Klein’s “Erlanger Programm”: do traces of it exist in physical theories?

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1 Introduction

Felix Klein’s “Erlanger Programm” of 1872 aimed at characterizing geometries by the invariants of simple linear transformation groups. It was reformulated by Klein in this way: “Given a manifold and a group of transformations of the same; to develop the theory of invariants relating to that group.” (1; 2, p. 28). As if he had anticipated later discussions about his program, a slightly different formulation immediately preceding this is: “Given a manifold and a group of transformations of the same; to investigate the configurations belonging to the manifoldness with regard to such properties as are not altered by the transformations of the group.” (3) A wide

1We have noticed the difficult relationship of Sophus Lie with regard to F. Klein or W. Killing in connection with priority issues (3, pp. 365-375). Klein acknowledged S. Lie as “the godfather of my Erlanger Programm” (2, p. 201). For the historical background of Lie groups, S. Lie and F. Klein cf. 4, 5.

2Es ist eine Mannigfaltigkeit und in derselben eine Transformationsgruppe gegeben. Man soll […] die Theorie der Beziehungen, welche relativ zur Gruppe invariant sind, untersuchen.” - The translation given is by M. W. Haskell and authorized by Klein; cf. New York Math. Soc. 2, 215-249 (1892/93).

3“Es ist eine Mannigfaltigkeit und in derselben eine Transformationsgruppe gegeben; man soll die der Mannigfaltigkeit angehörigen Gebilde hinsichtlich solcher Eigenschaften untersuchen, die durch die Transformationen der Gruppe nicht geändert werden.” (6, pp. 34-35) - In a later annotation reproduced in 7, he denied as too narrow an interpretation of his formulation strictly in the sense of looking only at algebraic invariants.
interpretation of a later time by a mathematician is: “According to F. Klein’s viewpoint thus geometrical quantities like distance, angle, etc. are not the fundamental quantities of geometry, but the fundamental object of geometry is the transformation group as a symmetry group; from it, the geometrical quantities only follow” ([8], p. 39). On the other hand, by a physicist Klein’s program is incorrectly given the expression “[..] each geometry is associated with a group of transformations, and hence there are as many geometries as groups of transformations” ([9], p. 2). The two quotations show a vagueness in the interpretation of F. Klein’s “Erlanger Programm” by different readers. This may be due to the development of the concepts involved, i.e., “transformation group” and “geometry” during the past century. Klein himself had absorbed Lie’s theory of transformation groups (Lie/Engel 1888-1893) when he finally published his Erlanger Program two decades after its formulation. Originally, he had had in mind linear transformations, not the infinitesimal transformations Lie considered.

F. Klein’s point of view became acknowledged in theoretical physics at the time special relativity was geometrized by H. Minkowski. Suddenly, the Lorentz (Poincaré) group played the role Klein had intended for such a group in a new geometry, i.e., in space-time. The invariants became physical observables. But, as will be argued in the following, this already seems to have been the culmination of a successful application to physical theories of his program. What has had a lasting influence on physical theories, is the concept of symmetry as expressed by (Lie-) transformation groups and the associated algebras with all their consequences. This holds particularly with regard to conservation laws. The reason is that in physical theories fields defined on the geometry are dominant, not geometry itself. Also, for many physical theories a geometry fundamental to them either does not exist or is insignificant. A case in sight is the theory of the fractional quantum Hall effect from which quasi-particles named “anyons” emerge. The related group is the braid group describing topological transformations [11]. What often prevails are geometrical models like the real line for the temperature scale, or Hilbert space, an infinite-dimensional linear vector space, housing the states of quantum mechanical systems. In place of geometries, differential geometrical “structures” are introduced. An example would be field

[Footnote 4: Important developments following the Noether theorems have been described by Y. Kosmann-Schwarzbach in her book about invariance and conservation laws [10].]
re-parametrization for scalar fields in space-time. The fields can be interpreted as local coordinates on a smooth manifold. In the kinetic term of the Lagrangian, a metric becomes visible which shows the correct transformation law under diffeomorphisms. The direct application of F. Klein’s classification program seems possible only in a few selected physical theories. The program could be replaced by a scheme classifying the dynamics of physical systems with regard to symmetry groups (algebras).

