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SINGULARITIES OF SCHUBERT VARIETIES, TANGENT CONES AND BRUHAT GRAPHS

By JAMES B. CARRELL and JOCHEN KUTTLER

Abstract. Let $G$ be a semi-simple algebraic group without $G_2$-factors over an algebraically closed field $k$ of characteristic $p \neq 2, 3$, and suppose $B$ is a Borel subgroup, $T \subset B$ is a maximal torus, and $P$ is a parabolic in $G$ containing $B$. In an earlier paper, the authors classified the singular $T$-fixed points $x$ of an arbitrary irreducible $T$-stable subvariety $X$ in $G/P$ in all characteristics, the key to this being the notion of a Peterson translate. In particular, we showed that if $X$ is Cohen-Macaulay, then $X$ is smooth at $x$ if and only if there exists a $T$-invariant curve in $X$ through $x$ which contains a smooth point of $X$ and dim $\Theta_t(X) = \dim X$, where $\Theta_t(X)$ is the linear span of the reduced tangent cone to $X$ at $x$. The purpose of this paper is to describe $\Theta_t(X)$ when $X$ is a Schubert variety in $G/P$ and $x$ is a maximal singular $T$-fixed point of $X$. In fact, we give two characterizations. We first show that in all characteristics, $\Theta_t(X)$ is the sum of all the Peterson translates at $x$. The second characterization involves further study of the Peterson translates, along the good $T$-invariant curves at $x$, for which the assumption char$(k) \neq 2, 3$ is needed. This leads to the following consequence: if $x$ is a maximal singularity of $X$ which is rationally smooth, then either the span of the tangent lines to the $T$-stable curves is not a module for the isotropy subgroup of $B$ at $x$, or there exist a pair of orthogonal $T$-invariant curves at $x$ which determine what we call a $B_2$-pair. This characterization gives a nonrecursive algorithm for finding the singular locus of an arbitrary Schubert variety in $G/P$ in terms of its Bruhat graph.

1. Introduction. Let $G$ be a semi-simple algebraic group over an arbitrary algebraically closed field $k$ without $G_2$-factors, and suppose $T \subset B \subset P$ are respectively a maximal torus, a Borel subgroup and an arbitrary standard parabolic in $G$. It is well known that the algebraic homogeneous spaces $G/P$, in particular the flag variety $G/B$, are projective $G$-varieties and that each $B$-orbit is an affine cell containing a unique $T$-fixed point. If $x \in (G/P)^T$, the set of $T$-fixed points in $G/P$, the Zariski closure $X(x)$ of $Bx$ is called the Schubert variety associated to $x$. The ordering of Schubert varieties in $G/P$ by inclusion thus gives a natural partial ordering on $(G/P)^T$. As is well known, putting $w = n_w B$, where $w = n_w T$, gives an identification between the Weyl group $W = N_G(T)/T$ and $(G/B)^T$ such that the Bruhat-Chevalley order on $W$ coincides with this ordering on $(G/B)^T$ [8].

Most Schubert varieties in a $G/P$ are singular, and describing the structure of their singular loci is an old problem considered by many authors. Since the singular locus of a Schubert variety $X$ is closed and $B$-stable, its irreducible components are also Schubert varieties. Thus, assuming $X$ isn’t smooth, there
exist unique points \( x_1, \ldots, x_k \in X^T \) such that the singular locus \( \text{Sing}(X) \) of \( X \) is

\[
X(x_1) \cup X(x_2) \cup \cdots \cup X(x_k),
\]

and \( X(x_i) \not\subset X(x_j) \) if \( i \neq j \). The \( x_i \) are called the maximal singular points of \( X \).

Before describing our main results, let us recall some basic terms and facts from [5] and [7]. A \( T \)-variety is an irreducible \( T \)-stable subvariety of some \( G/P \). A \( T \)-curve is the closure of a one-dimensional \( T \)-orbit. Every \( T \)-curve \( C \) in \( G/P \) contains two distinct \( T \)-fixed points, and if \( x \in C^T \), there exists a unique \( \alpha \) in the root system \( \Phi \) of the pair \( (G,T) \) such that \( C = U_\alpha x \), where \( U_\alpha \) is the root subgroup of \( G \) associated to \( \alpha \). It follows that every \( T \)-curve \( C \) is smooth, and \( C^T = \{x, y\} \) where either \( y < x \) or \( x < y \). Moreover, \( y = r_\alpha \), where \( r_\alpha \) is the reflection in \( W \) associated to \( \alpha \). Following [7], \( E(X,x) \) will denote the set of \( T \)-curves in \( X \) containing \( x \in X^T \), and \( TE(X,x) \) will be the \( T \)-subspace of \( T_x(X) \) spanned by the tangent lines \( T_x(C) \) as \( C \) varies through \( E(X,x) \). By [5], \( \dim TE(X,x) = \|E(X,x)\| \geq \dim X \).

Let \( X \) be a \( T \)-variety, and assume \( C \in E(X,x) \). The Peterson translate \( \tau_C(X,x) \) is the limit at \( x \) of the Zariski tangent spaces \( T_z(X) \) to \( X \) along the open orbit in \( C \). That is,

\[
\tau_C(X,x) = \lim_{z \to x} T_z(X) \quad (z \in C\setminus C^T).
\]

This limit always exists and is a \( T \)-stable subspace of \( T_x(X) \) [7]. An element \( C \) of \( E(X,x) \) is called good if \( C \) contains a smooth point of \( X \). If \( C \) is good, then \( \dim \tau_C(X,x) = \dim X \), and \( X \) is smooth at \( x \) if and only if \( \tau_C(X,x) = T_x(X) \). Finally, if \( x \in X^T \), let \( \Theta_x(X) \) be the linear span in \( T_x(X) \) of the reduced tangent cone to \( X \). Clearly, \( TE(X,x) \subset \Theta_x(X) \). Moreover, \( \Theta_x(X) = T_x(X) \) if \( X \) is smooth at \( x \). The classification of the smooth \( T \)-fixed points of a \( T \)-variety \( X \) in \( G/P \) is a corollary of the following two results, both proved in [7]. Note that Theorem 1.1 does not require the \( G_2 \)-restriction, but Theorem 1.2 does.

**Theorem 1.1.** Suppose \( \dim X \geq 2 \), and \( x \in X^T \). Then \( X \) is smooth at \( x \) if and only if \( E(X,x) \) contains at least two good \( T \)-curves \( C \) such that \( \tau_C(X,x) = TE(X,x) \). Moreover, if \( X \) is Cohen-Macaulay at \( x \), then smoothness at \( x \) is equivalent to \( \tau_C(X,x) = TE(X,x) \) for just a single good \( C \).

**Theorem 1.2.** For every good \( C \in E(X,x) \),

\[
\tau_C(X,x) \subset \Theta_x(X).
\]

Moreover, if \( C = U_\alpha x \), where \( \alpha \) is short, then

\[
\tau_C(X,x) \subset TE(X,x).
\]
Since Schubert varieties are Cohen-Macaulay, Theorem 1.2 says that if the reduced tangent cone to a Schubert variety $X$ at a $T$-fixed point $x$ is linear and $E(X,x)$ contains a good curve, then $X$ is smooth at $x$. This will be key to our result on the Bruhat graph stated below.

If $G$ is simply laced, then one already knows ([5, 6]) that $TE(X,x) = \Theta_x(X)$ for all Schubert varieties $X$ in $G/B$, hence in $G/P$ also. The first result in this paper is that $TE(X,x) = \Theta_x(X)$ indeed holds for all $T$-varieties (see § 3). We now state our first main result.

**Theorem 1.3.** Let $X$ be a Schubert variety in $G/P$, and suppose that $x \in X_T$ is either a smooth point or a maximal singularity of $X$. Then

$$\Theta_x(X) = \sum \tau_C(X,x),$$

where the sum is over all good $C \in E(X,x)$.

The proof is given in Section 4. Note that the $G_2$-restriction is necessary here. Indeed, we give an example of a Schubert variety in $G_2/B$ for which (2) fails (Example 4.8). If $G$ is simply laced, then Theorem 1.3 follows immediately from Theorem 1.2 and the fact that $\Theta_x(X) = TE(X,x)$.

