Vect(N) conformal fields and their exterior derivatives

T. A. Larsson*
Department of Theoretical Physics,
Royal Institute of Technology
100 44 Stockholm, Sweden
e-mail: tl@theophys.kth.se

(February 1992)

Abstract
Conformal fields are a recently discovered class of representations of the algebra of vector fields in N dimensions. Invariant first-order differential operators (exterior derivatives) for conformal fields are constructed.

PACS number: 02.10 11.30

* Supported by the Swedish Natural Science Research Council (NFR)
1. Introduction

In order to describe physical quantities, a coordinate system has to be introduced, but physics itself should not depend on how this is done. Therefore any sensible object must transform as a representation of the group of diffeomorphisms (coordinate transformations) in the relevant space, and conversely the representation theory of the diffeomorphism group amounts to a classification of inequivalent meaningful objects.

A sensible starting point is the Lie algebra of the diffeomorphism group; this is the algebra of vector fields. Moreover, we must first deal with the local properties of this algebra. Since any manifold is locally diffeomorphic to $\mathbb{R}^N$, the only important parameter is the dimensionality $N$, and it makes sense to talk about the algebra of vector fields in $N$ dimensions, $\text{Vect}(N)$. This algebra has recently attracted some interest by physicists\textsuperscript{1–6}; references some earlier mathematical literature can be found in Ref. 1. The one-dimensional case is special, because it admits a central extension (Virasoro algebra), which is a cornerstone of modern theoretical physics\textsuperscript{7}. From the arguments above, it is clear that $N$-dimensional local differential geometry may be considered as the representation theory of $\text{Vect}(N)$, and it is in fact straightforward to describe e.g. tensor fields and exterior derivatives in such terms.

In a recent paper\textsuperscript{1}, we discovered a new class of $\text{Vect}(N)$ representations which seem more natural than tensor fields, namely conformal fields. A tensor field can be considered as a scalar field decorated with indices from the rigid $gl(N)$ subalgebra. Similarly, a conformal field has indices from the “conformal” subalgebra $sl(N+1)$, which is obtained from $gl(N)$ by adding translations and “conformal” transformations. Because $gl(N) \subset sl(N + 1)$, a conformal field transforms nicely under a larger finite-dimensional subalgebra than a tensor field does.

This paper is organized as follows. In section 2 we recollect some relevant facts about tensor fields and exterior derivatives, formulated in a fashion which emphasizes the representation theory aspects. Section 3 contains the definition of conformal fields, in a slightly more streamlined notation than in Ref. 1. It also contains some minor new results. In section 4 we construct first-order differential operators that are compatible with the $\text{Vect}(N)$ action. These conformal exterior derivatives are considerably more abundant than their tensor counterparts. This leads to the development of a form language, whose applications are investigated in section 5. Because each conformal index can take one more value than the corresponding tensor index, we propose in the final section that this extra dimension can be interpreted as time.

2. Tensor fields

In some neighborhood of the origin, a vector field has the form $f(x)\partial^\mu$. By Fourier expansion we see that a basis of $\text{Vect}(N)$ is given by $L^\mu(m) = e^{m_\alpha} \partial^\mu$, where $m = (m^1, \ldots, m^N)$ belongs to an $N$-dimensional lattice $\Lambda$, and the brackets are

$$[L^\mu(m), L^\nu(n)] = n^\mu L^\nu(m + n) - m^\nu L^\mu(m + n) \quad (2.1)$$

Our convention is that the lattice is purely imaginary, which saves many explicit references.
to the imaginary unit \( i \). Of special interest are \( \Lambda = i \mathbb{Z}^N \) and \( \Lambda = i \mathbb{R}^N \). The \( \Lambda \)-gradation expresses momentum conservation.

An important class of \( \text{Vect}(N) \) representations are tensor fields with \( p \) upper and \( q \) lower indices and conformal weight \( \lambda \), which are constructed from \( \text{gl}(N) \) representations as follows. Assume that \( \{ T^\mu_\nu \}_{\mu, \nu=1}^N \) satisfies \( \text{gl}(N) \), i.e.

\[
[T^\mu_\sigma, T^\nu_\tau] = \delta^\nu_\sigma T^\mu_\tau - \delta^\mu_\sigma T^\nu_\tau.
\]

(2.2)

Then it is easy to check that

\[
L^\mu(m) = e^{m \cdot x} (\partial^\mu + m^\sigma T^\mu_\sigma)
\]

(2.3)

satisfies \( \text{Vect}(N) \). This means for each \( \text{gl}(N) \) representation we have a corresponding \( \text{Vect}(N) \) representation.

Instead of giving the matrix (2.3), we could equally well describe the representation by the action on a module. Let us elaborate on this trivial point, if nothing else because the signs have caused this author some confusion. Let \( J^a \) be the generators of some Lie algebra and \( M^a \) a representation matrix,

\[
[J^a, J^b] = f^{ab}_{\phantom{ab}c} J^c, \quad [M^a, M^b] = f^{ab}_{\phantom{ab}c} M^c.
\]

(2.4)

Then the module is a vector space with basis \( \phi \), and the Lie algebra acts as \( J^a \phi = M^a \phi \) (representation indices suppressed). Because \( J^a \) only acts on \( \phi \) and not on the numerical matrix \( M^a \),

\[
[J^a, J^b] \phi = J^a M^b \phi - a \leftrightarrow b = M^b M^a \phi - a \leftrightarrow b = -f^{ab}_{\phantom{ab}c} J^c \phi,
\]

(2.5)

so with this convention the structure constants change sign.

In particular, from the \( \text{gl}(N) \) action on a tensor \( T^p_\lambda(x) \),

\[
T^\mu_\sigma \phi_{\tau_1 \ldots \tau_q}^{\nu_1 \ldots \nu_p} = \lambda \delta^\mu_\sigma \phi_{\tau_1 \ldots \tau_q}^{\nu_1 \ldots \nu_p} + \sum_{i=1}^{p} \delta^\mu_\sigma \phi_{\tau_1 \ldots \tau_q}^{\nu_1 \ldots \mu \ldots \nu_p} - \sum_{j=1}^{q} \delta^\mu_\tau \phi_{\tau_1 \ldots \sigma \ldots \tau_q}^{\nu_1 \ldots \nu_p},
\]

