VORTEX MOTION FOR THE LAKE EQUATIONS

JUSTIN DEKEYSER AND JEAN VAN SCHAFTINGEN

Abstract. The lake equations
\[
\begin{aligned}
\nabla \cdot (b \mathbf{u}) &= 0 \quad \text{on } \mathbb{R} \times D, \\
\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\nabla h \quad \text{on } \mathbb{R} \times D, \\
\mathbf{u} \cdot \mathbf{\nu} &= 0 \quad \text{on } \mathbb{R} \times \partial D.
\end{aligned}
\]
model the vertically averaged horizontal velocity in an inviscid incompressible flow of a fluid in a basin whose variable depth \( b : D \to [0, +\infty) \) is small in comparison to the size of its two-dimensional projection \( D \subset \mathbb{R}^2 \). When the depth \( b \) is positive everywhere in \( D \) and constant on the boundary, we prove that the vorticity and energy of solutions of the lake equations whose initial vorticity concentrates at an interior point behaves asymptotically a multiple of a Dirac mass whose motion is governed by the depth function \( b \).

1. Introduction

The lake equations model an incompressible inviscid flow of a fluid in a lake whose velocity varies on distances whose scale is large compared to the depth (shallow water) and is small compared to the speed of gravity waves (small Froude number: \( Fr \ll 1 \)) \cite{8} \((1.1)\). Mathematically, the lake is modelled by its projection of its volume on a horizontal planar open set \( D \subset \mathbb{R}^2 \) and by a positive depth function \( b : D \to (0, +\infty) \); the velocity field \( \mathbf{u} : \mathbb{R} \times D \to \mathbb{R}^2 \) and the surface height \( h : \mathbb{R} \times D \to \mathbb{R} \) are governed by the system of equations
\[
\begin{aligned}
\nabla \cdot (b \mathbf{u}) &= 0 \quad \text{on } \mathbb{R} \times D, \\
\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\nabla h \quad \text{on } \mathbb{R} \times D, \\
\mathbf{u} \cdot \mathbf{\nu} &= 0 \quad \text{on } \mathbb{R} \times \partial D,
\end{aligned}
\]
where \( \mathbf{\nu} \) denotes the outgoing normal vector at the boundary \( \partial D \) of the domain \( D \). The equations \((1.1)\) express respectively the conservation of mass, the conservation of momentum and the impermeability of the boundary \( \partial D \). In particular when the depth \( b \) is constant on the domain

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D, the lake equations (1.1) reduce to the two-dimensional Euler equations of inviscid incompressible flows. The lake equations (1.1) can be derived formally from the three-dimensional Euler equations [8] and have been justified mathematically in the periodic case [42]. They appear in the mean-field limit for the Gross–Pitaevskii equation, which is the Schrödinger flow for the Ginzburg–Landau energy under forcing and pinning [19]. Weak solutions of the Cauchy problem for the lake equations (1.1) exist globally [24, 31, 33, 40, 41]; these solutions are unique [5, 31, 40] and as smooth as the data permits it ([24, 32, 41] and appendix B at the end of the present work).

The vorticity $\omega = \nabla \times u$ of a flow governed by the lake equations (1.1) obeys the vorticity equation

$$\partial_t \omega + b u \cdot \nabla \left( \frac{\omega}{b} \right) = 0 \quad \text{in } D.$$ 

For the planar Euler equation, corresponding to constant depth $b$, the vorticity equation (1.2) has been known since the works of Helmholtz, Kirchhoff and Routh to have singular vortex-point solutions whose vorticity is a linear combination of Dirac deltas whose position is governed by a dynamical system whose Hamiltonian is the Kirchhoff–Routh stream function [23, §5; 28, §23]. These vortex point solutions are merely distributional solutions of the Euler equations; since the works of Scheffer and Shirelman [47–49], the latter are known to exhibit unphysical behaviours in general. In a seminal work, Marchioro and Pulvirenti have proved mathematically that the singular vortex-point solutions are in fact the limits of solutions of the planar Euler equations whose initial data’s vorticity concentrates into Dirac masses [37].

For the lake equations (1.1), Richardson computed by formal matched asymptotics that the position $q: \mathbb{R} \to D$ of a vortex of vorticity $\Gamma$ and its typical radius $\varepsilon$ should evolve according to the law [45, (5.1)]

$$\dot{q}(t) \simeq \frac{\Gamma}{4\pi} \ln \frac{1}{\varepsilon} \left( \nabla_{\perp} \ln b \right) (q(t)),$$

where the orthogonal gradient is defined $\nabla_{\perp} \ln b = (\partial_2 \ln b, -\partial_1 \ln b)$. A similar law was derived from axisymmetric Euler flow and verified experimentally for vortex dipoles moving towards a planar sloping beach [9, 43] and was tested numerically on barred beaches [6] in order to understand the rip currents which represent a hazard to swimmers. As a consequence of the law (1.3), vortex points should follow at the leading order the level sets of the bathymetry $b$. In comparison with the planar Euler equation, the velocity of a vortex depends on its radius and the dominant term is local: it interacts at the leading order neither with the boundary nor with vortices that remain at a positive distance. These formal, experimental and numerical results raise the question whether the evolution law (1.3) is mathematically the limiting behaviour of families of solutions to the lake equations (1.1).

In the stationary case for the lake equations, where the velocity $u$ does not depend on the time $t$ (1.1), there exist families of stationary solutions concentrated at a point of maximal depth or at a point where the irrotational flow generated by a boundary condition of order $\ln \frac{1}{\varepsilon}$
balances the diverging motion of \((1.3)\) [14–16]. (Corresponding results were already known for the planar Euler equations \([3, 7, 50–52]\).) This approach also yields a rotating singular vortex pair in a rotation-invariant lake [14].

When \(D = (0, +\infty) \times \mathbb{R}\) and \(b(r, z) = r\), the lake equations \((1.1)\) is in fact the axisymmetric three-dimensional Euler equation. A single vortex ring is known formally to evolve according to \((1.3)\) since the work of Helmholtz and Kelvin [23, §6 and letter from Thompson], which is a particular case of Da Rios law of evolution of three-dimensional vortices by a binormal curvature flow rescaled by a factor \(\ln \frac{1}{\varepsilon}\) (Da Rios law [12, 44], see [1, §2.1] for a derivation in modern formalism). Benedetto, Caglioti and Marchioro have proved that axisymmetric flows whose initial vorticity concentrates on a vortex ring satisfy asymptotically this law [2]. For arbitrary filaments, Jerrard and Seis have proved the asymptotic binormal curvature flow under some hypotheses on the solution of the three-dimensional equation [25].

In order to state our main result describing flows whose initial vorticity is concentrating by their bathymetry, we rely on two conserved integral quantities of the flow: the vortex circulation of the flow at time \(t \in \mathbb{R}\)

\[
\Gamma(t) \triangleq \int_D \omega(t),
\]

and the kinetic energy at time \(t \in \mathbb{R}\),

\[
E(t) \triangleq \frac{1}{2} \int_D |u(t)|^2 b,
\]

which are independent of the time \(t \in \mathbb{R}\) when \(u\) is a classical solution of the lake equation and for which we will henceforth drop the time-dependence in the notation.

Our main result characterizes the asymptotic behaviour of solutions when the vorticity of the initial data shrinks to a Dirac mass:

**Theorem 1.1.** Let \(D \subseteq \mathbb{R}^2\) be a bounded domain of class \(C^2\) and \(b \in C^2(\bar{D}, (0, +\infty))\). Assume that \(b\) is constant on each component of \(\partial D\). If

(a) \((u^n, h^n)_{n>0} \in C^1(\mathbb{R} \times \bar{D}, \mathbb{R}^2) \times C(\mathbb{R} \times \bar{D})\) is a family of classical solutions to the lake equations \((1.1)\),

(b) \(\omega^n(0) \geq 0\) everywhere in \(D\),

(c) \(\frac{1}{\Gamma^n} \omega^n(0) \rightharpoonup \delta_{q_0}\) narrowly as measures for some \(q_0 \in D\),

(d) \(\omega^n(0) \leq C \Gamma^n \exp \left(\frac{8s}{\Gamma^n E^n(0)}\right)\) everywhere in \(D\), for some constant \(C > 0\) independent on \(n\),

then for every \(s \in \mathbb{R}\),

\[
\frac{1}{\Gamma^n} \omega^n(\Gamma^n \frac{E^n}{s}) \rightharpoonup \delta_{q(s)} \quad \text{and} \quad \frac{|u^n|^2}{E^n} \left(\Gamma^n \frac{E^n}{s}\right) \rightharpoonup \delta_{q(s)}
\]

narrowly as measures, where the function \(q \in C^1(\mathbb{R}, D)\) is the unique solution of the Cauchy problem

\[
\begin{cases}
q'(s) = -\left(\nabla^\perp b^{-1}\right)(q(s)) & s \in \mathbb{R}, \\
q(0) = q_0.
\end{cases}
\]
The assumptions of theorem 1.1 imply in particular that $E_n/\Gamma_n \to +\infty$ as $n \to \infty$.

Here above in theorem 1.1 the function $\Omega : D \to \mathbb{R}$ stands for the initial total vorticity defined for each $t \in \mathbb{R}$ by

$$\Omega(t) \equiv \int_D \omega(t) \, b.$$  

(1.7)

The narrow convergence means explicitly that we assume that for every test function $\varphi \in C(\bar{D})$

$$\lim_{n \to \infty} \frac{1}{\Gamma_n} \int_D \omega^n \varphi = \varphi(q_0);$$

and that we conclude that for every test function $\varphi \in C(\bar{D})$ and every $s \in \mathbb{R}$

$$\lim_{n \to \infty} \frac{1}{\Gamma_n} \int_D \omega^n \left( \frac{\Gamma_n s}{E^n} \right) \varphi = \varphi(q(s)).$$

The narrow convergence of the energy density is similar to the result obtained for two-dimensional incompressible Euler flow by Dávila, del Pino, Musso and Wei [13].

Examples of solutions satisfying the assumptions of theorem 1.1 are given by rescaling an initial boundary data since the lake equations (1.1) with smooth initial data admit classical solutions. Given a non-negative function $f \in C^\infty_c(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} f = 1$, a sequence of positive numbers $(\varepsilon_n)_{n \in \mathbb{N}}$ converging to 0, a point $q_0 \in D$ and a sequence $(\Gamma_n)_{n \in \mathbb{N}}$ such that for every $n \in \mathbb{N}$ $B(q_0, \varepsilon_n) \subset D$, we define a vorticity $\omega^n : \omega \to \mathbb{R}$, for each $n \in \mathbb{N}$ and $x \in D$

$$\omega^n(0, x) \equiv \frac{\Gamma_n}{(\varepsilon_n)^2} f \left( \frac{x - q_0}{\varepsilon_n} \right),$$

(1.8)

and it can be computed that as $n \to \infty$

$$E^n = b(q_0) \frac{(\Gamma_n)^2}{4\pi} \ln \frac{1}{\varepsilon_n} + O(1)$$

$$\Omega^n(0) = b(q_0) \Gamma_n \left( 1 + O(\varepsilon^n) \right),$$

and thus, by theorem 1.1

$$\frac{1}{\Gamma_n} \omega^n \left( \frac{4\pi s}{\Gamma_n b(q_0) \ln \frac{1}{\varepsilon_n}} \right) \rightharpoonup \delta_{q(s)}$$

narrowly as measures on $\bar{D}$, where the motion of $q$ is governed by (1.6). If we set $q_n(t) = q \left( \frac{\Gamma_n b(q_0) \ln \frac{1}{\varepsilon_n}}{4\pi} t \right)$ and observe that $b \circ q_n$ is constant, then $q_n$ satisfies the equation,

$$\begin{cases}
q'_n(t) = \frac{\gamma_n}{4\pi} \ln \frac{1}{\varepsilon_n} \nabla^\perp \ln b(q_n(t)) & t \in \mathbb{R}, \\
q(0) = q_0,
\end{cases}$$

that is, $q_n$ is governed by Richardson’s law (1.3).

The assumption that the depth $b$ is constant on each component of the boundary implies that the solution $q$ of the Cauchy problem obtained in the conclusion of theorem 1.1 remains inside the domain $D$ and is thus global; the assumption plays an important role in our method, but we do not see any reason for which it should be necessary for the convergence to hold on a time interval in which there is no collision with the boundary.
The assumption that the domain $D$ is simply-connected in theorem 1.1 yields a slightly simpler statement; it will be removed in the sequel under an additional condition that the circulations are controlled by the vortex circulation (see theorem 5.4 below). Similarly, our proof of theorem 1.1 also covers weak solutions of the lake equations in the vortex formulation.

When the depth $b$ is a constant function, theorem 1.1 implies that the vortex is stationary at the time scale $\Gamma_n(0)/E_n(0)$; this does contradict the classical planar vortex motion which occurs at a time scale of $1/\Gamma_n(0)$, which is much larger in the regime $E_n(0)/\Gamma_n(0)^2 \to +\infty$.

The description of the motion of vortices in theorem 1.1 can be formally written as

$$\dot{q}(t) \simeq -\frac{E}{\Gamma} \left( \nabla \cdot \frac{1}{b} \right)(q(t)). \quad (1.9)$$

An advantage of the formulation (1.9) is that the typical radius $\varepsilon$ of the vortex, which is not necessarily preserved or even well-defined a priori along the flow, is replaced by a conserved quantity.

A first step in the proof of theorem 1.1 is to prove that the vorticity of the solution $u_n(t)$ at any time $t \in \mathbb{R}$ concentrates as $n \to +\infty$. In contrast to other works for the planar Euler equations [38] or cylindrically symmetric Euler equations in the space [2, 38] in which the geometry of the vorticity region is constrained through its diameter or area, we rely on a typical length scale

$$\rho_n(t) \triangleq \exp \left( -\frac{4\pi E_n}{\Gamma_n \Omega_n(t)} \right), \quad (1.10)$$

which is defined in terms of integral quantities related to the flow: the energy $E_n(t)$ defined in (1.5) and the circulation $\Gamma_n(t)$ defined in (1.4), which are both conserved, and the total vorticity $\Omega_n(t)$ defined in (1.7) which satisfies

$$\left( \inf_D b \right) \Gamma \leq \Omega(t) \leq \left( \sup_D b \right) \Gamma. \quad (1.11)$$

The estimate (1.11) implies in particular that $\rho_n(t) \to 0$ uniformly as $n \to +\infty$.

In order to show that the vorticity effectively concentrates on balls of radius of the order $\rho_n(t)$ defined in (1.10), we rely on the assumption (d) of theorem 1.1 and on the fact that although the total vorticity $\Omega_n(t)$ is not conserved, it satisfies an estimate of the form

$$|\Omega_n(t) - \Omega_n(0)| \leq C(\Gamma_n)^2|t| = C|\Gamma_n|^2 \frac{E_n t}{E_n/\Gamma_n}, \quad (1.12)$$

The proof of (1.12) relies on the constancy of the depth $b$ on connected components of the boundary $\partial D$ (see proposition 5.3). Since $E_n/\Gamma_n^2 \to \infty$, the estimate (1.12) is stronger then the estimate (1.11) at time scale $\Gamma_n/E_n$.

Our strategy to obtain the equation of motion of the vortex, is to study the center of vorticity

$$q_n(t) \triangleq \frac{1}{\Gamma_n} \int_D x \omega_n(t, x) \, dx. \quad (1.13)$$

A formal derivation argument on (1.13) gives the formula

$$\dot{q}_n(t) \simeq \frac{1}{\Gamma_n} \int_D \frac{\nabla \cdot \frac{1}{b}}{b^2} \psi_n(t) \omega_n(t);$$
a suitable study of the asymptotics of the vorticity \( \omega_n(t) \) and of the stream function \( \psi_n(t) \) shows that the right-hand side behaves asymptotically as \( - (\nabla \cdot b^{-1}) E^n(t) / \Gamma^n(t) \). Unfortunately our derivation formula for \( q_n \) would require the identity to be constant on the boundary \( \partial D \); we bypass this technical obstacle by considering a modified version of the center of vorticity which is close to the center of vorticity thanks to concentration estimates and some repulsion properties of the boundary.

The sequel of the present work is organized as follows. In section 2 we precise the notion of weak solution of the lake equations in the vorticity formulation that we are using in the present work and we explain how the velocity can be reconstructed from the vorticity and the circulation around the boundary components and why the circulation \( \Gamma \) and energy \( E \) are preserved for weak solutions. In section 3 we expand the velocity construction formula in terms of the depth \( b \) and the Green function \( G_D \) for the classical Dirichlet problem on \( D \) at a level of precision required by the proof of our main result. These asymptotics are used in section 4 to obtain various concentration estimates on the vorticity. In section 5 we prove our main asymptotic result, after having obtained an asymptotic representation of derivatives of quantities and of the total vorticity \( \Omega \). Theorem 1.1 as a first result on the asymptotic behaviour of vortices for the lake equations, raises several open problems for future research that are presented in section 6.

In a first appendix, we state some variants of classical results for transport equations [17] for velocities preserving the density \( b \). The second appendix is devoted to a classical derivation of regularity results for the lake equations (1.1); this implies in particular that the classical solutions of the lake equations (1.1) appearing in theorem 1.1 exist for any smooth initial data.

