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C++ CODES OF IMPLICIT LU ALGORITHMS
FOR ABSDL01

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C++ CODES OF IMPLICIT LU ALGORITHMS FOR ABSDLL01

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Abstract: This report is devoted to some C++ codes implementing the implicit LU class algorithms for solving linear determined, and undetermined systems with \( n \) variables and \( m \) equations. A main program used in part of the numerical test is given in the last section.

Key words: ABS methods, Abaffian matrix, linearly system of equations, implicit LU algorithm, pivot, VC++, C++.

AMS Subject Classification (2000): 65F10, 68Q10, 90C30

1 Introduction

ABS methods were introduced by Abaffy, Broyden and Spedicato (1982/84) \cite{AbBS 82}, \cite{AbBS 84} originally for solving systems of linear equations. The basic ABS class was later generalized to the so-called scaled ABS class and, subsequently, applied to linear-squares, nonlinear equations and optimization problems, see for instance, Abaffy and Spedicato \cite{AbSp 89}, Spedicato \cite{Sped 97}, \cite{Sped 99}, Spedicato, Xia Z. and Zhang L. \cite{SpXZ 00} and Zhang L., Xia Z. and Feng E. \cite{ZhXF 99}. In this paper we first review the general scheme for solving linear determined or undetermined systems.

Let us consider the general linear systems, where rank(\( A \)) is arbitrary,

\[ Ax = b, \quad A \in \mathbb{R}^{m \times n} \]

or

\[ a_i^T x = b_i, \quad i = 1, \cdots, m \]

where

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The steps of the unscaled ABS class of algorithms are defined as follows:

**Basic (Unscaled) ABS Class of Algorithms:** [AbBS 84], Algorithm 1 [AbSp 89]: pp. 21-22

**Begin Algorithm**

(A) Initialization.

Give an arbitrary vector \( x_1 \in \mathbb{R}^n \), and an arbitrarily nonsingular matrix \( H_1 \in \mathbb{R}^{n,n} \).

Set \( i = 1 \) and iflag=0.

(B) Computer two quantities.

Compute

\[
\begin{align*}
    s_i &= H_i a_i \\
    \tau_i &= \tau^T e_i = a_i^T x_i - b^T e_i
\end{align*}
\]

(C) Check the compatibility of the system of linear equations.

If \( s_i \neq 0 \) then goto (D).

If \( s_i = 0 \) and \( \tau_i = 0 \) then set

\[
\begin{align*}
    x_{i+1} &= x_i \\
    H_{i+1} &= H_i
\end{align*}
\]

and goto (F), the \( i \)-th equation is a linear combination of the previous equations.

Otherwise stop, the system has no solution.

(D) Computer the search vector \( p_i \in \mathbb{R}^n \) by

\[
p_i = H_i^T z_i \tag{1.1}
\]

where \( z_i \), the parameter of Broyden, is arbitrary save that

\[
z_i^T H_i a_i \neq 0 \tag{1.2}
\]

(E) Update the approximation of the solution \( x_i \) by

\[
x_{i+1} = x_i - \alpha_i p_i \tag{1.3}
\]
where the stepsize $\alpha_i$ is computed by

$$\alpha_i = \tau_i / a_i^T p_i$$  \hspace{1cm} (1.4)$$

If $i = m$ stop; $x_{m+1}$ solves the system.

(F) Update the (Abaffian) matrix $H_i$. Compute

$$H_{i+1} = H_i - H_ia_iw_i^T H_i/w_i^T H_i a_i$$  \hspace{1cm} (1.5)$$

where $w_i \in \mathbb{R}^n$, the parameter of Abaffy, is arbitrary save for the condition

$$w_i^T H_i a_i = 1 \text{ or } \neq 0$$  \hspace{1cm} (1.6)$$

(G) Increment the index $i$ by one and goto (B).

