MAX-CLOSEDNESS, OUTER SUPPORT POINTS AND A VERSION OF THE
BISHOP-PHELPS THEOREM FOR CONVEX AND BOUNDED SETS OF
NONNEGATIVE RANDOM VARIABLES

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Abstract. We introduce the concepts of max-closedness and outer support points of convex sets
in $L_0^+$, the nonnegative orthant of the topological vector space $L^0$ of all random variables built
over a probability space, equipped with a topology consistent with convergence of sequences in
probability. Max-closedness asks that maximal elements of the closure of a set already lie on
the set. We show that outer support points arise naturally as optimizers of concave monotone
maximization problems. It is further shown that the set of outer support points of a convex, max-
closed and bounded set in $L_0^+$ is dense in the set of its maximal elements, which can be regarded as
a version of the celebrated Bishop-Phelps theorem in a space that even fails to be locally convex.

Discussion

Let $L_0$ denote the set of all (equivalence classes of real-valued) random variables built over a
probability space, equipped with a metric topology under which convergence of sequences coincides
with convergence in probability. Denote by $L_0^+$ the nonnegative orthant of $L_0$. In many problems
of interest—notably, in the field of financial mathematics—one seeks maximizers of a concave and
monotone (increasing) functional $U$ over convex set $C \subseteq L_0^+$. In order to ensure that such optimizers
exist, some closedness property of $C$ should be present. The monotonicity of $U$ a priori implies
that, if optimizers exist, they must be maximal elements in $C$ with respect to the natural lattice
structure of $L_0$; therefore, a natural condition to enforce is that maximal points of the closure of $C$
already lie in $C$. We then refer to the set $C$ as being max-closed, and the collection $C^{\text{max}}$ of all its
maximal elements is regarded as the “outer boundary” of $C$.

Concave maximization problems as the one described above are particularly amenable to first-
order analysis. Morally speaking, a maximizer of a concave functional $U$ over $C$ should also be
a maximizer of a nice nonzero linear functional over $C$. When nice means continuous, such an
element is called a support point of $C$ in traditional functional-analytic framework, and existence
of a supporting nonzero continuous linear functional is typically provided by an application of
the geometric form of the Hahn-Banach theorem. Unfortunately, $L_0$ is notorious for its barren
topological structure. More precisely, when the probability space is non-atomic:

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- $L^0$ fails to be locally convex, which implies that a rich body of results (including the Hahn-Banach theorem) cannot be used;
- the topological dual of $L^0$ contains only the zero functional [11, Theorem 2.2, page 18]; in particular, as the Namioka-Klee theorem [10] suggests, there is no real-valued nonzero positive linear functional on $L^0$.

In particular, convex sets in $L^0$ a fortiori lack support points according to the usual definition. In spite of this, it is possible to define the non-standard concept of an outer support point of a set $C \subseteq L^0_+$, by asking that it lies on the outer boundary $C^{\text{max}}$ and properly maximizes a functional of the form $L^0_+ \ni f \mapsto \int f \, d\mu \in [0, \infty]$, where $\mu$ is a $\sigma$-finite nonnegative measure that is absolutely continuous with respect to the underlying probability measure. (Proper maximization means that the value at the maximum should be strictly positive and finite.) Outer support points are exactly tailored to describe optimizers of the previously-introduced concave monotone maximization problems. To see this, note that the monotonicity of $U$ not only implies that such optimizers lie on the outer boundary $C^{\text{max}}$ of $C$, but also that a potential supporting “dual element” $\mu$ has to be “pointing outwards,” which in mathematical language translates to $\mu$ being a nonnegative measure. In fact, it will be rigorously illustrated, by means of the rather wide-encompassing Example 1.8, that optimizers for a large class of concave monotone maximization problems are indeed outer support points according to the previous definition.

A next natural step is to explore the richness of outer support points of a set $C \subseteq L^0_+$ on its outer boundary $C^{\text{max}}$. To ensure that the discussion is not void, it is established that if $C$ is convex, max-closed and bounded in $L^0$, then there exists at least one outer support point of $C$; this fact implies in particular that the outer boundary of $C$ is non-empty. It is further shown by an example in Subsection 1.4 that there exists a space $L^0$ and a convex and compact set $C \subseteq L^0_+$ containing an element in $C^{\text{max}}$ that is not an outer support point. (An infinite-dimensional space is required for such example. In finite-dimensional Euclidean spaces all boundary points of a closed and convex set are support points, and the same can be shown in the present non-standard set-up. Note also that in infinite-dimensional spaces there are examples of proper closed convex subsets that have no support points—see [8].) On the positive side, it is shown in Theorem 2.1 that outer support points of a convex, max-closed and bounded set $C \subseteq L^0_+$ are dense in the outer boundary $C^{\text{max}}$. In the context of Banach spaces, the celebrated result of Bishop and Phelps [1, Theorem 7.43] states that support points of closed and convex sets are dense on the boundary of the set. Therefore, Theorem 2.1 can be seen as a version of the Bishop-Phelps theorem in an extremely non-standard environment, where the topological space in question fails to even be locally convex.

The structure of the paper is simple. Section 1 introduces and discusses max-closedness and outer support points, while in Section 2 the aforementioned version of the Bishop-Phelps theorem is stated and proved. Certain technical results are gathered in Appendix A.
1. Max-Closedness and Outer Support Points

1.1. Preliminaries. Throughout the paper, $L^0$ denotes the set of all real-valued random variables over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The usual practice of not differentiating between a random variable and the equivalence class (modulo $\mathbb{P}$) it generates is followed. All relationships between elements of $L^0$ are to be understood in the $\mathbb{P}$-a.s. sense.

Define the nonnegative orthant $L^0_+ := \{ f \in L^0 \mid f \geq 0 \}$ of $L^0$. Further, define $M_+$ as the set of all nonnegative, $\sigma$-finite measures that are absolutely continuous with respect to $\mathbb{P}$. There exists a natural nonnegative bilinear pairing between $M_+$ and $L^0_+$, given by

$$M_+ \times L^0_+ \ni (\mu, f) \mapsto \langle \mu, f \rangle := \int_\Omega f \, d\mu \in [0, \infty].$$

When $Q \in M_+$ is a probability measure, the alternative notation $E_Q[f] \equiv \langle Q, f \rangle$ for $f \in L^0_+$ is used.

