Identification of source terms in heat equation with dynamic boundary conditions

El Mustapha Ait Ben Hassi | Salah-Eddine Chorfi | Lahcen Maniar

We study an inverse parabolic problem of identifying two source terms in heat equation with dynamic boundary conditions from a final time overdetermination data. Using a weak solution approach by Hasanov, the associated cost functional is analyzed, especially a gradient formula of the functional is proved and given in terms of the solution of an adjoint problem. Next, the existence and uniqueness of a quasi-solution are also investigated. Finally, the numerical reconstruction of some heat sources in a 1-D equation is presented to show the efficiency of the proposed algorithm.

KEYWORDS
adjoint problem, Fréchet gradient, inverse source problem, Lipschitz continuity, parabolic problem, quasi-solution

MSC CLASSIFICATION
35R30; 35K05; 49N45; 47A05

1 | INTRODUCTION

In this paper, we are interested in an inverse parabolic problem. It consists of identifying two source terms in a heat equation with dynamic boundary conditions, from a noisy measurement of the temperature at final time.

Let $T > 0$ be a fixed final time, and let $\Omega \subset \mathbb{R}^N$ be a bounded domain ($N \geq 2$ is an integer) with boundary $\Gamma = \partial \Omega$ of class $C^2$. We denote $\Omega_T = (0, T) \times \Omega$ and $\Gamma_T = (0, T) \times \Gamma$. We search for a pair of functions $y : \Omega_T \to \mathbb{R}$ and $y_\Gamma : \Gamma_T \to \mathbb{R}$ that fulfill the following heat equation

$$\begin{cases}
\partial_y y - d\Delta y + a(x)y = F(t, x), & \text{in } \Omega_T, \\
\partial_y y_\Gamma - \gamma \Delta y_\Gamma + d\partial_y y + b(x)y_\Gamma = G(t, x), & \text{on } \Gamma_T, \\
y(t, x) = y_\Gamma(t, x), & \text{on } \Gamma_T, \\
y(0) = y_\Gamma|_{t=0} = (y_0, y_0_{\Gamma}), & \Omega \times \Gamma
\end{cases} \tag{1}$$

for initial data $Y_0 := (y_0, y_0_{\Gamma}) \in L^2(\Omega) \times L^2(\Gamma)$ and heat sources $F \in L^2(\Omega_T)$ and $G \in L^2(\Gamma_T)$. The diffusion coefficients are positive constants $d, \gamma > 0$, and the spatial potentials are such that $a \in L^\infty(\Omega)$ and $b \in L^\infty(\Gamma)$. $\Delta = \Delta_x$ denotes the standard Laplace operator with respect to the space variable. By $y_\Gamma$, one denotes the trace of $y$, while the normal derivative is denoted by $\partial_y y := (\nabla y \cdot \nu)|_{\Gamma}$, where $\nu(x)$ stands for the unit outward normal vector to $\Gamma$ at $x$. The tangential gradient $\nabla_{\Gamma} y$ is given by $\nabla_{\Gamma} y = \nabla y - (\partial_y y) \nu$. Let $g$ be the natural Riemannian metric on $\Gamma$ inherited from $\mathbb{R}^N$. The Laplace–Beltrami operator $\Delta_{\Gamma}$ (with respect to $g$) is given locally by
\[ \Delta_{\Gamma} = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{N-1} \frac{\partial}{\partial x^i} \left( \sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j} \right), \]

where \( g = (g_{ij}) \) is the metric tensor corresponding to \( g \), \( g^{-1} = (g^{ij}) \) its inverse and \( |g| = \det (g_{ij}) \). In the sequel, we mainly use the following surface divergence formula

\[ \int_{\Gamma} \Delta_{\Gamma} y z \, dS = - \int_{\Gamma} \langle \nabla_{\Gamma} y, \nabla_{\Gamma} z \rangle_{\Gamma} \, dS, \quad y \in H^2(\Gamma), \quad z \in H^1(\Gamma), \quad (2) \]

where \( \langle \cdot, \cdot \rangle_{\Gamma} \) is the Riemannian inner product of tangential vectors on \( \Gamma \).

Parabolic equations with dynamic boundary conditions have received a lot of attention of many researchers in the last years,\(^1\)–\(^6\) since they appear in several fields of applications including chemical engineering such as chemical reactors, the chemistry of colloids, and special flows in hydrodynamics.\(^7\) The scope of applications also includes heat transfer problems where the diffusion takes place between a solid and a fluid in motion.\(^8\) We refer to the seminal paper,\(^9\) where the physical derivation and interpretation of dynamic boundary conditions are discussed.

A variety of analytical and numerical techniques have been proposed for solving inverse problems in partial differential equations (PDEs) with static boundary conditions (Dirichlet, Neumann, or Robin conditions); see, for instance other studies.\(^10\)-\(^13\) A well-known theoretical technique is the Carleman estimate, which is a priori weighted inequality estimating the solution of a PDE and its derivatives via the associated differential operator. Such an estimate was introduced, for the first time, in the study of multidimensional inverse problems by Bukhgeim and Klibanov,\(^10\) and became a powerful tool to establish uniqueness and stability results. Recently in Ait Ben Hassi et al.,\(^14\) the authors have applied this method in the study of an inverse source problem (ISP) for general parabolic equations with dynamic boundary conditions from interior measurements. They established a Lipschitz stability estimate for the source terms. In Ait Ben Hassi et al.,\(^15\) we have considered an inverse problem of radiative potentials and initial temperatures for the same equation. We have proven a Lipschitz stability estimate for the potentials and then obtained a logarithmic stability result for the initial conditions by a logarithmic convexity method. It is worthwhile to mention that Carleman estimates are also important in the numerical study of inverse problems.\(^16\)

In the context of numerical studies, Hasanov\(^12\) developed a weak solution approach to study the simultaneous determination of source terms in heat equation with static boundary conditions from the final overdetermination, Dirichlet or Neumann types output measured data.\(^17\),\(^18\) The underlying method relies on reformulating a quasi-solution problem as a minimization problem of Tikhonov functional, combined with an adjoint problem approach introduced by DuChateau.\(^19\),\(^20\) This approach provides a monotone iteration scheme to reconstruct unknown parameters in ill-posed problems, which implies fast numerical results. It has also been successfully applied to source terms in the cantilevered beam equation,\(^21\) Lotka Volterra system,\(^22\) and recently for thermal conductivity and radiative potential in heat equation from Dirichlet and Neumann boundary measured outputs.\(^23\) We refer the interested readers to the recent book by Hasanov Hasanoğlu and Romanov,\(^24\) which presents a systematic study of mathematical and numerical methods used in inverse problems within this framework.

In the present work, we extend the previous technique for the determination of two source terms acting in both, in the domain and on the boundary, in heat equation with dynamic boundary conditions. In this context, we mention Slodicka\(^25\) for the identification of a time-dependent source term from the knowledge of a space average using the backward Euler method. Also, in Ismailov,\(^26\) the author deals with a time-dependent source from an integral overdetermination condition in a 1-D heat equation by using the generalized Fourier method. These two papers dealt with a basic dynamic boundary condition that only contains time and normal derivatives on the boundary (known as Wentzell/Ventcel boundary condition). To the best of our knowledge, the study of such problem by a weak solution approach has not been addressed in the literature, and only equations with classical boundary conditions (Dirichlet, Neumann, and Robin) are considered. Here, we deal with a multidimensional heat system of two equations coupled through the boundary via the normal derivative, and which contains a surface diffusion on the boundary. Thus, the present problem requires a careful analysis and suitable combination between the two equations in order to obtain the desired results. We also study numerically the reconstruction for a heat source in the 1-D equation from a noisy terminal data.

