Analysis on Metric Space $\mathbb{Q}$*

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Abstract

In this paper, we show that the metric space $(\mathbb{Q}, G)$ is a positively-curved space (PC-space) in the sense of Alexandrov. We also discuss some issues like metric tangent cone and exponential map of $(\mathbb{Q}, G)$. Then we give a decomposition of this metric space according to the signature of points in $\mathbb{Q}$. Some properties of this decomposition are shown. The second part of this paper is devoted to some basic analysis on the space $(\mathbb{Q}, G)$, like the tensor sum and $L^p$ space, which can be of independent interest. In the end, we give another definition of derivative for multiple-valued functions, which is equivalent to the one used by Almgren. An interesting theorem about regular selection of multiple-valued functions which preserves the differentiability concludes this paper.

Contents

1 Introduction 2
2 Preliminaries 3
3 Metric Analysis on $\mathbb{Q}$ 4
  3.1 $(\mathbb{Q}, G)$ is a PC-space 4
  3.2 Metric Tangent Cone of $(\mathbb{Q}, G)$ 9
  3.3 Exponential Map 12
  3.4 A Decomposition of $(\mathbb{Q}, G)$ 13
4 Tensor Sum of Multiple-Valued Functions 17
5 Derivative of Multiple-Valued Functions 20
  5.1 Definition of Derivative 20
  5.2 A Regular Selection Theorem 22

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1 Introduction

The theory of multiple-valued functions in the sense of Almgren \cite{AF} has several applications in the framework of geometric measure theory. In deed, multiple-valued functions give a very useful tool to approximate some abstract objects arising from geometric measure theory. For example, Almgren (see \cite{AF}) used multiple-valued functions to approximate some rectifiable currents, hence successfully got the partial interior regularity of area-minimizing integral currents. Solomon (see \cite{SB}) succeeded in giving proofs of the closure theorem without using the structure theorem. His proofs rely on various facts about multiple-valued functions. There are also some other work concerning multiple-valued functions, see \cite{DGT, GJ, LC1, LC2, MP, ZW1, ZW2}. All these work raises the need of further studying of multiple-valued functions.

In \cite{AF}, Almgren gave an explicit bi-Lipschitzian correspondence between the metric space $(\mathbb{Q}, \mathcal{G})$ and a finite polyhedral cone $\mathbb{Q}^*$ in some higher dimensional Euclidean space. His analysis on multiple-valued functions are mainly based on this correspondence. In this paper, we are focusing on the metric space $(\mathbb{Q}, \mathcal{G})$ itself. By carefully studying the geodesics connecting any two points in $\mathbb{Q}$, we are able to claim:

**Theorem 1.1.** The space $\mathbb{Q}$ with metric $\mathcal{G}$ is positively-curved (a PC-space) in the sense of Alexandrov.

We also give an explicit description of the abstract tangent cone in this metric space $(\mathbb{Q}, \mathcal{G})$:

**Theorem 1.2.** For any point $A \in \mathbb{Q}$, such that $S(A) = (J, k_1, \cdots, k_J)$,

$$\text{Tan}_A(\mathbb{Q}) = \mathbb{Q}_{k_1}(\mathbb{R}^n) \times \mathbb{Q}_{k_2}(\mathbb{R}^n) \times \cdots \times \mathbb{Q}_{k_J}(\mathbb{R}^n)$$

with the product metric.

The second part of this paper is based on the following definition of tensor sum of multiple-valued functions. (Recall that there is no suitable notion of “addition” for arbitrary two multiple-valued functions)

**Definition 1.1.** Suppose $f(x) = \sum_{i=1}^p[[f_i(x)]]$, $g(x) = \sum_{j=1}^q[[g_j(x)]]$, where $p$ and $q$ are not necessarily the same. Define

$$(f \oplus g)(x) = \sum_{i,j}[[f_i(x) + g_j(x)]]$$

(i.e the tensor sum is a $pq-$valued function).

This definition is of limited uses in the sense that even if $p = q = Q$, the tensor sum of two $Q-$valued functions gives a $2Q-$valued functions, which makes it hard to talk about derivatives. We are expecting some good notion of “addition” which will enable us to define various things like integration, differential equation in the setting of multiple-valued functions.
The last part of this paper gives another definition of derivatives for multiple-valued functions if a priori the function is continuous. Unlike using linear approximation to define derivative in \[AF\], which avoids “subtraction”, our definition is more calculus-oriented:

**Definition 1.2.** Suppose \( f : \mathbb{R}^m \to \mathbb{Q} \) is continuous, fix \( x_0 \in \mathbb{R}^m \) and \( v \in \mathbb{R}^m \), the directional derivative of \( f \) at \( x_0 \) in the direction \( v \) is the following limit if it exists:

\[
L(v) := \lim_{t \to 0} \frac{f(x_0 + tv)(-)(f(x_0))}{t}.
\]

We will show that this definition is equivalent to the one in \[AF\].

We conclude this paper by a selection theorem for multiple-valued functions. One of the motivations for this kind of question is whether we can decompose a \( \mathbb{Q} \)-valued function into \( \mathbb{Q} \) single-valued functions which preserve some properties of the original function. There are already some important work by De Lellis, Grisanti and Tilli \[DGT\] and by Goblet \[GJ\].

As far as continuity is concerned, we have some positive and also some negative results, namely, a continuous \( \mathbb{Q} \)-\( \mathbb{Q} \)(\( \mathbb{R}^1 \))-valued function always has a continuous decomposition while this does not hold for a continuous \( \mathbb{Q} \)-\( \mathbb{Q} \)(\( \mathbb{R}^{n>1} \))-valued function. See the example in \[GJ\]. Nevertheless, every continuous multiple-valued function defined on a closed interval can always be split into continuous single-valued functions (see \[GJ\] proposition 5.2).

Now our question is whether differentiability can be preserved also. The example \([x] + [-x], x \in [-1,1]\) suggests that “affinely approximatable” (see \[AF\]) is not enough to guarantee a differentiable selection. Under stronger condition, we can prove:

**Theorem 1.3.** If \( f : [a,b] \subset \mathbb{R} \to \mathbb{Q} \) is a continuous function and \( x_0 \in (a,b) \). Suppose \( f \) is strongly affinely approximatable (see \[AF\]) at \( x_0 \), then there exist continuous functions \( f_1, f_2, \cdots, f_Q : [a, b] \to \mathbb{R}^n \) such that \( f = \sum_{i=1}^Q [f_i] \), and each \( f_i \) is differentiable at \( x_0 \).

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## 2 Preliminaries

Most of the notations, definitions and known results about multiple-valued functions that we need can be found in \[ZW1\]. The reader is also referred to \[AF\] for more details. We use standard terminology in geometric measure theory, all of which can be found on page 669-671 of the treatise *Geometric Measure Theory* by H. Federer \[FH\].

For reader’s convenience, here we state some useful results not included in \[ZW1\]. The proofs of them can be found in \[AF\].
Theorem 2.1 ([AF], §1.2). Suppose \( -\infty < r(1) \leq r(2) \leq \cdots \leq r(Q) < \infty \) and \( -\infty < s(1) \leq s(2) \leq \cdots \leq s(Q) < \infty \), then

\[
\sum_{i=1}^{Q} [r(i) - s(i)]^2 = \inf_{\sigma}\{\sum_{i=1}^{Q} [r(i) - s(\sigma(i))]^2 : \sigma \text{ is a permutation of } \{1, 2, \cdots, Q\}\}.
\]

Theorem 2.2 ([AF], §2.14). Suppose \( f : [-1, 1] \to \mathbb{Q} \) such that \( f([-1, 1]) \in \mathcal{Y}_2((-1, 1), \mathbb{Q}) \) is strictly defined and is Dirichlet minimizing. Then there exists \( J \in \{1, 2, \cdots, Q\} \), \( k_1, k_2, \cdots, k_J \in \{1, 2, \cdots, Q\} \) with \( Q = k_1 + k_2 + \cdots + k_J \), and \( f_1, f_2, \cdots, f_J \in \mathcal{A}(1, n) \) such that

1. Whenever \( -1 < x < 1 \), and \( i, j \in \{1, 2, \cdots, J\} \) with \( i \neq j \), \( f_i(x) \neq f_j(x) \).
2. For each \( -1 < x < 1 \), \( f(x) = \sum_{i=1}^{J} k_i[[f_i(x)]] \).

