HOMOGENEOUS ASYMPTOTIC LIMITS OF UNIFORM AVERAGES ON FUCHSIAN GROUPS

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Abstract. We show that averages on geometrically finite Fuchsian groups, when embedded via a representation into a space of matrices, have a homogeneous asymptotic limit under appropriate scaling. This generalizes some of the results of F. Maucourant to subgroups of infinite co-volume.

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1. Introduction

Let $G$ be a topological group and $\Gamma < G$ a discrete subgroup. The spectral and asymptotic properties of matrix elements of $\Gamma$ in various representations of $G$ are related to many different problems in mathematics, from expander graphs to Diophantine approximation. See for example [5] and [2] for a review.

One method of studying the asymptotic properties of matrix elements in linear groups was initiated by Maucourant [7], who considered the following question. Let $G$ be a non-compact connected semisimple Lie group with finite center and let $\rho: G \to \text{GL}(V)$ be a faithful representation of $G$ in a finite dimensional real vector space $V$. Let $d_g$ denote a Haar measure on $G$. Let $f$ be a compactly supported continuous function on $\text{End}(V)$.

Maucourant showed that for some rational number $d$, and a positive integer $e$ between 0 and $\text{rank}_R G - 1$, the measures

$$
\mu_{G,T}(f) = \frac{1}{T^d \log(T)^e} \int_G f\left(\frac{\rho(g)}{T}\right) \, dg, \quad f \in C_c(\text{End}(V)).
$$

on $\text{End}(V)$ converge weakly as $T$ tends to infinity. Moreover, $e$ and $d$ can be described explicitly, as is the limit measure $\mu_\infty$. Moreover, $\mu_\infty$ is homogeneous of degree $d$ in the sense that for any Borel set $E \subseteq \text{End}(V)$ we have $\mu_\infty(t E) = t^d \mu_\infty(E)$.

Let $\Gamma$ be an irreducible lattice in $G$, and for $T > 0$ consider the measure:

$$
\mu_{\Gamma,T} = \frac{1}{T^d \log(T)^e} \sum_{\gamma \in \Gamma} \delta_{\rho(\gamma)}.
$$

It is possible to compare $\mu_{\Gamma,T}$ with $\mu_{G,T}$ (see [7, section 3]) and get that as $T \to \infty$ the measures $\mu_{\Gamma,T}$ converge weakly to

$$
\lim_{T \to \infty} \mu_{\Gamma,T} = \frac{1}{\text{Vol}(\Gamma \backslash G)} \mu_\infty,
$$

where Vol $(\Gamma \backslash G)$ is the volume of a fundamental domain of $\Gamma$ with respect to the chosen Haar measure on $G$.

In this work we wish to study averaging operators similar to $\mu_{\Gamma,T}$ for $G = \text{SL}_2(\mathbb{R})$ and discrete subgroups $\Gamma < \text{SL}_2(\mathbb{R})$ which are not necessarily of finite co-volume. In this case it is not possible to compare the measures $\mu_{\Gamma,T}$ and $\mu_{\text{SL}_2(\mathbb{R}),T}$ for a general discrete subgroup $\Gamma$. The problem is that the number of points in $\Gamma$ might be small and in this case the normalization $T^{-d}$ is too large. For instance, denote
We then identify the limit set \(\Lambda(\Gamma)\) of \(\Gamma\) via fractional linear transformations in the upper half plane model. The stabilizer of \(F\) for the group \(\rho\) study the asymptotic distribution of the points \(\rho(\Gamma)\) the normalization should be proportional to the asymptotics of \(|\rho(\Gamma)\cap B_T|\). By a result of Lax and Phillips [6, theorem 1], if \(\Gamma\) is geometrically finite there is a value \(0 \leq \delta \leq 1\), called the critical exponent of \(\Gamma\), such that as \(T\) tends to infinity,

\[|\rho(\Gamma)\cap B_T| \asymp T^{\delta d}
\]

We can then consider the measures

\[\mu_{\Gamma,T} = \frac{1}{T^{\delta d}} \sum_{\gamma \in \Gamma} \delta_{\gamma(\cdot)}, \quad T > 0.
\]

In this paper we prove that if \(\delta > \frac{1}{2}\), the measures \(\mu_{\Gamma,T}\) defined above converge weakly to a nonzero measure \(\mu_\Gamma\) on \(\text{End}(V)\) as \(T \to \infty\). The measure \(\mu_\Gamma\) is homogeneous of degree \(\delta d\) and can be written explicitly in terms of \(\rho\) and the Patterson-Sullivan measure on the limit set of \(\Gamma\).

A study of the angular distribution of orbits in \(\mathbb{R}^2\) of finitely generated subgroup of \(\text{SL}_2(\mathbb{R})\) was carried by Maucourant and Schapira [3] using ergodic properties of the horocycle flow. The present work gives an alternative proof for some of their results corresponding to a similar normalization as the one we are considering.

Our proof relies on results of Bourgain, Kontorovich, and Sarnak [1]. The main idea is to decompose the sum (1.1) into averages over annuli depending on the norm of the elements in \(\rho(\gamma)\). Each individual average over an annulus can then be analyzed by decomposing the test function into harmonics according to its left and right SO\(_2(\mathbb{R})\)-type. This allows us to use the estimates of [1] to get the limit measure.

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### 2. The Main Results

Let \(\Gamma < \text{SL}_2(\mathbb{R})\) be a discrete subgroup containing \(-1\) such that the image \(\overline{\Gamma}\) of \(\Gamma\) in \(\text{PSL}_2(\mathbb{R})\) is a non-elementary geometrically finite Fuchsian group of the second kind with critical exponent \(\delta > \frac{1}{2}\). Let \(\mathbb{H}\) be the hyperbolic plane and let \(\overline{\mathbb{H}} = \mathbb{H} \cup \partial \mathbb{H}\) be its compactification with \(\partial \mathbb{H}\) the Gromov boundary. In the Poincaré disk model \(\partial \overline{\mathbb{H}}\) can be identified with the unit circle and in the upper half plane model with \(\mathbb{R} \cup \{\infty\}\).

For the group \(\Gamma\) and any two points \(x, y \in \overline{\mathbb{H}}\), Patterson [9] (see also [10]) defined a measure \(\mu_{x,y}\) on the limit set \(\Lambda(\Gamma)\) of \(\Gamma\). This limit set consists of all points in \(\partial \overline{\mathbb{H}}\) that lie in the closure of one (equivalently, all) orbit of \(\Gamma\) on \(\overline{\mathbb{H}}\). Since \(\Gamma\) has exponent \(\delta > \frac{1}{2}\), the measure \(\mu_{x,y}\) is obtained as the weak limit:

\[\mu_{x,y} = \lim_{s \to +\infty} \frac{1}{s^{\delta}} \mu_{s,x,y},
\]

of measures \(\mu_{s,x,y}\) on \(\overline{\mathbb{H}} = \mathbb{H} \cup \partial \mathbb{H}\) given by

\[\mu_{s,x,y} = \frac{1}{\sum_{\gamma \in \Gamma} e^{-s d(x,\gamma y)}} \sum_{\gamma \in \Gamma} e^{-s d(x,\gamma y)} \delta_{\gamma y},
\]

where \(d\) is the hyperbolic metric on \(\mathbb{H}\). Recall that \(\mathbb{H}\) carries a transitive action of \(\text{SL}_2(\mathbb{R})\) realized via fractional linear transformations in the upper half plane model. The stabilizer of \(i\) is \(K := \text{SO}_2(\mathbb{R})\).

We then identify \(\mathbb{H}\) with \(\text{SL}_2(\mathbb{R})/\text{SO}_2(\mathbb{R})\). Let \(o\) denote the point stabilized by \(K\).

Define:

\[K = \left\{ k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \middle| 0 \leq \theta < 2\pi \right\},
\]

\[N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in \mathbb{R} \right\},
\]

\[A^+ = \left\{ a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \middle| t \geq 0 \right\}.
\]
Recall the Cartan decomposition $\text{SL}_2(\mathbb{R}) = KA^+K$. Given $g = k_1a_1k_2$ for $g \in \text{SL}_2(\mathbb{R})$, the parameter $t$ is uniquely determined, and for $t \neq 0$ the elements $k_1, k_2$ are determined up to multiplication by $\pm I$, where $I \in \text{SL}_2(\mathbb{R})$ is the identity.

Let $\rho : \text{SL}_2(\mathbb{R}) \to \text{GL}(V)$ be a faithful representation on a finite dimensional real vector space $V$. Let

$$(V, \rho) = \bigoplus_i (V_k, \rho_k)$$

be the decomposition of $V$ into a direct sum of irreducible representations, where $\rho_k$ is the irreducible representation of highest weight $k_i$. Let $k$ be the highest weight occurring in $V$, so that $k = \max_i k_i$. We denote by $m$ the multiplicity of $V_k$ in $V$. Let $(\cdot, \cdot) : V \to \mathbb{R}$ be an inner product on $V$ which is $\rho(K)$-invariant and such that there exists an orthonormal basis of $V$ with respect to $(\cdot, \cdot)$ in which $\rho(A)$ is diagonal. Such an inner product always exists and for any $t \geq 0$ any such inner product satisfies (see 3.1.3 for details)

$$\|\rho(a)\|_{op} = \sup_{0 \neq v \in V} \frac{\|\rho(a)v\|}{\|v\|} = e^{\frac{2\pi t}{m}}.$$ 

We fix such an inner product and endow $\text{End}(V)$ with the operator norm, denoted $\|\| : \text{End}(V) \to \mathbb{R}$. Let $P_k : V \to V$ be the orthogonal projection onto the subspace of vectors in $V$ of weight $k$. That is, $P_k|_{V_k} = 0$ if $k_1 \neq k$, and on each copy of $V_k$ in $V$, $P_k|_{V_k}$ is the projection onto the one dimensional space consisting of highest weight vectors. For example, if $\rho$ is the standard representation of $\text{SL}_2(\mathbb{R})$, which is an irreducible representation of highest weight 1,

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

Let $\delta > \frac{1}{2}$ and define $V_\Gamma = 2\pi^2 \frac{\Gamma(\delta - 1/2)}{\Gamma(\delta + 1)}$, where $\Gamma(s)$ for $s \in \mathbb{C}$ on the right hand side is the value Gamma function at $s$. We are now ready to state our main result.

**Theorem 2.1.** Let $\Gamma, \rho, k,$ and $P_k$ be as above. There exists a measure $\mu_\Gamma$ on $\text{End}(V)$ such that for any compactly supported continuous function $f : \text{End}(V) \to \mathbb{C}$,

$$\lim_{T \to \infty} \frac{1}{T^2} \sum_{\gamma \in \Gamma} f\left(\frac{\rho(\gamma)}{T}\right) = \int_{\text{End}(V)} f \, d\mu_\Gamma.$$

That is, the measures

$$\mu_{\Gamma, \tau} = \frac{1}{T^2} \sum_{\gamma \in \Gamma} \frac{\delta}{T^2} \frac{\delta}{\mu_\Gamma},$$

converge to $\mu_\Gamma$ in the weak-* topology of $(C_c(\text{End}(V)))^*$. Moreover, $\mu_\Gamma$ is homogeneous of degree $\frac{2\delta}{k}$ and it is given by:

$$\mu_\Gamma(f) = \frac{\delta}{2k} \int_{K_\infty \times [0, \infty)} \int_{K \times [0, \infty) \times K} f(\rho(k_1) \cdot tP_k \cdot \rho(k_2)) t^{\frac{2\delta - 1}{k}} \mu(k_1) \, dt \, d\mu(k_2).$$

where $\mu$ is a symmetric lift of the Patterson-Sullivan measure $\mu_{o,o}$ from $\partial \mathbb{H}^n$ to $K$ (see Remark 2.2).

**Remark 2.2.** We parametrize $K = \text{SO}(2, \mathbb{R})$ by $[0, 2\pi]$ via

$$K = \left\{ k_\theta \mid k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \right\} | 0 \leq \theta \leq 2\pi \}.$$ 

This allows us to identify $\partial \mathbb{H}^2$ with $[0, \pi]$. This identification is consistent with the Cartan coordinates on $\text{SL}_2(\mathbb{R})$, since an elliptic transformation $k_\theta$ acts as a rotation by $\theta$ on hyperbolic space. The Patterson-Sullivan measure is constructed for discrete subgroups of $\text{PSL}_2(\mathbb{R})$. For $g \in \text{PSL}_2(\mathbb{R})$ we define the angles $\theta_1(g), \theta_2(g) \in [0, \pi]$ to be such that $g = \pm k_{\theta_1(g)} a_{t_1} k_{\theta_2(g)}$ for some $t \geq 0$. Then the Patterson-Sullivan measure $\mu_{o,o}$ can be identified with a measure on $[0, \pi]$. We extend $\mu_{o,o}$ to $[0, 2\pi]$ by setting it to be symmetric with respect to $\theta \mapsto \theta + \pi$. We denote the extended measure by $\mu$. This is consistent with the notation of Bourgain, Kontorovich, and Sarnak in [1]. In particular, if $\Gamma$ is a lattice then in this convention the Patterson-Sullivan measure on $K$ has total mass 2 and is proportional to the Lebesgue measure on $[0, 2\pi]$. 


We can also state a normalized version of the main theorem, which can be more useful in certain situations. It is easy to see that the two versions are equivalent.

**Theorem 2.3.** Let $\Gamma, \rho, P_k$ be as in [2,7]. Let $\|,\| : V \to \mathbb{R}$ be the norm on $V$ fixed above. Define:

$$\Gamma_T = \{ \gamma \in \Gamma | \| \rho(\gamma) \| \leq T \}.$$  

There exists a measure $\nu_T$ on $\text{End}(V)$ such that for any continuous $f : \text{End}(V) \to \mathbb{C},$

$$\lim_{T \to \infty} \frac{1}{T^{2s}} \sum_{\gamma \in \Gamma_T} f\left( \frac{\rho(\gamma)}{T} \right) = \int_{\text{End}(V)} f \, d\nu_T.$$

Moreover, $\nu_T$ is given by:

$$\nu_T(f) = \frac{\delta}{2k}|V|^2 \int_{K \times [0,1] \times K} f(\rho(k_1) \cdot tP_k \cdot \rho(k_2)) t^{\frac{s_1}{2} - 1} \, d\mu(k_1) \, dt \, d\mu(k_2).$$

A quantitative estimate for the error term in the theorems above can be given for Hölder continuous functions. Let

$$0 \leq \lambda_0 < \lambda_1 < \cdots < \lambda_N < \frac{1}{4}$$

be the eigenvalues of the Laplacian on $\Gamma \backslash \mathbb{H}$ below $\frac{1}{4}$ (see section 3.2). Write

$$\lambda_j = s_j (1 - s_j)$$

with $s_j > \frac{1}{2}$.

**Theorem 2.4.** (same assumptions as theorems [2,7] and [2,3]) Let $f : \text{End}(V) \to \mathbb{C}$ be Hölder continuous with exponent $\alpha \in (0,1]$ and constant $\|f\|_{\text{Lip}_{\alpha}}$.

1. Assume $f$ has compact support. Then, as $T$ tends to infinity,

$$\frac{1}{T^{2s}} \sum_{\gamma \in \Gamma_T} f\left( \frac{\rho(\gamma)}{T} \right) = \int_{\text{End}(V)} f \, d\mu_T + O\left( \|f\|_{\infty} + \|f\|_{\text{Lip}_{\alpha}} \right) \left( T^{\frac{s_1}{2} - \delta} + T^{\frac{1}{4} - \delta} \right) \log(T),$$

where $\|f\|_{\infty} = \sup_{x \in V} |f(x)|$, and the implied constants depend only on $\Gamma, \rho$ and the norm on $V$.

