Brane world models with bulk scalar fields

Éanna É. Flanagan*, S.-H. Henry Tye† and Ira Wasserman‡

Laboratory for Nuclear Studies and Center for Radiophysics and Space Research
Cornell University
Ithaca, NY 14853
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Abstract

We examine several different types of five dimensional stationary spacetimes with bulk scalar fields and parallel 3-branes. We study different methods for avoiding the appearance of spacetime singularities in the bulk for models with and without cosmological expansion. For non-expanding models, we demonstrate that in general the Randall-Sundrum warp factor is recovered in the asymptotic bulk region, although elsewhere the warping may be steeper than exponential. We show that nonsingular expanding models can be constructed as long as the gradient of the bulk scalar field vanishes at zeros of the warp factor, which are then analogous to the particle horizons found in expanding models with a pure AdS bulk. Since the branes in these models are stabilized by bulk scalar fields, we expect there to be no linearly unstable radion modes. As an application, we find a specific class of expanding, stationary solutions with no singularities in the bulk in which the four dimensional cosmological constant and mass hierarchy are naturally very small.

I. INTRODUCTION

The Randall-Sundrum (RS) model [1] for a warped 5-D geometry can account for the mass hierarchy, while providing a fine-tuning mechanism for canceling the 4-D cosmological constant without requiring a vanishing 5-D vacuum energy [1,2]. Variants of the model have been constructed in which the cosmological constant is exponentially small [3], and in which both the cosmological constant and the hierarchy problems may be solved simultaneously [4]. However, the RS model and similar models that solve the hierarchy problem are untenable: the radion-mediated interaction of matter on the visible brane dominates over the

* eef3@cornell.edu
† tye@mail.lns.cornell.edu
‡ ira@astro.cornell.edu
gravitational interaction \[3,4\]. This has the consequence that matter on the visible brane (with positive brane tension) gives a negative contribution to the square of the four dimensional Hubble parameter for homogeneous cosmological models \[4,6\]. Another manifestation of the problem is that the spacetimes are dynamically unstable: the 3-branes which they contain, whose positions must be arranged carefully to reproduce the mass hierarchy, move away from those special positions when slightly perturbed \[8,9\].

As suggested by Goldberger and Wise (GW) \[5\], the introduction of bulk scalar fields can solve these problems by giving a mass to the radion mode and stabilizing the positions of the branes. This is consistent with phenomenology if the radion mass is at or above the electroweak scale \[10\]. DeWolfe, Freedman, Gubser and Karch (DFGK) \[11\] showed how fully consistent stationary spacetimes could be constructed with bulk scalar fields, including the gravitational back-reaction of those fields; there have been further studies of spacetimes with bulk scalars along similar lines \[12,13\]. DFGK argued that the requirement that the induced metric on the brane be flat requires one fine-tuning of the parameters describing the solutions. Generalizing the class of solutions by allowing a non-zero effective 4-D cosmological constant $\Lambda_4$ relaxes this fine-tuning \[11\].

The purpose of this paper is to extend the work of DFGK \[11\] and Kakushadze \[13\], to discuss some general properties of 5-D spacetimes with a bulk scalar field, and to apply those properties to the solution of the cosmological constant and the hierarchy problems. The main points of this paper are as follows:

- The principal problem associated with the introduction of a bulk scalar field is that generic stationary solutions contain timelike curvature singularities in the bulk at finite distances from the branes. This occurs both for static solutions ($\Lambda_4 = 0$) and for stationary solutions with cosmological expansion ($\Lambda_4 \neq 0$). Such singularities have already been encountered in work on self-tuning of the cosmological constant \[14\]. It is possible to avoid these singularities in two ways. One well-known way is to simply orbifold or otherwise compactify the fifth dimension in such a way that the singularity is never encountered. A second way, which is perhaps preferable, is to carefully choose the scalar field potential in such a way that the occurrence of singularities is prevented. In Sec. 2 below, we discuss a simple condition (first considered by Kakushadze \[13\]) for identifying such preferred potentials for static spacetimes, and present several examples. One interesting example is obtained by compactifying a 11-D spacetime down to 5-D with a single radion field which acts as a bulk scalar, as well as a 7-form field strength in 11-D descends to a non-dynamical 5-form field strength in 5-D that generates a potential for the scalar field (Sec. III C 1 below). By adjusting the 5-form field strength, this model can be rendered nonsingular.

- In the RS model, the metric’s “warp factor” falls off exponentially as one moves away from the Planck brane, and this exponential fall-off underlies the RS solution of the hierarchy problem. In many models with scalar fields, the fall-off of the warp factor is faster than exponential, which facilitates solving the hierarchy problem (see the models in Secs. IIIA and IIIB below). This point was mentioned in passing for a specific model by DFGK.

- We show that allowing the four dimensional cosmological constant $\Lambda_4$ to be non-zero exacerbates the tendency to form spacetime singularities in the bulk. In particular,
models with bulk scalar fields that are nonsingular when $\Lambda_4 = 0$ should be expected to become singular for nonzero $\Lambda_4$, although the corresponding singularities are rather mild. The singularities are weak enough (see Eq. [10]) to be removed by simply requiring the bulk to become effectively AdS as the warp factor $A(y) \to 0$. In Sec. IV below we derive a general method of constructing nonsingular models with cosmological expansion based on this idea.

- We can now construct models without bulk curvature singularities. With us living on the visible brane, it is possible to account for the extreme smallness of the electroweak scale and the cosmological constant simultaneously (Sec. V below). These models are analogous to those of Refs. [3,4], with the additional feature that the stability problems of Refs. [3,4] have been cured.

