Highly accurate decoupled doubling algorithm for large-scale M-matrix algebraic Riccati equations

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Abstract

We consider the numerical solution of large-scale M-matrix algebraic Riccati equations with low-rank structures. We derive a new doubling iteration, decoupling the four original iteration formulae in the alternating-directional doubling algorithm. We prove that the kernels in the decoupled algorithm are small M-matrices. Illuminated by the highly accurate algorithm proposed in J.-G. Xue and R.-C. Li (2017) [32], we construct the triplet representations for the small M-matrix kernels in a highly accurate doubling algorithm. Illustrative numerical examples will be presented on the efficiency of our algorithm.

Keywords. decoupled form, highly accurate computation, large-scale problem, M-matrix, M-matrix algebraic Riccati equation, triplet representation

AMS subject classifications. 15A24, 65F30, 93C05

1 Introduction

Consider the M-matrix algebraic Riccati equation (MARE):

\[ XCX - XD - AX + B = 0, \]  
(1)

where

\[ W = \begin{bmatrix} D & -C \\ -B & A \end{bmatrix} \in \mathbb{R}^{(m+n) \times (m+n)} \]

is a nonsingular or an irreducible singular M-matrix, with \( A \in \mathbb{R}^{m \times m} \), \( D \in \mathbb{R}^{n \times n} \) and \( B, C^T, X \in \mathbb{R}^{m \times n} \). The MARE is a specific type of nonsymmetric algebraic Riccati equations, and there have been many associated studies from transport theory [4, 15]. The solvability of a highly structured MARE, with rank-1 updated \( A \) and \( D \), and rank-1 \( B \) and \( C \), has been established in [15, 16, 17]. A keen competition of various iterative methods then follows, mostly for the MARE with rank-1 structures but also its generations, lasting more than twenty years. This involves various iteration schemes [2, 10, 11, 12, 25, 26], Newton’s method [6, 23, 24], transformation to quadratic equations [7] and the structure-preserving doubling algorithm (SDA) [8, 13] and its generalizations [20, 21, 22, 30, 32], with theories benefitted from the associated Hamiltonian matrix and M-matrix

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The problem attracts such vast interests because of its interesting structures, as well as the diverse applications in transport theory [9, 15, 16, 17, 20, 21], Markov-modulated fluid queue theory [19, 29, 32] and the Wiener-Hopf decompositions [9].

Generally, the MARE (1) admits more than one solutions [18] owing to its nonlinearity. However, it is shown in [9, 11] that (1) has a unique minimal nonnegative solution \( \Theta - X \) interpreted componentwise which is of interest in practice. Here, by the minimal nonnegative solution we mean that \( \Theta - X \) is nonnegative for any other nonnegative solution \( \Theta \) of the MARE. The dual problem of (1) has the form, with \( Y \in \mathbb{R}^{n \times m} \):

\[
YBY - YA - DY + C = 0.
\]  (2)

The dual problem (2) is also an MARE and admits a unique minimal nonnegative solution \( Y \).

When solving the MAREs, one may particularly be interested in the accuracy of small entries in \( X \), in how small relative perturbations to the entries of \( A, D, B \) and \( C \) affect the entries in \( X \). Are they small, no matter how tiny the entries in \( X \) are? Thanks to the componentwise perturbation analysis in [33], small elements do not possess larger relative errors. Hence if implemented carefully several methods can compute \( X \) with high relative componentwise accuracy, including the fix-point iterations [9], the Newton method [10], the SDA [13] and the alternating-directional doubling algorithm (ADDA) [28, 32]. All the methods mentioned above are efficient for MAREs of small or medium sizes, in terms of execution time and memory requirements. However, in the SDA or ADDA, the nonsingular M-matrix kernels may become ill-conditioned, especially for the critical case. The inversions of these kernels may then cost the approximate solution its precision.

In this paper, we solve the large-scale MAREs with low-rank structures, and propose a highly efficient method for the MARE (1), generalizing the new decoupled form of the SDA [14] and adapting the highly accurate GTH-like algorithm for the M-matrix structures in the ADDA [32]. In detail, we show that the ADDA can be decoupled using the low-rank structures. Then we construct the triplet representations for the M-matrix kernels, for the GTH-like algorithm in high accuracy. Note that the highly accurate ADDA algorithm [32] produces efficiently these triplet representations for the kernels from that of \( W \), computing some required nonnegative vectors recursively. Our triplet representations are not straightforward adaptations of that given in [32]. To demonstrate the efficiency of our algorithm, we apply it to several test examples.

The ADDA is decoupled using the low-rank structures in Section 2.1. We prove the kernels in the iterations are M-matrices in Section 2.2, and in Section 2.3 we construct their triplet representations. We summarize our algorithm in Section 2.4. Illustrative numerical examples are presented in Section 3, before we conclude in Section 4.

**Notations**

By \( \mathbb{R}^{n \times n} \) we denote the set of all \( n \times n \) real matrices, with \( \mathbb{R}^n = \mathbb{R}^{n \times 1} \) and \( \mathbb{R} = \mathbb{R}^1 \). The \( n \times n \) identity matrix is \( I_n \) and we write \( I \) if its dimension is clear. The zero matrix is \( 0 \) and the superscript \((\cdot)^{T}\) takes the transpose. By \( 1_l \in \mathbb{R}^l \) and \( 1_{k \times l} \in \mathbb{R}^{k \times l} \), respectively, we denote the \( l \)-vector and \( k \times l \) matrix of all ones. The symbol \( M \otimes N \) is the Kronecker product of the matrices \( M \) and \( N \). For \( \Phi \in \mathbb{R}^{m \times n} \), \( \Phi_{ij} \) is its \((i, j)\) entry, and by \(|\Phi|\) we denote the matrix with elements \(|\Phi_{ij}|\). The inequality \( \Phi \leq \Psi \) holds if and only if \( \Psi_{ij} \leq \Phi_{ij} \), and similarly for \( \Phi < \Psi \), \( \Phi \geq \Psi \) and \( \Phi > \Psi \). In particular, \( \Phi \) is a nonnegative matrix means that \( \Phi_{ij} \geq 0 \). The submatrix of \( \Phi \), comprised of the rows \( k \) to \( m \) and columns \( l \) to \( n \), is written as \( \Phi_{k:m,l:n} \). We denote the triplet
representation of $W$ by
\[(N_W, [u_1^T, u_2^T], [v_1^T, v_2^T]),\]
where $N_W$ is the off-diagonal part of $-W$, with
\[W[u_1 \atop u_2] = [v_1 \atop v_2] \geq 0,\]
\[0 < u_1 \in \mathbb{R}^n, 0 < u_2 \in \mathbb{R}^m, 0 \leq v_1 \in \mathbb{R}^n \text{ and } 0 \leq v_2 \in \mathbb{R}^m.\]

## 2 Decoupled ADDA with high accuracy

The highly accurate alternating-directional doubling algorithm \[32\] (accADDA) is a variation of the SDA and shares the same doubling recursions:
\[
\begin{align*}
F_{k+1} &= F_k(I_m - H_kG_k)^{-1}F_k, & E_{k+1} &= E_k(I_n - G_kH_k)^{-1}E_k, \\
H_{k+1} &= H_k + F_k(I_m - H_kG_k)^{-1}H_kE_k, & G_{k+1} &= G_k + E_k(I_n - G_kH_k)^{-1}G_kF_k.
\end{align*}
\]

The pivotal difference from the SDA lies in the initial iterates with the two parameters $\alpha$ and $\beta$, with $D_\alpha := \alpha D + I$ and $A_\beta := 1 + I$,
\[
\begin{bmatrix}
E_0 & G_0 \\
H_0 & F_0
\end{bmatrix} = \begin{bmatrix}
D_\alpha & -\beta C \\
-\alpha B & A_\beta
\end{bmatrix}^{-1},
\]
where $0 \leq \alpha \leq \min_i a_{ii}^{-1}, 0 \leq \beta \leq \min_j d_{jj}^{-1}$, $\max\{\alpha, \beta\} > 0$, with $a_{ii}$ and $d_{jj}$ respectively being the diagonal entries of $A$ and $D$. The nonsingularity of the inverse in (4) is guaranteed as below.

**Lemma 2.1** (\[32, \text{Lemma 3.1}\]). Let $\alpha \geq 0$ and $\beta \geq 0$ with $\max\{\alpha, \beta\} > 0$, then \[\begin{bmatrix}
D_\alpha & -\beta C \\
-\alpha B & A_\beta
\end{bmatrix}\]
is a nonsingular $M$-matrix.

With $(F_0, E_0, H_0, G_0)$ from (4) as the initial iterates, by the doubling recursions in (3), \[32\] shows that the sequences $(F_k, E_k, H_k, G_k)$ satisfy

1. $E_k \geq 0$ and $F_k \geq 0$ ($k \geq 0$), and are uniformly bounded with respect to $k$;
2. $I - H_kG_k$ and $I - G_kH_k$ are nonsingular $M$-matrices for $k \geq 0$; and
3. $0 \leq H_k \leq H_{k+1} \leq X, 0 \leq G_k \leq G_{k+1} \leq Y$, implying that $\{H_k\}$ and $\{G_k\}$ respectively converge increasingly to $X$ and $Y$.

Actually, $\{H_k\}$ and $\{G_k\}$ converge quadratically except for the critical case \[8\], in which they converge linearly with a rate of 0.5. With the uniformly bounded property, Xue and Li \[32\] subtly devised the triplet representations of the nonsingular $M$-matrices $I - H_kG_k$ and $I - G_kH_k$ by tracking the difference
\[\begin{bmatrix}
u_1^T \atop u_2^T
\end{bmatrix} - \begin{bmatrix}
E_k & G_k \\
H_k & F_k
\end{bmatrix} \begin{bmatrix}
u_1^T \\
u_2^T
\end{bmatrix},\]
and proposed the accADDA for MAREs.

Recursively, one constructs the triplet representations of the $M$-matrices kernels $I - H_kG_k$ and $I - G_kH_k$ in a cancellation-free manner \[32\]. However, it is not suitable for large-scale MAREs with low-rank structures since all the iterates $F_k, E_k, H_k$ and $G_k$ are required, leading to a computational complexity of $\mathcal{O}(n^3 + m^3)$ in each iteration. To adopt the accADDA to large-scale MAREs, we firstly show that it can be decoupled when low-rank structure exists, as in our new.
dADDA. We prove the kernels in the dADDA are nonsingular M-matrices of small sizes. Then we construct the triplet representations of these nonsingular kernels, enabling the highly accurate computation with the GTH-like algorithm for the associated linear equations.

