Quantum Chernoff bound as a measure of nonclassicality for one-mode Gaussian states

Mădălina Boca, Iulia Ghiu, Paulina Marian, and Tudor A. Marian*
Centre for Advanced Quantum Physics, University of Bucharest,
P.O.Box MG-11, R-077125 Bucharest-Măgurele, Romania
(Dated: June 30, 2009)

We evaluate a Gaussian distance-type degree of nonclassicality for a single-mode Gaussian state of the quantum radiation field by use of the recently discovered quantum Chernoff bound. The general properties of the quantum Chernoff overlap and its relation to the Uhlmann fidelity are interestingly illustrated by our approach.

PACS numbers: 03.67.Mn; 42.50.Dv

Nonclassicality of a multimode state of the quantum radiation field is usually identified by inspection of its diagonal $P$ representation. States possessing a well-behaved $P$ representation are termed classical [1]. On the contrary, a negative or a highly singular $P$ representation (i.e., more singular than Dirac’s $\delta$) characterizes a nonclassical state. A quantitative measure of nonclassicality was first proposed by Hillery [2] as a properly defined distance between the given nonclassical state and the convex set of all classical states. However, the trace metric employed in Refs. [2] turned out to be difficult to deal with analytically. Therefore, the original definition of a nonclassical distance was subsequently modified twofold: first, by restricting the set of all classical states to a tractable subset identified by a classicality criterion, and second, by using more convenient distances between Gaussian states. Specifically, we mention the Hilbert-Schmidt [2] and the Bures metric [4, 5], as well as the relative-entropy measure in Refs. [4, 5]. Note also the recent work on defining a distance-type polarization degree in Refs. [6]. As shown in Refs. [4, 5], any good distance-type measure of nonclassicality has to satisfy several requirements, which proved to be of both principled and practical importance. The Bures metric [7] singled out among the proposed distance measures by its conspicuous distinguishability features [8] and its complete agreement with the Lee’s nonclassical depth [9].

The present work parallels previous papers [4, 5] in studying nonclassicality of one-mode Gaussian states by use of distance-type measures with remarkable distinguishability virtues. We here propose a nonclassicality measure built with the quantum Chernoff bound [10, 11], whose main properties we briefly recall in what follows.

In the symmetric quantum hypothesis, let $\rho$ and $\sigma$ be two equiprobable states of a quantum system. We denote by $P^{(n)}_{\min}(\rho, \sigma)$ the minimal error probability of discriminating these states in $n$ independent tests on identical copies of the system, all of them prepared in the same state, which is either $\rho$ or $\sigma$. The optimal asymptotic testing ($n \to \infty$) leads to an upper bound called the quantum Chernoff bound,

$$\xi_{QCB}(\rho, \sigma) := - \lim_{n \to \infty} \left\{ \frac{1}{n} \ln \left[ P^{(n)}_{\min}(\rho, \sigma) \right] \right\}. \quad (1)$$

Quite recently, it has been proven an intrinsic formula for this bound [10, 11]:

$$\xi_{QCB}(\rho, \sigma) = - \ln \min_{0 \leq \alpha \leq 1} \text{Tr}(\rho^s \sigma^{1-s}) \ . \quad (2)$$

The question of finding the above minimum is the quantum counterpart of a classical Bayesian probability problem formulated and solved long ago by Herman Chernoff [12]. The quantities

$$Q_s(\rho, \sigma) := \text{Tr}(\rho^s \sigma^{1-s}) \quad (3)$$

are the quantum analogues of the classical Rényi overlaps discussed in Ref. [8] as being distinguishability measures in their own right. According to Eq. (2), their minimum determines the quantum Chernoff bound:

$$Q(\rho, \sigma) := \min_{0 \leq s \leq 1} Q_s(\rho, \sigma) = \exp \{ - \xi_{QCB}(\rho, \sigma) \} \ . \quad (4)$$

In what follows, the non-negative function $Q(\rho, \sigma)$ defined in Eq. (4) will be termed the quantum Chernoff overlap of the states $\rho$ and $\sigma$. Notice that $Q(\rho, \sigma) \leq 1$ and its maximal value is reached when the states $\rho$ and $\sigma$ coincide.

