Generalized motion of level sets by functions of their curvatures on Riemannian manifolds

D. Azagra · M. Jiménez-Sevilla · F. Macià

Received: 18 July 2007 / Accepted: 21 January 2008 / Published online: 13 February 2008
© Springer-Verlag 2008

Abstract We consider the generalized evolution of compact level sets by functions of their normal vectors and second fundamental forms on a Riemannian manifold \( M \). The level sets of a function \( u : M \to \mathbb{R} \) evolve in such a way whenever \( u \) solves an equation \( u_t + F(Du, D^2u) = 0 \), for some real function \( F \) satisfying a geometric condition. We show existence and uniqueness of viscosity solutions to this equation under the assumptions that \( M \) has nonnegative curvature, \( F \) is continuous off \( \{Du = 0\} \), (degenerate) elliptic, and locally invariant by parallel translation. We then prove that this approach is geometrically consistent, hence it allows to define a generalized evolution of level sets by very general, singular functions of their curvatures. For instance, these assumptions on \( F \) are satisfied when \( F \) is given by the evolutions of level sets by their mean curvature (even in arbitrary codimension) or by their positive Gaussian curvature. We also prove that the generalized evolution is consistent with the classical motion by the corresponding function of the curvature, whenever the latter exists. When \( M \) is not of nonnegative curvature, the same results hold if one additionally requires that \( F \) is uniformly continuous with respect to \( D^2u \). Finally we give some counterexamples showing that several well known properties of the evolutions in \( \mathbb{R}^n \) are no longer true when \( M \) has negative sectional curvature.
1 Introduction

In the last 30 years there has been a lot of interest in the evolution of hypersurfaces of \( \mathbb{R}^n \) by functions of their curvatures. In this kind of problem one is asked to find a one parameter family of orientable, compact hypersurfaces \( \Gamma_t \), which are boundaries of open sets \( U_t \) and satisfy

\[
V = -G(v, \nu) \text{ for } t > 0, \ x \in \Gamma_t, \ \text{and} \\
\Gamma_t|_{t=0} = \Gamma_0
\]

(1.1)

for some initial set \( \Gamma_0 = \partial U_0 \), where \( V \) is the normal velocity of \( \Gamma_t \), \( \nu = \nu(t, \cdot) \) is a normal field to \( \Gamma_t \) at each \( x \), and \( G \) is a given (nonlinear) function.

Two of the most studied examples are the evolutions by mean curvature and by (positive) Gaussian curvature. In both cases, short time existence of classical solutions has been established. For strictly convex initial data \( U_0 \), it has been shown that \( U_t \) shrinks to a point in finite time, and moreover, \( \Gamma_t \) becomes spherical at the end of the contraction. See [3,14,15,20,21,23,24,34] and the references therein.

For dimension \( n \geq 3 \) it has been shown [19] that a hypersurface evolution \( \Gamma_t \) may develop singularities before it disappears. Hence it is natural to try to develop weak notions of solutions to (1.1) which allow to deal with singularities of the evolutions, and even with nonsmooth initial data \( \Gamma_0 \).

There are two mainstream approaches concerning weak solutions of (1.1): the first one uses geometric measure theory to construct (generally nonunique) varifold solutions, see [6,26], while the second one adapts the theory of second order viscosity solutions developed in the 1980s (see [8] and the references therein) to show existence and uniqueness of level-set weak solutions to (1.1).

In this paper we will focus on this second approach. The first works to develop a notion of viscosity level set solution to (1.1) were those of Evans and Spruck [10] and, independently developed, Chen et al. [7], [17]. This was followed by many important developments, which we find impossible to properly quote here; we refer the reader to the very comprehensive monograph [16] and the bibliography therein. This level set approach consists in observing that a smooth function \( u : [0, T] \times \mathbb{R}^n \to \mathbb{R} \) with \( Du := D_x u \neq 0 \) has the property that all its level sets evolve by (1.1) if and only if \( u \) is a solution of

\[
 u_t + F(Du, D^2u) = 0,
\]

(1.2)

where \( F \) is related to \( G \) in (1.1) through of the following formula:

\[
F(p, A) = |p| G \left( \frac{p}{|p|}, \frac{1}{|p|} \left( I - \frac{p \otimes p}{|p|^2} \right) A \right).
\]

(1.3)

The function \( F \) is assumed to be continuous off \( \{ p = 0 \} \) and (degenerate) elliptic, that is

\[
F(p, B) \leq F(p, A) \quad \text{whenever } A \leq B.
\]

(1.4)

Because of (1.3), \( F \) also has the following geometric property:

\[
F(\lambda p, \lambda A + \mu p \otimes p) = \lambda F(p, A) \quad \text{for all } \lambda > 0, \mu \in \mathbb{R}.
\]

(1.5)