SPIN CANONICAL INVARIANTS OF 4-MANIFOLDS AND ALGEBRAIC SURFACES

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0. Introduction

Donaldson's celebrated construction of polynomial invariants for 4-manifolds (see for example [D–K]) has served as a model for an enormous range of recent results and constructions relating algebraic geometry and differential topology. The short written version of Donaldson’s talk at the 1990 Arbeitstagung contained the construction of jumping instantons, and was the starting point for our constructions and applications of the new spin polynomial invariants of 4-manifolds and algebraic surfaces. I am very grateful to the Royal Society of London for the opportunity to make an extremely pleasant and stimulating 3 months visit to Warwick, Oxford and Cambridge in summer 1993, which has allowed me to discuss these constructions with Simon Donaldson.

I have also benefited from several conversations with Nigel Hitchin. His questions about what Yang–Mills connections look like from a geometric point of view (see for example [H]) has involved a journey to classical incidence geometry, or “algebraic protogeometry”.

To explain what I mean by this, suppose that we want to prove the following differentiable version of the Poincaré conjecture for \( \mathbb{CP}^2 \):

Conjecture (DPC for \( \mathbb{CP}^2 \)). The complex projective plane \( \mathbb{CP}^2 \) has a unique differentiable structure.

To prove this, we have to say what is a line or a nonsingular conic of \( \mathbb{CP}^2 \) in terms of the underlying differentiable structure of \( \mathbb{CP}^2 \). If we can do this, and check that the lines satisfy the classical incidence axioms, then we are done. If this can’t be done, then equipping \( \mathbb{CP}^2 \) with any Riemannian metric \( g \), we have to:

1. say what it means in terms of Riemannian geometry; and
2. describe the dependence on the metric \( g \).

Now the analog of a nonsingular conic in the language of Riemannian geometry is a \( g \)-instanton of topological type \((2, 0, 2)\) up to gauge equivalence, that is, a \( g \)-antiselfdual SU(2)-connections on a vector bundle \( E \) with \( c_2 = 2 \). This holds because of Donaldson’s identification

\[
\left\{ \text{instantons for the Fubini–Study metric on } \mathbb{CP}^2 \right\} = \{\text{stable holomorphic vector bundles}\},
\]
and Barth’s interpretation of stable bundles on $\mathbb{CP}^2$ (see [B]).

The next question is:

How many nonsingular conics can be inscribed in a general 5-gon?

This constant is the Donaldson polynomial $\gamma^5_{\mathbb{CP}^2}$ evaluated at the generator of $H^2(\mathbb{CP}^2, \mathbb{Z})$. Thus the Donaldson polynomials provide the information about the incidence correspondence between conics and lines.

My task today is to introduce new invariants of this geometric type. I avoid important technical details, referring the reader to more specialised articles, but my aim is to indicate why the following facts are actually true.

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1. Construction of spin polynomials invariants

For a smooth, compact 4-manifold $M$, the Stiefel–Whitney class is the characteristic vector $w_2(M) \in H^2(M, \mathbb{Z}/2)$ of the intersection form $q_M$ on $H^2(M, \mathbb{Z})$. Any vector $C \in H^2(M, \mathbb{Z})$ with $C \equiv w_2(M) \mod 2$ is called a Spin$^C$ structure of $M$. Thus a Spin$^C$ structure on $M$ is just a lifting of $w_2$ to an integer class.

For the rest of this paper, for $\sigma \in H^2(M, \mathbb{Z})$, we write $L_\sigma$ to denote a complex line bundle with first Chern class

$$c_1(L_\sigma) = \sigma.$$  

Any Riemannian metric $g$ and Spin$^C$ structure $C$ on $M$ defines a decomposition of the complexified tangent bundle $TM_C$ as a tensor product

$$TM_C = (W^-)^* \otimes W^+$$

of two rank 2 Hermitian vector bundles $W^\pm$ with

$$\bigwedge^2 W^\pm = L_C.$$  

Moreover, for any U(2)-bundle $E$ on $M$ and any Hermitian connection $a \in \mathcal{A}_h$ on $E$, putting any Hermitian connection $\nabla_0$ on $L_C$ gives a coupled Dirac operator

$$D^{g,C,\nabla_0}_a : \Gamma^\infty(E \otimes W^+) \to \Gamma^\infty(E \otimes W^-).$$  

Now the orbit space of irreducible connections modulo the gauge group

$$\mathcal{B}(E) = \mathcal{A}_h^*(E)/\mathcal{G}$$

contains the subspace

$$\mathcal{M}^g(E) \subset \mathcal{B}(E)$$  

(1.2)
of antiselfdual connections with respect to the Riemannian metric $g$.

For any positive integer $r$, we can consider the subspace of jumping bundles:

$$\mathcal{M}^g_C(E) = \{(a) \in \mathcal{M}^g(E) \mid \text{rank } \ker D^{g,C,\nabla_0}_a \geq r\} \subseteq \mathcal{M}^g(E).$$

The number $r$ is called the \textit{jumping level} of $E$. The collection of these subspaces defines a filtration:

$$\mathcal{M}^g(E) \supseteq \mathcal{M}^g_C(E) \supseteq \cdots \supseteq \mathcal{M}^g_{r-1}(E) \supseteq \cdots$$

The virtual (expected) codimension of $\mathcal{M}^g_{r}(E)$ is given by

$$v. \text{codim } \mathcal{M}^g_{r}(E) = 2r^2 - 2r\chi_C(E),$$

where $\chi_C(E)$ is the index of the coupled Dirac operator (1.1), which depends only on the Chern classes of $E$ and the Spin$^C$ structure $C$.

The analog of the Freed–Uhlenbeck theorem, that for generic metric $g$ the moduli space $\mathcal{M}^g(E)$ (2.1) is a smooth manifold of the expected dimension with regular ends (see [F–U], Theorem 3.13), was proved in [P–T], Chap. 2, §3 for the first step $\mathcal{M}^g_C(E)$ of the filtration (1.4). Moreover, $\mathcal{M}^g(E)$ admits a natural orientation (see [D–K]) inducing an orientation on $\mathcal{M}^g_{r}(E)$, because its normal bundle has a natural complex structure. This orientation is described in details in [P–T], Chap. 1, §5.

We need the usual restrictions on the topology of $M$: we suppose that

$$b_2^+(M) = 2p_g(M) + 1$$

is odd. Then both

$$v. \dim \mathcal{M}^g(E) = 2d \quad \text{and} \quad v. \dim \mathcal{M}^g_C(E) = 2d_r$$

must be even.

It is very natural here to use the Uhlenbeck compactification of the first step of our filtration. We get a filtration

$$\overline{\mathcal{M}^g(E)} \supseteq \overline{\mathcal{M}^g_C(E)} \supseteq \cdots \supseteq \overline{\mathcal{M}^g_{r}(E)} \supseteq \cdots$$

Actually, in applications, when $r$ is big (say $\geq 5$), we only consider cases with compact moduli spaces $\mathcal{M}^g_{r}(E)$.