II. SETUP AND GENERAL CONSIDERATIONS

In this section we outline the general framework, review the procedure introduced by DFGK for constructing static and stationary solutions, and derive criterion under which bulk singularities do not occur.

A. Basic equations

We consider 5-D gravity plus a bulk scalar field with parallel 3-branes, for which the action is

$$S = \int d^4x dy \sqrt{|g|} \left[ \frac{1}{2\kappa^2} R - \frac{(\nabla \phi)^2}{2} - V(\phi) \right] - \sum_b \int_{y_b} dy \sqrt{|\tilde{g}^b|} \sigma_b(\phi).$$  \(1\)

Here the coordinates are $(x^\mu, y)$ for $0 \leq \mu \leq 3$, the $b$th brane is located at $y = y_b$, $g_{ab}$ is the 5-D metric and $\tilde{g}^b_{\mu \nu}$ is the induced metric on the $b$th brane. The brane tensions $\sigma_b$ and potential $V$ are functions of the bulk scalar $\phi$, and $\kappa^2$ is the 5-D gravitational coupling constant.

We seek solutions to the field equations of the form

$$ds^2 = dy^2 + A(y) \left[ -dt^2 + e^{2Ht} \delta_{ij} dx^i dx^j \right],$$  \(2\)

$$\phi = \phi(y),$$  \(3\)

although we shall concentrate initially on the static case $H = 0$. For this ansatz, the Einstein and scalar field equations reduce to [11]

$$u' = -\frac{2\kappa^2(\phi')^2}{3} - \frac{2H^2}{A} - \frac{2\kappa^2}{3} \sum_b \sigma_b(\phi) \delta(y - y_b),$$

$$u^2 = \frac{\kappa^2(\phi')^2}{3} - \frac{2\kappa^2V(\phi)}{3} + \frac{4H^2}{A},$$

$$\phi'' + 2u\phi' = \frac{\partial V(\phi)}{\partial \phi} + \sum_b \left[ \frac{\partial \sigma_b}{\partial \phi} \right] \delta(y - y_b),$$  \(4\)
where
\[ u(y) = \frac{A'}{A} \]  \hspace{1cm} (5)
and primes denote derivatives with respect to \( y \). Only two of these three equations are independent: the scalar wave equation follows from the other two via the Bianchi identities.

The jump conditions at the branes are
\[ u|_{y = y_b^+} \equiv \lim_{y \to y_b^+} u - \lim_{y \to y_b^-} u = -2q_b, \quad \phi'|_{y = y_b^+} = \sigma'_{\phi}(\phi_b), \]  \hspace{1cm} (6)
where \( \phi_b = \phi(y_b) \) is the value of the bulk scalar on the \( b \)th brane and
\[ q_b = \frac{1}{3} \kappa^2 \sigma_b(\phi_b). \]  \hspace{1cm} (7)

The Ricci scalar is
\[ R = -4u' - 5u^2 + \frac{12H^2}{A} \]  \hspace{1cm} (8)
\[ = \kappa^2(\phi')^2 + \frac{10\kappa^2 V(\phi)}{3}. \]  \hspace{1cm} (9)

### B. Method of obtaining solutions

We now review the method of generating solutions introduced by DFGK. For any given solution of Eqs. (4), we can imagine inverting the relation \( \phi = \phi(y) \) to obtain \( y \) as a function of \( \phi \). Leaving aside questions of single-valuedness, we can imagine changing independent variables from \( y \) to \( \phi \). Since \( \phi'(y) \) is also a function of \( y \), we can likewise take \( \phi' \) to be a function of \( \phi \). By analogy with the practice in supergravity theory, let us define a function \( W(\phi) \) by
\[ \phi' \equiv \frac{1}{2} \frac{\partial W(\phi)}{\partial \phi}. \]  \hspace{1cm} (10)
In the \( H = 0 \) case, the first of Eqs. (4) then implies that, away from branes,
\[ \frac{\partial u}{\partial \phi} = -\frac{2\kappa^2}{3} \phi' = -\frac{\kappa^2}{3} \frac{\partial W(\phi)}{\partial \phi}, \]  \hspace{1cm} (11)
so that
\[ u = -\kappa^2 W(\phi)/3. \]  \hspace{1cm} (12)
The second of Eqs. (4) then becomes
\[ V(\phi) = \frac{1}{8} \left( \frac{\partial W(\phi)}{\partial \phi} \right)^2 - \frac{1}{6} \kappa^2 W(\phi)^2. \]  \hspace{1cm} (13)

The procedure for obtaining solutions in the \( H = 0 \) case is (i) choose a potential \( V(\phi) \) and superpotential \( W(\phi) \) that are related by Eq. (13); (ii) integrate Eq. (10) to obtain \( \phi \) as a function of \( y \); (iii) combine this with Eq. (12) to obtain \( u \) as a function of \( y \); and (iv) integrate Eq. (5) to obtain the warp factor \( A \) as a function of \( y \).
C. Occurrence of singularities in the bulk

Combining the definition (10) of the superpotential $W$ with the expression (9) for the Ricci scalar gives

$$R = \frac{k^2}{4} \left[ \frac{\partial W(\phi)}{\partial \phi} \right]^2 + \frac{10k^2 V(\phi)}{3}. \tag{14}$$

Eq. (14) is applicable to models with $H \neq 0$. When $H = 0$, substituting Eq. (13) into Eq. (14) yields

$$R = 2k^2 \left[ \frac{\partial W(\phi)}{\partial \phi} \right]^2 - \frac{5k^4 W^2(\phi)}{9}. \tag{15}$$

Eqs. (14) and (15) show why singularities are rampant in models with bulk scalar fields. For many choices of $W(\phi)$ and $V(\phi)$, it turns out that $|R(\phi)| \to \infty$ as $|\phi| \to \infty$. Examples include $V(\phi) = \mu^2 \phi^2/2$ or $V(\phi) = \lambda \phi^4/4$ or $V(\phi) = V_0 e^{-\phi/\phi_0}$, all of which are simple, and even well-motivated in some respects. Thus, unless $\phi$ is constrained to remain finite everywhere, a singularity will be encountered.

