Size Ramsey numbers of paths *

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Abstract
The size Ramsey number $\hat{R}(F, r)$ is the minimum integer $m$ such that there exists a graph $G$ on $m$ edges such that every coloring of the edges of $G$ with $r$ colors yields a monochromatic copy of $F$. Let $P_n$ be a path on $n$ vertices. In this paper, we prove that $\hat{R}(P_n, r) \geq \frac{(r+1)(r+2)}{4}n - \frac{3r^2+2r-2}{4}$ for sufficiently large $n$. In particular, $\hat{R}(P_n, 2) \geq 3n - 7$. Moreover, by using the pair model method, we prove that $\hat{R}(P_n, P_n, P_n) < 764.1n$ for sufficiently large $n$.

Keywords: Size Ramsey number; Pair model; Probabilistic method

Mathematics Subject Classification: 05C55; 05D10

1 Introduction

For graphs $G$ and $H$, we write $G \rightarrow (H)$, if for every coloring of the edges of $G$ with $r$ colors, there is a monochromatic copy of $H$ in some color $1 \leq i \leq r$. The Ramsey number $R_r(H)$ of $H$ is the minimum $n$ such that $K_n \rightarrow (H)_r$. Instead of minimizing the number of vertices, one can minimum number of edges. This naturally leads to the size Ramsey number $\hat{R}(H, r)$ introduced by Erdős, Faudree, Rousseau, Schelp [8] as follows:

$$\hat{R}(H, r) = \min\{E(G) : G \rightarrow (H)_r\}.$$ 

When $r = 2$, we denote $\hat{R}(H)$ instead by $\hat{R}(H, 2)$ for short. In the same paper, it was shown that $\hat{R}(K_n) = \binom{\hat{R}(K_n)}{2}$ and $\hat{R}(K_{n,n}) < \frac{3}{2}n^2$. In [10], Erdős and Rousseau gave a lower bound for $\hat{R}(K_{n,n})$, i.e. $\hat{R}(K_{n,n}) > \frac{1}{60}n^2$. There is no progress on the bounds on $\hat{R}(K_{n,n})$ since then.

*Supported in part by NSFC(11671088) and NSFFP(2016J01017).
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For the two color size Ramsey number of the path $P_n$ on $n$ vertices, Erdős [9] offered 100 dollars for a proof or disproof that

$$\hat{R}(P_n))/n \to \infty \text{ and } \hat{R}(P_n)/n^2 \to 0.$$ 

In 1983, Beck gained the 100 dollars prize by proving that $\hat{R}(P_n) < 900n$ for sufficiently large $n$. Bollobás [2] gave an improvement that $\hat{R}(P_n) < 720n$. Using a different and more elementary argument, Dudek and Prałat [6] obtained $\hat{R}(P_n) < 137n$. The argument was subsequently improved further by Letzer [12] to give $\hat{R}(P_n) < 91n$. Very recently, Dudek and Prałat [7] proved that $\hat{R}(P_n) < 74n$ by using expansion properties of random $d$-regular graphs, i.e. pair model, one can find more detailed introduction in Section 3.

For the lower bound of $\hat{R}(P_n)$, it is clear that $\hat{R}(P_n) \geq 2n - 1$ since every graph with at most $2n - 2$ edges has a red-blue coloring with at most $n - 1$ red edges and at most $n - 1$ blue edges. The progress on the lower bound for $\hat{R}(P_n)$ is extremely slow. The first nontrivial lower bound that $\hat{R}(P_n) \geq 9n/4$ was provided by Beck [4], which was improved by Bollobás [1] as that $\hat{R}(P_n) \geq (1 + \sqrt{2})n - 2$. Recently, Dudek and Prałat [7] improved the lower bound to $\hat{R}(P_n) \geq 5n/2 - 15/2$.

