Panel data segmentation under finite time horizon

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Abstract

We study the nonparametric change point estimation for common changes of means in panel data. The consistency of estimates is investigated when the number of panels tends to infinity but the sample size remains finite. Our focus is on weighted denoising and on the weighted CUSUM (cumulative sums) estimates. We propose extensions of existing weighting schemes and prove that they outperform the classical ones under the criterion of perfect estimation. This will be also demonstrated empirically in a simulation study. Additionally, we show that the results can be partly generalized to the estimation of spatial functional data.

Keywords
Panel data, Change point estimation, Segmentation, Nonparametric, CUSUM, Total variation denoising, LASSO, Serial dependence

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†Research partially supported by the Friedrich Ebert Foundation, Germany.
1 Introduction

Aim of this paper is to study the estimation of changes in the context of panel data. We focus on common changes, i.e., changes that occur simultaneously in many panels (but not necessarily in all) at the same time points and we consider an asymptotic framework where the number \( d \) of panels tends to infinity but the panel sample size \( n \) is fixed.

The analysis of change point estimation in panel data is subject of intensive research (in particular in econometrics) and, as discussed in Bai (2010), dates at least back to the works of Joseph and Wolfson (1992, 1993). However, the setting \( d \to \infty \), which we are looking at, is generally not studied much in the literature concerning change point analysis and the settings \( n \to \infty \) or \( n,d \to \infty \) are far more established. For the classical setting of \( n \to \infty \) we refer to Csörgő and Horváth (1997). In the context of panel data especially the setting \( n,d \to \infty \) is quite popular (cf., e.g., Bai (2010), Horváth and Huskova (2012) and Kim (2014)). Nevertheless, the assumption \( d \to \infty \) and \( n \) fixed is also quite natural in the context of panel data (cf., e.g., Bai (2010), Bleakley and Vert (2011a) and also Hadri et al. (2012)). It reflects the situation where the amount of panels, i.e., the dimensionality, is much larger than the sample size.

Bai (2010) and Bleakley and Vert (2011a) mention important applications in finance, biology and medicine where in particular the framework of common changes is appropriate: In finance such changes may occur simultaneously across many stocks e.g. due to a credit crisis or due to tax policy changes. In biology and medicine relevant applications are in the study of genomic profiles within classes of patients. As mentioned in Bleakley and Vert (2011a) the latter example fits particularly well in the \( n \) fixed and \( d \to \infty \) framework because the length of panels in genomic studies is fixed but the amount of panels can be increased by raising the number of patients.

The body of literature related to change point estimation in panel data is huge. Hence, we do not attempt to summarize it here and refer the reader instead to the reviews in Frick et al. (2012), Jandhyala et al. (2013), Aue and Horváth (2013) and Horváth and Rice (2014). Change point analysis in the \( d \to \infty \) and \( n \) fixed setting goes at least back to Bleakley and Vert (2010, 2011a) and to the aforementioned paper by Bai (2010). Therein estimation of common changes is studied independently from different perspectives. However, as will turn out by our analysis, the setups of Bleakley and Vert (2011a) and of Bai (2010) are closely related.

Bai (2010) considered a least squares estimate for independent panels of linear time series under a single change point assumption and Bleakley and Vert (2010, 2011a) developed a weighted total variation denoising approach for the multiple change point scenario. Furthermore, Bleakley and Vert (2010, 2011a) proposed a computationally efficient algorithm and implemented it in a convenient MATLAB package GFLseg\(^1\) which we also used for our simulations. For theoretical analysis they assumed independent panels of also independent Gaussian observations. In this article we follow the approach of Bleakley and Vert (2011a) and try to pick up and generalize their ideas. We shall also emphasize a (maybe surprising) connection between the total variation denoising approach of Bleakley and Vert (2011a), the least squares estimate of Bai (2010) and the well-studied weighted CUSUM (cumulative sums) estimates.

Our viewpoint will shed some new light on the weighting for the denoising estimate of Bleakley and Vert (2011a) and also for weighted CUSUM estimates in general. In particular we propose a modified weighting scheme. As will be demonstrated, this scheme improves estimation theoretically and practically for panels of dependent observations.

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\(^1\)Download is available at [http://cbio.ensmp.fr/GFLseg](http://cbio.ensmp.fr/GFLseg) and is licensed under the GNU General Public License.
with large dimension $d$ and small sample size $n$. Practically this approach outperforms the classical schemes even in random change point settings and even under rather moderate dimensions.

1.1 Basic setup

We observe $d$ panels $\{Y_{i,k}\}_{i=1,...,n}$ for $k = 1, \ldots, d$ in a signal plus noise model where

$$Y_{i,k} = m_{i,k} + \varepsilon_{i,k}. \quad (1.1)$$

Here, $\{m_{i,k}\}_{i,k} \in \mathbb{N}$ is an array of deterministic signals and $\{\varepsilon_{i,k}\}_{i,k} \in \mathbb{N}$ is a random centered noise sequence. We assume a (multiple) common change points scenario given by

$$m_{i,k} = \begin{cases} 
\mu_{1,k}, & i = 1, \ldots, u_1, \\
\mu_{2,k}, & i = u_1 + 1, \ldots, u_2, \\
\vdots & \\
\mu_{P+1,k}, & i = u_P + 1, \ldots, n, 
\end{cases} \quad (1.2)$$

where $u \in \{u_1, \ldots, u_P\} \subset \mathbb{N}$, $P \in \mathbb{N}$ are the change points. The $\mu_{j,k} \in \mathbb{R}$, $j = 1, \ldots, P+1$ describe the piecewise constant signals in each panel, i.e. equivalently the means of observations $Y_{i,k}$. In other words the means jump simultaneously from levels $\mu_{u,k}$ to levels $\mu_{u+1,k}$ in all panels $k = 1, \ldots, d$ at change points $u \in \{u_1, \ldots, u_P\}$. We do not require $\mu_{u,k} \neq \mu_{u+1,k}$ to hold for all $k = 1, \ldots, d$, i.e. the changes do not have to occur in all panels. Later on we will impose more specific conditions on the average magnitude of changes. We assume that $n \geq 3$ since otherwise the model (1.2) is not reasonable. For $n = 1$ the model may not contain any change and for $n = 2$ it holds trivially $u = 1$.

Following Bleakley and Vert (2011a) we will restrict our theoretical considerations to the single change point scenario, i.e. to $P = 1$. However, as will be shown, our findings do have practical implications on the multiple change point scenario as well and this is why we state the general model in (1.2).

1.2 Notation

We follow the compact matrix notation of Bleakley and Vert (2011a) and represent the model (1.2) as

$$Y = M + E,$$

with a deterministic matrix $M$ of means with $M_{i,k} = m_{i,k}$ and a random matrix of errors $E$ with $E_{i,k} = \varepsilon_{i,k}$. Now, let $X$ be any $n \times d$ matrix. To shorten the notation we write $X_{\bullet,j}$ for $[X_{1,j}, \ldots, X_{n,j}]^T$ and $X_{i,\bullet}$ for $[X_{i,1}, \ldots, X_{i,d}]$. For example $Y_{\bullet,k}$ represents the $k$-th panel and a common change at $u$ corresponds to

$$\Delta = M_{u+1,\bullet} - M_{u,\bullet} \neq 0. \quad (1.3)$$

$\|\cdot\|_F$ denotes the Frobenius norm and $\|\cdot\|_2$ stands for the Euclidean norm. We simply write $\|\cdot\|$ for the former when no confusion is possible, unless it is stated otherwise. The terms segmentation, estimation and identification will be used interchangeably.

We will consider functions $f(i)$ with a discrete support $i = 1, \ldots, n-1$. We say that a function $f$ is convex (or concave) if this holds true for the linear interpolation of points $f(i)$ on the interval $[1, n-1]$. 

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The paper is organized as follows. In Section 2 we discuss the denoising segmentation procedure proposed by Bleakley and Vert (2011a). To introduce the concept and notation we will repeat and summarize briefly the key facts of Bleakley and Vert (2011a) in Subsections 2.1 and 2.2. In Subsection 2.2 we will also analyze the estimation procedure with respect to different weighting schemes and propose a generalization of the existing approaches. In Subsection 2.3 we discuss estimates of the generalized weighting scheme. Finally, in Section 3 we demonstrate and confirm our theoretical results in a simulation study. All proofs are postponed to Section 4.

2 Segmentation of panel data

We start with a description of the denoising approach to change point estimation of Bleakley and Vert (2011a) and thereby we stay close to their exposition. Also, for an overview of the literature related to this approach we refer the reader to the references in Bleakley and Vert (2011a).

2.1 Total Variation Denoising

The total variation denoising approach to segmentation, i.e. to change point estimation, is to solve the convex minimization problem

$$\min_{U \in \mathbb{R}^{n \times d}} \frac{1}{2} \|Y - U\|_F^2 + \lambda \times \text{totvar}(U)$$

(2.1)

for an appropriate regularization parameter $\lambda \geq 0$ under a weighted total variation penalty term

$$\text{totvar}(U) = \sum_{i=1}^{n-1} \frac{\|U_{i+1, \cdot} - U_{i, \cdot}\|_2}{w(i,n)}$$

(2.2)

with positive position dependent weights $w(i,n) > 0$. We denote the solution of (2.1) by $\hat{U}(\lambda)$ and each column $\hat{U}_{\cdot, k}$ represents the best piecewise constant fit to the panel $Y_{\cdot, k}$ w.r.t. (2.1). A change in those fits, in the sense of $\hat{U}_{u+1, \cdot} \neq \hat{U}_{u, \cdot}$, is therefore assumed to identify a common change across panels at the time point $u$. Hence, the set $E$ of estimated change points is given by

$$E(\lambda) = \left\{ u \mid \hat{U}_{u, \cdot}(\lambda) \neq \hat{U}_{u+1, \cdot}(\lambda) \right\}.$$ 

(2.3)

The objective function in (2.1) is strictly convex, as a sum of convex functions and due to the strict convexity of the mapping $U \mapsto \|Y - U\|_F^2$. Therefore, a unique solution exists for any $\lambda \geq 0$. The penalty term $\text{totvar}(U)$ is designed in such a way that $\hat{U}(\lambda)$ has for $\lambda > 0$ a tendency to reduce the cardinality of (2.3), i.e. to reduce the amount of identified change points. Moreover, $E$ has a tendency to become smaller as $\lambda$ increases. Two extreme cases give some insight: For $\lambda \uparrow \infty$ the penalty term $\text{totvar}(U)$ dominates the minimization and forces the minimizer $\hat{U}$ to be constant across rows, i.e. we obtain $E = \emptyset$ and no change points are identified by this procedure at all. In contrast to this, if $\lambda = 0$ then $\hat{U}(0) = Y$ and therefore $E(0) = \{1, \ldots, n-1\}$, i.e. all points are now identified as change points.