To see how to avoid singularities in general [13], consider first a space time with a single brane, at $y = y_b$, and suppose that the value of the scalar field on that brane is $\phi_b$. From Eq. (10), we can solve for $y$ as a function of $\phi$:

$$y(\phi) = y_b + I(\phi_b, \phi), \tag{16}$$

where

$$I(\phi_b, \phi) \equiv \int_{\phi_b}^{\phi} \frac{d\phi}{\partial W(\phi)/\partial \phi}. \tag{17}$$

Now if there exist finite values $\phi_{-\infty}$ and $\phi_{+\infty}$ of the bulk scalar field such that $|I(\phi_b, \phi_{\pm\infty})| = \infty$, then the spacetime can be infinite in the fifth dimension with $\phi \to \phi_{\pm\infty}$ as $y \to -\infty$ and $y \to \infty$. In such cases, an infinite range in $y$ is mapped onto a finite range in $\phi$, thus potentially avoiding any singularities.

From Eq. (17), divergences of $I(\phi_b, \phi)$ will occur only at zeros of $W'(\phi)$. Suppose now that $W'(\phi)$ has a zeros at both $\phi > \phi_b$ and $\phi < \phi_b$, and that $W'(\phi)$ vanishes at those zeros at least linearly (but not like $|\phi - \phi_{\pm\infty}|^p$ with $p < 1$). If, in addition, neither $W'(\phi)$ nor $V(\phi)$ diverges in between these zeros, then the spacetime can extend infinitely in the fifth dimension away from the brane in either direction, without any singularities. A similar result can be obtained for spacetimes with more than one brane. In this case, the condition is that $I(\phi_{b_\pm}, \phi_{b'})$ be finite for any two branes $b, b'$, and that for the two bounding branes, $b_{\pm}$, there exist $\phi_{\pm\infty}$ such that $I(\phi_{b_{\pm}}, \phi_{\pm\infty}) = \pm \infty$ with no intervening infinities in $V(\phi)$ and $\partial W(\phi)/\partial \phi$.

In models where these conditions for avoiding singularities are met, $V(\phi)$ tends to a constant asymptotically, as a zero is approached. Thus, such models, for $H = 0$, tend toward the RS solution far from branes, provided that the scalar field potential is negative. However, there will also be regions in such solutions where the spacetime metric differs considerably
from the RS model, and where the warp factor drops far faster than exponentially. We shall show how this comes about in a specific model below. The $H \neq 0$ solutions may also tend asymptotically toward their counterparts with uniform bulk cosmological constant \[4\]. However, the relatively harmless particle horizons characteristic of those uniform bulk cosmological constant models will generically be transformed into curvature singularities, unless there are fortuitous cancellations, as can be seen from the $A \to 0$ limit of Eq. \[8\].

Models where singularities would be inevitable in an uncompactified geometry can be truncated by compactification to a region that does not include any singularities. Such models need not approach the RS model, even asymptotically, and can have warp factors that vary much more rapidly than exponentially throughout the compactified fifth dimension.

III. MODELS WITHOUT COSMOLOGICAL EXPANSION

In this section we discuss several different choices of superpotential $W(\phi)$ and the properties of the corresponding solutions in the static case $H = 0$.

A. Even superpotential

The RS model corresponds to $W(\phi) = \pm a$, where $a > 0$ is a constant; this model has $V = -\kappa^2 a^2/6$. DFGK showed that the GW stabilization mechanism can be implemented if instead $W(\phi) = \pm(a - b\phi^2)$, where $b > 0$ is another constant, for which

$$V(\phi) = -\frac{\kappa^2 a^2}{6} + \frac{\phi^2}{2} \left( b^2 - 2\kappa^2 ab \right) - \frac{\kappa^2 b^2 \phi^4}{6}. \tag{18}$$

Choosing the $+$ sign in $W(\phi)$ for $y > 0$ (and vice-versa), DFGK found $\phi(y) = \phi(0)e^{-by}$, and therefore (via Eq. \[4\]) $3u/\kappa^2 = -a + b\phi^2(0)e^{-2by}$, which implies a transition between two regions of constant $u$, one at small $by$ and the other at large $by$, where the model becomes equivalent to the RS model. This model has $\partial W(\phi)/\partial \phi = 0$ at $\phi = 0$, and is nonsingular as long as the maximum value of $|\phi|$ is finite. Thus, it avoids singularities by mapping an infinite range of $y$ to a finite range in $\phi$, as was discussed in Sec. II C.