Generally, for the multicolor size Ramsey number $\hat{R}(P_n, r)$, Dudek and Prałat [7] obtained that for $r \geq 2$,

$$\frac{(r + 3)r}{4} n - O(r^2) \leq \hat{R}(P_n, r) \leq 33r4^r n.$$ 

Recently, with the aid of affine planes, Krivelevich [11] proved that for $r \geq 3$ and $r - 2$ is a prime power, $\hat{R}(P_n, r) > (r - 2)^2 n - O(\sqrt{n})$ for all sufficiently large $n$; moreover, in the same paper, Krivelevich obtained that $\hat{R}(P_n, r) \leq 400^5 c r^{2+\frac{1}{2+c}} n$, where $c > 5$ is a constant depends only on $r$. This matches well when $r$ goes to infinity, however there still exists a large gap for small fixed $r \geq 2$ since the constant $400^5 c$ in the upper bound is very large.

In this paper, we shall consider the multicolor size Ramsey number $\hat{R}(P_n, r)$ for fixed integer $r \geq 2$. Our first result gives an improvement on the lower bound for $\hat{R}(P_n, r)$.

**Theorem 1** Let $r \geq 2$ be fixed. Then, for all sufficiently large $n$,

$$\hat{R}(P_n, r) \geq \frac{(r + 1)(r + 2)}{4} n - \frac{3r^2 + 9r - 2}{4}.$$ 

In particular, $\hat{R}(P_n) \geq 3n - 7$.

For sufficiently large $n$, the lower bound in Theorem 1 is better than that by Dudek and Prałat [7] for all $r \geq 2$ and that by Krivelevich [11] when $r$ is small. We also give an upper bound for $\hat{R}(P_n, P_n, P_n)$ as follows, which is better than that by Dudek and Prałat [7] and that by Krivelevich [11].

**Theorem 2** Let $c = 8.2919$ and $d = 82.1405$. Then, a.a.s $g_{cn, cn, d} \to (P_n)_3$, which implies that $\hat{R}(P_n, P_n, P_n) < 764.1n$ for sufficiently large $n$. 


2 Lower bounds on size Ramsey number of paths

In this section, we dedicate to give a lower bound for the size Ramsey number of path $P_n$. In order to establish our result, we need the following lower bound of the multicolor Ramsey number $R_r(P_n)$, one can find a slightly weaker bound by Davies, Jenssen and Roberts [5]. We use a similar construction by Sun et al. [14] in which the authors gave the lower bound for $R_r(C_n)$ for even cycle $C_n$.

Lemma 1 Let $r \geq 2$ and $n \geq 1$ be integers. We have

$$R_r(P_n) \geq 2(r - 1) \left( \left\lfloor \frac{n}{2} \right\rfloor - 1 \right) + 2.$$

Moreover, $R_r(P_n) \geq (r - 1)(n - 1) + 1$ if $n$ is odd.

Proof. Consider the complete graph $K_{2r-2}$ on vertices $\{0, 1, \ldots, 2r - 3\}$, and define a coloring $C$ of edges of $K_{2r-2}$ with $r$ colors as follows.

1. For $1 \leq i \leq r - 1$, color the edges from vertex $i$ to vertices $i + 1, \ldots, i + r - 2$ and the edges from vertex $i + r - 1$ to vertices $i + r, \ldots, i + 2r - 3$ (taken modulo $2r - 2$) with color $C_i$.

2. The remaining edges of the form $\{j, j + r - 1\}$ for $j = 0, \ldots, r - 2$ are colored with the final color $C_r$.

From the definition of color $C$, we know each color $C_1, \ldots, C_{r-1}$ consists two vertex-disjoint stars, each on $r - 1$ vertices. The final color forms a matching on $r - 1$ edges.

Now, we blow up each vertex $i$ of $K_{2r-2}$ into a set $V_i$ of $\lfloor n/2 \rfloor$ vertices in $K_{2r-2}$. Color the edges within $V_i$ with $C_i$, and color edges between $V_i$ and $V_j$ with the same color as the edge $\{i, j\}$ in $K_{2r-2}$. Let $N = 2(r - 1)(\lfloor n/2 \rfloor - 1) + 1$. This defines an $r$-coloring on edges of $K_N$.

It is easy to check that there is no monochromatic $P_n$ since each subgraph induced by edges in color $C_i$ for $1 \leq i \leq r - 1$ is bipartite with smallest part $\lfloor n/2 \rfloor - 1$, and the subgraph induced by color $C_r$ has less than $n$ vertices.