2. If $f$ is not necessarily of compact support, then, as $T \to \infty$

$$\frac{1}{T^{2s}} \sum_{\gamma \in \Gamma_T} f\left( \frac{\rho(\gamma)}{T} \right) = \int_{\text{End}(V)} f \, d\mu_T + O\left( \|f\|_{\infty,1} + \|f\|_{\text{Lip}_{\alpha}} \right) \left( T^{\frac{s_1}{2} - \delta} + T^{\frac{1}{4} - \delta} \right) \log(T),$$

where $\|f\|_{\infty,1} := \sup_{\|x\| \leq 1} |f(x)|$.

One application of the main theorem involves the study of the distribution of individual matrix elements of the group $\Gamma$. For example, let $\rho : SL_2(\mathbb{R}) \to \text{GL}_2(\mathbb{R})$ be the standard representation and let $\|,\|$ denote the operator norm induced by the Euclidean norm on $\mathbb{R}^2$. This norm is given by

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| = \max \{ |a|, |b| \},$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = k_\theta \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} k_\phi$$

for some $\theta, \phi \in [0,2\pi]$. Let $\Gamma_T$ denote the set of matrices in $\Gamma$ with norm smaller than $T$, where we identify $SL_2(\mathbb{R})$ with its image in $\text{GL}_2(\mathbb{R})$. For $T > 0$, consider the set of values

$$E_T = \left\{ \begin{pmatrix} a \\ c \\ d \end{pmatrix} \in \Gamma_T \right\}.$$

$E_T$ is a subset of the interval $[-1,1]$, and one can ask whether this set of values is equidistributed in $[-1,1]$, i.e., whether the limit

$$\lim_{T \to \infty} \frac{1}{|\Gamma_T|} \sum_{r \in E_T} f(r) = \lim_{T \to \infty} \frac{1}{|\Gamma_T|} \sum_{\gamma \in \Gamma_T} f\left( \frac{a}{T} \right),$$

exists for any continuous function $f : [-1,1] \to \mathbb{C}$. 
**Theorem 2.5.** For any continuous function \( f : [-1, 1] \to \mathbb{C} \)
\[
\lim_{T \to \infty} \frac{1}{|T|} \sum_{\gamma \in \Gamma_T} f \left( \frac{a}{T} \right) = \delta \int_{[0,2\pi] \times [0,1] \times [0,2\pi]} f \left( t \cos \theta_1 \cos \theta_2 \right) t^{2s-1} \, d\mu(\theta_1) \, dt \, d\mu(\theta_2).
\]

Proof. Applying theorem 2.3 to the function
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto f(a)
\]
on \text{End} (\mathbb{R}^2), we see that for any continuous \( f \) as above,
\[
\lim_{T \to \infty} \frac{1}{2^s} \sum_{\gamma \in \Gamma_T} f \left( \frac{a}{T} \right) = \delta \frac{1}{2} \int_{\mathbb{C}} f \left( k_{\theta_1} \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} k_{\theta_2} \right) t^{2s-1} \, d\mu(\theta_1) \, dt \, d\mu(\theta_2),
\]
where \( C = [0,2\pi] \times [0,1] \times [0,2\pi] \). Since the matrix elements are given by
\[
k_{\theta_1} \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} k_{\theta_2} = \begin{pmatrix} t \cos \theta_1 \cos \theta_2 & t \cos \theta_1 \sin \theta_2 \\ -t \sin \theta_1 \cos \theta_2 & -t \sin \theta_1 \sin \theta_2 \end{pmatrix},
\]
and the size of \( |\Gamma_T| \) by
\[
\lim_{T \to \infty} \frac{|\Gamma_T|}{2^s} = V_T,
\]
We get,
\[
\lim_{T \to \infty} \frac{1}{|T|} \sum_{\gamma \in \Gamma_T} f \left( \frac{a}{T} \right) = \delta \int_{[0,2\pi] \times [0,1] \times [0,2\pi]} f \left( t \cos \theta_1 \cos \theta_2 \right) t^{2s-1} \, d\mu(\theta_1) \, dt \, d\mu(\theta_2).
\]

Hence, \( \{E_T\} \) is equidistributed in \([-1, 1]\) with respect to the measure induced by the projection from the “limit cone”
\[
C(\Gamma, 1) = \left\{ k_{\theta_1} \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix} k_{\theta_2} \mid \theta_1, \theta_2 \in [0, 2\pi], 0 \leq t \leq 1 \right\}
\]
ono onto the \( a \)-axis. If \( \Gamma \) happens to be a lattice, the distribution is given by:
\[
\lim_{T \to \infty} \frac{1}{|T|} \sum_{\gamma \in \Gamma_T} f \left( \frac{a}{T} \right) = 2 \cdot \int_{[0,2\pi] \times [0,1] \times [0,2\pi]} f \left( t \cos \theta_1 \cos \theta_2 \right) t \, d\theta_1 \, dt \, d\theta_2.
\]

3. **Preliminaries**

This section reviews some of the results needed in the proof of the main theorem and sets up notation. We give a brief review of the sector estimates results of Bourgain, Kontorovich and Sarnak from [4] and recall some basic facts about \( \text{SL}_2(\mathbb{R}) \) and its finite dimensional representations.

3.1. **Finite dimensional representations of \( \text{SL}_2(\mathbb{R}) \).** For proofs of the facts stated here, see chapter II of [4].

Recall that we have defined:
\[
K = \left\{ k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \mid 0 \leq \theta < 2\pi \right\},
\]
\[
N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\},
\]
\[
A^+ = \left\{ a_t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix} \mid t \geq 0 \right\},
\]
We recall the Cartan decomposition for the particular case of \( \text{SL}_2(\mathbb{R}) \).

**Theorem.** (\( \text{KAK} \) decomposition) The map \( K \times A^+ \times K \to \text{SL}_2(\mathbb{R}) \) given by \( (k_\theta, a_t, k_\varphi) \mapsto k_\theta a_t k_\varphi \) is surjective. If \( t \neq 0 \), \( k_1 \) is uniquely determined up to multiplication on the right by \( \pm I \).
Even though the decomposition is not unique, we will call the coordinates \((\theta_1, t, \theta_2)\), with \(\theta_1, \theta_2 \in [0, 2\pi]\) and \(t \geq 0\), the *Cartan coordinates* on \(\text{SL}_2(\mathbb{R})\). We denote by \(g(\theta_1, t, \theta_2)\) the element \(k_0 a t k_0\). Recall that in terms of these coordinates a Haar measure on \(\text{SL}_2(\mathbb{R})\) is given by
\[
\mu(f) = \int_{K \times (0, \infty) \times K} f(k_0 a t k_0) \sinh(t) \, dt \, d\theta_1 \, d\theta_2.
\]

### 3.1.1. Irreducible representations

All finite dimensional irreducible representations of \(\text{SL}_2(\mathbb{R})\) can be obtained as follows. Fix an integer \(n \geq 0\) and let \(V_n\) be the complex vector space of polynomials in \(\mathbb{C}[z_1, z_2]\) which are homogeneous of degree \(n\). Define a representation \(\rho_n\) of \(\text{SL}_2(\mathbb{R})\) by
\[
\rho_n \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) P(z_1, z_2) = P(az_1 + bz_2, cz_1 + dz_2), \quad P \in V_n.
\]

Then \(\dim V_n = n + 1\) and \(\rho_n : \text{SL}_2(\mathbb{R}) \rightarrow \text{GL}(V_n)\) is a continuous representation of \(\text{SL}_2(\mathbb{R})\). In fact, \(\rho_n\) is irreducible and all irreducible finite dimensional representations of \(\text{SL}_2(\mathbb{R})\) are obtained in this way.

The representation \(\rho_n\) is called the representation of \(\text{SL}_2(\mathbb{R})\) of highest weight \(n\). Recall that the Lie algebra \(\mathfrak{sl}_2(\mathbb{R})\) of \(\text{SL}_2(\mathbb{R})\) is spanned by
\[
(3.1) \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]

which satisfy the relations
\[
[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.
\]

A smooth finite dimensional representation \(\rho : \text{SL}_2(\mathbb{R}) \rightarrow \text{GL}(V)\) of \(\text{SL}_2(\mathbb{R})\) induces a representation of \(\mathfrak{sl}_2(\mathbb{R})\) by
\[
\rho(X).v = \frac{d}{dt}
\]

\[\big|_{t=0} (\exp(tX)).v, \quad v \in V, X \in \mathfrak{sl}_2(\mathbb{R}).\]

On \(V_n\), \(\rho(h)\) acts as
\[
\rho(h) P(z_1, z_2) = \frac{d}{dt}
\]

\[\big|_{t=0} \left( \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right) P(z_1, z_2) = \frac{d}{dt}
\]

\[\big|_{t=0} P(e^t z_1, e^{-t} z_2).\]

Therefore, with respect to the basis given by the monomials
\[
\{z_1^n, z_1^{n-1} z_2, \ldots, z_2^n\},
\]

we have:
\[
\rho_n(h) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & n-2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & -n \end{pmatrix}.
\]

Hence, for \(a t \in \text{SL}_2(\mathbb{R})\), \(t \geq 0\), we have
\[
\rho_n(a t) = \begin{pmatrix} e^{\frac{at^2}{4}} & 0 & 0 & 0 \\ 0 & e^{\frac{(a-2)nt}{4}} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & e^{-\frac{at^2}{4}} \end{pmatrix}.
\]

For a general finite dimensional complex representation of \(\text{SL}_2(\mathbb{R})\) we have the following.

**Theorem 3.1.** *(Weyl)* Every finite dimensional complex representation of \(\text{SL}_2(\mathbb{R})\) is completely reducible.
3.1.2. Real finite dimensional representations of $\text{SL}_2(\mathbb{R})$. Let $V$ be a real vector space and $\rho : \text{SL}_2(\mathbb{R}) \to \text{GL}(V)$ a faithful representation. $\rho$ extends uniquely to a representation $\rho_C : \text{SL}_2(\mathbb{R}) \to \text{GL}(V \otimes \mathbb{C})$ of $\text{SL}_2(\mathbb{R})$ on the complexification $V_C := V \otimes \mathbb{C}$ of $V$. By Weyl's theorem $\rho_C$ decomposes to a direct sum of irreducible representations:

$$V_C = \bigoplus_{i=1}^{n} V_i.$$ 

The eigenvalues of $h$ on $V_C$ are real. Therefore, we can find highest weight vectors in each $V_i$ which are in $V$. An elementary argument using the relations shows that in fact each $V_i$ is stable under complex conjugation and $V \cap V_i$ is a real representation of $\text{SL}_2(\mathbb{R})$ with the same highest weight as $V_i$. This shows:

**Proposition 3.2.** Let $\rho : \text{SL}_2(\mathbb{R}) \to \text{GL}(V)$ be a faithful finite dimensional real representation of $\text{SL}_2(\mathbb{R})$ with $V_C = \bigoplus_{i=1}^{n} V_i$, with $V_i$ an irreducible representation of highest weight $k$. Then $V$ decomposes as a direct sum of real irreducible representations with highest weights $k_1, \ldots, k_n$. Let

$$A = \left\{ a_t = \begin{pmatrix} e^{\frac{2\pi i}{k} t} & 0 & 0 & 0 \\ 0 & e^{-\frac{2\pi i}{k} t} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & e^{-\frac{2\pi i}{k} t} \end{pmatrix} \right\} \quad | t \in \mathbb{R} \right\}.$$

There is a basis of $V$ in which $\rho(A)$ is simultaneously diagonalizable with

$$\rho(a_t) = \bigoplus \rho_k(a_t)$$

where,

$$\rho_k(a_t) = \begin{pmatrix} e^{\frac{2\pi i}{k} t} & 0 & 0 & 0 \\ 0 & e^{i\frac{2\pi i}{k} t} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & e^{-\frac{2\pi i}{k} t} \end{pmatrix}.$$

3.1.3. $K$-invariant norms.

**Definition 3.3.** Let $\rho : \text{SL}_2(\mathbb{R}) \to \text{GL}(V)$ be a faithful finite dimensional representation on a real vector space $V$. An inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ on $V$ is called $\rho$-standard if it is $\rho(K)$-invariant and if there exists an orthonormal basis in which $\rho(A)$ is diagonal. If $\|\cdot\| : V \to \mathbb{C}$ is the norm induced by a $\rho$-standard inner product on $V$, $\|\cdot\|$ is called a $\rho$-standard norm on $V$.

If $\|\cdot\| : V \to \mathbb{R}$ is $\rho$-standard, we can compute the operator norm of the operators in $\rho(A^+)$. Indeed, for $t \geq 0$, $\rho(a_t)$ is orthogonally diagonalizable with highest eigenvalue $e^{\frac{4\pi i}{k} t}$, where $k$ is the highest weight appearing in $\rho$. Therefore,

$$\|\rho(a_t)\|_{\text{op}} = \sup_{0 \neq x \in V} \frac{\|\rho(a_t) x\|}{\|x\|} = e^{\frac{4\pi i}{k} t}.$$ 

**Proposition 3.4.** For any finite dimensional real representation $\rho$ there exist a $\rho$-standard inner product on $V$.

**Proof.** First, assume $V = V_k$ is irreducible with highest weight vector $v_0 \in V$. We determine an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ on $V$ by setting the basis

$$\{v_0, v_1, \ldots, v_k\}$$

with

$$\rho(f) v_i = c_i v_{i+1}$$

$$c_i = \sqrt{(k-i)(i+1)}$$

to be orthonormal. With respect to this basis, the operators $\rho(a_t)$ are diagonal. Thus, to show that $\langle \cdot, \cdot \rangle$ is $\rho$-standard it remains to show that it is $\rho(K)$ invariant.

$K$ is the one parameter group generated by the element $e - f \in \mathfrak{sl}_2(\mathbb{R})$. To show that the inner product is $\rho(K)$-invariant, it is enough to show that $\rho(e - f)$ is antisymmetric with respect to this inner product. Since $\rho(e)$ must send a vector of weight $k - 2i$ to a vector of weight $k - 2i + 2$, one can verify by using $[\rho(e), \rho(f)] = \rho(h)$, that

$$\rho(e) v_i = d_i v_{i-1}$$
with \[ d_i = \sqrt{i(k-i+1)}. \]

For all \( 0 \leq i, j \leq 1 \), we have \[ \langle \rho(e) v_i, v_j \rangle = d_i \langle v_{i-1}, v_{j} \rangle = d_i \delta_{i-1,j}. \]

Note that \[ c_{i-1} = \sqrt{(k-(i-1))(i-1+1)} = \sqrt{(k-i+1)i} = d_i. \]

Therefore, \[ \langle v_i, \rho(f) v_j \rangle = c_j \langle v_i, v_{j+1} \rangle = c_j \delta_{i,j+1} = c_{i-1} \delta_{i-1,j} = d_i \delta_{i-1,j}. \]

Hence, for any \( 0 \leq i, j \leq k \), \[ \langle \rho(e) v_i, v_j \rangle = \langle v_i, \rho(f) v_j \rangle. \]

This means that \( \rho(e) \) and \( \rho(f) \) are adjoints with respect to \( \langle ., . \rangle \). Consequently, the operator \[ \rho(e) - \rho(f) = \rho(e-f) \]

is antisymmetric. Thus, for \( V_k \) irreducible, \( \rho \)-standard norms exist.

If \( V \) is not irreducible, \( V \) decomposes as a direct sum \( V = \bigoplus V_{k_i} \) of irreducible representations, and we can take an inner product which is the direct sum of \( \rho \)-standard inner products on each \( V_{k_i} \).

3.2. Sector estimates for Fuchsian groups (Bourgain, Kontorovich, and Sarnak). Let \( \Gamma \subset \text{PSL}_2(\mathbb{R}) \) be a non-elementary, geometrically finite Fuchsian group with critical exponent \( \delta \). As in the case of \( \text{SL}_2(\mathbb{R}) \), we define the coordinates \( t(g), \theta_1(g), \theta_2(g) \) as the values \( t(g) \geq 0 \) and \( 0 \leq \theta_1(g), \theta_2(g) < \pi \) such that \( g = \pm k_{\theta_1(g)\theta_2(g)}k_{\theta_1(g)}k_{\theta_2(g)} \). As mentioned in the introduction, a key step in the proof of our main theorem involves a decomposition of functions on \( \text{PSL}_2(\mathbb{R}) \) into harmonics with respect to the right and left \( K \)-types. This will lead us to consider expressions of the form

\[ \sum_{\gamma \in \Gamma, ||\gamma|| \leq T} e^{2\imath \theta_1(\gamma)} e^{2\imath \theta_2(\gamma)}. \]

These were analyzed by J. Bourgain, A. Kontorovich, and P. Sarnak in [1]. Their asymptotic distribution can be described in terms of the Patterson-Sullivan measure of \( \Gamma \).

