On particle collisions near Kerr’s black holes

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Abstract

Scattering of particles in the gravitational field of rotating black holes is considered. Expressions for scattering energy of particles in the centre of mass system are obtained. It is shown that scattering energy of particles in the centre of mass system can obtain very large values not only for extremal black holes but also for nonextremal ones. It is shown that for realizing of the collisions with infinite energy one needs the infinite interval not only of the coordinate time but also the infinite interval of the proper time of the free falling particle.

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1. Introduction

In [1] we put the hypothesis that active galactic nuclei can be the sources of ultrahigh energy particles in cosmic rays observed recently by the AUGER group (see [2]) due to the processes of converting dark matter formed by superheavy neutral particles into visible particles — quarks, leptons (neutrinos), photons. If active galactic nuclei are rotating black holes then in [1] we discussed the idea that “This black hole acts as a cosmic supercollider in which superheavy particles of dark matter are accelerated close to the horizon to the Grand Unification energies and can be scattering in collisions.” It was also shown [3] that in Penrose process [4] dark matter particle can decay on two particles, one with the negative energy, the other with the positive one and particles of very high energy of the Grand Unification order can escape the black hole. Then these particles due to interaction with photons close to the black hole will loose energy analogously up to the Greisen-Zatsepin-Kuzmin limit in cosmology [5].

First calculations of the scattering of particles in the ergosphere of the rotating black hole, taking into account the Penrose process, with the result that particles with high energy can escape the black hole, were made in [6, 7]. Recently in [8] it was shown that for the rotating black hole (if it is the critical one) the energy of scattering is unlimited. The result of [8] was criticized in [9] in the sense that it does not occur in nature. The authors of [9] claimed that if the black hole is not a critical rotating black hole so that its dimensionless angular momentum $A \neq 1$ but $A = 0.998$ then the energy is limited. In this paper we show that the energy of scattering in the centre of mass system can be still unlimited in the cases of multiple scattering. In the part [2] we calculate the energy of collisions in the centre mass system for the particles falling onto a rotating black hole. For the case of the nonrelativistic at infinity particles we reproduce the results of [8]. In the part [3] we consider the case of the nonextremal black holes and show that in some cases (multiple scattering) the results of [8] on the limitations of the scattering energy for the nonextremal black holes are not valid. In the part [4] the collisions inside black holes are considered. The limiting formulas are obtained and it is shown that the collisions with infinite energy can not be realized. In the part [5] we show that for realizing of the collisions with infinite energy near events horizon of black holes one needs the infinite interval of as coordinate as proper time of the free falling particle.

The system of units $G = c = 1$ is used in the paper.

2. The energy of collisions in the field of Kerr’s black hole

Let us consider particles falling on the rotating chargeless black hole. The Kerr’s metric of the rotating black hole in Boyer–Lindquist coordinates has the form

$$
\begin{align*}
\text{d}s^2 = \text{d}t^2 &= -\frac{2Mr (\Delta - \alpha a \sin^2 \theta \text{d}\phi)^2}{\Delta + a^2 \cos^2 \theta}, \\
\text{d}r^2 &= \frac{r^2 (\Delta - \alpha a \cos^2 \theta) \text{d}^2 \theta - (r^2 + a^2) \sin^2 \theta \text{d} \phi^2},
\end{align*}
$$

where

$$
\Delta = r^2 - 2Mr + a^2,
$$

$M$ is the mass of the black hole, $J = aM$ is angular momentum. In the case $a = 0$ the metric [1] describes the static chargeless black hole in Schwarzschild coordinates. The event horizon for the Kerr’s black hole corresponds to the value

$$
r = r_H \equiv M + \sqrt{M^2 - a^2}.
$$

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The Cauchy horizon is
\[ r = r_c \equiv M - \sqrt{M^2 - a^2}, \]  
(4)

The surface called “the static limit” is defined by the expression
\[ r = r_0 \equiv M + \sqrt{M^2 - a^2 \cos^2 \theta}. \]  
(5)

The region of space-time between the horizon and the static limit is ergosphere.