2. The lake model

2.1. Weak vortex formulation of the lake equation. A lake is represented by its projection on a bounded domain \( D \subseteq \mathbb{R}^2 \) of the horizontal plane endowed and a depth function \( b : D \to (0, +\infty) \). We assume that the domain \( D \) can be written as

\[
D = D_0 \setminus \bigcup_{i=1}^m I_i,
\]

where the set \( D_0 \subset \mathbb{R}^2 \) is simply-connected and its boundary is of class \( C^2 \) and the islands \( I_1, \ldots, I_m \subseteq D_0 \) are disjoint simply-connected compact sets whose boundary is of class \( C^2 \). We assume that \( b \in C^2(\bar{D}, (0, +\infty)) \). In particular, the depth \( b \) remains bounded away from 0 on the domain \( D \).
A weak solution of the vorticity formulation of the lake equations will satisfy weakly the following system

\[
\begin{aligned}
\nabla \cdot (b\mathbf{u}) &= 0 \quad \text{in } [0, +\infty) \times D, \\
\mathbf{u} \cdot \mathbf{\nu} &= 0 \quad \text{on } [0, +\infty) \times \partial D, \\
\partial_t \omega + \nabla \cdot (\mathbf{u} \omega) &= 0 \quad \text{in } [0, +\infty) \times D, \\
\nabla \times \mathbf{u} &= \omega \quad \text{in } [0, +\infty) \times D, \\
\omega(0, \cdot) &= \omega_0 \quad \text{on } D.
\end{aligned}
\]

More precisely, it will fulfill the following definition (see [24, Definition 2.2; 31, Definition 1.2]):

**Definition 2.1.** Given an initial pair \((\omega_0, \mathbf{u}_0)\) with \(\omega_0 \in L^\infty(D, \mathbb{R})\) and \(\mathbf{u}_0 \in L^\infty(D, \mathbb{R}^2)\) that satisfies weakly

\[
\begin{aligned}
\nabla \cdot (b\mathbf{u}_0) &= 0 \quad \text{in } D, \\
\mathbf{u}_0 \cdot \mathbf{\nu} &= 0 \quad \text{on } \partial D, \\
\nabla \times \mathbf{u}_0 &= \omega_0 \quad \text{in } D,
\end{aligned}
\]

a pair \((\omega, \mathbf{u}) \in L^\infty([0, +\infty) \times D, \mathbb{R}) \times L^\infty([0, +\infty), L^2(D, \mathbb{R}^2))\) is a weak solution of the lake equations in the vorticity formulation with initial condition \((\omega_0, \mathbf{u}_0)\) whenever

(i) for every test function \(\varphi \in C^1_c([0, +\infty) \times \bar{D})\), one has

\[
\int_0^{+\infty} \int_D b\mathbf{u} \cdot \nabla \varphi = 0,
\]

(ii) for every test function \(\varphi \in C^1_c([0, +\infty) \times \bar{D})\) such that for every \(t \in [0, +\infty)\), \(\varphi|_{\{t\} \times \partial D_0} = 0\)

and for every \(i \in \{1, \ldots, m\}\), \(\varphi|_{\{t\} \times \partial D_i}\) is constant, one has

\[
\int_0^{+\infty} \int_D (\mathbf{u} \cdot \nabla^i \varphi - \omega \varphi) = \int_0^{+\infty} \int_D (\mathbf{u}_0 \cdot \nabla^i \varphi - \omega_0 \varphi),
\]

(iii) for every test function \(\varphi \in C^1_c([0, +\infty) \times D)\), one has

\[
\int_D \omega \varphi(0, \cdot) + \int_0^{+\infty} \int_D \omega (\partial_t \varphi + \mathbf{u} \cdot \nabla \varphi) = 0.
\]

A pair \((\omega, \mathbf{u}) \in L^\infty(\mathbb{R} \times D, \mathbb{R}) \times L^\infty(\mathbb{R}, L^2(D, \mathbb{R}^2))\) is a weak solution of the vorticity formulation of the lake equations with initial condition \((\omega_0, \mathbf{u}_0)\) whenever the functions \(t \in [0, +\infty) \mapsto (\omega(t), \mathbf{u}(t))\) and \(t \in [0, +\infty) \mapsto (-\omega(-t), -\mathbf{u}(-t))\) are both weak solution to the vorticity formulation with initial condition \((\omega_0, \mathbf{u}_0)\).

Here and in the sequel, \(\mathbf{\tau}\) denotes the unit tangent vector to the boundary \(\partial D\) chosen so that det\((\mathbf{\nu}, \mathbf{\tau})\) = 1. The set \(C^1_c([0, +\infty) \times \bar{D})\) is the set of maps \(\varphi \in C^1([0, +\infty) \times \bar{D})\) such that there exists \(T > 0\) such that \(\varphi = 0\) in \((T, +\infty) \times D); C^1_c([0, +\infty) \times D)\) is the set of maps \(\varphi \in C^1([0, +\infty) \times \bar{D})\) such that there exists \(T > 0\) and a compact set \(K \subset D\) such that \(\varphi = 0\) in \(((0, +\infty) \times D) \setminus ([0, T] \times K)\).

Compared to [31, Definition 1.2], definition 2.1 considers fewer test functions in \(\mathbb{R}\) — this will not matter eventually (see proposition 2.7 below) — and incorporates the conservation of
circulation around the components of the boundary. Indeed, it follows from (ii) in definition 2.1 that the circulation \( \Gamma_i(t) \) along \( \partial I_i \) at time \( t \) can be defined by

\[
\Gamma_i(t) \triangleq \int_D (\mathbf{u}(t) \cdot \nabla \varphi - \omega(t) \varphi),
\]

for any function \( \varphi \in C^1(\bar{D}) \) such that \( \varphi = 1 \) on \( \partial I_i \) and for each \( j \in \{1, \ldots, m\} \setminus \{i\}, \varphi = 0 \) on \( \partial I_j \) (see also [31, (2.12)]). In view of (ii) in definition 2.1 we have \( \Gamma_i(t) = \Gamma_i(0) \) for almost every \( t \in \mathbb{R} \).

The surface height \( h \) does not appear in the weak formulation of definition 2.1, in accordance with the fact that \( \nabla h \) can be recovered in (1.1) from \( \mathbf{u} \).

The lake equations have at least one global weak solution in the vorticity formulation [5, Theorem 2.2 ii), 31 Lemma 2.11 & Theorem 1]; this solution is unique [5, Theorem 2.2 iii), 31 §2.3; 40, Theorem 1]. If moreover \( \omega_0 \in C^{k,\alpha}(\bar{D}) \), one has \( \omega \in C^{k,\alpha}([-T,T] \times D) \), \( \mathbf{u} \in C^{k,\alpha}([-T,T] \times D,\mathbb{R}^2) \) and \( \mathbf{u} \in L^\infty([-T,T],C^{k+1,\alpha}(\bar{D},\mathbb{R}^2)) \) ([24, Theorem 4.1] and proposition 3.1 below).

An alternative to definition 2.1 is the notion of weak solution for the velocity formulation [31, Proposition 2.13], based on (1.1). Under regularity assumptions on the domain \( D \), the depth function \( b \) and on the initial data \((\omega_0, \mathbf{u}_0)\), both notions are equivalent [31 Proposition A.1].

2.2. Velocity reconstruction. Given a vorticity \( \omega \in L^1(D) \) and circulations \( \Gamma_1, \ldots, \Gamma_m \), the velocity reconstruction problem consists in finding a vector field \( \mathbf{u} : D \to \mathbb{R}^2 \) such that

\[
\begin{cases}
\nabla \cdot (b \mathbf{u}) = 0 & \text{in } D, \\
\mathbf{u} \cdot \nu = 0 & \text{on } \partial D, \\
\nabla \times \mathbf{u} = \omega & \text{in } D, \\
\int_{\partial I_i} \frac{\mathbf{u} \cdot \tau}{b} = \Gamma_i & \text{for } i \in \{1, \ldots, m\},
\end{cases}
\]

weakly. That is, for every test function \( \varphi \in C^1(\bar{D}) \),

\[
\int_D b \mathbf{u} \cdot \nabla \varphi = 0,
\]

and for every \( \varphi \in C^1(\bar{D}) \) such that \( \varphi|_{\partial D_0} = 0 \) and \( \varphi|_{\partial I_i} = \lambda_i \in \mathbb{R} \),

\[
\int_D (\mathbf{u} \cdot \nabla \varphi - \omega \varphi) = \sum_{i=1}^m \Gamma_i \lambda_i.
\]

The system (2.1) corresponds thus for a fixed time \( t \) to the continuity equation (i) and the definition of vorticity (ii) in definition 2.1.

In view of the divergence-free condition \( \nabla \cdot (b \mathbf{u}) = 0 \) in \( D \) and of the boundary condition \( \mathbf{u} \cdot \nu = 0 \) on \( \partial D \), the solution can be written as \( \mathbf{u} = (\nabla \psi)/b \), where \( \psi : D \to \mathbb{R} \) is a scalar
stream function of the velocity field $u$ and the function $\psi$ satisfies the elliptic problem
\begin{equation}
\begin{cases}
-\nabla \cdot (b^{-1} \nabla \psi) = \omega & \text{in } D, \\
\psi = 0 & \text{on } \partial D_0, \\
\psi \text{ is constant} & \text{on } \partial I_i \text{ for each } i \in \{1, \ldots, m\}, \\
\int_{\partial I_i} b^{-1} \frac{\partial \psi}{\partial \nu} = \Gamma_i & \text{for each } i \in \{1, \ldots, m\}.
\end{cases}
\end{equation}

Since the ansatz $u = (\nabla^\perp \psi)/b$ only defines the stream function $\psi/b$ up to an additive constant, the boundary condition on $\partial D_0$ fixes the choice of a particular stream function.

The problem (2.2) can be handled by first solving the corresponding classical Dirichlet problem:
\begin{equation}
\begin{cases}
-\nabla \cdot (b^{-1} \nabla \psi) = \omega & \text{in } D, \\
\psi = 0 & \text{on } \partial D.
\end{cases}
\end{equation}

Since the function $b$ is smooth and bounded from above and from below on $D$, one has the classical result:

**Proposition 2.2.** For every $p \in (1, +\infty)$, there exists a linear continuous operator $G_b : L^p(D) \to W^{2,p}(D, \mathbb{R}^2)$ such that for every $\omega \in L^p(D)$, the function $G_b[\omega]$ is a weak solution of the problem (2.3).

**Proof.** See for example [21, theorem 9.15]. \qed

We now describe the solution to (2.2) in terms of (2.3) (see also [33, §3]; in the case of the planar Euler equations where $b$ is constant, see [29, §6; 35; 50, (45)].

**Proposition 2.3.** For every $i \in \{1, \ldots, m\}$, there exists a unique function $\psi_i \in C^2(\bar{D})$ that solves (2.2) with $\omega = 0$, $\Gamma_i = 1$ and $\Gamma_j = 0$ when $i \neq j$.

**Proof.** For every $i \in \{1, \ldots, m\}$, let $\varphi_i \in C^2(\bar{D})$ be a classical solution to the Dirichlet problem
\begin{equation}
\begin{cases}
-\nabla \cdot (b^{-1} \nabla \varphi_i) = 0 & \text{in } D, \\
\varphi_i = \delta_{ij} & \text{on } \partial I_j \text{ for each } j \in \{1, \ldots, m\}, \\
\varphi_i = 0 & \text{on } \partial D_0,
\end{cases}
\end{equation}

where $\delta_{ij}$ is the Kronecker delta, that is, $\delta_{ij} = 1$ whenever $i = j$ and $\delta_{ij} = 0$ otherwise. Since the functions $\varphi_1, \ldots, \varphi_m$ are by construction linearly independent and since the domain $D$ is connected, the matrix $(D_{ij})_{1 \leq i, j \leq m}$ defined by
\begin{equation}
D_{ij} \triangleq \int_D \frac{\nabla \varphi_i \cdot \nabla \varphi_j}{b}
\end{equation}

is positive-definite and thus invertible. Let $(D_{ij}^{-1})_{1 \leq i, j \leq m}$ denote the inverse of this matrix. For every $i \in \{1, \ldots, m\}$, we define the function $\psi_i : D \to \mathbb{R}$ by
\begin{equation}
\psi_i \triangleq \sum_{j=1}^m D_{ij}^{-1} \varphi_j.
\end{equation}
Moreover, there exists a smooth function $Q\psi$. The function where the functions $\phi_j$ and the definition of $\psi$:

$$
\int_{\partial I_j} \frac{1}{b} \frac{\partial \psi}{\partial \nu} = \sum_{i=1}^m D_{ij}^{-1} \int_{\partial J_i} \frac{\phi_j \frac{\partial \psi}{\partial \nu}}{b} = \sum_{i=1}^m D_{ij}^{-1} \int_D \frac{\nabla \phi_j \cdot \nabla \psi}{b} = \sum_{i=1}^m D_{ij}^{-1} D_{ij} = \delta_{ij}. \quad \square
$$

**Proposition 2.4.** For every $p \in (1, +\infty)$, there exists a linear continuous operator $K_b : L^p(D) \to W^{2,p}(D)$ such that for every $\omega \in L^p(D)$, the function $K_b[\omega]$ satisfies problem (2.2) weakly with $\Gamma_1 = \cdots = \Gamma_m = 0$.

Moreover, there exists a smooth function $Q_b \in C^2(\bar{D} \times \bar{D}, \mathbb{R}^2)$ such that for each $\omega \in L^p(D)$,

$$
K_b[\omega] = G_b[\omega] + \int_D Q_b(\cdot, y) \omega(y) \, dy.
$$

**Proof of proposition 2.4.** We define the function $Q_b : \bar{D} \times \bar{D} \to \mathbb{R}$ for every $x, y \in D$ by

$$
Q_b(x, y) \triangleq - \sum_{i,j=1}^m \phi_i(x) D_{ij}^{-1} \phi_j(y),
$$

where the functions $\phi_i$ and the matrix $D_{ij}^{-1}$ were defined in (2.4) and (2.5) in the proof of proposition 2.3 and for each $\omega \in L^p(D)$,

$$
K_b[\omega] \triangleq G_b[\omega] + \int_D Q_b(\cdot, y) \omega(y) \, dy.
$$

By linearity, we have that $-\nabla \cdot (b^{-1} \nabla K_b[\omega]) = -\nabla \cdot (b^{-1} \nabla G_b[\omega]) = \omega$, that $K_b[\omega] = G_b[\omega] = 0$ on $\partial D_0$ and that $K_b[\omega]$ is constant on each component of the boundary. Finally, we have for each $i \in \{1, \ldots, m\}$,

$$
\int_{\partial I_i} \frac{1}{b} \frac{\partial K_b[\omega]}{\partial \nu} = \int_{\partial D} \frac{1}{b} \frac{\partial G_b[\omega]}{\partial \nu} \phi_i - \sum_{j, \ell=1}^m D_{ij}^{-1} \int_{\partial D} \frac{1}{b} \phi_j \frac{\partial \phi_\ell}{\partial \nu} \int_D \phi_\ell \omega = \int_D \omega \phi_i - \sum_{j, \ell=1}^m D_{ij}^{-1} \int_D \frac{\nabla \phi_i \cdot \nabla \phi_\ell}{b} \omega \, \phi_\ell = 0. \quad \square
$$

We deduce from propositions 2.3 and 2.4 that for every $\omega \in L^p(D)$ and every $\Gamma_1, \ldots, \Gamma_m \in \mathbb{R}$, the solution $\psi : D \to \mathbb{R}$ to the problem (2.2) is given for each $x \in D$ by

$$
\psi(x) = K_b[\omega](x) + \sum_{i=1}^m \Gamma_i \phi_i(x) \quad (2.6)
$$

$$
= G_b[\omega](x) + \int_D Q_b(x, y) \omega(y) \, dy + \sum_{i=1}^m \Gamma_i \phi_i(x).
$$
The associated velocity field \( u : D \rightarrow \mathbb{R}^2 \) is then given for each \( x \in D \) by the relation (see also [31 Lemma 2.6; 33 Lemma 5])

\[
\begin{align*}
\mathbf{u}(x) &= \frac{1}{b(x)} \left( \nabla^\perp \mathbf{K}_b[\omega](x) + \sum_{i=1}^m \Gamma_i \nabla^\perp \psi_i(x) \right) \\
&= \frac{1}{b(x)} \left( \nabla^\perp \mathbf{G}_b[\omega](x) + \int_D \nabla^\perp Q_b(x, y) \omega(y) \, dy + \sum_{i=1}^m \Gamma_i \nabla^\perp \psi_i(x) \right),
\end{align*}
\]

where we adopt the convention that \( \nabla^\perp \) only acts on the first two-dimensional variable of the function \( Q_b \).

We conclude this section by showing how the kinetic energy defined by (1.5) at a fixed time can be computed in terms of the vorticity and the circulations.

