Generalised differences and multiplier operators in $L^2(\mathbb{R})$

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Abstract
Let $\alpha, \beta \in \mathbb{R}$ be given and let $s \in \mathbb{N}$ also be given. Let $\delta_x$ denote the Dirac measure at $x \in \mathbb{R}$, and let $\ast$ denote convolution. If $f \in L^2(\mathbb{R})$, and if there are $u \in \mathbb{R}$ and $g \in L^2(\mathbb{R})$ such that

$$f = \left[ \left( e^{iu(\alpha - \beta)} + e^{-iu(\alpha - \beta)} \right) \delta_0 - \left( e^{iu(\alpha + \beta)} \delta_u + e^{-iu(\alpha + \beta)} \delta_u \right) \right]^s \ast g,$$

then $f$ is called a generalised $(\alpha, \beta)$-difference of order $2s$, or simply a generalised difference. We denote by $D_{\alpha, \beta, s}(\mathbb{R})$ the vector space of all functions $f$ in $L^2(\mathbb{R})$ such that $f$ is a finite sum of generalised $(\alpha, \beta)$-differences of order $2s$. It is shown that every function in $D_{\alpha, \beta, s}(\mathbb{R})$ is a sum of $4s + 1$ generalised $(\alpha, \beta)$-differences of order $2s$. If we let $\hat{f}$ denote the Fourier transform of a function $f \in L^2(\mathbb{R})$, then $D_{\alpha, \beta, s}(\mathbb{R})$ is a weighted $L^2$-space under the Fourier transform, and its inner product $\langle \cdot, \cdot \rangle_{\alpha, \beta, s}$ is given by

$$\langle f, g \rangle_{\alpha, \beta, s} = \int_{-\infty}^{\infty} \left( 1 + \frac{1}{(x-\alpha)^2s(x-\beta)^2s} \right) \hat{f}(x) \overline{\hat{g}(x)} dx.$$

Letting $D$ denote differentiation and letting $I$ denote the identity operator, the operator $(D^2 - i(\alpha + \beta)D - \alpha \beta I)^s$ is bounded and invertible, mapping the Sobolev subspace of order $2s$ of $L^2(\mathbb{R})$ onto the Hilbert space $D_{\alpha, \beta, s}(\mathbb{R})$.

1 Introduction

Let $\mathbb{R}$ denote the set of real numbers, let $\mathbb{T}$ denote the set of complex numbers of modulus 1, and let $G$ denote either $\mathbb{R}$ or $\mathbb{T}$. Note that in some contexts $\mathbb{T}$ may be

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identified with the interval $[0,2\pi)$ under the mapping $t \mapsto e^{it}$ (some comments on this are in [9, page 1034]). Then $G$ is a group and its identity element we denote by $e$, so that $e = 0$ when $G = \mathbb{R}$ and $e = 1$ when $G = \mathbb{T}$. Let $\mathbb{N}$ denote the set of natural numbers, $\mathbb{Z}$ the set of integers, and let $s \in \mathbb{N}$. The Fourier transform of $f \in L^2(G)$ is denoted by $\hat{f}$, and is given by $\hat{f}(n) = (2\pi)^{-1} \int_0^{2\pi} f(e^{it})e^{-int} \, dt$ for $n \in \mathbb{Z}$ (in the case of $\mathbb{T}$), and by the extension to all of $L^2(\mathbb{R})$ of the transform given by $\hat{f}(x) = \int_{-\infty}^{\infty} e^{-ixu} f(u) \, du$ for $x \in \mathbb{R}$ (in the case of $\mathbb{R}$). Let $M(G)$ denote the family of bounded Borel measures on $G$. If $x \in G$ let $\delta_x$ denote the Dirac measure at $x$, and let $*$ denote convolution in $M(G)$.

We call a function $f \in L^2(G)$ a difference of order $s$ if there is a function $g \in L^2(G)$ and $u \in G$ such that $f = (\delta_u - \delta_u)^s g$. The functions in $L^2(G)$ that are a sum of a finite number of differences of order $s$ we denote by $\mathcal{D}_s(G)$. Note that $\mathcal{D}_s(G)$ is a vector subspace of $L^2(G)$. Now in the case of $\mathbb{T}$ it was shown by Meisters and Schmidt [3] that

$$\mathcal{D}_1(\mathbb{T}) = \left\{ f : f \in L^2(\mathbb{T}) \text{ and } \hat{f}(0) = 0 \right\},$$

and that every function in $\mathcal{D}_1(\mathbb{T})$ is a sum of 3 differences of order $s$. It was shown in [6] that, for all $s \in \mathbb{N}$,

$$\mathcal{D}_s(\mathbb{T}) = \mathcal{D}_1(\mathbb{T}) = \left\{ f : f \in L^2(\mathbb{T}) \text{ and } \hat{f}(0) = 0 \right\},$$

and that every function in $\mathcal{D}_s(\mathbb{T})$ is a sum of $2s + 1$ differences of order $s$. It was also shown in [6] that

$$\mathcal{D}_s(\mathbb{R}) = \left\{ f : f \in L^2(\mathbb{R}) \text{ and } \int_{-\infty}^{\infty} \frac{f(x)}{|x|^{2s}} \, dx < \infty \right\},$$

and again, that every function in $\mathcal{D}_s(\mathbb{R})$ is a sum of $2s + 1$ differences of order $s$. Further results related to the work of Meisters and Schmidt in [3] may be found in [1 2 4 7].

The Sobolev space of order $s$ in $L^2(G)$ is the space of all functions $f \in L^2(G)$ such that $D^s(f) \in L^2(G)$, where $D$ denotes differentiation in the sense of Schwartz distributions. Then, $D^s$ is a multiplier operator on $W^s(\mathbb{T})$ with multiplier $(in)^s$, in the sense that $D^s(f)(n) = (in)^s \hat{f}(n)$, for all $f \in W^s(\mathbb{T})$ and $n \in \mathbb{Z}$. Also, $D^s$ is a multiplier operator on $W^s(\mathbb{R})$ with multiplier $(ix)^s$, the sense that $D^s(f)(x) = (ix)^s \hat{f}(x)$, for all $f \in W^s(\mathbb{R})$ and $x \in \mathbb{R}$. Note that $W^s(\mathbb{T})$ is a Hilbert space whose norm $|| \cdot ||_{\mathbb{T}, s}$ derives from the inner product $\langle \cdot, \cdot \rangle_{\mathbb{T}, s}$ where

$$\langle f, g \rangle_{\mathbb{T}, s} = \sum_{n=-\infty}^{\infty} (1 + |n|^{2s}) \hat{f}(n) \hat{g}(n) \, dx.$$ 

