Quantum Chains of Hopf Algebras
with Order–Disorder Fields
and Quantum Double Symmetry

Florian Nill*

Institute for Theoretical Physics, FU Berlin
Arnimallee 14, D-1000 Berlin 33, Germany

Kornél Szlachányi**

Central Research Institute for Physics
H-1525 Budapest 114, P.O.B. 49, Hungary

Abstract: Given a finite dimensional $C^*$-Hopf algebra $H$ and its dual $\hat{H}$ we construct the infinite crossed product $\mathcal{A} = \ldots \triangleright H \triangleright \hat{H} \triangleright H \triangleright \ldots$ and study its representations. $\mathcal{A}$ is the observable algebra of a generalized spin model with $H$-order and $\hat{H}$-disorder symmetries. By pointing out that $\mathcal{A}$ possesses a certain compressibility property we can classify all DHR-sectors of $\mathcal{A}$ — relative to some Haag dual vacuum representation — and prove that their symmetry is described by the Drinfeld double $\mathcal{D}(H)$. Complete, irreducible, translation covariant field algebra extensions $\mathcal{F} \supset \mathcal{A}$ are shown to be in one-to-one correspondence with cohomology classes of 2-cocycles $u \in \mathcal{D}(H) \otimes \mathcal{D}(H)$.

* E-mail: NILL@omega.physik.fu-berlin.de
Supported by the DFG, SFB 288 "Differentialgeometrie und Quantenphysik"

** E-mail: SZLACH@rmki.kfki.hu
Supported by the Hungarian Scientific Research Fund, OTKA–1815.
1. Introduction and summary of the results

Quantum chains considered as models of 1 + 1-dimensional quantum field theory exhibit many interesting features that are either impossible or unknown in higher (2 + 1 or 3 + 1) dimensions. These features include integrability on the one hand and the emergence of braid group statistics and quantum symmetry on the other hand. The present paper deals with the latter, the problem of quantum symmetry of the superselection sectors in a wide class of quantum chains: the Hopf spin models.

Quantum chains in which a quantum group acts are well known for some time; for example the XXZ-chain with the action of $sl(2)_q$ [P,PS] or the lattice Kac–Moody algebras of [AFS]. For a recent paper on the general action of quantum groups on ultralocal quantum chains see [FNW]. However the discovery that quantum symmetries are described — even in the simplest models — by truncated quasi-Hopf algebras [MS,S] may constitute an obstruction towards such an approach because the field algebras are either non-associative or do not obey commutation relations with $c$-number coefficients, both properties being automatically assumed in any decent quantum chain.

Here we stress the point of view that an unbiased approach to reveal the quantum symmetry of a model must be based only on the knowledge of the quantum group invariant operators (the "observables") that obey local commutation relations. This is the approach of algebraic quantum field theory (AQFT) [H]. The importance of the algebraic method, in particular the DHR theory of superselection sectors [DHR], in low dimensional QFT has been realized by many authors (see [FRS,BMT,FröGab,F,R] and many others).

The implementation of the DHR theory to quantum chains has been carried out at first for the case of $G$-spin models in [SzV]. These models have an order-disorder type of quantum symmetry given by the double $D(G)$ of a finite group $G$ which generalizes the $Z(2) \times Z(2)$ symmetry of the lattice Ising model. Since the disorder part of the double (i.e. the function algebra $C(G)$) is always Abelian, $G$-spin models cannot be selfdual in the Kramers-Wannier sense, unless the group is Abelian. Non-Abelian Kramers-Wannier duality can therefore be expected only in a larger class of models.

Here we shall investigate the following generalization of $G$-spin models. On each lattice site there is a copy of a finite dimensional $C^*$-Hopf algebra $H$ and on each link there is a copy of its dual $\hat{H}$. Non-trivial commutation relations are postulated only between neighbor links and sites where $H$ and $\hat{H}$ act on each other in the "natural way", so as the link-site and the site-link algebras to form the crossed products ("Weyl algebras" in the terminology of [N]) $W(\hat{H}) \equiv \hat{H} \rtimes H$ and $W(H) \equiv H \rtimes \hat{H}$. The two-sided infinite crossed product $\ldots \rtimes H \rtimes \hat{H} \rtimes H \rtimes \hat{H} \rtimes \ldots$ defines the observable algebra $\mathcal{A}$ of the Hopf spin model. Its superselection sectors (more precisely those that correspond to charges localized within a finite interval $I$, the so called DHR sectors) can be created by localized amplimorphisms $\mu: \mathcal{A} \rightarrow \mathcal{A} \otimes \text{End } V$ with $V$ denoting some finite dimensional Hilbert space. The category of amplimorphisms $\text{Amp.} \mathcal{A}$ plays the same role in locally finite dimensional theories
as the category $\text{End} \mathcal{A}$ of endomorphisms in continuum theories. The symmetry of the superselection sectors can be revealed by finding the “quantum group” $\mathcal{G}$ the representation category of which is equivalent to $\text{Amp} \mathcal{A}$. In our model we find that $\mathcal{G}$ is the Drinfeld double (also called the quantum double) $\mathcal{D}(H)$ of $H$.

Finding all endomorphisms or all amplimorphisms of a given observable algebra $\mathcal{A}$ can be a very difficult problem in general. In the Hopf spin model $\mathcal{A}$ possesses a property we call *compressibility*, which allows to do so. Namely if $\mu$ is an amplimorphism creating some charge on an arbitrary large but finite interval then there exists an amplimorphism $\nu$ creating the same charge (i.e. $\nu$ is equivalent to $\mu$, written $\nu \sim \mu$) but within an interval $I$ of length 2 (i.e. $I$ consists of a neighbouring site–link pair). Therefore the problem of finding all DHR-sectors of the Hopf spin model is reduced to a finite dimensional problem, namely to find all amplimorphisms localized within an interval of length 2. In this way we have proven that all DHR-sectors of $\mathcal{A}$ can be classified by representations of the Drinfeld double.

An important role in this reconstruction is played by the so-called *universal* amplimorphisms in $\text{Amp} \mathcal{A}$. These are amplimorphisms $\rho: \mathcal{A} \to \mathcal{A} \otimes \mathcal{D}(H)$ such that for any object $\mu$ in $\text{Amp} \mathcal{A}$ there exists an — up to equivalence unique — representation $\beta_\mu$ of the double $\mathcal{D}(H)$ such that $\mu \sim (\text{id}_\mathcal{A} \otimes \beta_\mu) \circ \rho$. Moreover, we find that universal amplimorphisms $\rho$ can be chosen in such a way that they provide *coactions* of $\mathcal{D}(H)$ on $\mathcal{A}$, i.e. they satisfy the equations

\begin{align}
(\rho \otimes \text{id}) \circ \rho &= (\text{id}_\mathcal{A} \otimes \Delta_D) \circ \rho \quad (1.1a) \\
(\text{id}_\mathcal{A} \otimes \varepsilon_D) \circ \rho &= \text{id}_\mathcal{A} \quad (1.1b)
\end{align}

where $\Delta_D: \mathcal{D}(H) \to \mathcal{D}(H) \otimes \mathcal{D}(H)$ is the coproduct and $\varepsilon_D: \mathcal{D}(H) \to \mathcal{C}$ is the counit on $\mathcal{D}(H)$. The quasitriangular $R$-matrix can be determined by computing the statistics operator of $\rho$ giving

$$\epsilon(\rho, \rho) = \mathbb{1} \otimes P^{12} R \in \mathcal{A} \otimes \mathcal{D}(H) \otimes \mathcal{D}(H) \quad (1.2)$$

where $P^{12}$ is the usual transposition. The antipode $S_D$ can be recovered by studying conjugate objects $\tilde{\rho}$ and intertwiners $\rho \times \tilde{\rho} \to \text{id}_\mathcal{A}$. The statistical dimensions $d_r$ of the irreducible components $\rho_r$ of $\rho$ are integers: they coincide with the dimensions of the corresponding irreducible representation $D_r$ of $\mathcal{D}(H)$. The statistics phases can be obtained from the universal balancing element $s = S_D(R_2)R_1 \in \text{Center} \mathcal{D}(H)$ evaluated in the representations $D_r$.

We emphasize that having established the equivalence $\text{Amp} \mathcal{A} \cong \text{Rep} \mathcal{D}(H)$ does not mean that the double $\mathcal{D}(H)$ could be reconstructed from the observable algebra as a *unique* Hopf algebra. Only as a $C^\ast$-algebra (together with a distinguished 1-dimensional representation $\varepsilon$) it is uniquely determined. However the quasitriangular Hopf algebra structure on $\mathcal{D}(H)$ can be recovered only up to a twisting by a 2-cocycle: If $u \in \mathcal{G} \otimes \mathcal{G}$ is a 2-cocycle, i.e. a unitary satisfying

\begin{align}
(u \otimes \mathbb{1}) \cdot (\Delta_D \otimes \text{id})(u) &= (\mathbb{1} \otimes u) \cdot (\text{id} \otimes \Delta_D)(u), \\
(\varepsilon_D \otimes \text{id})(u) &= (\text{id} \otimes \varepsilon_D)(u) = \mathbb{1}
\end{align}

(1.3)
then the twisted double with data
\[
\Delta' = \text{Ad} u \circ \Delta \\
\varepsilon' = \varepsilon \\
S' = \text{Ad} q \circ S \\
R' = u^{op} Ru^*
\]
is as good for a symmetry as the original one. In fact, we prove in Section 4 that for all 2-cocycles \( u \) there is a universal coaction \( \rho' \) satisfying (1.1) with \( \Delta' \) instead of \( \Delta_D \). Vice versa, any universal coaction \( \rho' \) is equivalent to a fixed one by an isometric intertwiner \( U \in \mathcal{A} \otimes G \), \( \text{id}_A \otimes \varepsilon_D(U) = 1_1 \), \( \rho'(A) = U \rho(A) U^* \), \( A \in \mathcal{A} \), satisfying a twisted cocycle condition
\[
(U \otimes 1) \cdot (\rho \otimes \text{id}_G)(U) = (1_1 \otimes u) \cdot (\text{id}_A \otimes \Delta_D)(U), \tag{1.4}
\]
implies the identities (1.3) for \( u \). We point out that (1.4) is a generalization of the usual notion of cocycle equivalence for coactions where one requires \( u = 1_1 \otimes 1 \) [NaTa,BaSk,E].

This type of reconstruction of the quasitriangular Hopf algebra \( D(H) \) is a special case of the generalized Tannaka-Krein theorem [U,Maj1]. Namely, any faithful functor \( F: \mathcal{C} \rightarrow \text{Vec} \) of strict monoidal braided rigid \( C^* \)-categories to the category of finite dimensional vector spaces factorizes as \( F = f \circ \Phi \) to the forgetful functor \( f \) and to an equivalence \( \Phi \) of \( \mathcal{C} \) to the representation category \( \text{Rep} \mathcal{G} \) of a quasitriangular \( C^* \)-Hopf algebra \( \mathcal{G} \). In our case \( \mathcal{C} \) is the category \( \text{Amp} \mathcal{A} \) of amplimorphisms of the observable algebra \( \mathcal{A} \). The functor \( F \) to the vector spaces is given naturally by associating to the amplimorphism \( \mu: \mathcal{A} \rightarrow \mathcal{A} \otimes \text{End} V \) the vector space \( V \). Although the vector spaces \( V \) cannot be seen looking at only the abstract category \( \text{Amp} \mathcal{A} \), they are ”inherently” determined by the amplimorphisms and therefore by the observable algebra itself. In this respect using amplimorphisms one goes somewhat beyond the Tannaka-Krein theorem and approaches a Doplicher-Roberts [DR] type of reconstruction.

Section 4 is devoted to the construction and classification of field algebra extensions \( F \supset A \). Here a \( C^* \)-algebra extension \( F \supset A \) is called a complete irreducible field algebra over \( A \) if (I) \( F \) has sufficiently many fields in order to write any amplimorphism \( \mu: \mathcal{A} \rightarrow \mathcal{A} \otimes \text{End} V \) as \( \mu(A) = F(A \otimes 1_V) F^* \) for some unitary \( F \in F \otimes \text{End} V \), (II) the relative commutant of \( A \) in \( F \) is trivial, \( A' \cap F = C^* \mathbb{1} \), (III) there exists a conditional expectation \( \mathcal{E}: F \rightarrow A \) onto \( A \) with finite index, and (IV) \( F \) is minimal under conditions (I–III). For our Hopf spin model we classify all such field algebra extensions and find that, up to equivalence, they are in one-to-one correspondence with cohomology classes of 2-cocycles (1.3). All these field algebras arise as crossed products \( F = A \rtimes \hat{G} \) with respect to some coaction \( (\rho, \Delta) \), where \( \hat{G} \) is the Hopf algebra dual to \( (\mathcal{G}, \Delta) \). A convenient presentation for them can be given by a unitary master field \( F \in F \otimes \mathcal{G} \) satisfying
\[
F(A \otimes 1) = \rho(A) F \quad A \in \mathcal{A} \\
F^{01} F^{02} = (\text{id} \otimes \Delta)(F) \\
F^* = (\text{id} \otimes S)(F) \equiv F^{-1} \tag{1.5}
\]
where $F^{0i}$, for $i = 1, 2$, denotes the obvious embedding of $F$ into $A \otimes \mathcal{G} \otimes \mathcal{G}$. In other words the map $\hat{\mathcal{G}} \ni \xi \mapsto F(\xi) \equiv (\text{id}_A \otimes \xi)(F) \in \mathcal{F}$ provides a $*$-algebra inclusion. The fields $F(\xi)$ are precisely the generalizations of the order-disorder fields in [SzV].

The field algebras $\mathcal{F}$ carry a natural action $\gamma$ of the Hopf algebra $\mathcal{G}$ as a global gauge symmetry such that $A$ coincides with the subalgebra of $\mathcal{G}$-invariant elements:

$$\{ A \in \mathcal{F} \mid \gamma_X(A) = A \varepsilon(X), \forall X \in \mathcal{G} \} = A.$$ 

If $\xi = D^{ij}_{r} \in \hat{\mathcal{G}}$ are chosen to be the matrix elements in an irreducible representation of $\mathcal{G}$ then the fields $F^{ij}_{r} \equiv F(D^{ij}_{r})$ provide a $\mathcal{G}$-covariant matrix multiplet of field operators.

Studying the problem of translation covariance of field algebras we find that in the Hopf spin model all universal coactions are translation covariant, i.e. there exists an isometric intertwiner $U$ from $\rho$ to its translate $\rho^\alpha$ satisfying the cocycle condition (1.4) with $u = 1 \otimes 1$. This guarantees that on the corresponding field algebras $\mathcal{F}$ there exists an extension of the translation automorphism $\alpha$ commuting with the global symmetry action $\gamma$.

We also show that the master field $F$ and its translated images $F^\prime = (\alpha^n \otimes \text{id}_\mathcal{G})(F)$, $n \in \mathbb{N}$, satisfy braided commutation relations given by

$$F^\prime_{01}F_{02} = F_{02}F^\prime_{01}(1 \otimes R)$$

where the quasitriangular $R$-matrix is given by (1.2).

Our result showing that there are as many complete irreducible translation covariant field algebras as cohomology classes of 2-cocycles must be compared with the result of [DR] on the uniqueness of the field algebra. This apparent discrepancy may be explained by viewing group theory within the more general setting of Hopf algebra theory. In the higher dimensional situation with group symmetry one has a preferred choice of the coproduct and the $R$-matrix, while in our case one has to consider all possible ones and there is no analogue of the normal commutation relations of [DR]. This causes that we encounter a whole family of inequivalent field algebra extensions, among which one cannot make any “observable” distinction.

2. The structure of the observable algebra

2.1. $\mathcal{A}$ as an iterated crossed product

In this section we describe a canonical method by means of which one associates an observable algebra $\mathcal{A}$ on the 1-dimensional lattice to any $C^*$-Hopf algebra $H$. Although a good deal of our construction works for infinite dimensional Hopf algebras as well, we restrict the discussion here to the finite dimensional case.