Theorem 2.3 ([AF], §2.14). Suppose \( f_1, f_2, \cdots, f_Q \in \mathcal{Y}_2(\mathbb{B}^m_1(0), \mathbb{R}^n) \) are strictly defined, then \( f = \sum_{i=1}^{Q} [[f_i]] \in \mathcal{Y}_2(\mathbb{B}^m_1(0), \mathbb{Q}) \). Furthermore, in case \( f \) is Dirichlet minimizing, so is each \( f_i, i = 1, 2, \cdots, Q \).

3 Metric Analysis on \( \mathbb{Q} \)

3.1 \( (\mathbb{Q}, \mathcal{G}) \) is a PC-space

Theorem 3.1. \( (\mathbb{Q}, \mathcal{G}) \) is a geodesic length space, namely, given any two points \( A, B \in \mathbb{Q} \), there exists a curve \( \gamma \) connecting \( A \) and \( B \) such that the length of \( \gamma \) equals \( \mathcal{G}(A, B) \). This curve is called a geodesic.

Proof. Given \( A = \sum_{i=1}^{Q}[[a_i]], B = \sum_{i=1}^{Q}[[b_i]] \), from Theorem 2.2 in [AF] there is a curve \( \gamma : [0, 1] \to \mathbb{Q} \) such that \( \gamma(0) = A, \gamma(1) = B, \gamma \in \mathcal{Y}_2((-1, 1), \mathbb{Q}) \), \( \gamma \) is strictly defined and Dirichlet minimizing. We will show that this curve is a geodesic between \( A \) and \( B \). From Theorem 2.2, there exist \( J \in \{1, 2, \cdots, Q\}, k_1, k_2, \cdots, k_J \in \{1, 2, \cdots, Q\} \) with \( Q = k_1 + k_2 + \cdots + k_J \), and \( f_1, f_2, \cdots, f_J \in \mathcal{A}(1, n) \) such that

\[
\gamma(t) = \sum_{i=1}^{J} k_i[[f_i(t)]], \quad t \in [0, 1],
\]

and whenever \( 0 < t < 1 \), and \( i, j \in \{1, 2, \cdots, J\} \) with \( i \neq j \), \( f_i(t) \neq f_j(t) \).

Hence we can rewrite \( \gamma \) as:

\[
\gamma(t) = \sum_{i=1}^{J} k_i([(1-t)p_i + tq_i]]).
\]

Obviously, we have

\[
A = \sum_{i=1}^{Q}[[a_i]] = \sum_{i=1}^{J} k_i[[p_i]], B = \sum_{i=1}^{Q}[[b_i]] = \sum_{i=1}^{J} k_i[[q_i]].
\]
Claim 1: \( G^2(A, B) = \inf \{ \sum_{i=1}^{Q} |a_i - b_{\sigma(i)}|^2 \} = \sum_{i=1}^{J} k_i |p_i - q_i|^2 \).

This is because we choose any permutation \( \sigma \) of \( \{1, 2, \ldots, Q\} \), and define a multiple-valued function \( f_{\sigma} : [0, 1] \rightarrow \mathbb{Q} \) as:

\[ f_{\sigma}(t) = \sum_{i=1}^{Q} [(1-t)a_i + tb_{\sigma(i)}]. \]

Apparently, \( f_{\sigma}(0) = A, f_{\sigma}(1) = B \), and \( f_{\sigma} \in \mathcal{Y}_2((0, 1), \mathbb{Q}) \), strictly defined. Moreover

\[ \text{Dir}(f_{\sigma}; [0, 1]) = \int_0^1 |Df_{\sigma}|^2 dt = \int_0^1 \sum_{i=1}^{Q} |a_i - b_{\sigma(i)}|^2 dt = \sum_{i=1}^{Q} |a_i - b_{\sigma(i)}|^2. \]

Similarly,

\[ \text{Dir}(\gamma; [0, 1]) = \sum_{i=1}^{J} k_i |p_i - q_i|^2. \]

Those two equalities combined with the fact that \( \gamma \) is Dirichlet minimizing proves the first claim.

Claim 2: \( \gamma \) has constant speed, namely,

\[ G(\gamma(t), \gamma(s)) = (t-s)G(A, B), \forall 0 \leq s \leq t \leq 1. \]

Proof of Claim 2: Observe that \( \gamma|_{[s,t]} \) is also Dirichlet minimizing. Claim 1 says that the distance between two points is realized by matching components according to a Dirichlet minimizer connecting them. Therefore

\[ G^2(\gamma(t), \gamma(s)) = G^2(\sum_{i=1}^{J} k_i [(1-t)p_i + tq_i], \sum_{i=1}^{J} k_i [(1-s)p_i + sq_i]) \]

\[ = \sum_{i=1}^{J} k_i |(1-t)p_i + tq_i - (1-s)p_i - sq_i|^2 \]

\[ = (t-s)^2 \sum_{i=1}^{J} k_i |p_i - q_i|^2 \]

\[ = (t-s)^2 G^2(A, B). \]

We now can easily see that the length of \( \gamma \) equals \( G(A, B) \), i.e, \( \gamma \) is a geodesic connecting \( A \) with \( B \).

\[ \square \]

**Corollary 3.1.** Using the same notations, if \( n = 1 \), i.e, in \( \mathbb{Q}_Q(\mathbb{R}) \), both of the sequences \( \{p_i\}, \{q_i\} \) are non-decreasing or non-increasing. Hence geodesic in \( \mathbb{Q}_Q(\mathbb{R}) \) is unique. More precisely, two points are connected by matching components in the order of height.

Proof. It follows directly from Theorem 2.1. \( \square \)
Remark 3.1. (1) From the proof, we also can conclude that any geodesic connecting $A$ with $B$ must be a Dirichlet minimizer with boundary values $A$ and $B$.  
(2) Generally, this geodesic is not unique. For example, let $A = \{[(0,1)] + [(0,1)]\} \in \mathbb{Q}^2(\mathbb{R}^2), B = \{[(1,0)] + [(1,0)]\} \in \mathbb{Q}^2(\mathbb{R}^2)$. It is easy to check that both  
$$
\gamma_1(t) = \{[(t,t-1)] + [[(t,1-t)]] + [[(t,t-1)]]
\]$$
are geodesics connecting $A$ with $B$.  
(3) As we said, restriction of a geodesic always gives a geodesic by Dirichlet minimality. But an extension of a geodesic may not be a geodesic anymore. For example, considering $\gamma_1$ in the previous one. If we extend the domain of $\gamma_1$ to $[0,2]$, it is no longer a geodesic. In fact, the curve $\hat{\gamma}(t) = \{([(2t,1)] + [[(2t-1)]\}$ is the geodesic connecting $A$ with $[(-2,1)] + [[(2,1)]].$

**Theorem 3.2.** The space $\mathbb{Q}$ with metric $G$ is positively-curved (a PC-space) in the sense of Alexandrov.

**Proof.** Given any two points $A = \sum_{i=1}^{Q}[[a_i]], B = \sum_{i=1}^{Q}[[b_i]] \in \mathbb{Q}$, suppose $\gamma : [0,1] \rightarrow \mathbb{Q}$ is a geodesic connecting them. Take any point $C = \sum_{i=1}^{Q}[[c_i]] \in \mathbb{Q}$, we will show the following inequality:

$$
G^2(\gamma(t),C) \geq (1-t)G^2(A,C) + tG^2(B,C) - t(1-t)G^2(A,B), \text{ for any } t \in [0,1].
$$

Let’s write  
$$
\gamma(t) = \sum_{i=1}^{J}k_i[[((1-t)p_i + tp_i)], = \sum_{j=1}^{Q}[[((1-t)p_j + tp_j)],
$$
where in the second equality, we use the following convention:

$$
p_j' = p_i, q_j' = q_i \text{ if } k_1 + \cdots + k_{i-2} + 1 \leq j \leq k_1 + \cdots + k_i,
$$
and $k_0 = 0$.  