Theorem 1.3 gives a general geometric description of $\Theta_x(X)$ provided we know the good $T$-curves and how to compute Peterson translates. In fact, the good $T$-curves at a maximal singularity $x$ in $X$ are those $C \in E(X,x)$ such that $C \not\subset X(x)$, or, equivalently, $C^T = \{x,y\}$ where $y > x$. By Deodhar’s Inequality [5], $E(X,x)$ contains at least $\dim X - \dim X(x)$ good $T$-curves. In particular, every maximal singular point lies on at least one good $C$. In fact, since Schubert varieties are smooth in codimension one [8], each maximal singular point is on at least two good $C$.

Our problem is thus to describe $\sum \tau_C(X,x)$ as $C$ varies over the good $T$-curves. First of all, it suffices to assume $X$ is a Schubert variety in $G/B$. Indeed, the natural $G$-equivariant map $\pi : G/B \to G/P$ is a smooth closed morphism, so knowing $\Theta_x(X)$ at a maximal singular point $x$ for $X \subset G/P$ amounts to knowing $\Theta_y(Y)$, where $Y$ is the Schubert variety $\pi^{-1}(X) \subset G/B$, and $y$ is a maximal singularity of $Y$. Hence, we can focus our attention on characterizing $\Theta_x(X)$ at a maximal singularity $x$ in the $G/B$ setting. To do this, we use the algorithm for computing $\tau_C(X,x)$ stated in [7, §8]. This algorithm actually requires that the characteristic of $k$ be good (that is, $\text{char}(k) \neq 2, 3$), which was overlooked in [7], and, as well, [9]. See Remark 5.3 for further details.

To describe further what is needed, let $\Phi^+$ be the set of positive roots, i.e. those roots $\alpha$ of $(G, T)$ such that $U_\alpha \subset B$. Let $B_x \subset B$ be the isotropy subgroup of $x$ in $B$: namely the subgroup of $B$ generated by $T$ and all root subgroups $U_\alpha \subset B$ such that $U_\alpha x = x$ (equivalently, $x^{-1}(\alpha) > 0$). Clearly $\Theta_x(X)$ is a $B_x$-submodule of $T_x(X)$. The isotropy submodule of $X$ at $x$ is defined as the smallest $B_x$-module
$\mathbb{T}_x(X)$ such that
\[
(3) \quad TE(X, x) \subset \mathbb{T}_x(X) \subset \Theta_x(X).
\]

We will show that if $C \in E(X, x)$ is good, then the roots corresponding to $T$-lines in the $T$-module $\tau_C(x, x)/((\mathbb{T}_x(X) \cap \tau_C(x, x))$ arise from an orthogonal $B_2$-pair, which we define next. For each $\gamma \in \Phi$, let $g_\gamma$ denote the $T$-line of weight $\gamma$ in $g = \text{Lie}(G)$.

**Definition 1.4.** Let $X = X(w)$ be a Schubert variety in $G/B$, and assume $x < w$. Suppose $\mu$ and $\phi$ are long, positive orthogonal roots such that the following three conditions hold:

(i) $g_{-\mu} \oplus g_{-\phi} \subset TE(X, x)$ (hence $x < r_{\mu}x, r_{\phi}x \leq w$),

(ii) there exists a subroot system $\Phi'$ of $\Phi$ of type $B_2$ containing $\mu$ and $\phi$, and

(iii) if $\alpha$ and $\beta$ form the unique basis of $\Phi'$ contained in $\Phi^+ \cap \Phi'$ with $\alpha$ short and $\beta$ long, then

\[
r_\alpha x < x, \quad \text{and} \quad r_\alpha r_\beta x \leq w.
\]

Then we say that $\{\mu, \phi\}$ form an orthogonal $B_2$-pair for $X$ at $x$.

The notion of an orthogonal $B_2$-pair arises from the Schubert variety $X = X(r_\alpha r_\beta r_\alpha)$ in $B_2/B$, where $\alpha$ and $\beta$ are respectively short and long simple roots in $\Phi^+(B_2)$. The $T$-fixed point $x = r_\alpha$ is the unique maximal singularity of $X$. Now the weights of $TE(X, x)$ are $\alpha, -\beta$ and $-(\beta + 2\alpha)$. Furthermore, $B_x$ is generated by $T, U_{\beta}, U_{\alpha+\beta}$ and $U_{2\alpha+\beta}$, so it is easy to see that $TE(X, x)$ is already a $B_x$-submodule of $T_x(X)$. The point is that $\{\beta, \beta + 2\alpha\}$ is an orthogonal $B_2$-pair at $x$ such that $g_\gamma \subset \Theta_x(X)/TE(X, x)$, where $\gamma = -1/2(\mu + \phi) = -(\alpha + \beta)$. (See Example 5.2 and [7] for more details.)

Recall that the Bruhat graph $\Gamma(X)$ of a $T$-variety $X$ is the graph whose vertex set is $X^T$ such that two vertices $x, y$ are joined by an edge of $\Gamma(X)$ if and only if there exists a $T$-curve $C \subset X$ such that $C^T = \{x, y\}$. Figure 1 below shows the part of the Bruhat graph of the Schubert variety coming from a $B_2$-pair at $x$.

Our second characterization of $\Theta_x(X)$ at a maximal singularity goes as follows.

**Theorem 1.5.** Suppose the characteristic of $k$ is good and $x$ is a maximal singularity of a Schubert variety $X$ in $G/B$. Then for each $T$-weight $\gamma$ of the quotient $\Theta_x(X)/\mathbb{T}_x(X)$, there exists an orthogonal $B_2$-pair $\{\mu, \phi\}$ for $X$ at $x$ such that

\[
(4) \quad \gamma = -1/2(\mu + \phi).
\]

In other words, at a maximal singularity of $X$, every $T$-weight of $\Theta_x(X)$ not in $\mathbb{T}_x(X)$ arises from a $B_2$-pair at $x$ as in (4).
This is proved in § 5. In the course of the proof, we also obtain the following necessary and sufficient condition for a $T$-fixed point $x$ of a Schubert variety to be a smooth point.

**Theorem 1.6.** Assume the characteristic of $k$ is good, and let $X$ be a Schubert variety in $G/B$. Suppose $x \in X^T$ lies on a good $T$-curve. Then $X$ is smooth at $x$ if and only if the following three conditions simultaneously hold.

(i) $|E(X, x)| = \dim X$;
(ii) $T_x(X) = TE(X, x)$; and
(iii) if $\{\mu, \phi\}$ is an orthogonal $B_2$-pair for $X$ at $x$ and $\gamma = -1/2(\mu + \phi)$, then $\fr g_\gamma \subset TE(X, x)$ (i.e. $r_\gamma x \leq w$).

**Corollary 1.7.** Let $X \subset G/B$ be a Schubert variety with at most an isolated singularity $x$. Then $X$ is smooth if and only if $\dim T_x(X) = \dim X$.

**Proof.** Clearly an isolated singularity $x$ has to be the minimal element of $W$, i.e. $x = e$. But then there cannot be a $B_2$-pair at $x$. Hence conditions (i), (ii) and (iii) of the Theorem are trivially satisfied if and only if $\dim T_x(X) = \dim X$.

Theorem 1.6 can be formulated as an algorithm for locating the maximal singularities.

**Corollary 1.8.** Let $X \subset G/B$ be a Schubert variety. Then there exists a non-recursive algorithm involving only the Bruhat graph $\Gamma(X)$ and the root system $\Phi$ which classifies the smooth $T$-fixed points of $X$.

Let us describe the algorithm. Suppose we want to determine whether $X = X(w)$ is smooth at some $x \in X^T$. Consider any descending path

$$w > x_1 > x_2 > \cdots > x_m > x$$
in $\Gamma(X)$. If $X$ is singular at any $x_i$, then it is singular at $x$. Thus, we may assume $X$ is smooth at $x_i$. Then the edge joining $x_m$ and $x$ is a good $T$-curve $C$ in $X$, so it suffices to check the conditions of Theorem 1.6. Checking that $|E(X,x)| = \dim X$ is simply counting the edges of $\Gamma(X)$ at $x$. This is equivalent to showing $|\{\gamma > 0 \mid r_\gamma x \leq w]\} = \ell(w)$, where $\ell(w)$ is the length of $w$ with respect to $\Phi^+$, since $\ell(w) = \dim X(w)$. Verifying the second condition amounts to showing that $TE(X,x)$ is $B_x$-stable, which requires verifying that if $g_\gamma \subset TE(X,x)$, then $g_{\gamma+\alpha} \subset TE(X,x)$ for all $\alpha > 0$ such that $x^{-1}(\alpha) > 0$, $\gamma+\alpha \in \Phi$ and $x^{-1}(\gamma+\alpha) < 0$. The third condition is also verified by inspecting the Bruhat graph at $x$, so the algorithm involves only $\Phi$ and $\Gamma(X)$. The algorithm is nonrecursive since it only involves a single path from $w$ to $x$.