Now for any element of our filtration and for a general metric $g$, slant product defines a cohomological correspondence

$$\mu: H_i(M, \mathbb{Z}) \to H^{4-i}(\overline{\mathcal{M}^g_C(E)}, \mathbb{Z}),$$

and a collection of polynomials

$$\gamma^E_g, s\gamma^E_g, C, \ldots, s_r\gamma^E_g, C, \ldots \in S^*H^2(M, \mathbb{Z}),$$

where the first $\gamma^E_g$ is the Donaldson polynomial, the second $s\gamma^E_g$ is called the \textit{spin} polynomial, and the general term the spin polynomial of jumping level $r$. 
Since \( \overline{\mathcal{M}}^g(E \otimes L) = \overline{\mathcal{M}}^g(E) \) for any line bundle \( L \), it follows that \( \overline{\mathcal{M}}^g(E) \) depends only on \( c_1 \mod 2 \) and on the first Pontryagin number \( p_1 = c_1^2 - 4c_2 \), where \( c_i \) are the Chern classes of \( E \). Hence the Donaldson polynomial \( \gamma^E_g \) depends only on \( c_1 \mod 2 \) and \( p_1 \):

\[
\gamma^E_g = \gamma^{c_1 \mod 2, p_1}_g.
\]

Also, for the twisted bundle \( E \otimes L \), the coupled Dirac operator

\[
D^{g,C - 2c_1(L), \nabla_0}_a : \Gamma^\infty(E \otimes L \otimes W^+ \otimes L^*) \to \Gamma^\infty(E \otimes L \otimes W^- \otimes L^*)
\]

for the shifted \( \text{Spin}^C \) structure \( C - 2c_1(L) \) is precisely the Dirac operator (1.1). Hence the space \( \overline{\mathcal{M}}^{g,C}_r(E) \) depends only on \( c_1 + C \in H^2(M, \mathbb{Z}) \) and \( p_1 \). Also, the spin polynomial \( s_r\gamma^E_{g,C} \), of jumping level \( r \) depends only on \( c_1 + C \) and \( p_1 \):

\[
s_r\gamma^E_{g,C} = s_r\gamma^{p_1}_{g,C + c_1}.
\]

**Example: Algebraic surfaces.** If \( M \) is the underlying manifold of an algebraic surface \( S \), then there exists a canonical \( \text{Spin}^C \) structure given by the anticanonical class \( -K_S \) (we drop the lower index when there is no danger of confusion).

In this case, for any \( H \in \text{Pic} S \subset H^2(S, \mathbb{Z}) \), by the Donaldson–Uhlenbeck identification theorem, we have

\[
\overline{\mathcal{M}}^{g_H}(E) = M^H(2, c_1, c_2),
\]

where the right-hand side is the moduli space of holomorphic \( H \)-slope stable bundles on \( S \) with Chern classes \( c_1, c_2 \).

Making this identification \((a) = E\), we have identifications

\[
\ker D^{g_H, -K}_a = H^0(E) \oplus H^2(E) \quad \text{and} \quad \coker D^{g_H, -K}_a = H^1(E),
\]

where \( H^i(E) \) denote coherent cohomology groups. Thus the index of the coupled Dirac operator for this special \( \text{Spin}^C \) structure is

\[
\chi_{-K_S} D^{g_H, -K}_a = h^0(E) - h^1(E) + h^2(E).
\]

So the subspace \( \overline{\mathcal{M}}^{g_H, -K}_r(E) \) is the Brill–Noether locus

\[
\overline{\mathcal{M}}^{g_H, -K}_r(2, c_1, c_2) = \{ E \in M^H(r, c_1, c_2) \mid h^1(E) \geq -\chi(E) + r \}.
\]

But for surfaces, the last inequality can be rewritten

\[
h^1(E) \geq -\chi(E) + r \iff h^0(E) + h^2(E) \geq r.
\]

Hence we have a decomposition

\[
\overline{\mathcal{M}}^{g_H, -K}_r(2, c_1, c_2) = \bigcup M^H_{i,j}(2, c_1, c_2),
\]
where the components on the right-hand side are the algebraic subvarieties of $M^H(2, c_1, c_2)$ defined by

$$M^H_{i,j}(2, c_1, c_2) = \{ E \in M^H(2, c_1, c_2) \mid h^0(E) \geq i, h^2(E) \geq j \}. \quad (1.10)$$

On the other hand, sending $E \mapsto E^*(K) = E^* \otimes \mathcal{O}_S(K)$ identifies

$$M^H(2, c_1, c_2) = M^H(2, rK - c_1, c_2 - c_1 \cdot K - K^2),$$

and Serre duality gives

$$M^H_{i,j}(2, c_1, c_2) = M^H_{j,i}(2, rK - c_1, c_2 - c_1 \cdot K - K^2). \quad (1.11)$$

Now the Gieseker compactification $\overline{M^H(2, c_1, c_2)}$ (see [G]) gives the compactifications $\overline{M^H_{i,j}(r, c_1, c_2)}$.

The standard definition of $\mu$-homomorphism in the algebraic geometric context (see [T1] or [O’G]) gives the collection of polynomials

$$a_{\gamma_{i,j}}^H(2, c_1, c_2), \quad (1.12)$$

Now to compute the algebraic geometric version of the spin polynomial of jumping level $r$ (1.6), we must sum the individual polynomials (1.12)

$$a_{\gamma_r}^H(2, c_1, c_2) = \sum_{i+j=r} a_{\gamma_{i,j}}^H(2, c_1, c_2). \quad (1.13)$$

But here care is needed, because the natural orientations of the components (1.10) can be different (see [P–T], Chap. 1, §5).

One prove that the algebraic geometric polynomials (1.13) and spin polynomials (1.6) are equal using the same arguments as for the original Donaldson polynomials (see Morgan [M]). More precisely, for a Hodge metric $g^H$, if all the moduli spaces (1.10) have the expected dimension and avoid the reducible connections then

$$a_{\gamma_r}^H(2, c_1, c_2) = s_r \gamma_{g^H, c_1-K_S}^{p_1}; \quad (1.14)$$

see (1.13) and (1.7) and [T3], §6.

To finish this important example we must recall the description of the coupled Dirac operator (1.1) in terms of the complex structure of $S$. Namely

$$D_{a=E}^{g^H, c_1-K_S, \nabla_0} : \Omega^{0,0}(E) \oplus \Omega^{0,2}(E) \rightarrow \Omega^{0,1}(E) \quad (1.15)$$

is the convoluted Dolbeault complex of $E$.

In the following section, which is the core of this paper we describe the dependence of the spin polynomials on the choice of the Riemannian metric $g$. 


2. The dependence on the metric

Let $\mathcal{S}(M)$ be the space of Riemannian metrics on $M$. Every Riemannian metric $g$ provides

1) an identification

$$H^2(M, \mathbb{R}) = H_g$$

where $H_g$ is the space of harmonic 2-forms with respect to the metric $g$.