We define $n$ by $i$ matrices $A_i$, $W_i$ and $P_i$ by

$$A_i = (a_1, \cdots, a_i)^T, \quad W_i = (w_1, \cdots, w_i), \quad P_i = (p_1, \cdots, p_i)$$  \hspace{1cm} (1.7)$$

End Algorithm

Some properties of the above recursion, see for instance, Abaffy and Spedicato (1989), [AbSp 89], are listed below that are the basic formulae for use later on.

a. Implicit factorization property

$$A_i^T P_i = L_i$$  \hspace{1cm} (1.8)$$

with $L_i$ nonsingular lower triangular.

b. Null space characterizations

$$\mathcal{N}(H_{i+1}) = \mathcal{R}(A_i^T), \quad \mathcal{N}(H_{i+1}^T) = \mathcal{R}(W_i),$$  \hspace{1cm} (1.9)$$

$$\mathcal{N}(A_i) = \mathcal{R}(H_{i+1}^T)$$

where $\mathcal{N}=\text{Null}$ and $\mathcal{R}=\text{Range}.$

c. The linear variety containing all solutions to $Ax = b$ consists of the vectors of the form

$$x = x_{t+1} + H_{t+1}^T q$$  \hspace{1cm} (1.10)$$

where $q \in \mathbb{R}^n$ is arbitrary.
Much progress on the computational aspect of ABS algorithms have been made since the ABS algorithms were found in the early 1980’s. In the last several years lots of work has been done on enlarging, improving and completing ABSPACK that is a package of ABS algorithms in FORTRAN codes due to Spedicato and his collaborators [Bodo 89], [Bodo 90a], [Bodo 00a], [Bodo 00b], [Bodo 01a], [BoLS 00a], [BPLS 00a], [BPLS 00b], [BPLS 01]. In the last two years we investigated possibility of establishing C++/VC++ ABS software, i.e. ABS software written in C++/VC++. Part of work in the subject is presented in this paper concerning the implicit LU algorithms.

In this report a text of some codes implementing the implicit LU algorithms for solving linear determined and undetermined systems with \( n \) variables and \( m \) equations is given. The codes are written in C++ language using Matrix class (with CMatrix as its class name) and Vector class (with CVector as its class name) that are made by ourselves for constructing the ABS software. The paper is organized based on the following three algorithms:

1. Function \( iLUA \): the implicit LU method, where \( A \) is regular (i.e. all principal submatrices are nonsingular), without pivoting for determined and undetermined systems.

2. Function \( iLUA_{\text{PivotC}} \): the implicit LU method with column pivoting and explicit column interchanges for linear determined and undetermined systems. Since the column pivot cause that the ordering of components of solution is changed, it is necessary to recover the ordering of components of solution and related codes are given.

3. Function \( iLUA_{\text{PivotR}} \): the implicit LU method with row pivoting and explicit row interchanges for determined and undetermined systems.

In the next section three schemes of implicit LU algorithms and some general properties are given. Codes implementing these algorithms are listed in section 3. Finally a main program to test algorithms is presented.

2 Implicit LU Algorithms

2.1 Schemes of implicit LU algorithms

We give a short description of three versions of the implicit LU algorithms below:

**Implicit LU Algorithm Without Pivoting (iLUA):** \([\text{AbBS 82}], [\text{AbBS 84}], [\text{Sped 01}]\)

(iLUA is abbreviation for implicit LU algorithm without pivoting, under the assumption
of $A$ is regular

Set $x_1 = 0$, $H_1 = I$ and $i = 1$

For $i = 1$ to $m$ do

Set $s_i = H_ia_i$

$\quad d_i = s_i^T e_i$

$\quad x_{i+1} = x_i - (a_i^T x_i - b_i)/d_i H_i^T e_i$

if $i \leq m$, then set

$\quad H_{i+1} = H_i - s_i e_i^T H_i/d_i$

endo

**Implicit LU Algorithm With Column Pivoting (iLUaPivotC):** [AbSp 89], [Bodo 00a]

(iLUaPivotC is the abbreviation of "implicit LU algorithm with column pivoting" and column interchanges, without the regularity of $A$. The ordering of components of solution is changed because of the column pivot)

Set $x_1 = 0$, $H_1 = I$ and $i = 1$

For $i = 1$ to $m$ do

Set $s_i = H_ia_i$

(only $(i - 1)(n - i + 1)$ nonzero elements of $H_i$ are used)

Determine $d_i = |s_i^T e_k|$, such that $|s_i^T e_k| = \max\{|s_i^T e_j| \mid j = i, \cdots, n\}$

(only $n - i + 1$ nonzero elements of $s_i$ are used)

If $k_i \neq i$, then swap columns of $A$ and elements of $x_i$ and $s_i$ with these indices

