New spinor fields classes and applications

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Abstract. After revisiting the well-known Lounesto classification of spinor fields, wherefrom the bilinear covariants are considered as the landmark beyond Wigner classification, relevant features of regular and singular spinor fields are presented. Hence, non-standard spinor fields are scrutinised, together with their associated dynamics, paving recently found applications in Physics. The case of the new classes of spinors on 7-manifolds is revisited to provide concrete examples.

1. Introduction

The Lounesto classification of spinor fields [1] is much more recent than previous classifications, like the Wigner and the Cartan classifications [2]. The well-known Weyl, Dirac, and Majorana spinor fields occupy, in the Lounesto classification, a restricted – but not less important – place, among the whole panorama of possibilities comprised by the six disjoint classes of regular and singular spinors, in the Lounesto classification [1]. To fix the denomination of spinors in the six spinor field classes proposed by Lounesto, the standard – textbook – Dirac spinor is an eigenspinor of the parity operator and occupied a subset in the first class of regular spinors. The Weyl spinor is a very particular dipole spinor in the sixth class in Lounesto classification [1]. Possibly the most astonishing class of spinors is the fifth one, consisting of flagpole spinor fields. This class encompasses Majorana spinors, spinors that are solutions of the Dirac equation in a Kerr black hole background [3, 4, 5], and, moreover, also encompasses Elko (dark) spinor field – that are eigenspinors of the charge conjugation operator with dual helicity, having mass dimension one – as a flagpole spinor [6]. Hence, the fifth class in Lounesto classification has a subset of spinors that are neutral and another subset of spinors that are charged, whereas the fifth class of spinors has a subset of fermions of spin 1/2 that has mass dimension one and another subset of fermions that have canonical mass dimension one. To find the complementary subsets in this fifth class is still an open problem in the literature. In spite of those known examples in the field theoretical literature [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 20, 21], the dynamics ruling the spinor fields in this class still lacks, despite of the previously mentioned known spinor fields [22, 23, 24]. A bottom-up method consisting of searching for sharp examples. The methods designated in Refs. [7, 8] are successful efforts towards the derivation of the dynamics that govern each subset in each class of spinor fields, in the Lounesto spinor fields classification. Applications gravitation and field theory are abundant [9, 11, 10, 12]. Elko spinor fields were the first example of a (singular) spinor field that is beyond the Dirac, Majorana and Weyl spinor fields. Without going further in the second quantised fields, the 4-component Elko spinors are very similar to the Majorana spinors,
since both are eigenspinors of the charge conjugation operator, in a subset of neutral spinors in the class of flagpole spinors, in Lounesto classification. However, Majorana’s choice did not originally include both self-conjugate and anti-self-conjugate spinors. From first principles, Elko spinors consist of a four spinors family that, besides the charge conjugation operator, needs the helicity operator to act on them. Therefore Elko present dual helicity. Recently, flag-dipoles spinors, that are singular spinors in the fourth class of Lounesto classification, have been found to be solutions of the Dirac equation in a $f(R)$-background with torsion, in the context of Einstein-Sciama-Kibble (ESK) gravity. These kind of singular spinors work appropriately as a source of matter in ESK gravity, being the first explicit construction in Physics of flag-dipole spinors [8, 11]. Furthermore, solutions of the Dirac equation in a Kerr black hole background have been more recently shown to generate flag-dipoles and flagpoles spinor fields rather than the expected Dirac spinor field [3, 5]. Flagpoles comprise realisations of mass dimension one Elko spinors, which are candidates for solving the dark matter problem [9, 10, 12]. Far beyond the standard Dirac spinors paradigm, regular and singular spinors range new applications that encompass dark spinors Hawking radiation across black holes and black strings [17, 18], cosmology, gravity on extra-dimensional thick branes [29, 30], particle physics and condensed matter [9, 10]. In fact, this matter has been further explored in the context of black hole thermodynamics, where tunnelling methods were studied for the eigenspinors of the charge conjugation operator having dual helicity [15, 16], as special type of flagpoles [17, 19, 23]. Experimental signatures of type-5, flagpole, dark spinors in Lounesto classification are proposed as being a byproduct of the Higgs bosonic field at LHC [20]. In fact, the existence of spin 1/2 fermions with canonical mass dimension one, in the Lounesto classification, demands the Weinberg no go theorem must be eloped [15].