Fix any permutation $\sigma$ of $\{1,2,\cdots, Q\}$,

$$
\sum_{j=1}^{Q}|(1-t)p'_j + tq_j' - c_{\sigma(j)}|^2 = \sum_{j=1}^{Q}|(1-t)p'_j - (1-t)c_{\sigma(j)} + tq_j' - tc_{\sigma(j)}|^2
$$

$$
= \sum_{j=1}^{Q}(1-t)^2|p'_j - c_{\sigma(j)}|^2 + t^2|q'_j - c_{\sigma(j)}|^2 + 2t(1-t)\langle p'_j - c_{\sigma(j)}, q'_j - c_{\sigma(j)} \rangle
$$

$$
= \sum_{j=1}^{Q}(1-t)|p'_j - c_{\sigma(j)}|^2 + t|q'_j - c_{\sigma(j)}|^2 - t(1-t)|p'_j - q'_j|^2
$$

$$
\geq (1-t)G^2(A,C) + tG^2(B,C) - t(1-t)\sum_{j=1}^{Q}|p'_j - q'_j|^2
$$

6
By the definition of $p'_j, q'_j$ and the claim 1 in Theorem 3.1,

$$\sum_{j=1}^Q |p'_j - q'_j|^2 = G^2(A, B).$$

Hence

$$\sum_{j=1}^Q |(1 - t)p'_j + tq'_j - c_{\sigma(j)}|^2 \geq (1 - t)G^2(A, C) + tG^2(B, C) - t(1 - t)G^2(A, B).$$

Taking the infimum of $\sigma$ at the left side of the above inequality gives

$$G^2(\gamma(t), C) \geq (1 - t)G^2(A, C) + tG^2(B, C) - t(1 - t)G^2(A, B).$$

□

In the special case $n = 1$, we will show that in fact $Q_Q(\mathbb{R})$ is flat. Before we prove that, we need some lemma:

**Lemma 3.1.** Let $A = \sum_{i=1}^Q [a_i], B = \sum_{i=1}^Q [b_i]$ be any two points in $Q_Q(\mathbb{R})$. Suppose $\gamma$ is the geodesic connecting $A$ and $B$ and can be written as:

$$\gamma(t) = \sum_{i=1}^Q [(1 - t)a_i + tb_i].$$

Take any point $C = \sum_{i=1}^Q [c_i] \in Q_Q(\mathbb{R})$ and suppose $\sigma$ is a permutation of $\{1, 2, \cdots, Q\}$ such that

$$\sum_{i=1}^Q (a_i - c_{\sigma(i)})^2 = G^2(A, C),$$

then

$$\sum_{i=1}^Q (b_i - c_{\sigma(i)})^2 = G^2(B, C).$$

**Proof.** By Corollary 3.1 we may assume that $\{a_i\}$ and $\{b_i\}$ are both non-decreasing sequences. Applying the Theorem 2.1 to $G^2(A, C)$ we know $\{c_{\sigma(i)}\}$ is also a non-decreasing sequence. Therefore, applying the Theorem 2.1 again to $G^2(B, C)$ gives us the desired result. □

**Theorem 3.3.** $(Q_Q(\mathbb{R}), G)$ is a flat metric space in the sense of Alexandrov.

**Proof.** For any two points $A = \sum_{i=1}^Q [a_i], B = \sum_{i=1}^Q [b_i] \in Q_Q(\mathbb{R})$, suppose

$$\gamma(t) = \sum_{i=1}^Q [(1 - t)a_i + tb_i]$$
is the geodesic connecting $A$ with $B$.

From proof of Theorem 3.2, for any permutation $\sigma$ of \{1, 2, \ldots, Q\},
\[
\sum_{i=1}^{Q} |(1-t)a_i + tb_i - c_{\sigma(i)}|^2 =
\]
\[
= (1-t) \sum_{i=1}^{Q} |a_i - c_{\sigma(i)}|^2 + t \sum_{i=1}^{Q} |b_i - c_{\sigma(i)}|^2 - t(1-t)G^2(A, B).
\]

Now take a permutation $\sigma$ such that
\[
\sum_{i=1}^{Q} |a_i - c_{\sigma(i)}|^2 = G^2(A, C).
\]
By Lemma 3.1,
\[
\sum_{i=1}^{Q} |b_i - c_{\sigma(i)}|^2 = G^2(B, C).
\]
So for this permutation $\sigma$,
\[
\sum_{i=1}^{Q} |(1-t)a_i + tb_i - c_{\sigma(i)}|^2 = (1-t)G^2(A, C) + tG^2(B, C) - t(1-t)G^2(A, B).
\]

By the definition of $G^2(\gamma(t), C)$, we have
\[
G^2(\gamma(t), C) \leq (1-t)G^2(A, C) + tG^2(B, C) - t(1-t)G^2(A, B).
\]
The above inequality combined with Theorem 3.2 gives the following equality:
\[
G^2(\gamma(t), C) = (1-t)G^2(A, C) + tG^2(B, C) - t(1-t)G^2(A, B),
\]
which finishes the proof.

\textbf{Remark 3.2.} Generally the space $\mathbb{Q}$ is not flat, as shown by the following example.

Example: Consider $\mathbb{Q}_2(\mathbb{R}^2)$, $A = [[(0,1)],[[(0,0)],[[0,0)]]], B = [[(0,0)],[[0,1-1/2)]], C = [[(0,0)]] + [(-1,-1)].$ It is easy to compute that
\[
G^2(A, B) = 9/4, G^2(A, C) = 3, G^2(B, C) = 13/4,
\]
\[
\gamma(t) = [[(0,1-t)] + [(t,-t/2)].
\]
As for $G^2(\gamma(t), C)$, there are two permutations involved:
\[
|(0,0) - (0,1-t)|^2 + |(-1,-1) - (t,-t/2)|^2 = 9t^2/4 - t + 3,
\]
\[
|(0,0) - (t,-t/2)|^2 + |(-1,-1) - (0,1-t)|^2 = 9t^2/4 - 4t + 5.
\]
Hence when \( t \in [0, 2/3] \), \( G^2(\gamma(t), C) = 9t^2/4 - t + 3 \) and when \( t \in [2/3, 1] \),
\( G^2(\gamma(t), C) = 9t^2/4 - 4t + 5 \).

Now for any \( t \in (0, 2/3) \),
\[ G^2(\gamma(t), C) = 9t^2/4 - t + 3 > (1 - t)G^2(A, C) + tG^2(B, C) - t(1 - t)G^2(A, B) = 9t^2/4 - 2t + 3. \]

We conclude this section by a description of \((\mathbb{Q}, G)\) in terms of metric:

**Theorem 3.4.** \((\mathbb{Q}, G)\) is a complete, separable, path connected and locally compact metric space.

**Proof.** Taking a Cauchy sequence \( \{A_i\} \subset \mathbb{Q} \), we consider the sequence \( \{\xi(A_i)\} \subset \mathbb{Q}^* \subset \mathbb{R}^{PQ} \). Because Lip(\(\xi\)) < \(\infty\), we infer \( \{\xi(A_i)\} \) is a Cauchy sequence in \(\mathbb{R}^{PQ}\).

Due to the completeness of \(\mathbb{R}^{PQ}\), there is an element \( A \in \mathbb{R}^{PQ} \), such that
\[ \lim_{i \to \infty} \xi(A_i) = A. \]

Because \(\mathbb{Q}^*\) is closed in \(\mathbb{R}^{PQ}\), we conclude that \( A \in \mathbb{Q}^* \). Therefore \( \lim_{i \to \infty} A_i = \xi^{-1}(A) \) thanks to the fact that Lip(\(\xi^{-1}\)) < \(\infty\).

As for the separability, one can check that the set
\[ \{ \sum_{i=1}^{Q} [a_i], a_i \text{ is a rational point in } \mathbb{R}^n \} \]

is a countable dense subset of \(\mathbb{Q}\).

Take any two points \( A, B \in \mathbb{Q} \), any geodesic connecting them gives a path between them. Hence \(\mathbb{Q}\) is path connected.

As for the locally compactness, it is obviously once we notice the bi-Lipschitzian correspondence between \(\mathbb{Q}\) and \(\mathbb{Q}^* \subset \mathbb{R}^{PQ}\).

**Remark 3.3.** By Hopf-Rinow Theorem, any bounded closed set in \(\mathbb{Q}\) is compact.

### 3.2 Metric Tangent Cone of \((\mathbb{Q}, G)\)

**Definition 3.1.** For \( A, B, C \in \mathbb{Q} \), define
\[ \alpha(A; B, C) = \frac{G^2(A, B) + G^2(A, C) - G^2(B, C)}{2G(A, B)G(A, C)}, A \neq B, C. \]

**Theorem 3.5.** Let \( \gamma_1, \gamma_2 \) be geodesics starting from \( A \), then the function
\[ t, s \in (0, 1] \to \alpha(A; \gamma_1(t), \gamma_2(s)) \]

is nondecreasing in \( s, t \).

The angle \( \angle(\gamma_1, \gamma_2) \in [0, \pi] \) between \( \gamma_1 \) and \( \gamma_2 \) is thus defined by the formula
\[ \cos(\angle(\gamma_1, \gamma_2)) := \inf_{s, t} \alpha(A; \gamma_1(t), \gamma_2(s)) = \lim_{s, t \downarrow 0} \alpha(A; \gamma_1(t), \gamma_2(s)). \]
Proof. See [AGS] Lemma 12.3.4.

