Adversarial Bandits against Arbitrary Strategies

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Abstract

We study the adversarial bandit problem against arbitrary strategies, in which $S$ is the parameter for the hardness of the problem and this parameter is not given to the agent. To handle this problem, we adopt the master-base framework using the online mirror descent method (OMD). We first provide a master-base algorithm with simple OMD, achieving $\tilde{O}(S^{1/2}K^{1/3}T^{2/3})$, in which $T^{2/3}$ comes from the variance of loss estimators. To mitigate the impact of the variance, we propose using adaptive learning rates for OMD and achieve $\tilde{O}(\min\{E[\sqrt{SKT\rho_T(h^t)}], S\sqrt{KT}\})$, where $\rho_T(h^t)$ is a variance term for loss estimators.

1 Introduction

The bandit problem is a fundamental decision-making problem to deal with the exploration-exploitation trade-off. In this problem, an agent plays an action, “arm”, at a time, and receives loss or reward feedback for the option. The arm might be a choice of an item for a user in recommendation systems. In practice, it is often required to consider switching user preferences for items as time passes. This can be modeled by switching best arms.

In this paper, we focus on the adversarial bandit problem, where the losses for each arm at each time are arbitrarily determined. In such an environment, we consider that the target strategy is allowed to have any sequence of arms instead of the best arm in hindsight. Therefore, regret is measured by competing with any sequence of arms. We denote by $S$ the number of switches for the sequence of arms, which is referred to as hardness Auer et al. (2002). Importantly, we target any arbitrary strategies so that $S$ is not fixed in advance (in other words, the value of $S$ is not provided to the agent).

Competing with switching arms has been widely studied. In the expert setting with full information Cesa-Bianchi et al. (1997), there are several algorithms Daniely et al. (2015); Jun et al. (2017) that achieve near-optimal $\tilde{O}(\sqrt{ST})$ regret bound for $S$-switch regret (which is defined later) without information of switch parameter $S$. However, in the bandit problems, an agent cannot observe full information of loss at each time, which makes the problem more challenging compared with the full information setting. For stochastic bandit settings where each arm has switching reward distribution over time steps, referred to as non-stationary bandit problems, has been studied by Garivier and Moulines (2008); Auer et al. (2019); Russac et al. (2019);
Suk and Kpotufe (2022). Especially Auer et al. (2019); Suk and Kpotufe (2022) achieved near-optimal regret $\tilde{O}(\sqrt{SKT})$ without given $S$.

However, we cannot apply this method to the adversarial setting, where losses may be determined arbitrarily. For the adversarial bandit setting, EXP3-S Auer et al. (2002) achieved $\tilde{O}(\sqrt{SKT})$ with given $S$ and $\tilde{O}(S\sqrt{KT})$ without given $S$. It is also known that the Bandit-over-Bandit (BOB) approach achieved $\tilde{O}(\sqrt{SKT + T^{3/4}})$ Cheung et al. (2019); Foster et al. (2020) for the case when $S$ is not given. Recently, Luo et al. (2022) studied switching adversarial linear bandits and achieved $\tilde{O}(\sqrt{dST})$ with given $S$.

In this paper, we study the adversarial bandit problems against any arbitrarily switching arms (i.e. without given $S$). To handle this problem, we adopt the master-base framework with the online mirror descent method (OMD) which has been widely utilized for model selection problems Agarwal et al. (2017); Pacchiano et al. (2020); Luo et al. (2022). We first study a master-base algorithm with negative entropy regularizer-based OMD and analyze the regret of the algorithm achieving $\tilde{O}(S^{1/2}K^{1/3}T^{2/3})$. Nevertheless, this approach inadequately addresses the variance of estimators due to its use of a fixed learning rate throughout, resulting in a regret bound containing a term proportional to $T^{2/3}$.

Based on the analysis, we propose to use adaptive learning rates for OMD to control the variance of loss estimators and achieve $\tilde{O}(\min\{\mathbb{E}[\sqrt{SKT\rho_T(h^\dagger)}], S\sqrt{KT}\})$, where $\rho_T(h^\dagger)$ is a variance term for loss estimators. Importantly, instead of a negative entropy regularizer, we utilize a log-barrier regularizer, which allows us to control the worst case with respect to $\rho_T(h^\dagger)$. Lastly, we assess our algorithms in comparison to those proposed in earlier works, specifically Auer et al. (2002) and Cheung et al. (2019).

2 Problem statement

Here we describe our problem settings. We let $A = [K]$ be the set of arms and $l_t \in [0, 1]^K$ be a loss vector at time $t$ in which $l_t(a)$ is the loss value of the arm $a$ at time $t$. The adversarial environment is arbitrarily determined with a sequence of reward vectors $l_1, l_2, \ldots, l_T \in [0, 1]^K$ over the horizon time $T$. At each time $t$, an agent selects an arm $a_t \in [K]$, after which one observes partial feedback $l_t(a_t) \in [0, 1]$. In this adversarial bandit setting, we aim to minimize $S$-switch regret which is defined as follows. Let $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_T\} \in [K]^T$ be a sequence of actions. For a positive integer $S < T$, the set of sequence of actions with $S$ switches is defined as

$$B_S = \left\{ \sigma \in [K]^T : \sum_{t=1}^{T-1} \mathbb{I}\{\sigma_t \neq \sigma_{t+1}\} \leq S \right\}.$$

Then, we define the $S$-switch regret as

$$R_S(T) = \max_{\sigma \in B_S} \sum_{t=1}^{T} \mathbb{E}[l_t(a_t)] - l_t(\sigma_t).$$

We assume that $S$ is not given to the agent (or undermined). In other words, we aim to design
algorithms against any sequence of arms. Therefore, we need to design universal algorithms that achieve tight regret bounds for any non-fixed \( S \in [T - 1] \), in which \( S \) represents the hardness of the problem. It is noteworthy that this problem encompasses the non-stationary stochastic bandit problems without knowing a switching parameter Auer et al. (2019); Chen et al. (2019).