New classes of spinors and the Lounesto classification were primordially defined in four-dimensional, Minkowski, spacetime. On 2015, new surprises occurred, now consisting of the discovery of additional classes of spinors on higher dimensional spacetimes. Some of the spinors fields in this context have defined dynamics, and can be obtained from the analysis of the geometric Fierz identities, that constrain the bilinear covariants. This procedure turns possible to classify spinors fields on manifolds of arbitrary dimensions and signatures. Constructions on specific dimensions and signatures have been derived. In fact, the classification of spinor fields on seven-dimensional Riemannian [25] and Lorentzian [26] manifolds has been constructed, showing that the Lounesto classification in Minkowski spacetime can also exist in higher dimensional spacetimes [25, 26]. Particular interest on these new solutions has arisen in AdS$_5 \times S^7$ compactified setups, since the new classes of spinor fields solutions can represent new solutions in string theory. Besides, new classes of spinor fields have been also shown to exist on compactifications of type AdS$_5 \times S^5$, being further important to the analysis of fermion localisation on brane-worlds. Moreover, new fermionic field in a bulk correspond to the new spinor field classes that are evinced from the geometric Fierz identities, in five-dimensional Lorentzian spacetimes. Our main goal here is to present this framework and their applications, discussing the utility of considering spinors that are quite distinct of the usual Dirac, Majorana and Weyl constructions.

This paper is organised as follows: the bilinear covariants are the stage to revisit the classification of spinor fields in Minkowski spacetime, according to the Lounesto classification prescription, and the Fierz aggregate and its related boomerang are defined. The geometric Fierz identities are employed and, from the admissible pairings between spinor fields, the number of classes in the spinor field classification is constrained, by the geometric Fierz identities [27]. We study the case of spinor fields on Lorentzian 5-manifolds and conclude that five-dimensional spinors pertain to solely one class, through a similar approach to the one previously acquired in seven-dimensional [25, 26, 27]. Spinor fields on 5-manifolds can be then classified in eight classes, from the classification of quaternionic spinor fields. One of the new classes encompasses the
standard five-dimensional spinor fields, and six others non-trivial classes provide new candidates for physical solutions, for instance, in thick brane models and gravity on AdS$_5$.

2. Geometric Fierz Identities and Bilinear Covariants

In order to fix the notation, let an oriented manifolds $(M,g)$ endowed with the metric $g$, of signature $p-q$, has a spacetime structure \cite{[22]}, having an exterior bundle $\Omega(M) = \bigoplus_{k=0}^n \Omega^k(M)$ whereupon the endomorphisms can be usually defined, for a $k$-form $a \in \Omega^k(M)$, namely, the grade involution, $\hat{\alpha} = (-1)^ka$, that is an automorphism; the reversion, $\tilde{\alpha} = (-1)^{|k/2|}a$, where $|k|$ is the integer part of $k$, that is an anti-automorphism; and the conjugation anti-automorphism, defined by the commutativ composition between the grade involution and the reversion, denoted hereon by $\hat{\alpha} \equiv \hat{a} = \tilde{a}$. Those three morphism can be extended to the whole exterior bundle, by linearity and, further, to the Clifford bundle, likewise. By endowing the exterior bundle $\Omega(M)$ with the universal Clifford product $\psi \circ a = \psi \land a + u_\mu a$, for all form fields $\psi \in \sec \Omega^k(M)$, where $\land$ is the left contraction, the Clifford bundle can be, therefore, defined.

The spinor bundle in four-dimensional Minkowski spacetime corresponds to the bundle $\mathcal{P}_{\text{Spin}^1,3}(M) \times_\sigma \mathbb{C}^4$. Classical spinor fields $\psi \in \sec \mathcal{P}_{\text{Spin}^1,3}(M) \times_\sigma \mathbb{C}^4$ carry the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representations $\sigma$ of the Lorentz group. The bundle $\mathcal{P}_{\text{Spin}^1,3}(M)$ has fibers, whereupon the group Spin$_{1,3}$ acts. The bilinear covariants are sections of the exterior bundle $\Omega(M)$, whose basis is given by $\{e^\mu\}$ \cite{[1]} (hereupon $\mu, \nu = 0, 1, 2, 3$ are Minkowski spacetime indexes):

\[
\begin{align*}
\sigma &= \bar{\psi}\psi \in \sec \Omega^0(M), \quad (1a) \\
\mathbf{J} &= J_\mu e^\mu \in \sec \Omega^1(M), \quad (1b) \\
\mathbf{S} &= S_{\mu\nu} e^\mu \land e^\nu \in \sec \Omega^2(M), \quad (1c) \\
\mathbf{K} &= K_\mu e^\nu \in \sec \Omega^3(M), \quad (1d) \\
\omega &= \bar{\psi} \gamma_5 \psi \in \sec \Omega^4(M), \quad (1e)
\end{align*}
\]

