Derived equivalences for trigonometric double affine Hecke algebras

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Abstract

The trigonometric double affine Hecke algebra $H_c$ for an irreducible root system depends on a family of complex parameters $c$. Given two families of parameters $c$ and $c'$ which differ by integers, we construct the translation functor from $H_c$-Mod to $H_{c'}$-Mod and prove that it induces equivalence of derived categories. This is a trigonometric analogue of a theorem of Losev on the derived equivalences for rational Cherednik algebras.

Introduction

Trigonometric double affine Hecke algebras

Let $(\mathfrak{h}_\mathbb{R}, R)$ be an irreducible root system, where $\mathfrak{h}_\mathbb{R}$ is a euclidean vector space and $R \subset \mathfrak{h}_\mathbb{R}_R$ is the set of roots, the latter of which is assumed to span $\mathfrak{h}_\mathbb{R}$. Let $W$ be the Weyl group and $T^\vee$ the algebraic torus whose character lattice is the coroot lattice of $(\mathfrak{h}_\mathbb{R}, R)$. The trigonometric double affine Hecke algebra $H_c$, introduced by Cherednik in his works on Knizhnik–Zamolodchikov equations around 1995, is a flat deformation of the skew tensor product $\mathbb{C}W \rtimes \mathcal{O}(T^\vee \times \mathfrak{h})$, where $\mathfrak{h} = \mathfrak{h}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ and $\mathcal{O}(T^\vee \times \mathfrak{h})$ is the ring of regular functions on the algebraic variety $T^\vee \times \mathfrak{h}$. It depends on a family of parameters $c \in \mathfrak{P}$, where $\mathfrak{P} = \text{Map}(R/W, \mathbb{C})$. There is a remarkable polynomial subalgebra $\mathcal{O}(\mathfrak{h}) \subset H_c$, generated by Dunkl operators.

The representation theory of $H_c$ has been studied from different perspectives. Of great importance is the category of integrable $H_c$-modules, denoted by $\mathcal{O}(H_c)$. Recall that a $H_c$-module is called integrable if it is finitely generated and the Dunkl operators act locally finitely on it. The category $\mathcal{O}(H_c)$ is both noetherian and artinian, and its blocks are indexed by the set of orbits in $\mathfrak{h}$ under the reflection action of the affine Weyl group $\hat{W}$. For $[\lambda] \in \mathfrak{h}/\hat{W}$, let $O_\lambda(H_c) \subseteq \mathcal{O}(H_c)$ denote the corresponding block.

Let $K_t$ be the extended affine Hecke algebra for the dual root system $(\mathfrak{h}_\mathbb{R}_R, R^\vee)$, with a family of parameters $t : R/W \to \mathbb{C}^\times$, and let $K_t$-mod$_{fd}$ be the category of finite-dimensional $K_t$-modules. The blocks of $K_t$-mod$_{fd}$ are indexed by the set

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$T/W$, where $T$ is the dual torus of $T^\vee$; for $[\ell] \in T/W$, let $K_{\ell} \text{-mod}_\ell \subseteq K \text{-mod}_k$ denote the corresponding block. A main feature of the category $O(H_c)$ is the trigonometric Knizhnik–Zamolodchikov functor (KZ functor), constructed in [18] in terms of monodromy representation of trigonometric KZ equations. It is a quotient functor of abelian categories $V_c : O(H_c) \rightarrow K_{\ell} \text{-mod}_\ell$. Moreover, for each $[\lambda] \in \mathfrak{h}/W$, it restricts to a functor on the block $V_\lambda_c : O_{\lambda}(H_c) \rightarrow K_{\ell} \text{-mod}_\ell$, where $\ell = \exp(2\pi i \lambda) \in T$ and $t = \exp(2\pi i c)$.

The abelian category $O(H_c)$ has finite global dimension whereas $K_{\ell} \text{-mod}$ may have infinite global dimension, thus singular, in general. By analogy with algebraic geometry, the functor $V_\lambda_c$ can be viewed as a desingularisation of $K_{\ell} \text{-mod}_\ell$. If we are given another family of parameters $\ell' \in \mathfrak{g}$ such that $\ell' - c \in \mathfrak{g}_\mathbb{Z} = \text{Map}(R/W, \mathbb{Z})$, then $\ell'$ gives rise to the same parameters $t$ for $K_{\ell}$ and yields another quotient functor $V_{\lambda, c'} : O_{\lambda}(H_{c'}) \rightarrow K_{\ell} \text{-mod}_\ell$.

This leads to the question of the relation between these two "desingularisations" $V_\lambda_c$ and $V_{\lambda, c'}$. More precisely, one expects that there exists an equivalence on the bounded derived categories $D^b(O_{\lambda}(H_c)) \cong D^b(O_{\lambda}(H_{c'}))$ which intertwines $V_{\lambda, c}$ and $V_{\lambda, c'}$. Even better, one can expect that it extends to an equivalence $D^b(H_c \text{-Mod}) \cong D^b(H_{c'} \text{-Mod})$. The aim of this article is to prove a slight variant of this statement, where the KZ functor $V_\lambda_c$ is replaced by an algebraic version $V_\lambda_c$ introduced in [10] (it is conjectured that $V_\lambda_c$ and $V_{\lambda, c'}$ are isomorphic), see also §1.9.

### Derived equivalences

For $c, c' \in \mathfrak{g}$ such that $\ell' - c \in \mathfrak{g}_\mathbb{Z}$, we will construct a $(H_{c'}, H_c)$-bimodule $\mathcal{B}_c$. Let $T_{\ell' - c} = \mathcal{B}_c \otimes_{H_c} H_{c'} : D^b(H_c \text{-Mod}) \rightarrow D^b(H_{c'} \text{-Mod})$ be the derived tensor product. We call $T_{\ell' - c}$ the translation functor. The main theorem is the following:

**Theorem A (≈ Theorem 76+Proposition 64+Theorem 98).** The translation functor $T_{\ell' - c}$ is an equivalence of categories. Moreover, for each $[\lambda] \in \mathfrak{h}/W$, it restricts to an equivalence on subcategories $D^b(O_{\lambda}(H_c)) \cong D^b(O_{\lambda}(H_{c'}))$ which conserves the support on $T^\vee/W$ and intertwines the algebraic KZ functors $V_\lambda_c$ and $V_{\lambda, c'}$.

For the precise meaning of conservation of support on $T^\vee/W$, we refer the reader to §6.9. This theorem can be viewed as an analogue of a statement about rational Cherednik algebras conjectured by Rouquier [14] and proven by Losev [12, 13]. However, both the construction and the proof of equivalence require new elements for trigonometric double affine Hecke algebras.

Moreover, the bimodule $\mathcal{B}_c$ can be explicitly expressed as subspace of a certain nil-Hecke algebra.

**Example.** The trigonometric DAHA of type $A_1$ with parameter $c \in \mathbb{C}$ is defined by

$$H_c = \mathbb{C}(x, s_1, s_2)/(s_1^2 - 1, s_0^2 - 1, s_1x + xs_1 - c, s_0x - (1 - x)s_0 + c).$$

If we embed $H_c$ into the nil-Hecke algebra

$$H_c^{\text{nil}} = \mathbb{C}(x, \vartheta_1, \vartheta_2)/(\vartheta_1^2, \vartheta_0^2, \vartheta_1x + \vartheta_0x - (1 - x)\vartheta_0 + 1)$$

via the following injective ring map:

$$\rho_c : H_c \rightarrow H_c^{\text{nil}}, \quad \rho_c(x) = x, \quad \rho_c(1 - s_1) = (2x - c)\vartheta_1, \quad \rho_c(1 - s_0) = (1 - 2x - c)\vartheta_2,$$

then we have $c_{-1}\mathcal{B}_c = \{1, \vartheta_1, \vartheta_0\} \cdot \rho_c(H_c) \subseteq H_c^{\text{nil}}$. One verifies easily that $c_{-1}\mathcal{B}_c$ is an $(H_{c-1}, H_c)$-bimodule via $(\rho_{c-1}, \rho_c)$.
Geometric heuristic

Although our construction of the translation functor is purely algebraic, its idea arises from the realisation of the trigonometric DAHA as cohomological convolution algebra, such in the context of affine Springer theory [17] or Coulomb branches [4], and deserves a mention.

Suppose that the root system $(\mathfrak{h}_\mathbb{R}, R)$ is simply-laced so that $\mathfrak{g}_\mathbb{R} \cong \mathbb{C}$ and $H_c$ has only one parameter $c \in \mathbb{C}$. Let $(\mathfrak{h}_\mathbb{R}, \hat{R})$ be the affinisation of $(\mathfrak{h}_\mathbb{R}, R)$, so that we have $\mathfrak{h}_\mathbb{R} = \mathfrak{n} \oplus \mathbb{R} \delta$, where $\delta$ is the primitive imaginary root. Let $K = \mathbb{C}(\varpi)$, let $G = G(K)$ be the loop group and let $B \subseteq G$ be an Iwahori subgroup with pro-unipotent radical $U \subset B$. Set $N = G$, $N' = N \otimes K$ and $N^+ = \text{Lie}U$. Following [19, 4], one considers the following affine Springer resolution:

$$\pi : T = G \times_B N^+ \to N', \quad [g : x] \mapsto \text{Ad}_g(x).$$

Consider the homogeneous version of the trigonometric DAHA: $H_\delta$, whose underlying vector space is $O(\mathfrak{g} \times \mathfrak{h}) \otimes \mathbb{C}W$, where $\delta$ is the “quantisation parameter”. The algebra $H_c$ can be obtained from $H_\delta$ via specialisation of parameters to $\delta \mapsto 1$ and $c \in \mathfrak{g}_\mathbb{R}$. There a $(G \times \mathbb{C}^\times \times \mathbb{C}^\times)$-action on these spaces making $\pi$ equivariant and $H_\delta$ is isomorphic to the convolution algebra $H_c = \otimes \mathbb{C}^\times \times \mathbb{C}^\times (T \times_N T)$, defined an ad hoc way.

The coordinate ring of parameters $O(\mathfrak{g})$ is sent to the ring of coefficients $H_c^\times(\mathbb{C}^\times)(pt)$. One may twist this construction: for $d \in \mathbb{Z}$, let $\varpi^d T = G \times_B \varpi^d N^+$; then, the isomorphism $T \xrightarrow{\sim} \varpi^d T$ given by multiplication by $\varpi$ twists the $(G \times \mathbb{C}^\times \times \mathbb{C}^\times)$-action, which yields an isomorphism $H_\delta \xrightarrow{\sim} H_c^\times \times \mathbb{C}^\times \times \mathbb{C}^\times (\varpi^d T \times_N \varpi^d T)$ so that the restriction of this map to $O(\mathfrak{g}) \to H_c^\times(\mathbb{C}^\times)(pt)$ is shifted by $d$.

One will then define $B^{(d)} = H_c^\times \times \mathbb{C}^\times \times \mathbb{C}^\times (\varpi^d T \times_N T)$. It can be equipped with the structure of $H_\delta$-bimodule via the convolution product. When the parameters are specialised to $c \in \mathfrak{g}$ and $\delta$ is set to be $1$, the quotient of $B^{(d)}$ becomes a $(H_{c+d}, H_c)$-bimodule, denoted by $c+d B_c$. One will then show by geometric methods, notably via the equivariant localisation, that the derived tensor product $c+d B_c \otimes_{H_c}^{L}_{H_c} \Delta$ yields a derived equivalence $D^b(H_c - \text{Mod}) \cong D^b(H_{c+d} - \text{Mod})$.

When the root system $(\mathfrak{h}_\mathbb{R}, R)$ is not simply-laced, in order to allow the full generality for the parameters $c \in \mathfrak{g}$, one will need to replace the adjoint representation $N$. For type $BC$, one can use Kato’s exotic nilpotent cone [8]. For type $F_4$ and $G_2$, the representation $N$ can be chosen in characteristic $2$ and $3$ respectively (as it is told to the author by Kato).

Algebraic construction

Rather than working geometrically in the context of affine Springer theory, we have opted for a purely algebraic approach for three reasons: (i) the algebraic approach is uniform for all types of root systems (ii) due to the infinite-dimensional nature of the geometry, the geometric approach would require an ad hoc equivariant localisation theorem, which is a considerable technical complication (iii) the imaginary weights of the $G$-representation $N$ are embarrassing; the algebraic construction allows us get rid of the imaginary part.

The spherical polynomial representation of $H_\delta$ on $O(\mathfrak{g} \times \mathfrak{h})$ is faithful and yields an embedding of $H_\delta$ into the nil-Hecke algebra $H_{\text{nil}}$ for the affine Weyl group $\tilde{W}$. The geometric counterpart of this embedding is the map $z^* : H^\times_c \times \mathbb{C}^\times (\mathbb{T} \times_N \mathbb{T}) \hookrightarrow H^\times_c \times \mathbb{C}^\times (\mathbb{F} \times \mathbb{F})$, induced by the zero-section $z : \mathbb{F} \hookrightarrow \mathbb{T}$, where $\mathbb{F} = \mathbb{G}/B$ is the affine flag manifold; such a map has notably appeared in [4, 5(iv)].

Based on this embedding, we will construct a category $\mathbf{A}$, whose objects are chambers of certain hyperplane arrangement on the affine space $\mathfrak{h}_\mathbb{R}^1 = \{ z \in \mathfrak{h}_\mathbb{R} : \delta(z) = 1 \}$.
and whose hom-spaces are certain subspaces of $H^{\text{nil}}$. The composition of morphisms is given by the multiplication in $H^{\text{nil}}$. For each $d \in \mathbb{P}_{\mathbb{Z}}$, there is an object $\kappa_d \in \mathbf{A}$ and an isomorphism $H_\delta \cong \text{End}_A(\kappa_d)$. The bimodule $B^{(d)}$ will be defined as the hom-space $\text{Hom}_A(\kappa_0, \kappa_d)$, which is naturally an $H_\delta$-bimodule via the composition of morphisms. For each $d \in \mathbb{P}_{\mathbb{Z}}$, there is an object $\kappa_d \in \mathbf{A}$ and an isomorphism $H_\delta \cong \text{End}_A(\kappa_d)$.

Let us remark that similar constructions of the category $\mathbf{A}$ and the bimodule $\mathcal{B}_c$ have already been carried out by Webster in [21, 20] in the context of quantised Coulomb branches. However, our strategy to prove the equivalence is quite different from his.

**Strategy of the proof**

The essence of our approach to establish the derived equivalences consists of the following steps:

(i) We describe the completion of the algebra $H_c$ as well as the translation bimodule $\mathcal{B}_c$ at every maximal ideals $\lambda$ of the ring $\mathcal{O}(\mathfrak{h}^1)$ (called spectral completion) in terms of a category $\mathfrak{A}^{c,\lambda}$, imitating the equivariant localisation in the aforementioned cohomological approach; the category $\mathfrak{A}^{c,\lambda}$ is Morita equivalent to a semi-perfect Frobenius algebra over a commutative noetherian complete local ring.

(ii) We define in §6 a central hyperplane arrangement for each coset of $\mathfrak{P}_Q = \text{Map}(R/W, \mathbb{Q})$ in $\mathfrak{P}$; we prove in Proposition 86 that for two families of parameters $c$ and $c'$ which differ by integers and lie in the same stratum of the coset $c + \mathfrak{P}_Q$, the categories $H_c$-Mod and $H_{c'}$-Mod are canonically equivalent.

(iii) We prove in Proposition 89 that for two families of parameters $c$ and $c'$ which differ by integers and lie in antipodal strata, the translation bimodule $\mathcal{B}_c$ is a tilting module and induces a derived equivalence between $D^b(H_c$-Mod) and $D^b(H_{c'}$-Mod); this makes use of the spectral completion of $\mathcal{B}_c$ and a duality (Proposition 53) induced from the Frobenius structure on $\mathfrak{A}^{c,\lambda}$.

(iv) In Proposition 90, we apply an analogue of the degeneration technique of Harish-Chandra bimodules due to Bezrukavnikov–Losev [1] to show that when $c$ and $c'$ differ by integers and are situated in adjacent open strata, the translation bimodule yields a derived equivalence.

(v) In §5.6, we study the algebra $\bigoplus_{n \in \mathbb{N}} B^{(nd)}$ of sum of translation bimodules along the semigroup generated by $d \in \mathbb{P}_{\mathbb{Z}}$; a refinement of the degeneration technique allows us to prove in Proposition 92 the derived equivalence when both $c$ and $c'$ lie in open strata and differ by integers.

(vi) We treat in Proposition 94 the case where $c$ and $c'$ lies in non-open strata, again by the degeneration technique; this step relies crucially on the use of the algebraic KZ functor.

**Organisation**

The article is organised as follows:

In §1, we recollect some previously known results about trigonometric double affine Hecke algebras. Only the results of §1.7 are original.
In §2, we develop the basic theory of Harish-Chandra bimodules for trigonometric DAHAs. We prove that they form a Serre subcategory closed under $\text{Tor}$ and $\text{Ext}$. Moreover, we prove that the $\text{Tor}$ and $\text{Ext}$ of Harish-Chandra bimodules commute with spectral completion.

In §3, we define and study the category $\mathcal{A}$ and an embedding of the trigonometric DAHA in it.

In §4, we define and study the category $\mathcal{A}^{c,\lambda}$, which is a spectral-completed version of $\mathcal{A}$. The main result of this section is the comparison of $\mathcal{A}$ and $\mathcal{A}^{c,\lambda}$.

In §5, we construct the translation bimodule $\mathcal{B}^{(0)}$ as certain hom-space in $\mathcal{A}$ and prove that they are Harish-Chandra bimodules. We describe the spectral completion of $\mathcal{B}^{(0)}$ in terms of $\mathcal{A}^{c,\lambda}$. We then prove the noetherianity of the algebra $\bigoplus_{n \geq 0} \mathcal{B}^{(n)}$.

In §6, we define the translation functor $T_{c-e}$ and prove that it is a derived equivalence. We also prove that it preserves integrable modules and their support.

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1 Reminder on the trigonometric DAHA

In this section, we recollect results about the representations of trigonometric DAHA which will be needed later. Except §1.7 which is original, all other results presented here can be found in certain variant forms in [18, 10] for example.

1.1 Affine root systems

We fix the notation for affine root systems, which will be constantly used throughout this article.

Let $(\mathfrak{h}_R, R)$ be an irreducible root system, either reduced or not, where $\mathfrak{h}_R$ is a euclidean vector space and $R \subset \mathfrak{h}^*_R$ is the set of roots, which spans $\mathfrak{h}^*_R$. The affinisation $(\tilde{\mathfrak{h}}_R, \tilde{R})$ is an affine root system on the $\mathbb{R}$-vector space $\mathfrak{h}_R = \mathfrak{h}_R \oplus \mathbb{R} \partial$, whose dual can be written as $\tilde{\mathfrak{h}}^*_R = \mathfrak{h}^*_R \oplus \mathbb{R} \delta$ with the pairing

$$\langle \alpha + t \delta, \lambda + r \partial \rangle = \langle \alpha, \lambda \rangle + rt, \quad \text{for } \alpha \in \mathfrak{h}^*_R, \lambda \in \mathfrak{h}_R, \quad \text{and } r, s \in \mathbb{R}.$$ 

The set of affine roots is defined to be

$$\tilde{R} = \left\{ \alpha + n \delta \in \tilde{\mathfrak{h}}^*_R : \alpha \in R_{\text{red}}, n \in \mathbb{Z} \right\} \cup \left\{ \alpha + (2n + 1) \delta \in \mathfrak{h}^*_R : \alpha \in R \setminus R_{\text{red}}, n \in \mathbb{Z} \right\},$$

where $R_{\text{red}} \subseteq R$ is the subset of indivisible roots. However, for technical reasons, the following modification of $\tilde{R}$ will be more convenient for us:

$$\Phi = \left\{ \alpha + n \delta \in \mathfrak{h}^*_R : \alpha \in R_{\text{red}}, n \in \mathbb{Z} \right\} \cup \left\{ \alpha/2 + (n + 1/2) \delta \in \mathfrak{h}^*_R : \alpha \in R \setminus R_{\text{red}}, n \in \mathbb{Z} \right\}.$$ 

We fix a basis $\Delta = \{ \alpha_1, \cdots, \alpha_n \} \subset R$, which extends to $\tilde{\Delta} = \Delta \cup \{ \alpha_0 \}$, where $\alpha_0 \in \Phi$ is the affine root defined as follows: let $\theta \in R$ be the largest root with respect to $\Delta$; then $\alpha_0 = \delta - \theta$ if $\theta \in R_{\text{red}}$ and $\alpha_0 = (\delta - \theta)/2$ if $\theta \in R \setminus R_{\text{red}}$. 

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The affine Weyl group of \((\hat{h}, \Phi)\) is the Coxeter group \(\hat{W}\) generated by the simple reflections \(\{s_\alpha\}_{\alpha \in \Delta}\). Let \((m_{\alpha, \beta})_{\alpha, \beta \in \Delta}\) denote the Coxeter matrix, so that \(m_{\alpha, \beta} = \text{ord}(s_\alpha s_\beta)\).

**Example 1** (type \(BC_n\)). Let \(n \in \mathbb{Z}_{\geq 1}\). Let \(h = \mathbb{R}^n = \bigoplus_{i=1}^n \mathbb{R} \epsilon_i\) be the euclidean space with inner product \((\epsilon_i, \epsilon_j) = \delta_{ij}\). The root system of type \(BC_n\) is the datum \((h, R)\), where \(R = R_1 \cup R_2\) and \(R_2\) with

\[
R_1 = \{\pm i \leq i \leq n\}, \quad R_2 = \{\pm \epsilon_i \leq i < j \leq n\}, \quad R_0 = \{\pm 2 \epsilon_i \leq i \leq n\}.
\]

The indivisible roots are \(R_0 = R_1 \cup R_2\). The three subsets \(R_1, R_2\) and \(R_0\) (when \(n = 1\), the two sets \(R_1, R_2\)) are the \(W\)-orbits in \(R\) and they are characterised by the length of roots. We shall write \(R/W = \{\zeta, \frac{1}{2}\}\) \((R/W = \{\zeta\}\) when \(n = 1\)). The (standard) basis given by \(\Delta = \{\alpha_i\}_{1 \leq i \leq n}\) with \(\alpha_i = \epsilon_i - \epsilon_{i+1}\) for \(i = 1, \ldots, n - 1\) and \(\alpha_n = \epsilon_n\). The largest root is \(\theta = 2\epsilon_1\).

The affine root system of type \(BC_n\) is the datum \((h, \Phi)\), where \(\Phi = \Phi^s \cup \Phi^t \cup \Phi^s\) and \(\Phi^s = R_1 \times \mathbb{Z} \delta, \Phi^t = R_2 \times \mathbb{Z} \delta, \Phi = (1/2)R_0 \times (1/2 + \mathbb{Z}) \delta\). Let \(\alpha_0 = (\delta - \theta)/2\). The standard affine basis is \(\hat{\Delta} = \Delta \cup \{\alpha_0\}\). Note that \(\alpha_n \in \Phi^s, \alpha_0 \in \Phi^t\) and \(\alpha_i \in \Phi^s\) for \(i = 1, \ldots, n - 1\).

**Example 2** (type \(F_4\)). Let \(h = \mathbb{R}^4 = \bigoplus_{i=1}^4 \mathbb{R} \epsilon_i\) be the euclidean space. The root system of type \(F_4\) is the datum \((h, R)\), where \(R = R_3 \cup R_4\) with

\[
R_3 = \{\pm (\epsilon_i \pm \epsilon_j) \leq 1 \leq i < j \leq 4\}, \quad R_4 = \{\pm \epsilon_i \leq 1 \leq i \leq 4\} \cup \{(\pm \epsilon_i \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) / 2\}.
\]

The standard basis is given by \(\{\alpha_i\}_{1 \leq i \leq 4}\) with \(\alpha_1 = \epsilon_2 - \epsilon_3, \alpha_2 = \epsilon_3 - \epsilon_4, \alpha_3 = \epsilon_4, \alpha_4 = (\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4) / 2\). The largest root is \(\theta = \epsilon_1 + \epsilon_2\). The affine root system of type \(F_4\) is the datum \((h, \Phi)\), where \(\Phi = \Phi^s \cup \Phi^t\) and \(\Phi^t = R_3 \times \mathbb{Z} \delta, \Phi^t = R_4 \times \mathbb{Z} \delta\).

**Example 3** (type \(G_2\)). Let \(h = \text{span}_\mathbb{R}(\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3) \subset \bigoplus_{i=1}^2 \mathbb{R} \epsilon_i = \mathbb{R}^3\). The root system of type \(G_2\) is the datum \((h, R)\), where \(R = R_1 \cup R_2\) with

\[
R_1 = \{\pm (2 \epsilon_i - \epsilon_j - \epsilon_k) \leq 1 \leq i < j \leq 3\}, \quad R_2 = \{\pm (\epsilon_i - \epsilon_j) \leq 1 \leq i < j \leq 3\}.
\]

The standard basis is given by \(\{\alpha_i\}_{1 \leq i \leq 2}\) with \(\alpha_1 = \epsilon_1 - \epsilon_2\) and \(\alpha_2 = -2 \epsilon_1 + \epsilon_2 + \epsilon_3\). The largest root is \(\theta = -\epsilon_1 - \epsilon_2\). The affine root system of type \(G_2\) is the datum \((h, \Phi)\), where \(\Phi = \Phi^s \cup \Phi^t\) and \(\Phi^t = R_1 \times \mathbb{Z} \delta, \Phi^t = R_2 \times \mathbb{Z} \delta\).

### 1.2 Trigonometric DAHA

1.2.1 Fix an irreducible root system \((h, R)\) as above. We put \(h = \mathbb{R} \otimes \mathbb{R} \mathbb{C}\) and \(\mathbb{h} = \mathbb{R} \otimes \mathbb{R} \mathbb{C}\). Let \(C[c_\ast: \ast \in R/W]\) be the polynomial ring whose variables are indexed by the \(W\)-conjugacy classes of roots. We will write \(\mathfrak{P} = \text{Spm} C[c_\ast: \ast \in R/W]\) for its maximal spectrum, so that \(C[c_\ast: \ast \in R/W] = \mathcal{O}(\mathfrak{P})\). Let \(W = W \ltimes \hat{P}_R^\vee\) be the extended affine Weyl group, where \(P_R^\vee \subseteq h\) is the coweight lattice. There is a bijection \(R/W \cong \Phi/W\) which sends \(R_s, \Phi_s\) for \(s \in \{\zeta, \frac{1}{2}\}\) as in Example 1–Example 3. For \(s \in R/W\) and \(\alpha \in \Phi_s\), we will write \(c_\alpha = c_\ast\).

1.2.2 The (homogeneous) trigonometric double affine Hecke algebra for \((\hat{h}, \hat{\Delta})\), denoted by \(\mathcal{H}_\delta\), is the unital associated algebra over \(\mathcal{O}(\mathfrak{P})\) generated by the sets \(\{s_\alpha\}_{\alpha \in \Delta}\) and \(\{x^{\mu}\}_{\mu \in \mathbb{R}^\vee}\) subject to the following relations for \(\mu, \nu \in \mathbb{R}^\vee, a, b \in \mathbb{C}\) and \(\alpha, \beta \in \Delta\):

\[
x^{\mu + \nu} = ax^\mu + bx^\nu, \quad [x^\mu, x^\nu] = 0, \quad (s_\alpha s_\beta)^{m_{\alpha, \beta}} = 1, \quad s_\alpha x^\mu - x^\mu s_\alpha = c_\alpha (\mu, \alpha^\vee),
\]
where $\alpha' = \alpha / (\alpha, \alpha)$.

The family $\{x^\mu\}_{\mu \in \mathfrak{h}^*}$ generates a polynomial subalgebra $\text{Sym}(\mathfrak{h}^*) \cong \mathcal{O}(\mathfrak{h})$ whereas $\{s_\alpha\}_{\alpha \in \Delta}$ generates a subalgebra isomorphic to the group ring $\mathbb{C} W$ of the affine Weyl group. We will simply write $\mu = x^\mu$ for $\mu \in \mathfrak{h}^*$. Moreover, the centre of $H_\delta$ is the polynomial ring $\mathcal{O}(\mathfrak{p})[\delta]$. There is a decomposition of $H_\delta$ as vector space:

$$H_\delta = \mathcal{O}(\mathfrak{p} \times \mathfrak{h}) \otimes \mathbb{C} W.$$  

We set $S_\delta = \mathcal{O}(\mathfrak{p} \times \mathfrak{h}) \subseteq H_\delta$; it is a polynomial subalgebra.

**1.2.3** We put an $\mathbb{N}$-grading on $H_\delta$ by setting $\text{deg} \ \mathfrak{p}^* = \text{deg} \ \mathfrak{h}^* = 1$ and $\text{deg} \ \mathbb{C} W = 0$. Let $H = H_\delta/(\delta - 1)$ (resp. $S = S_\delta/(\delta - 1)$) be the quotient ring of $H_\delta$ (resp. $S_\delta$) by the two-sided ideal generated by $\delta - 1$. Let $\mathfrak{h}^\circ$ be the vanishing locus of $\delta - 1$ in $\mathfrak{h}$. We may identify $S$ with the coordinate ring $\mathcal{O}(\mathfrak{p} \times \mathfrak{h}^\circ)$. The ring $H$ is equipped with a filtration $F_{\text{can}}^\circ H$ induced from the grading on $H_\delta$, called canonical filtration of $H$. Since $H_\delta$ is graded-free over $\mathbb{C}[\delta]$, the homogeneous trigonometric DAHA $H_\delta$ is isomorphic to the Rees algebra of the canonical filtration $F_{\text{can}}^\circ H$ in a natural way.

Given a family of parameters $c \in \mathfrak{p}$, we will write $c_s = c_s(c)$ for $s \in R/W$ and define the specialisation of parameters $H_c = H/(c - c)$ and $S_c = S/(c - c)$.

**1.2.4** The following is well known, which can be established by the technique of degeneration of filtered rings, see [10, §2.2] for example:

**Proposition 4.** The rings $H$ and $H_c$ for $c \in \mathfrak{p}$ have finite left and right global dimension.  

**1.3** Integrable modules

**Definition 5.** A finitely generated $H$-module $M$ is called integrable if $M$ is, by restriction, a locally finite $S$-module.

We let $\mathcal{O}(H) \subset H$-mod denote the full subcategory of integrable $H$-modules.

