COMPRESSIBLE RELATIVISTIC MAGNETOHYDRODYNAMIC TURBULENCE IN MAGNETICALLY DOMINATED PLASMAS AND IMPLICATIONS FOR A STRONG-COUPLING REGIME

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ABSTRACT

In this Letter, we report compressible mode effects on relativistic magnetohydrodynamic (RMHD) turbulence in Poynting-dominated plasmas using three-dimensional numerical simulations. We decomposed fluctuations in the turbulence into 3 MHD modes (fast, slow, and Alfvén) following the procedure of mode decomposition in Cho & Lazarian, and analyzed their energy spectra and structure functions separately. We also analyzed the ratio of compressible mode to Alfvén mode energy with respect to its Mach number. We found the ratio of compressible mode increases not only with the Alfvén Mach number, but also with the background magnetization, which indicates a strong coupling between the fast and Alfvén modes. It also signifies the appearance of a new regime of RMHD turbulence in Poynting-dominated plasmas where the fast and Alfvén modes are strongly coupled and, unlike the non-relativistic MHD regime, cannot be treated separately. This finding will affect particle acceleration efficiency obtained by assuming Alfvénic critical-balance turbulence and can change the resulting photon spectra emitted by non-thermal electrons.

Key words: magnetic fields – magnetohydrodynamics (MHD) – plasmas – relativistic processes – turbulence

1. INTRODUCTION

Turbulence is ubiquitous in many plasma phenomena. Over the past few decades, a considerable number of studies have been conducted on the properties of non-relativistic magnetohydrodynamic (MHD) turbulence, and they have revealed many interesting and essential properties: for example, the critical-balance condition of strong turbulence (Goldreich & Sridhar 1995; Beresnyak & Lazarian 2010; Beresnyak 2014); the existence of weak turbulence (Lazarian & Vishniac 1999; Galtier et al. 2000; Meyrand et al. 2016); and the mode coupling between Alfvénic and compressible modes (Cho & Lazarian 2002, 2003). The interest in relativistic turbulence has recently been growing because of the development of high-power laser plasma devices and observation devices for high-energy astrophysical phenomena. In particular, the observation of high-energy astrophysical phenomena such as gamma-ray bursts and flares of relativistic jets, indicates that there will be strong turbulence which is necessary for acceleration of electrons emitting non-thermal photons observed from those phenomena (Hayashida et al. 2012; Asano & Hayashida 2015). Recent theoretical studies of those phenomena indicate that their background plasma is Poynting-energy dominated (Kennel & Coroniti 1984a, 1984b; Kino et al. 2015) in which the relativistic magnetization parameter, \( \sigma \equiv B_0^2/4\pi \rho hc^2 \gamma^2 \), is larger than unity where \( B_0 \) is the background magnetic field, \( \rho \) is the rest mass density, \( h \) is the specific enthalpy, \( c \) is the velocity of light, and \( \gamma \) is the Lorentz factor.³ Although there are several works investigating turbulence in relativistic MHD plasma (Inoue et al. 2011; Zrake & MacFadyen 2012, 2013; Radice & Rezzolla 2013; Takamoto et al. 2015) and force-free plasma (Thompson & Blaes 1998; Cho 2005; Cho & Lazarian 2014), many properties of the relativistic MHD turbulence and the dependence on the background \( \sigma \)-parameter are still unclear, in particular, in the case of Poynting-dominated relativistic magnetohydrodynamic (RMHD) turbulence.

In this Letter, we report our findings on RMHD turbulence, in particular, the properties of each MHD characteristic mode in low-\( \sigma \)–β plasma covering low-\( \sigma \) to high-\( \sigma \) plasma for the first time. We performed a series of numerical relativistic MHD simulations of turbulence and analyzed the spectral properties of each mode, their structure function, and eddy scale in terms of local mean field. We also discuss a possibility of a strong coupling between the fast and Alfvén modes and appearance of a new regime of RMHD turbulence in Poynting-dominated plasmas where the fast and Alfvén modes are strongly coupled and cannot be separated, different from the non-relativistic MHD case.