(2.6)

we find that the corresponding action of \( \text{Vect}(N) \) on the tensor field \( \phi_{\tau_1 \ldots \tau_q}^{\nu_1 \ldots \nu_p}(x) = \phi_{\tau_1 \ldots \tau_q}^{\nu_1 \ldots \nu_p} \otimes f(x) \) is

\[
L^\mu(m) \phi_{\tau_1 \ldots \tau_q}^{\nu_1 \ldots \nu_p}(x) = e^{m \cdot x} \left( (\partial^\mu + \lambda m^\mu) \phi_{\tau_1 \ldots \tau_q}^{\nu_1 \ldots \nu_p}(x) \right.
\]

\[
+ \sum_{i=1}^{p} m^\nu_i \phi_{\tau_1 \ldots \tau_q}^{\nu_1 \ldots \mu \ldots \nu_p}(x) - \sum_{j=1}^{q} \delta^\mu_\tau m^\sigma \phi_{\tau_1 \ldots \sigma \ldots \tau_q}^{\nu_1 \ldots \nu_p}\right).
\]

(2.7)

By abuse of notation, we also denote this \( \text{Vect}(N) \) module by \( T^p_\lambda(x) \). The parameter \( \lambda \) will be referred to as the (conformal) weight. Note that in one dimension the relevant quantity is \( \lambda + p - q \).
A slight generalization of tensor fields is obtained by shifting $\partial^\mu \to \partial^\mu + h^\mu$. The constant vector $h^\mu$ is of course only defined modulo $\Lambda$, because otherwise it could be absorbed into a relabelling of the Fourier components. E.g., the action on a scalar field now reads

$$L^\mu(m)\phi = e^{m_x}(\partial^\mu + \lambda m^\mu + h^\mu)\phi.$$  \hfill (2.8)

By forgetting the vector index we recognize this as the transformation law of a primary field in conformal field theory. In Ref. 1 it was suggested that $L^\mu(0)$ generates rigid translations in space-time, and thus $h^N$ could be interpreted as a mass. In view of the current section 6, this assumption seems incorrect. Moreover, for continuous $\Lambda$, $h^\mu$ can be eliminated altogether by substituting $\phi \to \exp(-h^x)\phi$.

Recall that a homomorphism, or module map, is a map between modules that preserves the Lie algebra structure. If the generic Lie algebra (2.4) had two matrix representations $M^a$ and $N^a$, $d$ is a homomorphism if

$$J^a(d\phi) = dJ^a\phi = dM^a\phi = N^a d\phi,$$

i.e. $dM^a = N^a d$.

Two important classes of homomorphisms can be defined for tensor fields. The first is pointwise multiplication, which is the map

$$T^p_\nu(\lambda \phi) \times T^p_\psi(\lambda \psi) \longrightarrow T^{p\phi+p\psi}_{\nu\psi}(\lambda \phi + \lambda \psi)$$

$$\phi(x) \times \psi(y) \mapsto (\phi\psi)(x) = \phi(x)\psi(x)$$ \hfill (2.10)

Note that this module is a much smaller than the tensor product $\phi(x)\psi(y)$. The result (2.10) follows from Leibniz' rule,

$$\partial^\mu = \partial^\mu \otimes 1 + 1 \otimes \partial^\mu, \quad T^\mu_\nu = T^\mu_\nu \otimes 1 + 1 \otimes T^\mu_\nu.$$  \hfill (2.11)

In the case that $\phi$ and $\psi$ are two scalar fields with non-zero conformal weights,

$$L^\mu(m)(\phi\psi) = (e^{m_x}(\partial^\mu + \lambda \phi m^\mu)\phi)\psi + \phi(e^{m_x}(\partial^\mu + \lambda \psi m^\mu)\psi)$$

$$= e^{m_x}(\partial^\mu + (\lambda \phi + \lambda \psi)m^\mu)(\phi\psi),$$ \hfill (2.12)

which proves that $\phi\psi$ is a new scalar field with weight $\lambda \phi + \lambda \psi$.

The second homomorphism is the exterior derivative. Tensor fields with several indices admit submodules which are symmetric or skew in some indices. Of particular interest is the module $\Omega^p \subset T^p_0(0)$ consisting of totally skew $p$-tensors (forms), because then we can define the sequence of maps $d_p$,

$$\Omega^0 \xrightarrow{d_0} \Omega^1 \xrightarrow{d_1} \ldots \Omega^N \xrightarrow{d_N} 0,$$ \hfill (2.13)

satisfying $d_{p+1}d_p = 0$. Explicitly, the maps are given by

$$(d_p\phi)^{\nu_1\ldots\nu_{p+1}} = \frac{1}{(p+1)!} \partial^{[\nu_1 \phi^{\nu_2\ldots\nu_{p+1}]},$$ \hfill (2.14)
where the bracketed indices are skew-symmetrized. To illustrate how to prove that $d_p$ are maps, and also to see what goes wrong if we apply it to tensor fields which are not forms, consider applying $d_0$ to a scalar field with conformal weight $\lambda$. The action of $\text{Vect}(N)$ is

$$L^\mu (d\phi)^\nu = \partial^\nu e^{m\mu} (\partial^\mu + \lambda m^\mu) \phi$$

$$= e^{m\mu} \left( (\partial^\mu + \lambda m^\mu) \partial^\nu \phi + m^\nu \partial^\mu \phi + \lambda m^\mu m^\nu \phi \right)$$

(2.15)

Thus, $(d\phi)^\nu$ transforms as a tensor apart from the last term. However, this term is absent if $\lambda = 0$, which is precisely the condition that $\phi \in \Omega^0$. The verification of the higher maps is straightforward.

We also have a dual sequence involving *chains* $\Omega_p \subset T^0_p(1)$.

$$\Omega_N \xrightarrow{\bar{d}_N} \Omega_{N-1} \xrightarrow{\bar{d}_{N-1}} \ldots \xrightarrow{\bar{d}_1} \Omega_0.$$  

(2.16)

These maps are given by

$$(d_\bar{p} \bar{\phi})_{\nu_1 \ldots \nu_{p-1}} = \partial^\sigma \bar{\phi}_{\nu_1 \ldots \nu_{p-1} \sigma}$$

(2.17)

and again $\bar{d}_{p-1} \bar{d}_p = 0$. Forms and chains are dual in the sense that an invariant pairing can be defined by

$$\langle \phi(x), \bar{\phi}(y) \rangle = \delta(x - y)$$

(2.18)

and $d_p$ and $\bar{d}_p$ are dual maps relative to this pairing.

Finally, pointwise multiplication endows forms with a ring structure which is compatible with the exterior derivative. The multiplication is given by the exterior product

$$\wedge : \Omega^p \times \Omega^q \to \Omega^{p+q}$$

$$\phi \wedge \psi_{\nu_1 \ldots \nu_{p+1} \ldots \nu_{p+q}} = \frac{1}{(p+q)!} \phi^{\nu_1 \ldots \nu_p} \psi^{\nu_{p+1} \ldots \nu_{p+q}}.$$  

(2.19)

The compatibility is expressed by Leibniz’ rule,

$$d_{p+q}(\phi \wedge \psi) = (d_p \phi) \wedge \psi + (-)^p \phi \wedge (d_q \psi).$$

(2.20)

No similar map can be constructed for chains.