**1.3.1** For each point $(c, \lambda) \in \mathfrak{p} \times \mathfrak{h}^\circ$, we let $m_{c, \lambda} \subseteq S$ denote the maximal ideal attached to $(c, \lambda)$. If $M$ is an integrable $H$-module, then there is a decomposition

$$M = \bigoplus_{(c, \lambda) \in \mathfrak{p} \times \mathfrak{h}^\circ} M_{c, \lambda}, \quad M_{c, \lambda} = \bigcup_{k \geq 0} \left\{ m \in M : m^k_{c, \lambda} m = 0 \right\}.$$  

The category $\mathcal{O}(H)$ can be decomposed into blocks

$$\mathcal{O}(H) = \bigoplus_{(c, [\lambda]) \in \mathfrak{p} \times \mathfrak{h}^\circ / \mathbb{W}} \mathcal{O}_{c, \lambda}(H), \quad \text{Obj}(\mathcal{O}_{c, \lambda}(H)) = \left\{ M \in \mathcal{O}(H) : M = \bigoplus_{\lambda' \in [\lambda]} M_{c, \lambda'} \right\}.$$  

**1.3.2** Given $c \in \mathfrak{p}$, we define $\mathcal{O}(H_c)$ to be the full subcategory of $H_c$-mod consisting of modules $M$ which, regarded as $H$-module, is in $\mathcal{O}(H)$. The block decomposition

$$\mathcal{O}(H_c) = \bigoplus_{[\lambda] \in \mathfrak{h}^\circ / \mathbb{W}} \mathcal{O}_{\lambda}(H_c)$$  

works in a similar way.
1.4 Spectral completion of $\mathbf{H}$

1.4.1 Define the following (non-unital) ring

$$\mathcal{J} = \bigoplus_{(c, \lambda) \in \mathfrak{P} \times \mathfrak{h}^1} S_{c, \lambda}^\wedge,$$

where $S_{c, \lambda}^\wedge$ is the completion of $S$ at the defining ideal $m_{c, \lambda} \subset S$ of $(c, \lambda) \in \mathfrak{P} \times \mathfrak{h}^1$. For each $(c, \lambda) \in \mathfrak{P} \times \mathfrak{h}^1$, the unit element of $S_{c, \lambda}^\wedge$ is an idempotent in $\mathcal{J}$, denoted by $1_{c, \lambda}$. Then, $\{1_{c, \lambda}\}_{(c, \lambda) \in \mathfrak{P} \times \mathfrak{h}^1}$ is a natural orthogonal family of central idempotents which is complete in the sense that $\mathcal{J} = \bigoplus_{(c, \lambda) \in \mathfrak{P} \times \mathfrak{h}^1} 1_{c, \lambda}\mathcal{J}$.

Lemma 6. The ring $\mathcal{J}$ is faithfully flat over $S$.

Proof. The flatness is due to the noetherianity of $S$, see [2, §3.4, Théorème 3.(ii)]. For the faithfulness, let $M$ be any $S$-module and let $a \in M \setminus \{0\}$. Choose a maximal ideal $m \subset S$ containing the annihilator $\text{ann}_S(a)$, so that the image of $a$ in the localisation $S_m \otimes_S M$ is non-zero. Since the completion $S_m^\wedge$ is faithfully flat over $S_m$ (the latter being noetherian and local), the image of $a$ in $S_m^\wedge \otimes_S M$ is non-zero. It follows that $S_m^\wedge \otimes_S M \neq 0$ and hence $\mathcal{J} \otimes_S M \neq 0$ holds, so $\mathcal{J}$ is faithfully flat over $S$. $\square$

1.4.2 For each element $w \in \hat{W}$, the action $w : \mathfrak{h}^1 \xrightarrow{\sim} \mathfrak{h}^1$ induces an isomorphism of complete local rings for each pair $(c, \lambda) \in \mathfrak{P} \times \mathfrak{h}^1$:

$$w(\cdot) : S_{c, \lambda}^\wedge \to S_{c, w\lambda}^\wedge, \quad f_\lambda \mapsto w(f_\lambda) \quad \text{for } f_\lambda \in S_{c, \lambda}^\wedge.$$  

For each affine root $\alpha \in \Phi$, define the Demazure operator $\vartheta_\alpha : \mathcal{J} \to \mathcal{J}$ to be the linear map satisfying

$$\vartheta_\alpha(f_\lambda) = \alpha^{-1}(f_\lambda - s_\alpha(f_\lambda)) \quad \text{for } (c, \lambda) \in \mathfrak{P} \times \mathfrak{h}^1 \text{ and } f_\lambda \in S_{c, \lambda}^\wedge.$$  

Note that the above expression makes sense: when $\alpha(\lambda) = 0$, we have $\alpha^{-1}(f_\lambda - s_\alpha(f_\lambda)) \in S_{c, \lambda}^\wedge$; when $\alpha(\lambda) \neq 0$, the element $\alpha$ is invertible in both $S_{c, \lambda}^\wedge$ and $S_{c, s_\alpha \lambda}^\wedge$, so we have $\alpha^{-1}f_\lambda - \alpha^{-1}s_\alpha(f_\lambda) \in S_{c, \lambda}^\wedge \oplus S_{c, s_\alpha \lambda}^\wedge$.

Let $\mathcal{H} = \mathcal{J} \otimes_S \mathbf{H}$. We extend the ring structures on $\mathcal{J}$ and $\mathbf{H}$ to $\mathcal{H}$ by the following rules:

$$(f \otimes 1)(g \otimes a) = fg \otimes a$$  

$$(f \otimes s_\alpha)(g \otimes a) = f : s_\alpha g \otimes s_\alpha a + c_\alpha \vartheta_\alpha(g) \otimes a$$

for $f, g \in \mathcal{J}$, $a \in \mathbf{H}$ and $\alpha \in \hat{\Delta}$. This ring structure induces an isomorphism:

$$\mathbf{H} \otimes_S \mathcal{J} \cong \mathcal{H} \otimes_S \mathbf{H} = \mathcal{H}, \quad a \otimes f \mapsto \sum_{(c, \lambda) \in \mathfrak{P} \times \mathfrak{h}^1} (1_{c, \lambda} \otimes a)(1 \otimes f).$$  

(7)

Note that the above summation must be finite. There is a natural $\mathbf{H}$-bimodule structure on $\mathcal{H}$.

Each element of $\mathcal{H}$ can be expressed as finite sum of the following form in a unique way:

$$\sum_{(c, \lambda) \in \mathfrak{P} \times \mathfrak{h}^1} \sum_{w \in \hat{W}} 1_{c, w\lambda} w f_{c, \lambda, w} 1_{c, \lambda} \quad \text{for } f_{c, \lambda, w} \in S_{c, \lambda}^\wedge.$$
1.4.3 The ring $\mathcal{H}$ admits a decomposition into direct sum (coproduct in the category of non-unital rings):

$$\mathcal{H} = \bigoplus_{(c, [\lambda]) \in \mathfrak{P} \times \mathfrak{h}^1 / \hat{W}} \mathcal{H}^{c, \lambda}, \quad \mathcal{H}^{c, \lambda} = \bigoplus_{\lambda' \in [\lambda]} S^\wedge_{c, \lambda'}.$$

(8)

Note that this decomposition is in accordance with that of the integrable modules given in §1.3.

1.4.4 Given $c \in \mathfrak{P}$, we define $\mathcal{H}_c = \mathcal{H} / (c - c)$ to be the specialisation of parameters at $c$. Similarly to (8), there is a decomposition $\mathcal{H}_c = \bigoplus_{\lambda' \in [\lambda]} S^\wedge_{c, \lambda'}$.

1.5 Properties of $\mathcal{H}$-modules

1.5.1 Module categories. Let $\mathcal{H}$-Mod denote the category of non-degenerate $\mathcal{H}$-modules (i.e. $\mathcal{H}$-modules $\mathcal{M}$ satisfying $\mathcal{M} = \bigoplus_{(c, \lambda) \in \mathfrak{P} \times \mathfrak{h}^1} 1_{(c, \lambda)}\cdot \mathcal{M}$). It admits a set of compact projective generators $\{ \mathcal{H}_1^{c, \lambda} \}_{(c, \lambda) \in \mathfrak{P} \times \mathfrak{h}^1}$. Let $\mathcal{H}$-mod $\subseteq$ $\mathcal{H}$-Mod denote the full subcategory of compact objects and let $\mathcal{H}$-mod fl $\subseteq$ $\mathcal{H}$-mod be the full subcategory of objects of finite length. The categories $\mathcal{H}_c$-Mod, $\mathcal{H}_c$-mod and $\mathcal{H}_c$-mod fl are defined similarly for each $c \in \mathfrak{P}$.

The decomposition of ring (8), induces a decomposition of the module category:

$$\mathcal{H}$-Mod $\cong \bigoplus_{(c, [\lambda]) \in \mathfrak{P} \times \mathfrak{h}^1 / \hat{W}} \mathcal{H}_c^{c, \lambda}$-Mod.

(9)

A similar decomposition holds for $\mathcal{H}_c$-Mod.

1.5.2 Centre. Let $(c, [\lambda]) \in \mathfrak{P} \times \mathfrak{h}^1$. We put $\mathcal{Z}_{c, \lambda} = (S^\wedge_{c, \lambda})^{\hat{W}_\lambda}$. It is a complete regular local ring since the stabiliser $\hat{W}_\lambda = \text{Stab}_{\hat{W}}(\lambda)$ is a finite reflection groups (see [3, V, §5.3, Th 3]). For each $\lambda' \in [\lambda]$, choose $w \in \hat{W}$ such that $\lambda' = w\lambda$ and consider the embedding

$$\mathcal{Z}_{c, \lambda} \hookrightarrow S^\wedge_{c, \lambda} \xrightarrow{w} S^\wedge_{c, \lambda'}, \quad z \mapsto z_{c, \lambda'} := wz,$

which is independent of the choice of $w$ and make $S^\wedge_{c, \lambda'}$ a free $\mathcal{Z}_{c, \lambda}$-module of rank $\# \hat{W}_\lambda$. It defines a $\mathcal{Z}_{c, \lambda}$-action on the identity functor of $\mathcal{H}_{c, \lambda}$-Mod by natural transformations:

$$\chi : \mathcal{Z}_{c, \lambda} \rightarrow \text{End}(\text{id}_{\mathcal{H}_{c, \lambda}$-Mod}), \quad \chi(z)(m) = \sum_{\lambda' \in [\lambda]} z_{c, \lambda'} m.$$

Consequently, the category $\mathcal{H}_{c, \lambda}$-Mod comes with a natural $\mathcal{Z}_{c, \lambda}$-linear structure (i.e. it is enriched over $\mathcal{Z}_{c, \lambda}$-Mod); moreover, the Hom-spaces between objects from $\mathcal{H}_{c, \lambda}$-mod (resp. $\mathcal{H}_{c, \lambda}$-proj) are finitely generated $\mathcal{Z}_{c, \lambda}$-modules (resp. free $\mathcal{Z}_{c, \lambda}$-modules of finite rank).

The same construction can be applied to $\mathcal{H}_c$, with $\mathcal{Z}_{c, \lambda}$ replaced by $\mathcal{Z}_c = \mathcal{Z}_{c, \lambda} / (c - c)$.

**Lemma 10.** Let $\mathcal{T} \subseteq \mathcal{H}_{c, \lambda}$-Mod be the full subcategory consisting of objects which are locally finite-dimensional as $\mathcal{Z}_{c, \lambda}$-modules. Then, the quotient category $\mathcal{H}_{c, \lambda}$-Mod / $\mathcal{T}$ is equivalent to (Frac $\mathcal{Z}_{c, \lambda}$)-Mod. A similar statement holds for $\mathcal{H}_c$ for $c \in \mathfrak{P}$.
Proof. Use the compact projective generators \( \{ \mathcal{H}^c\lambda_{1_c\lambda} \}_{\lambda \in [\lambda]} \) and observe that for each \( \lambda' \in [\lambda] \), the morphism

\[
- \cdot s_{\alpha} 1_{c, s_{\alpha} \lambda} : \mathcal{H}^c\lambda_{1_c\lambda} \to \mathcal{H}^c\lambda_{1_c, s_{\alpha} \lambda}'
\]
is injective with cokernel lying in \( \mathcal{F} \); it follows that this map becomes an isomorphism after passing to the quotient category \( \mathcal{H}^c\lambda_{-\text{Mod}} / \mathcal{F} \); therefore, the image of \( \mathcal{H}^c\lambda_{1_c\lambda} \) in \( \mathcal{H}^c\lambda_{-\text{Mod}} / \mathcal{F} \) is a compact projective generator. Let \( A = \text{End}_{\mathcal{H}^c\lambda_{1_c\lambda}}(\mathcal{H}^c\lambda_{1_c\lambda}) \otimes_{\mathcal{H}^c\lambda_{1_c\lambda}} \text{Frac} \mathcal{F}^c\lambda_{1_c\lambda} \), so that \( \mathcal{H}^c\lambda_{-\text{Mod}} / \mathcal{F} \) is equivalent to \( A_{-\text{Mod}} \) and

\[ A \cong 1_{c, \lambda} \mathcal{H}^c\lambda_{1_c, \lambda} \otimes_{\mathcal{F}^c\lambda_{1_c, \lambda}} \text{Frac} \mathcal{F}^c\lambda_{1_c, \lambda} \]
is isomorphic to the subalgebra of \( \text{End}_C(\text{Frac} \mathcal{S}_c^\lambda_{1\lambda}) \) generated by \( \text{Frac} \mathcal{S}_c^\lambda_{1\lambda} \) and the action of \( \hat{W}_\lambda \). Since \( \hat{W}_\lambda \) is a reflection group on \( \mathfrak{h}_R^1 \), the Galois theory implies that this subalgebra is equal to \( \text{End}_{\text{Frac} \mathcal{F}^c\lambda_{1_c\lambda}}(\text{Frac} \mathcal{S}_c^\lambda_{1\lambda}) \), which is a matrix algebra of rank \( \# \hat{W}_\lambda \) over the field \( \text{Frac} \mathcal{F}^c\lambda_{1_c, \lambda} \). The statement follows. \( \square \)

1.5.3 Coxeter complex. For a non-constant affine function \( \alpha \) on \( \mathfrak{h}_1^1 \), let \( H_\alpha = \{ h \in \mathfrak{h}_R^1 ; \alpha(h) = 0 \} \) be its zero locus in \( \mathfrak{h}_R^1 \). The affine hyperplane arrangement \( \{ H_\alpha \}_{\alpha \in \Phi} \) yields a simplicial decomposition of \( \mathfrak{h}_R^1 \), which can be identified with the Coxeter complex of \( \hat{W} \). The relative interior of simplices are called facets. The facets of maximal dimension are called alcoves. The fundamental alcove is defined by \( v_0 = \bigcap_{\alpha \in \Delta} \alpha^{-1}(\mathbb{R}_{>0}) \), where the affine root \( \alpha \) is viewed as affine function on \( \mathfrak{h}_R^1 \).

1.5.4 Clan decomposition. Fix \( (c, \lambda) \in \mathfrak{P} \times \mathfrak{h}^1 \). Consider the subfamily of affine roots \( \Phi_{c, \lambda} = \{ \alpha \in \Phi ; \alpha(\lambda) = c_\alpha \} \) and the hyperplane arrangement \( \{ H_\alpha \}_{\alpha \in \Phi_{c, \lambda}} \) on \( \mathfrak{h}_R^1 \).

Definition 11. The connected components of the complements \( \mathfrak{h}_R^1 \setminus \bigcup_{\alpha \in \Phi_{c, \lambda}} H_\alpha \) are called \( (c, \lambda) \)-clans.

Let \( \text{Cl}^{c\lambda}(\mathfrak{h}_R^1) \) denote the set of \( (c, \lambda) \)-clans in \( \mathfrak{h}_R^1 \). Given a \((c, \lambda)\)-clan \( \mathfrak{C} \subset \mathfrak{h}_R^1 \), we define its salient cone to be \( \kappa = \{ h \in \mathfrak{h}_R^1 ; \mathfrak{C} + h \subseteq \mathfrak{C} \} \). Its dual cone can be described as follows:

\[
\kappa^\vee = \sum_{\alpha \in \Phi_{c, \lambda} \atop \alpha(\mathfrak{C}) \subseteq \mathbb{R}_{>0}} \mathbb{R}_{\geq 0} \bar{\alpha} + \sum_{\alpha \in \Phi_{c, \lambda} \atop \alpha(\mathfrak{C}) \subseteq \mathbb{R}_{<0}} \mathbb{R}_{\leq 0} \bar{\alpha}, \quad \text{where} \quad \bar{\alpha} = \alpha \mid_{\mathfrak{h}_R^1} \in \mathfrak{h}_R^*,
\]

In other words, \( \kappa \) tells in which directions the clan \( \mathfrak{C} \) is unbounded. Each alcove of the Coxeter complex for \( \Phi \) is contained in a unique clan.

Definition 12. A \((c, \lambda)\)-clan \( \mathfrak{C} \in \text{Cl}^{c\lambda}(\mathfrak{h}_R^1) \) is said to be generic if its salient cone \( \kappa \) satisfies \( \dim \kappa = \dim \mathfrak{h}_R^1 \).

In other words, a \((c, \lambda)\)-clan \( \mathfrak{C} \in \text{Cl}^{c\lambda}(\mathfrak{h}_R^1) \) is generic if \( \bar{\alpha} \) does not vanish on \( \kappa \) for any \( \alpha \in \Phi_{c, \lambda} \).

1.5.5 Morita equivalence with complete noetherian algebra. Fix \( (c, \lambda) \in \mathfrak{P} \times \mathfrak{h}^1 \) as in the previous paragraph. The action of \( \hat{W}_\lambda = \text{Stab}_{\hat{W}}(\lambda) \) on \( \mathfrak{h}_R^1 \) preserves \( \mathfrak{h}_c\lambda \) and thus induces an action on \( \text{Cl}^{c\lambda}(\mathfrak{h}_R^1) \). Let \( \Sigma \subset \hat{W} \) be a subset such that each \( \hat{W}_\lambda \)-orbit in \( \text{Cl}^{c\lambda}(\mathfrak{h}_R^1) \) contains at least one alcove from the family \( \{ w^{-1}v_0 \}_{w \in \Sigma} \). It is shown in [10] that the idempotent \( 1_\Sigma := \sum_{w \in \Sigma} 1_{c, w\lambda} \in \mathcal{H}^{c\lambda}_{-\text{Mod}} \) yields a Morita equivalence:

\[
\mathcal{H}^{c\lambda}_{-\text{Mod}} \cong 1_\Sigma \mathcal{H}^{c\lambda}_{-\text{Mod}}, \quad M \mapsto 1_\Sigma M.
\]

(13)
Moreover, the natural map
\[ Z_{c, \lambda} \to 1_{\Sigma} \mathcal{H}^{c, \lambda} 1_{\Sigma}, \quad z \mapsto \sum_{w \in \Sigma} z_{c, w \lambda} \]
is an isomorphism onto the centre of \( 1_{\Sigma} \mathcal{H}^{c, \lambda} 1_{\Sigma} \).

**Proposition 14.** The following statements hold:

(i) Let \( M \in \mathcal{H}^{c, \lambda} \text{-Mod} \). Then, \( M \in \mathcal{H}^{c, \lambda} \text{-mod} \) holds if and only if \( 1_{\Sigma} \mathcal{H}^{c, \lambda} \) is a finitely generated \( Z_{c, \lambda} \)-module for each \( \lambda \in [\lambda] \). Moreover, \( M \in \mathcal{H}^{c, \lambda} \text{-mod} \) holds if and only if \( 1_{\Sigma} \mathcal{H}^{c, \lambda} \) is finite-dimensional for each \( \lambda \in [\lambda] \).

(ii) For \( M, N \in \mathcal{H}^{c, \lambda} \text{-mod} \), the \( Z_{c, \lambda} \)-module \( \text{Ext}^{n} \mathcal{H}^{c, \lambda}(M, N) \) is finitely generated for every \( n \in \mathbb{Z} \); moreover, it is finite-dimensional whenever \( M \in \mathcal{H}^{c, \lambda} \text{-mod} \) or \( N \in \mathcal{H}^{c, \lambda} \text{-mod} \) holds.

**Proof.** Making use of the Morita equivalence (13), it suffices to show these properties for \( 1_{\Sigma} \mathcal{H}^{c, \lambda} 1_{\Sigma} \). They hold since \( 1_{\Sigma} \mathcal{H}^{c, \lambda} 1_{\Sigma} \) is of finite rank over the complete noetherian subalgebra \( Z_{c, \lambda} \).

Similar statements hold for \( \mathcal{H}^{\lambda} \) for every \( (c, [\lambda]) \in \mathfrak{P} \times \mathfrak{h}^{1}/\hat{W} \).

### 1.6 Spectral completion of modules

1.6.1 For \( M \in \mathcal{H} \text{-Mod} \), define \( C(M) = \mathcal{I} \otimes_{S} M \). We equip \( C(M) \) with a \( \mathcal{H} \)-module structure by the following rules:

\[
(f \otimes 1)(g \otimes m) = fg \otimes m \\
(f \otimes s_{\alpha})(g \otimes m) = f \cdot s_{\alpha} g \otimes s_{\alpha} m + c_{\alpha} \vartheta_{\alpha}(g) \otimes m
\]

for \( f, g \in \mathcal{I} \), \( m \in M \) and \( \alpha \in \hat{\Delta} \).

**Definition 15.** The spectral completion is the following functor:

\[ C : \mathcal{H} \text{-Mod} \to \mathcal{H} \text{-Mod}, \quad M \mapsto \mathcal{I} \otimes_{S} M. \]

Alternatively, via the \( \mathcal{H} \)-bimodule structure on \( \mathcal{H} \), we may write \( C(M) = \mathcal{H} \otimes_{\mathcal{H}} M \). It is an exact conservative functor by Lemma 6. It induces \( C_{c} : \mathcal{H}_{c} \to \mathcal{H}_{c} \text{-Mod} \) for each \( c \in \mathfrak{P} \), which is also exact and conservative.

1.6.2 The decomposition (9) induces

\[ C = \bigoplus_{(c, [\lambda]) \in \mathfrak{P} \times \mathfrak{h}^{1}/\hat{W}} C^{c, \lambda}, \quad C^{c, \lambda} : \mathcal{H} \text{-Mod} \to \mathcal{H}^{c, \lambda} \text{-Mod}. \]

Similarly, we have \( C_{c} = \bigoplus_{[\lambda] \in \hat{\mathfrak{h}}^{1}/\hat{W}} C_{c}^{\lambda} \) with \( C_{c}^{\lambda} : \mathcal{H}_{c} \text{-Mod} \to \mathcal{H}_{c}^{\lambda} \text{-Mod} \).

### 1.7 Comparison of derived categories

1.7.1 We denote by \( \text{D}^{b}(\mathcal{H}) = \text{D}^{b}(\mathcal{H} \text{-Mod}) \) and \( \text{D}^{b}(\mathcal{H}) = \text{D}^{b}(\mathcal{H} \text{-Mod}) \) the corresponding bounded derived category of modules. Let \( \text{D}^{b}_{O}(\mathcal{H}) \) denote the full subcategory of \( \text{D}^{b}(\mathcal{H}) \) formed by the complexes \( K \) such that \( \mathcal{H}^{i}(K) \in O(\mathcal{H}) \) for each \( i \in \mathbb{Z} ; \text{D}^{b}_{O}(\mathcal{H}_{c}), \text{D}^{b}_{O}(\mathcal{H}_{c}) \) and \( \text{D}^{b}_{O}(\mathcal{H}) \) are defined similarly.
Theorem 16. The spectral completion yields equivalences of categories:

\[ \mathcal{C} : D^b_{\mathbb{H}}(\mathcal{H}) \xrightarrow{\sim} D^b_{\mathbb{N}}(\mathcal{H}), \quad \mathcal{C}_c : D^b_{\mathbb{H}}(\mathcal{H}_c) \xrightarrow{\sim} D^b_{\mathbb{N}}(\mathcal{H}_c). \]

and for each \((c, [\lambda]) \in \mathfrak{P} \times \mathfrak{h}^1/\mathbb{W}:

\[ \mathcal{C}^{c, \lambda} : D^b_{\mathbb{H}}(\mathcal{H}) \xrightarrow{\sim} D^b_{\mathbb{N}}(\mathcal{H}^{c, \lambda}), \quad \mathcal{C}^{\lambda}_c : D^b_{\mathbb{H}}(\mathcal{H}_c) \xrightarrow{\sim} D^b_{\mathbb{N}}(\mathcal{H}_c^{\lambda}). \]

Proof. Let \(c \in \mathfrak{P}\). We prove only the statement for \(\mathcal{H}_c\)-Mod; the proof for the case of \(\mathcal{C}\), \(\mathcal{C}^{c, \lambda}\) and \(\mathcal{C}^{\lambda}_c\) are similar.

We prove first that \(\mathcal{C}_c(M) \in \mathcal{H}_c\)-mod for \(M \in O(\mathbb{H}_c)\). Given such \(M\), we have \(M = \bigoplus_{\lambda \in \mathfrak{h}^1} M_{\lambda}\), where \(M_{\lambda}\) is the generalised \(\lambda\)-eigenspace for the \(O(\mathfrak{h}^1)\)-action. The condition \(M \in O(\mathbb{H})\) implies that there is an isomorphism \(M := \mathcal{C}_c M \cong M\) of vector spaces and the action maps \(\mathbb{H}_c \rightarrow \text{End}_{\mathbb{C}}(M)\) and \(\mathcal{H}_c \rightarrow \text{End}_{\mathbb{C}}(\mathcal{M}) \cong \text{End}_{\mathbb{C}}(M)\) have the same image; therefore, \(\mathcal{H}_c\)-module \(\mathcal{M}\) is compact.

By (9), we may decompose \(\mathcal{M}\):

\[ \mathcal{M} = \bigoplus_{[\lambda] \in \mathfrak{h}^1/\mathbb{W}} \mathcal{M}_{[\lambda]} ; \quad \mathcal{M}_{[\lambda]} = \bigoplus_{\lambda' \in [\lambda]} M_{\lambda'} \in \mathcal{H}_c^{\lambda'}\text{-mod}; \]

the summation is finite due to the compactness of \(\mathcal{M}\). The centre \(\mathfrak{Z}^\lambda = O(\mathfrak{h}^1)^{W_\lambda}\) acts locally finitely on \(\mathcal{M}_{[\lambda]}\); therefore, \(\mathcal{M}_{[\lambda]}\) lies in \(\mathcal{H}_c^{\lambda}\)-mod by Proposition 14. This proves that the spectral completion induces \(\mathcal{C}_c : D^b_{\mathbb{H}}(\mathcal{H}_c) \rightarrow D^b_{\mathbb{N}}(\mathcal{H}_c).

We prove the full faithfulness. Given \(M, N \in O(\mathbb{H}_c)\), choose a free resolution \((P_\bullet, \delta_\bullet)\) for \(M\), where \(P_i = \mathbb{H}_c^{\oplus m_i}\) and \(\delta_i \in \text{Hom}_{\mathbb{H}_c}(\mathbb{H}_c^{\oplus m_{i+1}}, \mathbb{H}_c^{\oplus m_i})\) for \(i \geq 0\). We may write \(\delta_i\) as right multiplication by a matrix \(m_i \in M_{m_i \times m_{i-1}}(\mathbb{H}_c)\). Then, we have

\[ \text{Hom}_{\mathbb{H}_c}(P_\bullet, N) = [0 \rightarrow N^{\oplus m_0} \overset{m_0}{\rightarrow} N^{\oplus m_1} \overset{m_1}{\rightarrow} \cdots ] \]

Via the generalised eigenspace decomposition \(N = \bigoplus_{(c, \lambda) \in \mathfrak{P} \times \mathfrak{h}^1} N_{\lambda}\), we may write

\[ m_i = \sum_{\lambda', \lambda'' \in \mathfrak{h}^1} m_{i, \lambda', \lambda''}, \quad m_{i, \lambda', \lambda''} : N^{\oplus m_\lambda} \rightarrow N^{\oplus m_{\lambda''+1}} \quad \text{for } \lambda', \lambda'' \in \mathfrak{h}^1. \]

Then, it follows that for \(i \in \mathbb{N}\), we have

\[ \text{Ext}^i_{\mathbb{H}_c}(M, N) = \bigoplus_{\lambda' \in \mathfrak{h}^1} \left( \bigcap_{\lambda'' \in \mathfrak{h}^1} \ker(m_{i, \lambda', \lambda''})/ \sum_{\lambda'' \in \mathfrak{h}^1} \text{im}(m_{i-1, \lambda'', \lambda''}) \right). \quad (17) \]

On the other hand, \((\mathcal{C}_c P_\bullet, \mathcal{C}_c \delta_\bullet)\) is a projective resolution of \(\mathcal{C}_c M\). We may write

\[ \text{Hom}_{\mathbb{H}_c}(\mathcal{C}_c P_\bullet, \mathcal{C}_c N) = \left[ 0 \rightarrow \prod_{\lambda' \in \mathfrak{h}^1} N^{\oplus m_0}_{\lambda'} \overset{(m_{0, \lambda', \lambda'}, \lambda', \lambda'')}{\rightarrow} \prod_{\lambda'' \in \mathfrak{h}^1} N^{\oplus m_1}_{\lambda''} \rightarrow \cdots \right]. \]

Therefore,

\[ \text{Ext}^i_{\mathbb{H}_c}(\mathcal{C}_c M, \mathcal{C}_c N) = \prod_{\lambda' \in \mathfrak{h}^1} \left( \bigcap_{\lambda'' \in \mathfrak{h}^1} \ker(m_{i, \lambda', \lambda''})/ \sum_{\lambda'' \in \mathfrak{h}^1} \text{im}(m_{i-1, \lambda'', \lambda''}) \right). \quad (18) \]

The chain map \(\text{Hom}_{\mathbb{H}_c}(P_\bullet, N) \rightarrow \text{Hom}_{\mathbb{H}_c}(\mathcal{C}_c P_\bullet, \mathcal{C}_c N)\) is given by the natural inclusion \(\bigoplus_{\lambda' \in \mathfrak{h}^1} \hookrightarrow \prod_{\lambda' \in \mathfrak{h}^1}\). This inclusion induces an isomorphism \(\text{Ext}^i_{\mathbb{H}_c}(M, N) \cong \text{Ext}^i_{\mathbb{H}_c}(\mathcal{C}_c M, \mathcal{C}_c N)\) for \(i \in \mathbb{Z}\) since all but finitely many terms in the sum (17)
and product (18) are zero by the finite dimensionality of \( \text{Ext}_c^i(C_cM, C_cN) \) (see Proposition 14 (ii)). This proves the full faithfulness.

It remains to show the essential surjectivity. We define a functor \( F : \mathcal{H}_c - \text{Mod} \to \mathcal{H}_c - \text{Mod} \) as follows: given \( M \in \mathcal{H}_c - \text{Mod} \), we equip \( M \) with an \( \mathcal{H}_c \)-action by the formula: 
\[
h \cdot m = \sum_{\lambda \in \mathfrak{h}} h_{1,c,\lambda} m \quad \text{for} \quad h \in \mathcal{H}_c \quad \text{and} \quad m \in M;
\]
then it is well-defined because \( 1_{c,\lambda} m \) is zero for all but finitely many \( \lambda \in \mathfrak{h}^1 \). For \( M \in \mathcal{H}_c - \text{mod}_\mathfrak{b} \), we have \( F(M) \in \mathcal{O}(\mathcal{H}_c) \) by the non-degeneracy of \( M \) as \( \mathcal{H}_c \)-module and by the finite dimensionality of \( 1_{c,\lambda} M = M_{\lambda} \) for each \( \lambda \in \mathfrak{h}^1 \). It is easy to see that \( \mathcal{C}_c F(M) \cong M \) for \( M \in \mathcal{O}(\mathcal{H}_c) \); therefore, \( \mathcal{O}(\mathcal{H}_c) \) lives in the essential image of \( \mathcal{C}_c \). Now, given \( \mathcal{H} \in \mathcal{D}_b^R(\mathcal{H}_c) \), we may suppose up to cohomological shift that \( \text{Hom}^n(\mathcal{H}) = 0 \) for \( n < 0 \); we may then prove by induction on \( n \in \mathbb{N} \) that the truncation \( \tau_{\leq n} \mathcal{H} \) lies in \( \mathcal{D}_b^R(\mathcal{H}_c) \) via the exactness and full faithfulness of \( \mathcal{C}_c \), whence the essential surjectivity.

