Quantitative estimates in Beurling–Helson type theorems

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Abstract. We consider the spaces $A_p(T)$ of functions $f$ on the circle $T$ such that the sequence of Fourier coefficients $\hat{f} = \{\hat{f}(k), k \in \mathbb{Z}\}$ belongs to $l^p$, $1 \leq p < 2$. The norm on $A_p(T)$ is defined by $\|f\|_{A_p} = \|\hat{f}\|_{l^p}$. We study the rate of growth of the norms $\|e^{i\lambda \varphi}\|_{A_p}$ as $|\lambda| \to \infty$, $\lambda \in \mathbb{R}$, for $C^1$-smooth real functions $\varphi$ on $T$. The results have natural applications to the problem on changes of variable in the spaces $A_p(T)$.

References: 17 items.

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Introduction

We consider Fourier series

$$f(t) \sim \sum_{k \in \mathbb{Z}} \hat{f}(k)e^{ikt}$$

of (integrable) functions $f$ on the circle $T = \mathbb{R}/2\pi\mathbb{Z}$ ($\mathbb{R}$ is the real line, $\mathbb{Z}$ is the set of integers).

Let $A_1(T)$ be the space of continuous functions $f$ on $T$ such that the sequence of Fourier coefficients $\hat{f} = \{\hat{f}(k), k \in \mathbb{Z}\}$ belongs to $l^1$. Let $A_p(T)$, $1 < p \leq 2$, be the space of integrable functions $f$ on $T$ such that $\hat{f}$ belongs to $l^p$. Endowed with the natural norms

$$\|f\|_{A_p(T)} = \|\hat{f}\|_{l_p} = \left(\sum_{k \in \mathbb{Z}} |\hat{f}(k)|^p\right)^{1/p},$$

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the spaces $A_p(\mathbb{T})$, $1 \leq p \leq 2$, are Banach spaces. The space $A(\mathbb{T}) = A_1(\mathbb{T})$ is a Banach algebra (with the usual multiplication of functions).

Suppose that we have a map of the circle into itself, i.e. a function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(t+2\pi) = \varphi(t)$ (mod 2\pi). According to the Beurling–Helson theorem [1] (see also [2], [3]), if $\|e^{in\varphi}\|_{A(\mathbb{T})} = O(1), \ n \in \mathbb{Z}$, then the function $\varphi$ is mod 2\pi-linear (affine) with integer tangent coefficient: $\varphi(t) = mt + t_0$(mod 2\pi), $m \in \mathbb{Z}$. This theorem yields the solution of the Levy problem on the description of endomorphisms of the algebra $A(\mathbb{T})$; all these endomorphisms prove to be trivial, i.e., have the form $f(t) \to f(mt + t_0)$. In other words only trivial changes of variable are admissible in $A(\mathbb{T})$. We note also another version of the Beurling–Helson theorem: if $U$ is a bounded translation-invariant operator from $l^1$ to itself such that $\|U^n\|_{l^1} = O(1), \ n \in \mathbb{Z}$, then $U = \xi S$, where $\xi$ is a constant, $|\xi| = 1$, and $S$ is a translation.

At the same time, although the Beurling–Helson theorem states unbounded growth of the norms $\|e^{in\varphi}\|_A$ for nonlinear maps $\varphi : \mathbb{T} \to \mathbb{T}$, the rate of growth of these norms as $|n| \to \infty$ is in general not clear. The same concerns the behavior of the norms $\|e^{in\varphi}\|_{A_p}$, $p > 1$.

Let $C^\nu(\mathbb{T})$ be the class of (complex-valued) functions on $\mathbb{T}$ with continuous derivative of order $\nu$. We have $C^1(\mathbb{T}) \subseteq A(\mathbb{T}) \subseteq A_p(\mathbb{T})$.

In this paper we mainly consider real $C^1$-smooth functions $\varphi$ on $\mathbb{T}$ and study the growth of the norms $\|e^{i\lambda\varphi}\|_{A_p}$ as $|\lambda| \to \infty$, $\lambda \in \mathbb{R}$. The only interesting case is certainly that of non-constant functions $\varphi$. The corresponding results on the behavior of the exponential functions $e^{in\varphi}$ for nonlinear maps $\varphi : \mathbb{T} \to \mathbb{T}$ and integer frequencies $n$ will be obtained as simple corollaries.

It is not difficult to show that if $\varphi \in C^1(\mathbb{T})$ is a real function (and moreover if $\varphi$ is an absolutely continuous real function with the derivative in $L^2(\mathbb{T})$) then for $1 \leq p < 2$ we have

$$\|e^{i\lambda\varphi}\|_{A_p(\mathbb{T})} = O(|\lambda|^{\frac{1}{p}-\frac{1}{2}}), \quad |\lambda| \to \infty, \quad \lambda \in \mathbb{R}, \quad (1)$$

(see [2, Ch. VI, § 3] in the case when $p = 1$; for $1 < p < 2$ the estimate follows immediately by interpolation between $l^1$ and $l^2$).

On the other hand, lower estimates for the norms of $e^{i\lambda\varphi}$ in $A_p$ for functions $\varphi$ of class $C^2$ have long been known. Suppose that $\varphi \in C^2(\mathbb{T})$ is a real non-constant function and $1 \leq p < 2$. Then

$$\|e^{i\lambda\varphi}\|_{A_p(\mathbb{T})} \geq c \ |\lambda|^{-\frac{1}{p}+\frac{1}{2}}, \quad \lambda \in \mathbb{R}, \quad (2)$$
where $c = c(p, \varphi) > 0$ is independent of $\lambda$. For $p = 1$ this estimate is contained implicitly in Leibenson’s paper [4] and was obtained in explicit form by Kahane [5] who used Leibenson’s approach. In the general case estimate (2) was obtained (by the same method) by Alpar [6]. A short and simple proof for $p = 1$ can be found in [2, Ch. VI, § 3] and in the general case in [7].

Thus, if $\varphi \in C^2(\mathbb{T})$ is a real function, $\varphi \neq \text{const}$, then

$$\|e^{i\lambda \varphi}\|_{A_p(\mathbb{T})} \simeq |\lambda|^{\frac{1}{p} - \frac{1}{2}}$$

for all $p$, $1 \leq p < 2$. In particular $\|e^{i\lambda \varphi}\|_A \simeq \sqrt{|\lambda|}$. (The sign $\simeq$ means that for all sufficiently large $|\lambda|$ the ratio of the corresponding quantities is contained between two positive constants).

We note that the proof of the Leibenson–Kahane–Alpar estimate (2) is based on the second part of the van der Corput lemma (see [8, Ch. V, Lemma 4.3]) and essentially uses the condition that the curvature of some arc of the graph of $\varphi$ is bounded away from zero, i.e., the condition $|\varphi''(t)| \geq \rho > 0$, $t \in I$, where $I$ is a certain interval.

In general, the norms $\|e^{i\lambda \varphi}\|_A$ can grow rather slowly. Kahane showed (see [2, Ch. VI, § 2]) that if $\varphi \neq \text{const}$ is a real continuous piecewise linear function on $\mathbb{T}$ (which means that $[0, 2\pi]$ is a finite union of some intervals such that $\varphi$ is linear on each of them) then $\|e^{i\lambda \varphi}\|_A \simeq \log |\lambda|$. It is not clear if there are non-trivial functions $\varphi$ that yield the growth of the norms of the exponential functions in $A(\mathbb{T})$ slower than logarithmic. Kahane conjectured [2], [3] that the Beurling–Helson theorem can be strengthened considerably. In particular, he posed the following question: is it true that if $\|e^{i\lambda \varphi}\|_A = o(\log |\lambda|)$, then $\varphi = \text{const}$. We do not know the answer even under the assumption that $\varphi \in C^1(\mathbb{T})$.

We also note that for any piecewise linear real function $\varphi$ on $\mathbb{T}$ we have $\|e^{i\lambda \varphi}\|_{A_p} = O(1)$ for all $p > 1$ (see [7]).

We recall the results known for the $C^1$-smooth case (besides estimate (1)).

In [7] (a joint work of the author and Olevskii) we constructed a real function $\varphi \in C^1(\mathbb{T})$, $\varphi \neq \text{const}$, such that $\|e^{i\lambda \varphi}\|_{A_p} = O(1)$ for all $p > 1$. In addition this function is nowhere linear, i.e., it is not linear on any interval (and thus, in a sense, it is essentially different from piecewise linear functions).

In [9] the author of the present paper showed that for functions $\varphi \in C^1$ the norms $\|e^{i\lambda \varphi}\|_A$ can grow rather slowly, namely: if $\gamma(\lambda) \geq 0$ and $\gamma(\lambda) \to$
\(+\infty\) as \(\lambda \rightarrow +\infty\), then there exists a nowhere linear real function \(\varphi \in C^1(\mathbb{T})\) such that
\[
\|e^{i\lambda \varphi}\|_A = O(\gamma(|\lambda|) \log |\lambda|).
\]

Thus, the case of \(C^1\)-smooth phase \(\varphi\) is essentially different from the \(C^2\)-smooth case.

As far as we know, the only lower estimate for the norms \(\|e^{i\lambda \varphi}\|_A\) obtained previously in the case when \(\varphi \in C^1\) but the twice differentiability of \(\varphi\) is not assumed is due to Leblanc [10]: if a real function \(\varphi \in C^1(\mathbb{T})\) is non-constant and its derivative \(\varphi'\) satisfies the Lipschitz condition of order \(\alpha, \ 0 < \alpha \leq 1\), then
\[
\|e^{i\lambda \varphi}\|_A \geq c \frac{|\lambda|^{\frac{\alpha}{1+\alpha}}}{(\log |\lambda|)^{\frac{1}{2}}}, \quad |\lambda| \geq 2.
\]

Now we shall briefly describe the results of the present work. In § 1 we prove Theorem 1 in which we give lower estimates for \(\|e^{i\lambda \varphi}\|_A\) for \(C^1\)-smooth real functions \(\varphi\) on \(\mathbb{T}\). In § 2 we obtain Theorem 2, which shows that the estimates of Theorem 1 are close to being sharp and in certain cases are sharp. Here, in the introduction, we omit the statements of the theorems and only note their corollaries.

From Theorem 1 it follows that if a real function \(\varphi \in C^1(\mathbb{T})\) is non-constant and \(\varphi'\) satisfies the Lipschitz condition of order \(\alpha, \ 0 < \alpha \leq 1\), then
\[
\|e^{i\lambda \varphi}\|_A \geq c_p |\lambda|^{\frac{1}{p+1+\alpha}}, \quad \lambda \in \mathbb{R},
\]
for all \(p, \ 1 \leq p < 1 + \alpha\) (see Corollary 1). In particular, putting \(p = 1\) here, we obtain the result, which is stronger than Leblanc’s estimate (3), namely,
\[
\|e^{i\lambda \varphi}\|_A \geq c |\lambda|^{\frac{1}{1+\alpha}}, \quad \lambda \in \mathbb{R}.
\]

For \(\varphi \in C^2\) we have \(\alpha = 1\) and from (4) we get the Leibenson–Kahane–Alpar estimate (2).

We note that the Leibenson–Kahane–Alpar estimate and the Leblanc estimate have local character; roughly speaking, they remain valid if we assume that \(\varphi\) is nonlinear on some interval and has an appropriate smoothness on this interval. Our lower estimates are of local character as well (see Theorem 1’).

