The shortest cut in brane cosmology

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Abstract

We consider brane cosmology studying the shortest null path on the brane for photons, and in the bulk for gravitons. We derive the differential equation for the shortest path in the bulk for a 1+4 cosmological metric. The time cost and the redshifts for photons and gravitons after traveling their respective path are compared. We consider some numerical solutions of the shortest path equation, and show that there is no shortest path in the bulk for the Randall-Sundrum vacuum brane solution, the linear cosmological solution of Binétruy, \textit{et al} for $\omega = -1, -\frac{2}{3}$, and for some expanding brane universes.

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1 Introduction

The possibility of using extra dimensions in order to explain features related to unified field theories has been advocated several decades ago by Kaluza and Klein. After a die out for many years such an idea was reestablished in the context of supergravity and string theory, especially in the latter, where extra dimensions are required in order that the theory is rendered well defined. Meanwhile other problems have been posed in the framework of unified theories. One of them is the huge hierarchy between the electro-weak scale ($\sim 100$ GeV) and the Planck scale ($\sim 10^{19}$ TeV). One possibility to explain that difference is based on the dynamics of supersymmetry, a very beautiful idea that has not, unfortunately, rendered due (and ripe) issues.

In the usual Kaluza-Klein, and also in the modern proposals to deal with extra dimensions, while the 1+3 (physical) dimensions open up to infinity, the extra dimensions are confined in a region of the size of the Planck length, namely $\sim 10^{-33}$ cm, staying beyond experimental verification, today or in the near future.

However, it has been recently shown that it is possible to explain the hierarchy between the electro-weak and the Planck scale by dimensional reduction without compactifying the extra dimensions. Moreover, the usual 1+3 dimensional Einstein theory of gravity can be reproduced on the macroscopic distance scale \cite{1}-\cite{5}. This is quite different from the standard approach, in which extra dimensions open up at short distances only, whereas above a certain length scale, physics is effectively described by 1+3 dimensional theories. Our 1+3 dimensional Universe would be a three dimensional brane living in a higher dimensional theory, thus displaying a certain number of additional dimensions. A further proposal to deal with the additional dimensions is to have them compactified in a submilimeter scale, unifying in a natural way the electro weak and Planck scales \cite{6}.

The possibility of relaxing the constraints on the size of the extra dimensions is very appealing. Such is the case of the Randall-Sundrum (RS) model \cite{1,2}, where the Universe is 1+4 dimensional and the Standard Model fields are localized on a 3-brane embedded in the 4-dimensional space. Only gravitational fields can propagate in all four space directions. At the phenomenological length scale the Kaluza-Klein zero-modes are responsible for the well-posed Einstein 1+3 dimensional theory of gravity and the excitations provide a correction. Due to the ”warp factor” of the brane, a mass
scale around that of Planck mass corresponds to a TeV mass scale in the visible brane. This explains the hierarchy problem. The cosmological consequence of this model is also under active investigation \[7\]-\[18\]. The model leads to new perspectives in many interesting aspects such as the question of the cosmological constant.

The construction of the brane-universe can be traced to the study of $E_8 \times E_8$ string theory, presumably 11-dimensional, with the field theory limit studied in \[13\], and where matter fields live in 10-dimensional branes at the edge of the space-time. The issue of higher dimensionality and its consequences for the early universe have been often discussed in the recent literature \[14\]. Problems related to higher derivative gravity \[15\] and on the cosmological constant problem \[16\] have also been studied, besides the AdS/CFT correspondence and Cardy formula \[17\].

In spite of the attractive aspects of the model, causality can be violated, as first noticed in \[19\] and \[20\]. We have two choices facing this situation. Either we accept the viewpoint that true causality should be defined by the null geodesics in the 1+4 universe instead of in the 1+3 brane spacetime or we find some mechanism to avoid such a violation on the brane. In the first case, the violation must be neglectable in low energy experiments, otherwise, it could have been already found. The question is whether it could be substantial in cosmology. If the answer is positive, it might help solving the well known horizon problem as discussed in \[19\] and \[20\]. In this paper, we consider the following problem. Suppose there are two observers $A$ and $B$ on the brane. $A$ can send series of photons or gravitons to $B$ in order to establish communication (see Fig. \[4\]). According to the brane cosmology, photons travel on the brane while gravitons may travel in the bulk. We consider the three questions: (i) what is the shortest path for gravitons, and whether it is on the brane or in the bulk; (ii) how earlier the gravitons can arrive at $B$; (iii) what is the difference of the redshift for photons and gravitons after they arrive at $B$.
2 Preliminaries