2This is the terminology of Bleakley and Vert (2011a).
2.1.1 Selection of $\lambda$ in the single change point scenario

In the single change point setup we aim to select $\lambda$ as large as possible such that the set $\mathcal{E}(\lambda)$ contains only one change point $[3]$ in which case $\mathcal{E} = \{\hat{u}\}$ and $\hat{u}$ denotes the estimate of $u$. Heuristically, this forces $\hat{u}$ to be the most reasonable selection of exactly one common change point according to the penalty term $\text{totvar}(U)$. The number of estimated change points does not necessarily decrease monotonously in $\lambda$ for $d > 1$ (cf. Figure 1 and Section 4 of Bleakley and Vert (2011a)). Further, it seems generally not clear whether any parameter $\lambda$ that identifies only one change point yields the same estimate. Hence, selection of $\lambda$ must be made more precise. As will be discussed in Proposition 2.3 below, one can identify under mild assumptions a random interval such that any $\lambda \in (\lambda_{\min},\lambda_{\max})$, $0 \leq \lambda_{\min} < \lambda_{\max}$ yields the same estimate $\hat{u}$ and such that any $\lambda \in [\lambda_{\max}, \infty)$ yields $\mathcal{E} = \emptyset$, i.e. no estimate. Thus, we will select any $\lambda \in (\lambda_{\min},\lambda_{\max})$ and the corresponding estimate $\hat{u}$ is then unambiguous.

Remark 2.1. In the case of multiple changes the selection of some reasonable $\lambda$ becomes more challenging and further meaningful statistical criteria are necessary. This becomes even more complicated if the number of changes is unknown in advance (which is a more realistic scenario). Reasonable criteria are discussed in Bleakley and Vert (2010, 2011a) (cf. also the references therein).

2.2 Theoretical analysis of the procedure

For theoretical investigations and also for practical purposes Bleakley and Vert (2010, 2011a) pick up the idea of Harchaoui and Lévy-Leduc (2008) and reformulate the minimization Problem (2.1) as a group fused LASSO (least absolute shrinkage and selection operator).

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3For the single change point scenario Bleakley and Vert (2011a) in their software GFLseg) perform a dichotomic search to find the “first” $\lambda$ such that $\mathcal{E}$ contains only one element.
2.2.1 Denoising restated as group fused LASSO

Using the weights $w(i, n)$, Bleakley and Vert (2011a) introduce a fixed design matrix $D \in \mathbb{R}^{n \times (n-1)}$ with

$$D_{i,j} = \begin{cases} w(j, n), & i > j, \\ 0, & \text{else} \end{cases}$$

and set:

$$\beta_i, \cdot = \frac{U_{i+1, \cdot} - U_{i, \cdot}}{w(i, n)}.$$  (2.4)

This can be compactly written as $U = 1U_{1, \cdot} + D\beta$ where $1 = [1, \ldots, 1]^T$. Thereby, the original problem (2.1) transforms to

$$\arg \min_{\beta \in \mathbb{R}^{(n-1) \times d}} \frac{1}{2} \|\bar{Y} - \bar{D}\beta\|_F^2 + \lambda \sum_{i=1}^{n-1} \|\beta_i, \cdot\|_2$$  (2.5)

with $\lambda \geq 0$, where $\bar{Y}$ and $\bar{D}$ are the columnwise centered matrices $Y$ and $D$, respectively. Let $\hat{\beta}(\lambda)$ denote a solution of (2.5). A solution $\hat{U}$ of (2.1) can be recovered via $\hat{U} = 1\hat{\gamma} + D\hat{\beta}$ with $\hat{\gamma} = 1^T(Y - D\hat{\beta})/n$. The indices of non-zero rows of matrix $\hat{\beta}$ correspond to the change point set $\mathcal{E}$ via (2.4) and it holds that $\mathcal{E}(\lambda) = \{u | \beta_u, \cdot \neq 0\}$.

2.2.2 KKT conditions

The crucial observation for a theoretical analysis is that $\beta$ minimizes (2.5), for any fixed $\lambda$, if it fulfills the necessary and sufficient Karush-Kuhn-Tucker (KKT) conditions:

$$G_i = \lambda B_i \quad \forall \beta_i, \cdot \neq 0,$$

$$\|G_i\| \leq \lambda \quad \forall \beta_i, \cdot = 0,$$  (2.6)

for all $i = 1, \ldots, n-1$ with vectors $G_i = \bar{D}_{i, \cdot}^T (\bar{Y} - \bar{D}\beta)$ and $B_i = \beta_i, \cdot / \|\beta_i, \cdot\|$. 

2.2.3 The single change point scenario

The subsequent theoretical analysis will be developed following Bleakley and Vert (2011a) under the next assumption.

Assumption 2.2. We consider a single change point scenario with a change at some time point $u \in \{1, \ldots, n-1\}$ where $n \geq 3$.

We start by formalizing the selection of the regularization parameter $\lambda$ which was already informally described in the previous Subsection 2.1.1.

Proposition 2.3. Consider the random matrix $\hat{c} = \bar{D}^T\bar{Y}$ and set $t_i = \|\hat{c}_{i, \cdot}\|$. Under Assumption 2.2 it holds that:

1. If $\lambda \geq t_{n,n}$ then $\hat{\beta} = 0$ solves the KKT system (2.6).

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Notice that $D$ does not have full rank. Hence, the convex minimization problem (2.5) is not strictly convex. Thus, in contrast to (2.1), the solutions of (2.5) are not necessarily unique. Here, $\arg \min$ denotes any argument at which the minimum is attained.
2. If \( t_{n-1} < \lambda < t_{n:n} \) then a random \( \lambda_{\min} \) exist such that for any \( \lambda_{\min} < \lambda < t_{n:n} \) a \( \hat{\beta} \) with rows

\[
\hat{\beta}_{i,\bullet} = \begin{cases} 
\hat{c}_{M,\bullet}, & i = M, \\
0, & i \neq M,
\end{cases}
\]

(2.7)

with \( t_M = t_{n:n} \) and \( \alpha = (t_M - \lambda)/(\hat{D}_M^T \hat{D}_M t_M) \) solves the KKT system (2.6). In this case \( \mathcal{E}(\lambda) = \{M\} \), i.e. \( \hat{u} = \arg \max_{i=1:...,n-1} \{t_i\} \).

It is not our goal to study the selection of \( \lambda \) in the situation of \( t_{n-1:n} = t_{n:n} \). Later on, we will discuss conditions that prevent this case with probability tending to 1 as \( d \to \infty \). However, to have a formally well defined estimate we simply set \( \hat{u} = \arg \max_{i=1:...,n-1} \{t_i\} \) in this situation. Here and subsequently, we mean by \( \arg \max \) the smallest argument such that the maximum is attained and the set of points at which the maximum is attained will be denoted by \( \arg \max^\circ \) to avoid confusion.

In the next corollary we observe, based on Proposition 2.3, a maybe surprising relation of the denosing approach to a well-known class of weighted CUSUM estimates. This allows us to study denoising and CUSUM estimates simultaneously and draw conclusions for situations where both methods differ.

**Corollary 2.4.** Under Assumption 2.2 it holds that

\[
\hat{u} = \arg \max_{i=1:...,n-1} t_i, \quad t_i = w(i,n) \sqrt{\sum_{k=1}^{d} \sum_{j=1}^{i} (Y_{j,k} - \bar{Y}_{n,k})^2}.
\]

(2.8)

This connection holds true only in case of a single change point and otherwise denoising and CUSUM estimates differ. Therefore, recall that the denosing segmentation approach yields \( P \) distinct change point estimates in case of \( P \) change points whereas the CUSUM in (2.8) always yields only one location\( \footnote{Nevertheless, one could use CUSUM estimate (2.8) via binary segmentation.} \).

Generally, in the field of change point analysis, testing with weighted CUSUM statistics

\[
\max_{i=1:...,n-1} w(i,n) \sqrt{\sum_{k=1}^{d} \sum_{j=1}^{i} (Y_{j,k} - \bar{Y}_{n,k})^2}
\]

and estimating changes via (2.8) received a considerable attention in the literature\( \footnote{When dealing with CUSUM estimates (under \( n \to \infty \) asymptotics) the observations \{\( Y_{j,\bullet} \)\} are usually additionally rescaled by the long run covariance matrix.} \). In particular the weighting scheme

\[
w_{\text{weighted}}(i,n) = ((i/n)(1-i/n))^{-\gamma},
\]

(2.9)

for \( 0 \leq \gamma \leq 1/2 \) is quite popular. A smaller \( \gamma \) is usually expected to increase the sensitivity of testing or estimation procedures towards change points in the middle of time series. Asymptotic properties of corresponding tests and estimates are well studied for \( n \to \infty \) (cf., e.g., Csörgő and Horváth (1997)). Complementary, to the existing literature, we want to study asymptotics for \( d \to \infty \) in (2.8) with weights according to (2.9).
Bleakley and Vert (2010, 2011a) already studied the weightings \( w_{\text{simple}}(i, n) \equiv 1 \), i.e. \( \gamma = 0 \), and
\[
w_{\text{standard}}(i, n) = ((i/n)(1 - i/n))^{-1/2},
\]
i.e. \( w_{\text{weighted}} \) with \( \gamma = 1/2 \). Notice that both schemes \( w_{\text{simple}} \) and \( w_{\text{standard}} \) can be considered as natural for the estimate (2.8). On the one hand, the former, \( w_{\text{simple}} \), appears to be the first choice from the point of view of the denoising approach (2.2). In fact, Bleakley and Vert (2010) started with this case first and studied \( w_{\text{standard}} \) later in Bleakley and Vert (2011a). Therein they showed that \( w_{\text{standard}} \) has better estimation properties for \( d \to \infty \) than \( w_{\text{simple}} \) does. On the other hand, the latter weighting, \( w_{\text{standard}} \), appears to be the natural choice from the CUSUM point of view, because it can be derived via a maximum-likelihood or a least-squares approach. The scheme \( w_{\text{standard}} \) has been studied by Bai (2010) w.r.t. \( d \to \infty \).

In the following we generalize the results of Bleakley and Vert (2011a) in two directions:

1. We introduce the notion of perfect estimation which formalizes that a relevant class of changes is estimated consistently as the number of panels \( d \to \infty \). Such perfect estimation is fulfilled for \( w_{\text{standard}} \) as shown in Bleakley and Vert (2011a) under independence and Gaussianity. We ask whether perfect estimation might hold true in more general settings. Therefore we consider weights \( w_{\text{exact}} \), as generalizations of \( w_{\text{standard}} \), and show that perfect estimation holds true under much weaker distributional assumptions than in Bleakley and Vert (2011a). In particular we allow for dependencies within and across panels. We show that \( w_{\text{exact}} \) are the only weights which ensure such perfect estimation.