Moreover, if $n$ is odd, then we blow up each vertex $i$ of $K_{2r-2}$ into a set $V_i$ of $\lfloor n/2 \rfloor$ vertices. In this case, the coloring is similar as above to get the desired lower bound. The proof is completed. \qed

We also need the following auxiliary result which generalizes that in [7, Claim 2.2].

Lemma 2 Let $k \geq 0$ be an integer. For any graph $G$, we have at least one of the following two properties holds:

(i) $G$ has $k$ edges $e_1, e_2, \ldots, e_k$ such that $G - \{e_1, e_2, \ldots, e_k\}$ contain no $P_n$;

(ii) $G$ contains $(k + 2)$ vertex-disjoint connected subgraphs of order at least $\lfloor n/2 \rfloor$ each.
Proof. We prove the statement by induction on \( k \). For \( k = 0 \), if (i) fails, then \( G \) contains a copy of \( P_n \), and we are done. This implies (ii) since we can split the path as equally as possible to get two subgraphs of the desired order as desired.

Let \( k \geq 0 \) and suppose that the statement holds for all \( \ell < k + 1 \). Again, assume that (i) fails for \( k + 1 \), i.e. for any choice of \( e_1, e_2, \ldots, e_{k+1} \), \( G - \{e_1, e_2, \ldots, e_{k+1}\} \) contains \( P_n \). We will show that (ii) must hold, i.e. \( G \) contains \((k + 3)\) vertex disjoint connected subgraphs of order at least \( \lceil n/2 \rceil \) each.

Clearly \( P_n \subseteq G \). Hence, let \( e \) be such that \( G - e \) consists of two subgraphs, \( G_1 \) and \( G_2 \), each of order at least \( \lceil n/2 \rceil \). By assumption that (i) fails for \( (k + 1) \), it follows that for any choice of \( k_1 \) edges \( e_1, e_2, \ldots, e_{k_1} \) in \( G_1 \) and \( k_2 \) edges \( f_1, f_2, \ldots, f_{k_2} \) in \( G_2 \) such that \( k_1 + k_2 = k \), either \( G_1 - \{e_1, e_2, \ldots, e_{k_1}\} \) or \( G_2 - \{f_1, f_2, \ldots, f_{k_2}\} \) contains \( P_n \).

If \( G_1 - \{e_1, e_2, \ldots, e_{k_1}\} \supseteq P_n \) and \( G_2 - \{f_1, f_2, \ldots, f_{k_2}\} \supseteq P_n \) for any choice of the edges, then by inductive hypothesis \( G_1 \) and \( G_2 \) have, respectively, \((k_1 + 2)\) and \((k_2 + 2)\) vertex-disjoint connected subgraphs of size \( \lceil n/2 \rceil \). Thus \( G \) has \( k_1 + k_2 + 4 \geq k + 3 \) vertex-disjoint connected subgraphs of order \( \lceil n/2 \rceil \). Consequently, without loss of generality, we may assume that \( G_2 - \{f_1, f_2, \ldots, f_{k_2}\} \nsubseteq P_n \) for some choice of \( f_1, f_2, \ldots, f_{k_2} \), where \( k_2 \) is as small as possible. Of course, this implies that \( G_1 - \{e_1, e_2, \ldots, e_{k_1}\} \supseteq P_n \) for any choice of the edges \( e_1, e_2, \ldots, e_{k_1} \). Now, we consider two cases as follows.

If \( k_2 = 0 \), then by inductive hypothesis \( G_1 \) has \((k_1 + 2)\) vertex disjoint connected subgraphs of order \( \lceil n/2 \rceil \) which, together with \( G_2 \) yield \((k + 3)\) desired large subgraphs in \( G \). On the other hand, if \( k_2 \geq 1 \), then, due to the minimality of \( k_2 \), we infer that for any choice of \( f_1, f_2, \ldots, f_{k_2-1} \), \( G_2 - \{f_1, f_2, \ldots, f_{k_2}\} \supseteq P_n \). Thus, again by inductive hypothesis, \( G \) has \((k_1 + 2) + (k_2 - 1 + 2) = k + 3 \) vertex-disjoint connected subgraphs of order \( \lceil n/2 \rceil \) as desired.