The problem of classifying the smooth points of a Schubert variety has consequences for the Schubert calculus. It is also related to the problem of determining the rationally smooth points of a Schubert variety. This has consequences in representation theory. If $G$ is defined over $\mathbb{C}$ and is simply laced, a result of D. Peterson (proved in [7]) tells us that every rationally smooth point of a Schubert variety in $G/P$ is in fact smooth. In general, however, the well-known criterion in terms of the Bruhat graph for locating the rationally smooth points ([5]) gives a recursive algorithm, since it requires that one calculate the number of edges in $\Gamma(X)$ at all vertices $y \geq x$. B. Boe and W. Graham have conjectured that a Schubert variety $X$ in $G/B$ is rationally smooth at $x \in X^T$ if and only if $|E(X,y)| = \dim X$ for all $y \in X^T$ such that either $y = x$ or $y > x$ and is on an edge of $\Gamma(X)$ containing $x$. Some special cases of the lookup conjecture are verified in [4], but the general conjecture is open. Theorem 1.6 says that as far as smoothness is concerned, one has to examine $\Gamma(X)$ along a single path two steps above and one step below a maximal $x$. This might be considered somewhat unexpected.

Finally, let us mention that this paper has connections with the work of S. Billey and A. Postnikov [3] and very likely also S. Billey and T. Braden [1]. However, unlike the situation in [3], our results do not say anything in the $G_2$ case, as noted in Remark 4.8.

2. Preliminaries. The terminology and notation of [7] (and that introduced in Section 1) will be used throughout the paper. The $G_2$-restriction is always in affect, although many statements we make are true without it.

Let us first mention a few standard facts concerning roots, weights, $T$-curves and so forth. The projection $\pi: G/B \rightarrow G/P$ is an equivariant, closed morphism, so $(G/P)^T$ may be identified with $W/W_P$, where $W_P$ is the parabolic subgroup of $W$ associated to $P$. The elements of $W/W_P$ thus parameterize the Schubert varieties in $G/P$. Every $T$-curve in a Schubert variety $X$ in $G/P$ containing an $x \in X^T$ has the form $C = U_\alpha x$ for a unique root $\alpha \in \Phi$, the root system of $(G,T)$. Moreover, $C^T = \{x, r_\alpha x\}$. If $X$ is a Schubert variety in $G/B$, say $X = X(w)$, then $C = U_\alpha x \subset X$ if and only if both $x, r_\alpha x \leq w$. By [5, LEMMA A], $|E(X,x)| \geq \dim X$ for every $T$-variety $X$. (This is one form of Deodhar’s
Inequality.) Furthermore, every $T$-curve in $G/P$ is the image of a $T$-curve in $G/B$ under the closed morphism $\pi : G/B \to G/P$. Also, recall that as $T$-modules,

$$T_x(G/B) = \bigoplus_{x^{-1}(\gamma) < 0} g_\gamma.$$

Two properties of $T$-varieties in $G/P$, used freely throughout the paper, are the following: first, each $T$-fixed point $x \in G/P$ is attractive in the sense that all the weights of the tangent space $T_x(G/P)$ lie on one side of a hyperplane in $X(T)$. Secondly, each fixed point $x$ has a $T$-stable open affine neighborhood. Since $X$ is irreducible and any $x \in X^T$ is attractive, the affine open $T$-stable neighborhood of $x$ is unique. It will be denoted by $X_x$. It is well known, and not hard to see, that there is a closed $T$-equivariant embedding of $X_x$ into the tangent space $T_x(X)$ of $X$ at $x$, thanks to the fact that $x$ is attractive.

Hence, we may assume $X_x \subset T_x(X)$. It follows that, for any $T$-stable line $L \subset T_x(X)$, we may choose a linear equivariant projection $T_x(X) \to L$ and restrict it to $X_x$. Identifying $L$ with $\mathbb{A}^1_k$ we thus obtain a regular function $f \in k[\tilde{x}_i]$, which is a $T$-eigenvector of weight $-\alpha$ if $L$ has weight $\alpha$. We will say $f$ corresponds to $L$ if it is obtained in this way.

### 3. Some General Results on $\Theta_\alpha(X)$.

The purpose of this section is to establish some general properties of an arbitrary $T$-variety $X$ in $G/P$, which are well known for Schubert varieties (see [5, 6]). In particular, we will prove that in the simply laced case, $\Theta_\alpha(X) = TE(X, x)$. Let $\mathfrak{T}_x(X)$ be the reduced tangent cone to $X$ at any $x \in X^T$, so $\Theta_\alpha(X) = \text{span}_k(\mathfrak{T}_x(X))$. As always, $G$ has no $G_2$-factors. We will assume, as in the previous section, that $X_x \subset T_x(X)$.

**Theorem 3.1.** Let $L = g_\omega \subset \Theta_\alpha(X)$ be a $T$-stable line with weight $\omega$. Then the following hold:

(i) If $\omega$ is long, then $L \subset TE(X, x)$. Otherwise, there exist roots $\alpha, \beta$ such that $g_\alpha, g_\beta \subset TE(X, x)$ and

$$\omega = \frac{1}{2}(\alpha + \beta).$$

(ii) In particular, if $G$ is simply laced, then $\Theta_\alpha(X) = TE(X, x)$.

(iii) If $X$ is a Schubert variety and $L$ does not correspond to a $T$-curve, then $\alpha$ and $\beta$ are long negative orthogonal roots in a copy of $B_2 \subset \Phi$.

**Proof.** Let $z \in k[X_x]$ be a $T$-eigenfunction corresponding to $L$, and let $x_1, x_2, \ldots, x_n \in k[X_x]$ be $T$-eigenfunctions which correspond to the $T$-curves $C_1, C_2, \ldots, C_n$ through $x$. Notice that since $X_x \subset T_x(X)$, each $T$-curve $C \in E(X_x, x)$ is in fact a coordinate line in $T_x(X)$. This follows from the fact that all $T$-curves are smooth and no two $T$-weights of $T_x(X)$ are proportional. Let $\tilde{x}_i$, resp. $\tilde{z}$ denote linear projections $T_x(X) \to T_x(C_i)$, resp. $T_x(X) \to L$, which restrict to $x_i, z \in k[X_x]$. 

Since the (restriction of the projection $X_x \to \bigoplus C T_x(C) = TE(X,x)$) has a finite fibre over 0, $k[x]_1$ is a finite $k[x_1, x_2, \ldots, x_n]$-module by the graded version of Nakayama’s Lemma. In particular $z \in k[x]_1$ is integral over $k[x_1, \ldots, x_n]$, so we obtain a relation

$$z^N = p_{N-1}z^{N-1} + p_{N-2}z^{N-2} + \cdots + p_1z + p_0,$$

where $N$ is a suitable integer and $p_i \in k[x_1, \ldots, x_n]$. Without loss of generality we may assume that every summand on the right hand side is a $T$-eigenvector with weight $N \omega$. Let $P_i \in k[\tilde{x}_1, \ldots, \tilde{x}_n]$ be a polynomial restricting to $p_i$, having the same weight $(N - i) \omega$ as $p_i$. Then every monomial $m$ of $P_i$ has this weight too. If for all $i$ every such monomial $m$ has degree $m > N - i$, then $p_i \tilde{z}^{N-i}$ is an element of $M^{N+1}$, where $M$ is the maximal ideal of $x$ in $k[x]_1$. This means that $\tilde{z}$ vanishes on the tangent cone of $X_x$, so $L \not\subset \Theta_x(X)$, which is a contradiction. Thus, there is an $i$ and a monomial $m$ of $P_i$, such that $\deg m \leq d = N - i$. Let $m = c\tilde{x}_1^{d_1} \tilde{x}_2^{d_2} \cdots \tilde{x}_n^{d_n}$, with integers $d_j$ and a nonzero $c \in k$. So $\sum_j d_j \leq d$. Let $\alpha_j$ be the weight of $\tilde{x}_j$. Then we have

$$d\omega = \sum d_j \alpha_j.$$

After choosing a new index, if necessary, we may assume that $d_j \neq 0$ for all $j$. Let $(\ , \ )$ be a Killing form on $X(T) \otimes \mathbb{R}$ which induces the length function on $\Phi$. We have to consider two cases. First suppose that $\omega$ is a long root, with length say $l$. Then $(\alpha_j, \omega) \leq l^2$ with equality if and only if $\alpha_j = \omega$. Thus, $dl^2 = \sum d_j(\alpha_j, \omega) \leq d \max_j (\alpha_j, \omega) \leq dl^2$, and so there is a $j$ with $\alpha_j = \omega$. This implies $\tilde{z} = \tilde{x}_j$. Hence, $L = \mathcal{C}_j$.

