AREA RIGIDITY FOR THE EQUATORIAL DISK IN THE BALL

EZÉQUIEL BARBOSA AND CELSO VIANA

Abstract. It is proved by Brendle in [3] that the equatorial disk $D^k$ has least area among $k$-dimensional free boundary minimal surfaces in the Euclidean ball $B^n$. By comparing the excess of free boundary minimal surfaces with the excess of the associated cones over their boundary, we prove the existence of a gap for the area.

1. Introduction

In these notes, we study the area of $k$-dimensional minimal surfaces in the Euclidean ball $B^n$ that meet $\partial B^n$ orthogonally. These surfaces are critical points of the area functional in the space of $k$-dimensional surfaces with boundary in $\partial B^n$. They are commonly known as free boundary minimal surfaces. The equatorial disk $D^k$ is the simplest example. Brendle [3] proved that $D^k$ is the least area free boundary minimal surface in $B^n$ (see also [5] for the case of 2-dimensional free boundary surfaces). More precisely,

**Theorem 1.1** (Brendle). Let $\Sigma^k$ be a $k$-dimensional free boundary minimal surface in $B^n$. Then

$$|\Sigma^k| \geq |D^k|$$

Moreover, the equality holds if, and only if, $\Sigma^k$ is contained in a $k$-dimensional plane in $\mathbb{R}^n$.

This result is the free boundary analogue of a classical result about closed minimal surfaces in the round sphere $\mathbb{S}^n$. Namely,

**Theorem 1.2.** There exists $\varepsilon(k, n) > 0$ so that whenever $\Sigma^k$ is a $k$-dimensional minimal surface in $\mathbb{S}^n$ which is not totally geodesic, then

$$|\Sigma^k| \geq |\mathbb{S}^k| + \varepsilon(k, n).$$

Despite the proofs of Theorem 1.1 and Theorem 1.2 both explore a monotonicity principle for minimal surfaces, they are quite different. Theorem 1.2, for instance, is only an application of the Monotonicity Formula for minimal surfaces together with the following smooth version of Allard’s Regularity Theorem:
Theorem 1.3 (Allard). There exist $\epsilon(k,n) > 0$, $C > 0$ and $r_0 > 0$ so that whenever $\Sigma$ is a $k$-dimensional minimal surface in $\mathbb{R}^{n+1}$ whose density satisfies

$$\theta(x,r) \leq 1 + \epsilon(k,n)$$

for every $x \in \Sigma$ and every $r < r_0$, then

$$\sup_{\Sigma} |A| \leq C.$$

Indeed, let $\Sigma_i$ be a sequence of $k$-dimensional minimal surfaces in $\mathbb{S}^n$ such that $\lim_{i \to \infty} |\Sigma_i| = A(k,n)$, where $A(k,n)$ is the infimum for the areas of free boundary minimal surfaces in $\mathbb{S}^n$. If $C\Sigma_i$ denotes the minimal cone over $\Sigma_i$ with vertex at 0 and if $y_i \in \Sigma_i$, then

$$\frac{|\Sigma_i|}{|\mathbb{S}^k|} = \lim_{r \to \infty} \frac{|C\Sigma_i \cap B_r(y_i)|}{|B^{k+1}|r^{k+1}} \geq \frac{|C\Sigma_i \cap B_r(y_i)|}{|B^{k+1}|r^{k+1}} = \theta(C\Sigma_i, y_i, r) \geq 1,$$

with equality if, and only if, $\Sigma_i$ is an equatorial sphere $\mathbb{S}^k$. The inequality follows from the monotonicity formula for minimal surfaces. Hence, $A(k,n) = |\mathbb{S}^k|$ and from Theorem 1.3 we conclude that $|A_{\Sigma_i}| \leq C$. By standard compactness results, $\Sigma_i$ converges graphically and with multiplicity one to $\mathbb{S}^k$. A comparison analysis between the Morse index of $\Sigma_i$ and $\mathbb{S}^k$ implies that $\Sigma_i$ is an equatorial sphere for $i$ large enough, see Section 3 below.

In view of Theorems 1.1 and 1.2, it is natural to expect similar gap phenomena also for the area of free boundary minimal surfaces in $B^n$. In contrast with Theorem 1.2, the smooth free boundary version of Allard’s regularity theorem does not readily apply to this end. It can be proved, however, that it follows from the strong Allard’s regularity theorem, proved by Grütter and Jost [8], together with the analysis developed in [3], which we also use here. Our first result is a direct and simpler proof of this fact:

Theorem 1.4. There exists $\epsilon(k,n) > 0$ such that whenever $\Sigma^k$ is a $k$-dimensional free boundary minimal surface in $B^n$ satisfying

$$|\Sigma^k| < |D^k| + \epsilon(k,n),$$

then $\Sigma^k$ is, up to ambient isometries, the equatorial disk $D^k$.