A particular case of Theorem 2 is the following statement: for each \(\alpha, \ 0 < \alpha < 1\), there exists a nowhere linear real function \(\varphi \in C^1(\mathbb{T})\) such that its derivative \(\varphi'\) satisfies the Lipschitz condition of order \(\alpha\) and we have
\[
\|e^{i\lambda \varphi}\|_A = O\left(|\lambda|^{\frac{1}{1+\alpha}} (\log |\lambda|)^{\frac{1}{1+\alpha}}\right), \quad |\lambda| \rightarrow \infty
\]
(see Corollary 2). For the same function \( \varphi \) we have

\[
\| e^{i \lambda \varphi} \|_{A_p} \simeq |\lambda|^{\frac{1}{p} - \frac{1}{1 + \alpha}} \quad \text{if} \quad 1 < p < 1 + \alpha
\]

and

\[
\| e^{i \lambda \varphi} \|_{A_p} \simeq 1 \quad \text{if} \quad 1 + \alpha < p < 2.
\]

In addition,

\[
\| e^{i \lambda \varphi} \|_{A_p} = O((\log |\lambda|)^{1/p}) \quad \text{for} \quad p = 1 + \alpha.
\]

From estimate (4) and the obvious estimate

\[
\| e^{i \lambda \varphi} \|_{A_p(\mathbb{T})} \geq \| e^{i \lambda \varphi} \|_{A_2(\mathbb{T})} = \| e^{i \lambda \varphi} \|_{L^2(\mathbb{T})} = 1, \quad 1 \leq p \leq 2,
\]

it follows that the function \( \varphi \) that we have constructed yields the slowest possible growth of the norms \( \| e^{i \lambda \varphi} \|_{A_p} \) for \( 1 < p < 2, \ p \neq 1 + \alpha \).

As we mentioned above, there exists a non-trivial \( C^1 \)-smooth function \( \varphi \) with extremely slow (arbitrarily close to logarithmic) growth of the norms \( \| e^{i \lambda \varphi} \|_A \). This result, which the author proved earlier, follows immediately from Theorem 2 (see Corollary 3).

In § 3 we consider \( C^1 \)-smooth maps of the circle into itself and give the corresponding versions of the results obtained in §§1 and 2. These versions have natural applications in the study of change of variable operators (superposition operators) \( f \to f \circ \varphi \) on the spaces \( A_p \). In particular, we show how smooth a nonlinear map \( \varphi : \mathbb{T} \to \mathbb{T} \) can be provided that \( f \circ \varphi \in \bigcap_{p>1} A_p \) whenever \( f \in A \).

By \( \omega(I, g, \delta) \) we denote the modulus of continuity of a function \( g \) on an interval \( I \subset \mathbb{R} \):

\[
\omega(I, g, \delta) = \sup_{\substack{|t_1-t_2| \leq \delta \\mid t_1, t_2 \in I}} |g(t_1) - g(t_2)|, \quad \delta \geq 0.
\]

If \( I = \mathbb{R} \), then we just write \( \omega(g, \delta) \). The class \( \text{Lip}_{\omega}(I) \) consists of all functions \( g \) on \( I \) with \( \omega(I, g, \delta) = O(\omega(\delta)), \ \delta \to +0 \), where \( \omega(\delta) \) is a given continuous non-decreasing function on \( [0, +\infty) \), \( \omega(0) = 0 \). The class \( C^{1, \omega}(I) \) consists of functions \( g \) on \( I \) with the derivative \( g' \in \text{Lip}_{\omega}(I) \). The class \( C^{1, \omega}(\mathbb{T}) \) consists of \( 2\pi \)-periodic functions that belong to \( C^{1, \omega}(\mathbb{R}) \). For \( 0 < \alpha \leq 1 \) we write \( C^{1, \alpha} \) instead of \( C^{1, \delta^{\alpha}} \). The class \( C^{1}(I) \) consists of continuously differentiable functions on \( I \). For a set \( E \subset \mathbb{T} \) we denote its characteristic function by \( 1_E \):
1_E(t) = 1 for t ∈ E, 1_E(t) = 0 for t ∈ T \ E and (if E is measurable) by |E| its (Lebesgue) measure. If f and E are a function on T and a set in T, respectively, we write supp f ⊆ E if f(t) = 0 for almost all t ∈ T \ E. We use the same notation for sets in [0, 2π] or in R and for functions defined on [0, 2π] or on R. In the standard way we identify integrable functions on [0, 2π] with integrable functions on T. We use c, c_1, c_p, c_p, c(p), etc. to denote various positive constants which may depend only on p and ϕ.

§ 1. Lower estimates

**Theorem 1.** Let 1 ≤ p < 2. Let φ be a real function on T. Suppose that φ is non-constant and φ ∈ C^{1,ω}(T). Then

\[ \| e^{i\lambda \varphi} \|_{A_p(T)} \geq c |\lambda|^{1/p} \chi^{-1}(1/|\lambda|), \quad \lambda \in \mathbb{R}, \quad |\lambda| \geq 1, \]

where χ^{-1} is the function inverse to χ(δ) = δω(δ) and c = c(p, ϕ) > 0 is independent of λ.

Certainly in view of the obvious estimate (5), the estimate in Theorem 1 makes sense only in the case when its right-hand side is growing unboundedly together with λ. (This is always the case when p = 1).

Theorem 1 immediately implies the following corollary.

**Corollary 1.** Let 0 < α ≤ 1. If φ is a real non-constant function on T and φ ∈ C^{1,α}(T) then for all p, 1 ≤ p < 1 + α, we have

\[ \| e^{i\lambda \varphi} \|_{A_p(T)} \geq c_p |\lambda|^{\frac{1}{p} - \frac{1}{1 + \alpha}}, \quad \lambda \in \mathbb{R}. \]

In particular, \( \| e^{i\lambda \varphi} \|_{A_p(T)} \geq c |\lambda|^{\frac{1}{p}} \).

As we mentioned in Introduction, a particular case of Corollary 1 is the Leibenson–Kahane–Alpar estimate. For p = 1 the estimate of Corollary 1 improves the result of Leblanc.

Theorem 1' below is a local version of Theorem 1.

Let I ⊂ R be an interval of length less then 2π. We say that a function f defined on I belongs to A_p(T, I) if there is a function F in A_p(T) such that its restriction F|_I to I coincides with f. We put

\[ \| f \|_{A_p(T, I)} = \inf_{F \in A_p(T), \, F|_I = f} \| F \|_{A_p(T)}. \]
It is clear that $A_p(\mathbb{T}, I)$ is a Banach space, $1 \leq p \leq 2$.

**Theorem 1′.** Let $1 \leq p < 2$. Let $\varphi$ be a real function on an interval $I \subset \mathbb{R}$, $|I| < 2\pi$. Suppose that $\varphi$ is nonlinear on $I$ and $\varphi \in C^{1, \omega}(I)$. Then

$$
\|e^{i\lambda \varphi}\|_{A_p(\mathbb{T}, I)} \geq c |\lambda|^{1/p} \chi^{-1}(1/|\lambda|), \quad \lambda \in \mathbb{R}, \quad |\lambda| \geq 1.
$$

The local version of Corollary 1 is obvious.

**Proof of Theorem 1.** Let

$$
m = \min_{t \in [0, 2\pi]} \varphi'(t), \quad M = \max_{t \in [0, 2\pi]} \varphi'(t).
$$

Since $\varphi$ is non-constant whereas every continuous $2\pi$-periodic function linear on $[0, 2\pi]$ is constant, we have $m < M$.

Fix $c > 0$ so that

$$
\omega(\varphi', \delta) \leq c\omega(\delta), \quad \delta \geq 0.
$$

Let $\lambda \in \mathbb{R}$. Without loss of generality we can assume that $\lambda > 0$ (complex conjugation does not affect the norm of a function in $A_p$). Define $\delta_\lambda > 0$ by

$$
\chi(2\delta_\lambda) = \frac{1}{2c\lambda}.
$$

For $0 < \varepsilon \leq \pi$ let $\Delta_\varepsilon$ be the “triangle” function supported on the interval $(-\varepsilon, \varepsilon)$, i.e. the function on $\mathbb{T}$, defined as follows:

$$
\Delta_\varepsilon(t) = \max \left(1 - \frac{|t|}{\varepsilon}, 0\right), \quad t \in [-\pi, \pi].
$$

For an arbitrary interval $J \subseteq [0, 2\pi]$ let $\Delta_J$ be the triangle function supported on $J$, namely: $\Delta_J(t) = \Delta_{|J|/2}(t - c_J)$, where $c_J$ is the center of $J$ (and $|J|$ is its length).

**Lemma 1.** Suppose that $\lambda > 0$ is sufficiently large. Then for every $k \in \mathbb{Z}$ satisfying $m\lambda < k < M\lambda$ there exists an interval $I_{\lambda,k} \subseteq [0, 2\pi]$ such that $|I_{\lambda,k}| = 2\delta_\lambda$ and

$$
|(\Delta_{I_{\lambda,k}} e^{i\lambda \varphi})^\wedge(k)| \geq \frac{\delta_\lambda}{4\pi}.
$$

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Proof. We assume \( \lambda \) to be so large, that
\[
2\delta_\lambda < 2\pi \tag{7}
\]
and
\[
\lambda > \frac{2}{M - m}. \tag{8}
\]

Take an arbitrary \( k \in \mathbb{Z} \) with \( m\lambda < k < M\lambda \) (see (8)). We can find a point \( t_{\lambda,k} \) in \((0, 2\pi)\) such that \( \varphi'(t_{\lambda,k}) = \frac{k}{\lambda} \). There exists an interval \( I_{\lambda,k} \subseteq [0, 2\pi] \) that contains the point \( t_{\lambda,k} \) and has length \( 2\delta_\lambda \) (see (7)). Consider the following linear function:
\[
\varphi_{\lambda,k}(t) = \varphi(t_{\lambda,k}) + \frac{k}{\lambda}(t - t_{\lambda,k}), \quad t \in [0, 2\pi].
\]

If \( t \in I_{\lambda,k} \) then for some point \( \theta \) that lies between \( t \) and \( t_{\lambda,k} \) we have
\[
\varphi(t) - \varphi(t_{\lambda,k}) = \varphi'((\theta)(t - t_{\lambda,k}).
\]

So,
\[
|\varphi(t) - \varphi_{\lambda,k}(t)| = |(\varphi(t) - \varphi(t_{\lambda,k})) - \frac{k}{\lambda}(t - t_{\lambda,k})|
= |\varphi'(\theta)(t - t_{\lambda,k}) - \varphi'(t_{\lambda,k})(t - t_{\lambda,k})|
= |t - t_{\lambda,k}||\varphi'(\theta) - \varphi'(t_{\lambda,k})| \leq 2\delta_\lambda \omega'(\varphi', 2\delta_\lambda) \leq 2\delta_\lambda c\omega(2\delta_\lambda) = c\chi(2\delta_\lambda).
\]

Hence, taking into account (6), we have
\[
|e^{i\lambda \varphi(t)} - e^{i\lambda \varphi_{\lambda,k}(t)}| \leq |\lambda \varphi(t) - \lambda \varphi_{\lambda,k}(t)| \leq \lambda c\chi(2\delta_\lambda) = \frac{1}{2}, \quad t \in I_{\lambda,k}. \tag{9}
\]

Using (9), we obtain
\[
|\left(\Delta_{t_{I_{\lambda,k}}} e^{i\lambda \varphi}\right)(k) - \left(\Delta_{t_{I_{\lambda,k}}} e^{i\lambda \varphi_{\lambda,k}}\right)(k)| \leq \frac{1}{2\pi} \int_0^{2\pi} \Delta_{t_{I_{\lambda,k}}}(t)|e^{i\lambda \varphi(t)} - e^{i\lambda \varphi_{\lambda,k}(t)}| \, dt
\leq \frac{1}{2} \cdot \frac{1}{2\pi} \int_0^{2\pi} \Delta_{t_{I_{\lambda,k}}}(t) \, dt = \frac{1}{2} \Delta_{t_{I_{\lambda,k}}}(0).
\]
At the same time,

\[ |(I_{\lambda,k} e^{i\lambda \phi})^\wedge(k)| = \left| \frac{1}{2\pi} \int_0^{2\pi} I_{\lambda,k}(t) e^{i(\lambda \phi - k t)} \, dt \right| = \frac{1}{2\pi} \int_0^{2\pi} I_{\lambda,k}(t) \, dt = \hat{I}_{\lambda,k}(0), \]

and thus

\[ |(I_{\lambda,k} e^{i\lambda \phi})^\wedge(k)| \geq \frac{1}{2} \hat{I}_{\lambda,k}(0) = \frac{\delta_{\lambda}}{4\pi}. \]

The lemma is proved.

