On the Difference of Coefficients of Bazilevič Functions

Nak Eun Cho¹ · Young Jae Sim² · Derek K. Thomas³

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Abstract
Let \( f \) be analytic in the unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \), and \( S \) be the subclass of normalized univalent functions given by \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) for \( z \in \mathbb{D} \). We give bounds for \( |a_3| - |a_2| \) for the subclass \( B(\alpha, i\beta) \) of generalized Bazilevič functions when \( \alpha \geq 0 \), and \( \beta \) is real.

Keywords Univalent function · Close-to-convex function · Bazilevič function · Difference of coefficients

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1 Introduction
Let \( A \) denote the class of analytic functions \( f \) in the unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) normalized by \( f(0) = 0 = f'(0) - 1 \). Then for \( z \in \mathbb{D} \), \( f \in A \) has the following representation

\[
 f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}
\]

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¹ Department of Applied Mathematics, Pukyong National University, Busan 48513, Korea
² Department of Mathematics, Kyungsung University, Busan 48434, Korea
³ Department of Mathematics, Swansea University, Bay Campus, Swansea SA1 8EN, UK
Let $S$ denote the subclass of all univalent (i.e., one-to-one) functions in $A$.

In 1985, de Branges [2] solved the famous Bieberbach conjecture by showing that if $f \in S$, then $|a_n| \leq n$ for $n \geq 2$, with equality when $f(z) = k(z) := z/(1-z)^2$, or a rotation. It was therefore natural to ask if for $f \in S$, the inequality $|a_{n+1} - a_n| \leq 1$ is true when $n \geq 2$. This was shown not to be the case even when $n = 2$ [4], and that the following sharp bounds hold.

$$-1 \leq |a_3| - |a_2| \leq 3/4 + e^{-\lambda_0} (2e^{-\lambda_0} - 1) = 1.029 \ldots ,$$

where $\lambda_0$ is the unique value of $\lambda$ in $0 < \lambda < 1$, satisfying the equation $4\lambda = e^\lambda$.

Hayman [6] showed that if $f \in S$, then $||a_{n+1} - a_n|| \leq C$, where $C$ is an absolute constant. The exact value of $C$ is unknown, best estimate to date being $C = 3.61 \ldots$ [5], which because of the sharp estimate above when $n = 2$, cannot be reduced to 1.

Denote by $S^*$ the subclass of $S$ consisting of starlike functions, i.e. functions $f$ which map $\mathbb{D}$ onto a set which is star-shaped with respect to the origin. Then it is well-known that a function $f \in S^*$ if, and only if, for $z \in \mathbb{D}$

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > 0.$$

It was shown in [8], that when $f \in S^*$, then $|a_{n+1} - a_n| \leq 1$ is true when $n \geq 2$.

Next denote by $K$ the subclass of $S$ consisting of functions which are close-to-convex, i.e. functions $f$ which map $\mathbb{D}$ onto a close-to-convex domain. Then again it is well-known that a function $f \in K$ if, and only if, there exists $g \in S^*$ such that for $z \in \mathbb{D}$

$$\text{Re} \left\{ \frac{zf'(z)}{g(z)} \right\} > 0. \quad (1.2)$$

Koeppf [7] showed that if $f \in K$, then $|a_{n+1} - a_n| \leq 1$, when $n = 2$, but establishing this inequality when $n \geq 3$ remains an open problem.

In 1955, Bazilevič [1] extended the notion of starlike and close-to-convex functions by showing that if $f \in A$, and is given by (1.1), then if $\alpha > 0$ and $\beta \in \mathbb{R}$, $f$ given by

$$f(z) = \left( (\alpha + i\beta) \int_0^z g^\alpha(t)p(t)ti^{\beta-1}dt \right)^{1/(\alpha+i\beta)}, \quad (1.3)$$

where $g \in S^*$, and $p \in P$, the class of functions with positive real part in $\mathbb{D}$, then functions defined by (1.3) form a subset of $S$. Such functions are known as Bazilevič functions.

We note that in the original definition of Bazilevič functions [1], Bazilevič assumed that $\alpha > 0$, however Sheil-Small [10], subsequently showed that when $\alpha = 0$, such functions also belong to $S$, and satisfy

$$zf'(z) \left( \frac{f(z)}{z} \right)^{i\beta} = p(z), \quad (1.4)$$

where $p \in P$.

\[ Springer \]
For $\alpha \geq 0$ and $\beta \in \mathbb{R}$, we denote functions defined as in (1.3) and (1.4) by $\mathcal{B}(\alpha, i\beta)$, and note that the class $\mathcal{B}(\alpha, 0) \equiv \mathcal{B}(\alpha)$ has been extensively studied, and that $\mathcal{B}(0, 0) \equiv S^*$ and $\mathcal{B}(1, 0) \equiv \mathcal{K}$.