The following discussion centers around finite-dimensional, continuous groups. Infinite-dimensional groups will be barely touched. (Cf. section 8.) Also, the important application to discrete groups in solid state and atomic physics (e.g., molecular vibration spectra) and, particularly, in crystallography are not dealt with. For the considerations to follow here, the question need not be posed whether a reformulation of Klein’s classifying idea appropriate to modern mathematics is meaningful.

2 Electrodynamics and Special Relativity

It is interesting that F. Klein admitted that he had overlooked the Galilei-group when writing up his “Erlanger Programm”: “Only the emergence of the Lorentz group has led mathematicians to a more correct appreciation of the Galilei-Newton group” ([2], p. 56). It turned out later that the Galilean “time plus space” of this group is more complicated than Minkowski’s space-time [16], [17].

What also had not been seen by F. Klein but, more than 30 years after the pronounciation of the “Erlanger Programm”, by mathematicians E. Cunningham and H. Bateman, was that the Maxwell equations in vacuum admit the 15-parameter conformal group as an invariance group [18], ([19], p. 409-436, here p. 423). However, this is a very specific case; if the electromagnetic field is coupled to matter, this group is no longer admitted, in general.

Special relativity, and with it Minkowski space, are thought to form a framework for all physical theories not involving gravitation. Hence, a branch

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5 A survey of the groups is given in [12]. For finite groups cf. also chapters 1 and 2 of [13]. For the history of the interaction of mathematics and crystallography cf. the book by E. Scholz [14].

6 Some material in this respect may be found in P. Cartier’s essay on the evolution of the concepts of space and symmetry [15].
of physics like relativistic quantum field theory in both its classical and quantized versions is included in this application of the “Erlanger Programm”. 7

In the beginning of string theory (Veneziano model), the string world sheet was likewise formulated in Minkowski space or in a Lorentz space of higher dimension.

We need not say much more concerning special relativity, but only recall Minkowski’s enthusiasm about his new find:

“For the glory of mathematicians, to the infinite astonishment of remaining humanity, it would become obvious that mathematicians, purely in their fantasy, have created a vast area to which one day perfect real existence would be granted - without this ever having been intended by these indeed ideal chaps.” (quoted from ([2], p. 77).8

3 General Relativity

The description of the gravitational field by a Lorentz-metric, in Einstein’s general relativity, was predestinated to allow application of Klein’s program. The exact solutions of Einstein’s field equations obtained at first like the Schwarzschild- and de Sitter solutions as well as the Einstein cosmos, defined geometries allowing 4- and 6-parameter Lie transformation groups as invariance groups. Most of the exact solutions could be found just because some invariance group had been assumed in the first place. Later, also algebraical properties of the metrics were taken to alleviate the solution of the non-linear differential equations. In the decades since, it has become clear, that the generic solution of Einstein’s field equations does not allow an invariance group - except for the diffeomorphisms Diff(M) of space-time M. As every physical theory can be brought into a diffeomorphism-invariant form, eventually with the help of new geometrical objects, the role of this group is

7We recall that, on the strictest mathematical level, an unambiguous union of quantum mechanics and special relativity has not yet been achieved. Note also that algebraic quantum field theory does not need full Minkowski space, but can get along with the weaker light-cone structure supplemented by the causality principle.

