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Jacobians and branch points of real analytic open maps

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Summary. The main result is that the Jacobian determinant of an analytic open map \( f: \mathbb{R}^n \to \mathbb{R}^n \) does not change sign. A corollary of the proof is that the set of branch points of \( f \) has dimension \( \leq n - 2 \).

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Introduction

The main object of this paper is to prove the following result:

**Theorem 1.** The Jacobian of a real analytic open map \( f: \mathbb{R}^n \to \mathbb{R}^n \) does not change sign.

One of the referees kindly pointed out that the special case of polynomial maps was proved by Gamboa and Ronga [3]:

**Theorem 2** (Gamboa and Ronga). A polynomial map in \( \mathbb{R}^n \) is open if and only if point inverses are finite and the Jacobian does not change sign.

The proof of Theorem 1 is very similar to methods in [3], which are easily adapted to analytic maps; but as Theorem 1 does not seem to be known, a direct proof may be useful.

\( f: \mathbb{R}^n \to \mathbb{R}^n \) denotes a (real) analytic map in Euclidean \( n \)-space. We always assume \( f \) is open, that is, \( f \) maps open sets onto open sets. Denote the Jacobian matrix of \( f \) at \( p \in \mathbb{R}^n \) by \( df_p = \left[ \frac{\partial f_i}{\partial x_j} (p) \right] \). The rank of \( df_p \) is called the rank of \( f \) at \( p \), denoted by \( \text{rk}_p f \); the determinant of \( df_p \) is the Jacobian of \( f \) at \( p \), denoted by \( Jf(p) \). When the analytic function \( Jf: \mathbb{R}^n \to \mathbb{R} \) is everywhere non-negative or everywhere non-positive (in a set \( X \)), we say \( Jf \) does not change sign (in \( X \)).

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The following sets are defined for any $C^1$ map $g: M \to N$ between $n$-manifolds (without boundary):

- the set $R_k = \{ p \in M : \text{rk}_p g \leq k \}$
- the critical set, $C = R_{n-1}$
- the branch set, $B = \{ p \in U : g$ is not a local homeomorphism at $p \}$

Note that $B \subset C$ by the inverse function theorem. When $g$ is analytic, we also define:

- the critical analytic hypersurface $H \subset C$, comprising those points having a neighborhood in $C$ that is an analytic submanifold of dimension $n - 1$
- the constant rank analytic hypersurface $V \subset H$, at which $g | H$ has locally constant rank

The following results are byproducts of the proof of Theorem 1:

**Theorem 3.**

(i) The restricted map $f | V$ has rank $n - 1$,
(ii) $f$ is a local homeomorphism at every point of $V$,
(iii) $B \subset R_{n-2}$,
(iv) $\text{dim } R_{n-2} \leq n - 2$.

When $n = 2$, conclusions (iii) and (iv) imply $B$ is a closed discrete set; thus in this case $f$ is light, i.e., point inverses are 0-dimensional. From Stolow [4], which topologically characterizes germs of light open surface maps, we obtain:

**Corollary 4.** When $n = 2$, the germ of $f$ at any point is topologically equivalent to the germ at 0 of the complex function $z^d$ for some integer $d \neq 0$.

A key role in our proofs is played by the following result, Theorem 1.4 of Church [2]:

**Theorem 5 (Church).** If $g: \mathbb{R}^n \to \mathbb{R}^n$ is $C^n$ and open with rank $\geq n - 1$ at every point, then $g$ is a local homeomorphism.

Our results are close to some of those obtained by Church for $C^n$ maps. It is interesting to compare Theorem 3 (iii) and Corollary 4 to the following results from paragraphs 1.5 to 1.8 of his paper [2]:

**Theorem 6 (Church).** Let $g: M \to N$ be a $C^n$ map between $n$-manifolds.

(i) If $M = N = \mathbb{R}^n$ and $g$ is light, the following conditions are equivalent:
   (a) $g$ is open,
   (b) $Jg$ does not change sign,
   (c) $B \subset R_{n-2}$.
(ii) If $M$ is compact and $g$ is open, then $g$ is light.
Lemma 7. Assume the critical set of $f$ is $C = Jf^{-1}(0) = \mathbb{R}^{n-1} \times \{0\}$, and $f|C$ has constant rank $k$, $0 \leq k \leq n - 1$. Then $f$ is a local homeomorphism, $k = n - 1$, and $Jf$ does not change sign in $\mathbb{R}^n$.

Proof. It suffices to prove that the conclusion holds in some neighborhood of each point, which we may take to be the origin.

It is convenient to denote points of $\mathbb{R}^n$ as $(y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

By the rank theorem we assume that in some open cubical neighborhood $N$ of the origin,

$$f_i(y, 0) \equiv 0, \quad i = k + 1, \ldots, n. \quad (1)$$

Identifying $N$ with $\mathbb{R}^n$ by an analytic diffeomorphism, we assume this holds for all $y \in \mathbb{R}^n$.