Another well studied subclass of $\mathcal{B}(\alpha, i\beta)$ is the class $\mathcal{B}_1(\alpha, i\beta)$, where $\beta = 0$ and the starlike function $g(z) \equiv z$, (see e.g. [11]). This class is usually denoted by $\mathcal{B}_1(\alpha)$. Although much is known about the initial coefficients of functions in $\mathcal{B}_1(\alpha)$, there appears to be no published information concerning the difference of coefficients. We also note that $\mathcal{B}_1(1, 0)$ reduces to the class of functions in $\mathcal{R}$ such that their derivatives have positive real part for $z \in \mathbb{D}$, and that the class $\mathcal{B}_1(1, i\beta)$ has been little studied.

In this paper we present some inequalities for $||a_3| - |a_2||$ when $f \in \mathcal{B}(\alpha, i\beta)$, obtaining sharp bounds when $f \in \mathcal{B}(\alpha)$, and $f \in \mathcal{B}_1(\alpha, i\beta)$ when $\alpha \geq 0$ and $\beta \in \mathbb{R}$.

We also give the sharp bounds for $||a_3| - |a_2||$, when $f \in \mathcal{B}(0, i\beta)$.

2 Preliminary Lemmas

Denote by $\mathcal{P}$, the class of analytic functions $p$ with positive real part on $\mathbb{D}$ given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{2.1}$$

We will use the following properties for the coefficients of functions $\mathcal{P}$, given by (2.1).

Lemma 2.1 [9] For $p \in \mathcal{P}$ and $v \in \mathbb{C}$,

$$\left| p_2 - \frac{v}{2} p_1^2 \right| \leq 2 \max \{|v - 1|; 1\},$$

and

$$\left| p_2 - \frac{1}{2} p_1^2 \right| \leq 2 - \frac{1}{2} |p_1|^2.$$

Both inequalities are sharp.

Lemma 2.2 [3] If $p \in \mathcal{P}$, then

$$p_1 = 2\xi_1 \tag{2.2}$$

and

$$p_2 = 2\xi_1^2 + 2(1 - |\xi_1|^2)\xi_2 \tag{2.3}$$

for some $\xi_i \in \mathbb{D}$, $i \in \{1, 2\}$. For $\xi_1 \in \mathbb{T}$, the boundary of $\mathbb{D}$, there is a unique function $p \in \mathcal{P}$ with $p_1$ as in (2.2), namely,
\[ p(z) = \frac{1 + \zeta_1 z}{1 - \zeta_1 z} \quad (z \in \mathbb{D}). \]

For \( \zeta_1 \in \mathbb{D} \) and \( \zeta_2 \in \mathbb{T} \), there is a unique function \( p \in \mathcal{P} \) with \( p_1 \) and \( p_2 \) as in (2.2) and (2.3), namely,

\[ p(z) = \frac{1 + (\bar{\zeta}_1 \zeta_2 + \zeta_1) z + \zeta_2 z^2}{1 + (\bar{\zeta}_1 \zeta_2 - \zeta_1) z - \zeta_2 z^2} \quad (z \in \mathbb{D}). \]

We will also need the following well-known result.

**Lemma 2.3** [7, Lem. 3] Let \( g \in S^* \) and be given by \( g(z) = z + \sum_{n=2}^{\infty} b_n z^n \). Then for any \( \lambda \in \mathbb{C} \),

\[ \left| b_3 - \lambda b_2^2 \right| \leq \max\{1; |3 - 4\lambda|\}. \]

The inequality is sharp when \( g(z) = k(z) \) if \( |3 - 4\lambda| \geq 1 \), and when \( g(z) = (k(z^2))^{1/2} \) if \( |3 - 4\lambda| < 1 \).

### 3 The class \( B(\alpha, i\beta) \)

We begin by proving the following inequalities for \( f \in B(\alpha, i\beta) \).

**Theorem 3.1** Let \( f \in B(\alpha, i\beta) \) and be given by (1.1). If \( 0 \leq \alpha \leq (\sqrt{17} - 1)/2 \) and \( \beta \in \mathbb{R} \), then

\[ -1 \leq |a_3| - |a_2| \leq \frac{2 + \alpha}{|2 + \alpha + i\beta|}. \quad (3.1) \]

**Proof** Recall that \( |a_2| - |a_3| \leq 1 \) for all \( f \in \mathcal{S} \) [4, Thm. 3.11]. So, since \( B(\alpha, i\beta) \subset \mathcal{S} \) for all \( \alpha \geq 0 \) and \( \beta \in \mathbb{R} \), it is sufficient to prove the upper bound in (3.1).

