Constructible functions on Artin stacks

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1 Introduction

Let $\mathbb{K}$ be an algebraically closed field, $X$ a $\mathbb{K}$-scheme, and $X(\mathbb{K})$ the set of closed points in $X$. A constructible set $C \subseteq X(\mathbb{K})$ is a finite union of subsets $Y(\mathbb{K})$ for finite type $\mathbb{K}$-subschemes $Y$ in $X$. A constructible function $f : X(\mathbb{K}) \to \mathbb{Q}$ has $f(X(\mathbb{K}))$ finite and $f^{-1}(c)$ constructible for all $0 \neq c \in f(X(\mathbb{K}))$. Write $CF(X)$ for the $\mathbb{Q}$-vector space of constructible functions on $X$.

Let $\phi : X \to Y$ and $\psi : Y \to Z$ be morphisms of $\mathbb{C}$-varieties. MacPherson [16, Prop. 1] defined a $\mathbb{Q}$-linear pushforward $CF(\phi) : CF(X) \to CF(Y)$ with $(CF(\phi)\delta_W)(y) = \chi_{an}(\phi^{-1}(y) \cap W)$ for subvarieties $W$ in $X$ and $y \in Y(\mathbb{C})$, where $\chi_{an}$ is the topological Euler characteristic in compactly-supported cohomology with the analytic topology, and $\delta_W$ the characteristic function of $W(\mathbb{C})$ in $X(\mathbb{C})$. It satisfies $CF(\psi \circ \phi) = CF(\psi) \circ CF(\phi)$, so that $CF$ is a functor from the category of $\mathbb{C}$-varieties to the category of $\mathbb{Q}$-vector spaces. This was extended to other fields $\mathbb{K}$ of characteristic zero by Kennedy [13].

This paper generalizes these results to $\mathbb{K}$-schemes and algebraic $\mathbb{K}$-stacks in the sense of Artin, for $\mathbb{K}$ of characteristic zero. We introduce a notion of pseudo-morphism $\Phi$ between locally constructible sets in $\mathbb{K}$-schemes or $\mathbb{K}$-stacks, generalizing morphisms. Pushforwards $CF(\Phi)$ exist, and pseudomorphisms seem very natural for constructible functions problems.

The motivation for this is my series of papers [8–11]. Let $\text{coh}(P)$ be the abelian category of coherent sheaves on a projective $\mathbb{K}$-scheme $P$, and $(\tau, T, \leq)$ a stability condition on $\text{coh}(P)$. Then the moduli space $\mathcal{Ob}^{\text{coh}}(\tau) \subseteq \text{coh}(P)$ of sheaves in $\text{coh}(P)$ is an Artin $\mathbb{K}$-stack, and the set $\text{Obj}^\alpha_{\tau}(\tau)$ of $\tau$-semistable sheaves in class $\alpha$ is a constructible subset in $\mathcal{Ob}^{\text{coh}}(\tau)$. We shall define invariants of $P, (\tau, T, \leq)$ as generalized Euler characteristics of $\text{Obj}^\alpha_{\tau}(\tau)$, and study identities they satisfy, and transformation laws under change of stability condition.

To carry out this programme requires a theory of constructible sets and functions in algebraic $\mathbb{K}$-stacks, and compatible notions of Euler characteristic and pushforward. As I could not find these tools in the literature, I develop them here. It seemed better to write a stand-alone paper that others could use, rather than include the material in the series [8–11].

Section 2 gives some background on schemes, varieties and stacks. In §3 we recall MacPherson’s constructible functions theory for $\mathbb{C}$-varieties, extend it to $\mathbb{K}$-schemes using $l$-adic cohomology in place of cohomology with the analytic topology for $\mathbb{C}$-varieties when $\mathbb{K}$ has characteristic zero, and define and
study pseudomorphisms between locally constructible sets in $\mathbb{K}$-schemes. We also explain why the theory cannot be extended to $\mathbb{K}$-schemes for $\mathbb{K}$ of positive characteristic.

Sections 4 and 5 extend these ideas to stacks. An important difference between stacks and schemes is that in an algebraic $\mathbb{K}$-stack $\mathfrak{F}$ points $x \in \mathfrak{F}(\mathbb{K})$ have stabilizer groups $\text{Iso}_\mathbb{K}(x)$, which are algebraic $\mathbb{K}$-groups, trivial if $\mathfrak{F}$ is a $\mathbb{K}$-scheme. It turns out that there are many different ways of including stabilizer groups when extending Euler characteristics $\chi$ and pushforwards $\text{CF}$ to stacks.

We highlight three interesting cases, the naive pushforward $\text{CF}^{na}$ which ignores stabilizer groups, the stack pushforward $\text{CF}^{stk}$ which is most natural in many stack problems, and the orbifold pushforward $\text{CF}^{orb}$, related to Deligne–Mumford stacks and their crepant resolutions. Each is associated with a notion of Euler characteristic $\chi^{na}, \chi^{stk}, \chi^{orb}$ of constructible sets in $\mathbb{K}$-stacks.

As $\chi^{stk}, \text{CF}^{stk}$ involve weighting by $1/\chi(\text{Iso}_\mathbb{K}(x))$, the obvious definitions fail when $\chi(\text{Iso}_\mathbb{K}(x)) = 0$. However, for representable 1-morphisms $\phi : \mathfrak{F} \to \mathfrak{G}$ we give a more subtle definition of $\text{CF}^{stk}(\phi) : \text{CF}(\mathfrak{F}) \to \text{CF}(\mathfrak{G})$ in §5.1, which is always well-defined, and suffices for the applications in [9–11]. We also define pullbacks $\psi^*$ by finite type 1-morphisms $\psi : \mathfrak{F} \to \mathfrak{G}$, and show pullbacks $\psi^*$ and pushforwards $\text{CF}^{stk}(\phi)$ commute in Cartesian squares.

A companion paper [7] studies ‘stack functions’ on Artin stacks, which are a universal generalization of constructible functions containing more information, and discusses how ‘motivic’ invariants of $\mathbb{K}$-varieties such as Euler characteristics and virtual Poincaré polynomials are best extended to Artin stacks.

All $\mathbb{K}$-schemes and $\mathbb{K}$-stacks in this paper are assumed locally of finite type.

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2 Schemes, varieties and stacks

Fix an algebraically closed field $\mathbb{K}$ throughout. There are four main classes of ‘spaces’ over $\mathbb{K}$ used in algebraic geometry, in increasing order of generality:

$$\mathbb{K}\text{-varieties} \subset \mathbb{K}\text{-schemes} \subset \text{algebraic } \mathbb{K}\text{-spaces} \subset \text{algebraic } \mathbb{K}\text{-stacks}.$$ 

Section 2.1 gives a few definitions and facts on $\mathbb{K}$-schemes and $\mathbb{K}$-varieties, and §2.2 introduces algebraic $\mathbb{K}$-stacks. Some good references for §2.1 are Hartshorne [5], and for §2.2 are Gómez [3] and Laumon and Moret-Bailly [15].

2.1 Schemes and varieties

We assume a good knowledge of $\mathbb{K}$-schemes and their morphisms, following Hartshorne [5]. We make the conventions that:

- All $\mathbb{K}$-schemes in this paper are locally of finite type.
- All $\mathbb{K}$-subschemas are locally closed, but not necessarily closed.
A \textit{\mathbb{K}\text{-variety}} is a reduced, irreducible, separated \mathbb{K}\text{-scheme of finite type.}

**Definition 2.1.** For a \mathbb{K}\text{-scheme }X, write \(X(\mathbb{K})\) for the set \(\text{Hom}(\text{Spec } \mathbb{K}, X)\) of morphisms of \mathbb{K}\text{-schemes }\text{Spec } \mathbb{K} \to X.\) Then \(X(\mathbb{K})\) is naturally identified with the subset of \textit{closed points} of the underlying topological space of \(X.\) Elements of \(X(\mathbb{K})\) are also called \textit{geometric points} or \textit{\mathbb{K}\text{-points}} of \(X.\)

There is a natural identification \((X \times Y)(\mathbb{K}) \cong X(\mathbb{K}) \times Y(\mathbb{K}).\) If \(\phi : X \to Y\) is a morphism of \mathbb{K}\text{-schemes}, composition \(\text{Spec } \mathbb{K} \to X \xrightarrow{\phi} Y\) gives a natural map \(\phi_* : X(\mathbb{K}) \to Y(\mathbb{K}).\) If \(X\) is a \mathbb{K}\text{-subscheme} of \(Y\) then \(X(\mathbb{K}) \subseteq Y(\mathbb{K}).\)

Much of the paper will involve cutting schemes or stacks into pieces. To do this we shall use two different notions of \textit{disjoint union}.

**Definition 2.2.** Let \(X\) be a \mathbb{K}\text{-scheme}, and \(\{X_i : i \in I\}\) a family of \mathbb{K}\text{-subschemes of }X.\) We say that \(X\) is the \textit{set-theoretic disjoint union} of the \(X_i\) for \(i \in I\) if \(X(\mathbb{K}) = \bigsqcup_{i \in I} X_i(\mathbb{K}).\)

If \(\{X_i : i \in I\}\) is a family of \mathbb{K}\text{-schemes, we define the \textit{abstract disjoint union} of the }X_i\text{ to be the }\mathbb{K}\text{-scheme }\langle X, \mathcal{O}_X \rangle,\text{ where }X\text{ is the disjoint union of the topological spaces }X_i,\text{ and }\mathcal{O}_X|_{X_i} = \mathcal{O}_{X_i}.\text{ Then }X\text{ exists and is unique up to isomorphism, and the }X_i\text{ are open and closed }\mathbb{K}\text{-subschemes of }X.\text{ Clearly, an abstract disjoint union is a set-theoretic disjoint union, but not necessarily vice versa. When we just say ‘disjoint union’ we mean set-theoretic disjoint union.}

Here is a useful result of Rosenlicht [21], on the existence of quotients of varieties by algebraic groups.

**Theorem 2.3.** Let \(\mathbb{K}\) be an algebraically closed field, \(X\) a \mathbb{K}\text{-variety}, \(G\) an algebraic \mathbb{K}\text{-group, and }\rho : G \times X \to X\text{ an algebraic action of }G\text{ on }X.\text{ Then there exists a dense, Zariski open subset }X'\text{ of }X,\text{ a }\mathbb{K}\text{-variety }Y,\text{ and a surjective morphism }\pi : X' \to Y\text{ inducing a bijection between }G\text{-orbits in }X'\text{ and }\mathbb{K}\text{-points in }Y,\text{ such that any }G\text{-invariant rational function on }X'\text{ defined at }x \in X'\text{ is the pull-back of a rational function on }Y\text{ defined at }\pi(x).
Definition 2.4. Let \( \mathcal{F} \) be a \( K \)-stack. Write \( \mathcal{F}(K) \) for the set of 2-isomorphism classes \([x]\) of 1-morphisms \( x : \text{Spec} \, K \to \mathcal{F} \). Elements of \( \mathcal{F}(K) \) are called \( K \)-points, or geometric points, of \( \mathcal{F} \). If \( \phi : \mathcal{F} \to \mathcal{G} \) is a 1-morphism then composition with \( \phi \) induces a map of sets \( \phi^* : \mathcal{F}(K) \to \mathcal{G}(K) \).

For a 1-morphism \( x : \text{Spec} \, K \to \mathcal{F} \), the stabilizer group \( \text{Iso}_K(x) \) is the group of 2-morphisms \( x \to x \). When \( \mathcal{F} \) is an algebraic \( K \)-stack, \( \text{Iso}_K(x) \) is an algebraic \( K \)-group. We say that \( \mathcal{F} \) has affine geometric stabilizers if \( \text{Iso}_K(x) \) is an affine algebraic \( K \)-group for all 1-morphisms \( x : \text{Spec} \, K \to \mathcal{F} \).

One important difference in working with 2-categories rather than ordinary categories is that in diagram-chasing one only requires 1-morphisms to be 2-isomorphic rather than equal. The simplest kind of commutative diagram is:

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\phi} & \mathcal{G} \\
\Downarrow{\chi} & & \Downarrow{\psi} \\
\mathcal{H} & \xleftarrow{\phi} & \mathcal{H},
\end{array}
\]

by which we mean that \( \mathcal{F}, \mathcal{G}, \mathcal{H} \) are \( K \)-stacks, \( \phi, \psi, \chi \) are 1-morphisms, and \( F : \psi \circ \phi \to \chi \) is a 2-isomorphism. Usually we omit \( F \), and mean that \( \psi \circ \phi \cong \chi \).

Definition 2.5. Let \( \phi : \mathcal{F} \to \mathcal{G}, \psi : \mathcal{G} \to \mathcal{H} \) be 1-morphisms of \( K \)-stacks. Then one can define the fibre product stack \( \mathcal{F} \times_{\phi, \mathcal{G}, \psi} \mathcal{H} \), or \( \mathcal{F} \times_{\mathcal{G}} \mathcal{H} \) for short, with 1-morphisms \( \pi_{\mathcal{F}}, \pi_{\mathcal{G}} \) fitting into a commutative diagram:

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\phi} & \mathcal{G} \\
\Downarrow{\pi_{\mathcal{F}}} & & \Downarrow{\pi_{\mathcal{G}}} \\
\mathcal{F} \times_{\mathcal{G}} \mathcal{H} & \xrightarrow{\phi} & \mathcal{H}.
\end{array}
\]

A commutative diagram

\[
\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\theta} & \mathcal{F} \\
\Downarrow{\eta} & & \Downarrow{\psi} \\
\mathcal{E} & \xleftarrow{\phi} & \mathcal{H}
\end{array}
\]

is a Cartesian square if it is isomorphic to (1), so there is a 1-isomorphism \( \mathcal{E} \cong \mathcal{F} \times_{\mathcal{G}} \mathcal{H} \). Cartesian squares may also be characterized by a universal property.

Here is a definition from Kresch [14, Def. 3.5.3], slightly modified.

