Shape and position of the shadow in the $\delta = 2$
Tomimatsu–Sato spacetime

Cosimo Bambi and Naoki Yoshida

Institute for the Physics and Mathematics of the Universe, The University of Tokyo,
Kashiwa, Chiba 277-8583, Japan

E-mail: cosimo.bambi@ipmu.jp and naoki.yoshida@ipmu.jp

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Abstract
Within 5–10 years, very long baseline interferometry facilities will be able to observe the ‘shadow’ of super-massive black hole candidates. This will allow us, for the first time, to test gravity in the strong field regime. In this paper, we numerically study the photon orbits in the $\delta = 2$ Tomimatsu–Sato spacetime. The $\delta = 2$ Tomimatsu–Sato spacetime is a stationary, axisymmetric and asymptotically flat exact solution of the vacuum Einstein equations. We compare the associated shadow with the one of Kerr black holes. The shape of the shadow in the $\delta = 2$ Tomimatsu–Sato spacetime is oblate and the difference between the two axes can be as high as 6% when viewed on the equatorial plane. We argue that future space sub-mm interferometers (e.g. VSOP-3) may distinguish the two cases, and thus are able to test the cosmic censorship conjecture.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Today gravity is relatively well tested in the weak field limit, while little or nothing is known when it becomes strong [1]. One of the most outstanding issues is the actual nature of the final product of the gravitational collapse. Currently there are many known ‘black hole candidates’, but we do not know if these objects are really the black holes predicted by general relativity and, more in general, if they have an event horizon. At the observational level, the situation may however change soon: the capability of very long baseline interferometry (VLBI) has improved significantly at short wavelength and it is now widely believed that within 5–10 years it will be possible to observe the direct image of the accretion flow around a black
hole with a resolution comparable to its event horizon [2, 3]. Such observations will allow us to test gravity in the strong field regime and investigate the nature of our black hole candidates.

One of the main targets of these VLBI experiments is the observation of the black hole ‘shadow’ [4–6], i.e. a dark area over a bright background. The exact shape and position of the shadow directly depend on the metric of the spacetime very close to the massive object, being determined by the innermost photon orbits. In light of this, it is important to understand how to use future observations to test general relativity and what questions can be addressed. For example, in [7, 8], it is shown that the shadow can be used to test the Kerr bound $|a_*| \leq 1$, where $a_*$ is the dimensionless spin parameter. In [9], the authors suggest the test of the no-hair theorem. In this paper we study the shadow in the $\delta = 2$ Tomimatsu–Sato (TS2) spacetime [10, 11], which is not a black hole solution. We argue that future sub-mm interferometers will be able to measure the shape of the shadow with an accuracy at the level of 1% and thus may distinguish the Kerr from the TS2 spacetime.

TS2 is a stationary, axisymmetric and asymptotically flat exact solution of the vacuum Einstein equations. The Kerr metric describes the gravitational field produced by a spinning mass, while the TS2 describes the gravitational field of a spinning and deformed mass. Despite such appealing features, the solution violates the cosmic censorship conjecture [12], which precludes the formation of naked singularities in the Cauchy development starting from regular initial data. In 4D general relativity, the Kerr spacetime is the only stationary and asymptotically flat solution with a regular event horizon [13, 14]. In the presence of naked singularities, there is no uniqueness theorem and the Tomimatsu–Sato spacetimes are just the simplest extension of the Kerr solution. In general, one can indeed expect that the massive object is described by an infinite number of free parameters [15]. However, it turns out that the introduction of a larger number of parameters do not change significantly the orbits of the spacetime: basically they represent corrections to higher order multipole moments. So, as a first approximation, deviations from the Kerr metric can be studied by considering the Tomimatsu–Sato spacetimes and here, for the sake of simplicity, we discuss the TS2 solution, even if this does not mean that TS2 is the only alternative to the Kerr spacetime when the cosmic censorship conjecture is relaxed. It is however remarkable that TS2 can be obtained from the Neugebauer–Kramer solution representing a superposition of two Kerr black holes, in the limit in which the centers of the two black holes coincide. This fact has been interpreted in [16] as an indication that TS2 can be a good candidate for the final stage of the gravitational collapse.