Consider $\mathbb{Z}$, the set of integers, as the set of cells of the 1-dimensional lattice: even integers represent lattice sites, the odd ones represent links. Let
\[ H = (H, \Delta, \varepsilon, S, \ast) \] be a finite dimensional \( C^* \)-Hopf algebra (see Appendix A). We denote by \( \hat{H} \) the dual of \( H \) which is then also a \( C^* \)-Hopf algebra. We denote the structural maps of \( \hat{H} \) also by \( \Delta, \varepsilon, S \), Elements of \( H \) will be typically denoted as \( a, b, \ldots \), while those of \( \hat{H} \) by \( \varphi, \psi, \ldots \). The canonical pairing between \( H \) and \( \hat{H} \) is denoted by \( a \in H, \varphi \in \hat{H} \mapsto \langle a, \varphi \rangle \equiv \langle \varphi, a \rangle \). There are natural actions of \( H \) on \( \hat{H} \) and \( \hat{H} \) on \( H \) given by the Sweedler’s arrows:

\[
\begin{align*}
    a \to \varphi &= \varphi(1) \langle \varphi(2), a \rangle \\
    \varphi \to a &= a(1) \langle a(2), \varphi \rangle
\end{align*}
\]  

(2.1)

Let us associate to each even integer \( 2i \) a copy \( A_{2i} \) of the \( C^* \)-algebra \( H \) and to each odd integer \( 2i+1 \) a copy \( A_{2i+1} \) of \( \hat{H} \). We specify isomorphisms \( a \in H \mapsto A_{2i}(a) \in A_{2i} \) and \( \varphi \in \hat{H} \mapsto A_{2i+1}(\varphi) \in A_{2i+1} \) such that \( A_{2i}(a) \mapsto A_{2j}(a), a \in H \) and \( A_{2i+1}(\varphi) \mapsto A_{2j+1}(\varphi), \varphi \in \hat{H} \) are isomorphisms for each integer \( i \) and \( j \). The algebra \( A \) of observables is — by definition — generated by the 1-point localized algebras \( A_i \),

\[ A = C^* \langle A_i, \; i \in \mathbb{Z} \rangle \]  

(2.2)

and subjected to the following commutation relations: On the one hand

\[
[A, B] = 0 \quad A \in A_i , \quad B \in A_j \quad \text{whenever} \quad |i - j| \geq 2 .
\]  

(2.3a)

On the other hand neighbour algebras are required to generate the crossed products \( A_i \rtimes A_{i+1} \cong H \rtimes \hat{H} \) if \( i \) is even and \( \cong \hat{H} \rtimes H \) if \( i \) is odd. These crossed products are understood with respect to the actions 2.1. More explicitely we postulate the relations

\[
\begin{align*}
    A_{2i+1}(\varphi)A_{2i}(a) &= A_{2i}(a(1))\langle a(2), \varphi(1) \rangle A_{2i+1}(\varphi(2)) \\
    A_{2i}(a)A_{2i-1}(\varphi) &= A_{2i-1}(\varphi(1))\langle \varphi(2), a(1) \rangle A_{2i}(a(2))
\end{align*}
\]  

(2.3b)

These relations allow to order any monomial in the \( A_i \)-s in increasing order with respect to their location \( i \). The existence of the antipode ensures that we can invert relations 2.3b and express everything in terms of decreasingly ordered monomials. The above relations define what can be called the iterated crossed product algebra

\[ A = \ldots \rtimes H \rtimes \hat{H} \rtimes H \rtimes \hat{H} \ldots \]  

(2.4)

where the dots include a \( C^* \)-inductive limit procedure. For an interval \( I = \{i, i+1, \ldots, i+n\} \) we denote by \( A(I) \) the subalgebra generated by the \( A_j \) with \( j \in I \). Elements of \( A(I) \) are called the observables localized within \( I \). An important property of these algebras is that \( A(I) \) is simple for all interval \( I \) of even length.

\[ A(I) \cong M_N^{\otimes |I|/2} , \; \; I \in \mathcal{I} , \; \; |I| = \text{even} , \]  

(2.5)

where \( N = \dim H \) and \( M_N \) denotes the algebra of \( N \times N \) complex matrices. For \( I \) of length 2 \((2.5)\) follows from the fact \([N]\) that the 2-point algebras \( A_i \rtimes A_{i+1} \) are isomorphic to \( \text{End} \, H \) or \( \text{End} \hat{H} \), respectively.
There is an other way to formulate the commutation relations 2.3b using the multiplicative unitaries of [BaSk]. Choosing a basis \( \{b^s\} \) of \( H \) and denoting by \( \{\beta^s\} \) the dual basis of \( \hat{H} \), i.e. \( \langle \beta^s, b^t \rangle = \delta^t_s \), we find that the unitary elements

\[
V_{2i,2i+1} := \sum_s A_{2i}(b^s) \otimes A_{2i+1}(\beta^s)
\]

\[
V_{2i-1,2i} := \sum_s A_{2i-1}(\beta^s) \otimes A_{2i}(b^s)
\]

of \( \mathcal{A} \otimes \mathcal{A} \) satisfy the relations

\[
V_{i,i+1}^{12} V_{i,i+1}^{23} = V_{i,i+1}^{23} V_{i,i+1}^{13} V_{i,i+1}^{12},
\]

\[
V_{i,i+1}^{13} V_{i-1,i}^{12} = V_{i-1,i}^{12} V_{i,i+1}^{23} V_{i,i+1}^{13}
\]

(2.7)

We also mention an other interesting property of the net \( \mathcal{A} \) related to Jones’ basic construction [J]. Let \( I \subset J \subset K \) be three intervals of length \( n, n+1, n+2 \), respectively, such that either the left or the right endpoints of all the three intervals coincide. Then the algebras \( (\mathcal{A}(I), \mathcal{A}(J), \mathcal{A}(K)) \) form a Jones’ triple, i.e. the algebra \( \mathcal{A}(K) \) arises as the basic construction associated to the inclusion \( \mathcal{A}(I) \subset \mathcal{A}(J) \).

2.2. \( \mathcal{A} \) as a Haag dual net

The local commutation relations (2.3) of the observables suggests that our Hopf spin model can be viewed in the more general setting of algebraic quantum field theory (AQFT) as a local net. More precisely we will use an implementation of AQFT appropriate to study lattice models in which the local algebras are finite dimensional. Although we borrow the language and philosophy of AQFT, the concrete mathematical notions we need on the lattice are quite different from the analogue notions one uses in QFT on Minkowski space.

Let \( \mathcal{I} \) denote the set of non-empty finite subintervals of \( \mathbb{Z} \). A net of finite dimensional \( C^* \)-algebras, or shortly a net is a correspondence \( I \mapsto \mathcal{A}(I) \) associating to each interval \( I \in \mathcal{I} \) a finite dimensional \( C^* \)-algebra together with unital inclusions \( \iota_{J,I} : \mathcal{A}(I) \to \mathcal{A}(J) \), whenever \( I \subset J \), such that for all \( I \subset J \subset K \) one has \( \iota_{K,J} \circ \iota_{J,I} = \iota_{K,I} \).

The inclusions \( \iota_{J,I} \) will be suppressed and for \( I \subset J \) we will simply write \( \mathcal{A}(I) \subset \mathcal{A}(J) \). If \( \Lambda \) is any (possibly infinite) subset of \( \mathbb{Z} \) we write \( \mathcal{A}(\Lambda) \) for the \( C^* \)-inductive limit of \( \mathcal{A}(I) \)-s with \( I \subset \Lambda \). Especially let \( \mathcal{A} = \mathcal{A}(\mathbb{Z}) \).

For \( \Lambda \subset \mathbb{Z} \) let \( \Lambda' = \{ i \in \mathbb{Z} \mid \text{distance}(i, \Lambda) \geq 2 \} \) which is the analogue of “spacelike complement” of \( \Lambda \). The net \( \{ \mathcal{A}(I) \} \) is called local if \( I \subset J' \) implies \( \mathcal{A}(I) \subset \mathcal{A}(J') \), where \( B' \) for a subalgebra \( B \) of the global algebra \( \mathcal{A} \) denotes its relative commutant within \( \mathcal{A} \).

The net \( \{ \mathcal{A}(I) \} \) is said to satisfy (algebraic) Haag duality if

\[
\mathcal{A}(I')' = \mathcal{A}(I) \quad \forall I \in \mathcal{I}.
\]

(2.8)
The net \( \{A(I)\} \) is called split if for all \( I \in \mathcal{I} \) there exists a \( J \in \mathcal{I} \) such that \( J \supset I \) and \( A(J) \) is simple.

If \( I = \{i, i+1, \ldots, j\} \) then we write \( A_{i,j} \) for \( A(I) \) and \( A_i \) for \( A_{i,i} \). Sometimes we use the convention \( A_{i,j} := C1 \) if \( i > j \). The net \( \{A(I)\} \) is called additive if \( A(I) \) is generated by \( \{A_i \mid i \in I\} \).

The local observable algebras \( \{A(I)\} \) of the Hopf spin model defined in subsection 2.1 provide an example of a local additive split net (see Eqns (2.2–5).) What is not so obvious that this net satisfies algebraic Haag duality. This follows from the \( \eta \)-property described below.

If \( \Lambda \) is finite we can choose a system \( \{E_{\alpha}^{ab}\} \) of matrix units for \( A(\Lambda) \) and define the map

\[
\eta_{\Lambda}(A) := \frac{1}{n_\alpha} \sum_{a,b=1}^{n_\alpha} E_{\alpha}^{ab} A E_{\alpha}^{ba}
\]

which can be seen to be a conditional expectation \( \eta_{\Lambda} : A \to A(\Lambda)' \). If \( tr \) is any trace state on \( A \) then

\[
tr(B \eta_{\Lambda}(A)) = tr(BA)
\]

for all \( A \in A \) and \( B \in A(\Lambda)' \). If the net is split then \( A \) is an UHF algebra and there is a unique faithful trace state \( tr \) on \( A \). Hence \( \eta_{\Lambda} \) is the orthogonal projection onto \( A(\Lambda)' \) with respect to the Hilbert–Schmidt scalar product \( \langle A|B \rangle = tr A^* B \). In order to prove Haag duality we need a kind of orthogonality between the ”hyperplanes” \( A(I) \) and \( A(J)' \) if none of the intervals \( I \) and \( J \) contains the other. More precisely we need the

\[\eta\text{-property ::}
\eta_i(A_{i+1,j}) = A_{i+2,j} & i < j
\eta_i(A_{j-1,i}) = A_{j-2,i} & i > j
\]

where \( \eta_i : A \to A_i' \) is the conditional expectation defined in (2.9).

**Proposition 2.1.** Let \( \{A(I)\} \) be a local net satisfying the \( \eta \)-property. Then the net satisfies Haag duality and wedge duality, i.e.

\[
A(I)' = A(I)
A(W)' = A(W)
\]

for all intervals \( I \in \mathcal{I} \) and for all wedge regions \( W = \{i, i+1, \ldots\} \) or \( W = \{\ldots, i-1, i\} \).

**Proof:** For \( \Lambda \) being a wedge \( W \) or an interval \( I \in \mathcal{I} \) we can define an \( \eta_{\Lambda}' \) as follows:

\[
\eta_{-\infty,i}(A) := \lim_{j \to -\infty} \eta_i \circ \eta_{i-1} \circ \ldots \circ \eta_j (A)
\eta_{i,\infty}(A) := \lim_{j \to \infty} \eta_i \circ \eta_{i+1} \circ \ldots \circ \eta_j (A)
\eta_I'(A) := \eta_{-\infty,i} \circ \eta_{j,\infty}(A) \quad \text{if } I = \{i+2, \ldots, j-2\}
\]

These infinite products of \( \eta_k \)-s exists on strictly local operators \( A \in A_0 := \cup_I A(I) \) because the sequences under \( \lim_j \) become eventually constants. Now the \( \eta \)-property implies that \( \eta_{\Lambda'}(A_0) \subset A(\Lambda) \). Since each \( \eta_k \) is positive and \( \eta_k(1) = 1 \),
the same hold for their limits \( \eta_{\Lambda'} \). Hence \( \eta_{\Lambda'} \) is continuous and can be extended to \( \mathcal{A} \). The extension also satisfies (2.10) by an \( \epsilon/3 \)-argument.

If \( B \in \mathcal{A}(\Lambda')' \) then for \( k \in \Lambda' \) we have \( B \in \mathcal{A}_k \) therefore \( \eta_k(B) = B \). Since \( \eta_{\Lambda'} \) is a product of \( \eta_k \)-s with \( k \in \Lambda' \), we find that \( \eta_{\Lambda'}(B) = B \). This proves \( \mathcal{A}(\Lambda')' \subset \mathcal{A}(\Lambda) \).

If \( B \in \mathcal{A}(\Lambda) \) then by locality \( B \in \mathcal{A}(\Lambda')' \). This proves \( \mathcal{A}(\Lambda) \subset \mathcal{A}(\Lambda')' \). Q.e.d.

In order to apply this result to the Hopf spin model we need to show that the \( \eta \)-property holds true in this case. The crossed product structure of the local algebras \( \mathcal{A}_{i,j} \) implies that every \( A \in \mathcal{A}_{i,j} \) is a linear combination of monomials

\[
A = A_i A_{i+1} \ldots A_j \quad \text{where} \quad A_k \in \mathcal{A}_k.
\]

(2.11)

In this situation the \( \eta \)-property is equivalent to

\[
\eta_i(\mathcal{A}_{i\pm 1}) = C'1.
\]

(2.12)

Let us prove (2.12) for \( i \) -even. (For odd \( i \)-s the proof is quite analogous.) Choose \( C^* \)-matrix units \( e_r^{ab} \) of the algebra \( H \). Then one can show that the coproduct of the integral (see Appendix A) \( z = e_0 \) takes the form

\[
\Delta(z) = \sum_r \frac{1}{n_r} \sum_{a,b} e_r^{ab} \otimes e_r^{ab}
\]

(2.13)

from which one recognizes that \( \eta_i \) evaluated on \( \mathcal{A}_{i\pm 1} \) is nothing but the adjoint action of the integral \( z \) on the dual Hopf algebra \( \hat{H} \). Consider the case of \( \mathcal{A}_{i+1} \):

\[
\eta_i(A_{i+1}(\varphi)) = \sum_r \frac{1}{n_r} \sum_{a,b} A_i(e_r^{ab}) A_{i+1}(\varphi) A_i(e_r^{ba})
\]

\[
= A_i(S(z(1))) A_{i+1}(\varphi) A_i(z(2)) = A_i(S(z(1)) z(2)) A(\varphi(2)) \langle \varphi(1) \vert z(3) \rangle
\]

\[
= A_{i+1}(\varphi \leftarrow z) = 1 \langle \varphi \vert z \rangle
\]

The case of \( \mathcal{A}_{i-1} \) can be handled similarly. This concludes the proof of the \( \eta \)-property for the Hopf spin model.

Summarizing: The local net \( \{ \mathcal{A}(I) \} \) of the Hopf spin model is an additive split net satisfying Haag duality and wedge duality. Furthermore the global observable algebra \( \mathcal{A} \) is simple, because the split property implies that \( \mathcal{A} \) is an UHF algebra and every UHF algebra is simple [Mu].

3. Amplimorphisms and comodule algebra actions

3.1. The categories \( \text{Amp} \, \mathcal{A} \) and \( \text{Rep} \, \mathcal{A} \)

In this subsection \( \{ \mathcal{A}(I) \} \) denotes a split net of finite dimensional \( C^* \)-algebras which satisfies algebraic Haag duality. Furthermore we assume that the the net
is translation covariant. That is the net is equipped with a *-automorphism $\alpha \in \text{Aut} A$ such that

$$\alpha(A(I)) = A(I + 2) \quad I \in \mathcal{I}. \quad (3.1)$$

At first we recall some notions introduced in [SzV]. An amplimorphism $\mu$ of $A$ is an injective C*-map

$$\mu : A \rightarrow A \otimes \text{End} V \quad (3.2)$$

where $V$ is some finite dimensional Hilbert space. If $\mu(1) = 1 \otimes 1_V$ then $\mu$ is called unital. Here we will restrict ourselves to unital amplimorphisms since the localized amplimorphisms in a split net are all equivalent to unital ones (see Thm. 4.13 in [SzV]). An amplimorphism $\mu$ is called localized within $I \in \mathcal{I}$ if

$$\mu(A) = A \otimes 1_V \quad A \in A(I^c) \quad (3.3)$$

where $I^c := \mathbb{Z} \setminus I$. The space of intertwiners from $\nu : A \rightarrow A \otimes \text{End} W$ to $\mu : A \rightarrow A \otimes \text{End} V$ is

$$(\mu|\nu) := \{T \in A \otimes \text{Hom}(W, V) | \mu(A)T = T\nu(A), \ A \in A\} \quad (3.4)$$

$\mu$ and $\nu$ are called equivalent, $\mu \sim \nu$, if there exists an isomorphism $U \in (\mu|\nu)$, that is an intertwiner $U$ satisfying $U^*U = 1 \otimes 1_W$ and $UU^* = 1 \otimes 1_V$. Let $\mu$ be localized within $I$. Then $\mu$ is called transportable if for all integer $a$ there exists a $\nu$ localized within $I + 2a$ and such that $\nu \sim \mu$. $\mu$ is called $\alpha$-covariant if $(\alpha^a \otimes \text{id}_V) \circ \mu \circ \alpha^{-a} \sim \mu$ for all $a \in \mathbb{Z}$.

Let $\text{Amp} A$ denote the category with objects the localized unital amplimorphisms $\mu$ and with arrows from $\nu$ to $\mu$ the intertwiners $T \in (\mu|\nu)$. This category has the following monoidal product:

$$\mu, \nu \mapsto \mu \times \nu := (\mu \otimes \text{id}_{\text{End} W}) \circ \nu : A \rightarrow A \otimes \text{End} V \otimes \text{End} W$$

$$T_1 \in (\mu_1|\nu_1), T_2 \in (\mu_2|\nu_2) \mapsto T_1 \times T_2 := (T_1 \otimes 1_{V_2})(\nu_1 \otimes \text{id}_{\text{Hom}(W, V_2)})(T_2) \quad (3.5)$$

with the monoidal unit being the trivial amplimorphism $\text{id}_A$. The monoidal product $\times$ is a bifunctor therefore we have $(T_1 \times T_2)(S_1 \times S_2) = T_1S_1 \times T_2S_2$, for all intertwiners for which the products are defined, and $1_\mu \times 1_\nu = 1_{\mu \times \nu}$ where $1_\mu := 1 \otimes 1_V$ is the unit arrow at the object $\mu : A \rightarrow A \otimes \text{End} V$.

$\text{Amp} A$ contains direct sums $\mu \oplus \nu$ of any two objects: $\mu \oplus \nu(A) := \mu(A) \oplus \nu(A)$ defines a direct sum for any orthogonal direct sum $V \oplus W$.

$\text{Amp} A$ has subobjects: If $P \in (\mu|\mu)$ is a Hermitian projection then there exists an object $\nu$ and an injection $S \in (\mu|\nu)$ such that $SS^* = P$ and $S^*S = 1_\nu$. The existence of subobjects is a trivial statement in the category of all, possibly non-unital, amplimorphisms because $S$ can be chosen to be $P$ in that case. In the category $\text{Amp} A$ this is a non-trivial theorem which can be proven [SzV] provided
the net is split. An amplimorphism $\mu$ is called irreducible if the only (non-zero) subobject of $\mu$ is $\mu$. Equivalently, $\mu$ is irreducible if $(\mu|\mu) = C1_\mu$. Since the selfintertwiner space $(\mu|\mu)$ of any localized amplimorphism is finite dimensional (use Haag duality to show that any $T \in (\mu|\mu)$ belongs to $A(I) \otimes \text{End} V$ where $I$ is the interval where $\mu$ is localized), the category $\text{Amp} A$ is fully reducible. That is any object is a finite direct sum of irreducible objects.

