INTEGRATION IN THE GHP FORMALISM III: FINDING ALL CONFORMALLY FLAT RADIATION METRICS AS AN EXAMPLE OF AN ‘OPTIMAL SITUATION’.

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Abstract.

Held has proposed an integration procedure within the GHP formalism built around four real, functionally independent, zero-weighted scalars. He suggests that such a procedure would be particularly simple for the ‘optimal situation’, when the formalism directly supplies the full quota of four scalars of this type; a spacetime without any Killing vectors would be such a situation. Wils has recently obtained a metric which he claims is the only conformally flat, pure radiation metric which is not a plane wave; this metric has been shown by Koutras to admit no Killing vectors, in general. Therefore, as a simple illustration of the GHP integration procedure, we obtain systematically the complete class of conformally flat, pure radiation metrics. Our result shows that the conformally flat, pure radiation metrics, which are not plane waves, are a larger class than Wils has obtained.

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1. INTRODUCTION.

About 20 years ago, soon after the introduction of the GHP formalism [1], Held proposed — in principle — a strikingly simple procedure for integration within this formalism: manipulate the complete system of GHP equations with the aim of reducing them to a complete involutive set of tables for the action of the four GHP operators \( \hat{p}, \hat{p}', \partial, \partial' \) on six real GHP quantities, [2]. If such an ‘optimal situation’ can be achieved Held emphasised that the problem of integrating the field equations was essentially solved.

To avoid misunderstandings, we point out that the ‘complete system of GHP equations’ consists of the GHP Ricci equations, the GHP Bianchi equations, and the GHP commutator equations, [3,4]: while the ‘GHP quantities’ consist of the GHP spin coefficients and the tetrad components of the Riemann tensor, together with any GHP operator-derivatives of these quantities which arise in the reduction procedure. We now envisage the ‘optimal situation’ more precisely as follows: a complete system of GHP equations reduces to a complete involutive set of tables for the action of the four GHP operators on four real, zero-weighted, functionally independent GHP quantities, and on one complex, ‘weighted GHP quantity’, [4,5]. We will refer to the four real, zero-weighted, functionally independent quantities as ‘coordinate candidates’ since they are the obvious, and often the most convenient, choice for the coordinates when a coordinate description of the geometry and an explicit metric is required; for a complex, ‘weighted GHP quantity’ we will require that neither spin nor boost weight of this complex quantity is zero, i.e. \( s \neq 0 \neq t \), or equivalently for GHP weights, \( p \neq \pm q \) in the usual notations, [1].

Once we have obtained such a situation the problem is indeed solved, since we can then use the tables to write down immediately a coordinate description of the GHP operators \( \hat{p}, \hat{p}', \partial, \partial' \) (equivalently the tetrad vectors \( \mathbf{l}, \mathbf{n}, \mathbf{m}, \mathbf{\bar{m}} \)) in terms of the coordinate candidates by applying the relationship for zero-weighted GHP quantities,

\[
\nabla_i = l_i \hat{p}' + n_i \hat{p} - m_i \partial' - \bar{m}_i \partial \tag{1.1}
\]

to each of the four coordinate candidates in turn. The complex weighted quantity will, of course, be cancelled out when the metric is constructed from the tetrad vectors by

\[
g_{ij} = l_i n_j + l_j n_i - m_i \bar{m}_j - m_j \bar{m}_i \tag{1.2}
\]

Moreover, if required, the explicit form of the tetrad vectors can be simplified by using the remaining freedom in the GHP formalism — usually by gauging the complex weighted quantity to unity.
Surprisingly, until now, it has not been possible to construct an example of such an optimal situation, and so we have no explicit confirmation that Held’s procedure works in practice. Early attempts, [2,5 - 9] to integrate the GHP equations always failed to generate the full quota of coordinate candidates; and so in these cases it was thought necessary to introduce coordinates from outside the GHP formalism, and to complete the integration by reverting to the NP formalism [10], and associated explicit coordinate techniques [11,12].

We now understand better why those difficulties occurred; those early investigations were carried out in specialised spaces, which usually meant that they contained at least one Killing vector. This, of course, leads to at least one cyclic coordinate, and so the shortage of explicit coordinate candidates in the GHP approach is not surprising. (We emphasise that we are not saying that the presence of a Killing vector always leads to a shortage of coordinate candidates; the precise relationship between Killing vectors, tetrad vectors and coordinate candidates needs careful consideration, which we will present, in detail, elsewhere.)

However, recently, in such ‘less than optimal’ situations where Killing vectors are present, it has been shown that it is possible to integrate the field equations completely within the GHP formalism, [4,13]; therefore, the coordinate- and gauge-free spirit of Held’s original procedure is upheld, although in such situations extra quantities — additional to the original GHP quantities — have to be introduced in order to generate the full quota of coordinate candidates. (We should also mention some recent work by Kolassis [14],[15] who has retained some of the spirit of Held’s approach by using zero-weighted tetrad vectors, although not zero-weighted coordinates, in general.)

