Isomorphisms using Dehn fillings: the splitting case.

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Abstract

We reduce the isomorphism problem for torsion free relatively hyperbolic groups, to a collection of algorithmic problems on the parabolic subgroups. Using the arithmetic nature of nilpotent groups we solve these algorithmic problems and thus solve the isomorphism problem for torsion free relatively hyperbolic groups with nilpotent parabolic subgroups.

In previous work, Guirardel and the first author initiated a strategy for solving the isomorphism problem for some rigid relatively hyperbolic groups by analyzing sequences of Dehn fillings. We generalize this result to classes of relatively hyperbolic groups that are not rigid, i.e. that have elementary splittings. New difficulties arise since the parabolic subgroups, which are no longer forced to be virtually abelian, may have complex conjugacy and automorphic structures. A main novelty to tackle these difficulties is to use hyperbolic Dehn fillings and congruence subgroups of the outer automorphism groups of the parabolic subgroups.

Contents

1 Introduction 2
   1.1 Acknowledgement ................................................. 6
   1.2 Organization ................................................... 6

2 Setting 6
   2.1 Peripheral structures, markings, and automorphisms .................. 6
   2.2 Relatively hyperbolic groups ..................................... 7
   2.3 Finding parabolic subgroups ..................................... 8
   2.4 Graphs of groups and trees ..................................... 9
      2.4.1 Splittings .................................................... 9
      2.4.2 Vocabulary on trees ....................................... 10
      2.4.3 Splittings of relatively hyperbolic groups ................ 11
1 Introduction

In a series of works, the isomorphism problem for many hyperbolic groups and relatively hyperbolic groups has been tackled. Sela [Sel95] introduced an elegant strategy for solving it for torsion-free rigid hyperbolic groups, based on the solvability of equations in these groups, and a Bestvina-Paulin-Rips rigidity criterion relating the finiteness of certain sets of solutions and the absence of splittings. Bumagin, Kharlampovich and Miasnikov [BKM07] solved the problem for the class limit groups, a class of non-rigid groups. Groves and the first author [DG08b], following and simplifying Sela’s strategy, gave a solution to the isomorphism problem for (possibly non-rigid) toral relatively hyperbolic groups, i.e. relatively hyperbolic groups that are torsion free with abelian...
parabolic subgroups. This class of groups also contains all torsion-free hyperbolic groups.

Guirardel and the first author [DG11], again following Sela’s strategy, gave a solution to the isomorphism problem for all hyperbolic groups, and showed [DG] that rigid relatively hyperbolic groups with residually finite parabolics are characterized by their hyperbolic Dehn fillings. The isomorphism problem for this class of rigid relatively hyperbolic groups was therefore reduced to the isomorphism for hyperbolic groups with torsion.

Except for the case of abelian parabolic subgroups, the status of the isomorphism problem for non-rigid relatively hyperbolic groups, remained open. There are two difficulties that must be overcome. Firstly there is the computation of the (canonical) JSJ splitting. In all previous results, this was done using equationnal techniques. The undecidability of the $\exists$-theories of nilpotent groups [Rom79] is an obstruction to generalization. Instead we compute the JSJ decomposition using work of the second author [Tou09] and the Guirardel-Levitt characterizations of JSJ splittings [GL11, GL10]; the resulting algorithm has a satisfying simplicity.

Once we have computed canonical JSJ splittings, it would be natural to assume that the isomorphism problem reduces to the isomorphism problem for the vertex groups, the latter being either parabolic, rigid, or quadratically hanging. However, the problem of how the vertex groups are assembled into a graph of groups introduces new subtleties. We illustrate this with an example.

Let $G_a, G_b$ be non-isomorphic rigid groups, with trivial outer automorphism group, and hyperbolic relative to $P_a, P_b$ respectively. Now let $P$ be a group in which $P_a$ and $P_b$ each embed in two different ways via $i_k : P_a \to P, j_k : P_b \to P, k = 1, 2$. Let $\Gamma_1, \Gamma_2$ be the amalgams

$$\Gamma_k = G_a \ast_{P_a} P \ast_{P_b} G_b$$

for the attaching maps $i_k, j_k ; k = 1, 2$. These groups are relatively hyperbolic, by the Combination Theorem, and the splitting of $\Gamma_k$ that is exhibited is the canonical JSJ splitting of $(\Gamma_k, P)$. It follows that $(\Gamma_1, P)$ is isomorphic to $(\Gamma_2, P)$ if and only if there is an automorphism $\alpha$ of $P$ such that the maps $\alpha \circ i_1$ is conjugate in $P$ to $i_2$ (i.e. they differ by postcomposition by an inner automorphism), and $\alpha \circ j_1$ is conjugate in $P$ to $j_2$. If we fix a tuple of generators for the edge groups, deciding the existence of such an automorphism is an instance of the mixed Whitehead problem in $P$.

If we allow $(G_a, P_a)$ and $(G_b, P_b)$ to have non-trivial outer automorphisms, the situation is further complicated: $(\Gamma_1, P)$ is isomorphic to $(\Gamma_2, P)$ if and only if there are automorphisms $\beta_a \in \text{Aut}(G_a), \beta_b \in \text{Aut}(G_b)$ and $\alpha \in \text{Aut}(P)$ such that the following hold:

$$\alpha \circ i_1 \circ \beta_a \sim_P i_2 \quad (1)$$

$$\alpha \circ j_1 \circ \beta_b \sim_P j_2 \quad (2)$$

where $\sim_P$ denotes conjugacy in $P$. Note that a choice of $\alpha$, to satisfy (1), depends
on $\beta_a$, and a choice of $\beta_2$, to satisfy (2), depends on $\alpha$; all three automorphisms are interdependent.

There is some comfort to be found in the fact the outer automorphism groups of rigid relatively hyperbolic groups are finite. This, combined with the fact that maximal parabolic subgroups of relatively hyperbolic groups are self-normalized, enables us to reduce the problem to only finitely many different choices of $\beta_a, \beta_b$; thus reducing the problem to finitely many instances of the mixed Whitehead problem.

Of course, although there are only finitely many possibilities, we must still compute a list of automorphisms that is guaranteed to give a representative of each orbit

$$\{[i_1 \circ \beta_a] \mid \beta_a \in \text{Aut}(G_a, P_a)\},$$

where the brackets $[]$ represent $\sim_p$ conjugacy classes. To do so we do not compute $\text{Out}(G_a)$ (we still do not know how to do this); instead we introduce the property of congruences subgroups effectively separating torsion in the outer automorphism groups which enables us to use hyperbolic Dehn fillings to certify that we have a sufficient list of representatives in $\text{Aut}(G_a)$. The use of congruence subgroups for the parabolic subgroups and Dehn fillings to solve orbit problems, is one of the novel ideas in this paper. We now state our main technical result.

**Theorem 1.1.** Let $\mathcal{C}$ be a class of algorithmically tractable (see Definition 2.19) and effectively coherent groups, satisfying the following properties:

- $\mathcal{C}$ is closed for taking subgroups, and contains virtually cyclic groups
- all groups in $\mathcal{C}$ are residually finite,
- the isomorphism problem is explicitly solvable in $\mathcal{C}$,
- in $\mathcal{C}$, congruences effectively separate the torsion (see Definition 5.5),
- the mixed Whitehead problem is effectively solvable in $\mathcal{C}$ (see Definition 6.16.)

There is an algorithm which decides if two explicitly given torsion-free relatively hyperbolic groups $(G, P)$, $(H, Q)$ whose peripheral subgroups belongs to $\mathcal{C}$, are isomorphic as groups with unmarked peripheral structure.

The overall strategy for this algorithm presents some similarities with previous works. Given two groups, we first compute their Grushko decompositions, then we compute canonical JSJ decompositions of the freely indecomposable factors. The vertex groups of the JSJ decompositions fall into three categories: parabolic, rigid or quadratically hanging. In all three cases we can verify if the vertex groups of the JSJ decompositions are isomorphic. The next step is to see if the vertex groups assemble to give isomorphic graphs of groups; the difficulties of doing this were illustrated in the examples above. As hinted earlier, we use hyperbolic Dehn fillings in which enough of the outer automorphism group survives to enable us to solve our orbit problem. A statement that gives a good idea of our approach is the following:
Proposition 1.2. Let $G$ be a finitely generated group hyperbolic relative to $P < G$. Assume that $P$ is residually finite, and that congruences separate the torsion in $\text{Out}(P)$. Let $S$ be a generating tuple of $P$. Then, there exists a finite index characteristic subgroup $N \trianglelefteq_c P$, such that the quotient map to the Dehn filling $G \to \overline{G} = G/\langle \langle N \rangle \rangle$ induces an isomorphism

$$\text{Out}(G, [P])/\text{Out}(G, [S]) \to \text{Out}(\overline{G}, [\overline{P}])/\text{Out}(\overline{G}, [\overline{S}]).$$

The fact that congruences subgroups separate torsion in general linear groups is the classical, and the mixed Whitehead problem for finitely generated free abelian groups is just an exercise in basic linear algebra. Our approach therefore re-proves the following:

Theorem 1.3 ([DG08b, Theorem A]). The isomorphism problem is solvable for toral relatively hyperbolic groups.

It is not clear whether the new proof is shorter or simpler than the former one. In fact, it depends on the solution of the isomorphism problem for all hyperbolic groups [DG11], hence ultimately on the solvability of equations in virtually free groups [DG10b], so it can hardly be qualified as simple or short, however it is certainly different from the former proof in several key aspects.

Second, we tackle the next most natural class of parabolic subgroups of relatively hyperbolic groups, namely nilpotent groups. This class of groups (in fact, the class of virtually polycyclic groups) is already known to be algorithmically tractable and effectively coherent [BCRS91].

Theorem 1.4 (c.f. theorems 6.1,6.2). Finitely generated nilpotent groups have congruences that effectively separate the torsion in their outer automorphism group, and have solvable mixed Whitehead problem.

Which by the main result and Corollary 2.10 immediately implies:

Theorem 1.5. There is an algorithm which, given two presentations $\langle X \mid R \rangle$, $\langle Y \mid S \rangle$ of torsion-free groups that are hyperbolic relative to a family of finitely generated nilpotent groups, decides if the presentations yield isomorphic groups.

Although this is certainly a new result it remains somewhat unsatisfying: our algorithm only works for torsion free groups, and we remain tantalized, with legitimate reason, by virtually nilpotent groups. The reason we cannot deal with torsion boils down to the fact that the algorithm in [Tou09], which is used to find JSJ decompositions, does not work in the presence of torsion. As for Theorem 1.4, the arguments we found do not generalize to virtually nilpotent groups.

We find it very likely, however, that Theorem 1.1 can be made to work in the presence of torsion, and that Theorem 1.4 could be generalized to the class of virtually polycyclic groups. Throughout the paper we will point out where generalizations could be made.
1.1 Acknowledgement

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1.2 Organization

In Section 2 we gather much of the setting, give terminology, as well as some basic results. In Section 3, we discuss canonical splittings of relatively hyperbolic groups, following Guirardel and Levitt general treatment. We explain how to compute, using the machinery of [Tou09], and arguments developed in [DG11], a Grushko decomposition as well as a canonical JSJ splitting for relatively hyperbolic groups. At the end of each subsection we will discuss the effect of torsion. In Section 4, we give a general discussion of the isomorphism problem for graph-of-groups, its relations with orbit problems in the automorphism groups of vertex groups, and formulate a useful reduction for the relatively hyperbolic setting (this reduction somewhat differs from that in [DG11].) In Section 5, we explain a key feature of this paper, namely how to use congruence subgroups and Dehn fillings to solve some orbit problems given in the previous section. In Section 6, we prove Theorem 1.4. This last section is somewhat different from the rest of the paper because it uses classical techniques for nilpotent groups, profinite methods, and the arithmeticity of nilpotent groups and their automorphism groups.

2 Setting

2.1 Peripheral structures, markings, and automorphisms

Let $G$ be a group and let $P_1, \ldots, P_n$ be a collection of subgroups. An unmarked peripheral structure on $G$ is a finite disjoint union of conjugacy classes:

$$\mathcal{P} = [P_1] \sqcup \ldots \sqcup [P_n]$$

where $[P_i] = \{gP_ig^{-1} \mid g \in G\}$. It is also convenient to specify an unmarked peripheral structure as follows $\mathcal{P} = \{[P_1], \ldots, [P_n]\}$. A subgroup $H \leq \mathcal{P}$ is called peripheral if $H \in \mathcal{P}$.

An unmarked ordered peripheral structure $\mathcal{P}_{uo}$ on $G$ is a tuple $\mathcal{P}_{uo} = ([P_1], \ldots, [P_n])$, where $P_i$ and $P_j$ are not conjugate in $G$ if $i \neq j$. Any unmarked ordered peripheral structure produces an unmarked peripheral structure via:

$$([P_1], \ldots, [P_n]) \mapsto [P_1] \sqcup \ldots \sqcup [P_n].$$

Conversely, an unmarked peripheral structure $[P_1] \sqcup \ldots \sqcup [P_n]$ produces a choice of $n!$ unmarked ordered peripheral structures.
A marked peripheral structure $\mathcal{P}_m = (S_1, \ldots, S_n)$ is a tuple of tuples of $G$ (each $S_i$ is a tuple of $G$). The underlying unmarked ordered peripheral structure $\mathcal{P}_{uo}$ is

$$\mathcal{P}_{uo} = ([\langle S_1 \rangle], \ldots, [\langle S_n \rangle]).$$

We also say that $\mathcal{P}_m$ is a marking of $\mathcal{P}_{uo}$.

Given a group $G$ with an unmarked ordered peripheral structure $\mathcal{P}_{uo}$, the group $\text{Aut}(G, \mathcal{P}_{uo})$ is the group of automorphisms of $G$ that preserves the tuple $\mathcal{P}_{uo}$. In other words, it is the group of automorphisms that send each $P_i$ to a conjugate of itself in $G$. Let us emphasize that the conjugators can be different for each $P_i$. Inner automorphisms of $G$ form obviously a normal subgroup of this group, and $\text{Out}(G, \mathcal{P}_{uo})$ is defined as usual by quotienting $\text{Aut}(G, \mathcal{P}_{uo})$ by this subgroup.

Given a marked peripheral structure $\mathcal{P}_m = (S_1, \ldots, S_n)$ in $G$, the group $\text{Aut}(G, \mathcal{P}_m)$ consists of the automorphisms of $G$ sending the marking tuple $S_i$ to a conjugate of itself (again conjugators can be different for different indices $i$). Again $\text{Out}(G, \mathcal{P}_m)$ is defined as usual. If $\mathcal{P}_u$ is the unmarked ordered structure underlying $\mathcal{P}_m$, there is a natural inclusion $\text{Out}(G, \mathcal{P}_m) \hookrightarrow \text{Out}(G, \mathcal{P}_u)$ as a subgroup.

## 2.2 Relatively hyperbolic groups

Let us briefly recall a definition of horoballs in a hyperbolic space (we use the convention of [Hru10, §2]). In a Gromov-hyperbolic space $X$, a function $h : X \to \mathbb{R}$ is a horofunction about a point $\xi \in \partial X$ if there is a constant $D_0$ such that the following holds. Let $\Delta(x, y, \xi)$ be a geodesic triangle, and let $w \in X$ be a center of the triangle. Then $h(x) + d(x, w) - h(y) - d(y, w) < D_0$. A closed subset $H \subset X$ is a horoball centred at $\xi$ if there is a horofunction $h$ about $\xi$ and a constant $D_1$ such that $h(x) \geq -D_1$ for all $x \in H$ and $h(x) \leq D_1$ for all $x \in X \setminus H$.

**Definition 2.1.** A pair $(G, \mathcal{P})$ is a relatively hyperbolic group (or we say that $\mathcal{P}$ is a relatively hyperbolic structure on $G$) if $G$ is a finitely generated group and $\mathcal{P}$ is a peripheral structure on $G$ such that there exists a proper geodesic hyperbolic space $X$ on which $G$ acts properly discontinuously, freely, by isometries, and a collection $\mathcal{H}$ of disjoint horoballs in $X$ such that

- $G$ acts co-compactly on the complement of $\mathcal{H}$ in $X$.
- The map sending a horoball in $\mathcal{H}$ to its stabilizer, is a bijection $\mathcal{H} \to \mathcal{P}$.

In the literature, the weaker assumption of a proper action of $G$ on $X$ is more common. Requiring the stronger property of a free action actually yields an equivalent definition. This is because Groves and Manning in [GM08], by attaching combinatorial horoballs onto Cayley graphs, construct a space $X$ on which any relatively hyperbolic group $(G, \mathcal{P})$ acts freely which still satisfies all the necessary axioms. We will not elaborate on this further since our main results are about torsion-free groups; thus proper actions are necessarily free actions.
Definition 2.2. If \((G, P)\) is relatively hyperbolic any subgroup \(H \leq G\) that is contained in a peripheral subgroup is called parabolic.

In particular maximal parabolic subgroups are the peripheral subgroups for the structure \(P\). It is classical that, in a relatively hyperbolic group, there are at most finitely many conjugacy classes of peripheral subgroups. We will also often consider that the peripheral structure given is unmarked ordered, thus making an implicit choice.

Proposition 2.3. Let \((G, P)\) be a relatively hyperbolic group. Then, each infinite group in \(P\) is its own normalizer.

Proof. Each infinite group \(P \in P\) is the stabilizer of one and only one horoball. Its normalizer has to fix it as well, hence it is \(P\).

2.3 Finding parabolic subgroups

Definition 2.4. A relatively hyperbolic group \((G, P)\) is barely given if we are given a finite presentation of \(G\), it is explicitly given if we are given a finite presentation of \(G\) and a marking of its peripheral structure tuple, and it is strongly explicitly given if it is explicitly given and a finite presentation over the marking is given for each representative of peripheral subgroup.

A class of groups is said to be recursive if there exists an algorithm enumerating precisely the finite presentations of the groups in this class. The following result is useful for all theoretical algorithmic purposes, that has to be solved from a mere presentation of a relatively hyperbolic group and a solution to its word problem.

Theorem 2.5 ([DG10a, Theorem 3]). Let \(C\) be a recursive class of finitely presented groups.

There is an algorithm that, given a presentation of a group, and a solution to its word problem, terminates if and only if the group is relatively hyperbolic group with respect to subgroups in the class \(C\), and outputs a finite generating set and a finite presentation of each representative of maximal parabolic group, and an isoperimetry constant for the relative isoperimetric inequality.

Since the parabolic subgroups will be assumed to be residually finite we can always solve the word problem:

Lemma 2.6 (c.f. [DG, Corollary 6.4]). Let \((G, P)\) be a relatively hyperbolic group with \(P\) a class of residually finite groups, then the word problem is solvable in \(G\), and even if \(G\) is barely given, the algorithm is explicit.

Proof. It is enough to show that for every \(g \in G \setminus \{1\}\) there is terminating procedure that certifies that \(g\) is non-trivial. Let \(g\) be some word in \(G = \langle X \mid R \rangle\) not representing the identity. We enumerate \(\tilde{G}_1, \tilde{G}_2, \ldots\), the quotients of \(G\). \(G\) is residually hyperbolic
by the Dehn filling theorem [Osi07, GM08] (see Theorem 5.2 below), so some $\bar{g}\in G_i$ will be hyperbolic and the image $\bar{g}$ will be non-trivial.

There is a procedure [Pap96] of Papasoglu's which given any finite group presentation $\langle Y \mid S \rangle$ will terminate if the group is hyperbolic and outputs a solution to its word problem. It follows that if, in parallel, we apply Papasoglu's procedure to each quotient we will eventually produce the required certificate.

In other words, for all theoretical purposes (under the suitable assumptions), from a barely given relatively hyperbolic groups we can get a strongly explicitly presentation. That being said, these explicitly given presentation may not be canonical. We therefore need the following extra results:

**Lemma 2.7** (Corollary of [Osi06b, Lemma 5.4]). Let $(G, P)$ be relatively hyperbolic then every subgroup $P \in P$ is undistorted in $G$.

**Theorem 2.8** ([DS05, Theorem 1.8]). Let $G = \langle S \rangle$ be a finitely generated group that is hyperbolic relative to subgroups $H_1, \ldots, H_n$. Let $G'$ be an undistorted finitely generated subgroup of $G$. Then $G'$ is relatively hyperbolic with respect to subgroups $H'_1, \ldots, H'_m$, where each $H'_i$ is one of the intersections $G' \cap gH_jg^{-1}$.

**Corollary 2.9.** Let $G$ be equipped with two relatively hyperbolic structures $\mathcal{P}, \mathcal{Q}$ that both consist of non-virtually cyclic nilpotent groups, then $\mathcal{P} = \mathcal{Q}$.

