A NON-ASYMPTOTIC ANALYSIS OF NETWORK INDEPENDENCE FOR DISTRIBUTED STOCHASTIC GRADIENT DESCENT∗
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Abstract. This paper is concerned with minimizing the average of n cost functions over a network, in which agents may communicate and exchange information with their peers in the network. Specifically, we consider the setting where only noisy gradient information is available. To solve the problem, we study the standard distributed stochastic gradient descent (DSGD) method and perform a non-asymptotic convergence analysis. For strongly convex and smooth objective functions, we not only show that DSGD asymptotically achieves the optimal network independent convergence rate compared to centralized stochastic gradient descent (SGD), but also explicitly identify the non-asymptotic convergence rate as a function of characteristics of the objective functions and the network. Furthermore, we derive the time needed for DSGD to approach the asymptotic convergence rate, which behaves as $K_T = \mathcal{O}(\frac{n}{(1 - \rho_w)^2})$, where $(1 - \rho_w)$ denotes the spectral gap of the mixing matrix of communicating agents.

Key words. distributed optimization, convex optimization, stochastic programming, stochastic gradient descent

AMS subject classifications. 90C15, 90C25, 68Q25

1. Introduction. In this paper, we consider the distributed optimization problem where a group of agents $N = \{1, 2, \ldots, n\}$ collaboratively look for $x \in \mathbb{R}^p$ that minimizes the average of n cost functions:

\begin{equation}
\min_{x \in \mathbb{R}^p} f(x) \left( = \frac{1}{n} \sum_{i=1}^{n} f_i(x) \right).
\end{equation}

Each local cost function $f_i : \mathbb{R}^p \rightarrow \mathbb{R}$ is known by agent $i$ only, and all the agents communicate and exchange information over a network. Problems in the form of (1.1) find applications in multi-agent target seeking [37, 9], distributed machine learning [17, 29, 14, 3, 51, 1, 5], and wireless networks [13, 24, 3], among other scenarios.

In order to solve (1.1), we assume each agent $i$ is able to obtain noisy gradient samples $g_i(x, \xi_i)$ satisfying the following assumption:

Assumption 1.1. For all $i \in N$ and all $x \in \mathbb{R}^p$, each random vector $\xi_i \in \mathbb{R}^m$ is independent, and

\begin{align}
\mathbb{E}_{\xi_i}[g_i(x, \xi_i) \mid x] &= \nabla f_i(x), \\
\mathbb{E}_{\xi_i}[\|g_i(x, \xi_i) - \nabla f_i(x)\|^2 \mid x] &\leq \sigma^2 \quad \text{for some } \sigma > 0.
\end{align}

This condition is satisfied for many distributed learning problems. For example, suppose $f_i(x) := \mathbb{E}_{\xi_i}[F_i(x, \xi_i)]$ represents the expected loss function for agent $i$, where $\xi_i$ are independent data samples gathered over time. Then for any $x$ and $\xi_i$, $g_i(x, \xi_i) := \nabla F_i(x, \xi_i)$ is an unbiased estimator of $\nabla f_i(x)$ satisfying Assumption 1.1. For another example, suppose the overall goal is to minimize an expected risk function $\mathbb{E}_{\xi} F(x, \xi)$,
and each agent has a single data sample $\zeta_i$. Then, the expected risk function can be approximated by $\frac{1}{n} \sum_{i=1}^{n} f_i(x)$, where $f_i(x) := F(x, \zeta_i)$. In this setting, the gradient estimation of $f_i(x)$ can incur noise from various sources such as approximation error and modeling and discretization errors.

Problem (1.1) has been studied extensively in the literature under various distributed algorithms [48, 30, 31, 23, 19, 20, 44, 15, 40, 28, 50, 39], among which the distributed gradient descent (DGD) method proposed in [30] has drawn the greatest attention. Recently, distributed implementation of stochastic gradient algorithms has received considerable interest [42, 46, 16, 4, 6, 47, 25, 7, 11, 12, 8, 27, 21, 22, 35, 36, 43, 45, 18, 38, 34, 49, 1]. Several works [10, 25, 11, 12, 22, 35, 26, 36, 2, 38, 34] have shown that distributed methods may compare with their centralized counterparts under certain conditions. For example, the paper [25] first proved that distributed stochastic approximation performs asymptotically as well as centralized schemes by means of a central limit theorem. In the constant stepsize regime, the work in [10, 11, 12] first showed that, with sufficiently small stepsize, a distributed stochastic gradient method achieves comparable performance to a centralized method in terms of the steady-state mean-square-error. A recent paper [34] discussed a distributed stochastic gradient method that asymptotically performs as well as the best bounds on centralized stochastic gradient descent (SGD) subject to possible message losses, delays, and asynchrony.

In this work, we perform a non-asymptotic analysis for the standard distributed stochastic gradient descent (DSGD) method adapted from DGD and the diffusion strategy [9]. In addition to showing that the algorithm asymptotically achieves the optimal convergence rate enjoyed by a centralized scheme, we precisely identify its non-asymptotic convergence rate as a function of characteristics of the objective functions and the network (e.g., spectral gap $(1-\rho_w)$ of the mixing matrix). Furthermore, we characterize the time needed for DSGD to achieve the optimal rate of convergence, demonstrated in the following corollary.

**Corollary 4.7.** It takes $K_T = O\left(\frac{n}{(1-\rho_w)^2}\right)$ time for DSGD to reach the asymptotic rate of convergence, i.e., when $k \geq K_T$, we have $\|\bar{x}(k) - x^*\|^2 \leq \frac{\sigma^2}{(2\theta-1)n\mu^2K}O(1)$.

Note that $\frac{\sigma^2}{(2\theta-1)n\mu}$ is the asymptotic convergence rate for SGD (see Theorem 4.6). Here $\rho_w$ denotes the spectral norm of $W - \frac{1}{n} 11^T$ with $W$ being the mixing matrix for all the agents. $\bar{x}(k)$ is the average solution at time $k$ and $x^*$ is the optimal solution. Stepsizes are set to be $\alpha_k = \frac{\theta}{\mu(k+K)}$ for some $\theta, K > 1$. These results are new to the best of our knowledge.

The rest of this paper is organized as follows. After introducing necessary notation in Section 1.1, we present the DSGD algorithm and some preliminary results in Section 2. In Section 3 we prove the sublinear convergence of the algorithm. Main convergence results and a comparison with centralized stochastic gradient method are demonstrated in Section 4. We conclude the paper in Section 5.

**1.1. Notation.** Vectors are column vectors unless otherwise specified. Each agent $i$ holds a local copy of the decision vector denoted by $x_i \in \mathbb{R}^p$, and its value at iteration/time $k$ is written as $x_i(k)$. Let

\[ x := [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^{n \times p}, \quad \bar{x} := \frac{1}{n} 1^T x \in \mathbb{R}^{1 \times p}, \]
where \( \mathbf{1} \) is the all one vector. Define an aggregate objective function

\[
F(\mathbf{x}) := \sum_{i=1}^{n} f_i(x_i),
\]

and let

\[
\nabla F(\mathbf{x}) := [\nabla f_1(x_1), \nabla f_2(x_2), \ldots, \nabla f_n(x_n)]^\top \in \mathbb{R}^{n \times p},
\]

\[
\bar{\nabla} F(\mathbf{x}) := \frac{1}{n} \mathbf{1}^\top \nabla F(\mathbf{x}).
\]

In addition, we denote

\[
\xi := [\xi_1, \xi_2, \ldots, \xi_n]^\top \in \mathbb{R}^{n \times p},
\]

\[
g(\mathbf{x}, \xi) := [g_1(x_1, \xi_1), g_2(x_2, \xi_2), \ldots, g_n(x_n, \xi_n)]^\top \in \mathbb{R}^{n \times p}.
\]

In what follows we write \( g_i(k) := g_i(x_i(k), \xi_i(k)) \) and \( g(k) := g(\mathbf{x}(k), \xi(k)) \) for short.

The inner product of two vectors \( a, b \) is written as \( \langle a, b \rangle \). For two matrices \( A, B \in \mathbb{R}^{n \times p} \), let \( \langle A, B \rangle := \sum_{i=1}^{n} \langle A_i, B_i \rangle \), where \( A_i \) (respectively, \( B_i \)) is the \( i \)-th row of \( A \) (respectively, \( B \)). We use \( \| \cdot \| \) to denote the 2-norm of vectors and the Frobenius norm of matrices.

A graph \( G = (\mathcal{N}, \mathcal{E}) \) has a set of vertices (nodes) \( \mathcal{N} = \{1, 2, \ldots, n\} \) and a set of edges connecting vertices \( \mathcal{E} \subseteq \mathcal{N} \times \mathcal{N} \). Consider agents interact in an undirected graph, i.e., \((i, j) \in \mathcal{E} \) if and only if \((j, i) \in \mathcal{E} \).

Denote the mixing matrix of agents by \( W = [w_{ij}] \in \mathbb{R}^{n \times n} \). Two agents \( i \) and \( j \) are connected if and only if \( w_{ij}, w_{ji} > 0 \) (\( w_{ij} = w_{ji} = 0 \) otherwise). Formally, we assume the following condition on the communication among agents:

**Assumption 1.2.** The graph \( G \) is undirected and connected (there exists a path between any two agents). The mixing matrix \( W \) is nonnegative and doubly stochastic, i.e., \( W \mathbf{1} = \mathbf{1} \) and \( \mathbf{1}^\top W = \mathbf{1}^\top \).

From Assumption 1.2, we have the following contraction property of \( W \) (see [40]):

**Lemma 1.3.** Let Assumption 1.2 hold, and let \( \rho_W \) denote the spectral norm of the matrix \( W - \frac{1}{n} \mathbf{1} \mathbf{1}^\top \). Then, \( \rho_W < 1 \) and

\[
\|W\omega - \overline{\omega}\| \leq \rho_W \|\omega - \overline{\omega}\|
\]

for all \( \omega \in \mathbb{R}^{n \times p} \), where \( \overline{\omega} := \frac{1}{n} \mathbf{1}^\top \omega \).