So we have the apparent paradox: Held’s approach promises to be simpler, at least in principle, for spaces without Killing vectors — something contrary to our experience with other approaches; on the other hand, there does not exist an explicit example to illustrate this supposedly simpler optimal situation.

Recently, Wils [16] has obtained a comparatively simple metric and Koutras [17] has pointed out that, in general, this spacetime contains no Killing vectors. In view of the discussion above, such a metric is of interest in the context of the GHP integration procedure.

Since Held’s optimal situation suggests the possibility of a simple procedure for investigating spacetimes without symmetries, we decided to see whether this procedure does, in fact, work in practice for the Wils metric. So the original purpose of this paper was to
rederive Wils’ metric, and hence give an explicit demonstration of the GHP integration procedure, in a comparatively simple, but non-trivial, optimal situation.

However, it turns out that — from the same starting point of conformally flat, pure radiation metrics — we obtain a more general metric than Wils; in fact our expression includes the metrics of Wils [16] and Koutras and McIntosh [18] respectively, as special cases. This illustrates precisely one advantage of our GHP integration method compared to the familiar NP [10 - 12] coordinate-based method. In the latter method, a careful account of coordinate and tetrad freedom is required, which is gradually used up, often in a considerable number of different steps, involving long tortuous calculations; not surprisingly, sometimes something is overlooked. For this particular case, as Wils has pointed out, Kramer et al. [19] mistakenly concluded that the only conformally flat pure radiation metric were the plane waves, found by McLenaghan et al. [20]; Wils [16], in turn, mistakenly concluded that his metric — from the Kundt class [21] but not representing plane waves — completed the class of conformally flat pure radiation metrics, which are not plane waves. However, because of the simplicity of our calculations, and their susceptibility to easy confirmation, we are able to state unambiguously that our more general metric completes this class of conformally flat, pure radiation metrics, which are not plane waves. Its determination, in the familiar way, from the Kundt form of the metric, has been reported in [22]. Independent confirmation that the metric presented here is indeed more general than the Wils metric [16] has been given by Skea [23], who has used the invariant classification of the CLASSI program [24]. The detailed classification by Skea of this metric, and of the Wils metric [16], is available at the on-line exact solutions database in Brazil and in Canada, [23].

In Section 2 we give a simple step-by-step illustration of our method, for conformally flat, pure radiation metrics which are not plane waves — in the generic case. In this situation we are able to choose all four of our coordinate candidates directly from the GHP quantities — precisely as Held had envisaged; but this of course involves the additional constraint that none of these four quantities is constant. In Section 3, by a slight modification, we find it is easy to obtain our complete class of these conformally flat pure radiation metrics — the generic case together with the excluded special cases — in one explicit expression. In this situation we are able to choose three of our coordinate candidates directly from the GHP quantities, and the remaining coordinate candidate indirectly.
In Section 4 we present the complete metric in the more familiar Kundt form, [19,21]; we also give an alternative version, in a form which retains some coordinate freedom. In Section 5 we discuss further the principles and advantages of our method.

As a preliminary step, at the beginning of the calculation, it is advantageous to fix (almost) completely the freedom of the two real null vectors \( l, n \); the details of this choice are given in the Appendix.

For those familiar with the NP coordinate integration procedures, [10 - 12] it may help to emphasise the fundamental differences in the approach illustrated for the GHP formalism in this paper. In order to obtain the metric for a particular class of spacetimes, the NP approach begins by making a tentative choice of a metric form, in a preferred coordinate system and tetrad frame (suggested by the geometry of the class), and determines what coordinate and tetrad freedom exists to retain this tentative metric form. Next, the three sets of NP equations — the Bianchi equations, the Ricci equations and the metric equations — are written down in these coordinates, and they are integrated step-by-step; to facilitate the integration, the coordinate and tetrad freedom is gradually used up, making the metric more precise, but still within the tentative form. Furthermore, the tetrad and coordinate freedom are linked together — a change in one, usually necessitates a change in the other.

In the approach in this paper, there is no tentative preliminary choice of metric nor of coordinate system; instead the three sets of GHP equations — the Bianchi equations, the Ricci equations and the commutator equations — are immediately simplified, within the operator notation — a process which is equivalent to integrating the field equations. Once this process is completed — in the form of six tables — the coordinates are chosen in terms of zero-weighted GHP quantities; in the optimal situation, these coordinates are chosen directly and uniquely, with no coordinate freedom. The metric only occurs at the very last step, and can be written down from the tables, in terms of these coordinates. *We emphasise that in our approach the metric form has been dictated by the nature of the calculations themselves, and that only zero-weighted GHP quantities are chosen as coordinates.* Furthermore, the remaining GHP tetrad freedom — built into the single weighted complex quantity of our analysis — is completely decoupled from our coordinate candidates, since they are zero-weighted; changing the GHP gauge does not affect the coordinate candidates.