For lattices, the asymptotic distribution of such harmonics was studied by Good [3]. Bourgain, Kontorovich, and Sarnak [1] have managed to provide estimates in the infinite volume case. There are two main differences between finite and infinite volume. The first is that for a general discrete group of infinite co-volume, the eigenvector with the lowest eigenvalue is no longer constant. If the critical exponent \( \delta \) satisfies \( \delta > \frac{1}{2} \) and \( \Gamma \) is geometrically finite, the lowest eigenvalue is

\[ \lambda_0 = \delta (1-\delta), \]

and it is of multiplicity 1. As was proven by Patterson in [9] theorem 3.1], a corresponding eigenvector is given in terms of the Patterson-Sullivan measure by

\[ \phi_0(x) = \int 1 d\mu_\pi = \int P(x, \xi) \delta d\mu_\pi, \]

where \( P(x, \xi) \) is the Poisson kernel. Therefore, the main term of the limit \( \sum_{\gamma \in \Gamma, ||\gamma|| \leq T} e^{2\imath \theta_1(\gamma)} e^{2\imath \theta_2(\gamma)} \)

will not distribute uniformly on the circle as in the case of a lattice. In fact, the angular distribution is given by the Patterson-Sullivan measure. Recall that choosing the angles \( \theta_1, \theta_2 \) to lie between 0 and \( \pi \) means that the boundary of the hyperbolic plane is identified with \([0, \pi]\) (see remark 2.2). In this convention, using Polar coordinates \( (r, \theta) \) on the unit disc with \( 0 \leq \theta < \pi \), the function \( \phi_0 \) can be written as:

\[ \phi_0(r, \theta) = \int_0^\pi \left( \frac{1-r^2}{|re^{2\imath \theta_1} - e^{2\imath \alpha}|^2} \right) d\mu(\alpha), \]

with \( \mu \) as in 2.2. The lift of \( \phi_0 \) to a function on \( \text{PSL}_2(\mathbb{R}) \) can be expressed in the coordinates \((\theta_1, r, \theta_2)\) with \( r(g) = \text{tanh}(t(g)) \), as

\[ \phi_0(\theta_1, r, \theta_2) = \int_0^\pi \left( \frac{1-r^2}{|re^{2\imath \theta_1} - e^{2\imath \alpha}|^2} \right) d\mu(\alpha). \]

The eigenfunction \( \phi_0 \) then gives the leading term in theorem 5.5. The precise result is the following ([1] theorem 1.5]).
Theorem 3.5. (J. Bourgain, A. Kontorovich, P. Sarnak) Let $\Gamma$ be a non-elementary geometrically finite Fuchsian group of the second kind with critical exponent $\delta > \frac{1}{2}$. Let

$$0 < \delta (1 - \delta) = \lambda_0 < \lambda_1 < \cdots < \lambda_N < \frac{1}{4}$$

be the eigenvalues of the Laplacian on $\Gamma \backslash \mathbb{H}$ below $\frac{1}{4}$. Write

$$\lambda_i = s_i (1 - s_i)$$

with $s_i > \frac{1}{2}$. Then, for integers $n$ and $k$ there are constants $c_1 (n, k), \ldots, c_N (n, k) \in \mathbb{C}$ such that

$$\sum_{\gamma \in \Gamma_T} e^{2i n \theta} e^{2i k \theta_2 (\gamma)} = \hat{\mu} (2n) \hat{\mu} (2k) \frac{x^{\delta}}{T^{2\delta}} + \sum_{i=1}^{N} c_i (n, k) T^{2s_i}$$

$$+ O \left( T^{\frac{1}{1+2\delta} + \log (T)^{\frac{1}{2}}} (1 + |n| + |k|)^{\frac{1}{4}} \right),$$

as $T \to \infty$. Here $|c_j (n, k)| \ll |c_1 (0, 0)|$ as $n$ and $k$ vary, and the implied constants depend only on $\Gamma$.

4. Proof of the main theorem

4.1. Reduction to the case of functions which are invariant under $\rho (-1)$. In order to work with $\text{PSL}_2 (\mathbb{R})$ instead of $\text{SL}_2 (\mathbb{R})$, we reduce to the case of functions which are invariant respect to translation by $\rho (-1)$. That is, functions with satisfy $f (\rho (-1) x) = f (x)$ for all $x \in \text{End} (\mathbb{R}^2)$. Note that if $V$ decomposes as

$$V = \bigoplus_i V_i,$$

into irreducible representations of $\text{SL}_2 (\mathbb{R})$, we have

$$\rho (-1) = \bigoplus_i \rho_{k_i} (-1),$$

with $\rho_{k_i} (-1)$ given by

$$\rho_{k_i} (-1) = \begin{cases} 
1 & k \text{ is even} \\
-1 & k \text{ is odd}
\end{cases}.$$

Since $\|\rho (-1)\| = 1$ and $-1$ lies in the center of $\text{SL}_2 (\mathbb{R})$, summation over $\Gamma$ or over $\Gamma_T$ for $T > 0$ is invariant under multiplication by $\rho (-1)$. Therefore, if $h \in C_c (\text{End} (V))$ is satisfies $h (\rho (-1) x) = -h (x)$ for all $x \in \text{End} (\mathbb{R}^2)$, we have

$$\mu_{\Gamma, T} (h) = \frac{1}{T^{2i/k}} \sum_{\gamma \in \Gamma} h \left( \frac{\rho (\gamma)}{T} \right) = \frac{1}{T^{2i/k}} \sum_{\gamma \in \Gamma} h \left( \rho (-1) \frac{\rho (\gamma)}{T} \right) = - \frac{1}{T^{2i/k}} \sum_{\gamma \in \Gamma} h \left( \frac{\rho (\gamma)}{T} \right)$$

so $\mu_{\Gamma, T} (h) = 0$. Let $f$ be a continuous function on $\text{End} (V)$. Then,

$$\mu_{\Gamma, T} (f) = \frac{1}{T^{2i/k}} \sum_{\gamma \in \Gamma} \frac{f \left( \frac{\rho (\gamma)}{T} \right) + f \left( \rho (-1) \frac{\rho (\gamma)}{T} \right)}{2} + \frac{f \left( \frac{\rho (\gamma)}{T} \right) - f \left( \rho (-1) \frac{\rho (\gamma)}{T} \right)}{2}$$

Assuming Theorem 3.1 is proved for functions which are even with respect to $\rho$, we get

$$\lim_{T \to \infty} \mu_{\Gamma, T} (f) = \frac{\delta}{2k} \cdot V_{\Gamma} \cdot \frac{1}{2} \cdot \int_{K \times [0, \infty) \times K} f (\rho (k_{\theta_1}) \cdot t P_k \cdot \rho (k_{\theta_2})) t^{\frac{2\delta}{k} - 1} \, d\mu (\theta_1) \, d\mu (\theta_2)$$

$$+ \frac{\delta}{2k} V_{\Gamma} \cdot \frac{1}{2} \cdot \int_{K \times [0, \infty) \times K} f (\rho (-1) \rho (k_{\theta_1}) \cdot t P_k \cdot \rho (k_{\theta_2})) t^{\frac{2\delta}{k} - 1} \, d\mu (\theta_1) \, d\mu (\theta_2).$$

Since $\mu$ is symmetric with respect to the transformation $\theta \mapsto \theta + \pi$ and

$$\rho (-1) \rho (k_{\theta_1}) = \rho (k_{\theta_1 + \pi}), \quad \theta_1 \in [0, 2\pi]$$
averages \nu_{x,y}

Partition of the sum into annuli. 4.2. Recall that we have defined even under the action of 2.1, 2.3, and 2.4 it suffices to prove theorem 2.3 and section (2) of theorem 2.4 for functions that are even under the action of \rho (-1). The rest of the chapter is devoted to the proof of these.

4.2. Partition of the sum into annuli. Let \( f : \text{End} (V) \to \mathbb{C} \) be continuous. We will analyze the averages \( \nu_T (f) \) by partitioning the sum into radial annuli. Recall that our fixed norm on \( \text{End} (V) \) is an operator norm. Therefore, for any \( x, y \in \text{End} (\mathbb{R}^2) \),

\[
\| xy \| \leq \| x \| \| y \|.
\]

Recall that we have defined

\[
\nu_T (f) = \frac{1}{T^N} \sum_{\gamma \in \Gamma} f \left( \frac{\rho(\gamma)}{T} \right).
\]

We will partition the sum \( \nu_T (f) \) into radial sections according to the norm of the elements in \( \rho (\Gamma) \). For \( N \in \mathbb{N} \) write

\[
\nu_T (f) = \frac{1}{T^N} \sum_{j=0}^{N-1} \sum_{\rho (\gamma) \in [jT, (j+1)T]} f \left( \frac{\rho(\gamma)}{T} \right).
\]

in order to simplify our notation, we define

\[
S_{T,j} = \left\{ \gamma \in \Gamma | \frac{j}{N} T < \| \rho (\gamma) \| \leq \frac{j+1}{N} T \right\}.
\]

Then, with this notation,

\[
\nu_T (f) = \frac{1}{T^N} \sum_{j=0}^{N-1} \sum_{\gamma \in S_{T,j}} f \left( \frac{\rho(\gamma)}{T} \right).
\]

Recall that the norm \( \| . \| \) is invariant under \( \rho (K) \) and \( \| \rho (a_i) \| = e^{\frac{\| a_i \|}{2}} \) for all \( t \geq 0 \). Hence, for \( \gamma \in \Gamma \) with \( \gamma = k_1 a_i k_2 \), we have

\[
\| \rho (\gamma) \| = \| \rho (k_1) \rho (a_i) \rho (k_2) \| = \| \rho (a_i) \| = e^{\frac{\| a_i \|}{2}}.
\]

Define, as in section 3.1.3 for \( \gamma \in \Gamma \),

\[
| \gamma | = e^{\frac{\| d(o, \gamma_o) \|}{2}},
\]

where \( d \) is the hyperbolic distance on \( \mathbb{H}^2 \) and \( o \in \mathbb{H}^2 \) is stabilized by \( K \). Then,

\[
S_{T,j} = \left\{ \gamma \in \Gamma | \left( \frac{j}{N} T \right)^{\frac{1}{2}} < | \gamma | \leq \left( \frac{j+1}{N} T \right)^{\frac{1}{2}} \right\}.
\]

The set \( S_{T,j, o} \subseteq \mathbb{H}^2 \) is then an annulus in hyperbolic space with center \( o \) and radii \( \left( \frac{j}{N} T \right)^{\frac{1}{2}} \) and \( \left( \frac{j+1}{N} T \right)^{\frac{1}{2}} \).

Lax and Phillips [11] theorem 1 have shown that

\[
\lim_{T \to \infty} \frac{\# \{ \gamma \in \Gamma | d(o, \gamma_o) < R \}}{e^{\frac{R}{2}}} = V_T.
\]

Therefore,

\[
\lim_{T \to \infty} \frac{|S_{T,j}|}{T^{\frac{2N}{2}}} = \lim_{T \to \infty} \frac{1}{T^N} \left( \left| \Gamma \left( \frac{j+1}{N} T \right)^{\frac{1}{2}} \right| - \left| \Gamma \left( \frac{j}{N} T \right)^{\frac{1}{2}} \right| \right) = V_T \frac{1}{N^{\frac{2N}{2}}} \left( (j+1)^{\frac{2N}{2}} - j^{\frac{2N}{2}} \right).
\]

The idea of the proof of our main theorems is to replace \( \frac{\rho (\gamma)}{T} \) in the argument of \( f \) with \( \frac{\rho (\gamma)}{\| \rho (\gamma) \|} \). As we shall see, this can be done when \( N \) is large enough.

Fix \( \epsilon > 0 \). We will show that there is a \( T (\epsilon) \) such that for \( T > T (\epsilon) \),

\[
| \nu_T (f) - \nu_T (f) | < \epsilon.
\]
The function $f$ is continuous in the closed unit ball in $\text{End}(V)$, which is compact. Let $N_0$ be such that for $N > N_0$, whenever $\|x - y\| < \frac{1}{N}$ with $x, y \in \text{End}(V)$ of norm $\|x\|, \|y\| \leq 1$ we have,

$$|f(x) - f(y)| < \epsilon.$$  

By the definition of $S_{T,j}$, $\frac{\rho(\gamma)}{N} < \frac{\rho(\gamma)}{T}$ for all $\gamma \in S_{T,j}$. Thus, for $\gamma \in S_{T,j}$,

$$\left\| \frac{\rho(\gamma)}{T} - \frac{j}{N} \right\| = \left\| \frac{\rho(\gamma)}{T} \right\| \left( \left\| \frac{\rho(\gamma)}{\|\rho(\gamma)\|} \right\| - \frac{j}{N} \right) \leq \left\| \frac{\rho(\gamma)}{T} \right\| - \frac{j}{N} \|\rho(\gamma)\| < \frac{1}{N}.$$  

We will use this estimate to replace $\frac{\rho(\gamma)}{N}$ with $\frac{j}{N}\|\rho(\gamma)\|$ when summing over $\gamma$ in the annulus $S_{T,j}$. Define

$$\tilde{\nu}_{T,N}(f) = \frac{1}{T^{2\delta}} \sum_{j=0}^{N-1} \sum_{\gamma \in S_{T,j}} f \left( \frac{j}{N} \frac{\rho(\gamma)}{\|\rho(\gamma)\|} \right).$$  

Then, for any $T > 0$,

$$|\nu_T(f) - \tilde{\nu}_{T,N}(f)| \leq \frac{C}{T^{2\delta}} \sum_{j=0}^{N-1} \sum_{\gamma \in S_{T,j}} \left| f \left( \frac{j}{N} \frac{\rho(\gamma)}{\|\rho(\gamma)\|} \right) - f \left( \frac{j}{N} \frac{\rho(\gamma)}{\|\rho(\gamma)\|} \right) \right| \leq \frac{|S_{T,j}|}{T^{2\delta}} \epsilon.$$  

Since $\frac{|S_{T,j}|}{T^{2\delta}}$ converges, $\frac{|S_{T,j}|}{T^{2\delta}}$ and $\frac{|S_{T,j}|}{T^{2\delta}}$ are bounded for all $T \geq 0$. Let $C_T$ denote a bound for both. Then, for $N > N_0$,

$$|\nu_T(f) - \tilde{\nu}_{T,N}(f)| \leq C_T \epsilon.$$  

We can therefore deal with the averages $\tilde{\nu}_{T,N}(f)$ instead of $\nu_T(f)$.