2. Under independence we characterize changes which are correctly estimated as \( d \to \infty \) for \( w_{\text{weighted}} \) and any \( 0 < \gamma < 1/2 \). The cases \( \gamma = 0 \) and \( \gamma = 1/2 \) correspond to \( w_{\text{simple}} \) and \( w_{\text{standard}} \) and are, as already mentioned, treated in Bleakley and Vert (2011a, Theorems 1,2). The latter is also studied in Bai (2010).

For our analysis we pick up and generalize the approach of Bleakley and Vert (2011a) under the following assumption, which imposes some homogeneous structure on the noise \( \{\varepsilon_{i,k}\} \) within and across panels.

**Assumption 2.5.** For all \( i = 1, \ldots, n \) and \( k \geq 1 \) we assume that

1. \( E(\varepsilon_{i,k}) = 0 \) and that the variances fulfill \( E(\varepsilon_{i,k})^2 = \sigma^2 \) for some \( 0 < \sigma^2 < \infty \), i.e. they are independent of \( i \) and \( k \).
2. The function \( V^2(i) = \text{Var}(S_{i,k}(\varepsilon))/\sigma^2 \) is independent of \( k \), where
\[
S_{i,k}(\varepsilon) = n^{-1/2} \sum_{j=1}^{i} (\varepsilon_{j,k} - \bar{\varepsilon}_{n,k})
\]
are the cumulated centered noises.

Sufficient conditions for part 2 of Assumption 2.5 are e.g. identical distribution of panels \( \{\varepsilon_{i,k}\}_{k \in \mathbb{N}} \) or simply uncorrelatedness within each panel in which case it holds that
\[
V^2(i) = (i/n)(1 - i/n).
\]

In the latter uncorrelated case the panels might have all different distributions.

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Both is straightforward to check.
The following deterministic critical functions are the cornerstone of our subsequent analysis:

\[ C_n(i; u, r) = w^2(i, n) \begin{cases} V^2(i) & 0 \leq r < \infty, \\ V^2(i) r & r = \infty, \end{cases} \quad (2.12) \]

with

\[ H(i, u) = \begin{cases} (i/n)(1 - u/n), & i = 1, \ldots, u, \\ (u/n)(1 - i/n), & i = u, \ldots, n - 1, \end{cases} \]

for \( u = 1, \ldots, n - 1 \). As before, \( u = u_1 \) is the change point and \( w(i, n) \) are the weights. The parameter

\[ r = \frac{1}{n} \frac{\sigma^2}{\Delta_\infty^2} \quad (2.13) \]

is the normalized noise to change ratio where

\[ \Delta_\infty^2 = \lim_{d \to \infty} \frac{\sum_{k=1}^d |\Delta_k|^2}{d} \]

is the average change across all panels at the change point \( u \). (Recall from (1.2) and (1.3) that \( \Delta_k = \mu_{2,k} - \mu_{1,k} \).) We stick to the convention \( r = \infty \) for \( \Delta_\infty = 0 \) and to \( r = 0 \) for \( \Delta_\infty = \infty \). For simplicity we will write \( C(i; u, n, r) = C_n(i; u, r) \).

Following the approach of Bleakley and Vert (2011a) we show that \( C(i; u, n, r) \) is in probability the limit of rescaled \( t_i \) for all \( i = 1, \ldots, n - 1 \) (cf. Proposition 2.3). The next theorem generalizes Bleakley and Vert (2011a, Lemma 1). Under Gaussianity and independence within panels the results coincide, up to a normalizing constant. Here, we do not require any distributional assumption within panels beyond Assumption 2.5.

**Theorem 2.6.** Let Assumptions 2.2 and 2.5 be fulfilled and \( \Delta_\infty > 0 \). Assume that it holds that, as \( d \to \infty \),

\[ \frac{1}{d^2} \text{Var} \left( \sum_{k=1}^d g_i(\varepsilon_{i,k}) \right) = o(1) \quad (2.14) \]

for \( i = 1, \ldots, n - 1 \), where

\[ g_i(\varepsilon_{i,k}) = (S_{i,k}(\varepsilon) - 2\Delta_k n^{1/2} H(i, u)) S_{i,k}(\varepsilon). \]

Then, it holds that, as \( d \to \infty \),

\[ P \left( \hat{u} \in \arg \max_{i=1,\ldots,n-1} C(i; u, n, r) \right) \to 1. \quad (2.15) \]

Notice that each \( g_i(\varepsilon_{i,k}) \) depends only on the noise in the \( k \)-th panel. Given strict stationarity, condition (2.14) constitutes a short range dependence condition itself. Obviously, (2.14) holds true if fourth moments of all \( \varepsilon_{i,k} \) are finite and uniformly bounded and if panels \{\( \varepsilon_{i,k} \)\} are i.i.d. or even \( m \)-dependent. One can formulate conditions for (2.14) in terms of some common dependence concepts but these conditions become too technical. Thus, to preserve readability we will not develop this point here.

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8The distribution of the \( \varepsilon_{i,k} \)'s, within each panel, might be entirely different.
Theorem 2.6 shows that the stochastic limit of $\hat{u}$ is entirely described by the deterministic critical function (2.12). Thus, the question of consistent estimation, $\hat{u} \overset{P}{\to} u$, as $d \to \infty$, can be reduced to an analytical problem. Assuming a single change point at $u$, consistent estimation holds true asymptotically e.g. if $C(i; u, n, r)$ has a unique maximum at $i = u$. Hence, we will speak of perfect estimation if

$$u = \arg \max_{i=1, \ldots, n-1} C(i; u, n, r)$$  \hspace{1cm} (2.16)

is valid for all possible change points $u = 1, \ldots, n-1$ ($n \geq 3$) and all possible ratios $0 \leq r < \infty$, i.e. according to the definition in (2.13) all possible variances $\sigma^2 \geq 0$ and sizes of changes $\Delta \geq 0$.

**Remark 2.7.** Assume no change, i.e. $\Delta = 0$. Following the proof of Theorem 2.6 it is clear that (2.15) also holds true. Furthermore, assume uncorrelatedness within panels, i.e. that (2.11) holds true. For $w_{\text{weighted}}$ this yields

$$\arg \max_{i=1, \ldots, n-1} C(i; u, n, r) = \begin{cases} \lfloor n/2 \rfloor, & \gamma \in [0, 1/2), \\ \{1, \ldots, n-1\}, & \gamma = 1/2, \end{cases}$$

i.e. estimation of spurious changes.

We proceed studying the existence of weights which ensure perfect estimation. The next theorem states necessary conditions in terms of weights $w(i, n)$ which are tacitly assumed not to depend on $u$.

**Theorem 2.8.** Let Assumptions 2.2 and 2.5 be fulfilled and assume further that $V(i) > 0$ for all $i = 1, \ldots, n-1$ and that $r \in \mathbb{R}$. A necessary condition for perfect estimation is that

$$\frac{w(u, n)}{w(i, n)} \geq \frac{V(i)}{V(u)}$$  \hspace{1cm} (2.17)

holds true for all $i, u = 1, \ldots, n-1$.

The first implication is that weights $w_{\text{exact}}(i, n) := \frac{\alpha}{V(i)}$, $\alpha > 0$, are one possible solution to the system of inequalities (2.17). Since the estimate $\hat{u}$ in (2.8) is independent of any scaling $\alpha > 0$ we may restrict ourselves to the case $\alpha = 1$ and consider $w_{\text{exact}}$ to be unique.

### 2.2.4 Theoretical properties of the exact weighting scheme

Our next theorem shows under rather mild assumptions that the weights (2.18) are the only candidates to provide perfect estimation.

**Theorem 2.9.** Under the Assumptions of Theorem 2.8 and given that $V(i) = V(n-i)$ for all $i = 1, \ldots, n-1$ the weights (2.18) are the only solution to (2.17).

Notice that $V(i) = V(n-i)$ for all $i = 1, \ldots, n-1$ is implied by

$$\text{Cov}(\epsilon_1, \ldots, \epsilon_n) = \text{Cov}(\epsilon_{n,1}, \ldots, \epsilon_{1,1}) = \text{Cov}(\epsilon_{n,1}, \ldots, \epsilon_{1,1})^T.$$  \hspace{1cm} (2.19)

A sufficient condition for (2.19) is weak stationarity of $\{\epsilon_i\}$. Moreover, we know that under uncorrelatedness within panels it holds that $w_{\text{exact}} = w_{\text{standard}}$ according to (2.11). Further, we also know that these weights yield perfect estimation due to...
Bleakley and Vert (2011a, Theorem 3). Observe that, if the noise is dependent in time, generally we have $w_{\text{exact}} \neq w_{\text{standard}}$. Hence, due to Theorem 2.9 weights $w_{\text{standard}}$ cannot ensure perfect detection in such cases (cf. (2.23) below for a specific counterexample). In the next theorem we state that indeed weights $w_{\text{exact}}$ ensure perfect estimation under additional (but again) quite mild assumptions.

**Theorem 2.10.** Let Assumptions 2.2 and 2.5 be fulfilled. Assume further that $V(i) > 0$ and $V(i) = V(n-i)$ for all $i = 1, \ldots, n-1$. For perfect estimation it is necessary and sufficient that

$$
\frac{V(i)}{V(u)} > \frac{i}{u} \quad (2.20)
$$

holds true for all $i, u = 1, \ldots, n-1$ with $i \neq u$ and that we use the weights $w_{\text{exact}}$ from (2.18). The former condition (2.20) is equivalent to $V(i)/i$ being strictly decreasing as a discrete function of $i = 1, \ldots, n-1$.

The next lemma provides, based on concavity, a condition for (2.20) which is sometimes easier to verify. It is not clear how to state a comparable condition under convexity.

**Lemma 2.11.** Assume that $V(i)$ is strictly concave on $i = 1, \ldots, n-1$. Then (2.20) holds true if

$$
V(1) > V(u)/u \quad (2.21)
$$

holds true for all $u = 2, \ldots, n-1$.

We show the applicability of Theorem 2.10 and Lemma 2.11 by using MA(1) panels as a toy example.