For each \( \lambda \in \mathbb{R} \) define a function \( g_{\lambda} \) by its Fourier expansion:

\[ g_{\lambda}(t) \sim \sum_{k \in \mathbb{Z}} |\hat{e}^{i\lambda \phi}(k)| e^{ikt}. \]

Obviously, \( g_{\lambda} \in L^2(\mathbb{T}) \).

It is well known that \( \Delta_{\varepsilon} \in A(\mathbb{T}) \) and the Fourier coefficients of \( \Delta_{\varepsilon} \) are non-negative. Since for an arbitrary interval \( J \) the function \( \Delta_J \) is obtained from \( \Delta_{|J|/2} \) by translation, we have \( |\hat{\Delta}_{J}(k)| = \hat{\Delta}_{|J|/2}(k), \ k \in \mathbb{Z} \). Thus, from Lemma 1 it follows that if \( \lambda \) is sufficiently large, then

\[ \frac{\delta_{\lambda}}{4\pi} \leq |(I_{\lambda,k} e^{i\lambda \phi})^\wedge(k)| \]

\[ = \left| \sum_{\nu \in \mathbb{Z}} \hat{\Delta}_{I_{\lambda,k}}(\nu) e^{ik \phi}(k - \nu) \right| \leq \sum_{\nu \in \mathbb{Z}} |\hat{\Delta}_{I_{\lambda,k}}(\nu)| |e^{ik \phi}(k - \nu)| \]

\[ = \sum_{\nu \in \mathbb{Z}} \Delta_{\delta_{\lambda}}(\nu) g_{\lambda}(k - \nu) = (\Delta_{\delta_{\lambda}} \cdot g_{\lambda})^\wedge(k), \ m\lambda < k < M\lambda. \quad (10) \]

Since \( \|\Delta_{\varepsilon}\|_A = \sum_k \hat{\Delta}_{\varepsilon}(k) = \Delta_{\varepsilon}(0) = 1 \), it follows that for every function \( f \in A_p(\mathbb{T}) \) we have \( \|\Delta_{\varepsilon} \cdot f\|_{A_p} \leq \|f\|_{A_p}, 1 \leq p \leq 2, 0 < \varepsilon \leq \pi \). So, using (10), we see that for all sufficiently large \( \lambda \)

\[ \left( \frac{1}{2}(M - m)\lambda \right)^{1/p} \frac{\delta_{\lambda}}{4\pi} \leq \left( \sum_{m\lambda < k < M\lambda} \left( \frac{\delta_{\lambda}}{4\pi} \right)^p \right)^{1/p}. \]

\[ ^2 \text{Direct calculation yields } \hat{\Delta}_{\varepsilon}(k) = \frac{2 \sin^2(\varepsilon k/2)}{\varepsilon k^2}, k \neq 0, \ \hat{\Delta}_{\varepsilon}(0) = \frac{\varepsilon}{2\pi}. \]
\[
\left( \sum_{m\lambda < k < M\lambda} |(\Delta_{\delta_{\lambda}} \cdot g_{\lambda})^{(k)}|^p \right)^{1/p} \leq \|\Delta_{\delta_{\lambda}} \cdot g_{\lambda}\|_{A_p} \leq \|g_{\lambda}\|_{A_p} = \|e^{i\lambda\varphi}\|_{A_p}.
\]

Note now that condition (6) yields
\[
1 = 2c\lambda 2\delta_{\lambda} \omega(2\delta_{\lambda}) \leq 4c\lambda \delta_{\lambda} \omega(4\delta_{\lambda}) \leq \lambda(c+1)4\delta_{\lambda} \omega((c+1)4\delta_{\lambda}) = \lambda \chi((c+1)4\delta_{\lambda}),
\]
and therefore,
\[
\delta_{\lambda} \geq \frac{1}{4(c+1)} \chi^{-1}(1/\lambda).
\]
Substituting this estimate in (11) we obtain the statement of the theorem.

**Proof of Theorem 1'**. We can obtain it by an obvious modification of the proof of Theorem 1. Namely, assuming that \( I \) is a closed subinterval of \([0, 2\pi]\) (this does not restrict generality), we put \( m = \min_{t \in I} \varphi'(t) \) and \( M = \max_{t \in I} \varphi'(t) \). Since \( \varphi \) is nonlinear on \( I \), we have \( m < M \). Fix \( p, 1 \leq p < 2 \).

For every \( \lambda \) fix a function \( F_{\lambda} \in A_p(T) \) which is a \( 2\pi \)-periodic extension of \( e^{i\lambda\varphi} \) from \( I \) to \( \mathbb{R} \) such that
\[
\|F_{\lambda}\|_{A_p(T)} \leq 2\|e^{i\lambda\varphi}\|_{A_p(T,I)}.
\]
In the proof of Theorem 1 one should replace \( \omega(\varphi', 2\delta_{\lambda}) \) by \( \omega(I, \varphi', 2\delta_{\lambda}) \) and, assuming that \( \lambda \) is sufficiently large, one should replace relation (7) by \( 2\delta_{\lambda} < |I| \). For \( m\lambda < k < M\lambda \) we choose a point \( t_{\lambda,k} \) with \( \varphi'(t_{\lambda,k}) = k/\lambda \) so that it lies in the interior of \( I \). We choose an interval \( I_{\lambda,k} \) of length \( 2\delta_{\lambda} \), which contains \( t_{\lambda,k} \), so that it lies in \( I \). Instead of \( e^{i\lambda\varphi} \) one should consider \( F_{\lambda} \).

**Remark 1**. As we mentioned in Introduction, we have
\[
\|e^{i\lambda\varphi}\|_{A_p(T)} = O\left(|\lambda|^\frac{1}{p} - \frac{1}{2}\right), \quad |\lambda| \to \infty,
\]
for any real function \( \varphi \in C^1(T) \). So from Corollary 1 we see that for any real function \( \varphi \in C^{1,1}(T), \ \varphi \neq \text{const} \), we have
\[
\|e^{i\lambda\varphi}\|_{A_p(T)} \simeq |\lambda|^\frac{1}{p} - \frac{1}{2}.
\]
Thus, the rate of growth of the norms \( \|e^{i\lambda\varphi}\|_{A_p} \) in the case of a \( C^{1,1} \)-smooth phase \( \varphi \) is the same as in the \( C^2 \)-smooth case.
Remark 2. A simple modification of the proof of Theorems 1, 1’ allows to obtain their multidimensional versions. Our results on behavior of the norms $\|e^{i\lambda \varphi}\|_{A_p(T^m)}$ for smooth functions $\varphi$ on the torus $T^m$, $m \geq 2$, will be presented elsewhere.

§ 2. Slow growth of $\|e^{i\lambda \varphi}\|_{A_p(T)}$

In this section, for each class $C^{1,\omega}$ (under a certain simple condition imposed on $\omega$), we shall construct a non-trivial function $\varphi \in C^{1,\omega}(T)$ such that the norms $\|e^{i\lambda \varphi}\|_{A_p(T)}$ have slow growth.

The case when $\omega(\delta) = \delta$ is described in Remark 1.

Recall that we say that a function is nowhere linear if it is not linear on any interval.

As above, $\chi^{-1}$ is the function inverse to $\chi(\delta) = \delta \omega(\delta)$.

Theorem 2. Suppose that $\omega(2\delta) < 2\omega(\delta)$ for all sufficiently small $\delta > 0$. There exists a nowhere linear real function $\varphi \in C^{1,\omega}(T)$ such that

(i) $\|e^{i\lambda \varphi}\|_{A(T)} \leq c \frac{|\lambda| \log |\lambda|}{\log |\lambda|} \chi^{-1}(|\lambda|^2)$, $\lambda \in \mathbb{R}$, $|\lambda| \geq 2$,

and

(ii) $\|e^{i\lambda \varphi}\|_{A_p(T)} \leq c_p \left( \int_1^{\frac{|\lambda|}{1/p}} \chi^{-1}(1/\tau)^p d\tau \right)^{1/p}$, $\lambda \in \mathbb{R}$, $|\lambda| \geq 2$,

for all $p$, $1 < p < 2$. The positive constants $c, c_p$ are independent of $\lambda$.

Put $\omega(\delta) = \delta^\alpha$, $0 < \alpha < 1$, in Theorem 2. Using Corollary 1 and the trivial estimate $\|e^{i\lambda \varphi}\|_{A_p} \geq 1$, $1 \leq p \leq 2$, we immediately obtain the following corollary.

Corollary 2. Let $0 < \alpha < 1$. There exists a nowhere linear real function $\varphi \in C^{1,\alpha}(T)$ such that

(i) $\|e^{i\lambda \varphi}\|_{A(T)} = O\left(|\lambda|^{\frac{\alpha}{1+\alpha}} (\log |\lambda|)^{\frac{1}{1+\alpha}}\right)$
and

\[(ii) \quad \|e^{i\lambda \varphi}\|_{A_p(T)} \simeq |\lambda|^{\frac{1}{p} - \frac{1}{p+\alpha}} \quad \text{for} \quad 1 < p < 1 + \alpha; \]

\[\|e^{i\lambda \varphi}\|_{A_p(T)} \simeq 1 \quad \text{for} \quad 1 + \alpha < p < 2; \]

\[\|e^{i\lambda \varphi}\|_{A_p(T)} = O((\log |\lambda|)^{1/p}) \quad \text{for} \quad p = 1 + \alpha. \]

Thus, for $p \neq 1$ the estimate in Corollary 1 is sharp. The function $\varphi \in C^{1,\alpha}$ from Corollary 2 yields the slowest possible growth of the norms $\|e^{i\lambda \varphi}\|_{A_p}$ for $1 < p < 2$, $p \neq 1 + \alpha$.

**Remark 3.** It is not clear if for $1 < p < 2$ there is a non-trivial function $\varphi \in C^{1,p-1}(T)$ such that $\|e^{i\lambda \varphi}\|_{A_p(T)} = O(1)$.

As we mentioned in Introduction, Theorem 2 implies also the result earlier obtained by the author in [9], namely, we have the following assertion.

**Corollary 3.** Let $\gamma(\lambda) \geq 0$ and $\gamma(\lambda) \to +\infty$ as $\lambda \to +\infty$. There exists a nowhere linear real function $\varphi \in C^1(T)$ such that

\[\|e^{i\lambda \varphi}\|_{A(T)} = O(\gamma(|\lambda|) \log |\lambda|), \quad |\lambda| \to \infty, \quad \lambda \in \mathbb{R}. \]

**Proof.** The right-hand side in estimate (i) of Theorem 2 does not exceed $c|\lambda|^{-1}(1/|\lambda|) \log |\lambda|$ and it remains to note that we can choose $\omega$ so that $|\lambda|^{-1}(1/|\lambda|)$ has an arbitrarily slow growth. The corollary is proved.

