NORMAL NUMBERS AND NESTED PERFECT NECKLACES

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Abstract. M. B. Levin used Sobol-Faure low discrepancy sequences with Pascal matrices modulo 2 to construct, for each integer \( b \), a real number \( x \) such that the first \( N \) terms of the sequence \( (b^n x \mod 1)_{n \geq 1} \) have discrepancy \( O(\log N)^2 / N \). This is the lowest discrepancy known for this kind of sequences. In this note we characterize Levin’s construction in terms of nested perfect necklaces, which are a variant of the classical de Bruijn necklaces. Moreover, we show that every real number \( x \) whose base-\( b \) expansion is the concatenation of nested perfect necklaces of exponentially increasing order satisfies that the first \( N \) terms of \( (b^n x \mod 1)_{n \geq 1} \) have discrepancy \( O(\log N)^2 / N \). For base 2 and the order being a power of 2, we give the exact number of nested perfect necklaces and an explicit method based on matrices to construct each of them.

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1. Introduction and statement of results

For a sequence \( (x_n)_{n \geq 0} \) of real numbers in the unit interval the discrepancy of the first \( N \) elements is

\[
D_N((x_n)_{n \geq 0}) = \sup_{0 \leq \alpha < \beta \leq 1} \left| \frac{1}{N} \# \left\{ n : 0 \leq n < N \text{ and } \alpha \leq x_n < \beta \right\} - (\beta - \alpha) \right|.
\]

In [10] Schmidt showed that there is a constant \( C \) such that for every sequence \( (x_n)_{n \geq 0} \) of real numbers there are infinitely many \( N \)s such that

\[
D_N((x_n)_{n \geq 0}) > C \frac{\log N}{N}.
\]

This is an optimal order of discrepancy since this lower bound is achieved by van der Corput sequences, see [7] [4] [2].

The property of Borel normality for real numbers can be defined in terms of uniform distribution. A sequence \( (x_n)_{n \geq 0} \) of real numbers in the unit interval is uniformly distributed exactly when \( \lim_{N \to \infty} D_N((x_n)_{n \geq 0}) = 0 \). We write \( \{x\} \) to denote \( x - \lfloor x \rfloor \), the fractional part of \( x \). For an integer \( b \) greater than 1, a real number \( x \) is normal to base \( b \) if the sequence \( (\{b^n x\})_{n \geq 0} \), is uniformly distributed. It is still unknown whether the optimal order of discrepancy can also be achieved by a sequence of the form \( (\{b^n x\})_{n \geq 0} \) for some real number \( x \) [6] [4] [2].

The lowest discrepancy known for sequences of this form is \( O((\log N)^2 / N) \) and it holds for a real number \( x \) constructed by Levin in [9]. Given an arbitrary integer base \( b \), Levin’s construction uses Sobol-Faure sequences with the Pascal triangle matrix modulo 2, see [5] [9] [5].

Our first result in this note is a characterization of Levin’s construction in terms of combinatorics of words, showing that it is a concatenation of what we call nested perfect necklaces of increasing order. Perfect necklaces were introduced in [11] as a variant of the classical de Bruijn necklaces [3]. Fix an alphabet \( A \). A word is a finite sequence of symbols and a necklace, or circular word, is the equivalence

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class of a word under rotations. For positive integers \( k \) and \( m \), we call a necklace 
\((k, m)\)-perfect if each word of length \( k \) occurs exactly \( m \) times at positions which 
are different modulo \( m \) for any convention on the starting point. The length of a 
\((k, m)\)-perfect necklace is \( m|A|^k \) where \( |A| \) denotes the cardinality of the alphabet \( A \). 
In this note we always consider the modulo \( m \) being a power of 2.

Notice that for \( m = 1 \), the \((k, m)\)-perfect necklaces are exactly the de Bruijn necklaces of order \( k \). For the binary alphabet the word 0011 is a \((1, 2)\)-perfect necklace. Both words 00110110 and 00011011 are \((2, 2)\)-perfect necklaces. The segments in Champernowne sequence which are the concatenation in lexicographic order of all words of length \( k \) is a \((k, k)\)-necklace. For instance the following word is a \((3, 3)\)-perfect necklace (the spacing is just for the readers convenience),

\[
000 \ 001 \ 010 \ 011 \ 100 \ 101 \ 110 \ 111
\]

More generally, every arithmetic sequence with difference coprime with the alphabet size yields a perfect necklace.

A word \( w \) is a \((k, m)\)-nested perfect necklace if for each integer \( \ell = 1, 2, \ldots, k \), each block of \( w \) of length \( m|A|^\ell \) which starts at a position congruent to 1 modulo \( m|A|^\ell \) is a \((\ell, m)\)-perfect necklace. An alternative recursive definition of nested perfect necklaces is as follows. A word \( w \) is a \((k, m)\)-nested perfect necklace if, first, it is a \((k, m)\)-perfect necklace; and, second, either \( k = 1 \) or whenever \( w \) is factorized \( w = w_1 \cdots w_{|A|} \) with each word \( w_i \) of length \( m|A|^{k-1} \), then each word \( w_i \) is a \((k-1, m)\)-nested perfect necklace.

Notice that each \((k, m)\)-nested perfect necklace is not an equivalence class closed under rotations, but it is a single word, with a unique initial position. The word 00110110 is a \((2, 2)\)-nested perfect necklace because it is a \((2, 2)\)-perfect necklace and both words 0011 and 0110 are \((1, 2)\)-perfect necklaces. The four words

\[
0000111010110110 \\
0011110001101001 \\
0011111001001011 \\
0010110110111000
\]

are \((2, 4)\)-nested perfect necklaces. Both the concatenation of the first two and the concatenation of the last two are \((3, 4)\)-nested perfect necklaces. The concatenation of all of them is a \((4, 4)\)-nested perfect necklace. The concatenation of all words of the same length in lexicographic order yields a perfect necklace that is not a nested perfect necklace.

The statement of our first result is as follows.

