficulty for string field theory - i.e. to naturally produce the fundamental domain.

8. Physical Behaviour of Strings at High Temperature

We now turn to an attempt to extract a better understanding of the physics of strings at high temperature from the above calculations [3,23]. To start with, assume that we have at our disposal a thermal bath that can interact with the string ensemble to heat it up. Within the context of fundamental strings this assumption is not necessarily valid as all matter is presumably made out of strings, but we make this initial assumption to aid us in isolating the source of any strange behavior that we may encounter; we will see that the assumption of the existence of such a reservoir doesn’t obviously appear to be related to any such behavior. Therefore suppose that we have a heat reservoir with a density of states $\Omega_r(E)$.

The total density of states for the string ensemble in contact with this reservoir is

$$\Omega_T(E) = \int_0^E dE_s \Omega(E_s) \Omega_r(E - E_s).$$

If we assume that the reservoir is large (in a sense to be made precise shortly), then most of the energy will be in the reservoir, i.e. $E \gg E_s$. In this case we can expand

$$\ln \Omega_r(E - E_s) \simeq \ln \Omega_r(E) - E_s \frac{\partial}{\partial E} \ln \Omega_r(E) + \frac{1}{2} E_s^2 \frac{\partial^2}{\partial E^2} \ln \Omega_r(E) + \cdots.$$

We identify $\frac{\partial}{\partial E} \ln \Omega_r(E) = \beta$ as the inverse temperature of the reservoir. We can drop the higher terms when the reservoir is sufficiently large; precisely, $E_s^2 \frac{\partial^2}{\partial E^2} \ln \Omega_r(E) \ll 1$ or

$$\frac{1}{\beta^2} C_{V_r} = -\frac{\partial E}{\partial \beta} \gg E_s^2. \quad (58)$$

Thus we can use the ensemble

$$\Omega_T(E) = \Omega_r(E) \int_0^\infty dE_s e^{-\beta E_s} \Omega(E_s) \quad (59)$$

as long as we ensure that the heat capacity of the reservoir is large enough in the sense of (58); replacing the upper limit by $\infty$ has no significant effect.

Now we are prepared to study the properties of the string gas. As we raise the temperature, (59) stays well defined all the way to $\beta_H$. Even at $\beta_H$ (59) is well defined.
because of the power law tail $\sim 1/E^{(d-\delta_D)}$. (Of course $\Omega(E)$ takes a different, convergent, form for low energy; since the string is a gas of massless particles at very low energies, we expect $\Omega(E) \sim \exp\left\{a^{D+1}E^{D/(D+1)}V_0^{1/(D+1)}\right\}$ there for some constant $a$). Furthermore, the mean energy density

$$\langle \mathcal{E} \rangle = \frac{\int_0^\infty dEe^{-\beta E}\mathcal{E}(E)}{\int_0^\infty dEe^{-\beta E}\Omega(E)} \quad (60)$$

is finite for $\beta = \beta_H$; $e^{-\beta_H E}\Omega(E)$ will presumably look something like Fig. 1 where the maximum $E_0$ is somewhere between the low energy and high energy domains where $\Omega(E)$ is known, i.e. $\mathcal{E}_0$ will be of order one in string scale units.

What happens when we attempt to raise the temperature further? It is apparent from (59) that for $\beta < \beta_H$ this ensemble is not defined; its mean energy becomes infinite. Therefore it seems that we cannot pass $\beta_H$. But in principle we could imagine adding more energy to the string system in an attempt to raise its temperature; where does this energy go? A clue was pointed out at the end of section five: at large energies the density of states is dominated by one (or a small number of) highly energetic, and therefore long and wiggly, strings. Thus it seems quite reasonable that any energy that we add to the system in an attempt to raise the temperature goes into making strings in very high oscillation modes. The low mass modes are populated with a thermal distribution at $\beta_H$, and the rest of the energy is in oscillation modes. It may be useful to think of this transition to long-string dominance as a higher-order phase transition with an infinite latent heat. The low temperature phase is what we know, and the ‘high temperature phase’ is the configuration where the fractional energy in long string is one, but this phase is inaccessible because it corresponds to infinite energy. (This is assuming that the fundamental degrees of freedom are not drastically modified at some finite energy density, as could well be the case.)

