The D2-D6 System
and
a Fibered AdS Geometry

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Abstract

The system of D2 branes localized on or near D6 branes is considered. The world-volume theory on the D2 branes is investigated, using its conjectured relation to the near-horizon geometry. The results are in agreement with known facts and expectations for the corresponding field theory and a rich phase structure is obtained as a function of the energy scale and the number of branes. In particular, for an intermediate range of the number of D6 branes, the IR geometry is that of an $AdS_4$ space fibered over a compact space. This D2-D6 system is compared to other systems, related to it by compactification and duality and it is shown that the qualitative differences have compatible explanations in the geometric and field-theoretic descriptions. Another system – that of NS5 branes located at D6 branes – is also briefly studied, leading to a similar phase structure.

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1. Introduction and Summary

In recent years, and especially since the discovery of the so-called AdS-CFT correspondence [1, 2, 3], it has become clear that many aspects of field theories can be studied by looking at the near-horizon region of brane geometries. In his pioneering work [1], Maldacena considered D3, M2 and M5 branes, as well as the D1 - D5 intersection and suggested a novel duality: on the one hand there is the field theory describing the low-energy dynamics of the brane configuration, when it decouples from the bulk stringy degrees of freedom; and it is conjectured to be dual to string/M theory in the geometry near the horizon of the branes. This is a strong-weak duality: whenever one of the dual descriptions is weakly coupled and can be treated semi-classically, the other is strongly coupled and the quantum corrections are large. Specifically, when the number of branes is large (and, in string theory, also when the string coupling is small) the string/M theory in the above cases is well approximated by classical supergravity and this can be used to learn about the dual, strongly-coupled, field theory.

For each of the systems mentioned above, the brane configuration has a near-horizon geometry of the $AdS_{n+1} \times S^m$ form, while the corresponding $n$ dimensional dual field theory is conformally invariant (hence the term AdS-CFT). One piece of evidence for the duality is the match between the isometries of the near-horizon geometry and the symmetries of the field theory. In particular, the isometries of the $AdS_{n+1}$ factor are identified with the conformal transformations in the field theory. The AdS-CFT correspondence continues to hold when the $S^m$ factor in the near-horizon geometry is replaced by another Einstein space $X_m$. The killing spinors on $X_m$ determine the amount of supersymmetry that is realized in the corresponding CFT and spheres lead to the maximally supersymmetric cases. A large class of examples with reduced supersymmetry can be obtained by orbifolding those spheres [4]-[16]. A general classification of possible Einstein spaces $X_m$, as well as the resulting supersymmetries, has been given in [17, 18]. However, as we will see below, this product structure is not the most general geometry that can correspond to a conformal field theory.

The original Maldacena conjecture has been generalized also to non-conformal theories. This was first done in [19], where it was shown how an intricate phase structure emerges for the various branes in type II string theory. There, the distance from the brane was identified with the energy scale in the world-volume field theory and it was found that the 10 dimensional classical geometry associated with the branes can only be trusted for a limited range of gauge theory energies. When going outside this range of validity two things can happen. On the one hand the dilaton may diverge, which signals a transition towards M-theory (for the type IIA branes) or to an S-dual regime (for the type IIB cases); on the other hand, the curvature may become large, in which case the geometric approach breaks down altogether. Fortunately, this precisely happens when the gauge theory becomes weakly coupled. In summary, one finds that for each energy scale there is precisely one of the available dual descriptions that can be treated semi-classically. Nowhere do inconsistencies arise, e.g. due to two dual descriptions being weakly coupled

\footnote{excluding the D3 brane, which corresponds to a \textit{conformal} field theory.}
Although the evidence supporting Maldacena's duality is, by now, quite impressive, there is still much to be understood. In order to extend our understanding, it would be useful to investigate more systems in which this duality exist and can be effectively checked. In this work we follow this route. We concentrate primarily on the D2-D6 system: a system of D2 branes located on or near D6 branes. One of our main results is the phase structure of the corresponding world-volume theory. It is described in figure 1 and we explain it below.

At low energies – i.e. when the D2 brane dynamics decouples from the bulk – the world-volume theory on the branes is described by 3 dimensional $N=4$ supersymmetric gauge theory, with gauge group $SU(N_2)$ and $N_6$ hypermultiplets in the fundamental representation (“quarks”), where $N_2$ and $N_6$ are the number of D2 and D6 branes, respectively. Following Maldacena’s conjecture, this theory is expected to be dual to string theory in the geometry near the horizon of the D2 branes. We argue that the relevant geometry is that of D2 branes localized on the D6 branes. This geometry is known only close enough to the D6 branes \[20\], or very far from them (where their influence is negligible). We find...
that the transition between these two regions can occur within the near-horizon range of the D2 branes and, therefore, it is a transition in the world-volume theory. Indeed, the region close to the D6 branes corresponds to the range of energy scales governed by the IR fixed point, as can be seen by noting that the near-horizon geometry we obtain is \(SO(3,2)\) invariant. We analyze quantitatively the geometric identification of the field-theoretic energy scale, and determine the bound \(E_{IR}\) on the range of the IR fixed point.

Far enough from the D6 branes, where their influence on the geometry can be neglected, one obtains the geometry of isolated D2 branes, which was analyzed in \[19\]. In particular, at a large enough distance from the D2 branes, the classical geometric description stops being valid, due to a large curvature, while the field-theoretical description becomes weakly coupled. Comparing the two transition points \(E_{IR}\) and \(E_{UV}\), one finds, as required, that they obey \(E_{IR} \leq E_{UV}\) whenever the geometric description is valid. In fact, this is a strict inequality, meaning that there is an intermediate phase. One should note that the relevance of the geometry without the D6 branes does not mean that in the world-volume theory, the quarks decouple at high energies. The energy is related to the distance from the D2 branes and at any such distance there are regions close to the D6 brane.

Restricting attention to the IR region, one can consider the dependence on the number of branes. As in all other systems, \(N_2\) must be large to obtain a reliable classical geometric description. Then, changing \(N_6\) (relative to \(N_2\)), the theory goes through several different phases. For small \(N_6\), the weakly coupled description is 11-dimensional and the geometry is \(AdS_4\) times the orbifold \(S_7/\mathbb{Z}_{N_6}\). This system was studied in \[8, 9, 12\]. By increasing \(N_6/N_2\) one goes to a 10-dimensional phase. The geometry still contains an \(AdS_4\) part, but now it is fibered over a 6-dimensional compact base manifold \(X_6\). This is called a warped product \(AdS_4 \times_w X_6\). It was shown some time ago \[21\] that the warped product \(AdS_{n+1} \times_w X_n\) is the most general metric with an \(SO(2, n)\) isometry. Since this isometry is required for a relation to an \(n\)-dimensional conformal field theory, the exploration of such spaces is important in the study of conformal field theories using Maldacena’s duality\(^3\).

When \(N_6 \gg N_2\) the 10-dimensional geometry becomes very highly curved, so we expect that, as in \[19\], this signals a transition towards a weakly coupled phase of the gauge theory. We check this possibility using field-theoretical considerations and present some evidence that for \(N_6 \gg N_2 \gg 1\), the theory indeed simplifies dramatically and may very-well be weakly coupled.

So far we described the phase structure when the D2 branes are on the D6 branes. Moving them off the D6 branes corresponds to a non-zero mass \(m_Q\) for the quarks. We analyze the corresponding geometry as well, and show that it describes correctly the renormalization group flow, from the conformal fixed point with \(m_Q = 0\), as described above, to another fixed point in which the quarks decouple. This second fixed point is

\(^3\)Warped products were also considered, for example, in \[22-28\]. Among those discussed so far in the context of the AdS-CFT duality, the one that was constructed explicitly \[24, 26, 27\] does not preserve any supersymmetry. Moreover, it was shown \[27\] to be unstable and, therefore, unlike the geometries studied in the present work, is not expected to be dual to a conformal field theory.
the IR limit of pure $N = 8$ SYM theory (the world-volume theory of D2 branes with no other branes), which is dual to M theory on $AdS_4 \times S^7$ – the near horizon geometry of M2 branes. For large $N_6$, there is also an intermediate phase, where the M2 branes are smeared over the 11th dimension.

In the above description of the phase structure, we implicitly assumed specific relations between the various parameters of the system, to display all the possible phases. This includes

$$g_s \ll N_2 g_s \ll 1.$$ 

We also assumed that the mass $m_Q$ of the quarks is smaller than the scale $g_{YM}^2$ set by the gauge coupling. Now one can increase the mass. Pictorially, this can be done by moving the horizontal axis in figure 1 up. The right-hand side of the figure will then deform smoothly to become as the left-hand side, reflecting the decoupling of the quarks at an increasingly higher energy. In this context, observe that the order of phases is the same on both sides: localized M2 branes, smeared M2 branes, D2 branes, field theory and, finally, the full string theory.

All the information that we obtain from the geometric description, including the phase structure described above, is in agreement with known facts about the gauge theory and also with many expectations. This is a further strong support for the duality conjecture. One of the old expectations is that, while a gauge theory with fields only in the adjoint representation should be described, in the large $N$ limit, by closed strings, the introduction of fields in the fundamental representation should correspond to the appearance of open strings. This is indeed realized in the present system: the D2 branes are replaced completely by their near-horizon geometry so, without other D branes, the theory is that of closed strings. On the other hand, the D6 branes do not disappear and a corresponding singularity in the geometry is a sign that the open strings ending on the D6 branes exist as additional degrees of freedom.

The structure of this work is as follows. In section 2 we briefly review some relevant information about the gauge theory. In section 3 we analyze the geometry of D2 branes located on D6 branes. We determine the phase structure and consider other issues that relate the geometry and the gauge theory: the UV-IR relation and the potential between quarks. The phase structure is then extended, in section 4, to D2 branes located off the D6 branes. In section 5 we consider the qualitative changes that occur when some of the directions in the geometry are compactified, and the corresponding relations to other systems. In particular, it is explained why in the D1-D5 system, unlike the present one, the geometric description is that of D1 branes smeared over the D5 branes. Appendix A reviews the construction of the geometry near D6 branes. Finally, in Appendix B we briefly analyze the system of NS5 branes located on D6 branes and obtain a phase structure very similar to that of the D2-D6 system.

**Note Added:**

After the completion of this work, the paper [29] appeared, where the question of localization of branes was considered. In accordance with our suggestion here (in section
it is shown there that smearing (delocalization) of branes in a classical supergravity solution corresponds in a dual field-theoretical description to quantum tunneling between classical vacua \[30\][31]. This, in particular, confirms our identification of the geometry of localized D2 branes as the one relevant for the description of the dynamics of the D2 branes.

2. The Field Theory on the D2 Branes

We will consider \(N_2\) D2 branes, extended in the \(x_0, x_1, x_2\) directions. In the absence of other branes, the low energy effective field theory on the D2 branes is 3D \(N = 8\) SYM theory with gauge group \(U(N_2)\). The fields are related to fundamental strings stretched among the D2 branes. The bosonic fields are a gauge field and 7 real scalar fields \(X_3 - X_9\), all transforming in the adjoint representation of \(U(N_2)\). The scalars describe transverse fluctuations of the D2 branes and (the eigenvalues of) their vev’s parametrize the moduli space of vacua.

By adding \(N_6\) D6 branes, extended in the \(x_0 - x_6\) directions, one breaks half of the supersymmetries, leaving \(N = 4\) supersymmetry on the D2 branes. The original \(N = 8\) gauge multiplet decomposes into an \(N = 4\) vector multiplet plus an adjoint hypermultiplet. In addition, one generates a set of hypermultiplets – “quarks” – that originate from strings stretched between a D6 brane and a D2 brane. These hypermultiplets, which we denote generically by \(Q\), transform in the fundamental representation of \(U(N_2)\) and also in the fundamental representation of a global \(U(N_6)\) symmetry. Each of them contains 4 scalars.