2) a Hodge $*$ decomposition

$$H_g = H_g^+ \oplus H_g^-$$

where $H_g^+$ is the subspace of selfdual forms and $H_g^-$ that of antiselfdual forms.

Fixing an orientation of each $H_g^+$, we get the period map:

$$\Pi: \mathcal{S}(M) \to \Omega \subset \text{Grass}_{\text{orient}}(b_2^+, H^2(M, \mathbb{R}))$$

where $\Omega$ is the subdomain of the Grassmanian of oriented subspaces in $H^2(M, \mathbb{R})$ of rank $b_2^+$ defined by:

$$\Omega = \{ H \in \text{Grass}_{\text{orient}}(b_2^+, H^2(M, \mathbb{R})) \mid q_M | H > 0 \}.$$ (2.2)

where $q_M$ is the intersection form of $M$.

Any class $e \in H^2(M, \mathbb{Z})$ with $e^2 < 0$ defines a subspace in $\Omega$:

$$W_e = \{ H \in \Omega \mid e \in H^\perp \}.$$ (2.3)

A metric $g \in \mathcal{S}$ is called $e$-irregular if

$$\Pi(g) \in W_e,$$ (2.4)

and the corresponding subset of metrics

$$\mathcal{S}_e = \{ g \in \mathcal{S} \mid \Pi(g) \in W_e \} \subset \mathcal{S}$$ (2.5)

is called a hurdle; a hurdle $\mathcal{S}_e$ is called a wall if its complement

$$\mathcal{S} \setminus \mathcal{S}_e$$ (2.6)

is disconnected.

The topological type $(c_1 \mod 2, p_1)$ of a SO(3)-bundle $E$ on $M$ defines a system \( \{e\}_E \) of vectors given by the conditions:

1) $e = c_1 \mod 2$

2) $e^2 = p_1$ (2.7)

and a system of hurdles:

$$\{ \mathcal{S}_e \} \quad \text{for} \quad e \in \{e\}_E.$$ (2.8)
Let $e \in \{e\}_E$ and suppose that $g$ is an $e$-irregular metric. What does this mean geometrically? First of all, by condition 1) of (2.7),

$$e = c_1 - 2\delta,$$

(2.7')

and $E$ splits topologically as a sum of line bundles:

$$E = L_\delta \oplus L_{c_1 - \delta}$$

(2.9)

**Remark.** The choice of $\delta$ or $c_1 - \delta$ defines an orientation of the hurdle $S_{c_1 - 2\delta}$ because of the equality

$$-e = c_1 - 2(c_1 - \delta).$$

So it is convenient to define an oriented hurdle by the class $\delta$.

Now if the line bundle $L_\delta$ has a $U(1)$-connection $\alpha$ and $L_{c_1 - 2\delta}$ a $U(1)$-connection $\alpha'$ such that the induced connection

$$\alpha' \otimes \alpha^* \text{ on } L_{c_1 - \delta} \otimes L_\delta^* \subset \text{ad } E$$

is $g$-antiselfdual, then the reducible connection $\alpha \oplus \alpha'$ on $E$ defines a singular point

$$\alpha \oplus \alpha' \in \text{Sing } \mathcal{M}^g(E).$$

(2.10)

Thus a metric $g$ is $e$-irregular if and only if the instanton space $\mathcal{M}^g(E)$ admits a singular point of type (2.9).

Now from (2.3) it is easy to see that $W_e$ has codimension

$$\text{codim } W_e = b_2^+$$

in the domain $\Omega$ (2.2). From this and the transversality conditions of the period map (2.1) along $W_e$, one can prove that if $b_2^+ > 1$ then a hurdle is never a wall, that is, the complement

$$S \setminus \bigcup_{e \in \{e\}_E} S_e$$

is connected.

One deduces from this, using the bordism arguments of [D–K], that if $b_2^+ > 0$ then the Donaldson and spin polynomials of any jumping level are independent of the metric, assumed to be regular in the sense of Uhlenbeck.

Now we would like to investigate when the singular point (2.10) is contained in the subspace $\mathcal{M}^{g,C}_r(E)$ of the instanton space $\mathcal{M}_g(E)$ (see (1.4)).

By definition, if

$$\alpha \oplus \alpha' \in \mathcal{M}^{g,C}_r(E)$$

(2.11)

then the operators

$$D^{g,C}_\alpha: \Gamma^\infty(L_\delta \otimes W^+) \to \Gamma^\infty(L_\delta \otimes W^-)$$

$$D^{g,C}_{\alpha'}: \Gamma^\infty(L_{c_1 - \delta} \otimes W^+) \to \Gamma^\infty(L_{c_1 - \delta} \otimes W^-),$$

(2.12)

have kernels satisfying

$$\text{rank ker } D^{g,C}_\alpha + \text{rank ker } D^{g,C}_{\alpha'} > r.$$

(2.13)
Now a metric \( g \in S \) is called Dirac regular, if
\[
\text{rank } \ker D^{g,C}_\alpha \geq r \implies \chi_C(L) \geq r
\] (2.14)
for the U(1)-connection \( \alpha \) on any line bundle \( L \); here \( \chi_C(L) \) is the index of the couple Dirac operator (2.12). It is proved that a generic metric \( g \in S \) is Dirac regular (see for example [P–T], Chap. 1, §1).

It is convenient to define a Spin\(^{\mathbb{C}}\) hurdle in terms of the class \( \delta \) (2.7′) and the decomposition (2.9):
\[
S_\delta = \{ g \in S_{c_1-2\delta} \mid g \text{ is Dirac irregular}\}. \tag{2.15}
\]
The main observation is the following

a Spin\(^{\mathbb{C}}\) hurdle \( S_\delta \) is a wall if and only if \( \chi_C(L_\delta) \geq r \).

Informally, this is easy, because both the \((c_1-2\delta)\)-irregularity condition and the Dirac irregularity condition are conditions of codimension \( \geq 1 \). If we can prove that they are independent conditions then we are done.

To carry out these arguments rigorously, we have to use a very simple trick. We consider a new space of parameters for our family of operators, larger than the space of all Riemannian metrics \( S \), namely the direct product
\[
S \times \Omega^2; \tag{2.16}
\]
that is, the set of pairs \((g, \nabla_0)\) consisting of any metric \( g \) and any Hermitian connection \( \nabla_0 \) on the determinant spin bundle \( L_C \) (see (1.1)), which we can view as a 2-form (for a fixed metric \( g \)).

Of course, there is no advantage for the lifted walls of a \((c_1-2\delta)\)-irregular metric, but we can use the new parameter \( \nabla_0 \) to regularise the Dirac irregularity. Namely on twisting the spinor bundles \( W^\pm \) by the line bundle \( L_\delta^* \), and changing the Spin\(^{\mathbb{C}}\) structure from \( C \) to \( C' = C - 2\delta \), the coupled Dirac operator \( D^{g,C}_\alpha \) (2.12) becomes the ordinary Dirac operator
\[
D^{g,C-2\delta}; \Gamma^\infty(W^+) \to \Gamma^\infty(W^-)
\]
of the metric \( g \) (and a new Spin\(^{\mathbb{C}}\) structure). So we must study how the jumping behaviour of the kernel of the ordinary Dirac operator of any Spin\(^{\mathbb{C}}\) structure changes on deforming the metric \( g \). But this was done in [P–T], Formulas (1.3.3) and (1.3.4), Proposition 3.1.1 and Corollary.