DFGK noted that they could also choose $W(\phi) = \pm(a + b\phi^2)$, with $b > 0$, for which

$$V(\phi) = -\frac{\kappa^2 a^2}{6} + \frac{\phi^2}{2} \left( b^2 + 2\kappa^2 ab \right) - \frac{\kappa^2 b^2 \phi^4}{6}. \tag{19}$$

Eqs. \[18\] and \[19\] can be rewritten in the form

$$V(\phi) = -\Lambda_b + \frac{\mu^2 \phi^2}{2} - \frac{\lambda_+ \phi^4}{4}, \tag{20}$$

where the $\mp$ signs apply to Eqs.\[18\] and \[19\] respectively, and where

$$\lambda_+ = \frac{2\kappa^2}{3} \left( \sqrt{\mu^2 + \frac{2\kappa^2 \Lambda_b}{3}} + \sqrt{\frac{2\kappa^2 \Lambda_b}{3}} \right)^2. \tag{21}$$
Despite appearances, the differences between the two models (18) and (19) are not minor. For the + model, we have \( W(\phi) = a + b\phi^2 \) for \( y > 0 \), so that \( \phi' = b\phi \), \( \phi(y) = \phi(0)e^{by} \) and \( 3u/\kappa^2 = -a - b\phi^2(0)e^{2by} \). This yields a warp factor \( A(y) \) which is an exponential of an exponential:

\[
\ln A(y) = \ln A(0) - \frac{1}{3}\kappa^2 ay - \frac{1}{6}\kappa^2\phi(0)^2 \left[ e^{2by} - 1 \right].
\] (22)

This super-warped version of the GW model, obtained from an apparently insignificant change to the potential (20), becomes singular if the fifth dimension is not compactified, because \( R(\phi) \to -\infty \) as \( \phi \to \infty \). Any compactification of the fifth dimension avoids the singularity by construction. The model can be altered slightly to become bulletproof against singularities by taking \( W(\phi) = \pm(a + b\phi^2 - b\phi^4/2\phi_\infty^2) \). For this choice of \( W \) we have

\[
\phi(y) = \frac{\phi(0)\phi_\infty e^{by}}{\sqrt{\phi_\infty^2 - \phi^2(0) + \phi^2(0)e^{2by}}},
\] (23)

so that \( \phi \to \phi_\infty \) and \( u \to -(\kappa^2/3)(a + b\phi_\infty^2/2) \) at large \( y \), reducing asymptotically to the RS model. Note that the total change in the logarithm of the warp factor in the super-warped regime is limited to \( \sim \kappa^2\phi_\infty^2 \), which may be large.

**B. Odd superpotential and Gaussian warp factor**

The DFGK realizations of the GW stabilization mechanism for the RS model are based on choices of superpotential \( W(\phi) \) that are even functions of \( \phi \). There are also simple models in which \( W(\phi) \) is odd, first considered in Ref [13]; the simplest is

\[ W(\phi) = \pm 2b\phi, \] (24)

with \( b > 0 \). (A constant term in \( W(\phi) \) can be absorbed into the definition of \( \phi \).) For the model (24), we find that

\[
\phi = \phi(0) + by,
\]

\[
u = \frac{-2\kappa^2 b\phi(0)}{3} - \frac{2\kappa^2 b^2 y}{3},
\]

\[
\ln A(y) = \ln A(0) - \frac{2\kappa^2 b\phi(0)y}{3} - \frac{\kappa^2 b^2 y^2}{3},
\] (25)

for \( y > 0 \). Thus, the warp factor in this model is Gaussian rather than exponential, and therefore falls off more rapidly than in the RS model. Note that \( A(y) \) has a maximum value, and all positive tension branes must be located away from that maximum.

This model contains a singularity at \( y \to \infty \) since \( \phi(y) \) diverges there. The singularity can be avoided by compactification, or can be removed altogether by setting \( W(\phi) = \pm 2b(\phi - \phi^3/3\phi_\infty^2) \), in which case

\[
\phi(y) = \phi_\infty \tanh \left[ \tanh^{-1} \left( \frac{\phi(0)}{\phi_\infty} \right) + \frac{by}{\phi_\infty} \right]
\] (26)
which tends to $\phi_\infty$ asymptotically. The warp factor for this model is given by

$$\ln A(y) = \ln A(0) - \frac{4\kappa^2\phi^2_\infty}{9} \left\{ \ln \left[ \cosh \left( \frac{by}{\phi_\infty} \right) + \frac{\phi(0)}{\phi_\infty} \sinh \left( \frac{by}{\phi_\infty} \right) \right] 
+ \frac{1}{4} \left( 1 - \frac{\phi^2(0)}{\phi^2_\infty} \right) \left[ 1 - \frac{1}{\cosh(by/\phi_\infty) + \phi(0) \sinh(by/\phi_\infty)/\phi_\infty} \right] \right\},$$

and tends toward $\ln A(y) \simeq -\frac{4\kappa^2 b\phi_\infty y}{9}$ for $by/\phi_\infty \gg 1$, which is the same scaling as in the RS model. The maximum change in the logarithm of the warp factor in the Gaussian-warped regime is of order $\sim \kappa^2 \phi^2_\infty$, similar to what we found for the model of Sec. III A above.

C. Exponential potential

The models considered so far become singular only in the asymptotic regime $y \to \infty$. Thus, whether or not these singularities merit a potential-altering cure is largely a matter of taste, since compactifying to any finite size in the fifth dimension removes the singularity without any sort of fine-tuning. However, there are other models which are “spontaneously singular” in the sense that they develop singularities at finite $y$. An example is when the superpotential is an exponential,

$$W(\phi) = 2ae^{-k\phi}. \quad (28)$$

For this model, we have

$$e^{-k\phi} = \left[ e^{k\phi(0)} - k^2 a y \right]^{-1}, \quad (29)$$

which diverges at $y = (k^2a)^{-1}e^{k\phi(0)}$, implying a divergence of the Ricci scalar. The singularity is removed by the following slight modification of the superpotential,

$$W(\phi) = 2a \left[ e^{-k\phi} - \frac{k}{q} e^{(q-k)\phi_\infty - q\phi} \right], \quad (30)$$

where $q > k$ is a constant. The associated potential is

$$V(\phi) = \frac{a^2e^{-2k\phi_\infty}}{2} \left[ e^{-2k(\phi - \phi_\infty)} \left( k^2 - \frac{4\kappa^2}{3} \right) - 2e^{-(k+q)(\phi - \phi_\infty)} \left( k^2 - \frac{4\kappa^2k}{3q} \right) 
+ e^{-2q(\phi - \phi_\infty)} \left( k^2 - \frac{4\kappa^2k^2}{3q^2} \right) \right]. \quad (31)$$