Note also that $W^s(\mathbb{R})$ is a Hilbert space whose norm $|| \cdot ||_{\mathbb{R}, s}$ derives from the inner product $\langle \cdot, \cdot \rangle_{\mathbb{R}, s}$ where

$$\langle f, g \rangle_{\mathbb{R}, s} = \int_{-\infty}^{\infty} (1 + |x|^{2s}) \hat{f}(x) \hat{g}(x) \, dx.$$
Using these observations, together with Plancherel’s Theorem, it is easy to verify that
\[ D^s(W^s(\mathbb{T})) = \left\{ f : f \in L^2(\mathbb{T}) \text{ and } \hat{f}(0) = 0 \right\}, \text{ and that} \]
\[ D^s(W^s(\mathbb{R})) = \left\{ f : f \in L^2(\mathbb{R}) \text{ and } \int_{-\infty}^{\infty} \frac{|\hat{f}(x)|^2}{|x|^{2s}} \, dx < \infty \right\}. \]  
(1.3)

(1.4)

In view of (1.3) and (1.4), (1.1) together with (1.2) can be regarded as describing the ranges of \( D^s \) upon \( W^s(\mathbb{T}) \) and \( W^s(\mathbb{R}) \) as spaces consisting of finite sums of differences of order \( s \). Corresponding results have been obtained in \( [8] \) for operators \((D^2 - i(\alpha + \beta)D - \alpha\beta I)^s\) acting on \( W^{2s}(\mathbb{T})\), where \( \alpha, \beta \in \mathbb{Z} \) and \( I \) denotes the identity operator. In this paper, the main aim is to derive corresponding results for the operator \((D^2 - i(\alpha + \beta) - \alpha\beta I)^s\), where \( \alpha, \beta \in \mathbb{R} \), for the non-compact case of \( \mathbb{R} \) in place of the compact group \( \mathbb{T} \). Note that, in general, the range of a multiplier operator depends upon the behaviour of Fourier transforms at or around the zeros of the multiplier of the operator, as in (1.3) and (1.4). Note also that \((D^2 - i(\alpha + \beta)D - \alpha\beta I)^s\) is a multiplier operator whose multiplier is, in the case of \( \mathbb{R} \), \(-(x - \alpha)(x - \beta)\) with zeros at \( \alpha \) and \( \beta \).

Given \( \alpha, \beta \in \mathbb{R} \) and \( s \in \mathbb{N} \), a generalised \((\alpha, \beta)\)-difference of order \( 2s \) is a function \( f \in L^2(G) \) such that for some \( g \in L^2(\mathbb{R}) \) we have
\[ f = \left[ \left( e^{i\alpha} + e^{-i\alpha} \right) \delta_0 - \left( e^{i\beta} + e^{-i\beta} \right) \delta_u + e^{-i\beta} \delta_{-u} \right]^s \ast g. \]
(1.5)

It may be called also an \((\alpha, \beta)\)-difference of order \( 2s \), or simply a generalised difference. In the case when \( G = \mathbb{T} \), we restrict \( \alpha \) and \( \beta \) to be in \( \mathbb{Z} \) and \( \mathbb{T} \) is identified with \([0, 2\pi]\). The vector space of functions in \( L^2(G) \) that can be expressed as some finite sum of \((\alpha, \beta)\)-differences of order \( 2s \) is denoted by \( D_{\alpha, \beta, s}(G) \). Thus, \( f \in D_{\alpha, \beta, s}(\mathbb{R}) \) if and only if there are \( m \in \mathbb{N}, u_1, u_2, \ldots, u_m \in \mathbb{R} \) and \( f_1, f_2, \ldots, f_m \in L^2(\mathbb{R}) \) such that
\[ f = \sum_{j=1}^{m} \left[ \left( e^{iu_j} + e^{-iu_j} \right) \delta_0 - \left( e^{iu_j} + e^{-iu_j} \right) \delta_u + e^{-iu_j} \delta_{-u} \right]^s \ast f_j. \]

We prove that if \( f \in L^2(\mathbb{R}), f \in D_{\alpha, \beta, s}(\mathbb{R}) \) if and only if \( \int_{-\infty}^{\infty} (x - \alpha)^{-2s}(x - \beta)^{-2s}|\hat{f}(x)|^2 < \infty \), in which case \( f \) is a sum of \( 4s + 1 \) \((\alpha, \beta)\)-differences of order \( 2s \). It follows that \( D_{\alpha, \beta, s}(\mathbb{R}) \) is the range of \((D^2 - i(\alpha + \beta) - \alpha\beta I)^s(\mathbb{R})\) on \( W^{2s}(\mathbb{R}) \), and is a weighted \( L^2 \)-space under the Fourier transform in which the inner product of \( f \) and \( g \) is \( \int_{-\infty}^{\infty} (1 + (x - \alpha)^{-2s}(x - \beta)^{-2s}) \hat{f}(x)\hat{g}(x) \, dx \).

Now, if \( \alpha, \beta \in \mathbb{Z} \), and if we take an \((\alpha, \beta)\)-difference \( f \) in \( L^2(\mathbb{T}) \) as in \( 1.7 \) with \( G = \mathbb{T} \), then \( \hat{f}(\alpha) = \hat{g}(\beta) = 0 \). In \( [8] \) it is proved that if \( f \in L^2(\mathbb{T}) \) and \( \hat{f}(\alpha) = \hat{g}(\beta) = 0 \), then \( f \) is a sum of \( 4s + 1 \) \((\alpha, \beta)\)-differences of order \( s \). Thus, the results obtained here extend the results obtained in \( [8] \), for the compact case of the circle group \( \mathbb{T} \), to the non-compact case of \( \mathbb{R} \). The techniques used here develop the approach in \( [8] \), so as to deal with the additional complexities in going from \( \mathbb{T} \) to \( \mathbb{R} \).
2 Further notations and background

First we need some notions relating to partitions of an interval.

**Definitions.** If $J$ is an interval, $\lambda(J)$ denotes its length. A closed-interval partition is a sequence $R_0, R_1, \ldots, R_{r-1}$ of closed intervals of positive length such that $r = 1$ or, when $r \geq 2$, the right hand endpoint of $R_j$ is the left hand endpoint of $R_{j+1}$ for all $j = 0, 1, \ldots, r-2$. We may refer to such a closed-interval partition as $\{R_0, R_1, \ldots, R_{r-1}\}$, where we understand that the intervals $R_j$ may be rearranged so as to get a sequence forming a closed-interval partition. In this case if we put $J = \bigcup_{j=0}^{r-1} R_j$ then $\{R_0, R_1, \ldots, R_{r-1}\}$ may be called a closed-interval partition of $J$. If $\{R_0, R_1, \ldots, R_{r-1}\}$ is a closed-interval partition and $\{S_0, S_1, \ldots, S_{s-1}\}$ is also a closed-interval partition, then $\{R_j \cap S_k : 0 \leq j \leq r-1, 0 \leq k \leq s-1 \text{ and } \lambda(R_j \cap S_k) > 0\}$ is a closed-interval partition, and we call it the refinement of the closed-interval partitions $\{R_0, R_1, \ldots, R_{r-1}\}$ and $\{S_0, S_1, \ldots, S_{s-1}\}$. Finally, if $A$ is a set, $A^c$ will denote its complement.