The topology on the vector space $L^0$ is the one induced by the translation-invariant metric $L^0 \times L^0 \ni (f, g) \mapsto E_\mathbb{P}[1 \wedge |f - g|]$, where “$\wedge$” is used to denote the minimum operation. With the above definition, $L^0$ becomes a complete metric space and $L^0_+$ its closed subspace. Convergence of sequences under this topology is convergence in $\mathbb{P}$-measure. (In fact, the topology only depends on the equivalence class of $\mathbb{P}$.) Unless explicitly stated otherwise, any topological property (closedness, etc.) pertaining to subsets of $L^0$ will be understood under the aforementioned topology.

A set $C \subseteq L^0_+$ will be called bounded if $\lim_{\ell \to \infty} \sup_{f \in C} \mathbb{P}[f > \ell] = 0$—as can be easily seen, the last property coincides with boundedness of $C$ when $L^0$ is viewed as a topological vector space [1, Definition 5.36]. An element $f \in C \subseteq L^0_+$ is called maximal in $C$ if the conditions $f \leq g$ and $g \in C$ imply $f = g$; the notation $C^{\text{max}}$ is used to denote the set of all maximal elements in $C$.

1.2. Max-closedness and the outer boundary. The next definition introduces a concept of closedness that additionally takes into account the lattice structure of $L^0$. It is exactly tailored for problems related to concave monotone maximization, as is shown in Proposition 1.3 below.

**Definition 1.1.** A set $C \subseteq L^0_+$ will be called max-closed if $\overline{C}^{\text{max}} \subseteq C$, where $\overline{C}$ is the closure of $C$.

The following example demonstrates that max-closedness, which asks that all maximal elements in the closure of $C \subseteq L^0_+$ are already in $C$, is a strictly weaker notion than closedness.

**Example 1.2.** Let $\Omega = (0, 1)$, $\mathcal{F}$ be the Borel $\sigma$-field on $\Omega$, and let $\mathbb{P}$ be Lebesgue measure on $(\Omega, \mathcal{F})$. Consider $C = \{ f \in L^0_+ \mid E_\mathbb{P}[f] = 1 \}$. The set-inclusion $\overline{C} \subseteq \{ f \in L^0_+ \mid E_\mathbb{P}[f] \leq 1 \}$ follows from Fatou’s lemma. Now, let $z_n := n^{-1} 1_{(0, n^{-1})}$, so that $z_n \in C$ for all $n \in \mathbb{N}$. Note that $\lim_{n \to \infty} z_n = 0$. For any $f \in L^0_+$ with $E_\mathbb{P}[f] \leq 1$, the $C$-valued sequence $(f + (1 - E_\mathbb{P}[f])z_n)_{n \in \mathbb{N}}$ converges to $f$, which shows that $f \in \overline{C}$. It follows that $\overline{C} = \{ f \in L^0_+ \mid E_\mathbb{P}[f] \leq 1 \}$; in particular, $C$ is not closed. However, note that $\overline{C}^{\text{max}} = \{ f \in L^0_+ \mid E_\mathbb{P}[f] = 1 \} = C^{\text{max}}$, which implies that $C$ is max-closed.
In this setting, note that \( S = \{ f \in \mathbb{L}^0_+ \mid f \leq g \text{ for some } g \in \mathcal{C} \} \) (the solid hull of \( \mathcal{C} \)) is equal to \( \{ f \in \mathbb{L}^0_+ \mid \mathbb{E}_F[f] \leq 1 \} \), which is closed. This did not happen by chance: it is shown in Lemma A.4 of Appendix A that the solid hull of a convex, max-closed and bounded set is always closed.

Let us make one more observation. With \( \partial \mathcal{C} \) denoting the topological boundary of \( \mathcal{C} \), it actually holds that \( \partial \mathcal{C} = \overline{\mathcal{C}} \). Indeed, for \( f \in \overline{\mathcal{C}} = \{ f \in \mathbb{L}^0_+ \mid \mathbb{E}_F[f] \leq 1 \} \) note that \( (f + 2z_n)_{n \in \mathbb{N}} \) is a \((\mathbb{L}^0_+ \setminus \overline{\mathcal{C}})\)-valued sequence which converges to \( f \).

In the example above, \( \overline{\mathcal{C}} \) turned out to be a much larger set than \( \mathcal{C} \). Even though \( \mathcal{C} \) is not closed, in many cases of interest the “important” elements of \( \mathcal{C} \) lie on the “outer boundary” \( \mathcal{C}^{\text{max}} \) of \( \mathcal{C} \). (As was seen in Example 1.2, the topological boundary of \( \mathcal{C} \subseteq \mathbb{L}^0_+ \) might be simply too provide useful information about optimal elements.) The next result demonstrates that the notion of max-closedness ties nicely together with concave monotone maximization.

**Proposition 1.3.** Suppose that \( \mathbb{U} : \mathbb{L}^0_+ \mapsto [\infty, \infty) \) be concave, upper semi-continuous and monotone, the latter meaning that \( \mathbb{U}(f) \leq \mathbb{U}(g) \) holds whenever \( f \leq g \). Let \( \mathcal{C} \subseteq \mathbb{L}^0_+ \) be convex, max-closed and bounded. Then, there exists \( g \in \mathcal{C}^{\text{max}} \) such that \( \sup_{f \in \mathcal{C}} \mathbb{U}(f) = \mathbb{U}(g) < \infty \).

**Proof.** If \( \overline{\mathcal{C}} \subseteq \mathbb{L}^0_+ \) is the closure of \( \mathcal{C} \), then \( \overline{\mathcal{C}} \) is clearly convex, closed and bounded. Since \( \mathbb{U} \) is concave and upper semi-continuous, [11] Lemma 4.3 implies the existence of \( g_0 \in \overline{\mathcal{C}} \) such that \( \mathbb{U}(g_0) = \sup_{f \in \overline{\mathcal{C}}} \mathbb{U}(f) \). Using again the fact that \( \overline{\mathcal{C}} \) is convex, closed and bounded, Lemma A.2 of Appendix A implies that there exists \( g \in \mathcal{C}^{\text{max}} \) such that \( g_0 \leq g \). Since \( \mathbb{U} \) is monotone, \( \mathbb{U}(g_0) \leq \mathbb{U}(g) \) holds, which means that \( \mathbb{U}(g) = \sup_{f \in \mathcal{C}} \mathbb{U}(f) \). Finally, since \( \mathcal{C} \) is max-closed, \( g \in \mathcal{C}^{\text{max}} \) follows. \( \square \)

**Remark 1.4.** Functions \( \mathbb{U} : \mathbb{L}^0_+ \mapsto [\infty, \infty) \) with the properties in the statement of Proposition 1.3 appear in problems of financial mathematics, where \( \mathbb{U} \) represents a utility functional.