In this particular application we fixed (almost) completely the two real null vectors \( l, n \) at the beginning of the calculation; but this is not essential, and the calculation can be
carried out with some, or all, of the freedom of these two null rotations; but in such cases the final metric may not be in its simplest or neatest form.
2. THE INTEGRATION PROCEDURE: THE GENERIC CASE.

2.1. Preliminary simplifications.
We consider the conformally flat spacetimes with energy momentum tensor given by

\[ T_{ij} = \Phi^2 l_i l_j \]  

(2.1)

where \( \Phi \) is a scalar function, and \( l_i \) is a null vector. If we identify this null vector with its counterpart in the usual null tetrad \( l, n, m, \bar{m} \) we obtain, in usual GHP notation,

\[ \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = \Psi_4 = 0 \]  

(2.2a)

\[ \Phi_{00} = \Phi_{11} = \Phi_{01} = \Phi_{02} = \Phi_{12} = \Lambda = 0, \quad \Phi_{22} = \Phi^2 \]  

(2.2b)

When these values are substituted into the GHP Bianchi equations, it follows immediately that

\[ \sigma = \rho = \kappa = 0 \]  

(2.3)

*At this stage it is necessary to subdivide our problem into two cases: (i) \( \tau = 0 \), (ii) \( \tau \neq 0 \). In this paper we shall restrict our explicit attention to the second of these cases. The first of these cases corresponds to a subset of the plane wave spacetimes [19,21].*

We still have the freedom to choose the tetrad vector \( n \) up to a null rotation about \( l \), and exploiting this freedom we can choose (see Appendix),

\[ \tau' = \sigma' = \rho' = 0 \]  

(2.4a)

\[ \Phi^2 - \tau \kappa' - \bar{\tau} \bar{\kappa}' = 0, \]  

(2.4b)

\[ \bar{p}'(\tau/\bar{\tau}) = 0, \]  

(2.4c)

Since the Bianchi equations for \( \Phi_{22} \) are identically satisfied under the substitution (2.4b), the only remaining equations from the Ricci and Bianchi equations GHP equations are,

\[ \bar{p}\tau = 0 \quad \bar{p}\kappa' = 0 \]

\[ \partial\tau = \tau^2 \quad \partial\kappa' = -\bar{\tau} \bar{\kappa}' \]  

(2.5)

\[ \partial'\tau = \tau\bar{\tau} \quad \partial'\kappa' = \bar{\tau}\kappa' \]
The commutators become
\[ [\mathbf{P}, \mathbf{P}'] = \bar{\tau} \partial + \tau \partial' \]
\[ [\partial, \partial'] = 0 \]
\[ [\mathbf{P}, \partial] = 0 = [\mathbf{P}, \partial'] \] \hspace{1cm} (2.6)
\[ [\mathbf{P}', \partial] = -\bar{\kappa}' \mathbf{P} - \tau \mathbf{P}' \]
\[ [\mathbf{P}', \partial'] = -\kappa' \mathbf{P} - \bar{\tau} \mathbf{P}' \]

It is very important to note that in order to extract all the information from these commutator equations, they must be applied explicitly to all four coordinate candidates, and to a weighted complex quantity, [3].

2.2. Finding four coordinate candidates, and extracting all the information from the complete system.

The spin coefficients and Riemann tensor components therefore supply four real quantities (from complex \( \tau, \kappa' \)) which can easily be rearranged into two real zero-weighted and one complex weighted quantity. The simplest zero-weighted quantities would appear to be \( \tau \bar{\tau} \) and \( \tau \kappa' / \bar{\kappa}' \), while an obvious weighted quantity — especially in view of the gauge choice (2.4c) — is \( (\tau / \bar{\tau}) \); these quantities satisfy respectively,

\[
\partial (\tau \bar{\tau}) = 2\tau^2 \bar{\tau} \\
\partial (\tau \kappa' / \bar{\kappa}') = -\tau \left( 1 + \tau \kappa' / \bar{\kappa}' \right) \\
\partial (\tau / \bar{\tau}) = 0
\] \hspace{1cm} (2.7)

and from these quantities it is straightforward to construct a set of tables. However, to keep the presentation of subsequent calculations to a minimum, it will be convenient to begin instead with the two real combinations of these two zero-weighted quantities,

\[
A = \frac{1}{\sqrt{2\tau \bar{\tau}}} \] \hspace{1cm} (2.8a)