**Theorem 1.** For each base \( b \) the number \( x \) defined by Levin in [9, Theorem 2] using the Pascal triangle matrix modulo 2 is obtained as the concatenation of the \((m, m)\)-nested perfect necklaces for \( m = 2^d \) with \( d = 0, 1, 2, \ldots \). Conversely, for every number \( x \) whose base \( b \) expansion is the concatenation of \((m, m)\)-nested perfect necklaces for \( m = 2^d \) with \( d = 0, 1, 2, \ldots \), the discrepancy \( D_N(\{b^n x\}_{n \geq 1}) \) is \( O((\log N)^2/N) \).

To state the second result, we consider the field with two elements \( \mathbb{F}_2 \) and we introduce a family of \( 2^{m-1} \) matrices of dimension \( m \times m \) over \( \mathbb{F}_2 \) obtained by rotating the columns of the Pascal triangle matrix modulo 2. We identify words of two symbols with vectors, for each matrix \( M \) of this family, we construct a nested perfect necklace by concatenating the words of the form \( Mw \oplus z \) where \( z \) is a fixed word over \( \mathbb{F}_2 \) of length \( m \) and \( w \) ranges over all words over \( \mathbb{F}_2 \) of length \( m \) in lexicographic order. Such a necklace is called an affine necklace. Our second result is as follows.
Theorem 2. For each $m = 2^d$ with $d = 0, 1, 2, \ldots$ there are $2^{2m-1}$ binary $(m, m)$-nested perfect necklaces and they are exactly the affine necklaces.

The rest of this note is devoted to the proofs. For the proof of Theorem 1 first notice that Levin’s construction in his Theorem 2 in [9] is the concatenation of blocks obtained using Pascal triangle modulo 2 for increasing $m = 2^d$ with $d = 0, 1, 2, \ldots$. Hence, each block is an $(m, m)$-affine necklace and, by Proposition 7, each block is an $(m, m)$-nested perfect necklace. Conversely, assume that the expansion in base $b$ of a given real $x$ can be split in consecutive blocks such that each block is an $(m, m)$-nested perfect necklace for $m = 2^d$ with $d = 0, 1, 2, \ldots$. Then, Levin’s chain of estimates in [9] yield the wanted discrepancy: his Lemma 5, Corollaries 1 and 2 and the end of the proof of his Theorem 2.

The proof of Theorem 2 follows from Propositions 5, 13 and 14.

2. Affine necklaces

In this note we consider transformations on words obtained as linear maps over the field $\mathbb{F}_2$ with two elements. We identify the words of length $n$ over $\mathbb{F}_2$ with the column vectors of dimension $n \times 1$ over $\mathbb{F}_2$. More precisely, we always identify the word $a_1 \cdots a_n$ with the column vector $(a_1, \ldots, a_n)^t \in (\mathbb{F}_2)_{n \times 1}$ where $^t$ denotes transpose of vectors and matrices. Suppose $w_1, \ldots, w_k$ is a sequence of words, each of them of length $n$ and $M$ is an $n \times n$-matrix over $\mathbb{F}_2$, we may consider the concatenation $(Mw_1)(Mw_2) \cdots (Mw_k)$. In this writing, the matrix $M$ is multiplied with each word $w_i$ considered as a column vector, and the resulting column vector is viewed again as a word of length $n$. Similarly, the component-wise sum of vectors in $\mathbb{F}_2$ is used directly on words of the same length. It is denoted by the symbol $\oplus$.

We assume that the alphabet is $\mathbb{F}_2 = \{0, 1\}$ and that the modulo $m$ is always a power of 2, namely $m = 2^d$ for some non-negative integer $d$. We now define a family of matrices that we will use to construct explicitly some nested perfect necklaces. We start by defining by induction on $d$ an $m \times m$-matrix $M_d$ for each $d \geq 0$ by

\[ M_0 = (1) \quad \text{and} \quad M_{d+1} = \begin{pmatrix} M_d & M_d \\ 0 & M_d \end{pmatrix}. \]

The matrices $M_1$ and $M_2$ are then

\[ M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]

The matrix $M_d$ is a variant of the Pascal triangle modulo 2 in rectangular form, we prove it in Lemma 5 below. This matrix is almost the one used by Levin in [9] because we have reversed the order of the columns. This definition of the matrix $M_d$ allows us to identify words with column vectors, which is not the case in [9].

We now introduce a family of matrices obtained by applying some rotations to columns of the matrix $M_d$. Let $\sigma$ be the function which maps each word $a_1 \cdots a_n$ to $a_n a_1 a_2 \cdots a_{n-1}$ obtained by moving the last symbol to the front. Since words over $\mathbb{F}_2$ are identified with column vectors, the function $\sigma$ can also be applied to a column vector.

Let $n_1, \ldots, n_m$ be a sequence of integers such that $n_m = 0$ and $n_{i+1} \leq n_i \leq n_{i+1} + 1$ for each integer $1 \leq i < m$. Let $C_1, \ldots, C_m$ be the columns of $M_d$, that is, $M_d = (C_1, \ldots, C_m)$. Define

\[ M_d^{n_1 \cdots n_m} = (\sigma^{n_1}(C_1), \ldots, \sigma^{n_m}(C_m)). \]
The following are the eight possible matrices $M_{d}^{n_{1} \cdots n_{m}}$ for $d = 2$ and $m = 2^2$.

\[
\begin{array}{cccc}
M_{d}^{0,0,0,0} & M_{d}^{1,0,0,0} & M_{d}^{1,1,0,0} & M_{d}^{2,1,0,0} \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
M_{d}^{1,1,1,0} & M_{d}^{2,1,1,0} & M_{d}^{2,2,1,0} & M_{d}^{3,2,1,0} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
\end{array}
\]

Let $m = 2^d$ for some $d \geq 0$ and let $k$ be some integer such that $1 \leq k \leq m$. Let $w_1, \ldots, w_{2^m}$ be the enumeration in lexicographic order of all words on length $m$ over $F_2$. Let $z$ be a word over $F_2$ of length $m$ and let $w_i' = w_i \oplus z$ for $1 \leq i \leq 2^m$. Let $M$ be a matrix a matrix $M_{d}^{n_{1} \cdots n_{m}}$ as above. Then, the concatenation

\[(M w_1')(M w_2') \cdots (M w_{2^m})\]

is called an $(k,m)$-affine necklace. In the sequel we refer to this necklace as the affine necklace obtained from the matrix $M_{d}^{n_{1} \cdots n_{m}}$ and the vector $z$. Note that setting $z' = Mz$ gives $M w_i' = M w_i \oplus z'$ which justifies the terminology. In Lemma 4 we will prove that each matrix $M_{d}^{n_{1} \cdots n_{m}}$ is invertible and therefore each vector $z'$ is equal to $Mz$ for some vector $z$.