The full subcategory $\text{Amp}^\text{tr} A$ of transportable amplimorphisms is a braided category. The braiding structure is provided by the statistics operators

$$\epsilon(\mu, \nu) \in (\nu \times \mu|\mu \times \nu)$$

defined by

$$\epsilon(\mu, \nu) := (U^* \otimes 1)(1 \otimes P)(\mu \otimes \text{id})(U)$$

where $U$ is any isomorphism from $\nu$ to some $\tilde{\nu}$ such that the localization region of $\tilde{\nu}$ lies to the left from that of $\mu$. The statistics operator satisfies

- naturality:
  $$\epsilon(\mu_1, \mu_2) (T_1 \times T_2) = (T_2 \times T_1) \epsilon(\nu_1, \nu_2)$$

- pentagons:
  $$\epsilon(\lambda \times \mu, \nu) = (\epsilon(\lambda, \nu)) (1_\mu \times \epsilon(\mu, \nu))$$

The relevance of the category $\text{Amp} A$ to the representation theory of the observable algebra $A$ can be summarized in the following theorem taken over from [SzV].

**Theorem 3.1.** Let $\pi_0$ be a faithful irreducible representation of $A$ that satisfies Haag duality:

$$\pi_0(A(I'))' = \pi_0(A(I)) \quad I \in I.$$

Let $\text{Rep} A$ be the category of representations $\pi$ of $A$ that satisfy the following selection criterion (analogue of the DHR-criterion):

$$\exists I \in I, n \in \mathbb{N} : \quad \pi|_{A(I')} \simeq n \cdot \pi_0|_{A(I')}$$

where $\simeq$ denotes unitary equivalence. Then $\text{Rep} A$ is isomorphic to $\text{Amp} A$. If we add the condition that $\pi_0$ is $\alpha$-covariant and denote by $\text{Rep}^\alpha A$ the full subcategory in $\text{Rep} A$ of $\alpha$-covariant representations then $\text{Rep}^\alpha A$ is isomorphic to the category $\text{Amp}^\alpha A$ of $\alpha$-covariant amplimorphisms.

In general $\text{Amp}^\alpha A \subset \text{Amp}^\text{tr} A \subset \text{Amp} A$. In the Hopf spin model we shall see that $\text{Amp}^\alpha A = \text{Amp} A$ (Section 5) and that $\text{Amp} A$ is equivalent to $\text{Rep} D(H)$ (subsection 3.5).

**3.2. Comodule algebra actions**

Let $\{\mu_r\}$ be a list of amplimorphisms in $\text{Amp} A$ containing exactly one from each equivalence class of irreducible objects. Then an object $\rho$ is called universal if it is equivalent to $\oplus_r \mu_r$. Define the $C^*$-algebra $G$ by

$$G := \oplus_r \text{End} V_r$$
then every universal object is a unital $C^*$-map $\rho: \mathcal{A} \to \mathcal{A} \otimes \mathcal{G}$. The identity morphism $\text{id}_\mathcal{A}$, as a subobject of $\rho$, determines a distinguished 1-dimensional block $\text{End} V_0 \cong C$ of $\mathcal{G}$ and also a $^*$-algebra map $\varepsilon: \mathcal{G} \to C$.

Universality of $\rho$ implies that the monoidal product $\rho \times \rho$ is quasi-equivalent to $\rho$. The question is whether there exists an appropriate choice of $\rho$ such that $\rho \times \rho = (\text{id}_\mathcal{A} \otimes \Delta) \circ \rho$ for some coproduct $\Delta: \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$. If $\rho$ can be chosen in such a way — which is probably the characteristic feature of Hopf algebra symmetry — then we arrive to the very useful notion of a comodule algebra action.

**Definition 3.2.** Let $\mathcal{G}$ be a $C^*$-bialgebra with coproduct $\Delta$ and counit $\varepsilon$. A **comodule algebra action** (or shortly a coaction) on $\mathcal{A}$ is an amplimorphism $\rho: \mathcal{A} \to \mathcal{A} \otimes \mathcal{G}$ that is also a comodule action on $\mathcal{A}$ with respect to the coalgebra $(\mathcal{G}, \Delta, \varepsilon)$. In other words: $\rho$ is a linear map satisfying the axioms:

\[
\begin{align*}
\rho(A)\rho(B) &= \rho(AB) \quad (3.12a) \\
\rho(\mathbb{1}) &= \mathbb{1} \otimes \mathbb{1} \quad (3.12b) \\
\rho(A^*) &= \rho(A)^* \quad (3.12c) \\
\exists I \in \mathcal{I} : \rho(A) &= A \otimes \mathbb{1} \quad A \in \mathcal{A}(I^c) \quad (3.12d) \\
\rho \times \rho &= (\rho \otimes \text{id}) \circ \rho = (\text{id} \otimes \Delta) \circ \rho \quad (3.12e) \\
(\text{id}_\mathcal{A} \otimes \varepsilon) \circ \rho &= \text{id}_\mathcal{A} \quad (3.12f)
\end{align*}
\]

$\rho$ is said to be **universal** if it is — as an amplimorphism — a universal object of $\text{Amp} \mathcal{A}$.

Examples of comodule algebra actions for the Hopf spin chain will be given in subsection 3.3. Later, in Sect.5, we will show that those comodule algebra actions are actually universal.

Every comodule algebra action $\rho: \mathcal{A} \to \mathcal{A} \otimes \mathcal{G}$ determines an action of the dual $\hat{\mathcal{G}}$ on $\mathcal{A}$, also denoted by $\rho$, as follows

\[
\rho_\xi: \mathcal{A} \to \mathcal{A} \quad \xi \in \hat{\mathcal{G}}
\]

\[
\rho_\xi(A) := (\text{id}_\mathcal{A} \otimes \xi)(\rho(A))
\]

The axioms for a localized action of the Hopf algebra $\hat{\mathcal{G}}$ on the $C^*$-algebra $\mathcal{A}$, that is

\[
\begin{align*}
\rho_\xi(AB) &= \rho_{\xi(1)}(A)\rho_{\xi(2)}(B) \quad (3.14a) \\
\rho_\xi(\mathbb{1}) &= \varepsilon(\xi)\mathbb{1} \quad (3.14b) \\
\rho_\xi(A^*) &= \rho_{\xi^*}(A^*) \quad (3.14c) \\
\exists I \in \mathcal{I} : \rho_\xi(A) &= A\varepsilon(\xi) \quad A \in \mathcal{A}(I^c) \quad (3.14d) \\
\rho_\xi \circ \rho_\eta &= \rho_{\xi\eta} \\
\rho_\mathbb{1} &= \text{id}_\mathcal{A} \quad (3.14e) \\
\rho_\mathbb{1} &= \text{id}_\mathcal{A} \quad (3.14f)
\end{align*}
\]

are equivalent to the condition that

\[
A \mapsto \rho(A) = \sum_s \rho_{\eta_s}(A) \otimes Y^s \in \mathcal{A} \otimes \mathcal{G}
\]
is a comodule algebra action, where \( \{ \eta_s \} \) and \( \{ Y^s \} \) denote a pair of dual bases of \( \hat{G} \) and \( G \), respectively. In (3.14c) we used the notation \( \xi \mapsto \xi^* \) for the antilinear involutive algebra automorphism defined by \( \langle \xi^* | a \rangle = \overline{\langle \xi | a^* \rangle} \). It is related to the antipode by \( \xi^* = S(\xi^*) \).

One can also check that \( \rho_{\xi} \) for \( \xi = D_{kl} \), the representation matrix of the unitary irrep \( r \), determines an ordinary matrix amplimorphism \( \rho_r : A \to A \otimes M_n \). Whether such a \( \rho_r \) is irreducible is not clear for the moment so we will call it a component of \( \rho \).

For a \( G \)-comodule algebra action \( \rho \) on \( A \) let \( \text{Amp}_\rho A \) denote the full subcategory of \( \text{Amp} A \) generated by objects that are direct sums of irreducibles occurring in \( \rho \) as a subobject.

If the bialgebra \( G \) also possesses an antipode \( S \) such that \((G, \Delta, \varepsilon, S)\) is a C*-Hopf algebra then the category \( \text{Amp}_\rho A \) becomes equipped with a rigidity structure: Identifying \( G \) with \( \sum_r \text{End} V_r \) and choosing evaluation and coevaluation maps \( \text{ev} : V \otimes V \to C, \text{coev} : C \to V \otimes V \), in the category of vector spaces, we obtain a transposition map

\[
T : \text{End} V \to \text{End} V, \\
x \in \text{End} V \mapsto x^T := (1 \otimes \text{ev})(1 \otimes x \otimes 1_V)(\text{coev} \otimes 1_V)
\]

such that

\[
\bar{\rho} := (\text{id}_A \otimes T \circ S) \circ \rho
\]

defines a conjugation functor in \( \text{Amp}_\rho A \) with evaluation and coevaluation \( \text{Ev} = 1 \otimes \text{ev} \) and \( \text{Coev} = 1 \otimes \text{coev} \), respectively.

The extremely simple form of the intertwiners \( \text{Ev} \) and \( \text{Coev} \), and also of the simple form of the basic intertwiners \( T_{pq}^{r\alpha} = 1 \otimes t_{pq}^{r\alpha} \) from the component \( \rho_r \) of \( \rho \) to the product \( \rho_p \times \rho_q \) — as it is suggested by (3.12e) — is in fact a general phenomenon of all amplimorphisms localized within an interval of length 2, provided the net \( A \) satisfies Haag duality and the intersection property. The latter one is defined as follows.

**Definition 3.3.** The net \( \{ A(I) \} \) is said to satisfy the intersection property if

\[
I, J \in \mathcal{I}, \ I \cap J = \emptyset \Rightarrow A(I) \cap A(J) = C1. \quad (3.15)
\]

Comodule algebra actions with a fixed augmented C*-algebra \((G, \varepsilon)\) but with varying coproduct will be denoted as a pair \((\rho, \Delta)\). In order to compare such comodule algebra actions one can introduce equivalences of three different kinds.

**Definition 3.4.** Let \((\rho, \Delta)\) and \((\rho', \Delta')\) be comodule algebra actions of \((G, \varepsilon)\) on \( A \). Then a pair \((U, u)\) of unitaries \( U \in A \otimes G \) and \( u \in G \otimes G \) is called a cocycle equivalence from \((\rho, \Delta)\) to \((\rho', \Delta')\) if

\[
U \rho(A) = \rho'(A) U \quad A \in A \quad (3.16a)
\]
\[
u \Delta(X) = \Delta'(X) u \quad X \in G \quad (3.16b)
\]
\[
U \times U = (1 \otimes u) \cdot (\text{id}_A \otimes \Delta)(U) \quad (3.16c)
\]
(U, u) is called a coboundary equivalence if in addition to (a–c)
\[ u = (x^{-1} \otimes x^{-1}) \Delta(x) \] (3.16d)
holds for some unitary \( x \in \mathcal{G} \). If \( u = 1 \otimes 1 \), \( \rho \) and \( \rho' \) are called equivalent.

Notice that as a consequence of (3.16c) the unitary \( u \) satisfies the cocycle condition
\[ (1 \otimes u)(\text{id} \otimes \Delta)(u) = (u \otimes 1)(\Delta \otimes \text{id})(u) \] (3.17)
which is precisely the condition for a twisting to preserve quasitriangularity of an \( R \)-matrix: \( R \mapsto R' = u^{op} R u^{-1} \).

**Lemma 3.5.** Let \( \{\mathcal{A}(I)\} \) be a net satisfying algebraic Haag duality and intersection property. Let furthermore \( \{\rho_r\} \) be a finite family of irreducible algebras \( \rho_r : \mathcal{A} \rightarrow \mathcal{A} \otimes \text{End} V_r \), closed under the monoidal product, and such that each one of them is localized within the same interval \( I \) of length \( |I| = 2 \).

Let \( \rho = \oplus_r \rho_r \) be localized within \( I \), too. Then there exists one and only one coproduct \( \Delta \) on \( \mathcal{G} = \oplus_r \text{End} V_r \) such that the pair \( (\rho, \Delta) \) is a comodule algebra action of \( \mathcal{G} \) on \( \mathcal{A} \).

**Proof:** Haag duality and intersection property imply that all intertwiners in \( T \in (\rho_p \times \rho_q|\rho_r) \) are scalars, i.e., have the form \( T = 1 \otimes t \) with some \( t : V_r \rightarrow V_p \otimes V_q \).

Choose an orthonormal basis \( \{t_{pq}^{r\alpha}\} \) in \( \text{Hom}(V_r, V_p \otimes V_q) \) and define
\[ \Delta(X) := \sum_{pqr} t_{pq}^{r\alpha} X t_{pq}^{r\alpha*} \], \( X \in \mathcal{G} \)

It is easy to verify that this map defines a coproduct on \( \mathcal{G} \) and furthermore that the intertwiners \( T_{pq}^{r\alpha} := 1 \otimes t_{pq}^{r\alpha} \) are complete and orthonormal in \( (\rho_p \times \rho_q|\rho_r) \). Therefore \( \rho \times \rho = (\text{id}_\mathcal{A} \otimes \Delta) \circ \rho \) follows.

Now assume that there exists an other coproduct \( \Delta' \) on \( \mathcal{G} \) such that \( (\rho, \Delta') \) is also a comodule algebra action. Then the "fusion coefficients" \( N_{pq}^{r\alpha} \) of \( \Delta' \) must be the same as those of \( \Delta \) since both of them are determined by the composition rules of the irreducible objects \( \rho_r \). Therefore there exists a twisting \( u \in \mathcal{G} \otimes \mathcal{G} \) such that \( \Delta' = \text{Ad}_u \circ \Delta \). Now multiplying the identity
\[ (\text{id}_\mathcal{A} \otimes \Delta') \circ \rho = (\text{id}_\mathcal{A} \otimes \Delta) \circ \rho \]
by \( u t_{pq}^{s\beta} \) from the right and by \( t_{pq}^{r\alpha*} \) from the left we obtain
\[ 1 \otimes t_{pq}^{r\alpha*} u t_{pq}^{s\beta} \in (\rho|\rho) \]
a selfintertwiner. Since \( \rho \) contains every irreducible only once, this selfintertwiner is a central element of \( \mathcal{G} \). That is there exist complex numbers \( c_{pq}^{r\alpha\beta} \) such that
\[ t_{pq}^{r\alpha*} u t_{pq}^{s\beta} = \delta_{rs} c_{pq}^{r\alpha\beta} \cdot e_r \], \( (e_r = \text{id}_{V_r}) \)
\[ u = \sum_{pqr} \sum_{\alpha\beta} t_{pq}^{r\alpha} c_{pq}^{r\alpha\beta} t_{pq}^{s\beta*} \]
Now it is easy to check that this \( u \) commutes with all \( \Delta(X) \), hence \( \Delta' = \Delta \). Q.e.d.

**Proposition 3.6.** Let \( \{ A(I) \} \) be as in the previous Lemma. Let \( (\rho_0, \Delta_0) \) be a comodule algebra action of the augmented algebra \( (\mathcal{G}, \varepsilon) \) that is localized within an interval of length 2. Then universal comodule algebra actions of \( \text{Amp}_{\rho_0} \) are unique up to cocycle equivalence.

**Proof:** Let \( (\rho, \Delta) \) be a universal comodule algebra action of \( \text{Amp}_{\rho_0} \). Then \( \exists U \in \text{Iso}(\rho_0|\rho) \). We have then two isometric intertwiners

\[
(id_A \otimes \Delta)(U) : \rho \times \rho \to (id_A \otimes \Delta) \circ \rho_0
\]

\[
U : \rho \times \rho \to \rho_0 \times \rho_0
\]

Since both \( (id_A \otimes \Delta) \circ \rho_0 \) and \( \rho_0 \times \rho_0 \) is localized within the same interval of length 2, the intertwiner \( (U \times U) \cdot (id_A \otimes \Delta)(U^*) \) between them must be a scalar: \( \mathbb{I} \otimes u \). We obtain

\[
U \times U = (\mathbb{I} \otimes u) \cdot (id_A \otimes \Delta)(U)
\]

\[
\rho_0 \times \rho_0 = (id_A \otimes \Delta') \circ \rho_0
\]

where \( \Delta' = \text{Ad}_u \circ \Delta \)

This proves that \( (U, u) \) is a twisted equivalence from \( (\rho, \Delta) \) to \( (\rho_0, \Delta') \). By Lemma 3.5 \( \Delta' \) is equal to \( \Delta_0 \), therefore every universal comodule algebra action is cocycle equivalent to the same \( (\rho_0, \Delta_0) \).

Q.e.d.

