Chirp Control of Sinusoidal Lattice Modes in Bose-Einstein Condensate

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Abstract.

A parametrically forced Bose-Einstein condensate (BEC) is studied in the mean field approach for the case of a general moving optical lattice. The interaction between the atoms in the condensate and the time dependent lattice potential leads to a novel propagating superfluid matter wave, which can be controlled through chirp management. This system, when placed in a trap, accelerates and undergoes rapid nonlinear compression, controlled by the chirp. The density achieves its maximum, precisely when the matter wave changes direction. A dynamical phase transition is identified, which takes the superfluid phase to an insulating state. The exact expression for energy is obtained and analyzed in detail to gain physical understanding of the chirp management of the sinusoidal excitations and also the dynamical phase transition.

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1. Introduction

Bose-Einstein condensate (BEC) in a periodic potential is an area of active research [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. It is an interdisciplinary field, which has connections with many areas of physics: electrons in crystal lattices, polarons and photons in optical fibers, gauge theories and exotic phase transitions [11, 12, 13], to mention a few. The nature of the phase transition is different for deep and shallow optical lattices (OL). In the former case, a metal to insulator quantum phase transition [12], driven by the quantum fluctuations can occur, whereas, in the latter case, the same can occur through a classical dynamical phase transition [14, 15, 16, 17]. The advantage of studying the condensate in optical lattice is that one can control and tune all the relevant parameters, such as lattice depth, its geometry and the interaction strength between the atoms [18, 19, 12, 20, 21]. Optical lattices (OL) can be made time dependent by temporal variations of the depth and the width of the lattice [22]. The lattice can be moved at a velocity \( v = (\lambda/2)\delta\nu \), where \( \lambda/2 \) is the spatial period of the periodic potential and \( \delta\nu \) is the stable detuning between the two laser beams, which also allows the lattice to accelerate. The amplitude of the lattice potential can be tuned through the following relation: \( V_0 \approx 2\hbar\Omega_R/E_R \), \( \Omega_R \) being the Rabi frequency and \( E_R = 2\hbar^2\pi^2/m\lambda^2 \) is the recoil energy.

A number of recent studies have investigated the condensate dynamics, in presence of time modulated optical lattices [23, 24, 25, 26]. Madison et al., studied quantum transport of atoms in presence of an accelerating lattice [23]. They observed resonances, when a small oscillatory trap is added to the system. The number of atoms decreases rapidly, when the resonances occur. The spatiotemporal dynamics of the condensate has been investigated in presence of moving optical lattice, using direct perturbation and Melnikov-function method [25]. The existence of subdiffractive solitons is shown in [26], for a time dependent shallow optical lattice. These authors also showed that the variation of the chemical potential can lead to a chaotic system, whereas, by controlling the intensity of the lattice potential, the chaos can be suppressed. It is worth pointing out that, Fallani et al., have observed dynamical instability in a one dimensional moving optical lattice [22].

We present here a detailed study of the dynamics of BEC in presence of a general time dependent optical lattice with both attractive and repulsive atom-atom interaction. The exact sinusoidal solutions are naturally chirped, which allows them to be compressed and accelerated in a controlled manner. We further consider the BEC in the combined presence of optical lattice and a harmonic oscillator (HO) trap. The presence of the trap does not affect the form of the solutions, but drastically changes the dynamics of the condensate. It is found that the BEC can accelerate and undergo rapid nonlinear compression, when the matter wave changes its direction. The density of the condensate is found to be maximum at this point. In order to gain a deeper insight in to the response of the condensate to various time varying parameters, we explicitly compute the energy of the system. In the combined presence of a harmonic trap (HO) and optical
lattice, BEC shows a resonant increase in energy at certain points in the scaled time variable. This is akin to a recently observed resonant behavior in an optical lattice, where energy transfer takes place between two bands \[27\]. In case of the expulsive oscillator, the energy initially increases with time and takes a maximum value, after which it decreases rapidly. In case of the moving lattice, as the BEC expands, the energy monotonically decreases with time. It is also shown that this system exhibit dynamical superfluid insulator transition (DSIT), where superfluidity breaks down and the condensate transits to an insulating phase. The stability of the obtained solutions are carried out using Vakitov-Kolokolov criterion (VKC) \[28, 29, 30\], where we found that solutions are stable for attractive atom-atom interaction.

