On the Cesàro average of the “Linnik numbers”

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Abstract
Let $\Lambda$ be the von Mangoldt function and

$$r_Q(n) = \sum_{m_1 + m_2^2 + m_3^2 = n} \Lambda(m_1)$$

be the counting function for the numbers that can be written as sum of a prime and two squares (that we will call Linnik numbers, for brevity). Let $N$ a sufficiently large integer. We prove that for $k > 3/2$ we have

$$\sum_{n \leq N} r_Q(n) \frac{(N-n)^k}{\Gamma(k+1)} = M(N, k) + O(N^{k+1})$$

where $M(N, k)$ is essentially a weighted sum, over non-trivial zeros of the Riemann zeta function, of Bessel functions of complex order and real argument. We also prove that with this technique the bound $k > 3/2$ is optimal.

1 Introduction

We continue the recent work of Languasco and Zaccagnini on additive problems with prime summands. In [9] and [10] they study the Cesàro weighted

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explicit formula for the Goldbach numbers (the integers that can be written as sum of two primes) and for the Hardy-Littlewood numbers (the integers that can be written as sum of a prime and a square). In a similar manner, we will study a Cesàro weighted explicit formula for the integers that can be written as sum of a prime and two squares. We will obtain an asymptotic formula with a main term and more terms depending explicitly on the zeros of the Riemann zeta function. The study of these numbers is classical. For example Hardy and Littlewood in [7] studied the number of solutions of the equation
\[ n = p + a^2 + b^2 \]
and Linnik in [13] derived an asymptotic formula for the number of representations of these numbers. Similar averages of arithmetical functions are common in literature, see, e.g., Chandrasekharan - Narasimhan [2] and Berndt [1] who built on earlier classical work. For our work we will need the Bessel functions \( J_v(u) \) of complex order \( v \) and real argument \( u \). For their definition and main properties we refer to Watson [15], but we recall that they were introduced by Daniel Bernoulli and they are the canonical solution of the differential equation
\[ u^2 d^2 J / du^2 + u dJ / du + (u^2 - v^2) J = 0 \]
for any complex number \( v \). In particular, equation (8) on page 177 of [15] gives the Sonine representation
\[ J_v(u) = (u/2)^v / 2\pi i \int_a e^{s} s^{-v-1} e^{-u^2/(4s)} ds \]
where the notation \( \int_a \) means \( \int_{a-i\infty}^{a+i\infty} \). The method we will use in this additive problem is based on a formula due to Laplace [11], namely
\[ 1 / 2\pi i \int_a v^{-s} e^v dv = 1 / \Gamma(s) \]
with \( \text{Re}(s) > 0 \) and \( a > 0 \) (see, e.g., formula 5.4 (1) on page 238 of [4]). As in [10], we combine this approach with line integrals with the classical methods dealing with infinite sum over primes and integers. Similarly as [10] the problem naturally involves the modular relation for the complex Jacobi \( \theta_3 \) function; the presence of the Bessel functions in our statement strictly depends on such modularity relation.
2 Preliminary definitions and Lemmas

Let
\[ r_Q(n) = \sum_{m_1 + m_2 + m_3 = n} \Lambda(m_1) \]
and let \( J_v(u) \) be the Bessel function of complex order \( v \) and real argument \( u \). Let \( z = a + iy, a > 0, \) and

\[
\theta_3(z) = \sum_{m \in \mathbb{Z}} e^{-m^2z}, \\
\tilde{S}(z) = \sum_{m \geq 1} \Lambda(m) e^{-mz}, \\
\omega_2(z) = \sum_{m \geq 1} e^{-m^2z},
\]
and we can see that

\[
\theta_3(z) = 1 + 2\omega_2(z).
\]
Furthermore we have the functional equation (see, for example, the proposition VI.4.3 of Freitag-Busam [5] page 340)

\[
\theta_3(z) = \left(\frac{\pi}{z}\right)^{1/2} \theta_3\left(\frac{\pi^2}{z}\right), \quad \text{Re}(z) > 0
\]
and so

\[
\omega_2^2(z) = \left(\frac{1}{2} \left(\frac{\pi}{z}\right)^{1/2} - \frac{1}{2}\right)^2 + \frac{\pi}{z} \omega_2^2 \left(\frac{\pi^2}{z}\right) + \left(\left(\frac{\pi}{z}\right)^{1/2} - 1\right) \left(\left(\frac{\pi}{z}\right)^{1/2} \omega_2 \left(\frac{\pi^2}{z}\right)\right).
\]

A trivial but important estimate is

\[
|\omega_2(z)| \leq \omega_2(a) \leq \int_0^\infty e^{-at^2} dt = \frac{\sqrt{\pi}}{2\sqrt{a}} \ll a^{-1/2}.
\]
Let us introduce the following

**Lemma 2.1.** Let \( z = a + iy, a > 0 \) and \( y \in \mathbb{R} \). Then

\[
\tilde{S}(z) = \frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) + E(a, y)
\]
where \( \rho = \beta + i\gamma \) runs over the non-trivial zeros of \( \zeta(s) \) and

\[
E(a, y) \ll |z|^{1/2} \begin{cases} 
1, & |y| \leq a \\
1 + \log^2(|y|/a), & |y| > a.
\end{cases}
\]
(For a proof see Lemma 1 of [9]. The bound for \( E(a, y) \) has been corrected in \([8]\)). So in particular, taking \( z = \frac{1}{N} + iy \) we have
\[
\left| \sum_{\rho} z^{-\rho} \Gamma (\rho) \right| = \left| \frac{1}{z} - \tilde{S}(z) + E\left(\frac{1}{N}, y\right) \right| \ll N + \frac{1}{|z|} + E\left(\frac{1}{N}, y\right).
\]
(2.9) \( \ll \begin{cases} N, & |y| \leq 1/N \\ N + |z|^{1/2} \log^2 (2N |y|), & |y| > 1/N. \end{cases} \)

Now we have to recall that the Prime Number Theorem (PNT) is equivalent, via Lemma 2.1, to the statement
\[
\tilde{S}(a) \sim a^{-1}, \text{ when } a \to 0^+
\]
(see Lemma 9 of [7]). For our purposes it is important to introduce the Stirling approximation
\[
|\Gamma (x + iy)| \sim \sqrt{2\pi e^{-\pi|y|/2}} |y|^{x-1/2},
\]
(see for example §4.42 of [14]) uniformly for \( x \in [x_1, x_2] \), \( x_1 \) and \( x_2 \) fixed, and the identity
\[
|z^{-w}| = |z|^{-\text{Re}(w)} \exp (\text{Im}(w) \arctan (y/a)).
\]

We now quote Lemmas 2 and 3 from [9]:

**Lemma 2.2.** Let \( \beta + i\gamma \) run over the non-trivial zeros of the Riemann zeta function and let \( \alpha > 1 \) be a parameter. The series
\[
\sum_{\rho, \gamma > 0} \gamma^{\beta-1/2} \int_{1}^{\infty} \exp (-\gamma \arctan (1/u)) \frac{dy}{u^{\alpha+\beta}}
\]
converges provided that \( \alpha > 3/2 \). For \( \alpha \leq 3/2 \) the series does not converge. The result remains true if we insert in the integral a factor \( \log^c (u) \), for any fixed \( c \geq 0 \).