9. X-Y Model and the Berezinskii-Kosterlitz-Thouless Transition

Consider the string action in the conformal gauge

$$S = \frac{1}{2\pi} \int d\sigma d\tau \left( \partial_\alpha X^\mu \partial^\alpha X^\mu \right), \quad (61)$$

where we take $\alpha' = 1/2$. Compactifying in the time direction, in order to account for the finite temperature, and using periodic boundary conditions for closed string $X^\mu(\sigma + \pi, \tau) = X^\mu(\sigma, \tau)$ and $X^\mu(\sigma, \tau + \pi) = X^\mu(\sigma, \tau) + n^\mu \delta_0^\mu$, we obtain the mode expansion for the time coordinate

$$X^0(\sigma, \tau) = \tilde{X}^0 + \frac{2\pi n\tau}{\beta} + \frac{m^0}{\pi} + \text{(oscillators)} \quad (62)$$
and
\[ \frac{1}{4} E^2 = N_L + \frac{1}{2} \left( \frac{n\pi}{\beta} + \frac{m\beta}{2\pi} \right)^2 - 1 + N_R + \frac{1}{2} \left( \frac{n\pi}{\beta} - \frac{m\beta}{2\pi} \right)^2 - 1 \] (63)
with the constraint \( N_L - N_R = m \cdot n \). The state \( N_L = N_R = 0 \) with \( n = 0 \) and \( m = \pm 1 \) has
\[ E(T)^2 = M(T)^2 = -8 + \frac{1}{\pi^2 T^2} \] (64)
For \( T < T_H = 1/2\pi\sqrt{2} \) its mass-squared is positive but at \( T = T_H \) it becomes massless and for \( T > T_H \) tachyonic. Call the \( m = \pm 1 \) winding states \( \varphi \) and \( \varphi^* \) respectively. Thus the Hagedorn divergence \( (T > T_H) \) corresponds to the existence of a tachyon and a divergence at the boundary \( \tau_2 \to \infty \) of the moduli space in the path-integral language \[ F(\beta) \sim (\text{const.}) \int d\tau_2 e^{-M^2(\beta)\tau_2} \to \infty \quad \text{for} \quad M^2(\beta) < 0. \] (65)
The action for the \( \varphi \) mode given by \( X^0 = \frac{\beta}{2\pi} \varphi \) with \( \varphi(\sigma, \tau + \pi) = \varphi(\sigma, \tau) + 2\pi \) is
\[ S_\varphi = \frac{\beta^2}{8\pi^3} \int (\partial \varphi)^2 = \frac{g}{2} \int (\partial \varphi)^2, \quad g \equiv \frac{\beta^2}{4\pi^3}. \] (66)
The action \( S_\varphi \) is that of the Villain model which is the continuum limit of that of the X-Y model (planar magnet)
\[ S = -\frac{g}{2} \sum_{<i,\delta>} \cos(\phi_i - \phi_{i+\delta}). \] (67)
The vortex solutions of the model are given by
\[ \varphi(z) = -\frac{i}{2} m \ln \left[ \frac{z - z_0}{\bar{z} - \bar{z}_0} \right], \] (68)
where \( z_0 \) is the position of the vortex and \( m \) is the winding number. The corresponding free energy is
\[ F_m = \left( \frac{\pi g}{2} m^2 - 1 \right) \ln \left( \frac{A}{a^2} \right), \] (69)
where \( a \) is the lattice spacing cutoff. The above model has an infinite order Berezinskii-Kosterlitz-Thouless phase transition at \( g_c = \frac{2}{\pi} \) or \( \beta_c = 2\sqrt{2}\pi = \beta_H! \) In the low temperature regime \( (g > g_c) \) the system is dominated by spin wave excitations and is in an unscreened scale invariant phase with power law spin-spin correlations and tightly bound
vortex-antivortex pairs. In the high temperature regime \((g < g_c)\) the topological order is destroyed, the vortices unbind and are free (vortex condensation). The phase is screened with exponentially decaying spin-spin correlations characterized by a coherence length \(\xi\) which breaks the scale invariance.

Consider a chemical potential \(\mu\) for vortices, where \(\mu\) corresponds to the energy of dissociation of a vortex-antivortex bound state. The RG flows for the system are in the fugacity \(y = e^{-\mu/kT}\) and \(T\) plane as shown in Fig.1.

For \(T < T_c\) and \(y\) small the system flows to \(y = 0\) or to \(\mu = \infty\) and vortices are bound (region I). For \(T > T_c\) and \(y\) small the system flows off to \(y\) large out of the domain of perturbation theory but \(\mu\) is effectively small and vortices can unbind.

