STRONG EXCEPTIONAL SEQUENCES OF VECTOR BUNDLES ON CERTAIN FANO VARIETIES

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Abstract. Exceptional sequences of vector bundles over a variety $X$ are special generators of the triangulated category $D^b(Coh X)$. Kapranov proved the existence of tilting bundles over homogeneous varieties for the general linear group. King conjectured the existence of tilting sequences of vector bundles on projective varieties which are obtained as quotients of Zariski open subsets of affine spaces.

The goal of this paper is to give further examples of strong exceptional sequences of vector bundles on certain projective varieties. These are obtained as geometric invariant quotients of affine spaces by linear actions of reductive groups, as appears in King’s conjecture.

INTRODUCTION

The concept of derived categories has been introduced by Grothendieck and developed further by Verdier. However, their work remained within a very general and abstract setting, and people wished to have concrete examples which arise from geometry. In algebraic geometry one of the essential objects associated to a projective variety is the (bounded) derived category of coherent sheaves over it. Its knowledge allows to recover all the cohomological data of the variety.

Beilinson made the first major step by proving that the line bundles $O_{P^n}, O_{P^n}(1),\ldots, O_{P^n}(n)$ generate $D^b(Coh P^n)$, and actually form a tilting sequence. Afterwards have appeared several other examples of varieties admitting (strong and complete) exceptional sequences of vector bundles. One of the most notable results in this direction has been obtained by Kapranov. He explicitly constructed tilting sequences of vector bundles over homogeneous varieties for $Gl(n)$, that is over Grassmannians and flag manifolds. Further examples, which are based on Kapranov’s result, have been obtained in [3].

In the unpublished preprint [10], King conjectured that there are tilting bundles over projective varieties which are obtained as invariant quotients of affine spaces for linear actions of reductive groups. Observe that flag varieties for $Gl(n,\mathbb{C})$, and toric varieties are special cases of such quotient varieties.

The answer to King’s conjecture is negative in general. Hille and Perling gave in [5] an example of a toric variety ($\mathbb{P}^2$ blown-up successively three times) with the property that it does not admit a tilting object formed by line bundles. However it is still a very interesting problem to find classes of examples for which the conjecture holds. In the paper [1], Altmann and Hille proved the existence of (partial) strong exceptional sequences on toric varieties arising from thin representations of quivers, but their construction gives sequences of very short length.

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The goal of this paper is to give further examples of strong exceptional sequences of vector bundles over certain Fano varieties. The varieties considered in this paper are obtained as geometric quotients of open subsets of affine spaces by linear actions of a reductive groups.

For the comfort of the reader, we recall that a sequence of vector bundles \((\mathcal{F}_1, \ldots, \mathcal{F}_z)\) over a variety \(Y\) is called strongy exceptional if the following two conditions are fulfilled:

(i) \(H^0(Y, \Hom(\mathcal{F}_j, \mathcal{F}_i)) = 0, \forall 1 \leq i < j \leq z;\)

(ii) \(H^q(Y, \Hom(\mathcal{F}_j, \mathcal{F}_i)) = 0, \forall i, j = 1, \ldots, z,\) and \(\forall q > 0.\)

(iii) A tilting sequence is a strongly exceptional sequence \((\mathcal{F}_1, \ldots, \mathcal{F}_z)\) with the property that \(\mathcal{F}_1, \ldots, \mathcal{F}_z\) generate \(D^b(\text{Coh} Y)\).

Consider an algebraically closed field \(K\) of characteristic zero, a connected, reductive group \(G\) over \(K\), and a representation \(\rho : G \to \text{Gl}(V)\). Let \(V := \text{Spec}(\text{Sym}^* V)\) be the affine space corresponding to \(V\). We denote \(\chi_{ac} = \chi_{ac}(G, V)\) the weight of the \(G\)-module \(\text{det} V\). We make the following assumptions:

(i) the ring of invariants \(K[V]^T = K\), where \(T\) is the maximal torus of \(G;\)

(ii) \(\text{codim}_{V} \mathcal{V}^{ss}(G, \chi_{ac}) \geq 2,\) and \(G\) acts freely on the semi-stable locus \(V^{ss}(G, \chi_{ac}).\)

We denote \(Y := V/\chi_{ac} G\) the invariant quotient. The main ingredient that we use for constructing exceptional sequences over \(Y\) is the set \(\mathcal{E}_1, \ldots, \mathcal{E}_N\) of ‘extremal’ nef vector bundles over \(Y\) (see section [3]). They enjoy good cohomology vanishing properties which are required by the definition of exceptional sequences. The first main result of this paper is the following:

**Theorem A** Let \(d_j := \text{rank}(\mathcal{E}_j)\), and write \(\chi_{ac} = \sum_{j=1}^{N} m_j \cdot \text{det}(\mathcal{E}_j)\), with \(m_j \geq 1\). We denote by \(\mathcal{Y}_{m,d}\) the set of Young diagrams with at most \(d\) rows and \(m\) columns, and consider

\[
\mathcal{E}S(Y) := \{ \text{the set of vector bundles occurring as direct summands in} \}
\]

\[
\text{the Schur powers } \mathbb{S}^{\lambda_{1}}(\mathcal{E}_1) \otimes \ldots \otimes \mathbb{S}^{\lambda_{N}}(\mathcal{E}_N), \chi^{(j)} \in \mathcal{Y}_{m_j-d_j,d_j},
\]

corresponding to irreducible \(G\)-modules.

Then the vector bundles \(\mathcal{E} \to Y, \mathcal{E} \in \mathcal{E}S(Y)\), form a strongly exceptional sequence over \(Y\) with respect to an appropriate order on \(\mathcal{E}S(Y)\).

Moreover, if the multiplicities of the isotypical components of \(V\) are sufficiently high, these vector bundles are slope semi-stable with respect to any polarization on \(Y\).

The estimates appearing in this theorem are not strong enough to recover Kapranov’s construction for partial flag varieties. We have to go on, and exploit the fibre bundle structure. The optimal result would be the following:

Consider a fibre bundle \(Y \overset{\phi}{\to} X\). Suppose that \((\mathcal{F}_i)_{i \in I}\) is a strongly exceptional sequence of vector bundles on \(X\), and that \((\mathcal{E}_j)_{j \in J}\) is a sequence of vector bundles on \(Y\) whose restriction to the fibres of \(\phi\) give rise to strong exceptional sequences relative to \(\phi\).

Then \((\phi^* \mathcal{F}_i \otimes \mathcal{E}_j)_{(i,j) \in I \times J}\) is a strongly exceptional sequence on \(Y\).

Unfortunately such a statement is overoptimistic in general. The content of our second main result is that the statement above becomes true under suitable restrictive hypotheses on the fibration \(\phi\). More precisely, we place ourselves in the following framework:

(i') There is a quotient group \(H\) of \(G\) with kernel \(G_0\), and a quotient \(H\)-module \(W\) of \(V\) with kernel \(V_0\), such that the natural projection \(\text{pr}_W^{V} : V \to W\) has the following property:

\[
\text{pr}_W^{V}( \mathcal{W}^{ss}(G, \chi_{ac}(G, V)) ) \subseteq \mathcal{W}^{ss}(G, \chi_{ac}(H, W)).
\]
We denote by $Y \xrightarrow{\phi} X$ the induced morphism at the quotient level.

(ii') The unstable loci have codimension at least two, and both quotients
\[
\mathcal{V}^g(G, \chi_{ac}(G, V)) \to Y \quad \text{and} \quad \mathcal{W}^g(H, \chi_{ac}(H, W)) \to X
\]
are principal bundles.

(iii') The nef cone of the total space $Y$ is the sum of the nef cones of the base $X$, and that of the fibre: $N_G(V, W) = N_H(W, H) + N_G(V_0, V_0)$. Denote $\mathcal{V}^g_{\mathcal{B}}(X)$ and $\mathcal{V}^g_{\mathcal{B}0}$ the corresponding sets of extremal nef vector bundles.

(iv') The main result in the relative case is the following:

**Theorem B** Let us denote $d_F := \dim F$, for $F \in \mathcal{V}^g_{\mathcal{B}}(X)$, and $d_E := \dim E$, for $E \in \mathcal{V}^g_{\mathcal{B}0}$. We write $\chi_{ac}(H, W) = \sum_{F \in \mathcal{V}^g_{\mathcal{B}}(X)} m_F \cdot \det F$ ($m_F \geq 0$), and $\chi_{ac}(G, V_0) = \sum_{E \in \mathcal{V}^g_{\mathcal{B}0}} m_E \cdot \det E$ ($m_E \geq 0$).

Suppose $(b_F)_{F \in \mathcal{V}^g_{\mathcal{B}}(X)}$ are integers such that for all $q > 0$, and for all Young diagrams $\beta^F$ of length $d_F$, with $\beta^F_{\min} \geq -b_F$, holds: $H^q \left( X, \bigotimes_{F \in \mathcal{V}^g_{\mathcal{B}}(X)} S^\beta^F F \right) = 0$.

Then the elements of the set $\mathcal{E}S(Y)$ defined below form a strong exceptional sequence of vector bundles over $Y$:
\[
\mathcal{E}S(Y) := \text{all the direct summands, corresponding to irreducible } G\text{-modules contained in } \phi^* \left( \bigotimes_{F \in \mathcal{V}^g_{\mathcal{B}}(X)} S^\beta^F F \right) \otimes \bigotimes_{E \in \mathcal{V}^g_{\mathcal{B}0}} S^\nu^E E,
\]
with $\lambda^F \in \mathcal{Y}_{b_F, d_F}$ and $\nu^E \in \mathcal{Y}_{m_E - d_E, d_E}$.

Moreover, it holds: $H^q \left( Y, \bigotimes_{F \in \mathcal{V}^g_{\mathcal{B}}(X)} \phi^* S^\beta^F F \otimes \bigotimes_{E \in \mathcal{V}^g_{\mathcal{B}0}} S^\alpha^E E \right) = 0$ for all $q > 0$, and all Young diagrams $\beta^F$ and $\alpha^E$ of length $d_F$ and $d_E$, with $\beta^F_{\min} \geq -b_F$ and $\alpha^E_{\min} \geq -(m_E - d_E)$ respectively.

We point out that in both cases it remains open the question under which hypothesis these sequences are/extend to tilting objects. However, we remark that, taking into account the example constructed in [5], a general answer concerning the (non-)existence of tilting vector bundles over quotients of affine spaces must be involved.

The definition of an exceptional set involves two conditions. Accordingly, the paper is divided in two main parts, each focusing on one of the two conditions:

- The sections 1 and 2 form the first part: we prove a stability result for associated vector bundles, and define an order on the set of irreducible $G$-modules for which there are no homomorphisms from a ‘larger’ vector bundle into a ‘smaller’ one (see theorem 2.3).

- The sections 3, 4 and 5 have a preparatory character: we introduce the ‘extremal’ nef vector bundles, and study their cohomological properties.

- The second part of the article consists of the sections 6 and 7: they contain the proofs of the main results. The main tool used for proving the vanishing of the higher cohomology groups is a result due to Manivel (see [11]), and Arapura (see [2]). However, this general result is not sufficient to address the relative case, and we have to dwell on our particular context. In theorem 5.3 we prove the following nefness property, which is an essential ingredient in the proof of Theorem B.
Theorem Suppose that $\mathcal{V} \to \mathcal{W}$ satisfies the properties (i') and (ii') above, and denote $Y \xrightarrow{\phi} X$ the morphism induced at the quotient level.

Let $\mathcal{E} \to Y$ be a nef vector bundle, associated to a $G$-module $E$. Then $R^q\phi_*\mathcal{E} = 0$ for all $q > 0$, and $\phi_*\mathcal{E} \to X$ is still a nef vector bundle.

Finally, in section 8 we illustrate the general theory. On one hand, we recover Kapranov’s construction for the Grassmannian and for flag varieties, by using our results. On the other hand, we give further examples of strong exceptional sequences over quiver varieties. The very pleasant feature is that we obtain these example by an almost algorithmic procedure, which applies to any quiver variety.

Some of the results have been presented at the HOCAT 2008 Conference, held at Centre de Recerca Matemàtica, Bellaterra, Spain.

1. A stability property

The symbol $\mathbb{Q}$ will always denote the field of rational numbers, and $K$ will be an algebraically closed field $K$ of characteristic zero. Throughout the paper, $G$ will always denote a connected, reductive group over $K$, and $T$ will be the maximal torus of $G$. We consider a faithful representation $\rho : G \to \text{Gl}(V)$, and denote by $\mathcal{V} := \text{Spec}(\text{Sym}^\bullet V^*)$ the corresponding affine space. We shall assume that the ring of invariants $K[\mathcal{V}]^T = K$; it follows automatically that $K[\mathcal{V}]^G = K$.

Lemma 1.1. Let $V$ be a non-zero $G$-module such that $K[\mathcal{V}]^T = K$. Then:

(i) There is a 1-PS $\lambda \in X_*(T)$ such that all its weights on $V$ are strictly positive.

(ii) $G$ is not semi-simple.

We fix once for all $l \in X_*(T) \otimes \mathbb{R}$ such that its weights on $V$ are all positive, and moreover it has ‘irrational slope’, that is $\text{Ker}(1 : X^*(T) \to \mathbb{R}) = \emptyset$.

Proof. (i) Let $\Phi$ denote the set of weights of the $T$-module $V$. Then the set of weights of the $T$ on $K[\mathcal{V}]$ is the ‘cone’ $\sum_{\eta \in \Phi} \mathbb{N}\eta$. Since $K[\mathcal{V}]^T = K$, this cone is strictly convex. Otherwise we can construct a non-trivial $T$-invariant monomial. It follows that there is $\lambda \in X_*(T)$ with $\langle \eta, \lambda \rangle > 0$ for all $\eta \in \Phi$.

