A regularization method for the parameter estimation problem in ordinary differential equations via discrete optimal control theory.

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Summary.

We present a parameter estimation method in Ordinary Differential Equation (ODE) models. Due to complex relationships between parameters and states the use of standard techniques such as nonlinear least squares can lead to the presence of poorly identifiable parameters. Moreover, ODEs are generally approximations of the true process and the influence of misspecification on inference is often neglected. Methods based on control theory have emerged to regularize the ill posed problem of parameter estimation in this context. However, they are computationally intensive and rely on a nonparametric state estimator known to be biased in the sparse sample case. In this paper, we construct criteria based on discrete control theory which are computationally efficient and bypass the presmoothing step of signal estimation while retaining the benefits of control theory for estimation. We describe how the estimation problem can be turned into a control one and present the numerical methods used to solve it. We show convergence of our estimator in the parametric and well-specified case. For small sample sizes, numerical experiments with models containing poorly identifiable parameters and with various sources of model misspecification demonstrate the accuracy of our method. We finally test our approach on a real data example.

Keywords: Ordinary differential equation; discrete optimal control; parametric estimation; semi parametric estimation; model uncertainty

1. Introduction

We are interested by parameter estimation in Ordinary Differential Equation (ODE) models of the form

\[
\begin{align*}
\dot{x}(t) &= f(t, x(t), \theta, \vartheta(t)) \\
x(0) &= x_0
\end{align*}
\]

(1)

where the state \(x\) is in \(\mathbb{R}^d\), \(f\) is a time-dependent vector field from \([0, T] \times \mathbb{R}^d \times \Theta \times \Theta_f\) to \(\mathbb{R}^d\), \(\theta\) is a parameter that belongs to a subset \(\Theta\) of \(\mathbb{R}^p\), \(\vartheta\) is a functional parameter from \([0, T]\) to \(\Theta_f \subseteq \mathbb{R}^{d_f}\) and \(x_0\) is the initial condition that belongs to a subset \(\chi\) of \(\mathbb{R}^d\). ODEs are much used in practice as they provide an efficient framework for analyzing and predicting...
complex systems (see e.g. Fall et al. (2002); Goldbeter (1997); Mirsky et al. (2009); Wu et al. (2014)). In particular, there has recently been focus on joint use of ODE models and control theory methods for the purpose of optimal treatment design at the individual Guo and Sun (2012); Orellana (2010) and population level Agusto and Adekunle (2014); Zhang and Xu (2016).

Our aim is to estimate the true parameters, denoted $\theta^*$ and $\vartheta^*$, starting from data $y_1, \ldots, y_n$, that are realizations of an observation process for $i = 1, \ldots, n$

$$Y_i = CX^*(t_i) + \epsilon_i$$

on the observation interval $[0, T]$ where $X^* := X_{\theta^*, \vartheta^*, x^*_0}$ is the solution of (1) for $\theta = \theta^*$, $\vartheta = \vartheta^*$ and $x_0 = x^*_0$, $C$ is a $d' \times d$ observation matrix and $\epsilon_i$ is centered observation noise. That is, we want to estimate the true parameters $(\theta^*, \vartheta^*)$ starting from discrete, partial and noisy observations of the true solution $X^*$ at observation times $0 = t_1 < t_2 \cdots < t_n = T$. If there is no functional parameter $\vartheta^*$, estimation of $\theta^*$ is a standard parametric nonlinear regression problem and can be solved by classical methods such as Nonlinear Least Squares (NLS), Maximum Likelihood Estimation (MLE), or Bayesian Inference Esposito and Floudas (2000); Li et al. (2005); Rodriguez-Fernandez et al. (2006); Wu et al. (2010). However, in the case of ODE models, there is a risk of an ill-posed inverse problem Engl et al. (2009); Stuart (2010).

To explain why, let us denote as $X_{\theta, x_0}$ the solution to (1). The Fisher information matrix which controls the Cramer-Rao bound is proportional to $\mathcal{I}_n(\theta, x_0) = \sum_{i=1}^n \left(C J_{\theta, x_0}^i \right)^T C J_{\theta, x_0}^i$ where $J_{\theta, x_0}^i$ is the sensitivity matrix of $X_{\theta, x_0}$ with respect to $(\theta, x_0)$ at time $t_i$. Instabilities in estimation arise when the matrices $C J_{\theta, x_0}^i$ are badly-conditioned because in this case the inverse problem is very sensitive to any source of perturbations and the objective function (NLS or MLE criteria) is nearly flat around its minimum. This practical identifiability problem can be measured by computing the spectrum $\mu_1 \geq \cdots \geq \mu_p$ of $\mathcal{I}_n(\theta, x_0)$ and the associated condition number $\kappa(\mathcal{I}_n) = \mu_1 / \mu_p$. The problem arises in part from the observation process, i.e. the observation matrix $C$, the sparsity and location of the observation times and also from the need to estimate the nuisance parameter $x^*_0$. Complication in ODEs also arises due to the complex geometry of the manifold $\{CX_{\theta, x_0}, \theta \in \Theta, x_0 \in \chi\}$ induced by the mapping $(\theta, x_0) \mapsto CX_{\theta, x_0}$ where there can be a small number (in comparison with $p$) of important directions of variation very skewed from the original parameter axes Gutenkunst et al. (2007); Transtrum et al. (2011, 2015). This situation is termed sloppiness and leads to a regular and widespread distribution of the eigenvalues $\mu_1, \ldots, \mu_p$ with no clear one to one correspondence between the eigenvectors of $\mathcal{I}_n(\theta, x_0)$ and the original ODE parametrization. Numerous ODEs used for example in systems biology Gutenkunst et al. (2007) and neuroscience Leary et al. (2015) have been identified as sloppy. Sloppiness in ODE models has been investigated in Tonsing et al. (2014) and shown to be mainly due to the sparse and block structure of $C J_{\theta, x_0}^i$, with highly correlated entries. Sloppiness is a phenomenon due to interactions between intrinsic system properties and the experimental design. Since we cannot clearly distinguish important parameters from the others, there
is no clear mechanism to suppress irrelevant parameters in the model. Moreover, methods based on optimal experimental design to circumvent sloppiness can lead to experiments which render important ignored parameters in the model and thus reduce accuracy and limit predictive ability [White et al. (2016)]. Despite that sloppiness and practical identifiability are not rigorously the same problem [White et al. (2016)], the former often induces the latter by making some subset of parameters unidentifiable. Thus, there is a need to improve estimation methods which use the sole training data set.

Another issue in ODE parameter estimation comes from the fact that the selected model is often derived after successive simplifying assumptions and approximations. One can think of regulation networks in systems biology, where interactions are modeled by pairwise products while higher order terms and the influence of external factors (forcing functions) are unknown or neglected. Moreover, many biological processes are known to be stochastic, and the justification of deterministic modeling comes from the approximation of stochastic processes by ODE solutions see [Kurtz (1970, 1978); Gillespie (2000); Kampen (1992)]. Hence, inference of the parameters has to be done while recognising that the model is false [Kirk et al. (2016); Brynjarsdottir and O'Hagan (2014)].

In this work, we propose a new estimation procedure to address these challenges, based on an approximate solution of the original ODE. The use of approximate solutions for statistical inference, such as the classical two-step approaches [Brunel (2008); Brunel and D'Alche-Buc (2014); Gugushvili and Klaassen (2011); Liang et al. (2010); Varah (1982); Dattner (2013), Generalized Profiling (GP) [G. Hooker and Earn (2011); Ramsay et al. (2007)] or even in a Bayesian framework [Chkrebtii et al. (2016); Jaeger and Lambert (2011)], has already proven to be useful for regularizing the inverse problem of parameter estimation.

Our proposed method here is seen as an attempt at improving the methods proposed in [Brunel and Clairon (2015); Clairon and Brunel (2018, 2017)], where an approximation \( X_{\theta,x_0,u} \) is a solution of the perturbed ODE \( \dot{x}(t) = f(t, x(t), \theta) + Bu(t) \) and where the perturbation \( t \mapsto Bu(t) \) captures different sources of model misspecification. After a pre-smoothing step to obtain a nonparametric curve estimator \( \hat{Y} \), the parametric estimator \( (\hat{\theta}, \hat{x}_0) \) is then defined as the minimizer of the cost \( C_\lambda(\theta, x_0, u) = \| CX_{\theta,x_0,u} - \hat{Y} \|_{L^2}^2 + \lambda \| u \|_{L^2}^2 \) profiled on the possible perturbations \( u: (\hat{\theta}, \hat{x}_0) = \arg \min_{(\theta,x_0)} S(\theta, x_0, u) \), where \( S(\theta, x_0) = \min_u C_\lambda(\theta, x_0, u) \). This estimator, called the Tracking Estimator (TE), is thus defined as the parameter which needs the smallest perturbation \( u \) in order to track \( \hat{Y} \), the balance between the two contrary objectives of data fidelity (i.e. \( \| CX_{\theta,x_0,u} - \hat{Y} \|_{L^2}^2 \)) and original model fidelity (i.e. \( \| u \|_{L^2}^2 \)) is done through the choice of an hyperparameter value \( \lambda > 0 \). For each value \( (\theta, x_0) \), the optimal control problem \( \min_u C_\lambda(\theta, x_0, u) \) is solved by using the Pontryagin maximum principle [Pontryagin et al. (1962)] in the nonlinear case [Clairon and Brunel (2017)] and the linear-quadratic theory [Sontag (1998)] for linear models [Brunel and Clairon (2015); Clairon and Brunel (2018)]. In comparison with GP and NLS, the TE generally has a lower variance and mean square error with the


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difference in performance even more marked in the presence of model misspecification. In the parametric case and for well-specified models, the TE has been proven to be consistent with a $\sqrt{n}$-convergence rate under very mild model regularity conditions and provided $\lambda > \lambda_1$, with $\lambda_1$ a positive model dependent bound. Moreover, an attractive feature of the tracking framework is the seamless estimation of finite-dimensional and time-varying parameters. The estimation of $t \mapsto \vartheta(t)$ in $\dot{x} = f(t, x, \theta, \vartheta)$ can be turned into an optimal control problem and our estimator $\hat{\vartheta}$ is a by-product of $\theta^*$ estimation which does not require the use of standard approximations such as sieves or basis expansions [Liang et al. (2010); Xue et al. (2010); G. Hooker and Earn (2011); Wang et al. (2014)]. However, two main limitations for the method given in Clairon and Brunel (2017). First, the computational time: solving the optimal control problem by using the Pontryagin maximum principle leads to a boundary value problem (BVP) for each new $(\theta, x_0)$ value and $x_0^*$ has to be estimated as nuisance parameter. Second, the method requires a nonparametric estimator $\hat{Y}$. In the sparse data case, the reconstruction $\hat{Y}$ can be biased and this nonparametric bias can then be spread to the parametric estimation. Here, while we keep the same formal approach as in Clairon and Brunel (2017), we solve the related optimal control problem by relying on discrete control theory and a numerical method inspired by Cimen and Banks (2004b). This allows us to construct an estimation method which:

(a) replaces the BVP by a sequence of finite difference equations, our procedure can be then applied to ODE systems of higher dimension than in Clairon and Brunel (2017);
(b) removes the pre-smoothing step, we can deal with sparse data cases which are consistent with most real observation framework;
(c) gives a consistent estimator with parametric convergence rate with only a strictly positive condition on the hyperparameter, i.e $\lambda > 0$;
(d) can be easily adapted to avoid estimation of $x_0^*$ if it is not required.

In order to define our estimator, we present in the next section the optimal control problem required to introduce our functional criteria which is again a profiled cost $S_n$. We also describe how a semi-parametric estimation problem can be turned into an optimal control problem similar to that used for parametric estimation. In Subsection 2.3, we derive a tractable form for $S_n$ and describe the related numerical procedure, which is based for linear models on discrete linear-quadratic theory and for nonlinear models on the work of Cimen and Banks (2004b,a). We present in Section 3 sufficient conditions to ensure $S_n$ is well-defined on the parameter space as well as consistency with $\sqrt{n}$-convergence in the parametric case and for well-specified models. In Section 4, we use Monte Carlo experiments in order to compare the Tracking, Nonlinear Least Squares and Generalized Profiling estimators on ODE examples from chemistry and biology with both well-specified and misspecified models. This section is concluded by simulations where we perform the joint estimation of the finite dimensional and time-varying parameters $\theta$ and $\vartheta$. In Section 5, we consider parameter estimation with real data in a model used to study microbiotal population evolution.
2. Model and methodology

We recall the aim of this work is to estimate \((\theta^*, \vartheta^*)\) from the data \(Y = \{y_1, \ldots, y_n\}\) defined as the minimizer of functional criteria \(S_n\). First, we derive \(S_n\) in the parametric case where there is no functional parameter \(\vartheta^*\).

2.1. Formal parametric estimator definition

We denote by \(X_{\theta, x_0}\) the solution of the Initial Value Problem (IVP):

\[
\begin{align*}
\dot{x}(t) &= f(t, x(t), \theta) \\
x(0) &= x_0. 
\end{align*}
\]  

(3)

First, we need to reformulate the model \(3\) into a pseudo-linear form:

\[
\begin{align*}
\dot{x}(t) &= A_{\theta}(x(t), t)x(t) \\
x(0) &= x_0. 
\end{align*}
\]  

(4)

This formulation is crucial for solving in a computationally efficient way the optimal control problem defining our estimator. Of course, linear models already fit in this formalism with \(A_{\theta}(t) = A_{\theta}(x(t), t)\). For nonlinear models, the pseudo-linear representation is not unique but always exists [Cimen and Banks (2004)]. Now, we introduce the solution \(X_{\theta, x_0, u}\) of the perturbed ODE:

\[
\begin{align*}
\dot{x}(t) &= A_{\theta}(x(t), t)x(t) + Bu(t) \\
x(0) &= x_0. 
\end{align*}
\]  

(5)

where the function \(t \mapsto Bu(t)\) is a linear perturbation, \(B\) is a \(d \times d_u\) matrix and \(u\) is in \(L^2([0, T], \mathbb{R}^{d_u})\). To proceed to parametric estimation, we consider a discretized version of the perturbed ODE \(5\). The discretization will be made at \(m + 1\) time points \(\{t^d_j\}_{0 \leq j \leq m}\) with \(t^d_0 = 0\) and \(t^d_m = T\). Letting \(\Delta_j = t^d_{j+1} - t^d_j\) being the mesh size between two discretization time-points and \(u = (u_0, \ldots, u_{m-1})\) the set of discrete values taken by the control at each time step, the discretized version is:

\[
\begin{align*}
x(t^d_{j+1}) &= \left( I_d + \Delta_j A_{\theta}(x(t^d_j), t^d_j) \right) x(t^d_j) + B\Delta_j u_j \\
x(0) &= x_0. 
\end{align*}
\]  

(6)

The set of discretization time-point has to contain the observation time points i.e. \(\{t_i\}_{0 \leq i \leq n} \subset \{t^d_j\}_{0 \leq j \leq m}\) but can be bigger, this is an important feature of the discretization scheme which allows us to accurately estimate \(X_{\theta, x_0, u}\) even when the observations are sparse on \([0, T]\). We denote:

- \(X^d_{\theta, x_0, u}(t^d_j)\), the solution of \(5\) for the parameter \(\theta\), initial condition \(x_0\) and the perturbation \(u\) at time \(t^d_j\).
Let $w_j = 1\{\exists t_i, s.t., t_i = t_j^d\}$ i.e. $w_i$ is equal to 1 if $t_j^d$ corresponds to an observation time $t_i$, otherwise $w_j = 0$.

- $y_j$ is equal to $y_i$ if $t_j^d$ corresponds to the observation time $t_i$ and 0 otherwise.

The weights $w_j$ and the set of extended data $\{y_i\}$ are introduced to have a vector of observation which has the same length as the discretization grid $\{t_j^d\}_{0 \leq j \leq m}$. Now, we can introduce the discretized cost we want to minimize:

$$C_T^d(Y; \theta, x_0, u, U) = \sum_{i=0}^{n} \Delta_i \left\| CX_{\theta,x_0,u}(t_i) - y_i \right\|_2^2 + \sum_{j=0}^{m-1} \Delta_j u_j^T U u_j$$

$$= \left\| CX_{\theta,x_0,u}(t_n) - y_n \right\|_2^2 + \sum_{j=0}^{m-1} \Delta_j \left( \left\| CX_{\theta,x_0,u}(t_j^d) - y_j \right\|_2^2 w_j + u_j^T U u_j \right)$$

and for each $(\theta, x_0)$ in $\Theta \times \chi$, the profiled cost:

$$S_n(Y; \theta, x_0, U) := \inf_{u \in L_u} C_T^d(Y; \theta, x_0, u, U)$$

where $L_u$ is the set of admissible perturbations defined as the set of controls generating trajectories bounded on $[0, T]$. Here $U$ is a symmetric definite positive matrix used as a weighting parameter balancing the amount of model and data fidelity. Similarly to Clairon and Brunel (2017), the tracking estimator (TE) is defined as:

$$\left( \hat{\theta}^T, \hat{x}_0^T \right) := \arg \min_{(\theta, x_0) \in \Theta \times \chi} S_n(Y; \theta, x_0, U),$$

i.e. as the parameter that gives the closest trajectory $CX_{\theta,x_0,u}^d$ to the observed data on $[0, T]$, while allowing a small divergence from (8). To compute $S_n$ in practice we need to solve the optimization problem:

$$\min_u C_T^d(Y; \theta, x_0, u, U)$$

such that $x(t_{j+1}^d) = \left( I_d + \Delta_j A_\theta(x(t_j^d), t_j^d) \right) x(t_i) + B \Delta_j u_j$

and $x(0) = x_0$.

The problem (10) is an optimal control one belonging to the subclass of tracking problems where the aim is to find the smallest control possible to apply to a given dynamical system in order to track a signal. For linear models, these problems have been efficiently solved as they fit into the framework of discrete linear-quadratic problems, which ensures the existence and uniqueness of the solution and gives a computationally efficient way to find it. For non-linear models, Cimen and Banks (2004b) proposes an iterative method to solve continuous time tracking problems, the main idea being to replace the original problem by a sequence of linear-quadratic ones. We will use the same method adapted to discrete models, but first, in the next subsection, we explain how the estimation of a time-varying parameter $\theta : [0, T] \rightarrow \mathbb{R}^d$ in ODEs $\dot{x}(t) = A_\theta(x(t), \vartheta(t), t)x$ is straightforward within our framework.
2.2. Semi-parametric estimation
For this, let us introduce the extended state \( x_e = (x, z_1, z_2) \) in \( \mathbb{R}^{d+2d_f} \), the extended pseudo-linear representation:

\[
A^e_\theta(x_e(t), t) = \begin{pmatrix}
A_\theta(x(t), z_1(t), t) & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

and the perturbed solution \( X^e_{\theta,x_0,u} \) of the parametric extended ODE:

\[
\begin{cases}
\dot{x}(t) = A^e_\theta(x(t), t)x_e(t) + B_{ext}u(t) \\
x_e(0) = x_0^e
\end{cases}
\]

with

\[
B_{ext} = \begin{pmatrix}
I_d & 0_{d,d_f} \\
0_{d_f,d} & 0_{d_f,d} \\
0_{d_f,d} & I_{d_f}
\end{pmatrix}
\]

Here, \( u \) is split into two parts, \( u(t) = (u_1(t), u_2(t)) \), and \( X^e_{\theta,x_0,u} \) is solution of

\[
\begin{align*}
\dot{x} &= A(t, x, z_1, \theta)x + u_1 \\
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= u_2.
\end{align*}
\]

One can see that \( z_1 \) plays the role of \( \vartheta \), and \( z_2 \) of \( \dot{\vartheta} \). If we get a state variable estimator \( \hat{X}^e \), which is the case in our method, \( \hat{X}^e \) being obtained as a byproduct of \( \theta^* \) estimation, then we define \( \hat{\vartheta} = \hat{z}_1 \). Let us introduce \( U = \begin{pmatrix} \lambda_1 I_d & 0_{d,d_f} \\
0_{d_f,d} & \lambda_2 I_{d_f} \end{pmatrix} \) and the cost

\[
C^d_{\vartheta}(Y; \theta, x_0^e, u, U) = \sum_{i=0}^{n} \Delta_i \left\| CX^e_{\theta,x_0,u}(t_i) - y_i \right\|_2^2 + \sum_{j=0}^{m-1} \Delta_j \left( \lambda_1 u_{1,j}^T u_{1,j} + \lambda_2 u_{2,j}^T u_{2,j} \right) \]

\[
= \sum_{i=0}^{n} \Delta_i \left\| CX^e_{\theta,x_0,u}(t_i) - y_i \right\|_2^2 + \sum_{j=0}^{m-1} \Delta_j \left( \lambda_1 u_{1,j}^T u_{1,j} + \lambda_2 z_{1,j}^T z_{1,j} \right).
\]

Here \( u_{1,j}^T u_{1,j} \) is used to quantify model discrepancy as in the parametric case, and the last term \( \lambda_2 z_{1,j}^T z_{1,j} \) is the standard penalty used for functional estimation. Thus, a good choice of hyperparameter for cost \((13)\) would be a large value for \( \lambda_1 \) (in order to select a small \( u \)), and \( \lambda_2 \) tending to 0 when the sample size \( n \) grows, as for standard nonparametric estimation.

2.3. Tractable form for \( S_n \)
In this subsection, we derive a tractable expression for \( S_n \). First, we deal with linear models then we extend the derived method to nonlinear models by following Cimen and Banks (2004b).
2.3.1. Linear models

Here, we focus on the linear case i.e. when $A_{\theta}(t) = A_{\theta}(x, t)$ in (3). For a given initial condition $x_0$, linear-quadratic theory ensures the existence and uniqueness of the optimal control $u_{\theta,x_0}^d = \arg\min_{u \in L_u} C_{\theta,x_0}^d(Y; \theta, x_0, u, U)$ and that $\inf_{u \in L_u} C_{\theta,x_0}^d(Y; \theta, x_0, u, U)$ can be computed by solving a discrete final value problem, denoted the Riccati equation. The formal computational details are left in supplementary materials (Section 2).

**Proposition 1.** For a given $(\theta, x_0)$ in $\Theta \times \chi$, the profiled cost value $S_n(Y; \theta, x_0, U)$ is equal to:

$$S_n(Y; \theta, x_0, U) = x_0^T R_{\theta,0}^d x_0 + 2h_{\theta,0}^d(Y)^T x_0 + \Delta_m y_m y_m^T + \sum_{j=0}^{n-1} \Delta_j \left( y_j^T y_j - h_{\theta,j+1}^d(Y)^T B G(R_{\theta,j+1}^d) B^T h_{\theta,j+1}^d(Y) \right)$$

with $G(R_{\theta,j+1}^d) := \left[ U + \Delta_j B^T R_{\theta,j+1}^d B \right]^{-1}$ and $(R_{\theta,j}^d, h_{\theta,j}^d(Y))$ for $1 \leq j \leq m$, the solution of the discrete Riccati equation:

$$R_{\theta,j}^d = R_{\theta,j+1}^d + \Delta_j w_j C^T C + \Delta_j \left( R_{\theta,j+1}^d A_{\theta}(t_j^d) + A_{\theta}(t_j^d)^T R_{\theta,j+1}^d \right)$$
$$+ \Delta_j^2 A_{\theta}(t_j^d)^T R_{\theta,j+1}^d A_{\theta}(t_j^d)$$
$$- \Delta_j (I_d + \Delta_j A_{\theta}(t_j^d)^T) R_{\theta,j+1}^d B G(R_{\theta,j+1}^d) B^T R_{\theta,j+1}^d (I_d + \Delta_j A_{\theta}(t_j^d))$$

$$h_{\theta,j}^d(Y) = h_{\theta,j+1}^d(Y) - \Delta_j y_j C^T y_j + \Delta_j A_{\theta}(t_j^d)^T h_{\theta,j+1}^d(Y)$$
$$- \Delta_j (I_d + \Delta_j A_{\theta}(t_j^d)^T) R_{\theta,j+1}^d B G(R_{\theta,j+1}^d) B^T h_{\theta,j+1}^d(Y)$$

with final condition $(R_{\theta,m}^d, h_{\theta,m}^d(Y)) = (\Delta_m C^T C, -\Delta_m C^T y_m)$. Moreover, the control $u_{\theta,x_0}^d$ which minimizes the cost (7) is unique and equal to:

$$u_{\theta,x_0}^d = -G(R_{\theta,j+1}^d) B^T R_{\theta,j+1}^d \left( I_d + \Delta_j A_{\theta}(t_j^d) \right) X_{\theta,x_0}^d(t_j^d) + h_{\theta,j+1}^d(Y)$$

where $X_{\theta,x_0}^d$ is the optimal trajectory, i.e. the solution of the initial value problem

$$X_{\theta,x_0}^d(t_{j+1}) = \left( I_d + \Delta_j A_{\theta}(t_j^d) \right) X_{\theta,x_0}^d(t_j^d)$$
$$- \Delta_j B G(R_{\theta,j+1}^d) B^T \left( R_{\theta,j+1}^d \left( I_d + \Delta_j A_{\theta}(t_j^d) \right) X_{\theta,x_0}^d(t_j^d) + h_{\theta,j+1}^d(Y) \right)$$
$$X_{\theta,x_0}^d(0) = x_0.$$

2.3.2. Non-linear models

Here, we adapt the method proposed by Cimen and Banks (2004b) to solve the tracking problem for discrete time models. The outline of the method is as follows. We replace the original problem (10) by a recursive sequence of control problems, with iteration $l$ defined
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by

\[
\min_{u} C_{T}^{d,l}(Y; \theta, x_0, u, U) := \Delta_m \left\| CX_{\theta,x_0,u}(t_m) - y \right\|_2^2 + \sum_{j=0}^{m-1} \Delta_j \left( \left\| CX_{\theta,x_0,u}(t_j^d) - y \right\|_2^2 w_j + u^T_j u_j \right) \tag{18}
\]

such that \(x(t_{j+1}) = (I_d + \triangle_j A_{\theta}(t_j^d)) x_j(t_j^d) + \triangle_j B_{\sigma} u_j \) and \(x(0) = x_0 \)

where \( A_{\theta}(t_j^d) := A_{\theta}(X_{\theta,x_0}^{l-1}(t_j^d), t_j^d) \) and \( A_{\theta}(t_j^d) := A_{\theta}(x_0, t_j^d) \). Here \( X_{\theta,x_0}^{l-1} \) is the optimal trajectory corresponding the optimal control problem \(\text{[1]}\) at iteration \(l-1\). Thus, for each \(l\) the problem \(\text{[1]}\) fits into the framework of our previous subsection, and for each \(l\), we have access to a solution of the Riccati equation \((R_{\theta}^{d,l}, h_{\theta}^{d,l}(Y))\), an optimal control \(u_{\theta,x_0}^{d,l}\), an optimal trajectory \(X_{\theta,x_0}^{d,l}\), and a profiled cost value \(S_n^{l}(Y; \theta, x_0, U)\). Moreover, the sequences \(\{R_{\theta}^{d,l}, h_{\theta}^{d,l}(Y)\}_{l \in \mathbb{N}}, \{u_{\theta,x_0}^{d,l}\}_{l \in \mathbb{N}}, \{X_{\theta,x_0}^{d,l}\}_{l \in \mathbb{N}}\) and \(\{S_n^{l}(Y; \theta, x_0, U)\}_{l \in \mathbb{N}}\) are uniformly convergent in \(l\) \cite{Cimen and Banks 2004a,b}. Thus, we can propose the following algorithm to compute \((R_{\theta}^{d,l}, h_{\theta}^{d,l}(Y))\), \(u_{\theta,x_0}^{d,l}, X_{\theta,x_0}^{d,l}\) and \(S_n(Y; \theta, x_0, U)\).

(a) Initialization phase: \(X_{\theta,x_0}^{d,0}(t_j^d) = x_0 \) and \(A_{\theta}^{0}(t_j^d) = A_{\theta}(x_0, t_j^d) \) for all \(j \in [0, m]\).

(b) At iteration \(l\): use Proposition \(\text{[1]}\) to obtain \((R_{\theta}^{d,l}, h_{\theta}^{d,l}(Y))\), \(u_{\theta,x_0}^{d,l}, X_{\theta,x_0}^{d,l}, S_n^{l}(Y; \theta, x_0, U)\).

(c) If \(\sum_{j=1}^{m} \left\| X_{\theta,x_0}^{d,l}(t_j) - X_{\theta,x_0}^{d,l-1}(t_j) \right\|_2^2 < \varepsilon_1 \) and

\[
\left| S_n^{l}(Y; \theta, x_0, U) - S_n^{l-1}(Y; \theta, x_0, U) \right| < \varepsilon_2, \tag{19}
\]

with \((\varepsilon_1, \varepsilon_2)\) two strictly positive constants, then step 4; otherwise return to step 2.

(d) Set \((R_{\theta}^{d,l}, h_{\theta}^{d,l}(Y)) = (R_{\theta}^{d,l}, h_{\theta}^{d,l}(Y))\), \(u_{\theta,x_0}^{d,l} = u_{\theta,x_0}^{d,l}, X_{\theta,x_0}^{d,l} = X_{\theta,x_0}^{d,l}\) and \(S_n(Y; \theta, x_0, U) = S_n^{l}(Y; \theta, x_0, U)\).

In this section, we have seen how to compute \(S_n\) in practice. However, before looking for its minimum \((\hat{\theta}^*, \hat{x}_0^T)\) by a numerical optimization method, we need to prove that the function \((\theta, x_0) \mapsto S_n(Y; \theta, x_0, U)\) is well defined on a parameter space containing \((\theta^*, x_0^*)\) and study its regularity. This achieved in the next section, where we also derive the conditions under which \((\hat{\theta}^*, \hat{x}_0^T)\) is consistent and even asymptotically normal with a \(\sqrt{n}\)-rate convergence rate. Before that, we describe an extension of our method which allows us to avoid estimation of \(x_0^*\) and reduces the dimension of the optimization problem \(\text{[1]}\), and which is expected to be more computationally efficient.
2.4. Profiling on $x_0$

We introduce a new estimator $\hat{\theta}^{T,CI}$ of the true parameter $\theta^*$:

$$\hat{\theta}^{T,CI} = \arg\min_{\theta \in \Theta} S_n^{CI}(Y; \theta, U)$$

We now profile the cost function on $x_0$ in addition to $u$. In equation (14), one can see that $S_n(Y; \theta, x_0, U)$ is a quadratic form with respect to $x_0$, hence the profiling is straightforward. Interestingly, the formal computation used to derive $S_n(Y; \theta, U)$ follows the same step as the deterministic Kalman Filter state estimator derivation Sontag (1998). We derive the following expression for $S_n^{CI}(Y; \theta, U)$ in the linear case:

**Proposition 2.** For a given $\theta$ in $\Theta$, $S_n^{CI}(Y; \theta, U)$ is equal to:

$$S_n^{CI}(Y; \theta, U) = -h^d_{\theta,0}(Y) + \Delta_mY_m^T \Delta_m + \sum_{j=0}^{m-1} \Delta_j \left( Y^T_j Y_j - h^d_{\theta,j+1}(Y)^T B G R^d_{\theta,j+1} B^T h^d_{\theta,j+1}(Y) \right).$$

where $(R^d_{\theta}, h^d_{\theta}(Y))$, $\overline{u^d_{\theta}}$ are given by equations (13), (10) and $\overline{X^d_{\theta}(0)}$ is given by equation (17) but with initial condition $X^d_{\theta}(0) = -\left( R^d_{\theta} \right)^{-1} h_{\theta,0}$.

In the non-linear case, the algorithm has to be adapted as follows:

(a) Initialization phase: $X^d_{\theta}(t^d_j) = x_0^r$ and $A^d_{\theta}(t^d_j) = A_{\theta}(x_0^r, t^d_j)$ for all $j \in [0, m]$ where $x_0^r$ is an arbitrary starting point.

(b) At iteration $l$: use Proposition 1 to obtain $(R^d_{\theta,l}, h^d_{\theta,l}(Y))$ and $\overline{u^d_{\theta,l}}$, then Proposition 2 to obtain $\overline{X^d_{\theta,l}}$ and $S_n^{CI,l}(Y; \theta, U)$.

(c) If $\sum_{j=1}^{m} \left\| X^d_{\theta,l}(t^d_j) - X^d_{\theta,l-1}(t^d_j) \right\|_2 < \varepsilon_1$ and $\left| S_n^{CI,l}(Y; \theta, U) - S_n^{CI,l-1}(Y; \theta, U) \right| < \varepsilon_2$, (22)

then step 4; otherwise return to step 2.

(d) Set $(R^d_{\theta}, h^d_{\theta}(Y)) = (R^d_{\theta,l}, h^d_{\theta,l}(Y))$, $\overline{u^d_{\theta}} = \overline{u^d_{\theta,l}}$, $\overline{X^d_{\theta}} = \overline{X^d_{\theta,l}}$ and $S_n^{CI}(Y; \theta, U) = S_n^{CI,l}(Y; \theta, U)$.

**Remark 3.** The state extension required for semi-parametric estimation involves the addition of new initial conditions $(\hat{\theta}(0), \dot{\hat{\theta}}(0))$ which need to be estimated in Clairon and Brunel (2013). Interestingly here, since we profile on $x_0$, our approach does not add nuisance parameters to estimate. However, we still need to consider a model of larger dimension than the original.
3. Theoretical analysis

3.1. Existence and regularity of $S_n$ and $S^{CI}_n$

First, we introduce the required conditions to ensure the existence and regularity of $\theta \mapsto (R^{d,l}_\theta, h^{d,l}_\theta(Y))$, $(\theta, x_0) \mapsto S^l_n(Y; \theta, x_0, U)$ and $\theta \mapsto S^{CI,l}_n(Y; \theta, U)$ for each $l \in \mathbb{N}$:

**Condition 1**: For all $t \in [0, T]$ and for all $\theta \in \Theta$, $x \mapsto A_\theta(x, t)$ has a compact support.

**Condition 2**: For all $x \in \Lambda$, $\theta \mapsto A_\theta(x, \cdot)$ is continuous on $\Theta$ and $\forall \theta \in \Theta$, $(x, t) \mapsto A_\theta(x, t)$ is continuous on $\Lambda \times [0, T]$.

Condition 1 ensures the existence of a unique bounded function $X_{\theta, x_0}$ defined on $[0, T]$ for all $(\theta \times x_0)$ in $\Theta \times \Lambda$. In practice, the tracked signal is always bounded, thus it is legitimate to focus on bounded solutions. Moreover, let us recall that, for any function $f$, we can construct a function $\tilde{f}$ with the same level of smoothness as $f$ such that for all $x \in \Lambda$, $\tilde{f}(x) = f(x)$ and for all $x$ such that $d(x, A) > \epsilon$, $\tilde{f}(x) = 0$ for any $\epsilon > 0$. Hence, any matrix $A_\theta(x, \cdot)$ can be replaced by a counterpart satisfying Condition 1 with a compact support restricted to relevant values for the state variables. From Condition 2 it is straightforward to derive by induction $\theta \mapsto (R^{d,l}_\theta, h^{d,l}_\theta(Y))$ continuity with respect to $\theta$.

In order to derive the same property for $(\theta, x_0) \mapsto S^l_n(Y; \theta, x_0, U)$ and $\theta \mapsto S^{CI,l}_n(Y; \theta, U)$, we need to ensure boundedness of $(R^{d,l}_\theta, h^{d,l}_\theta(Y))$ as well as $R^{d,l}_\theta$ invertibility for $S^{CI}_n$ which is why we need to introduce Condition 3a. The proofs are left in Section 3 of supplementary materials.

**Proposition 4.** Under conditions 1-2 for each $l \in \mathbb{N}$, $(\theta, x_0) \mapsto S^l_n(Y; \theta, x_0, U)$ is continuous on $\Theta \times \Lambda$. Moreover if $R^{d,l}_{\theta,0}$ is nonsingular then $\theta \mapsto S^{CI,l}_n(Y; \theta, U)$ is continuous on $\Theta$.

The $R^{d,l}_{\theta,0}$ nonsingularity condition is somewhat ad-hoc, so we present a necessary and sufficient testable criterion:

**Proposition 5.** Given $\theta \in \Theta$ and $l \in \mathbb{N}$, $R^{d,l}_{\theta,0}$ is invertible if and only the matrix

$$
O^{d,l}_{\theta}(T) = C^T C + \sum_{i=1}^{n-1} \prod_{j=0}^{i-1} \left( I_d + \Delta A_\theta(t^d_j, X_{\theta,x_0}^{d,j-1}(t^d_j)) \right) C^T C \prod_{j=0}^{i-1} \left( I_d + \Delta A_\theta(t^d_j, X_{\theta,x_0}^{d,j-1}(t^d_j)) \right)
$$

is invertible.

**Remark 6.** When the system is totally observed, $C$ is of full rank and the matrix $O^{d,l}_{\theta}(T)$ is always nonsingular for all $\theta$ in $\Theta$ and all $l \in \mathbb{N}$. Intuitively in the general case, $O^{d,l}_{\theta}(T)$ is invertible when the matrix $CA_\theta$ has a sufficient number of non-zeros entries among those corresponding to the unobserved section of the system. That is, $O^{d,l}_{\theta}(T)$ is invertible when the observed state variables give enough information on the whole system. This corresponds to the notion of observability in control theory (see [Sontag, 1998]).
with $X$.