We assume that \( B \) and \( C \) are of low-rank with the full rank factorizations \( B = B_lB_r^T \) and \( C = C_lC_r^T \), where \( 0 \leq B_l \in \mathbb{R}^{m \times p}, 0 \leq B_r \in \mathbb{R}^{n \times p}, 0 \leq C_l \in \mathbb{R}^{n \times q} \) and \( 0 \leq C_r \in \mathbb{R}^{m \times q} \).

### 2.1 Decoupled ADDA

Firstly, the initial iterates, specified in (4), can be rewritten as

\[
\begin{align*}
F_0 &= (A_{\beta} - \alpha \beta BD_{\alpha}^{-1}C)^{-1}(A_{-\alpha} + \alpha^2 BD_{\alpha}^{-1}C), \\
E_0 &= (D_{\alpha} - \alpha \beta CA_{\beta}^{-1}B)^{-1}(D_{-\beta} + \beta^2 CA_{\beta}^{-1}B), \\
H_0 &= \gamma(A_{\beta} - \alpha \beta BD_{\alpha}^{-1}C)^{-1}BD_{\alpha}^{-1}, \\
G_0 &= \gamma(D_{\alpha} - \alpha \beta CA_{\beta}^{-1}B)^{-1}CA_{\beta}^{-1},
\end{align*}
\]

where \( \gamma := \alpha + \beta > 0 \) and the nonsingularity of \( D_{\alpha} \) and \( A_{\beta} \) follows from that of \( W \). In fact, since \( W \) is a nonsingular or an irreducible M-matrix, \( A \) and \( D \) are nonsingular M-matrices, implying that \( D_{\alpha} \) and \( A_{\beta} \) are nonsingular. Furthermore, by Lemma 2.1 and the results in [27], we have further results for the Schur complements \( A_{\beta} - \alpha \beta BD_{\alpha}^{-1}C \) and \( D_{\alpha} - \alpha \beta CA_{\beta}^{-1}B \), given below.

**Lemma 2.2.** Let \( \alpha \geq 0 \) and \( \beta \geq 0 \) with \( \max\{\alpha, \beta\} > 0 \), then \( D_{\alpha} \) and \( A_{\beta} \) and the Schur complements \( A_{\beta} - \alpha \beta BD_{\alpha}^{-1}C \) and \( D_{\alpha} - \alpha \beta CA_{\beta}^{-1}B \) are nonsingular M-matrices.

Now substitute the full rank factorizations \( B = B_lB_r^T \) and \( C = C_lC_r^T \) into (5), by the Sherman-Morrison-Woodbury formula (SMWF), we obtain

\[
\begin{align*}
(A_{\beta} - \alpha \beta BD_{\alpha}^{-1}C)^{-1} &= (A_{\beta} - \alpha \beta BD_{\alpha}^{-1}C)^{-1} \\
&= A_{\beta}^{-1} + \beta U_0(I - Y_0Z_0)^{-1}Y_0V_0^T, \\
(D_{\alpha} - \alpha \beta CA_{\beta}^{-1}B)^{-1} &= (D_{\alpha} - \alpha \beta CA_{\beta}^{-1}B)^{-1} \\
&= D_{\alpha}^{-1} + \alpha W_0(I - Z_0Y_0)^{-1}Z_0Q_0^T, \\
\alpha \left[A_{\beta}^{-1} + \beta U_0(I - Y_0Z_0)^{-1}Y_0V_0^T \right] B_lB_r^T D_{\alpha}^{-1} C_l C_r^T &= U_0(I - Y_0Z_0)^{-1}Y_0C_r^T, \\
\beta \left[D_{\alpha}^{-1} + \alpha W_0(I - Z_0Y_0)^{-1}Z_0Q_0^T \right] C_l^T C_r A_{\beta}^{-1} B_l B_r^T &= W_0(I - Z_0Y_0)^{-1}Z_0B_r^T,
\end{align*}
\]

where

\[
\begin{align*}
U_0 &= A_{\beta}^{-1}B_l, & V_0 &= A_{\beta}^{-1}C_r, & W_0 &= D_{\alpha}^{-1}C_l, & Q_0 &= D_{\alpha}^{-1}B_r, \\
Y_0 &= \alpha B_l^TD_{\alpha}^{-1}C_l, & Z_0 &= \beta C_r^TA_{\beta}^{-1}B_l.
\end{align*}
\]

As a result, after defining

\[
\begin{align*}
A_{\alpha, \beta} &= A_{\beta}^{-1}A_{-\alpha}, & D_{\alpha, \beta} &= D_{\alpha}^{-1}D_{-\beta},
\end{align*}
\]


we obtain

\[ F_0 = A_\beta^{-1}A_{-\alpha} + \beta U_0(I - Y_0Z_0)^{-1}Y_0V_0^TA_{-\alpha} + \alpha U_0(I - Y_0Z_0)^{-1}Y_0C_\gamma^T \]
\[ = A_{\alpha,\beta} + \gamma U_0(I - Y_0Z_0)^{-1}Y_0V_0^T, \]
\[ E_0 = D_{\alpha}^{-1}D_{-\beta} + \alpha W_0(I - Z_0Y_0)^{-1}Z_0Q_0^TD_{-\beta} + \beta W_0(I - Z_0Y_0)^{-1}Z_0B_\gamma^T \]
\[ = D_{\alpha,\beta} + \gamma W_0(I - Z_0Y_0)^{-1}Z_0Q_0^T, \]
\[ H_0 = \gamma [A_\beta^{-1} + \beta U_0(I - Y_0Z_0)^{-1}Y_0V_0^T] B_1B_1^TD_{\alpha}^{-1} \equiv \gamma U_0(I - Y_0Z_0)^{-1}Q_0^T, \]
\[ G_0 = \gamma [D_{\alpha}^{-1} + \alpha W_0(I - Z_0Y_0)^{-1}Z_0Q_0^T] C_1C_1^TA_\beta^{-1} \equiv \gamma W_0(I - Z_0Y_0)^{-1}V_0^T. \]

Apparently, \( H_0 \) and \( G_0 \) are decoupled.

Next define \( S_0 := V_0^T U_0, T_0 := Q_0^T W_0, U_1 := A_{\alpha,\beta} U_0, Q_1 := D_{\alpha,\beta} Q_0 \) and \( L := I - Y_0Z_0 - \gamma^2 T_0(I - Z_0Y_0)^{-1} S_0 \), then we get

\[ (I - H_0 G_0)^{-1} = [I - \gamma U_0(I - Y_0Z_0)^{-1} T_0(I - Z_0Y_0)^{-1} V_0^T]^{-1} \equiv I + \gamma^2 U_0 L^{-1} T_0(I - Z_0Y_0)^{-1} V_0^T, \]
\[ (I - H_0 G_0)^{-1} H_0 \equiv \gamma U_0 L^{-1} Q_0^T. \]

Consequently, it holds that

\[ H_1 = H_0 + F_0(I - H_0 G_0)^{-1} H_0 E_0 \]
\[ = \gamma [U_0, U_1] \begin{bmatrix} I & (I - Y_0Z_0)^{-1} V_0 S_0 \\ 0 \end{bmatrix} \begin{bmatrix} I - Y_0Z_0 & L \\ 0 \end{bmatrix}^{-1} \begin{bmatrix} I - Z_0Y_0 \end{bmatrix}^{-1} Z_0 \begin{bmatrix} 0 \\ I \end{bmatrix} \begin{bmatrix} Q_0^T \\ Q_1^T \end{bmatrix} \]
\[ \equiv \gamma [U_0, U_1] (I - Y_1 Z_1)^{-1} \begin{bmatrix} Q_0^T \\ Q_1^T \end{bmatrix}, \]

where \( Y_1 = \begin{bmatrix} 0 \\ Y_0 \end{bmatrix} \gamma T_0 \) and \( Z_1 = \begin{bmatrix} 0 \\ Z_0 \end{bmatrix} \gamma S_0 \). Now denote \( V_1 := A_{\alpha,\beta}^T V_0 \) and \( W_1 := D_{\alpha,\beta} W_0 \), then a similar but tedious process produces

\[ G_1 = \gamma [W_0, W_1] (I - Z_1 Y_1)^{-1} \begin{bmatrix} V_0^T \\ V_1^T \end{bmatrix}, \quad F_1 = A_{\alpha,\beta}^2 + \gamma [U_0, U_1] (I - Y_1 Z_1)^{-1} Y_1 \begin{bmatrix} V_0^T \\ V_1^T \end{bmatrix}, \]
\[ E_1 = D_{\alpha,\beta}^2 + \gamma [W_0, W_1] (I - Z_1 Y_1)^{-1} Z_1 \begin{bmatrix} Q_0^T \\ Q_1^T \end{bmatrix}. \]

Clearly, \( H_1 \) and \( G_1 \) are decoupled. By a similar process with the help of the SMWF, we eventually obtain the sequences \((F_k, E_k, H_k, G_k)\) in the dADDA, as in the following theorem.

