CONFORMALLY EINSTEIN-MAXWELL KÄHLER METRICS AND STRUCTURE OF THE AUTOMORPHISM GROUP

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ABSTRACT. Let $(M, g)$ be a compact Kähler manifold and $f$ a positive smooth function such that its Hamiltonian vector field $K = J \nabla_g f$ for the Kähler form $\omega_g$ is a holomorphic Killing vector field. We say that the pair $(g, f)$ is conformally Einstein-Maxwell Kähler metric if the conformal metric $\tilde{g} = f^{-2} g$ has constant scalar curvature. In this paper we prove a reductiveness result of the reduced Lie algebra of holomorphic vector fields for conformally Einstein-Maxwell Kähler manifolds, extending the Lichnerowicz-Matsushima Theorem for constant scalar curvature Kähler manifolds. More generally we consider extensions of Calabi functional and extremal Kähler metrics, and prove an extension of Calabi’s theorem on the structure of the Lie algebra of holomorphic vector fields for extremal Kähler manifolds. The proof uses a Hessian formula for the Calabi functional under the set up of Donaldson-Fujiki picture.

1. Introduction.

Let $(M, J)$ be a compact complex manifold admitting a Kähler structure. A Hermitian metric $\tilde{g}$ on $(M, J)$ is said to be a conformally Kähler, Einstein-Maxwell (cKEM for short) metric if there exists a positive smooth function $f$ on $M$ such that $g = f^2 \tilde{g}$ is Kähler, that the Hamiltonian vector field $K = J \nabla_g f$ for the Kähler form $\omega_g$ is Killing for $g$ (and also for $\tilde{g}$ necessarily), and that the scalar curvature $s_{\tilde{g}}$ of $\tilde{g}$ is constant.

$K$ is necessarily a holomorphic vector field since any Killing vector field on a compact Kähler manifold is holomorphic. If $f$ is a constant function then $g$ is a Kähler metric of constant scalar curvature. When $f$ is not a constant function, typical known examples are conformally Kähler, Einstein metrics by Page [27] on the one-point-blow-up of $\mathbb{CP}^2$, by Chen-LeBrun-Weber [8] on the two-point-blow-up of $\mathbb{CP}^2$, by Apostolov-Calderbank-Gauduchon [1], [2] on 4-orbifolds and by Bérard-Bergery [4] on $\mathbb{P}^1$-bundles over Fano Kähler-Einstein manifolds.

In the more recent studies, non-Einstein cKEM examples are constructed by LeBrun [20], [21] showing that there are ambitoric examples on $\mathbb{CP}^1 \times \mathbb{CP}^1$ and the one-point-blow-up of $\mathbb{CP}^2$, and by Koca-Tønnesen-Friedman...
on ruled surfaces of higher genus. The authors [15] also extended Le-Brun’s construction on \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) to \( \mathbb{CP}^1 \times M \) where \( M \) is a compact constant scalar curvature Kähler manifold of arbitrary dimension.

In this paper we are interested in finding a Kähler metric \( g \) in a fixed Kähler class and a positive Hamiltonian Killing potential \( f \) such that \( \tilde{g} = f^{-2}g \) is a conformally Kähler, Einstein-Maxwell metric. This point of view is taken by Apostolov-Maschler [3]. Thus we may call such a Kähler metric \( g \) a conformally Einstein-Maxwell Kähler metric. Fixing a Kähler class, the primary question is which choice of a Killing vector field \( K \) is the right one. The answer is given by the volume minimization as shown by [15] in the same spirit as Kähler-Ricci solitons by Tian-Zhu [28] and Sasaki-Einstein metrics by Martelli-Sparks-Yau [26], see also [16]. The critical points of the volume functional considered in [15] are precisely those Killing vector fields such that the natural obstruction defined in [3] vanishes. This obstruction will be introduced in Remark 2.5 and Remark 2.6 in this paper, and is an extension of the one given for constant scalar curvature Kähler metrics in [11], [12]. The secondary question is whether the Lichnerowicz-Matsushima theorem ([24], [23]) asserting the reductiveness of the Lie algebra of holomorphic vector fields can be extended for conformally Einstein-Maxwell Kähler manifolds. The main theorem of this paper, Theorem 2.1, gives an affirmative answer to this question.

More generally we consider extensions of Calabi functional and extremal Kähler metrics and prove an extension of Calabi’s theorem for extremal Kähler manifolds. We call the extended metric an \( f \)-extremal metric where \( f \) stands for the Hamiltonian function of a Killing vector field \( K \). The proof uses a Hessian formula for the Calabi functional under the set up of Donaldson-Fujiki picture. The arguments in the finite dimensional setting is given by Wang [29]. Wang’s arguments were effectively employed in [14] to show an extension of Calabi’s structure theorem for extremal Kähler metrics to perturbed extremal Kähler metrics defined in [13]. The proof of the present paper follows closely the lines of arguments in [14], but we will not avoid overlaps with [14] so as to make the exposition self-contained.

After this introduction, in section 2, we consider the Donaldson-Fujiki type formulation of the problem, and compute the first variation of the scalar curvature of the conformal metrics \( \tilde{g} \). This computation and the resulting consequences have been obtained by Apostolov-Maschler [3]. A novelty, if any, is just that the derivation of the formulae is closer to the original style of Calabi’s computation [6], and this enables the computations in section 3 possible. In section 3, we prove a Hessian formula of the Calabi functional. As mentioned above, we follow the derivation of the formula given by Wang [29] and the first author [14]. As an example we consider the case of the one-point-blow-up of \( \mathbb{CP}^2 \). In section 4 we give an example of \( f \)-extremal Kähler metrics.

\[^{1}\text{A.Lahdili obtained independently a proof of Theorem 2.1 in 19.}\]
2. CALABI TYPE FORMULATION AND DONALDSON-FUJIKI SET-UP.

Let $\Omega \in H^2_{\text{DR}}(M, \mathbb{R})$ be a fixed Kähler class, and choose a Kähler metric $g$ with its Kähler form $\omega_g$ in $\Omega$. Denote by $\text{Aut}(M, \Omega)$ the group of all biholomorphisms of $M$ preserving $\Omega$. Its Lie algebra, denoted by $\mathfrak{h}(M)$, is the same as the Lie algebra of the full automorphism group $\text{Aut}(M)$ and consists of all holomorphic vector fields on $M$. Consider the Lie subalgebra $\mathfrak{g}$ of $\mathfrak{h}(M)$ defined as the kernel of the differential of the homomorphism of $\text{Aut}(M)$ to the Albanese torus induced by the Albanese map. Differential geometric expression of $\mathfrak{g}$ is given by

$$\mathfrak{g} = \{ X \in \mathfrak{h}(M) \mid \text{grad}' u = X \text{ for some } u \in C^\infty(M) \}$$

where $C^\infty(M)$ denotes the set of all complex valued smooth functions on $M$ and

$$\text{grad}' u = g^{i\bar{j}} \frac{\partial}{\partial z^j} \frac{\partial}{\partial \bar{z}^i},$$

refer to, for example, [22], [17]. The Lie subalgebra $\mathfrak{g}$ is independent of the choice of the Kähler metric as can be seen from the Albanese map description, and is referred to as the reduced Lie algebra of holomorphic vector fields. For the purpose of the present paper, it is convenient to take (1) to be the definition of $\mathfrak{g}$. Then the independence of the choice of the metric can also be seen from the standard fact that a holomorphic vector field $X$ belongs to $\mathfrak{g}$ if and only if $X$ has a zero, see [25], [7], [22], [17]. If $\omega_g = ig_i^j dz^i \wedge d\bar{z}^j$ is the Kähler form then $\text{grad}' u = X$ is equivalent to

$$i(X) \omega_g = i\partial u.$$

We shall write $X_u := \text{grad}' u$. The Lie algebra structure of $\mathfrak{g}$ is then given by

$$[X_u, Y_w] = X_{\{u,w\}}$$

where $\{u, w\} = \nabla^i u \nabla_i w - \nabla^i w \nabla_i u$, $\nabla^i = g^{i\bar{j}} \nabla_{\bar{j}}$.

