A Logical Approach to Decomposable Matroids

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Abstract. A notion of branch-width may be defined for matroids, which generalizes the one known for graphs. We first give a proof of the polynomial time model checking of $MSO_M$ on representable matroids of bounded branch-width, by reduction to $MSO$ on trees, much simpler than the one previously known. We deduce results about spectrum of $MSO_M$ formulas and enumeration on matroids of bounded branch-width. We also provide a link between our logical approach and a grammar that allows to build matroids of bounded branch-width. Finally we introduce a new class of non-necessarily representable matroids described by a grammar, on which $MSO_M$ is decidable in linear time.

1 Introduction

The model-checking of Monadic second order ($MSO$) formulas is a natural and extensively studied problem that is relevant to many fields of computer science such as verification or database theory. It is well-known that this problem is hard in general (since $MSO$ can express NP-complete properties like 3-colourability) but has been proved tractable on various structures. For example, this problem is decidable in linear time on trees [22] thanks to automata technique. It is also known that it remains linear time decidable [6] on the widely studied class of graphs of bounded tree-width. Since then, a lot of similar results have been found, either with similar notions of width or for extension of $MSO$ (see [11],[15]).

The central object of this paper are representable matroids. They may be seen as a direct generalization of both graphs and matrices. Monadic second order logic can be defined on matroids and is at least as powerful as $MSO$ on graphs. Natural notion of decomposition such as tree-width or branch-width can also be adapted in this context. It has been proved in [10] that a matroid is of bounded branch-width if and only if it is of bounded tree-width. It is also interesting to note that tree-width and branch-width on matroids are generalizations of the same notions on graphs. That is to say, the width of the cycle matroid is the same as the width of the graph, if it is simple and connected [13], therefore all theorems on matroids can be specialized to graphs.

Recently, the expressive power of $MSO$ on representable matroids of bounded branch-width has been studied and it has been shown that the model-checking problem for such structures is still decidable in linear time [13]. It has been subsequently extended to a broader but more abstract class of matroids in [19]. The proof of both results relies on an implicit construction of a decision automaton. It has some similarity in spirit with the – now classical – proofs on graph, but is long and technical. The first contribution of this paper is to introduce an alternative method to study these
matroids by a simple interpretation into MSO on labeled trees. For this purpose, we introduce the notion of signature over decomposable matroids which appear to be a useful general tool to study several classes of matroids. As a corollary of this method, we obtain the linear model checking of [13], but also the enumeration of all tuples of a MSO_M query with a linear delay or, in the vein of previous results on tree-like structures, that the spectrum (i.e. set of cardinality of models) of a MSO_M formula is ultimately periodic.

From this starting result, we derive a general way to build matroid grammars, inspired by the parse tree of [13]. We first provide a characterization of representable matroids of bounded branch-width. We then build a class of matroid thanks to series-parallel and abstract it further to amalgams on matroids which correspond to the categorical notion of pushout. As a decomposition measure, it appears to be distinct from the notion of branch-width. We give some useful insights about the structures of matroids in these classes and their relations. Our approach using the notion of signature, gives in a very uniform way the interpretation of these three classes in MSO on labeled trees, therefore MSO_M is still decidable in linear time on them. To our knowledge, it is the first such result that applies to non necessarily representable matroids.

2 Matroids and Branch-width

2.1 Matroids

Matroids have been designed to abstract the notion of dependence that appears, for example, in graph theory or in linear algebra. All needed informations about matroids can be found in the book Matroid Theory by J. Oxley [21].

Definition 1. A matroid is a pair \((S, \mathcal{I})\) where \(S\) is a finite set and \(\mathcal{I}\) is included in the power set of \(S\). Elements of \(\mathcal{I}\) are said to be independent sets, the others are dependent sets. A matroid must satisfy the following axioms:

1. \(\emptyset \in \mathcal{I}\)
2. If \(I \in \mathcal{I}\) and \(I' \subseteq I\), then \(I' \in \mathcal{I}\)
3. If \(I_1\) and \(I_2\) are in \(\mathcal{I}\) and \(|I_1| < |I_2|\), then there is an element \(e\) of \(I_2 - I_1\) such that \(I_1 \cup e \in \mathcal{I}\).

The most basic objects in a matroid are the bases which are maximal independent sets and the circuits which are minimal dependent sets. Let \(M\) be a matroid, \(S\) a subset of its elements, the restriction of \(M\) to \(S\), written \(M|S\) is the matroid on underlying set \(S\) whose independent sets are the independent sets of \(M\) that are contained in \(S\).

We can represent any finite matroid by giving the collection of its independent sets which can be exponential in the size of the ground set. One usual way to address this problem is to assume we have a black box deciding in constant time if a set is independent or not, see [18]. We also consider subclasses of matroids, for which we do not need the explicit set of independent sets, because we can decide if a set is independent or not in polynomial time. The two following examples are of this nature.
**Vector Matroid** The first concrete example of matroid is the vector matroid. Let $A$ be a matrix, the ground set $S$ is the set of the columns and a set of columns is independent if the vectors are linearly independent.

**Definition 2.** A matroid $M$ is representable over the field $\mathbb{F}$ if it is isomorphic to a vector matroid of a matrix $A$ with coefficients in $\mathbb{F}$. We also say that $M$ is represented by $A$.

The notion of representable matroid is central to this paper. Note that there are matrices, which are not similar but represent the same matroid.

**Example 1.**

$$
A = \begin{pmatrix}
1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 1
\end{pmatrix}
$$

The convention is to name a column vector by its position in the matrix. Here the set $\{1, 2, 4\}$ is independent while $\{1, 2, 3\}$ is dependent.

**Cycle Matroid** The second example is the cycle matroid or graphic matroid of a graph. Let $G$ be a graph, the ground set of its cycle matroid is the set of its edges. A set is said to be dependent if it contains a cycle. Here a base is a spanning tree of the graph and a circuit is a cycle.

**Example 2.** In Figure 1, the set $\{1, 2, 4\}$ is independent whereas $\{1, 2, 3, 4\}$ and $\{1, 2, 5\}$ are dependent.

**Remark 1.** Any cycle matroid is representable over $\mathbb{F}_2$. Chose an order on the edges and on the vertices then build the matrix $A$ such as $A_{i,j}=1$ iff the edge $j$ is incident to the vertex $i$. The dependence relation is the same over the edges and over the vectors representing the edges.

In the sequel we try to extend theorems on graphs to representable matroids, which seems to be a natural generalization.
2.2 Branch Decomposition

In this part we define the branch-width of a representable matroid, thanks to the more general notion of connectivity function, which also allows to define the branch-width of a graph. I follow the presentation of [11].

Let $S$ be a finite set and $\kappa : 2^S \to \mathbb{N}$ be a function called the connectivity function. A branch decomposition of $(S, \kappa)$ is a pair $(T, l)$ where $T$ is a binary tree and $l$ is a one to one labeling of the leaves of $T$ by the elements of $S$. We define the mapping $\tilde{l}$, from the vertices of the graph to the sets of $S$ recursively:

$$\tilde{l}(t) = \begin{cases} \{l(t)\} & \text{if } t \text{ is a leaf} \\ \tilde{l}(t_1) \cup \tilde{l}(t_2) & \text{if } t \text{ is an inner node with children } t_1, t_2 \end{cases}$$

The width of the branch decomposition of $(S, \kappa)$ is defined by

$$\text{width}(T, \tilde{l}) = \max \left\{ \kappa(\tilde{l}(t)) | t \in V(T) \right\}$$

The branch-width of $(S, \kappa)$ is the minimum of the width over all branch decompositions. In general the connectivity functions are symmetric and submodular. A function is symmetric if $\kappa$ is zero vector. A function is submodular if $\kappa(B) + \kappa(C) \geq \kappa(B \cup C) + \kappa(B \cap C)$.

The Matroid Case Let $M$ be a finite matroid with ground set $S$, $r$ its rank function and $B$ a set of elements. We define the connectivity function by $\kappa(B) = r(B) - r(S \setminus B)$, $\kappa$ is symmetric by construction and submodular because the rank function is submodular too.

In this paper, we study the branch-width of representable matroids only. In this case, the following holds : $\kappa(B) = \dim(<B \cap S \setminus B>)$ where $<B \cap S \setminus B>$ is the subspace generated by the elements of $B$.

Let $T$ be a branch decomposition tree of width $t$ of $A$ and let $s$ be a node of this tree. In the sequel we note $T_s$ the subtree of $T$ rooted in $s$ and $E_s$ the vector subspace generated by $\tilde{l}(s)$, that is to say the leaves of $T_s$. Let $E^c_s$ be the subspace generated by $S \setminus \tilde{l}(s)$ i.e. the leaves which do not belong to $T_s$. Let $B_s$ be the subspace $E_s \cap E^c_s$, it is the boundary between what is described inside and outside of $T_s$.

Remark 2. It can easily be checked that $\kappa(\tilde{l}(s)) = \dim(E_s \cap E^c_s) = \dim(B_s)$ and since $T$ is a branch decomposition tree of width $t$, $\dim B_s \leq t$.