Set $x_{i+1} = x_i - ((a_i^T x_i - b_i)/d_i) H_i^T e_i$

(only $i$ nonzero elements of $x_i$ are updated)

if $i \leq m$, then set $H_{i+1} = H_i - s_i e_i^T H_i/d_i$

(only $i(n - i)$ nonzero elements of $H_{i+1}$ are updated)

endo

**Implicit LU Algorithm With Row Pivoting (iLUaPivotR):**

(iLUaPivotR stands for implicit LU algorithm with row pivoting and explicit row interchanges, without the regularity of $A$ and change of components of solution vector)

Set $x_1 = 0$, $H_1 = I$ and $i = 1$

For $i = 1$ to $m$ do

Determine $d_i = |p_i^T a_k|$, such that $|p_i^T a_k| = \max\{|p_i^T a_j| \mid j = i, \cdots, n\}$

If $k_i \neq i$, then swap rows of $A$ and elements of $x_i$ and $b$ with these indices

Set $s_i = H_ia_i$
(only $(i - 1)(n - i + 1)$ nonzero elements of $H_i$ are used)
Set $x_{i+1} = x_i - ((a_i^T x_i - b_i)/d_i) H_i^T e_i$
(only $i$ nonzero elements of $x_i$ are updated)
if $i \leq m$, then set $H_{i+1} = H_i - s_i e_i^T H_i/d_i$
(only $i(n - i)$ nonzero elements of $H_{i+1}$ are updated)
enddo

2.2 Properties of general implicit LU algorithms

The general implicit LU algorithm is obtained by the parameter choices $H_1 = I$, $z_i = e_i$, $w_i = e_i$. Some properties of this class of algorithms is listed as followings:

(a) The algorithm is well defined iff $A$ is regular (i.e., all principal submatrices are nonsingular). Otherwise pivoting has to be performed.

(b) Since $W_i^T H_{i+1} = [I_i, 0]^T H_{i+1} = 0$, the first $i$ rows of the Abaffian matrix $H_{i+1}$ must be zero. More precisely, the Abaffian matrix has the following structure, with $K_i \in \mathbb{R}^{n-i,i}$

$$H_{i+1} = \begin{bmatrix} 0 & 0 \\ K_i & I_{n-i} \end{bmatrix}$$

(c) Only the first $i$ components of $p_i$ can be nonzero and the $i$-th is unity. Hence the matrix $P_i$ is unit upper triangular, so that the implicit factorization $A = L P^{-1}$ is of the LU type, with unit on the diagonal.

(d) Only $K_i$ has to be updated. The algorithm requires $nm^2 - 2m^3$ multiplications plus lower-order terms. Hence, for $m = n$ there are $n^3/3$ multiplications plus low-order terms, which is the same cost as for the classical LU factorization or Gaussian elimination (which are two essentially equivalent process).

3 Codes of Implicit LU Algorithms

3.1 iLUa code

```
function BOOL iLUa(CMatrix m_A, CVector v_b, CVector &v_x, double ep1, double ep2)
```
Under the condition of regularity of $A$, iLUa determines the solution of the linear systems $Ax = b$ ($A$ with dimension $m \times n$, $m \leq n$), using implicit LU algorithm without pivoting. The type of return value of function is BOOL, true if system has a solution, otherwise the return value is false.

$m_A$ = an object of CMatrix class, denote coefficient matrix
$v_b$ = an object of CVector class, denote right hand-side vector
$v_x$ = an object of CVector class, denote solution vector

(using reference operator to output the solution vector)

$ep1$ = a double type value of dependency control parameter
$ep2$ = a double type value of residual control parameter

{
    try
    {
        CVector v_x1;
        // declare an object of CVector class
        v_x1=v_x;

        CVector v_s;
        // declare an object of CVector class, denotes the vector $s_i = H_i a_i$
        CMatrix m_H(1.0,m_A.m_cols);
        // declare an object of CMatrix class, denotes the Abaffian matrix,
        // the initial matrix is unit matrix
        double ns;
        // declare a double precision number, denotes $||p||$, the norm of $p$
        double r;
        // declare a double precision number, $r_i = a_i^T x_i - b_i$
        int iflag=0;
        int i=1;