For equatorial \((\theta = \pi/2)\) geodesics in Kerr’s metric \(E\) one obtains \(\Delta t = e \Delta t\), \(\Delta \phi = e \Delta \phi\), \(\Delta z = e \Delta z\), \(\Delta t > 0\) in the centre of mass system of two colliding particles with the same rest mass \(m\) and equal angular momentum \(L\).

The energy in the centre of mass frame of two colliding particles with the same rest mass \(m\) is the specific energy: the particle with rest mass \(m\) has the energy \(E\) \(\Delta t\) in the gravitational field \(E\). \(Lm = \) const is the angular momentum of the particle relative to the axis orthogonal to the plane of movement.

Let us find the energy \(E_{cm}\) in the centre of mass system of two colliding particles with the same rest mass \(m\) in arbitrary gravitational field. It can be obtained from

\[ E_{cm, 0,0,0} = m u_1' i_1 + m u_2' i_2, \]  
(9)

where \(u_i = \frac{dx_i}{ds}\) and due to \(u'_1 u_1 = 1\) one obtains

\[ E_{cm} = m \sqrt{2} \sqrt{1 + u_i' u_i}. \]  
(10)

The scalar product does not depend on the choice of the coordinate frame so \(E_{cm}\) is valid in an arbitrary coordinate system and for arbitrary gravitational field.

We denote \(x = r/M, \ A = a/M, \ l_n = l_n/M, \ \Delta x = x^2 - 2x + A^2\) and

\[ x_H = 1 + \sqrt{1 - A^2} , \quad x_C = 1 - \sqrt{1 - A^2}. \]  
(11)

For the energy in the centre of mass frame of two colliding particles with \(\epsilon_1 = \epsilon_2 = \epsilon\) and angular momenta \(L_1, L_2\), which are moving in Kerr’s metric one obtains \(E_{cm}\) using \(E\) \(\Delta t\) and \(\Delta \phi\):

\[ E_{cm} = \frac{m}{\sqrt{2}} \sqrt{1 + u_i' u_i}, \]  
(10)

To find the limit \(r \to r_H\) for the black hole with a given angular momentum \(A\) one must take in \((12)\) \(x = x_H + \alpha\) with \(\alpha \to 0\) and do calculations up to the order \(\alpha^2\). Taking into account \(A^2 = x_H x_C, x_H + x_C = 2\), after resolution of uncertainties in the limit \(\alpha \to 0\) one obtains

\[ \frac{E_{cm}(r \to r_H)}{2m} = \sqrt{1 + \frac{(l_1 - l_2)^2 (4 + l_1^2)}{16(l_1 - l_1)(l_1 - l_2)}}, \]  
(13)

where

\[ l_H = \frac{2x_m H}{A} = \frac{2x(1 + \sqrt{1 - A^2})}{A}. \]  
(14)

is the limiting value of the angular momentum of the particle close to the horizon of the black hole. It can be obtained from the condition of positive derivative in \(\Delta t/d\tau > 0\), i.e. going “forward” in time:

\[ l < l_H \left( 1 + \frac{x_H + \alpha}{2} \right) + o(\alpha), \quad x = x_H + \alpha. \]  
(15)

So close to the horizon one has the condition \(l \leq l_H\).

For the extremal black hole \(A = x_H = 1, l_H = 2x\) and the expression \(\Delta t\) takes the form

\[ \frac{E_{cm}(r \to r_H)}{2m} = \sqrt{1 + \frac{1}{4} \frac{(l_1 - l_2)^2}{(2x - l_1)(2x - l_2)}}, \]  
(16)

is divergent when the dimensionless angular momentum of one of the falling into the black hole particles \(l = 2x\). The scattering energy in the centre of mass system is increasing without limit (for case \(\epsilon = 1\) it was established in \(\|\|\)). For example, if \(l_1 = l_H\) then one obtains from Eq. \((12)\)

\[ E_{cm}(r \to r_H) = \frac{l_H - l_2}{x - 1} \left( \frac{\sqrt{3x^2 - 1}}{2} \right), \quad x = 1. \]  
(17)

Can one get the unlimited high energy of this scattering energy for the case of nonextremal black hole?

