On fractionally linear functions over a finite field

V.M.Sidelnikov\(^1\), R.N.Mohan\(^2\) and Moon Ho Lee\(^3\)

University of Moscow, Russia\(^1\),
Sir CRR Institute of Mathematics\(^2\), Eluru-534007, AP, India
Institute of Information & Communication\(^3\), Chonbuk National University, Korea

Emails: siddd123@hotmail.com\(^1\), mohan420914@yahoo.com\(^2\), moonho@chonbuk.ac.kr\(^3\)

Abstract: In this note, by considering fractionally linear functions over a finite field and consequently developing an abstract sequence, we study some of its properties.

1. Introduction

The fractionally linear function, refer to [6, 7], means it is equal to the ratio of two linear functions like \( f(z) = \frac{az + b}{cz + d} \), where \( z \) is a variable possessing values in some finite field \( F \) and elements \( a, b, c, d \) are constants from the field \( F \) such that \( ad \neq bc \), i.e. \([a\ b]
\[c\ d]\neq0\). Then the function \( f(z) \) is called Mobius transformation or linear fractional function or transformation. For some details of the field under consideration refer to Sidelnikov [8].

In this paper we consider the group of all fractional linear transformations of the complex plane with coefficients (some constants) either in \( \mathbb{F}_q \) or in \( \mathbb{C} \). For detailed study of groups refer to Rotman [5]. This is essentially the same, up to a factor of 1, as \( SL(F) \), which is the group of 2-by-2 matrices with entries in \( \mathbb{F}_q \) and determinant 1. This \( SL(Z) \) is called as the modular group.

A special type of automorphic function, where the group involved is the modular group is being considered. Usually this term is applied in a more general way. We can define \( \varphi(x) \) as a function where \( \nu \) is an integer. Then the condition may be applied for certain subgroups of finite index in the modular group. We denote the group of fractionally linear functions by \( FL(F_q) \), where \( F = F_q \), the finite field with \( q \) elements and the order of \( FL(F_q) \) is equal to \( q(q^2 - 1) \). Note that \( SL(F_q) \) and \( FL(F_q) \) are different groups, which are at the same time isomorphic groups denoted by \( SL(F_q) \cong FL(F_q) \).

Now by considering such modular groups with automorphism, we develop a sequence and define correlation functions on that, and conclude by suggesting an application for the systems that are being developed in this paper.

2. Main results

Let \( \psi \in FL(F_q) \) be the fractionally linear function over the finite field \( F_q \). Suppose that \( \psi \) has no fixed point in \( F_q \) and the order of \( \psi \) as the element \( FL(F_q) \) is equal to \( q+1 \). In this case the set \( \{\alpha_0, \alpha_1, \ldots, \alpha_q\} = F_q \cup \{\infty\} \) can be represented in the following form:

\[ F_q \cup \{\infty\} = \{\psi^j(1)\mid j = 0, \ldots, q\} \]
Let $\phi$ be some function of group $FL\left(F_q\right)$. Then we can define the sequence $a_\phi$ as follows:

$$a_\phi = (\phi(\psi^0(1)), \phi(\psi^1(1)), \ldots, \phi(\psi^q(1))). \quad (1)$$

For further details of these sequences refer to [2, 6]. Then the cyclic shift $a_\phi^{(s)}$ on $s$ co-ordinates of sequence $a_\phi$ can be written in the form

$$a_\phi^{(s)} = (\phi(\psi^s(1)), \phi(\psi^{s+1}(1)), \ldots, \phi(\psi^{s+q}(1))).$$

Let $\chi(x)$ be a character of the multiplicative group of the finite field $F_q$ with $q$ elements. The finite fields have been studied by Sidelnikov [7]. Let us assume that $\chi(0) = \chi(\infty) = 1$. Then consider the above mentioned sequence with this character, which has the following form

$$a_\chi = (\chi(\phi(\psi^0(1))), \chi(\phi(\psi^1(1))), \ldots, \chi(\phi(\psi^q(1)))),$$

with entries from $\mathbb{F}_4$.

The sequence $a_\chi^{(s)}$ is the cyclic shift to the left of the sequence $a_\chi$ on $s$ coordinates and it can be represented in the following form

$$a_\chi^{(s)} = (\chi(\phi(\psi^s(1))), \chi(\phi(\psi^{s+1}(1))), \ldots, \chi(\phi(\psi^{s+q}(1)))).$$

The periodic auto correlation function of sequence $a_\chi$ can be defined as

$$T_s(a_\chi) = \langle a_\chi, a_\chi^{(s)} \rangle, \text{ where } s = 0, \ldots, q,$$

and $\langle \cdot, \cdot \rangle$ denotes the inner product in the unitary space. For further details regarding sequences refer to [3].

Now consider the sequence $a_\chi = (\chi(\phi(\psi^0(1))), \chi(\phi(\psi^1(1))), \ldots, \chi(\phi(\psi^q(1))))$,

then the cross correlation function of the sequences $a_\phi$ and $a_\phi'$, can be defined as

$$T_s(a_\phi, a_\phi') = \langle a_\phi, a_\phi^{(s)} \rangle, \text{ where } s = 0, \ldots, q.$$

The maximum non-trivial correlation $T_{\text{max}}$ of the sequence $a_\chi$ is defined as

$$T_{\text{max}}(\phi) = \max_{0 \leq s \leq q} \left| T_s(a_\chi) \right|.$$

The maximum non-trivial cross correlation of sequences $a_\chi$ and $a_\chi'$ is
\[ T_{\text{max}}(a_{\varphi}, a_{\varphi'}) = \max_{0 \leq s \leq q} |T_s(a_{\varphi}, a_{\varphi'})|, \]

In order to facilitate synchronization as well as to minimize interference due to the co-existence of other sequences, the maximal non-trivial auto-and cross-correlation values of this sequence family should be as small as possible. Welch bound, Sidelnikov bound and Sarwate bound, (for further details refer to Levenshtain \[4\]), provide lower bounds on the minimum possible value of the parameter \( T_{\text{max}} \) and commonly used to judge the merits of a particular sequence design. Now we will discuss them.

