Enhancements in F-theory models on moduli spaces of K3 surfaces with $ADE$ rank 17

Yusuke Kimura$^1$ and Shun’ya Mizoguchi$^{1,2}$

$^1$KEK Theory Center, Institute of Particle and Nuclear Studies, KEK, 1-1 Oho, Tsukuba, Ibaraki 305-0801, Japan
$^2$SOKENDAI (The Graduate University for Advanced Studies), 1-1 Oho, Tsukuba, Ibaraki 305-0801, Japan

E-mail: kimurayu@post.kek.jp, mizoguch@post.kek.jp

Abstract

We study the moduli of elliptic K3 surfaces with a section with the $ADE$ rank 17. While the Picard number of a generic K3 surface in such moduli space is 19, the Picard number is enhanced to 20 at special points in the moduli. K3 surfaces become attractive K3 surfaces at these points. Either of the following two situations occurs at such special points: i) the Mordell-Weil rank of an elliptic K3 surface is enhanced, or ii) the gauge symmetry is enhanced. The first case i) is related to the appearance of a $U(1)$ gauge symmetry. In this note, we construct the moduli of K3 surfaces with $ADE$ types $E_7D_{10}$ and $A_{17}$. We determine some of the special points at which K3 surfaces become attractive in the moduli of K3 surfaces with $ADE$ types $E_7D_{10}$ and $A_{17}$. We investigate the gauge symmetries in F-theory compactifications on attractive K3 surfaces which correspond to such special points in the moduli times a K3 surface. $U(1)$ gauge symmetry arises in some F-theory compactifications.
1 Introduction

It is now well recognized that F-theory \cite{1,2,3} offers a useful framework for particle physics model building in string theory. On one hand, F-theory can support not only $SU(N)$ or $SO(2N)$ supersymmetric gauge theories achieved in the pure D-brane+orientifold system but an exceptional-group gauge theory, which is made possible by identifying all the $(p,q)$ 7-branes up to the $SL(2,\mathbb{Z})$ monodromy. This is particularly relevant for the implementation of GUT because it then allows matter in spinor representations, in which one generation of quarks and leptons observed in nature perfectly fit. On the other hand, another advantage of F-theory is that it is basically formulated in type IIB string theory, where the scenarios of moduli stabilization and inflation have been well studied, although concrete implementations of these in F-theory still remain a challenge.

The rather recent development in F-theory model building has been mainly based on local models \cite{4,5,6,7}, in which the decoupling limit of gravity is taken. However, given the results of the recent Large Hadron Collider (LHC) experiment and the Planck observation, it is now essential to consider global models to deal with the issues of particle physics beyond the standard model and the early universe. In global models of F-theory, compactifications on spaces that admit a K3 fibration are often considered, in which a K3 surface as a fiber of the compactification space is assumed to allow (at least locally) an elliptic fibration. If stable degeneration is available, the sections of the two rational elliptic surfaces encode information on spectral covers of gauge bundles of the dual $E_8 \times E_8$ heterotic string theory \cite{8}. The Picard number of a rational elliptic surface is always 10, and the rank of the singularities and the rank of the Mordell–Weil (MW) group are complementary within $E_8$. The structure of the Mordell–Weil groups of the rational elliptic surfaces was classified by Oguiso-Shioda \cite{9}. On the other hand, the Picard numbers of K3 surfaces vary, and such a complementarity relation between the singularities and the MW group does not hold for K3 surfaces.

In this note, we study the moduli of K3 surfaces with the $ADE$ rank 17. The Néron–Severi lattice of an elliptic surface with a global section is generated by sections, a smooth elliptic fiber and components of the singular fibers \cite{10}. The Picard number is the rank of the Néron–Severi lattice, therefore, the following formula holds for an elliptic surface $S$ which admits a section, which is known as the Shioda–Tate formula \cite{10,11,12}:

$$\rho(S) = 2 + \text{rk } ADE + \text{rk MW}. \quad (1)$$

A generic K3 surface with the $ADE$ rank 17 has the Picard number 19, and the family of K3 surfaces with $ADE$ rank 17 with a fixed $ADE$ type constitutes a 1 dimensional moduli. The Picard number is enhanced to 20 at special points in this moduli, and K3 surfaces become attractive K3 surfaces \footnote{We follow the convention of the term in \cite{13} to refer to K3 surfaces with the Picard number 20 as attractive K3 surfaces. It is standard to refer to a K3 surface with the Picard number 20 as a singular K3 surface in mathematics.} at these points. Either of the following two situations occurs at such special points in the moduli:

i)the $ADE$ type is unchanged, and the Mordell–Weil rank is enhanced to 1.

ii)the $ADE$ type is changed, and the Mordell–Weil rank is enhanced to 2.

The rather recent development in F-theory model building has been mainly based on local models \cite{4,5,6,7}, in which the decoupling limit of gravity is taken. However, given the results of the recent Large Hadron Collider (LHC) experiment and the Planck observation, it is now essential to consider global models to deal with the issues of particle physics beyond the standard model and the early universe. In global models of F-theory, compactifications on spaces that admit a K3 fibration are often considered, in which a K3 surface as a fiber of the compactification space is assumed to allow (at least locally) an elliptic fibration. If stable degeneration is available, the sections of the two rational elliptic surfaces encode information on spectral covers of gauge bundles of the dual $E_8 \times E_8$ heterotic string theory \cite{8}. The Picard number of a rational elliptic surface is always 10, and the rank of the singularities and the rank of the Mordell–Weil (MW) group are complementary within $E_8$. The structure of the Mordell–Weil groups of the rational elliptic surfaces was classified by Oguiso-Shioda \cite{9}. On the other hand, the Picard numbers of K3 surfaces vary, and such a complementarity relation between the singularities and the MW group does not hold for K3 surfaces.

In this note, we study the moduli of K3 surfaces with the $ADE$ rank 17. The Néron–Severi lattice of an elliptic surface with a global section is generated by sections, a smooth elliptic fiber and components of the singular fibers \cite{10}. The Picard number is the rank of the Néron–Severi lattice, therefore, the following formula holds for an elliptic surface $S$ which admits a section, which is known as the Shioda–Tate formula \cite{10,11,12}:

$$\rho(S) = 2 + \text{rk } ADE + \text{rk MW}. \quad (1)$$

A generic K3 surface with the $ADE$ rank 17 has the Picard number 19, and the family of K3 surfaces with $ADE$ rank 17 with a fixed $ADE$ type constitutes a 1 dimensional moduli. The Picard number is enhanced to 20 at special points in this moduli, and K3 surfaces become attractive K3 surfaces \footnote{We follow the convention of the term in \cite{13} to refer to K3 surfaces with the Picard number 20 as attractive K3 surfaces. It is standard to refer to a K3 surface with the Picard number 20 as a singular K3 surface in mathematics.} at these points. Either of the following two situations occurs at such special points in the moduli:

i)the $ADE$ type is unchanged, and the Mordell–Weil rank is enhanced to 1.
ii) the $ADE$ rank is enhanced to 18.

A $U(1)$ gauge symmetry arises in F-theory compactifications in the first case, i), and the non-Abelian gauge symmetry in F-theory compactifications is enhanced in the second case, ii).

We particularly construct the moduli of K3 surfaces with a section with the $ADE$ types $E_7D_{10}$ and $A_{17}$ in this study.\footnote{We determine some of the special points in these moduli at which K3 surfaces become attractive. Furthermore, we discuss the structures of the gauge groups that arise in F-theory compactifications on these attractive K3 surfaces times a K3 surface.}

We particularly construct the moduli of K3 surfaces with a section with the $ADE$ types $E_7D_{10}$ and $A_{17}$ in this study.\footnote{We determine some of the special points in these moduli at which K3 surfaces become attractive. Furthermore, we discuss the structures of the gauge groups that arise in F-theory compactifications on these attractive K3 surfaces times a K3 surface.}

The following two attractive K3 surfaces, among others, appear at the special points in the moduli of K3 surfaces with the $ADE$ types $E_7D_{10}$ and $A_{17}$: the attractive K3 surfaces, whose transcendental lattices have the intersection matrices

\[
\begin{pmatrix}
2 & 1 \\
1 & 2 
\end{pmatrix}
\]  
(2)

and

\[
\begin{pmatrix}
2 & 0 \\
0 & 2 
\end{pmatrix}.
\]

(3)

The attractive K3 surfaces whose transcendental lattices have the intersection matrices \(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\) and \(\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\) are often denoted as $X_3$ and $X_4$, respectively, in the literature; we denote these attractive K3 surfaces as $X_3$ and $X_4$ in this note. The attractive K3 surfaces $X_3$ and $X_4$ are also referred to as the attractive K3 surface with the discriminant 3 and the attractive K3 surface with the discriminant 4, respectively.

Using a method, known as the Kneser-Nishiyama method, the $ADE$ types and the MW groups of the elliptic fibrations with a section of several K3 surfaces, including the attractive K3 surfaces $X_3$ and $X_4$, are classified in \[19\].

An elliptic K3 surface with a fixed complex structure generally has several different elliptic fibrations. Furthermore, an elliptic K3 surface with a fixed complex structure in general has genus-one fibrations without a section, as well as elliptic fibrations with a global section.\footnote{An elliptic K3 surface with a fixed complex structure generally has several different elliptic fibrations. Furthermore, an elliptic K3 surface with a fixed complex structure in general has genus-one fibrations without a section, as well as elliptic fibrations with a global section.} Some relevance of the $E_7$ point in particle physics model building was emphasized in \[14\].

F-theory compactifications on genus-one fibrations lacking a global section are studied, for example, in \[20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32\].

F-theory models with a section have been discussed, for example, in \[33, 34, 35, 36, 37, 38\].
However, it is known that the attractive K3 surfaces $X_3$ and $X_4$ admit only elliptic fibrations that have a global section. The Weierstrass equations of the elliptic fibrations of the attractive K3 surface $X_3$ are discussed, for example, in [39, 40, 41]. F-theory compactifications on elliptic fibrations of the K3 surfaces $X_3$ and $X_4$ times a K3 surface are discussed in [18, 30, 42].