3. The energy of collisions for nonextremal black hole

In this section we consider the case \(\epsilon = 1\), when the particles falling into the black hole are nonrelativistic at infinity. Formula \(\|\|\) leads to limitations on the possible values of the angular momentum of falling particles: the massive particle free falling in the black hole with dimensionless angular momentum \(A\) to achieve the horizon of the black hole must have angular momentum from the interval

\[ -2 \left( 1 + \sqrt{1 + A} \right) = l_1 \leq l \leq l_2 = 2 \left( 1 + \sqrt{1 - A} \right). \]  
(18)

Putting the limiting values of angular momenta \(l_1, l_2\) into the formula \(\|\|\) one obtains the maximal values of the collision energy of particles freely falling from infinity

\[ E_{cm}^{\text{inf}}(r \to r_H) = \frac{2m}{\sqrt{1 - A^2}} \sqrt{\frac{1 - A^2 + \left( 1 + \sqrt{1 + A} + \sqrt{1 - A} \right)^2}{1 + \sqrt{1 - A^2}}}. \]  
(19)
For \( A = 1 - \epsilon \) with \( \epsilon \to 0 \) formula (19) gives:

\[
E_{\text{cm}}^{\text{max}}(r \to r_H) \sim 2 \left( \frac{1}{2} + \frac{1}{4} \right) \frac{m}{\epsilon^{1/4}}.
\]  
(20)

So even for values close to the extremal \( A = 1 \) of the rotating black hole \( E_{\text{cm}}^{\text{max}}/m \) can be not very large as mentioned in [9]. So for \( A_{\text{max}} = 0.998 \) considered as the maximal possible dimensionless angular momentum of the astrophysical black holes (see [11]), from (19) one obtains \( E_{\text{cm}}^{\text{max}}/m \approx 18.97 \).

Does it mean that in real processes of particle scattering in the vicinity of the rotating nonextremal black holes the scattering energy is limited so that no Grand Unification or even Planckian energies can be obtained? Let us show that the answer is no! If one takes into account the possibility of multiple scattering so that the particle falling from infinity on the black hole with some fixed angular momentum changes its momentum in the result of interaction with particles in the accreting disc and after this is again scattering close to the horizon then the scattering energy can be unlimited.

From (8) one can obtain the permitted interval in \( r \) for particles with \( \epsilon = 1 \) and angular momentum \( l = l_H - \delta \). To do this one must put the left hand side of (8) to zero and find the root. In the second order in \( \delta \) close to the horizon one obtains

\[
l = l_H - \delta \Rightarrow x < x_d \approx x_H + \frac{\delta^2 x_C^2}{4x_H \sqrt{1 - A^2}}.
\]  
(21)

The effective potential for the case \( \epsilon = 1 \) defined by the right hand side of (8)

\[
V_{\text{eff}}(x, l) = \frac{1}{2} \left( \frac{dr}{d\tau} \right)^2 = -\frac{1}{x} + \frac{\ell^2}{2x^3} - \frac{(A - \ell)^2}{x^3}
\]  
(22)

(see, for example, Fig. 1) leads to the following behaviour of \( r_{\text{eff}} \)

\[
10^2 V_{\text{eff}}
\]  

Figure 1: The effective potential for \( A = 0.95 \) and \( l_H = 2.45 \), \( l = 2.5 \), \( l_H = 2.76 \). Allowed zones for \( l = 2.5 \) are shown by the green color.