**3.3. The special comodule actions and their charge transporters**

Let \( \mathcal{G} \) denote the Drinfeld Double \( \mathcal{D}(H) \) (See Appendix B). The formulae given below define amplimorphisms \( \rho_I : A \to A \otimes \mathcal{G} \) that are localized on an interval \( I \) of length 2:

\[
\rho_{2i,2i+1}(A_{2i}(a)A_{2i+1}(\varphi)) := A_{2i}(a_{(1)})A_{2i+1}(\varphi_{(2)}) \otimes \mathcal{D}(a_{(2)})\mathcal{D}(\varphi_{(1)})
\]

\[
\rho_{2i-1,2i}(A_{2i-1}(\varphi)A_{2i}(a)) := A_{2i-1}(\varphi_{(1)})A_{2i}(a_{(2)}) \otimes \mathcal{D}(\varphi_{(2)})\mathcal{D}(a_{(1)})
\]

(3.18)

The proof of that these expressions really determine amplimorphisms is straightforward and will be omitted. Likewise we left to the reader to check that (3.18) in fact define comodule algebra actions, that is

\[
\rho_{2i,2i+1} \times \rho_{2i,2i+1} = (id_A \otimes \Delta_D) \circ \rho_{2i,2i+1}
\]

\[
\rho_{2i-1,2i} \times \rho_{2i-1,2i} = (id_A \otimes \Delta_D^{op}) \circ \rho_{2i-1,2i}
\]

\[
(id_A \otimes \varepsilon_D) \circ \rho_{i,i+1} = id_A
\]

(3.19)

Hence \( \rho_{2i,2i+1} \) is a comodule action with respect to the coalgebra \( \mathcal{D}(H) = (\mathcal{G}, \Delta_D, \varepsilon_D) \) and \( \rho_{2i-1,2i} \) is one with respect to \( \mathcal{D}(H) = (\mathcal{G}, \Delta_D^{op}, \varepsilon_D) \). The formulæ (3.18) are manifestly translation covariant, so we have

\[
(\alpha \otimes \text{id}_\mathcal{G}) \circ \rho_{i,i+1} \circ \alpha^{-1} = \rho_{i+2,i+3}
\]

(3.20)
Of course, one expects that amplimorphisms related by translations (3.20) are actually equivalent, therefore there exists appropriate charge transporters between them. It is not clear, however, whether the amplimorphisms \( \rho_{2i,2i+1} \) and \( \rho_{2i-1,2i} \) create independent sectors or not. Let us define the charge transporter \( T_i \) as follows:

\[
T_i := \begin{cases} 
A_i(b_A) \otimes D(\beta^A) & i = \text{even} \\
A_i(\beta^A) \otimes D(b_A) & i = \text{odd} 
\end{cases}
\] (3.21)

Then we have

**Proposition 3.7.** The charge transporters \( T_i \) are unitary intertwiners from \( \rho_{i,i+1} \) to \( \rho_{i-1,i} \), i.e.

\[
T_i \rho_{i,i+1}(A) = \rho_{i-1,i}(A) T_i, \quad A \in \mathcal{A}
\] (3.22)

and satisfy the cocycle condition

\[
T_i \times T_i \equiv (T_i \otimes 1) \cdot (\rho_{i,i+1} \otimes \text{id})(T_i) = \\
= \begin{cases} 
(1 \otimes R) \cdot (\text{id} \otimes \Delta_D)(T_i) & i = \text{even} \\
(1 \otimes R^{op}) \cdot (\text{id} \otimes \Delta_D^{op})(T_i) & i = \text{odd} 
\end{cases}
\] (3.23)

**Proof:** By inspection.

In view of Definition 3.4 the above proposition claims that the pair \((T_i, R^{op})\) determines a cocycle equivalence between comodule actions of the form \((\rho_{i,i+1}, \Delta_D^{(op)})\). More precisely, we have the infinite sequence of cocycle equivalences

\[
\ldots (\rho_{2i,2i+1}, \Delta_D) \xleftarrow{(T_{2i+1}, R^{op})} (\rho_{2i+1,2i+2}, \Delta_D^{op}) \xrightarrow{(T_{2i+2}, R)} (\rho_{2i+2,2i+3}, \Delta_D) \ldots
\] (3.24)

Composing these two arrows we obtain a coboundary equivalence \((T_{2i+1}T_{2i+2}, R^{op}R)\) because \(R^{op}R = (s \otimes s)\Delta_D(s^{-1})\) according to (A.7) where \(s\) is given by (B.6). Likewise \((T_{2i}T_{2i+1}, RR^{op})\) yields a coboundary equivalence. Therefore introducing

\[
U_{i,i+1} := (1 \otimes s^{-1}) T_i T_{i+1} \in (\rho_{i-1,i} | \rho_{i+1,i+2})
\] (3.25)

we obtain a charge transporter localized within \(\{i, i+1\}\) that satisfies the cocycle condition

\[
U_{2i+1,2i+2} \times U_{2i+1,2i+2} = (\text{id}_A \otimes \Delta_D)(U_{2i+1,2i+2}) \\
U_{2i,2i+1} \times U_{2i,2i+1} = (\text{id}_A \otimes \Delta_D^{op})(U_{2i,2i+1})
\] (3.26)

The existence of such charge transporters means — by definition — that the comodule actions \(\rho_{i,i+1}\) are \(\alpha\)-covariant (and not only transportable).
If we want to see the components of the amplimorphism $\rho_{i,i+1}$ we can proceed as follows. Choose a system $\{E_{r}^{kl}\}$ of $C^*$-matrix units for $\mathcal{G}$ and dual basis $\{D_{r}^{kl}\}$ for $\hat{\mathcal{G}}$, i.e.

\[
E_{r}^{ij}E_{r}^{kl} = \delta_{pq}\delta^{ik}E_{r}^{il} \quad (E_{r}^{kl})^* = E_{r}^{lk}
\]

\[
\langle D_{r}^{ij}, E_{r}^{kl} \rangle \equiv D_{r}^{ij}(E_{r}^{kl}) = \delta_{pq}\delta^{ik}\delta^{jl}
\]

\[
\Delta(D_{r}^{kl}) = \sum_{m} D_{r}^{km} \otimes D_{r}^{ml}
\]

Then introducing the notation

\[
\rho^{kl}_{r} := \rho_{D_{r}^{kl}}
\]

one can verify that

\[
\rho^{kl}_{r}(AB) = \rho^{km}_{r}(A)\rho^{ml}_{r}(B)
\]

\[
\rho^{kl}_{r}(A)^* = \rho^{lk}_{r}(A^*)
\]

that is $\rho_{r}: A \rightarrow A \otimes M_{n_r}$ is a $*$-algebra map. One expects, of course, that $\rho_{r}$ is irreducible and that the intertwiners in $(\rho_{p} \times \rho_{q}|\rho_{r})$ are in one-to-one correspondence with the intertwiners in $(D_{p} \times D_{q}|D_{r})$. This is, however, not so trivial and we will return to it in subsection 3.5.

### 3.4. Outerness of $\rho$

Let $\rho$ denote one of the comodule actions defined in (3.18). Then the corresponding action $\rho_{\xi} = (\text{id}_A \otimes \xi) \circ \rho$ of $\hat{\mathcal{G}}$ is faithful in the following sense:

\[
\rho_{\xi}(A) = 0 \quad \forall A \in A \quad \Rightarrow \quad \xi = 0
\]

We prove the statement for the case $\rho = \rho_{2i,2i+1}$. Assume

\[
\rho_{\xi}(A_{2i}(a)A_{2i+1}(\varphi) = A_{2i}(a_{(1)})A_{2i+1}(\varphi_{(2)}) \otimes \xi(D(a_{(2)})D(\varphi_{(1)})) = 0
\]

for some $\xi \in \hat{\mathcal{G}}$ and for all $a \in H$, $\varphi \in \hat{H}$. Multiplying this equation from the left by $A_{2i+1}(\varphi_{(3)})A_{2i}(a_{(0)})$ we obtain

\[
1 \otimes \xi(D(a)D(\varphi)) = 0
\]

which immediately implies that $\xi = 0$.

Using faithfulness of the action $\rho_{\xi}$ we can completely determine the selfintertwiner space $(\rho|\rho)$ as follows.

Notice at first that for computing intertwiners it is not enough to know $\rho$ as a map to the abstract $C^*$-algebra $A \otimes \mathcal{G}$ but we have to specify $\mathcal{G}$ as a $*$-subalgebra of some $\text{End} V$. Our convention is that $\mathcal{G}$ is embedded into $\text{End} V$ as its defining representation, i.e. as the direct sum of its irreducible representations each of them with multiplicity 1. This remark completes the definition of the special
comodule algebra actions of the doubles $G = \mathcal{D}(H)$ or $G = \mathcal{D}(\hat{H})$, respectively, given in (3.18).

With this definition a selfintertwiner $T$ of a $\rho$ of (3.18) is necceassarily a scalar, i.e. has the form $T = \mathbb{1} \otimes t$ where $t \in \text{End} V$, because $\rho$ is localized on a 2-point interval. The intertwining property

$$(\mathbb{1} \otimes t) \cdot \rho(A) = \rho(A) \cdot (\mathbb{1} \otimes t) \quad A \in \mathcal{A}$$

can be equivalently written as

$$\rho_{\xi}(A) = 0 \quad \forall A \in \mathcal{A}$$

for all $\xi \in \hat{G}$ such that

$$\xi(X) = (u, (Xt - tX)v) \quad \text{for some } u, v \in V.$$ 

Now faithfulness of $\rho$ implies that $Xt - tX = 0$ for all $X \in \mathcal{G}$, that is $t \in \mathcal{G}' = \mathcal{G} \cap \mathcal{G}'$ by the definition of the embedding $\mathcal{G} \subset \text{End} V$. This proves that 2-point localized faithful comodule algebra actions $\rho$ of the $C^*$-Hopf algebra $\mathcal{G}$ on a Haag dual net $\mathcal{A}$ with intersection property have selfintertwiner spaces

$$(\rho|\rho) = \mathbb{1} \otimes \text{Center } \mathcal{G}. \quad (3.31)$$

In particular the irreducible components of such a $\rho$ are in one-to-one correspondence with minimal central projectors of $\mathcal{G}$. That is the components of $\rho$ are precisely its irreducible components.

The special form (3.31) of the selfintertwiner space implies the weaker but important property of outerness of $\rho$.

**Definition 3.8.** An amplimorphism $\mu: \mathcal{A} \rightarrow \mathcal{A} \otimes M_n$ is called inner if there exists a unitary $U \in \mathcal{A} \otimes M_n$ such that $U(A \otimes I_n)U^* = \mu(A)$ for $A \in \mathcal{A}$. The comodule algebra action $\rho: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{G}$ of the $C^*$-bialgebra $\mathcal{G}$ on the $C^*$-algebra $\mathcal{A}$ is outer iff none of the components of $\rho$ is inner except the trivial amplimorphism $(\text{id}_\mathcal{A} \otimes \varepsilon) \circ \rho = \text{id}_\mathcal{A}$. The action $\rho: \hat{\mathcal{G}} \otimes \mathcal{A} \rightarrow \mathcal{A}$ is outer iff the corresponding coaction is outer.

**Lemma 3.9.** Let $\mathcal{A}$ be a Haag dual net with intersection property and $\rho: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{G}$ be a coaction of the Hopf algebra $\mathcal{G}$ that is localized on an interval $I$ of length 2. Then faithfulness of the $\hat{\mathcal{G}}$-action $\rho: \hat{\mathcal{G}} \otimes \mathcal{A} \rightarrow \mathcal{A}$ implies outerness of the (co)action $\rho$.

**Proof:** From (3.31) we it follows that none of the non-trivial components $\rho_r = (\text{id}_\mathcal{A} \otimes D_r) \circ \rho$, $r \neq 0$, can have intertwiners to the trivial component $\rho_0 = \text{id}_\mathcal{A}$. Therefore $\rho_r$ is inner only for $r = 0$. \(Q.e.d.\)

Applying this Lemma to the Hopf spin chain we conclude that the comodule algebra actions defined in (3.18) are outer.
3.5. The equivalence of $\text{Amp}_\rho \mathcal{A}$ and $\text{Rep} \mathcal{G}$

We recall that $\text{Amp}_\rho \mathcal{A}$ denotes the full subcategory of $\text{Amp} \mathcal{A}$ generated by objects $\mu$ of the form $\mu \sim \oplus_s \mu_s$ where each $\mu_s$ is an irreducible subobject of $\rho$.

**Theorem 3.10.** Let $\mathcal{A}$ be a Haag dual net satisfying the intersection property. Let $\rho$ be a faithful comodule algebra action of the quasitriangular $C^*$-Hopf algebra $\mathcal{G}$ on $\mathcal{A}$ that is localized on an interval of length 2. Then the category $\text{Amp}_\rho \mathcal{A}$ and the category of $*$-representations $\text{Rep} \mathcal{G}$ of $\mathcal{G}$ are equivalent as strict monoidal, braided, rigid, $C^*$-categories.

**Proof:** We prove the equivalence $\text{Amp}_\rho \mathcal{A} \sim \text{Rep} \mathcal{G}$ as equivalence of monoidal categories. Other features such as rigidity ... etc. can be checked rather easily (cf. to the proof of Thm.4.16. in [SzV]) and therefore are omitted.

We need to construct a monoidal functor $\Phi: \text{Amp}_\rho \mathcal{A} \to \text{Rep} \mathcal{G}$ which is one-to-one on the equivalence classes of objects and is one-to-one between intertwiner spaces $(\mu|\nu) \to (\Phi(\mu)|\Phi(\nu))$ for each pair $\mu, \nu$ of amplimorphisms. Such a functor is readily obtained once we have established: (I) a one-to-one map $\rho_r \mapsto D_r$ between a complete family $\{\rho_r\}$ of irreducible objects of $\text{Amp}_\rho \mathcal{A}$ and a complete family $\{D_r\}$ of irreducible objects of $\text{Rep} \mathcal{G}$; and (II) one-to-one maps $(\rho_p \times \rho_q|\rho_r) \to (D_p \times D_q|D_r)$ between the basic intertwiner spaces.

Let $D_r: \mathcal{G} \to M_n$, be fixed irreducible *-representations of $\mathcal{G}$ and define $\rho_r := (\text{id}_A \otimes D_r) \circ \rho$. Since each $\rho_r$ is localized within $\{i, i+1\}$, the intersection property and Haag duality implies that all intertwiners $T \in (\rho_p \times \rho_q|\rho_r)$ have the form $T = \mathbf{1} \otimes t$ with some $t \in \text{Mat}(n_p n_q \times n_r, \mathbb{C})$. We claim that

$$\rho_r \mapsto D_r$$

$$T = \mathbf{1} \otimes t \in (\rho_p \times \rho_q|\rho_r) \mapsto t \in (D_p \times D_q|D_r)$$

defines the required functor $\Phi$. Let us show at first that $t \in (D_p \times D_q|D_r)$:

$$\rho_p \times \rho_q(A)(\mathbf{1} \otimes t) = (\mathbf{1} \otimes t)\rho_r(A)$$

$$(\text{id}_A \otimes D_p \otimes D_q) \circ (\text{id}_A \otimes \Delta) \circ \rho(A) \cdot (\mathbf{1} \otimes t) = (\mathbf{1} \otimes t) \cdot (\text{id}_A \otimes D_r) \circ \rho(A)$$

$$(\text{id}_A \otimes \xi) \circ \rho(A) = 0 \quad A \in \mathcal{A}$$

for all $\xi \in \hat{\mathcal{G}}$ which associates to $X \in \mathcal{G}$ one of the matrix elements of

$$(D_p \times D_q)(X) \cdot t - t \cdot D_r(X)$$

By faithfulness of $\rho$ all these $\xi$'s must be zero. Therefore $t \in (D_p \times D_q|D_r)$ as claimed.

In order to see surjectivity of the map $T \mapsto t$ let $t \in (D_p \times D_q|D_r)$. Then $\mathbf{1} \otimes t$ is in $(\rho_p \times \rho_q|\rho_r)$ by the very definition of the $\rho_p$'s. Hence $T \mapsto t$ is one-to-one.

Q.e.d.
4. Construction of field algebras

Field algebras $\mathcal{F}$ are $C^*$-algebra extensions $\mathcal{F} \supset A$ of the observables that satisfy the following physical requirements:

I. $\mathcal{F}$ contains all charge carrying fields: For any localized amplimorphism $\mu$ there exists $F_\mu \in \mathcal{F} \otimes \text{End} \, V_\mu$ such that $\mathcal{F}_\mu$ implements $\mu$, i.e. $F_\mu (A \otimes 1) = \mu(A) F_\mu$ for all $A \in A$.

II. The inclusion $A \subset \mathcal{F}$ must be irreducible: $A' \cap \mathcal{F} = \mathbb{C} \cdot 1$. In other words, an operator that commutes with all observables should be a symmetry and not a field.

III. There exists a conditional expectation $E: \mathcal{F} \to A$ of index finite type. This requirement comes from that we want $\mathcal{F}$ to carry an action of the quantum symmetry such that $A$ is the invariant subalgebra. $E$ will then be the quantum group average. The finiteness of the index is related to our interest in finite dimensional quantum symmetries.

IV. $\mathcal{F}$ is minimal under conditions (I–III). That is if $\mathcal{F}_1$ satisfies $A \subset \mathcal{F}_1 \subset \mathcal{F}$ and conditions (I–III) above then $\mathcal{F}_1 = \mathcal{F}$.

4.1. The field algebras $\mathcal{F}_\rho$

For a comodule action $(\rho, \Delta)$ we define the field algebra $\mathcal{F}_\rho$ as the crossed product

$$\mathcal{F}_\rho = A \rtimes \hat{G} = \text{Span}\{ AF_\xi \mid A \in A, \xi \in \hat{G} \} \quad (4.1)$$

$$F_\xi F_\eta = F_{\xi \eta}, \quad (4.1a)$$

$$F_\xi^* = F_{\xi^*}, \quad (4.1b)$$

$$F_\xi A = \rho_{\xi(1)}(A) F_{\xi(2)} \quad (4.1c)$$

There is an action $\gamma$ of the symmetry algebra $\mathcal{G}$ on $A$ with respect to which the invariant subalgebra $\mathcal{F}^\gamma$ is precisely the observable algebra:

$$\gamma_X( AF_\xi) := AF_{X \to \xi} \quad X \in \mathcal{G}, A \in A, \xi \in \hat{G}$$

$$\mathcal{F}^\gamma \equiv \{ F \in \mathcal{F} \mid \gamma_X(F) = \bar{\varepsilon}(X) F \} = A$$

If $h$ denotes the integral of the Hopf algebra $\mathcal{G}$ then $\mathcal{E} := \gamma_h$ defines a conditional expectation onto $A$, the “average” over $\mathcal{G}$. It can be seen to be of index finite type if $\mathcal{G}$ is finite dimensional. As a matter of fact let $\{ \eta_s \}$ be an orthonormal basis of $\hat{G}$ with respect to the scalar product $\langle \xi | \eta \rangle = \langle \xi^* | \eta, h \rangle$. Then $F_{\eta_s}$ provide us with a quasibasis [Wa] for the conditional expectation $\mathcal{E}$, i.e.

$$\sum_s F_{\eta_s} \mathcal{E} \left( F_{\eta_s}^* F \right) = F \quad \forall F \in \mathcal{F}.$$ 

For computing the index take into account that $\langle \xi, h \rangle$ is just the trace of $\xi$ in the left regular representation. Therefore the $\eta_s$ basis consists of appropriately normalized matrix units of $\hat{G}$. This gives $\text{Index} \mathcal{E} = \sum_s F_{\eta_s} F_{\eta_s}^* = \dim \mathcal{G} \cdot \mathbb{1}$. 

