Spherical Functions of Fundamental $K$-Types Associated with the $n$-Dimensional Sphere

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Abstract. In this paper, we describe the irreducible spherical functions of fundamental $K$-types associated with the pair $(G, K) = (\text{SO}(n+1), \text{SO}(n))$ in terms of matrix hypergeometric functions. The output of this description is that the irreducible spherical functions of the same $K$-fundamental type are encoded in new examples of classical sequences of matrix-valued orthogonal polynomials, of size 2 and 3, with respect to a matrix-weight $W$ supported on $[0, 1]$. Moreover, we show that $W$ has a second order symmetric hypergeometric operator $D$.

Key words: matrix-valued spherical functions; matrix orthogonal polynomials; the matrix hypergeometric operator; $n$-dimensional sphere

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1 Introduction

The theory of spherical functions dates back to the classical papers of É. Cartan and H. Weyl; they showed that spherical harmonics arise in a natural way from the study of functions on the $n$-dimensional sphere $S^n = \text{SO}(n+1)/\text{SO}(n)$. The first general results in this direction were obtained in 1950 by Gel'fand, who considered zonal spherical functions of a Riemannian symmetric space $G/K$. In this case we have a decomposition $G = KAK$. When the Abelian subgroup $A$ is one dimensional, the restrictions of zonal spherical functions to $A$ can be identified with hypergeometric functions, providing a deep and fruitful connection between group representation theory and special functions. In particular when $G$ is compact this gives a one to one correspondence between all zonal spherical functions of the symmetric pair $(G, K)$ and a sequence of orthogonal polynomials.

In light of this remarkable background it is reasonable to look for an extension of the above results, by considering matrix-valued irreducible spherical functions on $G$ of a general $K$-type. This was accomplished for the first time in the case of the complex projective plane $P_2(\mathbb{C}) = \text{SU}(3)/\text{U}(2)$ in [5]. This seminal work gave rise to a series of papers including [6, 7, 8, 10, 14, 15, 16, 17, 18, 19], where one considers matrix valued spherical functions associated to a compact symmetric pair $(G, K)$ of rank one, arriving at sequences of matrix valued orthogonal polynomials of one real variable satisfying an explicit three-term recursion relation, which are also eigenfunctions of a second order matrix differential operator (bispectral property).

The very explicit results contained in this paper are obtained for certain $K$-types, namely the fundamental $K$-types. Also, the detailed construction of sequences of matrix orthogonal polynomials out of these irreducible spherical functions, following the general pattern established in [5], gives new examples of classical sequences of matrix-valued orthogonal polynomials of size 2 and 3. For the general notions concerning matrix-valued orthogonal polynomials see [9].
Interesting generalizations of these sequences are given in [20], where the coefficients of the three term recursion relation satisfied by them is exhibited.

The present paper is an outgrowth of the results of [25, Chapter 5] and we are currently working on the extension of these results for the spherical functions of any $K$-type associated with the $n$-dimensional sphere. Using [23], one can obtain the corresponding results for the spherical functions of any $K$-type associated with $n$-dimensional real projective space. The starting point is to describe the irreducible spherical functions associated with the pair $(G, K) = (\text{SO}(n+1), \text{SO}(n))$ in terms of eigenfunctions of a matrix linear differential operator of order two. The output of this description is that the irreducible spherical functions of the same fundamental $K$-type are encoded in a sequence of matrix valued orthogonal polynomials.

Briefly the main results of this paper are the following. After some preliminaries, in Section 3 we study the eigenfunctions of an operator $\Delta$ on $G$, which is closely related to the Casimir operator. Every spherical function $\Phi$ has to be eigenfunction of this operator $\Delta$; considering the $KAK$-decomposition

$$\text{SO}(n+1) = \text{SO}(n)\text{SO}(2)\text{SO}(n)$$

and choosing an appropriate coordinate $y$ on an open subset of $A$, we translate the condition $\Delta \Phi = \lambda \Phi$, $\lambda \in \mathbb{C}$, into a matrix valued differential equation $\tilde{D}H = \lambda H$ on the open interval $(0,1)$, where $H$ is the restriction of $\Phi$ to $\text{SO}(2)$. The property of the spherical functions

$$\Phi(xgy) = \pi(x)\Phi(g)\pi(y), \quad g \in G, \quad x, y \in K,$$

tell us that $\Phi$ is determined by its $K$-type and the function $H$.

In Section 4 we first explicitly describe all the irreducible spherical functions of the symmetric pair $(G, K) = (\text{SO}(n+1), \text{SO}(n))$ with $M$-irreducible $K$-types, with $M = \text{SO}(n-1)$, the centralizer of the subgroup $A$ in $K$; we give these expressions in terms of the hypergeometric function $2F_1$.

In Section 5 the operator $\tilde{D}$ is studied in detail when the $K$-types correspond to fundamental representations. Certain $K$-fundamental types are $M$ irreducible, and therefore they were already considered in Section 4; besides, when $n$ is odd there is a particular fundamental $K$-type which has three $M$-submodules, this case is studied in the last section of this work. For the rest of the cases we considered separately when $n$ is even and when $n$ is odd. Although, in both cases we worked with the concrete realizations of the fundamental representations considering the exterior powers of the standard representation of $\text{SO}(n)$:

$$\Lambda^1(\mathbb{C}^n), \Lambda^2(\mathbb{C}^n), \ldots, \Lambda^{\ell-1}(\mathbb{C}^n),$$

with $n = 2\ell$ or $n = 2\ell + 1$.

In Section 6 we conjugate the operator $\tilde{D}$, by using the polynomial function

$$\Psi(y) = \begin{pmatrix} 2y - 1 & 1 \\ 1 & 2y - 1 \end{pmatrix},$$

whose columns correspond to irreducible spherical functions, in order to obtain a matrix-valued hypergeometric operator $D = \Psi^{-1}\tilde{D}\Psi$:

$$DP = y(1 - y)P'' + (C - yU)P' - VP,$$

with

$$C = \begin{pmatrix} (n/2 + 1) & 1 \\ 1 & (n/2 + 1) \end{pmatrix}, \quad U = (n + 2)I, \quad V = \begin{pmatrix} p & 0 \\ 0 & n - p \end{pmatrix}.$$
Then, we study all the possible eigenvalues corresponding to irreducible spherical functions and all the polynomial eigenfunctions of $D$.

In Section 7, for any fundamental $K$-type $(\Lambda^k(\mathbb{C}^n))$ with $1 \leq p \leq \ell - 1$, we find a matrix-weight $W$, which is a scalar multiple of

$$W = (y(1 - y))^{n/2 - 1} \left( \frac{p(2y - 1)^2 + n - p}{n(2y - 1)} \quad \frac{n(2y - 1)}{(n - p)(2y - 1)^2 + p} \right),$$

such that $D$ is a symmetric operator with respect to the inner product defined among continuous vector-valued functions on $[0, 1]$ by

$$\langle P_1, P_2 \rangle_W = \int_0^1 P_2^*(y)W(y)P_1(y)dy.$$

Also we prove that every spherical function gives a vector polynomial eigenfunction $P$ of $D$. Therefore we obtain the following explicit expression of $P$ in terms of the matrix hypergeometric function for any irreducible spherical function

$$P(y) = \sum_{j=0}^w \frac{y^j}{j!} [C; U; V + \lambda]_j P(0),$$

see Theorem 7.6.

In Section 8 for each pair $(n, p)$ we construct a sequence of matrix orthogonal polynomials $\{P_w\}_{w \geq 0}$ of size 2 with respect to the weight function $W$, which are eigenfunctions of the symmetric differential operator $D$. Namely,

$$DP_w = P_w \begin{pmatrix} \lambda(w, 0) & 0 \\ 0 & \lambda(w, 1) \end{pmatrix},$$

where

$$\lambda(w, \delta) = \begin{cases} -w(w + n + 1) - p & \text{if } \delta = 0, \\ -w(w + n + 1) - n + p & \text{if } \delta = 1. \end{cases}$$

Finally, in Section 9 we develop the same techniques in order to obtain analogous results for irreducible spherical functions of the particular $K$-fundamental type $\Lambda^\ell(\mathbb{C}^n)$ for which we have three $M$-submodules instead of only two. This only occurs when $n$ is of the form $2\ell + 1$.

It is worth to notice that, unlike the other cases, the $3 \times 3$ matrix-weight built here does reduce to a smaller size.

## 2 Preliminaries

### 2.1 Spherical functions

Let $G$ be a locally compact unimodular group and let $K$ be a compact subgroup of $G$. Let $\hat{K}$ denote the set of all equivalence classes of complex finite dimensional irreducible representations of $K$; for each $\delta \in \hat{K}$, let $\xi_{\delta}$ denote the character of $\delta$, $d(\delta)$ the degree of $\delta$, i.e. the dimension of any representation in the class $\delta$, and $\chi_{\delta} = d(\delta)\xi_{\delta}$. We shall choose once and for all the Haar measure $dk$ on $K$ normalized by $\int_K dk = 1$.