**2. Distributed Stochastic Gradient Descent.** We consider the following standard DSGD method adapted from DGD and the diffusion strategy [9]: at each step \( k \geq 0 \), every agent \( i \) independently performs the update:

\[
x_i(k + 1) = \sum_{j=1}^{n} w_{ij} (x_j(k) - \alpha_k g_j(k)),
\]

where \( \{\alpha_k\} \) is a sequence of non-increasing stepsizes. The initial vectors \( x_{i,0} \) are arbitrary for all \( i \in \mathcal{N} \). We can rewrite (2.1) in the following compact form:

\[
x_{k+1} = W(\mathbf{x}(k) - \alpha_k \mathbf{g}(k)).
\]
Throughout the paper, we make the following standing assumption regarding the objective functions $f_i$.

**Assumption 2.1.** Each $f_i : \mathbb{R}^p \rightarrow \mathbb{R}$ is $\mu$-strongly convex with $L$-Lipschitz continuous gradients, i.e., for any $x, x' \in \mathbb{R}^p$,

\begin{equation}
\langle \nabla f_i(x) - \nabla f_i(x'), x - x' \rangle \geq \mu \|x - x'\|^2, \\
\|\nabla f_i(x) - \nabla f_i(x')\| \leq L\|x - x'\|.
\end{equation}

Under Assumption 2.1, Problem (1.1) has a unique optimal solution $x^*$, and the following result holds (See [40] Lemma 10).

**Lemma 2.2.** For any $x \in \mathbb{R}^p$ and $\alpha \in (0, 2/L)$, we have

\[ \|x - \alpha \nabla f(x) - x^*\| \leq \lambda \|x - x^*\|, \]

where $\lambda = \max(|1 - \alpha \mu|, |1 - \alpha L|)$.

Denote $\overline{g}(k) := \frac{1}{n} 1^T g(k)$. The following two lemma will be useful for our analysis later.

**Lemma 2.3.** Under Assumption 1.1, for all $k \geq 0$,

\begin{equation}
\mathbb{E} \left[ \|\overline{g}(k) - \nabla F(x(k))\|^2 \right] \leq \frac{\sigma^2}{n}.
\end{equation}

**Proof.** By definitions of $\overline{g}(k)$, $\nabla F(x(k))$ and Assumption 1.1, we have

\[ \mathbb{E} \left[ \|\overline{g}(k) - \nabla F(x(k))\|^2 \right] = \mathbb{E} \left[ \left\| \frac{1}{n} 1^T g(k) - \frac{1}{n} 1^T \nabla F(x(k)) \right\|^2 \right] \]

\[ = \frac{1}{n^2} \sum_{i=1}^{n} \mathbb{E} \left[ \|g_i(k) - \nabla f_i(x_i(k))\|^2 \right] \leq \frac{\sigma^2}{n}. \]

**Lemma 2.4.** Under Assumption 2.1, for all $k \geq 0$,

\begin{equation}
\|\nabla f(\overline{x}(k)) - \nabla F(x(k))\| \leq \frac{L}{\sqrt{n}} \|x(k) - 1\overline{x}(k)\|.
\end{equation}

**Proof.** By definition,

\[ \|\nabla f(\overline{x}(k)) - \nabla F(x(k))\| = \left\| \nabla f(\overline{x}(k)) - \frac{1}{n} 1^T \nabla F(x(k)) \right\| \]

\[ = \left\| \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\overline{x}(k)) - \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(x_i(k)) \right\| \]

\[ \leq \frac{L}{\sqrt{n}} \sum_{i=1}^{n} \|\overline{x}(k) - x_i(k)\| \leq \frac{L}{\sqrt{n}} \|x(k) - 1\overline{x}(k)\|, \]

where the last relation follows from the Cauchy-Schwarz inequality.

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1The assumption can be generalized to the case where the agents have different $\mu$ and $L$. 

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2.1. Preliminary Results. In this section, we present some preliminary results concerning $\mathbb{E}[\|x(k) - x^*\|^2]$ (expected optimization error) and $\mathbb{E}[\|x(k) - 1\tau(k)\|^2]$ (expected consensus error). Specifically, we bound the two terms by linear combinations of their values in the last iteration. Throughout the analysis we assume Assumptions 1.1, 1.2 and 2.1 hold.

**Lemma 2.5.** Under Algorithm (2.2), for all $k \geq 0$, we have

\begin{equation}
\mathbb{E}[\|x(k+1) - x^*\|^2 | x(k)] \leq \|x(k) - \alpha_k \nabla f(x(k)) - x^*\|^2 + \frac{2\alpha_k L}{\sqrt{n}} \|x(k) - \alpha_k \nabla f(x(k)) - x^*\| \|x(k) - 1\tau(k)\| + \frac{\alpha_k^2 L^2}{n} \|x(k) - 1\tau(k)\|^2 + \frac{\alpha_k^2 \sigma^2}{n}.
\end{equation}

**Proof.** See Appendix A.1.

The next result is a corollary of Lemma 2.5.

**Lemma 2.6.** Under Algorithm (2.2), supposing $\alpha_k \leq \min\{\frac{1}{L}, \frac{1}{3\mu}\}$, then

\begin{equation}
\mathbb{E}[\|x(k+1) - x^*\|^2] \leq \left(1 - \frac{3}{2} \alpha_k \mu\right) \mathbb{E}[\|x(k) - x^*\|^2] + \frac{3\alpha_k L^2}{n \mu} \mathbb{E}[\|x(k) - 1\tau(k)\|^2] + \frac{\alpha_k^2 \sigma^2}{n}.
\end{equation}

**Proof.** See Appendix A.2.

Concerning the expected consensus error $\mathbb{E}[\|x(k) - 1\tau(k)\|^2]$, we have the following lemma.

**Lemma 2.7.** Under Algorithm (2.2), for all $k \geq 0$,

\begin{equation}
\mathbb{E}[\|x(k+1) - 1\tau(k+1)\|^2] \leq \left(\frac{1 + \rho_w^2}{2} + 2\alpha_k \rho_w^2 \frac{\rho_w^2}{L} + 2\alpha_k \rho_w^2 \frac{\rho_w^2}{L^2}\right) \mathbb{E}[\|x(k) - 1\tau(k)\|^2] + \rho_w^2 \left[\frac{2\alpha_k^2 L}{1 - \rho_w^2} \mathbb{E}[\|x(k) - x^*\|^2] + \frac{\alpha_k^2 \|\nabla F(1x^*)\|^2}{(1 - \rho_w^2)} + \alpha_k^2 \sigma^2\right].
\end{equation}

**Proof.** See Appendix A.3.

3. Analysis. We are now ready to derive some preliminary convergence results for Algorithm (2.2). First, we provide a uniform bound on the iterates generated by Algorithm (2.2) (in expectation) for all $k \geq 0$. Then based on the lemma established in Section 2.1, we prove the sublinear convergence rates $\mathbb{E}[\|x(k) - x^*\|^2] = O\left(\frac{1}{k}\right)$ and $\mathbb{E}[\|x(k) - 1\tau(k)\|^2] = O\left(\frac{1}{k^2}\right)$.

From now on we consider the following stepsize policy:

\begin{equation}
\alpha_k := \frac{\theta}{\mu(k + K)}, \quad \forall k,
\end{equation}

where $\theta > 1$ and

\begin{equation}
K := \left\lceil \frac{2\theta L^2}{\mu^2} \right\rceil.
\end{equation}

\[\text{\footnotesize{\textsuperscript{2}}\lceil \cdot \rceil \text{ denotes the ceiling function.}}\]
3.1. Uniform Bound. We derive a uniform bound on the iterates generated by Algorithm (2.2) (in expectation) for all \( k \geq 0 \).

**Lemma 3.1.** For all \( k \geq 0 \), we have

\[
\mathbb{E}[\|x(k)\|^2] \leq \max \left\{ \|x(0)\|^2, \sum_{i=1}^{n} R_i \right\},
\]

where

\[
R_i := \max_{q \in \mathcal{X}_i} \left\{ \left( 1 - \frac{\mu^2}{2L^2} \right) q + \frac{\mu}{L} \|\nabla f_i(0)\| \sqrt{q} + \frac{\mu^2}{4L^4} \left( 2\|\nabla f_i(0)\|^2 + \sigma^2 \right) \right\},
\]

and the sets \( \mathcal{X}_i \) are defined in (B.2).

**Proof.** See Appendix B.1.

We can further bound \( R_i \) as follows. From the definition of \( \mathcal{X}_i \),

\[
\max_{q \in \mathcal{X}_i} q \leq \frac{8\|\nabla f_i(0)\|^2}{\mu^2} + \frac{3\sigma^2}{4L^2}.
\]

Hence

\[
R_i = \max_{q \in \mathcal{X}_i} \left\{ q - \frac{\mu}{2L^2} \left[ \mu q - 2\|\nabla f_i(0)\| \sqrt{q} - \frac{\mu^2}{2L^2} \left( 2\|\nabla f_i(0)\|^2 + \sigma^2 \right) \right] \right\}
\leq \max_{q \in \mathcal{X}_i} q - \frac{\mu}{2L^2} \min_{q \in \mathcal{X}_i} \left\{ \mu q - 2\|\nabla f_i(0)\| \sqrt{q} - \frac{\mu^2}{2L^2} \left( 2\|\nabla f_i(0)\|^2 + \sigma^2 \right) \right\}
\leq \frac{8\|\nabla f_i(0)\|^2}{\mu^2} + \frac{3\sigma^2}{4L^2} + \frac{\mu}{2L^2} \left[ \frac{\|\nabla f_i(0)\|^2}{\mu} + \frac{\mu^2}{2L^2} \left( 2\|\nabla f_i(0)\|^2 + \sigma^2 \right) \right]
\leq \frac{9\|\nabla f_i(0)\|^2}{\mu^2} + \frac{\sigma^2}{L^2}.
\]

In light of Lemma 3.1 and inequality (3.5), further noticing that the choice of 0 is arbitrary in the proof of Lemma 3.1, we obtain the following uniform bound for \( \mathbb{E}[\|x(k) - 1x^*\|^2] \).

