APPROXIMATE FUNCTIONAL EQUATIONS FOR THE HURWITZ AND LERCH ZETA-FUNCTIONS

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ABSTRACT. As one of the asymptotic formulas for the zeta-function, Hardy and Littlewood gave asymptotic formulas called the approximate functional equation. In 2003, R. Garunkštis, A. Laurinčikas, and J. Steuding (in [1]) proved the Riemann-Siegel type of the approximate functional equation for the Lerch zeta-function \( \zeta_L(s, \alpha, \lambda) = \sum_{n=0}^{\infty} e^{2\pi in\lambda}(n + \alpha)^{-s} \). In this paper, we prove another type of approximate functional equations for the Hurwitz and Lerch zeta-functions. R. Garunkštis, A. Laurinčikas, and J. Steuding (in [2]) obtained the results on the mean square values of \( \zeta_L(\sigma + it, \alpha, \lambda) \) with respect to \( t \). We obtain the main term of the mean square values of \( \zeta_L(1/2 + it, \alpha, \lambda) \) using a simpler method than their method in [2].

1. Introduction and the statement of results

Let \( s = \sigma + it \) be a complex variable, and let \( 0 < \alpha \leq 1, 0 < \lambda \leq 1 \) be real parameters. The Hurwitz zeta-function \( \zeta_H(s, \alpha) \) and the Lerch zeta-function \( \zeta_L(s, \alpha, \lambda) \) are defined by

\[
\zeta_H(s, \alpha) = \sum_{n=0}^{\infty} \frac{1}{(n + \alpha)^s},
\]

\[
\zeta_L(s, \alpha, \lambda) = \sum_{n=0}^{\infty} \frac{e^{2\pi in\lambda}}{(n + \alpha)^s},
\]

respectively. There series are absolutely convergent for \( \sigma > 1 \). Also, if \( 0 < \lambda < 1 \), then the series (1.1) is convergent even for \( \sigma > 0 \).

As a classical asymptotic formula for the Riemann zeta-function, the following was proved by Hardy and Littlewood (§4 in [5]); we suppose that \( \sigma_0 > 0, x \geq 1 \), then

\[
\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma})
\]

uniformly for \( \sigma \geq \sigma_0, |t| < 2\pi x/C \), where \( C > 1 \) is a constant. Also, Hardy and Littlewood proved the following asymptotic formula (§4 in [5]); we suppose that \( 0 \leq \sigma \leq 1, x \geq 1, y \geq 1 \) and \( 2\pi xy = |t| \), then

\[
\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \chi(s) \sum_{n \leq y} \frac{1}{n^{1-s}} + O(x^{-\sigma}) + O(|t|^{1/2-\sigma} y^{\sigma-1}),
\]

where \( \chi(s) = 2\Gamma(1-s)\sin(\pi s/2)(2\pi)^{s-1} \) and note that \( \zeta(s) = \chi(s)\zeta(1-s) \) holds. This is called approximate functional equation.

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Further, there is a Riemann-Siegel type of the approximate functional equation for \( \zeta(s) \); suppose that \( 0 \leq \sigma \leq 1, x = \sqrt{t/2\pi} \), and \( N < Ct \) with a sufficiently small constant \( C \). Then

\[
\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \chi(s) \sum_{n \leq x} \frac{1}{n^{1-s}} + (1-x)^{-1} e^{\pi i(s-1)/2} (2\pi t)^{s/2-1/2} e^{it/2-i\pi/8} \\
\times \Gamma(1-s) \left( S_N + O \left( \left( \frac{CN}{t} \right)^{N/6} \right) + O(e^{-Ct}) \right),
\]

where

\[
S_N = \sum_{n=0}^{N-1} \sum_{\nu \leq n/2} \frac{n! \nu-n}{\nu!(n-2\nu)!2^\nu} \left( \frac{2}{\pi} \right)^{n/2-\nu} a_n (n-2\nu) \left( \frac{n}{\pi} - 2\lfloor x \rfloor \right),
\]

with \( a_n \) defined by

\[
\exp \left( (s-1) \log \left( 1 + \frac{z}{\sqrt{t}} \right) - iz\sqrt{t} - \frac{1}{2}iz^2 \right) = \sum_{n=0}^{\infty} a_n z^n,
\]

with \( a_0 = 1, a_n \ll t^{-n/2+[n/3]} \). R. Garunkštis, A. Laurinčikas, and J. Steuding proved an analogue of (1.4) for the Lerch zeta-function as follows;