The rest of the paper is organized as follows: in Section 2, we recall the wellposedness and regularity results of the system (1). In Section 3, we study the minimization problem associated to quasi-solutions. We also infer an explicit gradient formula for the Tikhonov functional via the solution of an appropriate adjoint problem. Then, we establish the Lipschitz
continuity of the gradient of the cost functional. In Section 4, we discuss the existence and uniqueness of a quasi-solution. Finally, in Section 5, the gradient formula is implemented via Landweber scheme for the numerical reconstruction of an unknown source term in 1-D equation. Section 6 is devoted to some conclusions.

2 WELLPOSEDNESS AND REGULARITY OF THE SOLUTION

We recall some results on the wellposedness and regularity of the solution to system (1) needed in the sequel. The reader can refer to Maniar et al\(^5\) for detailed proofs.

We first introduce the following real spaces

\[ \mathbb{L}^2 := L^2(\Omega, dx) \times L^2(\Gamma, dS) \quad \text{and} \quad \mathbb{L}^2_T := L^2(\Omega_T) \times L^2(\Gamma). \]

\(\mathbb{L}^2\) and \(\mathbb{L}^2_T\) are Hilbert spaces equipped with the scalar products given by

\[
\langle (y, y_T), (z, z_T) \rangle_{\mathbb{L}^2} = \langle y, z \rangle_{L^2(\Omega)} + \langle y_T, z_T \rangle_{L^2(\Gamma)},
\]

\[
\langle (y, y_T), (z, z_T) \rangle_{\mathbb{L}^2_T} = \langle y, z \rangle_{\mathbb{L}^2(\Omega_T)} + \langle y_T, z_T \rangle_{L^2(\Gamma)},
\]

respectively, where we denoted the Lebesgue measure on \(\Omega\) by \(dx\) and the surface measure on \(\Gamma\) by \(dS\). We also define the space

\[ \mathbb{H}^k := \{(y, y_T) \in H^k(\Omega) \times H^k(\Gamma) : y|_\Gamma = y_T\} \quad \text{for} \ k = 1, 2, \]

equipped with the standard product norm.

For the regularity of the solution, we consider the following spaces

\[ \mathbb{E}_1(t_0, t_1) := H^1(t_0, t_1; \mathbb{L}^2) \cap L^2(t_0, t_1; \mathbb{H}^2) \quad \text{for} \ t_1 > t_0 \ \text{in} \ \mathbb{R}, \]

\[ \mathbb{E}_2(t_0, t_1) := H^1(t_0, t_1; \mathbb{H}^2) \cap H^2(t_0, t_1; \mathbb{L}^2) \quad \text{for} \ t_1 > t_0 \ \text{in} \ \mathbb{R}. \]

In particular,

\[ \mathbb{E}_1 := \mathbb{E}_1(0, T) \quad \text{and} \quad \mathbb{E}_2 := \mathbb{E}_2(0, T). \]

We rewrite system (1) in the following abstract form

\[
\text{(ACP)} \quad \begin{cases} 
\partial_t Y = \mathcal{A} Y + \mathcal{F}, & 0 < t \leq T, \\
Y(0) = Y_0 := (y_0, y_0|_\Gamma).
\end{cases}
\]

where \(Y := (y, y_T)\), \(\mathcal{F} = (F, G)\) and the linear operator \(\mathcal{A} : D(\mathcal{A}) \subset \mathbb{L}^2 \to \mathbb{L}^2\) is given by

\[
\mathcal{A} = \begin{pmatrix} d\Delta - a & 0 \\
-\partial_\nu & \gamma \Delta_\Gamma - b \end{pmatrix}, \\
D(\mathcal{A}) = \mathbb{H}^2.
\]

The operator \(\mathcal{A}\) generates an analytic \(C_0\)-semigroup \((e^{\mathcal{A}t})_{t \geq 0}\) on \(\mathbb{L}^2\), see Maniar et al\(^5\) for more details. In the sequel, we will adopt the following notions for the solutions to system (1).

**Definition 1.** Let \(Y_0 \in \mathbb{L}^2\) and \((F, G) \in \mathbb{L}^2_T\).

(a) A strong solution of system (1) is a function \(Y := (y, y_T) \in \mathbb{E}_1\) fulfilling system (1) in \(L^2(0, T; \mathbb{L}^2)\).

(b) A mild solution of system (1) is a function \(Y := (y, y_T) \in C([0, T]; \mathbb{L}^2)\) satisfying, for \(t \in [0, T]\),

\[
Y(t, \cdot) = e^{\mathcal{A}t} Y_0 + \int_0^t e^{(t-s)\mathcal{A}} [F(r, \cdot), G(r, \cdot)] dr.
\]
A distributional (weak) solution of system (1) is a function \( Y := (y, y_T) \in L^2(0, T; \mathbb{R}^n) \) such that for all \( \varphi, \varphi_T \in E \) with \( \varphi(T, \cdot) = \varphi_T(T, \cdot) = 0 \) we have

\[
\int_{\Omega} y(-\partial_t \varphi - d \Delta \varphi + a \varphi) \, dx \, dt + \int_{\Gamma} y_T (-\partial_t \varphi_T - \gamma \Delta_T \varphi_T + d \partial_n \varphi + b \varphi_T) \, ds \, dt
\]

\[
= \int_{\Omega} F \varphi \, dx \, dt + \int_{\Gamma} G \varphi_T \, ds \, dt + \int_{\Omega} y_0 \varphi(0, \cdot) \, dx + \int_{\Gamma} y_0 T \varphi_T (0, \cdot) \, ds.
\]

The following proposition shows the \( L^2 \)-regularity for the system (1) and highlights the connections between different types of solutions. For the proof, see Maniaret al.\(^5\), Propositions 2.4 and 2.5

**Proposition 1.** Let \( F \in L^2(\Omega_T) \) and \( G \in L^2(\Gamma_T) \).

(i) For all \( Y_0 := (y_0, y_{0T}) \in \mathbb{H}^1 \), there exists a unique strong solution of system (1) such that \( Y := (y, y_T) \in E_1 \). Moreover, there exists a positive constant \( C = C(\|a\|_{\infty}, \|b\|_{\infty}) \) such that

\[
\|Y\|_{E_1} \leq C \left( \|F\|_{L^2(\Omega_T)} + \|G\|_{L^2(\Gamma_T)} + \|Y_0\|_{\mathbb{H}^1} \right).
\]

(ii) For all \( Y_0 := (y_0, y_{0T}) \in L^2 \), there exists a unique mild solution of system (1) \( Y := (y, y_T) \in C([0, T]; L^2) \) such that for all \( \tau \in (0, T) \),

\[
Y \in E_2(\tau, T) \coloneqq H^1(\tau, T; L^2) \cap L^2(\tau, T; \mathbb{H}^2).
\]

Moreover, if \( F = (F, G) \in H^1(0, T; L^2) \), then for all \( \tau \in (0, T) \), we have

\[
Y \in E_2(\tau, T) \coloneqq H^1(\tau, T; \mathbb{H}^2) \cap H^2(\tau, T; L^2),
\]

with initial data \( Y(\tau) \).

(iii) A function \( Y \) is a distributional solution of system (1) if and only if it is a mild solution.

### 3 | FRÉCHET DIFFERENTIABILITY AND GRADIENT FORMULA OF THE COST FUNCTIONAL

In this section, we consider the following ISP.

**ISP.** A couple of source terms \( F, G \in \mathbb{L}^2 \) in system (1) is unknown and needs to be recovered from the final temperature at \( T \), namely,

\[
Y_T := (y(T, \cdot), y_T(T, \cdot)) \in \mathbb{L}^2,
\]

which is not necessarily smooth due to the numerical noise.