**Theorem 2.1** (dADDA). Let \( U_j := A_{\alpha,\beta} U_{j-1}, V_j := A_{\alpha,\beta}^T V_{j-1}, W_j := D_{\alpha,\beta} W_{j-1} \) and \( Q_j := D_{\alpha,\beta}^T Q_{j-1} \) for \( j \geq 1 \). For \( k \geq 1 \), denote

\[ \hat{U}_k = [U_0, U_1, \cdots, U_{2^k-1}], \quad \hat{V}_k = [V_0, V_1, \cdots, V_{2^k-1}], \]
\[ \hat{W}_k = [W_0, W_1, \cdots, W_{2^k-1}], \quad \hat{Q}_k = [Q_0, Q_1, \cdots, Q_{2^k-1}], \]
and let
\[
Y_k = \begin{bmatrix} 0 & Y_k^{k-1} \\ Y_{k-1} & \gamma T_{k-1} \end{bmatrix}, \quad Z_k = \begin{bmatrix} 0 & Z_k^{k-1} \\ Z_{k-1} & \gamma S_{k-1} \end{bmatrix}
\]
with \( T_{k-1} = \hat{Q}_k^T W_{k-1} \) and \( S_{k-1} = \hat{V}_k^T U_{k-1} \). Then the iteration in (3) has the following decoupled form
\[
F_k = A_{\alpha, \beta}^{2^k} + \gamma \hat{U}_k (I - Y_k Z_k)^{-1} Y_k \hat{V}_k^T, \quad E_k = D_{\alpha, \beta}^{2^k} + \gamma \hat{W}_k (I - Z_k Y_k)^{-1} Z_k \hat{Q}_k^T,
\]
\[
H_k = \gamma \hat{U}_k (I - Y_k Z_k)^{-1} \hat{Q}_k^T, \quad G_k = \gamma \hat{W}_k (I - Z_k Y_k)^{-1} \hat{V}_k^T.
\]

The proof of Theorem 2.1 is similar to that for
\[
(\hat{F}_k, \hat{E}_k, \hat{H}_k, \hat{G}_k) = \left( (\alpha^{-1} \beta)^{2^k} F_k, (\beta^{-1} \alpha)^{2^k} E_k, H_k, G_k \right)
\]
given in [14], which we omit.

Remark 2.1. Select \( \alpha, \beta > 0 \), write
\[
\hat{F}_0 = \alpha^{-1} \beta F_0, \quad \hat{E}_0 = \beta^{-1} \alpha E_0, \quad \hat{H}_0 = H_0, \quad \hat{G}_0 = G_0,
\]
and by \((\hat{F}_k, \hat{E}_k, \hat{H}_k, \hat{G}_k)\) denote the ADDA sequences from \((\hat{F}_0, \hat{E}_0, \hat{H}_0, \hat{G}_0)\). Providing that \( B \) and \( C \) are of low-rank and possess the full rank factorizations \( B = B_i B_i^T \) and \( C = C_l C_l^T \), it has been demonstrated in [14] that the traditional ADDA sequences \((\hat{F}_k, \hat{E}_k, \hat{H}_k, \hat{G}_k)\) are decoupled.

2.2 Kernels are M-matrices

When applying the dADDA to solve large-scale MAREs with low-rank structures to obtain high accurate solution, two crucial issues have to be settled:

1. Are \( U_j, V_j, W_j \) and \( Q_j \) all nonnegative?

2. Are the kernels \( I - Y_k Z_k \) and \( I - Z_k Y_k \) nonsingular M-matrices?

For both questions, we need to show that \( A_{\alpha, \beta}, D_{\alpha, \beta}, (I - Y_k Z_k)^{-1} \) and \( (I - Z_k Y_k)^{-1} \) are nonnegative. In the following, we assume that \( \max \{ \alpha, \beta \} > 0 \), which can be satisfied by choice.

Lemma 2.3. It holds that \( A_{-\alpha} \geq 0 \) and \( D_{-\beta} \geq 0 \).

Proof. Since \( W \) is a nonsingular or an irreducible singular M-matrix, then \( A \) is a nonsingular M-matrix, implying \( a_{ij} \leq 0 \) with \( i \neq j \) and \( a_{ii} > 0 \). Thus in \( I - \alpha A \), the off-diagonal entries \(-\alpha a_{ij}\) are nonnegative. For the diagonal elements, because \( \alpha \leq \min_i a_{ii}^{-1} = (\max_i a_{ii})^{-1} \), then we have \( \alpha a_{ii} \leq a_{ii} (\max_i a_{ii})^{-1} \leq 1 \), showing that \( 1 - \alpha a_{ii} \geq 0 \). Hence \( I - \alpha A \) is nonnegative. Similarly, we can show that \( I - \beta D \) is nonnegative. \( \square \)

The nonnegativity of \( B_0, B_r, C_l, C_r, A^{-1}_\beta \) and \( D^{-1}_\alpha \) leads to that of \( U_0, Q_0, W_0 \) and \( V_0 \). Then by Lemmas 2.2 and 2.3, we know that \( A_{\alpha, \beta} = A^{-1}_\beta A_{-\alpha} \geq 0 \) and \( D_{\alpha, \beta} = D^{-1}_\alpha D_{-\beta} \geq 0 \). Furthermore, it holds that \( U_k = A_{\alpha, \beta}^{-1} U_0 \geq 0 \), \( V_k = (A_{\alpha, \beta}^T)^k V_0 \geq 0 \), \( W_k = D_{\alpha, \beta}^{-1} W_0 \geq 0 \) and \( Q_k = (D_{\alpha, \beta}^T)^k Q_0 \geq 0 \). Also, we have \( S_k \geq 0, T_k \geq 0, Y_k \geq 0 \) and \( Z_k \geq 0 \).
Lemma 2.4. The kernels $I - Y_0Z_0$ and $I - Z_0Y_0$ are nonsingular M-matrices.

Proof. It follows from Lemma 2.2 that

$$(A_\beta - \alpha \beta BD_\alpha^{-1}C)^{-1} \geq 0, \quad (D_\alpha - \alpha \beta CA_\beta^{-1}B)^{-1} \geq 0.$$  

Furthermore, by (6) and (7) it holds that

$$
0 \leq \beta C_r^T (A_\beta - \alpha \beta BD_\alpha^{-1}C)^{-1} B_l = \beta C_r^T \left[ A_\beta^{-1} + \beta U_0(I - Y_0 Z_0)^{-1} Y_0 V_0^T \right] B_l
= Z_0 + Z_0(I - Y_0 Z_0)^{-1} Y_0 Z_0 \equiv Z_0(I - Y_0 Z_0)^{-1},
$$
$$
0 \leq \alpha B_l^T (D_\alpha - \alpha \beta CA_\beta^{-1}B)^{-1} C_l = \alpha B_l^T \left[ D_\alpha^{-1} + \alpha W_0(I - Z_0 Y_0)^{-1} Z_0 Q_0^T \right] C_l
= Y_0 + Y_0(I - Z_0 Y_0)^{-1} Z_0 Y_0 \equiv Y_0(I - Z_0 Y_0)^{-1}.
$$

Because $Y_0 \geq 0$ and $Z_0 \geq 0$, we have $Y_0 Z_0(I - Y_0 Z_0)^{-1} \geq 0$ and $Z_0 Y_0(I - Z_0 Y_0)^{-1} \geq 0$, implying $I + Y_0 Z_0(I - Y_0 Z_0)^{-1} \equiv (I - Y_0 Z_0)^{-1} \geq 0$, $I + Z_0 Y_0(I - Z_0 Y_0)^{-1} \equiv (I - Z_0 Y_0)^{-1} \geq 0$.

Hence $I - Y_0 Z_0$ and $I - Z_0 Y_0$ are nonsingular M-matrices due to the fact that a nonsingular Z-matrix is an M-matrix if and only if its inverse is nonnegative [9].

The following theorem concerns the kernels $I - Y_k Z_k$ and $I - Z_k Y_k$ in the dADDA.

Theorem 2.2. The kernels $I - Y_k Z_k$ and $I - Z_k Y_k$ are nonsingular M-matrices for all $k \geq 0$.

Proof. We prove by induction, with the case for $k = 0$ from Lemma 2.4. Assume that the result holds for $k \geq 1$, that is $(I - Y_k Z_k)^{-1} \geq 0$ and $(I - Z_k Y_k)^{-1} \geq 0$, we then show that $(I - Y_{k+1} Z_{k+1})^{-1} \geq 0$. Since $I - H_k G_k$ is a nonsingular M-matrix, then with

$$K := [I - \gamma^2(I - Y_k Z_k)^{-1} T_k(I - Z_k Y_k)^{-1} S_k]^{-1},$$

we have

$$0 \leq (I - H_k G_k)^{-1} = [I - \gamma^2 \hat{U}_k (I - Y_k Z_k)^{-1} T_k(I - Z_k Y_k)^{-1} \hat{V}_k^T]^{-1}
= I + \gamma^2 \hat{U}_k [I - \gamma^2(I - Y_k Z_k)^{-1} T_k(I - Z_k Y_k)^{-1} \hat{V}_k^T \hat{U}_k]^{-1} (I - Y_k Z_k)^{-1} T_k(I - Z_k Y_k)^{-1} \hat{V}_k^T
= I + \gamma^2 \hat{U}_k K(I - Y_k Z_k)^{-1} T_k(I - Z_k Y_k)^{-1} \hat{V}_k^T
\equiv I + \gamma^2 \hat{U}_k K(I - Y_k Z_k)^{-1} T_k(I - Z_k Y_k)^{-1} \hat{V}_k^T,
0 \leq \hat{V}_k^T (I - H_k G_k)^{-1} \hat{U}_k = \hat{V}_k^T [I + \gamma^2 \hat{U}_k K(I - Y_k Z_k)^{-1} T_k(I - Z_k Y_k)^{-1} \hat{V}_k^T] \hat{U}_k
= S_k + \gamma^2 S_k K(I - Y_k Z_k)^{-1} T_k(I - Z_k Y_k)^{-1} S_k \equiv S_k K.$$

Moreover, it follows from $(I - Y_k Z_k)^{-1} \geq 0$, $T_k = Q_k^T \hat{W}_k \geq 0$ and $(I - Z_k Y_k)^{-1} \geq 0$ that

$$0 \leq I + \gamma^2(I - Y_k Z_k)^{-1} T_k(I - Z_k Y_k)^{-1} S_k K \equiv K,$$

leading to

$$M := [I - Y_k Z_k - \gamma^2 T_k(I - Z_k Y_k)^{-1} S_k]^{-1} \equiv K(I - Y_k Z_k)^{-1} \geq 0.$$
Substituting the expression (10) for $Y_{k+1}$ and $Z_{k+1}$ into $(I - Y_{k+1}Z_{k+1})^{-1}$, we obtain

$$(I - Y_{k+1}Z_{k+1})^{-1} = \left[ I - Y_k Z_k - \gamma Y_k S_k \right]^{-1}$$

$$= \left[ I - Y_k Z_k - \gamma^2 T_k (I - Z_k Y_k)^{-1} S_k + \gamma T_k Z_k (I - Y_k Z_k)^{-1} I \right].$$

Since $(I - Y_k Z_k)^{-1} \geq 0$, $Y_k \geq 0$, $S_k \geq 0$, $M \geq 0$, $T_k \geq 0$ and $Z_k \geq 0$, thus $(I - Y_{k+1}Z_{k+1})^{-1} \geq 0$, implying that $I - Y_{k+1}Z_{k+1}$ is a nonsingular M-matrix.

