Using a semi-analytic model based on the Thomas-Fermi approximation, we investigate the relevance of the quadrupole deformation of a trapped Bose-Einstein condensate for the nucleation of quantized vortices. For sufficiently high angular velocities $\Omega$ of the trap, the tendency of the system to exhibit spontaneous deformation is shown to lower the barrier which inhibits the nucleation of vortices at smaller $\Omega$. The corresponding value of the critical angular velocity $\Omega_c$ is calculated as a function of the deformation of the trap and of the chemical potential. The theoretical predictions for $\Omega_c$ refer to the case of a sudden switch-on of the deformed rotating trap and they are compared with recent experimental data.

I. INTRODUCTION

The problem of the nucleation of quantized vortices in Bose superfluids has been the object of recent experimental work with dilute gases in rotating traps. It now emerges clearly that the mechanism of nucleation depends crucially on the actual shape of the trap as well as on how the rotation is switched on. A first approach consists of a sudden switch-on of the deformation and rotation of the external potential which generates a non-equilibrium configuration. For sufficiently large values of the angular velocity of the rotating potential one observes the nucleation of one or more vortices. In a second approach either the rotation or the deformation of the trap are switched on slowly. In this way, the system is in conditions of equilibrium up to some critical velocity above which it exhibits a dynamical instability giving rise to the nucleation of vortices. In the case of a sudden switch-on of the deformation and rotation of the trap, the observed critical angular velocity turns out to be close to the value associated with the instability of the surface oscillations of the condensate. This occurs at the angular velocity

$$\omega_{ct}(\ell) = \frac{\omega(\ell)}{\ell},$$

where $\omega(\ell)/2\pi$ is the frequency of the surface oscillation with angular momentum $\ell$ and we have assumed that the deformation of the trap is negligibly small. The actual value of $\ell$ fixing the critical angular velocity depends explicitly on the shape of the rotating potential and, to some extent, can be selected in the experiment. Eq. (1) corresponds to the famous Landau criterion for superfluidity applied to the problem of rotations. In the Thomas-Fermi limit the critical angular velocity takes the simple form $\omega_{ct} = \omega_\perp/\sqrt{\ell}$, where $\omega_\perp$ is the transverse frequency of the harmonic confinement. The experimental evidence that Eq. (1) provides a good estimate for the critical angular velocity for the nucleation of quantized vortices points out the crucial role played by the shape deformation of the condensate, in agreement with the theoretical considerations developed in [7,10–13]. This is further confirmed by the direct observation of strong deformations occurring temporarily during the process of nucleation.

The purpose of this paper is to gain further insight into the mechanism underlying vortex nucleation when the deformation and rotation of the trap are switched on suddenly. So far, the explicit relation between the deformation instability and the vortex creation was not enough clear. In the absence of shape deformation the nucleation process is inhibited by the occurrence of a barrier located near the surface of the condensate. Using the Thomas-Fermi (TF) approximation to the Gross-Pitaevskii (GP) theory at zero temperature we will show that this barrier is lowered by the explicit inclusion of shape deformations and eventually disappears at sufficiently high angular velocities, making it possible for the vortex to nucleate. We will consider the simple but relevant case of quadrupole deformations which has already been the object of systematic experimental studies.

II. VORTICES IN AXISYMMETRIC CONFIGURATIONS

A quantized vortex is characterized by the appearance of a velocity field associated with a non-vanishing, quantized circulation. If we assume that the vortex can be described by a straight line the quantization of circulation takes the
simple form
\[ \nabla \times \mathbf{v}_{\text{vortex}} = \frac{2\pi \hbar}{m} \delta^{(2)}(\mathbf{r} - \mathbf{d}) \hat{z} \]  
(2)
for a vortex located at distance \( d \equiv |\mathbf{d}| \) from the \( z \)-axis. The general solution of (2) can be written in the form
\[ \mathbf{v}_{\text{vortex}} = \nabla (\varphi_d + S) \]  
(3)
where \( \varphi_d \) is the azimuthal angle around the vortex line at position \( \mathbf{d} \) and \( S \) is a single-valued function which gives rise to an irrotational component of the velocity field. The irrotational component may be important in the case of vortices displaced from the symmetry axis and its inclusion permits to optimize the energy cost associated with the vortex line [13]. Considering a straight vortex line is a first important assumption that we introduce in our description [10].

The inclusion of vorticity is accompanied by the appearance of angular momentum and by an energy cost. Under suitable conditions the system may nevertheless like to acquire the vortical configuration. This happens if there is a total energy gain in the rotating frame where the system is described by the Hamiltonian
\[ H(\Omega) = H - \Omega L_z. \]  
(4)
Here \( H \) is the Hamiltonian in the laboratory frame, \( L_z \) is the angular momentum, and \( \Omega \) is the angular velocity of the trap around the \( z \)-axis.