2. Sinusoidal excitation in time dependent optical lattice

In presence of a shallow optical lattice, the mean field Gross-Pitaevskii (GP) equation well captures the condensate dynamics:

\[
i\hbar \frac{\partial \Psi}{\partial t} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}} + U|\Psi|^2 - \tilde{\nu}(t) \right) \Psi.
\]  

(2.1)

Here, \(V_{\text{ext}} = V_{\text{HO}}(x,y) + V_1(z,t)\) is the external trapping potential, \(U\) is the strength of the atom-atom interaction and \(\tilde{\nu}(t)\) is the time dependent chemical potential. To have a cigar shaped BEC, one applies a strong oscillator trapping potential of frequency \(\omega_\perp\), \(V_{\text{HO}}(x,y) = \frac{1}{2}m\omega_\perp(x^2 + y^2)\), along the transverse direction. Assuming tight transverse confinement, the trial wavefunction is taken as, \(\Psi = \psi(z,t)\phi_0(x,y)\), where \(\phi_0\) is represented by a Gaussian ansatz \[31, 32, 33\]:

\[
\phi_0 = \sqrt{\frac{1}{\pi a_\perp^2}} e^{-\frac{x^2+y^2}{2a_\perp^2}},
\]  

(2.2)

with \(a_\perp = \sqrt{\frac{\hbar}{m\omega_\perp}}\). In the weak coupling regime, one finds that the quasi one dimensional GP equation takes the form \[34, 35, 36, 37, 38\]:

\[
i\frac{\partial \psi}{\partial t} = -\frac{1}{2} \frac{\partial^2 \psi}{\partial z^2} + V_1(z,t)\psi + g|\psi|^2\psi - \nu(t)\psi.
\]  

(2.3)

Here, \(V_1(z,t) = V_{\text{OL}}(z,t) = V_0(t)\cos^2(T)\) is the time dependent optical lattice potential, with \(T = A(t)z\) and \(\nu(t) = \mu A^2(t)\). The width \(A(t)\) can be controlled by the frequency of the laser beam. \(V_0(t)\) is the scaled time dependent amplitude of the lattice potential, which can be tuned by the intensity of the laser beam. The reduced two-body interaction is given by, \(g = \frac{m\omega}{2\pi\hbar} \left( \frac{m}{\hbar^2 k} \right) U\), where, \(k = 2\pi/\lambda\). The time, spatial coordinate and the wavefunction are, respectively, scaled as \(t \rightarrow m/(\hbar k^2)t\), \(z \rightarrow z/k\) and \(\psi \rightarrow \sqrt{k}\psi\). The chemical potential \(\nu(t)\) and the amplitude of the lattice potential have been normalized in terms of the recoil energy \(E_R = \hbar^2 k^2 / 2m\).

We assume the following ansatz for the BEC profile:

\[
\psi(z,t) = \sqrt{A(t)\sigma(T)}e^{i[\chi(z,t)+\phi(z,t)]}.
\]  