**Proposition 2.5.** If \( \omega \in L^p(D) \) for some \( p > 1 \), then \( u \in L^2(D, \mathbb{R}^2) \) and

\[
E = \frac{1}{2} \int_D |u|^2 \, b = \frac{1}{2} \int_D \omega \mathbf{K}_b[\omega] + \sum_{i=1}^m \Gamma_i \int_D \omega \psi_i + \sum_{i,j=1}^m \frac{\Gamma_i \Gamma_j}{2} \int_{D \times D} \frac{\nabla \psi_i \cdot \nabla \psi_j}{b}.
\]

**Proof.** In view of the representation formula for the velocity field (2.7), we have,

\[
E = \frac{1}{2} \int_D \left| \nabla \mathbf{K}_b[\omega] + \sum_{i=1}^m \Gamma_i \nabla \psi_i \right|^2
\]

\[
= \frac{1}{2} \int_D \omega \mathbf{K}_b[\omega] + \sum_{i=1}^m \Gamma_i \int_D \omega \psi_i + \sum_{i,j=1}^m \frac{\Gamma_i \Gamma_j}{2} \int_{D \times D} \frac{\nabla \psi_i \cdot \nabla \psi_j}{b},
\]

by integration by parts and by definition of \( \mathbf{K}_b[\omega] \) in proposition 2.3. \( \square \)

### 2.3. Additional regularity of weak solutions.

We apply the previous results to the regularity of stream functions of weak solutions (see [3 Theorem 1 i]):

**Proposition 2.6.** If \( (\omega, u) \in L^\infty(\mathbb{R} \times D, \mathbb{R}) \times L^\infty(\mathbb{R}, L^2(D, \mathbb{R}^2)) \) is a weak solution to the vortex formulation of the lake equations, then for every \( p \in [1, +\infty) \), we have

\[
u \in L^\infty((0, +\infty), W^{1,p}(D))
\]

and

\[
\mathbf{K}_b[\omega] \in W^{1,\infty}(0, +\infty) \times D, \mathbb{R}).
\]

**Proof.** We first observe that by proposition 2.3, for almost every \( t \in [0, +\infty) \), we have

\[
\|K_b[\omega(t)]\|_{W^{2,p}(D)} \leq C_1\|\omega(t)\|_{L^p(D)} \leq C_2\|\omega(t)\|_{L^\infty(D)} \leq C_2\|\omega\|_{L^\infty((0, +\infty) \times D)}.
\]

In particular, by taking \( p > 2 \), we have by the supercritical Sobolev embedding theorem that \( \mathbf{K}_b[\omega] \in L^\infty((0, +\infty), W^{1,\infty}(D)) \) and thus

\[
\|u\|_{L^\infty((0, +\infty), W^{1,p}(D))} \leq C_3\|\omega\|_{L^\infty((0, +\infty) \times D)}.
\]

Next, we have in view of the definition of \( \mathbf{K}_b \), for every \( \varphi \in C^\infty_c((0, +\infty) \times D) \),

\[
\int_0^{+\infty} \int_D \mathbf{K}_b[\omega] \partial_t \varphi = \int_0^{+\infty} \int_D \omega \partial_t \mathbf{K}_b[\varphi].
\]
Using the evolution equation for the vorticity $\omega$ (definition 2.1 (iii)), we obtain, since $\varphi = 0$ on $\{0\} \times D$ and thus $K_b[\varphi] = 0$ on $\{0\} \times D$,

$$
\left| \int_0^{+\infty} \int_D K_b[\omega] \partial_t \varphi \right| = \left| \int_0^{+\infty} \int_D \omega(t) \cdot \nabla K_b[\varphi] \right| \\
\leq C_4 \|\omega(t)\|_{L^\infty(D)} \int_{(0, +\infty) \times D} \|K_b[\varphi]\| \\
\leq C_5 \|\omega(t)\|^2_{L^\infty(D)} \int_{(0, +\infty) \times D} \|\varphi\|.
$$

This implies that the weak derivative $\partial_t K_b$ belongs to $L^\infty([0, +\infty) \times D, \mathbb{R})$. Therefore we deduce $K_b[\omega] \in W^{1,\infty}([1, +\infty) \times D)$.

The regularity that we have obtained so far implies that in fact the spatial boundary conditions on the test functions in definition 2.1 (iii) can be completely relaxed.

**Proposition 2.7.** If a pair $(\omega, u)$ with $\omega \in L^\infty([0, +\infty) \times D, \mathbb{R})$ and $u \in L^\infty([0, +\infty), L^2(D, \mathbb{R}^2))$ is a weak solution of the vorticity formulation of the lake equations with initial conditions $\omega_0 \in L^\infty(D)$ and $u_0 \in L^\infty(D, \mathbb{R}^2)$, then for every test function $\varphi \in C^1_{c}(\{0, +\infty) \times D)$, one has

$$
\int_D \omega_0 \varphi(0, \cdot) + \int_0^{+\infty} \int_D \omega(\partial_t \varphi + u \cdot \nabla \varphi) = 0.
$$

**Proof.** This follows from proposition 2.6 definition 2.1 (iii) and proposition A.1.

2.4. **Transport of the potential vorticity.** The vorticity equation (1.2) can be rewritten as

$$
\partial_t \omega + \nabla \cdot (u \cdot \omega) = 0
$$

and implies that the vortex circulation $\Gamma(t)$ defined by (1.4) of classical solutions of the lake equations (1.1) is conserved. The next proposition shows that this is still the case for weak solutions of the vorticity formulation of the lake equations (definition 2.1).

**Proposition 2.8.** If $(\omega, u) \in L^\infty(\mathbb{R} \times D, \mathbb{R}) \times L^\infty(\mathbb{R}, L^2(D, \mathbb{R}^2))$ is a weak solution to the vortex formulation of the lake equations, then $\omega \in C(\mathbb{R}, L^1(D))$ and for every $t \in \mathbb{R}$,

$$
\Gamma(t) = \int_D \omega(t) = \int_D \omega_0 = \Gamma(0).
$$

**Proof.** We follow [31 §2.3]. We observe that for every $t \in \mathbb{R}$, $u(t) \in W^{1,1}_{loc}(D)$ and thus if $\varphi \in C^1_{c}([0, +\infty) \times D)$, we have

$$
\int_0^{+\infty} \int_D \frac{\omega}{b} \left( \partial_t \varphi + \nabla \cdot (u \varphi) \right) + \int_D \frac{\omega_0}{b} \varphi(0, \cdot) \\
= \int_0^{+\infty} \int_D \omega \left( \partial_t \frac{\varphi}{b} + u \cdot \nabla \frac{\varphi}{b} \right) + \int_D \omega_0 \frac{\varphi(0, \cdot)}{b} = 0.
$$

By proposition 2.6, proposition A.2 is applicable to $f_0 = \omega_0 / b$ and gives the conclusion.
2.5. **Conservation of energy.** We now consider the total kinetic energy defined by (1.5). For classical solutions, one can show that the energy equation

\[ \partial_t \left( b \frac{|u|^2}{2} \right) + \nabla \cdot (b u \frac{|u|^2}{2}) = -\nabla \cdot (b u h) \]  

holds, and consequently, since \( b u \cdot \nu = 0 \) on the boundary, we have conservation of the total kinetic energy for classical solutions. The total kinetic energy still remains constant for weak solutions of the vortex formulation of the lake equations (definition 2.1).

**Proposition 2.9 (Conservation of energy).** If \((\omega, u) \in L^\infty(\mathbb{R} \times D, \mathbb{R}) \times L^\infty(\mathbb{R}, L^2(D, \mathbb{R}^2))\) is a weak solution to the vortex formulation of the lake equations, then for almost every \( t \in \mathbb{R} \),

\[
E(t) = E(0).
\]

The proof of proposition 2.9 relies on the following derivation formula.

**Lemma 2.10.** Given a weak solution of the vortex formulation of the lake equations \((\omega, u) \in L^\infty([0, +\infty) \times D, \mathbb{R}) \times L^\infty([0, +\infty), L^2(D, \mathbb{R}^2))\), we have for every \( \theta \in C_c^\infty([0, +\infty)) \),

\[
\frac{1}{2} \int_0^{+\infty} \left( \int_D \omega(t) K_b[\omega(t)] \right) \theta'(t) \, dt = -\frac{\theta(0)}{2} \int_0^\infty \omega_0 K_b[\omega_0] - \int_0^{+\infty} \left( \int_D \omega(t) \partial_t K_b[\omega](t) \right) \theta(t) \, dt.
\]

**Proof.** For every \( h \in (0, +\infty) \), we have by a change of variable

\[
\frac{1}{2} \int_0^{+\infty} \left( \int_D \omega(t) K_b[\omega(t)] \right) \frac{\theta(t + h) - \theta(t)}{h} \, dt = \frac{1}{2} \int_0^{+\infty} \left( \int_D \omega(t - h) K_b[\omega(t - h)] - \omega(t) K_b[\omega(t)] \right) \theta(t) \, dt = -\frac{1}{2h} \int_0^h \left( \int_D \omega(t) K_b[\omega(t)] \right) \theta(t) \, dt.
\]

By Lebesgue’s dominated convergence theorem, we have

\[
\lim_{h \to 0} \frac{1}{2} \int_0^{+\infty} \left( \int_D \omega(t) K_b[\omega(t)] \right) \frac{\theta(t + h) - \theta(t)}{h} \, dt = \frac{1}{2} \int_0^{+\infty} \left( \int_D \omega(t) K_b[\omega(t)] \right) \theta'(t) \, dt.
\]

Since \( \omega \in C([0, +\infty), L^1(D)) \) by proposition 2.7, we also have

\[
\lim_{h \to 0} \frac{1}{h} \int_0^h \left( \int_D \omega(t) K_b[\omega(t)] \right) \theta(t) \, dt = \frac{1}{2} \int_D \omega_0 K_b[\omega_0] \theta(0).
\]

For every \( t \in [0, +\infty) \), since

\[
\int_D \omega(t) K_b[\omega(t - h)] = \int_D \omega(t - h) K_b[\omega(t)],
\]

we have

\[
\lim_{h \to 0} \frac{1}{h} \int_0^h \left( \int_D \omega(t) K_b[\omega(t)] \right) \theta(t) \, dt = \frac{1}{2} \int_D \omega_0 K_b[\omega_0] \theta(0).
\]

Hence, we have conservation of energy for weak solutions.
we have
\[
\int_D \omega(t-h) K_b[\omega(t-h)] - \omega(t) K_b[\omega(t)] = \int_D (\omega(t-h) + \omega(t)) (K_b[\omega(t-h)] - K_b[\omega(t)]),
\]
and thus by the weak convergence of difference quotients to the weak derivative and by proposition 2.8 we obtain
\[
(2.17) \quad \lim_{h \to 0} \frac{1}{2} \int_0^{+\infty} \left( \int_D \frac{\omega(t-h) K_b[\omega(t-h)] - \omega(t) K_b[\omega(t)]}{h} \right) \theta(t) \, dt = \int_0^{+\infty} \left( \int_D \omega(t) \partial_t K_b[\omega](t) \right) \theta(t) \, dt.
\]
The conclusion follows from (2.14), (2.15), (2.16) and (2.17). \hfill \square

**Proof of proposition 2.9.** We consider a function \( \theta \in C_c^\infty([0, +\infty)) \). We want to prove that
\[
E(0) \theta(0) + \int_0^{+\infty} \theta'(t) E(t) \, dt = 0.
\]
We rely on the energy formula of proposition 2.5. We first have by lemma 2.10
\[
\frac{1}{2} \int_0^{+\infty} \int_D \omega(t) K_b[\omega(t)] \theta'(t) \, dt = \frac{\theta(0)}{2} \int_D \omega_0 K_b[\omega_0] + \int_0^{+\infty} \int_D \omega(t) \partial_t K_b[\omega](t) \theta(t) \, dt,
\]
since by Leibniz’s rule \( \partial_t (K_b[\omega])\theta(t) = \partial_t (K_b[\omega](t) \theta(t)) + K_b[\omega(t)] \theta'(t) \). By definition of weak solution of the lake equations in the vorticity formulation (definition 2.1 (iii)), we have then
\[
(2.18) \quad \frac{1}{2} \int_0^{+\infty} \int_D \omega(t) K_b[\omega(t)] \theta'(t) \, dt = -\frac{1}{2} \int_D \omega_0 K_b[\omega_0] - \int_0^{+\infty} \int_D \omega(t) \mathbf{u}(t) \cdot \nabla K_b[\omega(t)] \theta(t) \, dt,
\]
and, for every \( i \in \{1, \ldots, m\}, \)
\[
(2.19) \quad \int_0^{+\infty} \int_D \omega(t) \psi_i \theta'(t) \, dt = -\theta(0) \int_D \omega_0 \psi_i - \int_0^{+\infty} \int_D \omega(t) \mathbf{u}(t) \cdot \nabla \psi_i(t) \theta(t) \, dt.
\]
This implies thus, by combining proposition 2.5 with the identities (2.18) and (2.19) and the velocity reconstruction formula (2.7), that
\[
E(0) \theta(0) + \int_0^{+\infty} \theta'(t) E(t) \, dt = \int_0^{+\infty} \left( \int_D \omega(t) \mathbf{u}(t) \cdot \mathbf{u}(t) \right) \theta(t) \, dt = 0. \quad \square
\]

3. **Velocity reconstruction expansion**

In the sequel, we will need to understand the behaviour of the operator \( K_b \) appearing in the construction in proposition 2.4 of the stream function satisfying (2.2). In view of proposition 2.4 this can be done through the study of the operator \( G_b \) associated to the solution of the Dirichlet problem (2.3), whose existence was given in proposition 2.2.
3.1. Construction of the Green function. We represent the Green function of the Dirichlet problem (2.3) as a perturbation of the Green operator of the classical Laplacian on the same domain with Dirichlet boundary conditions.

Proposition 3.1. There exists a function $S_b \in C^{0,1}(D \times D)$ such that for every $\omega \in L^p(D)$ and every $x \in D$,

$$G_b[\omega](x) = \int_D \left( G_D(x,y) \sqrt{b(x)b(y)} + S_b(x,y) \right) \omega(y) \, dy.$$  

Here $G_D : D \times D \to \mathbb{R}$ is the Green function of the Laplacian $-\Delta$ with Dirichlet boundary conditions on the boundary $\partial D$, that is, if $f \in L^p(D)$ and if

$$u(x) = \int_D G_D(x,y)f(y) \, dy,$$

then

$$\begin{cases}
-\Delta u = f & \text{in } D, \\
u = 0 & \text{on } \partial D.
\end{cases}$$

In particular, proposition 3.1 implies that the weighted Dirichlet problem (2.3) has a Green function $G_b : D \times D \to \mathbb{R}$ defined for each $x, y \in D$ with $x \neq y$ by

$$G_b(x,y) \triangleq G_D(x,y) \sqrt{b(x)b(y)} + S_b(x,y),$$

and thus the stream function problem (2.2) also has a Green function in view of proposition 2.4.

The proof of proposition 3.1 will rely on the fundamental estimate, which is a classical consequence of the maximum principle for the Laplacian operator $-\Delta$.

Proposition 3.2. For every $x, y \in D$

$$0 \leq G_D(x,y) \leq \frac{1}{2\pi} \ln \frac{\text{diam}(D)}{|x-y|}.$$  

Proof of proposition 3.1. For each $y \in D$, let $S_b(\cdot,y) \in W_0^{1,2}(D)$ be the unique weak solution to the Dirichlet problem

$$\begin{cases}
-\nabla \cdot (b^{-1} \nabla S_b(\cdot,y)) = -G_D(\cdot,y) \sqrt{b(y)} \left( \frac{1}{\sqrt{b}} \right) & \text{in } D, \\
S_b(\cdot,y) = 0 & \text{on } \partial D.
\end{cases}$$

Since $b \in C^2(\overline{D},(0,\infty))$, by classical elliptic regularity estimates (see for example [21, theorem 9.15]), for every $y \in D$ we have $S_b(\cdot,y) \in W^{2,p}(D)$ for every $p \in (1,\infty)$ and

(3.1)  

$$\|S_b(\cdot,y)\|_{W^{2,p}(D)} \leq C_1\|G_D(\cdot,y)\|_{L^p(D)}.$$  

By proposition 3.2 we have

$$\|G_D(\cdot,y)\|_{L^p(D)} \leq \left( \int_{B(0, \text{diam}(D))} \left( \ln \frac{\text{diam } D}{|z|} \right)^p \, dz \right)^{\frac{1}{p}} \leq C_2.$$
It follows in particular by the classical Sobolev embedding theorem and by (3.1) that
\[
\sup_{y \in D} \| \nabla S_b(\cdot, y) \|_{L^\infty(D)} < +\infty.
\]

Finally, we observe that if \( \omega_1, \omega_2 \in L^p(D) \), we have
\[
\int_D \omega_1 G_b[\omega_2] - \iint_{D \times D} G(x, y) \omega_1(x) \omega_2(y) \sqrt{b(x) b(y)} \, dx \, dy = \int_D \omega_2 G_b[\omega_1] - \iint_{D \times D} G(x, y) \omega_2(x) \omega_1(y) \sqrt{b(x) b(y)} \, dx \, dy
\]
and therefore
\[
\iint_{D \times D} S_b(x, y) \omega_1(x) \omega_2(y) \, dx \, dy = \iint_{D \times D} S_b(x, y) \omega_2(x) \omega_1(y) \, dx \, dy.
\]
It follows that for every \( x, y \in D \), \( S_b(x, y) = S_b(y, x) \), and thus the function \( S_b \) is Lipschitz-continuous on \( D \times D \). □

As a consequence of proposition 3.1, the velocity field \( u \) admits the integral representation

**Proposition 3.3.** There exists a Lipschitz-continuous function \( R_b \in C^{0,1}(D \times D) \) such that for every \( \omega \in L^p(D) \), one has
\[
\mathcal{K}_b[\omega](x) = \int_D \left( G_D(x, y) \sqrt{b(x) b(y)} + R_b(x, y) \right) \omega(y) \, dy.
\]

**Proof of proposition 3.3.** This follows from proposition 2.4 and proposition 3.1 with \( S_b = R_b + Q_b \). □

### 3.2. Estimate on the Green function

We will also need a version of proposition 3.2 which is sharper close to the boundary.