To distinguish objects with tensor indices from those derived from the conformal fields introduced below, we sometimes prefix them with the word “tensor” and talk about tensor forms, tensor vielbeins, etc.
3. Conformal fields

The main result of Ref. 1 was the discovery of a new class of $Vect(N)$ representations called conformal fields. If $T_B^A$, $A, B = 0, 1, \ldots N$ are the generators of $gl(N + 1)$, i.e.

$$[T_B^A, T_C^D] = \delta_B^D T_C^B - \delta_C^B T_D^A,$$

(3.1)

then

$$L^\mu(m) = e^{mx}\left(\partial^\mu + (m^B + k^B)T_B^\mu + cm^\mu m^B T_B^A x_A\right).$$

(3.2)

satisfies $Vect(N)$. Here

$$m^A \equiv (m^0, m^\mu) = (-m, m^\mu),$$

$$x_B \equiv (x_0, x_\nu) = (1, x_\nu),$$

$$k^A \equiv (k^0, k^\mu) = (1, 0),$$

and $c$ is a c-number parameter.

By introducing the $N + 1$-dimensional derivative $\partial^A = (-x \partial, \partial^\mu)$, the relevant algebraic properties can be summarized as

$$x_A m^A = x_A \partial^A = 0, \quad k^A x_A = 1, \quad \partial^A x_A = N.$$

(3.4)

$$[\partial^A, \partial^B] = k^A \partial^B - k^B \partial^A \quad [\partial^A, m^B] = -k^B m^A$$

$$[\partial^A, x_B] = \delta_B^A - k^A x_B \quad [\partial^A, e^{mx}] = m^A e^{mx}$$

(3.5)

and all commutators between $m^A$, $x_B$ and $k^C$ vanish.

By substituting different representations of $gl(N + 1)$ into (3.2), we obtain various representations of $Vect(N)$. In particular, if \{ $\phi^{A_1 \ldots A_p}_{B_1 \ldots B_q}(x)$ $\}_{A_i, B_j = 1}^{N+1}$ is a basis for the $gl(N + 1)$ module $T_q^p(\lambda)$ (2.6), the action of $Vect(N)$ on $\phi^{A_1 \ldots A_p}_{B_1 \ldots B_q}(x) = \phi^{A_1 \ldots A_p}_{B_1 \ldots B_q} \otimes f(x)$ is given by

$$L^\mu(m) \phi^{A_1 \ldots A_p}_{B_1 \ldots B_q}(x) = e^{mx}\left(\partial^\mu + \lambda m^\mu\right) \phi^{A_1 \ldots A_p}_{B_1 \ldots B_q}(x)$$

$$\quad + \sum_{i=1}^p \left\{(m^{A_i} + k^{A_i}) \phi^{A_1 \ldots \mu \ldots A_p}_{B_1 \ldots B_q}(x) + cm^\mu m^{A_i} x_C \phi^{A_1 \ldots C \ldots A_p}_{B_1 \ldots B_q}(x)\right\}$$

$$\quad - \sum_{j=1}^q \left\{\delta^\mu_{B_j}(m^C + k^C) \phi^{A_1 \ldots A_p}_{B_1 \ldots C \ldots B_q}(x) + cm^\mu m^C x_{B_j} \phi^{A_1 \ldots A_p}_{B_1 \ldots C \ldots B_q}(x)\right\}$$

(3.6)

This defines the conformal field $C_q^p(\lambda, c)$.

The discovery (and name) of conformal fields arose in Ref. 1 from a study of of the largest finite-dimensional subalgebra of $Vect(N)$, which is the “conformal” algebra $sl(N + 1)$. For a scalar field, $J_B^A$ is given by

$$J_B^A \equiv \begin{pmatrix} J_0^0 & J_0^\mu \\ J_0^\nu & J_\nu^\mu \end{pmatrix} = \begin{pmatrix} -x \partial & -x_\nu x_\partial \\ x_\nu x_\partial & x_\nu \partial^\mu \end{pmatrix} = x_B \partial^A$$

(3.7)
Generally, the conformal generators are found by differentiating $L^\mu(m)$ with respect to $m$.

$$J_B^A = \begin{pmatrix}
-\frac{\partial L^\sigma(m)}{\partial m^\sigma}igr|_{m=0} & -\frac{\partial^2 L^\sigma(m)}{\partial m^\nu \partial m^\sigma}igr|_{m=0} \\
L^\mu(0) & \frac{\partial L^\mu(m)}{\partial m^\nu}igr|_{m=0}
\end{pmatrix}. \quad (3.8)$$

$sl(N+1)$ is not conformal in the usual sense that it preserves angles, because in order to define an angle a metric has to be introduced in an *ad hoc* fashion. Rather, it is a “conformal” algebra in the profane sense that the generators are of the form $\partial$, $x\partial$ and $x^2 \partial$.

The procedure to arrive at (3.8) may seem dubious, but it is merely a consequence of our particular basis. $Vect(N)$ can be formulated in terms of generators $L(f_\mu)$, $f_\mu(x)$ some vector-valued function.

$$[L(f_\mu), L(g_\mu)] = L(f\partial g_\mu) - L(g\partial f_\mu) \quad (3.9)$$

With such a basis, the conformal subalgebra is completely natural.

Conformal fields with $c = -1/(N+1)$ are singled out when the restriction to $sl(N+1)$ is considered. Using the formula

$$\frac{\partial m^A}{\partial m^\nu} = (-x_\nu, \delta_\nu^A) = \delta^A_\nu - k^A x_\nu \quad (3.10)$$

we find

$$J_B^A = x_B \partial^A + T_B^A - k^A x_B T_C^C \quad (3.11)$$

and in particular $J_A^A = 0$, as expected. For other values of $c$ we still of course obtain a representation of $sl(N+1)$, but it is more messy.

Just as tensor fields, conformal fields admit a pointwise tensor product. This is because Leibniz’ rule holds both for the derivative $\partial^\mu$ and for the $gl(N+1)$ generator $T_B^A$. Thus, we have the map

$${\mathbf{C}}_{q_1}^{p_1}(0) \times {\mathbf{C}}_{q_2}^{p_2}(0) \rightarrow {\mathbf{C}}_{q_1+q_2}^{p_1+p_2}(0)$$

$$\phi(x) \times \psi(y) \mapsto (\phi \psi)(x) = \phi(x) \psi(x). \quad (3.12)$$

A point worth noting is that the parameter $c$ must be the same in both participating fields. In this sense (3.12) is similar to a fusion rule in conformal field theory, and $c$ takes the role of the central charge. We believe that this similarity is not coincidental.

