THE DIEDERICH-FORNAESS INDEX AND THE GLOBAL REGULARITY OF THE $\bar{\partial}$-NEUMANN PROBLEM

STEFANO PINTON AND GIUSEPPE ZAMPIERI

Abstract. We describe along the guidelines of Kohn [11], the constant $E_s$ which is needed to control the commutator of a totally real vector field $T_\epsilon$ with $\bar{\partial}^*$ in order to have $H^s$ a-priori estimates for the Bergman projection $B_k$, $k \geq q - 1$, on a smooth $q$-pseudoconvex domain $D \subset \subset \mathbb{C}^n$. This statement, not explicit in [11], yields regularity results for $B_k$ in specific Sobolev degree $s$. Next, we refine the pseudodifferential calculus at the boundary in order to relate, for a defining function $r$ of $D$, the operators $(T_\epsilon + \delta) - 2\delta$ and $(-r)^{\frac{1}{2}}$. We are thus able to extend to general degree $k \geq 0$ of $B_k$, the conclusion of [11] which only holds for $q = 1$ and $k = 0$: if for the Diederich-Fornaess index $\delta$ of $D$, we have $(1 - \delta) \leq E_s$, then $B_k$ is $H^s$-regular.

MSC: 32F10, 32F20, 32N15, 32T25

1. Introduction

The regularity of the Bergman projection $B_k$ over forms of degree $k \geq 0$, as well as of the Neumann operator $N_k$ for $k \geq 1$ on a pseudoconvex domain $D \subset \subset \mathbb{C}^n$, has a long history. The first approach by Boas and Straube [2], [3] consists in requiring, for any $\epsilon$, the existence of a totally real vector field $T_\epsilon$ such that

$$\left\| \left( \partial r, [\partial_z, T_\epsilon] \right) \right\|_{bD} < \epsilon,$$

where $r$ is a defining function with $|\partial r| = 1$. This is referred to as “good vector fields” condition. In other terms, we are requiring that all the coefficients of the $T_\epsilon$ components of $[\bar{\partial}^*, T_\epsilon]$ are small (modulo “good” terms); cf. [14] Proposition 5.26. This can be weakened to a “multiplier” condition for $[\bar{\partial}^*, T_\epsilon]$. Thus, the regularity of $B_k$, $k \geq 0$ and $N_k$, $k \geq 1$, is in fact related to the existence, for any $\epsilon$, of a totally real vector field $T_\epsilon$, with $|T_\epsilon| \sim 1$, such that

$$\left\| \left[ \partial r, [\partial_z, T_\epsilon] \right] u \right\|^2 < \epsilon Q_1(u, u) + c_\epsilon \|u\|,$$

where $Q_1(u, u) = \|\bar{\partial}u\|^2_1 + \|\bar{\partial}^* u\|^2_1$. Indeed, in (1.1) and (1.2) one can make the weaker assumption that $T_\epsilon$ is “approximately tangential”, that is, $|T_\epsilon r| < \epsilon$; we refer for this point to the remarks after Theorem 5.22 of [14]. We deform the defining function $r$ to $r_\epsilon = g_\epsilon r$ and, accordingly, we deform the vector field $T = 2\text{Im} \sum_i r_i \partial_z_i$ to $T_\epsilon = 2\text{Im} \sum_i (r_\epsilon_i \partial_z_i)$. The condition of approximate tangentiality turns into $|\text{Im} g_\epsilon| < \epsilon$. These two deformations are
related by $[\bar{\partial}^* T_u, \cdot] \sim (\partial \bar{\partial} r_\epsilon \wedge \bar{\partial} r_\epsilon) T_u$ modulo an error whose restriction to $bD$ belongs to $T^1\theta bD \oplus T^{0,1} \mathbb{C}^n|_{bD}$; hence, the existence of $r_\epsilon$ such that

\begin{equation}
|\partial \bar{\partial} r_\epsilon \wedge \bar{\partial} r_\epsilon| \leq \epsilon Q + c_\epsilon \Lambda^{-1},
\end{equation}

for $|\partial r_\epsilon| \sim 1$, implies (1.2). (Here $\Lambda$ is the standard elliptic operator of order 1.) This is indeed the assumption under which Straube proves in [13] $H^s$-regularity for any $s$. In particular, this condition is fulfilled when there is a smooth defining function $H$.

It has been proved by Diederich-Fornaess in [4] that every domain possesses an index $\delta$ with $0 < \delta \leq 1$ such that $-(-r_\delta)\delta$ is plurisubharmonic. Again, $r_\delta$ is in the form $r_\delta = g_\delta r$ for some $g_\delta$. On the other hand, it has been proved by Barret [1] that given a Sobolev index $s \gg 0$, one can find a domain $D$ in which $B_k$ fails $H^s$-regularity; according to [4], for these domains, one has $\delta \gg 0$. So the relation between the index of regularity $s$ and the Diederich-Fornaess index $\delta$ is an attractive problem. Indeed, what is explicitly stated by Kohn and is by far the most interesting content of [11], is the way of obtaining $E_{s,g}$ out of $\delta$. This is described through the estimate of the Levi form

$$
(-r_\delta)^{\frac{1}{2}} |\partial \bar{\partial}(-(-r_\delta)^{\delta}) \wedge \bar{\partial} r_\delta| \leq (1 - \delta)^{\frac{1}{2}} Q_{(-r_\delta)^{\frac{1}{2}}}.
$$

(For an operator $\text{Op}$, such as $\text{Op}= (-r_\delta)^{\delta}$, we define $Q_{\text{Op}}$ by $Q_{\text{Op}}(u,u) = \|\text{Op}\bar{\partial}u\|^2 + \|\text{Op}\bar{\partial}^* u\|^2$.) In this estimate, one enjoiys the presence of the factor $(1 - \delta)^{\frac{1}{2}}$. When $(1 - \delta)^{\frac{1}{2}} \leq E_{s,g}$, one expects $s$-regularity by what has been said above, but this is not given for free because one encounters the unpleasant factor $(-r_\delta)^{\frac{1}{2}}$. It is well known that $(-r_\delta)^{\frac{1}{2}} \sim (T^*)^{-\frac{1}{2}}$ when the action is restricted to harmonic functions. For this reason, Kohn can prove regularity for the projection $B_0$ on 0-forms, since this factorizes through the projection over harmonic functions. The main task of the present paper is to develop an accurate pseudodifferential calculus at the boundary which relates the action of $(-r_\delta)^{\frac{1}{2}}$ and $(T^*)^{-\frac{1}{2}}$ over general functions by describing the error terms by means of $\Delta$. In this way, when
(1 - \delta)^{\frac{1}{2}} \leq E_{s,g}$, we get $H^s$-regularity of $B_k$ in general degree $k \geq 0$ (resp. $k \geq q - 1$) on a pseudoconvex (resp. $q$-pseudoconvex) domain.

Recent contribution to regularity of the Bergman projection by the method of the “multiplier” is given by Straube in the already mentioned paper [13] and Herbig-McNeal [6]; a combination of the “multiplier” and “potential” method (inspired to the “(P)-Property” by Catlin) is developed by Khanh [7] and Harrington [5].

Acknowledgements. We are grateful to Emil Straube for important advice.