For the particle. If the particle goes from infinity to the black hole it can achieve the horizon if the inequality (13) is valid. However the scattering energy in the centre of mass frame given by (19) is not large. But if the particle is going not from the infinity but from some distance defined by (21) then due to the form of the potential it can have values of \( l < l_H - \delta \) large than \( l_H \) and fall on the horizon. If the particle falling from infinity with \( l \leq l_H \) arrives to the region defined by (21) and here it interacts with other particles of the accretion disc or it decays into a lighter particle which gets an increased angular momentum \( l_1 = l_H - \delta \), then due to (13) the scattering energy in the centre of mass system is

\[
E_{\text{cm}} \approx \frac{m}{\sqrt{\delta}} \sqrt{\frac{2(l_H - l)}{1 - \sqrt{1 - A^2}}}
\]  
(23)

and it increases without limit for \( \delta \to 0 \). For \( A_{\text{max}} = 0.998 \) and \( l_2 = l_K \), \( E_{\text{cm}} \approx 3.85m/\sqrt{\delta} \).

Note that for rapidly rotating black holes \( A = 1 - \epsilon \) the difference between \( l_H \) and \( l_K \) is not large

\[
l_H - l_K = 2 \frac{\sqrt{1 - A}}{A} \left( \sqrt{1 + A} - \sqrt{1 + A - A} \right)
\]  
(24)

For \( A_{\text{max}} = 0.998 \), \( l_H - l_K \approx 0.04 \) so the possibility of getting small additional angular momentum in interaction close to the horizon seems much probable. The probability of multiple scattering in the accretion disc depends on its particle density and is large for large density. Second scattering surely can be not on one trajectory (which is improbable) with fixed angular momentum but on all trajectories with angular momenta from the interval [18].

Here we consider the model when the gravitation of the accretion disc is treated as some perturbation much smaller than the gravitation of the black hole so it is not taken into account. One must also mention that “particles” are considered as elementary particles and not macroscopic bodies. So their “large” energy is limited by a Planckian value and we neglect back reaction of it on the Kerr’s metric of the macroscopic black hole. Electromagnetic and gravitational radiation of the particle surely can change the picture but one needs exact calculations to see what will be the balance.

4. Collision of particles inside the rotating black hole

As one can see from formula (12) the infinite value of the collision energy in the centre of mass system can be obtained inside the horizon of the black hole on the Cauchy horizon (4). Indeed, the zeroes of the denominator in (12): \( x = x_H \), \( x = x_C \), \( x = 0 \).

Let us find the expression for the collision energy for \( x \to x_C \).

Denote

\[
l_c = \frac{2c x_C}{A} = \frac{2c}{A} \left( 1 - \sqrt{1 - A^2} \right).
\]  
(25)

Note that for \( \epsilon = 1 \) the Cauchy horizon can be crossed by the free falling from the infinity particle under the same conditions on the angular momentum (18) as in case of the event horizon and \( l_H < l_c \leq l_K \).

To find the limit \( r \to r_C \) for the black hole with a given angular momentum \( A \) one must take in (12) \( x = x_C + \alpha \) and do calculations with \( \alpha \to 0 \). The limiting energy has three different expressions depending on the values of angular momenta.