Now suppose $\omega$ is short, with its length also denoted $l$. In this case $(\alpha_j, \omega) \leq \hat{l}^2$. Since $dl^2 = d(\omega, \omega) = \sum d_j(\alpha_j, \omega)$ and since $\sum d_j \leq d$, it follows that all $\alpha_j$ satisfy $(\alpha_j, \omega) = \hat{l}^2$. If there is a $j$ such that $\alpha_j = \omega$, then, as above, we are done. Otherwise for each $j$, $\alpha_j$ is long, and $\alpha_j$ and $\omega$ are contained in a copy $B(\hat{j}) \subset \Phi$ of $B_2$. There is a long root $\beta_j \in B(\hat{j})$ with $\alpha_j + \beta_j = 2\omega$. We have to show that there are $j_0$ and $j_1$ so that $\beta_{j_0} = \alpha_{j_0}$. Fix $j_0 = 1$ and let $\alpha = \alpha_1$, $\beta = \beta_1$. Then $(\alpha, \beta) = 0$. This gives us the result $dl^2 = d(\omega, \beta) = 0 + \sum_{j \geq 1} d_j(\alpha_j, \beta)$. Now if all $(\alpha_j, \beta)$ are less or equal $\hat{l}^2$, this last equation cannot hold, since $\sum_{j \geq 1} d_j \leq d$. We conclude that there is a $j_1$ so that $(\alpha_{j_1}, \beta) = 2\hat{l}^2$ (the squared long root length), hence $\alpha_{j_1} = \beta$, and we are through with (i).

The proof of (ii) is obvious. For (iii), let $S$ be the slice (cf. [7]) to $X(w)$ at $x$. Then, locally, $X = S \times Bx$, where the weights of $TE(S,x)$ consist of the roots $\alpha < 0$ such that $x < r_{s,\alpha} \leq w$. Since $L \not\subset TE(X,x)$, the only possibility is that $L \subset \Theta_x(S)$ because $Bx$ is smooth (and so $TE(Bx,x) = \Theta_x(Bx)$) and $\Theta_x(X) = \Theta_x(S) \oplus \Theta_x(Bx)$.

Now we may apply part (i) to $S$. \qed
The following generalizes a well-known property of Schubert varieties.

**Corollary 3.2.** Suppose $L$ is a $T$-invariant line $\mathcal{T}_r(X)$. Then $L \subset T \mathcal{E}(X, x)$. 

**Proof.** We have already shown that in equation (6), some $P_i$ contains a monomial of degree at most $d = N - i$, and we have seen this is also the minimal degree possible. Taking homogeneous parts of degree $N$ in (6), we therefore get a homogeneous polynomial

$$f = \tilde{z}^N - \sum P_j \tilde{z}^j$$

vanishing on $\mathcal{T}_r(X)$. Hence $f(L) = 0$. But as $\tilde{z}(L) \neq 0$, this implies some $P_j(L) \neq 0$ as well, which means that $\tilde{z}$ occurs in a monomial of $P_j$, hence $L \subset T \mathcal{E}(X, x)$ by the construction of the $P_j$. □

Another interesting consequence is that the linear spans of the tangent cones of two $T$-varieties behave nicely under intersections.

**Corollary 3.3.** Suppose that $G$ is simply laced and that $X$ and $Y$ are $T$-varieties in $G/P$. Suppose also that $x \in (X \cap Y)^T$. Then

$$\Theta_x(X \cap Y) = \Theta_x(X) \cap \Theta_x(Y).$$

Consequently, if both $X$ and $Y$ are nonsingular at $x$, then $X \cap Y$ is nonsingular at $x$ if and only if $|E(X \cap Y, x)| = \dim (X \cap Y)$.

**Proof.** The first claim is clear since $E(X, x) \cap E(Y, x) = E(X \cap Y, x)$. For the second, note that if $X$ and $Y$ are nonsingular at $x$, then

$$T_x(X) \cap T_x(Y) = \Theta_x(X) \cap \Theta_x(Y) = \Theta_x(X \cap Y) \subset T_x(X \cap Y) \subset T_x(X) \cap T_x(Y).$$

Hence $\dim T_x(X \cap Y) = |E(X \cap Y)|$, and the result follows. □

For example, it follows that in the simply laced setting, the intersection of a Schubert variety $X(w)$ and a dual Schubert variety $Y(v) = B^{-y}$ is nonsingular at any $x \in [v, w]$ as long as $X(w)$ and $Y(v)$ are each nonsingular at $x$.

**4. $\Theta_x(X)$ at a Maximal Singularity.** The aim of this section is to prove Theorem 1.3. In fact, we will derive it as a consequence of a general result about
the relationship between \( \tau_C(X,x) \) and \( \Theta_s(X) \), when \( X \) is an arbitrary \( T \)-variety in \( G/P \) and \( x \) is at worst an isolated singularity. As usual, \( G \) has no \( G_2 \)-factors.

**Theorem 4.1.** Suppose \( X \subset G/P \) is a \( T \)-variety. Then for each \( x \in X^T \), we have

\[
\Theta_s(X) \subset \tau(X,x) := \sum_{C \in E(X,x)} \tau_C(X,x).
\]

In particular, if \( x \) is either smooth in \( X \) or an isolated singularity, then

\[
\Theta_s(X) = \sum_{C \in E(X,x)} \tau_C(X,x).
\]

Before proving Theorem 4.1, let us derive Theorem 1.3.

**Proof of Theorem 1.3.** The result is obvious if \( x \) is smooth, so assume \( x \) is a maximal singularity. Then there exists a slice representation \( X_x = S \times Bx \), where \( S \) has an isolated singularity at \( x \) and \( E(S,x) \) consists of the \( T \)-curves in \( X \) containing a smooth point of \( X_x \). To get the result, we apply Theorem 4.1 to \( S \) and use the fact that \( \Theta_s(X) = \Theta_s(S) \oplus \Theta_s(Bx) \). Indeed,

\[
\Theta_s(S) \oplus \Theta_s(Bx) = \sum_{C \in E(S,x)} \tau_C(S,x) \oplus TE(Bx,x),
\]

so it suffices to show that \( TE(Bx,x) \subset \tau_C(X,x) \) for any \( C \in E(S,x) \) since clearly \( \tau_C(S,x) \subset \tau_C(X,x) \). Let \( g_\gamma \subset TE(Bx,x) \). Then there is a curve \( D \subset Bx \) with \( g_\gamma = T_x(D) \). In fact, \( D = U_\gamma x \). Thus, the smooth \( T \)-stable surface \( \Sigma = C \times D \) is contained in \( X_x = S \times Bx \), and Proposition 3.4 of [7] implies \( g_\gamma \subset \tau_C(\Sigma,x) \subset \tau_C(X,x) \). \( \square \)

The proof of Theorem 4.1 will require several lemmas. To begin with, let \( R \) be a Noetherian graded commutative ring with irrelevant ideal \( I = \bigoplus_{d \geq 0} R_d \). Then \( \bigcap_{d \geq 0} I^d = 0 \). Thus, for each \( r \in R \setminus \{0\} \), there is an \( l > 0 \) such that \( r \in I^l \setminus I^{l+1} \).