are the respective components in Eqs. (1b) – (1d) and $\gamma_5 := i\gamma_0\gamma_1\gamma_2\gamma_3$, where the spinor conjugation is denoted by $\psi = \bar{\psi}^\dagger\gamma_0$ \cite{[2, 22]}. The gamma matrices satisfy the usual Clifford relation $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}I$, where $g_{\mu\nu}$ denotes the Minkowski spacetime metric components. The physical observables, exclusively for the Dirac electron theory, are realised by bilinear covariants. In fact, the current density $\mathbf{J}$, the spin density $\mathbf{S}$, and the chiral current $\mathbf{K}$, satisfy the Fierz identities \cite{[1]}

\[-(\omega + \sigma \gamma_5)\mathbf{S} = \mathbf{J} \land \mathbf{K}, \quad \mathbf{K} \cdot \mathbf{K} + \mathbf{J} \cdot \mathbf{J} = 0 = \mathbf{J} \cdot \mathbf{K}, \quad \mathbf{J} \cdot \mathbf{J} = \omega^2 + \sigma^2. \quad (5)\]

Lounesto derived, from the bilinear covariants, a classification of spinor fields \cite{[1]}. The expression $\mathbf{J} \neq 0$ holds for all spinors in Lounesto class. However, this condition that was firstly motivated by the Dirac electron theory can be circumvented in three additional classes that were recently derived in Minkowski spacetime \cite{[24]}, conjectured to consist of ghost spinors:

\[
\begin{align*}
1) \omega &\neq 0, \quad \sigma \neq 0, \quad \mathbf{K} \neq 0, \quad \mathbf{S} \neq 0 \quad (\text{regular spinors}) \quad (6a) \\
2) \omega &\neq 0, \quad \sigma \neq 0, \quad \mathbf{K} \neq 0, \quad \mathbf{S} \neq 0 \quad (\text{regular spinors}) \quad (6b) \\
3) \omega &\neq 0, \quad \sigma = 0, \quad \mathbf{K} \neq 0, \quad \mathbf{S} \neq 0 \quad (\text{regular spinors}) \quad (6c) \\
4) \omega &\neq 0, \quad \sigma = 0, \quad \mathbf{K} \neq 0, \quad \mathbf{S} \neq 0 \quad (\text{flag-dipole, singular spinors}) \quad (6d) \\
5) \omega &\neq 0, \quad \sigma = 0, \quad \mathbf{K} = 0, \quad \mathbf{S} \neq 0 \quad (\text{flagpole, singular spinors}) \quad (6e) \\
6) \omega &\neq 0, \quad \sigma = 0, \quad \mathbf{K} \neq 0, \quad \mathbf{S} = 0 \quad (\text{dipole, singular spinors}) \quad (6f)
\end{align*}
\]
Singular spinors consist of flag-dipole, flagpole, and dipole spinors, respectively in the fourth, fifth, and sixth classes in the just mentioned six classes. The standard Dirac spinor is an element of the set of regular spinors in class 1. Moreover, Majorana spinors – that have mass dimension 3/2 – and Elko spinor fields – presenting canonical mass dimension 1 – are neutral spinors that embrace particular realisations of flagpole spinors, that further includes a set of spinors that are solutions of the Dirac equation in a Kerr black hole background. The chiral Weyl spinors consist of a tiny subset of dipole spinors. Nevertheless, regular, flagpole, flag-dipole, and dipole spinor fields provide a plethora of opportunities that have not been explored in the literature whatsoever. Although dozen of explicit constructions, consisting of non-standard spinors, with defined dynamics, were derived in the last decade, like the flag-dipole spinors in Einstein-Sciama-Kibble theory of gravity [11], that was the first concrete example of flag-dipole spinor field in the literature, largely searched for by Lounesto itself [1], than those realisations.

The Fierz-Pauli-Kofink (FPK) identities (5) are well known not to be valid for the case of singular spinors. In this case, based upon a Fierz aggregate,

$$Z = \gamma_5(\omega + K) + iS + J + \sigma,$$

the FPK identities (5) can be replaced by the the most general equations

$$iZ = \frac{i}{\omega}Z\gamma_5Z = \frac{1}{S_{\mu\nu}}Z\gamma_\mu\gamma_\nu Z = -\frac{1}{K_{\mu}}Z\gamma_5\gamma_\mu Z,$$

that can be led to (5) when not both \(\sigma\) and \(\omega\) equal zero, namely, for the case of a regular spinor field.

