EFFICIENT ALGEBRAIC TWO-LEVEL SCHWARZ PRECONDITIONER FOR SPARSE MATRICES*
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Abstract. Domain decomposition methods are among the most efficient for solving sparse linear systems of equations. Their effectiveness relies on a judiciously chosen coarse space. Originally introduced and theoretically proved to be efficient for self-adjoint operators, spectral coarse spaces have been proposed in the past few years for indefinite and non-self-adjoint operators. This paper presents a new spectral coarse space that can be constructed in a fully-algebraic way unlike most existing spectral coarse spaces. We present theoretical convergence result for Hermitian positive definite diagonally dominant matrices. Numerical experiments and comparisons against state-of-the-art preconditioners in the multigrid community show that the resulting two-level Schwarz preconditioner is efficient especially for non-self-adjoint operators. Furthermore, in this case, our proposed preconditioner outperforms state-of-the-art preconditioners.

Key words. Algebraic domain decomposition, Schwarz preconditioner, sparse linear systems, diagonally dominant matrices.

1. Introduction. In this paper, we develop an algebraic overlapping Schwarz preconditioner for the linear system of equations

$$Ax = b,$$

for a sparse matrix $A \in \mathbb{C}^{n \times n}$ and a given vector $b \in \mathbb{C}^n$. Solving sparse linear systems of equations is omnipresent in scientific computing. Direct approaches, based on Gaussian elimination, have proved to be robust and efficient for a wide range of problems [23]. However, the memory required to apply sparse direct methods often scales poorly with the problem size, particularly for three-dimensional discretizations of partial differential equations (PDEs). Furthermore, the algorithms underpinning sparse direct software are poorly suited to parallel computation, which makes them difficult to adapt to emerging computing architectures.

Iterative methods for solving linear systems [45] have been an active research topic since early computers’ days. Their simple structure, at their most basic level requiring only matrix-vector multiplication and vector-vector operations, makes them attractive for tackling large-scale problems. However, since the convergence rate depends on the properties of the linear system, iterative methods are not, in general, robust. For the class of iterative methods known as Krylov subspace methods, we may alleviate this by applying a preconditioner, which transforms the problem into one with more favorable numerical properties. The choice of preconditioner is usually problem-dependent, and a wide variety of preconditioning techniques have been proposed to improve the convergence rate of iterative methods, see for example the recent survey [44] and the references therein.

Multilevel domain decomposition (DD) and multigrid methods are widely used preconditioners [21, 47, 52, 53, 54]. They have proved to be effective on a wide variety of matrices, but they are especially well suited to sparse Hermitian positive definite (HPD) matrices arising from the discretization of PDEs. Their efficiency stems from

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a judicious combination of a cheap fine-level solver with a coarse-space correction. In the last two decades, there has been a great advance in the development of spectral coarse spaces that yield efficient preconditioners. Spectral coarse spaces were initially proposed in the multigrid community for elliptic PDEs with self-adjoint operators [15, 19, 24, 34], and similar ideas were later picked up by the DD community for the same kind of problems [2, 3, 4, 7, 33, 32, 40, 48, 49]. The past three years have seen several approaches to tackle symmetric indefinite systems and non-self-adjoint problems. For example, spectral coarse spaces for least squares problems and symmetric indefinite saddle-point systems were proposed in [5, 39], where the problem is returned into the framework of self-adjoint operators. An exciting new development is that a number of multigrid methods and spectral coarse spaces have been suggested for problems with indefinite or non-self-adjoint operators [9, 10, 11, 12, 13, 14, 22, 36, 37]. These coarse spaces are mainly based on heuristics and show efficiency on several challenging model problems arising from discretized PDEs.

A variety of mathematical tools such as the fictitious subspace lemma [41] and local Hermitian positive semi-definite (HPSD) splitting matrices [2] are now available to analyze and propose effective coarse spaces for self-adjoint operators. However, these tools may not be directly used for indefinite or non-self-adjoint operators. An alternative approach to studying the convergence of DD methods in the indefinite or non-self-adjoint case is to use Elman’s theory [25] of GMRES convergence, see for example [8, 10, 18].

