computing the nonfree locus of the moduli space of arrangements and terao’s freeness conjecture

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abstract. in this paper, we show how to compute using fitting ideals the nonfree locus of the moduli space of arrangements of a rank 3 simple matroid, i.e., the subset of all points of the moduli space which parametrize nonfree arrangements. our approach relies on the so-called ziegler restriction and yoshinaga’s freeness criterion for multiarrangements. we use these computations to verify terao’s freeness conjecture for rank 3 central arrangements with up to 14 hyperplanes in any characteristic.

1. introduction

a (central) arrangement of hyperplanes \( A = \{H_1, \ldots, H_n\} \) is a finite collection of hyperplanes in a vector space \( V \) of dimension \( r \) over a field \( k \) containing the origin. denote by \( S \) the polynomial ring \( k[x_1, \ldots, x_r] \) where \( r = \dim V \). for each hyperplane \( H \in A \) we can fix a linear polynomial \( \alpha_H \in S \) such that \( H = V(\alpha_H) \) is the vanishing locus of \( \alpha_H \). then \( A = V(\prod_{H \in A} \alpha_H) \).

one of the most studied algebro-geometric invariants of \( A \) is the graded \( S \)-module of logarithmic derivations \( D(A) \) defined as

\[
D(A) := \{ \theta \in \text{Der}(S) | \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in A \},
\]

where \( \text{Der}(S) = \left\{ \theta = \sum_{i=1}^{r} \theta_i \frac{\partial}{\partial x_i} \bigg| \theta_i \in S \right\} \cong S^r \) is the module of all derivations on \( S \). if \( D(A) \) is a free \( S \)-module, \( A \) is called a free arrangement.

conjecture 1.1 (terao’s freeness conjecture). the freeness of an arrangement \( A \) defined over a field \( k \) only depends on the characteristic of the field and the intersection lattice \( \mathcal{L}(A) \) of hyperplanes, which is isomorphic to the lattice of flats of the underlying matroid.

recently, dimca, ibadula, and macinic confirmed terao’s conjecture for arrangements in \( \mathbb{C}^3 \) with up to 13 hyperplanes [DIM19]. in joint work with behrends, jefferson, and leuner, we confirmed terao’s conjecture for rank 3 arrangements with exactly 14 hyperplanes in arbitrary characteristic [BBJ+21].

the main application of the tools we develop in this paper is a common generalization of the two aforementioned results:

theorem 1.2. terao’s freeness conjecture is true for rank 3 arrangements with up to 14 hyperplanes in any characteristic.

in section 3 we use our database developed in [BBJ+21] to show that there are 9 matroids left to consider for the proof of theorem 1.2, which we defer to section 7. in appendix d we list the database keys of these 9 exceptional cases.

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an extended abstract of this paper appeared in the oberwolfach workshop report 5/2021 and in the computeralgebra rundbrief ausgabe 68.
We now describe the main tools we develop in this paper. Let \( \mathcal{R}(M) \) denote the space of all matrix representations of a matroid \( M \) (over an arbitrary field) (cf. Section 2.1). Since \( \mathcal{R}(M) \) is too big to compute with, we consider closed subsets \( \Sigma \subseteq \mathcal{R}(M) \) which
- have small dimension compared to \( \mathcal{R}(M) \),
- but still contain a representative of each equivalence class of arrangements representing \( M \).

In Section 2.3 we call any such \( \Sigma \) a \textbf{representation slice} of the matroid \( M \). In Section 2.4 we show how to compute a representation slice \( \Sigma \).

For rank 3 simple matroids, we then consider in Section 2.5 the so-called \textbf{nonfree locus} \( \text{NFL}_\Sigma(M) \) of a representation slice \( \Sigma \), which corresponds to the subset of nonfree arrangements in \( \Sigma \). We prove in Theorem 7.2 that \( \text{NFL}_\Sigma(M) \) is the vanishing locus of a Fitting ideal of a specific matrix, which relies on the embedding of \( \Sigma \subseteq \mathcal{R}(M) \). We construct this matrix in several steps. First we describe in Corollary 5.2 the graded module of logarithmic derivations of a multiarrangement as the kernel of a morphism between free graded modules. The specific multiarrangement we are interested in is the so-called Ziegler restriction which we briefly recall in Section 4. There we also recall Yoshinaga’s criterion, which states that the nonfree locus can be detected in a specific degree of the above mentioned morphism of free graded modules. We therefore describe in Appendix B how to compute homogeneous parts of morphisms of free graded modules.

The feasibility of the computation of this Fitting ideal relies on two techniques:
- An embedding of \( \Sigma \) in a smaller affine space as a quasi-affine set which we describe in Appendix A.
- In Section 6 we recall that the Fitting ideal of a matrix is an invariant of its cokernel module. Moreover in Appendix C we describe several heuristics how to find a smaller presentation matrix for this module.

We end the paper with further examples and a computationally motivated conjecture in Section 8.

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2. Matroids and their representation spaces

This section is an elaboration of [BBJ+21, Section 4].

**Definition 2.1.** A matroid \( M \) is a pair \((E, B)\), where \( E \) is a finite ground set and \( B \) is a nonempty set of subsets of \( E \), called bases, such that for any two bases \( B, B' \in B \) with \( i \in B \setminus B' \) there exists \( j \in B' \) with \( B \setminus \{i\} \cup \{j\} \in B \).

The size of \( M \) is the size of the ground set \( E \) and the common size of all bases is called the rank of \( M \). The matroid \( M \) is called simple if each pair of elements of \( E \) is contained in at least one basis.

Let \( M = (\{1, \ldots, n\}, B) \) be a simple rank \( r \) matroid. A \textbf{representation} of \( M \) over the field \( k \) is a matrix \( P \in k^{r \times n} \) such that
\[
\det P_B \neq 0 \iff B \in B,
\]
where \( P_B \) is the \( r \times r \)-submatrix consisting of the columns index by \( B \). The kernels of the linear forms given by the columns of \( A \) define an arrangement \( \mathcal{A} \) of \( n \) hyperplanes in \( k^r \) with an intersection lattice isomorphic to the lattice of flats of the matroid \( M \).
2.1. The space of matrix representations of a matroid. Condition (1) defines an ideal \( I' \) in the ring \( A' = A[d] \), where
\[
A = \mathbb{Z}[p_{ij} \mid i = 1, \ldots, r, j = 1, \ldots, n]
\]
given by
\[
I' = \langle \det(P_N) \mid N \subseteq E \text{ not a basis}, \vert N \vert = r \rangle + \left( 1 - d \prod_{B \in \mathcal{B}(M)} \det(P_B) \right) \subseteq A[d],
\]
where \( P = (p_{ij}) \in A^{r \times n} \).

The (possibly empty) space of matrix representations\(^1\) of a matroid \( M \) is an affine variety, namely the vanishing locus
\[
\mathcal{R}(M) := V(I') \subseteq A^{rn+1} = \text{Spec } A[d] \twoheadrightarrow \text{Spec } \mathbb{Z}.
\]
The matroid \( M \) is representable (over some field \( k \)) if and only if \( 1/\in \mathcal{R}(M) \), which is equivalent to the reduced Gröbner basis (over \( \mathbb{Z} \)) of \( I' \) being equal to \( \{1\} \). This is basically the algorithm suggested in [Oxl11].

2.2. A quasi-affine embedding of the space of matrix representations. However, it is computationally more efficient to represent \( \mathcal{R}(M) \) as a locally closed set
\[
\mathcal{R}(M) \cong V(\tilde{I}) \setminus V(\tilde{J}) \subseteq A^{rn} = \text{Spec } A \twoheadrightarrow \text{Spec } \mathbb{Z},
\]
where
\[
\tilde{I} = \sum \langle \det P_N \mid N \subseteq E \text{ not a basis}, \vert N \vert = r \rangle,
\]
\[
\tilde{J} = \prod \langle \det P_B \mid B \in \mathcal{B}(M) \rangle.
\]
In particular, \( \tilde{J} \) is a principal ideal\(^2\). It follows that
\[
\mathcal{R}(M) \cong V(\tilde{I}) \setminus V(\tilde{J})
\]
and \( M \) is representable (over some field \( k \)) if and only if \( \det(P_B) \notin \sqrt{\tilde{I}} \) for all \( B \in \mathcal{B}(M) \).