(2.4)

where, \(\chi(z,t)\) is a nontrivial phase, controlling the supercurrent of the condensate and the chirped phase \(\phi(z,t)\), is of the form: \(\phi(z,t) = -\frac{1}{2}c(t)z^2\).
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Current conservation yields, $\frac{\partial A}{\partial t} = \frac{2\delta}{\sigma}$, where $\delta = \frac{1}{4\kappa} \sqrt{(1 - 2\mu + 2\alpha)(1 - 2\mu)}$. The chirp parameter $c(t)$ satisfies $\frac{\partial c(t)}{\partial t} = c^2(t)$; from which all other parameters can be determined. The width $A(t)$ is related to $c(t)$ by,

$$\frac{\partial A(t)}{\partial t} = A(t)c(t). \quad (2.5)$$

For the solution to exist, the width and the depth of the lattice are related by, $V_0(t) = \alpha A^2(t)$ and the two-body interaction strength needs to be of the form: $g(t) = \kappa A(t)$. The real part of the GP equation, in terms of the density, then reduces to:

$$-\frac{1}{4}\sigma \frac{\partial^2 \sigma}{\partial T^2} - \frac{1}{8} \left( \frac{\partial \sigma}{\partial T} \right)^2 - \mu \sigma^2 + \kappa \sigma^3 + \alpha \cos^2(T) \sigma^2 + 2\delta^2 = 0. \quad (2.6)$$

For constant $\alpha$, one finds a self-similar solution of the form:

$$\sigma(T) = a + b \cos^2(T). \quad (2.7)$$

where, $a = \frac{2\mu - 1}{2\kappa}$ and $b = -\frac{\alpha}{\kappa}$. Notice that, for $\sigma(T)$ to be positive semi-definite, considering the atom-atom interaction to be repulsive, $\mu \geq \alpha + 1/2$, when $\alpha$ is positive. For, negative $\alpha$, $\mu \geq 1/2$. When the interaction is taken to be attractive, for positive and negative $\alpha$, $\mu \geq 1/2$ and $\mu \geq \alpha + 1/2$, respectively.

The periodicity of the density modulation is same as that of the lattice potential as seen in figure 1. Keeping in mind the role of chirp $c(t) = \frac{c_0}{1 - c_0 x}$, for effective pulse compression in optical fiber [39], we consider a time dependent chirp parameter, of the same form. Here, $A(t) = \frac{A_0}{1 - c_0 x}$, with the initial condition $c(t = 0) = c_0$ and $A(t = 0) = A_0$. The compression of the matter wave is shown in figure 1a. Chirping leads to a rapid compression of the BEC profile and the density tends to a singularity when $t \to 1/c_0$. The number of atoms per lattice site, in this case, is given by,

$$N = \frac{A(2\mu - \alpha - 1)}{2\kappa} - \frac{\alpha}{4\pi \kappa} \sin(2A\pi), \quad (2.8)$$

which can be controlled through chirp management.

Till now, we have restricted ourselves to the static optical lattice, whose width and amplitude have been allowed to vary with time. We now allow the lattice to move with a velocity $v = \frac{\partial l}{\partial t}$, with $T = A(t)(x - l(t))$. In this scenario, the solution retains its original form. The position of the matter wave can be found from the relation:

$$l(t) = l_0 e^{-\int_0^t c(t) dt'} = l_0(1 + c_0 t), \quad (2.9)$$

showing a constant velocity $v = l_0 c_0$. In the case, with $c(t = 0) = -1/c_0$, we find $c(t) = -1/(t + c_0)$ and $A(t) = A_0 c_0/(t + c_0)$. As time increases, the width, as well as amplitude of the matter wave decreases, leading to the spread of the BEC, as depicted in figure 1b.

3. Dynamics in presence of an optical lattice and harmonic trap

Keeping in mind the fact that most of the atomic systems are inhomogeneous due to the presence of the magnetic or optical traps, we now consider the same in a harmonic
Figure 1. Figure 1a depicts the sinusoidal BEC profile in presence of a static optical lattice, where the chirped BEC undergoes rapid compression. Figure 1b corresponds to a different parameter domain, where BEC undergoes expansion. Figure 1c represents the density in presence of a harmonic trap, where at the point of nonlinear compression, the matter wave changes its direction. Figure 1d corresponds to the expulsive oscillator, where the BEC spreads. In all these cases, the parameter values are: $c_0 = 1, A_0 = 0.5, \alpha = 0.15$ and $\kappa = 0.8$.

