DUFLO'S CONJECTURE FOR THE BRANCHING TO THE IWASAWA AN-SUBGROUP

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Abstract. The purpose of this paper is to prove Duflo's conjecture for \((G, \pi, AN)\) where \(G\) is a simple Lie group of Hermitian type and \(\pi\) is a discrete series of \(G\) and \(AN\) is the maximal exponential solvable subgroup for an Iwasawa decomposition \(G = KAN\). This is essentially reduced from the following general theorem we prove in this paper: let \(G\) be a connected semisimple Lie group. Then a strongly elliptic \(G\)-coadjoint orbit \(O\) is holomorphic if and only if \(p(O)\) is an open \(AN\)-coadjoint orbit, where \(p : g^* \rightarrow (a \oplus n)^*\) is the natural projection.

1. Introduction

Let \(H \subseteq G\) be real connected Lie groups of type I with Lie algebras \(h \subseteq g\). Let \(\pi\) be a unitary irreducible representation of \(G\). One fundamental problem in representation theory and harmonic analysis is to study the restriction of \(\pi\) to \(H\) i.e., the branching problem. For \(G\) exponential solvable, the branching problem was determined in ([Fu]). However, it is very hard to find an explicit branching laws for general \((G, \pi, H)\), especially for \(G\) reductive and \(H\) non reductive. When \(G, H\) are both reductive, good progress has been made, notably by work of Kobayashi ([Ko1], [Ko2], [Ko3], [KOP]) and recent work of Duflo-Vargas ([DV1], [DV2]).

A central problem in branching theory, initiated by Kobayashi, is to study when \(\pi|_H\) is \(H\)-admissible (in the sense of Kobayashi): i.e., \(\pi|_H\) is decomposed discretely with finite multiplicities.

Now let us consider the branching problem geometrically. Suppose that \(\pi\) is attached to a \(G\)-coadjoint orbit \(O\) in \(g^*\): i.e., \(\pi\) is a "quantization" of \(O\). Then \(O\), equipped with the Kirillov-Kostant-Souriau symplectic form \(\omega\), becomes a \(H\)-Hamiltonian space. The corresponding moment map is just the natural projection \(p : O \rightarrow h^*\). One might care whether the branching law \(\pi|_H\) can be studied via the \(H\)-Hamiltonian space \((O, \omega)\).

The answer is positive for \(G\) exponential solvable (by the work of Fujiwara [Fu]) or compact (by work of Heckman [He] and Guillemin-Sternberg [GS]). But for general \(G\), the answer is not that clear: for instance not all \(\pi \in \widehat{H}\) can be associated with a coadjoint orbit. Next even if \(\pi\) is attached to a certain orbit \(O\), it is not clear that each \(H\)-irreducible representation which appears in \(\pi|_H\) can be attached to a \(H\)-coadjoint orbit in \(h^*\). Nevertheless, for \(H \subseteq G\) which are almost algebraic Lie groups and \(\pi\) is a discrete series of \(G\), Duflo proposes a conjecture which relates the branching problem to the geometry of the moment map.
For $G$ semisimple and $\pi$ a discrete series of $G$ which is attached to a $G$-coadjoint orbit $O_\pi$ (in the sense of Duflo), Duflo’s conjecture states as follows:

(D1) $\pi|_H$ is $H$-admissible if and only if the projection $p$ is weakly proper.

(D2) If $\pi|_H$ is $H$-admissible, then each irreducible $H$-representation $\sigma$ which appears in $\pi|_H$ is attached to a strongly regular $H$-coadjoint orbit $\Omega$ (in the sense of Duflo) which is contained in $p(O_\pi)$.

(D3) If $\pi|_H$ is $H$-admissible, the multiplicity of each such $\sigma$ can be expressed geometrically on the reduced space of $\Omega$ (with respect to the moment map $p$).

Here ”weakly proper” in (D1) means that the preimage (for $p$) of each compact subset which is contained in $p(O_\pi) \cap \Upsilon_{sr}$ is compact in $O_\pi$. Here $\Upsilon_{sr}$ is the set of all strongly regular elements in $\mathfrak{h}^*$. For the definition of strongly regular elements (orbits) and more information on Duflo’s conjecture, we refer to [Liu].

As for (D2), let us remark that in the framework of Duflo’s orbit method, each discrete series of $G$ (resp. $H$) is attached to a strongly regular $G$ (resp. $H$)-coadjoint orbit. Moreover according to Duflo-Vargas’s work ([DV1], [DV2]), each irreducible $H$-representation $\tilde{\sigma}$ which appears in the integral decomposition of $\pi|_H$ (which is not necessarily $H$-admissible) is attached to a strongly regular $H$-coadjoint orbit. Note that $\tilde{\sigma}$ is not necessarily a discrete series. However, if $\pi|_H$ is $H$-admissible, then each $H$-irreducible representation appearing in $\pi|_H$ must be a discrete series. Thus (D2) has a good geometric meaning.

(D3) seems not very explicit, but at least it suggests a direction. It is obviously influenced by Guillemin-Sternberg’s work for compact Hamiltonian spaces and especially encouraged by Paradan’s work on non-compact Hamiltonian spaces. We will see in our case, this direction is also correct.

In this paper, we will prove Duflo’s conjecture for $(G, AN)$ where $G$ is a simple Lie group of Hermitian type and $AN$ is the maximal exponential solvable subgroup for an Iwasawa decomposition $G = KAN$.