**Lemma 3.2.** Under Algorithm (2.2), for all \( k \geq 0 \), we have

\[
\mathbb{E}[\|x(k) - 1x^*\|^2] \leq \hat{X} := \max \left\{ \|x(0) - 1x^*\|^2, \frac{9\sum_{i=1}^{n} \|\nabla f_i(x^*)\|^2}{\mu^2} + \frac{na^2}{L^2} \right\},
\]

3.2. Sublinear Rate. Denote

\[
U(k) := \mathbb{E} \left[ \|\mathbf{y}(k) - x^*\|^2 \right], \quad V(k) := \mathbb{E}[\|x(k) - 1\mathbf{y}(k)\|^2], \quad \forall k.
\]

Using Lemma 2.6 and Lemma 2.7 from Section 2.1, we show below that Algorithm (2.2) enjoys the sublinear convergence rate, i.e., \( U(k) = O\left( \frac{1}{k} \right) \) and \( V(k) = O\left( \frac{1}{k^2} \right) \).

Define a Lyapunov function:

\[
W(k) := U(k) + \omega(k)V(k), \quad \forall k,
\]

where \( \omega(k) > 0 \) is to be determined later.

For the ease of analysis, we define \( \tilde{U}(k) := U(k - K), \quad \tilde{V}(k) := V(k - K), \quad \tilde{W}(k) := W(k - K) \) for all \( k \geq K_1 + K \). In addition, we denote

\[
\tilde{k} := k + K.
\]
Lemma 3.3. Let

\[ K_1 := \left\lceil \frac{24L^2\theta}{(1-\rho_w^2)\mu^2} \right\rceil, \]

and

\[ \omega(k) := \frac{12\alpha_kL^2}{n\mu(1-\rho_w^2)}. \]

Under Algorithm (2.2), for all \( k \geq K_1 \), we have

\[ U(k) \leq \frac{\tilde{W}}{k}, \]

where

\[ \tilde{W} := \frac{K_1\tilde{X}}{n} + \frac{3}{(4\theta - 3)} \left( \frac{\sigma^2\sigma^2}{n\mu^2} + \frac{\sigma^2\rho^2_\omega\theta^2}{2\mu^2} \right) + \frac{6\|\nabla F(1x^*)\|^2}{(4\theta - 3)n\mu^2(1-\rho_w^2)} \rho^2_\omega \theta^2. \]

In addition,

\[ V(k) \leq \frac{\tilde{W} e^{-K_1} \tilde{X}}{k^2} + \frac{V_1}{k^2} + \frac{V_2}{k^3}, \]

where

\[ p_0 := \frac{3 + \rho^2_\omega}{4}, \]

and

\[ V_1 := \frac{8\theta^2\rho^2_\omega}{\mu^2(1-\rho_w^2)} \left\lceil \frac{4\|\nabla F(1x^*)\|^2}{(1-\rho_w^2)} + n\sigma^2 \right\rceil, \]

\[ V_2 := \frac{32\theta^2nL^2\rho^2_\omega}{\mu^2(1-\rho_w^2)^2} \tilde{W}. \]

**Proof.** See Appendix B.2.

Notice that the sublinear rates obtained in Lemma 3.3 are network dependent since \( \tilde{W} \) depends on the spectral gap \( (1-\rho_w) \), a function of the mixing matrix \( W \).

4. Main Results. In this section, we perform a non-asymptotic analysis of network independence for Algorithm (2.2). Specifically, in Theorem 4.2 and Corollary 4.4, we show that \( U(k) = \frac{\sigma^2\sigma^2}{(1-\rho_w)n\mu^2} + \mathcal{O}(\frac{1}{(1-\rho_w)^{1/2}}) \), where the first term is network independent and the second (higher-order) term depends on \( (1-\rho_w) \). In Theorem 4.5, we further improve the result and compare it with centralized stochastic gradient descent. We show that asymptotically, the two methods have the same convergence rate \( \frac{\sigma^2\sigma^2}{(2\theta-1)n\mu^2k} \). In addition, it takes \( K_T = \mathcal{O} \left( \frac{a}{(1-\rho_w)^2} \right) \) time for Algorithm (2.2) to reach this asymptotic rate of convergence.

We start with a useful lemma.

Lemma 4.1. For any \( a < k \) (\( a \in \mathbb{N} \)) and \( \gamma > 1 \),

\[ \prod_{t=a}^{k-1} \left( 1 - \frac{\gamma}{t} \right) \leq \frac{a^\gamma}{k^\gamma}. \]

**Proof.** See Appendix C.1.
The following Theorem demonstrates the asymptotic network independence property of Algorithm (2.2).

**Theorem 4.2.** Under Algorithm (2.2), suppose \( \theta > 2 \). We have for all \( k \geq K + K_1 \),

\[
U(k) \leq \frac{\theta^2 \sigma^2}{(1.5\theta - 1)n^2_k} + \left[ \frac{3\theta^2(1.5\theta - 3)\sigma^2}{(1.5\theta - 2)n^2_k} + \frac{6\theta L^2 V_1}{(1.5\theta - 2)n^2_k} \right] \frac{1}{k^2} + \frac{6\theta L^2 V_2}{(1.5\theta - 3)n^2_k} \frac{1}{k^3} + \left( \frac{K_{1.5\theta}^1 \tilde{X}}{\theta n} + \frac{6\theta L^2 K_{1.5\theta}^{1.5\theta - 1} \tilde{X}}{n^2(1 - p_0)} \right) \frac{1}{k^{1.5\theta}}.
\]

**Proof.** For \( k \geq K_1 \), in light of Lemma 2.6,

\[
U(k + 1) \leq \left( 1 - \frac{3}{2\alpha_k \mu} \right) U(k) + \frac{3\alpha_k L^2}{n \mu} V(k) + \frac{\sigma^2}{n} U(k).
\]

Recalling the definitions of \( \tilde{U}(k) \) and \( \tilde{V}(k) \),

\[
\tilde{U}(k + 1) \leq \left( 1 - \frac{3\theta}{2k} \right) \tilde{U}(k) + \frac{3\theta L^2}{n \mu^2} \tilde{V}(k) + \frac{\sigma^2}{n \mu^2} \frac{1}{k^2}.
\]

Therefore,

\[
\tilde{U}(k) \leq \prod_{t=K_1}^{k-1} \left( 1 - \frac{3\theta}{2t} \right) \tilde{U}(K_1) + \sum_{t=K_1}^{k-1} \left( \prod_{j=t+1}^{k-1} \left( 1 - \frac{3\theta}{2j} \right) \right) \left( \frac{\sigma^2}{n \mu^2} \frac{1}{t^2} + \frac{3\theta L^2}{n \mu^2} \tilde{V}(t) \right).
\]

From Lemma 4.1,

\[
\tilde{U}(k) \leq \frac{K_{1.5\theta}^1}{k^{1.5\theta}} \tilde{U}(K_1) + \sum_{t=K_1}^{k-1} \frac{(t + 1)^{1.5\theta}}{k^{1.5\theta}} \left( \frac{\sigma^2}{n \mu^2} \frac{1}{t^2} + \frac{3\theta L^2}{n \mu^2} \tilde{V}(t) \right)
\]

\[
= \frac{\sigma^2}{k^{1.5\theta}} \frac{1}{n \mu^2} \sum_{t=K_1}^{k-1} \frac{(t + 1)^{1.5\theta}}{t^2} \tilde{U}(K_1) + \sum_{t=K_1}^{k-1} \frac{(t + 1)^{1.5\theta}}{k^{1.5\theta}} \frac{3\theta L^2}{n \mu^2} \tilde{V}(t).
\]

In light of Lemma 3.3, when \( k \geq K + K_1 \),

\[
\tilde{V}(k) \leq p_0^{k-K_1} \tilde{X} + \frac{V_1}{k^2} + \frac{V_2}{k^3}.
\]

Hence,

\[
\tilde{U}(k) - \frac{\sigma^2}{k^{1.5\theta}} \frac{1}{n \mu^2} \sum_{t=K_1}^{k-1} \frac{(t + 1)^{1.5\theta}}{t^2} \tilde{U}(K_1) \leq \sum_{t=K_1}^{k-1} \frac{(t + 1)^{1.5\theta}}{k^{1.5\theta}} \frac{3\theta L^2}{n \mu^2} \frac{1}{t} \tilde{U}(K_1) + \frac{V_1}{k^2} + \frac{V_2}{k^3} + \tilde{X} \sum_{t=K_1}^{k-1} \frac{(t + 1)^{1.5\theta}}{t} p_0^{K-K_1}
\]

The condition \( \theta > 2 \) can be easily relaxed to the case where \( \theta > 1 \).
However, we have for any $b > a \geq K_1$, 
\[
\sum_{a}^{b} \frac{(t + 1)^{1.5\theta}}{t^2} \leq \sum_{a}^{b-2} \left[ \frac{(t + 1)^{1.5\theta}}{(t + 1)^2} + 3 \frac{(t + 1)^{1.5\theta}}{(t + 1)^3} \right] + \frac{b^{1.5\theta}}{(b - 1)^2} + \frac{(b + 1)^{1.5\theta}}{b^2} 
\leq \int_{a}^{b} \left( t^{1.5\theta - 2} + 3t^{1.5\theta - 3} \right) dt + \frac{2(b + 1)^{1.5\theta}}{b^2} \leq \frac{b^{1.5\theta - 1}}{1.5\theta - 1} + \frac{3b^{1.5\theta - 2}}{1.5\theta - 2} + 3b^{1.5\theta - 2},
\]
\[
\sum_{a}^{b} \frac{(t + 1)^{1.5\theta}}{t^3} \leq \int_{a}^{b} t^{1.5\theta - 3} dt \leq \frac{2b^{1.5\theta - 2}}{1.5\theta - 2} \quad \text{and} \quad \sum_{a}^{b} \frac{(t + 1)^{1.5\theta}}{t^4} \leq \frac{2b^{1.5\theta - 3}}{1.5\theta - 3}.
\]
and 
\[
\sum_{t=K_1}^{k-1} \frac{(t + 1)^{1.5\theta} p_{t-K_1}}{t} \leq 2 \int_{t=K_1}^{\infty} t^{1.5\theta - 1} p_{t-K_1} dt \leq \frac{2}{\ln p_0} \int_{t=K_1}^{\infty} \frac{(t^{1.5\theta - 1} p_{t-K_1})}{dt} \leq \frac{2K_1^{1.5\theta - 1}}{1 - p_0}.
\]

We have 
\[
(4.2) \quad \hat{U}(k) \leq \frac{\theta^2 \sigma^2}{(1.5\theta - 1)n\mu^2 k} + \frac{3\theta^2(1.5\theta - 1)\sigma^2}{(1.5\theta - 2)n\mu^2 k^2} + \frac{K_1^{1.5\theta}}{K^1.5\theta} \hat{U}(K_1) 
\]
\[
+ \frac{6\theta^2 V_1}{(1.5\theta - 2)n\mu^2 k^2} + \frac{6\theta^2 V_2}{(1.5\theta - 3)n\mu^2 k^3} + \frac{3\theta L^2 \hat{x}}{n\mu^2} \frac{2K_1^{1.5\theta - 1}}{1 - p_0} \frac{1}{k^{1.5\theta}}.
\]

Recalling Lemma B.1 and the definition of $\hat{U}(k)$ yields the desired result. □

Next, we estimate the constants appearing in Theorem 4.2 and derive their dependency on the network size $n$ and the spectral gap $(1 - \rho_w)$.