Let \( f \in B(\alpha, i\beta) \) be of the form (1.1). Then from (1.3) we have

\[ \left( \frac{zf'(z)}{f(z)} \right) \left( \frac{f(z)}{g(z)} \right)^\alpha \left( \frac{f(z)}{z} \right)^i\beta = p(z), \]

for some \( g \in S^* \) and \( p \in \mathcal{P} \). Writing

\[ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad \text{and} \quad p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \]

and equating the coefficients, we obtain

\[ a_2 = \frac{\alpha b_2 + p_1}{1 + \alpha + i\beta} \quad (3.2) \]
and
\[
a_3 = \frac{p_2}{2 + \alpha + i \beta} - \frac{(-1 + \alpha + i \beta) p_1^2}{2(1 + \alpha + i \beta)^2(2 + \alpha + i \beta)} + \frac{\alpha(3 + \alpha + i \beta) b_2 p_1}{(1 + \alpha + i \beta)^2(2 + \alpha + i \beta)} + \frac{\alpha b_3}{2 + \alpha + i \beta} + \frac{\alpha(-1 + \alpha - 2i \beta - i \alpha \beta + \beta^2) b_2^2}{2(2 + \alpha + i \beta)(1 + \alpha + i \beta)^2}.
\]

(3.3)

Let \( \mu_1 = (3 + \alpha + i \beta)/(2(2 + \alpha + i \beta)) \), and suppose that \(|a_2| \leq 1/|\mu_1| \). Then by Lemmas 2.1 and 2.3 we have
\[
|a_3 - \mu_1 a_2|^2 = \left| \frac{1}{2 + \alpha + i \beta} \left( p_2 - \frac{1}{2} p_1^2 + \alpha \left( b_3 - \frac{1}{2} b_2^2 \right) \right) \right|^2 \leq \frac{2 + \alpha}{|2 + \alpha + i \beta|}.
\]

(3.4)

Thus from (3.4) we obtain
\[
|a_3| - |a_2| \leq |a_3| - |\mu_1||a_2|^2 \leq |a_3 - \mu_1 a_2|^2 \leq \frac{2 + \alpha}{|2 + \alpha + i \beta|}.
\]

Now assume that \( 1/|\mu_1| \leq |a_2| \leq 2 \), and let \( \mu_2 = 1/(2 + \alpha + i \beta) \). Then
\[
a_3 - \mu_2 a_2^2 = \Psi_1 + \frac{1}{2 + \alpha + i \beta} \Psi_2,
\]

(3.5)

where
\[
\Psi_1 = \frac{\alpha b_3}{2 + \alpha + i \beta} - \frac{\alpha(1 + i \beta) b_2^2}{2(1 + \alpha + i \beta)(2 + \alpha + i \beta)},
\]

and
\[
\Psi_2 = \frac{\alpha b_2 p_1}{(1 + \alpha + i \beta)} - \frac{(\alpha + i \beta) p_1^2}{2(1 + \alpha + i \beta)} + p_2.
\]

Put \( \mu = (1 + i \beta)/(2(1 + \alpha + i \beta)) \). Then it is easily seen that \(|3 - 4\mu| = |1 + 3\alpha + i \beta|/|1 + \alpha + i \beta| \geq 1 \). Thus Lemma 2.3 gives
\[
|\Psi_1| \leq \frac{\alpha}{|2 + \alpha + i \beta|} |3 - 4\mu| = \frac{\alpha|1 + 3\alpha + i \beta|}{|2 + \alpha + i \beta||1 + \alpha + i \beta|}.
\]

(3.6)

Next use (2.2) and (2.3) in Lemma 2.2 to obtain
\[
\Psi_2 = \frac{2\alpha b_2 \xi_1}{1 + \alpha + i \beta} + \frac{2\xi_1^2}{1 + \alpha + i \beta} + 2 \left( 1 - |\xi_1|^2 \right) \xi_2.
\]
where $\zeta_i \in \mathbb{D} \,(i = 1, 2)$. The triangle inequality and $|b_2| \leq 2$ then gives

$$|\Psi_2| \leq \psi(|\zeta_1|), \quad (3.7)$$

where

$$\psi(x) = 2 + \frac{4\alpha}{|1 + \alpha + i\beta|} x + 2 \left( \frac{1 - |1 + \alpha + i\beta|}{|1 + \alpha + i\beta|} \right) x^2$$

with $x \in [0, 1]$.