Analogously, it holds that $I - Z_{k+1}Y_{k+1}$ is a nonsingular M-matrix. 

### 2.3 GTH-like algorithm and triplet representations

Firstly, we briefly sketch the GTH-like algorithm presented in [1], which solves the M-matrix linear system $Mx = b$ in high accuracy, where each entry of the solution $x$ have almost full relative accuracy. Given the triplet representation of the nonsingular M-matrix $M$, the GTH-like algorithm, a variation of Gaussian elimination without pivoting, computes the inverse $M^{-1}$ with high relative componentwise accuracy since computations are cancellation-free.

Let $M \in \mathbb{R}^{n \times n}$ be a nonsingular M-matrix and $(N_M, u_M, v_M)$ be its triplet representation, where $N_M$ is the off-diagonal part of $M$, $u_M > 0$ and $v_M = M u_M > 0$. Obviously, we have $N_M \geq 0$ and the diagonal part of $M$ can be determined in a cancellation-free way:

$$M_{ii} = \frac{v_M}{(u_M)_i} + \sum_{j \neq i} (N_M)_{ij} (u_M)_j. \quad (11)$$

It can be verified that $M^{(k)} \in \mathbb{R}^{(n-k) \times (n-k)}$, the coefficient matrix after $k$ Gaussian eliminations, is still a nonsingular M-matrix. Moreover, the triplet representation of $M^{(k)}$ can be constructed from that of $M^{(k-1)}$, with $M^{(0)} = M$. As a result, based on (11), one can compute the LU factorization of $M$ cancellation-free. We outline the GTH-like algorithm given in [1] in Algorithm 1.

Note that when $b \geq 0$, no subtraction occurs in the forward and backward substitutions, thus the whole solution process is cancellation-free, leading to full accuracy for all entries of $x$. For the detailed analysis for the GTH-like algorithm, please refer to [1, 31].

The computational complexity of Algorithm 1 is $O(n^3)$ and the dominant cost lies in line 6. Hence Algorithm 1 is efficient for M-matrix linear systems of medium sizes. However, for large-scale problems with some special structures, such as when $M$ is sparse or a rank one update of a nonsingular diagonal matrix, its complexity may be reduced when properly implemented. For example, where $M = D_M - ab^T$ with $D_M$ being diagonal and $a, b > 0$, the computational complexity will be reduced to $O(n^2)$ since for all $k \geq 0$ the off-diagonal part of $M^{(k)}$ are rank one updated.

In Section 2.2, we demonstrate that the kernels $I - Y_k Z_k$ and $I - Z_k Y_k$ are nonsingular M-matrices. To solve the associated M-matrix linear systems with the GTH-like algorithm [1, 28], one needs the triplet representations for those kernels.

From the triplet representation of $W$, we have $W \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, or equivalently

$$D u_1 = v_1 + C u_2, \quad A u_2 = v_2 + B u_1,$$
we then have or equivalently

\[ \text{Theorem 2.3.} \quad \text{It holds that} \]

\[ \text{Proof.} \quad \text{To obtain} \]

\[ \text{Remark 2.2.} \quad \text{Note that the GTH-like algorithm works via the LU factorizations of} \]

\[ \text{Consequently, with} \]

\[ \text{and the properties of M-matrices, it is obvious that} \]

\[ \text{These triplet representations enable the GTH-like algorithm [1, 28] to calculate} \]

\[ \text{Algorithm 1 GTH-like algorithm for solving} \]

\[ \text{Input:} \text{ the triplet representation} (N_M, u_M, v_M) \text{ and vector} \quad b. \]

\[ \text{Output:} \quad x = M^{-1}b. \]

\[ \text{1: set} \quad L = I_n, \quad U = -N_M \leq 0; \]

\[ \text{2: for} \quad k = 1 : 1 : n \text{ do} \]

\[ \text{3: } U_{k,k} = [(v_M)_k - U_{k,k+1:n}(u_M)_{k+1:n}]/(u_M)_k; \]

\[ \text{4: } L_{k+1:n} = U_{k+1:n}/U_{k,k}; \]

\[ \text{5: } U_{k+1:n,k} = 0; \]

\[ \text{6: } U_{k+1:n,k+1:n} = U_{k+1:n,k+1:n} - L_{k+1:n,k}U_{k,k+1:n}; \]

\[ \text{7: set the diagonal of} \quad U_{k+1:n,k+1:n} \text{ as} \quad 0; \]

\[ \text{8: } (v_M)_{k+1:n} = (v_M)_{k+1:n} - (v_M)_kL_{k+1:n,k}; \]

\[ \text{9: end} \]

\[ \text{10: solve} \quad Ly = b \text{ with forward substitution;} \]

\[ \text{11: solve} \quad Ux = y \text{ with backward substitution.} \]

which leads to

\[ D_\alpha u_1 = \alpha v_1 + u_1 + \alpha C u_2 \geq 0, \quad A_\beta u_2 = \beta v_2 + u_2 + \beta B u_1 \geq 0. \quad (12) \]

Consequently, with \( N_{\Theta} = \text{diag}(\Theta) - \Theta \), (12) gives the triplet representations of the nonsingular M-matrices \( D_\alpha \) and \( A_\beta \), respectively:

\[ (N_{D_\alpha}, u_1, \alpha v_1 + u_1 + \alpha C u_2), \quad (N_{A_\beta}, u_2, \beta v_2 + u_2 + \beta B u_1). \quad (13) \]

These triplet representations enable the GTH-like algorithm [1, 28] to calculate \( U_j, V_j, W_j, Q_j \) and \( T_j, S_j, Y_j, Z_j \) in high accuracy without cancellations, for \( j \geq 0 \). Recall from the structure of \( W \) and the properties of M-matrices, it is obvious that \( N_{D_\alpha} \) and \( N_{A_\beta} \) are nonnegative.

**Remark 2.2.** Note that the GTH-like algorithm works via the LU factorizations of \( D_\alpha \) and \( A_\beta \). To obtain \( Q_j \) and \( V_j \) accurately, there is no need for the triplet representations of \( D_\alpha^T \) and \( A_\beta^T \).

**Theorem 2.3.** It holds that \((I - Y_0Z_0)B_r^T u_1 \geq 0 \) and \((I - Z_0Y_0)C_r^T v_2 \geq 0 \).

**Proof.** Since

\[ W \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} D \\ -B \end{bmatrix} \begin{bmatrix} -C \\ A \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \]

we then have

\[ \begin{bmatrix} D_{-\beta} \\ -\alpha B \end{bmatrix} \begin{bmatrix} \alpha C \\ A_{-\alpha} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} D_{-\beta} \\ -\alpha B \end{bmatrix} \begin{bmatrix} \alpha C \\ A_{-\alpha} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \gamma \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \]

or equivalently

\[ \gamma D_{-\beta}^{-1}C u_2 + \gamma D_{-\beta}^{-1}v_1 = u_1 - D_{-\beta}^{-1}D_{-\beta} u_1, \quad (14) \]

\[ \gamma A_{-\beta}^{-1}B u_1 + \gamma A_{-\beta}^{-1}v_2 = u_2 - A_{-\beta}^{-1}A_{-\alpha} u_2. \quad (15) \]
Pre-multiplying $\alpha B_r^T$ and $\beta C_r^T$, respectively, on both sides of (14) and (15), then by (8) we have
\[
\begin{align*}
\gamma Y_0 C_r^T u_2 + \alpha \gamma Q_0^T v_1 &= \alpha B_r^T u_1 - \alpha B_r^T D_\alpha^{-1} D_{-\beta} u_1, \\
\gamma Z_0 B_r^T u_1 + \beta \gamma V_0^T v_2 &= \beta C_r^T u_2 - \beta C_r A_\beta^{-1} A_{-\alpha} u_2.
\end{align*}
\]
These are further equivalent to
\[
\begin{align*}
\gamma B_r^T u_1 - \gamma Y_0 C_r^T u_2 &= \beta B_r^T u_1 + \alpha \gamma Q_0^T v_1 + \alpha B_r^T D_\alpha^{-1} D_{-\beta} u_1, \quad (16) \\
\gamma C_r^T u_2 - \gamma Z_0 B_r^T u_1 &= \alpha C_r^T u_2 + \beta \gamma V_0^T v_2 + \beta C_r A_\beta^{-1} A_{-\alpha} u_2. \quad (17)
\end{align*}
\]
Now we rewrite (16) and (17) as
\[
\begin{align*}
\begin{bmatrix} -Y_0 & I \\ I & -Z_0 \end{bmatrix} \begin{bmatrix} C_r^T u_2 \\ B_r^T u_1 \end{bmatrix} &= \begin{bmatrix} \beta B_r^T u_1 + \alpha \gamma Q_0^T v_1 + \alpha B_r^T D_\alpha^{-1} D_{-\beta} u_1 \\ \alpha C_r^T u_2 + \beta \gamma V_0^T v_2 + \beta C_r A_\beta^{-1} A_{-\alpha} u_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\alpha k^T v_1}{\beta V_0^T v_2} + \frac{B_r^T D_\alpha^{-1} u_1}{C_r A_\beta^{-1} u_2} \end{bmatrix} = \begin{bmatrix} Q_0^T u_1 + \alpha Q_0^T v_1 \\ V_0^T u_2 + \beta V_0^T v_2 \end{bmatrix} \geq 0. \quad (18)
\end{align*}
\]
Pre-multiplying (18) by $\begin{bmatrix} I & Y_0 \\ Z_0 & I \end{bmatrix}$, it shows that
\[
\begin{bmatrix} -Y_0 & I \\ I & -Z_0 \end{bmatrix} \begin{bmatrix} C_r^T u_2 \\ B_r^T u_1 \end{bmatrix} = \begin{bmatrix} \alpha Q_0^T u_1 + \alpha Q_0^T v_1 \\ V_0^T u_2 + \beta V_0^T v_2 \end{bmatrix} \geq 0, \quad (19)
\]
implying the results we want to prove. \hfill \Box