**Lemma 2.1** Let $J$ be a closed interval with $\lambda(J) > 0$. Let $R_0, R_1, \ldots, R_{r-1}$ be $r$ intervals in a closed-interval partition of a closed interval $J$. Let $S_0, S_1, \ldots, S_{s-1}$ be $s$ intervals in a closed-interval partition of a closed interval $K$, and assume that $\lambda(R_j \cap S_k) \neq 0$ for at least one pair $j, k$. Then the refinement of $R_0, R_1, \ldots, R_{r-1}$ and $S_0, S_1, \ldots, S_{s-1}$ is a closed-interval partition of $J \cap K$ and it has at most $r+s-1$ elements.

**Proof.** It is easily checked that the refinement of $R_0, R_1, \ldots, R_{r-1}$ and $S_0, S_1, \ldots, S_{s-1}$ is a closed-interval partition of $J \cap K$. Now, for any $r, s$ and closed-interval partitions $\mathcal{P} = \{R_0, R_1, \ldots, R_{r-1}\}$ and $\mathcal{Q} = \{S_0, S_1, \ldots, S_{s-1}\}$ let’s put

$$\mathcal{A}(\mathcal{P}, \mathcal{Q}) = \{(j, k) : 0 \leq j \leq r-1, 0 \leq k \leq s-1 \text{ and } \lambda(R_j \cap S_k) > 0\},$$

and

$$\mathcal{B}(\mathcal{P}, \mathcal{Q}) = \{k : 0 \leq k \leq s-1 \text{ and } \lambda(R_r \cap S_k) > 0\}.$$

Let $\overline{\mathcal{A}(\mathcal{P}, \mathcal{Q})}$ and $\overline{\mathcal{B}(\mathcal{P}, \mathcal{Q})}$ denote the number of elements in $\overline{\mathcal{A}(\mathcal{P}, \mathcal{Q})}$ and $\overline{\mathcal{B}(\mathcal{P}, \mathcal{Q})}$ respectively. The statement in the lemma is thus equivalent to saying that $\overline{\mathcal{A}(\mathcal{P}, \mathcal{Q})} \leq r+s-1$. If $r = 1$, we have $J = R_0$ and we see that $\overline{\mathcal{A}(\mathcal{P}, \mathcal{Q})} \leq s = 1+s-1$, so in this case the result holds for any closed-interval partition $S_0, S_1, \ldots, S_{s-1}$. We proceed by induction on $r$, by assuming that, for some given $r \geq 2$, for every closed-interval partition $\mathcal{P} = \{R_0, R_1, \ldots, R_{r-1}\}$ and for any closed-interval partition $\mathcal{Q}$ having $s$ elements for arbitrary $s \in \mathbb{N}$, we have $\overline{\mathcal{A}(\mathcal{P}, \mathcal{Q})} \leq r+s-1$.

Now consider closed-interval partitions $\mathcal{P}' = \{R_0, R_1, \ldots, R_{r}\}$ and $\mathcal{Q}' = \{S_0, S_1, \ldots, S_{s-1}\}$. We may assume that $\overline{\mathcal{A}(\{R_0, R_1, \ldots, R_{r-1}\}, \mathcal{Q}')} \geq 1$, for otherwise we have $\overline{\mathcal{A}(\{R_0, R_1, \ldots, R_{r-1}\}, \mathcal{Q}')} = 0$, and then

$$\overline{\mathcal{A}(\{R_0, R_1, \ldots, R_{r}\}, \mathcal{Q}')} = \overline{\mathcal{A}(\{R_r\}, \mathcal{Q}')} \leq s \leq (r+1) + s-1,$$

as we have seen the lemma is true when one of the partitions has a single element. That is, in this case, the truth of the lemma for $r$ implies the truth of the lemma for $r+1$. 

4
Now, when \( A\{R_0, R_1, \ldots, R_{r-1}\}, Q' \) \( \geq 1 \), let \( s_1 \in \{1, 2, \ldots, s - 1\} \) be the maximum of all \( k \in \{1, 2, \ldots, s - 1\} \) such that \( \lambda(A_j \cap B_k) > 0 \) for some \( j \in \{0, 1, 2 \ldots, r-1\} \). By the inductive assumption, \( A(A_0, A_1, \ldots, A_{r-1}), Q' \) \( \leq r + s_1 \) . Also, as we go from \( r \) to \( r+1 \), the single interval \( R_r \) is adjoined to \( R_0, R_1, \ldots, R_{r-1} \) on the right. So, we see that assuming the inductive assumption holds for \( r \) it holds for \( r+1 \). Invoking induction completes the proof.

**Lemma 2.2** Let \( a, b, c, d \in \mathbb{R} \) with \( c < d \), \( a \leq d \) and \( b \geq c \). Put

(i) \( a' = a \) if \( a \in [c, d] \), and \( a' = c \) if \( a < c \);

(ii) \( b' = b \) if \( b \in [c, d] \), and \( b' = d \) if \( b < c \).

Then,

\[
| (u - a)(u - b) | \geq | (u - a')(u - b') | , \text{ for all } u \in [c, d].
\]

**Proof.** If \( a' = a \) or \( b' = b \), the result is easily checked. The only other case is when \( a' = c \) and \( b' = d \). In this case we have

\[
(u - a)(b - u) - (u - c)(d - u) = (a + b - c - d)u - ab + cd,
\]

and this expression is linear in \( u \) and non-negative for \( u = c \) and \( u = d \). Consequently, \( (u - a)(b - u) \geq (u - c)(d - u) \) for all \( u \in [c, d] \), and the proof is complete.

**Lemma 2.3** Let \( s, m \in \mathbb{N} \) with \( m \geq 4s + 1 \), and let \( V_1, V_2, \ldots, V_m \) be closed and bounded subintervals of \( \mathbb{R} \). Let \( c_1, c_2, \ldots, c_m, d_1, d_2, \ldots, d_m \in \mathbb{R} \) be such that \( c_j, d_j \in V_j \) for all \( j = 1, 2, \ldots, m \). Then, there is a number \( M > 0 \), depending upon \( s \) and \( m \) only and independent of the \( V_j, c_j \) and \( d_j \), such that

\[
\int \prod_{t=1}^{m} V_t \sum_{t=1}^{m} (u_t - c_t)^{2s} (u_t - d_t)^{2s} \leq M \left( \max \{ \mu(V_1), \mu(V_2), \ldots, \mu(V_m) \} \right)^{m - 4s}.
\]

**Proof.** See [3, Lemma 4.2].

**Theorem 2.4** Let \( f \in L^2(\mathbb{R}) \) and let \( \mu_1, \mu_2, \ldots, \mu_r \in M(\mathbb{R}) \). Then the following conditions (i) and (ii) are equivalent.