2.1 Regret Lower Bound

We can easily obtain the regret lower bound of this problem from the well-known regret lower bound of adversarial bandits. Let \( t_s \) be the time when the \( s \)-th switch of the best arm in hindsight happens for \( s \in [S] \) and \( t_{S+1} - 1 = T, t_0 = 1 \). We consider that \( t_s \)'s are equally distributed over \( T \). Then we have \( T_s := t_{s+1} - t_s = \Theta(T/S) \) for \( s \in [0, S] \). Then from Theorem 5.1 in Auer et al. (2002), for the best arm in hindsight over \( T_s \) time steps, we get the regret lower bound of \( \Omega(\sqrt{KT_s}) \). We can obtain that

\[
R(T) = \Omega \left( \sum_{s \in [0, S]} \sqrt{KT_s} \right) = \Omega \left( \sqrt{SKT} \right).
\]

However, determining the feasibility of a tighter regret lower bound under undetermined \( S \) remains an unresolved challenge.

3 Algorithms and regret analysis

To handle this problem, we suggest using the online mirror descent method integrated into the master-base framework.

3.1 Master-base framework

In the master-base framework, at each time, a master algorithm selects a base and the selected base selects an arm. For the undetermined switch value \( S \) in advance, we suggest tuning each base algorithm using a candidate of \( S \) as follows.

Let \( H \) represent the set of candidates of the switch parameter \( S \in [T - 1] \) for the bases such that:

\[
H = \{ T^0, T^{\lceil \log T \rceil}, T^{\lceil 2 \log T \rceil}, \ldots, T \}.
\]

Then, each base adopts one of the candidate parameters in \( H \) for tuning its learning rate. For simplicity, let \( H = |H| \) such that \( H = O(\log(T)) \) and let base \( h \) represent the base having the candidate parameter \( h \in H \) when there is no confusion. Also, let \( h^\dagger \) be the largest value \( h \leq S \) among \( h \in H \), which indicates the near-optimal parameter for \( S \). Then we can observe that

\[
e^{-1} S \leq h^\dagger \leq S.
\]
3.2 Online mirror descent (OMD)

Here we describe the OMD method Lattimore and Szepesvári (2020). For a regularizer function $F: \mathbb{R}^d \rightarrow \mathbb{R}$ and $p, q \in \mathbb{R}^d$, we define Bregman divergence as

$$D_F(p, q) = F(p) - F(q) - \langle \nabla F(q), p - q \rangle.$$ 

Let $p_t$ be the distribution for selecting an action at time $t$ and $P_d$ be the probability simplex with dimension $d$. Then with a loss vector $l$, using the online mirror descent we can get $p_{t+1}$ as follows:

$$p_{t+1} = \arg \min_{p \in P_d} \langle p, l \rangle + D_F(p, p_t).$$ (1)

The solution of (1) can be found using the following two-step procedure:

$$\tilde{p}_{t+1} = \arg \min_{p \in \mathbb{R}^d} \langle p, l \rangle + D_F(p, p_t),$$

$$p_{t+1} = \arg \min_{p \in P_d} D_F(p, \tilde{p}_{t+1}).$$ (2)

We use a regularizer $F$ that contains a learning rate (to be specified later). We note that, in the bandit setting, we cannot observe full information of loss at time $t$, but get partial feedback based on a selected action. Therefore, it is required to use an estimated loss vector for OMD.

3.3 Master-base OMD

We first provide a simple master-base OMD algorithm (Algorithm 1) with the negative entropy regularizer defined as

$$F_{\eta}(p) = (1/\eta) \sum_{i=1}^{d} (p(i) \log p(i) - p(i)),$$

where $p \in \mathbb{R}^d$, $p(i)$ denotes the $i$-index entry for $p$, and $\eta$ is a learning rate. We note that well-known EXP3 Auer et al. (2002) for the adversarial bandits is also based on the negative entropy function.

In Algorithm 1, at each time, the master selects a base $h_t$ from distribution $p_t$. Then following distribution $p_{t,h_t}$ for selecting an arm, the base $h_t$ selects an arm $a_t$ and receive a corresponding loss $l_t(a_t)$. Using the loss feedback, it gets unbiased estimators $l'_t(h)$ and $l''_{t,h}(a)$ for a loss from selecting each base $h \in H$ and each arm $a \in [K]$, respectively. Then using OMD with the estimators, it updates the distributions $p_{t+1}$ and $p_{t+1,h}$ for selecting a base and an arm from base $h$, respectively.

For getting $p_{t+1}$, it uses the negative entropy regularizer with learning rate $\eta$. The domain for updating the distribution for selecting a base is defined as a clipped probability simplex such that $P_{H}^{\alpha} = P_H \cap [\alpha, 1]^H$ for $\alpha > 0$. By introducing $\alpha$, it can control the variance of estimator $l'_t(h) = l_t(a_t) \mathbb{1}(h = h_t/p_t(h))$ by restricting the minimum value for $p_t(h)$. For getting $p_{t+1,h}$,
it also uses the negative entropy regularizer with learning rates depending on a value of $h$ for each base. The learning rate $\eta(h)$ is tuned by using a candidate value $h$ for $S$ in the base $h$ to control adaptation for switching such that

$$\eta(h) = h^{1/2}/(K^{1/3}T^{2/3}).$$

The domain for the distribution is also defined as a clipped probability simplex such that $\mathcal{P}_K = \mathcal{P}_K \cap [0, 1]^K$ for $\beta > 0$. The purpose of $\beta$ is to introduce some regularization in learning $p_{t+1,h}$ for dealing with switching best arms in hindsight, which is slightly different from the purpose of $\alpha$.

Now we provide a regret bound for the algorithm in the following theorem.