**Proof.** By Lemma 2.7 and Theorem 2.8 any $Q \in \mathcal{Q}$ must be hyperbolic with respect to its intersection with the groups in $\mathcal{P}$. Since nilpotent groups can only have the trivial relatively hyperbolic structure, $Q$ must be contained in a group in $\mathcal{P}$. By symmetry we get $\mathcal{Q} = \mathcal{P}$. □

In fact the same argument goes through for any class of so-called *not relatively hyperbolic* groups (c.f. [DS05]). Since we can recognize virtually cyclic groups, and nilpotent groups are residually finite and recursively enumerable, we immediately get:

**Corollary 2.10.** Let $\langle X \mid R \rangle$ be a presentation of a group $G$ that is known to have a relatively hyperbolic structure $\mathcal{P}$ consisting of non-virtually cyclic nilpotent groups, then $\mathcal{P}$ can be found effectively and explicitly.

### 2.4 Graphs of groups and trees

#### 2.4.1 Splittings

**Definition 2.11.** A graph $X$ consists of a set $V(X)$ of vertices and a set $E(X)$ of oriented edges, with a fix-point free involution $\bar{\cdot}: E(X) \to E(X)$ and incidence maps $t: E(X) \to V(X), o: E(X) \to V(X)$ that satisfy $o(e) = t(\bar{e})$. If $e$ is an edge, $t(e)$ is called its terminal vertex, and $o(e)$ is called its origin vertex. Given $v$ the set of edges $e$ such that $t(e) = v$ is denoted by $\text{lk}(v)$.
• A graph of groups structure $\mathbb{X}$ on a graph $X$ is the data of a group $\Gamma_v$ for each vertex $v$, and a group $\Gamma_e$ for each edge $e$ such that $\Gamma_e = \Gamma_{\bar{e}}$, and of a monomorphism $i_e : \Gamma_e \to \Gamma_{t(e)}$ for each edge $e$. We write $\mathbb{X} = (X, \{\Gamma_v, v \in V(X)\}, \{\Gamma_e, e \in E(X)\}, \{i_e, e \in E(X)\})$.

• The Bass group is the free product of the vertex groups and of the free groups on $E(X)$ subject to relations $\bar{e} = e^{-1}, \forall e \in E(X)$ and $ei_e(g)e^{-1} = i_{\bar{e}}(g)$.

• The fundamental group $\pi_1(\mathbb{X}, \tau)$ of a graph of groups $\mathbb{X}$ with maximal subtree $\tau \subset X$, is the quotient of the Bass group by the normal subgroup generated by the set of edges in $\tau$.

Classically, the groups $\Gamma_v, v \in V(X)$ are called vertex groups, the groups $\Gamma_e, e \in E(X)$ are edge groups, and the maps $i_e, e \in E(X)$ are the attaching maps. For convenience, for every vertex group $\Gamma_v$, we introduce the (unmarked) adjacency peripheral structure $(\Gamma_v, A_u)$, consisting of the conjugacy classes of the groups $i_e(\Gamma_e), e \in \text{lk}(v)$. For a choice of a generating tuple $S_e = S_{\bar{e}}$ of $\Gamma_e = \Gamma_{\bar{e}}$, we can define the marked peripheral structure $(\Gamma_v, A_m)$ induced by $i_e$.

We say that a group is explicitly given as a fundamental group of a graph of groups if a graph of groups is given, and a presentation of all vertex groups, a generating set of all edge groups, the image of the attaching maps on these generating sets, for which the group is isomorphic to the fundamental group of this graph of groups (once chosen a maximal subtree).

There is a universal covering $T_X$ of $\mathbb{X}$, a tree, called the Bass-Serre tree, on which the fundamental group $\pi_1(\mathbb{X}, \tau)$ acts, and $X \simeq \pi_1(\mathbb{X}, \tau) \backslash T_X$. Conversely, given only the abstract group $\pi_1(\mathbb{X}, \tau)$ and the action on $T_X$ we can recover the graph of groups $\mathbb{X}$. We refer to [Ser03] for further details, but we assume the reader is fluent in Bass-Serre theory.

### 2.4.2 Vocabulary on trees

We now define or recall a few adjectives and operations related to trees. A $G$-tree is reduced if any two adjacent vertices in different $G$-orbits have different stabilizers. A vertex of a $G$-tree is inessential if it has valence 1.

Given a group with a peripheral structure $(G, P)$, a $(G, P)$-tree is a tree with an action of $G$ such that each group in $P$ is elliptic. A splitting of $(G, P)$ is a splitting of $G$ such that the Bass-Serre tree is a $(G, P)$-tree. A $(G, P)$-tree is minimal if there is no proper invariant subtree. $(G, P)$ is said to be freely indecomposable if all $(G, P)$-tree with trivial edge stabilizers have a $G$-fixed point. We sometimes redundantly say relatively freely indecomposable to emphasise the role of the peripheral structure and avoid ambiguity.

Let $T$ be a $(G, P)$-tree. A collapse of $T$ is a $(G, P)$-tree $S$ with a $G$-equivariant map $T \to S$ such that each edge of $T$ is sent on an edge, or on a vertex of $S$, and no pair of edges of $T$ in different orbits are sent in the same edge (they may be sent on the same vertex though). A refinement of $T$ is a $(G, P)$-tree $S$ such that $T$ is a collapse of $S$. 
The following is well known but useful.

**Lemma 2.12.** If $T$ is a $G$-tree and $\hat{T}$ is a refinement of $G$, then any edge stabilizer in $T$ stabilises an edge in $\hat{T}$.

**Proof.** Consider $e$ an edge in $T$, $v, w$ its vertices, and $G_e$ its stabilizer. Let $T'_v, T'_w$ the preimages of $v, w$ by the collapse map $\hat{T} \to T$. These are trees, obviously disjoint, hence, let $\sigma$ be the segment joining them in $\hat{T}$. The collapse map sends $\sigma$ on $e$, hence, $G_e$ stabilizes $\sigma$ setwise, and also its ends points, since they have their $G_e$-orbit are respectively in $T'_v \cap \sigma$ and $T'_w \cap \sigma$ (which are singletons). Since $\sigma$ is a segment, all its edges are fixed by $G_e$. \qed

Two $(G, \mathcal{P})$-trees are *compatible* if they are collapses of a single $(G, \mathcal{P})$-tree (or equivalently if they have a common refinement). We say that a $(G, \mathcal{P})$-tree $T$ *dominates* a $(G, \mathcal{P})$-tree $T'$ if there is an equivariant map $T \to T'$. An equivalence class for the relation of mutual domination is a *deformation space*.

Let $E$ be a family of subgroups of $G$; we say that a $(G, \mathcal{P})$-tree is *over* $E$ if all edge stabilizers are in $E$. A $(G, \mathcal{P})$-tree over $E$ is *universally compatible* over $E$ relative to $\mathcal{P}$ if it is compatible with any other $(G, \mathcal{P})$-tree over $E$. In other words, such a tree $T$ is universally compatible if for any other $(G, \mathcal{P})$-tree $S$ over $E$, there exists a $(G, \mathcal{P})$-tree over $E$, denoted by $\hat{T}$, and (equivariant) collapse maps $\hat{T} \to T$ and $\hat{T} \to S$.

### 2.4.3 Splittings of relatively hyperbolic groups

We are now interested in splittings of relatively hyperbolic groups. Let $(G, \mathcal{P})$ be relatively hyperbolic, the class $\mathcal{E}$ of *elementary subgroups* of $(G, \mathcal{P})$ is the collection of virtually cyclic and parabolic subgroups. An *elementary* $(G, \mathcal{P})$-tree is a $(G, \mathcal{P})$-tree over $\mathcal{E}$.

Following [Bow01], we say that a $(G, \mathcal{P})$-tree (or a $(G, \mathcal{P})$-splitting) is *peripheral* if it is a bipartite with white and black vertices, such that the collection of black vertex stabilizers is $\mathcal{P}$. Every peripheral tree $S$ is a $(G, \mathcal{P})$-tree; conversely if a $(G, \mathcal{P})$-tree $T$ is obtained by collapsing edges adjacent to inessential (i.e. valence 1) vertices of a peripheral tree $S$, then $T$ is said to be *essentially peripheral* and $S$ is said to be a *peripheral refinement* of $T$; they both lie in the same deformation space. Peripheral splittings of relatively hyperbolic groups are elementary, and we have the following characterization.

**Lemma 2.13.** A splitting $X$ of $(G, \mathcal{P})$ is essentially peripheral if and only if it is bipartite, all black vertex groups being in the collection $\mathcal{P}$, and no white vertex groups are parabolic.

**Definition 2.14.** Let $T$ be an essentially peripheral $(G, \mathcal{P})$-tree, with the convention that black vertex stabilizers are the groups of $\mathcal{P}$. Let $w$ be a white vertex and $G_w$ its stabilizer in $G$. Then the *induced peripheral structure on* $G_w$ is

$$\mathcal{P}_T(G_w) = \{ P \cap G_w \mid P \in \mathcal{P}_T \}.$$
Note that this provides a well defined induced peripheral structure on a vertex group of a splitting. It may help to be more explicit, as in the following.

**Lemma 2.15.** Let \((G, \mathcal{P})\) be relatively hyperbolic. Let \(w\) a white vertex of a peripheral \((G, \mathcal{P})\)-tree, and \(\text{lk}(w)\) its link (the set of edges adjacent to \(w\)). The induced peripheral structure of the vertex stabilizer \(G_w\) is \(\mathcal{P}_T(G_w) = \{ P \in \mathcal{P}, P \lhd G_w \} \cup \{ G_e, e \in \text{lk}(w) \} \).

**Proof.** All these groups are clearly in \(\mathcal{P}_T(G_w)\), since the vertex at the other end of each \(e \in \text{lk}(w)\) is black. If \(P \in \mathcal{P}\), then it fixes a black vertex \(b\), and \(P \cap G_w\) stabilizes the first edge \(e\) of the segment \([w, b]\). Let \(b'\) be the other vertex of \(e\). Then \(G_{b'} \cap G_b\) is infinite, which, in a relatively hyperbolic group implies \(G_b = G_{b'}\) (both groups are peripheral).

Bowditch proved:

**Theorem 2.16** (First clause of [Bow01, Theorem 1.3]). Suppose that \((G, \mathcal{P})\) is a relatively hyperbolic group. Suppose \(\Gamma_v\) is some non-peripheral vertex group of some peripheral splitting \(\mathcal{X}\) of \(G\). Then \(\Gamma_v\) is hyperbolic relative the induced peripheral structure \(\mathcal{P}_X(\Gamma_v)\).

The reader can check that the same conclusion holds if \(\mathcal{X}\) is essentially peripheral instead of peripheral, by looking at the peripheral refinement. An elementary splitting of \((G, \mathcal{P})\) is not essentially peripheral, for example maybe an edge group is non-parabolic virtually cyclic. We therefore find it convenient to define the following augmented peripheral structure.

**Definition 2.17.** Let \(T\) be a bipartite \((G, \mathcal{P})\)-tree (with black and white vertices), and let \(\mathcal{B}_T\) denote the set of stabilizers in \(G\) of the black vertices of \(T\). We say that the augmented peripheral structure of \((G, \mathcal{P})\) induced by \(T\) is \(\mathcal{P}_T = \mathcal{P} \cup \mathcal{B}_T\). Let \(\mathcal{X}\) be a bipartite splitting (with black and white vertices) of \((G, \mathcal{P})\), the augmented peripheral structure \(\mathcal{P}_X\) of \((G, \mathcal{P})\) induced by \(\mathcal{X}\) is that induced by the Bass-Serre tree of \(T_X\).

If we are only increasing our peripheral structure to contain virtually cyclic groups, then the augmented peripheral structure is, yet again, relatively hyperbolic:

**Proposition 2.18.** Let \((G, \mathcal{P})\) be a relatively hyperbolic group. Let \(T\) be a bipartite elementary \((G, \mathcal{P})\)-tree with maximal elementary black vertices. Let \(G_v\) be a vertex stabilizer. Then \(G_v\) is hyperbolic relative to \(\mathcal{P}_T(G_v)\).

**Proof.** First, note that \(G\) itself is hyperbolic relative to \(\mathcal{P}_T\), the collection of maximal elementary subgroups stabilizing some edge, since it is obtained from \(\mathcal{P}\) by adding maximal virtually cyclic edge groups by [Osi06a, Corollary 1.7]. The result now follows from Theorem 2.16.

**Definition 2.19** (c.f. [Tou09]). A class of groups \(\mathcal{C}\) is algorithmically tractable if there is an algorithm enumerating triples of the form \((\langle S, R \rangle, \mathcal{C}, \text{Gen})\), where \(\langle S, R \rangle\)
ranges over finite presentations of elements of \( C \), and where \( CP \) and \( Gen \) are respectively solutions (Turing machines) to the conjugacy problem, and the generation problem over the presentation \( \langle S, R \rangle \).

A relatively hyperbolic group is **rigid** if there is no non-trivial, reduced elementary \((G, \mathcal{P})\)-splitting.

**Theorem 2.20** ([Tou09, Theorem C]). Let \( \mathcal{C} \) be an algorithmically tractable class of groups. There is an algorithm which takes as input an explicitly given torsion-free relatively hyperbolic group \((G, \mathcal{P})\), where groups in \( \mathcal{P} \) belong to \( \mathcal{C} \). The algorithm terminates and correctly states whether \((G, \mathcal{P})\) is rigid or not.

3 Canonical splittings of relatively hyperbolic groups

In this section we recall definitions and key properties of Grushko and JSJ decompositions, and we show how to compute such canonical splittings of relatively hyperbolic groups.

In comparison with previous works, [DG08b, DG11], we cannot rely on the so-called rigidity criterion in order to guaranty that our group has no splitting of a specific kind (or in other words, in order to compute a maximal splitting of a specific kind). Instead of this criterion, we use a machinery developed by the second author.

3.1 Grushko decompositions, and a look toward Dunwoody-Stallings decompositions

**Theorem 3.1** (Grushko’s Theorem). Let \( G \) be a finitely generated group with \( \mathcal{P} = \{P_1, \ldots, P_n\} \) a collection of subgroups then there exists a free decomposition called a Grushko decomposition relative to \( \mathcal{P} \):

\[
G = H_1 \ast \ldots \ast H_p \ast \mathbb{F}_q
\]

where \( \mathbb{F}_q \) is a free group of rank \( q \), each \( P_k \in \mathcal{P} \) is conjugate into some \( H_i \), and each \( H_i \) is freely indecomposable relative to \( \mathcal{P} \). Moreover this decomposition is canonical in the sense that if

\[
G = K_1 \ast \ldots \ast K_p' \ast \mathbb{F}_{q'}
\]

is another decomposition where \( \mathbb{F}_{q'} \) is a free group of rank \( q' \), each \( P_k \in \mathcal{P} \) is conjugate into some \( K_j \), and each \( K_j \) is freely indecomposable relative to \( \mathcal{P} \), then the Scott complexities \((p, q) = (p', q')\) are equal and each \( K_j \) is conjugate to some \( H_i \).

**Proposition 3.2** (Computing the Grushko decomposition). There is an algorithm that, provided with a finite presentation of a torsion free relatively hyperbolic group
(G, {[P_1], \ldots, [P_n]}) with residually finite parabolic subgroups \(P_1, \ldots, P_n\), produces an explicit relative Grushko free-product decomposition of \(G\) in which a factor is free (possibly trivial), and all other factors are freely indecomposable relative to the \(P_i\)'s, and relatively hyperbolic with respect to some explicit conjugates of some \(P_i\)'s.

**Proof.** By Lemma 2.6 since \(P_1, \ldots, P_n\) are residually finite, we have an explicit solution to the word problem of \(G\). The result now follows immediately by using [Tou09, Theorem B] with \(\kappa = 0\) and \(\mathcal{H} = \{P_1, \ldots, P_n\}\) and then following the proof of [Tou09, Theorem A].

**Proposition 3.3** (Reduction to the freely indecomposable case). If there exists an algorithm solving the isomorphism problem in the class of torsion free relatively hyperbolic groups with parabolics in \(\mathcal{C}\) that are freely indecomposable relative to \(\mathcal{P}\), then there exists an algorithm solving the isomorphism problem in the class of all torsion free relatively hyperbolic groups with parabolics in \(\mathcal{C}\). If the first algorithm is explicitly given, the second is also explicit.

**Proof.** By Theorem 3.1 two groups \(G, H\) are isomorphic if and only if they have the same Scott complexities and if there is a bijection between the freely indecomposable factors of the relative Grushko decompositions which sends each free factor to an isomorphic copy. It follows that if the Grushko decompositions are computable then the isomorphism problem reduces to finitely many instances of the isomorphism problem for freely indecomposable relatively hyperbolic groups. The result now follows by Proposition 3.2.

**Remark 3.4.** About torsion. Because of its reliance on [Tou09], Proposition 3.2 will not work in the presence of 2-torsion. However if we were to find a better algorithm, we would be able to handle groups with torsion, and obtain a generalization of Proposition 3.3.

The proof, however, would still be considerably complicated because in the presence of torsion we shouldn’t consider the Grushko decompositions, but Dunwoody-Stallings decompositions: maximal decompositions as fundamental groups of finite graphs-of-groups with finite edge groups. The vertex groups are necessarily (relatively) one-ended. Dunwoody accessibility ensures the existence of such decompositions for finitely presented groups, but unicity of such splittings is not ensured. The analogue of Proposition 3.3 would be a reduction of the isomorphism problem for relatively hyperbolic groups with parabolics in a suitable class \(\mathcal{C}\) to the isomorphism problem in the class of relatively one-ended relatively hyperbolic groups with parabolics in \(\mathcal{C}\).

We will now give a roadmap of how such a proof could go. First of all, the argument of [DG08a] can be used to detect if a relatively hyperbolic has infinitely many (relative) ends, provided the parabolic groups satisfy the technical conclusion of [DG08a, Lemma 2.15]. Specifically they must have a computable generating set and there must be a computable constant \(M\) such that, for any \(r > 0\), and elements \(a,b\) at distance \(\geq r\) from 1, there is a path of length \(\leq Md(a,b)\) from \(a\) to \(b\) avoiding the ball of
radius $r-1$ centered at 1. This is actually a sufficient condition in order to get [DG08a, Lemma 2.16], which is the only place where some specificity of parabolic subgroups is required. Many groups, including finitely generated nilpotent groups (non virtually cyclic), have linear divergence $\text{div}_2(n, 1/2)$ [DMS10, Definition 3.3, Proposition 1.1]. We found that it is possible to adapt the proof of [DG08a, Lemma 2.16] to the case of parabolic groups of linear divergence $\text{div}_2(n, 1/2)$.

Secondly, the argument of [DG11, §7] is generalizable, provided the marked isomorphism problem is solvable for relatively one-ended relatively hyperbolic groups with parabolic subgroups in a suitable class $C$. To fully explain these ideas would bring us far astray of the primary goal of this paper, but would be a good exercise for an interested reader.

### 3.2 JSJ decompositions

We now turn our attention to torsion-free relatively freely indecomposable relatively hyperbolic groups. Group theoretical JSJ theory (as initiated by Rips and Sela [RS97]) gives a complete description of their elementary splittings. We start by presenting some key points of this theory, following Guirardel and Levitt’s treatment. May the non-expert forgive the following cryptic sentence: the canonical JSJ decomposition will be the (collapsed) tree of cylinders for co-elementarity of the compatibility JSJ deformation space, as defined in [GL10]. We will provide a new characterization, Theorem 3.9, of this splitting that is particularly well-suited for computations.

We follow [GL11, GL10]; recall the terminology introduced in Sections 2.4.2 and 2.4.3. Let $(G, \mathcal{P})$ be a relatively freely indecomposable, relatively hyperbolic group, and let $\mathcal{E}$ be the class of its elementary subgroups. The tree of cylinders for co-elementarity $T_c$ of an elementary $(G, \mathcal{P})$-tree $T$ is constructed as follows ([GL11, Definition 4.3]).

Define an equivalence relation on the set of edges of $T$: two edges are equivalent if their stabilizers are subgroups of the same maximal elementary subgroup. The equivalence classes are the cylinders $\text{Cyl}(T)$ of $T$. The set of vertices of $T_c$ is the union of $V_1 = \text{Cyl}(T)$ and $V_2 \subset V(T)$ where $V_1$ is the set of cylinders, and $V_2$ is the set of vertices of $T$ in at least two cylinders. The set of edges of $T_c$ is given by the membership relation if a vertex in $w \in V_2$ is contained in a cylinder in $c \in V_1$, then we connect the vertices $w$ and $c$ of $T_c$ by an edge; thus $T_c$ is bipartite. The group $G$ acts naturally on $T_c$. The stabilizer of a vertex in $V_1$ is the global stabilizer of the cylinder. By [GL11, Proposition 6.1] the collapsed tree of cylinder for an elementary $(G, \mathcal{P})$-tree coincides with the tree of cylinders $T_c$ (the former tree is mentioned in some results we quote.)