Let $G$ be the connected subgroup of $\text{Aut}(M)$ corresponding to $\mathfrak{g}$, i.e. $\text{Lie}(G) = \mathfrak{g}$. $G$ is referred to as the reduced automorphism group. $G$ is then a subgroup of $\text{Aut}(M, \Omega)$ since $G$ is the connected group containing the identity. Let $T$ be a maximal torus of $G$, and $T^c$ be the complexification of $T$. Then $T^c$ is a subgroup of the maximal reductive subgroup $G_r$ of $G$.

Let $K \in \text{Lie}(T)$ be arbitrarily chosen. The problem is to find a Kähler form $\omega_g \in \Omega$ such that

(i) $f$ is a positive smooth function with $J \text{grad}_g f = K$, and that

(ii) $\tilde{g} = f^{-2} g$ is a conformally Kähler, Einstein-Maxwell metric.

At this point we could assume that $K$ is a critical point of the volume functional introduced in [15] so that the obstruction in [3] vanishes. But this assumption is needless for the arguments in what follows, and we do not assume it.

Denote by $\text{Isom}(M, g)$ the group of all isometries of $(M, g)$. The following is the main theorem of this paper.
Theorem 2.1. Let $\tilde{g}$ be a conformally Kähler, Einstein-Maxwell metric so that (i) and (ii) above are satisfied. Then the centralizer $G^K = \{ g \in G \mid Ad(g)K = K \}$ of $K$ in the reduced automorphism group $G$ is the complexification of $\text{Isom}(M, g) \cap G^K$. In particular $G^K$ is reductive.

Remark 2.2. It is shown in [3] that, under the conditions (i) and (ii), $\text{Isom}(M, \tilde{g}) \cap \text{Aut}(M, \Omega)$ is a subgroup of $\text{Isom}(M, g) \cap \text{Aut}(M, \Omega)$, and the Killing vector field $K$ is in the center of the Lie algebra of $\text{Isom}(M, \tilde{g}) \cap \text{Aut}(M, \Omega)$. The proof is as follows. Since $g = f^2 \tilde{g}$, $\text{Isom}(M, \tilde{g}) \cap \text{Aut}(M, \Omega)$ acts as conformal transformations of $(M, g)$. But, since $g$ is Kähler, $\text{Isom}(M, \tilde{g}) \cap \text{Aut}(M, \Omega)$ acts on $(M, g)$ as homotheties. Since $\text{Aut}(M, \Omega)$ preserves $\Omega$, $\text{Isom}(M, \tilde{g}) \cap \text{Aut}(M, \Omega)$ acts as isometries of $(M, g)$. This proves the first statement. Further, again since $g = f^2 \tilde{g}$, $\text{Isom}(M, \tilde{g}) \cap \text{Aut}(M, \Omega)$ preserves $f$. This implies $K$ is in the center of $\text{Isom}(M, \tilde{g}) \cap \text{Aut}(M, \Omega)$.

When $f$ is a constant function and $(M, g)$ has a constant scalar curvature, Theorem 2.1 is a part of Lichnerowicz-Matsushima theorem ([24], [23]) which asserts the reductiveness of the full automorphism group.

As in the case of the problem finding constant scalar curvature Kähler metrics, the Calabi type set-up fixing an integrable complex structure and varying Kähler forms in a fixed Kähler class can be turned through Moser’s theorem into Donaldson-Fujiki type set-up fixing a symplectic form and varying Kähler forms in a fixed Kähler class. We fix a Hamiltonian Killing vector field $K \in \text{Lie}(T)$. Let $T_K$ be the subtorus obtained as the closure in $T$ of $\{ \exp(tK) \mid t \in \mathbb{R} \}$. We set

$$g^K := \{ X \in g \mid [K, X] = 0 \} = \{ X \in g \mid X = \text{grad}u \text{ for some } u \in C^\infty_C(M), \ K u = 0 \}.$$ 

Now we consider $\omega$ as a fixed symplectic form on $M$. Then we can choose a fixed positive Hamiltonian function $f$ of $K$ by adding a positive constant if necessary. We may define

$$C^\infty(M)^K := \{ u \in C^\infty(M) \mid K u = 0 \},$$

$$C^\infty_C(M)^K := \{ u \in C^\infty_C(M) \mid K u = 0 \},$$

$$C^\infty(M)_0^K := \{ u \in C^\infty(M)^K \mid \int_M u f^{-2m-1} \omega^m = 0 \},$$

$$C^\infty_C(M)_0^K := \{ u \in C^\infty_C(M)^K \mid \int_M u f^{-2m-1} \omega^m = 0 \}.$$ 

We denote by $\text{Ham}(M)^K$ the group of Hamiltonian diffeomorphisms for $C^\infty(M)^K$ or equivalently $C^\infty_0(M)^K$ considered as Hamiltonian functions. Let $Z^K$ be the space of all $T_K$-invariant $\omega$-compatible integrable almost complex structures $J \in C^\infty(\text{End}(TM))$. Here, $J$ is said to be $\omega$-compatible if the following two conditions are satisfied:
(a) $\omega(JX, JY) = \omega(X, Y)$,
(b) $g_J := \omega(\cdot, J\cdot)$ is positive definite.

By (b), $g_J$ is a $T_K$-invariant Kähler metric for each $J \in Z^K$.