From this, the description of the normal cone to \( W_{c_1-2\delta} \) and the conditions of elliptic regularity provide the transversality conditions along a Spin\(^{\mathbb{C}}\) hurdle. Now we can specify conditions for a wall for the moduli space of jumping instantons of type \((r, C + c_1, p_1)\) (see (1.7)): a class \( \delta \in H^2(M, \mathbb{Z}) \) defines a wall if and only if
\[
\begin{align*}
1) & \quad 0 \geq (c_1 - 2\delta)^2 \geq p_1 \\
2) & \quad \chi_C(L_\delta) \geq r \quad \text{(respectively, } \chi_C(L_{(c_1-\delta)}) \geq r) \tag{2.17}
\end{align*}
\]

Remark 1. Why do we only consider the condition on one component of the decomposition (2.9)? The reason is the following: consider only the case \( \chi_C(E) \leq 0 \). If one component \( L_\delta \) has \( \chi_C(L_\delta) \geq 0 \) then the other component \( L_{(c_1-\delta)} \) has \( \chi_C(L_{(c_1-\delta)}) \leq 0 \), and vice versa.
Remark 2. Thus the crucial difference between the space of instantons \( \mathcal{M}^g \) and the subspace \( \mathcal{M}_g^{\delta,C} \) (1.4) is that \( \mathcal{M}^g \) depends only on the conformal class of \( g \) whereas the subspace of jumping instantons has as its full collection of indexes \( \mathcal{M}_g^{\delta,C,\nabla_0} \) (as usual we drop the final index).

The conditions 1) and 2) of (2.17) imply the following two inequalities:

1) \[ \frac{1}{4} c_1^2 \leq c_1 \cdot \delta - \delta^2 \] (2.18)

2) \[ \chi_C(L_\delta) = \frac{1}{2} \delta (\delta + C) + \frac{1}{8} (C^2 - I) \geq r \]

where \( I \) is the index of \( M \). (This is the Atiyah–Singer formula.)

The second inequality is equivalent to the following:

\[ -C \cdot \delta - \delta^2 \leq \frac{1}{4} C^2 - \frac{1}{4} I - 2r \] (2.18')

**Very important case.**

\[ c_1 = -C \] (2.19)

Then the right-hand side of the inequality 1) is equal to the left-hand side of the inequality (2.18') and we have the inequality

\[ 8r \leq -I \] (2.20)

Hence *if \( 8r > -I \) then the system of walls is empty.*

Remark. As the reader can probably guess, this paper is written for the sake of the preceding sentence. From it, using the bordism arguments of [D–K] one obtains that *the spin polynomials of jumping level \( 8r > -I \) are independent of the metric,* assumed to be regular in the sense of Uhlenbeck.

So the spin polynomial (1.7) with \( C + c_1 = 0 \), that is, \( s_r \gamma_{g,0}^{p_1} \) is called the *spin canonical polynomial of jumping level \( r \).*

These polynomials behave naturally under diffeomorphisms of \( M \). Namely, if \( 8r > -I \), then for any \( \sigma \in H^2(M, \mathbb{Z}) \) and any \( \varphi \in \text{Diff} M \) that preserves the orientation of a maximal positive subspace of \( H^2(M, \mathbb{R}) \), we have

\[ s_r \gamma_{g,0}^{p_1}(\sigma) = s_r \gamma_{g,0}^{p_1}(\varphi(\sigma)). \] (2.21)

This means that some aspects of the shape of these polynomials (and their coefficients, of course) are invariants of the smooth structure of 4-manifolds.

Remark. Of course, the same formula holds for every class \( (C+c_1) \) which is invariant under the group of diffeomorphisms \( \text{Diff} M \) preserving the orientation of a maximal positive subspace of \( H^2(M, \mathbb{R}) \). But it can happen, at least a priori, that the only invariant class is the trivial class 0.

To explain why the spin polynomials (1.7) are *canonical,* we return to our main example.
Example. **Algebraic surfaces.** If $M$ is the underlying manifold of an algebraic surface $S$, then the anticanonical class $-K_S$ gives a canonical Spin$^C$ structure, and the Very Important Case (2.19) predicts the equality

$$c_1 = K_S.$$  \hfill (2.22)

Then the standard numerical invariants of $S$, the topological Euler characteristic $c_2(S)$, and $K_S^2$ satisfy

$$\frac{1}{3} (2c_2(S) - K_S^2) = -I.$$  

Moreover, by the Noether formula,

$$-I + K_S^2 = 8 \left( \frac{1}{12} (K_S^2 + c_2(S)) \right) = 8(p_g + 1),$$

where $p_g$ is the geometric genus of $S$.

Hence the inequality $8r > -I$ is equivalent to the inequality

$$-K_S^2 < 8(r - 1) - 8p_g.$$  

Of course, we are interested in the case $p_g = 0$. Then our inequality is

$$-K_S^2 < 8(r - 1).$$  \hfill (2.23)

Therefore, for a minimal surface of general type, jumping level 1 is already enough to guarantee the invariance of the spin canonical polynomials.

Returning to the general case, it is convenient to write the pair of conditions (2.17) in the equivalent form:

1) $0 \geq (c_1 - 2\delta)^2 \geq p_1,$  
2) $(C + 2\delta)^2 \geq 8r + I,$ \hfill (2.17')

and to consider a new vector

$$\Delta = -C - 2\delta.$$  \hfill (2.24)

Then the pair of conditions (2.17) is equivalent to the pair

1) $0 \geq (\Delta + (c_1 + C))^2 \geq p_1,$  
2) $\Delta^2 \geq 8r + I.$  \hfill (2.25)

Now to convince you that spin canonical polynomials are useful in studying the geometry of differentiable 4-manifolds we have to prove that these invariants are nonvanishing. As usual, we have to restrict ourselves to our main example, the case of algebraic surfaces. Recall that Donaldson and Zuo proved that if $|p_1|$ is large then any Hodge metric $g_H$ on algebraic surface is generic in the sense of Freed and Uhlenbeck for the moduli space $M^9(c_1 \mod 2, p_1)$ (1.4). In spite of this, it is not true that every Hodge metric $g_H$ on an algebraic surface is regular for the moduli space of jumping instantons. Fortunately we can describe what we need to add to the standard arguments of Donaldson theory to use the new invariants. We do this in the following section.
3. HOW MUCH IS THE INDEPENDENCE?