**Theorem 1** (R. Garunkštis, A. Laurinčikas, and J. Steuding [1]). *Suppose that \( 0 < \alpha \leq 1, 0 < \lambda < 1 \) and \( 0 \leq \sigma \leq 1 \). Suppose that \( t \geq 1, x = \sqrt{t/2\pi}, N = \lfloor x \rfloor, M = \lfloor x - \alpha \rfloor \) and \( \beta = N - M \). Then*

\[
\zeta_L(s, \alpha, \lambda) = \sum_{m=0}^{M} \frac{e^{2\pi i m \lambda}}{(m+\alpha)^s} + \left( \frac{2\pi}{t} \right)^{s-1/2+it} e^{it+\pi i/4-2\pi i (\lambda) \alpha} \sum_{n=0}^{N} \frac{e^{-2\pi i n \lambda}}{(n+\lambda)^{1-s}} \\
+ \left( \frac{2\pi}{t} \right)^{s/2} e^{\pi i f(\lambda, \alpha, \sigma, t) \phi(2x - 2N + \beta - \{\lambda\} - \alpha)} + O(t^{(\sigma-2)/2}),
\]

where

\[
f(\lambda, \alpha, \sigma, t) = -\frac{t}{2\pi} \log \frac{t}{2\pi e} - \frac{7}{8} + \frac{1}{2}(\alpha^2 - \{\lambda\}^2) \\
- \alpha \beta + 2y(\beta + \{\lambda\} - \alpha) - \frac{1}{2}(N + M) - \{\lambda\}(\beta + \alpha).
\]

We prove an analogue of the approximate functional equation (1.3) for (1.1) and (1.2) (in Theorem 2), and gave another proof of the mean square formula for \( \zeta_L(1/2 + it, \alpha, \lambda) \) with respect to \( t \) (in Theorem 3).
Theorem 2. Let $0 < \alpha \leq 1$ and $0 < \lambda < 1$. Suppose that $0 \leq \sigma \leq 1$, $x \geq 1$, $y \geq 1$ and $2\pi xy = |t|$. Then

$$
\zeta_L(s, \alpha, \lambda) = \sum_{0 \leq n \leq x} \frac{e^{2\pi i n \lambda}}{(n + \alpha)^s} + \Gamma(1-s)(2\pi)^{(1-s)} \sum_{0 \leq n \leq y} \frac{e^{2\pi i (1-\alpha)}}{(n + \lambda)^{1-s}} \left\{ e^{(1-s)/2 - 2\alpha \lambda} \pi i \sum_{0 \leq n \leq y} \frac{e^{2\pi i n \alpha}}{(n + 1 - \lambda)^{1-s}} \right\} + O(x^{-\sigma}) + O(|t|^{1/2} y^{\sigma-1}). \tag{1.6}
$$

Also, in the case $\lambda = 1$ that is $\zeta_H(s, \alpha)$ it follows that

$$
\zeta_H(s, \alpha) = \sum_{0 \leq n \leq x} \frac{1}{(n + \alpha)^s} + \Gamma(1-s)(2\pi)^{(1-s)} \left\{ e^{2\pi i (1-s)} \sum_{n \leq y} \frac{e^{2\pi i (1-\alpha)}}{n^{1-s}} + e^{2\pi i (1-s)} \sum_{n \leq y} \frac{e^{2\pi i n \alpha}}{n^{1-s}} \right\} + O(x^{-\sigma}) + O(|t|^{1/2} y^{\sigma-1}). \tag{1.7}
$$

Remark 1. Theorem 2 can be proved by the method similar to the proof of Theorem 1 but results of Theorem 2 has advantage of choosing parameters $x$ and $y$ freely, only under the condition $2\pi xy = |t|$ as compared with the result of Theorem 1. Also for approximate functional equations (1.6) and (1.7), $\zeta_L(s, \alpha, \lambda)$ is a generalization of $\zeta_H(s, \alpha)$, but (1.6) in Theorem 2 does not include (1.7).

Theorem 3. Let $0 < \alpha \leq 1$, $0 < \lambda \leq 1$. Then,

$$
\int_1^T \left| \zeta_L \left( \frac{1}{2} + it, \alpha, \lambda \right) \right|^2 dt = T \log \frac{T}{2\pi} + \begin{cases} O(T(\log T)^{1/2}) & (0 < \alpha < 1), \\ O(T(\log T)^{3/4}) & (\alpha = 1), \end{cases} \tag{1.8}
$$

as $T \to \infty$.

Remark 2. The result of Theorem 3 has larger error term than the result already proved by R. Garunkštis, A. Laurinčikas and J. Steuding [2], and they proved using Theorem 1 (see [2]). However, the main term on the right-hand side of (1.8) can be obtained more simply than the method of [2] by using Theorem 2. We will describe the proof of Theorem 3 in Section 3.

2. Proof of Theorem 2

In this section, we prove Theorem 2. The basic tool of the proof is the same as the approximate functional equation for the Riemann zeta-function (1.3), that is the saddle point method.
Proof of Theorem 2 Let $M \in \mathbb{N}$ be sufficiently large. We have

$$
\zeta_L(s, \alpha, \lambda) = \sum_{n=0}^{M} \frac{e^{2\pi in\lambda}}{(n + \alpha)^s} + \sum_{n=M+1}^{\infty} \frac{e^{2\pi in\lambda}}{(n + \alpha)^s} + \frac{e^{2\pi iM}}{\Gamma(s)} \int_0^\infty t^{s-1} e^{(M+\alpha)t} dt - e^{2\pi i\lambda} \int_C z^{s-1} e^{(M+\alpha)z} dz,
$$

where $C$ is the contour integral path that comes from $+\infty$ to $\varepsilon$ along the real axis, then goes along the circle of radius $\varepsilon$ counter clockwise, and finally goes from $\varepsilon$ to $+\infty$.