In F-theory compactifications, the types of the singular fibers of the compactification space correspond to the non-Abelian gauge groups that form on the 7-branes [3, 43]. We list the types of the singular fibers and the corresponding singularity types of the compactification space in Table 1 below. The rank of the Mordell–Weil group gives the number of $U(1)$ gauge fields.

| Fiber type | Type of singularity |
|------------|---------------------|
| $I_1$      | none.               |
| $II$       | none.               |
| $I_N$ $(N \geq 2)$ | $A_{N-1}$ |
| $I_M^*$ $(M \geq 0)$ | $D_{4+M}$ |
| $III$      | $A_1$               |
| $IV$       | $A_2$               |
| $IV^*$     | $E_6$               |
| $III^*$    | $E_7$               |
| $II^*$     | $E_8$               |

Table 1: Types of singular fibers and the corresponding types of singularities.

See [44, 45] for classification of the types of the singular fibers of elliptic surfaces. Methods to determine the singular fibers of elliptic surfaces are discussed in [46, 47].

The outline of this note is as follows: we review the elliptic fibrations of the attractive K3 surfaces $X_3$ and $X_4$ in section 2. Applying a result in [48] and analyzing the structure of the Picard lattice [49, 50], it is shown that every elliptic fibration of the attractive K3 surfaces $X_3$ and $X_4$ has a global section [50]. This will be reviewed in section 2.3. We construct the moduli of K3 surfaces with a section with the $ADE$ types $E_7D_{10}$ and $A_{17}$ in section 3. In constructing the moduli of K3 surfaces with the $ADE$ type $E_7D_{10}$, we use the anomaly cancellation condition to determine the configurations of the singular fibers. The method to obtain the Weierstrass equation by gluing a pair of the identical rational elliptic surfaces is discussed in [42]. Applying this method, we obtain the moduli of K3 surfaces with the $ADE$ type $A_{17}$. The Weierstrass equations with parameters describe the moduli of K3 surfaces with the $ADE$ types $E_7D_{10}$ and $A_{17}$. When the parameters take special values, K3 surfaces become attractive in the moduli. We discuss some of these special points in section 3. Attractive K3 surfaces $X_3$ and $X_4$ appear at special enhanced points in these moduli. In section 4, we discuss the gauge groups in F-theory compactifications. We confirm that the anomaly cancellation
condition is satisfied when F-theory is compactified on the direct products of K3 surfaces. We also confirm that the tadpole can be cancelled with flux at some special points in the moduli at which K3 surfaces become attractive. We state the concluding remarks in section 5.

2 Attractive K3 surfaces $X_3$ and $X_4$

We discuss the Weierstrass equations of elliptic fibrations of the attractive K3 surfaces $X_3$ and $X_4$.

2.1 Weierstrass forms of elliptic fibrations of attractive K3 surface $X_3$

In [19], it was shown that the attractive K3 surface $X_3$ has 6 elliptic fibrations with a global section, and the $ADE$ types and Mordell–Weil groups of these fibrations were obtained. We list the results of these 6 elliptic fibrations obtained in [19] in Table 2 below. We discuss the Weierstrass forms of the 6 elliptic fibrations with a section of the K3 surface $X_3$.

| Fibration | $ADE$ type | MW group |
|-----------|------------|----------|
| No.1      | $E^2_8A_2$ | 0        |
| No.2      | $D_{16}A_2$| $\mathbb{Z}_2$ |
| No.3      | $E_7D_{10}$| $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| No.4      | $A_{17}$   | $\mathbb{Z} \oplus \mathbb{Z}_3$ |
| No.5      | $E^3_6$    | $\mathbb{Z}_3$ |
| No.6      | $D_7A_{11}$| $\mathbb{Z}_4$ |

Table 2: Elliptic fibrations of the K3 surface $X_3$, their $ADE$ types and the Mordell–Weil groups.

Fibration No.1 has the following Weierstrass form [51, 39]:

$$y^2 = x^3 + t^5 s^5 (t - s)^2.$$  \hspace{1cm} (4)

We have used the notation $[t : s]$ to denote the homogeneous coordinate on the base $\mathbb{P}^1$.

The $ADE$ type of fibration No.1 is $E^2_8A_2$. We see from Table 2 in [52] that the K3 surface $X_3$ is a unique extremal elliptic K3 surface $^5$ with $ADE$ type $E^2_8A_2$. We find from equation (4) that the fibration No.1 has 2 type $II^*$ fibers, at $[t : s] = [0 : 1]$ and $[1 : 0]$, and 1 type $IV$ fiber at $[t : s] = [1 : 1]$. We list the vanishing orders of zero of the coefficients, $f,g$, and the

$^5$An extremal K3 surface is an elliptically fibered attractive K3 surface, which admits a global section, with the Mordell–Weil rank 0.
discriminant $\Delta$, of the Weierstrass form $y^2 = x^3 + fx + g$ for the types of the singular fibers in Table 3.

| Fiber type | Ord($f$) | Ord($g$) | Ord($\Delta$) |
|------------|----------|----------|----------------|
| $I_0$      | $\geq 0$ | $\geq 0$ | 0              |
| $I_n$ ($n \geq 1$) | 0 | 0 | $n$ |
| $II$       | $\geq 1$ | 1 | 2              |
| $III$      | 1 | $\geq 2$ | 3              |
| $IV$       | $\geq 2$ | 2 | 4              |
| $I_0^*$    | $\geq 2$ | 3 | 6              |
| $2$        | $\geq 3$ |   |                |
| $I_m^*$ ($m \geq 1$) | 2 | 3 | $6 + m$ |
| $IV^*$     | 3 | $\geq 5$ | 9              |
| $III^*$    | $\geq 4$ | 5 | 10             |

Table 3: The vanishing orders of coefficients $f, g$ of the Weierstrass form $y^2 = x^3 + fx + g$, and the vanishing order of the discriminant $\Delta$, for types of the singular fibers.

The Weierstrass form of the fibration No.5 is given as follows:

$$y^2 = x^3 + (t - \alpha_1 s)^4(t - \alpha_2 s)^4(t - \alpha_3 s)^4.$$ (5)

In equation (5), $\alpha_i$, $i = 1, 2, 3$, are constants, and they are mutually distinct. From the vanishing orders of the coefficient, we find that the Weierstrass form (5) has three singular fibers of type $IV^*$. The fibration No.5 has the $ADE$ type $E_6^3$, and, as discussed in [31], it can be deduced from Table 2 in [32] that the K3 surface $X_3$ is a unique extremal K3 surface with the $ADE$ type $E_6^3$.

Smooth fibers of the fibrations No.1 and No.5 admit transformations into the Fermat curve. This can be seen as follows: in homogeneous coordinates, fibers of the fibrations No.1 and No.5 are described by equations of the following form:

$$y^2 z = x^3 + \beta z^3.$$ (6)
After some rescaling, equation (6) may be rewritten as
\[ x^3 + z(z^2 + y^2) = 0. \] (7)

Using the following transformation
\[ z = \frac{1}{\sqrt[4]{4}}(\bar{z} + \bar{y}) \] \[ y = \frac{\sqrt{3}}{\sqrt[4]{4}}(\bar{y} - \bar{z}), \]
equation (7) transforms into the Fermat curve:
\[ x^3 + \tilde{y}^3 + \tilde{z}^3 = 0. \] (9)

Fermat curves have j-invariant 0, therefore, smooth fibers of the fibrations No.1 and No.5 have the constant j-invariant 0 over the base \( \mathbb{P}^1 \). Thus, it follows that the singular fibers of both the fibrations No.1 and No.5 have j-invariant 0.

Each singular fiber type has a specific j-invariant. We list the correspondence of the fiber types and j-invariants \[ [44] \] in Table 4. We deduce from Table 4 that the fiber types that have j-invariant 0 are only \( II, IV, IV^*, I_0^* \) and \( II^* \) (fiber type \( I_0^* \) may take j-invariant 0; the j-invariant of this fiber type takes any finite value in \( \mathbb{C} \), depending on the situation.) The types of the singular fibers of the fibration No.1 are \( II^* \) and \( IV \), and the fibration No.5 has the singular fibers of type \( IV^* \). We confirm that these are in agreement with the constraints imposed by the j-invariant of the singular fibers \(^6\)

The fibration No.2 has the \( ADE \) type \( D_{16}A_2 \). According to Table 2 in \[ [52] \], an extremal K3 surface with the \( ADE \) type \( D_{16}A_2 \) is uniquely determined to be the K3 surface \( X_3 \). Therefore, it is enough to construct the Weierstrass form that has the \( ADE \) type \( D_{16}A_2 \) to determine the Weierstrass equation of the fibration No.2.

As discussed in \[ [53] \], the triple cover of the extremal rational elliptic surface \( [4*, 1, 1] \), \( X_{[4*, 1, 1]} \), is an attractive K3 surface with 1 \( I_{12}^* \) fiber, 1 \( I_3 \) fiber, and 3 \( I_1 \) fibers. (See No. 216 in Table 2 in \[ [53] \].) The extremal rational elliptic surface \( X_{[4*, 1, 1]} \) is the extremal rational elliptic surface with three singular fibers, of types \( I_{12}^*, I_1, \) and \( I_1 \). An extremal rational elliptic surface with such configuration of singular fibers is unique \[ [54] \]. We follow the notation used in \[ [42] \] to denote by \( X_{[4*, 1, 1]} \) the extremal rational elliptic surface with three singular fibers, of types

\(^6\)When elliptic fibers possess complex multiplications of orders 3 and 4, throughout the base, elliptic fibers have constant j-invariants 0 and 1728, respectively, over the base. The possible non-Abelian gauge groups that can form on the 7-branes in F-theory compactifications are strongly constrained by these symmetries\[ [18, 30, 31] \]. Fermat curve possesses complex multiplication of order 3.