Return to the Very Important Case of algebraic geometry (see (1.8–14), and (2.22–23)). This case is singular for the ad hoc reason that, by Serre duality,

$$M^H_{i,j}(2, K_S, c_2) = M^H_{j,i}(2, K_S, c_2)$$  \hspace{1cm} (3.1)

(see (1.19–11)). This means that even if the space $M^H_{i,j}(2, K_S, c_2)$ has the right dimension, this locus has nontrivial multiple structure; that is, as a subscheme, it is not reduced, it has nilpotents. Fortunately, we can describe the scheme-theoretic structure precisely. We do this in the simplest case $r = 1$, that is, for the minimal jumping level.

First of all, recall that in the regular case, the fibre of the normal bundle to the sublocus $M^H_{1,0}(2, c_1, c_2)$ in $M^H(2, c_1, c_2)$ at a point $E \in M^H_{1,0}(2, c_1, c_2)$ is given by

$$\left(N_{M^H_{1,0} \subset M^H}\right)_E = \text{Hom}(H^0(E), H^1(E)).$$  \hspace{1cm} (3.2)

So the codimension of $M^H_{1,0}(2, c_1, c_2)$ in $M^H(2, c_1, c_2)$ is given by

$$\text{v. codim } M^H_{1,0}(2, c_1, c_2) = h^0(E)(h^0(E) - \chi(E)).$$  \hspace{1cm} (3.3)

Moreover if $E \in M^H_{1,0}(2, c_1, c_2)$ is a singular point then in the space (3.2), we have the fibre of the normal cone of $M^H_{1,0}(2, c_1, c_2)$ in $M^H(2, c_1, c_2)$:

$$\left(C_{M^H_{1,0} \subset M^H}\right)_E \subset \text{Hom}(H^0(E), H^1(E)),$$  \hspace{1cm} (3.4)

defined as in [F].

Now in our Very Important Case, the spaces $H^i(E)$ involved in (3.2–4) admit additional structures. Namely, by Serre duality

$$H^0(E) = H^2(E)^* \quad \text{and} \quad H^1(E) = H^1(E)^*.$$  \hspace{1cm}

This means that the vector space $H^1(E)$ has a nondegenerate symmetric quadratic form

$$q_E : H^1(E) \rightarrow H^1(E)^*,$$  \hspace{1cm} (3.5)

and a light cone of isotropic vectors,

$$Q_E \subset H^1(E).$$  \hspace{1cm} (3.6)

Consider the simplest case

$$\text{rank } H^0(E) = h^0(E) = 1.$$  \hspace{1cm} (3.7)

A formal normal vector $n$ in the fibre of the formal bundle $\text{Hom}(H^0(E), H^1(E))$ is given as a nontrivial homomorphism

$$n : H^0(E) \rightarrow H^1(E).$$  \hspace{1cm} (3.8)

Then $n$ is contained in the fibre of normal cone (3.4) if and only if the image of the homomorphism (3.8) is contained in $Q_E$:

$$n(H^0(E)) \subset Q_E.$$  \hspace{1cm} (3.9)
that is, the image is isotropic with respect to the quadratic form $q_E$. So the normal cone (3.4) is given by

$$ (C_{M_1,0 \subset M})_E = Q_E $$  \hspace{1cm} (3.9')

This is almost obvious: the Dirac operator for a Hodge metric $g_H$ is the convoluted Dolbeault complex of $E$:

$$ D_{a=E}^{g_H, -K_S \nabla_0} : \Omega^{0,0}(E) \oplus \Omega^{0,2}(E) \xrightarrow{d'' \oplus (d'')^*} \Omega^{0,1}(E) $$

(see (1.15)). Now the arguments used to prove the transversality theorem in [P–T], Chap. 1, §3 make mathematical sense of the symbol

$$ \frac{\partial}{\partial g}(D_{a}^{g_H + \varepsilon g, -K_S \nabla_0}) $$  \hspace{1cm} (3.10)

as the line variation of a coupled Dirac operator with a jumping kernel with a fixed identification

$$ H^0(E) = \ker \frac{\partial}{\partial g}(D_{a}^{g_H + \varepsilon g, -K_S \nabla_0}) = \mathbb{C}, $$

$$ H^1(E) = \coker \frac{\partial}{\partial g}(D_{a}^{g_H + \varepsilon g, -K_S \nabla_0}). $$  \hspace{1cm} (3.11)

So we have the diagram

\[
\begin{array}{ccc}
\Omega^{0,0} & \xrightarrow{d''} & \Omega^{0,1} \\
\uparrow & & \downarrow \\
H^0(E) & \xrightarrow{n} & \ker \frac{\partial}{\partial g}(D_{a}^{g_H + \varepsilon g, -K_S \nabla_0}) = H^1(E)
\end{array}
\]

(3.12)

Now the Hermitian structure and Serre duality provide the diagram

\[
\begin{array}{ccc}
\Omega^{0,0}(E) & \xrightarrow{d''} & \Omega^{0,1}(E) & \xrightarrow{d''} & \Omega^{0,2} \\
\| & & \| \\
(\Omega^{0,1})^* & \xleftarrow{(d'')^*} & (\Omega^{0,2})^*
\end{array}
\]

(3.13)

But the Dolbeault complex is exact, hence the composite

$$ H^0(E) \xrightarrow{n} H^1(E) \xrightarrow{q_E} H^1(E)^* \xrightarrow{n^*} H^2(E) $$

(3.14)

is zero, where $q_E$ is the correlation (3.5). This means exactly (3.9).

**Example. The Barlow surface.** Let $S$ be the Barlow surface (see [K] for the definitions and motivation) with

$$ K^2_S = 1, $$

and consider the moduli space

$$ M^H(2, K, -1) $$

(3.15)
Then this moduli space is a finite set of vector bundles, so is compact. More precisely, the underlying reduced subscheme of $M^H(2, K_S, 1)$ is given by

\[ M^H(2, K_S, 1)_{\text{red}} = \text{Bs}|2K_S|, \]  

(3.16)

the set of base points of the bicanonical pencil. More precisely if

\[ \text{Bs}|K_S| = (2K_S)^2 = p_1 + p_2 + p_3 + p_4, \]  

(3.17)

is the quadruple of base points of $|K_S|$, then for every point $p_i$ there exist, up to the action of $\mathbb{C}^*$, only one nontrivial extension of type

\[ 0 \to \mathcal{O}_S \to E_i \to J_{P_i}(K_S) \to 0, \]  

(3.18)

and

\[ M^H(2, K_S, 1)_{\text{red}} = \{E_i\}, \quad \text{for } i = 1, \ldots, 4. \]  

(3.19)

Now for every $i = 1, \ldots, 4$, the Euler characteristic is $\chi(E_i) = 1$, and

\[ h^0(E_i) = h^1(E_i) = h^2(E_i) = 1. \]  

(3.20)

As we know, $H^1(E_i)$ has a nontrivial quadratic form (see (3.5)), for which zero is the only isotropic vector:

\[ Q_{E_i} = 0. \]

Consider the subscheme

\[ M^H_{1,0}(2, K_S, 1) \subset M^H(2, K_S, 1). \]  

(3.21)

Then

\[ M^H_{1,0}(2, K_S, 1)_{\text{red}} = M^H(2, K_S, 1)_{\text{red}} = \{E_i\}. \]  

(3.22)

Now using formula (3.9), we can see that

\[ M^H_{1,0}(2, K_S, 1) = M^H(2, K_S, 1) \]  

(3.23)

as schemes.