For a fixed $A \in \mathbb{Q}$ let us denote by $G(A)$ the set of all geodesics $\gamma$ starting from $A$ and parameterized in some interval $[0, T_\gamma]$; recall that the metric velocity of $\gamma$ is $|\gamma'| = G(\gamma(t), A)/t, t \in (0, T].$ We set

$$
\|\gamma\|_A := |\gamma'|, <\gamma_1, \gamma_2>_A := \|\gamma_1\|_A \|\gamma_2\|_A \cos(\angle(\gamma_1, \gamma_2)),
$$

$$
d_A^2(\gamma_1, \gamma_2) := \|\gamma_1\|_A^2 + \|\gamma_2\|_A^2 - 2 <\gamma_1, \gamma_2>_A.
$$

If $\gamma \in G(A)$ and $\lambda > 0$ we denote by $\lambda \gamma$ the geodesic

$$(\lambda \gamma)_t := \gamma_{\lambda t}, T_{\lambda \gamma} = \lambda^{-1}T_\gamma,$$

and we observe that for each $\gamma_1, \gamma_2 \in G(A), \lambda > 0$, it holds

$$
\|\lambda \gamma\|_A = \lambda \|\gamma\|_A, <\lambda \gamma_1, \gamma_2>_A = <\gamma_1, \lambda \gamma_2>_A = \lambda <\gamma_1, \gamma_2>_A.
$$

Recall that the restriction of a geodesic is still a geodesic; we say that $\gamma_1 \sim \gamma_2$ if there exists $\epsilon > 0$ such that $\gamma_1|_{[0, \epsilon]} = \gamma_2|_{[0, \epsilon]}$.

**Theorem 3.6.** If $\gamma_1, \gamma_2 : [0, T] \to \mathbb{Q}$ are two geodesics starting from $A$, we have

$$
d_A(\gamma_1, \gamma_2) = \lim_{t \to 0} \frac{G(\gamma_1(t), \gamma_2(t))}{t}.
$$

In particular, the function $d_A$ defined above is a distance on the quotient space $G(A)/\sim$. The completion of $G(A)/\sim$ is called the tangent cone $\text{Tan}_A(\mathbb{Q})$ at the point $A$.

Proof. It follows from the same argument as [AGS] Theorem 12.3.6.

Before we give an explicit representation of the abstract tangent cone $\text{Tan}_A(\mathbb{Q})$, we give several definitions concerning representation of elements in $\mathbb{Q}$.

**Definition 3.2.** Define $\sigma : \mathbb{Q} \to \{1, 2, \ldots, Q\}$ by

$$
\sigma(x) = \text{card}[\text{spt}(x)].
$$

**Remark 3.4.** It is easy to see that $\sigma$ is lower semi-continuous.

**Definition 3.3.** For any $x \in \mathbb{Q}$, which can be written as

$$
x = \sum_{i=1}^{J} k_i [x_i],
$$

for some $J \in \{1, 2, \ldots, Q\}, k_i \in \{1, 2, \ldots, Q\}, k_1 \leq k_2 \leq \cdots \leq k_J, \sum_{i=1}^{J} k_i = Q$ and $x_i$ distinct points in $\mathbb{R}^n$, define the signature of $x$ as:

$$
S(x) = (J, k_1, k_2, \cdots, k_J).
Remark 3.5. In $S(x)$, $J$ is exactly $\sigma(x)$.

Definition 3.4. For a fixed positive integer $Q$, $(J, k_1, k_2, \ldots, k_J)$ is called a permissible decomposition of $Q$ if

$$J \in \{1, 2, \ldots, Q\}, k_i \in \{1, 2, \ldots, Q\}, k_1 \leq k_2 \leq \cdots \leq k_J, \sum_{i=1}^{J} k_i = Q.$$  

The set of all permissible decompositions of $Q$ is denoted as $\mathcal{P}(Q)$.

Remark 3.6. From [15.8], $\card(\mathcal{P}(Q)) = p(n)$, where $p(n)$ is defined to be the number of unordered partitions of $n$.

Proposition 3.1. Suppose $S(A) = (J, k_1, \ldots, k_J)$. Let $\gamma_1, \gamma_2 \in G(A)$ such that

$$\gamma_1(t) = \sum_{i=1}^{J} \sum_{j=k_i+\cdots+k_{i-1}+1}^{k_i+\cdots+k_i} [[a_i + tw_j]],$$

$$\gamma_2(t) = \sum_{i=1}^{J} \sum_{j=k_i+\cdots+k_{i-1}+1}^{k_i+\cdots+k_i} [[a_i + tw_j]].$$

Then

$$d^2_A(\gamma_1, \gamma_2) = \sum_{i=1}^{J} \sum_{j=k_i+\cdots+k_{i-1}+1}^{k_i+\cdots+k_i} [[v_j]], \sum_{j=k_i+\cdots+k_{i-1}+1}^{k_i+\cdots+k_i} [[w_j]].$$

Proof. When $t$ small enough, the distance between $\gamma_1(t)$ and $\gamma_2(t)$ is obtained by: for each $i \in \{1, 2, \ldots, J\},$

$$\sum_{j=k_i+\cdots+k_{i-1}+1}^{k_i+\cdots+k_i} [[a_i + tw_j]]$$

matching with

$$\sum_{j=k_i+\cdots+k_{i-1}+1}^{k_i+\cdots+k_i} [[a_i + tw_j]].$$

Hence using Theorem 3.6 finishes the proof.

This proposition gives us the description of $\Tan_A(Q)$:

Theorem 3.7. For any point $A \in Q$, such that $S(A) = (J, k_1, \ldots, k_J)$, we have the isometry

$$\Tan_A(Q) \cong \mathbb{Q}_{k_1}(\mathbb{R}^n) \times \mathbb{Q}_{k_2}(\mathbb{R}^n) \times \cdots \times \mathbb{Q}_{k_J}(\mathbb{R}^n)$$

with the product metric.

Proof. Observe that any geodesic starting with $A$ is uniquely determined by the initial velocity, which is given by

$$\sum_{i=1}^{k_1} [[v_i]], \sum_{i=k_1+1}^{k_1+k_2} [[v_i]], \cdots, \sum_{i=k_1+\cdots+k_{j-1}+1}^{Q} [[v_i]].$$

Proposition 3.1 shows the metric on $\Tan_A(Q)$. We are done.

Remark 3.7. For any two points $x, y \in Q,$

$$\Tan_x(Q) \cong \Tan_y(Q) \iff S(x) = S(y).$$
3.3 Exponential Map

Given a point \( A = \sum_{i=1}^{J} k_i [a_i] \) of signature \( S(A) = (J, k_1, \ldots, k_J) \) and a nonzero element \( \vec{v} = (\sum_{i=1}^{k_1} [v_i], \ldots, \sum_{i=k_1+\cdots+k_{J-1}+1}^{Q} [v_i]) \in \text{Tan}_A(Q) \), there is a unique parameterized geodesic \( \gamma : (-\epsilon, \epsilon) \to Q \) of the form

\[
\gamma(t) = \sum_{i=1}^{J} k_1 + \cdots + k_i \sum_{j=k_1+\cdots+k_{i-1}+1}^{k_1+\cdots+k_i} [a_i + tv_j].
\]

To indicate the dependence of this geodesic on the vector \( \vec{v} \), we denote it by \( \gamma(t, \vec{v}) = \gamma \). Obviously we have

**Proposition 3.2.** If the geodesic \( \gamma(t, \vec{v}) \) is defined for \( t \in (-\epsilon, \epsilon) \), then the geodesic \( \gamma(t, \lambda \vec{v}) \), \( \lambda \in \mathbb{R}, \lambda > 0 \), is defined for \( t \in (-\epsilon/\lambda, \epsilon/\lambda) \), and \( \gamma(t, \lambda \vec{v}) = \gamma(\lambda t, \vec{v}) \).

**Definition 3.5.** If \( \vec{v} \in \text{Tan}_A(Q), \vec{v} \neq \vec{0}, \) is such that \( \gamma(|\vec{v}|, \vec{v}/|\vec{v}|) = \gamma(1, \vec{v}) \) is defined, we set \( \exp_A(\vec{v}) = \gamma(1, \vec{v}) \) and \( \exp_A(\vec{0}) = A \).

**Remark 3.8.** Exponential map is not necessarily one-to-one, as shown in Remark 3.1(1). However, if restricted in a small neighborhood of 0 in \( \text{Tan}_A(Q) \), it is an isometry.

**Theorem 3.8.** For any \( A \in Q \) with signature \( S(A) = (J, k_1, \ldots, k_J) \), there is a positive \( \epsilon \) such that \( \exp_A : B_\epsilon(0) \subset \text{Tan}_A(Q) \to Q \) is an isometry.

