Hattori-Stallings trace and character

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Dedicated to the memory of Dieter Happel

Abstract
It is shown that Hattori-Stallings trace induces a homomorphism of abelian groups, called Hattori-Stallings character, from the $K_1$-group of endomorphisms of the perfect derived category of an algebra to its zero-th Hochschild homology, which provides a new proof of Igusa-Liu-Paquette Theorem, i.e., the strong no loop conjecture for finite-dimensional elementary algebras, on the level of complexes. Moreover, the Hattori-Stallings traces of projective bimodules and one-sided projective bimodules are studied, which provides another proof of Igusa-Liu-Paquette Theorem on the level of modules.

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1 Introduction
Global dimension is a quite important homological invariant of an algebra or a ring. The (in)finiteness of global dimension plays an important role in representation theory of algebras. For instance, the bounded derived category of a finite-dimensional algebra has Auslander-Reiten triangles if and only if the algebra is of finite global dimension [10][11]. There are some well-known conjectures related to the (in)finiteness of global dimension, such as no loop conjecture, Cartan determinant conjecture — the determinant of the Cartan
matrix of an artin ring of finite global dimension is 1 (ref. [5]), Hochschild homology dimension conjecture — a finite-dimensional algebra is of finite global dimension if and only if its Hochschild homology dimension is 0 (ref. [8]). To a finite-dimensional elementary algebra $A$, we can associate a quiver $Q$, called its Gabriel quiver (ref. [1, Page 65]). The (in)finiteness of the global dimension of $A$ is closely related to the combinatorics of $Q$. If $Q$ has no oriented cycles then $\text{gl.dim} A < \infty$ (ref. [4]). Obviously, its converse is not true in general. Nevertheless, if $\text{gl.dim} A < \infty$ then $Q$ must have no loop, and 2-truncated cycle $[3]$. The former is due to the following conjecture:

**No loop conjecture.** *Let $A$ be an artin algebra of finite global dimension. Then $\text{Ext}^1_A(S,S) = 0$ for every simple $A$-module $S$.***

The no loop conjecture was first explicitly established for artin algebras of global dimension two [6, Proposition]. For finite-dimensional elementary algebras, which is just the case that loop has its real geometric meaning, as shown in [13], this can be easily derived from an earlier result of Lenzing [16]. A stronger version of no loop conjecture is the following:

**Strong no loop conjecture.** *Let $A$ be an artin algebra and $S$ a simple $A$-module of finite projective dimension. Then $\text{Ext}^1_A(S,S) = 0$.***

The strong no loop conjecture is due to Zacharia [13], which is also listed as a conjecture in Auslander-Reiten-Smalø’s book [1, Page 410, Conjecture (7)]. For finite-dimensional elementary algebras, and particularly, for finite-dimensional algebras over an algebraically closed field, it was proved in [14]. Some special cases were solved in [7, 15, 17, 18, 19, 21].

In this paper, we shall show that Hattori-Stallings trace induces a homomorphism of abelian groups, called Hattori-Stallings character, from the $K_1$-group of endomorphisms of the perfect derived category of an algebra to its zero-th Hochschild homology (see Section 2), which provides a neat proof of Igusa-Liu-Paquette Theorem, i.e., the strong no loop conjecture for finite-dimensional elementary algebras, on the level of complexes (see Section 3). Moreover, in Section 4, we shall study the Hattori-Stallings traces of projective bimodules and one-sided projective bimodules, which provides a simpler proof of Igusa-Liu-Paquette Theorem on the level of modules. A key point is the bimodule characterization of the projective dimension of a simple module.
2 Hattori-Stallings character

In this section, we shall show that Hattori-Stallings trace induces a homomorphism of abelian groups, called Hattori-Stallings character, from the $K_1$-group of endomorphisms of the perfect derived category of an algebra to its zero-th Hochschild homology.