For this model the relation between $\phi$ and $y$ is given by

$$\int_{e^{k(\phi(0) - \phi_\infty)}}^{e^{k(\phi - \phi_\infty)}} \frac{d\eta \eta^{q/k-1}}{\eta^{q/k-1} - 1} = k^2a e^{-k\phi_\infty} y, \quad (32)$$

which can be integrated numerically in general. The integral can be done analytically in special cases. For example, for $q = 2k$, $q = 3k$, and $q = 5k$ we find the results.
\[ e^{k(\phi - \phi_\infty)} + \ln \left[ e^{k(\phi - \phi_\infty)} - 1 \right] = e^{k(\phi(0) - \phi_\infty)} + \ln \left[ e^{k(\phi(0) - \phi_\infty)} - 1 \right] - k^2ae^{-k\phi_\infty} y, \]

\[ e^{k(\phi - \phi_\infty)} + \frac{1}{2} \ln \left[ \tanh \frac{k}{2}(\phi - \phi_\infty) \right] = e^{k(\phi(0) - \phi_\infty)} + \frac{1}{2} \ln \left[ \tanh \frac{k}{2}(\phi(0) - \phi_\infty) \right] - k^2ae^{-k\phi_\infty} y, \]

and

\[ e^{k(\phi - \phi_\infty)} + \frac{1}{4} \ln \left[ \tanh \frac{k}{2}(\phi - \phi_\infty) \right] - \frac{1}{2} \tan^{-1} \left( e^{k(\phi - \phi_\infty)} \right) = e^{k(\phi_\infty)} - \frac{1}{2} \tan^{-1} \left( e^{k(\phi - \phi_\infty)} \right) - k^2ae^{-k\phi_\infty} y, \]

respectively. For large values of \( y \) this model reduces to the RS model, since \( \phi \to \phi_\infty \) from above and \( 3u/k^2 \to -2ae^{-k\phi_\infty}(1 - k/q) \), a constant.

1. **Specific realization obtained from dimensional reduction**

A potential reminiscent of the potential [31] arises from compactification of an 11-D spacetime to 5-D; the result is

\[ V(\psi) = -\Lambda_b e^{-2\psi/3} - c_6 e^{-\psi} + \frac{\mathcal{E}^2}{2} e^{-2\psi}. \]  

(34)

Here it is assumed that the six compactified dimensions have a single associated radion field, \( \psi \), the curvature associated with the compactified dimensions is \( c_6 > 0 \), \( \Lambda_b \) is the descendant of an 11-D cosmological constant, and \( \mathcal{E} \) is a 5-form field strength (descended from a 7-form field strength in 11-D). Although the potentials [31] and [34] are similar, there are no choices of parameter values for which the potentials match up term by term. This is because the exponents in three terms in Eq. [34] are in the ratios 1 : 1.5 : 3, whereas the exponents in Eq. [31] are in the ratio 1 : \((k + q)/(2k) : q/k\). However, we can find parameter choices for which the potentials match if we allow one of the two parameters \( c_6 \) and \( \Lambda_b \) to vanish.

If we drop the term proportional to \( \Lambda_b \) and retain curvature in the six compactified dimensions, we find two possible identifications:

\[ q = 2k, \quad k^2 = 2\kappa^2/3, \quad \mathcal{E}^2 = c_6 = (\kappa^2a^2/3)e^{-2k\phi_\infty}, \]  

(35)

and

\[ q = 3k, \quad k^2 = 4\kappa^2/27, \quad \mathcal{E}^2 = c_6 = (16\kappa^2a^2/27)e^{-2k\phi_\infty}. \]  

(36)

If we set \( c_6 = 0 \) but retain \( \Lambda_b \neq 0 \), there are also two possibilities:

\[ q = 3k, \quad k^2 = 4\kappa^2/9, \quad \Lambda_b = (4\kappa^2a^2/9)e^{-2k\phi_\infty}, \quad \mathcal{E}^2 = (8k^2a^2/27)e^{-2k\phi_\infty}, \]  

(37)

and

\[ q = 5k, \quad k^2 = 4\kappa^2/75, \quad \Lambda_b = (12\kappa^2a^2/25)e^{-2k\phi_\infty}, \quad \mathcal{E}^2 = (32\kappa^2a^2/75)e^{-2k\phi_\infty}. \]  

(38)

Whichever possibility we choose, we can regard the combination \( \kappa^2a^2e^{-2k\phi_\infty} \) as a derived quantity, determined by either \( \Lambda_b \) or \( c_6 \), depending on which is nonzero. The 5-form field strength \( \mathcal{E} \), which involves a constant of integration, can then be adjusted to produce a nonsingular model. The 5-form fields therefore may play a central role in eliminating spacetime singularities from 5-D models derived by dimensional reduction from 11-D.
IV. MODELS WITH COSMOLOGICAL EXPANSION

A. Bulk singularities at zeros of the warp factor

When the bulk geometry is anti-deSitter (AdS) space, metrics of the form (3) have non-singular particle horizons at zeros of the warp factor $A(y)$. From the expression (8) for the Ricci scalar, one might anticipate a singularity from the term $\propto 1/A$, but that divergence is cancelled by divergences in the $u^2$ and $u'$ terms for an AdS bulk.