We wish to analyze the sums over the annuli $S_{T,j}$ individually. In order to accomplish that we need to normalize the sum over each $S_{T,j}$. Define

$$M_{T,j} = \left( \frac{T}{N} \right)^{2\delta} \left( j + 1 \right)^{2\delta} - j^{2\delta}.$$  

From (4.1) we see that $M_{T,j} \sim |S_{T,j}|$ as $T$ tends to infinity. We can write $\tilde{\nu}_{T,N}(f)$ as

$$\tilde{\nu}_{T,N}(f) = \frac{1}{N^{2\delta}} \cdot \frac{1}{T^{2\delta}} \sum_{\|\gamma\| \leq \frac{1}{N}} f \left( \frac{j}{N} \frac{\rho(\gamma)}{\|\rho(\gamma)\|} \right)$$

$$+ \sum_{j=1}^{N-1} \left( \frac{1}{N} \right)^{2\delta} \left( j + 1 \right)^{2\delta} - j^{2\delta} \frac{1}{M_{T,j}} \sum_{\gamma \in S_{T,j}} f \left( \frac{j}{N} \frac{\rho(\gamma)}{\|\rho(\gamma)\|} \right).$$  

Define

$$\nu_{T,N}(f) = \frac{2\delta}{k} \sum_{j=1}^{N-1} \left( \frac{1}{N} \right)^{2\delta} \left( j + 1 \right)^{2\delta-1} \frac{1}{M_{T,j}} \sum_{\gamma \in S_{T,j}} f \left( \frac{j}{N} \frac{\rho(\gamma)}{\|\rho(\gamma)\|} \right).$$  

In the average $\nu_{T,N}$, the sum over each annulus is normalized. The difference between $\nu_{T,N}(f)$ and $\tilde{\nu}_{T,N}(f)$ is then bounded by

$$|\tilde{\nu}_{T,N}(f) - \nu_{T,N}(f)| \leq \frac{1}{N^{2\delta}} \left( \left| S_{T,j}\right| \right)^{2\delta} \left| \Gamma_{2\delta} \right|$$

$$+ \frac{1}{T^{2\delta}} \sum_{j=1}^{N-1} \left( \frac{1}{N} \right)^{2\delta} \left( j + 1 \right)^{2\delta-1} \frac{2\delta}{k} \left| S_{T,j}\right| \frac{1}{M_{T,j}}.$$  

with $\|f\|_{\infty,1} = \max_{\|\gamma\| \leq \frac{1}{N}} |f(x)|$. As $j$ ranges over integers between 0 and $N$, if $j \neq \frac{1}{N}$ is always smaller than 1. Hence, if $j \neq 0$ there is some constant $C$, which does not depend on $j$, such that

$$\left| (j + 1)^{2\delta} - j^{2\delta} - \frac{2\delta}{k} \left( j^{2\delta} - j^{2\delta-1} \right) \right| \leq C \cdot j^{2\delta-2}.$$  

In particular,

$$\lim_{j \to \infty} \frac{(j + 1)^{2\delta} - j^{2\delta} - \frac{2\delta}{k} \left( j^{2\delta} - j^{2\delta-1} \right)}{j^{2\delta-1}} = \frac{2\delta}{k}.$$
Let $N_1 > N_0 > 0$ be such that for $N > N_1$, if $j > \epsilon N$, we have
\[
\left| \frac{(j+1)^{\frac{2k}{p}} - j^{\frac{2k}{p}}}{j^{\frac{2k}{p}-1}} \right| < \frac{2\delta}{k}
\]
We can write $|\tilde{\nu}_{T,N}(f) - \nu_{T,N}(f)|$ as
\[
|\tilde{\nu}_{T,N}(f) - \nu_{T,N}(f)| \leq \|f\|_{s,1} \cdot \frac{1}{N^{\frac{2k}{p}}} \left( \frac{N}{T} \right)^{\frac{2k}{p}} |\Gamma_{\frac{T}{N}}| + \|f\|_{s,1} \sum_{j=1}^{|\epsilon N|} \left( \frac{1}{N} \right)^{\frac{2k}{p}} j^{\frac{2k}{p}-1} \left| \frac{(j+1)^{\frac{2k}{p}} - j^{\frac{2k}{p}}}{j^{\frac{2k}{p}-1}} - \frac{2\delta}{k} \right| \frac{|S_{T,j}|}{M_{T,j}}
\]
\[
+ \|f\|_{s,1} \sum_{j=|\epsilon N|+1}^{N-1} \left( \frac{1}{N} \right)^{\frac{2k}{p}} j^{\frac{2k}{p}-1} \left| \frac{(j+1)^{\frac{2k}{p}} - j^{\frac{2k}{p}}}{j^{\frac{2k}{p}-1}} - \frac{2\delta}{k} \right| \frac{|S_{T,j}|}{M_{T,j}}.
\]
where $|\epsilon N|$ is the largest integer which is smaller than $\epsilon N$. Then, for $N > N_2 = \max \left\{ N_1, \epsilon^{-\frac{2k}{p}} \right\}$, and $j > \epsilon N$,
\[
\left( \frac{N}{T} \right)^{\frac{2k}{p}} |\Gamma_{\frac{T}{N}}| \leq C_T \frac{1}{N^{\frac{2k}{p}}} < \epsilon.
\]
\[
\frac{|S_{T,j}|}{M_{T,j}} \leq C_T
\]
\[
\left| \frac{(j+1)^{\frac{2k}{p}} - j^{\frac{2k}{p}}}{j^{\frac{2k}{p}-1}} - \frac{2\delta}{k} \right| < \epsilon.
\]
Also, the sequence
\[
\frac{(j+1)^{\frac{2k}{p}} - j^{\frac{2k}{p}}}{j^{\frac{2k}{p}-1}} = \frac{2\delta}{k}
\]
is bounded by some $S > 0$ for all $j \in \mathbb{N}$. In total, we get
\[
|\tilde{\nu}_{T,N}(f) - \nu_{T,N}(f)| \leq \|f\|_{s,1} \cdot \frac{1}{N^{\frac{2k}{p}}} C_T \|f\|_{s,1} \sum_{j=1}^{|\epsilon N|} \left( \frac{1}{N} \right)^{\frac{2k}{p}} j^{\frac{2k}{p}-1} \cdot S \cdot C_T
\]
\[
+ \|f\|_{s,1} \sum_{j=|\epsilon N|+1}^{N-1} \left( \frac{1}{N} \right)^{\frac{2k}{p}} j^{\frac{2k}{p}-1} C_T \epsilon.
\]
For $L > 0$, the sum $\sum_{j=1}^{L} \left( \frac{1}{N} \right)^{\frac{2k}{p}} j^{\frac{2k}{p}-1}$ can be written as $\sum_{j=1}^{L} \left( \frac{1}{N} \right)^{\frac{2k}{p}-1}$. Therefore, it is bounded by the integral
\[
\int_0^L x^{\frac{2k}{p}-1} \, dx = \frac{k}{2\delta} L^{\frac{2k}{p}}.
\]
Thus, for $N > N_2$ we get
\[
|\tilde{\nu}_{T,N}(f) - \nu_{T,N}(f)| \leq \|f\|_{s,1} C_T \epsilon + S \|f\|_{s,1} C_T \frac{k}{2\delta} \cdot \epsilon^{\frac{2k}{p}} + C_T \|f\|_{s,1} \cdot \frac{k}{2\delta} \cdot \epsilon
\]
\[\leq \left( 1 + \frac{k}{2\delta} (S+1) \right) C_T \|f\|_{s,1} \left( \epsilon + \epsilon^{\frac{2k}{p}} \right). \tag{4.4} \]
Combining (4.2) and (4.4) we see that for $N > N_2$ and $T > 0$,
\[
|\nu_T(f) - \nu_{T,N}(f)| \leq |\nu_T(f) - \tilde{\nu}_{T,N}(f)| + |\tilde{\nu}_{T,N}(f) - \nu_{T,N}(f)|.
\]
\[
\leq \left( 1 + \frac{k}{2\delta} (S+1) \right) C_T \|f\|_{s,1} + C_T \left( \epsilon + \epsilon^{\frac{2k}{p}} \right).
Put \( C_0 = (1 + \frac{k}{2\pi}(S+1))C_T\|f\|_{\infty,1} + C_T \). Then, for \( N > N_2 \) and every \( T > 0 \)

\[
(4.5) \quad |\nu_T(f) - \nu_{T,N}(f)| \leq C_0 \left( \epsilon + \epsilon^{\frac{4k}{T}} \right).
\]

Thus, to analyze the measures \( \nu_T(f) \) it is possible to consider instead the measures \( \nu_{T,N} \)

\[
\nu_{T,N}(f) = \frac{2\delta}{k} \sum_{j=1}^{N-1} \left( \frac{1}{N} \right)^{\frac{2\delta}{k}} j^{\frac{2\delta}{k} - 1} \frac{1}{M_{T,j}} \sum_{\gamma \in S_{T,j}} f \left( \frac{j}{N} \rho(\gamma) \right).
\]

**Remark 4.1.** Note that the sum \( \sum_{j=1}^{N} \left( \frac{1}{N} \right)^{\frac{2\delta}{k}} j^{\frac{2\delta}{k} - 1} \) looks like a “Riemann sum” of \( \int_{0}^{1} t^{\frac{2\delta}{k} - 1} dt \) (this function is not necessarily bounded). This will be the source of the integration over the interval [0, 1] in the expression of the limit measure \( \nu_T \).

We now turn our attention to the averages over individual annuli. Define:

\[
(4.6) \quad \nu_{T,N,j}(f) = \frac{1}{M_{T,j}} \sum_{\gamma \in S_{T,j}} f \left( \frac{j}{N} \rho(\gamma) \right).
\]

\( \nu_{T,N,j}(f) \) is a normalized average over the annuli \( S_{T,j} \). We have

\[
\nu_{T,N}(f) = \frac{2\delta}{k} \sum_{j=1}^{N-1} \left( \frac{1}{N} \right)^{\frac{2\delta}{k}} j^{\frac{2\delta}{k} - 1} \nu_{T,N,j}(f).
\]

In section 4.3 we will analyze the measures \( \nu_{T,N,j}(f) \) as \( T \) tends to infinity.

### 4.2.1. Error term for Hölder functions.

In order to get the error bound in theorem 2.4 we need a quantitative estimate of the difference between \( \nu_T(f) \) and \( \nu_{T,N}(f) \). In this subsection we assume \( f : \text{End}(V) \to \mathbb{C} \) is Hölder continuous with exponent \( \alpha \in (0, 1] \) and constant \( \|f\|_{\text{Lip}_\alpha} \) with respect our fixed norm, i.e.,

\[
|f(x) - f(y)| \leq \|f\|_{\text{Lip}_\alpha} \|x - y\|^\alpha, \quad x, y \in \text{End}(V).
\]

Recall that for \( 0 \leq j \leq N \) and \( \gamma \in S_{T,j} \),

\[
\left\| \rho(\gamma) - \frac{j}{N} \rho(\gamma) \right\| < \frac{1}{N}.
\]

Consequently,

\[
\left( \nu_T(f) - \frac{1}{T^{\frac{2\delta}{k}}} \sum_{j=0}^{N} \sum_{\gamma \in S_{T,j}} f \left( \frac{j}{N} \rho(\gamma) \right) \right) \leq \frac{1}{T^{\frac{2\delta}{k}}} \sum_{j=0}^{N-1} \sum_{\gamma \in S_{T,j}} \left| f \left( \frac{j}{N} \rho(\gamma) \right) - f \left( \frac{j}{N} \rho(\gamma) \right) \right| \leq \frac{1}{T^{\frac{2\delta}{k}}} \sum_{j=0}^{N-1} \sum_{\gamma \in S_{T,j}} \|f\|_{\text{Lip}_\alpha} N^{-\alpha} \leq \|f\|_{\text{Lip}_\alpha} \frac{\|T\|}{T^{\frac{2\delta}{k}}} N^{-\alpha}.
\]

We also need to estimate the error of replacing \( (j+1)^{\frac{2\delta}{k}} - j^{\frac{2\delta}{k}} \) with \( \frac{2\delta}{k} \cdot j^{\frac{2\delta}{k} - 1} \). Recall from 4.3 that for some constant \( C > 0 \),

\[
\left| (j+1)^{\frac{2\delta}{k}} - j^{\frac{2\delta}{k}} - \frac{2\delta}{k} j^{\frac{2\delta}{k} - 1} \right| \leq C j^{\frac{2\delta}{k} - 2},
\]

for all \( j \geq 1 \). Using this we can estimate the difference

\[
(4.7) \quad \frac{1}{T^{\frac{2\delta}{k}}} \sum_{j=0}^{N-1} \sum_{\gamma \in S_{T,j}} f \left( \frac{j}{N} \rho(\gamma) \right) - \frac{2\delta}{k} \sum_{j=1}^{N-1} \frac{1}{N} \left( \frac{j}{N} \right)^{\frac{2\delta}{k} - 1} \frac{1}{M_{T,j}} \sum_{\gamma \in S_{T,j}} f \left( \frac{j}{N} \rho(\gamma) \right).
\]

In the first sum, the term \( j = 0 \) is given by

\[
\frac{1}{T^{\frac{2\delta}{k}}} \sum_{\gamma \in S_{T,0}} f(0) = \frac{|S_{T,0}|}{T^{\frac{2\delta}{k}}} f(0).
\]
This term is bounded by
\[ \|f\|_{\infty,1} \frac{1}{N^{\frac{d}{2}}} \left| \frac{\Gamma_{T/N}}{N} \right|^\frac{d}{2} \leq C_T \frac{1}{N^{\frac{d}{2}}} \|f\|_{\infty,1} \cdot \]

The sum over the rest of the terms \(1 \leq j \leq N - 1\) is bounded by
\[ \frac{1}{N^{d/2}} \sum_{j=1}^{N-1} \left( (j+1)^{2d} - j^{2d} \right) \frac{1}{MT_j} \frac{1}{\gamma \in S_{T,j}} \sum_{\gamma \in S_{T,j}} f \left( \frac{j}{N} \rho(\gamma) \right) - \frac{2\delta}{k} \frac{1}{N} \sum_{j=1}^{N-1} \frac{1}{N} \sum_{\gamma \in S_{T,j}} f \left( \frac{j}{N} \rho(\gamma) \right) \cdot \]

Which is smaller or equal than the following:
\[ \sum_{j=1}^{N-1} \frac{1}{N^{d/2}} \left| (j+1)^{2d} - j^{2d} \right| \frac{1}{MT_j} \|f\|_{\infty,1} \cdot \]

This is then bounded by
\[ CC_T \|f\|_{\infty,1} \sum_{j=1}^{N-1} \frac{1}{N^{d/2}} j^{2d-2} = CC_T \|f\|_{\infty,1} \sum_{j=1}^{N-1} j^{2d-2}. \]

For \(\frac{1}{2} < \delta < 1\), we always have \(\frac{2d}{N} - 2 \neq 0, -1\), and so the sum \(\sum_{j=1}^{N-1} j^{2d-2}\) is bounded by the integral
\[ \sum_{j=1}^{N-1} j^{2d-2} \leq \int_{1}^{N} t^{2d-2} dt \leq C_{d,k} N^{2d-1}. \]

so, in this case, the difference \(17\) is bounded by
\[ C_T \|f\|_{\infty,1} N^{-2d/2} + C_{d,k} C_T \|f\|_{\infty,1} N^{-1}. \]

If \(\delta = 1\) and \(k > 2\), the same bound holds. If \(\delta = k = 1\), we have
\[ \frac{1}{N^{d/2}} \sum_{j=1}^{N-1} j^{2d-2} = \frac{1}{N^2} \sum_{j=1}^{N-1} j^{0} \leq \frac{1}{N}. \]

Finally, If \(\delta = 1\) and \(k = 2\), then \(\frac{2d}{N} - 2 = -1\) so
\[ \frac{1}{N} \sum_{j=1}^{N-1} j^{-1} \leq \frac{1}{N} \log N. \]

Hence, for any \(\frac{1}{2} < \delta \leq 1\) and \(k \in \mathbb{N}\),
\[ \frac{1}{N^{d/2}} \sum_{j=1}^{N-1} j^{2d-2} \leq C_1 \frac{1}{N} \log N, \]

For some constant \(C_1\) depending only on \(\Gamma\) and \(k\). Define \(C_1 = CC_T C_1\). \(C_1\) depends only on \(\Gamma\) and \(k\), and in total,
\[ |\nu_T(f) - \nu_{T,N}(f)| \leq C_1 \|f\|_{\infty,1} \left( N^{-\frac{d}{2}} + N^{-1} \log N \right) + \|f\|_{\text{Lip}} C_T N^{-\alpha}. \]

4.3. Analysis of averages over annuli. We now turn to analyze the averages \(\nu_{T,N,j}(f)\) over the individual annuli. Recall that
\[ \nu_{T,N,j}(f) = \frac{1}{MT_{T,j}} \sum_{\gamma \in S_{T,j}} f \left( \frac{j}{N} \rho(\gamma) \right), \quad j = 1, 2, \ldots, N - 1. \]

We will see that these averages converge as \(T\) tends to infinity. This will be done using the sector estimates obtained by Bourgain, Kontorovich, and Sarnak (Theorem 3.3).