2. Weak $s$-compactness and $H^s$-regularity

Let $D$ be a bounded smooth domain of $\mathbb{C}^n$ defined by $r < 0$ for $\partial r \neq 0$. We modify the defining function as $gr$ for $g \in C^\infty$ and use the notation $r_g$ or $r^g$ for $gr$. We use the lower scripts $i$ and $j$ to denote derivative in $\partial_{z_i}$ and $\partial_{\bar{z}_j}$ respectively and work with various vector fields such as

\begin{equation}
N_g = \sum_i r^g_i \partial_{z_i}, \quad L^g_j = \partial_{z_j} - r^g_j N_g, \quad T_g = -i (N_g - \bar{N}_g).
\end{equation}

The $L^g_j$'s are complex-tangential; $T_g$ is the complementary real-tangential vector field. We consider an orthonormal basis $\bar{\omega}_1, ..., \bar{\omega}_n$ of antiholomorphic 1-forms and general forms $u$ of degree $k$, that is, expressions of type $u = \sum_{\delta J = k} u_{J} \bar{\omega}_J$ where $J = j_1 < ... < j_k$ are ordered multiindices and $\bar{\omega}_J = \bar{\omega}_1 \wedge ... \wedge \bar{\omega}_k$. We use the notations

\[ S = \text{Span}\{L^g_j, \partial_{\bar{z}_j} \text{ for } j = 1, ..., n\}, \quad Q_s(u, u) = \|\partial u\|_s^2 + \|\bar{\partial} u\|_s^2. \]

We have (cf. [3] p. 83) for $u \in C^\infty(\overline{D})$,

\begin{equation}
\|Su\|_{s-1}^2 \leq Q_{s-1}(u, u) + \|u\|_s \|u\|_{s-1} \quad \text{for any } S \in S.
\end{equation}

Since $S \oplus CT_g = \mathbb{C} \otimes TC^n$, then (2.2) implies

\begin{equation}
\|u\|_s^2 \leq Q_{s-1}(u, u) + \|T^*_g u\|^2 + \|u\|_s \|u\|_{s-1}.
\end{equation}

With the notation $\bar{\theta}_j := -\frac{1}{\sum_i r^g_i r^g_j} \sum_i r^g_i r^g_j$, we define

\begin{equation}
\begin{cases}
\Theta_g u = \sum_{|K|=k-1} \sum_{ij} \left( \bar{\theta}^g_j u_{ik} - \bar{\theta}^g_i u_{jk} \right) + \text{error},
\Theta^*_g u = \sum_{|K|=k-1} \sum_j \theta^g_j u_{jK} + \text{error}.
\end{cases}
\end{equation}

We have the crucial commutation relation between $T_g$ and the Euclidean derivatives ([11] Lemma 3.33)

\begin{equation}
[\partial_{\bar{z}_j}, T_g] = \bar{\theta}_j T_g \quad \text{modulo } S.
\end{equation}

This implies

\begin{equation}
[\bar{\partial}, T_g] = \Theta_g T_g \quad \text{modulo } S.
\end{equation}
As for the commutation of the adjoint $\tilde{\partial}^*$, we need a modification of $T_g$ which preserves the condition of membership to $D_{\bar{\partial}}$. To this end, we define $\tilde{T}_g$ by

$$
(\tilde{T}_g u)_{jK} = T_g u_{jK} + \frac{r^q_j}{\sum_i |r_i|^2} \sum_i [T_g, r^q_i] u_{iK}.
$$

(2.7)

Thus $u \in D_{\bar{\partial}}$ implies $\tilde{T}_g u \in D_{\bar{\partial}}$. Note that $\tilde{T}_g$ differs from $T_g$ by a 0-order operator. With these preliminaries, (2.25) yields

$$
[\tilde{\partial}^*, \tilde{T}_g] = \tilde{\Theta}_g^* \tilde{T}_g \quad \text{modulo } S.
$$

**Definition 2.1.** Let $s$ be a positive integer and let $1 \leq q \leq n - 1$. We say that $T_g^s$ well commutes with $\tilde{\partial}^*$ in degree $\geq q$ when

$$
\|\tilde{\Theta}_g^* u\|^2 \leq \mathcal{E}_{s,g} Q(u, u) + c_g \|u\|^2_{s,1},
$$

(2.9)

for any $u$ of degree $\geq q$, and for $\mathcal{E}_{s,g} \leq c_1^2 e^{-2c_2 s} \inf \left( \frac{1}{|r|} \right)^{-1}$ or, alternatively, for $\mathcal{E}_{s,g} \leq c_1^2 e^{-2c_2 s} \inf \left( 1 + \frac{|q|}{|r|} \right)$, where $c_1$ is a small constant and $c_2$ is controlled by the $C^2$ norm of $r_g$.

We introduce the notion of $q$-pseudoconvexity of $D$; this consists in the requirement that, for the ordered eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{n-1}$ of the Levi form $\partial \bar{\partial} r|_{\partial r^{1}}$, we have $\sum_{j=1}^{q} \lambda_j \geq 0$. The basic estimates show that the complex Laplacian $\Box$ is invertible over $k$-forms for $k \geq q$. We denote by $N_k$ the inverse; we also denote by $B_k : L^{2,k} \rightarrow L^{2,k} \cap k e \bar{\partial}$ the Bergman projection. Recall Kohn’s formula $B_k = \text{Id} - \tilde{\partial}_{k+1}^* N_{k+1} \bar{\partial}_k$. We say that $B_k$ is regular, resp. $s$-exactly regular, when it preserves $C^\infty$, respectively $H^s$, the $s$-Sobolev space.

**Remark 2.2.** Assume that for any $s$ there is $r_g$ with $|\partial r_g| \sim 1$, that is $|g| \sim 1$, such that $|\tilde{\Theta}_g^* u| \leq c_1 e^{-c_2 s} \inf \left( 1 + \frac{|q|}{|r|} \right)_s$; then there is exact $s$-regularity for any $s$.

We recall from [2] that $s$-exact regularity of $N_k$ is equivalent to $s$-exact of the triplet $B_{k-1}$, $B_k$, $B_{k+1}$.

**Theorem 2.3.** Let $D$ be $q$-pseudoconvex and assume that for some $g$, $T_g^s$ well commutes with $\tilde{\partial}^*$ in degree $\geq q$. Assume also that this property of good commutation holds, with a uniform constant $\mathcal{E}_{s,g}$, for a strongly $q$-pseudoconvex exhaustion of $D$. Then for any form $f \in H^s$ we have that $B_k f \in H^s$ and

$$
\|B_k f\|_s \leq c \|f\|_s, \quad \text{for any } k \geq q - 1.
$$

(2.10)

The proof is intimately related to [3]. Formally, it follows the lines of [11] but also contains ideas taken from [7].

**Proof.** We first assume that we already know that $B_k$ is regular for any $k \geq q - 1$ and prove (2.10) for a constant $c$ which only depends on (2.9). In other terms, we show that (2.10)
the estimate and get, with the notation \( u \) and similarly for \( \bar{\Theta} \), prove that the same is true for \( B_{k-1} \). We use the notation \( f \) for the test form in our proof; the notation \( u \), which occurs in (2.9), will be reserved to \( \partial N_k f \). It suffices to estimate \( \| T^*_s B_{k-1} f \| \) since, by (2.3), this controls the full norm \( \| B_{k-1} f \|_s \). We have

\[
\| T^*_s B_{k-1} f \|^2 = (T^*_s B_{k-1} f, T^*_s f) - (T^*_s B_{k-1} f, T_g^* \bar{\partial} N_k \bar{\partial} f) = (T_g^* B_{k-1} f, T_g^* f) - (T_g^* T^*_s \bar{\partial} B_{k-1} f, N_k \bar{\partial} f) \tag{2.11}
\]

Now, (a) \( \leq sc \| T^*_s B_{k-1} f \|^2 + lc \| T_g^* f \|^2 \), whereas (b) = 0. The term which comes with small constant can be absorbed because we know a-priori that \( \| T^*_s B_{k-1} f \| < \infty \). As for the last term, we replace \( T_g^* \) by \( \bar{T}_g^* \) modulo an operator of order \( s - 1 \), that we regard as an error term, describe the commutator in the left of (c) by \( \bar{\Theta}_g \) according to (2.6), switch it to the right as \( \bar{\Theta}_g^* \) and end up with

\[
| (c) | \leq \left| (2s \bar{\Theta}_g \bar{T}_g^* B_{k-1} f, \bar{T}_g^* N_k \bar{\partial} f) \right| + \text{error}
\leq sc \| T^*_s B_{k-1} f \|^2 + lc s \| \bar{\Theta}_g^* T_g^* N_k \bar{\partial} f \|^2 + \text{error}. \tag{2.12}
\]

The error includes terms in \((s-1)\)-norm and terms in which derivatives belonging to \( S \) occur (cf. (2.2)). We use the hypothesis (2.8) under the choice \( \mathcal{E}_{s,g} \leq c^2 e^{-2c_2 s \text{diam}^2 D} \sup_{|g|^s} \frac{1}{g} \) and get, with the notation \( u = N_k \bar{\partial} f \)

\[
\| \bar{\Theta}_g^* \bar{T}_g^* u \|^2 \leq \sup \frac{1}{|g|^{2s}} \| \bar{\Theta}_g^* \bar{T}_g^* u \|^2 \leq \mathcal{E}_{s,g} \| T^*_s u \|^2 + \text{error}
\leq \mathcal{E}_{s,g} \sup_{|g|^{2s}} \left( Q_{\bar{T}_g^*} (u, u) + \| [\bar{\partial}, \bar{T}_g^*] u \|^2 + \| [\bar{\partial}^*, \bar{T}_g^*] u \|^2 \right) + \text{error}. \tag{2.13}
\]