20
An equivalent formulation of the defining relations (4.1.a–c) can be given by using the "master field"

\[ F := F_{ηs} \otimes Y^s \in F_ρ \otimes G \]  

(4.2)

as follows:

\[ F^01 F^02 = (\text{id} \otimes \Delta)(F) \]  

(4.2.a)

\[ F^* = F^{-1} = (\text{id} \otimes S)(F) \]  

(4.2.b)

\[ F(A \otimes \mathbf{1}) = ρ(A)F \]  

(4.2.c)

Now we turn to the question of how the crossed product depends on the comodule action.

**Theorem 4.1.** Two universal outer comodule algebra actions \((ρ, Δ)\) and \((ρ', Δ')\) of \(G\) on the observable algebra \(A\) of the Hopf spin model give rise to isomorphic crossed products \(F_ρ = A \rhd ρ G\) and \(F_ρ' = A \rhd ρ' G\) if and only if \(ρ\) and \(ρ'\) are coboundary equivalent.

Recall that under the "isomorphism of the crossed products \(F_ρ\) and \(F_ρ'\)" we mean that there exists a \(C^*\)-algebraic isomorphism between them which leaves the observable algebra \(A\) pointwise invariant.

**Proof:** Necessity: Let \((ρ, Δ), (ρ', Δ')\) be universal outer comodule algebra actions such that the corresponding crossed products \(F_ρ\) and \(F_ρ'\) are isomorphic. Let \(θ: F_ρ' \to F_ρ\) be such an isomorphism and let \(F\) and \(F'\) denote the master fields of the two field algebras. Pick up a twisted equivalence \((U, u) \in ((ρ', Δ') | (ρ, Δ))\), which exists by Proposition 3.6, and construct \(G = F^*U^*(θ \otimes \text{id}_G)(F')\). One checks easily that \(G\) commutes with all operators of the form \(A \otimes \mathbf{1}\) where \(A \in \mathcal{A}\). Thus \(G \in \mathcal{A}' \otimes G\). Since outernes of \(ρ\) implies that \(A\) has trivial relative commutant within \(F_ρ\) (see Proposition 4.2), it follows that \(G = \mathbf{1} \otimes x\), with some \(x \in G\). Hence we found that the two master fields are related by

\[ (θ \otimes \text{id}_G)(F') = UF(\mathbf{1} \otimes x). \]

Now we may compute the operator product

\[ (θ \otimes \text{id}_G \otimes \text{id}_G)(F_0' F_0') = (UF)_{01}(UF)_{02}(\mathbf{1} \otimes x \otimes x) = \]

\[ = (U \times U)F_{01}F_{02}(\mathbf{1} \otimes x \otimes x) = \]

\[ = (\text{id} \otimes Δ')(UF) \cdot (\mathbf{1} \otimes u(x \otimes x)) = \]

\[ = (θ \otimes \text{id}_G \otimes \text{id}_G) \circ (\text{id} \otimes Δ')(F') \cdot (\mathbf{1} \otimes uΔ(x^{-1})(x \otimes x)) \]

which immediately implies that \(u = (x^{-1} \otimes x^{-1})Δ(x)\), a coboundary.

Sufficiency: Let \((U, u)\) be a coboundary equivalence from \((ρ, Δ)\) to \((ρ', Δ')\). Then with \(x\) such that \(u = (x^{-1} \otimes x^{-1})Δ(x)\) we can define the map

\[ θ: F_ρ' \to F_ρ \]

\[ θ \otimes \text{id}_G: (A \otimes \mathbf{1})F' \mapsto (A \otimes \mathbf{1})UF(\mathbf{1} \otimes x) \]
Now it is not difficult to verify that $\theta$ is a $^*$-isomorphism which leaves $\mathcal{A}$ pointwise invariant. What one needs to do is only to check that $UF(\mathbb{1} \otimes x) \in \mathcal{F}_\rho \otimes \mathcal{G}$ is a master field associated to $(\rho', \Delta')$. Q.e.d.

4.2. The irreducibility of the inclusion $\mathcal{A} \subset \mathcal{F}$

If $\mathcal{F}$ had been constructed as a crossed product of $\mathcal{A}$ with the action $\rho$ of a group (instead of $\hat{\mathcal{G}}$) then we would conclude that irreducibility of $\mathcal{A} \subset \mathcal{F}$ is equivalent to the outerness of the action $\rho$: $\rho_g$ is an inner automorphism of $\mathcal{A}$ iff $g = 1$. In fact this conclusion holds also for any $C^*$-Hopf algebra action which is outer in the sense of Definition 3.8.

**Proposition 4.2.** Let $\mathcal{A}$ be a unital $C^*$-algebra and let $\mathcal{F}$ be the crossed product of $\mathcal{A}$ with respect to the coaction $\rho$ of the $C^*$-Hopf algebra $\mathcal{G}$. Then $\mathcal{A}' \cap \mathcal{F} = \mathcal{A}' \cap \mathcal{A}$ if and only if $\rho$ is outer. In particular if $\mathcal{A}$ has trivial center then the inclusion $\mathcal{A} \subset \mathcal{F}$ is irreducible if and only if the action $\rho$ is outer.

**Proof:** Let $C \in \mathcal{A}' \cap \mathcal{F}$ be a unitary which is not an observable. The identity $\gamma_x(C), A = \gamma_{x(1)}(C)\gamma_{x(2)}(A) - \gamma_{x(1)}(A)\gamma_{x(2)}(C) = \gamma_x([c, A]) = 0 \quad A \in \mathcal{A}$ shows that for each $x \in \mathcal{G}$, $\gamma_x(C) \in \mathcal{A}' \cap \mathcal{F}$ too. Since $C$ is not an observable, there exist an $r \neq 0$ such that $\gamma_{E_r}(C) \neq 0$. With this $r$ construct the fields

$$C_r^{ki} := \sum_{n, j} \gamma_{E_{r_ni}}(C)F_r^{kj}(t_r^{0\cdot})^{nj}.$$ 

where $t_{pq}^{r}: D_r \to D_p \times D_q$ denote orthonormal intertwiners (Clebsh-Gordan maps) for the representation theory of $\mathcal{G}$. The $F_r$ implements the amplimorphism $\rho_r$ therefore

$$C_r^{ki} A = \rho_r^{kj}(A)C_r^{ji}, \quad A \in \mathcal{A}.$$ 

On the other hand the $C_r^{ki}$ are observable since

$$\gamma_x(C_r^{ki}) = \sum_{n, j} \gamma_{E_{r_ni}}(C)F_r^{ki}D_r^{lj}(x(2))(t_r^{0\cdot})^{nj} = \sum_{n, j, l, m} \gamma_{E_{r_ni}}(C)F_r^{kl}D_r^{nm}(x(1))D_r^{lj}(x(2))(t_r^{0\cdot})^{nj} = \varepsilon(x)\sum_{l, m} \gamma_{E_{r_ni}}(C)F_r^{kl}(t_r^{0\cdot})^{ml} = \varepsilon(x)C_r^{ki}.$$ 

Therefore $\rho_r$ is inner.

Assume that for some $r \neq 0$ ($\rho_r|\mathcal{A} \neq 0$, i.e. 

$$\exists C_r \in \mathcal{A} \otimes M_n, C_r \neq 0 \text{ such that } C_r^{ki} A = \rho_r^{kj}(A)C_r^{ji} \quad a \in \mathcal{A}$$

22
Then \( S^{ji} := F^i F_2^* C^k_1 \) satisfies \( S^{ji} A = AS^{ji} \) for all \( A \in \mathcal{A} \) hence \( S^{ji} \in \mathcal{A}' \cap \mathcal{F} \).

We want to show that the \( S^{ji} \) are not observables.

\[
\gamma_X(S^{ji}) = \gamma_X(F^i F_2^* C^k_1) = \left[ \gamma_{S(X)}(F^i F_2^*) \right] C^k_1 = F^i_2 D^i_2(S(X)) C^k_1 = D^i_2(S(X)) F^i_2 C^k_1 = D^i_2(S(X)) S^{li}_1 \quad X \in \mathcal{G}.
\]

Therefore \( \mathcal{A}' \cap \mathcal{F} \) is not contained in \( \mathcal{A} \).

Q.e.d.

As a consequence of this proposition we can conclude that the field algebras \( \mathcal{F}_{i,i+1} \) associated to the special comodule actions \( \rho_{i,i+1} \) are irreducible.

4.3. Translation covariance

The problem of translation covariance of a field algebra extension consists of showing that the automorphism \( \alpha \) of \( \mathcal{A} \) extends to an automorphism \( \hat{\alpha} \) of the crossed product \( \mathcal{F} = \mathcal{A} \rtimes \mathcal{G} \). Further requirement is that \( \hat{\alpha} \) be geometric, that is \( \hat{\alpha} \) should commute with the internal symmetry \( \gamma \).

**Theorem 4.3.** Let \( \alpha \in \text{Aut} \mathcal{A} \) and let \( (\rho, \Delta) \) be a comodule action. Then there exists an extension \( \hat{\alpha} \in \mathcal{F}_\rho \) of \( \alpha \) commuting with the action \( \gamma \) of \( \mathcal{G} \) if and only if there exists \( U \in \text{Iso}(\rho^\alpha | \rho) \) satisfying the cocycle condition

\[
U \times U = (\text{id}_\mathcal{A} \otimes \Delta)(U) \quad (4.3)
\]

If this is the case then \( \hat{\alpha} \) is unique up to a central grouplike unitary \( g \in \mathcal{G} \) specified below.

**Proof:** We use the master field notation of (4.2). If \( \hat{\alpha} \) is an extension then introducing

\[
F^\alpha = (\hat{\alpha} \otimes \text{id})(F) \quad (4.4)
\]

\[
\rho^\alpha = (\alpha \otimes \text{id}_\mathcal{G}) \circ \rho \circ \alpha^{-1} \quad (4.5)
\]

a little calculation shows that

\[
F^\alpha(A \otimes 1) = \rho^\alpha(A) F^\alpha \quad (4.6a)
\]

\[
F^\alpha_{01} F^\alpha_{02} = (\text{id} \otimes \Delta)(F^\alpha) \quad (4.6b)
\]

\[
\rho^\alpha \times \rho^\alpha = (\text{id}_\mathcal{A} \otimes \Delta)(\rho^\alpha) \quad (4.6c)
\]

Therefore \( (\rho^\alpha, \Delta) \) is also a comodule algebra action and \( F^\alpha \) is the associated master field. Define \( U := F^\alpha F^* \in \mathcal{F}_\rho \otimes \mathcal{G} \) which is unitary. In order for this \( U \) to belong to \( \mathcal{A} \otimes \mathcal{G} \) we have to assume that \( \hat{\alpha} \) commutes with \( \gamma \). As a matter of fact

\[
(\gamma_X \otimes \text{id})(F) = F(1 \otimes X)
\]

\[
(\gamma_X \otimes \text{id})(F^\alpha) = F^\alpha(1 \otimes X)
\]

\[
(\gamma_X \otimes \text{id})(U) = F^\alpha(1 \otimes X(1))(1 \otimes S(X(2)))F^* = U \varepsilon(X)
\]

23
Now it is easy to check that \( U\rho(A) = \rho^\alpha(A)U \) hence \( U \in \text{Iso}(\rho^\alpha|\rho) \). Since \( F^\alpha = UF, \)

\[
F^\alpha_0 F^\alpha_2 = (U \times U) F_0 F_2 = \\
= (U \times U)(\text{id}_A \otimes \Delta)(U^*) \cdot (\text{id} \otimes \Delta)(F^\alpha)
\]

which, if compared to (4.6b), implies that \( U \) satisfies the cocycle condition (4.3).

Vice versa, if \( \alpha \in \text{Aut}_A \) is such that \( \exists U \in (\rho^\alpha|\rho) \) satisfying the cocycle condition then define the extension \( \hat{\alpha} \) as follows:

\[
\hat{\alpha}(AF_\xi) := \alpha(A) \cdot (\text{id} \otimes \xi)(UF) \equiv \alpha(A)F^\alpha_\xi
\]

where we have introduced \( F^\alpha = UF \). Now one can verify easily that this is indeed a *-automorphism. Also, \( \hat{\alpha} \) commutes with \( \gamma \), since

\[
(\hat{\alpha} \circ \gamma_X \otimes \text{id})(F) = F^\alpha(\mathbb{I} \otimes X)
\]

\[
(\gamma_X \circ \hat{\alpha} \otimes \text{id})(F) = UF(\mathbb{I} \otimes X)
\]

The possible ambiguity of the translation \( \hat{\alpha} \) lies in the ambiguity of choosing a \( U \in \text{Iso}(\rho^\alpha|\rho) \). If \( U' \) is an other isomorphism then \( U^*U' \) is a selfintertwiner of \( \rho \) and satisfies the cocycle condition, too. Therefore \( U^*U' = \mathbb{I} \otimes g \) with \( g \in \text{Center} \mathcal{G} \) grouplike: \( \Delta(g) = g \otimes g \).

It follows from this Theorem that the field algebras \( \mathcal{F}_{\rho_{2i,2i+1}} \) coincide for all \( i \in \mathbb{Z} \). Similarly the \( \mathcal{F}_{\rho_{2i+1,2i}}'s \) coincide for all \( i \). These two field algebras will be denoted respectively as \( \mathcal{F}_{\text{even}} \) and \( \mathcal{F}_{\text{odd}} \). They provide the simplest examples of complete covariant irreducible field algebra extensions of \( A \). They demonstrate also that such extensions are not unique, since by Theorem 4.1 the cocycle \( R \) of (3.23) ought to be a coboundary which is not the case for the \( R \)-matrix of the double (even for the simplest double \( D(Z(2)) = Z(2) \times Z(2) \)).

In order to study the commutation relations of fields at spacelike separation introduce the following notation. For \( \rho = \rho_{2i,2i+1} \) let \( F_{2i,2i+1} \) be the master field of \( \mathcal{F}_\rho \). Choose a unitary cocycle \( U \in \text{Iso}(\rho^\alpha|\rho) \) and denote it by \( U_{2i-1,2i} \). Let \( \hat{\alpha} \) be the translation automorphism of \( \mathcal{F}_{\text{even}} \) associated to \( U \). Then define \( F_{2j,2j+1} \in \mathcal{F}_{\text{even}} \otimes \mathcal{G} \) and \( U_{2j-1,2j} \in \mathcal{A} \otimes \mathcal{G} \) for \( j \in \mathbb{Z} \) by the recursions

\[
F_{2j,2j+1} = U_{2j+1,2j+2} F_{2j+2,2j+3}
\]

\[
U_{2j+1,2j+2} = (\alpha \otimes \text{id}_G)(U_{2j-1,2j})
\]

Analogously, for \( \rho = \rho_{2i-1,2i} \) one can define the fields \( F_{2j-1,2j} \in \mathcal{F}_{\text{odd}} \otimes \mathcal{G} \) and the associated charge transporters \( U_{2j,2j+1} \in (\rho_{2j-1,2j}|\rho_{2j+1,2j+2}) \). The following commutation relations can be obtained

\[
F^{01}_{2j,2j+1} F^{02}_{2k,2k+1} = \begin{cases} 
F^{02}_{2k,2k+1} F^{01}_{2j,2j+1} \cdot (\mathbb{I} \otimes R^{12}) & j > k \\
F^{02}_{2k,2k+1} F^{01}_{2j,2j+1} \cdot (\mathbb{I} \otimes R^{21}) & j < k 
\end{cases}
\]

\[
F^{01}_{2j,2j+1} F^{02}_{2k,2k+1} = \begin{cases} 
F^{02}_{2k,2k+1} F^{01}_{2j,2j+1} \cdot (\mathbb{I} \otimes R^{12}) & j > k \\
F^{02}_{2k,2k+1} F^{01}_{2j,2j+1} \cdot (\mathbb{I} \otimes R^{21}) & j < k 
\end{cases}
\]
Notice that these commutation relations are independent of the choice of the translation \( \hat{\alpha} \). If (4.9) hold true for the choice \( U_{i,i+1} = (I \otimes s^{-1})T_i T_{i+1} \), \( i \in \mathbb{Z} \), (cf. (3.25)) then they hold true for \( U \) replaced by \((I \otimes g)U\) for any central grouplike unitary \( g \in G \).

### 4.4. The "enveloping algebra" of field algebras

If we want to have operators \( Q(X), X \in G \) implementing the action \( \gamma \) of the double on a field algebra \( F_H \) we are lead to define the crossed product \( B_\rho = F_H \bowtie G \) generated by \( F \in F_H \) and new elements \( Q(X), X \in G \) satisfying

\[
Q(X)Q(Y) = Q(XY) \\
Q(X^*) = Q(X)^* \\
Q(X)A = A Q(X) \\
Q(X)F_\xi = F_{X(1)\rightarrow \xi} Q(X(2))
\]

for \( X \in G, A \in A, \) and \( \xi \in \hat{G} \). Hence \( B_\rho \) is the linear span of elements of the form \( AF_\xi Q(X) \). In terms of the master field \( F \) the implementation relation (4.10d) takes the form

\[
(Q(X) \otimes 1)F = F \cdot (Q \otimes \text{id}_G) \circ \Delta^{op}(X).
\]

\( B_\rho \) can also be viewed as the crossed product \( A \bowtie (\hat{G} \bowtie G) \) with the simple algebra \( \hat{G} \bowtie G \). It turns out that \( B_\rho \) is actually independent of the comodule action \( \rho \). More precisely, anticipating the result of section 5, that the special amplimorphism \( \rho_{i,i+1} : A \to A \otimes \hat{G} \), where \( G \) is the double \( D(H) \), is universal in the whole category \( \text{Amp} A \), we can say that \( B_\rho \) contains all field algebra extensions of \( A \) and is independent of \( \rho \). Therefore this algebra will be called the enveloping algebra of field algebras and be denoted by \( B \).