Indeed, a homomorphism $n: H^0(E_i) \to H^1(E_i)$ is an element of the normal cone of $M^H_{1,0}(2, K_S, 1)$ in $M^H(2, K_S, 1)$ if and only if $n = 0$. But this means exactly that the equality (3.23) holds. On the other hand, it means that every $E_i$ has multiplicity 2 in $M^H_{1,0}(2, K_S, 1) = M^H(2, K_S, 1)$. So we get a description of the nilpotent structure of $M^H_{1,0}(2, K_S, 1) = M^H(2, K_S, 1)$, and we can see that under a generic deformation of the Hodge metric $g_H$ on the underlying differentiable structure of $S$, the quadruple of points (3.17)=(3.19) bifurcate to 8 regular instantons. So the Donaldson–Kotschick polynomial of degree 0, which in this case equals half the spin canonical polynomial, is given by

\[ \gamma_{g}^{E_i} = \gamma_{g_H}^{K_S \mod 2,-3} = 8, \quad \text{and} \quad s\gamma_{g_H, 0}^{-3} = 16. \]  

(3.24)

Of course, we consider this example as an application of the description of the scheme structure of the jumping instantons locus in the algebraic geometric situation of the Very Important Case. In the following section we describe some applications of spin canonical polynomials.
4. Applications

We first recall the main constructions, results and conjectures concerning the smooth classification of algebraic surfaces. Every compact nonsingular algebraic surface $S$ over $\mathbb{C}$ defines three underlying structures: its topological class $tS$, its underlying differentiable 4-manifold $dS$, and its deformation class as an algebraic surface $vS$; compare the survey [T1], (5.44).

For any topological 4-manifold $X$, let $\text{diff}(X)$ be the set of differentiable 4-manifolds topologically equivalent to $X$. This set is discrete. Let

$$\text{alg}_k(X) = \{ M \in \text{diff}(X) \mid M = dS, \kappa(S) = k \}$$

be the subset of $\text{diff}(X)$ containing the underlying structures of the algebraic surfaces of Kodaira dimension $k$ and

$$\text{alg}(X) = \bigcup_{k=-\infty}^{2} \text{alg}_k(X) \subset \text{diff}(X)$$

be the subset of the underlying differentiable structures of algebraic surfaces.

**Van de Ven conjecture.**

$$k \neq k' \implies \text{alg}_k \cap \text{alg}_{k'} = \emptyset.$$  

This conjecture has recently been finally settled by Friedman and Qin [F–Q], and independently by Pidstrigach citeP.

Finally, we set

$$\text{var}(X) = \{ vS \mid tS = X \}$$

for the set of the deformation classes of algebraic surfaces of topological type $X$.

By definition there is a surjection

$$f: \text{var}(tS) \to \text{alg}(tS),$$

and the main question is to describe the fibres of $f$,

$$f^{-1}(dS) \subset \text{var}(tS).$$

We have some experience in estimating these sets:

$$\# \text{alg}(t\mathbb{CP}^2) = 1;$$

this is a Corollary of Yau’s theorem, proved independently in [T2].

**Remark.** This result should not be confused with the smooth Poincaré conjecture (see the Introduction), which says that

$$\# \text{diff}(t\mathbb{CP}^2) = 1.$$
see Friedman and Morgan, [F–M]. In the same vein, for every surface of Kodaira dimension 1, we have

\[ \# \text{alg}_1(tS) = \infty. \]

Finally, for surfaces of general type,

\[ \# \text{alg}_2(tS) < \infty \quad (4.8) \]

(see for example [7]).

The structures \((vS, dS, tS)\) corresponds to three groups

\[ \text{Mon} S \subset \text{Mod} S \subset O(qS) \quad (4.9) \]

where \(O(qS)\) is the orthogonal group of the lattice \(H^2(tS, \mathbb{Z})\) with the intersection form \(qS\) (recall that this quadratic form defines the topology type \(tS\) uniquely). \(\text{Mod} S\) is the image of the standard representation of the diffeomorphisms group of \(dS\) preserving the orientation to \(O(qS)\) and \(\text{Mon} S\) is the subgroup of \(O(qS)\) generated by all monodromy automorphisms of all algebraic families of the surface \(S\). The algebraic classification of surfaces is closely related to the differentiable classification of the underlying 4-manifolds (the fibre (4.6) is a measure of this relation) and the subgroup \(\text{Mon} S \subset \text{Mod} S\) can be large enough to describe this relation in some partial cases.

Our first application of the spin canonical polynomials is to (4.3).

**Application 1. Van de Ven conjecture.** Following the results of Friedman and Morgan [F–M], the only case of (4.3) that remains to prove is

\[ \text{alg}_2 \cap \text{alg}_{-\infty} = \emptyset. \quad (4.10) \]

It has been observed many times (see [T3] or [T4]) that all the spin canonical polynomials of a rational surfaces vanish. The reason for this is the following: if a polarisation \(H\) on \(S\) satisfies

\[ K_S \cdot H \leq 0 \]

then a torsion free sheaf \(F\) has either a section or a cosection \(F \to K_S\) that contradicts the stability of \(F\). Hence all the spaces (1.10) with \(c_1 = K_S\) are empty, and all the spin canonical polynomials vanish.

On the other hand, for a surface of general type \(S\), the map

\[ m: S \to S_{\text{min}} \]

to its minimal model \(S_{\text{min}}\), and its collection of exceptional rational curves

\[ \{l_i \subset S \mid l_i \cong \mathbb{P}^1, l_i^2 = -1\}, \quad \text{for } i = 1, \ldots, n \quad (4.11) \]

are uniquely determined. Let \(K_{\text{min}} = m^*(K_{S_{\text{min}}})\) be the pullback of the canonical class of the minimal model.

Consider first the case \(K_S^2 > 0\), that is, either \(S\) is minimal or the number of the blown points is less then \(K_{\text{min}}^2\). Then it is easy to see that a general nontrivial extension of type

\[ 0 \to \mathcal{O}_S \to E \to I(K_{\text{min}}) \to 0 \quad (4.12) \]


with $\xi \in \text{Hilb}^d S$ a general zero-dimensional subscheme of large degree $d$, is $H$-stable for any polarization $H$, and

$$E \in M_{1,0}^H(2, K_S, d) \implies M_{1,0}^H(2, K_S, d) \neq \emptyset$$

We can prove this using the same arguments as in the proof of [T3], Theorem 4.1. Thus, if

$$\dim M_{1,0}^H(2, K_S, d) = \nu \cdot \dim M_{1,0}^H(2, K_S, d)$$

then the spin canonical polynomial is nonzero:

$$s_r^\gamma K_S^{2-4d} \neq 0,$$

and spin canonical polynomials of jumping level 1 distinguish the smooth types of rational surfaces from surfaces of general type.