2. NUMERICAL SETUP

The plasma is modeled by the ideal RMHD approximation with the TM equation of state (Mignone et al. 2005), which allows us to simulate the relativistic perfect gas equation of state (Synge 1957) with less than 4% error. The equations are updated by the relativistic HLLD method (Mignone et al. 2009) in a conservative fashion through use of the constrained transport algorithm (Evans & Hawley 1988; Gardiner & Stone 2005). The initial background plasma is assumed to be uniform with a uniform magnetic field, \( B_0 \), density \( \rho_0 \), and temperature \( k_B T = 0.1mc^2 \), where \( k_B \) is the Boltzmann constant, \( T \) is the temperature, and \( m \) is the particle mass. We injected turbulence at the initial time-step, the so-called decaying turbulence. Similar to Takamoto et al. 2015, we injected the turbulence at large scales, \( l_{inj} = L/2, L/3, L/4 \), where \( L \) is the numerical box size, and the energy spectrum is assumed to be flat. Following Cho & Lazarian (2002, 2003), the turbulence is injected only with an Alfvén mode velocity component obtained through use of the method explained in the next section. We consider a cubic numerical domain that is divided by uniform meshes whose size is typically \( \Delta = L/512 \) to obtain fast to Alfvén mode power.

³ Note that the \( \sigma \)-parameter is originally defined as a ratio between Poynting-flux to particle energy flux. The form introduced in this Letter is a reduced one assuming MHD approximation \( E = \nu/c \times B \) that can be used in a static background flow. It also becomes the square of relativistic 4-Alfvén velocity in the fluid comoving frame, and is popular in high-energy astrophysics community.
Figure 1. Left: the ratio of fast to Alfvén mode velocity power in terms of the non-relativistic fast Mach number of the Alfvén mode component using three-velocity. This indicates that the compressible mode becomes more important with increasing $\sigma$-parameter. Right: the ratio of fast to Alfvén mode velocity power in terms of the background $\sigma$ parameter at $t = 2t_{\text{eddy}}$ and $\delta v_A/c_{\perp} = 0.16$ whose values are obtained by fitting of the curves of each $\sigma$-value by linear curves. Note that the error bar results from the fitting of the curves. This shows that the ratio is proportional with $\sqrt{1 + \sigma}$ when $\sigma > 1$.

Figure 2. Top: the kinetic energy spectra of Alfvén, fast, and slow modes. Bottom: the eddy scale of Alfvén, fast, and slow modes obtained by a second-order velocity structure function. All of the data are measured at one eddy turnover time. The initial Alfvén mode turbulence is injected at $k/2\pi = 3/L$ with velocity dispersion $\delta v/c_A = 0.6$ for $\sigma = 0.2$, 1, and $\delta v/c_A = 0.5$ for $\sigma = 3$.

ratio (Figure 1); we use higher resolutions, $\Delta = L/1024$, $L/2048$, to obtain the energy spectra and the structure functions provided in Figure 2 ($L/1024$ for $\sigma = 0.2$, 1 and $L/2048$ for $\sigma = 3$). The resolution is chosen as sufficient for resolving turbulent eddies responsible for the energy exchange between modes in our simulation.\footnote{“Ideal” RMHD means that no explicit dissipation processes are included, such as the viscosity, thermal conductivity, and resistivity. However, the explicit differential scheme we employ here always includes the numerical grid-scale dissipation, which allows the dissipation at the smallest eddy and the direct cascade of energy into smaller scale.}

3. MODE DECOMPOSITION OF RMHD TURBULENCE

Following Cho & Lazarian (2002, 2003), we consider the displacement vectors of slow and fast modes. Since there is no average velocity in the background flow, the displacement vectors reduce to

$$\hat{x}_{\text{slow}} \propto k_1\hat{k}_1 + \left[ \frac{k_{\text{slow}}}{c_A} \left( \frac{k}{k_1} \right)^2 - 1 \right] \left( \frac{k}{k_1} \right)^2 L_{\parallel} \hat{k}_\perp,$$  