Let $x_0 = \alpha/(|1 + \alpha + i\beta| - 1)$, so that $x_0 \in [0, 1]$, and $\psi$ has a unique critical point at $x = x_0$. Since $\psi$ has a negative leading coefficient, it follows from (3.7) that for all $x \in [0, 1]$,

$$|\Psi_2| \leq \psi(x_0) = 2 + \frac{2\alpha^2}{|1 + \alpha + i\beta|(|1 + \alpha + i\beta| - 1)} \quad (x \in [0, 1]). \quad (3.8)$$

Therefore from (3.5), (3.6) and (3.10) we obtain

$$|a_3 - \mu_2 a_2^2| \leq \frac{1}{|2 + \alpha + i\beta|} \left( 2 + \frac{\alpha|1 + 3\alpha + i\beta|}{|1 + \alpha + i\beta|} + \frac{2\alpha^2}{|1 + \alpha + i\beta|(|1 + \alpha + i\beta| - 1)} \right)$$

$$=: \Psi(\alpha, \beta).$$

Next write $y := |a_2|$, and assume that $y \in [1/|\mu_1|, \tilde{x}]$, where

$$\tilde{x} = \frac{2\alpha + 2}{|1 + \alpha + i\beta|}, \quad (3.9)$$

so that

$$|a_3| - |a_2| \leq |a_3 - \mu_2 a_2^2| + |\mu_2| |a_2|^2 - |a_2| \leq \Psi(\alpha, \beta) + \varphi(y), \quad (3.10)$$

where $\varphi$ is defined by

$$\varphi(y) = \frac{1}{|2 + \alpha + i\beta|} y^2 - y \quad (y \in [1/|\mu_1|, \tilde{x}]).$$

Since $\varphi$ is convex on $[1/|\mu_1|, \tilde{x}]$,

$$\varphi(y) \leq \max\{\varphi(1/|\mu_1|); \varphi(\tilde{x})\} \quad (3.11)$$

for all $y \in [1/|\mu_1|, \tilde{x}]$.

Thus in order to establish the upper bound in (3.1), we use (3.10) and (3.11), and need to show that

$$\Psi(\alpha, \beta) + \varphi\left( \frac{1}{|\mu_1|} \right) \leq \frac{2 + \alpha}{|2 + \alpha + i\beta|} \quad (3.12)$$
and
\[ \Psi(\alpha, \beta) + \varphi(\tilde{x}) \leq \frac{2 + \alpha}{|2 + \alpha + i \beta|}. \]  
(3.13)

We first obtain (3.12). Since
\[ \frac{4}{|3 + \alpha + i \beta|} - 2 < 0 \text{ and } \frac{|2 + \alpha + i \beta|}{|3 + \alpha + i \beta|} \geq \frac{2 + \alpha}{3 + \alpha}, \]
(3.12) holds provided
\[ A_1 := \frac{\alpha|1 + 3\alpha + i \beta|}{|1 + \alpha + i \beta|} + \frac{2\alpha^2}{|1 + \alpha + i \beta|||1 + \alpha + i \beta| - 1|} + \frac{4(2 + \alpha)|2 + \alpha + i \beta|}{(3 + \alpha)|3 + \alpha + i \beta|} - \alpha \leq \frac{2(2 + \alpha)|2 + \alpha + i \beta|}{3 + \alpha} =: A_2. \]

Clearly \( A_1 \leq A_2 \) is true when \( \alpha = 0 \). For \( \alpha > 0 \), using the inequalities
\[ \frac{|1 + 3\alpha + i \beta|}{|1 + \alpha + i \beta|} \leq \frac{1 + 3\alpha}{1 + \alpha}, \quad \frac{1}{|1 + \alpha + i \beta|} \leq \frac{1}{1 + \alpha} \]
and
\[ \frac{1}{|1 + \alpha + i \beta| - 1} \leq \frac{1}{\alpha}, \]

it follows that
\[ \frac{1}{2}(A_1 - A_2) \leq |2 + \alpha + i \beta| \left( \frac{\alpha}{|2 + \alpha + i \beta|} + \frac{2(2 + \alpha)}{(3 + \alpha)|3 + \alpha + i \beta|} - \frac{2 + \alpha}{3 + \alpha} \right). \]  
(3.14)

We next note that the following is valid provided \( \alpha \in [0, (\sqrt{17} - 1)/2] \).
\[ \frac{\alpha}{|2 + \alpha + i \beta|} + \frac{2(2 + \alpha)}{(3 + \alpha)|3 + \alpha + i \beta|} \leq \frac{\alpha}{2 + \alpha} + \frac{2(2 + \alpha)}{(3 + \alpha)^2} \leq \frac{2 + \alpha}{3 + \alpha}. \]  
(3.15)

Thus from (3.15) and (3.14), \( A_1 \leq A_2 \) and (3.12) is established, providing \( \alpha \in [0, (\sqrt{17} - 1)/2] \).