We shall consider a 5-dimensional metric describing brane cosmology. We thus set up a 5-dimensional action of the form\(^\text{[8]}\)

\[
S^{(5)} = -\frac{1}{2\kappa_5^2} \int d^5x \sqrt{-\tilde{g}} \tilde{R} + \int d^5x \sqrt{-\tilde{g}} \mathcal{L}_m . \tag{1}
\]

The constant \(\kappa_5\) is related to the Planck mass as \(\kappa_5^2 = M_{Pl}^{-2}\). The 5-dimensional metric is

\[
d s_5^2 = -n^2(t, y) dt^2 + a^2(t, y) \gamma_{kj} dx^k dx^j + b^2(\tau, y) dy^2 \tag{2}
\]

where \(\gamma_{kj}\) represents a maximally symmetric 3-metric. The energy-momentum appearing in the Einstein equation \(G_{AB} = \kappa_5^2 \mathcal{T}_{AB}\) is decomposed as

\[
\mathcal{T}_{AB} = \hat{\mathcal{T}}_{AB} + T_{AB} \tag{3}
\]
where $\hat{T}_{AB}$ is the energy-momentum tensor of the bulk matter (in the RS scenario it comes from the bulk cosmological constant $\Lambda$, that is, $\hat{T}_{AB} = -\Lambda \delta_B^A$) and $T_{AB}$ corresponds to the matter content on the brane located at $y = 0$. We are interested in the case where the energy-momentum tensor of the bulk matter can be expressed as

$$T_A^B = \frac{\delta(y)}{b} \text{diag}(-\rho - \sigma, p - \sigma, p - \sigma, p - \sigma, -\sigma).$$

(4)

Here, $\sigma$ is the brane tension in the RS scenario. The energy-density $\rho$ and the pressure $p$ come from the ordinary matter on the brane and are independent of the position. Assuming the $\mathbb{Z}_2$-symmetry and $\sigma = 0$, the Einstein equation permits the following exact cosmological brane solution [8] (corresponding to $\Lambda = 0$, $\sigma = 0$, $\gamma_{jk} = \delta_{jk}$)

$$a = a_0(t)(1 + \lambda |y|)$$

$$n = n_0(t)(1 + \mu |y|)$$

$$b = b_0$$

(5)

where $b_0$ is constant in time (a redefinition of $y$ renders it to be 1) and $n_0(t)$ is an arbitrary function (a suitable redefinition of $t$ fixes it to be 1). In the above,

$$\lambda = -\frac{\kappa_5^2}{6} b_0 \rho$$

(6)

$$\mu = \frac{\kappa_5^2}{2} \left( \omega + \frac{2}{3} \right) b_0 \rho$$

(7)

where $\kappa_5^2$ is related to the 5-dimensional Newton constant $G_5$ by $\kappa_5^2 = 8\pi G_5$, and the matter equation of state is $p = \omega \rho$ as usually.

For $\omega = -1$ we have the inflationary case,

$$a_0(t) = e^{Ht}, \quad H = \frac{\kappa_5^2}{6} \rho = \text{const.},$$

(8)

while for $\omega \neq -1$, the usual solution arises,

$$a_0 = t^q, \quad \kappa_5^2 \rho = \frac{6q}{t}, \quad q = \frac{1}{3(1 + \omega)}.$$ 

(9)

Remarkably, the exact solution in the RS model can also be obtained [21]. Note that the parameters $\rho_b$ and $p_b$ in [21] are related to the corresponding
ones here in this paper by the relations $\rho_b = \rho + \sigma$, $p_b = p - \sigma$. The solution can be written in terms of the function

$$a(t, y) = \left\{ \frac{1}{2} \left( 1 + \frac{\kappa_5^2 (\sigma + \rho)^2}{6 \Lambda} \right) a_0^2 + \frac{3C}{\kappa_5^2 \Lambda a_0^2} \right. \\
\left. + \left[ \frac{1}{2} \left( 1 - \frac{\kappa_5^2 (\sigma + \rho)^2}{6 \Lambda} \right) a_0^2 - \frac{3C}{\kappa_5^2 \Lambda a_0^2} \right] \cosh(\mu y) \\
\frac{-\kappa_5 (\sigma + \rho)}{\sqrt{-6\Lambda}} a_0^2 \sinh(\mu |y|) \right\}^{1/2}.$$  \hspace{1cm} (10)