**Algorithm 1 Master-base OMD**

1. **Given:** $T$, $K$, $\mathcal{H}$.
2. **Initialization:** $\alpha = K^{1/3}/(T^{1/3}H^{1/2})$, $\beta = 1/(KT)$, $\eta = 1/\sqrt{TK}$, $\eta(h) = h^{1/2}/(K^{1/3}T^{2/3})$, $p_t(h) = 1/H$, $p_{t,h}(a) = 1/K$ for $h \in \mathcal{H}$ and $a \in [K]$.
3. for $t = 1, \ldots , T$ do
4. **Select a base and an arm:**
5. Draw $a_t \sim$ probabilities $p_t(h)$ for $h \in \mathcal{H}$.
6. Draw $a_{t,h} \sim$ probabilities $p_{t,h}(a)$ for $a \in [K]$.
7. Pull $a_t = a_{t,h}$ and Receive $l_t(a_{t,h}) \in [0, 1]$.
8. **Obtain loss estimators:**
9. $l_t'(h) = l_t(a_{t,h})/p_t(h)$ and $l_t'(h) = 0$ for $h \in \mathcal{H}/\{h_t\}$.
10. $l_t'(a_{t,h}) = l_t'(h)/p_{t,h}(a_{t,h})$ and $l_t'(a) = 0$ for $h \in \mathcal{H}/\{h_t\}$, $a \in [K]/\{a_{t,h}\}$.
11. **Update distributions:**
12. $p_{t+1} = \arg \min_{p \in \mathcal{P}_K} \{p,l_t'\} + D_{\mathcal{F}_h}(p,p_t)$
13. $p_{t+1,h} = \arg \min_{p \in \mathcal{P}_K} \{p,l_t'(a_{t,h})\} + D_{\mathcal{F}_h}(p,p_{t,h})$ for all $h \in \mathcal{H}$
14. end for

**Theorem 1.** For any switch number $S \in [T-1]$, Algorithm 1 achieves a regret bound of

$$R_S(T) = \tilde{O}(S^{1/2}K^{1/3}T^{2/3})$$

**Proof.** Let $t_s$ be the time when the $s$-th switch of the best arm happens and $t_{S+1} = T$, $t_0 = 1$. Also let $t_{s+1} - t_s = T_s$. For any $t_s$ for all $s \in [0, S]$, the $S$-switch regret can be expressed as

$$R_S(T) = \sum_{t=1}^{T} \mathbb{E}[l_t(a_t)] - \sum_{s=0}^{S} \min_{k_s \in [K]} \sum_{t=t_s}^{t_{s+1}-1} l_t(k_s)$$

$$= \sum_{t=1}^{T} \mathbb{E}[l_t(a_{t,h_t})] - \sum_{t=1}^{T} \mathbb{E}[l_t(a_{t,h_{1}})] + \sum_{t=1}^{T} \mathbb{E}[l_t(a_{t,h_{1}})] - \sum_{s=0}^{S} \min_{k_s \in [K]} \sum_{t=t_s}^{t_{s+1}-1} l_t(k_s),$$

(3)
in which the first two terms are closely related with the regret from the master algorithm against the near optimal base $h^\dagger$, and the remaining terms are related with the regret from $h^\dagger$ base algorithm against the best arms in hindsight. We note that the algorithm does not need to know $h^\dagger$ in prior and $h^\dagger$ is brought here only for regret analysis.

First we provide a bound for the following regret from base $h^\dagger$:

$$
\sum_{t=s}^{t_s+1} E \left[ l_t(a_t, h^\dagger) \right] - \sum_{s=0}^{S} \min_{k_s \in [K]} \sum_{t=s}^{t_s+1-1} l_t(k_s).
$$

Let $k^*_s = \arg\min_{k \in [K]} \sum_{t=s}^{t_s+1-1} l_t(k)$ and $e_{j,K}$ denote the unit vector with 1 at $j$-index and 0 at the rest of $K-1$ indexes. Then, we have

$$
\begin{align*}
&\sum_{t=s}^{t_s+1-1} E \left[ l_t(a_t, h^\dagger) - l_t(k^*_s) \right] \\
&= \sum_{t=s}^{t_s+1-1} E \left[ \langle p_t, h^\dagger \rangle - e_{k^*_s,K}, l_t \rangle \right] \\
&\leq E \left[ \max_{p \in P^0_K} \sum_{t=s}^{t_s+1-1} \langle p - e_{k^*_s,K}, l_t \rangle + \max_{p \in P^0_K} \sum_{t=s}^{t_s+1-1} \langle p_t, h^\dagger - p, l_t \rangle \right] \\
&\leq \beta_T (K-1) + E \left[ \max_{p \in P^0_K} \sum_{t=s}^{t_s+1-1} \langle p_t, h^\dagger - p, l'_{t,h^\dagger} \rangle \right],
\end{align*}
$$

where the first term in the last inequality is obtained from the clipped domain $P^0_K$ and the second term is obtained from the unbiased estimator $l'_{t,h^\dagger}$ such that $E[l'_{t,h^\dagger} | \mathcal{F}_{t-1}] = E[l_t | \mathcal{F}_{t-1}]$ where $\mathcal{F}_{t-1}$ is the filtration. We can observe that the clipped domain controls the distance between the initial distribution at $t_s$ and the best arm unit vector for the time steps over $[t_s, t_{s+1} - 1]$. Let

$$
p_{t+1,h^\dagger} = \arg\min_{p \in \mathbb{R}^K} \langle p, l'_{t+1,h^\dagger} \rangle + D_{F_{\eta(h^\dagger)}}(p, p_{t,h^\dagger}).
$$

Then, by solving the optimization problem, we can get

$$
p_{t+1,h^\dagger}(k) = p_{t,h^\dagger}(k) \exp(-\eta(h^\dagger) l'_{t,h^\dagger}(k)),
$$

for all $k \in [K]$.

For the second term of the last inequality in (4), we provide a lemma in the following.