**Lemma 2.3.** Let \( \beta + i\gamma \) run over the non-trivial zeros of the Riemann zeta function, let \( z = a + iy \), \( a \in (0, 1) \), \( y \in \mathbb{R} \) and \( \alpha > 1 \). We have
\[
\sum_{\rho} |\gamma|^{\beta-1/2} \int_{\mathbb{Y}_1 \cup \mathbb{Y}_2} \exp \left( \gamma \arctan \left( \frac{y}{a} \right) - \frac{\pi}{2} |\gamma| \right) \frac{dy}{|z|^{\alpha+\beta}} \ll_{\alpha} a^{-\alpha}
\]
where \( \mathbb{Y}_1 = \{ y \in \mathbb{R} : \gamma y \leq 0 \} \) and \( \mathbb{Y}_2 = \{ y \in [-a, a] : y\gamma > 0 \} \). The result remains true if we insert in the integral a factor \( \log^c (|y|/a) \), for any fixed \( c \geq 0 \).
We now establish an important Lemma. We will use it to prove that there is a limitation in our technique. Essentially the lower bound of $k$ is linked to the number of squares in the problem. We have

**Lemma 2.4.** Let $\beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta-function, let $N, d$ be positive integers, $\| \|$ the euclidean norm in $\mathbb{R}^d$ and $k > 0$ be a real number. Then the series

$$
\sum_{\tau \in (0, \infty)^d} \sum_{\gamma > 0} \gamma^{-k-3/2} \int_0^\gamma e^{-N\|\tau\|^2} \frac{v^2}{\gamma^2} e^{-v} e^{\gamma k + \beta} dv,
$$

where

$$
\sum_{\tau \in (0, \infty)^d} = \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \cdots \sum_{l_d \geq 1},
$$

converges if $k > d - 1/2$ and this result is optimal.

**Proof.** From (2.4) we have that

$$
\omega_2^d (z) = \frac{1}{2d} \sum_{m=0}^d \binom{d}{m} (-1)^{d-m} \theta_3^m (z).
$$

Hence

$$
I = \sum_{\tau \in (0, \infty)^d} \sum_{\gamma > 0} \gamma^{-k-3/2} \int_0^\gamma e^{-N\|\tau\|^2} \frac{v^2}{\gamma^2} e^{-v} e^{\gamma k + \beta} dv
$$

$$
= \sum_{\gamma > 0} \gamma^{-k-3/2} \int_0^\gamma \omega_2^d \left( \frac{N v^2}{\gamma^2} \right) e^{-v} e^{\gamma k + \beta} dv
$$

$$
= \frac{1}{2d} \sum_{m=0}^d \binom{d}{m} (-1)^{d-m} \sum_{\gamma > 0} \gamma^{-k-3/2} \int_0^\gamma \theta_3^m \left( \frac{N v^2}{\gamma^2} \right) e^{-v} e^{\gamma k + \beta} dv.
$$

Now, using the functional equation (2.5) we have that

$$
I = \frac{1}{2d} \sum_{m=0}^d \binom{d}{m} (-1)^{d-m} \frac{\pi^{m/2}}{N^{m/2}} \sum_{\gamma > 0} \gamma^{m-k-3/2} \int_0^\gamma \theta_3^m \left( \frac{\pi^2 \gamma^2}{N v^2} \right) e^{-v} e^{\gamma k + \beta-m} dv
$$

$$
= \frac{1}{2d} \sum_{m=0}^d \binom{d}{m} (-1)^{d-m} \frac{\pi^{m/2}}{N^{m/2}} \sum_{\gamma > 0} \gamma^{m-k-3/2} I_{\gamma, m},
$$

say. Now we claim that

$$
\theta_3 \left( \frac{\pi^2 \gamma^2}{N v^2} \right) \asymp 1,
$$
where the notation \( f(x) \asymp g(x) \) means \( g(x) \ll f(x) \ll g(x) \), since \( \theta_3(x) \) is a continuous function in the interval \( \left[ \frac{\pi^2}{N}, \infty \right) \) (i.e. the range of \( 1/v^2 \)) and
\[
\lim_{x \to \infty} \theta_3(x) = 1.
\]

So we have
\[
I_{\gamma,m} \asymp \sum_{\gamma \in \mathbb{R}} \gamma^{m-k-3/2} \int_{0}^{\gamma} e^{-v(k+\beta-m)} dv
\]
and now, assuming \( k + \beta - m + 1 > 0 \), we get
\[
\int_{0}^{\gamma} e^{-v(k+\beta-m)} dv \asymp 1.
\]

Hence
\[
I_{\gamma,m} \asymp k \sum_{\gamma > 0} \gamma^{m-k-3/2}
\]
and the last series converges if \( k > m - 1/2 \). Since \( m = 0, \ldots, d \) for a global convergence we must have \( k > d - 1/2 \) and this result is optimal.

Let us introduce another lemma

**Lemma 2.5.** Let \( \rho = \beta + i\gamma \) run over the non-trivial zeros of the Riemann zeta function, let \( z = \frac{1}{N} + iy \), \( N > 1 \) natural number, \( y \in \mathbb{R} \) and \( \alpha > 3/2 \).