In this work, we propose a fully-algebraic spectral coarse space for the two-level Schwarz preconditioner for general sparse matrices. We review the overlapping DD framework in section 2, including a summary of the main features of local HPSD splitting matrices. For each subdomain, we introduce the local block splitting using lumping in the overlap of a sparse matrix in section 3. The coarse space is then constructed by solving locally and concurrently a generalized eigenvalue problem involving the local block splitting matrix and the local subdomain matrix. In the case where the matrix is HPD diagonally dominant, we prove that the local block splitting matrices are local HPSD splitting matrices, and in that case we show that one can bound the condition number of the preconditioned matrix from above by a user-defined number. Based on this heuristic, we generalize our approach for other cases. Unlike most existing spectral coarse spaces, especially those suggested for indefinite or non-self-adjoint operators, we obtain the matrices involved in the local generalized eigenvalue problem efficiently from the coefficient matrix; the preconditioner is therefore fully-algebraic. In order to assess the proposed preconditioner, we provide in section 4 a set of numerical experiments on problems arising from a wide range of applications including convection-diffusion equation and other linear systems from the SuiteSparse Collection [20]. Furthermore, we compare our proposed preconditioner against state-of-the-art preconditioners in the multigrid community. Finally, we give concluding remarks and future lines of research in section 5.

Notation. Let $1 \leq n \leq m$ and let $M \in \mathbb{C}^{m \times n}$ be a complex sparse matrix. Let $[1, p]$ denote the set of the first $p$ positive integers, and let $S_1 \subset [1, m]$ and $S_2 \subset [1, n]$. $M(S_1, :)$ is the submatrix of $M$ formed by the rows whose indices belong to $S_1$ and $M(:, S_2)$ is the submatrix of $M$ formed by the columns whose indices belong to $S_2$. $M(S_1, S_2)$ denotes the submatrix formed by taking the rows whose indices belong to $S_1$ and only retaining the columns whose indices belong to $S_2$. $[S_1, S_2]$ means the concatenation of any two sets of integers $S_1$ and $S_2$, where the order of the concatenation is important. $I_n$ is the identity matrix of size $n$, the transpose matrix of $M$ is denoted $M^\top$, and the adjoint of $M$, denoted $M^H$, is the conjugate transpose of $M$. 


of \( M \), i.e., \( M^H = M^\top \). \( \ker(M) \) and \( \text{range}(M) \) denote the null space and the range of \( M \), respectively.

2. Domain decomposition. Consider \( \mathcal{G}(A) \), the adjacency graph of the coefficient matrix in (1), and number its nodes, \( V \), from 1 to \( n \). Using a graph partitioning algorithm, we split \( V \) into \( N \ll n \) nonoverlapping subdomains, i.e., disjoint subsets \( \Omega_{I_i} \), \( i \in [1, N] \), of size \( n_{I_i} \). Let \( \Omega_{\Gamma_i} \) be the subset, of size \( n_{\Gamma_i} \), of nodes that are distance one in \( \mathcal{G}(A) \) from the nodes in \( \Omega_{I_i} \), \( i \in [1, N] \). We define the overlapping subdomain, \( \Omega_i \), as \( \Omega_i = [\Omega_{I_i}, \Omega_{\Gamma_i}] \), with size \( n_i = n_{\Gamma_i} + n_{I_i} \). The complement of \( \Omega_i \) in \([1, n]\) is denoted by \( \Omega_{c_i} \).

Associated with \( \Omega_{I_i} \) is a restriction (or projection) matrix \( R_{I_i} \in \mathbb{R}^{n_{I_i} \times n} \) given by \( R_{I_i} = I_{n(\Omega_{I_i},:)}. \) \( R_{I_i} \) maps from the global domain to subdomain \( \Omega_{I_i} \). Its transpose \( R_{I_i}^\top \) is a prolongation matrix that maps from subdomain \( \Omega_{I_i} \) to the global domain. Similarly, we define \( R_i = I_{n(\Omega_i,:)} \) as the restriction operator to the overlapping subdomain \( \Omega_i \).