However, when scalar fields are included, the surfaces where $A(y) = 0$ generically correspond to curvature singularities. This is the case even when we set $\phi' = \frac{1}{2} \partial W(\phi)/\partial \phi$ and choose $\partial W(\phi)/\partial \phi$ so that $\phi$ remains finite over the entire range of $y$. The simplest way to see this is to consider the first of Eqs. (4) in a neighborhood of a point where $A(y) = 0$. Suppose that $\phi'$ is finite and nonzero in that neighborhood, so that as long as we restrict attention to a small enough region, we can take it to be a constant. If $q^2 \equiv \kappa^2(\phi')^2/3$, then

$$u' = \left(\frac{A'}{A}\right)' = -2q^2 - \frac{2H^2}{A},$$

This equation would be exact for $\partial W(\phi)/\partial \phi = \text{constant}$, which led us to the Gaussian model in Sec. III. Multiply by $u = A'/A$ and integrate to find

$$u^2 = \left(\frac{A'}{A}\right)^2 = -4q^2 \ln |A| + \frac{4H^2}{A} + 4k^2.$$  \hspace{1cm} (40)

Here $k^2$ is a constant which may be positive or negative, although we shall take it to be positive without exception to recover the RS model when $q = 0 = H$. Substituting these results for $u'$ and $u^2$ into Eq. (8), we find

$$R = -20k^2 + 8q^2 + 20q^2 \ln |A|$$

as $A \to 0$. Thus, the Ricci scalar diverges at this point, which is a true singularity, unless $q^2 = 0$. The divergence is relatively mild (logarithmic). In fact, for $H = 0$ this is the same divergence as was found asymptotically in the Gaussian model of Sec. III.

B. Method of constructing singularity-free solutions

In the singularity-free models with $H = 0$, we saw that obtaining a mapping of the infinite $y$ domain to a finite range of $\phi$ required that $\phi' \to 0$ asymptotically. For $H = 0$ models the singularity is absent, because $A \to 0$ only asymptotically, and $q^2 \ln A \to 0$ asymptotically as well. For $H \neq 0$, nonsingular models can be constructed parametrically by the following procedure. Define a function $g(A)$ by

$$\frac{2\kappa^2(\phi')^2}{3} = 2A \frac{dg(A)}{dA} \geq 0,$$

with $g(0) = 0$. Then Eq. (40) generalizes to
Comparing this with the second of Eqs. (4) we obtain

$$\frac{2\kappa^2 V(\phi)}{3} = A \frac{dg(A)}{dA} + 4g(A) - 4k^2. \quad (44)$$

The procedure for obtaining a solution can now be summarized as follows: (i) pick a function $g(A)$ and the integration constant $k$; (ii) solve Eq. (43) to obtain $A$ as a function of $y$; (iii) solve Eq. (42) to obtain $\phi$ as a function of $y$; (iv) use Eq. (44) to compute the potential $V(\phi)$ corresponding to the solution just obtained. By construction, the Ricci scalar

$$R = -20k^2 + 8A \frac{dg}{dA} + 20g(A) \quad (45)$$

is finite at $A \to 0$.

Let us illustrate this procedure with the particular example

$$g(A) = q^2 A^p, \quad (46)$$

with $p > 0$. (For small $p$, we expect this model to resemble the singular Gaussian model.) For the choice (46) we find

$$\left( \frac{A'}{A} \right)^2 = 4k^2 - 4q^2 A^p + \frac{4H^2}{A}. \quad (47)$$

Apparently, there is a maximum value of $A(y)$ for such models; for small $H^2/k^2$ we find $A_{\text{max}} \simeq (k/q)^{2/p} + H^2 / pk^2$. Note that in this particular model, freedom to rescale $A(y)$ also implies that $q$ may be rescaled arbitrarily; this need not be true for $g(A)$ that are not scale-free. When $H = 0$, the solution to Eq. (47) is (apart from an overall ambiguity in the sign of $A'$ and hence $\phi$)

$$A(y) = \left( \frac{k}{q \cosh pky} \right)^{2/p}, \quad (48)$$

$$\phi \sqrt{\frac{k^2 p}{12}} = \tan^{-1} (e^{pk y}) - \frac{\pi}{4}, \quad (49)$$

$$\frac{2\kappa^2 V(\phi)}{3} = 4k^2 \left[ 1 + \left( 1 + \frac{p}{4} \right) \cos^2 \left( \phi \sqrt{\frac{k^2 p}{3}} \right) \right], \quad (50)$$

where we have located the maximum of $A(y)$ at $y = 0$ and chosen $\phi = 0$ at $y = 0$. As $y$ ranges from zero to infinity, $\phi \sqrt{k^2 p/3}$ ranges from 0 to $\pi/2$. Note that the maximum value of $V(\phi)$ is positive in this model, and occurs at the maximum of $A(y)$; this is already apparent from Eqs. (43) and (44). In a particular realization of this model, with branes at specific values of $\phi$, the maximum value of the warp factor may never be encountered. Finally, notice that for small values of $p$, the warp factor $A(y) \sim e^{-pky^2}$ for $pky \lesssim 1$, and, for any value of $p$, $A(y) \sim e^{-ky}$ for $pky \gtrsim 1$. These features are reminiscent of the modified nonsingular
Gaussian model of Sec. III, and the potential (50) therefore represents an alternative way of regularizing the Gaussian model to avoid singularities.