\(^7\)A rational elliptic surface is said to be extremal when the \( ADE \) type has the rank 8. The extremal rational elliptic surfaces were classified in \[ [54] \]. See, for example, \[ [42] \] for a review of the extremal rational elliptic surfaces.
| Fiber Type | j-invariant | Monodromy | order of Monodromy |
|------------|-------------|-----------|--------------------|
| $I_0^*$    | finite      | $-\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | 2 |
| $I_b$      | $\infty$   | $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ | infinite |
| $I_b^*$    | $\infty$   | $-\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ | infinite |
| $II$       | 0          | $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ | 6 |
| $II^*$     | 0          | $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ | 6 |
| $III$      | 1728       | $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ | 4 |
| $III^*$    | 1728       | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ | 4 |
| $IV$       | 0          | $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ | 3 |
| $IV^*$     | 0          | $\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$ | 3 |

Table 4: Fiber types and the corresponding j-invariants. “Finite” for type $I_0^*$ fiber means that j-invariant of $I_0^*$ fiber can take any finite value in $\mathbb{C}$. Monodromies around the singular fibers [44], and their orders are also included.

$I_1^*, I_1$, and $I_1$, in this study. The surface $X_{[4^*, 1,1]}$ is given by the following Weierstrass form [54]:

$$y^2 = x^3 - 3t^2(s^2 - 3t^2)x + t^3s(2s^2 - 9t^2).$$

(10)

The following linear transformation of the Weierstrass form (10) was considered in [53]:

$$(t, s) \rightarrow (t, 4s - 2t).$$

(11)

Under the linear transformation (11), equation (10) transforms into another Weierstrass form as:

$$y^2 = x^3 - 3t^2(t^2 - 16ts + 16s^2)x + 2t^3(t - 2s)(t^2 + 32ts - 32s^2).$$

(12)

We consider the following triple cover of the Weierstrass form (12) of the extremal rational elliptic surface $X_{[4^*, 1,1]}$:

$$(t, s) \rightarrow (t^3, s^3).$$

(13)
The resulting K3 surface is given by the following Weierstrass form:\[ y^2 = x^3 - 3t^2(t^6 - 16t^3s^3 + 16s^6)x + 2t^3(t^3 - 2s^3)(t^6 + 32t^3s^3 - 32s^6), \] (16) with the discriminant \[ \Delta \sim t^{18}s^3(t^3 - s^3). \] (17)

From the equations (16) and (17), we find that the K3 surface (16) has 1 type $I_{12}$ fiber at $[t : s] = [0 : 1]$, 1 type $I_3$ fiber at $[t : s] = [1 : 0]$, and 3 type $I_1$ fibers at the roots of $t^3 - s^3 = 0$.

Therefore, the K3 surface (16) has the $ADE$ type $D_{16}A_2$. Thus, we conclude that the Weierstrass equation (16) describes the fibration No.2.

We discuss the Weierstrass forms of the fibrations No.3 and No.4 separately, in sections 3.1.1 and 3.2.1 respectively. They correspond to special points in the moduli of K3 surfaces with the $ADE$ types $E_7D_{10}$ and $A_{17}$, respectively, at which the Mordell–Weil rank is enhanced to 1.

The general Weierstrass form of the fibration No.6 can be found in [40]; the general Weierstrass form is given by:

\[ y^2 + t^2xy = x^3 + 2tx^2 + t^2x. \] (18)

The general Weierstrass equation (18) transforms into the following Weierstrass form:

\[ y^2 = x^3 + t^2[s^6 - \frac{1}{48}(t^3 + 8s^3)^2]x + \frac{t^3}{27} \left( \frac{t^3}{4} + 2s^3 \right) \left( \frac{t^6}{8} + 2t^3s^3 - s^6 \right). \] (19)

The discriminant of the fibration No.6 (19) is

\[ \Delta \sim t^9(t^3 + 16s^3)s^{12}. \] (20)

The $I_3^*$ fiber is at $[t : s] = [0 : 1]$, the $I_{12}$ fiber is at $[t : s] = [1 : 0]$, and three $I_1$ fibers are at the roots of $t^3 + 16s^3 = 0$ [40].

The fibrations No.1 and No.5 are discussed in [42] in the context of the stable degenerations of K3 surfaces into pairs of rational elliptic surfaces. We construct the moduli of K3 surfaces

---

\footnote{When we take the base change (13), we obtain the following Weierstrass form:
\[ y^2 = x^3 - 3t^2(t^6 - 16t^3s^3 + 16s^6)x + 2t^3(t^3 - 2s^3)(t^6 + 32t^3s^3 - 32s^6). \] (14)

The Weierstrass equation (14) is not minimal; we consider the following minimalizing process:
\[ x \rightarrow t^2x, \quad y \rightarrow t^3y. \] (15)

Consequently, we obtain the minimal Weierstrass equation (16). Minimalizing process is discussed in [55] in the context of F-theory.}
with the $ADE$ type $A_{17}$ by gluing two identical rational elliptic surfaces, and taking a special limit, in section [3.2.1]. We find that the fibration No.4 in Table 2 corresponds to a special point in this moduli with the $ADE$ type $A_{17}$ at which the Mordell–Weil rank is enhanced to 1.

### 2.2 Weierstrass forms of elliptic fibrations of attractive K3 surface $X_4$

The $ADE$ types and the Mordell–Weil groups of the elliptic fibrations with a section of the attractive K3 surface with the discriminant 4, $X_4$, were classified in [19]. We list the results deduced in [19] in Table 5 below.

| Fibration | $ADE$ type | MW group |
|-----------|-------------|-----------|
| No.1      | $E_8^2A_1^2$ | 0         |
| No.2      | $E_8D_{10}$  | 0         |
| No.3      | $D_{16}A_1^2$ | $\mathbb{Z}_2$ |
| No.4      | $E_7^2D_4$   | $\mathbb{Z}_2$ |
| No.5      | $E_7D_{10}A_1$ | $\mathbb{Z}_2$ |
| No.6      | $A_{17}A_1$  | $\mathbb{Z}_3$ |
| No.7      | $D_{18}$     | 0         |
| No.8      | $D_{12}D_6$  | $\mathbb{Z}_2$ |
| No.9      | $D_8^2A_1^2$ | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ |
| No.10     | $A_{15}A_3$  | $\mathbb{Z}_4$ |
| No.11     | $E_6A_{11}$  | $\mathbb{Z} \oplus \mathbb{Z}_3$ |
| No.12     | $D^3_6$      | $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ |
| No.13     | $A_9^2$      | $\mathbb{Z}_5$ |

Table 5: Elliptic fibrations of the K3 surface $X_4$, their $ADE$ types and the Mordell–Weil groups.

We discuss some of the elliptic fibrations with a global section of the attractive K3 surface $X_4$.

The fibration No.1 in Table 5 has the following Weierstrass form [51, 39]:

$$y^2 = x^3 - 3t^4s^4x + t^5s^5(t^2 + s^2).$$  \hspace{1cm} (21)

The discriminant of the Weierstrass form (21) is

$$\Delta \sim t^{10}s^{10}(t + s)^2(t - s)^2.$$  \hspace{1cm} (22)

It can be seen from (21) and (22) that there are 2 type $II^*$ fibers, at $[t : s] = [1 : 0]$ and at $[0 : 1]$. There are 2 type $I_2$ fibers, at $[t : s] = [1 : 1]$ and at $[1 : -1]$. 
The fibration No.4 has the following Weierstrass form:

\[ y^2 = x^3 + (t - s)^2 t^3 s^3 x. \]  \hspace{1cm} (23)

The discriminant of the Weierstrass equation (23) is

\[ \Delta \sim t^9 s^9 (t - s)^6. \] \hspace{1cm} (24)

We find from (23) and (24) that there are 2 type III* fibers, at \([t : s] = [1 : 0]\) and at \([t : s] = [0 : 1]\). There is 1 type \(I_0^*\) fiber at \([t : s] = [1 : 1]\).

The fibration No.9 is given by the following Weierstrass equation [42]:

\[ y^2 = x^3 - 3 t^2 s^2 (t^4 + s^4 - t^2 s^2) x + (t^2 + s^2) t^3 s^3 (2t^4 - 5t^2 s^2 + 2s^4), \] \hspace{1cm} (25)

with the discriminant

\[ \Delta \sim t^{10} s^{10} (t - s)^2 (t + s)^2. \] \hspace{1cm} (26)

There are 2 type \(I_4^*\) fibers, at \([t : s] = [0 : 1]\) and \([1 : 0]\), and there are 2 type \(I_2\) fibers, at \([t : s] = [1 : 1]\) and \([1 : -1]\).

We will discuss the fibrations No.2, 5 and 6 in section 3 as special points of the moduli of K3 surfaces with the ADE types \(E_7\) and \(D_{10}\), and \(A_{17}\), at which the ADE ranks are enhanced.

The fibrations No.1, 4, 9 are discussed in [42] in the context of the stable degenerations of K3 surfaces into pairs of rational elliptic surfaces.

### 2.3 Absence of genus-one fibration without a section in K3 surfaces \(X_3\) and \(X_4\)

In [50], it is shown that every elliptic fibration of the K3 surfaces \(X_3\) and \(X_4\) has a global section. The argument goes as follows: If the K3 surface \(X_3\) has a genus-one fibration that lacks a global section, we can consider the Jacobian fibration \(J(X_3)\) of this genus-one fibration. The Jacobian fibration of a genus-one fibration is an elliptic fibration with a section, the types and locations of the singular fibers of which are identical to those of the singular fibers of the genus-one fibration.

We denote the smallest degree of the multisections that the genus-one fibration of the K3 surface \(X_3\) possesses by \(n\), with \(n > 1\). (The multisection of degree \(n = 1\) is a global section.) Then, the following relationship holds [50] between the determinants of the Picard lattice \(\text{Pic}(X_3)\) of the genus-one fibration, and the Picard lattice \(\text{Pic}(J(X_3))\) of the Jacobian fibration \(J(X_3)\):

\[ \det \text{Pic}(X_3) = n^2 \det \text{Pic}(J(X_3)). \] \hspace{1cm} (27)

\[^9\text{The smooth elliptic fibers of the Weierstrass equation} \text{[23] have complex multiplication of order 4.}\]
Since the transcendental lattice is the orthogonal complement of the Picard lattice in the K3 lattice, we have \[ \det \text{Pic} (X_3) = \det T(X_3). \] (28)

We have \[ \det T(X_3) = \det \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = 3, \] (29) therefore, we obtain \[ n^2 \det \text{Pic} (J(X_3)) = 3. \] (30)

Equation (30) implies that \( n^2 \) divides 3. However, \( n^2 \) divides 3 only when \( n = 1 \); \( n = 1 \) means that the genus-one fibration has a global section, which is a contradiction. Therefore, the K3 surface \( X_3 \) does not have a genus-one fibration without a global section.