We define the one-level Schwarz preconditioner as

\[
M_{\text{ASM}}^{-1} = \sum_{i=1}^{N} R_i^\top A_{ii}^{-1} R_i,
\]

where we assume \( A_{ii} = R_i A R_i^\top \) is nonsingular for \( i \in [1, N] \).

Applying this preconditioner to a vector involves solving concurrent local problems in each subdomain. Increasing \( N \) reduces the size of the subdomains, leading to smaller local problems and, correspondingly, faster computations. However, in practice, preconditioning by \( M_{\text{ASM}}^{-1} \) alone is often not be enough for convergence of the iterative solver to be sufficiently rapid. We can improve convergence, while still maintaining robustness with respect to \( N \), by applying a suitably chosen coarse space, or second-level [2, 8, 21, 27].

Let \( \mathcal{R} \subset \mathbb{C}^n \) be a subspace of dimension \( 0 < n_0 \ll n \) and let \( R_0 \in \mathbb{C}^{n_0 \times n} \) be a matrix such that the columns of \( R_0^H \) span the subspace \( \mathcal{R} \). Assuming that \( A_{00} = R_0 A R_0^H \) is nonsingular, we define the two-level Schwarz preconditioner as

\[
M_{\text{additive}}^{-1} = R_0^H A_{00}^{-1} R_0 + M_{\text{ASM}}^{-1}.
\]

Such preconditioners have been used to solve a large class of systems arising from a range of engineering applications (see, for example, [3, 5, 29, 35, 38, 47, 51] and references therein).

We denote by \( D_i \in \mathbb{R}^{n_{I_i} \times n_{I_i}}, \ i \in [1, N] \), any non-negative diagonal matrices such that

\[
\sum_{i=1}^{N} R_i^\top D_i R_i = I_n.
\]

We refer to \((D_i)_{1 \leq i \leq N}\) as an algebraic partition of unity.

Variants of one- and two-level preconditioners. The so-far presented Schwarz preconditioners are the additive one-level (2.1) and the additive two-level based on additive coarse space correction (2.2). It was noticed in [17] that scaling the one-level Schwarz preconditioner by using the partition of unity yields faster convergence. The resulting one-level preconditioner is referred to as restricted additive Schwarz and is
defined as:

\[
M_{\text{RAS}}^{-1} = \sum_{i=1}^{N} R_i^T D_i A_{ii}^{-1} R_i.
\]

Furthermore, there is a number of ways of how to combine the coarse space with a one-level preconditioner such as the additive, deflated, and balanced combinations, see for example [50]. Given a one-level preconditioner \(M^{-1}_*',\) where the subscript \(*\) stands for either ASM or RAS, the two-level preconditioner with additive coarse space correction is defined as

\[
M_{*,\text{additive}}^{-1} = R_H^H A_{00}^{-1} R_0 + M^{-1}_*.
\]

The two-level preconditioner based on a deflated coarse space correction is

\[
M_{*,\text{deflated}}^{-1} = R_H^H A_{00}^{-1} R_0 + M^{-1}_* (I - A R_H^H A_{00}^{-1} R_0).
\]

Due to its simple form, the additive two-level Schwarz based on the additive coarse space correction is the easiest to analyze. However, we observe that the deflated variant combined with the restricted additive Schwarz preconditioner has better performance in practice. The theory and presentation in this work employs the additive two-level Schwarz preconditioner using an additive coarse space correction, however, all numerical experiments involving the proposed preconditioner employ the restricted additive two-level Schwarz with deflated coarse space correction so that the two-level preconditioner used in section 4 reads as

\[
M_{*,\text{deflated}}^{-1} = R_H^H A_{00}^{-1} R_0 + M_{\text{RAS}}^{-1} (I - A R_H^H A_{00}^{-1} R_0).
\]

Note that the aforementioned variants are agnostic to the choice of the partitioning and the coarse space. That is, once the restriction operators to the subdomains and the coarse space are set, all these variants are available.