The paper is organized as follows. In section 2, assuming the results in section 5, we prove Duflo’s conjecture for the case mentioned above. In section 3, we state a general geometric theorem (Theorem 3.1) on strongly elliptic coadjoint orbits. The sections 4 and 5 are devoted to proving Theorem 3.1.

Notations and Conventions: $\mathbb{R}_+^* = \{x : x > 0\}$, $\mathbb{R}_-^* = \{x : x < 0\}$. For any Lie algebra $\mathfrak{g}$, $\mathfrak{g}^*$ denotes its algebraic dual.

2. Duflo’s conjecture for simple Lie groups of Hermitian type

Let $G = KAN$ be an Iwasawa decomposition of a connected semisimple Lie group $G$. Let $\mathfrak{a}$ (resp. $\mathfrak{n}$) be the Lie algebra of $A$ (resp. $N$). Let $\pi$ be a discrete series of $G$ with $O_\pi$ its associated coadjoint orbit (in the sense of Duflo). Suppose that there exists an open $AN$-coadjoint orbit in $(\mathfrak{a} \oplus \mathfrak{n})^*$. Then in Théorème 5.3 of [Liu], we proved
**Theorem 2.1.** The projection $p : \mathcal{O}_\pi \rightarrow (\mathfrak{a} \oplus \mathfrak{n})^*$ is weakly proper if and only if $p(\mathcal{O}_\pi)$ is an open $AN$-coadjoint orbit.

Now suppose that $G$ is simple of Hermitian type. In this section, we will prove Duflo’s conjecture for $(G, AN)$. It is well known that for such Lie group $G$, there exists an open $AN$-coadjoint orbit in $(\mathfrak{a} \oplus \mathfrak{n})^*$.

On the other hand it is known that $\pi|_{AN}$ is $AN$-admissible if and only if $\pi$ is holomorphic or anti-holomorphic (see [RiV] and the Theorem 4.6 of [ReV]).

Hence (D1) of Duflo’s conjecture follows from Theorem 2.1 and Carmona’s results (see section 4). For $\pi$ anti-holomorphic, it is treated exactly in the same way. Hence (D2) of Duflo’s conjecture is true. Note that in our case, strongly regular $AN$-subgroup 3.0. Let $\lambda \in \mathfrak{a}^*$ be the Blattner parameter of $\pi$, where $\rho_G$ (resp. $\rho_K$) is the half sum of positive roots (resp. compact positive roots). Let $\tau_\lambda \in \widehat{AN}\cdot h$ be the associated unitary irreducible representation whose highest weight is $\Lambda$ (with respect to $\Delta_+^\pm$). Then we have the following theorem due to Rossi-Vergne (see [RiV]).

**Theorem 2.2.** $\pi|_{AN} = \dim(\tau_\lambda).\pi_\Omega$.

However, according to section 5, we have $p(\mathcal{O}_\pi) = \Omega$. This is also directly deduced from Theorem 3.1 and Carmona’s results (see section 4). For $\pi$ anti-holomorphic, it is treated exactly in the same way. Hence (D2) of Duflo’s conjecture is true. Note that in our case, strongly regular $AN$-coadjoint orbits are nothing else but open orbits.

Below we will prove (D3) of Duflo’s conjecture.

**2.1. Reduced space and multiplicity.** Let $\tau_\lambda \in \widehat{K}$ be the unitary irreducible representation of $K$ with Harish-Chandra parameter $\lambda$ (with respect to $\Delta_+^\pm$). Then the highest weight of $\tau_\lambda$, $\Lambda' = \lambda - \rho_K$. As $\Lambda - \Lambda' = \rho_n$ which is a character of exp($\mathfrak{t}$), we have

**Observation:** $\dim(\tau_\lambda) = \dim(\tau_\lambda')$. 
Let $\varpi$ be the Kirillov-Kostant-Souriau form of $O_\pi$ and $X_\Omega$ be the reduced space of the open $AN$-orbit $\Omega = p(O_\pi)$. Since $AN$ is diffeomorphic to $\Omega$, we deduce that $X_\Omega$ is diffeomorphic to $K.f$. Then in particular $X_\Omega$ is a compact symplectic sub-manifold of $O_\pi$. Denote by $\varpi_\Omega$ the induced symplectic form of $X_\Omega$ (from $\varpi$), and $\beta_\Omega := \frac{\varpi_\Omega}{(2\pi)^l \cdot (l!)}$ the associated Liouville volume. Here $l = \dim X_\Omega = \dim K.f$.

Now we will prove the following theorem.

