3-manifolds represented by 4-regular graphs
with three Eulerian cycles

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We construct and study a new class of compact hyperbolic 3-manifolds with totally geodesic boundary. This class exhibits a number of remarkable properties. Members of this class are described by triples of Eulerian cycles in 4-regular graphs. Two Eulerian cycles are said to be compatible if there is no common pair of consecutive edges in these cycles. If a 4-regular graph $G$ contains a triple $\theta$ of pairwise compatible Eulerian cycles, we say that $G$ is 3-Eulerian and $\theta$ is a framing of $G$. Each finite 3-vertex-connected simple 4-regular graph is 3-Eulerian [1]. Let $G$ be a 3-Eulerian graph, and let $\theta$ be a framing of $G$. A polyhedral realization of the pair $(G, \theta)$ is a 2-dimensional polyhedron $P(G, \theta)$ obtained from $G$ by attaching a 2-cell along each cycle in $\theta$.

**Lemma 1.** The polyhedron $P(G, \theta)$ is a special spine of a 3-manifold with non-empty boundary.

**Proof.** As follows from the theory of special polyhedra [2], Chap. 1, it suffices to verify that the boundary curve of each 2-component $\xi$ of $P = P(G, \theta)$ is orientation preserving on the surface $P \setminus \xi$. The latter follows from the fact that the remaining 2-components of $P$ give a checkerboard colouring on $P \setminus \xi$ since their boundary curves are Eulerian cycles. $\square$

It is known ([2], Theorem 1.1.17) that each special spine of a 3-manifold $M$ with boundary determines $M$ uniquely. Denote by $M(G, \theta)$ the manifold (with boundary) determined by $P(G, \theta)$.

**Theorem 1.** Let $G$ be a 3-Eulerian graph with $n \geq 4$ vertices, and let $\theta$ be a framing of $G$. Then (1) $M(G, \theta)$ is hyperbolic with totally geodesic connected boundary; (2) the Matveev complexity of $M(G, \theta)$ is $n$; (3) the topological ideal triangulation $T(G, \theta)$ dual to $P(G, \theta)$ is minimal.

**Proof.** (1) The truncated triangulation $T^*$ of $M = M(G, \theta)$ dual to $P = P(G, \theta)$ consists of $n$ truncated tetrahedra (according to the number of true vertices of $P$) and has exactly three edges (according to the number of 2-components of $P$) such that each edge of $T^*$ is incident to exactly $2n$ dihedral angles of the tetrahedra. The last condition allows one (since $n \geq 4$) to realize topological truncated tetrahedra as congruent regular truncated hyperbolic tetrahedra, which gives a hyperbolic structure on $M$. A detailed description of this technique is given, for instance, in [3]. To prove that $\partial M$ is connected (which also holds for $n \in \{2, 3\}$) consider the graph $\Gamma$ whose vertices correspond to the components of $\partial M$ and whose edges

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correspond to the edges of \( T^* \). Analyzing the correspondence between the edges of any of the truncated tetrahedra in \( T^* \) and the edges of \( \Gamma \), we find that \( \Gamma \) is a wedge of three circles.

(2)–(3) The proof boils down to demonstrating that \( M \) has no special spine with fewer than \( n \) (true) vertices. Sufficiency follows by (1) since every compact hyperbolic 3-manifold with totally geodesic boundary has a special spine whose number of vertices is equal to its Matveev complexity ([2], Theorem 2.2.4). Suppose that \( M \) has a special spine \( Q \) with \( n' < n \) vertices. The equality of Euler characteristics \( \chi(Q) = \chi(M) = \chi(P) \) implies that the number of 2-components in \( Q \) is less than that in \( P \). From this we can deduce that \( Q \) contains at most one closed surface and \( P \) contains three. But this is impossible because \( H_2(Q; \mathbb{Z}_2) = H_2(M; \mathbb{Z}_2) = H_2(P; \mathbb{Z}_2) \), while the number of closed surfaces contained in an arbitrary special polyhedron \( S \) equals the number of non-trivial elements in the homology group \( H_2(S; \mathbb{Z}_2) \). \( \square \)

**Theorem 2.** Let \( G \) and \( G' \) be 3-Eulerian graphs with \( n \geq 4 \) and \( n' \geq 4 \) vertices and with framings \( \theta \) and \( \theta' \), respectively. Then the manifolds \( M(G, \theta) \) and \( M(G', \theta') \) are homeomorphic if and only if the framed graphs \( (G, \theta) \) and \( (G', \theta') \) are isomorphic.

**Proof.** This follows from assertion (1) of Theorem 1 and Mostow’s rigidity theorem because in the above construction with congruent regular truncated hyperbolic tetrahedra the polyhedron \( P(G, \theta) \) is homeomorphic to the cut locus of the corresponding hyperbolic manifold, and thus \( P(G, \theta) \) is uniquely determined by the manifold. \( \square \)

We denote the class of all manifolds of the form \( M(G, \theta) \) with complexity \( n \) by \( \mathcal{M}_n \).

**Theorem 3.** For each sufficiently large \( n \in \mathbb{N} \), we have

\[
n! < |\mathcal{M}_n| < n! 4^n.
\]

**Proof.** Since each Eulerian graph \( G \) is uniquely determined by the sequence of vertices of an Eulerian path in \( G \), it follows that the number of connected 4-regular \( n \)-vertex graphs with a distinguished Eulerian path is \((2n)!/(n! 2^n)\). Since an Eulerian cycle occurs in no more than \( 2^{n-1} \) framings, we have \(|\mathcal{M}_n| \leq (2n)!/(n! 2) < n! 4^n\).

The results of [4] and [5] imply that both the number of 4-vertex-connected asymmetric (that is, having no non-trivial automorphisms) simple 4-regular \( n \)-vertex unlabeled graphs and the number of all simple 4-regular \( n \)-vertex unlabeled graphs are asymptotic to \( e^{-15/4}(4n)!/((96)^n(2n)! n!) \). All finite 3-vertex-connected simple 4-regular graphs are 3-Eulerian [1]. A quick enumeration of the options shows that for each framing \( \theta \) of an arbitrary 3-Eulerian graph \( G \) and for each vertex \( v \) in \( G \), there is at least one way to change the cycles of \( \theta \) at \( v \) so as to obtain a framing of \( G \) that is distinct from \( \theta \). This implies that each asymmetric 3-Eulerian \( n \)-vertex graph admits at least \( 2^{n-1} \) framings. Combining this with the Stirling formula, we can deduce that \(|\mathcal{M}_n| > n! C \cdot 4^n/(3^n \cdot 2\pi n e^{15/4}\sqrt{2})\) for any \( C < 1 \) and for all sufficiently large \( n \). \( \square \)
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