A canonical basis of two-cycles on a $K3$ surface

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Abstract. We construct a basis of two-cycles on a $K3$ surface; in this basis, the intersection form takes the canonical form $2E_8(-1) \oplus 3H$. Elements of the basis are realized by formal sums of smooth submanifolds.

Bibliography: 8 titles.

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In this paper we give an explicit form of a canonical basis of two-cycles on a $K3$ surface. This basis is realized by formal sums of smooth submanifolds.

By analogy with the theory of Riemann surfaces, by a canonical basis we mean a basis of cycles in which the intersection form

$$H_2(X; \mathbb{Z}) \cap H_2(X; \mathbb{Z}) \rightarrow \mathbb{Z} = H_0(X; \mathbb{Z})$$

has the canonical form

$$E_8(-1) \oplus E_8(-1) \oplus H \oplus H \oplus H,$$  \hspace{1cm} (0.1)

where

$$E_8(-1) = (-1) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \hspace{1cm} (0.2)$$

$$H = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}. \hspace{1cm} (0.3)$$

A canonical basis is not defined uniquely, but only up to automorphisms of the lattice $H_2(X; \mathbb{Z})$ that preserve the intersection form (0.1). In this paper we give an explicit example of one such base and do not investigate the action of the automorphism group on the canonical bases.

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For four-dimensional simply connected manifolds, the Poincaré-Lefschetz-Pontryagin theory of intersections used in our paper (see Remark 3 below) degenerates into the intersection theory of two-cycles with values in $H_0$. However, this method can be used to evaluate the ring structure in the cohomology in a more general situation (for example, for multidimensional manifolds obtained using Kummer’s construction or its generalizations) in which the operation of intersection takes values in multidimensional homology groups.

Originally, the intersection form of a $K3$ surface was found in [1] using general considerations: according to Milnor’s theorem, which was proved in that paper (see also [2]), an even indefinite unimodular form over $\mathbb{Z}$ is defined uniquely by its rank $r$ and signature $\tau$ and has the form

$$\left(\frac{-\tau}{8}\right)E_8(-1) \oplus \left(\frac{r+\tau}{2}\right)H.$$ 

In our case the rank of the form coincides with the second Betti number $b_2(S) = 22$, which can be found by limiting ourselves to the rational cohomology, the signature of the manifold $\tau = -16$ can be obtained from Hodge theory, and the form is even because the second Stiefel-Whitney class $w_2(X)$ vanishes. We can see that the intersection form for the $K3$ surface $X$ is as given in (0.1) from this. For detailed proofs, see [3] and [4], for example.

The problem of constructing a canonical basis in $H_2(X;\mathbb{Z})$ explicitly, without using the abstract theory of symmetric forms was stated in recent lectures [5] on the theory of $K3$ surfaces. The author proposed to use Kummer surface to solve this problem. All $K3$ surfaces are not just pairwise homeomorphic but also diffeomorphic (for example, see [3]), and Kummer surfaces admit a simple topological description. Note that Pyatetskii-Shapiro and Shafarevich [6] showed that Kummer surfaces (and even special Kummer surfaces) are dense in the moduli space of (polarized) algebraic $K3$ surfaces and, using this fact, proved a Torelli-type theorem for algebraic $K3$ surfaces.

Now we turn to the definition of Kummer surfaces.

Consider the four-dimensional torus $T^4$ obtained as the quotient of the complex space $\mathbb{C}^2$ by the action (by shifts) of a lattice $\Lambda$ of rank 4: $T^4 = \mathbb{C}^2/\Lambda$. The reflection $\iota: \mathbb{C}^2 \to \mathbb{C}^2$, $\iota(z) = -z$, generates an action of $\mathbb{Z}_2$ on $T^4$. This $\mathbb{Z}_2$-action is not free and has 16 fixed points lying at half-periods of the lattice, that is, at points in $\frac{1}{2}\Lambda$. Every fixed point has an $\iota$-invariant disc neighbourhood $D^4$ with spherical boundary $\partial D^4 = S^3$. When passing to the quotient of $T^4$ by the action of $\iota$, every disc neighbourhood of this kind becomes a cone over the boundary $\partial D^4/\mathbb{Z}_2 = \mathbb{R}P^3$ with vertex at the fixed point. The Kummer surface $X$ is obtained from the quotient space $T^4/\mathbb{Z}_2$ by blowing up the singularities corresponding to the 16 fixed points of the involution $\iota$. The blowup ($\sigma$-process) consists in removing, for every singular point, the cone $D^4/\mathbb{Z}_2$ corresponding to it and pasting a smooth manifold $M^4$ with the same boundary to the resulting boundary surface $\mathbb{R}P^3$. We can assume that the fibration $M^4 \to S^2$ is embedded in a complex line bundle of the form $V^4 \xrightarrow{\mathbb{C}} \mathbb{C}P^1 = S^2$ which is the square $\gamma^2 = O(-2)$ of the standard line (or tautological) bundle $\gamma = O(-1)$ over $\mathbb{C}P^1$.