Before we proceed to the proof of Theorem 2 recall that, as we mentioned in Introduction, if $\varphi$ is a piecewise linear continuous functions then we have $\|e^{i\lambda \varphi}\|_A = O(\log |\lambda|)$ and $\|e^{i\lambda \varphi}\|_{A_p} = O(1)$ for $p > 1$. (The logarithmic growth in $A(T)$ is the slowest known.) So, if we want the norms $\|e^{i\lambda \varphi}\|_{A_p}$ to grow slowly for a $C^1$ -smooth non-trivial function $\varphi$, it seems natural to consider the primitives of Cantor staircase functions. These primitives resemble piecewise linear functions, although they are $C^1$ -smooth.
To make the proof of Theorem 2 more transparent we start by constructing a function \( \varphi \in C^1(\mathbb{T}) \), \( \varphi \neq \text{const} \) that satisfies estimates (i), (ii) of the theorem, without requiring nowhere linearity. In constructing this function we obtain certain lemmas that we shall also use to construct a nowhere linear function \( \varphi \) with the same properties.

Choosing (if necessarily) numbers \( a, b > 0 \) and replacing \( \omega(\delta) \) with \( a\omega(b\delta) \) (this only affects the constants \( c, c_p \) in estimates (i), (ii) in Theorem 2) we can assume that \( \omega(2\pi) = 1 \) and \( \omega(2\delta) < 2\omega(\delta) \) for \( 0 < \delta \leq 2\pi \).

Define positive numbers \( \rho_j, j = 0, 1, 2, \ldots \), by
\[
\omega(\rho_j) = 2^{-j}. \tag{12}
\]

We can assume that \( \rho_0 = 2\pi \). Since \( \omega(2\rho_{j+1}) < 2\omega(\rho_{j+1}) = 2^{-j} = \omega(\rho_j) \) we have \( 2\rho_{j+1} < \rho_j, j = 0, 1, 2, \ldots \).

We construct a symmetric perfect set \( E \subset [0, 2\pi] \) as follows (see [8, Ch. V, § 3], [11, Ch. XIV, § 19]). From the closed interval \([0, 2\pi] = [0, \rho_0] \) we remove a concentric open interval so that two remaining closed intervals have the same length equal to \( \rho_1 \). From each of the remaining closed intervals we remove the corresponding concentric open interval so that there remain four closed intervals of length \( \rho_2 \) etc. At the \( j \)-th step we obtain \( 2^j \) closed intervals \( I_{\nu}^j, \nu = 1, 2, \ldots, 2^j \), of length \( \rho_j \). Continuing the process we obtain the set of remaining points:
\[
E = \bigcap_{j=0}^{\infty} \bigcup_{\nu=1}^{2^j} I_{\nu}^j.
\]

Let \( \psi \) be the Cantor type staircase function related to the set \( E \), namely, a real continuous non-decreasing function on \([0, 2\pi] \) that takes constant values on the intervals complementary to \( E \) in \([0, 2\pi] \) (i.e. on the connected components of the compliment \([0, 2\pi] \setminus E \)) and increases by \( 2^{-j} \) on each closed interval \( I_{\nu}^j, \nu = 1, 2, \ldots, 2^j \), obtained at the \( j \)-th step of construction of \( E \). Note that condition (12) implies that \( \psi \in \text{Lip}_\omega([0, 2\pi]) \). This can be easily verified by nearly word for word repetition of the arguments used in [8, Ch. V, § 3] in the case when \( \omega(\delta) = \delta^a \).

By a modification of the staircase function it is easy to obtain a function \( \psi \) on \([0, 2\pi] \) with the following properties:

1) \( \psi \) takes constant values on the intervals complementary to \( E \) in \([0, 2\pi] \);
2) \( \psi \in \text{Lip}_\omega([0, 2\pi]) \);
3) \(\psi(0) = \psi(2\pi) = 0\);
4) \(\int_0^{2\pi} \psi(\theta) d\theta = 0\);
5) \(\max_{t \in [0, 2\pi]} |\psi(t)| = 1\);
6) \([0, 2\pi] = J_1 \cup J_2 \cup J_3\), where \(J_1, J_2, J_3\) are pairwise-disjoint intervals such that \(\psi\) is monotone on each of them.

Define a function \(\varphi\) on \(T\) by

\[
\varphi(t) = \int_0^t \psi(\theta) \, d\theta, \quad t \in [0, 2\pi].
\]

We have \(\varphi(0) = \varphi(2\pi) = \varphi'(0) = \varphi'(2\pi) = 0\). Thus, \(\varphi \in C^{1,\omega}(T)\). At the same time the function \(\varphi\) is non-constant on \(T\) and is linear on each interval complementary to \(E\) in \([0, 2\pi]\).

We claim that \(\varphi\) satisfies estimate (i) of Theorem 2.

We put

\[
\Theta(y) = \frac{y}{\log y} \log^{-1}\left(\frac{(\log y)^2}{y}\right), \quad y > 1.
\]

(13)

Lemma 2. Let \(g\) be a real function on \([0, 2\pi]\) linear on an interval \(\Delta \subseteq [0, 2\pi]\). Then for all \(y \geq 2\) we have

\[
\sum_{|k| \leq y} |\widehat{1_{\Delta} e^{ig}}(k)| \leq c \log y,
\]

where \(c > 0\) is independent of \(\Delta, g\) and \(y\).

Proof. Fix \(y \geq 2\). Assume that \(g(t) = at + b\) for \(t \in \Delta\), where \(a, b \in \mathbb{R}\). Direct calculation yields

\[
|\widehat{1_{\Delta} e^{ig}}(k)| = \left|\frac{\sin((k - a)|\Delta|/2)}{\pi(k - a)}\right| \leq \frac{1}{|k - a|}, \quad k \in \mathbb{Z}, \quad k \neq a.
\]

(14)

At the same time we have

\[
|\widehat{1_{\Delta} e^{ig}}(k)| \leq \frac{|\Delta|}{2\pi}, \quad k \in \mathbb{Z}.
\]

(15)

Thus,

\[
\sum_{|k| \leq y} |\widehat{1_{\Delta} e^{ig}}(k)| \leq \sum_{|k| \leq y} \min\left(\frac{1}{|k - a|}, 1\right).
\]

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Consider two cases: $|a| \geq 2y$ and $|a| < 2y$. In the first case for $|k| \leq y$ we have $|k - a| \geq y$. So,

$$\sum_{|k| \leq y} \min \left( \frac{1}{|k - a|}, 1 \right) \leq \sum_{|k| \leq y} \frac{1}{y} \leq 3.$$ 

In the second case for $|k| \leq y$ we have $|k - a| \leq 3y$. So,

$$\sum_{|k| \leq y} \min \left( \frac{1}{|k - a|}, 1 \right) \leq \sum_{|k-a| \leq 3y} \min \left( \frac{1}{|k - a|}, 1 \right) \leq \sum_{|k-a| \leq 2} 1 + \sum_{2 \leq |k-a| \leq 3y} \frac{1}{|k - a|} \leq 5 + \int_{1 \leq |x-a| \leq 3y} \frac{dx}{|x-a|} = 5 + 2 \log(3y).$$

The lemma is proved.

**Lemma 3.** Let $g$ be a real function on $[0, 2\pi]$ linear on each interval complimentary to $E$ in $[0, 2\pi]$. Let $\lambda \geq 2$. Then

$$\sum_{|k| \leq 2\lambda} |\hat{e}^{ig}(k)| \leq c \Theta(\lambda),$$

where $c > 0$ is independent of $\lambda$ and $g$.

**Proof.** Fix a positive integer $j$, which we shall specify later. The set $E$ can be covered by $2^j$ closed intervals of equal length $\rho_j$. Denote there union by $F_j$. The complement $G_j = [0, 2\pi] \setminus F_j$ is the union of $2^j - 1$ pairwise-disjoint open intervals such that on each of them $g$ is linear. Applying Lemma 2 to each of these intervals, we obtain

$$\sum_{|k| \leq 2\lambda} |\hat{1}_{G_j} e^{ig}(k)| \leq c_1 2^j \log \lambda. \tag{16}$$

At the same time, $|F_j| = 2^j \rho_j$, so (using the Cauchy inequality and the Parseval identity) we have

$$\sum_{|k| \leq 2\lambda} |\hat{1}_{F_j} e^{ig}(k)| \leq (4\lambda + 1)^{1/2} \left( \sum_{|k| \leq 2\lambda} |\hat{1}_{F_j} e^{ig}(k)|^2 \right)^{1/2}$$
$$\leq (4\lambda + 1)^{1/2} ||1_F e^{i g}||_{L^2(\Gamma)} = (4\lambda + 1)^{1/2} (|F_j|/(2\pi))^{1/2} \leq c_2 \lambda^{1/2} (2^j \rho_j)^{1/2}. \quad (17)$$

From (16), (17) we obtain

$$\sum_{|k| \leq 2\lambda} |\hat{e}^{ig}(k)| \leq c_1 2^j \log \lambda + c_2 \lambda^{1/2} (2^j \rho_j)^{1/2}. \quad (18)$$

The constants $c_1, c_2 > 0$ are independent of $\lambda, g$ and $j$.

Note that

$$\log y = o(\Theta(y)), \quad y \to +\infty. \quad (19)$$

So if $\lambda$ is large enough, say $\lambda \geq \lambda_0$, then

$$\frac{\Theta(\lambda)}{\log \lambda} \geq 1.$$ 

It suffices to obtain the estimate of the lemma under additional assumption that $\lambda \geq \lambda_0$.

Let $\lambda \geq \lambda_0$. We shall choose $j$ to minimize the right-hand side in (18).

Define a positive integer $j(\lambda)$ by condition

$$2^{j(\lambda) - 1} \leq \frac{\Theta(\lambda)}{\log \lambda} < 2^{j(\lambda)}. \quad (20)$$

It is easy to verify that

$$\frac{\log y}{\Theta(y)} = \omega \left( \frac{\log y}{y} \Theta(y) \right), \quad y > 1.$$ 

So, the right-hand side inequality in (20) yields (see (12))

$$\omega(\rho_j(\lambda)) = 2^{-j(\lambda)} \leq \frac{\log \lambda}{\Theta(\lambda)} = \omega \left( \frac{\log \lambda}{\lambda} \Theta(\lambda) \right),$$

whence

$$\rho_j(\lambda) \leq \frac{\log \lambda}{\lambda} \Theta(\lambda). \quad (21)$$

The left-hand side inequality in (20) and estimate (21) imply that for $j = j(\lambda)$ the right-hand side of (18) does not exceed $c \Theta(\lambda)$. The lemma is proved.
Lemma 4. Let $f$ be a real function on $[0, 2\pi]$. Let $I \subseteq [0, 2\pi]$ be an interval. Suppose that $f \in C^1(I)$, the derivative $f'$ is monotone on $I$, and $|f'(t)| \leq 1$ for $t \in I$. Let $\lambda > 0$. Then for all $k \in \mathbb{Z}$ such that $|k| \geq 2\lambda$, we have

$$|\widehat{1_I e^{i\lambda f}}(k)| \leq \frac{2}{|k|}.$$

Proof. According to the first part of the van der Corput lemma (see [8, Ch. V, Lemma 4.3]), if $I \subset \mathbb{R}$ is a bounded interval and $g \in C^1(I)$ is a real function such that its derivative $g'$ is monotone and $|g'(t)| \geq \rho > 0$ for all $t \in I$, then

$$\left| \int_I e^{i g(t)} dt \right| \leq \frac{2\pi}{\rho}.$$

Fix $\lambda > 0$ and $k \in \mathbb{Z}$, $|k| \geq 2\lambda$. Put $g(t) = \lambda f(t) - kt$. We see that the derivative $g'(t) = \lambda f'(t) - k$ is monotone on $I$ and $|g'(t)| \geq |k| - \lambda \geq |k|/2$. Hence,

$$|\widehat{1_I e^{i\lambda f}}(k)| = \left| \frac{1}{2\pi} \int_I e^{i g(t)} dt \right| \leq \frac{2}{|k|}.$$

The lemma is proved.

