Implicit regularization for convex regularizers

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Abstract

We study implicit regularization for over-parameterized linear models, when the bias is convex but not necessarily strongly convex. We characterize the regularization property of a primal-dual gradient based approach, analyzing convergence and especially stability in the presence of worst case deterministic noise. As a main example, we specialize and illustrate the results for the problem of robust sparse recovery. Key to our analysis is a combination of ideas from regularization theory and optimization in the presence of errors. Theoretical results are complemented by experiments showing that state-of-the-art performances are achieved with considerable computational speed-ups.

1 Introduction

In modern machine learning over-parameterized models are common. For example, in sparsity-based learning it is usual to consider linear models with a number of parameters vastly larger than the number of available data (Bühlmann and Van de Geer, 2012). In deep learning, non linear models with millions or more parameters are typical (Goodfellow et al., 2016). Considering over-parametrized models poses at least two orders of questions. On the one hand, since the corresponding problems are heavily under-determined, multiple solutions achieve perfect training error, among which a specific one must selected. On the other hand, potential instabilities with respect to noise and sampling must be controlled.

A classical way to achieve both goals is to consider explicitly penalized or constrained objective functions. In machine learning, this leads to regularized empirical risk minimization (Shalev-Shwartz and Ben-David, 2014). A more recent approach is based on the observation that this can be also achieved implicitly, by directly exploiting an optimization procedure. A classic example for linear models is gradient descent, which, if suitably initialized, converges to the minimal Euclidean norm solution (Engl et al., 1996). Further, its stability can be controlled along the iterative process (Engl et al., 1996; Yao et al., 2007; Raskutti et al., 2014). These ideas have gained attention for at least two reasons. On the one hand, implicit regularization seems to be one of the mechanisms at work to ensure generalization properties of deep learning methods (Neyshabur et al., 2017; Gunasekar et al., 2017; Soudry et al., 2018; Arora et al., 2019; Vaškevičius et al., 2020). On the other hand, this approach naturally blends modeling and numerical aspects, often improving computational efficiency, while retaining good prediction accuracy (Yao et al., 2007; Raskutti et al., 2014). There are a number of developments to this idea. One line of work has considered extensions to other gradient-based methods, such as stochastic and accelerated gradient descent (Moulines and Bach, 2011; Rosasco and Villa, 2015; Pagliana and Rosasco, 2019). Another line of work considered nonlinear models, such as deep neural networks (Kaltenbacher et al., 2008; Neyshabur et al., 2017), or classification problems (Gunasekar et al., 2017; Soudry et al., 2018).
In this work we consider over-parametrized linear models and design an implicit regularization procedure, biased towards solutions minimizing a general convex functional. A number of recent results have considered the case of strongly convex functionals. In this setting, linearized Bregman iterations/mirror descent can be used, and stability and convergence estimates are known (Burger et al., 2007; Gunasekar et al., 2018), even for accelerated algorithms (Matet et al., 2017). The more classical approach of considering a diagonal procedure have also been used (Garrigos et al., 2018) and paired with acceleration (Calatroni et al., 2019). In the general convex case however, even for linear models, the implicit regularization properties of optimization algorithms are much less understood. There have been studies for ADMM/Bregman iteration, but the optimization procedure requires solving a nontrivial optimization problem at each step (Burger et al., 2007). Stability and convergence results are proved in terms of the Bregman divergence, which in general is a weak result. Interestingly, there are various recent results for the specific case of sparse recovery, where one wants to select a sparse solution with minimal $\ell_1$ norm (Vaškevičius et al., 2019; Osher et al., 2016; Agarwal et al., 2012). Here, we consider general convex functionals beyond the $\ell_1$ norm. We adapt the Chambolle and Pock algorithm, popular in imaging (Chambolle and Pock, 2011), and study its implicit regularization properties. This is a first order primal-dual method, thus simple and requiring only matrix-vector multiplications and proximity operators. Our analysis extends convergence results to our setting and establishes stability both in terms of Bregman divergence and approximate feasibility, under worst case errors. Results from the analysis of primal-dual methods with errors are useful towards this end (Rasch and Chambolle, 2020). From our general results, several special cases can be derived and we discuss as an example sparse recovery, proving estimates in norm to the minimum $\ell_1$ norm solution. In the experimental section, we investigate the proposed method and show state of the art performances with significant computation savings.

**Notation**
The set of integers from 1 to $n$ is $[n]$. Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and $J : \mathbb{R}^p \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper, convex, and lower semicontinuous. The subdifferential of $J$ at $w \in \mathbb{R}^p$ is $\partial J(w)$. The Bregman divergence associated to $J$ is denoted $D^\theta_J(w, w') := J(w) - J(w') - \langle \theta, w - w' \rangle$, where $\theta \in \partial J(w')$. The Fenchel-Legendre conjugate of $f$ is $f^*(\theta) := \sup_w \langle w, \theta \rangle - f(w)$. The indicator function $\mathbb{I}_{\{y\}}$ is equal to zero if the argument equals $y$ and $+\infty$ otherwise.

## 2 Background: over-parameterization and implicit regularization

The basic problem of supervised learning is to find a relationship to predict outputs $y$ from inputs $x$,

$$x \mapsto f(x) \approx y ,$$

given a limited number of pairs $(x_i, y_i)_{i=1}^n$ with, e.g. $x_i \in \mathbb{R}^d$ and $y_i \in \mathbb{R}$. The search for a solution is typically restricted to a set of parameterized functions $f_w$, with $w \in \mathbb{R}^p$. A prototype example are linear models where $p = d$ and $f_w(x) = \sum_{j=1}^d w^j x_j$, or more generally $f_w(x) = \sum_{j=1}^p w^j \phi_j(x)$, for some dictionary $\phi_j : \mathbb{R}^d \rightarrow \mathbb{R}$, $j = 1, \ldots, p$ (Hastie et al., 2009; Shalev-Shwartz and Ben-David, 2014). In modern applications, it is often the case that the number of parameters $p$ is vastly larger than the number of available data points $n$, a regime called over-parameterized (Allen-Zhu et al., 2018). Excluding degenerate cases, one can then expect to find a solution $w$ capable of interpolating the data, that is satisfying,

$$f_w(x_i) = y_i , \quad \forall i \in [n] .$$

(1)

In fact, using over-parameterized models, one should expect to have multiple values of $w$ satisfying Equation (1), and both from the computational and modeling perspectives the question arises of how to appropriately select a meaningful solution. Further, when data are corrupted by noise the proposed selection should be stable. Both the selection
and stability property can be tackled explicitly introducing a bias (also called regularizer) \( J : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\} \), and considering either,

\[
\min_{w \in \mathbb{R}^p} J(w), \quad \text{s.t.} \quad f_w(x_i) = y_i, \quad \forall i \in [n],
\]

as a selection procedure or,

\[
\min_{w \in \mathbb{R}^p} J(w) + \lambda \sum_{i=1}^n (f_w(x_i) - y_i)^2,
\]

to account for stability, the positive scalar \( \lambda \) balancing stability and data-fitting. A variety of approaches can be used to solve Problem 3. In particular, gradient methods based on forward-backward splitting are natural given the composite form of the objective function (Combettes and Pesquet, 2011; Bach et al., 2012). In practice, however, tuning \( \lambda \) may be costly, since most procedures require solving Problem (3) for multiple values of \( \lambda \).

As mentioned in the introduction, implicit regularization provides an alternative, based on the fact that iterative optimization procedures on unpenalized problems can have an implicit bias. In the above setting, this means that they converge to the solution of Problem (2), rather than just fitting the data. This observation has a long history in regularization theory of inverse problems (Engl et al., 1996); the prototype example of implicit regularization is gradient descent on unpenalized least-squares,

\[
w_{t+1} = w_t - \gamma \sum_{i=1}^n (\langle w_t, x_i \rangle - y_i)x_i.
\]

Indeed, it is a classical fact that, for a suitable step-size choice \( \gamma \), gradient descent initialized at \( w_0 = 0 \) converges to the unique interpolating solution with minimal Euclidean norm, i.e. Problem (2) with \( J(\cdot) = \|\cdot\| \). Further, stability properties can be established under different noise models (Engl et al., 1996; Yao et al., 2007). In this paper we investigate the extension of these ideas when the bias \( J \) is a general convex functional.