To describe string theory in this phase one would have to look at the coupled \(\beta\)-functions for all the massless moduli at \(\beta = \beta_H\) which include \(g_{\mu\nu}, B_{\mu\nu}\), the dilaton \(\sigma\) and \(\varphi\). This might have solutions corresponding to a conformal fixed point for \(T > T_H\). Atick and Witten [6] suggested that \(\varphi\) gets a v.e.v. \(<\varphi>\) before \(T_H\) thus obviating the difficulties of the Hagedorn divergence. But they found the effective potential for \(\varphi\) obtained from its interactions with the dilaton \(\sigma\) is of the type

\[
V_{\text{eff}}(\varphi) \sim m^2(T)\varphi^*\varphi - \hat{\lambda}(\varphi^*\varphi)^2
\]

with

\[
\hat{\lambda} > 0 \quad \text{and} \quad m^2(T) = -8 + 1/\pi^2T^2.
\]

This implies a first-order phase transition. \(V_{\text{eff}}(\varphi)\) must, however, be stabilized by higher-order terms and \(<\varphi>\) is not calculable in perturbation theory.

Atick and Witten argued that, had \(\hat{\lambda}\) been negative, there would be a second order phase transition at \(T = T_H\). For \(T\) just greater than \(T_H\) then

\[
|<\varphi>|^2 = \frac{|m^2|}{2\hat{\lambda}}
\]

and the free energy is

\[
F = V(<\varphi>) = \frac{-m^2}{4\hat{\lambda}} = \frac{-m^2}{4\lambda g^2T}
\]

This is a genus zero (sphere) contribution to the free energy and implies the world-sheet is spontaneously tearing to become non-simply-connected. They then argue that this picture
is valid even for the first-order transition case $\hat{\lambda} > 0$. This is not, so far, justified by any calculation and fails in the weak coupling limit $g \to 0$.

**Appendix**

In this appendix we compute the asymptotic form of the level density $p(N)$ which led to (13). From (14) we have that

$$p(N) = \frac{1}{2\pi i} \oint dz \left( \frac{-\ln z}{2\pi} \right)^{1/2} \exp[g(z)],$$

(70)

with $g(z) = -\frac{\pi^2}{6zn_z} - (N + c) \ln z$ and $c = 23/24$. Then $g'(z) = \frac{\pi^2}{6\ln z^2} \cdot \frac{1}{z} - \frac{N+c}{z}$, which has a maximum at $\ln z_0 = -\frac{\pi}{\sqrt{6}} \cdot \frac{1}{\sqrt{N+c}}$. Then

$$g(z_0) = \frac{2\pi}{\sqrt{6}} \sqrt{N+c}.$$  

(71)

Thus the leading behavior of $p(N)$ is

$$p(N) \sim \exp \left( \pi \sqrt{\frac{2}{3}} \sqrt{N + \frac{23}{24}} \right)$$

$$\sim \exp \left( \beta_c \sqrt{N} \right)$$

(72)

with $\beta_c = \pi \sqrt{\frac{2}{3}}$. Corrections are obtained by looking at the quadratic fluctuations

$$g(z) = g(z_0) + \frac{1}{2}(z - z_0)^2g''(z_0) + \ldots,$$

(73)

where $g''(z_0) = \frac{2\sqrt{6}}{\pi}(N + c)^{3/2} \exp \left( \pi \sqrt{\frac{2}{3}} \sqrt{N+c} \right)$. Then (70) is a Gaussian integral which gives

$$p(N) = \frac{1}{4\sqrt{3}} \frac{1}{N} \exp \left( \pi \sqrt{\frac{2}{3}} \sqrt{N} \right).$$

(74)

For the string in $d$ dimensions (74) easily generalizes to

$$p(N) = (\text{const.})N^{-(d+1)/4} \exp \left( \pi \sqrt{\frac{2}{3}}(d-2)\sqrt{N} \right).$$

(75)

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Figure Captions

Fig. 1. R.G. flows for the Villain model. The $x$ axis is $T_c/T - 1$ and the $y$ axis the fugacity $e^{-\mu/kT}$. The $x$ axis is a line of trivial fixed points that can be reached from region I. Points from region II flow to large $y$ and out of the domain of validity of perturbation theory.