trap, of the form, $V_{HO}(z, t) = \frac{1}{2}M(t)z^2$, in addition to the lattice potential. Control of BEC and its solitonic and sinusoidal excitations is an area of significant current interest [40, 41, 42, 43, 44, 45, 46]. In a harmonic trap, this is achieved through the time dependencies of scattering length, transverse and longitudinal trap frequencies. It has been observed that, the center of mass of solitons, as well as its width and amplitude gets coupled to the trap parameters, through which they can be accelerated or compressed [47, 48], which is useful for coherent atom optics [49, 50, 51]. In the present case, all the parameters are controlled by the harmonic oscillator, although the solutions retain their self-similar form. The chirp parameter $c(t)$ can be found from the following Ricatti equation:

$$c_t(t) = c^2(t) + M(t).$$

(3.1)

We have analyzed the following cases, considering the fact that the oscillator can be both regular or expulsive.

(i) In the first case, we assume $M(t) = q^2$, representing a regular trap. Here, two
solutions are possible: (i) \( c(t) = q \tan qt \) and (ii) \( c(t) = -q \cot qt \). Consequently, the other parameters can be obtained. For (i), \( A(t) = A_0 / \cos qt \) and \( V_0(t) = V_0 / \cos^2 qt \) and for (ii), \( A(t) = A_0 / \sin qt \) and \( V_0(t) = V_0 / \sin^2 qt \), with \( V_0 = \alpha A_0^2 \). The position of the matter wave is oscillatory: (i) \( l(t) = l_0 \cos qt \) and (ii) \( l(t) = l_0 \sin qt \), respectively, with the initial position \( l_0 \). The density profile for this case is depicted in figure 1c. The sinusoidal wave propagates with a periodicity \( \pi \) and at the point of nonlinear compression, the density becomes maximum, wherein, the matter wave reverses its direction, as seen in figure 1c. Here, the supercurrent takes the form,

\[
J(t) = |\psi(z, t)|^2 \frac{\partial \chi}{\partial z} = 2\delta. \tag{3.2}
\]

(ii) If one considers \( M(t) = -q^2 \), which corresponds to an expulsive oscillator, the Riccati equation takes the form: \( c(t) = c^2(t) - q^2 \). Like the previous case, here also we have two solutions: (i) \( c(t) = -q \tanh qt \) and (ii) \( c(t) = -q \coth qt \). The width and lattice amplitude for (i) are: \( A(t) = A_0 / \cosh qt \), \( V_0(t) = V_0 / \cosh^2 qt \) and for (ii) \( A(t) = A_0 / \sinh qt \), \( V_0(t) = V_0 / \sinh^2 qt \). The density profile for this case is shown in figure 1d. As time increases, the amplitude of the propagating wave decreases, making the BEC spread out.

4. Energy of the excitation

To explore the dynamics of this parametrically driven system, we study the temporal behavior of energy, both in the absence and presence of the trap. In figure 2a, the same is shown for a static optical lattice, where the energy increases with time for this driven system. Figure 2b depicts the behavior of energy for the optical lattice moving with a constant velocity \( v \), where the energy is found to decrease rapidly as the BEC expands. Interestingly, in presence of a harmonic trap, the BEC shows a resonant increase in energy, at some suitable values of the scaled time variable, where it undergoes rapid nonlinear compression, as seen in figure 2c. The contribution of the chirping to the kinetic energy is solely responsible for this phenomenon. The matter wave changes its direction at the point of nonlinear compression. In case of an expulsive oscillator, as depicted in figure 3d, one can see that initially energy increases with time and reaches its maximum value after which it decreases rapidly. The number density in this case turns out to be:

\[
N = A \left( \frac{2\mu - 1}{2\kappa} - \frac{\alpha}{2\kappa} \right) - \frac{\alpha}{4\pi\kappa} \left( \sin(2Al) + \sin(2A(\pi - l)) \right). \tag{4.1}
\]

In case of moving lattice, the number of atoms decreases as a function of time, which is similar to the experimental observation in [22].