2.1 The Moduli Space of Vacua

Classically, the moduli space of vacua decomposes into branches, each being a product of two factors: a “Coulomb” factor and a “Higgs” factor. The “Coulomb” factor is parametrized by scalars belonging to vector multiplets: \(X_7 - X_9\) and the scalar \(X_\#\) which is the dual of the gauge field; the “Higgs” factor is parametrized by scalars belonging to hypermultiplets: \(X_3 - X_6\) and \(Q\). At a generic point in the moduli space, most of the gauge symmetry is broken and the remaining massless excitations are free. The origin of the moduli space is a singular point, where all the branches meet\[3\] and the full gauge symmetry is restored. The gauge coupling constant has a positive mass dimension, therefore the effective (dimensionless) coupling vanishes in the UV (asymptotic freedom) and diverges at low energy, apparently leading to a non-trivial IR fixed point at the origin.

This situation could be modified by quantum corrections, but non-renormalization theorems impose severe restrictions on such corrections (as explained, for example, in \[32\]). In particular, the singularity at the origin of the moduli space is not resolved.

\[3\]This is when all the fundamental hypermultiplets have the same mass, as is the case here, since we consider coinciding D6 branes. The origin of the moduli space corresponds to coinciding D2 branes, embedded in the D6 branes.
For the brane configuration, this means that the gauge theory on the coinciding D2 branes flows in the IR to an interacting conformal field theory. This is the theory which is conjectured to be dual to string theory in the geometry described in section 3.

2.2 Large Number of Flavors

In subsection 32, we will be led to expect that for \( N_6 \gg N_2 \), the field theory has a weakly coupled description. In this subsection we will provide support for this possibility, from field-theoretical considerations. The relevant limit is

\[
N_2 \to \infty \ , \quad g^2 N_2 = \text{const.} \ , \quad \nu \equiv N_6/N_2 = \text{const.} \ , \quad (2.1)
\]

with large \( \nu \). As was shown in 34, in the limit (2.1) the theory simplifies and only a small portion of the Feynman diagrams contributes. Specifically, a Feynman diagram can be shown to define a surface and the leading contribution to a given correlation function comes only from “planar” diagrams – those that define surfaces with the topology of a sphere. Moreover, considering correlation functions of gauge-invariant operators, one finds that all the \( n \)-point functions with \( n > 2 \) vanish in the limit (2.1). In non-supersymmetric QCD, such a behavior, together with the assumption of confinement, was used to argue that in the limit (2.1), the theory is indeed free, the interaction being suppressed by powers of \( 1/\sqrt{N} \) (see 36 and references therein).

In the present model we do not expect confinement and, in fact, the theory is not expected to be free for generic \( \nu = N_6/N_2 \) (since, for small \( \nu \), we will find a weakly coupled geometric dual). One should, therefore, look for something special that happens only for large \( \nu \). In fact, a further simplification does occur in this situation: the leading diagrams are restricted not only to be planar but also to have only “quark loops”, meaning that if the quark loops are contracted to points, one should obtain a tree diagram. This is illustrated in figure 2, where solid lines represent propagators of fields in the fundamental representation of the gauge group (quarks), the dashed ones represent propagators of fields in the adjoint representation and the stars “***” represent external vertices (insertions of gauge invariant operators). The structure of such diagrams is quite simple. In fact, the sum over all possible internal quark loops can be performed exactly (it is essentially a geometric sum), leading to a finite number of effective tree diagrams. This simplification is characteristic to the large \( N \) limit of vector models: models with a global symmetry group \( U(N) \), \( SO(N) \) or \( USp(N) \), in which all the fields are in the vectorial representation of the symmetry (or the trivial one). It typically leads to the conclusion that the theory is free in this limit. Presumably, this is what happens also in the present case, although we will not try to verify this here.

\[ ^4 \text{Note that in 3 dimensions, the gauge coupling } g_{YM} \text{ is not dimensionless. The parameter } g \text{ in (2.1) is the dimensionless ratio } g = g_{YM}/\sqrt{E}, \text{ where } E \text{ is the characteristic energy scale of the process considered.} \]

\[ ^5 \text{Unlike in 4 dimensions, here the number of flavors is not restricted by the requirement of asymptotic freedom, so } \nu \text{ can indeed be large.} \]

\[ ^6 \text{All other planar diagrams are suppressed by powers of } 1/\nu. \]

\[ ^7 \text{One such example is the Gross-Neveu model 35.} \]
3. D2 Branes Localized On D6 Branes

From now on, we turn our attention to the geometry of the D2-D6 system, and investigate its relation to the 3 dimensional gauge theory.

In this section we consider D2 branes localized on the D6 branes. This corresponds, in the field theory, to a vanishing mass for the fundamental hypermultiplets $Q$. We determine the corresponding geometry in 11 and 10 dimension, analyze the range of validity of the descriptions obtained and discuss the phase structure that emerges. We also estimate the entropy and show that it is consistent with smooth transitions between the field-theoretic and geometric phases. Then we consider other issues that relate the geometry and the gauge theory: the UV-IR relation and the potential between “quarks”.

3.1 The Geometry

There are several “D2-D6” geometries that one could try to relate to the field theory described in the previous section and, therefore, it is important to determine which is the correct one. Probably the most familiar one is the configuration in which the D2 branes are smeared along the directions of the D6 branes. However, the corresponding supergravity solution \cite{38}-\cite{41} is not the relevant one in the present context. One indication for this is that the near horizon geometry does not have the $SO(2,3)$ isometry group, necessary for a relation to a 3 dimensional conformal field theory. This issue will be discussed further.

\footnote{For reviews of (intersecting) brane solutions in supergravity, see \cite{42}-\cite{48}}
in section \[\text{5}\] where it will become clear that the correct configuration is that in which the D2 branes are fully localized inside the D6 branes (and of course also in the overall transverse space). The corresponding supergravity solution near the horizon of the D6 branes has been presented in \[\text{20}\] and can be understood by noting the following facts\[\text{9}\]: A set of \(N_6\) coinciding D6 branes corresponds in M theory to a Kaluza-Klein (KK) monopole \[\text{49}\]. The geometry is that of a \(\mathbb{R}^{6+1}\times\text{Taub-NUT}\) space and, close to the center, it is well-approximated by the orbifold \(\mathbb{R}^{6+1}\times\mathbb{R}^4/\mathbb{Z}_{N_6}\). D2 branes correspond in M theory to M2 branes, so to add them, one starts from the flat covering space of the orbifold, in which the M2 brane solution is well-known, and then makes the orbifold identifications.

In the covering space of the orbifold, there are \(N_6N_2\) images of M2 branes. When they all coincide (at the singularity), the corresponding metric for \textit{extremal} branes is \[\text{31}\]

\[
    ds^{2}_{11} = f_2^{-\frac{2}{3}}dx_\parallel^2 + f_2^{\frac{4}{3}}(dr^2 + r^2d\Omega_7^2), \tag{3.1}
\]

where

\[
    f_2 = 1 + \frac{32\pi^2N_6N_2l_p^6}{r^6}, \quad x_\parallel = \{x_0, x_1, x_2\}, \tag{3.2}
\]

and \(d\Omega_7^2\) denotes the metric on the unit 7-sphere. We split the directions transverse to the M2 branes into directions parallel to the singularity (along which in 10 dimensions the D6 branes will be extended) and directions transverse to it. This leads to the following parametrization of the 7-sphere:

\[
    d\Omega_7^2 = d\beta^2 + \cos^2\beta d\Omega_3^2 + \sin^2\beta d\Omega_3^2, \tag{3.3}
\]

The metrics \(d\Omega_3^2\), \(d\Omega_3^2\) describe two unit 3-spheres \(S^3_\parallel, S^3_\perp\), while the additional angle \(0 \leq \beta \leq \frac{\pi}{2}\) measures the orientation with respect to the singularity (\(\beta = 0\) denoting points at the singularity).

The orbifold projection acts only on \(S^3_\perp\), so the orbifold metric is obtained by replacing \(d\Omega_3^2\) by the metric \(d\tilde{\Omega}_3^2\) on \(S^3_\perp/\mathbb{Z}_{N_6}\):

\[
    d\Omega_3^2 \rightarrow d\tilde{\Omega}_3^2 = \frac{1}{4}d\tilde{\Omega}_3^2 + \left[\frac{1}{N_6}d\psi + \frac{1}{2}(1 - \cos \theta)d\varphi\right]^2, \tag{3.4}
\]

where \(0 \leq \theta \leq \pi\) and \(\varphi, \psi\) are periodic with period \(2\pi\)\[\text{10}\].

Next we reduce to 10 dimensions, identifying \(\psi\) as the coordinate of the compact 11th dimension \[\text{20}\]. The 10 dimensional metric \(ds^{2}_{10}\) (in the string frame) and the dilaton \(\phi\) are identified through

\[
    ds^{2}_{11} = e^{4(\varphi - \varphi_\infty)/3}(R_\#d\psi + A_\mu dx^\mu)^2 + e^{-2(\varphi - \varphi_\infty)/3}ds^{2}_{10}, \tag{3.5}
\]

\[\text{9}\]In order to make our paper self-contained, and also in order to establish our notations, these facts are reviewed in Appendix \[\text{A}\].

\[\text{10}\]As described in Appendix \[\text{A}\], the expression \(\frac{1}{4}d\Omega_3^2 + \left[\frac{1}{N_6}d\psi + \frac{1}{2}(1 - \cos \theta)d\varphi\right]^2\) parametrizes the unit 3-sphere when \(\psi\) ranges from 0 to \(2\pi N_6\). The \(\mathbb{Z}_{N_6}\) action leading to an orbifold is given by \(\psi \mapsto \psi + 2\pi\).
where $R_\#$ is the asymptotic radius of the 11th dimension. It is related to the string scale $l_s$ and string coupling $g_s = e^{\phi_\infty}$ by $R_\# = g_s l_s$ (see eq. (A.3)). This gives

$$d s_{10}^2 = e^{2(\phi - \phi_\infty)/3} \left\{ f_2^{-\frac{2}{3}} d x_\parallel^2 + f_2^{\frac{1}{3}} (d r^2 + r^2 d \tilde{\Omega}_6^2) \right\} ,$$

(3.6)

$$d \tilde{\Omega}_6^2 = d \beta^2 + \cos^2 \beta d \Omega_3^2 + \frac{1}{4} \sin^2 \beta d \Omega_2^2$$

and

$$e^\phi = g_s f_2^{\frac{1}{2}} \left( \frac{r \sin \beta}{N_6 g_s l_s} \right)^{\frac{1}{2}} .$$

(3.7)

The Near Horizon Geometry

Near the M2 brane horizon (when $f_2 \gg 1$), the metric (3.1) simplifies to

$$d s_{11}^2 = l_p^2 \left[ \frac{U^2}{(32\pi^2 N_6 N_2)^{\frac{1}{2}}} d x_\parallel^2 + (32\pi^2 N_6 N_2) \left( \frac{d U^2}{4 U^2} + d \tilde{\Omega}_7^2 \right) \right] ,$$

(3.8)

where

$$U = \frac{r^2}{l_p^2}$$

(3.9)

and $d \tilde{\Omega}_7^2$ stands for the metric on $S^7/\mathbb{Z}_{N_6}$, as described above. The metric (3.8) describes the product-space $AdS_4 \times S^7/\mathbb{Z}_{N_6}$, where the radius of the $AdS_4$ is given by

$$R^{(11)}_{AdS} = l_p \left( \frac{\pi^2 N_6 N_2^2}{2} \right)^{\frac{1}{16}} .$$

(3.10)

The relation between M-theory on this $AdS_4 \times S^7/\mathbb{Z}_{N_6}$ space and three dimensional field theory has been considered in [8][9][12].