As an example, we compute $E_{s_1}$ and $E^c_{s_1}$ to find $B_{s_1}$ which proves that the width of $s_1$ is 1 in Fig. 2. Note that the subspaces $B_s$, where $s$ is a leaf, are either equal to $E_s$, i.e. generated by $l(s)$, or trivial as for the left son of $s_3$ and then generated by the zero vector.

$$E_{s_1} = \langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \rangle; \ E^c_{s_1} = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \rangle; \ B_{s_1} = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rangle$$

Thanks to a result of Iwata, Fleischer and Fujishige [17] about minimalization of a submodular function, we know that we can find a branch decomposition which is almost optimal with a fixed parameter tractable (fpt) algorithm.
Fig. 2. A matrix $X$ and one of its branch decomposition of width 1

Theorem 1 (Oum and Seymour [20]). Let $\kappa$ be a tractable connectivity function and $k$ a parameter. There is an fpt algorithm which computes a branch decomposition of $(S, \kappa)$ of width at most $3k$ if $bw(S, \kappa) \leq k$. If $bw(S, \kappa) > k$, the algorithm halts without output.

In the case of a representable matroid, it can be done in cubic time [14].

3 Enhanced Branch Decomposition Tree

From now on, all matroids will be representable over a fixed finite field $F$. The results of the next part are false if $F$ is not finite, see [13]. As we do not know how to decide if a matroid is representable, when we say it is, we assume that it has been given as a matrix. Furthermore we assume that the matroids have no loops, to simplify the presentation, but this condition could easily be lifted.

Let $t$ be a fixed parameter representing the maximal branch-width of the considered matroids. Let $M$ be a matroid represented by the matrix $A$ and $T$ a branch decomposition of width less or equal to $t$. We will not distinguish between a leaf in a branch decomposition tree and the column vector it represents. Let $E$ be the vector space generated by the column vectors of $A$, we suppose that its dimension is the same as the length of the columns of $A$ denoted by $n$.

We now build inductively from leaves to root a matrix $C_s$ for each node $s$ of $T$. If $s$ is a leaf, $C_s$ is a base vector of the subspace $B_s$. If $s$ has two children $s_1$ and $s_2$ the matrix $C_s$ is divided in three parts $(C_1|C_2|C_3)$ where $C_1$, $C_2$ and $C_3$ are bases of $B_{s_1}$, $B_{s_2}$ and $B_s$ respectively. By induction hypothesis, one already knows the bases of $B_{s_1}$ and $B_{s_2}$ used to build $C_{s_1}$ and $C_{s_2}$ and we choose them for $C_1$ and $C_2$. We then choose any base of $B_s$ for $C_3$.

Matrices $C_s$ are of size $n \times t_1$, with $t_1 \leq 3t$, because of Remark 2 on the dimension of boundary subspaces. The characteristic matrix at $s$ is obtained by removing all linearly dependent rows of the matrix with Gaussian elimination. The result is the matrix $N_s = (N_1|N_2|N_3)$ of dimension $t_2 \times t_1$ with $t_2 \leq 3t$. The vectors in $N_s$ still
represent the bases of \( B_{s_1}, B_{s_2} \) and \( B_s \) in the same order but only carry the dependence informations. In fact, any linear dependence relation between the columns of \( C_s \) is a linear dependence relation on \( N_s \) with the same coefficients, and conversely.

**Definition 3 (Enhanced branch decomposition tree).** Let \( T \) be a branch decomposition tree of the matroid represented by \( A \), an enhanced branch decomposition tree is the same tree with, on each node, a label representing the characteristic matrix at this node.

Each label can be seen as a word of size polynomial in \( t \), this is the reason why we have chosen the matrix \( N \) instead of \( C \) which is of size linear in the matroid. Remark that given a matrix of size \( n \times m \) and a branch decomposition tree of width \( t \), one can transform this tree into an enhanced tree in polynomial time.

**Example 3.** Here we represent an enhanced tree constructed from the branch decomposition tree of Figure 2. Some intermediate computations needed to find the enhanced branch decomposition tree are given below. One easily check that the labels \( N_s \) of the tree are obtained by Gaussian elimination from the \( C_s \). Remark that, as it is a decomposition of branch-width 1, the subspaces \( B_s \) are of dimension 1 and are thus represented here by one vector.

![Enhanced tree diagram]

Fig. 3. An enhanced tree built from the branch decomposition tree of Figure 2

4 Decision on an enhanced tree

4.1 Signature

It is now shown how dependent sets of a matroid of bounded branch-width can be characterized using its enhanced tree. It will permit to define later the dependency
an ordered set. We say that $X$ is of signature $\lambda$ at $s$ if there exists a non trivial linear combination of the elements of $X$ equal to some $v$ in $B_s$ such that, $e_1, \ldots, e_t$ being the base of $B_s$ chosen to build $C_s$, $v = \sum_i \lambda_i e_i$. If there is no non trivial combination, the signature is $\emptyset$.

Note that a set $X$ may have more than one signature as a dependent set may satisfies several different dependency relations.

**Lemma 1.** Let $\bar{T}$ be an enhanced tree, $s$ one of its nodes with children $s_1$, $s_2$ and $N_s = (N_1|N_2|N_3)$ the label of $s$. $X_1$ and $X_2$ are respectively subsets of $E_{s_1}$ and $E_{s_2}$. $X = X_1 \cup X_2$ is of signature $\lambda$ at $s$ if and only if $X_1$ is of signature $\mu$ at $s_1$ and $X_2$ of signature $\gamma$ at $s_2$ and we have the relation

$$\sum_i \mu_i N^i_1 + \sum_j \gamma_j N^j_2 = \sum_k \lambda_k N^k_3$$  \hspace{1cm} (1)

*Proof.* A signature $\emptyset$ correspond to an empty sum of value 0 in Equality (1).

$(\Leftarrow)$ By construction of $N$, we know that $N_1$, $N_2$ and $N_3$ represent the bases $C_1$, $C_2$ and $C_3$ of $B_{s_1}$, $B_{s_2}$ and $B_s$ respectively, meaning that they satisfy the same linear dependence relations. Then Equation (1) implies

$$\sum_i \mu_i C^i_1 + \sum_j \gamma_j C^j_2 = \sum_k \lambda_k C^k_3$$

By definition of a signature, there exists a linear combination of elements of $X_1$ equal to $\sum_i \mu_i C^i_1$ and a linear combination of elements of $X_2$ equal to $\sum_j \gamma_j C^j_2$. Hence we have a linear combination of elements of $X_1 \cup X_2$ equal to $\sum_i \mu_i C^i_1 + \sum_j \gamma_j C^j_2$ which is equal to $\sum_k \lambda_k C^k_3$ by the previous equality.

$(\Rightarrow)$ By definition of a signature we have a linear combination of elements in $X$ equal to $v = \sum_k \lambda_k C^k_3$. As $v$ is in $E_s = < E_{s_1} \cup E_{s_2} >$, we can write $v = v_1 + v_2$ where $v_1 = \sum_{i \in X_1} \alpha_i x_i$ and $v_2 = \sum_{j \in X_2} \alpha_j x_j$ in $E_{s_2}$. Since $v_1 = v - v_2$ and $v$ in $B_s$, we have $v_1 \in < E_{s_2} \cup B_s >$. Moreover $B_s \subseteq E^c_2 \subseteq E^c_{s_1}$ and $E_{s_2} \subseteq E^c_{s_1}$ then $v_1 \in E^c_{s_1}$. Hence we have proven that $v_1$ is in $E_{s_1} \cap E^c_{s_1} = B_{s_1}$. As $v_1$ is equal to a linear combination of elements of $X_1$ by construction, we have proven that $X_1$ is of signature $(\mu_1, \ldots, \mu_k)$ at $s_1$ such that $v_1 = \sum_i \mu_i C^i_1$. $X_2$ plays a symmetric role,
then it is of signature \((\gamma_1, \ldots, \gamma_k)\) at \(s_2\) such that \(v_2 = \sum_j \gamma_j C^j_2\). Finally we have
\[
\sum_i \mu_i C^i_1 + \sum_j \gamma_j C^j_2 = \sum_{k} \lambda_k C^k_3
\]
and we can replace the \(C_i\) by \(N_i\).

We then derive a global result on the enhanced tree and signature.

Lemma 2. Let \(A\) be a matrix representing a matroid, \(\bar{T}\) one of its enhanced tree and \(X\) a set of column of \(A\). \(X\) is of signature \(\lambda\) at \(n\) if and only if there exist a signature \(\lambda_s\) for each node \(s\) of the tree \(\bar{T}\) such as :

1. for each node \(s\) with children \(s_1\) and \(s_2\) then \(\lambda_s, \lambda_{s_1}\) and \(\lambda_{s_2}\) satisfy Equation 1
2. the leaves of label \((0)\) are of signature \(\emptyset\) and those of label \((1)\) are of signature
   either \(\emptyset\) or \((1)\). The set of leaves of signature \(\emptyset\) is a non empty subset of \(X\)
3. \(\lambda_n = \lambda\)

Proof. The proof is by induction on the height of \(n\) in \(T\). For \(n\) a leaf the condition 2 gives the result and for the induction step we use conditions 1 and 3 with Lemma 1.

The following theorem is the key to the next part, it allows to test dependency of a set by checking local constraint on signatures.

Theorem 2 (Characterization of dependency). Let \(A\) be a matrix representing a matroid, \(\bar{T}\) one of its enhanced tree and \(X\) a set of column of \(A\). \(X\) is dependent if and only if there exist a signature \(\lambda_s\) for each node \(s\) of the tree \(\bar{T}\) such as :

1. if \(s_1\) and \(s_2\) are the children of \(s\) then \(\lambda_s, \lambda_{s_1}\) and \(\lambda_{s_2}\) satisfy Equation 1
2. the set of leaves of signature \(\emptyset\) is a non empty subset of \(X\)
3. the signature at the root is \((0, \ldots, 0)\)

Proof. Remark that \(X\) is of signature \((0, \ldots, 0)\) at the root means that a certain linear combination of the elements of \(X\) is equal to 0 which is exactly the definition of being a dependent set in a vector matroid. Then it is equivalent to be a dependent set or of signature \((0, \ldots, 0)\) at the root. The proof of the theorem follow from this remark and the previous lemma used at the root.