We now construct the remaining function

$$n(t, y) = \frac{\dot{a}(t, y)}{a_0(t)}.$$  \hspace{1cm} (11)

As for eq. (33) in [21], we also have

$$\dot{\rho} + 3 \frac{\dot{a}_0}{a_0} (\rho + p) = 0.$$  \hspace{1cm} (12)

Defining

$$\lambda = \sqrt{\frac{\Lambda}{6 \kappa_5^2} + \frac{\sigma^2}{36}},$$  \hspace{1cm} (13)

and assuming $\lambda \geq 0$ and $p = \omega \rho$, the Friedman equation can be solved in the case $C = 0, k = 0$. For $\lambda > 0$,

$$a_0(t) = a_* \rho_* \left\{ \frac{\sigma}{36 \lambda^2} \left[ \cosh(\kappa_5^2 \lambda t/q) - 1 \right] + \frac{1}{6 \lambda} \sinh(\kappa_5^2 \lambda t/q) \right\}^q.$$  \hspace{1cm} (14)

For $\lambda = 0$, which is the case of RS model,

$$a_0(t) = a_* (\kappa_5^2 \rho_*)^q \left( \frac{1}{72 q^2 \kappa_5^2 \sigma t^2 + \frac{1}{6 q}} \right)^q$$  \hspace{1cm} (15)

where $a_*, \rho_*$ are constant (the origin of time being chosen so that $a_0(0) = 0$).
3 The shortest cut and the redshift

Equation for the shortest cut.

We consider the generic metric $\mathbf{2}$ for $b = 1$. Consider two points, $r_A$ and $r_B$ on the brane. In general, there are more than one null geodesic connecting $r_A$ to $r_B$ in the 1+4 spacetime. The trajectories of photons must be on the brane and those of gravitons may be outside as assumed here. We consider the shortest path for both photons and gravitons. Since the 3-metric is spherically symmetric, we can omit the angular part and just consider the problem for

$$ds_3^2 = -n^2(t, y)dt^2 + a^2(t, y)f^2(r)dr^2 + dy^2$$  \hspace{1cm} (16)

The photon path is on the brane ($n(t, 0) = 1$), therefore

$$- dt^2 + a_0^2(t)f^2(r)dr^2 = 0, \hspace{1cm} (17)$$

which can be immediately integrated as

$$\int_{r_A}^{r_B} f(r')dr' = \int_{t_A}^{t_B} \frac{dt'}{a_0(t')}. \hspace{1cm} (18)$$

The graviton path is defined in terms of the geodesic equation

$$- n^2(t, y)dt^2 + a^2(t, y)f^2(r)dr^2 + dy^2 = 0 \hspace{1cm} (19)$$

We suppose that the path is parameterized by $y = y(t)$. Thus the relation $r = r(t)$ is obtained by

$$\int_{r_A}^{r_B} f(r')dr' = \int_{t_A}^{t_B} \sqrt{n^2(t, y) - \dot{y}^2(t)} \frac{dt'}{a(t, y)}dt \hspace{1cm} (20)$$

We are looking for the path for which, $t_B$ reaches its minimum when $r = r_B$. For this purpose, we consider the general case

$$\int_{r_A}^{r_B} f(r')dr' = \int_{t_A}^{t_B} \mathcal{L}[y(t), \dot{y}(t); t]dt \hspace{1cm} (21)$$

For an adjacent path $y = y(t) + \delta y(t)$, we have

$$\int_{r_A}^{r_B} f(r')dr' = \int_{t_A}^{t_B+\delta t} \mathcal{L}[y(t) + \delta y(t), \dot{y}(t) + \delta \dot{y}(t); t]dt \hspace{1cm} (22)$$
therefore we find the usual condition

\[-\delta t_B \mathcal{L}[y(t_B), \dot{y}(t_B); t_B] = \delta \int_{t_A}^{t_B} \mathcal{L}[y(t), \dot{y}(t); t]dt \] .