**Corollary 2.12.** Let Assumption 2.3 hold true. Further, we assume that all panels $\{\epsilon_{i,k}\}_{i=1}^{n}, k = 1, \ldots, d$ are identically distributed MA(1) time series with innovations given by

$$
\epsilon_{i,k} = (\eta_{i,k} + \phi \eta_{i-1,k}) + \theta (\eta_{i,k-1} + \phi \eta_{i-1,k-1}) \quad (2.22)
$$

for $i = 1, \ldots, n$ and $k = 1, \ldots, d$. We assume some common parameter $\phi \in \mathbb{R}$ and centered i.i.d. innovations $\{\eta_{i,k}\}_{k \in \mathbb{N}}$ with finite fourth moments and $E(\eta_{i,k}^2) = \tilde{\sigma}^2$, $0 < \tilde{\sigma}^2 < \infty$. In this case Assumption 2.5 is fulfilled with

$$
V^2(i)\sigma^2 = \tilde{\sigma}^2(1 + \theta^2) \left( \frac{\alpha(\phi)}{w_{\text{standard}}(i,n)^2} - 2\phi/n \right) \quad (2.23)
$$

with

$$
\alpha(\phi) = 1 + \phi^2 + 2\phi + 2\phi/n,
$$

where $\sigma^2 = \tilde{\sigma}^2(1 + \phi^2 + \theta^2 + \phi^2\theta^2)$ and (2.20) holds true. Hence, $w_{\text{exact}}$ fulfills all assumptions of Theorem 2.10 and therefore ensures perfect estimation for any parameters $\phi, \theta \in \mathbb{R}$.

Notice, that $V(i)$ is strictly concave in the i.i.d. case, i.e. for $\phi = 0$, but according to (2.23) the function $V(i)$ is even strictly convex for $\alpha(\phi) < 0$. In such cases the weightings $w_{\text{standard}}$ and $w_{\text{exact}}$ differ fundamentally.

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9 As a discrete function linearly interpolated on the interval $[1, n-1]$. 

11
2225 Theoretical properties of the weighted weighting scheme

We turn to the analysis of \( w_{\text{weighted}} \) and our aim is to identify noise to change ratios \( r \) for which \( (2.16) \) does or does not hold true. We expect more restrictive conditions in \( (2.16) \) on ratios \( r \) for changes \( u \) closer to the edges of the samples, and vice versa less restrictive conditions if \( u \) lies more centered. This is confirmed by the next theorem where the minimum is taken over smaller sets in \( (2.24) \) if \( u \) is closer to \( n/2 \).

In this Subsection we will assume \( \text{arg} \max \) in \( (2.16) \) to be defined as the smallest argument of a maximum.

**Theorem 2.13.** Let Assumptions 2.2 and part 1 of Assumption 2.5 hold true. Assume that \( \{\tilde{e}_{i,k}\}_{i\in\mathbb{N}} \) are uncorrelated in the time domain for any \( k = 1, \ldots, d \) and that \( 0 < \Delta_{\infty} < \infty \). Further, define \( R(u, i) \) as in \( (4.5) \), set

\[
u^* = \begin{cases} n - u, & u = 1, \ldots, \lfloor n/2 \rfloor, \\ u, & u = \lfloor n/2 \rfloor, \ldots, n - 1 \end{cases}
\]

and assume that \( w = w_{\text{weighted}} \) with \( \gamma \in [0, 1/2) \). Relation \( (2.16) \) holds true if

\[
0 \leq r < \mathcal{R} := \min_{n/2 \leq i < u^*} R(u^*, i), \tag{2.24}
\]

where \( \min \emptyset = \infty \). Relation \( (2.16) \) does not hold true for \( r > \mathcal{R} \). Furthermore, it always holds that \( \mathcal{R} = \infty \) for \( n = 3 \) for any \( u \) and also that \( \mathcal{R} = \infty \) for \( u = n/2 \) for any \( n \). Moreover, it always holds that \( \mathcal{R} > 0 \), i.e. that \( (2.16) \) always holds true for sufficiently small ratios \( r \).

The bound in \( (2.24) \) can be evaluated numerically but to gain more insight into the influence and the interaction of parameters in \( (2.24) \), it is desirable to get explicit representations and approximations to this expression. We will provide such representations and approximations where we first let \( d \to \infty \) and then consider \( n \to \infty \). Therefore, we have to introduce a boundary function

\[
B(\gamma) = \begin{cases} \frac{4\gamma^2 + 6\gamma^{3/2} - 3\gamma^{1/2} - 1}{2^{1/2}-2}, & \gamma \in [0, 1/2), \\ 1/2, & \gamma = 1/2. \end{cases} \tag{2.25}
\]

This function \( B(\gamma) \) is monotonously decreasing, continuous and \( B(\gamma) \geq 2^{-1/2} \) holds true which can be seen as follows. It holds that \( \frac{\partial B(\gamma)}{\partial \gamma} = -(1 + 2\gamma)^{-2} \) which in turn implies that \( \frac{\partial B(\gamma)}{\partial \gamma} < 0 \) holds true for any \( \gamma \in (0, 1/2) \). Applying l'Hôpital's rule to \( B(\gamma)^2 \) twice we get the continuity at \( \gamma = 1/2 \).

**Theorem 2.14.** Let the assumptions of Theorem 2.13 hold true, set \( B(\gamma) \) as in \( (2.25) \), set \( s = u^*/n \) and let \( \mathcal{R} = \mathcal{R}(s, \gamma) \) be as in \( (2.24) \).

1. If \( 1/2 < s \leq B(\gamma) \), then \( \mathcal{R}(s, \gamma) \) equals the unique solution to

\[
C(u^* - 1, u^*, n, r) = C(u^*, u^*, n, r). \tag{2.26}
\]

and it holds that,

\[
\lim_{u^* \to \infty} \mathcal{R}(s, \gamma) = s(a - s)(b - s)(s - c)^{-1} \tag{2.27}
\]

with \( a = 1, b = \frac{\gamma - 1}{2\gamma + 1}, c = 1/2 \) and \( \gamma \in [0, 1/2) \).

2. If \( B(\gamma) < s < 1 \), then the unique solution to \( (2.26) \) is larger than \( \mathcal{R}(s, \gamma) \).
In (2.27) the quantity \( R(s, \gamma) \), i.e. the range for \( r \) that ensures (2.16), becomes larger for parameters \( \gamma \) and \( s \) close to \( 1/2 \), which confirms our intuition. Notice that Theorem 2.14 is consistent with Bleakley and Vert (2011a, Theorem 2) since, as \( n \to \infty \), it holds
\[
R(s,0) = s(1-s)^2(s-1/2)^{-1} + o(1)
\]
and \( B(0) = 1 \).

The weighting with \( \gamma = 1/4 \) can be seen as a compromise between \( w_{\text{standard}} \) and \( w_{\text{simple}} \). For this particular \( \gamma \) we are able to compute \( \lim_{n \to \infty} R(s,1/4) \) for any \( s \in (0, 1) \). It would be interesting to know if such a formula could be derived for the remaining parameters \( \gamma \in (0, 1/4) \cup (1/4, 1/2) \) as well.

**Proposition 2.15.** Let the assumptions of Theorem 2.14 hold true, set \( \gamma = 1/4 \) and \( B(\gamma) \) as in (2.25), i.e. \( B(1/4) = 3/4 \), and set \( s = u^*/n \). Then it holds for \( R = R(s,1/4) \) that
\[
\lim_{n \to \infty} R(s,1/4) = \begin{cases} 
  s(1-s)(3/2-s)(s-1/2)^{-1}, & s \in (0,3/4], \\
  s(s^2+(s-1)/2)^{1/2}, & s \in [3/4,1). 
\end{cases} \tag{2.28}
\]
This function is continuously differentiable for \( s \in (1/2, 1) \). The second derivative does however not exist in \( s = 3/4 \).

### 2.3 Computation and estimation of exact weights

In this section we discuss estimates of \( w_{\text{exact}}(i,n) \) or equivalently of \( V(i) \). We stick to the single change point scenario.

Notice, that the partial sums (2.10) can be rewritten as
\[
S_{i,k}(\varepsilon) = \sum_{j=1}^{n} a_{i,j} \varepsilon_{j,k} \tag{2.29}
\]
with
\[
a_{i,j} = \frac{1}{n^{1/2}} \begin{cases} 
  1 - i/n, & j = 1, \ldots, i, \\
  -i/n, & j = i + 1, \ldots, n. 
\end{cases} \tag{2.30}
\]
Let Assumption 2.5 hold true and assume that all panels are independent. It is straightforward to check that
\[
V(i) = f_i(\Sigma)/\sigma, \tag{2.31}
\]
where \( \Sigma = \text{Cov}(Y_{1,1}) \), with functions \( f_i = (a_{i,\bullet} \Sigma a_{i,\bullet})^{1/2} \) for \( i = 1, \ldots, n-1 \). A natural estimate for \( \Sigma_{j,k} \) is given via \( \hat{\Sigma} \in \mathbb{R}^{n \times n} \) with
\[
\hat{\Sigma}_{j,k} = \frac{1}{d-1} \sum_{p=1}^{d} (Y_{j,p} - \bar{Y}_{j,d})(Y_{k,p} - \bar{Y}_{k,d}), \tag{2.32}
\]
where \( 1 \leq j, k \leq n \). Hence, \( V(i) \) simply may be estimated by \( \hat{V}(i) = f_i(\hat{\Sigma})/\sigma \) and the corresponding estimate for the weights will be denoted by \( \hat{w}_{\text{exact}} \). Here, \( \bar{Y}_{j,d} \) is the mean over all panels at timepoint \( j \). Recall also, that the change point estimate \( \hat{u} \) does not depend on the scaling of the weights \( w_{\text{exact}}(i,n) \). Thus, without loss
of generality we may assume here that $\sigma^2 = 1$ and technically we do not have to estimate this parameter in (2.31).

This estimate $\hat{\Sigma}_{j,k}$ is consistent as $d \to \infty$, e.g. if we assume the panels $\{\varepsilon_{i,k}\}_{k \in \mathbb{N}}$ to be independent and additionally if

$$M_{i,\bullet} = \begin{cases} c_1 1, & i = 1, \ldots, u, \\ c_2 1, & i = u + 1, \ldots, n, \end{cases} \quad (2.33)$$

holds true for some $c_1, c_2 \in \mathbb{R}$. The latter condition states that at all time points $i$ the means $m_{i,k}$ across all panels, are the same.

**Remark 2.16.** An important point to note here is that, in order to compute $\hat{w}_{\text{exact}}^{\text{est}} = 1/V(i)$, we need to ensure positiveness of $a_{i,\bullet}^\top a_{i,\bullet}$. This is ensured asymptotically with probability 1, as $d \to \infty$, because the estimate $\hat{\Sigma}$ is consistent and because of our assumption $V(i) > 0$. (Clearly, a sufficient condition for finite $d$ would be the positive definiteness of the estimate $\hat{\Sigma}$.)