**Proposition 3.4.** There exists a constant \( C \) such that for every \( x, y \in D \),
\[
G_D(x, y) \leq \frac{1}{4\pi} \ln \left( 1 + C \frac{\text{dist}(x, \partial D) \text{dist}(y, \partial D)}{|x - y|^2} \right).
\]

**Proof.** This can be obtained by observing that for the unit disk \( \mathbb{D}^2 \subset \mathbb{R}^2 \), one has for each \( x, y \in \mathbb{D}^2 \) such that \( x \neq y \),
\[
G_{\mathbb{D}^2}(x, y) = \frac{1}{4\pi} \ln \left( 1 + \frac{(1 - |x|^2)(1 - |y|^2)}{|x - y|^2} \right),
\]
and by applying conformal mapping techniques as in the proof of proposition 3.7. □
3.3. Gradient estimates of Green function of the Laplacian. Our goal now is to obtain estimates on the derivative of the Green function $G_D$. A first classical estimate is available [4].

**Proposition 3.5.** There exists a constant $C$ such that for every $x, y \in D$, one has
\[
|\nabla G_D(x, y)| \leq \frac{C}{|x - y|}.
\]

We will need a more refined directional information about the Green function of the Laplacian. We observe that in view of the definition of the regular part $H_D : D \times D \to \mathbb{R}$ for $x, y \in D$ such that $x \neq y$ as
\[
H_D(x, y) \triangleq G_D(x, y) - \frac{1}{2\pi} \ln \frac{1}{|x - y|},
\]
we have for every $x, y \in D$ such that $x \neq y$
\[
\nabla G_D(x, y) + \nabla G_D(y, x) = \nabla H_D(x, y) + \nabla H_D(y, x).
\]
Here above, $\nabla G_D$ denotes the gradient of $G_D$ with respect to its first variable.

**Proposition 3.6** (Interior symmetric gradient estimate). For every $\delta > 0$, there exist $C > 0$ such that if $\text{dist}(x, \partial D) + \text{dist}(y, \partial D) + |x - y| \geq \delta$,
\[
|\nabla G_D(x, y) + \nabla G_D(y, x)| \leq C.
\]

**Proof.** This follows from (3.3) and the smoothness of the regular $H_D$ part of the Green function defined in (3.2). □

We now investigate what the estimate of proposition 3.6 becomes near the boundary $\partial D$. We start by observing the Green function of the Laplacian on the half-plane $\mathbb{R}^2_+ = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 > 0\}$, which is given for each $x = (x_1, x_2), y = (y_1, y_2) \in D$ by
\[
G_{\mathbb{R}^2_+}(x, y) = \frac{1}{4\pi} \ln \left(1 + \frac{4x_2y_2}{|x - y|^2}\right).
\]
The gradient of this function with respect to its first variable, is then given by
\[
\nabla G_{\mathbb{R}^2_+}(x, y) = \frac{1}{\pi(|x - y|^2 + 4x_2y_2)} \left((0, y_2) - 2x_2y_2 \frac{x - y}{|x - y|^2}\right).
\]
One computes then that
\[
\nabla G_{\mathbb{R}^2_+}(x, y) + \nabla G_{\mathbb{R}^2_+}(y, x) = \frac{(0, x_2 + y_2)}{\pi(|x - y|^2 + 4x_2y_2)}.
\]
A notable feature of (3.4) is the vanishing of the tangential component.

We are going to extend formula (3.4) to any bounded domain, simply or non simply connected. To do this, we are going to show that (3.4) holds in a disk. From there, we are going to show that we can conformally transform $D_0$ to a disk, and find back similar estimates near $\partial D_0$, that is: near the boundary of $D$ that does not correspond to islands. The estimate would also hold in near every connected component of $\partial D$, after transformation via a conformal map of the form $z \in \mathbb{C} \setminus \{0\} \mapsto 1/z$. 

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We compute \( \frac{\nabla G_D(x, y) + \nabla G_D(y, x) - \frac{x - P_\partial D(x) + y - P_\partial D(y)}{\pi(|x - y|^2 + 4 \text{dist}(x, \partial D) \text{dist}(y, \partial D))}}{\pi(|x - y|^2 + 4 \text{dist}(x, \partial D) \text{dist}(y, \partial D))} \leq C. \)

**Proof.** We start by observing the Green function for the Laplacian on the disk \( \mathbb{D}^2 \subset \mathbb{R}^2 \), which is defined for each \( \tilde{x}, \tilde{y} \in \mathbb{D}^2 \) by
\[
G_{\mathbb{D}^2}(\tilde{x}, \tilde{y}) = \frac{1}{4\pi} \ln \left( 1 + \frac{(1 - |\tilde{x}|^2)(1 - |\tilde{y}|^2)}{|\tilde{x} - \tilde{y}|^2} \right).
\]
The gradient of this function, for fixed \( \tilde{y} \in \mathbb{D}^2 \), is then given by
\[
\nabla G_{\mathbb{D}^2}(\tilde{x}, \tilde{y}) = -\frac{1}{2\pi(|\tilde{x} - \tilde{y}|^2 + (1 - |\tilde{x}|^2)(1 - |\tilde{y}|^2))} \left( \tilde{x}(1 - |\tilde{y}|^2) + \frac{(1 - |\tilde{y}|^2)(1 - |\tilde{x}|^2)}{|\tilde{x} - \tilde{y}|^2} \right),
\]
and thus we have
\[
\nabla G_{\mathbb{D}^2}(\tilde{x}, \tilde{y}) + \nabla G_{\mathbb{D}^2}(\tilde{y}, \tilde{x}) = -\frac{\tilde{x}(1 - |\tilde{y}|^2) + \tilde{y}(1 - |\tilde{x}|^2)}{2\pi(|\tilde{x} - \tilde{y}|^2 + (1 - |\tilde{x}|^2)(1 - |\tilde{y}|^2))}.
\]
We observe that
\[
(1 - |\tilde{x}|^2)(1 - |\tilde{y}|^2) = (1 - |\tilde{x}|)(1 - |\tilde{y}|)(2 - (1 - |\tilde{x}|))(2 - (1 - |\tilde{y}|)) = 4(1 - |\tilde{x}|)(1 - |\tilde{y}|) + O((1 - |\tilde{x}|)^2 + (1 - |\tilde{y}|)^2)
\]
and
\[
\tilde{x}(1 - |\tilde{y}|^2) + \tilde{y}(1 - |\tilde{x}|^2) = (\tilde{x} - \tilde{y})(|\tilde{y}|^2 - |\tilde{x}|^2) + 2 \left( \frac{\tilde{x}}{|\tilde{x}|} + \frac{\tilde{y}}{|\tilde{y}|} \right) + \tilde{x}(|\tilde{x}| - 1)^2 \left( 1 + \frac{2}{|\tilde{x}|} \right) + \tilde{y}(|\tilde{y}| - 1)^2 \left( 1 + \frac{2}{|\tilde{y}|} \right)
\]
\[
= 2 \left( \frac{\tilde{x}}{|\tilde{x}|} + \frac{\tilde{y}}{|\tilde{y}|} \right) + O(|\tilde{x} - y|^2 + (1 - |\tilde{x}|)^2 + (1 - |\tilde{y}|)^2).
\]
It thus follows that when \(|x - y|^2 + (1 - |\tilde{x}|)^2 + (1 - |\tilde{y}|)^2\) is small enough,
\[
(3.5) \quad \left| \nabla G_{\mathbb{D}^2}(\tilde{x}, \tilde{y}) + \nabla G_{\mathbb{D}^2}(\tilde{y}, \tilde{x}) - \frac{\tilde{x} - \tilde{y}}{|\tilde{x} - \tilde{y}|^2 + 4(1 - |\tilde{x}|)(1 - |\tilde{y}|)} \right| \leq C_1.
\]

By the classical Riemann mapping theorem (see for example [30] theorems 4.0.1 and 5.2.1), there exists a map \( \Phi \in C^2(D_0, \mathbb{D}^2) \) which is a diffeomorphism up to the boundary and which is conformal map. For each \( i \in \{1, \ldots, m\} \), we have \( \Phi(I_i) \cap \partial \mathbb{D}^2 = \emptyset. \)

We define the function \( \tilde{G}_D \) for each \( x, y \in D \) as
\[
\tilde{G}_D(x, y) \triangleq G_{\mathbb{D}^2}(\Phi(x), \Phi(y)).
\]
We compute
\[
\nabla \tilde{G}_D(x, y) = (D\Phi(x))^* [\nabla G_{\mathbb{D}^2}(\Phi(x), \Phi(y))],
\]
and thus
\[
\nabla \tilde{G}_D(x, y) + \nabla \tilde{G}_D(y, x) = (D\Phi(x))^* [\nabla G_{D^2}(\Phi(x), \Phi(y)) + \nabla G_{D^2}(\Phi(y), \Phi(x))] + (D\Phi(y) - D\Phi(x))^*[\nabla G_{D^2}(\Phi(y), \Phi(x))].
\]

We observe that when \(|x - y| + \text{dist}(x, \partial D) + \text{dist}(y, \partial D) \to 0\), we have
\[
\Phi(x) - \frac{\Phi(x)}{|\Phi(x)|} = D\Phi(x)[x - P_{D^1}(x)] + O(\text{dist}(x, \partial D)^2),
\]
\[
\Phi(y) - \frac{\Phi(y)}{|\Phi(y)|} = D\Phi(x)[y - P_{D^1}(y)] + O(\text{dist}(y, \partial D)^2 + |x - y|^2),
\]
\[
|\Phi(x) - \Phi(y)|^2 = |D\Phi(x)|^2|x - y|^2 + O(|x - y|^3),
\]
\[
(1 - |\Phi(x)|) = \frac{\text{dist}(x, \partial D)}{|D\Phi(x)|} + O((\text{dist} x, \partial D)^2),
\]
\[
(1 - |\Phi(y)|) = \frac{\text{dist}(y, \partial D)}{|D\Phi(x)|} + O((\text{dist} y, \partial D)^2 + |x - y|^2),
\]

from which we deduce that
\[
(D\Phi(x))^*[\nabla G_{D^2}(\Phi(x), \Phi(y)) + \nabla G_{D^2}(\Phi(y), \Phi(x))] - \frac{x - P_{D^1}(x) + y - P_{D^1}(y)}{\pi(|x - y|^2 + 4 \text{dist}(x, \partial D) \text{dist}(y, \partial D))} \leq C_2.
\]

We also have immediately
\[
|(D\Phi(y) - D\Phi(x))^*[\nabla G_{D^2}(\Phi(y), \Phi(x))| \leq C_3.
\]

Now we draw the link between the transport \(\tilde{G}_D\) and the Green’s function \(G_D\) we target.

Since the map \(\Phi\) is conformal, the function \(G_{D^2}(\cdot, y)\) is harmonic in \(\mathbb{D}^2 \setminus \{y\}\) and there exists \(\delta > 0\) such that if \(\text{dist}(y, \partial D) \leq \delta\), then \(\Theta(\cdot, y) \equiv G_D(\cdot, y) - \tilde{G}_D(\cdot, y)\) is bounded uniformly in a neighborhood of \(\bigcup_{i=1}^m I_i\); this implies that for every \(x, y \in D\) such that \(\text{dist}(x, \partial D) + \text{dist}(y, \partial D) \leq \delta\), we have
\[
|\nabla \Theta(x, y)| \leq C_4.
\]

The conclusion in when \(\text{dist}(x, \partial D) + \text{dist}(y, \partial D) + |x - y| \leq \delta\) follows by combining the estimates (3.6), (3.7) and (3.8). The other components \(I_1, \ldots, I_m\) of the boundary can be reduced to this case by a suitable adaptation of the conformal mapping \(z \in \mathbb{C} \setminus \{0\} \mapsto 1/z\).

4. Vortex estimates

In this section we derive several estimates on the vorticity that govern the concentration of the vorticity.
In order to control the shape of the vortex, we will recurrently rely on the Lorentz norm \([36]\) of a vorticity \(\omega : D \to \mathbb{R}\) which will be defined as

\[
\|\omega\|_{\Lambda_{(ln)}+1} \triangleq \sup\left\{ \int_{\mathbb{R}^2} \left( \ln \frac{1}{|x|} \right) \omega(x) \, dx \mid \omega : \mathbb{R}^2 \to \mathbb{R} \text{ and for every } \lambda > 0 , \right. \\
\left. |\{x \in D \mid |\omega(x)| > \lambda\}| = |\{x \in \mathbb{R}^2 \mid |\tilde{\omega}(x)| > \lambda\}| \right\}.
\]

We use the Lebesgue measure in the definition, despite the fact that the flow transports the measure with density \(b\) of the potential vorticity \(\omega/b\).

By the Hardy–Littlewood rearrangement inequality (see for example \([34, \text{ theorem 3.4}]\)), the supremum in (4.1) is actually reached by the radially symmetric nonincreasing rearrangement \(\omega^*\) of \(\omega\), whose superlevel sets are balls centered on 0.

4.1. Stream function estimate. We first show how the Lorentz norm can be used to obtain a bound on the stream function.

**Proposition 4.1** (Boundedness of the stream function). There exists a constant \(C > 0\) that depends only on \(D\) and \(b\), such that for every non-negative function \(\omega \in L^p(D)\) and every \(\rho > 0\), we have

\[
\|K_b[\omega]\|_{L^\infty(D)} \leq \frac{1}{2\pi} \int_D \ln \frac{1}{\max(\rho, |x-y|)} \omega(y) b(y) \, dy + \frac{(\sup_D b) \rho^2 \|\omega(\rho)\|_{\Lambda_{(ln)}+1}}{2\pi} + C \int_D \omega.
\]

**Proof.** By writing the function \(K_b[\omega]\) in terms of integral kernels of proposition \([33]\), we have for each \(x \in D\),

\[
0 \leq K_b[\omega](x) = \int_D G_D(x,y) \omega(y) b(y) \, dy + \int_D R_b(x,y) \omega(y) \, dy \\
+ \int_D G_D(x,y) \omega(y) \left( \sqrt{b(x)} - \sqrt{b(y)} \right) \sqrt{b(y)} \, dy.
\]

By proposition \([3, 3]\) the function \(R_b\) is uniformly bounded. According to proposition \([3, 2]\) and the fact that the function \(\sqrt{b}\) is Lipschitz-continuous, the term

\[
G_D(x,y) \left( \sqrt{b(x)} - \sqrt{b(y)} \right) \sqrt{b(y)}
\]

is uniformly bounded as \((x,y) \in D \times D\). Moreover, using the direct estimate of proposition \([3, 2]\) we obtain for every \(x \in D\)

\[
\int_D G_D(x,y) \omega(y) b(y) \, dy \leq \frac{1}{2\pi} \int_D \ln \frac{\diam D}{|x-y|} \omega(y) b(y) \, dy.
\]

Since for every \(x, y \in \mathbb{R}^2\) such that \(x \neq y\),

\[
\ln \frac{\diam D}{|x-y|} = \ln \frac{\diam D}{\max(\rho, |x-y|)} + \left( \ln \frac{\rho}{|x-y|} \right)_+,
\]

\[
\ln \frac{\diam D}{|x-y|} = \ln \frac{\diam D}{\max(\rho, |x-y|)} + \left( \ln \frac{\rho}{|x-y|} \right)_+,
\]

\[
\int_D G_D(x,y) \omega(y) b(y) \, dy \leq \frac{1}{2\pi} \int_D \ln \frac{\diam D}{|x-y|} \omega(y) b(y) \, dy.
\]

Since for every \(x, y \in \mathbb{R}^2\) such that \(x \neq y\),
we have then for every $x \in D$,
\[
\mathcal{K}_b[\omega](x) \leq \frac{1}{2\pi} \int_D \ln \frac{1}{\max(\rho,|x-y|)} \omega(y) b(y) \, dy \\
+ \frac{(\sup_{D} b) \rho^2}{2\pi} \int_{x+\rho \in D} \left( \ln \frac{1}{|z|} \right) + \omega(x+\rho z) \, dz + C \int_D \omega,
\]
for some constant $C > 0$ that depends only on $D$ and $b$. The conclusion now follows from the definition of the Lorentz norm \[\|\cdot\|\] .