We shall denote by $V$ a finite dimensional vector space over the field $\mathbb{C}$ of complex numbers and by of all linear transformations of $V$ into $V$. Whenever we refer to a topology on such a vector space we shall be talking about the unique Hausdorff linear topology on it.
Definition 2.1. A spherical function $\Phi$ on $G$ of type $\delta \in \hat{K}$ is a continuous function on $G$ with values in $\text{End}(V)$ such that

i) $\Phi(e) = I$ ($I$ is the identity transformation);

ii) $\Phi(x)\Phi(y) = \int_K \chi\delta(k^{-1})\Phi(xky)dk$ for all $x, y \in G$.

The reader can find a number of general results in [21] and [4]. For our purpose it is appropriate to recall the following facts.

Proposition 2.2 ([21, Proposition 1.2]). If $\Phi : G \rightarrow \text{End}(V)$ is a spherical function of type $\delta$ then:

i) $\Phi(k_1gk_2) = \Phi(k_1)\Phi(g)\Phi(k_2)$, for all $k_1, k_2 \in K$, $g \in G$;

ii) $k \mapsto \Phi(k)$ is a representation of $K$ such that any irreducible subrepresentation belongs to $\delta$.

Concerning the definition, let us point out that the spherical function $\Phi$ determines its type univocally (Proposition 2.2) and let us say that the number of times that $\delta$ occurs in the representation $k \mapsto \Phi(k)$ is called the height of $\Phi$.

A spherical function $\Phi : G \rightarrow \text{End}(V)$ is called irreducible if $V$ has no proper subspace invariant by $\Phi(g)$ for all $g \in G$.

If $G$ is a connected Lie group, it is not difficult to prove that any spherical function $\Phi : G \rightarrow \text{End}(V)$ is differentiable ($C^\infty$), and moreover that it is analytic. Let $D(G)$ denote the algebra of all left invariant differential operators on $G$ and let $D(G)^K$ denote the subalgebra of all operators in $D(G)$ which are invariant under all right translations by elements in $K$.

In the following proposition $(V, \pi)$ will be a finite dimensional representation of $K$ such that any irreducible subrepresentation belongs to the same class $\delta \in \hat{K}$.

Proposition 2.3. A function $\Phi : G \rightarrow \text{End}(V)$ is a spherical function of type $\delta$ if and only if

i) $\Phi$ is analytic;

ii) $\Phi(k_1gk_2) = \pi(k_1)\Phi(g)\pi(k_2)$, for all $k_1, k_2 \in K$, $g \in G$, and $\Phi(e) = I$;

iii) $[D\Phi](g) = \Phi(g)[D\Phi](e)$, for all $D \in D(G)^K$, $g \in G$.

Moreover, we have that the eigenvalues $[D\Phi](e)$, $D \in D(G)^K$, characterize the spherical functions $\Phi$ as stated in the following proposition.

Proposition 2.4 ([21, Remark 4.7]). Let $\Phi, \Psi : G \rightarrow \text{End}(V)$ be two spherical functions on a connected Lie group $G$ of the same type $\delta \in \hat{K}$. Then $\Phi = \Psi$ if and only if $(D\Phi)(e) = (D\Psi)(e)$ for all $D \in D(G)^K$.

Let us observe that if $\Phi : G \rightarrow \text{End}(V)$ is a spherical function, then $\Phi : D \mapsto [D\Phi](e)$ maps $D(G)^K$ into $\text{End}_K(V)$ ($\text{End}_K(V)$ denotes the space of all linear maps of $V$ into $V$ which commutes with $\pi(k)$ for all $k \in K$) defining a finite dimensional representation of the associative algebra $D(G)^K$. Moreover, the spherical function is irreducible if and only if the representation $\Phi : D(G)^K \rightarrow \text{End}_K(V)$ is irreducible. We quote the following result from [19].

Proposition 2.5 ([19, Proposition 2.5]). Let $G$ be a connected reductive linear Lie group. Then the following properties are equivalent:

i) $D(G)^K$ is commutative;

ii) every irreducible spherical function of $(G, K)$ is of height one.
In this paper the pair \((G, K)\) is \((\text{SO}(n + 1), \text{SO}(n))\). Then, it is known that \(D(G)^K\) is an Abelian algebra; moreover, \(D(G)^K\) is isomorphic to \(D(G)^G \otimes D(K)^K\) (see in [13, Theorem 10.1] or [1]), where \(D(G)^G\) (resp. \(D(K)^K\)) denotes the subalgebra of all operators in \(D(G)\) (resp. \(D(K)\)) which are invariant under all right translations by elements in \(G\) (resp. \(K\)).

An immediate consequence of this is that all irreducible spherical functions of our pair \((G, K)\) are of height one.

Spherical functions of type \(\delta\) (see in [21, Section 3]) arise in a natural way upon considering representations of \(G\). If \(g \mapsto U(g)\) is a continuous representation of \(G\), say on a finite dimensional vector space \(E\), then

\[
P_\delta = \int_K \chi_\delta(k^{-1}) U(k) dk
\]

is a projection of \(E\) onto \(P_\delta E = E(\delta)\). If \(P_\delta \neq 0\) the function \(\Phi : G \rightarrow \text{End}(E(\delta))\) defined by

\[
\Phi(g) a = P_\delta U(g) a, \quad g \in G, \quad a \in E(\delta), \quad (2.1)
\]

is a spherical function of type \(\delta\). In fact, if \(a \in E(\delta)\) we have

\[
\Phi(x) \Phi(y) a = P_\delta U(x) P_\delta U(y) a = \int_K \chi_\delta(k^{-1}) P_\delta U(x) U(k) U(y) adk = \left( \int_K \chi_\delta(k^{-1}) \Phi(x ky) dk \right) a.
\]

If the representation \(g \mapsto U(g)\) is irreducible then the associated spherical function \(\Phi\) is also irreducible. Conversely, any irreducible spherical function on a compact group \(G\) arises in this way from a finite dimensional irreducible representation of \(G\).

### 2.2 Root space structure of \(\mathfrak{so}(n, \mathbb{C})\)

Let \(E_{ik}\) denote the square matrix with a 1 in the \(ik\)-entry and zeros elsewhere; and let us consider the matrices

\[
I_{ki} = E_{ik} - E_{ki}, \quad 1 \leq i, k \leq n.
\]

Then, the set \(\{I_{ki}\}_{i < k}\) is a basis of the Lie algebra \(\mathfrak{so}(n)\). These matrices satisfy the following commutation relations

\[
[I_{ki}, I_{rs}] = \delta_{ks} I_{ri} + \delta_{ri} I_{sk} + \delta_{is} I_{kr} + \delta_{rk} I_{is}.
\]

If we assume that \(k > i, r > s\) then we have

\[
[I_{ki}, I_{is}] = I_{sk}, \quad [I_{ki}, I_{rk}] = I_{ri}, \quad [I_{ki}, I_{ri}] = I_{kr}, \quad [I_{ki}, I_{ks}] = I_{is},
\]

and all the other brackets are zero. From this it easily follows that the set

\[
\{I_{p,p-1} : 2 \leq p \leq n\}
\]

generates the Lie algebra \(\mathfrak{so}(n)\).

**Proposition 2.6.** Given \(n \in \mathbb{N}\), we have that the operator

\[
Q_n = \sum_{1 \leq i, k \leq n} I_{ki}^2 \in D(\text{SO}(n))
\]

is right invariant under \(\text{SO}(n)\), i.e.

\[Q_n \in D(\text{SO}(n))^\text{SO}(n), \quad \forall n \in \mathbb{N}_0.\]
Proof. To prove that \( Q_n \) is right invariant under \( \text{SO}(n) \) it is enough to prove that \( \dot{I}_{p,p-1}(Q_n) = 0 \) for all \( 2 \leq p \leq n \). We have

\[
\dot{I}_{p,p-1}(Q_n) = \sum_{1 \leq i,k \leq n} ([I_{p,p-1}, I_{ki}]I_{ki} + I_{ki}[I_{p,p-1}, I_{ki}]).
\]

Then

\[
\dot{I}_{p,p-1}(Q_n) = \sum_{1 \leq i \leq n} (I_{ip}I_{p-1,i} + I_{p-1,i}I_{ip}) + \sum_{1 \leq k \leq n} (I_{k,p}I_{k,p-1} + I_{k,p-1}I_{k,p})
\]

\[
+ \sum_{1 \leq k \leq n} (I_{pk}I_{k,p-1} + I_{k,p-1}I_{pk}) + \sum_{1 \leq i \leq n} (I_{p-1,i}I_{p,i} + I_{p,i}I_{p-1,i}) = 0.
\]

This proves the proposition.\( \blacksquare \)