It is well-known that the radical ideal membership problem can be replaced by a saturated ideal membership problem, which we will utilize towards the end of this section.

2.3. A representation slices of a matroid. The algebraic group \( GL_r \times (GL_1 \wr S_n) \) acts on \( A^{rn} \) by
\[
(g, ((\lambda_1, \ldots, \lambda_n), \pi)) \cdot P = gP h^{-1},
\]
where \( g \in GL_r \) and \( h = (h_{ij}) = (\delta_{i,\pi(j)}) \lambda_j \) is the monomial matrix of
\[
((\lambda_1, \ldots, \lambda_n), \pi) \in GL_1 \wr S_n := (GL_1 \times \cdots \times GL_1) \rtimes S_n.
\]
This action induces an action on the invariant subscheme \( \mathcal{R}(M) \), the orbits of which are the equivalence classes of matrix representations of the matroid \( M \). The moduli space of representations of \( M \) is thus the global quotient stack
\[
\mathcal{M}(M) := \mathcal{R}(M) / (GL_r \times (GL_1 \wr S_n)).
\]
Instead of constructing the moduli space \( \mathcal{M}(M) \) it suffices to consider a closed subset
\[
\Sigma \subseteq \mathcal{R}(M)
\]
\(^1\)over some unspecified field \( k \)
\(^2\)One can recover the original affine description of \( \mathcal{R}(M) \) from the latter quasi-affine description by passing to the Rabinowitsch cover [BLH22, Example 6.1]
which intersects each orbit of the action of \( \text{GL}_r \times (\text{GL}_1 \wr S_n) \) on \( \mathcal{R}(M) \), i.e., where the
projection \( \pi : \mathcal{R}(M) \to \mathcal{M}(M) \) still restricts to a projection \( \pi|_\Sigma : \Sigma \to \mathcal{M}(M) \). We call any such \( \Sigma \) a representation slice\(^3\) of the matroid \( M \).

2.4. Computing a representation slice. Representation slices that can be embedded in affine spaces of smaller dimension are computationally favorable. In the ideal case the dimension of \( \Sigma \) should be equal to the dimension of the moduli space \( \mathcal{M}(M) \). This happens when \( \Sigma \) intersects each orbit in finitely many points.

We construct such a slice by fixing certain values of the matrix \( P \) to 0 or 1 as described in [Oxl11, p. 184]: Firstly, we choose a basis \( B \in \mathcal{B}(M) \) and fix the corresponding submatrix \( P_B \) to be the unit matrix. Without loss of generality we can assume \( B = \{1, \ldots, r\} \).

Secondly, we consider the fundamental circuits with respect to this basis \( B \), i.e., for each \( i \in E \setminus B \) let \( C(i, B) \) be the unique circuit of the matroid \( M \) contained in \( B \cup \{i\} \). The entries of \( P \) in the column \( i \in E \setminus B \) which do not appear in \( C(i, B) \) can be fixed to 0. Lastly, the first nonzero entry in every column and the first nonzero entry in every row of \( P \) can be taken as 1 by column and row scaling respectively. We have added this algorithm to \texttt{alcove} [Leu19]. This amounts to adding elements of the form \( p_{ij} \) or \( p_{ij} - 1 \) to the ideal \( \widetilde{I} \) defined in \( (\widetilde{I}) \) yielding the larger ideal

\[
(I^\Sigma) \quad I^\Sigma \subseteq R
\]

with

\[
\Sigma = V(I^\Sigma) \setminus V(\widetilde{J}) = V(I^\Sigma) \setminus V(I^\Sigma + \widetilde{J}),
\]

where \( \widetilde{J} \) is the ideal defined in \( (\widetilde{J}) \).

The ideal \( I^\Sigma \) can be replaced by the saturation

\[
(I) \quad I := I^\Sigma : \widetilde{J}^\infty = I^\Sigma : \det(P_{B_1})^\infty : \cdots : \det(P_{B_k})^\infty,
\]

when \( \mathcal{B}(M) = \{B_1, \ldots, B_k\} \). Likewise, the ideal \( \widetilde{J} \) can be replaced by the ideal

\[
(J) \quad J := \langle \text{NF}_{\mathcal{G}B(I)}(\det P_B) \mid B \in \mathcal{B}(M) \rangle,
\]

where \( \text{NF}_{\mathcal{G}B(I)}(f) \) is the normal form of the polynomial \( f \in R \) with respect to the Gröbner basis \( \mathcal{G}B(I) \) of the ideal \( I \). It follows that

\[
(\Sigma) \quad \Sigma = V(I) \setminus V(J).
\]

and \( M \) is representable (over some field \( k \)) if and only if \( 1 \notin I \), which is again equivalent to the reduced Gröbner basis (over \( Z \)) of \( I \) being equal to \( \{1\} \). For the Gröbner basis computations over \( Z \) we used \textsc{Singular} [DGPS19] from within the GAP package \textsc{ZariskiFrames} [BKLH19], which is part of the \textsc{Cap/homalg} project [hom20, BLH11, GPS18].

The final step is to embed the quasi-affine \( \Sigma \subseteq \mathcal{R}(M) \subseteq \mathbb{A}^m \) into an affine space of smaller dimension, which we explain in Appendix \( A \).

2.5. The nonfree locus of a representation slice. Each point \( p \in \mathcal{R}(M) \cong \text{Spec} \, A[d]/I' \) corresponds to a rank \( r \) arrangement \( A_{(p)} \) over the residue class field \( \kappa(p) := \text{Frac}(A[d]/p) \) and with an intersection lattice of hyperplanes isomorphic to the lattice of flats of the matroid \( M \). This allows us to define:

**Definition 2.2.** The nonfree locus of a closed subset \( \Sigma \subseteq \mathcal{R}(M) \) is the subset

\[
\text{NFL}_\Sigma(M) := \{ p \in \Sigma \mid D(A_{(p)}) \text{ is nonfree} \} \subseteq \Sigma.
\]

\(^3\)We do not call it a representation section since we do not require it to intersect each orbit in exactly one point. However, in many instances the representation slice we found is indeed a representation section.
Using any representation slice \( \Sigma \subseteq \mathcal{R}(M) \) we define the **nonfree locus of the moduli space** \( \mathcal{M}(M) \) as the image

\[
\text{NFL}(M) := \pi_{\Sigma}(\text{NFL}_{\Sigma}(M)).
\]

For the purpose of this paper it suffices to work with \( \text{NFL}_{\Sigma}(M) \) since \( \pi_{\Sigma} \), which we do not need to construct, is surjective onto \( \text{NFL}(M) \).

We will show in our main Theorem 7.2 how to compute \( \text{NFL}_{\Sigma}(M) \subseteq \Sigma \) for rank 3 matroids. In particular, we will see that the nonfree locus \( \text{NFL}_{\Sigma}(M) \) is a **closed** subset of \( \Sigma \) in this case.

### 3. The 9 Remaining Matroids

In [BBJ+21], we generated all 815 107 simple rank 3 matroids with up to 14 elements with integrally splitting characteristic polynomial and stored them in a public database [BK19]. To investigate Terao’s freeness conjecture it suffices to consider the matroids that are

- representable over some field,
- not essentially\(^4\) uniquely representable,
- not divisionally free, and
- are not unbalanced.

See [BBJ+21] for the detailed definitions of these properties. Somewhat surprisingly, it turns out that there are only 9 rank 3 integrally splitting matroids of size up to 14 satisfying all of these conditions. There is one matroid of size 9, one of size 11, two of size 12, and five of size 13. This already verifies Terao’s freeness conjecture for rank 3 arrangements with precisely 14 hyperplanes.