**Lemma 1** (Theorem 28.4 in Lattimore and Szepesvári (2020)). For any $p \in P^0_K$ we have

$$
\sum_{t=s}^{t_s+1-1} \langle p_t, h^\dagger - p_t, l'_{t,h^\dagger} \rangle \leq D_{F_{\eta(h^\dagger)}}(p_t, p_{t,s,h^\dagger}) + \sum_{t=s}^{t_s+1-1} D_{F_{\eta(h^\dagger)}}(p_t, p_{t,h^\dagger}, \tilde{p}_{t+1,h^\dagger}).
$$
In Lemma 1, the first term is for the initial point diameter at time \( t_s \) and the second term is for the divergence of the updated policy. Using the definition of the Bregman divergence and the fact that \( p_{t,s,h^1}(k) \geq \beta \), the initial point diameter term can be shown to be bounded as follows:

\[
D_{F_{\eta(h^1)}}(p,p_{t,s,h^1}) \leq \frac{1}{\eta(h^1)} \sum_{k \in [K]} p(k) \log(1/p_{t,s,h^1}(k)) \\
\leq \frac{\log(1/\beta)}{\eta(h^1)}.
\] (5)

Next, for the updated policy divergence term, using \( \tilde{P}_{t+1,h^1}(k) = p_{t,h^1}(k) \exp(-\eta(h^1)l''_{t,h^1}(k)) \) for all \( k \in [K] \), we have

\[
\sum_{t=t_s}^{t_{s+1}-1} \mathbb{E} \left[ D_{F_{\eta(h^1)}}(p_{t,h^1}, \tilde{P}_{t+1,h^1}) \right] \\
= \sum_{t=t_s}^{t_{s+1}-1} \sum_{k=1}^{K} \mathbb{E} \left[ \frac{1}{\eta(h^1)} p_{t,h^1}(k) \left( \exp(-\eta(h^1)l''_{t,h^1}(k)) - 1 + \eta(h^1)l''_{t,h^1}(k) \right) \right] \\
\leq \sum_{t=t_s}^{t_{s+1}-1} \sum_{k=1}^{K} \mathbb{E} \left[ \frac{\eta(h^1)}{2} p_{t,h^1}(k) l''_{t,h^1}(k)^2 \right] \\
\leq \sum_{t=t_s}^{t_{s+1}-1} \sum_{k=1}^{K} \mathbb{E} \left[ \frac{\eta(h^1)}{2p_t(h^1)} \right] \leq \frac{\eta(h^1)KT_s}{2\alpha},
\] (6)

where the first inequality comes from \( \exp(-x) \leq 1 - x + x^2/2 \) for all \( x \geq 0 \), the second inequality comes from \( \mathbb{E}[l''_{t,h^1}(k)^2 | p_{t,h^1}(k), p_t(h^1)] \leq 1/(p_t(h^1)p_{t,h^1}(k)) \), and the last inequality is obtained from \( p_t(h^1) \geq \alpha \) from the clipped domain. We can observe that the clipped domain controls the variance of estimators. Then from (4), Lemma 1, (5), and (6), by summing up over \( s \in [S] \), we have

\[
\sum_{t=1}^{T} \mathbb{E} \left[ l_t(a_{t,h^1}) \right] - \sum_{s=0}^{S} \min_{k \in [K]} \sum_{t=t_s}^{t_{s+1}-1} l_t(k_s) \leq \beta T(K - 1) + \frac{S \log(1/\beta)}{\eta(h^1)} + \frac{\eta(h^1)KT_s}{2\alpha}.
\] (7)

Next, we provide a bound for the following regret from the master:

\[
\sum_{t=1}^{T} \mathbb{E} \left[ l_t(a_{t,h^1}) \right] - \sum_{t=1}^{T} \mathbb{E} \left[ l_t(a_{t,h^1}) \right].
\]

Let \( \hat{p}_{t+1} = \arg\min_{p \in \mathcal{H}} \langle p, \hat{l}_t \rangle + D_{F_{\eta}}(p, p_t) \) and \( e_{h,H} \) denote the unit vector with 1 at base \( h \)-index and 0 at the rest of \( H - 1 \) indexes. For ease of presentation, we define \( \hat{l}_t(h_1) = l_t(a_{t,h_1}) \).
and \( \tilde{l}_t(h^1) = l_t(a_{t,h^1}) \). Then, we have
\[
\sum_{t=1}^{T} E \left[ l_t(a_{t,h^1}) - l_t(a_{t,h^1}) \right] = \sum_{t=1}^{T} E \left[ p_t - e_{h^1,1,1} \right] 
\leq E \left[ \max_{p \in P_H} \sum_{t=1}^{T} (p - e_{h^1,1,1}) + \max_{p \in P_H} \sum_{t=1}^{T} (p_t - p, \tilde{l}_t) \right] 
\leq \alpha T (H - 1) + E \left[ \max_{p \in P_H} \sum_{t=1}^{T} (p_t - p, \tilde{l}_t) \right].
\]
(8)

For bounding the second term in (8), we use the following lemma.

**Lemma 2** (Theorem 28.4 in Lattimore and Szepesvári (2020)).
\[
\max_{p \in P_H} \sum_{t=1}^{T} (p_t - p, \tilde{l}_t) \leq \max_{p \in P_H} \left( F_\eta(p) - F_\eta(p_1) + \sum_{t=1}^{T} D_{F_\eta}(p_t, \tilde{p}_{t+1}) \right).
\]

From (8) and Lemma 2, we have
\[
\sum_{t=1}^{T} E \left[ l_t(a_{t,h^1}) \right] - \sum_{t=1}^{T} E \left[ l_t(a_{t,h^1}) \right] 
\leq \alpha T (H - 1) + E \left[ \max_{p \in P_H} \left( F_\eta(p) - F_\eta(p_1) + \sum_{t=1}^{T} D_{F_\eta}(p_t, \tilde{p}_{t+1}) \right) \right] 
\leq \alpha T (H - 1) + \frac{\log(H)}{\eta} + \frac{\eta TK}{2},
\]
(9)
where the last inequality is obtained from the fact that
\[
F_\eta(p) - F_\eta(p_1) \leq -F_\eta(p_1) \leq \frac{\log(H)}{\eta}
\]
and
\[
E \left[ \sum_{t=1}^{T} D_{F_\eta}(p_t, \tilde{p}_{t+1}) \right] \leq \frac{\eta TK}{2}.
\]