4.2 Monadic second order logic

We recall the definition of \(MSO_M\), a logic adapted to matroids, which is inspired by the \(MSO_2\) logic over the graphs.

Definition 5. The first order variables (in lower case) range over the ground set, and the second order variables (in upper case) range over the subsets of this ground set. \(MSO_M\) is the set of Monadic Second Order formulas constructed from the usual quantifiers \(\exists, \forall\), the logical connectives \(\land, \lor\) and \(\neg\) and the relations :

1. \(=\), the equality between elements or sets of the matroid
2. \(\in\), the inclusion of a first order variable in a second order variable
3. \(\text{indep}\), such as \(\text{indep}(F)\) is true iff \(F\) is an independent set of the matroid
The structures are matroids represented by a finite ground set and the independence relation, which quite unusually, is defined over the subsets of the ground set (and not over tuples). In the sequel we will describe a way to deal with this predicate with a formula of the more traditional $MSO$ logic. For instance circuits are definable in this logic, $X$ is a circuit if and only if it satisfies

$$\neg \text{indep}(X) \land \forall Y \ (Y \not\subseteq X \lor X = Y \lor \text{indep}(Y))$$

We can also express that a matroid is connected, meaning that every pair of elements is in a circuit, a notion similar to 2-connectivity in graphs, by the formula

$$\forall x, y \exists X \ x \in X \land y \in X \land \text{Circuit}(X)$$

Matroid axioms given for the dependent sets or the circuits are also expressible in $MSO$. Finally, it has been proven in [13] that every $MSO_M$ formula over a graph can be translated into a $MSO_M$ formula over the cycle matroid of a graph defined from the first one. Thus, every property definable over a graph in $MSO$ can be defined in $MSO_M$.

4.3 From Matroids to Trees

The aim of this section is to translate $MSO_M$ formulas on a matroid into $MSO$ formulas on its enhanced tree. The main difficulty is to express the predicate $\text{indep}$ in $MSO$. To achieve that, we use the decomposition of a dependent set into a signature at each node of the enhanced tree as seen in Theorem 2.

We have to encode in $MSO$ a signature $\lambda$ of size less than $t$ at each node of an enhanced tree. It is represented by the set of second order variables $X_\lambda$ indexed by all the signatures of size less than $t$. The number of such variables is bounded by a function in $t$. $X_\lambda(s)$ holds if and only if $\lambda$ is the signature at $s$. The following formula states that there is one and only one value for the signature at each $s$.

$$\Omega(X_\lambda) = \forall s \bigg( \bigwedge_{\lambda} X_\lambda(s) \land \bigwedge_{\lambda' \neq \lambda} \neg X_{\lambda'}(s) \bigg)$$

The formula $\text{dep}(X)$ that represents the negation of the relation $\text{indep}$ is now built in three steps corresponding to the three conditions of Theorem 2.

1. Let $N$ be a matrix divided in three parts ($N_1|N_2|N_3$) and $\lambda_1$, $\lambda_2$ and $\lambda_3$ three signatures, that is to say ordered sets of elements in $F$. $\theta(N, \lambda_1, \lambda_2, \lambda_3)$ is the boolean that is true if and only if

$$\sum \lambda_1^i N_1^i + \sum \lambda_2^j N_2^j = \sum \lambda_3^k N_3^k$$

where $\lambda_j^i$ is the $i^{th}$ element of $\lambda_j$. If a signature $\lambda_j$ is bigger than the number of columns in $N_j$, then $\theta$ is false. The formula $\varphi_1$ ensures that the condition represented by the boolean $\theta$ is respected. It is a conjunction on all possible characteristic matrices $N$ and all signatures $\lambda$ which are in number bounded by a function in $t$. 

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\[ \Psi_1(s, \vec{X}_\lambda) = \exists s_1, s_2 lchild(s, s_1) \land rchild(s, s_2) \]

\[ \bigwedge_{\lambda_1, \lambda_2, \lambda, N} \text{label}(s) = N \land X_{\lambda_1}(s_1) \land X_{\lambda_2}(s_2) \land X_\lambda(s) \Rightarrow \theta(N, \lambda_1, \lambda_2, \lambda) \]

2. We define the formula \( \Psi_2(X, \vec{X}_\lambda) \) which means that the set of leaves with a non-\( \emptyset \) signature is non empty, included in \( X \) and all its signature are (1).

\[ \Psi_2(X, \vec{X}_\lambda) = \forall s \left[ (leaf(s) \land \neg X_{\emptyset}(s)) \Rightarrow (X(s) \land X(s)) \right] \land \exists u (\text{leaf}(u) \land \neg X_{\emptyset}(u)) \]

3. \( \Psi_3(\vec{X}_\lambda) \) states that the signature at the root is \((0, \ldots, 0)\).

\[ \Psi_3(\vec{X}_\lambda) = \exists s \text{ root}(s) \land X_{(0, \ldots, 0)}(s) \]

Thanks to Theorem 2 we know that the following formula is true if and only if \( X \) is a dependent set.

\[ \text{dep}(X) = \exists \vec{X}_\lambda \Omega(\vec{X}_\lambda) \land \Psi_2(X, \vec{X}_\lambda) \land \Psi_3(\vec{X}_\lambda) \land \forall s \neg \text{leaf}(s) \Rightarrow \Psi_1(s, \vec{X}_\lambda) \]

We call \( f \) the bijection between the leaves or set of leaves of an enhanced tree and the elements or set of elements of the matroid they represent. \( F(\phi(\vec{x})) \) is a formula of \( MSO \) defined inductively from \( \phi(\vec{x}) \) of \( MSO_\mathcal{M} \) by relativization to the leaves and use of the previously defined formula \( \text{dep} \) to replace the independence predicate. The relativization is needed because the elements of the matroid are in bijection with the leaves of its enhanced tree. We can now state the main theorem with the notations given above.

**Theorem 3.** Let \( M \) be a matroid of branch-width less than \( t \), \( \bar{T} \) one of its enhanced tree and \( \phi(\vec{x}) \) a \( MSO_\mathcal{M} \) formula with free variables \( \vec{x} \), we have

\[ (M, \vec{a}) \models \phi(\vec{x}) \iff (\bar{T}, f(\vec{a})) \models F(\phi(\vec{x})) \]

**Proof.** The demonstration is done by induction, every case is trivial except the translation of the predicate \( \text{indep} \) whose correction is given by Theorem 2.

The interest of the reduction to \( MSO \) over labeled tree is justified by its classical model checking theorem in linear time.

**Theorem 4 (Thatcher and Wright[22]).** Decision of \( MSO \) over labeled trees is solvable by a linear \( fpt \) algorithm, the parameter being the size of the formula.

Suppose we have a formula \( \phi \) of \( MSO_M \) and a representable matroid \( M \) of branch-width \( t \). We know that we can find a tree decomposition of width at most \( 3t \) in cubic time [14]. Then we build an enhanced tree also in cubic time and by Theorem 3 we know that we have only to decide the formula \( F(\phi) \) on the enhanced tree, which is done in linear time by Theorem 4 to decide \( \phi \) on \( M \). We have as a corollary the main result of [13].
Corollary 1 (Hliněný). The model checking problem for $MSO_M$ is decidable in time $f(t, k, l) \times P(n)$ over the set of representable matroids, where $f$ is a computable function, $P$ a polynomial, $k$ the size of the field, $t$ the branch-width and $l$ the size of the formula.

As we can decide dependency in a represented matroid of bounded branch width in linear time by only using one of its enhanced tree, they are a way to define completely a matroid and then to represent it. Moreover it is efficient as their size is $O(t^2 \times n)$, where $n$ is the size of the matroid, while the matrix, which usually defines it, will be of size $O(n^2)$.

5 Extensions and applications

In this section we show the scope of the result through extension of the logic or applications made easier by Theorem 3.

5.1 Logic extension

Colored matroids We can work with colored matroids, meaning that we add a finite number of unary predicates to the language which are interpreted by subsets of the ground set. Theorem 3 still holds for colored matroids except that we now have colored trees, on which decision of $MSO$ is still linear.

Example 4. Let $A$-Circuit be the problem to decide, given a matroid $M$ and a subset $A$ of its elements, if there is a circuit in which $A$ is included.

1. If $|A| = 1$, the problem is decidable in polynomial time.
2. If $|A| = 2$ and the matroid is a vector matroid then it is decidable in polynomial time [8], but for abstract matroids the question is open [18].
3. If $|A|$ is unbounded, even if the matroid is only a cycle matroid, the question is NP complete by reduction to Hamiltonian Path.

The problem $A$-Circuit is easily expressible in $MSO_M$ plus an unary predicate $A$ representing the subset by the formula $A - Circuit(X) = A \subseteq X \land Circuit(X)$. Then $A$-Circuit is decidable in polynomial time for matroids of branch-width $t$, it is an example of a NP complete problem over matroid which is made tractable with bounded branch-width.