Definition 2.6. Let \( \mathcal{F} \) be a finite type algebraic \( K \)-stack, and \( \mathcal{F}^{\text{red}} \) the associated reduced stack. We say that \( \mathcal{F} \) can be stratified by global quotient stacks if \( \mathcal{F}^{\text{red}} \) is the disjoint union of finitely many locally closed substacks \( U_i \) with each \( U_i \) 1-isomorphic to a stack of the form \([X_i/G_i]\), where \( X_i \) is a \( K \)-variety and \( G_i \) a smooth, connected, affine algebraic \( K \)-group acting linearly on \( X_i \). For a stack to be the disjoint union of a family of locally closed substacks is defined in [15, p. 22]. It implies that \( \mathcal{F}(K) = \mathcal{F}^{\text{red}}(K) = \bigsqcup_i U_i(K) \).
Kresch [14, Prop. 3.5.2(ii)] takes the $X_i$ to be *quasiprojective schemes*, rather than varieties, but this is equivalent to our definition. Kresch [14, Prop. 3.5.9] characterizes stacks stratified by global quotients.

**Theorem 2.7.** Let $\mathcal{X}$ be a finite type algebraic $K$-stack. Then $\mathcal{X}$ can be stratified by global quotient stacks if and only if $\mathcal{X}$ has affine geometric stabilizers.

### 3 Constructible functions on $K$-schemes

We now introduce *constructible sets and functions* on $K$-schemes, and the *pushforward* of constructible functions by morphisms. Section 3.1 defines (locally) constructible sets and functions on $K$-schemes. We explain the Euler characteristic and pushforwards over $\mathbb{C}$ in §3.2 and over other fields $K$ in §3.3. Section 3.4 defines *pseudomorphisms*, a notion of morphism for (locally) constructible sets, and pushforwards along pseudomorphisms.

Some references are Mumford [19, p. 51] and Hartshorne [5, p. 94] for constructible sets, and MacPherson [16], Viro [22] and Kennedy [13] for constructible functions and the pushforward. As far as the author can tell the ideas of §3.3–§3.4 are new, although elementary and probably obvious to experts.

#### 3.1 Constructible sets and functions on $K$-schemes

We define *constructible and locally constructible sets*.

**Definition 3.1.** Let $K$ be an algebraically closed field, and $X$ a $K$-scheme. A subset $C \subseteq X(K)$ is called *constructible* if $C = \bigcup_{i \in I} X_i(K)$, where $\{X_i : i \in I\}$ is a finite collection of finite type $K$-subschemes $X_i$ of $X$. We call $S \subseteq X(K)$ *locally constructible* if $S \cap C$ is constructible for all constructible $C \subseteq X(K)$.

This is easily seen to be equivalent to a stronger definition, where we take the union $C = \bigcup_{i \in I} X_i(K)$ to be disjoint, and the $X_i$ to be separated.

**Proposition 3.2.** Let $X$ be a $K$-scheme, and $C \subseteq X(K)$ a constructible subset. Then we may write $C = \coprod_{i \in I} X_i(K)$, where $\{X_i : i \in I\}$ is a finite collection of separated, finite type $K$-subschemes $X_i$ of $X$.

The following properties of constructible sets in $K$-varieties are well known, [5, p. 94], [19, p. 51]. Our extension to $K$-schemes is straightforward.

**Proposition 3.3.** Let $X, Y$ be $K$-schemes, $\phi : X \to Y$ a morphism, and $A, B \subseteq X(K)$ be constructible subsets. Then $A \cup B, A \cap B$ and $A \setminus B$ are constructible in $X(K)$, and $\phi_*(A)$ is constructible in $Y(K)$.

Note that showing $\phi_*(A)$ constructible, and the stack analogue in Proposition 4.5, are the only places we use the convention that $K$-schemes and $K$-stacks are *locally of finite type*. Next we define (locally) constructible functions.
Definition 3.4. Let $X$ be a $\mathbb{K}$-scheme and $S \subseteq X(\mathbb{K})$ be locally constructible. A constructible function on $S$ is a function $f : S \to \mathbb{Q}$ such that $f(S)$ is finite and $f^{-1}(c)$ is a constructible set in $S \subseteq X(\mathbb{K})$ for each $c \in f(S) \setminus \{0\}$. Note that we do not require $f^{-1}(0)$ to be constructible. Write $\text{CF}(S)$ for the $\mathbb{Q}$-vector space of constructible functions on $S$, and for brevity write $CF(X)$ for $CF(X(\mathbb{K}))$.

A locally constructible function on $S$ is a function $f : S \to \mathbb{Q}$ such that $f|_C$ is constructible for all constructible $C \subseteq S$. Equivalently, $f$ is locally constructible if $f^{-1}(c)$ is locally constructible for all $c \in \mathbb{Q}$, and $f(C)$ is finite for all constructible $C \subseteq S$. Write $\text{LCF}(S)$ for the $\mathbb{Q}$-vector space of locally constructible functions on $S$, and $\text{LCF}(X)$ for $\text{LCF}(X(\mathbb{K}))$.

Using Proposition 3.3 we see that products of (locally) constructible functions are (locally) constructible, so $\text{CF}(S)$ and $\text{LCF}(S)$ are commutative $\mathbb{Q}$-algebras, with $\text{CF}(S)$ an ideal in $\text{LCF}(S)$. Note that $1 \in \text{CF}(S)$ if and only if $S$ is constructible, so if it is not then $\text{CF}(S)$ is an algebra without identity.

Here are some remarks on this material:

- To define constructible functions $f : X(\mathbb{K}) \to \mathbb{Q}$ on $\mathbb{K}$-schemes $X$ which are not of finite type, or $f : S \to \mathbb{Q}$ for $S$ not constructible, we must allow $f^{-1}(0)$ to be non-constructible. If we did not there would be no constructible functions on $X$ or $S$, not even 0.

For $X$ not of finite type we can think of $X(\mathbb{K})$ as being ‘large’, or ‘unbounded’. Constructible functions $f : X(\mathbb{K}) \to \mathbb{Q}$ are nonzero only on small, bounded subsets of $X(\mathbb{K})$, and $f^{-1}(0)$ is the remaining, large, unbounded part of $X(\mathbb{K})$.

- We can also consider constructible functions with values in $\mathbb{Z}$, or any other abelian group, ring or field. But for simplicity we restrict to $\mathbb{Q}$.

3.2 Euler characteristics and pushforward for $\mathbb{C}$-schemes

We define the analytic Euler characteristic $\chi_{\text{an}}$.

Definition 3.5. Let $X$ be a separated $\mathbb{C}$-scheme of finite type. Then $X(\mathbb{C})$ is a Hausdorff topological space with the analytic topology. Write $\chi_{\text{an}}(X)$ for the Euler characteristic of $X(\mathbb{C})$, in compactly-supported cohomology.

The following properties of $\chi_{\text{an}}$ are well known.

Proposition 3.6. Let $X,Y$ be separated $\mathbb{C}$-schemes of finite type. Then

(i) If $Z$ is a closed subscheme of $X$ then $\chi_{\text{an}}(X) = \chi_{\text{an}}(Z) + \chi_{\text{an}}(X \setminus Z)$.
(ii) Suppose $X$ is the set-theoretic disjoint union of subschemes $U_1, \ldots, U_m$. Then $\chi_{\text{an}}(X) = \sum_{i=1}^m \chi_{\text{an}}(U_m)$.
(iii) $\chi_{\text{an}}(X \times Y) = \chi_{\text{an}}(X)\chi_{\text{an}}(Y)$.
(iv) If $\phi : X \to Y$ is a morphism which is a locally trivial fibration in the analytic topology with fibre $F$, then $\chi_{\text{an}}(X) = \chi_{\text{an}}(F)\chi_{\text{an}}(Y)$.
(v) $\chi_{\text{an}}(\mathbb{C}^m) = 1$ and $\chi_{\text{an}}(\mathbb{CP}^m) = m + 1$ for all $m \geq 0$. 
Now we can define pushforwards on \( \mathbb{C} \)-schemes.

**Definition 3.7.** Let \( X \) be a \( \mathbb{C} \)-scheme and \( C \subseteq X(\mathbb{C}) \) a constructible subset. Proposition 3.2 gives \( C = \bigsqcup_{i \in I} X_i(\mathbb{C}) \) for \( \{X_i : i \in I\} \) finitely many separated, finite type subschemes of \( X \). Define \( \chi_{\text{an}}(C) = \sum_{i \in I} \chi_{\text{an}}(X_i) \). If \( \{Y_j : j \in J\} \) is another choice from Proposition 3.2 then \( X_i \) is the set-theoretic union of \( X_i \cap Y_j \) for \( j \in J \), so Proposition 3.6(ii) gives \( \chi_{\text{an}}(X_i) = \sum_{j \in J} \chi_{\text{an}}(X_i \cap Y_j) \). Hence

\[
\sum_{i \in I} \chi_{\text{an}}(X_i) = \sum_{i \in I} \left[ \sum_{j \in J} \chi_{\text{an}}(X_i \cap Y_j) \right] = \sum_{j \in J} \left[ \sum_{i \in I} \chi_{\text{an}}(X_i \cap Y_j) \right] = \sum_{j \in J} \chi_{\text{an}}(Y_j),
\]

and \( \chi_{\text{an}}(C) \) is well-defined.

For \( f \in \text{CF}(X) \), define the **weighted Euler characteristic** \( \chi_{\text{an}}(X, f) \in \mathbb{Q} \) by

\[
\chi_{\text{an}}(X, f) = \sum_{c \in f(X(\mathbb{C})) \setminus \{0\}} \text{c} \chi_{\text{an}}(f^{-1}(c)).
\]

This is well-defined as \( f(X(\mathbb{C})) \) is finite and \( f^{-1}(c) \subseteq X(\mathbb{C}) \) is constructible for each \( c \in f(X(\mathbb{C})) \setminus \{0\} \). Clearly, \( f \mapsto \chi_{\text{an}}(X, f) \) is a linear map \( \text{CF}(X) \to \mathbb{Q} \).

Now let \( \phi : X \to Y \) be a morphism of \( \mathbb{C} \)-schemes, and \( f \in \text{CF}(X) \). Define the **pushforward** \( \text{CF}(\phi)f : Y(\mathbb{C}) \to \mathbb{Q} \) of \( f \) to \( Y \) by

\[
\text{CF}(\phi)f(y) = \chi_{\text{an}}(X, f \cdot \delta_{\phi^{-1}(y)}) \quad \text{for } y \in Y(\mathbb{C}).
\]

Here \( \phi_* : X(\mathbb{C}) \to Y(\mathbb{C}) \) is the induced map, \( \phi^{-1}(y) \subseteq X(\mathbb{C}) \) is the inverse image of \( \{y\} \) under \( \phi_* \), and \( \delta_{\phi^{-1}(y)} \) is its characteristic function. It is a locally constructible function, so \( f \cdot \delta_{\phi^{-1}(y)} \in \text{CF}(X) \), and \( \chi_{\text{an}}(X, f) \) is well-defined.

MacPherson [16, Prop. 1] gives an important property of the pushforward for algebraic \( \mathbb{C} \)-varieties. The extension to \( \mathbb{C} \)-schemes is straightforward. One can prove it by dividing \( X, Y \) into pieces upon which \( \phi \) is a locally trivial fibration in the analytic topology, and using Proposition 3.6(ii),(iv).

**Theorem 3.8.** Let \( X, Y, Z \) be \( \mathbb{C} \)-schemes, \( \phi : X \to Y \) and \( \psi : Y \to Z \) be morphisms, and \( f \in \text{CF}(X) \). Then \( \text{CF}(\phi)f \) is constructible, so \( \text{CF}(\psi) \circ \text{CF}(\phi) \) is a \( \mathbb{Q} \)-linear map. Also \( \text{CF}(\psi \circ \phi) = \text{CF}(\psi) \circ \text{CF}(\phi) \) as linear maps \( \text{CF}(X) \to \text{CF}(Z) \). Hence \( \text{CF} \) is a functor from the category of \( \mathbb{C} \)-schemes to the category of \( \mathbb{Q} \)-vector spaces.

Viro [22] gives an interesting point of view on constructible functions. One can regard the Euler characteristic as a **measure**, defined on constructible sets. Then \( \chi_{\text{an}}(X, f) \) in (2) is the integral of \( f \) with respect to this measure, and the pushforward \( \text{CF}(\phi)f \) integrates \( f \) over the fibres of \( \phi \).

### 3.3 Extension to other fields \( \mathbb{K} \)

To extend (3.2) to other fields \( \mathbb{K} \), we need a good notion of Euler characteristic \( \chi(X) \) for a separated \( \mathbb{K} \)-scheme \( X \) of finite type.
Definition 3.9. Let $K$ be an algebraically closed field of characteristic $p$, which may be zero, and fix a prime number $l \neq p$. Write $Q_l$ for the field of $l$-adic rationals. Let $X$ be a separated $K$-scheme of finite type. Then one may define the compactly-supported $l$-adic cohomology groups $H^i_{cs}(X, Q_l)$ of $X$, for $i \geq 0$.

The original reference for étale and $l$-adic cohomology is Grothendieck et al. [4], and a good book is Milne [18]. Define the Euler characteristic $\chi(X)$ of $X$ to be

$$\chi(X) = \sum_{i=0}^{\dim X} (-1)^i \dim_{Q_l} H^i_{cs}(X, Q_l).$$

(4)

Here are some properties of $\chi$, generalizing Proposition 3.9.

Theorem 3.10. Let $K$ be an algebraically closed field and $X, Y$ be separated $K$-schemes of finite type. Then

(i) If $Z$ is a closed subscheme of $X$ then $\chi(X) = \chi(Z) + \chi(X \setminus Z)$.

(ii) Suppose $X$ is the set-theoretic disjoint union of subschemes $U_1, \ldots, U_m$. Then $\chi(X) = \sum_{i=1}^m \chi(U_i)$.

(iii) $\chi(X \times Y) = \chi(X) \chi(Y)$.

(iv) $\chi(X)$ is independent of the choice of $l$ in Definition 3.11.

(v) When $K = \mathbb{C}$ we have $\chi(X) = \chi_{an}(X)$.

(vi) $\chi(K^n) = 1$ and $\chi(KP^n) = m + 1$ for all $m \geq 0$.