8“Es würde zum Ruhme der Mathematik, zum grenzenlosen Erstaunen der übrigen Menschheit offenbar werden, dass die Mathematiker rein in ihrer Phantasie ein großes Gebiet geschaffen haben, dem, ohne dass es je in der Absicht dieser so idealen Gesellen gelegen hätte, eines Tages die vollendete reale Existenz zukommen sollte.”
quite different from the one F. Klein had in mind. He was well aware of the changed situation and saved his program by reverting to infinitesimal point transformations. He expressed his regret for having neglected, at the time of the formulation of his “Erlanger Programm”, Riemann’s Habilitationsschrift of 1854 [24], and papers by Christoffel and Lipschitz. In fact, the same situation as encountered in general relativity holds already in Riemannian geometry: generically, no nontrivial Lie transformation group exists. Veblen had this in mind when he remarked:

“With the advent of Relativity we became conscious that a space need not be looked at only as a ‘locus in which’, but that it may have a structure, a field-theory of its own. This brought to attention precisely those Riemannian geometries about which the Erlanger Programm said nothing, namely those whose group is the identity. [...]” ([25], p. 181-182; quoted also by E. T. Bell [29], p. 443)

That general relativity allows only the identity as a Lie transformation group (in the sense of an isometry) to me is very much to the point. Perhaps, the situation is characterized best by H. Weyl’s distinction between geometrical automorphisms and physical automorphisms ([26], p. 17). For general relativity, this amounts to Diff(M) on the one hand, and to the unit element on the other. Notwithstanding the useful identities following from E. Noether’s second theorem, all erudite discussions about the physical meaning of Diff(M) seem to be adornments for the fact that scalars are its most general invariants possible on space-time. Usually, physical observables are transforming covariantly; they need not be invariants. While the space-time metric is both an intrinsically geometric quantity and a dynamical physical field, it is not a representation of a finite-dimensional Lie transformation group: F. Klein’s program just does not apply. If Einstein’s endeavour at a unified field theory...”

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9Since E. Kretschmann’s papers of 1915 and 1917 [20], [21], there has been an extended discussion about an eventual physical content of the diffeomorphism group in general relativity; cf. [22], [23]. It suffers from Einstein’s identification of coordinate systems and physical reference systems with the latter being represented by tetrads (frames). These can be adapted to matter variables.

10For the contributions of Lipschitz to the geometrization of analytical mechanics cf. ([25], pp. 29-31).

11The original quote from Veblen continues with “In such spaces there is essentially only one figure, namely the space structure as a whole. It became clear that in some respects the point of view of Riemann was more fundamental than that of Klein.”
built on a more general geometry had been successful, the geometrical quantities adjoined to physical fields would not have been covariants with regard to a transformation group in Klein’s understanding.

But F. Klein insisted on having strongly emphasized in his program: “that a point transformation $x_i = \phi(y_1...y_n)$ for an infinitely small part of space always has the character of a linear transformation $[..]$”\(^{12}\) (p. 108). A symmetry in general relativity is defined as an isometry through Killing’s equations for the infinitesimal generators of a Lie-algebra. Thus in fact, F. Klein’s original program is restricted to apply to the tangent space of the Riemannian (Lorentz-) manifold. This is how E. Cartan saw it: a manifold as the envelope of its tangent spaces; from this angle he developed his theory of groups as subgroups of $GL(n,R)$ with help of the concept of $G$-structure\(^{13}\). Cartan’s method for “constructing finitely and globally inhomogeneous spaces from infinitesimal homogeneous ones” is yet considered by E. Scholz as “a reconciliation of the Erlangen program(me) and Riemann’s differential geometry on an even higher level than Weyl had perceived” (p. 27)\(^{14}\).

An extension of general relativity and its dynamics to a Lorentz-space with one time and four space dimensions was achieved by the original Kaluza-Klein theory. Its dimensional reduction to space-time led to general relativity and Maxwell’s theory refurbished by a scalar field. Since then, this has been generalized in higher dimensions to a system consisting of Einstein’s and the Yang-Mills equations \(^{30}\), and also by including supersymmetry. An enlargement of general relativity allowing for supersymmetry is formed by supergravity theories. They contain a (hypothetical) graviton as bosonic particle with highest spin 2 and its fermionic partner of spin 3/2, the (hypothetical) gravitino; cf. also section \(^6\).

\(^{12}\)“dass eine Punkttransformation $[..]$ für eine unendlich kleine Partie des Raumes immer den Charakter einer linearen Transformation hat”.