### 2.3.1 Estimation based on a training period

The assumptions stated above are restrictive and may be questionable in applications. In the following we discuss (informally) estimates for more general situations.

We assume stationarity to hold in time, i.e. within each panel. Additionally, we assume to have a training period between $n_1$ and $n_2$, $1 \leq n_1 < n_2 \leq n$ where the above assumptions of $M_{i,\bullet} = c_1 1$ and $M_{j,\bullet} = c_2 1$ hold true for $i,j = n_1, \ldots, n_2$. Within this training period we can compute consistent estimates $\hat{\Sigma}_{j,k} \approx \text{Cov}(\varepsilon_{j,1}, \varepsilon_{k,1})$, according to (2.32), for $j,k = n_1, \ldots, n_2$ and then may average these estimates as follows

$$\xi_r = \frac{1}{n_2 - n_1} \sum_{j=n_1}^{n_2-r} \hat{\Sigma}_{j,j+r} \left( \approx \text{Cov}(\varepsilon_{1,1}, \varepsilon_{r,1}) \right) \quad (2.34)$$

for $r = 0, \ldots, n_2 - n_1$ to gain more stability. (These estimates are consistent as well.) Finally, the desired estimate $\hat{\Sigma}$ is obtained via

$$\hat{\Sigma}_{j,j+r} = \hat{\Sigma}_{j+r,j} = \begin{cases} \xi_r, & r = 0, \ldots, \min\{h, n-j\}, \\ 0, & r = \min\{h, n-j\} + 1, \ldots, n \end{cases} \quad (2.35)$$

for $j = 1, \ldots, n$ and with some bandwidth $h \in \{1, \ldots, n_2 - n_1\}$ to be chosen. Clearly, the corresponding estimate $V(i) = f_i(\hat{\Sigma})/\sigma$ is consistent for MA($q$) panels as $d \to \infty$ if $q \leq h \leq n_2 - n_1$ holds true. Heuristically, it yields also a reasonable approximation for $V(i) = f_i(\Sigma)/\sigma$ in case of stationary and weak dependent panels whenever the covariances $\text{Cov}(\varepsilon_{1,1}, \varepsilon_{r,1})$ for $r > h$ are negligible. The corresponding estimate for the weights will be denoted by $\hat{w}_{\text{exact}}^{\text{est}}$.

We may proceed as follows to deal with more complex situations when $M_{i,\bullet} = c_1 1$ and $M_{j,\bullet} = c_2 1$ do not hold true within a reasonable subsample. Again, we need to assume a training period between $n_1$ and $n_2$, $1 \leq n_1 < n_2 \leq n$, such that either $n_2 \leq u$ or $n_1 > u$ holds true, i.e. that the common change does not occur in this subsample. Further, we need to assume stationarity and the weak law of large numbers to hold in time, i.e. within each panel.

Now, in the first step, we center each panel based on means computed within the training period and for each panel separately, i.e.

$$Y_{i,k}^r := Y_{i,k} - \frac{1}{n_2 - n_1 + 1} \sum_{j=n_1}^{n_2} Y_{j,k}$$
for $i = 1, \ldots, n$ and $k = 1, \ldots, d$. In the next step (as before) we compute only estimates $\hat{\Sigma}_{i,k}$ for $n_1 \leq j \leq k \leq n_2$ but now based on the centered panels $\{Y_j\}$. Proceeding as under (2.34) and (2.35) we obtain $\hat{V}(i) = f_i(\hat{\Sigma})/\sigma$ and the corresponding weights $w_{\text{exact-cent}}$. These are heuristically reasonable for large $n$ and $d$ which is backed up by our simulations. However, to formalize this one would need to consider asymptotics $n, d \to \infty$ which is not in the scope of this paper.

### 2.4 Mixed panels of spatial functional data

In this section we want to emphasize the generality our results.\(^\text{10}\) We consider a natural extension of model (1.1) and (1.2) where for each panel the observations $Y_{i,k}$ (and the errors $\varepsilon_{i,k}$) are random elements in some separable Hilbert space $(H_k, \langle \cdot, \cdot \rangle_{H_k})$, $k \geq 1$ and where the $m_{i,k}$ are non-random functions in the same space. Recall, that the mean of any $Y_{i,k}$ is the unique element $m_{i,k} \in H_k$, such that $E(x, Y_{i,k})_{H_k}$ holds true for all $x \in H_k$ provided that $E\|Y_{i,k}\|_{H_k} < \infty$. (The norm $\|\cdot\|_{H_k}$ is induced by $\langle \cdot, \cdot \rangle_{H_k}$.)

For example, observations $Y_{i,k}(t)$ and the noise $\varepsilon_{i,k}(t)$ with $t \in [0, 1]^q$ could be random elements in the functional space $L^2([0, 1]^q)$ of square-integrable functions equipped with the usual inner product and the corresponding norm w.r.t. the Lebesgue measure on $[0, 1]^q$. Moreover, it is possible that panels are mixed, i.e. some are functional, some are univariate whereas others may be multivariate with e.g. different dimensionality.

We assume a common single change point scenario as in (1.2) with $P = 1$ and $u = u_1$. Here, the signals $\mu_{i,k}$ and $\mu_{2,k}$ are now also $H_k$-valued functions. We consider the following generalized CUSUM estimate as a natural extension of (2.8):

$$\hat{u} = \arg\max_{i = 1, \ldots, n-1} w(i, n) \sqrt{\sum_{k=1}^d \sum_{j=1}^n (Y_{j,k} - \bar{Y}_{n,k})^2}.$$  

(2.36)

We need to state an appropriate modification of Assumption 2.5.

**Assumption 2.17.** For all $i = 1, \ldots, n$ and $k \geq 1$ we assume that

1. $E\|\varepsilon_{i,k}(\cdot)\|_{H_k}^2 = \sigma^2$ for some $0 < \sigma^2 < \infty$,
2. $V^2(i) = E\|S_{i,k}(\varepsilon)\|_{H_k}^2/\sigma^2$ for the cumulated centered noises

$$S_{i,k}(\varepsilon) = n^{-1/2} \sum_{j=1}^i (\varepsilon_{j,k} - \bar{\varepsilon}_{n,k}).$$  

(2.37)

3. We set $\Delta_k = \mu_{2,k} - \mu_{1,k}$ and $\Delta_{\infty}^2 = \lim_{d \to \infty} d^{-1} \sum_{k=1}^d \|\Delta_k\|_{H_k}$.
4. $g_l(\varepsilon_{\bullet,k}) = \langle S_{i,k}(\varepsilon) - 2\Delta_k n^{1/2} H(i, n), S_{i,k}(\varepsilon) \rangle_{H_k}$, where $\varepsilon_{\bullet,k}$ represents formally the $k$-th panel.

Analogously to Theorem 2.6 we may state the following functional version.

**Theorem 2.18.** Let $n \geq 3$, assume a single change point scenario and let Assumption 2.17 be fulfilled. Assume that, as $d \to \infty$, it holds that

$$\frac{1}{d^2} \Var\left( \sum_{k=1}^d g_l(\varepsilon_{\bullet,k}) \right) = o(1)$$

(2.38)

\(\text{10}\) We could treat the Hilbertian case from the very beginning in (1.1) and (1.2). However, that would obscure the basic ideas and even more important - not fit into the denoising framework. Hence, to enhance readability, we discuss the Hilbertian case separately.
for all $i = 1, \ldots, n - 1$. Then, it holds that, as $d \to \infty$,

$$P\left( \hat{u} \in \arg \max_{i=1, \ldots, n-1} C(i; u, n, r) \right) \to 1. \quad (2.39)$$

Clearly, all considerations of Section 2.2 regarding condition (2.38) and perfect estimation remain valid in the functional setup. The estimation of $V(i)$ can be carried out along the lines of Section 2.3 but now with $\Sigma_{i,k} = E(\varepsilon_{i,1}, \varepsilon_{k,1}) u_i$.

## 2.5 Functional data interpreted as panel data

We would like to point out that the segmentation approach of [Bleakley and Vert (2011a)](Bleakley) also offers another possibility to estimate changes of means in time series of functional data in the multiple change point scenario.

We assume to have only one panel, i.e., $d = 1$, where the observations $Y_i(\cdot) = Y_{i,1}(\cdot)$ and the errors $\varepsilon_i(\cdot) = \varepsilon_{i,1}(\cdot)$ are random elements in the functional space $L^2([0,1])$ with $E|\varepsilon_{i,1}(\cdot)|^2 = \sigma^2$ for some $0 < \sigma^2 < \infty$ (cf. previous Subsection 2.4). Clearly, the means $m_i(\cdot) = m_{i,1}(\cdot)$ are also $L^2([0,1])$ valued. Further, we assume the sequence $\{\varepsilon_i(\cdot)\}$ to be i.i.d. and Gaussian.

Given any complete orthonormal system $\{v_1(\cdot), v_2(\cdot), \ldots\}$ in $L^2([0,1])$ we obtain the Fourier expansion

$$\varepsilon_i = \sum_{j=1}^{\infty} \langle \varepsilon_i, v_j \rangle v_j, \quad (2.40)$$

where the convergence is meant in the $L^2([0,1])$ sense. The random variables $\langle \varepsilon_i, v_j \rangle$ are independent in $i$, centered and Gaussian with variance $\lambda_j = \text{Var}(\varepsilon_i(\cdot), v_j(\cdot))$. We choose the orthonormal system which corresponds to the Karhunen-Loève expansion. Hence, all $\langle \varepsilon_i, v_j \rangle$ are uncorrelated in $j$ and therefore, due to Gaussianity, all

$$\varepsilon_{i,j} = \langle \varepsilon_i, v_j \rangle / \sqrt{\lambda_j} \quad (2.41)$$

are i.i.d. standard normally distributed. Here, we tacitly assume that $\lambda_j > 0$ holds true for all $j \geq 1$. Using expansion (2.40) and in view of (2.41) we set

$$Y_{i,k} = \langle Y_i, v_j \rangle / \sqrt{\lambda_j} = m_{i,k} + \varepsilon_{i,k}$$

where $m_{i,k} = \langle m_i, v_j \rangle / \sqrt{\lambda_j}$. Hence, we may interpret a functional time series as cross-sectional independent panel data with infinitely many panels which fulfill Assumption 2.5. Notice that any change in the time series $Y_i(\cdot)$ at time point $u$ induces a common change in panels at the same time point.

