ON REGULARITY FOR $J$-HOLOMORPHIC MAPS

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Abstract. We provide a short proof that an $L^2_1$ and $J$-holomorphic curve is in fact smooth. As an application, we deduce a removal of singularity theorem for curves of finite energy.

1. Statement of Results

Given a manifolds $\Sigma$. Let $L^p_k(\Sigma)$ denote the Sobolev space of functions with all derivatives up to order $k$ in $L^p$. Such functions may be valued in some finite dimensional vector space. Let $M$ be a compact smooth manifold with a smooth almost complex structure $J$. For convenience, we will fix an embedding $i : M \subset \mathbb{R}^N$ and assume that $J$ extends to smooth almost complex structure in a neighborhood of $M$. Let $\Sigma$ be a Riemann surface with complex structure $j$. We consider a map (defined almost everywhere) $u : \Sigma \to M$ to be $L^2_1$ if the corresponding map $i \circ u : \Sigma \to M$ is in $L^2_1$. Furthermore, such a map is said to be $J$-holomorphic if

$$du \circ j = J \circ du$$

almost everywhere.

Theorem 1. Suppose $u : \Sigma \to M$ is $L^2_1$ and $J$-holomorphic. We have $u \in C^\infty(\mathbb{R}^N)$.

Let $D$ be the unit open disk in the plane and let $D^*$ be the punctured disk. As a corollary, we deduce:

Theorem 2. Suppose $u : D^* \to M$ is $J$-holomorphic and has

$$E(u) = \frac{1}{2} \int_{D^*} |du|^2 < \infty$$

We have that $u$ extends smoothly to a $J$-holomorphic map on $D$.

There is a vast literature on $L^2_1$-regularity for harmonic maps going back to Morrey. See [3] for references.

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2. Proofs

Our proof of regularity will be based on the following application of Stokes’ theorem.

Lemma 1. Let $f, g \in L^2_1(\Sigma)$ be functions with compact support. We have $df \wedge dg \in L^2_{-1}(\Sigma)$.

Proof. Take test function $h \in L^2_1(\Sigma)$. Note that in since $df, dg \in L^2(\Sigma)$, a priori $df \wedge dg \in L^2(\Sigma)$ while $h \notin L^\infty(\Sigma)$. Therefore, it is not clear how to define $\int_\Sigma h \wedge df \wedge dg$.

Take a 3-manifold $Y$ with $\partial Y = \Sigma$. There exists $\tilde{f}, \tilde{g}, \tilde{h} \in L^2_{3/2}(Y)$ that extend the given $f, g, h$ from $\Sigma$ to $Y$. By solving the Dirichlet problem, such extensions can be taken to depend continuously on the given $f, g, h$. Since $L^2_{3/2}(Y) \to L^3(Y)$ in dimension 3, we have $d\tilde{h} \wedge d\tilde{f} \wedge d\tilde{g} \in L^1(Y)$. Finally, Stokes theorem implies that

$$\int_\Sigma h \wedge df \wedge dg = \int_Y d\tilde{h} \wedge d\tilde{f} \wedge d\tilde{g}$$

for smooth $h$ so we may set the RHS as the definition of the integral on the LHS. □

Remark. As a slight generalization, note that one may take the functions to be matrix valued. In addition, the $L^2_{-1}$-norm of $df \wedge dg$ depends only on the $L^2$-norms of $df, dg$ and not on $f, g$. Indeed, taking replacing $f$ with $f + \text{const}$ we may assume $\|f\|_{L^2_1} \leq C\|df\|$.

We now turn to the proof of the theorem. As the theorem is local in $\Sigma$ we will focus on the case of $D_r$ - a disk of radius $r$ in the plane.

Lemma 2. Given $u : D \to M$ as above. If $u \in L^2_{1,\text{loc}}(D)$, then $u$ is smooth.

Proof. Let $(s, t)$ be coordinates on $D$. We will establish that $u \in L^2_{3/2}$. Since $L^2_{3/2} \to L^p_1$ for $p > 2$, higher regularity is standard (see [1]) and follows from Sobolev multiplication theorems.

Let $\Delta = \partial_s^2 + \partial_t^2$. Since $\partial_s u + J(u)\partial_t u = 0$, we apply $\partial_s - J(u)\partial_t$ to deduce that

$$\Delta u + \partial_s J \partial_t u - J \partial_t J \partial_t u = 0$$

This equation holds in the weak sense. Now, using the fact that $J \partial_t J = -\partial_t J J$ and $J \partial_t u = -\partial_s u$ we deduce that

$$\Delta u + *dJ \wedge du = 0$$

where $*$ is the Hodge star operator on $\Sigma$.

We rewrite the equation as $\Delta u + T(u) = 0$. With $T(u) = *(dJ \wedge du)$. $T$ defines a continuous operator $L^2_{3/2} \to L^2_{-1/2}$ in view of the embedding

$$L^2_{1/2} \cdot L^2_{1/2} \cdot L^2 \to L^1$$
In addition, $T$ defines a continuous operator $L^2_1 \to L^2_{-1}$. This follows from the previous lemma. By rescaling the disk, we may assume that the $L^2$-norm of $dJ$ is as small as we like. Therefore, the norm of $T$ is as small as we like on the relevant Sobolev spaces.

Take a bump function $\phi$ with support on $D_{1/2}$. $\phi u$ satisfies $\Delta(\phi u) + T(\phi u) = g \in L^2$. Since $T$ is small, the solution to the Dirichlet problem implies there exists a unique $v \in L^2_{3/2}(D)$ such that $\Delta(v) + T(v) = g$ and $v = 0$ on $\partial D$. Viewed as an element of $L^2_1$, $v$ satisfies the same equation and thus $v = \phi u$. Therefore, $u \in L^2_{3/2}$ as desired. □

We now prove the removable singularities theorem. This follows from theorem 1 together with the following standard lemma:

**Lemma 3.** Given a bounded smooth $u : D^* \to \mathbb{R}^n$ with $E(u) < \infty$, we have that $u \in L^2_1$ as a map from $D$.

**Proof.** Since $u \in L^\infty$, we need to check that $du$ is the weak derivative of $u$ on $D$. Take a bump function $\phi : D \to \mathbb{R}$ with $\phi = 1$ outside the $1/2$-ball and equal to 0 near the origin. Let $\phi_\epsilon(s, t) = \phi(s/\epsilon, t/\epsilon)$. Set $u_\epsilon = \phi_\epsilon u$ and note that $u_\epsilon$ is smooth. Take a smooth test function $f$. We have

$$\langle u, df \rangle_D = \lim_{\epsilon \to 0} \langle u_\epsilon, df \rangle_D = \lim_{\epsilon \to 0} \langle du_\epsilon, f \rangle_D = \lim_{\epsilon \to 0} \langle (du)\phi_\epsilon + u d\phi_\epsilon, f \rangle_D$$

Since $\lim_{\epsilon \to 0} \langle (du)\phi_\epsilon, f \rangle_D = \langle (du), f \rangle_D$, we need only show that

$$\lim_{\epsilon \to 0} \langle u d\phi_\epsilon, f \rangle_D = 0$$

So see this, note that $d\phi_\epsilon$ has support on $D_\epsilon$ and is bounded by $C\epsilon^{-1}$ for some uniform $C > 0$. Since $u$ and $f$ are bounded, $\langle u d\phi_\epsilon, f \rangle_D \leq C'\pi \epsilon^2 \epsilon^{-1}$.

□

**References**

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