In order to prove independence of \( B \) on \( \rho \) choose an arbitrary comodule action \( \rho' \) which is equivalent to \( \rho \) as an amplimorphism. Then they are cocycle equivalent as comodule algebra actions by Proposition 3.6 and we may choose a cocycle equivalence \( (U, u) \) from \((\rho, \Delta)\) to \((\rho', \Delta')\). Let \( F \in F_\rho \otimes G \subset B \otimes \hat{G} \) be the master field of \( \rho \) and construct the unitary \( F' := U F(Q \otimes \text{id})(u^{op}) \).

A straightforward but lengthy calculation shows that \( F' \) satisfies the defining relations of the master field of \( \rho' \):

\[
F'^{01} F'^{02} = (\text{id} \otimes \Delta')(F') \\
F'(A \otimes 1) = \rho'(A) F' \\
F'^* = (\text{id} \otimes S')(F')
\]

with the primed structure maps on the RHS refering to the twisted double \((\hat{G}, \Delta' = \text{Ad}_v \circ \Delta, \varepsilon, S' = \text{Ad}_v \circ S)\), where \( v = u_1 S(u_2) \).

Now we can formulate our main result on the classification of field algebras.

**Theorem 4.4.** Equivalence classes of complete irreducible field algebra extensions of the observable algebra \( A \) of the Hopf spin model are in one-to-one
correspondence with cohomology classes of unitary cocycles (3.17). All such field algebra extensions are, up to equivalence, crossed products with respect to some coaction of $G$ on $A$. If $\mathcal{F}$ is one complete irreducible field algebra then the crossed product $\mathcal{B} = \mathcal{F} \rtimes \mathcal{G}$ is independent of the choice of $\mathcal{F}$ and contains all complete irreducible field algebras as unital $^*$-subalgebras. The translation automorphism $\alpha$ extends to $\mathcal{B}$ in such a way that its restriction to any one of the field algebras $\mathcal{F}$ in $\mathcal{B}$ is a translation $\hat{\alpha}$ in the sense of Theorem 4.3. Therefore all complete irreducible field algebras are translation covariant.

The proof of the theorem will use the following two lemmas.

**Lemma 4.5.** For the special comodule algebra action $(\rho, \Delta) = (\rho_{2i,2i+1}, \Delta_D)$ there exists an embedding $\Lambda: \hat{G} \to A$ of the double as a $C^*$-algebra into the observable algebra such that $\rho \circ \Lambda = (\Lambda \otimes \text{id}) \circ \Delta$. (4.12)

*Proof:* Define $\Lambda(D(a)) := A_{2i}(a)$ for $a \in H$ and $\Lambda(D(\varphi)) := A_{2i-1}(\varphi_{(2)})$ $A_{2i+1}(\varphi_{(1)})$ for $\varphi \in \hat{H}$ and verify by straightforward calculation that $\Lambda(D(a))$ and $\Lambda(D(\varphi))$ satisfy the defining relations (B.1) of the double and also the relations $\rho_{2i,2i+1} \circ \Lambda(X) = (\Lambda \otimes \text{id}) \circ \Delta_D(X)$ for the generators $X = D(a)$ and $D(\varphi)$. Q.e.d.

**Lemma 4.6.** Let $(\rho, \Delta)$ be a comodule algebra action and $u \in G \otimes G$ be a unitary $\Delta$-cocycle, i.e. satisfies (3.17). Then there exists a unitary $U \in A \otimes \hat{G}$ such that the pair $(U, u)$ is a cocycle equivalence, i.e. (3.16c) holds.

*Proof:* We need to consider only the case $(\rho, \Delta) = (\rho_{2i,2i+1}, \Delta_D)$. For general $(\rho, \Delta)$ the statement follows from the fact that cocycle equivalences can be composed.

Let $\Lambda$ be the embedding of $\hat{G}$ associated to $(\rho, \Delta)$ in the sense of Lemma 4.5. Then define

$$ U := (\Lambda \otimes \text{id})(u), \tag{4.13} $$

which gives

$$ U \times U = (U \otimes 1) \cdot (\rho \otimes \text{id})(U) = (\Lambda(u_1) \otimes u_2 \otimes 1) \cdot (\rho \circ \Lambda(u_1) \otimes u_2) = $$

$$ = (\Lambda \otimes \text{id} \otimes \text{id}) ((u \otimes 1) \cdot (\Delta \otimes \text{id})(u)) = $$

$$ = (\Lambda \otimes \text{id} \otimes \text{id}) ((1 \otimes u) \cdot (\text{id} \otimes \Delta)(u)) = $$

$$ = (\Pi \otimes u) \cdot (\text{id} \otimes \Delta)(U). $$

Q.e.d.

*Proof of Theorem 4.4.*: The field algebra $\mathcal{F}_{\text{even}}$ associated to the special comodule action $\rho_{2i,2i+1}$ was shown to be irreducible: $\mathcal{A} \cap \mathcal{F}_{\text{even}} = C\Pi$. Anticipating the results of Section 5, $\mathcal{F}_{\text{even}}$ is also complete, i.e. creates all sectors of $\mathcal{A}$, because $\rho_{2i,2i+1}$ is universal. Hence there exist complete irreducible field algebras. Let $(\rho, \Delta)$ be a fixed universal outer comodule algebra action and $\mathcal{F}$ be the
associated complete irreducible field algebra. Let $F^\sharp$ be any complete irreducible field algebra. Completeness implies that there exists a unitary $M \in F^\sharp \otimes G$ implementing a universal amplimorphism $\mu: M(A \otimes 1) = \mu(A)M \ A \in \mathcal{A}$. Although $\mu$ is not necessarily a comodule action there exists an isomorphism of amplimorphisms $U_0 \in \text{Iso}(\rho|\mu)$. Therefore $F_0 := U_0M$ is a unitary implementing $\rho$ and satisfying

$$F_0^{01}F_0^{02}(A \otimes 1 \otimes 1) = \rho \times \rho(A)F_0^{01}F_0^{02}$$

$$(\text{id} \otimes \Delta)(F_0)(A \otimes 1 \otimes 1) = (\text{id} \otimes \Delta) \circ \rho(A) \cdot (\text{id} \otimes \Delta)(F_0)$$

Therefore, by irreducibility of $F^\sharp$,

$$(\text{id} \otimes \Delta)(F_0^*) \cdot F_0^{01}F_0^{02} \in \mathbb{1} \otimes G \otimes G$$

that is, there exist a unitary $u \in G \otimes G$ such that

$$F_0^{01}F_0^{02} = (\text{id} \otimes \Delta)(F_0) \cdot (\mathbb{1} \otimes u) \ . \ (4.14)$$

Associativity of $F^\sharp$ implies that this $u$ is a $\Delta$-cocycle. (Equation (4.14) expresses the fact that $F^\sharp$ is a projective representation of the field algebra $F$ with cocycle $u$.) Now use Lemma 4.6 yielding a cocycle equivalence $(U, u)$ for the cocycle $u$. With this $U$ we can define $F^\sharp := UF_0$ which turns out to be a master field for the comodule algebra action $(\rho^\sharp = \text{Ad}_U \circ \rho, \Delta^\sharp = \text{Ad}_u \circ \Delta)$. Having found a master field for $(\rho^\sharp, \Delta^\sharp)$ within $F^\sharp \otimes G$ means that we have constructed a surjective $*$-homomorphism $\delta: F_{\rho^\sharp} \to F^\sharp$. In order to prove that $\delta$ is injective use assumption III that there exists a conditional expectation $\mathcal{E}^\sharp: F^\sharp \to A$ with finite index. For then $\mathcal{E} := \mathcal{E}^\sharp \circ \delta$ is a conditional expectation from the crossed product $F_{\rho^\sharp}$ onto $A$ and hence by irreducibility of $A \subset F_{\rho^\sharp}$ it must coincide with $\gamma_h$ of subsection 4.1. Now the existence of a quasibasis for $\mathcal{E} = \gamma_h$ shows that an ideal in $F_{\rho^\sharp}$, such as $\text{Ker} \delta$ which is annihilated by $\mathcal{E}$ ($\mathcal{E}(\text{Ker} \delta) = \{0\}$) must necessarily be zero:

$$\text{Ker} \delta = \sum_s F_{\eta_s} \mathcal{E}(F_{\eta_s}^* \text{Ker} \delta) = \{0\} \ .$$

In this way we have shown that $F^\sharp$ is a crossed product and have determined a map $F^\sharp \mapsto u$ from the set of complete irreducible field algebras to the set of $\Delta$-cocycles, for a fixed $\Delta$. This map is obviously surjective and by Theorem 4.1 equivalence classes of $F^\sharp$-s are mapped to coboundary equivalence classes of $u$-s.

That $B$ contains all crossed products $A \rtimes G$ have already been shown before the formulation of the Theorem. It remained to prove translation covariance of all complete irreducible field algebras. Since $F_{\text{even}}$ is translation covariant, we have an equivalence $(U, 1) \in ((\rho^\alpha, \Delta) | (\rho, \Delta))$ where $\rho = \rho_{2i, 2i+1}$ and $\Delta = \Delta_\mathcal{D}$. Let $(\rho', \Delta')$ be an arbitrary comodule algebra action. Choose a cocycle equivalence $(V, u) \in ((\rho', \Delta') | (\rho, \Delta))$. Hence the composition of cocycle equivalences

$$((\alpha \otimes \text{id})(V), u) \cdot (U, 1) \cdot (V^*, u^*) =: (U', 1) \in ((\rho'^\alpha, \Delta') | (\rho', \Delta'))$$

$$\in ((\rho'^\alpha, \Delta') | (\rho', \Delta'))$$

27
is an equivalence, showing that \((\rho', \Delta')\) is \(\alpha\)-covariant. Therefore the corresponding crossed product \(\mathcal{F}_{\rho'}\) is also \(\alpha\)-covariant. \(Q.e.d.\)

5. Universality of \(\rho\)

5.1. Compressibility of the net \(\mathcal{A}\)

Here we derive a property of the net \(\{\mathcal{A}(J)\}\) which will enable us to prove that the amplimorphisms constructed in subsection 3.3 actually exhaust all possible (localized) amplimorphisms up to equivalence.

Let \(J \in \mathcal{I}\) be a non-empty interval of length \(|J| = \text{even}\). Define a subset \(\mathcal{I}_J\) of the set of intervals \(\mathcal{I}\) by

\[
\mathcal{I}_J := \{I \in \mathcal{I} \mid \text{either } I \subset J' \text{ or } I \supset J\}
\]

and a surjective map \(\sigma_J : \mathcal{I}_J \rightarrow \mathcal{I}\) that can be described as the map arising when we discard \(J\) from the chain and reunite the two remaining parts again:

\[
\sigma_J(I) := \begin{cases} 
I & \text{if } I \subset J'_- \\
I - |J| & \text{if } I \subset J'_+ \\
(I \cap J^-_c) \cup (I \cap J^+_c) - |J| & \text{if } I \supset J
\end{cases} \tag{5.1}
\]

where \(J^\pm_\pm\) denote the two components of the complement \(J^c = \mathbb{Z} \setminus J\). \(\sigma_J\) is actually a bijection since it is the lift to intervals of a bijective map from \(\mathbb{Z} \setminus J\) to \(\mathbb{Z}\).

**Definition 5.1.** The relative net over \(J\) is defined by

\[
\mathcal{A}_J(I) = \mathcal{A}(\sigma_J^{-1}(I)) \cap \mathcal{A}(J)' \quad I \in \mathcal{I}
\]

providing a net structure for the relative commutant \(\mathcal{A}(J)' \cap \mathcal{A}\).

**Theorem 5.2.** (compressibility of the Hopf spin net) For any non-empty interval \(J\) of even length the relative net \(\{\mathcal{A}_J(I)\}\) over \(J\) is isomorphic to the original. That is there exists a *-isomorphism \(\kappa_J : \mathcal{A} \cap \mathcal{A}(J)' \rightarrow \mathcal{A}\) such that

\[
\kappa_J(\mathcal{A}_J(I)) = \mathcal{A}(I), \quad I \in \mathcal{I}. \tag{5.2}
\]

**Proof:** At first we point out that the algebras \(\mathcal{A}_J(I)\) and \(\mathcal{A}(I)\) are isomorphic for each \(I\). If \(\sigma_J^{-1}(I) \subset J'\) then this is a trivial consequence of translation covariance of the net and the fact that \(|J| = \text{even}\). If \(\sigma_J^{-1}(I) \supset J\) then the isomorphism follows from comparing the two inclusions \(\mathcal{A}(J) \subset \mathcal{A}(\sigma_J^{-1}(I))\) and \(C \cdot \mathbb{I} \subset \mathcal{A}(I)\). By the crossed product structure of the Hopf spin net and by simplicity of both \(\mathcal{A}(J)\) and \(C \cdot \mathbb{I}\) the two relative commutants \(\mathcal{A}(J)' \cap \mathcal{A}((\sigma_J^{-1}(I))\) and \(C \cdot \mathbb{I}' \cap \mathcal{A}\) can be represented by path algebras of one and the same inclusion graph, and therefore are isomorphic.

The proof that a global isomorphism \(\kappa_J\) exists will be carried out in two steps. At first one constructs \(\kappa_J\) for \(J\) of length \(|J| = 2\). This is a technically
involved calculation and will be presented afterwards. Then one shows that the powers of \( \kappa_J \) with \( |J| = 2 \) yield net isomorphisms \( A_J(I) \to A(I) \) for arbitrary even length intervals \( J \).

Now we proceed by assuming that the theorem is proven for \( |J| = 2 \). Construct the nested sequence of even length intervals \( J_1 = \{i, i + 1\}, \ldots , J_n = \{i, \ldots , i + 2n - 1\} \). Then the isomorphism \( \kappa_{J_1} \) exists. Notice that \( \sigma_{J_1}(J_{n+1}) = J_n \) and that for all \( n < m \)

\[
\sigma_{J_n}^{-1} \circ \sigma_{J_{n+1}}^{-1} = \sigma_{J_m}^{-1}.
\]

Therefore \( \sigma_{J_1}^{-1} \circ \sigma_{J_n}^{-1} = \sigma_{J_{n+1}}^{-1} \). It follows that

\[
A_{J_{n+1}}(I) = A(\sigma_{J_{n+1}}^{-1}(I)) \cap A(J_{n+1})' \subseteq A(\sigma_{J_1}^{-1} \circ \sigma_{J_n}^{-1}(I)) \cap A(J_1)' = A_J(\sigma_{J_n}^{-1}(I)) \quad \forall I \in I.
\]

On the other hand if \( A \in A_{J_{n+1}}(I) \) and \( B \in A(J_{n+1}) \) then \([A, B] = 0\) and especially for \( B \in A(J_{n+1}) \cap A(J_1)' \) in which latter case \([\kappa_{J_1}(A), \kappa_{J_1}(B)] = 0\) follows. Since

\[
\kappa_{J_1}(A(J_{n+1}) \cap A(J_1)') = \kappa_{J_1}(A(\sigma_{J_1}^{-1}(J_n)) \cap A(J_1)') = A(J_n),
\]

we obtain that

\[
\kappa_{J_1}(A_{J_{n+1}}(I)) \subseteq A(J_n)'.
\]

Finally (5.3) and (5.4) together with Definition 5.1 imply that

\[
\kappa_{J_1}(A_{J_{n+1}}(I)) \subseteq A_{J_n}(I) \quad I \in I.
\]

Now (5.5) for \( n = 0, 1, \ldots , m - 1 \) implies that the \( m \)-th power of \( \kappa_{J_1} \) yields a \( C^* \)-inclusion

\[
\kappa_{J_1}^m(A_{J_m}(I)) \subseteq A(I) \quad I \in I
\]

between isomorphic \( C^* \)-algebras and therefore is necessarily an isomorphism. Hence \( \kappa_{J_m} := \kappa_{J_1}^m \) is a net isomorphism required by the Theorem.

The case \( |J| = 2 \): Let us assume \( J = \{2l, 2l + 1\} \). (The case \( J = \{2l - 1, 2l\} \) can be handled analogously.) By additivity of the net it is enough to define \( \kappa_J \) on the 1-point algebras \( A_J(I), |I| = 1 \). If \( I = \{i\} \) and \( i < 2l - 1 \) or \( i > 2l \) then \( A_J(I) = A(I) \) or \( A_J(I) = A(I + 2) \), respectively, and we may define \( \kappa_J \) to be the restriction of \( \text{id}_A \), respectively \( \alpha \) onto \( A_J(I) \). The only non-trivial cases are \( I = \{2l - 1\} \) and \( I = \{2l\} \). To handle them the following Lemma will be useful.

**Lemma 5.3.** Let \( \varphi \in \hat{H} \), \( a \in H \), and the indices \( i, k \) run over \( 1, \ldots , N = \dim H \). We set \( D^{ik}(\varphi) = \langle b^i, \varphi \beta^k \rangle \) and \( D^{ik}(a) = \langle b^i, a \to \beta^k \rangle \) for the left regular representation of \( \hat{H} \) and \( H \), respectively, with a fixed pair of dual bases \( \{\beta^i\} \) and
Lemma is that the partial traces yield the relative commutants of hence providing isomorphisms of the two 3-point algebras with using the explicit representation of the matrix units $N$:

As follows. At first one computes the commutation relations

Therefore

The commutation relations between the neighbouring 1-point algebras. There are 3 non-trivial cases: the relations between $A_J(\{2l - 2\})$ and $A_J(\{2l - 1\})$, between $A_J(\{2l - 1\})$ and $A_J(\{2l\})$ and between $A_J(\{2l\})$ and $A_J(\{2l + 1\})$. The first and the third of these can be checked rather easily: For example

The proof of this Lemma is a rather elementary exercise with Hopf algebra identities, so will be omitted.