The 10D geometry becomes

$$d s_{10}^2 = l_s^2 \frac{\sin \beta}{N_6} \left[ \frac{U^2}{\sqrt{32\pi^2 N_6 N_2}} d x_\parallel^2 + \sqrt{32\pi^2 N_6 N_2} \left( \frac{d U^2}{4 U^2} + d \tilde{\Omega}_6^2 \right) \right] ,$$

(3.11)

$$e^\phi = \left( \frac{32\pi^2 N_2}{N_6^5} \right)^{\frac{1}{4}} (\sin \beta)^{\frac{1}{2}} ,$$

(3.12)

where $d \tilde{\Omega}_6^2$ is as in (3.6). Note that $(x_\parallel, U)$ still parametrize an $AdS_4$ space and the dilaton $\phi$ is independent of these coordinates, so this configuration has an $SO(2, 3)$ symmetry. However, the radius of the $AdS_4$ space (as well as the dilaton) depends on the orientation (the angle $\beta$):

$$R^{(10)}_{AdS} = l_s \left( \frac{\pi^2 N_2}{2 N_6} \right)^{\frac{1}{2}} \sqrt{\sin \beta} .$$

(3.13)

\[1\]In the Maldacena or decoupling limit $l_p \to 0$, the coordinate $U$ must be kept fixed in order to obtain a well-defined supergravity action [1].
Therefore the geometry is that of $AdS_4$ fibered over a compact manifold $X_6$ (parametrized by the coordinates of $dS^5[3]$).

As already mentioned at the beginning of this subsection, the emergence of a geometry with $SO(2,3)$ symmetry (which can, therefore, be dual to a conformal field theory) depends crucially on the fact that the D2 branes are localized on the D6 branes. Indeed, for D2 branes smeared in $k$ (compact) dimensions of the D6 branes ($1 \leq k \leq 4$), the corresponding harmonic function is $f_2 \sim r^{6-k}$ and the dilaton is no longer independent of $r$ (see eq. (3.7)). This will be discussed further in section 5.

Another symmetry of the theory on the branes that is realized geometrically is the R-symmetry. These are space-time rotations that leave the M2 brane invariant. In the absence of D6 branes, the R-symmetry is the $Spin(8)$ symmetry of the 7-sphere in eq. (3.1). With D6 branes, the 3D supersymmetry is reduced to $N = 4$ and, correspondingly, the R-symmetry is reduced to $Spin(4)$. In the present description, the D6 branes are represented by the $\mathbb{Z}_N$ orbifold and it breaks the $Spin(8)$ symmetry as follows: the orbifold plane breaks $Spin(8)$ to $Spin(4)_{\parallel} \otimes Spin(4)_{\perp}$, acting on the two 3-spheres in eq. (3.3). Decomposing further: $Spin(4) \equiv SU(2)_L \otimes SU(2)_R$, $\mathbb{Z}_N$ is a subgroup of one of the $SU(2)$ factors of $Spin(4)$, say $SU(2)_{\perp R}$, and it breaks this factor to $U(1)_{\perp R}$. Comparing to eq. (3.4), $SU(2)_{\perp L}$ is the symmetry of the 2-sphere parametrized by $\theta$ and $\varphi$ and $U(1)_{\perp R}$ acts as translations in $\psi$. Now, by analyzing the transformations of the various fields in the field theory under $SU(2)_{\parallel L} \otimes SU(2)_{\parallel R} \otimes SU(2)_{\perp L} \otimes U(1)_{\perp R}$, one identifies $SU(2)_{\parallel R} \otimes SU(2)_{\perp L}$ as the R-symmetry group, while $SU(2)_{\parallel L} \otimes U(1)_{\perp R}$ is a global symmetry.

### 3.2 The Phase Structure

In this subsection we consider the near-horizon geometries of section (3.1) and analyze the changes in the description of the corresponding theory as the parameters are varied. We will see, in particular, that the description changes as we vary the number of flavors relative to the number of colors.

The Classical Approximation and The Dependence on the Number of Branes

We obtained, in the previous subsection, two geometric descriptions – 10 and 11 dimensional – of which only in one the quantum corrections can be expected to be small. The transition between 11 dimensional and 10 dimensional geometries is governed by the radius $R_\psi$ of the circle parametrized by $\psi$ or, equivalently, by the dilaton $\phi$ given in eq. (3.12):

\[
\frac{R_\psi}{l_p} = e^{2\phi/3} = \left( \frac{32\pi^2 N_2}{N_6^5} \right)^{\frac{1}{6}} \sin \beta ,
\]

\[\text{As reviewed in the introduction, such a geometry is also called a “warped” product.}\]
so the geometry is 10 dimensional iff

\[ N_2 \ll N_6^5. \tag{3.14} \]

Another condition for the validity of the classical geometric description is small curvature. In the 11 dimensional geometry (3.8), the curvature \( R^{(11)} \) is

\[ l_p^2 R^{(11)} \sim (N_6 N_2)^{-\frac{1}{4}}, \]

so the classical description is reliable for large \( N_6 N_2 \). In the 10 dimensional geometry (3.11), we obtain

\[ l_p^2 R^{(10)} \sim \sqrt{\frac{N_6}{N_2 \sin^3 \beta}}, \tag{3.15} \]

and it is small away from the D6 branes iff \( N_6 \ll N_2 \). At the location \( \beta = 0 \) of the \( \hat{D}_6 \) branes, the curvature diverges and this singularity is identified as a sign that, in this geometric description, some degrees of freedom related to the D6 branes were effectively integrated out. Therefore, this singularity is expected to be resolved by adding these degrees of freedom\(^{14}\). It is clear what these degrees of freedom are: they are the open strings ending on the D6 branes. This is a realization of the original expectations about the relation between strings and gauge theory, namely, that pure Yang-Mills theory should be described by closed strings, while adding quarks corresponds to adding open strings. Indeed, here the pure SYM theory (with \( N_6 = 0 \)), is conjectured to be dual to type IIA string theory in the background of the near-horizon geometry of D2 branes \[^{19}\] \[^{51}\], which is a theory of closed strings\[^{14}\], while quarks appear as open strings ending on D6 branes. Observe that the D2 and D6 branes behave differently in this context: while the D2 branes “disappeared” and are fully represented by the geometry, the D6 branes remain as D branes, in addition to their influence on the geometry.

To summarize, to obtain a classical geometric description (i.e., with small curvature), \( N_2 \) must be large. When this is satisfied, there are three phases, depending on the relation between \( N_2 \) and \( N_6 \). For \( N_2^\frac{1}{3} \ll N_6 \ll N_2 \) there is a geometric description in 10 dimensions, for smaller \( N_6 \) there is such a description in 11 dimensions, and for larger \( N_6 \) there is no geometric description, as the curvature becomes large. In similar situations considered in the literature (e.g., refs. \[^{19} \][^51]), in the region of large curvature the field

\[^{13}\] Actually, the circle \( S^1 \) parametrized by \( \psi \) is contractible (at the orbifold singularity \( \beta = 0 \)), so one might worry that, like the polar angle in the plane, it cannot be identified as a small compact dimension. To exclude such a possibility, one could require that the radius of the circle \( R_\psi \) is much smaller than the distance \( \delta \) from the contraction point. However, whenever \( R_\psi / l_p \) is small, this condition is automatically satisfied almost everywhere, i.e. except at distances from the contraction point which are much smaller than \( l_p \). In the present case it is easy to see that \( R_\psi / \delta = 1 / N_6 \) so this is indeed small, due to (3.14).

\[^{14}\] The 11 dimensional geometry is also singular at \( \beta = 0 \) (although the singularity is milder – an orbifold singularity) and this is understood in the same way as in 10 dimensions. See also \[^{51}\] for a similar situation.

\[^{15}\] Although, at low energies, the coupling becomes large and the more appropriate description is M theory.
theory became weakly coupled. As was discussed in subsection 2.2, it is plausible that this is what happens also in the present model and for \( N_2 \ll N_6 \), there is a weakly coupled field-theoretical description.

It is interesting to see what the two types of quantum corrections in the 10 dimensional description correspond to in the field theory. The string loop expansion is in \( e^\phi \), holding the geometry fixed. Holding the metric (3.11) fixed means

\[
\nu \equiv \frac{N_6}{N_2} = \text{const.}
\]

and, using eq. (3.12), we see that the expansion is in small

\[
e^\phi \sim \left( \frac{N_2}{N_6} \right)^{\frac{1}{4}} \sim \frac{1}{N_2}.
\]

This is precisely the large \( N \) expansion considered in subsection 2.2 (see eq. (2.1)). Now, from eq. (3.13) we see that the curvature (\( \alpha' \)) corrections of string theory lead to an expansion in small \( \nu \). This is also in agreement with the field theory, as described in subsection 2.2; there, the sub-leading diagrams are suppressed by \( 1/\nu \) (see footnote 6), so \( \nu \) indeed controls the transition between the two descriptions. Finally, we note that \( 1/N_6 \) has in these expansions the same role as \( g_{YM}^2 \) in the consideration of D3 branes (where the expansion parameters are \( 1/N \) and \( 1/g_{YM}^2 N \) respectively, with \( N \) being the number of colors, as \( N_2 \)).

**The Near-Horizon Range**

Next, we analyze the implications of the fact that we consider only the near-horizon region. The near-horizon region of the D2 branes is characterized by the requirement \( f_2 \gg 1 \). Using the eqs. (3.2), (3.9) and (A.3) this translates into the requirement

\[
U^3 \ll \frac{N_6 N_2}{g_{YM}^2 l_s^4},
\]

where

\[
g_{YM}^2 = \frac{g_s}{l_s} = \frac{R_{\#}}{l_p^3}
\]

is the gauge coupling constant in the field theory on the D2 branes. This is the condition for the decoupling of the branes from the bulk and it can be satisfied for any given \( U \), by taking \( l_s \) small enough, while keeping \( g_{YM}^2 \) fixed.

The near-horizon region of the D6 branes is characterized by \( f_6 \gg 1 \), where \( f_6 \) is defined in eq. (A.2). By identifying \( r^2 \sin^2 \beta = l_3^2 U \sin^2 \beta \) with \( \rho^2 \) in eq. (A.3) and, using the relations (A.3), this requirement can be written as

\[
U \ll \left( \frac{N_6}{\sin \beta} \right)^2 g_{YM}^2.
\]
This restriction is non-trivial also for \( l_s \to 0 \), so it should be understood in the framework of the dynamics of D2 branes \textit{decoupled from the bulk, i.e.,} in the framework of the gauge theory. As will be explained below, the coordinate \( U \) is proportional to the energy scale in the field theory, so the restriction (3.18) means that the geometric description we consider corresponds in the gauge theory only to energy scales which are low compared to a scale \( E_{IR} \) set by the gauge coupling \( g_{YM} \) (and \( N_2, N_6 \)). In this context, note that the parameters \( g_s \) and \( R_\# \) do not appear in the geometry described in the previous section, so the geometry is independent of \( g_{YM} \), as expected for energy scales far below the scale set by \( g_{YM} \).

Going further from the D6 branes, their influence on the geometry becomes negligible and the geometry is approximately that of D2 branes in isolation. This system was analyzed in \cite{19}. The geometry is no longer \( SO(3, 2) \)-invariant, so this region corresponds, in the gauge theory, to energy scales above those governed by the IR fixed point. At a sufficiently high energy – in the range of the UV fixed point – the field theoretic description becomes weakly coupled\(^{16}\). The intermediate region, in which the gauge theory is strongly coupled but not conformally invariant, is bounded from above by

\[
E_{UV} \sim N_2 g_{YM}^2 ,
\]

(since the dimensionless parameter \( g_{eff} \) controlling the quantum corrections in the gauge theory is \( g_{eff}^2 = \frac{N_2 g_{YM}^2}{E} \)). To determine the lower bound \( E_{IR} \), as implied by eq. (3.18), we need the geometric identification of the field-theoretic energy scale \( E \). This will be discussed in some detail below (in subsections \( 3.4 \) and \( 4.2 \)), but already at this stage we can make some simple observations. First, realizing that \( U \) is the only dimensionfull quantity in the geometry (3.11), (3.12), we deduce that \( E \) must be proportional to \( U \) (as in any \( AdS \) geometry). Next, consider the large \( N \) limit (2.1). As discussed above, the metric (3.11) remains fixed in this limit. On the other hand, the limit is taken with a fixed energy, so the geometric identification of the energy should depend on \( N_2 \) and \( N_6 \) only through \( \nu \). This leads to the general form\(^{17}\)

\[
E \sim \frac{U}{N_6} \nu^\gamma ,
\]

so the lower bound implied by eq. (3.18) is

\[
E_{IR} \sim \nu^{\gamma + 1} N_2 g_{YM}^2 .
\]

In the next subsection we will find \( \gamma = 0 \) (in processes relevant to fundamental strings) or \( \gamma = \frac{1}{2} \) (in processes relevant to thermodynamics and supergravity fields). For both these identifications one obtains, as required, that \( E_{IR} < E_{UV} \) whenever \( \nu \ll 1 \) (\( i.e., \) in the

\(^{16}\)At the corresponding distances from the D2 branes, the curvature is large and the classical geometrical description is not reliable.