The 2-dimensional case in Theorem 1.4 was proved by Ketover in [10]. The key ingredients in the proof are an excess inequality for 2-dimensional free boundary surfaces in $B^n$, proved by Vokmann in [17] (see also [15]), and the classical Nitsche’s Uniqueness Theorem for free boundary minimal disks in $B^3$ (see also [6], for the generalization to high codimension). The excess inequality is particularly important
in proving curvature estimates for a sequence of free boundary minimal surfaces with area sufficiently close to the area of the equatorial disk. The main difficulty in implementing the arguments of [10] to \(k\)-dimensional surfaces in \(B^n\) is that neither the excess inequality in the form used in [10] nor Nitsche’s Theorem is readily available when \(k \geq 3\). To get around these difficulties, we consider a slightly more general quantity, originated in [3] and which also resemble an excess type formula, and compare it with that of the free boundary cones over the boundaries to obtain the necessary curvature estimates. Finally, we replace the use of Nitsche’s Theorem by an index of stability analysis.

**Remark 1.5.** We observe that the 2-dimensional proof of Theorem 1.4 given in [10] can be extended to constant mean curvature surfaces in \(B^3\). The quantity to consider in this case is the Willmore energy instead of area. Let \(\Sigma^2\) be a surface with boundary in \(\mathbb{R}^3\), the Willmore energy \(\mathcal{W}(\Sigma)\) is defined as

\[
\mathcal{W}(\Sigma) = \int_{\Sigma} H^2 \, d\Sigma + \int_{\partial\Sigma} k_g \, d\sigma.
\]

**\(\varepsilon\)-Regularity.** There exists \(\varepsilon > 0\) such that whenever \(\Sigma\) is a free boundary surface with constant mean curvature in \(B^3\) and satisfying

\[
\mathcal{W}(\Sigma) < 2\pi + \varepsilon,
\]

then \(\Sigma\) is either an equatorial disk or a spherical cap. The constant \(\varepsilon\) is independent of the value of the mean curvature.

Finally, we recall an unique continuation result which might be of independent interest in view of the discussion above. This result seems to be well known among experts but not clearly stated in the literature:

**Proposition 1.6.** Let \(\Sigma^k\) be a \(k\)-dimensional free boundary minimal surface in \(B^n\) which is smooth except possibly at the origin. If \(\partial\Sigma\) is a \((k-1)\)-minimal surface in \(S^{n-1}\), then \(\Sigma^k\) is the minimal cone \(C_1\partial\Sigma\).

# 2. Higher Dimension Free Boundary Minimal Surfaces

We start by recalling an excess inequality for free boundary minimal surfaces in the ball proved in [3]. More precisely, if \(\Sigma\) is a \(k\)-dimensional free boundary minimal surface in \(B^n\) and if \(y \in \partial\Sigma\), then

\[
\int_{\Sigma^k} \frac{|(x - y)|^2}{|x - y|^{k+2}} \, d\Sigma \leq |\Sigma^k| - |D^k|.
\]

This inequality, which implies Theorem 1.1, follows from a monotonicity argument obtained by an application of the Divergence Theorem to
the vector field $W_{t_0,y}(x)$ defined on $B^n - \{y\}$ and given by
\[
W_{t_0,y}(x) = \frac{x}{2} - \frac{x - y}{|x - y|^k} - \frac{k - 2}{2} \int_{t_0}^{\frac{|y|^2}{k}} \frac{t(x - y)}{|tx - y|^k} dt,
\]
where we assume that $t_0 \in \{0, 1\}$ if $y \in \partial\Sigma$ and $t_0 = |y|^2$ if $y \notin \partial\Sigma$. Note that the integrand is well defined by our definition of $t_0$. We will need a formula similar to (2.1) for when $y$ is not necessarily at the boundary. For this, we need to recall the techniques in [3] involved in the proof of (2.1).

**Lemma 2.1.** Let $\Sigma^k$ a free boundary surface in $B^n$ and $y \in \Sigma$. For $r$ sufficiently small, we have
\[
2 \int_{\Sigma \setminus B_r(y)} \frac{|(x - y)\perp|^2}{|x - y|^{k+2}} d\Sigma + \frac{k - 2}{k} \int_{\Sigma \setminus B_r(y)} \int_{t_0}^{\frac{|y|^2}{k}} \frac{t|(tx - y)\perp|^2}{|tx - y|^{k+2}} dt d\Sigma
\]
\[
= |\Sigma \setminus B_r(y)| - \frac{2}{k} \int_{\Sigma \cap \partial B_r(y)} \langle W_{t_0,y}(x), \nu(x) \rangle d\sigma
\]
\[
- \frac{2}{k} \int_{\partial\Sigma} \langle W_{t_0,y}, x \rangle d\sigma + 2 \int_{\Sigma \setminus B_r(y)} \langle \overrightarrow{H}, W_{t_0,y} \rangle d\Sigma,
\]
where $B_r(y)$ is a geodesic ball on $\Sigma^k$, $\nu$ is the outward co-normal to $\partial B_r(y)$ with respect to $\Sigma \setminus B_r(y)$, and $\overrightarrow{H}$ is the mean curvature vector of $\Sigma^k$.