Let $t > 0$ and $x \leq y$, so that $1 \leq x \leq \sqrt{t/2\pi}$. Let $\sigma \leq 1, M = [x], N = [y], \eta = 2\pi y$. We deform the contour integral path $C$ to the combination of the straight lines $C_1, C_2, C_3, C_4$ joining $\infty, c\eta + i\eta(1+c) + 2\pi i\lambda, -c\eta + i\eta(1-c) + 2\pi i\lambda, -c\eta - (2L+1)\pi i + 2\pi i\lambda$, $\infty$, where $c$ is an absolute constant, $0 < c \leq 1/2$. 

\begin{center}
\begin{tikzpicture}
\draw[->] (0,0) -- (6,0) node[anchor=north] {Re};
\draw[->] (0,0) -- (0,6) node[anchor=east] {Im};
\fill (0,0) circle (2pt) node[anchor=north] {O};
\fill (2,2) circle (2pt) node[anchor=south] {$c\eta + i(1+c)\eta + 2\pi i\lambda$};
\fill (2,-2) circle (2pt) node[anchor=north] {$-c\eta + i(1-c)\eta + 2\pi i\lambda$};
\fill (-2,-2) circle (2pt) node[anchor=east] {$-c\eta$};
\fill (-2,2) circle (2pt) node[anchor=west] {$i\eta + 2\pi i\lambda$};
\fill (2,-2) circle (2pt) node[anchor=north] {$-c\eta - (2L+1)\pi i + 2\pi i\lambda$};
\end{tikzpicture}
\end{center}
We calculate the residue of integrand of (2.1). Since
\[
\lim_{z \to 2\pi i(\lambda + n)} \left\{ z - 2\pi i(\lambda + n) \right\} \cdot \frac{z^{s-1}e^{-(M+\alpha)z}}{z^{2\pi i\lambda} - 1} = \lim_{z \to 2\pi i(\lambda + n)} \left( \frac{e^{z-2\pi i\lambda} - 1}{z - 2\pi i(\lambda + n)} \right)^{-1} e^{-(M+\alpha)z} \cdot z^{s-1} = e^{-2\pi i(M+\alpha)(\lambda + n)}(2\pi i(n + \lambda))^{s-1},
\]
we have
\[
\text{Res}_{z=2\pi i(\lambda + n)} \frac{z^{s-1}e^{-(M+\alpha)z}}{z^{2\pi i\lambda} - 1} = e^{-2\pi i(M+\alpha)(\lambda + n)}(2\pi i(n + \lambda))^{s-1}\]
\[
= \begin{cases} 
 e^{-2\pi i(M+\alpha)(\lambda + n)}(2\pi (n + \lambda)i)^{s-1} 
 & (n \geq 0) 
\end{cases} 
\]
\[
= \begin{cases} 
 e^{2\pi i(M+\alpha)(|n| - \lambda)}(2\pi (|n| - \lambda)e^{3\pi i/2})^{s-1} 
 & (n \leq -1) 
\end{cases} 
\]
and we have
\[
\sum_{n=-N+1}^{N} \text{Res}_{z=2\pi in} \frac{z^{s-1}e^{-(M+\alpha)z}}{z^{2\pi i\lambda} - 1} = \begin{cases} 
 - \frac{e^{\pi is}}{(2\pi)^{1-s}} \cdot e^{|(1-s)/2-2(M+\alpha)(\lambda)|\pi i} \sum_{n=0}^{N} \frac{e^{2\pi in(1-\alpha)}}{(n + \lambda)^{1-s}} & (n \geq 0) 
\end{cases} 
\]
\[
+ e^{-|\{(1-s)/2+2(M+\alpha)(1-\lambda)\}|\pi i} \sum_{n=0}^{N} \frac{e^{2\pi in\lambda}}{(n + 1 - \lambda)^{1-s}} 
\]
Therefore we obtain
\[
\zeta_{L}(s, \alpha, \lambda) = \sum_{n=0}^{M} \frac{e^{2\pi in\lambda}}{(n + \alpha)^{s}} 
\]
\[
+ \frac{\Gamma(1 - s)}{(2\pi)^{1-s}} \begin{cases} 
 e^{\{(1-s)/2-2\alpha\lambda\}\pi i} \sum_{n=0}^{N} \frac{e^{2\pi in(1-\alpha)}}{(n + \lambda)^{1-s}} 
\end{cases} 
\]
\[
+ e^{-\{(1-s)/2+2\alpha(1-\lambda)\}\pi i} \sum_{n=0}^{N} \frac{e^{2\pi in\lambda}}{(n + 1 - \lambda)^{1-s}} 
\]
\[
+ \frac{e^{2\pi i\lambda M\Gamma(1 - s)}}{2\pi i e^{\pi is}} \left( \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} \right) \frac{z^{s-1}e^{-(M+\alpha)z}}{z^{2\pi i\lambda} - 1} \, dz. \tag{2.2}
\]
From here, we consider the order of integral terms on right-hand side of (2.2).
First, we consider the integral path $C_4$. Let $z = u + iv = re^{i\theta}$ then $|z^{s-1}| = r^{\sigma-1}$, and since $\theta \geq 5\pi/4$, $r \gg \eta$, $|e^{z-2\pi i\lambda} - 1| \gg 1$, we have
\[
\int_{C_4} \frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda} - 1} \, dz = \int_{C_4} \frac{(re^{i\theta})^{\sigma+i\alpha-1}e^{-(M+\alpha)(u+iv)}}{e^{z-2\pi i\lambda} - 1} \, dz \\
\ll \eta^{\sigma-1}e^{-5\pi t/4} \int_{c\eta}^\infty e^{-(M+\alpha)u} \, du \\
= \eta^{\sigma-1}(M+\alpha)^{-1}e^{(M+\alpha)\eta-5\pi t/4} \\
\ll e^{(c-5\pi/4)t}.
\] (2.3)