An argument similar to that stated above shows that every elliptic fibration of the attractive K3 surface \( X_4 \) has a global section. This can be seen as follows: The intersection matrix of the transcendental lattice of an attractive K3 surface can be transformed into the following form under the \( GL_2(\mathbb{Z}) \) action:

\[
\begin{pmatrix}
2a & b \\
b & 2c
\end{pmatrix},
\]
(31)

where \( a, b, c \in \mathbb{Z} \), and \( a \geq c \geq b \geq 0 \). \( a \) and \( c \) are positive integers. Thus, the following inequality holds for the discriminant of an attractive K3 surface:

\[
\det \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} = 4ac - b^2 \geq 4c^2 - c^2 = 3c^2 \geq 3.
\] (32)

It follows that the minimum of the discriminants of the attractive K3 surfaces is 3. Suppose that the K3 surface \( X_4 \) admits a genus-one fibration lacking a global section. We consider the Jacobian fibration \( J(X_4) \) of this genus-one fibration. Since the K3 surface \( X_4 \) has the discriminant 4, we find from the relationship \([50]\) between the determinants of the Picard lattice \( \text{Pic}(X_4) \) of the genus-one fibration, and the Picard lattice \( \text{Pic}(J(X_4)) \) of the Jacobian fibration \( J(X_4) \), that the degree of the multisection of this genus-one fibration must be 2. This means that the Jacobian fibration \( J(X_4) \) has the discriminant 1, which contradicts the fact that the minimum of the discriminants of the attractive K3 surfaces is 3.

In particular, 6 elliptic fibrations with a section deduced in \([19]\) classify the elliptic fibrations of the attractive K3 surface \( X_3 \). Similarly, 13 elliptic fibrations with a section obtained in \([19]\) classify the elliptic fibrations of the K3 surface \( X_4 \).

Applying an argument similar to the proof stated above, we see that every elliptic fibration of the attractive K3 surface whose transcendental lattice has the intersection matrix \( \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \) has a section. Suppose that the attractive K3 surface whose transcendental lattice has the intersection matrix \( \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \) has a genus-one fibration without a global section. This attractive K3 surface has the discriminant 8, therefore, the degree of the genus-one fibration without a
section must be 2. This means that the discriminant of the Jacobian fibration of the genus-one fibration lacking a section is 2. However, the smallest discriminant of the attractive K3 surfaces is 3, which is a contradiction. Thus, every elliptic fibration of the attractive K3 surface whose transcendental lattice has the intersection matrix \(
abla \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \) has a global section.

3 Moduli of K3 surfaces with ADE types \(E_7 D_{10}\) and \(A_{17}\)

3.1 Moduli of K3 surfaces with ADE type \(E_7 D_{10}\)

3.1.1 Weierstrass equation of K3 surfaces with ADE type \(E_7 D_{10}\)

We determine the form of the Weierstrass equation of a general K3 surface with ADE type \(E_7 D_{10}\).

First, we specify the configuration of singular fibers of a K3 surface with ADE type \(E_7 D_{10}\). We consider F-theory compactification on a K3 surface with ADE type \(E_7 D_{10}\) times a K3. The form of the discriminant locus in the base space \(\mathbb{P}^1 \times \text{K3}\) is determined by the anomaly cancellation conditions to be 24 K3 surfaces. For example, see [18], for discussion of the form of the discriminant locus in F-theory compactification on the direct product of K3 surfaces. This imposes a constraint on the configuration of the singular fibers of K3 surface; the sum of the number of 7-branes associated with the singular fibers must be 24. The number of the 7-branes associated to a singular fiber is given by the Euler number of that fiber. The Euler numbers of the singular fibers are given in [45]. We list the numbers of 7-branes for the singular fiber types in Table 6. From the constraint that the total number of 7-branes is 24,

| Type of singular fiber | # of 7-branes (Euler number) |
|-----------------------|----------------------------|
| \(I_n\)              | \(n\)                      |
| \(I_0^*\)            | 6                          |
| \(I_m^*\)            | \(m+6\)                    |
| \(II\)               | 2                          |
| \(III\)              | 3                          |
| \(IV\)               | 4                          |
| \(IV^*\)             | 8                          |
| \(III^*\)            | 9                          |
| \(II^*\)             | 10                         |

Table 6: Types of singular fibers and the associated numbers of 7-branes.

we deduce that only two configurations of singular fibers are possible for ADE type \(E_7 D_{10}\):

i) 1 type \(III^*\) fiber, 1 type \(I_0^*\) fiber, and 3 type \(I_1\) fibers
ii) 1 type $III^*$ fiber, 1 type $I_6^*$ fiber, 1 type $II$ fiber, and 1 type $I_1$ fiber.

First, we discuss K3 surfaces with the second configuration, ii), of singular fibers. We may assume that the type $I_6^*$ fiber is at $[t:s] = [0:1]$, the type $III^*$ fiber is at $[t:s] = [1:0]$, and the type $II$ fiber is at $[t:s] = [1:1]$. We find from Table 3 that the coefficients $f, g$ of the Weierstrass form $y^2 = x^3 + f x + g$ of configuration ii) must be of the following form:

$$f = t^2 s^3 (t - s) h_2$$
$$g = t^3 s^5 (t - s) h_3.$$  \(33\)

In equation (33), we have used $h_2, h_3$ to denote homogeneous polynomials in $t, s$ of degrees 2 and 3, respectively. The discriminant is given as follows:

$$\Delta = t^6 s^9 (t - s)^2 \left[ 4(t - s) h_2^3 + 27 s h_3^2 \right].$$  \(34\)

To have a type $I_6^*$ fiber at $[t:s] = [0:1]$, $t^6$ must divide the term

$$4(t - s) h_2^3 + 27 s h_3^2$$  \(35\)

in the discriminant (34). The following polynomials $h_2, h_3$ satisfy this condition:

$$h_2 = \frac{\alpha}{6} ((i\sqrt{3} - 3)t^2 - 6i\sqrt{3}ts + 6s^2)$$  \(36\)

$$h_3 = \frac{\sqrt{\alpha}^3}{18} (-5\sqrt{3} + 3i)t^3 + 4(2\sqrt{3} - 3i)t^2s + 2(\sqrt{3} + 9i)ts^2 - 4\sqrt{3}s^3).$$

Under the following rescaling

$$x \rightarrow \sqrt{\alpha} x,$$
$$y \rightarrow \alpha^{3/4} y,$$

the coefficients $f, g$ of the Weierstrass form $y^2 = x^3 + f x + g$ transform as

$$f \rightarrow \frac{f}{\alpha},$$
$$g \rightarrow \frac{g}{\sqrt{\alpha}^3}.$$  \(38\)

Therefore, the complex structures of the elliptic K3 surfaces, the coefficients of the Weierstrass form of which are given by (33) where $h_2$ and $h_3$ are given by (36), in fact do not depend on the values of $\alpha$ in (36); we may fix the parameter $\alpha$ as $\alpha = 1$. The moduli of the elliptic K3 surfaces with a section, the singular fibers of which have configuration ii), is 0-dimensional, namely, it is discrete. Particularly, this means that neither enhancement of the singularity

\[^{10}\text{Elliptic K3 surfaces are elliptic fibrations over the base } P^1, \text{ therefore, the Weierstrass equations that we consider in this note are elliptic curves over the function field } C(u), \text{ where } u := t/s.\]
type nor enhancement of the Mordell–Weil rank occurs for the moduli of these elliptic K3 surfaces. We do not consider the elliptic K3 surfaces with a section, the singular fibers of which have configuration ii), in this study.

We discuss K3 surfaces with $ADE$ type $E_7D_{10}$ that have the first configuration, i), of the singular fibers: 1 type $III^*$ fiber, 1 type $I_6^*$ fiber, and 3 type $I_1$ fibers. We determine the form of the Weierstrass equation which has this configuration of singular fibers. We may assume that the type $I_6^*$ fiber is at $[t : s] = [0 : 1]$, and the type $III^*$ fiber is at $[t : s] = [1 : 0]$. The coefficients $f, g$ of the Weierstrass form $y^2 = x^3 + f x + g$ must be of the following form:

\[
\begin{align*}
  f &= t^2 s^3 \tilde{h}_3 \\
  g &= t^3 s^5 \tilde{h}_4.
\end{align*}
\]

(39)

We have used $\tilde{h}_3, \tilde{h}_4$ to denote homogeneous polynomials in $t, s$ of degree 3 and 4, respectively. The discriminant is given by

\[
\Delta = t^6 s^9 (4\tilde{h}_3^3 + 27s\tilde{h}_4^2).
\]

(40)

We have a type $I_6^*$ fiber at $[t : s] = [0 : 1]$; therefore, we impose the condition that $t^6$ divides the term

\[
4\tilde{h}_3^3 + 27s\tilde{h}_4^2
\]

(41)

in the discriminant (40). Solving this condition, we obtain two families of the Weierstrass equations that give the elliptic K3 surfaces with a global section with $ADE$ type $E_7D_{10}$.

One family of K3 surfaces with $ADE$ type $E_7D_{10}$ is given by the following Weierstrass form:

\[
y^2 = x^3 + t^2 s^3 (\alpha t^3 - \frac{1}{12} \beta^2 t^2 s + \beta ts^2 - 3s^3) x + t^3 s^5 [-\frac{\alpha \beta}{6} t^4 + (\alpha + \frac{\beta^3}{108}) t^3 s - \frac{1}{6} \beta^2 t^2 s^2 + \beta ts^3 - 2s^4].
\]

(42)

In the Weierstrass equation (42), $(\alpha, \beta)$ are parameters.\(^\text{11}\) The discriminant of the Weierstrass form (42) is

\[
\Delta \sim t^{12} s^9 (4\alpha t^3 - \frac{\beta^2}{4} t^2 s + 3\beta ts^2 - 9s^3).
\]

(43)

From equations (42) and (43), we confirm that the type $I_6^*$ fiber is at $[t : s] = [0 : 1]$, the type $III^*$ fiber is at $[t : s] = [1 : 0]$, and three type $I_1$ fibers are at the roots of $4\alpha t^3 - \frac{\beta^2}{4} t^2 s + 3\beta ts^2 - 9s^3$, for generic values of $(\alpha, \beta)$. Thus, we confirm that the $ADE$ type of K3 surface (42) is in fact $E_7D_{10}$.