Let \( Y(t, \cdot, F) \) be the mild solution of system (1) corresponding to the source terms \( F = (F, G) \in \mathbb{L}^2 \). We introduce the input–output operator \( \Psi : \mathbb{L}^2_T \to \mathbb{L}^2 \) defined as follows:

\[
(\Psi F)(\cdot) := Y_T(\cdot) \coloneqq Y(T, \cdot, F) \quad \text{on} \quad \Omega \times \Gamma.
\]

Then, the ISP can be reformulated as the following operator equation:

\[
\Psi F = Y_T, \quad Y_T \in \mathbb{L}^2. \tag{4}
\]

The following lemma is needed in the sequel.

**Lemma 1.** Let \( Y_0 \in \mathbb{L}^2 \) and let \( Y \) be the mild solution of system (1) corresponding to \( F = (F, G) \). Then, the solution map \( F \mapsto Y \) is continuous from \( H^1(0, T; \mathbb{L}^2) \) to \( C([0, T]; \mathbb{L}^2) \cap \mathbb{L}^2(0, T; \mathbb{H}^1) \).
Proof. Let $Y_0 \in H^2$ and $\delta F$ be a small variation of $F$ such that $F + \delta F \in U$. Consider $\delta Y := Y^\delta - Y$, where $Y^\delta$ is the mild solution of system $(1)$ corresponding to $F^\delta := F + \delta F$. Then $\delta Y \in C^1([0, T], L^2)$ and satisfies the following system:
\[
\begin{cases}
\partial_t(\delta y) - d\Delta(\delta y) + a(\chi)(\delta y) = \delta F(t, x), & \text{in } \Omega_T,
\partial_t(\delta y) - \gamma \Delta_T(\delta y_T) + d\partial_y + b(\chi)(\delta y_T) = \delta G(t, x), & \text{on } \Gamma_T,
\delta y_T(t, x) = (\delta y)_T(t, x), & \text{on } \Gamma_T,
(\delta y, \delta y_T)|_{t=0} = (0, 0), & \Omega \times \Gamma,
\end{cases}
\]
where
\[
\begin{align*}
\delta y(T, \cdot, F) &= y(T, \cdot, F + \delta F) - y(T, \cdot, F), \\
\delta y_T(T, \cdot, F) &= y_T(T, \cdot, F + \delta F) - y_T(T, \cdot, F).
\end{align*}
\]
Multiplying $(5)_1$ by $\delta y$, $(5)_2$ by $\delta y_T$ and using Green and surface divergence formulae, we obtain
\[
\begin{align*}
\frac{1}{2} \partial_t \left( \int_{\Omega} |\delta y|^2 \, dx \right) + d \int_{\Omega} |\nabla(\delta y)|^2 \, dx - \frac{1}{2} \int_{\Gamma} d\partial_y(\delta y)^2 \, dS + \int_{\Omega} a|\delta y|^2 \, dx &= \int_{\Gamma} \delta F \delta y \, dx, \\
\frac{1}{2} \partial_t \left( \int_{\Gamma} |\delta y_T|^2 \, dS \right) + \gamma \int_{\Gamma} |\nabla_T(\delta y_T)|^2 \, dS + \frac{1}{2} \int_{\Gamma} b|\delta y_T|^2 \, dS &= \int_{\Gamma} \delta G \delta y_T \, dS.
\end{align*}
\]
Adding up the last two equalities, we arrive at
\[
\begin{align*}
\frac{1}{2} \partial_t \left( \int_{\Omega} |\delta y|^2 \, dx + \int_{\Gamma} |\delta y_T|^2 \, dS \right) + d \int_{\Omega} |\nabla(\delta y)|^2 \, dx + \gamma \int_{\Gamma} |\nabla_T(\delta y_T)|^2 \, dS + \int_{\Omega} a|\delta y|^2 \, dx &= \int_{\Gamma} \delta F \delta y \, dx + \int_{\Gamma} \delta G \delta y_T \, dS.
\end{align*}
\]
Then, using the Cauchy–Schwarz inequality, we have
\[
\frac{1}{2} \frac{d}{dt} \left( \int_{\Omega} \|\delta Y(t)\|_{L^2}^2 + \min(d, \gamma) \|\nabla Y(t), \nabla_T y_T(t)\|_{L^2}^2 \right) \\
\leq \|\delta Y, \delta F\|_{L^2}^2 + \max(\|a\|_{\infty}, \|b\|_{\infty}) \|\delta Y(t)\|_{L^2}^2 \\
\leq \|\delta Y(t)\|_{L^2}^2 + \|\delta F(t)\|_{L^2}^2 + \max(\|a\|_{\infty}, \|b\|_{\infty}) \|\delta Y(t)\|_{L^2}^2 \\
\leq \frac{1}{2} \|\delta Y(t)\|_{L^2}^2 + \frac{1}{2} \|\delta F(t)\|_{L^2}^2 + \max(\|a\|_{\infty}, \|b\|_{\infty}) \|\delta Y(t)\|_{L^2}^2,
\]
where $F = (F, G)$. Consequently,
\[
\frac{d}{dt} \|\delta Y(t)\|_{L^2}^2 \leq (1 + 2 \max(\|a\|_{\infty}, \|b\|_{\infty})) \|\delta Y(t)\|_{L^2}^2 + \|\delta F(t)\|_{L^2}^2.
\]
By Gronwall inequality, we deduce
\[
\|\delta Y(t)\|_{L^2}^2 \leq e^{(1 + 2 \max(\|a\|_{\infty}, \|b\|_{\infty}))T} \left( \|\delta Y(0)\|_{L^2}^2 + \|\delta F\|_{L^2_T}^2 \right)
\]
for every $0 \leq t \leq T$. On the other hand, from Equations $(6)$ and $(7)$, we have
\[
\frac{d}{dt} \|\delta Y(t)\|_{L^2}^2 + 2 \min(d, \gamma) \|\nabla Y(t), \nabla_T y_T(t)\|_{L^2}^2 \\
\leq (1 + 2 \max(\|a\|_{\infty}, \|b\|_{\infty})) \|\delta Y(t)\|_{L^2}^2 + \|\delta F(t)\|_{L^2}^2 \\
\leq (1 + 2 \max(\|a\|_{\infty}, \|b\|_{\infty})) e^{(1 + 2 \max(\|a\|_{\infty}, \|b\|_{\infty}))T} \|\delta F\|_{L^2_T}^2 + \|\delta F(t)\|_{L^2}^2.
\]
Integrating between 0 and T and using the fact that $\delta Y(0) = 0$, we deduce
\[
\int_0^T \left\| (\nabla \gamma(t), \nabla \gamma_T(t)) \right\|_{L^2}^2 \, dt + 2 \min(d, \gamma) \left\| (\nabla \gamma(t), \nabla \gamma_T(t)) \right\|_{L^2}^2 \leq (1 + 2 \max(\|a\|_{\infty}, \|b\|_{\infty})) \left\| T e^{(1 + 2 \max(\|a\|_{\infty}, \|b\|_{\infty})) T} \delta F \right\|_{L^2}^2 + \| \delta F \|_{L^2}^2.
\]
Then, we arrive at
\[
\int_0^T \left\| (\nabla \gamma(t), \nabla \gamma_T(t)) \right\|_{L^2}^2 \, dt \leq C_T \| \delta F \|_{L^2}^2,
\]
with
\[
C_T := \frac{1 + 2 \max(\|a\|_{\infty}, \|b\|_{\infty})) T e^{(1 + 2 \max(\|a\|_{\infty}, \|b\|_{\infty})) T} + 1}{2 \min(d, \gamma)}.
\]
The latter inequality and Equation (7) imply that
\[
\sup_{t \in [0, T]} \left\| \delta Y(t) \right\|_{L^2}^2 + \int_0^T \left\| \delta Y(t) \right\|_{L^2}^2 \, dt \leq C \| \delta F \|_{L^2}^2,
\]
for some generic constant $C = C(T, \|a\|_{\infty}, \|b\|_{\infty}, d, \gamma) > 0$. Hence,
\[
\left\| \delta Y \right\|_{C([0, T]; L^2)}^2 + \left\| \delta Y \right\|_{L^2([0, T]; H^1)}^2 \leq C \| \delta F \|_{H^1([0, T]; L^2)}^2.
\]
Since $H^2$ is dense in $L^2$, the same inequality holds for any $Y_0 \in L^2$. This ends the proof.