\(^{13}\)H. Weyl with his concept of purely infinitesimal geometry in which a subgroup $G \subset SL(n, R)$ (generalized ”rotations”) acts on every tangent space of the manifold, separately, took a similar position (p. 24).

\(^{14}\)For a Lie group $G \subset L$, the homogeneous space corresponds to $I/\mathfrak{g} \cong T_p M$, where $I$ and $\mathfrak{g}$ are the respective Lie algebras.
4 Phase space

A case F. Klein apparently left aside, is phase space parametrized by generalized coordinates $q_i$ and generalized momenta $p_i$ of particles. This space plays a fundamental role in statistical mechanics, not through its geometry and a possibly associated transformation group, but because of the well known statistical ensembles built on its decomposition into cells of volume $\hbar^3$ for each particle, with $\hbar$ being Planck’s constant. For the exchange of indistinguishable particles with spin, an important role is played by the permutation group: only totally symmetric or totally anti-symmetric states are permitted. In 2-dimensional space, a statistics ranging continuously between Bose-Einstein and Fermi-Dirac is possible.

The transformation group to consider would be the abelian group of contact transformations (cf. [32]):

$$q_i' = f_i(q_i, p_j), \quad p_j' = g_j(q_i, p_j),$$

(1)

which however is of little importance in statistical mechanics. In some physics textbooks, no difference is made between contact and canonical transformations, cf. e.g., [34]. In others, the concept of phase space is limited to the cotangent bundle of a manifold with a canonical symplectic structure ([35], p. 341). An important subgroup of canonical transformations is given by all those transformations which keep Hamilton’s equations invariant for any Hamiltonian. Note that for the derivation of the Liouville equation neither a Hamiltonian nor canonical transformations are needed. In this situation, symplectic geometry can serve as a model space with among others, the symplectic groups $SP(n, R)$ acting on it as transformation groups. Invariance of the symplectic form $\Sigma^\omega_i dq_i \wedge dp_i$ implies the reduction of contact to canonical transformations. Symplectic space then might be viewed in the spirit of F. Klein’s program. He does not say this but, in connection with the importance of canonical transformations to “astronomy and mathematical physics”, he speaks of “quasi-geometries in a $R_{2n}$ as they were developed by Boltzmann and Poincaré [...]” ([36], p. 203).

\[15\] It is only loosely connected with Lie’s geometric contact transformation which transforms plane surface elements into each other. Manifolds in contact, i.e., with a common (tangential) surface element remain in contact after the transformation. A class of linear differential equations is left invariant; cf. [33], pp. 19-20.

\[16\] For contact transformations with higher derivatives cf. [31].
In analytical mechanics, Hamiltonian systems with conserved energy are studied and thus time-translation invariance is assumed. Unfortunately, in many systems, e.g., those named “dynamical systems”, energy conservation does not hold. For them, attractors can be interpreted as geometrical models for the “local asymptotic behavior” of such a system while bifurcation forms a “geometric model for the controlled change of one system into another” ([37], p. XI). Attractors can display symmetries, e.g., discrete planar symmetries [38], etc.

In statistical thermodynamics, there exist phase transitions between thermodynamic phases of materials accompanied by “symmetry breaking”. As an example, take the (2nd order) transition from the paramagnetic phase of a particle-lattice, where parallel and anti-parallel spins compensate each other to the ferromagnetic phase with parallel spins. In the paramagnetic state, the full rotation group is a continuous symmetry. In the ferromagnetic state below the Curie-temperature, due to the fixed orientation of the magnetization, the rotational symmetry should be hidden: only axial symmetry around the direction of magnetization should show up. However, in the Heisenberg model (spin 1/2) the dynamics of the system is rotationally invariant also below the Curie point. The state of lowest energy (ground state) is degenerate. The symmetry does not annihilate the ground state. By picking a definite direction, the system spontaneously breaks the symmetry with regard to the full rotation group. When a continuous symmetry is spontaneously broken, massless particles appear called Goldstone(-Nambu) bosons. They are corresponding to the remaining symmetry. Thus, while the dynamics of a system placed into a fixed external geometry can be invariant under a transformation group, in the lowest energy state the symmetry may be reduced. This situation seems far away from F. Klein’s ideas about the classification of geometries by groups.