The cosmic censorship conjecture is motivated by the fact that spacetimes with naked singularities present several kinds of pathologies. In the case of the Tomimatsu–Sato spacetimes, the most suspicious property is the existence of closed time-like curves close to the massive object. Nevertheless, one can even argue that the restriction imposed by such conjecture is more motivated by the limitations of our current knowledge of gravity rather than by true physical reasons. For example, the physical interpretation of spacetimes with causality violating regions has been recently investigated by a couple of authors in the framework of string theories [17–20]. In these works, it was found that such pathological regions go to a new phase and a domain wall forms. Across the domain wall, the metric is non-differentiable and the expected region with closed time-like curves arises from the naive continuation of the metric ignoring the domain wall. A similar mechanism could work for the Tomimatsu–Sato spacetimes. A domain wall as a surface of the astrophysical black hole candidates would also be consistent with the non-observation of x-ray bursts from x-ray binaries with a black hole candidate: the accreting matter could be converted to exotic stuff as soon as it hits the surface of the compact object and no thermonuclear reaction would occur [21].
The paper is organized as follows. In section 2, we review the basic properties of the TS2 spacetime. In section 3, we present our approach, and in section 4, we determine the shape and the position of the shadow in TS2 and we compare with the one of Kerr black holes. In section 5, we discuss our results and the possibility of detecting deviations from the Kerr metric with future experiments. Summary and conclusions are reported in section 6. Throughout the paper we use natural units $G_N = c = 1$ and metric with signature $(-+++)$.

2. The $\delta = 2$ Tomimatsu–Sato spacetime

The Tomimatsu–Sato spacetimes form a family of stationary, axisymmetric and asymptotically flat exact solutions of the vacuum Einstein equations. They are characterized by three parameters: the mass $M$, the spin $J$ and the deformation parameter $\delta$. The canonical form of the line element of a stationary and axisymmetric spacetime is

$$ds^2 = -f(d\tau - \omega d\phi)^2 + \frac{1}{f}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2],$$

where $f$, $\omega$ and $\gamma$ are the functions of the quasi-cylindrical coordinates $(\rho, z)$. The Tomimatsu–Sato solutions have

$$f = \frac{A}{B}, \quad \omega = \frac{2Mq(1 - y^2)C}{A}, \quad e^{2\gamma} = \frac{A}{p^{2\delta}(x^2 - 1)^2}.$$ (2)

Here $A$, $B$ and $C$ are the polynomials, respectively, of degrees $2\delta^2$, $2\delta^2$ and $2\delta^2 - 1$ in the prolate spheroidal coordinates $(x, y)$, defined by

$$\rho = \sigma \sqrt{(x^2 - 1)(1 - y^2)}, \quad z = \sigma xy.$$ (3)

The parameters $p$, $q$ and $\sigma$ are related to $M$, $J$ and $\delta$ as follows:

$$p^2 + q^2 = 1, \quad q = \frac{J}{M^2}, \quad \sigma = \frac{Mp}{\delta}.$$ (4)

The $\delta = 1$ solution is the Kerr spacetime. For $\delta = 2$, $A$, $B$ and $C$ are given by

$$A = p^4(x^2 - 1)^4 + q^4(1 - y^2)^4 - 2p^2q^2(x^2 - 1)(1 - y^2)
\times [2(x^2 - 1)^2 + 2(1 - y^2)^2 + 3(x^2 - 1)(1 - y^2)]$$ (5)

$$B = [p^2(x^2 + 1)(x^2 - 1) - q^2(1 + y^2)(1 - y^2) + 2px(x^2 - 1)]^2
+ 4q^2y^2[px(x^2 - 1) + (px + 1)(1 - y^2)]^2$$ (6)

$$C = -p^3x(x^2 - 1)[2(x^2 + 1)(x^2 - 1) + (x^2 + 3)(1 - y^2)]
- p^2(x^2 - 1)[4x^4(x^2 - 1) + (3x^2 + 1)(1 - y^2)] + q^2(px + 1)(1 - y^2)^3.$$ (7)