\[
B = \frac{i(\tau \kappa' - \bar{\kappa}' \bar{\tau})}{(\tau \kappa' + \bar{\kappa}' \bar{\tau})\sqrt{2\tau \bar{\tau}}} = iA \frac{-1 + (\tau \kappa' / \bar{\kappa}' \bar{\tau})}{1 + (\tau \kappa' / \bar{\kappa}' \bar{\tau})} \] \hspace{1cm} (2.8b)

and with the complex weighted quantity \((PQ)\) (with GHP weights \((0,1))\), given by,

\[
P = \sqrt{\frac{\tau}{2\bar{\tau}}} \quad \text{with} \quad P \bar{P} = \frac{1}{2}
\] \hspace{1cm} (2.9a)
\[ Q = \frac{\sqrt{\tau \kappa' + \bar{\tau} \bar{\kappa}'}}{\sqrt{2\tau \bar{\tau}}} \]  

(2.9b)

(Note that from (2.4b) the term $\tau \kappa' + \bar{\tau} \bar{\kappa}'$ is positive.)

These particular choices of $A, B, P, Q$ have been made because they enable us to replace (2.7) with the very simple equations

\[ \partial A = -P \quad \partial B = -iP \quad (2.10) \]

\[ \partial P = 0 \quad \partial Q = 0 \quad (2.11) \]

(We have already assumed from the beginning of this section that $\tau \neq 0$, and clearly in the gauge (2.4b), $\kappa' \neq 0$; therefore $A, B, P, Q$ will always be defined, and differ from zero.)

Complete tables for $A, B, P, Q$ can now be presented,

\[ \begin{align*}
\mathfrak{P}A &= 0 & \mathfrak{P}B &= 0 \\
\partial A &= -P & \partial B &= -iP \\
\partial' A &= -\bar{P} & \partial' B &= i\bar{P} \\
\mathfrak{P}' A &= Q\mathfrak{C}/A & \mathfrak{P}' B &= Q\mathfrak{E}/A \\
\mathfrak{P}' P &= 0 & \mathfrak{P}' Q &= 0 \\
\partial P &= 0 & \partial Q &= 0 \\
\partial' P &= 0 & \partial' Q &= 0 \\
\mathfrak{P}' P &= 0 & \mathfrak{P}' Q &= Q^2G/A
\end{align*} \]

(2.12)

by introducing the three real zero-weighted quantities $C, E, G$ respectively — as yet undetermined. (We have introduced the factor $\frac{Q}{A}$ in the above definitions simply for convenience in later calculations.)

Since neither $A$ nor $B$ can be constant, and also since

\[ \nabla A \neq \lambda \nabla B \]

(2.13)

for any scalar $\lambda$, clearly $A$ and $B$ are functionally independent, and can be adopted as coordinate candidates. Therefore, we next have to apply the commutators (2.6) to $A, B$, and also to the weighted $P, Q$. This results in the non-trivial equations respectively,

\[ \begin{align*}
\mathfrak{P}C &= -1/Q & \mathfrak{P}E &= 0 & \mathfrak{P}G &= 0 \\
\partial C &= 0 & \partial E &= 0 & \partial G &= 0 \\
\partial' C &= 0 & \partial' E &= 0 & \partial' G &= 0
\end{align*} \]

(2.14)
At this stage, we still require two more coordinate candidates, in addition to $A, B$, to make up our full quota. Since $C$ cannot be constant, and also since

$$\nabla C \neq \lambda \nabla B + \mu \nabla B$$  \hspace{1cm} (2.15)

for any scalars $\lambda, \mu$, clearly $C$ is functionally independent of $A$ and $B$, and can be adopted as the third coordinate candidate. So we therefore obtain a table for $C$,

$$\begin{align*}
\hat{p} C &= -1/Q \\
\partial C &= 0 \\
\partial' C &= 0 \\
\hat{p}' C &= QJ/A
\end{align*}$$  \hspace{1cm} (2.16)

which is completed with the real zero-weighted quantity $J$ — as yet undetermined.

When we apply the commutators (2.6) to the third coordinate candidate $C$ we obtain,

$$\begin{align*}
\hat{p} J &= G/Q \\
\partial J &= P(A + iB) \\
\partial' J &= \bar{P}(A - iB)
\end{align*}$$  \hspace{1cm} (2.17)

Rearranging we define the real zero-weighted quantity $S$ by

$$S = J + CG + (A^2 + B^2)/2$$  \hspace{1cm} (2.18)

so that

$$\begin{align*}
\hat{p} S &= 0 \\
\partial S &= 0 \\
\partial' S &= 0
\end{align*}$$  \hspace{1cm} (2.19)

The obvious choice for our fourth coordinate candidate is $E$; but of course that is only possible if $E$ is not constant.