With $B_r \geq 0$ and $C_r \geq 0$ of full column rank, we have $B_r^T u_1 > 0$ and $C_r^T u_2 > 0$. Hence from (19), we obtain the triplet representations of $I - Y_0 Z_0$ and $I - Z_0 Y_0$, respectively:
\[
\begin{align*}
(N_{I - Y_0 Z_0}, B_r^T u_1, Q_0^T u_1 + \alpha Q_0^T v_1 + Y_0 (V_0^T u_2 + \beta V_0^T v_2)), \\
(N_{I - Z_0 Y_0}, C_r^T u_2, V_0^T u_2 + \beta V_0^T v_2 + Z_0 (Q_0^T u_1 + \alpha Q_0^T v_1)). \quad (20)
\end{align*}
\]
Moreover, from the relationships between $Y_0$ and $Y_1$, and $Z_0$ and $Z_1$, the triplet representations in (20) provide further clues for the triplet representations of $I - Y_1 Z_1$ and $I - Z_1 Y_1$, and those of $I - Y_k Z_k$ and $I - Z_k Y_k$ for $k > 1$. Specifically, since $1_{2^k} \otimes B_r^T u_1 > 0$ and $1_{2^k} \otimes C_r^T u_2 > 0$, if
\[
(I - Y_k Z_k)(1_{2^k} \otimes B_r^T u_1) \geq 0, \quad (I - Z_k Y_k)(1_{2^k} \otimes C_r^T u_2) \geq 0, \quad (21)
\]
we have successfully found the triplet representations of $I - Y_k Z_k$ and $I - Z_k Y_k$. To verify (21), the following theorem is necessary.

**Theorem 2.4.** For the dADD with $u_1 > 0$ and $u_2 > 0$, it holds that
\[
\begin{align*}
(i) \quad & [Q_0, Q_1, \cdots, Q_{2^{k-1}}]^T u_1 - \gamma T_k (1_{2^k} \otimes C_r^T u_2) \geq 0; \quad \text{and} \\
(ii) \quad & [V_0, V_1, \cdots, V_{2^{k-1}}]^T u_2 - \gamma S_k (1_{2^k} \otimes B_r^T u_1) \geq 0.
\end{align*}
\]

**Proof.** Pre-multiplying $Q_i^T$ ($0 \leq i \leq 2^k - 1, k \geq 0$) and $D_{\alpha, \beta}$, respectively, on (14) yields
\[
\begin{align*}
\gamma Q_i^T W_0 C_r^T u_2 + \gamma Q_i^T D_{\alpha}^{-1} v_1 &= Q_i^T u_1 - Q_i^T v_1, \\
\gamma W_1 C_r^T u_2 + \gamma D_{\alpha, \beta} D_{\alpha}^{-1} v_1 &= D_{\alpha, \beta} u_1 - D_{\alpha, \beta}^2 u_1. \quad (22)
\end{align*}
\]
Thus the result in (ii).

Leading to the result in (i). Similarly, we get

\[
\begin{align*}
\gamma Q^T_1 W_2 C^T_r u_2 &+ \gamma Q^T_{i+2} D^{-1}_\alpha v_1 = Q^T_{i+1} u_1 - Q^T_{i+2} u_1, \\
\gamma W_2 C^T_r u_2 &+ \gamma D^2_{\alpha,\beta} D^{-1}_\alpha v_1 = D^2_{\alpha,\beta} u_1 - D^3_{\alpha,\beta} u_1.
\end{align*}
\] (23)

We pursue the same process as above on (23) and obtain

\[
\begin{align*}
\gamma Q^T_1 W_2 C^T_r u_2 &+ \gamma Q^T_{i+2} D^{-1}_\alpha v_1 = Q^T_{i+1} u_1 - Q^T_{i+3} u_1, \\
\gamma W_3 C^T_r u_2 &+ \gamma D^3_{\alpha,\beta} D^{-1}_\alpha v_1 = D^3_{\alpha,\beta} u_1 - D^4_{\alpha,\beta} u_1.
\end{align*}
\] (24)

Repeating the similar procedure as above, we eventually acquire

\[
\begin{align*}
\gamma Q^T_j W_j C^T_r u_2 &+ \gamma Q^T_{i+j} D^{-1}_\alpha v_1 = Q^T_{i+j} u_1 - Q^T_{i+j+1} u_1. 
\end{align*}
\] (25)

where \( j = 0, 1, \ldots, 2^k - 1 \). Summing (25) for \( j = 0, 1, \ldots, 2^k - 1 \), we have

\[
\begin{align*}
\gamma Q^T_i (W_0 + W_1 + \cdots + W_{2^k-1}) C^T_r u_2 &+ \gamma (Q^T_i + Q^T_{i+1} + \cdots + Q^T_{i+2^{k-1}}) D^{-1}_\alpha v_1 \\
&= Q^T_i u_1 - Q^T_{i+2^k} u_1.
\end{align*}
\] (26)

Rewrite (26) in matrix form \( (0 \leq i \leq 2^k - 1) \), we obtain

\[
\gamma T_k \begin{bmatrix} C^T_r u_2 \\ \vdots \\ C^T_r u_2 \end{bmatrix} + \gamma \begin{bmatrix} (Q^T_0 + \cdots + Q^T_{2^k-1}) D^{-1}_\alpha v_1 \\ \vdots \\ (Q^T_{2^k-1} + \cdots + Q^T_{2^{k+1}-1}) D^{-1}_\alpha v_1 \end{bmatrix} = \begin{bmatrix} Q^T_0 \\ \vdots \\ Q^T_{2^k} \end{bmatrix} u_1 - \begin{bmatrix} Q^T_{2^k} \\ \vdots \\ Q^T_{2^{k+1}-1} \end{bmatrix} u_1.
\]

This is equivalent to

\[
\begin{bmatrix} Q^T_0 \\ \vdots \\ Q^T_{2^{k+1}-1} \end{bmatrix} u_1 - \gamma T_k \begin{bmatrix} C^T_r u_2 \\ \vdots \\ C^T_r u_2 \end{bmatrix} = \gamma \begin{bmatrix} (Q^T_0 + \cdots + Q^T_{2^k-1}) D^{-1}_\alpha v_1 \\ \vdots \\ (Q^T_{2^k-1} + \cdots + Q^T_{2^{k+1}-1}) D^{-1}_\alpha v_1 \end{bmatrix} + \begin{bmatrix} Q^T_{2^k} \\ \vdots \\ Q^T_{2^{k+1}-1} \end{bmatrix} u_1,
\] (27)

leading to the result in (i). Similarly, we get

\[
\begin{bmatrix} V^T_0 \\ \vdots \\ V^T_{2^k-1} \end{bmatrix} u_2 - \gamma S_k \begin{bmatrix} B^T_r u_1 \\ \vdots \\ B^T_r u_1 \end{bmatrix} = \gamma \begin{bmatrix} (V^T_0 + \cdots + V^T_{2^k-1}) A^{-1}_\beta v_2 \\ \vdots \\ (V^T_{2^k-1} + \cdots + V^T_{2^{k+1}-1}) A^{-1}_\beta v_2 \end{bmatrix} + \begin{bmatrix} V^T_0 \\ \vdots \\ V^T_{2^{k+1}-1} \end{bmatrix} u_2 \geq 0,
\] (28)

thus the result in (ii).

The following part is devoted to the triplet representations of \( I - Y_k Z_k \) and \( I - Z_k Y_k \), for \( k \geq 1 \). We firstly compute the triplet representations of \( I - Y_1 Z_1 \) and \( I - Z_1 Y_1 \). Define

\[
P_1 = \begin{bmatrix} 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \in \mathbb{R}^{2(p+q) \times 2(p+q)}.
\]
Since

$$\begin{bmatrix} -Y_1 & I \\ I & -Z_1 \end{bmatrix} = P_1 \begin{bmatrix} -Y_0 & 0 & -Y_0 & I \\ 0 & 0 & I & -Z_0 \\ -Y_0 & I & 0 & 0 \\ I & -Z_0 & 0 & -\gamma S_0 \end{bmatrix} P_1^T$$

$$= P_1 \begin{bmatrix} 0 & 0 & -Y_0 & I \\ 0 & 0 & I & -Z_0 \\ -Y_0 & I & 0 & 0 \\ I & -Z_0 & 0 & -\gamma S_0 \end{bmatrix} P_1^T - P_1 \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \gamma T_0 & 0 & 0 & 0 \\ 0 & \gamma S_0 \end{bmatrix} P_1^T,$$

then by (18) and Theorem 2.4, it holds that

$$\begin{bmatrix} -Y_1 & I \\ I & -Z_1 \end{bmatrix} P_1 \begin{bmatrix} C_{r_2}^T u_2 \\ B_{r_2}^T u_1 \\ C_{r_2}^T u_2 \\ B_{r_2}^T u_1 \end{bmatrix} = P_1 \begin{bmatrix} 0 & 0 & -Y_0 & I \\ 0 & 0 & I & -Z_0 \\ -Y_0 & I & 0 & 0 \\ I & -Z_0 & 0 & -\gamma S_0 \end{bmatrix} P_1^T - P_1 \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \gamma T_0 & 0 & 0 & 0 \\ 0 & \gamma S_0 \end{bmatrix} P_1^T,$$