From (3.2) it immediately follows that $C^0_{0}(\lambda, c) \equiv T^0_{0}(\lambda)$. More generally, we may ask if it is possible to construct a conformal field from a tensor field. This construction is carried out for vector fields only, but it can be applied to arbitrary tensors by treating each index separately.

$$\phi^\mu \in T^0_{0}(0) \Rightarrow \phi^A = (\phi^\sigma, \delta^\mu_{\sigma}) \phi^\sigma \equiv (-x^\phi, \phi^\mu) \in C^1_{1}(0, c)$$

$$\psi \in T^0_{0}(0) \Rightarrow \psi_B = x_B \psi \equiv (\psi, x^\psi) \in C^1_{1}(0, c) \quad (3.13)$$
We say that such conformal fields are tensor derived. However, not every conformal field is tensor derived, because the definition (3.13) implies certain relations, namely
\[ x_A \phi^A = -x_0 x^\phi + x_\mu \phi^\mu = 0, \]
\[ m^B \psi_B = 0, \]
\[ k^B \psi_B \in T^0_0(0) \] (3.14)
The parameter \( c \) is irrelevant for tensor derived fields, because (3.14) guarantees that the term multiplying \( c \) vanishes. Let us prove that \( \psi_B \) does transform as claimed.
\[ L^\mu(m) \psi_B = x_B e^{m_\mu} \partial^\mu \psi \]
\[ = e^{m_\mu} (\partial^\mu (x_B \psi) - \delta^\mu_B \psi) \]
\[ = e^{m_\mu} (\partial^\mu \psi_B - \delta^\mu_B (m^C + k^C) \psi_C + cm^\mu x_B m^C \psi_C) \] (3.15)
This shows that tensor fields are intimately related to a particular class of conformal fields. Henceforth, we focus on conformal fields that are not tensor derived.

4. Conformal forms and exterior derivatives
In this section we construct the conformal analog of forms.
The \( Vect(N) \) module \( \Omega^p(\lambda, c) \subset C^p_0(\lambda, c) \) consists of conformal fields which are skew in all \( p \) upper indices. Similarly, the elements in the module \( \Omega_p(\lambda, c) \subset C_0(\lambda, c) \) are skew in all \( p \) lower indices. These objects will be referred to as positive and negative conformal forms, respectively. The form degree will often be indicated by the notation \( \phi_p \) and \( \bar{\phi}_q \), as shorthand for \( \phi_{A_1 \ldots A_p}(x) \) and \( \bar{\psi}_{A_1 \ldots A_p}(x) \), respectively. As in section 2, bracketed indices are anti-symmetrized. Our convention is that \( \phi^{[A_1 \ldots A_p]} = p! \phi_{A_1 \ldots A_p} \) for a field that is already skew.

**Theorem I**
There is a map (conformal exterior derivative)
\[ d_p(\lambda, c) : \Omega^p(\lambda, c) \rightarrow \Omega^{p+1}(\lambda, c) \]
\[ (\phi_p)^{A_1 \ldots A_p} \mapsto (d_p(\lambda, c) \phi_p)^{A_1 \ldots A_{p+1}} \equiv \frac{1}{(p+1)!} (\partial^{[A_1} + \gamma_p(\lambda, c) k_{A_1}^{[A_2} \ldots A_{p+1}]) (\phi_p)^{A_2 \ldots A_{p+1}]} \] (4.1)
where \( \gamma_p(\lambda, c) = \lambda/c - p. \)
There is a map
\[ \bar{d}_p(\lambda, c) : \Omega_p(\lambda, c) \rightarrow \Omega_{p-1}(\lambda, c) \]
\[ (\bar{\phi}_p)^{A_1 \ldots A_p} \mapsto (\bar{d}_p(\lambda, c) \bar{\phi}_p)^{A_1 \ldots A_{p-1}} \equiv (\bar{\partial}^{[A_1} + \bar{\gamma}_p(\lambda, c) k_{A_1}^{[A_2} \ldots A_{p-1}]} (\bar{\phi}_p)^{A_2 \ldots A_{p-1}]}, \] (4.2)
where \( \bar{\gamma}_p(\lambda, c) = (\lambda - 1)/c + p - N - 1. \)
These maps satisfy $d_{p+1}d_p = 0$ and $\tilde{d}_{p-1}\tilde{d}_p = 0$. Although $\Omega_0(\lambda, c) \equiv \Omega^0(\lambda, c)$, $d_0\tilde{d}_1 \neq 0$.

For $c = -1/(N+1)$, $\tilde{\gamma}_p(\lambda, c) = \gamma_{-p}(\lambda, c)$. Recall that this was the value of $c$ which simplified the inherited $sl(N+1)$ representation. The normalization is fixed by compatibility with the wedge product below.

The following diagram summarizes the situation.

$$
\begin{array}{cccccc}
\Omega^0(\lambda, c) & \xrightarrow{d_0} & \Omega^1(\lambda, c) & \xrightarrow{d_1} & \ldots & \xrightarrow{d_{N-1}} \Omega^{N-1}(\lambda, c) & \xrightarrow{d_N} \Omega^N(\lambda, c) & \xrightarrow{\tilde{d}_N} 0 \\
\xrightarrow{\Omega_0(\lambda, c)} & & \xrightarrow{\Omega_1(\lambda, c)} & & \ldots & & \xrightarrow{\Omega_{N-1}(\lambda, c)} & & \xrightarrow{\Omega_N(\lambda, c)}
\end{array}
$$  \hspace{1cm} (4.3)

Proof: Let $\phi \in \Omega^0(\lambda, c) \equiv C^0_0(\lambda, c)$. Then $(d\phi)^A$ transforms as

$$
L^\mu(m)(d\phi)^A = (\partial^A + \gamma k^A)e^{m-x}(\partial^\mu + \lambda m^\mu)\phi
$$

$$
= e^{m-x}\left(m^A(\partial^\mu + \lambda m^\mu)\phi + k^A\partial^\mu \phi + (\partial^\mu + \lambda m^\mu)(\partial^A + \gamma k^A)\phi\right)
$$

$$
= e^{m-x}\left((\partial^\mu + \lambda m^\mu)(d\phi)^A + (m^A + k^A)(d\phi)^\mu + \lambda m^\mu \frac{1}{\gamma} x_B(d\phi)^B\right)
$$

In the last line, we used that $(d\phi)^\mu = \partial^\mu \phi$ ($k^\mu = 0$) and $x_B(d\phi)^B = \gamma \phi$. Eq. (4.4) is the transformation law of $\Omega^1(\lambda, c)$ provided that we identify $c = \lambda/\gamma$.