(ii) Assume that $G$ is a semi-simple group. The previous step implies that $K[\mathcal{V}^m]^T = K$, hence $K[\mathcal{V}^m]^G = K$ for all $m \geq 1$. Since $G$ is semi-simple, it has an open orbit in $\mathcal{V}^m$. For large $m$ we get a contradiction. $\square$

Let $\theta \in X^*(G)$ be a character. We denote:

$K[\mathcal{V}]_0^G := \{ f \in K[\mathcal{V}] \mid f(g \times y) = \theta(g) \cdot f(y), \forall y \in \mathcal{V} \}$

(1.1) $K[\mathcal{V}]^{G, \theta} := K \oplus \bigoplus_{n \geq 1} K[\mathcal{V}]_{1^n}^G,$

$\mathcal{V}_{ss}(G, \theta) := \{ y \in \mathcal{V} \mid \exists n \geq 1 \text{ and } f \in K[\mathcal{V}]_{1^n}^G \text{ s.t. } f(y) \neq 0 \}.$

We say that $\theta$ is effective if there is $n \geq 1$ such that $K[\mathcal{V}]_{1^n}^G \neq 0$, that is $\mathcal{V}_{ss}(G, \theta) \neq \emptyset$.

Definition 1.2. We define the anti-canonical character of the $G$-module $V$ to be the character of the $G$-module $\text{det}V$. 

Explicitly: decompose $V = \bigoplus_{\omega \in X} M_{\omega}^{m_{\omega}}$ into its $G$-isotypical components. Let $\chi_\omega$ be the character by which $Z(G)^{x}$ acts on $M_\omega$, and denote $d_\omega := \dim M_\omega$. Then $\chi_{ac}(G,V) := \sum_{\omega \in X} m_\omega d_\omega \chi_\omega \in X^*(G)$. For shorthand, we will write $\chi = \chi_{ac}(G,V)$.

**Lemma 1.3.** Assume that $m_\omega \geq d_\omega$. Then the character $\chi_\omega$ is effective. Moreover, if $m_\omega > d_\omega$ for all $\omega$, then $\chi_{ac}$ is effective, and the $\chi_{ac}$-unstable locus has codimension at least two.

**Proof.** We view $V$ as $\bigoplus_{\omega \in X} \Hom(K_{m_\omega}, M_\omega)$. Since $m_\omega \geq d_\omega$, we can associate to an element $\Hom(K_{m_\omega}, M_\omega)$ the $d_\omega \times d_\omega$-minor corresponding to the first $d_\omega$ columns. This defines a regular function $f_\omega$ which is $d_\omega \chi_\omega$-equivariant; moreover, $f_\omega$ does not vanish on surjective homomorphisms. It follows that $d_\omega \chi_\omega$, and therefore $\chi_\omega$, is effective for all $\omega$.

If a point belongs to the unstable locus, then all the minors $f_\omega$ have to vanish. Since $m_\omega \geq d_\omega + 1$, this implies the vanishing of at least two independent minors. □

Now we prove a general stability result of independent interest. It is well known that the tangent bundle of the projective space is stable, and more generally the tautological bundles over Grassmannians are stable. Our goal is to generalize these facts.

We denote $\{G_j\}_{j \in J}$ the simple factors of $G$, and let $\gamma_j : G \to G_j$, be the corresponding quotient morphisms. Using the $\gamma_j$’s we extend the structural group of $\Omega \to Y$, and obtain the principal $G_j$-bundles $\Omega(G_j) \to Y$, $j \in J$, obtained by extending the structural group are semi-stable.

**Theorem 1.4.** Assume that $G$ acts freely on $\Omega := \mathcal{V}^\infty(G, \theta)$, for some $\theta \in X^*(G)$, and let $Y$ be the quotient. Assume that $m_\omega > \dim M_\omega$ holds for all $\omega \in X$. Then the principal $G_j$-bundles $\Omega(G_j) \to Y$, $j \in J$, obtained by extending the structural group are semi-stable.

**Proof.** We fix $j \in J$, and a maximal parabolic subgroup $P_j \subset G_j$; denote $P := \gamma_j^{-1}P_j$: it is a maximal parabolic subgroup of $G$. We observe that the associated homogeneous bundles $\Omega(G_j)(G_j/P_j)$ and $\Omega(G/P)$ are isomorphic.

We denote $H = \prod_{\omega} H_\omega := \prod_{\omega} \GL_K(m_\omega)$: it acts naturally on $\mathcal{V}$; the $G$- and $H$-actions on $\mathcal{V}$ commute. It follows that $H$ still acts on $\Omega(G/P)$ by $H \times \Omega(G/P) \to \Omega(G/P), \ h \times [y, gP] := [hy, gP]$. We will prove that whenever there is a reduction of the structural group $s : Y^o \to (\Omega(G_j))(G_j/P_j) = \Omega(G/P)$, with $Y^o \subset Y$ open and $\dim_Y(Y \setminus Y^o) \geq 2$, holds $\deg_Y (s^*\Omega(G_j)/P_j) \geq 0$. Equivalently, the reduction $s$ can be viewed as a $G$-equivariant morphism $S : \Omega^o = q^{-1}(Y^o) \to G/P$.

The idea is to move $s$ using the $H$-action on $\Omega(G/P)$. Let $\hat{y} \in Y$ be a generic point, and consider $y \in \Omega$ over $\hat{y}$. We define the following subgroups of $H$: $K_{\hat{y}} := \Stab_H(\hat{y})$, and $H_{\hat{y}} := \{ h \in H \ | \ \exists g_h \in G \text{ s.t. } h \times y = \rho(g_h^{-1})y \} = \prod_{\omega} H_{\omega, \hat{y}}$.

We observe that $K_{\hat{y}}$ does not depend on the choice of $y \in q^{-1}(\hat{y})$. Since $G$ acts freely on $\Omega$, the assignment $h \mapsto g_h$ defines a group homomorphism $\rho_{\hat{y}} : H_{\hat{y}} \to G$ whose kernel is $K_{\hat{y}}$. We move the section $s$ using the action of $H_{\hat{y}}$. For $h \in H_{\hat{y}}$ define a new section $s_h$ as follows: $s_h(\hat{x}) := [x, S(h^{-1} \times x)]$ (equivalently, $S_h(x) := S(h^{-1} \times x)$).

Observe that as $h \in H_{\hat{y}}$ varies, $s_h(\hat{y}) = h \times s(\hat{y})$ moves in the vertical direction.
Claim \( H_{\tilde{g}}/K_{\tilde{g}} \to G/Z(G)^0 \) is surjective. Write \( y = (y_\omega)_{\omega} \) w.r.t. the direct sum decomposition of \( V \); for each \( \omega \in X \), \( y_\omega = (y_{\omega 1}, \ldots, y_{\omega m_\omega}) \). Since \( y \in \Omega \) is chosen generically, and \( m_\omega > \dim M_\omega =: d_\omega \), we may assume that for each \( \omega \in X \) the vectors \( y_{\omega 1}, \ldots, y_{\omega m_\omega} \) span \( M_\omega \). Equivalently, we may view \( y_\omega \) as a surjective homomorphism \( K_{\omega} \to M_\omega \).

For \( g \in G \) holds \( p(g)y = (\rho_\omega(g) y_\omega)_{\omega} \). Using that \( m_\omega > d_\omega \), we deduce that for each \( \omega \in X \) there is \( h_\omega \in \text{GL}_K(m_\omega) \) such that \( h_\omega y_\omega = \rho_\omega(g^{-1})y_\omega \). For \( h := (h_\omega)_{\omega} \) we have \( hy = \rho(g^{-1})y \), that is \( g \in \text{Image}(H_{\tilde{g}}/K_{\tilde{g}} \to G) \).

Back to our proof: the infinitesimal action of \( H_{\tilde{g}} \) preserves the restriction to the fibre \( q^{-1}(\hat{y}) = \{ [y, gP] \mid g \in G \} \cong G/P \) of the relative tangent bundle \( T_{\Omega(G/P)/Y} \). By this isomorphism the relative tangent bundle corresponds to \( T_{G/P} \to G/P \). The claim implies that the infinitesimal action \( \text{Lie}(H_{\tilde{g}}) \to T_{\Omega(G/P)/Y} \) is surjective. Hence there is a section \( \sigma \in H^0(Y^0, s^*T_{\Omega(G/P)/Y}) \) which does not vanish at \( \hat{y} \). It follows \( \deg_Y(s^*T_{\Omega(G/P)/Y}) \geq 0 \).

**Corollary 1.5.** Assume \( \theta \in X^*(G) \) has the property that \( G \) acts freely on \( \Omega := \text{Vss}(G, \chi) \), and let \( Y \) be the quotient. Let \( E \) be an irreducible \( G \)-module, and denote by \( \mathcal{E} := \Omega(E) \) the associated vector bundle over \( Y \). Assume that \( m_\omega > \dim M_\omega \) holds for all \( \omega \in X \). Then \( \mathcal{E} \to Y \) is slope semi-stable with respect to the polarization induced by the character \( \theta \).

**Proof.** We may assume that \( G = Z(G)^0 \times \prod_{j \in J} G_j \). Since each \( \Omega(G_j) \) is semi-stable, \( \Omega \to Y \) itself is semi-stable. The homomorphism \( \rho_\omega : G \to \text{GL}(E) \) maps \( Z(G)^0 \) into the centre of \( \text{GL}(E) \). By using [14, theorem 3.18], we deduce that \( \mathcal{E} = \Omega(E) \to Y \) is semi-stable. \( \square \)

2. The \( H^0 \) spaces

Assume that \( E \) is a \( G \)-module. We will denote by \( \mathcal{E} \) the vector bundle over \( Y \) associated to it. More precisely, \( \mathcal{E} \) corresponds to the module of covariants \( (K[V] \otimes_K E^\vee)^G \).

The classical Schur lemma says that for two irreducible \( G \)-modules \( E \) and \( F \), the space \( \text{Hom}(E, F) \) consists either of scalars (if \( E = F \)), or vanishes (if \( E \neq F \)). In this section we will prove that a similar result holds for the associated vector bundles \( \mathcal{E} \) and \( \mathcal{F} \).

For warming-up, we start with a special case. We have proved in corollary [15] that \( \mathcal{E} \to Y \) is a semi-stable vector bundle w.r.t. any polarization on \( Y \), as soon as the multiplicities \( m_\omega > d_\omega \) for all \( \omega \). Its first Chern class equals \( \text{dim}(E) \cdot \chi_E \), where \( \chi_E \) denotes the character of \( Z(G)^0 \) on \( E \). Let \( \theta \in X^*(G) \) be an ample class on \( Y \); the slope of \( \mathcal{E} \) w.r.t. \( \theta \) equals

\[
\mu_\theta(\mathcal{E}) = \frac{\deg_\mathcal{E}}{\text{dim } E} = \{ \chi \cdot \theta^{\text{dim } Y - 1}, [Y] \}.
\]

**Definition 2.1.** Let \( \theta \) be a polarization of \( Y \). We define the order \( <_\theta \) on \( X^*(Z(G)^0) \) as follows: we declare that \( \chi <_\theta \eta \) if holds:

\[
\mu_\theta(\chi) := \{ \chi \cdot \theta^{\text{dim } Y - 1}, [Y] \} < \mu_\theta(\eta) := \{ \eta \cdot \theta^{\text{dim } Y - 1}, [Y] \}.
\]

Observe that, by the hard Lefschetz property, we can choose \( \theta \) in such a way that \( \chi = \eta \Leftrightarrow \mu_\theta(\chi) = \mu_\theta(\eta) \).

**Proposition 2.2.** We assume that \( m_\omega > d_\omega \) holds for all \( \omega \). Let \( E \) and \( F \) be two distinct irreducible \( G \)-modules, such that \( Z(G)^0 \) acts on them by two different characters \( \chi_E \) and \( \chi_F \) respectively, such that \( \mu_\theta(\mathcal{E}) < \mu_\theta(\mathcal{F}) \). Then \( H^0(Y, \text{Hom}(\mathcal{F}, \mathcal{E})) = 0 \).
Proof. This is an immediate consequence of the semi-stability property of $E$ and $F$. □

The proposition has two shortcomings: first, we have imposed the condition on the multiplicities; second, there are distinct representations $E$ and $F$ such that the characters $\chi_E$ and $\chi_F$ coincide. So we need to sharpen our result.

**Theorem 2.3.** Assume that $\text{codim}_Y \mathcal{V}^{\text{rus}}(G, \chi_{\text{ac}}) \geq 2$. Let $E$ be an irreducible $G$-module, and let $E \to Y$ be the associated vector bundle. Suppose that there is a weight $\varepsilon$ of $T$ on $E$ which is not $T$-effective (that is $\mathcal{V}^{\text{rus}}(T, \varepsilon) = \emptyset$). Then $H^0(Y, E) = 0$.

Proof. Recall that $H^0(Y, E) = \text{Mor}(\mathbb{V} \to E)^G$, where $$(g \times S)(y) = g \times S(g^{-1} \times y), \quad \forall g \in G \text{ and } \forall \overline{S}, E.$$ Assume that there is a non-zero $G$-equivariant morphism $S : \mathbb{V} \to E$. Then the linear span $\langle S \rangle := \langle S(y), y \in \mathbb{V} \rangle$ is actually a $G$-submodule of $E$. Since $E$ is irreducible and $S \neq 0$, we deduce $\langle S \rangle = E$.

