WHEN IS THE ÉTALE OPEN TOPOLOGY A FIELD TOPOLOGY?

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Abstract. We investigate the following question: Given a field $K$, when is the étale open topology $\mathcal{E}_K$ induced by a field topology? On the positive side, when $K$ is the fraction field of a local domain $R \neq K$, using a weak form of resolution of singularities due to Gabber, we show that $\mathcal{E}_K$ agrees with the $R$-adic topology when $R$ is quasi-excellent and henselian. Various pathologies appear when dropping the quasi-excellence assumption. For locally bounded field topologies, we introduce the notion of generalized t-henselianity (gt-henselianity) following Prestel and Ziegler. We establish the following: For a locally bounded field topology $\tau$, the étale open topology is induced by $\tau$ if and only if $\tau$ is gt-henselian and some non-empty étale image is $\tau$-bounded open. On the negative side, we obtain that for a pseudo-algebraically closed field $K$, $\mathcal{E}_K$ is never induced by a field topology.

1. Introduction

We continue the study of the étale open topology, initiated in [JTWY22] and continued in [WY23] and [JWY21]. Recall that the étale topology for a field $K$, also called $\mathcal{E}_K$, is given by a topology on the set of rational points $V(K)$ for every $K$-variety $V$ (a system of topologies in the terminology of [JTWY22]): concretely, the $\mathcal{E}_K$-topology on $V(K)$ is defined to have as a basis the collection of sets $f(W(K))$, where $W$ is another $K$-variety and $f: W \to V$ is an étale morphism.

The étale open topology is only interesting in the case of fields which are large in the sense of Pop (see [Pop14]) but not separably closed, since otherwise it degenerates to the discrete topology or the Zariski topology, respectively. Under this restriction, however, the abstract definition coincides with familiar topologies in many cases: Notably, over the fields $\mathbb{C}, \mathbb{R}, \mathbb{Q}_p$ we recover on each variety the Zariski topology, resp. real topology, resp. $p$-adic topology. In particular, for $\mathbb{R}$ and $\mathbb{Q}_p$, the étale open topology on every variety is induced by a Hausdorff non-discrete field topology on the ground field.

To generalize the phenomenon on $\mathbb{R}$ or $\mathbb{Q}_p$, consider a local domain $R \subset K$ with fraction field $K$, and recall that the $R$-adic topology on $K$ is the field topology with basis $\{aR + b: a \in K^*, b \in K\}$. Like any other field topology, this induces a topology on $V(K)$ for any $K$-variety $V$, which we also call the $R$-adic topology. If $R$ is a (non-trivial) valuation ring, then the $R$-adic topology is the usual valuation topology.

We now have the following facts relating $\mathcal{E}_K$ and $R$-adic topologies.

**Fact 1.1.** Let $R \subset K$ be a local domain with fraction field $K$.

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(1) [JWY21, Theorem 1.2] If \( R \) is henselian\(^1 \) then the \( R \)-adic topology refines \( \mathcal{E}_K \).

(2) [JTWY22, Theorem 6.15] If \( R \) is a valuation ring and the Henselization of \( K \) with respect to the corresponding valuation is not separably closed, then \( \mathcal{E}_K \) refines the \( R \)-adic topology.

(3) [JWY21, Theorem 1.2] If \( R \) is a regular (in the sense of commutative algebra) then \( \mathcal{E}_K \) refines the \( R \)-adic topology.

Here by the \( R \)-adic topology refining \( \mathcal{E}_K \) or vice versa we mean that the corresponding topologies on \( V(K) \) refine each other for every variety \( V/K \). (Note, however, that this is equivalent to merely saying that the same holds only on \( K^n = \mathbb{A}^n(K) \) for every \( n \), see Fact 2.3 below.)

The present paper is motivated by the following natural questions:

**Question 1.2.**

1. When is the \( \mathcal{E}_K \)-topology induced by a field topology?
2. When does the \( \mathcal{E}_K \)-topology agree with the \( R \)-adic topology for a local domain \( R \subset K \) with fraction field \( K \)?

We prove that the \( \mathcal{E}_K \) is not induced by a field topology when \( K \) is a pseudo-algebraically closed (PAC) field (Proposition 7.1 below), answering a question posed in [JTWY22, Section 8]. Since “most” algebraic extensions of \( \mathbb{Q} \) in a suitable sense are PAC, see [DF21, Proposition 1], this shows that the “generic” answer to Question 1.2(1) is negative.

In the other direction, we extend Fact 1.1 to quasi-excellent local domains, a wide class of non-pathological Noetherian domains:

**Theorem 1.3** (Theorem 4.5). If \( R \subset K \) is a quasi-excellent local domain with fraction field \( K \), then the \( \mathcal{E}_K \)-topology refines the \( R \)-adic topology.

Together with Fact 1.1(1), we deduce:

**Corollary 1.4.** If \( R \subset K \) is quasi-excellent henselian local domain (e.g. \( R \) a complete Noetherian local domain) with fraction field \( K \), then the \( R \)-adic topology coincides with the \( \mathcal{E}_K \)-topology.

In the case of a 1-dimensional Noetherian henselian local domain \( R \), we can even characterize precisely when the \( R \)-adic topology coincides with the étale open topology on the fraction field, see Corollary 3.5.

In Sections 5 and 6 we give examples of pathologies that can arise when the quasi-excellence assumption is dropped, exhibiting at the same time interesting behaviour of the étale open topology under finite field extensions.

Finally, to study Question 1.2 in much greater generality, we borrow the model-theoretic tools of [PZ78]. This allows to obtain comprehensive answers at least up to replacing the field \( K \) by a suitable elementary extension.

In this vein, it had previously been shown [JTWY22, Theorem B] that the \( \mathcal{E}_K \)-topology for \( K \) not separably closed is induced by a so-called V-topology on \( K \) if and only if \( K \) is a so-called t-henselian field, i.e. if and only if some elementary extension \( K^* \ni K \) carries a henselian valuation.

\(^1\)We recall the definition of a henselian local ring below. For a valuation ring, this agrees with the usual notion of henselianity.
We study a notion of gt-henselian field topologies, a natural generalization of the notion of a t-henselian field topology from \[\textit{PZ78}.\] In fact, this notion agrees with a different notion of henselianity for rings suggested (but hardly studied) in the literature, see Remark \[\textit{8.2}]. When \(\mathcal{E}_K\) is induced by a field topology, that topology must necessarily be gt-henselian (Lemma \[\textit{8.14}\]).

We then obtain the following answer to Question \[\textit{1.2}\] with the restriction to locally bounded field topologies:

**Theorem 1.5** (Proposition \[\textit{8.16}\]). Suppose that \(\tau\) is a locally bounded field topology on \(K\). Then \(\tau\) induces the \(\mathcal{E}_K\)-topology if and only if \(\tau\) is gt-henselian and some nonempty étale image in \(K\) is \(\tau\)-bounded.

**Theorem 1.6.** The \(\mathcal{E}_K\)-topology is induced by a locally bounded field topology if and only if there exists an elementary extension \(K^* \succ K\) and a henselian local domain \(R \subset K^*\) with fraction field \(K^*\) such that the \(R\)-adic topology induces \(\mathcal{E}_{K^*}\).

It remains open whether the \(\mathcal{E}_K\)-topology can ever be induced by a field topology which is not locally bounded.

### 2. Conventions and background

Throughout, \(K\) is a field and \(\text{Char}(K)\) its characteristic.

#### 2.1. Scheme theory

A \(K\)-variety is a separated \(K\)-scheme of finite type, not necessarily irreducible or reduced. (This is the convention of \[\textit{Poo17}, \text{Definition 2.1.1}\].) Throughout \(\mathbb{A}^n\) is \(n\)-dimensional affine space over \(K\), i.e. \(\mathbb{A}^n = \text{Spec} K[X_1, \ldots, X_n]\). We let \(V(K)\) be the set of \(K\)-points of a \(K\)-variety \(V\).

Given a scheme \(W\) we let \(\mathcal{O}_W\) be the structure sheaf of \(W\), \(\mathcal{O}_{W,p}\) be the local ring of \(W\) at \(p \in W\), and let \(\mathcal{O}_p = \mathcal{O}_{W,p}\) when \(W\) is clear.

**Fact 2.1.** Suppose that \(V \rightarrow W\) is a morphism of \(K\)-varieties. Then:

1. the induced map \(V(K) \rightarrow W(K)\) is \(\mathcal{E}_K\)-continuous.
2. if \(V \rightarrow W\) is étale then the induced map \(V(K) \rightarrow W(K)\) is \(\mathcal{E}_K\)-open.
3. the map \(K \rightarrow K, x \mapsto \alpha x + \beta\) is an \(\mathcal{E}_K\)-homeomorphism for any \(\alpha \in K^\times, \beta \in K\).
4. if \(n\) is prime to \(\text{Char}(K)\) then \(\{\alpha^n : \alpha \in K^\times\}\) is an étale open subset of \(K\).

**Proof.** (1), (2) is \[\textit{JTWY22}, \text{Lemma 5.2, 5.3}\], respectively. (3) follows from (1). Let \(\mathbb{G}_m = \text{Spec} K[X, X^{-1}]\) be the scheme-theoretic multiplicative group over \(K\). Then (4) follows from (2) as the morphism \(\mathbb{G}_m \rightarrow \mathbb{Q}_m, X \mapsto X^n\) is étale when \(n\) is prime to \(\text{Char}(K)\). \(\square\)

Suppose that \(L\) is an extension of \(K\) and \(V\) is a \(K\)-variety. We let \(V_L = V \times_{\text{Spec} K} \text{Spec} L\) be the base change of \(V\). Recall that \(V_L(L)\) is canonically identified with \(V(L)\), so we canonically equip \(V(L)\) with the \(\mathcal{E}_L\)-topology. Fact 2.2 below is \[\textit{JTWY22}; \text{Theorem 5.8}\].
Fact 2.2. Suppose that $L$ is an algebraic extension of $K$ and $V$ is a $K$-variety. Then the $\mathcal{E}_K$-topology on $V(K)$ refines the topology induced on $V(K)$ by the $\mathcal{E}_L$-topology on $V(L) = V_L(L)$, i.e. if $O \subseteq V(L)$ is $\mathcal{E}_L$-open then $O \cap V(K)$ is $\mathcal{E}_K$-open.

2.3. Ring topologies and field topologies. Our general reference for ring topologies and field topologies is [PZ78], and we follow its conventions. In particular, ring topologies are always taken to be Hausdorff and not discrete.

We have the following basic fact about comparisons between the étale open topology and a field topology, proven in [JTWY22, Lemma 4.8, Lemma 4.2].

Fact 2.3. Suppose that $\tau$ is a field topology on $K$. If the $\tau$-topology on each $K^n = \mathbb{A}^n(K)$ refines the $\mathcal{E}_K$-topology, then the $\tau$-topology on $V(K)$ refines the $\mathcal{E}_K$-topology for any $K$-variety $V$. If the $\mathcal{E}_K$-topology on $K$ refines $\tau$, then the $\mathcal{E}_K$-topology on $V(K)$ refines the $\tau$-topology for any $K$-variety $V$.

Let $R$ be a domain with fraction field $K = \text{Frac}(R)$, and assume $R \neq K$. The $R$-adic topology on $K$ is the topology with basis $\{aR + b: a \in K^\times, b \in K\}$. This is a ring topology. (Compare [PZ78] Example 1.2, although the name $R$-adic topology is not used there.) We are chiefly but not exclusively interested in the situation where $R$ is local.

We let $J(R)$ be the Jacobson radical of $R$. It is the intersection of all maximal ideals of $R$, or equivalently $J(R) = \{x \in R: 1 + xR \subseteq R^\times\}$.