(1)
\[ \hat{\xi}_{\text{fast}} \propto [\mu_{\text{fast}}^2 k^2 (1 + \sigma) - c_s^2 k^2 - k^2 \sigma] \left( \frac{1}{c_s k_l} \right)^2 k \hat{e}_l + k \hat{e}_l, \]

where \( c_s \) is the relativistic sound velocity, \( k \) is the absolute value of the wave vector \( \hat{k} \), and \( \mu_{\text{fast/slow}} \) are the characteristic velocities of fast and slow modes (Anile 1990). \( k, k_l \) are taken as the parallel and perpendicular to the background magnetic field direction \( (\mathbf{B}_0) \). Note that Equations (1) and (2) reduce to the Equations (1) and (2) of Cho & Lazarian (2002) if we take the non-relativistic limit and set \( c_s = 0.1 \) and \( c_A = 1 \), where \( c_A \) is the Alfvén velocity. Equations (1) and (2) allow us to obtain fast- and slow-mode velocity components by projecting the velocity field on the displacement vectors \( \hat{\xi}_{\text{fast/slow}} \). Similar to the non-relativistic case, the displacement vector of the Alfvén mode velocity component is given as \( \hat{\xi}_A = \hat{k} \times \hat{e}_l \) (Maron & Goldreich 2001). In the following, we write the velocity projection onto \( \hat{\xi}_{\{A, \text{fast}, \text{slow}\}} \) as \( \delta v_{\{A, f, s\}} \). Since the form of the linearized mass conservation law and induction equation are the same as the non-relativistic ones, the Fourier components of density and magnetic field can be obtained by exactly the same procedure in the non-relativistic case given in Cho & Lazarian (2002, 2003). We should stress that for turbulence, the mode decomposition is valid in a statistical sense as it discussed in Cho & Lazarian (2002). Moreover, the meaning of the decomposition is different for the case of weak and strong coupling of modes. When the transfer of energy between the modes is weak as it is the case of non-relativistic turbulence, the mode decomposition reveals the distinct cascades among Alfvén, fast, and slow modes (Cho & Lazarian 2002). In the case of relativistic turbulence, as the coupling increases, the decomposition reveals the transfer of energy between the cascades.

When we calculated the kinetic power, the energy spectrum, and the second-order structure functions of each mode, we projected the Fourier component of velocity onto the displacement vectors and performed inverse Fourier transformation of each decomposed component in real-space.

4. RESULTS

The left panel of Figure 1 is the ratio of fast to Alfvén mode velocity power at two eddy turnover time in terms of the fast Mach number of the Alfvén component of velocity. Each point corresponds to a simulation result with a fixed injection velocity. Note that the fast velocity \( c_{\text{fast.L}} \) in the horizontal axis is taken as that in the perpendicular direction to the background magnetic field. The curves in the panel show that the fast mode power increases nearly linearly with the fast Mach number as reported in the non-relativistic case (Cho & Lazarian 2002, 2003). In addition, it also shows that the fast component increases with \( \sigma \)-value. The right panel of Figure 1 is the ratio of fast to Alfvén mode velocity power in terms of the background \( \sigma \) value at \( t = 2 t_{\text{edd}} \) and \( \delta v_A/c_{\text{fast.L}} = 0.16 \). It shows that the ratio is nearly independent of the \( \sigma \)-parameter, when \( \sigma < 1 \) whose value is around 0.08. Note that this indicates the Alfvén to fast energy conversion is not more than 8% in the low-\( \sigma \) plasma, which is consistent with the result obtained by (Cho & Lazarian 2002). On the other hand, it increases approximately by \((1 + \sigma)^{1/2}\) when \( \sigma > 1 \), which is basically consistent with our previous result of the driven turbulence (Takamoto et al. 2015). This indicates that the ratio can be written as

\[ \frac{\langle \delta v_f \rangle^2}{\langle \delta v_A \rangle^2} \propto \frac{\langle \delta v_A \rangle}{c_{\text{fast.L}}}, \]

when \( \sigma \ll 1 \),

\[ (1 + \sigma)^{1/2} \langle \delta v_A \rangle / c_{\text{fast.L}}, \]

when \( \sigma \gg 1 \).