In order to proceed to asymptotic analysis, we introduce the asymptotic counterpart of $3.2.1$. Required conditions

Instead of the algorithm inspired by Cimen and Banks (2004b) and we can take $C_X$ this asymptotic framework, we have access to the true continuous signal $t \rightarrow Y^*(t) = CX_{\theta^*, x_0}(t)$ and so we can define the continuous cost:

$$C^*_T(\theta, x_0, u, U) = d^2 \sigma^2 + \int_0^T \left( \left\| CX^l_{\theta, x_0, u}(t) - Y^*(t) \right\|_2^2 + u(t)^TUu(t) \right) dt$$ (24)

with $X^l_{\theta, x_0, u}$ the solution of the ODE

$$\begin{cases}
\dot{X}^l_{\theta, x_0, u}(t) = A_\theta(X^l_{\theta, x_0, u}(t), t)X^l_{\theta, x_0, u}(t) + Bu(t) \\
X^l_{\theta, x_0, u}(0) = x_0.
\end{cases}$$ (25)

As in the discrete case, we introduce the cost sequentially profiled on $u$, $S^l(\theta, x_0, U) := \inf_u C^*_T(\theta, x_0, u, U)$, and we derive a closed-form expression

$$S^l(\theta, x_0, U) = x^T_0R_{\theta}(0)x_0 + 2x^T_0h_{\theta}(0) + \int_0^T \left( Y^*(t)^TY^*(t) + d^2 \sigma^2 - h_{\theta}(t)TBU^{-1}B^T h_{\theta}(t) \right) dt$$
by introducing \((R_\theta^l, h_\theta^l)\), the continuous time Riccati equation, and \(\overline{X}_{\theta,x_0}^l\) the optimal trajectory given by the ODE

\[
\begin{align*}
\dot{R}_\theta^l(t) &= -CTC - A_\theta(X_{\theta,x_0}^{l-1}(t), t)^T R_\theta^l(t) - R_\theta^l(t) A_\theta(X_{\theta,x_0}^{l-1}(t), t) \\
&\quad + R_\theta^l(t) BU^{-1} B^T R_\theta^l(t) \\
\dot{h}_\theta^l(t) &= CTY_\Lambda(\theta,x_0,t) - A_\theta(X_{\theta,x_0}^{l-1}(t), t)^T h_\theta^l(t) + R_\theta^l(t) BU^{-1} B^T h_\theta^l(t) \\
X_{\theta,x_0}^l(t) &= A_\theta(X_{\theta,x_0}^{l-1}(t), t)X_{\theta,x_0}^l(t) - BU^{-1} B^T(R_\theta^l(t)X_{\theta,x_0}^l(t) + h_\theta^l(t)) \\
\left(R_\theta^l(T), h_\theta^l(T), X_{\theta,x_0}^l(0)\right) &= \left(0_{d,d}, 0_{d,1}, x_0\right).
\end{align*}
\]

The previous functions correspond to \(n \to \infty\), but for non-linear models we need to consider asymptotics in \(l\) also, so we introduce the asymptotic cost

\[
C_T^\infty(\theta,x_0,u,U) = d^2 \sigma^2 + \int_0^T \left(\|CX_{\theta,x_0,u}^\infty(t) - Y_\Lambda^T(t)\|^2 + u(t)^T U u(t)\right) dt
\]

associated with ODE

\[
\begin{align*}
X_{\theta,x_0,u}^\infty &= A_\theta(X_{\theta,x_0,u}^\infty(t), t)X_{\theta,x_0,u}^\infty + Bu(t) \\
X_{\theta,x_0,u}^\infty(0) &= x_0,
\end{align*}
\]

together with the profiled cost value \(S^\infty(\theta,x_0,U) := \inf_u C_T^\infty(\theta,x_0,u,U)\). Again, thanks to the continuous LQ-theory, we characterize \(S^\infty\) with \(R_\theta^l, h_\theta^l, X_{\theta,x_0}^l\), the solution of (26) where \(l\) and \(l-1\) are replaced by \(\infty\). Now that \(S^l, R_\theta^l, h_\theta^l\) and \(S^\infty, R_\theta^\infty, h_\theta^\infty\) have been introduced, we can present the conditions required to derive \(\left(\hat{\theta}^T, \hat{x}_0^T\right)\) consistency with \(\sqrt{n}\)-convergence rate.

**Condition 3**: Matrix \(B\) has independent columns.

**Condition 4**: The true parameters \((\theta^*, x_0^*)\) belong to the interior of \(\Theta \times \chi\).

**Condition 5**: The solution \(X_{\theta,x_0}\) of (11) is such that if \(CX_{\theta,x_0}(t) = CX_{\theta^*,x_0}(t)\) for all \(t \in [0, T]\) then \((\theta, x_0) = (\theta^*, x_0^*)\).

**Condition 6**: For all \(x \in \Lambda, \theta \mapsto A_{\theta}(x,.)\) is differentiable on \(\Theta\), for all \(\theta \in \Theta, (x,t) \mapsto \frac{\partial A_{\theta}(x,t)}{\partial \theta}\) is continuous on \(\Lambda \times [0,T]\).

**Condition 7**: For all \(x \in \Lambda, \theta \mapsto A_{\theta}(x,.)\) is twice differentiable on \(\Theta\), for all \(\theta \in \Theta, (x,t) \mapsto \frac{\partial^2 A_{\theta}(x,t)}{\partial^2 \theta}\) is continuous on \(\Lambda \times [0,T]\).

**Condition 8**: The asymptotic hessian matrix \(\frac{\partial^2 S^\infty(\theta^*,x_0^*)}{\partial x^2(\theta,x_0)}\) is nonsingular.

Condition 3 is required for the uniform convergence of \(R_\theta^l, h_\theta^l\) to \(R_\theta^\infty, h_\theta^\infty\) and \(S^l\) to \(S^\infty\). Conditions 4 and 5 ensure \((\theta^*, x_0^*)\) constitutes a well-separated minimum of \(S^\infty\) and conditions 6 to 8 guarantee that \(\frac{\partial^2 S^\infty(\theta^*,x_0^*,U)}{\partial^2(x,x_0)}\) exists and the asymptotic variance-covariance of \(\theta^*\) is non singular.
Remark 7. Condition 3 on $B$ ranking is not as restrictive as it seems. Let us assume $\text{rank}(B) = r'$ with $r' < d_u$ and consider the singular value decomposition

$$B = V_1 \begin{pmatrix} \sigma_1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \cdots & \vdots \\ \vdots & \ddots & \sigma_{r'} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \cdots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} V_2^*$$

and the control reparametrization $v = V_2^*u$ with $V_1$ and $V_2$ two unitary matrices. The controlled ODE (29) becomes:

$$\dot{x}(t) = A_\theta(x(t), t)x(t) + V_1 \begin{pmatrix} \sigma_1 v_1 \\ \vdots \\ \sigma_{r'} v_{r'} \\ 0_{1, d_u-r'} \end{pmatrix}.$$  

From this one see the components $(v_{r'+1}, \ldots, v_{d_u})$ do not affect the system. Hence, a degenerate matrix $B$ underlines the presence of useless controls and can be replaced by a new full rank matrix $B_1$ with fewer columns for the same result.

3.2.2. Consistency

The estimator $(\hat{\theta}^T, \hat{x}_0^T)$ is defined as an M-estimator, so in order to derive consistency we need to show $S^\infty(\theta, x_0, U)$ has a global well-separated minimum at $(\theta, x_0) = (\theta^*, x_0^*)$ and that $S^l_\eta(\theta, x_0, U)$ converges uniformly to $S^\infty_\eta(Y; \theta, x_0, U)$ on $\Theta \times \chi$.

This is the point of the next two propositions.

Proposition 8. Under conditions C1 to C5 then $(\theta^*, x_0^*)$ is the unique global minimizer of $S^\infty(\theta, x_0, U)$ on $\Theta \times \chi$.

Proposition 9. Under conditions C1 to C5 we have

$$\sup_{(\theta, x_0) \in \Theta \times \chi} \left| S^\infty(\theta, x_0, U) - S^l_\eta(Y; \theta, x_0, U) \right| = o_l(1) + o_{p,n}(1).$$

From this, we use Theorem 5.7 in van der Vaart (1998) to conclude about the consistency.

Theorem 10. Under conditions C1 to C5, we have

$$\left(\hat{\theta}^T, \hat{x}_0^T\right) \longrightarrow (\theta^*, x_0^*) \text{ in probability}$$

when $(l, n) \longrightarrow \infty.$
Remark 11. Interestingly, in Clairon and Brunel (2017), for the weighting matrix under the form $U = \lambda I_d$, consistency proof for $\hat{\theta}^T$ requires the lower bound condition $\lambda > \lambda_1$ with $\lambda_1$ a positive model-dependent bound. Here, we just need to have $U$ positive definite.

3.2.3. Asymptotic normality
We show the asymptotic normality with $\sqrt{n}$-convergence rate in two steps. First, we derive a linear asymptotic representation of $(\hat{\theta}^T, \hat{x}_0^T) - (\theta^*, x_0^*)$ through a second order Taylor expansion of $\left( \theta, x_0 \right) \mapsto - \nabla_{(\theta,x_0)} S^l_n(Y; \theta, x_0, U)$, Second, we approximate this linear asymptotic representation in order to make explicit its dependence with respect to measurement noise.

Proposition 12. Under conditions C1 to C7, we have:

$$-\nabla_{(\theta,x_0)} S^l_n(Y; \theta^*, x_0^*) = \left( \frac{\partial^2 S^\infty(\theta^*, x_0^*, U)}{\partial^2 (\theta, x_0)} + o_p(1) + o_l(1) \right) \left( \hat{\theta}^T - \theta^*, \hat{x}_0^T - x_0^* \right).$$

Proposition 13. Under conditions C1 to C7, we have

$$-\nabla_v S^l_n(Y; \theta^*, x_0^*) = \left( \Delta \sum_{j=0}^{n-1} \epsilon_j^T \right) \left( K^l_{(\theta^*,x_0^*)} + o_n(1) \right) + L \left( \Delta \sum_{j=0}^{n-1} \epsilon_j \right) + o_p(n^{1/2}) + o_l(1)$$

with $K^l_{(\theta^*,x_0^*)} = 2CBU^{-1}B^T \int_0^T \frac{\partial h_1}{\partial (\theta^*,x_0^*)} dt$ and $L = \left( \begin{array}{c} 0_{p,d} \\ -2CT \end{array} \right)$.

From this, we recall the nonsingularity of $\frac{\partial^2 S^\infty(\theta^*, x_0^*, U)}{\partial^2 (\theta, x_0)}$ and the central limit theorem to obtain the following.

Theorem 14. Under conditions C1 to C8 and if $l$ is such that $l = O_n(\sqrt{\Delta})$, then $(\hat{\theta}, \hat{x}_0)$ is asymptotically normal and

$$(\hat{\theta}, \hat{x}_0) - (\theta^*, x_0^*) = o_p(1).$$

3.3. Asymptotic analysis of $\hat{\theta}^T, C^l$ for linear models in parametric case
For the asymptotic analysis of $\hat{\theta}^T, C^l$, we restrict to the linear models. Since $A_\theta$ does not depend then on $x$, we have $C_{T}^{l_1} = C_{T}^{l_2}$ and $C_T^l = C_T^{\infty}$ for all $l_1$ and $l_2$ belonging to $N$. Thus, there is no need to consider asymptotics in $l$ and we drop the dependence on $l$ in all quantities. The conditions we have derived for ensuring $\hat{\theta}^T, C^l$ consistency with $\sqrt{n}$-convergence rate are shown below.

Condition L1: For all $\theta \in \Theta$, $t \mapsto A_\theta(t)$ is differentiable on $[0, T]$.

Condition L2: $\theta \mapsto A_\theta$ is continuous on $\Theta$.

Condition L3: For all $\theta \in \Theta$, $R_\theta(0)$ is nonsingular, where $R_\theta$ is defined by ODE (20).
Condition L4: The true parameter \( \theta^* \) belongs to the interior of \( \Theta \).

Condition L5: The solution \( X_{\theta,x_0} \) of (1) is such that if \( CX_{\theta,x_0}(t) = CX_{\theta^*,x_0^*}(t) \) for all \( t \in [0, T] \) then \( (\theta, x_0) = (\theta^*, x_0^*) \).

Condition L6: \( \theta \mapsto A_\theta \) is \( C^2 \) on \( \Theta \).

Condition L7: The asymptotic hessian matrix \( \frac{\partial^2 S_{CI}(\theta^*)}{\partial \theta^2} \) is a nonsingular matrix.

The proofs follow the same steps as in the previous sections, hence we just present the theorems. The proofs are also detailed in supplementary materials, Sections 4 and 5.

**Theorem 15.** Under conditions LC1 to LC5, we have \( \hat{\theta}^{T,CI} \rightarrow \theta^* \) in probability when \( n \rightarrow \infty \).

**Theorem 16.** Under conditions LC1 to LC7, \( \hat{\theta}^{T,CI} \) is asymptotically normal and \( \hat{\theta}^{T,CI} - \theta^* = o_{p,n}(n^{-\frac{1}{2}}) \).

As in the discrete case, Condition L3 is an ad-hoc hypothesis, but here again we can derive a necessary and sufficient testable condition ensuring \( R_\theta(0) \) nonsingularity.

**Proposition 17.** Given \( \theta \in \Theta \), \( R_\theta(0) \) is invertible if and only if:
1) the matrix
\[
O_\theta(T) = \int_0^T (C\Phi_\theta(t,0))^T C\Phi_\theta(t,0) dt
\]
is invertible, where \( \Phi_\theta \) is the resolvant of \( \Phi_\theta \), and
2) the following holds:
\[
\|CX_{\theta,x_1} - CX_{\theta,x_2}\|_{L_2}^2 = 0 \implies x_1 = x_2.
\]

Interestingly, it is again equivalent to the notion of observability in control theory, but now for a continuous model [Sontag (1998)].

**Remark 18.** The difficulty in deriving the asymptotic behavior of \( \hat{\theta}^{T,CI} \) in all generality comes from the initialisation point \( x_0^* \) required by the algorithm. So far, we have been unable to analyze the mapping \( Q_\theta : x_0^* \mapsto X_\theta(.,x_0^*) \) where \( X_\theta(.,x_0^*) \) is the trajectory given by the algorithm in the limit case \( n = \infty \) and \( l = \infty \). If for \( \theta = \theta^* \), the true trajectory \( X^* \) is a global attractor of \( Q_\theta^* \), the demonstrations will be completed, but our attempts to prove it remain unfruitful.

4. Experiments

We use Monte-Carlo simulations on different models, for several sample sizes \( n \) and measurement noise variances \( \sigma^2 \). We compare four estimators: the ones presented here \( \hat{\theta}^T \) and \( \hat{\theta}^{T,CI} \), the classic nonlinear least square (NLS) estimator \( \hat{\theta}^{NLS} \) and the generalized
profiling (GP) estimator $\hat{\theta}^{\text{GP}}$ introduced in Ramsay et al. (2007). The latter is the regularization method of reference for the estimation problem in ODEs. As we have said in the Introduction, there are two main problems for parameter estimation, practical identifiability issues possibly due to the sloppiness phenomenon, and estimation inaccuracy due to model misspecifications. Thus, we compare $\hat{\theta}^T$, $\hat{\theta}^{T,CL}$, $\hat{\theta}^{NLS}$, $\hat{\theta}^{\text{GP}}$ on models facing practical identifiability problems in correctly and misspecified frameworks. For a given choice of $(n, \sigma)$, we compute the following by Monte Carlo based on $N_{MC} = 100$.

(a) The variance $V(\hat{\theta}_i)$ for each element $\theta_i$ of $\theta$ to analyze how each estimator behaves specifically for the components suffering from identifiability issues.

(b) The estimator variance-covariance norm $\left\| V(\hat{\theta}) \right\|_2$ to analyze how each estimator behaves for the whole parameter set.

(c) The componentwise mean square error $M(\hat{\theta}_i) = \left| \theta^*_i - \hat{E}[\hat{\theta}_i] \right|^2 + V(\hat{\theta}_i)$ and the global $M(\hat{\theta}) = \sum_{i=1}^{n} \left| \theta^*_i - \hat{E}[\hat{\theta}_i] \right|^2 + \left\| V(\hat{\theta}) \right\|_2$ to measure estimator accuracy, in particular its degradation when facing misspecification.

Since model parameters can have different orders of magnitude, the results will be given for normalized estimated values $\hat{\theta}./\theta^*$. Here the division has to be understood componentwise.

For each run, the observations are obtained by integrating the ODE with a Runge-Kutta algorithm (ode45 in Matlab), with added centered Gaussian noise of variance $\sigma^2$.

The GP method uses an approximate solution $\tilde{X}_\theta^\lambda$ of the ODE defined as the spline basis decomposition minimizing $\sum_{i=1}^{n} \left\| y_i - C\tilde{X}_\theta^\lambda(t_i) \right\|^2 + \lambda \left\| \frac{d}{dt}\tilde{X}_\theta^\lambda - f(\cdot, \tilde{X}_\theta^\lambda, \theta) \right\|^2_{L^2}$. GP requires a selection method for both the knots location and the hyperparameter $\lambda$ which has a similar role as in our method. The knots are placed on the observations and $\lambda$ is selected by using the method presented in D.A. Campbell and McAuley (2011); Qi and Zhao (2010): the value of $\lambda$ is increased until $\left\| \tilde{X}_{\theta_0}^{\text{GP}} - \tilde{X}_{\theta_0}^\lambda \right\|_{L^2}$ starts increasing, that is when $\tilde{X}_{\theta_0}^{\text{GP}}$ starts to differ significantly from the exact solution $\tilde{X}_{\theta_0}^\lambda$ where $\tilde{x}_0 = \tilde{X}_{\theta_0}^{\text{GP}}(0)$.

For $\hat{\theta}^T$ and $\hat{\theta}^{T,CL}$, we need to select both the discretization grid $\left\{ t^d_j, 0 \leq j \leq m \right\}$ and the matrix $U$. For the grid, we take $m = k_n n$ points and we place uniformly $k_n$ discretization points between two consecutive observation times. As in Clairon and Brunel (2017), for $U$ we consider scalar matrices $U = \lambda I_\lambda$. The $(k_n, \lambda)$ selection is done by minimizing the forward cross-validation method presented in G. Hooker and Earn (2011) among a model-specific trial of values. Let us denote $\tilde{\theta}_k^{T,\lambda}$ and $\tilde{\theta}_k^{T,CL}$, the tracking estimator obtained for a given $(k_n, \lambda)$ value. We split $[0,T]$ into $H$ subintervals $[t_h, t_{h+1}]$, such that $t_1 = 0$ and $t_H = T$ and we denote $X_{\theta}(\cdot, t_h, x_h)$ the solution of:

\[
\begin{aligned}
\dot{x}(t) &= f(t, x(t), \theta) \\
x(t_h) &= x_h
\end{aligned}
\]
defined on the interval \([t_h, t_{h+1}]\). The forward cross-validation uses the causal relation imposed on the data by the ODE to quantify the prediction error caused by \(\hat{\theta}_{kn,\lambda}\) (equal to \(\hat{\theta}^T_{kn,\lambda}\) or \(\hat{\theta}^{T,CI}_{kn,\lambda}\): 

\[
\text{EP}(k_n, \lambda) = \sum_{h=1}^{H} \sum_{i \in \{t_i, t_{i+1}\}} \left\| y_i - C X_{\hat{\theta}_{kn,\lambda}}(t_i, t_h, X_{\hat{\theta}_{kn,\lambda}}, \hat{\theta}_{kn,\lambda}, (t_h)) \right\|^2.
\]

Here, we choose \(H = 2\) subintervals and we now denote by \(\hat{T}\) and \(\hat{T,CI}\) the estimators corresponding to the value \((k_n, \lambda)\) minimizing EP.

### 4.1. \(\alpha\)-Pinene model

We begin with a linear ODE considered in Rodriguez-Fernandez et al. (2006) and used for modeling the isomerization of \(\alpha\)-Pinene. This is

\[
\begin{align*}
\dot{x}_1 &= -(\theta_1 + \theta_2)x_1 \\
\dot{x}_2 &= \theta_1 x_1 \\
\dot{x}_3 &= \theta_2 x_1 - (\theta_3 + \theta_4)x_3 + \theta_5 x_5 \\
\dot{x}_4 &= \theta_3 x_3 \\
\dot{x}_5 &= \theta_4 x_3 - \theta_5 x_5
\end{align*}
\]  

(33)

on the observation interval \([0, T] = [0, 100]\). Here

\[
A_\theta(t) = \begin{pmatrix}
-(\theta_1 + \theta_2) & 0 & 0 & 0 & 0 \\
\theta_1 & 0 & 0 & 0 & 0 \\
\theta_2 & 0 & -(\theta_3 + \theta_4) & 0 & \theta_5 \\
0 & 0 & \theta_3 & 0 & 0 \\
0 & 0 & \theta_4 & 0 & -\theta_5
\end{pmatrix}.
\]

The initial condition is \(x_0^* = (100, 0, 0, 0, 0)\) and the true parameter value is \(\theta^* = (5.93, 2.96, 2.05, 27.5, 4) \times 10^{-2}\). We plot in Figure 1 the solution of \(33\) corresponding to \(\theta^*\) and an example of simulated observations.

In Rodriguez-Fernandez et al. (2006), model \(33\) is used as a benchmark estimation comparison as many approaches fail to converge due to the difficulty of estimating \(\theta_4^*\) and \(\theta_5^*\) because of the high correlation between them. This is confirmed by the eigendecomposition of \(V^* D^* (V^*)^{-1} = I_p (\theta^*, x_0^*)\) for \(n = 10\), \(V^*\) being the matrix composed of the eigenvectors and \(D^*\) the diagonal matrix containing the eigenvalues \(\{\mu_i\}_{1 \leq i \leq 10}\). The lowest eigenvalues correspond almost exclusively to \(x_0^*\) and \((\theta_4^*, \theta_5^*)\) when one analyses the corresponding eigenvectors. We select \((k_n, \lambda)\) among the set \(\{30, 40, 50\} \times \{10^4, 5 \times 10^4\} \{-3 \leq i \leq 3\}\).

Influence of measurement noise  We consider one sample size \(n = 10\) and three levels of measurement noise \((\sigma = 2.5, \sigma = 5\) and \(\sigma = 10\)). Results are presented in Tables 1 and 2.
Discrete optimal control for estimation of ordinary differential equations

Fig. 1. Solution of (33) (blue) and corresponding noisy observations for $\sigma = 2.5$ (red).

Table 1. Scaled variance for $\alpha$–Pinene model.

| $\sigma$ | $\times 10^{-2}$ | $V(\hat{\theta}_1)$ | $V(\hat{\theta}_2)$ | $V(\hat{\theta}_3)$ | $V(\hat{\theta}_4)$ | $V(\hat{\theta}_5)$ | $\|\hat{\theta}\|_2$ |
|----------|-----------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 2.5      |                 |                   |                   |                   |                   |                   |                   |
| $\theta_{T,CI}$ | 0.06    | 0.12              | 0.30              | 0.20              | 0.29              | 0.65              |                   |
| $\theta^T$            | 0.06    | 0.12              | 0.30              | 0.25              | 0.37              | 0.79              |                   |
| $\theta^{NLS}$         | 0.05    | 0.10              | 0.12              | 0.34              | 0.48              | 0.89              |                   |
| $\theta^{GP}$          | 0.03    | 0.03              | 0.04              | 0.35              | 0.48              | 0.74              |                   |
| 5       |                 |                   |                   |                   |                   |                   |                   |
| $\theta_{T,CI}$ | 0.27    | 0.44              | 0.98              | 0.82              | 1.21              | 2.27              |                   |
| $\theta^T$            | 0.28    | 0.44              | 0.99              | 1.02              | 1.54              | 2.84              |                   |
| $\theta^{NLS}$         | 0.25    | 0.43              | 0.38              | 1.26              | 1.83              | 3.19              |                   |
| $\theta^{GP}$          | 0.17    | 0.15              | 0.13              | 1.27              | 1.63              | 2.92              |                   |
| 10      |                 |                   |                   |                   |                   |                   |                   |
| $\theta_{T,CI}$ | 0.61    | 1.22              | 2.63              | 3.40              | 5.36              | 9.04              |                   |
| $\theta^T$            | 0.62    | 1.22              | 2.64              | 4.08              | 6.21              | 10.7              |                   |
| $\theta^{NLS}$         | 0.60    | 1.49              | 1.37              | 5.48              | 8.37              | 14.3              |                   |
| $\theta^{GP}$          | 0.42    | 0.69              | 0.51              | 4.92              | 8.37              | 11.8              |                   |
Table 2. Scaled mean square error for $\alpha$–Pinene model.

| $\sigma$ | $10^{-2}$ | $M(\theta_1)$ | $M(\theta_2)$ | $M(\theta_3)$ | $M(\theta_4)$ | $M(\theta_5)$ | $M(\theta)$ |
|--------|-----------|----------------|----------------|----------------|----------------|----------------|-------------|
| 2.5    | 0.09      | 0.13           | 0.32           | 0.21           | 0.30           | 0.74           | $\hat{\theta}_{TE,CI}$ |
|        | 0.08      | 0.14           | 0.32           | 0.26           | 0.39           | 0.88           | $\hat{\theta}_{T}$          |
|        | 0.05      | 0.10           | 0.12           | 0.34           | 0.48           | 0.90           | $\hat{\theta}_{NLS}$        |
|        | 0.64      | 1.57           | 0.04           | 12.62          | 15.3           | 30.1           | $\hat{\theta}_{GP}$         |
| 5      | 0.37      | 0.50           | 1.01           | 0.82           | 1.22           | 2.46           | $\hat{\theta}_{TE,CI}$      |
|        | 0.37      | 0.44           | 0.99           | 1.02           | 1.54           | 3.02           | $\hat{\theta}_{T}$          |
|        | 0.25      | 0.43           | 0.38           | 1.27           | 1.85           | 3.22           | $\hat{\theta}_{NLS}$        |
|        | 0.80      | 1.75           | 0.14           | 14.0           | 17.7           | 34.0           | $\hat{\theta}_{GP}$         |
| 10     | 0.80      | 1.30           | 2.67           | 3.43           | 5.36           | 9.04           | $\hat{\theta}_{TE,CI}$      |
|        | 0.81      | 1.31           | 2.66           | 4.08           | 6.22           | 10.9           | $\hat{\theta}_{T}$          |
|        | 0.61      | 1.53           | 1.40           | 5.49           | 8.39           | 14.3           | $\hat{\theta}_{NLS}$        |
|        | 1.10      | 1.93           | 0.52           | 17.7           | 23.2           | 42.8           | $\hat{\theta}_{GP}$         |

For $\theta_4$ and $\theta_5$, we observe that $\hat{\theta}_{T}$ and $\hat{\theta}_{TE,CI}$ give the smallest variance followed by $\hat{\theta}_{GP}$ for $\sigma = 5$ and $\sigma = 10$. These approximate methods manage to regularize the estimation of parameters facing a practical identifiability problem in comparison with NLS. Moreover, we notice the same pattern for $\|V(\theta)\|_2$ which takes into account covariance among parameters. However, TE and GP are methods based on approximated solutions and so are likely to produce biased estimates. That is why we estimated the mean square error to verify that the price to pay to decrease the variance is not too high in terms of bias. Our methods have lower global mean square error than NLS which indicate a reasonable bias. GP on the other hand can have a very large mean square error. The reason, already been discussed in Clairon and Brunel (2017); Brunel and Clairon (2015), is linked to the limited ability of $\tilde{X}_\lambda$ to approach the true solution. This is contrary to our method where the mesh size can be arbitrarily small and thus $X_{d,u}^\theta$ can be arbitrarily close to the original ODE model.

**Influence of model misspecification** We still consider the sample size $n = 10$ with one level of measurement noise $\sigma = 2.5$. However, the observations are now generated by using the stochastically perturbed model:

$$
\begin{align*}
\frac{dx_1}{dt} &= -(\theta_1 + \theta_2)x_1 dt + c_tx_1 dt \\
\frac{dx_2}{dt} &= \theta_1x_1 dt + c_tx_2 dt \\
\frac{dx_3}{dt} &= (\theta_2x_1 - (\theta_3 + \theta_4)x_3 + \theta_5x_5) dt + c_tx_3 dt \\
\frac{dx_4}{dt} &= \theta_3x_3 dt + c_tx_4 dt \\
\frac{dx_5}{dt} &= (\theta_4x_3 - \theta_5x_5) dt + c_tx_5 dt
\end{align*}
$$

with $c_t \sim N(0, \sigma^2_t)$, we still estimate $\theta^*$ by using model (34), which is now a deterministic approximation of the true process. We plot in Figure 2 the solution of (33) and one realization of (34) for the sake of comparison. This experimental design has been chosen to mimic a real case of data analysis for chemical processes where the deterministic reaction
rate equations are used as an approximation of stochastic differential equations [Gillespie (2000)]. We study the effect of misspecification by varying the value of $\sigma_c^2$ and results are presented in Tables 3 and 4.

Here, the approximated methods manage to efficiently reduce the variance. However, only the TE estimators maintain a reasonable bias.

4.2. FitzHugh-Nagumo
We now consider the FitzHugh-Nagumo model

$$
\begin{align*}
\dot{V} &= c \left( V - \frac{V^3}{3} + R \right), \\
\dot{R} &= -\frac{1}{c} (V - a + bR),
\end{align*}
$$

which is a nonlinear ODE introduced to study the membrane potential evolution of neurons [FitzHugh (1961)]. Here $V$ is the neuron membrane potential, $R$ the synaptic conductance and we consider the partial observation framework where only $V$ is observed on $[0, T] = [0, 20]$. We take the parameter and initial condition values $a^* = b^* = 0.2$, $c^* = 3$ and $x_0^* = (V_0^*, R_0^*) = (-1, 1)$ given by [Ramsay et al. (2007)]. The related ODE solution exhibits a periodic behavior, as seen in Figure 3. Here, we choose

$$
A_0(x, t) = \begin{pmatrix}
c (1 - V^2/3) & cR & 0 \\
-\frac{1}{c} & -b & \frac{2}{c} \\
0 & 0 & 0
\end{pmatrix}
$$

where a third constant state variable $Z = 1$ is added to absorb the exogenous term $a/c$. This model has been chosen as it contains a parameter which is poorly identifiable under NLS, namely $b^*$. As in the $\alpha$-Pinene case, we confirm this by eigendecomposition of $\mathcal{L}_0(\theta^*, x_0^*)$.
Table 3. Scaled variance for misspecified $\alpha$--Pinene model.

| $\sigma^2_c$ | $\times 10^{-2}$ | $V(\hat{\theta}_1)$ | $V(\hat{\theta}_2)$ | $V(\hat{\theta}_3)$ | $V(\hat{\theta}_4)$ | $V(\hat{\theta}_5)$ | $\|V(\hat{\theta})\|_2$ |
|--------------|------------------|----------------------|----------------------|----------------------|----------------------|----------------------|------------------|
| 0.002        |                  |                      |                      |                      |                      |                      |                  |
| $\hat{\theta}_{T,CI}$ | 0.09 | 0.25 | 1.19 | 0.11 | 0.18 | 1.41 |                  |
| $\hat{\theta}_T$ | 0.09 | 0.29 | 1.32 | 0.13 | 0.22 | 1.61 |                  |
| $\hat{\theta}_{NLS}$ | 0.06 | 0.61 | 0.87 | 0.87 | 1.62 | 3.19 |                  |
| $\hat{\theta}_{GP}$ | 0.07 | 0.16 | 0.35 | 0.48 | 0.61 | 1.16 |                  |
| 0.004        |                  |                      |                      |                      |                      |                      |                  |
| $\hat{\theta}_{T,CI}$ | 0.12 | 0.43 | 2.17 | 0.15 | 0.27 | 2.51 |                  |
| $\hat{\theta}_T$ | 0.12 | 0.42 | 2.17 | 0.18 | 0.29 | 2.48 |                  |
| $\hat{\theta}_{NLS}$ | 0.12 | 1.78 | 1.99 | 1.52 | 2.93 | 6.95 |                  |
| $\hat{\theta}_{GP}$ | 0.12 | 0.38 | 0.51 | 0.94 | 1.26 | 2.45 |                  |
| 0.006        |                  |                      |                      |                      |                      |                      |                  |
| $\hat{\theta}_{T,CI}$ | 0.13 | 0.40 | 2.66 | 0.15 | 0.25 | 2.92 |                  |
| $\hat{\theta}_T$ | 0.12 | 0.41 | 2.45 | 0.19 | 0.30 | 2.83 |                  |
| $\hat{\theta}_{NLS}$ | 0.15 | 2.21 | 2.13 | 1.78 | 3.64 | 7.71 |                  |
| $\hat{\theta}_{GP}$ | 0.17 | 0.44 | 0.70 | 1.12 | 1.49 | 2.87 |                  |

Table 4. Scaled mean square error for misspecified $\alpha$--Pinene model.

| $\sigma^2_c$ | $\times 10^{-2}$ | $M(\theta_1)$ | $M(\theta_2)$ | $M(\theta_3)$ | $M(\theta_4)$ | $M(\theta_5)$ | $M(\theta)$ |
|--------------|------------------|----------------|----------------|----------------|----------------|----------------|-------------|
| 0.002        |                  |                |                |                |                |                |             |
| $\hat{\theta}_{T,CI}$ | 0.10 | 0.29 | 1.20 | 0.11 | 0.18 | 1.47 |             |
| $\hat{\theta}_T$ | 0.09 | 0.29 | 1.32 | 0.13 | 0.22 | 1.71 |             |
| $\hat{\theta}_{NLS}$ | 0.06 | 0.64 | 0.87 | 0.87 | 1.62 | 3.22 |             |
| $\hat{\theta}_{GP}$ | 0.06 | 1.79 | 0.35 | 12.9 | 16.1 | 31.3 |             |
| 0.004        |                  |                |                |                |                |                |             |
| $\hat{\theta}_{T,CI}$ | 0.14 | 0.45 | 2.17 | 0.15 | 0.27 | 2.55 |             |
| $\hat{\theta}_T$ | 0.15 | 0.43 | 2.17 | 0.18 | 0.29 | 2.52 |             |
| $\hat{\theta}_{NLS}$ | 0.12 | 1.78 | 2.00 | 1.52 | 2.94 | 6.97 |             |
| $\hat{\theta}_{GP}$ | 0.65 | 1.82 | 0.53 | 13.6 | 16.8 | 32.7 |             |
| 0.006        |                  |                |                |                |                |                |             |
| $\hat{\theta}_{T,CI}$ | 0.14 | 0.40 | 2.74 | 0.15 | 0.25 | 3.02 |             |
| $\hat{\theta}_T$ | 0.16 | 0.42 | 2.67 | 0.17 | 0.31 | 2.97 |             |
| $\hat{\theta}_{NLS}$ | 0.15 | 2.24 | 2.17 | 1.82 | 3.74 | 7.93 |             |
| $\hat{\theta}_{GP}$ | 0.63 | 1.99 | 0.71 | 14.0 | 17.4 | 33.7 |             |
for $n = 25$, which gives the eigenvectors

$$V^* \approx \begin{pmatrix}
0.04 & 0.08 & 0.56 & 0.12 & -0.82 \\
-0.97 & 0.10 & -0.12 & -0.09 & -0.14 \\
0.19 & 0.28 & -0.80 & 0.16 & -0.48 \\
-0.04 & 0.76 & 0.20 & 0.54 & 0.29 \\
-0.13 & -0.57 & -0.07 & 0.81 & 0.01
\end{pmatrix}$$

and the corresponding eigenvalues $\{\mu_l\}_{1 \leq l \leq 5} = \{0.9, 2.2, 15.7, 44.8, 256.6\}$. The lowest descend direction for the NLS criteria is nearly parallel to the $b$ direction in the parameter space. We select $(k_n, \lambda)$ among the set $\{50, 60, 70\} \times \{10^i, 5 \times 10^i\}_{-3 \leq i \leq -1}$.

**Influence of measurement noise** We take $n = 25$ and consider three levels of measurement noise ($\sigma = 0.01$, $\sigma = 0.03$ and $\sigma = 0.05$). Results are presented in Table 5.

Once more, we see the difference between the approximate methods and NLS for parameters facing practical identifiability issues. Here GP always gives the smallest variance for $b^*$ and for $\theta^*$ in two case out of three. It is directly followed by our approaches. In this finite sample case the TEs have a lower bias than NLS and GP estimators which explains their lower mean square error.

**Influence of model misspecification** We choose sample size $n = 25$ with variance $\sigma = 0.03$ for the measurement noise. The observations are now a realization of the hypoelliptic stochastic differential equation:

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{dV_t}{c} = c \left(V_t - \frac{V_t^3}{3} + R_t\right) dt \\
\frac{dR_t}{c} = -\frac{1}{c} (V_t - a + bR_t) dt + \sigma_d W_t
\end{array} \right. \tag{36}
\end{align*}
\]
Table 5. Scaled variance and mean square error for FHN model.

| \( \sigma \times 10^{-2} \) | \( V(\hat{\theta}) \) | \( V(b) \) | \( V(\hat{c}) \) | \( ||V(\hat{\theta})||_2 \) | \( M(\hat{\theta}) \) | \( M(b) \) | \( M(\hat{c}) \) | \( M(\theta) \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0.01 \( \theta^{T,CI} \) | 0.03 | 0.70 | 0.01 | 0.70 | 0.03 | 0.79 | 0.98 | 1.77 |
| \( \theta \) | 0.08 | 0.72 | 0.01 | 0.74 | 0.08 | 1.04 | 0.87 | 1.93 |
| \( \theta^{NLS} \) | 4.50 | 26.8 | 0.25 | 44.4 | 8.64 | 40.2 | 0.28 | 44.3 |
| \( \theta^{GP} \) | 0.01 | 0.52 | 0.01 | 0.52 | 1.58 | 49.9 | 2.90 | 54.3 |
| 0.03 \( \theta^{T,CI} \) | 0.45 | 12.2 | 0.19 | 12.2 | 0.46 | 12.3 | 0.93 | 13.1 |
| \( \theta \) | 1.33 | 11.2 | 0.08 | 11.8 | 1.34 | 11.9 | 0.90 | 13.4 |
| \( \theta^{NLS} \) | 5.25 | 28.4 | 0.25 | 42.7 | 11.4 | 58.4 | 0.30 | 64.9 |
| \( \theta^{GP} \) | 1.37 | 8.89 | 0.01 | 9.61 | 1.34 | 11.9 | 0.90 | 13.4 |
| 0.05 \( \theta^{T,CI} \) | 1.35 | 15.6 | 0.15 | 15.6 | 1.35 | 17.4 | 1.10 | 18.3 |
| \( \theta \) | 3.50 | 28.3 | 0.14 | 29.1 | 3.79 | 29.2 | 0.49 | 30.6 |
| \( \theta^{NLS} \) | 5.65 | 57.4 | 0.22 | 57.6 | 11.9 | 88.3 | 0.24 | 94.9 |
| \( \theta^{GP} \) | 2.11 | 12.3 | 0.06 | 33.7 | 2.17 | 30.6 | 2.53 | 33.7 |

with \( W_t \) a Wiener process and \( \sigma_r \) a diffusion parameter but \( \theta^* \) is still estimated by assuming the deterministic model (36) is true. This model has been proposed to include different sources of noise acting on the membrane of the cell, presynaptical currents etc. (Lindner and Schimansky-Geier (1999)). We plot in Figure 4 the solution of (35) and one realization of (36) for the sake of comparison. It has to be noted that dedicated methods for such hypoelliptic models have been developed (Ditlevsen and Samson (2017); Clairon and Samson (2017), but our point here is to show differences between approximate and exact estimation methods for ODEs in the presence of misspecification. We study the impact of misspecification by varying \( \sigma_r^2 \), with results presented in Table 6.

This example illustrate the benefit of using approximated methods in the presence of model error as TEs and GP have a lower variance than NLS and also a lower mean square error. By comparing \( \hat{\theta}^{T,CI} \) and \( \hat{\theta}^{T} \), we also notice the benefit of profiling on CI.

### 4.3. Repressilator

We present the Repressilator model proposed in Elowitz and Leibler (2000) for the study of a genetic regulation network. It is made of a feedback loop of 3 couples (mRNA, protein), denoted \((r_i, p_i)_{1\leq i \leq 3}\), in which each protein inhibits the next gene transcription in the loop:

\[
\begin{align*}
\dot{r}_i &= \frac{v_i k_{p_i}^{*}}{p_i^{*} + k_{p_i}^{*}}, \\
\dot{p}_i &= k_r r_i - k_p p_i.
\end{align*}
\]

In this model, we aim to estimate \( \theta^* = (v_1^*, v_2^*, k_1^*, k_{1,2}^*, k_{2,3}^*, k_2^*, k_3^*, k_1^{*}, k_2^{*}, k_3^{*}, n^*) = (50, 100, 80, 50, 30, 1, 1, 1, 1, 2, 3, 3) \) with true initial conditions \((r_{1,0}^*, r_{2,0}^*, r_{3,0}^*, p_{1,0}^*, p_{2,0}^*, p_{3,0}^*) = (60, 20, 6, 18, 27, 1)\). In order to reflect a real case observation framework, we consider that...
Fig. 4. Solution of (35) (blue) and a realization of (36) (red) for $\sigma_r^2 = 0.1$.