The geometric underlying content of the Fierz aggregate can be encompassed by physical considerations. The irreducible representation of the Lorentz group can be split into homogeneous components of the Fierz aggregate, as follows:

$$[(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)] \otimes [(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)] = \underbrace{(0, 0) \oplus (1, 0) \oplus (0, 1)}_{\sigma} \oplus \underbrace{(\frac{1}{2}, \frac{1}{2})}_{J} \oplus \underbrace{(\frac{1}{2}, 0)}_{S} \oplus \underbrace{(0, \frac{1}{2})}_{K} \oplus \underbrace{(0, 0)}_{\omega}$$

It is worth to emphasise that in classes 4 and 5, in Lounesto classification, the form fields \(\{J, K\}\) are not elements of a basis for the dual Minkowski spacetime cotangent bundle, since for these cases the linear span of the set \(\{J, K\}\) collapses into a null line with direction provided by \(J\). The 1-form field \(J\) is interpreted as being a pole, and flagpoles are consequently elements of the class 5 in Lounesto classification. In fact, for this case one finds by definition that \(K = 0\) and \(S \neq 0\). The 2-form field \(S\) can be realised as being a plaquette, namely, a flag-pole. Regarding now type-4 spinor fields, both the 2-form field \(S\) and the 1-form field \(K\) do not equal zero, thus constituting a flag-dipole structure. Both concepts encompass Penrose flagpoles [2]. Moreover, flagpole and dipoles are, by definition, limiting cases of flag-dipole spinor fields.

Generalising the FPK relations beyond the Minkowski spacetime setup has brought unexpectedly prominent panoramas for higher dimensional theories in Physics. Interesting features of newly found non-standard spinor fields, with respect to the textbook ones, have also unexpected applications. In this context, the standard FPK identities were extended and introduced for form-valued spinor bilinear covariants, for arbitrary spacetime dimensions and metric signatures [27, 28]. Indeed, Refs. [27, 28] show that the geometric Fierz identities strongly obstruct the number, and even the existence, of classes of pinor and spinor fields. We have recently found how to circumvent these constructions, to provide two new classes of spinors (on the spin bundle of a Riemannian 7-manifold) [25], three new classes of spinor fields (on the
spin bundle of a Lorentzian 7-manifold) [26], and some additional spinor classes (on the spin bundle of a Lorentzian 5-manifold, that serves as a 5-dimensional bulk) [5].

On manifolds of arbitrary dimensions having spin structure, the Kähler-Atiyah bundle (\(\sec \Omega(M, \omega)\)) is then employed, whose associated spin bundle \(S\) a module setup that is defined by the morphism \(\gamma : (\Omega(M, \omega)) \rightarrow (\text{End}(S), \ast)\) [2, 27, 28]. In addition, there is a splitting \(S = S(0) \oplus S(1)\) for an idempotent \(R \in \Gamma(\text{End}(S))\) in the space of the endomorphisms \(\text{sec End}(S)) [27]. The sets \(S(0)\) and \(S(1)\) are respectively determined by the eigenvalues \(\pm 1\) of \(R\) [2]. Spin projectors are defined by \(\Pi_{\pm} = \frac{1}{2}(I \pm R)\), providing the splitting \(S = \Pi_{+}(S) \oplus \Pi_{-}(S) \equiv S^+ \oplus S^-\). Sections of \(S^\pm\) are called Majorana spinors when \(p - q \equiv 7 \mod 8\) [25, 26]. The complex structure \(J \in \Gamma(\text{End}(S))\) further equips the Kähler-Atiyah bundle [27], and an endomorphism \(D\) on the spin bundle, that commutes with the grade involution, is defined [27], by \(D \circ D = (-1)^{\frac{p+q+1}{2}} I\). A non-degenerate bilinear pairing \(B : S \times S \rightarrow \mathbb{R}\) induces a more general bilinear form [27, 28], by the complexification procedure [25, 26]. In fact, smooth sections \(\psi, \psi' \in \Gamma(S)\) are employed to define a bilinear form \(\beta_0 : S \times S \rightarrow \mathbb{R}\), as [27, 28, 25, 26],

\[
\beta_0(\psi, \psi') = B(\Re(\psi) \Re(\psi')) - B(\Im(\psi) \Im(\psi')) + i\left[ B(\Re(\psi) \Im(\psi')) + B(\Im(\psi) \Re(\psi')) \right],
\]

where \((\Re \psi) = \frac{1}{2}(\psi + D(\psi)))\) [\(\Im \psi = \frac{1}{2}(\psi - D(\psi)))\] is the real [imaginary] parts of \(\psi\) [27].