Moreover, define

$$v_1^{(1)} = \alpha \begin{bmatrix} Q_1^T v_1 \\ Q_1^T v_1 \end{bmatrix} + \begin{bmatrix} Q_1^T \\ Q_1^T \end{bmatrix} u_1 + \gamma \begin{bmatrix} 0 \\ Q_0^T D_{-1} v_1 \end{bmatrix} \geq 0,$$

$$v_2^{(1)} = \beta \begin{bmatrix} V_0^T v_2 \\ V_0^T v_2 \end{bmatrix} + \begin{bmatrix} V_0^T \\ V_0^T \end{bmatrix} u_2 + \gamma \begin{bmatrix} 0 \\ V_0^T A_{-1} v_2 \end{bmatrix} \geq 0,$$

it then follows from (27) and (28) that

$$\begin{bmatrix} -Y_1 & I \\ I & -Z_1 \end{bmatrix} \begin{bmatrix} C_{r_2}^T u_2 \\ C_{r_2}^T u_2 \\ B_{r_2}^T u_1 \\ B_{r_2}^T u_1 \end{bmatrix} = P_1 \begin{bmatrix} 0 & 0 & -Y_0 & I \\ 0 & 0 & I & -Z_0 \\ -Y_0 & I & 0 & 0 \\ I & -Z_0 & 0 & -\gamma S_0 \end{bmatrix} P_1^T - P_1 \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \gamma T_0 & 0 & 0 & 0 \\ 0 & \gamma S_0 \end{bmatrix} P_1^T,$$

$$\begin{bmatrix} Q_1^T u_1 + \alpha Q_1^T v_1 - \gamma T_0 C_{r_2}^T u_2 \\ V_0^T u_2 + \beta V_0^T v_2 \\ Q_0^T u_1 + \alpha Q_0^T v_1 \\ V_0^T u_2 + \beta V_0^T v_2 \end{bmatrix} \equiv \begin{bmatrix} v_1^{(1)} \\ v_2^{(1)} \end{bmatrix} \geq 0.$$
which is equivalent to
\[(I - Y_1 Z_1) \begin{bmatrix} B^T u_1 \\ B^T r u_1 \end{bmatrix} = v_1^{(1)} + Y_1 v_2^{(1)} \geq 0, \quad (I - Z_1 Y_1) \begin{bmatrix} C^T u_2 \\ C^T r u_2 \end{bmatrix} = v_2^{(1)} + Z_1 v_1^{(1)} \geq 0. \quad (30)\]

Obviously, (30) yields the triplet representations of the nonsingular M-matrices \(I - Y_k Z_k\) and \(I - Z_k Y_k\) for \(k \geq 1\), as presented below.

**Theorem 2.5.** Define
\[v_1^{(k)} = \alpha(1_{2^k} \otimes Q_0^T v_1) + [Q_0, Q_1, \cdots, Q_{2^k-1}]^T u_1 + \gamma [0, Q_0, Q_0 + Q_1, \cdots, Q_0 + Q_1 + \cdots + Q_{2^k-1}]^T D_0^{-1} v_1,\]
\[v_2^{(k)} = \beta(1_{2^k} \otimes V_0^T v_2) + [V_0, V_1, \cdots, V_{2^k-1}]^T u_2 + \gamma [0, V_0, V_0 + V_1, \cdots, V_0 + V_1 + \cdots + V_{2^k-1}]^T D_1^{-1} v_2\]
for \(k \geq 1\). Then it holds that
\[\begin{bmatrix} -Y_k & I \\ I & -Z_k \end{bmatrix} \begin{bmatrix} 1_{2^k} \otimes C^T u_2 \\ 1_{2^k} \otimes B^T r u_1 \end{bmatrix} = \begin{bmatrix} v_1^{(k)} \\ v_2^{(k)} \end{bmatrix}. \quad (31)\]

Moreover, we have
\[(I - Y_k Z_k)(1_{2^k} \otimes B^T r u_1) = v_1^{(k)} + Y_k v_2^{(k)}, \quad (I - Z_k Y_k)(1_{2^k} \otimes C^T u_2) = v_2^{(k)} + Z_k v_1^{(k)}. \quad (32)\]

**Proof.** We will prove by induction. By (29) and (30), the result is valid for \(k = 1\). Now assume that the result holds for \(k \geq 2\), then by defining
\[P_2 = \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \in \mathbb{R}^{2k+1(p+q) \times 2k+1(p+q)},\]

it holds that
\[\begin{bmatrix} -Y_{k+1} & I \\ I & -Z_{k+1} \end{bmatrix} = P_2 \begin{bmatrix} -\gamma T_k & 0 & -Y_k & I \\ 0 & 0 & I & -Z_k \\ -Y_k & I & 0 & -Z_k \\ I & -Z_k & 0 & -\gamma S_k \end{bmatrix} P_2^T \]
\[= P_2 \begin{bmatrix} 0 & 0 & -Y_k & I \\ 0 & 0 & I & -Z_k \\ -Y_k & I & 0 & 0 \\ I & -Z_k & 0 & 0 \end{bmatrix} P_2^T - P_2 \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} [\gamma T_k 0 0 0; 0 0 0 \gamma S_k] P_2^T.\]
Furthermore, post-multiplying $P_2 \begin{bmatrix} 1_{2^k} \otimes C_r^T u_2 \\ 1_{2^k} \otimes B_r^T u_1 \end{bmatrix}$ yields

$$
\begin{bmatrix} -Y_{k+1} & I \\ I & -Z_{k+1} \end{bmatrix} \begin{bmatrix} 1_{2^k+1} \otimes C_r^T u_2 \\ 1_{2^k+1} \otimes B_r^T u_1 \end{bmatrix}
= P_2 \begin{bmatrix} 0 & 0 & -Y_k & I \\ 0 & 0 & I & -Z_k \\ -Y_k & I & 0 & 0 \\ I & -Z_k & 0 & 0 \end{bmatrix} \begin{bmatrix} 1_{2^k} \otimes C_r^T u_2 \\ 1_{2^k} \otimes B_r^T u_1 \\ 1_{2^k} \otimes C_r^T u_2 \\ 1_{2^k} \otimes B_r^T u_1 \end{bmatrix} - P_2 \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma T_k & 0 & 0 & 0 \\ 0 & \gamma S_k & 0 & 0 \end{bmatrix} \begin{bmatrix} 1_{2^k} \otimes C_r^T u_2 \\ 1_{2^k} \otimes B_r^T u_1 \\ 1_{2^k} \otimes C_r^T u_2 \\ 1_{2^k} \otimes B_r^T u_1 \end{bmatrix}
$$

Besides, it follows from the definition of $v_1^{(k)}$ and (27) that

$$
v_1^{(k)} - \gamma T_k(1_{2^k} \otimes C_r^T u_2) = \alpha(1_{2^k} \otimes Q_0^T v_1) + \begin{bmatrix} Q_{2^k}^T \\ Q_{2^k+1}^T \\ \vdots \\ Q_{2^k+1-1}^T \end{bmatrix} u_1 + \gamma \begin{bmatrix} Q_0^T + Q_1^T + \cdots + Q_{2^k}^T \\ Q_0^T + Q_1^T + \cdots + Q_{2^k}^T \\ \vdots \\ Q_0^T + Q_1^T + \cdots + Q_{2^k+1-2}^T \end{bmatrix} D_{1}^{-1} v_1,$$

indicating that

$$
\begin{bmatrix} v_1^{(k)} \\ v_1^{(k)} - \gamma T_k(1_{2^k} \otimes C_r^T u_2) \end{bmatrix} = \begin{bmatrix} \alpha(1_{2^k} \otimes Q_0^T v_1) \\ \alpha(1_{2^k} \otimes Q_0^T v_1) \end{bmatrix} + [Q_0, Q_1, \cdots, Q_{2^k-1}, Q_{2^k}, Q_{2^k+1}, \cdots, Q_{2^k+1-1}]^T u_1
+ \gamma [0, Q_0, \cdots, Q_0 + Q_1 + \cdots + Q_{2^k-1}, \cdots, Q_0 + Q_1 + \cdots + Q_{2^k+1-2}]^T D_{1}^{-1} v_1
= v_1^{(k+1)}.
$$

Analogously, with the definition of $v_2^{(k)}$ and (28), we have

$$
\begin{bmatrix} v_2^{(k)} \\ v_2^{(k)} - \gamma S_k(1_{2^k} \otimes B_r^T u_1) \end{bmatrix} \equiv v_2^{(k+1)}.
$$

Consequently, we obtain

$$
\begin{bmatrix} -Y_{k+1} & I \\ I & -Z_{k+1} \end{bmatrix} \begin{bmatrix} 1_{2^k+1} \otimes C_r^T u_2 \\ 1_{2^k+1} \otimes B_r^T u_1 \end{bmatrix} = \begin{bmatrix} 1_{2^k+1} \otimes C_r^T u_2 \\ 1_{2^k+1} \otimes B_r^T u_1 \end{bmatrix}.
$$

The proof for (31) by induction is complete. Pre-multiplying $\begin{bmatrix} I & -Y_k \\ Z_k & I \end{bmatrix}$ to (31) leads to (32). \qed
Obviously, \( v^{(k)}_1 \geq 0 \) and \( v^{(k)}_2 \geq 0 \). So (32) gives the triplet representations of the nonsingular M-matrices \( I - Y_k Z_k \) and \( I - Z_k Y_k \), which are respectively

\[
\left( N_{I - Y_k Z_k}, 1_2 \otimes B^T u_1, v^{(k)}_1 \right), \quad \left( N_{I - Z_k Y_k}, 1_2 \otimes C^T u_2, v^{(k)}_2 + Z_k v^{(k)}_1 \right),
\]

(33)

### 2.4 Algorithm for dADDA

With the triplet representations of \( D_\alpha, A_\beta \) and \( I - Y_k Z_k \), we can compute \( H_k \) in highly accurate using the GTH-like algorithm. Our proposed algorithm is summarized as Algorithm 2.

---

**Algorithm 2 dADDA**

**Input:** coefficients \( A, D, B_l, B_r, C_l, C_r \) and vectors \( u_1, u_2, v_1, v_2 \).

**Output:** the minimal nonnegative solution \( X \).

1. choose \( \alpha, \beta \) with \( 0 \leq \alpha \leq \min_i a_{ii}^{-1}, 0 \leq \beta \leq \min_j d_{jj}^{-1}, \max \{ \alpha, \beta \} > 0 \);
2. construct the triplet representations of \( D_\alpha \) and \( A_\beta \) by (12);
3. compute \( U_0, V_0, Q_0 \) and \( W_0 \) by the GTH-like algorithm, using the triplet representations of \( D_\alpha \) and \( A_\beta \) in (13);
4. compute \( Y_0 = \alpha Q_0^T C_l \) and \( Z_0 = \beta C_r^T U_0 \);
5. compute \( H_0 \) by the GTH-like algorithm, using the triplet representation in (20);
6. set \( k = 0 \);
7. repeat
   8. compute \( T_k \) and \( S_k \);
   9. set \( k = k + 1 \);
10. compute \( Y_k \) and \( Z_k \) by (10);
11. compute \( U_j, V_j, Q_j \) and \( W_j \) by the GTH-like algorithm for \( j = 2^{k-1}, \ldots, 2^{k-1} - 1 \);
12. compute \( v^{(k)}_1 \) and \( v^{(k)}_2 \) by Theorem 2.5;
13. compute \( H_k \) by the GTH-like algorithm, using triplet representation in (33);
14. until convergence
15. return the last \( H_k \) as the approximation to \( X \).