The simplest way to calculate the angular momentum associated with a displaced vortex is to assume axi-symmetric trapping and to work with the TF-approximation. In this limit the size of the vortex core is small compared to the radius of the condensate so that one can use the vortex-free expression \( n(r) = \mu[1 - (r/R_{\perp})^2 - (z/Z)^2]/g \) for the density profile of the condensate, where \( r_{\perp}^2 = x^2 + y^2 \) and \( g = 4\pi a \hbar^2/m \) is the coupling constant, fixed by the positive scattering length \( a \). Then, one can write the angular momentum in the form
\[ L_z = m \int dz \int_{r_{\perp}} d_{\perp} n(r_{\perp}, z) \int \mathbf{v}_{\text{vortex}} \cdot d\mathbf{l}, \]  
(5)
where the line integral is taken along a circle of radius \( r_{\perp} \). Use of Stokes’ theorem gives the result [17]
\[ L_z(d/R_{\perp}) = Nh \left[ 1 - \left( \frac{d}{R_{\perp}} \right)^2 \right]^{5/2}, \]  
(6)
where \( d \) is the distance of the vortex line from the symmetry axis. Eq. (6) shows that the angular momentum per particle is reduced from the value \( \hbar \) as soon as the vortex is displaced from the center.

To calculate the energy cost associated with the vortex line one should in principle start out with Eq. (6) and optimize the choice for the phase \( S \). In order to obtain a simplified description we will make use of the expression
\[ E_v(d/R_{\perp}, \mu) = E_v(d = 0, \mu) \left[ 1 - \left( \frac{d}{R_{\perp}} \right)^2 \right]^{3/2}, \]  
(7)
which generalizes the TF-result [18]
\[ E_v(d = 0, \mu) = \frac{4\pi n_0 \hbar^2}{3} Z \log \left( \frac{0.671 R_{\perp}}{\xi_0} \right) = Nh \omega_{\perp} \frac{g}{4} \log \left( \frac{1.342}{\Xi} \right). \]  
(8)
for the energy of an axi-symmetric vortex. Here, \( Z \) is the TF-radius in \( z \)-direction and \( \xi_0 \) is the healing length calculated with the central density \( n_0 \). The \( d \)-dependence contained in (7) is simply understood by noting that the factor \( 4n_0 Z [1 - (d/R_{\perp})^2]^{3/2}/3 \) corresponds to the column density \( \int dz \, n(d, z) \) evaluated with the TF-approximation. Expression (7) is expected to be correct within logarithmic accuracy (see also [11,13] and references therein). It could be improved by including an explicit \( d \)-dependence in the healing length inside the logarithm, in order to account for the density dependence of the size of the vortex core.

Eqs. (6) and (7) show that when \( d = R_{\perp} \) neither angular momentum nor excitation energy is carried by the system. Of course these estimates, being derived with the TF-approximation, are not accurate if we go too close to the border. If the surface of the condensate is described by a more realistic density profile both the angular momentum and the energy are nevertheless expected to vanish when the vortex line is sufficiently far outside the bulk region. Within the
simplifying assumptions made above we can conclude that the configuration with \( d = R_\perp \) corresponds to the absence of vortices, while the transition from \( d = R_\perp \) to \( d = 0 \) describes the path nucleating the vortex. The energy associated with each point of this path in the rotating frame is simply calculated using the Hamiltonian \((7)\). It is given by \((7)\):

\[
E_v(d/R_\perp, \Omega, \mu) = E_v(d = 0, \mu) \left[ 1 - \left( \frac{d}{R_\perp} \right)^2 \right]^{3/2} - \Omega N \hbar \left[ 1 - \left( \frac{d}{R_\perp} \right)^2 \right]^{5/2}.
\]

Some interesting features emerge from Eq. \((9)\). First one finds that the occurrence of a vortex at \( d = 0 \) is energetically favorable for angular velocities satisfying the condition \( \Omega \geq \Omega_v(\mu) = E_v(d = 0, \mu)/\hbar \). This is the well known criterion for the so called thermodynamic stability of the vortex. It is worth noticing that in the TF-limit one should have \( \Omega_v(\mu)/\omega_\perp \ll 1 \). In fact, using the result \( \mu = g n_0 \) for the chemical potential and the expression \( \mu = m \omega^2 R_\perp^2/2 \), one can write

\[
\frac{\Omega_v}{\omega_\perp} = \frac{5}{2} \left( \frac{a_\perp}{R_\perp} \right)^2 \log \left( \frac{0.671 R_\perp}{\xi_0} \right),
\]

which tends to zero when \( R_\perp \gg a_\perp \). Here, \( a_\perp = \sqrt{\hbar/m \omega_\perp} \) is the radial oscillator length. In the actual experiments the ratio \((10)\) is not very small. For example, in the case of Ref. \([1]\) \( \Omega_v \approx 0.35 \omega_\perp \).

A second interesting result following from \((9)\) concerns the behavior of the vortex line for small displacement \( d \). By expanding \( E_v(d/R_\perp, \Omega, \mu) \) one finds that the vortex solution at \( d = 0 \) is fully unstable if \( \Omega \leq 3 \Omega_v(\mu)/5 \), while it is metastable (local minimum) if \( 3 \Omega_v(\mu)/5 \leq \Omega \leq \Omega_v(\mu) \).