In the single changepoint scenario the condition $0 < \Delta^2_\infty < \infty$ for $\Delta^2_\infty = \lim_{d \to \infty} d^{-1} \sum_{k=1}^{d} |\Delta_k|^2$ with $\Delta_k = \mu_{2,k} - \mu_{1,k}$ depends not only on the change in the underlying functional time series but also on the distribution of $\varepsilon_1(\cdot)$ via the data dependent basis $\{v_j(\cdot)\}$. However, one can verify that this condition is fulfilled for many reasonable combinations of distributions and changes.

## 3 Simulations

For our simulations within the single changepoint scenario we have implemented the CUSUM estimates in Matlab. For demonstration purposes a Matlab application with a graphical user interface can be obtained from the author or from [www.mi.uni-koeln.de/~ltorgovi](http://www.mi.uni-koeln.de/~ltorgovi). For the simulations within the multiple changepoint scenario we work with the Matlab “GFLseg”-package of [Bleakley and Vert (2011b)](Bleakley). In particular, we use the group fused LASSO method implemented in gglassoK.m.
3.1 Estimation under dependence

We proceed by considering our MA(1) toy example (2.22) from Subsection 2.2.4. The Figure 2 shows (rescaled) critical functions for different weighting schemes. The curves denoted by “simple”, “standard”, “weighted”, “exact”, “exact-est” and “exact-center” show the quantities obtained using the corresponding weightings \( w_{\text{simple}}, w_{\text{standard}}, w_{\text{weighted}}, w_{\text{exact-est}}, w_{\text{exact-est-center}} \) (cf. Subsection 2.3 for the latter). We switch back to \( w_{\text{standard}} \) from \( w_{\text{exact-est}} \) or from \( w_{\text{exact-est-center}} \) whenever the latter estimates are not computable. This situation is described in Remark 2.16.

**Figure 2:** The left figure shows rescaled \( \tilde{t}_i \)'s from (2.8) and the right figure shows the rescaled critical curves \( \tilde{C}_n \) (i.e. the limit of \( \tilde{t}_i \)'s as \( d \to \infty \)) for different weighting schemes.

The vertical lines indicate the locations of maxima for different weightings, respectively. The underlying panels are MA(1) with independent Gaussian \( \{\eta_i,k\} \) where \( \tilde{\sigma}^2 = 100 \) and \( \phi = -3, \theta = 0 \). Further parameters are \( n = 100, d = 10000, u = 70 \) and \( \Delta_k = 1 \) with \( \mu_{1,k} = 0 \) for all \( k \). The training period is chosen as \( n_1 = 1 \) and \( n_2 = n \) with bandwidth \( h = 2 \) in case of \( \tilde{w}_{\text{exact-est}} \) and as \( n_2 = 30 \) with the same bandwidth in case of \( \tilde{w}_{\text{exact-est-center}} \). For comparison all curves are shifted and rescaled to \( \{1, \ldots, n-1\} \times [0, 1] \) by the following transformation:

\[
\tilde{f}(i) = f(i) - \frac{\min_{1 \leq j < n} f(j)}{\max_{1 \leq j < n} f(j) - \min_{1 \leq j < n} f(j)}.
\]

The Figures 3 - 11 demonstrate the performance of the estimates w.r.t. different weighting schemes. They compare the accuracy, i.e. \( P(\hat{u} = u) \), the means and the standard deviation for a range of dimensions \( d = 1, 100, 200, \ldots, 1000 \), a range of MA(1) parameters \( \phi = -2, -1, -0.5, 0, 1 \) and different changepoint locations \( u = 55, 70, 90 \). Those quantities are all simulated based on 100 repetitions. In all figures the underlying panels and parameters are the same as chosen for Figure 3 and only the changepoint location differs.

Overall the Figures 3 - 11 show that the changepoint estimate \( \hat{u} \) based on exact weighting scheme outperforms \( w_{\text{standard}} \). The former estimates all changes correctly for arbitrary \( \phi \) if we consider a sufficiently large number of panels \( d \). The exact scheme is less biased and also has less variation. Furthermore, we see that for chosen parameters the effect of estimation of \( w_{\text{exact}} \) by \( \tilde{w}_{\text{exact-est}} \) (even with a centering, i.e. with \( \tilde{w}_{\text{exact-est-center}} \)) is negligible.
Figure 3: The underlying panels are MA(1) with independent Gaussian innovations $\{\eta_{i,k}\}$ where $\tilde{\sigma}^2 = 25$ and $\theta = 0$. Further parameters are $n = 100$, $u = 55$ and $\Delta_k = 1$ with $\mu_{1,k} = 0$ for all $k$. The training period for estimation of $\hat{\Sigma}$ is chosen as $n_1 = 1$ and $n_2 = 20$ with bandwidth $h = 2$. 
Figure 4: Means for $u = 55$

Figure 5: Standard deviation for $u = 55$
Figure 6: Accuracy for $u = 70$

Figure 7: Means for $u = 70$
Figure 8: Standard deviation for $u = 70$

Figure 9: Accuracy for $u = 90$
Figure 10: Means for $u = 90$

Figure 11: Standard deviation for $u = 90$
3.2 Post processing weights using regression

Assume that we estimate changepoints using estimates $\hat{w}^{\text{exact}} = 1/\hat{V}(i)$ for $w^{\text{exact}}$ which is described in Subsection 2.3. We would like to mention a possible modification of the weighting schemes which seems to be beneficial and may serve as a motivation for further research.

Based on the results and the corresponding proofs of Section 2 it seems reasonable to expect that, if $V(i)$ is concave (convex), which is e.g. the case for i.i.d. or MA(1) panels, then estimates $\hat{V}(i)$ that are concave (convex) should increase the precision of the changepoint estimator. In the following we treat without loss of generality the convex case since the concave follows analogously. The estimates $\hat{V}(i)$ described in Subsection 2.3 are generally not convex due to randomness and fluctuate around the convex discrete function $V(i)$. Therefore, one may post-process the estimated function $\hat{V}(i)$, $i = 1, \ldots, n - 1$, using the well known least squares convex regression and work with a convex estimate $\hat{V}(i)$. The basic principle is that, given a regression model

$$\hat{V}(i) = V(i) + \varepsilon_i$$

with a convex function $V(i)$ and a noise sequence $\{\varepsilon_i\}$ for $i = 1, \ldots, n - 1$, we solve

$$\min_{i=1}^{n-1} \sum_{i=1}^{n-1} (\hat{V}(i) - \hat{V}(i))^2,$$

under the convexity restrictions

$$\hat{V}(j) \geq \hat{V}(i) + g_i(j - i). \quad (3.1)$$

Here, $g_i$ is an estimate for the sub-gradient of $V$ at $i$ and (3.1) holds true for $i, j = 1, \ldots, n - 1$ (cf., e.g., [Hannah and Dunson (2013)]). Clearly, in our situation, these estimates $\hat{V}(i)$ remain consistent for $d \to \infty$ if the underlying original estimates $\hat{V}(i)$ were consistent.

3.3 Estimation under dependence with varying locations

So far, we assumed a fixed changepoint at some timepoint $u$ across all panels. It seems more realistic to allow for some minor fluctuations (around timepoint $u$) of changepoints in different panels. Therefore, [Bleakley and Vert (2011a)] study theoretically and empirically how their procedure behaves under randomized change points. In this case they consider changes across panels $k = 1, \ldots, d$ that are located at random change points $u + U_k \in \{1, \ldots, n - 1\}$, where $\{U_k\}$ are some i.i.d. random variables describing the fluctuations. In [Bleakley and Vert (2011a), cf. Theorem 4 and Figure 3] they show under appropriate assumptions that the standard weighting works well in this setting in the sense that the probability $P(\hat{u} \in u + S)$ tends to 1 as $d \to \infty$, where $S$ is the support of $P_{U_k}$. We do not develop the theoretical analogue but show in Figure 13 empirically that, as should be expected, the exact weighting is also beneficial under dependencies. For this simulation we assume $P(U_k = \pm 2) = 0.5$ as in [Bleakley and Vert (2011a)] and use the term “accuracy” now for $P(\hat{u} \in u + S)$.  

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3.4 Multiple changepoints

4 Proofs

Proof of Proposition 2.3. If \( \lambda \geq t_{n:n} \) then \( \hat{\beta} = 0 \) solves the KKT system immediately because in this case only the second condition has to be fulfilled. Notice, that for \( \hat{\beta} = 0 \) this condition simplifies to \( \lambda \geq t_{n:n} \). If \( t_{n-1:n} < \lambda < t_{n:n} \) then \( \hat{\beta} \) is set according to (2.7). It holds that

\[
\hat{c}_{M,\bullet} = \left( \frac{\lambda}{\|\beta_{M,\bullet}\|} + D_{M}^{T}D_{M}\hat{\beta}_{M,\bullet} \right)\hat{\beta}_{M,\bullet}.
\]

The latter equality in (4.1) holds true since the former equality in (4.1) implies

\[
t_{M} = \|\hat{c}_{M,\bullet}\| = \lambda + D_{M}^{T}D_{M}\|\beta_{M,\bullet}\|.
\]

Hence, the first equation in (2.6) is also fulfilled. The second inequality is fulfilled because \( \hat{\beta} \to 0 \) as \( \lambda \uparrow t_{n:n} \) and this inequality is simply \( t_{n-1:n} \leq \lambda \) for \( \hat{\beta} = 0 \).

Proof of Corollary 2.4. After columnwise centering it holds that \( \bar{Y} = \bar{M} + \bar{E} \). Further, we may use the decomposition \( \bar{M} = D\hat{\beta} + \mathbb{I}M_{1,\bullet} \) with

\[
\beta_{i,\bullet} = (M_{i+1,\bullet} - M_{i,\bullet})/w(i,n).
\]

Clearly, \( \hat{u} \) must be the same according to (2.1) for any \( M_{1,\bullet} \) and we may assume w.l.o.g. that \( M_{1,\bullet} = 0 \), i.e. that \( \bar{Y} = D\hat{\beta} + \bar{E} \) and therefore

\[
D^{T}\bar{Y} = D^{T}D\hat{\beta} + D^{T}\bar{E}.
\]
holds true. Following the calculations in Bleakley and Vert (2011a, Lemma 1) we observe

\[(D^T D \beta)_{i,k} = w(i,n)w(u,n)\frac{\Delta_k}{w(u,n)} \begin{cases} (i/n)(1-u/n), & i = 1, \ldots, u, \\ (u/n)(1-i/n), & i = u, \ldots, n - 1 \end{cases} \]

and

\[(D^T E)_{i,k} = w(i,n)\left(\sum_{j=1}^{i}(i/n-1)\varepsilon_{j,k} + \sum_{j=i+1}^{n} (i/n)\varepsilon_{j,k}\right) = -w(i,n)\left(\sum_{j=1}^{i}(\varepsilon_{j,k} - \varepsilon_{n,k})\right)\]

for \(k = 1, \ldots, d\) and \(i = 1, \ldots, n - 1\). Hence, it holds that \((D^T E)_{i,\bullet} = (D^T E)_{\bullet,\bullet}\) for \(i = 1, \ldots, n - 1\), which together with (2.3) completes the proof.