On the other hand, $\varepsilon$ is a weight of $T$ on $E$ which is not effective. We choose a one dimensional $T$-submodule $E_\varepsilon \subset E$, and consider the function $S_\varepsilon := \text{pr}_{E_\varepsilon} \circ S$. Then $S_\varepsilon(t \times y) = \varepsilon(t) \cdot S_\varepsilon(y), \forall t \in T, y \in \mathbb{V}$.

Since $\varepsilon$ is not effective, the function $S_\varepsilon$ must vanish. This implies that the image of the morphism $S$, and consequently its linear span $\langle S \rangle$, is contained in the complement $E'$ of $E_\varepsilon$. The contradiction shows that $\langle S \rangle = E$. □

In order to check that a sequence of vector bundles forms an exceptional sequence, one has to prove that there are no non-trivial homomorphisms from ‘larger’ bundles into ‘smaller’ ones. Now we define the total order required for this property.

**Definition 2.4.** Consider $l \in X_* (T)$ as in lemma 1.21

(i) For any irreducible $G$-module, we define
$$l(E) := \max \{ \langle \eta, l \rangle \mid \eta \text{ is a weight of } T \text{ on } E \}.$$ Equivalently:
$$l(E) = \langle \eta_E, l \rangle,$$ where $\eta_E$ is the dominant weight of $E$ (with respect to $l$).

(ii) Let $E$ and $F$ be two irreducible $G$-modules. We say that $E <_l F$ if $l(E) < l(F)$.

Since $l$ has irrational slope, for any two irreducible $G$-modules $E$ and $F$ holds:
$$l(E) = l(F) \Rightarrow E = F.$$ Hence $<_l$ is a total order relation.

The following result can be viewed as a generalization of Schur’s lemma.

**Theorem 2.5.** Assume that $\text{codim}_Y \mathcal{V}^{\text{rus}}(G, \chi_{\text{ac}}) \geq 2$.

(i) Let $E$ be an irreducible $G$-module. Then $H^0(Y, \text{End}(E)) = K$.

(ii) Let $E$ and $F$ be two irreducible $G$-modules such that $E <_l F$. Then
$$H^0(Y, \text{Hom}(F, E)) = 0.$$ Proof. (i) A section $s \in H^0(Y, \text{End}(E))$ corresponds to a $G$-equivariant morphism $S : \mathbb{V} \to \text{End}(E)$, where the action on $\text{End}(E)$ is by conjugation. (Here we use the hypothesis on the codimension of the unstable set: regular maps defined on the semi-stable locus extend to the whole affine space.) We will prove that the morphism $S$ is a scalar multiple of the identity.

The origin $0 \in \mathbb{V}$ is fixed under $G$. Since $S$ is $G$-equivariant, the homomorphism $S_0 \in \text{End}(E)$ is $\text{Ad}_G$-invariant. Schur’s lemma implies that $S_0 = c \cdot \mathbb{1}_E$, with $c \in K$. By lemma
there is a 1-PS \( \lambda \in X_s(T) \) such that all its weights on \( V \) are strictly positive. In particular \( \lim_{t \to 0} \lambda(t)y = 0 \) for all \( y \in V \). The \( G \)-equivariance implies \( S_{\lambda(t)}y = \text{Ad}_{\lambda(t)} \circ S_y \), hence \( \lim_{t \to 0} \text{Ad}_{\lambda(t)} \circ S_y = S_0 = c1_E \).

The \( \lambda(t) \)-action on \( E \) can be diagonalized in an appropriate basis formed by weight vectors. We denote \( \{E_i\}_{i \in I} \) the weight spaces of \( E \). We order the elements of \( I \) in decreasing order, and consider the corresponding basis in \( E \). Then w.r.t. this basis, \( S_y \) has the following block-matrix shape:

\[
S_y = \begin{pmatrix} c1 & * & * \\
0 & c1 & * \\
0 & 0 & c1 \end{pmatrix}
\]
or equivalently \( S_y - c1 = \begin{pmatrix} 0 & * & * \\
0 & 0 & * \\
0 & 0 & 0 \end{pmatrix} \), \( \forall y \in V \)

Let \( \mathfrak{N}_\lambda \) be the vector space which is formed by matrices having this shape (\( \mathfrak{N}_\lambda \) is actually a nilpotent Lie algebra). Intrinsically,

\[
\mathfrak{N}_\lambda = \{ A \in \text{End}(E) \mid \lim_{t \to 0} \text{Ad}_{\lambda(t)} \circ A = 0 \}.
\]

We denote \( \text{Ker}(\mathfrak{N}_\lambda) := \bigcap_{N \in \mathfrak{N}_\lambda} \text{Ker}(N) \). By Engel’s theorem, \( \text{Ker}(\mathfrak{N}_\lambda) \) is a non-zero vector subspace of \( E \). Applying the \( G \)-equivariance once more, we deduce that for any \( g \in G \) holds:

\[
\text{Ad}_{g^{-1}} \circ (S_y - c1) = S_{g^{-1}y} - c1 \in \mathfrak{N}_\lambda.
\]

It follows that for all \( g \in G \),

\[
\text{Ker}(S_y - c1) \supset g \cdot \text{Ker}(\mathfrak{N}_\lambda) \implies \text{Ker}(S_y - c1) \supset \sum_{g \in G} g \cdot \text{Ker}(\mathfrak{N}_\lambda).
\]

Note that the right-hand-side is a non-zero \( G \)-submodule of \( E \). Since \( E \) is irreducible, it follows that \( \text{Ker}(S_y - c1) = E \), that is \( S_y = c1 \) for all \( y \in V \).

(ii) The \( G \)-module \( \text{Hom}(F, E) = F^\vee \otimes E \) contains the difference \( \varepsilon := \eta_E - \eta_F \) of the corresponding dominant characters. Since \( E <_1 F, l(E) - l(F) < 0 \), the weight \( \varepsilon \) is not \( T \)-effective.

The conclusion follows from theorem 2.3 \( \square \)

3. Numerical criteria for semi-stability

In this section we are reviewing some numerical criteria for semi-stability, needed later on. The following convention is used throughout this section: the letters \( E, V, W \) denote \( G \)-modules, while the symbols \( E, V, W \) denote the corresponding affine spaces: e.g. \( E := \text{Spec}(\text{Sym}^* E^\vee) \).

For a \( G \)-module \( W \), let \( \eta_1, \ldots, \eta_R \) be the weights of the maximal torus \( T \subset G \). We define:

\[
m : W \times X_s(G)_R \to \mathbb{R},
\]

\[
m(w, \lambda) := \min \left\{ j \mid \text{the } j^\text{th}-\text{isotypical component of } w \text{ w.r.t. } \lambda \text{ does not vanish} \right\}.
\]

Observe that for \( \lambda \in X_s(T) \) holds:

\[
m(w, \lambda) := \min \left\{ \langle \eta_j, \lambda \rangle \mid \text{the } \eta_j-\text{isotypical component of } w \text{ does not vanish} \right\}.
\]
We fix a norm $\| \cdot \|$ on $\mathcal{X}_s(T)$, invariant under the Weyl group of $G$. For a character $\theta \in \mathcal{X}^*(G)$, the Hilbert-Mumford criterion for $(G, \theta)$-(semi-)stability reads:

$$ w \in \mathbb{W}^{\text{ss}}(G, \theta) \iff m(w) := \inf \left\{ \frac{\langle \theta, \lambda \rangle}{|\lambda|} \mid m(w, \lambda) \geq 0 \right\} > 0 $$

(3.1)

$$ \iff \left[ m(w, \lambda) \geq 0 \iff \langle \theta, \lambda \rangle \geq 0 \right]. $$

For $w \in \mathbb{W}$ we define:

$$ S(w) := \{ \eta_j \mid \text{the } \eta_j\text{-isotypical component of } w \text{ does not vanish} \} $$

$$ \mathcal{C}_w = \sum_{\eta \in S(w)} \mathbb{R}_{\geq 0} \eta $$

$$ \Lambda_w^G := \{ \lambda \in \mathcal{X}_s(G) \mid m(w, \lambda) \geq 0 \} $$

$$ \Lambda_w^T := \{ \lambda \in \mathcal{X}_s(T) \mid m(w, \lambda) \geq 0 \} $$

$$ = \{ \lambda \in \mathcal{X}_s(T) \mid \langle \eta, \lambda \rangle \geq 0, \forall \eta \in \mathcal{C}_w \} = \mathcal{C}_w^\vee. $$

Note that $\mathcal{C}_w$ and $\Lambda_w^T$ are convex, polyhedral cones. Since there are finitely many $\eta$’s, only finitely many cones $\mathcal{C}_w$ and $\Lambda_w^T$ occur as $w$ varies in $\mathbb{W}^S(G, \theta)$. We are interested in the minimal cones $\mathcal{C}_w$.

**Definition 3.1.** Let $\theta$ be a character of $G$. A subset $S \subset \{ \eta_1, \ldots, \eta_R \}$ is minimal for $\theta$ if $\theta \in \sum_{\eta \in S} \mathbb{R}_{\geq 0} \eta$ and $\theta \not\in \sum_{\eta \in S} \mathbb{R}_{\geq 0} \eta$ for all $\eta_0 \in S$.

We denote $S_1, \ldots, S_z$ the (finitely many) minimal sets for $\theta$, and the corresponding cones by $\mathcal{C}_j$ and $\Lambda_j := \mathcal{C}_j^\vee$, $j = 1, \ldots, z$, respectively. The Weyl group of $G$ operates by permutations on them.

Observe that $\Lambda_w^G = \bigcup_{g \in G} \text{Ad}_{g^{-1}}(\Lambda_w^T)$. As $\theta$ is $\text{Ad}_G$-invariant, the numerical criterion can be reformulated as follows:

$$ \theta \in \mathcal{X}^*(G) \cap \text{int} \left( \bigcap_{w \in \mathbb{W}^S(G, \theta)} \mathcal{C}_w \right) = \text{int} \left( \mathcal{X}^*(G) \cap \mathcal{C}_1 \cap \cdots \cap \mathcal{C}_z \right). $$

(3.2)

For two $G$-modules $V, E$, we define the $K^\times \times G$-module $W_m := E \times V^m, m \geq 1$, with the module structure given by

$$(t, g) \times (\varphi, (v_j)_j) := (t \cdot (g \times \varphi), (g \times v_j)_j).$$

Consider $l > 0$, and define $\theta_m := l \chi_l + m \chi_{ac} \in \mathcal{X}^*(K^\times \times G)$. The numerical functions on $\mathbb{V}^m$ and $\mathbb{W}_m$ are the following:

$$ m(\varphi, \lambda) = \min_j m(v_j, \lambda), \quad \forall (v_j)_j \in \mathbb{V}^m, $$

$$ m((\varphi, \underline{v}, t^\varphi), \lambda) = \min_{1 \leq j \leq m} \{ \varepsilon + m(\varphi, \lambda), m(v_j, \lambda) \}, \quad \forall (\varphi, \underline{v}) \in \mathbb{W}_m. $$

The stability criterion for $\mathbb{W}_m$ reads: a point $w = (\varphi, \underline{v})$ is stable w.r.t. $(K^\times \times G, \theta_m)$ if and only if

$$ \begin{align*}
(A) \quad & m(\varphi, \lambda) \geq 0, m(\underline{v}, \lambda) \geq 0 \implies \langle \chi_{ac}, \lambda \rangle > 0; \\
(B) \quad & 1 + m(\varphi, \lambda) \geq 0, m(\underline{v}, \lambda) \geq 0 \implies l + m \cdot \langle \chi_{ac}, \lambda \rangle > 0; \\
(C) \quad & -1 + m(\varphi, \lambda) \geq 0, m(\underline{v}, \lambda) \geq 0 \implies -l + m \cdot \langle \chi_{ac}, \lambda \rangle > 0. 
\end{align*} $$

(3.3)
Note that \( \mathcal{C}_x(\varphi, \psi) = \mathcal{C}_\varphi + (\mathbb{R} \times \mathcal{C}_\psi) \) for all \((\varphi, \psi) \in \mathbb{W}_m\); moreover, for \( \psi = (v_1, \ldots, v_m) \), then 

\( \mathcal{C}_x = \mathcal{C}_{v_1} + \ldots + \mathcal{C}_{v_m} \). We deduce that as both \( m \) and \((\varphi, \psi) \in \mathbb{W}_m \) vary, there will be only finitely many dual cones:

\[
(3.4) \quad \Lambda(\varphi, \psi) = \Lambda_\varphi \cap (\mathbb{R} \times \Lambda_\psi) = \Lambda_\varphi \cap (\mathbb{R} \times (\Lambda_{v_1} \cap \ldots \cap \Lambda_{v_m})).
\]

We denote by \( \Lambda'_1, \ldots, \Lambda'_s \) the various intersections of \( \Lambda_1, \ldots, \Lambda_s \) defined above, corresponding to the fixed representation \( G \to \text{Gl}(V) \).

**Proposition 3.2.** Let us assume that the \( G \)-module \( V \) has the property:

\[
(\mathbb{V}^m)_{ss}(G, \chi_{ac}) = (\mathbb{V}^m)_{ss}(G, \chi_{ac}) \quad \text{for all} \quad m \geq 1.
\]

Then there is a constant \( a_0(E) \) depending on \( E \) such that for \( \frac{m}{T} > a_0(E) \):

\[
(\mathbb{E} \times \mathbb{V}^m)_{ss}(K^x \times G, l \chi + m \chi_{ac}) = (\mathbb{E} \setminus \{0\}) \times (\mathbb{V}^m)_{ss}(G, \chi_{ac}).
\]

Equivalently, \( \chi_t + r \chi_{ac} \) is an ample class on \( \mathbb{P}(\mathcal{E}) \) for \( r > a_0(E) \).