Another important consequence of \((9)\) is the appearance of a barrier. Even if \( \Omega \geq \Omega_v(\mu) \) and hence if \( E_v(d/R_\perp, \Omega, \mu) \) is negative at \( d = 0 \), the curve \((9)\) exhibits a maximum at intermediate values of \( d \) between 0 and \( R_\perp \) (see Fig. \([1]\)). The position \( d_B \) and height \( E_B \) of the barrier are given by the equations

\[
\left( \frac{d_B}{R_\perp} \right)^2 = 1 - \frac{3 \Omega_v(\mu)}{5 \Omega}
\]

and

\[
E_B = \frac{2}{5} E_v(d = 0, \mu) \left[ 1 - \left( \frac{d_B}{R_\perp} \right)^2 \right]^{3/2} = \frac{2}{5} E_v(d = 0, \mu) \left( \frac{3 \Omega_v(\mu)}{5 \Omega} \right)^{3/2}
\]

showing that the height of the barrier becomes smaller and smaller as \( \Omega \) increases, but never disappears. Since crossing the barrier costs a macroscopic amount of energy, the system will never be able to overcome it and the vortex cannot be nucleated.

It is finally interesting to rewrite the energy of the vortical configuration as a function of angular momentum rather than of the vortex displacement. This yields the expression

\[
E_v(L_z, \Omega, \mu) = N \hbar \left[ \Omega_v(\mu) \left( \frac{L_z}{N \hbar} \right)^{3/5} - \frac{\Omega L_z}{N \hbar} \right].
\]

Equation \((13)\) emphasizes the fact that the nucleation of the vortex is associated with an increase of angular momentum from zero (no vortex) to \( N \hbar \) (one centered vortex), accompanied by an initial energy increase (barrier) and a subsequent monotonous energy decrease. In this form the TF-result can be usefully compared with alternative approaches based on microscopic calculations of the vortex energy.

### III. ROLE OF QUADRUPOLE DEFORMATIONS

In the previous section we have shown that in order to nucleate the vortex the system has to overcome a barrier associated with a macroscopic energy cost. In the following sections we will show that the barrier disappears at sufficiently high angular velocities if we allow the system to take a quadrupolar deformation. The physical mechanism is especially clear for an axisymmetric trap. Above a given angular velocity, fixed by \((1)\) with \( \ell = 2 \), the system becomes energetically unstable against the excitation of quadrupole oscillations. This instability causes a continuous symmetry breaking which can favor the occurrence of alternative paths for the nucleation of the vortex. The appearance of this
The formalism of \cite{20} one finds the expression:

\[ \mathbf{v}_Q = \alpha \nabla (xy), \]

where \( \alpha \) is a parameter. Note that \( \mathbf{v}_Q \) is the velocity field in the laboratory frame expressed in terms of the coordinates of the rotating frame. The form \( \mathbf{v}_Q \) is suggested by the quadrupolar class of irrotational solutions exhibited by the time-dependent Gross-Pitaevskii equation in the rotating frame \cite{20}. In Ref. \cite{20} it has been shown that these solutions are associated with a quadrupole deformation of the density described by the deformation parameter

\[ \delta = \frac{(y^2 - x^2)}{(y^2 + x^2)}. \]

The parameters \( \delta \) and \( \alpha \) characterize the quadrupole degrees of freedom that we are including in our picture. In the following, in order to provide a simplified description, we will fix a relationship between these two parameters by requiring that the quadrupole velocity field satisfies the condition:

\[ \nabla \cdot [n(r)(\mathbf{v}_Q - \Omega \times \mathbf{r})] = 0, \]

where \( \Omega = \Omega \hat{z} \). As a consequence of the equation of continuity, this condition implies that in the rotating frame the density of the gas is stationary except for the motion of the vortex core which however involves the change of the density at small length scales. Eq.\( \mathbf{14} \) yields the relationship \( \alpha = -\Omega \delta \) \cite{20} which selects a natural class of paths that will be considered in the present investigation. In the presence of the quadrupole velocity field \( \mathbf{v}_Q \), the energy of the condensate in the rotating frame can then be expressed only in terms of the deformation parameter \( \delta \). Using the formalism of \cite{20} one finds the expression:

\[ E_Q(\delta, \bar{\Omega}, \varepsilon, \mu) = N\mu \left[ \frac{2}{7} \frac{1 - \varepsilon \delta - \bar{\Omega}^2 \delta^2}{\sqrt{1 - \varepsilon^2}} \frac{1 - \Omega^2}{\sqrt{1 - \delta^2}} + \frac{3}{7} \right]. \]  \hspace{1cm} (17)