## 3 Problem setting and proposed algorithms

Following the discussion above, we focus on iterative methods to robustly solve the problem

\[
\min_{w \in \mathbb{R}^p} J(w), \quad \text{s.t.} \quad \langle w, x_i \rangle = y_i, \quad i = 1, \ldots, n,
\]

where \( J : \mathbb{R}^p \to \mathbb{R} \cup \{+\infty\} \) is a proper, convex and lower semicontinuous functional. The constraints can be equivalently written as a linear system \( Xw = y \), where \( X \in \mathbb{R}^{n \times p} \) has rows \( \{x_1, \ldots, x_n\} \) and \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \). While all the results and proofs straightforwardly extend to linear and bounded Hilbert spaces, we consider a finite dimensional setting to ease the presentation. Throughout, we assume \( n < p \) and \( X \) to have rank \( n \), so that the set of constraints in Problem (4) is feasible. There are a number of examples where such a problem arises as we discuss next.

**Example 1 (Sparse recovery).** Choosing \( J = \|\cdot\|_1 \) corresponds to finding the minimal \( \ell_1 \)-norm solution to a linear system, and in this case Problem (4) is known as Basis Pursuit (Chen et al., 1998). The relaxed approach of Problem (3) in this case yields the Lasso (Tibshirani, 1996). \( \ell_1 \)-based approaches have had a tremendous impact in imaging, signal processing and machine learning in the last decades (Hastie et al., 2015).

**Example 2 (Low rank matrix completion).** In several applications, such as recommendation systems, it is useful to recover a low rank matrix, starting from the observation of a subset of its entries (Candès and Recht, 2009). A convex formulation of this problem is the following:

\[
\min_{W \in \mathbb{R}^{p_1 \times p_2}} \|W\|_*, \quad \text{s.t.} \quad W_{ij} = Y_{ij}, \quad \forall (i, j) \in \mathcal{D},
\]
where \(||\cdot||_w\) is the nuclear norm and \(\mathcal{D} \subset [p_1] \times [p_2]\) is the set of observed entries of the matrix \(Y\). In that case, \(X\) is a self-adjoint linear operator from \(\mathbb{R}^{p_1 \times p_2}\) to \(\mathbb{R}^{p_1 \times p_2}\), such that \((XW)_{ij}\) has value \(W_{ij}\) if \((i, j) \in \mathcal{D}\) and \(\theta\) otherwise; the constraints write \(XW = Y\).

**Example 3 (Total variation).** In many imaging tasks such as deblurring and denoising, regularization through total variation allows to simultaneously preserve edges whilst removing noise in flat regions (Rudin et al., 1992). The problem of Total Variation is:

\[
\min_{W \in \mathbb{R}^{p_1 \times p_2}} \|\nabla W\|_1 \quad \text{s.t.} \quad XW = Y,
\]

and can be reformulated as:

\[
\min \tilde{W} \Omega(\tilde{W}) \quad \text{s.t.} \quad \tilde{X}\tilde{W} = \tilde{Y},
\]

with \(\tilde{W} = \begin{pmatrix} W & U \end{pmatrix}, \quad \Omega(\tilde{W}) = \|U\|_1, \quad \tilde{X} = \begin{pmatrix} X & 0 & -\text{Id} \end{pmatrix} \) and \(\tilde{Y} = \begin{pmatrix} Y & 0 \end{pmatrix}\).

In the following, we are interested in iterative approaches that converge to a solution of Problem (4), but also into understanding their robustness to noise. Toward this end, we introduce a worst case deterministic noise model assuming the observations \(y\) to be unknown, and rather a vector \(y^\delta\) to be available such that \(||y - y^\delta|| \leq \delta\), where \(\delta \geq 0\) can be interpreted as the noise level. Next we present the algorithm and discuss related works.

### 3.1 Chambolle-Pock algorithm

Problem (4) consists in minimizing a convex, possibly non-smooth, function on a set defined by linear equalities. The convex optimization literature contains several algorithms to solve such a problem, thoroughly discussed in Appendix A. In this paper, we focus on the Chambolle and Pock (CP) algorithm (Chambolle and Pock, 2011), see Algorithm 1. It belongs to the class of primal-dual methods, designed to jointly solve Problem (4) (the primal problem), and its dual. To derive it, we rewrite Problem (4) as:

\[
\min_{w \in \mathbb{R}^p} \{J(w) + \iota_{\{y\}}(Xw)\},
\]

where \(\iota_{\{y\}}\) is the indicator function of \(\{y\}\), emphasizes its structured nature: the objective is the sum of the convex function \(J\) and the convex function \(\iota_{\{y\}} \circ X\). Therefore Fenchel-Rockafellar duality (Peynouquet, 2015, Sec. 3.6.2) can be applied to compute the dual problem; observing that \(\iota_{\{y\}}(\theta) = \langle y, \theta \rangle\), this dual reads:

\[
\min_{\theta \in \mathbb{R}^n} \left\{J^*(-X^\top \theta) + \iota_{\{y\}}(\theta)\right\} = \min_{\theta \in \mathbb{R}^n} \left\{J^*(-X^\top \theta) + \langle y, \theta \rangle\right\}, \quad \text{(8)}
\]

Assume that Problem (7) admits a solution \(\bar{w}\) satisfying the following qualification condition,

\[
(\exists \bar{\theta} \in \mathbb{R}^n) \quad -X^\top \bar{\theta} \in \partial J(\bar{w}). \quad \text{(QC)}
\]

Reasoning as in the proof of the Fenchel-Rockafellar duality theorem (Peynouquet, 2015, Thm. 3.51) it follows that strong duality holds, and \(\bar{\theta}\) is a solution of Problem (8). Primal-dual solutions are thus characterized by the first order conditions,

\[
-X^\top \bar{\theta} \in \partial J(\bar{w}) \quad \text{and} \quad X\bar{w} = y. \quad \text{(9)}
\]

The latter is also equivalent to require that \((\bar{w}, \bar{\theta})\) is a saddle-point for the Lagrangian \(\mathcal{L}(w, \theta) := J(w) + \langle Xw - y, \theta \rangle\), namely, to require that for every \((w, \theta) \in \mathbb{R}^p \times \mathbb{R}^n\),

\[
\mathcal{L}(\bar{w}, \bar{\theta}) \leq \mathcal{L}(w, \theta) \leq \mathcal{L}(w, \bar{\theta}). \quad \text{(10)}
\]

We stress the fact that we assume the existence of a solution \(\bar{w}\) satisfying (QC), but the primal solution is not necessarily unique. On the other hand, from strong duality we get also that, for every primal solution, there exists a dual one such that (QC) (and so Equations (9) and (10)) is verified.
The CP algorithm (recalled in its general form Appendix B), builds upon the Arrow-Hurwicz method (Arrow et al., 1958) for finding a saddle point of the Lagrangian.

Compared to the Arrow-Hurwicz method, it adds an extrapolation step, usually on the variable \( w \). In this work, we apply CP algorithm to the dual problem so that the extrapolation step is on \( \theta \). Note that this change of point of view between dual and primal does not affect the Lagrangian.

Thus, the quantities \( L \) implies that \( J \) is the fact that the duality gap (12) coincides with the Bregman divergence associated to \( X \). Lemma 4, proved in Appendix C.1, shows that this condition, when coupled with \( \|Xw - y\| \), implies that \( w \) is a solution of the primal problem. One key observation towards this end is the fact that the duality gap (12) coincides with the Bregman divergence associated to \( J \).

Thus, the quantities \( L(w', \bar{\theta}) - L(\bar{w}, \theta') \) and \( \|Xw' - y\| \), studied together, are a reasonable measure of optimality for \( w' \in \mathbb{R}^p \).