For arbitrary dimensions, a spin-invariant product induces a spinor conjugation, \(\bar{\psi} = \psi^b b^{-1}\), where \(b\) is some appropriate invertible element, uniquely defined for each spacetime dimension/signature, in the Clifford bundle, yielding the corresponding spin bilinear form on \(S\):

\[
\beta_k(\psi, \psi') = B(\psi, \gamma_{\rho_1...\rho_k} \psi') = \bar{\psi} \gamma_{\rho_1...\rho_k} \psi'.
\]

The particular case of Minkowski spacetime reads the well-known spinor conjugation \(\bar{\psi} = \psi^4 \gamma^0\) [2].

The important case studied in [27, 25, 26] consist of the signature \(p - q \equiv 7 \mod 8\) [27]. The real chiral spinor bundles \(S^\pm \equiv P_{\pm}(S)\) define an element \(\psi\) to be a Majorana spinor. Bilinear pairings can be restricted by the condition \(B(\psi, \gamma^{\rho_1...\rho_k} \psi) \neq 0\) if \(k\) is odd [27]. The endomorphisms \(A_{\psi|\psi'}\) of the spin bundle \(S\) were defined [27] as:

\[
A_{\psi_1|\psi_2}(\psi) := B(\psi, \psi_2)\psi_1, \quad \text{for all } \psi, \psi_1, \psi_2 \in \Gamma(S).
\]

It determines the geometric Fierz identities [27]

\[
A_{\psi_1|\psi} \ast A_{\psi_3|\psi_4} = B(\psi_3, \psi_2)A_{\psi_1|\psi_4}.
\]

Consider now the completeness relation \(A_{\psi|\psi'} \sim \sum_k \frac{1}{k!} (-1)^k B(\psi, \gamma_{\rho_1...\rho_k} \psi') e^{\rho_1...\rho_k}\), where the symbol "\(\sim\)" denotes an equivalence modulus a constant that equals either a suitable power of the constant 2 or two times this constant, that is spacetime signature dependent [27]. Moreover, every element in the space \(\Gamma(\text{End}(S))\), in particular the \(A_{\psi|\psi'}\), can be split uniquely as \(A_{\psi|\psi'} = D \circ A^{1}_{\psi|\psi'} + A^{0}_{\psi|\psi'}\) [27], where

\[
A^{0}_{\psi|\psi'} \sim \sum_k \frac{(-1)^k}{k!} B(\psi, \gamma_{\rho_1...\rho_k} \psi') e^{\rho_1...\rho_k},
\]

\[
A^{1}_{\psi|\psi'} \sim \sum_k \frac{1}{k!} (-1)^{k+1(p+q)} B(\psi, D \gamma_{\rho_1...\rho_k} \psi') e^{\rho_1...\rho_k}.
\]

The FPK identities (5) read, for generalised for arbitrary dimensions/signatures, hence [27]:

\[
\overline{A^0_{\psi_1|\psi_2}} \circ A^1_{\psi_3|\psi_4} + A^1_{\psi_1|\psi_2} \circ A^0_{\psi_3|\psi_4} = B(\psi_3, \psi_2)A^1_{\psi_1|\psi_4},
\]

\[
A^0_{\psi_1|\psi_2} \circ A^0_{\psi_3|\psi_4} + (-1)^{1(p+q)} \overline{A^0_{\psi_1|\psi_2}} \circ A^1_{\psi_3|\psi_4} = B(\psi_3, \psi_2)A^0_{\psi_1|\psi_4}.
\]
Hence, by defining \( \varphi_k = \frac{1}{k!} B(\psi, \gamma_{\rho_1...\rho_k} \psi) e^{\rho_1...\rho_k} \) for the Minkowski spacetime. As we are interested in determining the nature of Majorana spinor fields according to bilinear covariants in 7-manifolds, we focus in the particular, however important case of \( n = p + q = 7 + 0 \). Moreover, the expression

\[
\varphi_k = \frac{1}{k!} B(\psi, \gamma_{\rho_1...\rho_k} \psi) e^{\rho_1...\rho_k}
\]