TS2 has quite peculiar properties. Here we just mention briefly the main features, alerting the reader that there are numerous mistakes, typos and wrong sentences in the literature, and we refer to [16] for more details. The structure of TS2 is sketched in figure 1. On the equatorial plane, there is a ring singularity where $B$ vanishes. As discussed in [16], the ring singularity has a zero Komar mass and an infinite circumference, in the sense that there $g_{\phi\phi}$ diverges. For odd $\delta$, the segment $I$: $\rho = 0, |z| < \sigma$ (the surface $x = 1$) is an event horizon. In TS2, the spacetime cannot be extended to the region $\rho < 0$. It is a segment singularity with a quite exotic structure, see [16]. It seems to carry the gravitational mass of the system. At $\rho = 0$ and $z = \pm \sigma$ ($x = 1$ and $y = \pm 1$) there are two killing horizons, which are regular
except at the point of connection with the segment singularity. Around the segment singularity and inside the ring singularity $g_{\phi\phi} < 0$, and there are thus closed time-like curves. Such a causality violating region is the most suspicious property of TS2. Using the standard definition of multipole moments for stationary and asymptotically flat spacetimes [22], the quadrupole moment of the system is
\[ Q = \left( \frac{1}{4} + \frac{3}{4} \frac{J^2}{M^4} \right) M^3, \]
which is larger than the Kerr one $Q = J^2/M$ for $|q| < 1$. Like all the Tomimatsu–Sato spacetimes, TS2 reduces to the spacetime of an extreme Kerr black hole in the special case $|q| = 1$.

3. Motion of massless particles in TS2

In any stationary and axisymmetric spacetime, there are three conserved quantities: the mass of the test particle $\mu$, the energy at infinity $E$ and the angular momentum at infinity along the symmetry axis, say $L_z$. This fact allows for the separability with respect to the affine parameter $\lambda$ and the $t$ and $\phi$ coordinates of the Hamilton-Jacobi equation. Since the Kerr spacetime is of the Petrov type D, there is a fourth conserved quantity (the Carter constant $Q$) which leads to the additional separability with respect to the other two coordinates, at least in some coordinate systems. In general, however, this last step is not possible and one has to integrate (numerically) the associated second-order geodesic equations.

In order to compute the shadow of the massive object in TS2, we use the following set-up. First, we adopt the Schwarzschild coordinates $(r, \theta)$, which are related to the quasi-cylindrical coordinates $(\rho, z)$ by
\[ \rho = \sqrt{r^2 - 2Mr + q^2M^2} \sin \theta, \quad z = (r - M) \cos \theta. \]
Then, we consider an observer at infinity with inclination angle $i$. In the numerical integration, the radial coordinate of the observer is $r = 10^5 M$: at such a distance, the spacetime can be
Table 1. Photon capture radius in prolate spheroidal and Schwarzschild coordinates on the equatorial plane for corotating (+) and counterrotating (−) orbits in TS2 for a few values of |q|. b is the impact parameter. x, r and b are in units M = 1.

| q | x_{r}^+ | r_{r}^+ | b^+ | x_{r}^- | r_{r}^- | b^- |
|---|---------|---------|-----|---------|---------|-----|
| 0.0 | 4.01 | 3.17 | 5.38 | 4.01 | 3.17 | 5.38 |
| 0.1 | 3.81 | 3.08 | 5.19 | 4.26 | 3.29 | 5.56 |
| 0.3 | 3.45 | 2.84 | 4.79 | 4.87 | 3.47 | 5.91 |
| 0.5 | 3.17 | 2.56 | 4.34 | 5.86 | 3.65 | 6.24 |
| 0.7 | 2.95 | 2.22 | 3.80 | 7.66 | 3.80 | 6.55 |
| 0.9 | 2.82 | 1.72 | 3.07 | 13.39 | 3.94 | 6.85 |

Table 2. As in table 1, in the case of Kerr spacetime.