Let us estimate the norm $\|e^{i\lambda \varphi}\|_A$. We can assume that $\lambda \geq 2$. We shall separately estimate the sums of the moduli of the Fourier coefficients $\widehat{e^{i\lambda \varphi}}(k)$ over $k \in \mathbb{Z}$ in the ranges $|k| \leq 2\lambda$, $2\lambda < |k| \leq \lambda^2$, and $\lambda^2 < |k|$.

Applying Lemma 3 to the function $g = \lambda \varphi$, we obtain

$$\sum_{|k| \leq 2\lambda} |\widehat{e^{i\lambda \varphi}}(k)| \leq c_1 \Theta(\lambda). \quad (22)$$

Note that $|\varphi'(t)| = |\psi(t)| \leq 1$ on $[0, 2\pi]$ and the interval $[0, 2\pi]$ is a union of three pairwise-disjoint intervals such that the derivative $\varphi' = \psi$ is monotone on each of them. Applying Lemma 4 to each of these intervals, we see that

$$|\widehat{e^{i\lambda \varphi}}(k)| \leq \frac{6}{|k|}, \quad |k| > 2\lambda. \quad (23)$$

So,

$$\sum_{2\lambda < |k| \leq \lambda^2} |\widehat{e^{i\lambda \varphi}}(k)| \leq \sum_{2\lambda < |k| \leq \lambda^2} \frac{6}{|k|} \leq c_2 \log \lambda. \quad (24)$$

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Note then that
\[
\sum_{|k| > \lambda^2} |e^{i\lambda\varphi}(k)| = \sum_{|k| > \lambda^2} \frac{1}{|k|} |(e^{i\lambda\varphi})'(k)| \leq \left( \sum_{|k| > \lambda^2} \frac{1}{k^2} \right)^{1/2} \| (e^{i\lambda\varphi})' \|_{L^2(T)} \leq c_3.
\] (25)

Summing inequalities (22), (24), (25) and taking (19) into account, we obtain estimate (i) of Theorem 2.

We claim that the same function \( \varphi \) satisfies estimate (ii) of Theorem 2.

For \( p > 1 \) we put
\[
\Theta_p(y) = \left( \int_1^y (\chi^{-1}(1/\tau))^p \, d\tau \right)^{1/p}, \quad y > 1.
\] (26)

**Lemma 5.** Let \( 1 < p < 2 \). Let \( g \) be a real function on \([0, 2\pi]\) linear on an interval \( \Delta \subseteq [0, 2\pi] \). Then
\[
\| 1_\Delta e^{ig} \|_{A_p(T)} \leq c_p |\Delta|^{1/q},
\]
where \( 1/p + 1/q = 1 \) and the constant \( c_p > 0 \) is independent of \( g \) and \( \Delta \).

**Proof.** Assuming that \( g(t) = at + b \) for \( t \in \Delta \), from estimates (14), (15), used in the proof of Lemma 2, we obtain
\[
\| 1_\Delta e^{ig} \|_{A_p(T)}^p = \sum_{k \in \mathbb{Z}} \left| \hat{1_\Delta e^{ig}}(k) \right|^p \leq \sum_{|k-a| > 1/|\Delta|} \frac{1}{|k-a|} + \sum_{|k-a| \leq 1/|\Delta|} |\Delta|^p \leq c_p |\Delta|^{p-1}.
\]
The lemma is proved.

In what follows it will be convenient to use the analogues of the spaces \( A_p(T) \) for the functions defined on the line \( \mathbb{R} \). We use the same symbol \( \hat{\cdot} \) to denote the Fourier transform of tempered distributions on \( \mathbb{R} \). For \( 1 < p < \infty \) let \( A_p(\mathbb{R}) \) be the space of tempered distributions \( g \) on \( \mathbb{R} \) such that \( \hat{g} \) belongs to \( L_p(\mathbb{R}) \). We put
\[
\| g \|_{A_p(\mathbb{R})} = \| \hat{g} \|_{L_p(\mathbb{R})}.
\]

It is known (see, e.g., [12]) that for \( 1 \leq p \leq 2 \) the Fourier transform (as well as its inverse) is a bounded operator from \( L^p(\mathbb{R}) \) to \( L^q(\mathbb{R}) \), \( 1/p + 1/q = 1 \).
Thus, each distribution that belongs to $A_p(\mathbb{R})$, $1 < p \leq 2$, is a function in $L^q(\mathbb{R})$.

Let $1 < p \leq 2$. It is easy to verify (and is well known, see, e.g. [13, § 44]) that if $f$ is a $2\pi$-periodic function and $f^*$ is its restriction to $[0, 2\pi]$ extended by zero to $\mathbb{R}$, that is $f^* = f$ on $[0, 2\pi]$, $f^* = 0$ on $\mathbb{R} \setminus [0, 2\pi]$, then $f$ belongs to $A_p(\mathbb{T})$ if and only if $f^* \in A_p(\mathbb{R})$. The norms satisfy

$$c_1(p)\|f\|_{A_p(\mathbb{R})} \leq \|f\|_{A_p(\mathbb{T})} \leq c_2(p)\|f^*\|_{A_p(\mathbb{R})}.$$ 

We call this statement the transference principle.

Let $\Delta \subseteq \mathbb{R}$ be an arbitrary interval. It is known (see, e.g., [14]) that the operator $S_\Delta$ given by

$$S_\Delta(g) = (1_\Delta \cdot \hat{g})^\vee,$$

where $\vee$ means the inverse Fourier transform, is a bounded operator from $L^p(\mathbb{R})$ to itself for $1 < p < \infty$.

For an arbitrary family $\{\Delta_\nu\}$ of pairwise-disjoint intervals in $\mathbb{R}$ we define the Littlewood-Paley square function:

$$S(g) = \left( \sum_\nu |S_{\Delta_\nu}(g)|^2 \right)^{1/2}.$$ 

We recall the Rubio de Francia inequality [15]:

$$\|S(g)\|_{L^p(\mathbb{R})} \leq c_p\|g\|_{L^p(\mathbb{R})}, \quad g \in L^p(\mathbb{R}),$$

which holds for $2 < p < \infty$. The constant $c_p > 0$ is independent of $g$ and $\{\Delta_\nu\}$. By duality, for $1 < p < 2$ and for any function $g \in L^p(\mathbb{R})$ with

$$\text{supp } \hat{g} \subseteq \bigcup_\nu \Delta_\nu,$$  \hspace{1cm} (27)

we have

$$\|g\|_{L^p(\mathbb{R})} \leq c_p\|S(g)\|_{L^p(\mathbb{R})}. \hspace{1cm} (28)$$

The following lemma is a simple consequence of the Rubio de Francia inequality.
Lemma 6. Let $1 < p < 2$. Let $\Delta_\nu \subset \mathbb{T}$, $\nu = 1, 2, \ldots, N$, be a finite family of pairwise-disjoint intervals and let $f_\nu$, $\nu = 1, 2, \ldots, N$, be functions in $A_p(\mathbb{T})$ such that $\text{supp } f_\nu \subseteq \Delta_\nu$, $\nu = 1, 2, \ldots, N$. Then
\[
\left\| \sum_{\nu} f_\nu \right\|^p_{A_p(\mathbb{T})} \leq c_p \sum_{\nu} \left\| f_\nu \right\|^p_{A_p(\mathbb{T})},
\]
where $c_p > 0$ is independent of $N$ and the families $\{\Delta_\nu\}$, $\{f_\nu\}$.

Proof. Consider a finite family $\Delta_\nu$, $\nu = 1, 2, \ldots, N$, of pairwise-disjoint bounded intervals in $\mathbb{R}$. For $1 < p < 2$, taking into account the obvious inequality
\[
S(g) \leq \left( \sum_{\nu} |S_{\Delta_\nu}(g)|^p \right)^{1/p},
\]
we obtain from (28) that
\[
\|g\|^p_{L^p(\mathbb{R})} \leq c_p \sum_{\nu} \|S_{\Delta_\nu}(g)\|^p_{L^p(\mathbb{R})}
\]
for any function $g \in L^p(\mathbb{R})$ with condition (27).

Let $f_\nu$, $\nu = 1, 2, \ldots, N$, be functions that belong to $A_p(\mathbb{R})$, $1 < p < 2$, and have supports in $\Delta_\nu$, $\nu = 1, 2, \ldots, N$, respectively. Applying estimate (29) to the function
\[
g = \left( \sum_{\nu} f_\nu \right)^\vee,
\]
and taking into account that the direct and inverse Fourier transforms on $\mathbb{R}$ differ only in the sign of the variable, we see that
\[
\left\| \sum_{\nu} f_\nu \right\|^p_{A_p(\mathbb{R})} \leq c_p \sum_{\nu} \|f_\nu\|^p_{A_p(\mathbb{R})}.
\]
(Since the functions $f_\nu$ are in $L^q(\mathbb{R})$ and the intervals $\Delta_\nu$ are bounded, we have $f_\nu \in L^2(\mathbb{R})$, $\nu = 1, 2, \ldots, N$. So, $g \in L^2(\mathbb{R})$.)

It remains to use the transference principle. The lemma is proved.

Lemma 7. Let $1 < p < 2$ and let $g$ be a real function on $[0, 2\pi]$ linear on each interval complementary to $E$ in $[0, 2\pi]$. Let $\lambda \geq 2$. Then
\[
\sum_{|k| \leq 2\lambda} |\widehat{e^g}(k)|^p \leq c_p (\Theta_p(\lambda))^p,
\]

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where \( c_p > 0 \) is independent of \( \lambda \) and \( g \).

Proof. For an arbitrary \( j = 1, 2, \ldots \) the set \( E \) can be covered by \( 2^j \) closed intervals of equal length \( \rho_j \). Their union \( F_j \) is of measure \( |F_j| = 2^j \rho_j \). The compliment \( G_j = [0, 2\pi] \setminus F_j \) is a union of \( 2^j - 1 \) pairwise-disjoint open intervals. Denote these intervals by \( \Delta^j_{\nu}, \nu = 1, 2, \ldots, 2^j - 1 \). The function \( g \) is linear on each interval \( \Delta^j_{\nu} \), so, using Lemmas 6 and 5, we obtain that

\[
\|1_{G_j}e^{ig}\|_{A_p(T)}^p \leq c_p \sum_{\nu=1}^{2^j-1} \|1_{\Delta^j_{\nu}}e^{ig}\|_{A_p(T)}^p \leq c_{p,1} \sum_{\nu=1}^{2^j-1} |\Delta^j_{\nu}|^{p-1}. \tag{30}
\]

The family of intervals \( \Delta^j_{\nu}, \nu = 1, 2, \ldots, 2^j - 1 \), consists of one interval of length \( \rho_0 - 2\rho_1 < \rho_0 \), of two intervals of length \( \rho_1 - 2\rho_2 < \rho_1 \), etc., of \( 2^m \) intervals of length \( \rho_m - 2\rho_{m+1} < \rho_m \), where \( m = 0, 1, \ldots, j - 1 \). Hence we obtain (see (30)) that

\[
\sum_{|k| \leq 2\lambda} |1_{G_j}e^{ig}(k)|^p \leq \|1_{G_j}e^{ig}\|_{A_p(T)}^p \leq c_{p,1} \sum_{m=0}^{j-1} 2^m \rho_m^{p-1}. \tag{31}
\]

At the same time, using the Hölder inequality with \( p^* = 2/p \) and \( 1/p^* + 1/q^* = 1 \), we have

\[
\sum_{|k| \leq 2\lambda} |1_{F_j}e^{ig}(k)|^p \leq \left( \sum_{|k| \leq 2\lambda} |1_{F_j}e^{ig}(k)|^{pp^*} \right)^{1/p^*} \left( \sum_{|k| \leq 2\lambda} 1 \right)^{1/q^*} \\
\leq \left( \sum_{|k| \leq 2\lambda} |1_{F_j}e^{ig}(k)|^2 \right)^{p/2} (4\lambda + 1)^{1-\frac{p}{2}} \leq \|1_{F_j}e^{ig}\|_{L^2(T)}^p (4\lambda + 1)^{1-\frac{p}{2}} \\
= (|F_j|/(2\pi))^{p/2} (4\lambda + 1)^{1-\frac{p}{2}} \leq c_{p,2} (2^j \rho_j)^{p/2} \lambda^{1-\frac{p}{2}}. \tag{32}
\]

From (31), (32) we obtain that

\[
\left( \sum_{|k| \leq 2\lambda} |\widehat{e^{ig}}(k)|^p \right)^{1/p} \leq c_{p,3} \left( \sum_{m=0}^{j-1} 2^m \rho_m^{p-1} \right)^{1/p} + c_{p,4} (2^j \rho_j)^{1/2} \lambda^{1-\frac{p}{2}}, \tag{33}
\]

where \( c_{p,3}, c_{p,4} > 0 \) are independent of \( \lambda, g \) and \( j \).
It is easy to choose $j$ so that to minimize the expression on the right-hand side of (33) in the case when $\omega(\delta) = \delta^{\alpha}$. General case requires certain calculations.