**Theorem 2.3.** $\pi|_{AN} = (\int_{X_\Omega} \beta_\Omega) \cdot \pi_\Omega$.

**Proof.** According to the theorem of Rossi-Vergne (Theorem 2.2 above) and the observation above, it is sufficient to prove $\int_{X_\Omega} \beta_\Omega = \dim(\tau_{\Lambda'})$. Without loss of generality, we can assume that $f$ is also integral (i.e. there exists a unitary character $\chi_f$ of $G(f) = T := \exp \mathfrak{t}$, such that $d\chi_f = if$). Because otherwise, we can always choose a good covering $\tilde{\mathcal{G}}$ of $G$ such that $f$ is integral for $\tilde{\mathcal{G}}$ (and of course coverings do not change anything about multiplicities). As in ([Liu] for $SU(2,1)$ case), we can deduce that $\int_{X_\Omega} \beta_\Omega = \int_{K.f} \beta_K$, where $\beta_K$ is the Liouville volume of $K.f$ for the induced symplectic form $\varpi_K$ on $K.f$ (from the symplectic form $\varpi$ of $O_\pi$). However it is clear that $(K.f, \varpi_K)$ is isomorphic to the $K$-coadjoint orbit on $(K.f_K, \varpi_{f_K})$, where $f_K = f|_t \in \mathfrak{t}^*$ and $\varpi_{f_K}$ is the Kirillov-Kostant-Souriau symplectic form on $K.f_K$. Thus it is clear that $\int_{K.f} \beta_K = \int_{K.f_k} \beta_{f_k}$, where $\beta_{f_k}$ is the Liouville volume for $(K.f_k, \varpi_{f_k})$. On the other hand, it is clear that the associated irreducible unitary representation for $(K.f_K, \varpi_{f_K})$ is exactly $\tau_{\Lambda'}$. Hence according to Kirillov-Rossmann’s formula (see [DHV]), $\int_{K.f_k} \beta_{f_k} = \dim(\tau_{\Lambda'})$. Thus the theorem is proved. □

So the assertion (iii) is true according to the the above theorem.

**Remark.** From the previous theorem, we see that the $AN$-multiplicity equals a natural integral on the reduced space. However, it also equals ”very probably” an equivariant $Spin_c$-index on the reduced space which is reduced from the $Spin_c$-quantization of the $G$-orbit $O_\pi$. This equivariant index is the so-called reduction. In other words, in this situation, the principle quantization commutes with reduction holds. Hence this geometric principle is extended to Hamiltonian action of non-reductive Lie groups.

### 3. A GEOMETRIC THEOREM FOR STRONGLY ELLIPTIC COADJOINT ORBITS

Let $G$ be a real connected semisimple Lie group, $\mathfrak{g} = \text{Lie}(G)$. We let $G$ act on $\mathfrak{g}^*$ by coadjoint action. Recall that an element $f \in \mathfrak{g}^*$ is called strongly elliptic, if the Lie algebra of its stabilizer, $\mathfrak{g}(f)$ is compact.

Now let $f$ be strongly elliptic. Then $\mathfrak{g}(f)$ contains a compact Cartan sub-algebra $\mathfrak{t}$ (conversely, if $\mathfrak{g}$ has a compact Cartan sub-algebra, then the set of strongly elliptic elements is not empty). Since $\mathfrak{g} = \mathfrak{t} \oplus [\mathfrak{t}, \mathfrak{g}]$, and $f$ vanishes on $[\mathfrak{t}, \mathfrak{g}]$, we can regard $f \in \mathfrak{t}^*$. Let $\Delta$ be the root system with respect to $(\mathfrak{g}_C, \mathfrak{t}_C)$ and $G = K \exp(\mathfrak{p})$ be the associated Cartan decomposition. Let $\Delta_c$ (resp. $\Delta_n$) be the subset of compact (resp. noncompact) roots of $\Delta$. It is not hard to see that for each $\alpha \in \Delta_n$, we have $\langle f, i\alpha \rangle \neq 0$. Define the subset
\[ \Delta_+^n = \{ \alpha \in \Delta_n : \langle f, i\alpha \rangle > 0 \} \] where \( \langle , \rangle \) is the inner product over \( t^* \cong t \) deduced from the Killing form defined in section 2.

We say a strongly elliptic element \( f \in g^* \) is holomorphic, if \( \Sigma_{\alpha \in \Delta_+^n} g_{\alpha} \) is a (abelian) sub-algebra of \( p_C \). Here \( g_{\alpha} \) is the root space of \( \alpha \). Then it is well known that \( f \) is holomorphic if and only if \( \Delta_+^n \) is stable under the compact Weyl group \( W_K \). Notice that the existence of a strongly elliptic and holomorphic element implies that \( g \) is of Hermitian type.

A coadjoint orbit \( O \) (in \( g^* \)) is called strongly elliptic if an element (then each element) in \( O \) is strongly elliptic. A strongly regular orbit is called holomorphic, if an element (then each element) in it is holomorphic. Note that the subset of strongly elliptic (resp. strongly elliptic and holomorphic) elements is a \( G \)-invariant cone, if it is non-empty.

In the framework of orbit method, each discrete series \( \pi \) of \( G \) is associated to a (unique) coadjoint orbit \( O \) which is regular and strongly elliptic (in the sense of Duflo). Moreover \( \pi \) is holomorphic if and only if \( p(O_f) \) is an open AN-coadjoint orbit in \((a \oplus n)^* \).

**Theorem 3.1.** Let \( G = KAN \) be an Iwasawa decomposition of a connected semisimple Lie group \( G \) with Lie algebra \( g \). Let \( a = \text{Lie}(A) \) and \( n = \text{Lie}(N) \). Let \( p : g^* \longrightarrow (a \oplus n)^* \) be the natural projection. Assume that \( f \in g^* \) is a strongly elliptic element with coadjoint orbit \( O_f := G.f \). Then \( f \) is holomorphic if and only if \( p(O_f) \) is an open AN-coadjoint orbit in \((a \oplus n)^* \).

**Remark.** (1) If there exists no open AN-coadjoint orbit in \((a \oplus n)^* \), then it is clear that there is no holomorphic element in \( g^* \). Thus in this case Theorem 3.1 is true. Hence in order to prove Theorem 3.1, we can always assume the existence of an open AN-coadjoint orbit in \((a \oplus n)^* \).