where we expect that the increasing function Equation (4) can saturate around unity, as will be discussed later. Note that this is a relativistic extension of the Equation (6) in Cho & Lazarian (2002). This indicates that the nonlinearity of the Alfvén mode becomes strong enough to convert its energy into compressible mode at this value of fast Mach number as the electromagnetic field becomes relativistically strong, \( \sigma > 1 \). Qualitatively, this can be explained by the electric field that should be taken into account in the relativistic MHD equations because of the relativistic velocity: \( |\mathbf{E}| = |-(v/c) \times \mathbf{B}| \sim |\mathbf{B}| \). If we use the quasi-linear theory, that is, take into account the second-order terms in equations assuming first-order Alfvén mode perturbation, the force term in the equation of motion of RMHD can be written as

\[ F_x \equiv - \nabla_x \left[ \frac{B_0^2}{2} \left( 1 + \frac{c_s^2}{c_A^2} \right) \left( \frac{\delta v_A}{c_A} \right)^2 \right], \]

\[ F_y \equiv - \nabla_y \left[ \frac{B_0^2}{2} \left( 1 - \frac{c_s^2}{c_A^2} \right) \left( \frac{\delta v_A}{c_A} \right)^2 \right], \]

\[ F_z \equiv - B_0 \nabla_z \delta B_A. \]

where we set the background magnetic field as \( B_0 = B_0 \hat{e}_z \) and the Alfvén mode direction in \( z \)-direction. The \( z \)-component drives the usual Alfvén mode. Interestingly, the anisotropic nature of the electric field gives the weaker force in the \( y \)-direction and the stronger force in \( x \)-direction, or the parallel direction of the background magnetic field. In the Poynting-dominated regime, \( c_A \sim c \), the force is only in the direction parallel to the magnetic field direction, and we consider this is the origin of increasing fast mode conversion. Note that this also indicates that the mode coupling between fast and Alfvén modes becomes stronger, which is not observed in non-relativistic MHD turbulence (Cho & Lazarian 2002). Rewriting Equation (4) as \( \alpha \delta v_A \sim \sigma \), where \( \alpha \equiv 0.4 \delta v_A / \langle \delta v_A \rangle \) whose coefficient 0.4 is obtained by the fitting data in Figure 1, the completely coupling of fast and Alfvén modes, \( \delta v_p \sim \delta v_A \), occurs when \( \sigma \sim 1/\alpha^2 - 1 \); If we assume \( \delta v_A / \langle \delta v_A \rangle \sim 0.5 \), the necessary \( \sigma \) value becomes around 24. In this regime, we can expect an appearance of a new RMHD turbulence regime where the critical balance is invalid due to the strong coupling to the fast mode. Concerning the slow mode conversion, we found that slow modes also show similar behavior to the fast mode as shown in Figure 1, but their kinetic energy is approximately 1.5 times larger than the fast mode case. We take this to be due to the fact that the considered turbulence is in the regime of sub-Alfvénic but “super-slow” \( \delta v_s > c_{\text{slow}} \) turbulence. More detailed analysis will be reported in our forthcoming papers.

Importantly, the fast to Alfvén mode power ratio depends on a non-relativistic Mach number obtained with 3-fast velocity, \( c_{\text{fast}} \) not the relativistic Mach number obtained with 4-fast velocity. This means that in Poynting-dominated plasma the compressible...
mode turbulence becomes important even if the kinetic energy of the turbulence is much smaller than the background magnetic field energy; this is very different from the non-relativistic case whose kinetic energy of turbulence should be comparable to the background magnetic field energy, \( \rho_{\text{m}} \parallel \sim B^2 \). This will be important for the electron and cosmic-ray acceleration by MHD turbulence (Fermi 1949, 1954) in high-energy astrophysical phenomena, such as GRBs and blazars (see e.g., Asano & Hayashida 2015).\(^5\)

**Alfvén Mode.** The top-left panel of Figure 2 shows the kinetic energy spectrum of Alfvén mode in terms of the wave vector perpendicular to the background magnetic field. It indicates that the spectrum follows the Kolmogorov spectrum in its inertial region,

\[ E^A(k_{\parallel}) \propto k_{\perp}^{-5/3}, \]  

(8)

which is consistent with the critical balance predicted by Goldreich & Sridhar (1995), meaning that the energy-cascade time by Alfvén mode along the magnetic field is comparable to that by eddy interaction perpendicular to the magnetic field. This also means that the turbulent eddy becomes anisotropic.\(^6\)