**Corollary 19.** Given any \( M \in \mathcal{O}(\mathcal{H}) \), the generalised \( (c,\lambda) \)-weight space \( M_{c,\lambda} \subseteq M \) is finite-dimensional for every \( (c,\lambda) \in \mathfrak{p} \times \mathfrak{h}^1 \).

**Proof.** This is immediate from Proposition 14 (i) and Theorem 16.

**Corollary 20.** The categories \( \mathcal{H} - \text{Mod} \) and \( \mathcal{H}_c - \text{Mod} \) for \( c \in \mathfrak{p} \) have finite global dimension.

**Proof.** We prove only the statement for \( \mathcal{H} - \text{Mod} \). Since \( \mathcal{H} - \text{Mod} \) is compactly generated, it suffices to show that there exists \( d \in \mathbb{N} \) such that \( \text{Ext}^d_{\mathcal{H}}(M, N) = 0 \) holds for \( n > d \) and for \( M, N \in \mathcal{H} - \text{mod} \). For \( M \in \mathcal{H} - \text{mod} \), we have
\[
M \xrightarrow{\sim} \lim_{M' \subseteq M} M/M',
\]
where \( M' \) runs over submodules of \( M \) such that \( M/M' \in \mathcal{H} - \text{mod}_\mathfrak{b} \). It follows that
\[
\text{Ext}^n_{\mathcal{H}}(M, N) \cong \lim_{N' \subseteq N} \lim_{M' \subseteq M} \text{Ext}^n_{\mathcal{H}}(M/M', N/N')
\]
It results from Proposition 4 and Theorem 16 that there exists \( d \in \mathbb{N} \) such that \( \text{Ext}^n_{\mathcal{H}}(M/M', N/N') = 0 \) whenever \( n > d \), whence \( \text{Ext}^n_{\mathcal{H}}(M, N) = 0 \).

1.7.2 For \( A \in \{ \mathcal{H}, \mathcal{H}_c, \mathcal{H}, \mathcal{H}_c : c \in \mathfrak{p} \} \), let \( \text{D}^b_{\text{perf}}(A) \) be the full subcategory of \( \text{D}^b(A - \text{mod}) \) formed by perfect complexes.

**Proposition 21.** For \( A \in \{ \mathcal{H}, \mathcal{H}_c, \mathcal{H}, \mathcal{H}_c : c \in \mathfrak{p} \} \), the natural functor \( \text{D}^b(A - \text{mod}) \to \text{D}^b_{\text{perf}}(A) \) is an equivalence of categories.

**Proof.** That the embedding \( \text{D}^b(A - \text{mod}) \to \text{D}^b_{A - \text{mod}}(A - \text{Mod}) \) is an equivalence is a standard result in homological algebra. On the other hand, we have \( \text{D}^b_{A - \text{mod}}(A) = \text{D}^b_{\text{perf}}(A) \) by the finiteness of global dimension (Proposition 4 and Corollary 20).

**Proposition 22.** The natural functors \( \text{D}^b(\mathcal{H} - \text{mod}_\mathfrak{b}) \to \text{D}^b_{\mathfrak{b}}(\mathcal{H}) \) and \( \text{D}^b(\mathcal{H}_c - \text{mod}_\mathfrak{b}) \to \text{D}^b_{\mathfrak{b}}(\mathcal{H}_c) \) for \( c \in \mathfrak{p} \) are equivalences of categories.

**Proof.** We prove only the statement for \( \mathcal{H} \). By a dual statement of [16, 13.17.4], this is a consequence of the following property: given \( M \in \mathcal{H} - \text{mod} \) and \( N \subseteq M \) with \( N \in \mathcal{H} - \text{mod}_\mathfrak{b} \), there exists \( M' \subseteq M \) satisfying \( M' \cap N = 0 \) and \( M/M' \in \mathcal{H} - \text{mod}_\mathfrak{b} \) — this follows from the Morita equivalence (13) and the Artin–Rees lemma applied to the centre \( \mathcal{Z}^{c,\lambda} \) for every \( (c, [\lambda]) \in \mathfrak{p} \times \mathfrak{h}^1/\mathfrak{W} \).
Corollary 23. The natural functors $D^b(O(H_c)) \to D^b_O(H_c)$ and $D^b(O(H)) \to D^b_O(H)$ are equivalences of categories.

Proof. Consider the following commutative diagram of functors:

$$
\begin{array}{ccc}
D^b(O(H)) & \longrightarrow & D^b(O(H)) \\
\downarrow \phi & & \downarrow \phi \\
D^b(\mathcal{H}\text{-mod}_H) & \longrightarrow & D^b(\mathcal{H})
\end{array}
$$

By Theorem 16 and Proposition 22, the vertical arrows as well as the lower horizontal arrow are equivalences of categories; therefore, the upper horizontal arrow is also an equivalence. The statement for $C_c$ is similar. \hfill \square

Remark 24. Given an abelian category $A$ and a Serre subcategory $B \subseteq A$, the natural functor $D^b(B) \to D^b_B(A)$ may not be an equivalence in general.

1.8 Length filtration

We define another filtration $\{F^k_{lg}H\}_{k \in \mathbb{N}}$ on $H$, called length filtration, by setting

$$F^k_{lg}H = \bigoplus_{w \in \hat{W}, \ell(w) \leq k} wS.$$  

For each $w \in \hat{W}$, let $\varpi \in \text{gr}^k_{lg}H$ be its image under the map quotient. Then we have

$$\text{gr}^k_{lg}H = \bigoplus_{w \in \hat{W}, \ell(w) = k} \varpi S,$$

which is a $S$-bimodule, finitely generated both as left and right $S$-module.

Proposition 25. Let $M \in H\text{-mod}$. Then $M$ lies in $O(H)$ if and only if for some (equiv. for every) good filtration $\{F_k M\}_{k \in \mathbb{Z}}$ with respect to the length filtration of $H$, we have $\dim F_k M < \infty$ for every $k \in \mathbb{Z}$.

Proof. Suppose the good filtration $F_k M$ satisfies the condition of finiteness. Then, as the action of $S$ on $M$ preserves each degree $F_k M$, the action of $S$ on $M$ is locally finite; hence $M \in O(H)$. Conversely, suppose that $M \in O(H)$ and $\{F_k M\}_{k \in \mathbb{Z}}$ is a good filtration with respect to the length filtration $F^k_{lg}H$. Since $\text{gr}^F M$ is finitely generated over $\text{gr}^k_{lg}H$, we can find a finite-dimensional generating graded subspace $V \subset \text{gr}^F M$. Hence we have

$$\text{gr}^F_k M = \bigoplus_{j \in \mathbb{Z}} \text{gr}^F_j H(V \cap \text{gr}^F_{k-j} M) = \bigoplus_{j \in \mathbb{N}, \ell(w) = j} [w]S(V \cap \text{gr}^F_{k-j} M).$$

By the hypothesis that $M \in O(H)$, Corollary 19 implies that the space $S(V \cap \text{gr}^F_{k-j} M)$ is finite-dimensional. Since $\text{gr}^F_{k-j} M$ vanishes for $j \gg 0$ by the finite generation, it follows that $\dim \text{gr}^F_k M < \infty$ and, consequently, $\dim F_k M < \infty$. \hfill \square

1.9 Algebraic Knizhnik–Zamolodchikov functor

We review the “algebraic Knizhnik–Zamolodchikov (KZ) functor” introduced in [10].
1.9.1 Let \((c, \lambda) \in \mathfrak{P} \times \mathfrak{h}^1\). Consider the algebra \(\mathcal{H}_c^\lambda\) introduced in §1.4.4. It has been shown in loc. cit. that there exists an idempotent \(1_\mathbf{V} \in \mathcal{H}_c^\lambda\) such that the idempotent subalgebra \(1_\mathbf{V} \mathcal{H}_c^\lambda 1_\mathbf{V}\) is isomorphic to a block algebra of the affine Hecke algebra. The algebraic KZ functor is defined to be the idempotent truncation \(M \mapsto 1_\mathbf{V}M\) for \(M \in \mathcal{H}_c^\lambda\) - mod.

Let us recall the affine Hecke algebra and its completion. Let \(P = P_R \subseteq \mathfrak{h}_R^*\) be the weight lattice of the root system \((\mathfrak{h}_R, R)\). The extended affine Hecke algebra (for the dual root system \((\mathfrak{h}_R^*, R^\vee)\)), denoted \(\mathbf{K}\), is the associative algebra over the Laurent polynomial ring \(\mathbf{O}_c = \mathbb{C}[t^\pm 1, * \in R/W]\) generated by the two sets \(\{T_\alpha\}_{\alpha \in \Delta}\) and \(\{X^\mu\}_{\mu \in P}\) modulo the following relations for \(\mu, \nu \in P\) and \(\alpha, \beta \in \Delta\):

\[
X^0 = 1, \quad X^{\mu+\nu} = X^{\mu}X^{\nu}, \quad T_\alpha T_\beta T_\gamma \cdots = T_{\beta}T_\alpha T_{\beta} \cdots,
\]

and if \((\mathfrak{h}_R, R)\) is of type \(BC_n\) with \(n \geq 1\), with the notation of Example 1:

\[
\begin{align*}
&T_\alpha X^{\mu} - X^{s_\alpha(\mu)}T_\alpha = \begin{cases} 
(t_\alpha - 1)\frac{X^{\mu} - X^{s_\alpha(\mu)}}{1 - X^{-\alpha}} & \text{for } \alpha \in \{\alpha_1, \ldots, \alpha_{n-1}\}, \\
((t_\alpha t_\beta - 1) + (t_\beta - t_\alpha)X^{\alpha})\frac{X^{\mu} - X^{s_\alpha(\mu)}}{1 - X^{-\alpha}} & \text{for } \alpha = \alpha_n;
\end{cases}
\end{align*}
\]

otherwise:

\[
(T_\alpha - t_\alpha)(T_\alpha + 1) = 0, \quad T_\alpha X^{\mu} - X^{s_\alpha(\mu)}T_\alpha = (t_\alpha - 1)\frac{X^{\mu} - X^{s_\alpha(\mu)}}{1 - X^{-\alpha}}, \quad \text{for } \alpha \in \Delta.
\]

Given a family of parameters \(t = (t_\alpha)_{\alpha \in R/W} \in \mathbb{C}^\times\), we define \(\mathbf{K}_t = \mathbf{K} \otimes_{\mathbf{O}_c} \mathcal{O}_t/(t = t)\) to be the specialisation of parameters.

1.9.2 The elements \(\{X^\mu\}_{\mu \in P}\) generates inside \(\mathbf{K}\) a Laurent polynomial subalgebra isomorphic to the group ring \(\mathbb{Q}[\zeta]\). Let \(T = \text{Spm} \mathbb{C}P\) be its maximal spectrum. For each \(W\)-orbit \([\ell] \in T/W\), let \(\mathbf{K}_t\) - mod \(\subseteq \mathbf{K}_t\) - mod be the full subcategory consisting of finitely generated \(\mathbf{K}_t\) - modules \(M\) which admit the following decomposition:

\[
M = \bigoplus_{\ell' \in [\ell]} M_{\ell'}, \quad M_{\ell'} = \bigcup_{n \geq 0} \{x \in M : m^n_{\ell'}x = 0\}.
\]

According to a well-known result of Bernstein, the ring of invariant functions \(\mathcal{O}(T)^W\) constitutes the centre of \(\mathbf{K}_t\). Let \(\mathcal{H}_t^\ell = (\mathcal{O}(T)^W)^{[\ell]} \otimes_{\mathbf{O}(T)^W} \mathbf{K}_t\). The exact functor

\[(\mathcal{O}(T)^W)^{[\ell]} \otimes_{\mathbf{O}(T)^W} - : \mathbf{K}_t\text{-mod} \to \mathcal{H}_t^\ell\text{-mod}\]

induces an equivalence on subcategories \(\mathcal{H}_t^\ell\text{-mod} \approx \mathcal{H}_t^\ell\text{-mod}_{\text{id}}\).

1.9.3 Given \(c \in \mathfrak{P}\) and \(\lambda = (\lambda^0, 1) \in \mathfrak{h}^1\), consider the spectral completion \(\mathcal{H}_c^\lambda\).

Choose an element \(\gamma\) in the coroot lattice \(Q^\vee = \mathbb{Z}R^\vee\) such that \((\alpha, \gamma) \ll 0\) for every \(\alpha \in R^+\). In particular, \(w^{-1}v_0 - w^{-1}v_1\) lies in a generic clan (Definition 12) for each \(w \in W\). Set \(1_\mathbf{V} = \sum_{[w] \in W/W_\ell} 1_{c,w\lambda + \gamma}\), where \(\ell = \exp(2\pi i \lambda^0)\) in \(T\) and \(W_\ell = \text{Stab}_W(\ell)\). Moreover, we set \(t_* = \exp(2\pi i c_\alpha)\) for \(\alpha \in R/W\).

**Theorem 26** ([10]). Under the above assumptions, there exists an isomorphism \(1_\mathbf{V} \mathcal{H}_c^\lambda 1_\mathbf{V} \cong \mathcal{H}_t^\ell\). Moreover, the following functor

\[
\mathbf{V} : \mathcal{H}_c^\lambda\text{-mod} \to \mathcal{H}_t^\ell\text{-mod}, \quad M \mapsto 1_\mathbf{V}M
\]

is a quotient functor and satisfies the double centraliser property: the restriction of \(\mathbf{V}\) to the subcategory of compact projective objects \(\mathcal{H}_c^\lambda\text{-proj} \subset \mathcal{H}_c^\lambda\text{-mod}\) is fully faithful. \(\square\)
The functor $V$ above is called the algebraic KZ functor.

1.9.4 Let $\mathfrak{P}_2 = \text{Map}(R/W, \mathbb{Z}) \subseteq \mathfrak{P}$ be the set of maps from $R/W$ to $\mathbb{Z}$. Suppose we are given $c, c' \in \mathfrak{P}$ such that their difference $d := c' - c$ lies in $\mathfrak{P}_2$; set $t_s = e^{2\pi i c} = e^{2\pi i c'}$. By Theorem 26, we obtain the respective algebraic KZ functors:

$$\mathcal{H}^\lambda_{d'} - \text{mod} \xrightarrow{\phi^*} \mathcal{H}^\lambda_{d} - \text{mod}$$

In §5, we will construct the translation functor $T_{c' \to c}$ which intertwines the derived functors of $V_c$ and $V_{c'}$.

## 2 Harish-Chandra bimodules

In this section, we introduce the notion of Harish-Chandra (HC) bimodules for the trigonometric DAHA $\mathbf{H}$ and establish their basic properties.

### 2.1 Category of HC bimodules

Let $(A, \{F_k A\}_{k \in \mathbb{N}})$ and $(A', \{F_k A'\}_{k \in \mathbb{N}})$ be filtered rings such that $\text{gr}^F A$ and $\text{gr}^F A'$ are left and right noetherian. Let $M$ be an $(A, A')$-bimodule. We say that a filtration $\{F_k M\}_{k \in \mathbb{Z}}$ on $M$ is **good** if it is exhaustive and separable, $(F_k A)(F_k M)(F_k A') \subseteq F_{j+k+l} M$ holds for each $k, j, l \in \mathbb{Z}$ and $\text{gr}^F M$ is finitely generated as $(\text{gr}^F A, \text{gr}^F A')$-bimodule; we say that $\{F_k M\}_{k \in \mathbb{Z}}$ is **excellent** if it is good and $\text{gr}^F M$ is finitely generated both as left $(\text{gr}^F A)$-module and as right $(\text{gr}^F A')$-module.

**Definition 27.** Let $M$ be a $(\mathbf{H}, \mathbf{H})$-bimodule. A filtration $\{F_k M\}_{k \in \mathbb{Z}}$ on $M$ is called an **Harish-Chandra filtration (HC filtration)** if it is a good filtration with respect to the canonical filtration $F^\text{can}_k \mathbf{H}$ and the action of $\text{ad} z$ on $\text{gr}^F M$ is nilpotent for every $z \in \mathbb{Z}(\text{gr}^F \mathbf{H})$. We say $M$ is a **Harish-Chandra bimodule (HC bimodule)** if it admits a HC filtration.

**Proposition 28.** Every HC filtration is excellent. Consequently, if $M$ be a HC $\mathbf{H}$-bimodule. Then $M$ is finitely generated both as left and right $\mathbf{H}$-module.

**Proof.** See [11, 3.4.3].

Let $\mathbf{HC}(\mathbf{H})$ denote the category of HC $\mathbf{H}$-bimodules. Given $M \in \mathbf{HC}(\mathbf{H})$, if $\{F_k M\}_{k \in \mathbb{Z}}$ and $\{F_k'M\}_{k \in \mathbb{Z}}$ are good filtrations on $M$, then there exists $a \gg 0$ such that $F_{k-a} M \subseteq F_k'M \subseteq F_{k+a} M$ for every $k \in \mathbb{Z}$, see [7, D.1.3]. Consequently, every good filtration on $M$ is automatically HC and therefore excellent by Proposition 28.

**Proposition 29.** $\mathbf{HC}(\mathbf{H})$ is a Serre subcategory of the category of $\mathbf{H}$-bimodules.

**Proof.** It is obvious that the induced filtration of a HC filtration on a sub-bimodule or a quotient bimodule is again a HC filtration; thus $\mathbf{HC}(\mathbf{H})$ is closed under formation of sub-objects and quotient objects. Suppose that $M$ is an $\mathbf{H}$-bimodule, $M' \subset M$ is a sub-$\mathbf{H}$-bimodule and $M'' = M/M'$ such that $M'$ and $M''$ are in $\mathbf{HC}(\mathbf{H})$. Pick a good filtration $\{F_k M\}_{k \in \mathbb{Z}}$ for $M$ and let $\{F_k M'\}_{k \in \mathbb{Z}}$ and $\{F_k M''\}_{k \in \mathbb{Z}}$ be the induced filtrations. We have a short exact sequence

$$0 \to \text{gr}^F M' \to \text{gr}^F M \to \text{gr}^F M'' \to 0. \quad (30)$$

Since $M'$ and $M''$ are HC and since $\{F_k M'\}_{k \in \mathbb{Z}}$ and $\{F_k M''\}_{k \in \mathbb{Z}}$ are good (using (30)), it follows that they are also HC filtrations. We see that the filtration $\{F_k M\}$ is also HC by using (30) again.
**Proposition 31.** Let $M, N \in \text{HC}(\mathbf{H})$. Then, $\text{Tor}^H_i(M, N), \text{Ext}^i_H(M, N) \in \text{HC}(\mathbf{H})$ hold for each $i \in \mathbb{Z}$.

**Proof.** Let $\{F_k M\}_{k \in \mathbb{Z}}$ and $\{F_k N\}_{k \in \mathbb{Z}}$ be HC filtrations on $M$ and $N$ respectively. Consider the Rees bimodules

$$M_\delta = \bigoplus_{k \in \mathbb{Z}} \delta^k F_k M, \quad N_\delta = \bigoplus_{k \in \mathbb{Z}} \delta^k F_k N.$$  

They are finitely generated graded $\mathbf{H}_\delta$-bimodules flat over $\mathbb{C}[\delta]$. The extension group $\text{Ext}^k_{\mathbf{H}_\delta}(M_\delta, N_\delta)$ is a finitely generated graded $\mathbf{H}_\delta$-bimodule. We have an exact sequence

$$\text{Ext}^k_{\mathbf{H}_\delta}(M_\delta, N_\delta) \to \text{Ext}^k_{\mathbf{H}_\delta}(M_\delta, N_\delta) \to \text{Ext}^k_{\mathbf{H}_\delta}(M, N_\delta/\langle \delta \rangle).$$

By the flatness of $M_\delta$ over $\mathbb{C}[\delta]$, there is an isomorphism:

$$\text{Ext}^k_{\mathbf{H}_\delta}(M, N_\delta/\langle \delta \rangle) \cong \text{Ext}^k_{\mathbf{H}_\delta}((\mathbb{C}[\delta]/\langle \delta \rangle) \otimes_{\mathbb{C}[\delta]} M, N_\delta/\langle \delta \rangle) \cong \text{Ext}^k_{\mathbf{gr}^F_H}(\mathbf{gr}^F M, \mathbf{gr}^F N).$$

We obtain an embedding

$$\text{Ext}^k_{\mathbf{H}_\delta}(M_\delta, N_\delta)/\langle \delta \rangle \hookrightarrow \text{Ext}^k_{\mathbf{gr}^F_H}(\mathbf{gr}^F M, \mathbf{gr}^F N).$$

By the assumption, for every $z \in \mathbb{Z}(\mathbf{gr}^F \mathbf{H})$, the operator $(\text{ad } z)^d$ vanishes on $\mathbf{gr}^F M$ and $\mathbf{gr}^F N$ when $d \gg 0$; hence, it is also the case on $\text{Ext}^k_{\mathbf{gr}^F_H}(\mathbf{gr}^F M, \mathbf{gr}^F N)$ and on the sub-bimodule $\text{Ext}^k_{\mathbf{H}_\delta}(M_\delta, N_\delta)/\langle \delta \rangle)$. The $\mathbb{Z}$-grading on $\text{Ext}^k_{\mathbf{H}_\delta}(M_\delta, N_\delta)$ induces a filtration on the quotient

$$\text{Ext}^k_{\mathbf{H}_\delta}(M_\delta, N_\delta)/\langle \delta \rangle \cong \text{Ext}^k_{\mathbf{H}}(M, N),$$

denoted by $F$. It is easy to show that the associated graded $\mathbf{gr}^F \text{Ext}^k_{\mathbf{H}}(M, N)$ is a quotient of $\text{Ext}^k_{\mathbf{H}_\delta}(M_\delta, N_\delta)/\langle \delta \rangle$. It follows that $F$ is a HC filtration on $\text{Ext}^k_{\mathbf{H}}(M, N)$ and the latter is a Harish-Chandra $\mathbf{H}$-bimodule.

Let $\text{D}^b_{\text{HC}}(\mathbf{H})$ be the triangulated subcategory of $\text{D}^b(\mathbf{H} \otimes \mathbf{H}^{op}\text{-Mod})$ consisting of complexes $K$ such that $\mathbf{H}^k(K) \in \text{HC}(\mathbf{H})$ for every $k \in \mathbb{Z}$.

**Corollary 32.** Let $K, L \in \text{D}^b_{\text{HC}}(\mathbf{H})$. Then $K \otimes^L_{\mathbf{H}} L$ and $R\text{Hom}_{\mathbf{H}}(K, L)$ are also in $\text{D}^b_{\text{HC}}(\mathbf{H})$.

**Proof.** It results immediately from Proposition 4 and Proposition 31.

### 2.2 Functors from HC bimodules

For $B \in (\mathbf{H} \otimes \mathbf{H}^{op})\text{-Mod}$, we have the derived tensor product functor and the derived Hom functor:

$$B \otimes^L_{\mathbf{H}} - : \text{D}^b(\mathbf{H}\text{-Mod}) \to \text{D}^b(\mathbf{H}\text{-Mod}), \quad R\text{Hom}_{\mathbf{H}}(B, -) : \text{D}^b(\mathbf{H}\text{-Mod}) \to \text{D}^b(\mathbf{H}\text{-Mod}).$$

**Proposition 33.** For $B \in \text{HC}(\mathbf{H})$, the derived functors $B \otimes^L_{\mathbf{H}} -$ and $R\text{Hom}_{\mathbf{H}}(B, -)$ preserve the subcategories $\text{D}^b_{\text{perf}}(\mathbf{H})$ and $\text{D}^b_{\text{O}}(\mathbf{H})$.

**Proof.** We prove the statements for $R\text{Hom}_{\mathbf{H}}(B, -)$ and leave those for $B \otimes^L_{\mathbf{H}} -$ to the reader. Consider the case where $M = \mathbf{H}$. By Corollary 32, we have
\(\text{RHom}_H(B, H) \in \text{D}^b_{\text{HC}}(H)\). Since HC bimodules are finitely generated left \(H\)-modules, we have \(\text{RHom}_H(B, H) \in \text{D}^b_{\text{perf}}(H)\) when the right \(H\)-module structure is forgotten. Since \(H\) generates \(\text{D}^b_{\text{perf}}(H)\) as thick subcategory, we see that \(\text{RHom}_H(B, M) \in \text{D}^b_{\text{perf}}(H)\) for each \(M \in \text{D}^b_{\text{perf}}(H)\).

Suppose now that \(M \in \text{O}(H)\). We pick a good filtration \(\{F_k M\}_{k \in \mathbb{Z}}\) for \(M\) and, by Lemma 34 below, an excellent filtration \(\{F_k B\}_{k \in \mathbb{Z}}\) for \(B\) with respect to the length filtration \(F_{\bullet}^{\text{lg}} H\) introduced in §1.8. Set

\[
    H_\zeta = \bigoplus_{k \in \mathbb{N}} \zeta^k F_k^{\text{lg}} H, \quad B_\zeta = \bigoplus_{k \in \mathbb{Z}} \zeta^k F_k B, \quad M_\zeta = \bigoplus_{k \in \mathbb{Z}} \zeta^k F_k M.
\]

By Proposition 25, \(F_k M\) is a finite-dimensional \(S\)-module for each \(k \in \mathbb{Z}\). Moreover, \(B_\zeta\) is a \((H_\zeta, H_\zeta)\)-bimodule and finite as left \(H_\zeta\)-module. By taking a resolution of \(B_\zeta\) by finite graded-free left \(H_\zeta\)-modules, we deduce easily that \(\text{gExt}_{H_\zeta}(B_\zeta, M_\zeta)\) is a graded vector space and finite-dimensional in each degree and for each \(k \in \mathbb{Z}\); moreover, it has the structure of graded left \(H_\zeta\)-module coming from the right \(H_\zeta\)-module structure on \(B_\zeta\). By the finite dimensionality, \(S\) acts locally finitely on \(\text{gExt}_{H_\zeta}^k (B_\zeta, M_\zeta)\); this implies that the \(S\)-action on the quotient \(\text{gExt}_{H_\zeta}^k (B_\zeta, M_\zeta)/(\zeta - 1) \cong \text{Ext}_H^k (B, M)\) is also locally finite. It follows that \(\text{Ext}_H^k (B, M) \in \text{O}(H)\) for each \(k \in \mathbb{Z}\).

**Lemma 34.** Let \(B \in \text{HC}(H)\). Then, there exists an excellent filtration \(\{F_k B\}_{k \in \mathbb{Z}}\) for \(B\) with respect to the length filtration \(F_{\bullet}^{\text{lg}} H\).

**Proof.** The proof is modeled on that of [11, 5.4.3]. We first show that for each finite left \(S\)-submodule \(M \subseteq B\), the product \(N = MS\) remains finite as left \(S\)-module. Let \(\{F_k^N B\}_{k \in \mathbb{Z}}\) be a HC filtration (Definition 27) on \(B\) and set \(F_k^N M = M \cap F_k^N B\), \(F_k^N N = N \cap F_k^N B\). Then \(\text{gr}^{F_N} N = (\text{gr}^{F_B} M)(\text{gr}^{\text{can}} S)\) holds, where \(\text{gr}^{\text{can}} S\) is the associated graded of restriction of the canonical filtration (§1.2.3) to \(S\). Since the adjoint action of \((\text{gr}^{\text{can}} S)^W\) on \(\text{gr}^{F_B} B\) is nilpotent and since \(\text{gr}^{\text{can}} S\) is finite over \((\text{gr}^{\text{can}} S)^W\), it follows that \(\text{gr}^{F_N} N\) is also finite over \(\text{gr}^{\text{can}} S\). Hence, \(N\) is finite over \(S\).

Recall the length filtration \(F_{\text{lg}} H\) from §1.8. Now, let \(V \subseteq B\) be a finite-dimensional generating subspace as left \(H\)-module. Let \(\{\tilde{v}_j\}_{j=1}^{r}\) be a spanning set for \(\text{gr}^{\text{lg}} C W\) as left \(\text{gr}^{\text{lg}} (C V)^W\)-module. We assume \(\tilde{b}_k\) is homogeneous of degree \(d'_j > 0\) except that \(\tilde{b}_1 = 1\) and we choose a lifting \(b_j\) of \(\tilde{b}_j\) in \((C V)^W\). Let \(\{\tau_k\}_{k=1}^{r}\) be a set of homogeneous \(C\)-algebra generators for \(\text{gr}^{\text{lg}} (C V)^W\) of degree \(\deg \tau_k = d_k\) and we choose a lifting \(z_k\) of \(\tau_k\) in \((C V)^W\). Define

\[
    F_n B = \sum_{j=1}^{r} \sum_{n_0, n_1, \ldots, n_s \in \mathbb{N}} \text{gr}^{F_{n_0} H} ((\text{ad } z_1)^{n_1} \cdots (\text{ad } z_1)^{n_s} V) b_j S
\]

Let’s show that \(\text{gr}^{F} B\) is finitely generated as left \(\text{gr}^{F} H\)-module. Since the sum

\[
    V' = \sum_{j=1}^{r} \sum_{n_1, \ldots, n_s \in \mathbb{N}} S ((\text{ad } z_1)^{n_1} \cdots (\text{ad } z_s)^{n_s} V) b_j S
\]

is a finite left \(S\)-module by the first paragraph, we can find a finite subset of homogeneous polynomials \(I \subseteq S\) such that the finite-dimensional subspace

\[
    V'' = \sum_{f \in I} \sum_{j=1}^{r} \sum_{n_1, \ldots, n_s \in \mathbb{N}} \mathbb{C} ((\text{ad } z_1)^{n_1} \cdots (\text{ad } z_s)^{n_s} V) b_j f \subseteq V'
\]
Lemma 35. Moreover, this bimodule filtration with respect to $H^F$ is the initial step of the induction. Suppose $n > 0$ and the statement has been proven for smaller $n$. If $j \neq 1$, we have $d_j > 0$: the induction hypothesis applied to $b_j h$ in place of $h$ yields $(\text{ad } z_0)^{n_0} \cdots (\text{ad } z_1)^{n_1} v_0) b_j h \in F_{k+n} B$. Suppose now that $j = 1$. Let $i_0 = \max \{i : m_i > 0\}$. Then

$$(\text{ad } z_{i_0})^{n_0} \cdots (\text{ad } z_1)^{n_1} v_0) h = z_{i_0} ((\text{ad } z_{i_0})^{n_0-1} \cdots (\text{ad } z_1)^{n_1} v_0) h$$

then, the induction hypothesis yields the result. It follows that $F^n B$ is an $H$-bimodule filtration with respect to $F^{k^p} H$ and is good with respect to the left $H$-action. Analogously, we can construct an $H$-bimodule filtration $F^n B$ with respect to $F^{k^p} H$ and is good with respect to the right $H$-action. Since both filtrations are good over $H \otimes H^{op}$ with respect to the filtration $F^F \otimes F^F$, there exists $a > 0$ such that $F^n B \subseteq F_n B \subseteq F_{n+a} B$ for every $n \in \mathbb{Z}$, see [7, D.1.3]. It follows that $F^n B$ is also good for the right $H$-action, thus excellent.

2.3 Spectral completion of HC bimodules

Lemma 35. For each $S$-bimodule $B$ which is finite both as left and as right $S$-module, there are natural isomorphisms of $S$-bimodules

$$B \otimes S \mathcal{F} \cong \bigoplus_{(c, \lambda), (c', \lambda') \in \mathfrak{p} \times \mathfrak{m}_1} \lim_{\to} \oplus B \otimes \bigg( Bm^k_{c, \lambda} + m^l_{c', \lambda'} B \bigg) \cong \mathcal{F} \otimes_S B.$$

Proof. The proof, which we leave to the reader, is an easy exercise of commutative algebra involving the Chinese remainder theorem.