The following lemma highlights the compactness of the input-output operator $\Psi : L^2_T \rightarrow L^2$. By linearity, we may assume here that $Y_0 := (y_0, y_0_T) = (0, 0)$.

**Lemma 2.** The input–output operator
\[
\Psi : L^2_T \ni F \mapsto Y(T, \cdot, F) \in L^2
\]
is compact.

**Proof.** Consider $(F_n)_{n \in \mathbb{N}}$ a bounded sequence in $L^2_T$. By estimate problem (3), the sequence of associated strong solutions $(Y(\cdot, \cdot, F_n))_{n \in \mathbb{N}}$ of system (1) is bounded in $E_2$. Since the trace space of $E_1$ at $t = T$ equals $H^1$ (see Maniar et al. [5], Proposition 2.2), the sequence $(Y_n)_{n \in \mathbb{N}}$ defined by $Y_n := Y(T, \cdot, F_n)$ is bounded in $H^1$. By the compact embedding $H^1 \hookrightarrow L^2$ (see Maniar et al. [5]), there exists a subsequence of $(Y_n)_{n \in \mathbb{N}}$ which converges in $L^2$. Thus, the input–output operator $\Psi$ is compact.

It follows from Lemma 2 that the inverse problem (4) is ill-posed (in the sense of Hadamard). A quasi-solution $F_*$ of the ill-posed problem (4) is defined as a solution of the following minimization problem:
\[
J(F_*) = \inf_{F \in \mathcal{U}} J(F),
\]
\[
J(F) = \frac{1}{2} \| Y(T, \cdot, F) - Y_0 \|_{L^2}^2, \quad F \in \mathcal{U},
\]
where $Y_0 := (y_0^0, y_0^0_T)$ is a noisy data of $Y_T$ such that $\| Y_T - Y_0 \| \leq \delta$ for $\delta \geq 0$, and
\[
\mathcal{U} := \{ F = (F, G) \in H^1(0, T; L^2) : \| F \|_{H^1(0, T; L^2)} \leq R \}
\]
is the set of admissible source terms. Clearly, $\mathcal{U}$ is a bounded, closed and convex subset of $H^1(0, T; L^2)$, for any fixed $R > 0$. 

Due to the ill-posedness of Equation (4), we usually use a Tikhonov regularization approach and consider instead the following regularized functional:

\[
J_\epsilon(F) = \frac{1}{2} ||Y(T, \cdot, F) - Y_\Gamma^\delta||_{L_2}^2 + \frac{\epsilon}{2} ||F||_{L_2}^2, \quad F \in U,
\]

where \(\epsilon > 0\) is the regularizing parameter.

Next, we derive a gradient formula for \(J\) via the mild solution \(\Phi = (\varphi, \varphi_T)\) of an adjoint system.

**Proposition 2.** The cost functional \(J\) is Fréchet differentiable and its gradient at each \(F \in U\) is given by

\[
J'(F) = \Phi,
\]

where \(\Phi(t, \cdot, F) = (\varphi, \varphi_T)\) is the mild solution of the following adjoint system:

\[
\left\{
\begin{array}{ll}
-\partial_t \varphi - d\Delta \varphi + a(x)\varphi = 0, & \text{in } \Omega_T, \\
-\partial_t \varphi_T - \gamma \Delta_T \varphi_T + d\partial_x \varphi + b(x)\varphi_T = 0, & \text{on } \Gamma_T, \\
\varphi_T(t, x) = \varphi_T(t, x), & \text{on } \Gamma_T, \\
\varphi|_{t=\tau} = y(T, \cdot, F) - y^\delta_T, & \text{in } \Omega, \\
\varphi_T|_{t=\tau} = y_T(T, \cdot, F) - y^\delta_{T, \Gamma}, & \text{on } \Gamma.
\end{array}
\right.
\]

**Proof.** We assume that \(F, F + \delta F \in U\). Let us calculate the difference

\[
\delta J(F) := J(F + \delta F) - J(F).
\]

We have

\[
\delta J(F) = \frac{1}{2} ||Y(T, \cdot, F + \delta F) - Y_\Gamma^\delta||_{L_2}^2 - \frac{1}{2} ||Y(T, \cdot, F) - Y_\Gamma^\delta||_{L_2}^2,
\]

\[
\delta J(F) = \frac{1}{2} \left( ||y(T, \cdot, F + \delta F) - y^\delta_T||_{L_2(\Omega)}^2 + ||y_T(T, \cdot, F + \delta F) - y^\delta_{T, \Gamma}||_{L_2(\Gamma)}^2 \right)
\]

\[
- \frac{1}{2} \left( ||y(T, \cdot, F) - y^\delta_T||_{L_2(\Omega)}^2 + ||y_T(T, \cdot, F) - y^\delta_{T, \Gamma}||_{L_2(\Gamma)}^2 \right)
\]

\[
= \frac{1}{2} \int_\Omega \left[ (y(T, x, F + \delta F) - y^\delta_T(x))^2 - (y(T, x, F) - y^\delta_T(x))^2 \right] dx
\]

\[
+ \frac{1}{2} \int_\Gamma \left[ (y_T(T, x, F + \delta F) - y^\delta_{T, \Gamma}(x))^2 - (y_T(T, x, F) - y^\delta_{T, \Gamma}(x))^2 \right] dS.
\]

Using the identity

\[
\frac{1}{2} [(x - z)^2 - (y - z)^2] = (y - z)(x - y) + \frac{1}{2}(x - y)^2, \quad x, y \in \mathbb{R},
\]

in the last two terms, we obtain

\[
\delta J(F) = \int_\Omega (y(T, x, F) - y^\delta_T(x)) \delta y(T, x, F) dx + \frac{1}{2} \int_\Omega [\delta y(T, x, F)]^2 dx
\]

\[
+ \int_\Gamma (y_T(T, x, F) - y^\delta_{T, \Gamma}(x)) \delta y_T(T, x, F) dS + \frac{1}{2} \int_\Gamma [\delta y_T(T, x, F)]^2 dS,
\]

where

\[
\delta y(T, \cdot, F) = y(T, \cdot, F + \delta F) - y(T, \cdot, F),
\]

\[
\delta y_T(T, \cdot, F) = y_T(T, \cdot, F + \delta F) - y_T(T, \cdot, F).
\]
By linearity of the systems, $\delta Y = (\delta y, \delta y_r)$ is the mild solution of the following system:

\[
\begin{align*}
\begin{cases}
\partial_t(\delta y) - d\Delta(\delta y) + a(x)(\delta y) &= \delta F(t, x), & \text{in } \Omega_T, \\
\partial_t(\delta y_r) - \gamma \Delta_\Gamma(\delta y_r) + d\partial_r(\delta y) + b(x)(\delta y_r) &= \delta G(t, x), & \text{on } \Gamma_T, \\
\delta y_r(t, x) &= (\delta y_r)|_{t=0}, & \text{on } \Gamma_T, \\
\delta y, \delta y_r|_{t=0} &= (0, 0), & \Omega \times \Gamma.
\end{cases}
\end{align*}
\]  

(14)