**Proof.** A computation following Section 2 in [3] gives
\[
div_\Sigma W_{t_0,y} \frac{k}{2} - \frac{k}{2} \frac{|(x - y)\perp|^2}{|x - y|^{k+2}} - \frac{k - 2}{2} \int_{t_0}^{\frac{|y|^2}{k}} \frac{tk}{|tx - y|^{k+2}} dt.
\]
On the other hand, we have that
\[
div_\Sigma W_{t_0,y} = div_\Sigma W_{t_0,y}^T - k\langle W_{t_0,y}, \overrightarrow{H} \rangle.
\]
Integrating both sides above over $\Sigma \setminus B_r(y)$ and applying the Divergence Theorem we conclude the proof. \(\square\)

The next lemma deals with the second term in the right hand side of (2.2):

**Lemma 2.2.** Let $\Sigma^k$ be a free boundary surface in $B^n$ and let $\varphi(y) = 1$ if $y \in \partial\Sigma$ and $\varphi(y) = 2$ if $y \notin \partial\Sigma$. Then
\[
\lim_{r \to 0} \frac{2}{k} \int_{\Sigma \cap \partial B_r(y)} \langle W_{t_0,y}(x), \nu(x) \rangle = \varphi(y) |D^k|.
\]
Proof. The proof is essentially contained in Section 2 of [3]. More precisely, an application of [3, Lemma 8] gives

\[ W_{t_0,y}(x) = -\frac{x-y}{|x-y|^k} + o\left(\frac{1}{|x-y|^{k-1}}\right) \]

Note that this statement is trivial when \( y \notin \partial \Sigma \) or when \( y \in \partial \Sigma \) and \( t_0 = 1 \). The lemma now follows from the computations leading to equation (2) in [3]; the only minor difference comes from the possibility of \( y \notin \partial \Sigma \). \( \square \)

**Lemma 2.3.** If \( y \in \partial \Sigma \), then \( \langle W_{0,y}(x), x \rangle = 0 \) for every \( x \in \partial \Sigma - \{y\} \).

**Proof.** See Section 2 in [3]. \( \square \)

**Remark 2.4.** Applying Lemmas 2.1, 2.2, and 2.3, we obtain the general form of inequality (2.1) when \( y \in \partial \Sigma \):

\[
(2.3) \quad \int_\Sigma \frac{|(x-y)^\perp|^2}{|x-y|^{k+2}} + \frac{k-2}{2k} \int_\Sigma \int_0^1 \frac{tk|tx-y|^\perp|^2}{|tx-y|^{k+2}} - \int_\Sigma \langle \vec{H}, W_{0,y} \rangle = \frac{|\Sigma| - |D^k|}{2}.
\]

**Proposition 2.5.** Let \( \Sigma^k \) be a \( k \)-dimensional free boundary minimal surface in \( B^n \) and \( C_1 \partial \Sigma \) the cone with vertex at the origin and base \( \partial \Sigma \). If \( y \in \Sigma - C_1 \partial \Sigma \), then

\[
(2.4) \quad \int_\Sigma \frac{|(x-y)^\perp|^2}{|x-y|^{k+2}} = \int_{C_1 \partial \Sigma} \frac{|(x-y)^\perp|^2}{|x-y|^{k+2}} + \int_{C_1 \partial \Sigma} \langle \vec{H}_{C_1 \partial \Sigma}, x-y \rangle - |D^k|.
\]

If \( y \in \partial \Sigma \), then

\[
(2.5) \quad \int_\Sigma \frac{|(x-y)^\perp|^2}{|x-y|^{k+2}} = \int_{C_1 \partial \Sigma} \frac{|(x-y)^\perp|^2}{|x-y|^{k+2}} + \int_{C_1 \partial \Sigma} \langle \vec{H}_{C_1 \partial \Sigma}, x-y \rangle.
\]

**Proof.** For this proposition we choose \( t_0 = |y|^2 \). Hence, the vector field \( W_{t_0,y} \) becomes

\[ W_y = \frac{x}{2} - \frac{x-y}{|x-y|^k}. \]

First the case \( y \notin \partial \Sigma \). Applying Lemma 2.1 and Lemma 2.2 we obtain

\[
(2.6) \quad 2 \int_\Sigma \frac{|(x-y)^\perp|^2}{|x-y|^{k+2}} d_\Sigma = |\Sigma| - 2|D^k| - \frac{2}{k} \int_{\partial \Sigma} \langle W_y(x), x \rangle d\sigma.
\]