**Proof.** Suppose \( A = \sum_{i=1}^{J} k_i [a_i] \), and let

\[
\delta = 2^{-1} \inf \{|a_i - a_j|, i \neq j\}.
\]

Choose \( \epsilon = \delta/2 \). Let

\[
\vec{v} = (\sum_{i=1}^{k_1} [v_i], \ldots, \sum_{i=k_1+\cdots+k_{J-1}+1}^{Q} [v_i]) \in B_\epsilon(0) \subset \text{Tan}_A(Q),
\]

\[
\vec{w} = (\sum_{i=1}^{k_1} [w_i], \ldots, \sum_{i=k_1+\cdots+k_{J-1}+1}^{Q} [w_i]) \in B_\epsilon(0) \subset \text{Tan}_A(Q).
\]

Therefore,

\[
\gamma_1(t) = \sum_{i=1}^{J} k_1 + \cdots + k_i \sum_{j=k_1+\cdots+k_{i-1}+1}^{k_1+\cdots+k_i} [a_i + tv_j],
\]

\[
\gamma_2(t) = \sum_{i=1}^{J} k_1 + \cdots + k_i \sum_{j=k_1+\cdots+k_{i-1}+1}^{k_1+\cdots+k_i} [a_i + tw_j].
\]
\[ \exp_A(\vec{v}) = \sum_{i=1}^{J} \sum_{j=k_1+\cdots+k_{i-1}+1}^{k_1+\cdots+k_i} [[a_i + v_j]], \]
\[ \exp_A(\vec{w}) = \sum_{i=1}^{J} \sum_{j=k_1+\cdots+k_{i-1}+1}^{k_1+\cdots+k_i} [[a_i + w_j]]. \]

Then
\[ G^2(\exp_A(\vec{v}), \exp_A(\vec{w})) = d^2(\vec{v}, \vec{w}) = \sum_{i=1}^{J} G^2 \left( \sum_{j=k_1+\cdots+k_{i-1}+1}^{k_1+\cdots+k_i} [[v_j]], \sum_{j=k_1+\cdots+k_{i-1}+1}^{k_1+\cdots+k_i} [[w_j]] \right). \]

3.4 A Decomposition of \((Q, G)\)

We decompose \(Q\) according to the signature:

**Definition 3.6.** Define
\[ I_{J,k_1,k_2,\ldots,k_J} = \{ x \in Q : S(x) = (J, k_1, k_2, \ldots, k_J) \}, \]
for any \((J, k_1, k_2, \ldots, k_J) \in \mathcal{P}(Q)\).

**Definition 3.7.** Define
\[ \mathcal{I}_i = \{ x \in Q : \text{card}(spt(x)) = i \} = \bigcup_{(i,k_1,\ldots,k_i) \in \mathcal{P}(Q)} \mathcal{I}_{i,k_1,\ldots,k_i}, \]
for \(i \in \{1, 2, \ldots, Q\}\).

**Remark 3.9.** Geometrically, a point in \(\mathcal{I}_1\) is like the vertex point of a cone in the sense that the tangent cone at points in \(\mathcal{I}_1\) is isometric to \(Q\) itself. Points in \(\mathcal{I}_Q\) are like faces of a polyhedra cone in the sense that the tangent cone at points in \(\mathcal{I}_Q\) is isometric to \(\mathbb{R}^n\), namely, locally \(\mathcal{I}_Q\) is flat. We will rigorously prove the flatness of \(\mathcal{I}_Q\) in the sense of Alexandrov later.

**Theorem 3.9.** (1) \(\mathcal{I}_i\) only can be approximated by elements in \(\mathcal{I}_j\) for \(j \geq i\).

(2) \(\mathcal{I}_Q\) is open in \(Q\).

**Proof.** They both follow from the lower semi-continuity of the function \(\sigma\).

**Theorem 3.10.** \(\mathcal{I}_Q\) is path connected and dense in \(Q\).

**Proof.** We endow \(\mathbb{R}^n\) with the lexicographical order. Take any two points \(A, B \in \mathcal{I}_Q\). Write them as
\[ A = \sum_{i=1}^{Q} [[a_i]], B = \sum_{i=1}^{Q} [[b_i]], \]
with $a_1 < a_2 < \cdots < a_Q$, and $b_1 < b_2 < \cdots < b_Q$. Define a curve $\gamma : [0, 1] \to \mathbb{Q}$ as

$$\gamma(t) = \sum_{i=1}^{Q} \left[ (1-t)a_i + tb_i \right].$$

Obviously $\gamma$ connects $A$ with $B$. We will show that $\gamma \in \mathcal{I}_Q$. Suppose not, i.e., there are $i < j$ and $t \in (0, 1)$ such that

$$(1-t)a_i + tb_i = (1-t)a_j + tb_j.$$ 

Hence $a_i - a_j = \frac{t}{1-t} (b_j - b_i)$, which is impossible because $a_i < a_j, b_i < b_j$. 

$\mathcal{I}_Q$ is dense in $\mathbb{Q}$ because the set

$$\left\{ \sum_{i=1}^{Q} [a_i] : a_i \text{ is a rational point in } \mathbb{R}^n, \text{ and they are distinct} \right\}$$

is dense in $\mathbb{Q}$. \hfill \Box

**Theorem 3.11.** $\mathcal{I}_1$ is path connected.

**Proof.** Take any two points $A, B \in \mathcal{I}_1$. Write them as

$$A = Q[[a]], B = Q[[b]].$$

Then the curve $\gamma : [0, 1] \to \mathcal{I}_1, \gamma(t) := Q[((1-t)a + tb)]$ certainly works. \hfill \Box

**Remark 3.10.** Generally $\mathcal{I}_i$ is not path connected for $i \neq 1, Q$. For example, we consider $\mathbb{Q}_3(\mathbb{R})$. Let $A = [[1]] + 2[[2]], B = [[2]] + 2[[1]] \in \mathcal{I}_2$. Suppose there is a continuous curve $\gamma : [0, 1] \to \mathcal{I}_2$ connecting $A$ with $B$. Since $\gamma \in \mathcal{I}_2$, we can write $\gamma$ as

$$\gamma = [[\gamma_1]] + 2[[\gamma_2]],$$

for $\gamma_i : [0, 1] \to \mathbb{R}, i = 1, 2$ and $\gamma_1(t) \neq \gamma_2(t), \forall t \in [0, 1]$. Because $\gamma_1$ and $\gamma_2$ never cross, they both are continuous. Considering the initial conditions, we have

$$\gamma_1(0) = 1, \gamma_1(1) = 2,$$

$$\gamma_2(0) = 2, \gamma_2(1) = 1.$$ 

Therefore $\gamma_1$ and $\gamma_2$ must meet at some point in $(0, 1)$ due to elementary facts about continuous functions. This shows that $\mathcal{I}_2$ is not path connected.

However, we are able to prove the following theorem:

**Theorem 3.12.** Take $(J, k_1, k_2, \cdots, k_J) \in \mathcal{P}(Q)$ and $A \in \mathcal{I}_{J, k_1, k_2, \cdots, k_J}$. If $B \in \mathcal{I}_{J, k_1, k_2, \cdots, k_J}$, and $G(A, B)$ is small enough, then any geodesic $\gamma$ connecting them lies in $\mathcal{I}_{J, k_1, k_2, \cdots, k_J}$. 