Therefore, putting (3), (7), and (9) altogether, we have
\[
R_S(T) = \sum_{t=1}^{T} E \left[ l_t(a_t) \right] - \sum_{s=0}^{S} \min_{1 \leq k_s \leq K} \sum_{t=T_s+1}^{T_{s+1}-1} l_t(k_s) 
\leq \alpha T H + \frac{\log(H)}{\eta} + \frac{\eta TK}{2} + \beta T (K - 1) + \frac{S \log(1/\beta)}{\eta(h^1)} + \frac{\eta(h^1) KT}{2\alpha}
\]
\[
= \tilde{O}(S^{1/2} T^{2/3} K^{1/3}),
\]
8
where $\alpha = K^{1/3}/(T^{1/3}H^{1/2})$, $\beta = 1/(KT)$, $\eta = 1/\sqrt{TK}$, $\eta(h^1) = h^{1/2}/(K^{1/3}T^{2/3})$, $h^1 = \Theta(S)$, and $H = \log(T)$. This concludes the proof.

From Theorem 1, the regret bound of Algorithm 1 is tight with respect to $S$ compared to that of EXP3.S (Auer et al. 2002) which has a linear dependency on $S$. Therefore, when $S$ is large (specifically $S = \omega((T/K)^{1/3})$), Algorithm 1 performs better than EXP3.S. Also, compared with previous bandit-over-bandit approach (BOB) (Cheung et al. 2019) having a loose dependency on $T$ as $T^{3/4}$, our algorithm has a tighter regret bound with respect to $T$. Therefore, when $T$ is large (specifically $T = \omega(S^6K^4)$), Algorithm 1 achieves a better regret bound than BOB.

However, the achieved regret bound from Algorithm 1 still has $O(T^{2/3})$ rather than $O(\sqrt{T})$ due to the large variance of loss estimators from sampling twice at each time for a base and an arm. In the following, we provide an algorithm utilizing adaptive learning rates to control the variance of estimators.

### 3.4 Master-base OMD with adaptive learning rates

Here we propose Algorithm 2, which utilizes adaptive learning rates to control the variance of estimators. We first explain the base algorithm. For the base algorithm, we propose to use the negative entropy regularizer with adaptive learning rate $\eta_t(h)$ such that

$$F_{\eta_t}(p) = \frac{1}{\eta_t(h)} \sum_{i=1}^d (p(i) \log p(i) - p(i)).$$

The adaptive learning rate $\eta_t(h)$ is optimized using variance information for loss estimators at each time $t$ to control the variance such that

$$\eta_t(h) = \sqrt{h/(KT\rho_t(h))},$$

where $\rho_t(h)$ is a variance threshold term (to be specified later). This implies that if the variance of the estimators is small, then the learning rate becomes large.

For the master algorithm, we adopt the method of Corral Agarwal et al. (2017), in which, by using a log-barrier regularizer with increasing learning rates, it introduces a negative bias term to cancel a variance term from bases by handling the worst case with respect to $\rho_t(h^1)$. The log-barrier regularizer is defined as:

$$F_{\xi_t}(p) = -\sum_{i=1}^d \frac{\log p(i)}{\xi_t(i)},$$

with learning rates $\xi_t$ for the master algorithm.

Here we describe the update learning rates procedure for the master and bases in Algorithm 2; the other parts are similar with Algorithm 1. The variance of the loss estimator $l'_t(h)$ for base $h$ is $1/p_{t+1}(h)$. If the variance $1/p_{t+1}(h)$ for base $h$ is larger than a threshold $\rho_t(h)$, then it increases learning rate as $\xi_{t+1}(h) = \gamma \xi_t(h)$ with $\gamma > 1$ and the threshold is updated as
Algorithm 2 Master-base OMD with adaptive learning rates

Given: $T, K, \mathcal{H}$

Initialization: $\alpha = 1/(TH)$, $\beta = 1/(TK)$, $\gamma = e^{\log T}$, $\eta = \sqrt{H/T}$, $\rho_1(h) = 2H$, $\xi_1(h) = \eta$, $p_1(h) = 1/H$, $p_{1,h}(a) = 1/K$ for $h \in \mathcal{H}$ and $a \in [K]$.

for $t = 1, \ldots, T$ do

Select a base and an arm:
Draw $h_t \sim$ probabilities $p_t(h)$ for $h \in \mathcal{H}$.
Draw $a_{t,h_t} \sim$ probabilities $p_{t,h_t}(a)$ for $a \in [K]$.
Pull $a_t = a_{t,h_t}$ and Receive $l_t(a_{t,h_t}) \in [0,1]$.

Update loss estimators:
$l'_t(h_t) = \frac{l_t(a_{t,h_t})}{p_t(h_t)}$ and $l''_t(h_t) = 0$ for $h \in \mathcal{H}/\{h_t\}$.

$l''_{t,h}(a_{t,h_t}) = \frac{l''_{t,h}(h_t)}{p_{t,h}(a_{t,h_t})}$ and $l''_{t,h}(a) = 0$ for $h \in \mathcal{H}/\{h_t\}$, $a \in [K]/\{a_{t,h_t}\}$.

Update distributions:
$p_{t+1} = \arg \min_{p \in P_K} \left\{ D_{F_{\xi_t}}(p,p_1) + D_{F_{\rho_t}}(p,p) \right\}$
$p_{t+1,h} = \arg \min_{p \in P_K} \left\{ D_{F_{\rho_t}}(p,p_{t,h}) + D_{F_{\rho_t}}(p,p_{t,h}) \right\}$ for $h \in \mathcal{H}$

Update learning rates:
For $h \in \mathcal{H}$
If $\frac{1}{p_{t+1}(h)} > \rho_t(h)$, then
$\rho_{t+1}(h) = \frac{2}{p_{t+1}(h)}, \xi_{t+1}(h) = \gamma \xi_t(h)$.
 Else, $\rho_{t+1}(h) = \rho_t(h), \xi_{t+1}(h) = \xi_t(h)$.

end for

$\rho_{t+1}(h) = 2/p_{t+1}(h)$, which is also used for tuning the learning rate $\eta_t(h)$. Otherwise, it keeps the learning rate and threshold the same with the previous time step.