5 Gauge theories

Hermann Weyl’s positive thoughts about Klein’s program were expressed in a language colored by the political events in Germany at the time:

“The dictatorial regime of the projective idea in geometry was first broken by the German astronomer and geometer Möbius, but the classical document of the democratic platform in geometry, establishing the group of transformations as the ruling principle
in any kind of geometry, and yielding equal rights of independent
consideration to each and every such group, is F. Klein’s ‘Erlanger
Programm’. ” (quoted from Birkhoff & Bennet [39])

Whether he remembered this program when doing a very important step
for physics is not known: H. Weyl opened the road to gauge theory. He
associated the electromagnetic 4-potential with a connection, at first un-
successfully by coupling the gravitational and electrodynamic fields (local
scale invariance). A decade later then, by coupling the electromagnetic field
to matter via Dirac’s wave function; for the latter he expressly invented 2-
spinors. The corresponding gauge groups were $R$ and $U(1)$, respectively.
This development and the further path to Yang-Mills theory for non-abelian
gauge groups has been discussed in detail by L. O’Raifeartaigh and N. Strau-
mann [40], [41]. Weyl had been convinced about an intimate connection
of his gauge theory and general relativity: “Since gauge invariance involves
an arbitrary function $\lambda$ it has the character of ‘general’ relativity and can
naturally only be understood in that context” ([42], translation taken from
[41]). But he had not yet taken note of manifolds with a special mathe-
matical structure introduced since 1929, i.e., fibre bundles. Fibre bundles are
local products of a base manifold (e.g., space-time), and a group. The action
of the group creates a fibre (manifold) in each point of the base. Parallel
transport in base space corresponds to a connection defined in a section of
the bundle. In physics, the transformation group may be a group of “exter-
nal” symmetries like the Poincaré group or of “internal” symmetries like a
Yang-Mills (gauge) group. A well known example is the frame bundle of a
vector bundle with structure group $\text{GL}(n;R)$. It contains all ordered frames
of the vector space (tangent space) affixed to each point of the base man-
ifold. Globally, base and fibres may be twisted like the Möbius band is in
comparison with a cylindrical strip [18]. In 1929, Weyl had not been able to
see the gauge potential as a connection in a principal fibre bundle. Until this
was recognized two to three decades had to pass.

Comparing the geometry of principal fibre bundles with Riemannian
(Lorentzian) geometry, F. Klein’s program would be realized in the sense

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17 The original Yang-Mills gauge theory corresponded to $\text{SU}(2)$-isospin symmetry of the
strong interaction.

18 Since the introduction of fibre spaces by H. Seifert in 1932, at least five definitions of
fibre bundles were advanced by different researchers and research groups [43]. The first
textbook was written by Steenrod [44].
that a group has been built right into the definition of the bundle. On the other hand, the program is limited because the group can be any group. In order to distinguish bundles, different groups have to be selected in order to built, e.g., $SU(2)$-, $SU(3)$-bundles, etc.\footnote{Elementary particles are classified with regard to local gauge transformations $SU(3)_c \times SU(2)_L \times U(1)_Y$. The index $c$ refers to color-charge, $Y$ to weak hypercharge, and $L$ to weak isospin. For a review of the application of gauge theory to the standard model cf. \cite{45}.} This is a classification of bundle geometry in a similar sense as isometries distinguish different Lorentz-geometries. To classify different types of bundles is another story.

Moreover, in gauge theories, the relation between observables and gauge invariants is not as strong as one might have wished it to be. E.g., in gauge field theory for non-abelian gauge groups, the gauge-field strength (internal curvature) does not commute with the generators of the group: it is not an immediate observable. Only gauge-invariant polynomials in the fields or, in the quantized theory, gauge-invariant operators are observables. In contrast, the energy-momentum tensor is gauge-invariant also for non-abelian gauge groups.