Secondly, we consider the order of integral on $C_3$ of (2.2). Noting
\[
\arctan \varphi = \int_0^\varphi \frac{d\mu}{1+\mu^2} = \int_0^\varphi \frac{d\mu}{(1+\mu)^2} = \frac{\varphi}{1+\varphi}
\]
for $\varphi > 0$, we can write
\[
\theta = \arg z = \frac{\pi}{2} + \arctan \frac{c}{1-c} = \frac{\pi}{2} + c + A(c)
\]
on $C_3$, where $A(c)$ is a constant depending on $c$. Then we have
\[
|z^{s-1}e^{-(M+\alpha)z}| = r^{\sigma}e^{-t\theta+(M+\alpha)\eta} \\
\ll \eta^{\sigma-1}e^{-(\pi/2+c+A(c))t+(M+\alpha)\eta} \\
\ll \eta^{\sigma-1}e^{-(\pi/2+A(c))t}.
\]

Therefore, since $|e^{z-2\pi i\lambda} - 1| \gg 1$, we have
\[
\int_{C_3} \frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda} - 1} \, dz \ll \eta^{\sigma}e^{-(\pi/2+A(c))t}.
\] (2.4)

Thirdly, since $|e^{z-2\pi i\lambda} - 1| \gg e^u$ on $C_1$, we have
\[
\frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda} - 1} \ll \eta^{\sigma-1} \exp \left( -t \arctan \frac{(1+c)\eta}{u} - (M+\alpha+1)u \right).
\]
Since $M+\alpha+1 \geq x = t/\eta$, the term $(M+\alpha+1)u$ on the right-hand side of the above may be replaced by $tu/\eta$. Also, since
\[
\frac{d}{du} \left( \arctan \frac{(1+c)\eta}{u} + \frac{u}{\eta} \right) = -\frac{(1+c)\eta}{u^2 + (1+c)^2\eta^2} + \frac{1}{\eta} > 0
\]
we have
\[
\arctan \varphi = \int_0^\varphi \frac{d\mu}{1+\mu^2} < \int_0^\varphi d\mu = \varphi.
\]
Therefore
\[
\arctan \frac{(1+c)\eta}{u} + \frac{u}{\eta} \geq \arctan \frac{1+c}{c} + \pi = \frac{\pi}{2} - \arctan \frac{c}{1+c} + c > \frac{\pi}{2} + B(c)
\]
in $u \geq c\eta$, where $B(c) = c^2/(1+c)^2$. Then we have
\[
\frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i\lambda} - 1} \ll \eta^{\sigma-1} \exp \left( -\left( \frac{\pi}{2} + B(c) \right)t \right).
\]
Since

\[ \frac{z^{s-1} e^{-(M+\alpha)z}}{e^{z-2\pi i \lambda} - 1} \ll \begin{cases} \eta^{\sigma-1} \exp \left( - \frac{\pi}{2} + B(c) \right) t & (c \eta \leq u \leq \pi \eta), \\
\eta^{\sigma-1} \exp (-xu) & (u \geq \pi \eta), \end{cases} \]

we obtain

\[ \int_{C_1} \frac{z^{s-1} e^{-(M+\alpha)z}}{e^{z-2\pi i \lambda} - 1} \ll \eta^{\sigma-1} \left\{ \int_{c \eta}^{\pi \eta} e^{-(\pi/2+B(c))t} du + \int_{\pi \eta}^{\infty} e^{-xu} du \right\} \ll \eta^\sigma e^{-(\pi/2+B(c))t} + \eta^{\sigma-1} e^{-\pi \eta x} \ll \eta^\sigma e^{-(\pi/2+B(c))t}. \quad (2.5) \]