Proof. Part (i) comes from the long exact sequence [4, 4.XVII.5.1.16]:

$$\cdots \to H^i_{cs}(X \setminus Z, Q_l) \to H^i_{cs}(X, Q_l) \to H^i_{cs}(Z, Q_l) \to H^{i+1}_{cs}(X \setminus Z, Q_l) \to \cdots,$$

and (ii) follows from (i) by standard arguments. Part (iii) is a consequence of the Künneth formula [4, 4.XVII.5.4.3]. Part (iv) is proved for $X$ proper in [4, 5.VII.4.10]. The general case follows from (i) as we may write $X \cong \bar{X} \setminus Z$ for $\bar{X}$ a proper separated $K$-scheme of finite type, and $Z$ a closed subscheme. Part (v) follows from the comparison theorem [4, 4.XVI.4.1]. For (vi), calculation shows $H^i_{cs}(K^n, Q_l) = Q_l$ if $i = 2m$ and 0 otherwise, so $\chi(K^n) = 1$. Then $\chi(KP^n) = m + 1$ by (ii) and $K\mathbb{P}^n = \bigsqcup_{n=0}^m K^n$.

Here are the generalizations of Definition 3.9 and Theorem 3.10 to $K$.

Definition 3.11. Let $X$ be a $K$-scheme and $C \subseteq X(K)$ a constructible subset. Write $C = \bigsqcup_{i \in I} X_i(K)$ as in Proposition 3.2, and define $\chi(C) = \sum_{i \in I} \chi(X_i)$. This is well-defined as in Definition 3.7, using Theorem 3.10 (ii). For $f \in \text{CF}(X)$, define the weighted Euler characteristic $\chi(X, f) \in \mathbb{Q}$ by

$$\chi(X, f) = \sum_{c \in f(X(K)) \setminus \{0\}} c \chi(f^{-1}(c)).$$

(5)

Then $f \mapsto \chi(X, f)$ is a linear map $\text{CF}(X) \to \mathbb{Q}$. Let $\phi : X \to Y$ be a morphism of $K$-schemes. Define the pushforward $\text{CF}(\phi)f : Y(K) \to \mathbb{Q}$ of $f$ to $Y$ by

$$\text{CF}(\phi)f(y) = \chi(X, f \cdot \delta_{\phi^{-1}(y)}) \quad \text{for } y \in Y(K).$$

(6)

When $K = \mathbb{C}$ these definitions agree with Definition 3.7 by Theorem 3.10 (v).
There are several ways to prove the next theorem. One is to use results of Kennedy [13]. He defines pushforwards implicitly using intersections of Lagrangian cycles, but one can show using base change and comparison theorems for l-adic cohomology that his definition of \( \text{CF}(\phi) \) agrees with ours. Another is to use Katz and Laumon [12, Th. 3.1.2], which in characteristic zero relates pushforwards of constructible sheaves and functions, so functoriality of \( \text{CF} \) follows from that for sheaf pushforwards.

**Theorem 3.12.** Let \( \mathbb{K} \) be an algebraically closed field of characteristic zero, \( X, Y, Z \) be \( \mathbb{K} \)-schemes, \( \phi : X \to Y \) and \( \psi : Y \to Z \) be morphisms, and \( f \in \text{CF}(X) \). Then \( \text{CF}(\phi) f \) is constructible, so \( \text{CF}(\phi) : \text{CF}(X) \to \text{CF}(Y) \) is a \( \mathbb{Q} \)-linear map. Also \( \text{CF}(\psi \circ \phi) = \text{CF}(\psi) \circ \text{CF}(\phi) \), so \( \text{CF} \) is a functor.

The last part of the theorem is false for \( \mathbb{K} \) of characteristic \( p > 0 \), and I am grateful to Jörg Schürmann for the following explanation of why. The proof using constructible sheaves fails because if \( \mathcal{L} \) is a locally constant \( \mathbb{Q}_l \)-sheaf of rank \( r \) on a non-proper \( \mathbb{K} \)-scheme \( X \) for \( l \neq p \), we need the fact that

\[
\chi(H^*_{\text{et}}(X, \mathcal{L})) = r \cdot \chi(H^*_{\text{et}}(X, \mathbb{Q}_l)).
\]

This holds in characteristic zero, but not in characteristic \( p > 0 \) without extra conditions on \( \mathcal{L} \), which are studied in Illusie [6].

Here is a counterexample to Theorem 3.12 in positive characteristic. Let \( \mathbb{K} \) have characteristic \( p \geq 2 \), and \( \phi : \mathbb{K} \to \mathbb{K} \) be the Artin–Schreier morphism \( \phi : x \mapsto x^p - x \). It is a \( p \)-fold étale covering of \( \mathbb{K} \) by itself, so \( \text{CF}(\phi) 1 = p \) in \( \text{CF}(\mathbb{K}) \). Thus taking \( \psi : \mathbb{K} \to \text{Spec} \mathbb{K} \) to be the projection we have \( \text{CF}(\psi \circ \phi) 1 = 1 \) but \( \text{CF}(\psi) \circ \text{CF}(\phi) 1 = p \) in \( \text{CF}(\text{Spec} \mathbb{K}) = \mathbb{Q} \), so \( \text{CF}(\psi \circ \phi) \neq \text{CF}(\psi) \circ \text{CF}(\phi) \).

When \( Z = \text{Spec} \mathbb{K} \) and \( \psi : Y \to \text{Spec} \mathbb{K} \) is the projection we have \( \text{CF}(Z) = \mathbb{Q} \) and \( \text{CF}(\psi) g = \chi(Y, g) \) for \( g \in \text{CF}(Y) \). So \( \text{CF}(\psi \circ \phi) = \text{CF}(\psi) \circ \text{CF}(\phi) \) gives a relation between pushforwards and weighted Euler characteristics.

**Corollary 3.13.** Let \( \mathbb{K} \) have characteristic zero, \( X, Y \) be \( \mathbb{K} \)-schemes, \( \phi : X \to Y \) a morphism, and \( f \in \text{CF}(X) \). Then \( \chi(X, f) = \chi(Y, \text{CF}(\phi) f) \).

### 3.4 Extension to pseudomorphisms

We define pseudomorphisms, a notion of morphism between locally constructible sets that generalizes morphisms of schemes.

**Definition 3.14.** Suppose \( \mathbb{K} \) is an algebraically closed field, \( X, Y \) are \( \mathbb{K} \)-schemes and \( S \subseteq X(\mathbb{K}) \), \( T \subseteq Y(\mathbb{K}) \) are locally constructible. Let \( \Phi : S \to T \) be a map, and define the graph \( \Gamma_{\Phi} = \{ (s, \Phi(s)) : s \in S \} \) in \( X(\mathbb{K}) \times Y(\mathbb{K}) = (X \times Y)(\mathbb{K}) \). We call \( \Phi \) a pseudomorphism if \( \Gamma_{\Phi} \cap (C \times Y(\mathbb{K})) \) is constructible for all constructible \( C \subseteq X(\mathbb{K}) \). This implies \( \Gamma_{\Phi} \) is locally constructible.

A pseudomorphism \( \Phi \) is a pseudoisomorphism if \( \Phi \) is bijective and \( \Phi^{-1} : T \to S \) is a pseudomorphism. When \( S = X(\mathbb{K}) \) and \( T = Y(\mathbb{K}) \) we shall also call \( \Phi : X \to Y \) a pseudomorphism (pseudoisomorphism) from \( X \) to \( Y \).
When \( X, Y \) are \( \mathbb{K} \)-varieties, pseudomorphisms \( \Phi : X \to Y \) coincide with \textit{definable functions} in the model theory of algebraic geometry. See for instance Marker [17, §7.4], in particular [17, Lem. 7.4.7] which shows that \( \Phi \) equals a \textit{quasimorphism} of varieties on a nonempty open affine subset \( X_0 \) of \( X \), and a \textit{morphism} if \( \mathbb{K} \) has characteristic zero. Here are some basic properties of pseudomorphisms. They are easily proved using Proposition 3.15 and the projection morphisms \( X \times Y \to Y, X \times Y \times Z \to X \times Z \).

**Proposition 3.15.** Let \( \mathbb{K} \) be an algebraically closed field.

(a) Let \( \phi : X \to Y \) be a morphism (isomorphism) of \( \mathbb{K} \)-schemes. Then \( \phi_* : X(\mathbb{K}) \to Y(\mathbb{K}) \) is a pseudomorphism (pseudoisomorphism).

(b) Let \( X, Y \) be \( \mathbb{K} \)-schemes, \( S \subseteq X(\mathbb{K}), T \subseteq Y(\mathbb{K}) \) be locally constructible, \( \Phi : S \to T \) be a pseudomorphism, and \( C \subseteq S \) be constructible. Then \( \Phi(C) \) is constructible in \( Y(\mathbb{K}) \). Also, if \( t \in T \) then \( C \cap \Phi^{-1}(t) \) is constructible in \( X(\mathbb{K}) \). Hence, \( \Phi^{-1}(t) \) is locally constructible in \( X(\mathbb{K}) \).

(c) Let \( X, Y, Z \) be \( \mathbb{K} \)-schemes, \( S \subseteq X(\mathbb{K}), T \subseteq Y(\mathbb{K}), U \subseteq Z(\mathbb{K}) \) be locally constructible, and \( \Phi : S \to T, \Psi : T \to U \) be pseudo(iso)morphisms. Then \( \Psi \circ \Phi : S \to U \) is a pseudo(iso)morphism.

We define \textit{pushforwards} \( \text{CF}(\Phi) : \text{CF}(S) \to \text{CF}(T) \) along pseudomorphisms.

**Definition 3.16.** Let \( X, Y \) be \( \mathbb{K} \)-schemes, \( S \subseteq X(\mathbb{K}), T \subseteq Y(\mathbb{K}) \) be locally constructible, \( \Phi : S \to T \) a pseudomorphism, and \( f \in \text{CF}(S) \). Define the \textit{pushforward} \( \text{CF}(\Phi)f : T \to \mathbb{Q} \) by

\[
\text{CF}(\Phi)f(t) = \chi(S, f \cdot \delta_{\Phi^{-1}(t)}) \quad \text{for } t \in T.
\]

Here \( \delta_{\Phi^{-1}(t)} \) is the characteristic function of \( \Phi^{-1}(t) \subseteq S \) on \( S \). By Proposition 3.15(b) \( \delta_{\Phi^{-1}(t)} \in \text{LCF}(S) \), and \( f \in \text{CF}(S) \), so \( f \cdot \delta_{\Phi^{-1}(t)} \in \text{CF}(S) \). Thus \( \text{CF}(\Phi)f \) is well-defined, by Definition 3.11 If \( \phi : X \to Y \) is a morphism of \( \mathbb{K} \)-schemes then \( \phi_* : X(\mathbb{K}) \to Y(\mathbb{K}) \) is a pseudomorphism by Proposition 3.15(a), and \( \text{CF}(\phi) \) in Definition 3.11 coincides with \( \text{CF}(\phi_*) \) above.

Here is the generalization of Theorems 3.9 and 3.12 to pseudomorphisms.

**Theorem 3.17.** Let \( \mathbb{K} \) have characteristic zero, \( X, Y, Z \) be \( \mathbb{K} \)-schemes, \( S \subseteq X(\mathbb{K}), T \subseteq Y(\mathbb{K}), U \subseteq Z(\mathbb{K}) \) be locally constructible, \( \Phi : S \to T, \Psi : T \to U \) be pseudomorphisms, and \( f \in \text{CF}(S) \). Then \( \text{CF}(\Phi)f \) is constructible, so \( \text{CF}(\Phi) : \text{CF}(S) \to \text{CF}(T) \) is \( \mathbb{Q} \)-linear, and \( \text{CF}(\Psi \circ \Phi) = \text{CF}(\Psi) \circ \text{CF}(\Phi) \).

**Proof.** Define \( F_{XY} : X(\mathbb{K}) \times Y(\mathbb{K}) \to \mathbb{Q}, F_{XZ} : X(\mathbb{K}) \times Z(\mathbb{K}) \to \mathbb{Q}, F_{YZ} : Y(\mathbb{K}) \times \mathbb{Q} \).
\[ Z(\mathbb{K}) \to \mathbb{Q} \] and \( F_{XYZ} : X(\mathbb{K}) \times Y(\mathbb{K}) \times Z(\mathbb{K}) \to \mathbb{Q} \) by

\[
F_{XY}(x,y) = \begin{cases} 
  f(x), & x \in S \text{ and } y = \Phi(x), \\
  0, & \text{otherwise},
\end{cases}
\]

\[
F_{XZ}(x,z) = \begin{cases} 
  f(x), & x \in S \text{ and } z = \Psi \circ \Phi(x), \\
  0, & \text{otherwise},
\end{cases}
\]

\[
F_{YZ}(y,z) = \begin{cases} 
  (\mathcal{C}(\Phi)f)(y), & y \in T \text{ and } z = \Psi(y), \\
  0, & \text{otherwise}.
\end{cases}
\]

\[
F_{XYZ}(x,y,z) = \begin{cases} 
  f(x), & x \in S, \ y = \Phi(x) \text{ and } z = \Psi(y), \\
  0, & \text{otherwise}.
\end{cases}
\]

Write \( \Pi^X_Y : X \times Y \to Y \) for the projection morphism, and so on. It is easy to show \( F_{XY} \in \mathcal{C}(X \times Y) \), so \( \mathcal{C}(\Pi^X_Y)F_{XY} \in \mathcal{C}(Y) \) by Theorem 3.12. But comparing (6)–(8) shows \( \mathcal{C}(\Phi)f = (\mathcal{C}(\Pi^X_Y)F_{XY})|_T \). Therefore \( \mathcal{C}(\Phi)f \in \mathcal{C}(T) \), proving the first part. For the second part, \( F_{XZ}, F_{YZ}, F_{XYZ} \) are constructible on \( X \times Z, Y \times Z \) and \( X \times Y \times Z \) in the same way, and it is easy to prove that

\[
\mathcal{C}(\Psi \circ \Phi)f = (\mathcal{C}(\Pi^X_Z)F_{XZ})|_U, \quad \mathcal{C}(\Psi) \circ \mathcal{C}(\Phi)f = (\mathcal{C}(\Pi^Y_Z)F_{YZ})|_U, \\
\mathcal{C}(\Pi^{XY}_Z)F_{XYZ} = F_{XZ} \quad \text{and} \quad \mathcal{C}(\Pi^{XY}_Z)F_{XYZ} = F_{YZ}.
\]

But as \( \Pi^{XZ}_X \circ \Pi^{XY}_Z = \Pi^{XY}_X \circ \Pi^{XY}_Z \) Theorem 3.12 gives \( \mathcal{C}(\Pi^{XZ}_X) \circ \mathcal{C}(\Pi^{XY}_Z) = \mathcal{C}(\Pi^{XY}_X) = \mathcal{C}(\Pi^{XY}_Z) \circ \mathcal{C}(\Pi^{XY}_Z) \), and the result follows.

