The correction-to-scaling exponent in dilute systems

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The leading correction-to-scaling exponent \( \eta \) for the three-dimensional dilute Ising model is calculated in the framework of the \textit{el}d theoretic renormalization group approach. Both in the minimal subtraction scheme as well as in the \textit{el}d theory (resummed four loop expansion) excellent agreement with recent Monte Carlo calculations [Ballestatos H G et al. Phys. Rev. B 58, 2740 (1998)] is achieved. The expression of \( \eta \) as series in a power expansion up to \( O(\varepsilon^4) \) does not allow a reliable estimate for \( d = 3 \).

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From renormalization group (RG) theory one knows that the asymptotic regime the values of the critical exponents are universal and scaling laws between them hold. There the couplings of the model Hamiltonian describing the critical system have reached their fixed point values. In the nonasymptotic region deviations from the fixed point values are present. They die out according to a universal power law governed by the correction-to-scaling exponent \( \eta \). E.g. for the zero-eld susceptibility the approach from above to the critical temperature \( T_c \) is characterized by the so-called Wegner expansion \( \eta = 0 \),

\[ 0 + 1 + 1^2 + 2^2 + \ldots ; \tag{1} \]

where \( (T - T_c) = T_c \) and the \( i \) are the nonuniversal amplitudes, and are the asymptotic values of the susceptibility and correlation length critical exponents. The smaller the exponent \( \eta \), the larger is the region where corrections to the asymptotic power laws have to be taken into account. Being even further away from the fixed point it is necessary to consider the complete nonlinear crossover functions.

The implication of quenched dilution on the critical behavior is a long-standing problem attracting theoretical, experimental and numerical efforts. In the 3d-Ising model quenched disorder changes the asymptotic critical exponents compared to the pure one. In principle this statement should hold for arbitrary weak dilution. But in order to observe this change one should approach the critical point close enough. The width of this region turns out to be dilution dependent.

In particular Monte Carlo (MC) calculations of the critical exponents in the dilute 3d-Ising model are more difficult to perform than for the pure model since they need much larger sizes of lattices. Even then the exponents were found to be nonuniversal and varying continuously with dilution, i.e., they were effective ones. It became clear that a correction-to-scaling analysis is unavoidable and indeed universal exponents were found. Without it one still obtains concentration-dependent effective exponents.

The value of the correction-to-scaling exponent \( \eta \) found in MC calculations from an analysis invoking the first correction term in \( T - T_c \) turned out to be

\[ \eta = 0.37 \pm 0.06 ; \tag{2} \]

Thus it is almost half as large as its corresponding value in the pure model (see Table I) and this smallness of \( \eta \) in the dilute case explains its importance for an analysis of the asymptotic critical behavior. It is therefore highly desirable to have an independent quantitative theoretical prediction for the value of the correction-to-scaling exponent in the dilute system.

In theoretical calculations the value of \( \eta \) found by scaling \textit{el}d RG is \( \eta = 0.42 \). So far \textit{el}d theoretical RG studies mainly concentrated on the asymptotic values of the leading exponents. Correction-to-scaling exponents

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\textbf{Method} & \textbf{dilute} & \textbf{pure} \\
\hline
scaling \textit{el}d & 0.37 & 0.38 \\
\textit{el}d expansion & see text & 0.374 \\
\textit{el}d RG, \( d = 3 \) & 0.37 & 0.38 \textsuperscript{1} \\
\textit{el}d sub-RG, \( d = 3 \) & 0.39 & 0.42 \textsuperscript{1} \\
MC & 0.37 & 0.38 \textsuperscript{0} \\
\hline
\end{tabular}
\caption{Values of correction-to-scaling exponent \( \eta \) as obtained from different methods in dilute and pure 3d-Ising models.}
\end{table}
have been calculated within massive RG in two loop approximation in Ref. ([! = 0; 50]) and within the minimal subtraction scheme in three loop approximation in Ref. ([! = 0.36]). Here, we improve this value in the massive RG scheme up to four loop order with the result

\[ ! = 0.372 \, 0.005 \]  

in excellent agreement with \([2]\). In the minimal subtraction scheme we obtain \(! = 0.390 \, 0.04\) remaining within the band width of MC accuracy.

The critical behavior of the quenched weakly diluted Ising model in the Euclidian space of \(d = 4\) "dimensions" is governed by a Hamiltonian with two couplings \(\Lambda\):

\[ H = \frac{1}{2} \sum_{\alpha \beta} \phi_{\alpha} \phi_{\beta} + \phi_{\alpha} \phi_{\beta} + m_{\alpha} m_{\beta} \]

in replica limit \(! = 0\). Here \(\phi_{\alpha}\) are the components of order parameter; \(u_{\alpha} > 0, v_{\alpha} > 0\) are bare couplings; \(m_{\alpha}\) is bare mass.

We describe the long-distance properties of the model \([4]\) in the vicinity of the phase transition point using a "\(d\)-theoretical\(RG\) approach. The results presented in this paper are obtained on the basis of two different RG schemes; the renormalization conditions of massive renormalized theory at \(d = 3\) and the minimal subtraction scheme \([3]\). The last approach allows both protracted \(d = 3\) calculations \([3]\) as well as an "\(d\)-expansion.

In the RG method the change of the couplings \(u, v\) under renormalization is described by two functions \(u(\alpha; v) = \frac{\alpha u}{\alpha 0}; v(\alpha; u) = \frac{\alpha v}{\alpha 0} \)

where corresponds to the mass in the massive \(\alpha\)-theory approach and to the scale parameter in the minimal subtraction scheme. The subscript \([3]\) indicates that the derivatives are taken at constant unrenormalized parameters. The "functions at \(d\)-fixed \(\alpha\) RG scheme and in consequence the \(x\)ed \(\alpha\) point coordinates \(u, v\), defined by the simultaneous zeros of both \(\alpha\)-functions, are scheme dependent. The asymptotic critical exponents as well as the connection to scaling exponent do not depend on the RG scheme and take universal values.

