ASPECTS OF GAUSSIAN PROCESSES ON SPHERES

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In memory of Larry Shepp

Abstract. We review the Dudley integral for the Belyaev dichotomy applied to Gaussian processes on spheres, and discuss the approximate (or restricted) continuity of paths in the discontinuous case. In the continuous case, we investigate the link between the smoothness of paths and the decay rate of the angular power spectrum, following Tauberian work of the first author [Bin5,6], Malyarenko [Mal1,2], and Lang and Schwab [LangS]. We develop the work of the first author [Bin1] on projection from spheres of higher to those of lower dimension, transforming the numerically intractable series for the relevant density in [Bin1] into a numerically tractable double integral.

Key words. Belyaev dichotomy, Dudley integral, Gaussian processes, spheres, ultraspherical polynomials, Schoenberg’s theorem, Tauberian theorems, spherical functions, multiplication theorem, Feldheim-Vilenkin integral.

MSC Subject classification. 60B15, 60B99, 60G15, 60G60.

1. Belyaev’s dichotomy and the Dudley integral.

Let $X = \{X_t : t \in M\}$ be a real-valued zero-mean Gaussian process, on (defined on, indexed by) $M$. Here, $M$ will be the $d$-sphere $S^d \subset \mathbb{R}^{d+1}$ of radius 1; the motivating example is $d = 2$, with $M = S^2$ as Planet Earth. The law of $X$ is determined by either of the covariance or the incremental variance:

$$c(s, t) := \text{cov}(X_s, X_t) = E[X_s X_t], \quad i(s, t) := E[(X_t - X_s)^2]$$

(respectively positive and negative definite, or of positive and negative type); we can pass between them by

$$i(s, t) = c(s, s) + c(t, t) - 2c(s, t), \quad c(s, t) = \frac{1}{2}(i(s, o) + i(t, o) - i(s, t)),$$

with $o$ some base point (North Pole). We restrict attention to isotropic processes (those with stationary increments), where these are functions only
of the geodesic distance \(d(s,t)\), or of \(x := \cos d(s,t) \in [-1,1]\) \((s,t \in M)\):

\[c(s,t) = C(x), \quad i(s,t) = I(x).\]

We assume also that the covariance is continuous. We can then use reproducing-kernel Hilbert spaces and the Karhunen-Loève expansion, which we will need below ([MarR, p.203-207], [Adl, III.2, III.3]).

We need the Gegenbauer (ultraspherical) polynomials \(C_n^\lambda(x)\), normalised to take the value 1 at 1; for these we use Bochner’s notation \(W_n^\lambda(x)\):

\[W_n^\lambda(x) := C_n^\lambda(x)/C_n^\lambda(1) = C_n^\lambda(x).n!\Gamma(2\lambda)/\Gamma(n+2\lambda).\]

These are the orthogonal polynomials generated by the probability measure

\[G_\lambda(dx) := \frac{\Gamma(\lambda+1)}{\sqrt{\pi}\Gamma(\lambda+\frac{1}{2})}.(1-x^2)^{\lambda-\frac{1}{2}}dx \quad (x \in [-1,1]):\]

\[\int_{-1}^{1} W_n^\lambda(x)W_m^\lambda(x)G_\lambda(dx) = \delta_{mn}/\omega_n^\lambda, \quad \omega_n^\lambda := \frac{(n+\lambda)}{\lambda} \frac{\Gamma(n+2\lambda)}{\Gamma(2\lambda)}.\]

Half-integer values of the Gegenbauer index \(\lambda\) correspond to (integer) values of the Euclidean dimension \(d\) as above by

\[\lambda = \frac{1}{2}(d-1).\]

When we need two indices (as with the projections in §4) we will have

\[0 \leq \nu \leq \lambda \leq \infty\]

(again see §2 for the Hilbert-space case \(\lambda = \infty\)).

From the Bochner-Schoenberg theorem of 1940-42 ([Sch]; see [BinS1] for further references), \(C\) is then, to within a scale factor \(c \in (0, \infty)\) (reflecting physical units), a mixture (with mixing law \(a = (a_n)_{n=0}^{\infty}, a_n \geq 0, \sum a_n = 1\)) of ultraspherical polynomials \(W_n^\lambda(x)\) with \(\lambda := \frac{1}{2}(d-1)\):

\[C(x) = c \sum_{0}^{\infty} a_n W_n^\lambda(x), \quad I(x) = c \sum_{0}^{\infty} a_n (1 - W_n^\lambda(x)). \quad (B - S)\]

We recall Belyaev’s dichotomy for Gaussian processes ([Bel]; [MarR, Th. 5.3.10]): Gaussian paths are either very nice (continuous), or very nasty
(pathological: unbounded above and below on any interval, or set of positive measure). Much is known, by way of necessary conditions for continuity [MarR, §6.2], and sufficient conditions [MarR, §6.1]; see e.g. [MarS1,2], [Gar].

