Structure and diffusion time scales of disordered clusters

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Abstract

The eigenvalue spectra of the transition probability matrix for random walks traversing critically disordered clusters in three different types of percolation problems show that the random walker sees a developing Euclidean signature for short time scales as the local, full-coordination constraint is iteratively applied.

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Physical processes such as diffusion in porous media, electrical and thermal conduction in composite materials, and information flow in random networks may be modeled using random walks on disordered clusters. For random walks on uniform Euclidean systems, their mean-square displacement goes as $< R^2(t) > \sim t$, where $t$ is the number of steps. In contrast, constraining the random walker to traverse a critically disordered cluster such as an incipient infinite percolation cluster makes its mean-square displacement go as $< R^2(t) > \sim t^{2/d_w}$, where $d_w > 2$ is known as the walk dimension. This slowing down of the random walker, called anomalous diffusion (see, for reviews, [1,2]), is due to the delays it experiences by being trapped in or reflected from the irregular features of the cluster.

The disordered cluster where the random walker is constrained to traverse can be of many different types. For this article, we have chosen to investigate three closely-related models. These models are ordinary [3], fully coordinated (FC) [4], and iterated fully coordinated (IFC) percolation [5], all on the square lattice. Although the disordered clusters produced by each of these models look qualitatively different, it has been found previously that they exhibit

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Fig. 1. Plot of the eigenvalue spectra for ordinary (+), FC (○), and IFC (●) percolation. Each set of data points is obtained from 1200 lattice realizations of size 128 × 128. The solid curve shown is the smoothed plot of the eigenvalue spectrum for a uniform Euclidean system of size 120 × 120.

The eigenvalues \( \lambda \) of the transition probability matrix \( W \) directly provide the time-dependent behavior of the random walker [6,7]. Eigenvalues near \( \lambda = 1 \) characterize the long time behavior of the random walker while those at the region near \( \lambda = 0 \) only enter the characterization of the random walker’s short time behavior.

In Fig. 1 the eigenvalue spectra for a blind ant random walker traversing a critically disordered cluster constructed from ordinary, FC, and IFC percolation are shown together with the sketch for the eigenvalue spectrum for a random walk on a uniform Euclidean lattice. At the region near \( \lambda = 1 \), all three percolation models exhibit the same power-law behavior. Quantitatively, the eigenvalue density \( n(\lambda) \) near \( \lambda = 1 \) scales according to \( n(\lambda) \sim |\ln \lambda|^{d_w/2-1} \) [6]. For the above three models, therefore, the long time behavior of the random walker is the same. This agrees with the previous result that the three models exhibit the same walk dimension \( d_w \) [5].
Fig. 2. Eigenvalue spectra for (a) ordinary, (b) FC, and (c) IFC percolation, when FC and PC sites are culled. For (a), (b) and (c) the crosses (+) represent FC sites culled while the circles (◦) represent PC sites culled. Each set of data points is taken from 1200 realizations of lattices of size 128 \times 128.

The region near \( \lambda = 0 \) contributes to the behavior of the random walker only for short time scales. In this region the data for IFC percolation peaks at \( \lambda = 0 \), similar to the way the solid curve for the uniform Euclidean case peaks in that region [8]. Also, as we iterate from ordinary to FC and then to IFC percolation, this closeness to the uniform Euclidean case improves. Since the random walker perceives mostly the local structure of the cluster for short time scales, the local structure of the disordered cluster formed becomes increasingly more Euclidean as we iterate from ordinary to FC and then to IFC percolation. In particular, although critically disordered clusters formed in IFC percolation exhibit a fractal nature for long time scales they closely mimic the uniform Euclidean case for short time scales.

We further investigate the nature of the critically disordered clusters by culling either the fully-coordinated (FC) sites or the partially-coordinated (PC) sites. FC sites are sites with four neighbors also occupied while PC sites have at
least one neighbor that is empty. Culling PC sites would produce clusters consisting only of the FC sites and thus leaving only the part of the cluster that is highly compact.

Shown in Fig. 2 are the eigenvalue spectrum when either FC or PC sites are culled. For ordinary percolation, when PC sites are culled the eigenvalue spectrum is mostly zero. However, when FC sites are culled the eigenvalue spectrum still mimics the spectrum with no sites culled. This shows that PC sites are the dominant sites in clusters created from ordinary percolation in determining diffusion time scales, as they are also dominant structurally [5].

For FC percolation, culling either FC or PC sites will produce eigenvalue spectra that behave in a similar fashion. This also agrees with the previous result that neither FC nor PC sites dominate in the cluster’s structure produced by FC percolation [5].

For IFC percolation, the eigenvalue spectrum when FC sites are culled is mostly zero while the spectrum when PC sites are culled produces most of the features for the original spectrum. In particular, consider the features near $\lambda = 0$ and near $\lambda = 1$. The spectrum when PC sites are culled behaves in a similar fashion to those regions of the original spectrum where no culling is done (see Fig. 1). Thus, in IFC percolation FC sites are dominant not only structurally but also in dictating the dynamics of the Brownian motion.

In summary, although FC and IFC percolation exhibit the same fractal structure, short time diffusion characteristics become increasingly Euclidean as we iterate the full coordination requirement. Critically disordered clusters produced from IFC percolation, in particular, exhibit both a fractal structure for long time scales and a Euclidean structure for short time scales.

References

[1] S. Havlin, D. Ben-Avraham, Adv. Phys. 36 (1987) 695.
[2] H. Nakanishi, in: D. Stauffer (Ed.), Annual Reviews of Computational Physics I, World Scientific, Singapore, 1994.
[3] See, e.g., D. Stauffer, A. Aharony, Introduction to Percolation Theory, Taylor & Francis, London, 1994; J. Feder, Fractals, Plenum Press, New York, 1988.
[4] E. Cuansing, J. H. Kim, H. Nakanishi, Phys. Rev. E 60 (1999) 3670.
[5] E. Cuansing, H. Nakanishi, J. Stat. Phys. 105 (2001) 659.
[6] H. Nakanishi, S. Mukherjee, N. H. Fuchs, Phys. Rev. E 47 (1993) R1463.
[7] S. Mukherjee, H. Nakanishi, N. H. Fuchs, Phys. Rev. E 49 (1994) 5032.
In Euclidean (non-fractal) space, the density of states $\rho(E)$ for the corresponding classical vibration problem goes as $\rho(E) \sim E^{d/2-1}$ in $d$ dimensions. If we apply the correspondence $\lambda \sim e^{-E}$ (which is true at least for $\lambda$ close to 1) as an approximation, then the density in $\lambda$ space should behave as $\rho(\lambda) \sim \frac{1}{\lambda} |\ln \lambda|^{d/2-1}$. While this is not a rigorous argument, the $\frac{1}{\lambda}$ dependence is consistent with the observed peak in $\rho(\lambda)$ at $\lambda = 0$. 