2.1 Hattori-Stallings traces

Let $A$ be a ring with identity. Denote by $\text{Mod}A$ the category of right $A$-modules, and by $\text{proj}A$ the full subcategory of $\text{Mod}A$ consisting of all finitely generated projective right $A$-modules. Denote by $D(A)$ the unbounded derived categories of the complexes of right $A$-modules, and by $K^b(\text{proj}A)$ the homotopy category of the bounded complexes of finitely generated projective right $A$-modules, which is triangle equivalent to the perfect derived category of $A$.

For each $P \in \text{proj}A$, there is an isomorphism of abelian groups

$$\phi_P : P \otimes_A \text{Hom}_A(P,A) \rightarrow \text{End}_A(P)$$

defined by $\phi_P(p \otimes f)(p') = pf(p')$ for all $p, p' \in P$ and $f \in \text{Hom}_A(P,A)$. There is also a homomorphism of abelian groups

$$\psi_P : P \otimes_A \text{Hom}_A(P,A) \rightarrow A/[A,A]$$

defined by $\psi_P(p \otimes f) = f(p)$ for all $p \in P$ and $f \in \text{Hom}_A(P,A)$. Here, $[A,A]$ is the additive subgroup of $A$ generated by all commutators $[a,b] := ab - ba$ with $a,b \in A$. It is well-known that the abelian group $A/[A,A]$ is isomorphic to the zero-th Hochschild homology group $HH_0(A)$ of $A$. The homomorphism of abelian groups

$$\text{tr}_P := \psi_P \phi_P^{-1} : \text{End}_A(P) \rightarrow A/[A,A]$$

is called the Hattori-Stallings trace of $P$.

Hattori-Stallings trace has the following properties:

**Proposition 1.** (Hattori [12], Stallings [20], Lenzing [16]) Let $P, P', P'' \in \text{proj}A$.

1. (HS1) If $f \in \text{End}_A(P)$ and $g \in \text{Hom}_A(P,P')$ is an isomorphism then $\text{tr}_P(f) = \text{tr}_{P'}(gf)$. 
2. (HS2) If $f, f' \in \text{End}_A(P)$ then $\text{tr}_P(f + f') = \text{tr}_P(f) + \text{tr}_P(f')$. 

3
(HS3) If \( f = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \in \text{End}_A(P \oplus P') \) then \( \text{tr}_{P \oplus P'}(f) = \text{tr}_P(f_{11}) + \text{tr}_P'(f_{22}) \).

(HS4) If \( f \in \text{Hom}_A(P, P') \) and \( g \in \text{Hom}_A(P', P) \) then \( \text{tr}_P(gf) = \text{tr}_{P'}(fg) \).

(HS5) If \( 0 \rightarrow P' \rightarrow P \rightarrow \cdots \rightarrow P' \rightarrow 0 \) is a commutative diagram with exact rows then \( \text{tr}_P(f) = \text{tr}_{P'}(f') + \text{tr}_{P''}(f'') \).

(HS6) If \( l_a \in \text{End}_A(A) \) is the left multiplication by \( a \in A \) then \( \text{tr}_A(l_a) = \bar{a} \), the equivalence class of \( a \) in \( A/[A, A] \).

### 2.2 \( K_1 \)-groups of endomorphisms

Let \( \mathcal{C} \) be a category. Denote by \( \text{end} \mathcal{C} \) the category of endomorphisms of \( \mathcal{C} \), whose objects are all pairs \((C, f)\) with \( C \in \mathcal{C} \) and \( f \in \text{End}_\mathcal{C}(C) \) and whose Hom sets are \( \text{Hom}_{\text{end} \mathcal{C}}((C, f), (C', f')) := \{ g \in \text{Hom}_\mathcal{C}(C, C') | gf = f'g \} \). Obviously, if \( \mathcal{C} \) is a skeletally small category then so is \( \text{end} \mathcal{C} \).

For a skeletally small triangulated category \( \mathcal{T} \), we define its \( K_1 \)-group of endomorphisms (cf. [2, Chapter III]), denoted by \( K_1(\text{end} \mathcal{T}) \), to be the factor group of the free abelian group generated by all isomorphism classes of objects in \( \text{end} \mathcal{T} \) modulo the relations:

(K1) \([T, f + f'] = [(T, f)] + [(T, f')]\) for all \( T \in \mathcal{T} \) and \( f, f' \in \text{End}_\mathcal{T}(T) \).