One uses the Dudley metric (actually a pseudo-metric)

\[ d_X(s,t) := \sqrt{E[(X_s - X_t)^2]} \quad (s, t \in M). \]

For \( u > 0 \), write \( N(u) \) for the minimum number of \( d_X \)-balls of radius \( u \) needed to cover the parameter-space \( M \); then if \( H(u) := \log N(u), \ H := \{H(u) : u > 0\} \) is called the metric entropy. The Dudley integral is

\[ \int_0^\epsilon \sqrt{H(u)}du \quad (\epsilon > 0). \quad (Dud) \]

One obtains a clean necessary and sufficient condition for continuity, finiteness of \((Dud)\) [Dud1,2], only for \( X \) isotropic [MarR], which is why we restrict to this here. If \( \phi \) is a non-negative function increasing near 0 with

\[ d_X(s,t) \leq \phi(|s - t|) \quad (s, t \in M), \]

then the Dudley integral is finite if

\[ \int_M \phi(e^{-x^2})dx < \infty : \int_0^\epsilon \frac{\phi(u)}{\sqrt{-\log u}} \frac{du}{u} < \infty. \]

For isotropic processes on spheres, take

\[ \phi(u) := \sup\{\sqrt{I(cos v)} : v \leq u\} : \]

the condition for path continuity of \( X \) becomes ([Gar]; [Dud2, §7])

\[ \int_0^1 \sqrt{\frac{\sup_{v \leq u} I(cos v)}{-\log u}} \frac{du}{u} < \infty : \]

\[ \int_0^1 \sqrt{\frac{\sup_{v \leq u}(1 - \sum_0^\infty a_nW_n^{\lambda}(cos v))}{-\log u}} \frac{du}{u} < \infty. \quad (DudSph) \]

Despite the ‘pathological’ behaviour of the sample paths of the process in the discontinuous case of Belyaev’s dichotomy, there is a sense in which they
are ‘nearly continuous’: a ‘localisation of pathology’. For, by the Karhunen-
Loève expansion,

\[ X(t, \omega) = \sum_{n=0}^{\infty} \phi_n(t)Z_n(\omega), \quad (K - L) \]

with the \( Z_n \) independent standard normal random variables and the \( \phi_n \) continuous functions (a.s.: see [MarR, Remark 5.3.3]). In particular, a.s. (we may exclude the exceptional \( P \)-null \( \omega \)-set from our sample space and so omit this restriction), \( X(t) = X(t, \omega) \) is a measurable function of \( t \). So, by Lusin’s continuity theorem (or Lusin’s restriction theorem, of 1912: [Dud3, Th. 7.5.2], [Rud, §2.24]), \( X(t) \) becomes continuous in \( t \) when restricted to a time-set of \( t \) avoiding a set of arbitrarily small measure.

Remarks.

1. The oscillation function.
   The local behaviour of the paths of \( X(t, \omega) \) is governed by the oscillation function, a deterministic function, \( \alpha(t) \) say [MarR, p.209-211]. The continuous case of Belyaev’s dichotomy has \( \alpha \equiv 0 \), the discontinuous case has \( \alpha \equiv \infty \). Despite the ‘pathological’ appearance of this case, it is as well to note that a measurable function can have oscillation function \( \equiv \infty \), as here.

2. Approximate limits and limsups.
   The Lusin argument above from \((K - L)\) says that the paths, when slightly restricted, become continuous in some sense. The concepts of approximate limit, \( ap - \lim \) (and so of approximate limsup, \( ap - limsup \), and approximate derivative) and approximate continuity go back to Denjoy in 1916 and Khintchine in 1927 (see e.g. Saks [Sak, IX.10]). An approximate limit at a point \( t \) becomes an actual limit when the approach to \( t \) is made through a Borel set having \( t \) as a density point (in the sense of the Lebesgue density theorem, see e.g. [Rud, §7.2], [Dud3, p.422], [BinO2, Th. L], and of the density topology, below). For a number of probabilistic results on \( ap - limsup \), see Geman and Horowitz [GemH, §13] (cf. [Adl, IV.6]). For the relevant real-variable theory, see e.g. [GemH, §14], and the earlier paper by Smallwood [Sma].

3. The density topology.
   The density topology takes as its open sets the measurable sets all of whose points are density points. That this gives a topology, and the intimate link with Denjoy’s approximate continuity (and so with \( ap - lim \)), are due to Haupt and Pauc [HauP]. It has been much studied, by C. Goffman and others; for references see e.g. [BinO1,3].
2. Malyarenko’s theorem and the angular power spectrum

What \((DudSph)\) above says is that the paths are continuous if and only if the coefficients \(a_n\) in the mixture law \(a\) in \((B - S)\) – the angular power spectrum – decay fast enough. Slow decay means wild behaviour of the paths, but if the decay is fast enough, the paths become very smooth. As we shall see, if \(a_n = O(1/n^{1+\alpha})\) for \(\alpha > 0\), the paths are continuous (and become smoother with increasing \(\alpha\)).