We set \( in(r) = r + I^{l+1} \in I^l/I^{l+1} \subset gr R = gr_1 R \). Recall that for \( r, s \in R \), either in \( (r) \) in \( (s) \) or in \( (r) \) in \( (s) \) = 0. We say \( r \in R \) vanishes on the tangent cone if in \( (r) \) does, i.e. if in \( (r) \) is nilpotent. In the case that \( R \) is the coordinate ring of an affine variety \( Z \) with regular \( \mathbb{G}_m \)-action such that \( I \) corresponds to a maximal ideal and hence to an attractive \( \mathbb{G}_m \)-fixed point \( z \), then in \( (r) \) induces a function on the reduced tangent cone of \( Z \) at \( z \), and \( r \) vanishes on the tangent cone if and only if this function does. In what follows, we will consider closed and \( T \)-stable subvarieties of \( T_x(X) \). We therefore choose a one parameter subgroup \( \lambda \) of \( T \) such that \( \lim_{t \to 0} \lambda(t)v = 0 \) for all \( v \in T_x(X) \). Then the \( \mathbb{G}_m \)-action \( \lambda^{-1} \) induces a (positive) grading of \( k[T_x(X)] \) which carries over to any \( T \)-stable closed subvariety. (Note that the grading induced by \( \lambda \) is negative.)
For convenience, we allow $\Theta_*(Z)$ to be defined for reducible varieties. Notice that $\Theta_*(Z)$ may be canonically identified with $T_0(\mathfrak{T},(Z)) \subset T_*(Z)$. We wish to set up an induction on the dimension of $X$, so we need the following:

**Lemma 4.2.** Let $Z \subset T_*(X)$ be a closed $T$-stable subvariety with $Z = Z_1 \cup Z_2 \cup \cdots \cup Z_d$ its decomposition into irreducible components. Then

$$\Theta_0(Z) = \Theta_0(Z_1) + \Theta_0(Z_2) + \cdots + \Theta_0(Z_d).$$

*Proof.* Since every component $Z_i$ of $Z$ is $T$-stable, it has to contain 0. Therefore the proof is a simple consequence of the following well known fact: if a variety $Y = A \cup B$ is the union of two closed subvarieties, then for every point $x$ in the intersection $A \cap B$ we have $\mathfrak{T}_x(Y) = \mathfrak{T}_x(A) \cup \mathfrak{T}_x(B)$.

Let $Z \subset T_*(X)$ be an irreducible $T$-stable subvariety, and let $L \subset \Theta_0(Z)$ be a $T$-stable line with weight $\omega$. Moreover, suppose $\omega$ is short with respect to a Killing form $(\ ,\ )$ on $X(T)$. Denote by $z \in k[Z]$ the restriction of a linear $T$-equivariant projection $T_*(X) \to L \cong \mathbb{A}^1_k$.

**Lemma 4.3.** With the preceding notation, let $f \in k[Z]$ correspond to another $T$-equivariant linear projection onto some line $L' \subset T_*(X)$. Then $z$ vanishes on the tangent cone of $V(f)$ if and only if in $(z)^l = in(h) in(f)$ for some positive integer $l$ and a suitable $T$-eigenvector $h \in k[Z]$.

*Proof.* The sufficiency is clear, so suppose $z$ vanishes on the tangent cone of $V(f)$. By definition this means there is an integer $l$ and elements $g_1, g_2, \ldots, g_r \in I(V(f))$, the ideal of $V(f)$, such that in $(z)^l = a_1 in(g_1) + a_2 in(g_2) + \cdots + a_r in(g_r)$ for suitable $a_i \in grk[Z]$. Since in $(z)$ is homogeneous and since in $(I(V(f)))$ is a homogeneous ideal, we may assume that all of the $a_i$ are homogeneous as well. Moreover the $a_i$ and $g_i$ may be chosen to be $T$-eigenvectors. Omitting any indices $i$ for which $a_i in(g_i) = 0$, we may lift the $a_i$ equivariantly to $\bar{a}_i \in k[Z]$ such that in $(\bar{a}_i) = a_i$. Then $0 \neq in(\bar{a}_i) in(g_i) = in(a_i g_i)$. Leaving out degrees different from $l$, we may assume that $\sum in(\bar{a}_i) in(g_i) = in(\sum \bar{a}_i g_i)$. Now $\sum \bar{a}_i g_i$ is a $T$-eigenvector $g$ contained in the ideal of $V(f)$. A suitable $n$th power of $g$ is contained in $f k[Z]$. Now in $(z)$ is not nilpotent, and since in $(z)^l = in(g)$, in $(g)$ is also not nilpotent. Therefore in $(g)^n = in(g^n)$. Replacing $l$ by $nl$ we may assume that in $(z)^l = in(g)$ for some $g \in f k[Z]$. In other words in $(z)^l = in(hf)$ for some $T$-eigenvector $h \in k[Z]$. It remains to show that in $(hf) = in(h) in(f)$, which is equivalent to in $(h) in(f) \neq 0$. So suppose that in $(h) in(f) = 0$. This means that $h \notin M^{l-1}$ where $M$ is the maximal ideal of 0. For otherwise in $(h) in(f)$ would equal in $(hf)$ by definition, as their degrees would agree. We conclude that $h \in M^n$ for some $n < l - 1$. Thus there is a homogeneous polynomial $P$ in certain linear $T$-homogeneous coordinates $x_1, x_2, \ldots, x_m$ of $T_*(X)$ of the same $T$-weight as $h$ having degree $n$ such that, restricted to $Z$, $h = P$ modulo $M^{n+1}$. By
the definition of $f$, we may even assume that $x_1$ restricted to $Z$ is $f$. Replacing $P$ by any monomial of $P$ and letting $d_i$ be the degree of $x_i$ in $P$, we see that $l\omega = \alpha_1 + \sum d_i\alpha_i$, where $\alpha_i$ denotes the weight of $x_i$. Applying $(\ , \omega)$ on both sides gives $l(\omega, \omega) = (\alpha_1, \omega) + \sum d_i(\alpha_i, \omega)$. Since $(\alpha_i, \omega) \leq (\omega, \omega)$ for all $i$, this is impossible since $n = \sum d_i < l - 1$. This finishes the proof.

As an easy consequence we get:

**Lemma 4.4.** If $Z$ and $z$ are as above and $f$ corresponds to the projection to any other $T$-stable line of $T_x(X)$ with a short weight, then $z$ cannot vanish on the tangent cone of $V(f)$.

**Proof.** By the last lemma, we know that if $z$ vanishes on the tangent cone of $V(f)$, then there is a $T$-eigenvector $h \in k[Z]$ such that in $(z) = \langle h \rangle$ in $(f)$. Choosing a monomial as in the proof of the previous Lemma, we get a relation of the form $\omega = \alpha_1 + \sum d_i\alpha_i$ with $\sum d_i = l - 1$. But $(\alpha_1, \omega) < (\omega, \omega)$ because $\alpha_1$ is short and $(\alpha_i, \omega) \leq (\omega, \omega)$ for all $i$, so no such relation exists.

We now restrict our attention to varieties $Z$ in $T_x(X)$ such that $T_0(Z)$ contains exactly one $T$-stable line with a short weight. That is, $L$ is a short line.

**Lemma 4.5.** If $L \subset \Theta_0(Z)$ is the only short line in $T_0(Z)$, and if $C \in \mathcal{E}(Z, 0)$ is any $T$-curve, then $L \subset T_p(Z)$ for all $p \in C^0 = C \setminus \{0\}$.