4.2. **Energy concentration estimate.** The following proposition gives estimates on the kinetic energy outside a ball.

**Proposition 4.2.** There exists a constant $C > 0$ that depends only on $D$ and $b$, such that for every non-negative function $\omega \in L^p(D)$ and every $R, r, \rho > 0$ such that $R \geq r + \rho$ and $r \geq \rho$, we have

\[
\int_{D \setminus B(a, R)} |\mathbf{u}|^2 \leq C \left( |\Gamma| \ln \frac{1}{\rho} \int_{D \setminus B(a, r)} \omega + |\Gamma|^2 \ln \frac{1}{R-r} + \rho^2 \|\omega(\rho)\|_{\Lambda_{(\ln)+1}} |\Gamma| + |\Gamma|^2 \right).
\]

Here and in the sequel, we use the notation
\[
|\Gamma| \triangleq |\Gamma| + \sum_{i=1}^m |\Gamma_i|.
\]

**Proof of proposition 4.2.** We define the function $\phi \triangleq \mathcal{K}_b[\omega] + 2 \sum_{i=1}^m \Gamma_i \psi_i$. For every $x \in D \setminus B(a, R)$, we have if $y \in B(a, r)$, $|x-y| \geq |x-a| - |y-a| \geq R-r \geq \rho$, and thus by proposition 4.1,

\[
\phi(x) \leq \frac{1}{2\pi} \ln \frac{1}{\rho} \int_{D \setminus B(a, r)} \omega b + \frac{1}{2\pi} \ln \frac{1}{R-r} \int_{B(a, r)} \omega b + C_1(\rho^2 \|\omega(\rho)\|_{\Lambda_{(\ln)+1}} + |\Gamma|)
\]
\[
\leq \lambda \triangleq C_2 \left( \ln \frac{1}{\rho} \int_{D \setminus B(a, r)} \omega + |\Gamma| \ln \frac{1}{R-r} + \rho^2 \|\omega(\rho)\|_{\Lambda_{(\ln)+1}} + |\Gamma| \right).
\]

We have by the representation formula for the velocity field (2.7), as in (2.8) in the proof of proposition 2.3

\[
\int_{D \setminus B(a, R)} |\mathbf{u}|^2 \leq \int_{\phi^{-1}((0,\lambda))} |\mathbf{u}|^2 \\
= \frac{1}{2} \int_{\phi^{-1}((0,\lambda))} \nabla \mathcal{K}_b[\omega] \cdot \nabla \phi \bigg/ \frac{b}{b} + \sum_{i,j=1}^m \frac{\Gamma_i \Gamma_j}{2} \int_{\phi^{-1}((0,\lambda))} \nabla \psi_i \cdot \nabla \psi_j \bigg/ \frac{b}{b}.
\]

We have then,

\[
\frac{1}{2} \int_{\phi^{-1}((0,\lambda))} \nabla \mathcal{K}_b[\omega] \cdot \nabla \phi \bigg/ \frac{b}{b} = \frac{1}{2} \int_D \nabla \mathcal{K}_b[\omega] \cdot \nabla \max(\phi, \lambda) = \frac{1}{2} \int_D \omega \max(\phi, \lambda) \leq \frac{|\Gamma| \lambda}{2}.
\]

On the other hand, we have

\[
\sum_{i,j=1}^m \frac{\Gamma_i \Gamma_j}{2} \int_{\phi^{-1}((0,\lambda))} \nabla \psi_i \cdot \nabla \psi_j \bigg/ \frac{b}{b} \leq C_3 |\Gamma|^2.
\]
The conclusion follows from the combination of (4.5), (4.6) and (4.7), in view of the definition of \( \lambda \) in (4.4).

4.3. Vortex concentration. The next estimate shows that there is a characteristic radius \( \rho \) defined in terms of conserved quantities such that if the Lorentz norm at the scale \( \rho \) remains bounded, then the vorticity is concentrated in a region of radius comparable to \( \rho \).

**Proposition 4.3** (Concentration estimate). There exists a constant \( C > 0 \) that depends only on \( D, b \), such that for all \( R > 1 \), we have

\[
\inf_{\alpha \in D} \int_{D \setminus B(\alpha, R\rho)} \omega \leq \frac{C}{\ln(R)} \left( \rho^2 \| \omega(\rho) \|_{L(\ln)^{+},1} + \frac{\| \Gamma \|_2^2}{\| \Gamma \|} \right),
\]

where

\[
\rho \triangleq \exp \left( - \frac{4\pi E}{\Gamma \Omega} \right).
\]

Here and in the sequel, we reserve the symbol \( \rho \) to refer to the quantity defined in (4.8).

The proof of proposition 4.3 follows ideas introduced by Turkington [53] and Turkington & Friedmann [20] for the Euler equations. It was also used in chapters 5 and 6 in the study of steady solutions of the lake equations by energy maximization [14, 15].

**Proof of proposition 4.3.** We define the set

\[
A \triangleq \left\{ x \in D \mid \psi(x) \geq \frac{1}{\Gamma} \int_D \psi \omega \right\},
\]

in terms of the stream function \( \psi \triangleq \mathcal{K}_b[\omega] + \sum_{i=1}^k \Gamma_i \psi_i \). We observe that by definition of the vortex circulation \( \Gamma \) in (4.4), the set \( A \) is not empty.

By proposition 3.3 and proposition 3.2, since by assumption the function \( b \) is Lipschitz-continuous, we have for every \( x \in D \),

\[
\mathcal{K}_b[\omega](x) \leq \frac{1}{2\pi} \int_D \ln \frac{\text{diam } D}{|x-y|} \sqrt{b(x)} \sqrt{b(y)} \mathrm{d}y + \int_D R_b(x, y) \omega(y) \mathrm{d}y
\]

\[
\leq \frac{1}{2\pi} \ln \frac{1}{\rho} \int_D \omega b + \frac{1}{2\pi} \int_D \ln \frac{\rho}{|x-y|} \omega(y) b(y) \mathrm{d}y + C_1 \| \Gamma \|,
\]

and thus by definition of \( \psi \) and of \( \| \Gamma \| \) in (4.3),

\[
\psi(x) \leq \frac{1}{2\pi} \ln \frac{1}{\rho} \int_D \omega b + \frac{1}{2\pi} \int_D \ln \frac{\rho}{|x-y|} \omega(y) b(y) \mathrm{d}y + C_2 \| \Gamma \|.
\]

On the other hand, setting \( \rho \) to the value given by (4.8), we obtain for each \( x \in A \), in view of proposition 2.5 and the definition of the set \( A \) by (4.9)

\[
\psi(x) \geq \frac{2E}{\Gamma} - \sum_{i=1}^m \frac{\Gamma_i}{\Gamma} \int_D \omega \psi_i - \sum_{i,j=1}^m \frac{\Gamma_i \Gamma_j}{\Gamma} \int_D \nabla \psi_i \cdot \nabla \psi_j \geq \frac{1}{2\pi} \ln \frac{1}{\rho} \int_D \omega b - C_3 \frac{\| \Gamma \|_2^2}{\| \Gamma \|}.
\]

The combination of (4.10) and (4.11), shows that for each \( x \in A \),

\[
\frac{1}{2\pi} \int_D \ln \frac{|x-y|}{\rho} \omega(y) b(y) \mathrm{d}y \leq C_4 \frac{\| \Gamma \|_2^2}{\| \Gamma \|}.
\]
In order to conclude, we start from the inequality
\[
\ln(R) \inf_{D \setminus B(x, R\rho)} \omega(y) \, dy \leq \frac{1}{4\pi} \int_{D \setminus B(x, R\rho)} \ln \frac{|x - y|}{\rho} \omega(y) \, b(y) \, dy.
\] (4.13)

We also observe that
\[
\frac{1}{4\pi} \int_{D \cap B(x, R\rho)} \ln \frac{\rho}{|x - y|} \omega(y) \, b(y) \, dy \leq \frac{\rho^2 \sup_D b}{4\pi} \int_D \left( \ln \frac{1}{|z|} \right) + \omega(x + \rho z) \, dz \leq C_5 \rho^2 \|\omega(\rho \cdot \cdot)\|_{\Lambda_{(\ln)^+}},
\] (4.14)
in view of the definition of the Lorentz norm (4.1). Therefore, in view of (4.12), (4.13) and (4.14), we have for some constant \(C_6 > 0\) and for each \(x \in A\) and \(R > 1\):
\[
\int_{D \setminus B(x, R\rho)} \omega \leq \frac{C_6}{\ln(R)} \left( \rho^2 \|\omega(\rho \cdot \cdot)\|_{\Lambda_{(\ln)^+}} + \frac{\|\Gamma\|^2}{|\Gamma|} \right).
\]

4.4. **Boundary repulsion.** The next estimate shows that the vorticity \(\omega\) cannot be concentrated too much in a neighborhood of the boundary when \(\rho\) is small.

**Proposition 4.4.** If \(\rho\) satisfies (4.8), then
\[
\int_D \omega(x) \ln \frac{1}{\rho + C \text{dist}(x, \partial D)} \, dx \leq C \left( \rho^2 \|\omega(\rho \cdot \cdot)\|_{\Lambda_{(\ln)^+}} + \frac{\|\Gamma\|^2}{|\Gamma|} \right).
\]

**Proof.** By the energy identity of proposition 2.5, the decomposition of the Green function of proposition 3.3 and the upper bound on the Green function of the Laplacian of proposition 3.4, we have
\[
E = \frac{1}{2} \int_D \omega K_b[\omega] + \sum_{i=1}^m \Gamma_i \int_D \psi_i \omega + \sum_{i,j=1}^m \frac{\Gamma_i \Gamma_j}{2} \int_D \nabla \psi_i \cdot \nabla \psi_j \leq I + C_1 \|\Gamma\|^2,
\] (4.15)
where we have set
\[
I \triangleq \frac{1}{8\pi} \iint_{D \times D} \ln \left( 1 + C_2 \frac{\text{dist}(x, \partial D) \cdot \text{dist}(y, \partial D)}{|x - y|^2} \right) \omega(x) \omega(y) \sqrt{b(x) b(y)} \, dx \, dy
\]
which can be bounded as
\[
I \leq \sup_D b \frac{1}{8\pi} \iint_{D \times D} \ln \left( \rho^2 + C_2 \rho^2 \frac{\text{dist}(x, \partial D) \cdot \text{dist}(y, \partial D)}{|x - y|^2} \right) \omega(x) \omega(y) \, dx \, dy
\] (4.16)
\[
+ \frac{\ln \rho}{4\pi} \iint_{D \times D} \omega(x) \omega(y) \sqrt{b(x) b(y)} \, dx \, dy.
\]
The integral of the second term on the right-hand side of (4.16) can be bounded by the Cauchy–Schwarz inequality, as

\[
(4.17) \int_{D \times D} \omega(x) \omega(y) \sqrt{b(x) b(y)} \, dx \, dy 
\leq \left( \int_{D \times D} \omega(x) \omega(y) \, dx \, dy \right)^{1/2} \left( \int_{D \times D} \omega(x) \omega(y) b(x) b(y) \, dx \, dy \right)^{1/2} = \Omega \Gamma,
\]

which leads to

\[
(4.18) \frac{\ln \frac{1}{\rho}}{4\pi} \int_{D \times D} \omega(x) \omega(y) \sqrt{b(x) b(y)} \, dx \, dy \leq \frac{\Omega \Gamma}{4\pi} \ln \frac{1}{\rho} = E,
\]

by definition of \( \rho \) (see (4.8)). From (4.15), (4.16) and (4.18) we infer that

\[
(4.19) -C_1 \| \Gamma \|^2 \leq \frac{C}{8\pi} \int_{D \times D} \ln \left( \rho^2 + C_2 \frac{\rho^2 \text{dist}(x, \partial D) \text{dist}(y, \partial D)}{|x-y|^2} \right) \omega(x) \omega(y) \, dx \, dy.
\]

We split the integral of the right-hand side of (4.19) in two regions, depending on whether \(|x-y| \leq \rho \). For the first one we have, since the domain \( D \) is bounded, we have for every \( x, y \in D \),

\[
(4.20) |x-y|^2 + C_2 \text{dist}(x, \partial D) \text{dist}(y, \partial D) \leq C_3
\]

and thus by (4.20) and by the definition of the Lorentz norm in (4.1)

\[
(4.21) \int_{(x,y) \in D \times D, |x-y| \leq \rho} \ln \left( \rho^2 + C_2 \frac{\rho^2 \text{dist}(x, \partial D) \text{dist}(y, \partial D)}{|x-y|^2} \right) \omega(x) \omega(y) \, dx \, dy 
\leq 2 \int_{(x,y) \in D \times D, |x-y| \leq \rho} \ln \frac{\sqrt{C_3 \rho}}{|x-y|} \omega(x) \omega(y) \, dx \, dy 
\leq C_4 \left( \rho^2 \left\| \omega(\rho) \right\|_{L^1(\mathbb{R}^3)} + |\Gamma| \right) |\Gamma|;
\]
for the second part we have
\[
\iint_{(x,y)\in D\times D\atop |x-y|\geq \rho} \ln \left( \rho^2 + C_2 \frac{\rho^2 \text{dist}(x, \partial D) \text{dist}(y, \partial D)}{|x-y|^2} \right) \omega(x) \omega(y) \, dx \, dy
\]
\[
\leq \int_{D\times D} \ln(\rho^2 + C_2 \text{dist}(x, \partial D) \text{dist}(y, \partial D)) \omega(x) \omega(y) \, dx \, dy
\]
\[
\leq \int_{D\times D} \ln((\rho + \sqrt{C_2} \text{dist}(x, \partial D))(\rho + \sqrt{C_2} \text{dist}(y, \partial D))) \omega(x) \omega(y) \, dx \, dy
\]
\[
\leq 2 \int_{D\times D} \ln(\rho + \sqrt{C_2} \text{dist}(x, \partial D)) \omega(x) \omega(y) \, dx \, dy
\]
\[
= 2 |\Gamma| \int_{D} \ln(\rho + \sqrt{C_2} \text{dist}(x, \partial D)) \omega(x) \, dx.
\]
By (4.19), (4.21) and (4.22), we deduce then that
\[
\int_{D} \ln \frac{1}{\rho + \sqrt{C_2} \text{dist}(x, \partial D)} \omega(x) \, dx \leq C_5 \left( \rho^2 \|\omega(\rho)\|_{\Lambda(\ln+1)} + \frac{\|\Gamma\|^2}{|\Gamma|} \right),
\]
and the conclusion follows. \(\square\)

4.5. Center of vorticity. We define now the center of vorticity
\[
q \triangleq \frac{1}{\Gamma} \int_{D} x \omega(x) \, dx
\]
and we prove that concentration occurs in fact around the center of vorticity, as a consequence of proposition 4.3.

**Proposition 4.5** (Concentration around the center of vorticity). There exists constants \(C, C' > 0\) such that, for all \(R > 1\), we have
\[
\int_{D\setminus B(q, \rho_*(\rho, R))} \omega \leq \frac{C}{\ln(R)} \left( \rho^2 \|\omega(\rho)\|_{\Lambda(\ln+1)} + \frac{\|\Gamma\|^2}{|\Gamma|} \right),
\]
with \(\rho\) defined in (1.8) and
\[
\rho_*(\rho, R) = R\rho + \frac{C'}{\ln(R)} \left( \frac{\rho^2 \|\omega(\rho)\|_{\Lambda(\ln+1)}}{|\Gamma|} + \frac{\|\Gamma\|^2}{|\Gamma|^2} \right).
\]

**Proof.** Let \(R > 1\) be any number. By proposition 4.3 there exists a constant \(C_1\) (independent on \(R > 1\)) and some point \(a \in D\) such that
\[
\int_{D\setminus B(a, R\rho)} \omega \leq \frac{C_1}{\ln(R)} \left( \rho^2 \|\omega(\rho)\|_{\Lambda(\ln+1)} + \frac{\|\Gamma\|^2}{|\Gamma|} \right).
\]
We now compute
\[
|q - a| = \frac{1}{|\Gamma|} \int_D (x - a) \omega(x) \, dx \leq \frac{R \rho}{|\Gamma|} \int_{B(a, R \rho)} \omega + \frac{\text{diam} D}{|\Gamma|} \int_{D \setminus B(a, R \rho)} \omega
\]
(4.25)
\leq R \rho + C_2 \ln(R) \left( \frac{\rho^2 \|\omega(\rho)\|_{\Lambda_{(\ln)}^{(1,1)}} + \|\Gamma\|^2}{|\Gamma|^2} \right),

for some other constant \(C_2 > 0\) independent of \(R > 1\) and on \(\omega\). The conclusion follows from (4.25) and (4.24).

As a consequence of proposition 4.4 we estimate the distance between the center of vorticity \(q\) to the boundary \(\partial D\).