(K2) \([T, f] = [(T', f')] + [(T'', f'')]\) for every commutative diagram

\[
\begin{array}{ccc}
T' & \longrightarrow & T & \longrightarrow & T'' \\
\downarrow f' & & \downarrow f & & \downarrow f'' \\
T' & \longrightarrow & T & \longrightarrow & T''
\end{array}
\]

with triangles as rows.

Clearly, if two skeletally small triangulated categories are triangle equivalent then their \( K_1 \)-groups of endomorphisms are isomorphic.

### 2.3 Hattori-Stallings character

For any ring \( A \) with identity, both the exact category \( \text{proj} A \) and the triangulated category \( K^b(\text{proj} A) \) are skeletally small. So is \( \text{end} K^b(\text{proj} A) \).

The main result in this section is the following:
Theorem 1. Let $A$ be a ring with identity. Then the map

$$\text{tr}: K_1(\text{end}K^b(\text{proj}A)) \to A/[A,A], \quad [(P^*, f^*)] \mapsto \sum_{i \in \mathbb{Z}} (-1)^i \text{tr}_{P^i}(f^i),$$

is a homomorphism of abelian groups, called the Hattori-Stallings character of $A$, which satisfies the trace property (TP):

$$\text{tr}([(P^*, g^* \circ f^*)]) = \text{tr}([(P^*, f^*)])$$

for all $f^* \in \text{Hom}_{K^b(\text{proj}A)}(P^*, P^*)$ and $g^* \in \text{Hom}_{K^b(\text{proj}A)}(P^*, P^*)$.

Proof. Since $P^* \in K^b(\text{proj}A)$, $P^i$ is zero for almost all $i \in \mathbb{Z}$. Thus the sum $\sum_{i \in \mathbb{Z}} (-1)^i \text{tr}_{P^i}(f^i)$ makes sense.

**Step 1.** $\sum_{i \in \mathbb{Z}} (-1)^i \text{tr}_{P^i}(f^i)$ is independent of the choice of the representative $f^*$ of the homotopy equivalence class $f^*$. Indeed, if $f^* = \overline{f^*}$, then $f^* - f'^* = s^{i+1}d^i + d^{i-1}s^i$ for some homotopy map $s^*$, where $d^*$ is the differential of $P^*$. It follows from (HS2) and (HS4) that

$$\sum_{i \in \mathbb{Z}} (-1)^i \text{tr}_{P^i}(f^i) = \sum_{i \in \mathbb{Z}} (-1)^i \text{tr}_{P^i}(f'^i) = \sum_{i \in \mathbb{Z}} (-1)^i \text{tr}_{P^i}(s^{i+1}d^i + d^{i-1}s^i) = \sum_{i \in \mathbb{Z}} (-1)^i (\text{tr}_{P^i}(s^{i+1}d^i) + \text{tr}_{P^i}(d^{i-1}s^i)) = \sum_{i \in \mathbb{Z}} (-1)^i (\text{tr}_{P^i}(s^{i+1}d^i) + \text{tr}_{P^{i-1}}(s^i d^{i-1})) = 0.$$

**Step 2.** $\sum_{i \in \mathbb{Z}} (-1)^i \text{tr}_{P^i}(f^i)$ is also independent of the choice of the representative $(P^*, f^*)$ of the isomorphism class $[(P^*, f^*)]$. Indeed, if $(P^*, f^*) \cong (P'^*, f'^*)$ then there are morphisms $g^* \in \text{Hom}_{K^b(\text{proj}A)}(P^*, P'^*)$ and $g'^* \in \text{Hom}_{K^b(\text{proj}A)}(P'^*, P^*)$ such that $g'^* \circ g^* = \overline{1}$, $g^* \circ g'^* = \overline{1}$, and $f'^* \circ g^* = g'^* \circ f^*$. Thus $\overline{f^*} = g'^* \circ g^* \circ f^*$ and $\overline{g^*} \circ f^* \circ g'^* = f'^* \circ g^* = g'^*$. It follows from Step 1 and (HS4) that