\(^{17}\)This can be seen by observing that when the metric (3.11) is written in terms of \( V = \frac{U}{N_6} \) instead of \( U \), it indeed depends on \( N_2 \) and \( N_6 \) only through \( \nu \).
validity range of the classical geometric description). Note that this is a strict inequality, which means that an intermediate energy region does exist.

Finally we remark that both the restrictions on $U$ discussed above can be removed by taking first $l_s \to 0$ (the decoupling limit of the brane) and then $g_{\text{YM}} \to \infty$ (the IR limit of the gauge theory).

### 3.3 The Entropy and (no) Phase Transition

To check the possibility of a phase boundary between the geometric and non-geometric phases, we calculate here the entropy of the system, when it is at a non-vanishing temperature. For this we should consider non-extremal D2 branes. The non-extremal version of the metric (3.1) is

$$ds^2_{11} = f_2^{-\frac{1}{2}} (-f_h dt^2 + d\vec{x}_\parallel^2) + f_2^{-\frac{1}{2}} (f_h^{-1} dr^2 + r^2 d\tilde{\Omega}_7^2) ,$$

where

$$f_h = 1 - \frac{192\pi^4 l_s^9 \epsilon}{7 r^6} .$$

The parameter $\epsilon$ is the M2 brane tension above extremality and it is interpreted in the field theory as the energy density. Following the derivation in subsection 3.1 we obtain, in the near-horizon region of the D2 branes, the following string metric

$$ds^2_{10} = l_s^2 \sin \beta \left[ \frac{U^2}{\sqrt{32\pi^2 N_6 N_2}} (-f_h dt^2 + d\vec{x}_\parallel^2) + \sqrt{32\pi^2 N_6 N_2} \left(f_h^{-1} \frac{dU^2}{4U^2} + d\tilde{\Omega}_6 \right) \right] ,$$

with

$$f_h = 1 - \left( \frac{U_h}{U} \right)^3 , \quad U_h^3 = \frac{192\pi^4}{7} \epsilon .$$

The dilaton is independent of $\epsilon$ and, therefore, given by eq. (3.12).

In the geometric description, the entropy $S$ is related, by the Bekenstein-Hawking relation, to the area $A$ of the horizon (in the Einstein metric $ds^2_{\text{E}} = e^{(\phi - \phi_\infty)/2} ds^2_{10}$). The horizon is at $U = U_h$ and this gives, for the density of the entropy,

$$s \sim \frac{1}{V\parallel l_{10}^8} \sim (N_6 N_2) \frac{1}{2} U_h^2 \sim (N_6 N_2 \epsilon^2)^{\frac{1}{2}} ,$$

where $V\parallel = \int dx_\parallel^2$ is the (spatial) volume of the D2 brane and $l_{10} = g_s^{\frac{1}{2}} l_s$ is the 10-dimensional Planck length. On the other hand, when the field theory is weakly coupled, the entropy is approximately that of an ideal gas. This gives

$$s \sim (n \epsilon^2)^{\frac{1}{3}} ,$$

where $n$ is the number of degrees of freedom which in the present case is $n \sim N_6^2 + N_6 N_2$. As discussed in subsection 3.2, there are two situations where we expect weakly coupled
field theory. One is \( N_2 \ll N_6 \). At the transition point \( N_2 = N_6 \), the two expressions (3.23) and (3.24) for the entropy agree and this is consistent with a smooth transition between the two descriptions (i.e. with no phase boundary). The other weakly coupled region is at large energy scales, where the effective gauge coupling \( g_{\text{eff}} \) is small. For \( N_6 \ll N_2 \), \( n \sim N_2 \gg N_6 N_2 \) so (3.23) and (3.24) do not agree, however this is still consistent with a smooth transition since, as we saw, these phases are separated by an intermediate phase.

3.4 The UV-IR Relation

It was suggested in [53] (see also [54]) that a UV cutoff in the field theory – a minimal distance \( \delta x \) – is related to an IR cutoff in the geometric description – an upper bound \( U_{\text{max}} \) on the distance from the branes. We will check this suggestion in the present configuration.

The Field Equations

One of the approaches to a UV-IR relation is to consider solutions of the classical field equations. For a massless scalar field \( \phi \), the equation is

\[ \nabla^2 \phi = 0 \quad , \tag{3.25} \]

where \( \nabla^2 \) is the scalar Laplacian. For the metric (3.11), the Laplacian is

\[ \hat{t}_s^2 \sqrt{8\pi^2 \frac{N_2}{N_6}} \sin \beta \nabla^2 = \sin \beta \hat{\nabla}_6^2 + \nabla^2_{\text{AdS}} + 2(\partial_6 \ln \sin \beta) \partial_6 \quad , \tag{3.26} \]

where \( \hat{\nabla}_6^2 \) is the Laplacian of \( d\tilde{\Omega}_6^2 \) and \( \nabla^2_{\text{AdS}} \) is the Laplacian for the \( \text{AdS}_4 \) space with unit radius: \( \nabla^2_{\text{AdS}} = \hat{U}^2 \partial_\hat{U}^2 + 4\hat{U} \partial_\hat{U} + \frac{1}{\hat{U}^2} \partial^\alpha \partial_\alpha \quad , \quad \alpha = 0, 1, 2 \quad , \quad \hat{U} = \frac{U}{\sqrt{8\pi^2 N_6 N_2}} \).

Therefore, substitution of \( \phi = e^{ik \cdot x_\perp} \varphi(x_\perp) \) (where \( x_\perp \) represents the coordinates of the space transverse to the D2 branes) in eq. (3.25) gives the following functional dependence for the solution:

\[ \phi = e^{ik \cdot x_\perp} \varphi(N_6 N_2 \frac{k^2}{U^2}, \Omega) \]

(where \( \Omega \) represents the angles, including \( \beta \)). This suggests [54] the following relation

\[ \frac{1}{\delta x_\parallel} \sim k \sim \frac{U_{\text{max}}}{\sqrt{N_6 N_2}} \mathcal{E}_\phi(\beta) \quad . \tag{3.27} \]

Observe that we included in the above relation a dependence on \( \beta \), through a function \( \mathcal{E}_\phi \). Such a dependence cannot be excluded, since the geometry depends on \( \beta \). However, the present approach does not provide information on this dependence (i.e. on \( \mathcal{E}_\phi \)), since \( \beta \) enters independently in eq. (3.23).

\[^{18}\text{with the metric } ds_{\text{AdS}}^2 = \hat{U}^2 dx_\parallel^2 + \frac{\hat{t}_s^2}{\hat{U}^2} dx_\perp^2.\]
Thermodynamics

Another way to obtain a UV-IR relation is through thermodynamical considerations. For this we consider the geometry of non-extremal D2 branes, as derived in the previous subsection. The Hawking temperature $T$ is given by

$$T = \frac{1}{2\pi} \frac{d}{dU} \left. \frac{G_{tt}}{G_{UU}} \right|_{U=U_h} \sim \frac{d}{dU} \left. \left( \frac{U^2 f_h}{\sqrt{N_6 N_2}} \right) \right|_{U=U_h} \sim \frac{U_h}{\sqrt{N_6 N_2}}$$

(3.28)

and, assuming that the temperature is bounded by the UV cutoff $1/\delta x_\parallel$, we obtain

$$\frac{1}{\delta x_\parallel} \sim \frac{U_{\text{max}}}{\sqrt{N_6 N_2}}.$$  

(3.29)

The dependence on $U, N_6, N_2$ is as in eq. (3.27). Since the horizon is characterized by $U = \text{const.}$, the present approach suggests a UV-IR relation which is independent of $\beta$ (i.e., $\mathcal{E} = \text{const.}$ in eq. (3.27)).

A Fundamental String as a Charge

Finally, we consider a fundamental string with one end on the D2 branes and the other extending to infinity. This string corresponds, in the gauge theory on the branes, to a charge in the fundamental representation of the gauge group and its energy corresponds to the self energy of the charge. This is one example of an energy in the field theory which is related to a (static) string configuration in the geometric description. We will encounter such situations again later. Following [1],[55], we identify the energy $E$ (in the field theory) corresponding to a static string configuration with the (minimal) action of the string per unit time. In the present situation, the action is

$$S = \frac{1}{2\pi l_s^2} \int d\tau d\sigma \sqrt{\det G_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu}$$

so, for a static configuration $X^0 = \tau$, $\vec{X} = \vec{X}(\sigma)$, the corresponding energy is

$$E = \frac{1}{2\pi l_s^2} \int d\sigma \sqrt{G_{00} G_{ij} X^i X^j}.$$  

(3.30)

This is the length of the string in the effective metric

$$dE^2 = \frac{1}{4\pi^2 l_s^4} G_{00} ds^2$$

and, for extremal D2 branes,

$$dE^2 = \left( \frac{\sin \beta}{2\pi N_6} \right)^2 \left[ \frac{U^4}{32\pi^2 N_6 N_2} dx_\parallel^2 + \left( \frac{1}{4} dU^2 + U^2 d\Omega_6^2 \right) \right].$$

(3.31)

\[1\] It is obtained by considering the Euclidean continuation of the metric and demanding a periodicity of Euclidean time that eliminates the conical singularity at the horizon.
Observe that $N_6$ and $N_2$ can be “absorbed” in $dE$ and $x_\parallel$, so one can make the calculation with $N_2 = N_6 = 1$ and then recover $N_2$ and $N_6$ by

$$E \rightarrow N_6 E, \quad x_\parallel \rightarrow \frac{x_\parallel}{\sqrt{N_6 N_2}}.$$ 

We return now to the present case – a fundamental string with one end on the D2 branes and the other extending to infinity – where the corresponding energy is the self-energy $E_F$ of a charge. To obtain a finite energy, we need an IR cutoff in the geometry and a UV cutoff in the field theory, so this will lead to a relation between the two. An IR cutoff in the geometry means that the string extends to a finite point $(U, \Omega)$. By symmetry, for a minimal string, only $U$ and $\beta$ vary, so the effective metric (3.31) leads to

$$E_F = \int \frac{\sin \beta}{2 \pi N_6} \sqrt{\frac{1}{4} dU^2 + U^2 d\beta^2}.$$ 

We see that $E_F$ depends only on $U, \beta$ and $N_6$ and, moreover, $N_6 E_F$ is independent of $N_6$. Therefore, considering dimensions, we obtain for the self energy the following functional form

$$E_F = \mathcal{E}_F \frac{U}{N_6}.$$ 

(3.33)

$E_F$ depends on $\beta$ and, in particular, it vanishes for $\beta = 0$. However, it is not clear how to relate this dependence to the field theory on the D2 branes. The reason for this is that $\beta \neq \frac{\pi}{2}$ corresponds to a string which, if extended from the D2 branes, is not transverse to the D6 branes. This is not a stable configuration and it will “decay” to a string ending on the D6 branes. In fact, in the metric (3.11), the D6 branes totally coincide with the D2 branes (because of the $\sin \beta$ conformal factor), so the string for which we calculated the action may very well be one that ends on the D6 branes and not on the D2 branes. The vanishing of $E_F$ for $\beta \rightarrow 0$ supports this identification. If this is true, we should not identify such a string with a charge in the gauge theory, since this theory describes the dynamics of the D2 branes only. Because of the above complication, we will consider only $\beta = \frac{\pi}{2}$.