Now, we are ready to give a proof for Theorem 1.

Proof of Theorem 1. We prove the assertion by induction on \( k \geq 2 \). First we proved that for \( k = 2 \), i.e.

\[
\hat{R}(P_n) \geq 3n - 7.
\]

Suppose to contrary that \( \hat{R}(P_n) < 3n - 7 \). Let \( G = (V, E) \) be a graph of size \( \hat{R}(P_n) \), such that \( G \rightarrow (P_n)_2 \).

We apply Lemma 2 with \( k = 2 \). First, let us assume that property (i) in Lemma 2 holds, i.e. \( G - \{e_1, e_2\} \) contains no \( P_n \). Let us color all edges in \( G - \{e_1, e_2\} \) in red, and color \( e_1 \) and \( e_2 \) in blue, which gives a red/blue coloring of the edges of \( G \) such that there is no monochromatic \( P_n \). This leads to a contradiction.

Then assume the property (ii) in the Lemma 2 holds, i.e. \( G \) contains 4 vertex-disjoint connected subgraphs of order at least \( \lceil n/2 \rceil \) each. We color \( \lceil n/2 \rceil - 1 \) edges of each of subgraphs in red. Since the number of uncolored edges is at most

\[
\hat{R}(P_n) - 4(\lceil n/2 \rceil - 1) < 3n - 7 - 4(\lceil n/2 \rceil - 3/2) = n - 1,
\]

we can color the remaining edges in blue to get the desired contradiction that \( G \rightarrow (P_n)_2 \). This establishes the induction basis.
Now, assume that \( r \geq 2 \) and the statement holds for \( k = r \). We shall prove it also holds for \( k = r + 1 \). Suppose to contrary that it fails for \( k = r + 1 \), that is,

\[
\hat{R}(P_n, r + 1) < \frac{(r + 2)(r + 3)}{4} n - \frac{3r^2 + 15r + 10}{4}.
\]

Let \( G = (V, E) \) be a graph on \( N \) vertices and of size \( \hat{R}(P_n, r + 1) \), such that \( G \to (P_n)_{r+1} \). Clearly we may assume that \( G \) is connected, since otherwise the induced coloring of some of its connected components \( G' \) of smaller size also satisfies that \( G' \to (P_n)_{r+1} \), which means that

\[
\hat{R}(P_n, r + 1) = e(G') < e(G) = \hat{R}(P_n, r + 1),
\]
a contradiction. In the following, we separate the proof into two cases.

**Case 1.** \( N < \frac{r+2}{2} n - \frac{r}{2} - 2 \).

First, we claim that for fixed \( r \geq 2 \) and sufficiently large \( n \),

\[
R_{r+1}(P_n) > N.
\]

Indeed, from Lemma 1, we know for sufficiently large even \( n \),

\[
R_{r+1}(P_n) \geq 2r \left( \left\lceil \frac{n}{2} \right\rceil - 1 \right) + 2 = r(n - 2) + 2 > \frac{r+2}{2} n - \frac{r}{2} - 2 > N.
\]

Similarly, from Lemma 1, we know for sufficiently large odd \( n \),

\[
R_{r+1}(P_n) \geq r(n - 1) + 1 > \frac{r+2}{2} n - \frac{r}{2} - 2 > N.
\]

Therefore, the fact that we claimed holds. It follows that \( K_N \not\to (P_n)_{r+1} \), and hence \( G \not\subseteq K_N \), which yields a contradiction.