If \( \Phi : S \to T \) is a pseudoisomorphism then \( \Phi^{-1}(t) \) is a single point in \( T \), giving \( \mathcal{C}(\Phi)f(t) = f \circ \Phi^{-1}(t) \). We deduce:

**Corollary 3.18.** Let \( X, Y \) be \( \mathbb{K} \)-schemes, \( S \subseteq X(\mathbb{K}), T \subseteq Y(\mathbb{K}) \) be locally constructible, and \( \Phi : S \to T \) a pseudoisomorphism. Then \( \mathcal{C}(\Phi) : \mathcal{C}(S) \to \mathcal{C}(T) \) is an isomorphism, with \( \mathcal{C}(\Phi)f = f \circ \Phi^{-1} \) and \( \mathcal{C}(\Phi^{-1})g = g \circ \Phi \).

The moral is that pseudoisomorphic (locally) constructible sets are essentially the same from the point of view of constructible functions. So in problems involving constructible functions, we can work with (locally) constructible sets up to pseudoisomorphism, and pseudomorphisms between them.

## 4 Constructible functions on stacks

We now generalize to stacks. Sections 4.1 and 4.2 develop the basic definitions and properties of constructible sets and functions, and show that any finite type algebraic \( \mathbb{K} \)-stack \( \mathfrak{F} \) with affine geometric stabilizers is pseudoisomorphic to a finite type \( \mathbb{K} \)-scheme. This enables us to reduce to the scheme case of 3.

An important difference between stacks and schemes is that points \( x \in \mathfrak{F}(\mathbb{K}) \) in a \( \mathbb{K} \)-stack \( \mathfrak{F} \) have stabilizer groups \( \text{Isog}(x) \), which are trivial if \( \mathfrak{F} \) is a \( \mathbb{K} \)-scheme. There are many different ways of including stabilizer groups when extending...
Euler characteristics $\chi$ and pushforwards $\text{CF}$ to stacks. Section 4.3 studies the simplest of these, the naïve versions $\chi^{na}$, $\text{CF}^{na}$, which just ignore stabilizer groups. Given an allowable weight function $w$ upon affine algebraic $K$-groups, in 4.4 we modify $\chi^{na}$, $\text{CF}^{na}$ to get $\chi_w$, $\text{CF}_w$ by weighting by $w_S : x \mapsto w(\text{Isog}_S(x))$ on $\mathcal{F}(K)$. Two special cases are the stack versions $\chi^{stk}$, $\text{CF}^{stk}$ which are most natural in many problems, and the orbifold versions $\chi^{orb}$, $\text{CF}^{orb}$, related to Deligne–Mumford stacks and their crepant resolutions.

### 4.1 Basic definitions

We begin by giving analogues for stacks of the major definitions of 3.3.

**Definition 4.1.** Let $K$ be an algebraically closed field, and $\mathcal{F}$ an algebraic $K$-stack. We call $\mathcal{C} \subseteq \mathcal{F}(K)$ constructible if $\mathcal{C} = \bigcup_{i \in I} \mathcal{F}_i(K)$, where $\{\mathcal{F}_i : i \in I\}$ is a finite collection of finite type algebraic $K$-substacks $\mathcal{F}_i$ of $\mathcal{F}$. We call $\mathcal{S} \subseteq \mathcal{F}(K)$ locally constructible if $\mathcal{S} \cap \mathcal{C}$ is constructible for all constructible $\mathcal{C} \subseteq \mathcal{F}(K)$.

Here is a partial analogue of Proposition 3.3 proved in the same way.

**Lemma 4.2.** Let $\mathcal{F}$ be an algebraic $K$-stack and $\mathcal{A}, \mathcal{B} \subseteq \mathcal{F}(K)$ constructible subsets. Then $\mathcal{A} \cup \mathcal{B}$, $\mathcal{A} \cap \mathcal{B}$ and $\mathcal{A} \setminus \mathcal{B}$ are constructible in $\mathcal{F}(K)$.

**Definition 4.3.** Let $K$ be an algebraically closed field, $\mathcal{F}$ an algebraic $K$-stack, and $\mathcal{S} \subseteq \mathcal{F}(K)$ be locally constructible. Call a function $f : \mathcal{S} \rightarrow \mathbb{Q}$ constructible if $f(\mathcal{S})$ is finite and $f^{-1}(c)$ is a constructible set for each $c \in f(\mathcal{S}) \setminus \{0\}$. Call $f : \mathcal{S} \rightarrow \mathbb{Q}$ locally constructible if $f|_\mathcal{C}$ is constructible for all constructible $\mathcal{C} \subseteq \mathcal{S} \subseteq \mathcal{F}(K)$. Write $\text{CF}(\mathcal{S}), \text{LCF}(\mathcal{S})$ for the sets of (locally) constructible functions on $\mathcal{S}$. Using Lemma 4.2 we see that $\text{CF}(\mathcal{S}), \text{LCF}(\mathcal{S})$ are $\mathbb{Q}$-vector spaces. For brevity write $\text{CF}(\mathcal{F}), \text{LCF}(\mathcal{F})$ rather than $\text{CF}(\mathcal{F}(K)), \text{LCF}(\mathcal{F}(K))$.

As in Definition 3.3 using Lemma 4.2 we see that multiplication of functions makes $\text{CF}(\mathcal{S}), \text{LCF}(\mathcal{S})$ into commutative $\mathbb{Q}$-algebras, with $\text{CF}(\mathcal{S})$ an ideal in $\text{LCF}(\mathcal{S})$, and $\text{CF}(\mathcal{S})$ is an algebra without identity if $\mathcal{S}$ is not constructible.

Now let $\mathcal{F}, \mathcal{G}$ be algebraic $K$-stacks, and $\mathcal{S} \subseteq \mathcal{F}(K), \mathcal{T} \subseteq \mathcal{G}(K)$ be locally constructible. Then $\mathcal{F} \times \mathcal{G}$ is an algebraic $K$-stack with $(\mathcal{F} \times \mathcal{G})(K) = \mathcal{F}(K) \times \mathcal{G}(K)$. Let $\Phi : \mathcal{S} \rightarrow \mathcal{T}$ be a map, and define the graph $\Gamma_\Phi = \{(s, \Phi(s)) : s \in \mathcal{S}\}$. We call $\Phi$ a pseudomorphism if $\Gamma_\Phi \cap (C \times \mathcal{G}(K))$ is constructible in $(\mathcal{F} \times \mathcal{G})(K)$ for all constructible $C \subseteq \mathcal{F}(K)$. A pseudomorphism $\Phi$ is a pseudoisomorphism if $\Phi$ is bijective and $\Phi^{-1} : T \rightarrow S$ is a pseudomorphism.

These definitions agree with those of 3.3 when $\mathcal{F}, \mathcal{G}$ are $K$-schemes.

### 4.2 Constructible sets and pseudomorphisms in stacks

We now extend properties of constructible sets and pseudomorphisms in $K$-schemes to algebraic $K$-stacks with affine geometric stabilizers.

**Proposition 4.4.** Let $\mathcal{F}$ be a finite type algebraic $K$-stack with affine geometric stabilizers. Then there exist substacks $\mathcal{F}_1, \ldots, \mathcal{F}_n$ of $\mathcal{F}$ with $\mathcal{F}(K) = \bigsqcup_{a=1}^n \mathcal{F}_a(K)$, $K$-varieties $Y_1, \ldots, Y_n$, and 1-morphisms $\phi_a : \mathcal{F}_a \rightarrow Y_a$ with $(\phi_a)_* : \mathcal{F}_a(K) \rightarrow Y_a(K)$ bijective for $a = 1, \ldots, n$. 

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Proof. By Theorem 2.4, $\mathfrak{F}$ can be stratified by global quotient stacks. Thus there exist finitely many substacks $U_i$ of $\mathfrak{F}$ with $\mathfrak{F}(K) = \bigcup_i U_i(K)$, and such $U_i$ 1-isomorphic to a quotient stack $[X_i/G_i]$, for $X_i$ a $K$-variety and $G_i$ an affine algebraic $K$-group acting on $X_i$. Theorem 2.3 gives a dense open $G_i$-invariant $X'_i \subseteq X_i$, and a morphism of $K$-varieties $\pi_i : X'_i \rightarrow Y_i$ inducing a bijection $X'_i(K)/G_i \rightarrow Y_i(K)$. We may write $X_i \setminus X'_i$ as a disjoint union of finitely many $G_i$-invariant $K$-subvarieties $X_{ij}$ with $\dim X_{ij} < \dim X_i$. Applying Theorem 2.3 again gives a dense open $G_i$-invariant $X'_{ij} \subseteq X_{ij}$ and a morphism of $K$-varieties $\pi_{ij} : X'_{ij} \rightarrow Y_{ij}$ inducing a bijection $X'_{ij}(K)/G_{ij} \rightarrow Y_{ij}(K)$.

As the dimension decreases at each stage, this process eventually yields finitely many substacks $\mathfrak{F}_i$ of $\mathfrak{F}$ with $\mathfrak{F}(K) = \bigcup_{i \in I} \mathfrak{F}_i(K)$, such that $\mathfrak{F}_i$ is 1-isomorphic to $[X_a/G_a]$ for $K$-varieties $X_a$, $Y_a$ and $G_a$ an algebraic $K$-group acting on $X_a$, and morphisms $\pi_a : X_a \rightarrow Y_a$ inducing a bijection between $X_a(K)/G_a$ and $Y_a(K)$. The 1-isomorphisms $\mathfrak{F}_a \cong [X_a/G_a]$ and $\pi_a$ combine to give a 1-morphism $\phi_a : \mathfrak{F}_a \rightarrow Y_a$ with the properties we want.

We extend the last part of Proposition 3.3 to stacks.

Proposition 4.5. Let $K$ be an algebraically closed field, $\mathfrak{F}, \mathfrak{G}$ be algebraic $K$-stacks with affine geometric stabilizers, $\phi : \mathfrak{F} \rightarrow \mathfrak{G}$ be a 1-morphism, and $C \subseteq \mathfrak{F}(K)$ be constructible. Then $\phi_*(C)$ is constructible in $\mathfrak{G}(K)$.

Proof. By Definition 2.1 $C = \bigcup_{i \in I} \mathfrak{F}_i(K)$, where $\{ \mathfrak{F}_i : i \in I \}$ are finitely many finite type substacks $\mathfrak{F}_i$ of $\mathfrak{F}$. So by Lemma 2.2 it is enough to show each $\phi_*(\mathfrak{F}_i(K))$ is constructible. As by convention $\mathfrak{G}$ is locally of finite type it admits an open cover $\{ \mathfrak{G}_j : j \in J \}$ of finite type substacks $\mathfrak{G}_j$. By Proposition 4.4, for $a = 1, \ldots, n$, there exist substacks $\mathfrak{G}_{ja}$ of $\mathfrak{G}_j$, $K$-varieties $Y_{ja}$ and 1-morphisms $\psi_{ja} : \mathfrak{G}_{ja} \rightarrow Y_{ja}$ with $\psi_{ja}$, bijective, such that $\mathfrak{G}_j(K) = \bigcup_{a=1}^n \mathfrak{G}_{ja}(K)$.

Now $\{ \phi^{-1}(\mathfrak{G}_j) : j \in J \}$ covers $\mathfrak{F}_i$, which is quasicompact as it is of finite type. So there exists a finite subset $J_i \subseteq J$ such that $\{ \phi^{-1}(\mathfrak{G}_j) : j \in J_i \}$ covers $\mathfrak{F}_i$. Set $\mathfrak{F}_{ija} = \mathfrak{F}_i \cap \phi^{-1}(\mathfrak{G}_{ja})$ for $j \in J_i$ and $a = 1, \ldots, n_j$. Now $\mathfrak{F}_{ija}$ is a finite type $K$-substack, and $\phi_{ija} = \phi|_{\mathfrak{F}_{ija}} : \mathfrak{F}_{ija} \rightarrow \mathfrak{G}_{ja}$ a 1-morphism with $\phi_{ija}(\mathfrak{F}_{ija}(K)) = \bigcup_{j \in J_i} \bigcup_{a=1}^{n_j} (\psi_{ja})(\mathfrak{F}_{ija}(K))$. Hence by Lemma 1.2 it suffices to show $(\psi_{ja})_*((\mathfrak{F}_{ija}(K)))$ is constructible in $\mathfrak{G}_{ja}(K)$ for all $i, j, a$.

As $\mathfrak{F}_{ija}$ is finite type it has an atlas $u_{ija} : U_{ija} \rightarrow \mathfrak{F}_{ija}$ with $U_{ija}$ a finite type $K$-scheme. Then $\psi_{ja} \circ \phi_{ija} \circ u_{ija} : U_{ija} \rightarrow Y_{ja}$ is a morphism of $K$-schemes. But $U_{ija}(K)$ is constructible as $U_{ija}$ is of finite type, so Proposition 3.3 shows

$$((\psi_{ja} \circ \phi_{ija} \circ u_{ija})_*)(U_{ija}(K)) = (\psi_{ja})_* \circ (\phi_{ija})_* \circ (u_{ija})_* (U_{ija}(K))$$

is constructible in $Y_{ja}(K)$, where the second line follows since $(u_{ija})_*$ is surjective as $u_{ija}$ is an atlas. Now $\psi_{ja} : \mathfrak{G}_{ja} \rightarrow Y_{ja}$ is a finite type 1-morphism, so it pulls back constructible subsets to constructible subsets. Therefore

$$(\psi_{ja})_*^{-1}((\psi_{ja})_* \circ (\phi_{ija})_* (\mathfrak{F}_{ija}(K))) = (\phi_{ija})_* (\mathfrak{F}_{ija}(K))$$

is constructible in $\mathfrak{G}_{ja}(K)$, using $(\psi_{ja})_*$ a bijection in the second step.\]
Lemma 4.2 and Proposition 4.3 extend Proposition 3.3 to algebraic $K$-stacks with affine geometric stabilizers. As the proof of Proposition 3.15 depended only on Proposition 3.3, it extends to such stacks.