Each matrix $M_d$ is upper triangular, that is $(M_d)_{i,j} = 0$ for $1 \leq i < j \leq 2^d$. The following lemma states that the upper part of the matrix $M_d$ is the beginning of the Pascal triangle modulo 2 also known as the Sierpiński triangle.

**Lemma 3.** For any integers $d, i, j$ such that $d \geq 0$ and $1 \leq i, j < 2^d$, $(M_d)_{i,j} = (M_d)_{i+1,j+1} \oplus (M_d)_{i,j+1}$.

**Proof.** The proof is carried out by induction on $d$. If $d = 0$, the result trivially holds. Suppose that the result holds for $M_d$ and let $i, j$ be integers such that $1 \leq i, j < 2^{d+1}$. If $i$ and $j$ are different from $2^d$, the result follows directly from the induction hypothesis and the definition of $M_{d+1}$. If either $i = 2^d$ or $j = 2^d$, the result follows from the fact that $(M_d)_{i,j}$ is equal to 1 if either $i = 2^d$ or $j = 2^d$ and it is equal to 0 if $i = 0$ or $j = 0$ (and $i$ and $j$ different from $2^d$). This latter fact is easily proved by induction on $d$. \hfill \Box

### 3. Affine necklaces are nested perfect necklaces

\[
M = \begin{pmatrix}
\begin{array}{c}
\ell \\
k \\
\end{array}
\end{pmatrix}
\begin{pmatrix}
\begin{array}{c}
P \\
m-k \\
k \\
m \\
\end{array}
\end{pmatrix}
\]

**Figure 1.** Position of the sub-matrix $P$ in $M$ in Lemma 4.
Lemma 4. Let $M$ be a matrix $M_{d}^{n_{1},...,n_{m}}$. Let $\ell$ and $k$ be two integers such that $0 \leq \ell < \ell + k \leq 2^{d}$. Any sub-matrix obtained by selecting the $k$ rows $\ell + 1, \ell + 2, \ldots, \ell + k$ and the last $k$ columns $2^{d} - k + 1, \ldots, 2^{d}$ of $M$ is invertible.

Note that for $k = 2^{d}$ and $\ell = 0$, the sub-matrix in the statement of the lemma, is the whole matrix $M_{d}^{n_{1},...,n_{m}}$, which is invertible.

Proof. Let $m = 2^{d}$ be the number of rows and columns of $M$. By Lemma \[\text{Lemma} \] each entry $M_{i,j}$ for $1 \leq i, j < m$ of the matrix $M$ satisfies either $M_{i,j} = M_{i+1,j} \oplus M_{i,j+1}$ if $n_{j} = n_{j+1}$ (the column $C_{j}$ has been rotated as much as the column $C_{j+1}$) or $M_{i,j} = M_{i+1,j} \oplus M_{i+1,j+1}$ if $n_{j} = n_{j+1} + 1$ (the column $C_{j}$ has been rotated once more than the column $C_{j+1}$).

Let $P$ be the sub-matrix in the statement of the lemma, a picture appears as Figure \[\text{Figure} \] To prove that $P$ is invertible we apply transformations to make it triangular. Note that all entries of the last column are $1$. The first transformation applied to $P$ is as follows. The row $L_{3}$ is left unchanged and the row $L_{j}$ for $2 \leq j \leq k$ is replaced by $L_{j} \oplus L_{j-1}$. All entries of the last column but its top most one become zero. Furthermore, each entry is $P_{i,j}$ is either replaced by either $P_{i,j+1}$ or $P_{i+1,j+1}$ depending on the value $n_{j} = n_{j+1} + 1$. Note also that the new values of the entries still satisfy either $P_{i,j} = P_{i+1,j} \oplus P_{i,j+1}$ or $P_{i,j} = P_{i+1,j} \oplus P_{i+1,j+1}$ depending on the value $n_{j} = n_{j+1} + 1$. The second transformation applied to $P$ is as follows. The rows $L_{1}$ and $L_{2}$ are left unchanged and each row $L_{i}$ for $3 \leq i \leq k$ is replaced by $L_{i} \oplus L_{i-1}$. All entries of the second to last column but its two topmost ones are now zero. At step $n$ for $1 \leq n < k$, rows $L_{1}, \ldots, L_{n}$ are left unchanged and each row $L_{i}$ for $n + 1 \leq i \leq k$ is replaced by $L_{i} \oplus L_{i-1}$. After applying all these transformations for $1 \leq n < k$, each entry $P_{i,j}$ for $i + j = k + 1$ satisfies $P_{i,j} = 1$ and each entry $P_{i,j}$ for $i + j > k + 1$ satisfies $P_{i,j} = 0$. It follows that the determinant of $P$ is $1$ and that the matrix $P$ is invertible. \[\square \]

We now introduce the notions of upper and lower border of a matrix $M_{d}^{n_{1},\ldots,n_{m}}$. Let $m = 2^{d}$ for some $d \geq 0$ and let $M$ be one matrix $M_{d}^{n_{1},\ldots,n_{m}}$. An entry $M_{i,j}$ for $1 \leq i,j \leq m$ is said to be in the upper border (respectively lower border) of $M$ if $M_{i,j} = 1$ and $M_{i,j} = 0$ for all $k = 1, \ldots, i - 1$ (respectively below). For instance, the upper border of the matrix $M_{d}$ is the first row and its lower border is the main diagonal. The following pictures in boldface the upper and lower borders of the matrix $M_{3}^{3,3,2,1,1,1,0,0}$:

$$M_{3}^{3,3,2,1,1,1,0,0} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 
\end{pmatrix}$$