(i) There are \( f_1, f_2, \ldots, f_r \in L^2(\mathbb{R}) \) such that \( f = \sum_{j=1}^{r} \mu_j * f_j \).

(ii) \[
\int_{-\infty}^{\infty} \frac{| \hat{f}(x) |^2}{r} \sum_{j=1}^{r} | \hat{\mu_j}(x) |^2 \, dx < \infty.
\]

**Proof.** This is essentially proved in [3] pages 411-412, but see also [6] pages 77-88 and [7] page 23.
3 Main results

The main aim in this paper is to prove the following result.

**Theorem 3.1** Let \( s \in \mathbb{N} \) and let \( \alpha, \beta \in \mathbb{R} \). Let \( D_{\alpha,\beta,s}(\mathbb{R}) \) be the vector space of functions \( f \in L^2(\mathbb{R}) \) that can be expressed as some finite sum of \((\alpha, \beta)\)-differences of order \( 2s \). Then the following conditions (i) - (iii) are equivalent for a function \( f \in L^2(\mathbb{R}) \).

(i) \( \int_{-\infty}^{\infty} \frac{|\hat{f}(x)|^2}{(x-\alpha)^{2s}(x-\beta)^{2s}} \, dx < \infty \).

(ii) \( f \in D_{\alpha,\beta,s}(\mathbb{R}) \).

(iii) There are \( u_1, u_2, \ldots, u_{4s+1} \in \mathbb{R} \) and \( f_1, f_2, \ldots, f_{4s+1} \in L^2(\mathbb{R}) \) such that

\[
\hat{f} = \sum_{j=1}^{4s+1} \left[ \left( e^{iu_j(\frac{x-\alpha}{2})} + e^{-iu_j(\frac{\alpha-\beta}{2})} \right) \delta_0 - \left( e^{iu_j(\frac{\alpha+\beta}{2})} \delta_{u_j} + e^{-iu_j(\frac{\alpha+\beta}{2})} \delta_{-u_j} \right) \right]^s * f_j. \tag{3.1}
\]

When the preceding conditions hold for a given function \( f \in L^2(\mathbb{R}) \), for almost all \((u_1, u_2, \ldots, u_{4s+1}) \in \mathbb{R}^{4s+1}\), there are \( f_1, f_2, \ldots, f_{4s+1} \in L^2(\mathbb{R}) \) such that (3.1) holds. Also, \( D_{\alpha,\beta,s}(\mathbb{R}) \) is a Hilbert space with the inner product \( \langle , \rangle_{\alpha,\beta,s} \) given by

\[
\langle f, g \rangle_{\alpha,\beta,s} = \int_{-\infty}^{\infty} \left( 1 + \frac{1}{(x-\alpha)^{2s}(x-\beta)^{2s}} \right) \hat{f}(x) \overline{\hat{g}(x)} \, dx, \quad \text{for } f, g \in D_{\alpha,\beta,s}(\mathbb{R}).
\]

The operator \( D^2 - (\alpha + \beta)D - \alpha\beta I^s \) is a linear, bounded and invertible operator that maps \( W^{2s}(\mathbb{R}) \) onto \( D_{\alpha,\beta,s}(\mathbb{R}) \).

**Proof.** If (iii) holds it is clear that (ii) also holds.

Let (ii) hold. If \( u \in \mathbb{R} \), define \( \lambda_u \in M(\mathbb{R}) \) by

\[
\lambda_u = \frac{1}{2} \left[ e^{iu(\frac{\alpha-\beta}{2})} + e^{-iu(\frac{\alpha-\beta}{2})} \right] \delta_0 - \frac{1}{2} \left[ e^{iu(\frac{\alpha+\beta}{2})} \delta_u + e^{-iu(\frac{\alpha+\beta}{2})} \delta_{-u} \right]. \tag{3.2}
\]

The Fourier transform \( \hat{\lambda}_u \) of \( \lambda_u \) is given for \( x \in \mathbb{R} \) by

\[
\hat{\lambda}_u(x) = \sin (u(x-\alpha)) \sin (u(x-\beta)). \tag{3.3}
\]

So, if \( u \in \mathbb{R} \) and \( f, g \in L^2(\mathbb{R}) \) are such that \( f = \lambda_u^* * g \), we have

\[
\int_{-\infty}^{\infty} \frac{|\hat{f}(x)|^2}{(x-\alpha)^{2s}(x-\beta)^{2s}} \, dx = \int_{-\infty}^{\infty} \frac{\sin^{2s} (u(x-\alpha)) \sin^{2s} (u(x-\beta))}{(x-\alpha)^{2s}(x-\beta)^{2s}} |\hat{g}(x)|^2 \, dx < \infty.
\]

Using the definition of \( \lambda_u \) in (3.2), we deduce that (ii) implies (i).

Now, we assume that (i) holds, and we will prove that (iii) holds. Let \( c > 0 \) be given and let \( x \in \mathbb{R} \) also be given but with \( x \notin \{\alpha, \beta\} \). Put, for each \( k \in \mathbb{Z} \),

\[
a_k = \frac{k\pi}{|x-\alpha|}, \quad b_k = \frac{k\pi}{|x-\beta|}, \quad a'_k = \frac{(k-1/2)\pi}{|x-\alpha|} \quad \text{and} \quad b'_k = \frac{(k-1/2)\pi}{|x-\beta|}. \tag{3.4}
\]
Then put, again for each $k \in \mathbb{Z}$,

$$A_k = [a'_k, a'_{k+1}] \text{ and } B_k = [b'_k, b'_{k+1}].$$  \hspace{1cm} (3.5)$$

Note that $a_k$ is the mid-point of $A_k$ and $b_k$ is the mid-point of $B_k$. The points $a_k$ are the zeros of $u \mapsto \sin(u(x - \alpha))$, while the $b_k$ are the zeros of $u \mapsto \sin(u(x - \beta))$. It is immediate from the definitions that, for each $k \in \mathbb{Z}$,

$$\lambda(A_k) = \frac{\pi}{|x - \alpha|} \text{ and } \lambda(B_k) = \frac{\pi}{|x - \beta|}. \hspace{1cm} (3.6)$$

We will use the notation that $d_Z(x)$ denotes the distance from $x \in \mathbb{R}$ to the nearest integer. Note that $d_Z(x) = |x|$ if and only if $-1/2 \leq x \leq 1/2$. Note also that $|\sin(\pi x)| \geq 2d_Z(x)$ for all $x \in \mathbb{R}$ (for example see [7, page 89] or [10, page 233]).