Under this surgery every singular point $L \in T^4/\mathbb{Z}_2$ is replaced by the two-dimensional sphere $S^2_L = \mathbb{C}P^1$, and there is a natural projection $M^4 \setminus S^2_L \to D^4/\mathbb{Z}_2 \setminus L$,
Moreover, above, ponding classes of two-dimensional homology by permutations where these homology classes are equal to homology classes by the same symbols and note that the intersection indices of oriented basis (the vectors \( x \)) of tori, 2) classes coming from submanifolds diffeomorphic to Kummer manifolds are simply connected, and their rational homology is generated by classes of two types: 1) classes that can be pulled back to homology classes of \( S^2 \) is equal to \( \frac{1}{2} \). We modify the zero section \( \eta_0 \) of the tangent bundle \( TS^2 \) slightly in such a way that the resulting section \( \eta \) meets \( \eta_0 \) transversally. This section is a vector field \( \eta \) on \( S^2 \). In \( TS^2 \), on the one hand the index of intersection between \( \eta \) and \( \eta_0 \) is equal to the self-intersection index of \( S^2 \) and on the other hand it is equal to the index of the vector field \( \eta \), that is, the characteristic of \( S^2 \): \( \text{ind} \eta = 2 \). The Euler characteristic of the tangent bundle is \( \langle c_1(\gamma^2), [\mathbb{C}P^1] \rangle = -\langle c_1(\gamma^2), [\mathbb{C}P^1] \rangle \), and \( \langle c_1(\gamma^2), [\mathbb{C}P^1] \rangle \) is the self-intersection index of the \([L]\)-cycle: 
\[
[L] \cap [L] = -2.
\]

Multidimensional Kummer manifolds are obtained similarly as a result of resolving the singularities of the quotient spaces \( T^{2N}/\mathbb{Z}_2 \), which have \( 2^N \) singular points. Kummer manifolds are simply connected, and their rational homology is generated by classes of two types: 1) classes that can be pulled back to homology classes of tori, 2) classes coming from submanifolds diffeomorphic to \( \mathbb{C}P^{N-1} \) that replace singular points (for example, see [7]).

We will describe these generators for \( K3 \) surfaces in greater detail.

For simplicity, consider the case when \( \Lambda = \mathbb{Z}^4 \). We denote the Euclidean coordinates in \( \mathbb{R}^4 = \mathbb{C}^2 \) by \( x_1, x_2, x_3, x_4 \), the coordinates in \( \mathbb{C}^2 \) by \( z_1 = x_1 + ix_2 \) and \( z_2 = x_3 + ix_4 \), and the corresponding basis in \( \mathbb{R}^4 \), which simultaneously defines a basis in \( \Lambda \), by \( e_1, e_2, e_3, e_4 \).

We let \( T_{ij} \) denote the oriented two-dimensional torus in \( T^4 \) that is generated by the vectors \( e_i \) and \( e_j \) and assume that the orientation is defined by the positively oriented basis \( (e_i, e_j) \). Let \( [T_{ij}] \) be the corresponding homology class. If the torus \( T_{ij} \) does not pass through fixed points of the involution \( \iota \), then it is projected onto a torus in \( T^4/\mathbb{Z}_2 \) that defines a nontrivial homology class in \( X \). We denote these homology classes by the same symbols and note that the intersection indices of these homology classes are equal to
\[
[T_{ij}] \cap [T_{kl}] = 2 \varepsilon_{ijkl}, \tag{0.4}
\]
where \( \varepsilon_{ijkl} = 0 \) if at least two of the indices \( i, j, k \) and \( l \) coincide, \( \varepsilon_{ijkl} = 1 \) for even permutations \( \left( \begin{array}{llll} 1 & 2 & 3 & 4 \\ i & j & k & l \end{array} \right) \), and \( \varepsilon_{ijkl} = -1 \) if the permutation is odd.