(In case \( \mathcal{E}_{s,g} \leq c^2 e^{-2c_2 s \text{diam}^2 D (1 + \sup_{|g|} \frac{|g|}{|g|})} \) we have not to replace \( \bar{T}_g^* \) by \( \bar{T}^* \) and, instead, use the estimate

\[
| [\bar{T}_g^*, \bar{\partial}] v | \leq c_2 \sup (1 + \frac{|g|}{|g|}) |\bar{T}_g^* v| \text{ modulo } Sv \text{ for } S \in \mathcal{S}, \tag{2.14}
\]

and similarly for \( \bar{\partial} \) replaced by \( \bar{\partial}^* \); the proof will proceed similarly as below.)
Now,
\[ Q_{T_s}(u, u) \leq \|T^s f\|^2 + \|T^s B_{k-1} f\|^2 + \text{error}. \]
Next,
\[ \|[\bar{T}, T^s] u\|^2 \leq c_2 s^2 \|T^s N_{k} \bar{\partial} f\|^2 + \text{error}. \]
We now observe that
\[ N_k \bar{\partial} = B_k N_k \bar{\partial}(\text{Id} - B_{k-1}) \]
\[ = B_k e^{-\varphi_s} N_{k, \varphi_s} \partial e^{\varphi_s}(\text{Id} - B_{k-1}), \]
where \( N_{k, \varphi_s} \) is the \( \bar{\partial} \)-Neumann operator weighted by \( e^{-\varphi_s} = e^{-c_2 s|z|^2} \). Since \([D^s, \bar{\partial}]\) is an operator of degree \( s \) with coefficients controlled by \( sc_2 \) for \( c_2 \sim \|r\|_C^2 \), then \( N_{k, \varphi_s} \bar{\partial} \) is continuous in \( H^s_{\bar{\partial}r} \) with a continuity constant that we can assume to be unitary. We use that \( c_2 s^2 e^{-2c_2 s \text{diam}^2 D} \leq \inf_{z \in D} e^{-2c_2 s|z|^2} \) (for different \( c_2 \)) in order to remove weights from the norms. We also use the inductive assumption that (2.10) holds for \( B_k \). In this way, we end up with
\[ \mathcal{E}_{s, g} \sup_{|g|^2} \frac{1}{|g|^2} c_2 s^2 \|T^s \bar{\partial} N_{k} f\|^2 \leq c_1^2 (\|T^s f\|^2 + \|T^s B_{k-1} f\|^2) + \text{error} \]
\[ \leq c_1^2 (\|T^s f\|^2 + \|T^s B_{k-1} f\|^2) + \text{error}, \]
where the last inequality follows trivially from the fact that \( T_g = \frac{1}{g} T \) for \( \frac{1}{g} \gg 1 \). Here, \( \mathcal{E}_{s, g} \) takes care of \( \sup_{|g|^2} \frac{1}{|g|^2} \) and also of the constant which arises from removing weights owing to \( \mathcal{E}_{s, g} \leq c^2 e^{-2c_2 \text{diam}^2 D} \sup_{|g|^2} \frac{1}{|g|^2} \). Altogether, up to absorbable terms, \( \|T^s B_{k-1} f\|^2 \) has been estimated by \( lc \|T^s f\|^2 + \text{error} \). This concludes the proof of Theorem 2.3 if we are able to remove the assumption that we already know that (2.10) holds for some \( c' \gg c \). For this, we recall that we are assuming that there is a strongly \( q \)-pseudoconvex exhaustion \( D_{\rho} \hookrightarrow D \) which satisfies (2.3) uniformly with respect to \( \rho \). We observe that (2.10) holds over \( D_{\rho} \) for \( c' = c' \). What has been proved above shows that it holds in fact with \( c \) independent of \( \rho \). Passing to the limit over \( \rho \) we get (2.10) for \( D \).

\[ \square \]

**Theorem 2.4.** (Boas-Straube [3]) If there is a defining function \( r \) such that for the eigenvalues \( \mu_1 \leq \ldots \leq \mu_n \) of the full Levi form \( \partial \bar{\partial} r \) (not restricted to \( \partial r^\perp \)) we have \( \sum_{j=1}^n \mu_j \geq 0 \), then, \( B_k \) is exactly \( H^s \)-regular for any \( s \) and any \( k \geq q - 1 \).

**Proof.** The proof consists in proving that (2.3) holds for any \( \epsilon \) and uniformly over an exhaustion of \( D \). More precisely, we will show that for any \( \epsilon \), for \( \Theta^* \) independent of \( \epsilon \) (associated to a normalized defining function \( r \)), and for suitable \( c_\epsilon \), we have
\[ \|\Theta^* u\|^2 \leq \epsilon Q(u, u) + c_\epsilon \|u\|_{-1} \quad \text{for } u \text{ in degree } k \geq q; \]
moreover, we will prove that (2.17) holds for a strongly $q$-pseudoconvex exhaustion. (Here, the triplet $\|\cdot\|$ denotes the tangential norm (cf. [10]).)

(a) We begin by noticing that $\partial\bar{\partial}r + O(|r|)\text{Id} \geq 0$ over $k$-forms for $k \geq q$. We can then apply Cauchy-Schwartz inequality and get

\begin{equation}
(r_{ij})(u, \partial r) \leq (r_{ij})(u, u)^{\frac{1}{2}} + O(|r|^\frac{1}{2})|u|.
\end{equation}

(b) The Levi form is a \(\frac{1}{2}\)-subelliptic multiplier” (cf. [10]), that is

\begin{equation}
\|\left((r_{ij})(u, u)^{\frac{1}{2}}\right)\|_{\frac{2}{3}} \leq Q(u, u).
\end{equation}

This can be proved from the basic estimate

\[
\int_D (r_{ij})(Tu, u)dV \leq Q(u, u),
\]

by using the microlocalization $T^+$ and its decomposition $T^+ = (T^+)^{\frac{1}{2}}(T^+)^{\frac{1}{2}^*}$. (Here $dV$ is the element of volume.) Also, by Sobolev interpolation, we have

\begin{equation}
\|\left((r_{ij})(u, u)^{\frac{1}{2}}\right)\|^2 \leq \epsilon\|\left((r_{ij})(u, u)^{\frac{1}{2}}\right)\|^2_2 + c_\epsilon\|u\|^2_1
\leq \epsilon Q(u, u) + c_\epsilon\|u\|^2_1,
\end{equation}

where $c_\epsilon \sim \epsilon^{-1}|r|_{C^2}$. Finally, we estimate the norm of the last term in (2.18). We have

\begin{equation}
\|(-r)^{\frac{1}{2}}u\|^2 \leq \epsilon\|\zeta u\|^2_0 + \|\left(1 - \zeta\right)u\|^2_0
\leq \epsilon\|u\|^2_0 + \|\left(1 - \zeta\right)u\|^2_0 \leq \epsilon Q(u, u) + \|\left(1 - \zeta\right)u\|^2_0,
\end{equation}

where $\zeta$ is a cut-off outside of the $\epsilon$-strip such that $|\dot{\zeta}| \sim \epsilon$ (with $\zeta \equiv 1$ at $bD$). Moreover, we have

\begin{equation}
\|\left(1 - \zeta\right)u\|^2_0 \leq \epsilon^3\|\left(1 - \zeta\right)u\|^2_0 + c_\epsilon\|\left(1 - \zeta\right)u\|^2_0,
\end{equation}

and,

\begin{equation}
\epsilon^3\|\left(1 - \zeta\right)u\|^2_1 \leq \epsilon^3Q_0\left((1 - \zeta)u, (1 - \zeta)u\right)
\end{equation}

\begin{equation}
\sim \epsilon^3Q_0(u, u) + \epsilon^3\|\zeta u\|^2_0
\leq \epsilon^3Q_0(u, u) + \epsilon^3\|u\|^2_0
\end{equation}

\begin{equation}
\leq 2\epsilon Q_0(u, u),
\end{equation}

where (i) is Garding inequality applied to $(1 - \zeta)u|_{bD} \equiv 0$ and (ii) follows from applying the basic estimate to $\|u\|^2_0$. Putting together (2.18)--(2.23), we get (2.17).