We have
\[
\sum_{\rho} |\Gamma(\rho)| \int_{(1/N)}^{\gamma} |e^{Nz}| \left| z^{-\rho} \right| |z|^{-\alpha} |dz| \ll_{\alpha} N^\alpha.
\]

**Proof.** Put \( a = \frac{1}{N} \). Using the identity (2.11) and (2.10) we get that the left hand side in the statement above is
\[
\sum_{\rho} |\gamma|^{\beta-1/2} \int_{\mathbb{R}} \exp \left( \gamma \arctan \left( \frac{y}{a} \right) - \frac{\pi}{2} |\gamma| \right) \frac{dy}{z^{\alpha+\beta}}.
\]

and so by Lemma 2.3 (2.12) is \( \ll_{\alpha} a^{-\alpha} \) in \( \mathbb{V}_1 \cup \mathbb{V}_2 \). For the other part we can see that
\[
\sum_{\rho} \gamma^{\beta-1/2} \int_{a}^{\infty} \exp \left( -\gamma \arctan \left( \frac{a}{y} \right) \right) \frac{dy}{z^{\alpha+\beta}}
\]
\[
= a^{-\alpha-\beta+1} \sum_{\rho} \gamma^{\beta-1/2} \int_{1}^{\infty} \exp \left( -\gamma \arctan \left( \frac{1}{u} \right) \right) \frac{dy}{u^{\alpha+\beta}}
\]

since
\[
|z|^{-1} \asymp \begin{cases} a^{-1}, & |y| \leq a, \\ |y|^{-1}, & |y| \geq a, \end{cases}
\]
and so by Lemma 2.2 we have the convergence if \( \alpha > 3/2 \). \( \square \)
3 Settings

Using (2.1), (2.2) and (2.3) it is not hard to see that
\[
\tilde{S}(z) \omega_2^2(z) = \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \sum_{m_3 \geq 1} \Lambda(m_1) e^{-(m_1+m_2^2+m_3^2)z} = \sum_{n \geq 1} r_Q(n) e^{-nz}.
\]

Let \( z = a + iy, a > 0 \) and let us consider
\[
\frac{1}{2\pi i} \int_{(a)} e^{Nz} z^{-k-1} \tilde{S}(z) \omega_2^2(z) \, dz = \frac{1}{2\pi i} \int_{(a)} e^{Nz} z^{-k-1} \sum_{n \geq 1} r_Q(n) e^{-nz} \, dz.
\]

Now we prove that we can exchange the integral with the series. From (2.7) and the Prime Number Theorem in the form quoted above we have
\[
\sum_{n \geq 1} |r_Q(n) e^{-nz}| = \tilde{S}(a) \omega_2^2(a) \ll a^{-2}
\]
hence
\[
\int_{(a)} \left| e^{Nz} z^{-k-1} \right| \left| \tilde{S}(z) \omega_2^2(z) \right| \, dz \ll a^{-2} e^{Na} \left( \int_{-a}^{a} a^{-k-1} \, dy + 2 \int_{a}^{\infty} y^{-k-1} \, dy \right) \ll_k a^{-2-k} e^{Na}
\]
assuming \( k > 0 \). So finally we have
\[
(3.1) \quad \sum_{n \leq N} r_Q(n) \frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_{(a)} e^{Nz} z^{-k-1} \tilde{S}(z) \omega_2^2(z) \, dz.
\]

Now, using (2.8), we can write (3.1) as
\[
\sum_{n \leq N} r_Q(n) \frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_{(a)} e^{Nz} z^{-k-1} \left( \frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) \right) \omega_2^2(z) \, dz +
\]
\[
+ O \left( \int_{(a)} \left| e^{Nz} \right| \left| z \right|^{-k-1} \left| \omega_2^2(z) \right| \left| E(a,y) \right| \, dz \right)
\]
and the error term can be estimated, using Lemma 2.1, (2.7) and (2.13) as
\[
a^{-1} e^{Na} \left( \int_{-a}^{a} a^{-k-1} \, dy + \int_{a}^{\infty} y^{-k-1/2} \left( 1 + \log^2 \left( y/a \right) \right) \, dy \right) \ll_k e^{Na} a^{-k-1}
\]
assuming \( k > 1/2 \). Hereafter we will consider \( a = 1/N \). We have
\[
\sum_{n \leq N} r_Q(n) \frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-1} \left( \frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) \right) \omega_2^2(z) \, dz + O \left( N^{k+1} \right)
\]
and now, using the functional equation (2.6), we get

\[
\sum_{n \leq N} r_q(n) \frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{8\pi i} \int_{(1/N)} e^{Nz^{k-1}} \left( \frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) \right) \left( \frac{\pi}{z} \right)^{1/2} - 1 \right)^2 dz
\]

\[
+ \frac{1}{2\pi i} \int_{(1/N)} e^{Nz^{k-1}} \left( \frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) \right) \frac{\pi}{z} \omega_2 \left( \frac{\pi^2}{z} \right) dz
\]

\[
+ \frac{1}{2\pi i} \int_{(1/N)} e^{Nz^{k-1}} \left( \frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) \right) \left( \frac{\pi}{z} \right)^{1/2} - 1 \right) \left( \frac{\pi}{z} \right)^{1/2} \omega_2 \left( \frac{\pi^2}{z} \right) \right) dz
\]

\[+ O(N^{k+1})\]

\[= I_1 + I_2 + I_3 + O(N^{k+1}), \]

say.

4 Evaluation of $I_1$

From $I_1$ we will find the main terms $M_1(N,k)$ and $M_2(N,k)$ of our asymptotic formulae. We have

\[
I_1 = \frac{1}{8\pi i} \int_{(1/N)} e^{Nz^{k-2}} \left( \frac{\pi}{z} \right)^{1/2} - 1 \right)^2 dz
\]

\[- \frac{1}{8\pi i} \int_{(1/N)} e^{Nz^{k-1}} \sum_{\rho} z^{-\rho} \Gamma(\rho) \left( \frac{\pi}{z} \right)^{1/2} - 1 \right)^2 dz
\]

\[= I_{1,1} - I_{1,2}, \]

say. From $I_{1,1}$ we observe that

\[
I_{1,1} = \frac{\pi}{8\pi i} \int_{(1/N)} e^{Nz^{k-3}} dz + \frac{1}{8\pi i} \int_{(1/N)} e^{Nz^{k-2}} dz - \frac{\pi^{1/2}}{4\pi i} \int_{(1/N)} e^{Nz^{k-5/2}} dz
\]

so, if we put $Nz = s$, $ds = Ndz$ and use (1.2) we get immediately

\[
I_{1,1} = \frac{\pi N^{k+2}}{4} \int_{(1)} e^s s^{-k-3} ds + \frac{N^{k+1}}{4} \frac{1}{2\pi i} \int_{(1)} e^s s^{-k-2} ds - \frac{\pi N^{k+3/2}}{2\pi i} \int_{(1)} e^s s^{-k-5/2} ds
\]

\[= M_1(N,k). \]