Next we prove (3.13), which is satisfied if \( B_1 \leq B_2 \), where
\[ B_1 := \alpha(|1 + 3\alpha + i \beta| - |1 + \alpha + i \beta|) + \frac{2\alpha^2}{|1 + \alpha + i \beta| - 1|} + \frac{(2\alpha + 2)^2}{|1 + \alpha + i \beta|}. \]
and

\[ B_2 := 2(1 + \alpha)|2 + \alpha + i\beta|. \]

A similar process to the above gives

\[ B_1 \leq 2\alpha^2 + 2\alpha + \frac{(2\alpha + 2)^2}{1 + \alpha} = 2(1 + \alpha)(2 + \alpha) \leq B_2, \]

which proves inequality (3.13), and so the proof of Theorem 3.1 is complete. \(\square\)

When \(\beta = 0\), we deduce the following, noting that when \(\alpha = 1\), we obtain the inequality \(|a_3| - |a_2| | \leq 1 \) for \(f \in K\) obtained in [7].

**Corollary 3.1** Let \(f \in B(\alpha)\). Then \(|a_3| - |a_2| | \leq 1 \) provided \(0 \leq \alpha \leq (\sqrt{17} - 1)/2 \) = 1.56 . . .

The inequality is sharp when both the functions \(f\) and \(g\) are the Koebe function.

We end this section by noting from the definition, since \(B_1(0, i\beta) \equiv B(0, i\beta)\), the following is an immediate consequence of Theorem 4.1 below.

**Theorem 3.2** Let \(f \in B(0, i\beta)\), and be given by (1.1) with \(\beta \in \mathbb{R}\). Then

\[ -\frac{2}{\sqrt{|1 + i\beta|^2 + |3 + i\beta|}} \leq |a_3| - |a_2| \leq \frac{2}{|2 + i\beta|}. \]  

(3.16)

Both inequalities are sharp.

**4 The class \(B_1(\alpha, i\beta)\),**

We next consider the class \(B_1(\alpha, i\beta)\), recalling that \(f \in B_1(\alpha, i\beta)\) if, and only if, for \(\alpha \geq 0\) and \(\beta \in \mathbb{R}\),

\[ \text{Re} \left\{ \frac{zf'(z)}{f(z)} \left( \frac{f(z)}{z} \right)^{\alpha + i\beta} \right\} > 0 \quad (z \in \mathbb{D}). \]

We find the sharp upper and lower bounds of \(|a_3| - |a_2| | \) over the class \(B_1(\alpha, i\beta)\).

**Theorem 4.1** Let \(f \in B_1(\alpha, i\beta)\) for \(\alpha \geq 0\) and \(\beta \in \mathbb{R}\), and be given by (1.1). Then

\[ -\frac{2}{\sqrt{|1 + \alpha + i\beta|^2 + |3 + \alpha + i\beta|}} \leq |a_3| - |a_2| \leq \frac{2}{|2 + \alpha + i\beta|}. \]  

(4.1)

Both inequalities are sharp.
Proof From (3.2), (3.3) (with \(b_2 = b_3 = 0\)), and Lemma 2.2, we obtain
\[
a_2 = \frac{2\zeta_1}{1 + \alpha + i\beta}
\]
and
\[
a_3 = \left(\frac{2}{2 + \alpha + i\beta} - \frac{2(-1 + \alpha + i\beta)}{(1 + \alpha + i\beta)^2}\right)\zeta_1^2 + \frac{2}{2 + \alpha + i\beta} \left(1 - |\zeta_1|^2\right)\zeta_2
\]
for some \(\zeta_i \in \mathbb{D} \ (i = 1, 2)\). The triangle inequality gives
\[
|a_3| - |a_2| \leq \psi(|\zeta_1|), \quad (4.2)
\]
where
\[
\psi(x) = \kappa_2 x^2 + \kappa_1 x + \kappa_0 \quad (x \in [0, 1])
\]
with
\[
\kappa_2 = \left|\frac{2}{2 + \alpha + i\beta} - \frac{2(-1 + \alpha + i\beta)}{(1 + \alpha + i\beta)^2}\right| - \frac{2}{|2 + \alpha + i\beta|},
\]
\[
\kappa_1 = -\frac{2}{|1 + \alpha + i\beta|}, \quad \text{and} \quad \kappa_0 = \frac{2}{|2 + \alpha + i\beta|}.
\]