We gather now some easy facts about the upper and lower borders of a matrix $M_{d}^{n_{1},\ldots,n_{m}}$. Both borders start in the unique entry 1 of the first column. The upper border ends in the top most entry of the last column and the lower border ends in the bottom most entry of the last column. The upper border only uses either East or North-East steps and the lower border only uses either East or South-East steps. The upper border uses a East step from column $C_{j}$ to column $C_{j+1}$ if $n_{j} - n_{j+1} = 0$ and uses a North-East step if $n_{j} - n_{j+1} = 1$. Furthermore, whenever the upper border uses an East (respectively North-East) step to go from one columns to its
right neighbour, the lower border uses a South-East (respectively East) step. This is due to the fact that the distance from the upper border to the lower border in the $i$-th column is $i - 1$.

$$M = \begin{pmatrix} k & P & \cdots & 0 \\ k & & & \\ \vdots & & & \\ k & & & \\ m & & & \end{pmatrix}$$

**Figure 2.** Position of the sub-matrix $P$ in $M$ in Lemma 5.

Due to the symmetry in the matrix $M_d$ Lemma 4 applies also to the sub-matrices of $M_d$ obtained by selecting the first row. Since this symmetry is lost for matrices $M_{d^{n_1,\ldots,n_m}}$, we need the following lemma which accounts for the rotations made to the columns in $M_d$ to obtain $M_{d^{n_1,\ldots,n_m}}$.

**Lemma 5.** Let $M$ be one of the matrix $M_{d^{n_1,\ldots,n_m}}$. Let $k$ be an integer such that $1 \leq k \leq 2^d$. The $k \times k$-sub-matrix obtained by selecting $k$ consecutive rows and $k$ consecutive columns in such a way that such that its top right entry lies in the upper border of $M$ is invertible.

**Proof.** The proof is similar to that of Lemma 4. Let $P$ be the sub-matrix in the statement of the lemma, a picture appears in Figure 2. We apply transformations to the sub-matrix $P$ to put it in a nice form such that the determinant is easy to compute. Just to fix notation, we suppose that the sub-matrix $P$ is obtained by selecting rows $L_{r+1}, \ldots, L_{r+k}$ and columns $C_{s+1}, \ldots, C_{s+k}$. The hypothesis is that the entry $M_{r+k,s+k}$ is in the upper border of $M$. Note that the upper borders of $M$ and $P$ coincide inside $P$. We denote by $j_1, \ldots, j_t$ the indices of the columns of $P$ in $1, \ldots, k$ which are reached by a North-East step of the upper border. This means that $j_1, \ldots, j_t$ is the sequences of indices $j$ such that $n_{s+j} - n_{s+j+1} = 1$. By convention, we set $j_0 = 1$, that is, the index of the first column of $P$.

The first transformation applied to the matrix $P$ is the following. The columns $C_1, \ldots, C_{j_1-1}$ and $C_k$ are left unchanged and each column $C_j$ for $j_1 \leq j \leq k-1$ is replaced by $C_j$ and $C_{j+1}$. All entries of the first row but its right most one become zero.

Furthermore, each entry $P_{i,j}$ for $j_1 \leq j \leq k-1$ is replaced by $P_{i,j+1}$. The second transformation applied to the matrix $P$ is the following. The columns $C_1, \ldots, C_{j_1-1}$ and $C_{k-1}, C_k$ are left unchanged and each column $C_j$ for $j_1 \leq j \leq k-2$ is replaced by $C_j$ and $C_{j+2}$. The first row remains unchanged and all entries of the second row but the last two become $0$. We apply $t$ transformations like this using successively $j_1, j_1, \ldots, j_1$. Then $k-t-1$ further steps are made using then $k_0 = 1$ each time. After applying all these transformations, each entry $P_{i,j}$ for $i+j = \ell+1$ satisfies $P_{i,j} = 1$ and each entry $P_{i,j}$ for $i+j < \ell+1$ satisfies $P_{i,j} = 0$. It follows that the determinant of $P$ is $1$ and that the matrix $P$ is invertible. \[\square\]

For a word $w$ we write $w^n$ to denote the word given by concatenation of $n$ copies of $w$. The following lemma states that each $(k,m)$-nested perfect necklace can be transformed into another $(k,m)$-nested perfect necklace which starts with $0^n$.

**Lemma 6.** Let $w$ be a word of length $m2^k$ and let $z$ be a word of length $m$. The word $w$ is a $(k,m)$-nested perfect necklace if and only if the word $w \oplus z^{2^k}$ is a $(k,m)$-nested perfect necklace.
Proof. Note first that both words $u$ and $z^2$ have a length of $m2^k$. Let $w'$ be the word $w \oplus z^2$. Let $\ell$ be an integer such that $1 \leq \ell \leq k$ and let $v'$ be a block of $w'$ of length $m2^\ell$ starting at a position $j$ congruent 1 modulo $2^\ell$. The corresponding block of $w$ at the same position $j'$ is of course $v = v' \oplus z^2$. By hypothesis, this later block $v$ is a $(\ell, m)$-nested perfect necklace. We claim that $v'$ is also a $(\ell, m)$-nested perfect necklace.

Let $i$ be such that $1 \leq i \leq m$ and let $u'$ any word of length $\ell$. Let $t$ be the block of $zz$ of length $\ell$ starting at position $i$ and consider the word $u = u' \oplus t$. This word $u$ has an occurrence in the necklace $v$ at a position $j'$ congruent to $i$ modulo $m$. It follows that $u' = u \oplus t$ has an occurrence at the same position $j'$ in $v'$. Since each word $u$ has such an occurrence for each possible $i$ and $v'$ has length $m2^\ell$, $v'$ is a $(\ell, m)$-nested perfect necklace.

We can now prove that the all the $(k, m)$-affine necklaces are $(k, m)$-nested perfect necklaces.

Proposition 7. Let $k, m, d$ be integers such that $d \geq 0$, $m = 2^d$ and $1 \leq k \leq m$. Each $(k, m)$-affine necklace is a $(k, m)$-nested perfect necklace.