Fact 2.4. The $R$-adic topology on $K$ is a field topology if and only if $J(R) \neq \{0\}$.

Proof. The right to left implication is [Joh20, Proposition 3.1]. We prove the left to right implication. Suppose that the $R$-adic topology is a field topology. Then inversion gives a continuous map $K^\times \to K^\times$. Hence there is nonzero $\alpha \in R$ such that $(1 + \alpha R)^{-1} \subseteq R$. Thus $(1 + \alpha R) \subseteq R^\times$ and $\alpha \in J(R)$. □

Fact 2.5 follows from Fact 2.3, 2.4 and the definitions. We leave the details to the reader.

Fact 2.5. Suppose that $R$ has nonzero Jacobson radical (so the $R$-adic topology is a field topology.) The following are equivalent:

1. The $\mathcal{E}_K$-topology on $V(K)$ refines the $R$-adic topology for any $K$-variety $V$.
2. $R$ contains a nonempty $\mathcal{E}_K$-open subset of $K$.

Given a ring topology $\tau$ on $K$, a set $B \subseteq K$ is called bounded if for every neighbourhood $U$ of 0 there exists a neighbourhood $V$ of 0 such that $V \cdot B \subseteq U$. The topology $\tau$ is locally bounded if there exists a bounded neighbourhood of 0.

Fact 2.6. Let $\tau$ be a ring topology on $K$ and $S$ an open subring of $K$. Then $K = \text{Frac}(S)$.

Proof. Suppose that $\alpha \in K$ and $\alpha \notin \text{Frac}(S)$. Note that $S$ is a neighbourhood of zero. Then $S \cap \alpha S = \{0\}$, hence $\tau$ is discrete, contradiction. □

Fact 2.7. Let $\tau$ be a ring topology on $K$ and $S$ a bounded open subring of $K$. Then $\tau$ is the $S$-adic topology.

Therefore the $R$-adic topologies are exactly the ring topologies which admit bounded open subrings.
Fact 2.9. Suppose that regular, i.e. if all localizations of \( R \)-stalks are regular local rings, and a Noetherian ring \( R \) is a notion of non-singularity. A locally Noetherian scheme is defined to be regular if all its closure of \( L \)-algebra. Then \( R = Y \) if for any \( f \in R[X] \) and \( \alpha \in R \) with \( f(\alpha) = 0 \neq f'(\alpha) \pmod{m} \) there is \( \alpha^* \in R \) such that \( f(\alpha^*) = 0 \) and \( \alpha^* \equiv \alpha \pmod{m} \).

2.4. Commutative algebra. Let \( R \) be local with maximal ideal \( \mathfrak{m} \). Then \( R \) is henselian if for any \( f \in R[X] \) and \( \alpha \in R \) with \( f(\alpha) = 0 \) \( \neq f'(\alpha) \pmod{m} \) there is \( \alpha^* \in R \) such that \( f(\alpha^*) = 0 \) and \( \alpha^* \equiv \alpha \pmod{m} \).

Fact 2.8. The following are equivalent for a local domain \( R \) with maximal ideal \( \mathfrak{m} \).

1. \( R \) is henselian,
2. If \( a_0, \ldots, a_{n-1} \in \mathfrak{m} \) then \( X^n+X^n+a_{n-1}X^{n-1}+\ldots+a_1X+a_0 \) has a root in \( \mathfrak{m}+1 \).
3. If \( a_0, \ldots, a_{n-1} \in \mathfrak{m} \) then \( X^n+X^n+a_{n-1}X^{n-1}+\ldots+a_1X+a_0 \) has a root in \( \mathfrak{m}-1 \).

Proof. (1)\(\Leftrightarrow\) (2) is in [Gab92, Proposition 1]. (2)\(\Leftrightarrow\) (3) follows by considering the substitution \( Y = -X \).

We gather some more intricate notions from commutative algebra, for use in Sections 3, 4 for the definitions and to [Rot97] for a friendlier introduction, as well as [ILO14, Exposé I] for a comprehensive overview. The class of excellent rings excludes certain pathologies that can arise for general Noetherian rings, but nevertheless includes virtually all “naturally occurring” Noetherian rings.

Fact 2.9. Suppose that \( R \) is a one-dimensional Noetherian domain, \( K \) is the fraction field of \( R \), and \( S \) is the integral closure of \( R \) in \( K \). Then \( S \) is a regular ring.

Proof. By Krull-Akizuki [Sta20, Tag 00PG] \( S \) is Noetherian, and by [Sta20, Tag 00OK] \( S \) is one-dimensional. A one-dimensional Noetherian normal domain is a Dedekind domain, hence regular [Sta20, Tag 034X].

Let \( R \) be a domain with fraction field \( K \) and \( S \) the integral closure of \( R \) in \( K \). Then \( R \) is normal if \( R = S \), \( R \) is N-1 if \( S \) is a finite \( R \)-module, and \( R \) is Japanese (or N-2) if the integral closure of \( R \) in any finite field extension of \( K \) is a finite \( R \)-module. Non-Japanese Noetherian rings are viewed as pathologies.

We now discuss quasi-excellent rings, a class of Noetherian rings, and the related slightly more restrictive class of excellent rings. The definitions in full generality are somewhat technical, so we omit them. We direct the readers to [Sta20, Tag 07QT, 07GH, 07P7, 00NL] for the definitions and to [Rot97] for a friendlier introduction, as well as [ILO14, Exposé I] for a comprehensive overview. The class of excellent rings excludes certain pathologies that can arise for general Noetherian rings, but nevertheless includes virtually all “naturally occurring” Noetherian rings.

We give a definition of quasi-excellence for local rings. Suppose that \( L \) is a field and \( R \) is an \( L \)-algebra. Then \( R \) is geometrically regular if \( R \otimes_L L^{\text{alg}} \) is regular, where \( L^{\text{alg}} \) is an algebraic closure of \( L \). Regularity implies geometric regularity when \( L \) is perfect. A morphism \( R \to S \) of Noetherian rings is regular if \( R \to S \) is flat and \( S \otimes_\mathcal{O} \mathcal{O} \text{Spec}(R/p) \) is geometrically regular over \( \text{Frac}(R/p) \) for every prime ideal \( p \) in \( R \). In scheme-theoretic language \( R \to S \) is regular if it is flat and every scheme-theoretic fiber of \( \text{Spec} S \to \text{Spec} R \) is geometrically regular.
See the discussion in [Mat80, Section 34] or [IL014, Exposé I, Proposition 5.5.1 (ii)] for Fact 2.10(1) and [HRW04, Corollary 2.3] or [Sta20, Tag 0C2F] for Fact 2.10(2).

**Fact 2.10.** Let $S$ be a Noetherian local ring.

1. $S$ is quasi-excellent if and only if $S \to \hat{S}$ is regular, where $\hat{S}$ is the completion.
2. If $S$ is either normal or henselian, then $S$ is quasi-excellent if and only if it is excellent.

We may take regularity of $S \to \hat{S}$ to be the definition of quasi-excellence for Noetherian local rings. Note that complete local Noetherian rings are trivially excellent by this definition. We collect some general facts.

**Fact 2.11.**

1. The class of normal rings is closed under localizations.
2. The class of quasi-excellent rings is closed under finite extensions, localizations, and quotients.
3. Complete local rings are excellent.
4. Quasi-excellent rings are Japanese.
5. The class of henselian local rings is closed under quotients.
6. The Henselization of a quasi-excellent local ring is quasi-excellent.
7. If $R$ is $\mathbb{N}$-1 and $\text{Char}(K) = 0$ then $R$ is Japanese.

**Proof.** (1) is [Sta20, Tag 00GY] and (2) is [Sta20, Tag 07QU]. (3) follows from Fact 2.10. (4) is [Sta20, Tag 07QV]. (5) follows easily from the definitions. (6) is [Gro67, Corollaire 18.7.6]. (7) is [Sta20, Tag 032M]. □

**Remark 2.12.** We now give some examples of excellent (in particular quasi-excellent) henselian local rings, most of which arise as local rings in various kinds of tame spaces. Let $L$ be a field.

1. Henselizations of localizations of finitely generated $L$-algebras are excellent. In particular the local ring
   \[ L[[t_1, \ldots, t_n]]_{\text{alg}} = \{ p \in L[[t_1, \ldots, t_n]] : p \text{ algebraic over } L(t_1, \ldots, t_n) \}\]
   is excellent. (This is the Henselization of the localization of $L[t_1, \ldots, t_n]$ at the maximal ideal $(t_1, \ldots, t_k)$: henselianity of $L[[t_1, \ldots, t_n]]_{\text{alg}}$ follows immediately from henselianity of $L[[t_1, \ldots, t_n]]$, and conversely the Henselization of $L[t_1, \ldots, t_n]$ at $(t_1, \ldots, t_n)$ is algebraically closed in the completion [Nag75, Corollary 44.3].) When $L$ is real closed $L[[t_1, \ldots, t_n]]_{\text{alg}}$ is the ring of germs of $n$-variable Nash functions at the origin [BCR98, Corollary 8.1.6].
2. Complete Noetherian local rings, such as $L[[t_1, \ldots, t_k]]$ and its quotients, are excellent.
3. If $L$ is complete with respect to a norm the ring of convergent power series $L\{t_1, \ldots, t_n\}$ in $n$-variables is an excellent local ring. (See [Nag75, Theorem 45.5] for henselianity, [Mat80, (34.B)] for excellence in the case of $L = \mathbb{R}$ or $L = \mathbb{C}$, and [Duc09, Théorème 2.13] for excellence in the non-archimedean case.) Quotients of $\mathbb{C}\{t_1, \ldots, t_n\}$ arise as local rings of complex analytic varieties and when $L$ is non-archimedean quotients of $L\{t_1, \ldots, t_n\}$ arise as local rings of Berkovich spaces, see [Duc09].

**Fact 2.13.** Let $R$ be a Noetherian henselian local domain. Then the integral closure $S$ of $R$ in a finite extension $L$ of its fraction field $K$ is also a henselian local domain.
Proof. The integral closure $S$ is a direct limit of domains which are finite over $R$. Any domain finite over $R$ is itself a henselian and local by the characterization \cite[Tag 04GG (10)]{sta20} of henselinity, and the class of henselian local domains is closed under direct limits. \qed

2.5. Resolution of Singularities. A scheme $V$ is regular if all of its local rings are regular. A resolution of singularities of a reduced Noetherian scheme $W$ is given by a regular scheme $V$ and a proper birational morphism $V \to W$. A resolution of singularities of a Noetherian ring $R$ is a resolution of singularities of $\text{Spec } R$.

Fact 2.14 is related to the fact that a one-dimensional reduced $K$-variety admits a resolution of singularities.

**Fact 2.14.** Suppose that $R$ is a one-dimensional Noetherian domain and let $S$ be the integral closure of $R$ in $K = \text{Frac}(R)$. Then the following are equivalent:

1. $\text{Spec } R$ admits a resolution of singularities.
2. $R$ is $N$-1 (i.e. $S$ is a finite $R$-module).
3. the natural morphism $\pi : \text{Spec } S \to \text{Spec } R$ is a resolution of singularities for $\text{Spec } R$.

Recall that if $T$ is a finite extension of $R$ in $K$ then $\text{Spec } T \to \text{Spec } R$ is birational.

**Proof.** By Fact 2.9 $S$ is regular. If (2) holds then $\pi$ is finite, hence proper and birational. So (2) implies (3). Clearly (3) implies (1). See \cite[Section 2.4, p. 11, last paragraph before Exercise 2.15]{cut04} for a proof that (1) implies (2). \qed

Fact 2.15 is a famous theorem of Hironaka \cite{hir64} (cited as in \cite[1.2 (i)]{tem13}).

