Asymptotic distribution for the proportional covariance model

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Abstract

Asymptotic distribution for the proportional covariance model under multivariate normal distributions is derived. To this end, the parametrization of the common covariance matrix by its Cholesky root is adopted. The derivations are made in three steps. First, the asymptotic distribution of the maximum likelihood estimators of the proportionality coefficients and the Cholesky inverse root of the common covariance matrix is derived by finding the information matrix and its inverse. Next, the asymptotic distributions for the case of the Cholesky root of the common covariance matrix and finally for the case of the common covariance matrix itself are derived using the multivariate δ-method. As an application of the asymptotic distribution derived here, a hypothesis for homogeneity of covariance matrices is considered.

Keywords: Asymptotic distribution, Cholesky decomposition, maximum likelihood estimators, proportionality of covariance matrices.

1 Introduction

Let \( X_k \) be a generic \( p \)-variate random vector having a multivariate normal distribution \( N_p(\mu_k, \Sigma_k) \) for each \( k = 1, ..., K \), where the \( \mu_k \) are column vectors of the \( p \)-dimensional Euclidean space and the \( \Sigma_k \) are \( p \) by \( p \) positive definite covariance matrices. Let the \( S_k \) be the mutually independent unbiased estimators of the \( \Sigma_k \), each for a sample of size \( N_k \) and let \( n_k = N_k - 1 \). Then the \( n_k S_k \) are independently distributed according to Wishart distribution \( W_p(\Sigma_k, n_k) \).

The hypothesis that \( K \) covariance matrices are proportional to each other can be expressed as

\[
H_p : \Sigma_k = c_k \Sigma_1 \quad \text{for all } k = 2, ..., K
\]

where the \( c_k \) are unknown positive proportionality coefficients. In the two sample case, estimation for proportional covariance models was studied by Khatri(1967), Guttman, Kim and Olkin (1985) and Rao (1983). The proportional covariance model was adopted in

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classification by Owen (1984). An algorithm to find the maximum likelihood estimators was given independently by Owen (1984), Manly and Rainer (1987) and Eriksen (1987). The convergence of the algorithm and the uniqueness of maximum likelihood estimates were proved by Eriksen (1987). Using a parametrization of the spectral decomposition of the common covariance matrix, Flury (1986) considered maximum likelihood estimation of the proportionality coefficients and the common covariance matrix, and suggested an iterative method for performing computations. Under the same parametrization as in Flury (1986), Boente et al. (2007, 2009) studied influence diagnostics and robust estimation for proportional covariance models.

The Cholesky decomposition theorem states that there exists a lower triangular matrix $\mathbf{A}$ with positive diagonal elements such that

$$\Sigma_1 = \mathbf{A}\mathbf{A}^T,$$

and this expression is unique. We use $T$ to denote a transpose of a matrix. Then the model (1) is parametrized as

$$\Sigma_k = c_k\mathbf{A}\mathbf{A}^T \quad (1 \leq k \leq K).$$

For notational convenience, it is assumed that $c_1$ is identical with 1. The parametrization (2) can be considered as the special case of the model treated by Jöreskog (1971) and the easily accessible LISREL (linear structural relationships) program is available for getting the maximum likelihood estimates of the model parameters (Jöreskog and Sörbom, 1996).

Let $\mathbf{B}^T = \mathbf{A}^{-1}$. The matrix $\mathbf{B}$ is then an upper triangular matrix with positive diagonal elements. Then the parametrization (2) can be written as

$$\Sigma_k^{-1} = \frac{1}{c_k}\mathbf{B}\mathbf{B}^T \quad (1 \leq k \leq K).$$

In this work, we derive the asymptotic distribution of the maximum likelihood estimators of the proportionality coefficients and the common covariance matrix, using the well-known properties of the maximum likelihood estimators under regularity conditions (Rao, 1973, Chapter 6). The procedure for doing so is as follows. Firstly, the information matrix for the parameters $c_k$ and $\mathbf{B}$ in (3) is derived. Secondly, the asymptotic covariance matrix for the maximum likelihood estimators of the parameters $c_k$ and $\mathbf{B}$ in (2) is found by inverting the corresponding information matrix. Thirdly, we derive the asymptotic covariance matrix for the maximum likelihood estimators of the proportionality coefficients and the common covariance matrix is derived using the multivariate $\delta$-method. Finally, a hypothesis for homogeneity of covariance matrices is considered as an application of the asymptotic distribution derived here.
2  The information matrix for the parameters $c_k$ and $B$

In this section, we derive the information matrix for the $c_k$ and $B$. The likelihood function of the $\Sigma_k$ given the $S_k$ is

$$L(\Sigma_1, ..., \Sigma_K) = C \times \prod_{k=1}^{K} [\text{det} \Sigma_k \cdot (-\frac{n_k}{2}) \exp\{\text{tr}(\Sigma_k^{-1}S_k)\}],$$

where $C$ is a constant not depending on the parameters. Since we are interested in covariance matrices and a covariance matrix is location invariant, there is no restriction on mean vectors.