To obtain a UV-IR relation, we need the cutoff dependence of the self-energy of a charge in the field theory $E_F \propto \frac{1}{\delta x_\parallel}$. Comparing eqs. (3.29) and (3.33), we obtain the prediction

$$E_F \sim \sqrt{\frac{N_2}{N_6}} \frac{1}{\delta x_\parallel}.$$ 

(3.34)

Comparing to the corresponding result in the dynamics of D3 branes $E_F \sim \frac{\sqrt{N g_{YM}}}{\delta x_\parallel}$ [2], we see again that $1/N_6$ here is analogous to $g_{YM}^2$ there.

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20 This can be realized by placing at this point an additional D2 brane, parallel to the others. The additional D2 brane is considered as a probe and its influence on the geometry is neglected.

21 In particular, it preserves no supersymmetry.

22 Recall that we observed this analogy in the previous subsection, considering the quantum corrections to the geometric description.
Summary

In this subsection, we considered the UV-IR relation from various points of view and obtained the same results as obtained previously for D3 branes [53], [54]: one can identify a relation

\[ \frac{1}{\delta x} \propto U_{\text{max}} \]

between a UV cutoff \( \delta x \) in the gauge theory and an IR cutoff \( U_{\text{max}} \) in the geometry. This cutoff defines two energy scales: one

\[ \frac{1}{\delta x} \sim \frac{U}{\sqrt{N_6 N_2}} \]

relevant to thermodynamics and supergravity fields and another

\[ E_F \sim \sqrt{N_2} \frac{1}{N_6 \delta x} \sim \frac{U}{N_6} \]

relevant to fundamental strings.

3.5 The Potential between Fundamental Charges

A charge in the fundamental representation of the gauge group – a quark – can be realized by an additional D2 brane, considered as a probe, and by stretching a fundamental string between this probe and the \( N_2 \) D2 branes (see footnote 20). A configuration of a quark and an anti-quark can be realized by two such D2 branes. When, in the field theory, the quarks are very far from each other, the energy is the sum of the self energies of the two quarks. Geometrically, this energy corresponds to two fundamental strings (with opposite orientation), each connecting one of the additional D2 branes to the \( N_2 \) D2 branes. When the distance between the fundamental strings is decreased, it may be energetically favorable for the two strings to join to a single string, connecting directly the two D2 branes. In the field theory, this would correspond to an attractive potential between the charges [50] [53]. The potential is given by eq. (3.30), after subtracting the self energy (3.33) of the quarks. We are, therefore, led to consider a string stretched between \((x_1, U, \Omega_1)\) and \((x_2, U, \Omega_2)\), where \( \Omega \) represents the angular coordinates\(^{23}\). Scaling considerations, as in the derivation of eq. (3.33), lead to the following functional form for the energy:

\[ E = \sqrt{\frac{N_2}{N_6}} \frac{1}{\Delta x} \mathcal{E} \left( \frac{U \Delta x}{\sqrt{N_2 N_6}} \right), \]

where the function \( \mathcal{E} \) depends also on the angles. As explained above, we know that for \( \Delta x \to \infty \) (with a fixed cutoff \( U \)) the energy \( E \) remains finite, being the self energy of the

\(^{23}\text{Although everything said below is true for any } \beta, \text{ one should keep in mind that, as explained in subsection 3.4, we expect relevance to the D2 brane dynamics only for } \beta = \frac{\pi}{2}.\)
two quarks. Therefore, the expansion of $E$ for large $U\Delta x$ is:

$$E = \frac{U}{N_6} \mathcal{E}_1 + \sqrt{\frac{N_2}{N_6}} \frac{1}{\Delta x} \left[ \mathcal{E}_0 + \mathcal{O} \left( \frac{1}{U \Delta x} \right) \right].$$

The first term is the self energy (being independent of $\Delta x$ and proportional to $U$). Subtracting it and removing the cutoff (taking $U$ to infinity)\textsuperscript{24}, we obtain for the potential

$$V = \sqrt{\frac{N_2}{N_6}} \frac{1}{\Delta x} \mathcal{E}_0(\Omega_1, \Omega_2). \quad (3.35)$$

To find the function $\mathcal{E}_0$, one has to solve the geodesic equations of the metric \textsuperscript{(3.31)}, which we do not do here. However, much can be deduced already from the present form. For negative $\mathcal{E}_0$, the potential describes a Coulomb-like (attractive) force (as expected from conformal invariance), with an effective charge which depends on the angles. When $\mathcal{E}_0 > 0$, apparently corresponding to a repulsive potential, this actually means that the potential vanishes (classically), the minimal string configuration being the disconnected one – two strings ending on the $N_2$ D2 branes. The vanishing of the potential means that the force between the quarks is screened. From the point of view of the field theory, this is the expected result, since the theory contains massless dynamical matter in the fundamental representation of the gauge group – the excitations of the D2-D6 strings. Screening typically leads to a non-vanishing, exponentially-decreasing potential. In the present geometry such a potential is ruled out by conformal invariance but this only means that we are in the extreme IR, \textit{i.e.}, considering very large distances, where this potential is negligible (see in this context footnote \textsuperscript{24}). A vanishing potential was obtained also in \textsuperscript{57}, and there it was argued that an exponential correction arises as a quantum effect, related to a singularity in the geometry. Here such a correction can appear already classically, when considering higher energy scales, corresponding to regions farther from the D6 branes.

\section{D2 Branes Localized Near D6 Branes}

In this section we extend some of our analysis in the previous section to D2 branes which are at non-vanishing distance from the D6 branes. This corresponds to a non-vanishing mass $m_Q$ for the quarks $Q$. The solution we obtain interpolates between the $m_Q = 0$ solution – far from the D2 branes – and the $AdS_4 \times S^7$ geometry – close to the D2 branes. Since the distance from the brane represents the scale in the gauge theory, this is a geometric realization of the renormalization-group flow\textsuperscript{25} from a conformal fixed point.

\textsuperscript{24}Strictly speaking, the limit $U \to \infty$ is allowed only after taking $l_s \to 0$ and $g_{YM} \to \infty$ (see the end of subsection \textsuperscript{3.3}). Practically, what we want is to suppress the $\mathcal{O} \left( \frac{1}{U \Delta x} \right)$ terms and these will be suppressed also when $U$ is finite but $\Delta x$ is large enough. In physical terms, the restriction on $U$ corresponds to an effective UV cutoff in the field theory and this becomes irrelevant at large distances.

\textsuperscript{25}A geometric realization of an RG flow in field theory was also considered in \textsuperscript{20}, \textsuperscript{22}. 

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with massless quarks to a fixed point without the quarks. We also find, for \( N_6 \gg 1 \) an intermediate region, corresponding to energy scales where massive quarks are relevant. We combine this information with the results of subsection 3.2, to a unified phase structure, summarized in the introduction.

### 4.1 The Geometry

We consider (extremal) coinciding \( M_2 \) branes, positioned at

\[
r = r_Q \ , \quad \beta = \frac{\pi}{2} \ , \quad \theta = 0 \ , \quad \psi = 0 \mod 2\pi ,
\]

where \( r_Q \) is the distance from the singularity. In the covering space of the orbifold, where \( \psi \) ranges from 0 to \( 2\pi N_6 \), we have \( N_6 \) images of \( N_2 \) M2 branes, located at the positions \( \psi = 2\pi k \) for \( k = 1, \ldots, N_6 \) (see Appendix A for more details). The corresponding metric is, therefore,

\[
ds^{2}_{11} = f_2^{-2/3} dx_2^{2} + f_2^{4/3} (dr^2 + r^2 d\Omega_7^2) ,
\]

where

\[
f_2 = 1 + 32\pi^2 N_2 \ell_p^6 \sum_{k=1}^{N_6} \frac{1}{r_k^6} \]

and \( r_k \) is the distance (in the flat covering space) to the \( k \)'th image. In the Cartesian coordinates for the covering space:

\[
z_1 = r \sin \beta \cos \frac{\theta}{2} e^{i \frac{\psi}{N_6}} , \quad z_2 = r \sin \beta \sin \frac{\theta}{2} e^{i (\phi + \frac{\psi}{N_6})} ,
\]

the images are at

\[
z_1 = r_Q e^{\frac{2\pi i k}{N_6}} , \quad z_2 = 0 ,
\]

therefore,

\[
r_k^2 = (r \cos \beta)^2 + |z_2|^2 + |z_1 - r_Q e^{\frac{2\pi i k}{N_6}}|^2
\]

\[
= r^2 + r_Q^2 - 2rr_Q \sin \beta \cos \frac{\theta}{2} \cos \left( \frac{\psi - 2\pi k}{N_6} \right) .
\]

### The Near Horizon Geometry

Near the M2 brane horizon (when \( f_2 \gg 1 \)), the metric \( (4.1) \) simplifies to

\[
ds^{2}_{11} = \ell_p^2 \left[ \frac{U}{(32\pi^2 N_6 N_2)^{\frac{1}{3}}} dx_2^{2} + (32\pi^2 N_6 N_2)^{\frac{1}{3}} \frac{U}{U} \left( \frac{dU^2}{4U^2} + d\tilde{\Omega}_7^2 \right) \right] ,
\]

where

\[
\frac{1}{U^3} = \frac{1}{N_6} \sum_{k} \frac{1}{U_k^3}
\]
and $U$ and $d\tilde{\Omega}_4^2$ are as in (3.8), (3.9). Correspondingly, the 10D geometry becomes

$$
\begin{align*}
    ds_{10}^2 = l_s^2 \frac{\sin \beta}{N_6} \left[ \sqrt{\frac{U^3}{32\pi^2 N_6 N_2}} dx^2 + \sqrt{32\pi^2 N_6 N_2} \left( \frac{U}{U} \right)^{\frac{3}{4}} \left( \frac{dU^2 + d\tilde{\Omega}_4^2}{4U^2} \right) \right],
\end{align*}
$$

(4.6)

$$
    e^\phi = \left( \frac{32\pi^2 N_2}{N_6^3} \right)^{\frac{1}{4}} (\sin \beta)^{\frac{3}{4}} \left( \frac{U}{U} \right)^{\frac{3}{4}},
$$

(4.7)

where $d\tilde{\Omega}_6^2$ is as in (3.6).

4.2 The Phase Structure

The Renormalization Group Flow

The transition between the high and low energy regimes in the field theory occurs at energy scales which depend on the mass $m_Q$ of the quarks and this mass is related to the D2-D6 distance $r_Q$. Thus, before considering the RG flow, we first determine this relation. The quarks $Q$ in the gauge theory are related to the D2-D6 fundamental strings, so the mass $m_Q$ of these quarks is related to the energy carried by these strings. This is again a situation where an energy in the gauge theory corresponds to a string configuration, so we use eq. (3.30) to calculate it. The string extends from the singularity, $U \sin \beta = 0$, to the location of the D2 branes (the horizon): $U = U_Q$, $\sin \beta = \cos \theta = 1$. For a minimal string, only $U$ and $\beta$ vary, thus, eqs. (3.30) and (4.6) give

$$
    m_Q = \int \sin \beta \sqrt{\frac{1}{2\pi N_6} \sqrt{\frac{1}{4} dU^2 + U^2 d\beta^2}}.
$$

This expression is identical to that for the self energy considered in subsection 3.4 (eq. (3.32)), so the answer is given by eq. (3.33) with $U = U_Q$:

$$
    m_Q = \mathcal{E}_F \frac{U_Q}{N_6},
$$

(4.8)

We turn now to the RG flow. At low energies, the quarks $Q$ decouple and the theory flows to the IR fixed point of pure $N = 8$ SYM. In the geometric description, this should correspond to a region close to the M2 branes, where the existence of the D6 branes is irrelevant and the geometry is $AdS_4 \times S^7$. The condition for this is that the sum in eq. (4.4) is well approximated by $1/r_6^6$, where $r_*$ is the distance to the nearest D2 image, and this is the case iff $r_* \ll r_Q/N_6$ (since $r_Q/N_6$ is the distance, in the covering space, between neighboring images). We conclude that the regime of the above IR fixed point is $U_* \ll m_Q/N_6$ (where we define, as usual, $U_* = r_*^2/l_p^3$).