In terms of the symmetry\footnote{$SU(2) \times U(1)$ symmetry of electroweak interactions; approximate flavour $SU(3)$-symmetry of strong interactions.} gauge invariance is spontaneously broken, both in the case of electroweak and strong interactions.

General relativity with its metric structure is not a typical gauge theory: any external transformation group would not only act in the fibre but also in the tangent space of space-time as well. Thus, an additional structure is required: a soldering form gluing the tangent spaces to the fibres \cite{46}. Many gauge theories for the gravitational field were constructed depending on the group chosen: translation-, Lorentz-, Poincaré, conformal group etc.\footnote{A recent reader about gauge theories of gravitation is \cite{47}.} We will come back to a Poincaré gauge theory falling outside of this Lie-group approach in section \ref{sec:7}.

\section{Supersymmetry}

Another area in physics which could be investigated as a possible application of Klein’s program is supersymmetry-transformations and supermanifolds.
Supersymmetry expressed by super-Lie-groups is a symmetry relating the Hilbert spaces of particles (objects) obeying Bose- or Fermi-statistics (with integer or half-integer spin-values, respectively). In quantum mechanics, anti-commuting supersymmetry operators exist mapping the two Hilbert spaces into each other. They commute with the Hamiltonian. If the vacuum state (state of minimal energy) is annihilated by the supersymmetry operators, the 1-particle states form a representation of supersymmetry and the total Hilbert space contains bosons and fermions of equal mass. As this is in contradiction with what has been found, empirically, supersymmetry must be broken (spontaneously) in nature.

For an exact supersymmetry, the corresponding geometry would be supermanifolds, defined as manifolds over superpoints, i.e., points with both commuting coordinates as in a manifold with n space dimensions, and anti-commuting “coordinates” forming a Grassmann-algebra ζ, ¯ζ (“even” and “odd” elements). As a generalization of Minkowski space, the coset space Poincaré/Lorentz in which the super-Poincaré group acts, is called superspace ([50], Chapter 6), [51], p. 107). Superspace is a space with 8 “coordinates” \( z^A = (x^k, \theta^\mu, \bar{\theta}^\dot{\mu}) \), where \( x^k \) are the usual real space-time coordinates plus 4 real (anti-commuting) “fermionic” coordinates from a Weyl-spinor \( \theta^\mu \) and its conjugate \( \bar{\theta}^\dot{\mu} \).

A super-Lie group \( G \) is a Lie group with two further properties: 1) it is a supermanifold the points of which are the group elements of \( G \); 2) the multiplicative map \( F: G \to G \times G \) is differentiable ([49], p. 123). All classical Lie groups have extensions to super-Lie groups. Most important for quantum field theory is the super-Poincaré group and its various associated super-Lie-algebras. The super-Poincaré algebras contain both Lie-brackets and anti-commuting (Poisson) brackets. A superparticle (supermultiplet) corresponds to a reducible representation of the Poincaré algebra.

The geometry of supermanifolds seems to play only a minor role in physics. An example for its use would be what has been called the gauging of super-
Local super-Lie algebras are important because their representations constitute superfields by which the dynamics of globally or locally supersymmetric physical theories like *supergravity* are built. Supergravity containing no particle of spin larger than 2 can be formulated in Lorentz-spaces up to maximal dimension 11. In space-time, at least 7 supergravities can be formulated. Yet, a geometrical construct like a supermetric is of no physical importance.

This all too brief description is intended to convey the idea that, in physics, the role of supersymmetry primarily is not that of a transformation group in a supermanifold but of a group restricting the dynamics of interacting fields. By calling for invariants with regard to supersymmetry, the choice of the dynamics (interaction terms in the Lagrangian) is narrowed considerably. The supersymmetric diffeomorphism group can be used to formulate supersymmetric theories in terms of differential forms on superspace: “superforms” ([52], Chapter XII). Possibly, B. Julia envisioned the many occurring supersymmetry groups when drawing his illustration for supergravities “A theoretical cathedral” and attaching to the x-axis the maxim: GEOMETRY $\simeq$ GROUP THEORY ([54], p. 357). When the view is narrowed to F. Klein’s “Erlanger Programm” as is done here, then the conclusion still is that the program cannot fare better in supergravity than in general relativity.