When \( H = 0 \), Eq. (48) shows that \( A(y) \rightarrow 0 \) only when \( y \rightarrow \infty \). When \( H \neq 0 \), Eq. (47) shows that \(|A'/A|\) is larger than for \( H = 0 \), and we therefore expect the position of the zero of \( A(y) \) to move inward from \( y = \infty \) to finite \( y \). If \( y_0 \) is the zero, so that \( A(y_0) = 0 \), then \( A(y) \) may be found by inverting the equation

\[
2k(y_0 - y) = \int_0^A \frac{dA}{A\sqrt{1 - Ap/k^2 + H^2/k^2 A}}. 
\]

(51)

and the scalar field may be found from

\[
\phi(A) = \phi(0) - \left(\frac{3p}{4k^2}\right)^{1/2} \int_0^A \frac{dA Ap^{1/2} - 1}{\sqrt{k^2 - Ap + H^2/A}}. 
\]

(52)

Here \( \phi(0) \) is the value at \( A = 0 \), and we have assumed \( \phi' > 0 \). Eqs. (52) and (44) may be used to evaluate the potential \( V(\phi) \), which will have a nontrivial dependence on \( H \). For \( A \lesssim H^2/(1+p) \), the scalar field plays an insignificant role, and we recover the vacuum solution \( A(y) \simeq (H^2/k^2) \sinh[k(y - y_0)] \). For \( A \gtrsim H^2/(1+p) \), on the other hand, we can neglect \( H^2 \), and we recover the solution (18) for \( A(y) \). Corrections to these approximate solutions may be obtained from Eq. (51). In general, the behavior of models with scalar fields is subtle for small values of \( H^2 \), because nonsingular models require that the effects of the scalar field become insignificant as \( A(y) \rightarrow 0 \), so that the \( H^2 \) terms become important in these regions despite the smallness of \( H^2 \).

These nonsingular models appear to share common features with models developed previously without bulk scalar fields [4], particularly the occurrence of non-singular surfaces where \( A(y) = 0 \). Presumably, these surfaces are particle horizons similar to those which occur in the pure AdS case.

\section*{V. COSMOLOGICAL CONSTANT AND HIERARCHY PROBLEMS}

To examine the implications of these models for the value of the cosmological constant, focus again on the model (46) and rewrite Eq. (13) as

\[
u^2 = \left(\frac{A'}{A}\right)^2 = 4 \left(k^2 - \frac{\kappa^2(\phi')^2}{3p} + \frac{H^2}{A}\right).
\]

(53)

First, let us consider the \( S^1/Z_2 \) orbifolded model, with a brane at each end, where the warp factors at these branes are \( A_i \), \( i = 1, 2 \), and the brane tensions are \( \sigma_i \). Note that it is possible for both brane tensions to be positive, if one of them is located at \( y > y_0 \) and the other at \( y < y_0 \). We define the quantities \( Q_1 \) and \( Q_2 \) by

\[
4Q_i^2 = \left(\frac{\kappa^2\sigma_i}{3}\right)^2 + \frac{\kappa^2}{3p} \left(\frac{\partial \sigma_i}{\partial \phi}\right)_i^2.
\]

(54)

Using the jump conditions (3) along with Eq. (53), where the constant term in \( V(\phi) \) is adjustable (for example, via unimodular gravity), we find that
\[ k^2 = \frac{A_1 Q_1^2 - A_2 Q_2^2}{A_1 - A_2} \]
\[ H^2 = \frac{A_1 A_2 (Q_2^2 - Q_1^2)}{A_1 - A_2}. \]

(55)

In the regime \( A_2 \ll A_1 \), these formulae reduce to

\[ k^2 \approx Q_1^2 \]
\[ H^2 \approx A_2 (Q_2^2 - Q_1^2), \]

(56)

where we have assumed that \( Q_2^2 \sim Q_1^2 \). Note that Eq. (54) requires that \( Q_2^2 > Q_1^2 \). The result (56) implies that the four dimensional cosmological constant \( \Lambda_4 \propto H^2/A_1 \) is suppressed by the ratio \( A_2/A_1 \) of warp factors, which will be vanishingly small for suitable brane separations.

Now suppose that our Universe lives on a test brane located somewhere between the branes 1 and 2. Denote by \( A_U \) the warp factor on our brane, and let brane 1 be the Planck brane. Then, we find that the electroweak scale on our brane is given by \( m_{EW}^2 = \frac{A_1 A_2 (Q_2^2 - Q_1^2)}{m_P^2 A_U(A_1 - A_2)} \), where \( m_P \) is the Planck scale. The expansion rate on our brane is \( H_U^2 = H^2/A_U \), and therefore

\[ \frac{H_U^2}{m_{EW}^2} = \frac{A_1 A_2 (Q_2^2 - Q_1^2)}{m_P^2 A_U(A_1 - A_2)} \approx \frac{A_1 A_2 (Q_2^2 - Q_1^2)}{m_P^2 A_U^2}. \]

(57)

Assuming that \( Q_1^2 \sim m_P^2 \), we see that \( H_U^2/m_{EW}^2 \) can be exponentially small provided that \( A_U \gg \sqrt{A_1 A_2} \). Roughly speaking, this will be the case if the test brane on which our Universe resides is closer to the Planck brane, brane 1, than it is to brane 2. Turning on a small positive brane tension for the visible brane does not change the above qualitative features. Even for models in which singularities are not absolutely prevented, they can still be avoided in the orbifold case, which requires a negative tension brane at one fixed point. If one only wants to solve the mass hierarchy problem, this negative tension brane is the visible brane [1]. If one also wants to solve the cosmological constant problem, the visible brane must be identified with a third brane between the orbifold boundaries, as in the nonsingular models discussed above.