**Algorithm 1**: CP Algorithm (Dual)

| init : \( w_0 \in \mathbb{R}^p, \theta_0 = \theta_{-1} \in \mathbb{R}^n \) |
| param: \( \tau, \sigma \) s.t. \( \sigma \tau \|X\|_{op}^2 < 1 \) |
| \( k = 0, \ldots, K - 1 \) do |
| \( w_{k+1} = \text{pro}x_{w,f}(w_k - \tau X^T (2\theta_k - \theta_{k-1})) \) |
| \( \theta_{k+1} = \theta_k + \sigma (Xw_{k+1} - y) \) |
| return \( w_K, \theta_K \) |

As we will explain in Section 4, extrapolating on the dual variable is crucial to study stability. Finally, notice that the algorithm, even in the case of \( J(\cdot) = \|\cdot\|^2 \), does not reduce to the classical Landweber method.

**Other algorithms** As mentioned in the introduction, few other algorithms could be considered, e.g. ADMM/Bregman iteration. However, we are not aware of methods that can be efficiently implemented in our general setting. In Appendix A, we provide an extensive review discussing the connection with a number of different approaches and related works.

## 4 Theoretical analysis

In this section, we analyze the convergence properties of Algorithm 1 introduced above. First, we need to choose a suitable quantity to estimate the approximation properties of the iterates. In general, it is not reasonable to expect a rate of convergence for the distance between the iterates and the solution since the problem is just convex, and it is well known that under these assumptions the convergence can be arbitrarily slow. In Section 4.1, we explain why a reasonable choice is given by the duality gap together with the distance to feasibility (respectively, \( L(w_k, \theta) - L(\bar{w}, \theta_k) \) and \( \|Xw_k - y\| \)). For these two quantities, we derive:

- convergence rates in the exact case, i.e. when the data \( y \) is available (Proposition 5);
- early-stopping bounds in the inexact case, i.e. when the accessible data is only \( y^\delta \) with \( \|y^\delta - y\| \leq \delta \) (Proposition 6 and corollary 7).

In Section 4.4, we apply our analysis to the specific choice of \( J \) equal to the \( \ell_1 \)-norm. In this case, combining the previous results, we obtain bounds directly on \( \|w_k - \bar{w}\| \).

### 4.1 Measure of optimality

To discuss which “distance” is significant to study the convergence of the algorithm, we have to keep in mind the two classical ways to express the optimality (see (9) and (10)). We recall that, if

\[
\mathcal{L}(w', \theta) - \mathcal{L}(w, \theta') \leq 0
\]

for every \( (w, \theta) \in \mathbb{R}^p \times \mathbb{R}^n \), then \( (w', \theta') \) is a primal-dual solution. In general, it is difficult to prove that Equation (11) holds for every \( (w, \theta) \in \mathbb{R}^p \times \mathbb{R}^n \). Then, given a saddle-point \( (\bar{w}, \bar{\theta}) \) and a generic \( (w', \theta') \in \mathbb{R}^p \times \mathbb{R}^n \), we consider the surrogate (non-negative) quantity

\[
\mathcal{L}(w', \bar{\theta}) - \mathcal{L}(\bar{w}, \theta')
\]

To establish the optimality of \( (w', \theta') \), it is not enough to ensure \( \mathcal{L}(w', \bar{\theta}) - \mathcal{L}(\bar{w}, \theta') = 0 \). Lemma 4, proved in Appendix C.1, shows that this condition, when coupled with \( Xw' = y \), implies that \( w' \) is a solution of the primal problem. One key observation towards this end is the fact that the duality gap (12) coincides with the Bregman divergence associated to \( J \). Thus, the quantities \( \mathcal{L}(w', \bar{\theta}) - \mathcal{L}(\bar{w}, \theta') \) and \( \|Xw' - y\| \), studied together, are a reasonable measure of optimality for \( w' \in \mathbb{R}^p \).
Lemma 4. Let \((\bar{w}, \bar{\theta})\) be a primal-dual solution and \((w', \theta')\) a point in \(\mathbb{R}^p \times \mathbb{R}^n\) such that \(\mathcal{L}(w', \theta') - \mathcal{L}(\bar{w}, \bar{\theta}) = 0\) and \(Xw' = y\). Then \((w', \theta')\) is a primal-dual solution.

Note that the Bregman divergence is the error measure used in a series of papers dealing with regularization of inverse problems with general convex regularizers, see e.g. Burger et al. (2007). It is well known that if \(J\) is strongly convex than this quantity controls the distance in norm (Remark 8) and therefore is a proper measure of convergence. If \(J\) is only convex, this measure of error can be really weak. In Section 4.4 we point out the limitations of this quantity when dealing with \(J = \|\cdot\|_1\). For this choice of \(J\), \(D_J^{-\bar{X}^\top\bar{\theta}}(0, \bar{w})\) is always equal to 0; as illustrated on Figure 1, the Bregman divergence is 0 whenever the two vectors have the same support and sign.

![Figure 1: For \(J = \|\cdot\|_1\), convergence of primal-dual iterates \(w_k\) towards \(\bar{w}\), measured in norm, feasibility and Bregman divergence. The Bregman divergence quickly vanishes up to numerical errors (the iterates have the same sign as the solution); yet the iterates are still far from the solution.](image)

4.2 Exact case

First consider the iterates \((w_k, \theta_k)\) obtained by applying Algorithm 1 to the exact problem, namely where the data \(y\) is available. Let \((\bar{w}, \bar{\theta})\) be a saddle-point for the Lagrangian. Denoting the primal-dual variables by \(z = (w, \theta)\), we have \(z_k = (w_k, \theta_k)\) for the iterates of the algorithm and \(\bar{z} = (\bar{w}, \bar{\theta})\) for the saddle-point. For \(\tau\) and \(\sigma > 0\), define \(V\) as the following square weighted norm on \(\mathbb{R}^p \times \mathbb{R}^n\):

\[
V(z) := \frac{1}{2\tau} \|w\|^2 + \frac{1}{2\sigma} \|\theta\|^2 .
\]

(13)

For the ergodic iterates \(\bar{w}_k := \frac{1}{k} \sum_{t=1}^k w_t\) and \(\bar{\theta}_k := \frac{1}{k} \sum_{t=1}^k \theta_t\), we have the following rates.

Proposition 5 (Duality-gap and feasibility rates). Let \(\varepsilon \in (0, 1)\) and assume that the step-sizes are such that \(\sigma \tau \leq \varepsilon / \|X\|_{op}^2\). Then

\[
\mathcal{L}(\bar{w}_k, \bar{\theta}_k) - \mathcal{L}(\bar{w}, \bar{\theta}) \leq \frac{\sqrt{V(z_0 - \bar{z})}}{k} \quad \text{and} \quad \|Xw_k - y\|^2 \leq \frac{2(1 + \varepsilon) V(z_0 - \bar{z})}{\varepsilon \sigma (1 - \varepsilon) k} .
\]

The first result is classical (see Chambolle and Pock (2011)). Alternatively, it can be obtained by setting \(\delta = 0\) in Proposition 6, where we study the more general inexact case. To the best of our knowledge, the second bound is new and can also be derived by setting \(\delta = 0\) in Proposition 6. A similar result, in the more specific case of primal-dual coordinate descent, can be found in Fercoq and Bianchi (2019). Note that both results of Proposition 5 are true for every primal-dual solution. On the other hand, the left-hand-side in the second equation does not depend on the selection of \(\bar{z}\) and so the bound can be improved by taking the inf over all primal-dual solutions.

4.3 Inexact case

We now consider the iterates \((w_k, \theta_k)\), and their ergodic versions \((\bar{w}_k, \bar{\theta}_k)\), obtained by applying Algorithm 1 to the noisy problem, where \(y\) is replaced by \(y^\delta\) with \(\|y^\delta - y\| \leq \delta\).
In Proposition 6, we derive early-stopping bounds for the iterates, in terms of duality gap \( \mathcal{L}(w_k, \theta) - \mathcal{L}(\bar{w}, \bar{\theta}) \) and feasibility \( \|yw_k - y\| \). We highlight that, despite the error in the data \( y^3 \), both quantities are defined in terms of \( y \) and hence related to the noiseless problem. In particular, \((\bar{w}, \bar{\theta})\) is a saddle-point for the noiseless Lagrangian. We have the following estimates, whose proofs are given in Appendix C.3.