*In the remainder of this section we shall consider only the generic case where $E$ is not a constant.*

So we therefore complete a table for $E$,

$$\begin{align*}
\hat{p} E &= 0 \\
\partial E &= 0 \\
\partial' E &= 0 \\
\hat{p}' E &= QH/A
\end{align*}$$  \hspace{1cm} (2.20)
where the real, zero-weighted quantity $H$ is as yet undetermined.

A check on the determinant formed from the four tables for $A, B, C, E$ respectively shows that all four quantities are functionally independent, and so we can adopt $A, B, C, E$ as our four coordinate candidates.

The only information in the GHP field equations still unused is now obtained by applying the commutators to the last coordinate candidate $E$, obtaining

$$\begin{align*}
\mathfrak{p} H &= 0 \\
\partial H &= 0 \\
\partial' H &= 0
\end{align*}$$

(2.21)

So we have extracted all the information from the GHP Ricci, Bianchi and commutator equations.

2.3. The six tables.

We now have the following six tables,

$$\begin{align*}
\mathfrak{p} A &= 0 & \mathfrak{p} B &= 0 \\
\partial A &= -P & \partial B &= -iP \\
\partial' A &= -\bar{P} & \partial' B &= i\bar{P} \\
\mathfrak{p}' A &= QC/A & \mathfrak{p}' B &= QE/A \\
\mathfrak{p} C &= -1/Q & \mathfrak{p} E &= 0 \\
\partial C &= 0 & \partial E &= 0 \\
\partial' C &= 0 & \partial' E &= 0 \\
\mathfrak{p}' C &= Q(S - CG - \frac{1}{2} A^2 - \frac{1}{2} B^2)/A & \mathfrak{p}' E &= QH/A \\
\mathfrak{p} P &= 0 & \mathfrak{p} Q &= 0 \\
\partial P &= 0 & \partial Q &= 0 \\
\partial' P &= 0 & \partial' Q &= 0 \\
\mathfrak{p}' P &= 0 & \mathfrak{p}' Q &= Q^2 G/A
\end{align*}$$

(2.22)
where the real zero-weighted quantities $G, H, S$ satisfy
\[ \mathcal{P} G = 0 = \partial G \]
\[ \mathcal{P} H = 0 = \partial H \]
\[ \mathcal{P} S = 0 = \partial S \] (2.23)

Strictly speaking the six tables (2.22) are not involutive since they have to be supplemented by (2.23). However, it follows from (2.23) that $G, H, S$ are functions only of the one coordinate candidate, $E$ and so by stipulating that $G, H, S$ be functions only of $E$, we no longer need to write out (2.23) explicitly, and the tables are essentially involutive; hence the problem is essentially solved.

2.4. Using coordinate candidates as coordinates.
We now make an obvious choice of the coordinate candidates as the coordinates, $e, c, a, b$,
\[ e = E, \quad c = C, \quad a = A, \quad b = B \] (2.24)

Using (1.1) as follows
\[ l^e = l^i \nabla_i(e) = \mathcal{P}(E) \quad \text{etc.} \]
we can write down the tetrad vectors immediately in the $e, c, a, b$ coordinates from the respective tables as
\[ l^i = (0, -\frac{1}{Q}, 0, 0) \]
\[ n^i = \frac{Q}{a}(H, (S - Gc - \frac{1}{2}a^2 - \frac{1}{2}b^2), c, e) \] (2.25)
\[ m^i = P(0, 0, -1, -i) \]
\[ \bar{m}^i = \bar{P}(0, 0, -1, i) \]
and the metric is given by,
\[ g^{ij} = \begin{pmatrix} 0 & -H/a & 0 & 0 \\ -H/a & (2S + 2Gc + a^2 + b^2)/a & -c/a & -e/a \\ 0 & -c/a & -1 & 0 \\ 0 & -e/a & 0 & -1 \end{pmatrix} \] (2.26)

where $G, H, S$ are arbitrary functions of the coordinate $e$; clearly $H$ cannot be zero.

This has completed our integration procedure, and we have obtained the metric for all spacetimes satisfying (2.1,2), which are subject to the restrictions $\tau \neq 0$, and — with respect to the gauge chosen here — $E$ not a constant.
3. THE INTEGRATION PROCEDURE: THE COMPLETE METRIC.

3.1. Preliminaries.

In the previous section 2.2 we assumed that $E$ was not a constant, so that we were able to choose it as our fourth coordinate candidate. Next, we should look at the excluded case where $E$ is a constant. In such a situation, clearly $H$ is zero, but we still have the possibility of choosing $G$ or $S$ as our fourth coordinate. Once we make such a choice then we could continue in a similar manner as in the last section, building our tables, and hence the tetrad, around our four coordinate candidates. However, if all of the functions $E, G, S$ are constants, then it will not be possible to find the fourth coordinate candidate directly; we emphasise that in such circumstances no additional independent quantities can be generated by any direct manipulations of the tables and the commutators. In such a situation we still need a fourth coordinate candidate in order to extract all the information from the commutators. Clearly some indirect way is required to obtain this information. We shall now show that such an indirect approach to the fourth coordinate can in fact be used in general, so that we can obtain the complete metric as one expression.