Under the parametrization (3), we have

$$|\Sigma_k^{-1}| = c_k^{-p} \prod_{i=1}^{p} b_{ii}^2$$

$$\text{tr}(\Sigma_k^{-1}S_k) = \frac{1}{c_k} \sum_{i=1}^{p} b_i^T S_k b_i,$$

where $b_{ij}$ denotes the $(i,j)$th element of $B$ and $b_i$ the $i$th column of $B$. Thus the log-likelihood function, ignoring constant term, becomes

$$l(c, B) = n_+ \sum_{i=1}^{p} \log(b_{ii}) - \sum_{k=1}^{K} \frac{n_k}{2} \{p \log(c_k) + \frac{1}{c_k} \sum_{i=1}^{p} b_i^T S_k b_i\},$$

where $n_+ = \sum_{k=1}^{K} n_k$ and $c = (c_2, ..., c_K)^T$. It is easily shown that all the expectations of first derivatives of $l(c, B)$ with respect to the parameters are zero, and the expectations of second derivatives of the log-likelihood function are as follows:

$$-E\{\frac{\partial^2 l(c, B)}{\partial c_k^2}\} = \frac{pm_k}{2c_k^2} \quad (2 \leq k \leq K)$$

$$-E\{\frac{\partial^2 l(c, B)}{\partial b_{ii}^2}\} = n_+ (a_{ii}^2 + a_{(i)}^T a_{(i)}) \quad (1 \leq i \leq p)$$

$$-E\{\frac{\partial^2 l(c, B)}{\partial b_{ii} \partial c_k}\} = -\frac{n_k a_{ii}}{c_k} \quad (2 \leq k \leq K, \ 1 \leq i \leq p)$$

$$-E\{\frac{\partial^2 l(c, B)}{\partial b_{ji} \partial c_k}\} = \begin{cases} 0 & (2 \leq k \leq K, \ j < i) \\ n_+ a_{(j)}^T a_{(j')} & (1 \leq j, j' < i) \end{cases}$$

where $a_{ij}$ denotes the $(i,j)$th element of $A$ and $a_{(i)}$ the column vector formed by the elements in the $i$th row of $A$, and all the other cases have zero expectations.
Let $I(c, B)$ be the information matrix for the parameters $c$ and $B$, and let it be partitioned as

$$ I(c, B) = \begin{pmatrix} I_{11}(c, B) & I_{12}(c, B) \\ I_{21}(c, B) & I_{22}(c, B) \end{pmatrix}, $$

such that $I_{11}(c, B)$ is the information matrix for $c$ and $I_{22}(c, B)$ for the nonzero elements of $B$. Let $r_k = n_k/n_+$. Then $I_{11}(c, B)$ is a diagonal matrix of order $K - 1$:

$$ \frac{p}{2} \text{diag}(r_2/c_2, \ldots, r_K/c_K). $$

We are interested in nonzero elements of $B$, $b_{ij}$ ($i < j$) and the number of nonzero elements of $B$ is $p(p+1)/2$. Let $b_{i,j} = (b_{i1}, \ldots, b_{ji})^T$. Then we can notice that $b_{i,1} = b_{i1}$ and $b_{i,p} = b_{i}$. Let $\alpha = (r_2/c_2, \ldots, r_K/c_K)^T$. We write as $1_{p,i}$ the unit column vector of dimension $p$ having one in the $i$th position and zeroes in the other positions. Note that $1_{1,1} = 1$. Then the $(K - 1)$ by $p(p+1)/2$ matrix $I_{22}(c, B)$ is

$$ c \begin{pmatrix} b_{1,1}^T & b_{2,2}^T & \ldots & b_{p,p}^T \\ -a_{11}\alpha & -a_{22}\alpha 1_{2,2}^T & \ldots & -a_{pp}\alpha 1_{p,p}^T \end{pmatrix}. $$

Let $A_i$ be the leading principal submatrix of $A$ having order $i$. Note that $A_1 = a_{11}$ and $A_p = A$. Then the $p(p+1)/2$ by $p(p+1)/2$ matrix $I_{22}(c, B)$ is a block diagonal matrix whose $i$th block element for $b_{i,i}$ is given by

$$ a_{ii}^2 1_{i,i} 1_{i,i}^T + A_i A_i^T. $$

This $i$th block element is of order $i$.