Finally, we describe the evaluation of the integral on \( C_2 \). Rewriting \( z = i(\eta + 2\pi \lambda) + \xi e^{\pi i/4} \) (where \( \eta \in \mathbb{R} \) and \( |\eta| \leq \sqrt{2c\eta} \)), we have

\[
\begin{align*}
\eta^{s-1} &= \exp \left\{ (s-1) \left( \log \left( i(\eta + 2\pi \lambda) + \xi e^{-\pi i/4} \right) \right) \right\} \\
&= \exp \left\{ (s-1) \left( \frac{\pi i}{2} + \log \left( \eta + 2\pi \lambda + \xi e^{-\pi i/4} \right) \right) \right\} \\
&= \exp \left\{ (s-1) \left( \frac{\pi i}{2} + \log(\eta + 2\pi \lambda) + \frac{\xi}{\eta + 2\pi \lambda} e^{-\pi i/4} \right. \right. \\
&\quad \left. \left. -\frac{\xi^2}{2(\eta + 2\pi \lambda)^2} e^{-\pi i/2} + O \left( \frac{\xi^3}{\eta^3} \right) \right) \right\} \\
&\ll (\eta + 2\pi \lambda)^{s-1} \exp \left\{ \left( -\frac{\pi}{2} + \frac{\xi}{\sqrt{2}(\eta + 2\pi \lambda)} - \frac{\xi^2}{2(\eta + 2\pi \lambda)^2} + O \left( \frac{\xi^3}{\eta^3} \right) \right) t \right\}
\end{align*}
\]

as \( \eta \to \infty \). Also, since

\[
\frac{e^{-(M+\alpha)z}}{e^{z-2\pi i \lambda} - 1} = \frac{e^{-(M+\alpha-x)z}}{e^{z-2\pi i \lambda} - 1} \cdot e^{-xz}
\]

and

\[
\frac{e^{-(M+\alpha-x)z}}{e^{z-2\pi i \lambda} - 1} \ll \begin{cases} e^{(x-M-\alpha-1)u} & (u > \frac{\pi}{2}) \\
e^{(x-M-\alpha)u} & (u < -\frac{\pi}{2}) \end{cases},
\]

we have

\[
\frac{e^{-(M+\alpha)z}}{e^{z-2\pi i \lambda} - 1} \ll \left| e^{-xz} \right| = e^{-\xi t / \sqrt{2} \eta} \quad \left( |u| > \frac{\pi}{2} \right).
\]
Hence
\[
\int_{C_2 \cap \{ |u| > \pi/2 \}} \frac{z^{s-1} e^{-(M + \alpha)z}}{e^{2\pi i \lambda} - 1} \, dz \\
\ll \int_{C_2 \cap \{ |u| > \pi/2 \}} (\eta + 2\pi \lambda)^{\sigma - 1} \\
\times \exp \left\{ \left( -\frac{\pi}{2} + \frac{\xi}{\sqrt{2(\eta + 2\pi \lambda)}} - \frac{\xi^2}{2(\eta + 2\pi \lambda)^2} + O \left( \frac{\xi^3}{\eta^3} \right) \right) t \right\} \exp \left( -\frac{\xi t}{\sqrt{2\eta}} \right) \, d\xi
\]
\ll \int_{-\sqrt{2\pi \eta}}^{\sqrt{2\pi \eta}} (\eta + 2\pi \lambda)^{\sigma - 1} e^{-\pi t/2} \exp \left\{ \left( -\frac{\xi^2}{2(\eta + 2\pi \lambda)^2} + O \left( \frac{\xi^3}{\eta^3} \right) \right) t \right\} \, d\xi
\ll \int_{-\infty}^{\infty} (\eta + 2\pi \lambda)^{\sigma - 1} e^{-\pi t/2} \exp \left\{ \left( -\frac{\xi^2}{2(\eta + 2\pi \lambda)^2} + O \left( \frac{\xi^3}{\eta^3} \right) \right) t \right\} \, d\xi
\ll \eta^{\sigma - 1} e^{-\pi t/2} \int_{-\infty}^{\infty} \exp \left\{ -\frac{D(c)\xi^2 t}{\eta^2} \right\} \, d\xi
\ll \eta^{\sigma - 1} t^{-1/2} e^{\pi t/2},
\] (2.6)

where \( D(c) \) is a constant depending on \( c \). The argument can also be applied to the part \(|u| \leq \pi/2\) if \(|e^{z-2\pi i \lambda}| > A\). If not, that is the case when the contour goes too near to the pole at \( z = 2\pi i N + 2\pi i \lambda \), we take an arc of the circle \(|z - 2\pi i N - 2\pi i \lambda| = \pi/2\). On this arc we can write to \( z = 2\pi i N + 2\pi i \lambda + (\pi/2) e^{i\beta} \), and