(2) As we mentioned previously, for all semisimple Lie groups \( G = KAN \) of Hermitian type, there exists an open AN-coadjoint orbit in \((a \oplus n)^* \). However, there are also other semisimple Lie groups \( G \) of non-Hermitian type, for which there exists an open AN-coadjoint orbit in \((a \oplus n)^* \): such as the connected non compact Lie group \( G \) whose Lie algebra \( g \) is the split real form of the simple complex Lie algebra of type \( G_2 \).

(3) It is clear that the theorem (and the proof of the theorem) is independent of any choice of the Cartan decomposition and the subgroup \( AN \). In the extreme situation where \( AN \) is reduced to a point (in other words \( G \) is semisimple compact), it is clear that the theorem is correct. Thus in the following sections, we suppose that \( AN \) is not trivial (i.e. \( G \) is not compact).

4. Characterization of open AN-coadjoint orbits in \((a \oplus n)^* \)

In this section, we will give some results on open AN-orbits, which are essential for our proof of theorem 3.1. All these results can be found in ([Ca]).

Let \( G = KAN \) be an Iwasawa decomposition for a semisimple Lie group \( G \). Denote \( \mathfrak{a} := \text{Lie}(A) \) and \( \mathfrak{n} := \text{Lie}(N) \). Notice that a priori, we do not assume there exists an open AN-coadjoint orbit in \((a \oplus n)^* \). Let \( \mathfrak{h} = \mathfrak{h}_k \oplus \mathfrak{a} \) be a \( \theta \)-stable Cartan sub-algebra containing
Lemma 4.1. Let $\Omega$ be an open AN-orbit in $(\mathfrak{a} \oplus \mathfrak{n})^*$, then $\{h(X_j) : h \in \Omega\}$ is contained in $\mathbb{R}_+^r$ or $\mathbb{R}_+^+$. 

Proof. It is sufficient to notice that each $h \in \Omega$ is of the form $b.s$, where $b \in AN$ and $s$ is the unique element in $\Omega$ described previously. As $X_1$ is a highest root vector, it follows that for all $b \in AN$, $b.X_1 \in \mathbb{R}_+^+X_1$. \hfill $\square$

Remark. In general, this lemma is false for $X_j$ with $j \neq 1$.

5. PROOF OF THEOREM 3.1

From now on we assume the existence of an open AN-coadjoint orbit in $(\mathfrak{a} \oplus \mathfrak{n})^*$ (according to the previous remark, $\beta_j$ is the restriction of a unique real root in $\Phi^+_\mathfrak{a}$ and $\beta_j$ is strongly orthogonal to $\beta_i$ for $1 \leq i \neq j \leq r$, with $r = \text{dim}(\mathfrak{a})$. Thus the process of Cayley transforms applied to $\mathfrak{h}$ allows us to see that $t = \mathfrak{h}_k \oplus \bigoplus_{j=1}^r \mathbb{R}(X_j + \theta(X_j))$ is a $\theta$-stable compact Cartan sub-algebra (notice that $\theta(X_j) \in \mathfrak{g}_{-\beta_j}$). Moreover under the identification $t \cong t^*$ of section 2, $Y_j := X_j + \theta(X_j)$ is proportional to a non-compact root $\alpha_j$ with respect to the roots system $\Delta := \Delta(\mathfrak{g}_C, t_C)$. Especially, $\alpha_1$ is a long root, since $\beta_1$ is a long restricted root and the Cayley transforms preserve the length of roots. From now on, we will work on the compact Cartan sub-algebra $t$ constructed above.

Next we begin to prove the Theorem 3.1. It is clear that it is sufficient to prove it for $G$ simple connected:

Proof. We first prove the ”$\Leftarrow$” part.
Suppose that $p(G.f)$ is an open AN-orbit. Note that $p(G.f) = AN.p(K.f)$. Next it is clear that for each $k \in K$, we have $p(k.f)(X_1) = (k.f)(X_1)$. Hence we conclude that \{(k.f)(X_1) : k \in K\} is contained in $\mathbb{R}^*_+$ or $\mathbb{R}^*_-$. Further as $K.f \in \mathfrak{k}^*$ and $X_1 = \frac{X_1}{2} + \frac{X_1 - \theta(X_1)}{2}$, we have $p(k.f)(X_1) = (k.f)(X_1)$. This implies especially \{(f, w.Y_1) : w \in W_K\} is contained in $\mathbb{R}^*_+$ or $\mathbb{R}^*_-$.

On the other hand, we have seen that $Y_1$ is proportional to a long non-compact root $\alpha_1$. Thus the "\(\Leftarrow\)" part is a direct consequence of the lemma below.

\[\square\]

**Lemma 5.1.** Let $G = K \exp \mathfrak{p}$ be a Cartan decomposition (with respect to the Cartan involution $\theta$) of a connected simple Lie group $G$ with Lie algebra $\mathfrak{g}$. Suppose that $\mathfrak{t}$ is a $\theta$-stable compact Cartan sub-algebra. Let $f \in \mathfrak{t}^*$ be a strongly elliptic element such that $\mathfrak{t} \subseteq \mathfrak{g}(f)$. Suppose that there exists a long non-compact root $\beta$ in $\Delta = \Delta(\mathfrak{g}_C, \mathfrak{t}_C)$, such that \{(f, iw.\beta) : w \in W_K\} is contained in $\mathbb{R}^*_+$ or $\mathbb{R}^*_-$.

Then $f$ is holomorphic.