| q | r_{r}^+ | b^+ | r_{r}^- | b^- |
|---|---------|-----|---------|-----|
| 0.0 | 3.00 | 5.20 | 3.00 | 5.20 |
| 0.1 | 2.88 | 4.99 | 3.11 | 5.39 |
| 0.3 | 2.63 | 4.56 | 3.33 | 5.77 |
| 0.5 | 2.35 | 4.10 | 3.53 | 6.13 |
| 0.7 | 2.01 | 3.57 | 3.73 | 6.49 |
| 0.9 | 1.56 | 2.93 | 3.91 | 6.83 |

considered flat. The image plane of the observer is the plane orthogonal to his/her line of sight, see e.g. [4, 23]. The plane is provided by a system of the cartesian coordinates (X, Y). We consider the photons on the image plane with 3-momentum perpendicular to the plane, that is, with \( k = k \hat{Z} \), where \( \hat{Z} \) is the unit vector orthogonal to the image plane. Writing the position and the momentum of the photon in the prolate spheroidal coordinates, we solve the geodesic equation with a fourth-order Runge–Kutta–Nystrom method.

Some photons approach the object and then come back to infinity (non-captured orbits). Other photons are instead captured by the massive object (captured orbits): in the case of black hole, they would cross the event horizon, while in TS2 they can either cross one of the horizons at \( \rho = 0 \) and \( z = \pm \sigma \), or end at the segment or the ring singularities. In any case, these photons cannot come back to infinity. The shadow is the image formed on the image plane by the set of photons captured by the massive object. We have corotating orbits, when the photon angular momentum is parallel to the spin of the massive object, and counterrotating orbits, in the opposite case. The minimum radius from the object of the non-captured orbits is larger than the photon capture radius, which is the radius of the innermost photon orbit. In Table 1, we reported the photon capture radius for corotating and counterrotating orbits on the equatorial plane, for a few values of the parameter \( q \). In Table 2, there are the same quantities for the Kerr case. In Tables 1 and 2, we also show the impact parameter, \( b = \sqrt{X^2 + Y^2} \), of the orbits separating non-captured and captured photons.

4. Prediction of the shadow in TS2

The shadow of a black hole is the dark area over a bright background viewed by a distant observer when the black hole is in front of a planar light source [4]. In a realistic case, the emission occurs from the accreting gas surrounding the black hole, but one still observes the
same dark shape [5, 6]. The shadow can also be determined by studying the photon orbits leaving the image plane of the distant observer and intersecting the horizon.

In TS2, the shadow is formed by all the photons that leave the image plane and are captured by the gravitational field of the massive object. In the reference frame of the observer, the photon initial conditions are given by \((X, Y, Z = 0)\) and \((0, 0, k)\). In order to compute the shadow, we write the photon initial conditions in the prolate spheroidal coordinates. Then, we numerically integrate the standard second-order geodesic equations. As in the Kerr case, the shape and the position of the shadow depend only on the observer inclination angle, \(i\), and the dimensionless spin parameter of the object, \(q\).

The results of our study are summarized in figure 2, where we show the contour of the shadows for \(q = 0, 0.7\) and \(i = 90^\circ, 45^\circ\) and \(0^\circ\). The horizontal and vertical axes are respectively the \(X\) and \(Y\) axes of the image plane of the distant observer. The asymmetry of the shadow with respect to the \(Y\) axis is due to the difference of corotating and counterrotating orbits and is present either in Kerr or in TS2. For an observer on the equatorial plane \((i = 90^\circ)\), the shadow in TS2 is a bit flatter than in the Kerr case. The difference is however small: for \(|q| \lesssim 0.9\), the size of the shadow changes by about 3–4%, while, for a fast-rotating object, the difference is smaller, and becomes negligible when \(|q|\) is very close to 1 (in the special case \(|q| = 1\), TS2 reduces to an extreme Kerr black hole). For smaller and larger inclination angles, the two shadows are more and more similar and, for \(i = 0^\circ\) or \(180^\circ\), in both cases we find a circle, as one can easily argue from the axial symmetry of the spacetimes. In TS2, the circle is only slightly smaller than the one in the Kerr spacetime. For moderate values of the spin parameter, the radii of the two circles differ by about 2%. The contour of the shadow on the image plane of a distant observer can also be described by the polar coordinates \((R, \Theta)\).