If equality doesn’t hold in (4.14) then we can use the regularisation procedure described in detail in [T3], §7, and obtain the inequality (4.15), and this again distinguishes a surface of general type $S$ from rational surfaces.

On the other hand, if $K_S^2 < 0$ then for simple arithmetical reasons there exists a vector $\delta$ with

$$\delta \cdot (\delta - K_S) \geq 2(r - 1) \quad \text{and} \quad -K_S^2 < 8(r - 1),$$

and a polarization of the form

$$H = H_{\text{min}} - \sum_{i=1}^n a_i l_i, \quad \text{with } a_i > 0,$$

where $H_{\text{min}}$ is the inverse image of an almost canonical polarization on $S_{\text{min}}$, such that

$$2\delta \cdot H_{\text{min}} < K_S \cdot H_{\text{min}}.$$  

Recall (see [T5]) that a polarisation $H_0$ is almost canonical if the ray $\mathbb{R}^+ \cdot H$ in the projectivised Kähler cone of $S_{\text{min}}$ is close to the ray $\mathbb{R}^+ \cdot K_{\text{min}}$ in the Lobachevsky metric.

Then it is again easy to see that, for some suitable choice of degree $d$, and $\xi \in \text{Hilb}^d S$ a general zero-dimensional subscheme of degree $d$, a general nontrivial extension of type

$$0 \to \mathcal{O}_S(\delta) \to E \to \mathcal{J}_\xi(K_S - \delta) \to 0,$$

is $H$-stable for the polarization (4.17), and

$$E \in M_{r,0}^H(2, K_S, d) \implies M_{r,0}^H(2, K_S, d) \neq \emptyset,$$

in the same vein as (4.12). So if

$$\dim M_{r,0}^H(2, K_S, d) = \nu \cdot \dim M_{r,0}^H(2, K_S, d),$$

then the spin canonical polynomial is nonzero:

$$s_r^\gamma K_S^{2-4d} \neq 0,$$

and the spin canonical polynomial of jumping level $r > \frac{1}{8} (-K_S^2) + 1$ distinguishes a surface of general type $S$ from rational surfaces.

If the equality (4.14′) doesn’t hold, then we could use the regularisation again, although I have not carried this out in detail.
Application 2. The reducibility of \( \text{Mod}_{S} \).

We will consider \( \text{Mod}_{S} \) as a subgroup of the orthogonal group \( O(q_{S}) \), and its representation as a transformation group of \( H^{2}(S, \mathbb{Z}) \). We proved in [T4] that this representation is reducible: if \( p_{g} > 0 \), there exist a proper sublattice \( sV(S) \) invariant under diffeomorphisms. More precisely,

\[
\mathbb{Z} \cdot K_{S} \subset sV(S) \subset \langle K_{S}, C_{1}, \ldots, C_{N} \rangle,
\]

that is \( sV(S) \) contains \( K_{S} \), and is contained in the lattice generated by \( K_{S} \) and the classes of all effective curves \( \{ C_{i} \} \) satisfying the inequalities

\[
2C_{i} \cdot K_{\min} \leq K_{\min}^{2}
\]

These classes are algebraic; hence if \( p_{g} > 0 \) then \( sV(S) \) is a proper sublattice.

Now the main question is:

how close is the sublattice \( \mathbb{Z} \cdot K_{S} \) to \( sV(S) \)?

Other diffeomorphism invariant sublattices are known: perhaps the best approximation to \( \mathbb{Z} \cdot K_{S} \) is the Kronheimer–Mrowka–Witten sublattice

\[
L_{\text{KMW}} = \langle C_{1}, \ldots, C_{k} \rangle
\]

generated by the irreducible components of a general curve of the canonical linear system \( |K_{S}| \) (see [W]). It is easy to see that \( L_{\text{KMW}} \subset sV(S) \).

The method of proving diffeomorphism invariance is the following: the first step is to see the shape of the invariant polynomial. For example, if \( p_{g} > 0 \) then Kronheimer, Mrowka and essentially Witten proved that the Donaldson polynomials

\[
\gamma^{w_{2}(S), p_{1} < 0} \in S^{d}(q_{S}, C_{1}, \ldots, C_{k})
\]

are contained in the subring of \( S^{*}H^{2}(S, \mathbb{Z}) \) generated by the intersection form as a quadratic form and the classes \( C_{i} \) as linear forms. From this, the invariance of the polynomials implies the invariance of the sublattice generated by the collection of linear forms. To see that the same thing holds for the spin sublattice \( sV(S) \) (4.20) is much easier. So here we would like to explain why the spin canonical polynomials belong to the subring

\[
s^{p_{1} < 0} \in S^{d}(q_{S}, C_{1}, \ldots, C_{N}),
\]

where \( \{ C_{i} \} \) is the collection of classes (4.21). To do this informally, it is very convenient to use the vortex equation and the moduli space of stable pairs (see the introduction to [B–D]).

Recall that for any \( \text{U}(2) \)-vector bundle \( E \) on \( S \) (or on any compact Kähler manifold) there is a Yang–Mills–Higgs functional on the space of pairs \( (a, \varphi) \), where \( a \) is a \( \text{U}(2) \)-connection on \( E \) and \( \varphi \) is any section of \( E \), depending on a real parameter \( \tau \):

\[
\text{YMH}_{\tau}(a, \varphi) = \| F_{a} \|^{2} + \| d_{a}'' \varphi \|^{2} + \frac{1}{4} \| \varphi \|_{h}^{2} - \tau
\]

The pair \( (a, \varphi) \) is an absolute minimum of this functional if

\[
d'' \varphi = 0 \quad \text{and} \quad \bigwedge F_{a} = \frac{i}{2} |\varphi|_{h}^{2} + \frac{i}{2} \tau = 0.
\]
This system of differential equations is called the \( \tau \)-vortex equation, and the space \( \mathcal{M}_\tau \) of solutions up to gauge equivalence is called the moduli spaces of \( \tau \)-vortices.

For every class \( \sigma \in H^2(S, \mathbb{Z}) \), let

\[
\deg \sigma = \sigma \cdot [\omega]
\]

where \( \omega \) is the Kähler form.

Now it is convenient to change the parameter \( \tau \):

\[
\sigma = \frac{1}{4\pi} \tau \text{vol } S - \frac{1}{2} \deg E,
\]

(4.25)

(where \( \deg E = \deg c_1(E) \)). Then Bradlow’s Identification Theorem says that

\[
\mathcal{M}_\tau = MP_\sigma
\]

is the moduli space of \( \sigma \)-stable pairs \((E, s)\), where \( E \) is a holomorphic bundle and \( s \) is a holomorphic section of \( E \). Recall that a pair \((E, s)\) is \( \sigma \)-stable if

\[
\deg L < \begin{cases} \frac{1}{2} \deg E - \sigma & \text{if } s \in H^0(L), \\ \frac{1}{2} \deg E + \sigma & \text{otherwise,} \end{cases}
\]

for any line subbundle \( L \subset E \). (4.27)

Each moduli space \( MP_\sigma \) is a family of vector bundles, and hence slant product defines a cohomological correspondence

\[
\mu: H_i(S, \mathbb{Z}) \to H^{4-i}(MP_\sigma),
\]

and a collection of polynomials

\[
\gamma P_\sigma \in S^*H^2(S, \mathbb{Z}).
\]

(4.28)

**Remark.** Actually, to define these polynomials we must either construct the compactification of the moduli spaces \( MP_\sigma \) or use a trick due to Donaldson as in [T3], Lemma 2.1.