**Proof.** Firstly, if the condition in the lemma is satisfied, then $\langle f, i \sum_{w \in W_K} w.\beta \rangle \neq 0$. Hence $i \sum_{w \in W_K} w.\beta \neq 0$. But $i \sum_{w \in W_K} w.\beta$ is invariant under $W_K$. Thus it is in the center of $\mathfrak{t}$. This implies that the center of $\mathfrak{t}$ is non-trivial. Hence $\mathfrak{g}$ must be of Hermitian type. It follows that the Ad-representation of $K$ in $\mathfrak{p}_C$ decomposes into two irreducible components. Moreover in this case, the 2 irreducible components $\mathfrak{p}^+\mathfrak{C}$, $\mathfrak{p}^-\mathfrak{C}$ are abelian. Then without loss of generality, we can assume our $\beta \in \Delta^+_\mathbb{R}$, where $\sum_{\alpha \in \Delta^+_{\mathbb{R}}} \mathfrak{g}_\alpha = \mathfrak{p}^+\mathfrak{C}$.

However $\beta$ is a long root, thus an extreme weight for the Ad-representation of $K$. Hence according to a Kostant’s theorem, $\Delta^+_{\mathbb{R}}$ is contained in the convex hull of $W_K.\beta$, $\text{conv}(W_K.\beta)$. Then we deduce that \{(f, i\alpha) : \alpha \in \Delta^+_\mathbb{R}\} is contained in $\mathbb{R}^*_+$ or $\mathbb{R}^*_-$.

Hence the lemma is proved.

\[\square\]

Now we want to prove the "\(\Rightarrow\)" part of the Theorem 3.1. For this, we only need to treat the simple Lie groups of Hermitian type. But firstly we want to prove a general proposition for solvable Lie groups, then apply this proposition to our situation.

5.1. Open coadjoint orbits for solvable Lie groups.

**Proposition 5.1.** Let $S$ be a connected solvable Lie group with Lie algebra $\mathfrak{s}$. Suppose that

(i) $\mathfrak{s} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$, where $\mathfrak{s}_1$ is a Lie subalgebra and $\mathfrak{s}_2$ is an ideal of $\mathfrak{s}$.

(ii) $\mathfrak{s}_3 \subseteq \mathfrak{s}_2$ is an abelian ideal of $\mathfrak{s}$, which verifies $[\mathfrak{s}_2, \mathfrak{s}_2] \subseteq \mathfrak{s}_3$ and $[\mathfrak{s}_2, \mathfrak{s}_3] = \{0\}$.

(iii) $\dim(\mathfrak{s}_3) = \dim(\mathfrak{s}_1)$.

(iv) There exists an open $S$-coadjoint orbit in $\mathfrak{s}^*$.

Let $\lambda \in \mathfrak{s}^*$ and $\lambda_3 := \lambda|_{\mathfrak{s}_3}$. Then the coadjoint orbit $S.\lambda$ is open in $\mathfrak{s}^*$ if and only if $S.\lambda_3$ is an open orbit in $\mathfrak{s}_3^*$.

**Proof.** If $S.\lambda$ is open, it is obvious that $S.\lambda_3$ is open. Next we will prove "\(\Leftarrow\)".

Define $\tilde{\mathfrak{s}}_3 := \{\lambda \in \mathfrak{s}_3^* : S.\lambda$ is open in $\mathfrak{s}_3\}$ and $\tilde{\mathfrak{s}}_3 := \{\lambda_3 \in \mathfrak{s}_3^* : \text{there exists a regular element } \lambda \in \mathfrak{s}_2 \text{ such that } \lambda|_{\mathfrak{s}_3} = \lambda_3\}$ (recall that an element $\lambda \in \mathfrak{s}_2$ is called regular, if the Lie algebra of its stabilizer $\mathfrak{s}_2(\lambda)$ is of minimal dimension). Then $\tilde{\mathfrak{s}}_3$ is open and dense in $\mathfrak{s}_3^*$.

On the other hand, $[\mathfrak{s}_2, \mathfrak{s}_2] \subseteq \mathfrak{s}_3$ and $\mathfrak{s}_2$ is an ideal. Thus we deduce that $\tilde{\mathfrak{s}}_3$ is $S$-invariant.
and $\lambda_2 \in s_*^2$ is regular if and only if $\lambda_2|_{s^3} \in \tilde{s}_3^3$. Then the $S$-invariance and density of $\tilde{s}_3^3$ imply that each open $S$-orbit of $s_*^3$ is contained in $\tilde{s}_3^3$. In other words, we have $s_*^3 \subseteq \tilde{s}_3^3$.

Since $[s_2, s_3] = 0$, it is clear that for all $\lambda_2 \in s_*^3$, we have $s_3 \subseteq s_2(\lambda_2)$. Next we want to prove that for $\lambda_2$ regular in $s_*^3$, we have $s_3 = s_2(\lambda_2)$. Actually, according to our assumption, we can take a $\tilde{\lambda} \in s^*$ which lies in an open $S$-orbit. Denote $\tilde{\lambda}_2 := \tilde{\lambda}|_{s_2}$. Hence it is clear that $s_2(\tilde{\lambda}_2) = s_2 \cap s_2^\perp_{\tilde{\lambda}_2}$, where $s_2^\perp_{\tilde{\lambda}_2}$ is the orthogonal of $s_2$ in $s$ with respect to the Kirillov-Kostant-Souriau symplectic form $B_{\tilde{\lambda}} = \tilde{\lambda}(\cdot, \cdot)$. However, we have $\dim(s_2^\perp_{\tilde{\lambda}_2}) = \dim(s) - \dim(s_2) = \dim(s_3)$. Thus we have $s_2(\tilde{\lambda}_2) = s_3$. Hence $s_3 = s_2(\lambda_2)$, for all $\lambda_2$ regular in $s_*^3$.