equals zero except if \( k \) is even. Together with the symmetry of \( B \), this shows that the element \( A^{0,k} \equiv A^{0,k}_{\psi|\psi} \) vanishes except if \( k = 0, 3, 4, 7 \). Combining this with (16), it implies that, for \( \psi \) a Majorana spinor, the forms \( \varphi_k \) equal zero except when \( k = 0 \) or \( k = 4 \). By regarding \( \psi \) normalized such that \( B(\psi, \psi) = 1 \), it follows that \( A^{0,0} = 1 \) and the following bilinear can be defined [27]:

\[
\varphi_4 = \frac{1}{4!} B(\psi, \gamma_{\rho_1\rho_2\rho_3\rho_4} \psi) e^{\rho_1\rho_2\rho_3\rho_4},
\]

which are the components of the first generator \( A^0 = \frac{1}{16}(1 + \varphi_4) \) of the so called Fierz algebra represented in (14a, 14b) [27], where \( A^\lambda \equiv A^\lambda_{\psi|\psi} \). The product \( \Delta_k : \Omega(M) \times \Omega(M) \to \Omega(M) \) is iteratively defined by

\[
\theta_1 \Delta_{k+1} \theta_2 = \frac{1}{k+1} g_{ab}(e^a, \theta_1) \Delta_k (e^b, \theta_2), \quad \chi, \vartheta \in \sec \Omega(M),
\]

where the symbol \( g_{ab} \) stands for the metric tensor coefficients. One fixes \( \Delta_0 = \wedge \) as being the exterior product.

In the case here to be analyzed \( n = p + q = 7 \), we already know that \( \varphi_k = 0 \) except for the values \( k \in \{0, 3, 4, 7\} \). In addition, due to the restriction that \( B(\psi, J_\sigma \gamma_{\rho_1...\rho_k} \psi) = -B(\psi, J_\sigma D_\sigma \gamma_{\rho_1...\rho_k} \psi) = 0 \), unless if \( k \) is odd, one obtains the restriction \( B'(\psi, \gamma_{\rho_1...\rho_k} \psi) = 0 \), unless \( k = 3 \) or \( k = 7 \). It was shown in [27, 26, 25] that merely the forms \( \varphi_0 = B(\psi, \psi) \) and \( \varphi_4 = \frac{1}{4!} B(\psi, \gamma_{\rho_1\rho_2\rho_3\rho_4} \psi) e^{\rho_1\rho_2\rho_3\rho_4} \) are different of zero. The Fierz identities imply, in particular, that \( \varphi_4 \neq 0 \), corresponding to a single class of – Majorana – spinors:

\[
\varphi_0 \neq 0, \quad \varphi_1 = 0, \quad \varphi_2 = 0, \quad \varphi_3 = 0, \quad \varphi_4 \neq 0, \quad \varphi_5 = 0, \quad \varphi_6 = 0, \quad \varphi_7 = 0.
\]

Eq. (11) can be thus generalized, in order to encompass the complex case, providing the higher degree generalization of (10):

\[
\beta_k(\psi, \gamma_{\rho_1...\rho_k} \psi) = -B(\psi, \gamma_{\rho_1...\rho_k} (\text{Im}) \psi) + B(\psi, \gamma_{\rho_1...\rho_k} (\text{Re}) \psi) + i\left[ B(\psi, \gamma_{\rho_1...\rho_k} (\text{Im}) \psi) - B(\psi, \gamma_{\rho_1...\rho_k} (\text{Re}) \psi) \right].
\]

Hence, by defining \( \tilde{\phi}_k := \frac{1}{k!} \beta_k(\psi, \gamma_{\rho_1...\rho_k} \psi) e^{\rho_1...\rho_k} \), the non-trivial classes of spinor fields \( \psi \in \Gamma(S) \) read [25]:

\[
1) \quad \tilde{\phi}_0 = 0, \quad \tilde{\phi}_4 \neq 0, \quad \tilde{\phi}_8 = 0 \quad \tilde{\phi}_{12} \neq 0.
\]

All other \( \tilde{\phi}_k = 0 \) for \( k \neq 4 \). Hence, novel classes of spinors, under the geometric Fierz identities, was here illustrated, for the case of Riemannian 7-manifolds [25, 26]. Other cases, regarding new spinors in a five-dimensional bulk and in Lorentzian 7-manifolds [26] were found. Other few signatures can be, further, explored [27, 28].
2.1. Acknowledgments

RdR thanks FAPESP (grant No. 2015/10270-0) and CNPq (grant No. 303293/2015-2; No. 473326/2013-2), for partial financial support.

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