We put

$$a_m = \frac{1}{\chi(\rho_m)} , \quad m = 0, 1, 2, \ldots.$$  

The sequence $\{a_m\}$ is unbounded, strictly increases, and $a_0 = 1/(2\pi)$.

It is clear that it suffices to obtain the estimate of the lemma under additional condition $\lambda \geq a_1$.

Choose $j(\lambda) = 2, 3, \ldots$ so that

$$a_{j(\lambda) - 1} \leq \lambda < a_{j(\lambda)}.$$  

Let us estimate the first term on the right-hand side in (33). Note that for $m = 1, 2, \ldots$

$$\frac{a_m}{a_{m-1}} = \frac{\chi(\rho_{m-1})}{\chi(\rho_m)} = \frac{\rho_{m-1}\omega(\rho_{m-1})}{\rho_m \omega(\rho_m)} = \frac{\rho_{m-1}2^{-(m-1)}}{\rho_m 2^{-m}} = \frac{2\rho_{m-1}}{\rho_m} \geq 2,$$

whence

$$a_m \leq 2(a_m - a_{m-1}), \quad m = 1, 2, \ldots.$$  

So,

$$\sum_{m=1}^{j(\lambda)-1} 2^m \rho_m^{-1} = \sum_{m=1}^{j(\lambda)-1} \frac{\rho_{m-1}}{\omega(\rho_m)} = \sum_{m=1}^{j(\lambda)-1} \frac{\rho_{m-1}}{\chi(\rho_m)}$$

$$= \sum_{m=1}^{j(\lambda)-1} a_m (\chi^{-1}(1/a_m))^{p} \leq 2 \sum_{m=1}^{j(\lambda)-1} (a_m - a_{m-1})(\chi^{-1}(1/a_m))^{p}$$

$$\leq 2 \sum_{m=1}^{j(\lambda)-1} \int_{a_{m-1}}^{a_m} (\chi^{-1}(1/\tau))^{p} \, d\tau = 2 \int_{a_0}^{a_{j(\lambda)-1}} (\chi^{-1}(1/\tau))^{p} \, d\tau$$

$$\leq 2 \int_{a_0}^{\lambda} (\chi^{-1}(1/\tau))^{p} \, d\tau = c_{p,5} + 2(\Theta(\lambda))^{p}. \quad \text{(35)}$$

Let us estimate the second term on the right-hand side in (33). Put $t = \chi^{-1}(1/\lambda)$. Then (see (34))

$$\rho_{j(\lambda)} < t \leq \rho_{j(\lambda)-1}.$$  

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Hence, using the right-hand side inequality, we obtain that $\omega(t) \leq 2^{-(j(\lambda)-1)}$ and therefore

$$2^{j(\lambda)} \leq \frac{2}{\omega(t)} = \frac{2t}{\chi(t)} = 2\chi^{-1}(1/\lambda)\lambda.$$  

At the same time, the left-hand side inequality yields

$$\rho_{j(\lambda)} < t = \chi^{-1}(1/\lambda).$$

So,

$$(2^{j(\lambda)}\rho_{j(\lambda)})^{p/2}\lambda^{1-\frac{q}{p}} \leq 2^{p/2}\lambda(\chi^{-1}(1/\lambda))^{p}.$$  

Since $\lambda \geq 2$ and the function $\chi^{-1}$ is increasing, we obtain

$$\lambda(\chi^{-1}(1/\lambda))^{p} \leq 2(\lambda - 1)(\chi^{-1}(1/\lambda))^{p} \leq 2 \int_{1}^{\lambda} (\chi^{-1}(1/\tau))^{p}d\tau = 2(\Theta_{p}(\lambda))^{p}.$$  

Thus

$$(2^{j(\lambda)}\rho_{j(\lambda)})^{p/2}\lambda^{1-\frac{q}{p}} \leq c_{p,0}(\Theta_{p}(\lambda))^{p}.$$  

Using this inequality and (35), we see that for $j = j(\lambda)$ the right-hand side in (33) does not exceed $c_{p,1}(\Theta_{p}(\lambda))^{p}$. The lemma is proved.

Let us estimate the norms $\|e^{i\lambda\varphi}\|_{A_{p}}, p > 1$. Applying Lemma 7 to the function $g = \lambda\varphi$, we have (we can assume that $\lambda \geq 2$)

$$\sum_{|k| \leq 2\lambda} |\hat{e}^{i\lambda\varphi}(k)|^{p} \leq c_{p,1}(\Theta_{p}(\lambda))^{p}.$$  

At the same time, using estimate (23), we have

$$\sum_{|k| > 2\lambda} |\hat{e}^{i\lambda\varphi}(k)|^{p} \leq \sum_{|k| > 2\lambda} \left(\frac{6}{|k|}\right)^{p} \leq 6^{p} \sum_{|k| \geq 1} \frac{1}{|k|^{p}} = c_{p,2}.$$  

Thus,

$$\|e^{i\lambda\varphi}\|_{A_{p}(\mathbb{T})} \leq c_{p,1}(\Theta_{p}(\lambda))^{p} + c_{p,2},$$  

and we obtain (ii).

Proof of Theorem 2. For $m = 0, 1, 2, \ldots$ let $E_{m}$ be the portion of the set $E$ in the closed interval $[0, \rho_{m}]$, i.e., $E_{m} = E \cap [0, \rho_{m}]$. Certainly $E_{0} = E$.  

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Again let $\psi$ be the Cantor type staircase function related to the set $E$. For each $m = 0, 1, 2, \ldots$ the restriction of $\psi$ to $[0, \rho_m]$ belongs to $\text{Lip}_\omega([0, \rho_m])$. Thus, it is easy to see that there exist functions $\psi_m$, $m = 0, 1, 2, \ldots$, on $[0, 2\pi]$ (modified staircase functions) with the following properties:

1) $\psi_m$ takes constant values on the intervals complementary to $E_m$ in $[0, \rho_m]$;
2) $\psi_m \in \text{Lip}_\omega([0, \rho_m])$;
3) $\psi_m(0) = \psi_m(\rho_m) = 0$ and $\psi_m = 0$ on $[0, 2\pi] \setminus [0, \rho_m]$;
4) $\int_0^{\rho_m} \psi_m(\theta) d\theta = 0$;
5) $\max_{t \in [0, \rho_m]} |\psi_m(t)| = 1$;
6) $[0, \rho_m]$ is a union of three pairwise-disjoint intervals such that on each of them $\psi_m$ is monotone.

For $m = 0, 1, 2, \ldots$ we put

$$\varphi_m(t) = \int_0^t \psi_m(\theta) \, d\theta, \quad t \in [0, 2\pi].$$

It is clear that $\varphi_m(0) = \varphi_m(\rho_m) = \varphi'_m(0) = \varphi'_m(\rho_m) = 0$, $\varphi_m = 0$ on $[0, 2\pi] \setminus [0, \rho_m]$ and $\varphi'_m \in \text{Lip}_\omega[0, \rho_m]$, $m = 0, 1, 2, \ldots$.

Let $I \subseteq [0, 2\pi]$ be a closed interval. Let $E_m(I)$ be an affine copy of the set $E_m$ related to $I$, namely: $E_m(I) = l_{m,I}^{-1}(E_m)$, where $l_{m,I}$ is an affine map that maps $I$ onto $[0, \rho_m]$. Let $\varphi'_m$ be an affine copy of the function $\varphi_m$ related to $I$, namely a function on $[0, 2\pi]$ such that $\varphi'_m = \varphi_m \circ l_{m,I}$ on $I$ and $\varphi'_m = 0$ on $[0, 2\pi] \setminus I$.

Using induction we define a sequence of closed intervals $I_m \subseteq [0, 2\pi]$ with the condition

$$|I_m| \leq 2^{-m} \rho_m, \quad m = 0, 1, 2, \ldots, \quad (36)$$

and a sequence of sets $B_m \subset I_m$, $m = 0, 1, 2, \ldots$. Let $I_0 = [0, 2\pi]$ and $B_0 = E_0$. Once $I_0, I_1, I_2, \ldots, I_m$ and $B_0, B_1, B_2, \ldots, B_m$ have been defined, we define $I_{m+1}$ and $B_{m+1}$ as follows. Consider the union $\bigcup_{s=1}^m B_s$ and choose an interval of maximum length complementary to this union in $[0, 2\pi]$ (if there are several of them, we take any one). Denote this interval by $J$. Let $I_{m+1}$ be a closed interval contained in $J$, concentric with $J$, and of length $|I_{m+1}| \leq 2^{-(m+1)} \rho_{m+1}$. We put $B_{m+1} = E_{m+1}(I_{m+1})$.

Now we define functions $f_m$, $m = 0, 1, 2, \ldots$, by $f_m = \varphi'_m$. In what follows we consider $2\pi$-periodic extensions of functions $\varphi_m$, $f_m$, $m = 0, 1, 2, \ldots$, from $[0, 2\pi]$ to $\mathbb{R}$ and thus we regard these functions as functions
on $\mathbb{T}$. All these functions belong to $C^{1,\omega}(\mathbb{T})$. Each function $f_m$ vanishes on $[0, 2\pi] \setminus I_m$ and is linear on the intervals complimentary to $B_m$ in $I_m$. We also have

$$|\varphi'_m(t)| \leq 1, \quad t \in \mathbb{T}, \quad m = 0, 1, 2, \ldots,$$

and, since $l_m, I_m$ is a linear function on $[0, 2\pi]$ with tangent coefficient equal (up to a sign) to $\rho_m/|I_m|$, we see that

$$|f'_m(t)| \leq \rho_m/|I_m|, \quad t \in \mathbb{T}, \quad m = 0, 1, 2, \ldots$$

(37)

We put

$$\varphi = \sum_{m=0}^{\infty} \varepsilon_m f_m,$$

where numbers $\varepsilon_m > 0$ decrease to 0 so fast that $\varphi \in C^{1,\omega}(\mathbb{T})$.

It is clear that the function $\varphi$ is nowhere linear.