Let $\eta_B : B \otimes S \mathcal{F} \to \mathcal{F} \otimes_S B$ denote the isomorphism given by Lemma 35. The naturality of $\eta_B$ implies that $\eta_B$ can be extended to those $B$ which can be written as colimit of modules satisfying the condition of Lemma 35. Moreover, this isomorphism satisfies the property $\eta_{B \otimes B'} = (\eta_B \otimes \text{id}_{B'}) \circ (\text{id}_B \otimes \eta_{B'})$.

By Lemma 34, every HC bimodule $B \in HC(H)$ admits a filtration $\{F_k B\}$ such that $F_k B$ is a $S$-bimodule satisfying the condition of Lemma 35 for each $k \in \mathbb{Z}$. 

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The multiplication of \( \mathcal{H} \) can be alternatively described as follows:

\[
\mathcal{H} \otimes \mathcal{H} = \mathcal{I} \otimes \mathcal{H} \otimes \mathcal{S} \mathcal{I} \otimes \mathcal{H}
\]

where \( \mu_\mathcal{I} \) and \( \mu_\mathcal{H} \) are the multiplication map of \( \mathcal{I} \) and \( \mathcal{H} \), respectively. The \( \mathcal{H} \)-module structure on \( \mathcal{C}M \) for \( M \in \mathbf{H} \)-Mod can be described in a similar way.

**Proposition 36.** Let \( B \in \mathbf{HC}(\mathbf{H}) \). Then, the \( \mathbf{H} \)-bimodule structure on \( B \) induces an \( \mathcal{H} \)-bimodule structure on the spectral completion \( \mathcal{C}B \).

**Proof.** The left \( \mathcal{H} \)-module structure on \( \mathcal{C}B \) is described in the previous paragraph. The right \( \mathcal{H} \)-module structure is defined as follows:

\[
\mathcal{H} \mathcal{C} \otimes \mathcal{H} = \mathcal{I} \otimes \mathcal{H} B \otimes \mathcal{S} \mathcal{I} \otimes \mathcal{H}
\]

where \( \rho_B \) is the right \( \mathbf{H} \)-action on \( B \). It is easy to verify that the left and right actions on \( B \) commute.

For every \( P \in (\mathcal{H} \otimes \mathcal{H}^{\text{op}}) \)-Mod, let \( \text{Rhom}_{\mathcal{H}}(P, -) \) be the derived functor of the following functor:

\[
\text{hom}_{\mathcal{H}}(P, -) : \mathcal{H} \text{-Mod} \rightarrow \mathcal{H} \text{-Mod}, \quad N \mapsto \bigoplus_{(c, \lambda) \in \mathfrak{P} \times \mathfrak{h}^1} \text{Hom}_{\mathcal{H}}(P1_{c, \lambda}, N).
\]

Its left adjoint functor is given by \( P \otimes^L_{\mathcal{H}} - : D^b(\mathcal{H} \text{-Mod}) \rightarrow D^b(\mathcal{H} \text{-Mod}).

**Proposition 37.** Let \( B \in \mathbf{HC}(\mathbf{H}) \). Then, the following statements hold:

(i) The functor \( \mathcal{C}(B) \otimes^L_{\mathcal{H}} - \) preserves the subcategories \( D^b_{\text{perf}}(\mathcal{H}) \) and \( D^b_{\text{reg}}(\mathcal{H}) \).

(ii) There is a natural isomorphism

\[
\mathcal{C}(B \otimes^L_{\mathbf{H}} M) \cong \mathcal{C}(B) \otimes^L_{\mathcal{H}} \mathcal{C}(M)
\]

for \( M \in \mathbf{H} \text{-mod} \).

**Proof.** We have

\[
\mathcal{C}(B) \otimes^L_{\mathcal{H}} \mathcal{C}(M) \cong B \otimes_{\mathcal{H}} \mathcal{H} \otimes_{\mathcal{H}} \mathcal{H} \otimes_{\mathbf{H}} M
\]

\[
\cong B \otimes_{\mathcal{H}} \mathcal{H} \otimes_{\mathbf{H}} M \cong \mathcal{H} \otimes_{\mathbf{H}} B \otimes_{\mathbf{H}} M = \mathcal{C}(B \otimes_{\mathbf{H}} M).
\]

It is bi-functorial in \( B \) and \( M \). Taking the derived functors, we obtain the quasi-isomorphism:

\[
\mathcal{C}(B \otimes_{\mathbf{H}}^L M) \xrightarrow{\sim} \mathcal{C}(B) \otimes_{\mathcal{H}}^L \mathcal{C}(M).
\] (38)

Given any \( \mathcal{M} \in \mathcal{H} \text{-mod}_{\mathfrak{P}^1} \), we have \( \mathcal{M} \cong \mathcal{C}(M) \) for some \( M \in \mathbf{O}(\mathbf{H}) \) by Theorem 16. Hence, (38) yields \( \mathcal{C}(B) \otimes_{\mathcal{H}}^L \mathcal{M} \cong \mathcal{C}(B \otimes_{\mathbf{H}}^L M) \). By Proposition 33, \( B \otimes_{\mathbf{H}}^L M \) lies in \( D^b_{\text{reg}}(\mathcal{H}) \), so \( \mathcal{C}(B \otimes_{\mathbf{H}}^L M) \) is in \( D^b_{\text{reg}}(\mathcal{H}) \) by Theorem 16 again. Therefore, the functor \( \mathcal{C}(B) \otimes_{\mathcal{H}}^L - \) preserves \( D^b_{\text{reg}}(\mathcal{H}) \).

Given any \( (c, \lambda) \in \mathfrak{P} \times \mathfrak{h}^1 \), the projectivity of \( \mathcal{H}1_{c, \lambda} \) yield \( \mathcal{C}(B) \otimes_{\mathcal{H}}^L \mathcal{H}1_{c, \lambda} \cong \mathcal{C}(B)1_{c, \lambda} \). Let \( \{ b_i \}_{i \in I} \) be a finite generating set of \( B \) as \( \mathbf{H} \)-module. Then, there exists a finite subset \( \Sigma \subset \mathfrak{P} \times \mathfrak{h}^1 \) such that \( b_i1_{c, \lambda} \in \mathcal{C}(B)1_{c, \lambda} \) is decomposed as \( b_i1_{c, \lambda} = \cdots \).
$\sum_{(c', \lambda') \in \Sigma} 1_{c', \lambda'} b_i 1_{c, \lambda}$ for each $i \in I$. Then, the family $\{1_{c', \lambda'} b_i 1_{c, \lambda}\}_{i \in I, (c', \lambda') \in \Sigma}$ yields the following surjective morphism of $\mathcal{H}$-modules:

$$\bigoplus_{i \in I} \bigoplus_{(c', \lambda') \in \Sigma} \mathcal{H} 1_{c', \lambda'} \to \mathcal{C}(B) 1_{c, \lambda},$$

Since the left-hand side is compact, so is right-hand side. As the family $\{\mathcal{H} 1_{c, \lambda}\}_{(c, \lambda) \in \mathfrak{P} \times \mathfrak{h}^1}$ generates $D^b_{\text{perf}}(\mathcal{H})$ as thick subcategory, we see that the functor $\mathcal{C}(B) \otimes^L_{\mathcal{H}} -$ preserves $D^b_{\text{perf}}(\mathcal{H})$.

**Proposition 39.** Let $B \in \text{HC}(\mathcal{H})$. Then, the following statements hold:

(i) The functor $\text{RHom}_{\mathcal{H}}(\mathcal{C}B, -)$ preserves the subcategories $D^b_{\text{perf}}(\mathcal{H})$ and $D^b_{\mathcal{H}}(\mathcal{H})$.

(ii) There is a natural isomorphism

$$\mathcal{C}(\text{RHom}_{\mathcal{H}}(B, N)) \cong \text{RHom}_{\mathcal{H}}(\mathcal{C}(B), \mathcal{C}(N))$$

for $N \in D^b_{\text{perf}}(\mathcal{H})$.

**Proof.** We prove (ii). The morphism is constructed as follows: for $N \in \mathcal{H}\text{-Mod}$, we have

$$\text{Hom}_{\mathcal{H}}(B, N) \to \text{Hom}_{\mathcal{H}}(\mathcal{C}(B), \mathcal{C}(N))$$

by the functoriality of $\mathcal{C}$; it induces

$$\mathcal{C}\text{Hom}_{\mathcal{H}}(B, N) \to \bigoplus_{c, \lambda} 1_{c, \lambda} \text{Hom}_{\mathcal{H}}(\mathcal{C}(B), \mathcal{C}(N)) = \text{hom}_{\mathcal{H}}(\mathcal{C}(B), \mathcal{C}(N));$$

the natural morphism

$$\mathcal{C}\text{RHom}_{\mathcal{H}}(B, N) \to \text{RHom}_{\mathcal{H}}(\mathcal{C}(B), \mathcal{C}(N))$$

(40)

is obtained by passing to the derived functors.

Since $\text{RHom}_{\mathcal{H}}(B, \mathcal{H}) \in D^b_{\text{HC}}(\mathcal{H})$ by **Proposition 31**, the spectral completion $\mathcal{C}(\text{RHom}_{\mathcal{H}}(B, \mathcal{H}))$ is a complex of $\mathcal{H}$-bimodules by **Proposition 36**. Consider first the case $N \in \text{O}(\mathcal{H})$. We have for $(c, \lambda) \in \mathfrak{P} \times \mathfrak{h}^1$

$$1_{c, \lambda} \mathcal{C}(\text{RHom}_{\mathcal{H}}(B, N)) \cong \text{L lim}_{i} \text{RHom}_{\mathcal{S}} \left( S/m^i_{c, \lambda}, \text{RHom}_{\mathcal{H}}(B, N) \right)$$

(40) (Lemma 41)

$$\cong \text{L lim}_{i} \text{RHom}_{\mathcal{H}} \left( B \otimes_{\mathcal{S}} \left( S/m^i_{c, \lambda} \right), N \right)$$

(adjunction)

$$\cong \text{L lim}_{i} \text{RHom}_{\mathcal{H}} \left( \mathcal{C}(B) \otimes_{\mathcal{S}} \left( S/m^i_{c, \lambda} \right), \mathcal{C}(N) \right)$$

(56) (Theorem 16)

$$\cong \text{L lim}_{i} \text{RHom}_{\mathcal{S}} \left( S_{c, \lambda}^{\wedge} / m^i_{c, \lambda}, \text{RHom}_{\mathcal{H}}(\mathcal{C}(B) 1_{c, \lambda}, \mathcal{C}(N)) \right)$$

(adjunction)

$$\cong \text{RHom}_{\mathcal{H}}(\mathcal{C}(B) 1_{c, \lambda}, \mathcal{C}(N))$$

(40) (Lemma 41).

Thus, (40) is a quasi-isomorphism for $N \in \text{O}(\mathcal{H})$. Consider now the case $N = \mathcal{H}$. For $(c, \lambda), (c', \lambda') \in \mathfrak{P} \times \mathfrak{h}^1$, we have

$$1_{c, \lambda} \mathcal{C}(\text{RHom}_{\mathcal{H}}(B, \mathcal{H})) 1_{c', \lambda'}$$

$$\cong \text{R lim}_{i} 1_{c, \lambda} \mathcal{C}(\text{RHom}_{\mathcal{H}}(B, \mathcal{H})) 1_{c', \lambda'} \otimes_{\mathcal{S}_{c', \lambda'}^{\wedge}} (S/m^i_{c', \lambda'})$$

(40) (Lemma 41)

$$\cong \text{R lim}_{i} 1_{c, \lambda} \mathcal{C}(\text{RHom}_{\mathcal{H}}(B, \mathcal{H}) \otimes_{\mathcal{S}} (S/m^i_{c', \lambda'}))$$

$$\cong \text{R lim}_{i} \text{RHom}_{\mathcal{H}}(\mathcal{C}(B) 1_{c, \lambda} \mathcal{H} 1_{c', \lambda'} \otimes_{\mathcal{S}_{c', \lambda'}^{\wedge}} (S/m^i_{c', \lambda'}))$$

(case $N \in \text{O}(\mathcal{H})$)

$$\cong \text{RHom}_{\mathcal{H}}(\mathcal{C}(B) 1_{c, \lambda} \mathcal{H} 1_{c', \lambda'}) = \text{RHom}_{\mathcal{H}}(\mathcal{C}(B) 1_{c, \lambda} \mathcal{H}) 1_{c', \lambda'})$$

(Mittag-Leffler)
Thus, (40) is a quasi-isomorphism for $N = H$. Since $H$ generates $D^b_{\text{perf}}(H)$ as thick subcategory, this proves (ii).

We prove (i). Given any $\mathcal{N} \in \mathcal{H} \text{-mod}_R$, we have $\mathcal{N} \cong \mathcal{C}(N)$ for some $N \in \text{O}(H)$ by Theorem 16. Hence, (ii) yields $\text{RHom}_\mathcal{H}(\mathcal{C}(B), \mathcal{N}) \cong \mathcal{C}(\text{RHom}_H(B, N))$. By Proposition 33, $\text{RHom}_H(B, N)$ lies in $D^b_0(H)$, so $\mathcal{C}(\text{RHom}_H(B, N))$ lies in $D^b_b(\mathcal{H})$ by Theorem 16 again. Therefore, the functor $\text{RHom}_\mathcal{H}(\mathcal{C}(B), -)$ preserves $D^b_b(\mathcal{H})$.

Let $(c, \lambda) \in \mathfrak{g} \times h^1$. By (ii), we have

$$\text{RHom}_\mathcal{H}(\mathcal{C}(B), \mathcal{H}1_{c,\lambda}) \cong \mathcal{C}(\text{RHom}_H(B, H)1_{c,\lambda}) \cong \mathcal{C}(\text{RHom}_H(B, H)) \otimes_{\mathcal{H}} \mathcal{H}1_{c,\lambda}.$$ 

Since $\text{RHom}_H(B, H)$ is $D^b_{\text{fl}}(H)$ by Proposition 31, we have $\mathcal{C}(\text{RHom}_H(B, H)) \otimes_{\mathcal{H}} \mathcal{H}1_{c,\lambda}$ lies in $D^b_{\text{fl}}(\mathcal{H})$ by Proposition 37. Since the family $\{\mathcal{H}1_{c,\lambda}\}_{(c,\lambda) \in \mathfrak{g} \times h^1}$ generates $D^b_{\text{fl}}(\mathcal{H})$ as thick subcategory, it follows that functor $\text{RHom}_\mathcal{H}(\mathcal{C}(B), -)$ preserves $D^b_{\text{fl}}(\mathcal{H})$. This proves (i).

**Lemma 41.** Let $A$ be a complete noetherian local ring with maximal ideal $m$. Then the following natural morphisms for $K \in D^b_{\text{perf}}(A)$ and $L \in D^b_0(A)$ are quasi-isomorphisms:

$$K \to \text{R lim}_k(A/m^k \otimes_A K), \quad \text{L lim}_k \text{RHom}(A/m^k, L) \to L.$$ 

**Proof.** The first morphism is a quasi-isomorphism: as it is so for the regular $A$-module, also is it for every perfect complex $K \in D^b_{\text{perf}}(A)$ by dévissage. As for the second morphism, Nakayama’s lemma implies that $\text{lim}_k \text{Hom}_A(A/m^k, L) \to L$ is an isomorphism for $L \in A\text{-mod}_R$. Given $L \in A\text{-mod}_R$, let $L \to I$ be an injection with $I$ injective. We may find such $I$ whose finitely generated submodules are of finite length. Then, by the compactness of $A/m^k$, an easy argument of dévissage shows that $\text{lim}_k \text{Ext}^n_A(A/m^k, L) = 0$ for $n \geq 1$. 

3 Category of divided-difference calculus

In this section, we introduce a small $\mathbb{C}$-linear category $A$, which realises the trigonometric DAHA $H$ as the endomorphism ring of an object.

3.1 Chambers and galleries

Recall $\Phi$ from §1.1. Consider the affine space $\mathfrak{a} := \mathfrak{g} \times h^1_R$ and the family of functions:

$$\Psi = \left\{ \alpha - c_\alpha \in \mathfrak{g}^* \times \tilde{h}^*_\mathbb{Q} ; \alpha \in \Phi \right\},$$

where $\tilde{h}^*_\mathbb{Q}$ is the $\mathbb{Q}$-linear span of $\Phi$ and $\mathfrak{g}^* \times \tilde{h}^*_\mathbb{Q}$ is the $\mathbb{Q}$-linear span of $\{c_\alpha\}_{\alpha \in R/\Phi}$. For $\mu \in \Psi \cup \Phi$, let $H_\mu \subset \mathfrak{a}$ be its zero locus. Then, $\{H_\mu ; \mu \in \Psi \cup \Phi\}$ defines an affine hyperplane arrangement on $\mathfrak{a}$. The hyperplanes $H_\mu$ are called walls. Put

$$\text{Ch}(\mathfrak{a}) = \pi_0 \left( \mathfrak{a} \setminus \bigcup_{\mu \in \Psi \cup \Phi} H_\mu \right),$$

the elements of which are called chambers. For each chamber $C \in \text{Ch}(\mathfrak{a})$, let

$$\Phi^+_C = \left\{ \alpha \in \Phi ; \pm \alpha(C) \subset \mathbb{R}_{>0} \right\}, \quad \Psi^+_C = \left\{ \mu \in \Psi ; \pm \mu(C) \subset \mathbb{R}_{>0} \right\}.$$
For \( C_1, C_2 \in \text{Ch}(a) \), we write \( C_1 \upharpoonright \mu C_2 \) for \( \mu \in \Psi \) if \( C_1 \neq C_2 \) and they share a face with support \( H_\mu \) and \( \mu(C_1) < 0, \mu(C_2) > 0 \); we write \( C_1 \upharpoonright \alpha C_2 \) for \( \alpha \in \Phi \) if \( C_1 \neq C_2 \) and they share a face with support \( H_\alpha \) and \( \alpha(C_1) < 0, \alpha(C_2) > 0 \). We write \( C_1 \sim C_2 \) if \( C_1 \) and \( C_2 \) lie in the same connected component of \( a \setminus \bigcup_{\alpha \in \Phi} H_\alpha \).

A sequence of chambers \( G = (C_0, \ldots, C_n) \) is called a gallery (from \( C_0 \) to \( C_n \)) if, for \( i = 0, \ldots, n - 1 \), the chambers \( C_i \) and \( C_{i + 1} \) share a face and \( C_i \neq C_{i + 1} \); we denote \( \ell(G) = n \) and call it the length of \( G \). The distance \( d(C, C') \) is defined to be the minimum of the length of galleries from \( C \) to \( C' \). A gallery \( G = (C_0, \ldots, C_n) \) is called minimal if \( \ell(G) = d(C_0, C_n) \). Given chambers \( C, C' \in \text{Ch}(a) \), we define the interval between \( C \) and \( C' \) to be

\[
[C, C'] = \{ C'' \in \text{Ch}(a) ; \; d(C, C'') = d(C, C') + d(C'', C') \}.
\]

**Lemma 4.2.** The following statements hold:

(i) Let \( C, C', C'' \in \text{Ch}(a) \) be chambers. Then, \( C' \in [C, C''] \) holds if and only if the following conditions hold:

\[
\Phi^+_C \cap \Phi^+_{C''} \subseteq \Phi^+_C \triangleleft \Phi^+_C \cup \Phi^+_{C''}, \quad \Phi^+_C \cap \Phi^+_{C'} \subseteq \Phi^+_C \triangleleft \Phi^+_C \cup \Phi^+_{C'}.
\]

(ii) A gallery \( G = (C_0, \ldots, C_n) \) is minimal if and only if for each triplet \( 0 \leq i_1 < i_2 < i_3 \leq n \), we have \( C_{i_2} \in [C_{i_1}, C_{i_3}] \).

**Proof.** The first assertion results immediate from the fact that for \( C, C' \in \text{Ch}(a) \), the distance \( d(C, C') \) is the number of walls separating \( C \) and \( C' \). The second can be easily derived from the first one. \[ \square \]

If \( G = (C_0, \ldots, C_n) \) and \( G' = (C_n, \ldots, C_{n+m}) \) are galleries, their composite is defined to be \( GG' = (C_0, \ldots, C_{n+m}) \). The gallery opposite to a gallery \( G = (C_0, \ldots, C_n) \) is defined to be \( G^\circ = (C_n, \ldots, C_0) \); for \( w \in W \), the transport of \( G \) by \( w \) is defined to be \( wG = (wC_0, \ldots, wC_n) \).

### 3.2 The category \( A^o \)

#### 3.2.1 Recall \( S = \mathcal{O}(\mathfrak{g} \times \mathfrak{b}^+) \). Let \( A^o \) be the \( \mathbb{C} \)-linear category whose objects are \( \text{Ch}(a) \) and whose hom-spaces \( \text{Hom}_{A^o}(C, C') \) are subspaces of \( \text{End}_C(S) \) for \( C, C' \in \text{Ch}(a) \) to be defined below.

For each \( C, C' \in \text{Ch}(a) \) such that \( C' \upharpoonright _\alpha C \) for \( \alpha \in \Phi \), let \( \tau^\circ_{C', C} = \theta_\alpha \in \text{End}_C(S) \), where \( \theta_\alpha : f \mapsto \alpha^{-1}(f - \varphi \alpha f) \) is the Demazure operator.

For chambers \( C, C' \in \text{Ch}(a) \) such that \( C' \sim C' \), put \( \tau^\circ_{C, C'} = 1 \). For each gallery \( G = (C_0, \ldots, C_n) \), put \( \tau^G = \tau^\circ_{C'_n, C_n} \cdots \tau^\circ_{C_0, C_1} \in \text{End}_C(S) \). For \( C, C' \in \text{Ch}(a) \) the morphisms are given by

\[
\text{Hom}_{A^o}(C, C') = \sum_{w \in W} \sum_{G=(C,C_{i-1}, C_{i-1}, \ldots, C_{i-1}, w^{-1}C')} w \tau^G \mathcal{S} \subseteq \text{End}_C(S).
\]

The composition of morphisms is the usual composition of linear maps. For \( C, C' \in \text{Ch}(a) \), let \( 1_{C', C'} = 1_S \in \text{Hom}_{A^o}(C, C') \) denote the identity map.

#### 3.2.2 The canonical filtration on the hom-spaces of \( A^o \) is defined to be

\[
F^n_{\text{can}} \text{Hom}_{A^o}(C, C') = \{ f \in \text{Hom}_{A^o}(C, C') ; \; \forall m \in \mathbb{Z}, f(S_{\leq m}) \subseteq S_{\leq m+n} \}
\]
for $C, C' \in \text{Ch}(a)$ and $n \in \mathbb{Z}$, where $S_{\leq m}$ stands for polynomial functions on $\Phi \times \mathfrak{h}^1$ of order $\leq m$. The length filtration is defined to be

$$F_n^{\text{lg}} \text{Hom}_{A^w}(C, C') = \sum_{w \in W} \sum_{\ell(G) \leq n} w_{\tau_G}^0 S.$$ 

Both filtrations are compatible with composition of morphisms.

3.2.3 Let $C, C' \in \text{Ch}(a)$. For $w \in \hat{W}$, choose a minimal gallery $G_w$ from $C$ to $w^{-1}C'$ and let

$$\tau_{G_w}^0 = w \tau_{G_w}^0 \in F_{\ell(G)-1} \text{Hom}_{A^w}(C, C').$$

Let $\tau_{C,w}$ denote the image of $\tau_{G_w}^0$ in the quotient

$$\text{gr}_{d(C,w^{-1}C')}^0 \text{Hom}_{A^w}(C, C') = F_{d(C,w^{-1}C')}^0 \text{Hom}_{A^w}(C, C')/F_{d(C,w^{-1}C')-1}^0 \text{Hom}_{A^w}(C, C').$$

**Lemma 43.** Given $C, C' \in \text{Ch}(a)$, the following statements hold:

(i) if $G = (C, \cdots, w^{-1}C')$ is a non-minimal gallery, then $w \tau_G^0 \in F_{\ell(G)-1} \text{Hom}_{A^w}(C, C')$ holds.

(ii) $\tau_{C,w}^0$ is independent of the choice of the minimal gallery $G_w$ for $w \in \hat{W}$;

(iii) $(\tau_{C,w})_{w \in \hat{W}}$ forms a basis for $\text{gr}_{d(C,w^{-1}C')}^0 \text{Hom}_{A^w}(C, C')$ as left and right free $S$-module.

**Proof.** According to [9, 11.1.2], the Demazure operators $\vartheta_{\alpha} \in \text{End}(S)$ are square-zero $\vartheta_{\alpha}^2 = 0$ and satisfy the braid relations for $\alpha, \beta \in \Delta$, $\alpha \neq \beta$:

$$\vartheta_{\alpha} \vartheta_{\beta} \vartheta_{\alpha} \cdots = \vartheta_{\beta} \vartheta_{\alpha} \vartheta_{\beta} \cdots, \quad m_{\alpha, \beta} = \text{ord}(s_{\alpha}s_{\beta}).$$

For $w \in \hat{W}$, choose a reduced decomposition $w = s_{\beta_1} \cdots s_{\beta_k}$ for $w$ and put $\vartheta_w = \vartheta_{\beta_1} \cdots \vartheta_{\beta_k}$, which is, in view of the braid relations, independent of the choice of the reduced decomposition. Moreover, the family $\{\vartheta_w\}_{w \in \hat{W}}$ is free in $\text{End}_{\mathbb{C}}(S)$ both as left and as right $S$-modules, see [9, 11.1.3].

Let $\kappa_0 \in \text{Ch}(a)$ be the chamber which contains $\nu_0 \times \{0\}$, where $\nu_0 \subseteq \mathfrak{h}^1$ is the fundamental alcove. We deduce easily that $\tau_{C,w}^0 = y'y^{-1}_w y^{-1}_y y^{-1}$, where $y, y' \in \hat{W}$ are such that $y\kappa_0 \sim C$ and $y'\kappa_0 \sim C'$ hold. Now, given $w \in \hat{W}$ and a gallery $G = (C, \cdots, w^{-1}C')$, we have either $w \tau_G^0 = \tau_{C,w}^0$ or $\tau_G^0 = 0$. The assertions follow easily from this. $\square$

3.3 The category $A$

3.3.1 For $C, C' \in \text{Ch}(a)$, we define two invariants in $S$:

$$\vartheta(C, C') = \prod_{\mu \in \Psi^+_{C'} \cap \Psi_C} \mu, \quad \epsilon(C, C') = \prod_{\alpha \in \Phi^+_{C'} \cap \Phi_C} \alpha.$$ 

For $C, C' \in \text{Ch}(a)$ such that $C \sim C'$, let

$$\tau_{C,C'} \in \text{Hom}_{A^w}(C, C'), \quad f \mapsto f \vartheta(C, C').$$

For each $C, C' \in \text{Ch}(a)$ such that $C' \| C$ for $\alpha \in \Phi$, let

$$\tau_{C,C'} \in \text{Hom}_{A^w}(C, C'), \quad f \mapsto \vartheta_{\alpha}(f) = \alpha^{-1}(f - s_{\alpha} f).$$

Moreover, for $C \in \text{Ch}(a)$ and $w \in \hat{W}$, let $w_C \in \text{Hom}_{A^w}(C, wC)$ be given by $w_C : f \mapsto w(f)$.

For every gallery $G = (C_0, \cdots, C_n)$, we put $\tau_G = \tau_{C_{n-1},C_n} \cdots \tau_{C_0,C_1} \in \text{Hom}_{A^w}(C_0, C_n)$. Similarly, we put $\vartheta(G) = \prod_{0 \leq i \leq n-1} \vartheta(C_i, C_{i+1})$ and $\epsilon(G) = \prod_{0 \leq i \leq n-1} \epsilon(C_i, C_{i+1})$. 24
3.3.2 Let \( A \subset A^o \) be the subcategory with the same objects whose morphisms are given by

\[
\text{Hom}_A(C, C') = \sum_{w \in W} \sum_{G=(C,C_1,...,C_{n-1},w^{-1}C')} w \tau_G S. \tag{44}
\]

For \( n \in \mathbb{Z} \), let \( F_n^{\text{lg}} \text{Hom}_A(C, C') \subset \text{Hom}_A(C, C') \) be the sum in (44) with the second summation taken over galleries \( G \) with \( \ell(G) \leq n \).

The canonical filtration \( F_n^{\text{can}} \text{Hom}_A(C, C') \) for \( C, C' \in \text{Ch}(a) \) is defined to be the induced filtration from \( F_n^{\text{can}} \text{Hom}_{A^o}(C, C') \).

3.4 Basis theorem

Let \( C, C' \in \text{Ch}(a) \). For \( w \in \hat{W} \), choose a minimal gallery \( G_w \) from \( C \) to \( w^{-1}C' \) and let

\[
\tau_{C,C',w} = w \tau_{G_w} \in F_{d(C,w^{-1}C')}^{\text{lg}} \text{Hom}_A(C, C').
\]

Let \( \tau_{C,C',w} \) denote its image in the quotient:

\[
\text{gr}^{\text{lg}}_{d(C,w^{-1}C')} \text{Hom}_A(C, C') = F_{d(C,w^{-1}C')}^{\text{lg}} \text{Hom}_A(C, C')/F_{d(C,w^{-1}C')}^{\text{lg}}-1 \text{Hom}_A(C, C').
\]

**Theorem 45.** For \( C, C' \in \text{Ch}(a) \) and \( w \in \hat{W} \), the element \( \tau_{C,C',w} \) is independent of the choice of the minimal gallery \( G_w \); moreover, \( \{\tau_{C,C',w}\}_{w \in \hat{W}} \) forms a basis for \( \text{gr}^{\text{lg}} \text{Hom}_A(C, C') \) as left and right free \( S \)-module.

**Proof.** We will only prove the right freeness, the left freeness being similar. We prove by induction on \( n \in \mathbb{N} \) the following statements for all chambers \( C, C' \in \text{Ch}(a) \):

(i) if \( G = (C, C_1, \ldots, C_{n-1}, C') \) is a gallery, then

\[
\tau_G - \tau_{G-}^o \vartheta(G) \in F_{n-1}^{\text{lg}} \text{Hom}_{A^o}(C, C'), \quad \tau_G \vartheta(G) - 1_{C,C'} \vartheta(G) \in F_{n-1}^{\text{lg}} \text{Hom}_A(C, C');
\]

(ii) if \( G = (C, C_1, \ldots, C_{n-1}, C') \) is a non-minimal gallery, then \( \tau_G \in F_{n-1}^{\text{lg}} \text{Hom}_A(C, C') \);

(iii) if \( G = (C, C_1, \ldots, C_{n-1}, C') \) and \( G' = (C, C_1', \ldots, C'_{n-1}, C') \) are minimal galleries, then \( \tau_G - \tau_{G'} \in F_{n-1}^{\text{lg}} \text{Hom}_A(C, C') \);

(iv) the set \( \{\tau_{C,C',w}\}_{w \in \hat{W}} \) for \( d(C,w^{-1}C')=n \) forms a basis for \( \text{gr}^{\text{lg}} \text{Hom}_A(C, C') = F_n^{\text{lg}} \text{Hom}_A(C, C')/F_{n-1}^{\text{lg}} \text{Hom}_A(C, C') \) as right \( S \)-module;

(v) the quotient

\[
F_n^{\text{lg}} \text{Hom}_{A^o}(C, C')/F_n^{\text{lg}} \text{Hom}_A(C, C')
\]

is right \( \alpha \)-torsion-free for every \( \alpha \in \Phi \).