### 3 Asymptotic distribution of the maximum likelihood estimators of the $c_k$ and $B$

The asymptotic distribution of the maximum likelihood estimators of $c$ and $B$ is characterized by its asymptotic covariance matrix, and the asymptotic covariance matrix is just the inverse of the corresponding information matrix.

Let $V(c, B)$ be the asymptotic covariance matrix for the maximum likelihood estimators of the parameters $c$ and the nonzero elements of $B$, and let it be partitioned as

$$ V(c, B) = \begin{pmatrix} V_{11}(c, B) & V_{12}(c, B) \\ V_{21}(c, B) & V_{22}(c, B) \end{pmatrix}, $$

such that $V_{11}(c, B)$ is the covariance matrix for the maximum likelihood estimator of $c$ and $V_{22}(c, B)$ for the maximum likelihood estimators of the nonzero elements of $B$. 

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The following lemma gives the inverse of $I_{22}(c, B)$ and enables us to get $V_{11}(c, B)$.

**Lemma 3.1.** For each $i = 1, \ldots, p$, the inverse of $\alpha_{ii}^T 1_{i,i} 1_{i,i}^T + A_i A_i^T$ is

$$B_i B_i^T - \frac{1}{2} b_{i,i} b_{i,i}^T$$

where $B_i$ is the leading principal submatrix of $B$ having dimension $i$.

Lemma 3.1 is easily proved using the result (Sherman-Morrison formula) by Bartlett (1951) and the fact that $B_i^T = A_i^{-1}$.

**Lemma 3.2.** The matrix $I_{11}(c, B) - I_{12}(c, B) I_{22}(c, B)^{-1} I_{21}(c, B)$, $U_{11}(c, B)$ say, of dimension $K - 1$ is given by

$$\frac{p}{2} \{ \text{diag}(r_2, \ldots, r_K) - \alpha \alpha^T \}.$$  

**Proof of Lemma 3.2.** The $i$th block element of $I_{12}(c, B) I_{22}(c, B)^{-1}$ is a $(K - 1)$ by $i$ matrix for the pair $c$ and $b_{i,i}$ and is given by

$$-\frac{1}{2} \alpha b_{i,i}^T.$$  

Then $I_{12}(c, B) I_{22}(c, B)^{-1} I_{21}(c, B)$ is given by

$$\frac{p}{2} \alpha \alpha^T$$

and therefore the proof is completed.

The inverse of $U_{11}(c, B)$ is just $V_{11}(c, B)$ and is stated in the following lemma.

**Lemma 3.3.** The $(K - 1)$ by $(K - 1)$ matrix $V_{11}(c, B)$ is given by

$$\frac{2}{p} \{ \text{diag}(\frac{c_2^2}{r_2}, \ldots, \frac{c_K^2}{r_K}) + \frac{1}{r_1} c c^T \}.$$  

The matrix $V_{11}(c, B)$ is easily derived using the result by Bartlett (1951).

The matrix $I_{12}(c, B) I_{22}(c, B)^{-1}$ is derived in the proof of Lemma 3.2 and $V_{12}(c, B) = -V_{11}(c, B) I_{12}(c, B) I_{22}(c, B)^{-1}$. Hence we get easily the following lemma.

**Lemma 3.4.** The $(K - 1)$ by $p(p + 1)/2$ matrix $V_{12}(c, B)$ is given by the following matrix multiplied by $(1/p r_1)$:

$$\begin{pmatrix}
    b_{1,1}^T & b_{2,2}^T & \ldots & b_{p,p}^T \\
    c b_{1,1}^T & c b_{2,2}^T & \ldots & c b_{p,p}^T
\end{pmatrix}.$$
Since $V_{22}(c, B) = I_{22}(c, B)^{-1} - I_{22}(c, B)^{-1}I_{21}(c, B)V_{12}(c, B)$, Lemma 3.1, the proof of Lemma 3.2 and Lemma 3.4 yield $V_{22}(c, B)$.