We rewrite the first integral in the right-hand side of Equation (12) using $\Phi(t, \cdot, F)$ and $\delta Y(t, \cdot, F)$, the mild solutions of Equations (11) and (14), respectively. We have

\[
\int_{\Omega} (y(T, x, F) - y^0(x)) \delta y(T, x, F) \, dx = \int_{\Omega} \varphi(T, x, F) \delta y(T, x, F) \, dx
\]

\[
= \int_{\Omega} \left[ \int_{0}^{T} \partial_t(\varphi(t, x, F) \delta y(t, x, F)) \, dt \right] \, dx
\]

\[
= \int_{\Omega_T} \left[ (\partial_t \varphi) \delta y + \varphi \partial_t(\delta y) \right] \, dx \, dt
\]

\[
= \int_{\Omega_T} \left[ (-d\Delta \varphi + a(x) \varphi) \delta y + \varphi(d\Delta(\delta y) - a(x)(\delta y)) \right] \, dx \, dt
\]

\[
+ \int_{\Omega_T} \delta F(t, x) \varphi(t, x) \, dx \, dt
\]

\[
= \int_{\Omega_T} -d[(\Delta \varphi) \delta y - \Delta(\delta y) \varphi] \, dx \, dt + \int_{\Omega_T} \delta F(t, x) \varphi(t, x) \, dx \, dt
\]

\[
= \int_{\Gamma_T} -d[(\partial_r \varphi) \delta y_r - \varphi_r \partial_r(\delta y)] \, ds \, dt + \int_{\Gamma_T} \delta G(t, x) \varphi_r(t, x) \, ds \, dt
\]

Similarly, for the first integral in the right-hand side of Equation (13), we obtain

\[
\int_{\Gamma} (y_r(T, x, F) - y^0_r(x)) \delta y_r(T, x, F) \, dS
\]

\[
= \int_{\Gamma_T} -\gamma [(\Delta_\Gamma \varphi_r) \delta y_r - \Delta_\Gamma(\delta y_r) \varphi_r] \, ds \, dt
\]

\[
+ \int_{\Gamma_T} d[(\partial_r \varphi) \delta y_r - \varphi_r \partial_r(\delta y)] \, ds \, dt + \int_{\Gamma_T} \delta G(t, x) \varphi_r(t, x) \, ds \, dt
\]

\[
= \int_{\Gamma_T} d[(\partial_r \varphi) \delta y_r - \varphi_r \partial_r(\delta y)] \, ds \, dt + \int_{\Gamma_T} \delta G(t, x) \varphi_r(t, x) \, ds \, dt
\]

(16)

(17)

The first integral in the right-hand side of Equation (16) is null by the surface divergence formula (2). Adding up the two integrals (15) and (17), we obtain

\[
\int_{\Omega} (y(T, x, F) - y^0(x)) \delta y(T, x, F) \, dx + \int_{\Gamma} (y_r(T, x, F) - y^0_r(x)) \delta y_r(T, x, F) \, dS
\]

\[
= \int_{\Omega_T} \delta F(t, x) \varphi(t, x) \, dx \, dt + \int_{\Gamma_T} \delta G(t, x) \varphi_r(t, x) \, ds \, dt.
\]
For the second integrals in the right-hand side of Equations (12) and (13), the estimate (7) implies
\[
\frac{1}{2} \int_{\Omega} [\delta y(T, x, F)]^2 \, dx + \frac{1}{2} \int_{\Gamma} [\delta y_r(T, x, F)]^2 \, dS = O\left(\|\delta F\|_{L^2}^2\right).
\]

This completes the proof. \(\square\)

Next, we prove the Lipschitz continuity of the Fréchet gradient \(J'\), in particular, \(J \in C^1(U)\).

**Lemma 3.** Let \(F, \delta F \in U\). Then the Fréchet gradient \(J'\) is Lipschitz continuous,
\[
\|J'(F + \delta F) - J'(F)\|_{L^2_\Omega} \leq L \|\delta F\|_{L^2_\Omega},
\]
where the Lipschitz constant \(L > 0\) depends on \(T, \|a\|_\infty\) and \(\|b\|_\infty\) as follows
\[
L = \sqrt{2T\|1 + 4\max(\|a\|_\infty, \|b\|_\infty)\|T}.
\]

**Proof.** Let \(\delta \Phi(t, \cdot, F) := (\delta \varphi, \delta \varphi_T)\) be the strong solution of the adjoint system
\[
\begin{cases}
-\partial_t(\delta \varphi) - d\Delta(\delta \varphi) + a(x)(\delta \varphi) = 0, & \text{in } \Omega_T, \\
-\partial_t(\delta \varphi_T) - \gamma \Delta_T(\delta \varphi_T) + d\partial_\nu(\delta \varphi) + b(x)(\delta \varphi_T) = 0, & \text{on } \Gamma_T, \\
\delta \varphi_T(t, x) = (\delta \varphi)_T(t, x), & \text{on } \Gamma_T, \\
(\delta \varphi, \delta \varphi_T)|_{t=T} = (\delta y, \delta y_T)|_{t=T}, & \Omega \times \Gamma.
\end{cases}
\]

Using Proposition 2, we have
\[
\|J'(F + \delta F) - J'(F)\|_{L^2_\Omega}^2 = \int_{\Omega_T} |\delta \varphi|^2 \, dx \, dt + \int_{\Gamma_T} |\delta \varphi_T|^2 \, dS \, dt
\]
\[
\leq 2 \left(\|\delta \varphi\|_{L^2_\Omega(\Omega_T)}^2 + \|\delta \varphi_T\|_{L^2_\Gamma(\Gamma_T)}^2\right)
\]
\[
= 2\|\delta \Phi\|_{L^2_\Omega}^2.
\]

Next, we estimate the norm \(\|\delta \Phi\|_{L^2_\Omega}^2\). In the adjoint system (19), multiplying the first equation by \(\delta \varphi\) and the second by \(\delta \varphi_T\), we obtain the following identities
\[
-\frac{1}{2} \partial_t \left(\int_{\Omega} |\delta \varphi|^2 \, dx\right) + d \int_{\Omega} |\nabla(\delta \varphi)|^2 \, dx - \frac{1}{2} \int_{\Gamma} d\partial_\nu(\delta \varphi)^2 \, dS + \int_{\Omega} a|\delta \varphi|^2 \, dx = 0,
\]
\[
-\frac{1}{2} \partial_t \left(\int_{\Gamma} |\delta \varphi_T|^2 \, dS\right) + \gamma \int_{\Gamma} |\nabla_T(\delta \varphi_T)|^2 \, dS + \frac{1}{2} \int_{\Gamma} d\partial_\nu(\delta \varphi)^2 \, dS + \int_{\Gamma} b|\delta \varphi_T|^2 \, dS = 0,
\]
where we used integration by parts and the surface divergence formula (2). Adding up the last two equalities, we obtain
\[
\frac{1}{2} \partial_t \left(\int_{\Omega} |\delta \varphi|^2 \, dx + \int_{\Gamma} |\delta \varphi_T|^2 \, dS\right) - \int_{\Omega} a|\delta \varphi|^2 \, dx - \int_{\Gamma} b|\delta \varphi_T|^2 \, dS
\]
\[
d \int_{\Omega} |\nabla(\delta \varphi)|^2 \, dx + \gamma \int_{\Gamma} |\nabla_T(\delta \varphi_T)|^2 \, dS.
\]
Then,
\[
\frac{1}{2} \partial_t \left( \int_{\Omega} |\delta \varphi|^2 \, dx + \int_{\Gamma} |\delta \varphi_T|^2 \, dS \right) + \max(||a||_{\infty}, ||b||_{\infty}) \left( \int_{\Omega} |\delta \varphi|^2 \, dx + \int_{\Gamma} |\delta \varphi_T|^2 \, dS \right) \geq 0.
\]
This inequality implies that the function \( H \) defined by
\[
H(t) = e^{2 \max(||a||_{\infty}, ||b||_{\infty}) t} \left( \int_{\Omega} |\delta \varphi|^2 \, dx + \int_{\Gamma} |\delta \varphi_T|^2 \, dS \right)
\]
is nondecreasing on \([0, T]\). Using (19), it follows, for all \( t \in [0, T] \), that
\[
\|\delta \Phi\|_{L^2_t}^2 = \int_{\Omega} |\delta \varphi(t, x, F)|^2 \, dx \, dt + \int_{\Gamma} |\delta \varphi_T(t, x, F)|^2 \, dS \, dt
\]
\[
\leq T e^{2 \max(||a||_{\infty}, ||b||_{\infty}) T} \left( \int_{\Omega} |\delta \varphi(T, x, F)|^2 \, dx + \int_{\Gamma} |\delta \varphi_T(T, x, F)|^2 \, dS \right)
\]
\[
\leq T e^{2 \max(||a||_{\infty}, ||b||_{\infty}) T} \left( \int_{\Omega} |\delta y(T, x, F)|^2 \, dx + \int_{\Gamma} |\delta y_T(T, x, F)|^2 \, dS \right).
\]
Using this last inequality and inequality (7), we obtain
\[
2\|\delta \Phi\|_{L^2_t}^2 \leq 2T e^{2 \max(||a||_{\infty}, ||b||_{\infty}) T} e^{(1+2 \max(||a||_{\infty}, ||b||_{\infty})) T} \|\delta F\|_{L^2_t}^2
\]
\[
\leq 2T e^{(1+4 \max(||a||_{\infty}, ||b||_{\infty})) T} \|\delta F\|_{L^2_t}^2.
\]
With the estimate (20) this implies that
\[
\|J'(F + \delta F) - J'(F)\|_{L^2_t}^2 \leq 2T e^{(1+4 \max(||a||_{\infty}, ||b||_{\infty})) T} \|\delta F\|_{L^2_t}^2.
\]
This yields the desired result. \( \square \)