To deduce the more general Theorem 1.2, we will subsequently investigate the freeness of these 9 exceptional matroids in detail. We start by first describing the matroids in the rest of this section.

Additional details of these nine matroids are shown in Table 1. For further inspection of the properties of these matroids the reader is invited to retrieve them from our public database [BK19] using the keys shown Appendix D.

#### 3.1. The matroid \( M_9 \)

The matroid \( M_9 \) has size 9 and is known as the **affine geometry** \( AG(2, 3) \) (see for instance [Oxl11, Example 6.2.2]). A representation of \( M_9 \) over \( \mathbb{Q}(\zeta_3) \) is the reflection arrangement of the complex reflection group \( G(3, 3, 3) \). As such, it has the defining equation

\[
(x^3 - y^3)(x^3 - z^3)(y^3 - z^3).
\]

Based on this matroid, Ziegler already demonstrated that it is crucial to formulate Terao’s freeness conjecture for a fixed field: An arrangement \( \mathcal{A} \) over a field \( k \) with underlying matroid \( M_9 \) is free if and only if \( \text{char } k \neq 3 \) [Zie90].

The matroid \( M_9 \) has the representation slice

\[
\Sigma_{M_9} = V(a^2 - a + 1) \subseteq \text{Spec } \mathbb{Z}[a]
\]

which parametrizes all representations of \( M_9 \) given by the matrix:

\[
\begin{pmatrix}
1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & -a + 1 & -a + 1 & 1 \\
0 & 0 & 0 & 1 & 1 & -a + 1 & a & 1 & a
\end{pmatrix}
\]

#### 3.2. The matroid \( M_{11} \)

The matroid \( M_{11} \) is of size 11. It has a representation over \( \mathbb{Q}(\sqrt{5}) \) whose projectivization consists of the sides of a regular pentagon together with its five diagonals and the line at infinity. We call this arrangement the **pentagon arrangement.** It is

\(^4\)either uniquely representable or uniquely representable in a single characteristic up to Galois isomorphism.
depicted in Figure 1. It is known that the pentagon arrangement in characteristic zero is free but not inductively free [OT92, Example 4.59]. We will prove that it is free over a field \( k \) if and only if \( \text{char } k \neq 2 \).

The matroid \( M_{11} \) has the representation slice

\[
\Sigma_{M_{11}} = V(a^2 - a - 1) \subseteq \text{Spec } \mathbb{Z}[a]
\]

which parametrizes all representations of \( M_{11} \) given by the matrix:

\[
\begin{pmatrix}
  1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
  0 & 1 & 1 & a+1 & 0 & 0 & 0 & 1 & 1 & -a & -a \\
  0 & 0 & 0 & 0 & 1 & 1 & a & -1 & -a+1 & a+1 & a
\end{pmatrix}.
\]

Thus over characteristic 0, a point in \( \Sigma_{M_{11}} \) involves the golden ratio. Projecting \( \Sigma_{M_{11}} \) to \( \text{Spec } \mathbb{Z} \) we found that \( M_{11} \) admits a representation in all characteristics. In characteristic 5 for instance, the equation \( a^2 - a - 1 \) factors to \((a + 2)^2\) which means that \( M_{11} \) admits a representation over the prime field \( \mathbb{F}_5 \) in this characteristic.

3.3. The matroid \( M_{12}^1 \). The matroid \( M_{12}^1 \) is a matroid of size 12. Over \( \mathbb{C} \) is has a representation that is the reflection arrangement over the complex reflection group \( G(4, 4, 3) \) and is given by the following equation:

\[
(x^4 - y^4)(x^4 - z^4)(y^4 - z^4).
\]

As all reflection arrangements, this arrangement is free over \( \mathbb{C} \).

The matroid \( M_{12}^1 \) has the representation slice

\[
\Sigma_{M_{12}^1} = V(a^2 - 2a + 2) \setminus V(2) \setminus V(a - 1) \subseteq \text{Spec } \mathbb{Z}[a]
\]

which parametrizes all representations of \( M_{12}^1 \) given by the matrix:

\[
\begin{pmatrix}
  1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
  0 & 1 & 1 & -1 & 0 & -a+1 & -a+1 & -a+1 & 0 & 1 & 1 & -1 \\
  0 & 0 & 0 & 0 & 1 & 1 & a & -a+2 & 1 & -a+1 & -a+2 & a
\end{pmatrix}.
\]

3.4. The matroid \( M_{12}^2 \). The matroid \( M_{12}^2 \) is a matroid of size 12. It is only representable over field extensions of \( \mathbb{F}_4 \) as it has the representation slice

\[
\Sigma_{M_{12}^2} = V(2) \setminus V(a) \setminus V(a + 1) \subseteq \text{Spec } \mathbb{Z}[a]
\]
which parametrizes all representations of $M^2_{13}$ given by the matrix:
\[
\begin{pmatrix}
1 & 0 & 1 & 1 & 0 & 1 & 1 & a & +1 & 1 & 0 & a & +1 & 1 \\
0 & 1 & 1 & a^2 & +1 & 0 & a & +1 & a & +1 & a^2 & +1 & 0 & a & +1 & a & +1 & a^2 & +1
\end{pmatrix}.
\]

3.5. The matroid $M^1_{13}$. The matroid $M^1_{13}$ is of size 13. Its representations over $\mathbb{C}$ were first studied by Abe, Cuntz, Kawanoue, and Nozawa who proved that it is the smallest free arrangement in rank 3 over $\mathbb{C}$ that is free but not recursively free [ACKN16]. It also has a representation over $\mathbb{R}$ which is depicted in [ACKN16, Figure 1].

The matroid $M^1_{13}$ has the 1-dimensional representation slice
\[
\Sigma_{M^1_{13}} = V(a_1a_2^2 - 2a_1a_2 + a_1 - a_2) \setminus V(6a_2^3 + 4a_1^2 - 10a_1a_2 - a_2^2 - a_1 - 9a_2) \subseteq \text{Spec } \mathbb{Z}[a_1, a_2]
\]
which parametrizes all representations of $M^1_{13}$ given by the matrix:
\[
\begin{pmatrix}
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & -a^2 & + a_2 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -a & +1 & 1 & -a & +1 & a
\end{pmatrix}.
\]

3.6. The remaining four matroids. There are four more matroids of size 13 which we denote by $M^2_{13}, \ldots, M^5_{13}$. The prominence of these examples is not known to us.

$M^2_{13}$: The matroid $M^2_{13}$ has the representation slice
\[
\Sigma_{M^2_{13}} = V(a^2 + 1) \setminus V(2) \subseteq \text{Spec } \mathbb{Z}[a]
\]
which parametrizes all representations of $M^2_{13}$ given by the matrix:
\[
\begin{pmatrix}
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & -a & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.
\]

$M^3_{13}$: The matroid $M^3_{13}$ has the representation slice
\[
\Sigma_{M^3_{13}} = V(a^2 - a - 1) \setminus V(2) \setminus V(a) \subseteq \text{Spec } \mathbb{Z}[a]
\]
which parametrizes all representations of $M^3_{13}$ given by the matrix:
\[
\begin{pmatrix}
1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & -a & +1 & -a & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.
\]

$M^4_{13}$: The matroid $M^4_{13}$ has the representation slice
\[
\Sigma_{M^4_{13}} = V(2a^2 - 2a + 1) \setminus V(2a - 3) \setminus V(3a - 1) \subseteq \text{Spec } \mathbb{Z}[a]
\]
which parametrizes all representations of $M^4_{13}$ given by the matrix:
\[
\begin{pmatrix}
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & -2a & +1 & 1 & -a & 0 & 0 & 0 & 1 & a & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}.
\]

$M^5_{13}$: The matroid $M^5_{13}$ has the representation slice
\[
\Sigma_{M^5_{13}} = V(2a^2 + 2a + 1) \setminus V(2a + 3) \setminus V(a + 2) \subseteq \text{Spec } \mathbb{Z}[a]
\]
which parametrizes all representations of $M^5_{13}$ given by the matrix:
\[
\begin{pmatrix}
1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & -2a & -1 & -2a & 0 & 0 & 0 & 0 & 1 & 1 & -2a & -1 & -2a & -1 & -2a & -2 & 1 & 1
\end{pmatrix}.
\]

4. Zieglers’s restriction and Yoshinaga’s criterion

A (central) arrangement of hyperplanes $\mathcal{A}$ is a finite collection of hyperplanes in a vector space $V$ of dimension $r$ over a field $k$ containing the origin.
A generalization is a multiarrangement, which is defined to be an arrangement of hyperplanes \( \mathcal{A} \) with a multiplicity function \( m : \mathcal{A} \to \mathbb{Z}_{\geq 0} \). A multiarrangement was first defined by Ziegler in [Zie89] and is denoted by \((\mathcal{A}, m)\). Define \(|m| := \sum_{H \in \mathcal{A}} m(H)\).