For any fixed $J \in Z^K$ we have the decomposition

$$T^*M \otimes \mathbb{C} = T_{J'}^*M \oplus T_{J''}^*M$$

into type $(1,0)$-part and type $(0,1)$-part. Naturally $T_{J''}^*M = \overline{T_{J'}^*M}$. For another complex structure $J' \in Z^K$ we also have

$$T^*M \otimes \mathbb{C} = T_{J'}^*M \oplus T_{J''}^*M.$$

If $J'$ is close to $J$, $T_{J'}^*M$ is expressed as a graph over $T_{J''}^*M$ as

$$T_{J'}^*M = \{ \alpha + v(\alpha) | \alpha \in T_{J''}^*M \}$$

where

$$v \in C^\infty(\text{End}(T_{J''}^*M, T_{J''}^*M)) \cong C^\infty(T_{J'}^*M \otimes T_{J''}^*M) \cong C^\infty(T_{J'}^*M \otimes T_{J'}^*M)$$

and the last isomorphism uses the Kähler metric. As an element of the last term, $v$ is in the symmetric part $C^\infty(\text{Sym}(T_{J'}^*M \otimes T_{J'}^*M))$, see the proof of Lemma 2.1 in [13]. So, we have

$$T_J Z^K \subset C^\infty(\text{Sym}(T_{J'}^*M \otimes T_{J'}^*M)) \mathbb{R},$$

the last term being the underlying real vector space of $C^\infty(\text{Sym}(T_{J'}^*M \otimes T_{J'}^*M))$. The $L^2$-inner product with respect to the volume form $f^{-2m+1}\omega^m$ gives a Kähler structure on $Z^K$. If $J' = J + \delta J$ and

$$\alpha + \delta \alpha \in T_{J'}^*M \oplus T_{J''}^*M$$

is in $C^\infty(T_{J''}^*M)$, then

$$(J + \delta J)(\alpha + \delta \alpha) = i(\alpha + \delta \alpha), \quad J\alpha = i\alpha, \quad J\delta \alpha = -i\delta \alpha.$$ 

Thus, up to the first order,

(3) $$(\delta J)\alpha \equiv 2i\delta \alpha = -2J\delta \alpha,$$

(4) $$J^{-1}\delta J\alpha \equiv -2\delta \alpha.$$ 

If $J(t)$ is any smooth curve in $Z^K$ for $t \in (-\epsilon, \epsilon)$ with $J(0) = J$, (4) shows that we may regard

$$J^{-1}\dot{J} \in C^\infty(\text{End}(T_{J''}^*M, T_{J''}^*M)) \mathbb{R},$$

Here $\dot{J}$ denotes the derivative of $J(t)$ at $t = 0$. But (4) also shows

(5) $$J^{-1}\delta J\dot{\alpha} \equiv -2\ddot{\alpha}.$$ 

Thus, as a real tensor field, $J^{-1}\dot{J}$ is in the form of

(6) $$J^{-1}\dot{J} = 2\Re(v^i \frac{\partial}{\partial z^i} \otimes dz^k).$$

The corresponding curve of Kähler metrics $g(t) = \omega(\cdot, J(t)\cdot)$ may be expressed in the matrix form with respect to local coordinates as

$$g(t) = \omega J(t).$$
We thus have  
\begin{equation}
(7) \quad g^{-1} \dot{g} = J^{-1} \omega^{-1} \omega \dot{J} = J^{-1} \dot{J},
\end{equation}
and
\begin{equation}
(8) \quad (g^{-1})^* = -g^{-1} \dot{g} g^{-1} = -J^{-1} \dot{J} g^{-1}.
\end{equation}

**Lemma 2.3.** Let \( \Theta(t) = \partial_t (g(t)^{-1} \partial_t g(t)) \) be the curvature form of \( g(t) \) expressed in terms of time dependent local holomorphic coordinates. Then
\begin{equation}
(9) \quad \left. \frac{d}{dt} \right|_{t=0} \Theta(t) = d \nabla \nabla (J^{-1} \dot{J}).
\end{equation}

**Proof.** Consider \( \left. \frac{d}{dt} \right|_{t=0} \Theta(t) \) with normal coordinates with respect to \( g(0) = g \) at \( p \in M \) so that the derivative \( dg = 0 \) at \( p \). Let \( \theta_t = g_t^{-1} \partial_t g_t \) be the connection form. Then at \( p \) we have from (7)
\[ \dot{\theta} = g^{-1} \nabla \dot{g} = \nabla (g^{-1} \dot{g}) = \nabla (J^{-1} \dot{J}). \]
On the other hand, from \( \Theta = d\theta + \theta \wedge \theta \), we have
\[ (d\theta + \theta \wedge \theta)^* = d\dot{\theta} + \dot{\theta} \wedge \theta + \theta \wedge \dot{\theta} = d \nabla \dot{\theta}. \]
Thus
\[ \dot{\theta} = d \nabla \dot{\theta} = d \nabla (J^{-1} \dot{J}). \]
\[ \square \]

Let \( S(J, f) \) be the scalar curvature of the conformal metric \( f^{-2} g_J \). We often write \( S_J, f \) instead of \( S(J, f) \) and \( f^{-2} g_J \) when these are more convenient. The following theorem is due to Apostolov-Maschler [3], but we express and prove it in the form which fits to the aim of the present paper.

**Theorem 2.4** ([3]). For any smooth curve \( J(t), -\epsilon < t < \epsilon, \) in \( Z^K \) with
\begin{equation}
(10) \quad J^{-1} \dot{J} = 2 \Re (v^i_k \frac{\partial}{\partial z^i} \otimes dz^k) \in T^*_J Z^K,
\end{equation}
we put \( v := v^i_k \frac{\partial}{\partial z^i} \otimes dz^k \). Then for any smooth function \( h \in C^\infty(M) \) we have
\begin{equation}
(11) \quad \left. \frac{d}{dt} \right|_{t=0} \int_M S(J(t), f) h f^{-2m-1} \omega^m = 2 \Re \int_M (\nabla_i \nabla_j h) v^{ij} f^{-2m+1} \omega^m
\end{equation}
where \( v^{ij} = g^{jk} v^j_k \).

**Proof.** The Ricci form \( \text{Ric}_J \) of \( g_J \) is given by
\[ \text{Ric}_J = i \text{tr} \Theta_J. \]
Thus the scalar curvature \( S_J \) of \( g_J \) satisfies
\[ S_J \omega^m = 2 \text{Ric}_J \wedge \frac{\omega^{m-1}}{(m-1)!} = 2i \text{tr} \Theta_J \wedge \frac{\omega^{m-1}}{(m-1)!}. \]
In Lemma 2.3 the 1-form \( \dot{\theta} \) with values in \( \text{End}(T'M) \) is written as
\[ \Re (\nabla_j v^i_k (dz^j \otimes \frac{\partial}{\partial z^i}) \otimes dz^k). \]
Thus we have

\[ \dot{S}_J = 2\Re \nabla^j \nabla_i v_{ij} = 2\Re \nabla_j \nabla_i v^{ij}. \]

The scalar curvature \( S_{J,f} \) of \( g_{J,f} \) and the scalar curvature \( S_J \) of \( g_J \) are related by

\[ S_{J,f} = 2^{2m-1} \frac{1}{m-1} f^{m+1}(f^{-m+1}) + S_J f^2 \]

where \( \Delta_J = d^*_g d \) is the Hodge Laplacian, see e.g. [5]. Then

\[
\int_M S_{J,f} \frac{h}{f^{2m+1}} \omega^m = 2m(2m-1) \int_M h \frac{g^{-1}}{f^{2m+1}} (df,df) \omega^m \\
-2(2m-1) \int_M f^{-2m} g^{-1} (dh,df) \omega^m \\
+ \int_M S_J h f^{-2m+1} \omega^m.
\]

(12)