**Lemma 4.3.** Suppose $\|x(0) - x^*\|^2 = O(n), \|\nabla F(1x^*)\|^2 = O(n)$. Then,
\[
\hat{x} = O(n), \quad \hat{w} = O \left( \frac{1}{1 - \rho_w} \right), \quad V_1 = O \left( \frac{n}{(1 - \rho_w)^2} \right), \quad V_2 = O \left( \frac{n}{(1 - \rho_w)^3} \right).
\]

**Proof.** See Appendix C.2. □

In light of Lemma 3.2, Lemma 3.3 and Theorem 4.2, we have the following corollary.

**Corollary 4.4.** Under Algorithm (2.2) with $\theta > 2$, when $k \geq K_1$, 
\[
U(k) \leq \frac{\theta^2 \sigma^2}{(1.5\theta - 1)n\mu^2 k} + \frac{c_1}{k^2}, \quad \hat{v}(k) \leq \frac{c_2}{k^2},
\]
where 
\[
c_1 = O \left( \frac{1}{(1 - \rho_w)^2} \right), \quad c_2 = O \left( \frac{n}{(1 - \rho_w)^3} \right).
\]

**Proof.** See Appendix C.3. □

We improve the result of Theorem 4.2 and Corollary 4.4 with further analysis.

**Theorem 4.5.** Under Algorithm (2.2) with $\theta > 2$, when $k \geq K_1$,
\[
(4.3) \quad U(k) \leq \frac{\theta^2 \sigma^2}{(2\theta - 1)n\mu^2 k} + O \left( \frac{1}{\sqrt{n}(1 - \rho_w)} \right) \frac{1}{k^{1.5}} + O \left( \frac{1}{(1 - \rho_w)^2} \right) \frac{1}{k^2}.
\]
Proof. For $k \geq K_1$, in light of Lemma 2.2 and Lemma 2.5,

\[
U(k + 1) \\
\leq (1 - \alpha_k \mu)^2 U(k) + \frac{2\alpha_k L}{\sqrt{n}} \mathbb{E}[\|\mathbf{z}(k) - \mathbf{x}^*\| \|\mathbf{x}(k) - \mathbf{1}\mathbf{z}(k)\|] + \frac{\alpha_k^2 L^2}{n} V(k) + \frac{\alpha_k^2 \sigma^2}{n}.
\]

where the second inequality follows from the Cauchy-Schwarz inequality.

Recalling the definitions of $\tilde{U}(k)$ and $\tilde{V}(k)$, when $k \geq K + K_1$,

\[
\tilde{U}(k + 1) \leq \left(1 - \frac{2\theta}{k}\right) \tilde{U}(k) + \frac{\theta^2 \tilde{U}(k)}{k^2} + \frac{2\theta L \sqrt{\tilde{U}(k) \tilde{V}(k)}}{\sqrt{n} \mu} + \frac{\theta^2 L^2 \tilde{V}(k)}{n \mu^2} + \frac{\theta^2 \sigma^2}{n}.
\]

Therefore,

\[
\tilde{U}(k) \leq \left(\prod_{t=K_1}^{k-1} \left(1 - \frac{2\theta}{t}\right)\right) \tilde{U}(K_1) \\
+ \sum_{t=K_1}^{k-1} \left(\prod_{i=t+1}^{k-1} \left(1 - \frac{2\theta}{i}\right)\right) \left(\frac{\theta^2 \sigma^2}{n \mu^2 t^2} + \frac{\theta^2 \tilde{U}(t)}{t^2} + \frac{2\theta L \sqrt{\tilde{U}(t) \tilde{V}(t)}}{\sqrt{n} \mu} + \frac{\theta^2 L^2 \tilde{V}(t)}{n \mu^2 t^2}\right).
\]

From Lemma 4.1,

\[
(4.4) \quad \tilde{U}(k) \leq \frac{K_1^{2\theta}}{k^{2\theta}} \tilde{U}(K_1) \\
+ \sum_{t=K_1}^{k-1} \frac{(t + 1)^{2\theta}}{k^{2\theta}} \left(\frac{\theta^2 \sigma^2}{n \mu^2 t^2} + \frac{\theta^2 \tilde{U}(t)}{t^2} + \frac{2\theta L \sqrt{\tilde{U}(t) \tilde{V}(t)}}{\sqrt{n} \mu} + \frac{\theta^2 L^2 \tilde{V}(t)}{n \mu^2 t^2}\right)
\]

\[
= \frac{1}{k^{2\theta}} \frac{\theta^2 \sigma^2}{n \mu^2} \sum_{t=K_1}^{k-1} \frac{(t + 1)^{2\theta} t^{2\theta}}{k^{2\theta}} + \frac{K_1^{2\theta}}{k^{2\theta}} \tilde{U}(K_1) \\
+ \sum_{t=K_1}^{k-1} \frac{(t + 1)^{2\theta}}{k^{2\theta}} \left(\frac{\theta^2 \tilde{U}(t)}{t^2} + \frac{2\theta L \sqrt{\tilde{U}(t) \tilde{V}(t)}}{\sqrt{n} \mu} + \frac{\theta^2 L^2 \tilde{V}(t)}{n \mu^2 t^2}\right).
\]
Hence, by Corollary 4.4,

\[
\hat{U}(k) - \frac{1}{k^{2\theta}} \sum_{t=K_1}^{k-1} \frac{\theta^2 \sigma^2}{t^2} (t + 1)^{2\theta} \frac{K_1^{2\theta}}{k^2} \hat{U}(K_1) \\
\leq \frac{\theta^2}{k^{2\theta}} \sum_{t=K_1}^{k-1} \frac{(t + 1)^{2\theta}}{t^2} \left[ \frac{\theta^2 \sigma^2}{(1.5\theta - 1)n\mu^2 t} + \frac{c_1}{t^2} \right] \\
+ \frac{1}{k^{2\theta}} \frac{2\theta L}{\sqrt{n\mu}} \sum_{t=K_1}^{k-1} \frac{(t + 1)^{2\theta}}{t} \sqrt{\frac{\theta^2 \sigma^2 c_2}{(1.5\theta - 1)n\mu^2 t^3} + \frac{c_1 c_2}{t^2}} \\
+ \frac{1}{k^{2\theta}} \frac{\theta^2 L c_2}{n\mu^2} \sum_{t=K_1}^{k-1} \frac{(t + 1)^{2\theta}}{t^4} \\
\leq \frac{\theta^2}{k^{2\theta}} \sum_{t=K_1}^{k-1} \frac{(t + 1)^{2\theta}}{t^2} \left[ \frac{\theta^2 \sigma^2}{(1.5\theta - 1)n\mu^2 t} + \frac{c_1}{t^2} \right] \\
+ \frac{1}{k^{2\theta}} \frac{2\theta L}{\sqrt{1.5\theta - 1}n\mu^2} \sum_{t=K_1}^{k-1} \frac{(t + 1)^{2\theta}}{t^{2.5}} \\
+ \frac{1}{k^{2\theta}} \frac{\theta^4 \sigma^2}{(1.5\theta - 1)n\mu^2} + \frac{2\theta L \sqrt{c_1 c_2}}{\sqrt{n\mu}} \sum_{t=K_1}^{k-1} \frac{(t + 1)^{2\theta}}{t^3} \\
+ \frac{1}{k^{2\theta}} \left( \theta^2 c_1 + \frac{\theta^2 L c_2}{n\mu^2} \right) \sum_{t=K_1}^{k-1} \frac{(t + 1)^{2\theta}}{t^4}.
\]

Following a discussion similar to those in the proofs for Theorem 4.2 and Corollary 4.4, we have

\[
\hat{U}(k) \leq \frac{\theta^2 \sigma^2}{(2\theta - 1)n\mu^2 k} + O \left( \frac{1}{\sqrt{n}(1 - \rho_w)} \right) \frac{1}{k^{1.5}} + O \left( \frac{1}{(1 - \rho_w)^2} \right) \frac{1}{k^2} \\
+ O \left( \frac{1}{(1 - \rho_w)^2} \right) \frac{1}{k^3} + O \left( \frac{1}{(1 - \rho_w)^2} \right) \frac{1}{k^{2\theta}} \\
= \frac{\theta^2 \sigma^2}{(2\theta - 1)n\mu^2 k} + O \left( \frac{1}{\sqrt{n}(1 - \rho_w)} \right) \frac{1}{k^{1.5}} + O \left( \frac{1}{(1 - \rho_w)^2} \right) \frac{1}{k^2}.
\]

Noting that \( U(k) = \hat{U}(k + K) \), we obtain (4.3). \( \square \)

**4.1. Comparison with Centralized Implementation.** We compare the performance of DSGD and centralized stochastic gradient descent (SGD) stated below.

\[ x(k + 1) = x(k) - \alpha_k \hat{g}(k), \]

where \( \alpha_k := \frac{\theta}{n} (\theta > 1) \) and \( \hat{g}(k) := \frac{1}{n} \sum_{i=1}^{n} g(x(k), \xi_i(k)). \)

First, we derive the convergence rate for SGD which matches the optimal rate for such stochastic gradient methods (see [32, 41]). Our result relies on an analysis different from the literature that considered a compact feasible set and uniformly bounded stochastic gradients in expectation.