**Case 2.** \( N \geq \frac{r+2}{2} n - \frac{r}{2} - 2 \).

Let \( G_1 \) be any subgraph of \( G \) with \( e(G_1) \geq N - 1 \). We apply Lemma 2 to \( G_1 \) with \( k = r \). First, suppose that property \((i)\) in Lemma 2 holds, i.e. \( G_1 \) has \( r \) edges such that \( G_1 - \{e_1, e_2, \ldots, e_r\} \) contains no \( P_n \). Then we color \((N - 1) - r \) edges in \( G_1 - \{e_1, e_2, \ldots, e_r\} \) using the first color. The number of uncolored edges of \( G \) is

\[
\hat{R}(P_n, r + 1) - (N - r - 1) < \frac{(r + 2)(r + 3)}{4} n - \frac{3r^2 + 15r + 10}{4} - \left( \frac{r+2}{2} n - \frac{r}{2} - 2 \right) + r + 1
\]

\[
= \frac{(r^2 + 3r + 2)}{4} n - \frac{3r^2 + 9r - 2}{4} \leq \hat{R}(P_n, r),
\]

where the last inequality follows from the inductive hypothesis. Thus, we can color these
uncolored edges with remaining $r$ colours in such a way that there is no monochromatic $P_n$, which give us the desired contradiction.

Now, we suppose that property (ii) in Lemma 2 holds, i.e. $G_1$ contains $(r + 2)$ vertex-disjoint connected subgraphs of order at least $\lfloor n/2 \rfloor$ each. We color $\lfloor n/2 \rfloor - 1$ edges of each of the $(r + 2)$ subgraphs with the first color. The number of uncolored edges is

$$\hat{R}(P_n, r + 1) - (r + 2)(\lfloor n/2 \rfloor - 1)$$

$$< \frac{(r + 2)(r + 3)}{4}n - \frac{3r^2 + 15r + 10}{4} - (r + 2) \left( \frac{n}{2} - \frac{3}{2} \right)$$

$$= \frac{(r + 2)(r + 1)}{4}n - \frac{3r^2 + 9r - 2}{4}$$

$$\leq \hat{R}(P_n, r),$$

and this again yields a contradiction as before. The proof is completed now. $\square$

3 Upper bounds on Size Ramsey number of paths

In this section, we will focus on upper bounds for size Ramsey number of paths more than two colors. Here is a natural generalization of pair model in random bipartite graph. Recall that an event in a probability space holds asymptotically almost surely (a.a.s.) if the probability that it holds tends to 1 as $n$ goes to infinity. Since we aim for results that hold a.a.s., we will always assume that $n$ is large enough. For simplicity, we do not round numbers that are supposed to be integers either up or down, since these errors are negligible to the asymptomatic calculations we will make.

Our main results in this section refer to the probability space, the probability space of random $d$-regular balance bipartite graphs on $2cn$ vertices with uniform probability distribution. This space is denoted by $g_{cn, cn, d}$, where $d \geq 2$ is fixed and $n$ is large enough. Instead of working directly in $g_{cn, cn, d}$, we use the pairing model (also known as the configuration model) of random regular graphs, which was first introduced by Bollobás [3], here is a natural generalization of pair model in random bipartite graphs.

We consider $2cdn$ points partitioned into $2cn$ labeled buckets $v_1, v_2, \ldots, v_{2cn}$ each of which contains exactly $d$ points. A pairing of these points is a perfect matching into $cdn$ pairs. Given a pairing $P$, we may construct a $d$-regular multigraph $G(P)$, with loops and parallel edges allowed, as follows. The vertices of $G(P)$ are buckets $v_1, v_2, \ldots, v_{2cn}$, and a pair $\{x, y\}$ in $P$ corresponds to an edge $v_ixj$ in $G(P)$ if $x$ and $y$ are contained in the buckets $v_i$ and $v_j$, respectively. It can be easily seen that the probability that the random pairing yields a given simple graph is uniform, hence the restriction of the probability space of the random pairing to simple graphs is precisely $g_{cn, cn, d}$. Moreover, it is well known that a random pairing generates a simple graph with probability asymptotic to $e^{-(d^2-1)/4}$, depending on $d$. Thus, for fixed $d$, any event holding a.a.s. over the probability space of random pairing also holds a.a.s. over the corresponding space $g_{cn, cn, d}$. For more
information on this model, see, for instance, the survey of Wormald [13].

Before giving a proof for Theorem 2, we need the following useful lemma by Dudek and Pra\v{s}il\[7\], Lemma 3.7.