**Proposition 4.6.** Let $K$ be an algebraically closed field, and $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be algebraic $K$-stacks with affine geometric stabilizers.

(a) Let $\phi : \mathcal{F} \to \mathcal{G}$ be a 1-morphism (1-isomorphism). Then $\phi_* : \mathcal{F}(K) \to \mathcal{G}(K)$ is a pseudomorphism (pseudoisomorphism).

(b) Let $S \subseteq \mathcal{F}(K), T \subseteq \mathcal{G}(K)$ be locally constructible, $\Phi : S \to T$ be a pseudomorphism, and $C \subseteq S$ be constructible. Then $\Phi(C)$ is constructible in $\mathcal{G}(K)$. Also, if $t \in T$ then $C \cap \Phi^{-1}(t)$ is constructible in $\mathcal{F}(K)$. Hence, $\Phi^{-1}(t)$ is locally constructible in $\mathcal{F}(K)$.

(c) Let $S \subseteq \mathcal{F}(K), T \subseteq \mathcal{G}(K), U \subseteq \mathcal{H}(K)$ be locally constructible, and $\Phi : S \to T, \Psi : T \to U$ be pseudo(iso)morphisms. Then $\Psi \circ \Phi : S \to U$ is a pseudo(iso)morphism.

The next proposition allows results about constructible sets and functions on schemes to be easily extended to stacks.

**Proposition 4.7.** Let $K$ be an algebraically closed field, $\mathcal{F}$ an algebraic $K$-stack with affine geometric stabilizers, and $C \subseteq \mathcal{F}(K)$ be constructible. Then $C$ is pseudoisomorphic to $Y(K)$ for a separated, finite type $K$-scheme $Y$.

**Proof.** Write $C = \coprod_{i \in I} \mathcal{F}_i(K)$ for $\mathcal{F}_i, i \in I$ finitely many finite type substacks in $\mathcal{F}$. Proposition 3.14 gives substacks $\mathcal{F}_i$ in $\mathcal{F}_i$, $K$-varieties $Y_i$ and 1-morphisms $\phi_{ia} : \mathcal{F}_i \to Y_i$ for $a = 1, \ldots, n_i$, with $\mathcal{F}_i(K) = \coprod_{a=1}^{n_i} \mathcal{F}_{i a}(K)$, and $(\phi_{ia})_*$ bijective. Let $Y$ be the abstract disjoint union of the $Y_i$ for $i \in I$ and $a = 1, \ldots, n_i$, as in Definition 2.2. It is a separated, finite type $K$-scheme. Define $\Phi : C \to Y(K)$ by $\Phi|_{\mathcal{F}_i(K)} = (\phi_{ia})_*$ for all $i, a$. Then $\Phi$ is bijective, as $(\phi_{ia})_*$ is.

Proposition 4.10(a) shows $(\phi_{ia})_*$ is a pseudomorphism, so $\Phi$ is a pseudomorphism. As $\Phi$ is bijective and $C, Y(K)$ constructible, $\Phi$ is a pseudoisomorphism.

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**4.3 The naïve Euler characteristic and pushforward**

Fix an algebraically closed field $K$ of characteristic zero for the rest of the section. We consider the simplest generalization of $\chi(X), \chi(S, f), CF(\Phi)$ to $K$-stacks $\mathcal{F}$, which we call naïve as it ignores the stabilizer groups $\text{Iso}_K(x)$ for $x \in \mathcal{F}(K)$. Here is the analogue of Definitions 3.11 and 3.16.

**Definition 4.8.** Let $\mathcal{F}$ be an algebraic $K$-stack with affine geometric stabilizers, and $C \subseteq \mathcal{F}(K)$ be constructible. Then $C$ is pseudoisomorphic to $Y(K)$ for a separated, finite type $K$-scheme $Y$ by Proposition 4.7. Define the naïve Euler characteristic $\chi^\text{na}(C)$ by $\chi^\text{na}(C) = \chi(Y)$, where $\chi(Y)$ is as in Definition 3.9. If $Y'$ is another choice for $Y$ then $Y(K)$ is pseudoisomorphic to $Y'(K)$ by Proposition 4.10(c), so $\chi(Y) = \chi(Y')$. Thus $\chi^\text{na}(C)$ is well-defined. Now let...
As the unrestricted functions are zero outside \( B,C \) functions, these easily imply \( \beta,\gamma \). Theorem 3.17 gives \( \text{CF}(f) \) pseudoisomorphisms \( \alpha \).

By Proposition 4.7 there exist separated, finite type \( B,C \) also \( \delta \). Define \( \Phi : S \to T \) be a pseudomorphism, and \( f \in \text{CF}(S) \). Define the \textit{naive pushforward} \( \text{CF}^{na}(\Phi) : T \to Q \) of \( f \) to \( T \) by

\[
\text{CF}^{na}(\Phi) f(t) = \chi^{na}(S,f \cdot \delta_{\Phi^{-1}(t)}) \quad \text{for } t \in T.
\]

Here \( \delta_{\Phi^{-1}(t)} \) is the characteristic function of \( \Phi^{-1}(t) \subseteq S \) on \( S \). As \( \Phi^{-1}(t) \) is locally constructible by Proposition 4.6(b) we have \( \delta_{\Phi^{-1}(t)} \in \text{LCF}(S) \), and \( f \in \text{CF}(S) \), so \( f \cdot \delta_{\Phi^{-1}(t)} \in \text{CF}(S) \). Thus (9) is well-defined.

Here are the naive generalizations of Theorem 3.17 and Corollary 3.18.

**Theorem 4.9.** Let \( \mathcal{S}, \mathcal{G}, \mathcal{F} \) be algebraic \( K \)-stacks with affine geometric stabilizers, \( S \subseteq \mathcal{S}(K) \), \( T \subseteq \mathcal{G}(K) \) are locally constructible, and \( \Phi : S \to T \), \( \Psi : T \to U \) be pseudomorphisms. If \( f \in \text{CF}(S) \), then \( \text{CF}^{na}(\Phi) f \) is a constructible function on \( T \). Thus \( \text{CF}^{na}(\Phi) : \text{CF}(S) \to \text{CF}(T) \) is a \( Q \)-linear map.

Also \( \text{CF}^{na}(\Psi \circ \Phi) = \text{CF}^{na}(\Psi) \circ \text{CF}^{na}(\Phi) \) as linear maps \( \text{CF}(S) \to \text{CF}(U) \).

**Proof.** Define \( A = \text{supp}(f) \subseteq S \), \( B = \Phi(A) \subseteq T \) and \( C = \Psi(B) \subseteq U \). Then \( A \) is constructible by Definition 4.3 so \( B,C \) are constructible by Proposition 4.6(b). By Proposition 4.7 there exist separated, finite type \( K \)-schemes \( X,Y,Z \) and pseudoisomorphisms \( \alpha : A \to X(K) \), \( \beta : B \to Y(K) \) and \( \gamma : C \to Z(K) \). Then \( f \circ \alpha^{-1} \in \text{CF}(X) \) as \( f|_A \in \text{CF}(A) \). By Proposition 4.6(c), \( \beta \circ \Phi \circ \alpha^{-1} : X(K) \to Y(K) \) and \( \gamma \circ \Psi \circ \beta^{-1} : Y(K) \to Z(K) \) are pseudomorphisms of \( K \)-schemes, so Theorem 3.17 gives \( \text{CF}(\beta \circ \Phi \circ \alpha^{-1})(f \circ \alpha^{-1}) \in \text{CF}(Y) \) and

\[
\text{CF}^{na}(\Psi \circ \Phi) f|_C = \text{CF}^{na}(\Psi) \circ \text{CF}^{na}(\Phi) f|_C.
\]

As the unrestricted functions are zero outside \( B,C \), the theorem follows.

**Corollary 4.10.** Let \( \mathcal{S}, \mathcal{G} \) be algebraic \( K \)-stacks with affine geometric stabilizers, \( S \subseteq \mathcal{S}(K) \), \( T \subseteq \mathcal{G}(K) \) be locally constructible, \( \Phi : S \to T \) be a pseudomorphism, and \( f \in \text{CF}(S) \). Then \( \chi^{na}(S,f) = \chi^{na}(T,\text{CF}^{na}(\Phi) f) \).

### 4.4 Stabilizers \( \text{Iso}_K(x) \) and weight functions

We now discuss how to modify the naive Euler characteristic \( \chi^{na} \) and pushforward \( \text{CF}^{na} \) of 4.3 to take account of stabilizer groups \( \text{Iso}_K(x) \) for \( x \in \mathcal{S}(K) \). We do this by inserting a \textit{weight} \( w_X \) depending on \( \text{Iso}_K(x) \). We continue to fix \( K \) algebraically closed of characteristic zero.
Definition 4.11. Let \( w : \{ \text{affine algebraic } \mathbb{K}\text{-groups } G \} \to \mathbb{Q} \cup \{ \infty \}, G \mapsto w(G) \) be a map with \( w(G) = w(G') \) if \( G \cong G' \). If \( \mathfrak{X} \) is an algebraic \( \mathbb{K}\text{-stack} \) with affine geometric stabilizers, define \( w_\mathfrak{X} : \mathfrak{X}(\mathbb{K}) \to \mathbb{Q} \cup \{ \infty \} \) by \( w_\mathfrak{X}(x) = w(\text{Isog}_\mathbb{K}(x)) \). We call \( w \) an allowable weight function if \( w_\mathfrak{X} \) is a locally constructible function on \( \mathfrak{X} \) with values in \( \mathbb{Q} \cup \{ \infty \} \) for all \( \mathfrak{X} \). We also call \( w \) multiplicative if \( w(G \times H) = w(G)w(H) \) for all affine algebraic \( \mathbb{K}\text{-groups } G, H \).

Here are the weighted analogues \( \chi_w(C) \), \( \chi_w(S, f) \), \( CF_w(\Phi)f \) of \( \chi^a(C) \), \( \chi^a(S, f) \), \( CF^a(\Phi)f \). We allow \( w \) to take the values \( 0, \infty \) to accommodate the examples below. This means \( \chi_w(C), \chi_w(S, f), CF_w(\Phi)f \) are not always defined.

Definition 4.12. Let \( w \) be an allowable weight function, \( \mathfrak{X}, \mathfrak{G} \) algebraic \( \mathbb{K}\text{-stacks} \) with affine geometric stabilizers, \( C \subseteq \mathfrak{X}(\mathbb{K}) \text{ constructible}, S \subseteq \mathfrak{X}(\mathbb{K}), T \subseteq \mathfrak{G}(\mathbb{K}) \text{ locally constructible}, \) and \( \Phi : S \to T \) a pseudomorphism.

If \( w_\mathfrak{X} \neq \infty \) on \( C \), define the \( w \)-Euler characteristic \( \chi_w(C) = \chi^a(C, w_\mathfrak{X}|_C) \).

If \( w_\mathfrak{X}(c) = \infty \) for some \( c \in C \) we say \( \chi_w(C) \) is undefined. For \( f \in CF(S) \) with \( w_\mathfrak{X} \neq \infty \) on supp \( f \), define the weighted \( w \)-Euler characteristic \( \chi_w(S, f) \) by \( \chi_w(S, f) = \chi^a(S, w_\mathfrak{X}f) = \chi^a(\text{supp } f, w_\mathfrak{X}f) \), taking \( w_\mathfrak{X}f = 0 \) outside supp \( f \) even where \( w_\mathfrak{X} = \infty \). If \( w_\mathfrak{X}(s) = \infty \) for some \( s \in \text{supp } f \) we say \( \chi_w(S, f) \) is undefined. If \( w_\mathfrak{X} \neq \infty \) on \( S \) and \( w_\mathfrak{G} \neq 0 \) on \( T \) then \( w_\mathfrak{X} \in \text{LCF}(S) \) and \( w_\mathfrak{G}^{-1} \in \text{LCF}(T) \) by Definition 4.11. Define

\[
CF_w(\Phi)f = w_\mathfrak{G}^{-1} \cdot CF^a(\Phi)(w_\mathfrak{X}f) \quad \text{for } f \in CF(S). \tag{10}
\]

This is well-defined in \( CF(T) \) as \( w_\mathfrak{X}f \in CF(S) \), so \( CF^a(\Phi)(w_\mathfrak{X}f) \in CF(T) \). Therefore \( CF_w(\Phi) : CF(S) \to CF(T) \) is a \( \mathbb{Q} \)-linear map. If \( w_\mathfrak{X}(s) = \infty \) for some \( s \in S \) or \( w_\mathfrak{G}(t) = 0 \) for some \( t \in T \), we say \( CF_w(\Phi) \) is undefined.

Then \( \chi_w \) satisfies the following analogues of Theorem 3(ii),(iii):

Lemma 4.13. Let \( w \) be an allowable weight function, and \( \mathfrak{X}, \mathfrak{G} \) algebraic \( \mathbb{K}\text{-stacks} \) with affine geometric stabilizers. Then

(i) Suppose \( C, D_1, \ldots, D_m \subseteq \mathfrak{X}(\mathbb{K}) \) are constructible with \( C = \coprod_{i=1}^m D_i \). Then \( \chi_w(C) = \sum_{i=1}^m \chi_w(D_i) \) if either side is defined.

(ii) If \( w \) is multiplicative and \( C \subseteq \mathfrak{X}(\mathbb{K}), D \subseteq \mathfrak{G}(\mathbb{K}) \) are constructible then \( \chi_w(C \times D) = \chi_w(C) \chi_w(D) \) if both sides are defined.