**Proposition 6 (Stability for duality gap and feasibility).** Let \( \varepsilon \in (0, 1) \) and assume that the step-sizes are such that \( \sigma \varepsilon \leq \varepsilon / \|X\|_{op} \). Then,

\[
\mathcal{L}(\bar{w}^k, \bar{\theta}) - \mathcal{L}(\bar{w}, \bar{\theta}) \leq \frac{1}{k} \left( \sqrt{V(z_0 - \bar{z})} + \sqrt{2\sigma \delta k} \right)^2.
\]

and

\[
\|X\bar{w}^k - y\|^2 \leq \frac{2(1 + \varepsilon)}{\sigma \varepsilon (1 - \varepsilon)} \left[ \sqrt{2\sigma V(z_0 - \bar{z}) \delta} + \frac{\sigma \varepsilon}{1 - \varepsilon} \delta^2 + 2\sigma \delta^2 k + \frac{1}{k} V(z_0 - \bar{z}) \right].
\]

Note that, in the exact case \( \delta = 0 \), we recover the convergence results of the primal-dual algorithm stated in Proposition 5. Moreover, we have the following corollary.

**Corollary 7 (Early-stopping).** Under the assumptions of Proposition 6, choose \( k = c/\delta \) for some \( c > 0 \). Then there exist constants \( C, C' \) and \( C'' \) such that

\[
\mathcal{L}(w^k, \theta) - \mathcal{L}(\bar{w}, \bar{\theta}) \leq C \delta ; \quad \|Xw^k - y\|^2 \leq C' \delta + C'' \delta^2.
\]

The constants appearing in the Corollary are the ones from Proposition 6. In particular, they only depend on the saddle-point \( \bar{z} \), the initialization \( z_0 \) and the step-sizes \( \tau, \sigma \) (and so, implicitly, on the operator norm of \( X \)). We next add some remarks.

**Remark 8.** When \( J \) is \( \gamma \)-strongly convex, in particular when \( J(\cdot) = \frac{1}{2} \|\cdot\|^2 \), both the feasibility and the distance between the ergodic iterate and the solution can be controlled by \( \mathcal{L}(w^k, \theta) - \mathcal{L}(\bar{w}, \bar{\theta}) \). Indeed, recalling Equation (22),

\[
\|Xw^k - y\|^2 \leq \|X\|^2 \|w^k - \bar{w}\|^2 \leq \frac{2\|X\|^2}{\gamma} D_J X^\top \theta (w^k, \bar{w}) = \frac{2\|X\|^2}{\gamma} \left[ \mathcal{L}(w^k, \theta) - \mathcal{L}(\bar{w}, \bar{\theta}) \right].
\]

In particular, the previous early-stopping bounds for Algorithm 1 are of the same order of the ones obtained by dual gradient descent in Matet et al. (2017).

**Remark 9.** Similar estimates have been obtained in Burger et al. (2007), both for the Tikhonov variational scheme and for the Bregman iteration (also called inverse scale space method) with stopping-criteria given by the discrepancy principle. In the first case (see Theorem 3.1), for a suitable choice of the regularization parameter, the authors get similar estimates for the Tikhonov regularized solution \( w_\lambda \); \( D_J^+ (w_\lambda, \bar{w}) \leq C \delta \) and \( \|Xw_\lambda - y\|^2 \leq C' \delta^2 \), where \( D_J^+ \) is the symmetric Bregman divergence. For the Bregman iteration (see Theorem 4.2), they get an early-stopping bound on \( D_J^+(w^k, w_k) \), where \( p_k \in \gamma J(w_k) \). Note that they do not get any estimate for the quantity \( D_J^+(w^k, \bar{w}) \) neither for the feasibility. Moreover, the method requires to solve, at each iteration, an optimization problem with the same complexity of the original one.

**Proof Sketch** The proof of Proposition 6 is inspired by (Rasch and Chambolle, 2020). In this paper, the kind of errors allowed in the prox of the non-extrapolated step (Algorithm 1, line 3) are more general than the ones allowed for the extrapolated step (Algorithm 1, line 2). Here, we study stability properties of Algorithm 1 when \( y \) is replaced by \( y^3 \). This change can be read as an inexact proximity operator in the update of \( \theta \) computation; in order to have this error in the non-extrapolated step, we study Algorithm 1, that is CP algorithm applied to the dual problem. We summarize here the main steps.
In Lemma 13, we derive a “descent property” for every step $t$, which we then cumulate summing from $t = 1$ to $t = k$ and using two different approximations (Lemmas 14 and 15). The two bounds that we get are similar, but independent. The first one has the following form,

$$\frac{1}{2\sigma} \|\theta_k - \bar{\theta}\|^2 + \sum_{t=1}^{k} [\mathcal{L}(w_t, \bar{\theta}) - \mathcal{L}(\bar{w}, \theta_t)] \leq V(z_0 - \bar{z}) + \delta \sum_{t=1}^{k} \|\theta_t - \bar{\theta}\|. \quad (16)$$

We use the latter twice. First we combine it with Lemma 12, a discrete version of Bihari’s Lemma. This allows to estimate, for every $1 \leq t \leq k$, the quantity

$$\|\theta_t - \bar{\theta}\| \leq 2\sigma \delta k + \sqrt{2\sigma V(z_0 - \bar{z})}. \quad (17)$$

Then we use again Equation (16), joint with the previous information, to find a bound on $\sum_{t=1}^{k} [\mathcal{L}(w_t, \bar{\theta}) - \mathcal{L}(\bar{w}, \theta_t)]$. The second inequality (see Lemma 15) has the following form,

$$\frac{\sigma \alpha}{2\eta} \sum_{t=1}^{k} \|Xw_t - y\|^2 \leq V(z_0 - \bar{z}) + \delta \sum_{t=1}^{k} \|\theta_t - \bar{\theta}\| + \frac{1}{2} \sigma (\eta - 1) \delta^2 k. \quad (18)$$

Using again the bound on $\|\theta_t - \bar{\theta}\|$ and choosing $\eta = (1 + \varepsilon) / (1 - \varepsilon)$, we find an estimate for $\sum_{t=1}^{k} \|Xw_t - y\|^2$. In both cases, we get the claim on the ergodic iterates by Jensen’s inequality.

### 4.4 An example: sparse recovery

In the case of sparse recovery ($J = \|\cdot\|_1$), controlling the duality gap and the feasibility yields a bound on the distance to the minimizer, thanks to the following result (Grasmair et al., 2011, Lemma 3.10).

**Lemma 10.** Let $(\bar{w}, \bar{\theta})$ be such that $X\bar{w} = y$ and $-X^T \bar{\theta} \in \partial \|\cdot\|_1(\bar{w})$. With $\Gamma := \{j \in [p] : |X_j^T \bar{\theta}| = 1\}$, assume that $X_\Gamma$ (X restricted to columns whose indices lie in $\Gamma$) is injective. Let $m := \max_{j \in \Gamma} |X_j^T \bar{\theta}| < 1$. Then, for all $w \in \mathbb{R}^p$,

$$\|w - \bar{w}\| \leq \|X^{-1}\|_{op} \|Xw - y\| + \frac{1 + \|X^{-1}\|_{op} \|X\|_{op} D_{\|\cdot\|_1}^{-1}w - \bar{w}\|. \quad (19)$$

Note that, under the assumptions of Lemma 10, the primal solution to Problem (7) is unique (see (Grasmair et al., 2011, Thm 4.7)). Combining the latter with Corollary 7 yields a strong early-stopping result.

**Corollary 11 (Early-stopping for sparse recovery).** Under the assumptions of Proposition 6 and Lemma 10, choose $k = c/\delta$ for $c > 0$. Then there exist constants $C'$ and $C''$ such that

$$\|\bar{w}^k - \bar{w}\| \leq C' \sqrt{\delta} + C'' \delta. \quad$$

In this case, the constants $C', C''$ depend on the saddle-point $\bar{z}$, the initialization $z_0$, the step-sizes $\tau, \sigma$ and the norms of $X$ and $X^{-1}_\Gamma$. A completely different approach has been considered, for the same problem, in (Vaškevičius et al., 2019). A related approach, based on dynamical systems, has been proposed in Osher et al. (2016). Similar results for the Tikhonov regularization approach can be found in Schöpfer and Lorenz (2019b).