(c) We consider the exhaustion of $D$ by the domains $D_r$ defined by $r_\rho < 0$ for $r_\rho = r + \rho(1-|z|^2)$. by a suitable choice of $A$, these domains are strongly $q$-pseudoconvex. We remark that
\[ \partial \bar{\partial} r_\rho \geq -\|r\|_{C^2}|r_\rho| \text{Id} \geq -c|r_\rho| \text{Id} \] over \( k \) forms for \( k \geq q \). By Cauchy-Schwarz inequality we get

\[ (2.24) \quad (r^\rho_{ij})(u, \partial r) \leq (r^\rho_{ij})(u, u)^{\frac{1}{2}} + c|r_\rho|^\frac{1}{2}|u| \quad \text{for } u \text{ of degree } k \geq q. \]

The Levi form \((r^\rho_{ij})\) is a \( \frac{1}{2} \)-subelliptic multiplier (uniformly over \( \rho \)) and can be estimated as in (b) as well as the term with \( O(|r_\rho|^\frac{1}{2}) \). Altogether, for fixed \( \epsilon \) for any \( \rho \leq \rho_\epsilon \) and for \( \bar{\Theta}_\rho^* \) associated to the defining function \( r_\rho \), we have got

\[ (2.25) \quad \|\bar{\Theta}_\rho^* u\|^2 \leq \epsilon Q(D_\rho)(u, u) + c_\epsilon \|u\|^2_{-1}, \]

uniformly with respect to \( \rho \). Passing to the limit over \( \rho \), yields \((2.17)\). \( \square \)

**Theorem 2.5.** Let \( D \) be \( q \)-pseudoconvex and assume that for any \( \epsilon \) there is \( |g_\epsilon| \sim 1 \) such that

\[ (2.26) \quad |\bar{\Theta}_\epsilon^*(u)| \leq \epsilon \|u\|^2 \quad \text{on } bD \quad \text{for } u \text{ in degree } k \geq q. \]

Then \( B_k \) is exactly \( H^s \)-regular for any \( s \) and any \( k \geq q - 1 \).

**Proof.** \((2.26)\) readily implies

\[ (2.27) \quad \|\bar{\Theta}_\epsilon^* u\|^2 \leq \epsilon \|u\|^2 + \|g_\epsilon r\|_{C^2}(1 - \zeta_\epsilon)\|u\|^2 \quad \text{for } u \text{ in degree } k \geq q. \]

By plugging \((2.26)\) with the basic estimate \( \|u\|^2 \lesssim Q(u, u) \) and the Garding inequality \( \|g_\epsilon r\|_{C^2}(1 - \zeta_\epsilon)\|u\|^2 \lesssim \epsilon Q(u, u) + c_\epsilon \|u\|^2_{-1} \), we get

\[ (2.28) \quad \|\bar{\Theta}_\epsilon^* u\|^2 \leq \epsilon Q(u, u) + c_\epsilon \|u\|^2_{-1} \quad \text{for } u \in D_{\bar{\partial}}, \text{ of degree } k \geq q. \]

This would give the \( H^s \)-regularity of \( B_k \) if we were able to prove the stability of \((2.26)\) under a strongly \( q \)-pseudoconvex exhaustion. For this, we fix \( \epsilon_0 \) and \( g_{\epsilon_0} r \) and approximate \( D \) by \( D_\rho \) defined by \( g_{\epsilon_0} r + \rho e^{|z|^2} \); for suitable fixed \( A \), these are strongly \( q \)-pseudoconvex for any \( \rho \). Also, if we rewrite \( g_{\epsilon_0} r + \rho e^{|z|^2} = g_{\epsilon_0, \rho} r_\rho \) for a normalized equation \( r_\rho \) of \( D_\rho \), we have

\[
\begin{align*}
g_{\epsilon_0, \rho} &\to g_{\epsilon_0}, \\
r_\rho &\to r.
\end{align*}
\]

Hence

\[ \bar{\Theta}_{\epsilon_0, \rho}^*(u) \to \bar{\Theta}_{\epsilon_0}^*(u) \quad \text{uniformly over } u. \]

We then apply Theorem \(2.3\) to each \( \Omega_\rho \) and by uniformity of the estimate with respect to \( \rho \) we get that \( B_k f \) belongs to \( H^s \) and satisfies \((2.10)\). \( \square \)
Remark 2.6. We can give an alternative proof of Theorem 2.3 which uses Theorem 2.5. First, according to the lemma in [3], the existence of a plurisubharmonic defining function \( r \) implies the vector fields condition (1.1). (If \( r \) is only \( q \)-plurisubharmonic, (1.1) must be adapted by considering, similarly as in (2.26), the action over forms \( u \) of degree \( k \geq q \).) If we knew that the good vector fields \( T \) are of type \( T_g = -i(N_g - \bar{N}_g) \), then, by (2.8) we would get (2.26) and reach the conclusion from Theorem 2.5. In the general case, by [14] Proposition 5.26, the condition of good vector fields implies (2.26). (In that proposition, it is proved a generalization of (2.8). For any tangential vector field \( T \), not necessarily defined by (2.1), if we denote by \( g \) its \((N - \bar{N})\)-component, we have \([\bar{\partial}^* T], T\) \( bD = \bar{\Theta}_g|bD T \) modulo elliptic multipliers \((r, \partial r)\) and \( 1\)-subelliptic multipliers \((\partial \partial r)\).)

Remark 2.7. We point out that in [13], Straube proves that (2.28) suffices for exact \( H^s \)-regularity for any \( s \). This requires heavy work since, differently from (2.26), (2.28) is not transferred from \( \Omega \) to \( \Omega_\rho \).

3. Pseudodifferential calculus at the boundary

There is an important theory about the equivalence between \((-r)^\sigma\) and microlocal powers \( T^{-\sigma} \) over harmonic functions; we need to develop this theory and allow the action over general functions controlling errors coming from the Laplacian. In this discussion, we do not modify \( r \) to \( r_g \) and \( T \) nor \( T_g \). Also, we still write \( T \) but mean in fact its positive microlocalization \( T^+ \) which represents over \( v^+ \) the full elliptic standard operator \( \Lambda \); for this reason, negative and fractional powers of \( T \) make sense. We denote by \( U \) a neighborhood of \( bD \).

**Lemma 3.1.** We have

\[
\left\|( -r )^{\frac{\sigma}{2}} r^\sigma T^\sigma v \right\|_{lc} < \left\| (-r)^{\frac{\sigma}{2}} v \right\| + sc \left\| T^{-\frac{\sigma}{2}} v \right\| + sc \left\| -r T^{-1-\frac{\sigma}{2}} \Delta v \right\| \quad \text{for any} \ v \in C^\infty(\bar{D} \cap U) \ \text{and} \ \sigma > -\frac{1}{2}.
\]

This is a generalization of [11] Lemma 2.6 in which the extra terms with power \( \frac{\sigma}{2} \) do not occur.

**Proof.** We have

\[
\left\| ( -r )^{\frac{\sigma}{2}} r^\sigma T^\sigma v \right\|^2 = \left\| ( -r )^{\delta + 2\sigma} T^{2\sigma} v, v \right\|
\]

\[
= -\left\| \partial_r ( -r^{1+2\sigma+\delta} T^{2\sigma} v, v \right\|
\]

\[
= 2 \text{Re} \left( ( -r )^{1+2\sigma+\delta} \partial_r T^{2\sigma} v, v \right)
\]

\[
\leq lc \left\| ( -r )^{\frac{\sigma}{2}} v \right\|^2 + sc \left\| ( -r )^{1+2\sigma+\frac{\sigma}{2}} \partial_r T^{2\sigma} v, v \right\|^2
\]

\[
\leq lc \left\| ( -r )^{\frac{\sigma}{2}} v \right\|^2 + sc \left\| T^{-\frac{\sigma}{2}} v \right\|^2 + sc \left\| -r T^{-1-\frac{\sigma}{2}} \Delta v \right\|^2,
\]

where \((*)\) follows from [11] (2.4) applied for \( 1 + 2\sigma + \frac{\sigma}{2} > 0 \). \( \square \)
In [1] there is a result, Lemma 2.6, which applies to powers $>-\frac{1}{2}$ of $-r$; we need a variant, still for negative powers, for terms involving $\partial_r v$.