Counting $MSO$ The second generalization is to add to the language a finite number of second order predicates $Mod_{p,q}(X)$ which means that $X$ is of size $p$ modulo $q$. We obtain the logic called $CMSO_M$ for counting monadic second order. Theorem 3 also holds for counting $MSO$ except that the translated formula is now in $CMSO$, which is satisfying as the model checking of $CMSO$ is in linear time over trees [7]. We could also adapt Theorem 3 to $MSO_M$ problems with some optimization constraints, in the spirit of the $EMSO$ problems introduced in [1], which have been proven to be decidable in linear time over graph of bounded tree-width.
5.2 Spectra of $\text{MSO}_M$ formulas

In this section Theorem 3 is used to prove that the spectra of $\text{MSO}_M$ formulas are ultimately periodic.

Definition 6 (Spectrum). The spectrum of a formula $\phi$ is the set $\text{spec}(\phi) = \{ n | M \models \phi \text{ and } |M| = n \}$.

Definition 7 (Ultimately periodic). A set $X$ of integers is said to be ultimately periodic if there are two integers $a$ and $b$ such that, for $n > a$ in $X$ we have $n = a + k \times b$.

This kind of result has been proved for various restrictions of the second order logic, of the vocabulary or of the set of allowed models. The result holds for first order logic, finitely many unary relations and one unary function, cf. [9]. Then it has been generalized to second order logic with the same vocabulary in [12]. Fischer and Makowsky have proven theorems for different notions of width on graphs in [10] by reduction to labeled trees. We will use the same kind of method as matroid of bounded branch-width are reducible to enhanced trees by Theorem 3.

The following theorem is one of the simplest variant of the spectrum theorems. It comes from the pumping lemma on trees and the fact that recognizability and definability in $\text{MSO}$ are equivalent for a set of trees (see [4]).

Theorem 5. Let $\phi$ be a $\text{MSO}$ formula on trees, then $\text{spec}(\phi)$ is ultimately periodic.

We can now state the main theorem of this part, which is a corollary of Theorem 3 and its generalizations to other logics.

Theorem 6. Let $\phi$ a formula of $\text{MSO}_M$, $\text{CMSO}_M$ or $\text{MSO}_M$ plus unary predicate, then the spectrum of $\phi$ restricted to matroids of branch-width $t$ is ultimately periodic.

Proof. By Theorem 3 we have a formula $F(\phi) \in \text{MSO}$ such as for each enhanced tree $\bar{T}$ representing a matroid $M$, $\bar{T} \models F(\phi) \iff M \models \phi$. Remark that an enhanced tree is a full binary tree, that is to say each of its internal nodes have exactly two children. It is well known that such trees with $n$ leaves have exactly $2n - 1$ nodes.

By Theorem 5 the set of enhanced tree defined by $F(\phi)$ is ultimately periodic. Then there are $a$ and $b$ integers such that if $\bar{T}$ is a model of $\phi'$ and $|\bar{T}| > a$ then $|\bar{T}| = a + k \times b$. Note that the formula $F(\phi)$ forces the models satisfying it to be a proper enhanced tree, and then to be a full binary tree. As $|\bar{T}| = 2n - 1$, we have $n = \frac{a + 1}{2} + k \times \frac{b}{2}$.

A enhanced tree of size $2n - 1$ represents a matroid of size $n$ because each leaf represents an element of the matroid. Hence the set of the $n$ such as $|\bar{T}| = 2n - 1$ is the set of the sizes of the matroids represented by the $\bar{T}$ satisfying $F(\phi)$. It is then the set of the sizes of the matroids of branch-width $t$ satisfying $\phi$, therefore we have proved that the spectrum of the formula $\phi$ is ultimately periodic. The proof is for $\text{MSO}_M$ but is trivially generalizable to $\text{CMSO}_M$ and $\text{MSO}_M$ plus unary predicates.
5.3 Enumeration

Let first introduce the complexity notions adapted to enumeration. Let \( A \) be a binary predicate such as \( A(x, y) \) is decidable in polynomial time in \(|x|\) and \( Q \) a polynomial. Our problem is to find the set \( A(x) = \{ y | |y| \leq Q(|x|) \text{ and } A(x, y) \} \), and is noted \( \text{Enum} \cdot A \). The problems of this form are the class \( \text{Enum} \cdot \text{P} \) and correspond to \( \text{NP} \) for enumeration.

We want to control the dynamic of the enumeration, that is to say that we want to bound the delay between the production of two solutions.

**Definition 8.** A problem \( \text{Enum} \cdot A \) is decidable in incremental polynomial time \( \text{IncP} \), if there is an algorithm which on every instance \( x \) and every integer \( k \leq |A(x)| \) return the \( k \)th solution in time polynomial in \(|x|\) and \( k \).

As an example let consider the problem \( \text{Enum} \cdot A \cdot \text{Circuit} \), which given a matroid and a set of its elements require to enumerate every circuit containing this set. Two more well-known problems are special cases of this one when \(|A| = 1\). When the matroid is representable on \( \mathbb{F}_2 \), it is equivalent to find all the minimal (for the pointwise order) solutions of an affine formula, which is an affine variation of the circumscription problem studied in artificial intelligence. When the matroid is representable on a finite field, it is equivalent to enumerate all minimal solutions (for inclusion of the support) of a linear system. In these two cases, the matroid has an independence predicate decidable in polynomial time, then both are in \( \text{IncP} \).

Most problems which are considered to be efficiently solvable are in fact in the following class which allows only a shorter delay.

**Definition 9.** A problem \( \text{Enum} \cdot A \) is decidable with polynomial delay \( \text{DelayP} \), if there is an algorithm which on every instance \( x \) return \( A(x) \) using a time polynomial in \(|x|\) between two generated solutions.

A theorem of enumeration for \( \text{MSO} \) logic and tree is proved in [5] along with generalizations to other structures by \( \text{MSO} \) reduction to trees. For instance we have a similar theorem for tree of bounded tree-width [2].

**Theorem 7 (Courcelle [5]).** Let \( \phi(X_1, \ldots, X_m) \) be an \( \text{MSO} \) formula, there exists an enumeration algorithm which given a tree \( T \) of size \( n \) and of depth \( d \) enumerate the \( m \)-tuples \( B_1, \ldots, B_m \) such that \( T \models \phi(B_1, \ldots, B_m) \) with a linear delay and a preprocessing time \( O(n \times d) \).

The next corollary is a direct consequence of Theorem 7 and of the Theorem 3, which allows to translate \( \text{MSO}_M \) for matroid of branch-width \( t \) into \( \text{MSO} \) for trees.

**Corollary 2.** Let \( \phi(X_1, \ldots, X_m) \) be an \( \text{MSO}_M \) formula, for every matroid of branch-width \( t \), the enumeration of the sets satisfying \( \phi \) can be done with linear delay after a cubic preprocessing time.

*Proof.* Let \( \phi(X_1, \ldots, X_m) \) be an \( \text{MSO}_M \) formula, we compute the formula \( F(\phi(X_1, \ldots, X_m)) \) for matroids of branch-width \( t \) in linear time. Then given a matroid of branch-width \( t \) we compute its enhanced tree in cubic time. We run the
enumeration algorithm given by Theorem 7 on this enhanced tree and the formula $F(\phi(X_1, \ldots, X_m))$. The bijection between the leaves and the elements of the matroid can be computed in linear time, for example by storing its values in an array. Each time we find an $m$-tuple satisfying the formula, we translate it and output the result. This algorithm gives the solutions of $\phi(X_1, \ldots, X_m)$ in linear delay with a cubic preprocessing time.

This corollary also work if we extend $MSO_M$ with unary predicates or to $CMSO_M$. Applied to the formula $A\circ Circuit(X)$, it allows to solve the problem $ENUM\cdot A\circ Circuit$ with linear delay over representable matroids of bounded branch width, whereas the general problem is not known to be enumerable with polynomial delay even for $|A| = 1$.

It is possible to compute all the bases of a vector space with linear delay by a depth first search. Then from this simple algorithm, we derive an algorithm with polynomial delay for $ENUM\cdot A\circ Circuit$ with $|A| = 1$, if the matroid is representable by all the vectors of a vector space. This fact is not implied by the previous theorem because these matroids are of unbounded branch-width. It would be very interesting to find classes of representable matroids which are not either very sparse or very dense as the two previous examples, but on which the enumeration of circuit is still in $\text{DelayP}$.

6 Matroid Grammar

In this part we give several ways to build matroids through certain grammars, and we prove that decision is easy on these classes of matroids. Definitions and notations are taken from [13] but they are slightly more general as we will build several different parse trees.

Definition 10 (Boundaried matroid). A pair $(M, \gamma)$ is called a $t$ boundaried matroid if $M$ is a matroid and $\gamma$ is an injective function from $[1, t]$ to $M$ whose image is an independent set. The elements of the image of $\gamma$ are called boundary elements and the others are called internal elements.

The restriction of $M$ to its ground set minus the elements of the boundary is called the internal matroid of $(M, \gamma)$. We assume in the next definitions we have built an operator $\oplus$ which to two $t$ boundaried matroids $N_1$ and $N_2$ (or representations of these matroids) associates a matroid $N_1 \oplus N_2$ (or a representation). Every operator will satisfy that there is an injection of the internal matroids of $N_1$ and $N_2$ into $N_1 \oplus N_2$ which partitions it in two. The aim of the next sections will be to define several such operators designed to stick two matroids on their boundary.

A matroid $M$ which is partitioned in three independent sets $\gamma_i([1, t])$ with $t_i \leq t$ for $i = 1, 2, 3$ is called a 3-partitioned matroid. From $\oplus$ and $M$ a 3-partitioned matroid we define an operator $\odot_M$ which from two boundaried matroids gives a boundaried matroid. It is defined by two successive uses of $\oplus$ on the boundaries $\gamma_1$ and $\gamma_2$.