For \( n = 0 \), all these five statements are trivial. Let \( n \geq 1 \) and suppose that the statements are proven for \( 0, \ldots, n-1 \). We prove them for \( n \):

(i) Let \( G^- = (C_1, \ldots, C_{n-1}, C') \). We have

\[
\tau_G - \tau_{G^-}^o \vartheta(G) = (\tau_{G^-} - \tau_{G^-}^o \vartheta(G^-))\tau_{C,C_1} + \tau_{G^-} \vartheta(G^-) \tau_{C,C_1} - \tau_{C,C_1} \vartheta(G) = \tau_{C,C_1} \vartheta(G^-) + \vartheta(G^-) s_\alpha.
\]

The term \( \tau_{G^-} - \tau_{G^-}^o \vartheta(G^-) \) lies in \( F_{n-2}^{\text{lg}} \text{Hom}_{A^o}(C_1, C') \) by induction hypothesis. If \( C_1 |_\alpha C_1 \) for \( \alpha \in \Phi \), then \( \tau_{C,C_1} = \tau_{C,C_1} \vartheta = \vartheta_{\alpha}(\vartheta(G^-))s_\alpha \) is the Demazure operator and thus

\[
\vartheta(G^-) \tau_{C,C_1} = \tau_{C,C_1} \vartheta(G^-) + \vartheta_{\alpha}(\vartheta(G^-)) s_\alpha.
\]
Since $\partial(G) = \partial(G^-)$ and $\vartheta_\alpha(\partial(G^-)) s_\alpha \in F_{0}^{\text{lg}} \text{Hom}_{A^{\alpha}}(C, C')$, the second term $\tau_{G^-}^o(\partial(G^-) \tau_{C, C_1} - \tau_{C, C_1}^o \partial(G))$ lies in $F_{n-1}^{\text{lg}} \text{Hom}_{A^{\alpha}}(C, C')$. If $C|C_1$ or $C_1|C$ holds, then $\partial(G^-) \tau_{C, C_1} = \partial(G^-) \partial(C, C_1) = \partial(G) = \tau_{C, C_1}^o \partial(G)$, so $\tau_{G^-}^o(\partial(G^-) \tau_{C, C_1} - \tau_{C, C_1}^o \partial(G)) = 0$. It follows that $\tau_{G^-}^o \partial(G) \in F_{n-1}^{\text{lg}} \text{Hom}_{A^{\alpha}}(C, C')$ in all cases. The other statement can be proven similarly.

(ii) By (i), we have

$$\tau_G - \tau_{G^o}^o \partial(G) \in F_{n-1}^{\text{lg}} \text{Hom}_{A^{\alpha}}(C, C').$$

When $G$ is non-minimal, $\tau_{G^o}^o \partial(G) \in F_{n-1}^{\text{lg}} \text{Hom}_{A^{\alpha}}(C, C')$ holds by Lemma 43 (i); hence $\tau_G \in F_{n-1}^{\text{lg}} \text{Hom}_{A^{\alpha}}(C, C')$. On the other hand,

$$\tau_G e(G) \in F_{n-1}^{\text{lg}} \text{Hom}_{A}(C, C').$$

Let $\tau_{G}$ be the image of $\tau_G$ in $F_{n-1}^{\text{lg}} \text{Hom}_{A^{\alpha}}(C, C') / F_{n-1}^{\text{lg}} \text{Hom}_{A}(C, C')$. It follows that $\tau_G e(G) = 0$. By induction hypothesis, the space

$$F_{n-1}^{\text{lg}} \text{Hom}_{A^{\alpha}}(C, C') / F_{n-1}^{\text{lg}} \text{Hom}_{A}(C, C')$$

is $\alpha$-torsion-free for each $\alpha \in \Phi$ and therefore $e(G)$-torsion-free; hence $\tau_G = 0$ and subsequently $\tau_G \in F_{n-1}^{\text{lg}} \text{Hom}_{A}(C, C')$ holds.

(iii) By (i), we have $\tau_G - \tau_{G^o}^o \partial(G) \in F_{n-1}^{\text{lg}} \text{Hom}_{A^{\alpha}}(C, C')$ and $\tau_{G^o}^o \partial(G') \in F_{n-1}^{\text{lg}} \text{Hom}_{A^{\alpha}}(C, C')$. Since $G$ and $G'$ are minimal, we have $\partial(G) = \partial(G') = \partial(C, C')$. On the other hand, the minimality also implies $\tau_{G^o}^o = \tau_{G^o}^o$ by Lemma 43 (ii). Therefore

$$\tau_G - \tau_{G^o}^o \partial(G^o) \in F_{n-1}^{\text{lg}} \text{Hom}_{A^{\alpha}}(C, C')$$

holds. On the other hand, we have $e(G) = e(G') = e(C, C')$; therefore, (i) yields:

$$(\tau_G - \tau_{G^o}^o) e(C, C') = e(G) - \tau_{G^o}^o e(G') \in \tau_{G^o}^o \partial(G) - \tau_{G^o}^o \partial(G') + F_{n-1}^{\text{lg}} \text{Hom}_{A}(C, C');$$

hence the image of $\tau_G - \tau_{G^o}^o$ in $F_{n-1}^{\text{lg}} \text{Hom}_{A}(C, C') / F_{n-1}^{\text{lg}} \text{Hom}_{A^{\alpha}}(C, C')$ is annihilated by right multiplication by $e(C, C')$. The induction hypothesis implies that $F_{n-1}^{\text{lg}} \text{Hom}_{A^{\alpha}}(C, C') / F_{n-1}^{\text{lg}} \text{Hom}_{A}(C, C')$ is right $e(C, C')$-torsion-free, so $\tau_G - \tau_{G^o}^o \partial(G') \in F_{n-1}^{\text{lg}} \text{Hom}_{A}(C, C')$ holds.

(iv) By (ii) and (iii), the space $g_{n}^{\text{lg}} \text{Hom}_{A}(C, C')$ is spanned by $\{\tau_{C, C'}, w\}_{d(C, w^{-1} C') = n}$ as right $S$-module. This family is free — indeed, suppose there is a $S$-linear relation

$$\sum_w \tau_{C, C', w} f_w = 0,$$

then, we have $\sum_w \tau_{C, C', w} f_w \in F_{n-1}^{\text{lg}} \text{Hom}_{A^{\alpha}}(C, C')$; by (i), we have

$$\sum_w \tau_{C, C', w} \partial(C, w^{-1} C') f_w \in F_{n-1}^{\text{lg}} \text{Hom}_{A^{\alpha}}(C, C');$$

since the family $\{\tau_{C, C', w}\}_{d(C, w^{-1} C') = n}$ is free over $S$ by Lemma 43 (iii), we see that $\partial(C, w^{-1} C') f_w = 0$ for each $w$; hence $f_w = 0$. 

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Consider the following diagram:

\[
\begin{array}{c}
0 \rightarrow F^{b_k}_{n-1} \Hom_{A}(C, C') \rightarrow F^{b_k}_{n} \Hom_{A}(C, C') \rightarrow \gr_{n}^{b_k} \Hom_{A}(C, C') \rightarrow 0 \\
\downarrow \phi' \quad \downarrow \phi \quad \downarrow \phi'' \\
0 \rightarrow F^{b_k}_{n-1} \Hom_{A^\alpha}(C, C') \rightarrow F^{b_k}_{n} \Hom_{A^\alpha}(C, C') \rightarrow \gr_{n}^{b_k} \Hom_{A^\alpha}(C, C') \rightarrow 0.
\end{array}
\]

The maps \(\phi\) and \(\phi'\) are the natural inclusions. By (i) and (iv), \(\coker(\phi'')\) is isomorphic to

\[
\bigoplus_{w} \tau_{C',w} S/\phi(\tau_{C,C',w} S) \cong \bigoplus_{w} S/\hat{S}(C, w^{-1}C')
\]

and \(\phi''\) is injective. Since \(\hat{S}(C, w^{-1}C')\) is not divisible by any element of \(\Phi\), the cokernel \(\coker(\phi'')\) is right \(\alpha\)-torsion-free for \(\alpha \in \Phi\). The induction hypothesis implies that \(\coker(\phi')\) is right \(\alpha\)-torsion-free; hence, \(\coker(\phi)\) is also right \(\alpha\)-torsion-free by snake lemma.

\[\square\]

### 3.5 Translation symmetry

The lattice \(\mathfrak{P}_Z \subset \mathfrak{P}\) acts on \(\mathfrak{P}_Q \times \mathfrak{h}_Q\) as follows: for \(d \in \mathfrak{P}_Z\), let \(t_d : \mathfrak{P}_Q \times \mathfrak{h}_Q \rightarrow \mathfrak{P}_Q \times \mathfrak{h}_Q\) be the translation \((c, z) \mapsto (\delta(z)d + c, z)\). This action extends \(\mathbb{R}\)-linearly to \(\mathfrak{P}_R \times \mathfrak{h}_R\) and restricts to \(\mathfrak{a}\). The contragradient \(\mathfrak{P}_Z\)-action on the dual \(\mathfrak{P}_Q^* \times \mathfrak{h}_Q^*\) preserves the subsets \(\Phi\) and \(\Psi\). Consequently, the \(\mathfrak{P}_Z\)-action on \(\mathfrak{a}\) induces a \(\mathfrak{P}_Z\)-action on \(\operatorname{Ch}(\mathfrak{a})\).

Similarly, the scalar extension of contragradient action induces a \(\mathfrak{P}_Z\)-action on \(S \cong \operatorname{Sym}(\mathfrak{P}^* \times \mathfrak{h}^*)/\langle\delta - 1\rangle\) by ring automorphisms, denoted by \(t_d^*: S \rightarrow S\) for \(d \in \mathfrak{P}_Z\).

We define a \(\mathfrak{P}_Z\)-action on \(A\) as follows: define \(t_d : A^\alpha \rightarrow A^\alpha\) by setting

\[\operatorname{Ch}(\mathfrak{a}) \ni C \mapsto t_d(C) \in \operatorname{Ch}(\mathfrak{a})\]

\[\Hom_{A^\alpha}(C, C') \ni a \mapsto (t_d)_*(a) := (t_d^*)^{-1} \circ a \circ t_d \in \Hom_{A^\alpha}(t_d(C), t_d(C')).\]

The automorphism \(t_d\) preserves the subcategory \(A \subset A^\alpha\); indeed, we have \((t_d)_* \tau_G = \tau_{t_d(G)}\) for every gallery \(G\), \((t_d)_* w = w\) in \(\operatorname{End}_C(S)\) for every \(w \in \hat{W}\) and \((t_d)_*(f) = (t_d^*)^{-1}(f)\) for every \(f \in S\).

### 3.6 Embedding of \(H\)

Let \(\kappa_0 \in \operatorname{Ch}(\mathfrak{a})\) be the chamber which contains \(\nu_0 \times \{0\}\), where \(\nu_0 \subseteq \mathfrak{h}_R^1\) is the fundamental alcove. For \(d \in \mathfrak{P}_Z\), let \(\kappa_d = t_d(\kappa_0) = \kappa_0 + (d, 0) \in \operatorname{Ch}(\mathfrak{a})\).

**Theorem 46.** For each \(d \in \mathfrak{P}_Z\), there is an isomorphism of rings

\[t_d : H \cong \operatorname{End}_A(\kappa_d)\]

\[f \mapsto (t_d)_* f \quad \forall f \in S\]

\[1 - s_\alpha \mapsto (\alpha - (c_\alpha - d_\alpha)) \vartheta_\alpha \quad \forall \alpha \in \hat{A},\]

where \((t_d)_* f \in S\) is given by \(((t_d)_* f)(c, z) = f(c - d, z)\) for \((c, z) \in \mathfrak{P} \times \mathfrak{h}^1\).

**Proof.** Let us consider first the case \(d = 0\). Note that since \(((\alpha - c_\alpha) \vartheta_\alpha)^2 = 2(\alpha - c_\alpha) \vartheta_\alpha\) and \(\vartheta_\alpha f - s_\alpha (f) (\alpha - c_\alpha) \vartheta_\alpha = (\alpha - c_\alpha) \vartheta_\alpha (f)\), we have a homomorphism \(\iota_0 : H \rightarrow \operatorname{End}_C(S)\) given by the formula in the statement. We shall prove the following:

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Let $\alpha \in \hat{\Delta}$. We have $\mathfrak{d}(\kappa_0, s_\alpha \kappa_0) = -\alpha - c_\alpha$ and $\mathfrak{d}(s_\alpha \kappa_0, \kappa_0) = \alpha - c_\alpha$. The intersection $\overline{\mathfrak{r}} \cap H_\alpha \subseteq H$ is the closure of a face of the fundamental alcove $\nu_0$; let $z \in \overline{\mathfrak{r}} \cap H_\alpha$ be a point in its relative interior. For $\epsilon > 0$ small enough, the point $(z, \epsilon) \in a$ lies in a connected component of $a \setminus \bigcup_{\psi \in \Psi} H_\psi$, denoted by $Z$. The complement $Z \setminus H_\alpha$ consists of two chambers, one of which shares a wall with $\kappa_0$ with support $H_\alpha - c_\alpha$; denote this chamber by $C_\alpha \in \text{Ch}(a)$.

Then, it follows that $C_\alpha[\alpha - c_\alpha \kappa_0, s_\alpha(C_\alpha)]|_\alpha C_\alpha$ and $s_\alpha(C_\alpha)[\alpha - c_\alpha \kappa_0]$ hold, so $G_\alpha = (\kappa_0, C_\alpha, s_\alpha C_\alpha, s_\alpha \kappa_0)$ is a minimal gallery from $\kappa_0$ to $s_\alpha \kappa_0$. We have $s_\alpha \tau G_\alpha = s_\alpha(-\alpha - c_\alpha) \partial_\alpha = (\alpha - c_\alpha) \partial_\alpha$ and thus $(\alpha - c_\alpha) \partial_\alpha$ lies in $\text{End}_A(\kappa_0)$.

Given any $w \in W$, let $w = s_\beta_1 \cdots s_\beta_k$ be a reduced decomposition, where $\beta_i \in \hat{\Delta}$ are simple roots. Then, the composite

$$G_w := G_{\beta_1}(s_{\beta_1}G_{\beta_2})(s_{\beta_1}s_{\beta_2}G_{\beta_3}) \cdots (s_{\beta_1} \cdots s_{\beta_{k-1}}G_{\beta_k})$$

is a minimal gallery from $\kappa_0$ to $w \kappa_0$: indeed, if we put $\gamma_i = s_{\beta_1} \cdots s_{\beta_{i-1}} \beta_i$, then we have

$$\mathfrak{d}(s_{\beta_1} \cdots s_{\beta_{i-1}}G_{\beta_i}) = s_{\beta_1} \cdots s_{\beta_{i-1}} \partial(G_{\beta_i}) = \gamma_i - c_{\gamma_i}$$

and hence $\mathfrak{d}(G_w) = \prod_{i=1}^{k} (\gamma_i - c_{\gamma_i}) = \mathfrak{d}(\kappa_0, w \kappa_0)$ because $\Phi^+ \cap w \Phi^- = \{\gamma_1, \ldots, \gamma_k\}$; similarly, we have $\mathfrak{e}(G_w) = \mathfrak{e}(\kappa_0, w \kappa_0)$ and the same holds for $G_w^-$; it follows that the gallery $G_w$ is minimal by Lemma 42. Therefore, the element

$$w^{-1} \tau G_w = s_{\beta_k} \cdots s_{\beta_1} \tau G_{\beta_k} \cdots \tau G_{\beta_2} \tau G_{\beta_1} = s_{\beta_k} \tau G_{\beta_k} \cdots s_{\beta_2} \tau G_{\beta_2} s_{\beta_1} \tau G_{\beta_1}$$

lies in the subalgebra of $\text{End}_C(S)$ generated by $\{(\alpha - c_\alpha) \partial_\alpha = s_\alpha \tau G_\alpha\}_{\alpha \in \hat{\Delta}}$. By Theorem 45, it follows that $\iota_0$ is surjective. The fact that $\{w\}_{w \in W}$ forms a basis for $H$ as $S$-module implies the injectivity of $\iota_0$. Consequently, $\iota_0$ is an isomorphism.

For $d \in D_\Z$, we may write $t_d = (t_d)_0 \circ \iota_0$, which is clearly an isomorphism.

### 3.7 Homogeneous version

Just as $H$ admits a homogeneous version $H_\delta$ isomorphic to the Rees algebra of the canonical filtration $F_\text{can}^n H$, we can define a homogeneous version $A_\delta$ of $A$ by replacing $S$ with $S_\delta$ in the constructions presented in §3.1–§3.3. The category $A_\delta$ has for set of objects $\text{Ch}(a)$ and its hom-spaces are graded subspaces of $g \text{End}_C(S_\delta)$, where $S_\delta$ is graded by the degree. There are obvious analogues of Theorem 45 and Theorem 46 for $A_\delta$ and $H_\delta$ which can be proven with the same arguments. In particular, the hom-space $\text{Hom}_{A_\delta}(C, C')$ for $C, C' \in \text{Ch}(a)$ is graded-free over $C[\delta]$, so it can be identified with the Rees spaces:

$$\text{Hom}_{A_\delta}(C, C') \cong \bigoplus_{n \in \Z} \delta^n F_\text{can}^n \text{Hom}_A(C, C').$$

For $d \in D_\Z$, the translation $(t_d)_* : S_\delta \to S_\delta$ is given by $(t_d)_*(c_\alpha) = c_\alpha - \delta d_\alpha$ for $\alpha \in \Phi$ and $(t_d)_*(f) = f$ for $f \in O(\mathfrak{h})$.

### 4 Spectral completion of $A$

Fix $(c, \lambda) \in \mathfrak{P} \times \mathfrak{h}^1$. We define the following finite subsets

$$\Phi_\lambda = \{\alpha \in \Phi : \alpha(\lambda) = 0\}, \quad \Psi_{c, \lambda} = \{\mu \in \Psi : \mu(c, \lambda) = 0\}.$$

Recall that $S_{c, \lambda}$ is the completion of $S = O(\mathfrak{P} \times \mathfrak{h}^1)$ at the defining ideal of the point $(c, \lambda) \in \mathfrak{P} \times \mathfrak{h}^1$.
4.1 The category $\mathcal{A}_{c,\lambda}$

We shall define a category $\mathcal{A}_{c,\lambda}$ analogous to $\mathcal{A}$ by repeating the constructions from §3 with the following replacements: $\Phi \mapsto \Phi_\lambda$, $\Psi \mapsto \Psi_{c,\lambda}$, $\tilde{W} \mapsto \tilde{W}_\lambda$ and $S \mapsto S^\lambda_{c,\lambda}$.

4.1.1 The sets $\Phi_\lambda$ and $\Psi_{c,\lambda}$ define a finite hyperplane arrangement $\{H_\phi\}_{\phi \in \Phi_\lambda \cup \Psi_{c,\lambda}}$ on $a$. Chambers are defined to be connected components of $a \setminus \bigcup_{\phi \in \Phi_\lambda \cup \Psi_{c,\lambda}} H_\phi$. The set of chambers is denoted by $\text{Ch}_{c,\lambda}(a)$. The notion of galleries is the same as in §3.1.

For each chamber $C \in \text{Ch}_{c,\lambda}(a)$, define

$$\Phi^\perp_{\lambda,C} = \{\alpha \in \Phi_\lambda : \pm \alpha(C) \subseteq \mathbb{R}_{>0}\}, \quad \Psi^\perp_{c,\lambda,C} = \{\mu \in \Psi_{c,\lambda} : \pm \mu(C) \subseteq \mathbb{R}_{>0}\}.

4.1.2 For each $C, C' \in \text{Ch}_{c,\lambda}(a)$, we define two invariants in $S^\lambda_{c,\lambda}$:

$$\vartheta^\lambda_{c,\lambda}(C, C') = \prod_{\mu \in \Psi^\perp_{c,\lambda,C} \cap \Psi^\perp_{c,\lambda,C'}} \mu, \quad \epsilon^\lambda_{c,\lambda}(C, C') = \prod_{\alpha \in \Phi^\perp_{\lambda,C} \cap \Phi^\perp_{\lambda,C'}} \alpha.$$

For each $C, C' \in \text{Ch}_{c,\lambda}(a)$ such that $C$ and $C'$ lie in the same connected component of $a \setminus \bigcup_{\alpha \in \Phi_\lambda} H_\alpha$, let

$$\tau^\lambda_{C,C'} \in \text{End}_C(S^\lambda_{c,\lambda}), \quad f \mapsto f \vartheta^\lambda_{c,\lambda}(C, C').$$

For each $C, C' \in \text{Ch}_{c,\lambda}(a)$ such that $C' \parallel_C C$ for $\alpha \in \Phi_\lambda$, let

$$\tau^\lambda_{C,C'} \in \text{End}_C(S^\lambda_{c,\lambda}), \quad f \mapsto \vartheta_{\alpha}^{-1}(f - s_{\alpha}(f)).$$

For every gallery $G = (C_0, \cdots, C_n)$ in $\text{Ch}_{c,\lambda}(a)$, we put $\tau^\lambda_G = \tau^\lambda_{C_{n-1},C_n} \cdots \tau^\lambda_{C_0,C_1} \in \text{End}_C(S^\lambda_{c,\lambda})$. Similarly, we put $\vartheta^\lambda_{\text{G}(C)} = \prod_{0 \leq i \leq n-1} \vartheta^\lambda_{C_i, C_{i+1}}$ and $\epsilon^\lambda_{\text{G}(C)} = \prod_{0 \leq i \leq n-1} \epsilon^\lambda_{C_i, C_{i+1}}$.

4.1.3 Let $\mathcal{A}_{c,\lambda}$ be the small category with set of objects $\text{Ch}_{c,\lambda}(a)$ and with morphisms defined by

$$\text{Hom}_{\mathcal{A}_{c,\lambda}}(C, C') = \sum_{w \in \tilde{W}_\lambda} \sum_{G = (C, C_1, \cdots, C_{n-1}, w^{-1}C')} w \tau^\lambda_G S^\lambda_{c,\lambda} \subseteq \text{End}_C(S^\lambda_{c,\lambda}). \quad (47)$$

For $n \in \mathbb{Z}$, let $F^\text{lg}_n \text{Hom}_{\mathcal{A}_{c,\lambda}}(C, C') \subseteq \text{Hom}_{\mathcal{A}_{c,\lambda}}(C, C')$ be the sum in (47) with the second summation taken over galleries $G$ with $\ell(G) \leq n$.

4.1.4 Let $C, C' \in \text{Ch}_{c,\lambda}(a)$. For $w \in \tilde{W}_\lambda$, choose a minimal gallery $G_w$ from $C$ to $w^{-1}C'$ and let

$$\tau^\lambda_{C,C',w} = w \tau^\lambda_{G_w} \in F^\text{lg}_n \text{Hom}_{\mathcal{A}_{c,\lambda}}(C, C'), \quad n = d(C, w^{-1}C');$$

moreover, let $\tau^\lambda_{C,C',w}$ denote its image in the quotient:

$$\text{gr}^\text{lg}_n \text{Hom}_{\mathcal{A}_{c,\lambda}}(C, C') = F^\text{lg}_n \text{Hom}_{\mathcal{A}_{c,\lambda}}(C, C') / F^\text{lg}_{n-1} \text{Hom}_{\mathcal{A}_{c,\lambda}}(C, C').$$

**Theorem 48.** For $C, C' \in \text{Ch}_{c,\lambda}(a)$ and $w \in \tilde{W}_\lambda$, the element $\tau^\lambda_{C,C',w}$ is independent of the choice of the minimal gallery $G_w$; moreover, $\{\tau^\lambda_{C,C',w}\}_{w \in \tilde{W}_\lambda}$ forms a basis for $\text{gr}^\text{lg} \text{Hom}_{\mathcal{A}_{c,\lambda}}(C, C')$ as left and right free $S^\lambda_{c,\lambda}$-module.

**Proof.** The proof of Theorem 45 works mutatis mutandis. □
4.1.5 The translation action of $d \in \mathfrak{P}_Z$ induces $t_d^*: \Psi_Q^* \times \hat{b}_Q^* \to \Psi_Q^* \times \hat{b}_Q^*$, which sends $\Psi_{c+d,\lambda}$ to $\Psi_{c,\lambda}$ and fixes $\Phi_a$ pointwise. Hence, it induces a map $t_d: \mathcal{C}^{c,\lambda}(a) \to \mathcal{C}^{c+d,\lambda}(a)$ and an equivalence of categories $t_d: \mathcal{C}^{c,\lambda} \xrightarrow{\sim} \mathcal{C}^{c+d,\lambda}$.

4.2 Comparison

We shall define the spectral completion $\mathcal{C}^{c,\lambda}A$ of the category $A$ and compare it with $\mathcal{C}^{c,\lambda}$.

4.2.1 Let $\mathcal{C}^{c,\lambda} = \bigoplus_{\lambda \in [\lambda]} \mathcal{S}^\lambda_{c,\lambda}$, which is a non-unital ring. Define the spectral completion $\mathcal{C}^{c,\lambda} \operatorname{Hom}_A(C, C') = \mathcal{C}^{c,\lambda} \otimes_S \operatorname{Hom}_A(C, C')$.

The filtration $F_\bullet \operatorname{Hom}_A(C, C')$ implies that $\operatorname{Hom}_A(C, C')$ is the union of $\mathcal{S}$ submodules which are finite as left and right $\mathcal{S}$ modules. Similarly to Lemma 35, there is an isomorphism:

$$\mathcal{C}^{c,\lambda} \otimes_S \operatorname{Hom}_A(C, C') \xrightarrow{\sim} \operatorname{Hom}_A(C, C') \otimes \mathcal{C}^{c,\lambda};$$

therefore, for $C, C', C'' \in \operatorname{Ch}(a)$, there is a composition map:

$$\circ: \mathcal{C}^{c,\lambda} \operatorname{Hom}_A(C, C') \times \mathcal{C}^{c,\lambda} \operatorname{Hom}_A(C, C') \to \mathcal{C}^{c,\lambda} \operatorname{Hom}_A(C, C'').$$

4.2.2 The formula in §1.4.2 defines an action of the Demazure operators $\{\partial_\alpha\}_{\alpha \in \Phi}$ on $\mathcal{C}^{c,\lambda}$. Therefore, we obtain an embedding $\operatorname{Hom}_A(C, C') \hookrightarrow \operatorname{End}_{c}(\mathcal{C}^{c,\lambda})$.

It follows that the embedding $\operatorname{Hom}_A(C, C') \subseteq \operatorname{Hom}_A(C, C') \hookrightarrow \operatorname{End}_c(\mathcal{C}^{c,\lambda})$ extends to $\mathcal{C}^{c,\lambda} \operatorname{Hom}_A(C, C') \hookrightarrow \operatorname{End}_c(\mathcal{C}^{c,\lambda})$, compatible with the compositions.

4.2.3 For each $\lambda' \in [\lambda]$, let $1_{\lambda'} \in \mathcal{S}^\lambda_{c,\lambda}$ denote the unit element viewed as idempotent in $\mathcal{C}^{c,\lambda}$. Let $\mathcal{C}^{c,\lambda}A$ be the $C$-linear category with set of objects $\operatorname{Ch}(a) \times \mathcal{W}$ and with morphisms

$$\operatorname{Hom}_{\mathcal{C}^{c,\lambda}A}((C, w), (C', w')) = 1_{w,\lambda'} \mathcal{C}^{c,\lambda} \operatorname{Hom}_A(C, C') 1_{w,\lambda} \subseteq \operatorname{Hom}_C(\mathcal{S}^\lambda_{c,\lambda}, \mathcal{S}^\lambda_{c,\lambda}).$$

The composition of morphisms is the usual composition of linear maps.

**Theorem 50.** For each chamber $C \in \operatorname{Ch}(a)$, let $\tilde{C} \in \mathcal{C}^{c,\lambda}(a)$ denote the unique chamber satisfying $C \subseteq \tilde{C}$. Then, there is an equivalence of categories given by the following formula:

$$\rho: \mathcal{C}^{c,\lambda} \xrightarrow{\sim} \mathcal{C}^{c,\lambda}, \quad \rho(C, w) = \tilde{w}C, \quad \rho(\varphi) = \tilde{w}^{-1} \varphi \tilde{w}$$

for $\varphi \in \operatorname{Hom}_{\mathcal{C}^{c,\lambda}A}((C, w), (C', w'))$, where $w : \mathcal{S}^\lambda_{c,\lambda} \xrightarrow{\sim} \mathcal{S}^\lambda_{c,\lambda}$ and $w' : \mathcal{S}^\lambda_{c,\lambda} \xrightarrow{\sim} \mathcal{S}^\lambda_{c,\lambda}$ are induced via completion from the $\tilde{W}$-action on $\mathcal{S}$.

**Proof.** The morphisms of the category $\mathcal{C}^{c,\lambda}$ are generated by the following set:

$$\{ \tau_{C, C'} \in \operatorname{Hom}_{\mathcal{C}^{c,\lambda}}(C, C') \mid C, C' \in \mathcal{C}^{c,\lambda}(a) \text{ adjacent} \} \cdot \mathcal{S}^\lambda_{c,\lambda}$$

$$\cup \{ y \in \operatorname{Hom}_{\mathcal{C}^{c,\lambda}}(C, yC) \mid y \in \tilde{W}, C \in \mathcal{C}^{c,\lambda}(a) \} \cdot \mathcal{S}^\lambda_{c,\lambda}.$$

The morphisms of $\mathcal{C}^{c,\lambda}A$ are generated by the following set:

$$\bigcup_{w, w' \in \tilde{W}} \{ 1_{w,\lambda} \tau_{C, C'} 1_{w,\lambda} \in \operatorname{Hom}_{\mathcal{C}^{c,\lambda}A}((C, w), (C', w')) \mid C, C' \in \operatorname{Ch}(a) \text{ adjacent} \} \cdot \mathcal{S}^\lambda_{c,\lambda}$$

$$\cup \bigcup_{w \in \tilde{W}} \{ 1_{w,\lambda} y 1_{w,\lambda} \in \operatorname{Hom}_{\mathcal{C}^{c,\lambda}A}((C, w), (yC, yw)) \mid y \in \tilde{W}, C \in \operatorname{Ch}(a) \} \cdot \mathcal{S}^\lambda_{c,\lambda}. $$

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We show that $\rho$ sends the generators of morphisms of $\mathcal{C}^{c,\lambda}$ into $\mathcal{A}^{c,\lambda}$.