Table 6. Scaled variance and mean square error for misspecified FHN model.

| $\sigma_r^2 \times 10^{-2}$ | $V(\hat{a})$ | $V(\hat{b})$ | $V(\hat{c})$ | $\|V(\hat{\theta})\|_2$ | $M(\hat{a})$ | $M(\hat{b})$ | $M(\hat{c})$ | $M(\hat{\theta})$ |
|---------------------------|-------------|-------------|-------------|----------------|-------------|-------------|-------------|----------------|
| 0.1                       | $\theta^{L.CI}$ | 3.63 | 10.3 | 0.07 | 10.5 | 3.98 | 10.4 | 1.00 | 11.9 |
|                           | $\theta^T$   | 3.36 | 20.4 | 0.18 | 20.5 | 3.38 | 21.2 | 0.94 | 22.2 |
|                           | $\theta^{NLS}$ | 6.77 | 40.9 | 0.27 | 43.9 | 10.1 | 76.1 | 0.27 | 82.4 |
|                           | $\theta^{GP}$ | 6.11 | 28.7 | 0.48 | 29.7 | 7.62 | 43.4 | 0.49 | 45.8 |
| 0.15                      | $\theta^{L.CI}$ | 4.56 | 16.4 | 0.09 | 16.4 | 4.63 | 16.8 | 0.96 | 17.7 |
|                           | $\theta^T$   | 8.20 | 30.3 | 0.22 | 30.5 | 8.33 | 33.7 | 0.59 | 34.4 |
|                           | $\theta^{NLS}$ | 8.9  | 98.5 | 0.48 | 1.02 | 10.7 | 109  | 0.51 | 115  |
|                           | $\theta^{GP}$ | 5.88 | 67.3 | 0.39 | 40.0 | 5.88 | 67.3 | 0.39 | 67.6 |
| 0.2                       | $\theta^{L.CI}$ | 3.90 | 16.1 | 0.39 | 16.1 | 3.92 | 16.1 | 1.23 | 16.9 |
|                           | $\theta^T$   | 5.10 | 32.5 | 0.23 | 33.2 | 6.36 | 39.6 | 0.26 | 41.6 |
|                           | $\theta^{NLS}$ | 6.10 | 64.2 | 0.35 | 64.8 | 8.06 | 79.5 | 0.37 | 82.1 |
|                           | $\theta^{GP}$ | 10.4 | 32.4 | 0.52 | 34.1 | 10.6 | 37.6 | 0.68 | 39.3 |
only the mRNA concentrations are measured on \([0, T] = [0, 20]\) and for structural identifiability reasons we set \((k_{3,1}, k_1, k_2, k_3) = (40, 5, 6, 7)\). We plot in Figure 5 the solution of (37) corresponding to \(\theta^*\).

Here, we choose:

\[
A_\theta(r, p, t) = \begin{pmatrix}
-k_1^g & 0 & 0 & 0 & 0 & 0 & v_1 k_1^p & 0 & 0 & 0 & 0 & v_1 k_1^p \\
0 & -k_2^g & 0 & 0 & 0 & 0 & v_2 k_2^p & 0 & 0 & 0 & 0 & v_2 k_2^p \\
0 & 0 & -k_3^g & 0 & 0 & 0 & v_3 k_3^p & 0 & 0 & 0 & 0 & v_3 k_3^p \\
k_1 & 0 & 0 & -k_1^p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_2 & 0 & 0 & -k_2^p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_3 & 0 & 0 & 0 & -k_3^p & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

where, again, a constant artificial state variable \(Z = 1\) has been added. This model has been identified as sloppy in Gutenkunst et al. (2007) and for our particular experimental design, the eigendecomposition of \(I_n(\theta^*, x_0^*)\) for \(n = 25\) indicates the subset of parameters \((v_1^*, v_2^*, v_3^*, k_{1,2}^*, k_{2,3}^*)\) corresponds to the lowest eigenvalues. Henceforth, we separate \(\theta^*\) into \(\theta_1^* = (v_1^*, v_2^*, v_3^*, k_{1,2}^*, k_{2,3}^*)\) and \(\theta_2^* = (k_1^p, k_2^p, k_3^p, k_1^p, k_2^p, k_3^p, n^*)\) for presenting the estimation results and in particular analyze how the different methods behave with the poorly identifiable parameter set \(\theta_1^*\). In the following, \(V(\hat{\theta}_1), V(\hat{\theta}_2), \text{M}(\hat{\theta}_1)\) and \(\text{M}(\hat{\theta}_2)\) will denote the sum of the variance (resp. mean square error) of \(\theta_1\) and \(\theta_2\) components.

We select \((k_n, \lambda)\) among the set \(\{20, 25, 30\} \times \{10^i, 5 \times 10^j\} - 3 \leq i, j \leq 1\).

**Influence of measurement noise** We take \(n = 25\) and consider three levels of measurement noise \((\sigma = 9, \sigma = 10 \text{ and } \sigma = 11)\). Results are presented in Table 4.
Table 7. Scaled variance and mean square error for Repressilator model.

| σ  | V(\(θ_1\)) | V(\(θ_2\)) | V(\(\hat{θ}\)) | M(\(θ_1\)) | M(\(θ_2\)) | M(\(\hat{θ}\)) |
|----|------------|------------|----------------|-------------|-------------|--------------|
| 9  | 0.16       | 0.12       | 0.16           | 0.21        | 0.15        | 0.24         |
|    | 0.19       | 0.20       | 0.21           | 0.21        | 0.22        | 0.25         |
|    | 0.24       | 0.17       | 0.26           | 0.24        | 0.17        | 0.27         |
| 10 | 0.20       | 0.20       | 0.22           | 0.25        | 0.22        | 0.30         |
|    | 0.17       | 0.23       | 0.26           | 0.19        | 0.28        | 0.33         |
|    | 0.36       | 0.22       | 0.39           | 0.37        | 0.22        | 0.41         |
| 11 | 0.26       | 0.29       | 0.28           | 0.30        | 0.31        | 0.36         |
|    | 0.22       | 0.36       | 0.38           | 0.23        | 0.39        | 0.41         |
|    | 0.39       | 0.27       | 0.42           | 0.42        | 0.28        | 0.45         |

We were unable to obtain results for GP because of an important number of algorithmic failures during simulations (almost 80% of the runs) due to practical identifiability issues. Indeed, GP requires the introduction of nuisance parameters \(β\) needed for obtaining a smooth curve estimator \(\hat{X}_λ\) which can lead to overfitting with diverging parameter estimates. In a partially observed framework, even for a \(\hat{θ}_{GP}\) value far from \(θ^∗\), the observed part of the smooth curve \(\hat{X}_λ\) can remain close to the observations because the parameters \(\hat{β}_λ\) can counteract the effects of \(\hat{θ}_{GP}\). Clearly, our method improves the estimation of the subset of sloppy parameters. Even though it is at the expense of \(θ_2\) estimation for the highest noise level, our method globally improves the committed error when all parameters are simultaneously estimated, which is the recommended procedure in sloppy models Gutenkunst et al. (2007).

Influence of model misspecification. For the population size and the measurement noise variance, we set \((n, σ) = (25, 9)\). As in the α–Pinene case, the observations are now generated by a stochastically perturbed version of the original ODE:

\[
\begin{align*}
\mathrm{d}r_i &= \left(\frac{v_i k_{r i+1}}{k_{r i+1} + k_{p i+1}} - k_i r_i\right) \mathrm{d}t + c_i r_i \mathrm{d}t \\
\mathrm{d}p_i &= \left(k_i r_i - k_i^2 p_i\right) \mathrm{d}t + c_i p_i \mathrm{d}t
\end{align*}
\]

with \(c_i \sim N(0, σ_i^2)\). We plot in Figure 6 the solution of (37) and one realization of (38) for the sake of comparison. Results are presented in Table 8.

They confirm once again the advantages of using an estimation method based on a relaxation of the original model in the presence of model error.

4.4. FitzHugh-Nagumo with a functional parameter

We resume the experimental design presented in Section 4.2 but the true constant parameter \(a^∗\) is replaced by the function \(a^∗(t) = 0.2 (1 + \sin(\frac{t}{4}))\). We plot in Figure 7 the
Fig. 6. Solution of (37) (blue) and a realization of (38) (red) for $\sigma_c^2 = 0.5$.

Table 8. Scaled variance and mean square error for misspecified Repressilator model.

| $\sigma_c^2$ | $V(\hat{\theta}_1)$ | $V(\hat{\theta}_2)$ | $\|V(\hat{\theta})\|_2$ | $M(\hat{\theta}_1)$ | $M(\hat{\theta}_2)$ | $M(\hat{\theta})$ |
|--------------|---------------------|---------------------|--------------------------|---------------------|---------------------|---------------------|
| $\sigma_c^2 = 0.5$ | $\hat{\theta}^{T,CI}$ | 0.20 | 0.19 | 0.21 | 0.25 | 0.21 | 0.29 |
| | $\hat{\theta}^T$ | 0.18 | 0.24 | 0.25 | 0.18 | 0.26 | 0.27 |
| | $\hat{\theta}^{NLS}$ | 0.27 | 0.23 | 0.29 | 0.28 | 0.23 | 0.31 |
| $\sigma_c^2 = 1$ | $\hat{\theta}^{T,CI}$ | 0.21 | 0.25 | 0.24 | 0.26 | 0.28 | 0.32 |
| | $\hat{\theta}^T$ | 0.18 | 0.24 | 0.25 | 0.19 | 0.29 | 0.32 |
| | $\hat{\theta}^{NLS}$ | 0.26 | 0.28 | 0.33 | 0.27 | 0.29 | 0.35 |
| $\sigma_c^2 = 1.5$ | $\hat{\theta}^{T,CI}$ | 0.22 | 0.27 | 0.27 | 0.26 | 0.30 | 0.35 |
| | $\hat{\theta}^T$ | 0.24 | 0.25 | 0.28 | 0.25 | 0.29 | 0.33 |
| | $\hat{\theta}^{NLS}$ | 0.41 | 0.21 | 0.37 | 0.44 | 0.21 | 0.40 |
Discrete optimal control for estimation of ordinary differential equations

Fig. 7. Left: Solution of (35) for $a^*$ constant (blue) and $a^*$ time-varying (red). Right: Plot of time-varying parameter $a^*$.

solutions of (35) for $a^* = 0.2$ and $a^* = 0.2 \left(1 + \sin \left(\frac{t}{5}\right)\right)$. Here, we compare the variational approach presented in Section 2.2 with a classic basis decomposition method for the simultaneous estimation of $\theta^* = (b^*, c^*)$ and $\vartheta^* = a^*$. Our criteria is based on the cost (13) and thus requires the selection of three hyperparameters $k_n$, $\lambda_1$ and $\lambda_2$. We retain the value for $(k_n, \lambda_1, \lambda_2)$ which minimizes the prediction error $\text{EP}$ among the trial $\{20, 25\} \times \{10^i\}_{-4 \leq i \leq -2} \times \{10^i\}_{-4 \leq i \leq -1}$. To estimate $a^*$ with NLS and GP we use a finite basis decomposition, i.e. we approximate $a^*$ with $\hat{a}(t) \simeq \sum_{i=1}^{K_r} \beta_{i,K} p_i(t)$, here $\{p_i\}$ is a B-Spline basis with a uniform knot sequence. The additional $K_r$ parameters ($\beta_{r,1}, \ldots, \beta_{r,K_r}$) are estimated by introducing the extended set $\theta^{\text{ext}} = (\theta, \beta_{r,1}, \ldots, \beta_{r,K_r})$ and $K_r$ is selected by minimizing the Akaike Information: $\text{AIC}(\hat{\theta}^{\text{ext}}) = n \log \left(\sum_{i=1}^{n} (y_i - \mathbb{E}[X_{\theta^{\text{ext}}, \tilde{X}_0}(t)])^2\right) + 2K_r$, where $\tilde{X}_0$ is the standard initial condition estimator for NLS and for GP we choose $\tilde{X}_0 = \tilde{X}^{\theta_{\lambda\lambda}}_{\lambda\lambda}(0)$. For a given estimator $\hat{a}$, we quantify its accuracy by Monte-Carlo estimator of the integrated variance $V_f(\hat{a}) = \int_0^T \mathbb{E} \left[(\hat{a}(t))^2\right] - \mathbb{E} \left[\hat{a}(t)\right]^2 \right) dt$ and mean square error $M_f(\hat{a}) = \int_0^T \mathbb{E} \left[(\hat{a}(t) - a^*(t))^2\right] dt$.

Influence of measurement noise We take $n = 50$ and consider two levels of measurement noise ($\sigma = 0.03$ and $\sigma = 0.05$). Results are presented in Table 8.

Our estimators outperform GP and NLS both for parametric and functional estimation accuracy. The finite basis decomposition used to replace $a$ leads to use an approximated version of the original model for estimation purposes. This induced misspecification can explain the drop in accuracy for the NLS and GP estimators. Moreover, as pointed out in Clairon and Brunel (2017), the selection of a proper basis and knot location for semi-parametric estimation is complicated and model-specific. In our case, the extension of the parametric estimation method to the semi parametric framework is straightforward for hyperparameters selection.
Table 9. Scaled variance and mean square error for FHN model with functional parameter.

| $\sigma$ | $\times 10^{-2}$ | $V(\hat{b})$ | $V(\hat{c})$ | $\|V(\hat{\theta})\|_2$ | $V^f(\hat{\alpha})$ | $M(\hat{b})$ | $M(\hat{c})$ | $M(\hat{\theta})$ | $M^f(\hat{a})$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.03 | | | | | | | | | |
| $\hat{\theta}^{T, CI}$ | 2.50 | 0.05 | 2.50 | 1.06 | 2.87 | 0.07 | 2.88 | 1.77 |
| $\hat{\theta}^T$ | 3.50 | 0.06 | 3.52 | 0.77 | 4.25 | 0.06 | 4.27 | 1.08 |
| $\hat{\theta}^{NLS}$ | 5.32 | 3.55 | 7.94 | 2.26 | 5.91 | 3.72 | 8.71 | 2.41 |
| $\hat{\theta}^{GP}$ | 2.84 | 0.01 | 2.84 | 2.41 | 3.17 | 0.20 | 3.36 | 6.86 |
| 0.05 | | | | | | | | | |
| $\hat{\theta}^{T, CI}$ | 11.7 | 0.17 | 11.8 | 3.04 | 12.1 | 0.27 | 12.2 | 4.00 |
| $\hat{\theta}^T$ | 8.17 | 0.15 | 8.21 | 2.61 | 8.19 | 0.22 | 8.30 | 2.61 |
| $\hat{\theta}^{NLS}$ | 14.1 | 3.77 | 15.5 | 10.1 | 14.3 | 4.06 | 25.4 | 11.4 |
| $\hat{\theta}^{GP}$ | 13.9 | 0.04 | 13.6 | 16.7 | 13.9 | 0.17 | 14.0 | 18.6 |

Table 10. Scaled variance and mean square error for misspecified FHN model with functional parameter.

| $\sigma^2_r$ | $\times 10^{-2}$ | $V(\hat{b})$ | $V(\hat{c})$ | $\|V(\hat{\theta})\|_2$ | $V^f(\hat{\alpha})$ | $M(\hat{b})$ | $M(\hat{c})$ | $M(\hat{\theta})$ | $M^f(\hat{a})$ |
|---|---|---|---|---|---|---|---|---|---|
| 0.1 | | | | | | | | | |
| $\hat{\theta}^{T, CI}$ | 11.7 | 0.08 | 11.7 | 8.97 | 11.7 | 0.08 | 11.7 | 9.33 |
| $\hat{\theta}^T$ | 7.58 | 0.08 | 7.59 | 5.26 | 7.64 | 0.09 | 7.65 | 6.04 |
| $\hat{\theta}^{NLS}$ | 17.5 | 0.07 | 17.5 | 9.7 | 20.6 | 0.12 | 20.6 | 10.7 |
| $\hat{\theta}^{GP}$ | 65.4 | 0.11 | 65.4 | 42.8 | 65.8 | 0.14 | 65.8 | 43.9 |
| 0.15 | | | | | | | | | |
| $\hat{\theta}^{T, CI}$ | 13.7 | 0.10 | 13.7 | 9.8 | 13.7 | 0.12 | 13.7 | 10.5 |
| $\hat{\theta}^T$ | 9.50 | 0.09 | 9.50 | 6.75 | 9.56 | 0.10 | 9.58 | 7.43 |
| $\hat{\theta}^{NLS}$ | 32.3 | 16.7 | 43.2 | 36.9 | 35.8 | 21.3 | 51.3 | 47.7 |
| $\hat{\theta}^{GP}$ | 73.7 | 0.15 | 73.7 | 58.7 | 73.7 | 0.25 | 73.8 | 61.3 |

Influence of model misspecification We choose sample size $n = 50$ with variance $\sigma = 0.03$ for the measurement noise. The observations are now realization of the hypoelliptic stochastic differential equation (36) for two values of $\sigma^2_r$. Results are presented in Table 10.

Once again, our methods give better results than NLS and GP. The difference is even more striking here, possibly due to the accumulated effect of the different source of misspecifications on GP and NLS. For our approaches, the term $\lambda^T u_{1,j}^T u_{1,j}$ present in cost (13) takes into account model discrepancy and is expected to mitigate the effect of misspecification on estimation Brynjarsdottir and O’Hagan (2014); Kirk et al. (2016).

5. Real data case analysis

We focus on a model discussed by Stein et al. (2013) to study the impact on a microbiota ecosystem of the interaction between an antibiotic treatment and a pathogen inoculation. The model used by the authors is an 11-dimensional Generalized Lotka-Volterra
Regarding the parameter set \( \mu_t \mapsto x \), the interaction effect of \( \mu \) names of the microbial species as well as the values of \( v \) for each species. ODE: \( x_i = \mu_i x_i + x_i \sum_{j=1}^{11} M_{i,j} x_j + x_i s_i v(t) \) (39)

for \( i = 1, \ldots, 11 \). Each state variable \( x_i \) quantifies the presence of one microbial species and \( t \mapsto v(t) \) describes the perturbation due to antibiotic administration (here, clindamycin). Regarding the parameter set \( (\mu_i, M_{i,j}, s_i)_{1 \leq i,j \leq 11} \), \( \mu_i \) is the growth term for \( x_i \), \( M_{i,j} \) the interaction effect of \( x_j \) on \( x_i \) and \( s_i \) the susceptibility of \( x_i \) to the antibiotic treatment. The names of the microbial species as well as the values of \( (\mu_i, M_{i,j}, s_i)_{1 \leq i,j \leq 11} \) are provided by Stein et al. (2013) (Figure 2). Regarding the acquired data in Stein et al. (2013), they are divided in three groups of three subjects. Group 1 was exposed to the pathogen (here, \( C. difficile \)) by Stein et al. (2013) (Figure 2). Regarding the acquired data in Stein et al. (2013), they are divided in three groups of three subjects. Group 1 was exposed to the pathogen (here, \( C. difficile \)). Group 2 received a single dose of clindamycin and Group 3 received clindamycin and was exposed to \( C. difficile \) the day after. We focus on Group 3 for which the perturbation is the impulse function \( v(t) = 1_{\{t=0\}} \). In this restricted case, some microbical species have limited impacts on the whole ecosystem evolution. For parameter estimation we restrict ourselves to the simplified model:

\[ x_i^s = \mu_i^s x_i^s + x_i^s \sum_{j=1}^{7} M_{i,j}^s x_i^s + x_i^s s_i^s v(t) \] (40)

where \( x_i^s = x_i \) for \( 1 \leq i \leq 5 \), \( x_6^s = x_9 \) and \( x_7^s = x_{11} \) and the parameters \( \{\mu_i^s, M_{i,j}^s, s_i^s\} \) are defined and linked to the previous parametrization accordingly. For comparison, we plot in Figure 8 the solution of (39) and (40) for three initial condition values corresponding to the three subjects in Group 3.

Here, we focus on the \( M_{i,j} \) estimation for they give the nature of interaction between species. The different \( (\mu_i, s_i) \) are considered as known and we estimate the parameter set of dimension \( p = 31 \): \( \theta = \{M_{i,j}\}_{i \in \{1,3,4,5\}, j \in \{2,3,4,5\}}, \{M_{i,2}\}_{i \in \{1,2,3,4,5\}}, \{M_{i,3}\}_{i \in \{1,2,3\}}, \{M_{i,4}\}_{i \in \{1,2,4\}}, \{M_{i,5}\}_{i \in \{2,3,4,5\}}, \{M_{9,11}\}_{i \in \{3,5\}}, \{M_{11,11}\}_{i \in \{1,2,3,4,5\}} \). The other interaction terms are unidentifiable in practice when we use only data coming from Group 3. Before our real data analysis we perform first a simulation analysis in order to compare our approach with NLS.
5.1. Preliminary results on simulated data

To mimic the real case analysis, estimation is based on the observation of three individuals with different initial conditions $x_{1,0}^1$, $x_{2,0}^2$, and $x_{3,0}^3$. Both NLS and our approach requiring estimation of initial conditions have 52 unknown parameters. Hence, to save computational time, we restrict here to $\theta^NLS$ and $\theta^{T,CI}$. Because of the parameter dimension and similarly as in the Repressilator case, we split $\theta$ into two subgroups $\theta_1$ and $\theta_2$ according to the difficulty encountered by NLS to estimate them. To identify $\theta_2$, we rely once again on the eigendecomposition of $V^*D^*(V^*)^{-1} = I_n (\theta^*, x_{1,0}^1, x_{2,0}^2, x_{3,0}^3)$ for $n = 25$ where $D^* = \text{diag}(\mu_1, \ldots, \mu_{52})$ is the matrix composed of the 52 eigenvalues $\mu_i$ sorted in increasing order and $V^*$ is the matrix where each column $V^*_{i,}$ is the eigenvector related to $\mu_i$. The associated condition number is equal to $\kappa(I_n) \simeq 8 \times 10^{-10}$ which clearly indicates an ill-posed problem for NLS. Moreover, we have $\frac{\sigma_{\theta_{52}}}{\mu_{52}} = 2 \times 10^{-6}$, thus the first 25 eigenvectors correspond to directions of weak change for the NLS criteria. For each parameter $\theta^j$ in $\theta$, we compute $F(\theta^j) = \frac{\sum_{i=1}^{25} (V^*_{i,j})^2}{\sum_{i=1}^{52} (V^*_{i,j})^2}$ to quantify the impact of $\theta^j$ on NLS criteria. By doing so we identify 12 parameters such that $F(\theta^j) > 0.63$ which will compose $\theta_2$, the set of parameters poorly estimated by NLS. The choice of threshold for the eigenvalue rank and $F(\theta^j)$ value is somewhat arbitrary, but we will see in simulations that the variance and mean square error for $\theta^{NLS}$ come mainly from estimation of $\theta_2$. As in the Repressilator case, $V(\hat{\theta}_1)$, $M(\hat{\theta}_1)$ and $V(\hat{\theta}_2)$, $M(\hat{\theta}_2)$ will have to be understood as the sum of the variance and mean square error of the $\theta_1$ and $\theta_2$ elements. In the presented results, we also compare the ability of the different estimators to reconstruct the orientation of the interaction graph i.e. we estimate $I(\hat{\theta}) = \frac{1}{p} \sum_{i=1}^p I\{\text{sign}(\hat{\theta}_i) = \text{sign}(\theta_i)\}$ that is, the expected fraction of correctly retrieved interaction.

Influence of measurement noise  We take $n = 25$ and consider three levels of measurement noise ($\sigma = 0.01$, $\sigma = 0.02$ and $\sigma = 0.03$). Results are presented in Table 11. For both $\theta_1$ and $\theta_2$ our approach outperforms NLS. In particular, we retrieve the right orientation for the interaction graph, whereas the ability of $\theta^{NLS}$ to do so decreases with noise, especially for $\theta_2$. 

| $\sigma$ | $V(\hat{\theta}_1)$ | $V(\hat{\theta}_2)$ | $\| V(\hat{\theta}) \|_F$ | $\hat{M}(\hat{\theta}_1)$ | $M(\hat{\theta}_1)$ | $\hat{M}(\hat{\theta}_2)$ | $M(\hat{\theta}_2)$ | $I(\hat{\theta}_1)$ | $I(\hat{\theta}_2)$ | $I(\theta)$ |
|------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0.01 | 0.11            | 0.18            | 0.11            | 0.18            | 0.22            | 0.22            | 1               | 1               | 1               |
|      | $\hat{\theta}^{T,CI}$ |               |                 |                 |                 |                 |                 |                 |                 |                 |
|      | 0.07            | 1.86            | 2.04            | 0.68            | 1.89            | 2.09            | 0.99            | 0.98            | 0.99            |
| 0.02 | 0.22            | 0.51            | 0.28            | 0.29            | 0.54            | 0.38            | 1               | 1               | 1               |
|      | $\hat{\theta}^{T,CI}$ |               |                 |                 |                 |                 |                 |                 |                 |                 |
|      | 2.85            | 7.94            | 8.29            | 2.89            | 8.06            | 8.46            | 0.98            | 0.93            | 0.96            |
| 0.03 | 0.27            | 0.64            | 0.29            | 0.34            | 0.70            | 0.43            | 1               | 1               | 1               |
|      | $\hat{\theta}^{NLS}$ |               |                 |                 |                 |                 |                 |                 |                 |                 |
|      | 3.03            | 9.06            | 6.43            | 3.31            | 9.80            | 7.46            | 0.98            | 0.90            | 0.95            |
Table 12. Scaled variance, mean square error and I for misspecified Microbiota model

| σ   | V(θ₁) | V(θ₂) | V(θ) | M(θ₁) | M(θ₂) | M(θ) | I(θ₁) | I(θ₂) | I(θ) |
|-----|-------|-------|------|-------|-------|------|-------|-------|------|
| 0.01| θᵀ,CI | 0.30  | 2.56 | 1.61  | 0.69  | 3.10 | 2.54  | 1     | 0.98 | 1    |
|     | θᴺᴸS | 0.63  | 2.04 | 1.85  | 1.47  | 5.73 | 6.40  | 0.99  | 0.90 | 0.96 |
| 0.02| θᵀ,CI | 0.33  | 2.67 | 1.59  | 0.74  | 3.23 | 2.57  | 1     | 0.98 | 0.99 |
|     | θᴺᴸS | 1.98  | 6.35 | 4.78  | 2.90  | 9.97 | 9.32  | 0.98  | 0.88 | 0.95 |
| 0.03| θᵀ,CI | 0.35  | 2.76 | 1.63  | 0.77  | 3.37 | 2.59  | 0.99  | 0.98 | 0.99 |
|     | θᴺᴸS | 5.43  | 16.3 | 10.6  | 6.25  | 19.6 | 14.7  | 0.95  | 0.83 | 0.91 |

Fig. 9. Observed data for the three subjects belonging to group 3. Each line represents a microbiota species.

Influence of model misspecification

We choose the same sample size and measurement error levels as before but the observations are now generated by using model (39). We are interested in quantifying robustness of the different estimators with respect to misspecification due to neglected interactions, a common feature in the study of biological networks. In particular, we want to measure the ability of our estimator to retrieve the true interactions between two state variables despite the presence of unmeasured confounders. Results are presented in Table 12. The situation is quite similar to the well-specified case but with the additional feature that the capacity of \( \hat{θ}^{NLS} \) to retrieve the true interaction graph is more affected by model misspecification than \( \hat{θ}^{T,CI} \).

5.2. Real data analysis

In this section, we proceed to \( θ \) estimation in model (40) starting from the real data available in Stein et al. (2013) for the Group 3. The data are presented in Figure 9.

The original observation interval was \([0, 23]\) but here we restrict to \([0, 16]\) since no data are available on \([16, 23]\) and a first estimation attempt on the whole observation interval \([0, 23]\) lead to poor data fitting of the optimal trajectories \( \hat{X}_θ^k \) for any estimator \( \hat{θ} \) obtained for any \((k_n, λ)\) selected values. After some trial and error, we selected \( k_n = 20 \) and \( λ = 5 \times 10^{-6} \). This gave the mesh size small enough to accurately estimate the ODE perturbed model, an estimator \( \hat{θ}^{T,CI} \) close to the one obtained by Stein et al. (2013), and allowance for the possibility of model discrepancy.
Fig. 10. First, second and fifth component of $\hat{X}_{dT,CI}^d$ and corresponding raw data for subject 1 (blue), 2 (red), 3 (black) in Group 3

Table 13. Scaled (by $10^2$) values for Stein et al. (2013) estimator $\hat{\theta}^{Stein}$ and ours $\hat{\theta}^{T,CI}$

| $\hat{\theta}^{Stein}$ | -1 | -7 | 18 | -29 | 1 | -30 | 6 | -8 | 98 | -10 |
|-------------------------|----|----|----|-----|---|-----|---|----|----|-----|
| $\hat{\theta}^{T,CI}$  | -31| -28| -31| -20| -10| -31| 7 | -6 | 235| -7  |
| $\hat{\theta}^{Stein}$ | -19| 36 | -10| -23| -55| -16| -20| -79| 10 | 9   |
| $\hat{\theta}^{T,CI}$  | -59| -176| -6 | -86| -47| -30| -21| -80| 30 | 18  |
| $\hat{\theta}^{Stein}$ | -191| -28| -16| -50| 48 | 69 | 37 | 37 | 67 | -49 |
| $\hat{\theta}^{T,CI}$  | -458| -52| -33| -44| 1  | 27 | 64 | 38 | 28 | 36  |

Despite the use of a simplified model and the presence of outliers which render difficult a good data fitting of $\hat{X}_{dT,CI}^d$ (see Figure 10), we obtain an estimator consistent with the values obtained by Stein et al. (2013) when using a more accurate model. We obtained a graph orientation close to the one obtained in Stein et al. (2013) with only 4 out of the 31 estimated interaction parameters having a different sign (see Table 13). This confirms the benefit of using the approximated methods for real data analysis, where model uncertainty presence is the rule rather than the exception.

Our methodology copes with potential model misspecification by limiting its effect on estimation. However, our approach may also be useful for investigating the possibility of misspecification presence by analyzing the optimal control $\hat{u}_{dT,CI}$ values, which can be seen as residuals quantifying the discrepancy between the model and the actual system dynamic. We present in Figure 11 for each subject the components of $\hat{u}_{dT,CI}$ corresponding to the $\hat{X}_{dT,CI}^d$ components presented in Figure 10. One can see clearly there are shared patterns, for example in the first graph on $[3, 6]$, where the optimal controls present the same behavior for all subjects. Such features indicate that some deterministic elements of the actual system dynamic have been missed by the assumed model.
6. Conclusion

The advantages of using control theory to propose a method which regularizes the estimation problem in ODEs when facing practical identifiability issues and model misspecification have been pointed out in Clairon and Brunel (2018, 2017; Brunel and Clairon (2015). However, the existing procedures based on continuous control theory can be time consuming for high dimensional models and require a sufficient number of observations, given that these methods rely on a non-parametric estimator $\hat{Y}$ of the observed curve. The methodological and theoretical sections of this work demonstrate we can construct a consistent estimator with a parametric convergence rate based on discrete control theory which overcomes these two problems. They also expose how we can easily profile on the initial conditions to avoid the estimation of additional nuisance parameters. The experimental and real data analyses confirm the good performance of our method in comparison with non-linear least squares and generalized profiling for small sample case, where the asymptotic analysis results do not hold.

An under-exploited feature of the method so far is the obtained optimal control. Here, we only use it for a qualitative based analysis in the real data case, but we suspect that a full analysis of $u^d_{\hat{\theta}_{T,CI}}$ maybe be useful to construct a statistical test of misspecification at the derivative level, which is more relevant for such models than the test based on residuals Hooker and Ellner (2015). This is a subject for further work. A second point worth exploring in the future is the extension to mixed effect model in which several subjects are observed and despite that they present different trajectories it is assumed their dynamics are ruled by the same evolution law. It means each subject $i$ follows the equation $\dot{X} = f(t, X, \theta_i)$ where $f$ is common to the whole population but $\theta_i$ is an individual parameter defined as the realization of a random variable following a law $\rho$ depending to a population parameter $\theta$ i.e. $\theta_i \sim \rho(\theta)$. For these models, dedicated methods are necessary to incorporate inter-patient correlation in the estimation process Raftery and Bao (2010); Donnet and Samson (2006); M. Lavieille and Mentre (2011); M. Prague et al. (2013); Wang et al. (2014). For our method, it would be interesting to consider mixed-effect on the estimated optimal controls $u^d_i$ to take into account correlations
on the committed model error among the individuals.

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A regularization method for the parameter estimation problem in ordinary differential equations via discrete optimal control theory: Proofs

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1. Introduction

1.1. Model and objectives
We are interested in the estimation of the parameter $\theta^*$ from data $y_1, \ldots, y_n$, observed on the interval $[0, T]$, that are realizations of the observation process for $i = 0, \ldots, n$:

$$Y_i = CX^*(t_i) + \epsilon_i$$

where $X^*$ is the solution to the Initial Value Problem (IVP):

$$\begin{align*}
\dot{x}(t) &= A_\theta(x(t), t)x(t) \\
x(0) &= x_0
\end{align*}$$

for $(\theta, x_0) = (\theta^*, x^*_0)$.

Here, we assume the mesh-size $\Delta_i$ between points is uniform i.e $\Delta_i = \Delta = \frac{T}{n}$ and $\epsilon_i \sim N(0, \sigma^2 I_d)$ are i.i.d. For the purpose of estimation, we introduce the sequence of discretized costs $C_{T}^{d,l}$ at the observation points:

$$C_{T}^{d,l}(Y; \theta, x_0, u) = \sum_{i=0}^{n} \Delta \left\| CX_{G,\theta,x_0,u}^d(t_i) - Y_i \right\|^2 + \sum_{i=0}^{n-1} \Delta u_i^T U u_i$$

$$= \sum_{i=0}^{n} X_{d,l,e}^{\theta,x_0,u}(t_i) Y_i + \sum_{i=0}^{n-1} u_i^T U u_i$$

with $W_i = \begin{pmatrix} C^T C & -C^T Y_i \\ Y_i^T C & Y_i^T Y_i \end{pmatrix}$ and $U$ a positive definite matrix. Here $X_{\theta,x_0,u}^{d,l,e}$ is ruled by the finite difference equation:

$$\begin{align*}
X_{\theta,x_0,u}^{d,l,e}(t_{i+1}) &= A_{G,\theta,i}^{d,l,e} X_{\theta,x_0,u}^{d,l,e}(t_i) + B_1 \Delta u_i \\
X_{\theta,x_0,u}^{d,l,e}(0) &= X^e(0) = (x_0, 1)
\end{align*}$$
with $A_{G,\theta,i}^{d,l,e} = I_{d+1} + \Delta A_{G,\theta}(X_{\theta,x_0}^{d,l-1,e}(t_i), t_i)$, $A_{G,\theta}(x,t) = \begin{pmatrix} A_\theta(x,t) & 0_{d,1} \\ 0_{1,d} & 0 \end{pmatrix}$ and $B_1 = \begin{pmatrix} B \\ 0_{1,d_a} \end{pmatrix}$. In this case it is known (see Sontag (1998) for example) the cost $C_{T}^{d,l}(Y; \theta, x_0, u)$ has a global minimum in $u$, denoted $u_{\theta}^{d,l}$ and equal to:

$$
\overline{u}_{\theta,i}^{d,l} = -(U + \Delta B_1^T E_{\theta,i+1}^{d,l}(Y) B_1) - B_1^T E_{\theta,i+1}^{d,l}(Y) A_{G,\theta,i}^{d,l,e} X_{\theta,x_0}^{d,l,e}(t_i) \tag{5}
$$

with $E_{\theta,i}^{d,l}(Y)$ the solution of the discrete Riccati equation:

$$
\begin{cases}
E_{\theta,i}^{d,l}(Y) = \left(A_{G,\theta,i}^{d,l,e}\right)^T E_{\theta,i+1}^{d,l}(Y) A_{G,\theta,i}^{d,l,e} + \Delta W_i \\
- \left(A_{G,\theta,i}^{d,l,e}\right)^T E_{\theta,i+1}^{d,l}(Y) \Delta B_1 [U + \Delta B_1^T E_{\theta,i+1}^{d,l}(Y) B_1]^{-1} B_1^T E_{\theta,i+1}^{d,l}(Y) A_{G,\theta,i}^{d,l,e}
\end{cases} \tag{6}
$$

and $X_{\theta,x_0}^{d,l,e}$ the extended optimal trajectory i.e the solution of (4) for the control $u_{\theta}^{d,l}$. Moreover, the minimum cost value is equal to:

$$
S_{n}^{l}(Y; \theta, x_0) := \inf_u C_{T}^{d,l}(Y; \theta, x_0, u) = X^e(0)^T E_{\theta,0}^{d,l}(Y) X^e(0).
$$

From this, it is also easy to see that $\inf_u C_{T}^{d,l}(Y; \theta, x_0, u)$ is a quadratic form w.r.t $x_0$ and then we can profile $\inf_u C_{T}^{d,l}(Y; \theta, x_0, u)$ on $x_0$ without complexifying too much the computational problem. We also introduce the cost sequentially profiled on $u$ then $x_0$:

$$
S_{n}^{C_{T}^{d,l}(Y; \theta, x_0, u)} := \inf_{x_0} \inf_u C_{T}^{d,l}(Y; \theta, x_0, u) = \inf_{x_0} X^e(0)^T E_{\theta,0}^{d,l}(Y) X^e(0).
$$

Let us now assume we have access to the true continuous signal: $t \rightarrow Y^*(t) = CX_{\theta^*,x_0^*}(t)$ then we can define the sequence of continuous costs:

$$
C_{T}^{l}(\theta, x_0, u) = \left. d \sigma^2 + \int_0^T \left( \left\| CX_{\theta,x_0,u}^l(t) - Y^*(t) \right\|^2 + u(t)^T U u(t) \right) dt \right| \tag{7}
$$

with $W(t) = \begin{pmatrix} C^T C & -C^T Y^*(t) \\
-Y^*(t) C^T & Y^*(t) Y^*(t) + d \sigma^2 \end{pmatrix}$ and $X_{\theta,x_0,u}^{l,e}$ the solution of the extended ODE:

$$
\begin{cases}
X_{\theta,x_0,u}^{l,e}(t) = A_{G,\theta}(X_{\theta-x_0}^{l-1,e}(t), t) X_{\theta,x_0,u}^{l,e}(t) + B_1 u(t) \\
X_{\theta,x_0,u}^{l,e}(0) = X^e(0) = (x_0, 1)\).
\end{cases} \tag{8}
$$
As in the discrete case, the cost \( I \) has a unique minimum on \( u \), denoted \( u^*_\theta \) and equal to:

\[
\overline{u^*_\theta}(t) = -U^{-1}B_1^T \overline{E_\theta^t}(t)\overline{X^t_{\theta,x_0}}(t)
\]

with \( \overline{E_\theta^t} \) the solution of the continuous Riccati equation:

\[
\begin{align*}
\dot{\overline{E_\theta^t}}(t) &= -W(t) - A_{\theta, \theta}(X_{\theta,x_0}^{t-1,e}(t), t)^T \overline{E_\theta^t}(t) - \overline{E_\theta^t}(t)A_{\theta, \theta}(X_{\theta,x_0}^{t-1,e}(t), t) \\
+ &E_\theta^t(t) B_1 U^{-1} B_1^T \overline{E_\theta^t}(t) \\
\overline{E_\theta^t}(T) &= 0_{d+1,d+1}
\end{align*}
\]

(9)

and \( X_e^{t, \theta, x_0} \) the extended optimal trajectory i.e. the solution corresponding of \( I_\theta \) for the control \( u := \overline{u^*_\theta} \). We can also express the profiled cost on \( u \) then \( x_0 \) in a similar way as in the discrete case:

\[
\begin{align*}
S^t(\theta, x_0) := & \inf_u C_T^t(\theta, x_0, u) = X^e(0)^T \overline{E_\theta^t}(0) X^e(0) \\
S^{C,T, l}(\theta) := & \inf_{x_0} \inf_u C_T^t(\theta, x_0, u) = \inf_{x_0} X^e(0)^T \overline{E_\theta^t}(0) X^e(0).
\end{align*}
\]

Finally, we introduce the asymptotic cost:

\[
C_T^{\infty}(\theta, x_0, u) = d \sigma^2 + \int_0^T \left( \left\| CX_{\theta,x_0,u}^{\infty,e}(t) - Y^*(t) \right\|_2 + u(t)^T U u(t) \right) dt
\]

(10)

associated to the ODE:

\[
\begin{align*}
X_{\theta,x_0,u}^{\infty,e}(t) &= A_{\theta, \theta}(X_{\theta,x_0}^{\infty,e}(t), t)X_{\theta,x_0,u}^{\infty,e}(t) + B_1 u(t) \\
X_{\theta,x_0,u}^{\infty,e}(0) &= X^e(0) = (x_0, 1). 
\end{align*}
\]

(11)

Again, for a given couple \((\theta, x_0)\), the optimal control denoted \( \overline{u^*_\theta} \) is equal to:

\[
\overline{u^*_\theta}(t) = -U^{-1}B_1^T \overline{E_\theta^t}(t)\overline{X^t_{\theta,x_0}}(t)
\]

by introducing the continuous Riccati equation:

\[
\begin{align*}
\dot{\overline{E_\theta^\infty}}(t) &= -W(t) - A_{\theta, \theta}(X_{\theta,x_0}^{\infty,e}(t), t)^T \overline{E_\theta^\infty}(t) - \overline{E_\theta^\infty}(t)A_{\theta, \theta}(X_{\theta,x_0}^{\infty,e}(t), t) \\
+ &E_\theta^\infty(t) B_1 U^{-1} B_1^T \overline{E_\theta^\infty}(t) \\
\overline{E_\theta^\infty}(T) &= 0_{d+1,d+1}
\end{align*}
\]

(12)

and we can express the profiled cost values \( S^\infty(\theta, x_0) := \inf_u C_T^{\infty}(\theta, x_0, u) = X^e(0)^T \overline{E_\theta^\infty}(0) X^e(0) \) and \( S^{C,T, \infty}(\theta) := \inf_{x_0} X^e(0)^T \overline{E_\theta^\infty}(0) X^e(0) \). In the different costs definition, we dropped the dependance in \( U \) because no asymptotic behavior conditions are required on it for the next proofs.