---

For the implementation of Algorithm 2, we need to choose the stop criteria for convergence. With \( \varepsilon \) being a preselected tolerance, we may adopt one of the following criteria:

1. The normalized residual in norm:

\[
\frac{\| H_k C H_k - A H_k + B \|}{\| H_k C H_k \| + \| H_k D \| + \| A H_k \| + \| B \|} \leq \varepsilon,
\]

where \( \| \cdot \| \) is some matrix norm and for convenience one can use the Frobenius norm or the \( l_1 \) norm.

2. The relative change:

\[
| H_{k+1} - H_k | \leq \varepsilon H_{k+1},
\]

which is simple and cheap to use.
(3) The entrywise relative residual:
\[
\text{ERres}_k := \max_{i,j} \frac{|(H_k CH_k + N_A H_k + H_k N_D + C) - (\text{diag}(A) H_k + H_k \text{diag}(D))|_{(i,j)}}{|\text{diag}(A) H_k + H_k \text{diag}(D)|_{(i,j)}} \leq \varepsilon,
\]
which is the entrywise relative accuracy of \(H_k\) as an approximation to \(X\).

(4) The entrywise relative error:
\[
\text{ERerr}_k := \max_{i,j} \frac{|(H_k - X)_{(i,j)}|}{X_{(i,j)}} \leq \varepsilon,
\]
which is not generally available because \(X\) is unknown.

As we are interested in the accuracy of the entries of \(X\), thus for Algorithm 2 we recommend the entrywise relative residual \(\text{ERres}_k\) in the convergence control. In [30, 33], the Kahan’s stopping criteria was recommended for terminating iterations, which may lead premature termination without improvements in the approximate solution; please refer to [31] for details.

**Remark 2.3.** The dominant computational cost for Algorithm 2 involves \(U_j, V_j, Q_j, W_j\) and \(H_k\), which are obtained by solving M-matrix linear systems. More concretely, it follows from
\[
\begin{align*}
U_0 &= A_{\beta}^{-1} B_t, \quad V_0 = A_{\beta}^{-T} C_t, \quad W_0 = D_{\alpha}^{-1} C_t, \quad Q_0 = D_{\alpha}^{-T} B_t, \\
U_j &= A_{\alpha,\beta} U_{j-1} - A_{-}\alpha A_{\beta}^{-1} U_{j-1}, \quad V_j = A_{\alpha,\beta}^{T} V_{j-1} = A_{-\alpha}^{T} A_{\beta}^{-T} V_{j-1}, \\
W_j &= D_{\alpha,\beta} W_{j-1} - D_{-\beta} D_{\alpha}^{-1} W_{j-1}, \quad Q_j = D_{\alpha,\beta}^{T} Q_{j-1} = D_{-\beta}^{T} D_{\alpha}^{-T} Q_{j-1}
\end{align*}
\]

that it requires \(O(m^2(p + q) + n^2(p + q))\) flops as long as the LU factorizations of the M-matrices \(A_{\beta}\) and \(D_{\alpha}\) are known. For \(H_k\) we just need to solve the small M-matrix linear systems with the coefficient being \(I - Y_k Z_k \in \mathbb{R}^{2p \times 2p}\), which involves \(O(2^k p)\) arithmetic. Accordingly, in each iteration the computational complexity is \((m^2 + n^2)\) and the LU factorizations of \(A_{\beta}\) and \(D_{\alpha}\), obtained by performing Algorithm 1, dominate the whole computational cost. However by Algorithm 1, when the original \(A\) and \(D\) are sparse or rank one updates of some diagonal matrices, the complexities for calculating these LU factorizations can be reduced. It is worthwhile to point that in the accADDA proposed by Xue and Li [32], one cannot take advantage of special structures in \(A\) and \(D\), such as sparsity, because the doubling recursions (3) destroy these structures. Besides, in each iteration the accADDA requires solving two M-matrix linear systems whose sizes are \(m \times m\) and \(n \times n\), and thus the complexity is \(O(m^3 + n^3)\) per iteration.

**Remark 2.4.** One may also be interested in the dual solution \(Y\) to the dual MARE (2). In that case it is necessary to compute the triplet representation of the nonsingular M-matrix \(I - Z_k Y_k\) in (33). We can then compute \(G_k\) in high accuracy, adapting the GTH-like algorithm.

### 3 Numerical Examples

To illustrate the performance of the dADDA, we apply it to two test sets. One comes from stochastic fluid flows [3, 32] with 10 examples. The other originates from the transport theory, the one-group neutron transport equation [16]. For comparison we also apply the accADDA [32]...
to the test sets. In fairness, we do not utilize the special structures in A and D when computing the LU factorizations of $A_{\beta}$ and $D_{\alpha}$, which would benefit the dADDA greatly. Both algorithms are implemented in MATLAB 2017a on a 64-bit PC with an Intel Core i7 processor at 3.40 GHz and 16G RAM.

**Example 3.1** (Stochastic fluid flow). In this example, we have

$$A = nI_n, \quad D = (10^4n + m)I_n - 10^4I_{n \times n}, \quad B_l = 1_m, \quad B_r = 1_n, \quad C_l = 1_n, \quad C_r = 1_m,$$

which satisfies $W1_{n+m} = 0$, indicating $u1 = 1_n$, $u2 = 1_m$, $v1 = 0$ and $v2 = 0$. The minimal nonnegative solution is $X = \frac{1}{n}1_{m \times n}$. When taking $m = 2, n = 18$, it is exactly the example of a positive recurrent Markov chain displayed in [32, Example 6.1]. We set the tolerance for the entrywise relative residual $ER_{res_k}$ as $10^{-14}$ and the maximal number of iterations as 20.

Table 1 shows the numerical results produced by accADDA and dADDA. Besides $ER_{res_k}$ and $ER_{err_k}$, we also present the $\text{rank}(H_k)$, $\|H_k\|_F$, the numbers of iterations ($\#it$) required and also the respective execution times ($e\text{Time}$). From Table 1, in the same iterations, both algorithms produce comparable results on $ER_{res_k}$, $ER_{err_k}$, $\text{rank}(H_k)$ and $\|H_k\|_F$. However, the dADDA requires less execution time, especially for examples of medium and large sizes.

**Example 3.2** (Transport Theory). When using the Gauss-Legendre to discretize the integrodifferential equation satisfied by the scattering function [16], it leads to the MAREs with

$$A = \frac{1}{\beta(1 + \alpha)} \text{diag}(\omega_1^{-1}, \ldots, \omega_n^{-1}) - 1_nq^T, \quad D = \frac{1}{\beta(1 - \alpha)} \text{diag}(\omega_1^{-1}, \ldots, \omega_n^{-1}) - q1_n^T,$$

$$B_l = B_r = 1_n, \quad C_l = C_r = q,$$

where $\omega_1, \ldots, \omega_n$ are the Gauss-Legendre notes satisfying $0 < \omega_n < \omega_{n-1} < \cdots < \omega_1 < 1$, $q = \frac{1}{n} \text{diag}(\omega_1^{-1}, \ldots, \omega_n^{-1})c$, and $c$ is the weights vector with $\sum_{i=1}^n c_i = 1$ and $c_i > 0$.

In this test set, we will randomly generate $\alpha, \beta, \omega_i$ and $c_i$ as follows: $\alpha, \beta$ and $\omega_i$ follow the uniform distribution in the interval $(0, 1)$, and for $c$ we firstly obtain $c$ by the command $\text{randn}$ in MATLAB and then we normalize $c$ so that $\sum_{i=1}^n c_i = 1$. We have $W = \begin{bmatrix} D & -C \\ -B & A \end{bmatrix}$ being a rank one update of a nonsingular diagonal matrix. Its inverse can be computed cheaply with the help of the SMWF. For the triplet representation of $W$, we firstly generate $v_1 \geq 0$ and $v_2 \geq 0$ with the command $\text{rand}$, and then we compute $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = W^{-1} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, which obviously holds $u_1 > 0$ and $u_2 > 0$. Again, we take $ER_{res_k}$ as the stop criterion, with $\varepsilon$ setting as $10^{-13}$, and the maximal number of iterations as 100. Table 2 displays the numerical results produced by both algorithms for 10 examples. Within 100 iterations, the accADDA does not converge, while the dADDA produces satisfactory results within 11 iterations.