**Proposition 4.6** (Confinement of the center of vorticity). *There exists a constant \(C > 0\) such that for all \(R > 1\),
\[
C \left( \frac{\rho^2 \|\omega(\rho)\|_{\Lambda_{(\ln)}^{(1,1)}} + \|\Gamma\|^2}{|\Gamma|^2} \right) \left( \frac{1}{\ln(1 + \frac{1}{\rho + C_2 (\text{dist}(q, \partial D) + \rho_*(R, \rho))})} + \frac{1}{\ln R} \right) \geq 1,
\]
where \(\rho\) defined in (4.8) and \(\rho_*(R, \rho)\) is defined in proposition 4.5.*

**Proof of proposition 4.6.** By proposition 4.5 we have
\[
\int_{D \setminus B(q, \rho_*(R, \rho))} \omega \leq \frac{C_1}{\ln(R)} \left( \rho^2 \|\omega(\rho)\|_{\Lambda_{(\ln)}^{(1,1)}} + \|\Gamma\|^2 \right).
\]
On the other hand by proposition 4.1 we have
\[
\int_{B(q, \rho_*(R, \rho))} \omega \leq \frac{1}{\ln(1 + \frac{1}{\rho + C_2 (\text{dist}(q, \partial D) + \rho_*(R, \rho))})} \int_D \omega(x) \ln \frac{1}{\rho + C_2 \text{dist}(x, \partial D)} \, dx
\]
(4.27)
\leq C_3 \frac{\rho^2 \|\omega(\rho)\|_{\Lambda_{(\ln)}^{(1,1)}} + \|\Gamma\|^2}{|\Gamma|^2}.

The conclusion follows from (4.26), (4.27) and the definition of \(\Gamma\) in (1.4). □

**4.6. Transport of Lorentz norms.** The Lorentz norm that was defined in (4.1) and that appeared in all the concentration estimates of this section, is invariant under transformations that preserve the measure of level sets of \(\omega\), but by (1.2) the flow under the lake equations (1.1) preserves the measure with density \(b\) of level sets of \(\omega/b\) instead. The next proposition shows that the Lorentz norms can be kept in control.

**Proposition 4.7** (Transport of Lorentz norm). *Let \(\omega, \tilde{\omega} : D \to \mathbb{R}\). If for every \(\lambda > 0\),
\[
\int_{\omega(x) > \lambda b(x)} b(x) \, dx = \int_{\tilde{\omega}(x) > \lambda b(x)} b(x) \, dx,
\]
then
\[
\|\tilde{\omega}\|_{\Lambda_{(\ln)}^{(1,1)}} \leq \left( \frac{\sup_D b}{\inf_D b} \right)^2 \|\omega\|_{\Lambda_{(\ln)}^{(1,1)}}.
\]
The proof of proposition 4.7 is based on the following geometrical computation of the Lorentz norm.

**Lemma 4.8.** For every $\omega : \mathbb{R}^2 \to \mathbb{R}$, we have

$$\|\omega\|_{\Lambda_{(\ln^+)}^{1,1}} = 2\pi \int_0^{+\infty} \int_0^{\sqrt{\{[\omega] > \lambda\}/\pi}} r\left(\ln \frac{1}{r}\right) + drd\lambda.$$

**Proof.** By the Hardy–Littlewood rearrangement inequality, we have

$$\int_{\mathbb{R}^2} \left(\ln \frac{1}{|x|}\right)^+ |\omega|(x) dx \leq \int_{\mathbb{R}^2} \left(\ln \frac{1}{|x|}\right)^+ |\omega^*(x)| dx,$$

where $|\omega^*| : \mathbb{R}^2 \to \mathbb{R}$ is a radial function such that for every $\lambda > 0$, $\{|x \in \mathbb{R}^2 | |\omega^*(x) > \lambda|\} = \{|x \in \mathbb{R}^2 | |\omega(x) > \lambda|\}$. We compute then

$$\int_{\mathbb{R}^2} \left(\ln \frac{1}{|x|}\right)^+ |\omega^*(x)| dx = \int_0^{+\infty} \int_0^{\sqrt{\{[\omega^*] > \lambda\}/\pi}} r\left(\ln \frac{1}{r}\right) + drd\lambda$$

$$= 2\pi \int_0^{+\infty} \int_0^{\sqrt{\{[\omega] > \lambda\}/\pi}} r\left(\ln \frac{1}{r}\right) + drd\lambda.$$

**Proof of proposition 4.7.** We have for every $\lambda > 0$, by assumption

$$|\{\omega > \lambda\}| \leq \frac{1}{m} \int_{\{\omega/b > \lambda/M\}} b = \frac{1}{m} \int_{\{\omega/b > \lambda/M\}} b \leq \alpha |\{|\omega | > \lambda/\alpha|\},$$

where $m = \inf_D b$, $M = \sup_D b$ and $\alpha = M/m$.

By lemma 4.8 we have now

$$\|\tilde{\omega}\|_{\Lambda_{(\ln^+)}^{1,1}} = 2\pi \int_0^{+\infty} \int_0^{\sqrt{\{[\tilde{\omega}] > \lambda\}/\pi}} r\left(\ln \frac{1}{r}\right) + dr d\lambda$$

$$\leq 2\pi \int_0^{+\infty} \int_0^{\sqrt{\alpha([\omega] > \lambda/\alpha)/\pi}} r\left(\ln \frac{1}{r}\right) + dr d\lambda$$

$$= \alpha^2 \pi \int_0^{+\infty} \int_0^{\sqrt{\{[\omega^*] > \lambda^\prime\}}/\pi} r^\prime\left(\ln \frac{1}{r^\prime}\right) + dr^\prime d\lambda^\prime$$

$$\leq \alpha^2 2\pi \int_0^{+\infty} \int_0^{\sqrt{\{[\omega^*] > \lambda^\prime\}}/\pi} r^\prime\left(\ln \frac{1}{r^\prime}\right) + dr^\prime d\lambda^\prime.$$

Finally we estimate the behaviour of the Lorentz norm under rescaling on the domain.

**Proposition 4.9.** If $\sigma \in (0, +\infty)$ and $\omega : D \to \mathbb{R}$, then

$$\|\omega(\sigma \cdot)\|_{\Lambda_{(\ln^+)}^{1,1}} \leq \frac{1}{\sigma^2} \left(\|\omega\|_{\Lambda_{(\ln^+)}^{1,1}} + (\ln \frac{1}{\sigma}) + \|\omega\|_{L^1}\right).$$
Proof. We have, by lemma 4.8,
\[ \|\omega(\sigma)\|_{\Lambda(\ln)^+} = 2\pi \int_0^{+\infty} \int_0^{\sqrt{\frac{\ln r}{1}} - 1} r'(\ln \frac{1}{r} + 1) \, dr \, d\lambda \]
and by a change of variable:
\[ \|\omega(\sigma)\|_{\Lambda(\ln)^+} = 2\pi \sigma^2 \int_0^{+\infty} \int_0^{\sqrt{\frac{\ln r}{1}} - 1} r'(\ln \frac{1}{\sigma r} + 1) \, dr' \, d\lambda, \]
From this, we conclude
\[ \|\omega(\sigma)\|_{\Lambda(\ln)^+} \leq 2\pi \sigma^2 \left( \int_0^{+\infty} \int_0^{\sqrt{\frac{\ln r}{1}} - 1} r'(\ln \frac{1}{r} + 1) \, dr' \, d\lambda + \int_0^{+\infty} \int_0^{\sqrt{\frac{\ln r}{1}} - 1} r'(\ln \frac{1}{\sigma r} + 1) \, dr' \, d\lambda \right) \]
and the conclusion follows. □

5. Asymptotics evolution of vortices

5.1. Asymptotic representation of derivatives. In order to study the evolution of vortices, we will need to differentiate several quantities of the form
\[ \hat{D} \omega(t) \eta, \]
where \( \eta \in C^1(\bar{D}) \) is a given spatial test function.

**Proposition 5.1.** If \( (\omega, u) \in L^\infty(\mathbb{R} \times D) \times L^\infty(\mathbb{R}, L^2(D)) \) is a weak solution to the vorticity formulation lake equation and if \( \eta \in C^1(\bar{D}) \), then the function \( t \mapsto \hat{D} \omega(t) \eta \) is weakly differentiable and for almost every \( t \in \mathbb{R} \),
\[ \frac{d}{dt} \int_D \omega(t) \eta = - \int_D \omega(t) u(t) \cdot \nabla \eta - \sum_{i=1}^m \Gamma_i \int_D \omega(t) \nabla \psi_i \times \nabla \eta. \]

**Proof.** Given \( \theta \in C^1_c([0, +\infty)) \), we apply proposition 2.7 to the test function \( \theta \eta \) and we obtain
\[ \theta(0) \int_D \omega_0 \eta + \int_0^{+\infty} \int_D \omega(t) (\theta'(t) \eta + \theta(t) u(t) \cdot \nabla \eta) \, dt = 0. \]
The second identity follows then from (2.7).

Under the additional assumption that the function \( \eta \) is constant on each component of the boundary \( \partial D \), we obtain a representation in which the gradient of the stream function is replaced by the gradient of the depth function.

**Proposition 5.2.** If \( \eta \in C^2(\bar{D}) \) is constant on each component of \( \partial D \), then there exists a constant \( C > 0 \) such that for every \( \omega \in L^\infty(D) \),
\[ \left| \int_D \frac{\nabla (K_b[\omega])}{b} \times \nabla \eta - \frac{1}{2} \int_D \omega K_b[\omega] \frac{\nabla b}{b^2} \times \nabla \eta \right| \leq C |\Gamma|^2. \]
Proof of proposition 5.2. By proposition 3.3, we have the following identity:

\[
\hat{D} \omega \nabla \left( K b \left[ \omega \right] \right) b \times \nabla \eta - \frac{1}{2} \hat{D} \omega K b \left[ \omega \right] \nabla b b \times \nabla \eta = \int_D \nabla G_D(x,y) \times \nabla \eta(x) \omega(x) \omega(y) \sqrt{\frac{b(y)}{b(x)}} dx dy + \int_D \frac{\nabla R_b(x,y)}{b} x \nabla \eta(x) \omega(x) \omega(y) dx dy.
\]

For the second term on the right-hand side of (5.1), we have by the boundedness properties of the derivatives of the function \(R_b\),

\[
\left| \int_D \frac{\nabla R_b(x,y)}{b} x \nabla \eta(x) \omega(t,x) \omega(t,y) dx dy \right| \leq C_1 |\Gamma|^2.
\]

We now estimate the first term on the right-hand side of (5.1). By symmetry, we have

\[
\int_D \sqrt{\frac{b(y)}{b(x)}} \left( \nabla G_D(x,y) \times \nabla \eta(x) \right) \omega(x) \omega(y) dx dy = \frac{1}{2} \int_D \sqrt{b(x)b(y)} \left( \nabla G_D(x,y) \times \frac{\nabla \eta(x)}{b(x)} + \nabla G_D(y,x) \times \frac{\nabla \eta(y)}{b(y)} \right) \omega(x) \omega(y) dx dy.
\]

For every \(x, y \in D\), we have

\[
\nabla G_D(x,y) \times \frac{\nabla \eta(x)}{b(x)} + \nabla G_D(y,x) \times \frac{\nabla \eta(y)}{b(y)} = \nabla G_D(x,y) \times \left( \frac{\nabla \eta(x)}{b(x)} - \frac{\nabla \eta(x)}{b(x)} \right)
\]

\[
+ \left( \nabla G_D(x,y) + \nabla G_D(y,x) \right) \times \frac{\nabla \eta(y)}{b(y)}.
\]

By proposition 3.5, since by assumption \(\nabla \eta\) and \(b\) are both Lipschitz-continuous, we have

\[
\left| \nabla G_D(x,y) \times \left( \frac{\nabla \eta(x)}{b(y)} - \frac{\nabla \eta(x)}{b(x)} \right) \right| \leq C_2.
\]

For the other contribution proposition 3.7, we have if \(|y-x| + \text{dist}(x, \partial D) + \text{dist}(y, \partial D) \leq \delta,

\[
\left| \nabla G_D(x,y) + \nabla G_D(y,x) - 2 \pi \frac{x - P_{\partial D}(x)}{|x - y|^2 + 4 \text{dist}(x, \partial D) \text{dist}(y, \partial D)} \right| \leq C_3
\]

and

\[
(x - P_{\partial D}(x)) \times \nabla \eta(P_{\partial D}x) = 0,
\]
and thus
\[(5.6) \left| (\nabla G_D(x,y) + \nabla G_D(y,x)) \times \frac{\nabla \eta(y)}{b(y)} \right| \leq C_4 \left( 1 + \frac{\text{dist}(x,\partial D)^2}{|x-y|^2 + \text{dist}(x,\partial D)\text{dist}(y,\partial D)} \right) \leq C_5,\]
if \(\text{dist}(x,\partial D) \leq \text{dist}(y,\partial D)\). The case where \(\text{dist}(y,\partial D) \leq \text{dist}(x,\partial D)\) follows symmetrically.

If \(|y-x| + \text{dist}(x,\partial D) + \text{dist}(y,\partial D) \geq \delta\), by proposition 5.1, we have
\[(5.7) \left| (\nabla G_D(x,y) + \nabla G_D(y,x)) \times \frac{\nabla \eta(x)}{b(x)} \right| \leq C_6.\]
By combining (5.3), (5.4), (5.6), (5.5) and (5.7), we deduce that
\[(5.8) \left| \int_{D \times D} \frac{b(y)}{b(x)} (\nabla G_D(x,y) \times \nabla \eta(x)) \omega(x) \omega(y) \, dx \, dy \right| \leq C_7 |\Gamma|^2.\]

The conclusion follows from the combination of the identity (5.1) with the inequalities (5.2) and (5.8). \(\square\)

5.2. Asymptotic conservation of the total vorticity. For the lake equations (1.1), the total vorticity, defined by (1.7) is not conserved in general. Indeed, by proposition 5.1, one has for almost every \(t \in \mathbb{R}\)
\[\Omega'(t) = \int_D \omega(t) u(t) \cdot \nabla b,\]
and there is no reason for the right-hand side to vanish. On the other hand Richardson’s formal law (1.3) suggests that vorticity should follow level lines of the depth and thus one can hope the total vorticity to be asymptotically preserved. The following result gives a bound on the variation of the total vorticity during the motion.

**Proposition 5.3** (Asymptotic conservation of the total vorticity). If the function \(b \in C^2(\bar{D})\) is constant on each connected component of the boundary, then there exists a constant \(C > 0\) that depends only on \(D\) and \(b\), such that if \((\omega, u) \in L^\infty(\mathbb{R} \times D) \times L^\infty(\mathbb{R}, L^2(D))\) is a weak solution to the vorticity formulation lake equation then we have for almost every \(t \in \mathbb{R}\):
\[|\Omega(t) - \Omega(0)| \leq C |\Gamma| \|\Gamma\| |t|.\]

The conclusion of proposition 5.3 can be rewritten as
\[\frac{|\Omega(t) - \Omega(0)|}{|\Gamma|} \leq C \frac{|\Gamma| \|\Gamma\| E|t|}{|\Gamma|},\]
in the regime where \(\frac{E}{|\Gamma|} \rightarrow +\infty\), the above estimate may be interpreted as stating that at the time scale \(\Gamma/E\) the variations of the total vorticity are much smaller than the total circulation.