**Proof.** ‘\( \Rightarrow \)’ Let \((\varphi, \psi) \in (\mathbb{E} \setminus \{0\}) \times (\mathbb{V}^m)_{ss}(G, \chi_{ac}) \). By definition, this means: \( m(\psi, \lambda) \geq 0 \Rightarrow \langle \chi_{ac}, \lambda \rangle > 0 \).

The conditions \((A)\) and \((B)\) in \((3.3)\) are automatically fulfilled. We prove that for large \( m \) the condition \((C)\) holds too. Let \( \lambda_0 \) be such that \( m(\varphi, \lambda_0) \geq 1 \) and \( m(\psi, \lambda_0) \geq 0 \).

Recall that only finitely many cones \( \Lambda_\varphi \) will appear when both \( m \) and \( \psi \in (\mathbb{V}^m)_{ss} \) vary. On each such cone, the linear function \( \langle \chi_{ac}, \cdot \rangle \) is strictly positive. We choose \( a_1 > 0 \) such that \( \langle \chi_{ac}, \lambda \rangle \geq a_1 |\lambda| \), \( \forall \lambda \in \Lambda'_1 \cup \ldots \cup \Lambda'_s \).

For fixed \( \varphi \), the function \( m(\varphi, \cdot) \) is piecewise linear. As \( \varphi \) varies, \( m(\varphi, \cdot) \) depends only on the weights of \( T \) on \( E \). Overall we find a constant \( a_2(E) > 0 \) independent of \( \varphi \) such that \( |m(\varphi, \lambda)| \leq a_2(E) \cdot |\lambda| \) for all \( \lambda \in \mathcal{X}(T) \).

Back to the proposition:

\[
a_2(E) \cdot |\lambda_0| \geq m(\varphi, \lambda_0) \geq 1 \quad \Rightarrow \quad |\lambda_0| \geq \frac{1}{a_2(E)}.\]

Hence:

\[
-l + m \cdot \langle \chi_{ac}, \lambda_0 \rangle \geq -l + m \cdot a_1 |\lambda_0| \geq -l + m \cdot \frac{a_1}{a_2(E)}.
\]

We conclude that for \( \frac{m}{T} > \frac{a_2(E)}{a_1} \) the condition \((C)\) is satisfied.

‘\( \Leftarrow \)’ We prove that:

\[
(\mathbb{E} \times \mathbb{V}^m)_{ss}(K^x \times G, l \chi + m \chi_{ac}) \supset (\mathbb{E} \setminus \{0\}) \times (\mathbb{V}^m)_{ss}(G, \chi_{ac}) \quad \text{for} \quad m \gg 0.
\]

The conclusion follows from the hypothesis \((\mathbb{V}^m)_{ss}(G, \chi_{ac}) = (\mathbb{V}^m)_{ss}(G, \chi_{ac}) \).

Recall from \((3.4)\) that \( \psi \in (\mathbb{V}^m)_{ss}(G, \chi_{ac}) \) if and only if \( m(\psi) < 0 \). The value \( m(\psi) \) is reached at the ‘worst’ destabilizing \( \lambda \in \mathcal{X}(G) \) (see \((3)\)). For variable \( m \), there are only finitely many combinatorial strata in \((\mathbb{V}^m)_{ss}(G, \chi_{ac}) \), hence only finitely many possible values for \( m(\psi) \). It follows that:

\[
-\mu := \max \{ m(\psi) \mid m \geq 1, \psi \in (\mathbb{V}^m)_{ss}(G, \chi_{ac}) \} < 0.
\]

Now consider \((\varphi, \psi) \in (\mathbb{E} \setminus \{0\}) \times (\mathbb{V}^m)_{ss}(G, \chi_{ac}) \), and its worst destabilizing \( \lambda \in \mathcal{X}(G) \). After possibly moving \( \psi \) by an element in \( G \), we may assume that \( \lambda \in \mathcal{X}(T) \). Then holds:

\[
m(\varphi, \lambda) \geq 0, \quad \text{and} \quad \frac{\langle \chi_{ac}, \lambda \rangle}{|\lambda|} = m(\psi) \leq -\mu.
\]

We distinguish the following cases:

- If \( m(\varphi, \lambda) = 0 \) resp. \( > 0 \), then \((3.3)\)(A) and (B) imply that \((\varphi, \psi) \) is \( l \chi + m \chi_{ac} \) unstable.
If $m(\varphi, \lambda) < 0$, we normalize $\lambda$ such that $m(\varphi, \lambda) = -1$. We claim that $l + m(\chi_{ac}, \lambda) \leq 0$ for $m$ large enough. Otherwise we deduce:

$$
\mu |\lambda| \leq |(\chi_{ac}, \lambda)| < l/m \\
1 = |m(\varphi, \lambda)| \leq \frac{a_2(E) \cdot |\lambda|}{\mu} \\
\Rightarrow \frac{m}{l} < \frac{a_2(E)}{\mu}.
$$

\[\Box\]

### 4. The nef vector bundles

In this section we define a finite set of ‘extremal’ nef vector bundles, which will be the building blocks of the exceptional sequences. We continue the notations of the previous section. Consider the following Weyl group invariant cone:

\[ \mathcal{N} = \mathcal{N}(G, V) := \mathcal{C}_1 \cap \ldots \cap \mathcal{C}_2 = (\Lambda_1 + \ldots + \Lambda_2)^{\vee}. \]

When $G$ is a torus, $\mathcal{N}$ is the nef cone of the quotient, which is a toric variety. In our context, $\mathcal{N}$ can be viewed as the nef cone of $\mathcal{V}^{ss}(T, \chi_{ac})/T$. Its importance relies on the following:

**Proposition 4.1.** We consider a $G$-module $V$ which has the following property: $\mathcal{V}^{ss}(G, \chi_{ac}) = \mathcal{V}^s(G, \chi_{ac})$. Let $E$ be a $G$-module, and $E \to Y$ be its associated vector bundle.

Then $E$ is nef if and only if all the weights of $T$ on $E$ belong to the cone $\mathcal{N}$. We call a module with this property a nef module.

**Proof.** ($\Leftarrow$) Let us assume that the weights of $E$ belong to $\mathcal{N}$. We prove that, on $\mathbb{P}(\mathcal{E}^\vee)$, the class $\chi_t$ is nef, it means $\chi_t + r\chi_{ac}$ is ample for all $r > 0$. This translates into the following condition:

\[ (\mathcal{E}^\vee \times \mathcal{V})^s(K^\times \times G, \chi_t + r\chi_{ac}) = (\mathcal{E}^\vee \setminus \{0\}) \times \mathcal{V}^s(G, \chi_{ac}), \quad \forall r > 0. \]

‘$\supset$’ The conditions $(\mathcal{E}^\vee)(A)$ and $(B)$ are trivially satisfied. We show that the case $(\mathcal{E}^\vee)(C)$ does not occur.

Take $(\psi, v) \in (\mathcal{E}^\vee \setminus \{0\}) \times \mathcal{V}^s(G, \chi_{ac})$, and suppose that there is $\lambda_0$ with $m(\psi, \lambda_0) \geq 1$ and $m(v, \lambda_0) \geq 0$. Then $\lambda_0 \in \text{int}((\mathcal{C}_v)^{\vee}) \subseteq -\text{int} \mathcal{N}^{\vee}$ and also $\lambda_0 \in \mathcal{C}_v^{\vee} \subset \mathcal{N}^{\vee}$. Contradiction.

‘$\subset$’ For shorthand, we denote $\mathcal{S}_L$ resp. $\mathcal{S}_R$ the left- and the right-hand-side above. Note that the quotient $\mathcal{S}_R/K^\times \times G$ exists, and equals $\mathbb{P}(\mathcal{E}^\vee)$; let $Z := \mathcal{S}_L/(K^\times \times G)$ be the quotient.

By previous step, there is a morphism $\phi : \mathbb{P}(\mathcal{E}^\vee) \to Z$. Since $\phi$ is open and $\mathbb{P}(\mathcal{E}^\vee)$ is projective, $\phi$ is surjective. Recall from [13, Theorem 1.10], that $K^\times \times G$ acts with closed orbits on $\mathcal{S}_L$, and the quotient $\mathcal{S}_L \to Z$ is geometric. Since $\mathbb{P}(\mathcal{E}^\vee) \to Z$ is surjective, the inclusion $\mathcal{S}_L \supset \mathcal{S}_R$ must be an equality. Otherwise we find closed orbits in $\mathcal{S}_L$, which are not contained in $\mathcal{S}_R$.

($\Rightarrow$) Assume that $\mathcal{E} \to Y$ is nef, that means $\chi_t$ is a nef class on $\mathbb{P}(\mathcal{E}^\vee)$. By inspecting the conditions $(\mathcal{E}^\vee)$ we deduce:

\[ \exists \psi \in \mathcal{E}^\vee \setminus \{0\}, \ v \in \mathcal{V}^s(G, \chi_{ac}), \ \lambda \in \mathcal{X}_s(T) \text{ s.t. } \begin{cases} m(\psi, \lambda) \geq 1, \\ m(v, \lambda) \geq 0. \end{cases} \]

We choose $\psi = \varphi^\vee$, with $\varphi \in E$ of weight $\varepsilon$. The previous condition implies: $\exists \lambda \in \mathcal{X}_s(T)$ such that $(\varepsilon, \lambda) < 0$, and $\lambda \in (-\mathbb{R}_+ \varepsilon + \mathcal{N})^{\vee}$. This happens only for $\varepsilon \in \mathcal{N}. \quad \Box$

There is also an effective procedure to produce ‘the smallest’ such modules. Let us consider the set of weights:

\[ \mathcal{N}_1 = \mathcal{N}_1(G, V) := \left\{ \xi \mid \mathbb{R}_+ \xi \text{ is an extremal ray of } \mathcal{N}, \xi \text{ generates } \mathbb{R}_+ \xi \cap \mathcal{X}_s(T) \text{ over } \mathbb{Z}_{\geq 0} \right\}. \]
It is a Weyl-invariant set, and therefore it makes sense considering the irreducible $G$-modules whose dominant weights belong to $\mathcal{M}_1$. These modules will be the building blocks for constructing exceptional sequences. We denote

\begin{equation}
\mathcal{V}^+(Y) := \left\{ E \mid \text{the dominant weight of the } G\text{-module } E \text{ belongs to } \mathcal{M}_1 \right\}.
\end{equation}

Equivalently, denote $\mathcal{W}_G^+$ the closure of the positive Weyl chamber of $G$. Then $\mathcal{V}^+(Y)$ can be identified with

$$\mathcal{M}_1^+(G,V) := \mathcal{W}_G^+ \cap \mathcal{M}_1(G,V).$$

**Lemma 4.2.** The set $\mathcal{V}^+(Y)$ is finite. For any $E \in \mathcal{V}^+(Y)$, the weights of $T$ on $E$ belong to the cone $\mathcal{N}$.

**Proof.** As $\mathcal{M}_1$ is finite, $\mathcal{V}^+(Y)$ is the same. Let $\xi$ be the dominant weight of $E$. The weights of $T$ on $E$ belong to the convex hull of the images of $\xi$ under the Weyl group. But all of them generate rays of $\mathcal{N}$. Hence the convex hull of the images of $\xi$ is contained in $\mathcal{N}$. \hfill \Box

**Proposition 4.3.** Let $M$ be an irreducible, nef $G$-module. Then there are $E_1, \ldots, E_n \in \mathcal{V}^+(Y)$, and $c_1, \ldots, c_n \geq 1$ such that $M \subset \bigotimes_{j=1}^n \Sym^c E_j$. We say that $M$ is a positive combination of extremal nef modules.

**Proof.** Since the $G$-module $M$ is nef, its highest weight $\xi_M$ belongs to the cone $\mathcal{N}$. Then $\xi_M$ is a positive combination of $\xi_1, \ldots, \xi_n \in \mathcal{M}_1$:

$$\xi_M = \sum_{j=1}^n c_j \xi_j, \quad c_j \geq 1.$$  

Each $\xi_j$ is conjugated to some $\xi_j^+ \in \mathcal{M}_1^+$, since the Weyl group acts transitively on the Weyl chambers. The irreducible $G$-module $E_j$ with highest weight $\xi_j^+$ belongs to $\mathcal{V}^+(Y)$. Now observe that $\xi_M$ appears among the weights of $\bigotimes_{j=1}^n \Sym^c E_j$. Hence the whole module $M$ is contained in it. \hfill \Box

**Lemma 4.4.** Consider the set $\mathcal{V}^+(Y)$ of extremal nef vector bundles on $Y$, defined in (4.3). Then the anti-canonical character $\chi_{ac}(G,V)$ is a positive linear combination of $\det E$, with $E \in \mathcal{V}^+(Y)$:

$$\chi_{ac} = \sum_{E \in \mathcal{V}^+(Y)} m_E \cdot \det(E), \quad \text{with } m_E \geq 0.$$

**Proof.** Let $\{\xi_j\}_j$ be the elements of $\mathcal{M}_1$. Since $\chi_{ac}$ belongs to the interior of $\mathcal{N}$, there are positive numbers $c_j$ such that $\chi_{ac} = \sum_{j} c_j \xi_j = \sum_{j} c_j \xi_j^0 + \sum_{j} c_j \xi_j^\prime$. We decompose $\chi^*(T)_Q = \chi^*(Z(G)\circ) Q \oplus \chi^*(T')_Q$. Accordingly, each $\xi_j$ decomposes into $\xi_j = \xi_j^0 + \xi_j^\prime$, and each $\xi_j$ is conjugated to some $\xi_j^+ \in \mathcal{M}_1^+$. Let $E_j \in \mathcal{V}^+(Y)$ be the irreducible $G$-module with highest weight $\xi_j^+$. Note that $Z(G)\circ$ acts on $E_j$ by the character $\xi_j^0$. Since $\chi_{ac}$ is trivial on the semi-simple part of $G$, we deduce that $\chi_{ac} = \sum_{j} c_j \xi_j^0 = \sum_{j} c_j \dim E_j \det E_j$. \hfill \Box

5. COHOMOLOGICAL PROPERTIES OF NEF VECTOR BUNDLES

In section 4 we have introduced the set of nef vector bundles associated to representations of $G$. In this section we are going to study their cohomological properties.
Theorem 5.1. Let $E$ be a nef $G$-module. Then $H^q(Y, E) = 0$ for all $q > 0$.