**Fact 2.15.** Suppose that $R$ is a quasi-excellent local domain of residue characteristic zero. Then any reduced scheme of finite type over $R$ admits a resolution of singularities. In particular $R$ admits a resolution of singularities.

Fact 2.15 in positive residue characteristic is of course an open conjecture \cite[7.9.6]{gro65}. We use a weaker form of resolution of singularities due to Gabber. Suppose that $R$ is a Noetherian domain. An altered local uniformization of $R$ consists of regular integral schemes $V_1, \ldots, V_n$ and generically finite dominant morphisms $V_i \to \text{Spec } R$ of finite type such that every valuation ring $O$ containing $R$ can be prolonged to a valuation ring $O^*$ centered on some $V_i$, i.e. there exists a commutative diagram as follows:

$$
\text{Spec } O^* \longrightarrow V_i \\
\downarrow \quad \quad \quad \downarrow \\
\text{Spec } O \longrightarrow \text{Spec } R
$$

The valuative criterion for properness implies that a resolution of singularities is an altered local uniformization.

**Theorem 2.16** (Gabber). A quasi-excellent domain admits an altered local uniformization.

**Proof.** By \cite[Exposé VII, Théorème 1.1]{ilo14}, there are regular integral schemes $V_1, \ldots, V_n$ and finite type morphisms $\pi_i : V_i \to \text{Spec } R$ such that $\pi_1, \ldots, \pi_n$ are a covering family in the Grothendieck topology of alterations \cite[Exposé II, 2.3.3]{ilo14}. In particular, each $\pi_i$ is

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2The terminology is borrowed from \cite[1.2 (iv)]{tem13}.
dominant and generically finite. The prolongation property for valuation rings follows from [ILO14, Exposé IV, Théorème 4.2.1].

There are non-quasi-excellent Noetherian local domains which admit an altered local uniformization. For instance, this is trivially the case for regular local rings which are not quasi-excellent, see for example [Mat80, Chapter 13 (34.B)].

3. The one dimensional case

Let $R \subseteq K$ be a domain with fraction field $K$.

**Lemma 3.1.** Suppose that $R^*$ is a domain with $\text{Frac}(R^*) = K$. The following are equivalent:

1. The $R$-adic topology on $K$ refines the $R^*$-adic topology,
2. $R^*$ is $R$-adically open,
3. $\alpha R \subseteq R^*$ for some $\alpha \in K^\times$,
4. $R$ is bounded in the $R^*$-adic topology.

Hence the $R$-adic topology agrees with the $R^*$-adic topology if and only if there are $\alpha, \beta \in K^\times$ such that $\alpha R \subseteq R^*$ and $\beta R^* \subseteq R$, i.e. if $R, R^*$ is $R^*$-, $R$-adically bounded, respectively.

Lemma 3.1 follows easily from the definitions, so we leave it to the reader.

**Lemma 3.2.** Suppose that $R$ is Noetherian and $S$ is a subring of $K$ containing $R$. Then the following are equivalent:

1. $S$ is a finite $R$-module,
2. the $R$-adic and $S$-adic topologies on $K$ agree.

**Proof.** Suppose (2). By Lemma 3.1 there is $\alpha \in K^\times$ with $\alpha S \subseteq R$. As $R$ is Noetherian $\alpha S$ is a finite $R$-module, so $S$ is a finite $R$-module. Suppose (1). By Lemma 3.1 it is enough to show $\alpha S \subseteq R$ for some $\alpha \in K^\times$. We have $S = \beta_1 R + \ldots + \beta_n R$ for $\beta_1, \ldots, \beta_n \in K$. Fix $\alpha \in K$ with $\alpha \beta_i \in R$ for all $i$. Then $\alpha S = (\alpha \beta_1) R + \ldots + (\alpha \beta_n) R \subseteq R$. □

**Lemma 3.3.** Suppose that $R$ is Noetherian and $S$ is the integral closure of $R$ in $K$. Then the following are equivalent:

1. $R$ is $N$-1,
2. the $R$-adic and $S$-adic topologies on $K$ agree.

Proposition 3.4 is a partial converse to our theorem that if $R$ is an excellent henselian local domain then the $R$-adic and $E_K$-topologies agree. (Recall that an excellent ring is $N$-1.)

**Proposition 3.4.** Suppose that $R$ is a Noetherian, henselian, and local, and the $E_K$-topology agrees with the $R$-adic topology. Then $R$ is $N$-1.

**Proof.** Let $S$ be the integral closure of $R$ in $K$. As $R \subseteq S$, the $R$-adic topology refines the $S$-adic topology. By Fact 2.13, $S$ is a henselian local ring. By Fact 1.1(1) the $S$-adic topology on $K$ refines the $E_K$-topology. Hence the $S$-adic topology agrees with the $R$-adic topology. Apply Lemma 3.3 □

**Corollary 3.5.** Suppose that $R$ is one-dimensional, Noetherian, and henselian local. Then the following are equivalent:
(1) the $\mathcal{E}_K$-topology agrees with the $R$-adic topology.
(2) $R$ is $N$-1.
(3) Spec $R$ admits a resolution of singularities.

Proof. The equivalence of (2) and (3) is Fact 2.14. Proposition 3.3 shows that (1) implies (2). Suppose that $R$ is $N$-1 and let $S$ be the integral closure of $R$ in $K$. Then $S$ is itself local by Fact 2.13. By Lemma 3.3 it is enough to show that the $\mathcal{E}_K$-topology agrees with the $S$-adic topology. By Facts 2.13 and 1.1(1) the $S$-adic topology refines the $\mathcal{E}_K$-topology.

The equivalence of (2) and (3) is Fact 2.14. By Lemma 3.3 it is enough to show that the $\mathcal{E}_K$-topology agrees with the $S$-adic topology. By Facts 2.13 and 1.1(1) the $S$-adic topology refines the $\mathcal{E}_K$-topology.

\section{The quasi-excellent case}

4. The quasi-excellent case

Let again $R \subseteq K$ be a domain with fraction field $K$. The central result of this section is the following theorem, which generalizes Fact 1.1(3).

Theorem 4.1. Suppose that $R$ is local, normal, and Noetherian. If $R$ admits an altered local uniformization, then the étale open topology over $K$ refines the $R$-adic topology.

We gather some lemmas. Fact 4.2 is a slight generalization given in [JWY21 Lemma 4.3] of a result of Jensen and Lenzig [JL89, pg 52,55].

Fact 4.2. Suppose that $R$ is a regular local domain with maximal ideal $m$ and dim $R \geq 2$.

(1) If $\text{Char}(R/m) \neq 2$ and $\alpha, \beta \in K$ satisfy $1 + \alpha^4 = \beta^2$ then $\alpha \in R$ or $1/\alpha \in R$.

(2) If $\text{Char}(R/m) = 2$ and $\alpha, \beta \in K$ satisfy $1 + \alpha^3 = \beta^3$ then $\alpha \in R$ or $1/\alpha \in R$.

Lemma 4.3. Suppose that $R$ is a local domain, $O_1, \ldots, O_k$ are discrete valuation subrings of $K$ with maximal ideals $m_1, \ldots, m_k$, respectively. Let $a = m_1 \cap \ldots \cap m_k$. Suppose that one of the following holds:

(1) $\text{Char}(K) \neq 2$ and if $\alpha \in a, \beta \in K$ satisfy $1 + \alpha^4 = \beta^2$ then $\alpha \in R$,

(2) $\text{Char}(K) \neq 3$ and if $\alpha \in a, \beta \in K$ satisfy $1 + \alpha^3 = \beta^3$ then $\alpha \in R$.

Then the étale open topology over $K$ refines the $R$-adic topology.

Proof. Let us assume that $\text{Char}(K) \neq 2$. By Fact 2.5 it is enough to show that $R$ has interior in the $\mathcal{E}_K$-topology. Each $m_i$ is $\mathcal{E}_K$-open by Fact 1.1(2). (The condition there that the henselization of $K$ with respect to $O_i$ is not separably closed holds since $O_i$ is discrete; cf. also [JTMY22 Corollary 6.17].) Hence $a$ is $\mathcal{E}_K$-open. Let $B = \{\beta^2 : \beta \in K^\times\}$ and $f: K \to K$ be given by $f(\alpha) = 1 + \alpha^4$. By Fact 2.4 $f^{-1}(B)$ is $\mathcal{E}_K$-open. Note that $f^{-1}(B) \cap a$ is contained in $R$ by (1) in the assumption. Finally $f^{-1}(B) \cap a$ is nonempty as $f^{-1}(B)$ and each $m_i$ is an $\mathcal{E}_K$-neighbourhood of zero. The argument for $\text{Char}(K) \neq 3$ is analogous.

Lemma 4.4. Suppose that $R$ is normal, local, and Noetherian, and $\alpha \in K \setminus R$. Then there exists a valuation ring $\mathcal{O}$ of $K$ dominating $R$ with $\alpha \notin \mathcal{O}$.

Proof. By normality there is a height one prime ideal $p$ in $R$ such that $\alpha \notin \mathcal{O}_p$, see [Mat80 Chapter 7 (17.H) Theorem 38]. Then $\mathcal{O}_p$ is normal by Fact 2.11(1), and so $\mathcal{O}_p$ is a DVR since it is a one-dimensional normal local domain [Sta20, Tag 00PD]. By Chevalley’s extension theorem there is a valuation ring $\mathcal{O}^*$ of Frac($\mathcal{O}/p$) dominating the local ring $R/p$. Let $\mathcal{O}$ be the valuation ring corresponding to the composition of the places associated to $\mathcal{O}_p$ and $\mathcal{O}^*$. Then $\alpha \notin \mathcal{O}$ since $\mathcal{O} \subseteq \mathcal{O}_p$, and by construction $\mathcal{O}$ dominates $R$. \qed
We now prove Theorem 4.1.

Proof. Let $\Pi = (X_i \xrightarrow{\pi_i} \text{Spec } R : i \in \{1, \ldots, n\})$ be an altered local uniformization of $R$. We make some definitions and constructions for arbitrary $i \in \{1, \ldots, n\}$. Recall that $X_i$ is integral and let $K_i$ be the function field of $X_i$. Let $m$ be the maximal ideal of $R$. Then $m$ is the closed point of $\text{Spec } R$, hence $\pi_i^{-1}(m)$ is a proper closed subset of $X_i$ by dominance of $\pi_i$. The set $\pi_i^{-1}(m)$ has only finitely many irreducible components, each of which is contained in an irreducible codimension one subset of $X_i$. Let $A_i$ be a finite set of codimension one points in $X_i$ such that every point in $\pi_i^{-1}(m)$ is in the closure of some $p \in A_i$. By regularity $O_{X_i,p}$ is a DVR for every $p \in A_i$. Let $O_{i,p} = O_{X_i,p} \cap K$ for every $i \in \{1, \ldots, n\}$ and $p \in A_i$. Since the extension $K_i/K$ is finite, $O_{i,p}$ is a (non-trivial) DVR. Let $m_{i,p}$ be the maximal ideal of each $O_{i,p}$.