Now assume $\lambda \in s^*$ such that $\lambda_3 := \lambda|_{s_3} \in s_*^3$, i.e., $S.\lambda_3$ is open. Let $\lambda_2 := \lambda|_{s_2}$. Then according to what we have seen, $\lambda_2$ is regular. Now since $S.\lambda_3$ is open and $\dim(s) - \dim(s_3) = \dim(s_2)$, we have $\dim(s(\lambda_3)) = \dim(s_2)$. But it is clear that $s_2 \subseteq s(\lambda_3)$. Thus $s_2 = s(\lambda_3)$. Then we deduce that $s(\lambda) \subseteq s_2(\lambda_2)$. But we have proved $s_2(\lambda_2) = s_3$. Hence we deduce that $s(\lambda)$ equals the orthogonal of $s.\lambda_3 \subseteq s_*^3$. However, $s.\lambda_3 = s_*^3$. Hence $s(\lambda) = 0$. Then "$\leftarrow$" is proved.

\begin{proof}
\end{proof}

**Remark.** If $s_2 = s_3$, then we can drop the assumption that there exists an $S$-open orbit in $s$. This can be easily seen from the proof.

Let $G$ be simple of Hermitian type. Then the restricted roots system $\Phi_a$ is contained in $\{ \pm \frac{1}{2} (\beta_i + \beta_j) \}_{1 \leq i,j \leq r} \cup \{ \pm \frac{1}{2} (\beta_i - \beta_j) \}_{1 \leq i < j \leq r} \cup \{ \pm \frac{1}{2} \beta_i \}_{1 \leq i \leq r}$, where $r = \dim(a)$. Notice that the terms "$\frac{1}{2} \beta_i$" might not appear in $\Phi_a$. We denote the ideals of $a \oplus n$, $n_3 := \bigoplus_{1 \leq i,j \leq r} s_{\frac{1}{2}}(\beta_i + \beta_j)$ and $n_2 := n_3 \oplus \bigoplus_{1 \leq i \leq r} s_{\frac{1}{2}} \beta_i$. Then we have $n_3 \subseteq n_2$ and $a \oplus n = n_2 \oplus n_1$, where $n_1 = a \oplus \bigoplus_{1 \leq i < j \leq r} s_{\frac{1}{2}}(\beta_i - \beta_j)$. Hence the conditions of the previous proposition are satisfied: we replace "$s^*$" by $a \oplus n$ and "$s_1^*$" by $n_1$. Actually this can be easily seen for instance by the fact that there is a "$J$-algebra" structure in $a \oplus n$. Hence we have

**Corollary 5.1.** Let $\lambda \in (a \oplus n)^*$ and $\lambda_3 := \lambda|_{n_3}$. Then $AN.\lambda$ is an open $AN$-coadjoint orbit in $(a \oplus n)^*$ if and only if $AN.\lambda_3$ is an open $AN$-orbit in $n_3^*$.

5.2. strongly elliptic and holomorphic coadjoint orbits. Recall that we want to show: if $f \in g^*$ is strongly elliptic and holomorphic, then $p(O_f)$ is an open $AN$-coadjoint orbit. Corollary 5.1 tells us that it is sufficient to show that $p_1(O_f)$ is an open $AN$-orbit in $n_3^*$, where $p_1 : g^* \to n_3^*$ is the natural projection.

Firstly, we translate it into the adjoint picture. Identify $g$ with $g^*$ via the inner product $\langle .,. \rangle$. Here $\langle X,Y \rangle = -K(X,\theta(Y))$ for $X,Y \in g$. Then for $x \in G$ and $g^* \ni h = \langle .,X_h \rangle$, we have $Ad^*(x)h = \langle .,Ad(\theta(x))X_h \rangle$. Thus we still have $Ad^*(G).h \cong Ad(G).X_h$. Denote $pr_{n_3}$ the orthogonal projection of $g$ onto $n_3$ with respect to $\langle .,. \rangle$. Then we have the following lemma.

**Lemma 5.2.** The following diagram is commutative.
Proof. Let \( h \in \mathfrak{g}^* \) with \( h = \langle , X_h \rangle \). For any \( Y \in \mathfrak{n}_3 \), \( h(Y) = p_1(h)(Y) = \langle Y, X_h \rangle = \langle Y, \text{pr}_{\mathfrak{n}_3}(X_h) \rangle \). This completes the proof.

Fix \( f \in \mathfrak{t}^* \) a strongly elliptic and holomorphic element which corresponds to \( X_f \in \mathfrak{t} \). Recall that \( \mathfrak{t} \) always denotes the \( \theta \)-stable compact Cartan sub-algebra which is constructed at the beginning of the section. Let \( X_i \in \mathfrak{g}_{\beta_i} \) such that \( \langle f, X_i + \theta(X_i) \rangle > 0 \) (*). Let \( \mathfrak{n}_c := \bigoplus_{i<j} \mathfrak{g}_2(\beta_i, -\beta_j) \) and \( N_c := \exp(\mathfrak{n}_c) \).