**Proof of proposition 5.3.** By proposition 5.1 with \(\eta = b\), the function \(\Omega\) is weakly differentiable and for almost every \(t \in \mathbb{R}\),
\[(5.9) \Omega'(t) = -\int_D \frac{\nabla (K_b[\omega(t)])}{b} \cdot \nabla b \omega(t) - \sum_{i=1}^m \int_D \frac{\nabla \psi_i}{b} \cdot \nabla b \omega(t).\]
Since we have assumed the bathymetry function $b$ to be constant on the boundary, we apply proposition $5.2$ with $\eta = b$ and we obtain, since $\nabla b \times \nabla b = 0$,
\[
(5.10) \quad \left| \int_{\Omega} \frac{\nabla (K_b[\omega(t)]) \times \nabla b}{b} \omega(t) \right| \leq C_1 |\Gamma|^2.
\]
By the boundedness properties of the gradients of $b$ and $\psi_i$ and by definition of $\|\Gamma\|$ in (4.3), we also have
\[
(5.11) \quad \left| \sum_{i=1}^{m} \Gamma_i \int_{\Omega} \frac{\nabla \psi_i \times \nabla b}{b} \omega(t) \right| \leq C_2 (1 + \|\Gamma\|).
\]
By (5.9), (5.10) and (5.11) we deduce in view of proposition 5.1 that for almost every $t \in \mathbb{R}$,
\[
|\Omega'(t)| \leq C_3 |\Gamma| \|\Gamma\|.
\]
The conclusion then follows by integration. □

5.3. Evolution of a singular vortex. We are now in position to state and prove the main result of the present work.

**Theorem 5.4** (Evolution of the vortex core). Let $D \subseteq \mathbb{R}^2$ be a bounded domain of class $C^2$ and $b \in C^2(\bar{D}, (0, +\infty))$. Assume that $b$ is constant on each component of $\partial D$. If
(a) $(u^n)_{n>0}$ is family of weak solutions to the lake equations (1.1),
(b) $\omega^n(0) \geq 0$ almost everywhere on $D$,
(c) there exists $q_0 \in D$ such that for every $\eta \in C(D)$,
\[
\lim_{n \to \infty} \frac{1}{\Gamma^n} \int_{\Omega} \eta \omega^n(0) = \eta(q_0),
\]
(d) $\sup_{n \in \mathbb{N}} \frac{1}{\Gamma^n} \|((\rho^n)^2 \omega^n(0, \rho^n))\|_{L^1(D, \mathbb{R}^2)} < +\infty$ where $\rho^n = \exp\left(-\frac{4\pi E_n^{1/n}}{\Gamma^n}\right)$,
(e) $\sup_{n \in \mathbb{N}} \|\Gamma^n\| < +\infty$,
and let $q_* : \mathbb{R} \to D$ be the unique solution to the Cauchy problem
\[
\begin{cases}
\dot{q}_*(s) = -\nabla \perp \left(\frac{1}{b}\right)(q_*(s)) & \text{if } s \in \mathbb{R},
q_*(0) = q_0,
\end{cases}
\]
then one has, for every $\varphi \in C^1(\bar{D})$, uniformly in $s \in \mathbb{R}$ over compact subsets,
\[
\lim_{n \to \infty} \frac{1}{\Gamma^n} \int_{\Omega} \varphi \omega^n(\Gamma^n s/E^n) = \varphi(q_*(s))
\]
and
\[
\lim_{n \to \infty} \frac{1}{E^n} \int_{\Omega} \varphi |u^n|^2(\Gamma^n s/E^n) = \varphi(q_*(s)).
\]

Theorem 5.4 implies immediately theorem 1.1. Indeed, it suffices to observe that by (4.1) the Lorentz norm is controlled by the $L^\infty$ norm and that (4) is trivially satisfied since when $D$ is simply connected, $m = 0$ and $\|\Gamma\| = |\Gamma|$ by definition in (1.3).
Proof of theorem 5.4. We first note that by proposition 3.3 and (3.2) and by proposition 2.5 we have for every \( \delta > 0 \) such that \( \overline{B(q_0, \delta)} \subset D \),

\[
\frac{1}{4\pi} \ln \frac{1}{2\delta} \left( \int_{B(q_0, \delta)} \sqrt{b} \omega^n(0) \right)^2 \leq \int_{B(q_0, \delta)} \frac{1}{4\pi} \ln \frac{1}{|y - x|} \sqrt{b(x)b(y)} \omega^n(0, x) \omega^n(0, y) \, dy \, dx
\]

\[
\leq \frac{1}{2} \int_D \omega \mathcal{K}_b[\omega] + C_1 \|\Gamma^n\|^2 \leq E^n(0) + C_2 \|\Gamma^n\|^2.
\]

so that by the assumptions (c) and (d), for every \( \delta > 0 \) small enough we have

\[
\liminf_{n \to \infty} \frac{E^n}{\|\Gamma^n\|^2} \geq \frac{1}{4\pi} \ln \frac{1}{2\delta}
\]

and thus

(5.12) \[
\lim_{n \to \infty} \frac{E^n}{\|\Gamma^n\|^2} = +\infty.
\]

By definition of \( \rho^n(t) \) in (1.8), we observe that for each \( n \in \mathbb{N} \) and \( t \in \mathbb{R} \),

\[
\rho^n(t) = \exp \left( -\frac{4\pi E^n}{\|\Gamma^n\|^2} \Omega^n(t) \right) \leq \exp \left( -\frac{4\pi E^n}{\|\Gamma^n\|^2 \sup_D b} \right),
\]

and thus by (5.12), we have \( \rho^n \to 0 \) uniformly on \( \mathbb{R} \) as \( n \to \infty \). Moreover, we have by definition of \( \rho^n(t) \) and by proposition 5.3 for every \( n \in \mathbb{N} \) and \( t \in \mathbb{R} \),

(5.13) \[
\left| \ln \frac{\rho^n(t)}{\rho^n(0)} \right| = \frac{4\pi E^n \left| \Omega^n(t) - \Omega^n(0) \right|}{\|\Gamma^n\|^2 \Omega^n(t)} \leq C_3 \frac{E^n |t|}{\|\Gamma^n\|^2}.
\]

By proposition 4.9 by (5.13), and then proposition 4.7, this implies that for every \( n \in \mathbb{N} \) and \( t \in \mathbb{R} \)

\[
\rho^n(t)^2 \|\omega^n(t, \rho^n(t))\|_{A_{(n)}^{1,1}} \leq \rho^n(0)^2 \|\omega^n(t, \rho^n(0))\|_{A_{(n)}^{1,1}} + \left( \ln \frac{\rho^n(t)}{\rho^n(0)} \right) \|\omega^n\|_{L^1}
\]

\[
\leq \rho^n(0)^2 \|\omega^n(t, \rho^n(0))\|_{A_{(n)}^{1,1}} + C_3 E^n |t|
\]

\[
\leq C_4 \left( \rho^n(0)^2 \|\omega^n(0, \rho^n(0))\|_{A_{(n)}^{1,1}} + E^n |t| \right)
\]

\[
\leq C_5 \left( \|\Gamma^n\| \left( 1 + \frac{E^n |t|}{\|\Gamma^n\|^2} \right) \right),
\]

in view of our assumption (d). Since \( \rho^n(t) \) uniformly converges to 0 as \( n \to \infty \), one can choose \( R^n(t) \triangleq 1/\sqrt{\rho^n(t)} \) for sufficiently large \( n \in \mathbb{N} \) in proposition 4.6. For every \( S > 0 \), there exists then \( \delta > 0 \) such that if \( n \in \mathbb{N} \) is large enough and if \( E^n |t|/\|\Gamma^n\| \leq S \), then \( q^n(t) \in D \) and \( \text{dist}(q^n(t), \partial D) \geq \delta \).

Let \( \eta \in C^\infty(D) \) be a positive function bounded by 1, such that \( \eta(x) = 0 \) if \( \text{dist}(x, \partial D) \leq \delta/3 \) and \( \eta(x) = 1 \) if \( \text{dist}(x, \partial D) \geq 2\delta/3 \). We define the truncated center of vorticity \( \tilde{q}^n : \mathbb{R} \to \mathbb{R}^2 \) by setting for each \( t \in \mathbb{R} \) and \( n \in \mathbb{N} \),

\[
\tilde{q}^n(t) \triangleq \frac{1}{\|\Gamma^n\|} \int_D \eta(x) x \omega^n(t, x) \, dx.
\]
We observe that for every \( n \) large enough, \( \rho^n(t)(R^n(t), \rho^n(t)) \leq \frac{\delta}{2} \) and
\[
\int_{\text{dist}(x, \partial D) \leq \delta} \omega^n \leq \int_{D \setminus B(q(t), \rho^n(t)(R^n(t), \rho^n(t)))} \omega^n(t, x) \, dx.
\]

By proposition 5.3, we obtain
\[
|q^n(t) - \tilde{q}^n(t)| \leq \frac{1}{\Gamma^n} \int_{D \setminus B(q(t), \rho^n(t)(R^n(t), \rho^n(t)))} \omega^n(t, x) \, dx
\]
\[
\leq \frac{C_6}{\ln(R^n(t))} \left( \rho^n(t)^2 \frac{\|\omega^n(t, \rho^n(t))\|_{L^\infty(D)}}{\Gamma^n} + 1 \right).
\]  

(5.15)

By (5.14) and by the choice of (5.15), we have for almost every \( t \) by proposition 5.1, we have for all \( t \in \mathbb{R} \),
\[
\hat{q}^n(t) = \frac{1}{\Gamma^n} \left( \int_D \frac{\nabla^\perp K_b[\omega^n(t)] \cdot \nabla \xi}{b} \omega^n(t) + \sum_{i=1}^m \int_D \frac{\nabla^\perp \psi_i \cdot \nabla \xi}{b} \omega^n(t) \right),
\]

where the vector field \( \xi \in C_c^\infty(D, \mathbb{R}^2) \) is defined for each \( x \in D \) by \( \xi(x) \doteq x \eta(x) \). In view of proposition 5.2, we have
\[
\left| \hat{q}^n(t) - \frac{1}{2\Gamma^n} \int_D \frac{\nabla^\perp b \cdot \nabla \xi}{b^2} \omega^n(t) K_b[\omega^n(t)] \right| \leq C_7 \|\Gamma^n\|.
\]  

(5.16)

Now we observe that since \( \nabla b \) is Lipschitz-continuous and since \( \xi(x) = x \) if \( \text{dist}(x, \partial D) \geq \delta \), we have for every \( x \in D \) and \( y \) such that \( \text{dist}(y, \partial D) \geq \delta \),
\[
\left| \frac{\nabla^\perp b(x) \cdot \nabla \xi(x)}{b(x)^2} - \frac{\nabla^\perp b(y) \cdot \nabla \xi(y)}{b(y)^2} \right| \leq C_8 |x - y|
\]

and thus
\[
\left| \int_D \left( \frac{\nabla^\perp b \cdot \nabla \xi}{b^2} - \frac{\nabla^\perp b(q^n(t))}{b^2(q^n(t))} \right) \omega^n(t) K_b[\omega^n(t)] \right| \leq C_8 \left( \int_D |x - q^n(t)| \omega^n(t, x) \, dx \right) \|K_b[\omega^n(t)]\|_{L^\infty(D)}.
\]  

(5.17)

We have by a direct bound
\[
\int_{D \cap B(q^n(t), \rho^n(t)(R^n(t), \rho^n(t)))} |x - q^n(t)| \omega^n(t, x) \, dx \leq C_9 \rho^n(t)(R^n(t), \rho^n(t)) \|\Gamma^n\|
\]

(5.18)

and by proposition 5.2,
\[
\int_{D \setminus B(q^n(t), \rho^n(t)(R^n(t), \rho^n(t)))} \omega^n(t, x) \, dx \leq \frac{C_{10}}{\ln R^n(t)} \left( \rho^n(t)^2 \|\omega[\rho^n(t)]\|_{L^\infty(D)} + \Gamma^n \right).
\]  

(5.19)
Thus we have by (5.17), (5.18), (5.19) and by proposition 4.1,

\begin{equation}
\| \frac{\nabla^f b \cdot \nabla \xi}{b^2} - \frac{\nabla^f b(q^n(t))}{b^2(q^n(t))} \|_{L^2} \leq C_1 \left( \frac{1}{\ln R^n(t)} \left( \rho^n(t)^2 \| \omega^n(t, \rho^n(t)) \|_{\Lambda_{(1)}^+} + |\Gamma^n| \right) \right).
\end{equation}

Finally by proposition 2.5, we have

\begin{equation}
\left\| E^n - \omega^n(t) \int_D K_b[\omega^n(t)] \right\| \leq C_2 |\Gamma^n|^2.
\end{equation}

Summarizing (5.16), (5.20) and (5.21), we conclude that

\begin{equation}
\left\| \frac{\Gamma^n}{E^n q^n(\Gamma^n s / E^n)} - \frac{\nabla^f b(q^n(\Gamma^n s / E^n))}{b^n(q^n(\Gamma^n s / E^n))^2} \right\| \to 0,
\end{equation}

uniformly over $s \in [-S, S]$.

This implies in turn that $q^n(\Gamma^n s / E^n)$ converges uniformly on compact subsets of $\mathbb{R}$ to $q_s$. Finally, we conclude that $q^n(\Gamma^n s / E^n) \to q_s$ uniformly over compact subsets of $\mathbb{R}$. By (5.19) the narrow convergence of vorticity measures follows.

The convergence of energy density measures then follows from proposition 4.2. \qed

6. Open problems

The present work has given a first description of the asymptotic vortex dynamics for the lake equations (1.1). The setting in which we have been working does not cover the whole spectrum of physically relevant situations and suggests for future research some problems that we could not tackle with the techniques that we have developed here.

A first problem would be to determine whether theorem 5.4 holds when shore of the lake is a beach rather than a cliff, that is when $b$ goes smoothly to 0 on the boundary.

Open problem 1. Does a single vortex follow asymptotically the level lines of the depth $b$ when $\inf_D b = 0$?

Whereas in our proof the assumption that $\inf_D b > 0$ plays a role in the construction of the Green function and in keeping control on the Lorentz norm, stationary results cover the case of where the depth $b$ behaves like a power of the distance function close to the boundary \cite{11, 16, 16 §3} (see also chapters 5, 6).

Open problem 2. Does a single vortex follow asymptotically the level lines of the depth $b$ when the domain is unbounded?

The boundedness of the domain and of $b$ is used mainly in the construction and estimates on the Green functions.

The probably most accessible case would be when $D = \mathbb{R}^2$ and $b$ is constant outside a compact set; an interesting result would cover the case where $D = [0, +\infty) \times \mathbb{R}$, with $b(r, z) = r$, 

\[ b(r, z) = r, \]
corresponding to the construction of vortex rings for the three-dimensional Euler equations (see \cite{2}).

The solution of open problem \cite{2} would also show that the evolution of clifford tori in the binormal curvature flow \cite{27}.

Another problem would be the case of non-constant boundary values of \( b \).

**Open problem 3.** Does a single vortex follow asymptotically the level lines of the depth \( b \) when \( b \) is not constant on the boundary?

Currently, the constancy plays a crucial role in the proof and the application of proposition 5.2.

An issue with this setting is that the limiting equation would suggest vortices exiting the domain in finite time. This would not be consistent with the conservation of circulation. A possible solution to this paradox is that the interaction with the boundary at very short range perturbs strongly the asymptotics and makes the law of movement invalid.

When the lake has a flat bottom, that is when \( b \) is constant on some region, our results do not give an interesting description of the movement of the vortices, that occurs on a larger time scale. In analogy with the planar Euler equation, which corresponds to the case where \( b \) is constant on the whole domain, we expect this movement to occur at a time-scale of the order \( 1/\Gamma \).

**Open problem 4.** Describe the movement of a single vortex in a flat region of the lake at time scales of the order \( 1/\Gamma \).

We expect this to be described by some sort of Green function adapted to the problem. A similar second-order asymptotic description was already given for stationary vortex pairs \cite{15} (see chapter 6). One question is whether the movement depends only on the shape of the set on which \( b \) is flat or whether it depends fully on \( b \) and on \( D \). The first scenario would be consistent with results for an analogous Ginzburg–Landau problem with discontinuous pinning \cite{18}.

Finally, it would be natural to consider the problem where the vorticity concentrates in several regions.

**Open problem 5.** Do solutions whose initial vorticity concentrates at several points have these vortex patches following level lines of \( b \)?

This situation is not accessible to our proofs because we characterize the size of the vortex region by some global integral quantities.

An issue raised by this problem would be possible collision of vortices moving on the same line. They would probably interact at a small scale and produce potentially a vortex pair whose movement might be governed by a different equation and might have a different characteristic timescale. A similar related problem would be the description of vortex pairs.

**Appendix A. Weak solutions of the transport equation**

A first interesting fact is that for a transport equation with no flux through the boundary, it is equivalent to test the equation against compactly supported smooth functions or functions that are smooth up to the boundary.
Proposition A.1. Assume that \( u \in L^\infty(W^{1,1}(D)) \) and that \( u \cdot \nu = 0 \) on \( \partial D \) in the sense of traces. If \( f_0 \in L^\infty(D) \) and \( f \in L^\infty([0, +\infty) \times D) \) satisfy for every \( \varphi \in C^1_c([0, +\infty) \times D) \) the identity
\[
\int_0^{+\infty} \int_D f (\partial_t \varphi + u \cdot \nabla \varphi) + \int_D f_0 \varphi(0, \cdot) = 0,
\]
then the identity holds for every \( \varphi \in C^1_c([0, +\infty) \times \bar{D}). \)

Proof. We consider a map \( \theta \in C^1((0, +\infty)) \) such that \( \theta = 0 \) on \( (0, \frac{1}{2}) \) and \( \theta(t) = 1 \) on \( [1, +\infty) \) and we define for each \( n \in \mathbb{N} \), the function \( \chi_n : D \to \mathbb{R} \) for each \( x \in D \) by \( \chi_n(x) \triangleq \theta(n \text{ dist}(x, \partial D)) \). By the smoothness assumption on \( D \), \( \chi_n \in C^1_c(D) \). Since \( u \cdot \nu = 0 \) in the sense of traces, we have for every \( T \in [0, +\infty) \),
\[
\int_0^T \int_D |u \cdot \nabla \chi_n| \leq C_1 \int_0^T \int_{\text{dist}(x, \partial D) \leq \frac{1}{n}} |\nabla u|,
\]
and thus by Lebesgue’s dominated convergence theorem,
\[
(A.1) \quad \lim_{n \to \infty} \int_0^T \int_D |u \cdot \nabla \chi_n| = 0.
\]

For each \( \varphi \in C^1_c([0, +\infty) \times \bar{D}) \) and every \( n \in \mathbb{N} \), we take \( \chi_n \varphi \in C^1_c([0, +\infty) \times \bar{D}) \) as test function, and we obtain by assumption
\[
\int_0^{+\infty} \int_D \chi_n f (\partial_t \varphi + u \cdot \nabla \varphi) + \int_D \chi_n f_0 \varphi(0, \cdot) = -\int_0^{+\infty} \int_D \varphi f u \cdot \nabla \chi_n;
\]
the conclusion follows by letting \( n \to \infty \), and using \((A.1)\). \( \square \)

The flow \( u \) can be integrated following DiPerna and P.-L. Lions [17] in order to provide solutions to the corresponding transport problem.