Let us denote by $||A||_1 := \text{Tr}|A|$ the trace norm of a trace-class operator $A$. Originally, Hillery employed the trace metric $2T(\rho, \sigma) := ||\rho - \sigma||_1$ to define the nonclassical distance [2]. In the symmetric particular case $s = \frac{1}{2}$, Holevo proved the following pair of inequalities [13]:

$$1 - Q_{1/2}(\rho, \sigma) \leq T(\rho, \sigma) \leq \sqrt{1 - \left[ Q_{1/2}(\rho, \sigma) \right]^2}. \quad (5)$$

Equation (5) shows that the trace $Q_{1/2}(\rho, \sigma)$ is a measure of distinguishability as good as the trace metric $2T(\rho, \sigma)$. Having $Q(\rho, \sigma) \leq Q_{1/2}(\rho, \sigma)$, Holevo’s second inequality in Eq. (5) provides the upper bound of the Chernoff overlap $Q(\rho, \sigma)$ in terms of the trace distance, $Q^2 \leq 1 - T^2$. The lower bound $Q \geq 1 - T$ was proven in Refs. [11], so that the inequalities (5) still hold when replacing $Q_{1/2}(\rho, \sigma)$ by $Q(\rho, \sigma)$. We write them down below together with some other properties of the functions $Q_s$ and $Q$ proven in Refs. [11]:

*email: tudor.marian@g.unibuc.ro
1. **Relations to the trace distance:**

\[ 1 - Q(\rho, \sigma) \leq T(\rho, \sigma) \leq \sqrt{1 - |Q(\rho, \sigma)|^2}. \]

2. **Convexity in \( s \) of the trace \( Q_s(\rho, \sigma) \), Eq. \( \text{[3]} \).** As a consequence, \( Q(\rho, \sigma) \) is the unique minimum of the Rényi overlaps \( Q_s(\rho, \sigma) \).

3. **Joint concavity in \{ \rho, \sigma \} of the Rényi overlaps \( Q_s \) and of their lower bound \( T \).** This means that the Rényi overlaps between a given state, say \( \rho \), and an arbitrary set of states display a unique maximum.

4. **Multiplicativity of \( Q_s \),**

\[ Q_s(\rho_1 \otimes \rho_2, \sigma_1 \otimes \sigma_2) = Q_s(\rho_1, \sigma_1)Q_s(\rho_2, \sigma_2), \]

which is equivalent to the identity:

\[ \text{Tr} \left[ (\rho_1 \otimes \rho_2)^s (\sigma_1 \otimes \sigma_2)^{1-s} \right] = \text{Tr} \left( \rho_1^s \otimes \sigma_1^{1-s} \right) \text{Tr} \left( \rho_2^s \otimes \sigma_2^{1-s} \right). \]

5. **Invariance of \( Q(\rho, \sigma) \) under unitary transformations.**

6. **Monotonic increase of \( Q(\rho, \sigma) \) under completely positive, trace-preserving maps.**

In Refs.\[10, 14\] the relation between the Chernoff overlap and the Uhlmann fidelity is largely discussed. Recall that Uhlmann introduced the fidelity \( \mathcal{F}(\rho, \sigma) \) of two mixed states, \( \rho \) and \( \sigma \), as the maximal quantum-mechanical transition probability between all purifications of the given states \[15\]. Uhlmann wrote the distance \( d_B \) between two states discovered by Bures \[2\] in terms of the transition probability \( \mathcal{F} \) as \( |d_B(\rho, \sigma)|^2 = 2[1 - \sqrt{\mathcal{F}(\rho, \sigma)}] \), and succeeded in finding an explicit expression of the fidelity \[16\]:

\[ \mathcal{F}(\rho, \sigma) = \left( \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}} \right)^2. \] (6)

The above formula, \( \mathcal{F}(\rho, \sigma) = (||\sqrt{\rho} \sqrt{\sigma}||_1)^2 \), combined with a result of Fuchs and van de Graaf in Ref.\[17\] provides two important bounds:

\[ \mathcal{F}(\rho, \sigma) \leq Q(\rho, \sigma) \leq \sqrt{\mathcal{F}(\rho, \sigma)}. \] (7)