For those test examples we observe that different $\alpha$ and $\beta$ would make an impact on the number of iterations required by dADDA. Figures 1–6 illustrates this influence, where $\alpha$ and $\beta$ take 200 different values. In all six figures, the effects for different $\alpha$ are displayed on the left while that for $\beta$ are on the right. Besides the number of iterations, we also plot the numerical results for $ER_{res_k}$. It shows in all left figures that there are several “critical” points in $\alpha$, at which the number of iterations and $ER_{res_k}$ jump abruptly; and for all right figures it seems that one “critical” point exists for $\beta$, which grows as the size $n$ increases.
| $m = 18$, $n = 2$ | \(\text{ER}_{\text{Res}}\) | \(\text{ER}_{\text{Ref}}\) | \(\text{rank}(H_k)\) | \(\|H_k\|_F\) | \#it | eTime |
|---|---|---|---|---|---|---|
| accADDA | 7.7030 \times 10^{-16} | 5.8180 \times 10^{-12} | 1 | 3.333 \times 10^{-1} | 4 | 4.1238 \times 10^{-2} |
| dADDA | 7.7030 \times 10^{-16} | 5.8177 \times 10^{-12} | 1 | 3.333 \times 10^{-1} | 4 | 3.2198 \times 10^{-2} |
| $m = 50$, $n = 10$ | \(\text{ER}_{\text{Res}}\) | \(\text{ER}_{\text{Ref}}\) | \(\text{rank}(H_k)\) | \(\|H_k\|_F\) | \#it | eTime |
| accADDA | 1.1035 \times 10^{-15} | 5.8191 \times 10^{-12} | 1 | 3.333 \times 10^{-1} | 4 | 3.9930 \times 10^{-2} |
| dADDA | 1.1035 \times 10^{-15} | 5.8185 \times 10^{-12} | 1 | 3.333 \times 10^{-1} | 4 | 3.8817 \times 10^{-2} |
| $m = 100$, $n = 20$ | \(\text{ER}_{\text{Res}}\) | \(\text{ER}_{\text{Ref}}\) | \(\text{rank}(H_k)\) | \(\|H_k\|_F\) | \#it | eTime |
| accADDA | 9.1441 \times 10^{-16} | 5.8202 \times 10^{-12} | 1 | 3.333 \times 10^{-1} | 4 | 1.8318 \times 10^{-1} |
| dADDA | 4.5724 \times 10^{-15} | 5.8185 \times 10^{-12} | 1 | 3.333 \times 10^{-1} | 4 | 1.0650 \times 10^{-1} |
| $m = 200$, $n = 40$ | \(\text{ER}_{\text{Res}}\) | \(\text{ER}_{\text{Ref}}\) | \(\text{rank}(H_k)\) | \(\|H_k\|_F\) | \#it | eTime |
| accADDA | 3.8265 \times 10^{-15} | 5.8194 \times 10^{-12} | 1 | 3.333 \times 10^{-1} | 4 | 2.2336 |
| dADDA | 2.7332 \times 10^{-15} | 5.8183 \times 10^{-12} | 1 | 3.333 \times 10^{-1} | 4 | 4.5421 \times 10^{-1} |
| $m = 400$, $n = 80$ | \(\text{ER}_{\text{Res}}\) | \(\text{ER}_{\text{Ref}}\) | \(\text{rank}(H_k)\) | \(\|H_k\|_F\) | \#it | eTime |
| accADDA | 1.2746 \times 10^{-15} | 5.8157 \times 10^{-12} | 1 | 3.333 \times 10^{-1} | 4 | 1.1577 \times 10^{1} |
| dADDA | 2.7312 \times 10^{-15} | 5.8178 \times 10^{-12} | 1 | 3.333 \times 10^{-1} | 4 | 1.6188 |
| $m = 800$, $n = 160$ | \(\text{ER}_{\text{Res}}\) | \(\text{ER}_{\text{Ref}}\) | \(\text{rank}(H_k)\) | \(\|H_k\|_F\) | \#it | eTime |
| accADDA | 2.1838 \times 10^{-15} | 5.8162 \times 10^{-12} | 1 | 3.333 \times 10^{-1} | 4 | 1.3030 \times 10^{1} |
| dADDA | 3.4576 \times 10^{-15} | 5.8207 \times 10^{-12} | 1 | 3.333 \times 10^{-1} | 4 | 1.5391 \times 10^{1} |
| $m = 1600$, $n = 320$ | \(\text{ER}_{\text{Res}}\) | \(\text{ER}_{\text{Ref}}\) | \(\text{rank}(H_k)\) | \(\|H_k\|_F\) | \#it | eTime |
| accADDA | 5.8217 \times 10^{-16} | 5.8038 \times 10^{-12} | 1 | 3.333 \times 10^{-1} | 4 | 1.7404 \times 10^{3} |
| dADDA | 3.0928 \times 10^{-15} | 5.8157 \times 10^{-12} | 1 | 3.333 \times 10^{-1} | 4 | 1.6855 \times 10^{2} |
| $m = 3200$, $n = 640$ | \(\text{ER}_{\text{Res}}\) | \(\text{ER}_{\text{Ref}}\) | \(\text{rank}(H_k)\) | \(\|H_k\|_F\) | \#it | eTime |
| accADDA | 4.7295 \times 10^{-15} | 5.8430 \times 10^{-12} | 1 | 3.333 \times 10^{-1} | 4 | 1.4791 \times 10^{4} |
| dADDA | 3.6381 \times 10^{-15} | 5.8170 \times 10^{-12} | 1 | 3.333 \times 10^{-1} | 4 | 1.4503 \times 10^{3} |
| $m = 6400$, $n = 1280$ | \(\text{ER}_{\text{Res}}\) | \(\text{ER}_{\text{Ref}}\) | \(\text{rank}(H_k)\) | \(\|H_k\|_F\) | \#it | eTime |
| accADDA | 3.0922 \times 10^{-15} | 5.7604 \times 10^{-12} | 1 | 3.333 \times 10^{-1} | 4 | 5.0310 \times 10^{3} |
| dADDA | 8.1853 \times 10^{-15} | 5.8275 \times 10^{-12} | 1 | 3.333 \times 10^{-1} | 4 | 4.8325 \times 10^{3} |
| $m = 12800$, $n = 2560$ | \(\text{ER}_{\text{Res}}\) | \(\text{ER}_{\text{Ref}}\) | \(\text{rank}(H_k)\) | \(\|H_k\|_F\) | \#it | eTime |
| accADDA | 6.3962 \times 10^{-15} | 5.8536 \times 10^{-12} | 1 | 3.333 \times 10^{-1} | 4 | 9.3531 \times 10^{4} |
| dADDA | 9.8221 \times 10^{-15} | 5.8293 \times 10^{-12} | 1 | 3.333 \times 10^{-1} | 4 | 9.4298 \times 10^{3} |

Table 1: Numerical results for Example 3.1

4 Conclusions

The highly accurate alternating-directional doubling algorithm (accADDA) proposed by Xue and Li [32] is the most efficient method for computing the minimal nonnegative solution X of MAREs of small sizes, which keeps the accuracy of all entries in X, especially the tiny ones. Illumined by the accDDDA, we propose a highly accurate algorithm for solving large-scale MAREs with low-rank structures. Firstly we show that the iteration recursions given in [32] can be decoupled, enable the accADDA to solve large-scale MAREs. We prove the kernels in the decoupled form are M-matrices, and construct the novel triplet representations for these kernels, which is not a simple straightforward adaptation of that given in [32]. With these triplet representations, we develop the dADDA for large-scale MAREs with low-rank structures. Associated linear equations are solved in a cancellation-free manner, with the help of the GTH-like algorithm.

Unlike the accADDA, our dADDA just computes one iteration recursion for the solution X,
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making it possible to utilize the special structures that may exist in the original $A$ and $D$. More importantly it only requires $O(n^2 + m^2)$ operations in each iteration. Our proposed mADDA is also efficient for medium size MAREs with low-rank structures and numerical results illustrates the efficiency of the mADDA.

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| $n$ | ERres | rank($H_k$) | $\|H_k\|_F$ | #it | $\epsilon$Time |
|-----|-------|-------------|----------------|-----|---------------|
| 10  | accADDA | $2.9147 \times 10^{-6}$ | 7 | $3.1800 \times 10^{-2}$ | 100 | $1.9098 \times 10^{-1}$ |
|     | dADDA  | $1.3189 \times 10^{-15}$ | 6 | $3.1800 \times 10^{-2}$ | 4  | $5.0765 \times 10^{-2}$ |
| 20  | accADDA | $2.6358 \times 10^{-6}$ | 14 | $6.8249$ | 100 | $2.4467 \times 10^{-1}$ |
|     | dADDA  | $2.4856 \times 10^{-15}$ | 12 | $7.0425$ | 7  | $1.8044 \times 10^{-1}$ |
| 40  | accADDA | $9.1380 \times 10^{-7}$ | 14 | $5.9640$ | 100 | $2.9781 \times 10^{-1}$ |
|     | dADDA  | $1.9856 \times 10^{-15}$ | 11 | $5.7022$ | 6  | $8.9914 \times 10^{-2}$ |
| 80  | accADDA | $6.4166 \times 10^{-7}$ | 22 | $2.1559$ | 100 | $4.0659 \times 10^{-1}$ |
|     | dADDA  | $3.9555 \times 10^{-14}$ | 15 | $2.2523$ | 11 | $4.9627 \times 10^{1}$ |
| 150 | accADDA | $5.6262 \times 10^{-2}$ | 19 | $3.8084$ | 100 | $7.9389 \times 10^{-1}$ |
|     | dADDA  | $1.0065 \times 10^{-14}$ | 14 | $3.9519$ | 9  | $3.9953$ |
| 400 | accADDA | $8.0543 \times 10^{-1}$ | 9  | $8.0832 \times 10^{-1}$ | 100 | $1.4971$ |
|     | dADDA  | $4.3655 \times 10^{-14}$ | 8  | $8.0833 \times 10^{-1}$ | 5  | $1.0938 \times 10^{-1}$ |
| 600 | accADDA | $6.7672 \times 10^{-2}$ | 24 | $9.3653$ | 100 | $8.5334$ |
|     | dADDA  | $8.7930 \times 10^{-14}$ | 18 | $9.8271$ | 10 | $6.7998 \times 10^{1}$ |
| 800 | accADDA | $2.6961 \times 10^{-2}$ | 14 | $1.5130$ | 100 | $5.0015 \times 10^{4}$ |
|     | dADDA  | $7.5368 \times 10^{-14}$ | 10 | $1.5155$ | 10 | $2.2816 \times 10^{2}$ |
| 1000| accADDA | $7.4683 \times 10^{-4}$ | 14 | $9.9455$ | 100 | $1.6273 \times 10^{4}$ |
|     | dADDA  | $3.3671 \times 10^{-14}$ | 12 | $9.9643$ | 7  | $1.2134 \times 10^{4}$ |
| 1500| accADDA | $1.2776 \times 10^{-2}$ | 15 | $1.0030 \times 10^{2}$ | 100 | $4.0977 \times 10^{4}$ |
|     | dADDA  | $5.4891 \times 10^{-14}$ | 11 | $1.0049 \times 10^{2}$ | 9  | $2.2755 \times 10^{2}$ |

Table 2: Numerical results for Example 3.2
Figure 1: Iterations and ERres for $n = 10$

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Figure 5: Iterations and ERres for $n = 75$

(a) $\beta = 0.91508$

(b) $\alpha = 0.78181$

Figure 6: Iterations and ERres for $n = 100$

(a) $\beta = 0.80531$

(b) $\alpha = 0.79772$