By using the Sarwate’s result, we have

\[ \frac{|a_{\varphi}|^2}{N} = \frac{N-1}{N(M-1)} \frac{|a_{\varphi}|^2}{N} \geq 1, \]

where \( a_{\varphi} = (a_{\varphi,1}, \ldots, a_{\varphi,r}) \) and \(|a_{\varphi}| = \left( \sum_{i=0}^{q} |a_{\varphi,i}|^2 \right)^{1/2} \) is Euclidean norm or energy of the sequence and \( N = q + 1 \).

This Welch bound applies to complex-valued sequences in general, while Sidelnikov bound, applies only to those sequences whose symbols are complex \( q \)\( \text{th} \) roots of unity. In most of the cases, this Sidelnikov bound is tighter than that of Welch bound, restricting the symbols to the complex roots of unity. When \( M = N^u \), and \( N \equiv u \) and \( u \geq 1 \) is an integer, the Sidelnikov bounds can be well approximated by:

\[ M_A(n,d) \leq \frac{2n^3 - 2n(n-2d)^2}{3n - (n-2d)^2 - 2}, \quad \ldots \quad (2) \]

where \( M_A(n,d) \) is the number of elements of antipodal binary code.

Consider the remarkable binary Kerdok code \( K \) that has following parameters: The length \( n = 2^m \), where \( m \) is even, the code distance is equal to \( d = \frac{n-\sqrt{n}}{2} \) and the number of elements of \( K \) is equal to \( n^2 \). This Kerdok code meets the bound (2).

Let \( A = \{a_{\varphi_1}, \ldots, a_{\varphi_M}\} \) be the set of sequences, which determines by sequences \( \varphi_1, \ldots, \varphi_M \) by means of identity (1). The value of cross correlation of sequences from the set \( A \) is defined by

\[ T(A) = \max_{1 \leq s \leq j \leq M} \max_{s \neq j, \varphi_i \neq \varphi_j} |T_s(a_{\varphi_i}, a_{\varphi_j})| \]
Then in the considered case the Welch bound has the following form

$$T_{\text{max}}(A) \geq N \sqrt{\frac{M-1}{NM-1}}$$

Then we have the following

$$T_{\text{max}}^2 \geq \begin{cases} N \left(2u + 1 - \frac{1}{1.3.5...(2u-1)}\right) & \text{for binary sequences} \\ N \left(u + 1 - \frac{1}{1.2.3..u}\right) & \text{for non binary sequences of the present type} \end{cases}$$

These approximations may improve the performance of the sequences and when $M \geq N$ then consequently $u \geq 1$, (this $\geq$ indicates approximation). Then we can say that

$$T_{\text{max}} \geq \begin{cases} \sqrt{2N} & \text{for binary sequences} \\ \sqrt{N} & \text{for non binary sequences of the present type} \end{cases}$$

Let $a$ be sequence of the length $N$ with entries in $F_q$, i.e. $a \in F_q^N$. The linear span $L(a)$, of $a$ is the length of the shortest linear recursion over the field $F_q$ under consideration, which is generated sequence $a$. The linear span of a sequence is one measure of its predictability. Any “good” pseudo-random sequence should have large linear span compared with its length. If a sequence has little linear span then this sequence has predictability on its initial section.

If a sequence has a linear span $L = \ell$, then its linear recursion can be determined from any $2 \ell$ successive elements of this sequence by means of Berlekamp algorithm, for details refer to [1,2], which provides an elegant way to factor polynomials over a small finite field of order $q$. The remaining elements of the sequence can be produced from the recursion.

For an application point of view, refer to [6], these systems are being extensively used for control systems, besides by using Fuzzy logic and Fuzzy control systems.

**Acknowledgements:** This work was supported by the MIC (Ministry of Information and Communication), under the ITFSIP (IT Foreign Specialist Inviting Program) supervised by IITA, under ITRC supervised by IITA, and International Cooperation Research Program of the Ministry of Science & Technology, Chonbuk National University, Korea. So we convey our thanks to all the concerned.

**References**

[1] E.R. Berlekamp (1972) Factoring polynomials over large finite fields, Mathematics of computation, 24, PP. 713-735.
[2] E. Berlekamp, *Algebraic Coding Theory*, McGraw-Hill, New York, 1968.

[3] Pingzhi Fan and Michael Darnell (1996). *Sequence design for communications applications*, Research Studies Press LTD, John Wiley & Sons Inc, New York.

[4] V.I. Levenshtain (1998) Universal bounds for code designs., Handbook of coding theory, Elsevier Science, PP 439-448.

[5] Joseph J.Rotman (1995). *An Introduction to the Theory of Groups, Fourth Edition*, Springer-Verlag

[6] Robert Lea, Vladik Kreinovich, and Raul Trejo (1996). Optimal interval enclosures for fractionally linear functions, and their application to intelligent control, Reliable Computing, Vol. 2, No. 3, pp. 265-286.

[7] V.M. Sidelnikov, Moon Ho Lee, and R.N.Mohan (2006) Group on Linear Fractional Transformations. (submitted)

[8] V.M. Sidelnikov, (1988). On normal bases of a finite field, Matem.Co´opHNNK, Tom 133(175), Math.USSR Sbornik, Vol.61 No.2 PP 485-494.