When one of the states is pure, \( Q(\rho, \sigma) \) equals the fidelity \[14\] and has thus the significance of a transition probability: \( Q(\rho, \sigma) = \mathcal{F}(\rho, \sigma) = \text{Tr} \left( \rho \sigma \right) \). \( \sqrt{\mathcal{F}} \) shares with \( Q_s \) and \( Q \) the properties 1 and 3-6, which are precisely the demands for a genuine measure of nonclassicality, stated in Ref.\[4\]. In view of these properties, we introduce here an ideal Chernoff degree of nonclassicality for an arbitrary state \( \rho \):

\[ D^{(C)}(\rho) := \min_{\rho' \in \mathcal{C}} \left[ 1 - Q(\rho, \rho') \right], \] (8)

where \( \mathcal{C} \) is the set of all classical states. According to Eq. \[4\], the Chernoff degree of nonclassicality \[5\] vanishes when the given state \( \rho \) is classical. Definition \[8\] implies the maximization of the Chernoff overlap \( Q(\rho, \rho') \) over the whole set of classical states \( \rho' \in \mathcal{C} \).

We will focus on the nonclassicality of single-mode Gaussian states of the radiation field, which are especially useful in experiments. Taking advantage of their simple parametrization, we apply the definition \[8\] to evaluate a Gaussian degree of nonclassicality, just as in Refs.\[4, 5\]. Such a Gaussian approach consists in replacing the reference set \( \mathcal{C} \) of all classical one-mode states by its subset \( \mathcal{C}_0 \) consisting only of Gaussian ones.

Recall that any one-mode Gaussian state \( \rho_G \) can be parametrized as a displaced squeezed thermal state (DSTS) \[18\]:

\[ \rho_G = D(\alpha) S(r, \varphi) \rho_T S^\dagger(r, \varphi) D^\dagger(\alpha). \] (9)

Here \( D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) \) is a Weyl displacement operator with the coherent amplitude \( \alpha \), \( S(r, \varphi) = \exp \{ \frac{1}{2} r [e^{i\varphi} (a^\dagger)^2 - e^{-i\varphi} a^2] \} \) is a Stoler squeeze operator with the squeeze factor \( r \) and squeeze angle \( \varphi \), and

\[ \rho_T = \frac{1}{n + 1} \sum_{n=0}^{\infty} \left( \frac{n}{n + 1} \right)^n |n\rangle \langle n| \] (10)

is a thermal state with the Bose-Einstein mean occupancy \( \bar{n} = [\exp(\beta \hbar \omega) - 1]^{-1} \).

We consider a nonclassical single-mode Gaussian state \( \rho_G \) whose parameters are: \( \alpha, \varphi, \bar{n}, \) and \( r > r_c := \frac{1}{2} \ln(2\bar{n} + 1) \). Any classical one-mode state \( \rho_G' \in \mathcal{C}_0 \) is identified by its parameters \( \alpha', \varphi', \bar{n}', \) and \( r' \) subject to the classicality condition \( r' \leq r_c := \frac{1}{2} \ln(2\bar{n}' + 1) \). When employing Eq. \[8\], the Gaussian degree of nonclassicality of the state \( \rho_G \) reads:

\[ D_0^{(C)}(\rho_G) := 1 - \max_{\rho_G' \in \mathcal{C}_0} Q(\rho_G, \rho_G'). \] (11)

We evaluate the Rényi overlap \[8\] of the pair of states \( \{ \rho_G, \rho_G' \} \) as a Hilbert-Schmidt scalar product:

\[ Q_s(\rho_G, \rho_G') = \frac{1}{\pi} \int d^2\chi \chi^* \chi' \chi' (1 - s, \lambda). \] (12)