**Proof of Theorem 2.6** This is a special case of the proof of Theorem 2.18 and is therefore omitted.

**Proof of Theorem 2.8** The critical function \(C(i; u, n, r)\) has a maximum at \(i = u\) for all ratios \(r \geq 0\) if and only if

\[\frac{C(u; u, n, r)}{C(i; u, n, r)} \geq 1\]

for all \(r \geq 0\), \(i = 1, \ldots, n - 1\) and \(i \neq u\). This can be only fulfilled if

\[1 \leq \lim_{r \to \infty} \frac{C(u; u, n, r)}{C(i; u, n, r)} = \frac{[w(u,u)V(u)]^2}{w(i,u)V(i)}\]

holds true for any \(i = 1, \ldots, n - 1, i \neq u\) and the assertion follows.

**Proof of Theorem 2.4** We assume that \(V(u)/V(n-u) = 1\). According to the necessary condition (2.17) we obtain

\[\frac{w(k,n)}{w(n-k,n)} \geq \frac{V(n-k)}{V(k)} = 1 = \frac{V(n-k)}{V(k)} \geq \frac{w(k,n)}{w(n-k,n)}\]

for all \(k = 1, \ldots, n\). Hence, the weights fulfill \(w(k,n) = w(n-k,n)\), i.e. are necessarily symmetric. Moreover, due to (2.17) together with the symmetry of \(V(i)\) and the symmetry of the weights \(w(k,n)\) we obtain a system of inequalities

\[\frac{w(k,n)}{w(j,n)} \geq \frac{V(j)}{V(n-j)} = \frac{V(n-j)}{V(k)} \geq \frac{w(k,n)}{w(j,n)}\]

for any \(j,k = 1, \ldots, n - 1\). Hence, \(w(j+1,n) = w(j,n)V(j)/V(j+1)\) for \(j = 1, \ldots, n - 2\). Thus, on setting, e.g., \(w(1,n) = V(1)\) we obtain the remaining weights iteratively.

**Proof of Theorem 2.17** Since \(w(i,n) = 1/V(i)\), inequality (4.2) is equivalent to

\[0 < C(u; u, n, r) - C(i; u, n, r)\]

\[= \begin{cases} w^2(u,n)(u/n)^2(1-u/n)^2 - w^2(i,n)(i/n)^2(1-u/n)^2, & i < u, \\ w^2(u,n)(u/n)^2(1-u/n)^2 - w^2(i,n)(u/n)^2(1-i/n)^2, & i > u \end{cases}\]
for any \( r \geq 0 \). Due to the symmetry of \( V(i) \), this is equivalent to

\[
\frac{V(n-i)}{V(n-u)} = \frac{V(i)}{V(u)} = \frac{w(u,n)}{w(i,n)} = \begin{cases} \frac{i}{u}, & i \leq u, \\ \frac{(n-i)/(n-u)}{w(i,n)}, & i > u \end{cases}
\]

and the assertion follows. \( \square \)

**Proof of Lemma 2.11** Set \( f_u(i) = V(i)/V(u) \), \( g_u(i) = i/u \) and observe that 

\[ f_u(u) = V(u)/V(u) = 1 = u/u = g_u(u). \]

Now, assume that \( f_u(1) = V(1)/V(u) > 1/u = g_u(1) \). Since \( f_u(i) \) is concave in \( i \) and \( g_u(i) \) is linear in \( i \) the proof follows immediately. \( \square \)

**Proof of Corollary 2.13** According to (2.29) and due to independence we have

\[
\var(S_{i,k}(\varepsilon)) = \var\left( \sum_{j=1}^{n} a_{i,j} (\eta_{j,k} - \phi \eta_{j-1,k}) \right) + \theta^2 \var\left( \sum_{j=1}^{n} a_{i,j} (\eta_{j,k-1} - \phi \eta_{j-1,k-1}) \right).
\]

Straightforward calculations yield

\[
\frac{n}{\sigma^2} \var\left( \sum_{j=1}^{n} a_{i,j} (\eta_{j,k} - \phi \eta_{j-1,k}) \right) = (1 - i/n)^2 \left[ i(1 + \phi^2) + 2(i - 1)\phi \right] + (i/n)^2 \left[ (n - i)(1 + \phi^2) + 2((n - i) - 1)\phi \right] - 2(1 - i/n) (i/n) \phi
\]

\[
= (1 + \phi^2 + 2\phi) \left[ (1 - i/n)^2 i + (i/n)^2 (n - i) \right] - 2\phi \left[ (i/n)^2 + (i/n)^2 + (1 - i/n) (i/n) \right] = \alpha(\phi) (i (1 - i/n)) - 2\phi,
\]

where \( \alpha(\phi) \) is set in (2.23) and this implies

\[
V^2(i) \xi = \alpha(\phi) ((i/n) (1 - i/n)) - 2\phi/n
\]

with \( \xi = \sigma^2 / (\hat{\sigma}^2 (1 + \theta^2)) \). We have to distinguish the two cases \( \alpha(\phi) > 0 \) and \( \alpha(\phi) \leq 0 \). In the first case \( V^2(i) \) is strictly concave and we may use Lemma 2.11. Simple algebra shows

\[
c(V^2(1) - V^2(u)/u^2) = (un)^{-2}(u-1)((u + u\phi^2 - 2\phi)n + 2\phi u)
\]

\[
= (un)^{-2}(u-1)(\phi^2 nu - 2\phi(n - u) + nu)
\]

for some \( c > 0 \) and it is easy to check that \( V^2(1) - V^2(u)/u^2 > 0 \) holds true (for \( \phi \geq 0 \) we check via the first equality and for \( \phi < 0 \) via the second equality). The latter case, \( \alpha(\phi) \leq 0 \), occurs for

\[
-1 - \frac{1}{n} - \sqrt{\frac{2}{n} + \frac{1}{n^2}} \leq \phi \leq -1 - \frac{1}{n} + \sqrt{\frac{2}{n} + \frac{1}{n^2}} \quad (4.4)
\]
In this case it is sufficient to check that
\[ h(t) := \frac{\alpha(\phi) ((t/n)(1 - t/n)) - 2\phi/n}{t^2}, \]
i.e. \( h(i) = V^2(i)/i^2 \), is strictly decreasing on \( t \in [1, n - 1] \). It is
\[ nt^3 \left( \frac{\partial}{\partial t} h(t) \right) = -\alpha(\phi)t + 4\phi \]
and a sufficient condition for \( h(t) \) to be strictly decreasing is that \(-\alpha(\phi)t + 4\phi < 0\) holds true. This condition is fulfilled for any \( t \in [1, n - 1] \) whenever it is fulfilled for \( t = n - 1 \). In particular it is satisfied if \( \alpha(\phi) - 4\phi - 2\phi/n = 1 + \phi^2 - 2\phi \geq 0 \) holds true. Clearly, the latter is the case for any \( \phi \in \mathbb{R} \) since \( 1 \) is the only zero. \( \square \)

**Proof of Theorem 2.13.** Let \( w = w^{\text{weighted}} \). We assume that \( u > n/2 \), i.e. \( u^* = u \) and set \( w = w^{\text{weighted}} \). (The case \( u < n/2 \) follows by symmetry.) We consider the case \( i \leq u \) first and define
\[ C(x, r) = F(x)r + G(x) \]
with \( F(x) = w^2(x, n)V^2(x) \), \( V^2(x) = (x/n)(1 - x/n) \) and \( G(x) = w^2(x, n)((x/n)(1 - u/n))^2 \) on \([0, n)\), i.e. \( C(x, r) = C(x; u, n, r) \) for \( x \in \{1, \ldots, u\} \). For convenience we suppress the dependence on \( u \) and \( n \). It is easy to check that \( G(x) \) is strictly increasing with \( G(0) = 0 \), that \( F(x) \) is strictly concave on \([0, n]\) and symmetrical w.r.t. \( n/2 \) and that \( F(0) = 0 = F(n) \) holds true. Further, \( C(x, r) \), as a function of \( x \), has a unique maximum at \( u \) only if \( C(x, r) < C(u, r) \) for any \( x \neq u \). Now, \( C(x, r) < (>) C(y, r) \), \( x < y \) and \( n/2 < y \) is equivalent to
\[
\begin{aligned}
\left\{ \begin{array}{ll}
< (>) R(y, x), & n - y < x < y, \\
> (<) R(y, x), & 0 < x < n - y
\end{array} \right.
\end{aligned}
\]
and simply \( G(x) < (>) G(y) \) if \( x = n - u \), where
\[
R(y, x) = \begin{pmatrix}
G(y) - G(x) \\
F(y) - F(x)
\end{pmatrix}
\begin{pmatrix}
> 0, & n - y < x < y, \\
< 0, & 0 < x < n - y.
\end{pmatrix}
\]
(4.5)
The latter holds true because \( G(x) \) is strictly increasing on \([0, y]\), i.e. \( G(y) - G(x) > 0 \) is strictly decreasing in \( x \), and because of the properties of \( F \) described above. Since \( G(x) < G(y) \) we know that \( C(n - y, r) < C(y, r) \) for any \( r \) and since \( F \) is symmetric we also conclude that \( R(y, x) < R(y, n - x) \) for \( n - y < x < y \). Altogether, this implies that \( C(i, r) < C(u, r) \) for any \( 0 < i < u \) and \( r \geq 0 \) only if
\[
0 \leq r < R = \min_{n/2 \leq i < u} R(u, i).
\]
The function \( C(x; u, n, r) \) is obviously strictly decreasing for \( n/2 < u < x < n \) and the claim follows. (If \( u = n/2 \) the function \( C(i; u, n, r) \) has a maximum at \( u \) for any \( r \geq 0 \) due to concavity of \( H(i, u) \) and concavity of \( w^2(i, n)V(i) \). Hence, any \( r \) is admissible.) \( \square \)

**Proof of Theorem 2.14.** We restrict our considerations to \( u > n/2 \). The case \( u < n/2 \) follows by symmetry. In Bleakley and Vert [2013a] Theorem 2), i.e. for \( \gamma = 0 \), it is used that \( C(i; u, n, r) \) has a global maximum at \( u \) only if \( C(u; u, n, r) > C(u - 1; u, n, r) \). This does not hold true in the case of \( \gamma \in (0, 1/2) \) and a global maximum can differ from \( u \) even though \( C(u; u, n, r) > C(u - 1; u, n, r) \) holds true. However, we will see that this situation cannot occur if \( s \in (1/2, B(\gamma)] \).
We use the notation from the proof of Theorem 2.13. As mentioned there it is sufficient to consider the case \( i \leq u \). In this case we know from the proof of Theorem 2.13 that a possible local maximum of \( C(x, r) \) for \( x \in [0, u] \) can occur only at some \( x_{\text{max}} \in [n/2, u] \). Moreover, using basic analysis we know that

\[
\lim_{r \to \infty} \frac{C(x, r)}{F(y)} = \frac{F(x)}{F(y)}
\]

for any \( 0 < x, y < n \). Due to strict concavity of \( F \) we know that for any \( 0 < \delta < 1 \) it holds that \( F(n/2) > F(n/2 + \delta) \). That is, for sufficiently large \( r \), a local maximum of \( C(x, r) \) occurs within \( [n/2 - \delta, n/2 + \delta] \cap [n/2, u] = [n/2, n/2 + \delta] \).