**Lemma 3.2.** We have

$$\|(r)^{\sigma} \partial_r T^{\sigma-2} v \| \leq \|v\| + \| r T^{-1} \Delta v \| + \| T^{-2} \Delta v \|, \quad v \in C^\infty(\bar{D} \cap U), \quad \sigma > -\frac{1}{2}. \tag{3.2}$$

**Proof.** We have

$$\left( \partial_r (r)^{2\sigma+1} \partial_r T^{2\sigma-2} v, \partial_r v \right) = -2 \text{Re} \left( (r)^{2\sigma+1} \partial_r^2 T^{2\sigma-2} v, \partial_r v \right).$$

Write $\partial_r^2 = \Delta + T \partial_r + T^2$. For the three terms $\Delta$, $T^2$, and $T \partial_r$, we have the three relations below, respectively

$$\begin{aligned}
\left( T^{-2} \Delta v, (r)^{2\sigma+1} T^{2\sigma} \partial_r v \right) &\leq \| T^{-2} \Delta v \|^2 + \| v \|^2, \\
\left( (r)^{2\sigma+1} T^{2\sigma} v, \partial_r v \right) &= \left( (r)^{2\sigma+1} T^{2\sigma+1} v, \partial_r T^{-1} v \right) \\
&\leq \| v \|^2 + \| r T^{-1} \Delta v \|^2, \\
\left( (r)^{2\sigma+1} \partial_r T^{2\sigma-1} v, \partial_r v \right) &= \left( (r)^{2\sigma+1} \partial_r T^{(2\sigma+1)-1} v, \partial_r T^{-1} v \right) \\
&\leq \| v \|^2 + \| r T^{-1} \Delta v \|^2,
\end{aligned}$$

where the three inequalities come from Cauchy-Schwartz inequality combined with repeated use of [1] (2.4) (always under the choice $s = 0$ with the notations therein). Finally, we have to estimate the error term

$$\left( (r)^{2\sigma+1} [\Delta, T^{2\sigma-2}] v, \partial_r v \right). \tag{3.3}$$

We express the commutator in (3.3) as

$$[\Delta, T^{2\sigma-2}] = T^{2\sigma-1} + \partial_r T^{2\sigma-2}.$$  

Thus (3.3) splits into two terms to which the two inequalities below apply

$$\begin{aligned}
\left( (r)^{2\sigma+1} T^{2\sigma-1} v, \partial_r v \right) &= \left( (r)^{2\sigma+1} T^{(2\sigma+1)-1} v, T^{-1} \partial_r v \right) \\
&\leq \| v \|^2 + \| r T^{-1} \Delta v \|^2, \\
\left( (r)^{2\sigma+1} \partial_r T^{2\sigma-2} v, \partial_r v \right) &= \left( (r)^{2\sigma+1} \partial_r T^{2\sigma-1} v, T^{-1} \partial_r v \right) \\
&\leq \| v \|^2 + \| r T^{-1} \Delta v \|^2.
\end{aligned}$$

We are ready for the main technical tool in interchangeing powers of $-r$ and $T$.

**Proposition 3.3.** We have

$$\| T^{-\frac{3}{2}} v \| \leq \| (r)^{\frac{3}{2}} v \| + \| r T^{-1} \frac{3}{2} \Delta v \| + \| (r)^{\frac{3}{2}} T^{-2} \Delta v \|. \tag{3.4}$$
As for (ii) we have

\[ \sum_j \left| \left( (-r_\delta)^{\frac{\delta}{2}} \partial_{\bar{z}_j} T^{-1} v, (-r_\delta)^{\frac{\delta}{2}} \partial_{z_j} T^{-1} v \right) \right| \leq \left| \left( (-r_\delta)^{\frac{\delta}{2}} \Delta T^{-2} v, (-r_\delta)^{\frac{\delta}{2}} v \right) \right| \]

\[ + 2 \sum_j \left| \text{Re} \left( \left( (-r_\delta)^{\delta}, \partial_{\bar{z}_j} \right] \partial_{z_j} T^{-1} v, T^{-1} v \right) \right| \]

Now, the first and second terms in the right are good (in the right side of the estimate we wish to end with). As for the last, we have

\[ \sum_j \left( (-r_\delta)^{\frac{\delta}{2}} \partial_{\bar{z}_j} T^{-1} v, (-r_\delta)^{\frac{\delta}{2}} \partial_{z_j} T^{-1} v \right) \]

\[ \leq \left| \left( (-r_\delta)^{\frac{\delta}{2}} \Delta T^{-2} v, (-r_\delta)^{\frac{\delta}{2}} v \right) \right| \]

\[ + 2 \sum_j \left| \text{Re} \left( \left( (-r_\delta)^{\delta}, \partial_{\bar{z}_j} \right] \partial_{z_j} T^{-1} v, T^{-1} v \right) \right| \]

The first term in the right is estimated by

\[ \left| \left( (-r_\delta)^{\frac{\delta}{2}} \Delta T^{-2} v, (-r_\delta)^{\frac{\delta}{2}} v \right) \right| \leq l c \left\| (-r)^{\frac{\delta}{2}} v \right\| + s c \left\| (-r)^{\frac{\delta}{2}} (\partial_2^2 + \partial_r T + T^2) T^{-2} v \right\| \]

\[ \leq l c \left\| (-r)^{\frac{\delta}{2}} v \right\| + s c \left( \left\| (-r)^{\frac{\delta}{2}} T^{-2} \partial_2^2 v \right\| + \left\| (-r)^{\frac{\delta}{2}} \partial_r T^{-1} v \right\| \right) \]

\[ \leq l c \left\| (-r)^{\frac{\delta}{2}} v \right\| + s c \left( \left\| (-r)^{\frac{\delta}{2}} T^{-2} \Delta v \right\| + \left\| T^{-\frac{\delta}{2}} \Delta v \right\| \right). \]

The second term in the right of (3.5) has the estimate

\[ \left| \text{Re} \left( \left( (-r_\delta)^{\delta}, \partial_{\bar{z}_j} \right] T^{-1} v, T^{-1} v \right) \right| \leq \left| \left( (-r)^{-1+\delta+\epsilon} T^{-1+\frac{\delta}{2}+\epsilon} v, (-r)^{-\epsilon} T^{-\frac{\delta}{2}+\epsilon} v \right) \right| \]

\[ + \left| \left( (-r)^{-1+\delta} \partial_r T^{-1} v, T^{-1} v \right) \right| \]

To estimate (i), we write \(-1 + \delta + \epsilon = \frac{\delta}{2} + (-1 + \frac{\delta}{2} + \epsilon) = \frac{\delta}{2} + \sigma\) under the choice of \(\epsilon > \frac{1}{2} - \frac{\delta}{2}\) so that \(-1 + \frac{\delta}{2} + \epsilon > -\frac{1}{2}\). We then apply Lemma 3.11 and get the estimate of (i)

\[ (i) \leq l c \left\| (-r)^{\frac{\delta}{2}} v \right\|^2 + s c \left( \left\| T^{-\frac{\delta}{2}} v \right\|^2 + \left\| -r T^{-1-\frac{\delta}{2}} \Delta v \right\| \right). \]

As for (ii) we have

\[ (ii) = \left| \left( (-r)^{-1+\delta+(1-\frac{\delta}{2}-\epsilon)} \partial_r T^{-1} v, (-r)^{-1+\frac{\delta}{2}+\epsilon} T^{-1+\epsilon} v \right) \right| \]

\[ \leq s c \left( \left\| T^{-\frac{\delta}{2}} v \right\|^2 + \left\| -r T^{-1} \Delta v \right\| + \left\| T^{-2} \Delta v \right\| \right) \]

\[ + l c \left( \left\| (-r)^{\frac{\delta}{2}} v \right\|^2 + \left\| -r T^{-1-\frac{\delta}{2}} \Delta v \right\| \right). \]
In fact, the term with lc in the last line comes from Lemma 3.4 applied for \( \sigma = -1 + \epsilon \) (which requires \( \epsilon > \frac{1}{2} \)). The term with sc is estimated by the aid of Lemma 3.2

\[
\| (-r)^{-1+\delta+(1-\frac{\delta}{2}-\epsilon)} \partial_r T^{-1-\epsilon} v \| \leq \| (-r)^{\frac{\delta}{2}} \partial_r T^{-1+(\frac{\delta}{2}-\epsilon)} \| \| \partial_r T^{-1-\epsilon} v \|.
\]

We decompose now \( v = v^{(h)} + v^{(0)} \) where \( v^{(h)} \) is the harmonic extension and \( v^{(0)} := v - v^{(h)} \); note that \( v^{(0)}|_{bD} \equiv 0 \). We also recall the modification \( T \) of \( T \) defined by (2.7) and designed to preserve \( D_{b\ast} \).