2.3 The operator \( Q_{2\ell} \)

Let us assume that \( n = 2\ell \). We look at a root space decomposition of \( \mathfrak{so}(n) \) in terms of the basis elements \( I_{ki}, 1 \leq i < k \leq n \). The linear span

\[
\mathfrak{h} = \langle I_{21}, I_{43}, \ldots, I_{2\ell,2\ell-1} \rangle_C
\]

is a Cartan subalgebra of \( \mathfrak{so}(n, \mathbb{C}) \). To find the root vectors it is convenient to visualize the elements of \( \mathfrak{so}(n, \mathbb{C}) \) as \( \ell \times \ell \) matrices of \( 2 \times 2 \) blocks. Thus \( \mathfrak{h} \) is the subspace of all diagonal matrices of \( 2 \times 2 \) skew-symmetric blocks. The subspaces of all matrices \( A \) with a block \( A_{jk} \) of size two, \( 1 \leq j < k \leq \ell \), in the place \((j, k)\) and \( -A_{jk}^t \) in the place \((k, j)\) with zeros in all other places, are \( \text{ad}(\mathfrak{h}) \)-stable. Let

\[
H = i(x_1I_{21} + \cdots + x_\ell I_{2\ell,2\ell-1}) \in \mathfrak{h},
\]

for \( x_1, \ldots, x_\ell \in \mathbb{R} \). Then \([H, A] = \lambda(H)A, \forall H \in \mathfrak{h}\), if and only if for every \( A_{jk} \) we have

\[
x_j(H)iI_{2j,2j-1}A_{jk} - x_k(H)iA_{jk}I_{2k,2k-1} = \lambda(H)A_{jk}, \quad \forall H \in \mathfrak{h}.
\]

Up to a scalar, the nontrivial solutions of these linear equations are the following:

\[
A_{jk} = \begin{pmatrix} 1 & \pm i \\ \pm i & -1 \end{pmatrix} \quad \text{with corresponding} \quad \lambda = \mp(x_j + x_k),
\]

\[
A_{jk} = \begin{pmatrix} 1 & \mp i \\ \pm i & 1 \end{pmatrix} \quad \text{with corresponding} \quad \lambda = \mp(x_j - x_k).
\]

Let \( \epsilon_j \in \mathfrak{h}^* \) be defined by \( \epsilon_j(H) = x_j \) for \( 1 \leq j \leq \ell \). Then for \( 1 \leq j < k \leq \ell \), the following matrices are root vectors of \( \mathfrak{so}(2\ell, \mathbb{C}) \):

\[
X_{\epsilon_j + \epsilon_k} = I_{2k-1,2j-1} - I_{2k,2j} - i(I_{2k-1,2j} + I_{2k,2j-1}),
\]

\[
X_{-\epsilon_j - \epsilon_k} = I_{2k-1,2j-1} - I_{2k,2j} + i(I_{2k-1,2j} + I_{2k,2j-1}),
\]

\[
X_{\epsilon_j - \epsilon_k} = I_{2k-1,2j-1} + I_{2k,2j} - i(I_{2k-1,2j} - I_{2k,2j-1}),
\]

\[
X_{-\epsilon_j + \epsilon_k} = I_{2k-1,2j-1} + I_{2k,2j} + i(I_{2k-1,2j} - I_{2k,2j-1}).
\]

Thus, if we choose the following set of positive roots

\[
\Delta^+ = \{ \epsilon_j + \epsilon_k, \epsilon_j - \epsilon_k : 1 \leq j < k \leq \ell \},
\]

\[
\sum_{1 \leq i,k \leq n} \langle I_{p,p-1}, I_{ki} \rangle I_{ki} + I_{ki} \langle I_{p,p-1}, I_{ki} \rangle = 0.
\]

\[
\dot{I}_{p,p-1}(Q_n) = \sum_{1 \leq i \leq n} (I_{ip}I_{p-1,i} + I_{p-1,i}I_{ip}) + \sum_{1 \leq k \leq n} (I_{k,p}I_{k,p-1} + I_{k,p-1}I_{k,p})
\]

\[
+ \sum_{1 \leq k \leq n} (I_{pk}I_{k,p-1} + I_{k,p-1}I_{pk}) + \sum_{1 \leq i \leq n} (I_{p-1,i}I_{p,i} + I_{p,i}I_{p-1,i}) = 0.
\]

This proves the proposition.\( \blacksquare \)
then the Dynkin diagram of $\mathfrak{so}(2\ell,\mathbb{C})$ is $D_\ell$:

\[
\begin{align*}
\circ & \quad \epsilon_1 - \epsilon_2 \\
\circ & \quad \epsilon_2 - \epsilon_3 \\
& \quad \cdots \\
\circ & \quad \epsilon_{\ell-2} - \epsilon_{\ell-1} \\
\circ & \quad \epsilon_{\ell-1} + \epsilon_{\ell} \\
\end{align*}
\]

By looking at the $2 \times 2$ blocks $A_{jk}$ of the different roots, namely

\[
X_{\epsilon_j + \epsilon_k} = \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \quad X_{-\epsilon_j - \epsilon_k} = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},
\]

\[
X_{\epsilon_j - \epsilon_k} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \quad X_{-\epsilon_j + \epsilon_k} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix},
\]

it is easy to obtain the following inverse relations

\[
\begin{align*}
I_{2k-1,2j-1} & = \frac{1}{4}(X_{\epsilon_j + \epsilon_k} + X_{-\epsilon_j - \epsilon_k} + X_{\epsilon_j - \epsilon_k} + X_{-\epsilon_j + \epsilon_k}), \\
I_{2k,2j} & = \frac{1}{4}(-X_{\epsilon_j + \epsilon_k} - X_{-\epsilon_j - \epsilon_k} + X_{\epsilon_j - \epsilon_k} + X_{-\epsilon_j + \epsilon_k}), \\
I_{2k,2j-1} & = \frac{1}{4}(X_{\epsilon_j + \epsilon_k} - X_{-\epsilon_j - \epsilon_k} - X_{\epsilon_j - \epsilon_k} + X_{-\epsilon_j + \epsilon_k}), \\
I_{2k-1,2j} & = \frac{1}{4}(X_{\epsilon_j + \epsilon_k} - X_{-\epsilon_j - \epsilon_k} + X_{\epsilon_j - \epsilon_k} - X_{-\epsilon_j + \epsilon_k}).
\end{align*}
\]

From this it follows that

\[
I_{2k-1,2j-1}^2 + I_{2k,2j}^2 + I_{2k,2j-1}^2 + I_{2k-1,2j}^2 = \frac{1}{4}(X_{\epsilon_j + \epsilon_k} X_{-\epsilon_j - \epsilon_k} + X_{-\epsilon_j - \epsilon_k} X_{\epsilon_j + \epsilon_k} + X_{\epsilon_j - \epsilon_k} X_{-\epsilon_j + \epsilon_k} + X_{-\epsilon_j + \epsilon_k} X_{\epsilon_j - \epsilon_k}).
\]

Therefore

\[
Q_{2\ell} = \sum_{1 \leq j \leq \ell} I_{2j,2j-1}^2 + \frac{1}{4} \sum_{1 \leq j < k \leq \ell} (X_{\epsilon_j + \epsilon_k} X_{-\epsilon_j - \epsilon_k} + X_{-\epsilon_j - \epsilon_k} X_{\epsilon_j + \epsilon_k} + X_{\epsilon_j - \epsilon_k} X_{-\epsilon_j + \epsilon_k} + X_{-\epsilon_j + \epsilon_k} X_{\epsilon_j - \epsilon_k}).
\]

Now using the expressions in (2.2) we get

\[
\begin{align*}
[X_{\epsilon_j + \epsilon_k}, X_{-\epsilon_j - \epsilon_k}] & = -4i(I_{2j,2j-1} + I_{2k,2k-1}), \\
[X_{\epsilon_j - \epsilon_k}, X_{-\epsilon_j + \epsilon_k}] & = -4i(I_{2j,2j-1} - I_{2k,2k-1}).
\end{align*}
\]

Thus $Q_{2\ell}$ becomes

\[
Q_{2\ell} = \sum_{1 \leq j \leq \ell} I_{2j,2j-1}^2 - 2 \sum_{1 \leq j \leq \ell} (\ell - j)iI_{2j,2j-1} \\
+ \sum_{1 \leq j < k \leq \ell} \frac{1}{2} (X_{-\epsilon_j - \epsilon_k} X_{\epsilon_j + \epsilon_k} + X_{-\epsilon_j + \epsilon_k} X_{\epsilon_j - \epsilon_k}). \tag{2.3}
\]

### 2.4 The operator $Q_{2\ell+1}$

Now we look at a root space decomposition of $\mathfrak{so}(n)$ in terms of the basis elements $I_{ki}$, $1 \leq i < k \leq n$ when $n = 2\ell + 1$.