At the other extreme – high energy scales – the mass $m_Q$ is expected to be irrelevant. In the geometric description (eq. (4.4)), $r_Q$ enters through $U$. For $r_* \gg r_Q/N_6$, the sum
in (4.3) is well approximated by an integral, leading to
\[
\left(\frac{U}{U}\right)^3 = \frac{(1 + x^2)^2 + 2x^2y^2}{(1 + x^2 - 4x^2y^2)^{\frac{3}{2}}}, \quad x^2 = \frac{U_Q}{U}, \quad y = \sin \beta \cos \frac{\theta}{2}, \quad U_Q = \frac{r_Q^2}{l_p^3}.
\] (4.9)

Expanding for small \(x\), we obtain
\[
\left(\frac{U}{U}\right)^3 = 1 - 3x^2(1 - 4y^2) + O(x^4)
\] (4.10)

so, the mass is irrelevant when \(x \ll 0\), which means \(N_6 m_Q \ll U \approx U_*\). In this region, the geometry is as for \(m_Q = 0\), which was considered in the previous section. Comparing the two extremes, one observes that for large \(N_6\), there is an intermediate region \(m_Q/N_6 \ll U_* \ll N_6 m_Q\), in which the M2 branes are effectively smeared along the circle parametrized by \(\psi\).

In the field theory, the mass is expected to be negligible whenever it is small compared to the energy considered (i.e., one does not expect factors of \(N_2\) or \(N_6\) in this relation). With this in mind, the geometry suggests that when we consider quarks, the energy scale in the field theory, corresponding to the location \(U_*\), is
\[
E \sim \frac{U_*}{N_6}.
\] (4.11)

Note that for large \(U\), when the geometry is approximated by that of massless quarks, this identification coincides with that suggested in the previous section (see eq. (3.33)).

Turning now to the range of the \(AdS_4 \times S^7\) geometry, the identification (4.11) implies that the condition for the decoupling of the quarks is
\[
E \ll \frac{m_Q}{N_6^2}.
\]

The \(N_6\) dependence is reasonable: the influence of the quarks (as intermediate states) grows with \(N_6\), therefore, the energy at which they decouple gets smaller.

**Reduction to 10 Dimensions**

Next we check the transition between 11 dimensional and 10 dimensional geometries. As before, this is governed by the dilaton \(\phi\), which for the present case is given in eq. (4.7):
\[
\frac{R_{\psi}}{l_p} = e^{2\phi/3} = \left(\frac{32\pi^2 N_2}{N_6^5}\right)^{\frac{2}{3}} \sin \beta \sqrt{\frac{U}{U}}.
\]

For \(r_Q/N_6 \ll r_*, \ i.e., \) when it is valid to approximate the sum in eq. (1.2) by an integral (see eq. (1.3)), the geometry is independent of \(\psi\). This means that
\[
\cos \left(\frac{\psi - 2\pi k_*}{N_6}\right) \approx 1,
\]
so eq. (4.3) gives

$$\left( \frac{r_s}{r_Q} \right)^2 \approx \frac{\delta x^2 + 2(1 + \delta x)\delta y}{(1 + \delta x)^2}, \quad \delta x = x - 1, \quad \delta y = 1 - y,$$

with $x, y$ defined in (4.3). This implies

$$r_s \ll r_Q \iff \delta x^2, \delta y \ll 1.$$

Combining this with eq. (4.9) we obtain, for $r_Q/N_6 \ll r_s \ll r_Q$,

$$\left( \frac{U}{U_*} \right)^3 \approx 6 \left( \frac{r_Q}{2r_s} \right)^5 = \frac{6}{32} \left( \frac{U_Q}{U_*} \right)^{\frac{5}{2}},$$

which leads to

$$\frac{R_\psi}{l_p} \approx \left[ 6\pi^2 N_2 \left( \frac{m_Q}{N_6 U_*} \right)^{\frac{5}{2}} \right]^{\frac{1}{6}}. \quad (4.12)$$

Thus, as $U_*$ is decreased, $R_\psi$ increases. As discussed above, we consider $N_2 \gg 1$, therefore, by the time $U_*$ reaches $m_Q/N_6$ (where the M2 branes are not smeared anymore), $R_\psi/l_p$ is large and the geometry is 11 dimensional.

**Summary**

The phase structure that emerges from the discussion in this subsection and subsection 3.2, is described in figure 1 (in the introduction). It is assumed that $N_2 \gg 1$, otherwise, as we saw, there is no situation with small curvature. The coordinates are $N_6$ and $E = U_*/N_6$ and, as discussed above, we identify $E$ with the energy scale in the field theory.

At scales below $m_Q$ we arrive at the conformal theory of the pure $N = 8$ SYM, corresponding to the geometry $AdS_4 \times S^7$. Going up in energy, we pass through a region where the geometry is still 11 dimensional but the M2 branes are effectively smeared. This region exist only for large $N_6$ and corresponds to a regime in the field theory where the massive quarks have significant influence. At scales above $m_Q$, the mass is negligible and there are the three phases, described in subsection 3.2, depending on the relation between $N_2$ and $N_6$. Going further up in energy, one gets to the scale where the effective gauge coupling $g_{\text{eff}} = \frac{N_2 g_{YM}^2}{E}$ becomes small and there is a weakly coupled field theoretical description. Before this happens, $f_6$ stops being large and the geometric description includes also the geometry of isolated D2 branes. Finally, at a scale determined by the string scale, there is a further transition to an energy regime where stringy dynamics is important.

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26 Recall that this is the identification relevant to processes involving fundamental strings. Note also that this coincides with the identification used in [19] – the action (per unit of time) of a static string.
5 Relations to Other Systems

The D2-D6 system analyzed in this work is related, by compactification and U-duality, to several other systems. There are qualitative differences between some of these systems and in this section we check how these differences arise in the geometric and field-theoretic descriptions. We find satisfactory agreement and this should be considered as a (successful) consistency check of the proposed AdS-CFT duality. It is important to keep in mind that duality is, by definition, a relation between two descriptions of the same system, so U-duality cannot account for the differences mentioned above. The essential change is, therefore, compactification (in one of the dual descriptions).

Compactification and the RG Flow

We start by compactifying one of the directions of the D2 brane. The corresponding field theory is on $\mathbb{R}^2 \times S^1$ and it flows from a 3 dimensional theory in the UV to a 2 dimensional theory in the IR. In the geometric description, taking $x_2$ to be periodic with period $2\pi R_2$, the physical compactification radius is $\sqrt{G_{22}} R_2$. Since $G_{22} \sim U^2$ (see eq. (3.11)) and $U$ is proportional to the energy scale in the field theory, we indeed obtain a 3 dimensional world-volume at high energies and a 2 dimensional one at low energies. Note that we only used the fact that the metric components in directions parallel to the world-volume increase with $U$, which is true in all AdS geometries, so this correct realization of the RG flow will exist in any AdS-CFT relation.\(^{27}\)

The D1-D5 System and Localization

Performing T-duality in the (compactified) $x_2$ direction, one obtains D1 branes localized on D5 branes. However, the geometry obtained does not have the SO(2,2) symmetry required to identify it with a conformal theory on the D1 branes. Instead, such a geometry is obtained, when the D1 branes are smeared over the D5 branes. In the D2-D6 system, the situation is reversed and, as we saw in section 3, a geometry with SO(2,3) symmetry is obtained only when the D2 branes are fully localized on the D6 branes. This difference has a clear counterpart in field theory. For localized branes, there is a geometric moduli space of their possible positions, which corresponds, in the field theory, to the (classical) moduli space of vacua. In three dimensions (and more), the vacuum of a field theory corresponds to specific values of the moduli and, therefore, the corresponding geometry should be that of localized branes\(^{28}\). In two dimensions (and less), there is tunneling between continuously-connected classical ground states, leading to a single quantum ground state and this is reflected in the geometry by smearing. Note that the smearing is in directions parallel to the D5 branes, which correspond to the Higgs branch of the moduli space and there is no smearing in the other directions, corresponding to the Coulomb branch. This is in agreement with other evidence that the corresponding

\(^{27}\)Moreover, it also appears to exist in non-conformal Dp-branes analyzed in \cite{19}, although there the geometric identification of the energy scale is more problematic, as discussed in \cite{54}.

\(^{28}\)Continuously smeared branes were considered in \cite{58}, but this was meant as an approximation to a specific distribution of branes, which means, a specific point in the moduli space of vacua.
world-volume theory is in the Higgs branch.

**The D3-D7 System and a Limit on the Number of Flavors**

Next, we consider a compactification of a direction transverse to the D6 branes. This leaves two non-compact transverse directions and in such a situation, the number $N_6$ of D6 branes is limited. Such a bound exists also in the field-theoretical description: T-dualizing along the compact direction, one obtains D3 branes localized on D7 branes, with one of the D3 brane world-volume directions compactified on a dual circle. The corresponding field theory is a 4 dimensional gauge theory on $\mathbb{R}^3 \times S^1$. The number $N_6$ of D7 branes is the number of flavors in the gauge theory and requiring asymptotic freedom, one obtains a bound on $N_6$. To remove the bound, the radius of the dual circle must strictly vanish (since asymptotic freedom is a property of the small distance behavior) and this corresponds to D6 branes with 3 non-compact transverse directions. Therefore, we see that both approaches lead to the same kind of limit on $N_6$. To obtain a detailed agreement, further considerations are needed\[^{29}\] and this will not be done here.

**A Compactified Orbifold and Mirror Symmetry**

Finally, we consider the compactification of D6 directions which are transverse to the D2 brane. To be specific, let $x^6$ be periodic, with period $2\pi R_6$. When $R_6 \ll l_p$, one can return to 11 dimensions and identify $x^6$ as the small 11th dimension. This leads to a configuration of $N_2$ D2 branes on a $\mathbb{Z}_{N_6}$ orbifold. The field theory on the D2 branes is $\prod_{\alpha=1}^{N_6} U(N_2)_{\alpha}$ gauge theory with matter in the $(N_2, N_2)$ representation of $U(N_2)_{\alpha} \otimes U(N_2)_{\alpha+1}$ for each $\alpha$ (with cyclic identification $\alpha \sim \alpha + N_6$) and an additional fundamental for one of the $U(N_{\alpha})$ factors \[^{60}\]. Thus, for large $N_6$ and small $R_6$ there are two field theoretical descriptions of the same system and indeed, it is known that these two field theories are related by mirror symmetry \[^{32, 33}\].

**A Compact D6 brane**

When all the D6 directions (which are transverse to the D2 branes) are compactified, the dynamics of the D6 branes cannot be ignored and the corresponding field theory is a $U(N_2) \otimes U(N_6)$ gauge theory. The $U(N_6)$ gauge coupling is inversely proportional to $V^{-\frac{1}{2}}$ (where $V$ is the compactification radius) so, for infinite $V$, it vanishes and the $U(N_6)$ symmetry is a global symmetry. This seems as a smooth limit, but actually, there is a conceptual difference between global and gauge symmetries, the former being a real symmetry, while the later is a redundancy in the description. In the string theory context, this sharp distinction is reflected by the fact that global symmetries in the brane’s world-volume theory correspond to gauge symmetries (and gauge fields) in the bulk, while gauge symmetries on the brane are not seen in the bulk. We, therefore, return to our model and check how this distinction is seen.