Here, \( \varepsilon \) describes the deformation of the trapping potential in the \( x-y \) plane

\[ V_{\text{ext}}(r) = \frac{m}{2} \left[ (1 + \varepsilon) \omega_\perp^2 x^2 + (1 - \varepsilon) \omega_\perp^2 y^2 + \omega_\perp^2 z^2 \right], \]

where \( \omega_\perp \) is an average transverse oscillator frequency

\[ \omega_\perp^2 = \frac{\omega_x^2 + \omega_y^2}{2}, \]  \hspace{1cm} (19)

and \( \bar{\Omega} = \Omega / \omega_\perp \) is the angular velocity of the trap expressed in units of \( \omega_\perp \). Note that it is crucial to distinguish between the transverse trap deformation \( \varepsilon \) and the quadrupole deformation \( \delta \) of the condensate. In Eq.\( \mathbf{17} \) we have neglected the change of the central density caused by the velocity field \( \mathbf{v}_Q \). This assumption will be used throughout the paper \cite{20}.

It is useful to expand Eq. \( \mathbf{17} \) as a function of \( \delta \) in the case of axisymmetric trapping \( (\varepsilon = 0) \). One finds the result

\[ E_Q(\delta, \bar{\Omega}, \varepsilon = 0, \mu) \simeq N\mu \left[ \frac{5}{7} + \delta^2 \left( \frac{1}{7} (1 - 2\bar{\Omega}^2) \right) + \mathcal{O}(\delta^3) \right]. \]  \hspace{1cm} (20)

which explicitly shows that for \( \Omega > \omega_\perp / \sqrt{2} \) the symmetric configuration \( (\delta = 0) \) is energetically unstable against the occurrence of quadrupole deformations. As we will see in the next section this instability is crucial for the nucleation of the vortex.

Our aim now is to calculate the energy in the rotating frame combining the effects of the vortex and of the quadrupole deformation. The parameter \( \delta \) thereby emerges as a natural degree of freedom that can be used to optimize the path towards vortex nucleation. In the simultaneous presence of the vortex and of quadrupole deformation the energy of the system is not however simply given by the sum of Eqs. \( \mathbf{9} \) and \( \mathbf{17} \). In fact, on the one hand the kinetic energy contains the additional crossed term \( m \int \mathbf{d}r \left[ \mathbf{v}_{\text{vortex}}, \mathbf{v}_Q \right] n(r) \). On the other hand we have to take into account that the vortex energy \( \mathbf{6} \) and the angular momentum \( \mathbf{3} \) were calculated for an axisymmetric trap and in the absence of
quadrupole deformation. In the presence of a deformed trap the vortex energy in the laboratory frame (7) is simply generalized to

\[ E_v(d/R_x, \varepsilon, \mu) = E_v(d = 0, \varepsilon, \mu) \left[ 1 - \left( \frac{d}{R_x} \right)^2 \right]^{3/2}, \]  

(21)

where \( R_x \) is the TF-radius of the cloud along \( x \) and we have explicitly positioned the vortex on the \( x \)-axis. It is important to note that the first equality in (8) also holds for \( \varepsilon / R \) where \( d/R \) is the TF-radius of the cloud along \( x \).

\[ E_v(d = 0, \varepsilon, \mu) = N \hbar \omega_\perp \frac{5 \hbar \omega_\perp}{4} \sqrt{1 - \varepsilon^2} \log \left( 1.342 \frac{\mu}{\hbar \omega_\perp} \right), \]  

(22)

which points out its dependence on the trap deformation.

Finally, it is possible to also evaluate in a simple way the sum of the crossed term \( m \int \! \! d\mathbf{r} \left[ \mathbf{v}_{\text{vortex}} \cdot \mathbf{v}_Q \right] n(\mathbf{r}) \) entering the kinetic energy and of the angular momentum term \( -m \mathbf{\Omega} \cdot \int \! \! d\mathbf{r} \left[ \mathbf{r} \times \mathbf{v}_{\text{vortex}} \right] n(\mathbf{r}) \) due to the vortex. One finds

\[ E_3(d/R_x, \delta, \bar{\Omega}) = m \int \! \! d\mathbf{r} \mathbf{v}_{\text{vortex}} \cdot \left[ \mathbf{v}_Q - \mathbf{\Omega} \times \mathbf{r} \right] n(\mathbf{r}) = -N \hbar \omega_\perp \bar{\Omega} \sqrt{1 - \delta^2} \left[ 1 - \left( \frac{d}{R_x} \right)^2 \right]^{5/2}, \]  

(23)

which is consistent with (3) in the case of a symmetric condensate (\( \delta = 0 \)). In deriving result (23) we have used the condition (13) for the quadrupole velocity field and we have integrated by parts using the expression (3) for \( \mathbf{v}_{\text{vortex}} \). Note that the irrotational component \( \nabla S \) of \( \mathbf{v}_{\text{vortex}} \) does not contribute to (23).