For the analogue of Theorem 3(i) from 10 we have

\[
CF_w(\Psi \circ \Phi)f = w^{-1}_\mathfrak{G} \cdot CF^a(\Psi \circ \Phi)(w_\mathfrak{X}f) = w^{-1}_\mathfrak{G} \cdot CF^a(\Psi) \circ CF^a(\Phi)(w_\mathfrak{X}f)
\]

\[
= w^{-1}_\mathfrak{G} \cdot CF^a(\Psi) [w_\mathfrak{G} \cdot CF_w(\Phi)f] = CF_w(\Psi) \circ CF_w(\Phi)f
\]

by Theorem 4.10 provided everything is defined. So we deduce:

Corollary 4.14. Let \( w \) be an allowable weight function, \( \mathfrak{X}, \mathfrak{G}, \mathfrak{H} \) algebraic \( \mathbb{K}\text{-stacks} \) with affine geometric stabilizers, \( S \subseteq \mathfrak{X}(\mathbb{K}), T \subseteq \mathfrak{G}(\mathbb{K}), U \subseteq \mathfrak{H}(\mathbb{K}) \) locally constructible with \( w_\mathfrak{X} \neq \infty \) on \( S \), \( w_\mathfrak{G} \neq 0, \infty \) on \( T \), \( w_\mathfrak{H} \neq 0 \) on \( U \), and \( \Phi : S \to T, \Psi : T \to U \) be pseudomorphisms. Then \( CF_w(\Psi \circ \Phi) = CF_w(\Psi) \circ CF_w(\Phi) \).
As in Corollary 4.10 we have:

**Corollary 4.15.** Let $w$ be an allowable weight function, $\mathfrak{G}$ algebraic $K$-stacks with affine geometric stabilizers, $S \subseteq \mathfrak{G}(K)$, $T \subseteq \mathfrak{G}(K)$ locally constructible with $w \neq \infty$ on $S$, $w \neq 0, \infty$ on $T$, $\Phi : S \to T$ a pseudomorphism, and $f \in CF(S)$. Then $\chi_w(S, f) = \chi_w(T, CF_w(\Phi)f)$.

Here are two examples of multiplicative allowable weight functions.

**Proposition 4.16.** (a) Define $e : \{\text{affine algebraic } K\text{-groups}\} \to \mathbb{Z}$ by $e(G) = \chi(G)$. Then $e$ is a multiplicative allowable weight function.

(b) If $G$ is an affine algebraic $K$-group, define the adjoint action of $G$ on itself by $Ad(g) h = g h g^{-1}$. Then the quotient $[G/Ad(G)]$ is an algebraic $K$-stack of finite type. Define $o(G) = \chi^{na}([G/Ad(G)])$. Then $o : \{\text{affine algebraic } K\text{-groups}\} \to \mathbb{Z}$ is a multiplicative allowable weight function.

**Proof.** Clearly $e$ and $o$ are well-defined and multiplicative. Let $\mathfrak{G}$ be an algebraic $K$-stack with affine geometric stabilizers. We must show $e_{\mathfrak{G}}, o_{\mathfrak{G}} \in CF(\mathfrak{G})$, which holds provided $e_{\mathfrak{G}} = e_{\mathfrak{G}\circ \epsilon(\mathfrak{G})}$ and $o_{\mathfrak{G}} = o_{\mathfrak{G}\circ \epsilon(\mathfrak{G})}$ lie in $CF(\mathfrak{G})$ for all finite type $K$-substacks $\mathfrak{G}$ in $\mathfrak{G}$. Theorem 4.14 gives $\mathfrak{G}(K) = \bigcup_{i \in I} U_i(K)$, where $\{U_i : i \in I\}$ are finitely many substacks of $\mathfrak{G}$ with $U_i$ 1-isomorphic to $[X_i/G_i]$ for $X_i$ a $K$-variety and $G_i$ an affine algebraic $K$-group, acting on $X_i$ by $\rho_i : X_i \times G_i \to X_i$.

Write $\pi_i : X_i \to U_i$ for the projection 1-morphism. Let $Y_i$ be the inverse image under $id_{X_i} \times \rho_i : X_i \times G_i \to X_i \times X_i$ of the diagonal in $X_i \times X_i$. Then $Y_i$ is a finite type closed subscheme of $X_i \times G_i$. Let $\sigma_i : Y_i \to X_i$ be the restriction of the projection $X_i \times G_i \to X_i$. Then for each $x \in X_i(K)$, $\sigma_i^{-1}(x) = \{x\} \times Stab_x(G_i)$, where $Stab_x(G_i)$ is the stabilizer subgroup of $x$ in $G_i$.

Theorem 4.14 gives $CF(\sigma_i)1 \in CF(X_i)$, as $1 \in CF(Y_i)$. But for $x \in X_i(K)$

$$(CF(\sigma_i)1)(x) = \chi[Stab_x(G_i)] = \chi[\text{Isok}(((\pi_i)_*(x))]] = e_{\mathfrak{G}}((\pi_i)_*(x)),$$

as $\sigma_i^{-1}(x) = \{x\} \times Stab_x(G_i)$, and the stabilizer group $\text{Isok}(((\pi_i)_*(x))$ in $U_i \cong [X_i/G_i]$ is Stab$_x(G_i)$. Therefore $CF(\pi_i)1 = e_{\mathfrak{G}} \circ (\pi_i)_*$ as maps $X_i(K) \to \mathbb{Q}$. Since $(\pi_i)_*$ is surjective this implies that $e_{\mathfrak{G}|U_i(K)} \in CF(U_i)$, as $(\pi_i)_*$ takes constructible sets to constructible sets by Proposition 4.4. And $\mathfrak{G}(K) = \bigcup_{i \in I} U_i(K)$ and $I$ is finite, so $e_{\mathfrak{G}} \in CF(\mathfrak{G})$. This proves (a).

For (b), we form an algebraic $K$-stack $\mathfrak{H}$ with 1-morphisms $Y_i \xrightarrow{\alpha_i} \mathfrak{H} \xrightarrow{\beta_i} X_i$ with $\pi_i = \beta_i \circ \alpha_i$, such that if $x \in X_i(K)$ with $\text{Stab}_x(G_i) = H$, then $\pi_i^{-1}(x) = \{x\} \times H$, and $\alpha_i : \pi_i^{-1}(x) \to \beta_i^{-1}(x)$ is the projection $H \to [H/Ad(H)]$. Then $\alpha_i$ is an atlas, so $\mathfrak{H}$ is of finite type. Thus $1 \in CF(\mathfrak{H})$, so $CF^{na}(\beta_i)1 \in CF(X_i)$ by Theorem 4.14. But for $x \in X_i(K)$

$$(CF^{na}(\beta_i)1)(x) = \chi^{na}([\text{Stab}_x(G_i)/Ad(\text{Stab}_x(G_i))]$$

$$= \chi^{na}([\text{Isok}(((\pi_i)_*(x))]/Ad(\text{Isok}(((\pi_i)_*(x))))) = o_{\mathfrak{G}}((\pi_i)_*(x)).$$

The rest of the proof is as for (a). □

Other weight functions constructed from $e, o$ in a multiplicative way are also multiplicative and allowable, such as $e^k, o^k, |e|^k, |o|^k, \text{sign}(e), \text{sign}(o)$ and $e^k o^l$ for $k, l \in \mathbb{Z}$ with $k l > 0$. We give special names to two interesting cases.
defines known that for a complex orbifold $M/G$, Euler characteristics, see Roan [20]. In particular, it is believed and in many characteristic $\chi$ equivariant K-theory acting on a compact manifold $M$ be the natural notions for the problems in [9–11]. If $X$ $\chi$ scheme of $M/G$ stacks something already well understood for orbifolds. Let $x$ Cartesian squares It also has a universal property in $M$ where $stk$ the definition of CF $\chi$ as a subgroup, $c$ interesting situations, including everything in [9–11]. But in and this will be sufficient for the applications of [9–11]. Unfortunately, as $\chi$ takes values in $Z$, $\chi$ and $\chi$ are always defined, and $\chi$ is defined if $o(\text{Isog}(t)) \neq 0$ for all $t \in T$.

The stack Euler characteristic $\chi^{stk}$ and its pushforward CF$^{stk}$ turn out to be the natural notions for the problems in [9–11]. If $X$ is a $K$-variety and $G$ an algebraic $K$-group acting on $X$ with $\chi(G) \neq 0$, then $\chi^{stk}(X/G) = \chi(X)/\chi(G)$. It also has a universal property in Cartesian squares, in $\chi$. Unfortunately, as $\chi(G) = 0$ for any algebraic $K$-group $G$ with $K^x = K \setminus \{0\}$ as a subgroup, $\chi^{stk}(C), \chi^{stk}(S, f)$ and $\text{CF}^{stk}(\Phi)$ above are undefined in many interesting situations, including everything in [9–11]. But in [5], we will extend the definition of $\text{CF}^{stk}(\Phi)$ to $\text{CF}^{stk}(\phi)$ for $\phi : \mathcal{F} \to \mathcal{G}$ a representable 1-morphism, and this will be sufficient for the applications of [9–11].

For Deligne–Mumford stacks all stabilizer groups are finite, and for $G$ finite $\chi(G) = |G| > 0$, so that $\chi^{stk}, \text{CF}^{stk}$ are always defined. It is well-established that for enumerative problems on Deligne–Mumford stacks one counts a point $x \in \mathcal{F}(K)$ with weight $1/|\text{Isog}(x)|$, and $\chi^{stk}$ generalizes this approach.

The orbifold Euler characteristic is the author’s attempt to generalize to stacks something already well understood for orbifolds. Let $G$ be a finite group acting on a compact manifold $M$, so that $M/G$ is an orbifold. Dixon et al. [2, p. 684] observe the correct Euler characteristic of $M/G$ in String Theory is

$$\chi(M, G) = \frac{1}{|G|} \sum_{g, h \in G : gh = hg} \chi(M^{g, h}),$$

(11)

where $M^{g, h} = \{x \in M : g \cdot x = h \cdot x = x\}$.

Atiyah and Segal [1] later interpreted $\chi(M, G)$ as the Euler characteristic of equivariant K-theory $K_G(M)$. For a survey and further references on orbifold Euler characteristics, see Roan [20]. In particular, it is believed and in many cases known that for a complex orbifold $M/G$, $\chi(M, G)$ coincides with the Euler characteristic $\chi(X)$ of any crepant resolution of $M/G$.

Let $M$ be a $K$-scheme acted on by a finite group $G$. Then $M^{g, h}$ is a subscheme of $M$, and [11] makes sense. An easy calculation shows $\chi(M, G) =$
\( \chi^{na}([M/G], f) \), where \( f \in \text{CF}([M/G]) \) is given by

\[
f(x) = \frac{\{(g,h) \in \text{Iso}_G(x)^2 : gh = hg\}}{|\text{Iso}_G(x)|} = \sum_{g \in \text{Iso}_G(x)} \frac{1}{\text{Ad}(\text{Iso}_G(x))g} = \left| \text{Iso}_G(x)/\text{Ad}(\text{Iso}_G(x)) \right| = \alpha_{[M/G]}(x)
\]

for \( x \in [M/G]/(\mathbb{K}) \). Hence \( \chi(M, G) = \chi^{arb}([M/G]) \) by Definitions 4.11 and 4.17.

Thus, our orbifold Euler characteristic \( \chi^{arb}([M/G]) \) of the Deligne–Mumford stack \([M/G]\) agrees with the physicists’ orbifold Euler characteristic \( \chi(M, G) \) of the complex orbifold \( M/G \) when \( \mathbb{K} = \mathbb{C} \), but our notion \( \chi^{arb} \) is also defined over other fields \( \mathbb{K} \) and for more general stacks \( \mathcal{F} \). It would be interesting to know whether \( \chi(M, G) \) being the Euler characteristic of any crepant resolution over \( \mathbb{C} \) extends using \( \chi^{arb} \) to other fields, or to more general stacks.

## 5 Representable and finite type 1-morphisms

Next we study stack pushforwards \( \text{CF}^{\text{stk}}(\phi) \) by 1-morphisms \( \phi : \mathcal{F} \to \mathcal{G} \). Then \( \phi_* : \text{CF}^\text{stk}(\mathcal{F}) \to \text{CF}^\text{stk}(\mathcal{G}) \) is a pseudomorphism, so the obvious definition is \( \text{CF}^{\text{stk}}(\phi) = \text{CF}^{\text{stk}}(\phi_*) \). However, \( \text{CF}^{\text{stk}}(\phi_*) \) is undefined if \( x \in \mathcal{F}(\mathbb{K}) \) with \( \chi(\text{Iso}_G(x)) = 0 \).

Since \( \chi(G) = 0 \) for many affine algebraic \( \mathbb{K} \)-groups \( G \), this is a serious drawback. Instead, by using the extra data of the homomorphisms \( \phi_* : \text{Iso}_G(x) \to \text{Iso}_G(\phi_* (x)) \), in [5.1] we define \( \text{CF}^{\text{stk}}(\phi) \) in many cases when \( \text{CF}^{\text{stk}}(\phi_*) \) is undefined, in particular for all representable \( \phi \).

Section 5.2 defines the pullback \( \psi^* : \text{CF}(\mathcal{G}) \to \text{CF}(\mathcal{F}) \) for a finite type 1-morphism \( \psi : \mathcal{F} \to \mathcal{G} \), and proves pullbacks \( \psi^* \) and pushforwards \( \text{CF}^{\text{stk}}(\phi) \) commute in Cartesian squares. This will be an important tool in [9–11]. In 5.3 for finite type \( \phi : \mathcal{F} \to \mathcal{G} \) we extend \( \chi^{na}(\phi_*) \), \( \text{CF}^{\text{stk}}(\phi) \) to locally constructible functions, with the usual functorial property.

Fix an algebraically closed field \( \mathbb{K} \) of characteristic zero for all of this section.

### 5.1 Pushforwards by representable 1-morphisms

Here is our definition of the stack pushforward \( \text{CF}^{\text{stk}}(\phi) \) for a 1-morphism \( \phi \).