The linear span

\[
\mathfrak{h} = \langle I_{21}, I_{43}, \ldots, I_{2\ell,2\ell-1} \rangle_{\mathbb{C}}
\]

is a Cartan subalgebra of $\mathfrak{so}(n,\mathbb{C})$. To find the root vectors it is convenient to visualize the elements of $\mathfrak{so}(n,\mathbb{C})$ as $\ell \times \ell$ matrices of $2 \times 2$ blocks occupying the left upper corner of the
square matrices of size $2\ell + 1$, with the last column (respectively row) made up of $\ell$ columns (respectively rows) of size two and a zero in the place $(2\ell + 1, 2\ell + 1)$. The subspaces of all matrices $A$ with a block $A_{jk}$, $1 \leq j < k \leq \ell$, in the place $(j, k)$, with the block $-A^{t}_{jk}$ in the place $(k, j)$ and with zeros in all other places, are $\text{ad}(\mathfrak{h})$-stable. Also the subspaces of all matrices $B$ with a column $B_{j}$ of size two, $1 \leq j \leq \ell$, in the place $(j, \ell + 1)$, with the row $-B^{t}_{j}$ in the place $(\ell + 1, j)$ and with zeros in all other places, are $\text{ad}(\mathfrak{h})$-stable.

On the other hand $[H, B] = \lambda B$ if and only if

$$x_{j}iI_{2j,2j-1}B_{j} = \lambda B_{j}.$$ 

Up to a scalar this linear equation has two linearly independent solutions:

$$B_{j} = \begin{pmatrix} 1 \\
\pm i \end{pmatrix} \quad \text{with corresponding } \lambda = \mp x_{j},$$

Let $\epsilon \in \mathfrak{h}^{*}$ be defined by $\epsilon(H) = x_{j}$ for $1 \leq j \leq \ell$. Then for $1 \leq j < k \leq \ell$ and $1 \leq r \leq \ell$, the following matrices are root vectors of $\mathfrak{so}(2\ell + 1, \mathbb{C})$:

$$X_{\epsilon_{r} + \epsilon_{j}} = I_{2k-1,2j-1} - I_{2k,2j} - i(I_{2k-1,2j} + I_{2k,2j-1}),$$

$$X_{\epsilon_{r} - \epsilon_{j}} = I_{2k-1,2j-1} - I_{2k,2j} + i(I_{2k-1,2j} + I_{2k,2j-1}),$$

$$X_{\epsilon_{r} + \epsilon_{k}} = I_{2k-1,2j-1} + I_{2k,2j} - i(I_{2k-1,2j} - I_{2k,2j-1}),$$

$$X_{\epsilon_{r} - \epsilon_{k}} = I_{2k-1,2j-1} + I_{2k,2j} + i(I_{2k-1,2j} - I_{2k,2j-1}),$$

$$X_{\epsilon_{r}} = I_{n,2r-1} - iI_{n,2r},$$

$$X_{-\epsilon_{r}} = I_{n,2r-1} + iI_{n,2r}.$$

Thus, if we choose the following set of positive roots

$$\Delta^{+} = \{\epsilon_{r}, \epsilon_{j} + \epsilon_{k}, \epsilon_{j} - \epsilon_{k} : 1 \leq r \leq \ell, 1 \leq j < k \leq \ell\},$$

then the Dynkin diagram of $\mathfrak{so}(2\ell + 1, \mathbb{C})$ is $B_{\ell}$:

$$\circ \quad \circ \quad \circ \cdots \quad \circ \quad \circ \rightarrow \circ$$

$$\epsilon_{1} - \epsilon_{2} \quad \epsilon_{2} - \epsilon_{3} \quad \cdots \quad \epsilon_{\ell-1} - \epsilon_{\ell} \quad \epsilon_{\ell}$$

By looking at the $2 \times 1$ columns of the different roots, namely

$$X_{\epsilon_{j}} = \begin{pmatrix} 1 \\
-i \end{pmatrix}, \quad X_{-\epsilon_{j}} = \begin{pmatrix} 1 \\
i \end{pmatrix},$$

it is easy to obtain the following inverse relations

$$I_{n,2r-1} = \frac{1}{2}(X_{\epsilon_{r}} + X_{-\epsilon_{r}}), \quad I_{n,2r} = \frac{i}{2}(X_{\epsilon_{r}} - X_{-\epsilon_{r}}).$$

From this it follows that

$$I_{n,2r-1}^{2} + I_{n,2r}^{2} = \frac{1}{2}(X_{\epsilon_{r}}X_{-\epsilon_{r}} + X_{-\epsilon_{r}}X_{\epsilon_{r}}) = -iI_{2r,2r-1} + X_{-\epsilon_{r}}X_{\epsilon_{r}},$$

since $[X_{\epsilon_{r}}, X_{-\epsilon_{r}}] = -2iI_{2r,2r-1}$. Therefore we have that

$$Q_{2\ell+1} = \sum_{1 \leq j \leq 2\ell} I_{n,j}^{2} + Q_{2\ell} = \sum_{1 \leq r \leq 2\ell} (-iI_{2r,2r-1} + X_{-\epsilon_{r}}X_{\epsilon_{r}}) + Q_{2\ell}.$$

Then

$$Q_{2\ell+1} = \sum_{1 \leq j \leq \ell} I_{2j,2j-1}^{2} - \sum_{1 \leq j \leq \ell} (2(\ell - j) + 1)iI_{2j,2j-1}$$

$$+ \sum_{1 \leq j < k \leq \ell} \frac{1}{2}(X_{\epsilon_{j} - \epsilon_{k}}X_{\epsilon_{j} + \epsilon_{k}} + X_{-\epsilon_{j} + \epsilon_{k}}X_{\epsilon_{j} - \epsilon_{k}}) + \sum_{1 \leq r \leq 2\ell} X_{-\epsilon_{r}}X_{\epsilon_{r}}. \quad (2.4)$$
Spherical Functions of Fundamental $K$-Types Associated with the $n$-Dimensional Sphere

2.5 Gel’fand–Tsetlin basis

For any $n$ we identify the group $SO(n)$ with a subgroup of $SO(n+1)$ in the following way: given $k \in SO(n)$ we have

$$k \simeq \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix} \in SO(n+1).$$

Let $T_m$ be an irreducible unitary representation of $SO(n)$ with highest weight $m$ and let $V_m$ be the space of this representation. Highest weights $m$ of these representations are given by the $\ell$-tuples of integers $m = m_n = (m_1, \ldots, m_\ell)$ for which

$$m_1 \geq m_2 \geq \cdots \geq m_{\ell-1} \geq |m_\ell| \quad \text{if } n = 2\ell,$$

$$m_1 \geq m_2 \geq \cdots \geq m_\ell \geq 0 \quad \text{if } n = 2\ell + 1,$$

and $m_j$ are all integers.

The restriction of the representation $T_m$ of the group $SO(2\ell+1)$ to the subgroup $SO(2\ell)$ decomposes into the direct sum of all representations $T_{m'}$, $m' = m_n - 1 = (m_1, \ldots, m_{\ell-1})$ for which the betweenness conditions

$$m_1,2\ell \geq m_1,2\ell-1 \geq m_2,2\ell \geq \cdots \geq m_{\ell-1},2\ell \geq m_{\ell},2\ell \geq -m_{\ell},2\ell+1$$

are satisfied. For the restriction of the representations $T_m$ of $SO(2\ell)$ to the subgroup $SO(2\ell-1)$ the corresponding betweenness conditions are

$$m_1,2\ell \geq m_1,2\ell-1 \geq m_2,2\ell-1 \geq \cdots \geq m_{\ell-1},2\ell-1 \geq m_{\ell},2\ell-1 \geq |m_{\ell},2\ell|.$$ All multiplicities in the decompositions are equal to one (see [24, p. 362]).

If we continue this procedure of restriction of irreducible representations successively to the subgroups

$$SO(n-2) > SO(n-3) > \cdots > SO(2),$$

then we finally get one dimensional representations of the group $SO(2)$. If we take a unit vector in each one of these one dimensional representations we get an orthonormal basis of the representation space $V_m$. Such a basis is called a Gel’fand–Tsetlin basis. The elements of a Gel’fand–Tsetlin basis $\{v(\mu)\}$ of the representation $T_m$ of $SO(n)$ are labelled by the Gel’fand–Tsetlin patterns $\mu = (m_1, m_{n-1}, \ldots, m_3, m_2)$, where the betweenness conditions are depicted in the following diagrams.