From $I_{1,2}$ we have
\[ I_{1,2} = \frac{\pi}{8\pi i} \int_{(1/N)} e^{Nz^{-k-2}} \sum_{\rho} z^{-\rho} \Gamma(\rho) \, dz \]
\[ + \frac{1}{8\pi i} \int_{(1/N)} e^{Nz^{-k-1}} \sum_{\rho} z^{-\rho} \Gamma(\rho) \, dz \]
\[ - \frac{\pi^{1/2}}{4\pi i} \int_{(1/N)} e^{Nz^{-k-3/2}} \sum_{\rho} z^{-\rho} \Gamma(\rho) \, dz \]
\[ = I_1 + I_2 - I_3, \]
say. We observe that by Lemma 2.5 we have the absolute convergence of these integrals if, respectively, we have \( k > -1/2, k > 1/2 \) and \( k > 0 \). Hence for \( k > 1/2 \) we have
\[ I_1 = \frac{\pi}{4} \sum_{\rho} \Gamma(\rho) \frac{1}{2\pi i} \int_{(1/N)} e^{Nz^{-k-2-\rho}} dz = \frac{\pi}{4} \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(k+2+\rho)} N^{k+1+\rho} \]
\[ I_2 = \frac{1}{4} \sum_{\rho} \Gamma(\rho) \frac{1}{2\pi i} \int_{(1/N)} e^{Nz^{-k-1-\rho}} dz = \frac{1}{4} \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(k+1+\rho)} N^{k+\rho} \]
\[ I_3 = \frac{\pi^{1/2}}{2} \sum_{\rho} \Gamma(\rho) \frac{1}{2\pi i} \int_{(1/N)} e^{Nz^{-k-3/2-\rho}} dz = \frac{\pi^{1/2}}{2} \sum_{\rho} \frac{\Gamma(\rho)}{\Gamma(k+3/2+\rho)} N^{k+1/2+\rho}. \]

## 5 Evaluation of \( I_2 \)

We have
\[ I_2 = \frac{\pi}{2\pi i} \int_{(1/N)} e^{Nz^{-k-3}} \omega_2^2 \left( \frac{\pi^2}{z} \right) dz \]
\[ - \frac{\pi}{2\pi i} \int_{(1/N)} e^{Nz^{-k-2}} \sum_{\rho} z^{-\rho} \Gamma(\rho) \omega_2^2 \left( \frac{\pi^2}{z} \right) dz \]
\[ = I_{2,1} - I_{2,2}, \]
say.

### Evaluation of \( I_{2,1} \)

We have that
\[ I_{2,1} := \frac{\pi}{2\pi i} \int_{(1/N)} e^{Nz^{-k-3}} \omega_2^2 \left( \frac{\pi^2}{z} \right) dz = \frac{\pi}{2\pi i} \int_{(1/N)} e^{Nz^{-k-3}} \left( \sum_{l_1 \geq 1} e^{-l_1^2 \pi^2 / z} \right) \left( \sum_{l_2 \geq 1} e^{-l_2^2 \pi^2 / z} \right) dz; \]
so let us prove that we can exchange the integral with the series. Let us consider
\[ A_1 := \sum_{l_1 \geq 1} \int_{(1/N)} e^{Nz} |z|^{-k-3} e^{-l_1^2 \pi^2 \text{Re}(1/z)} \omega_2 \left( \frac{\pi^2}{z} \right) |dz|, \]
say. From
\[ \text{Re} \left( \frac{1}{z} \right) = \frac{N}{1 + N^2 y^2} \Rightarrow \begin{cases} N & |y| \leq 1/N \\ 1/(Ny^2) & |y| > 1/N \end{cases} \]
we have
\[ A_1 \ll \sum_{l_1 \geq 1} \int_0^{1/N} e^{-l_1^2 N} \frac{dy}{|z|^{k+3}} \omega_2 \left( \frac{N}{1 + N^2 y^2} \right) = U_1 + U_2 \]
hence, recalling (2.7) and (2.13),
\[ U_1 \ll N^{k+2} \omega_2^2 (N) \ll N^{k+1} \]
and from (2.13) (with \( a = 1/N \)) we get
\[ U_2 \ll N^{1/2} \sum_{l_1 \geq 1} \int_{1/N}^{\infty} \frac{e^{-l_1^2/(Ny^2)}}{y^{k+2}} \frac{dy}{|z|^{k+3}} \ll N^{k/2+1} \frac{1}{l_1^{k+1}} \int_0^{1/N} u^{k/2-1/2} e^{-u} du \leq \Gamma \left( \frac{k+1}{2} \right) N^{k/2+1} \frac{1}{l_1^{k+1}} \ll_k N^{k/2+1} \]
assuming \( k > 0 \). Now we have to study the convergence of
\[ A_2 := \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_{(1/N)} e^{Nz} |z|^{-k-3} e^{-l_1^2 \pi^2 \text{Re}(1/z)} e^{-l_2^2 \pi^2 \text{Re}(1/z)} |dz|, \]
say. Again from (2.13) we have
\[ A_2 \ll \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_0^{1/N} e^{-\left( l_1^2 + l_2^2 \right) N} |z|^{k+3} \frac{dy}{|z|^{k+3}} + \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_{1/N}^{\infty} e^{-\left( l_1^2 + l_2^2 \right)/(Ny^2)} \frac{dy}{y^{k+3}} \]
\[ = V_1 + V_2, \]
say. For \( V_1 \) we can repeat the same reasoning of \( U_1 \) thus getting
\[ V_1 \ll N^{k+2} \omega_2^2 (N) \ll N^{k+1} \]
and for \( V_2 \), assuming \( k > 1 \), we have
\[ V_2 \ll \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_{1/N}^{\infty} e^{-\left( l_1^2 + l_2^2 \right)/(Ny^2)} \frac{dy}{y^{k+3}} \ll_k N^{k/2+1/2}. \]
Then finally we have

$$I_{2,1} = \frac{\pi}{2\pi i} \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_{(1/N)} \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{1}{2\pi i} \int (1) \ e^{Nz} z^{-k-3} e^{-\left(\frac{l_1^2 + l_2^2}{2}\right)} dz = N^{k+2} \pi \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{1}{2\pi i} \int (1) \ e^{s} s^{-k-3} e^{-\left(\frac{l_1^2 + l_2^2}{2}\right)} N^{1/s} ds$$

from which, recalling the definition of the Bessel functions (1.1) we have, taking $u = 2\pi \left(\frac{l_1^2 + l_2^2}{2}\right)^{1/2} N^{1/2}$ and assuming $k > 1$, that

$$I_{2,1} = \frac{N^{k/2+1}}{\pi^{k+1}} \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} J_{k+2} \left(2\pi \left(\frac{l_1^2 + l_2^2}{2}\right)^{1/2} N^{1/2}\right).$$