Definition 11. Let $\overline{N_1} = (N_1, \gamma_1)$ and $\overline{N_2} = (N_2, \gamma_2)$ be respectively a $t_1$ and a $t_2$ boundaried matroids. $\overline{M} = \overline{N_1} \odot_M \overline{N_2}$ is a $t_3$ boundaried matroid defined by : $(\overline{N_1} \oplus (M, \gamma_1), \gamma_2) \oplus \overline{N_2}$ with boundary $\gamma_3$. 

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The operation $\odot$, boundaries represented in grey

Usually the operators $\oplus$ will be “associative” meaning that $\odot_M$ can also be defined by $N_1 \oplus ((M, \gamma'_2) \oplus N_2, \gamma'_1)$ with boundary $\gamma'_3$. These operators allow to define a grammar to build boundaried matroids.

**Definition 12 (Parse tree).** Let $\mathcal{L}$ a finite set of boundaried matroids and $\mathcal{M}$ a finite set of 3-partitioned matroids. Let $T$ be a tree labeled by elements of $\mathcal{L}$ at the leaves and elements of $\mathcal{M}$ at inner nodes. It is a parse tree if, given a node labeled by $\odot_M$, its left child labeled $\odot_{M_1}$ and its right by $\odot_{M_2}$, the third set of $M_1$ is of the same size as the first set of $M$, the third set of $M_2$ is of the same size as the second set of $M$.

**Definition 13.** Let $T$ be a parse tree, then the boundaried matroid parsed by $T$ is defined inductively:

- if $T$ is empty, it is the empty matroid
- if $T$ is a leaf labeled by $M \in \mathcal{M}$, it is $M$
- if $T$ has a root $s$ labeled by $M$ with children $s_1, s_2$ such that $N_1, N_2$ are parsed by $T_{s_1}, T_{s_2}$, then it parses $N_1 \odot_M N_2$ (which is well defined because of the size condition in the parse tree)

We often use the parse tree to represent the internal matroid of the parsed boundaried matroid rather than the boundaried matroid itself.

They are two simple generalizations of this definition we could allow for parse trees. First we can have a finite number of different operators $\oplus$ giving more possible $\odot$ operators labeling the inner nodes of the parse trees. Second the operator $\oplus$ can be defined on a representation of a boundaried matroid (a graph or a matrix for example), then the parse tree will be associated to a representation of a boundaried matroid.

In the next subsection we study the complexity to decide $\text{MSO}_M$ for the classes of matroids represented by a parse tree for different operators and leaves. Signature of a set still play a big role in these settings but depends on the operator $\oplus$ and will thus be defined in each case we study. It generally gives informations about the set and its relation to the boundary.
6.1 Boundaried parse trees

This section is an attempt to properly define a grammar over matroids similar to \cite{13} and to give the link between this grammar and enhanced trees. All the considered matroids will be vector matroids on the same finite field \( F \), as the operation \( \oplus \) we are going to define only works on these matroids. In fact, it is rather an operation on boundaried matrices than on boundaried matroids they represent. Let \( N_1 = (M_1, \gamma_1) \) and \( N_2 = (M_2, \gamma_2) \) two \( t \) boundaried representable matroids, they are represented by the set of vectors \( A_i \) in the vector space \( E_i \). \( E_1 \times E_2 \) is the direct product of the two vector spaces and \( \langle \{ \gamma_1(j) - \gamma_2(j) \} \rangle \) is the subspace generated by the elements of the form \( \gamma_1(j) - \gamma_2(j) \).

**Definition 14.** Let \( E \) be the quotient space of \( (E_1 \times E_2) \) by \( \langle \{ \gamma_1(j) - \gamma_2(j) \} \rangle \). There are natural injections from \( A_1 \) and \( A_2 \) into \( E_1 \times E_2 \) and then in \( E \). The elements of \( A = (A_1, \gamma_1) \oplus (A_2, \gamma_2) \) are the images of \( A_1 \) and \( A_2 \) by these injections minus the boundary elements. The dependence relation is the linear dependence in \( E \).

A may be seen as the representation of a matroid as it is a set of vectors in \( E \). The underlying operation before the elimination of the boundary is the pushout of two vector spaces, which generalizes the construction of parse trees for graphs of bounded branch-width, also obtained by a pushout in the category of graphs.

To have a more concrete idea of the action of \( \oplus \) and give examples, we extend the definition to matrices. Once a base is fixed, a set of vectors and a matrix are the same definition to matrices. Once a base is fixed, a set of vectors and a matrix are the same objects, therefore we only have to give an algorithm to build a base of \( E \). Let \( M_1 \) and \( M_2 \) be two matrices, we see them as representations of the set \( A_1 \) and \( A_2 \) of vectors of \( E_1 \) and \( E_2 \) in the canonical bases \( C_1 \) and \( C_2 \). We build a base \( B \) of \( E \) from \( C_1 \) and \( C_2 \). Let \( i \) be the canonical injection, \( B_0 = i(C_1) \) and \( B_{j+1} = B_j \cup \{ i(C_{j+1}) \} \) if this set is independent else \( B_{j+1} = B_j \). Let \( n \) be the size of \( C_2 \) and let \( B = B_n \). The matrix \( M = M_1 \oplus M_2 \) is the representation of \( A = (A_1, \gamma_1) \oplus (A_2, \gamma_2) \) in the base \( B \).

**Example 5.**

\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix} \oplus \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 1 & 0 \\
1 & 1 & 1 \\
0 & 0 & -1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix} \oplus \begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 2 & 0 \\
1 & 1 & 1 \\
0 & 0 & -1
\end{pmatrix}
\]

The boundary are the two first columns of the matrices, separated from the others by the symbol \( | \) for clarity. \( \{ e_1, e_2 \} \) is the image of the canonical base of \( E_1 \) in \( E \) and \( \{ e_3, e_4, e_5 \} \) the one of \( E_2 \). By identification of the first columns, we have \( e_1 = e_3 \) and for the second columns \( e_2 = e_4 + e_5 \). The basis built by the algorithm is thus \( \{ e_1, e_2, e_4 \} \).

The column \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) in the second matrix, first line is represented in the result by \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) because once injected in \( E \) it is equal to \( e_3 + e_4 + e_5 \) which is equal to \( e_1 + e_2 \) the sum of the two first vector of the base we have built.
Notice that the columns 1 and 2 of the result form a dependent set in the first example, and not in the second then the two results are distinct matroids. Yet the matrices we combine by $\oplus$ although different, represent the same matroids in both examples.

Example 5 shows that $\oplus$ cannot be seen as an operation on matroids because the result depends on the way the matroids are represented. We could also make this kind of construction by representing matroids by projective spaces as done in [13], but we would define essentially the same operation, which is still an operation over the projective spaces and not the matroids. Matrices of Example 5 can be seen as matrices over $\mathbb{F}_3$ or any bigger field, then the remark holds for all these fields. Nevertheless if the field on which the matroid is represented is $\mathbb{F}_2$, $\oplus$ properly defines an operation on matroids.

**Proposition 1.** Let $M_i$ be matroids of boundary $\gamma_i$ representable over $\mathbb{F}_2$, for every representation $A_i$ of $M_i$, $A_1 \oplus A_2$ represents the same matroid.

**Proof.** A circuit of $A_1 \oplus A_2$ is the union of internal elements of $A_1$ and $A_2$ denoted by $X$ and $Y$ such that $\sum_{x \in X} x + \sum_{y \in Y} y \in \langle \{\gamma_1(j) - \gamma_2(j)\} \rangle$ and $X \cup Y$ is minimal for this property. Then $X$ and $Y$ have to be completed by the same elements of their respective boundaries to obtain a circuit. Therefore a circuit of $A_1 \oplus A_2$ is obtained from two circuits with the same intersection with their respective boundaries. As the intersection with the boundary of a circuit of $A_i$ does not depend on the way $M_i$ is represented, the set of circuit of $M_1 \oplus M_2$ does not depend on the way $A_1$ and $A_2$ are represented. As the set of circuit entirely determines a matroid, the lemma is proved.

Behind this proof is hidden the notion of a signature characterizing a set in a boundaried matroid that we are going to use in the sequel. Let now introduce parse trees whose leaves are labeled by $\Upsilon_0$, $\Upsilon_1$ and inner nodes by the operators $\odot_A$ with $A$ a 3 partitioned matrix of size less than $3t$.

- $\Upsilon_0$ is the matrix \[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\].
- $\Upsilon_1$ is the matrix \[
\begin{pmatrix}
1 & 1
\end{pmatrix}
\].

These trees are called $t$ boundaried parse trees and define a matrix with boundary and therefore a represented boundaried matroid. Notations are chosen here to be as close as possible to [13] even if the object are slightly different. To study the parsed matroids, we now need to define the signature of a set $X$ in a way similar to the enhanced tree.

**Definition 15.** Let $M$ be a boundaried matroid represented by $T$, $s$ a node of $T$ and $X$ a subset of internal elements of $A_s$ the boundaried matrix parsed by $T_s$. We say that $X$ is of signature $\lambda = (\lambda_1, \ldots, \lambda_t)$ at $s$ if there exists a non trivial linear combination of the elements of $X$ plus the boundary equal to 0 in $A_s$ such that $\lambda$ is the set of coefficients of the elements of the boundary in this combination. If there is no non trivial combination, the signature is $\emptyset$. 17
We have exactly the same relation between signatures in an enhanced tree and in a \( t \)-boundaried parse tree.