Now suppose first that $\text{Char}(R/m) \neq 2$, hence $\text{Char}(K) \neq 2$. Suppose that $\alpha, \beta \in K$ satisfy $1 + \alpha^4 = \beta^2$ and $\alpha$ is in $\bigcap_{1 \leq i \leq n} \bigcap_{p \in A_i} m_{i,p}$. By Lemma 4.3 it is enough to show that $\alpha \notin R$. We suppose towards a contradiction that $\alpha \notin R$. By Lemma 4.4 there exists a valuation subring $O$ of $K$ dominating $R$ with $\alpha \notin O$. By the defining property of altered local uniformizations, there is $i \in \{1, \ldots, n\}$, $p \in A_i$, and a valuation subring $O^*$ of $K_i$ such that $O^*$ prolongs $O$ and $O^*$ dominates $O_{X_i,p}$. Thus $\alpha \notin O_{X_i,p}$. Fact 4.2 shows that $1/\alpha \in O_{X_i,p}$ when $\dim O_{X_i,p} \geq 2$; in fact the same holds if $\dim O_{X_i,p} = 1$, since then $O_{X_i,p}$ is a valuation ring. Since $\pi_i(p) = m$ as $O^*$ dominates $R$, by construction we can take $q \in A_i$ such that $O_{X_i,p} \subseteq O_{X_i,q}$. Then $1/\alpha \in O_{X_i,q} \cap K = O_{i,q}$, which is a contradiction as $\alpha \in m_{i,q}$.

Finally, suppose that $\text{Char}(R/m) = 2$, hence $\text{Char}(K) \neq 3$. Follow the same argument as above, replacing $1 + X^4 = Y^2$ with $1 + X^3 = Y^3$, and apply the second case of Lemma 4.3. \[\Box\]

Theorem 4.5. If $R$ is quasi-excellent local then the $\mathcal{E}_K$-topology refines the $R$-adic topology.

Proof. By Fact 2.11(4) $R$ is N-1, so the integral closure $S$ of $R$ in $K$ is finite over $R$. By Lemma 4.3 it is enough to show that the $\mathcal{E}_K$-topology refines the $S$-adic topology. It is enough to show that $S$ is $\mathcal{E}_K$-open. Since $S$ is finite over the local ring $R$, $S$ has only finitely many maximal ideals $m_1, \ldots, m_k$. Let $S_i$ be the localization of $S$ at $m_i$ for each $i \in \{1, \ldots, k\}$. Then $S = S_1 \cap \ldots \cap S_k$, so it is enough to fix $i \in \{1, \ldots, k\}$ and show that $S_i$ is $\mathcal{E}_K$-open. Note that $S_i$ is a localization of a finite extension of the quasi-excellent ring $R$ and $S_i$ is a localization of the normal ring $S$. By Fact 2.11 $S_i$ is quasi-excellent and normal. Theorem 4.4 (which applies by Theorem 2.16) shows that $S_i$ is $\mathcal{E}_K$-open. \[\Box\]

Remark 4.6. In [1189] Theorem 3.35, the henselian case of Fact 4.2 is used to prove that any henselian regular local domain is first-order definable in its fraction field. Whether the same holds for a henselian quasi-excellent local domain $R$ remains open.

We only obtain the weaker statement that the $R$-adic topology is definable in the fraction field $K$, i.e. there is a definable family of sets forming a basis for the $R$-adic topology: Indeed, we have shown in Lemma 4.3 that there is an étale image $\emptyset \neq U \subseteq K = \mathbb{A}_K^1(K)$ contained in $R$. Since $U$ is definable and open, the family $\{aU + b : a \in K^\times, b \in K\}$ is a definable basis for the $R$-adic topology.

Essentially the same argument shows that whenever $\mathcal{E}_K$ is induced by a locally bounded field topology $\tau$ (a situation which we shall study later in some detail), the topology $\tau$ is definable.
5. Behaviour of $\mathcal{E}_K$ under field extension

Suppose that $L/K$ is a finite field extension and let $[L : K] = d$. We briefly describe the extension $\text{Ext}_{L/K}(\mathcal{E}_K)$ of the $\mathcal{E}_K$-topology to $L$, see [JTWY22 Section 4.5] for details. After fixing a $K$-basis for $L$ we may identify each $L^n$ with $K^{dn}$. We declare the $\text{Ext}_{L/K}(\mathcal{E}_K)$-topology on $L^n$ to be the $\mathcal{E}_K$-topology on $K^{dn}$. This topology does not depend on the choice of the $K$-basis. More generally, given a quasi-projective $L$-variety $V$ the $\text{Ext}_{L/K}(\mathcal{E}_K)$-topology on $V(L)$ is the $\mathcal{E}_K$-topology on the $K$-points of the Weil restriction of $V$ (this set is canonically identified with $V(L)$.) Any variety is Zariski-locally quasi-projective, so we can define the $\text{Ext}_{L/K}(\mathcal{E}_K)$-topology on the $K$-points of an arbitrary $K$-variety in a natural way. For example $\text{Ext}_{\mathbb{C}/\mathbb{R}}(\mathcal{E}_R)$ is the usual complex analytic topology over $\mathbb{C}$.

Endowing all $V(L)$ with the $\text{Ext}_{L/K}(\mathcal{E}_K)$-topology gives a well-behaved system of topologies in the sense of [JTWY22 Definition 1.2], see the following consequence of [JTWY22 Proposition-Definition 4.17].

**Fact 5.1.** Suppose that $L/K$ is finite and $V \to W$ is a morphism of $L$-varieties. Then $V(L) \to W(L)$ is $\text{Ext}_{L/K}(\mathcal{E}_K)$-continuous. In particular $L \to L, x \mapsto ax + \beta$ is an $\text{Ext}_{L/K}(\mathcal{E}_K)$-homeomorphism for any $\alpha \in L^\times$, $\beta \in L$.

By [JTWY22 Proposition 5.7] $\text{Ext}_{L/K}(\mathcal{E}_K)$ refines $\mathcal{E}_L$ for any finite $L/K$. We would like to know when this refinement is strict.

Up to now we knew two examples. If $K$ is real closed and $L = K(\sqrt{-1})$ then $\mathcal{E}_L$ is the Zariski topology and $\mathcal{E}_K$ is the order topology, hence $\text{Ext}_{L/K}(\mathcal{E}_K)$ strictly refines $\mathcal{E}_L$. Recall that the following are equivalent by [JTWY22 Theorem C.1]:

1. $L$ is large,
2. the $\mathcal{E}_L$-topology on $L$ is not discrete,
3. the $\mathcal{E}_L$-topology on $V(L)$ is not discrete when $V$ is an $L$-variety with $V(L)$ infinite.

By [Sri19] there are non-large fields with large finite extensions. If $L$ is large and $K$ is not then the $\mathcal{E}_K$-topology on $K^d$ is discrete, hence the $\text{Ext}_{L/K}(\mathcal{E}_K)$-topology on $L$ is discrete, and the $\mathcal{E}_L$-topology on $L$ is not discrete.

We give a third example where $\text{Ext}_{L/K}(\mathcal{E}_K)$ strictly refines $\mathcal{E}_L$. This is also the first example where both $\mathcal{E}_K, \mathcal{E}_L$ are non-discrete field topologies.

**Theorem 5.2.** Let $R$ be a henselian regular local domain and $L$ a finite extension of the fraction field $K$ of $R$ such that the integral closure $S$ of $R$ in $L$ is not a finite $R$-module. Then $\text{Ext}_{L/K}(\mathcal{E}_K)$ strictly refines $\mathcal{E}_L$.

Note that any $R$ as in the theorem is by definition not Japanese and hence not quasi-excellent. Since regular local rings are normal, Fact [2.117] shows that the theorem is only ever applicable in positive characteristic.

Before proving the theorem, we give an important special case in the language of valued fields. See [Kuh11 Example 3.5] for an example of this situation.

**Corollary 5.3.** Let $v$ be a henselian discrete valuation on a field $K$, $L/K$ a finite extension and $v'$ the unique prolongation of $v$ to $L$. If $(L, v')/(K, v)$ is a defect extension, i.e. $e(v'/v)f(v'/v) \leq [L : K]$ where $e$ and $f$ are the relative ramification index and inertia degree, then $\text{Ext}_{L/K}(\mathcal{E}_K)$ strictly refines $\mathcal{E}_L$. 

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Proof. Let \( R \) and \( S \) be the valuation rings of \( v \) and \( v' \), respectively. Both are discrete valuation rings, in particular regular local domains. Furthermore, \( S \) is the integral closure of \( R \) in \( L \) [Bou06, Chap. V, §8, no 3, Remarque], and the defect condition implies that \( S \) is not a finite \( R \)-module [Bou06, Chap. V, §8, no 5, Théorème 2]. Hence Theorem 5.2 is applicable.

Proof of Theorem 5.2. Let \( b_1, \ldots, b_d \in L \) be a \( K \)-basis of \( L \). By scaling with a suitable element of \( R \), we may assume that for all \( i \), \( b_i \in S \). Hence \( R' := R[b_1, \ldots, b_d] \) is a finite \( R \)-module. The fraction field of \( R' \) is \( L \), and \( S \) is the normalization of \( R' \). By assumption, \( S \) is not a finite \( R' \)-module, and thus by Lemma 3.2 the \( R' \)-adic topology on \( L \) strictly refines the \( S \)-adic topology. The \( S \)-adic topology in turn refines \( \mathcal{E}_L \) (not necessarily strictly) by Fact 1.1(1), since \( S \) is henselian local by Fact 2.13.

Under the identification of \( L \) with \( K^d \) given by the basis \( b_1, \ldots, b_d \), the subgroup \( b_1 R + \cdots + b_d R \subseteq R' \subseteq L \) is identified with \( R^d \subseteq K^d \), which is open in the product topology of \( d \) copies of the \( R \)-adic topology on \( K \). Since the \( R \)-adic topology on \( K \) coincides with the \( \mathcal{E}_K \)-topology by Fact 1.1 this means that \( b_1 R + \cdots + b_d R \) and hence \( R' \) are \( \text{Ext}_{L/K}(\mathcal{E}_K) \)-open. By Fact 5.1 it follows that \( \text{Ext}_{L/K}(\mathcal{E}_K) \) refines the \( R' \)-adic topology on \( L \), which strictly refines \( \mathcal{E}_L \). \( \square \)

6. A LARGE COLLECTION OF INCOMPATIBLE TOPOLOGIES ON \( \mathbb{Q}_p \)

Fix a prime \( p \). In this section we produce \( 2^{2^{\phi_0}} \)-many henselian local subrings \( R \subsetneq \mathbb{Q}_p \) with fraction field \( \mathbb{Q}_p \) such that the corresponding \( R \)-adic topologies are pairwise incomparable. This is interesting in light of Corollary 1.4 which shows that this behaviour cannot occur for quasi-excellent \( R \), since in this case the \( R \)-adic topology induces the étale open topology. It is also in contrast to F. K. Schmidt’s theorem [EP05, Theorem 4.4.1], which shows that any two henselian valuation rings on a field which is not separably closed induce the same topology (compare also Fact 1.1(1, 2)).

Our approach is based on [JWY21 Section 5]. Given a \( \mathbb{Q} \)-derivation \( \partial : \mathbb{Q}_p \to \mathbb{Q}_p \) we let \( E_\partial \) be \( \{ \alpha \in \mathbb{Z}_p : \partial \alpha \in \mathbb{Z}_p \} \). It is easy to see that \( E_\partial \) is a subring of \( \mathbb{Z}_p \). Fact 6.1 is a summary of the statements of [JWY21 Section 5].

**Fact 6.1.** If \( \partial \) is not identically zero then:

1. \( E_\partial \) is a one-dimensional Noetherian henselian local ring with fraction field \( \mathbb{Q}_p \), and
2. the \( E_\partial \)-adic topology on \( \mathbb{Q}_p \) strictly refines the \( p \)-adic topology.

Furthermore \( \hat{E}_\partial \) is isomorphic to \( \mathbb{Z}_p[X]/(X^2) \), hence \( E_\partial \) is not excellent.