\[
\log (z^{s-1}) = (s - 1) \log \left( 2\pi i N + 2\pi i \lambda + \frac{\pi}{2} e^{i\beta} \right)
\]
\[
= (s - 1) \log e^{\pi i/2} \left( 2\pi N + 2\pi \lambda + \frac{\pi}{2} \cdot e^{i\beta} \right)
\]
\[
= (\sigma + it - 1) \left\{ \frac{\pi i}{2} + \log(2\pi(N + \lambda)) + \log \left( 1 + \frac{e^{i\beta}}{4(N + \lambda)i} \right) \right\}
\]
\[
= -\frac{\pi t}{2} + (s - 1) \log(2\pi(N + \lambda)) + \frac{te^{i\beta}}{4(N + \lambda)} + O(1).
\]

On the last line of the above calculations, we used \( N^2 \gg t \) which follows from the assumption \( x \leq y \). Then

\[
\begin{align*}
\int_{C_2 \cap \{ |u| > \pi/2 \}} \frac{z^{s-1} e^{-(M + \alpha)z}}{e^{2\pi i \lambda} - 1} \, dz \\
\ll \int_{C_2 \cap \{ |u| > \pi/2 \}} (\eta + 2\pi \lambda)^{\sigma - 1} \\
\times \exp \left\{ \left( -\frac{\pi}{2} + (s - 1) \log(2\pi(N + \lambda)) + \frac{te^{i\beta}}{4(N + \lambda)} - \frac{\pi}{2} (M + \alpha) e^{i\beta} + O(1) \right) \right\} \exp \left( -\frac{\xi t}{\sqrt{2\eta}} \right) \, d\xi
\end{align*}
\]

and since

\[
\frac{te^{i\beta}}{4(N + \lambda)} - \frac{\pi}{2} (M + \alpha) e^{i\beta} = \frac{2\pi xy - 2\pi(|x| + \alpha)(|y| + \lambda)}{4(N + \lambda)} e^{i\beta} = O(1)
\]

we have

\[
\begin{align*}
\int_{C_2 \cap \{ |u| > \pi/2 \}} \frac{z^{s-1} e^{-(M + \alpha)z}}{e^{2\pi i \lambda} - 1} \, dz \\
\ll \exp \left( -\frac{\pi}{2} + (s - 1) \log(2\pi(N + \lambda)) + O(1) \right) \\
\ll N^{\sigma - 1} e^{-\pi t/2}.
\end{align*}
\]
Hence, the integral on the small semicircle can be evaluated as $O(\eta^{-1}e^{-\pi t/2})$. Therefore together with (2.6), we have

$$
\int_{C_2} \frac{z^{s-1}e^{-(M+\alpha)z}}{e^{z-2\pi i \lambda} - 1} \, dz \ll \eta^\sigma t^{-1/2}e^{-\pi t/2} + \eta^{-1}e^{-\pi t/2}.
$$

(2.7)

Now, evaluation of all the integrals was done. Using the results (2.3), (2.4), (2.5), (2.7) and $e^{2\pi i(\lambda N - s/2)}\Gamma(1-s) \ll t^{1/2-\sigma}e^{\pi t/2}$, we see that the integral term of (2.2) is

$$
\ll t^{1/2-\sigma}e^{\pi t/2}\left\{ \eta^\sigma e^{-(\pi/2 + B(c))t} + \eta^\sigma t^{-1/2}e^{-\pi t/2} + \eta^{-1}e^{\pi t/2} + e^{t(\pi/2 + A(c))} + e^{(c+\varepsilon-5\pi/4)t} \right\}
$$

$$
\ll t^{1/2} \left( \frac{\eta}{t} \right)^\sigma e^{-(A(c)+B(c))t} + \left( \frac{\eta}{t} \right)^\sigma + t^{-1/2} \left( \frac{\eta}{t} \right)^{\sigma - 1} + t^{1/2-\sigma}e^{(c+\varepsilon-5\pi/4)t}
$$

$$
\ll e^{-\delta t} + x^{-\sigma} + t^{-1/2}x^{1-\sigma} \ll x^{-\sigma},
$$

where $\delta$ is a small positive real number. Therefore we have

$$
\zeta_L(s, \alpha, \lambda) = \sum_{0 \leq n \leq y} \frac{e^{2\pi in\lambda}}{(n + \alpha)^s} + \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \left\{ e^{((1-s)/2 - 2\alpha\lambda)\pi i} \sum_{0 \leq n \leq y} \frac{e^{2\pi in(1-\lambda)}}{(n + \lambda)^{1-s}} + e^{(-(1-s)/2 + 2\alpha(1-\lambda))\pi i} \sum_{0 \leq n \leq y} \frac{e^{2\pi in\lambda}}{(n + 1 - \lambda)^{1-s}} \right\} + O(x^{-\sigma}),
$$

(2.8)

that is, Theorem 2 in the case of $x \leq y$ has been proved.