$$\sum_{i \in \mathbb{Z}} (-1)^i \text{tr}_{P^i}(f^i) = \sum_{i \in \mathbb{Z}} (-1)^i \text{tr}_{P^i}(g^i f^i) = \sum_{i \in \mathbb{Z}} (-1)^i \text{tr}_{P^i}(g^i f^i) = \sum_{i \in \mathbb{Z}} (-1)^i \text{tr}_{P^{i}}(f'^i g'^i) = \sum_{i \in \mathbb{Z}} (-1)^i \text{tr}_{P^{i}}(f'^i g'^i).$$

Now we have shown that $\text{tr}$ is well-defined on the free abelian group generated by the isomorphism classes of $\text{end}K^b(\text{proj}A)$.

**Step 3.** $\text{tr}([(P^*, f^* + f'^*)]) = \text{tr}([(P^*, f^*)]) + \text{tr}([(P^*, f'^*)])$ for all $P^* \in K^b(\text{proj}A)$ and $f^*, f'^* \in \text{End}_{K^b(\text{proj}A)}(P^*)$. Indeed, this is clear by (HS2).
Step 4.  \( \text{tr}([([P^•, \overline{f}^•])]) = \text{tr}([([P'^•, \overline{f'}^•)])] + \text{tr}([([P''^•, \overline{f''}^•)])] \) for every commutative diagram

\[
\begin{array}{ccc}
P^• \xrightarrow{\overline{f}^•} P^• & \xrightarrow{\overline{f}^•} & P''^• \\
\downarrow{\overline{f}^•} & \downarrow{\overline{f}^•} & \downarrow{\overline{f}^•} \\
P'^• \xrightarrow{\overline{f'}^•} P'^• & \xrightarrow{\overline{f'}^•} & P''^•
\end{array}
\]

with triangles as rows. Indeed, in \( K^b(\text{proj} A) \) each triangle \( P^• \xrightarrow{\overline{f}^•} P^• \xrightarrow{\overline{f}^•} \) \( P''^• \rightarrow \) is isomorphic to a triangle

\[
P^• \xrightarrow{[1]} \text{Cyl}(u^•) \xrightarrow{[0 \ 1]} \text{Cone}(u^•)
\]

where \( \text{Cyl}(u^•) \) and \( \text{Cone}(u^•) \) are the cylinder and cone of the cochain map \( u^• : P^• \rightarrow P^• \) respectively. Thus, by Step 2, it is enough to consider the case that the following diagram

\[
\begin{array}{ccc}
P^• \xrightarrow{[1]} P^• \oplus P''^• \xrightarrow{[0 \ 1]} P''^• \\
\downarrow{\overline{f}^•} & \downarrow{\overline{f''}^•} & \downarrow{\overline{f''}^•} \\
P'^• \xrightarrow{[1]} P'^• \oplus P''^• & \xrightarrow{[0 \ 1]} P''^•
\end{array}
\]

with triangles as rows is commutative. In this case, by (HS3), we have
\[
\sum_{i \in \mathbb{Z}} (-1)^i \text{tr}_{P'^•}(f'^i) + \sum_{i \in \mathbb{Z}} (-1)^i \text{tr}_{P''^•}(f''^i) = \sum_{i \in \mathbb{Z}} (-1)^i \text{tr}_{P'^• \oplus P''^•}(\begin{bmatrix} f'^i & f''^i \\ 0 & f''^i \end{bmatrix}).
\]

Now we have shown that the Hattori-Stallings character \( \text{tr} \) is well-defined. Next, we prove that it satisfies trace property (TP).