### Evaluation of $I_{2,2}$

We have to calculate

$$I_{2,2} := \frac{\pi}{2\pi i} \int_{(1/N)} \ e^{Nz} z^{-k-2} \sum_{\rho} z^{-\rho} \Gamma (\rho) \left(\sum_{l_1 \geq 1} e^{-l_1^2 \pi^2 z}\right) \left(\sum_{l_2 \geq 1} e^{-l_2^2 \pi^2 z}\right) dz$$

and again we have to prove that is possible to exchange the integral with the series. So let us consider

$$A_3 := \sum_{l_1 \geq 1} \int_{(1/N)} \ e^{Nz} \left|z^{-k-2}\right| \sum_{\rho} z^{-\rho} \Gamma (\rho) \left|e^{-l_1^2 \pi^2 z}\right| \left|e^{-l_2^2 \pi^2 z}\right| |dz|,$$

say. Now using (2.9) and (2.7) we have

$$A_3 \ll N^{1/2} \sum_{l_1 \geq 1} \int_0^{1/N} \frac{e^{-y^2}}{|z|^{k+2}} dy + N^{3/2} \sum_{l_1 \geq 1} \int_0^{1/N} \frac{e^{-l_1^2/(Ny^2)}}{|z|^{k+2}} dy + N^{1/2} \sum_{l_1 \geq 1} \int_0^{\infty} \frac{y \log^2 (2Ny) e^{-l_1^2/(Ny^2)}}{|z|^{k+3/2}} dy$$

$$= W_1 + W_2 + W_3,$$

say. For $W_1$ and $W_2$ we can easily see that

$$W_1 \ll N^{k+3/2} \omega_2 (N) \ll N^{k+1}$$

and, taking $u = l_1^2 / (Ny^2)$, we obtain

$$W_2 \ll N^{3/2} \sum_{l_1 \geq 1} \int_{1/N}^{\infty} \frac{e^{-l_1^2/(Ny^2)}}{y^{k+1}} dy$$

$$\ll N^{k/2+3/2} \sum_{l_1 \geq 1} \frac{1}{l_1^{k+1}} \int_0^{l_1^2 N} e^{-u^{k/2-1}} du \ll_k N^{k/2+3/2}$$
assuming \( k > 1 \). We have now to check \( W_3 \). Taking again \( u = l_1^2 / (Ny^2) \) we have, assuming \( k > 3/2 \), that

\[
W_3 \ll N^{k/2-1/4} \sum_{l_1 \geq 1} \frac{1}{l_1-1/2} \int_0^{l_1^2/N} \log^2 \left( \frac{4Nl_1^2}{u} \right) e^{-u^{k/2-5/4}} du
\]

\[
\ll N^{k/2-1/4} \sum_{l_1 \geq 1} \frac{1}{l_1-1/2} \ll_k N^{k/2}.
\]

Let us consider

\[
A_4 := \sum_{l_1 \geq 1, l_2 \geq 2} \int_{1/N}^1 |e^{Nz}| |z^{k-2}| \left| \sum_{\rho} z^{-\rho} \Gamma (\rho) \right| e^{-l_1^2 \pi^2 \text{Re}(1/z)} e^{-l_2^2 \pi^2 \text{Re}(1/z)} |dz|,
\]

say. By (2.9) we get

\[
A_4 \ll N \sum_{l_1 \geq 1, l_2 \geq 2} \int_{1/N}^1 \frac{e^{-(l_1^2+l_2^2)/N}}{|z|^{k+2}} dy + \sum_{l_1 \geq 1, l_2 \geq 2} \int_{1/N}^1 \frac{e^{-(l_1^2+l_2^2)/(Ny^2)}}{|z|^{k+2}} dy
\]

\[
\quad + \sum_{l_1 \geq 1, l_2 \geq 2} \int_{1/N}^\infty \log^2 (2Ny) \frac{e^{-(l_1^2+l_2^2)/(Ny^2)}}{|z|^{k+3/2}} dy
\]

\[
= R_1 + R_2 + R_3,
\]

say. So we have immediately

\[
R_1 \ll N^{k+2} \omega^2 (N) \ll N^{k+1}
\]

and, if we take \( u = (l_1^2 + l_2^2) / (Ny^2) \), we obtain

\[
R_2 \ll \sum_{l_1 \geq 1, l_2 \geq 2} \int_1^{\infty} e^{-(l_1^2+l_2^2)/(Ny^2)} \frac{y^{k+2}}{y^{k+2}} dy \ll_k N^{(k+1)/2}
\]

for \( k > 1 \). So it remains to evaluate \( R_3 \). Again we take \( u = (l_1^2 + l_2^2) / (Ny^2) \) and we have

\[
R_3 \ll N^{k/2+1/4} \sum_{l_1 \geq 1, l_2 \geq 1} \log^2 \left( \frac{4N(l_1^2+l_2^2)}{(l_1^2+l_2^2)^{k/2+1/4}} \right) \int_0 \left( \frac{(l_1^2+l_2^2)^{1/2}}{N} \right)^{1/2} e^{-u^{k/2-3/4}} du
\]

\[
- N^{k/2+1/4} \sum_{l_1 \geq 1, l_2 \geq 1} \frac{1}{(l_1^2+l_2^2)^{k/2+1/4}} \int_0 \left( \frac{(l_1^2+l_2^2)^{1/2}}{N} \right)^{1/2} \log^2 (u) e^{-u^{k/2-3/4}} du
\]

and the convergence follows if \( k > 3/2 \). Note that the estimation of \( R_3 \) is optimal. For proving it, take \( c = (l_1^2 + l_2^2) / N \), assume \( k \leq 3/2 \) and \( y > 1 \). We have

\[
S := \sum_{l_1 \geq 1, l_2 \geq 1} \int_1^{\infty} \log^2 (2Ny) \frac{e^{-c/y^2}}{y^{k+3/2}} dy \geq \sum_{l_1 \geq 1, l_2 \geq 1} \int_1^{\infty} \log^2 (2Ny) \frac{e^{-c/y^2}}{y^{k+3/2}} dy.
\]
Now, since \( y \geq 1 \) we have \( \log^2 (2Ny) \geq \log^2 (2N) \) and since \( k \leq 3/2 \), we have

\[
S \geq \log (2N) \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_{1}^{\infty} \frac{e^{-c/y^2}}{y^{k+3/2}} dy \geq \log (2N) \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \int_{1}^{\infty} \frac{e^{-c/y^2}}{y^3} dy
\]

\[
= \log (2N) \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{1}{2c} (1 - e^{-c}) \geq \frac{N \log (2N) \left( 1 - e^{-2/N} \right)}{2} \sum_{l_1 \geq 1} \frac{1}{l_1^2 + l_2^2}.
\]