**Lemma 3.5.** The $p(p + 1)/2$ by $p(p + 1)/2$ matrix $V_{22}(c, B)$ is given by

$$
\begin{pmatrix}
b_{1,1}^T & b_{2,2}^T & \ldots & b_{p,p}^T \\
b_{1,1}B_1 + db_{1,1}b_{1,1}^T & c b_{1,1}b_{2,2}^T & \ldots & c b_{1,1}b_{p,p}^T \\
b_{2,2} & B_2 + d b_{2,2}b_{2,2}^T & \ldots & c b_{2,2}b_{p,p}^T \\
\vdots & \vdots & \ddots & \vdots \\
b_{p,p} & c b_{p,p}b_{1,1}^T & \ldots & B_p + d b_{p,p}b_{p,p}^T
\end{pmatrix},
$$

where $d = \{1 - (1 + p) r_1\}/2 pr_1$ and $e = (1 - r_1)/2 pr_1$.

We use hereafter the hat notation to denote the maximum likelihood estimator.

The general properties of the maximum likelihood estimators yield the asymptotic distribution of $\hat{c}$ and $\hat{B}$ under the proportional covariance model.

**Theorem 3.1.** As $n_+$ tends to infinity with each $r_k$ held fixed, the asymptotic distribution of

$$\sqrt{n_+}(\hat{c} - c, \hat{B} - B)$$

is a multivariate normal with mean zero and covariance matrix $V(c, B)$.

Lemma 3.3 shows that the asymptotic distribution of $\sqrt{n_+}(\hat{c}_2 - c_2, \ldots, \hat{c}_K - c_K)$ is free of $\Sigma_1$. Gutman et al. (1985) derived the asymptotic distribution of $\sqrt{n_+}(\hat{c}_2 - c_2)$ when $K = 2$ and Lemma 3.3 reduces to Theorem 4.1 of Gutman et al. (1985).

When $r_1 = 1$, that is, the single-population case, the coefficients $d$ and $e$ become $-1/2$ and zero, respectively. Hence Lemma 3.5 shows that the $\hat{b}_{i,i}$ ($i = 1, \ldots, p$) are asymptotically independent. By the invariance property of the maximum likelihood estimator, the Cholesky inverse root of $S_1$ is just $\hat{B}$, that is, $S_1^{-1} = \hat{B}\hat{B}^T$. Thus Lemma 3.5 gives the asymptotic distribution of the Cholesky inverse root of the sample covariance matrix.

**4 Asymptotic distribution of the maximum likelihood estimators of the $c_k$ and $A$**

The asymptotic distribution of the maximum likelihood estimators of the $c_k$ and $A$ is derived from that of the $c_k$ and $B$, using the multivariate $\delta$-method.
Let $V(c, A)$ be the asymptotic covariance matrix for the maximum likelihood estimators of the parameters $c$ and $A$. We write $V(c, A)$ as a partitioned form:

$$V(c, A) = \begin{pmatrix} V_{11}(c, A) & V_{12}(c, A) \\ V_{21}(c, A) & V_{22}(c, A) \end{pmatrix},$$

such that $V_{11}(c, A)$ is the covariance matrix for the maximum likelihood estimator of $c$ and $V_{22}(c, A)$ for the maximum likelihood estimators of the nonzero elements of $A$.

Since $B^T = A^{-1}$, we have the following lemma.

**Lemma 4.1.** The partial derivative of $a_{ji}$ with respect to $b_{hg}$ is

$$\frac{\partial a_{ji}}{\partial b_{hg}} = \begin{cases} -a_{jd}a_{hi} & (i \leq h \leq g \leq j) \\ 0 & \text{otherwise}. \end{cases}$$

Let $a_{i,j} = (a_{11}, \ldots, a_{ji})^T$ and $a_{i(-j)} = (a_{i+1,i}, \ldots, a_{pi})^T$. Note that $a_{i(-0)}$ and $a_{i,p}$ are equivalent to $a_i$, the $i$th column of $A$ and that $a_{i,1} = a_{1i}$. Then the Jacobian, $J(A, B)$, of $A$ with respect to $B$ is

$$\begin{pmatrix} b_{1.1}^T \\ b_{2.2}^T \\ \vdots \\ b_{p,p}^T \\ -a_{11}a_{1,1}^T \\ -a_{22}a_{1,2}^T \\ \vdots \\ -a_{p,p}a_{1,p}^T \\ -a_{2(-1)}a_{2,2}^T \\ -a_{3(-1)}a_{2,3}^T \\ \vdots \\ -a_{p(-1)}a_{2,p}^T \\ 0 \\ 0 \\ \vdots \\ -a_{p(-2)}a_{3,p}^T \end{pmatrix}.$$ 

Hence the Jacobian $J(A, B)$ is a block upper triangular matrix.

Under the transformation $B^T = A^{-1}$, the asymptotic distribution of $\hat{c}$ is not changed and in fact $V_{11}(c, A) = V_{11}(c, B)$.