While the condition \((DudSph)\) resolves the matter completely in principle, in practice implementing it is formidable, for the obvious three reasons: passage between the mixing law \((a_n)\) and the ultraspherical series \(\sum a_n W_n^\lambda\), the supremum, and the integration. The nub here is the first: the link between the decay of \(a_n\) for large \(n\), and the growth of \(1 - \sum_0^\infty a_n W_n^\lambda(\cos v)\) for small \(v > 0\).

While there is no definitive answer to this question (any more than there is in the classical case of Fourier series \([Zyg]\)), there is an answer in the principal case of practical interest, that when the angular power spectrum \((a_n)\) is regularly varying (see e.g. \([BinGT]\)). Here the results (whose proofs we sketch below) are due to Malyarenko \([Mal1,2]\), based on early work of the first author \([Bin5]\) (itself based on earlier work of Askey and Wainger \([AskW]\)):

**Theorem (Malyarenko).** For \(\ell\) slowly varying,

\[
A_n := \sum_0^\infty a_k \sim \ell(n)/n^\gamma \quad (n \to \infty) \quad (\gamma \in (0,2))
\]

iff

\[
I(v) = 1 - \sum_0^\infty a_n W_n^\lambda(\cos v) \sim \Gamma(\lambda + \frac{1}{\gamma} \frac{1}{\ell}, \frac{\Gamma(1 - \frac{1}{\gamma})}{2\Gamma(\lambda + \frac{1}{2} - \frac{1}{\gamma})} v^\gamma \ell(1/v) \quad (v \downarrow 0).
\]

**Proof.** The implication from \(A_n\) to \(I(v)\) is Abelian; the converse is Tauberian. One has

\[
I(v) = \sum_0^\infty a_n (1 - W_n^\lambda(v)) = \sum_0^\infty (A_n - A_{n+1})(1 - W_n^\lambda(v)),
\]

and writes this by partial summation as

\[
I(v) = \sum A_{n+1}(W_n^\lambda(v) - W_{n+1}^\lambda(v)).
\]
The difference of ultraspherical polynomials here may be expressed as one Jacobi polynomial (Erdélyi et al. [ErdMOT, Vol. II, 10.8(32)]. Recall that the Jacobi polynomials are a two-index family \( P_n^{(\alpha,\beta)} \) (\( \alpha, \beta \geq -\frac{1}{2} \); we take \( \alpha \geq \beta \)). When \( \alpha = \beta \), the Jacobi polynomials reduce to the ultraspherical polynomials, with (as above)

\[ \alpha = \beta = \lambda - \frac{1}{2} = \frac{1}{2}(d - 2). \]

We use the normalisation [Mal2, 4.3.1]

\[ R_n^{(\alpha,\beta)}(x) := P_n^{(\alpha,\beta)}(x)/P_n^{(\alpha,\beta)}(1). \]

Then ([Mal2, p.127], [ErdMOT II 10.8(32)])

\[ R_n^{\alpha,\beta}(\cos \theta) - R_{n+1}^{\alpha,\beta}(\cos \theta) = \frac{(2n + \alpha + \beta + 2)}{(\alpha + 1)} \sin \frac{1}{2} \theta \ R_{n+1}^{\alpha,\beta}(\cos \theta). \]

So

\[ I(\cos \theta) = \frac{2\sin^{\frac{1}{2}} \theta}{(\alpha + 1)} \sum (n + \alpha + 1) A_n R_n^{\alpha+1,\beta}(\cos \theta). \]

The \( \sin^{\frac{1}{2}} \theta \) (equivalently, \( \theta^2/4 \)) term on the right accounts for the upper limit 2 on \( \gamma \) in the result; that the incremental variance is non-negative accounts for the lower limit of 0. The results of [Bin5] now apply to the sequence \( (n + \alpha + 1) A_n = (n + \alpha + 1) \sum a_k \) with the \( \sigma \) there as \( 1 - \gamma \). The Tauberian conditions needed follow from \( a_n \geq 0 \) (so \( A_n \) is non-negative and non-decreasing). \( \Box \)

Malyarenko’s theorem is very similar to that of [Bin3] (proved more simply in [Bin7]) on Hankel transforms, the link being provided by Szegő’s Hilbert-type asymptotic formula for the Jacobi polynomials [Sze, Th. 8.21.12].