**Local HPSD splitting matrices of sparse HPD matrix.** A local HPSD matrix associated with subdomain \(i\) is any HPSD matrix of the form

\[
P_i \tilde{A}_i P_i^T = \begin{pmatrix}
A_{II,i} & A_{IG,i} \\
A_{GI,i} & \tilde{A}_{G,i}
\end{pmatrix},
\]

where \(P_i = I_n(\Omega_{ii}, \Omega_{ii}, \Omega_{ci}, \cdot)\) is a permutation matrix, \(A_{II,i} = A(\Omega_{ii}, \Omega_{ii}), A_{IG,i} = A_{GI,i} = A(\Omega_{ii}, \Omega_{Gi}),\) and \(\tilde{A}_{G,i}\) is any HPSD matrix such that the following inequality holds

\[
0 \leq u^T \tilde{A}_i u \leq u^T A u, \quad u \in \mathbb{C}^n.
\]

First presented and analyzed in [2], local HPSD splitting matrices provide a framework to construct robust two-level Schwarz preconditioners for sparse HPD matrices. Recently, this has led to the introduction of robust multilevel Schwarz preconditioners for finite element SPD matrices [3], sparse normal equations matrices [5], and sparse general SPD matrices [4].

**3. Two-level Schwarz preconditioner for sparse matrices.** We present in this section the construction of a two-level preconditioner for sparse matrices. First, we introduce a new local splitting matrix associated with subdomain \(i\) that uses local
values of $A$ to construct a preconditioner which is cheap to setup. In the special case where $A$ is HPD diagonally dominant, we prove that these local matrices are local HPD splitting matrices of $A$. We demonstrate that these matrices, when used to construct a GenEO-like coarse space, produce a two-level Schwarz preconditioner that outperforms existing two-level preconditioners in many applications, particularly in the difficult case where $A$ is not positive definite.

3.1. Local block splitting matrices of $A$ using lumping in the overlap.

**Definition 3.1.** Local block splitting matrix. Given the overlapping partitioning of $A$ presented in section 2, we have for each $i \in [1, N]$

$$P_i A P_i^T = \begin{pmatrix} A_{Ii} & A_{II} \\ A_{II} & A_{II} \\ A_{cI} & A_{ci} \end{pmatrix}$$

Let $s_i$ be the vector whose $j$th component is the sum of the absolute values of the $j$th row of $A_{cI}$, and let $S_i = \text{diag}(s_i)$. Define $\bar{A}_{Ii} = A_{II} - S_i$. The local block splitting matrix of $A$ associated with subdomain $i$ is defined to be $\bar{A}_i = R_i^T \bar{A}_{II} R_i$, where

$$\bar{A}_{II} = \begin{pmatrix} A_{Ii} & A_{II} \\ A_{II} & \bar{A}_{II} \end{pmatrix}.$$

Note that we only require the sum of the absolute values of each row in the local matrix $A_{cI}$ to construct the local block splitting matrix of $A$ associated with subdomain $i$. Then, each of these values is subtracted from the corresponding diagonal entry of the local matrix $A_{II}$. We can therefore construct $\bar{A}_i$ cheaply and concurrently for each subdomain.

The following lemma shows that if $A$ is HPD diagonally dominant, the local splitting matrices defined in Definition 3.1 are local HPD splitting matrices.

**Lemma 3.2.** Let $A$ be HPD diagonally dominant. The local block splitting matrix $\bar{A}_i$ defined in Definition 3.1 is local HPD splitting matrix of $A$ with respect to subdomain $i$.