Now we look at the last term in (2.6). Let \( C_1 \partial \Sigma \) be the free boundary cone over \( \partial \Sigma \) and vertice at 0. Applying Lemma 2.1 to \( C_1 \partial \Sigma \) and
observing that \(C_1 \partial \Sigma\) might not be a minimal surface, we obtain:

\[
2 \int_{C_1 \partial \Sigma \setminus B_r(0)} \frac{|(x - y)\perp|^2}{|x - y|^{k+2}} = |C_1 \partial \Sigma \setminus B_r(0)| - \frac{2}{k} \int_{C_1 \partial \Sigma \setminus \partial B_r(0)} \langle W_y, \nu \rangle \, d\sigma \\
- \frac{2}{k} \int_{\partial \Sigma} \langle W_y(x), x \rangle \, d\sigma + 2 \int_{C_1 \partial \Sigma \setminus B_r(0)} \langle \vec{H}_{C_1 \partial \Sigma}, W_y \rangle \, dC_1 \partial \Sigma.
\]

Taking the limit as \(r \to 0\) in above expression, we obtain

\[
2 \int_{C_1 \partial \Sigma} \frac{|(x - y)\perp|^2}{|x - y|^{k+2}} \, dC_1 \Sigma = |C_1 \partial \Sigma| - \frac{2}{k} \int_{\partial \Sigma} \langle W_y(x), x \rangle \, d\sigma \\
- 2 \int_{C_1 \partial \Sigma} \langle \vec{H}_{C_1 \partial \Sigma}, W_y \rangle \, dC_1 \partial \Sigma.
\]

Plugging (2.7) into (2.6), we obtain

\[
2 \int_{C_1 \partial \Sigma} \frac{|(x - y)\perp|^2}{|x - y|^{k+2}} \, dC_1 \Sigma = |\Sigma| - |C_1 \partial \Sigma| + 2 \int_{C_1 \partial \Sigma} \frac{|(x - y)\perp|^2}{|x - y|^{k+2}} \\
- 2 \int_{C_1 \partial \Sigma} \langle \vec{H}_{C_1 \partial \Sigma}, W_y \rangle \, dC_1 \partial \Sigma - 2|D^k|.
\]

The free boundary condition of \(\Sigma\) combined with the Divergence Theorem applied to the position vector \(X = \vec{x}\) give

\[
k|C_1 \partial \Sigma| = |\partial \Sigma| - k \int_{C_1 \partial \Sigma} \langle \vec{H}_{C_1 \partial \Sigma}, x \rangle \, d\sigma = |\partial \Sigma| = k|\Sigma|.
\]

The case \(y \in \partial \Sigma\) is done in similar manner with minor modifications. This completes the proof of the proposition. \(\square\)

### 3. Proof of Theorem 1.4

Following [16], we have for every \(X \in \mathcal{X}(\mathbb{R}^{n+1})\) the following expression for the second variation of area \(\Sigma^k\) in the direction of \(X\)

\[
\delta^2 \Sigma(X, X) = \int_{\Sigma} \left( |D^\perp X|^2 - |\langle A, X \rangle|^2 \right) \, d\Sigma + \int_{\partial \Sigma} \langle D_X X, \nu \rangle \, d\sigma,
\]

where \(A: \mathcal{X}(\Sigma) \times \mathcal{X}(\Sigma) \to \mathcal{X}^\perp(\Sigma)\) is the second fundamental form of \(\Sigma\).

**Theorem 3.1** (Fraser-Schoen [7]). If \(\Sigma^k\) is a free boundary minimal surface in \(B^n\) and \(v \in \mathbb{R}^n\), then

\[
\delta^2 \Sigma(v^\perp, v^\perp) = -k \int_{\Sigma} |v^\perp|^2 \, d\Sigma
\]
If $\Sigma$ is not contained in a cylinder $\Sigma_0 \times \mathbb{R}$ where $\Sigma_0$ is a free boundary minimal surface, then $\text{index}(\Sigma)$ is at least $n$. In particular, if $k = 2$ and $\Sigma$ is not a plane disk, its index is at least $n$.

Recall the $p$–th eigenvalue of $\delta^2 \Sigma$ has the min-max characterization:

$$\lambda_p = \inf_{W: \dim(W) = p} \sup_{X \in W} \frac{\int_{\Sigma} |X|^2 \, d\Sigma}{\int_{\Sigma} \delta^2 \Sigma(X, X)}$$

**Proof of Theorem 1.4.** Arguing by contradiction, we assume that $\{\Sigma_i\}$ is a sequence of non-totally geodesic $k$-dimensional free boundary minimal surfaces in $B^n$ satisfying

$$|\Sigma_i| \rightarrow |D^k|$$

(3.1)

Following the strategy in [10], we first show that (3.1) implies curvature estimates for $\Sigma_i$.