Denote by \( S \) the polynomial ring \( k[x_1, \ldots, x_r] \) where \( r = \dim V \). For each hyperplane \( H \) we can fix a linear defining equation \( \alpha_H \in S \). The \( S \)-module \( D(\mathcal{A}, m) \) is the module of logarithmic derivations of \((\mathcal{A}, m)\) defined as

\[
D(\mathcal{A}, m) := \left\{ \theta \in \text{Der}(S) \mid \theta(\alpha_H) = \alpha_H^m(H) S \text{ for all } H \in \mathcal{A} \right\},
\]

where \( \text{Der}(S) \cong S^e \) is the module of all derivations on \( S \). If \( D(\mathcal{A}, m) \) is a free \( S \)-module, we call \((\mathcal{A}, m)\) a free multiarrangement. In the case of a free multiarrangement \((\mathcal{A}, m)\) one can choose a homogeneous basis \( \theta_1, \ldots, \theta_r \) of \( D(\mathcal{A}, m) \). In this case we define \( \exp(\mathcal{A}, m) = (\deg \theta_1, \ldots, \deg \theta_r) \) to be the exponents of \((\mathcal{A}, m)\) where a derivation \( \theta \in \text{Der}(S) \) is homogeneous with \( \deg \theta = d \) if \( \theta(\alpha) \) is a homogeneous polynomial of degree \( d \) for all \( \alpha \in V^* \).

**Definition 4.1** (Ziegler restriction). Let \( \mathcal{A} \) be an arrangement and fix some \( H \in \mathcal{A} \). The **restricted arrangement** is defined as \( \mathcal{A}^H := \{H \cap L \mid L \in \mathcal{A} \setminus \{H\}\} \). A natural multiplicity function \( m^H \) on \( \mathcal{A}^H \) arises by counting how often a restricted hyperplane \( X \in \mathcal{A}^H \) appears as intersection of hyperplanes in \( \mathcal{A} \):

\[
m^H(X) := |\{L \in \mathcal{A} \setminus \{H\} \mid H \cap L = X\}|.
\]

We call the multiarrangement \( (\mathcal{A}^H, m^H) \) the **Ziegler restriction** of \( \mathcal{A} \) to the hyperplane \( H \).

**Example 4.2.** Consider again the arrangements realizing the matroid \( M_{11} \) parametrized by the matrix (4) in Section 3.2. The Ziegler restrictions onto the hyperplane \( \{z = 0\} \) are parametrized by the matrix

\[
\begin{pmatrix}
1 & 0 & 1 & 1 & -1 \\
0 & 1 & 1 & a + 1 & a
\end{pmatrix}
\]

with multiplicities \( (3, 3, 1, 1, 2) \) over the representation slice \( \Sigma_{M_{11}} = V(a^2 - a - 1) \subseteq \text{Spec} \mathbb{Z}[a] \).

Note that a multiarrangement \((\mathcal{A}, m)\) of rank 2 is always free for some exponents \((d_1, d_2)\) with \( d_1 + d_2 = |m| \) [Zie89].

Our first technical tool to compute the free locus within a representation slice of a matroid is the following remarkable theorem by Yoshinaga.

**Theorem 4.3.** [Yos05, Theorem 3.2] Let \( \mathcal{A} \) be an arrangement of rank 3 and assume the characteristic polynomial of \( \mathcal{A} \) factors as \( \chi_{\mathcal{A}}(t) = (t - 1)(t - d_2)(t - d_3) \) for some integers \( d_2 \leq d_3 \). Let \( H \) be any hyperplane in \( \mathcal{A} \) and assume that the Ziegler restriction \( (\mathcal{A}^H, m^H) \) is free with exponents \( \exp(\mathcal{A}, m) = (d_1', d_2') \) and \( d_1' \leq d_2' \). Then it holds that

\[
d_2d_3 \geq d_1'd_2'
\]

and \( \mathcal{A} \) is free if and only if (5) holds with equality.

**Remark 4.4.** Originally, Theorem 4.3 was formulated only for fields of characteristic 0 in [Yos05]. This assumption is however not essential, as it was dropped in the subsequent paper by Abe and Yoshinaga which generalized Theorem 4.3 to higher dimension [AY13].

**Corollary 4.5.** In the notation of Theorem 4.3 the arrangement \( \mathcal{A} \) is free if and only if

\[
D(\mathcal{A}^H, m^H)_{d_2 - 1} = 0.
\]

**Proof.** Since \( \mathcal{A} \) is an arrangement of rank 3, the Ziegler restriction \( (\mathcal{A}^H, m^H) \) is free with exponents \( \exp(\mathcal{A}, m) = (d_1', d_2') \) and \( d_1' \leq d_2' \). By definition of the Ziegler restriction we have \( d_2 + d_3 = |\mathcal{A}| - 1 = d_1' + d_2' \).
Theorem 4.3 therefore implies $d_2 \geq d'_1$ and $A$ is free if and only if $d_2 = d'_1$. Thus, $A$ is free if and only if the lowest degree derivation in $D(A^H, m^H)$ has degree $d_2$.

5. Freeness of Arrangements and Multiarrangements over a Field

The algorithm we use to compute $D(A, m)$ is a direct translation into the language of Gröbner bases of the following Proposition (cf. [BC12, Sec. 6.1]).

**Proposition 5.1.** $D(A, m)$ is the kernel of the coproduct morphism

$$d^{(A, m)} := (\pi_{\alpha^m_H} \circ d\alpha^H)_{H \in A} : \text{Der}(S) \to \bigoplus_{H \in A} S/\alpha^m_H S,$$

where $df : \text{Der}(S) \to S, \theta \mapsto \theta(f) = \sum_{i=1}^{r} \theta_i \frac{\partial f}{\partial x_i}$ is the exterior derivative of $f \in S$ and $\pi_f : S \to S/fS$ is the canonical projection.

**Proof.** The kernel of the universal morphism coincides with the intersection of the kernels of the individual maps $\pi_{\alpha^m_H} \circ d\alpha^H : \text{Der}(S) \to S/\alpha^m_H S$. The latter was the definition of $D(A, m)$.