The Belyaev integral is of course convergent in all these cases, and so Malyarenko’s theorem provides us with an ample range of examples of the continuous case in the Belyaev dichotomy (the pathological case being of course less common in practice). For, the supremum operation in (DudSph) (a reflection of the great mathematical difficulties in bridging the gap between the necessary and the sufficient conditions for finiteness of the Dudley integral) is harmless here: any regularly varying function of non-zero index is asymptotically monotone [BinGT, §1.5.2]).
One can extend to \( \gamma = 0 \) here, when the tail \( A_n \) of the mixing law is slowly varying, but convergence of the Dudley integral now hinges on the behaviour of \( \ell \) at infinity. This is shown by familiar examples such as \( \sum 1/(n\log n)^k \), \( \sum 1/(n \log n \log \log n)^k \), each convergent if \( k > 1 \), divergent if \( k \leq 1 \). One can also extend to the case \( \gamma = 2 \) [Bin5].

O-versions of these results are straightforward (cf. Korevaar [Kor, IV.10]).

The Hilbert sphere.

The Hilbert sphere \( S^\infty \) is not locally compact, and because of this one may expect very different behaviour for it from that on Euclidean spheres. Gaussian processes on \( S^\infty \) are discontinuous (Lévy [Lev]; Berman [Berm1,2]; Dudley [Dud2, §5]). More is true: such processes are locally deterministic (see [Lev, p.355], [Berm1, p.950] for the definition): the behaviour of the process locally determines it everywhere. This sounds reminiscent of the great smoothness shown by holomorphic functions in complex analysis, but is in fact diametrically opposite: the process is extremely wild, and ‘gets everywhere it will go immediately’.

The ultraspherical polynomials may be defined for \( \lambda = \infty \) by \( W^\infty_n(x) = x^n \) (see e.g. [Bin1]). But, as

\[
\frac{\Gamma(\lambda + \frac{1}{2} + \frac{1}{2} \gamma)}{\Gamma(\lambda + \frac{1}{2})} \sim \frac{\lambda^{\frac{1}{2} \gamma}}{\Gamma(1 - \frac{1}{2} \gamma)} (\lambda \to \infty),
\]

this case does not follow formally from Malyarenko’s theorem by letting \( \lambda \to \infty \). Instead, we have here:

**Proposition.** In the notation of Malyarenko’s theorem,

\[
A_n := \sum_{n} a_k \sim \ell(n)/n^{\frac{1}{2} \gamma} \quad (n \to \infty) \quad (\gamma \in (0, 2))
\]

iff

\[
1 - \sum a_n (\cos v)^n \sim \frac{\Gamma(1 - \frac{1}{2} \gamma)}{2^{\frac{1}{2} \gamma}} \quad (v \downarrow 0).
\]

**Proof.** The functions in \( P_\infty \) are the probability generating functions (in \( t \), say), or (putting \( t = e^{-s} \)) the Laplace-Stieltjes transforms. We can read off the relevant tail-behaviour here from e.g. [BinGT, Cor. 8.1.7]. Writing \( \cos v = e^{-s} \) here, we have \( s \sim \frac{1}{2} v^2 \) as \( s, v \downarrow 0 \), which gives the result. \( \square \)
One must expect the tails in the Hilbert case here (with the sphere non-compact) to be heavier than in the Euclidean case of Malyarenko’s theorem (with the sphere compact): now, the paths are wild rather than continuous, and there there are ‘more ways of going off to infinity’. Thus the relevant probability laws \((a_n)\) here have regularly varying tails in \((0,1)\), rather than in \((0,2)\) as in the Euclidean case of Malyarenko’s theorem – that is, they correspond to infinite mean rather than infinite variance.

The constants introduced (in going between the ‘Abelian’ and ‘Tauberian’ sides) in results of this type are the values, for \(s = \gamma\), of the Mellin transform

\[
\hat{k}(s) := \int_0^\infty u^s k(u) du / u \quad (s \in \mathbb{C})
\]

of the kernel \(k\) in the relevant Mellin-Stieltjes convolution (see e.g. [BinGT, Ch. 4, 5]). In the Hankel case of [Bin3, 7] the relevant transform is exactly of convolution type; here and in [Bin5] the ‘ultraspherical transform’ is only approximately so (cf. [BinGT, §4.2, 4.3, 4.10]). It is interesting to compare the Mellin transforms in these three cases.

Remarks.

1. **Besov paths.**

   Kerkyacharian et al. [KerOPP, §7.22] show that if the angular power spectrum satisfies \(a_n = O(1/n^{1+\gamma})\) for \(\gamma > 0\) (so \(A_n := \sum_n a_k = O(1/n^{\gamma})\)), then the sample paths of the process \(X\) are a.s. in the Besov space \(B^\infty_{\alpha,1}\) for all \(\alpha < \gamma\) (see Giné and Nickl [GinN] for the theory of Besov spaces in such contexts, Fukushima et al. [FukOT] for the necessary theory of Dirichlet structure on the index set, \(S^d\) here). Thus the faster the decay of the angular power spectrum, the smoother the paths of the process.

