Chord length distribution and the distance between two random points in a convex body in $\mathbb{R}^n$

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Abstract

In this article for $n$-dimensional convex body $D$ the relation between the chord length distribution function and the distribution function of the distance between two random points in $D$ was found. Also the relation between their moments was found.

Keywords: Integral geometry, Chord length distribution, Random points, Convex body.

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1. Introduction

The integral geometric concepts such as the distribution of the chord length, the distribution of the distance between two random points in a convex body $D$ and many others carry some information about $D$. In this article the relation between the chord length distribution function and the distribution function of the distance between two random points uniformly distributed in a convex body was found. By $\mathbb{R}^n$ we denote the $n$-dimensional Euclidean space (here we assume that $n > 2$, for case $n = 2$ see [1]).

Let $D \subset \mathbb{R}^n$ be a compact convex set (convex body) with boundary $\partial D$. By $V_n(D)$ we denote the $n$-dimensional volume of $D$. By $S^{n-1}$ we denote the unit sphere in $\mathbb{R}^n$ centered at the origin $O$ and let $e_{O,\xi}$ be the hyperplane containing $O$ and normal to $\xi \in S^{n-1}$. By $G^n$ we denote the set of all lines in $\mathbb{R}^n$. We use the usual parametrization of a line $\gamma = (\omega, P)$: where $\omega \in S^{n-1}$ is the direction of $\gamma$ and $P$ is the intersection point of $\gamma$ and $e_{O,\omega}$. By $[D]$ we denote the set of lines intersecting $D$. In $G^n$ we consider the invariant measure (with respect to the group of Euclidean motions) $\mu(\cdot)$. It is known that the element $d\gamma$ of the measure, up to a constant, has the following form ([2], [4], [6])

$$d\gamma = d\omega dP,$$

(1.1)

here $d\omega$ and $dP$ are elements of the Lebesgue measure on the sphere and the hyperplane, respectively. By $\chi(\gamma)$ we denote the length of a chord $D \cap \gamma$. We consider $\gamma$ as a random line intersecting $D$ with normalized measure $\frac{d\gamma}{\mu([D])}$. The distribution function $F_\chi(u)$ of $\chi(\gamma)$ is called the chord length distribution

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function of $D$. Thus we have

$$F_X(u) = \frac{1}{\mu([D])} \int_{[D]} I_{\chi(\gamma) < u} \, d\gamma.$$ (1.2)

Now let $Q_1 = (x_1, ..., x_n)$, $Q_2 = (y_1, ..., y_n)$ be random independent points uniformly distributed in $D$ and denote by $d$ distance between $Q_1$ and $Q_2$. By $F_d(t)$ we denote the distribution function of $d$. We have

$$F_d(t) = \frac{1}{V_n(D)^2} \int_{D \times D : |Q_1 - Q_2| < t} \, dQ_1 \, dQ_2, \quad t \in \mathbb{R},$$ (1.3)

here $dQ_i (i = 1, 2)$ is the usual Lebesgue’s measure in $\mathbb{R}^n$.

2. Main results

Now we present the main results. The following theorem describes the relation between $F_X(t)$ and $F_d(t)$.

**Theorem 2.1.** Let $D$ be a convex body. We have the following relation between $F_X(t)$ and $F_d(t)$

$$F_d(t) = \frac{V_{n-1}(\partial D) \cdot V_{n-2}(S^{n-2})}{(n-1) \cdot V_n(D)^2} \left( \frac{-t^{n+1}}{n+1} + \frac{(n-1) \cdot V_n(D) \cdot V_{n-1}(S^{n-1}) \cdot t^n}{n \cdot V_{n-1}(\partial D) \cdot V_{n-2}(S^{n-2})} + \right. \left. \frac{t^n}{n} \int_0^t F_X(u) \, du - \frac{1}{n} \int_0^t u^n F_X(u) \, du \right).$$ (2.1)

Also in the article was found the following relation.

**Theorem 2.2.** Let $D$ be a convex body. The following relation between the $k$-th moment of the distance between two random points and the moments of the chord length distribution of $D$ is valid

$$E d^k = \frac{V_{n-1}(\partial D) \cdot V_{n-2}(S^{n-2}) \cdot E \chi^{n+k+1}}{(n-1) (n+k) (n+k+1) \cdot V_n(D)^2}.$$ (2.2)

3. Preliminary results

In this section we need to prove the following lemmas.