The last double series diverges since

\[
\sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{1}{l_1^2 + l_2^2} \geq \sum_{l_1 \geq 1} \sum_{l_2 \leq l_1} \frac{1}{l_1^2 + l_2^2} \geq \frac{1}{2} \sum_{l_1 \geq 1} \frac{1}{l_1}.
\]

Now we have to estimate

\[
A_5 := \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho, \gamma > 0} | \Gamma (\rho) | \int_{1/N}^{\infty} \left| e^{Nz} \right| \left| z^{-k-2} \right| \left| z^{-\rho} \right| e^{-l_1^2 \pi^2 \text{Re}(1/z)} e^{-l_2^2 \pi^2 \text{Re}(1/z)} |dz|,
\]

say. Using (2.10) and (2.11) we have

\[
A_5 \ll \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho, \gamma > 0} e^{-\pi \gamma/2} e^{-\gamma - 1/2} \int_{1/N}^{\infty} \left| z^{-k-2} \right| \left| z^{-\beta} \right| \exp \left( \gamma \arctan (Ny) \right) e^{-l_1^2 \pi^2 \text{Re}(1/z)} e^{-l_2^2 \pi^2 \text{Re}(1/z)} |dz|.
\]

Let \( Q_k = \sup_{\beta} \{ \Gamma \left( \frac{k}{2} + \beta + \frac{1}{2} \right) \} \) and assume \( y < 0 \). Using the trivial bound \( \gamma \arctan (Ny) - \gamma \leq -\gamma \), we have

\[
A_5 \ll N^{k+1} \sum_{l_1 \geq 1} e^{-l_1^2 N} \sum_{l_2 \geq 1} e^{-l_2^2 N} \sum_{\rho, \gamma > 0} N^\beta e^{-\pi \gamma/2} e^{-\gamma - 1/2}
\]

\[
+ N^{(k+1)/2} Q_k \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \frac{1}{(l_1^2 + l_2^2)^{(k+1)/2}} \sum_{\rho, \gamma > 0} N^\beta e^{-\pi \gamma/2} e^{-\gamma - 1/2} \ll_k N^k.
\]

If \( y > 0 \) we have

\[
A_5 \ll \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho, \gamma > 0} e^{-\pi \gamma/2} e^{-\gamma - 1/2} \int_{0}^{1/N} N^{k+2+\beta} e^{-\left( l_1^2 + l_2^2 \right) N} dy
\]

\[
+ \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho, \gamma > 0} \gamma^{-1/2} \int_{1/N}^{\infty} \exp \left( \gamma \left( \arctan (Ny) - \frac{\pi}{2} \right) \right) \frac{e^{-\left( l_1^2 + l_2^2 \right) / (Ny^2)}}{y^{k+2+\beta}} dy
\]

and by a well-known trigonometric identity follows that

\[
A_5 \ll N^{k+1} + \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho, \gamma > 0} \gamma^{-1/2} \int_{1/N}^{\infty} \exp \left( -\gamma \arctan \left( \frac{1}{Ny} \right) \right) \frac{e^{-\left( l_1^2 + l_2^2 \right) / (Ny^2)}}{y^{k+2+\beta}} dy
\]

\[
\ll N^{k+1} + \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho, \gamma > 0} \gamma^{-1/2} \int_{1/N}^{\infty} \exp \left( -\gamma \frac{l_1^2 + l_2^2}{Ny^2} \right) y^{-k-2-\beta} dy
\]
and if we put \( \frac{\omega}{\sqrt{N}} = \nu \) we get

\[
A_5 \ll N^{k+1} + \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho \gamma > 0} \gamma^{\beta - 1/2} \int_{0}^{\gamma} e^{-\nu \left( N \nu^2 \left( \frac{l_1^2}{1} + \frac{l_2^2}{1} \right) \right)} \left( \frac{\gamma}{N \nu} \right)^{k-2-\beta} \frac{\gamma}{N \nu^2} \, d\nu
\]

(5.3)

\[
\ll N^{k+1} + \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} \sum_{\rho \gamma > 0} \gamma^{-k-3/2} \int_{0}^{\infty} e^{-\nu \left( N \nu^2 \left( \frac{l_1^2}{1} + \frac{l_2^2}{1} \right) \right)} \nu^{k+\beta} \, d\nu.
\]

Now we can observe that we are in the situation of Lemma 2.4 with \( d = 2 \) and so we can conclude immediately that we have the convergence for \( k > 3/2 \) and this result is optimal.

We studied the convergence, so we finally have, using again the identity (1.11), that

\[
I_{2,2} = \pi^{-k} N^{k/2+1/2} \sum_{\rho} \frac{\Gamma(\rho)}{\pi^{\rho}} N^{\rho/2} \sum_{l_1 \geq 1} \sum_{l_2 \geq 1} J_{k+1+\rho} \left( 2\pi \left( \frac{l_1^2}{1} + \frac{l_2^2}{1} \right)^{1/2} N^{1/2} \right) - \frac{\rho}{(l_1^2 + l_2^2)^{(k+1+\rho)/2}}.
\]

6 Evaluation of \( I_3 \)

We have

\[
I_3 = \frac{1}{2\pi i} \int_{(1/N)} e^{Nz} z^{-k-1} \left( \frac{\pi^{1/2}}{z^{3/2}} - \frac{\pi^{1/2}}{z^{1/2}} \sum_{\rho} z^{-\rho \Gamma(\rho)} \right) \left( \frac{\pi^{1/2}}{z} \omega_2 \left( \frac{\pi^2}{z} \right) \right) \, dz
\]

\[
= \frac{1}{2i} \int_{(1/N)} e^{Nz} z^{-k-3} \omega_2 \left( \frac{\pi^2}{z} \right) \, dz - \frac{1}{2i} \int_{(1/N)} e^{Nz} z^{-k-2} \sum_{\rho} z^{-\rho \Gamma(\rho)} \omega_2 \left( \frac{\pi^2}{z} \right) \, dz
\]

\[
- \frac{1}{2\pi^{1/2} i} \int_{(1/N)} e^{Nz} z^{-k-5/2} \omega_2 \left( \frac{\pi^2}{z} \right) + \frac{1}{2\pi^{1/2} i} \int_{(1/N)} e^{Nz} z^{-k-3/2} \sum_{\rho} z^{-\rho \Gamma(\rho)} \omega_2 \left( \frac{\pi^2}{z} \right) \, dz
\]

\[= I_{3,1} - I_{3,2} - I_{3,3} + I_{3,4}.
\]