**Lemma 3.5.** If $(G, \mathcal{P})$ is a relatively hyperbolic group, the stabilizer of a vertex in $V_1$ is the unique maximal elementary group containing the edge groups of the edges contained in the corresponding cylinder.

**Proof.** It is clear for non parabolic elementary edges, since the maximal elementary group is then virtually cyclic. In the case of a parabolic cylinder stabilizer $P_c$, let $P$
be the maximal parabolic group containing $P_c$. It is elliptic in $T$ by assumption, hence fixes a vertex $v$, which necessarily is in the cylinder $c$ of $P_c$, since otherwise $P_c$ fixes the segment jointing $v$ to $c$, forcing it to belong to the cylinder. If $P$ does not preserve $c$, up to changing the choice of $v$, it sends an edge $e$ of $c$ adjacent to $v$ to an edge $e'$ adjacent to $v$ outside $c$. But the stabilizer of $e'$ is a conjugate of that of $e$ by an element of $P$, hence both are subgroups of $P$, and coelementary. It follows that $e'$ is in $c$.

Recall that, by [GL10, Theorem 4.8] there exists a $(G, P)$-tree that is universally compatible over $E$ relative to $P$, and maximal for domination. We consider $T_3$ its tree of cylinders for co-elementarity. We call this tree (and the associated splitting) the canonical elementary JSJ tree (and splitting, or decomposition) of $(G, P)$ (but we often drop the word elementary). We record observations and results from [GL10] in the following statement.

**Proposition 3.6** (c.f. [GL10, Theorem 13.1, Corollary 13.2]). Let $(G, P)$ be a torsion free relatively freely indecomposable relatively hyperbolic group. The canonical JSJ decomposition $J$ of $(G, P)$ is an elementary splitting satisfying the following properties:

1. The underlying graph $X$ is bipartite with black and white vertices.
   (i) The black vertex groups $G_b$ are maximal elementary vertex groups.
   (ii) The white vertex groups $G_w$ fall into one of the following two categories:
     (a) $(G_w, P_j)$ is isomorphic to a (QH) group $(Q, P_Q)$, that is the fundamental group of a surface with boundary, with the peripheral structure of the boundary groups.
     (b) $(G_w, P_j)$ is rigid (in the sense previously defined that there is no non-trivial, reduced elementary $(G_w, P_j)$-splitting) and not isomorphic to a (QH) group, and for any two different adjacent edges, the image of the edge groups in $G_w$ are not conjugated into the same maximal elementary subgroup.

2. $J$ is universally compatible over $E$ relative to $P$.

3. Every edge stabilizer in $T_3$ is elliptic in any $(G, P)$-tree over $E$.

4. $(G_w, P_j)$ is essential, i.e. there are no valence-1 vertices in $T_3$.

5. $J$ is canonical for $(G, P)$.

**Proof.** By definition, $T_3$ is bipartite, and by Lemma 3.5, up to choice of colouring, the stabilizers of black vertex groups are maximal elementary. This proves (1i). The description of the white vertex groups (1ii) follows from [GL10, Corollary 13.2], and the rigidity of the of the non-QH white vertex groups is a consequence of the maximality for domination (for the universally compatible trees) of $T_3$. (2) follows from [GL10, Corollary 9.5].

By Lemma 2.12, every edge group in $J$ will remain elliptic after a refinement and, of course, also after a collapse. So, by universal compatibility, (3) follows. (4) is an
immediate consequence of requiring the action of $G$ on $T_J$ to be minimal, and (5) follows by construction (see the abstract of [GL11].)

**Definition 3.7.** An edge $e$ of a $G$-tree $T$ is called inessential if its stabilizer equals that of one of its vertices.

**Lemma 3.8 ((QH) groups are maximal).** Assume that, in $T$, $b$ is a black vertex of valence 2, whose neighbors both carry (QH) groups. Then at least one edge is essential.

**Proof.** Suppose towards a contradiction that both are inessential, and let us call $v$ and $w$ the white neighbours of $b$. We collapse one of the two edges (say the edge $(v, b)$) into a vertex $v'$, and call the remaining edge $e$, which is an edge between (QH) vertices $v', w$. Now we collapse the remaining edge $e$ into a vertex $z$. If $v' \neq w$, we are amalgamating two surfaces along a boundary component, thus yielding a surface group $(Q, P_Q)$, in which a (non-peripheral) simple closed curve $\gamma$ carries the conjugacy class of the group of $e$. If $v' = w$, we are making an HNN extension of a surface group over two boundary components. These boundary components are distinct, because otherwise, the HNN-extension of the boundary component group produces a rank 2 abelian group that is hyperbolic in $T_J$, which is prohibited. Thus this also yields a surface group $(Q, P_Q)$, in which a (non-peripheral) simple closed curve $\gamma$ carries the conjugacy class of the group of $e$.

In both cases, we can find a splitting of $(Q, P_Q)$ dual to a simple closed curve crossing $\gamma$. It follows that there is a splitting of $(G, P)$ in which $G_e$ is not elliptic contradicting (3) of Proposition 3.6.

**Theorem 3.9.** Let $(G, P)$ be a torsion free, relatively freely indecomposable, relatively hyperbolic group. The canonical JSJ splitting $T_J$ of $(G, P)$ is the unique essential splitting that satisfies (1) of Proposition 3.6 and for which, whenever $b$ is a black vertex of valence 2, whose neighbours both carry (QH) groups, at least one edge is essential.

The proof of this Theorem follows from the next two lemmas.

**Lemma 3.10.** If $T$ is a $(G, P)$-tree satisfying (1) of Proposition 3.6 and is such that whenever $b$ is a black vertex of valence 2, whose neighbors both carry (QH) groups, at least one edge is essential, then $T$ is in the same deformation space than $T_J$.

**Proof.** By [For02] it suffices to show that any vertex stabilizer of $T$ is elliptic in $T_J$, and that any vertex stabilizer of $T_J$ is elliptic in $T$. Actually, we will make a symmetric argument. We first use the universal compatibility of $T_J$: there exists a $(G, P)$-tree $\hat{T}$ collapsing on $T_J$ and on $T$. Now we are in the following setting: given two $(G, P)$-trees $T_1, T_2$ satisfying the assumption of the Lemma, and refined by a same tree $\hat{T}$, we must prove that vertex stabilizers in $T_1$ are elliptic in $T_2$.

Let $v$ be a vertex of $T_1$, and $G_v$ its stabiliser. Assume that $v$ is a white vertex of rigid type (in the sense of (1 (ii) (b)) of Proposition 3.6). In $\hat{T}$, the augmented peripheral
structure of $G_v$ is elliptic by Lemma 2.12. By rigidity, $G_v$ is thus elliptic in $\hat{T}$. It is therefore elliptic in $T_2$ as well, since $T_2$ is a collapse of $\hat{T}$.

Assume that $v$ is a white vertex of $(QH)$ type. The action of $G_v$ on its minimal subtree $\hat{T}_v$ in $\hat{T}$ is dual to a collection of separate, non-boundary simple curves in the underlying surface. Consider the collapse $\hat{T} \to T_2$ restricted to $\hat{T}_v$. Assume two distinct vertices with non-elementary stabiliser are mapped on distinct vertices $v_-, v_+$ (which are necessarily white). In $[v_-, v_+]$, we choose two white vertices at distance 2 apart, $v'_-, v'_+$, and denote by $T'_-, T'_+$ their preimages (obviously disjoint) in $\hat{T}$. The segment $\sigma$ between $T'_-$ and $T'_+$ is mapped to the segment $[v'_-, v'_+]$ in $T_2$, thus ensuring that it contains only vertices with elementary stabilisers. Since $\hat{T}_v$ corresponds to the pull back of a splitting of the underlying surface of $G_v$, this ensures that all edges of $\sigma$ are inessential, and therefore, both edges of $[v_-, v_+]$ are inessential. This contradicts our assumption on $T$. We have proved that all white vertex stabilisers in $T_1$ are elliptic in $T_2$.

Let $v$ be a black vertex. If its stabiliser is in $P$, by definition, it is elliptic in the $(G, P)$-tree $T_2$. If it is cyclic, it is virtually a subgroup of a white vertex stabiliser, hence elliptic in $T_2$ by our previous study. We have proved our claim, that all vertex stabilisers in $T_1$ are elliptic in $T_2$, hence the lemma.

Lemma 3.11. Under the assumptions of Lemma 3.10, the tree $T$ is its own tree of cylinders for co-elementarity. More precisely, let $(G, P)$ be a relatively hyperbolic group, and $T$ be a $(G, P)$ tree with infinite edge stabilizers. The following are equivalent.

(i) The tree $T$ is isomorphic to its tree of cylinders for co-elementarity.

(ii) $T$ is bipartite with all black vertex stabilizer being maximal elementary, black vertices in different orbits have non-conjugated stabilizers, all white vertex stabilizer being non-elementary.

(iii) $(G, P_T)$ is relatively hyperbolic, and $T$ is essentially peripheral for $P_T$, with non-elementary white vertex stabilizers, and black vertices in different orbits have non-conjugated stabilizers.

Proof. Since $T$ satisfies (ii), by the assumption (1(ii)b) of Proposition 3.6, we only need to prove the stated equivalence. The vertices in $V_2$ in the tree of cylinders are the vertices of $T$ belonging to at least two different cylinders, so their stabilizers are not elementary.

Lemma 3.5 ensures that vertices in $V_1$ have maximal elementary stabilizers. If two vertices in $V_1$ have the same stabilizer, they correspond to the same cylinder, hence are equal. This shows that (i) $\implies$ (ii). For (ii) $\implies$ (iii), note that $P_T$ is obtained from $P$ by adding maximal cyclic groups that are not parabolic. The relative hyperbolicity is thus preserved (see [Osi06a, Corollary 1.7], or [Dah03, Lemma 4.4]). The tree is essentially peripheral for $P_T$ by Lemma 2.13.

Finally, let us prove (iii) $\implies$ (i). It suffices to show that the cylinders of $T$ are exactly the stars of the black vertices. Consider two different edges adjacent to
a white vertex \( v_w \), and assume their stabilizers co-elementary. Their black vertices \( v_{b1}, v_{b2} \) have co-elementary stabilizers \( E_1, E_2 \). These groups are in \( \mathcal{PT} \) because \( T \) is essentially peripheral. But since \((G, \mathcal{PT})\) is relatively hyperbolic, these groups must be equal. The two edges are therefore in the same orbit and images of one another by an element \( g \) normalizing \( E = E_1 = E_2 \), therefore in \( E \) by relative hyperbolicity, since it stabilizes the unique horoball stabilized by \( E \) in the space of Definition 2.1. We get that \( g \) fixes \( v_{b1} \), hence \( v_{b1} = v_{b2} \) and both edges are equal. The claim follows.

Proof of Theorem 3.9. \( T_J \) was defined to be the tree of cylinders of a certain universally compatible tree over \( \mathcal{E} \) relative to \( \mathcal{P} \). Since the tree of cylinders of a deformation space is unique [GL11, Corollary 4.10], it follows that \( T_J \) is isomorphic to any \((G, \mathcal{P})\)-tree that is in the same deformation space and that is its own tree of cylinder. By our two previous lemmas, it is the case of any tree \( T \) satisfying the stated assumption.

**Theorem 3.12.** Let \((G, \mathcal{P})\) be relatively hyperbolic, and torsion free. If we are given

1. an algorithm to decide whether vertex groups of an elementary splitting of a relatively hyperbolic \((G, \mathcal{P})\) are rigid,
2. an algorithm which certifies whether a subgroup is maximal elementary, and
3. an algorithm which given the generating set of an elementary subgroup will produce a finite presentation (effective coherence);
then there is an algorithm to find the canonical JSJ decomposition of \((G, \mathcal{P})\).

**Proof.** The algorithm we propose is a branching process, we run subprocesses in parallel.

1. Enumerate all the presentations of \( G \).
2. For each presentation, check whether it is a graph-of-group presentation (by enumerating the partitions of the set of generators, and checking whether the presentation corresponds to presenting vertex groups, edge groups, attaching maps, and Bass relations, with respect to this partition).
3. For each graph of groups decomposition that is found try to certify that it is elementary (by enumerating the conjugates of the generating sets of edge groups (and the words representing these conjugates) until they fall as subgroups of the given peripheral subgroups).
4. For each elementary splitting that is found try to certify whether it is bipartite with maximal elementary black vertex groups (by using the algorithm given in assumption). If this is the case, use effective coherence to obtain finite presentations of the vertex groups.
5. For every bipartite splitting with maximal elementary black vertex groups, decide whether or not the white vertex groups are rigid (and non-(QH)) or pair-of-pants
group (this uses the splitting detection algorithm provided in the assumption, and the recognition of the free group of rank 2 with the usual “pair-of-pants” peripheral structure). This algorithm terminates on every white vertex.

For every white vertex group that is proved non-rigid, nor pair-of-pants, try to certify if it is \((QH)\), by enumerating the presentations until a classical presentation of a surface group with boundary is found.

6. For every bipartite splitting thus coloured by the previous step, and for which every white vertex group has been proved to be rigid (non-(\(QH\))), pair-of-pants, or \((QH)\), we check whether adjacent edge groups of white rigid vertices are conjugated into the same maximal elementary subgroup or not.

For that, we check which edge groups are parabolic, and which are cyclic, non-parabolic. For all those that are not parabolic, we consider the maximal cyclic group containing them (i.e. the adjacent black vertex group), and we use instances of the conjugacy problem to check whether they are conjugated or not.

For those that are parabolic, by enumeration, we may determine to which conjugacy class of \(\mathcal{P}\) they belong, and thus check whether they are all different or not.

7. For every bipartite splitting in which adjacent edge groups of white rigid vertices are not conjugated into the same maximal elementary subgroup, verify that there is no black vertex of valence 2, with cyclic vertex group, around which both adjacent edges are inessential, and with both white neighbours carrying \((QH)\) groups (including pair of pants groups).

If \((G, \mathcal{P})\) is freely indecomposable then the algorithm will terminate with a splitting that will be certified by Theorem 3.9 to be the canonical JSJ. \(\square\)

**Remark 3.13. About torsion.** The reader may have noticed that the results [GL10, Theorem 13.1, Corollary 13.2] are not limited to torsion-free groups. In fact Proposition 3.6 could be restated in the presence of torsion, the main difference being that instead of \((QH)\) subgroups, which are simply the fundamental groups of compact surfaces with boundary, we would have to deal with \((QH)_f\) groups, which are finite extensions of fundamental groups of compact 2-orbifolds. With the appropriate analogue of Lemma 3.8, which requires a combination theorem for \((QH)_f\) groups, the proof of Theorem 3.9 goes through because it doesn’t explicitly take torsion into account.

Such a characterization of the canonical JSJ splitting is worthwhile in its own right, however we encounter a new problem when trying to generalize the algorithm that actually computes the JSJ splitting: that of ”fake” rigid groups. In the torsion-free case there is only one possibility: the fundamental group of a pair of pants. In the absence of 2-torsion there are infinitely many: fundamental groups of basic orbisockets (c.f. [DG11, Definition 5.1, Lemma 5.6]). When there is 2-torsion, we have found many new families of groups that are rigid, but which can be glued onto \((QH)_f\) groups to enlarge them.
Although the classification and algorithmic detection of these "fake" rigid groups appears to be rather involved, it certainly seem feasible that a meta-algorithm to construct a JSJ decomposition, modulo a splitting detector that works in the presence of torsion, could be given. We have chosen not to do so because no such rigidity detector is currently known to exist. We do wish to stress, however, that the only obstruction to removing the torsion-free assumption of the main result, Theorem 1.1, is the computation of canonical splittings.

4 Graph of groups isomorphisms

The following definition of a graph of groups isomorphisms follows [Bas93, §2.3]; this recapitulation was also recorded in [DG11, §2.7.2, 2.7.3]. Recall Definition 2.11 of a graph of groups.

**Definition 4.1.** Given two graphs of groups on the same abstract graph $X$, $X = (X, \{\Gamma_v, v \in V(X)\}, \{\Gamma_e, e \in E(X)\})$, and $X' = (X, \{\Gamma'_v\}, \{\Gamma'_e, e \in E(X)\})$, an isomorphism of graph of groups $\Phi : X' \rightarrow X$ is a tuple $\{\varphi_v, v \in V(X)\}, \{\varphi_e, e \in E(X)\}, \{\gamma_e, e \in E(X)\}$ where:

1. for all $v \in V(X)$, $\varphi_v : \Gamma'_v \rightarrow \Gamma_v$ is an isomorphism,
2. for all $e \in E(X)$ $\varphi_e : \Gamma'_e \rightarrow \Gamma_e$ is an isomorphism, and $\varphi_e = \varphi_{\bar{e}}$
3. for all $e \in E(X)$, $\gamma_e$ is an element of $G_{t(e)}$ such that the diagrams commute:

\[
\begin{array}{ccc}
\Gamma'_{t(e)} & \xrightarrow{i'_e} & \Gamma'_e \\
\varphi_{t(e)} \downarrow & & \downarrow \varphi_e \\
\Gamma_{t(e)} & \xleftarrow{\text{ad}_{\gamma_e}} & \Gamma_{t(e)} \xrightarrow{i_e} \Gamma_e.
\end{array}
\]

(3)

The maps $\varphi_v$ are called the vertex maps, the maps $\varphi_e$ are the edge maps, and the elements $\gamma_e$ are the attaching elements.

4.1 The extension problem

Suppose we are given two graphs groups $X, X'$ and a set of isomorphisms between their edge groups and their vertex groups. We are interested in necessary and sufficient conditions to extend these to an isomorphism $X \sim X'$. The following is immediate from the definition.

**Proposition 4.2.** Given two structures of graphs of groups $X', X$ on the same graph $X$, and an isomorphism of graph of groups $\Phi : X' \rightarrow X$ as above, each vertex map of $\Phi$ induces an isomorphism of unmarked peripheral structures $\varphi_v : (\Gamma'_v, A'_u) \rightarrow (\Gamma_v, A_u)$, where $A'_u$ and $A_u$ are the adjacency peripheral structures induced by the graphs-of-groups.
We now define extension adjustments, which will relate an arbitrary collection of unmarked peripheral structures between the vertex groups, to an isomorphism of graph of groups.

**Definition 4.3.** Let $X'$ and $X$ be two graphs of groups on the same graph $X$ let $\Psi$ be a collection of isomorphisms of groups $\Psi = \{ \psi_v : \Gamma'_v \to \Gamma_v, v \in V(X) \}$. An extension adjustment on $(X', X)$ with respect to $\Psi$ is a collection $\{ \alpha_v : \Gamma_v \to \Gamma_v, v \in V(X) \}$ of automorphisms, and a collection $\{ g_e \in \Gamma_t(e), e \in E(X) \}$ such that:

1. $g_e$ conjugates $\alpha_{t(e)} \circ \psi_{t(e)} \circ i'_e(\Gamma'_e)$ to $i_e(\Gamma_e)$ in $\Gamma_t(e)$,
2. the following diagram (in which we abuse notation: the bottom line is not defined everywhere, but it is on the image of the vertical arrows, by the previous point), is a commutative diagram:

\[
\begin{array}{ccc}
\Gamma'_{t(e)} & \xrightarrow{i'_e} & \Gamma'_{t(e)} \\
\psi_{t(e)} \downarrow & & \downarrow \psi_{t(e)} \\
\Gamma_{t(e)} & \xrightarrow{i_e} & \Gamma_{t(e)} \\
\alpha_{t(e)} \downarrow & & \downarrow \alpha_{t(e)} \\
\Gamma_{t(e)} & \xrightarrow{\text{ad}_{g_e}} & \Gamma_{t(e)} \\
\end{array}
\] (4)

Note that the first point tells that the composition $\alpha_v \circ \psi_v$ is an isomorphism of groups with unmarked peripheral structure (their adjacency peripheral structure) $(\Gamma'_v, A'_{\psi_v}) \to (\Gamma_v, A_\alpha)$. Our motivation here is to express the isomorphism of graph of groups avoiding the “choice” of a good isomorphism of edge groups.

**Proposition 4.4.** Let $X$ be a finite graph, and let $X$, and $X'$ be two structures of graph of groups on $X$.

Assume that there exists a collection of isomorphisms $\Psi = \{ \psi_v : \Gamma'_v \to \Gamma_v, v \in V(X) \}$. Then, the graphs of groups $X'$ and $X$ are isomorphic if, and only if there exists an extension adjustment on $(X', X)$ with respect to $\Psi$.