**Lemma 3.** Let \( M \) a matroid represented by a \( t \) parse tree \( T \) and \( s \) a node of this tree labeled by \( \odot_N \) with children \( s_1 \) and \( s_2 \). \( N \) is partitioned into \( N_1 \), \( N_2 \) and \( N_3 \). Let \( X \) be a set of leaves i.e. of internal elements of \( M \), whose intersection with the leaves of \( T_{s_1} \) is \( X_1 \) and \( X_2 \) with the leaves of \( T_{s_2} \). \( X \) is of signature \( \lambda \) if and only if \( X_1 \) and \( X_2 \) are of signature \( \mu \) and \( \gamma \) and we have the relation

\[
\sum_i \mu_i N_1^i + \sum_j \gamma_j N_2^j = \sum_k \lambda_k N_3^k
\]

**Proof.** Just apply two times the definition of \( \oplus \) and follow the demonstration of Lemma 1.

From this lemma we derive a theorem identical to Theorem 2 as we can do as soon as we have a relation between the signature of a node and the ones of its children. We could then obtain a theorem of translation of \( MSO_M \) into \( MSO \) over boundaried parse trees with the same formula \( \text{dep}(X) \) but we now give a more precise result.

**Theorem 8.** A finitely representable matroid has a \( t \)-boundaried parse tree if and only if it is of branch-width less than \( t \).

**Proof.** Consider the following bijection between boundaried parse trees and enhanced trees. Let \( \overline{T} \) a boundaried parse tree, replace each label \( \odot_N \) on internal node by \( N \), each \( \Upsilon_0 \) on a leaf by \( (0) \) and \( \Upsilon_1 \) by \( (1\|1) \). This defines a enhanced tree \( T \) and we see easily how to do the reverse.

As we have exactly the same characterization theorem of dependent sets by signature on boundaried parse tree and enhanced tree then the matroids defined by \( T \) and \( \overline{T} \) are the same.

![Fig. 5. The parse tree associated to the enhanced tree of Fig. 3](image-url)
Example 6. The matrices below represent the boundaried matroid computed at each node of the parse tree. The boundary is on the left of the matrix.

\[
\begin{align*}
M_{s_3} &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \\
M_{s_4} &= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\
M_{s_2} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\
M_{s_5} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}
\end{align*}
\]

The matrix \( M_s \) represents the same matroid as the matrix \( X \) of Fig. 2 which was used to find an enhanced tree and then a parse tree as explained in the proof of the previous theorem.

6.2 Series and parallel connection

Here we consider one of the most simple operation on matroids, called the series and parallel connections. They extend well-known graph operations, on which they characterize the graphs of tree-width 2 \[3\]. The following definition and theorem are taken from \[21\].

Definition 16. Let \( M_1 \) and \( M_2 \) be two 1 boundaried matroids with ground sets \( S_1 \) and \( S_2 \) boundaries \( \{p_1\} \) and \( \{p_2\} \). We note \( C(M) \) the collection of circuit of the matroid \( M \). Let \( E \) be the set \( S_1 \cup S_2 \cup \{p\} \setminus \{p_1, p_2\} \), we define two collection of sets

\[
C_S = \left\{ \begin{array}{c} 
C(M_1 \setminus \{p_1\}) \cup C(M_2 \setminus \{p_2\}) \\
\cup \{C_1 \setminus \{p_1\} \cup C_2 \setminus \{p_2\} \cup \{p\} \mid p_i \in C_i \in C(M_i) \} 
\end{array} \right\}
\]

\[
C_P = \left\{ \begin{array}{c} 
C(M_1 \setminus \{p_1\}) \cup C(M_2 \setminus \{p_2\}) \\
\cup_{i=1,2} \{C_i \setminus \{p_1\} \cup \{p\} \mid p_i \in C_i \in C(M_i) \} \\
\cup \{C_1 \setminus \{p_1\} \cup C_2 \setminus \{p_2\} \mid p_i \in C_i \in C(M_i) \} 
\end{array} \right\}
\]

Theorem 9. The sets \( C_S \) and \( C_P \) are collection of circuits of a matroid on \( E \).

\( C_P \) defines the parallel connection of the two matroids and \( C_S \) defines the series connection. We define the operators \( \oplus_p \) (called 2 sum in \[21\]) and \( \oplus_s \), by doing either the parallel or series connection and removing the element \( p \). A short analysis shows that \( \oplus_s \) is only the direct sum of the two matroids without their boundary then the operator \( \odot \) which derives from \( \oplus_s \) may be simulated by the operator \( \odot \) which derives from \( \oplus_p \).

Definition 17 (\( k \) parallel parse tree). A \( k \) parallel parse tree is a parse tree whose leaves are labeled by abstract boundaried matroids of size \( k \) and inner nodes by 3-partitioned matroids of size 3.

As leaves are abstract matroids, they define matroids which have not been already studied because they are not representable. They are two interesting matroids \( M \) of size 3 regarding the action of \( \odot_M \), \( N_1 \) whose only dependent set is the set of size 3 and \( N_2 \) whose sets of size 2 are all dependents. The other operators give a direct sum plus some modifications on the boundary.
Fig. 6. Example of series and parallel connections over graph with boundaries represented by a dotted line

- $M_1 \circ N_1 M_2$ is the matroid given by the series connection of $M_1$ and $M_2$ with boundary $p$.
- $M_1 \circ N_2 M_2$ is the matroid given by the parallel connection of $M_1$ and $M_2$ with boundary $p$.

Note that in a parallel parse tree the boundary is always the same, then we can prove that the set of matroids generated by $k$ parallel parse trees is properly included in the closure of the abstract matroids of size $k$ by series and parallel connections. It would be interesting to study the complexity to decide $MSO_M$ on this broader class. As we allow abstract matroids for the leaves, the $k$ parallel parse trees define different matroids from the matroids of branch-width $t$ studied before, which are all representable. Nevertheless there is a relation between the operations $\oplus_p$ and $\oplus$ and therefore between branch-width and parallel parse tree as illustrated by the next proposition.

**Proposition 2.** Let $M_i$ be a matroid of boundary $\{p_i\}$ represented by the sets of vectors $A_i$ in $E_i$ for $i \in \{1, 2\}$. Then $A_1 \oplus A_2$ represents the matroids $M_1 \oplus_p M_2$.

**Proof.** Let $D_1$ and $D_2$ be dependent sets of $M_1$ and $M_2$ with linear dependency relations $\lambda_1 p_1 + \sum \alpha_i A_i^1 = 0$ and $\lambda_2 p_2 + \sum \beta_i A_i^2 = 0$. By definition of $\oplus_p$, the set $D$ union of $D_1 \setminus \{p_1\}$ and $D_2 \setminus \{p_2\}$ is a dependent set of $M_1 \oplus_p M_2$. If $\lambda_1$ or $\lambda_2$ are zero, $D$ is dependent in $A_1 \oplus A_2$ too, then we can assume they are both non-zero. By linear combination of the two previous equalities we get

$$\lambda_1 (p_1 - p_2) + \sum \alpha_i A_i^1 + \sum -\lambda_1 \lambda_2^{-1} \beta_i A_i^2 = 0$$

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Once we inject this equality in $A_1 \oplus A_2$, we have \( \sum \alpha_i A_i^1 + \sum -\lambda_1 \lambda_2^{-1} \beta_i A_i^2 = 0 \) meaning that the representation of $D$ in $A_1 \oplus A_2$ is also dependent. The converse is easy and left to the reader.

Of course for every generalization of $\oplus_p$ to a boundary bigger than one, the previous lemma will fail because $\oplus$ is not a operation of matroid as seen in Example 5 with a boundary of size two.

**Corollary 3.** A matroid defined by a $k$ parallel parse tree with leaves labeled by representable matroids is of branch-width at most $k$.

**Proof.** The leaves are representable matroids of size at most $k$, then they are of branch-width at most $k$. By the previous lemma and Theorem 8 the result is proved.

We now define a very general notion of signature to use the previously introduced technique and illustrate it in these settings. They represent which elements of the boundary makes the set dependent. Note also that the signature is unique.

**Definition 18 (Signature).** Let $M$ be a matroid, $T$ be one of its parse tree and $X$ a set of elements of $M$. The signature $\lambda_s$ of the set $X$ at $s$ a node of $T$ is the set of all the subsets $A$ of the boundary such as $X \cup A$ is a dependent set in the matroid parsed by $T_s$.

In these case, the boundary is of size one, then we have only three different cases :

1. $X$ is dependent then it is of signature $\{\{\},\{1\}\}$
2. $X$ is dependent only when we add the boundary element then it is of signature $\{\{1\}\}$
3. $X$ is independent even with the boundary element then it is of signature $\emptyset$

Remarks that an empty set is of signature $\emptyset$, as long as the boundary is an independent set.

**Lemma 4.** There is a relation $R_p(\mu, \gamma, \lambda, N)$, the first three arguments being signatures and $N$ a 3-partitioned matroid, which have the following property. Let $T$ be a $k$ parallel parse tree, $s$ a node of label $\odot_N$ and children $s_1, s_2$. $X_1$ is a set of elements of the matroid parsed by $T_{s_1}$ of signature $\mu$ and $X_2$ a set of elements of the matroid parsed by $T_{s_2}$ of signature $\gamma$. $X_1 \cup X_2$ is of signature $\lambda$ at $s$ if and only if $R_p(\mu, \gamma, \lambda, N)$.