Evaluation of \( I_{3,1} \)

We have

\[
I_{3,1} := \frac{1}{2i} \int_{(1/N)} e^{Nz} z^{-k-3} \omega_2 \left( \frac{\pi^2}{z} \right) \, dz = \frac{1}{2i} \int_{(1/N)} e^{Nz} z^{-k-3} \sum_{m \geq 1} e^{-m^2 \pi^2 / z} \, dz
\]

hence we have to establish the convergence of

\[
A_6 := \sum_{m \geq 1} \int_{(1/N)} |e^{Nz}| |z|^{-k-3} e^{-m^2 \text{Re}(1/z)} |dz|,
\]

\[
\]
say. Using (2.7), (2.13) and (5.1) we have

\[ A_6 \ll N^{k+3/2} + \sum_{m \geq 1} \int_0^\infty y^{-k-3} e^{-m^2/(Ny^2)} dy \ll_k N^{k+3/2} \]

for \( k > -1 \). So we obtain, recalling (1.1), that

\[ J_{3,1} = \frac{N^{k/2+1}}{\pi k+1} \sum_{m \geq 1} J_{k+2} \left( \frac{2m \pi N^{1/2}}{m^{k+2}} \right). \]

**Evaluation of \( I_{3,3} \)**

We have

\[ I_{3,3} := \frac{1}{2 \pi^{1/2} i} \int_{(1/N)} e^{Nz} z^{-k-5/2} \sum_{m \geq 1} e^{-m^2 \pi^2/z} dz \]

so we have to establish the convergence of

\[ \sum_{m \geq 1} \int_{(1/N)} |e^{Nz}| |z|^{-k-5/2} e^{-m^2 \text{Re}(1/z)} |dz|. \]

Arguing as for \( I_{3,1} \), we have the convergence for \( k > -1/2 \). Summing up, we obtain

\[ I_{3,3} = \frac{N^{k/2+3/4}}{\pi k+1} \sum_{m \geq 1} J_{k+3/2} \left( \frac{2m \pi N^{1/2}}{m^{k+3/2}} \right). \]

**Evaluation of \( I_{3,2} \)**

We have to establish the convergence of

\[ A_7 := \sum_{m \geq 1} \int_{(1/N)} |e^{Nz}| |z|^{-k-2} \left| \sum_{\rho} z^{-\rho} \Gamma(\rho) \right| e^{-m^2 \pi^2/z} |dz|, \]

say. Using (2.7), (2.13), (5.1) and (2.9) we get

\[ A_7 \ll N^{k+1/2} + N \sum_{m \geq 1} \int_{1/N}^\infty y^{-k-2} e^{-m^2/(Ny^2)} dy \]

\[ + \log^2 (2N) \sum_{m \geq 1} \int_{1/N}^\infty y^{-k-3/2} e^{-m^2/(Ny^2)} dy \]

\[ + \sum_{m \geq 1} \int_{1/N}^\infty \log^2 (y) y^{-k-3/2} e^{-m^2/(Ny^2)} dy. \]
Now if we put \( m^2/(Ny^2) = u \) we have
\[
N \sum_{m \geq 1} \int_{1/N}^{\infty} y^{-k-2} e^{-m^2/(Ny^2)} dy \ll N^{k/2+3/2} \Gamma \left( \frac{k+1}{2} \right) \sum_{m \geq 1} m^{-k-1}
\]
which converges if \( k > 0 \). With the same substitution we get
\[
\log^2 (2N) \sum_{m \geq 1} \int_{1/N}^{\infty} y^{-k-3/2} e^{-m^2/(Ny^2)} dy \ll \log^2 (2N) N^{k/2+1/4} \Gamma \left( \frac{k}{2} + \frac{1}{4} \right) \sum_{m \geq 1} m^{-k-1/2}
\]
which converges for \( k > 1/2 \). For the estimation of the last integral in the bound of \( A_7 \) we observe that if we take \( \epsilon > 0 \) we have
\[
\sum_{m \geq 1} \int_{1/N}^{\infty} \log^2 (y) y^{-k-3/2} e^{-m^2/(Ny^2)} dy \ll \sum_{m \geq 1} \int_{1/N}^{\infty} y^{-k-3/2+\epsilon} e^{-m^2/(Ny^2)} dy
\]
and so, arguing analogously as we did for (6.1), we get
\[
\ll N^{k/2+1/4-\epsilon/2} \Gamma \left( \frac{k}{2} + \frac{1}{4} - \frac{\epsilon}{2} \right) \sum_{m \geq 1} m^{-k-1/2+\epsilon}
\]
and for the arbitrariness of \( \epsilon \) we have the convergence for \( k > 1/2 \). We have now to study
\[
A_8 := \sum_{m \geq 1} \sum_{\rho} |\Gamma (\rho)| \int_{(1/N)} |e^{nz}| |z^{-k-2}| |z^{-\rho}| e^{-m^2/|z|^2} |dz|
\]
say. By symmetry we may assume that \( \gamma > 0 \). If \( y \leq 0 \) we have \( \gamma \arctan (y/a) - \frac{\pi}{2} \gamma \leq -\frac{\pi}{2} \gamma \) and so using (2.10) and (2.11) we get
\[
A_8 \ll \sum_{m \geq 1} \sum_{\gamma > 0} \gamma^{-1/2} \exp \left( -\frac{\pi}{2} \gamma \right) \left( \int_{-1/N}^{0} N^{k+2+\beta} e^{-m^2 N} dy + \int_{-\infty}^{-1/N} e^{-m^2/(Ny^2)} dy \right)
\]
\[
\ll_k N^{k+3/2} + N^{k/2+1/2} Q_k \sum_{m \geq 1} \frac{1}{m^{k+1}} \sum_{\gamma > 0} N^{3/2} \gamma^{-1/2} \frac{\exp \left( -\frac{\pi}{2} \gamma \right)}{m^\beta} \ll_k N^{k+3/2}
\]
provided that \( k > 0 \) and \( Q_k = \sup_{\beta} \left\{ \Gamma \left( \frac{k}{2} + \frac{1}{2} + \frac{\beta}{2} \right) \right\} \). Let \( y > 0 \). We have
\[
A_8 \ll \sum_{m \geq 1} \sum_{\gamma > 0} \gamma^{-1/2} \exp \left( -\frac{\pi}{4} \gamma \right) \int_{0}^{1/N} N^{k+2+\beta} e^{-m^2 N} dy
\]
\[
+ \sum_{m \geq 1} \sum_{\gamma > 0} \int_{1/N}^{\infty} \exp \left( \gamma \arctan (Ny) - \frac{\pi}{2} \gamma \right) e^{-m^2/(Ny^2)} dy
\]
\[
= L_1 + L_2,
\]
say. From (2.7) and (2.13) we have