**Remark 1.5.** Taking \( \mathbb{U} \equiv 0 \) in the setting of Proposition 1.3 it follows that \( \mathcal{C}^{\text{max}} \neq \emptyset \) holds whenever \( \mathcal{C} \) is convex, max-closed and bounded. The existence of maximal elements for convex, max-closed and bounded subsets of \( \mathbb{L}^0_+ \) can be also shown directly by means of Zorn’s lemma.

### 1.3. Outer support points

In the context and notation of Proposition 1.3 optimizers of \( \mathbb{U} \) over \( \mathcal{C} \) exist and lie on \( \mathcal{C}^{\text{max}} \). (Since \( \mathbb{U} \) may not be strictly monotone, it is possible for optimizers of \( \mathbb{U} \) to exist in \( \mathcal{C} \setminus \mathcal{C}^{\text{max}} \) as well. However, these optimizers should be regarded as suboptimal, since there always exist optimizers of \( \mathbb{U} \) in \( \mathcal{C}^{\text{max}} \) that dominate them with respect to the lattice structure of \( \mathbb{L}^0_+ \) in view of Lemma A.2 in Appendix A.) As mentioned in the introductory discussion, additional analysis using first order conditions suggests that optimizers on the outer boundary should support the convex set \( \mathcal{C} \), with the corresponding supporting functional being a measure in \( \mathbb{M}_+ \). Although the bilinear mapping of (1.1) fails to be continuous in the second argument (in view of Fatou’s lemma it is, however, lower semi-continuous) and may give infinite value to certain pairs in \( \mathbb{M}_+ \times \mathbb{L}^0_+ \), we anyway proceed with a definition of such non-standard outer support points, and then illustrate
by means of Example\ref{ex:outer-support-points} the way that outer support points naturally appear in the setting of the concave monotone maximization problem of Proposition\ref{prop:max-closedness}.

**Definition 1.6.** For $C \subseteq \mathbb{L}^0_+$ with $C \neq \{0\}$, $g \in C$ will be called an *outer support point* of $C$ if $g \in C^{\text{max}}$ and there exists $\mu \in \mathcal{M}_+$ such that $0 < \sup_{f \in C} \langle \mu, f \rangle = \langle \mu, g \rangle < \infty$. The set of outer support points of $C$ is denoted by $C^{\text{osp}}$. As a matter of convention, we also set $\{0\}^{\text{osp}} = \{0\}$.

**Remark 1.7.** The requirement $0 < \langle \mu, g \rangle < \infty$ when $C \neq \{0\}$ in Definition\ref{defn:outer-support-points} ensures that the trivial cases $\langle \mu, f \rangle = 0$ or $\langle \mu, f \rangle = \infty$ for all $f \in C$ are excluded—both situations are possible.

As promised, the following example demonstrates the usefulness of outer support points.

**Example 1.8.** Consider a utility random field $U : \Omega \times (0, \infty) \mapsto \mathbb{R}$, such that $U(\cdot, x) \in \mathbb{L}^0$ for all $x \in (0, \infty)$ and $U(\omega, \cdot) : (0, \infty) \mapsto \mathbb{R}$ is a nondecreasing, concave and continuously differentiable function for all $\omega \in \Omega$. Define the derivative (with respect to the spatial variable $x$) random field $U' : \Omega \times (0, \infty) \mapsto \mathbb{R}_+$ in the obvious way. By means of continuity, the definition of $U$ and $U'$ is extended so that $U(\cdot, 0) := \lim_{x \downarrow 0} U(\cdot, x)$ and $U'(\cdot, 0) := \lim_{x \downarrow 0} U'(\cdot, x)$—note that the latter random variables may take with positive probability the values $-\infty$ and $\infty$, respectively. Assume that $\mathbb{E}_P [0 \lor U(\infty)] < \infty$ holds, where $U(\infty) := \lim_{x \to \infty} U(\cdot, x)$ and "$\lor$" denotes the maximum operation. Define the functional $\mathbb{U} : \mathbb{L}^0_+ \mapsto [-\infty, \infty]$ via $\mathbb{U}(f) = \mathbb{E}_P [U(f)]$, where for $f \in \mathbb{L}^0_+$ $U(f) : \Omega \mapsto [-\infty, \infty]$ is defined via $(U(f))(\omega) = U(\omega, f(\omega))$ for $\omega \in \Omega$. Clearly, $\mathbb{U}$ is concave and monotone. A combination of $\mathbb{E}_P [0 \lor U(\infty)] < \infty$ and Fatou’s lemma implies that $\mathbb{U}$ is upper semi-continuous.