If $\phi^A \in \Omega^1(0, c) \equiv C^1_0(1, c)$, $(d\phi)^{AB}$ transforms as

$$
L^\mu(m)(d\phi)^{AB}
$$

$$
= \frac{1}{2}(\partial^A + \gamma k^A)e^{m-x}\left((\partial^\mu + \lambda m^\mu)\phi^B + (m^B + k^B)\phi^\mu + cm^\mu m^B x_C\phi^C\right) - A \leftrightarrow B
$$

$$
= \frac{1}{2}e^{m-x}\left(m^A((\partial^\mu + \lambda m^\mu)\phi^B + (m^B + k^B)\phi^\mu + cm^\mu m^B x_C\phi^C) + k^A\partial^\mu \phi^B - k^B m^A \phi^\mu
$$

$$
+ cm^\mu(-k^B m^A x_C\phi^C + m^B \phi^A - m^B k^A x_C\phi^C) + (\partial^\mu + \lambda m^\mu)(\partial^A + \gamma k^A)\phi^B
$$

$$
+ (m^B + k^B)(\partial^A + \gamma k^A)\phi^\mu + cm^\mu m^B x_C(\partial^A + \gamma k^A)\phi^C\right) - A \leftrightarrow B
$$

$$
= e^{m-x}\left((\partial^\mu + \lambda m^\mu)(d\phi)^{AB} + (m^A + k^A)(d\phi)^\mu B + (m^B + k^B)(d\phi)^A B
$$

$$
+ \frac{c}{2} m^\mu m^B (x_C(\partial^A + \gamma k^A)\phi^C + (1 - \frac{\lambda}{c})\phi^A - A \leftrightarrow B)\right).
$$

If we now realize that

$$
2x_C(d\phi)^{AC} = x_C(\partial^A + \gamma k^A)\phi^C - \gamma \phi^A,
$$

the last term becomes $cm^\mu m^B x_C(d\phi)^{AC} - A \leftrightarrow B$ provided that $\gamma + 1 - \lambda/c = 0$, which precisely is the condition for $\gamma_1$. 

9
Let \( \phi_A \in \Omega_1(\lambda, c) \equiv C^0_1(\lambda, c) \). Then \( \tilde{d}\phi \) transforms as
\[
L^\mu(m)(\tilde{d}\phi) = (\partial^A + \tilde{\gamma}k^A)e^{m\cdot x}
\left( (\partial^\mu + \lambda m^\mu)\phi_A - \delta^\mu_A(m^B + k^B)\phi_B - cm^\mu x_A m^B \phi_B \right)
\]
\[
= e^{m\cdot x}
\left( m^A ((\partial^\mu + \lambda m^\mu)\phi_A - \delta^\mu_A(m^B + k^B)\phi_B - cm^\mu x_A m^B \phi_B ) + k^A \partial^\mu \phi_A \\
- \delta^\mu_A(-m^A)k^B\phi_B - cm^\mu(Nm^B\phi_B - x_A m^A k^B\phi_B) + (\partial^\mu + \lambda m^\mu)\tilde{d}\phi \\
- \delta^\mu_A(m + k)^B(\partial^A + \tilde{\gamma}k^A)\phi_B - cm^\mu x_A m^B(\partial^A + \tilde{\gamma}k^A)\phi_B \right)
\]
\[
= e^{m\cdot x}
\left( (\partial^\mu + \lambda m^\mu)\tilde{d}\phi + m^\mu(\lambda - 1 - Nc - \tilde{\gamma}c)m^A \phi_A \right).
\]
(4.7)

This is the transformation law of \( \Omega_0(\lambda, c) \) provided that the last term vanishes, which yields the condition on \( \tilde{\gamma} \).

The same brute force method can be applied to higher conformal forms. When this is done it is realized that the structure is such that the higher exterior derivatives are module maps. However, a simpler proof can be constructed, at least for positive forms, but it must be postponed until Leibniz’ rule has been verified.

Two consecutive maps give zero. If \( \phi \in \Omega^p \),
\[
(d_{p+1}d_p\phi)^{A_1\ldots A_p} \propto (\partial^A + \gamma_{p+1}k^A)(\partial^B + \gamma_p k^B)\phi^{C_1\ldots C_p}
\]
(4.8)

Most terms vanish because of anti-symmetry in \( A \) and \( B \), but two remain:
\[
(\partial^A, \partial^B) = k^A \partial^B - k^B \partial^A
\]
(4.9)
and
\[
\gamma_{p+1}k^A \partial^B + \gamma_p \partial^A k^B = (\gamma_{p+1} - \gamma_p)k^A \partial^B
\]
(4.10)

However, because \( \gamma_{p+1} - \gamma_p = -1 \), these terms cancel and \( d_{p+1}d_p\phi = 0 \).

Similarly, if \( \phi \in \Omega_{p+2} \),
\[
(d_{p+1}\tilde{d}_{p+2}\phi)^{C_1\ldots C_p} = (\partial^A + \tilde{\gamma}_{p+2}k^A)(\partial^B + \tilde{\gamma}_{p+1}k^B)\phi^{C_1\ldots C_p AB},
\]
(4.11)
which is identically zero because \( \tilde{\gamma}_{p+2} - \tilde{\gamma}_{p+1} = 1 \). Remains to prove that \( d_0\tilde{d}_1 \neq 0 \). But this is obvious, because in the expression
\[
(d_0\tilde{d}_1\phi)^A = (\partial^A + \gamma_0 k^A)(\partial^B + \tilde{\gamma}_1 k^B)\phi_B
\]
(4.12)
nothing cancels.

Conformal forms can be endowed with a ring structure by introducing the following products.
\[
\wedge : \quad \Omega^p(\lambda_\phi, c) \times \Omega^q(\lambda_\psi, c) \longrightarrow \Omega^{p+q}(\lambda_\phi + \lambda_\psi, c)
\]
\[
(\phi_\mu \wedge \psi_\eta)^{A_1\ldots A_p B_1\ldots B_q}(x) = \frac{1}{(p+q)!} \phi^{[A_1\ldots A_p}(x) \psi^{B_1\ldots B_q]}(x)
\]
(4.13)
\[
\cdot : \quad \Omega^p(\lambda_\phi, c) \times \Omega_q(\lambda_\psi, c) \longrightarrow \Omega_{q-p}(\lambda_\phi + \lambda_\psi, c) \quad (p \leq q)
\]
\[
(\phi_\mu \psi_{-\eta})^{A_1\ldots A_{q-p}}(x) = \phi^{B_1\ldots B_p}(x) \psi^{A_1\ldots A_{q-p}B_1\ldots B_p}(x)
\]
The wedge product $\wedge$ is clearly associative, and the contraction product $:.$ associates with $\wedge$ in the following sense.

$$(\phi_p \wedge \psi_q):\tilde{\theta}_{-r} = \phi_p:(\psi_q:\tilde{\theta}_{-r}) = (-)^{pq}\psi_q:(\phi_p:\tilde{\theta}_{-r})$$  \hspace{1cm} (4.14)

where $p + q \leq r$. Conformal forms commute as usual.

$$\psi_q \wedge \phi_p = (-)^{pq}\phi_p \wedge \psi_q$$ \hspace{1cm} (4.15)

Note that our notation differs from Ref. 1, where the double dots were used to indicate contraction of arbitrary conformal indices. Here its use is restricted to forms.