**Theorem 4.6.** Under centralized stochastic gradient descent (4.5), suppose \( k \geq K_2 := \left[ \frac{2\theta L}{\mu} \right] \). We have

\[
\mathbb{E}[||x(k) - x^*||^2] \leq \frac{\theta^2 \sigma^2}{(2\theta - 1)n\mu^2 k} + O \left( \frac{1}{n} \right) \frac{1}{k^2}.
\]
Proof. Noting that \( \alpha_k \leq 1/L \) when \( k \geq K_2 \), we have

(4.6)

\[
\mathbb{E}[\|x(k+1) - x^*\|^2 \mid x(k)] = \mathbb{E}[\|x(k) - \alpha_k \tilde{g}(k) - x^*\|^2 \mid x(k)] = \|x(k) - \alpha_k \nabla f(x(k)) - x^*\|^2 + \alpha_k^2 \mathbb{E}[\|\nabla f(x(k)) - \tilde{g}(k)\|^2]
\]

\[
\leq (1 - \alpha_k \mu)^2 \|x(k) - x^*\|^2 + \frac{\alpha_k^2 \sigma^2}{n}
\]

\[
= \left(1 - \frac{\theta}{k}\right)^2 \|x(k) - x^*\|^2 + \frac{\theta^2 \sigma^2}{n \mu^2} \frac{1}{k^2}
\]

\[
= \left(1 - \frac{2\theta}{k}\right) \|x(k) - x^*\|^2 + \frac{\theta^2}{k^2} \|x(k) - x^*\|^2 + \frac{\theta^2 \sigma^2}{n \mu^2} \frac{1}{k^2}.
\]

It can be shown first that \( \mathbb{E}[\|x(k) - x^*\|^2] \leq \frac{c}{n} \) for \( k \geq K_2 \), where \( c_3 = \mathcal{O}(\frac{1}{n}) \). Then from relation (4.6), when \( k \geq K_2 \),

\[
\mathbb{E}[\|x(k) - x^*\|^2] \leq \left(\prod_{t=K_2}^{k-1} \left(1 - \frac{2\theta}{t}\right)\right) \mathbb{E}[\|x(K_2) - x^*\|^2]
\]

\[
+ \sum_{t=K_2}^{k-1} \left(\prod_{i=t+1}^{k-1} \left(1 - \frac{2\theta}{i}\right)\right) \left(\frac{\theta^2 \sigma^2}{n \mu^2 t^2} + \frac{\theta^2 c_3}{t^3} \right).
\]

From Lemma 4.1,

\[
\mathbb{E}[\|x(k) - x^*\|^2] \leq \frac{K_2^{2\theta}}{k^{2\theta}} \mathbb{E}[\|x(K_2) - x^*\|^2] + \sum_{t=K_2}^{k-1} \frac{(t + 1)^{2\theta}}{k^{2\theta}} \left(\frac{\theta^2 \sigma^2}{n \mu^2 t^2} + \frac{\theta^2 c_3}{t^3} \right)
\]

\[
= \frac{\theta^2 \sigma^2}{(2\theta - 1) n \mu^2} \sum_{t=K_2}^{k-1} \frac{(t + 1)^{2\theta}}{k^{2\theta}} + \frac{K_2^{2\theta}}{k^{2\theta}} \mathbb{E}[\|x(K_2) - x^*\|^2] + \frac{\theta^2 c_3}{k^{2\theta}} \sum_{t=K_2}^{k-1} \frac{(t + 1)^{2\theta}}{k^{2\theta}}
\]

\[
= \frac{\theta^2 \sigma^2}{(2\theta - 1) n \mu^2} + \mathcal{O} \left(\frac{1}{k}\right) \frac{1}{k^2} \quad \Box
\]

Comparing the results of Theorem 4.5 and Theorem 4.6, we can see that asymptotically, DSGD and SGD have the same convergence rate \( \frac{\theta^2 \sigma^2}{(2\theta - 1) n \mu^2} \). The next corollary identifies the time needed for DSGD to achieve this rate.

**Corollary 4.7** (Transient Time). It takes \( K_T = \mathcal{O} \left(\frac{n}{1 - \rho_w} \right) \) time for Algorithm (2.2) to reach the asymptotic rate of convergence, i.e., when \( k \geq K_T \), we have \( U(k) \leq \frac{\theta^2 \sigma^2}{(2\theta - 1) n \mu^2} k \mathcal{O}(1) \).

**Proof.** From (4.3),

\[
U(k) \leq \frac{\theta^2 \sigma^2}{(2\theta - 1) n \mu^2} \left[1 + \mathcal{O} \left(\frac{\sqrt{n}}{1 - \rho_w}\right) \frac{1}{k^{0.5}} + \mathcal{O} \left(\frac{n}{(1 - \rho_w)^2}\right) \frac{1}{k}\right]
\]

Let \( K_T \) be such that

\[
\mathcal{O} \left(\frac{\sqrt{n}}{1 - \rho_w}\right) \frac{1}{K_T^{0.5}} + \mathcal{O} \left(\frac{n}{(1 - \rho_w)^2}\right) \frac{1}{K_T} = \mathcal{O}(1).
\]

\[\text{The argument here is similar to that in the proof for Lemma 3.3.}\]
We then obtain that
\[ K_T = \mathcal{O}\left(\frac{n}{(1 - \rho_w)^2}\right). \]

**Remark 4.8.** In general, if we adopt the Lazy Metropolis rule for choosing the weights \([w_{ij}]\) (see [33]), then \(\frac{1}{1 - \rho_w} = \mathcal{O}(n^2)\), and hence \(K_T = \mathcal{O}(n^5)\).

5. Conclusions. This paper is devoted to the non-asymptotic analysis of network independence for distributed stochastic gradient descent (DSGD). We show that the algorithm asymptotically achieves the optimal network independent convergence rate compared to SGD, and identify the non-asymptotic convergence rate as a function of characteristics of the objective functions and the network. In addition, we compute the time needed for DSGD to reach its asymptotic rate of convergence. Future work will consider more general problems such as nonconvex objectives and constrained optimization.

**Appendix A. Proofs for Section 2.**

A.1. Proof of Lemma 2.5. By the definitions of \(\bar{x}(k), \bar{g}(k)\) and relation (2.2), we have
\[ \bar{x}(k + 1) = \bar{x}(k) - \alpha_k \bar{g}(k). \]

Hence,
\[ ||\bar{x}(k + 1) - x^*||^2 = ||\bar{x}(k) - \alpha_k \bar{g}(k) - x^*||^2 \]
\[ = ||\bar{x}(k) - \alpha_k \nabla F(x(k)) - x^* + \alpha_k \nabla F(x(k)) - \alpha_k \bar{g}(k)||^2 \]
\[ = ||\bar{x}(k) - \alpha_k \nabla F(x(k)) - x^*||^2 + 2\alpha_k \langle \bar{x}(k) - \alpha_k \nabla F(x(k)) - x^*, \nabla F(x(k)) - \bar{g}(k) \rangle \]
\[ + \alpha_k^2 \|
\n\n\]
\nNoting that \(E[\bar{g}(k) | x(k)] = \nabla F(x(k))\) and \(E[||\bar{g}(k) - \nabla F(x(k))||^2 | x(k)] \leq \frac{\sigma^2}{n}\) from Lemma 2.3,
\[ E[||\bar{x}(k + 1) - x^*||^2 | x(k)] \leq ||\bar{x}(k) - \alpha_k \nabla F(x(k)) - x^*||^2 + \frac{\alpha_k^2 \sigma^2}{n}. \]

We next bound the first term on the right-hand-side.
\[ ||\bar{x}(k) - \alpha_k \nabla F(x(k)) - x^*||^2 \]
\[ = ||\bar{x}(k) - \alpha_k \nabla f(\bar{x}(k)) - x^* + \alpha_k \nabla f(\bar{x}(k)) - \alpha_k \nabla F(x(k))||^2 \]
\[ \leq ||\bar{x}(k) - \alpha_k \nabla f(\bar{x}(k)) - x^*||^2 + 2\alpha_k \|
\n\n\]
\nwhere we used the Cauchy-Schwarz inequality. By Lemma 2.3,
\[ ||\nabla f(\bar{x}(k)) - \nabla F(x(k))|| \leq \frac{L^2}{n}||x(k) - 1\bar{x}(k)||^2. \]

Then, we have
\[ ||\bar{x}(k) - \alpha_k \nabla F(x(k)) - x^*||^2 \leq ||\bar{x}(k) - \alpha_k \nabla f(\bar{x}(k)) - x^*||^2 \]
\[ + \frac{2\alpha_k L}{\sqrt{n}} ||\bar{x}(k) - \alpha_k \nabla f(\bar{x}(k)) - x^*|| ||x(k) - 1\bar{x}(k)|| + \frac{\alpha_k^2 L^2}{n} ||x(k) - 1\bar{x}(k)||^2. \]
The conclusion follows.

**A.2. Proof of Lemma 2.6.** Since $\alpha_k \leq \frac{1}{2\tau}$, in light of Lemma 2.2,

$$\|\mathbf{r}(k) - \alpha_k \nabla f(\mathbf{x}(k)) - x^*\|^2 \leq (1 - \alpha_k \mu)^2 \|\mathbf{r}(k) - x^*\|^2.$$  

The above relation and (2.6) imply that

$$\mathbb{E} \left[\|\mathbf{r}(k + 1) - x^*\|^2 | \mathbf{x}(k)\right] \leq (1 - \alpha_k \mu)^2 \|\mathbf{r}(k) - x^*\|^2 + \frac{2\alpha_k L}{\sqrt{n}} (1 - \alpha_k \mu) \|\mathbf{r}(k) - x^*\| \|\mathbf{x}(k) - \mathbf{1r}(k)\|$$

$$+ \frac{\alpha_k^2 L^2}{n} \|\mathbf{x}(k) - \mathbf{1r}(k)\|^2 + \frac{\alpha_k^2 \sigma^2}{n},$$

where $c > 0$ is arbitrary.