To deal with all the averages \(\nu_{T,N,j}(f)\) at once and to simplify notation we will normalize the parameters \(N\) and \(j\). The general setting is the following. Fix \(\beta \in (0, 1)\) and define:
\[ S_T = \{ \gamma \in \Gamma | T < |\rho(\gamma)| \leq (1 + \beta) T \} \]
\[ MT_T = T^{\frac{d}{2}} \left( (1 + \beta)^{\frac{d}{2}} - 1 \right). \]
In the case of the measures $\nu_{T,N,j}$, $T$ is replaced by $\frac{L}{N}$ and $\beta$ by \frac{1}{T}$. Fix a constant $0 < a \leq 1$ ($a$ plays the role of $\frac{L}{N}$). We will consider the measures $\nu_T$ on $\text{End}(V)$ given by:

$$\lambda_T (f) = \frac{1}{M_T} \sum_{\gamma \in S_T} f \left( a \cdot \frac{\rho (\gamma)}{||\rho (\gamma)||} \right), \quad f \in C (\text{End} (V)).$$

Define a function $h : \text{PSL}_2 (\mathbb{R}) \to \mathbb{C}$ by

$$h (g) = f \left( a \cdot \frac{\rho (\tilde{g})}{||\rho (\tilde{g})||} \right), \quad g \in \text{PSL}_2 (\mathbb{R})$$

where $\tilde{g} \in \text{SL}_2 (\mathbb{R})$ has image $g$ in $\text{PSL}_2 (\mathbb{R})$. $h$ is well defined since by assumption $f$ is invariant under translation by $\rho (1)$. Let $\gamma \in S_T$ be the image of $S_T$ in $\text{PSL}_2 (\mathbb{R})$. Since $\Gamma$ contains $-1$,

$$\lambda_T (f) = \frac{1}{M_T} \sum_{\gamma \in S_T} f \left( a \cdot \frac{\rho (\gamma)}{||\rho (\gamma)||} \right) = \frac{2}{M_T} \sum_{\gamma \in S_T} h (\gamma).$$

Let $\theta_1 (g), \theta_2 (g)$ be the Cartan coordinates on $\text{PSL}_2 (\mathbb{R})$. Instead of the standard coordinate $t (g)$ on $A^+$, it will be more useful to us to use the coordinate $r (g)$ obtained from $t$ by $r = \tanh (\frac{2}{k})$ so that $r \in [0, 1)$.

**Lemma 4.2.** For any $0 \leq \theta_1, \theta_2 \leq \pi$ the limit

$$\lim_{r \to 1^+} h \left( k (\theta_1) a_t (r) k (\theta_2) \right)$$

exists and is uniform in $\theta_1, \theta_2$. Consequently, the function $\tilde{h} : [0, \pi] \times [0, 1] \times [0, \pi] \to \mathbb{C}$ given by

$$\tilde{h} (\theta_1, r, \theta_2) = h (k_{\theta_1} a_t (r) k_{\theta_2})$$

can be extended to a continuous function on $[0, \pi] \times [0, 1] \times [0, \pi]$.

**Proof.** Taking the limit $r \to 1^+$ is equivalent to taking the limit $t \to \infty$.

$$h (k_{\theta_1} a_t k_{\theta_2}) = f (a \cdot \rho (k_{\theta_1}) \rho (a_t) \rho (k_{\theta_2})) = f (a \rho (k_{\theta_1}) \rho (a_t) \rho (k_{\theta_2}) \rho (k_{\theta_3})).$$

Write the decomposition of $V$ as

$$V = \bigoplus_i V_{k_i}$$

with $V_{k_i}$ irreducible of weight $k_i$, and such that all $V_{k_i}$ with $k_i = k (k = \max k_i)$ appear in the first $m$ summands. Recall from 3.1.3 that

$$\rho (a_t) = \begin{pmatrix} \rho_{k_1} (a_t) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \rho_{k_m} (a_t) \end{pmatrix}.$$ 

with

$$\rho_{k_i} (a_t) = \begin{pmatrix} e^{\frac{k_{i-1} t}{2}} & 0 & 0 & 0 \\ 0 & e^{\frac{(k_i-2) t}{2}} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & e^{-\frac{k_{i} t}{2}} \end{pmatrix}.$$ 

Since $||\rho (a_t)|| = e^{\frac{k t}{2}}$, we see that

$$\lim_{t \to \infty} \frac{\rho_{k_i} (a_t)}{||\rho (a_t)||} = P_k.$$ 

Therefore,

$$\lim_{r \to 1^+} \tilde{h} (\theta_1, r, \theta_2) = \lim_{t \to \infty} h (k_{\theta_1} a_t k_{\theta_2}) = f (\rho (k_{\theta_1}) \cdot a P_k \cdot \rho (k_{\theta_2})).$$

Since the image of $K$ in $\text{End} (V)$ is compact and $f$ is continuous, the limit is uniform in $\theta_1, \theta_2$. □
We can therefore extend $\hat{h}$ to a continuous function on $[0, \pi] \times [0, 1] \times [0, \pi]$. The proof of the lemma shows that the values of $\hat{h}$ for $r = 1$ are given by

$$\hat{h}(k_1, 1, k_2) = f(\rho(k_{\theta_1}) \cdot \alpha P_k \cdot \rho(k_{\theta_2})).$$

We are now ready to study the averages $\lambda_T(f)$ in the general case of a continuous function $f$, and give an error estimate when $f$ satisfies the Hölder condition.

4.3.1. The case of continuous functions. For continuous functions, the convergence result will be obtained following the proposition.

**Proposition 4.3.** Let $\psi : SL_2(\mathbb{R}) \to \mathbb{C}$ be a continuous function such that the function $\tilde{\psi} : [0, \pi] \times [0, 1] \times [0, \pi]$ is given by

$$\tilde{\psi}(\theta_1, r, \theta_2) = \psi(k_{\theta_1}, \alpha_t(r), k_{\theta_2}),$$

extends to a continuous function on $[0, \pi] \times [0, 1] \times [0, \pi]$. Let $\mu$ be the lift of the Patterson-Sullivan to a measure on $\mathcal{K}$ as in remark 3.2. Then,

$$\lim_{T \to \infty} \frac{1}{M_T} \sum_{\gamma \in S_T} \psi(\gamma) = \frac{1}{2} V_T \int_{[0, \pi] \times [0, \pi]} \tilde{\psi}(\theta_1, 1, \theta_2) \, d\mu(\theta_1) \, d\mu(\theta_2).$$

**Proof.** By the Stone-Weierstrass theorem, the algebra spanned by functions of the form

$$e^{2in\theta_1} \varphi(r) e^{2rn\theta_2}, \quad n, m \in \mathbb{Z}, \varphi \in C([0, 1]),$$

is dense in the algebra of all continuous functions on $[0, \pi] \times [0, 1] \times [0, \pi]$ with respect to the maximum norm. Therefore, it is enough to prove the claim for functions $\psi$ of the form

$$\psi(g) = e^{2in\theta_1(g)} \xi(r(g)) e^{2rn\theta_2(g)},$$

with $n, m \in \mathbb{Z}$ and $\xi \in C([0, 1])$. For functions of this form,

$$\tilde{\psi}(k_1, 1, k_2) = e^{2in\theta_1(1)} e^{2rn\theta_2}.$$

Now,

$$\frac{V_T}{2} \int_{[0, \pi] \times [0, \pi]} e^{2in\theta_1(1)} e^{2rn\theta_2} \, d\mu(\theta_1) \, d\mu(\theta_2) = \frac{V_T}{2} \xi(1) \int_0^\pi e^{2in\theta_1} \, d\mu(\theta_1) \int_0^\pi e^{2rn\theta_2} \, d\mu(\theta_2)$$

$$= \frac{1}{2} V_T \xi(1) \hat{\mu}(2n) \hat{\mu}(2m)$$

with $\hat{\mu}$ defines as in [1] (see [5.2]). For $\gamma \in S_T$,

$$e^{2\pi i \hat{T}(\gamma)} = \|\rho(\gamma)\| \geq T$$

Therefore, as $T \to \infty$,

$$r(\gamma) \equiv r(t(\gamma)) \to 1$$

uniformly over all $\gamma \in S_T$. Since $\xi$ is continuous, $\xi(r(\gamma)) \to \xi(1)$ uniformly on $S_T$ as $T \to \infty$. Therefore,

$$\lim_{T \to \infty} \frac{1}{M_T} \sum_{\gamma \in S_T} \psi(\gamma) = \lim_{T \to \infty} \frac{1}{M_T} \sum_{\gamma \in S_T} e^{2in\theta_1(\gamma)} \xi(1) e^{2rn\theta_2}.$$

Put $|\gamma| = e^{\frac{1}{2}d(0, \gamma, 0)}$ as before. By theorem 5.20,

$$\lim_{T \to \infty} \frac{1}{T^{2\beta}} \sum_{|\gamma| \leq T} e^{2in\theta_1(\gamma)} e^{2rn\theta_2(\gamma)} = \frac{V_T}{2} \hat{\mu}(2n) \hat{\mu}(2m).$$

By the definition of $S_T$, we have $S_T = \{ \gamma \in T \mid |\gamma|^k \leq (1 + \beta) T \}$. Hence,

$$\sum_{\gamma \in S_T} \psi(\gamma) = \sum_{|\gamma| \leq (1+\beta)T} e^{2in\theta_1(\gamma)} \xi(1) e^{2rn\theta_2(\gamma)} - \sum_{|\gamma| \leq T} e^{2in\theta_1(\gamma)} \xi(1) e^{2rn\theta_2(\gamma)}.$$
Therefore, by \(4.13\),
\[
\lim_{T \to \infty} \frac{1}{M_T} \sum_{\gamma \in \mathcal{S}_T} \psi(\gamma) = \lim_{T \to \infty} \frac{1}{T^{\frac{d_H}{2}} (1 + \beta)^{\frac{d_H}{2}} - 1} \sum_{\gamma \in \mathcal{S}_T} \psi(\gamma)
\]
\[
= \lim_{T \to \infty} \frac{1}{T^{\frac{d_H}{2}} (1 + \beta)^{\frac{d_H}{2}} - 1} \sum_{|\gamma| \leq (1 + \beta)T^{\frac{d_H}{2}}} e^{2i \theta_1(\gamma)} \xi(1) e^{2i \theta_2(\gamma)}
\]
\[
= \lim_{T \to \infty} \frac{1}{T^{\frac{d_H}{2}} (1 + \beta)^{\frac{d_H}{2}} - 1} \sum_{|\gamma| \leq T^{\frac{d_H}{2}}} e^{2i \theta_1(\gamma)} \xi(1) e^{2i \theta_2(\gamma)}
\]
\[
= \frac{V_T \xi(1)}{2} \mu(2n) \mu(2m).
\]
Thus, by \(4.12\),
\[
\lim_{T \to \infty} \frac{1}{M_T} \sum_{\gamma \in \mathcal{S}_T} \psi(\gamma) = \frac{V_T}{2} \int_{[0, \pi] \times [0, \pi]} e^{2i \theta_1} \xi(1) e^{2i \theta_2} \, d\mu(\theta_1) \, d\mu(\theta_2)
\]
\[
= \frac{V_T}{2} \int_{[0, \pi] \times [0, \pi]} \psi(\theta_1, 1, \theta_2) \, d\mu(\theta_1) \, d\mu(\theta_2).
\]

**Corollary 4.4.** For \(h\) defined in \(4.9\),
\[
\lim_{T \to \infty} \frac{1}{M_T} \sum_{\gamma \in \mathcal{S}_T} h(\gamma) = \frac{1}{8} V_T \int_{[0, 2\pi] \times [0, 2\pi]} f(\rho(k_{\theta_1}) \cdot a P_k \cdot \rho(k_{\theta_2})) \, d\mu(\theta_1) \, d\mu(\theta_2).
\]

**Proof.** Lemma \(4.2\) shows that \(\tilde{h}\) can be extended to a continuous function on \([0, \pi] \times [0, 1] \times [0, \pi]\). This means that \(h\) satisfies the conditions of proposition \(4.3\). Consequently,
\[
\lim_{T \to \infty} \frac{1}{M_T} \sum_{\gamma \in \mathcal{S}_T} h(\gamma) = \frac{1}{2} V_T \int_{[0, \pi] \times [0, \pi]} \tilde{h}(\theta_1, 1, \theta_2) \, d\mu(\theta_1) \, d\mu(\theta_2).
\]

From \(4.11\) we see that the values of \(\tilde{h}\) for \(r = 1\) are given by
\[
\tilde{h}(\theta_1, 1, \theta_2) = f(\rho(k_{\theta_1}) \cdot a P_k \cdot \rho(k_{\theta_2})).
\]

Therefore, since \(f\) is even under \(\rho(-1)\),
\[
\lim_{T \to \infty} \frac{1}{M_T} \sum_{\gamma \in \mathcal{S}_T} h(\gamma) = \frac{1}{2} V_T \int_{[0, \pi] \times [0, \pi]} f(\rho(k_{\theta_1}) \cdot a P_k \cdot \rho(k_{\theta_2})) \, d\mu(\theta_1) \, d\mu(\theta_2)
\]
\[
= \frac{1}{8} V_T \int_{[0, 2\pi] \times [0, 2\pi]} f(\rho(k_{\theta_1}) \cdot a P_k \cdot \rho(k_{\theta_2})) \, d\mu(\theta_1) \, d\mu(\theta_2).
\]

**Corollary 4.5.** For \(j = 1, \ldots, N - 1,\)
\[
\lim_{T \to \infty} \nu_{T, N, j}(f) = \frac{1}{4} V_T \int_{K \times K} f\left(\rho(k_1) \cdot \frac{j}{N} P_k \cdot \rho(k_2)\right) \, d\mu(k_1) \, d\mu(k_2).
\]

**Proof.** Recall from \(4.10\) that
\[
\frac{2}{M_T} \sum_{\gamma \in \mathcal{S}_T} h(\gamma) = \frac{1}{M_T} \sum_{\gamma \in \mathcal{S}_T} f\left(a \cdot \frac{\rho(\gamma)}{\|\rho(\gamma)\|}\right).
\]

By corollary \(4.3\),
\[
\lim_{T \to \infty} \frac{1}{M_T} \sum_{\gamma \in \mathcal{S}_T} f\left(a \cdot \frac{\rho(\gamma)}{\|\rho(\gamma)\|}\right) = \frac{1}{4} V_T \int_{K \times K} f(\rho(k_1) \cdot a P_k \cdot \rho(k_2)) \, d\mu(k_1) \, d\mu(k_2).
\]

By setting \(T = \frac{T^j}{N}, \beta = j^{-1}\), and \(a = \frac{j}{N}\) we get the desired result.
4.3.2. The case of Hölder continuous functions. In this subsection, we assume that \( f \) is Hölder continuous with exponent \( \alpha \) and Hölder constant \( \| f \|_{\text{Lip}} \). The quantitative estimate for the convergence of the annuli averages \( \nu_{T,N,j} \) will be obtained using the following proposition.

**Proposition 4.6.** Let \( h \) be as in [4.9]. There exists some \( T_0 > 0 \) such that for every \( R \in \mathbb{N} \), and \( T > T_0 \) we have:

\[
\left| \frac{1}{MT} \sum_{\gamma \in \mathcal{F}_T} h(\gamma) - \frac{1}{8} V_T \int_{K \times K} f(\rho(k_1) \cdot aP_k \cdot \rho(k_2)) \, d\mu(k_1) \, d\mu(k_2) \right| \\
\leq C_2 \| f \|_{\text{H}^{\frac{\alpha}{2}}} \left( R^2 T^{\frac{\alpha}{2}(\alpha - \delta)} + R^2 (1 + 2R)^{\frac{\alpha}{2}} T^{\frac{\alpha}{2}(1 - 2\delta)} \log(T)^{\frac{\alpha}{2}} \right) \\
+ C_2 \| f \|_{\text{Lip}} \left( R^{-\alpha} \log(R) + T^{-\alpha} \right)
\]

for some constant \( C_2 > 0 \) which does not depend on \( f \).