(23)

The problem is transformed into the Euler-Lagrange problem

\[\delta \int_{t_A}^{t_B} \mathcal{L}[y(t), \dot{y}(t); t]dt = 0 \] .

(24)

In our case,

\[\mathcal{L}[y(t), \dot{y}(t); t] = \sqrt{n^2(t, y) - \dot{y}^2(t)} \]

(25)

and we have

\[\frac{\partial \mathcal{L}}{\partial y} = -a^{-2}a'(n^2 - \dot{y}^2)^{1/2} + a^{-1}(n^2 - \dot{y}^2)^{-1/2}nn'\]

\[\frac{\partial \mathcal{L}}{\partial \dot{y}} = -a^{-1}(n^2 - \dot{y}^2)^{-1/2} \dot{y} \] .

(26)

The Euler-Lagrange equation thus reads

\[-\ddot{y} + (\dot{a} \frac{\dot{a}}{a} + \dot{n})\dot{y} + \left(\frac{2n'}{n} - \frac{\dot{a}}{a}\right)\dot{y}^2 - \frac{\dot{a}}{an^2} \dot{y}^3 + \left(\frac{\dot{a}'}{a}n^2 - nn'\right) = 0 \] .

(27)

From this equation we can see that the shortest path is on the brane only when

\[\frac{\dot{a}'}{a}n^2 - nn' = 0 \] ,

(28)

i.e.

\[\partial_y(\frac{a}{n}) = 0 \] .

(29)

Further, if there exists a solution, when \(y\) reaches its maximum, where \(\dot{y} = 0\) and \(\ddot{y} < 0\), we have

\[-\ddot{y} + \left(\frac{\dot{a}'}{a}n^2 - nn'\right) = 0 \] .

(30)

Thus, \(\frac{\dot{a}'}{a}n^2 - nn'\), i.e. \(\partial_y(an^{-1})\) must be negative at this point.
The equation is a very difficult nonlinear ordinary differential equation. There is no guarantee for the existence of the required solutions. In order to obtain a solution with both two ends on the brane, we can make the Fourier expansion

\[ y(t) = \sum_{l=1}^{+\infty} y_l \sin \left( \frac{l\pi}{t_{gB} - t_A} (t - t_A) \right), \tag{31} \]

\[ a(t, y) = A(y) + \sum_{l=1}^{+\infty} \left[ a_l^e(y) \sin \left( \frac{l\pi}{t_{gB} - t_A} (t - t_A) \right) + a_l^c(y) \cos \left( \frac{l\pi}{t_{gB} - t_A} (t - t_A) \right) \right] \tag{32} \]

\[ n(t, y) = N(y) + \sum_{l=1}^{+\infty} \left[ n_l^e(y) \sin \left( \frac{l\pi}{t_{gB} - t_A} (t - t_A) \right) + n_l^c(y) \cos \left( \frac{l\pi}{t_{gB} - t_A} (t - t_A) \right) \right] \tag{33} \]

and then substitute back into the differential equation to obtain the coefficients \( y_l \). Here \( t_{gB} \) is the time when the graviton arrives at \( r_B \), which is different from the time \( t_{\gamma B} \) when the photon arrives at \( r_B \). It should be determined self-consistently by the equation

\[ \int_{r_A}^{r_B} f(r')dr' = \int_{t_A}^{t_B} \frac{\sqrt{n^2(t, y) - \dot{y}^2(t)}}{a(t, y)} dt \tag{34} \]

once the solution is obtained.