Proposition A.2. Let \( b \in C^1(\bar{D}, (0, +\infty)) \) and assume that the velocity field satisfies \( u \in L^1_\text{loc}([0, +\infty), W^{1,1}(D) \cap L^\infty(D)) \). If \( u \cdot \nu = 0 \) in the sense of traces and if \( \nabla \cdot (b u) = 0 \) in \( D \) almost everywhere, then there is a unique Borel-measurable function \( X : [0, +\infty) \times [0, +\infty) \times D \to D \), such that

(i) the map \( (s, t) \in [0, +\infty)^2 \mapsto X(s, t, \cdot) \) is continuous for the convergence in measure,
(ii) for every \( r, s, t \in [0, +\infty) \) and almost every \( x \in D \), one has \( X(s, t, x) = X(s, r, X(r, t, x)) \),
(iii) for every function \( f_0 \in L^\infty(D) \) and every \( s, t \in [0, +\infty) \), one has
\[
\int_D f_0(X(s, t, x)) b(x) \, dx = \int_D f_0(x) b(x) \, dx,
\]
(iv) for almost every \( x \in D \),
\[
X(s, t, x) = x + \int_t^s u(r, X(r, t, x)) \, dr.
\]
Moreover, for every $f_0 \in L^\infty(D)$, $f_0 \circ X(0, \cdot)$ is the unique function $f \in L^\infty([0, +\infty) \times D)$ that satisfies for every $\varphi \in C^1_c([0, +\infty) \times D)$,

$$\int_0^{+\infty} \int_D f(\partial_t \varphi + u \cdot \nabla \varphi) + \int_D f_0 \varphi(0, \cdot) = 0$$

and $f \in C([0, +\infty), L^1(D))$.

As a corollary of the above representation formula, the potential vorticity $\omega(t)/b$ at any time $t \geq 0$ is a rearrangement of the initial potential vorticity $\omega_0/b$, in the sense of the weighted Lebesgue measure $d\mu(x) = b(x) \, dx$.

Note that, a priori, the statements only make sense when the function $f_0$ is Borel measurable; the proposition implies then that $X(s, t, \cdot)$ preserves Lebesgue null sets and thus allows one to extend the statement to Lebesgue-measurable functions.

Proof of proposition A.2. We first observe that $\nabla \cdot u = u \cdot \nabla (\ln b)$ almost everywhere on $D$ and thus $\nabla \cdot u \in L^\infty(D)$. The existence and the properties (i), (ii) and (iv) of $X$ follow from the DiPerna–Lions theory [17, Theorem III.2], as does the characterization of solutions to the transport equations and the continuity of the latter [17, Corollary II.2]. By proposition A.1, the transport equation holds for each test function $\varphi \in C^1_c([0, +\infty) \times \bar{D})$.

Given $f_0 \in L^\infty(D)$ as an initial data, we observe that the function $f : [0, +\infty) \times D \to \mathbb{R}$ defined for each $(t, x) \in [0, +\infty) \times D$ by $f(t, x) \triangleq f_0(X(t, 0, x))$ satisfies the transport equation. By taking $\varphi \in C^1_c([0, +\infty) \times \bar{D})$ defined for each $t, x \in [0, +\infty) \times D$ by $\varphi(t, x) \triangleq b(x) \theta(t)$, with $\theta \in C^1_c((0, +\infty))$ as test function we have

\[
(A.2) \quad \int_0^{+\infty} \theta'(t) \left( \int_D f_0(X(t, 0, \cdot)) \, b \right) \, dt + \theta(0) \int_D f_0 \, b = - \int_0^{+\infty} \left( \int_D f(X(t, 0, \cdot)) \nabla \cdot (b u(t)) \right) \, dt = 0,
\]

since $\nabla \cdot (b u(t)) = 0$ almost everywhere in $D$ for almost every $t \in [0, +\infty)$ and the conclusion follows.

Appendix B. Regularity of solutions with smooth initial data

We prove that when the initial vorticity is smooth enough, then weak solutions of the vorticity formulation of the lake equations have some regularity.

Proposition B.1. Assume that $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, that $D$ is of class $C^{k+1}$ and that $b \in (C^2 \cap C^{k+1, \alpha})(D)$. If $\omega_0 \in C^{k, \alpha}(\bar{D}, \mathbb{R})$ and if $(\omega, u) \in L^\infty([0, +\infty) \times D, \mathbb{R}) \times L^\infty([0, +\infty), L^2(D, \mathbb{R}^2))$ is a weak solution to the vortex formulation of the lake equations, then for every $T > 0$, $\omega \in C^{k, \alpha}([0, T] \times \bar{D})$ and $u \in L^\infty([0, T], C^{k+1, \alpha}(\bar{D}, \mathbb{R}^2)) \cap C^{k, \alpha}([0, T] \times \bar{D}, \mathbb{R}^2)$.

When $k = 0$, proposition B.1 is due to Huang [24, Theorem 4.1].

Our proof follows the same strategy as proofs of the regularity of solutions of the planar Euler equations [39, §2.4] (see also [26, §3.1]).
The first tool that we use is the fact that the velocity field generated by a bounded vorticity field satisfies a bound known as quasi-Lipschitz bound \cite{26} Lemma 1.4; \cite{39} Lemma 3.1] or logarithmically Lipschitz \cite{22}.

**Lemma B.2.** There exists a constant $C > 0$ such that for every $\omega \in L^\infty(D)$ and every $x, y \in D$, one has

$$|\nabla K_b[\omega](x) - \nabla K_b[\omega](y)| \leq C|y - x| \ln \frac{2 \text{diam } D}{|y - x|}.$$  

**Proof.** By proposition \[5.3\], we have

\begin{equation}
\nabla K_b[\omega](x) = \int_D \nabla G_D(x, z) \omega(z) \sqrt{b(x) b(z)} \, dz + \frac{1}{2} \int_D G_D(x, z) \omega(z) \sqrt{\frac{b(z)}{b(x)}} \nabla b(x) \, dz + \int_D \nabla R_b(x, z) \omega(z) \, dz.
\end{equation}

We first have the estimate

\begin{equation}
\left| \int_D \nabla G_D(y, z) \omega(z) \sqrt{b(y) b(z)} \, dz - \int_D \nabla G_D(x, z) \omega(z) \sqrt{b(x) b(z)} \, dz \right| \leq C_1 \|\omega\|_{L^\infty(D)} |y - x| \ln \frac{2 \text{diam } (D)}{|y - x|}.
\end{equation}

(see \cite{39} Lemma 2.3.1 and Appendix 2.3]). Next, we have

\begin{equation}
\left| \int_D G_D(y, z) \omega(z) \sqrt{b(z)} \nabla b(y) \, dz - \int_D G_D(x, z) \omega(z) \sqrt{\frac{b(z)}{b(x)}} \nabla b(x) \, dz \right| 
\leq \frac{|\nabla b(y) - \nabla b(x)|}{\sqrt{b(x)}} \int_D G_D(x, z) |\omega(z)| \sqrt{b(z)} \, dz 
+ C_2 \frac{|\nabla b(x)|}{\sqrt{b(x)}} \int_D |G_D(y, z) - G_D(x, z)| \|\omega(z)\| \sqrt{b(z)} \, dz.
\end{equation}

We compute now by proposition \[5.3\]

\begin{equation}
\int_D |G_D(y, z) - G_D(x, z)| \, dz \leq \int_0^1 \int_D |\nabla G_D((1-s)x + sy, z)| |y-x| \, dz \, dt 
\leq \int_0^1 \int_D \frac{C_3 |y-x|}{|(1-s)x + sy - z|} \, dz \, ds \leq C_4 |y-x|.
\end{equation}

By (B.3) and (B.4), we deduce, since the derivative of $\nabla b$ is bounded, that the gap

\begin{equation}
\left| \int_D G_D(y, z) \omega(z) \sqrt{\frac{b(z)}{b(y)}} \nabla b(y) \, dz - \int_D G_D(x, z) \omega(z) \sqrt{\frac{b(z)}{b(x)}} \nabla b(x) \, dz \right|
\end{equation}

is bounded by

$C_5 \|\omega\|_{L^\infty(D)} |y - x|$.

In order to control the variation of $\nabla R_b(\cdot, z)$, we recall that by the proof of proposition \[3.3\] $R_b = S_b + Q_b$ for some function $Q_b \in C^2(D \times D)$ defined in proposition \[2.4\] and for some
function $S_b$ constructed in the proof of proposition 3.1 in such a way that for each $z \in D$, the function $S_b(\cdot, y) \in W^{1,2}_0(D)$ is the unique solution of the elliptic problem

$$
\begin{cases}
-\nabla \cdot (b^{-1} \nabla S_b(\cdot, z)) = -G_D(\cdot, z) \sqrt{b(z)} \left( \Delta \frac{1}{\sqrt{b}} \right) & \text{in } D, \\
S_b(\cdot, z) = 0 & \text{on } \partial D.
\end{cases}
$$

Hence, in order to conclude the proof, it is sufficient to focus on the $S_b$-term. We recall that for all $z \in D$ the function $S_b$ admits the integral representation

$$
S_b(x, z) = -\sqrt{b(z)} \int_D G_b(y, x) G_D(y, z) \left( \Delta \frac{1}{\sqrt{b}} \right)(y) \, dy;
$$

and moreover, $S_b$ is continuous and symmetric on $D \times D$ (proposition 3.1). Therefore, we have for all $x, z \in D$:

$$
S_b(x, z) = S_b(z, x) = -\sqrt{b(x)} \int_D G_b(y, z) G_D(y, x) \left( \Delta \frac{1}{\sqrt{b}} \right)(y) \, dy,
$$

or equivalently, using the symmetry of the Green’s function $G_D$:

$$
S_b(x, z) = -\int_D G_b(y, z) \left( \sqrt{b} \, G_D(\cdot, y) \right)(x) \left( \Delta \frac{1}{\sqrt{b}} \right)(y) \, dy.
$$

In particular, a direct application of Fubini’s theorem shows that, for almost-every $x \in D$, we have

$$(B.6) \quad \nabla \int_D S_b(x, y) \omega(y) \, dy = \int_D \omega(y) \left( \int_D G_b(z, y) \left( \Delta \frac{1}{\sqrt{b}} \right)(z) \nabla \left( \sqrt{b} \, G_D(\cdot, z) \right)(x) \, dz \right) \, dy.$$

Since the $L^p$-norms of Green’s functions are uniformly bounded on as $y$ varies in $D$ [51], we may apply estimates (B.2) and (B.5) to obtain

$$
\left| \int_D \nabla S_b(x, z) \omega(z) \, dz - \int_D \nabla S_b(x, z) \omega(z) \, dz \right| \leq C_6 \|\omega\|_{L^\infty(D)} |y - x| \ln \frac{2 \text{diam}(D)}{|y - x|},
$$

and therefore

$$(B.7) \quad \left| \int_D \nabla R_b(x, z) \omega(z) \, dz - \int_D \nabla R_b(x, z) \omega(z) \, dz \right| \leq C_7 \|\omega\|_{L^\infty(D)} |y - x| \ln \frac{2 \text{diam}(D)}{|y - x|}.
$$

The conclusion follows from (B.1), (B.2), (B.5) and (B.7). \qed

The next tool is Grönwall type estimate for a logarithmic perturbation of linear growth.

**Lemma B.3.** Let $A, B, C \in [0, +\infty)$, $A < C$ and let $f$ be a continuous function from $[0, +\infty)$ to $(0, C)$, that is: $f \in C([0, +\infty), (0, C))$. If for every $t \in [0, +\infty)$,

$$
f(t) \leq A + B \int_0^t f(s) \ln \frac{C}{f(s)} \, ds,
$$

then for every $t \in [0, +\infty)$,

$$
f(t) \leq C \exp\left(-\ln \frac{C}{A} e^{-Bt}\right).
$$
Proof. We observe that if the function \( u \in C^1(\mathbb{R}, (0, +C)) \) satisfies for every \( t \in I \) the equation
\[
 u'(t) = Bu(t) \ln \frac{C}{u(t)},
\]
then
\[
 u(t) = C \exp(-e^{-Bt} \ln \frac{C}{u(t)}).
\]
and the conclusion follows then by comparison. \qed

We finally rely on the next classical regularity property of Lagrangian flows.

Lemma B.4. If \( u \in C^{k-1}([0, +\infty) \times \bar{D}, \mathbb{R}^2) \cap C([0, +\infty), C^k(\bar{D}, \mathbb{R}^2)) \) and if the function \( X \in C^1([0, +\infty), C(\bar{D}, \bar{D})) \) satisfies
\[
 \begin{cases}
 \partial_t X(t, x) = u(t, X(t, x)) & \text{if } t \in [0, +\infty) \text{ and } x \in D, \\
 X(0, x) = x & \text{if } x \in D,
\end{cases}
\]
then \( X \in C^k([0, +\infty) \times D). \) If moreover \( u \in C^{k-1,\alpha}([0, +\infty) \times \bar{D}, \mathbb{R}^2) \cap L^{\infty}([0, +\infty), C^{k,\alpha}(\bar{D}, \mathbb{R}^2)), \) then \( X \in C^{k,\alpha}([0, +\infty) \times D). \)

Proof. The first part is classical (see for example [10 §1.7]). For the second part, we first have by the first part \( X \in C^k(\mathbb{R} \times \bar{D}, \bar{D}) \) and thus by the chain rule for Hölder continuous functions (see for example [11 Theorem 16.31]) \( \partial_t X \in C^{k-1,\alpha}(\mathbb{R} \times \bar{D}, \bar{D}). \)

Next we observe that for every \( T > 0 \) and \( t \in [0, T], \) we have
\[
 |D_x^k X(t, y) - D_x^k X(t, x)| \leq \int_0^t |D^k f(X(s, y))| [D_x^k X(s, y)] - D^k f(X(s, x)][D_x^k X(s, x)]| ds + C_1 |y - x|
\]
\[
 \leq C_2 \int_0^t |D_x^k X(s, y) - D_x^k X(s, x)| ds + C_3 |y - x|^\alpha,
\]
and it follows then from the classical Grönwall inequality that
\[
 |D_x^k X(t, y) - D_x^k X(t, x)| \leq C_4 |y - x|^\alpha. \quad \square
\]

Proof of proposition B.1. By proposition 2.6, proposition A.2 is applicable to \( f_0 = \omega_0/b \) and implies that for every \( t \in \mathbb{R}, \)
\[
 \omega(t, x) = \frac{b(x)}{b(X(t, x))} \omega_0(X(t, x)).
\]
By proposition A.2 and lemma B.2 we have for every \( x, y \in D, \)
\[
 |X(t, y) - X(t, x)| \leq |y - x| + C_1 \left( (\|\omega_0\|_{L^\infty(D)} + \|\Gamma\|) \int_0^t |X(s, y) - X(s, x)| \ln \frac{2 \text{diam } D}{|X(s, y) - X(s, x)|} ds \right).
\]
It follows then by lemma $3.3$ that
\[
|X(t, y) - X(t, x)| \leq C_2 \exp\left(-\alpha e^{-C_3 t} \ln \frac{2 \text{diam } D}{|y - x|}\right) = C_3 \left(\frac{|y - x|}{2 \text{diam } D}\right)^\alpha \exp(-C_3 t).
\]
This implies thus that
\[
|\omega(t, y) - \omega(t, x)| \leq C_5 \left(\frac{|y - x|}{2 \text{diam } D}\right)^\alpha \exp(-C_3 t).
\]
By the representation formula for the velocity (2.7) and classical regularity estimates \cite{21} Theorem 6.8, it follows then that we have the inclusion $u \in C([0, +\infty), C^1(D))$. By classical regularity theory of the Lagrangian flow, this implies that $X \in C^1([0, +\infty) \times \bar{D}, \bar{D})$ and thus by composition $\omega \in C^{0,\alpha}(0, T) \times \bar{D}$). By regularity estimates \cite{21} Theorem 6.8 we have then $u \in L^\infty([0, T], C^{1,\alpha}(D))$ and $w \in C^{0,\alpha}(0, T) \times \bar{D})$.

We assume now that $u \in L^\infty([0, T], C^{k,\alpha}(D)) \cap C^{k-1,\alpha}(0, T) \times \bar{D})$ and that $\omega \in C^{k,\alpha}(\bar{D})$. By regularity of the Lagrangian flow (lemma $1.4$), we have $X \in C^{k,\alpha}(D)$ and thus $\omega \in C^{k,\alpha}(0, T) \times \bar{D})$. By classical regularity estimates, this implies that $u \in L^\infty([0, T], C^{k+1,\alpha}(\bar{D}) \cap C^{k,\alpha}(0, T) \times \bar{D})$. \hfill $\square$

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