The Weierstrass form of the other family of elliptic K3 surfaces with a section with the $ADE$ type $E_7D_{10}$ is given as:

\[
y^2 = x^3 + t^2 s^3 \left[ \frac{-1}{144} (12\gamma \delta + \delta^3) t^3 + \gamma t^2 s + \delta ts^2 - 3s^3 \right] x + t^3 s^5 \left[ (-\frac{\gamma^2}{12} + \frac{\delta^4}{1728}) t^4 - (\frac{\gamma \delta}{4} + \frac{5}{432} \delta^3) t^3 s + (\gamma - \frac{\delta^2}{12}) t^2 s^2 + \delta ts^3 - 2s^4 \right].
\]

(44)

\(^\text{11}\)The solution to the condition that $t^6$ divides the term (41) has three parameters. Using the rescaling similar to (37) and (38), one of the three parameters can be fixed. Consequently, we obtain the family (42).
(γ, δ) are parameters.

The discriminant of the Weierstrass equation (44) is given as follows:

\[
\Delta \sim t^{12} s^9 \left[ -\frac{1}{324} (12\gamma \delta + \delta^3) \delta^2 t^3 + \frac{1}{48} (4\gamma + \delta^2) (36\gamma + \delta^2) t^2 s \right.
\]
\[
+ \frac{1}{2} \delta (4\gamma + \delta^2) t s^2 - \frac{1}{3} \left( 24\gamma + 5\delta^2 \right) s^3 \right].
\]

(45)

We find from the discriminant (45) that there is 1 type III* fiber at \([t : s] = [1 : 0]\), 1 type I* fiber at \([t : s] = [0 : 1]\), and there are 3 type I_fibers at the roots of \(-\frac{1}{324} (12\gamma \delta + \delta^3) \delta^2 t^3 + \frac{1}{48} (4\gamma + \delta^2) (36\gamma + \delta^2) t^2 s + \frac{1}{2} \delta (4\gamma + \delta^2) t s^2 - \frac{1}{3} \left( 24\gamma + 5\delta^2 \right) s^3\), for generic values of \((\gamma, \delta)\).

The Weierstrass equations (42) and (44) have the Mordell–Weil rank 0 for generic values of the parameters. For special values of the parameters, the Mordell–Weil rank is enhanced to 1, or the ADE rank is enhanced to 18. K3 surfaces (42) and (44) become attractive in these situations. Either of the equations (42) and (44) has two parameters, since there is a parameter which determines the locations of the type I_fibers, in addition to the parameter which determines the complex structure.

### 3.1.2 Enhancement to models with \( U(1) \)

We particularly consider the case \( \beta = 0 \) of the Weierstrass equation (42). The Weierstrass equation (42), in this case, reduces to the following Weierstrass form:

\[
y^2 = x^3 + t^2 s^3 (\alpha t^3 - 3s^3) x + t^3 s^6 (\alpha t^3 - 2s^3),
\]

(46)

with the discriminant

\[
\Delta \sim t^{12} s^9 (4\alpha t^3 - 9s^3).
\]

(47)

The parameter \( \alpha \) parameterizes the locations of 3 type I_fibers. The complex structure of the attractive K3 surface (46) does not depend on the parameter \( \alpha \), however, we must require that

\[
\alpha \neq 0.
\]

(48)

After some computation, we find that the Weierstrass form (46) admits a section

\[
[X : Y : Z] = [-ts^3 : 0 : 1],
\]

(49)

in addition to a constant zero section

\[
[X : Y : Z] = [0 : 1 : 0].
\]

(50)

Section (49) is a torsion section that generates a \( \mathbb{Z}_2 \) group.

The general form of the Weierstrass equation of an elliptic fibration with the Mordell–Weil rank 1 is determined in [33]. By comparing the coefficients of the Weierstrass form (46) with the general form obtained in [33], we deduce that the Weierstrass form (46) admits another section, which generates the group \( \mathbb{Z} \):

\[
[X : Y : Z] = [2ts^3 : \sqrt{3\alpha} t^3 s^3 : 1].
\]

(51)
Thus, we conclude that the Weierstrass equation (46) has the Mordell–Weil rank 1.

The torsion parts of the Mordell–Weil groups of the elliptic K3 surfaces with a section were determined in [57]. According to Table 1 in [57], the torsion part of the Mordell–Weil group of a K3 surface with ADE type $E_7D_{10}$ is either 0 or $\mathbb{Z}_2$ (No.2421 in Table 1 in [57]). Thus, we conclude that the $\mathbb{Z}_2$ group generated by the torsion section (49) gives the whole torsion part of the Mordell–Weil group of the K3 surface (46). Therefore, we deduce that the Mordell–Weil group of the K3 surface (46) is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$.

Thus, we conclude that the Weierstrass equation (46) describes an attractive K3 surface with the ADE type $E_7D_{10}$ with the Mordell–Weil group $\mathbb{Z} \oplus \mathbb{Z}_2$. Comparing the equation (46) with the general Weierstrass equation of the fibration No.3 given in [41], we deduce that the equation (46) gives the fibration No.3 in Table 2 of the attractive K3 surface $X_3$.

Therefore, we find that the Mordell–Weil rank is enhanced to 1 at the point $\beta = 0$ in moduli of K3 surfaces with the ADE type $E_7D_{10}$ (42). The K3 surface becomes attractive at this point. F-theory compactification on the K3 elliptic fibration (46) times a K3 surface has a $U(1)$ gauge field.

Using the Kneser-Nishiyama method, we find that the attractive K3 surfaces that admit an elliptic fibration with a section with ADE type $E_7D_{10}$ include: attractive K3 surfaces, the intersection matrices of the transcendental lattices of which are

- $\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$,
- $\begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$,
- $\begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$,
- $\begin{pmatrix} 8 & 1 \\ 1 & 2 \end{pmatrix}$,
- $\begin{pmatrix} 12 & 0 \\ 0 & 2 \end{pmatrix}$,

and $\begin{pmatrix} 12 & 0 \\ 0 & 2 \end{pmatrix}$ in addition to the attractive K3 surface $X_3$. These attractive K3 surfaces correspond to special points in the moduli of K3 surfaces (42) and (44) at which the Mordell–Weil ranks are enhanced. We particularly discuss the attractive K3 surface whose transcendental lattice has the intersection matrix $\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$.

The general Weierstrass equation of the elliptic fibration with ADE type $E_7D_{10}$ of the attractive K3 surface whose transcendental lattice has the intersection matrix $\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$ is given in [58] as:

$$y^2 = x^3 - (5t^2 + t)x^2 - t^5 x.$$  \hspace{1cm} (52)

The general Weierstrass equation (52) transforms into the following Weierstrass equation:

$$y^2 = x^3 - t^2 s^3 \left[ t^3 + \frac{1}{3}s(5t + s)^2 \right] x - \frac{1}{27} t^3 (5t + s) \left( 9t^3 + 50t^2 s + 20ts^2 + 2s^3 \right) s^5.$$ \hspace{1cm} (53)

\[12\]Elliptic fibrations with a global section, and their Weierstrass forms, of an attractive K3 surface whose transcendental lattice has the intersection matrix $\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$ are classified in [58]. Elliptic fibrations with a global section of an attractive K3 surface whose transcendental lattice has the intersection matrix $\begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}$ are studied in [59].
Under the following rescaling
\[ x \rightarrow \frac{x}{(\sqrt{3})^2} \]
\[ y \rightarrow \frac{y}{(\sqrt{3})^3}, \]
the coefficients \( f, g \) of the Weierstrass form \( y^2 = x^3 + f x + g \) transform as
\[ f \rightarrow 3^2 f \]
\[ g \rightarrow 3^3 g. \]

Thus, under the rescaling (54), the Weierstrass form (53) further transforms into another Weierstrass form as:
\[ y^2 = x^3 - t^2 s^3 \left[ 9t^3 + 3s(5t + s)^2 \right] x - t^3 (5t + s) (9t^3 + 50t^2 s + 20ts^2 + 2s^3) s^5. \] (56)

Weierstrass equation (56) is a member of the family of the Weierstrass equations with \( ADE \) type \( E_7D_{10} \) (42), with
\[ \alpha = -9 \]
\[ \beta = -30. \] (57)

Thus, we conclude that the elliptic fibration with the \( ADE \) type \( E_7D_{10} \) of the attractive K3 surface whose transcendental lattice has the intersection matrix \( \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \) is described by the equation (42) with \( (\alpha, \beta) = (-9, -30) \).

The Mordell–Weil group of the fibration (52) is isomorphic to \( \mathbb{Z} \oplus \mathbb{Z}_2 \). A \( U(1) \) gauge field arises in F-theory compactification on the K3 elliptic fibration (56) times a K3 surface.

3.1.3 Enhancement to rank 18

The \( ADE \) rank of a K3 surface in the moduli (42) and (44) is enhanced to 18 at special points. The K3 surface becomes attractive at these points. The Mordell–Weil rank remains to be 0 in these situations. We discuss some such points.

When \( \delta = 0 \) in the moduli (44), the Weierstrass equation (44) becomes:
\[ y^2 = x^3 + t^2 s^4 (\gamma t^2 - 3s^2) x + t^3 s^5 \left( -\frac{\gamma^2}{12} t^4 + \gamma t^2 s^2 - 2s^4 \right). \] (58)

The discriminant of the Weierstrass equation (58) is
\[ \Delta \sim t^{12} s^{10} (3\gamma t^2 - 8s^2). \] (59)

The \( ADE \) type of the Weierstrass equation (58) is \( E_8D_{10} \): a type \( II^* \) fiber is at \( [t : s] = [1 : 0] \), and a type \( I_6^* \) fiber is at \( [t : s] = [0 : 1] \). The K3 surface (58) is attractive with the Mordell–Weil rank 0. We find from Table 2 in (52) that the attractive K3 surface (58) is in fact the
attractive K3 surface with the discriminant 4, \( X_4 \) (No.320 in Table 2 in [52]). Therefore, we deduce that the Weierstrass equation (58) gives the fibration No.2 in Table 5. (In the equation (58), we require that \( \gamma \neq 0 \).)