In the linear case since \( A_{\theta} \) does not depends on \( x \) anymore, we have \( C_T^{d,l_1} = C_T^{d,l_2} \) and \( C_T^{l_1} = C_T^{l_2} = C_T^{\infty} \) for all \( l_1 \) and \( l_2 \) belonging to \( \mathbb{N} \). Thus, there is no need to consider the asymptotic in \( l \) in this case and we drop the dependance in \( l \) in all quantities of interest i.e. \( C_T^{d} := C_T^{d,l} \), \( C_T := C_T^{l} \) and \( S_n^{C,T} := S_n^{C,T,l} \) and for the solution of the Riccati equation.
1.2. Hypothesis & Notations

1.2.1. Hypothesis for \( \hat{\theta}^T \) in the general case.

**Condition 1:** For all \( t \in [0, T] \) and for all \( \theta \in \Theta \), \( x \mapsto A_\theta(x, t) \) has a compact support \( \Lambda \).

**Condition 2:** For all \( x \in \Lambda \), \( \theta \mapsto A_\theta(x, \cdot) \) is continuous on \( \Theta \) and \( \forall \theta \in \Theta \), \( (x, t) \mapsto A_\theta(x, t) \) is continuous on \( \Lambda \times [0, T] \).

**Condition 3:** Matrix \( B \) has independent columns.

**Condition 4:** The true parameters \( (\theta^*, x_0^*) \) belong to the interior of \( \Theta \times \chi \).

**Condition 5:** The solution \( X_{\theta,x_0} \) of (2) is such that if \( CX_{\theta,x_0}(t) = CX_{\theta^*,x_0^*}(t) \) for all \( t \in [0, T] \) then \( (\theta, x_0) = (\theta^*, x_0^*) \).

**Condition 6:** For all \( x \in \Lambda \), \( \theta \mapsto A_\theta(x, \cdot) \) is differentiable on \( \Theta \), for all \( \theta \in \Theta \), \( (x, t) \mapsto \frac{\partial A_\theta(x,t)}{\partial \theta} \) is continuous on \( \Lambda \times [0, T] \).

**Condition 7:** For all \( x \in \Lambda \), \( \theta \mapsto A_\theta(x, \cdot) \) is twice differentiable on \( \Theta \), for all \( \theta \in \Theta \), \( (x, t) \mapsto \frac{\partial^2 A_\theta(x,t)}{\partial^2 \theta} \) is continuous on \( \Lambda \times [0, T] \).

**Condition 8:** The asymptotic hessian matrix \( \frac{\partial^2 S^\infty(\theta^*,x^*_0)}{\partial^2 \theta} \) is nonsingular.

1.2.2. Hypothesis for \( \hat{\theta}^T, CI \) in the linear case

**Condition L1:** For all \( \theta \in \Theta \), \( t \mapsto A_\theta(t) \) is differentiable on \([0, T]\).

**Condition L2:** \( \theta \mapsto A_\theta \) is continuous on \( \Theta \).

**Condition L3:** For all \( \theta \in \Theta \), \( R_\theta(0) \) is nonsingular, where \( R_\theta \) is defined by ODE (13).

**Condition L4:** The true parameter \( \theta^* \) belongs to the interior of \( \Theta \).

**Condition L5:** The solution \( X_{\theta,x_0} \) of (2) is such that if \( CX_{\theta,x_0}(t) = CX_{\theta^*,x_0^*}(t) \) for all \( t \in [0, T] \) then \( (\theta, x_0) = (\theta^*, x_0^*) \).

**Condition L6:** \( \theta \mapsto A_\theta \) is \( C^2 \) on \( \Theta \).

**Condition L7:** The asymptotic hessian matrix \( \frac{\partial^2 S^{C^1}(\theta^*)}{\partial \theta} \) is a nonsingular matrix.

1.3. Notation

We denote:

(a) \( \overline{A} = \sup_{(\theta,x,t) \in \Theta \times \Lambda \times [0,T]} \| A_\theta(x,t) \|_2 \) .
The interest is twofold, first it reduces the computational burden for w.r.t to the solutions of simplified versions of the original Riccati equations (6, 9 and 12). 

In this section, we derive the expressions of  

\[ \hat{\theta} \text{(1)} \text{ an arbitrary function } g \text{ (possibly vector or matrix valued) such that } \lim_{t \to \infty} g(t) = 0. \]

\[ o_n(f(\Delta)) \text{ (resp } O_n(f(\Delta)) \text{ ) an arbitrary function } g \text{ (possibly vector or matrix valued) such that } \lim_{n \to \infty} \frac{g(\Delta)}{f(\Delta)} = 0. \text{ (resp } \lim_{n \to \infty} \frac{g(\Delta)}{f(\Delta)} = L \text{ with } L \text{ constant and finite).} \]

\[ o_{p,n}(f(\Delta)) \text{ (resp } O_{p,n}(f(\Delta)) \text{ ) a random variable } g \text{ such that } \frac{g(\Delta)}{f(\Delta)} \text{ tends to } 0 \text{ in probability when } n \to +\infty \text{ ( resp } \frac{g(\Delta)}{f(\Delta)} \text{ is bounded in probability when } n \to +\infty). \]

For the proof of estimator consistency and asymptotic normality in the nonlinear case, we often use the notation \( v = (\theta, x_0) \) as well as \( Y = \Theta \times \chi \).

For the sake of notation we denote \( \hat{\theta} \) for both \( \hat{\theta}^T \) and \( \hat{\theta}^{T,CI} \), the ambiguity being clarified by the context.

2. Profiled costs: alternative representations, well-definednesses and regularities

In this section, we derive the expressions of \( S^l_n, S^l \) and \( S^\infty \) (resp. \( S_n^{CI,l}, S^{CI,l} \) and \( S^{CI,\infty} \)) w.r.t to the solutions of simplified versions of the original Riccati equations [6, 9] and [12]. The interest is twofold, first it reduces the computational burden for \( S^l \) and \( S_n^{CI,l} \) optimization and make the theoretical asymptotic analysis of our estimator easier by specifying precisely where the measurement errors intervene in \( S^l_n \) and \( S_n^{CI,l} \). In this section, we denote \( X^l_{\nu} \) and \( X^d,l_{\nu} \) the discrete and continous optimal trajectories whether or not we profile on the initial conditions.

**Proposition 1.** We have:

\[
\begin{align*}
S^l(\theta, x_0) &= x_0^T R_{\theta}^l(0)x_0 + 2x_0^T \sigma^2 B^T R_{\theta}^l(0) \sigma^2 B^T R_{\theta}^l(0) + \int_0^T (Y^*(t)^TY^*(t) + \sigma^2 B^T R_{\theta}^l(0) \sigma^2 B^T R_{\theta}^l(0) dt \\
S^{CI,l}(\theta) &= -h_\theta^l(0)^T R_{\theta}^l(0)^{-1} h_\theta^l(0) + \int_0^T (Y^*(t)^TY^*(t) + \sigma^2 B^T R_{\theta}^l(0) \sigma^2 B^T R_{\theta}^l(0) dt
\end{align*}
\]

with:

\[
\begin{align*}
\dot{R}_{\theta}^l(t) &= -C T C - A_\theta (X_{\nu}^{l,-1}(t), t)^T R_{\theta}^l(t) - R_{\theta}^l(t) A_\theta (X_{\nu}^{l,-1}(t), t) + R_{\theta}^l(t) B^T h_\theta^l(t) + R_{\theta}^l(t) B U^{-1} B^T h_\theta^l(t) \\
h_{\theta}^l(t) &= C T Y^*(t) - A_\theta (X_{\nu}^{l,-1}(t), t)^T h_\theta^l(t) + R_{\theta}^l(t) B U^{-1} B^T h_\theta^l(t) \\
X_{\nu}^l(t) &= A_\theta (X_{\nu}^{l,-1}(t), t) X_{\nu}^l(t) - B U^{-1} B^T (R_{\theta}^l(t) X_{\nu}^l(t) + h_\theta^l(t)) \\
(R_{\theta}^l(T), h_\theta^l(T)) &= (0_{d,d}, 0_{d,1})
\end{align*}
\]

where \( X_{\nu}^l(0) = x_0 \) for \( S^l(\theta, x_0) \) computation and \( X_{\nu}^l(0) = -R_{\theta}^l(0)^{-1} h_\theta^l(0) \) for \( S^{CI,l}(\theta) \).
Similarly, we have the expressions:

\[
\begin{align*}
S^\infty(\theta, x_0) &= x_0^T R_\theta^\infty(0)x_0 + 2x_0^T h_\theta^\infty(0) \\
&\quad + \int_0^T (Y^*(t)^TY^*(t) + d' \sigma^2 - h_\theta^\infty(t)^T BU^{-1}B^T h_\theta^\infty(t)) \, dt \\
S^{CI, \infty}(\theta) &= -h_\theta^\infty(0)^T R_\theta^\infty(0)^{-1} h_\theta^\infty(0) \\
&\quad + \int_0^T (Y^*(t)^TY^*(t) + d' \sigma^2 - h_\theta^\infty(t)^T BU^{-1}B^T h_\theta^\infty(t)) \, dt
\end{align*}
\]

with:

\[
\begin{align*}
R_\theta^\infty(t) &= -C^T C - A_\theta(\overline{X}_0^\infty(t), t)^T R_\theta^\infty(t) - R_\theta^\infty(t)A_\theta(\overline{X}_0^\infty(t), t) \\
&\quad + R_\theta^\infty(t)BU^{-1}B^T R_\theta^\infty(t) \\
h_\theta^\infty(t) &= C^TY^*(t) - A_\theta(\overline{X}_0^\infty(t), t)^T h_\theta^\infty(t) + R_\theta^\infty(t)BU^{-1}B^T h_\theta^\infty(t) \\
\overline{X}_0^\infty(0) &= x_0 \\
(\overline{X}_0^\infty(T), h_\theta^\infty(T)) &= (0_{d,d}, 0_{1,1})
\end{align*}
\]

where \( \overline{X}_0^\infty(0) = x_0 \) for \( S^\infty(\theta, x_0) \) and \( \overline{X}_\theta^\infty(0) = -R_\theta^\infty(0)^{-1} h_\theta^\infty(0) \) for \( S^{CI, \infty}(\theta) \).

**Proof.** It is easy to verify that \( E_\theta^I(t) \) is symmetric and thus can be decomposed under the form \( E_\theta^I(t) = \begin{pmatrix} R_\theta^I(t) & h_\theta^I(t) \\ h_\theta^I(t)^T & \alpha_\theta^I(t) \end{pmatrix} \). We now re-inject this expression into the ODE (9) to obtain:

\[
\begin{align*}
\dot{E}_\theta(t) &= -\begin{pmatrix}
C^T C & -C^TY^*(t) \\
-Y^*(t)^TC & Y^*(t)^TY^*(t) + d' \sigma^2
\end{pmatrix}
\begin{pmatrix}
R_\theta^I(t) \\
h_\theta^I(t)
\end{pmatrix}
- \begin{pmatrix}
A_\theta(\overline{X}_0^{-1}(t), t)^T R_\theta^I(t) \\
h_\theta^I(t)^T A_\theta(\overline{X}_0^{-1}(t), t)
\end{pmatrix}
\begin{pmatrix}
0_{d,d,1} \\
0
\end{pmatrix}
+ \begin{pmatrix}
R_\theta^I(t) \\
h_\theta^I(t)
\end{pmatrix}
\begin{pmatrix}
B \\
0_{1,d_u}
\end{pmatrix}
U^{-1}
\begin{pmatrix}
B \\
0_{1,d_u}
\end{pmatrix}^T
\begin{pmatrix}
R_\theta^I(t) \\
h_\theta^I(t)
\end{pmatrix}
\begin{pmatrix}
\alpha^I(t) \\
\alpha^I(t)
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
&= -\begin{pmatrix}
C^T C \\
-Y^*(t)^TY^*(t) + d' \sigma^2
\end{pmatrix}
\begin{pmatrix}
R_\theta^I(t) \\
h_\theta^I(t)
\end{pmatrix}
- \begin{pmatrix}
A_\theta(\overline{X}_0^{-1}(t), t)^T R_\theta^I(t) \\
h_\theta^I(t)^T A_\theta(\overline{X}_0^{-1}(t), t)
\end{pmatrix}
\begin{pmatrix}
0_{d,d,1} \\
0
\end{pmatrix}
+ \begin{pmatrix}
R_\theta^I(t) \\
h_\theta^I(t)
\end{pmatrix}
\begin{pmatrix}
BU^{-1}B^T \\
0_{d,d,1}
\end{pmatrix}
\begin{pmatrix}
R_\theta^I(t) \\
h_\theta^I(t)
\end{pmatrix}
\begin{pmatrix}
\alpha^I(t) \\
\alpha^I(t)
\end{pmatrix}
\end{align*}
\]
From this we can derive the ODE followed by each element of $E_0^l$,

\[
\begin{cases}
\dot{R}_0^l(t) = -C^T C - A_0 (X_{d,l}^{-1}(t), t)^T R_0^l(t) - R_0^l(t) A_0 (X_{d,l}^{-1}(t), t) \\
+ R_0^l(t) B U^{-1} B^T R_0^l(t) \\
\dot{h}_0^l(t) = C Y^*(t) - A_0 (X_{d,l}^{-1}(t), t)^T h_0^l(t) + R_0^l(t) BU^{-1} B^T h_0^l(t) \\
\alpha_0^l(t) = -Y^*(t) Y^*(t) - d' \sigma^2 + h_0^l(t) B U^{-1} B^T h_0^l(t)
\end{cases}
\]

in particular, $\alpha_0^l(0) = \int_0^T (Y^*(t) Y^*(t) + d' \sigma^2 - h_0^l(t) B U^{-1} B^T h_0^l(t)) \, dt$ and:

\[
\inf_a C_T^l(Y; \theta, x_0, u) = X^*(0)^T E_0^l(0) X^*(0)
\]

\[
= \begin{pmatrix} x_0^T & 1 \end{pmatrix} \begin{pmatrix} R_0^l(0) & h_0^l(0) \\ h_0^l(0)^T & \alpha_0^l(0) \end{pmatrix} \begin{pmatrix} x_0 \\ 1 \end{pmatrix}
\]

\[
= x_0^T R_0^l(0) x_0 + 2 x_0^T h_0^l(0) + \alpha_0^l(0)
\]

\[
= x_0^T R_0^l(0) x_0 + 2 x_0^T h_0^l(0) + \int_0^T (Y^*(t) Y^*(t) + d' \sigma^2 - h_0^l(t) B U^{-1} B^T h_0^l(t)) \, dt
\]

hence the expression for $S^l(\theta, x_0)$. One can see $\inf_a C_T^l(Y; \theta, x_0, u)$ is quadratic w.r.t to $x_0$ with the minimum reached for $x_{\theta,0}^l = -R_0^l(0)^{-1} h_0^l(0)$, from this we easily obtain:

\[
S_{C1,l}(\theta) = \inf_{x_0} \inf_a C_T^l(Y; \theta, x_0, u)
\]

\[
= -h_0^l(0)^T R_0^l(0)^{-1} h_0^l(0)
\]

\[
+ \int_0^T (Y^*(t) Y^*(t) + d' \sigma^2 - h_0^l(t) B U^{-1} B^T h_0^l(t)) \, dt.
\]

The optimal cost becomes

\[
\overline{u_0}(t) = -U^{-1} B_1^T E_0^l(t) \overline{X_0^l}(t) = -U^{-1} B_1^T (R_0^l(t) \overline{X_0^l}(t) + h_0^l(t))
\]

and from this we derive the ODE followed by $\overline{X_0^l}$. For $S^\infty$ (resp. $S_{C1,\infty}$), the formal computation is almost the same as in $S^l$ (resp. $S_{C1,l}$) case and thus is omitted.

**Proposition 2.** For all $(l, i) \in \mathbb{N} \times [0, n]$, $E_{\theta, i}^{d,l}(Y)$ is symmetric and can be written

\[
E_{\theta, i}^{d,l}(Y) = \begin{pmatrix} R_{\theta, i}^{d,l} & h_{\theta, i}^{d,l}(Y) \\ h_{\theta, i}^{d,l}(Y)^T & \alpha_{\theta, i}^{d,l}(Y) \end{pmatrix}.
\]

Moreover, each element is ruled by the finite difference
\[
R_{\theta,i}^{d,l} = R_{\theta,i+1}^{d,l} + \Delta CT C + \Delta \left( R_{\theta,i+1}^{d,l} A_\theta \overline{X_{v,i}^{d,l-1}}(t_i, t_i) + A_\theta \overline{X_{v,i}^{d,l-1}}(t_i, t_i)^T R_{\theta,i+1}^{d,l} \right) \\
+ \Delta^2 A_\theta \overline{(X_{v,i}^{d,l-1}(t_i, t_i))^T R_{\theta,i+1}^{d,l} A_\theta (X_{v,i}^{d,l-1}(t_i, t_i))} \\
- \Delta R_{\theta,i+1}^{d,l} BG(R_{\theta,i+1}^{d,l})^T R_{\theta,i+1}^{d,l} (I_d + \Delta A_\theta (X_{v,i}^{d,l-1}(t_i, t_i))) \\
- \Delta^2 A_\theta \overline{(X_{v,i}^{d,l-1}(t_i, t_i))^T R_{\theta,i+1}^{d,l} BG(R_{\theta,i+1}^{d,l})^T R_{\theta,i+1}^{d,l} (I_d + \Delta A_\theta (X_{v,i}^{d,l-1}(t_i, t_i)))} \\
\]

Proof. We prove that by using the finite difference equation (8) and reversed time induction. It is obvious the property holds for \( R_{\theta,n}^{d,l+1} \). Now let us assume it holds for \( i+1 \) i.e.

that: \( E_{\theta,i+1}^{d,l+1} = \left( \begin{array}{c}
R_{\theta,i+1}^{d,l+1} \\
h_{\theta,i+1}^{d,l+1}(Y) \\
de_{\theta,i+1}^{d,l+1}(Y)
\end{array} \right) \). Now let us compute \( A_{G,\theta,i}^{d,l,e} \) and \( A_{G,\theta,i}^{d,l,e} \) we have:

\[
\left( A_{G,\theta,i}^{d,l,e} \right)^T E_{\theta,i+1}^{d,l+1}(Y) A_{G,\theta,i}^{d,l,e} \\
= \left( \begin{array}{c}
R_{\theta,i+1}^{d,l+1} \\
h_{\theta,i+1}^{d,l+1}(Y) \\
de_{\theta,i+1}^{d,l+1}(Y)
\end{array} \right)^T \left( \begin{array}{c}
A_\theta \overline{X_{v,i}^{d,l-1}}(t_i, t_i) + A_\theta \overline{X_{v,i}^{d,l-1}}(t_i, t_i)^T R_{\theta,i+1}^{d,l} A_\theta \overline{X_{v,i}^{d,l-1}}(t_i, t_i) + A_\theta \overline{X_{v,i}^{d,l-1}}(t_i, t_i)^T R_{\theta,i+1}^{d,l} A_\theta \overline{X_{v,i}^{d,l-1}}(t_i, t_i)
\end{array} \right) \\
+ \Delta \left( \begin{array}{c}
A_\theta \overline{X_{v,i}^{d,l-1}}(t_i, t_i)^T R_{\theta,i+1}^{d,l} A_\theta \overline{X_{v,i}^{d,l-1}}(t_i, t_i) \\
de_{\theta,i+1}^{d,l+1}(Y) \\
0
\end{array} \right)
\]

Now let us assume it holds for \( i+1 \) i.e.

\[
S_n^l(Y; \theta, x_0) = x_0^T R_{\theta,0}^{d,l} x_0 + 2h_{\theta,0}^{d,l}(Y)^T x_0 + \Delta Y_n T Y_n + \Delta \sum_{i=0}^{n-1} \left( Y_i^T Y_i - h_{\theta,i+1}^{dl}(Y)^T BG(R_{\theta,i+1}^{d,l})^T h_{\theta,i+1}^{dl}(Y) \right)
\]

Moreover, the profiled cost can be expressed as

\[
\overline{X_{v}^{d,l}}(t_{i+1}) = \left( \begin{array}{c}
I_d + \Delta A_\theta \overline{X_{v,i}^{d,l-1}}(t_i, t_i) \\
\overline{X_{v,i}^{d,l-1}}(t_i)
\end{array} \right) - \Delta BG(R_{\theta,i+1}^{d,l})^T \left( \begin{array}{c}
R_{\theta,i+1}^{d,l} (I_d + \Delta A_\theta \overline{X_{v,i}^{d,l-1}}(t_i, t_i)) \\
\overline{X_{v,i}^{d,l-1}}(t_i)
\end{array} \right)
\]

(15)

with final condition \( (R_{\theta,n}^{d,l}, h_{\theta,n}^{d,l}(Y)) = (\Delta CT C, -\Delta CT Y_n) \) and \( G(R_{\theta,i+1}^{d,l}) := \left[ U + \Delta B^T R_{\theta,i+1}^{d,l} B \right]^{-1} \).
where the symmetric matrix $F$ is easily derivable by identification and the $d$ dimensional square matrix $F_1$ only depends on $R_{\theta,i+1}^{d,l}$. Now let us compute the expression in $[U + \Delta B_1^T E_{\theta,i+1}^{d,l}(Y) B_1]^{-1}$, we obtain:

\[
[U + \Delta B_1^T E_{\theta,i+1}^{d,l}(Y) B_1]^{-1} = \left[ U + \Delta \left( B^T R_{\theta,i+1}^{d,l} B R_{\theta,i+1}^{d,l}(Y) \right) \left( \begin{array}{c} B \\
0_{1,d_u} \end{array} \right) \right]^{-1} 
\]

\[
= \left[ U + \Delta \left( B^T R_{\theta,i+1}^{d,l} B R_{\theta,i+1}^{d,l}(Y) \right) \left( \begin{array}{c} B \\
0_{1,d_u} \end{array} \right) \right]^{-1} 
\]

\[
= \left[ U + \Delta B_1^T R_{\theta,i+1}^{d,l} B \right]^{-1} := G(R_{\theta,i+1}^{d,l}) 
\]

with $G$ a $d_u$ symmetric (as the inverse of a symmetric matrix square matrix depending only of $R_{\theta,i+1}^{d,l}$. Moreover, we have:

\[
\left( A_{G,\theta,i}^{d,l,e} \right)^T E_{\theta,i+1}^{d,l}(Y) B_1 = \left( I_{d+1} + \Delta \left( A_{\theta}(X_v^{d,l-1}(t_i), t_i) 0_{d,1} \right) 0_{1,d} \right)^T \left( \begin{array}{c} P_{\theta,i+1}^{d,l} F_{\theta,i+1}^{d,l}(Y) \\
0_{1,d_u} \end{array} \right) 
\]

\[
= \left( I_{d+1} + \Delta \left( A_{\theta}(X_v^{d,l-1}(t_i), t_i) 0_{d,1} \right) 0_{1,d} \right)^T \left( \begin{array}{c} P_{\theta,i+1}^{d,l} B \\
0_{1,d_u} \end{array} \right) 
\]

so we can compute:

\[
\left( A_{G,\theta,i}^{d,l,e} \right)^T E_{\theta,i+1}^{d,l}(Y) B_1 G(R_{\theta,i+1}^{d,l}) B_1^T E_{\theta,i+1}^{d,l}(Y) A_{G,\theta,i}^{d,l,e} 
\]

\[
\left( R_{\theta,i+1}^{d,l} B + \Delta A_{\theta}(X_v^{d,l-1}(t_i), t_i) T R_{\theta,i+1}^{d,l} B \right) G(R_{\theta,i+1}^{d,l}) \left( R_{\theta,i+1}^{d,l} B + \Delta A_{\theta}(X_v^{d,l-1}(t_i), t_i) T R_{\theta,i+1}^{d,l} B \right)^T 
\]

\[
= \left( I_{d+1} + \Delta A_{\theta}(X_v^{d,l-1}(t_i), t_i) T R_{\theta,i+1}^{d,l} B \right) G(R_{\theta,i+1}^{d,l}) \left( B^T R_{\theta,i+1}^{d,l} B \right) 
\]

By re-injecting all the derived expression in (6), we obtain:

\[
E_{\theta,i}^{d,l}(Y) = \left( \begin{array}{c} F_1(R_{\theta,i+1}^{d,l}) \\
F_2(h_{\theta,i+1}^{d,l}(Y)) \\
F_3(h_{\theta,i+1}^{d,l}(Y)) \end{array} \right) + \Delta \left( \begin{array}{c} C^T C \\
-C^TY_i \\
-Y_i^T C \end{array} \right) 
\]

and $E_{\theta,i}^{d,l}(Y)$ is symmetric and has indeed the required form, hence the recursion. We also
obtain the following finite difference equation:

\[
R_{\theta,i}^{d,l} = R_{\theta,i+1}^{d,l} + \Delta C^T C + \Delta \left( R_{\theta,i+1}^{d,l} A_\theta(\overline{X_{\theta,i}^{d,l-1}(t_i)}, t_i) + A_\theta(\overline{X_{\theta,i}^{d,l-1}(t_i)}, t_i)^T R_{\theta,i+1}^{d,l} \right)
+ \Delta^2 A_\theta(\overline{X_{\theta,i}^{d,l-1}(t_i)}, t_i)^T R_{\theta,i+1}^{d,l} A_\theta(\overline{X_{\theta,i}^{d,l-1}(t_i)}, t_i)
+ \left( \frac{\partial}{\partial x} \right)^2 A_\theta(\overline{X_{\theta,i}^{d,l-1}(t_i)}, t_i)^T R_{\theta,i+1}^{d,l} A_\theta(\overline{X_{\theta,i}^{d,l-1}(t_i)}, t_i)
\]

From these equation, we easily derive that:

\[
\begin{align*}
R_{\theta,i}^{d,l} h_{\theta,i}^{d,l}(Y) &= \left( R_{\theta,i}^{d,l} A_\theta(\overline{X_{\theta,i}^{d,l-1}(t_i)}, t_i) + A_\theta(\overline{X_{\theta,i}^{d,l-1}(t_i)}, t_i)^T R_{\theta,i+1}^{d,l} \right) \overline{X_{\theta,i}^{d,l-1}(t_i)} + h_{\theta,i}^{d,l}(Y)
\quad \text{and}
\end{align*}
\]

Proposition 3. Given \( \theta \in \Theta \) and \( l \in \mathbb{N}, R_\theta(0) \) is invertible if and only if:

1) the matrix

\[
O_\theta^l(T) = \int_0^T \left( C \Phi_\theta^l(t, 0) \right)^T C \Phi_\theta^l(t, 0) dt
\]

is invertible, where \( \Phi_\theta^l \) is the resolvent of \( L \),

2) the following implication holds:

\[
\left\| CX_{\theta,x_0} - CX_{\theta, x_0}^1 \right\|^2_{L^2} = 0 \implies x_0^1 = x_0^2.
\]
PROOF. Similarly as in the discrete case, we have \( \min_u \tilde{C}_T^l(\theta, z_0, u) = z_0^T \tilde{E}_\theta^l(Y)z_0 \) with:
\[
\tilde{C}_T^l(\theta, z_0, u) = \int_0^T Z_{\theta,z_0,u}(t)\left(\begin{array}{cc}
C^T & 0_{d,1} \\
0_{1,d} & 0
\end{array}\right) Z_{\theta,z_0,u}(t) + u(t)^T U u(t) dt
\]
linked to the finite difference equation:
\[
\begin{align*}
\dot{Z}_{\theta,z_0,u}(t) &= A_G,\theta(X_{\theta}^{l-1}(t), t)Z_{\theta,z_0,u}(t) + B_1 u(t) \\
Z_{\theta,z_0,u}(0) &= z_0
\end{align*}
\] (18)
here \( \tilde{E}_\theta^l \) is the solution of the Riccati equation (21) where the weight function \( t \mapsto W(t) \) has been replaced by \( \left(\begin{array}{cc}
C^T & 0_{d,1} \\
0_{1,d} & 0
\end{array}\right) \), henceforth \( \tilde{E}_\theta^l \) and \( E_\theta^l \) share the same component \( R_\theta^l \). We define \( \bar{x}_0 \) s.t \( \bar{x}_0^T R_\theta^l(0) \bar{x}_0 = 0 \) and \( \tilde{z}_0 = \left(\begin{array}{c}
\bar{x}_0 \\
0
\end{array}\right) \). From this, we derive \( \min_u \tilde{C}_T^l(\theta, \tilde{z}_0, u) = 0 \) which implies \( u_0^* = 0 \). The ODE corresponding to the optimal trajectory becomes:
\[
\dot{Z}_{\theta,z_0}^l(t) = A_G,\theta(X_{\theta}^{l-1}(t), t)Z_{\theta,z_0}^l(t)
\]
Hence the optimal trajectory is \( Z_{\theta,z_0}^l(t) = \left(\begin{array}{c}
\Phi^l_\theta(t, 0) \bar{x}_0 \\
0
\end{array}\right) \). Thus, the minimal cost has the simpler expression
\[
\min_u \tilde{C}_T^l(\theta, \tilde{z}_0, u) = \int_0^T \bar{x}_0^T \Phi^l_\theta(t, 0)^T (C^T C) \Phi^l_\theta(t, 0) \bar{x}_0 dt
\]
so we have: \( \bar{x}_0^T R_\theta^l(0) \bar{x}_0 = \min_u \tilde{C}_T^l(\theta, z_0, u) = \bar{x}_0^T O_\theta^l(T) \bar{x}_0 = 0 \) and we can conclude.

We now demonstrate \( 17 \) \( \implies \) \( 16 \), we choose an arbitrary initial condition \( \bar{x}_0 \) respecting \( \bar{x}_0^T O_\theta^l(T) \bar{x}_0 = \bar{x}_0^T \left( J_0 \Phi^l_\theta(t, 0)^T C^T C \Phi^l_\theta(t, 0) dt \right) \bar{x}_0 = 0 \). We denote \( X_{\theta,x_0}^l \) and \( X_{\theta,x_0}^l \) the solutions of \( \dot{X} = A(X_{\theta}^{l-1}(t), t)X \) respectively with initial conditions \( x_0^l \) and \( x_0^l \). By superposition principle for linear ODE solution we have \( \Phi^l_\theta(t, 0) \bar{x}_0 = X_{\theta,x_0}^l(t) - X_{\theta,x_0}^l(0) \), hence \( 17 \) imposes that \( X_{\theta,x_0}^l(0) = X_{\theta,x_0}^l(0) \) and so \( \bar{x}_0 = 0 \).

To demonstrate the implication \( 16 \) \( \implies \) \( 17 \), we take two functions \( X_{\theta,x_0}^l \) and \( X_{\theta,x_0}^l \) assumed to verify \( \| C X_{\theta,x_0}^l - C X_{\theta,x_0}^l \|_{L^2} = 0 \) solutions of \( \dot{X} = A(X_{\theta}^{l-1}(t), t)X \) with respective initial conditions \( x_0^l \) and \( x_0^l \). We know the function difference \( X_{\theta,x_0}^l(t) := X_{\theta,x_0}^l(t) - X_{\theta,x_0}^l(t) \) is equal to \( \Phi^l_\theta(t, 0) \bar{x}_0 \) and so \( \bar{x}_0^T O_\theta^l(T) \bar{x}_0 = 0 \), \( O_\theta^l(T) \) invertibility gives us \( x_0^l - x_0^l = 0 \).

**Proposition 4.** Under conditions C1-C2, we have \( \| E_{\theta,i}^l(Y) \|_2 = O_{p,n}(1) \) and \( \| E_{\theta}^l(t) \|_2 = O_{n}(1) \) respectively uniformly on \( \mathbb{N} \times [0, n] \times \Theta \) and on \( \mathbb{N} \times [0, T] \times \Theta \).
PROOF. By property of the solution of Riccati equation, we have

\[
Z_k^{d,l} E_{\theta,k}^{d,l}(Y) z_k = \min_{u=\{u_k, \ldots, u_{n-1}\}} \left\{ \Delta \sum_{i=k}^{n} Z_{\theta, z_k, u}^{d,l}(t_i) T W_i Z_{\theta, z_k, u}^{d,l}(t_i) + \Delta \sum_{i=k}^{n-1} u_i T U u_i \right\} \geq 0
\]

where \(Z_{\theta, z_k, u}^{d,l}\) is the solution of

\[
\begin{align*}
Z_{\theta, z_k, u}^{d,l}(t_{i+1}) &= A^{d,l} Z_{\theta, z_k, u}^{d,l}(t_i) + B_1 \Delta \tau_i \\
Z_{\theta, z_k, u}^{d,l}(t_k) &= z_k
\end{align*}
\]

where \(\tau = \{\tau_i\}_{i\in[k, n-1]}\) is the optimal control. This holds for every possible values \(z_k\).

In particular, we have the bound \(z_k^T E_{\theta,k}^{d,l}(Y) z_k \leq \Delta \sum_{i=k}^{n} Z_{\theta, z_k, u}^{d,l}(t_i) T W_i Z_{\theta, z_k, u}^{d,l}(t_i)\) with \(Z_{\theta, z_k}\) solution of:

\[
\begin{align*}
Z_{\theta, z_k}^{d,l}(t_{i+1}) &= A^{d,l} Z_{\theta, z_k}^{d,l}(t_i) \\
Z_{\theta, z_k}^{d,l}(t_k) &= z_k
\end{align*}
\]

According to the discrete Gronwall lemma \[16\] and conditions 1-2, we have the bound \(Z_{\theta, z_k}^{d,l}(t_i) \leq e^{T\tau} z_k\) for \(i \in [k, n-1]\), thus \(z_k^T E_{\theta,k}^{d,l}(Y) z_k \leq \Delta \sum_{i=k}^{n} (e^{T\tau} z_k)^T W_i e^{T\tau} z_k \leq \Delta \sum_{i=0}^{n} (e^{T\tau} z_k)^T W_i e^{T\tau} z_k\). Using Cauchy-Schwarz inequality we have \(z_k^T E_{\theta,k}^{d,l}(Y) z_k \leq \|z_k\|^2 e^{2T\tau} \Delta \sum_{i=0}^{n} \|W_i\|_2\) for all \(z_k\).