Now

$$u \in A_j \implies \frac{(j - 1/2)\pi}{|x - \alpha|} \leq u \leq \frac{(j + 1/2)\pi}{|x - \alpha|} \implies -1/2 \leq |x - \alpha| \frac{u - \frac{j}{\pi}}{|x - \alpha|} \leq 1/2.$$  

So, for $u \in A_j$,

$$|\sin(u(x - \alpha))| = \left| \sin \left( \pi \frac{|x - \alpha|}{u - \frac{j}{\pi}} \right) \right| \geq 2d_Z \left( \frac{|x - \alpha|}{u - \frac{j}{\pi}} \right) = 2|x - \alpha| \left| \frac{u}{\pi} - \frac{j}{|x - \alpha|} \right| = \frac{2}{\pi} |x - \alpha| \left| u - \frac{j\pi}{|x - \alpha|} \right|. \hspace{1cm} (3.7)$$

Similarly, for $u \in B_k$,

$$|\sin(u(x - \beta))| \geq \frac{2}{\pi} |x - \beta| \left| u - \frac{k\pi}{|x - \beta|} \right|. \hspace{1cm} (3.8)$$

So, for $u \in A_j \cap B_k$ we have

$$|\sin(u(x - \alpha)) \sin(u(x - \beta))| \geq \frac{4}{\pi^2} |(x - \alpha)(x - \beta)| \left| u - \frac{j\pi}{|x - \alpha|} \right| \left| u - \frac{k\pi}{|x - \beta|} \right|. \hspace{1cm} \text{That is, for } u \in A_j \cap B_k \text{ we have} \hspace{1cm} (3.9)$$

where $a_j$ and $b_k$ are the points as given in [334], with $a_j$ the midpoint of $A_j$ and $b_k$ the midpoint of $B_k$.  

7
Now let the intervals $A_j$ such that $\lambda(A_j \cap [-c, c]) > 0$ be $A_{m_1}, \ldots, A_{m_1+r-1}$, and let the intervals $B_k$ such that $\lambda(B_k \cap [-c, c]) > 0$ be $B_{m_2}, \ldots, B_{m_2+s-1}$. Put

$$A'_j = A_{m_1+j}, \text{ for } j = 0, 1, \ldots, r-1,$$

$$B'_k = B_{m_2+k}, \text{ for } k = 0, 1, \ldots, s-1. \quad (3.10)$$

Then, putting

$$P_1 = \{A'_0, A'_1, \ldots, A'_{r-1}\} \text{ and } P_2 = \{B'_0, B'_1, \ldots, B'_{s-1}\}, \quad (3.11)$$

we see that $P_1$ and $P_2$ are closed-interval partitions. If we put

$$A = \{(j, k) : 0 \leq j \leq r-1, 0 \leq k \leq s-1, \text{ and } \lambda(A'_j \cap B'_k) > 0\},$$

and if we let $P$ be the refinement of $P_1$ and $P_2$, we have

$$P = \{A'_j \cap B'_k : (j, k) \in A\} \text{ and } [-c, c] \subseteq \bigcup_{(j, k) \in A} A'_j \cap B'_k. \quad (3.12)$$

Now, from (3.6) we see that all lengths of the $r$ intervals in the closed-interval partition $P_1$ equal $\pi/|x - \alpha|$, so that $(r-2)\pi/|x - \alpha| < 2c$. Hence,

$$1 \leq r < \frac{2c|x - \alpha|}{\pi} + 2 = \frac{2c}{\pi} \left(1 + \frac{\pi}{c|x - \alpha|}\right) |x - \alpha|. \quad (3.13)$$

Let $0 < \delta < 1/2$. Then, if $|x - \alpha| > \pi\delta/c$, we have from (3.13) that

$$1 \leq r < \frac{2c}{\pi} \left(1 + \frac{1}{\delta}\right) |x - \alpha|. \quad (3.14)$$

On the other hand, if $|x - \alpha| \leq \pi\delta/c$, as $0 < \delta < 1/2$ we have $2c < \pi/|x - \alpha|$, and it follows from (3.6) that $[-c, c] \subseteq A_0$, so that $m_0 = 0$ and

$$r = 1. \quad (3.15)$$

Again let $0 < \delta < 1/2$. Then, as in the preceding argument, but with $\beta$ replacing $\alpha$, if $|x - \beta| > \pi\delta/c$ we have

$$1 \leq s < \frac{2c}{\pi} \left(1 + \frac{1}{\delta}\right) |x - \beta|, \quad (3.16)$$

while if $|x - \beta| \leq \pi\delta/c$, we have

$$s = 1. \quad (3.17)$$

Now, again let $0 < \delta < 1/2$. Assume that either $|x - \alpha| > \pi\delta/c$ or $|x - \beta| > \pi\delta/c$, with both perhaps holding. In the case that $|x - \alpha| > \pi\delta/c$ and $|x - \beta| \leq \pi\delta/c$, we
have from (3.14) and (3.17) that

\[
\begin{align*}
    r + s - 1 & \leq 2 \max\{r, s\} < 2 \max\left\{ \frac{2c}{\pi} \left( 1 + \frac{1}{\delta} \right) |x - \alpha|, 1 \right\} \\
    & \leq 2 \max\left\{ \frac{2c}{\pi} \left( 1 + \frac{1}{\delta} \right) |x - \alpha|, \frac{2c}{\pi} \left( 1 + \frac{1}{\delta} \right) |x - \beta| \right\} \\
    & = \frac{4c}{\pi} \left( 1 + \frac{1}{\delta} \right) \max\{|x - \alpha|, |x - \beta|\}. \quad (3.18)
\end{align*}
\]

The argument that produced (3.18) is symmetric in \( \alpha \) and \( \beta \), and we see from (3.14), (3.15), (3.16) and (3.17) that in all cases when either \( |x - \alpha| > \pi \delta/c \) or \( |x - \beta| > \pi \delta/c \) we have

\[
    r + s - 1 \leq 2 \max\{r, s\} \leq \frac{4c}{\pi} \left( 1 + \frac{1}{\delta} \right) \max\{|x - \alpha|, |x - \beta|\}. \quad (3.19)
\]

Also, observe that if \( 0 < \delta < 1/2, |x - \alpha| \leq \pi \delta/c \) and \( |x - \beta| \leq \pi \delta/c \), we have from (3.15) and (3.17) that

\[
    r = s = 1. \quad (3.20)
\]

Note that in the above, \( a_k, b_k, A_k, B_k \) and so on, depend upon \( x \). In particular, \( r \) and \( s \) depend upon \( x \).