Note that this is also consistent with the results by Zrake & MacFadyen (2012, 2013) and Radice & Rezzolla (2013) who performed not the mode decomposition but the Helmholtz decomposition of velocity, although the incompressible mode obtained by the Helmholtz decomposition is essentially different from the Alfvén mode. The bottom-left panel of Figure 2 shows the values of \( r_1 \) and \( r_\perp \) with the same second-order structure function for the velocity. The distance \( r_1 \) and \( r_\perp \) is determined using the local magnetic field direction following Cho & Vishniac (2000). Note that it is essential to measure \( r_1 \) and \( r_\perp \) in terms of the local mean magnetic field because the critical balance proposed by Goldreich & Sridhar (1995) is applicable only in the local system of reference, which was established by Lazarian & Vishniac (1999), Cho & Vishniac (2000), Maron & Goldreich (2001), and Cho et al. (2002). It shows that the eddy size scales as \( k_1 \propto k_{\perp}^{2/3} \) as predicted by Goldreich & Sridhar (1995) and Thompson & Blaes (1998). Our simulation results indicate that the critical balance is still valid in high-\( \sigma \) regime which is consistent with works assuming the force-free approximation (Thompson & Blaes 1998; Cho 2005; Cho & Lazarian 2014).\(^7\)

**Fast Mode.** The top-middle panel of Figure 2 shows the kinetic energy spectrum of fast mode in terms of the wave vector perpendicular to the background magnetic field. It indicates that the energy spectrum of their inertial region can be written as

\[ E^f(k) \propto k^{-3/2} \quad \text{(when } \sigma < 1), \]  

(9)

\[ \propto \sigma^{-1.86} \quad \text{(when } \sigma > 1). \]  

(10)

The value of the spectrum index 1.86 was previously obtained by Zrake & MacFadyen (2012, 2013) which did not perform mode decomposition but considered the compressible component (potential component) \( u_\parallel = \nabla \phi \) where \( \phi \) is a scalar function. Note that our results indicate that the index becomes slightly larger than 1.86, and an increasing function of \( \sigma \). The bottom-middle panel of Figure 2 shows the values of \( r_1 \) and \( r_\perp \) of fast mode turbulent eddy. It indicates that the eddy is nearly isotropic, \( r_1 \sim r_\perp \) independent of its scale, similar to the non-relativistic case.

**Slow Mode.** The top-right panel of Figure 2 shows the kinetic energy spectrum of slow mode in terms of the wave vector perpendicular to the background magnetic field. The indicated energy spectrum indicates the non-power law behavior even as the resolution is increased from \( L/512 \) to \( L/2048 \). This may signify that the energy exchange between slow mode and Alfvén becomes stronger than in the case of non-relativistic turbulence in Cho & Lazarian (2002). The bottom-right panel of Figure 2 shows the values of \( r_1 \) and \( r_\perp \) of the slow mode turbulent eddy. It indicates that the slow mode eddy size also follows the critical condition, similar to the Alfvén mode. We take these results to indicate that the slow mode or the pseudo-Alfvén mode do not cascade their energy for themselves as is known to occur in the non-relativistic case (Lithwick & Goldreich 2001; Cho & Lazarian 2002).

5 The enhancement of compressibility effects was mentioned in Radice & Rezzolla (2013) in the relativistic temperature case, but in the paper we quantify this effects covering Poynting-energy dominated regime.

6 Note that it is still very difficult to judge if the spectrum index is \(-5/3 \) or \(-1.5 \) due to the insufficient numerical resolution. The effects of the bottleneck (Beresnyak & Lazarian 2010; Beresnyak 2014; Beresnyak & Lazarian 2015) might also distort the measured spectra.

7 Note that we observed only the strong turbulence regime, though we injected sub-Alfvénic turbulence that induces the weak turbulence in the large-scale regime as first pointed out in Lazarian & Vishniac (1999). We take this to indicate that the resolution in the direction perpendicular to the background magnetic field is not enough to observe the weak turbulence cascade. It may also be because the generation of compressible modes hinders the Alfvén mode energy in cascading in the perpendicular direction (pointed out by Sébastien Galier). Recent findings of the weak turbulence regime in numerical simulations are reported by Meyrand et al. (2016).
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