Since \(\Delta \sum_{i=1}^{n} Y_i = \Delta \sum_{i=1}^{n} \left( Y_i - CX_{\theta, \tau, z}^2(t_i) \right) + CX_{\theta, \tau, z}^2(t_i)\)

\[= \frac{1}{n} \sum_{i=1}^{n} \left( Y_i - CX_{\theta, \tau, z}^2(t_i) \right) + \Delta \sum_{i=1}^{n} CX_{\theta, \tau, z}^2(t_i)\]

\[= \mathbb{E}_{(\theta, \tau, z)} [\epsilon_1] + \int T CX_{\theta, \tau, z}^2(t) dt = \int T CX_{\theta, \tau, z}^2(t) dt\]

almost surely (The first term is obtained by using the law of large number, the second as the limit of a Riemann sum) and

\[
\begin{align*}
\Delta \sum_{i=1}^{n} \|Y_i\|^2_2 &= \Delta \sum_{i=1}^{n} \|Y_i - CX_{\theta, \tau, z}^2(t_i)\|^2_2 + \Delta \sum_{i=1}^{n} \|CX_{\theta, \tau, z}^2(t_i)\|^2_2 \\
&+ 2 \Delta \sum_{i=1}^{n} (Y_i - CX_{\theta, \tau, z}^2(t_i))^T CX_{\theta, \tau, z}^2(t_i) \\
&= \frac{1}{n} \sum_{i=1}^{n} \epsilon_1^2 + \Delta \sum_{i=1}^{n} \|CX_{\theta, \tau, z}^2(t_i)\|^2_2 + 2 \frac{1}{n} \sum_{i=1}^{n} \epsilon_1^T CX_{\theta, \tau, z}^2(t_i) \\
&+ \sigma^2 + \int T \|CX_{\theta, \tau, z}^2(t)\|^2_2 dt
\end{align*}
\]

by using Prohorov’s theorem (van der Vaart (1998) theorem 2.4), we know that \(\Delta \sum_{i=0}^{n} \|W_i\|_2 = O_{p,n}(1)\). Henceforth \(\left\| E_{\theta,k}^{d,l}(Y) \right\|_2 = O_{p,n}(1)\) uniformly on \(N \times [0, n] \times \Theta\). Similarly as in the discrete case, we derive the following bound in the continuous case:

\[
z_t^T E_{\theta}^{l}(t) z_t \leq Z_{\theta, z_t}^{l}(T)^T Q Z_{\theta, z_t}^{l}(T) + \int_t^T Z_{\theta, z_t}(s)^T W(s) Z_{\theta, z_t}(s) ds
\]
with
\[
\begin{align*}
Z^i_{\theta, z_t}(s) &= A_{G, \theta}(X^i_{\theta}(s), s) Z^i_{\theta, z_t}(s) \\
Z^i_{\theta, z_t}(t) &= z_t
\end{align*}
\]
for all \(z_t\). Using the continuous version of the Gronwall lemma we obtain \(Z^i_{\theta, z_t}(s) \leq e^{Tz_t}z_t\) and so \(z_t^T E^i_\theta(t) z_t \leq \|z_t\|^2 e^{2Tz_t} (\|Q\|_2 + \int_0^T \|W(s)\|_2 \, ds)\). This allows us to conclude \(\|E^i_\theta(t)\|_2 = O_n(1)\) uniformly on \(N \times [0, T] \times \Theta\).

3. Well-posedness nature of control problem (10)-(11)

In order to derive asymptotic properties of \(\hat{\theta}^T\) and \(\hat{\theta}^T, C_1\), we need to ensure the well-posedness nature of the optimal control problems defining our estimators. That is, the existence of an admissible solution for the problem (10)-(11) for each possible value \((\theta, x_0)\).

For linear models this is a classic outcome of Linear-Quadratic theory. In the non-linear case however it depends on the cost function and the vector field regularities w.r.t state and control. That is why we prove here the existence of \(\min_{u \in \mathcal{U}_{\theta, x_0}} C^\infty_T(\theta, x_0, u)\) where \(\mathcal{U}_{\theta, x_0}\) is the set of feedback controls:

\[
\mathcal{U}_{\theta, x_0} = \left\{ u_v(t) = U^{-1} B^T (R_v(t) X_{\theta, u_v}(t) + h_v(t)), (R_v, h_v) \in L^2([0, T], \mathbb{R}^{d \times d} \times \mathbb{R}^d) \right\}
\]

The proof is almost similar as the one presented in [Clairon and Brunel (2017)] but with the additional requirement that \(\overline{u}_{\theta, x_0}\) belongs to \(\mathcal{U}_{\theta, x_0}\) which in turn calls for the introduction of C3.

**Theorem 5.** Under conditions C1-C2-C3 for all signals \(Y \in L^2([0, T], \mathbb{R}^d)\) and for all \(\theta \in \Theta\), the asymptotic control problem (10)-(11) admits at least one solution. It exists a control \(\overline{u}_\theta\) belonging to \(\mathcal{U}_{\theta, x_0}\) that minimizes the cost, i.e. \(C^\infty_T(\theta, x_0, \overline{u}_\theta, x_0) = \min_{u \in \mathcal{U}_{\theta, x_0}} C^\infty_T(\theta, x_0, u)\).

**Proof.** Thanks to model regularity conditions, we know it exists admissible controls \(u_v \in L^2([0, T], \mathbb{R}^d)\). Thus, we can consider an admissible minimizing sequence \(\{u^i_v\}_{i \in \mathbb{N}}\). Since we have \(\lambda \|u^i_v\|^2_{L^2} \leq C^\infty_T(u, u^i_v)\) with \(\lambda\) the maximum eigenvalue of \(U\), the sequence \(\{u^i_v\}\) is uniformly bounded in \(L^2([0, T], \mathbb{R}^d)\), a reflexive Banach space, according to Theorem III.27 in [Brezis (1983)], it exists a subsequence converging weakly to a limit \(\overline{u}_\theta, x_0\). Using Hölder inequality \(\|fg\|_{L^1} \leq \|f\|_{L^2} \|g\|_{L^2}\), we derive the subsequence boundedness in \(L^1([0, T], \mathbb{R}^d)\). (For the sake of notation, we still denote the subsequence by \(\{u^i_v\}\)). For the following, \(\overline{u}_v\) denotes the upper bound of the sequence \(\{u^i_v\}\).

Knowing that:

\[
\begin{align*}
\|X_{\theta, u_v^i}(t) - X_v(t)\|_2 &\leq \|A_{\theta}(X_{\theta, u_v^i}(t), t)X_{\theta, u_v^i}(t) - A_{\theta}(X_v(t), t)X_v(t)\|_2 \\
&\quad + \|Bu_v^i(t)\|_2 \\
&\leq (\overline{A} + \overline{A}_x) \|X_{\theta, u_v^i}(t) - X_v(t)\|_2 + \|B\|_2 \|u_v^i(t)\|_2
\end{align*}
\]
here $\overline{A}_x < \infty$ is the Lipschitz constant of $A$ w.r.t state, which existence is ensured by C2 as a continuous function on a compact subset. Gronwall’s lemma gives us:

$$\|X_{\theta,x_0,u^i}(t) - X_{u}(t)\|_2 \leq \|B\|_2 \int_0^t e^{(\overline{A}_x + \overline{A}_v)(t-s)} \|u^i(s)\|_2 \, ds$$

and so:

$$\|X_{u,u^i}(t)\|_2 \leq \|X_{u,u^i}(t) - X_{u}(t)\|_2 + \|X_{u}(t)\|_2 \leq \|B\|_2 e^{(\overline{A}_x + \overline{A}_v)t} \int_0^t \|u^i(s)\|_2 \, ds + \sup_{t \in [0,T]} \|X_{u}(t)\|_2 \cdot$$

The control $u^i$ being bounded in $L^1([0,T], \mathbb{R}^d)$, $X_{u,u^i}$ (modulo a subsequence) is uniformly bounded on $[0,T]$ and since:

$$\|\dot{X}_{u,u^i}(t)\|_2 \leq \|\dot{X}_{u,u^i}(t) - \dot{X}_{u}(t)\|_2 + \|\dot{X}_{u}(t)\|_2 \leq \|A\|_2 (X_{u,u^i}(t) - X_{u}(t)) \|X_{u}(t)\|_2 + \|B u^i(t)\|_2 \leq (\overline{A} + \overline{A}_x) \|X_{u,u^i}(t) - X_{u}(t)\|_2 + \|B u^i(t)\|_2$$

$\dot{X}_{u,u^i}$ is also bounded in $L^2([0,T], \mathbb{R}^d)$, hence (again modulo a subsequence) $\dot{X}_{\theta,x_0,u^i}$ converges weakly to a limit $\overline{X}_v$.

Since the sequence $X_{u,u^i}$ is equicontinuous because $\|X_{u,u^i}(t) - X_{u,u^i}(t')\|_2 \leq \overline{A} |t - t'| + \|B\|_2 \overline{u} \sqrt{|t - t'|}$, we can invoke Arzela-Ascoli theorem to obtain the uniform convergence (modulo a subsequence) of $X_{u,u^i}$ toward a continuous function $\overline{X}_v$ on $[0,T]$. Using the identity, $X_{u,u^i}(t) = x_0 + \int_0^t \dot{X}_{u,u^i}(s) \, ds$ and by taking the limit we know $\overline{X}_v$ is an absolutely continuous function with $\dot{X}_v(t) = \dot{X}_{\overline{v}}(t)$ a.e.

Since $B$ has independent columns then $(U^{-1}B^T)^+ = (BU^{-2}B^T)^-\overline{B}U^{-1}$ is the Moore-Penrose inverse of $U^{-1}B^T$ and is a left inverse of $U^{-1}B^T$, and reminding that the sequences $\{X_{u,u^i}\}_{i \in \mathbb{N}}$ and $\{u^i\}_{i \in \mathbb{N}}$ are bounded and convergent, we can construct a bounded sequence $\{R^i_v, h^i_v\}$ which converges to a limit $\{\overline{R}_v, \overline{h}_v\}$ such that it respect the relation $R^i_v(t)X_{u,u^i}(t) + h^i_v(t) = (U^{-1}B^T)^+ u^i(t)$ and $\overline{R}_v(t)\overline{X}_v + \overline{h}_v(t) = (U^{-1}B^T)^+ \overline{u}_v$. For example, we could take for $R^i_v$ a continuous bounded function and $h^i_v = -R^i_v(t)X_{u,u^i}(t) + (U^{-1}B^T)^+ u^i(t)$ and thus, $\overline{u}_v \in U_v$.

We respect the hypothesis of THEOREM 6.38 in Clarke (2013) and we derive from that:

$$C^\infty_T(v, \overline{u}_v) \leq \liminf_{i \to \infty} C^\infty_T(v, u^i) = \inf_{u} C^\infty_T(v, u).$$

We now demonstrate $\overline{u}_v$ is an admissible process (thus the infimum is reached). Using uniform convergence we have $\overline{X}_v(0) = x_0$. The last thing left to show is that $\overline{X}_v$ is a trajectory corresponding to $\overline{u}_v$, thus $\overline{X}_v = X_{v,\overline{u}_v}$. For any measurable subset $S$ of $[0,T]$ we have:

$$\int_S \left( \dot{X}_{v,u^i}(t) - A\theta(X_{v,u^i}(t), t)X_{v,u^i}(t) - Bu^i(t) \right) \, dt = 0$$
by weak convergence we directly obtain \( \int_S \dot{X}_{v,u}(t)dt \rightarrow \int_S X_v dt \) and \( \int_S B\nu(t)dt \rightarrow \int_S B\pi_v(t)dt \). Using continuity of the vector field on a compact and invoking dominated convergence theorem: \( \int_S A_\theta(X_{v,u}(t), t)X_{v,u}(t)dt \rightarrow \int_S A_\theta(X_v(t), t)X_v(t) \). By taking the limit we obtain:

\[
\int_S \left( \dot{X}_v(t) - A_\theta(X_v(t), t)X_v(t) - B\pi_v(t) \right) dt = 0.
\]

Hence, we have indeed demonstrate \( \pi_v \in L^2([0, T], \mathbb{R}^d) \) and

\[
\left\{ \begin{align*}
\dot{X}_v(t) &= A(X_v(t), t)X_v(t) + B\pi_v(t) \text{ a.e on } [0, T] \\
X_v(0) &= x_0
\end{align*} \right.
\]

which finishes the proof.

4. Consistency

4.1. \( \hat{\theta}^T \), general case

**Theorem 6.** Under conditions C1 to C5, we have

\[
\left( \hat{\theta}, \hat{x}_0 \right) \rightarrow (\theta^*, x_0^*)
\]

in probability when \( (l, n) \rightarrow \infty \).

**Proof.** First, we decompose the difference \( S^\infty(v) - S^l_h(Y; v) \) in two terms we can analyse separately:

\[
S^\infty(v) - S^l_h(Y; v) = S^\infty(v) - S^l(v) + S^l(v) - S^l_h(Y; v).
\]

By using the continuous profited costs formula given by proposition \( \text{[1]} \) we obtain for the first term

\[
S^\infty(v) - S^l(v) = x_0^T R^\infty(0)x_0 + 2x_0^T h^\infty(0) + \int_0^T \left( Y^*(t)^T Y^*(t) + d^1 \sigma^2 - h^\infty_1(t)^T BU^{-1} BT h^\infty(t) \right) dt \\
- x_0^T R^l(0)x_0 + 2x_0^T h^l(0) - \int_0^T \left( Y^*(t)^T Y^*(t) + d^1 \sigma^2 - h^l(t)^T BU^{-1} BT h^l(t) \right) dt \\
= x_0^T \left( R^\infty(0) - R^l(0) \right) x_0 + 2x_0^T \left( h^\infty(0) - h^l(0) \right) \\
+ \int_0^T \left( h^\infty_1(t)^T BU^{-1} BT (h^\infty(t) - h^\infty(t)) + (h^l(t) - h^\infty(t))^T BU^{-1} BT h^\infty(t) \right) dt.
\]

By using lemma \( \text{[3]} \) we conclude that \( \sup_{v \in \mathcal{T}} \left| S^\infty(v) - S^l(v) \right| = o_1(1) \). Now, we control the difference \( S^l(v) - S^l_h(Y; v) \) by using the form given by proposition \( \text{[1]} \) and \( \text{[2]} \):

\[
S^l(v) - S^l_h(Y; v) = x_0^T \left( R^l(0) - R^l_{v, i+1}(Y) \right) x_0 + 2x_0^T \left( h^l_1(0) - h^l_{v, i+1}(Y) \right) \\
+ \int_0^T \left( Y^*(t)^T Y^*(t) + d^1 \sigma^2 \right) dt - \sum_{i=0}^{n-1} Y^T_i \tilde{Y}_i \\
- \left( \int_0^T h^l_1(t)^T BU^{-1} BT h^l(t) dt - \sum_{i=0}^{n-1} h^l_{v, i+1}(Y)^T BG(R^l_{v, i+1}) BT h^l_{v, i+1}(Y) \right).
\]
Lemma [17] gives us $\sup_{v \in \mathcal{T}} \left\| R^{d \times d}_{v, i} - R^{d \times d}_v(t_i) \right\|_2 = o_{p,n}(1)$, $\sup_{v \in \mathcal{T}} \left\| h^{d \times d}_{v, i} - h^{d \times d}_v(t_i) \right\|_2 = o_{p,n}(1)$, so

$$x_0^T \left( R^{d \times d}_v(0) - R^{d \times d}_{v, 0} \right) x_0 + 2x^T_0 \left( h^{d \times d}_v(0) - h^{d \times d}_{v, 0}(Y) \right) = o_{p,n}(1)$$

uniformly on $\mathcal{Y}$. Regarding the second term, we have:

$$J_0^T \left( Y^*(t) Y^*(t) + d' \sigma^2 \right) dt - \Delta \sum_{i=1}^{n_i} \sum_{i=0}^{n_i} Y_i T Y_i$$

with $\Delta \sum_{i=1}^{n_i} Y^*(t_i) T Y_i \sim N(0, \sigma^2 \Delta^2 \sum_{i=1}^{n_i} Y^*(t_i) T Y^*(t_i)) = o_{p,n}(1)$ and $\Delta \sum_{i=0}^{n_i} (d' \sigma^2 - \varepsilon_i^T \varepsilon_i) = o_{p,n}(1)$ by using the strong law of large numbers. Thus,

$$\int_0^T \left( Y^*(t) Y^*(t) + d' \sigma^2 \right) dt - \Delta \sum_{i=0}^{n_i} Y_i T Y_i = o_{p,n}(1).$$

Regarding the third term, we have:

$$J_0^T h^{d \times d}_v(t) B U^{-1} B^T h^{d \times d}_v(t) dt - \Delta \sum_{i=0}^{n_i} h^{d \times d}_{v, i+1} Y_i T B G(R^{d \times d}_{v, i+1}) B^T h^{d \times d}_{v, i+1}(Y)$$

uniformly on $\Gamma$. The third equality has been obtained by using lemma [17] and the fifth one by using proposition [1] From this, we have $\sup_{v \in \mathcal{T}} \left| S^I(v) - S^I_n(Y; v) \right| = o_{p,n}(1)$ and triangular inequality gives us:

$$\sup_{v \in \mathcal{T}} \left| S^{\infty}(v) - S^I_n(Y; v) \right| = o_{p,n}(1).$$

Application of proposition [7] and the fact that $v \mapsto S^{\infty}(v)$ is continuous on the compact $\mathcal{T}$ gives us the identifiability criteria required to apply theorem 5.7 in [van der Vaart, 1998].

**Proposition 7.** Under conditions C1 to C5 $v^*$ is the unique global minimizer of $S^{\infty}(v)$ on $\mathcal{T}$.

We have shown in theorem [5] the control problem defining our estimator is well posed i.e. it exists a control $u^{\infty}_v$ such that $u^{\infty}_v = \arg \min_{u \in U} C^\infty_T (v, u)$ where $U$ is the set of feedback controls:

$$U = \left\{ u(v) = U^{-1} B T (R_v(t) X_{v, u}(t) + h(v), (R_v, h_v) \in L^2([0, T], \mathbb{R}^d \times \mathbb{R}^d) \right\} .$$

By using theorem 2 in [Cimen and Banks, 2004], we derive $u^{\infty}_v = \arg \min_{u \in U} C^\infty_T (v, u)$ uniformly converges to $u^{\infty}_v$, hence:

$$S^{\infty}(v) = d' \sigma^2 + \inf_{u \in \mathcal{U}_v} \left\{ \int_0^T \left( \left\| CX^{\infty}_v u(t) - Y^*(t) \right\|_2^2 + u(t) T U u(t) \right) \right\} .$$
from which we derive the lower bound \( S^{\infty}(v) \geq d' \sigma^2 \) for all \( v \in \mathcal{T} \). We now demonstrate this lower bound \( d' \sigma^2 \) can only be attained by \( v^* \). The associated ODE to the profiled cost \( S^{\infty}(v) \) is:

\[
\begin{align*}
X^{\infty}_{\nu,\nu^*} &= A_\theta(X^{\infty}_{\nu,\nu^*}(t), t)X^{\infty}_{\nu,\nu^*} + B^{\infty}_\nu \\
X^{\infty}_{\nu,\nu^*}(0) &= x_0.
\end{align*}
\]

For all \( v \in \mathcal{T} \), let us consider the functions \( R^0 \) and \( h^0 \) such that \( R^0(t)X^{\infty}_{\nu,\nu^*}(t) + h^0(t) = 0 \) for all \( t \in [0, T] \), in this case the corresponding control is \( u_{\nu}^0(t) = 0 \) for all \( t \in [0, T] \) and it belongs to \( \mathbb{U}_\nu \). The solution \( X^{\infty}_{\nu} \) of the corresponding ODE:

\[
\begin{align*}
\dot{X}^{\infty}_{\nu} &= A_\theta(X^{\infty}_{\nu,\nu^*}(t), t)X^{\infty}_{\nu} \\
X^{\infty}_{\nu}(0) &= x_0
\end{align*}
\]

gives us the upper bound \( S^{\infty}(v) \leq d' \sigma^2 + \int_0^T \| CX^{\infty}_{\nu}(t) - Y^*(t) \|^2_2 \, dt \). From this, it is obvious that for \( v^* \), \( u_{\nu}^0 \) is also the optimal control since in this case the ODE becomes:

\[
\begin{align*}
\dot{X}^{\infty}_{v^*} &= A_\theta(X^{\infty}_{v^*}(t), t)X^{\infty}_{v^*} \\
X^{\infty}_{v^*}(0) &= x_0
\end{align*}
\]

which is the true model and so \( S^{\infty}(v^*) \leq d' \sigma^2 + \int_0^T \| CX^{\infty}_{v^*}(t) - Y^*(t) \|^2_2 \, dt = d' \sigma^2 \). Moreover thanks to the identifiability condition, we know that \( \int_0^T \| CX_\nu(t) - CX_{v^*}(t) \|^2_2 \, dt = 0 \) if and only if \( v = v^* \). Thus \( S^{\infty}(v) = d' \sigma^2 \) if and only if \( v = v^* \).

4.2. \( \hat{\theta}^{T,CI} \), linear case

Theorem 8. Under conditions LC1-LC2-LC3-LC4-LC5, we have \( \hat{\theta} \rightarrow \theta^* \) in probability when \( n \rightarrow \infty \).

Proof. As in the nonlinear case, we derive:

\[
S^{CI}(\theta) - S^{CI}_n(Y; \theta) = h^d_{\theta,0}(Y)^T \left( R^d_{\theta,0} \right)^{-1} \left( h^d_{\theta,0}(Y) - h_\theta(0) \right) + h^d_{\theta,0}(Y)^T \left( \left( R^d_{\theta,0} \right)^{-1} - R_\theta(0)^{-1} \right) h^d_{\theta,0}(Y)
\]

\[+(h^d_{\theta,0}(Y) - h_\theta(0)) \left( R^d_{\theta,0} \right)^{-1} h_\theta(0)\]

\[+ \int_0^T \left( Y^*(t)^T Y^*(t) + d' \sigma^2 \right) dt - \Delta \sum_{i=0}^n Y_i Y_i \]

\[= \left( \int_0^T h_\theta(t)^T BU^{-1} B^T h_\theta(t) dt - \Delta \sum_{i=0}^{n-1} h^d_{\theta, i+1}(Y)^T B^T R^d_{\theta, i+1} B h^d_{\theta, i+1}(Y) \right)
\]

by using propositions \([1,2]\). Lemma \([17]\) gives us \( \sup_{\theta \in \Theta} \left\| R^d_{\theta, i} - R_\theta(t_i) \right\|_2 = o_{p,n}(1) \) and \( \sup_{\theta \in \Theta} \left\| h^d_{\theta, i} - h_\theta(t_i) \right\|_2 = o_{p,n}(1) \). Since \( R^d_{\theta,0} = R_\theta(0) + o_{p,n}(1) \) and LC3 holds, it exists \( n' \in \mathbb{N} \) such that LC3disc holds i.e. \( R^d_{\theta,0} \) is invertible for all \( n \geq n' \) and from equation (189) in the Matrix Cookbook, we derive \( \sup_{\theta \in \Theta} \left\| \left( R^d_{\theta,0} \right)^{-1} - R_\theta(0)^{-1} \right\|_2 = o_{p,n}(1) \). So
We also introduce measured observations i.e. without measurement noise $Y$. In this section the notation

5. Asymptotic normality

Using the identifiability condition we derive $S$ the previous upper bound to $\int h_{\hat{\theta},0}(Y)^T \left( R_{\hat{\theta},0} \right)^{-1} \left( h_{\hat{\theta},0}(Y) - h_{\theta}(0) \right) = o_p, n(1)$, $(h_{\theta,0}(Y) - h_{\theta}(0)) \left( R_{\theta,0} \right)^{-1} h_{\theta}(0) = o_p, n(1)$ and $h_{\theta,0}(Y)^T \left( R_{\theta,0} \right)^{-1} - R_{\theta,0}^{-1} h_{\theta,0}(Y) = o_p, n(1)$ uniformly on $\Theta$. In theorem 5, we already derived:

$$\int_0^T \left( Y^*(t)Y^*(t) + d^2 \sigma^2 \right) dt - \Delta \sum_{i=0}^{n} Y_i^T Y_i = o_p, n(1)$$

and

$$\int_0^T h_{\theta}(t)^T BU^{-1} B^T h_{\theta}(t) dt - \Delta \sum_{i=0}^{n-1} h_{\theta,i+1}(Y)^T BG(R_{\theta,i+1}) B^T h_{\theta,i+1}(Y) = o_p, n(1)$$

uniformly on $\Theta$ and so $\sup_\theta \left| S_C^1(\theta) - S_n^C(Y; \theta) \right| = o_p, n(1)$. Similarly as in theorem 6, application of proposition 9 and $\theta \mapsto S_C^1(\theta)$ continuity on $\Theta$ gives us the identifiability criteria required to apply theorem 5.7 in van der Vaart (1998).

**Proposition 9.** Under conditions LC1-LC2-LC3-LC4-LC5, $\theta^*$ is the unique global minimizer of $S_C^1(\theta)$ on $\Theta$.

By definition, $S_C^1(\theta) = d^2 \sigma^2 + \inf_{x_0} \inf_u \left\{ \int_0^T \left( \| CX_{\theta,x_0,u}(t) - Y^*(t) \|^2 + u(t)^T U u(t) \right) dt \right\}$, so $S_C^1(\theta) \geq d^2 \sigma^2$ for all $\theta \in \Theta$. As in proposition 7, we now demonstrate this lower bound $d^2 \sigma^2$ can only be attained by $\theta^*$. The null control $u_0^\delta(t) = 0$ for all $t \in [0, T]$ gives us the upper bound $S_C^1(\theta) \leq d^2 \sigma^2 + \inf_{x_0} \inf_u \left\{ \int_0^T \| CX_{\theta,x_0}(t) - Y^*(t) \|^2 dt \right\}$. Since $\int_0^T \| CX_{\theta,x_0}^*(t) - Y^*(t) \|^2 dt = \int_0^T \| CX_{\theta,x_0}^*(t) - CX_{\theta,x_0}(t) \|^2 dt = 0$, we can further refine the previous upper bound to $S_C^1(\theta^*) \leq d^2 \sigma^2$ and conclude that $S_C^1(\theta^*) = d^2 \sigma^2$. Again, by using the identifiability condition we derive $S_C^1(\theta) = d^2 \sigma^2$ only if $\theta = \theta^*$.

5. Asymptotic normality

In this section the notation $Y^{dl}$ will often appear; it denotes the set of discrete and perfectly measured observations i.e. without measurement noise $Y^{dl} := \{ CX^* (t_0), \ldots, CX^* (t_n) \}$. We also introduce $X^{dl} := X_{\hat{\theta}}^l, R^{dl} := R^l_{\hat{\theta}}, h^{dl} := h^l_{\hat{\theta}}, (\text{resp. } X^{dl,d} := X^{dl}_{\hat{\theta}}, R^{dl,d} := R^{dl}_{\hat{\theta}}, h^{dl,d} := h^{dl,d}_{\hat{\theta}}(Y^{dl}))$ the solution of the ODE (resp. the finite difference equation) evaluated along the noiseless continuous signal $Y^*$ (resp. discrete signal $Y^{dl}$).

5.1. $\hat{\theta}^T$, general case

**Theorem 10.** Under conditions C1 to C8 and if $l$ is such that $l = O(n \sqrt{\Delta})$, then $(\hat{\theta}, \hat{x}_0)$ is asymptotically normal and $(\hat{\theta}, \hat{x}_0) - (\theta^*, x_0^*) = o_p, n^{1/2}$.
PROOF. By merging the proposition 11 and proposition 12 we obtain the following asymptotic representation between \( \hat{\nu} \) and \( \nu^* \):

\[
\left( \frac{\partial^2 S^\infty(\nu^*)}{\partial \nu^2} \right) + o_l(1) + o_{p,n}(1) \left( \hat{\nu} - \nu^* \right) = \left( \Delta \sum_{j=0}^{n} \epsilon_j^T \right) (K_{\nu}^l + o_n(1)) + L \left( \Delta \sum_{j=0}^{n} \epsilon_j \right) + o_{p,n}(\sqrt{\Delta}) + o_l(1)
\]

with \( K_{\nu}^l \) and \( L \) defined in proposition 12. So if we choose \( l \) such that \( l = O_n(\sqrt{\Delta}) \) and use condition 8 which ensures the matrix \( \frac{\partial^2 S^\infty(\nu^*)}{\partial \nu^2} \) is nonsingular, then \( o_l(1) + o_{p,n}(1) \) tends to a nonsingular one with probability 1, we can use the central limit theorem to conclude.

PROPOSITION 11. Under conditions C1 to C7, we have

\[
-\nabla \nu S_n^l(Y; \nu^*) = \left( \Delta \sum_{j=0}^{n} \epsilon_j^T \right) (K_{\nu}^l + o_n(1)) + L \left( \Delta \sum_{j=0}^{n} \epsilon_j \right) + o_{p,n}(\sqrt{\Delta}) + o_l(1)
\]

with \( K_{\nu}^l = 2CBU^{-1}B^T \int_0^T \frac{\partial h(\nu(t))}{\partial \nu} dt \) and \( L = \left( \begin{array}{c} 0_{p,d^\prime} \\ -2C^T \end{array} \right) \).

PROOF. First of all, we use the following decomposition:

\[
-\nabla \nu S_n^l(Y; \nu^*) = \nabla \nu S_n^l(Y^{d^*}; \nu^*) - \nabla \nu S_n^l(Y; \nu^*) + \nabla \nu S^\infty(\nu^*) - \nabla \nu S^l(\nu^*)
\]

since first order conditions impose \( \nabla \nu S^\infty(\nu^*) = 0 \). Moreover lemma 20 and 19 gives us respectively \( \nabla \nu S^\infty(\nu^*) - \nabla \nu S^l(\nu^*) = o_l(1) \) and \( \nabla \nu S^l(\nu^*) - \nabla \nu S_n^l(Y^{d^*}; \nu^*) = O_{p,n}(\Delta) \), so the previous asymptotic decomposition becomes:

\[
-\nabla \nu S_n^l(Y; \nu^*) = \nabla \nu S_n^l(Y^{d^*}; \nu^*) - \nabla \nu S_n^l(Y; \nu^*) + o_l(1) + O_{p,n}(\Delta).
\]

Now, we analyze the asymptotic behavior of \( \nabla \nu S_n^l(Y^{d^*}; \nu^*) - \nabla \nu S_n^l(Y; \nu^*) \). We denote \( R_{i,d}^l := R_{i,d}^l(\nu^*; \nu), h_{i,d}^l := h_{i,d}^l(\nu^*) \) and \( X_{d,d}^l := X_{d,d}^l(\nu^*) \). The differences \( R_{i,d}^l - R_{i,d}^l, h_{i,d}^l - h_{i,d}^l \) and \( X_{d,d}^l(t_{i+1}) - X_{d,d}^l(t_{i+1}) \) respect the equations:

\[
R_{i,d}^l - R_{i,d}^l = \left( I_d + \Delta A_{\nu^*}(X_{d,d}^{1-1}(t_i), t_i) - \Delta R_{i+1}^lBU^{-1}B^T \right) \left( R_{i+1}^l - R_{i+1}^l \right) + \Delta \left( R_{i+1}^l - R_{i+1}^l \right) \left( BU^{-1}B^T R_{i+1}^l + A_{\nu^*}(X_{d,d}^{1-1}(t_i), t_i) \right)
\]

\[
+ \Delta \left( A_{\nu^*}(X_{d,d}^{1-1}(t_i), t_i) - \Delta A_{\nu^*}X_{d,d}^{1-1}(t_i), t_i) \right) ^T R_{i+1}^l
\]

\[
+ \Delta \left( R_{i+1}^l - R_{i+1}^l \right) \left( \right) + O_{p,n}(\Delta^2)
\]

\[
h_{i,d}^l - h_{i,d}^l = \left( I_d + \Delta A_{\nu^*}(X_{d,d}^{1-1}(t_i), t_i) + \Delta R_{i+1}^lBU^{-1}B^T \right) \left( h_{i+1}^l - h_{i+1}^l \right) - \Delta C^T \epsilon_i
\]

\[
+ \Delta \left( A_{\nu^*}(X_{d,d}^{1-1}(t_i), t_i) - \Delta A_{\nu^*}X_{d,d}^{1-1}(t_i), t_i) \right) ^T h_{i+1}^l
\]

\[
+ \Delta \left( R_{i+1}^l - R_{i+1}^l \right) BU^{-1}B^T h_{i+1}^l + O_{p,n}(\Delta^2)
\]
\[ X^{d,l}(t_{i+1}) - X^{d,l}(t_i) = (I_d + \Delta A_\theta^\ast(X^{d,l-1}(t_i), t_i) + \Delta B U^* T H^{d,l}(t_{i+1}) \sqrt{O_p,n(\triangle^2)}) \chi^{d,l}(t_{i+1}) - X^{d,l}(t_i) \\
+ \Delta B U^* T (R^{d,l}(t_{i+1}) - R^{d,l}(t_i)) X^{d,l}(t_i) + h^{d,l}(t_{i+1}) - h^{d,l}(t_i) \\
+ \Delta (A_\theta^\ast(X^{d,l-1}(t_i), t_i) - A_\theta^\ast(X^{d,l-1}(t_i), t_i)) X^{d,l}(t_i) + O_p,n(\triangle^2). \]

From these equation, we prove by induction \( h^{d,l}_i - h^{d,l}_{i+1} = -\Delta C T \sum_{j=i+1}^{n} \epsilon_j + O_p,n(\triangle) \) for all \( l \in \mathbb{N} \), indeed \( h^{d,l}_i - h^{d,l}_{i+1} = -\Delta C T \epsilon_n \), and by assuming \( h^{d,l}_i - h^{d,l}_{i+1} = -\Delta C T \sum_{j=i+1}^{n} \epsilon_j + O_p,n(\triangle) \) we have:

\[
\begin{align*}
\frac{\partial R^{d,l}}{\partial x} - \frac{\partial R^{d,l}}{\partial t} &= \frac{\partial R^{d,l}}{\partial x} + \Delta \left( \frac{\partial R^{d,l}}{\partial x} A_\theta^\ast(X^{d,l-1}(t_i), t_i) - \frac{\partial R^{d,l}}{\partial t} A_\theta^\ast(X^{d,l-1}(t_i), t_i) \right) \\
&+ \Delta \left( R^{d,l} \frac{\partial A_\theta}{\partial x} (X^{d,l-1}(t_i), t_i) - R^{d,l} \frac{\partial A_\theta}{\partial t} (X^{d,l-1}(t_i), t_i) \right) \\
&+ \Delta \left( R^{d,l} \frac{\partial A_\theta}{\partial x} (X^{d,l-1}(t_i), t_i) \frac{\partial X^{d,l-1}(t_i)}{\partial \nu} - R^{d,l} \frac{\partial A_\theta}{\partial t} (X^{d,l-1}(t_i), t_i) \frac{\partial X^{d,l-1}(t_i)}{\partial \nu} \right) \\
&+ \Delta \left( A_\theta^\ast(X^{d,l-1}(t_i), t_i) \frac{\partial R^{d,l}}{\partial x} - A_\theta^\ast(X^{d,l-1}(t_i), t_i) \frac{\partial R^{d,l}}{\partial t} \right) \\
&+ \Delta \left( \frac{\partial R^{d,l}}{\partial \nu} - \frac{\partial R^{d,l}}{\partial \nu} \right) + O_p,n(\Delta^2) \\
- \Delta \left( \frac{\partial R^{d,l}}{\partial x} B U^* T H^{d,l} - \frac{\partial R^{d,l}}{\partial \nu} B U^* T H^{d,l} \right) \\
&- \Delta \left( \frac{\partial R^{d,l}}{\partial x} - \frac{\partial R^{d,l}}{\partial \nu} \right) + O_p,n(\Delta^2) \\
&= \frac{\partial R^{d,l}}{\partial x} - \frac{\partial R^{d,l}}{\partial t} = O_p,n(\Delta), \frac{\partial h^{d,l}_i}{\partial x} - \frac{\partial h^{d,l}_i}{\partial \nu} = O_p,n(\Delta). \]

For notation clarity we treat the case \( d = 1 \), by using these approximations, we obtain for
\[ \nabla_\theta S^t_n(Y^{d*}; v^*) - \nabla_\theta S^t_n(Y; v^*) \]

\[
\begin{align*}
\nabla_\theta S^t_n(Y^{d*}; v^*) - \nabla_\theta S^t_n(Y; v^*) &= (x_0^*)^T \left( \frac{\partial K_i^{d,t}}{\partial \theta} - \frac{\partial K_i^t}{\partial \theta} \right) x_0^* + 2 (x_0^*)^T \left( \frac{\partial h_i^{d,t}}{\partial \theta} - \frac{\partial h_i^t}{\partial \theta} \right) \\
&- 2\Delta \sum_{i=0}^{n-1} \left( \left( h_i^{d,t} \right)^T B U^{d,t - 1} B^T \frac{\partial h_i^{d,t}}{\partial \theta} - \left( h_i^{d,t} \right) B U^{d,t - 1} B^T \frac{\partial h_i^{d,t}}{\partial \theta} \right) + o_p(n) \Delta \\
&= (x_0^*)^T \left( \frac{\partial K_i^{d,t}}{\partial \theta} - \frac{\partial K_i^t}{\partial \theta} \right) x_0^* + 2 (x_0^*)^T \left( \frac{\partial h_i^{d,t}}{\partial \theta} - \frac{\partial h_i^t}{\partial \theta} \right) \\
&- 2\Delta \sum_{i=0}^{n-1} \left( h_i^{d,t} \right)^T B U^{d,t - 1} B^T \frac{\partial h_i^{d,t}}{\partial \theta} + o_p(n) \Delta \\
&= -2\Delta \sum_{i=0}^{n-1} \left( h_i^{d,t} \right) B U^{d,t - 1} B^T \frac{\partial h_i^{d,t}}{\partial \theta} + o_p(n) \Delta \\
&= -2\Delta \sum_{i=0}^{n-1} \left( -\Delta C^T \sum_{j=i+1}^{n-1} \epsilon_j + o_p(n) \right) B U^{d,t - 1} B^T \frac{\partial h_i^{d,t}}{\partial \theta} + o_p(n) \Delta \\
&= 2\Delta^2 \sum_{i=0}^{n-1} \left( \sum_{j=i+1}^{n-1} \epsilon_j \right) C B U^{d,t - 1} B^T \frac{\partial h_i^{d,t}}{\partial \theta} + o_p(n) \Delta.
\end{align*}
\]

Let us denote \( K_i^t(\beta) = 2 C B U^{d,t - 1} B^T \frac{\partial h_i^{d,t}}{\partial \beta} \) and decompose the right hand side term:

\[
\begin{align*}
\Delta^2 \sum_{i=0}^{n-1} \left( \sum_{j=i+1}^{n} \epsilon_j \right) K_i^t(\theta) &= \Delta^2 \sum_{i=0}^{n-1} \left( \sum_{j=0}^{n} \epsilon_j \right) K_i^t(\theta) - \Delta^2 \sum_{i=0}^{n-1} \left( \sum_{j=0}^{i} \epsilon_j \right) K_i^t(\theta).
\end{align*}
\]

By definition of \( K_i^t(\theta) \), \( \Delta \sum_{i=0}^{n-1} K_i^t(\theta) \) converges to the limit \( K^t(\theta) = 2 C B U^{d,t - 1} B^T \int_0^T \frac{\partial h_i(t)}{\partial \theta} dt \) as a Riemann sum and so \( \Delta^2 \sum_{j=0}^{n} \epsilon_j \sum_{i=0}^{n-1} K_i^t(\theta) = \Delta \sum_{j=0}^{n} \epsilon_j (K^t(\theta) + o_p(1)) \). Moreover,

\[
\begin{align*}
\Delta^2 \sum_{i=0}^{n-1} \left( \sum_{j=0}^{i} \epsilon_j \right) K_i^t(\theta) &= \Delta \frac{\Delta}{2} \sum_{i=0}^{n-1} \sqrt{\Delta} \left( \sum_{j=0}^{i} \epsilon_j \right) K_i^t(\theta) \\
&= \Delta \sum_{i=0}^{n-1} K_i^t(\theta) + o_p(n(\sqrt{\Delta}) \\
&= \Delta \sum_{i=0}^{n} K_i^t(\theta) + o_p(n(\sqrt{\Delta}) = o_p(n(\sqrt{\Delta})
\end{align*}
\]

since \( \sqrt{\Delta} \sum_{j=0}^{i} \epsilon_j = o_p(n(1)) \) for each \( i \). Thus:

\[
\begin{align*}
\Delta^2 \sum_{i=0}^{n-1} \left( \sum_{j=i+1}^{n} \epsilon_j \right) K_i^t(\theta) &= \left( \Delta \sum_{j=0}^{n} \epsilon_j \right) (K^t(\theta) + o_p(1)) + o_p(n(\sqrt{\Delta}).
\end{align*}
\]

and we derive from that

\[
\nabla_\theta S^t_n(Y^{d*}; v^*) - \nabla_\theta S^t_n(Y; v^*) = \left( \Delta \sum_{j=0}^{n} \epsilon_j \right) (K^t(\theta) + o_p(1)) + o_p(n(\sqrt{\Delta}).
\]
Now let us focus on $\nabla \sigma^l_n(Y^d;v^*) - \nabla \sigma^l_n(Y;v^*)$:

$$
\nabla \sigma^l_n(Y^d;v^*) - \nabla \sigma^l_n(Y;v^*) = \left( R_{0}^{s,dl} - R_{0}^{d,l} \right) x_n + \left( x_n \right)^T \left( \frac{\partial R_{0}^{s,dl}}{\partial x_0} - \frac{\partial R_{0}^{d,l}}{\partial x_0} \right) x_0
$$

$$+ 2 \left( h_{d,l}^{s,dl} - h_{d,l}^{d,l} \right) + 2 \left( x_n \right)^T \left( \frac{\partial h_{d,l}^{s,dl}}{\partial x_0} - \frac{\partial h_{d,l}^{d,l}}{\partial x_0} \right) - 2 \Delta \sum_{i=0}^{n-1} \left( h_{d,l}^{s,dl} - h_{d,l}^{d,l} \right) T\left( \frac{\partial h_{d,l}^{d,l}}{\partial \theta} - \frac{\partial h_{d,l}^{s,dl}}{\partial \theta} \right) + O_{p,n}(\Delta)
$$

Similarly as in $\nabla \sigma^l_n(Y^d;v^*) - \nabla \sigma^l_n(Y;v^*)$ case, we reformulate this expression as:

$$
\nabla \sigma^l_n(Y^d;v^*) - \nabla \sigma^l_n(Y;v^*) = -2CT \left( \Delta \sum_{j=0}^{n-1} \epsilon_j \right) \left( K^l(x_0) + o_p(1) \right) + o_p(\sqrt{\Delta}).
$$

**Proposition 12.** Under conditions C1 to C7, we have $-\nabla \sigma^l_n(Y;v^*) = \frac{\partial S^\infty(Y^d)}{\partial v} + o_p(1)$.