The proof of Proposition 7 follows and extends that of [9] Lemma 5 but we use a different notation.

Proof. It suffices of course to prove the result for $k = m$. By Lemma 6 it may be assumed that the vector $z$ in the definition of affine necklaces is the zero vector. Let $M$ be one of the matrices $M_{d^1 \ldots d^m}$, let $w_1, \ldots, w_{2^m}$ be the enumeration in lexicographic order of all words of length $m$ over $\mathbb{F}_2$ and suppose that the $(m, m)$-affine necklace $w$ is the concatenation $(Mw_1)(Mw_2) \cdots (Mw_{2^m})$. Let $k$ be an integer such that $1 \leq k \leq m$ and let $w'$ be a block of $w$ of length $m2^k$ starting at a position congruent to 1 modulo $m2^k$. The word $w'$ is thus equal to a concatenation of the form $(Mw_{p2^k+1}) \cdots (Mw_{(p+1)2^k})$ for some fixed integer $p$ such that $0 \leq p \leq 2^{m-k} - 1$. We claim that $w'$ is a $(k, m)$-perfect necklace.

To prove the claim, it must be shown that for each integer $\ell$ such that $0 \leq \ell < m$, each word $u$ of length $k$ has exactly one occurrence in $w$ with a starting position congruent to $\ell + 1$ modulo $m$ (we write $\ell + 1$ rather than $\ell$ because positions are numbered from 1). We now suppose that the word $u$ and the integer $\ell$ such that $0 \leq \ell < m$ are fixed. We distinguish two cases depending on whether $\ell + k \leq m$ or not.

We first suppose that $k + \ell \leq m$. It follows that the wanted occurrence of $u$ must be fully contained in a single word $Mw_{p2^k+q}$ for $1 \leq q \leq 2^k$. More precisely it must lie in the positions $\ell + 1, \ldots, \ell + k$ of $Mw_{p2^k+q}$. In that case, the claim boils down to showing that there is exactly one integer $q$ such that $u$ occurs in positions $\ell + 1, \ldots, \ell + k$ of $Mw_{p2^k+q}$. Let us recall that $p$ is fixed and that $q$ ranges in $1, \ldots, 2^k$. Since $w_i$ is the base 2 expansion of $i - 1$ with $m$ bits, $w_{p2^k+q}$ can factorized $x_p y_q - 1$ where $x_p$ and $y_q - 1$ are the base 2 expansions of $p$ and $q - 1$ with $m - k$ and $k$ bits.

$$M = \begin{pmatrix} N & P \end{pmatrix}$$

The occurrence of $u$ in $w_{p2^k+q}$ is now translated into linear equations by introducing the following two matrices $N$ and $P$ (see above). Let $N$ and $P$ be the following sub-matrices of the matrix $M$. The $k \times m - k$ matrix $N$ is obtained by
selecting the $k$ rows $L_{ℓ+1}, \ldots, L_{ℓ+k}$ and the $m - k$ columns $C_1, \ldots, C_{m-k}$. The $k \times k$ matrix $P$ is obtained by selecting the same $k$ rows $L_{ℓ+1}, \ldots, L_{ℓ+k}$ and the $k$ columns $C_{m-k+1}, \ldots, C_m$. The word $u$ occurs in the positions $ℓ + 1, \ldots, ℓ + k$ of $Mw_{p2^k+q}$ if and only if $u = Nx_p + Py_{q-1}$ where words $u$, $x_p$ and $y_{q-1}$ are considered as columns vectors of respective dimensions $k$, $m - k$ and $k$. Since $x_p$ is fixed and $P$ is invertible by Lemma 3 there is exactly one solution for $y_{q-1}$ and thus one solution for $q$. This proves the claim when $k + ℓ \leq m$.

We now suppose that $ℓ + k > m$. The wanted occurrence of $u$ must then overlap two consecutive words $Mw_{p2^k+q}$ and $Mw_{p2^k+q+1}$ where $p2^k + q + 1$ should be understood as $p2^k + 1$ if $q = 2^k$. Let us write $u = u_1u_2$ where $u_1$ and $u_2$ have length $m - ℓ$ and $ℓ + k - m$. The wanted occurrences exist if $u_1$ occurs at positions $ℓ + 1, \ldots, m$ of $Mw_{p2^k+q}$ and $u_2$ occurs at positions $1, \ldots, ℓ + k - m$ of $Mw_{p2^k+q+1}$ with the same convention for $p2^k + q + 1$. As in the previous case, these occurrences are translated into linear equations. For that purpose, we introduce the following four matrices.

$$M = \begin{pmatrix} N_2 & P_2 \\ N_1 & P_1 \end{pmatrix}$$

The matrices $N_1$ and $P_1$ are obtained by selecting the rows $L_{ℓ+1}, \ldots, L_m$ and the columns $C_1, \ldots, C_{m-k}$ for $N_1$ and $C_{m-k+1}, \ldots, C_m$ for $P_1$. The matrices $N_2$ and $P_2$ are obtained by selecting the rows $L_1, \ldots, L_{ℓ+k-m}$ and the columns $C_1, \ldots, C_{m-k}$ for $N_2$ and $C_{m-k+1}, \ldots, C_m$ for $P_2$ (see above). The two words $w_{p2^k+q}$ and $w_{p2^k+q+1}$ are then factorized $w_{p2^k+q} = x_p y_{q-1}$ and $w_{p2^k+q+1} = x_p y_{q}$ where $x_p$ is the base 2 expansion of $p$ with $2^m - k$ bits and words $y_{q-1}$ and $y_q$ are the base 2 expansions of $q - 1$ and $q$ (understood as 0 if $q = 2^k$) with $2^k$ bits. The occurrences of $u_1$ and $u_2$ do exist as wanted if and only if these two equalities hold,

$$u_1 = N_1 x_p + P_1 y_{q-1},$$
$$u_2 = N_2 x_p + P_2 y_q.$$

Notice that the first equation involves $y_{q-1}$ while the second one involves $y_q$. These two words are strongly related in the sense that each one determines the other. For each $i$ such that $0 \leq i \leq k$, the $i$ right most bits of either $y_{q-1}$ or $y_q$ determine the $i$ right most bits of the other. This is due to the fact that either adding or subtracting 1 can be performed on the bits from right to left. For that reason, we will show that the equations $u_1 = N_1 x_p + P_1 y_{q-1}$ and $u_2 = N_2 x_p + P_2 y_q$ have a unique solution in $q$ by successively computing the bits of $q - 1$ and $q$ from right to left.