**Proof.** Assume that $\psi_v, g_e$ and $\alpha_v$ are given as in the statement. Then we set $\varphi_v = \alpha_v \circ \psi_v$, and $\varphi_e$ to be the map $\Gamma'_e \to \Gamma_e$ given by (4), i.e. we have:

\[
\begin{array}{ccc}
\Gamma'_{t(e)} & \xrightarrow{i'_e} & \Gamma'_{t(e)} \\
\varphi_v \downarrow \psi_{t(e)} & & \downarrow \psi_{t(e)} \\
\Gamma_{t(e)} & \xrightarrow{i_e} & \Gamma_{t(e)} \\
\alpha_{t(e)} \downarrow \alpha_{t(e)} & & \downarrow \alpha_{t(e)} \\
\Gamma_{t(e)} & \xrightarrow{\text{ad}_{g_e}} & \Gamma_{t(e)} \\
\end{array}
\] (5)
This way, we have $\varphi_e = \varphi_e$. To get a graph of groups isomorphism in the sense of Definition 4.1, it remains to check that Diagram (3) commutes for some $\gamma_e \in \Gamma_v$. It is immediate from (5) that $\gamma_e = g_e^{-1}$ is the required element.

In the other direction, assume that the graphs of groups are isomorphic, and let $\Phi = (\{\varphi_v, v \in V(X)\}, \{\varphi_e, e \in E(X)\}, \{\gamma_e, e \in E(X)\})$ be an isomorphism. Now for the given isomorphisms $\psi_v$, the elements $\alpha_v = \varphi_v \circ \psi_v^{-1}$, and $g_e = \gamma_e^{-1}$ are immediately seen to be extension adjustments w.r.t $\Psi$ by comparing diagrams (3) and (5).

### 4.2 Reduction to orbit problems

Recall that a marking of an unmarked peripheral structure $\mathcal{P}_u$ is a marked peripheral structure inducing $\mathcal{P}_u$.

**Proposition 4.5** (Reduction to algorithmic problems in the vertex groups). Let $\mathcal{W}$ be a class of groups with unmarked ordered peripheral structures, and $\mathcal{B}$ be a class of groups. Assume that

1. for all $(G, \mathcal{P}_u)$ in the class $\mathcal{W}$,
   
   (a) the orbit of a given marking of the unmarked ordered peripheral structure $\mathcal{P}_u$ under the action of $\text{Out}(G, \mathcal{P}_u)$, is finite, effectively computable,
   
   (b) the isomorphism problem (for groups with unmarked ordered peripheral structure) is effectively decidable in the class $\mathcal{W}$;

2. in the class $\mathcal{B}$,
   
   (a) the orbit problem of $\text{Aut}(G)$ on tuples of conjugacy classes of tuples of $G$, i.e. the mixed Whitehead problem, is effectively solvable;
   
   (b) the isomorphism problem for groups (without peripheral structure) is effectively decidable.

Consider a bipartite graph $X$ (with black and white vertices: $V(X) = BV(X) \sqcup WV(X)$), and choose an order on each (oriented) link of each vertex. Consider the class of graphs-of-groups on $X$ with black vertex groups in $\mathcal{B}$ and white vertex groups in $\mathcal{W}$, and edge groups such that the adjacency peripheral structure on a white vertex group is that given by $\mathcal{W}$.

Then, the graph-of-groups isomorphism problem is solvable for this class of graphs of groups.

Note that in the assumption on the black vertices, we forget about the peripheral structure. Note also that in the case of the white, the assumption is stronger: not only the orbit problem on such objects is solvable, but each orbit is finite (and computable).

**Proof.** If both graphs of groups are isomorphic, this will be discovered by enumeration. We need an algorithmic certificate that they are not isomorphic. If there is $w \in WV(X)$ such that the groups with adjacency (unmarked ordered) peripheral structures
(Γ_w, A_w) and (Γ'_w, A'_w) are not isomorphic, this will be discovered assumption (1b) on W. If there is b ∈ BV(X) such that Γ_b and Γ'_b are not isomorphic, this will be discovered by assumption (2b) on B. Hence we assume that they are isomorphic, and, after an enumeration chase, that we know some ψ_w : (Γ'_w, A'_w) → (Γ_w, A_w) for each w ∈ WV(X) and ψ_b : Γ'_b → Γ_b for each b ∈ BV(X). Let Ψ be this collection.

By Proposition 4.4, the two graphs of groups fail to be isomorphic if and only if there is no extension adjustment for Ψ. Note that for white vertices, the automorphisms α_w must preserve the unmarked peripheral structure in order to be part of an extension adjustment, but that this is not the case for black vertices.

By assumption (1a), for each white vertex w, one can compute a finite list of automorphisms \{α_{w,i}\} of the vertex group Γ_w realizing the orbit of any given marking A_m(w) of A_w under Out(Γ_w, A_w). For each choice of representatives, one for each w ∈ WV(X), from the finite collections of automorphisms \{α_{w,i}, w ∈ WV(X)\}, we need to decide whether there exists a collection of automorphisms \{α_b, b ∈ BV(X)\} that can complete an extension adjustment (for a certain choice of conjugating elements g_e, e ∈ E(X)).

For that we use the solution given to the orbit problem on tuples of tuples under automorphisms of black vertex groups given by assumption (2a). Indeed, given a black vertex b, and e an adjacent edge (t(e) = b) whose other end is a white vertex w = t(\bar{e}), once chosen a generating tuple for Γ'_e, and an automorphism α_w = α_{t(\bar{e})} (there are only finitely many of these to consider), one obtains a generating tuple of Γ_e by chasing down the right side of Diagram (4) that is well defined up to conjugacy; this is because \bar{g}_e has not been fixed yet.

Obtaining α_b = α_{t(e)} so that the diagram commutes is exactly solving the orbit problem of the conjugacy class of this tuple in Γ_b. However, the automorphism α_b should make the diagrams associated to all edges adjacent to b commute; this is exactly solving the orbit problem of a tuple of conjugacy classes of tuples, or the mixed Whitehead problem, in Γ_w.

5 Orbits of markings in rigid relatively hyperbolic groups

In this section we show how assumption (1a) of Proposition 4.5 is satisfied by relatively hyperbolic groups with residually finite parabolic groups in which congruences effectively separate the torsion in the outer automorphism groups. This will reduce the isomorphism problem for relatively hyperbolic groups to algorithmic problems in the parabolic subgroups; thus yielding the main result.
5.1 Computing orbits via Dehn fillings and congruences

Consider the class $\mathcal{G}$ of groups with an unmarked ordered peripheral structure, $(G, P_{uo})$, such that

- $G$ is finitely presented, and $(G, P_{uo})$ is relatively hyperbolic,
- groups of $P_{uo}$ lie in a class of residually finite groups, with congruences effectively separating torsion (see Definition 5.5),
- $(G, P_{uo})$ admits no splitting over an elementary subgroup.

Our main objective here is the following. Recall the terminology introduced in Section 2.1.

**Proposition 5.1.** There is an algorithm that, given $(G, P_{uo})$ in the class $\mathcal{G}$ and any marking $P_m$ of the peripheral structure $P_{uo}$, computes the orbit of $P_m$ under $\text{Out}(G, P_{uo})$, or equivalently computes a set of coset representatives of

$$\text{Out}(G, P_{uo})/\text{Out}(G, P_m)$$

in $\text{Out}(G, P_{uo})$.

In order to compute a system of representatives of $\text{Out}(G, P_{uo})/\text{Out}(G, P_m)$ in $\text{Out}(G, P_{uo})$, we will prove that there is a Dehn filling of $G$, cofinite in the parabolic subgroups, that induces a bijection $\text{Out}(G, P_{uo})/\text{Out}(G, P_m) \rightarrow \text{Out}(\bar{G}, \bar{P}_{uo})/\text{Out}(\bar{G}, \bar{P}_m)$. This helps because we can compute the various groups associated to automorphism of hyperbolic groups, in particular we can explicitly find $\text{Out}(\bar{G}, \bar{P}_{uo})/\text{Out}(\bar{G}, \bar{P}_m)$.

The stages of the proof will be as follows. First, we need to define the map

$$\pi : \text{Out}(G, P_{uo})/\text{Out}(G, P_m) \rightarrow \text{Out}(\bar{G}, \bar{P}_{uo})/\text{Out}(\bar{G}, \bar{P}_m).$$

Surjectivity will be achieved by invoking [DG], and the limit argument there. Our main task is to show injectivity, and for that, we will push the situation into the parabolic groups, see Diagram (6). The assumption of congruences separating the torsion will be sufficient to ensure injectivity for sufficiently deep Dehn fillings, the map $\pi$ will be injective.

The next difficulty is to show that the vertical arrows of the diagram are well defined. The left vertical arrow is well defined because the maximal parabolic subgroups of a relatively hyperbolic group are self-normalized. In a hyperbolic Dehn filling, however, since the group is hyperbolic relative to a collection of finite groups, the situation is not so clear. We will show that this is also the case for sufficiently deep Dehn fillings by examining the geometry more closely.

5.1.1 The map $\pi : \text{Out}(G, P_{uo})/\text{Out}(G, P_m) \rightarrow \text{Out}(\bar{G}, \bar{P}_{uo})/\text{Out}(\bar{G}, \bar{P}_m)$

The operation of Dehn fillings is an important tool. In the context of relatively hyperbolic groups, it was first studied by Osin, and by Groves and Manning.
Theorem 5.2 ([Osi07, GM08, Dehn filling theorem]). Let $(G, \{[P_1], \ldots, [P_k]\})$ be a relatively hyperbolic group. There exists a finite set $X$ in $G \setminus \{1\}$ such that whenever $N_i \triangleleft P_i$ avoids $X$, the image of $P_i$ in the group $G/\langle \langle \bigcup_i N_i \rangle \rangle$ is canonically $P_i/N_i$, and $G/\langle \langle \bigcup_i N_i \rangle \rangle$ is hyperbolic relative to $\{[P_1/N_1], \ldots, [P_k/N_k]\}$.

In this case, we say that $G/\langle \langle \bigcup_i N_i \rangle \rangle$ is a Dehn filling of $G$ by $K = \langle \langle \bigcup_i N_i \rangle \rangle$.

A sequence of Dehn fillings by $K_n = \langle \langle \bigcup_i N_i,n \rangle \rangle$ is said cofinal if $\bigcap_n K_n = \{1\}$. A convenient property of Dehn fillings is that the cofinality of $K_n$ is equivalent to the cofinality of each $N_i,n$: $\forall i, \bigcap_n N_i,n = \{1\}$ (see for instance [DGO11, Theorem 7.9]).

In the following, $(G, \{[P_1], \ldots, [P_k]\})$ will be a relatively hyperbolic group. $S_i$ will be a finite generating set of $P_i$, so that $\mathcal{P}_{uo} = ([P_1], \ldots, [P_k])$, and $\mathcal{P}_m = ([S_1], \ldots, [S_k])$ are respectively the unmarked ordered peripheral structure, and the marked peripheral structure on $G$ associated to $\mathcal{P}$ (and the implicit choices). We first set the stage and define $\pi$.

Lemma 5.3. Let $N_i \triangleleft P_i$ be a characteristic subgroup of $P_i$. Let $\bar{G} = G/\langle \langle \bigcup_i N_i \rangle \rangle$, $\mathcal{P}_{uo} = ([P_1/N_1], \ldots, [P_k/N_k])$, and $\mathcal{P}_m = ([S_1], \ldots, [S_k])$. Then, the quotient map $G \to \bar{G}$ induces a morphism $\pi : \text{Out}(G, \mathcal{P}_{uo}) \to \text{Out}(G, \mathcal{P}_{uo})$, that sends $\text{Out}(G, \mathcal{P}_m)$ inside $\text{Out}(G/\mathcal{P}_m)$.

Proof. If an automorphism $\varphi$ of $G$ preserves $[P_1]$, it has the same class in $\text{Out}(G)$ than an automorphism preserving $P_1$, and therefore $N_i$, since the later is characteristic in $P_i$. It follows that $\varphi$ preserves $[N_i]$. If this happens for all $i$, $\varphi$ thus defines an element $\bar{\varphi}$ of $\text{Out}(\bar{G})$, and it preserves $\mathcal{P}_{uo}$. The map $\pi : \text{Out}(G, \mathcal{P}_{uo}) \to \text{Out}(G/\mathcal{P}_m)$ hence defined is clearly a homomorphism. If $\varphi$ preserves $[S_i]$, $\bar{\varphi}$ preserves $[S_i]$. \hfill \qed

Lemma 5.4. Assume that $(G, \mathcal{P}_{uo})$ is a relatively hyperbolic group that does not split over an elementary subgroup. Then, there exists a finite set $X'$ in $G \setminus \{1\}$ such that whenever one chooses, for each $i$, $N_i \triangleleft P_i$ avoiding $X'$, characteristic and of finite index in $P_i$, the homomorphism $\pi : \text{Out}(G, \mathcal{P}_{uo}) \to \text{Out}(G, \mathcal{P}_{uo})$ is surjective.

Proof. We invoke [DG, Theorem 5.1] (applied to $G = G'$) which says that if $(G, \mathcal{P}_{uo})$ is a relatively hyperbolic group that does not split over an elementary subgroup, then for any cofinal sequences of Dehn fillings by $K_n$, and any sequence of automorphism $\varphi_n : (G/K_n, \mathcal{P}_n) \to (G/K_n, \mathcal{P}_n)$, there is an automorphism $\varphi : (G, \mathcal{P}) \to (G, \mathcal{P})$ that induces, for infinitely many $n$, a composition of $\varphi_n$ with a conjugation.

Let $X'_n, (n \in \mathbb{N})$ be an exhaustion of $G \setminus \{1\}$ by finite sets. By assumption, there is an associated Dehn filling by characteristic finite index subgroups of $P_i$, such that the homomorphism $\pi : \text{Out}(G, \mathcal{P}_{uo}) \to \text{Out}(G_n, (\mathcal{P}_{uo})_n)$ is not surjective, and in which $X'_n$ survives. One can produces a sequence $\varphi_n$ of automorphisms of $(G_n, (\mathcal{P}_{uo})_n)$, by choosing an automorphism not in the image, for each $n$. [DG, Theorem 5.1], that we just recalled provides an automorphism $\varphi : (G, \mathcal{P}) \to (G, \mathcal{P})$ that induces, for infinitely many $n$, a composition of $\varphi_n$ with a conjugation. That is a contradiction. \hfill \qed
5.1.2 Injectivity of $\pi$: Congruences separating the torsion

The next lemma will serve to introduce the separation of torsion by congruences. If $P_0$ is a finite index characteristic subgroup, we call $P_0$-congruence the quotient map $P \to P/P_0$. Note that, because $P_0$ is characteristic, this quotient map induces a homomorphism $\text{Out}(P) \to \text{Out}(P/P_0)$. Its kernel is called a $P_0$-congruence subgroup of $\text{Out}(P)$ (in generalization of the case $P = \mathbb{Z}^n$).

The desirable property is the existence of torsion-free congruence subgroups.

**Definition 5.5.** Given a group $P$, and a characteristic finite index subgroup $P_0$, we say that the $P_0$-congruence separates the torsion of $\text{Out}(P)$ if the $P_0$-congruence subgroup of $\text{Out}(P)$ is torsion free. In a class of groups, congruences separate the torsion if for each group $P$ in this class, there is a congruence in $P$ that separates the torsion of $\text{Out}(P)$. In a class of groups, congruences effectively separate the torsion if there is an algorithm that, given a presentation of a group in the class, produces a congruence separating the torsion.

**Remark 5.6.** Let us emphasize that this is a property of outer automorphisms groups. In the abelian group $\mathbb{Z}^n$, congruences separate the torsion of $\text{GL}(n, \mathbb{Z})$, and more precisely, that the kernels of the quotient on $\text{GL}(n, \mathbb{Z}/p\mathbb{Z})$ are torsion free for $p \geq 3$, by a classical theorem of Minkowski.

**Lemma 5.7.** Let $P$ be a group in which congruences separate the torsion. Whenever one chooses $N \triangleleft P$ characteristic, of finite index, inside a congruence group separating the torsion, the quotient map $P \to P/N$ induces a map $c : \text{Out}(P) \to \text{Out}(P/N)$ with torsion free kernel.

**Proof.** The map $c$ is well defined since $N$ is characteristic. By definition of congruences separating the torsion, there exists $N_0$ characteristic, of finite index such that $c_0 : \text{Out}(P) \to \text{Out}(P/N_0)$ has torsion free kernel. If $N \triangleleft N_0$ is characteristic in $G$, the map $c_0 : \text{Out}(P) \to \text{Out}(P/N_0)$ factorizes through $\text{Out}(P) \xrightarrow{c} \text{Out}(P/N) \to \text{Out}(P/N_0)$. It follows that $\ker c \subset \ker c_0$ is torsion free. \hfill \Box

5.1.3 Injectivity of $\pi$: the diagram.

Our aim is to obtain a well defined commutative diagram

$$
\begin{array}{ccc}
\text{Out}(G, P_{\text{uo}})/\text{Out}(G, P_{m}) & \xrightarrow{\pi} & \text{Out}(\bar{G}, \bar{P}_{\text{uo}})/\text{Out}(\bar{G}, \bar{P}_{m}) \\
\downarrow & & \downarrow \\
\prod \text{Out}(P_i) & \xrightarrow{\oplus c_i} & \prod \text{Out}(\bar{P}_i).
\end{array}
$$

in which the vertical arrows are injective.

The reader can already see how the diagram will help to prove that $\pi$ is a bijection, under suitable assumptions. Indeed, we established the surjectivity of $\pi$ (in deep enough Dehn fillings), and the torsion-freeness of the kernel of $\oplus c_i$. If the left vertical arrow is injective, and $\text{Out}(G, P_{\text{uo}})$ finite, the bijectivity of $\pi$ will follow.
**Proposition 5.8.** Let $G$ be a group, and $P < G$. Assume that $P$ is its own normalizer in $G$. Then, the inclusion $P \to G$ induces a homomorphism $r : \text{Out}(G, ([P])) \to \text{Out}P$. Moreover, the kernel of this homomorphism is precisely $\text{Out}(G, ([S]))$ for some (any) tuple $S$ generating $P$.

**Proof.** Take any automorphism $\alpha$ of $G$ preserving the conjugacy class of $P$. There is $g$ such that, if $\text{ad}_g$ denotes the conjugation by $g$, the automorphism $\text{ad}_g \circ \alpha$ preserves $P$. There is no uniqueness of $g$, but for any other choice $g'$, the element $g^{-1}g'$ normalizes $P$, hence is in $P$ by assumption. It follows that the automorphisms $\text{ad}_g \circ \alpha$ and $\text{ad}_{g'} \circ \alpha$ coincide in $\text{Out}(P)$. It is then straightforward to check that this map $\text{Aut}(G, [P]) \to \text{Out}(P)$ is a homomorphism.

The kernel of this homomorphism is precisely the subgroup of $\text{Aut}(G, [P])$ that induces some conjugation by an element of $G$ on $P$. The claims follow.

The left vertical arrow of Diagram (6) is therefore well defined and given by:

**Corollary 5.9.** If $(G, \mathcal{P}_{uo})$ is a relatively hyperbolic group, with $\mathcal{P}_{uo} = ([P_1], \ldots, [P_k])$, then for all $i$, the restriction induces a homomorphism $r_i : \text{Out}(G, \mathcal{P}_{uo}) \to \text{Out}(P_i)$, such that $\ker(\oplus r_i)$ is $\text{Out}(G, \mathcal{P}_m)$. In particular, the restriction induces an injective homomorphism

$$\text{Out}(G, \mathcal{P}_{uo})/\text{Out}(G, \mathcal{P}_m) \to \prod_i \text{Out}(P_i)$$

**Proof.** By Proposition 2.3, in a relatively hyperbolic group, peripheral subgroups are their own normalizers, and Proposition 5.8 can be applied.

### 5.1.4 Injectivity of $\pi$: the last arrow of the diagram.

The last brick of the proof is to obtain the right vertical arrow of Diagram (6), and for that, again by Propositions 5.8, it suffices to show that, for sufficiently deep Dehn fillings, $P_i/N_i$ is its own normalizer in $\bar{G}$.

For that, we need to examine the geometry of the Dehn fillings more closely. Let us introduce the group $K_n$ that is the kernel of the Dehn filling by the groups $N_i(n)$ that are intersections of all index $\leq n$ subgroups of $P_i$. If the groups $P_i$ are residually finite, the Dehn fillings by the groups $N_i(n)$ satisfy the assumption of Theorem 5.2 for $n$ large enough, and in particular, $P_i \cap K_n = N_i(n)$. We sometimes write $\{[P_1/N_1(n)], \ldots, [P_k/N_k(n)]\} = \mathcal{P}/N(n)$ for brevity.