We now take \( m \in \mathbb{N} \) with \( m \geq 4s + 1 \), and we estimate the integral

\[
    \int_{[-c, c]^m} \frac{du_1 du_2 \ldots du_m}{\sum_{j=1}^{m} \sin^{2s} u_j (x - \alpha) \sin^{2s} u_j (x - \beta)}
\]

allowing for the different values \( x \) may be, but recall that \( x \notin \{\alpha, \beta\} \). We let \( \mathcal{P}_1, \mathcal{P}_2 \) be the closed-interval partition as given in (3.11) and let \( \mathcal{P} \) be their refinement as given in (3.12). We have, using the definitions and (3.4), (3.9), (3.10) and (3.12),

\[
    \int_{[-c, c]^m} \frac{du_1 du_2 \ldots du_m}{\sum_{j=1}^{m} \sin^{2s} u_j (x - \alpha) \sin^{2s} u_j (x - \beta)}
\]

\[
    \leq \sum_{(j_1, k_1), (j_2, k_2), \ldots, (j_m, k_m) \in \mathcal{A}} \int_{A'_{j_1} \cap B'_{k_1}} \frac{du_1 du_2 \ldots du_m}{\sum_{j=1}^{m} \sin^{2s} u_j (x - \alpha) \sin^{2s} u_j (x - \beta)}
\]

\[
    \leq \frac{\pi^{4s}}{2^{4s} (x - \alpha)^{2s} (x - \beta)^{2s}} \sum_{(j_1, k_1), \ldots, (j_m, k_m) \in \mathcal{A}} \int_{A'_{j_1} \cap B'_{k_1}} \frac{du_1 du_2 \ldots du_m}{(u_j - a_{m_1 + j_1})^{2s} (u_j - b_{m_2 + k_1})^{2s}}. \quad (3.21)
\]

Now in (3.21), the points \( a_{m_1 + j_1}, b_{m_2 + k_1} \) do not necessarily belong to \( A'_{j_1} \cap B'_{k_1} \).

However, suppose that \( a_{m_1 + j_1}, b_{m_2 + k_1} \notin A'_{j_1} \cap B'_{k_1} \) with \( a_{m_1 + j_1} \leq b_{m_2 + k_1} \) and with
both \(a_{m_1+j_1}\) and \(b_{m_2+k_1}\) lying to the left of \(A'_{j_1} \cap B'_{k_1}\). Let \(y\) be the left endpoint of \(A'_{j_1} \cap B'_{k_1}\). Then, \(a_{m_1+j_1} \in A_{m_1+j_1}\) so we see that \([a_{m_1+j_1}, y] \subseteq A_{m_1+j_1}\). Similarly, \([b_{m_2+k_1}, y] \subseteq B_{m_2+k_1}\). We deduce that

\[
[b_{m_2+k_1}, y] \subseteq A_{m_1+j_1} \cap B_{m_2+k_1} = A'_{j_1} \cap B'_{k_1},
\]

so that \(b_{m_2+k_1} \in A'_{j_1} \cap B'_{k_1}\), and this contradicts the fact that \(b_{m_2+k_1} \not\in A'_{j_1} \cap B'_{k_1}\).

This argument, repeated for other cases, means we can say that if \(a_{m_1+j_1}, b_{m_2+k_1} \not\in A'_{j_1} \cap B'_{k_1}\), then \(a_{m_1+j_1}\) is on the left of \(A'_{j_1} \cap B'_{k_1}\) and \(b_{m_2+k_1}\) is on the right of \(A'_{j_1} \cap B'_{k_1}\), or vice versa.

Now, let \(t \in \{1, 2, \ldots, m\}\) and \((j_t, k_t) \in A\) be given. In Lemma 2.2, let’s take \([c, d]\) to be \(A'_{j_t} \cap B'_{k_t}\), \(a\) to be \(\min\{a_{m_1+j_1}, b_{m_2+k_1}\}\), and \(b\) to be \(\max\{a_{m_1+j_1}, b_{m_2+k_1}\}\). The observation at the end of the preceding paragraph implies that we must have \(a \leq d\) and \(b \geq c\). Also, note that either \(a = a_{m_1+j_1}\) and \(b = b_{m_2+k_1}\) or vice versa.

Then, from Lemma 2.2 we have

\[
|(u-a_{m_1+j_1})(u-b_{m_2+k_1})| \geq [(u-a'_{m_1+j_1})(u-b'_{m_2+k_1})], \quad \text{for all} \ u \in A'_{j_t} \cap B'_{k_t}. \tag{3.22}
\]

Now let \(0 < \delta < 1/2\) and assume that we have either \(|x-\alpha| > \pi \delta / c\) or \(|x-\beta| > \pi \delta / c\). Then from Lemma 2.1, the right hand side of (3.19) gives an upper bound for the number of elements in \(P\). Using (3.19), and using (3.9), (3.12), (3.20), (3.21), (3.22), Lemma 2.3 and the assumption that \(m \geq 4s + 1\), we have in this case that

\[
\int_{[-c,c]^m} \frac{du_1 du_2 \ldots du_m}{\sum_{j=1}^{m} \sin^{2s} u_j (x-\alpha) \sin^{2s} u_j (x-\beta)} 
\leq \frac{\pi^{4s}}{2^{4s}(x-\alpha)^{2s}(x-\beta)^{2s}} \sum_{(j_1, k_1), \ldots, (j_m, k_m) \in A} \int_{A'_{j_1} \cap B'_{k_1}} \frac{du_1 du_2 \ldots du_m}{|u_j - a'_{m_1+j_1}|^{2s}|u_j - b'_{m_2+k_1}|^{2s}} 
\leq \frac{\pi^{4s} M}{2^{4s}(x-\alpha)^{2s}(x-\beta)^{2s}} \sum_{(j_1, k_1), \ldots, (j_m, k_m) \in A} \left( \max \left\{ \lambda(A'_{j_1} \cap B'_{k_1}), \ldots, \lambda(A'_{j_m} \cap B'_{k_m}) \right\} \right)^{m-4s},
\]

where \(M\) is the constant in Lemma 2.3,

\[
\leq \frac{\pi^{4s} \delta^{2m-4s} c^m (\delta + 1)^m M}{\delta^m (x-\alpha)^{2s}(x-\beta)^{2s}} \max \left\{ |x-\alpha|^m, |x-\beta|^m \right\} \min \left\{ \frac{\pi^{m-4s}}{|x-\alpha|^{m-4s}}, \frac{\pi^{m-4s}}{|x-\beta|^{m-4s}} \right\},
\]

using (3.0),

\[
\leq Q \max \left\{ \frac{(x-\alpha)^{2s}}{(x-\beta)^{2s}}, \frac{(x-\beta)^{2s}}{(x-\alpha)^{2s}} \right\}, \tag{3.23}
\]

for all \(x \not\in \{\alpha, \beta\}\) with either \(|x-\alpha| > \pi \delta / c\) or \(|x-\beta| > \pi \delta / c\). Note that the constant \(Q\) in (3.23) is independent of \(x\).