**Claim 3.2.** Let $A_{\Sigma_i}$ be the second fundamental form of $\Sigma_i$. Then, there exists $C > 0$ such that

$$\sup_{x \in \Sigma_i} |A_{\Sigma_i}(x)| \leq C.$$  

(3.2)

Let us show first that the index estimate in Theorem 3.1 combined with the Claim 3.2 imply the theorem:

By the Claim 3.2, the second fundamental form of $\{\Sigma_i\}$ is uniformly bounded. Theorem 6.1 in [12] (see also [1]) implies that $\Sigma_i$ converges smoothly up to the boundary to a free boundary minimal surface $\Sigma_\infty$. Since $|\Sigma_\infty| = |D^k|$, we conclude by Theorem 1.1 that $\Sigma_\infty$ is an equatorial disk. In particular, $\Sigma_i$ is diffeomorphic to a disk and has trivial normal bundle for $i$ large enough. By Lemma 3.1, $\text{index}(\Sigma_i) \geq n$ unless $\Sigma_i$ is contained in the cylinder $\Sigma_i \times \mathbb{R}$. By our assumptions $\Sigma_i$ is a free boundary minimal surface smooth perturbation of an equatorial disk $\mathbb{D}^{k-1}$. This suggest applying an induction argument. Note that the first step corresponds to two-dimensional minimal surface $\Sigma_i$ in $B^n$, and by Lemma 3.1, $\text{index}(\Sigma_i)$ is at least $n$ unless totally geodesic. The discussion for both the case $k = 2$ and the induction step $k = n + 1$ are, hence, the same. Therefore, without loss of generality, it suffices assuming that $\text{index}(\Sigma_i) \geq n$ unless $\Sigma_i$ is an equatorial disk $\mathbb{D}^k$. This implies that there exist at least $n$ mutually orthonormal eigenvector fields $X_j$ of the quadratic form $\delta^2 \Sigma(\cdot, \cdot)$ defined on $\mathcal{X}^+(\Sigma_i)$ satisfying

$$\Delta^X + \sum_{j,l} \langle A(e_j, e_l), X \rangle A(e_j, e_l) + \lambda_X X = 0,$$

$$\lambda_X < 0.$$
Note that the vector fields $X$ are not necessarily equal to $v^\perp$ from Theorem 3.1. As $i \to \infty$, these eigenvectors converge to eigenvectors of the Jacobi operator on $\Sigma_\infty \subset B^n$ and none of these eigenvector have eigenvalue zero since by the observation after Theorem 3.1, $\lambda_X < -k$.

This is contradiction since $\text{index}(\Sigma_\infty) = n - k$. \hfill \Box

**Proof of Claim 3.2.** Arguing by contradiction, we assume that

$$\text{Area}(\Sigma_i) \to |D^k| \quad \text{and} \quad \lambda_i = \sup_{x \in \Sigma_i} |A_i|^2(x) \to \infty.$$ 

For each $i$ choose $x_i \in \Sigma_i$ with the property that $\sup_{\Sigma_i} |A_i|^2 = |A_i|^2(x_i)$. Note that $\lim_{i \to \infty} |x_i| = 1$. Indeed, the excess inequality (2.1) implies that $\Sigma_i$ converges with multiplicity one to $\mathbb{D}^k$ as a varifold. Hence, in $B^n(1) \setminus \mathbb{D}^k$, the surface $\Sigma_i$ satisfy $\theta(\Sigma_i, x, r) \leq 1 + \varepsilon$ for every $i$ large enough and $r$ small enough. If $\lim_{i \to \infty} |x_i| = 1$, then we would get a contradiction with the smooth version of Allard’s regularity theorem.

Now we consider the surface

$$\hat{\Sigma}_i = \lambda_i(\Sigma_i - x_i).$$

One can check that $\hat{\Sigma}_i$ satisfies

$$(3.4) \quad \sup_{x \in \Sigma_i} |A(x)| \leq 1 \quad \text{and} \quad |A_{\Sigma_i}|(0) = 1$$

and it is a free boundary minimal surface in $\lambda_i(B^n_{i+1}(0) - x_i)$. It follows from Theorem 6.1 in [12](see also [1]) that, after passing to a subsequence, $\hat{\Sigma}_i$ converges smoothly and locally uniformly to $\Sigma_\infty$. $\Sigma_\infty$ is either complete without boundary minimal surface or it is a free boundary minimal surface in a half space. Moreover, (3.4) implies that

$$(3.5) \quad |A_{\Sigma_\infty}|(0) = 1.$$ 

On the other hand, by the scale invariance of the excess, we have that

$$\int_{\Sigma_\infty} \frac{\varepsilon^{\perp}}{|z|^{k+2}} d\Sigma_\infty \leq \liminf_{i \to \infty} \int_{\Sigma_i} \frac{\varepsilon^{\perp}}{|z|^{k+2}} d\Sigma_i = \liminf_{i \to \infty} \int_{\Sigma_i} \frac{|(x - x_i)^{\perp}|^2}{|x - x_i|^{k+2}} d\Sigma_i$$

We want to prove that the last term above goes to zero as $i \to \infty$.