Since $V_{12}(c, A) = V_{12}(c, B)J(A, B)^T$, we get the following lemma.

**Lemma 4.2.** The $(K - 1)$ by $p(p + 1)/2$ matrix $V_{12}(c, A)$ is given by the following matrix multiplied by $-1/pr_1$:

$$c \begin{pmatrix} a_1^T & a_{2(-1)}^T & \cdots & a_{p(-p+1)}^T \\ ca_1^T & ca_{2(-1)}^T & \cdots & ca_{p(-p+1)}^T \end{pmatrix}.$$
Proof of Lemma 4.2. By Lemma 3.4, $V_{12}(c, A)$ has as its $i$th block element for the covariance matrix of $\hat{c}$ and $\hat{a}_{i(-i+1)}$

$$-\frac{c}{pp'} \sum_{j=1}^{p} b_{j,j} T \hat{a}_{i,j} T \hat{a}_{j(-j+1)} = -\frac{1}{pp'} c \hat{a}_{i(-i+1)} T$$

because $B^T A = I_p$.

Let $A_{-i}$ be the matrix formed by deleting the first $i$ columns and the first $i$ rows of $A$ simultaneously. Note that $A_{-i} = A$ and $A_{-p+1} = a_{p(-p+1)} = a_{pp}$.

Lemma 4.3. The $p(p+1)/2$ by $p(p+1)/2$ matrix $V_{22}(c, A)$ is given by

$$
\begin{pmatrix}
    a_1^T \\
    a_2(-1)^T \\
    \vdots \\
    a_{p(-p+1)}^T \\
\end{pmatrix} \begin{pmatrix}
    \AA^T + da_1 a_1^T & e a_1 a_2(-1)^T & \cdots & e a_1 a_{p(-p+1)}^T \\
    e a_2(-1) a_1^T & A_{-1} A_{-1}^T + da_2(-1) a_{2(-1)}^T & \cdots & e a_2(-1) a_{p(-p+1)}^T \\
    \vdots & \vdots & \ddots & \vdots \\
    e a_{p(-p+1)} a_1^T & e a_{p(-p+1)} a_{2(-1)}^T & \cdots & (1 + d) a_{p(-p+1)} a_{p(-p+1)}^T \\
\end{pmatrix},
$$

where $d$ and $e$ are defined as in Lemma 3.5.

Proof of Lemma 4.3. The $(i, j)$th block element of $V_{22}(c, A)$ is the covariance matrix of $\hat{a}_{i(-i+1)}$ and $\hat{a}_{j(-j+1)}$, and is given by

$$\sum_{l=i}^{p} \sum_{m=j}^{p} a_{i(-i+1)} a_{l,l}^T V_{22}[l, m] a_{j,m} a_{m(-j+1)}^T \quad (i \leq j),$$

where $V_{22}[l, m]$ denotes the $(l, m)$th block element of $V_{22}(c, B)$ which is the covariance matrix of $\hat{b}_{l,l}$ and $\hat{b}_{m,m}$. Let $\delta_{ij}$ be the Kronecker delta. Since $a_{l,l}^T b_{l,l} = \delta_{il}$ and $a_{l,l}^T B_l = 1_{l,l}^T$, the $(i, j)$th block element of $V_{22}(c, A)$ becomes

$$e(1 - \delta_{ij}) a_{i(-i+1)} a_{j(-j+1)}^T + \delta_{ij}(A_{-i+1} A_{-i+1}^T + d a_{i(-i+1)} a_{i(-i+1)}^T),$$

which completes the proof.

Lemmas 3.3, 4.2 and 4.3 yield the asymptotic distribution of $\hat{c}$ and $\hat{A}$ under the proportional covariance model.

Theorem 4.1. The asymptotic distribution of

$$\sqrt{m}(\hat{c} - c, \hat{A} - A)$$

is
is a multivariate normal with mean zero and covariance matrix $\mathbf{V}(\mathbf{c}, \mathbf{A})$, as $n_+ \to \infty$ with each $r_k$ kept fixed.

Consider the single-population case, $r_1 = 1$. Since the coefficients $d$ and $e$ become $-1/2$ and zero, respectively, Lemma 4.3 shows that the $\hat{a}_{i(-i+1)}$ ($1 \leq i \leq p$) are asymptotically independent. In this single-population case, $\mathbf{S}_1 = \hat{\mathbf{A}}\hat{\mathbf{A}}^T$ and therefore Lemma 4.3 gives the asymptotic distribution of the Cholesky root of the sample covariance matrix when the underlying distribution is a multivariate one. The asymptotic distribution of the sample generalized variance being equal to the product of the $\hat{a}^2_{ii}$ is easily derived from Lemma 4.3 and it can be found also in Theorem 7.5.4 of Anderson (1984).