14
Proof. Let

\[ A = \sum_{i=1}^{J} k_i [a_i], B = \sum_{i=1}^{J} k_i [b_i], \]

for distinct \(a_i\)'s and \(b_i\)'s. Let

\[ \delta = 2^{-1} \inf \{|a_i - a_j|, i \neq j\}. \]

Choose \(B\) close enough with \(A\) such that

\[ |b_i - a_i| < \delta/4, i \in \{1, 2, \ldots, Q\}. \]

Obviously, \(G^2(A, B) = \sum_{i=1}^{J} k_i |a_i - b_i|^2\). Therefore, according to proof of Theorem 3.1, any geodesic \(\gamma: [0, 1] \to Q\) connecting \(A\) with \(B\) can be written as:

\[ \gamma(t) = \sum_{i=1}^{J} k_i [(1-t)a_i + tb_i]. \]

We will show that \(\gamma \in I_{J,k_1,k_2,\ldots,k_J}\). If not, i.e., there are \(i < j\) and \(t \in (0, 1)\) such that

\[ (1-t)a_i + tb_i = (1-t)a_j + tb_j. \]

Subtract \(a_i\) from both sides,

\[ t(b_i - a_i) = t(b_j - a_j) + a_j - a_i. \]

The absolute value of LHS is less than \(\delta/4Q\), while the absolute value of RHS is great than \(\delta - \delta/4Q = (1 - 4^{-Q})\delta\). A contradiction. □

As we promised, we will show \(I_Q\) is locally flat:

**Theorem 3.13.** \(I_Q\) is locally flat in the sense of Alexandrov.

Proof. Take any point \(A = \sum_{i=1}^{Q} a_i \in I_Q\). Let

\[ \delta = 2^{-1} \inf \{|a_i - a_j|, i \neq j\}, \]

and

\[ U = \{x \in Q: G(x, A) < \delta/2\} \subseteq I_Q. \]

Take any point \(B \in U\) and \(\gamma: [0, 1] \to I_Q\) be a geodesic connecting \(A\) with \(B\) (Theorem 3.12 guarantees that \(\gamma \in I_Q\) once \(U\) is small enough). We write \(\gamma\) as

\[ \gamma(t) = \sum_{i=1}^{Q} [(1-t)a_i + tb_i], \]

which means

\[ G^2(A, B) = \sum_{i=1}^{Q} |a_i - b_i|^2. \]
Take $C \in U$ and suppose
\[ G^2(A, C) = \sum_{i=1}^{Q} |a_i - c_i|^2, \]
which guarantees that
\[ G^2(B, C) = \sum_{i=1}^{Q} |b_i - c_i|^2, G^2(\gamma(t), C) = \sum_{i=1}^{Q} |(1-t)a_i + tb_i - c_i|^2, \forall t. \]

Then we have the following equality
\[ G^2(\gamma(t), C) = (1-t)G^2(A, C) + tG^2(B, C) - t(1-t)G^2(A, B), \forall t \in [0,1]. \]

The proof suggests a one-to-one correspondence between a neighborhood of $A$ with an open set in $\mathbb{R}^{nQ}$. Moreover, this correspondence turns out to be an isometry:

**Theorem 3.14.** For any point $A \in \mathcal{I}_Q$, there is an open neighborhood $U$ of $A$ in $\mathcal{I}_Q$ and an bijective isometry between $U$ with some open set in $\mathbb{R}^{nQ}$.

**Proof.** Using the same notations as Theorem 3.13, for any $B, C \in U$, we have
\[ G^2(A, B) = \sum_{i=1}^{Q} |a_i - b_i|^2, G^2(A, C) = \sum_{i=1}^{Q} |a_i - c_i|^2, \]
\[ G^2(B, C) = \sum_{i=1}^{Q} |b_i - c_i|^2. \]

That means that the distance between any two points in $U$ is obtained in a unique way. Define $V = \{x \in \mathbb{R}^{nQ} : |x - (a_1, a_2, \ldots, a_Q)| < \delta/2\} \subset \mathbb{R}^{nQ}$ and define
\[ \phi : U \rightarrow V, \phi(\sum_{i=1}^{Q} [b_i]) = (b_1, b_2, \ldots, b_Q). \]

It is easy to check that is is a bijection and an isometry.

**Remark 3.11.** We can use the lexicographic order to relate $\mathbb{Q}$ with $\mathbb{R}^{nQ}$. But it is not very useful because is is not necessarily an isometry. For example, we endow $\mathbb{R}^n$ with the lexicographical order. Then take any $A = \sum_{i=1}^{Q} [a_i] \in \mathbb{Q}$, we can order $a_i$ as following
\[ a_1 \leq a_2 \leq \cdots \leq a_Q \]
in a unique way. Hence we are able to define

\[ \psi : \mathbb{Q} \to \mathbb{R}^{nQ}, \psi(\sum_{i=1}^{Q}[[a_i]]) = (a_1, a_2, \cdots, a_Q). \]

Unless \( n = 1 \), \( \psi \) will not be an isometry. For example, we consider \( \mathbb{Q}_2(\mathbb{R}^2) \). For \( 0 < \epsilon < 1/2 \), let \( A_\epsilon = [[(1, 1)] + [[(1 + \epsilon, 2)]] \), \( B = [[(1 + \epsilon, 1)] + [[(1, 2)]] \). Under the map \( \psi \), we have

\[ \psi(A_\epsilon) = (1, 1, 1 + \epsilon, 2), \psi(B_\epsilon) = (1, 2, 1 + \epsilon, 1). \]

Therefore, \( G(A_\epsilon, B_\epsilon) = \sqrt{2} \epsilon, |\psi(A_\epsilon) - \psi(B_\epsilon)| = \sqrt{2}. \) Letting \( \epsilon \downarrow 0 \) shows that \( \psi \) is not even a Lipschitz map.

4 Tensor Sum of Multiple-Valued Functions

**Definition 4.1.** Suppose \( f(x) = \sum_{i=1}^{p}[[f_i(x)]] \), \( g(x) = \sum_{j=1}^{q}[[g_j(x)]] \), where \( p \) and \( q \) are not necessarily the same. Define

\[ (f \oplus g)(x) = \sum_{i,j}[[f_i(x) + g_j(x)]]. \]

(i.e the Tensor sum is a \( pq \)-valued function).

**Remark 4.1.** It is easy to check:

\[ \eta(f \oplus g) = \eta(f) + \eta(g) \]

**Remark 4.2.** If \( f \) and \( g \) are both Dirichlet minimizing, although their tensor sum \( f \oplus g \) may not be minimizing, the average function \( \eta(f \oplus g) \) is harmonic. This is because both \( \eta(f) \) and \( \eta(g) \) are harmonic. To see that \( f \oplus g \) may not be minimizing, let \( f(x) = [[x]] + [[-1]], g(x) = [[1 - x]] + [[-1]]. \) Their domains are both \([0, 1]\). Both of the functions are Dirichlet minimizers. But their tensor sum \( f \oplus g = [[x - 1]] + [[-x]] + [[1]] + [[-2]] \) is no longer minimizing since there is a branch point for the function \( f \oplus g \).

**Theorem 4.1 (Weighted Triangular Inequality).** Suppose \( f(x) = \sum_{i=1}^{p}[[f_i(x)]] \), \( g(x) = \sum_{j=1}^{q}[[g_j(x)]] \), then

\[ \mathcal{G}(f \oplus g, pq[[0]]) \leq p^{1/2}\mathcal{G}(f, p[[0]]) + q^{1/2}\mathcal{G}(g, q[[0]]). \]
Proof. By definition,

\[ G(f \oplus g, pq[[0]])^2 = \sum_{i=1}^{p} \sum_{j=1}^{q} |f_i(x) + g_j(x)|^2 \]

\[ = \sum_{i=1}^{p} \sum_{j=1}^{q} (|f_i(x)|^2 + |g_j(x)|^2 + 2f_i(x) \cdot g_j(x)) \]

\[ = qG(f, p[[0]])^2 + pG(g, q[[0]])^2 + 2 \sum_{i=1}^{p} \sum_{j=1}^{q} f_i(x) \cdot g_j(x) \]

\[ = qG(f, p[[0]])^2 + pG(g, q[[0]])^2 + 2pq \eta(f) \cdot \eta(g) \]

Since for each \( S, T \in \mathcal{Q} \), \( |\eta(S) - \eta(T)| \leq Q^{-1/2} G(S, T) \),

\[ |\eta(f)| \leq p^{-1/2} G(f, p[[0]]), |\eta(g)| \leq q^{-1/2} G(g, q[[0]]). \]

Therefore,

\[ G(f \oplus g, pq[[0]])^2 \leq qG(f, p[[0]])^2 + pG(g, q[[0]])^2 \]

\[ + 2p \cdot q \cdot p^{-1/2} G(f, p[[0]]) \cdot q^{-1/2} G(g, q[[0]]) \]

\[ = [q^{1/2}G(f, p[[0]]) + p^{1/2}G(g, q[[0]])]^2. \]

\[ \square \]

**Theorem 4.2.** Suppose \( f(x) = \sum_{i=1}^{p} [[f_i(x)]], g(x) = \sum_{j=1}^{q} [[g_j(x)]] \), where \( f_i, g_j \in \mathcal{V}_2(B_{m}^{m})(0), \mathbb{R}^n \) are strictly defined, then

\[ \text{Dir}(f \oplus g) = q\text{Dir}(f) + p\text{Dir}(g) + 2pq \int < apD(\eta \circ f), apD(\eta \circ g) > dH^m. \]

Proof. By Theorem 2.3, \( f \oplus g \in \mathcal{V}_2(B_{1}^{m}(0), \mathbb{Q}_{pq}(\mathbb{R}^n)) \). It is easy to see that \( \text{Dir}(f \oplus g) = \sum_{j=1}^{q} \text{Dir}(f \oplus g_j) \).