**IV. CRITICAL ANGULAR VELOCITY FOR VORTEX NUCLEATION**

The total energy of a quadrupolar deformed condensate in the presence of a vortex is given by the sum of Eqs. (21), (17) and (23):

\[ E_{\text{tot}}(d/R_x, \delta, \bar{\Omega}, \varepsilon, \mu) = E_v(d/R_x, \varepsilon, \mu) + E_Q(\delta, \bar{\Omega}, \varepsilon, \mu) + E_3(d/R_x, \delta, \bar{\Omega}), \]  

(24)

where we have explicitly indicated the dependence of the three contributions on the various physical parameters. The values of the angular velocity \( \bar{\Omega} \), the trap anisotropy \( \varepsilon \), the average transverse oscillator frequency \( \omega_\perp \), and the chemical potential \( \mu \) are fixed by the experimental conditions. Hence the degrees of freedom of the system with which one can play in order to identify the optimal path for vortex nucleation are the vortex displacement \( d \) and the condensate deformation \( \delta \). Note that each of the three energy contributions has a different dependence on the chemical potential resulting in a non-trivial dependence of the critical angular velocity on the relevant parameters of the system (see Eq. (20) below). Result (24) generalizes the one given in Ref. [17], which holds only in the limit of small \( \bar{\Omega} \) where \( \delta \sim \varepsilon \).

We assume that the system is initially in the state \( d/R_x = 1, \delta = 0 \). This assumption adequately describes an experiment in which \( \bar{\Omega} \) and \( \varepsilon \) are switched on suddenly. In this case, the condensate is initially axisymmetric (\( \delta = 0 \)) and vortex-free (\( d/R_x = 1 \)). Of course this configuration is not stationary and will evolve in time. In the following we will make use of energetic considerations in order to explore the possible paths followed by the system towards the nucleation of the vortex line. These paths should be associated with a monotonous decrease of the energy. In Figs. 2 and 3 we have plotted the energy surface \( E_{\text{tot}} \) for two different values of the angular velocity \( \bar{\Omega} \) and fixed \( \varepsilon \) and \( \mu/\hbar \omega_\perp \). In both cases the angular velocity was chosen high enough to make the vortex state a global energy minimum. This minimum is surrounded by an energy ridge which, for \( \delta = 0 \), forms a barrier between the initial state \( (d/R_x = 1) \) and the vortex state \( (d = 0) \), as discussed in section 1. Moreover, the energy ridge exhibits a saddle point at non-zero deformation \( \delta \). The height of this saddle depends on \( \bar{\Omega}, \varepsilon, \) and \( \mu/\hbar \omega_\perp \). In Fig. 4 the energy at the saddle point is higher than the energy of the initial state and the ridge can not be surpassed. However, at higher angular velocities \( \bar{\Omega} \) the situation changes. In Fig. 3 the saddle lies lower than the initial state. Hence in this case the system can bypass the barrier by crossing the saddle. The corresponding path is always associated with the occurrence of a strong intermediate deformation of the condensate.

The critical angular velocity for the nucleation of vortices naturally emerges as the angular velocity \( \bar{\Omega}_c \) at which the energy on the saddle point \( (d/R_x)_{sp}, \delta_{sp} \) is the same as the energy of the initial state \( (d/R_x = 1, \delta = 0) \):

\[ E((d/R_x)_{sp}, \delta_{sp}, \bar{\Omega}_c, \varepsilon, \mu) = E(d/R_x = 1, \delta = 0, \bar{\Omega}_c, \varepsilon, \mu). \]  

(25)
It is worth mentioning that crossing the saddle point is not the only possibility for the system to lower its energy. In fact, Figs. 2 and 3 show the existence of stationary deformed vortex-free states which can be reached starting from the initial state. These are the states predicted in [20] and experimentally studied in [33,44] through an adiabatic increase of either \( \Omega \) or \( \varepsilon \) instead of doing a rapid switch-on. The energy ridge separates these configurations from the vortex state. Still, under certain conditions this stationary vortex-free state becomes dynamically unstable and a vortex can be nucleated starting out from it [33,44]. The study of this type of vortex nucleation is beyond the scope of the present paper.

The actual value of the critical angular velocity \( \bar{\Omega}_c \) for vortex nucleation depends on the parameters \( \varepsilon \) and \( \mu/\hbar \omega_\perp \). Fig. 4 shows the dependence of \( \bar{\Omega}_c \) on \( \varepsilon \) for different choices of \( \mu/\hbar \omega_\perp \). To lowest order in \( \varepsilon \) and \( \hbar \omega_\perp/\mu \) the dependence is given by

\[
\bar{\Omega}_c(\varepsilon) = 1/\sqrt{2} \approx 1/\sqrt{2} \left[ (\eta A)^{1/2}/4 - \varepsilon/(\eta A)^{1/4} \right],
\]

where

\[
\eta = \left[ \log \left( 1.342 \frac{\mu}{\hbar \omega_\perp} \right) \right]^{5/2} \left( \frac{\hbar \omega_\perp}{\mu} \right)^{7/2}
\]

and \( A = 2^{-5/4} 21 \sqrt{3} \). This formula shows that the relevant parameters of the expansion are \( \eta^{1/2} \) and \( \varepsilon/\eta^{1/4} \). Fig. 3 demonstrates that it is applicable also at rather small values of the chemical potential.