Consider a convex, max-closed and bounded $C \subseteq \mathbb{L}^0_+$, and assume that there exists $h \in C$ such that $\mathbb{P}[h = 0] = 0$. Proposition\ref{prop:max-closedness} provides the existence of $g \in C^{\text{max}}$ such that $\mathbb{U}(g) = \sup_{f \in C} \mathbb{U}(f) < \infty$. In order to avoid unnecessary technical complications, a final couple of assumptions involving the optimizer $g$ are introduced: we ask that $\mathbb{U}(n^{-1}g) > -\infty$ holds for all $n \in \mathbb{N}$, and that $\mathbb{P}[g > 0, \mathbb{U}(g) < \mathbb{U}(\infty)] > 0$. (The former assumption is rather mild; the latter is satisfied in all cases where $U$ is strictly increasing in the spatial variable.) In particular, $\mathbb{U}(n^{-1}g) > -\infty$ for all $n \in \mathbb{N}$ implies that the function $(0, \infty) \ni a \mapsto \mathbb{U}(ag)$ is concave, nondecreasing and $\mathbb{R}$-valued. Such mapping must have a finite (right-hand-side) derivate; then, a use of the monotone convergence theorem gives that $\mathbb{E}_P [U'(ag)g1_{\{g > 0\}}] < \infty$ holds for all $a \in (0, \infty)$. Define the convex set $C_g = \{f \in C \mid f \geq n^{-1}g \text{ for some } n \in \mathbb{N}\}$. Note that $g \in C_g$; furthermore, since for all $f \in C$ the $C_g$-valued sequence $((1 - n^{-1})f + n^{-1}g)_{n \in \mathbb{N}}$ converges to $f$, $C_g$ is dense in $C$. Fix $f \in C_g$ and let $n \in \mathbb{N}$ be such that $f \geq n^{-1}g$. Since $\mathbb{U}(f) \geq \mathbb{U}(n^{-1}g) > -\infty$, it holds that $\mathbb{P}[U(f) = -\infty] = 0$, i.e., $U(f) \in \mathbb{L}^0$. Similarly, $\mathbb{E}_P(g) > -\infty$ implies $U(g) \in \mathbb{L}^0$. For $\epsilon \in (0, 1)$, define

$$f_\epsilon := (1 - \epsilon)g + \epsilon f, \text{ and } \Delta(f_\epsilon \mid g) := \frac{U(f_\epsilon) - U(g)}{\epsilon} \in \mathbb{L}^0.$$ 

The optimality of $g$ gives $\mathbb{E}_P[\Delta(f_\epsilon \mid g)] \leq 0$, for all $\epsilon \in (0, 1)$. Note that $\Delta(f_\epsilon \mid g) \geq 0$ holds on $\{f_\epsilon \geq g\}$; furthermore, for all $\epsilon \in (0, 1)$, $f_\epsilon \geq n^{-1}g$ implies that $\Delta(f_\epsilon \mid g) \geq -U'(f_\epsilon)(g - f) \geq$
\(-U'(n^{-1} g)gI_{\{g>0\}}\) holds on \(\{f_e < g\}\). Since \(\mathbb{E}_P[U'(n^{-1} g)gI_{\{g>0\}}] < \infty\), and \(\lim \inf_{t \downarrow 0} \Delta(f_e | g) = U'(g)(f - g)\) holds in the \(\mathbb{P}\)-a.s. sense, Fatou’s lemma implies that \(\mathbb{E}_P[U'(g)(f - g)] \leq 0\), where in particular \(\mathbb{P}[U'(g)(f - g) = -\infty] = 0\) is implied. Recall that by assumption there exists \(h \in \mathcal{C}\) with \(\mathbb{P}[h > 0] = 1\). It can be assumed without loss of generality that \(h \in \mathcal{C}_g\); therefore, \(\mathbb{P}[U'(g)(h - g) = -\infty] = 0\) implies that \(\{g = 0\} \subseteq \{U'(0) < \infty\}\), i.e., \(U'(g) \in \mathbb{L}_+^0\). The fact that \(\mathbb{E}_P[U'(g)g] = \mathbb{E}_P[U'(g)gI_{\{g>0\}}] < \infty\) holds allows to write \(\mathbb{E}_P[U'(g)(f - g)] \leq 0\) as \(\mathbb{E}[U'(g)f] \leq \mathbb{E}[U'(g)g]\) for all \(f \in \mathcal{C}_g\). In other words, upon defining \(\mu \in \mathcal{M}_+\) via \(d\mu = U'(g)d\mathbb{P}\), and noting that \(\mathbb{P}[g > 0, U(g) < U(\infty)] > 0\) implies \(\mathbb{E}_P[U'(g)g] > 0\), it follows that \(0 \leq \sup_{f \in \mathcal{C}_g} \langle \mu, f \rangle = \langle \mu, g \rangle < \infty\) holds. As \(\mathcal{C}_g\) is dense in \(\mathcal{C}\), Fatou’s lemma implies that \(0 < \sup_{f \in \mathcal{C}} \langle \mu, f \rangle = \langle \mu, g \rangle < \infty\). Therefore, \(g \in \mathcal{C}^{\text{osp}}\).

Note that if the Inada condition \(\mathbb{P}[U'(0) = \infty] = 1\) additionally holds, then \(\mathbb{P}[g = 0] = 0\).

Furthermore, if \(U'\) is strictly increasing in the spatial variable in a \(\mathbb{P}\)-a.s. sense, then \(\mu\) is actually equivalent to \(\mathbb{P}\). However, none of these extra conditions are enforced on \(U\).

**Remark 1.9.** In Example 1.8, under certain assumptions on the utility random field \(U\), the set \(\mathcal{C}\) and the optimizer \(g \in \mathcal{C}^{\max}\), it is concluded that \(g \in \mathcal{C}^{\text{osp}}\). The most restrictive assumption is the boundedness from above of the utility random field, encoded in the requirement \(\mathbb{E}_P[0 \vee U(\infty)] < \infty\). This assumption is there to ensure that \(U\) is \([-\infty, \infty]\)-valued and upper semi-continuous, in order to allow the invocation of Proposition 1.3 and obtain existence of an optimizer \(g \in \mathcal{C}^{\max}\). However, if existence of an optimizer \(g \in \mathcal{C}\) can be obtained with other methods, in which case Lemma A.2 from Appendix A ensures that it can be additionally assumed that \(g \in \mathcal{C}^{\max}\), the discussion in Example 1.8 goes through even without enforcing boundedness conditions on \(U\). (The other, milder, assumptions should of course still be satisfied.) There has been a significant body of work in the field of financial mathematics where existence of optimizers for such types of expected utility maximization problems is established using convex duality methods; for successful examples, see [9] in the case of deterministic \(U\) and [5] for the case where \(U\) may actually be a random field.

### 1.4. Maximal points versus outer support points.

By definition, \(\mathcal{C}^{\text{osp}} \subseteq \mathcal{C}^{\max}\) holds for all \(\mathcal{C} \subseteq \mathbb{L}_+^0\). As Theorem 2.1 later will show, \(\mathcal{C}^{\text{osp}} \neq \emptyset\) holds whenever \(\mathcal{C}\) is convex, max-closed and bounded. However, the inclusion \(\mathcal{C}^{\text{osp}} \subseteq \mathcal{C}^{\max}\) can be strict, as will be shown here.