**Proof.** For notation clarity we treat the case $d = 1$, if $\theta \mapsto A_{\theta}$ is C1 on $\Theta$, then $v \mapsto S^l_n(Y;v)$ is C1 as well with

$$
\nabla \sigma^l_n(Y;v) = x_0^T \frac{\partial R_{0}^{d,l}}{\partial \theta} x_0 + 2x_0^T \frac{\partial h_{d,l}^{d,l}}{\partial \theta} (Y)
$$

$$- \frac{\partial}{\partial \theta} \left( \sum_{i=0}^{n-1} h_{d,l}^{d,l} (Y) B G (R_{d,l}^{d,l}) BT h_{d,l}^{d,l} (Y) \right)
$$

$$= x_0^T \frac{\partial R_{0}^{d,l}}{\partial \theta} x_0 + 2x_0^T \frac{\partial h_{d,l}^{d,l}}{\partial \theta} (Y)
$$

$$- 2\Delta \sum_{i=0}^{n-1} h_{d,l}^{d,l} (Y) B G (R_{d,l}^{d,l}) BT \frac{\partial h_{d,l}^{d,l}}{\partial \theta} (Y)
$$

$$- \Delta \sum_{i=0}^{n-1} h_{d,l}^{d,l} (Y) B G (R_{d,l}^{d,l}) BT \frac{\partial h_{d,l}^{d,l}}{\partial \theta} (Y)
$$

and

$$
\nabla \sigma^l_n(Y;v) = 2R_{d,l}^{d,l} x_0 + x_0^T \frac{\partial R_{d,l}^{d,l}}{\partial x_0} x_0 + 2x_0^T \frac{\partial h_{d,l}^{d,l}}{\partial x_0}
$$

$$- 2\Delta \sum_{i=0}^{n-1} h_{d,l}^{d,l} (Y) B G (R_{d,l}^{d,l}) BT \frac{\partial h_{d,l}^{d,l}}{\partial x_0} (Y)
$$

$$- \Delta \sum_{i=0}^{n-1} h_{d,l}^{d,l} (Y) B G (R_{d,l}^{d,l}) BT \frac{\partial h_{d,l}^{d,l}}{\partial x_0} (Y)
$$

If now $\theta \mapsto A_{\theta}$ is C2 on $\Theta$, from proposition 17 we derive $v \mapsto S^l_n(Y;v)$ is C2 as well.
with components equal to:

\[
\frac{\partial^2 S_n^i(Y;v)}{\partial^2 x_0} = x_0^T \frac{\partial^2 R_{i,0}^d}{\partial^2 x_0} x_0 + 2x_0^T \frac{\partial^2 R_{i,0}^d}{\partial^2 x_0^2} x_0 + 2x_0^T \frac{\partial^2 h_{i,0}^d}{\partial^2 \theta} (Y)
\]

\[-2\Delta \sum_{i=0}^{n-1} \frac{\partial}{\partial x_0} \left( h_{v,i+1}^d(Y)^T B G(R_{v,i+1}^d) B^T \frac{\partial h_{v,i+1}^d}{\partial x_0} (Y) \right)
\]

\[-\Delta \sum_{i=0}^{n-1} \frac{\partial}{\partial x_0} \left( h_{v,i+1}^d(Y)^T B G(R_{v,i+1}^d) B^T \frac{\partial h_{v,i+1}^d}{\partial \theta} (Y) \right)
\]

\[= x_0^T \frac{\partial^2 R_{i,0}^d}{\partial^2 x_0} x_0 + 2x_0^T \frac{\partial^2 h_{i,0}^d}{\partial^2 \theta} (Y)
\]

By using Taylor’s theorem we derive that it exists ̄v on the line segment between ̂v and v* such that:

\[
\nabla_v S_n^i(Y;\tilde{v}) - \nabla_v S_n^i(Y;v^*) = -\nabla_v S_n^i(Y;v^*) = \frac{\partial^2 S_n^i(Y;\tilde{v})}{\partial^2 v}^T (\tilde{v} - v^*)
\]
since first order optimality condition imposes $\nabla_v S^l_n(Y; \tilde{v}) = 0$. If $\theta \mapsto A_\theta$ is $C^2$ on $\Theta$, from lemma \[77\] we derive that $v \mapsto S^l(v)$ is $C^2$ and we have:

$$
\frac{\partial^2 S^l(v)}{\partial^2 v} = x_0^T \frac{\partial^2 R^l(0)}{\partial y} x_0 + 2x_0^T \frac{\partial^2 h^l_1(0)}{\partial y} - 2 \int_0^T \frac{\partial}{\partial y} \left( h^l_1(t) B^T U^T \frac{\partial h^l_1(t)}{\partial y} \right) dt
$$

$$
= x_0^T \frac{\partial^2 R^l(0)}{\partial y} x_0 + 2x_0^T \frac{\partial^2 h^l_1(0)}{\partial y} - 2 \int_0^T \left( \frac{\partial h^l_1(t)}{\partial y} (t) B^T U^T \frac{\partial h^l_1(t)}{\partial y} + h^l_1(t) B^T U^T \frac{\partial^2 h^l_1(t)}{\partial y} \right) dt
$$

which gives us the difference:

$$
\frac{\partial^2 S^l(v(t))}{\partial^2 y} - \frac{\partial^2 S^l(Y,v)}{\partial^2 y} = x_0^T \left( \frac{\partial^2 R^l(0)}{\partial y} x_0 + 2x_0^T \frac{\partial^2 h^l_1(0)}{\partial y} \right) - 2 \left( \int_0^T \frac{\partial h^l_1(t)}{\partial y} (t) B^T U^T \frac{\partial h^l_1(t)}{\partial y} dt - \Delta \sum_{i=0}^{n-1} \frac{\partial h^d_{v_1+1} Y}{\partial y} \frac{\partial h^d_{v_1+1}}{\partial y} (Y) \right)
$$

$$
- 2 \left( \int_0^T \frac{\partial h^l_1(t)}{\partial y} (t) B^T U^T \frac{\partial h^l_1(t)}{\partial y} dt - \Delta \sum_{i=0}^{n-1} h^d_{v_1+1} Y \frac{\partial h^d_{v_1+1}}{\partial y} (Y) B^T U^T \frac{\partial h^d_{v_1+1}}{\partial y} (Y) \right) + O_{p,n}(\Delta)
$$

$$
= x_0^T \left( \frac{\partial^2 R^l(0)}{\partial y} x_0 + 2x_0^T \frac{\partial^2 h^l_1(0)}{\partial y} \right) - 2 \left( \sum_{i=0}^{n-1} \frac{\partial h^d_{v_1+1} Y}{\partial y} \frac{\partial h^d_{v_1+1}}{\partial y} (Y) \right) B^T U^T \frac{\partial h^d_{v_1+1}}{\partial y} (Y)
$$

$$
- 2 \left( \sum_{i=0}^{n-1} h^d_{v_1+1} Y \frac{\partial h^d_{v_1+1}}{\partial y} (Y) B^T U^T \frac{\partial h^d_{v_1+1}}{\partial y} (Y) \right) + O_{p,n}(\Delta)
$$

again, from proposition \[77\] we know

$$
\frac{\partial h^l_1(t_{i+1})^T}{\partial y} B^T U^T \frac{\partial h^l_1(t_{i+1})}{\partial y} - \frac{\partial h^d_{v_1+1} Y}{\partial y} \frac{\partial h^d_{v_1+1}}{\partial y} (Y)
$$

$$
= o_{p,n}(1) + o_{p,n}(1) + o_{p,n}(1) - \frac{\partial h^d_{v_1+1}}{\partial y} (Y) B^T U^T \frac{\partial h^d_{v_1+1}}{\partial y} (Y)
$$

and:

$$
\frac{\partial h^l_1(t_{i+1})^T}{\partial y} B^T U^T \frac{\partial h^l_1(t_{i+1})}{\partial y} - h^d_{v_1+1} Y \frac{\partial h^d_{v_1+1}}{\partial y} (Y)
$$

$$
= o_{p,n}(1) + o_{p,n}(1) + o_{p,n}(1) - h^d_{v_1+1} Y \frac{\partial h^d_{v_1+1}}{\partial y} (Y)
$$

from this we can derive that $\frac{\partial^2 S^l(v)}{\partial y^2} - \frac{\partial^2 S^l(Y,v)}{\partial y^2} = o_{p,n}(1)$, in the same way we obtain $\frac{\partial^2 S^l(v)}{\partial x_0^2} - \frac{\partial^2 S^l(Y,v)}{\partial x_0^2} = o_{p,n}(1)$ and $\frac{\partial^2 S^l(v)}{\partial x_0 \partial x_0} - \frac{\partial^2 S^l(Y,v)}{\partial x_0 \partial x_0} = o_{p,n}(1)$, and by using the continuous mapping theorem, we derive

$$
\frac{\partial^2 S^l(Y,v)}{\partial y^2} = \frac{\partial^2 S^l(Y,v)}{\partial y^2} + o_{p,n}(1)
$$

and thus conclude the proof.
5.2. $\hat{\theta}^{T,CI}$, linear case

**Theorem 13.** Under conditions $LC1-LC2-LC3-LC4-LC5-LC6-LC7$, $\hat{\theta}$ is asymptotically normal and $\hat{\theta} - \theta^* = o_{p,n}(n^{-\frac{1}{2}})$.

**Proof.** Since $R_{\theta,0}^d = R_{\theta}(0) + o_{p,n}(1)$ and LC3 holds, it exists $n' \in \mathbb{N}$ such that LC3disc holds i.e. $R_{\theta,0}^d$ is invertible for all $n \geq n'$. By merging the proposition and proposition we obtain the following asymptotic representation between $\hat{\theta}$ and $\theta^*$:

$$
\left(\frac{\partial^2 S^{CI}(\theta^*)}{\partial \theta^2} + o_{p,n}(1)\right) \left(\hat{\theta} - \theta^*\right) = \Delta^2 \sum_{i=0}^{n-1} \left(\sum_{j=i+1}^{n} \epsilon_j^T\right) K_i + (H + o_n(1)) \left(\Delta \sum_{j=1}^{n} \epsilon_j\right) + \left(\Delta \sum_{j=1}^{n} \epsilon_j\right)^T \left(J + o_n(1)\right) + O_{p,n}(\Delta)
$$

with $K_i$, $H$ and $J$ defined in proposition. Let us decompose the first right hand side term:

$$
\Delta^2 \sum_{i=0}^{n-1} \left(\sum_{j=i+1}^{n} \epsilon_j^T\right) K_i = \Delta^2 \sum_{i=0}^{n-1} \left(\sum_{j=1}^{n} \epsilon_j^T - \sum_{j=1}^{i} \epsilon_j^T\right) K_i = \Delta^2 \sum_{i=0}^{n-1} \left(\sum_{j=1}^{i} \epsilon_j^T\right) K_i - \Delta^2 \sum_{i=0}^{n-1} \left(\sum_{j=1}^{i} \epsilon_j\right) K_i.
$$

By definition of $K_i$, $\Delta \sum_{i=0}^{n-1} K_i$ converges to a limit $K$ as a Riemann sum and so $\Delta^2 \sum_{j=1}^{n} \epsilon_j^T \sum_{i=0}^{n-1} K_i = \Delta \sum_{j=1}^{n} \epsilon_j^T (K + o_n(1))$. Similarly as in proposition we derive

$$
\Delta^2 \sum_{i=0}^{n-1} \left(\sum_{j=1}^{i} \epsilon_j^T\right) K_i = o_{p,n}(\sqrt{\Delta})
$$

Thus:

$$
\Delta^2 \sum_{i=0}^{n-1} \left(\sum_{j=1}^{i} \epsilon_j^T\right) K_i = \left(\Delta \sum_{j=1}^{n} \epsilon_j^T\right) (K + o_n(1)) + o_{p,n}(\sqrt{\Delta}).
$$

We can now reformulate the asymptotic representation:

$$
\left(\frac{\partial^2 S^{CI}(\theta^*)}{\partial \theta^2} + o_{p,n}(1)\right) \left(\hat{\theta} - \theta^*\right) = \left(\Delta \sum_{j=1}^{n} \epsilon_j^T\right) (K + o_n(1)) + (H + o_n(1)) \left(\Delta \sum_{j=1}^{n} \epsilon_j\right) + \left(\Delta \sum_{j=1}^{n} \epsilon_j\right)^T \left(J + o_n(1)\right) + o_{p,n}(\sqrt{\Delta}).
$$

By using L7 which ensures $\frac{\partial^2 S^{CI}(\theta^*)}{\partial \theta^2} + o_{p,n}(1)$ tends to a nonsingular matrix with probability 1, we can use the central limit theorem to conclude.

**Proposition 14.** Under conditions $LC1-LC2-LC3disc-LC3b-LC4-LC5-LC6$, we have

$$
-\nabla_{\theta} S_n^{CI}(Y; \theta^*) = (H + o_n(1)) \left(\Delta \sum_{j=1}^{n} \epsilon_j\right) + \left(\Delta \sum_{j=1}^{n} \epsilon_j\right)^T (J + o_n(1)) + \Delta^2 \sum_{i=0}^{n-1} \left(\sum_{j=i+1}^{n} \epsilon_j^T\right) K_i + O_{p,n}(\Delta)
$$

with $H = - \left( h^*(0)^T R^*(0)^{-1} \frac{\partial R^*(0)}{\partial \theta} R^*(0)^{-1} + 2 \left(\frac{\partial h^*(0)}{\partial \theta}\right)^T R^*(0)^{-1}\right)$, $J = CR^*(0)^{-1} \frac{\partial R^*(0)}{\partial \theta} R^*(0)^{-1} h^*(0)$ and $K_i = 2CBU^{-1}B^T \frac{\partial h^*}{\partial \theta}$.
\[ \nabla_\theta S_n^C(Y; \theta^*) = \nabla_\theta S_n^C(Y^{d_s}; \theta^*) - \nabla_\theta S_n^C(Y; \theta^*) + \nabla_\theta S^C(Y; \theta^*) - \nabla_\theta S_n^C(Y^{d_s}; \theta^*) \]

since:

\[
\begin{align*}
\nabla_\theta S_n^C(Y^{d_s}; \theta^*) - \nabla_\theta S_n^C(Y; \theta^*) &= \left( h_0^{d_0} \right)^T \left( R_0^{d_0} \right)^{-1} \frac{\partial R_0^{d_0}}{\partial \theta} \left( R_0^{d_0} \right)^{-1} h_0^{d_0} - h^*(0) R^*(0)^{-1} \frac{\partial R^*(0)}{\partial \theta} R^*(0)^{-1} h^*(0) \\
&+ 2 \left( \frac{\partial h_0^{d_0}}{\partial \theta} \right)^T \left( R_0^{d_0} \right)^{-1} h_0^{d_0} - 2 \left( \frac{\partial h_0^{d_0}}{\partial \theta} \right)^T \left( R_0^{d_0} \right)^{-1} h_0^{d_0} \\
&- 2 \sum_{i=0}^{1} \left( f_i^{d_1} h_0^{d_1} TBU^{-1} B^T \frac{\partial h_0^{d_1}}{\partial \theta} dt - \Delta \left( h_0^{d_1} \right)^T BU^{-1} B^T \frac{\partial h_0^{d_1}}{\partial \theta} \right) + O_n(\Delta) \\
&= -2 \sum_{i=0}^{1} \left( f_i^{d_1} h_0^{d_1} TBU^{-1} B^T \frac{\partial h_0^{d_1}}{\partial \theta} dt - \Delta \left( h_0^{d_1} \right)^T BU^{-1} B^T \frac{\partial h_0^{d_1}}{\partial \theta} \right) + O_n(\Delta)
\end{align*}
\]

we derive as in lemma \[19\] that \( \nabla_\theta S_n^C(Y^{d_s}; \theta^*) = O_{p,n}(\Delta) \), so the previous asymptotic decomposition becomes \( -\nabla_\theta S_n^C(Y; \theta^*) = \nabla_\theta S_n^C(Y^{d_s}; \theta^*) - \nabla_\theta S_n^C(Y; \theta^*) + O_{p,n}(\Delta) \). We already derive in proposition \[11\] that \( h_0^{d_0} - h_0^d = -\Delta C^T \sum_{j=1}^n \epsilon_j + O_{p,n}(\Delta) \), \( R_0^{d_0} - R_0^d = O_{p,n}(\Delta) \), \( \frac{\partial R_0^{d_0}}{\partial \theta} - \frac{\partial R_0^d}{\partial \theta} = O_{p,n}(\Delta) \) and \( \frac{\partial h_0^{d_0}}{\partial \theta} - \frac{\partial h_0^d}{\partial \theta} = O_{p,n}(\Delta) \). By using these approximations, we obtain for \( \nabla_\theta S_n^C(Y^{d_s}; \theta^*) - \nabla_\theta S_n^C(Y; \theta^*) \):

\[
\begin{align*}
\nabla_\theta S_n^C(Y; \theta^*) &= \left( h_0^{d_0} \right)^T \left( R_0^{d_0} \right)^{-1} \frac{\partial R_0^{d_0}}{\partial \theta} \left( R_0^{d_0} \right)^{-1} h_0^{d_0} - h^*(0) R^*(0)^{-1} \frac{\partial R^*(0)}{\partial \theta} R^*(0)^{-1} h^*(0) \\
&+ 2 \left( \frac{\partial h_0^{d_0}}{\partial \theta} \right)^T \left( R_0^{d_0} \right)^{-1} h_0^{d_0} - 2 \left( \frac{\partial h_0^{d_0}}{\partial \theta} \right)^T \left( R_0^{d_0} \right)^{-1} h_0^{d_0} \\
&- 2 \sum_{i=0}^{1} \left( f_i^{d_1} h_0^{d_1} TBU^{-1} B^T \frac{\partial h_0^{d_1}}{\partial \theta} dt - \Delta \left( h_0^{d_1} \right)^T BU^{-1} B^T \frac{\partial h_0^{d_1}}{\partial \theta} \right) + O_{p,n}(\Delta) \\
&= \left( h_0^{d_0} \right)^T \left( R_0^{d_0} \right)^{-1} \frac{\partial R_0^{d_0}}{\partial \theta} \left( R_0^{d_0} \right)^{-1} h_0^{d_0} - h^*(0) R^*(0)^{-1} \frac{\partial R^*(0)}{\partial \theta} R^*(0)^{-1} h^*(0) \\
&+ 2 \left( \frac{\partial h_0^{d_0}}{\partial \theta} \right)^T \left( R_0^{d_0} \right)^{-1} h_0^{d_0} - 2 \left( \frac{\partial h_0^{d_0}}{\partial \theta} \right)^T \left( R_0^{d_0} \right)^{-1} h_0^{d_0} \\
&- 2 \sum_{i=0}^{1} \left( f_i^{d_1} h_0^{d_1} TBU^{-1} B^T \frac{\partial h_0^{d_1}}{\partial \theta} dt - \Delta \left( h_0^{d_1} \right)^T BU^{-1} B^T \frac{\partial h_0^{d_1}}{\partial \theta} \right) + O_{p,n}(\Delta)
\end{align*}
\]
The first term is equal to
\[
(h_0^d)^T (R_0^d)^{-1} \frac{\partial R_0^d}{\partial \theta} (R_0^d)^{-1} + 2 \left( \frac{\partial h_0^d}{\partial \theta} \right)^T (R_0^d)^{-1} (h_0^d - h_0^d)
\]
\[
= - \left( h_0^d \right)^T (R_0^d)^{-1} \frac{\partial R_0^d}{\partial \theta} (R_0^d)^{-1} + 2 \left( \frac{\partial h_0^d}{\partial \theta} \right)^T (R_0^d)^{-1} C^T \left( \triangle \sum_{j=1}^n \epsilon_j \right) + O_{p,n}(\Delta)
\]
\[
= (H + o_n(1)) \left( \triangle \sum_{j=1}^n \epsilon_j \right) + O_{p,n}(\Delta)
\]
	hanks{For the second term, we have:
\[
\left( h_0^d - h_0^d \right)^T (R_0^d)^{-1} \frac{\partial R_0^d}{\partial \theta} (R_0^d)^{-1} h_0^d
\]
\[
= \left( \triangle \sum_{j=1}^n \epsilon_j \right)^T C (R_0^d)^{-1} \frac{\partial R_0^d}{\partial \theta} (R_0^d)^{-1} h_0^d + O_{p,n}(\Delta)
\]
\[
= \left( \triangle \sum_{j=1}^n \epsilon_j \right)^T (J + o_n(1)) + O_{p,n}(\Delta)
\]
\]
from which we derive the expression:
\[
\nabla_\theta S_n^{CI}(Y^d; \theta^*) - \nabla_\theta S_n^{CI}(Y; \theta^*)
\]
\[
\nabla_\theta S_n^{CI}(Y; \theta) = (\nabla_\theta S_n^{CI}(Y^d; \theta^*) - \nabla_\theta S_n^{CI}(Y; \theta^*))
\]
\[
= (H + o_n(1)) \left( \triangle \sum_{j=1}^n \epsilon_j \right) + \left( \triangle \sum_{j=1}^n \epsilon_j \right)^T (J + o_n(1))
\]
\[
+ 2 \triangle^2 \sum_{i=0}^{n-1} \left( \sum_{j=i+1}^n \epsilon_j \right)^T BU^{-1}B^T \frac{\partial h_0^d}{\partial \theta} + O_{p,n}(\Delta)
\]

and we can conclude the proof.

**Proposition 15.** Under conditions LC1-LC2-LC3disc-LC3-LC4-LC5-LC6, we have \(-\nabla S_n^{CI}(Y; \theta^*) = \left( \frac{\partial^2 S_n^{CI}(\theta^*)}{\partial \theta^2} + o_{p,n}(1) \right) \left( \hat{\theta} - \theta^* \right)\).

**Proof.** For notation clarity we treat the case \(d = 1\), if \(\theta \rightarrow A_\theta\) is \(C^1\) on \(\Theta\), then \(\theta \rightarrow S_n^{CI}(Y; \theta)\) is \(C^1\) as well and
\[
\nabla_\theta S_n^{CI}(Y; \theta) = \left( h_{\theta,0}^d \right)^T (R_{\theta,0}^d)^{-1} \frac{\partial R_{\theta,0}^d}{\partial \theta} (R_{\theta,0}^d)^{-1} h_{\theta,0}^d - 2 \left( \frac{\partial h_{\theta,0}^d}{\partial \theta} (Y) \right)^T (R_{\theta,0}^d)^{-1} h_{\theta,0}^d
\]
\[
- 2 \triangle \sum_{i=0}^{n-1} h_{\theta,i+1}^d (Y)^T BG(R_{\theta,i+1}^d)B^T \frac{\partial h_{\theta,i+1}^d}{\partial \theta} + o_{p,n}(1) \left( \hat{\theta} - \theta^* \right)
\]
\[
- \triangle \sum_{i=0}^{n-1} h_{\theta,i+1}^d (Y)^T B^T \frac{\partial h_{\theta,i+1}^d}{\partial \theta} + o_{p,n}(1) \left( \hat{\theta} - \theta^* \right)
\].
If now $\theta \rightarrow A_\theta$ is $C^2$ on $\Theta$, then $\theta \mapsto S_n^{CI}(Y; \theta)$ is also $C^2$ and equal to

$$
\frac{\partial^2 S_n^{CI}(Y, \theta)}{\partial \theta^2} = \frac{\partial}{\partial \theta} \left( h_{\theta,0}^T \left( R_{\theta,0}^{-1} \frac{\partial R_{\theta,0}}{\partial \theta} \right) \left( R_{\theta,0}^{-1} h_{\theta,0}^T \right) \right) - 2 \frac{\partial}{\partial \theta} \left( \frac{\partial h_{\theta,0}^T}{\partial \theta} (Y) \right) \left( R_{\theta,0}^{-1} h_{\theta,0}^T \right)
$$

$$
-2 \Delta \sum_{i=0}^{n-1} \frac{\partial}{\partial \theta} \left( h_{\theta,i+1}^T (Y) B G (R_{\theta,i+1}^d) B \frac{\partial G}{\partial \theta} (R_{\theta,i+1}^d) B h_{\theta,i+1} (Y) \right)
$$

$$
-2 \Delta \sum_{i=0}^{n-1} h_{\theta,i+1}^T (Y) B \frac{\partial G}{\partial \theta} (R_{\theta,i+1}^d) B \frac{\partial^2 G}{\partial \theta^2} (Y)
$$

By using Taylor’s theorem we derive that it exists $\hat{\theta}$ on the line segment between $\tilde{\theta}$ and $\theta^*$ such that:

$$
\nabla_\theta S_n^{CI}(Y; \tilde{\theta}) - \nabla_\theta S_n^{CI}(Y; \theta^*) = -\nabla_\theta S_n^{CI}(Y; \theta^*) = \frac{\partial^2 S_n^{CI}(Y, \tilde{\theta})}{\partial \theta^2} (\tilde{\theta} - \theta^*)
$$

since first order optimality condition imposes $\nabla_\theta S_n^{CI}(Y; \tilde{\theta}) = 0$. If $\theta \mapsto A_\theta$ is $C^2$ on $\Theta$, from lemma [17] we derive that $\theta \mapsto S^{CI}(\theta)$ is $C^2$ and

$$
\frac{\partial^2 S^{CI}(\theta)}{\partial \theta^2} = \frac{\partial}{\partial \theta} \left( h_{\theta}(0) R_{\theta}(0)^{-1} \frac{\partial (R_{\theta}(0))}{\partial \theta} \left( R_{\theta}(0)^{-1} h_{\theta}(0) \right) \right) - 2 \frac{\partial}{\partial \theta} \left( \frac{\partial h_{\theta}(0)^T}{\partial \theta} R_{\theta}(0)^{-1} h_{\theta}(0) \right)
$$

$$
-2 \int_0^T \left( \frac{\partial h_{\theta}(t)^T}{\partial \theta} (Y) B U^{-1} B \frac{\partial^2 h_{\theta}(t)}{\partial \theta^2} + h_{\theta}(t)^T (Y) B U^{-1} B \frac{\partial^2 h_{\theta}(t)}{\partial \theta^2} \right) dt
$$
which gives us the difference:

\[
\frac{\partial^2 S_{CI}(\theta)}{\partial \theta^2} - \frac{\partial^2 S_{CI}(Y, \theta)}{\partial \theta^2} = \frac{\partial}{\partial \theta} \left( h_\theta(0) R_\theta(0) \frac{1}{\partial \theta} \frac{1}{\partial \theta} R_\theta(0) - h_\theta(0) \right) - \frac{\partial}{\partial \theta} \left( \left( R_{\theta,0}^d \right)^T \left( R_{\theta,0}^d \right) \right) - \frac{\partial h_{\theta}(0)}{\partial \theta} \frac{1}{\partial \theta} \frac{1}{\partial \theta} R_\theta(0) - h_{\theta}(0)
\]

+ 2 \frac{\partial}{\partial \theta} \left( \left( \frac{\partial h_{\theta}(0)}{\partial \theta} \right)^T \left( R_{\theta,0}^d \right)^T h_{\theta,0}^d \right) - 2 \frac{\partial}{\partial \theta} \left( \frac{\partial h_{\theta}(0)}{\partial \theta} \right)^T R_\theta(0) - h_{\theta}(0)

\]

which gives us the difference:

\[
\frac{\partial^2 S_{CI}(\theta)}{\partial \theta^2} - \frac{\partial^2 S_{CI}(Y, \theta)}{\partial \theta^2} = \frac{\partial}{\partial \theta} \left( h_\theta(0) R_\theta(0) \frac{1}{\partial \theta} \frac{1}{\partial \theta} R_\theta(0) - h_\theta(0) \right) - \frac{\partial}{\partial \theta} \left( \left( R_{\theta,0}^d \right)^T \left( R_{\theta,0}^d \right) \right) - \frac{\partial h_{\theta}(0)}{\partial \theta} \frac{1}{\partial \theta} \frac{1}{\partial \theta} R_\theta(0) - h_{\theta}(0)
\]

\[
+ 2 \frac{\partial}{\partial \theta} \left( \left( \frac{\partial h_{\theta}(0)}{\partial \theta} \right)^T \left( R_{\theta,0}^d \right)^T h_{\theta,0}^d \right) - 2 \frac{\partial}{\partial \theta} \left( \frac{\partial h_{\theta}(0)}{\partial \theta} \right)^T R_\theta(0) - h_{\theta}(0)
\]

thanks to lemma 17. From this we derive that \( \frac{\partial^2 S_{CI}(\theta)}{\partial \theta^2} - \frac{\partial^2 S_{CI}(Y, \theta)}{\partial \theta^2} = o_{p,n}(1) \) similarly as in proposition 12. Since \( \tilde{\theta} \) is consistent, we can use the continuous mapping theorem to conclude.

6. Useful lemma

6.1. Discrete Gronwall Lemma

Here, we just recall the discrete Gronwall lemma, in a form particularly convenient for us.

**Lemma 16.** Let introduce the positive sequences \( f_n, \lambda_n, \epsilon_n \) linked by the recursive inequality \( f_n \leq (1 + \lambda_{n-1}) f_{n-1} + \epsilon_{n-1} \), then we have:

\[
f_n \leq e^{\sigma \sum_{i=1}^{n-1} \lambda_i} f_0 + \sum_{i=1}^{n-1} e^{\sigma \sum_{j=i+1}^{n-1} \lambda_j} \epsilon_i
\]

in particular for \( \lambda = \lambda_1 = \ldots = \lambda_n \) we have \( f_n \leq e^{\sigma \lambda} f_0 + \sum_{i=1}^{n-1} e^{(n-i)\lambda} \epsilon_i \).

**Proof.** Let us prove it recursively. For \( n = 1 \), we have \( f_1 \leq (1 + \lambda_0) f_0 + \epsilon_0 \), by remembering that \( e^x \geq 1 + x \), for all \( x \geq 0 \), the initialization is easy to make. Now let us assume the property holds for \( n \), we have:

\[
f_{n+1} \leq (1 + \lambda_n) f_n + \epsilon_n
\]

\[
\leq e^{\lambda_n} f_n + \epsilon_n
\]

\[
\begin{align*}
&\leq e^{\lambda_n} \left( e^{\sum_{i=1}^{n-1} \lambda_i} f_0 + \sum_{i=1}^{n-1} e^{\sum_{j=i+1}^{n-1} \lambda_j} \epsilon_i \right) + \epsilon_n \\
&\leq e^{\sum_{i=1}^{n-1} \lambda_i} f_0 + \sum_{i=1}^{n-1} e^{\sum_{j=i+1}^{n-1} \lambda_j} \epsilon_i + \epsilon_n \\
&= e^{\sum_{i=1}^{n-1} \lambda_i} f_0 + \sum_{i=1}^{n-1} e^{\sum_{j=i+1}^{n-1} \lambda_j} \epsilon_i
\end{align*}
\]

Hence the conclusion.
By using equation (15), we obtain the next equality where terms of order $O_p, n$ is continuous on $\mathcal{Y}$ and $\sup_{v \in \mathcal{Y}} \left\| R^{d,l}_v(t) - R^{d,l}_v(t_i) \right\|_2 = o_p, n(1)$, \[ \sup_{v \in \mathcal{Y}} \left\| X_v^u(t_i) - X_v^u(t_i) \right\|_2 = o_p, n(1). \]

Under conditions C1 to C3 and C6 for each $l \in \mathbb{N}, v \mapsto \left( X_v^u, R_v^l, h_v^l \right)$ is $C^1$ on $\mathcal{Y}$ and \[ \sup_{v \in \mathcal{Y}} \left\| \frac{\partial R_v^l(t)}{\partial v} - \frac{\partial R_v^l(t)}{\partial v} \right\|_2 = o_p, n(1), \sup_{v \in \mathcal{Y}} \left\| \frac{\partial h_v^l(t)}{\partial v} - \frac{\partial h_v^l(t)}{\partial v} \right\|_2 = o_p, n(1) \]
and \[ \sup_{v \in \mathcal{Y}} \left\| \frac{\partial X_v^u(t)}{\partial v} - \frac{\partial X_v^u(t)}{\partial v} \right\|_2 = o_p, n(1). \]

**Proof.** By integrating equation (13), we obtain:

\[
R_v^l(t) = R_v^l(t_{i+1}) + \int_{t_i}^{t_{i+1}} \left( C^T C + A \theta(X_v^u(t), t) A^T(t) \right) dt + R_v^l(t) A \theta(X_v^u(t), t) - R_v^l(t) B U^{-1} B^T \]

\[
\left. \right\|_{i \in \mathbb{N}, v \in \mathcal{Y}} R_v^l(t_{i+1}) + \Delta C^T C + A \theta(X_v^u(t), t) A^T(t) \right) dt + R_v^l(t_{i+1}) + \Delta R_v^l(t_{i+1}) A \theta(X_v^u(t), t_{i+1})
- \Delta R_v^l(t_{i+1}) B U^{-1} B^T R_v^l(t_{i+1}) + O_n(\Delta^2)

h_v^l(t) = h_v^l(t_{i+1}) + \int_{t_i}^{t_{i+1}} \left( -C^T Y^*(t) + A \theta(X_v^u(t), t) h_v^l(t) - R_v^l(t) B U^{-1} B^T \right) dt

\[
\left. \right\|_{i \in \mathbb{N}, v \in \mathcal{Y}} h_v^l(t_{i+1}) - \Delta C^T Y^*(t_i) + \Delta A \theta(X_v^u(t_i), t_i) h_v^l(t_{i+1}) - R_v^l(t_{i+1}) B U^{-1} B^T h_v^l(t_{i+1}) + O_n(\Delta^2). \]