Step 5.  \( \text{tr}([([P^•, \overline{f}^• \circ \overline{g}^•])]) = \text{tr}([([P'^•, \overline{f'}^• \circ \overline{g'}^•)])] \) for all morphisms \( \overline{f^•} \in \text{Hom}_{K^b(\text{proj} A)}(P^•, P'^•) \) and \( \overline{g^•} \in \text{Hom}_{K^b(\text{proj} A)}(P'^•, P'') \). Indeed, this is clear by Step 2 and (HS4).

3 Igusa-Liu-Paquette Theorem

In this section, we shall apply Hattori-Stallings character to give a new proof of Igusa-Liu-Paquette Theorem on the level of complexes. From now
on, let $k$ be a field and $A$ a finite-dimensional elementary $k$-algebra, i.e., $A/J \cong k^n$ for some natural number $n$, where $J$ denotes the Jacobson radical of $A$.

### 3.1 Projective dimension

Some homological properties on modules can be characterized by those of bimodules. For instance, Happel showed that for a finite-dimensional $k$-algebra $A$, $\text{gl.dim} A = \text{pd}_A e$ (ref. [9]). In this subsection, we shall give a bimodule characterization of the projective dimension of a simple module. For this, we need the following well-known result, which implies that $\text{top}_A = A/J$ is a “testing module” of the projective dimension of an $A$-module:

**Lemma 1.** Let $A$ be an artin algebra, and $M \neq 0$ a finitely generated left $A$-module. Then $\text{pd}_AM = \sup\{i|\text{Ext}_A^i(M, A/J) \neq 0\} = \sup\{i|\text{Tor}_A^i(A/J, M) \neq 0\}$.

**Proof.** Let $P^\bullet$ be a minimal projective resolution of the left $A$-module $M$. Then all the differentials of the complex $\text{Hom}_A(P^\bullet, A/J)$ are zero. Thus $\text{Ext}_A^i(M, A/J) = \text{Hom}_A(P^{-i}, A/J)$. Hence, $\text{pd}_AM = \sup\{i|P^{-i} \neq 0\} = \sup\{i|\text{Hom}_A(P^{-i}, A/J) \neq 0\} = \sup\{i|\text{Ext}_A^i(M, A/J) \neq 0\}$.

Similarly, all the differentials of the complex $A/J \otimes_A P^\bullet$ are zero. Thus $\text{Tor}_A^i(A/J, M) = A/J \otimes_A P^{-i}$. Therefore, $\text{pd}_AM = \sup\{i|P^{-i} \neq 0\} = \sup\{i|A/J \otimes_A P^{-i} \neq 0\} = \sup\{i|\text{Tor}_A^i(A/J, M) \neq 0\}$. 

A key point of this paper is the following observation:

**Lemma 2.** Let $A$ be a finite-dimensional elementary $k$-algebra, $S = Ae/Je$ the left simple $A$-module corresponding to a primitive idempotent $e$ in $A$, and $\bar{A} := A/(1-e)A$. Then $\text{pd}_AS = \text{pd}_{A \otimes_k \bar{A}^{\text{op}}} \bar{A}$.

**Proof.** We have isomorphisms $\text{Tor}_A^i(A/J, S) \cong H^{-i}(A/J \otimes_A^L S) \cong H^{-i}(A/J \otimes_{\bar{A} \otimes_k \bar{A}^{\text{op}}} \bar{A}) \cong H^{-i}((A/J \otimes_k S) \otimes_{A \otimes_k \bar{A}^{\text{op}}} \bar{A}) \cong \text{Tor}_i^A(A/J \otimes_k S, \bar{A})$. Applying Lemma 1 twice, we obtain $\text{pd}_AS = \text{pd}_{A \otimes_k \bar{A}^{\text{op}}} \bar{A}$, since $A/J \otimes_k S = \text{top}(A \otimes_k \bar{A})$.

### 3.2 A new proof of Igusa-Liu-Paquette Theorem

**Theorem 2.** (Igusa-Liu-Paquette [14]) Let $A$ be a finite-dimensional elementary $k$-algebra, $S$ a left simple $A$-module, and $\text{pd}_AS < \infty$. Then $\text{Ext}_A^1(S, S) = 0$. 