3.2. Finding a fourth coordinate candidate indirectly, and extracting all the information from the complete system.

The results in Section 2 up to (2.19) apply, and therefore when we write out our tables explicitly we obtain (2.22) — except that the table for the quantity $E$ is missing. Clearly we do not have our full quota of four coordinate candidates, but we do not wish to use any of the remaining quantities in the five tables, since it would involve the additional condition of that quantity being non-constant. However, we know that we have not yet extracted all the information from the commutators (2.6), since they have only been applied to three zero-weighted coordinate candidates. So we examine the commutators,

\[
\begin{align*}
[\tilde{P}, \tilde{P}'] &= (\bar{P}/A)\partial + (P/A)\partial' \\
[\partial, \partial'] &= 0 \\
[\tilde{P}, \partial] &= 0 = [\tilde{P}, \partial'] \\
[\tilde{P}', \partial] &= (PQ^2(A + iB)/A)\tilde{P} - (P/A)\tilde{P}' \\
[\tilde{P}', \partial'] &= (PQ^2(A - iB)/A)\tilde{P} - (\bar{P}/A)\tilde{P}'
\end{align*}
\]

(3.1)

to determine whether they suggest the existence of a fourth zero-weighted quantity, functionally independent of the first three coordinate candidates, whose table is consistent with
the commutators. In fact, we get a strong hint from the previous section, and consider the possibility of the existence of a real zero-weighted quantity $T$, which satisfies the table

\begin{align*}
\bar{p}T &= 0 \\
\partial T &= 0 \\
\partial' T &= 0 \\
\bar{p}'T &= Q/A
\end{align*}

(3.2)

It is straightforward to confirm that such a choice is consistent with the commutators and the other five tables in (2.22). Furthermore, a check on the determinant formed from the four tables for $A, B, C, T$ respectively, shows that all four quantities are functionally independent.

### 3.3 The six tables.

Therefore, in the set of tables (2.22), we can replace the table for $E$ with the table (3.2) for $T$, and the real zero-weighted quantities $E, G, S$ satisfy

\begin{align*}
\bar{p}E &= 0 = \partial E \\
\bar{p}G &= 0 = \partial G \\
\bar{p}S &= 0 = \partial S
\end{align*}

(3.3)

### 3.4 Using coordinate candidates as coordinates.

We now make the obvious choice of the coordinate candidates as the coordinates,

\begin{align*}
t &= T, \quad c = C, \quad a = A, \quad b = B
\end{align*}

(3.4)

where the only coordinate freedom is for $t$ up to an additive constant. We can write down the tetrad vectors immediately in the $t, c, a, b$ coordinates from the respective tables as

\begin{align*}
l^i &= (0, -\frac{1}{Q}, 0, 0) \\
n^i &= \frac{Q}{a} \left(1, (S - Gc - \frac{1}{2}a^2 - \frac{1}{2}b^2), c, E\right) \\
m^i &= P(0, 0, -1, -i) \\
\bar{m}^i &= \bar{P}(0, 0, -1, i)
\end{align*}

(3.5)
and therefore the metric is given by,

\[
g^{ij} = \begin{pmatrix}
0 & -1/a & 0 & 0 \\
-1/a & -2S + 2Gc + a^2 + b^2)/a & -c/a & -E/a \\
0 & -c/a & -1 & 0 \\
0 & -E/c & 0 & -1 \\
\end{pmatrix}
\]  

(3.6)

where \( E, G, S \) are arbitrary functions of the coordinate \( t \). This form now includes the possibility of \( E \) (or \( G \), or \( S \)) being constant.
4. ALTERNATIVE FORMS FOR THE COMPLETE METRIC.

Although the expression (3.6) is in a concise form, we can make a coordinate transformation,

\[ u = U(t) \quad v = -ca/V(t) \quad x = -a \quad y = b + W(t)/2 \]  \quad (4.1)

which will take the metric into the familiar Kundt form [19,21], in coordinates \( u, v, x, y \),

\[
g^{ij} = \begin{pmatrix}
0 & \dot{U}/V & 0 & 0 \\
\dot{V}/V & Z & -2v/x & (2E+W)/2V \\
0 & -2v/x & -1 & 2V/0 \\
0 & (2E+W)/2V & 0 & -1
\end{pmatrix}
\]  \quad (4.2)

where

\[ Z = -3v^2/x^2 - 2v/V^2(\dot{V} + VG) - x/V^2(-2S + (W^2/4) - W y + x^2 + y^2) \]

and \( \cdot \) denotes differentiation with respect to \( t \).