At \( \varepsilon = 0 \) the vortex nucleation, according to the present scenario, is possible only at angular velocities slightly higher than the value \( \omega_\perp/\sqrt{2} \). For non-vanishing \( \varepsilon \) the preferable configuration will be always deformed even for small values of \( \bar{\Omega} \) where \( \delta \) depends linearly on \( \varepsilon \). At higher \( \bar{\Omega} \) the condensate gains energy by increasing its deformation in a nonlinear way (see [33]). Eq. (23) and Fig. 3 shows that for non-vanishing \( \varepsilon \) the saddle point on the energy ridge can be surpassed at angular velocities smaller than \( \omega_\perp/\sqrt{2} \).

The above scenario seems to be in reasonable agreement with experiments. In particular, evidence for critical angular velocities smaller than \( \omega_\perp/\sqrt{2} \) occurring when \( \varepsilon \neq 0 \) has emerged from the experiments reported in [33]. A more systematic comparison with the expected dependence of \( \bar{\Omega}_c \) on the trap deformation \( \varepsilon \) and on the chemical potential \( \mu \) would be crucial in order to assess the validity of the model [24].

### V. Stability of a Vortex-Configuration Against Quadrupole Deformation

Once the energy barrier is bypassed, the vortex moves to the center of the trap (\( d/R_x = 0 \)) where the energy has a minimum. In an axisymmetric trap (\( \varepsilon = 0 \)) this configuration will be in general stable against the formation of quadrupole deformations of the condensate unless the angular velocity \( \bar{\Omega} \) of the trap becomes too large. The criterion for instability is easily obtained by studying the \( \delta \)-dependence of the energy of the system in the presence of a single quantized vortex located at \( d/R_x = 0 \). Considering the total energy (24) we find

\[
E_{\text{tot}}(d/R_x = 0, \delta, \bar{\Omega}, \varepsilon = 0, \mu) \simeq E_{\text{tot}}(d/R_x = 0, \delta = 0, \bar{\Omega}, \varepsilon = 0, \mu) + \delta^2 N \mu \left( \frac{1}{4} (1 - 2\bar{\Omega}^2) + \frac{\bar{\Omega} \hbar \omega_\perp}{2 \mu} \right) + \mathcal{O}(\delta^3). \tag{27}
\]

Comparing Eqs. (27) with the analog expression (20) holding in the absence of the vortex line one observes that in the presence of the vortex the instability against quadrupole deformation occurs at a higher angular velocity given by

\[
\bar{\Omega} = \omega_\perp \left( \frac{1}{\sqrt{2}} + \frac{7 \hbar \omega_\perp}{8 \mu} \right) \tag{28}
\]

If the angular velocity is smaller than (25) the vortex is stable in the axisymmetric configuration while at higher angular velocities the system prefers to deform, giving rise to new stationary configurations. The critical angular velocity (25) can also be obtained by applying the Landau criterion (4) to the quadrupole collective frequencies in the presence of a quantized vortex. These frequencies were calculated in [24] using a sum rule approach. For the \( m = \pm 2 \) quadrupole frequencies the result reads

\[
\omega_{\pm 2} = \omega_\perp \sqrt{2 \pm \frac{\Delta}{2}}, \tag{29}
\]
where

\[ \Delta = \omega_\perp \left( \frac{\hbar \omega_\perp}{2 \mu} \right) \]  

(30)

is the frequency splitting between the two modes. Applying the condition (1) to the \( m = +2 \) mode one can immediately reproduce result (28) for the onset of the quadrupole instability in the presence of the quantized vortex.

In an anisotropic trap (\( \varepsilon \neq 0 \)), the stable vortex state will generally be associated with a non-zero deformation \( \delta \) of the condensate. It is interesting to note that \( \delta \) increases with the angular velocity of the trap and easily exceeds \( \varepsilon \) (see Figs. 2 and 3). This behaviour is analogous to the properties of the deformed stationary states in the absence of vortices [20].

VI. FINAL COMMENTS AND CONCLUSIONS

In this paper we have developed a semi-analytic model to describe the nucleation of a quantized vortex induced by the sudden switch-on of the trap deformation and rotation. The trap was assumed to be of quadrupolar shape. The model includes the distance of the vortex line from the principal axis and the quadrupole deformation of the condensate as the key degrees of freedom of the system. We have shown that, for sufficiently high angular velocities, the instability exhibited by a condensate with respect to surface quadrupole deformations gives rise to a path for vortex nucleation. The energy diagram is characterized by the occurrence of a saddle point whose height can favor or inhibit the nucleation, depending on the value of the angular velocity, the deformation of the trap and the chemical potential. The scenario for the nucleation emerging from the present approach agrees with the experimental results recently obtained in [1,2,4,5,8] as well as with the predictions based on the numerical solution of the time-dependent Gross-Pitaevskii equation [13]. In particular, the nucleation is always associated with an intermediate configuration exhibiting a significant quadrupole deformation of the condensate, even in the presence of a very small deformation of the trap. Furthermore the nucleation is favored by increasing the values of the trap deformation and the value of the chemical potential.