**Proof.** First, note that the $j$th diagonal element of $\bar{A}_i$ is

$$\bar{A}_i(j, j) = \begin{cases} A(j, j) & \text{if } j \in \Omega_{II}, \\
A(j, j) - s_i(j) & \text{if } j \in \Omega_{II}, \\
0 & \text{if } j \in \Omega_{cI}, \end{cases}$$

where $s_i$ is the vector whose $j$th component is the sum of the absolute values of the $j$th row of $A_{cI}$, and $S_i = \text{diag}(s_i)$. Therefore, by construction, $\bar{A}_i$ is Hermitian diagonally dominant, hence HPD. Furthermore, $A - \bar{A}_i$ is Hermitian and diagonally dominant, hence HPD. By the local structure of $\bar{A}_i$, we conclude it is HPD splitting of $A$ with respect to subdomain $i$. 

3.2. Coarse space. In this section we present a coarse space for the two-level Schwarz preconditioner. For each $i \in [1, N]$, given the local nonsingular matrix $A_{ii}$, the local splitting matrix $\bar{A}_{ii} = R_i \bar{A}_{II} R_i^T$, and the partition of unity matrix $D_i$, let $L_i = \ker(D_i A_{ii} D_i)$ and $K_i = \ker(\bar{A}_{II})$. Now, define the following generalized eigenvalue problem:

$$\Pi_i D_i A_{ii} D_i \Pi_i u = \lambda \bar{A}_{ii} u,$$

(3.1)
where $\Pi_i$ is the projection on $\text{range} \left( \tilde{A}_{ii} \right)$.

Given a number $\tau > 0$, the coarse space we propose is defined to be the space generated by the columns of the matrix

$$R^H_i = \begin{bmatrix} R^T_1 D_1 Z_1 & \cdots & R^T_N D_N Z_N \end{bmatrix},$$

where $Z_i$ is the matrix whose columns form a basis of the subspace

$$(L_i \cap K_i)^{\perp_{K_i}} = \text{span} \left\{ u \mid \Pi_i D_i A_{ii} D_i \Pi_i u = \lambda \tilde{A}_{ii} u, |\lambda| > \frac{1}{\tau} \right\},$$

where $(L_i \cap K_i)^{\perp_{K_i}}$ is the complementary subspace of $(L_i \cap K_i)$ inside $K_i$.

Note that in the case where $A$ is sparse HPD diagonally dominant, $\tilde{A}_{ii}$ is a local HPSD splitting matrix of $A$, and the definition of the coarse space matches the one defined in [2]. Therefore, the two-level Schwarz preconditioner using the coarse space defined guarantees an upper bound on the condition number of the preconditioned matrix

$$\kappa_2 \left( M^{-1}_{\text{ASM, additive}} A \right) \leq (k_c + 1) \left( 2 + (2k_c + 1) \frac{k_m}{\tau} \right),$$

where $k_c$ is the number of colors required to color the graph of $A$ such that each two neighboring subdomains have different colors and $k_m$ is the maximum number of overlapping subdomains sharing a row of $A$. Therefore, when $A$ is sparse HPD diagonally dominant, the upper bound on $\kappa_2 \left( M^{-1}_{\text{ASM, additive}} A \right)$ is independent of $N$ and can be controlled by using the value $\tau$.

4. Numerical experiments. In this section, we validate the effectiveness of the two-level method when compared to other preconditioners. Table 1 presents a comparison between four preconditioners: $M^{-1}_{\text{deflated}}$ (2.4), BoomerAMG [26], GAMG [1], and AGMG [42]. The results are for the right-preconditioned GMRES [46] with a restart parameter of 30 and a relative tolerance set to $10^{-8}$. We highlight the fact that our proposed preconditioner can handle unstructured systems, not necessarily stemming from standard PDE discretization schemes, by displaying some nonzero patterns in Figure 1. For preconditioners used within PETSc [6] (all except AGMG), the systems are solved using 256 MPI processes. After loading them from disk, their symmetric part $A^T + A$ is first renumbered by ParMETIS [31]. The resulting permutation is then applied to $A$ and the corresponding linear systems are solved using a random right-hand side. The initial guess is always zero. The code that implements these steps is given in Appendix A. For our DD method, we leverage the PCHIPDDM framework [30] which is used to assemble spectral coarse spaces using (3.1). The new option `-pc_hpdmm_block_splitting`, introduced in PETSc 3.17, is used to compute the local splitting matrices of $A$ from subsection 3.1. At most 60 eigenpairs are computed on each subdomain and the threshold parameter $\tau$ from (3.2) is set to 0.3. These parameters were found to provide good numerical performance after a very quick trial-and-error approach on a single problem. We did not want to adjust them for each problem individually, but it will be shown next that they are fine overall without additional tuning.