If $n = 2\ell + 1$

$$\mu = \begin{bmatrix}
m_1 & m_2 & \cdots & m_\ell & -m_\ell \\
m_1 & m_1 & \cdots & m_{\ell-1} & m_{\ell-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
m_5 & m_4 & m_3 & m_2 & m_1 \\
\end{bmatrix}.$$

If $n = 2\ell$

$$\mu = \begin{bmatrix}
m_1 & m_2 & \cdots & m_\ell & -m_\ell \\
m_1 & m_1 & \cdots & m_{\ell-1} & m_{\ell-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
m_5 & m_4 & m_3 & m_2 & m_1 \\
\end{bmatrix}.$$
The chain of subgroups \( \text{SO}(n-1) > \text{SO}(n-2) > \cdots > \text{SO}(2) \) defines the orthonormal basis \( \{v(\mu)\} \) uniquely up to multiplication of the basis elements by complex numbers of absolute value one.

### 2.6 An explicit expression for \( \dot{\pi}(Q_n) \)

Since \( Q_n \in D(\text{SO}(n)^{\text{SO}(n)}) \), given \( \dot{\pi} \in \hat{\text{SO}}(n) \) it follows that \( \dot{\pi}(Q_n) \) commutes with \( \pi(k) \) for all \( k \in \text{SO}(n) \). Hence, by Schur’s Lemma \( \dot{\pi}(Q_n) = \lambda I \). From expressions (2.3) and (2.4) we can give the explicit value of \( \lambda \) in terms of the highest weight of \( \pi \), by computing \( \dot{\pi}(Q_n) \) on a highest weight vector.

**Proposition 2.7.** Let \( (\pi, V_{\pi}) \) be an irreducible representation of \( \text{SO}(2\ell) \) of highest weight \( m = (m_1, m_2, \ldots, m_{\ell}) \). Then, \( \dot{\pi}(Q_{2\ell}) = \lambda I \), with

\[
\lambda = \sum_{1 \leq j \leq \ell} (-m_j^2 - 2(\ell - j)m_j). \tag{2.5}
\]

**Proposition 2.8.** Let \( (\pi, V_{\pi}) \) be an irreducible representation of \( \text{SO}(2\ell + 1) \) of highest weight \( m = (m_1, m_2, \ldots, m_{\ell}) \). Then, \( \dot{\pi}(Q_{2\ell+1}) = \lambda I \), with

\[
\lambda = \sum_{1 \leq j \leq \ell} (-m_j^2 - (2(\ell - j) + 1)m_j). \tag{2.6}
\]

### 3 The differential operator \( \Delta \)

We shall look closely at the left invariant differential operator \( \Delta \) of \( \text{SO}(n + 1) \) defined by

\[
\Delta = \sum_{j=1}^{n} I_{2n+1,j}^2,
\]

in order to study its eigenfunctions and eigenvalues. Later we will use all this to understand the irreducible spherical functions of fundamental \( K \)-types associated with the pair \( (G, K) = (\text{SO}(n + 1), \text{SO}(n)) \).

**Proposition 3.1.** Let \( G = \text{SO}(n + 1) \) and \( K = \text{SO}(n) \). Let us consider the following left invariant differential operator of \( G \)

\[
\Delta = \sum_{j=1}^{n} I_{2n+1,j}^2.
\]

Then \( \Delta \) is also right invariant under \( K \).

**Proof.** This is a direct consequence of the identity

\[
Q_{n+1} = Q_n + \Delta
\]

and Proposition 2.6. \( \blacksquare \)

Let us define the one-parameter subgroup \( A \) of \( G \) as the set of all elements of the form

\[
a(s) = \begin{pmatrix}
I_{n-1} & 0 & 0 \\
0 & \cos s & \sin s \\
0 & -\sin s & \cos s
\end{pmatrix}, \quad -\pi \leq s \leq \pi,
\]

\[
(3.1)
\]
where \( I_{n-1} \) denotes the identity matrix of size \( n - 1 \), and let \( M = \text{SO}(n - 1) \) be the centralizer of \( A \) in \( K \).

Now we want to get the expression of \( [\Delta \Phi](a(s)) \) for any smooth function \( \Phi \) on \( G \) with values in \( \text{End}(V_\pi) \) such that \( \Phi(kgk') = \pi(k)\Phi(g)\pi(k') \) for all \( g \in G \) and all \( k, k' \in K \).

We have
\[
[I^2_{n+1,j}\Phi](a(s)) = \frac{\partial^2}{\partial t^2} \Phi(a(s) \exp tI_{n+1,j}) \bigg|_{t=0}.
\]

Hence, we will use the decomposition \( G = KAK \) to write \( a(s) \exp tI_{n+1,j} = k(s,t)a(s,t)h(s,t) \), with \( k(s,t), h(s,t) \in K \) and \( a(s,t) \in A \).

Let us take on \( A \setminus \{a(\pi)\} \) the coordinate function \( x(a(s)) = s \), with \(-\pi < s < \pi\), and let
\[
F(s) = F(x(a(s))) = \Phi(a(s)).
\]

From now on we will assume that \(-\pi < s, t, s + t < \pi\).

If \( j = n \) we have \( a(s) \exp tI_{n+1,n} = a(s)a(t) = a(s + t) \). Thus we may take
\[
a(s,t) = a(s+t), \quad k(s,t) = h(s,t) = e.
\]

Since \( x(a(s+t)) = s + t \), we obtain
\[
[I^2_{n+1,a}\Phi](a(s)) = \frac{\partial^2}{\partial t^2} \Phi(a(s) \exp tI_{n+1,n}) \bigg|_{t=0} = \frac{\partial^2}{\partial t^2} \Phi(a(s + t)) \bigg|_{t=0} = \frac{\partial^2}{\partial t^2} F(s + t) \bigg|_{t=0} = F''(s).
\]

For \( 1 \leq j \leq n - 1 \), when \( s \notin \mathbb{Z}\pi \), we may take
\[
k(s,t) = \begin{pmatrix}
I_{j-1} & 0 & 0 & 0 \\
0 & \frac{\sin s \cos t}{\sqrt{1 - \cos^2 s \cos^2 t}} & 0 & \frac{\sin t}{\sqrt{1 - \cos^2 s \cos^2 t}} \\
0 & 0 & I_{n-j-1} & 0 \\
0 & \frac{\sin s \cos t}{\sqrt{1 - \cos^2 s \cos^2 t}} & 0 & \frac{\sin t}{\sqrt{1 - \cos^2 s \cos^2 t}} \\
0 & 0 & 0 & 1
\end{pmatrix},
\]
\[
h(s,t) = \begin{pmatrix}
I_{j-1} & 0 & 0 & 0 \\
0 & \frac{\sin s}{\sqrt{1 - \cos^2 s \cos^2 t}} & 0 & \frac{\cos s \sin t}{\sqrt{1 - \cos^2 s \cos^2 t}} \\
0 & 0 & I_{n-j-1} & 0 \\
0 & \frac{\sin s}{\sqrt{1 - \cos^2 s \cos^2 t}} & 0 & \frac{\cos s \sin t}{\sqrt{1 - \cos^2 s \cos^2 t}} \\
0 & 0 & 0 & 1
\end{pmatrix},
\]
\[
a(s,t) = \begin{pmatrix}
I_{n-1} & 0 & 0 & 0 \\
0 & \frac{\cos s \cos t}{\sqrt{1 - \cos^2 s \cos^2 t}} & 0 & \frac{\sin s}{\sqrt{1 - \cos^2 s \cos^2 t}} \\
0 & \frac{\cos s \cos t}{\sqrt{1 - \cos^2 s \cos^2 t}} & 0 & \frac{\sin s}{\sqrt{1 - \cos^2 s \cos^2 t}} \\
0 & -\frac{\sqrt{1 - \cos^2 s \cos^2 t}}{\cos s \cos t} & \frac{\sin s}{\sqrt{1 - \cos^2 s \cos^2 t}} & \cos s \cos t
\end{pmatrix}.
\]

Then, for \( 0 < s < \pi \), we have \( x(a(s,t)) = \arccos(\cos s \cos t) \) and
\[
\frac{\partial}{\partial t} x(a(s,t)) = \frac{\cos s \sin t}{\sqrt{1 - \cos^2 s \cos^2 t}}.
\]

From here we get
\[
\frac{\partial}{\partial t} x(a(s,t)) \bigg|_{t=0} = 0 \quad \text{and} \quad \frac{\partial^2}{\partial t^2} x(a(s,t)) \bigg|_{t=0} = \frac{\cos s}{\sin s}.
\]
Thus
\[
\frac{\partial}{\partial t} \Phi(a(s,t)) \bigg|_{t=0} = F'(s) \frac{\partial}{\partial t} x(a(s,t)) \bigg|_{t=0} = 0 \quad \text{and} \quad \frac{\partial^2}{\partial t^2} \Phi(a(s,t)) \bigg|_{t=0} = \frac{\cos s}{\sin s} F''(s).
\]