**Proof.** We may assume that $S$ is the left simple $A$-module $Ae/Je$ corresponding to a primitive idempotent $e$ in $A$ and $\bar{A} := A/A(1-e)A$.

We have the following commutative diagram in $D(A)$:

\[
\begin{array}{ccc}
J^{j+1} & \longrightarrow & J^j \\
\downarrow l_a & & \downarrow l_a \\
J^{j+1} & \longrightarrow & J^j / J^{j+1}
\end{array}
\]

with triangles as rows for all $a \in J$, the Jacobson radical of $A$, and $0 \leq j \leq t - 1$ where $t$ is the Loewy length of $A$. Applying the derived tensor functor $- \otimes^L_A \bar{A}$ to the commutative diagram above, we obtain the following commutative diagram in $D(\bar{A})$:

\[
\begin{array}{ccc}
J^{j+1} \otimes^L_A \bar{A} & \longrightarrow & J^j \otimes^L_A \bar{A} \\
\downarrow l_a \otimes^L_A \bar{A} & & \downarrow l_a \otimes^L_A \bar{A} \\
J^{j+1} \otimes^L_A \bar{A} & \longrightarrow & J^j \otimes^L_A \bar{A}
\end{array}
\]

with triangles as rows for all $a \in J$ and $0 \leq j \leq t - 1$. By the assumption $\text{pd}_A S < \infty$ and Lemma 2, we have a bounded finitely generated projective $A\bar{A}$-bimodules resolution $P^\bullet$ of $\bar{A}$. Thus we have the following commutative diagram in $K^b(\text{proj} \bar{A})$:

\[
\begin{array}{ccc}
J^{j+1} \otimes_A P^\bullet & \longrightarrow & J^j \otimes_A P^\bullet \\
\downarrow l_a & & \downarrow l_a \\
J^{j+1} \otimes_A P^\bullet & \longrightarrow & J^j \otimes_A P^\bullet
\end{array}
\]

with triangles as rows for all $a \in J$ and $0 \leq j \leq t - 1$. Therefore, for any $\bar{a} \in \bar{J}$, the Jacobson radical of $\bar{A}$, the equivalence class of $\bar{a}$ in $\bar{A}/[\bar{A}, \bar{A}]$

\[
\bar{a} = \text{tr}([\bar{A}, l_a]) = \text{tr}([\bar{A}, l_a]) = \text{tr}([J^0 \otimes_A P^\bullet, l_a]) = \text{tr}([J^1 \otimes_A P^\bullet, l_a]) = \ldots
\]

Hence, $\bar{J} \subseteq [\bar{A}, \bar{A}]$.

Let $A' := \bar{A}/\bar{J}^2$ and $J' = \bar{J}/\bar{J}^2$ its Jacobson radical. Then $A'$ is a local algebra with radical square zero, and thus commutative. Since $\bar{J} \subseteq [\bar{A}, \bar{A}]$, we have $J' \subseteq [A', A'] = 0$, i.e., $J' = 0$. Hence, $\text{Ext}^{1}_{\bar{A}}(S, S) \cong eJe/eJ^2e \cong J' = 0$. □
4 Hattori-Stallings traces of bimodules

In this section, we shall study the Hattori-Stallings traces of projective bimodules and one-sided projective bimodules, which provides another proof of Igusa-Liu-Paquette Theorem on the level of modules.

Firstly, we consider the Hattori-Stallings traces of finitely generated projective bimodules.

**Proposition 2.** Let $A$ and $B$ be finite-dimensional $k$-algebras, and $P$ a finitely generated projective $A$-$B$-bimodule. Then $\text{tr}_{P}(l_a) = 0$ for all $a \in J$, the Jacobson radical of $A$.