In addition we require that the sequence $\{\varepsilon_m\}$ satisfies

$$\sum_{m=0}^{\infty} \varepsilon_m \rho_m/|I_m| = 1 \quad (38)$$

and decreases so fast that

$$\delta_j(\lambda) = O(1/\lambda), \quad \lambda \to +\infty, \quad (39)$$

where

$$\delta_j = \sum_{m=j+1}^{\infty} \varepsilon_m \rho_m/|I_m| \quad (40)$$

and $j(\lambda)$ is the positive integer defined for each sufficiently large $\lambda$ by condition (20) (it is clear that $j(\lambda) \to \infty$ as $\lambda \to +\infty$, see (19)).

Let us estimate the norms $\|e^{i\lambda \varphi}\|_A$. We can assume that $\lambda \geq 2$.

Put

$$S_j = \sum_{m=0}^{j} \varepsilon_m f_m, \quad j = 0, 1, 2, \ldots$$

For each $j$ the set $E_m, m \leq j$, can be covered by $2^{j-m}$ closed intervals of length $\rho_j$. Since $B_m$ is obtained from $E_m$ by contraction (see (36)), we
see that the set $B_m, \ m \leq j$, can be covered by $2^{j-m}$ closed intervals whose length does not exceed $\rho_j$. Therefore, the set

$$\bigcup_{m=0}^{j} B_m$$

(41)

can be covered by

$$2^j + 2^{j-1} + 2^{j-2} + \ldots + 2^1 + 1 \leq 2^{j+1}$$
closed intervals of length $\rho_j$. Denote the union of these closed intervals by $F_j$. We have

$$|F_j| \leq 2^{j+1} \rho_j.$$  

We put $G_j = [0, 2\pi] \setminus F_j$. It is clear that $G_j$ is a union of at most $2^{j+1} + 1$ pairwise-disjoint intervals. It is clear that the function $S_j$ is linear on each interval complementary to the union (41) in $[0, 2\pi]$. Therefore the function $\lambda S_j$ is linear on the intervals that form $G_j$. Applying Lemma 2 to each of these intervals, we have

$$\sum_{|k| \leq 2\lambda} |1_{G_j} e^{\lambda S_j}(k)| \leq c_1 2^j \log \lambda.$$  

(42)

At the same time,

$$\sum_{|k| \leq 2\lambda} |1_{F_j} e^{\lambda S_j}(k)| \leq (4\lambda + 1)^{1/2} \left( \sum_{|k| \leq 2\lambda} |1_{F_j} e^{\lambda S_j}(k)|^2 \right)^{1/2}$$

$$\leq (4\lambda + 1)^{1/2} \|1_{F_j} e^{\lambda S_j}\|_{L^2(T)} = (4\lambda + 1)^{1/2} (|F_j|/(2\pi))^{1/2}$$

$$\leq c_2 \lambda^{1/2} (2^j \rho_j)^{1/2}.$$  

(43)

By summing estimates (42) and (43) we obtain

$$\sum_{|k| \leq 2\lambda} |e^{\lambda S_j}(k)| \leq c_1 2^j \log \lambda + c_2 \lambda^{1/2} (2^j \rho_j)^{1/2},$$

(44)

where $c_1, c_2 > 0$ are independent of $\lambda$ and $j$.

We note that (see (37), (38)),

$$|S_j'(t)| \leq \sum_{m=0}^{j} \varepsilon_m \frac{\rho_m}{|I_m|} \leq 1, \quad t \in T.$$  

(45)
We also note that for every \( j = 0, 1, 2, \ldots \) the interval \([0, 2\pi]\) is a union of three pairwise disjoint intervals such that the derivative \( f'_j \) of \( f_j \) is monotone on each of them. In addition, \( f'_j \) is supported in a certain interval on which the derivative \( f'_m \) of any function \( f_m \), \( m < j \), is constant. So, for every \( j = 0, 1, 2, \ldots \) the interval \([0, 2\pi]\) is a union of at most \( 4j + 3 \) pairwise disjoint intervals such that on each of them the derivative of \( S_j \) is monotone (this can be easily verified by induction with respect to \( j \)).

Taking into account estimate (45) and applying Lemma 4 to each interval on which \( S'_j \) is monotone, we obtain for \( |k| > 2\lambda \)

\[
|e^{i\lambda S_j}(k)| \leq \frac{8j + 6}{|k|},
\]

whence

\[
\sum_{2\lambda < |k| \leq \lambda^2} |e^{i\lambda S_j}(k)| \leq c_3 (8j + 6) \log \lambda. \tag{46}
\]

Note then, that we have \( |S'_j(t)| \leq 1 \) for \( t \in \mathbb{T} \) (see (45)), so,

\[
\sum_{|k| > \lambda^2} |e^{i\lambda S_j}(k)| = \sum_{|k| > \lambda^2} \frac{1}{|k|} |(e^{i\lambda S_j})'(k)| \leq \left( \sum_{|k| > \lambda^2} \frac{1}{|k|^2} \right)^{1/2} \| (e^{i\lambda S_j})' \|_{L^2(\mathbb{T})} \leq c_4. \tag{47}
\]

From (44), (46), (47) we obtain

\[
\| e^{i\lambda S_j} \|_{A(\mathbb{T})} \leq c_5 2^j \log \lambda + c_6 \lambda^{1/2} (2^j \rho_j)^{1/2}, \tag{48}
\]

where \( c_5, c_6 > 0 \) are independent of \( \lambda, j \).

We put now

\[
r_j = \sum_{m=j+1}^{\infty} \varepsilon_m f_m.
\]

Obviously, for any function \( g \in C^1(\mathbb{T}) \) we have

\[
\| g \|_{A(\mathbb{T})} \leq c \| g \|_{C^1(\mathbb{T})}, \tag{49}
\]

where

\[
\| g \|_{C^1(\mathbb{T})} = \max_{t \in \mathbb{T}} |g(t)| + \max_{t \in \mathbb{T}} |g'(t)|,
\]

and \( c > 0 \) is independent of \( g \). Since

\[
|r'_j(t)| \leq \delta_j, \quad t \in \mathbb{T}
\]

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(see (37), (40)), we obtain
\[ \| e^{i\lambda r_j} \|_{A(T)} \leq c \| e^{i\lambda r_j} \|_{C^1(T)} \leq c(1 + \lambda \delta_j). \] (50)

We note now that the expression on the right-hand side in (48) is of the same form as in (18). We minimize it in the same way as in the proof of Lemma 3, namely, assuming that \( \lambda \) is sufficiently large we choose \( j = j(\lambda) \) satisfying condition (20). We see that for this \( j \) the right-hand side in (48) does not exceed \( c_7 \Theta(\lambda) \). Thus,
\[ \| e^{i\lambda S_j(\lambda)} \|_{A(T)} \leq c_7 \Theta(\lambda). \] (51)

At the same time, from (50), taking into account condition (39), we obtain
\[ \| e^{i\lambda r_j(\lambda)} \|_{A(T)} \leq c_8. \] (52)

Since \( \varphi = S_j + r_j \) for any \( j \), we have
\[ \| e^{i\lambda \varphi} \|_{A(T)} \leq \| e^{i\lambda S_j} \|_{A(T)} \| e^{i\lambda r_j} \|_{A(T)}. \]

Thus, from (51), (52) we obtain estimate (i) of Theorem 2.

Let us estimate the norms \( \| e^{i\lambda \varphi} \|_{A_p}, \ p > 1 \). We need two simple lemmas.

**Lemma 8.** Let \( 1 < p < 2 \). Let \( I, J \) be two intervals in \([0, 2\pi]\) and let \( U, V \) be two functions in \( A_p(T) \) vanishing on \([0, 2\pi] \setminus I \) and \([0, 2\pi] \setminus J \) respectively. Suppose that \( U(t) = V \circ l(t) \) for \( t \in I \), where \( l \) is an affine map of \( I \) onto \( J \). Then \( \| U \|_{A_p(T)} \leq c_p |a|^{-1/q} \| V \|_{A_p(T)} \), where \( a \) is the tangent coefficient of \( l \), \( |a| = |J|/|I| \), and \( 1/p + 1/q = 1 \). The constant \( c_p > 0 \) is independent of \( U, V, I, J \).

**Proof.** If \( l(t) = at + b, \ t \in \mathbb{R}, \ a \neq 0 \), then for an arbitrary function \( g \in L^1(\mathbb{R}) \cap A_p(\mathbb{R}) \) direct calculation yields
\[ |\hat{g} \circ l(u)| = \left| \frac{1}{a} \hat{g} \left( \frac{u}{a} \right) \right|, \]
and therefore,
\[ \| g \circ l \|_{A_p(\mathbb{R})} = \frac{1}{|a|^{1/q}} \| g \|_{A_p(\mathbb{R})}. \]
It remains to use the transference principle. The lemma is proved.

**Lemma 9.** Let \(1 < p < 2\) and let \(l\) be a real \(2\pi\)-periodic function linear on \((0, 2\pi)\). Then for any function \(f \in A_p(\mathbb{T})\) we have \(e^{il}f \in A_p(\mathbb{T})\) and \(\|e^{il}f\|_{A_p(\mathbb{T})} \leq c_p\|f\|_{A_p(\mathbb{T})}\) where \(c_p > 0\) is independent of \(f\) and \(l\).

**Proof.** For an arbitrary linear function \(l(t) = at + b\) on \(\mathbb{R}\) and for an arbitrary function \(g \in L^1(\mathbb{R}) \cap A_p(\mathbb{R})\) we have

\[
|\hat{e^{il}g}(u)| = |\hat{g}(u - a)|, \quad u \in \mathbb{R},
\]

whence \(e^{il}g \in A_p(\mathbb{R})\) and \(\|e^{il}g\|_{A_p(\mathbb{R})} = \|g\|_{A_p(\mathbb{R})}\). It remains to use the transference principle. The lemma is proved.

Let \(\lambda \geq 2\). For each \(m = 0, 1, 2, \ldots\) the function \(\varphi_m\) is linear on the intervals complementary to \(E_m\) in \([0, 2\pi]\) and therefore (since \(E_m \subseteq E_0 = E\)), on the intervals complementary to \(E\) in \([0, 2\pi]\). Using Lemma 7, we obtain

\[
\sum_{|k| \leq 2\lambda} |\hat{e^{i\lambda \varphi_m}}(k)|^p \leq c_{p,1}(\Theta_p(\lambda))^p, \quad \lambda \geq 2, \quad m = 0, 1, 2, \ldots \quad (53)
\]

For each \(m = 0, 1, 2, \ldots\) we have \(|\varphi'_m(t)| \leq 1, \ t \in [0, 2\pi]\), and the interval \([0, 2\pi]\) can be partitioned into three intervals so that the derivative \(\varphi'_m\) is monotone on each of them. Using Lemma 4 we obtain

\[
|\hat{e^{i\lambda \varphi_m}}(k)| \leq \frac{6}{|k|}
\]

for \(|k| > 2\lambda\). So,

\[
\sum_{|k| > 2\lambda} |\hat{e^{i\lambda \varphi_m}}(k)|^p \leq c_{p,2}, \quad \lambda \geq 2, \quad m = 0, 1, 2, \ldots
\]

Together with (53) this yields

\[
\|e^{i\lambda \varphi_m}\|_{A_p(\mathbb{T})} \leq c_{p,3}\Theta_p(\lambda), \quad \lambda \geq 2, \quad m = 0, 1, 2, \ldots \quad (54)
\]

Let now \(0 \leq \lambda < 2\). We obtain that (see (49))

\[
\|e^{i\lambda \varphi_m}\|_{A_p(\mathbb{T})} \leq \|e^{i\lambda \varphi_m}\|_{A(\mathbb{T})} \leq c\|e^{i\lambda \varphi_m}\|_{C^1(\mathbb{T})} \leq c(1 + \lambda) \leq 3c
\]

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\[ 0 \leq \lambda < 2, \quad m = 0, 1, 2, \ldots \quad (55) \]