**Definition 5.1.** Let \( \mathcal{F}, \mathcal{G} \) be algebraic \( \mathbb{K} \)-stacks with affine geometric stabilizers and \( \phi : \mathcal{F} \to \mathcal{G} \) a 1-morphism. Then for any \( x \in \mathcal{F}(\mathbb{K}) \) we have a morphism \( \phi_* : \text{Iso}_G(x) \to \text{Iso}_G(\phi_* (x)) \) of affine algebraic \( \mathbb{K} \)-groups. The kernel \( \text{Ker} \phi_* \) is an affine algebraic \( \mathbb{K} \)-group in \( \text{Iso}_G(x) \), so \( \chi(\text{Ker} \phi_*) \) is defined. The image \( \phi_* (\text{Iso}_G(x)) \) is an affine algebraic \( \mathbb{K} \)-group closed in \( \text{Iso}_G(\phi_*(x)) \), so the quotient \( \text{Iso}_G(\phi_*(x))/\phi_*(\text{Iso}_G(x)) \) is a quasiprojective \( \mathbb{K} \)-variety. Thus \( \chi(\text{Iso}_G(\phi_*(x))/\phi_*(\text{Iso}_G(x))) \) is also defined.

Suppose \( \chi(\text{Ker} \phi_*) \neq 0 \) for all \( x \in \mathcal{F}(\mathbb{K}) \). Define \( m_\phi : \mathcal{F}(\mathbb{K}) \to \mathbb{Q} \) by

\[
m_\phi(x) = \frac{\chi(\text{Iso}_G(\phi_*(x))/\phi_*(\text{Iso}_G(x)))}{\chi(\text{Ker} \phi_* : \text{Iso}_G(x) \to \text{Iso}_G(\phi_*(x)))} \text{ for } x \in \mathcal{F}(\mathbb{K}).
\]
An argument similar to Proposition\[4.10\] shows \(m_\phi \in LCF(\mathcal{F})\). Define the stack pushforward \(CF^{stk}(\phi) : CF(\mathcal{F}) \to CF(\mathcal{G})\) by
\[
CF^{stk}(\phi)f = CF^{na}(\phi_\ast)(m_\phi f).
\]
(13)

Here \(m_\phi \cdot f \in CF(\mathcal{F})\) as \(m_\phi \in LCF(\mathcal{F})\) and \(f \in CF(\mathcal{F})\), so \(13\) is well-defined, and \(CF^{stk}(\phi) : CF(\mathcal{F}) \to CF(\mathcal{G})\) is \(\mathbb{Q}\)-linear.

This agrees with the previous definition of \(CF^{stk}(\phi_\ast)\) when it is defined, regarding \(\phi_\ast : \mathcal{F}(\mathbb{K}) \to \mathcal{G}(\mathbb{K})\) as a pseudomorphism by Proposition\[4.6(a)\].

**Lemma 5.2.** In Definition\[4.17\] if \(CF^{stk}(\phi_\ast)\) is defined in Definition\[4.6\] then \(CF^{stk}(\phi) : CF(\mathcal{F}) \to CF(\mathcal{G})\) is defined and \(CF^{stk}(\phi_\ast) = CF^{stk}(\phi)\).

**Proof.** Suppose \(CF^{stk}(\phi_\ast)\) is defined. Then \(1/e_\mathcal{F} \neq \infty\) in \(\mathcal{F}(\mathbb{K})\), so \(\chi(ISo_\mathcal{F}(x)) \neq 0\) for \(x \in \mathcal{F}(\mathbb{K})\). Now \(\ker \phi_\ast\) is normal in \(ISo_\mathcal{F}(x)\), with quotient \(ISo_\mathcal{F}(x)/\ker \phi_\ast\) naturally isomorphic to \(\phi_\ast(ISo_\mathcal{G}(x))\). Hence
\[
\chi(ISo_\mathcal{G}(x)) = \chi(Ker \phi_\ast) \cdot \chi(\phi_\ast(ISo_\mathcal{G}(x)));
\]
by general properties of \(\chi\). As \(\chi(ISo_\mathcal{G}(x)) \neq 0\) this implies \(\chi(Ker \phi_\ast) \neq 0\) for \(x \in \mathcal{F}(\mathbb{K})\), so \(CF^{stk}(\phi) : CF(\mathcal{F}) \to CF(\mathcal{G})\) is defined. Similarly, we have
\[
\chi(ISo_\mathcal{G}(\phi_\ast(x))) = \chi(\phi_\ast(ISo_\mathcal{G}(x))) \cdot \chi(ISo_\mathcal{G}((\phi_\ast) / \phi_\ast(ISo_\mathcal{G}(x))).
\]

Dividing this equation by the previous one for \(x \in \mathcal{F}(\mathbb{K})\), which is valid as \(\chi(ISo_\mathcal{G}(x)) \neq 0\), and using \(12\) gives
\[
m_\phi(x) = \frac{\chi(ISo_\mathcal{G}(\phi_\ast(x)))}{\chi(ISo_\mathcal{G}(x))} = \frac{e_\mathcal{G}(\phi_\ast(x))}{e_\mathcal{F}(x)} \quad \text{for } x \in \mathcal{F}(\mathbb{K}).
\]
It follows immediately from Definition\[4.17\] that \(CF^{stk}(\phi_\ast) = CF^{stk}(\phi)\).

The functorial behaviour of Theorem\[4.9\] holds for \(CF^{stk}(\phi)\).

**Theorem 5.3.** Let \(\mathcal{F}, \mathcal{G}, \mathcal{H}\) be algebraic \(\mathbb{K}\)-stacks with affine geometric stabilizers, and \(\phi : \mathcal{F} \to \mathcal{G}\), \(\psi : \mathcal{G} \to \mathcal{H}\) 1-morphisms. Suppose the kernels of \(\phi_\ast : ISo_\mathcal{F}(x) \to ISo_\mathcal{G}(\phi_\ast(x))\) for \(x \in \mathcal{F}(\mathbb{K})\) and \(\psi_\ast : ISo_\mathcal{G}(x) \to ISo_\mathcal{H}(\psi_\ast(y))\) for \(y \in \mathcal{G}(\mathbb{K})\) have nonzero Euler characteristics. Then \(CF^{stk}(\psi_\ast) = CF^{stk}(\psi) \circ CF^{stk}(\phi)\) as well-defined linear maps \(CF(\mathcal{F}) \to CF(\mathcal{H})\).

**Proof.** Let \(x \in \mathcal{F}(\mathbb{K})\), and set \(y = \phi_\ast(x)\) and \(z = \psi_\ast(y)\). Write
\[
G_x = ISo_\mathcal{F}(x), \quad G_y = ISo_\mathcal{G}(y), \quad G_z = ISo_\mathcal{H}(z), \quad \phi_\ast = \phi : G_x \to G_y,
\]
\[
\psi_y = \psi_\ast : G_y \to G_z, \quad K_{\phi_\ast} = Ker(\phi_\ast), \quad K_{\psi_\ast} = Ker(\psi_\ast), \quad K_{\psi_\ast \phi_\ast} = Ker(\psi_\ast \phi_\ast),
\]
\[
I_{\phi_\ast} = \Image(\phi_\ast), \quad I_{\psi_\ast} = \Image(\psi_\ast), \quad I_{\psi_\ast \phi_\ast} = \Image(\psi_\ast \phi_\ast).
\]
Then \(K_{\phi_\ast}\) is normal in \(K_{\psi_\ast \phi_\ast}\), and the quotient \(K_{\psi_\ast \phi_\ast} / K_{\phi_\ast}\) is isomorphic to \(I_{\phi_\ast} \cap K_{\psi_\ast}\). So general properties of \(\chi\) give
\[
\chi(K_{\psi_\ast \phi_\ast}) = \chi(K_{\phi_\ast}) \cdot \chi(I_{\phi_\ast} \cap K_{\psi_\ast}).
\]
(14)
The inclusions $I_{\phi,x} \cap K_{\psi,y} \subseteq K_{\psi,y}$ and $I_{\psi,\phi,x} \subseteq I_{\psi,y} \subseteq G_z$ imply that

$$
\chi(K_{\psi,y}) = \chi(K_{\psi,y}/(I_{\phi,x} \cap K_{\psi,y})) \cdot \chi(I_{\phi,x} \cap K_{\psi,y})
$$

and

$$
\chi(G_z/I_{\psi,\phi,x}) = \chi(G_z/I_{\psi,y}) \cdot \chi(I_{\psi,y}/I_{\psi,\phi,x}).
$$

By assumption $\chi(K_{\phi,x}), \chi(K_{\psi,y}) \neq 0$ for $x \in \mathcal{F}(K)$ and $y \in \mathcal{G}(K)$, so $\text{CF}^{\text{stk}}(\phi) : \text{CF}(\mathcal{F}) \to \text{CF}(\mathcal{G})$ and $\text{CF}^{\text{stk}}(\psi) : \text{CF}(\mathcal{G}) \to \text{CF}(\mathcal{F})$ are defined. As $\chi(K_{\psi,y}) \neq 0$ equation \ref{eq:chiK} gives $\chi(I_{\phi,x} \cap K_{\psi,y}) \neq 0$, and this, $\chi(K_{\phi,x}) \neq 0$ and \ref{eq:chiK} show that $\chi(K_{\phi,\phi,x}) \neq 0$, which holds for all $x \in \mathcal{F}(K)$. Hence $\text{CF}^{\text{stk}}(\psi \circ \phi) : \text{CF}(\mathcal{F}) \to \text{CF}(\mathcal{F})$ is defined.

Now $\psi_x^{-1}(I_{\psi,\phi,x})$ is an algebraic group with $I_{\phi,x} \subseteq \psi_x^{-1}(I_{\psi,\phi,x}) \subseteq G_y$, so

$$
\chi(G_y/I_{\phi,x}) = \chi(G_y/\psi_x^{-1}(I_{\psi,\phi,x})) \cdot \chi(\psi_x^{-1}(I_{\psi,\phi,x})/I_{\phi,x}).
$$

But $\psi_x$ and $\gamma(I_{\phi,x} \cap K_{\psi,y}) \to \gamma I_{\phi,x}$ induce isomorphisms of homogeneous spaces

$$
G_y/\psi_x^{-1}(I_{\psi,\phi,x}) \cong I_{\psi,\phi,x}/I_{\psi,\phi,x} \cap K_{\psi,y} \cong \psi_x^{-1}(I_{\psi,\phi,x})/I_{\phi,x}.
$$

Therefore the last two equations give

$$
\chi(G_y/I_{\phi,x}) = \chi(I_{\psi,y}/I_{\psi,\phi,x}) \cdot \chi(K_{\psi,y}/(I_{\phi,x} \cap K_{\psi,y})).
$$

Combining equations \ref{eq:m} and \ref{eq:chiK} yields

$$
m_{\psi,\phi}(x) = \frac{\chi(G_z/I_{\psi,\phi,x})}{\chi(K_{\psi,\phi})} \cdot \frac{\chi(G_z/I_{\psi,y})}{\chi(K_{\psi,y})} \cdot \frac{\chi(I_{\psi,z}/I_{\psi,\phi,x})}{\chi(I_{\psi,y}/I_{\psi,\phi,x})} = m_{\psi}(y) \cdot m_{\phi}(x).
$$

This identity is easily seen to be the extra ingredient needed to modify the proof of Theorem \ref{thm:tensor} to prove that $\text{CF}^{\text{stk}}(\psi \circ \phi) = \text{CF}^{\text{stk}}(\psi) \circ \text{CF}^{\text{stk}}(\phi)$.

If $\phi : \mathcal{F} \to \mathcal{G}$ is representable then $\phi_* : \text{Iso}_G(x) \to \text{Iso}_G(\phi_* (x))$ is an injective morphism of algebraic $K$-groups for all $x \in \mathcal{F}(K)$. Thus $\text{Ker} \phi_* = \{1\}$, so $\chi(Ker \phi) = 1 \neq 0$ for all $x \in \mathcal{F}(K)$, and $\text{CF}^{\text{stk}}(\phi)$ is defined. This gives:

**Theorem 5.4.** Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be algebraic $K$-stacks with affine geometric stabilizers, and $\phi : \mathcal{F} \to \mathcal{G}$, $\psi : \mathcal{G} \to \mathcal{H}$ representable 1-morphisms. Then $\psi \circ \phi : \mathcal{F} \to \mathcal{H}$ is representable, and $\text{CF}^{\text{stk}}(\phi) : \text{CF}(\mathcal{F}) \to \text{CF}(\mathcal{G})$, $\text{CF}^{\text{stk}}(\psi) : \text{CF}(\mathcal{G}) \to \text{CF}(\mathcal{H})$ and $\text{CF}^{\text{stk}}(\psi \circ \phi) : \text{CF}(\mathcal{F}) \to \text{CF}(\mathcal{H})$ are well-defined linear maps with $\text{CF}^{\text{stk}}(\psi \circ \phi) = \text{CF}^{\text{stk}}(\psi) \circ \text{CF}^{\text{stk}}(\phi)$.

Also, for $\phi$ representable $m_\phi$ in \ref{eq:m} takes values in $\mathbb{Z}$, so $\text{CF}^{\text{stk}}(\phi)$ maps $\mathbb{Z}$-valued functions $\text{CF}(\mathcal{F})_\mathbb{Z} \subset \text{CF}(\mathcal{F})$ to $\mathbb{Z}$-valued functions $\text{CF}(\mathcal{G})_\mathbb{Z} \subset \text{CF}(\mathcal{G})$. 

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5.2 Pullbacks by finite type 1-morphisms

For finite type $\phi: \mathcal{F} \to \mathcal{G}$ we can pull back constructible functions from $\mathcal{G}$ to $\mathcal{F}$.