Take $c = \frac{2}{5} \alpha_k \mu$. Noting that $\alpha_k \leq \frac{1}{2\tau}$, we have $(1 + c)(1 - \alpha_k \mu)^2 \leq 1 - \frac{3}{5} \alpha_k \mu$, and $(1 + \frac{1}{\tau}) \alpha_k \leq \frac{3}{\tau}$. Thus,

$$\mathbb{E} \left[\|\mathbf{r}(k + 1) - x^*\|^2 | \mathbf{x}(k)\right] \leq \left(1 - \frac{3}{2} \alpha_k \mu\right) \|\mathbf{r}(k) - x^*\|^2 + \frac{3\alpha_k L^2}{n \mu} \|\mathbf{x}(k) - \mathbf{1r}(k)\|^2$$

$$+ \frac{\alpha_k^2 \sigma^2}{n},$$

**A.3. Proof of Lemma 2.7.** Given that

$$\mathbf{x}(k + 1) - \mathbf{1r}(k + 1) = \mathbf{W}(\mathbf{x}(k) - \alpha_k \mathbf{g}(k)) - \mathbf{1r}(k) - \alpha_k \mathbf{1g}(k))$$

$$= \left(\mathbf{W} - \frac{11^T}{n}\right) \left[(\mathbf{x}(k) - \mathbf{1r}(k)) - \alpha_k (\mathbf{g}(k) - \mathbf{1g}(k))\right],$$

we have

$$\|\mathbf{x}(k + 1) - \mathbf{1r}(k + 1)\|^2 \leq \rho_w^2 \|((\mathbf{x}(k) - \mathbf{1r}(k)) - \alpha_k (\mathbf{g}(k) - \mathbf{1g}(k))\|^2$$

$$= \rho_w^2 \left[\|\mathbf{x}(k) - \mathbf{1r}(k)\|^2 + \alpha_k^2 \|\mathbf{g}(k) - \mathbf{1g}(k)\|^2 - 2\alpha_k \langle \mathbf{x}(k) - \mathbf{1r}(k), \mathbf{g}(k) - \mathbf{1g}(k) \rangle\right].$$

Since $\mathbb{E}[\mathbf{g}(k) | \mathbf{x}(k)] = \nabla F(\mathbf{x}(k))$ and $\mathbb{E}[\mathbf{1g}(k) | \mathbf{x}(k)] = \nabla F(\mathbf{x}(k))$,

$$\mathbb{E}[\|\mathbf{g}(k) - \mathbf{1g}(k)\|^2 | \mathbf{x}(k)]$$

$$= \mathbb{E}[\|\nabla F(\mathbf{x}(k)) - 1\nabla F(\mathbf{x}(k)) - \nabla F(\mathbf{x}(k)) + 1\nabla F(\mathbf{x}(k)) + \mathbf{g}(k) - \mathbf{1g}(k)\|^2 | \mathbf{x}(k)]$$

$$\leq \|\nabla F(\mathbf{x}(k)) - 1\nabla F(\mathbf{x}(k))\|^2$$

$$+ \mathbb{E}[\|\nabla F(\mathbf{x}(k)) - \mathbf{g}(k) - (1\nabla F(\mathbf{x}(k)) - \mathbf{1g}(k))\|^2 | \mathbf{x}(k)]$$

$$\leq \|\nabla F(\mathbf{x}(k)) - 1\nabla F(\mathbf{x}(k))\|^2 + \mathbb{E}[\|\nabla F(\mathbf{x}(k)) - \mathbf{g}(k)\|^2 | \mathbf{x}(k)]$$

$$\leq \|\nabla F(\mathbf{x}(k)) - 1\nabla F(\mathbf{x}(k))\|^2 + n\sigma^2,$$

where the last inequality follows from Assumption 1.1, and

$$\mathbb{E} \left[\langle \mathbf{x}(k) - \mathbf{1r}(k), \mathbf{g}(k) - \mathbf{1g}(k) \rangle | \mathbf{x}(k)\right] = \langle \mathbf{x}(k) - \mathbf{1r}(k), \nabla F(\mathbf{x}(k)) - \mathbf{1g}(\mathbf{x}(k))\rangle.$$
Therefore,
\[
\frac{1}{\rho_k^2} \mathbb{E}\left[ \|x(k + 1) - 1\tau(k + 1)\|^2 \mid x(k) \right] \\
\leq \|x(k) - 1\tau(k)\|^2 + \alpha_k^2 \|\nabla F(x(k)) - 1\nabla F(x(k))\|^2 + \alpha_k^2 n\sigma^2 \\
- 2\alpha_k \|x(k) - 1\tau(k), \nabla F(x(k)) - 1\nabla F(x(k))\| \\
\leq \|x(k) - 1\tau(k)\|^2 + \alpha_k^2 \|\nabla F(x(k))\|^2 + \alpha_k^2 n\sigma^2 + 2\alpha_k \|x(k) - 1\tau(k)\| \|\nabla F(x(k))\|.
\]

Noting that by Assumption 2.1,
\[
\|\nabla F(x(k))\| \leq \|\nabla F(x(k)) - \nabla F(1\tau(k))\| + \|\nabla F(1\tau(k)) - \nabla F(x^*)\| + \|\nabla F(x^*)\| \\
\leq L\|x(k) - 1\tau(k)\| + \sqrt{n}L\|\tau(k) - x^*\| + \|\nabla F(x^*)\|,
\]
and so that
\[
\|\nabla F(x(k))\|^2 \leq 2L^2\|x(k) - 1\tau(k)\|^2 + 4nL^2\|\tau(k) - x^*\|^2 + 4\|\nabla F(x^*)\|^2,
\]
we have
\[
\frac{1}{\rho_k^2} \mathbb{E}\left[ \|x(k + 1) - 1\tau(k + 1)\|^2 \mid x(k) \right] - \alpha_k^2 n\sigma^2 \\
\leq \|x(k) - 1\tau(k)\|^2 + \alpha_k^2 (2L^2\|x(k) - 1\tau(k)\|^2 + 4nL^2\|\tau(k) - x^*\|^2 + 4\|\nabla F(x^*)\|^2) \\
+ 2\alpha_k \|x(k) - 1\tau(k)\| (L\|x(k) - 1\tau(k)\| + \sqrt{n}L\|\tau(k) - x^*\| + \|\nabla F(x^*)\|) \\
(1 + 2\alpha_k L + 2\alpha_k^2 L^2) \|x(k) - 1\tau(k)\|^2 + \alpha_k^2 (4nL^2\|\tau(k) - x^*\|^2 + 4\|\nabla F(x^*)\|^2) \\
+ 2\alpha_k \|x(k) - 1\tau(k)\| (\sqrt{n}L\|\tau(k) - x^*\| + \|\nabla F(x^*)\|) \\
\leq (1 + 2\alpha_k L + 2\alpha_k^2 L^2 + c) \|x(k) - 1\tau(k)\|^2 + \alpha_k^2 (4nL^2\|\tau(k) - x^*\|^2 + 4\|\nabla F(x^*)\|^2) \\
+ \frac{\alpha_k^2}{c} (\sqrt{n}L\|\tau(k) - x^*\| + \|\nabla F(x^*)\|)^2 \\
\leq (1 + 2\alpha_k L + 2\alpha_k^2 L^2 + c) \|x(k) - 1\tau(k)\|^2 + \alpha_k^2 (4nL^2\|\tau(k) - x^*\|^2 + 4\|\nabla F(x^*)\|^2) \\
+ \frac{\alpha_k^2}{c} (2\alpha_k^2 (2L^2\|\tau(k) - x^*\|^2 + 2\|\nabla F(x^*)\|^2) \\
= (1 + 2\alpha_k L + 2\alpha_k^2 L^2 + c) \|x(k) - 1\tau(k)\|^2 \\
+ \alpha_k^2 (2 + \frac{1}{c}) (2L^2\|\tau(k) - x^*\|^2 + 2\|\nabla F(x^*)\|^2),
\]
where \(c > 0\) is arbitrary. Letting \(c = \frac{1 - \rho_k^2}{\alpha_k^2 \rho_k^2}\) leads to
\[
\mathbb{E}\left[ \|x(k + 1) - 1\tau(k + 1)\|^2 \mid x(k) \right] \\
\leq \frac{1}{2} + 2\alpha_k \rho_k^2 L + 2\alpha_k^2 \rho_k^2 L^2 \|x(k) - 1\tau(k)\|^2 \\
+ \rho_k^2 \left[ \frac{\alpha_k^2 (2L^2)}{(1 - \rho_k^2)} \|\tau(k) - x^*\|^2 + \alpha_k^2 \frac{4\|\nabla F(x^*)\|^2}{(1 - \rho_k^2)} + \alpha_k^2 n\sigma^2 \right].
\]
Taking full expectation on both sides of the inequality gives the result. 

**Appendix B. Proofs for Section 3.**
B.1. Proof of Lemma 3.1. By Assumption 1.1,\(^{5}\)

\[
\mathbb{E} \left[ \|x_i(k) - \alpha_k g_i(k)\|^2 \mid x_k \right] = \|x_i(k) - \alpha_k \nabla f_i(x_i(k))\|^2 + \alpha_k^2 \mathbb{E} \left[ \|\nabla f_i(x_i(k)) - g_i(k)\|^2 \mid x_k \right] \\
\leq \|x_i(k)\|^2 - 2\alpha_k \langle \nabla f_i(x_i(k)), x_i(k) - 0 \rangle + \alpha_k^2 \|\nabla f_i(x_i(k))\|^2 + \alpha_k^2 \sigma^2.
\]

From the strong convexity and Lipschitz continuity of \(f_i\), we know that

\[
\langle \nabla f_i(x_i(k)), x_i(k) \rangle = \langle \nabla f_i(x_i(k)) - \nabla f_i(0), x_i(k) - 0 \rangle + \langle \nabla f_i(0), x_i(k) \rangle \\
\geq \mu \|x_i(k)\|^2 + \langle \nabla f_i(0), x_i(k) \rangle,
\]

and

\[
\|\nabla f_i(x_i(k))\|^2 = \|\nabla f_i(x_i(k)) - \nabla f_i(0) + \nabla f_i(0)\|^2 \leq 2L^2 \|x_i(k)\|^2 + 2\|\nabla f_i(0)\|^2.
\]

Hence,

\[
\mathbb{E} \left[ \|x_i(k) - \alpha_k g_i(k)\|^2 \mid x_k \right] \\
\leq \|x_i(k)\|^2 - 2\alpha_k \left[ \mu \|x_i(k)\|^2 + \langle \nabla f_i(0), x_i(k) \rangle \right] + 2\alpha_k^2 \left( L^2 \|x_i(k)\|^2 + \|\nabla f_i(0)\|^2 \right) \\
+ \alpha_k^2 \sigma^2 \\
\leq (1 - 2\alpha_k \mu + 2\alpha_k^2 L^2) \|x_i(k)\|^2 + 2\alpha_k \|\nabla f_i(0)\| \|x_i(k)\| + \alpha_k^2 \left( 2\|\nabla f_i(0)\|^2 + \sigma^2 \right).
\]