We denote the fixed points of the involution \( \iota \) by \( L_1, \ldots, L_{16} \) and the corresponding classes of two-dimensional homology by \([L_1], \ldots, [L_{16}]\). As we have shown above,
\[
[L_i] \cap [L_j] = -2 \delta_{ij}. \tag{0.5}
\]
Moreover,
\[
[T_{ij}] \cap [L_k] = 0 \quad \text{for all } i, j, k. \tag{0.6}
\]
Hence the cycles \([L_i], i = 1, \ldots, 16,\) and \([T_{ij}], 1 \leq i < j \leq 4,\) form a basis in the rational homology group \(H_2(X; \mathbb{Q}) = \mathbb{Q}^{22},\) and the lattice generated by these cycles is a sublattice of index 2^{11} in \(H_2(X; \mathbb{Z}),\) since the intersection form on this sublattice is not unimodular, and the determinant of the form is equal to \(-2^{22}.\)

In order to find a canonical basis in \(H_2(X; \mathbb{Z})\) we must consider other cycles, which can be expressed linearly over the field of rationals in terms of \([L]-\) and \([T]-\)cycles.

Consider a two-torus \(T^2 \subset T^4\) that passes through fixed points. Since it contains a fixed point (a half-period) and is generated by two generators of the period lattice \(\Lambda,\) it is obvious that it must pass through four fixed points. We denote the torus passing through the points \(L_i, L_j, L_k\) and \(L_m\) by \(T_{ijkm}.\) It is invariant under the involution \(\iota.\) Consider the square (on the torus \(T_{ijkm}\)) with vertices at the points \(L_i, L_j, L_k\) and \(L_m,\) which are ordered so that in going along the boundary of the square we go through these vertices in succession. We assume that making this circuit defines the positive orientation of the boundary of the square and, with it, the compatible orientations of the square and the torus \(T_{ijkm}.\)

We perform surgery on the quotient spaces \(T_{ijkm}/\mathbb{Z}_2\) in such a way that, after blowing up the singularities we obtain smooth manifolds whose intersection indices with \(L_n\) are either zero or one. To do this we consider two cases.

**Case 1.** Suppose that \(T_{ijkm}\) is obtained in \(\mathbb{C}^2/\mathbb{Z}^4\) from the complex line \(\mathbb{C} = \{z_2 = \text{const}\}\) by taking the quotient by the lattice \(\mathbb{Z}^2.\) In this case, under the resolution of a singularity, a two-dimensional disc (a two-dimensional cone with the vertex at the singular point) is removed from the two-sphere of the form \(T_{ijkm}/\mathbb{Z}_2\) and replaced by a two-disc that lies in a fibre of the bundle \(\gamma^2: V^4 \to \mathbb{CP}^1 = S^2\) and meets \(L_n, n \in \{i, j, k, m\},\) at a unique point, the zero vector of the fibre. We denote the smooth manifold thus obtained by \(S_{ijkm}.\) Since both this manifold and \(L_n\) are complex curves, it follows that their intersection index is positive and hence equal to one.

**Case 2.** Suppose that the torus \(T_{ijkm}\) is distinct from a complex submanifold and that it passes through a fixed point \(L.\) The torus intersects the boundary of the disc neighbourhood \(D^4(L)\) of this point along a curve \(\gamma_0\) whose orientation agrees with that of the torus and which is \(\iota-\)invariant. We construct a homotopy \(\gamma_t, 0 \leq t \leq 1,\) of this curve that consists of the simultaneous application of a rotation about \(L\) and a homothety with coefficient \((1 - t/2)\) centred at \(L\) so that the oriented contour \(\gamma_1\) lies in the plane \(z_2 = \text{const}\), where it bounds a positively oriented two-disc \(D_0(L).\) The union of the cylinder swept by the contour under the homotopy \(\gamma_t\) and the disc \(D_0(L)\) is an \(\iota-\)invariant disc \(D_1(L).\) We remove the disc \(T_{ijkm} \cap D^4(L)\) from the torus \(T_{ijkm}\) and replace it by \(D_1(L).\) Of course, making a small perturbation of the homotopy \(\gamma_t\) we can smooth this construction to obtain an \(\iota-\)invariant smooth submanifold. Now we apply this construction to all the intersection points of the torus \(T_{ijkm}\) with the fixed point set of the involution \(\iota\) and obtain an \(\iota-\)invariant smooth submanifold \(T'_{ijkm}.\) The blowups of the singularities of the quotient space \(T'_{ijkm}/\mathbb{Z}_2\) have the same form as in Case 1, and we obtain a smooth submanifold \(S_{ijkm} \subset X\) such that its intersection indices with submanifolds of the form \(L_n\) are equal to either one (for \(n \in \{i, j, k, m\}\)) or zero (otherwise).
The submanifolds $S_{ijkm}$ are clearly diffeomorphic to the two-sphere $S^2$, and each $T'_{ijkm} = T^2 \rightarrow S_{ijkm}$ is a two-dimensional covering of the sphere branched at four points.