**Proof.** Choose an equivariant embedding $Z \subset T_0(Z)$. If $C = L$ as a subset of $T_0(Z)$, there is nothing to show. Otherwise $C$ is a coordinate line of $T_0(Z)$ having a long $T$-weight. Call this weight $\alpha$. If $L \not\subset T_p(Z)$ for a $p \in C^0$, there is a $T$-eigenfunction $f$ in the ideal of $Z$ in $k[T_0(Z)]$, such that $df_p(L) \neq 0$. We may assume that $k[T_0(Z)] = k[z, x_1, x_2, \ldots, x_n]$ with $z$ as above corresponding to $L$ and the $x_i$ corresponding to the long lines of $T_0(Z)$. Then we write $f = P_0 + P_1z + P_2z^2 + \cdots + P_dz^d$, where the $P_i$ are $T$-eigenvectors and polynomials in the $x_i$ only. Without loss of generality, we may assume $P_1z^l$ has the same weight as $f$. It follows that $df_p = dP_0p + P_1(p)dz_p$ because $z$ vanishes on $C$. By assumption $P_1(p)$ is nonzero, implying that there is a monomial of the form $x^l$ contained in $P_1$, where $x$ is the coordinate corresponding to $C$ and $l \geq 1$. Thus, the $T$-weight of $f$ is $l\alpha + \omega$. On the other hand $P_0$ is nonzero. For if $P_0 = 0$, then $f$ is divisible by $z$, and therefore $f = hz$ for some $h$. But $Z$ is irreducible and clearly $z$ does not vanish on $Z$, so $h$ vanishes on $Z$. Now $z$ and $h$ vanish at $p$ forcing $df_p$ to be zero as well, which is a contradiction. With $P_0$ being nonzero, it follows that there is a monomial in the $x_i$ having weight $l\alpha + \omega$. This clearly shows that $\omega = (l\alpha + \omega) - l\alpha$ is contained in the $Z$-submodule of $X(T)$ generated by all long weights of $T_0(Z)$. The next lemma shows that this is impossible and therefore finishes the proof.
LEMMA 4.6. Let $\Gamma$ be a $\mathbb{Z}$-submodule of $X(T)$ generated by long roots. If the Killing form is normalized so that $(\omega, \omega) = 1$ is the short root length, then the function $f \colon \Gamma \to \mathbb{Q}$ given by $f(\gamma) = (\gamma, \gamma)$ takes its values in $2\mathbb{Z}$.

Proof. If $\alpha$ and $\beta$ are long roots, then $(\alpha, \beta) \in \mathbb{Z}$. Indeed, $(\alpha, \beta) \in \{0, \pm 1, \pm 2\}$ by general properties of root systems. Hence, $(\gamma, \delta) \in \mathbb{Z}$ for all $\gamma, \delta \in \Gamma$ as well. Moreover $f(\alpha) = 2$ for all long roots. Now $f(\alpha + \beta) = f(\alpha) + f(\beta) + 2(\gamma, \delta) \in 2\mathbb{Z}$ provided $f(\gamma), f(\delta) \in 2\mathbb{Z}$. We can now induct on the length of a shortest representation $\gamma = \sum n_i \alpha_i$, where the $n_i \in \mathbb{Z}$ and $\alpha_1, \alpha_2, \ldots$ are the long generators of $\Gamma$. By the length of such a representation, we mean $\sum |n_i|$). So, if $n_1$ is nonzero and positive, then $\gamma = \alpha_1 + (n_1 - 1)\alpha_1 + \sum_{i>2} n_i \alpha_i$. The induction hypothesis for $\alpha_1$ and $(n_1 - 1)\alpha_1 + \sum_{i>2} n_i \alpha_i$ gives the result for $\gamma$ by the above arguments. If $n_1$ is negative we may use $-\gamma$, since $f(\gamma) = f(-\gamma)$. Finally, if $n_1$ is zero, we may replace $\alpha_1$ with any other $\alpha_i$ such that $n_i \neq 0$. 

We are now in a position to prove the Theorem 4.1.

Proof. We proceed by induction on $\dim Z$ for an irreducible $T$-stable subvariety $Z \subset T_0(X)$. Of course there is nothing to show when $\dim Z \leq 1$. If $\dim Z > 1$, let $L \subset \Theta_0(Z)$ be any $T$-stable line that has a short weight $\omega$, say. Let $z$ be a corresponding function of $k[Z]$. Suppose there is another line with short weight in $T_0(Z)$. By the previous lemma, if $f$ is a corresponding function, $z$ does not vanish on the tangent cone of $\mathcal{V}(f)$. Thanks to Lemma 4.2, $z$ does not vanish on the tangent cone of at least one irreducible component $Z'$ of $\mathcal{V}(f)$. In particular this implies that $L$ is contained in $\Theta_0(Z')$. By induction $L \subset \tau(Z', 0) \subset \tau(Z, 0)$. This concludes the case that there is a short root line in $T_0(Z)$ different from $L$. So suppose $L$ is the only line in $T_0(Z)$ with a short weight. Then $L \subset T_0(Z)$ for all $p \in C'$ and any curve $C \in \mathcal{E}(Z, 0)$. For each such $C$ it then follows that $L \subset \tau(C, 0)$. By Theorem 3.1 all the lines in $\Theta_0(Z)$ with long $T$-weights are tangent to $T$-curves, so they are contained in $\tau(Z, 0)$.

We complete this section with an example that shows the $G_2$-restriction is necessary. We will need the following general fact about $\Theta_0(X)$ proved in [6].

PROPOSITION 4.7. Suppose $X$ is a Schubert variety in $G/B$ and $x \in X^T$. Let $\mathcal{H}$ denote the convex hull in $\Phi \otimes \mathbb{R}$ of the $T$-weights of $TE(X, x)$. Then every $T$-weight of $\Theta_0(X)$ lies in $\mathcal{H}$.

Example 4.8. Now suppose $\alpha$ and $\beta$ are the short and long simple roots in the root system of $G_2$, and consider the Schubert variety $X$ in $G_2/B$ corresponding to $w = r_{\beta} r_{\alpha} r_{\beta} r_{\alpha}$. By [2, p. 168], the singular locus of $X$ is the Schubert variety $X(r_{\beta} r_{\alpha})$, so $x = r_{\beta} r_{\alpha}$ is a maximal singularity. By a direct check, the $T$-weights of $TE(X, x)$ are $-\alpha, \beta, \alpha + \beta$ and $-\lambda$, where $\lambda = 3\alpha + 2\beta$ is the longest root. Thus the weights in $\mathcal{H}$ are $-\alpha, \beta, \alpha + \beta, -(2\alpha + \beta)$, and $-\lambda$. The good $T$-curves in $E(X, x)$ correspond to $-\alpha$ and $-\lambda$. We claim that $-(3\alpha + \beta)$ is a weight in
\( \tau_C(X, x) \), where \( C \) corresponds to \(-\lambda\). Indeed, put \( y = r_\lambda x \). Then one sees that the weights of \( TE(X, y) \) are \( \beta, \alpha + \beta, -(2\alpha + \beta) \) and \( \lambda \). By inspection, \( TE(X, y) \) is a \( g_{-\lambda} \)-submodule of \( T_\gamma(G_2/B) \), so, by the algorithm in [7, §3] (summarized in Remark 5.3 below), the weights of \( \tau_C(X, x) \) are obtained by reflecting the weights of \( TE(X, y) \) by \( r_\lambda \). Thus \( \tau_C(X, x) \) has weights

\[
\begin{align*}
r_\lambda(\beta) &= -(3\alpha + \beta), \\
r_\lambda(\alpha + \beta) &= -(2\alpha + \beta), \\
r_\lambda(-(2\alpha + \beta)) &= \alpha + \beta, \\
\end{align*}
\]

Since \(-(3\alpha + \beta)\) isn’t in \( \mathcal{H} \), Theorem 1.3 fails without the \( G_2 \)-restriction.

5. The proofs of Theorems 1.5 and 1.6. Let \( X = X(w) \) be a Schubert variety in \( G/B \) where, as usual, \( G \) does not contain any \( G_2 \)-factors. We will assume henceforth that \( \text{char}(k) \neq 2, 3 \) (see Remark 5.3).

The goal of this section is to study the \( T \)-weights in \( \tau_C(X, x) \) when \( C \) is good. Assume \( C^T = \{x, y\} \), where \( y > x \), and note that \( X \) is smooth at \( y \). Thus we can write \( C = U_{-\mu} x \), where \( \mu > 0 \), and \( y = r_\mu x \). By Theorem 1.1, if \( \mu \) is short, then \( \tau_C(X, x) \subset TE(X, x) \). Hence we can ignore this case and suppose \( \mu \) is long. Also, if \( g_\gamma \subset \Theta_\tau(X) \) and \( \gamma \) is long, then \( g_\gamma \subset TE(X, x) \) [6].

To begin, we need a result similar to Theorem 3.1 for \( \tau_C(X, x) \).

**Lemma 5.1.** Suppose \( \gamma \) is a short root such that \( g_\gamma \subset \tau_C(X, x) \). If \( g_\gamma \nsubseteq TE(X, x) \), then there exists a long root \( \phi \) orthogonal to \( \mu \) such that \( g_\phi \subset TE(X, x) \), and

\[
\gamma = -\frac{1}{2}(\mu + \phi).
\]

In addition, the roots \( \gamma, \mu, \phi \) lie in a copy of \( B_2 \) contained in \( \Phi \).