The contraction product could naturally be extended to the case that combined form degree is positive. A simple example would be

$$(\phi_2:\tilde{\psi}_{-1})^A = \phi^{AB}\tilde{\psi}_B$$ \hspace{1cm} (4.16)

However, it is easy to see that this map would violate associativity.

$$(\phi_p \wedge \psi_q):\tilde{\theta}_{-r} \neq \phi_p:(\psi_q:\tilde{\theta}_{-r})$$ \hspace{1cm} (4.17)

if $p + q > r$. Moreover, it turns out that this extension would not comply with Leibniz’ rule, e.g.

$$d_0(\phi_1:\tilde{\psi}_{-1}) \neq (d_1\phi_1):\tilde{\psi}_{-1} - \phi_1:(d_1\tilde{\psi}_{-1})$$ \hspace{1cm} (4.18)

To obtain a higher degree of symmetry between positive and negative forms, it would also be tempting to introduce a multiplication rule for negative forms. However, we have failed to do so. The obvious candidate,

$$\vee : \Omega_p(\lambda_\phi, c) \times \Omega_q(\lambda_\psi, c) \longrightarrow \Omega_{p+q}(\lambda_\phi + \lambda_\psi, c)$$

$$(\tilde{\phi}_{-p} \vee \tilde{\psi}_{-q})_{A_1...A_pB_1...B_q}(x) = \tilde{\phi}_{[A_1...A_p}(x)\tilde{\psi}_{B_1...B_q]}(x)$$ \hspace{1cm} (4.19)

is certainly a module map, but Leibniz’ rule does not hold for the exterior derivative and this product.

**Theorem II**

*(Leibniz’ rule)* The exterior derivative is compatible with the wedge and contraction products in the following sense.

$$d_{p+q}(\lambda_\phi + \lambda_\psi, c)(\phi_p \wedge \psi_q) = d_p(\lambda_\phi, c)\phi_p \wedge \psi_q + (-)^p\phi_p \wedge d_q(\lambda_\psi, c)\psi_q$$

$$\tilde{d}_{q-p}(\lambda_\phi + \lambda_\psi, c)(\phi_p:\tilde{\psi}_{-q}) = d_p(\lambda_\phi, c)\phi_p:\tilde{\psi}_{-q} + (-)^p\phi_p:\tilde{d}_{q}(\lambda_\psi, c)\tilde{\psi}_{-q}$$ \hspace{1cm} (4.20)

**Proof:** We want to prove that

$$\frac{1}{(p+q+1)!}(\partial^{[C} + \gamma_{p+q}(\lambda_\phi + \lambda_\psi, c)k^{[C}})(\phi^{A_1...A_p\psi B_1...B_q]})$$

$$= \frac{1}{(p+q+1)!}(\partial^{[C} + \gamma_{p+q}(\lambda_\phi + \lambda_\psi, c)k^{[C}})(\phi^{A_1...A_p\psi B_1...B_q])$$ \hspace{1cm} (4.21)
follows that (4.24) and (4.25) agree provided that

\[ \gamma_{p+q}(\lambda_\phi + \lambda_\psi, c) = \gamma_p(\lambda_\phi, c) + \gamma_q(\lambda_\psi, c). \]  \hspace{1cm} (4.23)

To prove the second part, we note that

\[ (d(\phi_p; \tilde{\psi}_{-q}))_{A_1 \ldots A_{q-p-1}} = (\partial^C + \tilde{\gamma}_{q-p}(\lambda_\phi + \lambda_\psi, c)k^C)(\phi^{B_1 \ldots B_p} \tilde{\psi}_{A_1 \ldots A_{q-p-1} CB_1 \ldots B_p}) \]  \hspace{1cm} (4.24)

whereas

\[ (d\phi_p; \tilde{\psi}_{-q})_{A_1 \ldots A_{q-p-1}} + (-)^p(\phi_p; d\tilde{\psi}_{-q})_{A_1 \ldots A_{q-p-1}} \]
\[ = \frac{1}{(p+1)!}(\partial^C + \gamma_p(\lambda_\phi, c)k^C)(\phi^{B_1 \ldots B_p} \tilde{\psi}_{A_1 \ldots A_{q-p-1} CB_1 \ldots B_p}) \]
\[ + (-)^p \phi^{B_1 \ldots B_p} (\partial^C + \tilde{\gamma}_q(\lambda_\psi, c)k^C)\tilde{\psi}_{A_1 \ldots A_{q-p-1} CB_1 \ldots B_p} \]  \hspace{1cm} (4.25)

We now use that \( \tilde{\psi}_{A_1 \ldots A_{q-p-1} B_1 \ldots B_p} = (-)^p \tilde{\psi}^{A_1 \ldots A_{q-p-1} CB_1 \ldots B_p} \) and that the first term consists of \((p+1)!\) identical terms, which cancels the factor in the denominator. It then follows that (4.24) and (4.25) agree provided that

\[ \tilde{\gamma}_{q-p}(\lambda_\phi + \lambda_\psi, c) = \gamma_p(\lambda_\phi, c) + \tilde{\gamma}_q(\lambda_\psi, c). \]  \hspace{1cm} (4.26)

From the definition of \( \gamma \) and \( \tilde{\gamma} \) in Theorem I it is clear that this equation holds identically. \[\square\]

As promised in the proof of Theorem I, Leibniz’ rule can be utilized to establish the existence of the exterior derivative for higher forms. Let \( \phi_p \in \Omega^p(\lambda, c) \) and \( \psi_q \in \Omega^q(0, c) \), which means that \( \phi_p \wedge \psi_q \in \Omega^{p+q}(\lambda, c) \). Assume that we have established that \( d_p \phi_p \) and \( d_q \psi_q \) transform as conformal fields. Then

\[ d_{p+q}(\phi_p \wedge \psi_q) = (d_p \phi_p) \wedge \psi_q + (-)^p \phi_p \wedge (d_q \psi_q) \]  \hspace{1cm} (4.27)

must transform as a conformal field of the appropriate type because both terms to the right do so. This argument extends to linear combinations of the same type. Since any form is a linear combination of exterior products and the base case is clear from the explicit calculation (4.5), we conclude by induction that higher positive forms do exist.