**Proposition 3.4.** We have

\[
\| [\tilde{T}^{s-\frac{\delta}{2}}, \tilde{\partial}^*] v^{(h)} \| \lesssim \| (-r)^{\frac{\delta}{2}} [\tilde{T}^{s}, \tilde{\partial}^*] v^{(h)} \|, \quad v \in C^\infty (D \cap U).
\]

**Remark 3.5.** In turn, by (2.8), we have \([\tilde{T}^{s}, \tilde{\partial}^*] = s\tilde{\Theta} \tilde{T}^{s} \), and therefore (3.6) implies

\[
\| [\tilde{T}^{s-\frac{\delta}{2}}, \tilde{\partial}^*] v^{(h)} \| \lesssim \| (-r)^{\frac{\delta}{2}} s\tilde{\Theta} \tilde{T}^{s} v^{(h)} \|.
\]

**Proof.** In fact, Jacobi identity yields

\[
[\tilde{T}^{s}, \tilde{\partial}^*] = -\tilde{T}^{s-\frac{\delta}{2}} [\tilde{T}^{\frac{\delta}{2}}, \tilde{\partial}^*] + \tilde{T}^{\frac{\delta}{2}} [\tilde{T}^{s-\frac{\delta}{2}}, \tilde{\partial}^*] + [\tilde{T}^{s-\frac{\delta}{2}} [\tilde{T}^{\frac{\delta}{2}}, \tilde{\partial}^*]]
\]

It follows

\[
\tilde{T}^{\frac{\delta}{2}} [\tilde{T}^{s-\frac{\delta}{2}}, \tilde{\partial}^*] = [\tilde{T}^{s}, \tilde{\partial}^*] + \tilde{T}^{s-\frac{\delta}{2}} [\tilde{T}^{\frac{\delta}{2}}, \tilde{\partial}^*] - [\tilde{T}^{s-\frac{\delta}{2}} [\tilde{T}^{\frac{\delta}{2}}, \tilde{\partial}^*]].
\]

We apply \( T^{-\frac{\delta}{2}} \) to both sides of (3.8) and use Proposition 3.3. The conclusion will follow once we are able to show that \( -rT^{-1-\frac{\delta}{2}}[\Delta, [\tilde{T}^{s}, \tilde{\partial}^*]] \) and \( (-r)^{\frac{\delta}{2}} T^{-2}[\Delta, T^{-1} \tilde{T}^{s} \tilde{\partial}^*] \) are error terms. In fact, we write

\[
[\Delta, [\tilde{T}^{s}, \tilde{\partial}^*]] = [\partial^2 + \partial_r \text{Tan} + \text{Tan}^2, \text{Tan}^s + \partial_r \text{Tan}^{s-1}]
\]

\[
= \text{Tan}^{s-1} + \partial_r \text{Tan}^s \sim \tilde{T}^{s+1} + \partial_r \tilde{T}^s \quad \text{modulo S}.
\]

It follows

\[
\begin{align*}
\| -rT^{-1-\frac{\delta}{2}}[\Delta, [\tilde{T}^{s}, \tilde{\partial}^*]] v^{(h)} \| & \leq \| -rT^{s-\frac{\delta}{2}} v^{(h)} \| + \| -r\partial_r T^{s-1-\frac{\delta}{2}} v^{(h)} \| \quad \leq \| T^{s-1-\frac{\delta}{2}} v^{(h)} \|, \quad \text{(2.4)} \\
\| (-r)^{\frac{\delta}{2}} T^{-2}[\Delta, [\tilde{T}^{s}, \tilde{\partial}^*]] v^{(h)} \| & \leq \| (r)^{\frac{\delta}{2}} T^{-1-\frac{\delta}{2}} v^{(h)} \| + \| (-r)^{\frac{\delta}{2}} \partial_r T^{-2} v^{(h)} \| \quad \leq \| T^{s-1-\frac{\delta}{2}} v^{(h)} \|. \quad \text{(2.4)}
\end{align*}
\]

\( \square \)
4. Non-smooth plurisubharmonic defining functions

**Definition 4.1.** We say that $D$ has a Diederich-Fornaess index $\delta = \delta_s$ for $0 < \delta \leq 1$ which controls the commutators of $\partial$ and $\partial^*$ with $D^s$ over forms in degree $k \geq q$, when there is $r_\delta = g_\delta r$ for $g_\delta \in C^\infty$, $g_\delta \neq 0$, such that

$$\begin{cases}
-(-r_\delta)^\delta \text{ is } q\text{-plurisubharmonic, that is, the sum of the first } \\
q \text{ eigenvalues of } \partial\bar\partial(-(-r_\delta)^\delta) \text{ is non-negative}
\end{cases}$$

(4.1)

where $\mathcal{E}_{s,g}$ can be chosen so that $\mathcal{E}_{s,g} \leq c_1 e^{-c_2 s \text{ diam} D^s \sup \left( \frac{1}{|g|^2} \right)^{-1}}$ or, alternatively, $\mathcal{E}_{s,g} \leq c_1 e^{-c_2 s \text{ diam} D^s \sup \left( 1 + \frac{|z|^2}{|g|^2} \right)}$.

Related to the above notion, is the condition

$$\|(-r_\delta)^{\frac{1}{2}} \Theta_{s}^{*} u\|^2 \leq \mathcal{E}_{s,g} Q(-r_\delta)^{\frac{1}{2}}(u, u),$$

for $\delta \leq 1$.

**Theorem 4.2.** If $D$ is $q$-pseudoconvex and has a Diederich-Fornaess index $\delta = \delta_s$ which controls the commutators of $(\partial, \partial^*)$ with $D^s$ in degree $k \geq q$, then $B_k$ is $s$-regular for $k \geq q$.

**Remark 4.3.** The proof consists in showing that (4.1) implies (4.2) (points (a) and (b) below) and then showing that (4.2) implies the conclusion. Note that, when $\delta = 1$, we have in fact the better conclusion contained in Theorem 2.4

**Proof.** We decompose a form as $u = u^* + u^\nu$ where $u^*$ and $u^\nu$ are the tangential and normal component respectively. We have

$$\begin{cases}
\|u^\nu\|_1^2 \leq \sum_i \|\partial_s u^\nu\|_0^2 \leq Q(u, u) \\
Q(u^*, u^*) \leq Q(u, u) + Q(u^\nu, u^\nu) \\
Q(u, u) + \|u^\nu\|_1^2 \\
\leq Q(u, u).
\end{cases}$$

(4.3)

Hence it suffices to prove (4.2). The same conclusion also applies to the decomposition $u = u^{(h)} + u^{(0)}$ and, in general, to any decomposition in which either of the two terms is 0 at $bD$.