To obtain the geometry, one starts, as before, in the (11 dimensional) covering space, and considers an array of M2 branes located at the points of the lattice $\Gamma$ defined by the

\[^{29}\text{Note that a straight-forward application of the asymptotic freedom requirement in the present case would rule out any } N_6 > 0.\]
compactification (i.e., $\mathbb{R}^4/\Gamma$ is the compactification torus). The corresponding metric is

$$ds_{11}^2 = f_2^{-\frac{2}{3}} dx^2_\parallel + f_2^{\frac{4}{3}} dr^2, \quad f_2 = 1 + 32\pi^2 N_6 N_2 l_p^6 \sum_{k \in \Gamma} \frac{1}{r_k^6}, \tag{5.13}$$

where $dr^2$ is the (flat) metric of the transverse covering space, and $r_k$ is the distance to the lattice point $k$. Following the usual procedure, we obtain the 10 dimensional near-horizon geometry

$$ds_{10}^2 = \frac{l_s^2}{N_6} \sqrt{\frac{U}{U}} \sin \beta \left[ \frac{U^2}{\sqrt{32\pi^2 N_6 N_2}} dx^2_\parallel + \sqrt{32\pi^2 N_6 N_2 U} \left( \frac{dU^2}{4U^2} + d\tilde{S}_0^2 \right) \right], \tag{5.14}$$

$$e^\phi = \left[ \frac{32\pi^2 N_2}{U N_6^5} \left( \frac{U}{U} \right)^3 \sin^6 \beta \right]^{\frac{1}{2}}, \tag{5.15}$$

where

$$U = \frac{r^2}{l_s^2}, \quad \frac{1}{U^3} = \sum_k \frac{1}{U_k^3}. \tag{5.16}$$

An $AdS$ geometry is obtained only for $r$ small compared to the compactification length’s, (when $\overline{U} \approx U$), which corresponds to low energy scales in the world-volume theory. This agrees with field-theoretical expectations, since at larger scales, the 6-dimensional physics is no longer negligible. The $AdS$ geometry (for small $U$) is locally the same as for the non-compact D6 branes. In particular, there is a singularity at $\beta = 0$ and, as always, one expects that it corresponds to additional massless degrees of freedom living there. For non-compact D6 branes we identified them as the open strings ending on these branes, and the implied $U(N_6)$ gauge symmetry in the bulk agreed with the global symmetry in the world-volume theory. However, in the present situation, the $U(N_6)$ symmetry is a gauge symmetry in the world-volume theory, so the singularity must have a different origin. To find what it is, one observes that the length of the compact directions vanishes as $\beta$ is decreased to 0 (because of the $\sin \beta$ factor in the metric), so the massless degrees of freedom at the singularity are closed strings winding around these directions. When only part of the 4 extra dimensions of the D6 branes are compactified, the singularity is due to both closed and open strings. In all these cases we have agreement between the global symmetry in the world-volume theory and the gauge symmetry in the bulk.

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\[30\] We use here, for the transverse space, the same spherical coordinates as in the previous section. To analyze distances larger than the compactification lengths, one should use other coordinates, but this will not be done here explicitly.

\[31\] In this context, “bulk” means “off the D2 branes”; of course, the $U(N_6)$ gauge fields are on the D6 branes and not in the 10 dimensional bulk.
Appendix A. The Near-Horizon Geometry of Coinciding D6 Branes

D6 branes are identified in M theory with KK monopoles [49]. The corresponding transverse metric – the Taub-NUT metric – for $N_6$ coinciding D6 branes is

$$ds_{TN}^2 = f_6[dr_6^2 + r_6^2d\Omega_2^2] + f_6^{-1}R_#^2[d\psi + \frac{1}{2}N_6(1 - \cos \theta)d\varphi]^2,$$  \hspace{1cm} (A.1)

where

$$f_6 = 1 + \frac{N_6R_#}{2r_6}, \quad R_# > 0,$$ \hspace{1cm} (A.2)

d$\Omega_2^2$ is the metric on the unit 2-sphere

$$d\Omega_2^2 = d\theta^2 + \sin^2 \theta d\varphi^2,$$

$(0 \leq \theta \leq \pi; \varphi$ is periodic with period $2\pi)$ and the angle $\psi$ parametrizes the compact Taub-NUT direction, normalized to have a periodicity of $2\pi$.

For large $r_6$ (far from the D6 branes; where $f_6 \approx 1$), the 11D metric approaches that of flat $\mathbb{R}_{6+1}^4 \times \mathbb{R}^4 \times S^1$, $R_#$ being the asymptotic radius of the compact dimension. M theory on this space is identified with type IIA string theory [61], and the string scale $l_s$ and the string coupling $g_s$ are given in terms of the asymptotic radius $R_#$ and the 11-dimensional Planck length $l_p$ by

$$l_s^2 = \frac{l_p^3}{R_#^3}, \quad g_s^2 = \left(\frac{R_#}{l_p}\right)^3.$$ \hspace{1cm} (A.3)

For small $r_6$ (near the D6 branes; where $f_6 \gg 1$), the transverse space (A.1) becomes an ALE-space with $A_{N_6-1}$ singularity [62]. The corresponding metric reads

$$ds_{ALE}^2 = d\rho^2 + \rho^2d\tilde{\Omega}_3^2,$$ \hspace{1cm} (A.4)

where

$$\rho^2 = 2N_6R_#r_6,$$ \hspace{1cm} (A.5)

and

$$d\tilde{\Omega}_3^2 := \frac{1}{4}d\Omega_2^2 + \left[\frac{1}{N_6}d\psi + \frac{1}{2}(1 - \cos \theta)d\varphi\right]^2.$$  \hspace{1cm} (A.6)

This metric describes an orbifold $\mathbb{R}^4/\mathbb{Z}_{N_6}$. To demonstrate this, one changes coordinates

$$\theta = 2\hat{\theta}, \quad \varphi = \hat{\varphi}_2 - \hat{\varphi}_1, \quad \psi = N_6\hat{\varphi}_1,$$ \hspace{1cm} (A.6)

obtaining

$$d\tilde{\Omega}_3^2 := d\hat{\theta} + \cos^2 \hat{\theta}d\hat{\varphi}_1^2 + \sin^2 \hat{\theta}d\hat{\varphi}_2^2.$$ \hspace{1cm} (A.7)
Extending the periodicity of $\psi$ to $2\pi N_6$, one obtains that $0 \leq \hat{\theta} \leq \frac{\pi}{2}$ and $\hat{\varphi}_i$ are periodic with period $2\pi$. So, defining

$$z_1 = \rho \cos \hat{\theta} e^{i\hat{\varphi}_1}, \quad z_2 = \rho \sin \hat{\theta} e^{i\hat{\varphi}_2},$$

the metric (A.4) becomes that of flat $\mathbb{C}^2$:

$$ds^2 = |dz_1|^2 + |dz_2|^2.$$ 

Since the range of $\psi$ is $2\pi$ (and not $2\pi N_6$) $\mathbb{C}^2$ covers our space $N_6$ times, with the following identifications:

$$(z_1, z_2) \sim (\alpha z_1, \alpha z_2) \quad \forall \alpha^{N_6} = 1. \quad (A.9)$$

**Appendix B. NS5 Branes Localized On D6 Branes**

The NS5 branes decouple from the stringy bulk degrees of freedom when the string coupling $g_s$ vanishes [63].

The resulting world-volume theory is expected to be a non-local 6 dimensional theory without gravity [63]. It has stringy excitations (which can be thought of as boundaries of D2 branes ending on the NS5 branes) and is called “little string theory”. The tension of these strings is proportional to $1/l_s^2$. In the absence of D6 branes, the theory has (2,0) supersymmetry. It was analyzed, in the context of the AdS-CFT correspondence, in [19][64]. The D6 branes break the supersymmetry to (1,0) and add 3 dimensional degrees of freedom – the boundaries of D4 branes extending between the NS5 branes and the D6 branes.

In this appendix we consider briefly the world-volume theory of NS5 branes localized on D6 branes, as described above. One can repeat most of the analysis of the D2-D6 system, performed in the main text, also for the D5-D6 system and very similar results are obtained. In particular, also here the geometry is that of a warped product. So far, there is not much that is known about the world-volume theory of decoupled NS5 branes, therefore, the information provided by the present approach, as sketched below, should be useful in the further study of this system.

**The Geometry**

The geometry near the horizon of D6 branes, in the presence of NS5 branes was determined in [20]. To reproduce it, we follow the same steps as in section 3. For simplicity, we consider extremal NS5 branes on the D6 branes. A NS5 brane corresponds to an M5 brane in 11 dimensions. The metric corresponding to $N_6 N_5$ coinciding images of M5 branes in the (flat) covering space of the orbifold is

$$ds^2_{11} = f_5^{-\frac{1}{3}} dx_\parallel^2 + f_5^{\frac{2}{3}} (dr^2 + r^2 d\Omega_4^2), \quad (B.1)$$
\[ f_5 = 1 + \frac{\pi N_6 N_5 \ell_5^3}{r^3}, \quad d\Omega_4^2 = d\beta^2 + \sin^2 \beta d\Omega_{3\perp}^2, \quad (B.2) \]

where \( d\Omega_4^2 \) is the metric on the unit 4-sphere, parametrized using the metric \( d\Omega_{3\perp}^2 \) of a unit 3-sphere and an additional angle \( 0 \leq \beta \leq \pi \). The orbifold metric is obtained, as in section 3, by replacing \( d\Omega_{3\perp}^2 \) by the metric \( d\tilde{\Omega}_3^2 \) on \( S^3/\mathbb{Z}_N \) (see eq. (3.4)). The reduction to 10D gives

\[ ds_{10}^2 = e^{2(\phi - \phi_\infty)/3} \left\{ f_5^{-\frac{1}{3}} dx_\parallel^2 + f_5^{\frac{2}{3}} \left[ dr^2 + r^2 (d\beta^2 + \frac{1}{4} \sin^2 \beta d\Omega_2^2) \right] \right\} (B.3) \]

and

\[ e^\phi = g_s f_5^{\frac{1}{3}} \left( \frac{r \sin \beta}{\ell_6 R_{\#}} \right)^{\frac{2}{3}}. (B.4) \]

Near the M5 brane horizon (when \( f_5 \gg 1 \)), the metric (B.1) simplifies to

\[ ds_{11}^2 = l_p^2 \left[ \frac{U^2}{\pi N_6 N_5} dx_\parallel^2 + \left( \pi N_6 N_5 \right)^{\frac{2}{3}} \left( 4 \frac{dU^2}{U^2} + d\tilde{\Omega}_4^2 \right) \right], \quad U^2 = \frac{r}{\ell_p^2} (B.5) \]

where \( d\tilde{\Omega}_4^2 \) is the metric on \( S^4/\mathbb{Z}_N \), as described above. This is the space \( AdS_7 \times S^4/\mathbb{Z}_N \).