Consider the probability space \((\Omega, \mathcal{F}, \mathbb{P})\), where \(\Omega = (0, \infty)\), \(\mathcal{F}\) the Borel \(\sigma\)-field over \((0, \infty)\), and \(\mathbb{P}\) is a probability equivalent to Lebesgue measure on \((0, \infty)\). Define \(\xi : \Omega \mapsto (0, \infty)\) via \(\xi(\omega) = \omega\) for all \(\omega \in (0, \infty)\). Furthermore, define \(K := \{(\alpha, \beta) \in \mathbb{R}^{2} \mid 0 \leq \beta \leq \sqrt{\alpha} \leq 1\}\), and note that \(K\) is a convex and compact subset of \(\mathbb{R}^{2}_+\). Let \(\mathcal{C} := \{1 - \alpha + (\alpha + \beta)\xi \mid (\alpha, \beta) \in K\}\). Being the image of \(K\) via a continuous linear mapping, \(\mathcal{C}\) is a convex and compact subset of \(\mathbb{L}_+^0\)—therefore, it is closed (in particular, max-closed) and bounded.

Note that \(\mathbb{P}[\xi \leq \epsilon] > 0\) and \(\mathbb{P}[\xi^{-1} \leq \epsilon] > 0\) hold for all \(\epsilon \in (0, \infty)\); given this, \(\mathcal{C}^{\max} = \{1 - \alpha + (\alpha + \sqrt{\alpha})\xi \mid \alpha \in [0, 1]\}\) follows in a rather straightforward way. In particular, it holds
that $1 \in C^{\max}$. However, we claim that $1 \notin C^{\osp}$. To this end, it is enough to show that there cannot exist a probability $Q \in M_+$ such that $E_Q[f] \leq 1$ holds for all $f \in C$. If such a probability $Q$ existed, $E_Q[1 - \alpha + (\alpha + \sqrt{\alpha})\xi] \leq 1$ for all $\alpha \in [0,1]$ would follow. Rearranging, $E_Q[\xi] \leq \alpha/\alpha = \sqrt{\alpha/\sqrt{\alpha} + 1}$ would hold for all $\alpha \in (0,1)$. This would imply that $E_Q[\xi] = 0$, i.e., $Q[\xi > 0] = 0$ which, in view of $P[\xi > 0] = 1$, contradicts the fact that $Q$ is a probability which is absolutely continuous with respect to $P$.

In fact, one can say more: in this example, it holds that $C^{\osp} = C^{\max}\setminus \{1\}$. Indeed, fix $\gamma \in (0,1]$ and define $g_\gamma := 1 - \gamma + (\gamma + \sqrt{\gamma})\xi$; we shall show that $g_\gamma \in C^{\osp}$. Note that the law of the random variable $1/g_\gamma$ under $P$ is equivalent to Lebesgue measure on $(0, (1 - \gamma)^{-1}]$. Therefore, setting $c_\gamma := (1 + 2\sqrt{\gamma}) (1 + \sqrt{\gamma})^{-2}$, the strict inequality $c_\gamma < 1 \leq (1 - \gamma)^{-1}$ implies that there exists a probability $Q_\gamma \in M_+$ such that $E_{Q_\gamma}[1/g_\gamma] = c_\gamma$. The straightforward calculation $\xi/g_\gamma = \gamma^{-1/2} (1 + \sqrt{\gamma})^{-1} - \gamma^{-1/2} (1 - \sqrt{\gamma}) (1/g_\gamma)$ implies $E_{Q_\gamma}[(\xi/g_\gamma) = \gamma^{-1/2} (1 + \sqrt{\gamma})^{-1} - \gamma^{-1/2} (1 - \sqrt{\gamma}) c_\gamma = 2\sqrt{\gamma} (1 + \sqrt{\gamma})^{-2}$. Therefore, it follows that

$$E_{Q_\gamma} \left[ \frac{1 - \alpha + (\alpha + \beta)\xi}{g_\gamma} \right] = \frac{1 + 2\sqrt{\gamma} - \alpha + 2\sqrt{\gamma}\beta}{(1 + \sqrt{\gamma})^2} \leq \frac{1 + 2\sqrt{\gamma} - \alpha + 2\sqrt{\gamma}\alpha}{(1 + \sqrt{\gamma})^2},$$

for all $(\alpha, \beta) \in K$.

It is easily seen that the latter expression, as a function of $(\alpha, \beta) \in K$, is maximized when $(\alpha, \beta) = (\gamma, \sqrt{\gamma})$, and that the maximum is $1$. Defining $\mu_\gamma \in M_+$ via $d\mu_\gamma = (1/g_\gamma) dQ_\gamma$, it holds that $\sup_{f \in C} \langle \mu_\gamma, f \rangle \leq \langle \mu_\gamma, g_\gamma \rangle = 1$, i.e., $g_\gamma \in C^{\osp}$.

Before abandoning this example, a final observation is in order. Even though $1 \in C^{\max}\setminus C^{\osp}$, note that the $C^{\osp}$-valued sequence $\{(1 - n^{-1}) + n^{-1} + n^{-1/2}\xi\}_{n \in N}$ actually converges to $1$. Theorem 2.1 below states that $C^{\osp}$ is dense in $C^{\max}$ whenever $C \subseteq L^0_+$ is convex, max-closed and bounded.

2. A Version of the Bishop-Phelps Theorem

2.1. The result. What follows can be regarded as a version of the Bishop-Phelps theorem for convex, max-closed and bounded sets of $L^0_+$. The setting is by all means non-standard, especially since $L^0_+$ typically fails to be locally convex.

**Theorem 2.1.** Let $C \subseteq L^0_+$ be convex, max-closed and bounded. Then, $C^{\osp} \neq \emptyset$, and $C^{\osp}$ is dense in $C^{\max}$.