We will also need to manipulate a good space on which $G/K_n$ acts. In order to comply with some conventions used in some constructions in [DGO11], we introduce $\delta_e$ a certain positive constant, whose value can be found in [DGO11, §5], but which is actually irrelevant for us. Note that, in the definition of relative hyperbolicity, one can, up to rescaling, assume that the space $X$ is $\delta_e$-hyperbolic.

Thus, start with $\delta \leq \delta_e$, and with a $\delta$-hyperbolic proper space $X$ as in Definition 2.1, and endow it with a $10^5\delta$-separated invariant family of horoballs $\mathcal{H}$. Given $R$, we
construct the parabolic cone-off space $\hat{X}_R$ as in [DGO11, §7.1, Definition 7.2], which is as follows.

First choose $\mathcal{H}_R$ an invariant system of $10^5\delta + R$-separated horoballs, obtained by taking sub-horoballs of those in $\mathcal{H}$. For each horoball $H$ in $\mathcal{H}_R$, let $\partial H$ be its horosphere, $\partial H = H \setminus \hat{H}$. Fix $r_0 \geq r_U$, where $r_U$ is a constant whose value can be found in [DGO11, §5.3] (it is $3 \times 10^9 + 1$). Let $X$ be the cone-off of $X$ along each $H \in \mathcal{H}_R$, defined in [DGO11, §5.3], for the parameter $r_0$, and let $A$ be its set of apices.

For $a \in A$, let $\hat{H}_a$ its corresponding horoball in $X$, and $\hat{H}_a$, the cone on $H_a$, is a subset of $X$. For each pair of points $p, q$ in $\partial H_a$ and each geodesic $[p, q]$ of $H_a$ (for the intrinsic metric) avoiding $a$, consider the set $T_{[p, q]} \subset \hat{H}_a$ to be the convex hull of $[a, p] \cup [p, q] \cup [q, a]$ in $\hat{H}_a$. For every other pair of points, i.e. for any connecting geodesic $a \in [p, q]$ , then $T_{[p, q]} = [p, q]$. We define $B_{H_a}$ to be the union of all $T_{[p, q]}$, for $p, q$ in $\partial H_a$ and $[p, q]$ a geodesic.

The parabolic cone-off, as defined in [DGO11, Definition 7.2], is then

$$\hat{X}_R = \left( X \setminus \bigcup_{a \in A} \hat{H}_a \right) \cup \left( \bigcup_{a \in A} B_{H_a} \right)$$

The space $\hat{X}_R$ is geodesic and locally compact everywhere except at the apices (see the comment after [DGO11, Definition 7.2]). Also, by [DGO11, Lemma 7.4], the space $\hat{X}_R$ is $16\delta_i$-hyperbolic, for some universal constant $\delta_i$.

Fix $R > 0$. Then, by [DGO11, Proposition 6.7], for $n$ large enough, $K_n$ is the group of a so-called separated very rotating family on $\hat{X}_R$, and by [DGO11, Proposition 5.20] the quotient of $\hat{X}_R$ by $K_n$ is $4000\delta$-hyperbolic, and the action of $G/K_n$ is metrically proper, co-bounded.

**Lemma 5.10.** For $n \geq 1$, the action of $G/K_n$ on $(\hat{X}_R/K_n) \setminus \hat{A}$ is free.

**Proof.** Assume that $\bar{g}$ fixes a point $\bar{x} \in \hat{X}_R/K_n$ that is not an apex, and choose a preimage $x$ of $\bar{x}$. Then, $\bar{g}$ has a preimage $g$ that fixes $x \in \hat{X}_R$, hence also the (unique) projection of $x$ on $X$. Since the action on $X$ is free, $g$ is trivial \( \square \)

**Lemma 5.11.** There exists an integer $n_0$ such that, for all ball $B \subset X \setminus \mathcal{H}$, of radius $1000\delta$, and any orbit $Gx$, the set $Gx \cap B$ has at most $n_0$ elements. Moreover, for any $n \geq 1$, and for any ball $\tilde{B} \subset (X \setminus \mathcal{H})/K_n \subset \hat{X}_n$ of radius $1000\delta$, and any orbit $(G/K_n)\bar{x}$, the set $(G/K_n)\bar{x} \cap \tilde{B}$ has at most $n_0$ elements.

**Proof.** The first claim follows from the fact that the action of $G$ on $X \setminus \mathcal{H}$ is metrically proper. For the second claim, one can lift $\tilde{B}$ as a ball $B$ in $X \setminus \mathcal{H}$, and the full preimage of the orbit $(G/K_n)\bar{x}$ is itself an orbit. The estimation follows. \( \square \)

**Proposition 5.12.** Let $(G, \{[P_1], \ldots, [P_k]\})$ be a relatively hyperbolic group. Assume that each $P_i$ is infinite, residually finite. For all $n$, let $K_n$ be the $n$-th Dehn kernel as above. Then, for $n$ large enough, the groups $\hat{P}_i = P_i/(K_n \cap P_i)$ are their own...
normalizers in $G/K_n$. Moreover, there is an algorithm that, given $n$, will terminate if and only if $G/K_n$ is hyperbolic and the groups $P_i$ are their own normalizers.

Proof. By definition of the parabolic subgroups, the fixator of the apex $a_i \in \hat{X}_R$ associated to the horball preserved by $P_i$ is precisely $P_i$ in $G$. Therefore the fixator of $\bar{a}_i$ is exactly $\bar{P}_i = P_i/(K_n \cap P_i)$. The first step is to show the other inclusion. This next lemma implies that $\bar{P}_i$ is its own normalizer, since the normalizer preserves the set of fixed points.

Lemma 5.13. $\bar{P}_i$ fixes only one point in $\hat{X}_n$.

Proof. Let $\bar{p}$ be non trivial in $\bar{P}_i$, of maximal order. We claim that if $n$ is large enough, $\bar{p}$ is of order greater than $n_0$, the constant given by Lemma 5.11.

The proof of the claim is easier if $P_i$ has an infinite order element $p$: by residual finiteness of $P_i$, the $n_0$-th power of $p$ in $P_i$ survives in a certain finite quotient $\bar{P}_i$, forcing the maximal order of elements of $\bar{P}_i$ to be larger than $n_0$. If $P_i$ has no infinite order element, since $P_i$ is finitely generated, infinite, residually finite, the solution to the restricted Burnside problem states that there is no upper bound on the order of its elements. A similar argument then concludes; the claim is established.

Assume that $\bar{p}$ fixes $\bar{x} \in \hat{X}_n$, different from $\bar{a}_i$. Recall (Lemma 5.10) that $G/K_n$ acts freely on $\hat{X}_n \setminus \mathcal{A}$, and therefore, $\bar{x}$ has to be the image of an apex. We may choose $\bar{x}$ so that $d(\bar{x}, \bar{a}_i)$ does not exceed $\inf_{x'} d(\bar{x}', \bar{a}_i) + R/100$, for $x'$ ranging over the set of points fixed by $\bar{p}$.

Consider a geodesic segment $[\bar{a}_i, \bar{x}]$. Let us show that it contains no apices. If $\bar{z}$ was such an apex, then $\bar{p}\bar{z}$ would be another image of an apex on the geodesic $\bar{p}[\bar{a}_i, \bar{x}]$, which is another geodesic joining $[\bar{a}_i, \bar{x}]$. Using that $\hat{X}_n$ is $200\delta$-hyperbolic, one easily obtains that $d_{\hat{X}_n}(\bar{z}, \bar{p}\bar{z}) \leq 500\delta$, but image of apices are $R/2$-separated, which implies that $\bar{z} = \bar{p}\bar{z}$. Moreover, being on the geodesic, $\bar{z}$ is at least $R/2$-closer to $\bar{a}_i$ than $\bar{x}$. This contradicts the minimality of $d_{\hat{X}_n}(\bar{a}_i, \bar{x})$ among the fixed points of $\bar{p}$. Thus, there is no other image of apex in $[\bar{a}_i, \bar{x}]$.

Now we may lift $[\bar{a}_i, \bar{x}]$ as a connected path in $\hat{X}_R$ starting from $a_i$ (there is a choice of first edge, in an orbit under $K_n \cap P_i$, but once this first edge is chosen, the lift is determined, since one never pass through an apex, the only points whose stabilizers non-trivially intersect $K_n$). Let $x$ the end point of this path $[a_i, x]$, which is necessarily a geodesic, since any shorter path gives a shorter path in $\hat{X}_n$.

By separation of the horballs in $X$, there is, on the geodesic $[a_i, x]$, a point $y$ that is in the original space $X \setminus \mathcal{H}$. Let $\bar{y}$ be its image in $[\bar{a}_i, \bar{x}]$. Since for all $k$, $\bar{p}^k$ fixes both $\bar{a}_i$ and $\bar{x}$, it has to move $\bar{y}$ at distance at most $5\delta$ from itself, where $\delta$ is the hyperbolicity constant of $\hat{X}_n$. However, $\delta \leq 1000\delta$ where $\delta$ is the hyperbolicity constant of $X$. Therefore, the sequence $\bar{p}^k\bar{y}$ stays in the ball centred at $\bar{y}$ of radius $1000\delta$, which, by the local injectivity property of the quotient map, is isometric to $B(y, 1000\delta) \subset X$, and in which the injectivity radius of $G/K_n$ is the same as $G$. It follows that, by definition of $n_0$, there is some power $k \leq n_0$ such that $\bar{p}^k\bar{y} = \bar{y}$. However, $\bar{y}$ is not the image of an
apex, and by Lemma 5.10, \( \bar{\rho}^k = 1 \), thus contradicting that the order of \( \bar{\rho} \) was greater than \( n_0 \).

The lemma proves the first assertion of the proposition. For the final assertion of the proposition, let us invoke Papasoglu’s algorithm [Pap96] that will terminate if \( G/K_n \) is hyperbolic, and if so, [DG11, Lemma 2.5] will compute generating sets of the normalizers of each \( P_i \) (which are easily checked to belong or not to belong to the finite group \( P_i \)).

**Proposition 5.14.** Assume that \( \text{Out}(G, \mathcal{P}_{uo})/\text{Out}(G, \mathcal{P}_m) \) is finite, and that groups in \( \mathcal{P}_{uo} \) have congruences separating the torsion. Consider a Dehn filling of \( G \) by \( K = \langle \bigcup_i N_i \rangle \). Assume \( \bar{G} = G/K \) is hyperbolic, and \( P_i/N_i \) is its own normalizer in \( G/K \). If all \( N_i \) are characteristic, of finite index, and in congruence subgroups of \( P_i \) separating torsion, then the homomorphism \( \pi : \text{Out}(G, \mathcal{P}_{uo})/\text{Out}(G, \mathcal{P}_m) \to \text{Out}(\bar{G}, \mathcal{P}_{uo})/\text{Out}(\bar{G}, \mathcal{P}_m) \) induced by the quotient map is an injective.

**Proof.** By Lemma 5.3, the homomorphism \( \pi \) is well defined. By Lemma 5.4 it is surjective. By Corollary 5.9, the left vertical arrow of Diagram (6), say \( \rho \), is injective, and induced by restriction. Since \( \text{Out}(G, \mathcal{P}_{uo})/\text{Out}(G, \mathcal{P}_m) \) is finite, and \( \ker \oplus c_i \) is torsion free (by Lemma 5.7), the composition \( \oplus c_i \circ \rho \) is injective. The right vertical arrow is well defined, by Proposition 5.12 and 5.8. The diagram is clearly commutative, hence it follows that \( \pi \) is injective.

This proves our structural feature, Proposition 1.2.

**Corollary 5.15.** Assume that \( \text{Out}(G, \mathcal{P}_{uo})/\text{Out}(G, \mathcal{P}_m) \) is finite, and that groups in \( \mathcal{P}_{uo} \) are residually finite, and have congruences separating the torsion. Then there is a Dehn filling by groups \( N_i \) characteristic in \( P_i \), of finite index, such that the map \( \pi \) is an isomorphism.

**Proof.** Lemma 5.4 guaranties that deep enough Dehn fillings will make \( \pi \) surjective. On the other hand, Proposition 5.12 ensures that for deep enough Dehn fillings, Proposition 5.14 can be applied, and this ensures the injectivity.

**5.1.5 Proof of Proposition 5.1**

We can now prove Proposition 5.1.

**Proof of Proposition 5.1.** On one hand, the algorithm search for classes of automorphisms in \( \text{Out}(G, \mathcal{P}_{uo}) \) modulo \( \text{Out}(G, \mathcal{P}_m) \), and enumerates them in a list \( L \). On the other hand, the algorithm computes a number \( n_0 \) so that congruence by characteristic subgroups of index larger than \( n_0 \) separate the torsion, in the parabolic subgroups (this is possible thanks to the assumption of effective separation of torsion by congruences). Then, in parallel, for incrementing \( n \geq n_0 \), the algorithm tries to certify that \( n \) is
suitable for applying Proposition 5.14 (this is done by the last point of Proposition 5.12). It may happen that for some \( n \), the certification procedure does not terminate, but we know by the first point of Proposition 5.12 that it will terminate if \( n \) is large enough (hence the reason why we perform this certification in incrementing parallel).

For each \( n \) which is certified suitable for the conclusion of Proposition 5.14, the algorithm computes \((G/K_n, P_{uo}/P(n))\) (which is thus certified hyperbolic), computes its outer automorphism group by [DG11, Theorem 8.1].

Proposition 5.14 guarantees that for all certified \( n \), and all state of enumeration of the list \( \mathcal{L} \) (in particular when it is complete), \( \mathcal{L} \) naturally embeds (injectively) in \( \text{Out}(G/K_n, P_{uo}/P(n))/\text{Out}(G/K_n, P_m/P(n)) \). Lemma 5.4, on the other hand, guarantees that, for some \( n \), this embedding is a bijection. When that happens, we are done: the list covers representatives of the cosets \( \text{Out}(G/,P_{uo})/\text{Out}(G/,P_m) \).

5.2 The proof of Theorem 1.1

We now prove the main theorem of the introduction, Theorem 1.1. Given two torsion-free relatively hyperbolic groups \((G_1, P_1), (G_2, P_2)\), with a finite generating set for each representative of maximal parabolic subgroup, we need to check whether they are isomorphic as groups with unmarked peripheral structure.

By Proposition 3.3, and by assumption of solvability of isomorphism problem for elementary groups, we may assume that we are given non-elementary torsion-free relatively hyperbolic groups, freely indecomposable relative to their peripheral structures.

Next, let \( \mathcal{R} \) be the class of torsion-free relatively hyperbolic groups with parabolic groups in the given class \( \mathcal{C} \). Since \( \mathcal{C} \) is stable for taking subgroups, by Proposition 2.18, \( \mathcal{R} \) is stable for taking subgroups that are vertex groups of elementary splittings (relative to the peripheral structure).

By the assumptions of Theorem 1.1, we may use the algorithm of Theorem 2.20 in order to determine whether a group in \( \mathcal{R} \) is rigid or not. By Theorem 2.5, we have also an algorithm that terminates only when a given (infinite) group is maximal elementary. Therefore, we may use Theorem 3.12, to find a bipartite canonical JSJ decompositions \( X_i \) for \( (G_i, P_i) \), with maximal elementary black vertex groups, and white vertex groups are either rigid or \( (QH) \).

By criterion (iii) of Lemma 3.11, \( G_i \) is hyperbolic relative to the augmented peripheral structure \((P_i)_{X_i}\), as well, and \( X_i \) is essentially peripheral for this structure. To avoid handling too many peripheral structures at different places, it is convenient to turn this splitting into a peripheral one, and we do that canonically as follows. On each white vertex \( v \) of \( X_i \), we may compute the induced peripheral structure on its group \( \Gamma_v \) as follows. We know the images of the adjacent edge groups, and for each representative of group in \( P_i \), by enumeration, we may find a vertex containing one of its conjugates, hence, once this is done for all, we may find the list of peripherals that have a conjugate in \( \Gamma_v \). According to Lemma 2.15, this is the induced peripheral structure.
For each conjugacy class of peripheral subgroups $P < \Gamma_v$ that are not yet conjugated to an edge group, we may refine $X_i$ by appending one black vertex, with an edge to this white vertex $v$, both whose groups are $P$ with the natural attaching maps. This produces peripheral splittings $X'_i$ of $(P_1)_{X_i}$. By Proposition 3.6 (5) this splitting is canonical; therefore $(G_1, P_1)$ and $(G_2, P_2)$ are isomorphic if and only if $X'_1$ and $X'_2$ are isomorphic as graphs-of-groups.

We may list all the isomorphisms of the underlying graphs of $X'_1$ and $X'_2$, and the possible orderings of the unmarked structures $P_1$ and $P_2$, and proceed in parallel for each choice of them. This way, we reduce to the case of two structures of graphs of groups on the same underlying graph.

According to Proposition 4.5, in order to check whether these decompositions are isomorphic it is enough to solve, in black (elementary) vertex groups, the mixed Whitehead problem, and the isomorphism problem, and, in the white vertex groups, the isomorphism problem (of groups with unmarked ordered peripheral structures) and the orbit problem for markings. The assumptions on the parabolic groups readily guarantee that the two first problems (on black vertex groups) are solvable.

For white vertex groups, let us differentiate whether the groups to compare are rigid or $(QH)$ non rigid (whether they belong to a class or the other can be decided by Theorem 2.20, but actually, at this stage, we know which vertex groups are rigid). Let us consider first rigid groups. By Bowditch’s Theorem 2.16, since $X'_i$ is peripheral, these rigid vertex groups are hyperbolic relative to the peripheral structure induced by $(P_i)_{X_i}$. To show that $(1b)$ in Proposition 4.5 (the isomorphism problem) is satisfied, we use one of the main result of [DG].

**Theorem 5.16 ([DG]).** There is an algorithm which decides if two explicitly given finitely presented relatively hyperbolic groups $(G, \mathcal{P})$, $(H, \mathcal{Q})$ (by finite presentation, and finite generating sets of conjugacy representatives of the peripheral structure) are isomorphic (as groups with unmarked peripheral structure), provided $\mathcal{P}$, $\mathcal{Q}$ consist of residually finite groups and that $(G, \mathcal{P})$, $(H, \mathcal{Q})$ are rigid.

In [DG, Theorem 6.8], the algorithm is given (not just proved to exist). As for the requirement $(1a)$ of Proposition 4.5 (the orbit problem for markings), the assumption on the congruences separating torsion allows us to use Proposition 5.1, which solves this algorithmic problem.

Let now us consider $(QH)$ white vertex groups. Deciding isomorphism is a matter of identifying the surface, and the orbit problems for markings is easy, since the peripheral groups are infinite cyclic; two different markings are either equal, or opposite in each boundary components (in which case, they are in the same orbit under the automorphism group), or they differ on certain but not all boundary components, in which case they are in the same orbit if and only if the surface is not-orientable.

Thus, one may apply Proposition 4.5, and decide whether $X'_1$ and $X'_2$ are isomorphic graphs of groups. As already discussed, this allows to decide whether $(G_1, \mathcal{P}_1)$ and $(G_2, \mathcal{P}_2)$ are isomorphic. \[\square\]
Remark 5.17. About torsion In this proof, aside from the computation of the JSJ splitting, the only other place where the presence of torsion would make a difference is for the non-rigid white vertex groups. Deciding isomorphisms of \((QH)_f\) groups follows from the main result of \cite{DG11}, and computing the finite orbits of markings also appears to be possible (c.f. \cite[§8]{DG11}).

6 Algorithmic problems in nilpotent groups

Coming up with a new method to solve the isomorphism problem for relatively hyperbolic groups is really super; it is even better when this new method can be used to prove new results!

Let \(N\) be a finitely generated nilpotent group, such groups will be called \(\mathfrak{N}\)-groups. Let \(\text{Out}_0(N)\) denote the finite order elements of \(\text{Out}(N)\). We write \(H \trianglelefteq G\) to denote a characteristic subgroup and \(H \triangleleft G\) to denote a normal subgroup. Recall (Definition 5.5) that a finite index characteristic subgroup \(P \trianglelefteq N\) such that the elements of \(\text{Out}_0(N)\) do not vanish via the natural map \(\text{Out}(N) \to \text{Out}(N/P)\), is said to separate torsion in \(\text{Out}(N)\).

The goal of this section is to prove the following two results:

**Theorem 6.1.** Let \(N\) be an \(\mathfrak{N}\)-group. Then given a finite presentation of \(N\) we can algorithmically construct a generating set of a subgroup \(P\) which separates torsion in \(\text{Out}(N)\).

**Theorem 6.2.** There is a uniform algorithm to solve the mixed Whitehead problem for \(\mathfrak{N}\)-groups.

Which, with our main result, gives us a Theorem 1.5, that The isomorphism problem is solvable for the class of torsion-free relatively hyperbolic groups with finitely generated nilpotent parabolics.