We first prove the upper bound in (4.1).
If \(\kappa_2 \leq 0\), then since \(\kappa_1 < 0\), we have \(\psi'(x) = 2\kappa_2 x + \kappa_1 < 0\) for all \(x \in [0, 1]\). Thus
\[
\psi(x) \leq \psi(0) = \kappa_0 \quad (x \in [0, 1]). \tag{4.3}
\]

Suppose next that \(\kappa_2 > 0\). We now note that \(\kappa_2 + \kappa_1 \leq 0\), since
\[
\frac{1}{2}(\kappa_2 + \kappa_1) \leq \left|\frac{-1 + \alpha + i\beta}{|1 + \alpha + i\beta|^2} - \frac{1}{|1 + \alpha + i\beta|}\right| = \frac{1}{|1 + \alpha + i\beta|} \left(\frac{|-1 + \alpha + i\beta|}{|1 + \alpha + i\beta|} - 1\right)
\]
and \(|1 + \alpha + i\beta| \geq |-1 + \alpha + i\beta|\).
Since \(\kappa_2 > 0\), \(\psi\) is a quadratic function with positive leading coefficient, and \(\psi(1) = \kappa_2 + \kappa_1 + \kappa_0 \leq \kappa_0 = \psi(0)\), it follows that
\[
\psi(x) \leq \max\{\psi(0); \psi(1)\} = \psi(0) = \kappa_0 \quad (x \in [0, 1]). \tag{4.4}
\]
Thus from (4.2), (4.3) and (4.5) we obtain

$$|a_3| - |a_2| \leq \kappa_0 = \frac{2}{|2 + \alpha + i\beta|}.$$  

We next prove the lower bound in (4.1).

Write

$$|a_3| - |a_2| = \frac{2}{|2 + \alpha + i\beta|} \Psi,$$

where

$$\Psi = \left| R_1 e^{i\theta} \xi_1^2 + (1 - \xi_1^2) \xi_2 - R_2 \xi_1 \right|$$

with

$$R_1 = \left| \frac{3 + \alpha + i\beta}{(1 + \alpha + i\beta)^2} \right|, \quad \theta = \arg \left( \frac{3 + \alpha + i\beta}{(1 + \alpha + i\beta)^2} \right)$$

and

$$R_2 = \left| \frac{2 + \alpha + i\beta}{1 + \alpha + i\beta} \right|,$$

so that we need to show that

$$\Psi \geq -\frac{R_2}{\sqrt{R_1 + 1}}.$$

Since both $B_1(\alpha, i\beta)$ and $P$ are rotationally invariant, we may assume that $\xi_1 \in [0, 1]$.

Now write $\xi_2 = se^{i\varphi}$ with $s \in [0, 1]$ and $\varphi \in \mathbb{R}$, so that

$$\Psi = \left| R_1 e^{i(\theta - \varphi)} \xi_1^2 + (1 - \xi_1^2)s - R_2 \xi_1 \right|.$$  

Then

$$\Psi = \sqrt{R_1^2 \xi_1^4 + 2R_1 \xi_1^2 (1 - \xi_1^2)s \cos(\theta - \varphi) + (1 - \xi_1^2)^2 s^2} - R_2 \xi_1 \geq \left| R_1 \xi_1^2 - (1 - \xi_1^2)s - R_2 \xi_1 \right|$$

with equality when $\cos(\theta - \varphi) = -1$.  

\(\square\) Springer
Suppose that $R_1 \zeta_1^2 - (1 - \zeta_1^2)s \leq 0$, then $\zeta_1 \leq \sqrt{s/(R_1 + s)} =: \eta_1$, and so by (4.6) it follows that
\[
\Psi \geq -(R_1 + s)\zeta_1^2 - R_2 \zeta_1 + s \\
\geq -(R_1 + s)\eta_1^2 - R_2 \eta_1 + s \\
= -R_2 \sqrt{\frac{s}{R_1 + s}} \\
\geq \frac{-R_2}{\sqrt{R_1 + 1}},
\]

since $s \leq 1$.