To prove Theorem 2 in the case $x \geq y$, we use the following functional equation of the Lerch zeta-function;

$$
\zeta_L(s, \alpha, \lambda) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \left\{ e^{((1-s)/2 - 2\alpha\lambda)\pi i} \zeta_L(1 - s, \lambda, 1 - \alpha) + e^{(-(1-s)/2 + 2\alpha(1-\lambda))\pi i} \zeta_L(1 - s, 1 - \lambda, \alpha) \right\},
$$

(2.9)
Applying \((2.8)\) to \(\zeta_L(1-s, \lambda, 1-\alpha)\) and \(\zeta_L(1-s, 1-\lambda, \alpha)\), and substitute these into \((2.9)\), we have

\[
\zeta_L(s, \alpha, \lambda) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \left\{ \sum_{0 \leq n \leq x} \frac{e^{2\pi in\lambda}}{(n+\lambda)^{1-s}} + e^{\{s/2-2\alpha(1-\lambda)\}\pi i} \sum_{0 \leq n \leq y} \frac{e^{2\pi in(1-\lambda)}}{(n+\alpha)^{1-s}} \right\}
\]

Interchanging \(x\) and \(y\), we obtain the theorem with \(x \geq y\). Combining this equation with \((2.8)\), we obtain the proof of \((1.6)\).

The proof of \((1.7)\) is similar. However, the four integral path \(C_1, C_2, C_3\) and \(C_4\) are different from the proof of \((1.6)\), that is, as follows: The straight lines \(C_1, C_2, C_3, C_4\) joining \(\infty, c\eta + i\eta(1+c), -c\eta + i\eta(1-c), -c\eta - (2L+1)\pi i, \infty\), where \(c\) is an absolute constant, \(0 < c \leq 1/2\). Also, in the proof for the case \(x \geq y\), we use the functional equation

\[
\zeta_H(s, \alpha) = \frac{\Gamma(1-s)}{(2\pi)^{1-s}} \left\{ e^{(1-s)\pi i/2} \zeta_L(1-s, 1, 1-\alpha) + e^{-\{1-s\}\pi i/2} \zeta_L(1-s, 1, \alpha) \right\},
\]

but this equation is not included in the functional equation \((2.9)\). Noticing these points, we can prove \((1.7)\) by a similar method. This completes the proof of Theorem 2. 

\[\Box\]

3. Proof of Theorem 3

In this section, using Theorem 2, we give the proof of Theorem 3.

**Proof of Theorem 3** Let

\[
x = \frac{t}{2\pi \sqrt{\log t}}, \quad y = \sqrt{\log t}
\]
and we assume \( t > 0 \) satisfies \( x \geq 1 \) and \( y \geq 1 \). Use the Stirling formula

\[
\Gamma(1 - s)e^{((1-s)/2 - 2\lambda \pi i) \pi i} \ll 1, \quad \Gamma(1 - s)e^{(-(1-s)/2 - 2\alpha(1-\lambda)) \pi i} \ll 1.
\]

Then if \( 0 < \lambda < 1 \), using (1.6) we have

\[
\zeta_L \left( \frac{1}{2} + it, \alpha, \lambda \right) = \sum_{0 \leq n \leq x} \frac{e^{2\pi in\lambda}}{(n + \alpha)^{1/2+it}} + O \left( \sum_{0 \leq n \leq y} \frac{e^{2\pi in(1-\alpha)}}{(n + \lambda)^{1/2-it}} + \sum_{0 \leq n \leq y} \frac{e^{2\pi in\alpha}}{(n + 1 - \lambda)^{1/2-it}} \right) + O(t^{-1/2}(\log t)^{1/4}) + O((\log t)^{-1/4}),
\]

and if \( \lambda = 1 \), using (1.7) we have

\[
\zeta_H \left( \frac{1}{2} + it, \alpha \right) = \sum_{0 \leq n \leq x} \frac{1}{(n + \alpha)^{1/2+it}} + O \left( \sum_{n \leq y} \frac{e^{2\pi in(1-\alpha)}}{n^{1/2-it}} + \sum_{n \leq y} \frac{e^{2\pi in\alpha}}{n^{1/2-it}} \right) + O(t^{-1/2}(\log t)^{1/4}) + O((\log t)^{-1/4}).
\]

(i) In the case \( 0 < \lambda < 1 \) and \( 0 < \alpha < 1 \), since

\[
\sum_{n=0}^{\infty} \frac{e^{2\pi in(1-\alpha)}}{(n + \lambda)^{1/2}}, \quad \sum_{n=0}^{\infty} \frac{e^{2\pi in\alpha}}{(n + 1 - \lambda)^{1/2}}
\]

are convergent, and \( t^{-1/2}(\log t)^{1/4} = o(1), (\log t)^{-1/4} = o(1) \), we have

\[
\zeta_L \left( \frac{1}{2} + it, \alpha, \lambda \right) = \sum_{0 \leq n \leq x} \frac{e^{2\pi in\lambda}}{(n + \alpha)^{1/2+it}} + O(1).
\]