6.1 Congruences effectively separate the torsion in finitely generated nilpotent groups

The algorithm we shall present that finds a finite index characteristic subgroup of \(N\) that separates the torsion in \(\text{Out}(N)\) will be by recursion on the upper central series length. It is almost completely elementary. Let \(\nu_1 N\) denote the centre of \(N\) and suppose we have found finite index subgroups in \(\nu_1 N\) and \(N/\nu_1 N\) that separate torsion in \(\text{Out}(\nu_1 N)\) and \(\text{Out}(N/\nu_1 N)\) respectively. In section 6.1.1 we will show how to construct a finite index subgroup \(P_0\) of \(N\) that is “good enough” in the sense that it will separate the torsion in \(\text{Out}(N)\) which can be seen by its natural images in \(\text{Out}(\nu_1 N)\) and \(\text{Out}(N/\nu_1 N)\). The finite order elements of \(\text{Out}(N)\) which are not separated by \(P_0\) are called elusive. In Section 6.1.2 we give a complete description of these elements and in Section 6.1.3 we give a procedure to find all of them. Finally in
Section 6.1.4, with the help of Proposition 6.15 (due to Dan Segal), we assemble all these components to give a proof of Theorem 6.1

6.1.1 A partial inductive step

Let \( \nu_0 = \{1\} \trianglelefteq \nu_1 \trianglelefteq \cdots \trianglelefteq \nu_m \trianglelefteq N = N \) denote the upper central series. In particular \( \nu_1 = Z(N) \), i.e. it is the centre, and \( \nu_{i+1} \trianglelefteq N \) is the preimage of \( Z(N/\nu_i N) \). The proof of Theorem 6.1 will be by induction on the length of the upper central series of \( N \), denoted \( l_{u.c.s}(N) \).

Consider the short exact sequence

\[
1 \to \nu_1 N \to N \to N/\nu_1 N \to 1,
\] (7)

then in particular \( l_{u.c.s}(\nu_1 N) = 1 \) and \( l_{u.c.s}(N/\nu_1 N) = l_{u.c.s}(N) - 1 \). Let \( \beta \in \text{Aut}(N) \) and denote by \([\beta]\) its class in Out\( (N) \). We have two homomorphisms from Out\( (N) \) the first one:

\[
\begin{array}{ccc}
\text{Out}(N) & \overset{p}{\longrightarrow} & \text{Out}(N/\nu_1 N) \\
[\beta] & \mapsto & [\beta]
\end{array}
\] (8)

follows from the fact that \( \nu_1 \trianglelefteq N \) and the second one:

\[
\begin{array}{ccc}
\text{Aut}(N) & \overset{r}{\longrightarrow} & \text{Aut}(\nu_1 N) \\
\downarrow & & \downarrow \\
\text{Out}(N) & \overset{\text{r}}{\longrightarrow} & \text{Out}(\nu_1 N)
\end{array}
\] (9)

follows from the fact that, on hand restriction gives a homomorphism \( \text{Aut}(N) \to \text{Aut}(\nu_1 N) \) and on the other hand since \( \nu_1 \trianglelefteq N \) is free abelian, all its automorphisms are outer. The following proposition needed for our induction.

**Proposition 6.3** (Good enough). Let \( P \) be a finitely presented and subgroup separable group and let \( H \trianglelefteq c \trianglelefteq P \) be a finitely presented characteristic subgroup. Suppose there are given finite index subgroups \( H_0 \trianglelefteq c \trianglelefteq H \) and \( K_0 \trianglelefteq c \trianglelefteq P/H \). Then there is a finite index subgroup \( P_0 \trianglelefteq c \trianglelefteq P \) such that:

(a) \( P_0 H/H \leq K_0 \) and \( P_0 H/H \trianglelefteq P/H \).

(b) \( P_0 \cap H \leq H_0 \) and \( P_0 \cap H \trianglelefteq H \).

Moreover, given a finite presentation of \( H \) and \( P \) and an explicit homomorphism \( H \hookrightarrow P \) given by the images of the generators of \( H \), generating sets for \( H_0 \leq H \) and \( K_0 \leq P/H \); \( P_0 \) can be constructed algorithmically.
Proof. Because $[H : H_0]$ is finite we have the finite partition
\[ H = H_0 \cup g_1H_0 \cup \ldots \cup g_nH_0 \] (10)
with $g_i \notin H_0$. Since $P$ is subgroup separable we can find a finite index $P^* \leq P$ such that $P^* \geq H_0$ and $P^* \cap \{g_1, \ldots, g_n\} = \emptyset$. We therefore have
\[ P^* \cap H = H_0. \]

W.l.o.g. $P^*$ can be characteristic by being the intersection of all finite index subgroups $H$ of minimal index in $P$ such that $H \cap H = H_0$. We now look at
\[ \pi : P \twoheadrightarrow P/H. \]

Since $K_0 \trianglelefteq P/H$ and is finite index we have $\pi^{-1}(K_0) \trianglelefteq P$, which is also finite index. Finally we take
\[ P_0 = P^* \cap \pi^{-1}(K_0). \]

Since it is the intersection of two finite index characteristic subgroups of $P$ we find that $P_0 \trianglelefteq P$ and $[P : P_0] < \infty$, and $P_0$ is easily seen to satisfy the requirements of the proposition.

We are given finite presentations of $H$ and $P/H$. We then use the Todd-Coxeter algorithm for $H_0 \leq H$ to produce the list of cosets (10). Since we have an explicit homomorphism $H \hookrightarrow P$ we have a generating set for $H_0$ in $P$ as well as the images of the coset representatives $g_1, \ldots, g_n$ in $P$. We identify these with their image in $P$.

We now enumerate the finite index subgroups $P = P_1, P_2, \ldots$ of $P$ in a way such that if $i < j$ then $[P : P_i] \leq [P : P_j]$. Eventually we will find some subgroup $P_k \leq P$ of index $d$ such that:

(i) $H_0 \leq P_k$, and
(ii) $\{g_1, \ldots, g_n\} \cap P_k = \emptyset$.

If we take the intersection $P^*$ of all subgroups of index $d$ satisfying (i) and (ii), then we have a finite index characteristic subgroup of $P$ with the desired properties.

Now given $K_0 \leq P/H$ we can easily find $\pi^{-1}(K_0)$ as the only finite index subgroup of $P$ that contains $H$ and maps onto $K_0$ via $\pi$ (the subgroup membership problem for $K_0 \leq P/H$ is decidable because $[P/H : K_0] < \infty$). Finally computing the intersection of finite index subgroups of finitely presented groups is straightforward so the desired subgroup $P_0$ can be constructed algorithmically. \(\square\)

**Definition 6.4.** A subgroup $P_0 \trianglelefteq P$ that satisfies (a) and (b) of Proposition 6.3 is said to be good enough w.r.t $K_0$ and $H_0$.

Suppose that we were able to find subgroups $N_0 \trianglelefteq \nu_1N$ and $K_0 \trianglelefteq N/\nu_1N$ that separated torsion in $\text{Out}(\nu_1N)$ and $\text{Out}(N/\nu_1N)$ respectively and recall the homomorphisms $p, r$ given in (8) and (9).
Definition 6.5. We call elusive any element \([\beta] \in \text{Out}_0(N)\) that lies in \(\ker(r) \cap \ker(p) \setminus \{1\}\).

Lemma 6.6. Let \(N_0 \trianglelefteq \nu_1 N\) and \(K_0 \trianglelefteq N/\nu_1 N\) be deep enough and let \(P_0 \trianglelefteq N\) be good enough w.r.t. \(N_0\) and \(K_0\), then every \([\beta] \in \text{Out}_0(N)\) that isn't elusive survives in the homomorphism
\[
\text{Out}(N) \to \text{Out}(N/P_0).
\]

Proof. By construction we have the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccccccc}
1 & 1 & 1 & \ & \ & \ & \ \\
\downarrow & \downarrow & \downarrow & \ & \ & \ & \ \\
1 \rightarrow \nu_1 N/(P_0 \cap \nu_1 N) \rightarrow N/P_0 \rightarrow \frac{N}{\nu_1 N} \rightarrow \frac{P_0 \nu_1 N}{\nu_1 N} \rightarrow 1 \\
\downarrow & \downarrow & \downarrow & \ & \ & \ & \ \\
1 \rightarrow \nu_1 N \rightarrow N \rightarrow \frac{N}{\nu_1 N} \rightarrow 1 \\
\downarrow & \downarrow & \downarrow & \ & \ & \ & \ \\
1 \rightarrow P_0 \cap \nu_1 N \rightarrow P_0 \rightarrow \frac{P_0 \nu_1 N}{\nu_1 N} \rightarrow 1 \\
\downarrow & \downarrow & \downarrow & \ & \ & \ & \ \\
1 & 1 & 1 & \ & \ & \ & \ 
\end{array}
\]  
(11)

Now note that by (a),(b) of Proposition 6.3 there are epimorphisms
\[
\nu_1 N/(P_0 \cap \nu_1 N) \rightarrow \nu_1 N/N_0 \quad \text{and} \quad \frac{N}{\nu_1 N} \rightarrow \frac{N}{\nu_1 N}{\frac{P_0 \nu_1 N}{\nu_1 N}}.
\]
Moreover, since \(P_0 \trianglelefteq N\) there are well defined natural maps
\[
\overline{r} : \text{Out}(N) \rightarrow \text{Out}(\nu_1 N/(P_0 \cap \nu_1 N)) \\
\overline{p} : \text{Out}(N) \rightarrow \text{Out}(\frac{N}{\nu_1 N}{\frac{P_0 \nu_1 N}{\nu_1 N}}).
\]
It is now possible check that for \([\beta] \in \text{Out}_0(N)\) if either \(r([\beta]) \neq 1\) or \(p([\beta]) \neq 1\), since \(K_0, N_0\) separated torsion in \(\text{Out}(N/\nu_1 N), \text{Out}(\nu_1 N)\) respectively, then \([\beta]\) will not vanish in the natural homomorphism
\[
\text{Out}(N) \to \text{Out}(N/P_0).
\]

6.1.2 An algebraic characterization elusive elements

It turns out that elusive elements have a very natural algebraic characterization. We will first describe some algebraic constructions and then give a description of elusive
elements in terms of these constructions. Recall that \( \nu_1 N \leq \nu_2 N \triangleleft N \) is the maximal subgroup such that

\[
\nu_2 N/\nu_1 N = Z(N/\nu_1 N) \tag{12}
\]

**Lemma 6.7.** Each \( \xi \in \nu_2 N \) induces a homomorphism \( z_\xi : N \to \nu_1 N \) given by the mapping

\[
x \mapsto [x, \xi].
\]

Moreover the mapping

\[
\Phi : \nu_2 N \to \text{Hom}(N, \nu_1 N)
\]

is in fact a homomorphism where \( \text{Hom}(N, \nu_1 N) \) is viewed as an abelian group with equipped with the standard \( \mathbb{Z} \)-module addition.

**Proof.** By (12) for all \( x \in N, \xi \in \nu_2 N \) we have \( [x, \xi] = z_\xi(x) \in \nu_1 N \). Let \( x, y \in N \) then with the commutator convention \( [x, y] = x^{-1}y^{-1}xy \) we can observe that on one hand \( (xy)\xi = \xi(x, y) = \xi(xy)z_\xi(xy) \) and on the other hand (recall that \( [z, \xi] \) is always central):

\[
xy\xi = x\xi y[y, \xi] = \xi[x, \xi][y, \xi] = \xi(xy)[x, \xi][y, \xi] = \xi(xy)z_\xi(x)z_\xi(y)
\]

so the map \( x \mapsto z_\xi(x) \) is a homomorphism.

Let now \( \xi, \zeta \in \nu_2 N \) and let \( x \in N \) one hand setting \( [x, \xi \zeta] = z_{\xi \zeta}(x) \) we have

\[
x(\xi \zeta) = (\xi \zeta)x(z_\zeta(x))
\]

and on the other hand we have:

\[
xz_{\xi \zeta}(x) = \xi xz_{\zeta}(x) = (\xi \zeta)xz_{\zeta}(x)z_{\zeta}(x)
\]

which gives the formula:

\[
z_{\xi \zeta}(x) = z_{\zeta}(x) + z_{\xi}(x)
\]

so the map \( \Phi \) is a homomorphism. \( \square \)

We now have something like converse to this result.

**Lemma 6.8.** Let

\[
\text{Hom}^*(N, \nu_1 N) = \{ f \in \text{Hom}(N, \nu_1 N) \mid \nu_1 N \leq \ker(f) \}.
\]

For each \( f \in \text{Hom}^*(N, \nu_1 N) \) the map \( x \mapsto xf(x) \) is an automorphism \( \Psi(f) \in \text{Aut}(N) \). Moreover this map \( \Psi : \text{Hom}^*(N, \nu_1 N) \to \text{Aut}(N) \) is a homomorphism.

**Proof.** We first check that \( \Psi(f) \) is an endomorphism. Indeed \( \Psi(f)(xy) = xyf(xy) \) and, since \( f \) is a homomorphism and \( f(x) \in \nu_1 N \), we have

\[
xyf(xy) = xyf(x)f(y) = xf(x)yf(y) = \Psi(f)(x)\Psi(f)(y).
\]
We can also immediately check that by our hypothesis \( \Psi(f) \circ \Psi(-f) = 1 \) so \( \Psi(f) \) is an automorphism.

Now note that because we are only interested in \( f,g \in \text{Hom}(N,\nu_1 N) \) in which elements of \( \nu_1 N \) vanish we have the following

\[
\Psi(f) \circ \Psi(g)(x) = \Psi(f)(xg(x)) = xg(x)f(x) = \Psi(g + f)(x)
\]

therefore \( \Psi \) is a homomorphism. \( \square \)

We leave the verification of this next lemma to the reader:

**Lemma 6.9.** For all \( \xi \) in \( \nu_2 N \) we have the following equality of automorphisms:

\[
\Psi \circ \Phi(\xi) = \text{Ad}_\xi.
\]

Having described the homomorphisms \( \Psi : \text{Hom}^*(N,\nu_1 N) \to \text{Aut}(N) \) and \( \Phi : \nu_2 N \to \text{Hom}^*(N,\nu_1 N) \) we can now give a precise characterization of the elusive elements of \( \text{Out}_0(N) \).

**Proposition 6.10.** The set of elusive elements of \( \text{Out}_0(N) \) coincides exactly with the set

\[
\{ [\beta] \in \text{Out}(N) \mid (\exists \beta \in [\beta]) \beta \in \Psi(\hat{S} \setminus S) \}
\]

where \( S = \Phi(\nu_2 N) \) and \( S \leq \hat{S} \) is the set consisting of \( f \in \text{Hom}^*(N,\nu_1 N) \) such that

\[
d \cdot f = f + \cdots + f \in S = \Phi(\nu_2 N)
\]

for some \( d \in \mathbb{Z}_{\geq 0} \).

**Proof.** We first show \( \text{Out}_0(N) \cap \ker(r) \cap \ker(p) \setminus \{1\} \subset \Phi(\hat{S} \setminus S) \), where \( r,p \) are defined in (9), (8). Let \( \alpha \in [\beta] \) be an automorphism. Consider the canonical projection:

\[
\begin{align*}
N &\twoheadrightarrow N/\nu_1 N \\
w &\mapsto \bar{w}.
\end{align*}
\]

and denote by \( \overline{\alpha} \) the natural image of \( \alpha \) in \( \text{Aut}(N/\nu_1 N) \). Since \( [\beta] \in \ker(p) \), \( \overline{\alpha} \) must in fact be an inner automorphism of \( N/\nu_1 N \), so there exists some \( u_\alpha \in N \) such that for all \( w \in N \)

\[
\overline{\alpha}(\bar{w}) = \overline{u_\alpha w u_\alpha^{-1}}.
\]

We therefore take

\[
\beta = \text{Ad}_{u_\alpha} \circ \alpha \in [\beta]
\]

where \( \text{Ad}_x \) denotes conjugation by \( x \) and we get that \( \overline{\beta} = 1 \). This means that for all \( w \in N \) we have

\[
\beta(w) = wz_\beta(w)
\]

39
where $z_\beta(w) \in \nu_1N$. Now since $\beta : N \to N$ is a homomorphism we have for all $u, w \in N$

$$wuz_\beta(w)z_\beta(u) = \beta(w)\beta(u) = \beta(wu) = wuz_\beta(wu)$$

which means that $z_\beta : N \to \nu_1N \subseteq \text{Hom}(N, \nu_1N)$ and since $[\beta] \in \ker(r)$, we have that $\beta(\xi) = \xi$ for all $\xi \in \nu_1N$ and therefore $z_\beta(\xi) = 1$ so $z_\beta \in \text{Hom}^*(N, \nu_1N)$ and

$$\beta = \Psi(z_\beta).$$ (13)

obviously since $[\beta] \in \text{Out}(N) \setminus \{1\}$, $z_\beta \notin S$.

Since $[\beta] \in \ker(r)$ we have that for all $\xi \in \nu_1N$ $\beta(\xi) = \xi$, which gives

$$\beta^n(w) = wz_\beta(w)^n$$ (14)

for all $w \in N, n \in \mathbb{Z}$.

Finally since $[\beta] \in \text{Out}_0(N)$, and say of order $d$, $\beta^d$ must be an inner automorphism so there exists some $\xi_\beta$ such that for all $w \in N$

$$\beta^d(w) = Ad_{\xi}(w) = wz_\beta(w)^d$$ (15)

which implies $[w, \xi_\beta] = w^{-1}\xi^{-1}_\beta w\xi_\beta = z_\beta(w)^d$. This in particular implies that $\xi_\beta \in Z(N/\nu_1N)$ so $\xi_\beta \in \nu_2N$ and by (15) we have $\Phi(\xi_\beta) = d \cdot z_\beta$. This means that $z_\beta \in \hat{S} \setminus S$ and we have the first desired inclusion.

We now prove that every element of $\Psi(\hat{S} \setminus S)$ is elusive. We first note that every automorphism in the image of $\Psi$ (see the statement of Lemma 6.8) fixes $\nu_1N$ pointwise and therefore every corresponding outer automorphisms lies in $\ker(r)$. Similarly every such outer automorphism lies in $\ker(p)$. Now suppose that for some $s \in \hat{S} \setminus S$, $\Phi(s) = Ad_{\xi}$ for some $\xi \in N$. Then for all $w \in N$ we have

$$\xi^{-1}w\xi = ws(w) \Rightarrow [w, \xi] = s(w) \in \nu_1N$$

which means that $\xi \in Z(N/\nu_1N)$, i.e. $\xi \in \nu_2N$ so $s \in \Phi(\xi) = S$ — contradiction. Therefore for all $f \in \hat{S} \setminus S$, $[\Phi(f)] \in \text{Out}(N) \setminus \{1\}$.

Finally since $\Psi$ is a homomorphism and since for every $s \in \hat{S} \setminus S$ there is some minimal $d \in \mathbb{Z}$ such that $d \cdot s \in S$ we have

$$\Phi(d \cdot s) = \Phi(s)^d = Ad_{\xi}$$

where $\xi \in \nu_2N$ so the element $[\Phi(s)] \in \text{Out}(N) \setminus \{1\}$ has order $d$. We have therefore shown the other inclusion.

The following is a basic exercise in $\mathbb{Z}$-modules that we leave to the reader.

**Lemma 6.11.** The submodule $\hat{S} \leq \text{Hom}^*(N, \nu_1N)$ given in Proposition 6.10 is the smallest direct summand of $\text{Hom}^*(N, \nu_1N)$ containing $S$ and the torsion subgroup. We will call it the isolator of $S$. 40
6.1.3 Computing the list of conjugacy classes of elusive elements

For this section as in the statement of Proposition 6.10, let $S$ denote the image of $\Phi(\nu_2 N) \leq \text{Hom}^*(N, \nu_1 N)$ and let $\hat{S}$ denote smallest direct summand of $\text{Hom}^*(N, \nu_1 N)$ containing $S$ and the torsion elements. Since we are working over finitely generated $\mathbb{Z}$ modules it follows that $[\hat{S} : S] < \infty$.

**Proposition 6.12.** There is a surjection from the set of cosets $\hat{S}/S$ onto the set of elusive elements in $\text{Out}(N)$. In particular there are finitely many elusive elements.

**Proof.** Let $\xi \in \nu_2 N$ and let $t = \Phi(\xi) \in S$ and let $x \in \hat{S}$ be arbitrary then we need only verify the formula $\Psi(t + x) = Ad_\xi \circ \Psi(x)$.