**Corollary 5.2.** $D(A, m)$ is isomorphic to the kernel of a morphism $\psi^{(A, m)}$ of free graded modules of finite rank. More precisely, $D(A, m)$ is isomorphic to the pullback $K$ in the pullback diagram

$$K \to \bigoplus_{H \in A} S(-m(H)) \to \bigoplus_{H \in A} S$$

where

$$j : \text{Der}(S) \to \bigoplus_{H \in A} S$$

is given by the Jacobi matrix and

$$\mu : \bigoplus_{H \in A} S(-m(H)) \to \bigoplus_{H \in A} S$$

is an embedding. The pullback $K$ (which is isomorphic to $D(A, m)$) can in turn be computed as the kernel of the coproduct morphism

$$\psi^{(A, m)} := \left(\frac{j}{\mu}\right) : \text{Der}(S) \oplus \bigoplus_{H \in A} S(-m(H)) \to \bigoplus_{H \in A} S.$$
Example 5.3. We again consider the matroid $M_{11}$. The morphism $\psi^{(A,m)}$ of the Ziegler restriction described in Example 4.2 over the representation slice $\Sigma_{M_{11}} = V(a^2 - a - 1)$ is given by the matrix

$$
\begin{pmatrix}
1 & 0 & 1 & 1 & -1 \\
0 & 1 & 1 & a + 1 & a \\
x^3 & 0 & 0 & 0 & 0 \\
y^3 & 0 & x + y & 0 & 0 \\
0 & 0 & 0 & x + (a + 1)y & 0 \\
0 & 0 & 0 & 0 & (x - ay)^2
\end{pmatrix},
$$

as a map $S^2 \oplus S(-3)^2 \oplus S(-1)^2 \oplus S(-2) \to S^5$ where $S = \mathbb{Z}[a]/(a^2 - a - 1)[x,y]$.

Corollary 5.2 is a special case of the following result valid in any Abelian category:

Proposition 5.4. Let $\iota$ be the (necessarily monic) pullback of a monic $\mu : T \hookrightarrow D$ along a morphism $\iota : S \to D$ in an Abelian category. Further consider the cokernel projection $\varepsilon : D \to C$ and the kernel embedding $\kappa$ of $\varepsilon \circ \jmath$.

Then $\kappa : K' \hookrightarrow S$ and $\iota : K \hookrightarrow S$ define the same subobject in $S$, i.e., these two monos are mutually dominating.

Proof. We need to construct a unique lift of $\iota$ along $\kappa$ and vice versa, necessarily unique since both morphisms are monos. First note that $\iota$ is monic since pullbacks of a monos are monos in any category. The morphism $\iota$ lifts along the kernel embedding $\kappa = \ker(\varepsilon \circ \jmath)$ since $(\varepsilon \circ \jmath) \circ \iota = \varepsilon \circ \mu \circ \lambda = 0$. Conversely, the unique lift of $\kappa$ along $\iota$ can constructed as follows: First construct the unique kernel lift $\chi : K' \to T$ of $\jmath \circ \kappa : K' \to D$ along the kernel embedding $\mu : T \hookrightarrow D$. The desired lift $K' \to K$ of $\kappa$ along $\iota$ is now the universal morphism of the pullback. \hfill \Box

In the notebook [BK21] we demonstrate a completely mechanical proof of a generalization of Proposition 5.4 following Posur’s impressive paper [Pos22].

Proof of Corollary 5.2. $D(A,m)$ is by Proposition 5.1 the kernel of the composition $\varepsilon \circ \jmath$, where $\varepsilon$ is the cokernel projection of $\mu$. The first isomorphism in

$$
D(A,m) \cong K \cong \ker \psi^{(A,m)}
$$

is now nothing but Proposition 5.4 and the second is the well-known fact stated in Corollary 5.2. \hfill \Box

Finally, one computes the kernel of the morphism $\psi^{(A,m)}$ in the category of free graded modules by computing syzygies.

\[^{5}\text{deg } a = 0 \text{ and } \deg x = \deg y = 1.\]
6. FITTING IDEALS

**Definition 6.1.** Let \( A \) be a commutative nonzero unital ring and \( \varphi : U \to W \) a morphism of free \( A \)-modules of finite rank. After choosing sets of free generators for \( U \) and \( W \) one can identify \( \varphi \) with a matrix in \( A^{\text{rk}_A U \times \text{rk}_A W} \). For \( \ell \in \mathbb{Z} \) set \( m := \text{rk}_A W - \ell \) and define the \( \ell \)-th Fitting ideal

\[
F\text{itt}_\ell \varphi := \begin{cases} A, & m \leq 0, \\ \text{the ideal generated by all } m \times m \text{ minors of } \varphi, & 0 < m \leq \min\{\text{rk}_A U, \text{rk}_A W\}, \\ \{0\}, & m > \min\{\text{rk}_A U, \text{rk}_A W\}, \end{cases}
\]

or, equivalently, for index \( \varphi := \text{rk}_A W - \text{rk}_A U \)

\[
F\text{itt}_\ell \varphi := \begin{cases} A, & \ell \geq \text{rk}_A W, \\ \text{the ideal generated by all } m \times m \text{ minors of } \varphi, & \max\{\text{index } \varphi, 0\} \leq \ell < \text{rk}_A W, \\ \{0\}, & \ell < \max\{\text{index } \varphi, 0\}. \end{cases}
\]

**Theorem 6.2 (Fitting’s Lemma, [Eis95, Cor.-Def. 20.4]).** The \( \ell \)-th Fitting ideal of \( \varphi \) only depends on the isomorphism type of \( \text{coker } \varphi \). In particular, it does not depend on the choice of free bases in Definition 6.1.

**Remark 6.3.** In particular, in order to compute \( \text{Fitt}_\ell \varphi \) one might pass from the matrix \( \varphi \) over \( A \) (interpreted as a free presentation of \( \text{coker } \varphi \)) to another matrix \( \varphi' \), preferably of smaller dimensions such that either

(a) \( \text{coker } \varphi' \cong \text{coker } \varphi \) and hence, by Theorem 6.2

\[
\text{Fitt}_\ell \varphi = \text{Fitt}_\ell \varphi'
\]

(b) or \( \text{coker } \varphi \cong \text{coker } \varphi' \oplus A^{\oplus \mathbb{Z}} \) and hence, by Definition 6.1

\[
\text{Fitt}_\ell \varphi = \text{Fitt}_{\ell-\mathbb{Z}} \varphi'.
\]

These are the two major tricks which allow us to compute Fitting ideals. In Appendix C we describe several heuristics for finding smaller presentations of finitely presented modules over computable rings.

**Remark 6.4.** Let \( \varphi : U \to W \) be a morphism of finite dimensional vector spaces over some field \( k \). Then the following statements are equivalent for \( c \in \mathbb{Z} \):

(a) \( \dim_k \ker \varphi \leq c \);
(b) \( m := \dim_k U - c \leq \text{rank}_k \varphi \);
(c) There exists a nonzero \( m \times m \) minor of \( \varphi \);
(d) \( \text{Fitt}_\ell \varphi \neq 0 \) for

\[
\ell := \dim_k W - m = c + \text{index } \varphi,
\]

where \( \text{index } \varphi := \dim_k W - \dim_k U \).

**Proof.** This follows from the dimension formula \( \dim_k \ker \varphi + \text{rank}_k \varphi = \dim_k U \) and the definition of the Fitting ideals.

This remark implies:

**Corollary 6.5.** Let \( A \) be a commutative nonzero unital ring, \( \varphi : U \to W \) a morphism of free \( A \)-modules of finite rank. Denote by index \( \varphi := \text{rk}_A W - \text{rk}_A U \) and for \( c \in \mathbb{Z} \) define \( \ell := c + \text{index } \varphi \geq 0 \). Then

\[
V(\text{Fitt}_\ell \varphi) = \{ p \in \text{Spec } A \mid \dim_{\kappa(p)} \ker \varphi_{\kappa(p)} > c \},
\]

where \( \varphi_{\kappa(p)} : U_{\kappa(p)} \to W_{\kappa(p)} \) is the specialization of \( \varphi \) at \( p \) and \( \kappa(p) := \text{Frac}(A/p) \) is the residue class field of \( p \).
7. Proof of Theorem 1.2

We described in Section 2.4 how to compute an affine or quasi-affine representation slice.