### 5 Empirical analysis

The code is in the supplementary material, and additional experiments are in Appendix D.
Sparse recovery. Random data for this experiment are generated as follows: \((n,p) = (50,100)\), columns of \(X\) are Gaussian with \(\text{cov}(X_{i,j}) = 0.2^{|i-j|}\), entries of \(y\) are i.i.d. sampled from a standard Gaussian, scaled such that \(\|y\| = 20\). Note that the linear system \(Xw = y\) has solutions, since \(X\) is full-rank. The noiseless solution \(\bar{w}\) is determined by running Algorithm 1 up to convergence, on \(y\). For the considered values of \(\delta\), \(y^\delta\) is created by adding i.i.d. Gaussian noise to \(y\), so that \(\|y - y^\delta\| = \delta\). We denote by \(w^\delta_k\) the iterates of Algorithm 1 ran on \(y^\delta\).

Existence of stopping time. In the first experiment, we highlight the existence of a stopping time in terms of distance to \(\bar{w}\), i.e. \(k^* = \arg \min_k \|w^\delta_k - \bar{w}\| < +\infty\). Figure 2 shows semi-convergence: before converging to their limit, the iterates get close to \(\bar{w}\). As expected, as \(\delta\) decreases, the stopping time increases and the early-stopped iterate is closer to \(\bar{w}\).

Dependency of empirical stopping time on \(\delta\). For each value of \(\delta\) we generate 200 values of \(y^\delta\), We run Algorithm 1 on \(y^\delta\) and determine the empirical best stopping times as above. Because of the noisy, oscillatory aspect of convergence curves, we remove the 10% highest and 10% lowest values of stopping times. Figure 3a shows the mean and std. of the inverse empirical stopping time as a function of \(\delta\), where a clear linear trend \((k = c/\delta)\) appears as suggested by Proposition 6.

Budget comparison with Tikhonov approach. The most popular approach to address stability is to solve Problem (3) (here, the Lasso) for, typically\(^1\) 100 values of \(\lambda\) geometrically chosen as \(\lambda_t = 10^{-3t/99}\) \(\|X^Ty\|_\infty\) for \(t = 0, \ldots, 99\). We compare the quality of the solutions obtained with the two methods: \(w^{\text{best}}\) denotes the closest value to \(\bar{w}\) on the Lasso path (resp. the best early-stopped iterate of Algorithm 1). We use ISTA/Forward-Backward to solve the Lasso, with warm-start (using the solution for \(\lambda_{t-1}\) as initialization for problem with \(\lambda_t\)). For each \(\lambda_t\), we perform \(K \in \{1,10,100,1000\}\) iterations and report the best value obtained on the grid. On Figure 3b, one can see that the best solutions along the regularization and optimization paths have similar distances to \(\bar{w}\) for \(K \geq 10\). The budgets however differ strongly: \(K \times 100\), to be compared to the between 25 and 100 iterations (Figure 3a) needed with the implicit regularization approach.

\(^1\)default grid in scikit-learn (Pedregosa et al., 2011) and GLMNET (Friedman et al., 2010) packages
6 Conclusion

In this paper, we have studied algorithms for implicit regularization with convex bias, not necessarily strongly convex or smooth. In particular, we proposed to use the Chambolle-Pock algorithm of which we analyzed both convergence and stability to deterministic worst case noise. Our general analysis was specialized, as an example, to the problem of sparse recovery. The proposed approach was investigated empirically both for sparse recovery and matrix completion showing promising results. Our results suggest a number of possible future developments, e.g. considering variants of the algorithms, such as stochastic or accelerated versions. Another example is considering other noise models and in particular sequential and statistical prediction problems. Finally, considering nonlinear models would be very interesting and possibly help the understanding of deep neural networks.
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A Detailed discussion of related works

The idea of exploiting the implicit regularizing properties of optimization algorithms is not new, and has been studied in three different related areas, often under the name of iterative regularization: inverse problems, image restoration, and machine learning. The related results can roughly be divided in those assuming strong convexity of $J$ and those assuming only convexity of $J$. Related approaches to implicit regularization include diagonal strategies and exact regularization approaches. Extensions to general data fits and non-convex/non-linear problems have been considered. In the following we briefly review existing results.

- Gradient and stochastic descent. The study of implicit regularization properties of gradient descent, known in the inverse problem community as Landweber method, goes back to the 50s (Engl et al., 1996). The classical result shows that gradient descent applied to least squares and initialized at 0 converges to the minimal norm solution of the linear equation (1). Accelerated versions have been also studied under the name of $\nu$-method (Engl et al., 1996). Generalization towards more general regularizers, apart from $p$ norms with $p > 1$, has not been considered much by this community, while there is a rich literature in the non-convex setting for nonlinear inverse problems (Kaltenbacher et al., 2008). These ideas have been extended to machine learning considering regularizing properties of gradient descent (Yao et al., 2007) and its stochastic versions (Moulines and Bach, 2011; Rosasco and Villa, 2015).

- Linearized Bregman and Mirror descent. The interest in more general regularizers has been mainly motivated by imaging applications and total variation regularization. Starting from Osher et al. (2005) there is an entire line of work devoted to iterative regularization for general convex regularizers (see e.g. Burger et al. (2007) and references therein). We briefly review the available algorithms and their advantages and limitations. If strong convexity of $J$ is assumed, the algorithm of choice is mirror descent (Nemirovski and Yudin, 1983; Teboulle and Beck, 2003). It has been popularized in the inverse/imaging problems community under the name of linearized Bregman iteration (Yin et al., 2008). It has been shown that this algorithm in combination with a discrepancy type stopping rule regularizes ill posed problems. The stability and regularization properties of the accelerated variant of the algorithm have been studied also in Matet et al. (2017), using a different approach, based on the interpretation of the method as a gradient descent applied to the dual problem (8). Similar ideas can be found in (Schöpfer and Lorenz, 2019a).

- Bregman iteration and ADMM. If the regularizer $J$ is not strongly convex, but only convex, as in our case, the algorithm above cannot be applied. The algorithm of choice is in this context ADMM Boyd et al. (2010), which has been studied in the imaging community under the name of Bregman iteration. Its regularization properties can be found in Burger et al. (2007). However, this method has a main drawback: at each iteration the solution of a nontrivial optimization problem of the form $\min_{w} \{ \| Xw - y \|^2 + J(w) + \langle w, \eta \rangle \}$, for $\eta \in \mathbb{R}^p$ is required, and in general a subroutine is needed at each iteration. In the setting where $n$ is big, this can have a high computational cost. The extension of this approach to nonlinear inverse problems has been considered in Bachmayr and Burger (2009).

- Bregmanized Operator Splitting and linearized/preconditioned ADMM. These are variants of Bregman iteration and ADMM very similar to the CP algorithm: they rely on preconditioning to avoid the solution of a difficult optimization problem at each iteration. These have been used empirically as regularizing procedures in the context of inverse and imaging problems (Zhang et al., 2011, 2010). We are not aware of any theoretical quantitative stability result.

- Diagonal approaches. The implicit regularization techniques described above are well-suited for problems where the quadratic data fit is appropriate. If other losses are used, this approach completely neglect them. A way to circumvent this problem is to use a diagonal strategy. The idea is to combine an optimization algorithm with a sequence of approximations of the original problem (7) which change at each iteration (Bahraoui and Lemaire, 1994). Convergence rates and stability of diagonal approaches for inverse problems
have been considered in Garrigos et al. (2018); Calatroni et al. (2019).

- **Sparse recovery and compressed sensing** In the context of sparse recovery the implicit regularization approach has been considered in Osher et al. (2016), and also in Vaskevičius et al. (2019). Matching pursuit Mallat and Zhang (1993) is a computational procedure which can be used to select relevant components, but it is not clear from the theoretical point of view how to early stop the iterations.

- **Exact regularization** Another possible approach is to use the notion of exact regularization (Friedlander and Tseng, 2008; Schopfer, 2012). The latter refers to solving $\min_w J(w) + \alpha Q(w)$, s.t. $Xw = y$, where $Q$ is strongly convex and to showing that there exists a value of $\alpha$ such that this new problem and Problem (2) have the same minimizer. Then, known iterative regularization results of the strongly convex case (Matet et al., 2017) can be applied.