(a) We have

$$\left| \partial \bar\partial r_\delta(u^*, \partial r_\delta) \right| \leq (1 - \delta)^{\frac{1}{2}} (\frac{1}{2})^\delta (\partial \bar\partial (-(-r_\delta)^\delta) (u^*, u^*))^{\frac{1}{2}}.$$  

(4.4)

To see it, we start from

$$\partial \bar\partial (-(-r_\delta)^\delta) = \delta (-r_\delta)^{\delta-1} \partial \bar\partial r_\delta + (-r_\delta)^{\delta-2} \delta (1 - \delta) \partial r \otimes \bar\partial r.$$
In particular,
\[ \partial \bar{\partial} r_\delta = \frac{1}{\delta} (-r_\delta)^{1-\delta} \partial \bar{\partial} (-(-r_\delta)^\delta) - (-r_\delta)^{-1} (1-\delta) \partial r \otimes \bar{\partial} r. \]

We suppose that \( \delta \) disregard it in the following. We have
\[ \partial \bar{\partial} r_\delta (u, \partial r_\delta) \sim (-r_\delta)^{1-\delta} \partial \bar{\partial} (-(-r_\delta)^\delta) (u, \partial r_\delta) - (-r_\delta)^{-1} (1-\delta) \partial r_\delta \otimes \bar{\partial} r_\delta (u, \partial r_\delta) \]
\[ \leq (-r_\delta)^{1-\delta} \left( \partial \bar{\partial} (-(-r_\delta)^\delta) (u, u) \right)^{\frac{1}{2}} \left( (-r_\delta)^{-2+\delta} (1-\delta) |\partial r_\delta|^2 + O((-r_\delta)^{-1+\delta}) \right)^{\frac{1}{2}} \]
\[ + (1-\delta) |\partial r_\delta|^2 (-r_\delta)^{-1} |\partial r_\delta \cdot u| \]
\[ \leq \left( (1-\delta)^{\frac{1}{2}} (-r_\delta)^{-\frac{1}{2}} + O((-r_\delta)^{\frac{1}{2}}) \right) \left( \partial \bar{\partial} (-(-r_\delta)^\delta) (u, u) \right)^{\frac{1}{2}} + (1-\delta) |\partial r_\delta|^2 (-r_\delta)^{-1} |\partial r_\delta \cdot u|. \]

Evaluation for \( u = u^\tau \), yields \((4.4)\).

(b) We prove now \((4.2)\) using the basic estimates. Generally, these apply to smooth plurisubharmonic defining functions. However, in \([11]\), Kohn has a version for Hölder continuous plurisubharmonic functions such as \(-(-r_\delta)^\delta\). This implies the inequality \((\ast)\) below

\[ \| (-r_\delta)^{\frac{1}{2}} \Theta_g^* u^\tau \|^2 \simeq \int_D (-r_\delta)^{\delta} \left| \partial \bar{\partial} r_\delta (u^\tau, \partial r_\delta) \right|^2 dV \]
\[ \leq (1-\delta) \int_D \partial \bar{\partial} (-(-r_\delta)^\delta) (u^\tau, u^\tau) dV \]
\[ \leq (1-\delta) Q_{(-r_\delta)^{\frac{1}{2}}} (u^\tau, u^\tau) \]
\[ \leq \mathcal{E}_{s,g} Q_{(-r_\delta)^{\frac{1}{2}}} (u^\tau, u^\tau). \]

This proves \((4.2)\)

(c) We are therefore in the same situation as in Definition \((2.1)\) apart from the term \((-r_\delta)^\delta\) which occurs in the integral in the left of \((1.5)\) and in \( Q_{(-r_\delta)^{\frac{1}{2}}} \). As above, we continue to write \( T \) but take in fact its positive microlocalization \( T^* \) which represents the full action of \( \Lambda \) over \( u^\tau \). To carry on the proof, we suppose from now on that \( f \in C^\infty (\bar{D}) \) and that \( B_k \) is \( H^s \) regular for some continuity constant \( c' \); we prove that this implies continuity for a constant \( c \) which is solely related to the constants which occur in \([?]\). An exhaustion by domains endowed with \( H^s \)-regular projections \( B_k \), \( k \geq q \), will be discussed only at the end. We start from \((2.11)\)

\[ \| T_g^{s-\frac{1}{2}} B_{k-1} f \| \leq sc \| T_g^{s-\frac{1}{2}} B_{k-1} f \|^2 + lc \| T_g^{s-\frac{1}{2}} f \|^2 \]
\[ + lc \| [\partial^*, T_g^{s-\frac{1}{2}}] N_k \bar{\partial} f \|. \]
At this point, we need to convert $T^{s-\frac{1}{2}}_g$ into $(-r_\delta)^{\frac{s}{2}} T^{s}_g$ in the last term of (4.6) in order to enjoy (4.2). We also replace $N_k \bar{\partial} f$ by $(N_k \bar{\partial} f)^{(h)}$ where the subscript $(h)$ denotes the harmonic extension. We apply the crucial estimate (3.6) to the last term in (4.6), regard as errors the terms which come in $(s-1)$-norm or in which vector fields of $S$ occur, and get

\begin{equation}
\left\| \left[ \bar{\partial}^s, T^{s-\frac{1}{2}}_g \right] (\bar{\partial} N_k f)^{(h)} \right\| \leq \left\| (-r_\delta)^{\frac{s}{2}} \left[ \tilde{T}^{s}_g, \bar{\partial}^s \right] (\bar{\partial} N_k f)^{(h)} \right\|^2 \\
\leq s^2 \left\| (-r_\delta)^{\frac{s}{2}} \tilde{T}^{s}_g (\bar{\partial} N_k f)^{(h)} \right\|^2 + \text{error} \\
\leq s^2 \left\| (-r_\delta)^{\frac{s}{2}} \tilde{T}^{s}_g (\bar{\partial} N_k f)^{(h)} \right\|^2 + \text{error} \\
\leq s^2 \sup_{|g|^{2s}} \left\{ \left| (-r_\delta)^{\frac{s}{2}} \partial N_k f \right| \right\} + \left\| (-r_\delta)^{\frac{s}{2}} \tilde{T}^{s}_g (\bar{\partial} N_k f)^{(h)} \right\|^2 + \mathcal{E}(0) + \text{error} \\
\leq s^2 \mathcal{E}_{s,g} \sup_{|g|^{2s}} \left\{ \left| (-r_\delta)^{\frac{s}{2}} \partial N_k f \right| \right\} + \left\| (-r_\delta)^{\frac{s}{2}} \tilde{T}^{s}_g (\bar{\partial} N_k f)^{(h)} \right\|^2 + \mathcal{E}(0) + \text{error} \\
\leq s^2 \mathcal{E}_{s,g} \sup_{|g|^{2s}} \left\{ \left| (-r_\delta)^{\frac{s}{2}} \partial N_k f \right| \right\} + \left\| (-r_\delta)^{\frac{s}{2}} \tilde{T}^{s}_g (\bar{\partial} N_k f)^{(h)} \right\|^2 + \mathcal{E}(0) + \text{error} \\
\leq s^2 \mathcal{E}_{s,g} \sup_{|g|^{2s}} \left\{ \left| (-r_\delta)^{\frac{s}{2}} \tilde{T}^{s}_g (\bar{\partial} N_k f) \right|^2 + \mathcal{E}(0) + \text{error} \right\}. \tag{4.1}
\end{equation}

where we have used the notation $\mathcal{E}(0) := \left\| (-r_\delta)^{\frac{s}{2}} \tilde{T}^{s}_g (\bar{\partial} N_k f)^{(h)} \right\|^2$. Here in (4.1) we have chosen the first alternative $s^2 \mathcal{E}_{s,g} e^{cs^2} \sup_{|g|^{2s}} \left( \frac{1}{|g|} \right) \leq c_1 = sc$ (for a new $c_2$). (The other alternative $\mathcal{E}_{s,g} c^{cs^2} \sup_{|g|^{2s}} \left( \frac{1}{|g|} \right) \leq c_1 = sc$ can be handled similarly as in Theorem 2.3 without replacing $T_g$ by $T$. It is at this point, where the continuity of $B_k$ in $H^s$, not just in $C^\infty$, is needed; in fact, in formula (2.15), $N^s_\varphi$ is $H^s$, not $C^\infty$, continuous. We have to reconvert now $(-r_\delta)^{\frac{s}{2}}$ into $T^{-\frac{s}{2}}$. We first suppose that we had started from $f^{(h)}$ and wished
to prove the regularity for $B_{k-1}f^{(h)}$. We have

$$\| (-r_{\delta})^{\frac{3}{2}} T^s \bar{\partial}^* \bar{\partial} N_k f^{(h)} \| \lesssim (2.4) \left( \begin{array}{c} \| T^s \bar{\partial}^* \bar{\partial} N_k f^{(h)} \| \\ \| -r T^s \bar{\partial}^* \bar{\partial} N_k f^{(h)} \| \end{array} \right).$$

Now,

$$(i) \lesssim \| T^s \bar{\partial}^* f^{(h)} \|^2 + \| T^s \bar{\partial}^* B_{k-1} f^{(h)} \|^2,$$

where the first term in the right is good and the second can be absorbed since it comes, inside (4.7), with sc. As for (ii),

$$(ii) = \| -r T^s \bar{\partial}^* \bar{\partial} \bar{\partial}^* f^{(h)} \| + \text{error}$$

$$= \| -r T^s \bar{\partial}^* \bar{\partial} \bar{\partial}^* f^{(h)} \| + \text{error}$$