        // iteration
        while(i<=m_A.m_rows)
        {
        }
// compute $s_i = H_i a_i$
if (i == 1)
    v_s = m_A.GetRow(i).Trans();
else
{
    v_s[i-1] = 0.0;
    for (int i1 = i; i1 <= m_A.m_cols; i1++)
    {
        double sum1 = 0.0;
        for (int j1 = 1; j1 < i; j1++)
            sum1 = sum1 + m_H.GetValue(i1, j1) * m_A.GetValue(i, j1);
        v_s[i1] = sum1 + m_A.GetValue(i, i1);
    }
}

r = Dot(m_A.GetRow(i), v_x1) - v_b[i];
ns = v_s.Module();
if (ns <= ep1)
{
    if (fabs(r) <= ep2)
        // $i$-th row is linearly dependent with first $i-1$ rows
        {
            iflag = iflag + 1;
            i = i + 1;
        }
    else
    {
        // system is incompatible
        iflag = -i;
        AfxMessageBox("No Solution");
        break;
    }
}
else
{
}

// update solution $x_i$
double temp=0;
temp=r/v_s[i];
for(int i2=1;i2<=i;i2++)
    v_x1[i2]=v_x1[i2]-temp*m_H.GetValue(i,i2);

// update projection matrix $H_i$
if(i<m_A.m_cols)
{
    for(int i3=i+1;i3<=m_A.m_cols;i3++)
    {
        for(int j3=1;j3<=i;j3++)
        {
            double tt;
            tt=(v_s[i3]*m_H.GetValue(i,j3))/v_s[i];
            tt=m_H.GetValue(i3,j3)-tt;
            m_H.SetValue(i3,j3,tt);
        }
    }
    for(i3=1;i3<=i;i3++)
        m_H.SetValue(i,i3,0.0);
    i++;
}

if(iflag>=0)
{
    v_x=v_x1;
    return(true);
}
else
    return(false);

}
AfxMessageBox(e->GetErrorInfo());
e->Delete();
exit(1);}
}

3.2 iLUaPivotC code

function BOOL iLUaPivotC(CMatrix m_A, CVector v_b, CVector &v_x, double ep1, 
 double ep2)

// iLUaPivotC determines the solution of the linear systems \( Ax = b \) (\( A \) with
// dimension \( m \times n \), \( m \leq n \) ) using implicit LU algorithm with column pivoting
// and explicit column interchanges, and the order of components solution
// vector are changed
// the type of return value of function is BOOL, true if system has a solution
// otherwise the return value is false

// \( m_A \) = an object of CMatrix class, denote coefficient matrix
// \( v_b \) = an object of CVector class, denote right hand-side vector
// \( v_x \) = an object of CVector class, denote solution vector
// (using reference operator to output the solution vector)
// \( \text{ep1} \) = double type value of dependency control parameter
// \( \text{ep2} \) = double type value of residual control parameter

{
    try
    {
        CVector v_x1;
        // declare an object of CVector class
        v_x1=v_x;

        CVector v_s;
        // declare an object of CVector class, denotes the vector \( s_i = H_i a_i \)
        CMatrix m_H(1.0,m_A.m_cols);
        // declare an object of CMatrix class, denotes the Abaffian matrix,
// the initial matrix is unit matrix
int ki;
// declare an integer number, denotes the ordering of pivot
double r;
int iflag=0;
double d;

int* Index;
// declare a pointer to type integer
Index=new int[m_A.m_cols];
// use new operator to allocate a pointer to an array of integer with
// dimension m_A.m_cols
for(int ii=1;ii<=m_A.m_cols;ii++)
    Index[ii]=ii;
// initialize the array of integer
int i=1;

// iteration
while(i<=m_A.m_rows)
{
    // compute $s_i = H_i a_i$
    if(i==1)
        v_s=m_A.GetRow(i).Trans();
    else
    {
        v_s[i-1]=0.0;
        for(int i1=i;i1<=m_A.m_cols;i1++)
        {
            double sum1=0.0;
            for(int j1=1;j1<i;j1++)
                sum1=sum1+m_H.GetValue(i1,j1)*m_A.GetValue(i,j1);
            v_s[i1]=sum1+m_A.GetValue(i,i1);
        }
    }
}
// pivoting

d=-1;
for(int j2=i;j2<=m_A.m_cols;j2++)
{
    if(d<fabs(v_s[j2]))
    {
        d=fabs(v_s[j2]);
        ki=j2;
    }
}