This result can be generalized to other \( H \neq 0 \) spacetimes with no bulk curvature singularities, and remains valid for uncompactified, multi-brane models as well. It can also be generalized to cases where the branes are charged under some 4-form potentials [4]. In this case, the constant term (and/or other coefficients) in \( V(\phi) \) depends on the 5-form field strengths, and the background values of these field strengths are determined by satisfying the jump conditions at the branes. With the warp factor normalized to unity at the Planck brane, \( \Lambda_4 \) is proportional to \( A_p \), the warp factor at the nearest particle horizon. This factor can easily be exponentially small. Since the visible brane must be closer to the Planck brane than the particle horizon, the warp factor \( A_v \) at the visible brane can easily be exponentially small as well, providing a solution to the hierarchy problem [4], but \( A_v >> A_p \), as observed in nature.
VI. DISCUSSION

In this paper, we have constructed several different models of 5-D brane worlds with bulk scalar fields. We gave a general procedure for constructing static ($H = 0$) models with no curvature singularities in the bulk. A specific model was constructed corresponding to compactifying from an 11-D theory with a 7-form field strength down to 5-D, where the 7-form field descends to a 5-form field strength. By adjusting the 5-form field strength, we were able to render the model nonsingular. In general, we found that nonsingular models are more warped than the RS model on relatively small distance scales, but tend towards simple exponential warping on large scales. Thus, the RS behavior is robust in the asymptotic regime, although stabilization of branes via the introduction of bulk scalar fields may require the branes to reside where the fields produce substantial extra warping of spacetime.

For expanding ($H \neq 0$) models, we found that singularities occur even for those scalar field potentials that do not give rise to singularities in the $H = 0$ case. We therefore were compelled to explore a different method for finding nonsingular models with $H \neq 0$. A subclass of these models incorporates a non-singular surface on which the warp factor $A(y)$ vanishes, analogous to the particle horizons that occur in the pure AdS case [4]. One specific class of such models, which corresponds to a sinusoidally varying potential in the nonexpanding case, was treated in some detail. For this case we constructed an orbifolded model, with one bounding brane being the Planck brane, and the other bounding brane (on which the warp factor is exponentially smaller) having either positive or negative tension. In this model, we live on a test brane between the two boundaries branes, and the expansion rate on our brane is exponentially smaller than the local electroweak scale provided that our brane is located somewhat closer to the Planck brane than to the other brane. This feature does not appear to depend in an essential way on details of our particular model, and even can be shown to hold when singularities are avoided (by orbifolding) rather than prevented outright by choice of potential. Thus, there are models in which the cosmological constant and mass hierarchy problems can be explained simultaneously.

Two fine-tunings of the parameters of the RS orbifold model are required to get flat 4-D spacetime [1]. Allowing non-zero $\Lambda_4$ relaxes one of these fine-tunings, while the other can be eliminated by allowing the bulk cosmological constant to depend on 5-form field strengths, whose piecewise constant values are determined by boundary conditions at the branes. When the branes are relatively far apart, $\Lambda_4$ turns out to be exponentially small [4]. However, the models of Ref. [4] were dynamically unstable. Bulk scalar fields can be introduced to fix brane positions. When $\Lambda_4 = 0$, there are two sources of fine tuning in models with bulk scalar fields, one associated with expressing the potential $V(\phi)$ in terms of a superpotential, $W(\phi)$ [Eqs. (11) and (13)], and the other associated with adjusting $W(\phi)$ so that the solution becomes nonsingular. Given a potential $V(\phi)$, there is no guarantee that a $W(\phi)$ can be found that satisfies Eq. (13) without tuning the parameters [11]. For example, a general quartic potential of the form $V(\phi) = a_0 + a_2 \phi^2/2 + a_4 \phi^4/4$ contains three parameters, whereas Eq. (15) or (19) contains only two, implying that a relationship among $a_0$, $a_2$ and $a_4$ is needed in order for $V(\phi)$ to arise from a superpotential $W(\phi)$. Moreover, Eq. (18) corresponds to a nonsingular model, whereas Eq. (19) is singular; the two independent parameters in a quartic potential have to be specifically adjusted to prevent singularities. The adjustment can be achieved by the introduction of adjustable 5-form field strengths, and we may speculate
that, in an evolving bulk, the 5-form field strengths may relax naturally (e.g. via bubble nucleation) to values that prevent the occurrence of singularities. (A specific example of the role that 5-form field strengths might play is given in Eqs. (35)-(38) of Sec. III C 1.) In general, additional fine-tunings required to prevent singularities need not be invoked in an orbifold model, where compactification may simply avoid the presence of singularities that could otherwise arise in an uncompactified model.

When $\Lambda_4 \neq 0$, some fine tuning is still needed, apparently, for singularities to be absent, but we had to resort to a different method, not based on a superpotential, to construct nonsingular solutions in this case. The problem arises because singularities can only be prevented if $\phi' \to 0$ when $A(y) \to 0$, a condition that does not follow readily from a description in terms of a superpotential, and therefore represents a different kind of tuning. This suggests that the bulk alone may be supersymmetric, making a description in terms of a superpotential $W(\phi)$ natural, but that the presence of branes breaks the supersymmetry completely, and leads to nonzero $\Lambda_4$. As in the nonexpanding case, it is also possible to preserve models based on a superpotential without any special tuning of the parameters in $W(\phi)$ in a appropriately compactified or orbifolded model, but it is never possible to relate $V(\phi)$ directly to $W(\phi)$ when $\Lambda_4 \neq 0$.

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