We have

$$\begin{cases} \bar{\partial}^* \bar{\partial} = \tan^2 + \partial_r \tan + \partial_r^2 \sim T^2 + \partial_r T + \partial_r^2, \\ \partial_r^2 = \Delta + \tan^2 + \partial_r \tan \sim \Delta + T^2 + \partial_r T, \end{cases}$$

which implies

$$\bar{\partial}^* \bar{\partial} \sim T^2 + \partial_r T + \Delta.$$ 

It follows

$$\| -r T^s \bar{\partial}^* \bar{\partial} f^{(h)} \| = \| -r T^s \bar{\partial}^* \bar{\partial} \Delta f^{(h)} \|$$

$$\lesssim \| -r T^s \bar{\partial}^* \bar{\partial} f^{(h)} \| + \| -r T^s \bar{\partial}^* \bar{\partial} T f^{(h)} \|$$

$$\lesssim \| T^s \bar{\partial}^* f^{(h)} \|,$$

which is good. As for the term $f^{(0)}$, the regularity of $B_{k-1}f^{(0)}$ follows readily, without using the machinery (a)–(c) above, from elliptic regularity

$$(4.9) \quad \| T^s N_{k-1} f^{(0)} \| \lesssim \| T^s f^{(0)} \|.$$ 

(Note that $N_{k-1}$ makes sense even for $k-1 = 0$ when acting on $f^{(0)}|_{\partial D} \equiv 0$ because $\Box$ is, under this restriction, invertible.)

We pass to the term which has been omitted in the estimate of $\bar{\Theta}_g^*$, that is, $E^{(0)}$. The use of elliptic regularity is different here and applies to $(\bar{\partial} N_{k-1}) f^{(0)}$ instead of $f^{(0)}$; it then
passes though \( Q \) instead of \( \Box \) and through Boas-Straube formula. We have

\[(4.10)\]

\[
\|(-r_\delta)^{\frac{1}{2}} \hat{T}_g^s T_s^*(\overline{\partial} N_k f)^{(0)}\|^2 \leq \sup_{|q|^{2s}} \frac{1}{|q|} \|(-r_\delta)^{\frac{1}{2}} \hat{T}_g^s T_s^*(\overline{\partial} N_k f)^{(0)}\|^2
\]

\[
\leq \mathcal{E}_{s,g} \sup_{|q|^{2s}} \frac{1}{|q|} \left( Q(-r_\delta)^{\frac{1}{2}} T_s^*(\overline{\partial} N_k f)^{(0)} + \text{error} \right)
\]

\[
+ \|(-r_\delta)^{\frac{1}{2}} [\overline{\partial}, T_s^*] (\overline{\partial} N_k f)^{(0)}\|^2 + \|(-r_\delta)^{\frac{1}{2}} [\overline{\partial}^*, T_s^*] (\overline{\partial} N_k f)^{(0)}\|^2 \right) + \text{error}
\]

This is the same as \((4.7)\) with the advantage that in the last line the Sobolev indices have decreased by \(-1\) since terms with superscript \(0\) vanish at \(\partial D\); these are therefore error terms. Also there remain to control \(\|T^* \hat{T}_g^s T_s^*(\overline{\partial} N_k f)^{(0)}\|\) and \(\|(-r_\delta)^{\frac{1}{2}} \hat{T}_g^s T_s^*(\overline{\partial} N_k f)^{(0)}\|\); but these are controlled by elliptic regularity as in \((4.10)\). Summarizing up, we have proved that for a suitable \(c\), only related to the constants in \((4.1)\), we have

\[(4.11)\]

\[
\|B_k f\|_s \leq c \|f\|_s
\]

if we knew that it holds for some \(c' \gg c\). We show now that we can exhaust \(D\) by domains \(D_\rho\) endowed with continuous projections \(B_k\), \(k \geq q - 1\) for some \(c'\) and which inherit the assumption of Theorem 4.2 with uniform constants with respect to \(\rho\). For this, we define \(D_\rho = \{ z : r_\delta(z) + \rho < 0 \}\). We first notice that, \(\partial D_\rho\) being also defined by \(-(-r_\delta)^{\frac{1}{2}} + \rho^{\frac{1}{2}} < 0\), it has a smooth \(q\)-plurisubharmonic defining function. Hence, by Theorem 2.4 \(B_k\) is \(H^s\)-regular for any \(k \geq q - 1\). Coming back to the initial defining function \(r_\delta + \rho\), this satisfies \(\partial \overline{\partial}(-(-r_\delta + \rho)^{\delta}) \geq \partial \overline{\partial}(-(-r_\delta)^{\delta})\); thus the Diederich-Fornaess index of \(D_\rho\) is \(\geq \delta\). Also, if for the new boundary we rewrite \(r_\delta + \rho = g_{\delta,\rho} r_\delta\), for a normalized equation \(r_\rho\) of \(D_\rho\), and if \(\mathcal{E}_{s,g,\rho}\) are the constants which occur in \((4.1)\), then

\[
\begin{align*}
g_{\delta,\rho} &\rightarrow g_\delta, \\
\mathcal{E}_{s,g,\rho} &\rightarrow \mathcal{E}_{s,g}.
\end{align*}
\]

Thus, the estimate \((4.11)\) passes from the \(D_\rho\)'s (in which it has been proved thanks to the regularity of the \(B_k\) (for a different \(c')\) to the initial domain \(D\).

The proof is complete.

}\]

---

**References**

[1] **D.E. Barret**—Behavior of the Bergman projection on the Diederich-Fornaess worm, *Acta Math.* **168** (1992), 1-10
[2] **H. P. Boas and E. J. Straube** — Equivalence of regularity for the Bergman projection and the \( \bar{\partial} \)-Neumann operator, *Manuscripta Math.* 67 (1990) 25–33.

[3] **H. P. Boas and E. J. Straube** — Sobolev estimates for the \( \bar{\partial} \)-Neumann operator on domains in \( \mathbb{C}^n \) admitting a defining function that is plurisubharmonic on the boundary, *Math. Z.* 206 (1) (1991) 81–88.

[4] **K. Diederich and J.E. Fornaess** — Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions, *Invent. Math.* 39 (1977), 129–141

[5] **P. Harrington** — Global regularity for the \( \bar{\partial} \)-Neumann operator and bounded plurisubharmonic exhaustion functions, Preprint (2011)

[6] **A. K. Herbig and J. D. McNeal** — Regularity of the Bergman projection on forms and plurisubharmonicity conditions. *Math. Ann.* 336 (2006), 2, 335–359.

[7] **T.V. Khanh** — Global hypoellipticity of the Kohn-Laplacian \( \square_b \) on pseudoconvex CR manifolds, (2010) Preprint.

[8] **J.J. Kohn and L. Nirenberg** — Non-coercive boundary value problems, *Comm. Pure Appl. Math.* 18 (1965), 443–492

[9] **J. J. Kohn** — Global regularity for \( \bar{\partial} \) on weakly pseudo-convex manifolds, *Trans. of the A.M.S.* 181 (1973), 273–292.

[10] **J.J. Kohn** — Subellipticity of the \( \bar{\partial} \)-Neumann problem on pseudoconvex domains: sufficient conditions, *Acta Math.* 142 (1979), 79–122

[11] **J.J. Kohn** — Quantitative estimates for global regularity, *Analysis and geometry in several complex variables*, Trend Math. Birkhäuser Boston (1999), 97–128

[12] **S. Krantz and M. Peloso** — Analysis and geometry on worm domains, *J. Geom. Anal.* 18 (2008), 478-510

[13] **E. Straube** — A sufficient condition for global regularity of the \( \bar{\partial} \)-Neumann operator, *Adv. in Math.* 217 (2008), 1072–1095

[14] **E. Straube** — Lectures on the \( L^2 \)-Sobolev theory of the \( \bar{\partial} \)-Neumann problem, *ESI Lect. in Math. and Physics* (2010)

Dipartimento di Matematica, Università di Padova, via Trieste 63, 35121 Padova, Italy

E-mail address: pinton@math.unipd.it, zampieri@math.unipd.it