if(ki!=i)
// swap i-th column and k_i-th column of A
// swap i-th component and k_i-th component of x_i and s_i
// save ordering of pivot in Index
{
    m_A=m_A.SwapCol(ki,i);
    v_x=v_x.SwapElement(ki,i);
    v_s=v_s.SwapElement(ki,i);
    int it;
    it=Index[i];
    Index[i]=Index[ki];
    Index[ki]=it;
}

r=Dot(m_A.GetRow(i),v_x1)-v_b[i];
if(fabs(v_s[i])<=ep1)
{
    if(fabs(r)<=ep2)
    // i-th row is linearly dependent with first i−1 rows
    {
        iflag=iflag+1;
        i=i+1;
    }
    else

// system is incompatible
{
    iflag=-i;
    AfxMessageBox("No Solution");
    break;
}
}
else
{
    // update solution $x_i$
    double temp=0;
    temp=r/v_s[i];
    for(int i2=1;i2<=i;i2++)
        v_x1[i2]=v_x1[i2]-temp*m_H.GetValue(i,i2);
    // update projection matrix $H_i$
    if(i<m_A.m_cols)
    {
        for(int i3=i+1;i3<=m_A.m_cols;i3++)
        {
            for(int j3=1;j3<=i;j3++)
            {
                double tt;
                tt=(v_s[i3]*m_H.GetValue(i,j3))/v_s[i];
                tt=m_H.GetValue(i3,j3)-tt;
                m_H.SetValue(i3,j3,tt);
            }
            for(i3=1;i3<=i;i3++)
                m_H.SetValue(i,i3,0.0);
        }
        i++;
    }
    if(iflag>=0)
// exchange components of solution to adapt the original system
{
    for(int i4=1;i4<=m_A.m_cols;i4++)
    {
        for(int j4=1;j4<=m_A.m_cols;j4++)
        {
            if(Index[j4]==i4)
                v_x[i4]=v_x1[j4];
        }
    }
    return(true);
}
else
    return(false);
}
catch(CErrorException *e)
{
    AfxMessageBox(e->GetErrorInfo());
    e->Delete();
    exit(1);
}

3.3 iLUaPivotR code

function BOOL iLUaPivotR(CMatrix &m_A, CVector &v_b, CVector &v_x, double ep1,
                          double ep2)

    // iLUaPivotC determines the solution of the linear systems $Ax = b$ ($A$ with
    // dimension $m \times n$, $m \leq n$) using implicit LU algorithm with column pivoting
    // and explicit column interchanges
    // the type of return value of function is BOOL true if system has a solution
    // otherwise the return value is false

    // m_A = an object of CMatrix class, denote coefficient matrix
// v_b = an object of CVector class, denote right hand-side vector  
// v_x = an object of CVector class, denote solution vector (using reference operator to output the solution vector)  
// ep1 = double type value of dependency control parameter  
// ep2 = double type value of residual control parameter

{
    try
    {
        CVector v_x1;  
        // declare an object of CVector class
        v_x1=v_x;

        CVector v_s;
        // declare an object of CVector class, denotes the vector s_i = H_i a_i
        double r;
        // declare a double precision number, r_i = a_i^T x_i - b_i

        CMatrix m_H(1.0,m_A.m_cols);
        // declare an object of CMatrix class, denotes the Abaffian matrix, the initial matrix is unit matrix
        int ki;
        // declare an integer number, denotes the ordering of pivot
        double d;
        int iflag=0;
        int i=1;

        // iteration
        while(i<=m_A.m_rows)
        {
            // pivoting
            double mpt=0
            d=-1;
            for(int j2=i;j2<=m_A.m_rows;j2++)
{
    mpt = Dot(m_H.GetRow(i), m_A.GetRow(j2));
    if (d < fabs(mpt))
    {
        d = fabs(mpt);
        ki = j2;
    }
}

if (ki != i)
    // swap i-th row and k_i-th row of A
    // swap i-th component and k_i-th component of x_i and s_i
    // save ordering of pivot in Index
{
    m_A = m_A.SwapRow(ki, i);
    v_x = v_x.SwapElement(ki, i);
    v_b = v_b.SwapElement(ki, i);
}