We denote the homology cycles corresponding to the submanifolds $S_{ijkm}$ by $[S_{ijkm}]$. We go over directly to the construction of a canonical basis of cycles.

We split all $[L]$-cycles into two groups corresponding to fixed points with $x_4 = 0$ and fixed points with $x_4 = 1/2$.

We consider the first group and enumerate the cycles in it using the following rule (on the right-hand side we indicate the coordinates of the corresponding fixed points):

$$[L_1] \leftrightarrow (0, 0, 0), \quad [L_2] \leftrightarrow (0, 0, \frac{1}{2}), \quad [L_3] \leftrightarrow \left(\frac{1}{2}, 0, 0\right),$$

$$[L_4] \leftrightarrow \left(\frac{1}{2}, 0, \frac{1}{2}\right), \quad [L_5] \leftrightarrow \left(\frac{1}{2}, \frac{1}{2}, 0\right), \quad [L_6] \leftrightarrow \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right),$$

$$[L_7] \leftrightarrow \left(0, \frac{1}{2}, 0\right), \quad [L_8] \leftrightarrow \left(0, \frac{1}{2}, \frac{1}{2}\right).$$

By construction, the cycles $[S_{1357}], [S_{2156}], [S_{5643}]$, and $[S_{3487}]$ have nonnegative indices of intersection with cycles of the form $L_n$ and, decomposing them over the field of rationals with respect to the basis $\{[L_i], [T_{ij}]\}$ and taking (0.4), (0.5) and (0.6) into account we obtain

$$[S_{1357}] = -\frac{1}{2}([L_1] + [L_3] + [L_5] + [L_7]) + \frac{1}{2}[T_{12}],$$

$$[S_{2156}] = -\frac{1}{2}([L_1] + [L_2] + [L_5] + [L_6]) + \frac{1}{2}([T_{13}] + [T_{23}]),$$

$$(0.7)$$

$$[S_{5643}] = -\frac{1}{2}([L_3] + [L_4] + [L_5] + [L_6]) + \frac{1}{2}[T_{23}],$$

$$[S_{3487}] = -\frac{1}{2}([L_3] + [L_4] + [L_7] + [L_8]) + \frac{1}{2}([T_{13}] - [T_{23}]).$$

Now, we define the following homology cycles:

1) the cycles $w_1, \ldots, w_8$,

$$w_1 = -[S_{1357}] - [L_3] - [L_5] - [L_7] = \frac{1}{2}([L_1] - [L_3] - [L_5] - [L_7]) - \frac{1}{2}[T_{12}],$$

$$w_2 = -[S_{2156}] = \frac{1}{2}([L_1] + [L_2] + [L_5] + [L_6]) - \frac{1}{2}([T_{13}] + [T_{23}]),$$

$$w_3 = ([T_{13}] + [T_{23}]) - [S_{2156}] - [L_1] - [L_2]$$

$$= \frac{1}{2}(-[L_1] - [L_2] + [L_5] + [L_6]) + \frac{1}{2}([T_{13}] + [T_{23}]),$$

$$w_4 = -[L_6],$$

$$w_5 = -[S_{5643}] - [L_5] = \frac{1}{2}([L_3] + [L_4] - [L_5] + [L_6]) - \frac{1}{2}[T_{23}],$$

$$w_6 = -[L_4],$$

$$w_7 = -[S_{3487}] - [L_7] = \frac{1}{2}([L_3] + [L_5] - [L_7] + [L_6]) - \frac{1}{2}[T_{13}],$$

$$w_8 = -[L_8].$$
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\[ w_6 = -[L_4], \]
\[ w_7 = -[S_{3487}] - [L_5] = \frac{1}{2} \left( -[L_3] + [L_4] + [L_7] + [L_8] \right) - \frac{1}{2}(T_{13} - T_{23}), \]
\[ w_8 = -[L_8]; \]