Next, we consider the Landweber method given by the following iteration
\[
F_{k+1} = F_k - a_k J'(F_k), \ k = 0, 1, 2, \ldots ,
\]
where \( F_0 \in U \) is a given initial iteration and \( a_k \) is a relaxation parameter defined by the minimization problem
\[
h_k(a) := \inf_{a \geq 0} h_k(a), \ h_k(a) := J(F_k - a J'(F_k)), \ k = 0, 1, 2, \ldots ,
\]
We refer to Engl et al\(^{27}\) for a detailed exposition of this method.

**Remark 1.** At this level, some remarks should be made:

- The Lipschitz continuity of the gradient \( J' \) implies that the sequence \((J(F_k))\) is decreasing, where \((F_k)\) is defined by Equation (21). This fact yields fast numerical results; see Hasanov Hasanoğlu and Romanov, \(^{24}\) Lemma 3.4.4 for more details.
- In some situations, the choice of the iteration parameter \( a_k > 0 \) is difficult. However, the Lipschitz continuity of \( J' \) allows us to estimate this parameter by the Lipschitz constant \( L \) via the estimate
\[
0 < \lambda_0 \leq a_k \leq \frac{2}{L + 2\lambda_1},
\]
for arbitrary parameters $\lambda_0, \lambda_1 > 0$; see Hasanov, 12, Section 3.4.3

- If $\alpha_k = \gamma = \text{const} > 0$ for all $k$, the optimal value of $\alpha$ (which corresponds to $\lambda_0 = \frac{1}{L}$ and $\lambda_1 = \frac{L}{2}$) is $\alpha_s = \frac{1}{L}$. Hence, if $L$ is large, the step parameter $\alpha_s$ will be small. This fact illustrates the importance of a sharp Lipschitz constant $L$.

The next lemma follows the same ideas in Corollary 4.1 and Theorem 4.1 in Hasanov 12 (one can also refer to Vasil’ev 28). We denote by $U_*$ the set of all quasi-solutions of Equations (8)–(9).

**Lemma 4.** Let $(F_k) \subset U^*$ be the sequence defined by Equation (21). If the iteration parameter $\alpha_k = \alpha_s$ for all $k$, then, the following assertions hold.

(i) The sequence $(J(F_k))$ is monotone decreasing and convergent. Moreover,

$$\lim_{k \to \infty} \|J'(F_k)\|_{L^2} = 0.$$

Moreover,

$$\|F_{k+1} - F_k\|_{L^2}^2 \leq \frac{2}{\lambda} [J(F_k) - J(F_{k+1})], \quad k = 1, 2, 3, \ldots,$$

where $L$ is the Lipschitz constant in Equation (18);

(ii) for any given initial iteration $F_1 \in U^*$, the sequence $(F_k)$ converges weakly in $L^2$ to a quasi-solution $F_* \in U_*$ of (ISP).

The rate of convergence of $(J(F_k))$ can be estimated as follows:

$$0 \leq J(F_k) - J_* \leq 2L \frac{\beta^2}{k}, \quad k = 1, 2, 3, \ldots,$$

where $J_* := \lim_{k \to \infty} J(F_k)$ and $\beta := \sup\{\|F_k - F_*\|_{L^2} : F_k \in U, F_* \in U_*\}$.

### 4 EXISTENCE AND UNIQUENESS OF THE SOLUTION TO (ISP)

In the following, we use some tools from the calculus of variations to study the existence and uniqueness of the solution to (ISP). First, we prove the following lemma.

**Lemma 5.** For the cost functional $J \in C^1(U^*)$, the following formula holds

$$\langle J'(F + \delta F) - J'(F), \delta F \rangle_{L^2} = 2\|\delta Y(T, \cdot, F)\|_{L^2}^2, \quad \forall F, \delta F \in U^*,$$

where $\delta Y(T, \cdot, F)$ is the solution of Equation (14).

**Proof.** Let $(\delta \varphi, \delta \varphi T)$ be the solution of Equation (19). By the gradient formula (10), we have

$$\langle J'(F + \delta F) - J'(F), \delta F \rangle_{L^2} = \int_{\Omega_T} \delta F \delta \varphi \, dx \, dt + \int_{\Gamma_T} \delta G \delta \varphi T \, dS \, dt$$

$$= \int_{\Omega_T} \partial_t (\delta \varphi) \delta \varphi \, dx \, dt + \int_{\Gamma_T} \partial_y (\delta \varphi T) \delta \varphi \, dS \, dt + \int_{\Omega_T} [-d \Delta (\delta \varphi) + a(x)(\delta \varphi) \] \delta \varphi \, dx \, dt$$

$$+ \int_{\Gamma_T} [-\gamma \Delta (\delta \varphi T) + d \delta \varphi T (\delta \varphi) + b(x)(\delta \varphi T) \] \delta \varphi T \, dS \, dt$$

$$= \|\delta y(T, \cdot)\|_{L^2(\Omega)}^2 + \|\delta y_T(T, \cdot)\|_{L^2(\Gamma)}^2 - \int_{\Omega_T} \partial_t (\delta \varphi) \delta y \, dx \, dt - \int_{\Gamma_T} \partial_y (\delta \varphi T) \delta y_T \, dS \, dt$$

$$+ \int_{\Omega_T} [-d \Delta (\delta \varphi) + a(x)(\delta \varphi) \] \delta y \, dx \, dt + \int_{\Gamma_T} [-\gamma \Delta (\delta \varphi T) + d \delta \varphi T (\delta \varphi) + b(x)(\delta \varphi T) \] \delta y_T \, dS \, dt$$

$$= \|\delta y(T, \cdot)\|_{L^2(\Omega)}^2 + \|\delta y_T(T, \cdot)\|_{L^2(\Gamma)}^2,$$
where we used integration by parts with respect to $t$, the Green formula in $\Omega$, and the surface divergence formula on $\Gamma$, together with the system (14).

The monotonicity of $J' : U \to L^2_{\gamma}L^2_{\gamma}$ in Lemma 5 implies the convexity of the functional $J$. As a direct consequence of Lemma 1 and Zeidler,29 Theorem 25.C we have the following existence result.