By using equation (15), we obtain the next equality where terms of order $O_p, n(\Delta^2)$ or
higher have been neglected:

\[ R^{d,l}_{v,i} = R^{d,l}_{v,i+1} + \Delta C^TC + \Delta \left( R^{d,l}_{v,i+1} \bar{A}_\theta (X^{d,l-1}_{v,i}(t), t_i) + \bar{A}_\theta (X^{d,l-1}_{v,i}(t), t_i)^T R^{d,l}_{v,i+1} \right) \]

\[ + \Delta^2 \bar{A}_\theta (X^{d,l-1}_{v,i}(t), t_i)^T \bar{R}^{d,l}_{v,i+1} \bar{A}_\theta (X^{d,l-1}_{v,i}(t), t_i) \]

\[- \Delta (I_d + \Delta \bar{A}_\theta (X^{d,l-1}_{v,i+1}(t), t_{i+1})^T R^{d,l}_{v,i+1} B \bar{G}(R^{d,l}_{v,i+1}) B^T R^{d,l}_{v,i+1} (I_d + \Delta \bar{A}_\theta (X^{d,l-1}_{v,i+1}(t), t_{i+1}) \right) = R^{d,l}_{v,i+1} + \Delta C^TC + \Delta \bar{R}^{d,l}_{v,i+1} \bar{A}_\theta (X^{d,l-1}_{v,i}(t), t_i) + \bar{A}_\theta (X^{d,l-1}_{v,i}(t), t_i)^T R^{d,l}_{v,i+1} \]

\[- \Delta R^{d,l}_{v,i+1} BU^{-1} B^T R^{d,l}_{v,i+1} + O_{p,n}(\Delta^2) \]

\[ h^{d,l}_{v,i+1}(Y) = h^{d,l}_{v,i+1}(Y) + O_{p,n}(\Delta), \quad \bar{A}_\theta (X^{d,l-1}_{v,i+1}(t), t_i) = O_{p,n}(1), \quad \left\| E^{d,l-1}_{v,i+1}(Y) \right\|_2 = O_{p,n}(1) \text{ and } \left\| E^{d,l-1}_{v,i+1}(t) \right\|_2 = O_{p,n}(1) \text{ (proposition 4). By making the subtraction between the two last finite difference equations, we obtain:} \]

\[ R^{d,l}_{v,i}(t_i) - R^{d,l}_{v,i+1} \]

\[ = R^{d,l}_{v,i+1} + \Delta C^TC + \Delta \bar{A}_\theta (X^{d,l-1}_{v,i}(t), t_i)^T \bar{R}^{d,l}_{v,i+1} + \Delta R^{d,l}_{v,i+1} \bar{A}_\theta (X^{d,l-1}_{v,i}(t), t_i) \]

\[- \Delta \bar{R}^{d,l}_{v,i+1} BU^{-1} B^T \bar{R}^{d,l}_{v,i+1} \]

\[ - R^{d,l}_{v,i+1} - \Delta C^TC + \Delta \bar{A}_\theta (X^{d,l-1}_{v,i+1}(t), t_i)^T R^{d,l}_{v,i+1} - \Delta R^{d,l}_{v,i+1} \bar{A}_\theta (X^{d,l-1}_{v,i+1}(t), t_i) \]

\[ + \Delta R^{d,l}_{v,i+1} BU^{-1} B^T R^{d,l}_{v,i+1} + O_{p,n}(\Delta^2) \]

\[ = R^{d,l}_{v,i+1} + \Delta \bar{A}_\theta (X^{d,l-1}_{v,i+1}(t), t_i)^T \bar{R}^{d,l}_{v,i+1} + \Delta \bar{A}_\theta (X^{d,l-1}_{v,i+1}(t), t_i)^T R^{d,l}_{v,i+1} \]

\[ + \Delta \bar{R}^{d,l}_{v,i+1} \bar{A}_\theta (X^{d,l-1}_{v,i+1}(t), t_i) - \Delta R^{d,l}_{v,i+1} \bar{A}_\theta (X^{d,l-1}_{v,i+1}(t), t_i) \]

\[ + \Delta R^{d,l}_{v,i+1} \bar{A}_\theta (X^{d,l-1}_{v,i+1}(t), t_i) - \Delta R^{d,l}_{v,i+1} \bar{A}_\theta (X^{d,l-1}_{v,i+1}(t), t_i) \]

\[ - \left( \bar{R}^{d,l}_{v,i+1} BU^{-1} B^T \bar{R}^{d,l}_{v,i+1} + O_{p,n}(\Delta^2) \right) \]

\[ = R^{d,l}_{v,i+1} + \Delta \left( \bar{A}_\theta (X^{d,l-1}_{v,i+1}(t), t_i) - \bar{A}_\theta (X^{d,l-1}_{v,i+1}(t), t_i) \right)^T \bar{R}^{d,l}_{v,i+1} \]

\[ + \Delta \left( \bar{A}_\theta (X^{d,l-1}_{v,i+1}(t), t_i) - \bar{A}_\theta (X^{d,l-1}_{v,i+1}(t), t_i) \right) \]

\[ + \Delta R^{d,l}_{v,i+1} \left( \bar{A}_\theta (X^{d,l-1}_{v,i+1}(t), t_i) - \bar{A}_\theta (X^{d,l-1}_{v,i+1}(t), t_i) \right) \]

\[ - \Delta \left( R^{d,l}_{v,i+1} - R^{d,l}_{v,i+1} \right) BU^{-1} B^T R^{d,l}_{v,i+1} - \Delta R^{d,l}_{v,i+1} BU^{-1} B^T \left( R^{d,l}_{v,i+1} - R^{d,l}_{v,i+1} \right) + O_{p,n}(\Delta^2) \]
and:
\[
\begin{align*}
& h_{v,i}^{d,l}(Y) - h_i^l(t_i) \\
& = h_{v,i+1}^{d,l}(Y) - h_i^l(t_i+1) - \Delta C^T \epsilon_i + \Delta A_\theta(X_{v}^{d,l-1}(t_i), t_i) h_{v,i+1}^{d,l}(Y) \\
& - \Delta R_{v,i+1}^{d,l} \text{BU}^{-1}B^T h_{v,i+1}^{d,l}(Y) \\
& - \Delta A_\theta(X_{v}^{d,l-1}(t_i), t_i) h_{v,i+1}^{d,l}(Y) + R_{v,i+1}^{d,l} \text{BU}^{-1}B^T h_{v,i+1}^{d,l}(Y) + O_{\text{p,n}}(\Delta^2) \\
& = h_{v,i+1}^{d,l}(Y) - h_i^l(t_i+1) + \Delta \left( A_\theta(X_{v}^{d,l-1}(t_i), t_i) - A_\theta(X_{v}^{d,l-1}(t_i), t_i) \right)^T h_{v,i+1}^{d,l}(Y) \\
& + \Delta A_\theta(X_{v}^{d,l-1}(t_i), t_i)^T \left( h_{v,i+1}^{d,l}(Y) - h_i^l(t_i+1) \right) - \Delta R_{v,i+1}^{d,l} \text{BU}^{-1}B^T h_{v,i+1}^{d,l}(Y) - h_i^l(t_i+1)) \\
& - \Delta (R_{v,i+1}^{d,l} - R_{v,i+1}^{d,l}) \text{BU}^{-1}B^T h_{v,i+1}^{d,l}(Y) - \Delta C^T \epsilon_i + O_{\text{p,n}}(\Delta^2).
\end{align*}
\]

Triangular inequality gives us:
\[
\begin{align*}
& \left\| R_{v,i}^{d,l}(t_i) - R_{v,i+1}^{d,l}(t_i+1) \right\|_2 \\
& \leq \left\| R_{v,i}^{d,l}(t_i+1) - R_{v,i+1}^{d,l}(t_i+1) \right\|_2 + 2\Delta \left\| R_{v,i}^{d,l}(t_i+1) - R_{v,i+1}^{d,l}(t_i+1) \right\|_2 \\
& + \Delta \left( \left\| R_{v,i+1}^{d,l}(t_i+1) \right\|_2 + \left\| R_{v,i+1}^{d,l}(t_i+1) \right\|_2 \right) \left\| BU^{-1}B^T \right\|_2 \left\| R_{v,i}^{d,l}(t_i+1) - R_{v,i+1}^{d,l}(t_i+1) \right\|_2 \\
& + \Delta \left( \left\| \overbrace{R_{v}^{d,l}(t_i+1)} + \left\| R_{v,i+1}^{d,l}(t_i+1) \right\|_2 \right) \left\| A_\theta(X_{v}^{d,l-1}(t_i), t_i) - A_\theta(X_{v}^{d,l-1}(t_i), t_i) \right\|_2 + O_{\text{p,n}}(\Delta^2) \\
& \leq \left( 1 + \Delta \left( 2A + \left( \left\| R_{v,i+1}^{d,l}(t_i+1) \right\|_2 + \left\| R_{v,i+1}^{d,l}(t_i+1) \right\|_2 \right) \left\| BU^{-1}B^T \right\|_2 \right) \right) \left\| R_{v,i+1}^{d,l} - R_{v,i+1}^{d,l}(t_i+1) \right\|_2 \\
& + \Delta \left( \left\| R_{v,i+1}^{d,l}(t_i+1) \right\|_2 + \left\| R_{v,i+1}^{d,l}(t_i+1) \right\|_2 \right) \left\| A_\theta(X_{v}^{d,l-1}(t_i), t_i) - A_\theta(X_{v}^{d,l-1}(t_i), t_i) \right\|_2 + O_{\text{p,n}}(\Delta^2) \\
& \leq (1 + \Delta (2A + O_{\text{p,n}}(1))) \left\| R_{v,i+1}^{d,l} - R_{v,i+1}^{d,l}(t_i+1) \right\|_2 \\
& + O_{\text{p,n}}(\Delta) \left\| A_\theta(X_{v}^{d,l-1}(t_i), t_i) - A_\theta(X_{v}^{d,l-1}(t_i), t_i) \right\|_2 + O_{\text{p,n}}(\Delta^2)
\end{align*}
\]

and:
\[
\begin{align*}
& \left\| X_{v}^{d,l}(t_i+1) - X_{v}^{d,l}(t_i+1) \right\|_2 \\
& \leq \left\| X_{v}^{d,l}(t_i) - X_{v}^{d,l}(t_i) \right\|_2 + \Delta \left\| X_{v}^{d,l}(t_i) \right\|_2 \left\| A_\theta(X_{v}^{d,l-1}(t_i), t_i) - A_\theta(X_{v}^{d,l-1}(t_i), t_i) \right\|_2 \\
& + \Delta \left( \left\| A_\theta(X_{v}^{d,l-1}(t_i), t_i) \right\|_2 + \left\| BU^{-1}B^T \right\|_2 \left\| R_{v,i+1}^{d,l}(t_i+1) \right\|_2 \right) \left\| X_{v}^{d,l}(t_i) - X_{v}^{d,l}(t_i) \right\|_2 \\
& + \Delta \left( \left\| BU^{-1}B^T \right\|_2 \left\| h_i^l(t_i+1) - h_i^l(t_i+1) \right\|_2 + O_{\text{p,n}}(\Delta^2) \\
& \leq (1 + O_{\text{p,n}}(\Delta)) \left\| X_{v}^{d,l}(t_i) - X_{v}^{d,l}(t_i) \right\|_2 \\
& + \Delta \left( \left\| BU^{-1}B^T \right\|_2 \left\| X_{v}^{d,l}(t_i) \right\|_2 \left\| R_{v,i+1}^{d,l}(t_i+1) \right\|_2 + \left\| h_i^l(t_i+1) - h_i^l(t_i+1) \right\|_2 \right) \\
& + \Delta \left\| X_{v}^{d,l}(t_i) \right\|_2 \left\| A_\theta(X_{v}^{d,l-1}(t_i), t_i) - A_\theta(X_{v}^{d,l-1}(t_i), t_i) \right\|_2 + O_{\text{p,n}}(\Delta^2)
\end{align*}
\]

From these last equations describing the behavior of $h_{v,i}^{d,l}(Y) - h_i^l(t_i)$, $R_{v,i}^{d,l}(t_i) - R_{v,i+1}^{d,l}(t_i)$, and $X_{v}^{d,l}(t_i+1) - X_{v}^{d,l}(t_i+1)$, we prove now by induction $\sup_{t_i \in T} \left\| R_{v,i}^{d,l} - R_{v,i+1}^{d,l}(t_i) \right\|_2 = o_{\text{p,n}}(1)$, $\sup_{t_i \in T} \left\| h_i^l(t_i) \right\|_2 = o_{\text{p,n}}(1)$, and $\sup_{t_i \in T} \left\| X_{v}^{d,l}(t_i) - X_{v}^{d,l}(t_i) \right\|_2 = o_{\text{p,n}}(1)$ for each $l \in \mathbb{N}$.

First of all, let us start the initialization with $l = 1$, we already know $\sup_{t_i \in T} \left\| X_{v}^{d,0}(t_i) - X_{v}^{d,0}(t_i) \right\|_2$
\[\|x_0 - x_0\|_2 = 0, \text{ the inequality respected by } \|R_v^1(t_i) - R_v^{d,1}\|_2 \text{ becomes } \|R_v^1(t_i) - R_v^{d,1}\|_2 \leq (1 + O_o(\Delta)) \|R_v^{d,1}(t_{i+1}) - R_v^{d,1}(t_{i})\|_2 + O_p,n(\Delta^2). \text{ Discrete Gronwall lemma gives us } \|R_v^1(t_i) - R_v^{d,1}\|_2 \leq \sum_{j=1}^{n-i} e^{O_p,n(1)} O_p,n(\Delta^2) = O_p,n(\Delta) \text{ since } \|R_v^1(t_n) - R_v^{d,1}\|_2 = O_p,n(\Delta), \text{ from this we can derive the uniform bound } \|R_v^{d,1}(t_{i+1}) - R_v^1(t_i)\|_2 = o_p,n(1) \text{ for all } i \in [0, n] \text{ and } \theta \in \Theta. \]

By using this and \(R_v^{d,1}(t_{i+1}) = O_p,n(1), h_v^1(t_{i+1}) = O_n(1), \) we can simplify the expression of \(h_{v,i}^{d,1}(Y) - h_v^1(t_i): \)

\[h_{v,i}^{d,1}(Y) - h_v^1(t_i) = h_{v,i+1}^{d,1}(Y) - h_v^1(t_{i+1}) + \Delta \theta \beta(x_0, t_i) \sum_{j=i+1}^n \epsilon_j + O_p,n(\Delta) \]

\[= (I_d + O_p,n(\Delta)) (-\Delta CT \sum_{j=i+1}^n \epsilon_j + O_p,n(\Delta)) - \Delta CT \epsilon_i + O_p,n(\Delta) \]

\[= -\Delta CT \sum_{j=i+1}^n \epsilon_j + O_p,n(\Delta) - \Delta CT \epsilon_i + O_p,n(\Delta) \]

\[= -\Delta CT \sum_{j=i}^n \epsilon_j + O_p,n(\Delta) \]

from this we conclude \(\|h_{v,i}^{d,1}(Y) - h_v^1(t_i)\|_2 = o_p,n(1) \text{ for all } i \in [0, n] \text{ and } v \in \mathcal{Y}. \)

From these inequalities, we derive:

\[\|X_v^i(t_{i+1}) - X_v^{d,t}(t_{i+1})\|_2 \leq (1 + O_o(\Delta)) \|X_v^i(t_i) - X_v^{d,t}(t_i)\|_2 \]

and by using the discrete Gronwall lemma\[\[\text{to obtain } \|X_v^i(t_{i+1}) - X_v^{d,t}(t_{i+1})\|_2 \leq \sum_{j=1}^{n-i} e^{O_o(n)} O_p,n(\Delta) = o_p,n(1). \]

Now let us assume the property holds for \(l - 1, \) by using the induction hypothesis, we can simplify the inequality respected by \(\|R_v^{d,l}(t_i) - R_v^{d,l}(t_i)\|_2 \) which becomes \(\|R_v^1(t_i) - R_v^{d,l}(t_i)\|_2 \leq (1 + O_p,n(\Delta)) \|R_v^{d,l}(t_{i+1}) - R_v^1(t_{i+1})\|_2 + O_p,n(\Delta) \) again, by using the discrete Gronwall lemma we have \(\|R_v^{d,l}(t_i) - R_v^1(t_i)\|_2 = o_p,n(1), \) with the help of this result, we have

\[h_{v,i}^{d,l}(Y) - h_v^1(t_i) = (I_d + O_p,\Delta) (h_{v,i+1}^{d,l}(Y) - h_v^1(t_{i+1})) - \Delta CT \epsilon_i + o_p,n(\Delta) \]

and by induction we derive the expression \(h_{v,i}^{d,l}(Y) - h_v^1(t_i) = -\Delta CT \sum_{j=i}^n \epsilon_j + O_p,n(\Delta) \) and thus \(\|h_{v,i}^{d,l}(Y) - h_v^1(t_i)\|_2 = o_p,n(1). \) By proceeding the same way as in the \(l = 1 \) case, we
Again, if \( R_{d}(x, \cdot) \) is \( C^{1} \) on \( \Theta \), then \( t \rightarrow R_{v}(t) \) and \( t \rightarrow h_{v}(t) \) are differentiable for all \( v \in \mathcal{Y} \). By differentiating and integrating (13), we have:

\[
\frac{\partial R_{v}(t)}{\partial v_{i}} = \frac{\partial R_{v+1}(t)}{\partial v_{i}} = \int_{t_{i}}^{t_{i+1}} \left( \frac{\partial R_{v}(t)}{\partial v_{i}} B U^{-1} B^{T} R_{v}(t) + R_{v}(t) B U^{-1} B^{T} \frac{\partial R_{v}(t)}{\partial v_{i}} \right) dt
\]

\[
+ \int_{t_{i}}^{t_{i+1}} \left( \frac{\partial A_{v}(X_{v}^{-1}(t), t)}{\partial v_{i}} + R_{v}(t) \frac{\partial A_{v}(X_{v}^{-1}(t), t)}{\partial v_{i}} - \frac{\partial A_{v}(X_{v}^{-1}(t), t)}{\partial v_{i}} - \frac{\partial A_{v}(X_{v}^{-1}(t), t)}{\partial v_{i}} \right) \frac{\partial X_{v}^{-1}(t)}{\partial v_{i}} dt
\]

\[
- \frac{\partial R_{v}(t)}{\partial v_{i}} B U^{-1} B^{T} R_{v}(t) = \int_{t_{i}}^{t_{i+1}} \frac{\partial h_{v}(t)}{\partial v_{i}} dt
\]

\[
\frac{\partial h_{v}(t)}{\partial v_{i}} = \frac{\partial h_{v+1}(t)}{\partial v_{i}} = \int_{t_{i}}^{t_{i+1}} \left( \frac{\partial h_{v}(t)}{\partial v_{i}} B U^{-1} B^{T} h_{v}(t) + h_{v}(t) B U^{-1} B^{T} \frac{\partial h_{v}(t)}{\partial v_{i}} \right) dt
\]

\[
+ \int_{t_{i}}^{t_{i+1}} \left( \frac{\partial A_{v}(X_{v}^{-1}(t), t)}{\partial v_{i}} + \frac{\partial A_{v}(X_{v}^{-1}(t), t)}{\partial v_{i}} - \frac{\partial A_{v}(X_{v}^{-1}(t), t)}{\partial v_{i}} - \frac{\partial A_{v}(X_{v}^{-1}(t), t)}{\partial v_{i}} \right) \frac{\partial h_{v}^{-1}(t)}{\partial v_{i}} dt
\]

\[
- \frac{\partial h_{v}(t)}{\partial v_{i}} B U^{-1} B^{T} h_{v}(t) = \int_{t_{i}}^{t_{i+1}} \frac{\partial X_{v}^{-1}(t)}{\partial v_{i}} dt
\]

\[
\frac{\partial X_{v}^{-1}(t)}{\partial v_{i}} = \frac{\partial X_{v+1}(t)}{\partial v_{i}} = \int_{t_{i}}^{t_{i+1}} \left( \frac{\partial X_{v}^{-1}(t)}{\partial v_{i}} B U^{-1} B^{T} X_{v}^{-1}(t) + X_{v}^{-1}(t) B U^{-1} B^{T} \frac{\partial X_{v}^{-1}(t)}{\partial v_{i}} \right) dt
\]

\[
+ \int_{t_{i}}^{t_{i+1}} \left( \frac{\partial A_{v}(X_{v}^{-1}(t), t)}{\partial v_{i}} + \frac{\partial A_{v}(X_{v}^{-1}(t), t)}{\partial v_{i}} - \frac{\partial A_{v}(X_{v}^{-1}(t), t)}{\partial v_{i}} - \frac{\partial A_{v}(X_{v}^{-1}(t), t)}{\partial v_{i}} \right) \frac{\partial X_{v}^{-1}(t)}{\partial v_{i}} dt
\]

\[
- \frac{\partial X_{v}^{-1}(t)}{\partial v_{i}} B U^{-1} B^{T} X_{v}^{-1}(t) = \int_{t_{i}}^{t_{i+1}} \frac{\partial X_{v}(t)}{\partial v_{i}} dt
\]

Again, if \( \theta \rightarrow A_{\theta} \) is \( C^{1} \) on \( \Theta \), then we derive by induction that \( v \rightarrow R_{v,i}^{d} \) and \( v \rightarrow \)
\( h_{v,i}^d(Y) \) are \( C^1 \) as well for all \( i \in [0, n] \) and ruled by the following finite difference equations:

\[
\frac{\partial R_{v,i}^d}{\partial t}(Y) = \frac{\partial R_{v,i+1}^d}{\partial t}(Y) + \Delta \frac{\partial R_{v,i+1}^d}{\partial t} A_0(\mathbf{X}_v^{d,1}(t_i), t_i) + \Delta R_{v,i+1}^d \frac{\partial A_{\theta}}{\partial t} (\mathbf{X}_v^{d,1}(t_i), t_i) + \Delta R_{v,i+1}^d \frac{\partial A_{\theta}}{\partial t} (\mathbf{X}_v^{d,1}(t_i), t_i) T R_{v,i+1}^d + \Delta A_0(\mathbf{X}_v^{d,1}(t_i), t_i) T R_{v,i+1}^d - \Delta^2 \frac{\partial}{\partial t} A_0(\mathbf{X}_v^{d,1}(t_i), t_i) R_{v,i+1}^d \frac{\partial R_{v,i+1}^d}{\partial t} (\mathbf{X}_v^{d,1}(t_i), t_i) + \Delta A_0(\mathbf{X}_v^{d,1}(t_i), t_i) T R_{v,i+1}^d \frac{\partial R_{v,i+1}^d}{\partial t} (\mathbf{X}_v^{d,1}(t_i), t_i)
\]

Supplementary Materials
ence between these last equations, we obtain:

\[
\begin{align*}
\frac{\partial R^d_i(t_i)}{\partial \nu} - \frac{\partial R^d_i(t_{i+1})}{\partial \nu} &= \frac{\partial R^d_i(t_{i+1})}{\partial \nu} - \frac{\partial R^{d,l}_i(t_{i+1})}{\partial \nu} \\
+ \Delta \frac{\partial R^d_i(t_{i+1})}{\partial \nu} \left( R^d_i(t_{i+1}) - R^{d,l}_i(t_{i+1}) \right) + \frac{\partial R^d_i(t_{i+1})}{\partial \nu} \left( \frac{\partial \Delta \theta}{\partial \nu} X^{d,l-1}_v(t_i, t_i) - \frac{\partial \Delta \theta}{\partial \nu} X^{d,l-1}_v(t_i, t_i) \right) R^{d,l}_v(t_{i+1}) \\
+ \Delta \left( \frac{\partial \Delta \theta}{\partial \nu} (X^{d,l-1}_v(t_i, t_i) - \frac{\partial \Delta \theta}{\partial \nu} (X^{d,l-1}_v(t_i, t_i)) \right) T \frac{\partial \Delta \theta}{\partial \nu} R^{d,l}_v(t_{i+1}) \\
+ \Delta A_\theta (X^{d,l-1}_v(t_i, t_i) T \left( \frac{\partial R^d_i(t_{i+1})}{\partial \nu} - \frac{\partial R^{d,l}_i(t_{i+1})}{\partial \nu} \right) + \Delta \left( A_\theta (X^{d,l-1}_v(t_i, t_i) - A_\theta (X^{d,l-1}_v(t_i, t_i)) \right) \frac{\partial \Delta \theta}{\partial \nu} (X^{d,l-1}_v(t_i, t_i)) \\
+ \Delta \left( \frac{\partial \Delta \theta}{\partial \nu} (X^{d,l-1}_v(t_i, t_i) - \frac{\partial \Delta \theta}{\partial \nu} (X^{d,l-1}_v(t_i, t_i))) \right) \frac{\partial \Delta \theta}{\partial \nu} (X^{d,l-1}_v(t_i, t_i)) \\
+ \Delta \left( \frac{\partial \Delta \theta}{\partial \nu} (X^{d,l-1}_v(t_i, t_i) - \frac{\partial \Delta \theta}{\partial \nu} (X^{d,l-1}_v(t_i, t_i))) \right) \frac{\partial \Delta \theta}{\partial \nu} (X^{d,l-1}_v(t_i, t_i)) \\
+ \Delta \left( \frac{\partial \Delta \theta}{\partial \nu} (X^{d,l-1}_v(t_i, t_i) - \frac{\partial \Delta \theta}{\partial \nu} (X^{d,l-1}_v(t_i, t_i))) \right) \frac{\partial \Delta \theta}{\partial \nu} (X^{d,l-1}_v(t_i, t_i)) \\
+ \Delta \left( R^d_v(t_{i+1}) - R^{d,l}_v(t_{i+1}) \right) A_\theta (X^{d,l-1}_v(t_i, t_i) \frac{\partial \Delta \theta}{\partial \nu} (X^{d,l-1}_v(t_i, t_i)) \\
+ \Delta \frac{\partial R^d_i(t_{i+1})}{\partial \nu} BU^{-1} B^T \left( R^{d,l}_v(t_{i+1}) - R^d_v(t_{i+1}) \right) + \Delta \left( \frac{\partial R^d_i(t_{i+1})}{\partial \nu} - \frac{\partial R^{d,l}_i(t_{i+1})}{\partial \nu} \right) BU^{-1} B^T R^{d,l}_v(t_{i+1}) \\
- \Delta R^d_v(t_{i+1}) BU^{-1} B^T \left( \frac{\partial R^d_i(t_{i+1})}{\partial \nu} - \frac{\partial R^{d,l}_i(t_{i+1})}{\partial \nu} \right) + \Delta \left( R^{d,l}_v(t_{i+1}) - R^d_v(t_{i+1}) \right) BU^{-1} B^T \frac{\partial R^{d,l}_i}{\partial \nu} \\
+ O_{n,n} (\Delta^2)
\end{align*}
\]
\[
\begin{align*}
\frac{\partial X^{l}(t_i + 1)}{\partial v} - \frac{\partial X^{d,l}(t_i)}{\partial v} &= \frac{\partial X^{l}(t_i)}{\partial v} - \frac{\partial X^{d,l}(t_i)}{\partial v} \\
+ \Delta \frac{\partial A_{\theta}}{\partial x} (X^{l-1}_v(t_i), t_i) (X^{l}_v(t_i) - X^{d,l}_v(t_i)) \\
&+ \left( \frac{\partial A_{\theta}}{\partial x} (X^{l-1}_v(t_i), t_i) - \Delta \frac{\partial A_{\theta}}{\partial x} (X^{d,l-1}_v(t_i), t_i) \right) X^{d,l}_v(t_i) \\
+ \Delta \frac{\partial A_{\theta}}{\partial x} (X^{l-1}_v(t_i), t_i) \left( \frac{\partial X^{l-1}_v(t_i)}{\partial v} (X^{l}_v(t_i) - X^{d,l}_v(t_i)) + \left( \frac{\partial X^{l-1}_v(t_i)}{\partial v} - \frac{\partial X^{d,l-1}_v(t_i)}{\partial v} \right) X^{d,l}_v(t_i) \right) \\
+ \Delta \left( \frac{\partial A_{\theta}}{\partial x} (X^{l-1}_v(t_i), t_i) - \Delta \frac{\partial A_{\theta}}{\partial x} (X^{d,l-1}_v(t_i), t_i) \right) \frac{\partial X^{d,l-1}_v(t_i)}{\partial v} X^{d,l}_v(t_i) \\
+ \Delta \frac{\partial A_{\theta}}{\partial x} (X^{l-1}_v(t_i), t_i) \left( \frac{\partial X^{l-1}_v(t_i)}{\partial v} - \frac{\partial X^{d,l}_v(t_i)}{\partial v} \right) \\
- \Delta B U^{-1} B^T \left( \frac{\partial R^{d,l}_l(t_{i+1})}{\partial v} (X^{l}_v(t_i) - X^{d,l}_v(t_i)) + \left( \frac{\partial R^{d,l}_l(t_{i+1})}{\partial v} - \frac{\partial R^{d,l}_l(t_{i+1})}{\partial v} \right) X^{d,l}_v(t_i) \right) \\
- \Delta B U^{-1} B^T \left( \frac{\partial h^{d,l}_l(t_{i+1})}{\partial v} - \frac{\partial h^{d,l}_l(t_{i+1})}{\partial v} \right) + O_{p,n}(\Delta).
\end{align*}
\]

From this, we can see that \( \frac{\partial R^{d,l}_l(t_{i+1})}{\partial v} = O_{p,n}(1) \) and \( \frac{\partial h^{d,l}_l(Y)}{\partial v} = O_{p,n}(1) \) for all \( i \in [0, n] \) and \( v \in \mathcal{Y} \). By using that \( \|X^{d,l}_{v,i+1}(Y) - R^{l}_l(t_i)\|_2 = O_{p,n}(1) \), \( \|h^{d,l}_{v,i}(Y) - h^{l}_{v,i}(t_i)\|_2 = O_{p,n}(1) \) and \( \|X^{l}_v(t_i) - X^{l}_v(t_i)\|_2 = O_{p,n}(1) \), we can simplify the previous equations:

\[
\begin{align*}
\frac{\partial r^{l}_l(t_{i+1})}{\partial v} &= \left( 1 + 2 \Delta A_{\theta} (X^{l-1}_v(t_i), t_i) - 2 \Delta R^{l}_l(t_{i+1}) B U^{-1} B^T \right) \left( \frac{\partial R^{d,l}_l(t_{i+1})}{\partial v} - \frac{\partial R^{d,l}_l(t_{i+1})}{\partial v} \right) \\
+ \Delta A_{\theta} (X^{l-1}_v(t_i), t_i) B^T \left( \frac{\partial X^{l-1}_v(t_i)}{\partial v} - \frac{\partial X^{d,l}_v(t_i)}{\partial v} \right) R^{d,l}_l(t_{i+1}) + O_{p,n}(\Delta)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial h^{l}_l(t_{i+1})}{\partial v} &= \left( I_p + \Delta A_{\theta} (X^{l-1}_v(t_i), t_i) - R^{l}_l(t_{i+1}) B U^{-1} B^T \right) \left( \frac{\partial h^{d,l}_l(t_{i+1})}{\partial v} - \frac{\partial h^{d,l}_l(t_{i+1})}{\partial v} \right) \\
+ \Delta A_{\theta} (X^{l-1}_v(t_i), t_i) B^T \left( \frac{\partial X^{l-1}_v(t_i)}{\partial v} - \frac{\partial X^{d,l}_v(t_i)}{\partial v} \right) h^{d,l}_l(t_{i+1}) \\
- \Delta \left( \frac{\partial R^{d,l}_l(t_{i+1})}{\partial v} - \frac{\partial R^{d,l}_l(t_{i+1})}{\partial v} \right) B U^{-1} B^T h^{d,l}_l(t_{i+1}) + O_{p,n}(\Delta)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial X^{l}_v(t_{i+1})}{\partial v} - \frac{\partial X^{d,l}_v(t_i)}{\partial v} &= \left( 1 + 2 \Delta A_{\theta} (X^{l-1}_v(t_i), t_i) X^{d,l}_v(t_i) + \Delta B U^{-1} B^T R^{l}_l(t_{i}) \right) \left( \frac{\partial X^{l}_v(t)}{\partial v} - \frac{\partial X^{d,l}_v(t)}{\partial v} \right) \\
- \Delta B U^{-1} B^T \left( \frac{\partial R^{d,l}_l(t_{i+1})}{\partial v} - \frac{\partial R^{d,l}_l(t_{i+1})}{\partial v} \right) X^{d,l}_v(t_i) + \left( \frac{\partial h^{d,l}_l(t_{i+1})}{\partial v} - \frac{\partial h^{d,l}_l(t_{i+1})}{\partial v} \right) \right) + O_{p,n}(\Delta).
\end{align*}
\]
Because of various inequalities and the discrete Gronwall lemma [16] that
\[\frac{\partial R_{t}^{i}}{\partial \nu} - \frac{\partial R_{t+1}^{i}}{\partial \nu} \leq (1 + \Delta \tilde{A} + \Delta \| R_{t}^{i}(t) \|_{2} \| BU^{-1} B^{T} \|_{2} \| \frac{\partial R_{t}^{i}(t)}{\partial \nu} - \frac{\partial R_{t+1}^{i}}{\partial \nu} \|_{2} + O_{p,n}(\Delta) \]
\[\frac{\partial h^{i}_{t}(t)}{\partial \nu} - \frac{\partial h^{i}_{0}(Y)}{\partial \nu} \leq (1 + \Delta \tilde{A} + \Delta \| R_{t}^{i}(t) \|_{2} \| BU^{-1} B^{T} \|_{2} \| \frac{\partial R_{t}^{i}(t)}{\partial \nu} - \frac{\partial R_{t+1}^{i}}{\partial \nu} \|_{2} + O_{p,n}(\Delta) \]
+ O_{p,n}(\Delta) \]
\[\frac{\partial X^{i}_{t}(t)}{\partial \nu} - \frac{\partial X^{i}_{t+1}(t)}{\partial \nu} \leq (1 + \Delta \tilde{A}) \| X^{i}_{t}(t) \|_{2} + \Delta \| R_{t}^{i}(t) \|_{2} \| BU^{-1} B^{T} \|_{2} \| \frac{\partial X^{i}_{t}(t)}{\partial \nu} - \frac{\partial X^{i}_{t+1}(t)}{\partial \nu} \|_{2} + O_{p,n}(\Delta) \]
+ O_{p,n}(\Delta) \]
\[\| \frac{\partial X^{i}_{t}(t)}{\partial \nu} - \frac{\partial X^{i}_{t+1}(t)}{\partial \nu} \|_{2} = 0 , \text{ we can prove again by induction, by using the previous inequalities and the discrete Gronwall lemma [16] that} \]
\[\| \frac{\partial h^{i}_{t}(t)}{\partial \nu} - \frac{\partial h^{i}_{0}(Y)}{\partial \nu} \|_{2} = o_{p,n}(1) \text{ and} \]
\[\| \frac{\partial X^{i}_{t}(t)}{\partial \nu} - \frac{\partial X^{i}_{t+1}(t)}{\partial \nu} \|_{2} = o_{p,n}(1). \]

Despite the formal computation, there are no theoretical difficulties to derive under condition C8 that we can differentiate again the sensitivity equation and obtain
\[\| \frac{\partial^{2} R_{t}^{i}(t)}{\partial \nu^{2}} - \frac{\partial^{2} R_{t+1}^{i}}{\partial \nu^{2}} \|_{2} = o_{p,n}(1), \| \frac{\partial^{2} h^{i}_{t}(t)}{\partial \nu^{2}} - \frac{\partial^{2} h^{i}_{0}(Y)}{\partial \nu^{2}} \|_{2} = o_{p,n}(1). \]

**Lemma 18.** Under conditions C1 to C3, the uniform convergence of \( \overline{X}^{i}_{t} \) for each \( \nu \in \mathcal{T} \) leads to \( \sup_{\nu \in \mathcal{T}} \| R^{i}_{t} - R^{i}_{\infty} \|_{L^{2}} \rightarrow 0, \sup_{\nu \in \mathcal{T}} \| h^{i}_{t} - h^{i}_{\infty} \|_{L^{2}} \rightarrow 0 \) and \( \sup_{\nu \in \mathcal{T}} \| \overline{X}^{i}_{t} - \overline{X}^{i}_{\infty} \|_{L^{2}} \rightarrow 0 \).
0 when \( l \to \infty \).

**PROOF.** Let us consider \( R_{v}^{l}, h_{v}^{l}, X_{v}^{l} \), and \( R_{v}^{l'}, h_{v}^{l'}, X_{v}^{l'} \), respectively the solutions of (13) for a given \( l \in \mathbb{N} \) and \( l' \in \mathbb{N} \). By making the difference of the ODEs ruling the reversed time functions \( \tilde{R}_{v}^{l} := R_{v}^{l}(T - \cdot), \tilde{h}_{v}^{l} := h_{v}^{l}(T - \cdot), X_{v}^{l} := X_{v}^{l}(T - \cdot) \) and \( \tilde{R}_{v}^{l'} := R_{v}^{l'}(T - \cdot), \tilde{h}_{v}^{l'} := h_{v}^{l'}(T - \cdot), X_{v}^{l'} := X_{v}^{l'}(T - \cdot) \) we obtain:

\[
\begin{align*}
&\tilde{R}_{v}^{l}(t) - \tilde{R}_{v}^{l'}(t) \\
&= A_{\theta}(X_{v}^{\tilde{l}-1}(t), t)^{T} \left( R_{v}^{l}(t) - \tilde{R}_{v}^{l'}(t) \right) + \left( A_{\theta}(X_{v}^{\tilde{l}-1}(t), t) - A_{\theta}(X_{v}^{\tilde{l}'-1}(t), t) \right)^{T} \tilde{R}_{v}^{l'}(t) \\
&+ \tilde{R}_{v}^{l}(t) \left( A_{\theta}(X_{v}^{\tilde{l}-1}(t), t) - A_{\theta}(X_{v}^{\tilde{l}'-1}(t), t) \right) + \left( \tilde{R}_{v}^{l}(t) - \tilde{R}_{v}^{l'}(t) \right) A_{\theta}(X_{v}^{\tilde{l}'-1}(t), t) \\
&+ \tilde{R}_{v}^{l}(t) BU^{-1} B^{T} \left( R_{v}^{l}(t) - \tilde{R}_{v}^{l'}(t) \right) + \left( \tilde{R}_{v}^{l}(t) - \tilde{R}_{v}^{l'}(t) \right) BU^{-1} B^{T} \tilde{R}_{v}^{l'}(t) \\
&= A_{\theta}(X_{v}^{\tilde{l}-1}(t), t)^{T} \left( \tilde{h}_{v}^{l}(t) - \tilde{h}_{v}^{l'}(t) \right) + \left( A_{\theta}(X_{v}^{\tilde{l}-1}(t), t) - A_{\theta}(X_{v}^{\tilde{l}'-1}(t), t) \right)^{T} \tilde{h}_{v}^{l'}(t) \\
&+ \tilde{R}_{v}^{l}(t) BU^{-1} B^{T} \left( \tilde{h}_{v}^{l}(t) - \tilde{h}_{v}^{l'}(t) \right) + \left( \tilde{R}_{v}^{l}(t) - \tilde{R}_{v}^{l'}(t) \right) BU^{-1} B^{T} \tilde{h}_{v}^{l'}(t)
\end{align*}
\]

with \( \tilde{R}_{v}^{(0)}(0) - \tilde{R}_{v}^{l'}(0) = 0 \) and \( \tilde{h}_{v}^{l}(0) - \tilde{h}_{v}^{l'}(0) = 0 \). Here by taking the norm and by using proposition [1] we know \( \| R_{v}^{l}(t) \|_{2}, \| R_{v}^{l'}(t) \|_{2}, \| h_{v}^{l}(t) \|_{2}, \| h_{v}^{l'}(t) \|_{2} \) are uniformly bounded on \( \mathbb{N} \times [0, T] \times \mathcal{T} \) and we obtain:

\[
\begin{align*}
\frac{d}{dt} \left\| R_{v}^{l}(t) - \tilde{R}_{v}^{l'}(t) \right\|_{2} &\leq O_{n}(1) \left\| R_{v}^{l}(t) - \tilde{R}_{v}^{l'}(t) \right\|_{2} + O_{n}(1) \left\| A_{\theta}(X_{v}^{\tilde{l}-1}(t), t) - A_{\theta}(X_{v}^{\tilde{l}'-1}(t), t) \right\|_{2} \\
\frac{d}{dt} \left\| h_{v}^{l}(t) - \tilde{h}_{v}^{l'}(t) \right\|_{2} &\leq O_{n}(1) \left\| h_{v}^{l}(t) - \tilde{h}_{v}^{l'}(t) \right\|_{2} + O_{n}(1) \left\| A_{\theta}(X_{v}^{\tilde{l}-1}(t), t) - A_{\theta}(X_{v}^{\tilde{l}'-1}(t), t) \right\|_{2} \\
&\quad + O_{n}(1) \left\| R_{v}^{l}(t) - \tilde{R}_{v}^{l'}(t) \right\|_{2}.
\end{align*}
\]

By using the continuous Gronwall lemma, we easily obtain from the first inequality \( \| R_{v}^{l}(t) - \tilde{R}_{v}^{l'}(t) \|_{2} \leq O_{n}(1) \int_{0}^{T} \left\| A_{\theta}(X_{v}^{\tilde{l}-1}(t), t) - A_{\theta}(X_{v}^{\tilde{l}'-1}(t), t) \right\|_{2}^{2} dt. Since \( X_{v}^{\tilde{l}-1} - X_{v}^{\tilde{l}'-1} \to 0 \) on \( C \left( [0, T], \mathbb{R}^{d} \right) \) and \( \forall \theta \in \Theta, (x, t) \to A_{\theta}(x, t) \) is continuous on \( \Lambda \times [0, T] \), we have

\[
\sup_{v \in \mathcal{T}} \int_{0}^{T} \left\| A_{\theta}(X_{v}^{\tilde{l}-1}(t), t) - A_{\theta}(X_{v}^{\tilde{l}'-1}(t), t) \right\|^{2}_{2} dt \to 0
\]

when \( (l, l') \to +\infty \) and so \( \sup_{v \in \mathcal{T}} \left\| R_{v}^{l} - R_{v}^{l'} \right\|^{2}_{L^{2}} \to 0 \). From this limit, we derive the uniform convergence of the sequence \( \{ R_{v}^{l} \}_{l \in \mathbb{N}} \), and \( \{ h_{v}^{l} \}_{l \in \mathbb{N}} \) as well. Now let us control the difference \( X_{v}^{l} - X_{v}^{l'} \), by integrating and taking the norm, we obtain:

\[
\begin{align*}
\left\| X_{v}^{l}(t) - X_{v}^{l'}(t) \right\|_{2} &\leq O_{n}(1) \int_{0}^{t} \left\| X_{v}^{l'}(s) - X_{v}^{l'}(s) \right\|_{2} ds + O_{n}(1) \int_{0}^{t} \right\| h_{v}^{l'}(s) - h_{v}^{l'}(s) \right\|_{2} ds + O_{n}(1) \int_{0}^{t} \right\| h_{v}^{l'}(s) - h_{v}^{l'}(s) \right\|_{2} ds
\end{align*}
\]
by taking the norm, using Gronwall lemma and the limits sup \( v \in T \left\| \frac{\partial h^\ell_i}{\partial v} - \frac{\partial R^\ell_i}{\partial v} \right\|_{L^2}^2 \rightarrow 0 \), sup \( v \in T \left\| h^\ell_i - h^\ell_i \right\|_{L^2}^2 \rightarrow 0 \) we conclude sup \( v \in T \left\| \frac{X^\ell_i}{\ell_i} \right\|_{L^2}^2 \rightarrow 0 \).