It follows that

\[
\mathbb{E} \left[ \|x_i(k) - \alpha_k g_i(k)\|^2 \right] \leq (1 - 2\alpha_k \mu + 2\alpha_k^2 L^2) \mathbb{E} \|x_i(k)\|^2 \\
+ 2\alpha_k \|\nabla f_i(0)\| \sqrt{\mathbb{E} \|x_i(k)\|^2} + \alpha_k^2 \left( 2\|\nabla f_i(0)\|^2 + \sigma^2 \right).
\]

From the definition of \(K\), \(\alpha_k \leq \frac{\mu}{4L^2}\) for all \(k \geq 0\). Hence,

\[
\text{(B.1) } \mathbb{E} \left[ \|x_i(k) - \alpha_k g_i(k)\|^2 \right] \leq (1 - \alpha_k \mu) \mathbb{E} \|x_i(k)\|^2 + 2\alpha_k \|\nabla f_i(0)\| \sqrt{\mathbb{E} \|x_i(k)\|^2} + \alpha_k^2 \left( 2\|\nabla f_i(0)\|^2 + \sigma^2 \right) \\
= \mathbb{E} \|x_i(k)\|^2 - \alpha_k \left[ \mu \mathbb{E} \|x_i(k)\|^2 - 2\|\nabla f_i(0)\| \sqrt{\mathbb{E} \|x_i(k)\|^2} - \frac{\mu}{2L^2} \left( 2\|\nabla f_i(0)\|^2 + \sigma^2 \right) \right].
\]

Let’s define the following set:

\[
\mathcal{X}_i := \left\{ q \geq 0 : \mu q - 2\|\nabla f_i(0)\| \sqrt{q} - \frac{\mu}{2L^2} \left( 2\|\nabla f_i(0)\|^2 + \sigma^2 \right) \leq 0 \right\},
\]

which is non-empty and compact. If \(\mathbb{E} \|x_i(k)\|^2 \notin \mathcal{X}_i\), we know from inequality (B.1) that

\[
\mathbb{E} \left[ \|x_i(k) - \alpha_k g_i(k)\|^2 \right] \leq \mathbb{E} \|x_i(k)\|^2.
\]

\(^{5}\)The following arguments are inspired by [27].
Otherwise,
\[
\mathbb{E} [\| x_i(k) - \alpha_k g_i(k) \|^2 ] \\
\leq \max_{q \in \mathcal{X}_i} \left\{ q - \mu \frac{L^2}{2} \left( \| q - 2 \| \nabla f_i(0) \| \sqrt{q} - \frac{\mu}{2L^2} \left( 2 \| \nabla f_i(0) \|^2 + \sigma^2 \right) \right) \right\} \\
= \max_{q \in \mathcal{X}_i} \left\{ \left( 1 - \frac{\mu^2}{2L^2} \right) q + \frac{\mu}{L^2} \| \nabla f_i(0) \| \sqrt{q} + \frac{\mu^2}{4L^4} \left( 2 \| \nabla f_i(0) \|^2 + \sigma^2 \right) \right\} \\
= R_i.
\]

Note that
\[
\| x(k+1) \|^2 \leq \| W \|^2 \| x(k) - \alpha_k g(k) \|^2 \leq \| x(k) - \alpha_k g(k) \|^2.
\]
The previous arguments imply that for all \( k \geq 0 \),
\[
\mathbb{E}[\| x(k) \|^2] \leq \max \left\{ \| x(0) \|^2, \sum_{i=1}^{n} R_i \right\}.
\]

\[\Box\]

**B.2. Proof of Lemma 3.3.** We first bound \( U(K_1 - K) \), \( V(K_1 - K) \), and \( W(K_1 - K) \).

**Lemma B.1.** We have
\[
U(K_1 - K) \leq \frac{\hat{X}}{n}, \quad V(K_1 - K) \leq \hat{X}, \quad W(K_1 - K) \leq \frac{\hat{X}}{n}.
\]

**Proof.** First, by definitions of \( U(K_1 - K) \), \( V(K_1 - K) \), and Lemma 3.2,
\[
U(K_1 - K) = \mathbb{E}[\| \mathbf{\tau}(K_1 - K) - x^* \|^2] \leq \frac{1}{n} \mathbb{E}[\| x(K_1 - K) - 1x^* \|^2] \leq \frac{\hat{X}}{n},
\]
and
\[
V(K_1 - K) = \mathbb{E}[\| x(K_1 - K) - 1\mathbf{\tau}(K_1 - K) \|^2] \leq \mathbb{E}[\| x(K_1 - K) - 1x^* \|^2] \leq \hat{X}.
\]

Second, since
\[
K_1 = \left\lfloor \frac{24\theta L^2}{(1 - \rho_{\omega}^2)\mu^2} \right\rfloor,
\]
we have
\[
W(K_1 - K) = U(K_1 - K) + \omega(K_1 - K)V(K_1 - K) \leq U(K_1 - K) + \frac{12\omega K_1 - K L^2}{n\mu(1 - \rho_{\omega}^2)} V(K_1 - K)
\]
\[
\leq U(K_1 - K) + \frac{V(K_1 - K)}{2n}
\]
\[
= \mathbb{E}[\| \mathbf{\tau}(K_1 - K) - x^* \|^2] + \frac{\mathbb{E}[\| x(K_1 - K) - 1\mathbf{\tau}(K_1 - K) \|^2]}{2n}
\]
\[
\leq \frac{\mathbb{E}[\| x(K_1 - K) - 1x^* \|^2]}{n} \leq \frac{\hat{X}}{n}.
\]
\[\Box\]
From Lemma 2.6 and Lemma 2.7, we know that when \( k \geq K_1 \),

\[
W(k + 1) \leq \left( 1 - \frac{3}{2} \alpha_k \mu \right) U(k) + \frac{3 \alpha_k L^2}{n \mu} V(k) + \frac{\alpha_k^2 \sigma^2}{n} \omega(k) \left[ \left( 1 + \frac{\rho_w^2}{2} \right) + 2 \alpha_k \rho_w^2 \right] V(k) + \frac{\alpha_k^2}{(1 - \rho_w^2)} 4nL^2 \rho_w^2 U(k) + \frac{\alpha_k^2}{(1 - \rho_w^2)} 4\|\nabla F(1x^*)\|^2 \rho_w^2 + \frac{\alpha_k^2}{(1 - \rho_w^2)} n\sigma^2 \rho_w^2 \right].
\]

We show the following inequalities hold for all \( k \geq K_1 \):

\[
\begin{align*}
\text{(B.6a)} & \quad 1 - \frac{3}{2} \alpha_k \mu + \omega(k) \frac{\alpha_k^2}{(1 - \rho_w^2)} 4nL^2 \rho_w^2 \leq 1 - \frac{4}{3} \alpha_k \mu, \\
\text{(B.6b)} & \quad \frac{3 \alpha_k L^2}{n \mu} + \omega(k) \left( \frac{1 + \rho_w^2}{2} + 2 \alpha_k \rho_w^2 \right) \leq \left( 1 - \frac{4}{3} \alpha_k \mu \right) \omega(k).
\end{align*}
\]

Noticing that \( K_1 \) satisfies

\[
2\alpha_{K_1} \rho_w^2 + 2\alpha_{K_1}^2 \rho_w^2 + \frac{4}{3} \alpha_{K_1} \mu \leq \frac{1 - \rho_w^2}{4},
\]

it is sufficient that

\[
\text{(B.8a)} & \quad \omega(k) \leq \frac{(1 - \rho_w^2) \mu}{24nL^2 \rho_w^2} \frac{1}{\alpha_k}, \\
\text{(B.8b)} & \quad \omega(k) \geq \frac{12 \alpha_k L^2}{n \mu (1 - \rho_w^2)}.
\]

Since the sequence \( \{ \alpha_k \} \) is non-increasing and by (3.10),

\[
\alpha_{K_1} \leq \frac{(1 - \rho_w^2) \mu}{12\sqrt{2}L^2 \rho_w},
\]

condition (B.8) is satisfied with \( \omega(k) \) defined in (3.11).

Then, from (B.6), for all \( k \geq K_1 \), we have

\[
W(k + 1) \leq \left( 1 - \frac{4}{3} \alpha_k \mu \right) W(k) + \frac{\alpha_k^2 \sigma^2}{n} \omega(k) \left[ \frac{4 \|\nabla F(1x^*)\|^2 \rho_w^2}{(1 - \rho_w^2)} + \frac{\alpha_k^2}{(1 - \rho_w^2)} n\sigma^2 \rho_w^2 \right].
\]

Given that \( \tilde{W}(k) = W(k - K) \) for all \( k \geq K_1 + K \) and \( \alpha_k = \frac{\theta}{\mu(k + K)} \),

\[
\tilde{W}(k + 1) \leq \left( 1 - \frac{4\theta}{3k} \right) \tilde{W}(k) + \frac{\sigma^2 \theta^2}{n \mu^2 k^2} + \frac{c_0 \theta^3}{k^3},
\]

where

\[
\text{(B.9)} & \quad c_0 := \frac{12 L^2 \rho_w^2}{n \mu^2 (1 - \rho_w^2)} \left[ \frac{4 \|\nabla F(1x^*)\|^2}{(1 - \rho_w^2)} + n\sigma^2 \right].
\]

Then,

\[
\tilde{W}(k) \leq \left( \prod_{t=K_1}^{k-1} \left( 1 - \frac{4\theta}{3t} \right) \right) \tilde{W}(K_1) + \sum_{t=K_1}^{k-1} \left( \prod_{j=t+1}^{k-1} \left( 1 - \frac{4\theta}{3j} \right) \right) \left( \frac{\sigma^2 \theta^2}{n \mu^2 t^2} + \frac{c_0 \theta^3}{t^3} \right).
\]
By induction we obtain
\[
\tilde{W}(k) \leq \frac{1}{k} \left[ K_1 \tilde{W}(K_1) + \frac{3}{(4\theta - 3)} \left( \frac{\sigma^2 \theta^2}{n\mu^2} + \frac{c_0 \theta^3}{K_1} \right) \right]
\]
\[
\leq \frac{1}{k} \left[ K_1 \tilde{W}(K_1) + \frac{3}{(4\theta - 3)} \left( \frac{\sigma^2 \theta^2}{n\mu^2} + \frac{\sigma^2 \rho^2_w \theta^2}{2\mu^2} + 6\|\nabla F(1x^*)\|^2 \frac{\rho^2_w \theta^2}{(4\theta - 3)n\mu^2(1 - \rho^2_w)} \right) \right],
\]
where the second inequality follows from (3.10) and (B.9). Since \( \tilde{W}(K_1) = W(K_1 - K) \leq \frac{\tilde{W}}{n} \) from Lemma B.1, and \( U(k) \leq W(k) = \tilde{W}(k + K) \) by definition, we obtain relation (3.12).