If we want to find the path for a graviton so that it can reach the farthest within a given time interval \([t_A, t_B]\), we can also use the Euler-Lagrange equation. Then the length difference between geodesics for photons and gravitons within a given time interval can be evaluated

\[ \int_{r_A}^{r_g} f(r')dr' = \int_{t_A}^{t_B} \frac{\sqrt{n^2(t, y) - \dot{y}^2(t)}}{a(t, y)} dt \tag{35} \]

\[ \int_{r_A}^{r_\gamma} f(r')dr' = \int_{t_A}^{t_B} \frac{dt'}{a_0(t')}, \tag{36} \]

**Photon and graviton redshift.**
In general, if \( A \) sends out massless signals at \( x_A^\mu \) and \( x_A^\mu + dx_A^\mu \), these signals will reach \( B \) at \( x_B^\mu \) and \( x_B^\mu + dx_B^\mu \). The relation of \( x_A^\mu \), \( x_A^\mu + dx_A^\mu \) and \( x_B^\mu \), \( x_B^\mu + dx_B^\mu \) can be obtained by solving the geodesic equation. Then the redshift of the signal is \[ \nu_B^2 = \frac{g_{00}(x_B)}{g_{00}(x_A)} \frac{dx_A^\mu}{dx_B^\mu} (37) \]

For a static metric such as the Schwarzschild case, it can be shown that \( dx_A^0 = dx_B^0 \), therefore,
\[ \frac{\nu_B}{\nu_A} = \sqrt{\frac{g_{00}(x_A)}{g_{00}(x_B)}} . \quad (38) \]

For the time-dependent RW metric we have
\[ \frac{dx_A^0}{dx_B^0} = \frac{R(x_A^0)}{R(x_B^0)} , \quad (39) \]

in which case the redshift is given by
\[ \frac{\nu_B}{\nu_A} = \frac{R(x_A^0)}{R(x_B^0)} . \quad (40) \]

Thus, in the geometric-optics limit, the redshifts in the two cases can be systematically discussed.

Here, we consider that another graviton starts traveling from \( r_A \) at a later time \( t_A + \delta t_A \). Its shortest path is in general different from the previous one. Let us denote it as \( y_\ast = y_\ast(t) \). Then the time when it arrives at \( r_B \) will be a later time \( t_{gB} + \delta t_{gB} \)
\[ \int_{t_A}^{t_B} f(r')dr' = \int_{t_A + \delta t_A}^{t_B + \delta t_{gB}} \sqrt{n^2(t, y_\ast) - \dot{y}_\ast^2(t)} a(t, y_\ast) dt . \quad (41) \]

Therefore we have the equality
\[ \int_{t_A}^{t_B} \sqrt{n^2(t, y) - \dot{y}^2(t)} \frac{1}{a(t, y)} dt = \int_{t_A + \delta t_A}^{t_B + \delta t_{gB}} \sqrt{n^2(t, y_\ast) - \dot{y}_\ast^2(t)} \frac{1}{a(t, y_\ast)} dt . \quad (42) \]
For infinitesimal $dt_A$ and $dt_B$, we have

$$dt_B \left( \frac{\sqrt{n^2(t, y) - \dot{y}^2(t)}}{a(t, y)} \right) \bigg|_B = dt_A \left( \frac{\sqrt{n^2(t, y) - \dot{y}^2(t)}}{a(t, y)} \right) \bigg|_A \quad (43)$$

Thus, the graviton redshift is given by

$$\frac{\nu_{gB}}{\nu_{gA}} = \frac{a_0(t_A)}{a_0(t_B)} \sqrt{\frac{1 - \dot{y}_2(t_B)}{1 - \dot{y}_2(t_A)}} \quad (44)$$

while for the photon we have

$$\frac{\nu_{gB}}{\nu_{gA}} = \frac{a_0(t_A)}{a_0(t_B)} \quad (45)$$

4 Examples

RS vacuum solution[1] [2].

In this case

$$n(y, t) = a(y, t) = e^{-k|y|} \quad (46)$$

Eq. (27) turns out to be

$$\ddot{y} + ky^2 = 0 \quad (47)$$

It has two possible solutions, one is $y = y_A = 0$, and the other is $y = y_0 + k \ln(t - t_0)$. The second solution does not meet our requirement because it will not end on the brane. So the shortest path must be on the brane. This agrees with the conclusion in [19].

The linear cosmological solution.