The 10D geometry becomes

\[ ds_{10}^2 = \frac{l_s^2}{N_6} \sin \beta \left[ U^2 dx_\parallel^2 + \pi N_6 N_5 \left( 4 \frac{dU^2}{U^2} + d\beta^2 + \frac{1}{4} \sin^2 \beta d\Omega_2^2 \right) \right], \quad (B.6) \]

\[ e^\phi = \sqrt{\frac{N_5}{N_6}} \sin \beta. (B.7) \]

This is the geometry of an \( AdS_7 \) fibered over a compact manifold \( X_3 \) (parametrized by \( \beta \) and the coordinates of \( d\Omega_2^2 \)), where the radius of the \( AdS_7 \) space depends on orientation (the angle \( \beta \))

\[ R_{AdS}^{(10)} = 2l_s \sqrt{\frac{N_5}{N_6}} \sin \beta. (B.8) \]

**The Phase Structure**

The dependence on the number of branes is qualitatively the same as in the D2-D6 system: to obtain a classical geometric description (i.e., with small curvature), \( N_5 \) must be large. When this is satisfied, there are three phases, depending on the relation between \( N_5 \) and \( N_6 \). For \( \sqrt{N_5} \ll N_6 \ll N_5 \) there is a geometric description in 10 dimensions and for smaller \( N_6 \) there is such a description in 11 dimensions. For larger \( N_6 \) there is no geometric description, possibly indicating a simplification in some non-geometric description. It would be interesting to consider this limit in other approaches to this system.
Considering the dependence on energy, one first observes that, as in any conformally-invariant geometry, the energy scale is proportional to $U$. The restriction to the near-horizon range of the NS5 branes implies

$$U^6 \ll \frac{N_6 N_5}{g_s^{16/3}} ,$$

which is satisfied in the decoupling limit $g_s \rightarrow 0$, so also in this case the near horizon condition has the same meaning as in other systems. The restriction to the near-horizon range of the D6 branes implies

$$U^2 \ll \frac{N_6}{l_s^2 \sin \beta} ,$$

which is, as in the D2-D6 system, an independent condition (i.e. non-trivial for $g_s \rightarrow 0$). Its meaning is also the same: it determines the domain of the IR fixed point which is, as expected, bounded by the (little) string scale. Finally, one can take the NS5 branes off the D6 branes and realize an RG flow between two different fixed points, as was done for the D2-D6 system.

**References**

[1] J.M. Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity*, Adv.Theor.Math.Phys. 2 (1998) 231, [hep-th/9711200].

[2] S.S. Gubser, I.R. Klebanov, A.M. Polyakov, *Gauge Theory Correlators from Non-Critical String Theory*, Phys.Lett. B428 (1998) 105, [hep-th/9802109].

[3] E. Witten, *Anti De Sitter Space And Holography*, Adv.Theor.Math.Phys. 2 (1998) 253, [hep-th/9802150].

[4] S. Kachru, E. Silverstein, *4d Conformal Field Theories and Strings on Orbifolds*, Phys.Rev.Lett. 80 (1998) 4855, [hep-th/9802183].

[5] M. Berkooz, *A Supergravity Dual of a (1,0) Field Theory in Six Dimensions*, Phys.Lett. B437 (1998) 315, [hep-th/9802193].

[6] S. Ferrara, A. Zaffaroni, *N=1,2 4D Superconformal Field Theories and Supergravity in AdS\(_5\)*, Phys.Lett. B431 (1998) 49, [hep-th/9803060].

[7] A. Lawrence, N. Nekrasov, C. Vafa, *On Conformal Theories in Four Dimensions*, Nucl.Phys. B533 (1998) 199, [hep-th/9803017].

[8] S. Ferrara, A. Kehagias, H. Partouche, A. Zaffaroni, *Membranes and Fivebranes with Lower Supersymmetry and their AdS Supergravity Duals*, Phys.Lett. B431 (1998) 42, [hep-th/9803109].
[9] J. Gomis, *Anti de Sitter Geometry and Strongly Coupled Gauge Theories*, Phys.Lett. B435 (1998) 299, hep-th/9803119.

[10] Y. Oz, J. Terning, *Orbifolds of AdS$_5 \times S^5$ and 4d Conformal Field Theories*, Nucl.Phys. B532 (1998) 163, hep-th/9803167.

[11] E. Halyo, *Supergravity on AdS$_{5/4} \times$ Hopf Fibrations and Conformal Field Theories*, hep-th/9803193.

[12] R. Entin, J. Gomis, *Spectrum of Chiral Operators in Strongly Coupled Gauge Theories*, Phys.Rev. D58 (1998) 105008, hep-th/9804060.

[13] C. Ahn, K. Oh, R. Tatar, *Orbifolds AdS$_7 \times S^4$ and Six Dimensional (0, 1) SCFT*, Phys.Lett. B442 (1998), hep-th/9804093.

[14] S. Gukov, *Comments on N=2 AdS Orbifolds* Phys.Lett. B439 (1998) 23, hep-th/9806180.

[15] D.-E. Diaconescu, J. Gomis, *Neveu-Schwarz Five-Branes at Orbifold Singularities and Holography*, hep-th/9810132.

[16] S. Gukov, M. Rangamani, E. Witten, *Dibaryons, Branes, And Strings In AdS Orbifold Models*, hep-th/9811048.

[17] B.S. Acharya, J.M. Figueroa-O’Farrill, C.M. Hull, B. Spence, *Branes at conical singularities and holography*, hep-th/9808014.

[18] D.R. Morrison, M.R. Plesser, *Non-Spherical Horizons I*, hep-th/9810201.

[19] N. Itzhaki, J.M. Maldacena, J. Sonnenschein, S. Yankielowicz, *Supergravity and The Large N Limit of Theories With Sixteen Supercharges*, Phys.Rev. D58 (1998) 46004, hep-th/9802042.

[20] N. Itzhaki, A.A. Tseytlin, S. Yankielowicz, *Supergravity solutions for branes localized within branes*, Phys.Lett. B432 (1998) 298, hep-th/9803103.

[21] P. van Nieuwenhuizen, Les Houches 1983 lectures, B.S. de Witt, R. Stora eds., North Holland.

[22] B. de Wit, H. Nicolai, *A New SO(7) Invariant Solution of d = 11 Supergravity*, Phys.Lett. 148B (1984) 60.

[23] P. Van Nieuwenhuizen and N. Warner, *New Compactifications of Ten Dimensional and Eleven Dimensional Supergravity on Manifolds Which Are Not Direct Products*, Comm.Math.Phys. 99 (1985) 141.
[24] M. Günaydin, L.J. Romans and N.P. Warner, *Compact and Noncompact Gauged Supergravity Theories in Five Dimensions*, Nucl.Phys. B272 (1986) 598.

[25] C.M. Hull and N.P. Warner, *Noncompact Gaugings From Higher Dimensions*, Class.Quant.Grav. 5 (1988) 1517.

[26] L. Girardello, M. Terini, M. Porrati, A. Zaffaroni, *Novel Local CFT and Exact Results on Perturbations of N = 4 Super Yang Mills From AdS Dynamics*, hep-th/9810126.

[27] J. Distler, F. Zamora, *Non-supersymmetric Conformal Field Theories From Stable Anti-de Sitter Spaces*, hep-th/9810206.

[28] A. Khavaev, K. Pilch, N.P. Warner, *New Vacua of Gauged N=8 Supergravity*, hep-th/9812035.

[29] D. Maldorf and A. Peet, *Brane Baldness vs. Superselection Sectors*, hep-th/9903213.

[30] N.D. Mermin and H. Wagner, Phys. Rev. Lett 17 (1966) 1133.

[31] S. Coleman, Commun. Math. Phys. 31 (1973) 259.

[32] K. Intriligator, N. Seiberg, *Mirror Symmetry in Three Dimensional Gauge Theories*, Phys.Lett. B387 (1996) 513, hep-th/9607207.

[33] J. de Boer, K. Hori, H. Ooguri, Y. Oz, *Mirror Symmetry in Three-Dimensional Gauge Theories, Quivers and D-branes*, Nucl.Phys. B493 (1997) 101, hep-th/9611063.

[34] G. ’t Hooft, *A Planar Diagram Theory For Quark Confinement*, Nucl. Phys. B72 (1974) 461.

[35] G. Veneziano, *Some Aspects Of A Unified Approach To Gauge, Dual And Gribov Theories*, Nucl.Phys. B117 (1976) 519.

[36] E. Witten, *Baryons In The 1/N Expansion*, Nucl.Phys. B160 (1979) 57.

[37] D.J. Gross, A. Neveu, *Dynamical Symmetry Breaking In Asymptotically Free Field Theories*, Phys.Rev. D10 (1974) 3235.

[38] G. Papadopoulos, P.K. Townsend, *Intersecting M-branes*, Phys.Lett. B380 (1996) 273, hep-th/9603087.

[39] A.A. Tseytlin, *Harmonic superpositions of M-branes*, Nucl.Phys. B475 (1996) 149, hep-th/9604035.

No-force condition and BPS combinations of p-branes in 11 and 10 dimensions, Nucl.Phys. B487 (1997) 141, hep-th/9609212.

[40] J.P. Gauntlett, D.A. Kastor and J. Traschen, *Overlapping Branes in M-Theory*, Nucl.Phys. B478 (1996) 544, hep-th/9604179.
[41] R. Argurio, F. Englert, L. Houart, *Intersection Rules for p-Branes*, Phys.Lett. **B398** (1997) 61, hep-th/9701042.

[42] M.J. Duff, R.R. Khuri, J.X. Lu, *String solitons*, Phys. Rept. **259** (1995) 213, hep-th/9412184.

[43] K. S. Stelle, *Lectures on Supergravity p-branes*, in "Trieste 1996, High energy physics and cosmology" 287, hep-th/9701088;
*BPS Branes in Supergravity*, in "Trieste 1997, High energy physics and cosmology" 29, hep-th/9803110.

[44] J.P. Gauntlett, *Intersecting branes*, lectures APCTP Winter School on Dualities of Gauge and String Theories, Feb 1997, Seoul and Sokcho, Korea, hep-th/9705011.

[45] D. Youm, *Black Holes and Solitons in String Theory*, hep-th/9710046.

[46] P.K. Townsend, *M-theory from its superalgebra*, Cargese lectures 1997, hep-th/9712009.

[47] R. Argurio, *Brane Physics in M-theory*, PhD thesis, hep-th/9807171.

[48] K. Skenderis, *Black holes and branes in string theory*, Erice lectures 1998, hep-th/9901050.

[49] P.K. Townsend, *The Eleven-dimensional Supermembrane Revisited*, Phys.Lett. **B350** (1995) 184, hep-th/9501068.

[50] M.J. Duff, K.S. Stelle, *Multimembrane Solutions Of D = 11 Supergravity*, Phys.Lett. **B253** (1991) 113.

[51] O. Aharony, A. Fayyazuddin, J. Maldacena, *The Large N Limit of \( \mathcal{N} = 2 \) Field Theories from Threebranes in F-theory*, JHEP 9807 (1998) 013, hep-th/9806159.

[52] I.R. Klebanov, A.A. Tseytlin, *Entropy of Near-Extremal Black p-branes*, Nucl.Phys. **B475** (1996) 164, hep-th/9604089.

[53] L. Susskind, E. Witten, *The Holographic Bound in Anti-de Sitter Space*, hep-th/9805113.

[54] A.W. Peet, J. Polchinski, *UV/IR Relations in AdS Dynamics*, hep-th/9809022.

[55] J. M. Maldacena, *Wilson loops in large N field theories*, Phys.Rev.Lett. **80** (1998) 4859, hep-th/9803002.

[56] S.-J. Rey, J. Yee, *Macroscopic Strings as Heavy Quarks of Large N Gauge Theory and Anti-de Sitter Supergravity*, hep-th/9803001.
[57] D.J. Gross, H. Ooguri, *Aspects of large N Gauge Theory Dynamics as seen by String Theory*, hep-th/9805129.

[58] P. Kraus, F. Larsen, S.P. Trivedi, *The Coulomb Branch of Gauge Theory from Rotating Branes*, hep-th/9811120.

[59] E. Witten, *On The Conformal Field Theory Of The Higgs Branch*, JHEP 9707 (1997) 003, hep-th/9707093.

[60] M.R. Douglas, G. Moore, *D-branes, Quivers, and ALE Instantons*, hep-th/9603167.

[61] E. Witten, *String Theory Dynamics In Various Dimensions*, Nucl.Phys. B443 (1995) 85, hep-th/9503124.

[62] G.W. Gibbons, S.W. Hawking, *Gravitational Multi-Instantons*, Phys.Lett. B78 (1978) 430.

[63] N. Seiberg, *Matrix Description of M-theory on T^5 and T^5/Z_2*, Phys.Lett. B408 (1997) 98, hep-th/9705221.

[64] O. Aharony, M. Berkooz, D. Kutasov, N. Seiberg, *Linear Dilatons, NS5-branes and Holography*, JHEP 9810 (1998) 004, hep-th/9808149.