## B Chambolle-Pock algorithm

Consider the generic optimization problem

$$\min_x \{ f(x) + g(Kx) \},$$  \hspace{1cm} (20)

with Fenchel-Rockafellar dual problem given by

$$\min_y \{ f^*(-K^Ty) + g^*(y) \}. \hspace{1cm} (21)$$

In this general case, the Chambolle-Pock’s algorithm (with interpolation parameter equal to 1) is given by

$$y_{k+1} = \prox_{\tau g^*}(y_k + \tau K(2\theta_k - \theta_{k-1})), \hspace{1cm} x_{k+1} = \prox_{\sigma f}(x_k - \sigma K^Ty_{k+1}).$$

Notice that the CP algorithm, except for the interpolation, threatens the primal and the dual problem in a symmetric way. In particular, we can cast the method both for Problems (20) and (21). In order to apply the latter to our dual problem, we set $f = \langle y, \cdot \rangle$, $g = J^*$ and $K = -X^T$. Then $g^* = J$, $\prox_{\sigma f}(\theta) = \theta - \sigma y$ and we recover Algorithm 1:

$$w_{k+1} = \prox_{\tau g^*}(w_k - \tau X^T(2\theta_k - \theta_{k-1})), \hspace{1cm} \theta_{k+1} = \theta_k + \sigma (Xw_{k+1} - y^\delta).$$

The latter uses, in the update of the variable $w$, an interpolation of $\theta$ with the value at the previous step.

As we already remarked, we could also apply the CP algorithm directly to the primal problem, setting $f = J$, $g = \iota_{\{y\}}$ and $K = X$. Then $g^* = \langle y, \cdot \rangle$ and $\prox_{\tau g^*}(\theta) = \theta - \tau y$, leading to the following method:

$$\theta_{k+1} = \theta_k + \tau (X(2w_k - w_{k-1}) - y^\delta), \hspace{1cm} w_{k+1} = \prox_{\tau f}(w_k - \tau X^T\theta_{k+1}).$$

In this case, in the update of the variable $\theta$, we use an interpolation of $w$. In general, the two versions should not differ in a significant manner. Nevertheless, the error we consider affects the data $y^\delta$ and so its nature is not symmetric. Then, a different choice for the interpolation can play a role. In this work, we put emphasis in Algorithm 1 because it is the one for which we have proximal errors in the non-extrapolated step.
C Proofs

C.1 Lemmas

Lemma 4. Let \((\bar{w}, \bar{\theta})\) be a primal-dual solution and \((w', \theta')\) a point in \(\mathbb{R}^p \times \mathbb{R}^n\) such that \(\mathcal{L}(w', \theta) - \mathcal{L}(\bar{w}, \theta') = 0\) and \(Xw' = y\). Then \((w', \theta)\) is a primal-dual solution.

Proof.
Step 1: the duality gap is the Bregman divergence. Indeed, using \(-X^T \bar{\theta} \in \partial J(\bar{w})\) and \(X\bar{w} = y\):

\[
\mathcal{L}(w', \theta) - \mathcal{L}(\bar{w}, \theta') = J(w') - J(\bar{w}) + \langle \bar{\theta}, Xw' - y \rangle - \langle \theta', X\bar{w} - y \rangle
= J(w') - J(\bar{w}) + \langle X^T \bar{\theta}, w' - \bar{w} \rangle = D^J_{\bar{\theta}} X^T (w', \bar{w}) ,
\]

(22)

Step 2: Zero duality gap plus feasibility implies primal optimality We show that if \(\bar{v} \in \partial J(\bar{w})\) and \(D^J_{\bar{\theta}}(w', \bar{w}) = 0\), then \(\bar{v} \in \partial J(w')\). Indeed, \(J(w') - J(\bar{w}) - \langle \bar{v}, w' - \bar{w} \rangle = 0\) and so, for all \(z \in \mathbb{R}^p\),

\[
J(z) \ge J(\bar{w}) + \langle \bar{v}, z - \bar{w} \rangle = J(w') - \langle \bar{v}, w' - \bar{w} \rangle + \langle \bar{v}, z - \bar{w} \rangle = J(w') + \langle \bar{v}, z - w' \rangle .
\]

(23)
The statement follows by applying step 2 with \(\bar{v} = -X^T \bar{\theta}\).

Next, we recall the result that allows us to control the non-vanishing error. It is a discrete version of Bihari’s Lemma and a particular case of Lemma 1 in Schmidt et al. (2011), where the proof can be found.

Lemma 12. Assume that \((u_j)\) is a non-negative sequence and that \(\lambda \ge 0, S \ge 0\) with \(S \ge u^2_0\). If \(u^2 \le S + \lambda \sum_{j=1}^n u_j\), then

\[
u \le \frac{\lambda t}{2} + \left[ S + \left( \frac{\lambda t}{2} \right)^2 \right]^{\frac{1}{2}} .
\]

So, in particular,

\[
u \le \lambda t + \sqrt{S} .
\]

C.2 Preliminary estimates

Lemma 13 (One step estimate). Defining \(\hat{\theta}_k := 2\theta_k - \theta_{k-1}\), the updates of Algorithm 1 for the noisy problem read as:

\[
w_{k+1} = \text{prox}_{\tau J} \left( w_k - \tau X^T \hat{\theta}_k \right) ,
\]

(24)

\[
\theta_{k+1} = \theta_k + \sigma \left( Xw_{k+1} - y^\delta \right).
\]

(25)

Then, for any \((w, \theta) \in \mathbb{R}^p \times \mathbb{R}^n\), we have the following estimate:

\[
V(z_{k+1} - z) - V(z_k - z) + V(z_{k+1} - z_k) + [\mathcal{L}(w_{k+1}, \theta) - \mathcal{L}(w, \theta_{k+1})] + \langle \theta_{k+1} - \theta, y^\delta - y \rangle
+ \langle \theta_{k+1} - \hat{\theta}_k, X(w - w_{k+1}) \rangle \le 0 .
\]

(26)
Proof. Consider first Equation (24) and the firm non-expansiveness of the proximal-point. Then we get that, for any \( w \in \mathbb{R}^p \),
\[
0 \geq \|w_{k+1} - w\|^2 - \left( \left\| w_k - \tau X^\top \hat{\theta}_k \right\| - w \right)^2 + \left\| w_{k+1} - \left( w_k - \tau X^\top \hat{\theta}_k \right) \right\|^2 + 2\tau \left[ J(w_{k+1}) - J(w) \right] \\
= \|w_{k+1} - w\|^2 - \|w_k - w\|^2 + \|w_{k+1} - w_k\|^2 + 2\tau \left[ J(w_{k+1}) - J(w) \right] + 2\tau \langle X^\top \hat{\theta}_k, w_k - w \rangle + 2\tau \langle X^\top \hat{\theta}_k, w_{k+1} - w_k \rangle \\
= \|w_{k+1} - w\|^2 - \|w_k - w\|^2 + \|w_{k+1} - w_k\|^2 + 2\tau \left[ J(w_{k+1}) - J(w) \right] + 2\tau \langle \hat{\theta}_k, X(w_{k+1} - w) \rangle.
\]

Now consider Equation (24) and notice that the dual update can be re-written as \( \theta_{k+1} = \text{prox}_{\sigma\langle y^\delta, \cdot \rangle} (\theta_k + \sigma Xw_{k+1}) \). Similarly as before, for any \( \theta \in \mathbb{R}^n \),
\[
0 \geq \|\theta_{k+1} - \theta\|^2 - \|\theta_k + \sigma Xw_{k+1} - \theta\|^2 + \|\theta_{k+1} - (\theta_k + \sigma Xw_{k+1})\|^2 + 2\sigma \left[ \langle y^\delta, \theta_{k+1} \rangle - \langle y^\delta, \theta \rangle \right] \\
= \|\theta_{k+1} - \theta\|^2 - \|\theta_k - \theta\|^2 + \|\theta_{k+1} - \theta\|^2 + 2\sigma \langle \theta_{k+1} - \theta, y^\delta \rangle \\
- 2\sigma \langle \theta_k - \theta, Xw_{k+1} \rangle - 2\sigma \langle \theta_{k+1} - \theta_k, Xw_{k+1} \rangle \\
= \|\theta_{k+1} - \theta\|^2 - \|\theta_k - \theta\|^2 + \|\theta_{k+1} - \theta_k\|^2 + 2\sigma \langle \theta_{k+1} - \theta_k, y^\delta - Xw_{k+1} \rangle.
\]