This lemma is very similar to Lemma 4 but we do not precise the relation, it is not useful for the sequel. In fact, the nature of the relation $R_p$ is not relevant, what matters is that it only depends on $\mu$, $\gamma$, $\lambda$ and $N$, not on $X_1$ and $X_2$ or $T$. If we want to implement the formula $dep$ we have to compute this relation, meaning we have to consider the action of the operator $\odot_N$ for all $N$ of size 3, something we could still do manually. This lemma is implied by Lemma 4 in the next part, where we consider an extension of this construction (and then of this relation).

The other difference with boundaried parse trees is that leaves now represent abstract matroids of size less than $t$ and not elements of the matroid, thus the parallel
parse tree must be modified for our proof. Each leaf labeled by an abstract $k$ bound- 
aried matroid $M$ is replaced by a binary tree with as much leaves as internal elements 
of $M$. Each leaf of this subtree is labeled by a number between 1 and $t$ to determine 
which element of $M$ it represents. We have then a bijection $f$ between the leaves of 
this new tree and the internal elements of the matroid parsed by the original parallel 
parse tree.

**Theorem 10 (Characterization of dependency).** Let $T$ be a parallel parse tree 
parsing the matroid $M$ modified as previously explained and $X$ a set of elements of $M$. 
$X$ is dependent if and only if there exists a signature $\lambda_s$ for each node $s$ of the tree $T$ 
such that :

1. if $s_1$ and $s_2$ are the children of $s$ of label $\odot N$ then $R_p(\lambda_{s_1}, \lambda_{s_2}, \lambda_s, N)$
2. if $s$ is labeled by an abstract boundaryed matroid $N$, then $X \cap N$ is a set of signature $\lambda_s$
3. the set of nodes labeled by an abstract matroid of signature non $\emptyset$ is non empty
4. the signature at the root is $\{\emptyset, \{1\}\}$

$F(\phi(\vec{x}))$ is again a formula of MSO defined inductively from $\phi(\vec{x})$ of $MSO_M$ by 
relativization to the leaves and use of the formula $dep$ which is given in the proof of 
the next theorem to replace the independence predicate. The construction of $F(\phi(\vec{x}))$ 
can be made in time linear in the size of $\phi(\vec{x})$ for a fixed $k$.

**Theorem 11.** Let $M$ be a matroid, $T$ one of its $k$ parallel parse tree and $\phi(\vec{x})$ an 
$MSO_M$ formula then $M \models \phi(\vec{a}) \iff T \models F(\phi(f(\vec{a})))$.

**Proof.** The demonstration is done by the construction of a formula $dep(X)$ satisfying 
the conditions of the characterization theorem. We use the formulas defined in the 
proof of Theorem 3 condition 1 is implemented by formula $\Psi_1$ except that $\theta$ is now 
the relation $R_p$. Conditions 3 and 4 are obtained by slight modifications of $\Psi_2$ and $\Psi_3$.

To enforce condition 2, we define a formula $\Gamma(X, \vec{X}, s)$ which is true if and only 
if the signature at internal node labeled by an abstract boundaryed matroid $N$ is a 
signature of the intersection of $X$ with $N$. The relation $\gamma(s, \vec{X}, S, N)$ is true if and only 
if the signature represented by $\vec{X}$ at $s$ is the one of $S$ subset of the matroid $N$.

$$\Gamma(X, \vec{X}, s) = \bigwedge_{N, S \subseteq N} (\text{label}(s) = N \land X \cap N = S) \Rightarrow \gamma(s, \vec{X}, S, N)$$

This formula is a conjunction on all boundaryed matroids $N$ of size $k$ and subset of 
these matroids $S$ which are in number bounded by a function in $k$. $X \cap N = S$ is easily 
implemented by a MSO formula but not developed here for readability.

By combining the aforementioned formulas we build $dep(X)$ which is true in $T$ a $k$ 
parallel parse tree if and only if $X$ represent a dependent set of the matroid parsed by 
$T$. The proof of correctness of $F(\phi)$ is achieved by an easy induction, the hard case 
being proved by the characterization theorem.

Thanks to this theorem, we know that the class of matroids defined by $k$ parallel 
parse tree has a dependence relation decidable in linear time. The other consequences 
will be given in the next section which generalizes this construction.
6.3 Pushout for abstract matroids

In this section we give a generalization of the parallel connection to a boundary of an arbitrary size. We define $M_1 \oplus M_2$ as a universal object, more precisely the limit of the diagram of Figure 7 in the category of finite matroids whose arrows are morphisms preserving the dependence relation.

\[
\begin{array}{ccc}
B & \xrightarrow{\gamma_1} & M_1 \\
\gamma_2 & & \downarrow \\
M_2 & \xrightarrow{i_2} & M_1 \oplus M_2
\end{array}
\]

The set $B$ is an independent set of size $t$. $i_1 \circ \gamma_1 = i_2 \circ \gamma_2$ where $\gamma_1$ and $\gamma_2$ are injective their images are the boundaries and therefore independent sets of $M_1$ and $M_2$.

Fig. 7. The diagram of the pushout

A set $D$ is the set of dependent sets of a matroid if it satisfies:

- $(A_1) : D_1, D_2 \in D^2, e \in D_1 \cap D_2 \Rightarrow D_1 \cup D_2 \setminus \{e\} \in D$, where $D$ is the set of its dependent sets.
- $(A_2) : D \in D, D \subset D' \Rightarrow D' \in D$

Let $S$ be a set of subsets, we note $\overline{S}$ the closure of $S$ by the axiom $(A_1)$. We obtain this set by an inductive construction, $S_0 = S$ and $S_{n+1}$ is the set of all $D_1 \cup D_2 \setminus \{e\}$ with $(D_1, D_2) \in S_n^2$.

**Definition 19.** Let $M_1$ and $M_2$ two $t$ boundaried matroids with ground sets $S_i$ and boundaries $\gamma_i$. We introduce the elements $\{e_1, \ldots, e_t\}$, which are disjoint from $S_1$ and $S_2$. Let $E = S_1 \cup S_2 \cup \{e_1, \ldots, e_t\} \setminus \{\gamma_1([1, t]) \cup \gamma_2([1, t])\}$. Let $D$ be the set $\{D_1 \cup D_2 \mid D_1$ dependent in $M_1$ or $D_2$ dependent in $M_2\}$ where $\gamma_1(i)$ and $\gamma_2(i)$ are changed in $e_i$. Then $M_1 \oplus M_2 = (E, D)$.

**Lemma 5.** With the above notation, $t$ being the size of the boundary, we have $D_t = \overline{D}$.

*Proof.* Let us prove by induction that $D_i$ contains already all the sets of $\overline{D}$ which contain more than $t-i$ elements of the boundary. The base case is true by construction. Moreover all dependent sets in $D_{i+1}$ and not in $D(i)$ are obtained by the axiom $(A_1)$ which removes a boundary element. This proves the induction step. Then $D(t)$ contains all sets of $\overline{D}$ which contain more than 0 elements of the boundary that is to say $D_t = \overline{D}$.

This definition is a generalization of the parallel composition to a $t$ boundary. Note that in the case of the parallel connection for which the boundary is of size 1, we have to complete the dependent sets by the axiom $(A_1)$ only $t = 1$ time.

**Theorem 12.** $M_1 \oplus M_2$ is the limit of Diagram 7 that is to say it is a matroid and for every matroid $N$ which satisfies the diagram there is a morphism from $M_1 \oplus M_2$ to $N$. 
Proof. Existence:
To prove that \( M_1 \oplus M_2 \) is a matroid, it is enough to prove that it satisfies axioms \((A_1)\) and \((A_2)\). The set \( D \) satisfies the axiom \((A_1)\) by construction. We prove by induction that the set \( D \) satisfies the axiom \((A_2)\). By definition \( D_0 = D \) satisfies \((A_2)\). Let \( C \in D_{n+1} \) then we have \( D_1 \) and \( D_2 \) in \( D_n \) with \( e \in D_1 \cup D_2 \) such that \( C = D_1 \cup D_2 \setminus \{e\} \). Let \( C \subseteq C' \in D_{n+1} \), if \( e \in C' \) then \( D_1 \in C \) and by induction hypothesis, \( C' \in D_n \).

Else, we can write \( C' = D'_1 \cup D'_2 \setminus \{e\} \) with \( D_1 \subseteq D'_1 \) and \( D_2 \subseteq D'_2 \). By induction hypothesis, \( D'_1 \) and \( D'_2 \) are in \( D_n \), and we deduce that \( C' \in D_{n+1} \).

The maps \( i_1 \) and \( i_2 \) are the inclusions, then by construction of the ground set \( E \) of \( M_1 \oplus M_2 \), we have \( i_1 \circ \gamma_1 = i_2 \circ \gamma_2 \).