Furthermore, a single subdomain is mapped to each process, i.e., $N = 256$ in (2.3). Eventually, exact subdomain and second-level operator LU factorizations are computed.
Table 1

Preconditioner comparison. Iteration counts are reported. $M_{\text{deflated}}^{-1}$ is the restricted two-level overlapping Schwarz preconditioner as in (2.4). No value denotes iteration count exceeds 100. † denotes either a failure in constructing the preconditioner or a breakdown in GMRES. ‡ denotes the problem is complex valued and the preconditioner is unavailable. Matrix identifiers that are emphasized correspond to symmetric matrices, otherwise matrices are non-self-adjoint.

| Identifier   | n   | nnz(A) | AGMG | BoomerAMG | GAMG | $M_{\text{deflated}}^{-1}$ | $n_\text{ig}$ |
|--------------|-----|--------|------|-----------|------|-----------------|-------------|
| light_in_tissue | 29,282 | 406,084 | 15   | †         | 53   | 6               | 7,230       |
| finan512     | 74,752 | 596,992 | 9    | 7         | 8    | 6               | 2,591       |
| consph       | 83,334 | 6,010,480 |         |           |      | 93              | 31,136      |
| Dubcova3     | 146,689 | 3,636,643 | 72   | 71        | 7    | 21,047          |             |
| CO           | 221,119 | 7,666,057 |      | 25        |      | 26              | 56,135      |
| nxp1         | 414,604 | 2,655,880 | □    | □         | □    | 20              | 19,707      |
| CoupCons3D   | 416,800 | 17,277,420 | □    | 26        | 20   | 28,925          |             |
| parabolic_fem| 525,825 | 3,674,625 | 12   | 8         | 16   | 5               | 24,741      |
| Chevrom4     | 711,450 | 6,376,412 | □    | □         | □    | 5               | 22,785      |
| apache2      | 715,176 | 4,817,870 | 14   | 11        | 35   | 8               | 45,966      |
| tm1_sym      | 726,713 | 5,080,961 | 14   | 10        | 17   | 5               | 28,253      |
| tm1_unsym    | 917,825 | 4,584,801 | 23   | 13        | 18   | 6               | 32,947      |
| ecology2     | 999,999 | 4,995,991 | 18   | 12        | 18   | 6               | 34,080      |
| thermal2     | 1,228,045 | 8,580,313 | 18   | 14        | 20   | 26              | 40,098      |
| atmosmodj    | 1,270,432 | 8,814,880 | □    | 8         | 17   | 7               | 76,368      |
| G3_circuit   | 1,585,478 | 7,660,826 | 25   | 12        | 35   | 8               | 71,385      |
| Transport    | 1,062,111 | 23,487,281 | 18   | 10        | 98   | 9               | 76,800      |
| memchip      | 2,707,524 | 13,343,948 | □    | 15        | □    | 36              | 57,942      |
| circuit5Mdc  | 3,523,317 | 14,965,409 | □    | 5         | □    | 7               | 8,629       |

A convection-diffusion problem will now be investigated. It reads:

$$\nabla \cdot (Vu) - \nu \nabla \cdot (\kappa \nabla u) = 0 \text{ in } \Omega$$

(4.1)

$$u = 0 \text{ in } \Gamma_0$$

$$u = 1 \text{ in } \Gamma_1.$$  

The problem is SUPG-stabilized [16] and discretized by FreeFEM [28]. It is important to keep in mind that the proposed preconditioner is algebraic, thus there is no specific transfer of information from the discretization kernel to solver backend. The domain $\Omega$ is either the unit square or the unit cube meshed semi-structurally to account for boundary layers, see an example of such a mesh in Figure 3a. The value of $\nu$ is
Table 2