**Lemma 3.1.** For the invariant measure of the lines intersecting $D$ we have

$$\mu([D]) = \frac{V_{n-1}(\partial D) \cdot V_{n-2}(S^{n-2})}{2 \cdot (n-1)}.$$ (3.1)

**Proof of lemma 3.1.** By definition we have

$$\mu([D]) = \int_{[D]} d\gamma = \int_{[D]} d\omega dP = \frac{1}{2} \int_{S^{n-1}} d\omega \int_{D_\omega} dP = \frac{1}{2} \int_{S^{n-1}} V_{n-1}(D_\omega) \, d\omega.$$ (3.2)
where $D_\omega$ is the orthogonal projection of $D$ onto hyperplane $e_{O,\omega}$. In this article, we consider a convex body $D$ with positive Gaussian curvature at every point of $\partial D$. For $\xi \in S^{n-1}$ we denote by $s(\xi)$ the point on $\partial D$ the outer normal of which is $\xi$. It is known that (see [3], [5])

$$V_{n-1}(D_\omega) = \frac{1}{2} \int_{\partial D} |\cos(\omega, \xi)| ds(\xi),$$

(3.3)

here $ds(\xi)$ is the element of $n-1$-dimensional Lebesgue’s measure on $\partial D$. Substituting (3.3) into (3.2) and using the Fubini’s theorem we obtain

$$\mu([\partial D]) = \frac{1}{4} \int_{S^{n-1}} \int_{\partial D} |\cos(\omega, \xi)| ds(\xi) d\omega = \frac{1}{4} \int_{\partial D} \int_{S^{n-1}} |\cos(\omega, \xi)| d\omega ds(\xi).$$

(3.4)

For any $\xi \in S^{n-1}$ we have (see [3])

$$\int_{S^{n-1}} |\cos(\omega, \xi)| d\omega = \frac{2V_{n-2}(S^{n-2})}{n-1}$$

(3.5)

thus

$$\mu([\partial D]) = \frac{1}{4} \int_{\partial D} \int_{S^{n-1}} |\cos(\omega, \xi)| d\omega ds(\xi) = \frac{V_{n-2}(S^{n-2})}{2(n-1)} \int_{\partial D} ds(\xi) = \frac{V_{n-1}(\partial D)V_{n-2}(S^{n-2})}{2(n-1)}. $$

(3.6)

For a line $\gamma$ intersecting a convex body $D$ we have the following lemma.

**Lemma 3.2.** Let $\chi(\gamma)$ be the length of the chord $D \cap \gamma$. We have

$$\int_{[D]} \chi(\gamma) d\gamma = \frac{V_n(D)V_{n-1}(S^{n-1})}{2}.$$  

(3.7)

**Proof of Lemma 3.2.** By definition we have ($\gamma = (\omega, P)$)

$$\int_{[D]} \chi(\gamma) d\gamma = \frac{1}{2} \int_{S^{n-1}} d\omega \int_{D_\omega} \chi(\omega, P) dP.$$  

(3.8)

For any $\omega \in S^{n-1}$ it is obvious that $\chi(\omega, P) dP$ is the element of $n$-dimensional volume of $D$, hence the integrating by $dP$ over $D_\omega$ we get $V_n(D)$.

$$\int_{[D]} \chi(\gamma) d\gamma = \frac{1}{2} \int_{S^{n-1}} d\omega \int_{D_\omega} \chi(\omega, P) dP = \frac{V_n(D)}{2} \int_{S^{n-1}} d\omega = \frac{V_n(D)V_{n-1}(S^{n-1})}{2}.$$  

(3.9)

It is known that a pair of points $Q_1, Q_2$ in $\mathbb{R}^n$ can be represented by the line $\gamma = (\omega, P)$ passing through the points and pair of one dimensional coordinates $(t_1, t_2)$ (see [6]). Thus

$$(Q_1, Q_2) = (\gamma, t_1, t_2) = (\omega, P, t_1, t_2).$$  

(3.10)
Lemma 3.3. The Jacobian of the transform (3.10) is

\[ dQ_1 dQ_2 = |t_1 - t_2|^{n-1} dt_1 dt_2 d\omega dP. \] (3.11)

Proof of lemma 3.3. For a fixed \( Q_1 \) we represent \( Q_2 \) by polar coordinates with respect to \( Q_1 \). It is known that

\[ dQ_2 = r^{n-1} dr d\omega \] (3.12)

where \( r = |Q_1 - Q_2| \) and \( \omega \) is the direction of \( Q_2 - Q_1 \). For a fixed \( \omega \) the point \( Q_1 \) can be represented by \( P \) and \( t_1 \). Thus

\[ dQ_1 = dt_1 dP \] (3.13)

and by multiplying (3.12) and (3.13) and taking into account that \( r = |t_1 - t_2| \) we get

\[ dQ_1 dQ_2 = |t_1 - t_2|^{n-1} dt_1 dt_2 d\omega dP. \] (3.14)

4. Proof of theorem 2.1

By definition of the distribution function we have

\[ F_d(t) = \Pr\{d < t\} = \frac{1}{V_n(D)^2} \int_{d < t} dQ_1 dQ_2. \] (4.1)

Using (3.11) from lemma 3.3 we get

\[ F_d(t) = \frac{1}{V_n(D)^2} \int_{[D]} \int_{|t_1 - t_2| < t} |t_1 - t_2|^{n-1} dt_1 dt_2 d\gamma. \] (4.2)

We represent the integral by the form

\[ \int_{[D]} \int_{|t_1 - t_2| < t} |t_1 - t_2|^{n-1} dt_1 dt_2 d\gamma = \int_{[D]} I_{\{\chi(\gamma) \geq t\}} J_1(t) \, d\gamma \] (4.3)

where

\[ J_1(t) = \int_{|t_1 - t_2| < t} |t_1 - t_2|^{n-1} dt_1 dt_2. \] (4.4)