Let \( \mathcal{D} \) be the set of derivations \( \mathbb{Q}_p \to \mathbb{Q}_p \) which are not constant zero. We say that \( \partial, \partial^* \in \mathcal{D} \) are constant multiples of each other if \( \lambda \partial = \partial^* \) for some \( \lambda \in \mathbb{Q}_p \). We prove:

**Theorem 6.2.** If \( \partial, \partial^* \in \mathcal{D} \) are not constant multiples of each other, then the \( E_\partial \)-adic topology does not refine the \( E_{\partial^*} \)-adic topology and vice versa. There is \( I \subset \mathcal{D} \) such that \( |I| = 2^{2^{\phi_0}} \) and if \( \partial, \partial^* \in I \), \( \partial \neq \partial^* \) then the \( E_\partial \)-adic topology does not refine the \( E_{\partial^*} \)-adic topology.

Thus there are \( 2^{2^{\phi_0}} \)-distinct \( E_\partial \)-adic topologies on \( \mathbb{Q}_p \). We explain how the second claim follows from the first. Let \( B \) be a transcendence basis for \( \mathbb{Q}_p \). By the usual rules for extending derivations to separable field extensions [FJ05 Section 2.8], it is easy to see that any function \( B \to \mathbb{Q}_p \) uniquely extends to a derivation \( \mathbb{Q}_p \to \mathbb{Q}_p \). Since \( |B| = 2^{\phi_0} \), this shows \( |\mathcal{D}| = 2^{2^{\phi_0}} \). As every element of \( \mathcal{D} \) is a constant multiple of precisely \( |\mathbb{Q}_p^\times| = 2^{\phi_0} \) other
elements of $D$, this shows that there are $2^{2^{|Q|}}$ classes of elements of $D$ under the equivalence relation of being a constant multiple of one another, and we may take $I \subseteq D$ to be a set of representatives for this equivalence relation.

**Lemma 6.3.** Suppose that $\partial, \partial^*: Q_p \to Q_p$ are derivations and neither is a constant multiple of the other. Then $\{(\alpha, \partial \alpha, \partial^* \alpha) : \alpha \in Q_p\}$ is $p$-adically dense in $Q_p^3$.

**Proof.** As $\partial, \partial^*$ are not constant multiples of each other there are $s, t \in Q_p$ such that $(\partial s, \partial t)$ and $(\partial^* s, \partial^* t)$ are not scalar multiples of each other in $Q_p^2$. Any derivation $Q_p \to Q_p$ is $Q$-linear and vanishes on $Q$, hence for $\alpha = a + sb + tc$ with $a, b, c \in Q$ we have

$$(\alpha, \partial \alpha, \partial^* \alpha) = (a + sb + tc, b\partial s + c\partial t, b\partial^* s + c\partial^* t).$$

We let $T: Q_p^3 \to Q_p^3$ be the $Q$-linear transformation given as follows:

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + sy + tz \\ (\partial s)y + (\partial t)z \\ (\partial^* s)y + (\partial^* t)z \end{pmatrix}.$$  

Note that $T(Q^3) \subseteq \{(\alpha, \partial \alpha, \partial^* \alpha) : \alpha \in Q_p\}$, so it is enough to show that $T(Q^3)$ is dense in $Q_p^3$. As $Q^3$ is dense in $Q_p^3$ and $T$ is linear it is sufficient to note that $T$ is invertible since

$$\det(T) = \det \begin{pmatrix} 1 & s & t \\ 0 & \partial s & \partial t \\ 0 & \partial^* s & \partial^* t \end{pmatrix} = \det \begin{pmatrix} \partial s & \partial t \\ \partial^* s & \partial^* t \end{pmatrix} \neq 0. \quad \Box$$

**Proof of Theorem 6.2.** Suppose $\partial, \partial^*$ are not constant multiples of each other. We show that the $E_\alpha$-adic topology does not refine the $E_\alpha$-adic topology. By Lemma 3.1 is enough to show that $aE_\alpha \not\subseteq E_\alpha$ for any $a \in Q_p^\times$. Let

$$U = \{(b, b', b'') \in Z_p \times Q_p \times Z_p : ab' + (\partial a)b \in Q_p \setminus Z_p\}.$$  

Then $U$ is open and nonempty as $(0, (pa)^{-1}, 0) \in U$. By Lemma 6.3 we have $(b, \partial b, \partial^* b) \in U$ for some $b \in Q_p$. Then $b, \partial^* b \in Z_p$, so $b \in E_\alpha$, and $\partial(ab) = a(\partial b) + b(\partial a) \notin Z_p$, so $ab \notin E_\alpha$. \quad \Box

7. The étale open topology on pseudo-algebraically closed fields

Recall that a field $K$ is pseudo-algebraically closed (PAC) if every geometrically integral $K$-variety has a $K$-point.

**Proposition 7.1.** Let $K$ be a PAC field. Then the étale open topology on varieties over $K$ is not induced by a field topology on $K$.

**Proof.** Suppose for a contradiction that there is a field topology $\tau$ on $K$ inducing $\mathcal{E}_K$. We consider the morphism $\alpha: \text{PGL}_{2,K} \times \mathbb{P}^1_K \to \mathbb{P}^1_K$ given by the natural group action, as well as the projection morphisms $\pi_1: \text{PGL}_{2,K} \times \mathbb{P}^1_K \to \text{PGL}_{2,K}$ and $\pi_2: \text{PGL}_{2,K} \times \mathbb{P}^1_K \to \mathbb{P}^1_K$. For later use we observe that the morphism $(\pi_1, \alpha): \text{PGL}_{2,K} \times \mathbb{P}^1_K \to \text{PGL}_{2,K} \times \mathbb{P}^1_K$ is an isomorphism, since it has an obvious inverse given by acting with the inverse group element. In particular, the morphism $\alpha = \pi_2 \circ (\pi_1, \alpha)$ is smooth since $\pi_2$ is smooth (as it is a base change of the smooth morphism $\text{PGL}_{2,K} \to \text{Spec} K$).
The étale open topology on $\text{PGL}_{2,K}(K) \times \mathbb{P}^1_K(K)$ is the product topology of the étale open topologies on $\text{PGL}_{2,K}(K)$ and $\mathbb{P}^1_K(K)$, since the analogous statement is true for the $\tau$-topology and the two topologies agree on the $K$-points of every variety. Let $\emptyset \neq U \subseteq \mathbb{P}^1(K)$ be open. We show that $U$ is necessarily cofinite.

The group scheme action $\alpha$ induces a map $\text{PGL}_{2,K}(K) \times \mathbb{P}^1(K) \to \mathbb{P}^1(K)$ on $K$-points, which we also denote by $\alpha$. It is continuous by the defining properties of the étale open topology, and so there exist non-empty étale open subsets of $\text{PGL}_{2,K}(K)$ and $\mathbb{P}^1(K)$ whose product is contained in the preimage of $U$ under $\alpha$. In other words, there exist two $K$-varieties $X$ and $Y$ with étale maps $X \to \text{PGL}_{2,K}$, $Y \to \mathbb{P}^1_K$ such that $X(K), Y(K) \neq \emptyset$ and $U$ contains the image of $X(K) \times Y(K)$ under the composite

$$g: X \times Y \to \text{PGL}_{2,K} \times \mathbb{P}^1_K \overset{\alpha}{\to} \mathbb{P}^1_K.$$ 

The $K$-schemes $X$ and $Y$ are smooth since they are étale over the smooth $K$-schemes $\text{PGL}_{2,K}$ and $\mathbb{P}^1_K$, respectively. Passing to a connected component of $X$ and $Y$ if necessary, we may additionally assume that both $X$ and $Y$ are connected (as schemes, i.e. not in relation to the topologies $\mathcal{E}_K$ or $\tau$). Since they both have a $K$-point, $X$ and $Y$ are then geometrically connected [Poo17 Proposition 2.3.24] and hence (by smoothness) geometrically integral $\text{PGL}_{2,K}$. We claim that the generic fibre of $g$ is geometrically integral (as a variety over the function field $K(\mathbb{P}^1_K)$), i.e. that the function field $K(X \times Y)$ is a regular extension of the function field $K(\mathbb{P}^1_K)$ via the map $g$. Let us defer the proof of this claim for the moment. By [Sta20, Tags 0578 and 0559], all but finitely many fibres of $g$ are geometrically integral. In particular, for all but finitely many $x \in \mathbb{P}^1(K)$, the $K$-variety $g^{-1}(x)$ has a $K$-point by the PAC property, and thus $x \in g(X(K) \times Y(K)) \subseteq U$.

This shows that $U$ is cofinite. Thus the étale open topology on $K = A^1_K(K) \subseteq \mathbb{P}^1_K(K)$, and therefore the topology $\tau$ is the cofinite topology. Since the cofinite topology is not a field topology on any infinite field, this yields the desired contradiction.

It remains to prove the claim. This is purely a matter of algebraic geometry, so the topologies $\mathcal{E}_K$ and $\tau$ no longer intervene. As a consequence of Zariski’s Main Theorem, we can embed $X$ and $Y$ as open subschemes of normal integral schemes $\overline{X}$, $\overline{Y}$ with finite morphisms $p_1: \overline{X} \to \text{PGL}_{2,K}$, $p_2: \overline{Y} \to \mathbb{P}^1_K$ extending the étale morphisms from $X$ respectively $Y$. (See for instance [Poo17, Theorem 3.5.52 (c)] (recalling that $X$ and $Y$ are separated by our convention on varieties), where $\overline{X}$ and $\overline{Y}$ are described concretely as normalisations of $\text{PGL}_{2,K}$ (respectively $\mathbb{P}^1_K$) in the function field of $X$ (respectively $Y$).)

Via the dominant morphism $\overline{X} \times \overline{Y} \overset{p_1 \times p_2}{\to} \text{PGL}_{2,K} \times \mathbb{P}^1_K \overset{\alpha}{\to} \mathbb{P}^1_K$, which restricts to the morphism $g$ considered earlier on $X \times Y$, we can consider $\overline{K}(X \times Y) = \overline{K}(\overline{X} \times \overline{Y})$ as an extension field of $K(\mathbb{P}^1_K)$. Let $F \subseteq K(X \times Y)$ be the relative algebraic closure of $K(\mathbb{P}^1_K)$ therein. Then $F/K$ is regular since $K(X \times Y)/K$ is regular, due to the geometric integrality of $X$ and $Y$. Let $C \to \mathbb{P}^1_K$ be the normalisation of $\mathbb{P}^1_K$ in $F$. Thus $C/K$ is a geometrically integral normal projective curve and $C \to \mathbb{P}^1_K$ is a finite morphism. We shall show using a ramification argument that in fact $C \to \mathbb{P}^1_K$ is an isomorphism.
Let us consider the following diagram:

\[
\begin{array}{ccc}
\overline{X} \times \overline{Y} & \xrightarrow{p_1 \times p_2} & \text{PGL}_2, K \times \mathbb{P}^1_K \\
\downarrow & & \downarrow (\pi_1, \alpha) \\
\text{PGL}_2, K \times C & \longrightarrow & \text{PGL}_2, K \times \mathbb{P}^1_K
\end{array}
\]

All varieties occurring are geometrically integral and normal, the vertical morphism on the right is an isomorphism, the top horizontal morphism is finite and generically étale, and the bottom morphism (given by the identity on PGL\(_2, K\) and the previous map \(C \to \mathbb{P}^1_K\)) is finite. We can complete the diagram by a finite morphism on the left side, shown as a dashed arrow: Observe first that by construction, the function field \(K(\overline{X} \times \overline{Y})\) is an extension of the function field of PGL\(_2, K \times C\), i.e. we can find a rational function on the left side making the diagram commute. In particular, we then have a normalisation of PGL\(_2, K \times C\) in the function field \(K(\overline{X} \times \overline{Y})\) (see for instance [Lin02, Definition 4.1.24]), which is also a normalization of PGL\(_2, K \times \mathbb{P}^1_K\) in this field by construction. However, the morphism \((\pi_1, \alpha) \circ (p_1 \times p_2)\) already describes \(\overline{X} \times \overline{Y}\) as the normalisation of PGL\(_2, K \times \mathbb{P}^1_K\) within \(K(\overline{X} \times \overline{Y})\); therefore, by uniqueness of normalisations, \(\overline{X} \times \overline{Y}\) must already be the normalisation of PGL\(_2, K \times C\) in \(K(\overline{X} \times \overline{Y})\), and the morphism on the left side of the diagram making it commutative is none other but the normalisation morphism.