We first consider the case $\omega = -\frac{2}{3}$ so that from (7) $\mu = 0$, $a(t, y) = t - y$, $\lambda = -\frac{1}{t}$. The equation is

$$- (t - y) \ddot{y} + \dot{y} + \dot{y}^2 - \dot{y}^3 - 1 = 0 \quad (48)$$

Let $t - y = u$, then

$$u \dddot{u} + u^{\ddot{3}} - 2u^{\dddot{2}} = 0 \quad , \quad (49)$$

or

$$\frac{1}{2u^{\dddot{2}} - u^{\dddot{3}}} \frac{d}{dt} u^{\dddot{2}} = \frac{2\dddot{u}}{u} \quad . \quad (50)$$
Therefore,
\[ \int \frac{du}{2\dot{u} - \dot{u}^2} = \int \frac{du}{u}, \]  \hspace{1cm} (51)
\[ \frac{\dot{u}}{2 - \dot{u}} = cu^2. \]  \hspace{1cm} (52)

We can obtain the solution \( t_0 \) and \( c \) are two integration constants
\[ y = t \pm \sqrt{(t - t_0)^2 + \frac{1}{c}}. \]  \hspace{1cm} (53)

It is obvious that this path can not end on the brane either. Furthermore, we consider the case \( \omega = -1, \lambda = \mu = \text{const.} \ a_0(t) = e^{Ht}. \) So \( \partial_y (a/n) = 0. \) Therefore the shortest path is on the brane.

The general linear cosmological solution [8].

Consider the case \( \omega \neq -1 \)
\[ a_0(t) = t^q, \quad \lambda = -\frac{q}{t}, \quad \mu = w \frac{q}{t}, \quad w = 2 + 3\omega \]  \hspace{1cm} (54)
\[ a(t, y) = t^q - qt^q -1 y, \quad n(t, y) = 1 + \frac{q}{t} y \]  \hspace{1cm} (55)
\[ \dot{a}(t, y) = qt^{q-1} - q(q-1)t^{q-2} y, \quad a'(t, y) = -qt^{q-1} \]  \hspace{1cm} (56)
\[ \dot{n}(t, y) = -q\omega t^{-2} y, \quad n'(t, y) = q\omega t^{-1} \]  \hspace{1cm} (57)

Letting \( y = tf(t) \) in (27), we get a nonlinear differential equation
\[ \begin{align*}
-\left[1 + (2q\omega - q)f + (q^2\omega^2 - 2q^2\omega^2)f^2 - q^3\omega^2 f^3\right] & (t^2 \ddot{f} + 2t \dot{f}) \\
\quad + [q + (2q\omega - q^2 + q - q\omega)] f \\
\quad + (q^2\omega - q^2\omega^2 - 2q\omega^2 + 2q^2\omega^2)f^2 \\
\quad + (2q^3\omega^2 - q^4\omega^2)f^3 (t \ddot{f} + f) \\
\quad + [2q\omega - q] + q^3\omega^2 f^2 (t \ddot{f} + f)^2 \\
\quad - [q - q(q-1)f] (t \ddot{f} + f)^3 \\
\quad + [(q^2 - 4q^2 - 3q^2\omega^2)f \\
\quad + (3q^3\omega - 6q^3\omega^2 - 3q^3\omega^3)f^2 \\
\quad + (q^3 - 3q^3 - 3q^4\omega^4)f^3 \\
\quad + (q^5\omega^3 - q^5\omega^4)f^4] &= 0. \\
\end{align*} \]  \hspace{1cm} (58)
The analysis of such a differential equation is beyond our capability. We leave it as it stands and pass to a discussion of some simple cases where numerical analysis can be performed.

The case considered by Binétruy et al. [21] is that of a 3-brane universe in the 5-dimensional space time with a cosmological constant. For an equation of state \( p = \omega \rho \) they found explicit solutions which we use in order to study the question of the existence of shortcuts. The solutions are very involved, and we first disentangle the equations using a MAPLE program, and further on numerically solve the differential equations. We shall consider the matter dominated (\( \omega = 0 \)) and radiation dominated (\( \omega = 1/3 \)) cases.