We observe that \(k(s,0) = h(s,0) = e\) and that \(a(s,0) = a(s)\). Then
\[
[I_n^2 \Phi](a(s)) = \frac{\partial^2}{\partial t^2} \pi(k(s,t)) \bigg|_{t=0} \Phi(a(s)) + 2 \frac{\partial}{\partial t} \pi(k(s,t)) \bigg|_{t=0} \frac{\partial}{\partial t} \Phi(a(s)) \bigg|_{t=0}
+ 2 \frac{\partial}{\partial t} \pi(k(s,t)) \bigg|_{t=0} \Phi(a(s)) \bigg( \frac{\partial}{\partial t} \pi(h(s,t)) \bigg|_{t=0} + \frac{\partial^2}{\partial t^2} \Phi(a(s)) \bigg|_{t=0}
+ 2 \frac{\partial}{\partial t} \Phi(a(s)) \bigg|_{t=0} \frac{\partial}{\partial t} \pi(h(s,t)) \bigg|_{t=0} + \Phi(a(s)) \bigg( \frac{\partial}{\partial t^2} \pi(h(s,t)) \bigg|_{t=0}.
\]

We also have
\[
\frac{\partial}{\partial t} \pi(k(s,t)) \bigg|_{t=0} = \pi \left( \frac{\partial}{\partial t} k(s,t) \bigg|_{t=0} \right) = \frac{1}{\sin s} \hat{\pi}(I_{n,j}),
\]
and
\[
\frac{\partial}{\partial t} \pi(h(s,t)) \bigg|_{t=0} = \hat{\pi} \left( \frac{\partial}{\partial t} h(s,t) \bigg|_{t=0} \right) = -\frac{\cos s}{\sin s} \hat{\pi}(I_{n,j}).
\]

We will need the following proposition, whose proof appears in the Appendix and its idea is taken from [5].

**Proposition 3.2.** If \(A(s,t) = k(s,t)\) or \(A(s,t) = h(s,t)\), then in either case for \(0 < s < \pi\), we have
\[
\frac{\partial^2 (\pi \circ A)}{\partial t^2} \bigg|_{t=0} = \hat{\pi} \left( \frac{\partial A}{\partial t} \bigg|_{t=0} \right)^2.
\]

Moreover in each case, for \(1 \leq j \leq n - 1\) and \(0 < s < \pi\), we have
\[
\frac{\partial^2}{\partial t^2} \pi(k(s,t)) \bigg|_{t=0} = \frac{1}{\sin^2 s} \hat{\pi}(I_{n,j})^2, \quad \frac{\partial^2}{\partial t^2} \pi(h(s,t)) \bigg|_{t=0} = \frac{\cos^2 s}{\sin^2 s} F(s) \hat{\pi}(I_{n,j})^2.
\]

Now we obtain the following corollaries.

**Corollary 3.3.** Let \(\Phi\) be any smooth function on \(G\) with values in \(\text{End}(V_\pi)\) such that \(\Phi(kg') = \pi(k)\Phi(g)\pi(k')\) for all \(g \in G\) and all \(k, k' \in K\). Then, if \(F(s) = \Phi(a(s))\), for \(0 < s < \pi\) we have
\[
[\Delta \Phi](a(s)) = F''(s) + (n - 1) \frac{\cos s}{\sin s} F'(s) + \frac{1}{\sin^2 s} \sum_{j=1}^{n-1} \hat{\pi}(I_{n,j})^2 F(s)
- 2 \frac{\cos s}{\sin^2 s} \sum_{j=1}^{n-1} \hat{\pi}(I_{n,j}) F(s) \hat{\pi}(I_{n,j}) + \frac{\cos^2 s}{\sin^2 s} F(s) \sum_{j=1}^{n-1} \hat{\pi}(I_{n,j})^2.
\]

**Corollary 3.4.** Let \(\Phi\) be an irreducible spherical function on \(G\) of type \(\pi \in \hat{K}\). Then, if \(F(s) = \Phi(a(s))\), we have
\[
F''(s) + (n - 1) \frac{\cos s}{\sin s} F'(s) + \frac{1}{\sin^2 s} \sum_{j=1}^{n-1} \hat{\pi}(I_{n,j})^2 F(s)
- 2 \frac{\cos s}{\sin^2 s} \sum_{j=1}^{n-1} \hat{\pi}(I_{n,j}) F(s) \hat{\pi}(I_{n,j}) + \frac{\cos^2 s}{\sin^2 s} F(s) \sum_{j=1}^{n-1} \hat{\pi}(I_{n,j})^2 = \lambda F(s),
\]
for some \(\lambda \in \mathbb{C}\) and \(0 < s < \pi\).
Notice that the expression in Corollary 3.4 generalizes the very well known situation when the $K$-type is the trivial one, as we state in the following corollary (cf. [11, p. 403, equation (10)]).

**Corollary 3.5.** Let $\Phi$ be an irreducible spherical function on $G$ of the trivial $K$-type. Then, for $F(s) = \Phi(a(s))$ we have

$$F'''(s) + (n - 1) \frac{\cos s}{\sin s} F'(s) = \lambda F(s),$$

for some $\lambda \in \mathbb{C}$ and $0 < s < \pi$.

Let us make the change of variables $y = (1 + \cos s)/2$, with $0 < s < \pi$; then $0 < y < 1$. We also have $\cos s = 2y - 1$, $\sin^2 s = 4y(1 - y)$ and $\frac{dy}{ds} = -\frac{\sin s}{2}$. If we let $H(y) = F(s)$, i.e.

$$H(y) = \Phi(a(s)), \quad \text{with} \quad \cos s = 2y - 1,$$

we obtain

$$F'(s) = -\frac{\sin s}{2} H'(s), \quad F''(s) = \frac{\sin^2 s}{4} H''(y) - \frac{\cos s}{2} H'(y).$$

In terms of this new variable Corollary 3.4 becomes

**Corollary 3.6.** Let $\Phi$ be an irreducible spherical function on $G$ of type $\pi \in \hat{K}$. Then, if $H(y) = \Phi(a(s))$ with $y = (1 + \cos s)/2$, we have

$$y(1 - y)H''(y) + \frac{1}{2} n(1 - 2y)H'(y) + \frac{1}{4y(1 - y)} \sum_{j=1}^{n-1} \hat{\pi}(I_{n,j})^2 H(y)$$

$$+ \frac{(1 - 2y)}{2y(1 - y)} \sum_{j=1}^{n-1} \hat{\pi}(I_{n,j})H(y) \hat{\pi}(I_{n,j}) + \frac{(1 - 2y)^2}{4y(1 - y)} H(y) \sum_{j=1}^{n-1} \hat{\pi}(I_{n,j})^2 = \lambda H(y),$$

for some $\lambda \in \mathbb{C}$ and $0 < y < 1$.

**Remark 3.7.** Let us notice that, for any $y \in (0, 1)$, $H(y)$ is a scalar linear transformation when restricted to any $M$-submodule, see Proposition 2.2. Therefore, if $m$ is the number of $M$-submodules contained in $(V, \pi)$, we consider the vector valued function $H : (0, 1) \to \mathbb{C}^m$ whose entries are given by those scalar values that $H(y)$ takes on every $M$-submodule.

If the $\text{End}(V)$-valued function $H$ satisfies the differential equation given in Corollary 3.6, then the vector valued function $H$ satisfies

$$y(1 - y)H''(y) + \frac{1}{2} n(1 - 2y)H'(y) + \frac{1}{4y(1 - y)} N_1 H(y)$$

$$+ \frac{(1 - 2y)}{2y(1 - y)} E H(y) + \frac{(1 - 2y)^2}{4y(1 - y)} N_2 H(y) = \lambda H(y),$$

where $E, N_1$ and $N_2$ are matrices of size $m \times m$.

Even more, since $\sum_{j=1}^{n-1} I_{n,j}^2 = Q_n - Q_{n-1}$, Proposition 2.6 implies $\sum_{j=1}^{n-1} I_{n,j}^2 \in D(\text{SO}(n))^{\text{SO}(n-1)}$, therefore $\sum_{j=1}^{n-1} \hat{\pi}(I_{n,j})^2$ is scalar valued when restricted to any $M$-submodule. Hence, $N_1 = N_2$ and the equation above is equivalent to

$$y(1 - y)H''(y) + \frac{n}{2} (1 - 2y)H'(y) + \frac{(1 - 2y)}{2y(1 - y)} E H(y) + \frac{1 + (1 - 2y)^2}{4y(1 - y)} N H(y) = \lambda H(y), \ (3.2)$$

where $N$ is a diagonal matrix of size $m \times m$. To obtain an explicit expression of $E$ for any $K$-type is a very serious matter; in the following sections we shall find explicitly the expressions of $E$ and $N$, for certain $K$-types.
Remark 3.8. It is worth to observe that from (2.5) and (2.6) we can immediately obtain every entry of the diagonal matrix $N$.