5 Asymptotic distribution of the maximum likelihood estimators of the $c_k$ and $\Sigma_1$

Now, we are in a position to derive the asymptotic distribution of the maximum likelihood estimators of the proportionality coefficients and the common covariance matrix, and it will be done using the multivariate $\delta$-method.

Let $\sigma_{ij}$ be the $(i, j)$th element of $\Sigma_1$. The common covariance matrix $\Sigma_1$ has $p(p+1)/2$ independent parameters. Let $\mathbf{I}_{p,i}$ be the $(p-i)$ by $p$ matrix in which the first $i$ columns are zero vectors and the remaining $p-i$ columns form the unit matrix of order $p-i$. When $i = 0$, $\mathbf{I}_{p,0}$ becomes the unit matrix of order $p$ and is denoted by $\mathbf{I}_p$. For the $i$th column $\sigma_i$ of $\Sigma_1$, $\sigma_{i(-j)}$ is defined similarly to $a_{i(-j)}$. 

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Since $\sigma_{ij} = a^T_{(i)} a_{(j)}$, we get the Jacobian, $J(\Sigma_1, A)$, of $\Sigma_1$ with respect to $A$ as follows:

\[
\begin{pmatrix}
  a_1^T & a_2^T & a_3^T & \cdots & a_{p(-p+1)}^T \\
  a_{11}I_p + 0 & 0 & 0 & \cdots & 0 \\
  a_{12}I_p + a_{21}I_{p,1} & a_{22}I_{p-1} & 0 & \cdots & 0 \\
  a_{13}I_p + a_{23}I_{p-2} & a_{32}I_{p-1,1} & a_{33}I_{p-2} & \cdots & 0 \\
  a_{1p}I_{p,p-1} & a_{2p}I_{p-1,p-2} & a_{3p}I_{p-2,p-3} & a_{pp} & \cdots & + \\
  a_{p1}I_{p,p-1} & a_{p2}I_{p-1,p-2} & a_{p3}I_{p-2,p-3} & a_{pp} & \cdots & + \\
  a_{1(-p+1)}I_{p,p} & a_{2(-p+1)}I_{p-1,p-1} & a_{3(-p+1)}I_{p-2,p-2} & a_{p(-p+1)} & \cdots & +
\end{pmatrix}
\]

Hence the Jacobian $J(\Sigma_1, A)$ is a block lower triangular matrix.

Let $V(c, \Sigma_1)$ be the asymptotic covariance matrix for the maximum likelihood estimators of the parameters $c$ and $\Sigma_1$, and let it be partitioned as

\[
V(c, \Sigma_1) = \begin{pmatrix}
  V_{11}(c, \Sigma_1) & V_{12}(c, \Sigma_1) \\
  V_{21}(c, \Sigma_1) & V_{22}(c, \Sigma_1)
\end{pmatrix},
\]

such that $V_{11}(c, \Sigma_1)$ is the covariance matrix for the maximum likelihood estimators of $c$ and $V_{22}(c, \Sigma_1)$ for the maximum likelihood estimator of the independent parameters of $\Sigma_1$.

The covariance matrix of $\hat{c}$, $V_{11}(c, \Sigma_1)$, is equivalent to $V_{11}(c, B)$ given in Lemma 3.3.

We need some useful formulas to get $V_{12}(c, \Sigma_1)$ which are stated in the following lemma which is easily proved.

**Lemma 5.1.** For $j \leq i$, we have

(i) $I_{p-j+1,i-j} a_{j(-j+1)} = a_{j(-i+1)}$

(ii) $a^T_{j(-j+1)} I_{p-j+1,i-j+1} = a_{ij}$

(iii) $\sigma_{i(-i+1)} = \sum_{j=1}^{i} a_{ij} a_{j(-i+1)}$. 

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The covariance matrix \( V_{12}(c, \Sigma_1) \) of \( \hat{c} \) and \( \hat{\Sigma}_1 \) is given by \( V_{12}(c, A)J(\Sigma_1, A)^T \) as in the following lemma.