The existence of negative forms can be motivated, although not strictly proved, by using Leibniz’ rule backwards. The relation

\[ d_{q-p}(\phi_p; \tilde{\psi}_{-q}) = (d_p \phi_p; \tilde{\psi}_{-q}) + (-)^p \phi_p; (d_q \tilde{\psi}_{-q}) \]  \hspace{1cm} (4.28)

can be viewed as an equation for \( d_q \tilde{\psi}_{-q} \). Because all other entities in (4.28) are conformal fields, this is a strong hint that the solution is conformal; that it is skew is manifest. We have explicitly verified that \( d_2 \) is a module map.
5. Covariant derivatives and conformal vielbeins

Having established a form language and the existence of exterior derivatives, we can now apply the standard machinery of local differential geometry\cite{8,9} to conformal forms. Since the algebraic manipulations leading to the formulas in this section are standard, we do not spell out each of them in detail. However, it should be stressed that the results themselves are new, because the formalism is applied to conformal forms rather than to tensor forms.

Consider the algebra of maps from $N$-dimensional space to a finite-dimensional Lie algebra $\mathfrak{g}$, $\text{Map}(N, \mathfrak{g})$ (algebra of gauge transformations).

\[
[J^a(m), J^b(n)] = f^{abc} J^c(m+n), \tag{5.1}
\]

where $f^{abc}$ are the totally skew-symmetric structure constants of $\mathfrak{g}$ and $m, n \in \Lambda$. Moreover, we only consider Lie algebras with a non-degenerate Killing metric $\delta^{ab}$, so there is no need to distinguish between upper and lower $\mathfrak{g}$ indices. The gauge algebra must be compatible with $\text{Vect}(N)$ in the sense that

\[
[L^\mu(m), J^b(n)] = n^\mu J^c(m+n) \tag{5.2}
\]

An explicit representation of (5.2) is given by

\[
J^a(m) = e^{mx} M^a, \tag{5.3}
\]

where $M^a$ are matrices in a finite-dimensional $\mathfrak{g}$ irrep. In other words, we have a $\text{Map}(N, \mathfrak{g})$ module consisting of a $\mathfrak{g}$-representation-valued conformal field, and action

\[
J^a(m) \phi(x) = e^{mx} M^a \phi(x) \tag{5.4}
\]

The representation (5.4) is compatible with $\text{Vect}(N)$.

As usual, the exterior derivative does not commute with the action of the gauge algebra, so it has to be covariantized. A gauge connection is a $C^1_0(0,c)$ conformal field transforming under $\mathfrak{g}$ as

\[
[J^a(m), \omega^Bb(x)] = e^{mx} \left(-f^{abc} \omega^Bc(x) - m^B \delta^{ab}\right) \tag{5.5}
\]

Thus it transforms according to the adjoint representation apart from an inhomogeneous term. The gauge connection can be viewed as a conformal 1-form $\omega^b_1 \in \Omega^1(0,c)$. In form language, (5.5) takes the form

\[
[J^a(m), \omega^b_1] = e^{mx} \left(-f^{abc} \omega^b_1 - \delta^{ab} m_1\right), \tag{5.6}
\]

where the constant $m^A$ is regarded as a 1-form $m_1$.

The covariant derivative of a form transforming as (5.3) is

\[
D_p(\omega, \lambda, c) \phi_p(x) = d_p(\lambda, c) \phi_p(x) + M^b \omega^b_1 \wedge \phi_p(x).
\]

\[
\tilde{D}_p(\omega, \lambda, c) \tilde{\psi}_{-p}(x) = \tilde{d}_p(\lambda, c) \tilde{\psi}_{-p}(x) + M^b \omega^b_1 \tilde{\psi}_{-p}(x). \tag{5.7}
\]
E.g., for 0-forms we have explicitly
\[(D_0 \phi)^A = (\partial^A + \frac{\lambda}{c} k^A) \phi + M^b \omega^{A\bar{b}} \phi \] (5.8)

That this is a \( \text{Vect}(N) \) map is clear because both terms to the right are separately so.
Remains to check the gauge part.
\[ J^a(m) D_p \phi_p = (d_p + \omega_1^b M^b e^{m\bar{x}} M^a \phi_p + e^{m\bar{x}} (- f^{abc} \omega_1^c - \delta^{ab} m_1) \wedge M^b \phi_p \]
\[ = e^{m\bar{x}} (M^a D_p \phi_p - M^a m_1 \wedge \phi_p) + M^a [d_p, e^{m\bar{x}}] \wedge \phi_p \] (5.9)

Noting that \([d_p, e^{m\bar{x}}] = e^{m\bar{x}} m_1\) for every exterior derivative constructed in the previous section, we obtain the claimed result.

The combination of two covariant derivatives gives
\[ D_{p+1}(\omega, \lambda, c) D_p(\omega, \lambda, c) \phi_p = M^a R^a_2(\omega, c) \wedge \phi_p, \] (5.10)

where the field strength is the \( \Omega_2(0, c) \) form
\[ R^a_2(\omega, c) = D_1(\omega, 0, c) \omega_1^a = d_1(0, c) \omega_1^a - \frac{1}{2} f^{abc} \omega_1^b \wedge \omega_1^c. \] (5.11)

It is subject to the Bianchi identity
\[ D_2(\omega, 0, c) R^a_2(\omega, c) \equiv 0. \] (5.12)

A frame algebra is a special type of gauge algebra that admits a conformal vielbein, which is a \( \mathbf{C}_0^{1}(0, c) \) field \( e^{A_i} \) with a frame vector index. The vielbein is defined by the non-trivial property of having a two-sided inverse \( \tilde{e}_A^i \) everywhere.
\[ e^{A_i}(x) \tilde{e}_B^i(x) = \delta^A_B, \quad e^{A_i}(x) \tilde{e}_A^j(x) = \eta^{ij}, \] (5.13)

where \( \eta^{ij} \) is the constant metric. From the definition follows that these conformal fields can not be tensor derived in the sense of (3.13). Namely, \( x_A e^{A_i} \tilde{e}_B^i = x_B \) so \( x_A e^{A_i} \neq 0 \), and a similar relation holds for the inverse.

For \( N \)-dimensional conformal fields, the appropriate frame algebra is \( so(N+1) \) (or possibly \( so(N, 1) \), since the zeroth direction is special), because only square matrices can have an inverse. It would also be possible to use \( sl(N+1) \) as frame algebra, but then the metric could not be constant. Moreover, the description of frame spinors would be unnatural, which is a problem if applications to physics are a concern. The possible frame algebras for tensor fields are of course \( so(N) \) and \( gl(N) \).

Under the frame algebra \( Map(N, so(N+1)) \),
\[ [J^{ij}(m), J^{kl}(n)] = \eta^{ik} J^{jl}(m+n) - \eta^{il} J^{jk}(m+n) - \eta^{jk} J^{il}(m+n) + \eta^{il} J^{ik}(m+n), \] (5.14)

the vielbein transforms as a vector,
\[ J^{ij}(m) e^{A_k} = \eta^{jk} e^{A_i} - \eta^{ik} e^{A_j}, \] (5.15)
as does $\tilde{e}^k_A$.