Now, we compute the rescaled first derivative of \( C(x, r) \) on \( [0, n] \) which will be denoted by

\[
P(x, r) := \frac{(x/n)(1 - x/n))^{2\gamma + 1}}{x} \left( \frac{\partial C(x, r)}{\partial x} \right).
\]

\( P(x, r) \) can be evaluated to a second order polynomial in \( x \) and for \( x \in (0, n) \) we know that \( \frac{\partial C(x, r)}{\partial x} = 0 \) if and only if \( P(x, r) = 0 \) with \( x \in (0, n) \). Furthermore, since \( C(0, r) = 0 \) and \( C(x, r) \to \infty \) as \( x \uparrow n \) for any \( r \geq 0 \) we may have either only a saddle-point or a maximum and a minimum must occur simultaneously at some \( 0 < x_{\text{max}} < x_{\text{min}} < n \). We also know from previous considerations that \( x_{\text{max}} \geq n/2 \).

The discriminant, \( D(r) \), of \( P(x, r) \) is a second order polynomial in \( r \) with roots

\[
r_{1,2} = \frac{-2(2\gamma + 2) \pm 4\gamma^{1/2}(1 - u/n)^2}{2\gamma - 1}.
\]

\( D(r) \), as a second order polynomial, must be positive for \( r > r_2 \), where \( r_2 \) denotes the larger root. Recall that, \( C(x, r) \) has a local maximum within \( [n/2, n/2 + \delta] \) for any \( 0 < \delta < 1 \) and all sufficiently large \( r \). Otherwise, \( D(r) \) would be negative for \( r > r_2 \) and we would have no extrema of \( C(x, r) \) in case of large \( r \).

The solution of \( P(x, r_2) = 0 \) is given by \( x^* = nB(\gamma) \in (n/2, n) \) which is unique and therefore must be a saddle-point of \( C(x, r_2) \). For \( r_1 < r < r_2 \) a real solution to \( P(x, r) = 0 \) does not exist and therefore \( C(x, r) \) does not have any extrema on \( (0, n) \). For \( 0 \leq r' \leq r_1 \) real solutions do exist but the corresponding roots of \( P(x, r') \) cannot correspond to a maximum or a minimum of \( C(x, r') \) on \( (0, n) \) as discussed in the following. Assume that it is not a saddle-point then we would have a maximum and a minimum because they must occur simultaneously at some \( n/2 \leq x_{\text{max}} < x_{\text{min}} < n \), i.e. \( C(x_{\text{max}}, r') > C(x_{\text{min}}, r') \). This implies \( r' \geq R(x_{\text{max}}, x_{\text{min}}) \) and therefore \( C(x_{\text{max}}, r) > C(x_{\text{min}}, r) \) for any \( r > r' \), which contradicts the fact of no extrema for \( r_1 < r < r_2 \). We did not exclude the possibility of a saddle-point at \( x_s \in (0, n) \) because for our conclusions it will be no problem as long as the function remains strictly increasing on \( (0, n)/\{x_s\} \).

Assume that \( \frac{\partial C(x, 0, r_0)}{\partial x} = 0 \) for some \( n/2 < x_0 < n \), \( r_0 > 0 \). For any \( \varepsilon \geq 0 \) we have

\[
\frac{\partial C(x, r_0 + \varepsilon)}{\partial x} = \frac{\partial C(x, r_0)}{\partial x} + \varepsilon \frac{\partial F(x)}{\partial x}
\]

with \( \frac{\partial F(x)}{\partial x} < 0 \) for \( x \in (n/2, n) \) and

\[
\frac{\partial C(x, r_0)}{\partial x} = \begin{cases} 0, & x = x_0, \\ > 0, & 0 < x < n, \quad x \neq x_0 \end{cases}
\]

and \( x_0 \) is a saddle point.
The first equality ensures that \( \frac{\partial C(x_0, r_0 + \varepsilon)}{\partial x} < 0 \) for any \( \varepsilon > 0 \) and that \( C(x, r_0 + \varepsilon) \) has local extrema \( n/2 \leq x_{\max} < x_0 < x_{\min} \). The second inequality ensures for saddle-points \( x_0 \) that, for any \( 0 < \delta < 1 \) we can find an \( \varepsilon > 0 \) such that \( x_{\max}, x_{\min} \in (x_0 - \delta, x_0 + \delta) \). Moreover, we know that \( x_{\max} \downarrow n/2 \) and that \( x_{\min} \uparrow n \) as \( r \uparrow \infty \).

The properties discussed above ensure that given \( n/2 < u \leq nB(\gamma) \) it suffices to compare \( C(u; u, u, n, r) \) and \( C(u; u - 1; u, n, r) \) to decide whether a maximum is at \( u \) or not. The remaining assertions follow now by simple analysis.

Proof of Theorem 2.18. As before, we restrict our considerations to \( u > n/2 \) and the case \( u < n/2 \) follows by symmetry. Using the notation of Theorem 2.13 we consider the quantities \( R(u, x) \), with a continuous argument \( x \in (n - u, u) \). If \( nB(\gamma) < u < n \) the properties in the previous theorem ensure that, for \( \gamma \in (0, 1/2) \), the differentiable function \( R(u, x) \) in \( x \) must have a local minimum at \( x_{\min} \in [n/2, nB(\gamma)] \) such that \( C(u, r) > C(x, r) \), for any \( x < u \) in case of \( r < R(u, x_{\min}) \) and \( C(u, r) < C(x, r) \) for some \( x < u \) in case of \( r > R(u, x_{\min}) \). Some tedious but straightforward calculations for \( \gamma = 1/4 \) allow us to solve \( \frac{\partial R(u, x)}{\partial x} = 0 \) explicitly and to identify the minimum at \( x_{\min} = n/2 + (n^2 - u^2)^{1/2} \). Since, \( x_{\min} \) is not necessarily in \( \mathbb{N} \) we calculate

\[
\lim_{n \to \infty} R(sn, x_{\min} + \delta) \quad (4.6)
\]

for any \( \delta \in [-1, 1] \) and see that all limits are (2.28). This convergence is uniform in \( \delta \). Therefore, \( R(sn, |x_{\min}|) \) has the same limit and the assertion follows. Finally, the smoothness properties follow on applying l’Hôpital’s rule.

Proof of Proposition 2.15. As before, we write \( \| \cdot \| = \| \cdot \|_{H_k} \) and \( \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{H_k} \). Using the notation of (2.10) we consider

\[
S_{i,k}(Y) = \frac{n^{-1/2}}{n^{-1/2}} \sum_{j=1}^{i} (Y_{j,k} - \bar{Y}_{n,k})
\]

\[
= n^{-1/2} \sum_{j=1}^{i} (\varepsilon_{j,k} - \varepsilon_{n,k}) - n^{1/2} H(i, u) \Delta_k
\]

\[
= S_{i,k}(\varepsilon) - n^{1/2} H(i, u) \Delta_k.
\]

It holds that

\[
\|S_{i,k}(Y)\|^2 = \|S_{i,k}(\varepsilon)\|^2 - 2\Delta_k \langle n^{1/2} H(i, u), S_{i,k}(\varepsilon) \rangle + nH^2(i, u) \|\Delta_k\|^2
\]

\[
= g_i(k, \varepsilon) + nH^2(i, u) \|\Delta_k\|^2.
\]

Since \( ES_{i,k}(\varepsilon) = 0 \) for any \( i \) and \( k \), we get

\[
E\|S_{i,k}(Y)\|^2 = E\|S_{i,k}(\varepsilon)\|^2 + nH^2(i, u) \|\Delta_k\|^2.
\]

Due to part 2 of Assumption 2.17 (or part 2 Assumption 2.5 in case of Theorem 2.6) it holds that \( \frac{1}{d} \sum_{k=1}^{d} E\|S_{i,k}(\varepsilon)\|^2 = V^2(i) \sigma^2 \). Therefore, we get that, as \( d \to \infty \),

\[
E \left( \frac{1}{d} \sum_{k=1}^{d} \|S_{i,k}(Y)\|^2 \right) = V^2(i) \sigma^2 + nH^2(i, u) \left( \frac{1}{d} \sum_{k=1}^{d} \|\Delta_k\|^2 \right)
\]

\[
\to V^2(i) \sigma^2 + nH^2(i, u) \Delta_{\infty}^2.
\]
for each $i = 1, \ldots, n - 1$. Taking Assumption (2.14) into account we obtain, via Chebyshev’s inequality, as $d \to \infty$,

$$
\frac{1}{n} \left( \frac{w(i, n)}{\Delta_\infty} \right)^2 \frac{1}{d} \sum_{k=1}^{d} \|S_{i,k}(Y)\|^2 \overset{P}{\longrightarrow} C(i; u, n, r)
$$

for each $i = 1, \ldots, n - 1$. Due to the continuous mapping theorem we obtain

$$
\max_{i \in S} \frac{1}{n} \left( \frac{w(i, n)}{\Delta_\infty} \right)^2 \frac{1}{d} \sum_{k=1}^{d} \|S_{i,k}(Y)\|^2 \overset{P}{\longrightarrow} \max_{i \in S} C(i; u, n, r)
$$

for any $S \subset \{1, \ldots, n - 1\}$ and a similar statement holds true for $\max_{i \in S^c}$ which completes the proof. 

\begin{proof}
\end{proof}

Acknowledgment

The author wishes to thank Prof. J. Steinebach for helpful comments and Christoph Heuser for suggestions to the proof of Theorem 2.14. This research was partially supported by the Friedrich Ebert Foundation, Germany.
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