6.3. Asymptotic normality

Lemma 19. Under conditions C1 to C6, we have \( h^\ell_i = h^\ell_i(t_i) + O_n(\Delta) \), \( \frac{\partial h^\ell_i}{\partial v} = \frac{\partial h^\ell_i(t_i)}{\partial v} + O_n(\Delta) \) and \( \nabla_v s_n^l(Y_{dx}; v^*) - \nabla_v s^l(v^*) = O_n(\Delta) \).

Proof. By formal computations similar as in lemma [7] we obtain the finite difference equations ruling \( R^\ell_i - R^\ell_i(t_i), h^\ell_i, X^\ell_i(t_{i+1}) - X^\ell_i(t_i) \):

\[
\begin{align*}
R^\ell_i &= R^\ell_i(t_i) + \Delta A_{\theta^*}(X^{\ell_i-1}(t_i), t_i)^T (R^\ell_i - R^\ell_i(t_i)) \\
&+ \Delta \left( A_{\theta^*}(X^{\ell_i-1}(t_i), t_i) - A_{\theta^*}(X^{\ell_i-1}(t_i), t_i) \right) R^\ell_i(t_i) \\
&+ \Delta R^\ell_i \left( R^\ell_i(t_i) - R^\ell_i(t_{i+1}) \right) A_{\theta^*}(X^{\ell_i-1}(t_i), t_i) \\
&+ \Delta \left( R^\ell_i(t_{i+1})BU^{-1}B^T (R^\ell_i(t_{i+1}) - R^\ell_i(t_i)) \right) \\
&+ O_n(\Delta^2)
\end{align*}
\]

\[
\begin{align*}
h^\ell_i &= h^\ell_i(t_i) + \Delta A_{\theta^*}(X^{\ell_i-1}(t_i), t_i)^T (h^\ell_i - h^\ell_i(t_i)) \\
&+ \Delta \left( A_{\theta^*}(X^{\ell_i-1}(t_i), t_i) - A_{\theta^*}(X^{\ell_i-1}(t_i), t_i) \right) h^\ell_i(t_i) \\
&+ \Delta R^\ell_i(t_i)BU^{-1}B^T (h^\ell_i(t_{i+1}) - h^\ell_i(t_i)) + \Delta \left( R^\ell_i(t_i) - R^\ell_i(t_{i+1}) \right) BU^{-1}B^T h^\ell_i(t_{i+1}) + O_n(\Delta^2)
\end{align*}
\]

Now, let us prove by induction that \( \left\| R^\ell_i(t_i) - R^\ell_i(t_i) \right\|_2 = O_n(\Delta) \), \( \left\| h^\ell_i - h^\ell_i(t_i) \right\|_2 = O_n(\Delta) \) and \( \left\| X^{\ell_i-1}(t_i) - X^{\ell_i-1}(t_i) \right\|_2 = O_n(\Delta) \). For initialisation, let us consider the case \( l = 1 \). We have \( \left\| X^{\ell_i-1}(t_i) - X^{\ell_i-1}(t_i) \right\|_2 = \left\| x_0^* - x_0^* \right\| = 0 \) for all \( i \in [0, n] \). From this, equations ruling \( R^\ell_i(t_i), h^\ell_i(t_i) \) and \( X^\ell_i(t_{i+1}) - X^\ell_i(t_i) \) become:

\[
\begin{align*}
R^\ell_i - R^\ell_i(t_i) &= R^\ell_i(t_i) + \Delta A_{\theta^*}(x_0^*, t_i)^T (R^\ell_i(t_i) - R^\ell_i(t_i)) \\
&+ \Delta \left( R^\ell_i(t_i) - R^\ell_i(t_{i+1}) \right) A_{\theta^*}(x_0^*, t_i)^T (R^\ell_i(t_{i+1}) - R^\ell_i(t_i)) \\
&+ \Delta \left( R^\ell_i(t_{i+1}) - R^\ell_i(t_{i+1}) \right) BU^{-1}B^T R^\ell_i(t_{i+1}) + O_n(\Delta^2)
\end{align*}
\]
\[ h_{i+1}^{d,1} - h^s(t_i) = h_{i+1}^s - h^s(t_{i+1}) + \Delta A_{t_i} (x_{i}^{\delta}, t_i)^T \left( h_{i+1}^{d,1} - h^s(t_{i+1}) \right) + \Delta R^s(t) BU^{-1} B^T \left( h^s(t_{i+1}) - h_{i+1}^{d,1} \right) + \Delta \left( R^s(t) - R_{i+1}^s \right) \]

By using triangular inequality, we derive:

\[ \left\| R_{i+1}^{s,1} - R^s(t_{i+1}) \right\|_2 \leq (1 + O_n(\Delta)) \left\| R_{i+1}^{s,1} - R^s(t_{i+1}) \right\|_2 + O_n(\Delta^2) \]

\[ \left\| h_{i+1}^{s,1} - h^s(t_{i+1}) \right\|_2 \leq (1 + O_n(\Delta)) \left\| h_{i+1}^{s,1} - h^s(t_{i+1}) \right\|_2 + O_n(\Delta) \left\| R_{i+1}^{s,1} - R^s(t_{i+1}) \right\|_2 + O_n(\Delta^2) \]

and since \( \left\| R_{0,1}^{s,1} - R^s(0) \right\|_2 = \left\| h_{0}^{s,1} - h^s(0) \right\|_2 = 0 \), we can use discrete Gronwall lemma \( \text{[16]} \) to derive:

\[ \left\| R_{i+1}^{s,1} - R^s(t_{i+1}) \right\|_2 = O_n(\Delta), \left\| h_{i+1}^{s,1} - h^s(t_{i+1}) \right\|_2 = O_n(\Delta), \left\| X^{s,1}(t_i) - X^s(t_i) \right\|_2 = O_n(\Delta). \]

Now let us assume the property holds up to \( l - 1 \), triangular inequality gives in the general case:

\[ \left\| R_{i+1}^{s,l} - R^s(t_{i+1}) \right\|_2 \leq (1 + 2\Delta \overline{T} + \Delta) \left\| R_{i+1}^{s,l} \right\|_2 + \Delta \left\| R_{i+1}^{s,l} \right\|_2 \left\| BU^{-1} B^T \right\|_2 \left\| R_{i+1}^{s,l} - R^s(t_{i+1}) \right\|_2 \]

\[ + \Delta \left\| R_{i+1}^{s,l} \right\|_2 + \left\| R^s(t_{i+1}) \right\|_2 \left\| A_{t_i} (X^{s,l-1}(t_i), t_i) - A_{t_i} (X^s(t_i), t_i) \right\|_2 + O_n(\Delta^2) \]

By using the induction hypothesis and \( (x, t) \mapsto A_{t_i} \) continuity, we have \( \left\| A_{t_i} (X^{s,l-1}(t_i), t_i) - A_{t_i} (X^s(t_i), t_i) \right\|_2 = O_n(\Delta) \) and we can use again lemma \( \text{[16]} \) to prove:

\[ \left\| R_{i+1}^{s,l} - R^s(t_{i+1}) \right\|_2 = O_n(\Delta), \] from this we derive:

\[ \left\| h_{i+1}^{s,l} - h^s(t_{i+1}) \right\|_2 \leq (1 + \Delta \overline{T} + \Delta \left\| R_{i+1}^{s,l} \right\|_2 \left\| BU^{-1} B^T \right\|_2 \left\| h_{i+1}^{s,l} - h_{i+1}^{s,d,l} \right\|_2 \]

\[ + \Delta \left\| BU^{-1} B^T \right\|_2 \left\| h_{i+1}^{s,l} \right\|_2 \left\| R^s(t_{i+1}) \right\|_2 \]

\[ + \Delta \left\| h^s(t_{i+1}) \right\|_2 \left\| A_{t_i} (X^{s,l-1}(t_i), t_i) - A_{t_i} (X^s(t_i), t_i) \right\|_2 + O_n(\Delta^2) \]

which leads to \( \left\| h_{i+1}^{s,l} - h^s(t_{i+1}) \right\|_2 = O_n(\Delta) \). As in the initialisation phase, we easily derive \( \left\| X^{s,l}(t_i) - X^s(t_i) \right\|_2 = O_n(\Delta) \) and we can conclude the induction proof. Thanks to condition 7, we can derive the sensitivity equations of \( R^{s,l}, h^{s,d,l} - h^s \) and \( X^{s,d,l} - X^{s,l-1} \).
\[
\frac{\partial}{\partial t_i} \left( \mathbf{R}^{s,d,l} - \mathbf{R}^s(t_i) \right) = \frac{\partial}{\partial t_i} \left( \mathbf{R}^{s,d,l} - \mathbf{R}^s(t_{i+1}) \right)
\]
\[
+ \Delta \left( \frac{\partial A^{s,d,l}}{\partial x_i} \left( \mathbf{X}^{s,d,l-1}(t_i), t_i \right)^T + \frac{\partial A^{s,d,l}}{\partial x_i} \left( \mathbf{X}^{s,d,l-1}(t_i), t_i \right) \frac{\partial X^{s,d,l-1}(t_i)}{\partial t_i} \right) \left( \mathbf{R}^{s,d,l} - \mathbf{R}^s(t_{i+1}) \right)
\]
\[
+ \Delta A_{\theta^*} \left( \mathbf{X}^{s,d,l-1}(t_i), t_i \right) \frac{\partial}{\partial t_i} \left( \mathbf{R}^{s,d,l} - \mathbf{R}^s(t_{i+1}) \right)
\]
\[
+ \Delta \left( \frac{\partial A_{\theta^*}}{\partial \xi} \left( \mathbf{X}^{s,d,l-1}(t_i), t_i \right) - \frac{\partial A_{\theta^*}}{\partial \xi} \left( \mathbf{X}^s(t_i), t_i \right) \right)^T \mathbf{R}^s(t_{i+1})
\]
\[
+ \left( \frac{\partial A_{\theta^*}}{\partial x} \left( \mathbf{X}^{s,d,l-1}(t_i), t_i \right) - \frac{\partial A_{\theta^*}}{\partial x} \left( \mathbf{X}^s(t_i), t_i \right) \right) \left( \frac{\partial X^{s,d,l-1}(t_i)}{\partial t_i} - \frac{\partial X^s(t_i)}{\partial t_i} \right) \left( \mathbf{R}^{s,d,l} - \mathbf{R}^s(t_{i+1}) \right)
\]
\[
+ \Delta \left( A_{\theta^*} \left( \mathbf{X}^{s,d,l-1}(t_i), t_i \right) - A_{\theta^*} \left( \mathbf{X}^s(t_i), t_i \right) \right)^T \frac{\partial R^s}{\partial \xi(t_{i+1})}
\]
\[
+ \Delta \frac{\partial R_{\theta^{*}}^{s,d,l}}{\partial \xi(t_{i+1})} \left( A_{\theta^*} \left( \mathbf{X}^{s,d,l-1}(t_i), t_i \right) - A_{\theta^*} \left( \mathbf{X}^s(t_i), t_i \right) \right),
\]
\[
+ \Delta R_{\theta^{*}}^{s,d,l} \left( \frac{\partial A_{\theta^*}}{\partial \xi} \left( \mathbf{X}^{s,d,l-1}(t_i), t_i \right) - \frac{\partial A_{\theta^*}}{\partial \xi} \left( \mathbf{X}^s(t_i), t_i \right) \right) \left( \frac{\partial X^{s,d,l-1}(t_i)}{\partial t_i} - \frac{\partial X^s(t_i)}{\partial t_i} \right)
\]
\[
+ \Delta \frac{\partial}{\partial \xi(t_{i+1})} \left( \mathbf{R}^{s,d,l} - \mathbf{R}^s(t_{i+1}) \right) A_{\theta^*} \left( \mathbf{X}^s(t_i), t_i \right) + \Delta \left( \mathbf{R}^{s,d,l} - \mathbf{R}^s(t_{i+1}) \right) \frac{\partial A_{\theta^*}}{\partial \xi} \left( \mathbf{X}^s(t_i), t_i \right)
\]
\[
+ \Delta \frac{\partial R_{\theta^{*}}^{s,d,l}}{\partial \xi(t_{i+1})} \left( \mathbf{R}^{s,d,l} - \mathbf{R}^s(t_{i+1}) \right) \left( \mathbf{R}^{s,d,l} - \mathbf{R}^s(t_{i+1}) \right) + \Delta \left( \mathbf{R}^{s,d,l} - \mathbf{R}^s(t_{i+1}) \right) \frac{\partial A_{\theta^*}}{\partial \xi} \left( \mathbf{X}^s(t_i), t_i \right)
\]
\[
+ \Delta \frac{\partial}{\partial \xi(t_{i+1})} \left( \mathbf{R}^{s,d,l} - \mathbf{R}^s(t_{i+1}) \right) \mathbf{B}^{t_i} \mathbf{B}^{t_i+1} + \Delta \left( \mathbf{R}^{s,d,l} - \mathbf{R}^s(t_{i+1}) \right) \mathbf{B}^{t_i} \mathbf{B}^{t_i+1} \frac{\partial R_{\theta^{*}}^{s,d,l}}{\partial \xi(t_{i+1})} + O_n(\Delta^2).
\]
Supplementary Materials

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\[ \frac{\partial}{\partial x} \left( X^{s, l}(t_{i+1}) - X^{s,l}(t_i) \right) = \frac{\partial}{\partial x} \left( X^{s,d,l}(t_i) - X^{s,l}(t_i) \right) \]

+ \Delta \left( \frac{\partial A_+}{\partial x} X^{s,d,l-1}(t_i, t_i) + \frac{\partial A_-}{\partial x} \left( X^{s,d,l-1}(t_i, t_i) - \frac{\partial X^{s,d,l-1}(t_i)}{\partial x} \right) \right) \left( X^{s,d,l}(t_i) - X^{s,l}(t_i) \right)

+ \Delta A_+ \left( X^{s,d,l-1}(t_i, t_i) + \frac{\partial A_+}{\partial x} \left( X^{s,l-1}(t_i, t_i) \right) \right) X^{s,l}(t_i)

+ \Delta A_- \left( X^{s,d,l-1}(t_i, t_i) - \frac{\partial A_-}{\partial x} \left( X^{s,l-1}(t_i, t_i) \right) \right) X^{s,l}(t_i)

+ \Delta \left( A_+ \left( X^{s,l-1}(t_i, t_i) \right) - A_- \left( X^{s,l-1}(t_i, t_i) \right) \right) \frac{\partial X^{s,l}(t_i)}{\partial x}

+ \Delta \left( R^{s,d,l} + R^{s,l+1} \right) \left( X^{s,l}(t_i) \right)

+ \Delta \left( R^{s,d,l} + R^{s,l+1} \right) \left( X^{s,l}(t_i) \right)

+ \Delta \left( R^{s,d,l} + R^{s,l+1} \right) \left( X^{s,l}(t_i) \right)

+ O_n(\Delta^2).

Application of triangular inequality gives us:

\[ \left\| \frac{\partial}{\partial x} \left( X^{s,d,l}(t_{i+1}) - X^{s,l}(t_i) \right) \right\|_2 \leq (1 + O_n(\Delta)) \left\| \frac{\partial}{\partial x} \left( X^{s,d,l}(t_{i+1}) - X^{s,l}(t_i) \right) \right\|_2 + O_n(\Delta) \left\| X^{s,d,l}(t_{i+1}) - X^{s,l}(t_i) \right\|_2 \]

+ O_n(\Delta) \left\| \frac{\partial A_+}{\partial x} \left( X^{s,l-1}(t_i, t_i) \right) - \frac{\partial A_-}{\partial x} \left( X^{s,l-1}(t_i, t_i) \right) \right\|_2 \left\| X^{s,l-1}(t_i, t_i) \right\|_2

+ O_n(\Delta) \left\| \frac{\partial A_+}{\partial x} \left( X^{s,l-1}(t_i, t_i) \right) - \frac{\partial A_-}{\partial x} \left( X^{s,l-1}(t_i, t_i) \right) \right\|_2 + O_n(\Delta^2).

Again from these inequalities, we can prove by induction \( \left\| \frac{\partial}{\partial x} \left( X^{s,d,l}(t_{i+1}) - X^{s,l}(t_i) \right) \right\|_2 = O_n(\Delta) \),

\( \left\| \frac{\partial}{\partial x} \left( h^{s,d,l}(t_{i+1}) - h^{s,l}(t_i) \right) \right\|_2 = O_n(\Delta) \) and \( \left\| \frac{\partial}{\partial x} \left( X^{s,d,l}(t_{i+1}) \right) \left( X^{s,l}(t_i) \right) \right\|_2 = O_n(\Delta) \).

Since

\[ \nabla \theta S^T(v^*) = (x^*_0)^T \frac{\partial R^{s,l+1}(t)}{\partial y} x^*_0 + 2 (x^*_0)^T \frac{\partial h^{s,l+1}(t)}{\partial y} - 2 J^T h^{s,l}(t) BU^{-1} B^T \frac{\partial h^{s,l+1}(t)}{\partial y} dt \]
we have:
\[
\nabla_\theta S_n^I(Y^{d*}; v^*) = (x_0^*)^T \frac{\partial R^{s,d,l}_0}{\partial \theta} x_0^* + 2 (x_0^*)^T \frac{\partial h^{s,d,l}_0}{\partial \theta} + 2 \Delta \sum_{i=0}^{n-1} \left( h^{s,d,l}_{i+1} \right)^T \left( B \frac{\partial G(R^{s,d,l}_0)B^T}{\partial \theta} h^{s,d,l}_{i+1} \right) \]
\[
= (x_0^*)^T \frac{\partial R^{s,d,l}_0}{\partial \theta} x_0^* + 2 (x_0^*)^T \frac{\partial h^{s,d,l}_0}{\partial \theta} - 2 \Delta \sum_{i=0}^{n-1} \left( h^{s,d,l}_{i+1} \right)^T \left( BU^{-1} B^T \frac{\partial h^{s,d,l}_0}{\partial \theta} + O_n(\Delta) \right)
\]
we derive:
\[
\nabla_\theta S_n^I(Y^{d*}; v^*) = \nabla_\theta S_n(Y^{d*}; v^*)
\]
and because we can approximate uniformly the terms in the last sum by:
\[
\int_{t_i}^{t_i+1} h^{s,l}(t)^T BU^{-1} B^T \frac{\partial h^{s,l}(t)}{\partial \theta} dt - \Delta \left( h^{s,d,l}_{i+1} \right)^T \left( BU^{-1} B^T \frac{\partial h^{s,d,l}_0}{\partial \theta} + O_n(\Delta) \right) = O_n(\Delta)
\]
we conclude \( \nabla_\theta S_n^I(Y^{d*}; v^*) = O_n(\Delta) \). Regarding \( \nabla_x S_n^I(v^*) \) and \( \nabla_x S_n^I(Y^{d*}; v^*) \), we have:
\[
\nabla_x S_n^I(Y^{d*}; v^*) = 2 R^{s,l}(0) x_0^* + (x_0^*)^T \frac{\partial R^{s,l}(0)}{\partial x_0} x_0^* + 2 h^{s,l}(0) + 2 (x_0^*)^T \frac{\partial h^{s,l}_0}{\partial x_0}
\]
\[
= 2 R^{s,l}(0) x_0^* + (x_0^*)^T \frac{\partial R^{s,l}(0)}{\partial x_0} x_0^* + 2 h^{s,l}(0) + 2 (x_0^*)^T \frac{\partial h^{s,l}_0}{\partial x_0} + O_n(\Delta)
\]
so their difference is given by:
\[
\nabla_x S_n^I(Y^{d*}; v^*) = \nabla_x S_n(Y^{d*}; v^*)
\]
and we derive from this \( \nabla_{x_0} S_n^l(Y^d; u^*) - \nabla_{x_0} S_l^l(v^*) = O_n(\Delta) \), hence the conclusion for \( \nabla_{x_0} S_n^l(Y^d; u^*) - \nabla_{x_0} S_l^l(v^*) \).

**Lemma 20.** Under conditions C1 to C7, we have \( \frac{\partial R^l}{\partial v} = \frac{\partial R^\infty}{\partial v} + o_l(1) \), \( \frac{\partial h^l}{\partial v} = \frac{\partial h^\infty}{\partial v} + o_l(1) \) and

\[
\left\{ \begin{array}{c}
\nabla_{v^*} S_l^l(v^*) = \nabla_{v^*} S^\infty(v^*) + o_l(1)
\end{array} \right.
\]

PROOF. As in lemma 18 we derive the differences \( X^l - X^{\infty} \), \( R^l - R^{\infty} \) and \( h^l - h^{\infty} \) are ruled by the equations:

\[
\frac{d}{dt} (X^l(t) - X^\infty(t)) = A_{\theta^*}(X^{l-1}(t), t)X^l(t) + BU^T B^T (R^l(t)) (X^l(t) + h^l(t))
\]

\[
= A_{\theta^*}(X^\infty(t), t)X^\infty(t) - BU^T B^T (R^\infty(t)) (X^\infty(t) + h^\infty(t))
\]

\[
= (A_{\theta^*}(X^{l-1}(t), t) + BU^T B^T R^l(t)) (X^l(t) - X^\infty(t))
\]

\[
+ (A_{\theta^*}(X^{l-1}(t), t) - A_{\theta^*}(X^\infty(t), t)) X^\infty(t)
\]

\[
+ BU^T B^T (R^l(t) - h^\infty(t) + (R^l(t) - R^\infty(t)) X^\infty(t))
\]

\[
\frac{d}{dt} (R^l(t) - R^\infty(t)) = (A_{\theta^*}(X^\infty(t), t) - R^l(t) BU^T B^T (R^l(t) - R^\infty(t))
\]

\[
+ R^\infty(t) ((A_{\theta^*}(X^\infty(t), t) - A_{\theta^*}(X^{l-1}(t), t)) R^l(t)
\]

\[
+ (R^l(t) - R^\infty(t)) (BU^T B^T R^\infty(t) - A_{\theta^*}(X^{l-1}(t), t))
\]

\[
+ (A_{\theta^*}(X^\infty(t), t) T - R^l(t) BU^T B^T (R^\infty(t) - h^\infty(t))
\]

\[
+ (A_{\theta^*}(X^\infty(t), t) - A_{\theta^*}(X^{l-1}(t), t)) T h^l(t)
\]

\[
+ (R^l(t) - R^\infty(t)) BU^T B^T h^\infty(t).
\]

Differentiate the last two equations gives us:

\[
\frac{d}{dt} \frac{\partial R^l}{\partial v} (R^l(t) - R^\infty(t))
\]

\[
= \left( \frac{\partial}{\partial v} (A_{\theta^*}(X^\infty(t), t) - R^l(t) BU^T B^T (R^l(t) - R^\infty(t))
\]

\[
+ (A_{\theta^*}(X^\infty(t), t) - A_{\theta^*}(X^{l-1}(t), t)) T R^l(t)
\]

\[
+ (R^l(t) - R^\infty(t)) (BU^T B^T R^\infty(t) - A_{\theta^*}(X^{l-1}(t), t))
\]

\[
+ (A_{\theta^*}(X^\infty(t), t) - A_{\theta^*}(X^{l-1}(t), t)) T h^l(t)
\]

\[
+ (R^l(t) - R^\infty(t)) BU^T B^T h^\infty(t)
\]

\[
= \left( \frac{\partial}{\partial v} (A_{\theta^*}(X^\infty(t), t) - A_{\theta^*}(X^{l-1}(t), t)) T R^l(t)
\]

\[
+ (R^l(t) - R^\infty(t)) (BU^T B^T R^\infty(t) - A_{\theta^*}(X^{l-1}(t), t))
\]

\[
+ (R^l(t) - R^\infty(t)) BU^T B^T h^\infty(t).
\]

\[
\frac{d}{dt} \frac{\partial h^l}{\partial v} (h^l(t) - h^\infty(t))
\]

\[
= \left( \frac{\partial}{\partial v} (X^\infty(t), t) - (X^{l-1}(t), t)) T R^l(t)
\]

\[
+ (R^l(t) - R^\infty(t)) (BU^T B^T R^\infty(t) - A_{\theta^*}(X^{l-1}(t), t))
\]

\[
+ (R^l(t) - R^\infty(t)) BU^T B^T h^\infty(t).
\]
Taking the norm and using triangular inequality gives us:

\[
\frac{d}{dt} \left\| \frac{\partial}{\partial \theta} (R^l(t) - R^{\infty}(t)) \right\|_2 \leq \left\| \frac{\partial}{\partial \theta} \left( A_{\theta^*} (X^*\infty(t), t)^T - R^l(t)BU^{-1}B^T \right) \right\|_2 \left\| h^*\infty(t) - h^l(t) \right\|_2 + \left\| \frac{\partial}{\partial \theta} \left( BU^{-1}B^T h^*\infty(t) \right) \right\|_2 \left\| \frac{\partial}{\partial \theta} (R^l(t) - R^{\infty}(t)) \right\|_2.
\]

By using lemma [18] and proposition [4], we can simplify these inequalities:

\[
\frac{d}{dt} \left\| \frac{\partial}{\partial \theta} (R^l(t) - R^{\infty}(t)) \right\|_2 \leq o_1(1) + O(1) \left\| \frac{\partial}{\partial \theta} \left( R^l(t) - R^{\infty}(t) \right) \right\|_2.
\]

Since \( \frac{\partial}{\partial \theta} R^l(t) = 0 \), \( R^{\infty}(0) = 0 \), and \( \frac{\partial}{\partial \theta} h^l(0) = \frac{\partial}{\partial \theta} h^*\infty(0) = 0 \), the continuous version of the Gronwall lemma successively gives us \( \left\| \frac{\partial}{\partial \theta} \left( R^l(t) - R^{\infty}(t) \right) \right\|_{L^2} = o_1(1) \) and \( \left\| \frac{\partial}{\partial \theta} \left( h^l(t) - h^{\infty}(t) \right) \right\|_{L^2} = o_1(1) \). In theorem [6], we already derived the expression of \( S^{\infty}(v) - S^l(v) \), from which we
obtain
\[ \nabla_{\theta} S^x(v^*) - \nabla_{\theta} S^l(v^*) = (x_0^*)^T \frac{\partial}{\partial \theta} (R^x(0) - R^l(0)) x_0^* + 2 (x_0^*)^T \frac{\partial}{\partial \theta} (h^x(0) - h^l(0)) \\
+ \int_0^T \left( \frac{\partial h^x_0}{\partial \theta}(t) + \frac{\partial h^x}{\partial \theta}(0) \right) B U^{-1} B^T (h^x(t) - h^*(t)) \, dt \\
+ \int_0^T (h^x(t) + h^*(t)) B U^{-1} B^T \left( \frac{\partial h^x_0}{\partial \theta}(t) - \frac{\partial h^x}{\partial \theta}(t) \right) \, dt. \]

and
\[ \nabla_{\theta} S^x(v^*) - \nabla_{\theta} S^l(v^*) = 2 (R^x(0) - R^l(0)) x_0^* + 2 (x_0^*)^T \frac{\partial}{\partial \theta} (h^x(0) - h^l(0)) \\
+ \int_0^T \left( \frac{\partial h^x_0}{\partial \theta}(t) + \frac{\partial h^x}{\partial \theta}(0) \right) B U^{-1} B^T (h^x(t) - h^*(t)) \, dt \\
+ \int_0^T (h^x(t) + h^*(t)) B U^{-1} B^T \left( \frac{\partial h^x_0}{\partial \theta}(t) - \frac{\partial h^x}{\partial \theta}(t) \right) \, dt. \]

and since we know \( \frac{\partial R^x}{\partial \theta} = \frac{\partial R^x}{\partial \theta} + o(1), \frac{\partial h^x}{\partial \theta} = \frac{\partial h^x}{\partial \theta} + o(1) \), we have \( \nabla_{\theta} S^x(v^*) - \nabla_{\theta} S^l(v^*) = o(1) \). As sensitivity equations, the ODEs ruling the functions \( \frac{\partial}{\partial \theta} (R^x(t) - R^l(t)), \frac{\partial}{\partial \theta} (h^x(t) - h^*(t)) \) are linear. By using C6 and classic existence and regularity results for linear ODEs, we know these functions are differentiables w.r.t to \( \theta \) and \( \frac{\partial}{\partial \theta} (R^x(t) - R^l(t)), \frac{\partial}{\partial \theta} (h^x(t) - h^*(t)) \) are defined on \([0, T]\). From the previous derived expressions, it is straightforward to see \( \frac{\partial R^x}{\partial \theta}, \frac{\partial h^x}{\partial \theta} \) (resp. \( \frac{\partial R^x}{\partial \theta}, \frac{\partial h^x}{\partial \theta} \)) are ruled by ODEs of the form
\[ V^l = F^l(t, v^*) V^l + G^l(t, v^*) \]
resp. \( V^x = F^x(t, v^*) V^x + G^x(t, v^*) \) with \( F^l, F^x, G^l, G^x \) continuous w.r.t \( t \) and \( \|V^l - V^x\|_{L^2} = o(1), \|F^l(t, v^*) - F^x(t, v^*)\|_{L^2} = o(1), \)
\( \|G^l(t, v^*) - G^x(t, v^*)\|_{L^2} = o(1), \)
\( \frac{\partial G^l}{\partial \theta}(t, v^*) - \frac{\partial G^x}{\partial \theta}(t, v^*) \) \( = o(1) \). Here \( V \) arbitrarily stands for \( \frac{\partial R^x}{\partial \theta}, \frac{\partial h^x}{\partial \theta} \). By differentiation, we obtain:
\[ \frac{d}{dt} \frac{\partial}{\partial \theta} (V^l(t) - V^x(t)) \]
\[ = \frac{\partial F^l}{\partial \theta}(t, v^*) V^l(t) + F^l(t, v^*) \frac{\partial V^l}{\partial \theta}(t) + \frac{\partial G^l}{\partial \theta}(t, v^*) V^x(t) - F^x(t, v^*) \frac{\partial V^x}{\partial \theta}(t) - \frac{\partial G^x}{\partial \theta}(t, v^*) V^x(t) \\
+ \left( \frac{\partial F^l}{\partial \theta}(t, v^*) - \frac{\partial F^x}{\partial \theta}(t, v^*) \right) V^l(t) + \frac{\partial G^l}{\partial \theta}(t, v^*) V^x(t) - \frac{\partial G^x}{\partial \theta}(t, v^*) V^x(t) \\
+ \left( \frac{\partial F^l}{\partial \theta}(t, v^*) - \frac{\partial F^x}{\partial \theta}(t, v^*) \right) \frac{\partial V^l}{\partial \theta}(t) + F^x(t, v^*) \left( \frac{\partial V^x}{\partial \theta}(t) - \frac{\partial V^x}{\partial \theta}(t) \right) \\
+ \frac{\partial G^l}{\partial \theta}(t, v^*) - \frac{\partial G^x}{\partial \theta}(t, v^*). \]

By taking the norm and by using triangular inequality, we obtain:
\[ \frac{d}{dt} \frac{\partial}{\partial \theta} \|V^l(t) - V^x(t)\|_2 \]
\[ \leq \left\| \frac{\partial F^l}{\partial \theta}(t, v^*) - \frac{\partial F^x}{\partial \theta}(t, v^*) \right\|_2 \|V^l(t)\|_2 + \left\| \frac{\partial F^l}{\partial \theta}(t, v^*) - \frac{\partial F^x}{\partial \theta}(t, v^*) \right\|_2 \|V^x(t) - V^x(t)\|_2 \\
+ \left\| \frac{\partial G^l}{\partial \theta}(t, v^*) - \frac{\partial G^x}{\partial \theta}(t, v^*) \right\|_2 \|V^l(t)\|_2 + \left\| \frac{\partial G^l}{\partial \theta}(t, v^*) - \frac{\partial G^x}{\partial \theta}(t, v^*) \right\|_2 \|V^x(t) - V^x(t)\|_2 \\
+ \left\| \frac{\partial G^l}{\partial \theta}(t, v^*) - \frac{\partial G^x}{\partial \theta}(t, v^*) \right\|_2 \|V^x(t) - V^x(t)\|_2 \\
\leq o(1) + O(1) \left\| \frac{\partial V^l}{\partial \theta}(t) - \frac{\partial V^x}{\partial \theta}(t) \right\|_2 \]
and since $\frac{\partial V^i(0)}{\partial x_0} = \frac{\partial V^\infty(0)}{\partial x_0} = 0$, we can conclude by using the continuous Gronwall lemma that $\frac{\partial^2 R^i}{\partial t \partial x_0} = \frac{\partial^2 R^\infty}{\partial t \partial x_0} + o_1(1)$, $\frac{\partial^2 h^i}{\partial t \partial x_0} = \frac{\partial^2 h^\infty}{\partial t \partial x_0} + o_1(1)$. By differentiating $\nabla_\theta S^\infty(v^*) - \nabla_\theta S^i(Y; v^*)$ with respect to $\theta$, we obtain:

$$
\frac{\partial^2 S^\infty(v^*)}{\partial \theta \partial \theta} - \frac{\partial^2 S^i(v^*)}{\partial \theta \partial \theta} = 2 \frac{\partial}{\partial \theta} \left( R^{\infty}(0) - R^i(0) \right) x_0^* + 2 \left( x_0^* \right)^T \frac{\partial^2}{\partial \theta \partial x_0} \left( R^{\infty}(0) - R^i(0) \right) x_0^* \\
+ 2 \int_0^T \left( \frac{\partial h^i}{\partial t \partial \theta}(t) + \frac{\partial h^\infty}{\partial t \partial \theta}(t) \right) BU^{-1} BT \left( h^i(t) - h^\infty(t) \right) dt \\
+ 2 \int_0^T \left( \frac{\partial h^i}{\partial \theta}(t) + \frac{\partial h^\infty}{\partial \theta}(t) \right) BU^{-1} BT \left( \frac{\partial h^i}{\partial t \partial \theta}(t) - \frac{\partial h^\infty}{\partial t \partial \theta}(t) \right) dt \\
+ \int_0^T \left( h^i(t) + h^\infty(t) \right) BU^{-1} BT \left( \frac{\partial h^i}{\partial \theta}(t) - \frac{\partial h^\infty}{\partial \theta}(t) \right) dt.
$$

and by differentiating $\nabla_{x_0} S^\infty(v^*) - \nabla_{x_0} S^i(v^*)$ with respect to $\theta$, we obtain:

$$
\frac{\partial^2 S^\infty(v^*)}{\partial x_0 \partial \theta} - \frac{\partial^2 S^i(v^*)}{\partial x_0 \partial \theta} = 2 \frac{\partial}{\partial x_0} \left( R^{\infty}(0) - R^i(0) \right) x_0^* + \left( x_0^* \right)^T \frac{\partial^2}{\partial x_0 \partial \theta} \left( R^{\infty}(0) - R^i(0) \right) x_0^* \\
+ 2 \int_0^T \left( \frac{\partial h^i}{\partial \theta}(t) + \frac{\partial h^\infty}{\partial \theta}(t) \right) BU^{-1} BT \left( h^i(t) - h^\infty(t) \right) dt \\
+ 2 \int_0^T \left( \frac{\partial h^i}{\partial \theta}(t) + \frac{\partial h^\infty}{\partial \theta}(t) \right) BU^{-1} BT \left( \frac{\partial h^i}{\partial \theta}(t) - \frac{\partial h^\infty}{\partial \theta}(t) \right) dt \\
+ \int_0^T \left( h^i(t) + h^\infty(t) \right) BU^{-1} BT \left( \frac{\partial h^i}{\partial \theta}(t) - \frac{\partial h^\infty}{\partial \theta}(t) \right) dt.
$$

and:

$$
\frac{\partial^2 S^\infty(v^*)}{\partial x_0 \partial \theta} - \frac{\partial^2 S^i(v^*)}{\partial x_0 \partial \theta} = 2 \frac{\partial}{\partial x_0} \left( R^{\infty}(0) - R^i(0) \right) x_0^* + \left( x_0^* \right)^T \frac{\partial^2}{\partial x_0 \partial \theta} \left( R^{\infty}(0) - R^i(0) \right) x_0^* \\
+ 2 \frac{\partial}{\partial x_0} \left( R^{\infty}(0) - R^i(0) \right) x_0^* + \left( x_0^* \right)^T \frac{\partial^2}{\partial x_0 \partial \theta} \left( R^{\infty}(0) - R^i(0) \right) x_0^* \\
+ 2 \frac{\partial}{\partial x_0} \left( h^\infty(0) - h^i(0) \right) + 2 \left( \frac{\partial}{\partial x_0} \left( h^\infty(0) - h^i(0) \right) + \left( x_0^* \right)^T \frac{\partial^2}{\partial x_0 \partial \theta} \left( h^\infty(0) - h^i(0) \right) \right) \\
+ \int_0^T \left( \frac{\partial h^i}{\partial \theta}(t) + \frac{\partial h^\infty}{\partial \theta}(t) \right) BU^{-1} BT \left( h^i(t) - h^\infty(t) \right) dt \\
+ 2 \int_0^T \left( \frac{\partial h^i}{\partial \theta}(t) + \frac{\partial h^\infty}{\partial \theta}(t) \right) BU^{-1} BT \left( \frac{\partial h^i}{\partial \theta}(t) - \frac{\partial h^\infty}{\partial \theta}(t) \right) dt \\
+ \int_0^T \left( h^i(t) + h^\infty(t) \right) BU^{-1} BT \left( \frac{\partial h^i}{\partial \theta}(t) - \frac{\partial h^\infty}{\partial \theta}(t) \right) dt.
$$

from this we can conclude that $\frac{\partial^2 S^i(v^*)}{\partial x_0 \partial \theta} = \frac{\partial^2 S^\infty(v^*)}{\partial x_0 \partial \theta} + o_1(1)$.

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