Noting \( \nabla_i \nabla_j f = \nabla_i \nabla_j f = 0 \)

since \( K \) is a holomorphic vector field, and using \( [5] \) and \( v^{ij} = v^{ji} \) we can compute the derivative

\[
\frac{d}{dt}_{t=0} \int_M S_{J,f} \frac{h}{f^{2m+1}} \omega^m = -2m(2m-1) \int_M h \frac{g^{-1}}{f^{2m+1}} (i(J^{-1}) df, df) \omega^m \\
+2(2m-1) \int_M f^{-2m} g^{-1} (i(J^{-1}) dh, df) \omega^m \\
+ \int_M \dot{S}_J h f^{-2m+1} \omega^m.
\]

\[
= 2\Re \left[ -2m(2m-1) \int_M h \frac{g^{k\ell}}{f^{2m+1}} g^{i\ell} v^i_j \nabla_i f \nabla_k f \omega^m \\
+ 2(2m-1) \int_M f^{-2m} g^{k\ell} v^i_j \nabla_i h \nabla_j f \omega^m \\
+ \int_M \nabla^i v^i_j h f^{-2m+1} \omega^m \right]
\]

\[
= 2\Re \left[ \int_M \nabla_i v^i_j h v^{ij} f^{-2m+1} \omega^m \right].
\]

This completes the proof of Theorem 2.4. \[\square\]

**Remark 2.5.** Recall we fixed \( K \in \text{Lie}(T) \), the symplectic form \( \omega \) and the positive Hamiltonian function \( f \) of \( K \) with respect to \( \omega \). Let \( \mathfrak{g}_\mathbb{R} \) be the subalgebra of \( \mathfrak{g} \) consisting of all \( \mathfrak{X} = \text{grad} u \) with real smooth function \( u \). The linear map \( \text{Fut}_f : \mathfrak{g}_\mathbb{R} \to \mathbb{R} \) defined by

\[
\text{Fut}_f(\mathfrak{X}) = \int_M S(J,f) u \frac{\omega^m}{f^{2m+1}}
\]

(13)
is independent of the choice of $J \in \mathbb{Z}$ where $\text{grad} \, u_X = X$ with
\begin{equation}
\int_M u_X \, f^{-2m-1} \omega^m = 0.
\end{equation}
This follows by taking $h = u_X$ in Theorem 2.4

**Remark 2.6.** By taking $h = 1$ in Theorem 2.4 we see
\begin{equation}
c_{J,f} = \int_M S(J,f) \, f^{-2m-1} \omega^m / \int_M f^{-2m-1} \omega^m
\end{equation}
is independent of $J \in \mathbb{Z}$. Thus, we may remove the normalization (14) and define $\text{Fut}_f$ by
\begin{equation}
\text{Fut}_f(X) = \int_M (S(J,f) - c_{J,f}) \, u_X \frac{\omega_g^m}{f^{2m+1}}.
\end{equation}
In the Calabi type setup, when $K$ is fixed, $f$ varies as $\omega_g$ varies in $\Omega$ in the unique manner with a normalization
\begin{equation}
\int_M f \, \omega_g^m = a.
\end{equation}
When this normalization is satisfied we shall write $f$ as $f_{K,g,a}$. Then $\text{Fut}_{K,a} : \mathfrak{g}_\mathfrak{R} \to \mathbb{R}$ defined by
\begin{equation}
\text{Fut}_{K,a}(X) = \int_M (S(\omega_g,f_{K,g,a}) - c_{\Omega,K,a}) \, u_X \frac{\omega_g^m}{f_{K,g,a}^{2m+1}}
\end{equation}
with
\begin{equation}
c_{\Omega,K,a} := \frac{\int_M S(\omega_g,f_{K,g,a}) \left( \frac{1}{f_{K,g,a}} \right)^{2m+1} \omega_g^m}{\int_M \left( \frac{1}{f_{K,g,a}} \right)^{2m+1} \omega_g^m / m!},
\end{equation}
is independent of the choice of $\omega_g \in \Omega$ where $S(\omega_g,f)$ is the scalar curvature of $\tilde{g} = f^{-2}g$. If there exists a Kähler metric $g$ with its Kähler form $\omega_g$ in $\Omega$ and with $S(\omega_g,f)$ constant then $\text{Fut}_{K,a} = 0$. Namely, $\text{Fut}_{K,a}$ is an obstruction to the existence of conformally Einstein-Maxwell Kähler metric in the Kähler class $\Omega$.

Since the Hermitian inner product of the tangent space $T_J Z^K$ of $Z^K$ at $J$ is given by the $L^2$-inner product
\begin{equation}
(\lambda, \nu)_{L^2(f^{2m+1})} := \int_M \overline{\lambda} \nu^j f^{-2m+1} \omega^m,
\end{equation}
the symplectic structure $\Omega_{J,f}$ on $Z^K$ is given at $J$ by
\begin{equation}
\Omega_{J,f}(\lambda, \nu) = \Re(\lambda, \sqrt{-1} \nu) = \Re \int_M \overline{\lambda}_{ij} \sqrt{-1} \nu^{ij} f^{-2m+1} \omega^m.
\end{equation}
Proposition 2.7 (3). The $(\cdot, \cdot)_{L^2(f^{-2m-1})}$-dual of the scalar curvature $S(J, f)$ gives an equivariant moment map for the action of the group $\text{Ham}(M)^K$ of Hamiltonian diffeomorphisms generated by $C_0^\infty(M)^K$ where

$$(\varphi, \psi)_{L^2(f^{-2m-1})} = \int_M \varphi \psi \, f^{-2m-1} \omega^m.$$  

More precisely, the equality (23) below holds and the left hand side is equivariant under the action on $Z^K$ of the group of Hamiltonian diffeomorphisms generated by $C_0^\infty(M)^K$

Proof. If $X = X' + X''$ is a smooth complex vector field where $X'$ and $X''$ are type $(1, 0)$ and $(0, 1)$ part respectively, then by Lemma 2.3 in [13]

$$(22) \quad L_X J = 2\sqrt{-1} \nabla'_{j} u^i \frac{\partial}{\partial z^j} \otimes \overline{dz}^k - 2\sqrt{-1} \nabla'_{j} u^i \frac{\partial}{\partial z^j} \otimes dz^k.$$  

In particular, if $X_u$ is the Hamiltonian vector field of $u \in C_0^\infty(M)^K$, we have

$$(23) \quad L_{X_u} J = 2\sqrt{-1} \nabla'_{j} u^i \frac{\partial}{\partial z^j} \otimes \overline{dz}^k - 2\sqrt{-1} \nabla'_{j} u^i \frac{\partial}{\partial z^j} \otimes dz^k.$$  

Hence

$$\chi_u := 4\Re(\sqrt{-1} \nabla'_{j} u^i \frac{\partial}{\partial z^j} \otimes \overline{dz}^k)$$  

defines a tangent vector in $T_J Z^K$. For $\alpha = \alpha_i dz^i$, we have

$$\chi_u(\alpha) = 2\sqrt{-1} \nabla'_{j} u^i \alpha_i \overline{dz}^j.$$  

If $J^{-1} J = v$ then by Theorem [24] we have

$$(24) \quad L_{J^{-1} J} J = -2 \nabla'_{j} u^i \alpha_i \overline{dz}^j$$  

and the corresponding tangent vector $J^{-1} J \in T_J Z^K$ is expressed as

$$(25) \quad (J^{-1} J) \alpha = -2(\nabla'_{j} u^i) \alpha_i \overline{dz}^j$$  

for $\alpha = \alpha_i dz^i$. 