**Lemma 5.3.** \( p_1(\text{Ad}(AN)^*f) \) corresponds to the subset \( \text{Ad}(\theta(N_c))\sum_{j=1}^r \mathbb{R}_+^*X_j \) in \( \mathfrak{n}_3 \).

**Proof.** We can write \( f := X_f = \sum_{j=1}^r c_j(X_j + \theta(X_j)) + X_0 \in \mathfrak{t} \), where \( X_j \in \mathfrak{g}_{\beta_j} \) is the same as the ones in (*) and \( c_j > 0 \) and \( X_0 \in \mathfrak{m} \). Here \( \mathfrak{m} \) is the centralizer of \( \mathfrak{a} \) in \( \mathfrak{k} \).

Now let \( a \in A, n \in N \) and \( Y \in \mathfrak{n}_3 \). Then \( \text{Ad}^*(an)f(Y) = f(\text{Ad}(an)^{-1}Y) = \langle X_f, \text{Ad}(an)^{-1}Y \rangle \).

On the other hand \( \text{Ad}(an)^{-1}Y \in \mathfrak{n}_3 \) and \( \theta(X_j) \) and \( X_0 \) are orthogonal to \( \mathfrak{n}_3 \) (actually even to \( \mathfrak{n} \)). Then \( \text{Ad}^*(an)f(Y) = \langle \sum_{j=1}^r c_jX_j, \text{Ad}(an)^{-1}Y \rangle \). Hence we deduce that

\[
p_1(\text{Ad}(AN)^*f) \cong \text{pr}_{\mathfrak{n}_3}(\text{Ad}(\theta(N))\sum_{j=1}^r c_jX_j) = \text{pr}_{\mathfrak{n}_3}(\text{Ad}(\theta(N))\sum_{j=1}^r \mathbb{R}_+^*X_j).
\]

However \( N = N_3 \cdot N_2 \cdot N_c \) with \( N_3 := \exp(\mathfrak{n}_3) \) and \( N_2 := \exp(\bigoplus_{1 \leq j \leq r} \mathfrak{g}_2(\beta_j)) \). Then we have

\[
\text{pr}_{\mathfrak{n}_3}(\text{Ad}(\theta(N))\sum_{j=1}^r \mathbb{R}_+^*X_j) = \text{pr}_{\mathfrak{n}_3}(\text{Ad}(\theta(N_3))\text{Ad}(\theta(N_2))\text{Ad}(\theta(N_c))\sum_{j=1}^r \mathbb{R}_+^*X_j).
\]

Nevertheless, it is clear that for any \( Y \in \mathfrak{n}_3 \), \( \text{pr}_{\mathfrak{n}_3}(\text{Ad}(\theta(N_3))\text{Ad}(\theta(N_2))Y) = Y \). Then the proof follows.

It is known that \( \mathfrak{n}_3 \) carries the structure of an Euclidean Jordan-algebra. Let \( \Omega^+ \) be (up to sign) the associated open convex cone. Recall the construction of \( \Omega^+ \). For that let \( \mathfrak{g}_0 := \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{i \neq j} \mathfrak{g}_2(\beta_i, -\beta_j) \) with \( \mathfrak{m} \) the centralizer of \( \mathfrak{a} \) in \( \mathfrak{k} \). Let \( G_0 := \exp(\mathfrak{g}_0) \). Then

\[
\Omega^+ = \text{Ad}(G_0)\sum_{j=1}^r X_j = \text{Ad}(\theta(N_c))\sum_{j=1}^r X_j = \text{Ad}(\theta(N_c))(\sum_{j=1}^r \mathbb{R}_+^*X_j).
\]

Then we have the following.

**Corollary 5.2.** \( p_1(\text{Ad}(AN)^*f) \) corresponds to \( \Omega^+ \).
Next we will prove our main theorem based on the fine geometry of convex cones in the simple Lie algebra of Hermitian type.

Thus let $\Delta_+^n$ be one of the two holomorphic subsets of non compact roots.

Define $c_{\text{max}} := \{ X \in t : \forall \alpha \in \Delta_+^n, i\alpha(X) > 0 \}$. Then $X_f \in \pm c_{\text{max}}$. It is known that $C_{\text{max}} := \text{Ad}(G)c_{\text{max}}$ is a proper maximal Ad$(G)$-invariant open convex cone in $\mathfrak{g}$ (see [Ne]). Without loss of generality, we can assume $\mathcal{O}_{X_f} := \text{Ad}(G).X_f \subseteq C_{\text{max}}$. recall that $X_j \in \mathfrak{g}_{\beta_j}$ are those in (*). Since $X_1$ is a highest weight vector for Ad-representation of $G$ on $\mathfrak{g}$, we have the following characterization of $C_{\text{max}}$ due to Paneitz-Vinberg (see Theorem 2.1.21 in [HO]).

$$C_{\text{max}} = \{ X \in \mathfrak{g} : \langle X, \text{Ad}(g).X_1 \rangle > 0, \forall g \in G \}.$$

Hence as each $X_j$ is conjugate to $X_1$ via Weyl group (up to a positive scalar), we deduce the following

**Corollary 5.3.** For each $Y \in C_{\text{max}}$, we have $\langle Y, \Omega^+ \rangle > 0$.

Now in order to conclude $p_1(\mathcal{O}_f) = \Omega^+$ (then our theorem is proved) we prove the following.