If

\[
(l_1 - l_c)(l_2 - l_c) > 0 \, ,
\]  
(26)
i.e. \( l_1, l_2 \) are either both larger than \( l_C \), or both smaller than \( l_C \), then
\[
E_{c.m}(r \to r_C) = \frac{\sqrt{1 + \frac{(l_1 - l_2)^2(4 + l_C^2)}{16(l_c - l_1)(l_c - l_2)}}}{2m} \tag{27}
\]
This formula is similar to \((13)\) if everywhere \( H \leftrightarrow C \). If
\[(l_1 - l_c)(l_2 - l_c) = 0, \tag{28}\]
for example, \( l_1 = l_c \), then
\[
E_{c.m} \approx \sqrt{\frac{l_2 - l_c)}{(2l_2 - l_c)}(\varepsilon^2l_C + 2l_H), \quad x \to x_C. \tag{29}\]
If
\[(l_1 - l_c)(l_2 - l_c) < 0, \tag{30}\]
i.e. \( l_2 \in (l_c, l_1), \ l_1 \in (l_c, l_0) \) (or the opposite), then
\[
E_{c.m} \approx \sqrt{\frac{x_H(l_1 - l_c)(l_2 - l_c)}{x_C(x_H - x_C)(x - x_C)}}, \quad x \to x_C. \tag{31}\]
It is seen that the limits of \((29)\) and \((31)\) is infinite for all values of angular momenta \( l_1, l_2 \) \((28)\) and \((30)\). However, from Eq. \((6)\) we can see
\[
\frac{dt}{dx} (x \to x_C + 0) = \begin{cases} +\infty, & \text{if } l > l_c, \\ -\infty, & \text{if } l < l_c. \end{cases} \tag{32}\]
That is why the collisions with infinite energy can not be realized (see also \((12)\)).

5. The time of movement before the collision with unbounded energy

Let us show that in order to get the unboundedly growing energy one must have the time interval from the beginning of the falling inside the black hole to the moment of collision also growing infinitely. This is connected with the fact of infinity of coordinate interval of time needed for a freely particle to cross the horizon of the black hole (for Schwarzschild metric this question was considered in \((13)\)).

From Eq. \((12)\) one can see that the collision energy can get large values in the centre of mass system only if the collision occurs close to the horizon. Unboundedly large energy of collisions outside of a black hole is possible only for collisions on horizon \( x \to x_H \) (see \((17)\)).

From equation of the equatorial geodesic \((6)\), \((8)\) for a particle with dimensionless angular momentum \( l \) and specific energy \( \varepsilon = 1 \) (i.e. the particle is non relativistic at infinity) falling on the black hole with dimensionless angular momentum \( A \) one obtains
\[
\frac{dr}{dt} = \frac{(x - x_H)(x - x_C)}{\sqrt{x}} \sqrt{\frac{2x^2 - l^2 x + 2(A - l)^2}{x^3 + A^2 x + 2(A - l)^2}}. \tag{33}\]
So the coordinate time (proper time of the observer at rest far from the black hole) of the particle falling from some point \( r_0 = x_0M \) to the point \( r_f = x_fM > r_H \) is equal to
\[
\Delta t = M \int_{x_f}^{x_0} \frac{\sqrt{\frac{x^3 + A^2 x + 2(A - l)^2}{(x - x_H)(x - x_C)}}}{\sqrt{2x^2 - l^2 x + 2(A - l)^2}} dx. \tag{34}\]
In case of the extremal rotating black hole \((A = 1, \ x_c = x_H = 1)\) and the limiting value of the angular momentum \( l = 2 \) the integral \((34)\) is equal to
\[
\Delta t = M \left[ \frac{2 \sqrt{x^3 + 8x - 15}}{3(x - 1)} + 5 \ln \frac{\sqrt{x - 1} - 1}{\sqrt{x + 1} + 1} \right]_{x_f}^{x_0} \tag{35}\]
and it diverges as \((x_f - 1)^{-1}\) for \( x_f \to 1 \).

For the interval of proper time of the free falling to the black hole particle one obtains from \((34)\)
\[
\Delta \tau = M \int_{x_f}^{x_0} \frac{\sqrt{x^x} dx}{\sqrt{2x^2 - l^2 x + 2(A - l)^2}} \tag{36}\]
For \( A = 1, \ l = 2 \) the integral \((36)\) is equal to
\[
\Delta \tau = \frac{M}{3} \left[ 2 \sqrt{x(3 + x) + 3 \ln \sqrt{x - 1} - \sqrt{x + 1}} \right]_{x_f}^{x_0} \tag{37}\]
and it diverges logarithmically when \( x_f \to 1 \). So to get the collision with infinite energy one needs the infinite interval not only of the coordinate time but also the infinite interval of the proper time of the free falling particle.

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