After calculating \( J_1 \) we obtain that

\[ J_1(t) = \frac{-2t^{n+1}}{n+1} + \frac{2t^n}{n} \chi(\gamma) \] (4.5)

for \( \chi(\gamma) \geq t \) and

\[ J_1(t) = \frac{2(\chi(\gamma))^{n+1}}{n(n+1)} \] (4.6)

for \( \chi(\gamma) < t \).
Now we denote by $J_2$, $J_3$, $J_4$ the following integrals

$$J_2(t) = \int_{[D]} I_{\{X(\gamma) \geq t\}} d\gamma,$$

$$J_3(t) = \int_{[D]} I_{\{X(\gamma) \geq t\}} X(\gamma) d\gamma,$$

$$J_4(t) = \int_{[D]} I_{\{X(\gamma) < t\}} (X(\gamma))^{n+1} \, d\gamma.$$

Using the same technique we can calculate (see (3.7) from lemma 3.2) then denote

$$J_5(t) = \int_{[D]} I_{\{X(\gamma) < t\}} X(\gamma) d\gamma.$$

and calculate it instead. First we calculate the derivative of $J_5$ and then integrate from 0 to $t$ ($J_5(0) = 0$).

Using the first mean value theorem for definite integrals we have

$$\frac{d}{dt} J_5(t) = \lim_{\Delta t \to 0} \frac{J_5(t + \Delta t) - J_5(t)}{\Delta t} = \frac{1}{\Delta t} \int_{[D]} I_{\{t \leq X(\gamma) < t+\Delta t\}} X(\gamma) d\gamma =$$

$$\lim_{\Delta t \to 0} V_{n-1}(\partial D) V_{n-2} \left( S^{n-2} \right) t \left( F_X(t + \Delta t) - F_X(t) \right) =$$

$$\frac{V_{n-1}(\partial D) V_{n-2} \left( S^{n-2} \right)}{2(n-1)} tf_X(t).$$

and after integrating that we get

$$J_5(t) = \frac{V_{n-1}(\partial D) V_{n-2} \left( S^{n-2} \right)}{2(n-1)} \left( t F_X(t) - \int_0^t F_X(u) \, du \right).$$

For $J_3$ we have

$$J_3(t) = \frac{V_n(D) V_{n-1} \left( S^{n-1} \right)}{2} -$$

$$\frac{V_{n-1}(\partial D) V_{n-2} \left( S^{n-2} \right)}{2(n-1)} \left( t F_X(t) - \int_0^t F_X(u) \, du \right).$$

Using the same technique we can calculate $J_4$ and get that

$$J_4(t) = \frac{V_{n-1}(\partial D) V_{n-2} \left( S^{n-2} \right)}{2(n-1)} \left( t^{n+1} F_X(t) - (n+1) \int_0^t u^n F_X(u) \, du \right).$$
After substituting (4.11), (4.16), (4.17) into (4.7) finally we obtain that
\[
F_d(t) = \frac{V_{n-1}(\partial D)}{(n-1)V_n(D)^2} \left( \frac{t^{n+1}}{n+1} + \frac{(n-1)V_n(D)V_{n-1}(S^{n-1})}{nV_{n-1}(\partial D)} \right) + \frac{t^n}{n} \int_0^t F_X(u) \, du - \frac{1}{n} \int_0^t u^n F_X(u) \, du .
\] (4.18)

Differentiating the distribution function \(F_d(t)\) we get the following expression for the density function \(f_d(t)\) of \(d\)
\[
f_d(t) = \frac{V_{n-1}(\partial D)}{(n-1)V_n(D)^2} \left( -t^n + \frac{(n-1)V_n(D)V_{n-1}(S^{n-1})}{V_{n-1}(\partial D)V_{n-2}(S^{n-2})} t^{n-1} \right) + \frac{t^{n-1}}{n} \int_0^t F_X(u) \, du .
\] (4.19)

Note that in \(\mathbb{R}^2\) formula (4.19) first was found in [1]. Now we are going to prove theorem 2.2.

5. **Proof of theorem 2.2**

We can prove that theorem by just putting the (4.19) in moments formula
\[
E d^k = \int_{-\infty}^{\infty} t^k f_d(t) \, dt
\] (5.1)
but we will do that by the following way
\[
E d^k = \frac{1}{V_n(D)^2} \int_{Q_1, Q_2 \in D} |Q_1 - Q_2|^k \, dQ_1 \, dQ_2 = \frac{1}{V_n(D)^2} \int_{|D|} \frac{X(\gamma)}{0} \int_{0}^{X(\gamma)} |t_1 - t_2|^{n+k-1} \, dt_1 \, dt_2 \, d\gamma = \frac{2}{V_n(D)^2(n+k)(n+k+1)} \int_{|D|} (X(\gamma))^{n+k+1} \, d\gamma = \frac{V_{n-1}(\partial D)V_{n-2}(S^{n-2})}{(n-1)(n+k)(n+k+1)V_n(D)^2} E X^{n+k+1} .
\] (5.2)

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