Let $C, C' \in \text{Ch}(\mathfrak{a})$ be a pair of chambers which share a common wall. We show that for each $w, w' \in W$, the following holds:

$$
\rho(1_{w'}\tau_{C,C'}1_w) \in \text{Hom}_{\mathfrak{g}^{c,\lambda}}(D, D'), \quad \text{where } D = \widetilde{w}^{-1}C, D' = \widetilde{w'}^{-1}C'.
$$

(51)

Suppose that $C'|\_a C'$ for some $\alpha \in \Phi$. Then, $\tau_{C,C'} = \vartheta_{\alpha}$. If $\alpha(w\lambda) = 0$, then $w^{-1}\alpha \in \Phi_\lambda$ and $D|_{w^{-1}\alpha}D'$ in $\text{Ch}^{c,\lambda}(\mathfrak{a})$ hold; therefore,

$$
\rho(1_{w'}\tau_{C,C'}1_w) = \begin{cases}
\vartheta_{w^{-1}\alpha} = \tau_{D,D'}^{c,\lambda} & \text{if } w\lambda = w', \\
0 & \text{otherwise}.
\end{cases}
$$

If $\alpha(w\lambda) \neq 0$, then $w^{-1}\alpha$ is invertible in $S^\wedge_{c,\lambda}$ and $S^\wedge_{c,s\alpha,\lambda}$; therefore, the Demazure operator induces

$$
\vartheta_{\alpha} : S^\wedge_{c,w\lambda} \to S^\wedge_{c,w\lambda} \oplus S^\wedge_{c,s\alpha,w\lambda}, \quad f \mapsto \alpha^{-1}f - \alpha^{-1}s\alpha f.
$$

and thus

$$
\rho(1_{w'}\tau_{C,C'}1_w) = \begin{cases}
(w^{-1}\alpha)^{-1} - 1 & \text{if } w'\lambda = w\lambda, \\
(w^{-1}\alpha)^{-1}w'^{-1}s\alpha w & \text{if } w'\lambda = s\alpha w, \\
0 & \text{otherwise}.
\end{cases}
$$

Note that in the second case above, we have $w'^{-1}s\alpha w \in W_\lambda$. It follows that (51) holds in each of the cases. Similar arguments show (51) holds also in the case $C'|_\psi C'$ or $C'|\_1\psi C$ for some $\psi \in \Psi$ and that $\rho(1_{yw}\lambda y1_w\lambda)$ lies in $\text{Hom}_{\mathfrak{g}^{c,\lambda}}(D, D')$ for all $w, w', y \in W$. Since the formula for $\rho$ respects the composition, it follows that $\rho$ is a well-defined functor.

The same kind of arguments as above shows that $\rho$ is full. The faithfulness of $\rho$ is due to the fact that $\varphi \mapsto w'^{-1}\varphi w$ gives an isomorphism from $\text{Hom}_C(S^\wedge_{c,w\lambda}, S^\wedge_{c,w'\lambda})$ to $\text{End}_C(S^\wedge_{c,\lambda})$ and that $\text{Hom}_{\mathfrak{g}^{c,\lambda}}(\mathcal{C}, (C, w), (C', w')) \subseteq \text{Hom}_C(S^\wedge_{c,w\lambda}, S^\wedge_{c,w'\lambda})$ is a subspace. The essential surjectivity is obvious. \qed

### 4.3 Duality

#### 4.3.1

Let $C, C' \in \text{Ch}^{c,\lambda}(\mathfrak{a})$. We say that $C$ and $C'$ are antipodal if the following conditions hold:

$$
\Phi_+^{C,C} = \Phi^-_{\lambda,C'}, \quad \Psi_+^{-C,\lambda} = \Psi^-_{c,\lambda,C'}.
$$

Suppose that $C, C'$ are antipodal. By Theorem 48, $d(C, C')$ is the highest degree of $\text{gr}^{1\mathfrak{g}}\text{Hom}_{\mathfrak{g}^{c,\lambda}}(C, C')$ and the graded piece of degree $d(C, C')$ is $S^\wedge_{c,\lambda} \tau_{C,C'}^{c,\lambda}$.

Let $\text{tr}_{C,C'} : \text{Hom}_{\mathfrak{g}^{c,\lambda}}(C, C') \to S^\wedge_{c,\lambda}$ be the following composition:

$$
\text{Hom}_{\mathfrak{g}^{c,\lambda}}(C, C') \to \text{gr}^{1\mathfrak{g}}\text{Hom}_{\mathfrak{g}^{c,\lambda}}(C, C') \to S^\wedge_{c,\lambda} \tau_{C,C'}^{c,\lambda} \to S^\wedge_{c,\lambda}.
$$

Note that $\text{tr}$ satisfies $\text{tr}(a\varphi) = a\text{tr}(\varphi) = \text{tr}(\varphi a)$ for $a \in S^\wedge_{c,\lambda}$ and $\varphi \in \text{Hom}_{\mathfrak{g}^{c,\lambda}}(C, C')$ because $[S^\wedge_{c,\lambda}, \tau_{C,C'}^{c,\lambda}] = 0$.

**Lemma 52.** Suppose that $C, C' \in \text{Ch}^{c,\lambda}(\mathfrak{a})$ are antipodal chambers. Then, given any chamber $C'' \in \text{Ch}^{c,\lambda}(\mathfrak{a})$, the following is a $S^\wedge_{c,\lambda}$-bilinear perfect pairing:

$$
\text{Hom}_{\mathfrak{g}^{c,\lambda}}(C'', C') \times \text{Hom}_{\mathfrak{g}^{c,\lambda}}(C, C'') \to S^\wedge_{c,\lambda}, \quad (\phi, \psi) \mapsto \text{tr}(\phi\psi).
$$

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Theorem 48. Given §1.5.2 and the fact that \( \tau \) implies \( \tau \) holds. The inclusion \( \tau \) holds.

Proof. Since \( C \) and \( C' \) are antipodal, it is easy to see that \( [C, C'] = \text{Ch}^{c,\lambda}(a) \); namely, \( d(C, C') = d(C, C'') + d(C'', C') \) holds for every \( C'' \in \text{Ch}^{c,\lambda}(a) \). In particular, for each \( w \in \hat{W} \), there exists a minimal gallery from \( C \) to \( C' \) passing through \( w^{-1}C'' \).

It follows from Theorem 48 that

\[
\tau_{C', w^{-1}C, C''} = \tau_{w^{-1}C', C''} \tau_{C', w} = \tau_{C, C'}
\]

holds.

Pick any liftings \( \tau_{C', w} \) and \( \tau_{C, C''} \) for each \( w \in \hat{W} \); then, Theorem 48 implies that \( \{ \tau_{C', w} \}_{w \in \hat{W}} \) and \( \{ \tau_{C, C''} \}_{w \in \hat{W}} \) are free bases over \( S_{c,\lambda}^a \). If we order the elements \( w \) of \( \hat{W} \) by the distance \( d(C, w^{-1}C'' \), then the matrix \( (\text{tr}(\tau_{C', w} \tau_{C, C''}, w))_{w \in \hat{W}} \)

is triangular with 1 in the diagonal. Hence, the pairing is perfect. \( \square \)

4.3.2 Recall the ring \( \mathcal{Z}^{c,\lambda} \) from §1.5.2. The inclusion \( \mathcal{Z}^{c,\lambda} \hookrightarrow S_{c,\lambda}^a \) induces an action of \( \mathcal{Z}^{c,\lambda} \) on \( \text{End}_{\mathcal{Z}^{c,\lambda}}(C, C') \subseteq \text{End}_{\mathcal{B}}(S_{c,\lambda}^a) \) by multiplication for each \( C, C' \in \text{Ch}^{c,\lambda}(a) \). This yields \( \mathcal{Z}^{c,\lambda} \rightarrow \text{End}(\mathcal{B}_{\mathcal{Z}^{c,\lambda}}) \). It is not hard to show that this is an isomorphism.

It is known that \( S_{c,\lambda}^a \) is a symmetric algebra over \( \mathcal{Z}^{c,\lambda} \) with trace form \( \text{tr}_{S_{c,\lambda}^a} : S_{c,\lambda}^a \rightarrow \mathcal{Z}^{c,\lambda} \) given by \( \text{tr}_{S_{c,\lambda}^a} = \vartheta_{\beta_1} \cdots \vartheta_{\beta_r} \), where \( w_0 = s_{\beta_1} \cdots s_{\beta_r} \) is a reduced expression for the longest element \( w_0 \) of \( \hat{W}_\lambda \) with respect to any basis for the finite root system \( (\hat{h}_R, \Phi_\lambda) \). In particular, the hom-spaces in \( \mathcal{Z}^{c,\lambda} \) are free \( \mathcal{Z}^{c,\lambda} \)-modules due to Theorem 48.

Proposition 53. The pairing from Lemma 52 composed with \( \text{tr}_{S_{c,\lambda}^a} \) is a \( \mathcal{Z}^{c,\lambda} \)-bilinear perfect pairing

\[
\text{Hom}_{\mathcal{Z}^{c,\lambda}}(C'', C') \times \text{Hom}_{\mathcal{Z}^{c,\lambda}}(C, C'') \rightarrow \mathcal{Z}^{c,\lambda}, \ (\phi, \psi) \mapsto \text{tr}_{S_{c,\lambda}^a} \text{tr}(\phi \psi).
\]

Proof. The pairing is perfect by Lemma 52 and the fact that \( \text{tr}_{S_{c,\lambda}^a} \) induces an isomorphism \( S_{c,\lambda}^a \cong \text{Hom}_{\mathcal{Z}^{c,\lambda}}(S_{c,\lambda}^a, \mathcal{Z}^{c,\lambda}) \).

\( \square \)

4.4 Generic chambers

Given \( d \in \mathcal{P}_R \), let \( a_d = \{ d \} \times \hat{h}_R^\lambda \subseteq a \) and let \( \text{Ch}^{c,\lambda}(a_d) \subseteq \text{Ch}^{c,\lambda}(a) \) be the subset of chambers \( C \in \text{Ch}^{c,\lambda}(a) \) such that \( C \cap a_d \neq \emptyset \). We say that a chamber \( C \in \text{Ch}^{c,\lambda}(a_d) \) is generic if no non-constant affine function on \( a_d \) is bounded on \( C \cap a_d \).

Lemma 54. If \( C \in \text{Ch}^{c,\lambda}(a_d) \) is a generic chamber, then \( C \in \text{Ch}^{c,\lambda}(a_{d'}) \) for every \( d' \in \mathcal{P}_R \) (in other words, \( C \) is mapped surjectively onto \( \mathcal{P}_R \) via the projection \( a \rightarrow \mathcal{P}_R \)).

Proof. Let \( \overline{\mathcal{P}}_{c,\lambda,C}^\pm \) (resp. \( \overline{\mathcal{P}}_{c,\lambda,C}^\pm \)) be the image of \( \mathcal{P}_{c,\lambda,C}^\pm \) (resp. \( \mathcal{P}_{c,\lambda,C}^\pm \)) in \( \hat{h}_Q^\lambda \times \mathcal{P}_Q \). The genericity implies there exists \( \rho \in \hat{h}_Q \) such that \( \pm \varphi(\rho) > 0 \) for every \( \varphi \in \overline{\mathcal{P}}_{c,\lambda,C}^\pm \). It follows that for every pair \( (d', z) \in \mathcal{P}_R \times \hat{h}_R^\lambda = a \), there exists \( r > 0 \) such that \( (d', z + \rho) \in C \). Hence, \( C \in \text{Ch}^{c,\lambda}(a_{d'}) \) for every \( d' \in \mathcal{P}_R \).

\( \square \)

5 Translation bimodules

5.1 Definition

For \( d \in \mathcal{P}_Z \), set \( B^{(d)} := \text{Hom}_{\mathcal{A}}(\kappa_0, a_d) \). In particular, we have \( B^{(0)} = \text{End}_{\mathcal{A}}(\kappa_0) \cong H \) by Theorem 46. Given \( d, d' \in \mathcal{P}_Z \), we define the multiplication \( * : B^{(d')} \otimes B^{(d)} \rightarrow \)
Theorem 45. By \( \phi \), making it filtered \( \tau \). Put \( \text{Theorem } 45 \) is equivalent to the Definition 56.

It is easy to show that \( \ast \) is associative. In particular, \( \mathcal{B}^{(d)} \) has an \( \mathbf{H} \)-bimodule structure and \( \ast \) factorises through \( \mathcal{B}^{(d)} \oplus \mathbb{H} \mathcal{B}^{(d)} \). Note that \( \mathcal{B}^{(d)} \) satisfies
\[
(c_{\alpha} + d_{\alpha}) \ast a = a \ast c_{\alpha} \quad \text{for } a \in \mathcal{B}^{(d)} \text{ and } \alpha \in \Phi. \tag{55}
\]

**Definition 56.** For \( d \in \mathfrak{H}_Z \), the \( \mathbf{H} \)-bimodule \( \mathcal{B}^{(d)} \) is called the translation bimodule.

Let \( c, c' \in \mathfrak{H} \) be such that \( d = c - c' \in \mathfrak{H}_Z \). By (55), we have \( \mathcal{B}^{(d)} m_c = m_c \mathcal{B}^{(d)} \subset \mathcal{B}^{(d)} \), where \( m_c \subset \mathcal{O}(\mathfrak{H}) \) is the defining ideal of \( c \). Put \( \mathcal{B}_c = \mathcal{B}^{(d)} / \mathcal{B}^{(d)} m_c \). It is an \( (\mathbf{H}_{c'}, \mathbf{H}_c) \)-bimodule.

### 5.2 Length filtration

Recall the length filtration \( F^*_{\text{lg}} \mathcal{B}^{(d)} = F^*_{\text{lg}} \text{Hom}_A(\kappa_0, \kappa_d) \) defined in §3.3.2. By Theorem 45, the associated graded space has a free basis:
\[
\text{gr}^*_{\text{lg}} \mathcal{B}^{(d)} = \bigoplus_{w \in W} \mathbf{S} \tau_{\kappa_0, \kappa_d, w}, \quad \deg \tau_{\kappa_0, \kappa_d, w} = d(\kappa_0, w^{-1} \kappa_d).
\]

For simplicity of notation, denote \( \gamma^{d,w} := \tau_{\kappa_0, \kappa_d, w} \in \text{gr}^*_{\text{lg}} \mathcal{B}^{(d)} \).

**Lemma 57.** The multiplication \( \ast \) descends to the associated graded \( \text{gr}^*_{\text{lg}} \mathcal{B}^{(d)} \) and satisfies following rule:
\[
\gamma^{d,w} \ast \gamma^{e,y} = \begin{cases} 
\gamma^{d+e,wy} & \text{if } y^{-1} \kappa_d \in [\kappa_0, (wy)^{-1} \kappa_{d+e}] \\
0 & \text{otherwise}
\end{cases}, \tag{58}
\]

for \( w, y \in W \) and \( d, e \in \mathfrak{H}_Z \).

**Proof.** The condition \( y^{-1} \kappa_d \in [\kappa_0, (wy)^{-1} \kappa_{d+e}] \) (see Lemma 42) is equivalent to the following: if \( G \) (resp. \( G' \)) is a minimal gallery from \( \kappa_0 \) to \( y^{-1} \kappa_d \) (resp. from \( y^{-1} \kappa_d \) to \( (wy)^{-1} \kappa_{d+e} \)), then \( GG' \) is a minimal gallery from \( \kappa_0 \) to \( (wy)^{-1} \kappa_{d+e} \). The statement follows immediately from Theorem 45. \( \square \)

### 5.3 Harish-Chandra property

The translation bimodule \( \mathcal{B}^{(d)} \) is equipped with the canonical filtration \( F^*_{\text{can}} \mathcal{B}^{(d)} \) from §3.3.2 making it filtered \( \mathbf{H} \)-bimodule. It is the motivating example of our definition of Harish-Chandra bimodules:

**Proposition 59.** For each \( d \in \mathfrak{H}_Z \), the canonical filtration \( F^*_{\text{can}} \mathcal{B}^{(d)} \) is a HC filtration, so we have \( \mathcal{B}^{(d)} \in \text{HC}(\mathbf{H}) \).

**Proof.** Recall the homogeneous version \( \mathcal{B}^{(d)}_\delta = \text{Hom}_A(\kappa_0, \kappa_d) \) defined in §3.7. Put \( \mathcal{B}^{(d)} := \mathcal{B}^{(d)}_\delta / (\delta) \cong \text{gr}^*_{\text{can}} \mathcal{B}^{(d)} \). The inclusion \( \text{gr}^*_{\text{can}} \text{End}_A(\kappa_0) \hookrightarrow \text{gr}^*_{\text{can}} \text{End}_A(\kappa_0) \) becomes an isomorphism after inverting the images of elements of \( \Psi \) in \( \mathbf{S}_\delta / (\delta) \). In particular, it preserves the centre:
\[
Z(\text{gr}^*_{\text{can}} \text{End}_A(\kappa_0)) = Z(\text{gr}^*_{\text{can}} \text{End}_A(\kappa_0)) \cap \text{gr}^*_{\text{can}} \text{End}_A(\kappa_0).
\]

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Similarly, we have $\mathcal{B}^{(d)} \to \text{gr}^{\text{can}} \text{Hom}_A(\kappa_0, \kappa_d) \cong \text{gr}^{\text{can}} \text{End}_A(\kappa_d)$. Therefore, for any $x \in Z(\text{gr}^{\text{can}} \text{End}_A(\kappa_d))$ and $b \in \mathcal{B}^{(d)}$, we have $x * b = ((t_d)_*) b = xb = bx = b * x$; the second equation is due to the fact that $(t_d)_*$ is the reduced to the identity map on $\text{gr}^{\text{can}} S = O(\mathfrak{g} \times \mathfrak{h})$. This together with Lemma 60 below implies that the canonical filtration on $\mathcal{B}^{(d)}$ is a HC filtration. \qed

**Lemma 60.** $\mathcal{B}^{(d)}_\delta$ is finitely generated as graded left $H_\delta$-module.

**Proof.** It suffices to show that $\text{gr}^{\text{bk}} \mathcal{B}^{(d)}_\delta$ is finitely generated as graded left $\text{gr}^{\text{bk}} H_\delta$-module. We first show that for each $w \in \hat{W}$ such that $\ell(w) > 0$, there exists $\alpha \in \Delta$ such that $\gamma^{0,s_\alpha} \star \gamma^{d,s_\alpha w} = \gamma^{d,w}$. This will imply the required statement because $\text{gr}^{\text{bk}} \mathcal{B}^{(d)}_\delta$ is spanned by $\{\gamma^{d,w}\}_{w \in \hat{W}}$ as graded left $S_\delta$-module.

Let $w \in \hat{W}$ and suppose that $\ell(w) > 0$. By the decomposition $\hat{W} = Q^\vee \rtimes W$, we can write $w = t\mu w_1$ for $w_1 \in W$ and $\mu \in Q^\vee$. Moreover, there exists positive integers $\{e_{\alpha}\}_{\alpha \in \Delta}$ satisfying $\sum_{\alpha \in \Delta} e_{\alpha} \alpha = \delta$, so that

$$\sum_{\alpha \in \Delta} e_{\alpha} \langle \alpha, \mu \rangle = \langle \delta, \mu \rangle = 0;$$

it implies that when $\ell(w) > 0$, it happens that $\ell(t\mu) \geq \ell(w) - \#W > 0$ and there exists $\alpha \in \Delta$ such that $\langle \alpha, \mu \rangle \ll 0$; it follows that for such $\alpha$, we have

$$(t\mu)_{\alpha} - c_{\alpha} = (\alpha - c_{\alpha})(w_1 \kappa_d) = \langle \alpha, \mu \rangle \subseteq \mathbb{R}_{<0}$$

and thus $\alpha - c_{\alpha} \in (t\mu)_{\alpha}$. Similarly, $-\alpha - c_{\alpha} \in \Psi^+_w$. Let $w' = s_{\alpha} w$. We have thus

$$\Psi^+_w \cap \Psi^+_w = (\Psi^+_w \cap s_{\alpha} \Psi^+_w) \cup \{\alpha - c_{\alpha}\},$$

for $w' \in \Psi^+_w$. The first equation implies $\Psi^+_w \cap \Psi^+_w \subseteq \Psi^+_w$, and the second implies $\Psi^-_w \cap \Psi^-_w \subseteq \Psi^-_w$. Similar arguments show that $\Phi^+_w \cap \Phi^+_w \subseteq \Phi^+_w$ or equivalently $\Psi^-_w \cup \Psi^+_w \subseteq \Psi^-_w$ holds. By Lemma 42, these together imply that the condition $w' \kappa_d \in [\kappa_0, w_0 \kappa_d]$ in Lemma 57 is satisfied and therefore $\gamma^{0,s_\alpha} \star \gamma^{d,s_\alpha w} = \gamma^{d,w}$ holds. \qed

### 5.4 Spectral completion of translation bimodules

The translation bimodule has been defined in terms of the category $A$. We shall describe its spectral completion in terms of the completed category $\mathcal{A}^{c,\lambda}$, introduced in §4.1, and establish a relation with the algebraic KZ functor $V$.

#### 5.4.1 Let $[\lambda] \in \mathfrak{h}^1/\hat{W}$ be a $\hat{W}$-orbit let $c, c' \in \mathfrak{g}$ be such that $d = c - c' \in \mathfrak{g}$. Set $c, \mathcal{B}^{c,\lambda} = \mathcal{L}^{c,\lambda}(c, \mathcal{B}c)$.

**Proposition 61.** There is a natural isomorphism

$$c, \mathcal{B}^{c,\lambda} \cong \bigoplus_{[w], [w'] \in \hat{W}/\hat{W}_\lambda} \text{Hom}_{\mathcal{A}^{c,\lambda}}(w^{-1} \kappa_0, w'^{-1} \kappa_d)/(c - c).$$

(62)

In particular, when $c' = c$, we have

$$\mathcal{H}^{c,\lambda} \cong \bigoplus_{[w], [w'] \in \hat{W}/\hat{W}_\lambda} \text{Hom}_{\mathcal{A}^{c,\lambda}}(w^{-1} \kappa_0, w'^{-1} \kappa_d)/(c - c).$$

(63)
Theorem 50 implies that \( \alpha \) denoted by \( C \) for each \( w \). Moreover, if we are given \( w, w' \in W \):

\[
1_{w'} \cdot c^B_c \cdot 1_w \cong \text{Hom}_{\mathcal{A}^{c',\lambda}}(w^{-1}_{K-d}, w'^{-1}_{K_0})/(c' - c).
\]

Applying the translation equivalence \( t_d : \mathcal{A}^{c',\lambda} \rightarrow \mathcal{A}^{c,\lambda} \) from \( \S 4.1.5 \), we obtain

\[
\text{Hom}_{\mathcal{A}^{c',\lambda}}(w^{-1}_{K-d}, w'^{-1}_{K_0})/(c' - c) \cong \text{Hom}_{\mathcal{A}^{c',\lambda}}(w^{-1}_{K_0}, w'^{-1}_{K_d})/(c - c).
\]

Taking summation over cosets \([w], [w'] \in \tilde{W}/\tilde{W}_\lambda\), we obtain the isomorphism (62).

\[\square\]

### 5.4.2

Let \( (c, \lambda) \in \mathcal{P} \times \mathfrak{h}^1 \). Recall the algebraic KZ functor \( V \) and the completed affine Hecke algebra \( \mathcal{H}_\mathcal{C}^d \) from \( \S 1.9 \). Let \( \mathcal{P}^\lambda_c = V(\mathcal{H}_c^\lambda) \); it is a \( (\mathcal{H}_\mathcal{C}^d, \mathcal{H}_c^\lambda) \)-bimodule.

**Proposition 64.** For \( \lambda \in \mathfrak{h}^1 \) and \( c, c' \in \mathcal{P} \) such that \( d = c - c' \in \mathcal{P}_\Sigma \), there is a natural isomorphism

\[
\alpha_{c',c} : \mathcal{P}^\lambda_c \otimes \mathcal{H}_c^{c'} \rightarrow \mathcal{P}^\lambda_c \cong \mathcal{P}^\lambda_c.
\]

Moreover, if we are given \( c'' \in \mathcal{P} \) such that \( d' = c' - c'' \in \mathcal{P}_\Sigma \), then the following square is commutative:

\[
\begin{array}{ccc}
\mathcal{P}^\lambda_c \otimes \mathcal{H}_c^{c''} \otimes \mathcal{P}^\lambda_c \otimes \mathcal{H}_c^{c'} & \xrightarrow{1 \otimes \cdot} & \mathcal{P}^\lambda_c \otimes \mathcal{H}_c^{c''} \otimes \mathcal{P}^\lambda_c \\
\mathcal{P}^\lambda_c \otimes \mathcal{H}_c^{c''} \otimes \mathcal{P}^\lambda_c & \xrightarrow{\alpha_{c',c} \otimes 1} & \mathcal{P}^\lambda_c \\
\end{array}
\]

(65)

**Proof.** The algebraic KZ functor is given by \( V : M \mapsto 1_V M \), where \( 1_V \in \mathcal{H}_c^\lambda \) is the sum \( 1_V = \sum_{w \in \Sigma} 1_{\Sigma}, w \lambda \) for some finite set \( \Sigma \subseteq \tilde{W} \) such that \( w^{-1} \nu_0 \) is in a generic clan for each \( w \in \Sigma \), or equivalently, \( \emptyset \times w^{-1} \nu_0 \subseteq \mathfrak{a} \) lies in a generic chamber, denoted by \( C(w^{-1}) \). Thus, (63) implies that we can express

\[
\mathcal{P}^\lambda_c = \bigoplus_{[w] \in \tilde{W}/\tilde{W}_\lambda} \bigoplus_{y \in \Sigma} \text{Hom}_{\mathcal{A}^{c',\lambda}}(w^{-1}_{K_0}, y^{-1}_{K_0})/(c - c).
\]

On the other hand, \( \text{Lemma 54} \) implies that \( C(y^{-1}) \cap (\{d\} \times \mathfrak{h}^1_\mathbb{R}) \neq \emptyset \) for each \( y \in \Sigma \). We may assume that \( \{d\} \times y^{-1} \nu_0 \subseteq C(y^{-1}) \) for \( y \in \Sigma \) by choosing \( \Sigma \) such that each \( y^{-1} \nu_0 \) is far in the Weyl chamber. Hence, \( y^{-1} \nu_0 = C(y^{-1}) = y^{-1} \nu_0 \) holds for each \( y \in \Sigma \) and (62) yields isomorphisms

\[
\mathcal{P}^\lambda_c \otimes \mathcal{H}_c^{c''} \otimes \mathcal{P}^\lambda_c \cong \bigoplus_{[w] \in \tilde{W}/\tilde{W}_\lambda} \bigoplus_{y \in \Sigma} \text{Hom}_{\mathcal{A}^{c',\lambda}}(w^{-1}_{K_0}, y^{-1}_{K_0})/(c - c)
\]

\[
= \bigoplus_{[w] \in \tilde{W}/\tilde{W}_\lambda} \bigoplus_{y \in \Sigma} \text{Hom}_{\mathcal{A}^{c',\lambda}}(w^{-1}_{K_0}, y^{-1}_{K_0})/(c - c) \cong \mathcal{P}^\lambda_c.
\]

The isomorphism \( \alpha_{c',c} \) is then defined to be the composite. The commutativity of (65) results from the associativity of the composition of morphisms in \( \mathcal{A}^{c',\lambda} \). \( \square \)
5.5 Double centraliser property

Let $c, c' \in \mathcal{P}$ be such that $c - c' \in \mathcal{P}_Z$.

**Proposition 66.** The following ring homomorphisms induced by the bimodule structure

$$H_c^{op} \to \text{End}_{H_c}(c, B_c), \quad H_{c'} \to \text{End}_{H_{c'}}(c, B_c)$$

are isomorphisms.

**Proof.** We will only show that the first map of (67) is an isomorphism, the second map being similar. Since the $S$-algebra $\mathcal{I}$ is faithfully flat by Lemma 6, it suffices to show that (67) becomes an isomorphism after applying the spectral completion $\mathcal{C}$. Since both sides of (67), viewed as left $\mathcal{O}(\mathcal{P})$-modules, are punctually supported on $c \in \mathcal{P}$, it suffices to show that the map (67) becomes an isomorphism after applying $\mathcal{C}$ for each $[\lambda] \in \mathfrak{h}^1/\mathbb{W}$. By Proposition 39, the spectral completion becomes

$$(\mathcal{C}^{\lambda}_c)^{op} \to \text{hom}_{\mathcal{C}^{\lambda}}(c, B^{\lambda}_c, c, B^{\lambda}_c).$$

This map is an isomorphism due to Lemma 69 below. $\square$

**Lemma 69.** Let $[\lambda] \in \mathfrak{h}^1/\mathbb{W}$. Then, given compact projective modules $P, Q \in \mathcal{H}^{\lambda}_c$-proj, the following natural map

$$\text{Hom}_{\mathcal{H}^{\lambda}_c}(P, Q) \to \text{Hom}_{\mathcal{H}^{\lambda}}(c, B^{\lambda}_c \otimes \mathcal{H}^{\lambda}_c, P, c, B^{\lambda}_c \otimes \mathcal{H}^{\lambda}_c, Q)$$

is an isomorphism.

**Proof.** For simplicity of notation, we denote $H = \mathcal{H}^{\lambda}_c$, $H' = \mathcal{H}^{\lambda}_c$ and $B = c, B^{\lambda}_c$. We apply the algebraic KZ functor $V$ from Theorem 26 and consider the composition:

$$\text{Hom}_H(P, Q) \xrightarrow{(70)} \text{Hom}_{B^{\lambda}_c}(B \otimes B^{\lambda}_c, P, B \otimes B^{\lambda}_c, Q) \xrightarrow{V_*} \text{Hom}_K(V(B \otimes B^{\lambda}_c, P, V(B \otimes B^{\lambda}_c, Q)).$$

By Proposition 64, there is a natural isomorphism $V(B \otimes B^{\lambda}_c, P, V(B \otimes B^{\lambda}_c, Q)$ and the composite of (71) can be identified with the map of $V$ on the hom-space:

$$\text{Hom}_H(P, Q) \to \text{Hom}_H(V(P), V(Q)),$$

which by the double centraliser property of $V$ (see Theorem 26), is an isomorphism; consequently, the map $V_*$ in (71) is surjective. To prove that (70) is an isomorphism, it remains to show that $V_*$ is injective.

Recall the ring $\mathcal{Z}_{c}^{\lambda} = (S_c^{\lambda} / (c - c))^{\lambda}$ from §1.5.2 Given a $\mathcal{Z}_{c}^{\lambda}$-module $M$, let $M_F = M \otimes \mathcal{Z}_{c}^{\lambda} F$ denote the base change to the field of fractions $F := \text{Frac} \mathcal{Z}_{c}^{\lambda}$. We have a commutative square

$$\begin{array}{ccc}
\text{Hom}_H(B \otimes B^{\lambda}_c, P, B \otimes B^{\lambda}_c, Q) & \xrightarrow{V_*} & \text{Hom}_K(V(B \otimes B^{\lambda}_c, P, V(B \otimes B^{\lambda}_c, Q)) \\
\downarrow & & \downarrow \\
\text{Hom}_H((B \otimes B^{\lambda}_c)_F, (B \otimes B^{\lambda}_c)_F, (B \otimes B^{\lambda}_c)_F) & \sim & \text{Hom}_K((V(B \otimes B^{\lambda}_c)_F, (V(B \otimes B^{\lambda}_c)_F)_F)
\end{array}$$

where the vertical maps are injective (because $B$, $H$, $P$ and $Q$ are free over $\mathcal{Z}_c^{\lambda}$; see §4.3.2); the lower horizontal map is an isomorphism because $(V(B \otimes B^{\lambda}_c)_F = \text{Hom}_H((H_F 1_V, P, H_F 1_V)$ and $H_F 1_V$ is a compact projective generator of $H_F$-$\text{Mod}$, which is equivalent to $F$-$\text{Mod}$ by Lemma 10; hence, the upper horizontal map $V_*$ is injective. This concludes the proof. $\square$
5.6 Translation bimodules along a semigroup

We provide here some results on the \( \mathbb{N} \)-graded ring \( B^{\otimes} \) formed by the translation bimodules along the semigroup generated by an element \( d \in \mathfrak{P}_{\mathbb{Z}} \). We prove the noetherianity of \( B^{\otimes} \). It serves as preparation for the key technical Lemma 93 of the generic freeness over the parameter space \( \mathfrak{P} \) of certain Harish-Chandra bimodules.