Let us show that the morphism of curves \(C \to \mathbb{P}^1_K\) is unramified. First observe that the only prime divisors of PGL\(_2, K \times \mathbb{P}^1_K\) which ramify under the map \(p_1 \times p_2\) are of the form \(D \times \mathbb{P}^1_K\) or PGL\(_2, K \times D'\), where \(D\) ramifies under \(p_1\) or \(D'\) ramifies under \(p_2\). Since \(\alpha\) is a transitive group action, the image of such a prime divisor under the automorphism \((\pi_1, \alpha)\) of PGL\(_2, K \times \mathbb{P}^1_K\) is never of the form PGL\(_2, K \times \{x\}\) for a closed point \((\text{i.e., prime divisor})\) \(x\) of \(\mathbb{P}^1_K\). In other words, for every closed point \(x\) of \(\mathbb{P}^1_K\), the prime divisor PGL\(_2, K \times \{x\}\) does not ramify along the map \((\pi_1, \alpha) \circ (p_1 \times p_2)\): \(\overline{X} \to \overline{Y} \to \text{PGL}_2, K \times \mathbb{P}^1_K\). Due to the commutative diagram above, it follows that the prime divisor in question cannot ramify along PGL\(_2, K \times C\) \(\to\) PGL\(_2, K \times \mathbb{P}^1_K\) either, and so \(x\) is not a branch point of \(C \times \mathbb{P}^1_K\). Since \(x\) was arbitrary, this shows that \(C \to \mathbb{P}^1_K\) is unramified. Since \(C\) is a geometrically integral projective curve and \(\mathbb{P}^1_K\) is geometrically simply connected (see [Lin02, Corollary 7.4.20]), it follows that the map \(C \to \mathbb{P}^1_K\) is an isomorphism, and so \(F = K(C) = K(\mathbb{P}^1_K)\). In other words, the field \(K(\mathbb{P}^1_K)\) is relatively algebraically closed in \(K(\overline{X} \times \overline{Y})\).

Finally, the morphism \(g\) is smooth, since it factors as the composition of the étale morphism \(X \times Y \to \text{PGL}_2, K \times \mathbb{P}^1_K\), and the smooth morphism \(\alpha\). Smoothness of \(g\) at the generic point means that \(K(\overline{X} \times \overline{Y})/K(\mathbb{P}^1_K)\) is a separable field extension, so (together with relative algebraic closedness) we have shown that it is a regular field extension. This finishes the proof of the claim that the generic fibre of \(g\) is geometrically integral.

\[\square\]

**Remark 7.2.** The precise choice of the morphism PGL\(_2, K \times \mathbb{P}^1_K \to \mathbb{P}^1_K\) in the proof above is not very important. We only used that it is a transitive group action on a geometrically simply connected variety. In characteristic zero, one can instead use the simpler addition action \(A^1_K \times A^1_K \to A^1_K\), but in positive characteristic \(A^1_K\) is not geometrically simply connected.
8. GT-HENSELIAN FIELD TOPOLOGIES

8.1. **Background on topological fields.** We develop the basics of a theory of gt-henselian field topologies extending the Prestel-Ziegler theory of t-henselian field topologies. Recall our convention that all field topologies are Hausdorff and non-discrete. Throughout, we fix such a field topology \( \tau \) on the field \( K \).

**Definition 8.1.** We say that \( \tau \) is generalized (topologically) henselian, for short gt-henselian, if for every \( n \) and every neighbourhood \( P \subseteq K \) of \(-1\) there is a neighbourhood \( O \subseteq K \) of zero such that the polynomial \( X^{n+1} + X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0 \) has a root in \( P \) for any \( a_0, \ldots, a_{n-1} \in O \).

As the terminology suggests, gt-henselianity generalizes t-henselianity. For more on t-henselianity, see [PZ78, Section 7].

**Remark 8.2.** In fact, the field topology \( \tau \) is gt-henselian if and only if \( K \) is \( \tau \)-henselian in the sense considered in [Pop14, Examples 1.7], as follows from the characterization we give in Proposition 8.4 below. However, Pop’s notion of \( \tau \)-henselian rings does not seem to have been studied in any depth in the literature. We prefer the name gt-henselianity to stress the link with [PZ78].

Another notion of henselianity for rings in the literature is given by the henselian semi-normed rings of [FT11] (on which Pop’s definition of weak \( \tau \)-henselianity is modelled), but there do not appear to be interesting examples of field topologies obtained in this way, except in the well-known case of a field with an absolute value.

Recall from [PZ78, Theorem 7.2 a)] (which we may as well take as a definition) that the field topology \( \tau \) on \( K \) is t-henselian if and only if it is a V-topology (see [PZ78, Section 3]) and for every \( n \geq 1 \) there is a \( \tau \)-neighbourhood \( U \) of \( 0 \) such that any polynomial \( f = X^n + X^{n-1} + a_{n-2}X^{n-2} + \ldots + a_1X + a_0 \in K[X] \) with \( a_{n-2}, \ldots, a_0 \in U \) has a zero in \( K \). We show that gt-henselianity generalises t-henselianity.

**Proposition 8.3.** The topology \( \tau \) is t-henselian if and only if it is gt-henselian and a V-topology.

For the proof we need the following fact, a special case of the polynomial implicit function theorem for t-henselian fields [PZ78, Theorem 7.4]. We can also prove a polynomial implicit function theorem for locally bounded gt-henselian field topologies, but we will not do so here.

**Fact 8.4.** Suppose that \( \tau \) is t-henselian, \( f \in K[Y_1, \ldots, Y_n, X] \), and \((\alpha, \beta) \in K^n \times K \) is such that \( \partial f / \partial X f(\alpha, \beta) \) is nonzero. Then there are \( \tau \)-neighbourhoods \( U_1 \subseteq K^n \), \( U_2 \subseteq K \) of \( \alpha, \beta \), respectively, and a \( \tau \)-continuous function \( g : U_1 \to U_2 \) such that

\[
\{(a, g(a)) : a \in U_1\} = \{(a, b) \in U_1 \times U_2 : f(a, b) = 0\}
\]

**Proof of Proposition 8.3.** It follows directly from the definitions that a gt-henselian V-topology is t-henselian. Suppose that \( \tau \) is t-henselian. Then \( \tau \) is necessarily a V-topology. Fix \( n \geq 1 \) and a neighbourhood \( P \subseteq K \) of \(-1\). We let \( f \in K[Y_1, \ldots, Y_{n-1}, X] \) be the polynomial \( X^{n+1} + X^n + Y_{n-1}X^{n-1} + \ldots + Y_1X + Y_0 \). Then we have \( f(0, \ldots, 0, -1) = 0 \neq \partial f / \partial X f(0, \ldots, 0, -1) \). Let \( U_1 \), \( U_2 \), and \( g \) be as in Fact 8.4. Let \( O = g^{-1}(P \cap U_2) \). Then \( O \) is a neighbourhood of zero. By construction, if \( a_0, \ldots, a_{n-1} \in O \) then \( X^{n+1} + X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0 \) has a root in \( P \). □
A significant set of examples for gt-henselian field topologies is furnished by $R$-adic topologies for $R$ henselian.

**Proposition 8.5.** Let $R \subseteq K$ be a henselian local domain with fraction field $K$. Then the $R$-adic topology is gt-henselian.

**Proof.** Let $P \subseteq K$ be an $R$-adic neighbourhood of $-1$. Then $P$ contains $-1 + \alpha R$ for some $\alpha \in K^\times$. By multiplying $\alpha$ with a suitable element of $R$, we may assume that $\alpha \in R$ and $\alpha$ is not a unit. It now suffices to show that for every $n$ and all $a_0, \ldots, a_{n-1} \in \alpha R$, the polynomial $X^{n+1} + X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ has a root in $-1 + \alpha R$. This precisely means that $(R, \alpha R)$ is a henselian pair (see the characterization in [Sta20, Tag 02GT]), which follows from [Sta20, Tag 09XI (5)] as $\alpha R$ is a henselian pair (where $\mathfrak{m}$ is the maximal ideal of $R$). □

We let $\text{Pol}_n$ be the $K$-variety parameterizing degree $n$ monic polynomials, so $\text{Pol}_n$ is just a copy of $\mathbb{A}^n$. Recall that $\alpha \in K$ is a **simple root** of $f \in K[X]$ if $f(\alpha) = 0$ and $f'(\alpha) \neq 0$.

**Proposition 8.6.** The following are equivalent:

1. $\tau$ is gt-henselian.
2. For any $n$ and neighbourhood $P$ of $1$ there is a neighbourhood $O$ of $0$ such that if $a_0, \ldots, a_{n-1} \in O$ then $X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ has a root in $P$.
3. For any $n$ and neighbourhood $P$ of $-1$ there is a neighbourhood $O$ of $0$ such that if $c_2, \ldots, c_n \in O$ then $1 + X + c_2X^2 + \cdots + c_nX^n$ has a root in $P$.
4. If $\alpha \in K$ is a simple root of a monic polynomial $f \in K[X]$, $\deg f = n$, and $P \subseteq K$ is a neighbourhood of $\alpha$ then there is a neighbourhood $O \subseteq \text{Pol}_n(K)$ of $f$ such that every $f^* \in O$ has a simple root in $P$.
5. $V(K) \rightarrow W(K)$ is $\tau$-open for any étale morphism $V \rightarrow W$.
6. $V(K) \rightarrow W(K)$ is $\tau$-open for any smooth morphism $V \rightarrow W$.

**Definition 8.8.** A basic standard étale morphism is a morphism $\pi: V \rightarrow W$ where $W$ is an affine $K$-variety, $V$ is the subvariety of $W \times \mathbb{A}^1$ given by $f = 0 \neq g$ for $f, g \in (K[W])[X]$ such that $f$ is monic, $\delta f/\delta X \neq 0$ on $V$, and $\pi$ is the restriction of the projection $W \times \mathbb{A}^1 \rightarrow W$ to $V$. A standard étale morphism is a morphism $\pi: V \rightarrow W$ of $K$-varieties such that there is a $K$-variety isomorphism $\rho: V^* \rightarrow V$ with $\pi \circ \rho: V^* \rightarrow W$ basic standard étale.

**Fact 8.8.** Any étale morphism of $K$-varieties is locally standard étale. That is, if $V \rightarrow W$ is an étale morphism of $K$-varieties and $p \in V$ then there is a Zariski open neighbourhood $V^* \subseteq V$ of $p$ and an affine Zariski open neighbourhood $W^* \subseteq W$ of $f(p)$ such that $f(W^*) \subseteq V^*$ and $V^* \rightarrow W^*$ is standard étale.

In the following proof, we work with respect to $\tau$ throughout.