4 The $K$-types which are $M$-irreducible

Let $K = \text{SO}(n)$, $M = \text{SO}(n-1)$, with $n = 2\ell + 1$, and let $m_n = (m_{1n}, \ldots, m_{\ell n})$ be a $K$-type such that $V_m$ is irreducible as $M$-module. The highest weights $m_{n-1}$ of the $M$-submodules of $V_m$ are those that satisfies the following intertwining relations

$$m_{1n} \quad m_{2n} \quad \cdots \quad m_{\ell n} \quad -m_{\ell n}$$

$$m_{1,n-1} \quad \cdots \quad m_{\ell,n-1}$$

Since $V_m$ is irreducible as $M$-module it follows that $m_{1n} = \cdots = m_{\ell,n} = 0$. The converse is also true, therefore $V_m$ is $M$-irreducible if and only if it is the trivial representation.

Let now consider the case $K = \text{SO}(n)$, $M = \text{SO}(n-1)$, with $n = 2\ell$ and let $m_n = (m_{1n}, \ldots, m_{\ell n})$ be a $K$-type such that $V_m$ is irreducible as $M$-module. The highest weights $m_{n-1}$ of the $M$-submodules of $V_m$ are those that satisfies the following intertwining relations

$$m_{1n} \quad m_{2n} \quad \cdots \quad m_{\ell n} \quad -m_{\ell n}$$

$$m_{1,n-1} \quad \cdots \quad m_{\ell,n-1}$$

Since $V_m$ is irreducible as $M$-module it follows that $m_{1n} = \cdots = m_{\ell-1,n} = d$ and $m_{\ell,n} = d - j$ with $0 \leq j \leq 2d$, since $m_{\ell-1,n} \geq |m_{\ell n}|$. This implies that $m_{1,n-1} = \cdots = m_{\ell-2,n-1} = d$ and $m_{\ell-1,n-1} = q$ with $d \geq q \geq \max\{d-j, \ell-d\}$. Thus, if $0 \leq j \leq d$ we have $d \geq q \geq d - j$ and by irreducibility we must have $j = 0$. Similarly if $d \leq j \leq 2d$ we have $d \geq q \geq j - d$ and by irreducibility we must have $j = 2d$. Therefore $m_n = d\alpha$ or $m_n = d\beta$, where

$$\alpha = (1, \ldots, 1), \quad \beta = (1, \ldots, 1, -1).$$

The converse is also true, therefore $V_m$ is $M$-irreducible if and only if $m_n = d\alpha$ or $m_n = d\beta$ for any $d \in \mathbb{N}_0$.

If $\Phi$ is an irreducible spherical function on $\text{SO}(n+1)$ of type $\pi$, whose highest weight is $m_n = d\alpha$ or $m_n = d\beta$, then from Corollary 3.6 we get that the associated function $H$ satisfies

$$y(1-y)H''(y) + (\ell(1-2y))H'(y) + \frac{1}{y} \sum_{j=1}^{n-1} \hat{\pi}(I_{nj})^2 H(y) = \lambda H(y).$$

To compute $\sum_{j=1}^{n-1} \hat{\pi}(I_{nj})^2$ we write $\sum_{j=1}^{n-1} \hat{\pi}(I_{nj})^2 = \hat{\pi}(Q_n - Q_{n-1})$.

Let us first consider $m_n = d\alpha$. If $v \in V_{m_n}$ is a highest weight vector, then

$$\hat{\pi}(Q_n)v = -d(\ell + \ell - 1)v \quad \text{and} \quad \hat{\pi}(Q_{n-1})v = -d(\ell - 1)(\ell + \ell - 1)v,$$

see (2.5) and (2.6). Therefore

$$\sum_{j=1}^{n-1} \hat{\pi}(I_{nj})^2 v = -d(d + \ell - 1)v.$$

Let us now consider $m_n = d\beta$. If $v \in V_{m_n}$ is a highest weight vector, then $\hat{\pi}(Q_n)v = -2d(\ell + \ell - 1)v$ as before, and $\hat{\pi}(Q_{n-1})v = -2d(\ell - 1)(\ell + \ell - 1)v$ as before because in both cases $m_{n-1}$ is the same.
Therefore if $m_n = (d, \ldots, d, \pm d)$ we have
\[
\sum_{j=1}^{n-1} \pi(I_{nj})^2 v = -d(d + \ell - 1)v.
\]

Hence, if $\Phi$ is an irreducible spherical function on $SO(n + 1)$, $n = 2\ell$, of type $m_n = (d, \ldots, d, \pm d) \in \mathbb{C}^\ell$, then the associated scalar value function $H = h$ satisfies
\[
y(1 - y)h''(y) + \ell(1 - 2y)h'(y) - \frac{d(d + \ell - 1)(1 - y)}{y} h(y) = \lambda h(y).
\]

Let us now compute the eigenvalue $\lambda$ corresponding to the spherical function of type $\pi \in S\Omega(2\ell)$, of highest weight $m_n = d\alpha$, associated with the irreducible representation $\tau \in SO(2\ell + 1)$, of highest weight $m_{n+1} = (w, d, \ldots, d) \in \mathbb{C}^\ell$. If $v \in V_{m_{n+1}}$ is a highest weight vector, then from (2.6) we have
\[
\hat{\tau}(Q_{n+1})v = - (w(w + 2\ell - 1) + d(\ell - 1)(d + \ell - 1)) v.
\]

If $v \in V_{m_n}$ is a highest weight vector, then from (2.5) we have
\[
\hat{\tau}(Q_n)v = \hat{\pi}(Q_n)v = -d\ell(d + \ell - 1)v.
\]

Since $\Delta = Q_{n+1} - Q_n$ it follows that
\[
\lambda = -w(w + 2\ell - 1) + d(d + \ell - 1).
\]

To solve (4.1) we write $h = y^\alpha f$. Then we get
\[
y(1 - y)y'' + (2\alpha(1 - y) + \ell(1 - 2y))y' + (\alpha(\alpha - 1)(1 - y) + \ell\alpha(1 - 2y) - d(d + \ell - 1)(1 - y))y^\alpha f = \lambda y^\alpha f.
\]

Thus the indicial equation is $\alpha(\alpha - 1) + \ell\alpha - d(d + \ell - 1) = 0$ and $\alpha = d$ is one of its solutions. If we take $h = y^df$, then we obtain
\[
y(1 - y)f'' + (2d + \ell - 2(d + \ell)y)f' - d\ell f = \lambda f.
\]

If we replace $\lambda = -w(w + 2\ell - 1) + d(d + \ell - 1)$ we get
\[
y(1 - y)f'' + (2d + \ell - 2(d + \ell)y)f' - (d - w)(2\ell + d + w - 1)f = 0.
\]

Let $a = d - w$, $b = 2\ell + d + w - 1$, $c = 2d + \ell$ then the above equation becomes
\[
y(1 - y)f'' + (c - (1 + a + b)y)f' - abf = 0.
\]

A fundamental system of solutions of this equation near $y = 0$ is given by the following functions
\[
_{2}F_{1}\left(\begin{array}{c} a, b \\ c \end{array}; y \right), \quad \ y^1-c_{2}F_{1}\left(\begin{array}{c} a-c+1, b-c+1 \\ 2-c \end{array}; y \right).
\]

Since $h = y^df$ is bounded near $y = 0$ it follows that
\[
h(y) = uy^d_{2}F_{1}\left(\begin{array}{c} d-w, 2\ell + d + w - 1 \\ 2d + \ell \end{array}; y \right),
\]

where the constant $u$ is determined by the condition $h(1) = 1$. 

Remark 4.1. Let $h_w = h_w(y)$, $w \geq d$, be the function $h$ above. Then $h_w$ is a polynomial of degree $w$. Moreover observe that the function $y^d$ used to hypergeometrize (4.1) is precisely $h_d$.

Let us now compute the eigenvalue $\lambda$ corresponding to the spherical function of type $m_n = d \beta$ associated with an irreducible representation $\tau$ of $SO(n + 1)$ of highest weight $m_{n+1} = (w, d, \ldots, d) \in \mathbb{C}^\ell$. If $v \in V_{m_{n+1}}$ is a highest weight vector, we obtain $\hat{\tau}(Q_{n+1})v = -(w(w + 2\ell - 1) + d(\ell - 1)(d + \ell - 1))v$.

If $v \in V_{m_n}$ is a highest weight vector, then $\hat{\tau}(Q_n)v = -d(\ell + d - \ell - 1)v$ as above, because $Q_nv$ does not depend on the sign of the last coordinate of $m_n$. Since $\Delta = Q_{n+1} - Q_n$ we also have

$$\lambda = -w(w + 2\ell - 1) + d(d + \ell - 1).$$

Therefore we have proved the following result.

**Theorem 4.2.** The scalar valued functions $H = h$ associated with the irreducible spherical functions on $SO(n + 1)$, $n = 2\ell$, of $SO(n)$-type $m_n = (d, \ldots, d, \mp d) \in \mathbb{C}^\ell$, are parameterized by the integers $w \geq d$ and are given by

$$h_w(y) = uy^d_2F_1 \left( \frac{d - w, 2\ell + d + w - 1}{2d + \ell}; y \right)$$

where the constant $u$ is determined by the condition $h_w(1) = 1$.