Indeed we have for all $w \in N$

$$\Psi(t + x)(w) = wt(w)x(w) = \xi^{-1}w\xi x(w) = \xi^{-1}wx(w)\xi = Ad_\xi \circ \Psi(x)(w)$$

It follows that if in $\hat{S}$, $s \equiv s' \pmod{S}$ then $[\Psi(s)] = [\Psi(s')]$. The result now follows from Proposition 6.10.

This next result is an immediate consequence of Theorems 4.1 and 6.5 of [BCRS91].

**Proposition 6.13.** Given a presentation $\langle X | R \rangle$ of a $\mathcal{R}$-group $N$, we can find finite generating sets $\{u_i(X)\}$ and $\{v_j(X)\}$ for $\nu_1 N$ and $\nu_2 N$ respectively. Moreover we can construct an explicit isomorphism of the finitely generated abelian group

$$\nu_1 N \sim \mathbb{Z}^s \oplus (\bigoplus_{i=1}^t \mathbb{Z}/n_i \mathbb{Z})$$

**Corollary 6.14.** We can construct a finite set $\{\beta_1, \ldots, \beta_v\} \in \text{Aut}(N)$ such that $\{[\beta_1], \ldots, [\beta_v]\} \subset \text{Out}_0(N) \setminus \{1\}$ is the complete set of elusive elements.

**Proof.** We first note that since $\nu_1 N \approx \mathbb{Z}^s \oplus (\bigoplus_{i=1}^t \mathbb{Z}/n_i \mathbb{Z})$ is finitely generated abelian we have a natural isomorphism $\text{Hom}(N, \nu_1 N) \approx \text{Hom}(N/[N,N], \nu_1 N)$. Seeing as $N/[N,N] = \mathbb{Z}^r \oplus (\bigoplus_{i=1}^t \mathbb{Z}/m_i \mathbb{Z})$ is also a finitely generated $\mathbb{Z}$-module it follows that $\text{Hom}(N/[N,N], \nu_1 N)$ is also a finitely generated $\mathbb{Z}$-module.

The isomorphism

$$\rho : \text{Hom}(N, \nu_1 N) \sim \text{Hom}(\mathbb{Z}^r \oplus (\bigoplus_{i=1}^s \mathbb{Z}/m_i \mathbb{Z}), \mathbb{Z}^s \oplus (\bigoplus_{i=1}^t \mathbb{Z}/n_i \mathbb{Z}))$$

is computable in the following sense: given a map $f^*$ from the generators of $N$ to the generators of $\nu_1 N$ defining a homomorphism $f$, we can find the corresponding element
\( \rho(f) \) and conversely given some \( x \in \text{Hom} \left( \mathbb{Z}^r \oplus (\oplus_{i=1}^s \mathbb{Z}/m_i \mathbb{Z}), \mathbb{Z}^s \oplus (\oplus_{i=1}^t \mathbb{Z}/n_i \mathbb{Z}) \right) \) we can find a map \( \rho^{-1}(x)^* \) from the generators of \( N \) to \( \nu_1 N \).

By Proposition 6.13 we can find a generating set for \( \nu_1 N \) and therefore find the submodule \( \rho(\text{Hom}^*(N,\nu_1 N)) = T \). Similarly we can and we can find the submodule \( \rho(\Phi(\nu_2 N)) = \rho(S) \leq T \). We can then compute \( U \geq \rho(S) \), the minimal direct summand of \( T \) containing \( \rho(S) \) and the torsion elements of \( T \). We must have \( U = \rho(\hat{S}) \). We can now finally find a complete finite list of representatives \( \{b_1, \ldots, b_n\} \subset \text{Hom} \left( \mathbb{Z}^r \oplus (\oplus_{i=1}^s \mathbb{Z}/m_i \mathbb{Z}), \mathbb{Z}^s \oplus (\oplus_{i=1}^t \mathbb{Z}/n_i \mathbb{Z}) \right) \) of cosets of \( U/\rho(S) \). The result now follows from Proposition 6.12.

### 6.1.4 The proof of Theorem 6.1

The final ingredient we need is an argument graciously communicated to us by Dan Segal.

**Proposition 6.15** (Segal). Let \( P \) be a virtually polycyclic group and denote by \( P_n \triangleleft P \) be a sequence of finite index subgroups which eventually lie inside any fixed finite index subgroup. For every finite order \( [\alpha] \in \text{Out} \left( P \right) \) there exists some \( j \) such that for every \( k \geq j \) the image \( [\alpha]_k \in \text{Out} \left( P/P_k \right) \) is non-trivial.

**Proof.** Let \( q \) be the order of \( [\alpha] \) in \( \text{Out} \left( P \right) \) and chose a representative \( \alpha \in \text{Aut} P \). We assume that \( q > 1 \). Let \( E = \langle \alpha \rangle \rtimes P \) be the natural semidirect product, i.e. endowed with the following multiplication

\[
(\alpha^{n_1}; p_1) \cdot (\alpha^{n_2}; p_2) = (\alpha^{n_1+n_2}; \alpha^{n_2}(p_1)p_2),
\]

where \( p_1, p_2 \in P \). \( E \) is obviously virtually polycyclic.

Suppose towards a contradiction that \( [\alpha]_n \) (which is in \( \text{Out} \left( P/P_n \right) \)) is trivial for infinitely many \( n \). Then there exists \( \bar{x}_n \in P/P_n \) such that the image of \( \alpha \) in \( \text{Aut} \left( P/P_n \right) \) satisfies \( \bar{x}_n = \text{ad}_{\bar{x}_n} \). This means that \( \alpha(g) \in g^{-1} P_n \) for all \( g \in P \), which implies that in \( E \),

\[
[\{1\} \rtimes P; (\alpha, x)] \leq [\{1\} \rtimes P_n].
\]

Indeed for each \( g \in P \) we verify by successive applications of (16):

\[
[(1; g), (\alpha; x)] = (1; g^{-1}) \cdot (\alpha^{-1}; \alpha^{-1}(x_n^{-1})) \cdot (1; g) \cdot (\alpha; x) = (1; g^{-1}) \cdot (\alpha^{-1}; \alpha^{-1}(x_n^{-1})) \cdot (\alpha; \alpha(g)x) = (1; g^{-1}) \cdot (1; x_n^{-1}) \alpha(g)x_n = (1; g^{-1} x_n^{-1} \alpha(g)x_n) \in [\{1\} \rtimes P_n]
\]

Go to the projective limit of finite quotients (i.e. the profinite completion), up to diagonal extraction, there exists \( x \in \bar{P} \) such that \( x \mapsto \bar{x}_n \in P/P_n \) for infinitely many \( n \).

By definition, \( (\alpha; x) \in \bar{E} \) centralizes \( [\{1\} \rtimes P \) in \( \bar{E} \), i.e. \( (\alpha; x) \in C_{\bar{E}}([\{1\} \rtimes P) \). We now apply a deep result, [RSZ98, Prop 3.3a], to infer that in fact \( (\alpha; x) \in C_{\bar{E}}([\{1\} \rtimes P) \).
Moreover, \( C_E(\{1\} \times P) \triangleleft P \rtimes \langle \alpha^q \rangle \). Indeed, for conjugation we have the formula
\[
(\alpha^{-m}, \alpha^{-m}(y^{-1}))(1; g)(\alpha^m; y) = (1; \text{ad}_y \circ \alpha^m(g))
\]
so if \( (\alpha^m; y) \) centralizes \( \{1\} \rtimes P \) the image \([\alpha^m]\) is the identity in \( \text{Out}(P) \), and \( q \mid m \).

This now gives us that on one hand \((\alpha; x) \in \langle \alpha^q \rangle \rtimes \overline{P} \).

On the other hand, \((1; x) \in \langle \alpha^q \rangle \rtimes \overline{P} \), since \( \langle \alpha^q \rangle \rtimes \overline{P} \) is a subgroup \((\alpha; 1) \) also lies in \( \langle \alpha^q \rangle \rtimes \overline{P} \). Since \((\alpha; 1) \in \langle \alpha \rangle \rtimes P \), we have
\[
(\alpha; 1) \in \langle \alpha^q \rangle \rtimes \overline{P} \cap \langle \alpha \rangle \rtimes P = \langle \alpha^q \rangle \rtimes P.
\]

It follows that \( q = 1 \) — contradiction. \( \square \)

**Proof of Theorem 6.1.** We proceed by induction on \( l_{u.c.s}(N) \). If \( l_{u.c.s}(N) = 1 \) then \( N \) is finitely generated abelian and by the remark 5.6 we can find \( P \triangleleft \alpha \) \( N \) such that the map \( \text{Out}(N) = \text{GL}(N) \rightarrow \text{GL}(N/P) \) doesn’t kill any finite order elements.

Suppose now that \( l_{u.c.s}(N) = m \) and that the result held for all \( \alpha \)-groups \( M \) such that \( l_{u.c.s}(M) < m \), i.e. that we can compute a finite index subgroup \( L \triangleleft \alpha \) \( M \) that is deep enough. We have a short exact sequence
\[
1 \to \nu_1 N \to N \to N/\nu_1 N \to 1
\]
with \( l_{u.c.s}(\nu_1 N) = 1 \) and \( l_{u.c.s}(N/\nu_1 N) = m - 1 \). By the induction hypothesis we can construct subgroups \( N_0 \triangleleft \nu_1 N \) and \( K_0 \triangleleft \alpha \) \( N/\nu_1 N \) that separate torsion in \( \text{Out}(\nu_1 N) \) and \( \text{Out}(N/\nu_1 N) \) respectively.

The elements of \( \text{Out}_0(N) \) are either elusive or they aren’t. By Proposition 6.3 we can compute \( P_0 \triangleleft \alpha \) \( N \) that is good enough w.r.t. \( N_0, K_0 \) so by Lemma 6.6 all the non elusive elements \( [\beta] \in \text{Out}_0(N) \) survive in the natural map \( \text{Out}(N) \to \text{Out}(N/P_0) \). By Corollary 6.14 we can construct a finite set \( \{\beta_1, \ldots, \beta_n\} \) of automorphisms such that \( \{[\beta_1], \ldots, [\beta_n]\} \subset \text{Out}_0(N) \setminus \{1\} \) is the complete set elusive elements.

We now start iteratively constructing a chain of finite index subgroups
\[
N \triangleright \alpha \ P_0 \triangleright \alpha \ N_1 \triangleright \alpha \ N_2 \triangleright \alpha \ldots
\]
that converge to \( \{1\} \). By Proposition 6.15, for all \( j \) sufficiently large, the set \( \{\beta_1, \ldots, \beta_n\} \) is mapped monomorphically via the map \( \text{Out}(N) \to \text{Out}(N/N_j) \). Since \( P_0 \triangleright \alpha \ N_j \) we have that non-elusive elements of \( \text{Out}_0(N) \) survive in \( \text{Out}(N) \to \text{Out}(N/N_j) \), therefore for each new \( N_i \) we construct , it is enough to check that each element of our finite set \( \{[\beta_1], \ldots, [\beta_n]\} \) survives in \( \text{Out}(N) \to \text{Out}(N/N_i) \). Eventually this will be the case and we will have found \( P \triangleleft \alpha \) \( N \) that separates the torsion in \( \text{Out}(N) \). \( \square \)

**6.2 The mixed Whitehead problem for finitely generated nilpotent groups**

Let \( G \) be a finitely generated nilpotent group. We shall denote a tuple \( S = (s_1, \ldots, s_r) \), if \( S \) is a tuple of elements of a group \( H \) we shall abuse notation and write \( S \in H \). If
\[ S = (s_1, \ldots, s_r), T = (t_1, \ldots, t_r) \] are tuples in \( H \) and \( h \in H \) we will write \( S^h = T \) if
\[ h^{-1}s_ih = t_i \]
for \( i = 1, \ldots, r \). If \( \sigma \in \text{Aut} (H) \) and \( S = (s_1, \ldots, s_r) \in H \) then we denote
\[ \sigma(S) = (\sigma(s_1), \ldots, \sigma(s_r)). \]

**Definition 6.16** (The mixed Whitehead problem). Let \((S_1, \ldots, S_k), (T_1, \ldots, T_k)\) be tuples of elements in \( G \). The mixed Whitehead problem consists in deciding whether there exists \( \sigma \in \text{Aut} (G) \) and elements \( g_1, \ldots, g_k \in G \) such that
\[ \sigma(S_i) = T_{g_i}^i \]
for \( i = 1, \ldots, k \). If such is the case we say the tuples of tuples \((S_1, \ldots, S_k), (T_1, \ldots, T_k)\) are Whitehead equivalent.

The terminology mixed Whitehead problem first occurs in [BV11] where the mixed Whitehead problem is solved for torsion-free hyperbolic groups. The goal of this section is to give an algorithm that solves the mixed Whitehead problem for finitely generated nilpotent-groups.

It should be noted that the instances of the mixed Whitehead problem for a single pair of tuples, i.e. find if there is some \( \sigma \in \Gamma \) such that \( \sigma(S) = T \), are solvable by the results in [GS80b] and applying Algorithm A of [GS80a]. The situation here is more complicated because it is a two quantifier problem. We will remedy this by re-expressing the mixed Whitehead problem as an orbit problem for a certain semidirect product. As in [GS80a, GS80b] we must enter the realm of matrix groups.

### 6.2.1 Algebraic groups, arithmetic groups, \( \exp, \log \), etc.

We denote by \( \Sigma \) the class of finitely generated torsion-free nilpotent groups and call such groups \( \Sigma \)-groups.

We will always assume that the inclusions \( GL(n, \mathbb{Z}) < GL(n, \mathbb{Q}) < GL(n, \mathbb{C}) \) hold. There is a natural identification of the set of \( n \times n \) matrices with \( \mathbb{C} \)-coefficients with \( \mathbb{C}^{n^2} \), which we may regard as affine space. \( GL(n, \mathbb{C}) \) can therefore be viewed as an open algebraic subvariety of \( \mathbb{C}^{n^2} \), i.e. the set of matrices whose determinant is non-zero.

If \( S \) is some subset of \( \mathbb{C}[x_1, \ldots, x_n] \) we denote by \( V(S) \) the corresponding algebraic variety and if \( V \) is an algebraic variety we denote by \( I(V) \) the corresponding ideal.

**Definition 6.17.** [Algebraic group] A subgroup \( \mathcal{H} \leq GL(n, \mathbb{C}) \) is called algebraic if it is a closed algebraic subvariety \( V(S) \), i.e. it consists of matrices whose entries satisfy some set \( S \) of polynomial equations.

- \( S \) is a set of polynomials that have \( \mathbb{Q} \)-coefficients then \( \mathcal{H} \) is a \( \mathbb{Q} \)-defined algebraic group.
• If $S$ is explicitly given then $\mathcal{H}$ is an explicitly given $\mathbb{Q}$-defined algebraic group.

**Definition 6.18.** [Arithmetic group] Write $\mathcal{H}_\mathbb{Z} = \mathcal{H} \cap GL(n, \mathbb{Z})$. A subgroup $\Delta \leq GL(n, \mathbb{C})$ that is commensurable with $\mathcal{H}_\mathbb{Z}$ for some $\mathbb{Q}$-defined algebraic group $\mathcal{H}$ is called an arithmetic group. We say that $\Delta \leq \mathcal{H}_\mathbb{Z}$ is an explicitly given arithmetic subgroup if we are given the following:

1. We are given the system of polynomial $S$ so that $\mathcal{H} = V(S)$.
2. We are given a finite upper bound for the index $[\mathcal{H}_\mathbb{Z} : \Delta] \leq k$.
3. There is an effective procedure so that for each $g \in \mathcal{H}_\mathbb{Z}$ we can decide if $g \in \Delta$.

Let $\text{Tr}_1(n, R)$ denote the set of upper triangular $n \times n$ matrices with coefficients in the ring $R$ with 1s on the diagonal. We observe that $\text{Tr}_1(n, \mathbb{Z})$ and $\text{Tr}_1(n, \mathbb{C})$ are arithmetic and algebraic groups respectively. The connection to $\mathfrak{T}$-groups starts with the following result which is a rewording and weakening of Algorithm E and the supplement to Algorithm E in [GS80b].

**Theorem 6.19 ([GS80b]).** Given a finite presentation $\langle X \mid R \rangle$ of a $\mathfrak{T}$-group $G$ we can effectively find a suitable $n$ and an embedding

$$\Theta_G : G \hookrightarrow \text{Tr}_1(n, \mathbb{Z})$$

such that the natural map

$$N_{GL(n, \mathbb{Z})}(\Theta_G(G)) \rightarrow \text{Aut}(G)$$

is surjective.

**Definition 6.20.** A subgroup $H \leq \text{Tr}_1(n, \mathbb{Z})$ which occurs as the image $\Theta_G(G)$ of some $\mathfrak{T}$-group is called a $\Theta$-subgroup.

The discussion that will now follow is essentially contained in Chapter 6 of [Seg83] the reader may refer to this for more details and proofs.

We denote by $\text{Tr}_0(n, R)$ the set of upper triangular $n \times n$ matrices with coefficients in the ring $R$ with 0s on the diagonal. $\text{Tr}_0(n, R)$ equipped with matrix addition and the Lie bracket

$$(u, v) = uv - vu$$

is an $R$-Lie algebra. There are celebrated maps $\exp : \text{Tr}_0(n, \mathbb{C}) \rightarrow \text{Tr}_1(n, \mathbb{C})$ and $\log : \text{Tr}_1(n, \mathbb{C}) \rightarrow \text{Tr}_0(n, \mathbb{C})$ which are in fact mutually inverse isomorphisms in the category of algebraic varieties. Moreover these restrict to bijections between $\text{Tr}_1(n, \mathbb{Q})$ and $\text{Tr}_0(n, \mathbb{Q})$.

**Definition 6.21.** A subgroup $H \leq \text{Tr}_1(n, \mathbb{Q})$ is called radicable if for every $h \in H, n \in \mathbb{Z}_{\geq 1}$ there is some element $\sqrt[n]{h}$ such that $(\sqrt[n]{h})^n = h$.  

45
It is a result of Mal’cev [Mal49] that a \( T \)-group \( H \) embeds into a unique up to isomorphism radicable hull, or Mal’cev completion. We denote this by \( \sqrt{H} \).

A proof of the following result follows from the discussion in Chapter 6 of [Seg83] leading up to and including Theorem 2.

**Theorem 6.22 ([Seg83]).** \( \log \) sends radicable subgroups of \( \text{Tr}_1(n, \mathbb{Q}) \) to Lie subalgebras, i.e. linear subspaces closed under the Lie bracket, of \( \text{Tr}_0(n, \mathbb{Q}) \) bijectively. \( \exp \) does the inverse.

This bijective correspondence is far deeper than merely set-theoretic. Let \( H \leq \text{Tr}_1(n, \mathbb{Q}) \) and let \( \mathbb{Q} \log(H) \) be the smallest \( \mathbb{Q} \)-Lie subalgebra of \( \text{Tr}_0(n, \mathbb{Q}) \) containing \( \log(H) \) then (c.f. Theorem 2 of Chapter 6 of [Seg83])

\[
\exp(\mathbb{Q} \log(H)) = \sqrt{H}.
\]

In particular \( \mathbb{Q} \log(H) \) is a \( \mathbb{Q} \)-linear subspace of \( \text{Tr}_0(n, \mathbb{Q}) \) and it follows that:

**Proposition 6.23.** For any group \( H \leq \text{Tr}_1(n, \mathbb{Q}) \) there is a \( \mathbb{Q} \)-defined algebraic group \( \mathcal{H} \) such that:

\[
\mathcal{H} \cap \text{GL}(n, \mathbb{Q}) = \mathcal{H}_\mathbb{Q} = \sqrt{H}.
\]

The group \( \mathcal{H}_\mathbb{Z} \) is therefore the \( \text{GL}(n, \mathbb{Z}) \)-isolator of \( H \). We now have the following result which immediately follows from the results in Section 9 of [GS80b] up to Lemma 9.1.3.

**Lemma 6.24 ([GS80b]).** There is an algorithm which takes as input \( \langle X \mid R \rangle \) a finite presentation of a \( T \)-group \( G \) and outputs a positive integer \( m_H \) such that if \( H = \Theta_G(G) \leq \text{Tr}_1(n, \mathbb{Z}) \) is the isomorphic image constructed by Theorem 6.19 then the following holds: \( g \in H \) if and only if \( g \in \mathcal{H}_\mathbb{Z} \) and \( \pi_{m_H}(g) \in \pi_{m_H}(H) \) where \( \pi_{m_H} \) is the canonical epimorphism

\[
\pi_{m_H} : \text{GL}(n, \mathbb{Z}) \twoheadrightarrow \text{GL}(n, \mathbb{Z}/m_H \mathbb{Z})
\]

**Corollary 6.25.** There is an algorithm which takes as input \( \langle X \mid R \rangle \) a finite presentation of a \( T \)-group \( G \) and represents \( H = \Theta_G(G) \) as an explicitly given arithmetic group.