In the following theorem, we provide a regret bound of Algorithm 2.

Theorem 2. For any switch number $S \in [T - 1]$, Algorithm 2 achieves a regret bound of

$$R_S(T) = \tilde{O} \left( \min \left\{ E \left[ \sqrt{SKT \rho_T(h')} \right], S \sqrt{KT} \right\} \right) .$$

Proof. Let $t_s$ be the time when the $s$-th switch of the best arm happens and $t_{S+1} - 1 = T$, $t_0 = 1$. Also let $t_{s+1} - t_s = T_s$. For any $t_s$ for all $s \in [0, S]$, the $S$-switch regret can be expressed as

$$R_S(T) = \sum_{t=1}^{T} E\left[l_t(a_t)\right] - \sum_{s=0}^{S} \min_{k_s \in [K]} \sum_{t=t_s}^{t_{s+1}-1} l_t(k_s)$$

$$= \sum_{t=1}^{T} E\left[l_t(a_t,h_t)\right] - \sum_{t=1}^{T} E\left[l_t(a_t,h_t)\right] + \sum_{t=1}^{T} E\left[l_t(a_{t,h},t)\right] - \sum_{s=0}^{S} \min_{k_s \in [K]} \sum_{t=t_s}^{t_{s+1}-1} l_t(k_s),$$

(10)
in which the first two terms are closely related with the regret from the master algorithm against the near optimal base \( h^\dagger \), and the remaining terms are related with the regret from \( h^\dagger \) base algorithm against the best arms in hindsight.

First we provide a bound for the following regret from base \( h^\dagger \). From (4), we can obtain

\[
\sum_{t=1}^{t+1} \mathbb{E} \left[ l_t(a_{t,h^\dagger}) \right] - \sum_{s=0}^{S} \min_{k_s \in [K]} \sum_{t=1}^{t+1} l_t(k_s) \leq \beta T_s K + \mathbb{E} \left[ \max_{p \in p^a_K} \sum_{t=1}^{t+1} \langle p_{t,h^\dagger} - p, \ell'_t \rangle \right].
\]

(11)

Then for the second term of the last inequality in (11), we provide a following lemma.

**Lemma 3.** For any \( p \in p^a_K \) we can show that

\[
\sum_{t=1}^{t+1} \mathbb{E} \left[ \langle p_{t,h^\dagger} - p, \ell'_t \rangle \right] \leq \mathbb{E} \left[ 2 \log(1/\beta) \sqrt{KT \rho_{T}(h^\dagger)/h^\dagger} + \frac{T_s}{2} \sqrt{SK \rho_{T}(h^\dagger)/T} \right].
\]

Proof. For ease of presentation, we define the negative entropy regularizer without a learning rate as

\[
F(p) = \sum_{i=1}^{K} (p(i) \log p(i) - p(i))
\]

and define learning rate \( \eta_0(h^\dagger) = \infty \). From the first-order optimality condition for \( p_{t+1,h^\dagger} \) and using the definition of the Bregman divergence,

\[
\langle p_{t+1,h^\dagger} - p, \ell'_t \rangle \leq \frac{1}{\eta_t(h^\dagger)} \langle p - p_{t+1,h^\dagger}, \nabla F(p_{t+1,h^\dagger}) - \nabla F(p_{t,h^\dagger}) \rangle \\
= \frac{1}{\eta_t(h^\dagger)} \left( D_F(p_t, p_{t+1,h^\dagger}) - D_F(p_{t+1,h^\dagger}, p_t) \right).
\]

(12)

Also, we have

\[
\langle p_{t,h^\dagger} - p_{t+1,h^\dagger}, \ell'_t \rangle = \frac{1}{\eta_t(h^\dagger)} \langle p_{t,h^\dagger} - p_{t+1,h^\dagger}, \nabla F(p_{t,h^\dagger}) - \nabla F(p_{t+1,h^\dagger}) \rangle \\
= \frac{1}{\eta_t(h^\dagger)} \left( D(p_{t+1,h^\dagger}, p_t) + D(p_{t,h^\dagger}, \hat{p}_{t+1,h^\dagger}) - D(p_{t+1,h^\dagger}, \hat{p}_{t+1,h^\dagger}) \right) \\
\leq \frac{1}{\eta_t(h^\dagger)} \left( D(p_{t+1,h^\dagger}, p_t) + D(p_{t,h^\dagger}, \hat{p}_{t+1,h^\dagger}) \right).
\]

(13)
Then, we can obtain

\[
\sum_{t=t_s}^{t_{s+1}-1} \langle p_{t,h}, \lambda'' \rangle \leq \sum_{t=t_s}^{t_{s+1}-1} \langle p_{t,h}, \lambda'' \rangle + \sum_{t=t_{s+1}}^{t_{s+1}-1} \frac{1}{\eta_t(h^\dagger)} \left( D(p, p_{t,h}) - D(p, p_{t+1,h}) - D(p_{t+1,h}, p_{t,h}) \right) \]

\[
= \sum_{t=t_s}^{t_{s+1}-1} \langle p_{t,h}, \lambda'' \rangle + \sum_{t=t_{s+1}}^{t_{s+1}-1} D_F(p, p_{t+1,h}) \left( \frac{1}{\eta_t(h^\dagger)} - \frac{1}{\eta_{t+1}(h^\dagger)} \right) \]

\[
\leq 2 \log(1/\beta) \eta_T(h^\dagger) + \sum_{t=t_s}^{t_{s+1}-1} \frac{D_F(p_{t+1,h}', p_{t+1,h})}{\eta_T(h^\dagger)} + \sum_{t=t_s}^{t_{s+1}-1} \frac{D_F(p_{t+1,h}', p_{t+1,h})}{\eta_T(h^\dagger)},
\]

where the first inequality is obtained from (12) and the last inequality is obtained from (13), \( D(p, p_{t+1,h}) \leq \log(1/\beta) \), and \( \eta_t(h^\dagger) \geq \eta_T(h^\dagger) \) from non-decreasing \( \rho(h^\dagger) \).