Proof. Using the projection formula, $H^q(Y, E) = H^q(\mathbb{P}(E'), \mathcal{O}_\mathbb{P}(1))$, and $\mathcal{O}_\mathbb{P}(1) \to \mathbb{P}(E')$ is a nef line bundle. The vanishing of the latter cohomology group is a consequence of the Hochster-Roberts theorem (see [4]).

We place ourselves in the following framework:

\begin{equation}
\begin{cases}
\text{(i) There is a quotient group } H \\ 
\text{with kernel } G_0 \\ 
\text{and a quotient } H\text{-module } W \text{ of } V \\
\text{with kernel } \mathcal{V}_0 \\
\text{such that the natural projection } \pi_W : V \to W \\
\text{has the property } \pi_W^* (\mathbb{V}^s(G, \chi_{ac}(G, V))) \subseteq \mathbb{V}^s(H, \chi_{ac}(H, W)).
\end{cases}
\end{equation}

(5.1)

We denote $Y \to X$ the induced morphism.

(ii) Both unstable loci have codimension at least two.

(iii) $G$ and $H$ act freely on $\mathbb{V}^s(G, \chi_{ac}(G, V))$ and $\mathbb{V}^s(H, \chi_{ac}(H, W))$ respectively.

Now let us study the positivity properties of direct images of nef vector bundles.

Lemma 5.2. Suppose that we are in the situation (5.1), and that $E$ is a $G$-module such that its associated vector bundle $E \to Y$ is nef. Then $\phi_\ast E \to X$ is a vector bundle, and it is associated to the $H$-module $\phi_\ast E := \text{Mor}(\mathcal{V}_0, E)^{G_0} = H^0(\mathcal{V}_0/\chi_{ac} G_0, E)$.

Proof. The restriction of $E$ to the fibres of $\phi$ are nef. By applying theorem 5.1 we obtain that $R^q \phi_\ast E = 0$ for all $q > 0$, and therefore $\phi_\ast E \to X$ is locally free. Observe that both $V_0$ and $H$ are actually $G$-modules, and $V = V_0 \oplus W$; the kernel $G_0$ is acting trivially on $W$. For an $H$-invariant open set $O \subseteq \mathcal{W}$, holds:

\[
H^0(O/\mathcal{H}, \phi_\ast E) = H^0(\mathcal{V}_0 \times O/\mathcal{G}, E) = \text{Mor}(\mathcal{V}_0 \times O, E)^H
\]

\[
= (\text{Mor}(\mathcal{V}_0 \times O, E)^{G_0})^H = \text{Mor}(O, \text{Mor}(\mathcal{V}_0, E)^{G_0})^H.
\]

Theorem 5.3. Assume that (5.1) holds, and let $E$ be a nef $G$-module. Then the $H$-module $\phi_\ast E$ is still nef. (The direct image $\phi_\ast E \to X$ is a nef vector bundle.)

Mourougane proves in [12] a similar statement for adjoint bundles. The proof below follows ad litteram his proof (loc. cit. section 3), with the necessary changes.

Proof. By lemma 5.2, $\phi_\ast E \to X$ is locally free.

Step 1: Construct the tensor powers $(\phi_\ast E)^{\otimes n}$.

Let $Y^{(n)} = Y \times_X \ldots \times_X Y$ be the fibre product, and $\phi^{(n)} : Y^{(n)} \to X$ be the projection. Note that the vector bundle $\mathcal{E}^{(n)} := \mathcal{E} \times_X \ldots \times_X \mathcal{E}$ on $Y^{(n)}$ is nef. Its direct image is $\phi^{(n)} \mathcal{E} = (\phi_\ast E)^{\otimes n}$. Moreover, $Y^{(n)}$ is the quotient of the affine space $\mathbb{V}^{(n)}$ by the action of the group $G^{(n)}$, and $\mathcal{E}^{(n)}$ is associated to the $G^{(n)}$-module $E^{\otimes n}$.

- $V^{(n)} := \{(v_1, \ldots, v_n) \in V^{\otimes n} | \pi_W^{v_1} = \ldots = \pi_W^{v_n} \}$;
- The group $G^{(n)} := G \times \mu \ldots \times \mu G$ is still reductive.

Step 2: Let $A \to X$ be a very ample line bundle, associated to some character of $H$. Then $(\phi_\ast E)^{\otimes n} \otimes A^{\dim X + 1}$ is globally generated.

We replace $Y$ by $Y' := Y^{(n)}$, $\phi$ by $\phi' := \phi^{(n)}$, and $\mathcal{E}$ by $\mathcal{E}' := \mathcal{E}^{(n)}$.

By the Castelnuovo-Mumford criterion, in order to prove that $\phi_\ast \mathcal{F} \otimes A^{\dim X + 1}$ is globally generated, it is enough to check that $H^q(X, \phi_\ast \mathcal{E}' \otimes A^{\dim X + 1-q}) = 0$ for all $q > 0$. Since the higher direct images of $\mathcal{E}'$ vanish, the projection formula gives:
deduce that its higher cohomology groups vanish. 

But $Y'$ is still a quotient of an affine space, $E'$ is associated to a nef $G$-module, and $(\phi')^*A$ corresponds to a nef character of $G$. We apply theorem 5.3 to $E' \otimes (\phi')^*A^{\dim X+1-\eta}$, and deduce that its higher cohomology groups vanish.

**Step 3:** According to the previous step $(\phi, E)^{\otimes n} \otimes A^{\dim X+1}$ is globally generated for all $n > 0$, and therefore $\phi, E$ is nef. 

We use this result to describe more precisely the nef cone $\mathcal{N}(G,V)$. We consider the projective variety

$$\text{Flag}(Y) := \Omega_G/B = (\Omega_G \times (G/B))/G,$$

and denote $\pi : \text{Flag}(Y) \to Y$ the projection. It is a $G/B$-fibre bundle over $Y$, justifying the notation $\text{Flag}(Y)$.

For any $\xi \in \mathcal{X}^*(T) = \mathcal{X}^*(B)$, we denote by $L_\xi \to \text{Flag}(Y)$ the line bundle $(\Omega_G \times K)/B$, where $B$ acts on $K$ by $\xi$.

**Corollary 5.4.** Let $\xi \in \mathcal{X}^*(T)$ be a dominant character, and let $E_\xi$ be the corresponding irreducible $G$-module. Then holds:

(i) $E_\xi = \pi_*L_\xi$;

(ii) $E_\xi \to Y$ is nef if and only if $L_\xi \in \text{Pic}^+(\text{Flag}(Y)) := \text{the nef cone of Flag}(Y)$.

**Proof.** (i) The equality is a direct consequence of the Borel-Weil theorem, which says that $H^0(G/B, L_\xi) = E_\xi$.

(ii) Assume that $L_\xi$ is nef. The Borel-Weil theorem implies that the higher direct images $R^i\pi_*L_\xi = 0$. By the same argument of the theorem 5.3 we deduce that $E_\xi = \pi_*L_\xi \to Y$ is still nef.

Conversely, assume that $E_\xi$ is nef, hence $\mathcal{V}^{ss}(T, \chi_{ac}) \subset \mathcal{V}^{ss}(T, \xi)$. We claim that some tensor power of $L_\xi$ is globally generated, and therefore $L_\xi$ is nef. Let $B$ be the Borel subgroup of $G$ for which $\xi$ is dominant. Our hypothesis implies that

$$\mathcal{V}^{ss}(G, \chi_{ac}) = \bigcap_{g \in G} g^{-1}\mathcal{V}^{ss}(T, \chi_{ac}) \subset \bigcap_{b \in B} b^{-1}\mathcal{V}^{ss}(T, \chi_{ac}) \subset \bigcap_{b \in B} b\mathcal{V}^{ss}(T, \xi) = \mathcal{V}^{ss}(B, \xi).$$

Observe that $B$ is solvable, not reductive, and therefore the standard invariant theory does not apply. The $B$-semi-stable locus $\mathcal{V}^{ss}(B, \xi)$ is defined exactly as in (1.1), in terms of the algebra $K[V]^B, \xi$. Its finite generacy has been proved by Grosshans (see e.g. 4.1 Corollary 9.5).

We deduce that for some $n > 0$, $\mathcal{V}^{ss}(B, \xi)$ can be covered by a finite number of sets $\{y \mid f(y) \neq 0\}$, with $f \in K[V]_{\xi^n}$. Altogether, we find at each point $y \in \mathcal{V}^{ss}(G, \chi_{ac})$ a function which is $(B, \xi^n)$-equivariant, and does not vanish at $y$. Hence $L_\xi^n$ is globally generated. 

**Corollary 5.5.** Suppose that (5.1) holds. Let $E$ be a nef $G$-module, and $M$ an irreducible $H$-submodule of $\phi_*E$. Then $M$ is a direct summand in a $H$-module of the form $\bigotimes_{F \in \mathcal{V}^+(X)} \text{Sym}^{c_F} F$.

**Proof.** The push-forward $\phi_*E \to X$ is nef, and therefore all its weights belong to the cone $\mathcal{N}(H,W)$. We deduce that $M$ is nef too, and the conclusion follows from proposition 1.3.

**Example 5.6.** Consider the Grassmannian $X := \text{Grass}(K^m, d)$ of $d$-dimensional quotients, and denote $Q$ the tautological quotient on it. Note that the variety $\text{Flag}(X)$ is the variety of full quotient flags of $Q$. The cone $\mathcal{M}^+ \cap \text{Pic}^+(\text{Flag}(X))$ is generated by $d$ elements which
correspond to the characters $\tau_1, \tau_1 + \tau_2, \ldots, \tau_1 + \ldots + \tau_d$ (here the $\tau_j$'s denote the obvious characters of the maximal torus in $\text{GL}(d)$).

We deduce that for any nef $\text{GL}(d)$-module $F$, its associated vector bundle $F \rightarrow \text{Grass}(K^m, d)$ is a direct summand in a tensor product of the form

$$\text{Sym}^c_1(Q) \otimes \text{Sym}^c_2(\bigwedge Q) \otimes \ldots \otimes \text{Sym}^c_d(\bigwedge Q).$$

This is in agreement with the fact that this tensor product contains the Schur power $S_\alpha(Q)$, where $\alpha = (\alpha_1 \geq \ldots \geq \alpha_d \geq 0)$, and the positive integers $c_j$ satisfy $\alpha_j = c_j + \ldots + c_d$ for $j = 1, \ldots, d$.

6. The main result: the absolute case

In this section we prove our first main result. We consider a $G$-module $V$, and the character $\chi = \chi_{ac}(G, V)$. Assume that the codimension of the $\chi_{ac}$-unstable locus is at least two, and $G$ acts freely on the semi-stable locus. It follows that $Y := V \sslash \chi_{ac} G = V^{\text{ss}}/(\chi_{ac}(G, V))$ is a projective Fano variety. Observe that lemma [13] implies that $\chi_{ac} = \chi_{ac}(G, V)$ is effective as soon as $m_\omega > d_\omega$ for all $\omega \in X$ (the result below does not require this hypothesis).

We define a Young diagram $\lambda$ of length $d$ to be an array of decreasing integers $(\lambda_1 \geq \ldots \geq \lambda_d)$. We denote $\lambda_{\text{max}} := \lambda_1$, $\lambda_{\text{min}} := \lambda_d$, $\text{length}(\lambda) := d$. For arrays consisting of positive integers, we visualize the Young diagrams, and the parameters as in the figure:

```
  | λ_{max} |
---|--
  | λ |
  | λ_{min} |
  \lambda_{1} \ldots \lambda_{d}

  length(λ)
```

We introduce the following shorthand notation: for a Young diagram $\lambda$, let $\lambda \pm [c]$ be the diagram obtained by adding/subtracting the integer $c$ to/from the entries of $\lambda$. For a vector space $E$ and a Young diagram $\lambda$ of length $\text{dim} E$, we will denote $S^\lambda E$ its usual Schur power (for $\lambda_{\text{min}} \geq 0$), or $S^{\lambda - \lambda_{\text{min}}} \otimes (\det E)^{\lambda_{\text{min}}}$ (for arbitrary $\lambda$).