2. **Fractional calculus on spheres.**

   Following Askey and Wainger [AskW, Part I Section III], a theory of fractional integration and differentiation on spheres was given by Bavinck [Bav]. This is based on the expansion into spherical harmonics \(S_{l,m}\) of order \(l\) and type \(m \in \{-l, \ldots, l\}\) (see e.g. [Mul], [AndAR, Ch. 9]); these are eigenfunctions of the spherical Laplacian \(\Delta\) (Laplace-Beltrami operator on the sphere), with eigenvalues \(-l(l + \lambda)\) (or \(-l(l + \alpha + \beta + 1)\) in the Jacobi case):

\[
\Delta S_{l,m} = -l(l + \lambda) S_{l,m}, \quad (1 - \Delta) S_{l,m} = (1 + l(l + \lambda)) S_{l,m}.
\]

In terms of the fractional Laplacian (for which see e.g. Künn and Schilling
[KuhS] and the references cited there), applying \((1 - \Delta)^{\sigma/2}\) introduces multipliers \((1 + l(l + \lambda))^{\sigma/2}\) into the expansion. For \(\sigma > 0\), this corresponds to (fractional) differentiation of order \(\sigma\) (\(\Delta\) being a second-order differential operator), or (fractional) integration if \(\sigma\) is negative (recall: the faster the angular power spectrum coefficients decay, the smoother the paths of the process, and the slower, the rougher).

This has the semigroup property

\[
I_{\alpha + \beta} = I_\alpha \circ I_\beta.
\]

This desirable property is not shared by previous definitions of spherical fractional integration (see [AskW] for references), nor by analogues in the literature on ‘dimension walks’; see e.g. [BinS2] for references.

Note that \(\lambda\) here may be a continuous parameter, and is not restricted to the half-integer values implied by \(\lambda = \frac{1}{2}(d - 1)\) with \(d\) the dimension of the sphere (as a Riemannian manifold). In §4 below, we will deal with two such continuous parameters \(\nu < \lambda\), corresponding when both are half-integers to projection from a higher-dimensional to a lower-dimensional sphere.

§3. Integrability and path-continuity

The question of path-continuity of the process is addressed in the work of Lang and Schwab [LangS] (cf. [AndL]) and Lan, Marinucci and Xiao [LanMX]. The picture is much as above: the faster the decay of the angular power spectrum, the better: the more regular the paths of the process (and, as in §2, the faster the decay of the incremental variance at the origin).

In [LangS, §4, Assumption 4.1], Lang and Schwab assume a decay condition on the angular power spectrum measured by a summability condition (rather than by rate of decay as in §2): in our notation, they assume

\[
\sum a_n n^\gamma < \infty \quad (\gamma > 0).
\]  

\[(Int)\]

In view of \((\ast)\) above, we re-write this by partial summation as

\[
\sum A_n n^{\gamma - 1} < \infty : \quad \sum (n + \alpha + 1) A_n n^{\gamma - 2} < \infty.
\]

As in §2, and in [LangS §4], the case \(\gamma \in (0, 2)\) is specially important, so we begin with that. Then the summability condition \((Int)\) may [Bin6, Th. 1] be translated into a corresponding integrability condition on the incremental
variance at the origin: \((Int)\) implies
\[
\int_{0+}^{\pi/2} I(\cos \theta) \theta^{-\gamma} d\theta / \theta < \infty.
\] (Int')

As \(\int_{0+} d\theta / \theta\) diverges, this gives in particular that
\[
I(\cos \theta) = o(\theta^\gamma) \quad (\theta \downarrow 0).
\]

This strengthens the result of [LangS, Lemma 2] from \(O(.)\) to \(o(.)\) (though in view of the ‘\(\epsilon\)-gap’ in [LangS, Th. 4.7], where it is used, this does not matter).

This leads quickly to the path-regularity result ([LangS]; cf. [LanMX]):

**Theorem (Lang and Schwab, [LangS Th. 4.7]).** Under the summability condition \((Int)\) on the angular power spectrum, for any \(\delta < \gamma/2\) the process has a \(C^\delta\)-valued modification: for \(k\) the integer part of \(\gamma/2\), the modification is \(k\) times continuously differentiable, with \(k\)th derivative Hölder continuous with exponent \(\delta - k\).

The proof involves the following:

(i) For \(n \in \mathbb{N}, \, x, y \in S^d\)
\[
E[|X(x) - X(y)|^{2n}] \leq C_{\gamma, n} d(x, y)^{\gamma n},
\]
with \(d(\ldots)\) geodesic distance as before [LangS, Lemma 4.3].

(ii) The Kolmogorov-Chentsov theorem on manifolds [AndL] gives the result for \(\gamma \in (0, 2]\).

(iii) For \(\gamma > 2\), \(k\)-fold fractional differentiation (see §3 Remark 2) reduces to the range above.