The vielbein and its inverse can be regarded as forms $e_1^i \in \Omega^1(0, c)$ and $\tilde{e}^{-i}_1 \in \Omega^1(0, c)$. The corresponding gauge connection $\omega_1^{ij}$ is usually called spin connection in the tensorial case. For completeness we write down the conformal analog of Cartan’s structure equations for the torsion $T^i_2$ and the curvature $R^{ij}_2$.

\[
T^i_2 = D_1(\omega, 0, c)e_1^i = d_1(0, c)e_1^i + \omega_1^{ij} \wedge e_1^j \\
R^{ij}_2 = D_1(\omega, 0, c)\omega_1^{ij} = d_1(0, c)\omega_1^{ij} + \omega_1^{ik} \wedge \omega_1^{kj} \\
D_2(\omega, 0, c)T^i_2 = R^{ij}_2 \wedge e_1^j \\
D_2(\omega, 0, c)R^{ij}_2 = 0 \tag{5.16}
\]

The vielbein enables a translation between conformal and frame indices. If $\phi_A \in C^0_1(\lambda, c)$ is a frame scalar, $\phi^i \equiv e^{Bi}\phi_B \in C^0_0(\lambda, c)$ is a frame vector, etc. Using the exterior derivative on scalar fields, maps from arbitrary conformal fields can now be constructed, e.g.,

\[
\nabla^A\phi_B = \tilde{e}^i_B(\partial^A + \frac{\lambda}{c}k^A)e^C_i\phi_C + \tilde{e}^i_B\omega^{Ai}e^C_j\phi_C \tag{5.17}
\]

From this relation the transformation law for conformal Christoffel symbols can be extracted, but it is not very illuminating.

6. The zeroth dimension

From the previous section it is clear that conformal fields in $N$ dimensions in many ways behave as tensor fields in $N + 1$ dimensions, simply because of the number of components. One striking effect is the $so(N + 1)$ frame algebra. It is tempting to take this feature seriously and interpret the extra zeroth dimension as time. The frame algebra would then be a gauged Lorentz symmetry which could act as the arena for flat space physics. This possibility is investigated in this section. However, although our language is colored by this physical interpretation, which may or may not be correct, the mathematical results are perfectly sensible.

There is an obvious extension of (3.2) to an algebra of vector fields in $N + 1$ dimensions; simply replace the tensor index $\mu$ by a conformal index $A$. However, using the building blocks of (3.3), there could also be a term proportional to $k^A$ which vanishes when $A = \mu$. Hence the most general expression reads

\[
L^A(m) = e^{m-\alpha}(\partial^A + (m^B + k^B)T^A_B + k^A(\alpha m^B + \beta k^B)T^C_B x_C + cm^A m^B T^C_B x_C), \tag{6.1}
\]

where the parameters $\alpha$ and $\beta$ are arbitrary.

The $L^A(m)$ do not generate all of $Vect(N + 1)$, which is clear already from the fact that $m^0$ is not independent of $m^\mu$. This is not really physically surprising, because we do not expect the time component of the momentum to be independent of its space components.
To see what kind of algebra that the $L^A(m)$ generate, we limit ourselves to the scalar representation $T^A_B = 0$; other representations yield more complicated expressions.

\[ [L^A(m), L^B(n)] = (n^A + k^A)L^B(m + n) - (m^B + k^B)L^A(m + n) \]
\[ [L^A(m), n^B] = -k^B n^A e^{nx} \]
\[ [L^A(m), e^{nx}] = n^A e^{(m+n)x} \]  

(6.2)

and all other brackets vanish. This algebra is similar to, but distinct from, $Vect(N+1)$, and our “time” is therefore not a dimension in the same sense as the $N$ space dimensions.

By setting $m = 0$ in (6.1), we obtain the momentum operator

\[ P^A_\beta = \partial^A + k^B T^A_B + \beta k^A k^B T^C_B x_C. \]  

(6.3)

The momenta satisfy the same relations as the derivatives $\partial^A$, namely

\[ [P^A_\beta, P^B_{\beta'}] = k^A P^B_{\beta'} - k^B P^A_{\beta'}, \quad [P^A_\beta, m^B] = -k^B m^A \]
\[ [P^A_\beta, x_B] = \delta^A_B - k^A x_B \quad [P^B_\beta, e^{mx}] = m^A e^{mx} \]  

(6.4)

where $\beta''$ is arbitrary. The Hamiltonian is

\[ P^0_\beta = -x \partial + \beta T^0_0 x_\sigma + (1 + \beta) T^0_0 \]  

(6.5)

It commutes with the $gl(N)$ generators $J^\mu_\nu$ (3.8), but the remaining bracket $[P^0_\beta, K_\nu]$ is complicated.

A scalar conformal field $\phi(x) \in C^0_0(\lambda, c)$ satisfies

\[ P^A_\beta \phi(x) = (\partial^A + \lambda(1 + \beta)k^A)\phi(x), \]  

(6.6)

and in the particular case that it is translationally and scale invariant ($\partial^A \phi = 0$), it is a momentum eigenvector. The only non-zero eigenvalue is $\lambda(1 + \beta)$ in the “time” direction, and thus we think of this quantity as the mass. This is closely related to the dilatation operator

\[ D = \frac{\partial L^\mu(m)}{\partial m^\mu} \bigg|_{m=0} = x \partial + T^\mu_\mu, \]  

(6.7)

with eigenvalue is $\lambda$.

The value $\alpha = \beta = -1$ is special, for two related reasons. The “mass” is zero independent of the value of $\lambda$, and $x_A L^A(m) \equiv 0$. In this case, the following relation holds.

\[ [L^\mu(m), P^A_{-1}] = m^A L^\mu(m) - e^{mx} P^A_{-1} \]  

(6.8)

Otherwise this bracket is complicated.
References
1. T. A. Larsson, to appear in Int. J. Mod. Phys. A (1992)
2. E. Ramos and R. E. Shrock, Int. J. Mod. Phys. A 4 (1989) 4295.
3. E. Ramos, C. H. Sah and R. E. Shrock, J. Math. Phys. 31 (1989) 1805.
4. F. Figueirido and E. Ramos, Int. J. Mod. Phys. A 5 (1991) 771.
5. T. A. Larsson, Phys. Lett. B 231 (1989) 94.
6. T. A. Larsson, J. Phys. A 25 (1992) 1177.
7. P. Goddard and D. Olive, Int. J. Mod. Phys. A 1 (1986) 303.
8. T. Eguchi, P. B. Gilkey and A. J. Hansson, Phys. Rep. 66 (1980) 213.
9. C. Nash and S. Sen, Topology and geometry for physicists, (London: Academic Press, 1983).