2) the cycles \( w_9, \ldots, w_{16} \) are constructed in a similar way; they correspond to submanifolds contained in the hyperplane \( x_4 = \frac{1}{2} \); in this case the arguments can be repeated verbatim, except that the numerical subscripts in the notation must be increased by 8 for \( w \)-, \([L]\)- and \([S]\)-cycles and must be preserved for \([T]\)-cycles.

3) the cycles \( w_{17}, \ldots, w_{22} \)

\[ w_{17} = [T_{12}], \quad w_{19} = [T_{13}], \quad w_{21} = [T_{23}], \]
\[ w'_{18} = [S_{129(10)}] + [L_2] + [L_{10}] = -\frac{1}{2} \left( [L_1] - [L_2] + [L_9] - [L_{10}] \right) + \frac{1}{2}[T_{34}], \]
\[ w'_{20} = [S_{719(15)}] = -\frac{1}{2} \left( [L_1] + [L_7] + [L_9] + [L_{15}] \right) - \frac{1}{2}[T_{24}], \]
\[ w'_{22} = [S_{13(11)}9] = -\frac{1}{2} \left( [L_1] + [L_3] + [L_9] + [L_{11}] \right) + \frac{1}{2}[T_{14}]. \]

We denote the lattice generated by the cycles \( w_1, \ldots, w_{16} \) by \( \Lambda_1 \) and that generated by the cycles \( w_{17}, w'_{18}, w_{19}, w'_{20}, w_{21} \) and \( w'_{22} \) by \( \Lambda_2 \).

It follows from (0.4) and (0.5) that

1) the lattices \( \Lambda_1 \) and \( \Lambda_2 \) are orthogonal with respect to the intersection form:

\[ u \cap v = 0 \quad \text{for all} \quad u \in \Lambda_1, \ v \in \Lambda_2; \]

2) in the basis \( w_1, \ldots, w_{16} \), the restriction of the intersection form to \( \Lambda_1 \) becomes

\[
\begin{pmatrix}
E_8(-1) & 0 \\
0 & E_8(-1)
\end{pmatrix},
\]

where the form \( E_8(-1) \) is as in (0.2);

3) in the basis \( w_{17}, w'_{18}, w_{19}, w'_{20}, w_{21}, w'_{22} \), the restriction of the intersection form to \( \Lambda_2 \) becomes

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & -1 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 1 & -2 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & -1 & 1 & -2
\end{pmatrix};
\]

this restriction is of the form \( H \oplus H \oplus H \) in the basis \( w_{19}, \ldots, w_{22} \), where the cycles \( w_{17}, w_{19} \) and \( w_{21} \) are as above and

\[ w_{18} = w'_{18} + w_{17}, \quad w_{20} = w'_{20} + w_{17} + w_{19}, \quad w_{22} = w'_{22} + w_{17} + w_{19} + w_{21}. \]

By definition, the cycles \( w_1, \ldots, w_{22} \) are realized as formal sums of smooth submanifolds and belong to \( H_2(X; \mathbb{Z}) \). Since the intersection form is given by (0.1) on
the lattice $\Lambda$ generated by these cycles, it follows, in particular, that this form is unimodular, and therefore the lattice $\Lambda$ coincides with the whole of $H_2(X; \mathbb{Z})$.

This completes the proof of the following theorem.

**Theorem 1.** In the basis $w_1, \ldots, w_{22}$, the intersection form in $H_2(X; \mathbb{Z})$ takes the canonical form (0.1).