Choosing

\[ W(e) = -2 \int Ede, \quad V(e) = \exp(- \int Gde), \quad U(e) = \int Vde, \]  \quad (4.3)

we find the metric in the \( u, v, x, y \) coordinates becomes,

\[
g^{ij} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & -2f(u)x(x^2 + y^2 + g(u)y + h(u)) - 3v^2/x^2 & -2v/x & 0 \\
0 & -2v/x & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]  \quad (4.4)

where \( f(\neq 0), g, h \) are arbitrary functions of the coordinate \( u \). (There are of course other possible ways to exploit this coordinate transformation.)

We emphasise that in our method we do not always have to fix our coordinates completely as we did in Section 3 and above. Instead, we could have introduced the fourth coordinate with some freedom via the table for \( \tilde{T} \), given by

\[ \mathcal{P}\tilde{T} = 0 \]
\[ \partial\tilde{T} = 0 \]
\[ \partial'\tilde{T} = 0 \]
\[ \mathcal{P}'\tilde{T} = Q\tilde{H}/A \]  \quad (4.5)
where $\tilde{H}$ is an arbitrary quantity, as yet undetermined. By this approach we could have obtained the complete metric in $\tilde{t}(= \tilde{T})$, $c, a, b$ coordinates,

$$g^{ij} = \begin{pmatrix} 0 & -\tilde{H}(\tilde{t})/a & 0 & 0 \\ -\tilde{H}(\tilde{t})/a & \left(-2\tilde{S}(\tilde{t}) + 2\tilde{G}(\tilde{t})a + b^2\right)/a & -c/a & -E(\tilde{t})/a \\ 0 & -c/a & -1 & 0 \\ 0 & -E(\tilde{t})/a & 0 & -1 \end{pmatrix}$$ (4.6)

where $E, \tilde{G}, \tilde{H}, \tilde{S}$ are now arbitrary functions of the coordinate $\tilde{t}$. There remains the freedom $\tilde{t} \rightarrow F(\tilde{t})$ where $F$ is an arbitrary function of $\tilde{t}$ which can be used to choose: (i) $\tilde{t} = E(\tilde{t})$ (if $E$ is not constant, as in Section 2); or (ii) $\tilde{t}$ as some other of the (non-constant) quantities $\tilde{G}, \tilde{S}, \tilde{H}$; or (iii) $\tilde{H} = 1$ (as in Section 3); or (iv) other convenient possibilities.

For completeness we present the covariant form of (4.4) in $u, v, x, y$ coordinates,

$$g_{ij} = \begin{pmatrix} 2f(u)x\left(x^2 + y^2 + g(u)y + h(u)\right) - v^2/x^2 & 1 & -2v/x & 0 \\ 1 & 0 & 0 & 0 \\ -2v/x & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$ (4.7)

This metric — subject to a difference in sign convention — agrees with the form given in [22], and Wils’ special case [16] is obtained when $g = 0 = h$. 

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5. DISCUSSION.

The metric given in (3.6) — equivalently (4.4) — represents the complete class of all conformally flat, pure radiation metrics, which are not plane waves.

However, the primary purpose of this paper was to illustrate our method, and demonstrate how it differs from the more familiar NP tetrad approach. One important aspect of this approach is the crucial role of the commutators as field equations in their own right, and the need to extract all the information from them in a systematic manner; in fact, in this example very little information came directly from the Ricci and Bianchi equations, and the major part via the commutators. In addition, we wished to emphasise the fact that in this method, ideally, the coordinates are fixed directly and completely in terms of zero-weighted GHP quantities — probably the most significant difference from the NP approach; this was illustrated explicitly in Section 2 in the choice of all four coordinate candidates. In Section 3, the situation was a little different because we choose not to equate our fourth coordinate candidate directly to a GHP quantity; rather, the fourth coordinate candidate $T$ had to be introduced as a ‘potential’ (essentially an integral) of GHP quantities. But, also in this case, the coordinate candidate was chosen uniquely — without any coordinate freedom, except for an additive constant.

To emphasise that in our method we are not always bound to fix our coordinates completely, we showed in Section 4 how we can permit a measure of coordinate freedom; although with the dangers and complications inherent in introducing such coordinate freedom, it is often better to fix the coordinates where possible.

In this paper we have retained the term ‘coordinate candidates’ for our four real functionally independent scalars, although, in fact, we do eventually choose them as our coordinates. We use this term because we prefer to distinguish between the role of the coordinate candidates in extracting all the information from the commutators, and their possible additional and optional role as coordinates in the final explicit statement of the metric. Especially in less than optimal situations, it may not be convenient to choose the coordinate candidates as the eventual coordinates.