We actually describe a strategy for solving the two equations. This strategy is based of the upper and lower borders of the matrix $M$. The main ingredient is that between two consecutive columns $C_j$ and $C_{j+1}$, one of the two borders uses a step which is not horizontal, that is, either North-East for the upper border or South-East for the lower border.

The rightmost bit of $y_{q-1}$ and $y_q$ can be found as follows. Either the upper border or the lower border makes a non horizontal step from $C_{m-1}$ to $C_m$. It means that either the first row or the last row of $M$ has the form $(0, \ldots, 0, 1)$. This row can be used to find the right most bit of $y_{q-1}$ and $y_q$ as it is the first row the equation $u_1 = N_1 x_p + P_1 y_{q-1}$ or the last row of the equation $u_2 = N_2 x_p + P_2 y_q$. The second right most bit of $y_{q-1}$ and $y_q$ can be found as follows. Either the upper
have been used. Suppose that the left most rows of the first equation have not been used. Considering the form stated in the lemma. And for each matrix \( M \) of the form \( M^{d_1\ldots d_n} \), \( Mu_0 \) and \( Mw_1 \) are respectively equal to \( 0^n \) and \( 1^n \). This proves the last claim.

4. Nested perfect necklaces are affine necklaces

We shall now show that all \((k, m)\)-nested perfect necklaces are \((k, m)\)-affine necklaces. Since the other inclusion has been already proved, it suffices to show that they have the same cardinality. The next lemma shows that for \( k = 1 \), they coincide.

**Lemma 8.** The \((1, m)\)-nested perfect necklaces are the words of the form \( \frac{ww'}{w} \) where \( w \) and \( w' \) are two words of length \( m \) satisfying \( w' = w \oplus 1^m \). Furthermore, they are all affine.

**Proof.** It is straightforward that \((1, m)\)-nested perfect necklaces are the words of the form stated in the lemma. And for each matrix \( M \) of the form \( M^{d_1\ldots d_n} \), \( Mu_0 \) and \( Mw_1 \) are respectively equal to \( 0^n \) and \( 1^n \). This proves the last claim.

The next lemma provides the number of \((m, m)\)-affine necklaces. It shows that \((m, m)\)-affine necklaces obtained by the different choices of the matrix \( M^{d_1\ldots d_n} \) and of the vector \( z \) are indeed different.

**Lemma 9.** Let \( m = 2^d \) for some \( d \geq 0 \). There are exactly \( 2^{2m-1} \) different \((m, m)\)-affine necklaces.

**Proof.** There are exactly \( 2^{m-1} \) matrices \( M^{d_1\ldots d_n} \). Indeed, the sequence \( n_1, \ldots, n_m \) is fully determined by the sequence \( n_1 = n_2 = \ldots = n_{m-1} = n_m \) of \( m-1 \) differences which take their value in \( \{0, 1\} \). There are also \( 2^m \) possible values for the word \( z \) in \( \mathbb{F}_2^m \). This proves that the number of \((m, m)\)-affine necklaces is bounded by \( 2^{2m-1} \).

It remains to show that two \((m, m)\)-affine necklaces obtained for two different pairs \((M, z)\) and \((M', z')\) are indeed different. Let \( w_1, \ldots, w_{2^d} \) be the enumeration in lexicographic order of all words of length \( m \) over \( \mathbb{F}_2 \). Let \( M \) and \( M' \) be two matrices of the form \( M^{d_1\ldots d_n} \). Let \( z \) and \( z' \) be two words over \( \mathbb{F}_2 \) of length \( m \) and let \( u_i = w_i \oplus z \) and \( u'_i = w_i \oplus z' \) for \( 1 \leq i \leq 2^m \). Let \( w \) and \( w' \) be the two concatenations \((Mu_1)\cdots(Mu_{2^d})\) and \((M'u'_1)\cdots(M'u'_{2^d})\). We claim that if \( w = w' \), then \( M = M' \) and \( z = z' \).

We suppose that \( w = w' \). Since both matrices \( M \) and \( M' \) are invertible by Lemma 3 \( Mu_i \) (respectively \( M'u'_i \)) is the zero vector if and only if \( u_i \) (respectively \( u'_i \)) is the zero vector, that is, \( z = w_i \) (respectively \( z' = w_i \)). It follows then that \( z = z' \) and thus \( u_i = u'_i \) for \( 1 \leq i \leq 2^m \). Note that the vector \( u_i \) ranges over all possible vectors of length \( m \). If \( Mu_i = M'u_i \) for all \( 1 \leq i \leq 2^m \), then \( M = M' \).

Lemma will show how \((k, m)\)-affine necklaces can be concatenated with \((k, m)\)-affine necklaces to get \((k + 1, m)\)-perfect necklaces. The next two lemmas are
intermediate steps towards the proof. The first states that each rotation of a column of $M_d$ is a linear combination of some columns to its right.

**Lemma 10.** Let $d \geq 0$ be integer and let $(C_1, \ldots, C_{2d})$ be the columns of the matrix $M_d$. For any integers $i, k$ such that $1 \leq i \leq 2^d$ and $k \geq 0$, the vector $\sigma^k(C_i) + C_i$ is equal to a linear combination $\bigoplus_{j=i+1}^{2^d} b_j C_j$ where $b_j \in \mathbb{F}_2$.