**Remark 1.** When defining the submanifolds $S_{ijkm}$ we transformed the tori $T_{ijkm}$ into the tori $T'_{ijkm}$ by surgeries in neighbourhoods of fixed points of the involution $\iota$ in such a way that the intersection indices of $S_{ijkm}$ with $L_n$, $n \in \{i, j, k, n\}$, became equal to one. The homotopy $\kappa_t$ entering the surgery can also be chosen so that the resulting contour $\kappa_1$ bounds a negatively oriented disc $D_0(L)$ in the plane $z_2 = \text{const}$. In this case the index of intersection of the submanifold $S$ (obtained after blowing up the singularities) with $L$ is $[S] \cap [L] = -1$, and the relation

$$[S] = [S_{ijkm}] + [L]$$

holds in the homology group. However, in this case it is certainly impossible to realize $[S]$ by complex submanifolds, since $[S]$ has negative intersection index with the complex projective line $L$. Moreover, we can choose another orientation of the submanifold $S_{ijkm}$. This implies that if

$$[S_{ijkm}] = -\frac{1}{2}([L_i] + [L_j] + [L_k] + [L_m]) + \frac{1}{2}[T],$$

where $[T]$ is an integer combination of cycles of the form $[T_{pq}]$, then every cycle of the form

$$[S] = -\frac{1}{2}(\varepsilon_i[L_i] + \varepsilon_j[L_j] + \varepsilon_k[L_k] + \varepsilon_m[L_m]) + \frac{1}{2}\varepsilon_t[T],$$

where $\varepsilon_i, \varepsilon_j, \varepsilon_k, \varepsilon_m, \varepsilon_t \in \{1, -1\}$, is realized by a submanifold homeomorphic to a sphere.

**Remark 2.** By the Hurewicz theorem for simply connected manifolds, the natural homomorphism

$$\pi_2(X) \to H_2(X; \mathbb{Z})$$

is an isomorphism. By the previous remark, the cycles $w_1, \ldots, w_{16}$ belong to the image of this homomorphism. Formula (0.7) shows how cycles of the form $[T_{pq}]$ can be represented as sums of spherical cycles. For example, if

$$[S_{1357}] = -\frac{1}{2}([L_1] + [L_3] + [L_5] + [L_7]) + \frac{1}{2}[T_{12}],$$

then it follows that

$$[T_{12}] = 2[S_{1357}] + [L_1] + [L_3] + [L_5] + [L_7].$$

**Remark 3.** For a closed oriented $N$-dimensional smooth manifold $X$ the operation of intersection of cycles

$$H_k(X; \mathbb{Z}) \times H_l(X; \mathbb{Z}) \xrightarrow{\cap} H_{k+l-N}(X; \mathbb{Z})$$
is dual to the cohomology product. Namely, if the cycles \( u \in H_k(X; \mathbb{Z}) \) and \( v \in H_l(X; \mathbb{Z}) \) are realized by smooth submanifolds \( Y \) and \( Z \) that intersect transversally, then the intersection of \( Y \) and \( Z \) is a smooth submanifold of \( W \) of dimension \( k + l - N \), and

\[
Du \cup Dv = Dw,
\]

where \( w \) denotes the cycle realizable by the submanifold \( W \) and

\[
D : H_i(X; \mathbb{Z}) \to H^{N-i}(X; \mathbb{Z}), \quad i = 0, \ldots, N,
\]

stands for the Poincaré duality. This holds also for the homology of non-oriented manifolds over the field of coefficients \( \mathbb{Z}_2 \). It is known that not all cycles are realized by smooth submanifolds; however, this duality is generalized to arbitrary manifolds by introducing a special class of chains. For a careful and complete presentation of this construction, see [8], where the homology group \( H_*(X) \) with the intersection operation is called the Lefschetz (intersection) ring and, in particular, using an isomorphism \( D \) of this ring to the ring \( H^*(X) \), we can establish the topological invariance of the intersection ring.

**Remark 4.** For the eight-dimensional Kummer manifold, which was obtained by blowing up the quotient space \( T^8/\mathbb{Z}_2 \), an analogue of the cycles of half the dimension, \( S_{ijkm} \), is given by the four-dimensional cycles generated by tori \( T^4 \subset T^8 \). These four-dimensional tori pass through the 16 fixed points of the involution \( \iota \), and after the blowup the singularities go over to submanifolds diffeomorphic to \( K3 \) surfaces. By analogy with the case of \( K3 \) surfaces (see Remarks 1 and 2), we can conclude that all four-dimensional cycles are realized by linear combinations of embedded \( K3 \) surfaces and complex projective planes \( \mathbb{C}P^2 \).

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