In those special circumstances when the formalism does not directly yield four coordinate candidates — for instance when $E,G,H$ are all constants in (3.6) — the spacetime admits a Killing vector, since in this case, the fourth coordinate candidate is cyclic. Of course, this does not mean that we can conclude that Killing vectors are absent in other cases, for the general metric. The explicit links between Killing vectors, tetrad vectors and the
existence of coordinate candidates in the GHP formalism will be considered elsewhere.

Although this particular example, which we have chosen to illustrate our GHP integration method, underlines the simplicity and conciseness of the method compared to the NP coordinate approach, we emphasise that these are not our only reasons for developing this method. As Held has pointed out, and demonstrated in [6,7], such a method has the potential to extract additional information when other methods have been brought to a stop. In addition, the procedure followed has much in common with aspects of the Karlhede classification of spacetimes [25]; in fact, once a spacetime has been obtained by the method in this paper, its Karlhede classification — by a GHP approach similar to that introduced by Collins et al. [26] — is a comparatively simple undertaking.

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Assuming (2.2,3), under the null rotation
\[ \begin{align*}
l^i & \rightarrow l^i \\
m^i & \rightarrow m^i + Zl^i \\
m^i & \rightarrow \bar{m}^i + Zl^i \\
n^i & \rightarrow n^i + Zm^i + \bar{\bar{m}} + Zl^i
\end{align*} \]
the zero valued spin coefficients \( \kappa, \rho, \sigma \) do not change, while
\[ \begin{align*}
\tau & \rightarrow \tau \\
\tau' & \rightarrow \tau' - \bar{\Phi}Z \\
\rho' & \rightarrow \rho' - \partial Z - \bar{Z}(\tau' - \bar{\Phi}Z) \\
\sigma' & \rightarrow \sigma' - \partial' Z - Z(\tau' - \bar{\Phi}Z) \\
\kappa' & \rightarrow \kappa' - \bar{\Phi}'Z - Z(\rho' - \partial Z) - \bar{Z}(\sigma' - \partial' Z) - \bar{Z}(\tau' - \bar{\Phi}Z) - Z^2 \tau
\end{align*} \]
\( \Phi_{22} \rightarrow \Phi_{22} \)

We could therefore choose \( Z \) such that \( \tau', \rho', \sigma', \kappa' \) are all zero, providing the choices
\[ \begin{align*}
\tau' - \bar{\Phi}Z = 0 \\
\rho' - \partial Z = 0 \\
\sigma' - \partial' Z = 0 \\
\kappa' - \bar{\Phi}'Z - Z^2 \tau = 0
\end{align*} \]
are consistent with the commutator equations for \( Z \); but this is not the case.

However, a careful examination of the calculations leads us to note that we can choose
\[ \begin{align*}
\tau' - \bar{\Phi}Z = 0 \\
\rho' - \partial Z = 0 \\
\sigma' - \partial' Z = 0 \\
\Phi_{22} - \tau \kappa' - \bar{\tau} \bar{\kappa}' + \tau \bar{\Phi}'Z + \bar{\tau} \Phi'Z + Z^2 \tau^2 + Z^2 \bar{\tau}^2 = 0
\end{align*} \]
since these choices are consistent with the relevant commutators for \( Z \).

(As an illustration, we consider the commutator
\[ [\bar{\Phi}, \partial]Z = \bar{\tau}' \bar{\Phi}Z \]
and the substitution of the first two equations of (A4) results in
\[ \bar{P}\rho' - \partial \tau' + \bar{\tau}' \tau = 0 \]
which is one of the Ricci identities. For all other commutators, we also get one of the Ricci identities.)

There is clearly some freedom left in our choice for \( Z \), and noting that,
\[ \bar{P}' \rightarrow \bar{P}' + Z\partial + \bar{Z}\partial' + \bar{Z}Z\bar{P} - pZ\tau - q\bar{Z}\bar{\tau} \quad (A5) \]
we see that
\[ \bar{P}'(\tau/\bar{\tau}) \rightarrow \bar{P}'(\tau/\bar{\tau}) - 2(Z\tau - \bar{Z}\bar{\tau})\tau/\bar{\tau} \quad (A6) \]
If we choose \( Z \) such that
\[ \bar{P}'(\tau/\bar{\tau}) - 2(Z\tau - \bar{Z}\bar{\tau})\tau/\bar{\tau} = 0 \quad (A7) \]
we find that this choice is consistent with the previous choices (A4), for \( Z \).

We have therefore fixed the behaviour of all four operators on the type \((-2,0)\) quantity \( Z \), as well as fixing part of \( Z \) itself by (A7); the result being that we can choose
\[ \tau' = \rho' = \sigma' = 0 \]
\[ \Phi_{22} - \tau\kappa' - \bar{\tau}\bar{\kappa}' = 0 \]
\[ \bar{P}'(\tau/\bar{\tau}) = 0 \quad (A8) \]
(In an appropriate gauge, we have essentially fixed \( Z \) up to a real constant.)

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