In what follows we assume that \( \lambda \geq 2 \). It is clear that \( 0 < \varepsilon_m \leq 1 \) for all \( m \) (see (36), (38)). Applying estimates (54), (55) in the cases when \( \lambda \varepsilon_m \geq 2 \) and \( \lambda \varepsilon_m < 2 \) respectively, we have

\[ \| e^{i\lambda \varepsilon_m f} \|_{A_p(T)} \leq c_{p, 4} \Theta_p(\lambda), \quad \lambda \geq 2, \quad m = 0, 1, 2, \ldots \quad (56) \]

whence

\[ \| e^{i\lambda \varepsilon_m f} \|_{A_p(T)} \leq c_{p, 5} \Theta_p(\lambda), \quad \lambda \geq 2, \quad m = 0, 1, 2, \ldots \quad (57) \]

For a fixed \( m \) we apply Lemma 8 to the closed intervals \( I = I_m, \quad J = [0, \rho_m] \) and the functions \( U = e^{i\lambda \varepsilon_m f_m} - 1, \quad V = e^{i\lambda \varepsilon_m \varphi_m} - 1 \). For the corresponding tangent coefficient \( a \) we have \( |a| = \rho_m / |I_m| \geq 2^m \) (see (36)), so, from (57) we obtain that

\[ \| e^{i\lambda \varepsilon_m f_m} - 1 \|_{A_p(T)} \leq c_{p, 6} 2^{-m/q} \| e^{i\lambda \varepsilon_m \varphi_m} - 1 \|_{A_p(T)} \leq c_{p, 7} 2^{-m/q} \Theta_p(\lambda), \quad \lambda \geq 2, \quad m = 0, 1, 2, \ldots \quad (58) \]

We note now that for \( j = 0, 1, 2, \ldots \)

\[ e^{i\lambda S_{j+1}} - e^{i\lambda S_j} = 1_{I_{j+1}}(e^{i\lambda S_{j+1}} - e^{i\lambda S_j}) = 1_{I_{j+1}} e^{i\lambda S_{j+1}}(e^{i\lambda S_{j+1}} - 1), \]

and the function \( S_j \) is linear on \( I_{j+1} \). So, using Lemma 9, we have

\[ \| e^{i\lambda S_{j+1}} - e^{i\lambda S_j} \|_{A_p(T)} \leq c_{p, 8} \| 1_{I_{j+1}}(e^{i\lambda S_{j+1}} - 1) \|_{A_p(T)}, \]

and, since \( f_{j+1} = 0 \) on \([0, 2\pi] \setminus I_{j+1}\), we see that

\[ \| e^{i\lambda S_{j+1}} - e^{i\lambda S_j} \|_{A_p(T)} \leq c_{p, 8} \| (e^{i\lambda S_{j+1}} - 1) \|_{A_p(T)}. \]

Therefore (see (58)),

\[ \| e^{i\lambda S_{j+1}} - e^{i\lambda S_j} \|_{A_p(T)} \leq c_{p, 9} 2^{-j/q} \Theta_p(\lambda), \quad \lambda \geq 2, \quad j = 0, 1, 2, \ldots \]

In addition, since \( S_0 = \varepsilon_0 f_0 = \varepsilon_0 \varphi_0 \), we have (see (56))

\[ \| e^{i\lambda S_0} \|_{A_p(T)} = \| e^{i\lambda \varepsilon_0 \varphi_0} \|_{A_p(T)} \leq c_{p, 4} \Theta_p(\lambda), \quad \lambda \geq 2. \]

Thus,

\[ \| e^{i\lambda S} \|_{A_p(T)} \leq \| e^{i\lambda S_0} \|_{A_p(T)} + \sum_{j=0}^{\infty} \| e^{i\lambda S_{j+1}} - e^{i\lambda S_j} \|_{A_p(T)} \leq c_{p} \Theta_p(\lambda). \]
The theorem is proved.

Remark 4. The derivative $\phi'$ of the function $\phi$ constructed in Theorem 2 is of bounded variation on $\mathbb{T}$. This follows from (37), (38) since for any $m$ the interval $[0, 2\pi]$ can be partitioned into three intervals so that the derivative $f'_m$ is monotone on each of them.

§ 3. Superposition operator

In this section we consider the maps of the circle $\mathbb{T}$ into itself, i.e., the functions $\phi : \mathbb{R} \to \mathbb{R}$ satisfying

$$\phi(t + 2\pi) = \phi(t) \pmod{2\pi}.$$

(59)

If such a function is continuous or of class $C^1(\mathbb{R})$, or of class $C^{1, \omega}(\mathbb{R})$, then $\phi$ is respectively continuous or $C^1$-smooth, or $C^{1, \omega}$-smooth map of the circle into itself. If a function $\phi$ satisfying (59) is continuous and is one-to-one mod $2\pi$, then $\phi$ is a homeomorphism of the circle. If a homeomorphism of the circle $\phi$ and its inverse $\phi^{-1}$ are $C^1$-smooth, then $\phi$ is a diffeomorphism of the circle. If a diffeomorphism $\phi$ is $C^{1, \omega}$-smooth, then $\phi$ is called $C^{1, \omega}$-diffeomorphism (it is easy to verify that in this case the inverse map $\phi^{-1}$ is $C^{1, \omega}$-smooth as well).

Theorems 3, 4 below follow from Theorems 1, 2 respectively and give their versions for maps of the circle into itself and integer frequencies.

**Theorem 3.** Let $1 \leq p < 2$. Let $\phi$ be a nonlinear $C^{1, \omega}$-smooth map of the circle into itself. Then

$$\|e^{in\phi}\|_{A_p(\mathbb{T})} \geq c_p |n|^{1/p} \chi^{-1}(1/|n|), \quad n \in \mathbb{Z}, \quad n \neq 0.$$

**Proof.** We have $\phi(t + 2\pi) = \phi(t) + 2\pi k$. It is clear that $k \in \mathbb{Z}$ is independent of $t$. Put

$$\phi_0(t) = \phi(t) - kt.$$

We obtain $\phi_0(t + 2\pi) = \phi_0(t)$ and thus $\phi_0$ is a real function of class $C^{1, \omega}(\mathbb{T})$. The function $\phi_0$ is non-constant. It remains to note that $\|e^{in\phi}\|_{A_p(\mathbb{T})} = \|e^{in\phi_0}\|_{A_p(\mathbb{T})}$ and apply Theorem 1 to $\phi_0$. The theorem is proved.
We put $\Theta_1 = \Theta$, where $\Theta$ is the function defined by (13). For $p > 1$ the functions $\Theta_p$ are defined by (26).

**Theorem 4.** Suppose that $\omega(2\delta) < 2\omega(\delta)$ for all sufficiently small $\delta > 0$. There exists a nowhere linear $C^{1,\omega}$-diffeomorphism $h$ of the circle $\mathbb{T}$ such that

$$\|e^{inh}\|_{A_p(\mathbb{T})} = O(\Theta_p(|n|)), \quad |n| \to \infty, \quad n \in \mathbb{Z},$$

for all $p$, $1 \leq p < 2$.

**Proof.** Take the function $\varphi$ from Theorem 2 and put

$$h(t) = t + \varepsilon\varphi(t),$$

where $0 < \varepsilon \leq 1$ is sufficiently small. We obtain a nowhere linear diffeomorphism $h$ of the circle $\mathbb{T}$ of class $C^{1,\omega}$. It remains to note that

$$\|e^{inh}\|_{A_p(\mathbb{T})} = \|e^{in\varphi}\|_{A_p(\mathbb{T})} \leq c_p\Theta_p(|n|\varepsilon) \leq c_p\Theta_p(|n|).$$

The theorem is proved.

The corresponding versions of Corollaries 1–3 (as well as of estimate (1)) are obvious. It is also clear that we have $\|e^{in\varphi}\|_{A_p(\mathbb{T})} \approx |n|^{\frac{1}{p} - \frac{1}{2}}$ for any nonlinear $C^{1,1}$-smooth map $\varphi : \mathbb{T} \to \mathbb{T}$ (see Remark 1).

We shall give now certain natural and obvious applications of Theorems 3, 4 to the problem on changes of variable in the spaces $A_p$.

Let $1 < p < 2$ and let $\varphi : \mathbb{T} \to \mathbb{T}$ be a continuous map such that for any $f \in A(\mathbb{T})$ we have $f \circ \varphi \in A_p(\mathbb{T})$. In this case we say that $\varphi$ acts from $A$ to $A_p$. Standard arguments (the closed graph theorem) show that this is the case if and only if the superposition operator $f \to f \circ \varphi$ is a bounded operator from $A(\mathbb{T})$ to $A_p(\mathbb{T})$. This, in its turn, is equivalent to the condition

$$\|e^{inh}\|_{A_p(\mathbb{T})} = O(1), \quad n \in \mathbb{Z}.$$

Using Theorem 3, we see that if $\omega(\delta) = o(\delta^{p-1})$, then every $C^{1,\omega}$-smooth map $\varphi$ of the circle into itself that acts from $A$ to $A_p$ is linear. At the same time from Theorem 4 it follows that if $p > 1$, then for any $\varepsilon > 0$ there exists a nowhere linear $C^{1,p-1-\varepsilon}$-diffeomorphism $h$ of the circle $\mathbb{T}$ such that

$$\|e^{inh}\|_{A_p(\mathbb{T})} = O(1)$$

and therefore $h$ acts from $A$ to $A_p$. It is unknown to the author if one can take here $\varepsilon = 0$ (see Remark 3).

Similarly it is easy to show that the existence of non-trivial $C^{1,\omega}$-smooth changes of variable that transfer functions from $A(\mathbb{T})$ to $\bigcap_{p>1} A_p(\mathbb{T})$ is
equivalent to the condition that \( \omega(\delta) \) decreases to zero slower than any power, i.e. to the condition

\[
\lim_{\delta \to +0} \frac{\omega(\delta)}{\delta^\varepsilon} = \infty \quad \text{for all } \varepsilon > 0.
\]

(60)

The necessity of this condition follows from Theorem 3. The sufficiency follows from Theorem 4 (condition (60) implies that \( \Theta_p(y) = O(1), \ y \to \infty \), for all \( p, \ 1 < p < 2 \)). Moreover, if condition (60) holds, then there is a nowhere linear \( C^{1, \omega} \)-diffeomorphism of the circle \( h \) such that the corresponding superposition operator is bounded from \( A(T) \) to \( A_p(T) \) for all \( p > 1 \).

Concerning this diffeomorphism we note that the superposition operator \( f \to f \circ h \), which it generates, is bounded from \( A_p(T) \) to \( A_{p+\varepsilon}(T) \) for any \( p, \ 1 \leq p < 2 \), and any \( \varepsilon > 0 \). Indeed, this superposition operator is a bounded operator from \( A_2(T) = L^2(T) \) to itself (every homeomorphism whose inverse satisfies the Lipschitz condition of order 1 generates a bounded superposition operator from \( L^2(T) \) to itself) and it remains to interpolate \( l^p \) between \( l^1 \) and \( l^2 \). At the same time we note, that, as we showed earlier jointly with Olevskii [7], every \( C^1 \)-smooth map of the circle into itself that for some \( p \neq 2 \) generates a bounded superposition operator from \( A_p \) to itself is linear.

In conclusion we recall one open problem that we posed in [9]: is there a non-trivial map \( \varphi : T \to T \) such that

\[
\|e^{int\varphi}\|_{A(T)} = O(1)
\]

where \( n_k, \ k = 1, 2, \ldots, \) is some (unbounded) sequence of integers. As the author showed [9], such a map can not be absolutely continuous. 

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3There is a minor inconsistency in the proof (see [9, Remark]). Everywhere in the proof one should replace integer \( n \) by real \( \lambda \).
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