**Corollary 1.** The cost functional $J$ is continuous and convex on $U$. There exists then a minimizer $\hat{F} \in U$ such that

$$J(\hat{F}) = \min_{F \in U} J(F).$$

Since the strict convexity of $J$ is characterized by the strict monotonicity of $J'$, the equality (22) yields a sufficient condition for uniqueness.

**Lemma 6.** If the positivity condition

$$\int_\Omega (\delta_y(T,x,F))^2 \, dx + \int_\Gamma (\delta_y(T,x,F))^2 \, dS > 0, \quad \forall F \in U,$$

holds on a closed convex subset $V \subset U$, then the problem (ISP) admits at most one solution in $V$.

The above lemma is a consequence of the uniqueness theorem for strictly convex functionals defined on convex sets (see, e.g., Zeidler29, Corollary 25.15).

## 5 | NUMERICAL SIMULATION FOR 1-D INTERNAL HEAT SOURCE

In this section, we analyze the numerical reconstruction for an unknown source term that depends only on the space variable. More precisely, we consider the reconstruction of $f(x)$ in the following 1-D heat equation with dynamic boundary conditions

$$\begin{align*}
  y_t(t,x) - y_{xx}(t,x) &= f(x)r(t,x), \quad (t,x) \in (0,T) \times (0,\ell), \\
  y_t(t,0) - y_x(t,0) &= 0, \quad t \in (0,T), \\
  y_t(t,\ell) + y_x(t,\ell) &= 0, \quad t \in (0,T), \\
  y(0,x) &= y_0(x), \quad (y(0,0), y(0,\ell)) = (a,b), \quad x \in (0,\ell),
\end{align*}$$

(23)

where $T > 0$ is a final time, $\ell > 0$ and $(y_0, a, b) \in L^2(0,\ell) \times \mathbb{R}^2$ is an initial condition. The function $r \in C^1([0,T]; C([0,\ell]))$ is assumed to be known.

The 1-D heat equation with dynamic boundary conditions has attracted special attention as a model of heat conduction problems for a metal bar of length $\ell$. We mention Kumpf and Nickel,30 where the authors have studied the approximate controllability problem from the boundary.

We shall apply the Landweber iteration scheme discussed in a previous section to system (23). Let $Y(t,x,f) := (y(t,x), y(t,0), y(t,\ell))$ be the solution of system (23). The input–output operator $\Psi : L^2(0,\ell) \to L^2(0,\ell) \times \mathbb{R}^2$ is given by

$$(\Psi f)(x) := Y(T,x,f) = (y(T,x), y(T,0), y(T,\ell)), \quad x \in (0,\ell),$$

and the Tikhonov functional by

$$J_\varepsilon(f) = \frac{1}{2} \| Y(T,\cdot,f) - Y^\delta_T \|^2_{L^2(0,\ell) \times \mathbb{R}^2} + \frac{\varepsilon}{2} \| f \|^2_{L^2(0,\ell)}, \quad f \in L^2(0,\ell),$$

$$= \frac{1}{2} \left( \| y(T,\cdot) - y^\delta_T \|^2_{L^2(0,\ell)} + \| y(T,0) - y_0^\delta \|^2_{L^2(0,\ell)} + \| y(T,\ell) - y_\ell^\delta \|^2_{L^2(0,\ell)} + \varepsilon \| f \|^2_{L^2(0,\ell)} \right),$$
where \( Y_T^\delta := (y_T^\delta, y_0^\delta, y_T^\varepsilon) \in L^2(0, \ell) \times \mathbb{R}^2 \). The adjoint system corresponding to Equation (11) is given by

\[
\begin{aligned}
\varphi_t(t, x) + \varphi_x(t, x) &= 0, & (t, x) \in (0, T) \times (0, \ell), \\
\varphi(t, 0) + \varphi_x(t, 0) &= 0, & t \in (0, T), \\
\varphi(t, \ell) - \varphi_x(t, \ell) &= 0, & t \in (0, T), \\
\varphi(T, x) &= y(T, x) - y_T^\delta(x), & x \in (0, \ell), \\
(\varphi(T, 0), \varphi(T, \ell)) &= (y(T, 0) - y_0^\delta, y(T, \ell) - y_T^\varepsilon).
\end{aligned}
\]

(24)

Similarly to calculations in Section 3, the gradient of \( J_\varepsilon \) is given by

\[
J_\varepsilon'(f) = \int_0^T \varphi(t, x, f) \kappa(t, x) \, dt + \varepsilon f, \quad f \in L^2(0, \ell), \quad x \in (0, \ell).
\]

The gradient formula (25) for the Tikhonov functional \( J_\varepsilon \) allows us to implement classical versions of the Conjugate Gradient Algorithm with different conjugation coefficients. Here, we shall restrict ourselves to the following basic algorithm.

**Algorithm 1** Landweber iteration scheme

1. Set \( k = 0 \) and choose an initial source \( f_0 \)
2. Solve the direct problem (23) to obtain \( Y(t, x, f_k) \)
3. Knowing the computed \( Y(T, x, f_k) \) and the measured \( Y_T^\delta \), solve the adjoint problem (24) to obtain \( \varphi(t, x, f_k) \)
4. Compute the descent direction \( p_k = J_\varepsilon'(f_k) \) using problem (25)
5. Solve the direct problem (23) with source \( p_k \) to get the solution \( \Psi p_k \)
6. Compute the relaxation parameter \( \alpha_k = \frac{||Y_T^\delta||_{L^2(0, \ell) \times \mathbb{R}^2}}{||\Psi p_k||_{L^2(0, \ell) \times \mathbb{R}^2}} \) (see Hasanov Hasanoglu and Romanov, 2014, Lemma 3.4.1)
7. Find the next iteration \( f_{k+1} = f_k - \alpha_k p_k \)
8. Stop the iteration process if the stopping criterion \( J_\varepsilon(f_{k+1}) < \varepsilon_j \) holds. Otherwise, set \( k := k + 1 \) and go to Step 2

Since we deal with dynamic boundary conditions that contain time derivative, the numerical solutions for the direct problem (23) and the adjoint problem (24) will be obtained by using the method of lines. This method is implemented in our case with help of the Mathematica system via the function NDSolve for solving ordinary differential systems. The noisy terminal data will be generated from the exact output data as follows

\[
Y_T^\delta(x) = Y_T(x) + p \times ||Y_T||_{L^2(0, \ell) \times \mathbb{R}^2} \times \text{RandomReal[]},
\]

where \( p \) stands for the percentage of the noise level and the function RandomReal[] produces random real numbers.

In the subsequent numerical tests, we will choose, for simplicity, the following values for the known parameters

\[
T = 1, \quad \ell = 1, \quad r = 1, \quad y_0 = 0, \quad a = b = 0.
\]

**Example 1.** We consider the following basic source term \( f(x) = x(1 - x), \quad x \in (0, 1) \). First, we compute the numerical solution \( Y(t, \cdot, f) \) to generate the data \( Y(1, \cdot, f) \). Next, we plot the corresponding solution (Figures 1 and 2).

The initial iteration is chosen as \( f_0 = 0 \) with regularization parameter and stopping parameter \( \varepsilon = \varepsilon_j = 10^{-6} \). The algorithm stops at iterations \( k \in \{2, 2, 2\} \), for \( p \in \{1\%, 3\%, 5\%\} \), respectively.

**Remark 2.** The optimal regularization parameter \( \varepsilon^\text{opt} \) can be defined from the conditions \( \varepsilon < 1 \) and \( \frac{\varepsilon}{\varepsilon^\text{opt}} < 1 \) depending on the noise level \( \delta \). The optimal stopping parameter \( \varepsilon_j^\text{opt} \) can be defined by analyzing the behavior of the convergence error and the accuracy error defined respectively by

\[
E(k, f_k) := ||Y_T - Y_T^\delta||_{L^2(0, 1) \times \mathbb{R}^2},
\]

\[
E(k, f_k) := ||f - f_k||_{L^2(0, 1)}.
\]
Example 2. We take the exact source term as \( f(x) = \sin(\pi x), \ x \in (0, 1) \) (Figure 3).