If $R_1 \zeta_1^2 - (1 - \zeta_1^2)s \geq 0$, then $\zeta_1 \geq \eta_1$, and define $\phi$ by
\[
\phi(x) = (R_1 + s)x^2 - R_2 x - s,
\]
and let
\[
\eta_2 = \frac{R_2}{2(R_1 + s)}
\]
be the unique critical point of $\phi$. Then by (4.6) we have
\[
\Psi \geq \phi(\zeta_1). \tag{4.7}
\]

The condition $\eta_2 \geq \eta_1$ is equivalent to the inequality $4s^2 + 4R_1 s - R_2^2 \geq 0$, which holds for $0 \leq s \leq \lambda$, where
\[
\lambda = \lambda_{\alpha, \beta} := \frac{1}{2} \left(-R_1 + \sqrt{R_1^2 + R_2^2}\right).
\]

It is easily seen that $\lambda < 1$ since
\[
R_2^2 = \frac{(2 + \alpha)^2 + \beta^2}{(1 + \alpha)^2 + \beta^2} \leq \left(\frac{2 + \alpha}{1 + \alpha}\right)^2 \leq 4 < 4 + R_1,
\]
for $\alpha \geq 0$, and $\beta \in \mathbb{R}$.

We also note that $R_2 - 2R_1 < 2$, since
\[
R_2 - 2R_1 < R_2 \leq \frac{2 + \alpha}{1 + \alpha} \leq 2.
\]

We consider next the case $R_2 \leq 2R_1$, where $\eta_1 \leq 1$ for all $s \in [0, 1]$, and distinguish two sub-cases, $\eta_2 \leq \eta_1$, and $\eta_2 \geq \eta_1$.

When $s \in [\lambda, 1]$, we have $\eta_2 \leq \eta_1$, and so from (4.7) we obtain
\[
\Psi \geq \phi(\eta_1) = -R_2 \sqrt{\frac{s}{R_1 + s}} \geq \frac{-R_2}{\sqrt{R_1 + 1}} \tag{4.8}
\]
since \( s \in [0, 1] \). When \( s \in [0, \lambda] \), we have \( \eta_2 \geq \eta_1 \). This, and (4.7), implies that

\[
\Psi \geq \phi(\eta_2) = -\left( s + \frac{R_2^2}{4(R_1 + s)} \right) = -\frac{1}{4}h(s),
\]  

(4.9)

where \( h \) is defined by

\[
h(x) = 4x + \frac{R_2^2}{R_1 + x}.
\]  

(4.10)

Differentiating \( h \) gives

\[
(R_1 + x)^2 h'(x) = 4x^2 + 8R_1 x + 4R_1^2 - R_2^2.
\]

Since \( 4R_1^2 - R_2^2 = (2R_1 + R_2)(2R_1 - R_2) \geq 0 \), \( h \) is increasing on the interval \([0, \lambda]\), and so from (4.9) we have

\[
\Psi \geq -\frac{1}{4}h(\lambda) = -\left( \lambda + \frac{R_2^2}{4(R_1 + \lambda)} \right).
\]  

(4.11)

Next note that

\[
\frac{R_2}{\sqrt{R_1 + 1}} \geq \lambda + \frac{R_2^2}{4(R_1 + \lambda)},
\]  

(4.12)

since

\[
\lambda + \frac{R_2^2}{4(R_1 + \lambda)} \leq \frac{R_2}{\sqrt{R_1 + \lambda}},
\]

provided \( \sqrt{\lambda(R_1 + 1)} \leq \sqrt{R_1 + \lambda} \) which is valid for all \( \alpha \geq 0 \) and \( \beta \in \mathbb{R} \) since \( \lambda < 1 \).

Thus it follows from (4.8), (4.11) and (4.12) that

\[
\Psi \geq -\frac{R_2}{\sqrt{R_1 + 1}}
\]

is true provided \( R_2 \leq 2R_1 \).

Next assume that \( R_2 \geq 2R_1 \). In this case there exists \( s \in [0, 1] \), such that \( \eta_2 \geq 1 \). Setting \( \hat{\lambda} = (R_2 - 2R_1)/2 \) it follows that \( 0 < \hat{\lambda} < \lambda < 1 \).