To bound \( V(k) \), from Lemma 2.7 and the definitions of \( K_1 \) and \( \tilde{V}(k) \), we know when \( k \geq K + K_1 \),
\[
\tilde{V}(k + 1) \leq \frac{(3 + \rho^2_w)}{4} \tilde{V}(k) + \frac{4\theta^2 n L^2 \rho^2_w}{\mu^2(1 - \rho^2_w)} \left( \frac{1}{k^2} \right) \tilde{W} + \frac{\theta^2}{\mu^2} \left[ \frac{4\|\nabla F(1x^*)\|^2}{(1 - \rho^2_w)} + n\sigma^2 \right] \frac{\rho^2_w}{k^2} \left( \frac{1}{k^2} \right) = p_0 \tilde{V}(k) + \frac{p_1}{k^2} + \frac{p_2}{k^2},
\]
where \( p_0 \) is given in (3.14) and
\[
(B.10) \quad p_1 := \frac{\theta^2}{\mu^2} \left[ \frac{4\|\nabla F(1x^*)\|^2}{(1 - \rho^2_w)} + n\sigma^2 \right] \frac{\rho^2_w}{k^2}, \quad p_2 := \frac{4\theta^2 n L^2 \rho^2_w}{\mu^2(1 - \rho^2_w)} \tilde{W}.
\]
It follows that
\[
\tilde{V}(k) \leq \left( \prod_{i=K_1}^{k-1} p_i \right) \tilde{V}(K_1) + \sum_{i=K_1}^{k-1} \left( \prod_{i=K_1+1}^{i} p_0 \frac{p_1}{k^2} + \frac{p_2}{k} \right)
\]
\[
= p_0^{-K_1} \tilde{V}(K_1) + \sum_{i=K_1}^{k-1} p_0^{-1-t} \left( \frac{p_1}{k^2} + \frac{p_2}{k} \right) = p_0^{-K_1} \tilde{V}(K_1) + T_1(k) + T_2(k),
\]
where the auxiliary variables \( T_1(k) \) and \( T_2(k) \) \((k \geq K_1)\) are defined as follows:
\[
T_1(K_1) = T_2(K_1) = 0,
\]
\[
T_1(k) := \sum_{i=K_1}^{k-1} p_0^{-1-t} \frac{p_1}{k^2}, \quad T_2(k) := \sum_{i=K_1}^{k-1} p_0^{-1-t} \frac{p_2}{k}, \quad \forall k \geq K_1.
\]
Note that
\[
T_1(k + 1) = \sum_{i=K_1}^{k} p_0^{-1-t} \frac{p_1}{k^2} = p_0 T_1(k) + \frac{p_1}{k^2},
\]
\[
T_2(k + 1) = \sum_{i=K_1}^{k} p_0^{-1-t} \frac{p_2}{k} = p_0 T_2(k) + \frac{p_2}{k},
\]
By induction,
\[
T_1(k) \leq \frac{1}{k^2} \frac{p_1}{K_1^2 / (K_1 + 1)^2 - p_0}, \quad T_2(k) \leq \frac{1}{k^3} \frac{p_2}{K_1^3 / (K_1 + 1)^3 - p_0}.
\]
We can verify from the definition of $K_1$ that
\[
\frac{K_1^2}{(K_1 + 1)^2} - p_0 \geq \frac{K_1^3}{(K_1 + 1)^3} - p_0 \geq \frac{1}{2}(1 - p_0) = \frac{1}{8}(1 - \rho_w^2).
\]
Hence,
\[
\tilde{V}(k) \leq p_0^{k-K_1} \tilde{V}(K_1) + \frac{1}{k^2} \frac{8p_1}{1 - \rho_w^2} + \frac{1}{k^3} \frac{8p_2}{1 - \rho_w^2} = p_0^{k-K_1} \tilde{V}(K_1) + \frac{V_1}{k^2} + \frac{V_2}{k^3}.
\]

Recalling the definition of $\tilde{V}(k)$ and Lemma $B.1$, we conclude that
\[
V(k) \leq p_0^{k-K_1} \tilde{X} + \frac{V_1}{k^2} + \frac{V_2}{k^3}.
\]

\[\square\]

**Appendix C. Proofs for Section 4.**

**C.1. Proof of Lemma 4.1.** Denote $G(k) := \prod_{i=a}^{k-1} (1 - \gamma i)$. Suppose $G(k) \leq \frac{M}{k^\gamma}$ for some $M > 0$. Then,
\[
G(k + 1) = \left(1 - \frac{\gamma}{k}\right) G(k) \leq \left(1 - \frac{\gamma}{k}\right) \frac{M}{k^\gamma} \leq \frac{M}{(k + 1)\gamma}.
\]
To see why the last inequality holds, note that
\[
\left(\frac{k}{k + 1}\right)^\gamma \geq 1 - \frac{\gamma}{k}.
\]
Taking $M = a\gamma$, $G(a) \leq \frac{M}{a\gamma}$ since $G(a) = 1$. The desired relation then holds for all $k > a$.

**C.2. Proof of Lemma 4.3.** From Lemma 3.2,
\[
\tilde{X} \leq E[||x(K_1) - 1x^*||^2] + \frac{9||\nabla F(1x^*)||^2}{\mu^2} + \frac{n\sigma^2}{L^2}.
\]
Since $E[||x(K_1) - 1x^*||^2] = \mathcal{O}(n)$ and $||\nabla F(1x^*)||^2 = \mathcal{O}(n)$, we have
\[
\tilde{X} = \mathcal{O}(n).
\]
From the definition of $\tilde{W}$ in (3.13) and relation (B.5),
\[
\tilde{W} = K_1 W(K_1) + \frac{3}{4(\theta - 3)} \left( \frac{\sigma^2 \theta^2}{n\mu^2} + \frac{\sigma^2 \rho_w^2 \theta^2}{2\mu^2} \right) + \frac{6||\nabla F(1x^*)||^2 \rho_w^2 \theta^2}{(4\theta - 3)n\mu^2(1 - \rho_w^2)} \leq K_1 \tilde{X} + \frac{3}{4(\theta - 3)} \left( \frac{\sigma^2 \theta^2}{n\mu^2} + \frac{\sigma^2 \rho_w^2 \theta^2}{2\mu^2} \right) + \frac{6||\nabla F(1x^*)||^2 \rho_w^2 \theta^2}{(4\theta - 3)n\mu^2(1 - \rho_w^2)} \leq \frac{K_1 \tilde{X}}{n} + \frac{3}{4(\theta - 3)} \left( \frac{\sigma^2 \theta^2}{n\mu^2} + \frac{\sigma^2 \rho_w^2 \theta^2}{2\mu^2} \right) + \frac{6||\nabla F(1x^*)||^2 \rho_w^2 \theta^2}{(4\theta - 3)n\mu^2(1 - \rho_w^2)}.
\]
Noting that $K_1 = \mathcal{O}(\frac{1}{1 - \rho_w})$ and $\tilde{X} = \mathcal{O}(n)$, we have
\[
\tilde{W} = \mathcal{O}\left(\frac{1}{2\theta - \rho_w}\right).
\]
From the definition of $V_1$ and $V_2$ in (3.15),

$$V_1 = \frac{8\theta^2\rho_w^2}{\mu^2(1-\rho_w^2)} \left[ \frac{4|\nabla F(1x^*)|^2}{(1-\rho_w^2)} + n\sigma^2 \right] = O\left( \frac{n}{(1-\rho_w)^2} \right),$$

$$V_2 = \frac{32\theta^2nL^2\rho_w^2}{\mu^2(1-\rho_w^2)} W = O\left( \frac{n}{(1-\rho_w)^3} \right).$$

\[\square\]

**C.3. Proof of Corollary 4.4.** From Theorem 4.2 and Lemma 4.3, when $k \geq K_1 = O\left(\frac{1}{\rho_w} \right)$,

$$U(k) \leq \frac{\theta^2\sigma^2}{(1.5\theta - 1)n\mu^2k} + \left[ \frac{3\theta^2(1.5\theta - 1)\sigma^2}{(1.5\theta - 2)n\mu^2} + \frac{6\theta L^2V_1}{(1.5\theta - 2)n\mu^2} \right] \frac{1}{k^2}$$

$$+ \frac{6\theta L^2V_2}{(1.5\theta - 3)\mu^2k} + \left[ \frac{K_{\theta}^{1.5\theta}}{n} + \frac{6\theta L^2K_{\theta}^{1.5\theta - 1}X}{n\mu^2(1-\rho_w)} \right] \frac{1}{k^{1.5\theta}}$$

$$= \frac{\theta^2\sigma^2}{(1.5\theta - 1)n\mu^2k} + O\left( \frac{1}{(1-\rho_w)^2} \right) \frac{1}{k^2} + O\left( \frac{1}{(1-\rho_w)^3} \right) \frac{1}{k^{1.5\theta}}$$

$$+ O\left( \frac{1}{(1-\rho_w)^{1.5\theta}} \right) \frac{1}{k}\cdot$$

From Lemma 3.3, when $k \geq K_1$,

$$V(k) \leq \rho_0^{\frac{k-1}{K}} X + \frac{V_1}{k^2} + \frac{V_2}{k^3} = \rho_0^{\frac{k-1}{K}} O(n) + O\left( \frac{n}{(1-\rho_w)^2} \right) \frac{1}{k^2}$$

$$= O\left( \frac{n}{(1-\rho_w)^2} \right) \frac{1}{k^2}.$$

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