Iteration counts of the proposed preconditioner for solving the two- and three-dimensional convection-diffusion problem from (4.1) with order \( k \) Lagrange finite element space. The number of subdomains is \( N \) and the size of the discrete system is \( n \). After each iteration count for each \( \nu \), the size of second-level operator is typeset between parentheses.

| Dimension | \( k \) | \( N \) | \( n \) | \( \nu \) | \( 10^{-1} \) | \( 10^{-2} \) | \( 10^{-3} \) | \( 10^{-4} \) |
|-----------|------|------|------|-----|------|------|------|------|
| 2         | 1    | 1.024 | 6.3 \( \cdot \) 10^6 | 23 (52,675) | 20 (52,672) | 19 (52,759) | 20 (47,497) | 21 (28,235) |
| 3         | 2    | 4096  | 8.1 \( \cdot \) 10^6 | 18 (1.8 \( \cdot \) 10^5) | 14 (1.8 \( \cdot \) 10^5) | 11 (1.6 \( \cdot \) 10^5) | 16 (97,077) | 29 (76,633) |

Table 3

Iteration counts of GAMG for solving the two- and three-dimensional convection-diffusion problem from (4.1). \( \dagger \) denotes either a failure to converge or a breakdown in GMRES.

| Dimension | \( n \) | \( \nu \) | \( 10^{-1} \) | \( 10^{-2} \) | \( 10^{-3} \) | \( 10^{-4} \) |
|-----------|------|-----|------|------|------|------|
| 2         | 6.3 \( \cdot \) 10^6 | 42 | 48 | 88 | \( \dagger \) | \( \dagger \) |
| 3         | 8.1 \( \cdot \) 10^6 | 40 | 38 | 65 | \( \dagger \) | \( \dagger \) |

constant in \( \Omega \). The value of \( \kappa \) is given in Figure 3b. The value of the velocity field \( V \) is either:

\[
V(x, y) = \begin{pmatrix} x(1-x)(2y-1) \\ -y(1-y)(2x-1) \end{pmatrix}
\]

or

\[
V(x, y, z) = \begin{pmatrix} 2x(1-x)(2y-1)z \\ -y(1-y)(2x-1) \\ -z(1-z)(2x-1)(2y-1) \end{pmatrix},
\]

in 2D and 3D, respectively. These are standard values taken from the literature [43]. The definition of \( \Gamma_0 \) and \( \Gamma_1 \) may be inferred by looking at the two- and three-dimensional solutions in Figures 3c to 3e and Figures 3f to 3h, respectively. The iteration counts reported in Table 2 show that the proposed preconditioner handles this problem, even as \( \nu \) tends to zero. In 2D, the operator, resp. grid, complexity is of at most 1.008, resp. 1.43. In 3D, these figures are 1.02 and 1.7, respectively. For comparison, GAMG and BoomerAMG iteration counts are also reported in Table 3 and Table 4, respectively.

5. Conclusion. We presented in this work a fully-algebraic two-level Schwarz preconditioner for large-scale sparse matrices. The proposed preconditioner combines a classic one-level Schwarz preconditioner with a spectral coarse space. The latter is constructed efficiently by solving concurrently in each subdomain a local generalized eigenvalue problem whose pencil matrices are obtained algebraically and cheaply from the local coefficient matrix. Convergence results were obtained for diagonally dominant HPD matrices. The proposed preconditioner was compared to state-of-the-art multigrid preconditioners on a set of challenging matrices arising from a wide range of applications including a convection-dominant convection-diffusion equation. The numerical results demonstrated the effectiveness and robustness of the proposed preconditioner especially for highly non-symmetric matrices.

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Fig. 2. (a) Mesh, (b) diffusivity coefficient, and solutions of some of the (c)–(e) two- and (f)–(h) three-dimensional test cases from Table 2.