**Remark 2.2.** For a convex and bounded $C \subseteq L^0_+$, there exists a probability $Q$, equivalent to $P$, such that $\sup_{f \in C} E_Q[f] < \infty$. Indeed, [6, Theorem 1.1(4)] implies the existence of $h \in C$ such that $\sup_{f \in C} E_P[f/(1 + h)] \leq 1$; defining $Q$ via $dQ = \kappa(1 + h)^{-1} \gamma dP$, where $\kappa = (E_P[(1 + h)^{-1}])^{-1} \in (0, \infty)$, it follows that $Q$ is equivalent to $P$ and $\sup_{f \in C} E_Q[f] = \kappa \sup_{f \in C} E_P[f/(1 + h)] \leq \kappa < \infty$. This fact seems to provide hope that one could use the classical version of the Bishop-Phelps theorem by applying $L^1(Q)$-$L^\infty$ duality. In fact, under the assumptions of Theorem 2.1 it is not hard to see that, if $C \subseteq L^1_0(Q)$, then $C^{\max}$ is actually contained in the $L^1(Q)$-topological boundary of $C$. However, for a given $g \in C^{\max}$, it is not at all clear that the sequence of (usual) support
points that approximates $g$ is $C^{\max}$-valued; even if this hurdle is surpassed, the positivity of the corresponding supporting linear functionals of the approximating sequence is not evident. As the previous issues do not appear \textit{a priori} trivial, a bare-hands alternative route is taken in the proof of Theorem \ref{thm:main}, given in Subsection \ref{subsec:proof} below.

**Remark 2.3.** Convexity and closedness are also needed for the statement of the Bishop-Phelps theorem in Banach spaces \cite[Theorem 7.43, statement 1]{zbMATH03528587}. In the setting of Theorem \ref{thm:main} bound edness of $C \subseteq L^0$ is further needed in order to ensure that $C^{\ospp} \neq \emptyset$. In fact, if $C$ is convex and closed but fails to be bounded, \cite[Theorem 1.3 and Lemma 2.3]{zbMATH03528587} imply that $C^{\max} = \emptyset$. In this case, $C^{\ospp} \subseteq C^{\max} = \emptyset$ is trivially dense in $C^{\max}$, but the statement is completely uninformative.

### 2.2. Proof of Theorem \ref{thm:main}

Throughout the proof, there will be use of technical tools from Appendix \ref{app:technical}.

The first line of business is to show that one can assume a bit more in the statement of Theorem \ref{thm:main}. Let $C \subseteq L^0_+$ be convex, max-closed and bounded, and define its solid hull $S = \{ f \in L^0_+ | f \leq g \text{ for some } g \in C \}$. By Lemma \ref{lem:solidhull} $S$ convex, closed and bounded. Of course, $S$ is also solid, meaning that $g \in S$ and $0 \leq f \leq g$ implies $f \in S$. The equality $C^{\max} = S^{\max}$ trivially holds, which also easily implies that $C^{\ospp} = S^{\ospp}$. In view of the previous remarks, \textit{from now onwards it is assumed that $C$ is solid, convex, closed and bounded}. Furthermore, since $C^{\ospp} = C^{\max} = \{0\}$ when $C = \{0\}$, we shall be assuming that $C \neq \{0\}$.

Under the above extra assumptions on $C$, the fact that $C^{\ospp} \neq \emptyset$ follows immediately from Lemma \ref{lem:ospp}. Until the end of the proof, fix $g \in C^{\max}$; since $C \neq \{0\}$, $\mathbb{P}[g > 0] > 0$. Theorem \ref{thm:main} will be proved if we show that there exists a $C^{\ospp}$-valued sequence $(g_n)_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} g_n = g$.

Note that there exists a probability $Q$, equivalent to $\mathbb{P}$, such that $\mathbb{E}_Q[g^2] < \infty$ holds—for example, take $Q$ to be such that $dQ = \left( \mathbb{E}_P \left[ (1 + g)^{-2} \right] \right)^{-1} (1+g)^{-2} d\mathbb{P}$. Since the topology on $L^0$ only depends on the equivalence class of $\mathbb{P}$, we shall assume from the outset that $\mathbb{E}_P[g^2] < \infty$, i.e., that $g \in L^2(\mathbb{P})$.

For any $n \in \mathbb{N} \cup \{\infty\}$, let $\phi_n = (1 + n^{-1})g$, with the understanding that $\phi_\infty = g$. The set $C \cap L^2(\mathbb{P})$ is nonempty (recall that $C$ is solid), convex and $L^2(\mathbb{P})$-closed, the latter in view of the fact that it is $L^0$-closed. Since $\phi_n \in L^2(\mathbb{P})$ for all $n \in \mathbb{N} \cup \{\infty\}$, the nearest point theorem in Hilbert spaces \cite[Theorem 6.53]{zbMATH03528587} implies that there exists $f_n \in C \cap L^2(\mathbb{P})$ such that
\[ \mathbb{E}_P \left[ |f_n - \phi_n|^2 \right] \leq \mathbb{E}_P \left[ |f - \phi_n|^2 \right] \text{ holds for all } f \in C \cap L^2(\mathbb{P}). \]

Of course, $f_\infty = \phi_\infty = g \in C$. Since the projection operator on convex and $L^2(\mathbb{P})$-closed subsets of $L^2(\mathbb{P})$ is a contraction \cite[Lemma 6.54(d)]{zbMATH03528587}, we obtain
\[ \mathbb{E}_P \left[ |f_n - g|^2 \right] = \mathbb{E}_P \left[ |f_n - f_\infty|^2 \right] \leq \mathbb{E}_P \left[ |\phi_n - \phi_\infty|^2 \right] = n^{-2} \mathbb{E}_P \left[ g^2 \right]. \]

In particular, upon summation over all $n \in \mathbb{N}$, it follows that $\sum_{n \in \mathbb{N}} |f_n - g|^2 < \infty \mathbb{P}$-a.s. holds. The last fact implies that $\lim_{n \to \infty} f_n = g$ holds (actually, in the stronger $\mathbb{P}$-a.s. sense).
Therefore, $P_{f_n} < f_n$ for all $n \in \mathbb{N}$ in view of the fact that $C$ is solid. Furthermore, $E[P_{f_n} - f_n] = E[E[P_{f_n} - f_n]]$. Therefore, $P_{f_n} > \phi_n > 0$ would contradict the optimality of $f_n$ for $n \in \mathbb{N}$. It then follows that $f_n \leq \phi_n = (1 + n^{-1})g$ holds for all $n \in \mathbb{N}$.