Note in passing that (22) shows
Consider the Calabi functional $\Phi : Z^K \to \mathbb{R}$ defined by

$$\Phi(J) = \int_M S_{J,f}^2 \omega^m.$$ 

If $J$ is a critical point of $\Phi$, the Kähler metric $g = \omega J$ is called an $f$-extremal Kähler metric. From Theorem 2.4 we see

$$(26) \quad \frac{d}{dt} \int_M S_{J(t),f}^2 f^{-2m-1} \omega^m = 8 \Re \int_M \nabla_i \nabla_j S_{J,f} \sqrt{-1} \nabla^i \nabla^j u f^{-2m+1} \omega^m$$

when

$$(27) \quad J^{-1} J = \chi_u$$

$$= 2 \sqrt{-1} \nabla_i u \frac{\partial}{\partial z^i} \otimes d\bar{z} - 2 \sqrt{-1} \nabla_k u \frac{\partial}{\partial z^i} \otimes d\bar{z}^k.$$ 

Thus we obtain the following.

**Lemma 3.1.** The Kähler metric $g = \omega J$ is an $f$-extremal Kähler metric if and only if $\nabla^i \nabla^j \nabla_i \nabla_j u$ is a holomorphic vector field.

We define the fourth order elliptic differential operator $L : C^\infty_C(M) \to C^\infty_C(M)$ by

$$(28) \quad (w, Lu)_{L^2(f^{-2m+1})} = (\nabla^i \nabla^j w, \nabla^i \nabla^j u)_{L^2(f^{-2m+1})}.$$ 

We further define the fourth order elliptic differential operator $\overline{L} : C^\infty_C(M) \to C^\infty_C(M)$ by

$$(29) \quad \overline{L} u = \overline{Lu}.$$ 

**Lemma 3.2.** If $J^{-1} J = 2 \sqrt{-1} \nabla_i \nabla^j u \frac{\partial}{\partial z^k} \otimes d\bar{z}^j$ for a real valued smooth function $u \in C^\infty(M)$, we have

$$\frac{d}{dt}|_{t=0} S_{J(t),f} = (Lu + \overline{Lu}).$$ 

**Proof.** For any real smooth function $w$ we see from Theorem 2.4

$$\frac{d}{dt}|_{t=0} \int_M w S_{J(t),f} f^{-2m-1} \omega^m$$

$$= \int_M ((\nabla^i \nabla^j w, \nabla^i \nabla^j u) + \nabla_i \nabla_j w, \nabla_i \nabla_j u) f^{-2m+1} \omega^m$$

$$= (w, Lu) + (w, \overline{Lu})$$

$$= (w, Lu) + (\overline{w}, \overline{Lu}) = (w, Lu + \overline{Lu}).$$

This completes the proof. $\square$

**Lemma 3.3.** For real valued smooth functions $u$ and $w$ in $C^\infty(M)^K$ we have

$$\Omega(\chi_u, \chi_w) = - \int_M \{w, u\} S_{J,f} f^{-2m-1} \omega^m.$$
Proof. This lemma is simply a restatement of Proposition 2.7. Let \( \sigma \) be in the Hamiltonian diffeomorphisms generated by the Hamiltonian vector field of \( w \in C^\infty(M)^K \). Since \( S_{J,f} \) gives a \( \text{Ham}(M)^K \)-equivariant moment map we have

\[
\int_M u S(\sigma J, f) f^{-2m-1} \omega^m = \int_M u \circ \sigma^{-1} S_{J,f} f^{-2m-1} \omega^m.
\]

Taking the time differential of \( \sigma \) we obtain the lemma by (23). \( \square \)

Lemma 3.4. For any smooth complex valued smooth function \( u \in C^\infty(M)^K \) we have

\[
(\mathcal{T} - L)u = \frac{i}{2} \{u, S(J, f)\} = \frac{1}{2} (u^\alpha S(J, f)_\alpha - S(J, f)^\alpha u_\alpha)
\]

where \( u^\alpha = g^{\alpha\beta} \partial u / \partial z^\beta \) for local holomorphic coordinates \( z^1, \ldots, z^m \).

Proof. It is sufficient to prove when \( u \) is real valued. For any real valued smooth function \( w \in C^\infty(M)^K \) it follows from Lemma 3.3 that

\[
(w, \mathfrak{L}u - Lu)_{L^2(f^{-2m-1})} = \frac{i}{2} \Omega(X_v, \chi_u)
\]

\[
= \frac{i}{2} \{w, \{u, S(J, f)\}\}_{L^2(f^{-2m-1})}
\]

\[
= \frac{i}{2} \{w, X_u S(J, f)\}_{L^2(f^{-2m-1})}
\]

\[
= \frac{i}{2} \{w, \omega(X_u, J \text{grad} S_{J,f})\}_{L^2(f^{-2m-1})}
\]

\[
= \frac{i}{2} \{w, du(J \text{grad} S_{J,f})\}_{L^2(f^{-2m-1})}
\]

\[
= (w, S(J, f)_\alpha u^\alpha - S(J, f)^\alpha u_\alpha)_{L^2(f^{-2m-1})}.
\]

This completes the proof of Lemma 3.4. \( \square \)

Lemma 3.5. If \( u \in C^\infty(M)^K \) and \( J^{-1} \hat{J} = 2 \Re \nabla^i \nabla^j u \frac{\partial}{\partial z^i} \otimes dz^j \in T_J Z^K \), then

\[
\frac{d}{dt} \bigg|_{t=0} \int_M S(J(t), f)^2 f^{-2m-1} \omega^m = 4(u, \mathfrak{L}S(J, f))_{L^2(f^{-2m-1})}
\]

\[
= 4(u, \mathfrak{L}S(J, f))_{L^2(f^{-2m-1})}.
\]
Proof. Since $S(J, f) \in C^\infty(M)^K$ we can apply Theorem 2.4 to show that the left hand side of (31) is equal to

$$4\Re(\nabla''\nabla'' S(J, f), \nabla''\nabla'' u)_{L^2(f^{2m+1})}$$

$$= 2((\nabla''\nabla'' S(J, f), \nabla''\nabla'' u)_{L^2(f^{2m+1})} + (\nabla''\nabla'' u, \nabla''\nabla'' S(J, f))_{L^2(f^{2m+1})})$$

$$= 2(u, L\nabla''\nabla'' S(J, f))_{L^2(f^{2m-1})} + 2(u, \nabla''\nabla'' S(J, f))_{L^2(f^{2m-1})}.$$

But Lemma 3.4 implies

$$L\nabla''\nabla'' S(J, f) = L\nabla''\nabla'' S(J, f).$$

Hence the left hand side of (31) is equal to

$$4(u, L\nabla''\nabla'' S(J, f))_{L^2(f^{2m-1})} = 4(u, L\nabla''\nabla'' S(J, f))_{L^2(f^{2m-1})}.$$