In Eq. \[12\], \( \chi(s, \lambda) \) and \( \chi'(1 - s, \lambda) \) are the weight functions in the Weyl expansions of the one-mode Gaussian operators \( (\rho_G)^s \) and \( (\rho_G')^{1-s} \), respectively. We write down the explicit expression

\[ \chi(\lambda) = \frac{1}{(n + 1)^s - \bar{n}^s} \exp \left\{ - \left[ f(s, \bar{n}) + \frac{1}{2} \right] |\bar{\lambda}|^2 \right\} \times \exp[\lambda \alpha^* - \lambda^* \alpha], \] (13)

with the notations:

\[ f(s, \bar{n}) := \frac{\bar{n}^s}{(n + 1)^s - \bar{n}^s}, \quad (0 < s < 1); \]

\[ \bar{\lambda} := \cosh(r) \lambda - e^{i\varphi} \sinh(r) \lambda^*. \]
As shown in Ref. [18], Appendix A, the Gaussian integral in the r. h. s. of Eq. (12) can be readily performed to get the formula

\[ Q_s(\rho_G, \rho'_G) = \frac{1}{n^s(n')^{1-s}} f(s, \bar{n}) f(1-s, \bar{n}') \frac{1}{\sqrt{K^2 - |L|^2}} \]

\[ \times \exp \left[ - \frac{K|\mathcal{M}|^2 + \Re(L^* \mathcal{M}^2)}{K^2 - |L|^2} \right] , \]  

(14)

where we have denoted:

\[ K := f(s, \bar{n}) + \frac{1}{2} \cosh(2r) + f(1-s, \bar{n}') + \frac{1}{2} \cosh(2r') , \]

\[ L := f(s, \bar{n}) + \frac{1}{2} e^{i\varphi} \sinh(2r) \]

\[ + f(1-s, \bar{n}') + \frac{1}{2} e^{i\varphi'} \sinh(2r') , \]

\[ M := \alpha' - \alpha . \]  

(15)

We mention that an equivalent expression of the Rényi overlap \( Q_s(\rho_G, \rho'_G) \) has already been found in Ref. [19] as Eq. (91) thereof. Maximization of \( Q_s(\rho_G, \rho'_G) \), Eqs. (14) and (15), with respect to the displacement \( \alpha' \) and the squeeze angle \( \varphi' \) yields obvious values of these parameters for the closest classical state \( \tilde{\rho}'_G \): \( \tilde{\alpha}' = \alpha \) and \( \tilde{\varphi}' = \varphi \). It turns out that nonclassicality is invariant under classical operations such as translations and rotations. A natural assumption is that \( \tilde{\rho}'_G \) belongs to the boundary of the set \( \mathcal{C} \) of all classical one-mode Gaussian states: \( \tilde{\rho}'_G \in \partial \mathcal{C} \). This means that the closest classical state \( \tilde{\rho}'_G \) is at the classicality threshold specified by the condition \( r' = r'_e := \frac{1}{2} \ln(2\bar{n} + 1) \Leftrightarrow \bar{n}' = e^{r'} \sinh(r') \). After introducing all these findings into Eqs. (14) and (15), the Rényi overlap \( Q_s(\rho_G, \rho'_G) \) becomes a two-variable function:

\[ Q_G(s, r') = \left\{ [ (\bar{n} + 1)^s \sinh^2(r') - \bar{n}^s \cosh^2(r') ]^2 \right. \]

\[ + \left. \cosh^{2(1-s)}(r') - \sinh^{2(1-s)}(r') \right\} \]

\[ \times \left[ (\bar{n} + 1)^{2s} - \bar{n}^{2s} \right] \cos^2(r - r') \right\}^{1/2} \]

\[ \times \exp(1 - s) r' \right] . \]  

(16)

Let us denote by \( \tilde{Q}_G \) the maximum of the Chernoff overlap in the r. h. s. of Eq. (11):

\[ \tilde{Q}_G := \max_{\rho'_G \in \mathcal{C}_0} \min_{0 \leq s \leq 1} Q_s(\rho_G, \rho'_G) = \min_{0 \leq s \leq 1} \max_{\rho'_G \in \mathcal{C}_0} Q_s(\rho_G, \rho'_G) . \]  

(17)