Because $f$ is analytic and open, there is a dense open set $\Lambda \subset \mathbb{R}^{n-1}$ such that for every $y \in \Lambda$, the map $t \mapsto f_n(y, t)$ is not constant on any interval. For each $y \in \Lambda$ there exists a maximal integer $\mu(y) \geq 0$ such that

$$0 < j < \mu(y) \implies \left(\frac{\partial}{\partial t}\right)^j f_n(y, 0) = 0,$$

Fix $y_* \in \Lambda$ such that the function $\mu: \Lambda \to \mathbb{N}$ takes its minimum value $m$ at $y_*$. Then $\mu = m$ in a precompact open neighborhood $W \subset \Lambda$ of $y_*$. By Taylor’s theorem there exists $\epsilon > 0$ such that for $(y, t)$ in the open set $N = W \times ] - \epsilon, \epsilon[ \subset \mathbb{R}^{n-1} \times \mathbb{R}$ we have

$$f_n(y, t) = t^m H(y, t), \quad H(y, t) \neq 0. \quad (2)$$

Claim. If $k \leq n - 2$ and $(y_0, t_0) \in N$ is such that $f_n(y_0, t_0) = 0$, then $f_{n-1}(y_0, t_0) = 0$. For $t_0 = 0$ by (2), and $k \leq n - 2$ implies $f_{n-1}(y_0, 0) = 0$ by (1).

Now we assume $k \leq n - 2$ and reach a contradiction. Since $f(N)$ is open and contains

$$f(y_*, 0) = (a_1, \ldots, a_{n-2}, 0, 0),$$

$f(N)$ also contains points $(a_1, \ldots, a_{n-2}, \delta, 0)$ with $\delta > 0$. But this contradicts the claim.

As $f$ has rank $n - 1$ at every point of the critical set, $f$ must be a local homeomorphism by Theorem 5. Therefore for every $p$, the induced homomorphism of homology groups

$$\mathbb{Z} = H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{p\}) \to H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{f(p)\}) = \mathbb{Z}$$

is an isomorphism, hence is multiplication by a number $\delta(p) \in \{+1, -1\}$. 


Homology theory implies that each of the two level sets of \(\delta: \mathbb{R}^n \to \{+1, -1\}\) is open. As \(\mathbb{R}^n\) is connected, \(\delta(p)\) is constant. As \(\delta(p)\) is the sign of \(Jf(p)\) if \(Jf(p) \neq 0\), we have proved \(Jf\) does not change sign.

**Proof of Theorem 1.** For any set \(Y \subset \mathbb{R}^n\), we say the local theorem holds in \(Y\) if every point of \(Y\) has a neighborhood in \(Y\) in which \(Jf\) does not change sign.

**Lemma 8.** If the local theorem holds in a connected set \(Y\), then \(Jf\) does not change sign in the closure \(\overline{Y}\).

**Proof.** It suffices to prove \(Jf\) does not change sign in \(Y\), because \(Jf\) is continuous. Define \(Y_+, Y_-\) to be the subsets of \(Y\) where \(Jf\) is respectively \(\geq 0\) and \(\leq 0\). These sets are closed in \(Y\) by continuity of \(Jf\), and open in \(Y\) by hypothesis. As \(Y\) is connected, either \(Y = Y_+\) or \(Y = Y_-\).

The local theorem obviously holds in the set \(\mathbb{R}^n \setminus C\) of noncritical points. By Lemma 7, it also holds in the relatively open analytic hypersurface \(V \subset C\) defined in the introduction. It remains to prove that every point of \(C \setminus V\) has a neighborhood in which \(Jf\) does not change sign.

**Lemma 9.** Every point \(p \in C \setminus V\) has a neighborhood \(X_p \subset C \setminus V\) that is an analytic variety of dimension \(\leq n-2\).

**Proof.** Write

\[
C \setminus V = (C \setminus H) \cup (H \setminus V)
\]

Suppose \(p \in C \setminus H\). In this case we take \(X_p\) to be the union of the variety \(C_{\text{sing}}\) of singular points of \(C\) and those connected components of \(C \setminus C_{\text{sing}}\) having dimension \(\leq n-2\).

Suppose \(p \in H \setminus V\), or equivalently: \(p \in H\) and some minor determinant of \(df\) vanishes at \(p\) but not identically in any neighborhood of \(p\) in \(H\). We take \(X_p\) to be the intersection of \(C\) with the union of the zero sets of such minors.

Now consider any point \(p \in C \setminus V\). By Lemma 9, \(p\) has a neighborhood \(X_p \subset C \setminus V\) that is the union of finitely many smooth submanifolds of \(\mathbb{R}^n\) having dimensions \(\leq n-2\).

Choose a connected open neighborhood \(N_p \subset \mathbb{R}^n\) of \(p\) such that \(N_p \cap (C \setminus V) = N_p \cap X_p\). Then \(N_p \setminus X_p \subset (\mathbb{R}^n \setminus C) \cup V\). Therefore the local theorem holds in \(N_p \setminus X_p\).

Now \(N_p \setminus X_p\) is connected, by a standard general position argument. Therefore from Lemma 8, with \(Y = N_p \setminus X_p\), we infer that \(Jf\) does not change sign in \(N_p \setminus X_p\), which equals \(N_p\) because \(X_p\) is nowhere dense. This completes the proof of Theorem 1.
Proof of Theorem 3. Parts (i) and (ii) of Theorem 2 are proved by applying Lemma 7 locally. Lemma 9 implies (iii), because \( B \subset C \setminus V \) by (ii). For (iv), suppose \( \dim R_{n-2} = n - 1 \). Then the variety \( R_{n-2} \) contains an analytic hypersurface, which must meet \( V \). As \( R_{n-2} \subset C \), this implies \( R_{n-2} \cap V \neq \emptyset \), contradicting (i).

\[ \square \]

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