The contour of every shadow can thus be written as \(R = R(\Theta)\). In figure 3, we plot the quantity \(R_{TS2}/R_{Kerr}\) as a function of \(\Theta\), for the cases shown in figure 2.

5. Discussion

The shadows in the Kerr and TS2 spacetimes are surprisingly similar. In particular, we do not see any feature associated with the ring singularity of TS2. The reason is that the shadow is determined by the photon capture radius, which is always larger than the radius of the ring singularity. In particular, for \(q = 0\) the ring singularity is on the singular surface \(x = 1\), while the photon capture radius is at \(x_\gamma \approx 4\) (see table 1). For \(q = 0.7\), the ring singularity is at \(x \approx 1.09\); here the photon capture radius is at \(x_\gamma \approx 2.95\). Some confusion may be also generated by the use of the prolate spheroidal coordinates \((x, y)\), which allows for writing the metric in analytic form. This point can be understood better if we consider the limit \(|q| \rightarrow 1\), in which TS2 reduces to an extreme Kerr black hole. In the prolate spheroidal coordinates, the radius of the ring singularity increases as \(|q|\) increases, and goes to infinity when \(|q| = 1\). In the Schwarzschild coordinates, the radius of the ring singularity decreases as \(|q|\) increases, and is ‘absorbed’ by the event horizon when \(|q| = 1\).

The ring singularity has several unphysical features. In particular, it is a curvature singularity which marks the boundary of the causality violating region. We can thus expect that new physics, if any, appears before reaching the ring singularity and that here the spacetime is quite different from the one predicted by the TS2 metric. Nevertheless, one may also consider an exact TS2 spacetime and wonder how the ring singularity appears to a distant observer if, for example, it is a bright source of photons. The distant observer can see multiple images,
Figure 2. Contour of the shadow in $\delta = 2$ Tomimatsu–Sato spacetimes (solid lines) and in Kerr spacetimes (dotted lines) in the $XY$-plane of the distant observer, for spin parameter $q = 0$ (left column) and $q = 0.7$ (right column). The viewing angle is respectively $i = 90^\circ$ (top panels), $45^\circ$ (central panels) and $0^\circ$ (bottom panels). Horizontal and vertical axes in units $M = 1$. 
in the sense that a single point of the ring singularity produces a brighter primary image and less bright secondary images. The primary image is associated with the shortest photon orbit connecting the point of the ring singularity with the image plane. The other images are associated with photon orbits turning around the massive objects before reaching the distant observer. On the basis of the symmetries of the spacetime, one can immediately say that an observer on the equatorial plane sees the ring singularity as a segment at $Y = 0$ with the two ends at the boundary of the shadow. An observer with $i = 0^\circ$ or $180^\circ$ sees instead a set of circles inside the shadow. The primary image is the circle with smaller radius. For $q = 0$, the radius of the primary image on the image plane of the observer is about $2.8 \, M$. For rotating objects, this radius is a bit smaller: for example, when $q = 0.7$, the radius is about $2.6 \, M$. For different inclination angles, the observer sees ellipses or arcs.