For two positive numbers $m, d$ we define the following sets:

$$\tilde{Y}_d := \{ \lambda \mid \text{length}(\lambda) = d \};$$

$$\mathcal{Y}_{m,d} := \{ \lambda \in \tilde{Y}_d \mid 0 \leq \lambda_{\text{min}} \leq \lambda_{\text{max}} \leq m \};$$

$$\mathcal{Y}_d := \bigcup_{m \geq 0} \mathcal{Y}_{m,d}; \quad \mathcal{Y}_d^+ := \bigcup_{m \geq 0} \{ \lambda \in \mathcal{Y}_{m,d} \mid \lambda_d \geq \text{length}(\lambda - [\lambda_d]) \}. $$

Roughly speaking, our main result is that certain Schur powers of the extremal nef bundles on $Y$ form a strong exceptional sequence.

The main technical tool that will be used is the following cohomology vanishing theorem, proved by Manivel for Kählerian varieties (see [11]), and Arapura for projective ones (see [2]).

**Theorem** Let $Y$ be a smooth projective variety, and $\{E_1, \ldots, E_N\}$ be a set of nef vector bundles over $Y$. Choose a set $\{\lambda^{(1)}, \ldots, \lambda^{(N)}\}$ of Young diagrams such that $\lambda^{(j)} \in \mathcal{Y}_{\text{rank}(E_j)}^+$.
for all $j$. Consider an ample line bundle $A \to Y$. Then holds:

$$H^q(Y, S^{\lambda(1)} \mathcal{E}_1 \otimes \ldots \otimes S^{\lambda(N)} \mathcal{E}_N \otimes A \otimes \kappa_Y) = 0.$$ 

Next comes our first main result.

**Theorem 6.1.** Let $V$ be a $G$-module such that $K[V]^T = K$. Assume that the unstable locus has codimension at least two, and that $G$ acts freely on $\mathbb{V}^{ss}(G, \chi_{\text{ac}})$; we denote by $Y := \mathbb{V}^{ss}(G, \chi_{\text{ac}})/G$ the quotient. We consider the order $<_1$ defined in 2.4.

Let $E_1, \ldots, E_N$ be the elements of $\mathbb{V}^{ss}(Y)$, and denote $d_j := \dim E_j$. We write $\chi_{\text{ac}} = \sum_{j=1}^N m_j \cdot \det(E_j)$, with $m_j \geq 0$ as in lemma 4.4 and assume that all the numbers $m_j$ are integers. Consider the set

$$\mathcal{E}(Y) := \text{the set of all irreducible } G\text{-modules contained in}$$

$$S^{\lambda(1)} \mathcal{E}_1 \otimes \ldots \otimes S^{\lambda(N)} \mathcal{E}_N,$$

where $\lambda(j) \in \mathcal{Y}_{m_j-d_j,d_j}$.

Then the vector bundles $\mathcal{E} \to Y$ associated to the modules $E \in \mathcal{E}(Y)$ form a strong exceptional sequence over $Y$ w.r.t. the order $<_1$.

**Proof.** The condition on $H^0(\text{Hom}(\mathcal{U}', \mathcal{U}''))$ for two elements $\mathcal{U}', \mathcal{U}'' \in \mathcal{E}(Y)$ is implied by theorem 2.5. It remains to prove the vanishing of the higher cohomology groups. First of all we observe that, by definition, the vector bundles $\mathcal{U}', \mathcal{U}''$ are direct summands of $S^{\lambda_{\mathcal{E}}} \mathcal{E}_*$. Therefore it is enough to prove that vanishing of $H^q(Y, \text{Hom}(S^{\lambda_{\mathcal{E}}} \mathcal{E}_*, S^{\mu_{\mathcal{E}}} \mathcal{E}_*))$, $q > 0$. Using the Littlewood-Richardson rules, we decompose

$$\text{Hom}(S^{\lambda_{\mathcal{E}}} \mathcal{E}_*, S^{\mu_{\mathcal{E}}} \mathcal{E}_*) = \bigoplus_{\alpha^*=(\alpha^{(1)}, \ldots, \alpha^{(N)})} S^{\alpha^*_{\mathcal{E}}} \mathcal{E}_*,$$

and observe that $\alpha^{(j)} = (m_j - d_j \geq \alpha_{d_j}^{(j)} \geq \ldots \geq \alpha_{d_j}^{(j)} \geq -m_j + d_j)$. For each direct summand holds:

$$H^q(Y, S^{\alpha^*_{\mathcal{E}}} \mathcal{E}_*) = H^q\left(Y, \kappa_Y \otimes \bigotimes_{j=1}^N (S^{\alpha^{(j)}_{\mathcal{E}}} \mathcal{E}_j \otimes \det(\mathcal{E}_j)^{m_j})\right)$$

$$= H^q\left(Y, \kappa_Y \otimes \bigotimes_{j=1}^N S^{\alpha^{(j)}_{\mathcal{E}}} \mathcal{E}_j \otimes \det(\mathcal{E}_j)^{m_j}\right).$$

Note that $\alpha^{(j)} + \underbrace{m_j}_{\tilde{\alpha}_{d_j}^{(j)}} = \alpha^{(j)} + \underbrace{-\alpha_{d_j}^{(j)} + d_j - 1}_{\tilde{\alpha}_{d_j}^{(j)}} + \overbrace{\alpha_{d_j}^{(j)} + m_j - d_j + 1}^\alpha$ and

$$\left\{ \begin{array}{c}
\tilde{\alpha}_{d_j}^{(j)} = d_j - 1 \geq \text{length} \left( \alpha^{(j)} - \frac{d_j}{d_j - 1} \right) \\
\tilde{\alpha}_{d_j}^{(j)} := \alpha_{d_j}^{(j)} + m_j - d_j + 1 \geq 1.
\end{array} \right.$$}

Since $E_1, \ldots, E_N$ are all the extremal nef bundles, it follows that the $A := \bigotimes_{j=1}^N \det(\mathcal{E}_j)^{\tilde{\alpha}_{d_j}^{(j)}}$ is an ample line bundle over $Y$. The theorem cited above implies that the higher cohomology of $S^{\alpha^*_{\mathcal{E}}} \mathcal{E}_*$ vanishes. \hfill $\square$

**Corollary 6.2.** Assume that the $G$-module $V$ has the property that the multiplicities $m_\omega > d_\omega$ for all $\omega \in \chi$. Then the exceptional sequence constructed above is formed by semi-stable vector bundles.

**Proof.** It is an immediate consequence of the corollary 4.4. \hfill $\square$
Remark 6.3. It is important to observe that $\kappa_Y^{-1}$ is ample, and it becomes increasingly positive as we increase the multiplicities $m_\omega$ of the isotypical components of $V$. It follows that the effect of increasing the $m_\omega$’s is that of simultaneously increasing the dimension of the quotient, and that of the length of the exceptional sequence. In other words, for our construction we will always have a lower bound for

$$\frac{\text{length of exceptional sequence on } Y}{\text{Euler characteristic of } Y}.$$

Compare this construction with the one discussed in subsection 8.3.

7. The main result: the relative case

Theorem 6.1 is too weak for fibred varieties. By applying it directly, one loses many terms of the exceptional sequences (see subsections 8.1 and 8.2). The goal of this section is to address the relative case described in (5.1). The additional hypothesis which will be imposed in (7.1) may look overabundant, but in many concrete cases they are naturally fulfilled (especially for quiver representations).

Definition 7.1. Denote $T_0$ and $T_H$ the maximal tori of $G_0$ and $H$ respectively. The exact sequence $1 \to G_0 \to G \to H \to 1$ induces a natural splitting $\mathcal{X}^*(T)_{\mathbb{Q}} = \mathcal{X}^*(T_0)_{\mathbb{Q}} \oplus \mathcal{X}^*(T_H)_{\mathbb{Q}}$.

We will denote by $\mathcal{N}(G_0, V_0)$ respectively $\mathcal{N}(H, W)$ the nef cones of the $G_0$-module $V_0$ and $H$-module $W$, corresponding to $\chi_{\text{ac}}(G_0, V_0) = \chi_{\text{ac}}(G, V)|_{G_0}$ and $\chi_{\text{ac}}(H, W)$.

Throughout this section we will assume:

$$\begin{align*}
(i) & \text{ The situation described in (5.1) holds.} \\
(ii) & \mathcal{N}(G, V) = \mathcal{N}(G_0, V_0) + \mathcal{N}(H, W). \\
(\text{We use the shorthand notation } \mathcal{N} = \mathcal{N}_0 + \mathcal{N}_H.) \\
(iii) & \text{ The maximal torus } T_0 \subset G_0 \text{ has exactly } \dim T_0 \text{ weights on } V_0.
\end{align*}$$

(7.1)

Remark 7.2. Let us make a few comments related to the assumptions:

- The condition (ii) means that there is a partition
  $$\mathcal{VB}^+(Y) = \mathcal{VB}^+(X) \cup \mathcal{VB}^+(\text{fibre}).$$

The set $\mathcal{VB}^+(X)$ can always be viewed as a subset of $\mathcal{VB}^+(Y)$ via the pull-back $V \overset{\phi}{\to} W$. What we assume is that the ‘extremal’ nef bundles on the fibres extend to ‘extremal’ nef bundles on the whole $Y$. For shorthand, we will write $\mathcal{VB}_0^+ := \mathcal{VB}^+(\text{fibre}).$

- $T_0$ has always at least $\dim T_0$ linearly independent weights on $V_0$. The assumption (iii) is equivalent to any of the following:
  
  (iii') For any $\xi \in \mathcal{X}^*(T_0)$, $\xi$ is $T_0$-nef on $V_0$ if and only if $\xi$ is $T_0$-effective on $V_0$;

(iii'') The quotient $V_0/T_0$ is a product of projective spaces.

Observe that by lemma 4.3 we can express

$$\chi_{\text{ac}}(H, W) = \sum_{F \in \mathcal{VB}^+(X)} m_F \cdot \det F \quad (m_F \geq 0), \quad \text{and}$$

$$\chi_{\text{ac}}(G_0, V_0) = \sum_{E \in \mathcal{VB}_0^+} m_E \cdot \det E \quad (m_E \geq 0).$$
Proposition 7.3. Assume that (7.1) holds, and denote $d_F := \dim F$, and $d_E := \dim E$.

(i) Suppose that $(a_E)_{E \in V B^+_0}$ and $(b_F)_{F \in V B^+(X)}$ are integers having the following property: for all $q > 0$, and all Young diagrams $\alpha^E \in \overline{\mathcal{Y}}_{d_E}$ resp. $\beta^F \in \overline{\mathcal{Y}}_{d_F}$, such that $\alpha_{\min}^E \geq -a_E$ and $\beta_{\min}^F \geq -b_F$, holds:

\[
H^q \left( V_0 \big/ _{\chi ac} (G_0, V_0), \bigotimes_{E \in V B^+_0} S^{\alpha^E} \mathcal{E} \right) = 0, \tag{7.2}
\]

\[
H^q \left( X, \bigotimes_{F \in V B^+(X)} S^{\beta^F} \mathcal{F} \right) = 0. \tag{7.3}
\]

Then $H^q \left( Y, \bigotimes_{F \in V B^+(X)} \phi^* S^{\beta^F} \mathcal{F} \otimes \bigotimes_{E \in V B^+_0} S^{\alpha^E} \mathcal{E} \right) = 0$ for all $q > 0$, and for all Young diagrams $\beta^F \in \overline{\mathcal{Y}}_{d_F}$ and $\alpha^E \in \overline{\mathcal{Y}}_{d_E}$ with $\beta_{\min}^F \geq -b_F$ and $\alpha_{\min}^E \geq -a_E$ respectively.

(ii) The condition (7.2) is fulfilled for $a_E := m_E - d_E$, $\forall E \in V B^+_0$.

(ii) The condition (7.3) is fulfilled for $b_F := m_F - d_F$, $\forall F \in V B^+(X)$.

**Proof.** (i) The hypothesis implies that the higher direct images of $\bigotimes_{E \in V B^+_0} S^{\alpha^E} \mathcal{E}$ vanish. By using the projection formula we deduce:

\[
H^q \left( Y, \bigotimes_{F \in V B^+(X)} \phi^* S^{\beta^F} \mathcal{F} \otimes \bigotimes_{E \in V B^+_0} S^{\alpha^E} \mathcal{E} \right) = H^q \left( X, \bigotimes_{F \in V B^+(X)} S^{\beta^F} \mathcal{F} \otimes \phi_* \left( \bigotimes_{E \in V B^+_0} S^{\alpha^E} \mathcal{E} \right) \right).
\]

Let us write $\mathcal{V}^0 := \bigotimes_{E \in V B^+_0} S^{\alpha^E} \mathcal{E}$, and decompose it into the direct sum corresponding to the irreducible $G$-modules appearing in the tensor product: $\mathcal{V}^0 = \bigoplus_{j} \mathcal{V}^0_j$. The cohomology group breaks up into the direct sum of the ‘smaller’ cohomology groups. For each component $\mathcal{V}^0_j$ there are two possibilities:

**Case 1** There is a weight of $T_0$ on $V^0_j$ which is not effective. In this case $\phi_* \mathcal{V}^0_j = 0$ (c.f. theorem 2.13), and we discard it from the direct sum.

**Case 2** All the weights of $T_0$ on $V^0_j$ are effective. In this case the hypotheses (7.1) (ii)+(iii) imply that the weights of $V^0_j$ are nef, and therefore $\mathcal{V}^0_j \rightarrow Y$ is nef itself. Using theorem 3.3 and proposition 4.3 we deduce that $\phi_* \mathcal{V}^0_j \rightarrow X$ is nef, and is actually contained in $\bigotimes_{F \in V B^+(X)} S^{\beta^F} \mathcal{F}$, with $c_F \geq 0$.