**Corollary 5.4.** $pr_{\mathfrak{n}_3}(C_{\text{max}}) = \Omega^+$.

**Proof.** Firstly, $pr_{\mathfrak{n}_3}(C_{\text{max}}) \supseteq \Omega^+$ follows from Corollary 5.2 and Lemma 5.2.

Next it is known that the closure of $\Omega^+$, $\overline{\Omega^+}$ is self-dual (see [FK]): i.e., $X \in \overline{\Omega^+}$ if and only if $\langle X, \overline{\Omega^+} \rangle \geq 0$. Then according to the previous corollary, we have $pr_{\mathfrak{n}_3}(C_{\text{max}}) \subseteq \overline{\Omega^+}$. But $C_{\text{max}}$ is open and $pr_{\mathfrak{n}_3}$ is an open map. Hence we deduce that $pr_{\mathfrak{n}_3}(C_{\text{max}}) \subseteq \Omega^+$. □

**Remark.** Since $G$ is simple of Hermitian type, the set of strongly elliptic and holomorphic elements has two connected components $\pm \Psi^+$ (actually $\Psi^+ \cong C_{\text{max}}$). Since $\Psi^+$ is union of strongly elliptic and holomorphic $G$-orbits, a simple topological argument implies that $p(\Psi^+) = \Omega_+$, where $\Omega_+$ is an open $AN$-orbit in $(\mathfrak{a} \oplus \mathfrak{n})^*$. In other words, among many open $AN$-orbits in $(\mathfrak{a} \oplus \mathfrak{n})^*$, there are only two and exactly two opposite open orbits onto which the cone of strongly elliptic and holomorphic elements in $\mathfrak{g}^*$ are projected.

**Acknowledgements.** I would particularly like to thank Prof. Krötz for his crucial help for convex cone theory which is essential for our general proof. I also thank Prof. Duflo with whom I had useful discussions for communicating Carmona’s unpublished paper to us. I would like to thank Prof. Torasso for useful discussions, some of his ideas are helpful for our work. Finally, my thanks goes to Prof. Hilgert for comments and discussions on a preliminary version of the paper.

**References**

[Ca] J. Carmona, *Structure symplectiques sur les orbites ouvertes de certains groups résolubles et espaces hermitiens symétriques*, unpublished paper.
M. Duflo, G. Heckman, M. Vergne, *Projection d’orbites, formule de Kirillov et formule de Blattner*, Harmonic analysis on Lie groups and symmetric spaces (Kleebach, 1983). Mm. Soc. Math. France (N.S.) No. 15 (1984), 65128.

M. Duflo, J.A. Vargas, *Proper map and multiplicity*, 2007, preprint.

M. Duflo, J.A. Vargas, *Branching laws for square integrable representations*, Proc. Japan Acad. Ser. A Math. Sci. 86 (2010), no. 3, 4954.

J. Faraut, A. Korányi, *Analysis on symmetric cones*, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1994. xii+382 pp. ISBN: 0-19-853477-9.

H. Fujiwara, *Sur les restrictions des représentations unitaires des groupes de Lie resolubles exponentiels*, Invent. Math. 104 (1991), no. 3, 647-654.

V. Guillemin, S. Sternberg, *Geometric quantization and multiplicities of group representations*, Invent. Math. 67 (1982), no. 3, 515538.

G.J. Heckman, *Projections of orbits and asymptotic behavior of multiplicities for compact connected Lie groups*, Invent. Math. 67 (1982), no. 2, 333356.

J. Hilgert, G. Ólafsson, *Causal symmetric spaces. Geometry and harmonic analysis*, Perspectives in Mathematics, 18. Academic Press, Inc., San Diego, CA, 1997. xvi+286 pp. ISBN: 0-12-525430-X.

G. Liu, *Restriction des séries discrètes de SU(2,1) un sous-groupe exponentiel maximal et à un sous-groupe de Borel*, Ph.D thesis, Université de Poitiers.

A.W. Knapp *Lie groups beyond an introduction*, Progress in Mathematics, 140. Birkhäuser Boston, Inc., Boston, MA, 1996. xvi+604 pp. ISBN: 0-8176-3926-8.

T. Kobayashi, *Discrete decomposability of the restriction of $A_q(\lambda)$ with respect to reductive sub-groups and its applications*, Invent. Math. 117 (1994), 181-205.

T. Kobayashi, *Discrete decomposability of the restriction of $A_q(\lambda)$ with respect to reductive subgroups. III. Restriction of Harish-Chandra modules and associated varieties*, Invent. Math. 131 (1998), no. 2, 229-256.

T. Kobayashi, *Discrete decomposability of the restriction of $A_q(\lambda)$ with respect to reductive subgroups. II. Micro-local analysis and asymptotic K-support*, Ann. of Math. (2) 147 (1998), no. 3, 709-729.

T. Kobayashi, B. Ørsted, M. Pevzner, *Geometric analysis on small unitary representations of GL(N, R)*, J. Funct. Anal. 260 (2011), no. 6, 16821720.

K.-H. Neeb, *Holomorphy and convexity in Lie theory*, de Gruyter Expositions in Mathematics, 28. Walter de Gruyter & Co., Berlin, 2000. xxii+778 pp. ISBN: 3-11-015669-5.

J. Rosenberg, M. Vergne, *Harmonically induced representations of solvable Lie groups*, J. Funct. Anal. 62 (1985), no. 1, 8-37.

H. Rossi, M. Vergne, *Representations of certain solvable Lie groups on Hilbert spaces of holomorphic functions and the application to the holomorphic discrete series of a semisimple Lie group*, J. Functional Analysis 13 (1973), 324-389.

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