From [AF]§2.3, we have

\[ \text{Dir}(f \oplus g_j) = \text{Dir}(f) + 2p \int < apD(\eta \circ f), apD(g_j) > dH^m + p\text{Dir}(g_j). \]

Sum them up, we get

\[ \text{Dir}(f \oplus g) = q\text{Dir}(f) + \sum_{j=1}^{q} 2p \int < apD(\eta \circ f), apD(g_j) > dH^m + p\text{Dir}(g) \]

\[ = q\text{Dir}(f) + 2p \int < apD(\eta \circ f), apD(\sum_{j=1}^{q} g_j) > dH^m + p\text{Dir}(g) \]

\[ = q\text{Dir}(f) + p\text{Dir}(g) + 2pq \int < apD(\eta \circ f), apD(\eta \circ g) > dH^m. \]

\[ \square \]
**Definition 4.2.** For a \( Q \)-valued function \( f \), its \( L^k \) norm \((0 < k < \infty)\) is defined to be
\[
\|f\|_k = \left( \int G(f(x), Q[[0]])^k d\mathcal{H}^m \right)^{1/k}.
\]
Moreover, its \( L^\infty \) norm is defined to be
\[
\|f\|_\infty = \inf_M \{ G(f(x), Q[[0]]) \leq M, \text{a.e.} \}.
\]

**Theorem 4.3 (Weighted Minkowski Inequality).** Suppose \( f \) is a \( p \)-valued function and \( g \) is a \( q \)-valued function, \( 1 \leq k \leq \infty \), then
\[
\|f \oplus g\|_k \leq q^{1/2} \|f\|_k + p^{1/2} \|g\|_k.
\]

**Proof.** When \( k = 1 \), or \( \infty \), it follows easily from the Weighted Triangular Inequality.
When \( 1 < k < \infty \),
\[
\int G(f \oplus g, pq[[0]])^k d\mathcal{H}^m = \int G(f \oplus g, pq[[0]])^{k-1} G(f \oplus g, pq[[0]]) d\mathcal{H}^m \\
\leq \int G(f \oplus g, pq[[0]])^{k-1} q^{1/2} G(f, p[[0]]) d\mathcal{H}^m + \int G(f \oplus g, pq[[0]])^{k-1} p^{1/2} G(g, q[[0]]) d\mathcal{H}^m.
\]
Now applying the Hölder inequality for parameters \( \frac{k}{k-1} \), \( k \):
\[
\int G(f \oplus g, pq[[0]])^k d\mathcal{H}^m \leq \left( \int G(f \oplus g, pq[[0]])^k \right)^{\frac{k-1}{k}} \left( \int q^{1/2} G(f, p[[0]])^k \right)^{1/k} \\
+ \left( \int G(f \oplus g, pq[[0]])^k \right)^{\frac{k-1}{k}} \left( \int p^{1/2} G(g, q[[0]])^k \right)^{1/k} \\
= \|f \oplus g\|_{k}^{k-1} q^{1/2} \|f\|_k + \|f \oplus g\|_{k}^{k-1} p^{1/2} \|g\|_k
\]
If \( \|f \oplus g\|_k \) is zero, we are done. Otherwise, divide both sides of the above inequality by \( \|f \oplus g\|_{k}^{k-1} \).

\[\blacksquare\]

**Theorem 4.4.** Suppose \( f \) is a \( p \)-valued function and \( g \) is a \( q \)-valued function, \( 0 < k < 1 \), then
\[
\|f \oplus g\|_k^k \leq q^{k/2} \|f\|_k^k + p^{k/2} \|g\|_k^k.
\]

**Proof.** Using the inequality \((a + b)^k \leq a^k + b^k, a \geq 0, b \geq 0, 0 < k < 1 \) and the Weighted Triangular Inequality, we have
\[
G(f \oplus g, pq[[0]])^k \leq (q^{1/2} G(f, p[[0]]) + p^{1/2} G(g, q[[0]]))^k \\
\leq [q^{1/2} G(f, p[[0]])]^k + [p^{1/2} G(g, q[[0]])]^k.
\]
Hence
\[
\| f \oplus g \|^k = \int G(f \oplus g, pq[0])^k d\mathcal{H}^m
\]
\[
\leq \int q^{k/2} G(f, p[0])^k d\mathcal{H}^m + \int p^{k/2} G(g, q[0])^k d\mathcal{H}^m
\]
\[
= q^{k/2} \| f \|^k + p^{k/2} \| g \|^k.
\]

5 Derivative of Multiple-Valued Functions

5.1 Definition of Derivative

For reader’s convenience, we include here the definition of derivative used in [AF].

Definition 5.1. (a) \( f \) is called affine if there are \( A_1, \ldots, A_Q \) where each \( A_i \) is an affine map from \( \mathbb{R}^m \) to \( \mathbb{R}^n \), such that
\[
f(x) = \sum_{i=1}^{Q} [A_i(x)].
\]
(b) \( f \) is called affinely approximatable at \( x_0 \) if there are affine maps \( A_1, \ldots, A_Q \) from \( \mathbb{R}^m \) to \( \mathbb{R}^n \) such that
\[
\lim_{|x-x_0| \to 0} \frac{G(f(x), \sum_{i=1}^{Q} [A_i(x)])}{|x-x_0|} = 0.
\]
(c) \( f \) is strongly affinely approximatable at \( x_0 \) if (b) holds for \( f \) at \( x_0 \) and \( A_i = A_j \) if \( A_i(x_0) = A_j(x_0) \).

Remark 5.1. If \( f \) is affinely approximatable at \( x_0 \) with \( \sum_{i=1}^{Q} [A_i] \) as its affine approximation, then obviously \( f(x_0) = \sum_{i=1}^{Q} [A_i(x_0)] \) and \( A_i(x) = A_i(x_0) + L_i(x-x_0) \) with \( L_i \in \text{Hom}(\mathbb{R}^m, \mathbb{R}^n) \).

Definition 5.2. If \( f \) is affinely approximatable at \( x_0 \), then
(a) \( \sum_{i=1}^{Q} [L_i] \in Q(\text{Hom}(\mathbb{R}^m, \mathbb{R}^n)) \), denoted by \( Df(x_0) \), is defined as the differential of \( f \) at \( x_0 \). We let \( |Df(x_0)|^2 = \sum_{i=1}^{Q} |L_i|^2 \), where \( |L| \) is the Euclidean norm of the matrix associated with any \( L \in \text{Hom}(\mathbb{R}^m, \mathbb{R}^n) \).
(b) \( \sum_{i=1}^{Q} [L_i(v)] \) is defined as the derivative of \( f \) at \( x_0 \) in the direction \( v \) and is denoted by \( D_v f \in Q \). Let \( |D_v f(x_0)|^2 = \sum_{i=1}^{Q} |L_i(v)|^2 \).

In this section we will define derivative in a more natural way, as we do in one variable calculus:
\[
f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.
\]
The major difficulty is due to the fact that “subtraction” between two elements in $Q$ is generally not well-defined except for some special cases. For example, we can define the subtraction $(\cdot)$: $Q \times I_1 \rightarrow Q$, for any $z = \sum_{i=1}^{Q} [z_i] \in Q$ and $q = Q[[q_0]] \in I_1$ by setting $z(\cdot)q = \sum_{i=1}^{Q} [z_i - q_0]$. Actually, as we will see, as long as two points are close enough with each other, we can define “subtraction” between them by matching corresponding components.

**Definition 5.3.** For $q = \sum_{i=1}^{J} k_i [q_i] \in Q \sim I_1$, with $S(q) = (J, k_1, \ldots, k_J)$, define

$$P(q;r) = \{ \sum_{j=1}^{Q} [z_j] : z_1, \ldots, z_Q \in \mathbb{R}^n \text{ with } \text{card}\{ j : z_j \in B_n(r_i) \} = k_i \}
$$

for each $i = 1, \ldots, J$ and $r < r_0 = 2^{-i} \inf\{|q_i - q_j| : 1 \leq i < j \leq J\}$.

**Remark 5.2.** Any element $z \in P(q;r)$ can be expressed as $\sum_{i=1}^{J} \sum_{j=1}^{k_i} [z_j^{(i)}]$, where $z_j^{(i)} \in B_n(r_i)$ for all fixed $i$ and $j = 1, 2, \ldots, k_i$.