**Proof.** The result is proved by the induction on the difference $2^d - i$. If $i = 2^d$, the result holds trivially because $\sigma(C_{2^d}) = C_{2^d}$. Assume that $i < 2^d$ is fixed. The proof is now by induction on the integer $k$. The result for $k = 0$ is void. By Lemma 3 applied to the column $C_{i+2^d}$ of the matrix $M_{d+1}$, the equality $\sigma(C_i) + C_i = C_{i+1}$ holds. We apply Lemma 3 to the column $C_{i+2^d}$ of the matrix $M_{d+1}$ because this column has period $2^d$ and its first half is the column $C_i$ of $M_d$. This proves the result for $k = 1$. Suppose now that the result is true for some $k \geq 1$. Applying $\sigma$ to both terms of the equality and replacing first $\sigma(C_i)$ by the value $C_i + C_{i+1}$ and second each $\sigma(C_j)$ by the value given by the induction hypothesis gives the result for $k + 1$. \qed

The next lemma shows for each $(k, m)$-affine necklace, there is just one possible way of rotating it to get another $(k, m)$-affine necklace.

**Lemma 11.** Let $d, m, k$ and $p$ be integers such that $d \geq 0$, $m = 2^d$, $1 \leq k \leq m$ and $p \geq 0$. Let $w$ be a $(k, m)$-affine necklace. If $m$ divides $p$ and $\sigma^p(w)$ is also a $(k, m)$-affine necklace, then $p \equiv m2^k - 1 \mod |w|$.

**Proof.** Since $|w| = m2^k$ and $\sigma^{|w|}(w) = w$, we may assume that $0 \leq p \leq m2^k$. The result holds if either $p = 0$ or $p = m2^k$. Therefore we assume that $1 \leq p \leq m2^k - 1$. Let $w_1, \ldots, w_{2^d}$ be the enumeration in lexicographic order of all words over $\mathbb{F}_2$ of length $m$. Since $w$ is an affine necklace, it is a concatenation $(M_{u_1}) \cdots (M_{u_{2^d}})$ where $M$ is a matrix $M_{u_1}^{u_{2^d}}$ and $u_i$ is equal to $w_i \oplus z$ for each integer $1 \leq i \leq 2^d$ and for some fixed vector $z$. Since $\sigma^p(w)$ is also an affine necklace, it is a concatenation $(M'_{u_1}) \cdots (M'_{u_{2^d}})$ where $M'$ is a matrix $M_{u_1}^{u_{2^d}}$ and $u_i'$ is equal to $w_i \oplus z'$ for each integer $1 \leq i \leq 2^d$ and for some other fixed vector $z'$. For each $\ell$ such that $0 \leq \ell < d$, the vector $M'_{u_1} \oplus M'_{u_{1+2^\ell}}$ is equal to the column $C_{m-\ell}$ of the matrix $M'$. Since $M$ and $M'$ are two matrices of the form $M_{u_1}^{u_{2^d}}$, the column $C_{m-\ell}$ of $M'$ is equal to $\sigma^\ell(C_{m-\ell})$ where $C_{m-\ell}$ is the corresponding column of $M$ and $t$ is some integer. Since $m$ divides $p$, the necklace $\sigma^p(w)$ is equal to
\[(Mw_1') \cdots M(w_{2^d}')(Mw_1') \cdots (Mw_{1-1}')\]
where $i = 1 + p/m$. We consider the word $w_i$ which is the base 2 expansion of $i - 1$ with $m$ bits. Let $\ell$ be the greatest integer such that $2^\ell$ divides $i - 1 = p/m$. The integer $i - 1$ is equal to $2^\ell(2r + 1)$ for some non-negative integer $r$. We claim that $\ell = k - 1$.

Suppose by contradiction that $\ell < k - 1$. The integer $r$ satisfies thus $r \geq 1$. The word $w_i$ is then equal to $0^{m-k}u10^r$ where $0^{m-k}$ is the block leading zeros due to $i \leq 2^k$ and $u$ is the base 2 expansion of $r$ with $k - \ell - 1$ digits. We now consider the word $w_{i+2^\ell}$. This word is equal to $0^{m-k}u10^{\ell+1}$ where $u$ is the base 2 expansion of $r + 1$ with $k - \ell - 1$ digits. Computing $Mw_i \oplus Mw_{i+2^\ell}$ gives $C_{2^\ell+1} + R$ where $R$ is a non-zero linear combination of $C_{m-k}, \ldots, C_{m-\ell-1}$. This linear combination $R$ cannot be equal to zero because the words $u$ and $u'$ are different. The vector $Mw_i \oplus Mw_{i+2^\ell}$ is also equal to $M'_{u_1} \oplus M'_{u_{1+2^\ell}} = C_{m-\ell} = \sigma^\ell(C_{m-\ell})$. By Lemma 10 this vector is equal to $C_{m-\ell} + R'$ where $R'$ is a linear combination of $C_{m-\ell+1}, \ldots, C_m$. This is a contradiction: the equality $R = R'$ is impossible because, by Lemma 4 the matrix $M$ is invertible. \qed
We are now ready to show that each \((k, m)\)-affine necklace can be extended by at most two \((k, m)\)-affine necklaces to get a \((k + 1, m)\)-perfect necklace.

**Lemma 12.** Let \(m = 2^d\) for some \(d \geq 0\) and let \(k\) be an integer such that \(1 \leq k \leq m\). Let \(w\) be a \((k, m)\)-affine necklace. There are at most two \((k, m)\)-affine necklaces \(w'\) such that \(ww'\) is a \((k + 1, m)\)-perfect necklace.

**Proof.** We use the characterization of \((k, m)\)-perfect necklaces as cycles in appropriate graphs \(G_k\) (which variant of de Bruijn graphs) given in \[\text{I}\]. Consider the directed graph \(G_k\) whose vertex set is \(\mathbb{F}_2^k \times \{1, \ldots, m\}\) and whose transitions are defined as follows. There is an edge in \(G_k\) from \((u, i)\) to \((u', i')\) if first there are two symbols \(a\) and \(b\) in \(\mathbb{F}_2\) such that \(ua = bu'\) and second \(i' \equiv i + 1 \mod m\). The condition on \(u\) and \(u'\) means that \(u\) and \(u'\) are respectively the prefix and the suffix of length \(k\) of the word \(v = au = bu'\) of length \(k + 1\). Therefore, the edges of the graph \(G_k\) can be identified with the words of length \(k + 1\) over \(\mathbb{F}_2\). Note that each vertex of \(G_k\) has two incoming and two outgoing edges. It follows from the definition of the graph \(G_k\), that each \((k, m)\)-nested perfect necklace is identified to a Hamiltonian cycle in \(G_k\) and that each \((k + 1, m)\)-nested perfect necklaces is identified to an Eulerian cycle in \(G_k\).