□

Lemma 3.6. Suppose that $(\omega, J, f)$ is an $f$-extremal Kähler metric so that $J\nabla' S(J, f)$ is a holomorphic vector field. If $J^{-1}J = 2\Re\nabla\nabla'' u \frac{\partial}{\partial z} \otimes d\bar{z}$ for some real smooth function $u \in C^\infty(M)^K$ then we have

$$\left(\frac{d}{dt}\right)_{t=0} L(J, \omega) S(J, f) = L(\mathcal{L} - L) u$$

Proof. First note that if

$$i(X_u)\omega = du$$

then

$$L_{\frac{1}{2}JX_u} \omega = i\partial\bar{\partial} u.$$

Let $\{f_s\}$ be the flow generated by $-\frac{1}{2}JX_u$. Let $S$ be a smooth function on $M$ such that $\text{grad}' S$ is a holomorphic vector field. We shall compute $\left(\frac{d}{dt}\right)_{t=0} L(f_s J, \omega) S$, and apply to $S = S(J, f)$, and obtain the conclusion of Lemma 3.6. Let $\{S_s\}$ be a family of smooth functions such that $S_0 = S$, that

$$\text{grad}' S_s = \text{grad}' S,$$

where $\text{grad}' s$ denotes the gradient with respect to $f^*_{-s}\omega$, and that

$$\int_M S_s(f^*_{-s}\omega)^m = \int_M S\omega^m.$$

This implies

$$L(f_s J, \omega) f^*_s S_s = f^*_s (L(J, f^*_{-s}\omega) S_s) = 0.$$

On the other hand, in general, if $f^*_{-s}\omega = \omega + i\partial\bar{\partial} s$ then $S_s = S + S^a \varphi_a$. Therefore, since $L_{\frac{1}{2}JX_u} \omega = i\partial\bar{\partial} u$ by (32) we have

$$S_s = S + s S^a u_a + O(s^2).$$
Thus taking the derivative of (34), we obtain

\[ (d \frac{d}{ds}|_{s=0}L)S + L(-\frac{1}{2}(JX_u)S + S^\alpha u_\alpha) = 0. \]

By an elementary computation we see

\[ (JX_u)S = g(JX_u, \nabla S) = \omega(X_u, \nabla S) = du(\nabla S) = (\partial u + \bar{\partial}u)(\nabla' S + \nabla'' S) = u_\alpha S^\alpha + u^\alpha S_\alpha. \]

Thus, from (35) and the above computation, we obtain

\[ (\frac{d}{ds}|_{s=0}L)S = \frac{1}{2}(u_\alpha S^\alpha + u^\alpha S_\alpha) - S^\alpha u_\alpha = 0 \]
\[ = \frac{1}{2}L(u^\alpha S_\alpha - u_\alpha S^\alpha) \]
\[ = L(\bar{L} - L)u. \]

This completes the proof of Lemma 3.6.

To express the Hessian formula, for a real smooth function \( u \in C^\infty(M)^K \), we identify \( J^{-1}J = 2\mathbb{R}\nabla^i \nabla^j_{\bar{j}} u \frac{\partial}{\partial z^i} \otimes d\bar{z}^j \) with \( \nabla'' \nabla'' u \).

**Theorem 3.7.** Let \( J \) be a critical point of \( \Phi \) so that \( (\omega, J, f) \) is an \( f \)-extremal Kähler metric. Let \( u \) and \( w \) be real smooth functions in \( C^\infty(M)^K \). Then the Hessian \( \text{Hess}(\Phi)_J \) at \( J \) is given by

\[ \text{Hess}(\Phi)_J(\nabla'' \nabla'' u, \nabla'' \nabla'' w) = 8(u, L\bar{L}w) = 8(u, \bar{L}Lw). \]

In particular, \( L\bar{L} = \bar{L}L \) on \( C^\infty(M)^K \) at any critical point of \( \Phi \).

**Proof.** Suppose \( J^{-1}J = 2\mathbb{R}\nabla^i \nabla^j_{\bar{j}} w \frac{\partial}{\partial z^i} \otimes d\bar{z}^j \), or \( \nabla'' \nabla'' w \) by our identification. Then by Lemma 3.5, Lemma 3.6 and Lemma 3.2 we obtain

\[ \text{Hess}(\Phi)_J(\nabla'' \nabla'' u, \nabla'' \nabla'' w) = \frac{d}{dt}|_{t=0}4(u, L\bar{L}(J, f))_{L^2(f^{2m-1})} \]
\[ = 4(u, \frac{d}{dt}|_{t=0}L\bar{L}(J, f))_{L^2(f^{2m-1})} \]
\[ = 4(u, L(\bar{L} - L)w + L(L + \bar{L})w)_{L^2(f^{2m-1})} \]
\[ = 8(u, L\bar{L}w)_{L^2(f^{2m-1})}. \]

Similarly, we obtain

\[ \text{Hess}(\Phi)_J(\nabla'' \nabla'' u, \nabla'' \nabla'' w) = \frac{d}{dt}|_{t=0}4(u, \bar{L}L(J, f))_{L^2(f^{2m-1})} \]
\[ = 4(u, L\bar{L}(J, f))_{L^2(f^{2m-1})} \]
\[ = 4(u, \bar{L}(L - \bar{L})w + \bar{L}(L + \bar{L})w)_{L^2(f^{2m-1})} \]
\[ = 8(u, \bar{L}Lw)_{L^2(f^{2m-1})}. \]

This completes the proof of Theorem 3.7.

\[ \square \]
The following theorem extends a theorem of Calabi [6] for extremal Kähler metrics.

**Theorem 3.8.** If $g = \omega J$ is an $f$-extremal Kähler metric with $K = J \text{grad} f$ then the centralizer $g^K$ of $K$ in the reduced Lie algebra $g$ of holomorphic vector fields has the following structure:

(a) $g^K_0 := (i(M) \cap g^K) \otimes \mathbb{C}$ is the maximal reductive subalgebra of $g^K$ where $i(M)$ denotes the real Lie algebra of all Killing vector fields.

(b) $\text{grad}' S_{J,f} = g^J \frac{\partial S_{J,f}}{\partial z^j}$ is in the center of $g^K_0$.

(c) $g^K = g^K_0 + \sum_{\lambda \neq 0} g^K_{\lambda}$ where $g^K_{\lambda}$ is the $\lambda$-eigenspace of $\text{ad}(\text{grad}' S_{J,f})$.

Moreover, we have $[g^K_{\lambda}, gn] \subset g^K_{\lambda + n}$.