The initial iteration is chosen as \( f_0 = 0 \) with regularization parameter and stopping parameter \( \epsilon = e_1 = 10^{-8} \). The algorithm stops at iterations \( k \in \{3, 4, 4\} \), for \( p \in \{1\%, 3\%, 5\%\} \), respectively.

Example 3. We take the exact source term as \( f(x) = \exp\left(-8\left(x - \frac{1}{2}\right)^2\right), \ x \in (0, 1) \) (Figure 4 & Table 1).

The initial iteration is chosen as \( f_0 = 0 \) with regularization parameter and stopping parameter \( \epsilon = e_1 = 10^{-8} \). The algorithm stops at iterations \( k \in \{5, 5, 35\} \) for \( p \in \{1\%, 3\%, 5\%\} \), respectively.
Remark 3. The above numerical experiments show that the Landweber scheme yields stable, accurate, and fast results for the reconstruction of unknown source terms in heat equation with dynamic boundary conditions. We clearly see that the recovery of the source $f(x)$ becomes more accurate as the noise level $p$ decreases. This approach can be adapted for the simultaneous recovery of internal and boundary source terms that depend on both time and space, but it might require a large number of iterations. A performance analysis for this case will be treated in a forthcoming paper.

6 | CONCLUSIONS

In this paper, we have considered an inverse problem for determining internal and boundary source terms from final time data in heat equation with dynamic boundary conditions. Adapting the weak solution approach, a minimization problem for the Tikhonov functional is analyzed, and a gradient formula of the functional is established via the solution of an appropriate adjoint system. Then, the Lipschitz continuity of the Fréchet gradient is proved. Using calculus of variations techniques, the existence and the uniqueness of a quasi-solution are proved. In particular, a sufficient condition for the uniqueness is presented. Finally, some numerical tests are provided for recovering an internal heat source in the 1-D case.

ACKNOWLEDGEMENT

We would like to thank the anonymous reviewers for their invaluable comments. There are no funders to report for this submission.

CONFLICT OF INTEREST

This work does not have any conflicts of interest.

ORCID

Salah-Eddine Chorfi https://orcid.org/0000-0002-7707-9445

REFERENCES

1. Boutaayamou I, Chorfi SE, Maniar L, Oukdach O. The cost of approximate controllability of heat equation with general dynamical boundary conditions. Portugal Math. 2021;78:65-99.
2. Favini A, Goldstein JA, Goldstein GR, Romanelli S. The heat equation with generalized Wentzell boundary condition. J Evol Equ. 2002;2:1-19.
3. Khoutaibi A, Maniar L. Null controllability for a heat equation with dynamic boundary conditions and drift terms. Evol Equ and Cont Theo. 2020;9:535-559.
4. Khoutaibi A, Maniar L, Mugnolo D, Rhandi A. Parabolic equations with dynamic boundary conditions and drift terms. To appear in Mathematische Nachrichten.
5. Maniar L, Meyries M, Schnaubelt R. Null controllability for parabolic equations with dynamic boundary conditions of reactive-diffusive type. *Evol Equ and Cont Theo*. 2017;6:381-407.
6. Miranville A, Zelik S. Exponential attractors for the Cahn-Hilliard equation with dynamic boundary conditions. *Math Meth Appl Sci*. 2005;28:709-735.
7. Vold RD, Vold MJ. *Colloid and Interface Chemistry*. Addison-Wesley, Reading Mass; 1983.
8. Langer RE. A problem in diffusion or in the flow of heat for a solid in contact with a fluid. *Tohoku Math J*. 1932;35:260-275.
9. Goldstein GR. Derivation and physical interpretation of general boundary conditions. *Adv Diff Equ*. 2006;11:457-480.
10. Bukhgeim AL, Klibanov MV. Global uniqueness of class of multidimensional inverse problems. *Soviet Math Dokl*. 1981;24:244-247.
11. Erdem A. A simultaneous approach to inverse source problem by Green's function. *Math Meth Appl Sci*. 2015;38:1393-1404.
12. Hasanov A. Simultaneous determination of source terms in a linear parabolic problem from the final overdetermination: Weak solution approach. *J Math Anal Appl*. 2007;330:766-779.
13. Isakov V. Inverse source problems. *Mathematical Surveys and Monographs*, Vol. 34. Providence, RI: Amer Math Soc; 1990.
14. Ait Ben Hassi EM, Chorfi SE, Maniar L, Oukdach O. Lipschitz stability for an inverse source problem in anisotropic parabolic equations with dynamic boundary conditions. https://doi.org/10.3934/eect.2020094; 2020.
15. Ait Ben Hassi EM, Chorfi SE, Maniar L. An inverse problem of radiative potentials and initial temperatures in parabolic equations with dynamic boundary conditions. https://doi.org/10.1515/jiip-2020-0067; 2021.
16. Klibanov MV. Carleman estimates for global uniqueness, stability and numerical methods for coefficient inverse problems. *J Inverse Ill-Posed Probl*. 2013;21:477-560.
17. Hasanov A. An inverse source problem with single Dirichlet type measured output data for a linear parabolic equations. *Appl Math Lett*. 2011;24:1269-1273.
18. Hasanov A, Otelbaev M, Akpayev B. Inverse heat conduction problems with boundary and final time measured output data. *Inverse Probl Sci Eng*. 2011;19:985-1006.
19. DuChateau P. Introduction to inverse problems in partial differential equations for engineers, Physicists and Mathematicians. *In Parameter Identification and Inverse Problems in Hydrology, Geology and Ecology*. Dordrecht: Springer; 1996:3-38.
20. DuChateau P, Thelwell R, Butters G. Analysis of an adjoint problem approach to the identification of an unknown diffusion coefficient. *Inverse Probl*. 2004;20:601-625.
21. Hasanov A. Identification of an unknown source term in a vibrating cantilevered beam from overdetermination. *Inverse Probl*. 2009;25:115015.
22. Gnanavel S, Barani Balan N, Balachandran K. Identification of source terms in the Lotka-Volterra system. *J Inverse Ill-Posed Probl*. 2012;20:287-312.
23. Hasanov A. Simultaneously identifying the thermal conductivity and radiative coefficient in heat equation from Dirichlet and Neumann boundary measured outputs. *J Inverse Ill-Posed Probl*. 2021;29:81-91.
24. Hasanov Hasanoglu A, Romanov VG. *Introduction to Inverse Problems for Differential Equations*. New York: Springer; 2017.
25. Slodkowska M. A parabolic inverse source problem with a dynamical boundary condition. *Appl Math Comput*. 2015;256:529-539.
26. Ismailov MI. Inverse source problem for heat equation with nonlocal Wentzell boundary condition. *Results Math*. 2018;73:68.
27. Engl HW, Hanke M, Neubauer A. *Regularization of Inverse Problems*. Kluwer Academic Publishers; 2000.
28. Vasil’ev FP. *Methods for Solving Extremal Problems*. Nauka: Moscow; 1981.
29. Zeidler E. *Nonlinear Functional Analysis and Its Applications, II/B Nonlinear Monotone Operators*. New York: Springer; 1990.
30. Kumpf M, Nickel G. Dynamic boundary conditions and boundary control for the one-dimensional heat equation. *J Dynam Control Syst*. 2004;10:213-225.
31. Wolfram S. *The Mathematica Book*. Cambridge, UK: Wolfram Media; 2005.

**How to cite this article:** Ait Ben Hassi EM, Chorfi S-E, Maniar L. Identification of source terms in heat equation with dynamic boundary conditions. *Math Meth Appl Sci*. 2022;45:2364-2379. doi:10.1002/mma.7933