When \( s \in [\hat{\lambda}, 1] \), we have \( \eta_2 \leq \eta_1 \), and a similar method to that used in the case \( R_2 \leq 2R_1 \) gives

\[
\Psi \geq -\frac{R_2}{\sqrt{R_1 + 1}}.
\]

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When \( s \in [\hat{\lambda}, \lambda] \), we have \( \eta_2 \geq \eta_1 \), and so the function \( h \), defined by (4.10), is increasing on \([\hat{\lambda}, \lambda]\) since

\[
(R_1 + x)^2 h'(x) = 4x^2 + 8R_1x + 4R_1^2 - R_2^2 \\
\geq 4\hat{x}^2 + 8R_1\hat{x} + 4R_1^2 - R_2^2 = 0 \quad (x \in [\hat{\lambda}, \lambda]).
\]

Thus from (4.11) and (4.12), we have

\[
\Psi \geq -\frac{1}{4} h(\lambda) \geq -\frac{R_2}{\sqrt{R_1 + 1}}.
\]

When \( s \in [0, \hat{\lambda}] \), we have \( \eta_2 \geq 1 \), which implies

\[
\Psi \geq \phi(1) = R_1 - R_2.
\]  

(4.13)

Finally from (4.13), in order to establish the left hand inequality in (4.1), it is enough to show that

\[
\frac{R_2}{\sqrt{R_1 + 1}} \geq R_2 - R_1.
\]  

(4.14)

Since

\[
R_1 - R_2 + \frac{R_2}{\sqrt{R_1 + 1}} = R_1R_2 \left( \frac{1}{R_2} - \frac{1}{R_1 + 1 + \sqrt{R_1 + 1}} \right),
\]

and since \( R_1 > 0 \) and \( R_2 > 0 \), (4.14) is satisfied, if for \( \alpha \geq 0 \) and \( \beta \in \mathbb{R} \)

\[
\sqrt{R_1 + 1} > R_2 - R_1 - 1.
\]  

(4.15)

Since

\[
R_2 - R_1 - 1 = \frac{1}{|1 + \alpha + i\beta|} \left( |2 + \alpha + i\beta| - |1 + \alpha + i\beta| - \frac{|3 + \alpha + i\beta|}{|1 + \alpha + i\beta|} \right)
\]

and

\[
|2 + \alpha + i\beta| \leq |1 + \alpha + i\beta| + 1 < |1 + \alpha + i\beta| + \frac{|3 + \alpha + i\beta|}{|1 + \alpha + i\beta|},
\]

it follows that \( R_2 - R_1 - 1 < 0 < \sqrt{R_1 + 1} \), which establishes (4.15), and hence (4.14).

Thus the proof of the inequalities for \(|a_3| - |a_2|\) is complete.

In order to show that the inequalities are sharp, first let the function \( f_1 \) be defined by (1.3) with \( g(z) = z \) and \( p(z) = (1 + z^2)/(1 - z^2) \). Then \( f_1 \in B_1(\alpha, i\beta) \) with
\[ f_1(z) = z + \frac{2}{2 + \alpha + i \beta} z^3 + \ldots. \]

Thus the upper bound in (4.1) is sharp.

Next put \( \zeta_1 = 1/\sqrt{R_1 + 1} \), and \( \zeta_2 = se^{i\varphi} \) with \( s = 1 \) and \( \varphi = \theta - \pi \). Then
\[
\Psi = \left| R_1 e^{i(\theta-\varphi)} \zeta_1^2 + (1 - \zeta_1^2)s \right| - R_2 \zeta_1 = -\frac{R_2}{\sqrt{R_1 + 1}}.
\]

Since \( \zeta_1 \in \mathbb{D} \) and \( \zeta_2 \in \mathbb{T} \), it follows from Lemma 2.2 that the function \( \hat{p} \) defined by
\[
\hat{p}(z) = \frac{1 + (\zeta_1 \zeta_2 + \zeta_1)z + \zeta_2 z^2}{1 + (\zeta_1 \zeta_2 - \zeta_1)z - \zeta_2 z^2} = \frac{\sqrt{R_1 + 1} + (e^{i\varphi} + 1)z + \sqrt{R_1 + 1}e^{i\varphi}z^2}{\sqrt{R_1 + 1} + (e^{i\varphi} - 1)z - \sqrt{R_1 + 1}e^{i\varphi}z^2}
\]

belongs to \( \mathcal{P} \). Now let the function \( f_2 \) be defined by (1.3) with \( g(z) = z \) and \( p = \hat{p} \). Then \( f_2 \in \mathcal{B}_1(\alpha, i \beta) \). From (4.5) and (4.16), we obtain
\[
|a_3| - |a_2| = \frac{2}{|2 + \alpha + i \beta|} \Psi = -\frac{2}{\sqrt{|1 + \alpha + i \beta|^2 + |3 + \alpha + i \beta|}^2},
\]

which shows that the left hand equality in (4.1) is sharp.

This completes the proof of Theorem 4.1.

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