Continuing with the proof of the Theorem notice that a consequence of this Lemma is that the partial traces $E(\varphi) = \sum_k E_{kk}(\varphi)$ and $E(a) = \sum_k E_{kk}(a)$ yield the relative commutants of $A(J)$ within the two 3-point algebras:

Therefore $\kappa_J$ is defined by setting $\kappa_J(E(\varphi)) := A_{2l-1}(\varphi)$ and $\kappa_J(E(a)) := A_{2l}(a)$.

In order to show that $\kappa_J$ is a $\ast$-homomorphism we need to check only the commutation relations between the neighbouring 1-point algebras. There are 3 non-trivial cases: the relations between $A_J(\{2l - 2\})$ and $A_J(\{2l - 1\})$, between $A_J(\{2l - 1\})$ and $A_J(\{2l\})$ and between $A_J(\{2l\})$ and $A_J(\{2l + 1\})$. The first and the third of these can be checked rather easily: For example

The commutation relations between $A_J(\{2l - 1\})$ and $A_J(\{2l\})$ can be obtained as follows. At first one computes the commutation relations

using the explicit representation of the matrix units $N$:

$$E^{ik} = \sum_j A_{2l}(b^j S^{-1}(b^j)) A_{2l+1}(\beta^j \beta^k).$$  

(5.6)
Then one obtains
\[
E(a)E(\varphi) = \frac{1}{N^2} \sum_{ijkmn} E^{ij} D^{jk}(a(1)) A(a(2)) E^{km} A(\varphi(1)) D^{mn}(\varphi(2)) E^{ni}
\]
\[
= \frac{1}{N^2} \sum_{ijkmn} \sum_{k'm'} E^{ij} D^{jk}(a(1)) A(a(2)) A(\varphi(1)) D^{kk'}(\varphi(2)) E^{k'm'}
\]
\[
D^{m'm}(S(\varphi(3))) D^{mn}(\varphi(4)) E^{ni} = \sum_{ijkmn} E^{ij} A(\varphi(1)) \sum_{k} D^{jk}(a(1)) D^{km}(\varphi(2)) A(a(2)) E^{ni}
\]
\[
= \frac{1}{N} \sum_{ijm} E^{ij} A(\varphi(1)) \sum_{k} D^{jk}(\varphi(2)) \langle \varphi(3), a(1) \rangle D^{km}(a(2)) A(a(3)) E^{ni}
\]

where in the last line we used the identity
\[
\sum_{k} D^{jk}(a) D^{km}(\varphi) = \sum_{k} D^{jk}(\varphi(1)) \langle \varphi(2), a(1) \rangle D^{km}(a(2))
\]
expressing the fact that \( D \) is a representation of the whole Weil algebra \( \mathcal{W}(H) = \hat{H} \rtimes H \). Now using the commutation relations

\[
E^{ik} A_{2l+2}(a) = A_{2l+2}(a(3)) E^{i'k'} \cdot D^{i'i'}(a(2)) D^{k'k}(S(a(1)))
\]

we can compute the expression
\[
E(\varphi(1)) \langle \varphi(2), a(1) \rangle E(a(2)) = \frac{1}{N^2} \sum_{ijkmn} E^{ij} A(\varphi(1)) D^{jk}(\varphi(2)) E^{km} \langle \varphi(3), a(1) \rangle D^{mn}(a(2)) A(a(3)) E^{ni} = \]
\[
= \frac{1}{N^2} \sum_{ijkmn} \sum_{k'm'} E^{ij} A(\varphi(1)) D^{jk}(\varphi(2)) \langle \varphi(3), a(1) \rangle A(a(5)) E^{k'm'} D^{kk'}(a(4))
\]
\[
D^{m'm}(S(a(3))) D^{mn}(a(2)) E^{ni} = \frac{1}{N} \sum_{ijkm} E^{ij} A(\varphi(1)) D^{jk}(\varphi(2)) \langle \varphi(3), a(1) \rangle D^{km}(a(2)) A(a(3)) E^{ni}
\]

which, when compared to (5.7), yields finally
\[
E(a)E(\varphi) = E(\varphi(1)) \langle \varphi(2), a(1) \rangle E(a(2)). \quad (5.8)
\]

In this way we have constructed a *-homomorphism \( \kappa_J \) mapping \( A_J(I) \) into \( A(I) \) for all \( I \). To see that it is actually an isomorphism one simply constructs its inverse by defining it on the 1-point algebras in the obvious way. \( Q.e.d. \)

5.2. Compressibility of the amplimorphisms
The following theorem will show that compressibility of the chain $A$ has very strong consequences on the structure of amplimorphisms of $A$.

**Theorem 5.4. (Compressibility of the amplimorphisms)** Let $\mu$ be a localized amplimorphism of the Hopf spin chain. Then $\mu$ is equivalent to an amplimorphism $\mu_0$ that is localized within an interval of length 2.

**Proof:** Let $\mu: A \to A \otimes \text{End} V$ be a localized amplimorphism. Choose an interval $I$ of length $|I| = \text{even}$ and $|I| \geq 4$ such that $\mu$ is localized within $I$. Define the interior of $I$ by $\text{Int} I := (I^c)'$. Then by Haag duality $\mu(A(\text{Int} I)) \subset A(\text{Int} I) \otimes \text{End} V$ and by the split property $A(\text{Int} I)$ is simple. Since any amplimorphism of a simple (finite dimensional) algebra is inner, there exists a unitary $U \in A(\text{Int} I) \otimes \text{End} V$ such that

$$\mu(A) = U(A \otimes 1_V)U^*, \quad A \in A(\text{Int} I). \quad (5.9)$$

Let $\mu$ denote the amplimorphism $\text{Ad}_{U^*} \circ \mu$, then $\mu$ acts as the amplified identity on $A(\text{Int} I)$. It follows that for all interval $J \supset \text{Int} I$ we have

$$\mu(J) \cap A(\text{Int} I)' \subset (A(J) \cap A(\text{Int} I)') \otimes \text{End} V$$

Therefore $\mu$ can be restricted to the relative net $A_{\text{Int} I} = A \cap A(\text{Int} I)'$ to yield an amplimorphism $\mu_0$ localized on the interval $I_0 = \sigma_{\text{Int} I}(I)$ (cf. Def.5.1) of length 2 and satisfying

$$\mu_0 \circ \kappa^{-1} = (\kappa^{-1} \otimes \text{id}) \circ \mu$$

where $\kappa$ denotes the compressing isomorphism $\kappa: A_{\text{Int} I} \to A$ constructed in Theorem 5.2.

The map $\mu \mapsto \mu_0$, from the set of ”smeared” amplimorphisms $\mu$ to the 2-point amplimorphisms $\mu_0$, can be inverted. As a matter of fact, using additivity of the net, we can define $\mu$ on the subalgebra $A(\text{Int} I)'$ as $(\kappa \otimes \text{id}) \circ \mu_0 \circ \kappa^{-1}$ and extend it to $A$ by letting it to act on $A(\text{Int} I)$ as the trivial amplification.

Therefore there is a one-to-one correspondence between amplimorphisms localized within $I$ and smeared over $\text{Int} I$ and the 2-point amplimorphisms localized within $I_0$. Since the intertwiners $T \in (\mu|\nu)$ between two smeared amplimorphisms belong to the commutant of both $A(I')$ and $A(\text{Int} I)$, they are scalars. Therefore they are mapped bijectively onto the intertwiner space $(\mu_0|\nu_0)$. This proves that the category of amplimorphisms localized within $I$ is equivalent to the category of amplimorphisms localized within $I_0$, the latter one being a subcategory of the former. In other words all the charges that can be created on a finite interval can also be created on an interval of length 2. \[Q.e.d.\]

An other striking consequence of compressibility is that all amplimorphisms (i.e. localized $C^*$-maps from $A$ to $A \otimes \text{End} V$) are transportable in the following very strong sense: If $\mu$ is localized within some $I \in I$ and $J \in I$ is an arbitrary interval of length at least 2 then there exists a $\nu$ localized within $J$ that is equivalent to $\mu$. The proof goes as follows. At first we remark that there is a left analogue of the shrinking map $\sigma_J$ of (5.1) that uses translation on the left.
hand side of $J$. Combining the left and right $\sigma_J$-s we have a slightly different relative net but can construct a compression isomorphism $\kappa_J$ as in Theorem 5.2. Consequently Theorem 5.4 will imply that an amplimorphism localized within $\{i, \ldots, i + 2n + 1\}$, $i \in \mathbb{Z}$, $n \in \mathbb{N}$, is equivalent to an amplimorphism localized within any 2-point subintervals of the form $\{i + 2m, i + 2m + 1\}$. Now the general transportability property easily follows. Choose an interval $K$ of length even which contains both $I$ and $J$. Let $\mu$ be the smearing of $\mu$ over $K$ and let $\mu_0$ be the 2-point amplimorphism localized within $K_0$ arising as the compression of $\mu$. Obviously $\mu \sim \mu_0$. By the above remark $K$ and $K_0$ can be chosen in such a way that $K_0 \subset J$. Hence $\mu$ is equivalent to an amplimorphism localized within $J$.

In order to be able to conclude that all superselection sectors of the Hopf spin model arise from the application of the special comodule action $\rho_{i,i+1}$ of subsection 3.3 (i.e. that the special comodule action is universal in the category $\text{Amp}_A$) we only have to find the general form of an amplimorphism localized within a 2-point interval $\{i, i + 1\}$. This will be done in the next subsection.

5.3. The 2-point amplimorphisms

**Proposition 5.4.** Let $\mu: A \to A \otimes M_n$ be an amplimorphism localized within an interval $I$ of length 2 and let $\rho$ denote the comodule action $\rho_I$ defined in (3.18) which is localized within the same interval. Then there exists a non-zero intertwiner $T \in (\rho|\mu)$.

**Proof:** We restrict ourselves to the case $I = \{2i, 2i + 1\}$ from Haag duality of the net it follows that $\mu(A_{2i}) \subset A_{2i} \otimes M_n$ and $\mu(A_{2i+1}) \subset A_{2i+1} \otimes M_n$. Hence we may define the $^*$-algebra maps $\overline{\mu}: \hat{H} \to M_n \otimes \hat{H}$ and $\overline{\mu}: H \to H \otimes M_n$ by

$$
\overline{\mu}(\varphi) = \tau_{01} \circ \mu(A_{2i+1}(\varphi)) \quad \overline{\mu}(a) = \mu(A_{2i}(a)) \quad (5.10)
$$

Using commutation relations with $A_{2i+2}$ and $A_{2i-1}$ respectively, we can write

$$
(I_n \otimes a) \overline{\mu}(\varphi) = \overline{\mu}(a_{(1)} \to \varphi)(I_n \otimes a_{(2)}), \quad \overline{\mu}(a)(\varphi \otimes I_n) = (\varphi_{(1)} \otimes I_n)\overline{\mu}(a \leftarrow \varphi_{(2)}),
$$

$$(\text{id}_{M_n} \otimes a \to)(\overline{\mu}(\varphi)) = \overline{\mu}(a \to \varphi), \quad (\leftarrow \varphi \otimes \text{id}_{M_n})\overline{\mu}(a) = \overline{\mu}(a \leftarrow \varphi),$$

$$(\text{id}_{M_n} \otimes \Delta_{\hat{H}}) \circ \overline{\mu} = (\overline{\mu} \otimes \text{id}_{\hat{H}}) \circ \Delta_{\hat{H}}, \quad (\Delta_H \otimes \text{id}_{M_n}) \circ \overline{\mu} = (\text{id}_H \otimes \overline{\mu}) \circ \Delta_H,$$

Applying $\text{id}_{M_n} \otimes \varepsilon_{\hat{H}} \otimes \text{id}_{\hat{H}}$ and $\text{id}_H \otimes \varepsilon_H \otimes \text{id}_{M_n}$ respectively

$$
\overline{\mu}(\varphi) = \tilde{\mu}_\varepsilon(\varphi_{(1)}) \otimes \varphi_{(2)}, \quad \overline{\mu}(a) = a_{(1)} \otimes \tilde{\mu}_\varepsilon(a_{(2)})
$$

with $\tilde{\mu}_\varepsilon: \hat{H} \to M_n$, $\tilde{\mu}_\varepsilon: H \to M_n$ being unital $C^*$-maps. Therefore

$$
\mu(A_{2i+1}(\varphi)) = A_{2i+1}(\varphi_{(2)}) \otimes \tilde{\mu}_\varepsilon(\varphi_{(1)}) \quad \varphi \in \hat{H}
$$

$$
\mu(A_{2i}(a)) = A_{2i}(a_{(1)}) \otimes \tilde{\mu}_\varepsilon(a_{(2)}) \quad a \in H
$$

(5.11)
It remained to investigate the mutual commutations of \( \mu(A_{2i}) \) and \( \mu(A_{2i+1}) \).

\[
\mu(A_{2i+1}(\varphi))\mu(A_{2i}(a)) = \\
A_{2i}(a(1)) \langle a(2), \varphi(2) \rangle A_{2i+1}(\varphi(3)) \otimes \tilde{\mu}_{\varphi}(\varphi(1)) \tilde{\mu}_{\varphi}(a(3))
\]

\[
\mu(A_{2i+1}(\varphi)A_{2i}(a)) = \\
A_{2i}(a(1)) \langle a(3), \varphi(1) \rangle A_{2i+1}(\varphi(3)) \otimes \tilde{\mu}_{\varphi}(a(2)) \tilde{\mu}_{\varphi}(\varphi(2))
\]

Multiplying both of these equations by \( A_{2i}(S(a(0))) \) from the left and by \( A_{2i+1}(S(\varphi(4))) \) from the right we obtain

\[
\langle a(1), \varphi(2) \rangle \tilde{\mu}_{\varphi}(\varphi(3)) \tilde{\mu}_{\varphi}(a(2)) = \langle a(2), \varphi(1) \rangle \tilde{\mu}_{\varphi}(a(3)) \tilde{\mu}_{\varphi}(\varphi(2))
\]

which is but the defining relation (B.1c) of the double \( \mathcal{G} \). Hence

\[
\mathcal{D}(a)\mathcal{D}(\varphi) \in \mathcal{G} \quad \mapsto \quad \tilde{\mu}_{\varphi}(a) \tilde{\mu}_{\varphi}(\varphi) \in M_{a}
\]

defines a non-zero *-algebra homomorphism by means of which we can express \( \mu \) as

\[
\mu = (\text{id}_A \otimes \theta) \circ \rho \quad \tag{5.12}
\]

which obviously implies the existence of a non-zero \( T \in (\rho|\mu) \). As a matter of fact, since \( \theta \neq 0 \), there exists a non-zero \( t \in \text{Hom}(C^n, V) \) such that \( xt = t\theta(x), \ x \in \mathcal{G} \). Then \( T = 1 \otimes t \) is the required intertwiner. \( \text{Q.e.d.} \)

This Proposition together with Theorem 5.4 imply that amplimorphisms \( \rho_{2i,2i+1} \) of (3.18) are universal in the whole category \( \text{Amp}_A \). Therefore Theorems 3.1 and 3.10 yield finally the equivalence of \( \text{Rep}\mathcal{D}(H) \) with the category \( \text{Rep}_A \) of all DHR-representations of \( A \). As a byproduct we obtain that all amplimorphisms are \( \alpha \)-covariant since \( \rho_{2i,2i+1} \) was shown to be \( \alpha \)-covariant by (3.25–26) which immediately gives \( \text{Amp}^\alpha A = \text{Amp}_A \).

**Appendix A: Finite dimensional \( C^* \)-Hopf algebras**

There is an extended literature on Hopf algebra theory the nomenclature of which, however, is by far not unanimous [Sw, ES, Dr1-2, BaSk]. Therefore we summarize in this appendix some standard notions in order to fix our conventions and notations.

A linear space \( B \) over \( C \) together with linear maps

\[
m: B \otimes B \rightarrow B \quad \text{(multiplication)}, \quad \Delta: B \rightarrow B \otimes B \quad \text{(comultiplication)},
\]

\[
\iota: C \rightarrow B \quad \text{(unit)}, \quad \varepsilon: B \rightarrow C \quad \text{(counit)}
\]
is called a **bialgebra** and denoted by \( B(m, \iota, \Delta, \varepsilon) \) if the following axioms hold:

\[
m \circ (m \otimes \text{id}) = m \circ (\text{id} \otimes m), \quad (\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta
\]

\[
m \circ (\iota \otimes \text{id}) = m \circ (\text{id} \otimes \iota) = \text{id}, \quad (\varepsilon \otimes \text{id}) \circ \Delta = (\text{id} \otimes \varepsilon) \circ \Delta = \text{id}
\]

\[
\varepsilon \circ m = \varepsilon \otimes \varepsilon, \quad \Delta \circ \iota = \iota \otimes \iota
\]

\[
\Delta \circ m = (m \otimes m) \circ \tau_{23} \circ (\Delta \otimes \Delta)
\]
where \( \tau_{23} \) denotes the permutation of the tensor factors 2 and 3. We use Sweedler’s notation \( \Delta(x) = x_{(1)} \otimes x_{(2)} \), where the right hand side is understood as a sum \( \sum_i x_{(i)}^i \otimes x_{(i)}^i \in B \otimes B \). For iterated coproducts we write \( x_{(1)} \otimes x_{(2)} \otimes x_{(3)} := \Delta(x_{(1)}) \otimes x_{(2)} \equiv x_{(1)} \otimes \Delta(x_{(2)}) \), etc. The image under \( \iota \) of the number 1 \( \in C \) is the unit element of \( B \) and denoted by 1. The linear dual \( \hat{B} \) becomes also a bialgebra by transposing the structural maps \( m, \iota, \Delta, \varepsilon \) by means of the canonical pairing \( \langle \ , \rangle : \hat{B} \times B \to C \).