Now from the definition it is easy to see that

\[
MP_\sigma \neq \emptyset \implies \frac{1}{2} \deg E \geq \sigma \geq 0,
\]

(4.29)

and for obvious numerical reasons the \( \sigma \)-stability condition (4.27) remains the same (and implies proper stability) for any

\[
\sigma \in (\max(0, \frac{1}{2} \deg E - i - 1), \frac{1}{2} \deg E - i).
\]

(4.30)

So for \( \sigma \) in this interval we get a fixed moduli space \( MP_\sigma \), and as \( \sigma \) varies we have a chain of birational transformations or flips (see [R])

\[
\text{MP}_{\text{max}} \leftrightarrow \cdots \leftrightarrow \text{MP}_1 \leftrightarrow \text{MP}_0.
\]

(4.31)

You can recognise the situation described by Thaddeus and Bertram for the \( \sigma \)-scale of the moduli space of stable pairs on an algebraic curve \( C \) of genus \( g \) (see [Th], Lemma 2.1).
Recall that, in this classical case, the moduli space on the extreme left has a birational regular map

\[ \text{MP}_{\text{max}} \to M_{C, \xi} \]

to the moduli space of stable bundle with fixed determinant \( \xi \) of degree \( 2g - 1 \). In our case we have a map of the same type (now \( c_1(E) = K_S \))

\[ \text{MP}_{\text{max}} \to M^H_{1,0}(2, K_S, d), \quad (4.32) \]

and from this it is easy to see that the polynomial (4.28)

\[ \gamma P_{\text{max}} = s\gamma_0^{K_S^2 - 4d} \quad (4.33) \]

is our spin canonical polynomial of jumping level 1.

Now in the classical case, the moduli space \( \text{MP}_0 \) on the right-hand end of (4.31) is a projective space of dimension \( 3g - 3 \).

\[ \text{MP}_0 = \mathbb{P}^{3g - 3}. \]

In our case \( \text{MP}_0 \) is not so simple. Namely,

\[ \text{MP}_0 = \text{GAM} \]

is the space of all nontrivial extensions of type

\[ 0 \to \mathcal{O}_S \to E \to \mathcal{J}_\xi(K_S) \to 0 \quad (4.34) \]

for all \( \xi \in \text{Hilb}^d S \), up to the action of \( \mathbb{C}^* \) (see [T3, T4, T4]). For large \( d \), this space is birational to the direct product

\[ \text{Hilb}^d S \times \mathbb{P}^n. \]

It is not hard to compute the polynomial \( \gamma P_0 \), and to see that

\[ \gamma P_0 \in S'(q_S, K_S). \quad (4.35) \]

Now we can compute the increment

\[ \gamma P_1 - \gamma P_0, \quad (4.36) \]

For this, we have to blow up the space \( \text{MP}_0 \) (4.34) in the subvariety of all extensions admitting a diagram of type

\[ 0 \to \mathcal{O} \to E \to \mathcal{J}_\xi(K_S) \to 0, \quad (4.37) \]
where $C_1$ is a curve of degree 1, and to carry out the elementary transformation of the universal extension as in [T4], §6. Computing the slant product again for the new family of vector bundles, we get the shape of the increment (4.36)

$$\gamma P_1 - \gamma P_0 \in S^*(q_S, K_S, C_1).$$

Therefore

$$\gamma P_1 \in S^*(q_S, K_S, \{C_1\}),$$

(4.38)

where $\{C_1\}$ is the set of all curves of degree 1.

Continuing this procedure gives the shape of the spin canonical polynomial (4.33).

In fact, we can’t do this construction rigorously because there isn’t any rigorous treatment of the compactification of moduli spaces of $\sigma$-stable pairs on a Kähler surface, and there are other technical problems. But in the classical case, there is the pure algebraic geometric procedure imitating this Kähler geometric procedure proposed by Bertram in [Be]. We can do the same for an algebraic surface (see [T4]).

Moreover, Huybrechts and Lehn in [H–L] proposed a beautiful algebraic geometric theory of stable pairs on a surface. First of all, they construct compactifications of moduli spaces of stable pairs by torsion free sheaves. After this, in place of the real number $\sigma$ (4.25), it is very natural to consider a polynomial $\delta(z)$ with rational coefficients such that $\delta > 0$ for all $z \gg 0$ (in fact $\sigma$ is the first coefficient of $\delta$). Now we have an exact order in the sequence of flips (4.31), and the elementary transformations of the universal sheaves such as (3.37).

Our last remark is the following: we can prove the reducibility of $\text{Mod} S$ even if $p_g = 0$, provided that $K_S^2 > 0$ and the sublattice $sV(S)$ (4.20) is proper. For this, it is enough to verify that the sublattice $\langle K_S, C_1, \ldots, C_N \rangle$ (see (4.20–21)) is proper.

**Example. The Barlow surface.** The Barlow surface $S$ is minimal, and has four smooth rational $-2$-curves $C_1, C_2, C_3, C_4$. Thus

$$C \cdot K_S \leq K_S^2 = 1 \implies C = \sum_{i=1}^{4} n_i C_i, \quad \text{with} \quad n_i \geq 0.$$ 

But $\text{rank Pic} S = 9$, and therefore

$$\langle K_S, C_1, C_2, C_3, C_4 \rangle \subset \text{Pic} S = H^2(S, \mathbb{Z})$$

is a proper sublattice.

**Remark.** The diffeomorphism invariant sublattice $sV(S)$ (4.20) is a sublattice of $\langle K_S, C_1, C_2, C_3, C_4 \rangle$, a priori, a proper sublattice. This holds for the Barlow surface. Actually, the exact computation shows that

$$sV(S) = \mathbb{Z} \cdot K_S$$

(compare the computations for $-2$-curves in [P–T], Chap. 4, §3). Thus the canonical class of the Barlow surface is diffeomorphism invariant. This means that the following conjecture is now verified for all currently known algebraic surfaces.
Conjecture. 1) Every simply connected surface of general type $S$ has a proper sublattice $sV(S) \subset H^2(S, \mathbb{Z})$ invariant under diffeomorphisms; that is,

$$dS' \in \text{diff}(dS) \implies sV(S) = sV(S').$$

Moreover, $sV(S)$ satisfies

$$2) \quad sV(S) \subset \bigcap_{dS' \in \text{diff}(dS)} \text{Pic} S';$$

and

$$3) \quad \bigcup_{dS' \in \text{diff}(dS)} K_{S'} \subset sV(S).$$

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