CASE I: \(\alpha \neq \beta\).
In this case, choose \( \delta \) so that
\[
0 < \delta < \min \left\{ \frac{1}{2}, \frac{c|\alpha - \beta|}{2\pi} \right\}.
\]
(3.24)

Then, define disjoint intervals \( J, K \) by putting
\[
J = \left[ \alpha - \frac{\pi \delta}{c}, \alpha + \frac{\pi \delta}{c} \right] \quad \text{and} \quad K = \left[ \beta - \frac{\pi \delta}{c}, \beta + \frac{\pi \delta}{c} \right].
\]

Clearly, there is \( C_1 > 0 \) such that
\[
\max \left\{ \frac{(x - \alpha)^2}{(x - \beta)^2} : \frac{(x - \beta)^2}{(x - \alpha)^2} \right\} \leq C_1, \quad \text{for all} \quad x \in (J \cup K)^c.
\]
(3.25)

As well, \((x - \beta)^{-2s}\) is bounded on \( J \), so we see that there is \( C_2 > 0 \) such that
\[
\max \left\{ \frac{(x - \alpha)^2}{(x - \beta)^2} : \frac{(x - \beta)^2}{(x - \alpha)^2} \right\} (x - \beta)^2 \leq C_2, \quad \text{for all} \quad x \in J \cap \{ \alpha \}^c.
\]
(3.26)

And, as \((x - \alpha)^{-2s}\) is bounded on \( K \), there is \( C_3 > 0 \) such that
\[
\max \left\{ \frac{(x - \alpha)^2}{(x - \beta)^2} : \frac{(x - \beta)^2}{(x - \alpha)^2} \right\} (x - \alpha)^2 \leq C_3, \quad \text{for all} \quad x \in K \cap \{ \beta \}^c.
\]
(3.27)

We now have from (3.23), (3.25), (3.26) and (3.27), that
\[
\int_{-\infty}^{\infty} \left( \int_{[-c,c]^m} \frac{du_1 du_2 \ldots du_m}{\sin^2 u_j (x - \alpha) \sin^2 u_j (x - \beta)} \right) |\hat{f}(x)|^2 \, dx
\]
\[
\leq C_1 Q \int_{(J \cup K)^c} |\hat{f}(x)|^2 \, dx + C_2 Q \int_{J} \frac{|\hat{f}(x)|^2}{(x - \alpha)^{2s}} \, dx + C_3 Q \int_{K} \frac{|\hat{f}(x)|^2}{(x - \beta)^{2s}} \, dx
\]
\[
< \infty,
\]
(3.28)
as we are assuming that \( \int_{-\infty}^{\infty} |\hat{f}(x)|^2 (x - \alpha)^{-2s} (x - \beta)^{-2s} < \infty \).

CASE II. \( \alpha = \beta \).

Let’s assume that \( \alpha \in (-c, c) \) and that
\[
\delta < \min \left\{ \frac{1}{2}, \frac{c(c - |\alpha|)}{\pi} \right\}.
\]
(3.29)

Put \( L = (\alpha - \pi \delta/c, \alpha + \pi \delta/c) \), and observe that because of (3.29), \( L \subseteq (-c, c) \). If \( x \in L \), we have \(|x - \alpha| < \pi \delta/c\) and it follows from (3.12) and (3.15) that \( r = 1 \). Now
as \( r = 1 \) and as \( A_0 \cap [-c, c] \neq \emptyset \), we see that \([c, -c] \subseteq A_0 = A'_0 = B'_0\). In this case, in (3.7) we must have \( k = 0 \), and as \( L \subseteq [-c, c] \subseteq A'_0 \), we deduce from (3.7) that
\[
|\sin(u(x - \alpha))| \geq \frac{2}{\pi} |u| \cdot |x - \alpha|, \text{ for all } x \in L \text{ and } u \in (-c, c).
\]

Let \( C > 0 \) be such that
\[
\sum_{j=1}^{m} u_j^{4s} \geq C \left( \sum_{j=1}^{m} u_j^2 \right)^{2s}, \text{ for all } (u_1, u_2, \ldots, u_m) \in \mathbb{R}^m.
\]

We now have from (3.30) and (3.31) that if \( m \geq 4s + 1 \) and \( x \in L \),
\[
\int_{[-c, c]^m} \frac{du_1 du_2 \ldots du_m}{\sum_{j=1}^{m} \sin^{4s} u_j (x - \alpha)} \leq \frac{\pi^{4s}}{2^{4s} (x - \alpha)^{4s}} \int_{[-c, c]^m} \frac{du_1 du_2 \ldots du_m}{\sum_{j=1}^{m} u_j^{4s}}
\]
\[
\leq \frac{1}{C} \cdot \frac{\pi^{4s}}{2^{4s} (x - \alpha)^{4s}} \int_{[-c, c]^m} \frac{du_1 du_2 \ldots du_m}{\left( \sum_{j=1}^{m} u_j^2 \right)^{2s}}
\]
\[
\leq \frac{D}{C} \cdot \frac{\pi^{4s}}{2^{4s} (x - \alpha)^{4s}} \int_{0}^{c \sqrt{m}} r^{m-4s-1} dr,
\]
for some \( D > 0 \), by [10] pages 394-395,
\[
\leq \frac{G}{(x - \alpha)^{4s}},
\]
for some \( G > 0 \) that is independent of \( x \in L \cap \{\alpha\}^c\).

On the other hand, if \( x \notin L \) we have \( |x - \alpha| \geq \pi \delta / c \), so that if we apply (3.23) with \( \alpha = \beta \) we have
\[
\int_{[-c, c]^m} \frac{du_1 du_2 \ldots du_m}{\sum_{j=1}^{m} \sin^{4s} u_j (x - \alpha)} \leq Q < \infty.
\]
Assuming that $|\alpha| < c$, we now have, using (3.32) and (3.33) and the fact that $\mathbb{R} = L \cup L^c$,

$$
\begin{align*}
\int_{-\infty}^{\infty} \left( \int_{[-c,c]^m} \frac{du_1 du_2 \ldots du_m}{\sum_{j=1}^{m} \sin^{4s} u_j (x - \alpha)} \right) |\hat{f}(x)|^2 dx \\
\leq G \int_{L} |\hat{f}(x)|^2 dx + Q \int_{L^c} |\hat{f}(x)|^2 dx < \infty,
\end{align*}
$$

as $\alpha = \beta$ and we are assuming that $\int_{-\infty}^{\infty} |\hat{f}(x)|^2 (x - \alpha)^{-2s} (x - \beta)^{-2s} dx < \infty$.