**Definition 7.1.** Let $\Sigma = \{A_p \mid p \in \text{Spec } A\} \equiv \text{Spec } A$ be an affine representation slice of arrangements representing a rank 3 matroid with integrally splitting characteristic polynomial $\chi(t) = (t-1)(t-d_2)(t-d_3)$ for some roots $d_2, d_3 \in \mathbb{Z}_{>0}$ with $d_2 \leq d_3$. We view the family $\{A_p \mid p \in \text{Spec } A\}$ as an arrangement $A$ over the ring $A$. For a fixed $H \in A$ define the degree $d_2-1$ part of the morphism $\psi_{d_2-1}(A^H, m^H)$ (defined in Corollary 5.2) as the morphism

$$\varphi^M := \psi_{d_2-1}(A^H, m^H) : U \to W$$

of free $A$-modules

$$U := \left(\text{Der}(S) \oplus \bigoplus_{X \in A^H} S(-m^H(X))\right)_{d_2-1}, \quad W := \left(\bigoplus_{X \in A^H} S\right)_{d_2-1}.$$

Finally we define

$$\text{index } \varphi^M := \text{rk}_A W - \text{rk}_A U.$$

We show in Appendix B how to compute homogeneous parts of morphisms of free graded modules of finite rank.

**Theorem 7.2.** In the notation of Definition 7.1 the following are equivalent for $p \in \text{Spec } A$:

(a) $A_p$ is not free.
(b) $D(A_p, m_p)_{d_2-1}$ does not vanish.
(c) $p$ is in the locus $V(\text{Fitt}_\ell \varphi^M)$ for $\ell = \text{index } \varphi^M$.

In particular

$$\text{NFL}_\Sigma(M) = V(\text{Fitt}_\ell \varphi^M).$$

**Proof.**

(a)$\Leftrightarrow$(b): This the statement of Corollary 4.5.
(b)$\Leftrightarrow$(c): This the combined statement of Corollary 5.2 and Corollary 6.5 for $c = 0$. \qed

**Corollary 7.3.** Let $M$ be a rank 3 simple matroid with a representation slice $\Sigma$. Then $\text{NFL}_\Sigma(M)$ is a closed subvariety of $\Sigma$. In particular, freeness is an open condition.

The last statement, restricted to the various fibers of $\Sigma \to \text{Spec } \mathbb{Z}$, implies Yuzvinsky’s openness result [Yuz93] for the case of rank 3 arrangements.

Before discussing the proof of Theorem 1.2 we continue the discussion of the $M_{11}$ matroid underlying the pentagon arrangement.

**Example 7.4.** Since the characteristic polynomial of $M_{11}$ is $\chi(t) = (t-1)(t-5)^2$ the morphism $\varphi^M$ is the degree 4 part of the morphism given by matrix (6). The morphism $\varphi^M$ is defined by a $25 \times 25$ matrix over $R = \mathbb{Z}[a]/(a^2 - a - 1)$.

As the index of this morphism is zero, the nonfree locus $\text{NFL}_{\Sigma_{M_{11}}}(M_{11})$ is the vanishing locus of the determinant of this square matrix within the representation slice $\Sigma_{M_{11}} = V(a^2 - a - 1)$ of $M_{11}$. Using the heuristics described in Appendix C we can replace this matrix by a $1 \times 1$ matrix $\varphi^M_{\text{red}} = (4) \in R^{1 \times 1}$ with $\text{coker}(\varphi^M_{\text{red}}) \cong \text{coker } \varphi^M$. Hence

$$\text{NFL}_{\Sigma_{M_{11}}}(M_{11}) = V(\text{Fitt}_0(\varphi^M)) = V(\text{Fitt}_0(\varphi^M_{\text{red}})) = V\left(\sqrt{\text{Fitt}_0(\varphi^M_{\text{red}})}\right)$$

$$= V(2, a^2 - a - 1) \subset V(a^2 - a - 1) = \Sigma_{M_{11}} \subset \text{Spec } \mathbb{Z}[a].$$
Therefore, a representation of $M_{11}$ is free if and only if the underlying field is not of characteristic 2.

There are in fact nonfree representations of $M_{11}$ over $\mathbb{F}_4$. A posteriori, this can also be theoretically explained through another freeness criterion of Yoshinaga which applies to arrangements over finite fields [Yos06]. Hence, the “pentagon” matroid $M_{11}$ is another example witnessing the dependence of freeness on the underlying field.

**Proof of Theorem 1.2.** As explained in Section 3, verifying Terao’s freeness conjecture for the 9 exceptional matroids $M_9, M_{11}, M_{12}, M_{12}^1, M_{13}^1, \ldots, M_{13}^5$ completes the proof.

Analogous to Example 7.4 we computed the free locus of each of these matroids using the techniques of Section 6 and report the results in Table 1. Over a fixed characteristic, the representations of a given matroid turn out to be either all free or all nonfree. Notice, however that the freeness depends on the characteristic for representations of $M_9$ and $M_{11}$; all representations are free except in characteristic 3 and 2, respectively. □

| $M$ | $|M|$ | roots | $|\text{Aut}_{\Sigma_M}|$ | the representation slice $\Sigma_M$ within $\text{Spec} \mathbb{Z}[a]$ | $\varphi^M$ | $\varphi^M_{\text{NFL}}$ | NFL$_{\Sigma_M}$$(M) \subseteq \Sigma_M$ |
|-----|------|-------|--------------------------|---------------------------------|--------|-------------|---------------------|
| $M_9$ | 9    | (4, 4) | 432 | $V(a^2 - a + 1) \cong \text{Spec} \mathbb{Z}[a] = \text{Spec} \mathbb{Z}[\omega]$, $\omega = \frac{1+\sqrt{5}}{2}$ | $1 \times 1$ | $V(3)$ | $\emptyset$ |
| $M_{11}$ | 11   | (5, 5) | 20 | $V(a^2 - a - 1) \cong \text{Spec} \mathbb{Z}[a]$, $\omega = \frac{1-\sqrt{5}}{2}$ | $25 \times 25$ | $1 \times 1$ | $V(2)$ |
| $M_{12}^1$ | 12   | (5, 6) | 192 | $V(a^2 - 2a + 2) \setminus V(2) \setminus V(a - 1)$ | $24 \times 25$ | $8 \times 4$ | $\emptyset$ |
| $M_{12}^2$ | 12   | (5, 6) | 576 | $V(2) \setminus V(a) \setminus V(a + 1)$ | $24 \times 25$ | $4 \times 4$ | $\Sigma_{M_{12}^2}$ |
| $M_{13}^1$ | 13   | (6, 6) | 18 | $(\ast)$ | $36 \times 36$ | $9 \times 2$ | $\emptyset$ |
| $M_{13}^2$ | 13   | (6, 6) | 16 | $V(a^2 + 1) \setminus V(2)$ | $36 \times 36$ | $8 \times 5$ | $\Sigma_{M_{13}^2}$ |
| $M_{13}^3$ | 13   | (6, 6) | 8 | $V(a^2 - a - 1) \setminus V(2) \setminus V(a)$ | $36 \times 36$ | $2 \times 2$ | $\Sigma_{M_{13}^3}$ |
| $M_{13}^4$ | 13   | (6, 6) | 2 | $V(2a^2 - 2a + 1) \setminus V(2a - 3) \setminus V(3a - 1)$ | $36 \times 36$ | $2 \times 1$ | $\emptyset$ |
| $M_{13}^5$ | 13   | (6, 6) | 2 | $V(2a^2 + 2a + 1) \setminus V(2a + 3) \setminus V(a + 2)$ | $36 \times 36$ | $2 \times 1$ | $\emptyset$ |

Table 1. The 9 exceptional matroids together with the two nontrivial roots of its characteristic polynomial. We also describe the size of its automorphism group, the representation slice $\Sigma_M$, and the nonfree locus NFL$_{\Sigma_M}$$(M) \subseteq \Sigma_M$.