**Remark 4.7.** The parameter \( R \) in the proposition will measure the degree of the trigonometric approximation of \( h \). After computing the overall error in the proof of the main theorem, \( R \) will be chosen (depending on \( T \)) in an optimal way to make the error small.

Before we turn to the proof of proposition 4.6, we draw the conclusion for the convergence of the averages \( \nu_{T,N,j} \).

**Corollary 4.8.** There exists a \( T_0 > 0 \) and \( C_2 > 0 \) which does not depend on \( f \), such that for \( T > T_0 N \) and for all \( R \in \mathbb{N} \),

\[
\left| \nu_{T,N,j}(f) - \frac{1}{4} V_T \int_{K \times K} f(\rho(k_1) \cdot \frac{j}{N} P_k) \, d\mu(k_1) \, d\mu(k_2) \right| \\
\leq C_2 \| f \|_{\text{H}^{\frac{\alpha}{2}}} \left( R^2 \left( \frac{T j}{N} \right)^{\frac{\alpha}{2}(\alpha - \delta)} + R^2 (1 + 2R)^{\frac{\alpha}{2}} \left( \frac{T j}{N} \right)^{\frac{\alpha}{2}(1 - 2\delta)} \log(T)^{\frac{\alpha}{2}} \right) \\
+ C_2 \| f \|_{\text{Lip}} \left( R^{-\alpha} \log(R) + \left( \frac{T j}{N} \right)^{-\alpha} \right)
\]

for all \( 1 \leq j \leq N \).

**Proof.** By applying [4.6] to the values \( a = \frac{T}{N} \), \( \beta = \frac{1}{2} \) and \( T' = T \frac{T}{N} \), we get the desired inequality for any \( T \) such that \( \frac{T j}{N} > T_0 \). Since \( j \geq 1 \), it is enough to take \( T > NT_0 \).

Our goal in the rest of this section is to prove proposition 4.6. First, we can utilize the Hölder condition in order to let the radial part of \( h(\gamma) \) tend to infinity. Indeed, for any \( t \geq 0 \), the largest possible eigenvalue of \( \rho(aT) \) which is smaller than \( e^{\frac{t}{2}} \) is \( e^{\frac{(k-1)t}{2}} \). Since \( \rho(aT) \) and \( P_k \) are both diagonal in some orthonormal basis,

\[
\left\| \frac{\rho_k(aT)}{\| \rho(aT) \|} - P_k \right\| \leq e^{-\frac{t}{2}}.
\]

Note that this is the case only if \( V_{k-1} \) also appears in \( V \). Otherwise, the second largest entry is \( e^{\frac{(k-2)t}{2}} \). This means that for \( \gamma \in S_T \), with \( \gamma = k_1 \rho(aT) k_2 \),

\[
\left\| a \frac{\rho(\gamma)}{\| \rho(\gamma) \|} - \rho(k_1) \cdot aP_k \cdot \rho(k_2) \right\| = a \left\| \frac{\rho(aT)}{\| \rho(aT) \|} - P_k \right\| \leq ae^{-\frac{t}{2}}.
\]

Hence, by the Hölder property of \( f \), for \( t > 0 \) such that \( e^{\frac{t}{2}} \geq T \), we have

\[
|h(\gamma) - f(\rho(k_1) \cdot aP_k \cdot \rho(k_2))| \leq \| f \|_{\text{Lip}} T^{-\alpha}.
\]

Recall that we denoted the image of \( S_T \) in \( \text{PSL}_2(\mathbb{R}) \) by \( \overline{S_T} \). Then, for any \( T > 0 \),

\[
\left| \frac{1}{MT} \sum_{\gamma \in S_T} h(\gamma) - \frac{1}{MT} \sum_{\gamma \in S_T} f(\rho(k_1) \cdot aP_k \cdot \rho(k_2)) \right| \leq \frac{\| f \|_{\text{Lip}} T^{-\alpha}}{MT} \leq C_T \| f \|_{\text{Lip}} T^{-\alpha}.
\]

Define a function \( \phi : [0, \pi] \times [0, \pi] \to \mathbb{C} \) by

\[
\phi(\theta_1, \theta_2) = f(\rho(\theta_1) \cdot aP_k \cdot \rho(k_2)).
\]
We extend $\varphi$ to a function on $[0, 2\pi] \times [0, 2\pi]$ by setting
$$
\varphi(\theta_1 + \pi, \theta_2) = \varphi(\theta_1, \theta_2 + \pi) = \varphi(\theta_1, \theta_2).
$$
We denote the extended function by $\varphi$ as well. On $[0, 2\pi] \times [0, 2\pi] \cong T^2$ we fix the metric
defined by
$$
d(\{(\theta_1, \theta_2), (\phi_1, \phi_2)\}) = \max \{|\theta_1 - \phi_1|, |\theta_2 - \phi_2|\}, \quad \theta_1, \phi_1 \in [0, 2\pi].
$$

**Lemma 4.9.** $\varphi : T^2 \to \mathbb{C}$ is Hölder continuous with exponent $\alpha$ and constant $b \cdot \|f\|_{\text{Lip}}$, where $b$ is some constant which depends only on $\rho$ and the norm $\|\cdot\|$ on $V$.

**Proof.** Consider the function $q : [0, 2\pi] \times [0, 2\pi] \to \text{End}(V)$ given by
$$
q(\theta_1, \theta_2) = \rho(k(\theta_1)) \cdot aP_k \cdot \rho(k(\theta_2)), \quad \theta_1, \theta_2 \in [0, 2\pi].
$$
Then, $\varphi = f \circ q$. As we have seen in 3.1, matrix elements of dimensional representations of $\text{SL}_2(\mathbb{R})$ are polynomials in the entries of the matrices in $\text{SL}_2(\mathbb{R})$ of degree smaller than or equal to $2$. Therefore, the elements of the matrix $q(\theta_1, \theta_2) \in \text{End}(V)$ are polynomials in $\cos \theta_i, \sin \theta_i, \; i = 1, 2$ of degree at most $\dim(V)$. The derivative of $q(\theta_1, \theta_2)$ is then bounded by some constant $b$ which depends on $\rho$ and the norm we have fixed. Consequently, $q$ is Lipschitz continuous with Lipschitz constant $b$. Given that $f$ is $(\|f\|_{\text{Lip}}, \alpha)$-Hölder, $\varphi$ is Hölder continuous with exponent $\alpha$ and constant $b \cdot \|f\|_{\text{Lip}}$ as the composition $f \circ q$.

We now turn to the proof of proposition 4.6. As we have seen in 4.15

$$
(4.16) \quad \left| \frac{1}{M_T} \sum_{\gamma \in \mathcal{S}_T} h(\gamma) - \frac{1}{M_T} \sum_{\gamma \in \mathcal{S}_T} \varphi(\theta_1(\gamma), \theta_2(\gamma)) \right| \leq C_T \|f\|_{\text{Lip}} T^{-\alpha}.
$$

We will study the averages $\frac{1}{M} \sum_{\gamma \in \mathcal{S}_T} \varphi(\theta_1(\gamma), \theta_2(\gamma))$ by using the quantitative result in theorem 3.5. In order to do that, we need to quantify the approximation of $\varphi$ by trigonometric polynomials.

We will approximate $\varphi$ by its Fejér means. The $R$-th Fejér mean of $\varphi$ will be denoted by $\sigma_R$. It is defined by

$$
\sigma_R(\theta_1, \theta_2) = \frac{1}{4\pi^2} \int_{T^2} \varphi(\theta_1 - u, \theta_2 - v) F_R(u, v) \, du dv,
$$

where

$$
F_R(u, v) = F_R(u) F_R(v) = \frac{1}{R} \left( \frac{\sin(\frac{Ru}{2})}{\sin(\frac{R}{2})} \right)^2 \frac{1}{R} \left( \frac{\sin(\frac{Rv}{2})}{\sin(\frac{R}{2})} \right)^2
$$

is a product of one dimensional Fejér kernels. $\sigma_R$ is a trigonometric polynomial of degree $R$. It is of the form

$$
\sigma_R(\theta_1, \theta_2) = \sum_{n,m=-R}^{R} \varphi_{n,m} e^{2ni\theta_1 + 2mi\theta_2},
$$

with $|\varphi_{n,m}| \leq \sup_{\theta_1, \theta_2} |\varphi(\theta_1, \theta_2)|$. Only even frequencies appear in $\sigma_R$ since $\varphi$ is symmetric under $\theta \mapsto \theta + \pi$ in both coordinates. Since $\varphi$ is Hölder continuous with exponent $\alpha$ and constant $b \cdot \|f\|_{\text{Lip}}$, there is some constant $C_0 > 0$ such that for all $R \in \mathbb{N}$ and for all $\theta_1, \theta_2 \in [0, 2\pi]$,

$$
(4.17) \quad |\varphi(\theta_1, \theta_2) - \sigma_R(\theta_1, \theta_2)| \leq C_0 b \|f\|_{\text{Lip}} R^{-\alpha} \log R.
$$

For the proof, see Appendix 4.4.2. We have,

$$
\frac{1}{M_T} \sum_{\gamma \in \mathcal{S}_T} \sigma_R(\theta_1(\gamma), \theta_2(\gamma)) = \frac{1}{M_T} \sum_{\gamma \in \mathcal{S}_T} \sum_{n,m=-R}^{R} \varphi_{n,m} e^{2ni\theta_1(\gamma) + 2mi\theta_2(\gamma)}
$$

$$
= \sum_{n,m=-R}^{R} \varphi_{n,m} \frac{1}{M_T} \sum_{\gamma \in \mathcal{S}_T} e^{2ni\theta_1(\gamma) + 2mi\theta_2(\gamma)}.
$$

Now, by the definition of $\mathcal{S}_T$,

$$
\frac{1}{M_T} \sum_{\gamma \in \mathcal{S}_T} e^{2ni\theta_1(\gamma) + 2mi\theta_2(\gamma)} = \frac{1}{T^{2\pi}} \left( \frac{1 + \beta}{1 + \beta} - 1 \right) \sum_{T^{2\pi} \leq |\gamma| \leq (1 + \beta)T^{2\pi}} e^{2ni\theta_1(\gamma) + 2mi\theta_2(\gamma)}.
$$
By theorem 5.5 there exists a $T_0 > 0$ and a constant $\tilde{C}_3$ which does not depend on $n, m$ such that for $T' > T_0$,
\[
\left| \frac{1}{(T')^{2\delta}} \sum_{|\gamma| \leq T'} e^{i(2n\theta_1(\gamma) + 2m\theta_2(\gamma))} - \frac{1}{2} V_T \mu(2n) \mu(2m) \right| \leq \tilde{C}_3 \left( (T')^{2(s_1 - \delta)} + (T')^{\frac{1}{2}(1 - 2\delta)} \log (T')^{\frac{1}{2}} (1 + |n| + |m|)^{\frac{3}{2}} \right).
\]

Therefore, taking $T' = T^\frac{1}{T}$, and setting $T_0 = \left( \frac{T_0}{T} \right)^{\frac{1}{T}}$, we get that there is a constant $C_3 > 0$ such that for $T > T_0$,
\[
(4.18)
\]
\[
\frac{1}{M_T} \sum_{\gamma \in \Gamma_T} e^{i(2n\theta_1(\gamma) + 2m\theta_2(\gamma))} - \frac{1}{2} V_T \mu(2n) \mu(2m) \leq C_3 \left( T^{\frac{1}{T}(s_1 - \delta)} + T^{\frac{1}{2}(1 - 2\delta)} \log (T)^{\frac{1}{2}} (1 + |n| + |m|)^{\frac{3}{2}} \right)
\]

From 4.18 we see that we can bound the difference
\[
(4.19)
\]
\[
\left| \frac{1}{M_T} \sum_{\gamma \in \Gamma_T} \sigma_R(\theta_1(\gamma), \theta_2(\gamma)) - \frac{1}{2} \sum_{n, m = -R}^R \varphi_{n, m} V_T \mu(2n) \mu(2m) \right|
\]
by the sum
\[
(4.20)
\]
\[
\sum_{n, m = -R}^R \varphi_{n, m} C_3 \left( T^{\frac{1}{T}(s_1 - \delta)} + T^{\frac{1}{2}(1 - 2\delta)} \log (T)^{\frac{1}{2}} (1 + |n| + |m|)^{\frac{3}{2}} \right)
\]

Since $\varphi_{n, m}$ is bounded by the maximum of $\varphi$ on $[0, 2\pi] \times [0, 2\pi]$ for any $n, m \in \mathbb{Z}$, it is bounded by $\|f\|_{\infty, 1}$. Also,
\[
(1 + |n| + |m|)^{\frac{3}{2}} \leq (1 + 2R)^{\frac{3}{2}}.
\]

Therefore, 4.20 is bounded by
\[
4R^2C_3 \|f\|_{\infty, 1} \left( T^{\frac{1}{T}(s_1 - \delta)} + (1 + 2R)^{\frac{3}{2}} T^{\frac{1}{2}(1 - 2\delta)} \log (T)^{\frac{1}{2}} \right).
\]

On the other hand, we have:
\[
\int_{K \times K} \sigma_R(\theta_1, \theta_2) \, d\mu(\theta_1) \, d\mu(\theta_2) = \int_{\mathbb{T}^2} \sigma_R(\theta_1, \theta_2) \, d\mu(\theta_1) \, d\mu(\theta_2)
\]
\[
= \int_{\mathbb{T}^2} \sum_{n, m = -R}^R \varphi_{n, m} e^{2in\theta_1} e^{2im\theta_2} \, d\mu(\theta_1) \, d\mu(\theta_2)
\]
\[
= \sum_{n, m = -R}^R \varphi_{n, m} \left( \int_{\mathbb{T}^2} e^{2in\theta_1} \, d\mu(\theta_1) \right) \left( \int_{\mathbb{T}^2} e^{2im\theta_2} \, d\mu(\theta_2) \right).
\]

Recall that by the definition of the Fourier coefficients of the measure $\mu$ in 5.2,
\[
\int_0^{2\pi} e^{2in\theta_1} \, d\mu(\theta_1) = 2 \int_0^\pi e^{2in\theta_1} \, d\mu(\theta_1) = 2\mu(2n).
\]

Then,
\[
\int_{K \times K} \sigma_R(\theta_1, \theta_2) \, d\mu(\theta_1) \, d\mu(\theta_2) = 4 \sum_{n, m = -R}^R \varphi_{n, m} \mu(2n) \mu(2m).
\]

This gives
\[
\frac{1}{8} V_T \int_{K \times K} \sigma_R(\theta_1, \theta_2) \, d\mu(\theta_1) \, d\mu(\theta_2) = \frac{1}{2} \sum_{n, m = -R}^R \varphi_{n, m} V_T \mu(2n) \mu(2m).
\]

Recall that for any $\theta_1, \theta_2 \in [0, 2\pi]$,
\[
|\sigma_R(\theta_1, \theta_2) - \varphi(\theta_1, \theta_2)| \leq C_0 b \|f\|_{\text{Lip}_a} R^{-\alpha} \log R,
\]
and that, by definition $\varphi(\theta_1, \theta_2) = f(\rho(k_1) \cdot aP_k \cdot \rho(k_2))$. Therefore, the difference

$$\left| \frac{1}{MT} \sum_{\gamma \in G_T} \varphi(\theta_1(\gamma), \theta_2(\gamma)) - \frac{1}{8} V_T \int_{K \times K} f(\rho(k_1) \cdot aP_k \cdot \rho(k_2)) \, d\mu(k_1) \, d\mu(k_2) \right|$$

is bound by,

$$(C_0b + C_1) \|f\|_{\text{Lip}} R^{-\alpha} \log R + 4R^2C_3 \|f\|_{\infty, 1} \left(T^{\frac{1}{4}(s_1 - \delta)} + (1 + 2R)^{\frac{3}{4}} T^{\frac{1}{4}(1 - 2\delta)} \log(T) \right).$$

In total, for $T > T_0$

$$\left| \frac{1}{MT} \sum_{\gamma \in G_T} h(\gamma) - \frac{1}{8} V_T \int_{K \times K} f(\rho(k_1) \cdot aP_k \cdot \rho(k_2)) \, d\mu(k_1) \, d\mu(k_2) \right| \leq (C_0b + C_1) \|f\|_{\text{Lip}} R^{-\alpha} \log R + 4R^2C_3 \|f\|_{\infty, 1} \left(T^{\frac{1}{4}(s_1 - \delta)} + (1 + 2R)^{\frac{3}{4}} T^{\frac{1}{4}(1 - 2\delta)} \log(T) \right) + C_1 \|f\|_{\text{Lip}} T^{-\alpha}.$$

This ends the proof of proposition 4.10.