5. Dynamical superfluid-insulator transition

We now examine the possibility of dynamical phase transition in this system. In case of shallow lattice, a dynamical superfluid-insulator transition (DSIT), driven
Figure 2. Figure 2a shows the rapid increase in energy with time, for a static lattice. In figure 2b, the energy decreases with time due to expansion of BEC. A resonant increase in energy is observed in figure 2c, where the matter changes its direction. In presence of expulsive oscillator, depicted in figure 2d, energy increases and takes a maximum value, after which it decreases rapidly as a function of time.

by modulational instability, was predicted by Smerzi et al., using the mean field discrete nonlinear Schrödinger equation [14]. In the following year, Cataliotti et al., experimentally observed this classical phase transition in presence of a stationary optical lattice [15]. It has been found [17] that, DSIT can also occur when both two and three-body interactions are present. Recently, Fallani et al., have observed this dynamical instability in a one dimensional moving optical lattice [22]. Since the dynamical phase transition occurs at the point, where the energy of the system becomes non-analytic (NA), we have investigated the same by computing the the exact energy for all the three cases, separately. The expression for energy is rather too lengthy to be reported here, we will only concentrate on the non-analytic term, which for all the three cases is given by,

\[ E_{NA} = -\frac{4A^2(a^2a(a+b) - \delta^2)}{\sqrt{a(a+b)}} \tan^{-1} \frac{\sqrt{a} \tan Al}{\sqrt{a+b}}, \]  

(5.1)

In the presence of a harmonic trap, the addition contribution to the energy due to the trap arises, is analytic. It is non-analytic at \( a+b = 0 \) and \( a = 0 \), which respectively, lead to \( 2\mu + 2\alpha - 1 = 0 \) and \( 2\mu - 1 = 0 \). Both of these give \( \delta = 0 \). For the density to be a finite positive quantity, we exclude the second point. Taking into account the first condition, one can see that the supercurrent vanishes at this point and the superfluid
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phase transits to an insulating phase:

$$\psi_I(z,t) = \sqrt{\frac{A(t)}{\kappa}} \sin (A(t)z)e^{i\phi}. \quad (5.2)$$

We now check the stability of the above solution by using VKC, which states that, if \( \frac{\partial N}{\partial \mu} > 0 \), the solutions are unstable and are marginally stable if \( \frac{\partial N}{\partial \mu} = 0 \). Stable solutions correspond to \( \frac{\partial N}{\partial \mu} < 0 \). For all the three possible cases, we found:

$$\frac{\partial N}{\partial \mu} = \frac{A(t)}{\kappa}. \quad (5.3)$$

Therefore, one can note that the stability of the sinusoidal solution depends on the chirp, as well as on the nature of the atom-atom interaction. The solutions are stable for attractive atom-atom interaction.

6. Conclusion

In summary, we presented a detailed study of the parametrically forced GP equation in general time dependent optical lattices, with both attractive and repulsive two-body interaction. The interaction between atoms and the lattice leads to sinusoidal solutions, which can be controlled by chirp management. Both for stationary and the time dependent optical lattice scenario, a dynamical phase transition occurs, which takes the superfluid phase to the insulating phase. The BEC can be made to accelerate and undergo rapid compression by incorporating harmonic trap, apart from the lattice. A resonant increase in energy is observed at a certain point of time, when the system couples to the harmonic trap. The combined effect of lattice and trap leads to a nonlinear compression of the condensate, at certain intervals of time, where the density becomes maximum. In case of expulsive trap, the energy increases with time initially and achieves its maximum value, after which, it decreases rapidly. We have carried out the stability analysis using VKC, and found that the stability of these solutions depends on the chirp, as well as on the nature of the interaction. We hope that these observations will find experimental realization in the present laboratory setup.

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