Universality:
Let \( N \) be a matroid satisfying the diagram of Figure 8 for the morphisms \( g \) and \( f \) with set of dependent sets noted \( D(N) \). Let \( i \) be the natural map from \( M_1 \oplus M_2 \) to \( N \) defined by associating the image of an element of \( M_i \) in \( M_1 \oplus M_2 \) to the image of this same element in \( M \). To prove that \( i \) is a morphism of matroid is equivalent to prove that if \( D \) is dependent in \( M_1 \oplus M_2 \) then \( i(D) \in D(N) \). Again we prove this property by induction. Let \( D \in D_0 \), then \( D = i_1(D_1) \cup i_2(D_2) \) with let say \( D_1 \) dependent in \( M_1 \). The set \( f(D_1) \) is dependent in \( N \) because \( f \) is a morphism and by construction \( f(D_1) \subseteq i(D) \) which is then a dependent set in \( N \). Let \( D = D_1 \cup D_2 \setminus \{e\} \) with \( D_1, D_2 \) elements of \( D_n \) and \( e \in D_1 \cap D_2 \), by induction hypothesis \( i(D_1) \) and \( i(D_2) \) are in \( D(N) \). By \((A_1)\), \( i(D_1) \cup i(D_2) \setminus \{i(e)\} \) is in \( D(N) \) and as this set is included in \( i(D) \), it holds that \( i(D) \in D(n) \).

![Fig. 8. Universality of \( M_1 \oplus M_2 \)](image)

Let \( M_1 \oplus M_2 \) be the restriction of the matroid \( M_1 \oplus M_2 \) to \( E \setminus \{b_1, \ldots, b_t\} \). The operation \( \oplus_M \), we now use in parse tree, comes from this operation \( \oplus \).

Definition 20. A matroid is in \( \mathcal{M}_{k,t} \), if it can be represented by parse trees labeled at internal nodes by abstract matroids of size less than \( 3t \) and abstract matroids of size less than \( k \) as leaves.

The set \( \mathcal{M}_{k,1} \) is the set of matroids represented by a \( k \) parallel parse tree.

Characterization of dependent sets. We need to understand how to build a dependent set of \( M_1 \oplus M_2 \) from dependent sets of \( M_1 \) and \( M_2 \) and how to represent this construc-
tion with an object of constant size (dependent only on \( t \)). This will permit to define dependency in MSO with parse tree as input.

We consider trees labeled at the leaves by dependent sets of \( M_1 \) and \( M_2 \) and at internal nodes by an element of the boundary. An example is given in the left part of Figure 9. We associate inductively a dependent set to a tree:

- to a leaf we associate the set which labels it
- to an internal node \( s \) labeled by \( b \) with children \( s_1 \) and \( s_2 \) associated to the sets \( D_1 \) and \( D_2 \), we associate \( D_1 \cup D_2 \setminus \{ b \} \) if \( b \in D_1 \cup D_2 \) else the tree does not defines a set.

By definition of \( \oplus \) we know that every dependent set of \( M_1 \oplus M_2 \) and then of \( \tilde{M}_1 \oplus M_2 \) is represented by such a tree. By Lemma 5 we also know it is of depth at most \( t \). Let \( T \) be a tree representing a dependent set of \( M_1 \oplus M_2 \), \( X_1 \) (and \( X_2 \)) the internal elements of the union of all dependent sets of \( M_1 \) (respectively \( M_2 \)) labeling the leaves of \( T \).

Let consider the tree built from \( T \) where each leaf labeled by a dependent set \( A_i \) of \( M_i \) is replaced by \( A_i \cup X_i \). This tree is associated to the same dependent set as \( T \). Then the only useful informations to define a dependent set are \( X_1, X_2 \) and the tree \( T \), from which we have removed every internal element in the sets labeling the leaves but add a symbol to know if it was a set from \( M_1 \) or \( M_2 \). This kind of tree is called a dependent set tree and an example is given in the right part of Figure 9.

![Fig. 9. Two ways of representing \( \{ d_1^1, d_1^2, d_1^3, d_2^1, d_2^2 \} \) a dependent set of \( \tilde{M}_1 \oplus M_2 \)](image)

Let \( s \) be a node of a parse tree \( T \) with children \( s_1 \) and \( s_2 \). Let \( X_1, X_2 \) be sets of internal elements of the matroid parsed by \( T_{s_1}, T_{s_2} \) and \( \lambda_1, \lambda_2 \) their signature. \( X_1 \cup X_2 \) is dependent if and only if there exists a dependent set tree such that every leaf labeled by 1 (respectively 2), corresponding to the matroid parsed by \( T_{s_1} \) (respectively \( T_{s_2} \)), is also labeled by a set of boundary element which appears in \( \lambda_1 \) (respectively \( \lambda_2 \)). This holds since in this section signatures \( \lambda_1 \) and \( \lambda_2 \) are the sets of boundary elements which extend \( X_1 \) and \( X_2 \) into dependent sets. From this remark we obtain a lemma.
which generalizes Lemma 4 with the relation $R$ extending $R_p$ to more signatures and 3 partitioned matroids.

**Lemma 6.** There is a relation $R(\mu, \gamma, \lambda, N)$, the first three arguments being signatures and the last a 3 partitioned matroid, which have the following property. Let $T$ be a parse tree, $s$ a node of label $\odot_N$ and children $s_1, s_2$. $X_1$ is a set of elements of the matroid parsed by $T_{s_1}$ of signature $\mu$ and $X_2$ a set of elements of the matroid parsed by $T_{s_2}$ of signature $\gamma$. $X_1 \cup X_2$ is of signature $\lambda$ at $s$ if and only if $R(\mu, \gamma, \lambda, N)$.

We do the same modifications on the parse tree as in the parallel parse tree case to take care of the leaves representing abstract matroids.

**Theorem 13 (Characterization of dependency).** Let $T$ be a tree parsing the matroid $M$ and $X$ a set of elements of $M$. $X$ is dependent if and only if there exist a signature $\lambda_s$ for each node $s$ of the tree $T$ such that:

1. if $s_1$ and $s_2$ are the children of $s$ of label $\odot_N$ then $R(\lambda_{s_1}, \lambda_{s_2}, \lambda_s, N)$
2. if $s$ is labeled by an abstract matroid $N$, then the intersection of $X$ with the elements of $N$ is a set of signature $\lambda_s$
3. the set of nodes labeled by an abstract matroid of signature non $\emptyset$ is non empty
4. the signature at the root contains the empty set

**Proof.** Remark that by definition of a signature a set with the empty set in its signature is a dependent set and conversely. The theorem follow by an induction using Lemma 6.

Thanks to this characterization we can define a formula $dep(X)$ exactly as in the previous case. As the boundary is of size at most $t$, there are at most $2^t$ elements in the signature and therefore $2^{2t}$ signatures, then the size of the formula is bounded by a function in $t$ and $k$. The relation $R$ is a finite object which can be then computed once for every $t$. It plays the role of the boolean $\theta$ in the construction of the formula $dep$ for enhanced trees.

**Theorem 14.** Let $M \in \mathcal{M}_{k,t}$ be a matroid given by a parse tree $T$ and $\phi(\vec{x})$ an $MSO_M$ formula, then $M \models \phi(\vec{a}) \iff T \models F(\phi(f(\vec{a})))$.

All extensions and applications given in the section of this name, can be adapted to this case, the next corollaries being some examples.

**Corollary 4.** The model checking problem for $MSO_M$ is decidable in time $f(k, t, l) \times P(n)$ over $\mathcal{M}_{k,t}$, where $f$ is a computable function, $P$ an affine function and $l$ the size of the formula.

**Corollary 5.** Let $\phi(X_1, \ldots, X_n)$ be an $MSO_M$ formula, for every matroid in $\mathcal{M}_{k,t}$, the enumeration of the sets satisfying $\phi$ can be done with linear delay after a linear preprocessing time.

**Corollary 6.** Let $\phi$ a formula of $MSO_M$, its spectrum restricted to $\mathcal{M}_{k,t}$, is ultimately periodic.
7 Discussion

In this article we have introduced a systematic way of constructing a reduction from $MSO_M$ over a class of matroid to $MSO$ over trees. The main tool is the notion of signature, which characterize locally in the trees, the potentiality to be a dependent set. This method has the advantage to give some insights about the classes of matroid of bounded branch-width and of matroid in $M_{k,t}$, explaining how the dependent sets are constructed.

It is still an open question to find an efficient algorithm which decides if a matroid is in $M_{k,t}$. Then we cannot see this as a decomposition measure but rather as a grammar which allows to build abstract matroids, with the good properties that they are representable in linear space and the independence predicate is decidable in linear time. Note that we can check in linear time that a tree is really a parse tree.

The method we introduce could work with any operation derived from a pushout over a class of matroids. For instance with graphical matroids we would have yet another demonstration of Courcelle theorem \[6\] in an algebraic flavor. Another good candidate would be to introduce an operation over the algebraic matroids. Although different, it is possible that the powerful method of \[19\] may be adapted to obtain similar results.

Moreover we could try this method on structures which are not matroids but which are also given by a particular set of subset. For instance we can work with oriented matroids by using a predicate circuit$(A^+, A^-)$ which represents the fact that $A^+ \cup A^-$ is a circuit with positive elements in $A^+$ and negative in $A^-$. We could also try with greedoids which are a generalization of the matroids. In both case we can define a pushout, then a parse tree and a generalized signature.

Finally we can combine the different operators $\oplus$ we have defined (and any other to be invented) to have more general parse trees. The formula $dep$ will involve as many signatures as different operators and we have to write a formula which ensure that the different signatures at the same node are compatible. For instance we can build two matroids of bounded branch-width by boundaried parse trees and then combine them as an abstract boundaried matroid.

Acknowledgements Thanks to Bruno Courcelle for very helpful suggestions and to David Duris for his support and corrections to this article. I am also very grateful to Brice Minaud and Pierre Clairambault for discussions we had respectively about matroid representations and categories. Last but not the least, I wish to thank Arnaud Durand for his numerous good remarks which have increased a lot the contents and the quality of this paper.

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