**Claim 3.3.** There exist $C > 0$ such that for every $y \in \partial \Sigma_i$

$$|A_{\partial \Sigma_i}|(y) \leq C$$

**Proof.** Let $w_i \in \partial \Sigma_i$ such that $\sup_{w \in \partial \Sigma_i} |A_{\Sigma_i}(w)| = |A_{\partial \Sigma_i}(w_i)| = \beta_i$. Take the sequence $\bar{\Sigma}_i = \beta_i \Sigma'_i$, where $\Sigma'_i = \{C \partial \Sigma_i - w_i\}$. By the cone excess (2.5) and the excess (2.1):

$$\int_{\Sigma_i} \frac{|(x - w_i)^{\perp}|^2}{|x - w_i|^{k+2}} + \int_{\Sigma_i} \langle H, \frac{x - w_i}{|x - w_i|} \rangle = \int_{\Sigma_i} \frac{|(x - w_i)^{\perp}|^2}{|x - w_i|^{k+2}} \leq \frac{|\Sigma_i| - |D^k|}{2}$$
By compactness, we obtain that $\tilde{\Sigma}_i$ converge $C^{1,\alpha}$ to a submanifold $V_\infty$ which is free boundary on a hyperplane. By the scaling invariance, we obtain

$$\int_{V_\infty} \frac{|Z^\perp|^2}{|Z|^{k+2}} + \int_{V_\infty} \langle \vec{H}, \frac{Z}{|Z|^k} \rangle = 0.$$

Applying a reflection symmetry with respect to the hyperplane and the Classical Monotonicity formula for varifolds with bounded mean curvature, see [4, Theorem 2.1] we obtain that the density at the origin is $\lim_{r \to +\infty} \theta(V_\infty, 0, r) = 1$. Therefore, $V_\infty$ is totally geodesic. This contradicts the choice of $w_i \in \Sigma_i$. \hfill \square

In particular, it follows from the Claim 3.3 that the second fundamental form of $\partial \Sigma_i$ in $\mathbb{R}^{n+1}$ is uniformly bounded. Thus, up to subsequence, $\partial \Sigma_i$ converges in the $C^{1,\alpha}$ topology to $\partial D^k \subset S^{n-1}$. Equivalently, $C_1 \partial \Sigma_i \cap (B^n - B_s(0))$ converges in the $C^{1,\alpha}$ topology to $D^k$. Without loss of generality, we can assume that $x_i \notin C_1 \partial \Sigma_i$ since $\Sigma_i$ cannot coincide with $C_1 \partial \Sigma_i$ near $x_i$. Applying (2.4) in Proposition 2.5, we obtain

$$\int_{\Sigma_\infty} \frac{|z^\perp|^2}{|z|^{k+2}} d\mu \leq \lim_{i \to \infty} \int_{C_1 \partial \Sigma_i} \frac{|x - x_i|^2}{|x - x_i|^{k+2}} d\mu + \int_{C_1 \partial \Sigma_i} \langle \vec{H}_{C_1 \partial \Sigma_i}, \frac{x - x_i}{|x - x_i|^k} \rangle d\mu - |D^k|.$$

Now we explore that $C_1 \partial \Sigma_i$ converges graphically to $D^k$ to prove the following claim:

**Claim 3.4.**

$$\lim_{i \to \infty} \left( \int_{C_1 \partial \Sigma_i} \frac{|x - x_i|^2}{|x - x_i|^{k+2}} + \int_{C_1 \partial \Sigma_i} \langle \vec{H}_{C_1 \partial \Sigma_i}, \frac{x - x_i}{|x - x_i|^k} \rangle \right) \leq |D^k|.$$

**Proof.** Since $\Sigma_i$ converges weakly to $D^k$, we assume that $x_i \to y \in \partial D^k$. First note that

$$\lim_{i \to \infty} \int_{C_1 \partial \Sigma_i - B_s(y)} \frac{|x - x_i|^2}{|x - x_i|^{k+2}} + \int_{C_1 \partial \Sigma_i - B_s(y)} \langle \vec{H}_{C_1 \partial \Sigma_i}, \frac{x - x_i}{|x - x_i|^k} \rangle = 0$$