For the second term in the last inequality in (14), using \( \hat{p}_{t+1,h}(k) = p_{t,h}(k) \exp(-\eta(h^\dagger)\lambda''(k)) \) for all \( k \in [K] \), we have

\[
\sum_{t=t_s}^{t_{s+1}-1} \mathbb{E} \left[ \frac{D_F(p_{t+1,h}', \hat{p}_{t+1,h})}{\eta_T(h^\dagger)} \right] = \sum_{t=t_s}^{t_{s+1}-1} \sum_{k=1}^{K} \mathbb{E} \left[ \frac{1}{\eta_t(h^\dagger)} p_{t,h}(k) \left( \exp(-\eta_t(h^\dagger)\lambda''(k)) - 1 + \eta_t(h^\dagger)\lambda''(k) \right) \right] \]

\[
\leq \sum_{t=t_s}^{t_{s+1}-1} \sum_{k=1}^{K} \mathbb{E} \left[ \eta_t(h^\dagger) p_{t,h}(k) \lambda''(k) \right] \]

\[
\leq \sum_{t=t_s}^{t_{s+1}-1} \sum_{k=1}^{K} \mathbb{E} \left[ \eta_t(h^\dagger) \rho_t(h^\dagger) \right] \]

\[
\leq \sum_{t=t_s}^{t_{s+1}-1} \sum_{k=1}^{K} \mathbb{E} \left[ \frac{1}{2} \rho_t(h^\dagger) \right] \]

\[
\leq T \frac{h^\dagger K \mathbb{E} \left[ \rho_T(h^{1/2}) \right]}{2},
\]

(15)
where the first inequality comes from $\exp(-x) \leq 1 - x + x^2/2$ for all $x \geq 0$, the second inequality comes from $E[l''_{t,h^1}(k)^2 \mid p_t(h^1), p_t(h^1)] \leq 1/(p_t(h^1)p_t(h^1))$, and the third inequality is obtained from $1/p_t(h^1) \leq \rho_t(h^1)$.

Then from (8) and Lemma 3, we have

$$\sum_{t=1}^{T} E[l_t(a_{t,h^1})] - \sum_{s=0}^{S} \min_{1 \leq k_s \leq K} \sum_{t=T_s}^{T_{s+1}-1} l_t(k_s) \leq \beta T(K - 1) + E \left[2S \log(1/\beta) \sqrt{KT \rho_T(h^1) \rho_T(h^1)} \right] + \frac{1}{2} \sqrt{TSK \rho_T(h^1)} \right]. \quad (16)$$

Next, we provide a bound for the regret from the master in the following lemma.

**Lemma 4** (Lemma 13 in Agarwal et al. (2017)).

$$\sum_{t=1}^{T} E[l_t(a_{t,h^1})] - \sum_{t=1}^{T} E[l_t(a_{t,h^1})] \leq O \left( \frac{H \log(T)}{\eta} + T\eta \right) - E \left[ \frac{\rho_T(h^1)}{40 \eta \log T} \right] + \alpha T(H - 1).$$

The negative bias term in Lemma 4 is derived from the log-barrier regularizer and increasing learning rates $\xi_t(h)$. This term is critical to bound the worst case regret which will be shown soon. Also, $H \log(T)/\eta$ is obtained from $H \log(1/(H \alpha))/\eta$ considering the clipped domain.
Then, putting (10) and Lemmas 3 and 4 altogether, we have  
\begin{align*}
R_S(T) &= \sum_{t=1}^{T} \mathbb{E}[l_t(a_t)] - \sum_{s=0}^{S} \min_{\tilde{a}_s \leq K} \sum_{t=T_s}^{T_{s+1}-1} l_t(k_s) \\
&\leq O\left(\frac{H \log T}{\eta} + T\eta\right) - \mathbb{E}\left[\frac{\rho_T(h^1)}{40\eta \log T}\right] \\
&+ \alpha T(H - 1) + \beta T(K - 1) \\
&+ \mathbb{E}\left[2S \log(1/\beta)\sqrt{\frac{KT\rho_T(h^1)}{h^1}} + \frac{1}{2}\sqrt{4SKT\rho_T(h^1)}\right] \\
&= \tilde{O}\left(\mathbb{E}\left[\sqrt{SKT\rho_T(h^1)}\right]\right) - \mathbb{E}\left[\frac{\rho_T(h^1)\sqrt{TK}}{40\sqrt{H \log(T)}}\right],
\end{align*}
where \(\alpha = 1/(TH)\), \(\beta = 1/(TK)\), \(\eta = \sqrt{MT/T}\), \(\eta(h^1) = \sqrt{h^1/(KT\rho_T(h^1))}\), \(H = \log(T)\), and \(h^1 = \Theta(S)\). Then we can obtain
\begin{align*}
R_S(T) &= \tilde{O}\left(\min\left\{\mathbb{E}\left[\sqrt{SKT\rho_T(h^1)}\right], S\sqrt{KT}\right\}\right),
\end{align*}
where \(\tilde{O}(S\sqrt{KT})\) is obtained from the worst case of \(\rho_T(h^1)\). The worst case can be found by considering a maximum value of the concave bound of the last equality in (17) with variable \(\rho_T(h^1) > 0\) such that \(\rho_T(h^1) = \Theta(S)\). This concludes the proof.  