Recall that \( z := (w, \theta) \) and the definition of \( V \) in Equation (13). Divide the first inequality by \( 2\tau \), the second one by \( 2\sigma \) and sum-up, to get
\[
0 \geq V(z_{k+1} - z) - V(z_k - z) + \left[ J(w_{k+1}) - J(w) \right] + \langle \hat{\theta}_k, X(w_{k+1} - w) \rangle + \langle \theta_{k+1} - \theta, y^\delta - Xw_{k+1} \rangle.
\]

To conclude, compute
\[
\left[ J(w_{k+1}) - J(w) \right] + \langle \hat{\theta}_k, X(w_{k+1} - w) \rangle + \langle \theta_{k+1} - \theta, y^\delta - Xw_{k+1} \rangle \\
= \left[ \mathcal{L}(w_{k+1}, \theta) - \mathcal{L}(w, \theta_{k+1}) \right] - \langle \theta, Xw_{k+1} - y \rangle + \langle \theta_{k+1}, Xw - y \rangle \\
+ \langle \hat{\theta}_k, X(w_{k+1} - w) \rangle + \langle \theta_{k+1} - \theta, y^\delta - Xw_{k+1} \rangle \\
= \left[ \mathcal{L}(w_{k+1}, \theta) - \mathcal{L}(w, \theta_{k+1}) \right] + \langle \theta_{k+1} - \theta, y^\delta - Xw_{k+1} \rangle \\
- \langle \theta, Xw_{k+1} \rangle + \langle \theta_{k+1}, Xw \rangle + \langle \hat{\theta}_k, Xw_{k+1} \rangle - \langle \theta_{k+1} - \theta, Xw_{k+1} \rangle \\
= \left[ \mathcal{L}(w_{k+1}, \theta) - \mathcal{L}(w, \theta_{k+1}) \right] + \langle \theta_{k+1} - \theta, y^\delta - y \rangle + \langle \theta_{k+1} - \theta, X(w - w_{k+1}) \rangle.
\]

Lemma 14 (First cumulating estimate). Define \( \omega := 1 - \tau \sigma \|X\|_{op}^2 \). Then we have the following estimate:
\[
\frac{\omega}{2\tau} \|w_k - \tilde{w}\|^2 + \frac{1}{2\tau} \|\theta_k - \tilde{\theta}\|^2 - V(z_0 - \bar{z}) + \sum_{t=1}^{k} \left[ \mathcal{L}(w_t, \bar{\theta}) - \mathcal{L}(\tilde{w}, \bar{\theta}_t) \right] + \frac{\omega}{2\tau} \sum_{t=1}^{k} \|w_t - w_{t-1}\|^2 \\
\leq \delta \sum_{t=1}^{k} \|\theta_t - \bar{\theta}\|.
\]

Proof. We start from Equation (26), switching the index from \( k \) to \( t \) and evaluating \( (w, \theta) \)
at the saddle-point \((\bar{w}, \bar{\theta})\). Recall that \(\hat{\theta}_{t} := 2\theta_{t} - \theta_{t-1}\), to get

\[
V(z_{t+1} - \bar{z}) - V(z_t - \bar{z}) + V(z_{t+1} - z_t) + \left[\mathcal{L}(w_{t+1}, \bar{\theta}) - \mathcal{L}(\bar{w}, \bar{\theta}_{t+1})\right]
\leq \langle \theta_t + 2\bar{\theta}_{t-1} - \theta_{t-1}, w_{t+1} - w_t \rangle - \langle \bar{\theta}_{t+1} - \bar{\theta}, y - y \rangle
\leq -\langle \theta_t + 2\bar{\theta}_{t-1} - \theta_{t-1}, w_{t+1} - w_t \rangle - w_t - w_t + \delta \|w_t - w_{t-1}\| + \delta \|\theta_t - \bar{\theta}_{t+1}\|
\leq -\langle \theta_t + 2\bar{\theta}_{t-1} - \theta_{t-1}, w_{t+1} - w_t \rangle - \delta \|\theta_t - \bar{\theta}_{t+1}\|
\leq -\langle \theta_t + 2\bar{\theta}_{t-1} - \theta_{t-1}, w_{t+1} - w_t \rangle - \delta \|\theta_t - \bar{\theta}_{t+1}\| + \frac{1}{2\sigma} \|\theta_t - \theta_{t-1}\|^2 + \frac{\sigma}{2} \|X\|^2_{\text{op}} \|w_{t+1} - w_t\|^2 + \delta \|\theta_{t+1} - \bar{\theta}\|,
\]

where in the last estimate we used Cauchy-Schwartz and Young inequalities, the latter with parameter \(\sigma\). Then, using the definition of \(\omega := 1 - \tau \sigma \|X\|_{\text{op}}\), we have

\[
V(z_{t+1} - \bar{z}) - V(z_t - \bar{z}) + \mathcal{L}(w_{t+1}, \bar{\theta}) - \mathcal{L}(\bar{w}, \bar{\theta}_{t+1})
\leq -\langle \theta_t + 2\bar{\theta}_{t-1} - \theta_{t-1}, w_{t+1} - w_t \rangle - \delta \|\theta_t - \bar{\theta}_{t+1}\|
\leq -\langle \theta_t + 2\bar{\theta}_{t-1} - \theta_{t-1}, w_{t+1} - w_t \rangle - \delta \|\theta_t - \bar{\theta}_{t+1}\| + \frac{1}{2\sigma} \|\theta_t - \theta_{t-1}\|^2 + \frac{\sigma}{2} \|X\|^2_{\text{op}} \|w_{t+1} - w_t\|^2 + \delta \|\theta_{t+1} - \bar{\theta}\|,
\]

where in the last inequality we used again Cauchy-Schwartz and Young inequalities with parameter \(\sigma\). Reordering, we obtain the claim.

**Lemma 15 (Second cumulative estimate).** For \(\varepsilon > 0\) and \(\eta = \frac{1 + \varepsilon}{1 - \varepsilon} \geq 1\), define \(\omega := \varepsilon - \sigma \varepsilon \|X\|_{\text{op}}^2\). Then we have

\[
V(z_{k} - \bar{z}) - V(z_0 - \bar{z}) + \frac{\omega}{2\tau} \sum_{t=0}^{k-1} \|w_{t+1} - w_t\|^2 + \frac{\sigma \varepsilon}{2\tau} \sum_{t=1}^{k} \|Xw_t - y\|^2
\leq \delta \sum_{t=1}^{k} \|\theta_t - \theta\| + \frac{\sigma (\eta - 1) \delta^2 k}{2}.
\]  

(28)

**Proof.** In a similar fashion as in the previous proof, we start again from Equation (26), switching the index from \(k\) to \(t\) and evaluating \((w, \theta)\) at the saddle-point \((\bar{w}, \bar{\theta})\). Since \(\bar{\theta}_{t} = \theta_{t} + (\theta_{t} - \theta_{t-1}) = \theta_{t} + \sigma (Xw_{t} - y)\) and \(\theta_{t+1} - \theta_{t} = \sigma (Xw_{t+1} - y)\), we get

\[
V(z_{t+1} - \bar{z}) - V(z_t - \bar{z}) + \frac{1}{2\tau} \|w_{t+1} - w_t\|^2 + \frac{\sigma}{2} \|Xw_{t+1} - y\|^2 + \|\mathcal{L}(w_{t+1}, \bar{\theta}) - \mathcal{L}(\bar{w}, \bar{\theta}_{t+1})\|
\leq \langle \theta_{t+1} - \theta_{t} - \sigma (Xw_{t} - y), Xw_{t+1} - y \rangle + \langle \theta_{t+1} - \bar{\theta}, y - y \rangle
\leq \sigma \langle X(w_{t+1} - w_{t}), Xw_{t+1} - y \rangle + \langle \theta_{t+1} - \bar{\theta}, y - y \rangle.
\]
Now compute
\[ \frac{\sigma}{2} \| Xw_{t+1} - y^\delta \|^2 = \frac{\sigma}{2} \| Xw_{t+1} - y \|^2 + \frac{\sigma}{2} \| y^\delta - y \|^2 - \sigma \langle Xw_{t+1} - y, y^\delta - y \rangle. \]