**Proof.** This follows from [7, Lemma 5.1 and Proposition 5.2]. \[\square\]

We will see below that if \( g_\gamma \nsubseteq TE(X, x) \), then \( \phi > 0 \). As noted in the Introduction, the notion of an orthogonal \( B_2 \)-pair arises from the following illuminating example worked out in detail in [7, Example 8.4].

**Example 5.2.** Let \( G \) be of type \( B_2 \), and let \( w = r_\alpha r_\beta r_\alpha \), where \( \alpha \) is the short simple root and \( \beta \) is the long simple root. Put \( X = X(w) \). The singular set of \( X \) is \( X(r_\alpha) \), so \( x = r_\alpha \) is \( X \)'s unique maximal singular point. There are two good \( T \)-curves at \( x \), namely \( C = U_{-\beta} x \) and \( D = U_{-(2\alpha + \beta)} x \). Suppose \( y = r_\beta x \) and \( z = r_{2\alpha + \beta} x \). Then

\[
T_y(X) = g_{-\alpha} \oplus g_{\alpha + \beta} \oplus g_\beta \quad \text{and} \quad T_z(X) = g_\alpha \oplus g_{-(\alpha + \beta)} \oplus g_{2\alpha + \beta}.
\]

Thus (cf. Remark 5.3),

\[
\tau_C(X, x) = g_\alpha \oplus g_{-(\alpha + \beta)} \oplus g_\beta \quad \text{and} \quad \tau_D(X, x) = g_\alpha \oplus g_{-(\alpha + \beta)} \oplus g_{-(2\alpha + \beta)}.
\]
Note that the weight at \( x \) that does not give a \( T \)-curve, namely \(-(\alpha + \beta)\), is in both Peterson translates.

**Remark 5.3.** We will use the algorithm for Peterson translates in [7, §3] in several places, so let us briefly recall how it works. The reason for assuming \( \text{char}(k) \neq 2, 3 \) is mentioned below. Unfortunately, this assumption was omitted in both references [7] and [9]. Suppose \( C = U_{-\mu}X \), where \( \mu > 0 \) and \( y = r_{\mu}x \). Consider the weights of the form \( \nu + h\mu \) in \( T_y(X) \), and form a (possibly partial) \( \mu \)-string consisting of roots of the form \( \kappa - j\mu \), where \( 0 \leq j \leq r \), such that \( y^{-1}(\kappa - j\mu) < 0 \) for each \( j \), but \( y^{-1}(\kappa - (r + 1)\mu) > 0 \), and \( r \) is the number of elements of \( \kappa + \mathbb{Z}\mu \) which are weights in \( T_y(X) \). Then the roots \( r_{\mu}(\kappa - j\mu) \) occur as weights in \( \tau_C(X, x) \), and every weight occurring in \( \tau_C(X, x) \) arises in this way. This follows from the fact that \( \tau_C(X, x) \) is a \( g_\mu \)-module and \( \{g_{\mu}, g_\alpha\} = g_{\mu + \alpha} \), provided \( \text{char}(k) \neq 2, 3 \).

The next result extends the above example to the general case. Let \((, )\) be a \( W \)-invariant inner product on \( X(T) \otimes \mathbb{R} \).

**Theorem 5.4.** Suppose \( \gamma \) is a short root such that \( g_\gamma \subset \tau_C(X, x) \). If either \((\gamma, \mu) \geq 0\), or in the expression (7) one has \( \phi < 0 \), then \( g_\gamma \subset TE(X, x) \). On the other hand, if \( g_\gamma \nsubseteq TE(X, x) \), then the following statements hold:

(a) \( \gamma < 0 \).

(b) \((\gamma, \mu) < 0\), hence \( \delta := \gamma + \mu \subset \Phi \).

(c) if \( x^{-1}(\delta) < 0 \), then \( g_\delta \subset \tau_C(X, x) \cap TE(X, x) \) (and, of course, conversely), and

(d) \( \phi > 0 \).

**Remark 5.5.** Example 5.2 shows that one can have \( g_\gamma \subset \tau_C(X, x) \cap TE(X, x) \) yet still have \((\gamma, \mu) < 0\).

**Proof.** If \((\gamma, \mu) \geq 0\), it follows immediately from Lemma 5.1 that \( g_\gamma \subset TE(X, x) \). Suppose \( \gamma \) has the form (7), where \( \phi < 0 \), and put \( \delta = \gamma + \mu \). Since \((\gamma, \mu) < 0\), \( \delta \subset \Phi \). Moreover, since \( \phi < 0 \), we have \( \delta > 0 \). Now if \( \gamma > 0 \), then \( r_\gamma x < x \), since \( x^{-1}(\gamma) < 0 \). Thus \( g_\gamma \subset TE(X, x) \) if \( \gamma > 0 \).

Next, suppose \( \gamma < 0 \). We will consider the two cases \( x^{-1}(\delta) < 0 \) and \( x^{-1}(\delta) \geq 0 \) separately. Assume first that \( x^{-1}(\delta) < 0 \). Since \( \tau_C(X, x) \) is a \( g_\mu \)-submodule of \( T_x(X) \) (cf. [7, §3]) and \( g_\gamma \subset \tau_C(X, x) \), we therefore know that

\[ g_\delta \oplus g_\gamma \subset \tau_C(X, x) \]

Since \( \mu \) is long and there are no \( G_2 \)-factors, Proposition 8.1 [7] implies

\[ g_\delta \oplus g_\gamma \subset T_y(X) \]

Since \( \gamma < 0 \), we therefore get the inequality \( y < r_\gamma y \leq w \), and hence \( X \) is also nonsingular at \( r_\gamma y \). Moreover, since \( \phi < 0 \) and \( x^{-1}(\phi) = y^{-1}(\phi) > 0 \), it also
follows that $g_{-\phi} \subset TE(X, y)$, which equals $T_\gamma(X)$ since $X$ is smooth at $y$. Since there are no $G_2$-factors, $\mu$, $\delta$, $-\phi$ constitute a complete $\gamma$-string occurring as $T$-weights of $T_\gamma(X)$. Letting $E$ be the good $T$-curve in $X$ such that $E^T = \{y, r_\gamma y\}$, we have $T_\gamma(X, y) = T_\gamma(X)$, so the string $\mu$, $\delta$, $-\phi$ also has to occur in the $T$-weights of $T_{r_\gamma y}(X)$. In particular, $g_{-\phi} \subset TE(X, r_\gamma y) = T_{r_\gamma y}(X)$, and hence $r_{\phi} r_\gamma y \leq w$. But this means

$$r_\gamma x = r_\gamma r_\mu y = r_\gamma r_\mu r_\gamma y = r_\phi r_\gamma y \leq w,$$

so $g_\gamma \subset TE(X, x)$.

Next, assume $x^{-1}(\delta) > 0$. Since $\mu$ is long, $r_\mu(\delta) = \delta - \mu = \gamma$, hence $y^{-1}(\delta) = x^{-1}(\gamma) < 0$. Thus, since $\delta > 0$, $g_\delta \subset T_\gamma(X)$. Furthermore,

$$y^{-1}( - \gamma) = -x^{-1} r_\mu(\gamma) = -x^{-1}(\delta) < 0,$$

so $g_{-\gamma} \subset T_\gamma(X)$. It follows that $r_\gamma y < y$. As $-\phi > 0$, $U_{-\phi} r_\gamma y \subset X$ as well. We claim $U_{-\phi} r_\gamma y \neq r_\gamma y$, which then proves that $r_{\phi} r_\gamma y \leq w$. But

$$(r_\gamma y)^{-1}( - \phi) = y^{-1}(r_\gamma( - \phi)) = y^{-1}(\mu) < 0,$$

hence we get the claim. Finally, we note that $r_{\phi} r_\gamma r_\mu = r_\gamma$, so it follows that $r_\gamma x \leq w$. Therefore, if $\phi < 0$, we get $g_\gamma \subset TE(X, x)$.