### 4.4. The conclusion of the main theorems.

#### 4.4.1. The main theorem for continuous functions.

We now turn to the proof of theorem 2.3. We wish to estimate the difference between $\nu_T(f)$ and $\nu_T(f)$. As we have seen, $\nu_T(f)$ can be approximated by the averages $\nu_{T,N}(f)$. The first step is to establish the convergence of $\nu_{T,N}(f)$ as $T \to \infty$.

**Lemma 4.10.** For all $N \in \mathbb{N}$,

$$\lim_{T \to \infty} \nu_{T,N}(f) = \frac{\delta}{2k} V_T \sum_{j=1}^{N-1} \frac{1}{N} \left( \frac{j}{N} \right)^{\frac{4d-1}{8}} \int_{K \times K} f\left( \rho(k_1) \cdot \frac{j}{N} P_k \cdot \rho(k_2) \right) \, d\mu(k_1) \, d\mu(k_2).$$

**Proof.** Recall that

$$\nu_{T,N}(f) = \frac{2\delta}{k} \sum_{j=1}^{N-1} \frac{1}{N} \left( \frac{j}{N} \right)^{\frac{4d-1}{8}} \nu_{T,N,j}(f).$$

Then,

$$\lim_{T \to \infty} \nu_{T,N}(f) = \frac{2\delta}{k} \sum_{j=1}^{N-1} \frac{1}{N} \left( \frac{j}{N} \right)^{\frac{4d-1}{8}} \lim_{T \to \infty} \nu_{T,N,j}(f).$$

By corollary 4.5

$$\lim_{T \to \infty} \nu_{T,N,j}(f) = \frac{1}{4} V_T \int_{K \times K} f\left( \rho(k_1) \cdot \frac{j}{N} P_k \cdot \rho(k_2) \right) \, d\mu(k_1) \, d\mu(k_2).$$

Therefore, the limit of $\nu_{T,N}$ is given by

$$\lim_{T \to \infty} \nu_{T,N}(f) = \frac{\delta}{2k} V_T \sum_{j=1}^{N-1} \frac{1}{N} \left( \frac{j}{N} \right)^{\frac{4d-1}{8}} \int_{K \times K} f\left( \rho(k_1) \cdot \frac{j}{N} P_k \cdot \rho(k_2) \right) \, d\mu(k_1) \, d\mu(k_2).$$

$\square$

We wish to study the right hand side of the limit from lemma 4.10 as $N \to \infty$. We can write the expression in the limit as the integral $\int_0^1 G_N(t) \, dt$, where,

$$G_N(t) = \frac{\delta}{2k} V_T \sum_{j=1}^{N-1} \chi_{[\frac{1}{4}, \frac{3}{4}]}(t) \left( \frac{j}{N} \right)^{\frac{4d-1}{8}} \int_{K \times K} f\left( \rho(k_1) \cdot \frac{j}{N} P_k \cdot \rho(k_2) \right) \, d\mu(k_1) \, d\mu(k_2),$$

where $\chi_{[a,b]}$ is the indicator function of the interval $[a, b]$ for $a, b \in \mathbb{R}$. For every $t \in (0, 1)$,

$$\lim_{N \to \infty} G_N(t) = \frac{\delta}{2k} V_T t^{\frac{1}{4d-1}} \int_{K \times K} f\left( \rho(k_1) \cdot t P_k \cdot \rho(k_2) \right) \, d\mu(k_1) \, d\mu(k_2).$$
The functions \( \{G_N\}_{N \in \mathbb{N}} \) are uniformly bounded by
\[
|G_N(t)| \leq \frac{\delta}{2k} V_T t^{\frac{2}{k}-1} \sup_{|x| \leq 1} |f(x)| \mu(K) \mu(K) = \frac{2\delta}{k} V_T t^{\frac{2}{k}-1} \|f\|_{\infty,1}.
\]
The function \( t^{\frac{2}{k}-1} \) is integrable on \([0,1]\). Hence, by the dominant convergence theorem applied to the sequence \( \{G_N\}_{N \in \mathbb{N}} \), we get:
\[
\lim_{N \to \infty} \int_0^1 G_N(t) \, dt = \frac{\delta}{2k} V_T \int_0^1 t^{\frac{2}{k}-1} \left( \int_{K \times K} f(\rho(k_1) \cdot tP_k \cdot \rho(k_2)) \, d\mu(k_1) \, d\mu(k_2) \right) \, dt
\]
\[= \frac{\delta}{2k} V_T \int_{K \times [0,1] \times K} t^{\frac{2}{k}-1} f(\rho(k_1) \cdot tP_k \cdot \rho(k_2)) \, d\mu(k_1) \, dt \, d\mu(k_2). \tag{4.21}
\]
The right hand side of (4.21) equals \( \int f \, dv_T \). Thus, by lemma 4.10
\[
\lim_{N \to \infty} \left( \lim_{T \to \infty} \nu_{T,N}(f) \right) = \int f \, dv_T. \tag{4.22}
\]
To finish the proof of theorem 2.3 recall that we have seen in 4.5 that there are \( N_2 > 0 \) and \( C_0 > 0 \) such that for \( N > N_2 \) and \( T > 0 \),
\[
|\nu_T(f) - \nu_{T,N}(f)| \leq C_0 \left( \epsilon + \epsilon^{\frac{2}{k}} \right).
\]
From (4.22) there is some \( N_3 > 0 \) such that for \( N \geq N_3 \),
\[
\left| \lim_{T \to \infty} \nu_{T,N}(f) - \int f \, dv_T \right| < \epsilon.
\]
In particular, the inequality holds for \( N_4 = \max\{N_3, N_2\} \). Let \( T_0 > 0 \) be such that for all \( T > T_0 \),
\[
|\nu_{T,N_4}(f) - \lim_{T \to \infty} \nu_{T,N_4}(f)| < \epsilon.
\]
Then, for all \( T > T_0 \),
\[
|\nu_T(f) - \int f \, dv_T| \leq |\nu_T(f) - \nu_{T,N_4}(f)| + |\nu_{T,N_4}(f) - \lim_{T \to \infty} \nu_{T,N_4}(f)| + |\lim_{T \to \infty} \nu_{T,N_4}(f) - \int f \, dv_T|
\]
\[< C_0 \left( \epsilon + \epsilon^{\frac{2}{k}} \right) + \epsilon + \epsilon \leq (2 + C_0) \left( \epsilon + \epsilon^{\frac{2}{k}} \right).
\]
This concludes the proof of theorem 2.3

4.4.2. The quantitative theorem for Hölder continuous functions. Recall from 4.8 that we have
\[
|\nu_T(f) - \nu_{T,N}(f)| \leq C_1 \|f\|_{\infty,1} \left( N^{-\frac{2}{k}} + N^{-1} \log N \right) + \|f\|_{\text{Lip}} C_T N^{-\alpha}. \tag{4.23}
\]
By corollary 4.8 there exists a \( T_0 > 0 \) such that for \( T > NT_0 \) and \( R \in \mathbb{N} \),
\[
|\nu_{T,N,j}(f) - \frac{1}{4} V_T \int_{K \times K} f(\rho(k_1) \cdot \frac{j}{N} P_k \cdot \rho(k_2)) \, d\mu(k_1) \, d\mu(k_2) | \leq C_2 \|f\|_{\infty,1} \left( R^2 \left( \frac{T_j}{N} \right)^{\frac{2}{k}(1-\delta)} + R^2 \left( 1 + 2R \right)^{\frac{2}{k}} \left( \frac{T_j}{N} \right)^{\frac{2}{k}(1-\delta)} \log (T)^{\frac{2}{k}} \right)
\]
\[+ C_2 \|f\|_{\text{Lip}} \left( R^{-\alpha} \log (R) + \left( \frac{T_j}{N} \right)^{-\alpha} \right).
\]
Thus, for $T > NT_0$,
\[
|\nu_{T,N}(f) - \frac{\delta}{2k} V_T \sum_{j=1}^{N-1} \frac{1}{N} \left( \frac{j}{N} \right)^{\frac{2k+1}{k}} \int_{K \times K} f \left( \rho(k_1) \cdot \frac{j}{N} P_k \cdot \rho(k_2) \right) d\mu(k_1) d\mu(k_2) |
\leq \frac{2\delta}{k} \sum_{j=1}^{N-1} \frac{1}{N} \left( \frac{j}{N} \right)^{\frac{2k+1}{k}} \left| \nu_{T,N,j}(f) - \frac{1}{V_T} \int_{K \times K} f \left( \rho(k_1) \cdot \frac{j}{N} P_k \cdot \rho(k_2) \right) d\mu(k_1) d\mu(k_2) \right|
\leq \frac{2\delta}{k} \sum_{j=1}^{N-1} \frac{1}{N} \left( \frac{j}{N} \right)^{\frac{2k+1}{k}} \cdot C_2 \|f\|_{L^1} \left( R^2 \left( \frac{Tj}{N} \right)^{\frac{2(s_1-\delta)}{s_1-\delta}} + R^2 \left( 1 + 2R \right)^{\delta} \left( \frac{Tj}{N} \right)^{\frac{1}{2}(1-2\delta)} \log(T) \right)
\]
\[
+ \frac{2\delta}{k} \sum_{j=1}^{N-1} \frac{1}{N} \left( \frac{j}{N} \right)^{\frac{2k+1}{k}} \cdot C_2 \|f\|_{L^\infty,1} \left( R^{\alpha} \log(R) + \left( \frac{Tj}{N} \right)^{-\alpha} \right).
\]
Since $f$ is Hölder continuous and $t^{\frac{2k+1}{k}}$ is integrable on $[0, 1]$, the difference between the sum
\[
\sum_{j=1}^{N-1} \frac{1}{N} \left( \frac{j}{N} \right)^{\frac{2k+1}{k}} \int_{K \times K} f \left( \rho(k_1) \cdot \frac{j}{N} P_k \cdot \rho(k_2) \right) d\mu(k_1) d\mu(k_2)
\]
and the integral
\[
\int_0^1 t^{\frac{2k+1}{k}} \left( \int_{K \times K} f \left( \rho(k_1) \cdot tP_k \cdot \rho(k_2) \right) d\mu(k_1) d\mu(k_2) \right) dt
\]
is bounded by a constant $C$ times $\|f\|_{L^\infty,1} N^{-\alpha}$. Thus, for $T > NT_0$,
\[
|\nu_{T,N}(f) - \frac{\delta}{2k} V_T \int_{K \times \{0,1\} \times K} t^{\frac{2k+1}{k}} f \left( \rho(k_1) \cdot tP_k \cdot \rho(k_2) \right) dt d\mu(k_1) d\mu(k_2) |
\leq C \|f\|_{L^\infty,1} N^{-\alpha}
\]
\[
+\frac{2\delta}{k} \sum_{j=1}^{N-1} \frac{1}{N} \left( \frac{j}{N} \right)^{\frac{2k+1}{k}} \cdot C_2 \|f\|_{L^1} \left( R \cdot \left( \frac{Tj}{N} \right)^{\frac{2(s_1-\delta)}{s_1-\delta}} + R \left( 1 + 2R \right)^{\delta} \left( \frac{Tj}{N} \right)^{\frac{1}{2}(1-2\delta)} \log(T) \right)
\]
\[
+ \frac{2\delta}{k} \sum_{j=1}^{N-1} \frac{1}{N} \left( \frac{j}{N} \right)^{\frac{2k+1}{k}} \cdot C_2 \|f\|_{L^\infty,1} \left( R^{\alpha} \log(R) + \left( \frac{Tj}{N} \right)^{-\alpha} \right).
\]
Since
\[
\int f \, d\nu_T = \frac{\delta}{2k} V_T \int_{K \times \{0,1\} \times K} t^{\frac{2k+1}{k}} f \left( \rho(k_1) \cdot tP_k \cdot \rho(k_2) \right) dt d\mu(k_1) d\mu(k_2)
\]
the inequality above combined with (4.23) gives, for $T > NT_0$ and $R \in \mathbb{N}$,
\[
|\nu_T(f) - \int f \, d\nu_T| \leq C_1 \|f\|_{L^1} \left( N^{-\frac{2k}{k}} + N^{-1} \log N \right) + (C_1 + C) \|f\|_{L^\infty,1} N^{-\alpha}
\]
\[
+ \frac{2\delta}{k} \sum_{j=1}^{N-1} \frac{1}{N} \left( \frac{j}{N} \right)^{\frac{2k+1}{k}} \cdot C_2 \|f\|_{L^1} \left( R \cdot \left( \frac{Tj}{N} \right)^{\frac{2(s_1-\delta)}{s_1-\delta}} + R \left( 1 + 2R \right)^{\delta} \left( \frac{Tj}{N} \right)^{\frac{1}{2}(1-2\delta)} \log(T) \right)
\]
\[
+ \frac{2\delta}{k} \sum_{j=1}^{N-1} \frac{1}{N} \left( \frac{j}{N} \right)^{\frac{2k+1}{k}} \cdot C_2 \|f\|_{L^\infty,1} \left( R^{\alpha} \log(R) + \left( \frac{Tj}{N} \right)^{-\alpha} \right). 
\]
We have,
\[
\sum_{j=1}^{N-1} \frac{1}{N} \left( \frac{j}{N} \right)^{\frac{2k+1}{k}} = \sum_{j=0}^{N-1} \frac{1}{N} \left( \frac{j}{N} \right)^{\frac{2k+1}{k}} = \sum_{j=0}^{N-1} \frac{1}{N} \left( \frac{j}{N} \right)^{\frac{2k+1}{k}}.
\]
The last expression is bounded by the integral $\int_0^1 t^{\frac{2k+1}{k}} dt$, which is finite. Similarly,
\[
\sum_{j=1}^{N-1} \frac{1}{N} \left( \frac{j}{N} \right)^{\frac{2k+1}{k}} \cdot \left( \frac{j}{N} \right)^{\frac{2(s_1-\delta)}{s_1-\delta}} = \sum_{j=1}^{N-1} \frac{1}{N} \left( \frac{j}{N} \right)^{\frac{2k+1}{k}(1-2\delta)}.
\]
which is bounded by the integral $\int_0^1 T^{\frac{1+4\alpha}{4\alpha}-1} \text{d}t$, which is also finite. Then, there is some constant $C_3$ which depends only on $k$ and $\Gamma$ such that the second row of $[4.24]$ is bounded by

$$\frac{2\delta}{k} C_3 C_2 \|f\|_{\infty,1} \left( R^2 T^{\frac{\alpha}{2} (s_1 - \delta)} + R^2 (1 + 2R)^{\frac{\alpha}{2}} T^{\frac{1}{4\alpha}(1-2\delta)} \log (T)^{\frac{1}{4\alpha}} \right).$$

Since $0 < \frac{1}{N} \leq \frac{1}{T}$, the third row of $[4.24]$ is bounded by

$$\frac{2\delta}{k} \sum_{j=1}^{N-1} \frac{1}{N} \left( \frac{j}{N} \right)^{\frac{1}{4\alpha}-1} C_2 \|f\|_{\text{Lips}} \left( R^{-\alpha} \log (R) + N^\alpha T^{-\alpha} \right).$$