So,
\[
V(z_{t+1} - \bar{z}) - V(z_t - \bar{z}) + \frac{1}{2\tau} \| w_{t+1} - w_t \|^2 + \frac{\sigma}{2} \| Xw_{t+1} - y \|^2 + \left[ \mathcal{L}(w_{t+1}, \bar{\theta}) - \mathcal{L}(\bar{w}, \theta_{t+1}) \right]
\leq \sigma \langle X(w_{t+1} - w_t), Xw_{t+1} - y \rangle + \langle \theta_{t+1} - \bar{\theta}, y - y^\delta \rangle + \sigma \langle Xw_{t+1} - y, y^\delta - y \rangle - \frac{\sigma}{2} \| y^\delta - y \|^2
\leq \frac{\sigma \| X \|^2_{op}}{2\tau} \| w_{t+1} - w_t \|^2 + \frac{\sigma \| X \|^2}{2\eta} \| Xw_{t+1} - y \|^2 + \delta \| \theta_{t+1} - \bar{\theta} \|^2 - \frac{\sigma}{2} \| y^\delta - y \|^2
+ \frac{\sigma}{2} \| Xw_{t+1} - y \|^2 + \frac{\sigma \eta}{2} \| y^\delta - y \|^2.
\]

In the last inequality we used three times Cauchy-Schwartz inequality, the bound on the error given by \( \| y^\delta - y \| \leq \delta \) and two times Young inequality with parameters \( \varepsilon > 0 \) and \( \eta = \frac{\delta}{1 + \delta} > 0 \). Then, re-ordering and recalling the definitions of \( \omega := \varepsilon - \sigma \| X \|^2_{op} \), we obtain
\[
V(z_{t+1} - \bar{z}) - V(z_t - \bar{z}) + \frac{\omega}{2\tau \varepsilon} \| w_{t+1} - w_t \|^2 + \frac{\sigma \varepsilon}{2\eta} \| Xw_{t+1} - y \|^2 + \left[ \mathcal{L}(w_{t+1}, \bar{\theta}) - \mathcal{L}(\bar{w}, \theta_{t+1}) \right]
\leq \delta \| \theta_{t+1} - \bar{\theta} \|^2 + \frac{\sigma (\eta - 1) \delta^2}{2}.
\]

Summing-up the latter from \( t = 0 \) to \( t = k - 1 \), by telescopic property, we get
\[
V(z_k - \bar{z}) - V(z_0 - \bar{z}) + \frac{\omega}{2\tau \varepsilon} \sum_{t=0}^{k-1} \| w_{t+1} - w_t \|^2 + \frac{\sigma \varepsilon}{2\eta} \sum_{t=0}^{k-1} \| Xw_{t+1} - y \|^2
+ \sum_{t=0}^{k-1} \left[ \mathcal{L}(w_{t+1}, \theta) - \mathcal{L}(w, \theta_{t+1}) \right]
\leq \delta \sum_{t=0}^{k-1} \| \theta_{t+1} - \bar{\theta} \|^2 + \frac{\sigma (\eta - 1) \delta^2 k}{2}.
\]

By trivial manipulations, we get the claim.

### C.3 Proof of Proposition 6

**Proposition 6 (Stability for duality gap and feasability).** Let \( \varepsilon \in (0, 1) \) and assume that the step-sizes are such that \( \sigma \tau \leq \varepsilon / \| X \|^2_{op} \). Then,

\[
\mathcal{L}(\bar{w}^k, \bar{\theta}) - \mathcal{L}(\bar{w}, \bar{\theta}) \leq \frac{1}{k} \left( \sqrt{V(z_0 - \bar{z})} + \sqrt{2\sigma \delta k} \right)^2. \tag{14}
\]

and

\[
\| Xw^k - y \|^2 \leq \frac{2(1 + \varepsilon)}{\sigma \varepsilon (1 - \varepsilon)} \left[ \sqrt{2\sigma V(z_0 - \bar{z})} \delta + \frac{\sigma \varepsilon}{1 - \varepsilon} \delta^2 + 2\sigma \delta^2 k + \frac{1}{k} V(z_0 - \bar{z}) \right]. \tag{15}
\]

**Proof.** Inequality in Equation (27) holds true for every \( k \geq 1 \). Then, recalling that \( \mathcal{L}(w, \theta) - \mathcal{L}(\bar{w}, \bar{\theta}) \geq 0 \) for every \( (w, \theta) \in \mathbb{R}^k \times \mathbb{R}^n \) and that \( \omega \geq 0 \) by assumption, for every \( t \geq 1 \) we have that
\[
\| \theta_t - \bar{\theta} \|^2 \leq 2\sigma V(z_0 - \bar{z}) + 2\sigma \delta \sum_{j=1}^{t} \| \theta_j - \bar{\theta} \|. \tag{29}
\]

Apply Lemma 12 to Equation (29) with \( u_j = \| \theta_j - \bar{\theta} \|, \ S = 2\sigma V(z_0 - \bar{z}) \) and \( \lambda = 2\sigma \delta \), to get
\[
\| \theta_t - \bar{\theta} \| \leq 2\sigma \delta t + \sqrt{2\sigma V(z_0 - \bar{z})}.
\]
In particular, for $1 \leq t \leq k$, we have

$$\|\theta_t - \bar{\theta}\| \leq 2\sigma \delta k + \sqrt{2\sigma V(z_0 - \bar{z})}.$$  \hspace{1cm} (30)

Insert the latter in Equation (27), to obtain

$$\sum_{t=1}^{k} \left[ \mathcal{L}(w_t, \bar{\theta}) - \mathcal{L}(\bar{w}, \theta_t) \right] \leq V(z_0 - \bar{z}) + \delta \sum_{t=1}^{k} \left( 2\sigma \delta k + \sqrt{2\sigma V(z_0 - \bar{z})} \right)$$

$$= V(z_0 - \bar{z}) + \delta k \sqrt{2\sigma V(z_0 - \bar{z})} + 2\sigma \delta^2 k^2$$

$$\leq \left( \sqrt{V(z_0 - \bar{z}) + \sqrt{2\sigma \delta k}} \right)^2 .$$

By Jensen’s inequality, we get the claim.

For the second result, recall that, from Equation (30), we have

$$\delta \sum_{t=1}^{k} \|\theta_t - \bar{\theta}\| \leq \sqrt{2\sigma V(z_0 - \bar{z})} \delta k + 2\sigma \delta^2 k^2 .$$

Inserting the latter in Equation (28), we get

$$\frac{\sigma \varepsilon}{2\eta} \sum_{t=1}^{k} \|Xw_t - y\|^2 \leq \delta \sum_{t=1}^{k} \|\theta_t - \bar{\theta}\| + \frac{(\eta - 1) \delta^2 k^2 + V(z_0 - \bar{z})}{2}$$

$$\leq \sqrt{2\sigma V(z_0 - \bar{z})} \delta k + 2\sigma \delta^2 k^2 + \frac{(\eta - 1) \delta^2 k^2}{2} + V(z_0 - \bar{z}).$$

By Jensen’s inequality, rearranging the terms, and taking $\eta = \frac{1 + \epsilon}{1 - \epsilon}$, we get the claim.

### D Additional experiments

#### D.1 Low-rank matrix completion

In this application of Example 2 $w$ and $y$ are matrices, of size chosen $d \times d$ here. We select a random mask $D \in [d] \times [d]$ of $d^2/2$ observed values, on which $y$ has standard Gaussian values – values outside the mask are not taken into account in the constraint. In order to have a meaningful range of values for $\delta$, we set $\|y\|$ to 20.

We observe that the curves are flatter than in the sparse recovery experiments, but the algorithm can still be early stopped since the curve become quite flat after some iterations. As in the sparse recovery example, the stopping time (here, the iteration after which $\|w_k - \bar{w}\|$ stop decreasing) are very low: less than 100 iterations.

![Figure 4: Convergence behavior for $d = 10$ (left) and $d = 50$ (right). When the dimension increases, the semi-convergence is less visible, but the algorithm can still be stopped way before its convergence.](image-url)