5.6.1 Fix \( d \in \mathfrak{P}_{\mathbb{Z}} \). Form the following ring:

\[
B^{\otimes} = \bigoplus_{n \geq 0} B^{(nd)},
\]

whose multiplication is given by the map \( \ast \) from §5.1. It has a \( \mathbb{Z}^2 \)-filtration given by

\[
F_{n,m}B^{\otimes} = \bigoplus_{k \geq 0} F_{n,m}F_{n,m}B^{(kd)},
\]

where the first filtration on the right-hand side is the length filtration (§5.2) and the second is the canonical filtration (§5.1). Note that \( F_{n,m}B^{\otimes} = 0 \) whenever \( n < 0 \) or \( m < 0 \).

**Proposition 72.** The graded ring \( B^{\otimes} \) is left and right noetherian.

**Proof.** Consider the \( \mathbb{N}^2 \)-graded Rees ring

\[
B_{\zeta,\delta} = \bigoplus_{n,m \geq 0} \zeta^n \delta^m F_{n,m}B^{\otimes}.
\]

By the technique of associated graded [7, Appendix D], it suffices to show that the associated quotient \( \overline{B}^{\otimes} := B_{\zeta,\delta}/(\zeta, \delta) \) is left and right noetherian. **Lemma 73** below ensures this property.

**Lemma 73.** Let \( \overline{B}^{\otimes} \) be as in the proof of **Proposition 72**. Then, \( \overline{B}^{\otimes} \) contains a commutative graded subring \( A \) of finite type over \( \mathbb{C} \) such that \( B^{\otimes} \) is finitely generated as left and right \( A \)-module.

**Proof.** By **Theorem 45**, the ring \( \overline{B}^{\otimes} \) admits the following decomposition:

\[
\overline{B}^{\otimes} = \bigoplus_{n \in \mathbb{N}} \bigoplus_{w \in W} S^{\gamma^{nd,w}},
\]

where \( S = S_{\delta}/(\delta) \) and \( \gamma^{nd,w} \) is the image of \( \gamma^{nd,w} \in \text{gr} F B^{\otimes} \) in \( \overline{B}^{\otimes} \). We claim that the subring

\[
A = \bigoplus_{n \in \mathbb{N}} \bigoplus_{\mu \in Q^\vee} S^{\gamma^{nd,t\mu}} \subset \overline{B}^{\otimes}
\]

satisfies the requirements.

Indeed, for each chamber \( U \subset \mathbb{R}_{\geq 0}d \times h_{\mathbb{R}} \) of the hyperplane arrangement given by the finite set of linear functions \( \{ \psi \mid_{\mathbb{R}d \times h_{\mathbb{R}}} : \psi \in \Psi \} \), let \( \Sigma_U \) be a finite generating subset of the semigroup \( (Nd \times Q^\vee) \cap U \). It is easy to show that for pairs \( (jd, \mu), (kd, \nu) \in (Nd \times Q^\vee) \cap U \), we have \( t_\mu \kappa_d \in [\kappa_0, t_\mu + \nu \kappa_{(j+k)d}] \), so that

\[
\gamma^{jd,t\mu} \times \gamma^{kd,t\nu} = \gamma^{(j+k)d,t_{\mu + \nu}}
\]

holds by **Lemma 57**. Put \( \Sigma = \bigcup_U \Sigma_U \). Then, \( \{ \gamma^{kd,t\mu} \} \) \((kd, \mu) \in \Sigma\) is a generating subset of the ring \( A \). The proof of finite generation of \( \overline{B}^{\otimes} \) over \( A \) is more sophisticated and will be completed in §5.6.3. \( \square \)
5.6.2 Let $a' := \mathbb{R}^d \times h_R^1 \subseteq a$. The hyperplane arrangement $\{ H_{\psi} \}_{\psi \in \Phi \cup \Psi}$ introduced in §3.1 can be restricted to $a'$. Let $\text{Ch}(a')$ denote the set of chambers of this restriction.

We choose any euclidean metric $\| \cdot \|$ for $h_R^1$ and extend it to $a'$ by setting $\| (rd, y) - (sd, y') \| = \| y - y' \|^2 + |r - s|^2$. Let $Q^\vee \subset h_R^1$ be the coroot lattice of the finite root system $R$. The lattice $Z^d \times Q^\vee$ acts on $a'$ by translation and the quotient $\Omega = a'/Z^d \times Q^\vee$ is compact.

**Lemma 74.** Let $C_0, C_1 \in \text{Ch}(a')$ be chambers. Let $\varepsilon > 0$, $p_0 \in C_0$, $p_1 \in C_1$ be such that the open ball $B_{\varepsilon}(p_1) \subseteq a'$ is contained in $C_i$ for $i \in \{0, 1\}$. Suppose that the following inequality holds:

$$\frac{\| p_1 - p_0 \|}{\varepsilon} \geq \frac{\text{Vol}(\Omega)}{\text{Vol}(B_{\varepsilon/2}(p_0))} + 1.$$ 

Then, there exists $\mu \in Q^\vee$ and $l \in \mathbb{Z}$ such that $(l, \mu) \neq (0, 0)$ and there exists $p_2 \in C_0 + (ld, \mu)$ which lies in the interval $[p_0, p_1]$.

**Proof.** Let $m \in \mathbb{N}$ be an integer satisfying $\text{Vol}(\Omega)/\text{Vol}(B_{\varepsilon/2}(p_0)) < m \leq \| p_1 - p_0 \|/\varepsilon$. Put $\bar{\varepsilon} = \varepsilon(p_1 - p_0)/\| p_1 - p_0 \|$. Consider the family of open balls $\{ B_{\varepsilon/2}(p_0 + kn\bar{\varepsilon}) \}_{k=0}^m$. Since the sum of the volume of these balls exceed the volume $\text{Vol}(\Omega)$, there exists $n_1, n_2 \in \mathbb{N}$ satisfying $m \geq n_1 > n_2 \geq 0$ such that $B_{\varepsilon/2}(p_0 + n_1\bar{\varepsilon})$ and $B_{\varepsilon/2}(p_0 + n_2\bar{\varepsilon})$ overlap in the quotient $\Omega$. In other words, there exists $\mu \in Q^\vee$ and $l \in \mathbb{Z}$ such that $\| (n_1 - n_2)\bar{\varepsilon} - (ld, \mu) \| < \varepsilon$. Consequently, the point $p_0 + (n_1 - n_2)\bar{\varepsilon} - (ld, \mu) \in B_\varepsilon(p_0)$ lies in the interior of $C_0$, or equivalently, the point $p_2 := p_0 + (n_1 - n_2)\bar{\varepsilon}$ lies in the interior of $C_0 + (ld, \mu)$. The point $p_2$ satisfies the requirement: we have $(ld, \mu) \neq (0, 0)$ because $\| (n_1 - n_2)\bar{\varepsilon} \| > \varepsilon$; in addition, $p_2$ lies in the interval $[p_0, p_1]$ because $n_1 - n_2 \leq \| p_1 - p_0 \|/\varepsilon$. \square

**Lemma 75.** Let $p_0, p_1, p_2 \in a$ such that $p_2 \in [p_0, p_1]$. For $i \in \{0, 1, 2\}$, put

$$R_i = \{ \alpha \in \Phi : \alpha(p_i) > 0 \}, \quad S_i = \{ \mu \in \Psi : \mu(p_i) > 0 \}.$$ 

Then, we have $R_0 \cap R_1 \subseteq R_2 \subseteq R_0 \cup R_1$ and $S_0 \cap S_1 \subseteq S_2 \subseteq S_0 \cup S_1$.

**Proof.** The proof is straightforward and left to the reader. \square

5.6.3 Let’s show that $\overline{B}^\otimes$ is finitely generated left $A$-module. Let $\varepsilon > 0$ be a number such that for each chamber $C \in \text{Ch}(a')$, there exists $p_C \in C$ satisfying $B_{\varepsilon}(p_C) \subseteq C$ — such $\varepsilon$ exists due finiteness of $(Q^\vee \times \mathbb{Z}d)$-orbits in $\text{Ch}(a')$. Set $N' = \left( \frac{\text{Vol}(\Omega)}{\text{Vol}(B_{\varepsilon/2}(p_0))} + 1 \right) \varepsilon$ and let $N \geq N'$ be an integer satisfying the following property: for each $w \in W$ such that $\ell(w) > N$ and for each $(p, p') \in \nu_0 \times w\nu_0$, the condition $\| p' - p \| > N'$ holds. We claim that $\Upsilon = \{ \nu^{kd,u} : w \in W, \ell(u) \leq N, 0 \leq k \leq N \}$ is a generating set of $\overline{B}^\otimes$ both as left and as right $A$-module.

Let $w \in W$ and $n \in \mathbb{N}$. We prove by induction on $\ell(w) + n$ that $\nu^{kd,w}w \in A\Upsilon$ holds. Suppose $\max(\ell(w), n) \leq N$, then $\nu^{kd,w}w \in \Upsilon$ holds trivially. Suppose then $\max(\ell(w), n) > N$. Let $C_0 = \kappa_0 \cap a' \subseteq a'$ be the chamber containing $\{0\} \times \nu_0$ and let $C_1 \subseteq a'$ be the chamber containing $\{nd\} \times w\nu_0$. Let $p_0 \in C_0$ and $p_1 \in C_1$ be such that $B_{\varepsilon}(p_i) \subseteq C_i$ for $i \in \{0, 1\}$. By our choice of $N$, it follows that $\| p_0 - p_1 \| > N'$. Therefore, the hypothesis of Lemma 74 is satisfied, which yields $(l, \mu) \in Z \times Q^\vee$ and $p_2 \in [p_0, p_1] \cap C_0 + (ld, \mu)$; moreover, $0 \leq l \leq n$ holds. Let $C_2 = C_0 + (ld, \mu) \in \text{Ch}(a')$. Lemma 75 applied to $(p_0, p_1, p_2)$ yields $S_0 \cap S_1 \subseteq S_2 \subseteq S_0 \cup S_1$ and $R_0 \cap R_1 \subseteq R_2 \subseteq R_0 \cup R_1$, which implies that $C_2 \subseteq [C_0, C_1]$ holds by Lemma 42 (i).
It allows us to apply Lemma 57 and obtain that $\gamma_{d,t,\mu} \ast \gamma_{(n-l)d,t,\mu} = \gamma_{nd,w}$. Since $\ell(t_\mu) + \ell(t_\mu w) = \ell(w)$ holds by the condition $C_2 \subset [C_0, C_1]$, we have $\ell(t_\mu w) + (n-l) < \ell(w) + n$. By induction hypothesis, $\gamma_{(n-l)d,t,\mu} \in \gamma_{nd,w}$ holds and hence $\gamma_{nd,w} \in \gamma_A$ also holds. This completes the induction step. Similarly, one can show that $\gamma_{nd,w} \in \gamma_A$ by exchanging $C_0$ and $C_1$ in the arguments.

Finally, as the family $\{\gamma_{nd,w}\}_{w \in W, n \in \mathbb{N}}$ spans $\mathcal{B}^\circ$ both as left and right $\mathcal{S}$-module and $\mathcal{S} \subseteq A$, we have $\gamma_A = \mathcal{B}^\circ = \gamma_A$. This completes the proof of Lemma 73.

6 Derived equivalences

Let $c, c' \in \mathfrak{P}$ be such that $d := c - c' \in \mathfrak{P}_2$ holds. The $(\mathcal{H}_{c'}, \mathcal{H}_c)$-bimodule $c'B_c$ from §5.1 induces the following translation functor:

$$T_{c' \leftarrow c} = c'B_c \otimes_{\mathcal{H}_c}^L : \mathcal{D}^b(\mathcal{H}_c) \to \mathcal{D}^b(\mathcal{H}_{c'}).$$

Since $\mathcal{H}$ is flat over $\mathcal{O}(\mathfrak{P})$, we can also express $T_{c' \leftarrow c} = \mathcal{B}^{\{a\}} \otimes_{\mathcal{H}}^L -$. The aim of this section is to establish the following theorem:

Theorem 76. The translation functor $T_{c' \leftarrow c}$ is an equivalence of categories between $\mathcal{D}^b(\mathcal{H}_c)$ and $\mathcal{D}^b(\mathcal{H}_{c'})$.

6.1 Stratification of $\mathfrak{P}$

6.1.1 We shall introduce a family $\mathfrak{M}$ of linear functions on $\mathfrak{P}_Q$. Let $\overline{\mathfrak{P}}$ be the image of $\Psi \subseteq \mathfrak{h}_Q^\times \times \mathfrak{P}_Q^s$ under the projection $\mathfrak{h}_Q^\times \times \mathfrak{P}_Q^s \to \mathfrak{h}_Q^\times \times \mathfrak{P}_Q^s$; concretely, it is given by

$$\overline{\mathfrak{P}} = \{\alpha - c_\alpha : \alpha \in R_{red}\} \cup \{\alpha/2 - c_\alpha : \alpha \in R \setminus R_{red}\}.$$  

The linear functions in $\overline{\mathfrak{P}}$ define a hyperplane arrangement $\{H_\mu\}_{\mu \in \overline{\mathfrak{P}}}$ on the space $\mathfrak{a}^0 = \mathfrak{P}_R \times \mathfrak{h}_R$, where $H_\mu$ is the zero locus of $\mu \in \overline{\mathfrak{P}}$. A subset $\sigma \subseteq \Psi$ is called circuit if it is minimal with the following property: there are coefficients $\{d_\mu\}_{\mu \in \sigma}$ with $d_\mu \in \mathbb{Q}$ such that

$$\mu_\sigma = \sum_{\mu \in \sigma} d_\mu \mu \in \mathfrak{P}_Q^s \setminus \{0\}.$$  

If $\sigma \subseteq \overline{\mathfrak{P}}$ is a circuit, then such family $\{d_\mu\}_{\mu \in \sigma}$ is unique up to scaling by $\mathbb{Q}^\times$; in other words, $[\mu_\sigma] := [\mathfrak{P}_Q^s, \mu_\sigma]$ is an element of the projectivisation $\mathfrak{P}(\mathfrak{P}_Q^s)$ independent of the choice of $[d_\mu]_{\mu \in \sigma}$. Define

$$\mathfrak{M} = \{[\mu_\sigma] \in \mathfrak{P}(\mathfrak{P}_Q^s) : \sigma \subseteq \overline{\mathfrak{P}} \text{ is a circuit}\}.$$  

Proposition 77. When $(\mathfrak{h}_R, R)$ is simply-laced, we have $\mathfrak{P}_Q^s = \mathbb{Q}c_\Psi$ and

$$\mathfrak{M} = \{[c_\Psi]\}.$$  

When $(\mathfrak{h}_R, R)$ is of type $BC_1$, we have $\mathfrak{P}_Q^s = \mathbb{Q}c_\Psi \oplus \mathbb{Q}c_\Psi$ and

$$\mathfrak{M} = \{[c_\Psi], [c_\Psi]\} \cup \{[c_\Psi + \epsilon c_\Psi] : \epsilon \in \{\pm 1\}\}.$$  

When $(\mathfrak{h}_R, R)$ is of type $BC_n (n \geq 2)$, we have $\mathfrak{P}_Q^s = \mathbb{Q}c_\Psi \oplus \mathbb{Q}c_\Psi \oplus \mathbb{Q}c_\Psi$ and

$$\mathfrak{M} = \{[c_\Psi], [c_\Psi], [c_\Psi]\}$$

$$\cup \{[c_\Psi + 2\epsilon c_\Psi], [c_\Psi + 2\epsilon c_\Psi], [c_\Psi + \epsilon c_\Psi] : \epsilon \in \{\pm 1\}\}$$  

$$\cup \{[c_\Psi + \epsilon_1 c_\Psi + \epsilon_2 c_\Psi] : \epsilon_1, \epsilon_2 \in \{\pm 1\}\}.$$  

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When \((h_{\mathbb{R}}, R)\) is of type \(F_4\), we have \(\mathcal{P}_Q^* = \mathbb{Q}c_1 \oplus \mathbb{Q}c_2\) and
\[
\mathcal{M} = \{[c_1], [c_2]\} \cup \{[c_1 + \varepsilon c_2], [c_2 + 2\varepsilon c_1]\}_{\varepsilon \in \{\pm 1\}}.
\]
When \((h_{\mathbb{R}}, R)\) is of type \(G_2\), we have \(\mathcal{P}_Q^* = \mathbb{Q}c_2 \oplus \mathbb{Q}c_4\) and
\[
\mathcal{M} = \{[c_2], [c_4]\} \cup \{[c_2 + \varepsilon c_4], [c_4 + 2\varepsilon c_2]\}_{\varepsilon \in \{\pm 1\}}.
\]

**Proof.** The proof is elementary and left to the reader. \(\square\)

6.1.2 For each parameter \(c \in \Psi\) we introduce a sub-family of \(\mathcal{M}\):
\[
\mathcal{M}_c = \{[\mu] \in \mathcal{M} : \mu(c) \in \mathbb{Q}\}.
\]
Then, the zero loci of the elements of \(\mathcal{M}_c\) define a hyperplane arrangement on \(\mathcal{P}_{\mathbb{R}}\), which decomposes \(\mathcal{P}_{\mathbb{R}}\) into a union of (open) facets, called c-facets.

For \(c \in \Psi\) and \(\lambda \in h^*_\mathbb{R}\), consider the subfamily \(\Psi_{c,\lambda} \subseteq \Psi\) introduced in §4.1. Let \(\widetilde{\mathcal{V}}_{c,\lambda} \subseteq \widetilde{\mathcal{V}}\) be its image in \(\mathcal{P}_Q^* \times h^*_\mathbb{R}\). Let \(\text{Ch}^{c,\lambda}(a^0)\) be the set of chambers of the hyperplane arrangement \(\{H_{\mu}\}_{\mu \in \Psi_{c,\lambda}}\) on \(a^0\).

**Lemma 78.** Let \(C \in \text{Ch}^{c,\lambda}(a^0)\) be a chamber. Then, the image of \(C\) under the projection \(p : a^0 \to \mathcal{P}_{\mathbb{R}}\) is a union of c-facets in \(\mathcal{P}_{\mathbb{R}}\).

**Proof.** Since the closure \(\overline{C}\) of \(C\) is a rational convex polyhedral cone of maximal dimension in \(a^0\), the same is true for its image \(p(\overline{C})\) in \(\mathcal{P}_{\mathbb{R}}\); moreover, \(p(C)\) is the interior of \(p(\overline{C})\). Let \(F\) be the closure of a face of \(p(\overline{C})\). Since \(p(\overline{C})\) is a rational convex polyhedral cone, there exists a linear form \(\mu_F \in \mathcal{P}_Q^*\) such that \(p(\overline{C}) \subseteq \mu_F^{-1}(\mathbb{R}_{\geq 0})\) and \(F = \mu_F^{-1}(0) \cap p(\overline{C})\). It suffices to show that \([\mu_F] = Q^\lambda \mu_F\) lies in \(\mathcal{M}_c\) — indeed, this will imply that \(p(C)\) is the intersection of the open half-spaces \(\mu_F > 0\) over such \(F\).

The intersection \((\mu_F \circ p)^{-1}(0) \cap \overline{C}\) is a union of facets of \(\overline{C}\) because \(\overline{C} \subseteq (\mu_F \circ p)^{-1}(\mathbb{R}_{\geq 0})\). Let
\[
\sigma' = \{a \in \widetilde{\mathcal{V}}_{c,\lambda} : (\mu_F \circ p)^{-1}(0) \cap \overline{C} \subseteq a^{-1}(0)\}.
\]
It is clear the intersection \(\bigcap_{a \in \sigma'} a^{-1}(0)\) is the subspace spanned by the polytope \((\mu_F \circ p)^{-1}(0) \cap \overline{C}\). In particular, we have \((\mu_F \circ p)(\bigcap_{a \in \sigma'} a^{-1}(0)) = 0\); therefore, \(\sigma'\) contains a circuit. Let \(\sigma \subseteq \sigma'\) be a circuit. Then, it follows that \([\mu_F] = [\mu_\sigma]\), where \(\mu_\sigma = \sum_{a \in \sigma} d_a a \in \mathcal{P}_Q^* \setminus \{0\}\) for some \(d_a \in \mathbb{Q}^+\). For each \(a \in \sigma\), let \(\tilde{a} \in \Psi_{c,\lambda}\) be a lifting of \(a\) through the projection \(\Psi_{c,\lambda} \to \widetilde{\mathcal{V}}_{c,\lambda}\). It follows that \(\mu_\sigma \in \sum_{a \in \sigma} d_a \tilde{a} + \mathbb{Q}\delta\) and hence \(\mu_\sigma(c) \in \sum_{a \in \sigma} d_a \tilde{a}(c, \lambda) + \mathbb{Q} = \mathcal{M}_c\); in other words, \([\mu_F] = [\mu_\sigma] \in \mathcal{M}_c\) holds. \(\square\)

6.1.3 For each \(c \in \Psi\), put \(\mathcal{P}_{Q,c} = c + \mathcal{P}_{Q}\). Given a c-facet \(F \subseteq \mathcal{P}_{\mathbb{R}}\) of this arrangement, we call \(F \cap \mathcal{P}_{Q,c}\) a stratum of \(\mathcal{P}_{Q,c}\). We say that \(F \cap \mathcal{P}_{Q,c}\) is open if \(F\) is a chamber.

6.2 Criteria of equivalence

6.2.1 Let \(c, c' \in \Psi\) be such that \(d := c - c' \in \mathbb{Z}\).

**Lemma 79.** The following conditions are equivalent:

(i) The functor \(T_{c \leftrightarrow c'} : D^b(H_c) \to D^b(H_{c'})\) is an equivalence of categories.

(ii) The left \(H_{c'}\)-module \(cB_c\) is tilting, namely, \(cB_c\) generates \(D^b_{\text{perf}}(H_{c'})\) as thick triangulated subcategory and \(\text{Ext}^2_{H_{c'}}(cB_c, cB_c) = 0\) holds.
(iii) The following adjoint unit and counit are quasi-isomorphisms
\begin{align}
\epsilon : & \mathbf{H}_c \to \text{RHom}_{\mathbf{H}_c}(\mathbf{c}B_c, \mathbf{c}B_c) \\
\eta : & \mathbf{c}B_c \otimes_{\mathbf{H}_c} \text{RHom}_{\mathbf{H}_c}(\mathbf{c}B_c, \mathbf{c}B_c) \to \mathbf{H}_c.
\end{align}

(i’’) For each \( \lambda \in \mathfrak{h}^1 \), the functor
\[ T_{\langle c \rightarrow c \rangle}^{'(c)} = \mathbf{c}B_c \otimes_{\mathbf{H}_c} \text{RHom}_{\mathbf{H}_c}(\mathbf{c}B_c, \mathbf{c}B_c) \to \mathbf{H}_c \]
is an equivalence of categories.

(ii’’) For each \( \lambda \in \mathfrak{h}^1 \), the left \( \mathcal{H}_c^\lambda \)-module \( \mathbf{c}B_c \) is tilting.

(iii’’) For each \( \lambda \in \mathfrak{h}^1 \), the following morphisms are quasi-isomorphisms
\[ \mathcal{H}_c^\lambda \to \text{Rhom}\mathcal{H}_c^\lambda(\mathbf{c}B_c, \mathbf{c}B_c) \]
\[ \mathbf{c}B_c \otimes_{\mathbf{H}_c} \text{Rhom}\mathcal{H}_c^\lambda(\mathbf{c}B_c, \mathbf{c}B_c) \to \mathcal{H}_c^\lambda. \]

Proof. The equivalence (i) \( \Leftrightarrow \) (ii) follows from [6, 1.6], the condition \( \mathbf{H}_c \cong \text{End}_{\mathbf{H}_c}(\mathbf{c}B_c) \) being ensured by Proposition 66. The assertion (i) \( \Rightarrow \) (iii) is trivial. The assertion (iii) \( \Rightarrow \) (ii) follows from the fact that \( \mathbf{H}_c \) generates \( D^b(\mathcal{H}_c^\lambda) \) as thick triangulated subcategory due to Proposition 4. The equivalences (i’) \( \Leftrightarrow \) (ii’) \( \Leftrightarrow \) (iii’) can be proven similarly. The equivalence (ii’) \( \Leftrightarrow \) (iii’) follows from the faithful flatness of \( \mathcal{F} \) over \( \mathfrak{S} \) proven in Lemma 6 and the formulæ Proposition 37 (ii) and Proposition 39 (ii).

6.2.2 Consider the following condition for \( c, c' \in \mathfrak{P} \) such that \( c - c' \in \mathfrak{P}_2 \):
\[ \text{Eq}(c, c') : \ \text{the multiplication induces isomorphisms} \]
\[ \mathbf{c}B_c \otimes_{\mathbf{H}_c} \mathbf{c}B_c \cong \mathbf{H}_c, \ \mathbf{c}B_c \otimes_{\mathbf{H}_c} \mathbf{c}B_c \cong \mathbf{H}_c. \]

Lemma 84. Suppose that \( \text{Eq}(c, c') \) holds, then

(i) \( T_{\langle c \rightarrow c \rangle}^{'(c)} \) and \( T_{\langle c \rightarrow c \rangle}^{'(c)} \) are t-exact equivalences and inverse to each other;

(ii) \( T_{\langle c \rightarrow c \rangle}^{'(c)} \circ T_{\langle c \rightarrow c \rangle}^{'(c)} \cong T_{\langle c \rightarrow c \rangle}^{'(c)} \) and \( T_{\langle c \rightarrow c \rangle}^{'(c)} \circ T_{\langle c \rightarrow c \rangle}^{'(c)} \cong T_{\langle c \rightarrow c \rangle}^{'(c)} \) hold for each \( c'' \in c + \mathfrak{P}_2 \).

Proof. The conditions imply that the functors
\[ \mathbf{c}B_c \otimes_{\mathbf{H}_c} \mathbf{H}_c \to \mathbf{H}_c, \ \mathbf{c}B_c \otimes_{\mathbf{H}_c} \mathbf{H}_c \to \mathbf{H}_c \]
are equivalences of categories and inverse to each other. In particular, \( \mathbf{c}B_c \) is projective as left \( \mathbf{H}_c \)-module and as right \( \mathbf{H}_c \)-module and subsequently, we have
\[ \mathbf{c}B_c \otimes_{\mathbf{H}_c} \mathbf{H}_c \otimes_{\mathbf{H}_c} \mathbf{c}B_c \cong \mathbf{H}_c. \]

The associativity of the multiplication yields a commutative diagram
\[ \mathbf{c}B_c \otimes_{\mathbf{H}_c} \mathbf{H}_c \otimes_{\mathbf{H}_c} \mathbf{c}B_c \cong \mathbf{H}_c \]
\[ \mathbf{c}B_c \otimes_{\mathbf{H}_c} \mathbf{c}B_c \cong \mathbf{H}_c \]
\[ \mathbf{c}B_c \otimes_{\mathbf{H}_c} \mathbf{c}B_c \cong \mathbf{H}_c. \]

We have seen that the left vertical arrow is an isomorphism; in particular, the upper horizontal arrow admits a left inverse and the right vertical arrow admits a right inverse. Applying \( \mathbf{c}B_c \otimes_{\mathbf{H}_c} \mathbf{H}_c \) to the diagram, we see that the following morphism induced by multiplication:
\[ m : \mathbf{c}B_c \otimes_{\mathbf{H}_c} \mathbf{c}B_c \otimes_{\mathbf{H}_c} \mathbf{c}B_c \to \mathbf{c}B_c \otimes_{\mathbf{H}_c} \mathbf{c}B_c. \]
admits a right inverse. Interchanging $c$ and $c'$, we deduce that the multiplication $m$ also admits a left inverse. Therefore, $m$ is a quasi-isomorphism. Applying $cB_{c'} \otimes_{H_{c'}} H_{c'} \to m$, and use the isomorphism $cB_{c'} \otimes_{H_{c'}} cB_c \cong H_c$, we conclude that $cB_{c'} \otimes_{H_{c'}} cB_{c''} \cong cB_{c''}$ and hence $T_{cc'} \circ T_{c'c''} \cong T_{cc''}$. The other isomorphism is similar.

By the associativity of the multiplication and the above lemma, $\text{Eq}(-, -)$ defines an equivalence relation.

**Lemma 85.** The condition $\text{Eq}(c, c')$ holds if and only if for each $\lambda \in \mathfrak{h}^1$, the following maps induced from multiplication are isomorphisms:

$$cB_c \otimes_{\mathcal{H}_c^\lambda} cB_c \cong \mathcal{H}_c^\lambda, \quad cB_c \otimes_{\mathcal{H}_c^\lambda} cB_c \cong \mathcal{H}_c^\lambda.$$

**Proof.** This follows immediately from the formula Proposition 37 (ii) and the faithful flatness of $\mathcal{S}$ over $\mathcal{S}$ due to Lemma 6.

### 6.3 Intrastratal equivalence

**Proposition 86.** Suppose that $c, c' \in \mathcal{P}$ satisfy $d = c - c' \in \mathcal{P}_Z$ and lie in the same stratum of $\mathcal{P}_{Q,c}$. Then, the condition $\text{Eq}(c, c')$ holds.

**Remark 87.** The converse may not be true in general.

**Proof of Proposition 86.** By Lemma 85, it suffices to verify the condition thereof. Let $\lambda \in \mathfrak{h}^1$. Consider the category $\mathcal{A}^c$ introduced in §4.1. Recall the isomorphisms (62) and (63) from §5.4. Choose $(\overline{a}, \overline{\lambda}) \in \mathcal{P}_Q \times \mathfrak{h}_Q^*$ satisfying the following property:

$$\mu(\overline{a}, \overline{\lambda}) = \mu(c, \lambda)$$

holds for each $\mu \in (\mathcal{P}_Q \times \mathfrak{h}_Q)^*$ such that $\mu(c, \lambda) \in \mathcal{Q}$.

Set $\overline{c} = \overline{\sigma} - d \in \mathcal{P}_Q$. By hypothesis, $-\overline{c}$ and $-\overline{\sigma}$ lie in a common $c$-facet $F \subseteq \mathcal{P}_R$. For each $w \in \widehat{W}$, the image of chamber $w^{-1}K_d - (\overline{a}, \overline{\lambda}) \in \text{Ch}_{c^\lambda}(a^0)$ under the projection $a^0 \to \mathcal{P}_R$ contains $-\overline{c}$. By Lemma 78, this image should contain $F$, to which $-\overline{c}$ belongs. In other words, the image of the chamber $w^{-1}K_d \in \text{Ch}_{c^\lambda}(a)$ under the projection $a \to \mathcal{P}_R$ contains 0. Therefore, there exists $w' \in \widehat{W}$ such that $w'^{-1}K_d = w'^{-1}K_0$. It follows that the direct factor

$$\bigoplus_{[y] \in \widehat{W}/\widehat{W}_d} \text{Hom}_{A_{c^\lambda}}(w^{-1}K_d, y^{-1}K_0)/(c - c) \subseteq c\mathcal{B}_c^\lambda.$$

is also a direct factor of $\mathcal{H}_c^\lambda$, which is projective. This implies that $c\mathcal{B}_c^\lambda$ is a projective $\mathcal{H}_c^\lambda$-module. Similar arguments show that $c\mathcal{B}_c^\lambda$ is a projective right $\mathcal{H}_c^\lambda$-module. The projectivity implies that the map

$$c\mathcal{B}_c^\lambda \otimes_{\mathcal{H}_c^\lambda} c\mathcal{B}_c^\lambda \to \mathcal{H}_c^\lambda, \quad a \otimes b \mapsto a \ast b$$

is an isomorphism. Exchanging $c$ and $c'$, we obtain the other isomorphism; hence, the condition of Lemma 85 holds.
6.4 Antipodal wall-crossing

Let $S = F \cap \mathcal{P}_{Q,c}, S' = F' \cap \mathcal{P}_{Q,c}$ be strata of $\mathcal{P}_{Q,c}$ (in the notation of §6.1.3). We say that $S$ and $S'$ are antipodal if $F' = -F$.