## 7 Enlarged Lie algebras

We now come back to space-time and to a generalization of (Lie)-transformation groups acting on it. As insinuated before, for the classification of structures in physical theories the attention should lie rather on the algebras associated with the groups; geometrical considerations intimately related to groups are of little concern. Lie algebras have been generalized in a number of ways. One new concept is “soft”, “open” or “nonlinear” Lie algebras, in which the structure constants are replaced by structure functions depending on the generators themselves. They can also be interpreted as infinite-dimensional Lie algebras ([55], pp. 60-61). An example from physics are local supersymmetry

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25Superfields can be defined as functions on superspace developed into power series in the nilpotent Grassmann-variables in superspace; the power series break off after the term $\theta\bar{\theta}\theta\bar{a}(x)$. Local supersymmetric theories are theories invariant under *supergauge*-transformations.
transformations (defined to include diffeomorphisms, local Lorentz and local supersymmetry transformations) which form an algebra with structure functions. They depend on the symmetry generators themselves ([53], p. 140). Another generalization is “local Lie algebras” which arise as the Lie algebras of certain infinite-dimensional Lie groups. The structure of the Lie algebra in given by:

\[ [f_1, f_2] = \sum_{i,j,k} c_{ij}^k x^k \partial_i f_1 \partial_j f_2 \]

where \( f_1, f_2 \) are smooth functions on a smooth manifold, \( \partial_k \) the partial derivatives with respect to local coordinates on \( M \), and \( c_{ij}^k \) the structure constants of an n-dimensional Lie algebra (cf. [56], section 7). This seems to be a rather special kind of algebra.

Recently, a further enlargement has been suggested called “extended Lie algebras” and in which the structure constants are replaced by functions of the space-time coordinates. In the associated groups, the former Lie group parameters are substituted by arbitrary functions [57]. The Lie algebra elements form an “involutive distribution”, a smooth distribution \( V \) on a smooth manifold \( M \). The Lie brackets constitute the composition law; the injection \( V \hookrightarrow TM \) functions as the anchor map. Thus, this is a simple example for a tangent Lie algebroid. In addition to the examples from physics given in [57], the Poincaré gauge theory of F.-W. Hehl et al. seems to correspond to the definition of an extended Lie algebra. In this theory, the difference with the Lie algebra of the Poincaré group is that the structure functions now contain the frame-metric and the gauge fields, i.e., curvature as rotational and torsion as translational gauge field, all dependent on the space-time coordinates [59].

8 Conclusions

In the course of ranging among physical theories with an eye on F. Klein’s “Erlanger Programm”, we noticed that the focus had to be redirected from groups and geometry to algebras and the dynamics of fields. In particular, with regard to infinite-dimensional groups, the discussion within physical theories of Klein’s program would have been easier had it been formulated in terms of algebras. Then, also Virasoro- and Kac-Moody algebras, appearing among others in conformal (quantum) field theory and in string
theory could have been included in the discussion.\textsuperscript{26} Hopf-algebras occurring in non-commutative geometry could have formed another example. With the mentioned change in focus included, the application of Klein’s program to physical theories is far more specific than a loosely defined methodological doctrine like the “geometrization of physics” (cf. \textsuperscript{60}). While both, general relativity and gauge theory, can be considered as geometrized, they only partially answer F. Klein’s “Erlanger Programm”. In physical theories, the momentousness of Lie’s theory of transformation groups easily surpasses Klein’s classification scheme.

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\textsuperscript{26}The Virasoro group is an infinite dimensional group related to conformal (quantum) field theory in 2 dimensions. It is defined as $\text{Diff}(S^1)$ where $S^1$ is the unit circle, and its geometry the infinite dimensional complex manifold $\text{Diff}(S^1)/S^1$. In string theory, the Virasoro-algebra appears. It is a central extension of a Witt-algebra providing unitary representations. The Witt-algebra is the Lie-algebra of smooth vector fields on $S^1$. 

14
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