We aim to find the value \( \tilde{Q}_G \) of the function (10) that is reached for a pair of optimal values of its variables, hereafter denoted by \( \tilde{s} \) and \( \tilde{r}' \): \( \tilde{Q}_G = Q_G(\tilde{s}, \tilde{r}') \). An analytic solution can be found only if \( \rho_G \) is a pure state, namely, a displaced squeezed vacuum state. Indeed, when setting \( \bar{n} = 0 \) into Eq. (16), we readily get the corresponding solution: \( \tilde{s} = 0 \), \( \tilde{r}' = 0 \), \( \tilde{Q}_G = \text{sech}(r) \). For any mixed nonclassical state \( \rho_G \), the optimal parameters \( \tilde{s} \) and \( \tilde{r}' \) cannot be determined analytically. This situation is similar to that encountered when using the relative entropy as a measure of nonclassicality for single-mode Gaussian states in Ref. [5]. According to Eq. (17), \( \tilde{Q}_G = Q_G(\tilde{s}, \tilde{r}') \) is a saddle point of the function (10).

The saddle-point numerical results can be seen in Fig. 1, where the Rényi overlaps \( Q_G(s, r') \) are plotted versus the variables \( s \) and \( r' \).

**FIG. 1:** (Color online) Displaying the saddle-point evaluations. The function \( Q_G(s, r') \), Eq. (15), is plotted versus \( r' \) and \( s \) for two nonclassical Gaussian states having the parameters: a) \( r = 2, \bar{n} = 1 \), for which we get the saddle-point values: \( \tilde{Q}_G = 0.617, \tilde{r}' = 1.265, \tilde{s} = 0.283 \); b) \( r = 1, \bar{n} = 0.5 \), with the saddle-point results: \( \tilde{Q}_G = 0.893, \tilde{r}' = 0.653, \tilde{s} = 0.387 \).

It is now interesting to compare the present results with similar ones, found previously by using the Bures metric to quantify the nonclassicality of one-mode Gaussian states. In Ref. [3], a Gaussian degree of nonclassicality has been defined as follows:

\[ D_{0}^{(B)}(\rho_G) := 1 - \max_{\rho'_G \in \mathcal{C}_0} \sqrt{\mathcal{F}(\rho_G, \rho'_G)} . \]  

(18)

Maximization of the fidelity could be performed analytically to give the simple result

\[ \tilde{\mathcal{F}} := \max_{\rho'_G \in \mathcal{C}_0} \mathcal{F}(\rho_G, \rho'_G) = \text{sech}(r - r_c) . \]  

(19)
We present in Fig. 2 the degrees of nonclassicality (11) and (18) as functions of the mixedness parameter \( \bar{n} \) at a fixed squeeze factor \( r > 0 \). They have close graphs over the whole nonclassicality domain of the squeezed thermal state \( \rho_G \); \( 0 \leq \bar{n} < \bar{n}_c := e^r \sinh(r) \). Also plotted are the corresponding saddle-point parameters \( \tilde{s} \) and \( \tilde{r}' \). Both of them are increasing functions of the variable \( \bar{n} \), starting from the pure-state values \( \tilde{s} = 0 \) and \( \tilde{r}' = 0 \), and ending at the threshold values \( \tilde{s} = \frac{1}{2} \) and \( \tilde{r}' = r \), respectively. Making use of Eq. (16), we have proven that if \( r = r'_c \), then the optimal \( \tilde{s} \) tends to \( \frac{1}{2} \). The curves in Fig. 2 are in fact calibration graphs for an easy reading of the nonclassicality properties of the squeezed mixed state \( \rho_G \). In addition, the bounds in Eq. (7) are illustrated in Fig. 3 by plots of the optimal values \( \tilde{Q}_G \), \( \tilde{F} \), and \( \sqrt{\tilde{F}} \) versus the thermal mean occupancy \( \bar{n} \) of the nonclassical state \( \rho_G \). Inequalities (7) are clearly displayed in this figure: \( \tilde{Q}_G \) coincides with \( \tilde{F} \) for pure states (\( \bar{n} = 0 \)) and becomes rather close to \( \sqrt{\tilde{F}} \) as the degree of mixing increases.