**Proof of Proposition 8.6.** The equivalence of (1) and (3) is clear by considering the substitution $Y = 1/X$. The equivalence of (1) and (2) is likewise clear by considering the substitution $Y = -X$. The implication from (6) to (5) is clear since étale morphisms are smooth, and the converse holds since a smooth morphism is locally the composition of an étale morphism and a product projection [Sta20, Tag 054L], see also [WY23, Proposition 3.1].
We show that (4) implies (5). Suppose (4) and let $\pi: V \to W$ be étale. We show that $V(K) \to W(K)$ is $\tau$-open. By Fact 8.8 we may suppose that $\pi$ is basic standard étale. Let $\pi$, $f$, and $g$ be as in Definition 8.7. Given $\alpha \in W(K)$ let $f_\alpha \in K[X]$ be given by evaluating $f$ at $\alpha$ and let $t: W(K) \to \text{Pol}_n(K)$ be $t(\alpha) = f_\alpha$. Note that $t$ is continuous with respect to $\tau$. It is enough to fix $(\alpha, \beta) \in V(K)$ and a neighbourhood $P \subseteq W(K) \times K$ of $(\alpha, \beta)$ and show that $\pi(V(K) \cap P)$ is a neighbourhood of $\alpha$. We may suppose that $P$ is contained in the open subvariety of $W \times \mathbb{A}^1$ given by $g \neq 0$. As the $\tau$-topology on $W(K) \times K$ is the product topology we suppose that $P = O^* \times U$ for a neighbourhood $O^* \subseteq W(K)$ of $\alpha$ and a neighbourhood $U \subseteq K$ of $\beta$. Note that $\beta$ is a simple root of $f_\alpha$ as $\partial f/\partial X$ does not vanish at $(\alpha, \beta)$. Hence there is a neighbourhood $O \subseteq \text{Pol}_n(K)$ such that every $f^* \in O$ has a simple root in $U$. We show that $O^* \cap t^{-1}(O)$ is contained in $\pi(V(K) \cap P)$, note that $O^* \cap t^{-1}(O)$ is a neighbourhood of $\alpha$. Fix $\gamma \in O^* \cap t^{-1}(O)$. Then $f_\gamma \in O$, hence $f_\gamma$ has a simple root $\eta$ in $U$. We show that $(\gamma, \eta) \in V(K) \cap P$. Note $f(\gamma, \eta) = f_\gamma(\eta) = 0$. As $\gamma \in O^*$ and $\eta \in U$ we have $(\gamma, \eta) \in P$, so $g(\gamma, \eta) \neq 0$, hence $(\gamma, \eta) \in V(K)$.

We show that (5) implies (4). Suppose (5) and fix $n \geq 2$ (4 is trivial for $n = 1$). Let $V$ be the subvariety of Spec $K[Y_1, \ldots, Y_n, X] = \mathbb{A}^n \times \mathbb{A}^1$ given by $X^n + Y_{n-1}X^{n-1} + \ldots + Y_1X + Y_0 = 0$ and $(\partial/\partial X)[X^n + Y_{n-1}X^{n-1} + \ldots + Y_1X + Y_0] \neq 0$. Let $\pi: V \to \mathbb{A}^n$ be the projection. Then $\pi$ is standard étale, hence the projection $V(K) \to K^n$ is open. Suppose $a = (a_0, \ldots, a_{n-1}) \in K^n$, $f(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0 \in K$ is a simple root of $f$, and $P \subseteq K$ is a neighbourhood of $b$. Note that $(a, b) \in V(K)$. Let $O = \pi([K^n \times P] \cap V(K))$, so $O$ is a neighbourhood of $a$. It is easy to see that $f^*(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0^*$ has a simple root in $P$ for any $(a_0^*, \ldots, a_{n-1}^*) \in O$.

We show that (4) implies (1). Let $P$ be a neighbourhood of $-1$. Note that $-1$ is a simple root of $X^{n+1} + X^n$. Hence there is a neighbourhood $O \subseteq K^n$ of $(1, 0, \ldots, 0)$ such that if $a = (a_n, \ldots, a_0) \in O$ then $X^{n+1} + a_nX^n + \ldots + a_1X + a_0$ has a root in $P$. Fix a neighbourhood $Q \subseteq K$ of $0$ such that $\{1\} \times Q^n \subseteq O$. Then $X^{n+1} + X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0$ has a root in $P$ for all $a_0, \ldots, a_{n-1} \in Q$. Hence $\tau$ is gt-henselian.

We finish by showing that (3) implies (4). Suppose (3). Let $f \in K[X]$ be monic of degree $n$, and let $\alpha \in K$ be a simple root of $f$. The change of variables $Y = X - \alpha$ induces an automorphism of $\text{Pol}_n(K)$, so we may assume without loss of generality that $\alpha = 0$. Thus $f = a_1X + \cdots + a_{n-1}X^{n-1} + X^n$ with coefficients $a_i \in K$, $a_1 = f'(0) \neq 0$. Let $P \subseteq K$ be a neighbourhood of $0$. Let $P' \subseteq P$ be a smaller neighbourhood of $0$ such that $-1 \notin P'$ and $a_1^{-1}P' \cdot (1 + P')^{-1}(-1 + P') \subseteq P$. By (3) there exists a neighbourhood $O$ of $0$ such that every polynomial $1 + X + c_2X^2 + \cdots + c_nX^n$ with $c_i \in O$ has a root in $-1 + P'$. By shrinking $O$ and $P'$, we may assume that any root in $-1 + P'$ of any such polynomial is simple.

Let $O^*$ be the set of polynomials $b_0 + b_1X + \cdots + b_{n-1}X^{n-1} + X^n \in \text{Pol}_n(K)$ with $b_0 \in P'$, $b_1 \in a_1(1 + P')$, and $b_0b_1^{-1}b_1^{-i} \in O$ for all $i = 2, \ldots, n$. This is a neighbourhood of $f$ in $\text{Pol}_n(K)$.

Let us show that every $g = b_0 + b_1X + \cdots + b_{n-1}X^{n-1} + X^n \in O^*$ has a simple root in $P$. If $b_0 = 0$, then $0$ is a simple root of $g$. Otherwise, consider the polynomial $h = b_0^{-1}g(b_0b_1^{-1}X) \in K[X]$. By construction, $h$ has the form $1 + X + c_2X^2 + \cdots + c_nX^n$ with $c_i \in O$, and thus has a simple zero in $-1 + P'$. Therefore $g$ has a simple zero in $b_0b_1^{-1}(-1 + P') \subseteq P$, as desired. □
Remark 8.9. We have seen in Section 6 that the field $\mathbb{Q}_p$ carries $2^{2^{\aleph_0}}$ many pairwise incomparable locally bounded $\text{gt}$-henselian topologies. This is in marked contrast to $\text{t}$-henselian topologies, where a field which is not separably closed can admit at most one such (PZ78 Theorem 7.9), essentially F. K. Schmidt’s theorem on independent henselian valuations). Therefore, while it is sensible to speak of $\text{t}$-henselian fields and the $\text{t}$-henselian topology on one such (forbidding separably closed fields), we avoid the analogous terminology in the $\text{gt}$-henselian case.

The analysis of the topological field $(K, \tau)$ simplifies when $\tau$ is $\omega$-complete, i.e. it the collection of neighbourhoods of 0 is closed under countable intersections. Using an ultrapower argument, Prestel-Ziegler in [PZ78, Theorem 1.1] show that every $(K, \tau)$ may be replaced by some $(K^*, \tau^*)$ which is “locally equivalent” to $(K, \tau)$ and such that $\tau^*$ is $\omega$-complete. Here local equivalence means that $(K, \tau)$ and $(K^*, \tau^*)$ satisfy the same sentences in a certain logic extending first-order logic in the language of rings, allowing restricted second-order quantification over neighbourhoods of 0. See [PZ78, Section 1] for details on this formalism.

Lemma 8.10. Let $(K^*, \tau^*)$ be locally equivalent to $(K, \tau)$. Then $\tau^*$ is $\text{gt}$-henselian ($\text{t}$-henselian) if and only if $\tau$ is $\text{gt}$-henselian ($\text{t}$-henselian).

Proof. It is immediate from the definition that $\text{gt}$-henselianity is expressed by a collection of local sentences. The same holds for $\text{t}$-henselianity (as is already expressed in [PZ78, Corollary 7.3]).

For $\omega$-complete field topologies, we have the following.

Fact 8.11. Suppose that $\tau$ is $\omega$-complete. Then $\tau$ is locally bounded if and only if $\tau$ is the $S$-adic topology for a local subring $S$ of $K$ with $K = \text{Frac}(S)$. Furthermore $\tau$ is a $V$-topology if and only if $\tau$ is the $S$-adic topology for a valuation subring $S$ of $K$ and $\tau$ is $\text{t}$-henselian if and only if $\tau$ is the $S$-adic topology for a henselian valuation subring $S$ of $K$.

We note that Fact 8.11 can fail without $\omega$-completeness. For instance, it fails for the usual topology on $\mathbb{R}$ or $\mathbb{C}$.

Proof of Fact 8.11. The first claim is in the proof of [PZ78, Theorem 2.2 (b)], the second is [PZ78, Lemma 3.3], and the third follows from [PZ78, Theorem 7.2].

A subset of $K$ is a henselian ideal if it is the maximal ideal of a henselian local subring of $K$ with fraction field $K$. We say that $\tau$ is induced by a henselian local ring if $\tau$ is the $R$-adic topology for a henselian local subring $R$ of $K$ with Frac$(R) = K$.

Proposition 8.12. Suppose $\tau$ is $\omega$-complete. The following are equivalent:

1. $\tau$ is $\text{gt}$-henselian.
2. $\tau$ admits a neighbourhood basis at zero consisting of henselian ideals.

If $\tau$ is also locally bounded then $\tau$ is $\text{gt}$-henselian if and only if $\tau$ is induced by a henselian local ring.

Hence an $\omega$-complete $\text{gt}$-henselian field topology is a union of henselian field topologies. For the proof of the proposition, we partly follow the proof of [PZ78, Theorem 2.2], see also [PZ78, Theorem 7.2].
Proof. Let us first assume that (2) holds and show that this implies (1). Any $\tau$-neighbourhood $P \subseteq K$ of $-1$ contains a set $-1 + I$, where $I$ is a henselian ideal which is a $\tau$-neighbourhood of $0$. Applying condition (3) from Fact 2.8 we see that $O = I$ satisfies the condition from Definition 8.1. Thus $\tau$ is gt-henselian.

For the converse direction, let us suppose that $\tau$ is gt-henselian. We wish to show that (2) holds. We fix a neighbourhood $Q$ of zero and construct an open henselian ideal $P \subseteq K$ which is contained in $Q$. We use Fact 2.8 to show that $P$ is a henselian ideal. Let $K_{pr}$ be the prime subfield of $K$.

Claim. Suppose that $O$ is a neighbourhood of $-1$. Then there is a neighbourhood $P \subseteq Q$ of zero such that:

1. $X^{n+1} + X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0$ has a root in $O$ when $a_0, \ldots, a_{n-1} \in P$,
2. $K_{pr} + P$ is a local subring of $K$ with fraction field $K$ and maximal ideal $P$.