The Littlewood-Richardson rules imply that the tensor product

\[
\bigotimes_{F \in V B^+(X)} S^{\beta^F} \mathcal{F} \otimes \bigotimes_{F \in V B^+(X)} S^{\beta^F} \mathcal{F}
\]

breaks up into the direct sum of $\bigotimes_{F \in V B^+(X)} S^{\beta^F} \mathcal{F}$, with $\beta_{\min}^F \geq \beta_{\min}^F + c_F \geq b_F$. By the hypothesis, their higher cohomology vanishes.
Corollary 7.5. Assume the following assumptions hold:

\( \alpha_{\min} \geq d_E - m_E \) for all \( E \). It holds:

\[
\bigotimes_{E \in \mathcal{V}^+} S^{\alpha E} \mathcal{E} \otimes \kappa_{Y/X}^{-1} \bigg|_{\text{fibre}} = \bigotimes_{E \in \mathcal{V}^+} \left( S^{\alpha E} \mathcal{E} \otimes (\det \mathcal{E})^{m_E} \right) \bigg|_{\text{fibre}}
\]

and \( \alpha^E + m_E = \alpha^E + d_E - m_E - d_E + 1 \) with

\[
\left\{ \begin{array}{l}
\bar{a}_E^E = d_E - 1 \geq \text{length}(\alpha^E - d_E - 1), \\
\bar{a}^E_{\min} = \alpha^E_{\min} + m_E - d_E + 1 \geq 1.
\end{array} \right.
\]

Manivel and Arapura’s theorem implies that \( R^q \phi_*(S^{\alpha} \mathcal{E}_*) = 0 \), for all \( q > 0 \).

(ii) Consider Young diagrams \( (\beta^F)_{F \in \mathcal{V}^+(X)} \) with \( \beta^F_{\min} \geq d_F - m_F \) for all \( F \). Then holds:

\[
\bigotimes_{F \in \mathcal{V}^+(X)} \mathcal{S}^{\beta^F} \mathcal{F} \otimes \kappa_X^{-1} \bigg|_{\text{fibre}} = \bigotimes_{F \in \mathcal{V}^+(X)} \left( \mathcal{S}^{\beta^F} \mathcal{F} \otimes (\det \mathcal{F})^{m_F} \right) + \bigotimes_{F \in \mathcal{V}^+(X)} \mathcal{S}^{\beta^F + \bar{a}_E^E} \mathcal{F}.
\]

We deduce the vanishing of the higher cohomology as in (iii).

Theorem 7.4. Assume that the conditions (7.1) are satisfied, and that there are integers \( (b_F)_{F \in \mathcal{V}^+(X)} \) which fulfill the property (7.3). Then the elements of the set \( \mathcal{E} \mathcal{S}(Y) \) defined below form a strong exceptional sequence of vector bundles over \( Y \):

\( \mathcal{E} \mathcal{S}(Y) := \{ \text{all direct summands, corresponding to irreducible } \}
\]

\( \mathcal{G} \text{-modules contained in } \phi^* (\mathcal{S}^{\lambda} \mathcal{F}_*) \otimes \mathcal{S}^{\nu} \mathcal{E}_* := \phi^* \left( \bigotimes_{F \in \mathcal{V}^+(X)} \mathcal{S}^{\lambda^F} \mathcal{F} \right) \otimes \bigotimes_{E \in \mathcal{V}^+} \mathcal{S}^{\nu^E} \mathcal{E} \),

with \( \lambda^F \in \mathcal{Y}_{b_F, d_F} \), and \( \nu^E \in \mathcal{Y}_{m_E - d_E, d_E} \).

Moreover, it holds: \( H^q \left( Y, \bigotimes_{F \in \mathcal{V}^+(X)} \phi^* \mathcal{S}^{\beta^F} \mathcal{F} \otimes \bigotimes_{E \in \mathcal{V}^+} \mathcal{S}^{\nu^E} \mathcal{E} \right) = 0 \) for all \( q > 0 \), and all Young diagrams \( \beta^F \in \widetilde{\mathcal{Y}}_{d_F} \) and \( \alpha^F \in \mathcal{Y}_{d_E} \), with \( \beta^F_{\min} \geq -b_F \) and \( \alpha^F_{\min} \geq -(m_E - d_E) \) respectively.

Proof. Let \( \mathcal{U}' \) and \( \mathcal{U}'' \) be two elements of \( \mathcal{E} \mathcal{S}(Y) \). The condition on the \( H^0(\text{Hom}(\mathcal{U}', \mathcal{U}'')) \) follows again from theorem (2.5).

It remains to prove the vanishing of \( H^q(\text{Hom}(\mathcal{U}', \mathcal{U}'')) \), for \( q \geq 1 \). By using the Littlewood-Richardson rules, we deduce that \( \text{Hom}(\mathcal{U}', \mathcal{U}'') \) is direct summand in \( \bigoplus_{\alpha^*, \beta^*} \phi^* (\mathcal{S}^{\beta^F} \mathcal{F}_*) \otimes \mathcal{S}^{\nu^E} \mathcal{E}_* \), with

\[
\left\{ \begin{array}{l}
b_F \geq \beta^F_{\max} \geq \beta^F_{\min} \geq -b_F, \\
m_E - d_E \geq \alpha^E_{\max} \geq \alpha^E_{\min} \geq -m_E + d_E.
\end{array} \right.
\]

The conclusion of the theorem follows from proposition (7.3)(ii).

An immediate consequence of the previous theorem is the following:

Corollary 7.5. Assume the following assumptions hold:

(i) There is a sequence of quotients \( G \rightarrow G_1 \rightarrow \cdots \rightarrow G_k \rightarrow 1 \), with \( \Gamma_j := \text{Ker}(G_j \rightarrow G_{j+1}) \).
(ii) \( V = W_1 \oplus \ldots \oplus W_k \), where \( W_j \) is a \( G_j \)-module for all \( j \). We define \( V_j := W_j \oplus \ldots W_k \) for all \( j \).

(iii) The projections \( \text{pr}_j : V_j \to V_{j+1} \) satisfy the conditions (7.1). The induced morphisms are denoted by

\[
\phi_j : V_j \rightarrow V_{j+1},
\]

for all \( 1 \leq j \leq k - 1 \).

Let us write \( \chi_{ac}(\Delta_j, W_j) = \sum_{E \in \mathcal{V}B^+} m_{j,E} \cdot \det E \) (c.f. [4,4]), and denote \( \mathcal{V}B^+ := \mathcal{V}B^+(\mathcal{V}_j//\Delta_j) \).

Then the elements of the set \( \mathcal{E}(Y) \) defined below form a strong exceptional sequence of vector bundles over \( \mathcal{V}//H \):

\[
\mathcal{E}(Y) := \text{all the direct summands, corresponding to irreducible } G\text{-modules contained in } \bigotimes_{j=1}^k \left( \bigotimes_{E \in \mathcal{V}B^+} S_{j,E}^{q_j} \mathcal{E} \right),
\]

with \( \alpha^j, E \in \mathcal{V}_{m_{j,E} - d_E, d_E} \).

Assume moreover that the multiplicity condition in corollary (1.5) is fulfilled. Then \( \mathcal{E}(Y) \) consists of semi-stable vector bundles over \( Y \).

8. Examples

In this section we are going to present a few particular cases, in order to illustrate the general discussion. We concentrate on quiver varieties because they are a source of infinitely many examples, and are also very convenient: for generic choices of the dimension vector, the semi-stability and stability concepts agree. Therefore the quotients which will appear are geometric, as we wish. Even more remarkably, the procedure of constructing exceptional sequences of vector bundles over quiver varieties is almost algorithmic.

Let \( Q = (Q_0, Q_1, h, t) \) be a quiver, and \( q = (d_a)_{q \in Q_0} \) be a dimension vector. We adopt the following convention: suppose that \( q, q' \) are two vertices, and there is (at least) one arrow from \( q \) to \( q' \); then we draw only one arrow \( a \), and we denote by \( m_a \) its multiplicity (that is how many times the arrow is repeated). In other words, we consider the group \( G = \bigotimes_{q \in Q_0} \text{Gl}(d_q) \), and the \( G \)-module \( V = \bigoplus_{a \in Q_1} \text{Hom}(K^{d_t(a)}, K^{d_h(a)})^\oplus m_a \). The construction of exceptional sequences involves the following steps:

**Step 1** Compute the anti-canonical character:

\[
\chi_{ac} = \sum_{a \in Q_1} m_a \cdot (d_t(a) \det h(a) - d_h(a) \det t(a))
\]

\[
= \sum_{q \in Q_0} \left( \sum_{a \in Q_1^+(q)} m_a d_t(a) - \sum_{a \in Q_1^-(q)} m_a d_h(a) \right) \cdot \det(q).
\]

Note that the multiplicative group, embedded diagonally in \( G \), acts trivially on \( V \), and the quotient \( G/(K^\times)_{\text{diag}} \) acts effectively on \( V \). Moreover, for generic choices of the multiplicities \( m_a \) (w.r.t. the dimension vector \( q \)), the \( \chi_{ac}\)-semi-stable locus of \( V \) coincides with the stable locus (see e.g. [9, proposition 3.1]). For such a generic choice, there is a natural ‘Euler sequence’ over the quotient \( Y \):

\[
0 \rightarrow \mathcal{O}_Y^{\dim C} \rightarrow \bigoplus_{a \in Q_1} \mathcal{H}om(\mathcal{E}_t(a), \mathcal{E}_h(a))^{\oplus m_a} \rightarrow T_Y \rightarrow 0.
\]
It follows that the anti-canonical class of the quotient is \( \kappa_{Y}^{-1} = \chi_{ac} \).

**Step 2** It consists in determining the ‘extremal bundles’ in the set \( \mathcal{VB}^+(Y) \) (see (4.3)), and expressing \( \chi_{ac} \) as a positive combination of their determinants (see lemma [4.4]). Actually this step is responsible for the use of the word ‘almost’ above: the computation of the extremal nef bundles is algorithmic, but involves the maximal torus of \( G \), and is therefore tedious.

**Step 3** Denote \( E_1, \ldots, E_N \) the extremal bundles above, and take tensor products of their Schur powers \( S^{\lambda_1} \otimes \cdots \otimes S^{\lambda_N} E_N \). The third step consists in determining the sizes of the Young diagrams \( \lambda_1, \ldots, \lambda_N \) which fulfill the requirements of theorem 6.1.

**Step 4 (Optional)** Search for fibrations coming from a sub-quiver. More precisely, we are looking for a sub-quiver \( R \subset Q \) having the property:

\[
\forall (A_a)_{a \in Q_1} \in \mathcal{Vss}(G, \chi_{ac}(V)) \quad \Rightarrow \quad (A_a)_{a \in R_1} \in \mathcal{Wss}(H, \chi_{ac}(W)),
\]

\[
G = \prod_{v \in Q_0} \text{Gl}(v) \quad \quad \quad H = \prod_{v \in R_0} \text{Gl}(v).
\]

Here \( V \) and \( W \) denote the representation spaces of \( Q \) and \( R \) respectively. In such a situation there is a natural projection map \( Y \to X \) between the corresponding quotients. Moreover, if \( R \) is chosen appropriately, the numerous hypotheses in (7.1) are naturally fulfilled.

Very often one obtains better bounds for the sizes of the Young diagrams involved in the Schur powers than those which are obtained by applying the step 3 directly (see subsections 8.1 and 8.2 below).

### 8.1. Kapranov’s construction

Let us start by reviewing Kapranov’s examples of tilting bundles over the Grassmannian, and over the flag variety for \( \text{Gl}(m) \). We show that by using our approach we automatically recover the vector bundles which appear in the tilting objects constructed by Kapranov over the Grassmannian, and over partial flag manifolds.

They are the quiver varieties associated respectively to:

\[
\begin{array}{ccc}
\begin{array}{cccc}
 m & \to & B & \to & d \\
 A_k & \to & d_k & \to & \cdots & \to & A_1 & \to & d_1
\end{array}
\end{array}
\]

with \( m > d \).

A doubled circle means that the corresponding linear group acts at that entry (we have factored out the diagonal \( K^X \)-action).

#### 8.1.1. The case of the Grassmannian

Let us consider the Grassmannian \( Y := \text{Grass}(\mathbb{C}^m, d) \) of \( d \)-dimensional quotients of \( K^m \). Its anti-canonical class is \( \kappa_{\text{Grass}(K^m, d)}^{-1} = (\det Q)^m \), where \( Q \) denotes the universal quotient bundle. The cone \( \mathcal{N} \) is generated by the characters \( t_1, \ldots, t_d \) of \( \text{Gl}(d) \), and \( \mathcal{N}_1^+ = \{ t_1 \} \). Hence the set \( \mathcal{VB}^+(Y) \) of extremal nef bundles \( \mathcal{VB}^+(Y) \) consists of \( Q \) only.

Theorem 6.1 says that the elements of the set \( \{ S^\lambda Q \mid \lambda \in \mathcal{Y}_{m-d,d} \} \) form a strong exceptional sequence of vector bundles on \( \text{Grass}(K^m, d) \). Indeed, this is what Kapranov proves in loc. cit.. Let us remark that he actually proves that they form a tilting sequence.
8.1.2. The case of flag manifolds. We denote by $\mathbb{F}_k := \text{Flag}(K^m, d_k, \ldots, d_1)$ the variety of quotient $k$-flags of $K^m$. Let $Q_1, \ldots, Q_k$ be the tautological quotient bundles over $\mathbb{F}_k$ with rank $Q_j = d_j$.