since $C_1 \partial \Sigma_i \to D^k$ in the $C^{1,\alpha}$ topology, here $B_s(y)$ denotes an Euclidean ball. Hence, it is enough to focus on $\Sigma_i \cap B_s(y)$. The convergence $C_1 \partial \Sigma_i \to D^k$ also implies that we can choose $s < 1$ very small so that $T_x C_1 \partial \Sigma_i$ is uniformly close to $T_y D^k$ for every $x \in C_1 \partial \Sigma_i \cap B_s(y)$. In particular, we can write $C_1 \partial \Sigma_i$ as a graph over $T_y D^k$ and we have that $dvol_{C_1 \partial \Sigma_i} = dvol_D(1 + o_i(1))$. Let $z_i \in T_x C_1 \partial \Sigma_i$ a point which realizes the distance $r_i = d(T_x C_1 \partial \Sigma_i, x_i)$. Hence, $|(x - x_i)^\perp|^2 = r_i^2$ and
\[ x - x_i = x - z_i + u_i \] for every \( x \in C_1 \partial \Sigma_i \cap B_s(x_i) \) where \( u_i \perp T_{x} C_1 \partial \Sigma_i \) and \( |u_i| = r_i \). In particular, \( u_i \perp (x - z_i) \). Therefore,

\[
\lim_{i \to \infty} \int_{C_1 \partial \Sigma_i \cap B_s(y)} \frac{|x - x_i|^{k+2}}{|x - x_i|^{k+2}} = \lim_{i \to \infty} \int_{C_1 \partial \Sigma_i \cap B_s(y)} \frac{r_i^2}{|x - z_i + u_i|^{k+2}} = \lim_{i \to \infty} \int_{C_1 \partial \Sigma_i \cap B_s(y) - z_i} \frac{r_i^2}{|x - z_i + u_i|^{k+2}} = \lim_{i \to \infty} \frac{1}{\int_{C_1 \partial \Sigma_i \cap B_s(y) - z_i} (|w|^{2} + 1)^{\frac{k+2}{2}}} = \frac{1}{\int_{D^k} (s^2 + 1)^{\frac{k+2}{2}}} ds = |\partial D^k| \int_{0}^{\infty} \frac{s^{k-1}}{(s^2 + 1)^{\frac{k+2}{2}}} ds = \frac{|\partial D^k|}{k} = |D^k|,
\]

where \( P_1 \) is either \( \mathbb{R}^k \) or a half space \( \mathbb{R}^k_+ = \{ x \in \mathbb{R}^k : \langle x, e_k \rangle \leq a \} \). Similarly,

\[
\lim_{i \to \infty} \int_{C_1 \partial \Sigma_i \cap B_s(y)} \left( \frac{x - x_i}{|x - x_i|} \right) dC_1 \partial \Sigma_i \leq \lim_{i \to \infty} \sup_{C_1 \partial \Sigma_i \cap B_s(y)} \left| \left( \frac{x - x_i}{|x - x_i|} \right) \right| \int_{C_1 \partial \Sigma_i \cap B_s(y)} \frac{1}{|x - x_i|^{k-1}} dC_1 \partial \Sigma_i
\]

\[
= \lim_{i \to \infty} O(1 - s) \int_{C_1 \partial \Sigma_i \cap B_s(y)} \frac{1}{|x - z_i + u_i|^{k-1}} dC_1 \partial \Sigma_i
\]

\[
= \lim_{i \to \infty} O(1 - s) \int_{C_1 \partial \Sigma_i \cap B_s(y) - z_i} \frac{r_i^{1-k}}{|x - z_i + u_i|^{k-1}} dC_1 \partial \Sigma_i
\]

\[
= \lim_{i \to \infty} O(1 - s) \int_{\frac{1}{2} (T_{z_i} C_1 \partial \Sigma_i \cap B_s(y) - z_i)} \frac{r_i}{|w + v_i|^{k-1}} d\mathcal{H}^k (|v_i| = 1)
\]

\[
\leq \lim_{i \to \infty} O(1 - s) \int_{\mathbb{R}^k \cap B_{\frac{r_i}{2}}(z_i)} \frac{r_i}{|w + v_i|^{k-1}} dw (1 + o_i(1))
\]

\[
\leq \lim_{i \to \infty} O(1 - s) \int_{\mathbb{R}^k \cap B_{\frac{r_i}{2}}(z_i)} \frac{r_i}{|w|^{k-1}} dw (1 + o_i(1))
\]

\[
\leq \lim_{i \to \infty} O(1 - s) \int_{0}^{\frac{1}{r_i}} r_i \frac{s^{k-1}}{s^{k-1}} ds (1 + o_i(1)) = O(s).
\]

We used that \( u_i \perp (x - z_i) \) to obtain \( |w| \leq |w + v_i| \) above. Making \( s \to 0 \), we conclude the proof of the Claim 3.4. \( \square \)
Using the Claim above, we obtain that
\[
\int_{\Sigma_{\infty}} \frac{|z^\perp|^2}{|z|^{k+2}} d\Sigma_{\infty} \leq |D^k| - |D^k| = 0.
\]
Since this contradicts (3.5), the Claim 3.2 is proved.