Here we provide regret bound comparison with other approaches. For simplicity in the comparison, we use the fact that \(\mathbb{E}[\sqrt{\rho_T(h^1)}] \leq \sqrt{\mathbb{E}[\rho_T(h^1)]}\) for the regret bound in Theorem 2 such that
\begin{align*}
R_S(T) &= \tilde{O}\left(\min\left\{\sqrt{SKT\mathbb{E}[\rho_T(h^1)]}, S\sqrt{KT}\right\}\right).
\end{align*}
The regret bound in Theorem 2 depends on \(\rho_T(h^1)\) which is closely related with variance of loss estimators \(l_t'(h^1)\) for \(t \in [T - 1]\). Even though the regret bound depends on the variance term, it is of interest that the worst case bound is always bounded by \(\tilde{O}(S\sqrt{KT})\), which implies that the regret bound of Algorithm 2 is always tighter than or equal to that of EXP3.S. Algorithm 2 has a tight regret bound \(O(\sqrt{T})\) with respect to \(T\). Therefore, when \(T\) is large such that \(T = \omega(S^4K^2)\), Algorithm 2 shows a better regret bound compared with BOB. We note that the value of \(\mathbb{E}[\rho_T(h^1)]\) depends on the problems, and further analysis for the term would be an interesting avenue for future research.

**Remark 1.** For implementation of our algorithms, we describe how to update policy \(p_t\) using OMD in general. Let \(\hat{l}_t(a)\) be a loss estimator for action \(a \in [d]\). For the negative entropy regularizer, by solving the optimization in (2), from Lattimore and Szepesvári (2020), we have
\begin{align*}
p_{t+1}(a) &= \frac{\exp\left(-\eta \sum_{s=1}^{t} \hat{l}_s(a)\right)}{\sum_{b \in [d]} \exp\left(-\eta \sum_{s=1}^{t} \hat{l}_s(b)\right)}.
\end{align*}
In the case of the log-barrier regularizer, we have \( p_{t+1}(a) = (\eta \sum_{s=1}^{t} \hat{t}_s(a) + Z)^{-1} \), where \( Z \) is a normalization factor for a probability distribution Luo et al. (2022). Also, a clipped domain with \( 0 < \epsilon < 1 \) in the distribution can be implemented by adding a uniform probability to the policy such that \( p_{t+1}(a) \leftarrow (1 - \epsilon)p_{t+1}(a) + \epsilon/d \) for all \( a \in [d] \).

**Remark 2.** The regret bounds of Theorems 1 and 2 can also apply for the non-stationary stochastic bandit problems without knowing a switching parameter where reward distributions are switching over time steps. This is because adversarial bandit problems encompass stochastic bandit problems.

### 4 Conclusion

In this paper, we studied adversarial bandits against any sequence of arms with \( S \)-switch regret without given \( S \). We proposed two algorithms that are based on a master-base framework with the OMD method. We propose Algorithm 1 based on simple OMD achieving \( \tilde{O}(S^{1/2}K^{1/3}T^{2/3}) \). Then by using adaptive learning rates, Algorithm 2 achieved \( \tilde{O}(\min\{\mathbb{E}[\sqrt{SKT_{\text{PT}}(h^3)}], S\sqrt{KT}\}) \).

It is still an open problem to achieve the optimal regret bound for the worst case.

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### References

Agarwal, A., Luo, H., Neyshabur, B., and Schapire, R. E. (2017). Corralling a band of bandit algorithms. In *Conference on Learning Theory*, pages 12–38. PMLR.

Auer, P., Cesa-Bianchi, N., Freund, Y., and Schapire, R. E. (2002). The nonstochastic multi-armed bandit problem. *SIAM journal on computing*, 32(1):48–77.

Auer, P., Gajane, P., and Ortner, R. (2019). Adaptively tracking the best bandit arm with an unknown number of distribution changes. In *Conference on Learning Theory*, pages 138–158.

Cesa-Bianchi, N., Freund, Y., Haussler, D., Helmbold, D. P., Schapire, R. E., and Warmuth, M. K. (1997). How to use expert advice. *Journal of the ACM (JACM)*, 44(3):427–485.

Chen, Y., Lee, C.-W., Luo, H., and Wei, C.-Y. (2019). A new algorithm for non-stationary contextual bandits: Efficient, optimal and parameter-free. In *Conference on Learning Theory*, pages 696–726. PMLR.

Cheung, W. C., Simchi-Levi, D., and Zhu, R. (2019). Learning to optimize under non-stationarity. In *The 22nd International Conference on Artificial Intelligence and Statistics*, pages 1079–1087.
Daniely, A., Gonen, A., and Shalev-Shwartz, S. (2015). Strongly adaptive online learning. In *International Conference on Machine Learning*, pages 1405–1411.

Foster, D. J., Krishnamurthy, A., and Luo, H. (2020). Open problem: Model selection for contextual bandits. In *Conference on Learning Theory*, pages 3842–3846. PMLR.

Garivier, A. and Moulines, E. (2008). On upper-confidence bound policies for non-stationary bandit problems.

Jun, K.-S., Orabona, F., Wright, S., and Willett, R. (2017). Improved strongly adaptive online learning using coin betting. In *Artificial Intelligence and Statistics*, pages 943–951. PMLR.

Lattimore, T. and Szepesvári, C. (2020). *Bandit algorithms*. Cambridge University Press.

Luo, H., Zhang, M., Zhao, P., and Zhou, Z.-H. (2022). Corralling a larger band of bandits: A case study on switching regret for linear bandits. *arXiv preprint arXiv:2202.06151*.

Pacchiano, A., Phan, M., Abbasi-Yadkori, Y., Rao, A., Zimmert, J., Lattimore, T., and Szepesvari, C. (2020). Model selection in contextual stochastic bandit problems. *arXiv preprint arXiv:2003.01704*.

Russac, Y., Vernade, C., and Cappé, O. (2019). Weighted linear bandits for non-stationary environments. In *Advances in Neural Information Processing Systems*, pages 12017–12026.

Suk, J. and Kpotufe, S. (2022). Tracking most significant arm switches in bandits. In *Conference on Learning Theory*, pages 2160–2182. PMLR.