A bialgebra \( H(m, \iota, \Delta, \varepsilon) \) is called a Hopf algebra \( H(m, \iota, S, \Delta, \varepsilon) \) if there exists an antipode \( S : H \to H \), i.e. a linear map satisfying

\[
m \circ (S \otimes \mathrm{id}) \circ \Delta = m \circ (\mathrm{id} \otimes S) \circ \Delta = \varepsilon \circ \iota
\]

(A.1)

Using the above notation equ. (A1) takes the form \( S(x_{(1)})x_{(2)} = x_{(1)}S(x_{(2)}) = \varepsilon(x)1 \), which in connection with the coassociativity of \( \Delta \) is often applied in formulas involving iterated coproducts like, e.g., \( x_{(1)} \otimes x_{(2)}S(x_{(3)})x_{(4)} = x_{(1)} \otimes x_{(2)} \). All other properties of the antipode, i.e. \( S(xy) = S(y)S(x), \Delta \circ S = (S \otimes S) \circ \Delta_{\text{op}} \) and \( \varepsilon \circ S = \varepsilon \), as well as the uniqueness of \( S \) are all consequences of the axiom (A.1) [Sw]. The dual bialgebra \( \hat{H} \) of \( H \) is also a Hopf algebra with the antipode defined by

\[
\langle S(\varphi), x \rangle := \langle \varphi, S(x) \rangle \quad \varphi \in \hat{H}, \ x \in H.
\]

(A.2)

A \(*\)-Hopf algebra \( H(m, \iota, S, \Delta, \varepsilon, \ast) \) is a Hopf algebra \( H(m, \iota, S, \Delta, \varepsilon) \) together with an antilinear involution \( \ast : H \to H \) such that \( H(m, \iota, \ast) \) is a \(*\)-algebra and \( \Delta \) and \( \varepsilon \) are \(*\)-algebra maps. It follows that \( \overline{S} := \ast \circ S \circ \ast \) is the antipode in the Hopf algebra \( H_{\text{op}} \) (i.e. with opposite multiplication) and therefore \( \overline{S} = S^{-1} \) [Sw].

The dual of a \(*\)-Hopf algebra is also a \(*\)-Hopf algebra with \(*\)-operation defined by

\[
\langle \varphi^\ast, x \rangle := \overline{\langle \varphi, S(x) \rangle}.
\]

(A.3)

Let \( \mathcal{A} \) be a \(*\)-algebra and \( H \) be a \(*\)-Hopf algebra. A (Hopf module) left action of \( H \) on \( \mathcal{A} \) is a linear map \( \gamma : H \otimes \mathcal{A} \to \mathcal{A} \) satisfying the following axioms: For \( A, B \in \mathcal{A}, x, y \in H \)

\[
\begin{align*}
\gamma_x \circ \gamma_y(A) &= \gamma_{xy}(A) \\
\gamma_x(AB) &= \gamma_{x(1)}(A)\gamma_{x(2)}(B) \\
\gamma_x(A^\ast) &= \gamma_{S(x)}^\ast(A)^\ast
\end{align*}
\]

(A.4)

A right action of \( H \) is a left action of \( H_{\text{op}} \). Important examples are the action of \( H \) on \( \hat{H} \) and that of \( \hat{H} \) on \( H \) given by the Sweedler’s arrows:

\[
\begin{align*}
\gamma_x(\varphi) &= x \to \varphi := \varphi_{(1)} \langle x, \varphi_{(2)} \rangle \\
\gamma_{\varphi}(x) &= \varphi \to x := x_{(1)} \langle \varphi, x_{(2)} \rangle
\end{align*}
\]

(A.5a, b)

A left action is called inner if there exists a \(*\)-algebra map \( i : H \to \mathcal{A} \) such that \( \gamma_x(A) = i(x_{(1)}) A i(S(x_{(2)})) \). Left \( H \)-actions \( \gamma \) are in one-to-one correspondence
with right $\hat{H}$-coactions (often denoted by the same symbol) $\gamma : A \to A \otimes \hat{H}$ defined by

$$\gamma(A) := \gamma_{b_i}(A) \otimes \xi_i, \quad A \in A$$

where $\{b_i\}$ is a basis in $H$ and $\{\xi_i\}$ is the dual basis in $\hat{H}$ and where for simplicity we assume from now on $H$ to be finite dimensional. Conversely, we have $\gamma_x = (\text{id}_A \otimes x) \circ \gamma$. The defining properties of a coaction are given in equs. (3.11a-e).

Given a left $H$-action (right $\hat{H}$-coaction) $\gamma$ one defines the crossed product $A\rtimes_{\gamma} H$ as the $C$-vector space $A \otimes H$ with $*$-algebra structure

$$\gamma(A \otimes x)(B \otimes y) := A\gamma_{x(1)}(B) \otimes x(2)y \quad \quad (A.6a)$$

$$\gamma(A \otimes x)^* := (1_A \otimes x^*)(A^* \otimes 1_H) \quad \quad (A.6b)$$

An important example is the ”Weyl algebra” $\mathcal{W}(\hat{H}) := \hat{H} \rtimes_{\gamma} H$, where the crossed product is taken with respect to the natural left action (A.5a). We have $\mathcal{W}(\hat{H}) \cong \text{End} \hat{H}$ where the isomorphism is given by (see [N] for a review)

$$w : \psi \otimes x \mapsto Q^+(\psi)P^+(x). \quad \quad (A.7)$$

Here we have introduced $Q^+(\psi), \psi \in \hat{H}$ and $P^+(x), x \in H$ as operators in End $\hat{H}$ defined on $\xi \in \hat{H}$ by

$$Q^+(\psi)\xi := \psi \xi$$

$$P^+(x)\xi := x \to \xi$$

Any right $H$-coaction $\beta : A \to A \otimes H$ gives rise to a natural left $H$-action $\gamma$ on $A\rtimes_{\beta} \hat{H}$

$$\gamma_x(A \otimes \psi) := A \otimes (x \to \psi) \quad \quad (A.8)$$

The resulting double crossed product $(A\rtimes_{\beta} \hat{H})\rtimes_{\gamma} H$ contains $\mathcal{W}(\hat{H}) \cong \text{End} \hat{H}$ as the subalgebra given by $1_A \otimes \psi \otimes x \cong Q^+(\psi)P^+(x)$, $\psi \in \hat{H}, x \in H$. Moreover, by the Takesaki duality theorem [NaTa] the double crossed product $(A\rtimes_{\beta} \hat{H})\rtimes_{\gamma} H$ is canonically isomorphic to $A \otimes \text{End} \hat{H}$. In fact, defining the representation $L : H \to \text{End} \hat{H}$ by

$$L(x)\xi := \xi \leftarrow S^{-1}(x) \equiv \langle \xi(1), S^{-1}(x)\rangle\xi(2) \quad \quad (A.9)$$

one easily verifies that $T : (A\rtimes_{\beta} \hat{H})\rtimes_{\gamma} H \to A \otimes \text{End} \hat{H}$

$$T(A \otimes 1_{\hat{H}} \otimes 1_H) := (\text{id}_A \otimes L)(\beta(A)) \quad \quad (A.10a)$$

$$T(1_A \otimes \psi \otimes x) := 1_A \otimes Q^+(\psi)P^+(x) \quad \quad (A.10b)$$

defines a $*$-algebra map. $T$ is surjective since $w$ is surjective and therefore $1_A \otimes \text{End} \hat{H} \subset \text{Im} T$ and

$$A \otimes 1_{\text{End} \hat{H}} \equiv A_{(0)} \otimes L(A_{(1)}S(A_{(2)}))$$

$$= T(A_{(0)} \otimes 1_{\hat{H}} \otimes 1_H)(1_A \otimes L(S(A_{(1)})))$$

$$\in \text{Im} T$$
for all \( A \in \mathcal{A} \). Here we have used the notation \( A_{(0)} \otimes A_{(1)} = \beta(A) \),
\[
A_{(0)} \otimes A_{(1)} \otimes A_{(2)} = (\beta \otimes \text{id}_H)(\beta(A)) \equiv (\text{id}_A \otimes \Delta)(\beta(A))
\]
(including a summation convention) and the identity \( (\text{id}_A \otimes \varepsilon) \circ \beta = \text{id}_A \), see equ. (3.11d,e). The inverse of \( \mathcal{T} \) is given by
\[
\mathcal{T}^{-1}(1_A \otimes W) = 1_A \otimes w^{-1}(W) \\
\mathcal{T}^{-1}(A \otimes 1_{\text{End} \hat{H}}) = A_{(0)} \otimes w^{-1}(L(S(A_{(1)})))
\]
for \( W \in \text{End} \hat{H} \) and \( A \in \mathcal{A} \).

A left(right) integral in \( \hat{H} \) is an element \( \chi^L(\chi^R) \in \hat{H} \) satisfying
\[
\varphi \chi^L = \chi^L \cdot \varepsilon(\varphi) \quad \chi^R \varphi = \varepsilon(\varphi) \cdot \chi^R
\]
for all \( \varphi \in \hat{H} \) or equivalently
\[
\chi^L \rightarrow x = \langle \chi^L, x \rangle 1, \quad x \leftarrow \chi^R = \langle \chi^R, x \rangle 1
\]
for all \( x \in H \). Similarly one defines left(right) integrals in \( H \).

If \( H \) is finite dimensional and semisimple then so is \( \hat{H} \) [LaRa] and in this case they are both unimodular, i.e. left and right integrals coincide and are all given as scalar multiples of a unique one dimensional central projection
\[
e_{\varepsilon} = e^*_\varepsilon = e^2_\varepsilon = S(e_\varepsilon)
\]
which is then called the Haar integral.

For \( \varphi, \psi \in \hat{H} \) and \( h \equiv e_\varepsilon \in H \) the Haar integral define the hermitian form
\[
\langle \varphi | \psi \rangle := \langle \varphi^* \psi, h \rangle
\]
Then \( \langle \cdot | \cdot \rangle \) is nondegenerate [LaSw] and it is positive definite — i.e. the Haar integral \( h \) provides a positive state (the Haar “measure”) on \( \hat{H} \) — if and only if \( \hat{H} \) is a \( C^* \)-Hopf algebra. These are the “finite matrix pseudogroups” of [Wo]. They also satisfy \( S^2 = \text{id} \) and \( \Delta(h) = \Delta_{op}(h) \) [Wo]. If \( \hat{H} \) is a finite dimensional \( C^* \)-Hopf algebra then so is \( H \), since \( H \ni x \rightarrow P^+(x) \in \text{End} \hat{H} \) defines a faithful \(*\)-representation on the Hilbert space \( \mathcal{H} \equiv L^2(\hat{H}, h) \). Hence finite dimensional \( C^* \)-Hopf algebras always come in dual pairs. Any such pair serves as a building block for our Hopf spin model.

**Appendix B: The Drinfeld Double**

Here we list the basic properties of the Drinfeld double \( \mathcal{D}(H) \) (also called quantum double) of a finite dimensional \(*\)-Hopf algebra \( H \) [Dr1-2, Maj2]. Although most of them are well known in the literature, the presentation (B.1) by generators and relations given below seems to be new.
As a ∗-algebra \( \mathcal{D}(H) \) is generated by elements \( \mathcal{D}(a), \ a \in H \) and \( \mathcal{D}(\varphi), \ \varphi \in \hat{H} \) subjected to the following relations:

\[
\begin{align*}
\mathcal{D}(a)\mathcal{D}(b) &= \mathcal{D}(ab) & (B.1a) \\
\mathcal{D}(\varphi)\mathcal{D}(\psi) &= \mathcal{D}(\varphi\psi) & (B.1b) \\
\mathcal{D}(a_{(1)}) \langle a_{(2)}, \varphi_{(1)} \rangle \mathcal{D}(\varphi_{(2)}) &= \mathcal{D}(\varphi_{(1)}) \langle \varphi_{(2)}, a_{(1)} \rangle \mathcal{D}(a_{(2)}) & (B.1c) \\
\mathcal{D}(a)^* &= \mathcal{D}(a^*), \mathcal{D}(\varphi)^* = \mathcal{D}(\varphi^*) & (B.1d)
\end{align*}
\]

The relation (B.1c) is equivalent to any one of the following two relations

\[
\begin{align*}
\mathcal{D}(a)\mathcal{D}(\varphi) &= \mathcal{D}(\varphi_{(2)})\mathcal{D}(a_{(2)}) \langle a_{(1)}, \varphi_{(3)} \rangle \langle S^{-1}(a_{(3)}), \varphi_{(1)} \rangle & (B.2a) \\
\mathcal{D}(\varphi)\mathcal{D}(a) &= \mathcal{D}(a_{(2)})\mathcal{D}(\varphi_{(2)}) \langle \varphi_{(1)}, a_{(3)} \rangle \langle S^{-1}(\varphi_{(3)}), a_{(1)} \rangle & (B.2b)
\end{align*}
\]

These imply that as a linear space \( \mathcal{D}(H) \cong H \otimes \hat{H} \) and also that as a ∗-algebra \( \mathcal{D}(H) \) and \( \mathcal{D}(\hat{H}) \) are isomorphic. This ∗-algebra will be denoted by \( \mathcal{G} \).

The Hopf algebraic structure of \( \mathcal{D}(H) \) is given by the following coproduct, counit, and antipode:

\[
\begin{align*}
\Delta_{\mathcal{D}}(\mathcal{D}(a)) &= \mathcal{D}(a_{(1)}) \otimes \mathcal{D}(a_{(2)}) & \Delta_{\mathcal{D}}(\mathcal{D}(\varphi)) &= \mathcal{D}(\varphi_{(2)}) \otimes \mathcal{D}(\varphi_{(1)}) & (B.3a) \\
\varepsilon_{\mathcal{D}}(\mathcal{D}(a)) &= \varepsilon(a) & \varepsilon_{\mathcal{D}}(\mathcal{D}(\varphi)) &= \varepsilon(\varphi) & (B.3b) \\
S_{\mathcal{D}}(\mathcal{D}(a)) &= \mathcal{D}(S(a)) & S_{\mathcal{D}}(\mathcal{D}(\varphi)) &= \mathcal{D}(S^{-1}(\varphi)) & (B.3c)
\end{align*}
\]

It is straightforward to check that equs. (B.3) provide a ∗-Hopf algebra structure on \( \mathcal{D}(H) \). Moreover, \( \mathcal{D}(\hat{H}) = (\mathcal{D}(H))_{\text{cop}} \) (i.e. with opposite coproduct) by (B.3a).

If \( H \) and \( \hat{H} \) are \( C^* \)-Hopf algebras then so is \( \mathcal{D}(H) \). To see this one checks that

\[
\mathcal{D}(h)\mathcal{D}(\chi) = \mathcal{D}(\chi)\mathcal{D}(h) =: h_{\mathcal{D}}
\]

provides the Haar integral in \( \mathcal{D}(H) \) and that the positivity of the Haar states \( h \in H \) and \( \chi \in \hat{H} \) implies the positivity of the state \( h_{\mathcal{D}} \) on \( \hat{\mathcal{D}(H)} \).

The dual \( \hat{\mathcal{D}(H)} \) of \( \mathcal{D}(H) \) has been studied by [PoWo]. As a coalgebra it is \( \mathcal{G} \) and coincides with the coalgebra \( \mathcal{D}(\hat{H}) \). The latter one, however, as an algebra differs from \( \hat{\mathcal{D}(H)} \) in that the multiplication is replaced by the opposite multiplication.

The remarkable property of the double construction is that it always yields a quasitriangular Hopf algebra [Dr1-2]. By definition this means that there exists a unitary \( R \in \mathcal{D}(H) \otimes \mathcal{D}(H) \) satisfying the hexagonal identities \( R^{13}R^{12} = (\text{id} \otimes \Delta)(R) \), \( R^{13}R^{23} = (\Delta \otimes \text{id})(R) \), and the intertwining property \( R\Delta(x) = \Delta_{\text{op}}(x)R \), \( x \in \mathcal{D}(H) \), where \( \Delta_{\text{op}}: x \mapsto x_{(2)} \otimes x_{(1)} \).

If \( \{b_A\} \) and \( \{\beta^A\} \) denote bases of \( H \) and \( \hat{H} \), respectively, that are dual to each other, \( \langle \beta^A, b_B \rangle = \delta_A^B \), then

\[
R \equiv R_1 \otimes R_2 := \sum_A \mathcal{D}(b_A) \otimes \mathcal{D}(\beta^A)
\]
is independent of the choice of the bases and satisfies the above identities.

An important theorem proven by Drinfeld [Dr1] claims that in a quasitriangular Hopf algebra \( G(m, \iota, S, \Delta, \varepsilon, R) \) there exists a canonically chosen element \( s \in G \) implementing the square of the antipode, namely \( s = S(R_2)R_1 \). Its coproduct is related to the \( R \)-matrix by the equation
\[
\Delta(s) = (R^{op}R)^{-1}(s \otimes s) = (s \otimes s)(R^{op}R)^{-1}
\]
which turns out to mean that \( s \) defines a universal balancing element in the category of representations of \( G \).

The universal balancing element \( s \) of \( \mathcal{D}(H) \) takes the form
\[
s := S_\mathcal{D}(R_2)R_1 \equiv \mathcal{D}(S^{-1}(\beta^A))\mathcal{D}(b_A)
\]
and is a central unitary of \( G \). Its inverse can be written simply as
\[
s^{-1} = R_1R_2 = R_2R_1.
\]

The existence of \( s \) satisfying (B.6) is needed in Section 4.1 to prove that in the Hopf spin model the two-point amplimorphisms (and therefore, by Lemma 3.16, all universal amplimorphisms) are strictly translation covariant.

Summarizing, the 5-plet \( \mathcal{D}(H) = (G, \Delta_\mathcal{D}, \varepsilon_\mathcal{D}, S_\mathcal{D}, R) \) defines a quasitriangular \(*\)-Hopf algebra. If we compare this structure with the double of the dual Hopf algebra \( \hat{H} \) we find \( \mathcal{D}(\hat{H}) = (G, \Delta^{op}_\mathcal{D}, \varepsilon_\mathcal{D}, S^{-1}_\mathcal{D}, R^{op}) \).

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