**Proof.** We abuse notation and consider \( G = H \). We can find a generating set \( X' = \langle x_1, \ldots, x_h \rangle \) of \( H \) which witnesses the fact that \( H \) is polycyclic (here \( h \) is the Hirsh length). Equipped with \( X' \), the maps \( \Theta_G \) and \( \log \) we can find elements \( \log(x_1), \ldots, \log(x_n) \) which generate \( \mathbb{Q} \log(H) \) (Lemma 9.1.1. of [GS80b]) as a \( \mathbb{Q} \)-Lie subalgebra so we can effectively find a system of linear equations \( S \) which define the algebraic variety \( \mathcal{C} \log(H) \). Seeing as \( \exp : \text{Tr}_0(n, \mathbb{C}) \rightarrow \text{Tr}_1(n, \mathbb{C}) \) is an isomorphism of algebraic varieties with inverse \( \log \) (both explicitly given) we can effectively find a system of polynomial equations \( \exp_*(S) \) which define the algebraic group \( \exp(\mathcal{C} \log(H)) = \mathcal{H} \leq \text{GL}(n, \mathbb{C}) \). This gives item 1. of Definition 6.18.

Items 2. and 3. of Definition 6.18 follow immediately from the criterion of Lemma 6.24. \( \square \)
Another Corollary (this is essentially Lemma 9.1.4. of [GS80b]) of Lemma 6.24 is:

**Corollary 6.26.** Let $H \leq \text{Tr}_1(n, \mathbb{Z})$ be a $\mathfrak{T}$-group and let $m_H$ be as in Lemma 6.24. For any $g \in \text{GL}(n, \mathbb{Z})$ the following are equivalent:

- $g^{-1}Hg = H$.
- $g^{-1}\mathcal{H}_Zg = \mathcal{H}_Z$ and $\pi_{m_H}(g^{-1})\pi_{m_H}(H)\pi_{m_H}(g) = \pi_{m_H}(H)$.

From which we deduce:

**Proposition 6.27.** There is an algorithm which takes as input $\langle X \mid R \rangle$ a finite presentation of a $\mathfrak{T}$-group $G$ and represents the $\text{GL}(n, \mathbb{Z})$-normalizer $N_{\text{GL}(n, \mathbb{Z})}(H) = \hat{\Gamma}$ of $H = \Theta_G(G) \leq \text{GL}(n, \mathbb{Z})$ as an explicitly given arithmetic group.

**Proof.** There is an explicitly given algebraic group $\mathcal{A}(H) \leq \text{GL}(m, \mathbb{C})$ given in (7) on page 613 of [GS80b], which satisfies:

$$\mathcal{A}(H)_\mathbb{Z} = N_{\text{GL}(n, \mathbb{Z})}(\mathcal{H}_\mathbb{Z})$$

by Corollary 6.26 the explicitly given arithmetic subgroup

$$\hat{\Gamma} = \{ g \in \mathcal{A}(H)_\mathbb{Z} \mid \pi_{m_H}(g^{-1})\pi_{m_H}(H)\pi_{m_H}(g) = \pi_{m_H}(H) \}$$

is exactly the normalizer $N_{\text{GL}(n, \mathbb{Z})}(H)$. \qed

### 6.2.2 The mixed Whitehead problem as an orbit problem

We first fix some notation. Let $H, K$ be groups and let

$$\varphi : K \to \text{Aut}(H)$$

$k \mapsto \varphi_k$

be a homomorphism. We denote the associated right semidirect product $K \ltimes_{\varphi} H$ with the multiplication rule

$$(k_1; h_1) \cdot (k_2; h_2) = (k_1k_2; \varphi_{k_2}^{-1}(h_1)h_2)$$

It is a standard result that if some group $G$ has subgroups $H, K$ with $H \vartriangleleft G$, $H \cap K = \{1\}$, and $G = KH$ then $G \cong K \ltimes_{\varphi} H$ where $\varphi$ is induced by conjugation.

Let $G$ be some group and let $\Gamma = \text{Aut}(G)$. There is a well defined right action of the right semidirect product $\Gamma \ltimes G^r$ on the set of $r$-tuples of tuples given by

$$(S_1, \ldots, S_r) \cdot (\sigma; (g_1, \ldots, g_r)) = (\sigma^{-1}(S_1)^{g_1}, \ldots, \sigma^{-1}(S_r)^{g_r}). \quad (17)$$

It immediately follows that
Proposition 6.28. Let \((S_1, \ldots, S_k), (T_1, \ldots, T_k)\) be tuples of elements in \(G\). They are Whitehead equivalent if and only if they lie in the same orbit under the \(\Gamma \ltimes G^r\)-action given in (17).

Our goal is to use the Grunewald-Segal orbit algorithm (we slightly changed the terminology to make it consistent with this paper):

Theorem 6.29 ([GS80a] Algorithm A). There exists an algorithm which takes as input:

- an explicitly given \(\mathbb{Q}\)-defined algebraic group \(G\),
- \(\rho\), an explicitly given rational action on some subset \(W \subset \mathbb{C}^n\),
- \(\Gamma\), an explicitly given arithmetic subgroup of \(G\), and
- two points \(a, b \in W \cap \mathbb{Q}^n\)

The algorithm decides whether there is some \(\gamma \in \Gamma\) such that \(\rho(\gamma) \cdot a = b\) and if so produces such a matrix \(\gamma\).

Remark 6.30. We will not give the full definition of a rational action but the only example we will consider is the action of \(GL(n, \mathbb{C})\) on \(GL(n, \mathbb{C}) \subset \mathbb{C}^{n^2}\) given by conjugation.

Definition 6.31. Let \(g_0, \ldots, g_m\) be \(n \times n\), matrices the \(mn \times mn\) matrix \(\text{diag}(g_0, \ldots, g_m)\) is the corresponding block-diagonal matrix.

Let \(H \leq GL(n, \mathbb{C})\) and let \(K \leq N_{GL(n, \mathbb{C})}(H)\) then we have a natural map \(\varphi : K \to \text{Aut}(H)\) given by \(\varphi_k(h) = khhk^{-1}\).

Lemma 6.32. Let \(H, K, \varphi\) be as above. Then the set of matrices

\[
T = \{\text{diag}(k, k_1h_1, \ldots, k_rh_r) \mid k \in K; h_1, \ldots, h_n \in H\}
\]

equipped with the induced multiplication is isomorphic to \(K \ltimes_{\varphi^r} H^r\) via

\[
(k; (h_1, \ldots, h_r)) \mapsto \text{diag}(k, k_1h_1, \ldots, k_rh_r).
\]

Proof. We first note that

\[
\text{diag}(k, k_1h_1, \ldots, k_rh_r) \cdot \text{diag}(k', k'_1h'_1, \ldots, k'_rh'_r) = \text{diag}(kk', kh_1k'h'_1, \ldots, k_rk_rh_r) = \text{diag}(kk', k'h'_1\varphi_{k'}^{-1}(h_1)h'_1, \ldots, k'_r\varphi_{k'}^{-1}(h_r)h'_r).
\]

The map (18) is therefore a homomorphism, which is clearly bijective.

Convention 6.33. If, \(H, K\) are as above we will abuse notation and denote the linear group \(T\) given in Lemma 6.32 simply as \(K \ltimes H^r\).

The following is obvious
Lemma 6.34. There is a well defined right rational action of $K \ltimes H^r$ on $\bigoplus_{i=1}^r H^{r_i}$ given by

$$(\oplus_{i=1}^r (h_{i,1},\ldots,h_{i,n_i})) \cdot \text{diag}(k,kh_1,\ldots,kh_r) = \oplus_{i=1}^r (h_i^{-1}k^{-1}h_{i,1}kh_i,\ldots,h_i^{-1}k^{-1}h_{i,n_i}kh_i)$$

$$= \oplus_{i=1}^r (\varphi_k^{-1}(h_{i,1})^k h_i,\ldots,\varphi_k^{-1}(h_{i,n_i})^k).$$

The final missing ingredient is an explicit arithmeticity result

Lemma 6.35. Let $G = K \ltimes \varphi H = KH$ and let $H' < H, K' < K$ be such that $[K : K'] = d, [H : H'] = e$. Suppose moreover that $H'$ is $\varphi(K')$-invariant. Then $K'H' = G'$ is a subgroup that satisfies $[G : G'] \leq ed$.

Proof. Since $G/H \approx K$ and $[K : K'] = d$ we immediately get $[KH : K'H] = d$. Now we have cosets

$$H = H'h_1 \sqcup \cdots \sqcup H'h_e$$

which means that

$$K'H = K'H'h_1 \sqcup \cdots \sqcup K'H'h_e$$

so $[K'H : K'H'] \leq e$ and the desired inequality follows. \hfill \Box

Proposition 6.36. Let $H \leq \text{Tr}_1(n,\mathbb{Z})$ be a $\Theta$-group, obtained from a finite presentation $(X | R)$, and let $\hat{\Gamma} = N_{GL(n,\mathbb{Z})}(H)$. Then the semidirect product $\hat{\Gamma} \ltimes H^r$ can be realized as an explicitly arithmetic subgroup of $GL((r+1)n,\mathbb{C})$.

Proof. Let $S_{\mathcal{X}}$ and $S_{\mathcal{A}(H)}$ be the explicit systems of polynomial equations defining the $\mathbb{Q}$-algebraic groups $\mathcal{X}$ and $\mathcal{A}(H)$ containing $H$ and $\hat{\Gamma}$ as arithmetic subgroups respectively. Let $\mathcal{S}$ be the set of matrices $\text{diag}(g_0,\ldots,g_r)$ such that the entries of $g_0$ satisfy $S_{\mathcal{A}(H)}$ and for $i = 1,\ldots,r$ the entries of $g_0^{-1} g_i$ satisfy $S_{\mathcal{X}}$. It immediately follows from the definition that

$$\mathcal{S} = S_{\mathcal{A}(H)} \ltimes S_{\mathcal{X}}^T.$$

By Lemma 6.35 we have the upper bound $[\mathcal{S} : \hat{\Gamma} \ltimes H^r] \leq e'd$ where $[A(H) : \hat{\Gamma}] = d$ and $[\mathcal{H}, H] = e$. Finally given some $\text{diag}(\alpha, g_1,\ldots,g_r) \in \mathcal{S}$ we can decide if $\alpha \in \hat{\Gamma}$ using the criterion of Corollary 6.26 and we can decide whether each $\alpha^{-1} g_i \in H$ with the criterion given by Lemma 6.24. \hfill \Box

We now give our solution to the mixed Whitehead problem for $\mathcal{X}$-groups.

Proposition 6.37. There is a uniform algorithm to solve the mixed Whitehead problem for $\mathcal{X}$-groups.

Proof. We are given a finite presentation $(X | R)$ of a $\mathcal{X}$-group $G$ and a pair of tuples of tuples $(S_1,\ldots,S_k), (T_1,\ldots,T_k)$.
**Step 1:** Construct the embedding $\Theta_G : G \to GL(n, \mathbb{Z})$ given by Theorem 6.19 and represent $H = \Theta_G(G)$ and $\hat{\Gamma} = N_{GL(n, \mathbb{Z})}(H)$ explicitly as arithmetic groups.

**Step 2:** Via $\Theta_G$ represent the tuples of tuples $(S_1, \ldots, S_k), (T_1, \ldots, T_k)$ as points $s, t$ in $\bigoplus_{i=1}^k GL(n, \mathbb{Z})^{n_i}$.

**Step 3:** Represent $\hat{\Gamma} \rtimes H^k$ explicitly as an arithmetic subgroup of $GL((k + 1)n, \mathbb{C})$.

By Theorem 6.19 since the natural map $\hat{\Gamma} \to \text{Aut } (H)$ is surjective, Proposition 6.28 implies that $(S_1, \ldots, S_k)$ and $(T_1, \ldots, T_k)$ are Whitehead equivalent if and only if the points $s, t$ are in the same $\hat{\Gamma} \rtimes H^r$ orbit via the action given in Lemma 6.34.

**Step 4:** Use [GS80a] Algorithm A to decide if $s, t \in \bigoplus_{i=1}^k GL(n, \mathbb{Z})^{n_i}$ are in the same $\hat{\Gamma} \rtimes H^r$-orbit via the action given in Lemma 6.34.

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### 6.2.3 Finitely generated nilpotent groups with torsion

If $G$ is a finitely generated nilpotent group, i.e. an $\mathfrak{N}$-group. Then the approach is essentially the same. We will construct an embedding $\hat{G}$ in to some $GL(m, \mathbb{C})$ as an arithmetic group and this embedding will be such that conjugation gives a surjection $N_{GL(m, \mathbb{C})}(G) \to \text{Aut } (G)$, the latter also being explicitly given as an arithmetic group. The proof will then go through immediately as in the proof of Proposition 6.37.

We now follow [GS80b] section 7. Let $G$ be a $\mathfrak{N}$-group, it is classical that $\tau(G)$, the set of finite order elements of $G$, in fact form a finite characteristic subgroup of $G$ and that $\bar{G} = G/\tau(G)$ is a $\mathfrak{T}$-group.

**Lemma 6.38** (c.f. [GS80b, Section 7]). There is an algorithm which given a finite presentation of a $\mathfrak{N}$-group $G$ we can algorithmically find the following:

- a finite set of words representing the set $\tau(G)$,
- an integer $m$ such that $G^m$ is a characteristic subgroup and $G^m \cap \tau(G) = \{1\}$ (here $G^m$ denotes set of $m$-th powers of $G$),

Let $\bar{G} = G/\tau(G)$ and let $\tilde{G} = G/G^m$ where $m$ is given as Lemma 6.38. By choice of $m$ have an embedding $\iota : G \hookrightarrow \bar{G} \times \tilde{G}$.

**Corollary 6.39** (c.f. [GS80b, Section 7]). There is an algorithm which given a finite presentation of a $\mathfrak{N}$-group $G$, will

- find a finite presentation for $\bar{G}$,
- write out a multiplication table for $\tilde{G}$,
- construct an explicit embedding $\iota : G \hookrightarrow \bar{G} \times \tilde{G}$.

Denote $\Gamma = \text{Aut } (\bar{G})$ and $\Delta = \text{Aut } (\tilde{G})$. Let $\Gamma \times \Delta$ act on $\bar{G} \times \tilde{G}$ componentwise.
Lemma 6.40 (c.f. [GS80b, Section 7]). Let $\pi : \tilde{G} \times \tilde{G} \to \tilde{G}/(\tilde{G})^m \times \tilde{G}$ be the canonical epimorphism.

- An element $(\tilde{g}, \tilde{h}) \in \tilde{G} \times \tilde{G}$ lies in $\iota(G)$ if and only if $\pi((\tilde{g}, \tilde{h})) \in \pi \circ \iota(G)$.
- The canonical map $\alpha : \text{Aut}(G) \to \Gamma \times \Delta$ is an embedding and an element $(\gamma, \delta) \in \Gamma \times \Delta$ lies in $\alpha(\text{Aut}(G))$ if and only if its canonical image $(\tilde{\gamma}, \tilde{\delta}) \in \text{Aut}(\tilde{G}/(\tilde{G})^m) \times \text{Aut}(\tilde{G})$ maps $\pi \circ \iota(G)$ to itself.

From which we immediately conclude

Corollary 6.41. The subgroups $\iota(G) \leq \tilde{G} \times \tilde{G}$ and $\alpha(\text{Aut}(G)) \leq \Gamma \times \Delta$ are finite index, and this index is computable. Moreover there is an effective procedure to decide the membership problem for these subgroups.

The next step is to explicitly present $\Gamma \times \Delta$ and $\tilde{G} \times \tilde{G}$ as arithmetic groups. We start with a sensible linear representation of finite groups.

Let $F$ be a finite group and denote by $S_F$ the set of permutations of the set $F$. Let $e_i$ denote the $i$-th standard basis vector of $C_{|F|}$ and let $f \mapsto e_i(f)$ be some (set-theoretic) embedding $F \hookrightarrow C_{|F|}$. This gives a natural faithful representation $\rho : S_F \hookrightarrow \text{GL}(|F|, \mathbb{C})$ where $\sigma \in S_F$ is sent to the matrix that permutes the set of basis vectors $\{e_1, \ldots, e_{|F|}\}$ accordingly.

We can embed $F \hookrightarrow S_F$ via the map $g \mapsto l_g$ where $l_g$ is the permutation of $F$ induced by left multiplication. Each element of $\text{Aut}(F)$ is already a permutation of $F$ so we put $\text{Aut}(F) \leq S_F$. Note that for all $g, k \in F$ and $\delta \in \text{Aut}(F)$ we have

$$
\delta \circ l_g \circ \delta^{-1}(k) = \delta \circ l_g(\delta^{-1}(k)) = \delta(g\delta^{-1}(k)) = \delta(g)k = l_{\delta^{-1}(g)}(k)
$$

So in this representation the elements of $\text{Aut}(F)$ act naturally on the image of $F$ via conjugation.

We can now embed $\tilde{G} \times \tilde{G}$ into a group of matrices as follows. Let $\Theta_{\tilde{G}} : \tilde{G} \hookrightarrow \text{GL}(|\tilde{G}|, \mathbb{C})$ be the embedding given in Theorem 6.19 and denote $H = \Theta_{\tilde{G}}(\tilde{G})$. Let $\rho : \tilde{G} \hookrightarrow \text{GL}(|\tilde{G}|, \mathbb{C})$ be the permutation representation described above. Then the map

$$
\psi : \tilde{G} \times \tilde{G} \mapsto \text{GL}(n + |\tilde{G}|, \mathbb{C})
$$

$$(\tilde{g}, \tilde{h}) \mapsto \text{diag}(\Theta_{\tilde{G}}(\tilde{g}), \rho(\tilde{h}))$$

has explicitly arithmetic image by Corollary 6.25 and the fact that $\tilde{G}$ is a finite group.

Similarly if $\hat{\Gamma}$ is as given in Proposition 6.27 then we also have an embedding:

$$
\mu : \hat{\Gamma} \times \Delta \mapsto \text{GL}(n + |\tilde{G}|, \mathbb{C})
$$

$$(k, \delta) \mapsto \text{diag}(k, \rho(\delta))$$
where \( \rho : \Delta \rightarrow GL(|\tilde{G}|, \mathbb{C}) \) is the permutation representation described above. \( \mu \) also has an explicitly arithmetic image.

**Lemma 6.42.** The image \( \psi \circ \iota(G) = K \) and the subgroup \( \hat{\Sigma} \) of \( \mu(\hat{\Gamma} \times \Delta) \) that normalizes \( K \) and naturally surjects onto \( \text{Aut}(G) \) via conjugation on \( K \) are explicitly given arithmetic groups.

**Proof.** By Corollary 6.41, to show that \( K \) is an explicitly given arithmetic group, it is enough to show there is an effective procedure to determine if for some \( g \in \psi(G \times \tilde{G}) \) whether \( g \in K = \psi \circ \iota(G) \).

By construction we have a finite presentation \( \langle Z \mid T \rangle \) of \( G \times \tilde{G} \) so by exhaustive searching we will find some word \( g(Z) \) such that \( \psi(g(Z)) = g \), and by Corollary 6.41 we can check whether \( g(Z) \in \iota(G) \) so we are done.

Obviously \( \mu(\hat{\Gamma} \times \Delta) \) normalizes \( \psi(G \times \tilde{G}) \) and by Theorem 6.19 and Proposition 6.27 there a surjection \( \mu(\hat{\Gamma} \times \Delta) \rightarrow \Gamma \times \Delta \) induced conjugation on \( \psi(G \times \tilde{G}) \). Now \( \text{Aut}(G) \) sits inside \( \Gamma \times \Delta \) as a finite index subgroup whose index is algorithmically bounded by Corollary 6.41 thus the index of \( \hat{\Sigma} \) in \( \mu(\hat{\Gamma} \times \Delta) \) also has this bound. So again we only need to check membership.

Let \( \sigma \in \mu(\hat{\Gamma} \times \Delta) \) and let \( \{g_1, \ldots, g_s\} \) be the image in \( \psi(G \times \tilde{G}) \) of a generating set of \( G \times \tilde{G} \), then again by exhaustive searching we can find preimages of \( \{g_1^\sigma, \ldots, g_s^\sigma\} \) in \( \tilde{G} \times \tilde{G} \) and thus by Corollary 6.41 decide if \( \sigma \) normalized \( K \) and hence lies in \( \hat{\Sigma} \).

Finally we have:

**Proof of Theorem 6.2.** By Lemma 6.42 after substituting \( \hat{\Sigma} \) in place of \( \hat{\Gamma} \) and \( K \) in place of \( H \) all the arguments from Lemma 6.32 to Proposition 6.37 go through.

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