To conclude, in this work we have shown that the remarkable properties of quantum Chernoff bound can be used to discriminate between a (nonclassical) state and a set of (classical) states. We have chosen the class of one-mode Gaussian states, for which an explicit expression of the Rényi overlap is recovered as Eqs. (14) and (13). In general, the Chernoff overlap \( Q(\rho_G, \rho'_G) \) could be computed only numerically, while analytic expressions of the corresponding fidelity \( F(\rho_G, \rho'_G) \) are at hand for a long time (20). However, the numerical calculation of the Chernoff degree of nonclassicality by saddle-point methods is straightforward and can be performed with great accuracy. Our present results are consistent with the analogous ones obtained previously by use of the Bures metric, in accordance with the general relations between the Chernoff overlap and the Uhlmann fidelity.

This work was supported by the Romanian Ministry of Education and Research through Grant No. IDEI-995/2007 for the University of Bucharest.

[1] U. M. Titulaer and R. J. Glauber, Phys. Rev. 140, B676 (1965).
[2] M. Hillery, Phys. Rev. A 31, 338 (1985); M. Hillery, ibid., 35, 725 (1987); M. Hillery, ibid., 39, 2994 (1989).
[3] V. V. Dodonov, O. V. Man’ko, V. I. Man’ko, and A. Wünsche, Physica Scripta 59, 81 (1999); V. V. Dodonov, O. V. Man’ko, V. I. Man’ko, and A. Wünsche, J. Mod. Optics 47, 633 (2000).
[4] Paulina Marian, T. A. Marian, and H. Scutaru, Phys. Rev. Lett. 88, 153601 (2002).
[5] Paulina Marian, T. A. Marian, and H. Scutaru, Phys. Rev. A 69, 022104 (2004). Here the relative entropy of one-mode Gaussian states has been first obtained.
[6] A. B. Klimov, L. L. Sánchez-Soto, E. C. Yustas, J. Söderholm, and G. Björk, Phys.Rev. A 72, 033813 (2005); A. Luis, ibid., 73, 063806 (2006); L. L. Sánchez-Soto, E. C. Yustas, G. Björk, and A. B. Klimov, ibid., 76, 043820 (2007).
[7] D. Bures, Trans. Am. Math. Soc. 135, 199 (1969).
[8] C. A. Fuchs, Ph. D. thesis, University of New Mexico, 1995, (quant-ph/9601020/1996).
[9] C. T. Lee, Phys. Rev. A 44, R2775 (1991).
[10] M. Nussbaum and A. Szkoła: quant-ph/0607216 (2006).
[11] K. M. R. Audenaert, J. Calsamiglia, R. Muñoz-Tapia, E. Bagan, Ll. Masanes, A. Acín, and F. Verstraete, Phys. Rev. Lett. 98, 160501 (2007). See also the recent comprehensive work: K. M. R. Audenaert, M. Nussbaum, A. Szkoła, and F. Verstraete, Commun. Math. Phys. 279, 251 (2008).
[12] H. Chernoff, Ann. Math. Stat. 23, 493 (1952).
[13] A. S. Holevo, Theor. Math. Phys. 13, 184 (1972).
[14] V. Kargin, Ann. Statist. 33, 959 (2005).
[15] A. Uhlmann, Rep. Math. Phys. 9, 273 (1976); ibid., 24, 229 (1986).
[16] We have chosen to preserve the physical significance of the fidelity as a transition probability between optimal purifications. In Refs. [8, 10, 14], fidelity is defined as the square root of Uhlmann’s fidelity, Eq. (6).

[17] C. A. Fuchs and J. van de Graaf, IEEE Trans. Inf. Theory, 45, 1216 (1999).

[18] Paulina Marian and T. A. Marian, Phys. Rev. A 47, 4474 (1993).

[19] J. Calsamiglia, R. Muñoz-Tapia, Ll. Masanes, A. Acin, and E. Bagan, Phys. Rev. A 77, 032311 (2008).

[20] J. Twamley, J. Phys. A 29, 3723 (1996); H. Scutaru, ibid., 31, 3659 (1998).