Proof. By gt-henselianity, for every $n \geq 2$ we may fix a neighbourhood $U_n$ of $0$ such that if $a_0, \ldots, a_{n-1} \in U_n$ then $X^{n+1} + X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0$ has a root in $K$. By $\omega$-completeness there is a neighbourhood $U$ of zero such that $U \subseteq U_n$ for all $n$. We may suppose that $U \subseteq O$ and that $U$ does not contain $1$. Let $r_1, r_2, \ldots$ be an enumeration of $K_{pr}$. Construct a descending sequence $(P_i : i \in \mathbb{N})$ of open neighbourhoods of zero such that $P_0 = U$, and for all $i \geq 1$ the sets $P_i + P_i$, $P_i - P_i$, $P_i \cdot P_i$, and $r_1P_i, \ldots, r_iP_i$ are all contained in $P_{i-1}$ and $(1 + P_i)^{-1} \subseteq 1 + P_{i-1}$. By $\omega$-completeness $P := \bigcap_{i \in \mathbb{N}} P_i$ is a neighbourhood of zero. The proof of [PZ78, Theorem 2.2] shows that $K_{pr} + P$ is a local subring of $K$ with maximal ideal $P$. Finally, $K_{pr} + P$ is open so Fact 2.6 shows that $K = \text{Frac}(K_{pr} + P)$. □

Inductively construct sequences $(P_i : i \in \mathbb{N})$, $(O_i : i \in \mathbb{N})$ of open neighbourhoods of $0$, $-1$, respectively such that $P_0 \subseteq Q$, $O_0 \subseteq O$ and for each $i \in \mathbb{N}$ and $n \geq 1$:

1. $X^{n+1} + X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0$ has a root in $O_i$ for any $a_0, \ldots, a_{n-1} \in P_i$,
2. $K_{pr} + P_i$ is a local ring with maximal ideal $P_i$ and fraction field $K$,
3. $P_i \subseteq O_{i+1} - 1$ and $O_{i+1} \subseteq P_i + 1$.

Let $P := \bigcap_{i \in \mathbb{N}} P_i$ and $O := \bigcap_{i \in \mathbb{N}} O_i$. Note that $O = P + 1$. By $\omega$-completeness $P$ is a neighbourhood of $0$. Let $R = K_{pr} + P$. Since $K_{pr} + P_i$ is a local ring with maximal ideal $P_i$ for each $i$, we easily check that $R$ is a ring and $P$ is an ideal with residue field $K_{pr}$. Furthermore, every element of $1 + P$ is invertible in $R$, since we have $(1 + P_i)^{-1} \subseteq 1 + P_{i-1} \subseteq R$ for every $i$. It follows that $R$ is a local ring with maximal ideal $P$. As $P$ is a neighbourhood of zero, $R$ is open. By Fact 2.6 $K = \text{Frac}(K_{pr} + P)$. Note that $X^{n+1} + X^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0$ has a root in $P + 1$ for every $a_0, \ldots, a_{n-1} \in P$. Hence $R$ is henselian by Fact 2.8.

We now suppose that $\tau$ is locally bounded. We take $Q$ in the construction above to be bounded, hence $P$ is bounded. Then $\alpha + P$ is bounded for all $\alpha \in K_{pr}$, so $K_{pr} + P$ is bounded as a countable union of bounded sets, since $(K, \tau)$ is $\omega$-complete.

An application of Fact 2.7 shows that $\tau$ is the $R$-adic topology. □

Corollary 8.13. Suppose that $\tau$ is locally bounded. Let $(K^*, \tau^*)$ be locally equivalent to $(K, \tau)$ and $\omega$-complete. Then $\tau$ is gt-henselian if and only if $\tau^*$ is induced by a henselian local ring.

Proof. Suppose $\tau$ is gt-henselian. By Lemma 8.10 $\tau^*$ is gt-henselian and by Proposition 8.12 $\tau^*$ is induced by a henselian local ring. Conversely, if $\tau^*$ is induced by a henselian local ring, then $\tau^*$ is gt-henselian, hence $\tau$ is gt-henselian by Lemma 8.10. □
8.2. When is the étale open topology induced by a locally bounded field topology?

**Lemma 8.14.** If $\tau$ is gt-henselian then $\tau$ refines the étale open topology. If $\tau$ induces the étale open topology then $\tau$ is gt-henselian.

**Proof.** Suppose $\tau$ is gt-henselian and $V$ is a $K$-variety. By Proposition 8.6 any étale image in $V(K)$ is $\tau$-open, and so $\tau$ refines the $E_K$-topology on $V(K)$. If $\tau$ induces the étale open topology then Fact 2.1(2) shows that if $V \to W$ is an étale morphism of $K$-varieties then $V(K) \to W(K)$ is $\tau$-open. Now again apply Proposition 8.6. □

**Corollary 8.15.** If $K$ admits a gt-henselian field topology then $K$ is large.

See [Pop14] for a definition and an account of largeness. Corollary 8.15 generalizes the theorem that the fraction field of a henselian local domain is large [Pop10]. Corollary 8.15 was also observed in slightly more general form in [Pop14, Theorem 1.8].

**Proposition 8.16.** Suppose that $\tau$ is locally bounded. Then the following are equivalent:

1. $\tau$ induces the étale open topology over $K$.
2. $\tau$ is gt-henselian and some nonempty étale image in $K$ is bounded.

In particular the following are equivalent when $R$ is a local domain with fraction field $K$:

3. The $R$-adic topology agrees with the étale open topology.
4. The $R$-adic topology is gt-henselian and $R$ contains a nonempty étale image.

**Proof.** The second equivalence follows easily from the first and the definitions. We prove the first equivalence. Suppose (1). Then $\tau$ is gt-henselian by Lemma 8.14. As $\tau$ is locally bounded we may fix a bounded open $U \subseteq K$. Then $U$ contains a nonempty étale image, which is also bounded. Thus (2) holds. Now suppose (2). By Lemma 8.14 and Fact 2.3 it suffices to show that the $E_K$-topology on $K$ refines $\tau$. Let $f : V \to A^1$ be an étale morphism of $K$-varieties such that $U = f(V(K))$ is bounded. By Proposition 8.6 $U$ is $\tau$-open, hence $(\alpha U + \beta : \alpha \in K^\times, \beta \in K)$ is a basis for $\tau$. Fact 2.1(3) shows that each $\alpha U + \beta$ is $E_K$-open. □

**Corollary 8.17.** Suppose that $K$ is perfect and $\tau$ is a locally bounded field topology on $K$. Then the following are equivalent:

1. $\tau$ induces the étale open topology over $K$.
2. $\tau$ is gt-henselian and $f(V(K))$ is bounded for some $K$-variety morphism $f : V \to A^1$ with $f(V(K))$ infinite.

In particular the following are equivalent when $R$ is a local domain with fraction field $K$:

1. The $R$-adic topology agrees with the étale open topology.
2. The $R$-adic topology is gt-henselian and $f(V(K)) \subseteq R$ for some $K$-variety morphism $f : V \to A^1$ with $f(V(K))$ infinite.

**Proof.** This follows from Proposition 8.16 and the fact that if $K$ is perfect and $f : V \to A^1$ is a $K$-variety morphism then $f(V(K))$ is the union of a definable $E_K$-open set and a finite set, see [WY23, Theorem B]. □

**Lemma 8.18.** Suppose $K \equiv K'$ and $E_K$ is induced by some locally bounded field topology $\tau$. Then $E_{K'}$ is induced by a locally bounded field topology $\tau'$. Furthermore, $(K, \tau)$ is locally equivalent to $(K', \tau')$. 21
Proof. We first argue that \( E_{K'} \) on \( K' = \mathbb{A}^1_{K'}(K') \) is a locally bounded field topology. For every \( d > 0 \), let

\[
\mathcal{U}_d := \{ U \subseteq K : U = \{ x \in K : \exists y \in K, f(x, y) = 0, g(x, y) \neq 0, \frac{\partial f}{\partial Y}(x, y) \neq 0 \} \text{ for some } f, g \in K[X, Y] \text{ of total degree } \leq d \}.
\]

Every \( U \in \mathcal{U}_d \) is an étale open subset of \( K = \mathbb{A}^1_K(K) \), and by Fact 8.8 every étale open subset of \( K \) is a union of elements of \( \bigcup_d \mathcal{U}_d \), so that the étale open topology on \( K = \mathbb{A}^1_K(K) \) has \( \bigcup_d \mathcal{U}_d \) as a basis. Let \( \mathcal{V}_d \) be the collection of \( V \in \mathcal{U}_d \) such that \( V \neq \emptyset \) and every \( U \in \mathcal{U}_d \) is a union of scaled translates of \( V \), i.e. sets of the form \( aV + b \) with \( a \in K^\times \), \( b \in K \).

There exists some \( \tau \)-bounded étale image \( 0 \in U \subseteq K \), and by possibly shrinking \( U \) we may assume that \( U \in \mathcal{U}_d \) for some \( d \) which we now fix. The scaled translates of \( U \) form a basis for the étale open topology on \( K = \mathbb{A}^1_K(K) \), i.e. a basis of \( \tau \). In particular \( U \in \mathcal{V}_d \), so \( \mathcal{V}_d \) is not empty. Note that by definition for any other element \( V \in \mathcal{V}_d \) we may write \( U \) as a union of scaled translates of \( V \).

Let \( \mathcal{U}'_d, \mathcal{V}'_d \) be the collection of subsets of \( K' \) defined analogously as \( \mathcal{U}_d \) and \( \mathcal{V}_d \) in \( K \). Since these are definable families and \( K' \equiv K \), there exists \( U' \in \mathcal{V}'_d \). Moreover, we may assume the scaled translates of \( U' \) form a basis of a field topology \( \tau' \) on \( K' \).

Since the families \( \mathcal{V}_d \) and \( \mathcal{V}'_d \) of subsets of \( K \) resp. \( K' \) are defined by the same (parameter-free) formula, and are bases of \( \tau \) resp. \( \tau' \), we have local equivalence of \( (K, \tau) \) and \( (K', \tau') \). In particular, \( (K', \tau') \) is gt-henselian. By Proposition 8.16 \( \tau' \) induces the étale open topology on \( K' \).

We now want to analyse the situation of a locally bounded \( \omega \)-complete topology.

**Theorem 8.19.** Suppose that \( \tau \) is locally bounded and induces the \( E_K \)-topology. Let \( (K^*, \tau^*) \) be locally equivalent to \( (K, \tau) \) and \( \omega \)-complete. Then \( \tau^* \) is induced by a henselian local ring and \( \tau^* \) induces the étale open topology over \( K^* \).

**Proof.** By Lemma 8.18 there is a locally bounded field topology \( \tau' \) on \( K^* \) inducing \( E_{K^*} \). Since \( (K^*, \tau^*) \) is gt-henselian by local equivalence, \( \tau^* \) refines \( \tau' \). On the other hand, there exists an étale image in \( K' \) which is \( \tau'-\)bounded, since the same holds in \( (K, \tau) \). Thus \( \tau^* = \tau' \) by [PZ78] Lemma 2.1(f)]. The statement now follows from Corollary 8.13. \( \square \)

Theorem 8.19 reduces the question “When is the étale open topology induced by a locally bounded field topology?” to the question “When does the étale open topology agree with the \( R \)-adic topology for a henselian local ring \( R \)?”.

**Proposition 8.20.** Suppose that \( K \) is \( \aleph_1 \)-saturated and suppose that the étale open topology over \( K \) is induced by a locally bounded field topology on \( K \). Then there is a henselian local subring \( R \) of \( K \) such that the étale open topology over \( K \) agrees with the \( R \)-adic topology.

A field is \( \aleph_1 \)-saturated if any descending sequence of nonempty definable sets has nonempty intersection. Such fields can for instance be produced using the ultrapower construction, see [CK90] Theorem 6.1.1).

**Proof.** Let \( \tau \) be the locally bounded field topology inducing the étale open topology over \( K \).

By Theorem 8.19 it is enough to show that \( \tau \) is \( \omega \)-complete. Fix a \( \tau \)-bounded étale image \( U \) in \( K \) which contains 0. Then \( B = (\alpha U : \alpha \in K^\times) \) forms a neighbourhood basis for \( \tau \) at zero.
consisting of definable sets, see [PZ78, Lemma 2.1 (e)]. By \( \aleph_1 \)-saturation any intersection of countably many elements of \( B \) contains an element of \( B \). Hence \( \tau \) is \( \omega \)-complete. \( \square \)

Theorem 1.6 from the introduction follows from the preceding proposition together with Lemma 8.18.

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