Defining $\gamma_n := \phi_n - f_n$, it holds from the discussion above that $\gamma_n \in L^0_+$ for all $n \in \mathbb{N}$. Furthermore, since $f_n$ is the $L^2(\mathbb{P})$-projection of $\phi_n$ on $C \cap L^2(\mathbb{P})$, $L^2$ to $L^2$, a convexly compact set $g \in C$ and $\gamma_n \in L^0_+$, then $\mu < g$ for all $n \in \mathbb{N}$; therefore, $E[\gamma_n f_n] \leq E[\gamma_n g] > 0$, i.e., that $\gamma_n \leq \phi_n$ and $f_n \leq \phi_n$ hold,

$$E_P[\gamma_n f_n] \leq E_P[\gamma_n g] > 0, \quad \text{for all } n \in \mathbb{N}. \quad (2.1)$$

But $\gamma_n \leq \phi_n$ and $f_n \leq \phi_n$ hold, therefore, $E_P[\gamma_n f_n] \leq E_P[\gamma_n \phi_n] = (1 + n^{-1})^2E_P[\phi_n] < \infty$.

For all $n \in \mathbb{N}$, apply Lemma A.2 (with $f$ in Lemma A.2 being $f_n$) to obtain $g_n \in C_{\max}$ such that $f_n \leq g_n$. By (2.1), $f_n \leq g_n \in C$, $\langle \mu_n, g_n \rangle = \langle \mu_n, f_n \rangle$ has to hold for all $n \in \mathbb{N}$; therefore, $0 < \sup_{f \in C} \langle \mu_n, f \rangle = \langle \mu_n, g_n \rangle < \infty$, which implies that $(g_n)_{n \in \mathbb{N}}$ is a $C_{\max}$-valued sequence. Since $\lim_{n \to \infty} f_n = g \in C_{\max}$ and $f_n \leq g_n \in C$ holds for all $n \in \mathbb{N}$, statement (2) of Lemma A.5 gives $\lim_{n \to \infty} g_n = g$. This concludes the proof.

APPENDIX A. AUXILIARY RESULTS INVOLVING CONVEX SETS OF $L^0_+$

In the sequel we collect results sets that are used in the main body of the paper. They all involve convex, closed and bounded sets in $L^0_+$; following [11], we call such sets convexly compact.

**Lemma A.1.** For a convexly compact $C \subseteq L^0_+$, $C_{\max} \neq \emptyset$.

**Proof.** If $C = \{0\}$, the result is trivial. Therefore, assume that $C$ is convexly compact and $C \neq \{0\}$. In this case, [9] Theorem 1.1(4)] implies that there exists $g \in C$ such that $P [g = 0, f > 0] = 0$ and $E[(f/g)1_{\{g > 0\}}] \leq E[P \geq 0]$ holds for all $f \in C$. Note that $C \neq \{0\}$ implies that $P[g > 0] > 0$. Define $\mu \in M_+$ via $d\mu = \frac{1}{g}1_{\{g > 0\}}d\mathbb{P}$. Then, $0 < \sup_{f \in C} \langle \mu, f \rangle = \langle \mu, g \rangle < \infty$, which demonstrates that $g \in C_{\max}$.

**Lemma A.2.** Let $C \subseteq L^0_+$ be convexly compact. Then, for $f \in C$, there exists $h \in C_{\max}$ with $f \leq h$.

**Proof.** Fix $f \in C$ and define $Z := \{g \in C \mid f \leq g\}$. It is clear that $Z$ is convexly compact. Lemma A.1 implies that there exists $h \in Z_{\max} \subseteq Z_{\max}$. Now, let $g \in C$ be such that $h \leq g$. Since $f \leq h$, $g \in Z$. Then, $h \leq g$ implies that $h = g$ because $h \in Z_{\max}$. This establishes that $h \in C_{\max}$. 


For the remaining results, a definition is required. Let \((f_n)_{n \in \mathbb{N}}\) be a sequence in \(\mathbb{L}^0_+\). Any sequence \((g_n)_{n \in \mathbb{N}}\) with the property that \(g_n\) is in the convex hull of \(\{f_n, f_{n+1}, \ldots\}\) for all \(n \in \mathbb{N}\) will be called a sequence of forward convex combinations of \((f_n)_{n \in \mathbb{N}}\).

The next result follows in a straightforward way from \([3, \text{Lemma A.1}]\).

**Lemma A.3.** Let \(C\) be convexly compact, and suppose that \((f_n)_{n \in \mathbb{N}}\) is a \(C\)-valued sequence. Then, there exists a sequence \((g_n)_{n \in \mathbb{N}}\) of forward convex combinations of \((f_n)_{n \in \mathbb{N}}\) that converges \(\mathbb{P}\)-a.s. to some \(g \in C\).

We now associate convexity, max-closeness, boundedness, solidity, and closedness.

**Lemma A.4.** Let \(C \subseteq \mathbb{L}^0_+\) be convex, max-closed and bounded, and define its solid hull \(S = \{f \in \mathbb{L}^0_+ \mid f \leq g \text{ for some } g \in C\}\). Then, \(S\) is solid and convexly compact.

**Proof.** Let \((f_n)_{n \in \mathbb{N}}\) be a \(S\)-valued sequence converging to \(f \in \mathbb{L}^0_+\); we shall establish that \(f \in S\). By passing to a subsequence if necessary, assume that \((f_n)_{n \in \mathbb{N}}\) converges \(\mathbb{P}\)-a.s. to \(f\). (The importance of \(\mathbb{P}\)-a.s. convergence is that any sequence of forward convex combinations of \((f_n)_{n \in \mathbb{N}}\) also converges to \(f\), which will be tacitly used in the proof later on—\(\mathbb{R}\) is a locally convex space, while \(\mathbb{L}^0\) is not.) For each \(n \in \mathbb{N}\), there exists \(g_n \in C\) such that \(f_n \leq g_n\). Note that \(\overline{C}\) (the closure of \(C\)) is a convexly compact set; then, Lemma A.3 implies the existence of a sequence of forward convex combinations of \((g_n)_{n \in \mathbb{N}}\) that \(\mathbb{P}\)-a.s. converges to some \(g \in \overline{C}\). Since \((f_n)_{n \in \mathbb{N}}\) converges \(\mathbb{P}\)-a.s. to \(f\) and \(f_n \leq g_n\) for all \(n \in \mathbb{N}\), it follows that \(f \leq g\). Now, invoking Lemma A.2, it follows that there exists \(h \in C^{\max}\) such that \(g \leq h\). As \(C^{\max} \subseteq C\) and \(f \leq g\), we obtain that \(f \leq h \in C\), which implies that \(f \in S\). □

The final result is concerned with convergence of sequences to points of the outer boundary.