Now suppose $g_\gamma \not\subset TE(X, x)$. Then (a) is immediate and (b) follows from the first statement of the Theorem. Assume that $x^{-1}(\delta) < 0$. Since $T_\gamma(X, x)$ is a $g_\mu$-submodule of $T_\gamma(X)$, we get that $g_\delta \subset T_\gamma(X, x)$. As $\delta = \gamma + \mu$ and $\mu$ is long, we also get that $(\delta, \mu) \geq 0$. Applying the first part of the Theorem again, we see that $g_\delta \subset TE(X, x)$. This establishes (c). The assumption that $g_\gamma \not\subset TE(X, x)$ immediately implies that $\phi$ is positive, giving (d).

\[
\text{Remark 5.6.} \text{ Let } X \text{ be a Schubert variety, and suppose } x \in X^T \text{ is a maximal singularity such that } |E(X, x)| = \dim X. \text{ In this case, the second author has shown that the multiplicity } \tau_\gamma(X) \text{ of } X \text{ at } x \text{ is exactly } 2^d, \text{ where }
\]

$$d = |\{\alpha \in x(\Phi^-) \mid g_\alpha \subset \tau_\gamma(X, x) \text{ and } r_\alpha x \not\leq w\}|,$$

for any good $C \in E(X, x)$ [9].

**Theorem 5.7.** Suppose $C = U_{-\mu} x$ is a good $T$-curve, where $\mu > 0$, and let $y = r_{\mu} x$. Assume $g_\gamma \subset \tau_\gamma(X, x)$ but $g_\gamma \not\subset \tau_\gamma(X)$. Then there exists a positive root $\phi$ such that $\{\mu, \phi\}$ is an orthogonal $B_2$-pair for $X$ at $x$ such that $\gamma = -1/2(\mu + \phi)$. Conversely, suppose that for some $\phi > 0$, $\{\mu, \phi\}$ is an orthogonal $B_2$-pair for $X$ at $x$, and $\gamma = -1/2(\mu + \phi)$. Then $g_\gamma \subset \tau_\gamma(X, x)$.

**Proof.** Suppose $g_\gamma \subset \tau_\gamma(X, x)$ but $g_\gamma \not\subset \tau_\gamma(X)$. By Theorem 3.1, there exists a long positive root $\phi$ orthogonal to $\mu$ such that $\gamma = -1/2(\mu + \phi)$. Put $y = r_{\mu} x$,
and note $X$ is smooth at $y$. To show that $\{\mu, \phi\}$ is an orthogonal $B_2$-pair, we have to consider two cases.

Case 1. $\mu$ is simple. Then $\alpha = \gamma + \phi$ is the short simple root. We have to show that if $g_\gamma \not\subset T_x(X)$, then $r_\alpha x < x$ and $r_\alpha r_\mu x \leq w$. But $g_\gamma \not\subset T_x(X)$ implies $x^{-1}(\alpha) < 0$, since if $x^{-1}(\alpha) > 0$, then the fact that $\gamma = -\phi + \alpha$ would say $g_\gamma \subset T_x(X)$. Hence $r_\alpha x < x$.

Since $r_\mu(\alpha) = -\gamma$, it follows that $y^{-1}(\alpha) = x^{-1}(-\gamma) > 0$, so $g_{-\alpha} \subset T_y(G/B)$. But $y^{-1}(\gamma) = x^{-1}(-\alpha) > 0$, hence $g_\gamma \not\subset T_y(G/B)$. Hence, by the algorithm for computing the Peterson translate and the fact that $g_\gamma \subset \tau_C(x, x)$, we infer that $g_{-\alpha} \subset T_y(X)$. Therefore, $r_\alpha y = r_\alpha r_\mu x \leq w$, as was to be shown.

Case 2. $\phi$ is simple. Here $\alpha = \gamma + \mu$ is the short simple root, and $r_\mu(\gamma) = \alpha$. As in Case 1, $x^{-1}(\alpha) < 0$, so $r_\alpha y < x$. Now $y^{-1}(\alpha) = x^{-1}(\gamma) < 0$, so $r_\alpha y < y$ and hence $g_\alpha \subset T_y(X)$. Also, $y^{-1}(\gamma) = x^{-1}(\alpha) < 0$, so $g_\gamma \subset T_y(G/B)$. Thus the algorithm for $\tau_C(X, x)$ says that $g_\alpha \subset \tau_C(X, x)$. But as $g_\gamma \subset \tau_C(X, x)$ too, we have to conclude that $g_\gamma \subset T_y(X)$, due to the fact that $\gamma$ and $\alpha$ comprise a $\mu$-string. Hence $r_\gamma y \leq w$. But since we are in a $B_2$ where $\alpha$ and $\phi$ are the simple roots, $r_\gamma r_\mu = r_\alpha r_\phi$. Hence $r_\alpha r_\phi x \leq w$, so Case 2 is finished.

To prove the converse, we need to consider Cases 1 and 2 again with the assumption that $x^{-1}(\alpha) < 0$, which follows from the condition that $r_\alpha x < x$. The argument is, in fact, very similar to the above, but we will outline it anyway. Assume first that $\mu = \beta$, i.e. $\mu$ is simple. As $r_\alpha r_\beta x \leq w$, we see that $r_\alpha y \leq w$. But $y^{-1}(-\alpha) = x^{-1}(\gamma) < 0$, consequently $g_{-\alpha} \subset T_y(X)$. Also, $y^{-1}(\gamma) = x^{-1}(-\alpha) > 0$, so $g_\gamma \not\subset T_y(G/B)$. Thus, by the algorithm for computing $\tau_C(x, x)$, the weight $r_{\beta}(\gamma)$ occurs in $\tau_C(X, x)$. Hence $g_\gamma \subset \tau_C(X, x)$.

On the other hand, if $\phi$ is simple, then $\mu = \beta + 2\alpha$. Thus, $y^{-1}(\alpha) = x^{-1}(\gamma) < 0$, so $r_\alpha y < y$, hence $g_\alpha \subset T_y(X)$. But $r_\alpha r_\phi x \leq w$ means $r_\gamma r_\mu x \leq w$, that is, $r_\gamma y \leq w$. As $y^{-1}(\gamma) = x^{-1}(\alpha) < 0$, $g_\alpha \oplus g_\gamma \subset T_y(Y)$. Since $\alpha$ and $\gamma$ make up a $\beta + 2\alpha$-string in $B_2$, $g_\alpha \oplus g_\gamma \subset \tau_C(X, x)$ also. This finishes the proof.

We now prove Theorems 1.5 and 1.6.

Proof of Theorem 1.5. Suppose $g_\gamma \subset \Theta_\gamma(X)$. Since $x$ is either smooth or a maximal singularity, Theorem 1.3 implies that $g_\gamma \subset \tau_C(X, x)$ for some good $C$. If $C$ is short, then $\tau_C(X, x) \subset TE(X, x)$, by Theorem 1.1, hence $\tau_C(X, x) \subset T_x(X)$. Thus we can suppose $C$ is long. But then, by Theorem 5.7, either $g_\gamma \subset T_x(X)$ or there exists a $B_2$-pair $\{\mu, \phi\}$ for $X$ at $x$ such that $\gamma = -1/2(\mu + \phi)$. Hence Theorem 1.5 is proven.

Proof of Theorem 1.6. Suppose $C \subset E(X, x)$ is good and $\dim TE(X, x) = \dim T_x(X) = \dim X$. If $C$ is short, then $X$ is smooth at $x$ by Theorem 1.1. Hence we may suppose $C$ is long. Suppose there exists a $T$-line $g_\gamma$ in $\tau_C(X, x)$ which is not in $T_x(X)$. Then by Theorem 5.7, there is an orthogonal $B_2$-pair $\{\mu, \phi\}$ for
X at x for which \( \gamma = -1/2(\mu + \phi) \). But then by assumption, \( g_\gamma \subset TE(X,x) \). This contradicts the choice of \( g_\gamma \), so \( \tau_C(X,x) \subset T(x) = TE(X,x) \). Applying Theorem 1.1 again, we see that X is smooth at x.

For the converse, suppose X is smooth at x. Then conditions (1) and (2) of Theorem 1.5 clearly hold. Suppose \( \{\mu, \phi\} \) is a \( B_2 \)-pair for X at x and \( \gamma = -1/2(\mu + \phi) \). By the converse assertion of Theorem 5.7, \( g_\gamma \subset \tau_C(X,x) \), where C \( \in E(X,x) \) is the T-curve of weight \( \mu \) at x. Since x is smooth, \( \tau_C(X,x) = TE(X,x) \), so \( g_\gamma \subset TE(X,x) \). \( \Box \)

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