Let \(w\) be a \((k, m)\)-affine necklace. Then \(w\) determines a Hamiltonian cycle \(C\) in \(G_k\). Since \(C\) visits each node of \(G_k\) exactly once, it uses one outgoing edge of each node. Any \((k, m)\)-nested perfect necklace \(w'\) such that \(ww'\) is a \((k + 1, m)\)-nested perfect necklace induces an Hamiltonian \(C'\) cycle which cannot use an edge of \(C\). Otherwise, it would not be possible to build an Eulerian cycle from \(C\) and \(C'\) and \(ww'\) would not be a \((k + 1, m)\)-nested perfect necklace. If there is no \((k, m)\)-affine necklace \(w'\) such that \(ww'\) is a \((k + 1, m)\)-nested perfect necklace the lemma trivially holds. Suppose now that there exists at least one such \(w'\). Since the graph \(G_k \setminus C\) has only one outgoing edge from any vertex, any \((k, m)\)-nested perfect necklace \(w''\) such that such that \(ww''\) is a \((k + 1, m)\)-nested perfect necklace must be of the form \(\sigma^p(w')\) for some integer \(p \geq 0\). Since both Hamiltonian cycles \(C\) and \(C''\) induced by \(w'\) and \(w''\) must start from a vertex in \(\mathbb{F}_2^k \times \{1\}\), it follows that \(m\) divides \(m\). By Lemma \[\text{II}\] the only possible choices for \(p\) are 0 and \(m2^{k-1}\). This proves that there is at most one such \(w''\) different from \(w'\). \(\square\)

We can now give the number of \((k, m)\)-affine necklaces.

**Proposition 13.** Let \(m = 2^d\) for some \(d \geq 0\). For each integer \(k\) such that \(1 \leq k \leq m\), the number of \((k, m)\)-affine necklaces is exactly \(2^{k+m-1}\).

**Proof.** We assume the integer \(m\) to be fixed and we let \(t_k\) denote the number of \((k, m)\)-affine necklaces. By Lemma \[\text{S}\] \(t_1\) is equal to \(2^m\) and by Lemma \[\text{H}\] \(t_m\) is equal to \(2^{2m-1}\). It follows from Lemma \[\text{I2}\] that \(t_{k+1} \leq 2t_k\) for each integer \(k\) such that \(1 \leq k < m\). None of these inequalities can be strict because otherwise \(t_m\) would be strictly less than \(2^{2m-1}\). So, for each integer \(k\) such that \(1 \leq k \leq m\), \(t_{k+1} = 2t_k\), hence, \(t_k = 2^{k+m-1}\). \(\square\)

The results above allows us to prove the wanted inclusion.

**Proposition 14.** Each \((k, m)\)-nested perfect necklace is a \((k, m)\)-affine necklace.

**Proof.** Fix \(m\), let \(s_k\) be the number of \((k, m)\)-nested perfect necklaces and let \(t_k\) be the number of \((k, m)\)-affine necklaces. By Proposition \[\text{S}\] \(s_k \leq t_k\) holds for each integer \(k\) such that \(1 \leq k \leq m\). To prove the statement, it suffices to prove that \(s_k = t_k\) for each integer \(k\) such that \(1 \leq k \leq m\). We prove it by induction on \(k\). By Lemma \[\text{S}\] \(s_1 = t_1 = 2^m\). We now suppose \(s_k = t_k\) and we prove that \(s_{k+1} = t_{k+1}\). Each \((k + 1, m)\)-nested perfect necklace can be written as \(ww'\) where \(w\) and \(w'\) are two \((k, m)\)-nested perfect necklaces. Since \(s_k = t_k\), \(w\) and \(w'\) are also \((k, m)\)-affine
there are at most two possible choices of $w'$ for each $w$. This proves that $t_{k+1} \leq 2t_k$. Since $s_{k+1} = 2s_k$ by Proposition 13 and $s_{k+1} \leq t_{k+1}$, the equality $s_{k+1} = t_{k+1}$ holds. □

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References

[1] N. Álvarez, V. Becher, P. A. Ferrari, and S. A. Yuhjtman. Perfect necklaces. Advances of Applied Mathematics, 80:48–61, 2016.
[2] Y. Bugeaud. Distribution modulo one and Diophantine approximation, volume 193 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 2012.
[3] N. G. de Bruijn. A combinatorial problem. Koninklijke Nederlandse Akademie v.Wetenschappen, 49:758–764, 1946. Indagationes Mathematicae 8 (1946) 461-467.
[4] M. Drmota and R. Tichy. Sequences, Discrepancies and Applications. Lecture Notes in Mathematics, Vol. 1651. Springer-Verlag, 1997.
[5] H. Faure. Discr´eance de suites associ´ees `a un syst`eme de num´eration (en dimension $s$). Acta Arithmetica, 41, 1982.
[6] N.M. Korobov. Numbers with bounded quotient and their applications to questions of diophantine approximation. Izv. Akad. Nauk SSSR, Ser. Mat., 19:361–380, 1955.
[7] L. Kuipers and H. Niederreiter. Uniform distribution of sequences. Dover Publications, Inc., New York, 2006.
[8] M. B. Levin. On the upper bounds of discrepancy of completely uniform distributed and normal sequences. Abstracts American Mathematical Society, 16:556–557, 1995. AMS-IMU joint meeting, Jerusalem, Israel, May 24–26, 1995.
[9] M. B. Levin. On the discrepancy estimate of normal numbers. Acta Arithmetica, 88(2):99–111, 1999.
[10] W. Schmidt. Irregularities of distribution. vii. Acta Arithmetica, 21:45–50, 1972.

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