We have considered the cases $\alpha \neq \beta$ and $\alpha = \beta$. The dénouement results from using Fubini’s Theorem, (3.28) and (3.34). We see that provided $|\alpha| < c$ and $m \geq 4s + 1$, in both cases we have

$$
\begin{align*}
\int_{[-c,c]^m} \left( \int_{-\infty}^{\infty} \frac{|\hat{f}(x)|^2 dx}{\sum_{j=1}^{m} \sin^{2s} u_j (x - \alpha) \sin^{2s} u_j (x - \beta)} \right) du_1 du_2 \ldots du_m < \infty.
\end{align*}
$$

We conclude from this that for almost all $(u_1, u_2, \ldots, u_m) \in [-c, c]^m$,

$$
\begin{align*}
\int_{-\infty}^{\infty} \frac{|\hat{f}(x)|^2 dx}{\sum_{j=1}^{m} \sin^{2s} (u_j (x - \alpha)) \sin^{2s} (u_j (x - \beta))} < \infty.
\end{align*}
$$

By letting $c$ tend to $\infty$ through a sequence of values, we deduce that, in fact, the inequality in (3.35) holds for almost all $(u_1, u_2, \ldots, u_m) \in \mathbb{R}^m$. But then, using (3.33) and Theorem 2.4, we see that provided $m \geq 4s + 1$, for almost all $(u_1, u_2, \ldots, u_m) \in \mathbb{R}^m$ there are $f_1, f_2, \ldots, f_m \in L^2(\mathbb{R})$ such that

$$
f = \sum_{j=1}^{m} \left[ \left( e^{ib_j (\alpha - \beta)} + e^{-ib_j (\alpha + \beta)} \right) \delta_0 - \left( e^{ib_j (\alpha + \beta)} \delta_{b_j} + e^{-ib_j (\alpha + \beta)} \delta_{-b_j} \right) \right]^s f_j.
$$

We deduce that (i) implies (ii) in Theorem 3.1 and, by taking $m = 4s + 1$, we see that (i) implies (iii).

We have now proved that (i), (ii) and (iii) are equivalent. Also, the statement that (iii) is possible for almost all $(u_1, u_2, \ldots, u_{4s+1}) \in \mathbb{R}^{4s+1}$ has been proved.

Finally, put $T = (D^2 - i(\alpha + \beta)D - \alpha \beta I)^s$. Then, if $g \in L^2(\mathbb{R})$ we have $T(g) (x) =$
(−1)\(s\)(x−\(α\))\(s\)(x−\(β\))\(\hat{g}(x)\). Consequently, \(\int_{-\infty}^{\infty}(x−\alpha)^{-2s}(x−\beta)^{-2s}|(T(g))(x)|^2\,dx < \infty\). As the multiplier of \(T\) is (−1)\(s\)(x−\(α\))\(s\)(x−\(β\))\(−2s\), it is easy to see that there is \(K > 0\) such that

\[||T(g)||_{\alpha,\beta,s}^2 = \langle T(g), T(g)\rangle_{\alpha,\beta,s} \leq K \int_{-\infty}^{\infty}(1 + x^{2s})|\hat{g}(x)|^2\,dx \leq K ||g||_{R,2s}^2 < \infty,\]

and it follows that \(T\) is bounded from \(W^{2s}(\mathbb{R})\) into \(D_{\alpha,\beta,s}(\mathbb{R})\). As the multiplier of \(T\) vanishes only at the two points \(\alpha\) and \(\beta\), \(T\) is injective on \(W^{2s}(\mathbb{R})\). Finally, if \(h \in L^2(\mathbb{R})\) is such that \(\int_{-\infty}^{\infty}(x−\alpha)^{-2s}(x−\beta)^{-2s}|\hat{h}(x)|^2\,dx < \infty\), we may let \(g \in L^2(\mathbb{R})\) be the function such that \(\hat{g}(x) = (−1)^s(x−\alpha)^{-s}(x−\beta)^{-s}\hat{h}(x)\). It is easy to see that \(g \in W^{2s}(\mathbb{R})\) and that \(T(g) = h\). Consequently, \(T\) maps \(W^{2s}(\mathbb{R})\) onto \(D_{\alpha,\beta,s}(\mathbb{R})\), and it follows that \(T\) is a bounded invertible linear operator from \(W^{2s}(\mathbb{R})\) onto \(D_{\alpha,\beta,s}(\mathbb{R})\). This completes the proof of Theorem 3.1.

Note that an alternative proof of Theorem 3.1 for the special case \(\alpha = \beta\) may be derived from the identity (1.2), which was proved originally in [6] and [7]. In [8] Meisters and Schmidt showed that every translation invariant linear form on \(L^2(\mathbb{T})\) is continuous, but in [9] it was shown that there are discontinuous translation invariant linear forms on \(L^2(\mathbb{R})\), and this latter result may also be deduced from the subsequent identity (1.2).

**Definition.** Let \(\alpha, \beta \in \mathbb{R}\) and let \(s \in \mathbb{N}\). Then a linear form \(T\) on \(L^2(G)\) is called \((\alpha, \beta, s)\)-invariant if, for all \(f \in L^2(\mathbb{R})\) and \(u \in L^2(\mathbb{R})\),

\[T\left(\left[\left(e^{i\frac{\alpha}{2}u} + e^{-i\frac{\alpha}{2}u}\right)\delta_0 - \left(e^{i\frac{\alpha}{2}u} - e^{-i\frac{\alpha}{2}u}\right)\delta_u + e^{-iu_j}\left(\frac{\alpha}{2}\delta_{u_j}\right)\right]s \ast f\right) = T(f).\]

When \(\alpha, \beta \in \mathbb{Z}\), we may also introduce the notion of \((\alpha, \beta, s)\)-invariant linear forms on \(L^2(\mathbb{T})\). It was shown in [8] that an \((\alpha, \beta, 1)\)-invariant linear form on \(L^2(\mathbb{T})\) is continuous and, in fact, any \((\alpha, \beta, s)\)-invariant linear form on \(L^2(\mathbb{T})\) is continuous (proved using the same technique as in [8] for the case \(s = 1\)). Together with the preceding comments, the following corollary to Theorem 3.1 shows that the situation pertaining to translation invariant linear forms on \(L^2(\mathbb{T})\) and \(L^2(\mathbb{R})\) is mirrored by that for \((\alpha, \beta, s)\)-invariant linear forms on \(L^2(\mathbb{T})\) and \(L^2(\mathbb{R})\).

**Corollary 3.2** Let \(\alpha, \beta \in \mathbb{R}\) and let \(s \in \mathbb{N}\). Then, there are discontinuous \((\alpha, \beta, s)\)-invariant linear forms on \(L^2(\mathbb{R})\).

**Proof.** We see from the definitions that if \(T\) is a linear form on \(L^2(\mathbb{R})\), then \(T\) is \((\alpha, \beta, s)\)-invariant if and only if \(T\) vanishes on \(D_{\alpha,\beta,s}(\mathbb{R})\). However, it is consequence of Theorem 3.1 that \(D_{\alpha,\beta,s}(\mathbb{R})\) has infinite algebraic codimension in \(L^2(\mathbb{R})\). Consequently there are discontinuous linear forms on \(L^2(\mathbb{R})\) that vanish on \(D_{\alpha,\beta,s}(\mathbb{R})\), and such forms are also \((\alpha, \beta, s)\) invariant. \(\square\)

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