We find from equations (44) and (45) that when the parameters of the Weierstrass equation (44) satisfy
\[
24\gamma + 5\delta^2 = 0, \quad (60)
\]
the ADE type is enhanced to \( E_7D_{11} \). (We exclude the case \( \gamma = \delta = 0 \).) We deduce from Table 2 in [52] that, in this situation, the K3 surface (44) becomes the attractive K3 surface whose transcendental lattice has the intersection matrix \( \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \) (No.291 in Table 2 in [52]). Therefore, we conclude that the K3 surface becomes the attractive K3 surface whose transcendental lattice has the intersection matrix \( \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \) in the moduli (44) when the parameters satisfy the condition (60). The Mordell–Weil group of this K3 elliptic fibration is isomorphic to 0 [52].

We find from the equations (42) and (43) that when \( (\alpha, \beta) = (1, 6\sqrt{3}) \), the ADE type is enhanced to \( E_7D_{10}A_1 \). The K3 surface (42) becomes attractive in this situation. We deduce from Table 2 in [52] that this attractive K3 surface is the attractive K3 surface \( X_4 \) (No.290 in Table 2 in [52]). Therefore, we find that a K3 surface becomes the surface \( X_4 \) at \( (\alpha, \beta) = (1, 6\sqrt{3}) \) in the moduli (42). The Weierstrass equation (42) with \( (\alpha, \beta) = (1, 6\sqrt{3}) \) describes the fibration No.5 in Table 5.

3.2 Moduli of K3 surfaces with ADE type \( A_{17} \)

3.2.1 Weierstrass equation of K3 surfaces with ADE type \( A_{17} \)

The method to deduce the Weierstrass equation of a K3 surface from the Weierstrass equation of a rational elliptic surface, by gluing two isomorphic rational elliptic surfaces, is discussed in [42]. Using this method, we construct a family of Weierstrass equations that describes K3 surfaces with ADE type \( A_{17} \).

Gluing two identical rational elliptic surfaces along smooth fibers, we generically obtain a K3 surface, the singular fibers of which are twice the singular fibers of the rational elliptic surface. This corresponds to the quadratic base change of the rational elliptic surface [42].

We consider the extremal rational elliptic surface with four singular fibers, of types \( I_9, I_1, I_1, \) and \( I_1 \). Following the notation used in [42], we denote this extremal rational elliptic surface by \( X_{[9,1,1,1]} \). The Weierstrass form of the extremal rational elliptic surface \( X_{[9,1,1,1]} \) is given as follows [54]:
\[
y^2 = x^3 - 3t (t^3 + 24s^3) x + 2(t^6 + 36t^3s^3 + 216s^6). \quad (61)
\]
We consider the gluing of two copies of the extremal rational elliptic surface \( X_{[9,1,1,1]} \).
The limit of the quadratic base change of rational elliptic surfaces, at which singular fibers of the same type collide, is also discussed in [42]. We consider the limit of the gluing of two identical extremal rational elliptic surfaces at which two $I_9$ fibers collide; they are enhanced to an $I_{18}$ fiber in this limit. This limit is given by the following base change:

\[
\begin{align*}
t & \rightarrow t^2 + \alpha s^2 \\
s & \rightarrow s^2.
\end{align*}
\]

In transformation (62), $\alpha$ is a parameter. The transformation of $t$, in general, can take the form $t \rightarrow t^2 + \alpha s^2 + \gamma ts$. However, by completing the square in $t$, we find that $\gamma$ is a redundant parameter. The K3 surface as a result of the base change (62) has the following Weierstrass form:

\[
y^2 = x^3 - 3(t^2 + \alpha s^2) [(t^2 + \alpha s^2)^3 + 24s^6] x + 2 [(t^2 + \alpha s^2)^6 + 36(t^2 + \alpha s^2)^3 s^6 + 216s^{12}].
\]

(63)

The discriminant of the Weierstrass equation (63) is given by

\[
\Delta \sim s^{18} [(t^2 + \alpha s^2)^3 + 27s^6].
\]

(64)

From equations (63) and (64), we confirm that the K3 surface (63) has 1 type $I_{18}$ fiber at $[t : s] = [1 : 0]$, and 6 type $I_1$ fibers at the roots of $(t^2 + \alpha s^2)^3 + 27s^6$, for generic values of $\alpha$.

For generic values of $\alpha$, the Mordell–Weil group of the Weierstrass equations (63) has rank 0; for special values of $\alpha$, the Mordell–Weil rank is enhanced, becoming 1, or the $ADE$ rank is enhanced to 18. For these special values of $\alpha$, the K3 surface (63) becomes attractive.

### 3.2.2 Enhancement to models with $U(1)$

We particularly consider the case $\alpha = 0$. For this case, the equation (63) becomes the following Weierstrass form:

\[
y^2 = x^3 - 3t^2 (t^6 + 24s^6) x + 2 (t^{12} + 36t^6 s^6 + 216s^{12}).
\]

(65)

It can be confirmed that the Weierstrass equation (65) has the following sections:

\[
[X : Y : Z] = [t^4 : 12\sqrt{3} s^6 : 1],
\]

\[
[t^4 : -12\sqrt{3} s^6 : 1].
\]

(66)

Sections (66) are torsion sections, and they generate the group $\mathbb{Z}_3$. The torsion part of the Mordell–Weil group of an elliptic K3 surface with $ADE$ type $A_{17}$ is either 0 or $\mathbb{Z}_3$ [57]. (See No.2787 in Table 1 in [57].) Therefore, we conclude that $\mathbb{Z}_3$ group generated by torsion sections (66) gives the whole torsion part of the Mordell–Weil group of the K3 surface (65).

Some computation shows that, in addition to the torsion sections (66), the K3 surface (65) admits another section:

\[
[X : Y : Z] = [t^4 + 12ts^3 : 12\sqrt{3} (t^3 s^3 + s^6) : 1].
\]

(67)
Section 67 does not belong to the torsion part $\mathbb{Z}_3$. This shows that the Mordell–Weil group of the K3 surface (65) has a free part. Thus, we conclude that the K3 surface (65) has the Mordell–Weil rank 1.

The Weierstrass equation (65) gives an attractive K3 surface with the $ADE$ type $A_{17}$ with the Mordell–Weil group isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_3$.

We find that the equation (65) describes the fibration No.4 of the attractive K3 surface $X_3$ in Table 2 [41]. F-theory compactification on the K3 elliptic fibration (65) times a K3 surface has a $U(1)$ gauge field.

We find via the Kneser-Nishiyama method that the attractive K3 surfaces that admit an elliptic fibration with a section with the $ADE$ type $A_{17}$ include: attractive K3 surfaces, the intersection matrices of the transcendental lattices of which are

\[ \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 8 & 1 \\ 1 & 2 \end{pmatrix}, \text{ and } \begin{pmatrix} 12 & 0 \\ 0 & 2 \end{pmatrix}, \]

in addition to the attractive K3 surface $X_3$.

We show that the Weierstrass equation (63) with

\[ \alpha = 5 \quad (68) \]

gives the attractive K3 surface whose transcendental lattice has the intersection matrix \( \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \). The general Weierstrass form of elliptic fibration with the $ADE$ type $A_{17}$ of the attractive K3 surface whose transcendental lattice has the intersection matrix \( \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \) is given in [58] as:

\[ y^2 + (t^2 + 5) xy + y = x^3. \quad (69) \]

The general Weierstrass form (69) transforms into the following Weierstrass form:

\[ y^2 = x^3 - \frac{1}{2} (t^2 + 5 s^2) \left[ \frac{1}{24} (t^2 + 5 s^2)^3 - s^6 \right] x + \frac{2}{12^3} (t^2 + 5 s^2)^6 - \frac{1}{24} (t^2 + 5 s^2)^3 s^6 + \frac{s^{12}}{4} \quad (70) \]

Under the following rescaling

\[ x \rightarrow \frac{x}{(\sqrt{12})^2} \quad (71) \]
\[ y \rightarrow \frac{y}{(\sqrt{12})^3} \]

the Weierstrass form (70) further transforms into another Weierstrass form as

\[ y^2 = x^3 - 3 (t^2 + 5 s^2) \left[ (t^2 + 5 s^2)^3 - 24 s^6 \right] x + 2 \left[ (t^2 + 5 s^2)^6 - 36 (t^2 + 5 s^2)^3 s^6 + 216 s^{12} \right]. \quad (72) \]
The Weierstrass equation (72) corresponds to the following base change of the rational elliptic surface $X_{[9,1,1,1]}$ (61):

$$
\begin{align*}
t & \to -(t^2 + 5s^2) \\
s & \to s^2.
\end{align*}
$$

When we replace $t$ with $it$ and $\alpha$ with $-\alpha$, $-(t^2 + \alpha s^2)$ is replaced with $t^2 + \alpha s^2$. Therefore, the minus sign of $-(t^2 + 5s^2)$ in the base change (73) may be replaced by a positive sign. Thus, we deduce that the K3 surface (63) becomes an attractive K3 surface whose transcendental lattice has the intersection matrix $\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$ when $\alpha$ takes the value 5.

The Mordell–Weil group of the elliptic fibration (69) is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_3$ (58). Thus, F-theory compactification on the K3 elliptic fibration (72) times a K3 surface has a $U(1)$ gauge field.