**Lemma 5.2.** The \((K-1)\) by \(p(p+1)/2\) matrix \( V_{12}(c, \Sigma_1) \) is given by the following matrix multiplied by \(-2/pr_1\):

\[
\begin{pmatrix}
\sigma_1^T & \sigma_2^T \circ (-1) & \cdots & \sigma_p^T \circ (-p+1) \\
 c\sigma_1^T & c\sigma_2^T \circ (-1) & \cdots & c\sigma_p^T \circ (-p+1)
\end{pmatrix}
\]

**Proof of Lemma 5.2.** The covariance matrix of \( \hat{c} \) and \( \hat{\sigma}_{i(-i+1)} \) which is the \( i \)th block element of \( V_{12}(c, \Sigma_1) \) is given by

\[
-\frac{1}{pr_1} \sum_{j=1}^{i} a_{j(-j+1)}^{T}(a_{ij}I_{p-j+1,i-j} + 1_{p-j+1,i-j+1}a_{j(-i+1)})
\]

which becomes by Lemma 5.1

\[
-\frac{2}{pr_1} c\sigma_{i(-i+1)}^T .
\]

Let \( \Sigma_{(i,j)} = (\sigma_{j(-i+1)}, \ldots, \sigma_{p(-i+1)}) \). Then \( \Sigma_{(i,j)} \) is a \((p-i+1)\) by \((p-j+1)\) matrix consisting of the elements in the lower right corner of \( \Sigma_1 \) starting at \( \sigma_{ij} \). The \((p-i+1)\) by \((p-j+1)\) matrix \( A_{(i,j)} \) is defined similarly to \( \Sigma_{(i,j)} \).

**Lemma 5.3.** The \(p(p+1)/2\) by \(p(p+1)/2\) matrix \( V_{22}(c, \Sigma_1) \) has as its \((i,j)\) block element

\[
\sigma_{ij} \Sigma_{(i,j)} + 4e\sigma_{i(-i+1)}\sigma_{j(-j+1)}^T + \sigma_{j(-i+1)}\sigma_{i(-j+1)}^T \quad (i \leq j)
\]

which is the covariance matrix of \( \hat{\sigma}_{i(-i+1)} \) and \( \hat{\sigma}_{j(-j+1)} \), where \( d \) and \( e \) are defined as in Lemma 3.5.

Since \(3 + 4d = 4e + 1\), the covariance matrix of \( \hat{\sigma}_{i(-i+1)} \) becomes

\[
\sigma_{ii} \Sigma_{(i,i)} + (3 + 4d)\sigma_{i(-i+1)}\sigma_{i(-i+1)}^T .
\]

More formulas are needed to prove Lemma 5.3. Let \( a_{(i)(-j)} = (a_{i,j+1}, \ldots, a_{i,p})^T \). Then we get the following lemma together with Lemma 5.1 useful for proving Lemma 5.3.

**Lemma 5.4.** For \( i \leq j, 1 \leq m \leq i \) and \( 1 \leq l \leq j \), we have

(i) \( 1_{p-m+1,i-m+1}A_{-m+1} = a_{(i)(-m+1)}^T \)

(ii) \( 1_{p-m+1,i-m}A_{-m+1} = A_{(i,m)} \)
Proof of Lemma 5.3. For \( i \leq j \), the \((i,j)\)th block element of \( V_{22}(c, \Sigma_1) \) which is the covariance matrix of \( \hat{\sigma}_{i(-i+1)} \) and \( \hat{\sigma}_{j(-j+1)} \) is given by

\[
\sum_{l=1}^{j} \sum_{m=1}^{i} \left\{ \left[ a_{im} \mathbf{1}_{p-m+1,i-m} + a_{m(-i+1)} \mathbf{1}_{p-m+1,i-m+1} \right] V_{22}[m,l] \right\} (a_{lj} \mathbf{1}_{p-l+1,j-l} + \mathbf{1}_{p-l+1,j-l+1} \mathbf{1}_{a_{i(-i+1)}}, \sigma_{i(-i+1)}^T \sigma_{j(-j+1)}^T)
\]

where \( V_{22}[m,l] \) denotes the \((m,l)\)th block element of \( V_{22}(c, \mathbf{A}) \). By Lemmas 5.1 and 5.4, summing the summands in (4) over \( 1 \leq l = m \leq i \) gives

\[
\sum_{m=1}^{i} \left\{ a_{im} a_{jm} \mathbf{A}_{(i,m)}^T \mathbf{A}_{(j,m)} + 4d a_{im} a_{jm} a_{m(-i+1)} \mathbf{a}_{m(-i+1)}^T \right\} + a_{jm} a_{m(-i+1)} \mathbf{a}_{(i,m)}^T \mathbf{A}_{(j,m)} + a_{im} \mathbf{A}_{(i,m)} a_{j(-m+1)} \mathbf{a}_{m(-i+1)}^T + (\sum_{u=m}^{i} a_{iu} a_{jm} a_{m(-i+1)} \mathbf{a}_{m(-i+1)}^T)
\]

and taking a summation of the summands in (4) over \( l \neq m \) yields

\[
4 \epsilon \sigma_{i(-i+1)} \sigma_{j(-j+1)}^T - 4 \epsilon \sum_{m=1}^{i} a_{im} a_{jm} a_{m(-i+1)} \mathbf{a}_{m(-i+1)}^T.
\]