Now we are able to define subtraction between $q \in Q \sim I_1$ and $z \in P(q;r)$ as follows:

**Definition 5.4.** For $q \in Q \sim I_1$ and $z \in P(q;r)$, define

$$q(\cdot)z = \sum_{i=1}^{J} \sum_{j=1}^{k_i} [q_i - z_j^{(i)}],$$

and

$$z(\cdot)p = \sum_{i=1}^{J} \sum_{j=1}^{k_i} [z_j^{(i)} - q_i].$$

We are ready to introduce the quotient-based-derivative of multiple-valued functions. Since if a multiple-valued function is differentiable (in the sense of $AF$) at some point, then it must be continuous there, we may just assume that the multiple-valued function $f$ is continuous throughout its domain.

For simplicity, let’s first consider the one-dimensional domain. We fix $x_0 \in \mathbb{R}$. Case 1: $f(x_0) \in I_1$.

**Definition 5.5.** Suppose $f(x_0) \in I_1$. $f$ is said to be differentiable at $x_0$ if the following limit exists (i.e, each component has a finite limit):

$$\lim_{x \rightarrow x_0} \frac{f(x)(\cdot)f(x_0)}{x - x_0}.$$  

If so, the limit is denoted as $f'(x_0) \in Q$, called the derivative of $f$ at $x_0$.

**Remark 5.3.** It is easy to check that this definition is equivalent to the one in Definition 5.1 (b).
Case 2: \( f(x_0) = \sum_{i=1}^{J} k_i[[q_i]] \in \mathbb{Q} \sim I_1 \), with \( S(f(x_0)) = (J, k_1, \ldots, k_J) \).

Since \( f \) is continuous at \( x_0 \), \( f(x) \in \mathbb{P}(f(x_0); r) \) when \( x \) is close enough with \( x_0 \). So the quotient

\[
\frac{f(x) - f(x_0)}{x - x_0}
\]

makes sense when \( |x - x_0| \) small enough.

**Definition 5.6.** Suppose \( f(x_0) \in \mathbb{Q} \sim I_1 \). \( f \) is said to be differentiable at \( x_0 \) if the following limit exists (i.e., each component has a finite limit):

\[
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.
\]

If so, the limit is denoted as \( f'(x_0) \in \mathbb{Q} \), called the derivative of \( f \) at \( x_0 \).

**Remark 5.4.** It is easy to check that this definition is also equivalent to the one in Definition 5.1 (b).

For higher dimensional domain, we define the directional derivative as follows:

**Definition 5.7.** Suppose \( f : \mathbb{R}^m \to \mathbb{Q} \) is continuous, fix \( x_0 \in \mathbb{R}^m \) and \( v \in \mathbb{R}^m \), the directional derivative of \( f \) at \( x_0 \) in the direction \( v \) is the following limit if it exists:

\[
L(v) := \lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t}.
\]

### 5.2 A Regular Selection Theorem

**Theorem 5.1 (GJ Proposition 5.2).** Let \( f : [a, b] \subset \mathbb{R} \to \mathbb{Q} \) be a continuous function, then there exist continuous functions \( f_1, f_2, \ldots, f_Q : [a, b] \to \mathbb{R}^n \) such that \( f = \sum_{i=1}^{Q} [f_i] \) on \([a, b] \).

**Remark 5.5.** Generally a continuous multiple-valued function does not necessarily admit a continuous decomposition as noticed by GJ. Another interesting question is whether the decomposition shown above preserves the differentiability of \( f \). The following example suggests that being "affinely approximatable" is not enough:

\[
f : [-1, 1] \to \mathbb{Q}(\mathbb{R}), f(x) = [[x]] + [[-x]].
\]

It is affinely approximatable but not strongly affinely approximatable at 0. The decompositions \( f_1 = [x], f_2 = -[x] \) certainly are not differentiable at 0. With the assumption of being strongly affinely approximatable, we can prove:

**Theorem 5.2.** If \( f : [a, b] \subset \mathbb{R} \to \mathbb{Q} \) is a continuous function and \( x_0 \in (a, b) \). Suppose \( f \) is strongly affinely approximatable at \( x_0 \), then there exist continuous functions \( f_1, f_2, \ldots, f_Q : [a, b] \to \mathbb{R}^n \) such that \( f = \sum_{i=1}^{Q} [f_i] \), and each \( f_i \) is differentiable at \( x_0 \).
Proof. Let \( f_i \) be a decomposition from Theorem 5.1. We also let \( g_1, g_2, \cdots, g_Q \in \mathbb{A}(1, n) \) such that

\[
f'(x_0) = Af(x_0) = \sum_{i=1}^{Q} [g_i] = g.
\]

Consider \( f(x_0) \) which is the same as both \( \sum_{i=1}^{Q} [f_i(x_0)] \) and \( \sum_{i=1}^{Q} [g_i(x_0)] \). Suppose \( \mathcal{S}(f(x_0)) = (J, k_1, \cdots, k_J) \), and reordering \( f_i's \) if necessary such that

\[
f(x) = \sum_{i=1}^{Q} [f_i(x)],
\]

\[
f_1(x_0) = f_2(x_0) = \cdots = f_{k_1}(x_0),
\]

\[
f_{k_1+1}(x_0) = f_{k_1+2}(x_0) = \cdots = f_{k_1+k_2}(x_0),
\]

\[
\vdots
\]

\[
f_{k_1+k_2+\cdots+k_{J-1}+1}(x_0) = f_{k_1+k_2+\cdots+k_{J-1}+2}(x_0) = \cdots = f_Q(x_0).
\]

By the assumption that \( f \) is strongly affinely approximatable at \( x_0 \), we may rewrite \( g \) as

\[
g = \sum_{i=1}^{J} k_i [g_i],
\]

for \( J \) non-crossing affine maps \( g_i \).

By continuity of \( f \), apparently we have

\[
\mathcal{G}^2(f(x), g(x)) = \sum_{i=1}^{J} \mathcal{G}^2(\sum_{j=k_{i-1}+1}^{k_i} |f_j(x)|, k_i [g_i(x)])
\]

\[
= \sum_{i=1}^{J} \sum_{j=k_{i-1}+1}^{k_i} |f_j(x) - g_i(x)|^2,
\]

when \( |x - x_0| \) small enough. The assumption that

\[
\lim_{x \to x_0} |x - a|^{-1} \mathcal{G}(f(x), g(x)) = 0
\]

shows that each \( f_j \) is differentiable at \( x_0 \) and

\[
f'_j(x_0) = g_i(x_0), j = k_1 + \cdots + k_{i-1} + 1, \cdots, k_1 + \cdots + k_{i}, i = 1, 2, \cdots, J.
\]
References

[AF] Frederick J. Almgren, Jr. Q-valued functions minimizing dirichlet’s integral and the regularity of the area-minimizing rectifiable currents up to codimension 2, World Scientific 2000

[AGS] Luigi Ambrosio, Nicola Gigli, Giuseppe Savare, Gradient flows in metric spaces and in the space of probability measures, Lectures in Mathematics, ETH, Zurich, Birkhauser, 2005

[BGP] G. Burago, M. Gromov, G. Perelman, A.D.Aleksandrov spaces with curvatures bounded below, Uspekhi Mat. Nauk, 47 (1992), pp. 3-51, 222

[DGT] Camillo De Lellis, Carlo Romano Grisanti, Paolo Tilli, Regular selections for multiple-valued functions, Annali di Matematica, 183, 79-95 (2004)

[FH] Herbert Federer, Geometric measure theory, Springer-Verlag, New York, 1969

[GJ] Jordan Goblet, A selection theory for multiple-valued functions in the sense of Almgren, Annales Academiae Scientiarum Fennicae Mathematica, Vol. 31. 2006, 297-314

[JF] Frank Jones, Lebesgue integration on euclidean space, Jones & Bartlett, 2001

[JJ] Jurgen Jost, Nonpositive curvature: geometric and analytic aspects, Lectures in Mathematics, ETH, Zurich, Birkhauser, 1997

[LC1] Chun-Chi Lin, Variational problems with multiple-valued functions and mappings, PhD Thesis, Rice University

[LC2] Chun-Chi Lin, On the regularity of two-dimensional stationary-harmonic multiple-valued functions, preprint

[LW] J.H.van Lint, R.M.Wilson, A course in combinatorics, second edition, Cambridge University Press

[MP] Pertti Mattila, Lower semicontinuity, existence and regularity theorems for elliptic variational integrals of multiple valued functions, Transactions of the American Mathematical Society, Vol. 280, No. 2, 1983

[SB] Bruce Solomon, A new proof of the closure theorem for integral currents, Indiana Univ. Math. J. 33 no.3 (1984) 393-418

[ZW1] Wei Zhu, An energy reducing flow for multiple-valued functions, arXiv:math.AP/0606478

[ZW2] Wei Zhu, A regularity theory for multiple-valued Dirichlet minimizing maps, preprint