The anti-canonical class is $\kappa_{\mathbb{F}_k}^{-1} = \bigotimes_{j=1}^{k} (\det Q_j)^{d_j+1-d_j-1}$. The cone $\mathcal{N}$ is generated by the characters $t_1^{(j)}, \ldots, t_{d_j}^{(j)}$, $j = 1, \ldots, k$, and $\mathcal{N}_1^+ = \{t_1^{(1)}, \ldots, t_1^{(k)}\}$. We deduce that $\mathcal{VB}^+(\mathbb{F}_k) = \{Q_1, \ldots, Q_k\}$. By applying theorem 6.1 directly, we obtain that the elements of

$$\left\{S^{\lambda^*}Q^\vee := S^{\lambda^*}Q_k \otimes \ldots \otimes S^{\lambda_1}Q_1, \quad \lambda^* = (\lambda_k, \ldots, \lambda_1),\right\}$$

with $\lambda^* \in \mathcal{Y}_{m-d_k-d_{k-1},d_k} \times \ldots \times \mathcal{Y}_{d_3-d_2-d_1,d_2} \times \mathcal{Y}_{d_2-d_1,d_1}$, form a strong exceptional sequence over $\mathbb{F}_k$. The problem is that these bounds are very weak, and this set can be empty!

At this point Step 4 becomes useful. There is a natural projection from the $k$-flag onto the $(k-1)$-flag variety

$$\mathbb{F}_k \xrightarrow{\phi} \mathbb{F}_{k-1}, \quad [A_k, \ldots, A_2, A_1] \longmapsto [A_k, \ldots, A_2].$$

One checks easily that all the conditions of (7.1) are fulfilled. By applying corollary 7.3 we deduce that the elements of the set

$$\left\{S^{\lambda^*}Q^\vee := S^{\lambda^*}Q_k \otimes \ldots \otimes S^{\lambda_1}Q_1^\vee, \quad \lambda^* = (\lambda_k, \ldots, \lambda_1) \in \mathcal{Y}_{m-d_k,d_k} \times \ldots \times \mathcal{Y}_{d_3-d_2-d_1,d_2} \times \mathcal{Y}_{d_2-d_1,d_1}\right\}$$

form a strong exceptional sequence of vector bundles over $\mathbb{F}_k$.

8.2. $A_3$-type quiver with multiple arrows. Interesting phenomena occur already for $A_3$-type quivers, as soon as we increase the multiplicities of the arrows. Consider the quiver

$$V = (K^{d_2})^{\oplus m} \oplus \text{Hom}(K^{d_2}, K^{d_1})^{\oplus n}, \quad G = \text{Gl}(d_1) \times \text{Gl}(d_2).$$

Let $Y := \mathbb{N}///_{\chi_{ac}} G$ be the corresponding quiver variety. The flag variety Flag($K^m, d_2, d_1$) corresponds to the case $n = 1$. We denote the vector bundles over $Y$ associated to the $G$-modules $K^{d_1}$ and $K^{d_2}$ by $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively. The anti-canonical character is

$$\chi_{ac} = nd_2 \cdot \text{det}_1 + (m - nd_1) \cdot \text{det}_2 = n \cdot [d_2 \cdot \text{det}\mathcal{E}_1 + (r - d_1) \cdot \text{det}\mathcal{E}_2], \quad r := \frac{m}{n}.$$ 

We are going to see that the effect of introducing the parameter $n$ is that of obtaining several types of quotients. Observe that for generic choices of $m$ and $n$, the semi-stable and the stable loci coincide; this happens for

$$\gcd(nd_2, m - nd_1) = \gcd(nd_2, m - nd_1, nd_2) = 1.$$ 

For details about semi-stability criteria for quiver representations, the reader may consult [9].
8.2.1. Case $r > d_1$. The $\chi_{ac}$-semi-stability condition for $(B, \mathbb{A}) \in V$ is:
\[
\begin{align*}
\{ & U_2 \subset K^{d_2} \text{ and } U_1 \subset K^{d_1} \text{ s.t. } \mathbb{A}(U_2) \subset U_1 \\
\dim(U_2) = d'_2 & \text{ and } \dim(U_1) = d'_1
\end{align*}
\]
\[
\implies d'_2 d'_1 + (r - d_1) d'_2 \geq r d_2 \text{ for } (d'_2, d'_1) \neq (d_2, d_1).
\]

The set of extremal nef vector bundles is $\mathcal{VB}^+(Y) = \{ \mathcal{E}_1, \mathcal{E}_2 \}$, and the anti-canonical class is $\kappa_Y^{-1} = (\det \mathcal{E}_2)^{m-n d_1} \otimes (\det \mathcal{E}_1)^{n d_2}$. Theorem 6.1 implies that the elements of the set
\[
\{ S^\lambda \mathcal{E}_1 \otimes S^\mu \mathcal{E}_2 \mid \lambda \in \mathcal{Y}_{n d_2 - d_1, d_1} \text{ and } \mu \in \mathcal{Y}_{m - n d_1 - d_2, d_2} \}
\]
form a strong exceptional sequences of vector bundles over $Y$.

We illustrate again the role of Step 4 described at the beginning of this section: by using an appropriate fibre bundle structure on $Y$, we will increase the number of elements in the exceptional sequence.

Observe that both $B$ and $\mathbb{A} \in \text{Hom}(K^{d_2} \otimes K^n, K^{d_1})$ are surjective, for any $\chi_{ac}$-semi-stable point $(B, \mathbb{A})$. Indeed: by inserting $d'_1 = d_1$ we obtain $d'_2 \geq d_2$, and by inserting $d'_2 = d_2$ we obtain $d'_1 \geq d_1$. It follows that there is a natural projection $\phi : Y \to \text{Grass}(K^{d_2}, d_2)$, whose fibres are isomorphic to Grass$(K^{n d_2}, d_1)$. The group $\text{Gl}(m) \times \text{Gl}(n)$ acts on $Y$, and the projection is equivariant for the $\text{Gl}(m)$-action. However $Y$ is not the 2-flag variety.

We observe that the projection $V \to \text{Hom}(K^{d_2}, K^{d_2})$ fulfills the conditions (7.1), and moreover $\mathcal{VB}^+(\text{Grass}(K^{d_2}, d_2)) = \{ \mathcal{E}_2 \}$, and $\mathcal{VB}^+_0 = \{ \mathcal{E}_1 \}$. Applying corollary 7.5 to $\phi$ we deduce that the elements of the following set form a strong exceptional set of vector bundles over $Y$:
\[
\{ S^\lambda \mathcal{E}_1 \otimes S^\mu \mathcal{E}_2 \mid \lambda \in \mathcal{Y}_{n d_2 - d_1, d_1} \text{ and } \mu \in \mathcal{Y}_{m - n d_1 - d_2, d_2} \}
\]

8.2.2. Case $r < d_1$. The $\chi_{ac}$-semi-stability condition for $(B, \mathbb{A}) \in V$ is:
\[
\begin{align*}
\{ & U_2 \subset K^{d_2} \text{ and } U_1 \subset K^{d_1} \text{ s.t. } \mathbb{A}(U_2) \subset U_1 \\
\dim(U_2) = d'_2 & \text{ and } \dim(U_1) = d'_1
\end{align*}
\]
\[
\implies \begin{cases} 
(i) & d'_2 d'_1 - (d_1 - r)d'_2 \geq 0 \text{ for } (d'_2, d'_1) \neq (0, 0), \\
(ii) & d'_2 d'_1 - (d_1 - r)d'_2 \geq r d_2 \text{ for } (d'_2, d'_1) \neq (d_2, d_1).
\end{cases}
\]

Now we determine the extremal nef vector bundles. Since $r - d_1 < 0$, the situation differs from the previous case; now we will have $\mathcal{VB}^+(Y) = \{ \mathcal{E}_2, \mathcal{H} \}$, with $\mathcal{H} := \text{Hom}(\mathcal{E}_2, \mathcal{E}_1)$. We express the anti-canonical class as a positive combination of the extremal bundles: $\kappa_Y^{-1} = (\det \mathcal{E}_2)^m \otimes (\det \mathcal{H})^n$. Theorem 6.1 implies that
\[
\{ S^\lambda \mathcal{E}_2 \otimes S^\mu \mathcal{H} \mid \lambda \in \mathcal{Y}_{m - d_2, d_2} \text{ and } \mu \in \mathcal{Y}_{n - d_1 - d_2, d_1} \}
\]
is a strong exceptional sequence of vector bundles over $Y$.

Let us interpret the result. We consider the sub-quiver formed by the two rightmost vertices, and let $W := \text{Hom}(K^{d_2} \otimes K^n, K^{d_1})$ be its representation space. The anti-canonical character is $\chi_{ac}(W) = d_2 \text{det}_1 - d_1 \text{det}_2$. The symmetry group which is acting (effectively) is $G/((K^\times)_{\text{diag}}$. The condition (8.3)(i) implies that if $(B, \mathbb{A})$ is $\chi_{ac}$-semi-stable, then $\mathbb{A}$ is $\chi_{ac}(W)$-semi-stable. Hence there is a natural morphism
\[
Y \overset{\phi}{\to} X := \text{Hom}(K^{d_2} \otimes K^n, K^{d_1})/_{\chi_{ac}(W)} G,
\]
which is a projective bundle, with fibre isomorphic to $\mathbb{P}(K^{m d_2})$. The conditions (7.1) are fulfilled, and we may apply corollary 7.5 However, in this case we do not improve the previous bound.
8.3. Altman and Hille’s examples. In the article [1] the authors present the following construction: consider a quiver $Q$ without oriented cycles, and a thin and faithful representation space $V$ of it. This means that the dimension vector of the representation space is $\underline{d} = (1)_{q \in Q_0}$, and the symmetry group which is acting is the torus $T = \prod_{q \in Q_0} K^\times/(K^\times)_{\text{diag}}$.

**Theorem** (Altman, Hille) Assume that $\mathcal{V}^s(T, \chi_{\text{ac}}) = \mathcal{V}^s_{(0)}(T, \chi_{\text{ac}})$. Then the tautological line bundles $(\mathcal{L}_q)_{q \in Q_0}$ form an exceptional sequence over the toric variety $Y := \mathcal{V}^s(T, \chi_{\text{ac}})/T$.

We wish to remark that this construction fits into a more general framework: we consider a quiver $Q = (Q_0, Q_1, h, t)$ without oriented cycles, and we fix a dimension vector $\underline{d} = (d_q)_{q \in Q_0}$; we denote $V$ the corresponding representation space. For $m \geq 1$, we denote $Q^{(m)}$ the quiver obtained from $Q$ by multiplying each arrow $m$ times. The representation space of $Q^{(m)}$ with dimension vector $\underline{d}$ is $V^m$, and the symmetry group which is acting is $G = \prod_{q \in Q_0} \text{Gl}(d_q)/K^\times_{\text{diagonal}}$.

**Proposition 8.1.** Assume that $(\mathcal{V}^m)^s(G, \chi_{\text{ac}}) = (\mathcal{V}^m)^s(G, \chi_{\text{ac}})$, and denote $Y_m$ the quotient by the $G$-action. For $q \in Q_0$, we denote $\mathcal{E}_q$ the tautological bundle over $Y_m$, associated to $G \to \text{Gl}(d_q)$.

Then there is a constant $m(Q) \geq 1$ such that for all $m > m(Q)$, the set $\{\mathcal{E}_q\}_{q \in Q_0}$ is a strong exceptional sequence of vector bundles over $Y_m$ (with respect to an appropriate ordering). Moreover, these vector bundles are semi-stable.

**Proof.** For two vertices $p, q \in Q_0$, let $E_{pq} := \text{Hom}(E_p, E_q)$, and $E_{pq}$ the associated vector bundle over $Y_m$, and let $e_{pq} := \dim E_{pq} = d_p d_q$.

The condition on $H^0(Y_m, E_{pq})$ follows from theorem 2.5. It remains to prove the vanishing of the higher cohomology. We compute $H^{n-i}(Y_m, \mathcal{E}_{pq})$, $n = \dim Y$, by using the relative duality for $\mathbb{P}(\mathcal{E}_{pq}) \cong Y_m$; it equals:

$$H^{(e_{pq} - 1) + i}(\mathbb{P}(\mathcal{E}_{pq}), \text{pr}^*(\kappa_{Y_m} \otimes (\det \mathcal{E}_{pq})^{-1}) \otimes O_{\mathbb{P}(\mathcal{E}_{pq})}(e_{pq} - 1))^\vee.$$  

The Kodaira vanishing theorem implies that $H^j(Y_m, \mathcal{E}_{pq})$ vanishes for all $j \geq 1$, as soon as $\text{pr}^*(\kappa_{Y_m} \otimes (\det \mathcal{E}_{pq})^{-1}) \otimes O_{\mathbb{P}(\mathcal{E}_{pq})}(e_{pq} + 1)$ is ample over $\mathbb{P}(\mathcal{E}_{pq})$. By proposition 3.2 there is a number $m_{pq}$ such that this property holds for all $m > m_{pq}$. Consider now $m(Q) := \max\{m_{pq} : p, q \in Q_0\}$.

The isotypical components of $V^m$ are $\text{Hom}(E_{t(a), E_{h(a)}})$, $a \in Q_1$. Note that $m > m(Q)$ implies $m > \dim \text{Hom}(E_p, E_q)$, and the semi-stability of the tautological bundles $\mathcal{E}_q$ follows from corollary 1.5.

We wish to point out the following shortcoming: in this construction the length of the exceptional sequence equals the number of vertices of $Q$, which is independent of the multiplicity $m$. It follows that for large $m$ this sequence is certainly not a tilting object for $Y_m$ (compare this with remark 6.3).

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