Applying Lemma 5.4 again to (4) and (6) completes the proof since \( 4d + 3 = 1 + 4\epsilon \).

From Lemmas 3.3, 5.2 and 5.3, we get the asymptotic distribution of \( \hat{c} \) and \( \hat{\Sigma}_1 \) under the proportional covariance model (11).
Theorem 5.1. The statistic

$$\sqrt{n_+}(\hat{c} - c, \hat{\Sigma}_1 - \Sigma_1)$$

is asymptotically distributed according to a multivariate normal distribution with mean zero and covariance matrix $V(c, \Sigma_1)$ as $n_+$ tends to infinity with each $r_k$ held fixed.

The asymptotic distribution of $\hat{c}$ is free of $\Sigma_1$ as can be seen in Lemma 3.3, and that of $\hat{\Sigma}_1$ is independent of $c$. However, $\hat{c}$ and $\hat{\Sigma}_1$ are not asymptotically independent.

When $r_1 = 1$, the $(i, j)$th block element of $V_{22}(c, \Sigma_1)$ becomes

$$\sigma_{ij} \Sigma_{(i,j)} + \sigma_{j(-i+1)} \sigma^T_{i(-j+1)}$$  \hspace{1cm} (7)

because $e$ is zero. In this case, the maximum likelihood estimator of $\Sigma_1$ becomes $S_1$, derived from the Wishart distribution, which is an unbiased estimator of $\Sigma_1$. However, the maximum likelihood estimator of $\Sigma_1$ based on the multivariate normal distribution is in fact $(n_1/N_1)S_1$. Since $S_1$ and $(n_1/N_1)S_1$ have the same limiting distribution, this case ($r_1 = 1$) yields the asymptotic distribution of the maximum likelihood estimator of $\Sigma_1$. We can easily check if (7) is identical with (15) in Anderson (1984, p.82).

6 A hypothesis for homogeneity of covariance matrices

As an application of Theorem 5.1, consider a hypothesis for homogeneity of covariance matrices

$$H_0 : c_2 = ... = c_K = 1. \hspace{1cm} (8)$$

The inverse $U_{11}(c, B)$ of $V_{11}(c, \Sigma_1)$ is needed to test the hypothesis (8), which is given in Lemma 3.2.

The asymptotic distribution of

$$n_+(\hat{c}_2 - 1, ... , \hat{c}_K - 1)U_{11}(c, B)(\hat{c}_2 - 1, ... , \hat{c}_K - 1)^T$$

under (8) is a chi-squared distribution with degrees of freedom $K - 1$. However, $U_{11}(c, B)$ is still unknown and can be estimated by replacing $c$ with the corresponding maximum likelihood estimator. We denote the estimated $U_{11}(c, B)$ by $\hat{U}_{11}(c, B)$. Since $\hat{c}$ is a consistent estimator of $c$, $\hat{U}_{11}(c, B)$ converges in probability to $U_{11}(c, B)$ and therefore a test statistic for testing the hypothesis (8) is given by

$$n_+(\hat{c}_2 - 1, ... , \hat{c}_K - 1)\hat{U}_{11}(c, B)(\hat{c}_2 - 1, ... , \hat{c}_K - 1)^T \hspace{1cm} (9)$$
whose asymptotic distribution under (8) is a chi-squared distribution with $K - 1$ degrees of freedom (Rao, 1973, Chapter 6). The test statistic (9) can be expressed in a simple form

$$n_+\{\frac{pr_1(1 - r_1)}{2} - pr_1\sum_{k=2}^{K} \frac{r_k}{c_k} + \frac{p}{2}\sum_{k=2}^{K} \frac{r_k}{c_k^2} - (\sum_{k=2}^{K} \frac{r_k}{c_k})^2\}$$

$$= \frac{n_+p}{2}\left[\sum_{k=1}^{K} \frac{r_k}{c_k^2} - (\sum_{k=1}^{K} \frac{r_k}{c_k})^2\right],$$

where $c_1 \equiv 1$.

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