**Proof.** By Theorem 3.7, $\overline{LL} = LL$ on $C^\infty(M)^K$. Therefore $L$ maps $\text{ker} L$ to $\text{ker} L$, and we have the direct sum decomposition

$$\text{ker} L = \sum \lambda E_\lambda$$

into the eigenspaces of $2\overline{L}$. Further by Lemma 3.4

$$\lambda u = 2\overline{L} u = 2(\overline{L} - L) u = S(J, f)^{\alpha} u_{\alpha} - u^{\alpha} S(J, f)_{\alpha}.$$  

This shows

$$[\text{grad}' S(J, f), \text{grad}' u] = \lambda \text{grad}' u.$$  

Thus we have $E_\lambda = g^K_{\lambda}$. For $\lambda = 0$ we have

$$g^K_0 = E_0 = \text{ker} L \cap \text{ker} \overline{L}.$$  

Since $\overline{L} u = 0$ is equivalent to $L \overline{u} = 0$, if $u \in g^K_0$ then $u$ satisfies both $Lu = 0$ and $L \overline{u} = 0$. This implies $L \Re u = 0$ and $L \Im u = 0$. In general if $\text{grad}' u$ is a holomorphic vector field for a real smooth function $u$ then $J \text{grad} u$ is a Killing vector field. Hence we obtain

$$g^K_0 = (i(M) \cap g^K_0) \otimes \mathbb{C}.$$  

Now we are in a position to prove Theorem 2.1

**Proof of Theorem 2.1.** If $g = \omega J$ is a conformally Einstein-Maxwell Kähler metric, then $g$ is an $f$-extremal Kähler metric with $S(J, f)$ is a constant function. Therefore, in this case

$$g^K = g^K_0 = (i(M) \cap g^K) \otimes \mathbb{C}.$$  

Since $\text{Isom}(\tilde{M}, g)$ is compact, $g^K$ is reductive. This completes the proof of Theorem 2.1. \qed
Example 3.9. We consider the case of the one-point-blow-up \( \mathbb{CP}^2 \) of \( \mathbb{CP}^2 \). Let \( \Delta_p \) be the convex hull of \((0,0),(p,0),(p,1-p),(0,1)\) in \((\mu_1,\mu_2)\)-plane. Then for each \( p \), \( \Delta_p \) determines a Kähler class of \( \mathbb{CP}^2 \). The Hamiltonian function \( f \) of a holomorphic Killing vector field is an affine linear function \( f = a\mu_1 + b\mu_2 + c \), which is determined uniquely up to the choice of \( c \). Then \( f \) is positive on \( \Delta_p \) if and only if

\[
c, b + c, (1 - p)b + pa + c, pa + c > 0.
\]

In [15] we showed that the obstruction \( \text{Fut}_f \) in Remark 2.9 vanishes if and only if \( K = J\text{grad}f \) gives a critical point of the volume functional defined in Theorem 1.1 in [15]. The Hamiltonian functions \( f \) which correspond to critical points are determined in section 4.3 in [15]. The following is the list of critical points.

1. \( a = \frac{p+2\sqrt{1-p^2}}{2p^2}, b = 0, 0 < p < 1.\)
2. \( a = -\frac{\sqrt{9p^2-8p+8}}{4p^2}, b = 0, \frac{8}{9} < p < 1.\)
3. \( a = \frac{\sqrt{9p^2-8p-8}}{4p^2}, b = 0, \frac{8}{9} < p < 1.\)
4. \( a = -\frac{\sqrt{p^4-4p^3+16p^2-16p+4-p^2+4p-2}}{2p^3-4p^2+12p-8}, b = -\frac{\sqrt{p^4-4p^3+16p^2-16p+4}}{2p^3-4p^2+12p-8}.\)
5. \( a = \frac{\sqrt{p^4-4p^3+16p^2-16p+4+p^2-4p+2}}{2p^3-4p^2+12p-8}, b = \frac{\sqrt{p^4-4p^3+16p^2-16p+4}}{p^3-2p^2+6p-4}.\)
6. \( a = \frac{2\sqrt{-9p^2+16}}{(21p^2+1)p^2+(1-16p^2)p+4b^2-1+3b^2+(1-2b)p}.\)
7. \( a = -\frac{2\sqrt{-9p^2+16}}{(21p^2+1)p^2+(1-16p^2)p+4b^2-1-3b^2+(2b-1)p}.\)

Here, \( \alpha \) is the smallest positive root of \( p^4 - 4p^3 + 16p^2 - 16p + 4 = 0 \). In the cases (1), (2) and (3) we have \( b = 0 \) which shows the solution has to have \( U(2) \)-symmetry. In fact LeBrun [21] constructed a solution in each cases. \( K \) and the centralizer \( G^K \) are then of the form

\[
\begin{pmatrix}
\alpha & 0 & 0 \\
0 & \beta & 0 \\
0 & 0 & \beta
\end{pmatrix}, \quad \left\{ \begin{pmatrix}
* & 0 & 0 \\
0 & * & * \\
0 & * & *
\end{pmatrix} \right\}
\]

in \( \text{PGL}(3,\mathbb{C}) \) where \( \alpha \neq \beta, \alpha 
eq 0 \) and \( \beta 
eq 0 \). In the cases (4), (5), (6) and (7), the existence is not known at this moment of writing, but if a solution exists it must have \( U(1) \times U(1) \)-symmetry. \( K \) and the centralizer \( G^K \) are then of the form

\[
\begin{pmatrix}
\alpha & 0 & 0 \\
0 & \beta & 0 \\
0 & 0 & \gamma
\end{pmatrix}, \quad \left\{ \begin{pmatrix}
* & 0 & 0 \\
0 & * & 0 \\
0 & 0 & *
\end{pmatrix} \right\}
\]

in \( \text{PGL}(3,\mathbb{C}) \) where \( \alpha, \beta \) and \( \gamma \) are non-zero and mutually distinct.
4. Construction of $f$-extremal Kähler metrics

In this section we give a construction of $f$-extremal Kähler metrics on $\mathbb{C}P^1 \times M$ when $M$ is an $(m-1)$-dimensional compact complex manifold with a Kähler metric $g_2$ of constant scalar curvature $s_{g_2} = c$. This is an extension of a construction of conformally Kähler Einstein-Maxwell metrics given in section 3 of \[15\].

Let $g_1$ be an $S^1$-invariant metric on $\mathbb{C}P^1$. Using the action-angle coordinates $(t, \theta) \in (a, b) \times (0, 2\pi]$, the $S^1$-invariant metric $g_1$ can be written as

$$g_1 = \frac{dt^2}{\Psi(t)} + \Psi(t)d\theta^2$$

for some smooth function $\Psi(t)$ where the Hamiltonian function of the generator of the $S^1$-action is $t$. Therefore $f = t$ in this case. We put $g = g_1 + g_2$. We wish to construct $\Psi$ such that the gradient vector field with respect to $g$ of the scalar curvature $s(\tilde{g})$ of the Hermitian metric $\tilde{g} = g/t^2$ on $\mathbb{C}P^1 \times M$ is a holomorphic vector field. This happens to be the case if

$$s(\tilde{g}) = dt + e$$

for some constants $d$ and $e$.

Since the scalar curvature of $g_1$ is given by

$$s_1 = \Delta_{g_1} \log \Psi = -\Psi''(t),$$

the scalar curvature of $g$ is given by

$$s = s_1 + s_2 = c - \Psi''(t).$$

It follows that the equation (36) is equivalent to

$$dt + e = 2\left(\frac{2m-1}{m-1}\right)t^{m+1} \Delta_{g_1}(t^{1-m}) + (c - \Psi''(t))t^2. \quad (37)$$

Using

$$\Delta_{g_1} \left( \frac{1}{t^{m-1}} \right) = (m-1) \left( \frac{\Psi}{t^m} \right)' ,$$

the equation (37) reduces to the ODE

$$t^2 \Psi'' - 2(2m-1)t \Psi' + 2m(2m-1)\Psi = ct^2 - dt - e. \quad (38)$$

The general solution of the equation (38) is

$$\Psi = At^{2m} + Bt^{2m-1} + \frac{c}{2(m-1)(2m-3)}t^2 - \frac{d}{2(m-1)(2m-1)}t - \frac{e}{2m(2m-1)} \quad (39)$$

with the requirement of

$$\Psi(t) > 0$$

on $(a, b)$. The boundary conditions are

$$\Psi(a) = \Psi(b) = 0, \Psi'(a) = -\Psi'(b) = 2,$$
which reduce to a simultaneous linear equation for $A, B, c, d$ and $e$. The space of solutions are 1-dimensional. If we express it in terms $B$ we have the following expression.