### 3.2.3 Enhancement to rank 18

We find from the equations (63) and (64) that the $ADE$ type is enhanced to $A_{17}A_1$ when $\alpha = -3$. The type $I_{18}$ fiber is at $[t : s] = [1 : 0]$, and the type $I_2$ fiber is at $[t : s] = [0 : 1]$. The K3 surface (63) becomes attractive in this situation. According to Table 1 in (57), the Mordell–Weil group of an attractive K3 surface with $ADE$ type $A_{17}A_1$ is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z}_3$. K3 surface (63) always has torsion sections:

$$
[X : Y : Z] = \left[ (t^2 + \alpha s^2)^2 : 12\sqrt{3} s^6 : 1 \right],
\left[ (t^2 + \alpha s^2)^2 : -12\sqrt{3} s^6 : 1 \right].
$$

The torsion sections (74) generate the group $\mathbb{Z}_3$. Thus, we conclude that the Mordell–Weil group of the attractive K3 surface (63) with $\alpha = -3$ is isomorphic to $\mathbb{Z}_3$. Therefore, the K3 surface (63) with $\alpha = -3$ has the $ADE$ type $A_{17}A_1$ with the Mordell–Weil group $\mathbb{Z}_3$. We find from Table 2 in (52) that the K3 surface (63) with $\alpha = -3$ is the attractive K3 surface $X_4$. (No.111 in Table 2 in (52)) Therefore, K3 surface in the moduli (63) becomes the attractive K3 surface $X_4$ at $\alpha = -3$. The Weierstrass equation (63) with $\alpha = -3$ describes the fibration No.6 in Table 5.

### 4 Gauge symmetries in F-theory compactifications

#### 4.1 Gauge symmetries in F-theory models on special points in the moduli of K3 surfaces with $ADE$ types $E_7D_{10}$ and $A_{17}$

In section 3 we discussed special points in the moduli of K3 surfaces with $ADE$ types $E_7D_{11}$ and $A_{17}$, at which the Picard number is enhanced to 20 and the K3 surfaces become attractive. We consider F-theory compactifications on these attractive K3 surfaces times a K3 surface.
The $ADE$ types and Mordell–Weil groups of these attractive K3 surfaces were determined in section 3. The global structures of the gauge groups in F-theory compactifications follow from these results. We summarize the structures of the gauge symmetries in Tables 7 and 8.

| Weierstrass equation | Complex Str. | Mordell-Weil group | Gauge group |
|----------------------|--------------|-------------------|-------------|
| equation (46)        | $X_3$        | $\mathbb{Z} \oplus \mathbb{Z}_2$ | $E_7 \times SO(20)/\mathbb{Z}_2 \times U(1)$ |
| equation (42) with $(\alpha, \beta) = (-9, -30)$ | (4 0) (0 2) | $\mathbb{Z} \oplus \mathbb{Z}_2$ | $E_7 \times SO(20)/\mathbb{Z}_2 \times U(1)$ |
| equation (44) with $24\gamma + 5\delta^2 = 0$ | (4 0) (0 2) | 0 | $E_7 \times SO(22)$ |
| equation (58)        | $X_4$        | 0                 | $E_8 \times SO(20)$ |
| equation (42) with $(\alpha, \beta) = (1, 6\sqrt{3})$ | $X_4$ | $\mathbb{Z}_2$ | $E_7 \times SO(20) \times SU(2)/\mathbb{Z}_2$ |

Table 7: Some attractive K3 surfaces that appear at special enhanced points in the moduli of K3 surfaces with $ADE$ type $E_7D_{10}$, and the structures of the gauge symmetries in F-theory compactifications on these attractive K3 surfaces times a K3 surface.

### 4.2 Cancellation of anomaly

In section 4.1, we considered F-theory compactified on spaces which are built as the direct products of K3 surfaces. Such compactifications give a four-dimensional theory with $N = 2$ supersymmetry. As we stated in section 3.1.1 it can be deduced from the anomaly cancellation condition that 7-branes are wrapped on K3 surfaces, and there are 24 7-branes in F-theory on $K3 \times K3$. We determined the configurations of the singular fibers of the K3 surfaces with the $ADE$ type $E_7D_{10}$ from the anomaly cancellation condition in section 3.1.1. We confirm that the anomaly cancellation condition is satisfied for F-theory compactifications on K3 surfaces with the $ADE$ type $A_{17}$ (63) times a K3. We determined in section 3.2.1 that the K3 surface with the $ADE$ type $A_{17}$ (63) has 1 type $I_{18}$ and 6 type $I_1$ fibers. According to Table 6, the number of associated 7-branes is 24. This satisfies the anomaly cancellation condition. As we saw in section 3, the K3 surface becomes attractive at special points in the moduli of K3 surfaces with the $ADE$ types $E_7D_{10}$ and $A_{17}$. An argument similar to that stated above shows that the anomaly cancellation condition is satisfied at these points.

We also discuss the cancellation of the tadpole with 4-form flux turned on $[60]$ at special points at which K3 surfaces become attractive. By including 4-form flux $[61, 60, 62, 63]$,
| Weierstrass equation | Complex Str. | Mordell-Weil group | Gauge group |
|----------------------|--------------|--------------------|-------------|
| equation (65)        | $X_3$        | $\mathbb{Z} \oplus \mathbb{Z}_3$ | $SU(18)/\mathbb{Z}_3 \times U(1)$ |
| equation (63)        | (4 0)        | $\mathbb{Z} \oplus \mathbb{Z}_3$ | $SU(18)/\mathbb{Z}_3 \times U(1)$ |
| with $\alpha = 5$    | (0 2)        |                    |             |
| equation (63)        | $X_4$        | $\mathbb{Z}_3$     | $SU(18) \times SU(2)/\mathbb{Z}_3$ |
| with $\alpha = -3$   |              |                    |             |

Table 8: Some attractive K3 surfaces that appear at special enhanced points in the moduli of K3 surfaces with $ADE$ type $A_{17}$, and the structures of the gauge symmetries in F-theory compactifications on these attractive K3 surfaces times a K3 surface.

F-theory compactifications on the direct products of K3 surfaces become four-dimensional theory with $N = 1$ supersymmetry. We saw in section 3 that the attractive K3 surfaces $X_3$, $X_4$ and the attractive K3 surfaces, the intersection matrices of the transcendental lattices of which are

$$
\begin{pmatrix}
4 & 0 \\
0 & 2
\end{pmatrix}, \quad
\begin{pmatrix}
6 & 0 \\
0 & 2
\end{pmatrix}, \quad
\begin{pmatrix}
4 & 1 \\
1 & 2
\end{pmatrix}, \quad
\begin{pmatrix}
8 & 1 \\
1 & 2
\end{pmatrix}, \quad
\begin{pmatrix}
12 & 0 \\
0 & 2
\end{pmatrix},
$$

appear in the moduli of K3 surfaces with the $ADE$ types $E_7 D_{10}$ and $A_{17}$. We deduce from Table 1 in [64] and Table 2 in [17] that for these attractive K3 surfaces, when they are paired with some attractive K3 surfaces with appropriate complex structures, the tadpole can be cancelled in F-theory compactifications on these attractive K3 surfaces times such appropriately chosen attractive K3 surfaces. Discussions of the cancellation of the tadpole in F-theory flux compactifications on $K3 \times K3$ can be found in [18, 30].

5 Conclusion

We constructed the moduli of K3 surfaces with the $ADE$ types $E_7 D_{10}$ and $A_{17}$. The Picard number is enhanced to 20 at special points in these moduli; the K3 surfaces become attractive at such points. At these points, either the Mordell–Weil rank increases, or the non-Abelian gauge symmetry on the 7-branes in F-theory compactification is enhanced. We determined some of the special points at which K3 surfaces become attractive in the moduli of K3 surfaces with the $ADE$ types $E_7 D_{10}$ and $A_{17}$. We also studied the gauge groups in F-theory compactifications on the attractive K3 surfaces, which correspond to the special points in the moduli times a K3 surface. A $U(1)$ gauge symmetry arises in F-theory compactifications at the points in the moduli where the Mordell–Weil rank is enhanced.
Acknowledgments

We would like to thank Taro Tani for discussions. We are also grateful to the referee for improving this manuscript. YK is partially supported by Grant-in-Aid for Scientific Research #16K05337 from the Ministry of Education, Culture, Sports, Science and Technology of Japan. SM is supported by Grant-in-Aid for Scientific Research #16K05337 from the Ministry of Education, Culture, Sports, Science and Technology of Japan.

References

[1] C. Vafa, “Evidence for F-theory”, Nucl. Phys. B 469 (1996) 403 [arXiv:hep-th/9602022].
[2] D. R. Morrison and C. Vafa, “Compactifications of F-theory on Calabi-Yau threefolds. 1”, Nucl. Phys. B 473 (1996) 74 [arXiv:hep-th/9602114].
[3] D. R. Morrison and C. Vafa, “Compactifications of F-theory on Calabi-Yau threefolds. 2”, Nucl. Phys. B 476 (1996) 437 [arXiv:hep-th/9603161].
[4] R. Donagi and M. Wijnholt, “Model Building with F-Theory,” Adv. Theor. Math. Phys. 15, 1237 (2011) [arXiv:0802.2969 [hep-th]].
[5] C. Beasley, J. J. Heckman and C. Vafa, “GUTs and Exceptional Branes in F-theory - I,” JHEP 0901, 058 (2009) [arXiv:0802.3391 [hep-th]].
[6] C. Beasley, J. J. Heckman and C. Vafa, “GUTs and Exceptional Branes in F-theory - II: Experimental Predictions,” JHEP 0901, 059 (2009) [arXiv:0806.0102 [hep-th]].
[7] R. Donagi and M. Wijnholt, “Breaking GUT Groups in F-Theory,” Adv. Theor. Math. Phys. 15, 1523 (2011) [arXiv:0808.2223 [hep-th]].
[8] R. Friedman, J. Morgan and E. Witten, “Vector bundles and F theory”, Commun. Math. Phys. 187 (1997) 679–743 [arXiv:hep-th/9701162].
[9] K. Oguiso and T. Shioda, “The Mordell-Weil lattice of a rational elliptic surface”, Comment. Math. Univ. St. Pauli 40 (1991) 83.
[10] T. Shioda, “On elliptic modular surfaces”, J. Math. Soc. Japan 24 (1972), 20–59.
[11] J. Tate, “Algebraic cycles and poles of zeta functions”, in Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963), Harper & Row (1965), 93–110.
[12] J. Tate, “On the conjectures of Birch and Swinnerton-Dyer and a geometric analog”, Séminaire Bourbaki 9 (1964–1966), Exposé no. 306, 415–440.
[13] G. W. Moore, “Les Houches lectures on strings and arithmetic”, [arXiv: hep-th/0401049].
[14] S. Mizoguchi, “F-theory Family Unification,” JHEP 1407, 018 (2014) [arXiv:1403.7066 [hep-th]].
[15] I. I. Piatetski-Shapiro and I. R. Shafarevich, “A Torelli theorem for algebraic surfaces of type K3”, Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971), 530–572.
[16] T. Shioda and H. Inose, “On Singular K3 surfaces”, in W. L. Jr. Baily and T. Shioda (eds.), Complex analysis and algebraic geometry, Iwanami Shoten, Tokyo (1977), 119–136.