The representation slice $(\ast)$ of $M_{13}^1$ is

$$\Sigma_{M_{13}^1} = V(a_1a_2^2 - 2a_1a_2 + a_1 - a_2) \setminus V(6a_2^3 + 4a_1^2 - 10a_1a_2 - a_2^3 - a_1 - 9a_2) \subseteq \text{Spec} \mathbb{Z}[a_1, a_2].$$

### 8. Further Examples with More than 14 Lines

The two exceptional matroids $M_9$ and $M_{12}^1$ underlie the arrangements corresponding to the complex reflection groups $G(3, 3, 3)$ and $G(4, 4, 3)$, respectively. It is known that the reflection arrangements over characteristic 0 stemming from $G(n, n, 3)$ are free for all $n \geq 1$. We determined the obstruction variety of their underlying matroids for $n < 10$ and found the following: If $n$ is a prime number, a representation is free if and only if the characteristic of the field is not $n$. If $n$ is not a prime number, the obstruction variety was empty in all considered cases. Therefore, we pose the following conjecture:

**Conjecture 8.1.** An arrangement $A$ over a field $k$ which is combinatorially equivalent to the reflection arrangement $G(n, n, 3)$ for some $n \geq 3$ is free if and only if $n$ is not a prime number or if the characteristic of $k$ is not $n$ in the case that $n$ is a prime number.
APPENDIX A. EMBEDDING OF AFFINE SETS IN SMALLER AFFINE SPACES

The goal of this section is to list some simple heuristics we use to embed the representation slice \( \Sigma \subseteq \mathcal{R}(M) \) into a smaller affine space. Since the representation space \( \mathcal{R}(M) \) has an affine representation given by \( \mathcal{R}(M) \subseteq \mathbb{A}^{rn+1} \) in (2) and a quasi-affine representation \( \mathcal{R}(M) \subseteq \mathbb{A}^{rn} \) given by (3), the same holds true for the closed subset \( \Sigma \subseteq \mathcal{R}(M) \). Recall that in (\( \Sigma \)) we have described

\[
\Sigma = V(I) \setminus V(J),
\]

with \( I, J \leq A \).

What we describe now works for any ideal \( I \) in a polynomial ring \( A = \mathbb{Z}[x_1, \ldots, x_t] \).\(^6\) Our goal is to find a subset \( \{z_1, \ldots, z_u\} \subseteq \{x_1, \ldots, x_t\} \) and an ideal \( I_\infty \leq A_\infty = \mathbb{Z}[z_1, \ldots, z_u] \), such that the natural morphism \( A_\infty \to A/I \) yields an isomorphism \( A_\infty/I_\infty \cong A/I \), i.e., to replace the natural projection \( \pi : A \to A/I \) by an equivalent surjective morphism \( \pi_\infty : A \to A_\infty/I_\infty \).

\[
\begin{array}{ccc}
A & \xrightarrow{\pi} & A/I \\
\downarrow{\pi_\infty} & & \downarrow{?} \\
A_\infty/I_\infty & & \\
\end{array}
\]

which describes an embedding

\[
A^u \supseteq V(I_\infty) \cong V(I) \hookrightarrow A^t.
\]

We achieve this by computing successive isomorphisms \( A/I \cong A_1/I_1 \cong \ldots \cong A_\infty/I_\infty \).

(a) Let \( G \) be a Gröbner basis of \( I \). We call an indeterminate \( x_i \) standard if \( \text{NF}_G(x_i) = x_i \), where \( \text{NF}_G(f) \) denotes the normal form of the polynomial \( f \in A \) with respect to \( G \). For all other indeterminates the normal form will only contain terms in the polynomial subring \( A_1 = \mathbb{Z}[y_1, \ldots, y_s] \) generated by the standard indeterminates \( \{y_1, \ldots, y_s\} \subseteq \{x_1, \ldots, x_t\} \). Let \( I_1 \) denote the kernel of the surjective ring morphism \( A_1 \to A/I, y_i \mapsto y_i + I \), the composition of the embedding \( A_1 \hookrightarrow A \) and the natural projection \( \pi : A \to A/I \). A Gröbner basis \( G_1 \) of \( I_1 \) can be computed using an elimination order.\(^7\) Composing the ring morphism

\[
\tilde{\pi}_1 : A \to A_1, x_i \mapsto \text{NF}_G(x_i)
\]

with the natural surjection \( A_1 \to A_1/I \) yields the surjective morphism

\[
\pi_1 : A \to A_1/I_1, x_i \mapsto \text{NF}_G(x_i) + I
\]

which describes the embedding \( A^s \supseteq V(I_1) \cong V(I) \hookrightarrow A^t \).

(b) Inspect all elements of the Gröbner basis \( G_1 \) of \( I_1 \) for elements of the form \( g = u y_i - f(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_s) \) for some \( 1 \leq i \leq s \) and \( u \) is a unit in \( A_1 \) (here \( u = \pm 1 \)). The ring morphism

\[
\tilde{\psi}_2 : \begin{cases} 
A_1 &\to A_2 = \mathbb{Z}[y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_s], \\
y_j &\mapsto y_j, \quad j \neq i, \\
y_i &\mapsto f/u.
\end{cases}
\]

yields the isomorphism \( A_1/I_1 \cong A_2/I_2 \) with \( I_2 := \tilde{\psi}_2(I_1) \). Composing \( \tilde{\pi}_2 := \tilde{\psi}_2 \circ \tilde{\pi}_1 : A \to A_2 \) with the natural surjection \( A_2 \to A_2/I_2 \) we get the surjective morphism \( \pi_2 : A \to A_2/I_2 \) which describes the embedding \( A^{s-1} \supseteq V(I_2) \cong V(I) \hookrightarrow A^t \).

---

\(^6\) \( A = \mathbb{Z}[p_{ij} \mid i = 1, \ldots, r, j = 1, \ldots, n] \) in Section 2.1.

\(^7\) Or in fact any global monomial order [BLHP17].
(c) Iterate the last step until no element of the Gröbner basis of the ideal \( I_i \) has the above mentioned form. Then set \( A_{∞} := A_i \), \( I_{∞} := I_i \) and \( π_∞ : A \rightarrow A_{∞}/I_{∞} \).

For any other ideal \( J \leq A \) we can define \( J_{∞} := \pi_∞(J) \) and obtain the embedding
\[
A^u \supseteq V(I_{∞}) \setminus V(J_{∞}) \cong V(I) \setminus V(J) \hookrightarrow A^t.
\]

These heuristics are implemented in the package MatricesForHomalg [hom20] and interpreted geometrically in ZariskiFrames [BKLH19].

APPENDIX B. COMPUTING THE DEGREE \( d \) PART OF \( ϕ \)

Here we show how to compute homogeneous parts of morphisms of free graded modules of finite rank which we use to compute the degree-(\( d_2 - 1 \))-part \( ϕ^M := \left( ϕ(A^{H}, m^H) \right)_{d_2 - 1} \) of the morphism \( ϕ(A^{H}, m^H) \) considered in Section 7.

Let \( A \) be a commutative ring and \( S = A[x_1, \ldots, x_r] \) the free polynomial \( A \)-algebra, equipped with the standard grading \( \deg(x_i) = 1 \) and \( \deg(a) = 0 \) for all \( a \in A \setminus \{0\} \). Consider the category \( A^@ \) of free \( A \)-modules of finite rank and the \( (A^@\text{-enriched}) \) category \( S^@ \) of free graded modules of finite rank consisting of the objects
\[
M = \bigoplus_{i=1}^h S(-m_i) \quad \text{for } h \in \mathbb{N}, \forall i=1 m_i \in \mathbb{Z}.
\]

For \( d \in \mathbb{Z} \) the degree-\( d \)-part is the additive functor
\[
(-)^d := \text{Hom}_{A^@}(S(-d), -) : S^@ \rightarrow A^@.
\]

The value of the functor on an object \( M = \bigoplus_{i=1}^h S(-m_i) \in S^@ \) is given by
\[
M_d = \text{Hom}_{S^@}(S(-d), M) = \text{Hom}_{S^@}\left( S(-d), \bigoplus_{i=1}^h S(-m_i) \right) = \bigoplus_{i=1}^h \text{Hom}_{S^@}(S(-d), S(-m_i)) = \bigoplus_{i=1}^h S(-m_i)_d = \bigoplus_{i=1}^h S_{d-m_i},
\]
where \( S_a \) is the degree-\( a \)-part of the polynomial ring \( S \), which is a free \( A \)-module of rank \( (r^{-1+a}) \).