**Lemma 88.** Let $\lambda \in h^1$ and $c, c' \in \mathfrak{p}$ such that $d := c - c' \in \mathcal{P}_Z$ and $c, c'$ belong to antipodal strata in $\mathcal{P}_{Q,c}$. Then, each $w \in \hat{W}$, there exists $y \in W$ such that the chambers $w^{-1}\kappa_0, y^{-1}\kappa_d \in \text{Ch}^{c,\lambda}(a)$ are antipodal in the sense of §4.3.

**Proof.** Choose $(\overline{c}, \overline{\lambda}) \in \mathcal{P}_Q \times h^1_\mathfrak{q}$ as in the proof of Proposition 86. Set $\overline{c'} = \overline{c}-d \in \mathcal{P}_Q$. By hypothesis, $-\overline{c}$ and $-\overline{c'}$ lie in antipodal facets: $-\overline{c} \in F$ and $-\overline{c'} \in -F$ for a $c$-facet $F \subseteq P_\mathfrak{g}$. For each $w \in \hat{W}$, the image of chamber $C = w'^{-1}\kappa_d - (\overline{c}, \overline{\lambda}) \in \text{Ch}^{c,\lambda}(a^0)$ under the projection $a^0 \rightarrow P_\mathfrak{g}$ contains $-\overline{c'}$. By Lemma 78, this image should contain $F$. Consequently, the image of the antipode $-C \in \text{Ch}^{c,\lambda}(a^0)$ contains the antipode $-F$. In other words, the image of the chamber $-C + (\overline{c}, \overline{\lambda}) \in \text{Ch}^{c,\lambda}(a)$ under the projection $a \rightarrow P_\mathfrak{g}$ contains 0. Therefore, there exists $y \in \hat{W}$ such that $-C + (\overline{c}, \overline{\lambda}) = y^{-1}\kappa_0$ is antipodal to $w'^{-1}\kappa_d$. \hfill $\square$

Recall the centre $Z^\lambda_c$ from §1.5.2. We say that a compact module $M \in \mathcal{H}_c^\lambda$-mod is relatively injective if

(i) the $Z^\lambda_c$-module structure on $M$ introduced in §1.5.2 is free and

(ii) $\text{Ext}^{>0}_{\mathcal{H}_c^\lambda}(M, N) = 0$ for every $N \in \mathcal{H}_c^\lambda$-mod satisfying the condition (i).

The functor $M \mapsto \bigoplus_{\lambda \in \mathcal{A}} \text{Hom}_{Z^\lambda_c}(M1_{c,\lambda}, Z^\lambda_c)$ induces a bijection between the isomorphism classes of compact projective right $\mathcal{H}_c^\lambda$-modules and compact injective left $\mathcal{H}_c^\lambda$-modules.

**Proposition 89.** Let $c, c' \in \mathfrak{p}$ such that $d := c - c' \in \mathcal{P}_Z$ and $c, c'$ belong to antipodal open strata of $\mathcal{P}_{Q,c}$. Then, the functor $T_{c \rightarrow c'}$ is an equivalence of categories.

**Proof.** By Lemma 79, it suffices to show that $T_{\lambda \rightarrow c}$ is an equivalence for $\lambda \in h^1$.

Let $w \in \hat{W}$. By Lemma 88, there exists $y \in \hat{W}$ such that $\overline{w^{-1}\kappa_0}$ and $\overline{y^{-1}\kappa_d}$ are antipodal. Set

$$\mathcal{I}_w = \bigoplus_{w' \in \hat{W}/W_\lambda} \text{Hom}_{Z^\lambda_c}(w'^{-1}\kappa_d, w^{-1}\kappa_0)/(c - c)$$

$$\mathcal{P}_w = \bigoplus_{w' \in \hat{W}/W_\lambda} \text{Hom}_{Z^\lambda_c}(y^{-1}\kappa_d, w'^{-1}\kappa_0)/(c - c).$$

Then, by Proposition 53, there is a $\mathcal{H}_c^\lambda$-bilinear perfect pairing

$$\mathcal{I}_w \times \mathcal{P}_w \rightarrow Z^\lambda_c$$

which induces $\mathcal{I}_w \cong \text{Hom}_{Z^\lambda_c}(\mathcal{P}_w, Z^\lambda_c)$; the right $\mathcal{H}_c^\lambda$-module $\mathcal{P}_w$ being a direct factor of $\mathcal{H}_c^\lambda$, is projective. It follows that $\mathcal{I}_w$ is a relatively injective left $\mathcal{H}_c^\lambda$-module. Similar arguments show that the sum $\mathbb{P}_w^\lambda = \bigoplus_{w \in \hat{W}/W_\lambda} \mathcal{I}_w$ contains every indecomposable injective relatively injective left $\mathcal{H}_c^\lambda$-module as direct summand. Since the direct sum of indecomposable relative injective modules is a tilting module, the condition (ii’) of Lemma 79 is satisfied. \hfill $\square$
6.5 Simple wall-crossing

Let $c, c' \in \mathcal{P}$ such that $d = c - c'$ lies in $\mathcal{P}_Z$ and $c, c'$ belong to different open strata of $\mathcal{P}_{\mathbb{Q}, c}$ which share a common wall, say $[\mu] \in \mathfrak{M}_c$.

**Proposition 90.** Under the above hypothesis, the functor $T_{c'\gets c}$ is an equivalence of categories.

**Proof.** We prove this proposition by the degeneration technique from [1]. Let $D \subset \mathcal{P}$ be the complex hyperplane parallel to $h = 0$ such that $c \in D$ and put $\mathbf{B}_D = \mathbf{B}^{\langle d \rangle} \otimes_{\mathcal{O}(\mathcal{P})} \mathcal{O}(D)$. Let $U \subset D$ be the subset of $\mathbb{C}$-points of $D$ defined by

$$U = \{ u \in D : \mu'(u) \notin \mathbb{Q}, \forall [\mu'] \in \mathfrak{M}_c \setminus \{ [\mu] \} \}.$$ 

It follows that $\mathfrak{M}_u = \{ [\mu] \}$ consists of a single wall, so that the pair $(u, u')$ belong to antipodal open strata. Applying Proposition 89 to $u \in U$ and setting $u' = u - d$, we see that $\mathbf{u}_u \mathbf{B}_u \otimes \mathbb{H}_u$ induces an equivalence of categories. Consequently, the following adjoint unit and counit are quasi-isomorphisms:

$$\varepsilon_u : \mathbf{H}_u \to R\text{Hom}_{\mathbf{H}_u}(\mathbf{u}_u \mathbf{B}_u, \mathbf{u}_u \mathbf{B}_u), \quad \eta_u : \mathbf{u}_u \mathbf{B}_u \otimes \mathbf{H}_u \to \mathbf{H}_u.$$ 

Set $D' = D - d \subset \mathcal{P}$, $\mathbf{H}_D = \mathbf{H} \otimes_{\mathcal{O}(\mathcal{P})} \mathcal{O}(D)$, $\mathbf{H}_D' = \mathbf{H} \otimes_{\mathcal{O}(\mathcal{P})} \mathcal{O}(D')$ and

$$K = \text{Cone}(\mathbf{H}_D \to R\text{Hom}_{\mathbf{H}_D'}(\mathbf{B}_D, \mathbf{B}_D)), \quad K' = \text{Cone}(\mathbf{B}_D \otimes \mathbf{H}_D, \mathbf{B}_D \otimes \mathbf{H}_D' \to \mathbf{H}_D').$$

It follows that $K \otimes \mathcal{O}(D) \kappa(u) \cong \text{Cone}(\varepsilon_u) = 0$ and $K' \otimes \mathcal{O}(D) \kappa(u) \cong \text{Cone}(\varepsilon_u) = 0$ hold for each $u \in U$ and hence $\mathbf{H}^*(K)_u = 0 = \mathbf{H}^*(K')_u$.

Applying the lemma of generic freeness Lemma 91 below to $\mathbf{H}^*(K)$ viewed as $(\mathbf{H}, \mathcal{O}(D))$-bimodule, there exists a Zariski-open dense subset $V \subset D$ containing $U$ such that $K|_V = 0$. Similarly, we may assume that $K'|_V = 0$. Since the subset

$$D' = \{ z \in D \cap (c + \mathcal{P}_Z) : (c, z) \text{ same stratum}, (c', z - d) \text{ same stratum} \}$$

is Zariski-dense in $D$, we may choose $z \in D' \cap V \neq \emptyset$, so that $K_z = 0$ and $K'_z = 0$. Set $z' = z - d$, the translation functor $T_{z'\gets z}$ is an equivalence of categories. By Proposition 86, the conditions Eq$(c, z)$ and Eq$(c', z')$ hold, so it follows from Lemma 84 that $T_{c'\gets c} \cong T_{c'\gets z + d} \circ T_{z'\gets z} \circ T_{z\gets c}$ and it is also an equivalence of categories. \hfill $\square$

**Lemma 91.** Let $E$ be a noetherian integral commutative $\mathbb{C}$-algebra and let $M$ be a finitely generated $(\mathbf{H}, E)$-bimodule. Then, there exists a non-zero element $f \in E$ such that the localisation $M_f = M \otimes_E E_f$ is free as $E_f$-module, where $E_f = E[f^{-1}]$.

**Proof.** The proof is modeled on [1, 5.7, 5.8]. We extend the filtration $F^\text{can}_n \mathbf{H}$ to $\mathbf{H} \otimes E$ by setting $F^\text{can}_n(\mathbf{H} \otimes E) = (F^\text{can}_n \mathbf{H}) \otimes E$. Pick a good filtration $\{ F_k M \}_{k \in \mathbb{Z}}$ for $M$ with respect to $F^\text{can}_n(\mathbf{H} \otimes E)$. Then, the associated graded $\text{gr}^F M$ is finitely generated over $(\text{gr}^\text{can}_n \mathbf{H}) \otimes E$ and so is it finitely generated over the centre $Z(\text{gr}^\text{can}_n \mathbf{H}) \otimes E$. The lemma of generic freeness [3, 6.9.2] implies that there exists a non-zero element $f \in E$ such that $(\text{gr}^F M)_f$ is free over $E_f$. Since the filtration $\{ F_k M \}_{k \in \mathbb{Z}}$ is bounded below, $M_f$ is isomorphic to $(\text{gr}^F M)_f$ as free module over $E_f$. \hfill $\square$
6.6 General wall-crossing

**Proposition 92.** Let \( c, c' \in \mathcal{P} \) be such that \( d = c - c' \in \mathcal{P}_\mathbb{Z} \). Suppose that \( c \) and \( c' \) lie in open strata \( S \) and \( S' \) respectively. Then, the functor \( T_{c' \leftarrow c} \) is an equivalence of categories.

**Proof.** We prove the statement by induction on the number \( k \in \mathbb{N} \) of walls in \( \mathfrak{M}_c \) separating \( S \) and \( S' \). The case \( k = 0 \) is covered by Proposition 86. Let \( k > 0 \) and assume that the statement holds for smaller \( k \). Let \( S'' \) be an open stratum which is separated from \( S \) by one wall and from \( S' \) by \( k - 1 \) walls. Let \( [h] \in \mathfrak{M}_c \) be the unique element such that \( h^{-1}(0) \) separates \( S \) and \( S'' \). Choose \( c'' \in S'' \cap (c + \mathcal{P}_\mathbb{Z}) \) and put \( d'' = c - c'' \). Let \( D \subset \mathcal{P} \) be the complex hyperplane parallel to \( h = 0 \) such that \( c \in D \). Choose an element \( d' \in \mathcal{P}_\mathbb{Z} \cap S' \) and put \( \mathbf{B}^\circ = \bigoplus_{n \geq 0} \mathbf{B}^{(nd)} \) and \( \mathbf{C}^\circ = \bigoplus_{n \geq 0} \mathbf{B}^{(nd + d')} \). A variant of the proof of Lemma 73 shows that \( \mathbf{C}^\circ \) is a finitely generated left \( \mathbf{B}^\circ \)-module. The multiplication \( \star \) yields a morphism

\[ m : \mathbf{B}^\circ \otimes_{\mathbf{L}} \mathbf{B}^{(d'')} \to \mathbf{C}^\circ, \]

which induces

\[ m_D : \mathbf{B}^\circ \otimes_{\mathbf{L}} \mathbf{B}^{(d'')} \otimes_{\mathbf{L}(\mathcal{P})} \mathcal{O}(D) \to \mathbf{C}^\circ \otimes_{\mathbf{L}(\mathcal{P})} \mathcal{O}(D). \]

Put \( K = \text{Cone}(m_D) \in D^b(\mathbf{B}^\circ \text{-mod} - \mathcal{O}(D)) \). Let \( U \subset D \) be the subset of \( \mathcal{C} \)-points of \( D \) defined by

\[ U = \{ u \in D : f(u) \notin \mathbb{Q}, \ \forall \{ f \} \in \mathfrak{M}_c \setminus \{ [h] \} \}. \]

We have \( K \otimes_{\mathcal{O}(D)} K(u) = 0 \) for each \( u \in U \) by Proposition 86 (ii), because \( \mathfrak{M}_u = \{ [h] \} \) and \( u - d'' \) and \( u - d'' - nd' \) belong to the same open stratum for all \( n \in \mathbb{N} \). By the generic freeness Lemma 93 below, there exists \( f \in \mathcal{O}(D) \setminus \{ 0 \} \) such that \( K \otimes_{\mathcal{O}(D)} \mathcal{O}(D)[f^{-1}] = 0 \). Hence, the density implies that there exists \( z \in D \cap (c + \mathcal{P}_\mathbb{Z}) \) such that \( z \in S, \ z'' := z - d'' \in S'' \) and \( K \otimes_{\mathcal{O}(D)} K(z) = 0 \). Pick \( n_0 \gg 0 \) such that \( z' := z - d'' - nd' \in S' \). The condition \( K \otimes_{\mathcal{O}(D)} K(z) = 0 \) implies that \( z \mathbf{B}_{z''} \otimes_{\mathbf{L}(\mathcal{P}')} \mathbf{B}_{z'} \to z \mathbf{B}_{z} \) and hence \( T_{z' \leftarrow z''} \circ T_{z'' \leftarrow z} \to T_{z' \leftarrow z} \). Since \( S'' \) and \( S' \) are separated by \( k - 1 \) hyperplanes in \( \mathfrak{M}_c \), the induction hypothesis implies that \( T_{z' \leftarrow z''} \) is an equivalence of categories. On the other hand, Proposition 90 implies that \( T_{z'' \leftarrow z} \) is also an equivalence of categories; hence, so is \( T_{z' \leftarrow z} \) an equivalence of categories. Proposition 86 implies that the conditions \( \text{Eq}(c, z) \) and \( \text{Eq}(c', z') \) hold, so by Lemma 84 there is an isomorphism:

\[ T_{c' \leftarrow c} \circ T_{z' \leftarrow z} \circ T_{z \leftarrow c} \cong T_{c' \leftarrow c}. \]

Therefore, \( T_{c' \leftarrow c} \) is also an equivalence of categories. \( \square \)

**Lemma 93.** Let \( E \) be a noetherian integral commutative \( \mathcal{C} \)-algebra. For every finitely generated \( (\mathbf{B}^\circ, E) \)-bimodule \( M \), there exists \( f \in E \setminus \{ 0 \} \) such that \( M_f = M \otimes_E E_f \) free over \( E_f \).

**Proof.** The arguments are similar to the proof of Lemma 91: find a good filtration \( \{ F_{j,k} \} \) with respect to the \( \mathbb{N}^2 \)-filtration \( F_{\bullet, \bullet} \mathbf{B}^\circ \otimes E \); pass to the associated quotient \( \text{gr}^F M \); the restriction of \( \text{gr}^F M \) is finitely generated over \( A \otimes E \) thanks to Lemma 73, where \( A \) is a commutative \( \mathcal{C} \)-algebra of finite type; apply the generic freeness to deduce that when \( E \) is integral, there exists a non-zero element \( f \in E \) such that \( \text{gr}^F M_f \) is a free \( E_f \)-module. It follows that \( M_f \) is also free over \( E_f \). \( \square \)
6.7 Exceptional case

In this subsection, we deal with the case where \( c \) lies in a non-open stratum of \( \mathcal{P}_{Q,c} \).

6.7.1 Consider the most exceptional case, where the parameter \( c \in \mathcal{P} \) lies in the intersection of all the hyperplanes appearing in \( \mathcal{M}_c \). The following proposition shows that, in this case, all the hyperplanes in \( \mathcal{M}_c \) become irrelevant and the translation functors are abelian equivalences.

**Proposition 94.** Suppose that \( c \in \mathcal{P} \) satisfies \( \mu(c) = 0 \) for every \( \mu \in \mathcal{M}_c \). Then, \( \text{Eq}(c, c') \) holds for each \( c' \in c + \mathcal{P}_Z \).

This will result from the following lemma:

**Lemma 95.** Let \( c \) and \( c' \) be as in Proposition 94. Then, the algebraic KZ functor \( \mathbf{V} : \mathcal{H}^c_{\lambda} - \text{Mod} \to \mathcal{H}^{c'}_{\lambda} - \text{Mod} \) is an equivalence of categories for each \( \lambda = (\lambda^0, 1) \in \mathfrak{h}^1 \) and \( \ell = \exp(2\pi i \lambda^0) \in T \).

**Proof.** Consider first the case where \( c' = c \). We show that:

for each \( \lambda \in \mathfrak{h}^1 \), every \((c, \lambda)\)-clan is generic (see Definition 12).

We prove this assertion by contradiction. Suppose there exists a non-generic clan \( \mathcal{C} \subset \mathcal{C}^{c,\lambda}(\mathfrak{h}^1) \). Then, the set

\[
\Sigma = \{ \alpha \in \Psi_{c,\lambda} \cup -\Psi_{c,\lambda} : \alpha(\mathcal{C}) \subseteq \mathbb{R}_{>0} \}
\]

contains a positive \( \mathbb{Q} \)-linear relation \( \sum_{\alpha \in \Sigma'} d_\alpha \overline{\pi} = 0 \), where \( \Sigma' \subseteq \Sigma \) is a non-empty subset, \( \overline{\pi} = \alpha \mid_{\mathbb{Q} \times \mathcal{P}_Z} \) and \( d_\alpha \in \mathbb{Q}_{>0} \) for each \( \alpha \in \Sigma' \). We assume that \( \Sigma' \) is minimal with this property and let \( \sigma = \{ \overline{\pi} \in \mathbb{Q}^* \times \mathcal{P}_Z^*: \alpha \in \Sigma' \} \). Let \( h \in \mathcal{C} \), so that \( \alpha(h) > 0 \) for each \( \alpha \in \Sigma' \). It follows that

\[
0 < \sum_{\alpha \in \Sigma'} d_\alpha \alpha(h) = \sum_{\alpha \in \Sigma'} (d_\alpha c_\alpha + (d_\alpha \overline{\pi}, h - \lambda)) = \sum_{\alpha \in \Sigma'} d_\alpha c_\alpha.
\]

By the minimality of \( \Sigma' \), the set \( \sigma \) is a circuit, so that \( \mu_\sigma \in \mathcal{M}_c \) holds for \( \mu_\sigma = \sum_{\alpha \in \Sigma} d_\alpha c_\alpha \). This contradicts the assumption that \( \mu_\sigma(c) = 0 \). The claim is proven.

Since \( \mathcal{H}^c_{\lambda} 1_{\mathbf{V}} \) is a direct sum of such \( \mathcal{H}^c_{\lambda} 1_{w_{\lambda}} \) for \( w \in \hat{W} \) such that \( w^{-1} \nu_0 \) lies in a generic clan, the previous paragraph shows that \( \mathcal{H}^c_{\lambda} 1_{\mathbf{V}} \) is a compact projective generator of \( \mathcal{H}^c_{\lambda} - \text{Mod} \), so the functor \( \mathbf{V} = \text{Hom}_{\mathcal{H}^c_{\lambda}}(\mathcal{H}^c_{\lambda} 1_{\mathbf{V}}, -) \) is an equivalence of categories. Consequently, the category \( \mathcal{H}^{c'}_{\lambda} - \text{Mod} \) has finite global dimension. Since \( \mathcal{H}^{c'}_{\lambda} \) is a Frobenius algebra over the centre \( \mathcal{Z}^{c'} = Z(\mathcal{H}^{c'}_{\lambda}) \cong \mathcal{Z}^{c} \) (see [10, 9.3]), all \( \mathcal{H}^{c'}_{\lambda} \)-modules which are projective over \( \mathcal{Z}^{c'} \) must be projective.

Now, let \( c' \in c + \mathcal{P}_Z \). Since \( \mathbf{V} \) sends \( \mathcal{Z}^{c} \)-projective objects to \( \mathcal{Z}^{c'} \)-projective objects, it preserves projective objects. The double centraliser property (see Theorem 26) implies that \( \mathbf{V} \) is an equivalence of categories.

**Proof of Proposition 94.** By Lemma 85, it suffices to verify the conditions thereof. Let \( \lambda \in \mathfrak{h}^1 \). Lemma 95 implies that it suffices to show that

\[
\mathcal{P}^c \otimes_{\mathcal{H}^c_{\lambda}} c \mathcal{P}^c \otimes_{\mathcal{H}^{c'}_{\lambda}} c \mathcal{P}^{c'} \xrightarrow{\alpha \otimes \alpha} \mathcal{P}^c \otimes_{\mathcal{H}^{c'}_{\lambda}} c \mathcal{P}^{c'} \xrightarrow{\alpha \otimes \alpha} \mathcal{P}^c \otimes_{\mathcal{H}^c_{\lambda}} c \mathcal{P}^{c'} \xrightarrow{\alpha \otimes \alpha} \mathcal{P}^{c'}
\]

are isomorphisms. By Proposition 64, these maps can be factorised as \( \alpha_{c,c'} \circ (\alpha_{c,c'} \otimes \alpha_{c,c'}) \) and \( \alpha_{c,\lambda} \circ (\alpha_{c,\lambda} \otimes \alpha_{c,\lambda}) \), which are isomorphisms. \( \square \)
6.7.2 Let \( c \in \mathfrak{P} \) and let \( S \subset \mathfrak{P}_{Q,c} \) be the stratum which contains \( c \). Pick an open stratum \( S' \subset \mathfrak{P}_{Q,c} \) whose closure contains \( S \) and pick a point \( c' \in S' \cap (c + \mathfrak{P}_Z) \) and set \( d = c - c' \).

**Proposition 96.** Under the above hypothesis, the condition \( \text{Eq}(c, c') \) holds.

**Proof.** Let \( E \subseteq \mathfrak{P} \) be the \( \mathbb{C} \)-linear span of \( S - c \), let \( D = c' + E \), let \( \mathcal{M}_c = \{ h \in \mathcal{M}_c \mid h(c) \neq 0 \} \) and let \( U \subset D \) be the subset defined by

\[
U = \{ u \in D : f(u) \notin \mathbb{Q}, \forall [f] \in \mathcal{M}_c \}.
\]

Put \( B_D = B^{(d)} \otimes_{\mathcal{O}(\mathfrak{P})} \mathcal{O}(D) \). For each \( u \in U \), we have \( \mathcal{M}_u = \mathcal{M}_c \setminus \mathcal{M}_{c'} \), so the condition \( \text{Eq}(u, u + d) \) holds by Proposition 94. Again, the degeneration technique as in the proof of Proposition 90 implies that there exists \( z \in D \cap S \) such that \( z' := z - d \in S' \) and \( \text{Eq}(z, z') \) holds. Proposition 86 implies that \( \text{Eq}(c, z) \) and \( \text{Eq}(c', z') \) hold. Hence, \( \text{Eq}(c, c') \) holds. \( \Box \)

### 6.8 Proof of Theorem 76

Let \( S, S' \subset \mathfrak{P}_{Q,c} \) be open strata such that \( c \in \overline{S} \) and \( c' \in \overline{S'} \). Let \( z \in S \cap (c + \mathfrak{P}_Z) \) and \( z' \in S' \cap (c + \mathfrak{P}_Z) \). Then, \( T_{z'\leftarrow z} \) is an equivalence of categories by Proposition 92; \( T_{z'\leftarrow z'} \) and \( T_{z\leftarrow z'} \) are equivalences of category and \( T_{z'\leftarrow c} \cong T_{c'\leftarrow z'} \circ T_{z'\leftarrow z} \circ T_{z\leftarrow c} \) holds by Proposition 96 and Lemma 84 (ii); hence, \( T_{z'\leftarrow c} \) is an equivalence of categories. \( \Box \)

### 6.9 Derived equivalences on integrable modules

We prove that the derived equivalence established in Theorem 76 preserves the integrable modules, their blocks and their support.

6.9.1 Consider the subring \( \mathcal{O}(T^\vee) \cong \mathbb{C}Q^\vee \subseteq H \), where \( T^\vee \) is the torus whose character lattice is the coroot lattice \( Q^\vee \) of \((\mathfrak{g}_\mathbb{R}, R)\). Given any \( M \in \mathcal{O}(H) \), by the triangular decomposition \( H = \mathbb{C}Q^\vee \otimes CW \otimes \mathcal{O}(\mathfrak{P} \times \mathfrak{h}) \), it is clear that \( M \) is finitely generated over \( \mathbb{C}Q^\vee \). Let \( \text{Supp}_{T^\vee/W} M \subseteq T^\vee/W \) be the (set-theoretic) support of \( M \) considered as \( \mathcal{O}(T^\vee)^W \)-module by restriction.

**Lemma 97.** Let \( B \in \text{HC}(H) \) and \( M \in \text{O}(H) \). Then, \( \text{Supp}_{T^\vee/W} \text{Tor}^i_H(B, M) \subseteq \text{Supp}_{T^\vee/W} M \) and \( \text{Supp}_{T^\vee/W} \text{Ext}^i_H(B, M) \subseteq \text{Supp}_{T^\vee/W} M \) hold for each \( i \in \mathbb{Z} \).

**Proof.** Choose a HC filtration \( \{ F_k B \}_{k \in \mathbb{Z}} \) for \( B \) and a good filtration \( \{ F_k M \}_{k \in \mathbb{Z}} \) for \( M \) with respect to the canonical filtration \( F^\text{an}_BB \). Note that the subspaces \( F_k B \) and \( F_k M \) are stable under multiplication by \( \mathcal{O}(T^\vee) \) for each \( k \in \mathbb{Z} \). Since \( \text{gr}^F B \) is finitely generated over the centre \( Z(H_\delta/(\delta)) \) and the adjoint action of the latter is nilpotent on \( \text{gr}^F B \), there exists a number \( m > 0 \) such that \( (\text{ad} Z(H_\delta/(\delta)))^m \text{gr}^F B = 0 \). Note that \( \mathcal{O}(T^\vee)^W \subseteq Z(H_\delta/(\delta)) \). Let \( I = \text{ann}_{\mathcal{O}(T^\vee)^W} M \) be the annihilator of \( M \). Then, the ideal \( I^m \) annihilates \( \text{Ext}^i_H(\text{gr}^F B, \text{gr}^F M) \).

As noted in the proof of Proposition 31, there is an induced good filtration on \( \text{Ext}^i_H(B, M) \), denoted by \( F^i \), such that \( \text{gr}^F \text{Ext}^i_H(B, M) \) is a subquotient of \( \text{Ext}^i_H(\text{gr}^F B, \text{gr}^F M) \). Hence, \( I^m \) annihilates \( \text{gr}^F \text{Ext}^i_H(B, M) \) and thus \( \text{Supp}_{T^\vee/W} \text{Ext}^i_H(B, M) \subseteq \text{Supp}_{T^\vee/W} M \). The result for Ext follows. The statement for Tor can be proven in a similar way. \( \Box \)
6.9.2 For any closed subcategory \( Z \subseteq T^\vee/W \) and \((c,[\lambda]) \in \mathcal{P} \times B^1/\hat{W}\), let \( \mathcal{O}_{\lambda,Z}(H_c) \) be the full subcategory of \( \mathcal{O}_\lambda(H_c) \) consisting of objects \( M \) satisfying \( \text{Supp}_{T^\vee/W}(M) \subseteq Z \) and let \( D^b_O(\mathcal{O}_\lambda(H_c)) \) (resp. \( D^b_{O_{\lambda,Z}}(H_c) \)) be the full subcategory of \( D^b(\mathcal{O}_\lambda(H_c)) \) (resp. of \( D^b(H_c) \)) consisting of complexes \( K^* \) such that \( H^k(K^*) \in \mathcal{O}_{\lambda,Z}(H_c) \) for every \( k \in \mathbb{Z} \).

Theorem 98. Under the hypothesis of Theorem 76, the translation functor \( T_{c'\leftrightarrow c} \) induces to an equivalence \( D^b_O(\mathcal{O}_\lambda(H_c)) \xrightarrow{\sim} D^b_O(\mathcal{O}_\lambda(H_{c'})) \) for each \([\lambda] \in B^1/\hat{W}\) and each closed subset \( Z \subseteq T^\vee/W \).

Proof. Since there is natural isomorphism \( \mathcal{O}_{[\lambda],[\beta]}^{[\alpha],b_1} \otimes \mathcal{O}(b_1)^c \mathcal{B}_c \cong \mathcal{B}_c \otimes \mathcal{O}(b_1)^c \mathcal{O}_{[\lambda],[\beta]}^{[\alpha],b_1} \) (see §5.4), by Theorem 16, Proposition 37, the restriction of \( T_{c'\leftrightarrow c} \) to \( D^b_O(\mathcal{O}_\lambda(H_c)) \) is given by

\[
\left( \mathcal{C}_c^\lambda \right)^{-1} \circ T_{c'\leftrightarrow c} \circ \mathcal{C}_c^\lambda : D^b_O(\mathcal{O}_\lambda(H_c)) \to D^b_O(\mathcal{O}_\lambda(H_{c'})).
\]

By Lemma 97, it restricts to a functor from from \( D^b_{O_{\lambda,Z}}(H_c) \) to \( D^b_{O_{\lambda,Z}}(H_{c'}) \). Similarly, the right adjoint functor of \( T_{c'\leftrightarrow c} \) restricts to a functor from \( D^b_{O_{\lambda,Z}}(H_{c'}) \) to \( D^b_{O_{\lambda,Z}}(H_c) \). By Theorem 76, they are inverse to each other, thus equivalences of categories. Applying Corollary 23, we obtain

\[
D^b_O(\mathcal{O}_\lambda(H_c)) \cong D^b_{O_{\lambda,Z}}(H_c) \xrightarrow{T_{c'\leftrightarrow c}} D^b_{O_{\lambda,Z}}(H_{c'}) \cong D^b_O(\mathcal{O}_\lambda(H_{c'})).
\]

\[\square\]

Remark 99. With slight modifications, all results and arguments in the present article work over arbitrary field of characteristic 0 instead of \( \mathbb{C} \).

Remark 100. In [12], it is shown that certain wall-crossing derived equivalences obtain from translation bimodules are perverse in the sense of Rouquier [15]. We expect the derived equivalences in Theorem 98 to be perverse as well.

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