We refer for full detail to [LangS], [AndL]. □

**Remarks.**

1. **Sphere cross line: spatio-temporal random fields on spheres.**

   In practice, we observe not a static field on the sphere but a dynamic one, evolving in time. This motivates the study of Gaussian processes on the sphere-cross-line, \(S^d \times \mathbb{R}\), where the real line models the temporal component. It turns out that the covariances of these spatio-temporal fields have a very
similar structure to the Bochner-Schoenberg form \((B-S)\) in the spatial case,

\[
c \sum_{0}^{\infty} a_n P_n^\lambda(x), \quad c > 0, \ a_n \geq 0, \ \sum a_n = 1
\]

(to within a constant multiple, a mixture of ultraspherical polynomials; see e.g. [BinS1]), but now with

\[
c \sum_{0}^{\infty} a_n P_n^\lambda(x) \phi_n(t), \quad c > 0, \ a_n \geq 0, \ \sum a_n = 1,
\]

with the \(\phi_n\) characteristic functions of probability laws on the line (Berg-Porcu/Ma theorem; see [BerP], [Ma2], [BinS1]). Many results proved for the class \(\mathcal{P}(S^d)\) thus carry over to \(\mathcal{P}(S^d \times \mathbb{R})\).

Interest in path properties of Gaussian processes \(S^d \times \mathbb{R}\) is still developing; see [ClaAP] for an early contribution. Using the framework of [LangS], [ClaAP] describe regularity properties of spatio-temporal processes using Sobolev and interpolation spaces, aided by two new spectral expansions for spatio-temporal processes: a double Karhunen-Loève expansion and a decomposition of the \(\phi_n(t)\) above into their Hermite expansions \(\sum_k b_{n,k} H_k(t)\), with \(H_k\) the normalised Hermite polynomial of degree \(k\).

2. **Strong local non-determinism.**

Using the concept of strong local determinism, Lan, Marinucci and Xiao [LanMX] improve the Lang-Schwab result above, obtaining an exact modulus of continuity (and so avoiding an \(\epsilon\)-gap), for the case \(d = 2\) and with the angular power spectrum coefficients bounded above and below by constant multiples of powers. This condition holds, for example, for spherical fractional Brownian motion (Lan and Xiao [LanX]).

3. **Vector data.**

Often data on spheres are vectors, as several different quantities are measured (temperature, wind speed, humidity etc.); the relevant covariances are then matrices. See e.g. [Ma1], where a number of applications are given.

4. **Statistics.**

One extremely important application for the theory of Gaussian random fields on spheres is of course the study of cosmic microwave background (CMB) radiation; see [MariP] for a monograph treatment. For statistical estimation in this and related areas, see e.g. Durastanti, Lan and Marinucci [DurLM], Leonenko, Taqqu and Terdik [LeoTT].
5. *Stochastic partial differential equations (SPDEs).*
   For the stochastic heat equation on the sphere, see Lang and Schwab [LangS, §7].

6. *Regular variation and function spaces.*
   To avoid the ‘$\epsilon$-gap’ in the Lang-Schwab theorem above, one needs a finer scale of spaces than is provided by the powers (in particular, one that is sensitive to logarithmic factors, etc.) One such is provided by the *Orlicz spaces*; see e.g. Krasnoselkii and Rutickii [KraR].

7. *Integrability theorems for Fourier series.*
   Much is known about integrability conditions for Fourier series. For detail, see the two monographs on the subject, by Boas [Boa] and Yong [Yon] (as well as [Bin6] and the references cited there).

§4. *Projections*
The spheres $S^d$ in Euclidean space are of course ordered by dimension, corresponding to projection from a higher dimension to a lower one. Correspondingly, the classes $\mathcal{P}(S^d)$ are similarly ordered but with the inclusions reversed:

$$\mathcal{P}(S^1) \supset \mathcal{P}(S^2) \supset \ldots \supset \mathcal{P}(S^d) \supset \mathcal{P}(S^\infty),$$

with all inclusions strict, and (Schoenberg [Sch]; [Bin4]; Gneiting [Gne])

$$\bigcap_{1}^{\infty} \mathcal{P}(S^d) = \mathcal{P}(S^d).$$

The Poisson kernel for the Jacobi polynomials reduces in the ultraspherical case to the generating function

$$\sum_{0}^{\infty} \omega_n r x W_n(x) = (1 - r^2)/(1 - 2rx + x^2)^{\nu+1} \quad (r \in (-1, 1)). \quad (GF)$$

See Bailey [Bai, 102], Watson [Wat2], [Bin1, (2.1)] (this is not the usual generating function for the ultraspherical polynomials [Sze, (4.7.23) p.83]).