The energy diagram employed to calculate the critical angular velocity was based on the Thomas-Fermi approximation to the vortex energy. This approximation is expected to become worse and worse as the vortex line approaches the surface region [2,3]. Since the main mechanism of nucleation takes place near the surface, we expect that the quantitative predictions concerning the dependence of the critical angular velocity on the trap deformation and on the chemical potential would be improved by using a microscopic evaluation of the vortex energy, beyond the Thomas-Fermi approximation. This should include, in particular, the density dependence of the size of the vortex core, as well as the proper inclusion of image vortex effects. Work in this direction is in progress.

Useful discussions with Y. Castin, F. Dalfovo, J. Dalibard, D. Feder, E. Hodby, O. Maragò and A. Recati are acknowledged.
\[ \Omega = \frac{3}{2} \Omega_v(\mu) \]

FIG. 1. Vortex excitation energy in the rotating frame (9) for an axisymmetric configuration as a function of the reduced vortex displacement \( d/R_{\perp} \) from the center. The curves refer to two different choices for the angular velocity of the trap: \( \Omega = \Omega_v(\mu) \) (solid line), and \( \Omega = \frac{3}{2} \Omega_v(\mu) \) (dotted line), where \( \Omega_v(\mu) = E_v(d = 0, \mu)/N\hbar \). The initial vortex-free state corresponds to \( d/R_{\perp} = 1 \). For \( \Omega > \Omega_v(\mu) \), the state with a vortex at the center \( (d/R_{\perp} = 0) \) is preferable. However, in this configuration the nucleation of the vortex is inhibited by a barrier separating the vortex-free state \( (d/R_{\perp} = 1) \) from the energetically favored vortex state \( (d/R_{\perp} = 0) \).
FIG. 2. Below the critical angular velocity of vortex nucleation: the plot shows the dependence of the total energy $E_{\text{tot}} - E(d/R_x = 1, \delta = 0)$ on the quadrupolar shape deformation $\delta$ and on the vortex displacement $d/R_x$ from the center. Energy is given in units of $N\hbar \omega_\perp$. The dashed line corresponds to $E_{\text{tot}} - E(d/R_x = 1, \delta = 0) = 0$, while the solid curve refers to $E_{\text{tot}} - E(d/R_x = 1, \delta = 0) = 0.015N\hbar \omega_\perp$. This plot has been obtained by setting $\varepsilon = 0.04$, $\mu = 10 \hbar \omega_\perp$ and $\Omega = 0.64 \omega_\perp$. The initial state is indicated with $\oplus$, while $\otimes$ corresponds to the energetically preferable centered vortex state. The barrier $\oslash$ inhibits vortex nucleation in a non-deforming condensate ($\delta = 0$). The saddle point $\ominus$ lies lower than the barrier $\oslash$. However, at the chosen $\Omega$ the energy on the saddle is still higher than the one of the initial state $\oplus$. Note that the preferable vortex state is associated with a shape deformation $\delta > \varepsilon$ (see section V). Note also the existence of a favorable deformed and vortex-free state labeled by $\vartriangledown$. 

Note also the existence of a favorable deformed and vortex-free state labeled by $\vartriangledown$. 

$\vartriangledown$
FIG. 3. Above the critical angular velocity of vortex nucleation: the plot shows the dependence of the total energy \[ E_{\text{tot}}(d/R_x = 1, \delta = 0) \] minus the energy of the initial non-deformed vortex-free state \( E(d/R_x = 1, \delta = 0) \) on the quadrupolar shape deformation \( \delta \) and on the vortex displacement \( d/R_x \) from the center. Energy is given in units of \( N \hbar \omega_\perp \). The dashed line corresponds to \( E_{\text{tot}} - E(d/R_x = 1, \delta = 0) = 0 \), while the solid curve refers to \( E_{\text{tot}} - E(d/R_x = 1, \delta = 0) = 0.015 N \hbar \omega_\perp \). In this plot \( \varepsilon = 0.04 \), \( \mu = 10 \hbar \omega_\perp \), as in Fig. 2, and \( \Omega = 0.7 \omega_\perp \). Important states are indicated as in Fig. 2. At the chosen \( \Omega \), the saddle \( \odot \) lies lower than the initial state \( \oplus \) allowing the system to bypass the barrier \( \odot \) by taking a quadrupolar deformation \( \delta \) and reach the preferable vortex state \( \otimes \). Note also the existence of a favorable deformed and vortex-free state labeled by \( \nabla \).
\[ \mu = 9 \hbar \omega_{\perp} \quad \mu = 18 \hbar \omega_{\perp} \]

\[ \Omega_{c}/\omega_{\perp} \]

**FIG. 4.** Critical angular velocity of vortex nucleation in units of \( \omega_{\perp} \) as a function of the trap deformation \( \varepsilon \). The long dashed and short dashed curves correspond to the numerical calculation satisfying condition (25) for \( \mu = 9 \hbar \omega_{\perp} \) and \( \mu = 18 \hbar \omega_{\perp} \) respectively, while the solid line is the analytic prediction (26) evaluated with \( \mu = 9 \hbar \omega_{\perp} \). The arrow indicates the angular velocity at which the quadrupole surface mode becomes unstable in the case \( \varepsilon = 0 \). The value \( \mu = 9 \hbar \omega_{\perp} \) is close to the experimental setting of [1].