The correction-to-scaling exponent \(\gamma\) is defined by the smallest eigenvalue of the matrix of derivatives of the \(\alpha\)-functions

\[
\frac{\partial \alpha}{\partial u} \frac{\partial \alpha}{\partial v} \]

taken at the stable \(x\)ed point. For the stable \(x\)ed point both eigenvalues of this \(\epsilon\) matrix have a positive real part.

Our results for the correction-to-scaling exponent are based on the known high order expansions for the functions \(u, v\). In the massive scheme they are known in four loop approximation \([3]\). In the minimal subtraction scheme one can obtain these functions in \(\epsilon\)-loop approximation in the replica limit from those of a cubic model \([3]\). In the limiting case of the pure model only the coupling \(u\) is present. The corresponding \(\alpha\)-functions results from putting \(v = 0\) in \(u(\alpha; v)\) and the correction-to-scaling exponent is simply the derivative \(\frac{\partial \alpha}{\partial u}(u, 0) = \gamma_u\) as taken at the stable \(x\)ed point \(u\). Note that for the pure model the \(\alpha\)-functions in the massive scheme are known in six loop approximation \([3]\) and the \(\epsilon\)-loop results for the RG functions in the minimal subtraction scheme agree with those recovered from Ref. \([3]\).

It is known that the series obtained in the perturbational RG approach are at best asymptotic for the dilute model see however Ref. \([3]\). An appropriate resummation procedure has to be applied to the \(\alpha\)-functions in order to obtain reliable information. The choice of the resummation procedure depends on the information about the high order behavior of the expansion series. This information is not available for the case of the \(\alpha\)-functions \([3]\). In this situation the most appropriate way to proceed is to use the Padé-Borel resummation \([3]\) generalized for the two variable case \([4]\).

The steps which we follow in the calculation of the correction-to-scaling exponent \(\gamma\) are the following: First the \(\alpha\)-functions \([3]\) are resummed, and the system of equations for the \(x\)ed \(\alpha\) points \(u(x; \gamma) = 0, v(x; \gamma) = 0\) is solved. Then the \(\epsilon\) matrix of derivatives \([3]\) is calculated for the resummed \(\epsilon\)-functions. The stability of the \(x\)ed points is checked. The \(x\)ed point with both \(u \geq 0\) and \(v \leq 0\) is the stable one at \(d = 3\) and the smallest eigenvalue gives the desired correction-to-scaling exponent. Note that the eigenvalues might be complex, in this case both have the same positive real part de ning \(!\).

In Fig. \([\epsilon]\) we present our results for the exponent \(!\) obtained in successive orders of perturbation theory in number of loops. To perform the resummation the Borel transform of the truncated \(1\)th order perturbation theory expansion for the \(\alpha\)-functions were presented in the form of \([\epsilon = 1]\) rational approximants of two variables \([3]\). This form of rational approximants appeared to give the most reliable results. The four loop results for the exponent \(!\) obtained in both RG schemes are given in the
second column of Tab. 1. For the uncertainty of the four loop results we simply take the difference between the four loop and the three loop result. This is suggested by the behavior of $\gamma$ in successive number of loops shown in Fig. 1. Although both RG schemes lead to comparable values for $\gamma$, the convergence of the values in the massive scheme is much faster (however the accidentally very small error for $\gamma$ derived from this procedure does not hold for other quantities). Note that the result for $\gamma$ combined with the corresponding four loop results for the asymptotic critical exponents $\nu$ and $\Delta$ confirms the conjectured inequality, $\nu < \Delta < 0$, for the random model.

As it was noted above, the loop results for the minimal subtraction scheme are available. In particular applying the resummation scheme to the pure Ising model case, $\nu = 0$, we get the following values for $\nu$ in increasing number of loops starting from two loop: $\nu = 0.566; 0.852; 0.756; 0.791$. This leads to an improvement in accuracy of the previously calculated $d = 3$ critical exponent $\nu$ since $\nu = 0.791 \pm 0.036$, the value and its uncertainty now is comparable to the six loop calculation within the massive scheme (see the third column of Tab. 1).

The degeneracy of the dilute Ising model functions on the one loop level leads to the $\tilde{\gamma}$-expansion. For the critical exponents this expansion is known up to $O(e^{8})$. Starting from the four loop results of Ref. 7 in the replica limit we get the following expansions for the eigenvalues $\lambda_{1}$ and $\lambda_{2}$ of the stability matrix $\lambda$: in the $\tilde{\gamma}$-expansion $\lambda_{1} = 2^{n+1} + 3704011194 a^{32} + 1138073837 a^{62}; \lambda_{2} = 0.6729265850 a^{12} + 1.925509085 a^{62} + 0.572521806 a^{32} + 1.393125952 a^{2} (7)$.

From naively adding the successive perturbational contributions one observes that already in three loop approximation $\lambda_{1}$ becomes negative and therefore no stable fixed point exists in strict $\tilde{\gamma}$-expansion. Even the resummation procedure we applied above, do not change this picture. This can be considered as indirect evidence that the $\tilde{\gamma}$-expansion is not Borel summable, as may be expected from Ref. 7. A physical reason might be the existence of the $\tilde{\gamma}$-th singularities caused by the zeros of the partition function of the pure system. The $\tilde{\gamma}$-expansion approach, both within the minimal subtraction scheme, seems to be the only reliable way to study critical behavior of the model by means of RG technique.

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