Table 4
Iteration counts of BoomerAMG for solving the two- and three-dimensional convection-diffusion problem from (4.1). † denotes either a failure to converge or a breakdown in GMRES.

| Dimension | n     | $\nu = 1$ | $10^{-1}$ | $10^{-2}$ | $10^{-3}$ | $10^{-4}$ |
|-----------|-------|-----------|-----------|-----------|-----------|-----------|
| 2         | $6.3 \cdot 10^5$ | 50 | 49 | 19 | 7 | † |
| 3         | $8.1 \cdot 10^6$ | 12 | 9 | 7 | † | † |
Appendix A. Code reproducibility.

```c
#include <petsc.h>

static char help[] = "Solves a linear system after having repartitioned its symmetric part."

int main(int argc, char **args)
{
    Vec b;
    Mat A, perm;
    KSP ksp;
    IS is, rows;
    PetscBool flg;
    PetscViewer viewer;
    char name[ PETSC_MAX_PATH_LEN ];
    MatPartitioning mpart;

    ierr = PetscInitialize(& argc, & args, NULL, help); if (ierr) return ierr;
    ierr = MatCreate(PETSC_COMM_WORLD,&A); CHKERRQ(ierr);
    ierr = PetscOptionsGetString(NULL, NULL, "-mat_name ", name, sizeof(name), & flg);
    CHKERRQ(ierr);
    if (! flg) SETERRQ(PETSC_COMM_WORLD, PETSC_ERR_USER, "Missing -mat_name ");

    ierr = PetscViewerBinaryOpen(PETSC_COMM_WORLD, name, FILE_MODE_READ, & viewer );
    CHKERRQ(ierr);
    ierr = MatLoad(A, viewer ); CHKERRQ(ierr);
    ierr = PetscViewerDestroy(& viewer ); CHKERRQ(ierr);
    ierr = KSPCreate(PETSC_COMM_WORLD,& ksp); CHKERRQ(ierr);
    ierr = MatPartitioningCreate(PETSC_COMM_WORLD, & mpart ); CHKERRQ(ierr);

    Mat B, T;
    ierr = MatTranspose(A, MAT_INITIAL_MATRIX, & T); CHKERRQ(ierr);
    ierr = MatDuplicate(A, MAT_COPY_VALUES, & B); CHKERRQ(ierr);
    ierr = MatAXPY(B, 1.0, T, DIFFERENT_NONZERO_PATTERN); CHKERRQ(ierr);

    ierr = MatPartitioningSetAdjacency(mpart, B); CHKERRQ(ierr); // partition A^T+A
    ierr = MatPartitioningSetFromOptions(mpart); CHKERRQ(ierr);
    ierr = MatPartitioningApply(mpart, & is); CHKERRQ(ierr);
    ierr = MatDestroy(& B); CHKERRQ(ierr);
    ierr = MatDestroy(& T); CHKERRQ(ierr);

    ierr = MatPartitioningDestroy(& mpart); CHKERRQ(ierr);
    ierr = ISBuildTwoSided(is, NULL, & rows ); CHKERRQ(ierr);
    ierr = ISDestroy(& is); CHKERRQ(ierr);
    ierr = MatCreateSubMatrix(A, rows, rows, MAT_INITIAL_MATRIX, & perm); CHKERRQ(ierr);

    ierr = ISDestroy(& rows ); CHKERRQ(ierr);
    ierr = MatHeaderReplace(A, & perm); CHKERRQ(ierr);

    ierr = VecSetRandom(b, NULL); CHKERRQ(ierr);
    ierr = KSPSolve(ksp, b, b); CHKERRQ(ierr);
    ierr = VecDestroy(& b); CHKERRQ(ierr);
    ierr = VecDestroy(& b); CHKERRQ(ierr);
    ierr = MatDestroy(& A); CHKERRQ(ierr);
    ierr = PetscFinalize();
    return ierr;
}
```

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