The key ingredient to test new physics is the accurate determination of the shape, rather than of the absolute size, of the shadow. The size is indeed determined by the mass and the distance of the massive objects, which can be unlikely measured with good precision. This means, in particular, that for viewing angles close to $0^\circ$ or $180^\circ$ it is impossible to distinguish the two metrics, since in both cases we have a circle with a radius which is slightly different (see figure 2). In the non-rotating case and for a viewing angle $i = 90^\circ$ the ratio between the larger and smaller axes of the shadow in TS2 is about $1.06$. So, if we consider the supermassive black hole candidate at the center of the Galaxy, SgrA*, whose shadow should have a diameter $\sim 50 \, \mu$ as, we need at least a resolution of a few $\mu$ as in order to rule out the TS2 metric. For the supermassive black hole candidate in M87, the angular size of its shadow should be about $20 \, \mu$ as and we would thus need a resolution around $1 \, \mu$ as. Unfortunately, near future experiments on the Earth or space missions such as VSOP-2 [24, 25] will not be able to distinguish the Kerr metric from the TS2 one. Indeed, we already know that such experiments will not be able to measure the spin of a black hole [5, 6], assuming the Kerr metric, where the size of the shadow along its axis of symmetry changes even by about $14\%$ (for $i = 90^\circ$) between a Schwarzschild and a maximally rotating black hole, while the one perpendicular to its axis of symmetry is very similar. However, more advanced space missions may distinguish the two cases. Sub-mm space interferometers (e.g. VSOP-3) will be capable of measuring the shape of the shadow of nearby super-massive black hole candidates at the level of $1\%$, which may be enough to distinguish the Kerr from the TS2 case, at least for $|q| \lesssim 0.9$ and for a viewing angle $\sim 90^\circ$. In less favorable circumstances, more advanced experiments, such as x-ray interferometers, might be necessary.

Figure 3. $R_{TS2}/R_{Kerr}$ as a function of the polar coordinate $\Theta$ for the shadows shown in figure 2. Left panel: spin parameter $q = 0$. Right panel: spin parameter $q = 0.7$. 

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TS2 is just a special solution in the Tomimatsu–Sato family, with no particular properties except the fact it can be obtained as the superposition of two Kerr black holes. It is worth mentioning what happens for higher values of the deformation parameter $\delta$. The expression of the metric becomes more and more complicated as $\delta$ increases. Nevertheless, we can argue that the shadow becomes only slightly more flattened than the case $\delta = 2$. Indeed, the shadow, being determined by the photon orbits around the massive object, can be seen as a measurement of the multipole moments associated with the spacetime. After the mass $M$ and the spin $J$, the most important term is the quadrupole moment $Q$, which is given by

$$Q = \left( q^2 + \frac{\delta^2 - 1}{3\delta^2} p^2 \right) M^3. \quad (11)$$

In the simplest case of $q = 0$, the quadrupole moment is $Q = 0$ for $\delta = 1$, $Q = M^3/4$ for $\delta = 2$, and approaches the limit $Q = M^3/3$ as $\delta$ increases. So, if an observer on the equatorial plane sees a shadow whose symmetry axes differ by about 6% for $\delta = 2$, the difference becomes only a bit larger, approximately 8%, when the value of the deformation parameter is very large.

6. Conclusions

Within the next decade, VLBI techniques will likely be able to image the surrounding environment of some super-massive black hole candidates, with resolution at the level of the black hole event horizon. These experiments will open the possibility of studying strong gravity and the actual nature of the final product of the gravitational collapse. One of the main goals is the observation of the shadow, whose shape and position directly depend on the metric of the spacetime around the massive object.

In this paper, we have investigated the shape and the position of the shadow which would be observed in a $\delta = 2$ Tomimatsu–Sato spacetime, and compared with the one expected in the Kerr case. The $\delta = 2$ Tomimatsu–Sato spacetime is a stationary, axisymmetric and asymptotically flat exact solution of the vacuum Einstein equations which violates the cosmic censorship conjecture. It is not a black hole solution. Despite several significant differences between the two spacetimes, the associated shadows are quite similar. In the $\delta = 2$ Tomimatsu–Sato spacetime, the shadow is more flattened and an accurate measurement of its shape, at the level of $\sim 1\%$, may observe deviations from the Kerr metric, at least for certain values of the spin parameter $q$ and the viewing angle $i$. So, near future ground-based experiments or space missions like VSOP-2 will not be able to detect any difference. With more advanced space missions in the future, it is still challenging, but not out of reach.

However, our conclusion that we have to wait for 15–20 years may be too pessimistic. Since around naked singularities the gravitational force can be repulsive [26] and outflows of gas can be produced [27]. Similar phenomena are among the science goals of experiments like VSOP-2.

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