**Definition 5.5.** Suppose $\phi: \mathcal{F} \to \mathcal{G}$ is a finite type 1-morphism of algebraic $\mathbb{K}$-stacks and $C \subseteq \mathcal{G}(\mathbb{K})$ is constructible. Then $C = \bigcup_{i \in I} \mathcal{G}_i(\mathbb{K})$, where $\{ \mathcal{G}_i : i \in I \}$ are finitely many substacks of $\mathcal{G}$. Set $\mathfrak{F}_i = \phi^*(\mathcal{G}_i)$, a $\mathbb{K}$-substack of $\mathcal{F}$. Then $\mathfrak{F}_i$ is of finite type, as $\phi, \mathcal{G}_i$ are. Hence $\phi^{-1}(C) = \bigcup_{i \in I} \phi_i^{-1}(\mathcal{G}_i(\mathbb{K})) = \bigcup_{i \in I} \mathfrak{F}_i(\mathbb{K})$ is constructible in $\mathcal{F}$. That is, $\phi_*: \mathcal{F}(\mathbb{K}) \to \mathcal{G}(\mathbb{K})$ pulls back constructible sets to constructible sets. Thus if $f \in \text{CF}(\mathcal{G})$ then $f \circ \phi_*$ lies in $\text{CF}(\mathcal{F})$. Define the pullback $\phi^*: \text{CF}(\mathcal{G}) \to \text{CF}(\mathcal{F})$ by $\phi^*(f) = f \circ \phi_*$. Pullbacks commute with multiplication of functions, that is, $\phi'^* (fg) = \phi'^* (f) \phi'^* (g)$. If also $\psi: \mathcal{G} \to \mathcal{H}$ is a finite type 1-morphism, it is immediate that $(\psi \circ \phi)^* = \phi^* \circ \psi^*: \text{CF}(\mathcal{F}) \to \text{CF}(\mathcal{H})$.

It is an interesting question how pullbacks $\psi^*$ and pushforwards $\text{CF}^{\text{stk}}(\phi)$ are related. The next theorem shows they commute in Cartesian squares, as in Definition 2.5. It will be an important tool in [9, 11]. The theorem would not hold if we replaced $\text{CF}^{\text{stk}}(\eta), \text{CF}^{\text{stk}}(\phi)$ in (13) by $\text{CF}^{\text{na}}(\eta), \text{CF}^{\text{na}}(\phi)$, or pushforwards defined using some other weight function. This supports our claim that $\text{CF}^{\text{stk}}$ is the most natural pushforward in many stack problems.

**Theorem 5.6.** Let $\mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers. If

$$\begin{array}{ccc}
\mathcal{E} & \xrightarrow{\eta} & \mathcal{G} \\
\downarrow{\theta} & \text{is a Cartesian square with} & \downarrow{\phi^*} \\
\mathcal{F} & \xrightarrow{\psi} & \mathcal{H}
\end{array}$$

$\eta, \phi$ representable and $\theta, \psi$ of finite type, then

the following commutes:

$$\begin{array}{ccc}
\text{CF}^{\text{na}}(\mathcal{E}) & \xrightarrow{\eta^*} & \text{CF}^{\text{na}}(\mathcal{G}) \\
\downarrow{\phi^*} & \text{CF}^{\text{stk}}(\mathcal{E}) & \downarrow{\psi^*} \\
\text{CF}^{\text{na}}(\mathcal{F}) & \xrightarrow{\phi^*} & \text{CF}^{\text{na}}(\mathcal{H})
\end{array}$$

Proof. Let $C \subseteq \mathcal{F}(\mathbb{K})$ be constructible, and $\delta_C \in \text{CF}(\mathcal{F})$ be its characteristic function. We shall prove that

$$\text{(CF}^{\text{stk}}(\eta) \circ \theta^*) \delta_C = (\psi^* \circ \text{CF}^{\text{stk}}(\phi)) \delta_C. \tag{19}$$

As $\text{CF}^{\text{stk}}(\eta) \circ \theta^*, \psi^* \circ \text{CF}^{\text{stk}}(\phi)$ are linear and such $\delta_C$ generate $\text{CF}(\mathcal{F})$, this implies $\text{CF}^{\text{stk}}(\eta) \circ \theta^* = \psi^* \circ \text{CF}^{\text{stk}}(\phi)$, as we want.

Define $B = \theta^{-1}(C)$. Since $\theta$ is of finite type $B$ is constructible, as in Definition 5.5, and $\theta^* (\delta_C) = \delta_B$, the characteristic function of $B$. Let $x \in \mathcal{E}(\mathbb{K})$, and define $y = \psi_*(x), B_x = B \cap \eta_x^{-1}(\{x\})$ and $C_y = C \cap \phi_y^{-1}(\{y\})$. Then $B_x, C_y$ are constructible, as $B, C$ are and $\eta_x^{-1}(\{x\}), \phi_y^{-1}(\{y\})$ are locally constructible. Write $\delta_{B_x}, \delta_{C_y}, \delta_{\eta_x^{-1}(\{x\})}, \delta_{\phi_y^{-1}(\{y\})}$ for the characteristic functions. Then

$$\left( (\text{CF}^{\text{stk}}(\eta) \circ \theta^*) \delta_C \right)(x) = \left( (\text{CF}^{\text{stk}}(\eta)) (\delta_C \circ \theta_*) \right)(x) = \left( \text{CF}^{\text{stk}}(\eta) \delta_B \right)(x) = \left( \text{CF}^{\text{na}}(\eta)(m_\eta \cdot \delta_B) \right)(x) = \chi^{\text{na}}(\mathcal{E}, m_\eta \cdot \delta_{B_x}) = \chi^{\text{na}}(\mathcal{F}, \text{CF}^{\text{na}}(\theta)(m_\eta \cdot \delta_{B_x})), \tag{20}$$

by (9), (13) and Corollary 5.10 where $m_\eta$ is defined in (12). Similarly we have

$$((\psi^* \circ \text{CF}^{\text{stk}}(\phi)) \delta_C)(x) = \chi^{\text{na}}(\mathcal{F}, m_\phi \cdot \delta_{C_y}). \tag{21}$$
We shall prove that
\[ \text{CF}^{\text{na}}(\theta)(m_\eta \cdot \delta_{B_x}) = m_\phi \cdot \delta_{C_y} \quad \text{in } \text{CF}(\mathfrak{F}). \]  
(22)

If \( z \in \mathfrak{F}(\mathbb{K}) \setminus C_y \) then both sides of (22) are zero at \( z \). So let \( z \in C_y \). Then
\[ \theta^{-1}(\{z\}) \cap B_x = \theta^{-1}(\{z\}) \cap \eta^{-1}_z(\{x\}), \]
so by (11) equation (22) at \( z \) reduces to
\[ \chi^{\text{na}}(\mathfrak{E}, m_\eta \cdot \delta_{\eta^{-1}_z(x)} \cdot \delta_{\eta^{-1}_z(z)}) = m_\phi(z). \]  
(23)

Define \( G_x = \text{Iso}_x(x) \), \( G_y = \text{Iso}_y(y) \) and \( G_z = \text{Iso}_z(z) \), as algebraic \( \mathbb{K} \)-groups. Since \( \psi_x(x) = y \) and \( \phi_x(z) = y \) we have homomorphisms \( \psi_x : \text{Iso}_x(x) \to \text{Iso}_y(y) \) and \( \phi_x : \text{Iso}_y(z) \to \text{Iso}_y(y) \). Write these as \( \psi_x : G_x \to G_y \) and \( \phi_z : G_z \to G_y \). Then \( \phi_z \) is injective, as \( \phi \) is representable, so \( \chi(\text{Ker } \phi_z) = \{1\} \) and (12) gives
\[ m_\phi(z) = \chi(G_y/\phi_z(G_z)). \]  
(24)

As (13) is Cartesian \( \mathfrak{E} \) is 1-isomorphic to \( \mathfrak{F} \times_B \mathfrak{G} \) by Definition 25. By definition of fibre products we find \( \eta^{-1}_z(\{x\}) \cap \theta^{-1}_z(\{z\}) \) is naturally isomorphic to \( \psi_x(G_x) \setminus G_y/\phi_z(G_z) \), a \textit{biquotient}. The stabilizer groups are given by
\[ \text{Iso}_x(\psi_x(G_x)\beta\phi_z(G_z)) = \{ (\alpha, \gamma) \in G_x \times G_y : \psi_x(\alpha)\beta = \beta\phi_z(\gamma) \} \quad \text{for } \beta \in G_y, \]
and the group homomorphism \( \eta_x : \text{Iso}_x(\psi_x(G_x)\beta\phi_z(G_z)) \to \text{Iso}_x(x) = G_x \) is given by \( (\alpha, \gamma) \mapsto \alpha \). It is injective as \( \phi_z \) is injective. Thus (12) yields
\[ m_\eta(\psi_x(G_x)\beta\phi_z(G_z)) = \chi(G_x/\{\alpha \in G_x : \psi_x(\alpha)\beta = \beta\phi_z(G_z)\}). \]  
(25)

Let \( \Pi_{x,y,z} : G_y/\phi_z(G_z) \to \psi_x(G_x) \setminus G_y/\phi_z(G_z) \) be the natural projection. Then the fibre of \( \Pi_{x,y,z} \) over \( \psi_x(G_x)\beta\phi_z(G_z) \) is isomorphic to \( G_x/\{\alpha \in G_x : \psi_x(\alpha)\beta = \beta\phi_z(G_z)\} \). So (25) implies that \( \text{CF}^{\text{na}}(\Pi_{x,y,z})1 = m_\eta \) in \( \text{CF}(\psi_x(G_x) \setminus G_y/\phi_z(G_z)) \). Therefore
\[ \chi^{\text{na}}(\mathfrak{E}, m_\eta \cdot \delta_{\eta^{-1}_x(x)} \cdot \delta_{\eta^{-1}_z(z)}) = \chi^{\text{na}}(\psi_x(G_x) \setminus G_y/\phi_z(G_z), m_\eta) = \]
\[ \chi^{\text{na}}(\psi_x(G_x) \setminus G_y/\phi_z(G_z), \text{CF}^{\text{na}}(\Pi_{x,y,z})1) = \chi^{\text{na}}(G_y/\phi_z(G_z), 1) = m_\phi(z), \]
by Corollary 4.10 and (24). This proves (23), and hence (22). Equations (21)–(22) then give (19) at \( x \), as we have to prove.

Note that by [15, Rem. 4.14.1 & Lem. 3.11 & Rem. 4.17(2)], in a Cartesian square (15) of algebraic \( \mathbb{K} \)-stacks, if \( \phi \) is representable then \( \eta \) is representable, and if \( \psi \) is of finite type then \( \theta \) is of finite type. Thus it is enough to suppose only that \( \phi \) is representable and \( \psi \) of finite type in (15).

5.3 Pushforwards of locally constructible functions

Next we observe that if \( \phi : \mathfrak{F} \to \mathfrak{G} \) is of \textit{finite type} then the definitions of \( \text{CF}^{\text{na}}(\phi_*)_f, \text{CF}^{\text{et}}(\phi)_f \) in (15), (15) make sense for \( f \) only \textit{locally} constructible.
Definition 5.7. Let $\phi : \mathfrak{F} \to \mathfrak{G}$ be a finite type 1-morphism of algebraic $\mathbb{K}$-stacks with affine geometric stabilizers. For $f \in \text{LCF}(\mathfrak{F})$, define $\text{LCF}^{\text{na}}(\phi)f$ by

$$\text{LCF}^{\text{na}}(\phi)f(x) = \chi^{\text{na}}(\mathfrak{F}, f \cdot \delta_{\phi^{-1}(x)}) \quad \text{for } x \in \mathfrak{F}(\mathbb{K}),$$

following (26). This is well-defined as $\phi^{-1}(\{x\})$ is constructible since $\phi$ is of finite type. Thus $\delta_{\phi^{-1}(x)} \in \text{CF}(\mathfrak{F})$ and $f \in \text{LCF}(\mathfrak{F})$, giving $f \cdot \delta_{\phi^{-1}(x)} \in \text{CF}(\mathfrak{F})$. If $\phi$ is also representable, define $\text{LCF}^{\text{stk}}(\phi)f = \text{LCF}^{\text{na}}(\phi)(m_\phi \cdot f)$ as in (13).

Suppose $C \subseteq \mathfrak{G}(\mathbb{K})$ is constructible, and let $B = \phi^{-1}(C)$. Then $B$ is constructible as $\phi$ is of finite type, by Definition 5.5. Write $\delta_B, \delta_C$ for the characteristic functions of $B, C$. Then $f \cdot \delta_B \in \text{CF}(\mathfrak{F})$, and it follows easily that

$$(\text{LCF}^{\text{na}}(\phi)f)\delta_C = \text{CF}^{\text{na}}(\phi_*)(f \cdot \delta_B) \quad \text{and} \quad (\text{LCF}^{\text{stk}}(\phi)f)\delta_C = \text{CF}^{\text{stk}}(\phi)(f \cdot \delta_B).$$

Therefore $(\text{LCF}^{\text{na}}(\phi)f)|_C, (\text{LCF}^{\text{stk}}(\phi)f)|_C$ are constructible by Theorems 4.9 and 5.4, for any constructible $C \subseteq \mathfrak{G}(\mathbb{K})$. Hence $\text{LCF}^{\text{na}}(\phi)f, \text{LCF}^{\text{stk}}(\phi)f$ are locally constructible, and $\text{LCF}^{\text{na}}(\phi), \text{LCF}^{\text{stk}}(\phi)$ are linear maps $\text{LCF}(\mathfrak{F}) \to \text{LCF}(\mathfrak{G})$. From Theorems 4.9 and 5.4 we deduce:

Theorem 5.8. Let $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, and $\phi : \mathfrak{F} \to \mathfrak{G}, \psi : \mathfrak{G} \to \mathfrak{H}$ finite type 1-morphisms. Then so is $\psi \circ \phi$, and $\text{LCF}^{\text{na}}(\psi \circ \phi) = \text{LCF}^{\text{na}}(\psi) \circ \text{LCF}^{\text{na}}(\phi)$. If $\phi, \psi$ are representable then so is $\psi \circ \phi$, and $\text{LCF}^{\text{stk}}(\psi \circ \phi) = \text{LCF}^{\text{stk}}(\psi) \circ \text{LCF}^{\text{stk}}(\phi)$.

The locally constructible analogue of Theorem 5.6 also holds.

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