The solution of the gravity equations reads [21]

\[
a(t, y) = \left\{ \frac{1}{2} \left( 1 + \frac{\kappa^2 \rho_B^2}{6 \rho_B} \right) + \frac{1}{2} \left( 1 - \frac{\kappa^2 \rho_B^2}{6 \rho_B} \right) \cosh(\mu y) - \frac{\kappa \rho_B}{\sqrt{-6 \rho_B}} \sinh(\mu |y|) \right\}^{\frac{1}{2}} a_0(t),
\]

Figure 2: Diagram for \( y \sim 0.3 \ell_p \).
Figure 3: The same diagram as before, with \( y \) beginning at the brane.

\[
\begin{align*}
n(t, y) &= \frac{\dot{a}(t, y)}{\dot{a}_0(t)} , \\
a_0(t) &= a_* (\kappa^2 \rho_*)^{1/q} \left( \frac{a^2}{72} \kappa^2 \rho \Lambda t^2 + \frac{a t}{6} \right)^{1/q} , \\
\mu &= \sqrt{-\frac{2\kappa^2}{3} \rho_B}
\end{align*}
\]
the parameters according to the discussion in Binétruy et al. \(8\)

\[
\rho_b = \rho_\Lambda + \rho ,
\]

where \(\rho\) stands for the ordinary energy density in cosmology given by

\[
\rho = \rho_* (a_0/a_*)^{-q} , \quad q = 3(1 + \omega) .
\]

The intrinsic tension of the brane, \(\rho_\Lambda\), has to be identified with Newton’s constant in order to recover the standard cosmology, that is

\[
8\pi G = \frac{\kappa^4 \rho_\Lambda}{6} ,
\]

when \(\rho \ll \rho_\Lambda\).

Moreover the 5 dimensional coupling constant \(\kappa\), the 5-dimensional Newton constant \(G_{(5)}\), and the Planck mass \(M_{(5)}\) are related by

\[
\kappa^2 = 8\pi G_{(5)} = M_{(5)}^{-3} .
\]

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Furthermore, we follow Randall and Sundrum and relate the bulk energy density $\rho_B$ and the cosmological constant density $\rho_\Lambda$ by

$$\rho_B = \frac{\kappa^2 \rho_\Lambda^2}{6}.$$  \hspace{1cm} (65)

At this point all constants are defined in terms of the Planck mass, and our discussion of the evolution of gravity signs can be established.

For the matter dominated case, $\omega = 0$, we experimented using different initial conditions. In general, we prefer to start with $y \neq 0$ in order to avoid any spurious solution in the differential equation, which is rather singular. We thus suppose that $y$ starts at the order of the Planck length. Pictures 2 to 4 show some results. We have chosen to plot the adimensional function $z(x) = \mu y(x)$, where $\mu$ corresponds to twice Planck mass units $M_P$ and $x = t/t_0$, $t_0$ being the present age of the universe.
Figure 6: Same as before, with vanishing initial position with respect to the brane.

Each graph contains a set of curves corresponding to three typical velocities, whose values are shown in the legend of each graph, producing similar behaviors. In figures 2 and 3 we use negative initial velocities and, independently of the chosen initial point $y$, the curve decays and escapes, never returning to the same brane. In the case of positive initial velocities, picture 4 shows three curves from which we can notice that the greater initial velocity is, the further away from the brane the object will travel.

Summarizing, these graphs show that the gravity wave always “tries to follow the brane”, since the $y$ coordinate either drops fast to zero, or drives away, which means that the final point reached is far from the original brane.

We thus conjecture, based on these results, that the shortest path is inside the brane, being the one followed by light. However, there is certainly room for further paths due to the extremely complicated character...
of the differential equation involved in the problem. Moreover, there seems to be some attractors in the differential equation, which further complicate the matter, rendering a possible solution even more obscure, while opening further possibilities of shortcuts, especially in cases where the bulk density becomes important.

Such complications actually do not arise in full in the matter dominated case, but can be clearly seen in the radiation dominated era. In these cases, solutions are shown in figures 5 through 7. Again, we have plotted the adimensional function \( z(x) \), where \( x = M_P t \) in this case.

Pictures 5 and 6 show a plateau behavior for low positive initial velocities; however, there is a threshold velocity for which the curve decouples and escapes to infinity. Picture 7 shows curves for three negative initial velocities. Again, the wave tries to follow the brane from a distance depending on the initial velocity value as we had seen in matter dominated case.

Figure 7: Same as before, with negative initial velocity
In the radiation dominated era, $\omega = \frac{1}{3}$, attractors are more clearly formed. Their meaning is not known and in some cases, where we can avoid dropping into them using special initial conditions, it is natural to foresee solutions which return to the brane after a roundabout in the bulk, although we have to stress that no such solution has been found so far. We leave this more difficult numerical problem for a future publication.

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