mpt = Dot(m_H.GetRow(i), m_A.GetRow(i));
// compute s_i = H_i a_i
if (i == 1)
    v_s = m_A.GetRow(i).Trans();
else
{
    v_s[i-1] = 0.0;
    for (int i1 = i; i1 <= m_A.m_cols; i1++)
    {
        double sum1 = 0.0;
        for (int j1 = 1; j1 < i; j1++)
            sum1 = sum1 + m_H.GetValue(i1, j1) * m_A.GetValue(i, j1);
        v_s[i1] = sum1 + m_A.GetValue(i, i1);
    }
}

r = Dot(m_A.GetRow(i), v_x1) - v_b[i];
if(fabs(mpt)<=ep1)
{
  if(fabs(r)<=ep2)
  // i-th row is linearly dependent with first \( i-1 \) rows
  {
    iflag=iflag+1;
    i=i+1;
  }
  else
  // system is incompatible
  {
    iflag=-i;
    AfxMessageBox("No Solution");
    break;
  }
}
else
{
  // update solution \( s_i \)
  double temp=0;
  temp=r/mpt;
  for(int i2=1;i2<=i;i2++)
    v_x1[i2]=v_x1[i2]-temp*m_H.GetValue(i,i2);
  // update projection matrix \( H_i \)
  if(i<m_A.m_cols)
  {
    for(int i3=i+1;i3<=m_A.m_cols;i3++)
    {
      for(int j3=1;j3<=i;j3++)
      {
        double tt;
        tt=(v_s[i3]*m_H.GetValue(i,j3))/mpt;
        tt=m_H.GetValue(i3,j3)-tt;
      }
4 A Main Program for LU Algorithm

To make ABS algorithms be used more widely, we encapsulate Matrix class, Vector class and the ABS algorithms modules into the library modules ABSDL1.dll. A main program for the use of module(function) iLUaPivotC that solves linear system $Ax = b$ (with Micchelli-Fiedler matrix as coefficient matrix) is as following:

```c
#include "stdafx.h"

m_H.SetValue(i3,j3,tt);
}
}
for(i3=1;i3<=i;i3++)
    m_H.SetValue(i,i3,0.0);
}
i++;
}

if(iflag>=0)
{
    v_x=v_x1;
    return(true);
}
else
    return(false);
}
catch(CErrorException *e)
{
    AfxMessageBox(e->GetErrorInfo());
    e->Delete();
    exit(1);
}
}

4 A Main Program for LU Algorithm

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```c
#include "stdafx.h"

m_H.SetValue(i3,j3,tt);
}
}
for(i3=1;i3<=i;i3++)
    m_H.SetValue(i,i3,0.0);
}
i++;
}

if(iflag>=0)
{
    v_x=v_x1;
    return(true);
}
else
    return(false);
}
catch(CErrorException *e)
{
    AfxMessageBox(e->GetErrorInfo());
    e->Delete();
    exit(1);
}
```
#include "absalg.h"
#include "iostream.h"

void main()
{
    try
    {
        int m=1000;
        int n=1000;

        HINSTANCE hLib=AfxLoadLibrary("ABSDLL.dll");

        CMatrix a(m,n);
        CVector b(m);

        for(int i=1;i<=m;i++)
        {
            for(int j=1;j<=n;j++)
            {
                a.SetValue(i,j,abs(i-j));
            }
        }

        for(int k=1;k<=m;k++)
            b[k]=k;

        CVector xx(0.0,n);
        if(!iLUaPivotC(a,b,xx,1.0e-7,1.0e-7))
        {
            cout<<("Do you want a least-square solution?\endl");
            char c;
            cin>>c;
            if(c=='y'||c=='Y')
                // calling the modules for solving the least-squares problem
            else
exit(0);
}

else
{
    CVector r;
    r=a*xx-b;
    er=r.Module();
cout<<("errorbound=%f\n",er);

    AfxFreeLibrary(hLib);
}

}

catch(CErrorException *e)
{
    AfxMessageBox(e->GetErrorInfo());
    e->Delete();
}

 Remarks
As for numerical experiments, we have made some tests, for example, matrices with elements
\[ a_{ij} = |i - j|, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n \] (Micchelli-Fiedler matrix) and matrices with elements
\[ a_{ij} = |i - j|^2, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n. \] The results show that the algorithms are efficient. Further numerical experiments are in progress.

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