**Proof of Proposition 1.6.** Let $C_1 \partial \Sigma$ be the free boundary cone over the boundary $\partial \Sigma$. Since $\partial \Sigma$ is minimal in $S^{n-1}$, $C_1 \partial \Sigma$ is a minimal submanifold in $\mathbb{R}^n$. Moreover, $\Sigma$ and $C_1 \partial \Sigma$ coincide up to first order at $\partial \Sigma$. More precisely, if we write locally $\Sigma = \text{graph}(u)$ and $C_1 \partial \Sigma = \text{graph}(v)$ near $x \in \partial \Sigma$, where $u, v : U \subset T_x \Sigma \to \mathbb{R}^{n-k}$, then $w = v - u$ satisfies $w = \nabla w = 0$ at $\partial \Sigma$. Moreover, $w$ satisfies, for each $l = 1, \ldots, n-k$, the second order linear elliptic system of equations:
\[
a_{ij}(\nabla u) \frac{D_j(w_l)}{\sqrt{1 + |\nabla u|^2}} + \sum_{m=1}^p b^m_j(\nabla u, \nabla v) D_j(w_m) = 0,
\]
for some smooth functions $a_{ij}(\nabla u)$ and $b^m_j(\nabla u, \nabla v)$. By a result of Simon-Hardt [9] (see also [2, Lemma 2.3] for the generalization to systems of equations), $w$ vanishes at infinite order at $\partial \Sigma$. On the other hand, [11, Lemma 1] and Morrey [13, Theorem 6.7.6] imply that the map $w$ is analytic. Therefore, $w \equiv 0$ and $\Sigma = C_1 \partial \Sigma$.

**References**

[1] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*. I, Comm. Pure Appl. Math. 12 (1959), 623-727.

[2] E. Barbosa and C. Viana, *A remark on a curvature gap for free boundary minimal surfaces in the ball*, Math. Z., vol. 294, (2019) 713-720.

[3] S. Brendle, *A sharp bound for the area of minimal surfaces in the unit ball*, Geom. Funct. Anal. vol 22 (2012) 621-626.

[4] C. De Lellis, *Allard’s interior regularity theorem: an invitation to stationary varifolds*. Nonlinear analysis in geometry and applied mathematics. Part 2, 23-49, Harv. Univ. Cent. Math. Sci. Appl. Ser. Math., 2, Int. Pr ess, Somerville, MA, 2018.

[5] A. Fraser and R. Schoen, *The first Steklov eigenvalue, conformal geometry, and minimal surfaces*. Adv. Math. 226, 4011-4030 (2011).

[6] A. Fraser and R. Schoen, *Uniqueness theorems for free boundary minimal disks in space forms*, Int. Math. Res. Not. IMRN (2015), no. 17, 8268-8274.

[7] A. Fraser and R. Schoen, *Sharp eigenvalue bounds and minimal surfaces in the ball*, Invent. Math. 203 (2016), pp. 823-890.

[8] M. Grütter and J. Jost, *Allard type regularity results for varifolds with free boundaries*, Ann. Scuola Nor Sup. Pisa Cl. Sci. (4) 13 (1986), no. 1, 129-169.

[9] R. Hardt and L. Simon, *Nodal sets for solutions of elliptic equations*. J. Differential Geom. Volume 30, Number 2 (1989), 505-522.
[10] D. Ketover, *Free boundary minimal surfaces of unbounded genus*, arXiv:1612.08691 [math.DG].

[11] D. Leung, *The reflection principle for minimal submanifolds of Riemannian symmetric spaces*, J. Differential Geometry 8 (1973) 153-160.

[12] M. Li, Q. Guang, and X. Zhou, *Curvature estimates for stable free boundary minimal hypersurfaces*, Journal fur die reine und angewandte Mathematik (Crelle’s Journal), vol. 2020, no. 759, 2020, pp. 245-264.

[13] C. Morrey, *Multiple Integrals in the Calculus of Variations*. Springer-Verlag, Berlin-Heidelberg-New York, (1966).

[14] J. C. C. Nitsche, *Stationary partitioning of convex bodies*, Arch. Rational Mech. Anal. 89, (1985), pp. 1-19.

[15] A. Ros and E. Vergasta, *Stability for Hypersurfaces of Constant Mean Curvature with Free Boundary*. Geometriae Dedicata. 56 (1995), 19-33.

[16] R. Schoen, *Minimal submanifolds in higher codimension*, Matematica Contemporanea 30 (2006), 169-199.

[17] A. Volkmann, *A monotonicity formula for free boundary surfaces with respect to the unit ball*, Comm. Anal. Geom., Vol. 24, Number 1 (2016), 195-221.

**Universidade Federal de Minas Gerais (UFMG), Caixa Postal 702, 30123-970, Belo Horizonte, MG, Brazil**

*Email address: ezequiel@mat.ufmg.br*

**Universidade Federal de Minas Gerais (UFMG), Caixa Postal 702, 30123-970, Belo Horizonte, MG, Brazil**

*Email address: celso@mat.ufmg.br*