**Lemma 3 (Dudek and Pra\v{s}il\[7\])** Let $r \geq 2$ and $G = (V_1 \cup V_2, E)$ be a balanced bipartite graph of order $cn$ for some $c > 2^r - 1$. Assume that for any subsets $S \subseteq V_1$ and $T \subseteq V_2$ with $|S| = |T| = ((c + 1)/2^r - 1)n/2$, we have $e(S, T) \neq 0$. Then, $G \rightarrow (P_n)_r$.

We will use the following form of Lemma 3.

**Lemma 4** Let $G = (V_1 \cup V_2, E)$ be a balanced bipartite graph on $2cn$ $(c > 3.5)$ vertices. Assume that for any subsets $S \subseteq V_1$ and $T \subseteq V_2$ with $|S| = |T| = n(2c - 7)/16$, we have $e(S, T) \neq 0$. Then, $G \rightarrow (P_n)_3$.

**Proof of Theorem 2.** Consider $g_{c_n, c_n, d}$ for some $c \in (3.5, \infty)$ and $d \in N$. Our goal is to show that, for some suitable choice of $c$ and $d$, the expected number of pairs of two disjoint sets $S$ and $T$, where $S \subseteq V_1$ and $T \subseteq V_2$ such that $|S| = |T| = n(2c - 7)/16$ and $e(S, T) = 0$ tends to zero as $n \rightarrow \infty$. This together with the first moment principle, implies that a.a.s. no such pair exists and so, by Lemma 4 , we get that a.a.s. $g_{c_n, c_n, d} \rightarrow (P_n)_3$. As a result, $\hat{R}(P_n, P_n, P_n) \leq (cd + o(1))n$.

Let $c_1 = (2c - 7)/16$ and $X(c, d)$ be the expected number of pairs of two disjoint sets $S, T$ such that $|S| = |T| = n(2c - 7)/16, e(S, T) = 0$, and $e(S, V_2 \setminus T) = c_1 dn$, $e(T, V_1 \setminus S) = c_1 dn$. Using the pair model, it is clear that we have

$$X(c, d) = \binom{cn}{c_1n} \binom{c_2n}{c_1dn} \binom{(c - c_1)dn}{c_1dn} \binom{(c - c_1)dn}{c_1dn} (c_1dn)! (c_1dn)!$$

$$\times M((c - 2c_1)dn, (c - 2c_1)dn)/M(cdn, cdn),$$

where $M(t, t)$ is the number of perfect matchings on balanced bipartite graph of order $2t$ vertices, that is, $M(t, t) = t!$.

Using Stirling’s formula $t! \sim \sqrt{2\pi t}(t/e)^t$, after simplification we get

$$X(c, d) = f(c, d)e^{g(c, d)n},$$

where

$$f(c, d) = \frac{1}{2\pi} \cdot \frac{1}{c_1n} \sqrt{\frac{c}{c - 2c_1}}$$

and

$$g(c, d) = 2c \ln c + 2(c - c_1)d \ln(c - c_1) - 2c_1 \ln c_1$$

$$- 2(c - c_1) \ln(c - c_1) - (c - 2c_1)d \ln(c - 2c_1) - cd \ln c.$$
It remains to choose suitable $c$ and $d$ such that $g(c, d) \leq 0$ and $cd$ as small as possible. By using Matlab, we can take $c = 8.2919$ and $d = 82.1405$, it follows that

$$
\hat{R}(P_n, P_n, P_n) \leq cdn < 764.1n
$$

for sufficiently large $n$. This completes the proof of Theorem 2.

\[\square\]

### 4 Concluding Remark

Indeed, we can use the same method as in Theorem 2 to get the upper bounds for $r = 4, 5$ that are listed in Table 1 as follows, in which $n$ is sufficiently large. We omit the proofs of these results since they are quite similar to that of Theorem 2.

| $r$ | $P_n$ | $P_n$ | $P_n$ |
|-----|-------|-------|-------|
| 3   | 764.1  | 5167.7| 56110 |
| 4   | 6336   | 33792 | 168960|
| 5   |        |       |       |

Table 1. Comparing upper bounds with Dudek and Prałat [7]

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