**Lemma A.5.** Let \(C \subseteq \mathbb{L}^0_+\) be convexly compact. Let \((f_n)_{n \in \mathbb{N}}\) be a \(C\)-valued sequence that converges to \(g \in C^{\max}\). Then,

1. All sequences of forward convex combinations of \((f_n)_{n \in \mathbb{N}}\) also converge to \(g\).
2. Any \(C\)-valued sequence \((g_n)_{n \in \mathbb{N}}\) such that \(f_n \leq g_n\) holds for all \(n \in \mathbb{N}\) also converges to \(g\).

**Proof.** In the sequel, \((f_n)_{n \in \mathbb{N}}\) is a \(C\)-valued sequence that converges to \(g \in C^{\max}\).

Suppose that \((g_n)_{n \in \mathbb{N}}\) is a sequence of forward convex combinations of \((f_n)_{n \in \mathbb{N}}\) that converges to \(h \neq g\). By \([7, \text{Theorem 1.3}]\), this would contradict the fact that \(g \in C^{\max}\). Therefore, any convergent sequence of forward convex combinations of \((f_n)_{n \in \mathbb{N}}\) must have the same limit \(g\) that \((f_n)_{n \in \mathbb{N}}\) has. Again, by \([7, \text{Theorem 1.3}]\), it follows that all sequences of forward convex combinations of \((f_n)_{n \in \mathbb{N}}\) converge to \(g\), which establishes statement (1).

Now, pick any \(C\)-valued sequence \((g_n)_{n \in \mathbb{N}}\) such that \(f_n \leq g_n\) holds for all \(n \in \mathbb{N}\), and define \(\zeta_n := g_n - f_n\) for \(n \in \mathbb{N}\); then, \(\zeta_n \in \mathbb{L}^0_+\) for all \(n \in \mathbb{N}\). If \(\lim_{n \to \infty} \zeta_n = 0\) is established, \(\lim_{n \to \infty} f_n = g\) will imply \(\lim_{n \to \infty} g_n = g\). For \(n \in \mathbb{N}\), let \(T_n\) denote the closure of the convex
hull of \( \{ \zeta_k \mid k = n, n + 1, \ldots \} \), and set \( \mathcal{T}_\infty := \bigcap_{n \in \mathbb{N}} \mathcal{T}_n \). For \( \psi \in \mathcal{T}_\infty \), there exists a sequence \( (\psi_n)_{n \in \mathbb{N}} \) of forward convex combinations of \( (\zeta_n)_{n \in \mathbb{N}} \) such that \( \lim_{n \to \infty} \psi_n = \psi \). Since \( g \in C^{\max} \), \( \mathcal{T}_\infty \) cannot contain any \( \psi \in L^0_+ \) with \( \mathbb{P}[\psi > 0] > 0 \); indeed, if this was the case, using statement (1) of Lemma A.5 that was just proved, one would be able to construct a \( C \)-valued sequence \( (h_n)_{n \in \mathbb{N}} \) with \( \lim_{n \to \infty} h_n = g + \psi \), which would mean that \( g + \psi \in C \) and would contradict \( g \in C^{\max} \). On the other hand, as each \( \mathcal{T}_n \), \( n \in \mathbb{N} \), is convexly compact and \( (\mathcal{T}_n)_{n \in \mathbb{N}} \) is a non-increasing sequence, it follows from [11] that \( \mathcal{T}_\infty \neq \emptyset \). We conclude that \( \mathcal{T}_\infty = \{0\} \) — in other words, all convergent sequences of forwards convex combinations of \( (\zeta_n)_{n \in \mathbb{N}} \) converge to zero. Then, another application of [7, Theorem 1.3] (for the special case of zero limit) implies that \( \lim_{n \to \infty} \zeta_n = 0 \), completing the proof of statement (2). \( \square \)

**References**

[1] C. D. Aliprantis and K. C. Border, *Infinite dimensional analysis*, Springer, Berlin, third ed., 2006. A hitchhiker’s guide.

[2] W. Brannath and W. Schachermayer, *A bipolar theorem for \( L^0_+ (\Omega, \mathcal{F}, \mathbb{P}) \)*, in Séminaire de Probabilités, XXXIII, vol. 1709 of Lecture Notes in Math., Springer, Berlin, 1999, pp. 349–354.

[3] F. Delbaen and W. Schachermayer, *A general version of the fundamental theorem of asset pricing*, Math. Ann., 300 (1994), pp. 463–520.

[4] N. J. Kalton, N. T. Peck, and J. W. Roberts, *An \( F \)-space sampler*, vol. 89 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 1984.

[5] I. Karatzas and G. Žitković, *Optimal consumption from investment and random endowment in incomplete semimartingale markets*, Ann. Probab., 31 (2003), pp. 1821–1858.

[6] C. Kardaras, *Numéraire-invariant preferences in financial modeling*, Ann. Appl. Probab., 20 (2010), pp. 1697–1728.

[7] C. Kardaras and G. Žitković, *Forward-convex convergence in probability of sequences of nonnegative random variables*. Forthcoming in the Proceedings of the AMS; electronic preprint available at [http://arxiv.org/abs/1002.1889](http://arxiv.org/abs/1002.1889) 2011.

[8] V. Klee, *On a question of Bishop and Phelps*, Amer. J. Math., 85 (1963), pp. 95–98.

[9] D. Kramkov and W. Schachermayer, *Necessary and sufficient conditions in the problem of optimal investment in incomplete markets*, Ann. Appl. Probab., 13 (2003), pp. 1504–1516.

[10] I. Namioka, *Partially ordered linear topological spaces*, Mem. Amer. Math. Soc. no., 24 (1957), p. 50.

[11] G. Žitković, *Convex compactness and its applications*, Math. Financ. Econ., 3 (2010), pp. 1–12.

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