[1] K.W. Madison, F. Chevy, W. Wohlleben, and J. Dalibard, Phys. Rev. Lett. 84, 806 (2000).
[2] J.R. Abo-Shaeer, C. Raman, J.M. Vogels, and W. Ketterle, *Science* **292**, 476 (2001). Published on-line March 22, 2001; 10.1126/science.1060182 (Science Express Research Articles).
[3] K.W. Madison, F. Chevy, V. Bretin, and J. Dalibard, Phys. Rev. Lett. **86**, 4443 (2001).
[4] C. Raman, J.R. Abo-Shaeer, J.M. Vogels, K. Xu, and W. Ketterle, *cond-mat/0106233*.
[5] E. Hodby, G. Hechenblaikner, S.A. Hopkins, O.M. Marag`o, and C.J. Foot, *cond-mat/0106263*.
[6] F. Dalfovo, S. Giorgini, M. Guilleumas, L.P. Pitaevskii, and S. Stringari, Phys. Rev. A **56**, 3840 (1997).
[7] F. Dalfovo and S. Stringari, Phys. Rev. A **63**, 011601(R) (2001).
[8] F. Chevy, K.W. Madison, V. Bretin, and J. Dalibard, e-print *cond-mat/0104218*. To be published in *Proceedings of trapped particles and fundamental physics workshop* (Les Houches 2001) edited by S. Atutov, K. Kalabrese, and L. Moi.
[9] S. Stringari, Phys. Rev. Lett. **77**, 2360 (1996).
[10] T. Isoshima and K. Machida, Phys. Rev. A **60**, 3313 (1999).
[11] S. Sinha and Y. Castin, e-print *cond-mat/0101292*.
[12] D.L. Feder, C.W. Clark, and B.I. Schneider, Phys. Rev. A **61**, 011601(R) (1999); D.L. Feder, A.A. Svidzinsky, A.L. Fetter, and C.W. Clark, Phys. Rev. Lett. **86**, 564 (2001).
[13] M. Tsubota, K. Kasamatsu, and M. Ueda, e-print *cond-mat/0104523*.
[14] F. Dalfovo, S. Giorgini, L. Pitaevskii and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
[15] For a uniform superfluid confined in a cylinder, this extra irrotational velocity field is crucial in order to satisfy the proper boundary conditions and its effects can be exactly accounted for by the inclusion of an image vortex located outside the cylinder. See for example G.B. Hess, Phys. Rev. **161**, 189 (1967).
[16] The inclusion of curvature effects in the description of quantized vortices in trapped condensates has been the subject of recent theoretical studies. See, for example, A.A. Svidzinsky and A.L. Fetter, Phys. Rev. A **62**, 063617 (2000); J.J. Garcia-Ripoll and V.M. Perez-Garcia, Phys. Rev. A **63**, 041603(R) (2001). See also [19] and references therein.
[17] A.A. Svidzinsky and A.L. Fetter, Phys. Rev. Lett. **84**, 5919 (2000).
[18] E. Lundh, C.J. Pethick, and H. Smith, Phys. Rev. A **55**, 2126 (1997).
[19] A.L. Fetter and A.A. Svidzinsky, J. Phys. Condens. Matter **13**, R135 (2001).
[20] A. Recati, F. Zambelli, and S. Stringari, Phys. Rev. Lett. 86, 377 (2000).

[21] Imposing that the energy (17) be stationary with respect to $\delta$ yields the solutions derived in [20] apart from small corrections due to the changes of the central density not accounted for in the present formalism.

[22] Our results do not change if $d/R_x$ is replaced by $d/R_y$. We expect that predicting the preferable direction for a vortex to enter the condensate demands the calculation of the vortex energy beyond logarithmic accuracy.

[23] A first experimental curve of $\bar{\Omega}_c(\varepsilon)$ in qualitative agreement with our predictions has recently been reported in [3]. However the measured values of $\bar{\Omega}_c$ are systematically shifted upwards, probably because the sudden switch-on condition was not satisfied in this experiment.

[24] F. Zambelli and S. Stringari, Phys. Rev. Lett. 81, 1754 (1998).

[25] F. Dalfovo, L.P. Pitaevskii, and S. Stringari, Phys. Rev. A 54, 4213 (1996).