(i) $A$ is given by

$$A = \frac{-2ab^{2m-1}}{m + 2b^{2m}m + 2a^{2m-1}bm - 2a^{2m}m - 2b^{2m} + 2a^{2m}} A_1 B$$

with

$$A_1 = 2b^{2m-1}m - 2ab^{2m-2}m + 2a^{2m-2}bm - 2a^{2m-1}m - 3b^{2m-1} + ab^{2m-2} - a^{2m-2}b + 3a^{2m-1}.$$ 

(ii) $c$ is given by

$$c = \frac{(m - 1)(2m - 3)P}{Q} - \frac{4(m - 1)(2m - 3)}{b - a}$$

with

$$P = (2a^{m+1}b^{m+1}m - 2a^{m+1}b^{m}m - a^{m}b^{m+1} - ab^{2m} + a^{m+1}b^{m} + a^{2m}b)$$

$$Q = ab(ab^{m+2}m - 2a^{2}b^{2m+1}m + a^{3}b^{2m}m + a^{2m}b^{3}m - 2a^{2m+1}b^{2}m$$

$$+ a^{2m+2}bm - ab^{2m+2} + a^{2}b^{2m+1} + a^{2m+1}b^{2} - a^{2m+2}b).$$

(iii) $d$ is given by

$$d = \frac{(2m - 1)(2m - 1)R}{S} - \frac{4(a + b)(m - 1)(2m - 1)}{b - a}$$

with

$$R = 2a^{2m}b^{2m+3}m^{2} - 2a^{2m+1}b^{2m+2}m^{2} - 2a^{2m+2}b^{2m+1}m^{2} + 2a^{2m+3}b^{2m}m^{2}$$

$$- 3a^{2m}b^{2m+3}m^{2} + 3a^{2m+1}b^{2m+2}m + 3a^{2m+2}b^{2m+1}m$$

$$- 3a^{2m+3}b^{2m}m + a^{2m}b^{2m+3} - a^{3}b^{4m} + a^{2m+3}b^{2m} - a^{4m}b^{3},$$

$$S = ab(ab^{m+2}m - 2a^{2}b^{2m+1}m + a^{3}b^{2m}m + a^{2m}b^{3}m - 2a^{2m+1}b^{2}m + a^{2m+1}b^{2}m$$

$$- ab^{2m+2} + a^{2}b^{2m+1} + a^{2m+1}b^{2} - a^{2m+2}b).$$

(iv) $e$ is given by

$$e = -\frac{m(2m - 1)U}{b - a} + \frac{4abm(2m - 1)}{b - a}$$

with

$$T = 4a^{2m+1}b^{2m+3}m^{2} - 8a^{2m+2}b^{2m+2}m^{2} + 4a^{2m+3}b^{2m+1}m^{2} - 8a^{2m+1}b^{2m+3}m$$

$$+ 16a^{2m+2}b^{2m+2}m - 8a^{2m+3}b^{2m+1}m + 4a^{2m+1}b^{2m+3} - 6a^{2m+2}b^{2m+2}$$

$$+ 4a^{2m+3}b^{2m+1} - a^{4}b^{4m} - a^{4m}b^{4},$$

$$U = ab(ab^{m+2}m - 2a^{2}b^{2m+1}m + a^{3}b^{2m}m + a^{2m}b^{3}m - 2a^{2m+1}b^{2}m + a^{2m+2}bm$$

$$- ab^{2m+2} + a^{2}b^{2m+1} + a^{2m+1}b^{2} - a^{2m+2}b).$$
We used Maxima to obtain the above result. For example, if we put $B = 0$ then we obtain

\[
\begin{align*}
A &= 0, \\
c &= -\frac{4(m - 1)(2m - 3)}{b - a}, \\
d &= -\frac{4(a + b)(m - 1)(2m - 1)}{b - a}, \\
e &= \frac{4abm(2m - 1)}{b - a}
\end{align*}
\]  
(40)

and

\[
\Psi(t) = -\frac{2(t - a)(t - b)}{b - a}.
\]

This $\Psi$ is positive on $(a, b)$ and satisfies the boundary conditions.

As another example, we set $a = 1, b = 2$ for simplicity, and put

\[
\begin{align*}
A &= 2 \left(\frac{2^{m+1} - 5 \cdot 2^m + 8m + 4}{(-2^{m+1} + 2^m + 2^m - 2)(2^{m+1} + 2^m - 2m - 2)}\right), \\
B &= \frac{-8 \left(2^{2m} - 2^{m+1} + 2m + 2\right)}{(-2^{m+1} + 2^m + 2^m - 2)(2^{m+1} + 2^m - 2m - 2)}, \\
c &= 0, \\
d &= \frac{4(m - 1)(2m - 1) \left(-3m2^{m+1} + 24^m + 3 \cdot 2^m - 4\right)}{(-2^{m+1} + 2^m + 2^m - 2)(2^{m+1} + 2^m - 2m - 2)}, \\
e &= \frac{4m(2m - 1) \left(-2^{2m+3} + 3 \cdot 2^{m+1} + 2^{4m} - 8\right)}{(-2^{m+1} + 2^m + 2^m - 2)(2^{m+1} + 2^m - 2m - 2)}
\end{align*}
\]  
(41)

Then $\Psi$ satisfies the boundary conditions. Further $\Psi > 0$ on the interval $(1, 2)$ because of $Ad < 0$ and the following Lemma.

**Lemma 4.1.** Let $m$ be an integer greater than 1, and suppose $0 < a < b$. If the real valued function

\[
f(t) = \alpha t^{2m} + \beta t^{2m-1} + \gamma t^2 + \delta t + \epsilon
\]

with $\alpha \delta > 0$ satisfies the boundary conditions

\[
f(a) = f(b) = 0, \quad f'(a) > 0, \quad -f'(b) > 0
\]

then $f$ is positive on the interval $(a, b)$.

**Proof.** Suppose that there is a $c \in (a, b)$ such that $f(c) \leq 0$. Then by the boundary conditions $f$ has at least three critical points on $(a, b)$. On the other hand, we have

\[
\frac{f'(t)}{t} = 2m \alpha t^{2m-2} + (2m - 1) \beta t^{2m-3} + 2 \gamma + \delta / t,
\]

and thus

\[
\left(\frac{f'(t)}{t}\right)' = t^{2m-4} \left(2m(2m - 2) \alpha t + (2m - 1)(2m - 3) \beta - \frac{\delta}{t^{2m-2}}\right).
\]
From $\alpha \delta > 0$, the right hand side of (42) changes sign only once on the region $t > 0$. This implies $f'/t$ has at most two zeros on the interval $(a,b)$. This is a contradiction, and completes the proof of Lemma 4.1.

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