Askey and Fitch [AskF] showed that for $x, y \in [-1, 1], r \in (-1, 1), 0 \leq \nu < \lambda \leq \infty$, the series

$$\sum_{n=0}^{\infty} \omega_n^{\nu} r^n W_n^\lambda(x) W_n^\nu(y) \quad (AF - r)$$
converges to a non-negative sum-function, which leads to a corresponding probability measure \( M^\lambda_\nu(x) \) satisfying

\[
W^\lambda_n(x) = \int_{-1}^{1} W^n_\nu M^\lambda_\nu(x; dy) \quad (n = 0, 1, 2, \ldots).
\]

Here [Bin1] we may take \( 0 \leq \nu \leq \lambda \leq \infty, x \in [-1,1] \). Of course, some cases give Dirac laws: if \( x = \pm 1 \), \( M^\lambda_\nu(\pm 1) = \delta_\pm 1 \) (as \( W^\lambda_n(\pm 1) = (\pm 1)^n \)). If \( \lambda = \nu \), then \( M^\lambda_\nu(x) = \delta_x \) (as there is no projection to be done); so we may now restrict to \( \nu < \lambda \) as before. The first author [Bin1, Lemma 1] showed that one may carry out the Abel-limit operation here explicitly: for \( x, y \in (-1,1) \), one may take \( r = 1 \) here to get

\[
m^\lambda_\nu(x; y) := \sum_{n=0}^{\infty} \omega^n_\nu W^\lambda_n(x)W^n_\nu(y) \geq 0 \quad (AF - 1)
\]
a non-negative function in \( L_1(G_\nu) \), finite-valued unless \( x = y \) and \( \nu < \lambda \leq \nu + 1 \) (this exception is of no importance here, as all integrals are Lebesgue).

It is in fact the Radon-Nikodym derivative \( dM^\lambda_\nu(x; dy)/G_\nu(dy) \):

\[
M^\lambda_\nu(x; dy) = G_\nu(dy).m^\lambda_\nu(x; y) = G_\nu(dx) \sum_{n=0}^{\infty} \omega^n_\nu W^\lambda_n(x)W^n_\nu(y). \quad (RN)
\]

There is no explicit formula for the series \( (AF - 1) \); one needs to evaluate it numerically. But the series itself is ill-suited for this purpose, in view of the wild oscillation shown by polynomials of high degree. Accordingly, our purpose here is to obtain an alternative form, as a double integral, which is much more tractable numerically. This also shows the dependence on the higher index, \( \lambda \), in a more convenient and structurally revealing way.

Following [Bin1], for \( \lambda > \nu \) as above write \( H^\lambda_\nu \) for the probability measure of Beta type on \([0,1]\) given by the Sonine law

\[
H^\lambda_\nu(dx) := \frac{2\Gamma(\lambda + \frac{1}{2})}{\Gamma(\nu + \frac{1}{2})\Gamma(\lambda - \nu)} x^{2\nu}(1 - x^2)^{\lambda - \nu - \frac{1}{2}} dx.
\]

This occurs in Sonine’s first finite integral for the Bessel function (Sonine, 1880; [Wat1, 373]): for

\[
\Lambda_\mu(t) := \Gamma(\nu + 1)J_\nu(t)/(\frac{1}{2} t)^\mu,
\]
\[ \Lambda_{\lambda - \frac{1}{2}}(t) = \int_0^1 \Lambda_{\nu - \frac{1}{2}}(ut)H_\nu^\lambda(du) \]  

(S)

(the drop by a half-integer in parameter here reflects the drop in dimension in \( S^d \subset \mathbb{R}^{d+1} \); see Remark 2 below).

To cope with the product of \( W_n \) terms in \((AF-1)\), we need \textit{Gegenbauer's multiplication theorem} for the ultraspherical polynomials (Gegenbauer, 1874; [Wat1, 369], [Vil2, IX.4 p.474]),

\[ W_\nu^n(x)W_\nu^n(y) = \int_{-1}^1 W_\nu^n(xy + \sigma \sqrt{1-x^2} \sqrt{1-y^2})G_{\nu - \frac{1}{2}}(d\sigma). \]  

(G)

To cope with the drop in index (dimension) in \((AF-1)\), we need the \textit{Feldheim-Vilenkin integral} ([Bin1, (2.11)]; [Fel], [Vil1], [AskF]),

\[ W_\lambda^n(x) = \int_0^1 \frac{[x^2 - x^2u^2 + u^2]^{\nu n}}{\sqrt{x^2 - x^2u^2 + u^2}}du. \]  

(FV)

\textbf{Theorem.} (i) For \( r \in (-1,1) \), the sum of the Askey-Fitch series \((AF-1)\) above is given by the integral (**) below:

\[ \int_0^1 H_\nu^\lambda(du) \int_{-1}^1 G_{\nu - \frac{1}{2}}(dv) \left[ 1 - r^2(x^2 - x^2u^2 + u^2) \right] \nu^{\nu+1}, \]  

(**)

where \( I \) is given by

\[ I := 1 - 2r. \frac{xy + uv \sqrt{1-x^2} \sqrt{1-y^2}}{\sqrt{x^2 - x^2u^2 + u^2}} + \frac{(xy + uv \sqrt{1-x^2} \sqrt{1-y^2})^2}{(x^2 - x^2u^2 + u^2)^2}. \]

(ii) This holds also for \( r = 1 \) unless \( \mu < \lambda \leq \mu + 1 \).