The value of the functor on a morphism \( ϕ : M \rightarrow N \) in \( S^@ \) can be constructed as follows. For \( a \in \mathbb{Z} \) define the column matrix \( τ_a \in S^{(r^{-1+a})} \) consisting of all degree-\( a \) monomials in \( S \). The embedding of the \( S \)-submodule \( (M_d) \leq M = \bigoplus_{i=1}^h S(-m_i) \) generated by the degree-\( d \)-part \( M_d \) is given by the block-diagonal matrix
\[
τ_{M,d} : \text{diag}(τ_{d-m_1}, \ldots, τ_{d-m_h}) : (M_d) \hookrightarrow M,
\]
interpreted as a morphism in \( S^@ \). The degree-\( d \)-part of \( ϕ \) can now be computed by computing the lift \( ⟨ϕ_d⟩ \)

\[
\begin{CD}
⟨M_d⟩ @>{τ_{M,d}}>> M \\
⟩⟩⟨⟩\downarrow \quad \downarrow ϕ\\n⟨N_d⟩ @>>{τ_{N,d}}>> N
\end{CD}
\]
as a matrix over \( S \) (using Gröbner bases over \( S \)). Due to degree reasons the matrix \( ⟨ϕ_d⟩ \) is in fact a matrix \( ϕ_d \) over \( A \), which defines the desired morphism \( ϕ_d : M_d \rightarrow N_d \). Note that the lift is unique as a lift (of the composition \( ϕ \circ τ_{M,d} \)) along the monomorphism \( τ_{N,d} \).
The above functor is implemented in the GAP-packages GradedModules [BLHLM20] and IntrinsicGradedModules [BS21]. The former is based on homalg [BLH11] and the latter is a reimplementation based on CAP [GPS18, Pos21].

APPENDIX C. FINDING A SMALLER PRESENTATION OF A FINITELY PRESENTED MODULE OVER AN AFFINE RING AND COMPUTING FITTING IDEALS

Let $A$ be a commutative nonzero unital ring and $\varphi : U \to W$ a morphism of free $A$-modules of finite rank. After choosing free bases of $U$ and $W$ one can identify the morphism $\varphi$ with a matrix $\varphi \in A^{\operatorname{rk}_A U \times \operatorname{rk}_A W}$. Using Remark 6.3 we now describe several heuristics which start with the matrix $\varphi$ and try to produce a matrix $\varphi'$ of smaller dimensions, and where the Fitting ideals of $\varphi$ can be computed in terms of the Fitting ideals of $\varphi'$. 

(a) The rows of $\varphi$ describe a generating set of relations among the generators of the $A$-module $\coker \varphi$. Hence, the matrix $\varphi$ can be replaced with any other $? \times \operatorname{rk}_A W$ matrix $\varphi'$ having the same row span (= the $A$-submodule of relations in the free $A$-module $A^{1 \times \operatorname{rk}_A W}$). In particular, one can remove from $\varphi$ zero rows, duplicate rows, etc. More generally, if $A$ is a ring with a Gröbner basis notion as in our application, then one can replace $\varphi$ by a reduced Gröbner basis of the rows, at least if this results in a smaller generating system of the row span.

(b) Let $\rho := \operatorname{row-syz}(\varphi)$ be a matrix of syzygies among the rows of $\varphi$. If any row-syzygy (i.e., row of $\rho$) has a unit at the $i$-th column, then the $i$-th row of $\varphi$ is a linear combination of the remaining rows and can be deleted as in (a).

(c) A unit $\varphi_{i,j} \in A^\times$ means that the $i$-th row expresses the $j$-th generator of $\coker \varphi$ as an $A$-linear combination of the remaining generators. In this case define $\varphi'$ as follows: Turn the $i$-th row into the $j$-th unit vector by dividing the $j$-th column by $\varphi_{i,j}$ and then use the resulting 1 to clean up the rest of the $i$-th row. This corresponds to multiplying $\varphi$ with the invertible matrix $\chi$ defined as the $\operatorname{rk}_A W \times \operatorname{rk}_A W$ identity matrix with the $j$-th row replaced by 

$$
\begin{pmatrix}
-\varphi_{i,1} & \cdots & -\varphi_{i,j-1} & 1 & -\varphi_{i,j+1} & \cdots & -\varphi_{i,\operatorname{rk}_A W}
\end{pmatrix}.
$$

The $i$-th row of the resulting matrix $\varphi \chi$ is indeed the $j$-th unit vector in $A^{1 \times \operatorname{rk}_A W}$. Multiplying with $\chi$ can be interpreted as a change of the generating system, where the $j$-th unit vector (now occurring as the $i$-th relation) states that in the new generating system of $\coker \varphi \chi \cong \coker \varphi$ the $j$-th generator is zero. This means that the $j$-th column of $\varphi \chi$ (containing the coefficients of this new zero generator) can be deleted. The $i$-th row of the resulting matrix is zero and can also be deleted as in (a). Hence, the matrix $\varphi'$ defined by deleting the $j$-th column and the $i$-th row in $\varphi \chi$ obviously satisfies $\coker \varphi' \cong \coker \varphi$.

(d) If $\varphi$ contains a zero column, then $\coker \varphi$ admits a free direct summand $A$. More precisely, if the $j$-th column of $\varphi$ is zero then $\coker \varphi \cong \coker \varphi' \oplus A$, where $\varphi'$ results from $\varphi$ by deleting the zero $j$-th column. It follows from Definition 6.1 that 

$$
\text{Fitt}_\ell \varphi = \text{Fitt}_{\ell-1} \varphi'.
$$

(e) Let $\Sigma := \operatorname{col-syz}(\varphi)$ be a matrix of syzygies among the columns of $\varphi$. If any columnsyzygy (i.e., column of $\Sigma$) has a unit at the $j$-th entry then the $j$-th column of $\varphi$ is an $A$-linear combination of the remaining columns. This syzygy can be used to replace the $j$-th generator of $\coker \varphi$ by a free generator of a free direct summand $A$, i.e., a new generator for which in the transformed relation matrix $\bar{\varphi}$ the $j$-th column is zero.

8We use the row-convention.
Then
\[ \text{Fitt}_\ell(\varphi) = \text{Fitt}_\ell(\tilde{\varphi}) = \text{Fitt}_{\ell-1}(\varphi'), \]
where \( \varphi' \) is the matrix \( \varphi \) (or \( \tilde{\varphi} \)) with the \( j \)-th column deleted.

**APPENDIX D. THE 9 EXCEPTIONAL MATROIDS IN THE DATABASE**

| \( M \) | the key of \( M \) in the database |
|---|---|
| \( M_9 \) | d8ffce083ea534e34556086ac2416ca48cc0483 |
| \( M_{11} \) | ba024dadc693ecf1dcfffc3b1b080b174123ef456 |
| \( M_{12} \) | ba067e5f970cd1f1f61c6f8da9819b11518a676 |
| \( M_2 \) | 0df50b71bd5adf05683022f4b5dac3deff13df93 |
| \( M_1 \) | 118c9bab77e406eb53043ac399bf851a012830 |
| \( M_2 \) | f6326c481d3f3cdf408d7e1a57beac74611e5b4 |
| \( M_3 \) | cc85717959738f046a5a3bc4e225df535310d8 |
| \( M_4 \) | 8b5726983dbde6b2c794225ce124b1e35271399 |
| \( M_5 \) | 3582712ca004e5a91726c695b5a991af7c1b5 |

**Table 2.** The hashed keys of the 9 exceptional matroids in the database [BK19].

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