[17] A. P. Braun, Y. Kimura and T. Watari, “The Noether-Lefschetz problem and gauge-group-resolved landscapes: F-theory on $K3 \times K3$ as a test case”, *JHEP* 04 (2014) 050 [arXiv:1401.5908 [hep-th]].

[18] Y. Kimura, “Gauge Groups and Matter Fields on Some Models of F-theory without Section”, *JHEP* 03 (2016) 042 [arXiv:1511.06912 [hep-th]].

[19] K.-I. Nishiyama, “The Jacobian fibrations on some K3 surfaces and their Mordell-Weil groups”, *Japan. J. Math.* 22 (1996), 293–347.

[20] V. Braun and D. R. Morrison, “F-theory on Genus-One Fibrations”, *JHEP* 08 (2014) 132 [arXiv:1401.7844 [hep-th]].

[21] D. R. Morrison and W. Taylor, “Sections, multisections, and $U(1)$ fields in F-theory”, *J. Singularities* 15 (2016) 126–149 [arXiv:1404.1527 [hep-th]].

[22] L. B. Anderson, I. Garcia-Etxebarria, T. W. Grimm and J. Keitel, “Physics of F-theory compactifications without section”, *JHEP* 12 (2014) 156 [arXiv:1406.5180 [hep-th]].

[23] D. Klevers, D. K. Mayorga Pena, P. K. Oehlmann, H. Piragua and J. Reuter, “F-Theory on all Toric Hypersurface Fibrations and its Higgs Branches”, *JHEP* 01 (2015) 142 [arXiv:1408.4808 [hep-th]].

[24] I. Garcia-Etxebarria, T. W. Grimm and J. Keitel, “Yukawas and discrete symmetries in F-theory compactifications without section”, *JHEP* 11 (2014) 125 [arXiv:1408.6448 [hep-th]].

[25] C. Mayrhofer, E. Palti, O. Till and T. Weigand, “Discrete Gauge Symmetries by Higgsing in four-dimensional F-Theory Compactifications”, *JHEP* 12 (2014) 068 [arXiv:1408.6831 [hep-th]].

[26] C. Mayrhofer, E. Palti, O. Till and T. Weigand, “On Discrete Symmetries and Torsion Homology in F-Theory”, *JHEP* 06 (2015) 029 [arXiv:1410.7814 [hep-th]].

[27] V. Braun, T. W. Grimm and J. Keitel, “Complete Intersection Fibers in F-Theory”, *JHEP* 03 (2015) 125 [arXiv:1411.2615 [hep-th]].

[28] M. Cvetič, R. Donagi, D. Klevers, H. Piragua and M. Poretschkin, “F-theory vacua with $Z_3$ gauge symmetry”, *Nucl. Phys.* B898 (2015) 736–750 [arXiv:1502.06953 [hep-th]].

[29] L. Lin, C. Mayrhofer, O. Till and T. Weigand, “Fluxes in F-theory Compactifications on Genus-One Fibrations”, *JHEP* 01 (2016) 098 [arXiv:1508.00162 [hep-th]].

[30] Y. Kimura, “Gauge symmetries and matter fields in F-theory models without section-compactifications on double cover and Fermat quartic K3 constructions times K3”, [arXiv:1603.03212 [hep-th]].
[31] Y. Kimura, “Gauge groups and matter spectra in F-theory compactifications on genus-one fibered Calabi-Yau 4-folds without section - hypersurface and double cover constructions”, arXiv:1607.02978 [hep-th].

[32] Y. Kimura, “Discrete Gauge Groups in F-theory Models on Genus-One Fibered Calabi-Yau 4-folds without Section”, JHEP 04 (2017) 168 [arXiv:1608.07219 [hep-th]].

[33] D. R. Morrison and D. S. Park, “F-Theory and the Mordell-Weil Group of Elliptically-Fibered Calabi-Yau Threefolds”, JHEP 10 (2012) 128 [arXiv:1208.2695 [hep-th]].

[34] C. Mayrhofer, E. Palti and T. Weigand, “U(1) symmetries in F-theory GUTs with multiple sections”, JHEP 03 (2013) 098 [arXiv:1211.6742 [hep-th]].

[35] V. Braun, T. W. Grimm and J. Keitel, “New Global F-theory GUTs with U(1) symmetries”, JHEP 09 (2013) 154 [arXiv:1302.1854 [hep-th]].

[36] J. Borchmann, C. Mayrhofer, E. Palti and T. Weigand, “Elliptic fibrations for SU(5) × U(1) × U(1) F-theory vacua”, Phys. Rev. D88 (2013) no.4 046005 [arXiv:1303.5054 [hep-th]].

[37] M. Cvetič, D. Klevers and H. Piragua, “F-Theory Compactifications with Multiple U(1)-Factors: Constructing Elliptic Fibrations with Rational Sections”, JHEP 06 (2013) 067 [arXiv:1303.6970 [hep-th]].

[38] C. Lawrie, S. Schäfer-Nameki and J.-M. Wong, “F-theory and All Things Rational: Surveying U(1) Symmetries with Rational Sections”, JHEP 09 (2015) 144 [arXiv:1504.05593 [hep-th]].

[39] T. Shioda, “K3 surfaces and sphere packings”, J. Math. Soc. Japan 60 (2008) 1083–1105.

[40] M. Schütt, “Arithmetic of a singular K3 surface”, Michigan Math. J. 56 (2008) 513–527 [arXiv:math/0605560 [math.NT]].

[41] K. Utsumi, “Jacobian fibrations on the singular K3 surface of discriminant 3”, J. Math. Soc. Japan 68 (2016) 1133–1146 [arXiv:1405.3577 [math.AG]].

[42] Y. Kimura, “Structure of stable degeneration of K3 surfaces into pairs of rational elliptic surfaces”, JHEP 03 (2018) 045 [arXiv:1710.04984 [hep-th]].

[43] M. Bershadsky, K. A. Intriligator, S. Kachru, D. R. Morrison, V. Sadov and C. Vafa, “Geometric singularities and enhanced gauge symmetries”, Nucl. Phys. B 481 (1996) 215 [arXiv:hep-th/9605200].

[44] K. Kodaira, “On compact analytic surfaces II”, Ann. of Math. 77 (1963), 563–626.

[45] K. Kodaira, “On compact analytic surfaces III”, Ann. of Math. 78 (1963), 1–40.

[46] A. Néron, “Modèles minimaux des variétés abéliennes sur les corps locaux et globaux”, Publications mathématiques de l’IHÉS 21 (1964).

[47] J. Tate, “Algorithm for determining the type of a singular fiber in an elliptic pencil”, in Modular Functions of One Variable IV, Springer, Berlin (1975), 33–52.
[48] S. Mukai, “On the moduli space of bundles on K3 surfaces. I”, in Vector bundles on algebraic varieties, Tata Institute of Fundamental Research, Bombay, 1987, 341–413.
[49] S.-M. Belcastro, “Picard lattices of families of K3 surfaces”, Ph.D. Thesis at University of Michigan (1997).
[50] J.-H. Keum, “A note on elliptic K3 surfaces”, Trans. Amer. Math. Soc. 352 (2000) 2077–2086.
[51] T. Shioda, “Kummer sandwich theorem of certain elliptic K3 surfaces”, Proc. Japan Acad. 82 Ser. A (2006) 137–140.
[52] I. Shimada and D.-Q. Zhang, “Classification of extremal elliptic K3 surfaces and fundamental groups of open K3 surfaces”, Nagoya Math. J. 161 (2001), 23–54, arXiv:math/0007171.
[53] M. Schütt, “Elliptic fibrations of some extremal K3 surfaces”, Rocky Mount. J. Math. 37 (2007) 609–652 [arXiv:math/0412049 [math.AG]].
[54] R. Miranda and U. Persson, “On extremal rational elliptic surfaces”, Math. Z. 193 (1986) 537–558.
[55] D. R. Morrison and D. S. Park, “Tall sections from non-minimal transformations”, JHEP 10 (2016) 033 [arXiv:1606.07444 [hep-th]].
[56] V. V. Nikulin, “Integral symmetric bilinear forms and some of their applications”, Math. USSR-Izv. 14 (1980) 103–167.
[57] I. Shimada, “On elliptic K3 surfaces”, Michigan Math. J. 47 (2000) 423–446 [arXiv:math/0505140 [math.AG]].
[58] M. J. Bertin and O. Lecacheux, “Elliptic fibrations on the modular surface associated to \(\Gamma_1(8)\)”, in Arithmetic and geometry of K3 surfaces and Calabi-Yau threefolds, 153–199, Fields Institute Commun. 67, Springer (2013) [arXiv:1105.6312 [math.AG]].
[59] M. J. Bertin, A. Garbagnati, R. Hortsch, O. Lecacheux, M. Mase, C. Salgado and U. Whitcher, “Classifications of elliptic fibrations of a singular K3 surface”, in Women in Numbers Europe, 17–49, Association for Women in Mathematics Series, Springer (2015) [arXiv:1501.07484 [math.AG]].
[60] S. Sethi, C. Vafa and E. Witten, “Constraints on low dimensional string compactifications”, Nucl. Phys. B 480 (1996) 213–224, [arXiv:hep-th/9606122].
[61] K. Becker and M. Becker, “M theory on eight manifolds”, Nucl. Phys. B477 (1996) 155–167 [arXiv:hep-th/9605053].
[62] E. Witten, “On flux quantization in M theory and the effective action”, J. Geom. Phys. 22 (1997) 1–13 [arXiv:hep-th/9609122].
[63] K. Dasgupta, G. Rajesh and S. Sethi, “M theory, orientifolds and G-flux”, JHEP 08 (1999) 023 [arXiv:hep-th/9908088].
[64] P. S. Aspinwall and R. Kallosh, “Fixing all moduli for M-theory on K3×K3”, JHEP 10 (2005) 001 [arXiv:hep-th/0506014].

27