\[ L_1 \ll N^{k+1} \sum_{m \geq 1} e^{-m^2 N} \sum_{\gamma > 0} N^{\beta} \gamma^{\beta - 1/2} \exp \left( -\frac{\pi \gamma}{4} \right) \ll_k N^{k+3/2} \]

and again by a well-known trigonometric identity and taking \( v = m / (N^{1/2} y) \) we have

\[ L_2 \ll \sum_{m \geq 1} \sum_{\gamma > 0} \gamma^{\beta - 1/2} \int_{1/N}^{\infty} \exp \left( -\frac{\gamma}{N y} - \frac{m^2}{N y^2} \right) \frac{dy}{y^{k+2+\beta}} \]

\[ = N^{(k+1)/2} \sum_{m \geq 1} \frac{1}{m^{k+1}} \sum_{\gamma > 0} \frac{N^{\beta/2}}{m^{\beta}} \gamma^{\beta - 1/2} \int_{0}^{\frac{m\sqrt{N}}{N}} \exp \left( -\frac{\gamma v}{N^{1/2} m} - v^2 \right) v^{k+\beta} dv. \]

Using \( e^{-v^2} v^k = O_k(1) \) if \( k > 0 \), we have, taking \( s = \gamma v / (N^{1/2} m) \), that

\[ \ll N^{k/2+1} \sum_{m \geq 1} \frac{1}{m^{k}} \sum_{\gamma > 0} N^{\beta} \gamma^{-3/2} \int_{0}^{\infty} \exp (-s) s^\beta ds \ll_k N^{k/2+2} \]

for \( k > 1 \). Now we can exchange the series with the integral and so we have

\[ I_{3,2} = \pi^{-k} N^{(k+1)/2} \sum_{\rho} \pi^{-\rho} N^{\rho/2} \Gamma (\rho) \sum_{m \geq 1} \frac{J_{k+1+\rho} \left( 2m \pi \sqrt{N} \right)}{m^{k+1+\rho}}. \]

**Evaluation of \( I_{3,4} \)**

We have to establish the convergence of

\[ I_{3,4} := \frac{1}{2\pi^{1/2} i} \int_{(1/N)} e^{Nz} z^{-k-3/2} \sum_{\rho} z^{-\rho} \Gamma (\rho) \omega_2 \left( \frac{\pi^2}{z} \right) dz. \]

Arguing analogously as we did for estimating \( I_{3,2} \) we obtain the condition \( k > 1 \). We can exchange the series with the integral and obtain

\[ I_{3,4} = \pi^{-k} N^{k/2+1/4} \sum_{\rho} \pi^{-\rho} N^{\rho} \Gamma (\rho) \sum_{m \geq 1} \frac{J_{k+1/2+\rho} \left( 2m \pi \sqrt{N} \right)}{m^{k+1/2+\rho}}. \]
Defining

\( M_1 (N, k) = \frac{\pi N^{k+2}}{4 \Gamma (k+3)} + \frac{N^{k+1}}{4 \Gamma (k+2)} - \frac{\pi^{1/2} N^{k+3/2}}{2 \Gamma (k+5/2)}, \)

\( M_2 (N, k) = -\frac{\pi}{4} \sum_{\rho} \frac{\Gamma (\rho)}{\Gamma (k+2+\rho)} N^{k+1+\rho} - \frac{1}{4} \sum_{\rho} \frac{\Gamma (\rho)}{\Gamma (k+1+\rho)} N^{k+\rho} \)

\( + \frac{\pi^{1/2}}{2} \sum_{\rho} \frac{\Gamma (\rho)}{\Gamma (k+3/2+\rho)} N^{k+1/2+\rho}, \)

\( M_3 (N, k) = \frac{N^{k/2+1}}{\pi k+1} \sum_{l_1 \geq 1, l_2 \geq 1} \sum_{l_1 \geq l_2} \frac{J_{k+2} \left( 2 \pi \left( l_1^2 + l_2^2 \right)^{1/2} N^{1/2} \right)}{(l_1^2 + l_2^2)^{k/2+1}} \)

\( M_4 (N, k) = \frac{N^{k/2+1}}{\pi k+1} \sum_{m \geq 1} \frac{J_{k+2} \left( 2 m \pi N^{1/2} \right)}{m^{k+2}} - \frac{N^{k/2+3/4}}{\pi k+1} \sum_{m \geq 1} \frac{J_{k+3/2} \left( 2 m \pi N^{1/2} \right)}{m^{k+3/2}} \)

\( -\pi^{-k} N^{(k+1)/2} \sum_{\rho} \frac{\pi^{-\rho} N^{\rho/2} \Gamma (\rho)}{\pi^{-k+1}} \sum_{m \geq 1} \frac{J_{k+1+\rho} \left( 2 m \pi \sqrt{N} \right)}{m^{k+1+\rho}} \)

\( +\pi^{-k} N^{k/2+1/4} \sum_{\rho} \pi^{-\rho} N^{\rho/2} \Gamma (\rho) \sum_{m \geq 1} \frac{J_{k+1/2+\rho} \left( 2 m \pi \sqrt{N} \right)}{m^{k+1/2+\rho}}, \)

we have proved the following

**Main Theorem 6.1.** Let \( N \) be a sufficiently large integer. We have

\[
\sum_{n \leq N} r_Q (n) \frac{(N - n)^k}{\Gamma (k+1)} = M_1 (N, k) + M_2 (N, k) + M_3 (N, k) + M_4 (N, k) + O \left( N^{k+1} \right)
\]

for \( k > 3/2 \), where \( \rho \) runs over the non-trivial zeros of the Riemann zeta function \( \zeta (s) \) and \( J_v (u) \) is the Bessel function of complex order \( v \) and real argument \( u \). Furthermore the bound \( k > 3/2 \) is optimal using this technique.

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On the Cesàro average of the “Linnik numbers”

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