If we take $C_3$ to be also larger than the integral $\int_0^1 T^{\frac{1}{4\alpha}-1}$, we get that this expression is bounded by

$$\frac{2\delta}{k} C_3 \|f\|_{\text{Lips}} \left( R^{-\alpha} \log (R) + N^\alpha T^{-\alpha} \right).$$

In total, for $T > NT_0$, we have,

$$\left| \nu_T(f) - \int f \text{d}\nu_T \right| \leq C_1 \|f\|_{\infty,1} \left( N^{-\frac{2}{4\alpha}} + N^{-1} \log N \right) + (C_T + C) \|f\|_{\text{Lips}} N^{-\alpha}$$

$$+ \frac{2\delta}{k} C_3 C_2 \|f\|_{\infty,1} \left( R^2 T^{\frac{\alpha}{2} (s_1 - \delta)} + R^2 (1 + 2R)^{\frac{\alpha}{2}} T^{\frac{1}{4\alpha}(1-2\delta)} \log (T)^{\frac{1}{4\alpha}} \right)$$

$$+ \frac{2\delta}{k} C_3 \|f\|_{\text{Lips}} \left( R^{-\alpha} \log (R) + N^\alpha T^{-\alpha} \right).$$

Write this as

$$\left| \nu_T(f) - \int f \text{d}\nu_T \right| \leq C_1 \|f\|_{\infty,1} \left( N^{-\frac{2}{4\alpha}} + N^{-1} \log N \right) + (C_T + C) \|f\|_{\text{Lips}} N^{-\alpha}$$

$$+ \frac{2\delta}{k} C_3 \|f\|_{\text{Lips}} N^{-\alpha} T^{-\alpha}$$

$$+ \frac{2\delta}{k} C_3 C_2 \|f\|_{\infty,1} \left( R^2 T^{\frac{\alpha}{2} (s_1 - \delta)} + R^2 (1 + 2R)^{\frac{\alpha}{2}} T^{\frac{1}{4\alpha}(1-2\delta)} \log (T)^{\frac{1}{4\alpha}} \right)$$

$$+ \frac{2\delta}{k} C_3 \|f\|_{\text{Lips}} \left( R^{-\alpha} \log (R) \right).$$

Taking the optimal choice

$$N = T^{\frac{1}{4\alpha}}$$

The condition $T > NT_0$ becomes $T > T_0^\alpha$. Then, there is some constant $C_4$ depending only on $\Gamma$ and $k$ such that, for $T > T_0^\alpha$ and for $R > 0$, we have:

$$\left| \nu_T(f) - \int f \text{d}\nu_T \right| \leq C_4 \left( \|f\|_{\infty,1} \left( T^{-\frac{\alpha}{2}} + T^{-\frac{1}{2}} \log T \right) + \|f\|_{\text{Lips}} T^{-\frac{\alpha}{2}} \right)$$

$$+ \frac{2\delta}{k} C_3 C_2 \|f\|_{\infty,1} \left( R^2 T^{\frac{\alpha}{2} (s_1 - \delta)} + R^2 (1 + 2R)^{\frac{\alpha}{2}} T^{\frac{1}{4\alpha}(1-2\delta)} \log (T)^{\frac{1}{4\alpha}} \right)$$

$$+ \frac{2\delta}{k} C_3 \|f\|_{\text{Lips}} \left( R^{-\alpha} \log (R) \right).$$

If $s_1 - \delta > \frac{1}{2} (1 - 2\delta)$, we take $R = T^{-\frac{\alpha}{2} (s_1 - \delta)}$ and in this case the second and third rows are bounded by a constant times

$$\left( \|f\|_{\infty,1} + \|f\|_{\text{Lips}} \right) \left( T^{\frac{\alpha}{2} (s_1 - \delta)} + T^{\frac{1}{4\alpha}(1-2\delta)} \log (T) \right).$$

Since $\alpha \in (0, 1]$, we always have $\frac{\alpha}{2} (s_1 - \delta) > \frac{1}{2} (s_1 - \delta)$ (recall that $s_1 - \delta < 0$). Therefore, the total bound we get in this case is

$$\left( \|f\|_{\infty,1} + \|f\|_{\text{Lips}} \right) T^{\frac{1}{4\alpha}(1-2\delta)} \log (T).$$

If $s - \delta \leq \frac{1}{2} (1 - 2\delta)$, we take $R = T^{-\frac{\alpha}{4\alpha}(1-2\delta)}$, and then the second and third rows are bounded by some constant times

$$\left( \|f\|_{\infty,1} + \|f\|_{\text{Lips}} \right) \left( T^{\frac{1}{4\alpha}(1-2\delta)} \log (T)^{\frac{1}{4\alpha}} + T^{\frac{1}{4\alpha}(1-2\delta)} \log (T) \right).$$

Since $\alpha \in (0, 1]$, the total bound in this case is

$$\left( \|f\|_{\infty,1} + \|f\|_{\text{Lips}} \right) T^{\frac{1}{4\alpha}(1-2\delta)} \log (T).$$
Combining all these estimates, we get that there is a constant $C_5 > 0$, depending only on $\Gamma$ and $\rho$, such that for $T > T_0^2$,

$$\left| \nu_T(f) - \int f \, d\nu_T \right| \leq C_5 \left( \|f\|_{\infty,1} \left( T^{-\frac{\delta}{2}} + T^{-\frac{\delta}{2}} \log T \right) + \|f\|_{\text{Lip}} T^{-\frac{\delta}{2}} \right)$$
$$+ C_5 \left( \|f\|_{\infty,1} + \|f\|_{\text{Lip}} \right) \left( T^{\frac{1}{2}\alpha (s_1 - \delta)} + T^{\frac{1}{2\alpha} (1-2\delta)} \right) \log (T).$$

Since $\frac{1}{2} < \delta \leq 1$ and $0 < \alpha \leq 1$, we have

$$\left| \frac{\alpha}{16k} (1 - 2\delta) \right| \leq \frac{\alpha}{16}.$$

Which is smaller than the exponents $\frac{\alpha}{2k}$, and $\frac{1}{2}$. Also, $\delta > \delta - s_1 > 0$ and therefore

$$\left| \frac{\alpha}{2k} (s_1 - \delta) \right| \leq \frac{\delta}{k}.$$

This means that the error $\left( T^{\frac{1}{2\alpha} (s_1 - \delta)} + T^{\frac{1}{2\alpha} (1-2\delta)} \right) \log (T)$ is always larger than $T^{-\frac{\delta}{2}} + T^{-\frac{\delta}{2}} \log T + T^{-\frac{\delta}{2}}$. We deduce that as $T$ tends to infinity,

$$\nu_T(f) = \int f \, d\nu_T + O \left( \left( \|f\|_{\infty,1} + \|f\|_{\text{Lip}} \right) \left( T^{\frac{1}{2\alpha} (s_1 - \delta)} + T^{\frac{1}{2\alpha} (1-2\delta)} \right) \log (T) \right),$$

where the implied constant depends only on $\Gamma$ and $\rho$. This concludes the proof of theorem 2.4.
Appendix - Convergence rate of Fejér means

In this appendix we derive a quantitative estimate for the approximation of a Hölder continuous function on $\mathbb{T}^2$ by its Fejér means. This result is certainly well known but the author could not find the statements used in this paper in the literature for the two dimensional case. The proof is included here for convenience.

Let $\psi : [0, 2\pi] \times [0, 2\pi] \to \mathbb{C}$ be Hölder continuous with exponent $\alpha \in (0, 1]$ and constant $C_{\psi}$. We wish to approximate $\psi$ by trigonometric polynomials in a quantitative fashion. We will approximate $\psi$ by its Fejér means, which are Cesàro means of Fourier series.

For $N, M \in \mathbb{N}$, define the $(N, M)$-th Fourier partial sum by

$$s_{N,M}(\theta_1, \theta_2) = \sum_{n=-N}^{N} \sum_{m=-M}^{M} \hat{\psi}(n, m) e^{i n \theta_1} e^{i m \theta_2},$$

where

$$\hat{\psi}(n, m) = \frac{1}{4\pi^2} \int_{\mathbb{T}^2} \psi(\theta_1, \theta_2) e^{-(in \theta_1 + im \theta_2)} d\theta_1 d\theta_2.$$

The $N$-th Dirichlet kernel $D_N : [0, 2\pi] \to \mathbb{R}$ is defined by

$$D_N(\theta) = \sum_{k=-N}^{N} e^{ik\theta} = \sin \left( (N - 2) \theta \right) \sin \left( \frac{\theta}{2} \right).$$

It is easy to verify that $s_{N,M} \psi$ is given by the convolution of $\psi$ and the product of Dirichlet kernels as,

$$s_{N,M}(\theta_1, \theta_2) = \frac{1}{4\pi^2} \int_{\mathbb{T}^2} \psi(\theta_1, \theta_2) D_N(\theta_1 - u) D_M(\theta_2 - v) du dv.$$

Define:

$$\sigma_R(\theta_1, \theta_2) = \frac{1}{R^2} \sum_{N,M=0}^{R-1} s_{N,M}(\theta).$$

$\sigma_R$ is the Cesàro sum of the Fourier developments of $\psi$ with $N, M \leq R$. It is called the $R$-th Fejér mean of $\psi$. We can also write this as

$$\sigma_R(\theta_1, \theta_2) = \sum_{n,m=-R}^{R} A_{n,m} \hat{\psi}(\theta_1, \theta_2) e^{i(n \theta_1 + m \theta_2)}$$

for some constants $A_{n,m}$ such that $0 \leq A_{n,m} \leq 1$ for all $m, n \in \mathbb{Z}$. Then, using \[4.25\]

$$\sigma_R(\theta_1, \theta_2) = \frac{1}{R^2} \sum_{N,M=0}^{R-1} \frac{1}{4\pi^2} \int_{\mathbb{T}^2} \psi(u, v) D_N(\theta_1 - u) D_M(\theta_2 - v) du dv$$

$$= \frac{1}{4\pi^2} \int_{\mathbb{T}^2} \psi(\theta_1 - u, \theta_2 - v) \frac{1}{R^2} \sum_{n,m=0}^{R-1} D_N(u) D_M(v) du dv.$$

Thus, $\sigma_R$ is obtained by a convolution of $\psi$ with the kernel:

$$\frac{1}{R^2} \sum_{n,m=0}^{R-1} D_N(u) D_M(v) = \left( \frac{1}{R} \sum_{N=0}^{R-1} D_N(u) \right) \left( \frac{1}{R} \sum_{M=0}^{R-1} D_M(v) \right).$$

The function

$$F_R(u) = \frac{1}{R} \sum_{N=0}^{R-1} D_N(u),$$

is called the Fejér kernel. It is known (see \[11\]) that

$$F_R(u) = \frac{1}{R} \left( \frac{\sin \left( \frac{Ru}{4} \right)}{\sin \left( \frac{u}{4} \right)} \right)^2.$$ 

Since $F_R(u) \geq 0$ it follows that $\int_{\mathbb{T}^2} |F_R| = 1$ for all $R$. We can now give an estimate for

$$\|\sigma_R - \psi\| = \sup_{\theta_1, \theta_2 \in [0, 2\pi]} |\sigma_R(\theta_1, \theta_2) - \psi(\theta_1, \theta_2)|.$$
For all \(\theta_1, \theta_2 \in [0, 2\pi]\),
\[
|\sigma_R(\theta_1, \theta_2) - \psi(\theta_1, \theta_2)| \leq \frac{1}{4\pi^2} \int_{\mathbb{T}^2} |\psi(\theta_1 - u, \theta_2 - v) - \psi(\theta_1, \theta_2)| F_R(u) F_R(v) \, du \, dv \\
\leq \frac{1}{4\pi^2} \int_{\mathbb{T}^2} C_\psi \max\{|u|, |v|\}^\alpha F_R(u) F_R(v) \, du \, dv.
\]
Using \((\max\{|u|, |v|\})^\alpha \leq |u|^\alpha + |v|^\alpha\) and the symmetry between \(v\) and \(u\),
\[
|\sigma_R(\theta_1, \theta_2) - \psi(\theta_1, \theta_2)| \leq \frac{C_\psi}{4\pi^2} \int_{\mathbb{T}^2} |u|^\alpha F_R(u) F_R(v) \, du \, dv \\
+ \frac{C_\psi}{4\pi^2} \int_{\mathbb{T}^2} |v|^\alpha F_R(u) F_R(v) \, du \, dv \\
= \frac{C_\psi}{2\pi^2} \int_{\mathbb{T}^2} |u|^\alpha F_R(u) F_R(v) \, du \, dv.
\]
Since \(\int F_R(v) \, dv = 1\), we get
\[
|\sigma_R(\theta_1, \theta_2) - \psi(\theta_1, \theta_2)| \leq \frac{C_\psi}{2\pi^2} \int_0^{2\pi} |u|^\alpha F_R(u) \, du.
\]
Since \(|u|^\alpha F_R(u)\) is symmetric under \(\theta \mapsto \theta + \pi\), we have
\[
|\sigma_R(\theta_1, \theta_2) - \psi(\theta_1, \theta_2)| \leq \frac{C_\psi}{\pi^2} \int_0^{\pi} u^\alpha F_R(u) \, du
\]
If we split the integral over \([0, \pi]\) to the intervals \([0, \frac{\pi}{2}]\) and \([\frac{\pi}{2}, \pi]\), we get
\[
\int_0^{\pi} u^\alpha F_R(u) \, du = \int_0^{\frac{\pi}{2}} u^\alpha F_R(u) \, du + \int_{\frac{\pi}{2}}^{\pi} u^\alpha F_R(u) \, du.
\]
Now, for all \(0 \leq x \leq \frac{\pi}{2}\), \(|\sin(x)| \geq \frac{|x|}{\pi}\). So, for \(0 \leq \theta \leq \pi\), we have that \(|\sin(\frac{\theta}{2})| \geq \frac{|\theta|}{2\pi}\). Then, since \(|\sin(\frac{R\theta}{2})| \leq 1\) for all \(u\),
\[
\int_0^{\pi} u^\alpha F_R(u) \, du \leq \int_0^{\frac{\pi}{2}} R^{-\alpha} F_R(u) \, du + \int_{\frac{\pi}{2}}^{\pi} u^\alpha \frac{1}{R} \left( \frac{\sin(\frac{R\theta}{2})}{\sin(\frac{\theta}{2})} \right)^2 \, du \\
\leq R^{-\alpha} \int_0^{\frac{\pi}{2}} F_R(u) \, du + \frac{1}{R} \int_{\frac{\pi}{2}}^{\pi} u^\alpha \frac{4\pi^2}{u^2} \, du \\
= R^{-\alpha} + \frac{4\pi^2}{R} \int_{\frac{\pi}{2}}^{\pi} u^{\alpha-2} \, du.
\]
For \(\alpha \in (0, 1)\),
\[
\int_{\frac{\pi}{2}}^{\pi} u^{\alpha-2} \, du = \frac{1}{(\alpha - 1)} \left(1 - R^{\alpha-1}\right).
\]
and for \(\alpha = 1\),
\[
\int_{\frac{\pi}{2}}^{\pi} u^{-1} \, du = \log(\pi R).
\]
In total,
\[
\int_0^{\pi} u^\alpha F_R(u) \, du = \begin{cases} 
O(R^{-\alpha}) & \alpha \in (0, 1) \\
O\left(\frac{\log R}{R}\right) & \alpha = 1
\end{cases}
\]
We have proved:

**Proposition 4.11.** If \(\psi : \mathbb{T}^2 \to \mathbb{C}\) is Hölder continuous with exponent \(\alpha\) and constant \(C_\psi\), then for \(R \in \mathbb{N}\) and \(\sigma_R\) the \(R\)-th Fejér mean of \(\psi\), there is some constant \(C_0\) such that
\[
\|\sigma_R - \psi\|_{\infty} \leq C_\psi C_0 R^{-\alpha} \log R.
\]
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