\textbf{Proof.} We sum the series by reducing it to the generating function \((GF)\). There are two steps: reduction of \( \lambda \) to \( \mu \) by the Feldheim-Vilenkin integral \((FV)\), and reduction of two \( W_n \) terms to one by Gegenbauer’s multiplication theorem \((G)\).

We follow [Bin1]. As there, we may substitute for \( W_\lambda^n(x) \) from \((FV)\) into the series \((AF-1)\) and integrate term-wise, to re-write \((AF-1)\) as

\[ \int_0^1 H_\nu^\lambda(du) \sum_0^\infty \omega_n(r[x^2 - x^2u^2 + u^2]^{\frac{1}{2}})^nW_\nu^n(y)W_\nu^n \left( \frac{x}{\sqrt{x^2 - x^2u^2 + u^2}} \right). \]

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We use \((G)\) with 
\[ r \mapsto r \sqrt{x^2 - x^2 u^2 + u^2}. \]
and replace the product of \(W_{\nu}^n\) factors in the above, at the cost of another integration over \(G_{\nu - \frac{1}{2}}(dv)\), by a single \(W_{\nu}^n\) term, with argument
\[
\frac{xy}{\sqrt{x^2 - x^2 u^2 + y^2}} + v \sqrt{1 - y^2} \sqrt{1 - \frac{x^2}{x^2 - x^2 u^2 + u^2}} = \frac{xy + uv \sqrt{1 - x^2} \sqrt{1 - y^2}}{\sqrt{x^2 - x^2 u^2 + u^2}}.
\]
The result now follows from \((GF)\). \(\square\)

**Remarks.**

1. **Symmetric spaces and spherical functions.**
   The \(d\)-sphere \(S^d\) may be identified with the homogeneous space \(SO(d + 1)/SO(d)\), and is the prime example of a compact symmetric space of rank 1 (constant positive curvature). For background on symmetric spaces, including their classification, see e.g. [Hel]. The ultraspherical polynomials are (or may be identified with) the spherical functions (again, see [Hel]). For their probabilistic relevance, see e.g. [Bin2], [BinMS] and the references cited there. The Bochner-Schoenberg theorem of §1 is a special case of the Bochner-Godement theorem ([AskB], [BinS1]).

2. **Multiplication theorems and hypergroups.**
   Gegenbauer’s multiplication theorem for \(W_{\nu}^n\), \((G)\) above, is a special case of Harish-Chandra’s formula for the spherical function in the case of a symmetric space of rank one and compact type. It plays a key role in the theory of random walks on spheres [Bin2]. The relevant operation on measures gives a hypergroup [BloH], the Bingham (or Bingham-Gegenbauer) hypergroup.
   
   The analytic and probabilistic similarities between this setting and that of Kingman’s random walks with spherical symmetry [Kin] reflect the similarity between the rank-1 symmetric spaces of constant positive curvature (compact type) and those of zero curvature (Euclidean type). The relevant hypergroup there is the Kingman (or Kingman-Bessel) hypergroup.
   
   Gegenbauer’s multiplication theorem for Bessel functions (Gegenbauer, 1875, [Wat1, 367] – specialising from cylinder to Bessel functions [Wat1, §3.9]; [Kin, Proof, Th. 1]) is also a special case of Harish-Chandra’s formula, for this setting.

3. **Dimension walks.**
   The Feldheim-Vilenkin integral is relevant to a projection (drop in dimension) from a sphere of higher to one of lower dimension. This relates to
recent literature on ‘dimension walks’; see e.g. [BinS2] and the references cited there.

4. Projections again.

The Sonine law is relevant both to projections between spheres and to projections between Euclidean spaces preserving spherical symmetry. It was re-derived recently by Daley and Porcu [DalP, Th. 4], unaware of the previous work of Sonine, Watson and Kingman here.

Acknowledgements. The second author acknowledges financial support from the EPSRC Centre for Doctoral Training in the Mathematics of Planet Earth [EP/L016613/1]. Both authors thank Adam Ostaszewski for discussions, and the editors for the invitation to contribute to the Larry Shepp Memorial Issue.

Postscript. It is a pleasure for the first author to record here his happy memories of all his dealings with Larry Shepp, and of the excellent conference in his memory at Rice University, 25-29 June 2018, so ably organised by Philip Ernst. The title of his talk there was ‘Four themes from the work of Larry Shepp’, the first of which was Gaussian processes, as here.

It is also a pleasure for both authors to record here the great debt that they and their colleagues owe to the work of Larry Shepp.

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