**Proof.** We have the following commutative diagram in $\text{Mod}A$:

$$
\begin{array}{cccccc}
0 & \longrightarrow & J^{j+1} & \longrightarrow & J^j & \longrightarrow & J^j/J^{j+1} & \longrightarrow & 0 \\
0 & \longrightarrow & J^{j+1} & \longrightarrow & J^j & \longrightarrow & J^j/J^{j+1} & \longrightarrow & 0 \\
\end{array}
$$

with exact rows for all $a \in J$ and $0 \leq j \leq t - 1$ where $t$ is the Loewy length of $A$. Since $P$ is a finitely generated projective $A$-$B$-bimodule, we have the following commutative diagram in $\text{proj}B$:

$$
\begin{array}{cccccc}
0 & \longrightarrow & J^{j+1} \otimes_A P & \longrightarrow & J^j \otimes_A P & \longrightarrow & (J^j/J^{j+1}) \otimes_A P & \longrightarrow & 0 \\
0 & \longrightarrow & J^{j+1} \otimes_A P & \longrightarrow & J^j \otimes_A P & \longrightarrow & (J^j/J^{j+1}) \otimes_A P & \longrightarrow & 0 \\
\end{array}
$$

with exact rows for all $a \in J$ and $0 \leq j \leq t - 1$. It follows from (HS5) that $\text{tr}_{P}(l_a) = \text{tr}_{J^0 \otimes_A P_B}(l_a) = \text{tr}_{J^1 \otimes_A P_B}(l_a) = \cdots = \text{tr}_{J^{t-1} \otimes_A P_B}(l_a = 0) = 0$ for all $a \in J$.

Secondly, we consider the Hattori-Stallings traces of finitely generated one-sided projective bimodules.

**Proposition 3.** Let $A$ and $B$ be finite-dimensional $k$-algebras, $M$ a finitely generated $A$-$B$-bimodule which is projective as a right $B$-module, and $P^*$ a finitely generated projective $A$-$B$-bimodule resolution of $M$. Then

$$
\text{tr}_{M_B}(l_a) = (-1)^i \text{tr}_{\Omega_i(M)}(l_a)
$$

for all $a \in J$ and $i \in \mathbb{N}$, where $\Omega_i(M)$ is the $i$-th syzygy of $M$ on $P^*$.  


Proof. Since $M_B$ is projective, all $\Omega_i(M)_B$’s are projective. We have the following commutative diagrams in $\text{proj} B$:
\[
\begin{array}{c}
0 & \rightarrow & \Omega_i(M) & \rightarrow & P^{i+1} & \rightarrow & \Omega_{i-1}(M) & \rightarrow & 0 \\
\downarrow l_a & & \downarrow l_a & & \downarrow l_a & & \downarrow l_a & & \\
0 & \rightarrow & \Omega_i(M) & \rightarrow & P^{i+1} & \rightarrow & \Omega_{i-1}(M) & \rightarrow & 0
\end{array}
\]
with exact rows for all $a \in J$ and $i \geq 1$. By Proposition 2 and (HS5), we obtain $\text{tr}_{\Omega_i(M)}(l_a) = -\text{tr}_{\Omega_{i-1}(M)}(l_a)$, thus $\text{tr}_{M_B}(l_a) = \text{tr}_{\Omega_0(M)}(l_a) = -\text{tr}_{\Omega_1(M)}(l_a) = \cdots = (-1)^i \text{tr}_{\Omega_i(M)}(l_a)$ for all $a \in J$ and $i \in \mathbb{N}$.

Finally, we provide another proof of Igusa-Liu-Paquette Theorem, i.e., Theorem 2, on the level of modules.

Proof. By the assumption $\text{pd}_{A^eS} < \infty$ and Lemma 2, we have $\text{pd}_{A_{kA^eA^e}} A < \infty$. It follows from Proposition 3 that, for any $\bar{a} \in J$, the equivalence class of $\bar{a}$ in $A/[A,A]$, $\bar{a} = \text{tr}_A(l_a) = \text{tr}_{\bar{A}}(l_a) = (-1)^i \text{tr}_{\Omega_i(M)}(l_a)$ which equals 0 for $i > \text{pd}_{A_{kA^eA^e}} A$. Thus $J \subseteq [A,A]$. Then we may continue as the last paragraph of the proof of Theorem 2 in Section 3.2. 

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