(ii) In the case \( 0 < \lambda < 1 \) and \( \alpha = 1 \), the second term on right-hand side of (1.6) is

\[
\ll \int_0^y \frac{1}{(u + \lambda)^{1/2}} du = O(\sqrt{y}) = O((\log t)^{1/4}),
\]

so we have

\[
\zeta_L \left( \frac{1}{2} + it, 1, \lambda \right) = \sum_{n \leq x} \frac{e^{2\pi in\lambda}}{n^{1/2+it}} + O((\log t)^{1/4}).
\]

(iii) In the case \( \lambda = 1 \) and \( 0 < \alpha < 1 \), consider similarly as in the case of (i) to obtain

\[
\zeta_L \left( \frac{1}{2} + it, \alpha, 1 \right) = \sum_{0 \leq n \leq x} \frac{1}{(n + \alpha)^{1/2+it}} + O(1).
\]

(iv) In the case \( \lambda = 1 \) and \( \alpha = 1 \), since \( \zeta_L(s, 1, 1) = \zeta(s) \) we obtain

\[
\zeta_L \left( \frac{1}{2} + it, 1, 1 \right) = \sum_{n \leq x} \frac{1}{n^{1/2+it}} + O((\log t)^{1/4})
\]

(see Chap. VII in [5]).

Let

\[
\Sigma(\alpha, \lambda) = \sum_{0 \leq n \leq x} \frac{e^{2\pi in\lambda}}{(n + \alpha)^{1/2+it}},
\]
and calculate as

$$|\Sigma(\alpha, \lambda)|^2 = \sum_{0 \leq m, n \leq x} \frac{e^{2\pi i (m-n)\lambda}}{(m+\alpha)^{1/2}(n+\alpha)^{1/2}} \left( \frac{n+\alpha}{m+\alpha} \right)^it$$

$$= \sum_{0 \leq n \leq x} \frac{1}{n+\alpha} + \sum_{0 \leq m, n \leq x \atop m \neq n} \frac{e^{2\pi i (m-n)\lambda}}{(m+\alpha)^{1/2}(n+\alpha)^{1/2}} \left( \frac{n+\alpha}{m+\alpha} \right)^it.$$ 

Also $T_1 = T_1(m, n)$ is a function in $m, n$ satisfying

$$\max\{m, n\} = \frac{T_1}{2\pi \sqrt{\log T_1}}.$$

Let $X = T/2\pi \sqrt{\log T}$, then

$$\int_1^T |\Sigma(\alpha, \lambda)|^2 dt = \sum_{0 \leq n \leq X} \frac{1}{n+\alpha} \{T - T_1(n, n)\}$$

$$+ O\left( \sum_{0 \leq m < n \leq X} \frac{e^{2\pi i (m-n)\lambda}}{(m+\alpha)^{1/2}(n+\alpha)^{1/2}} \left( \log \frac{n+\alpha}{m+\alpha} \right)^{-1} \right).$$

Here, since

$$n \sqrt{\log n} = \frac{T_1}{2\pi \sqrt{\log T_1}} \left( \log \frac{T_1}{2\pi \sqrt{\log T_1}} \right)^{1/2} \sim \frac{1}{2\pi} T_1(n, n)$$

and

$$\sum_{0 \leq m < n \leq X} \frac{e^{2\pi i (m-n)\lambda}}{(m+\alpha)^{1/2}(n+\alpha)^{1/2}} \left( \log \frac{n+\alpha}{m+\alpha} \right)^{-1} \ll X \log X \ll T(\log T)^{1/2}$$

(see Lemma 3 in [2] or Lemma 2.6 in [4]), (3.3) can be rewritten as

$$\int_1^T |\Sigma(\alpha, \lambda)|^2 dt = T \log \frac{T}{2\pi} + O(T(\log T)^{1/2}).$$

Therefore from (i), (ii), (iii), (iv) and (3.4), and the Cauchy-Schwarz inequality, we obtain

$$\int_1^T \left| \zeta_L \left( \frac{1}{2} + it, \alpha, \lambda \right) \right|^2 dt$$

$$= \int_1^T |\Sigma(\alpha, \lambda)|^2 dt + \begin{cases} O(T^{1/2}(\log T)^{1/4})) + O(T) \quad (0 < \alpha < 1), \\
O(T(\log T)^{3/4}) + O(T(\log T)^{1/2}) \quad (\alpha = 1) \end{cases}$$

$$= T \log \frac{T}{2\pi} + \begin{cases} O(T(\log T)^{1/2}) \quad (0 < \alpha < 1), \\
O(T(\log T)^{3/4}) \quad (\alpha = 1). \end{cases}$$

Thus we obtain the proof of Theorem 3. □
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