### 5 The operator $\Delta$ for fundamental $K$-types

We are interested in finding a more explicit expression of the differential equation given in Corollary 3.6:

$$y(1 - y)H''(y) + \frac{1}{2}n(1 - 2y)H'(y) + \frac{1}{4y(1 - y)}\sum_{j=1}^{n-1}(\hat{\pi}(I_{n,j}))^2H(y)$$

$$+ \frac{(1 - 2y)}{2y(1 - y)}\sum_{j=1}^{n-1}(\hat{\pi}(I_{n,j})(h(y))\hat{\pi}(I_{n,j}) + \frac{(1 - 2y)^2}{4y(1 - y)}H(y)\sum_{j=1}^{n-1}(\hat{\pi}(I_{n,j}))^2 = \lambda H(y),$$

for certain representations $\pi \in \hat{SO}(n)$, including those that are fundamental.

The obvious place to start to look for irreducible representations of $SO(n)$ is among the exterior powers of the standard representation of $SO(n)$. It is known that $\Lambda^p(\mathbb{C}^2\ell)$ are irreducible $SO(2\ell)$-modules for $p = 1, \ldots, \ell - 1$, and that $\Lambda^{\ell}(\mathbb{C}^2\ell)$ splits into the direct sum of two irreducible submodules. While in the odd case $\Lambda^p(\mathbb{C}^2\ell+1)$ are irreducible $SO(2\ell+1)$-modules for $p = 1, \ldots, \ell$.

See Theorems 19.2 and 19.14 in [3].

Moreover, $\Lambda^p(\mathbb{C}^n)$ and $\Lambda^{n-p}(\mathbb{C}^n)$ are isomorphic $SO(n)$-modules. In fact, if $\{e_1, \ldots, e_{n}\}$ is the canonical basis of $\mathbb{C}^n$, then the linear map $\xi : \Lambda^p(\mathbb{C}^n) \rightarrow \Lambda^{n-p}(\mathbb{C}^n)$ defined by

$$\xi(e_{u_1} \wedge \cdots \wedge e_{u_p}) = (-1)^{u_1 + \cdots + u_p}e_{u_1} \wedge \cdots \wedge e_{v_{n-p}},$$

where $u_1 < \cdots < u_p$ and $v_1 < \cdots < v_{n-p}$ are complementary ordered set of indices, is an $SO(n)$-isomorphism.

All these statements can be established directly upon observing that the elements $I_{ki} = E_{ki} - E_{ik}$ with $1 \leq i < k \leq n$ form a basis of the Lie algebra $so(n)$, and that

$$I_{ki}e_k = e_i, \quad I_{ki}e_i = -e_k \quad \text{and} \quad I_{ki}e_j = 0 \quad \text{if} \quad j \neq k, i.$$  

We will refer to the irreducible $SO(2\ell)$-modules $\Lambda^p(\mathbb{C}^2\ell)$ for $p = 1, \ldots, \ell - 1$, respectively, the irreducible $SO(2\ell+1)$-modules $\Lambda^p(\mathbb{C}^{2\ell+1})$ for $p = 1, \ldots, \ell$, as the fundamental $SO(2\ell)$-modules, respectively, as the fundamental $SO(2\ell+1)$-modules, for reasons that will be clarified in the following Sections 5.1 and 5.2.
5.1 The even case: \( K = \text{SO}(2\ell) \)

First we will study the case \( n = 2\ell \), with \( \ell > 2 \). The fundamental weights of \( \mathfrak{so}(2\ell, \mathbb{C}) \) are

\[
\lambda_p = \epsilon_1 + \cdots + \epsilon_p, \quad 1 \leq p \leq \ell - 2, \\
\lambda_{\ell - 1} = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{\ell - 1} - \epsilon_\ell), \quad \lambda_\ell = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{\ell - 1} + \epsilon_\ell).
\]

We will also consider the fundamental \( K \)-modules \( \Lambda^1(\mathbb{C}^n), \Lambda^2(\mathbb{C}^n), \ldots, \Lambda^{\ell - 1}(\mathbb{C}^n) \).

Here we will consider the fundamental \( K \)-modules

\[
\Lambda^1(\mathbb{C}^n), \Lambda^2(\mathbb{C}^n), \ldots, \Lambda^{\ell - 1}(\mathbb{C}^n).
\]

We will show that the highest weight of \( \Lambda^p(\mathbb{C}^n) \) is \( \epsilon_1 + \cdots + \epsilon_p \) for \( 1 \leq p \leq \ell - 1 \). Observe that \( \lambda_{\ell - 1} \) and \( \lambda_\ell \) are not analytically integral and therefore they will not be considered, although we will also consider the \( K \)-module with highest weight \( \lambda_{\ell - 1} + \lambda_\ell = \epsilon_1 + \cdots + \epsilon_{\ell - 1} \). Notice that we have already considered the cases \( 2\lambda_{\ell - 1} \) and \( 2\lambda_\ell \) in Section 4, which are \( M \)-irreducible. We will also show that the fundamental \( K \)-modules are direct sum of two irreducible \( M \)-submodules.

In order to obtain the explicit expression of \( E \) in (3.2) for a given irreducible representation \( \pi \) of \( K = \text{SO}(n) \), of highest weight \( \epsilon_1 + \cdots + \epsilon_p \), we are interested to compute

\[
\sum_{j=1}^{n-1} \hat{\pi}(I_{nj})P_s \hat{\pi}(I_{nj})|_{V_r} = \lambda(r,s)I_{V_r},
\]

with \( r, s = 0, 1 \) corresponding to the two \( M \)-submodules \( V_0 \) and \( V_1 \) of the representation \( \pi \), associated with \( m_{n-1} = (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^{\ell - 1} \) with \( p - 1 \) and \( p \) ones, respectively (see the betweeness conditions in Section 2.5); being \( P_0 \) and \( P_1 \) the respective projections.

Let us consider the standard action of \( K = \text{SO}(n) \) on \( V = \mathbb{C}^n \), and take the canonical basis \( \{e_1, \ldots, e_n\} \). Then we have the irreducible \( K \)-module \( \Lambda^p(V) \) for \( 1 \leq p \leq \ell - 1 \). The vector \((e_1 - ie_2) \wedge (e_3 - ie_4) \wedge \cdots \wedge (e_{2p-1} - ie_{2p})\) is the unique, up to a scalar, dominant vector and its weight is \((1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^\ell \) with \( p \) ones. Then, if \( V' \) is the subspace generated by \( \{e_1, \ldots, e_{n-1}\} \), \( \Lambda^p(V) \) is the direct sum of two \( M \)-submodules, namely

\[
\Lambda^p(V) = V_0 \oplus V_1 = \Lambda^{p-1}(V') \wedge e_n \oplus \Lambda^p(V')
\]

(5.1)

whose highest weights are \((1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^{\ell - 1} \) with \( p - 1 \) ones and \((1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^{\ell - 1} \) with \( p \) ones, respectively. It is easy to see that \((e_1 - ie_2) \wedge (e_3 - ie_4) \wedge \cdots \wedge (e_{2p-3} - ie_{2p-2}) \wedge e_n\) is an \( M \)-highest weight vector in \( \Lambda^{p-1}(V') \wedge e_n \) and that \((e_1 - ie_2) \wedge (e_3 - ie_4) \wedge \cdots \wedge (e_{2p-1} - ie_{2p})\) is an \( M \) highest weight vector in \( \Lambda^p(V') \).

To get \( \lambda(0,0) \) it is enough to compute

\[
\sum_{j=1}^{n-1} \hat{\pi}(I_{nj})P_0 \hat{\pi}(I_{nj})(e_1 \wedge \cdots \wedge e_{p-1} \wedge e_n).
\]

Since we have that \( \hat{\pi}(I_{nj})(e_1 \wedge \cdots \wedge e_{p-1} \wedge e_n) = e_1 \wedge \cdots \wedge e_{p-1} \wedge e_j \) we obtain \( P_0 \hat{\pi}(I_{nj})(e_1 \wedge \cdots \wedge e_{p-1} \wedge e_n) = 0 \) and \( \lambda(0,0) = 0 \).

To get \( \lambda(0,1) \) it is enough to compute

\[
\sum_{j=1}^{n-1} \hat{\pi}(I_{nj})P_1 \hat{\pi}(I_{nj})(e_1 \wedge \cdots \wedge e_{p-1} \wedge e_n).
\]

We have

\[
P_1 \hat{\pi}(I_{nj})(e_1 \wedge \cdots \wedge e_{p-1} \wedge e_n) = \begin{cases} 
0 & \text{if } 1 \leq j \leq p - 1, \\
e_1 \wedge \cdots \wedge e_{p-1} \wedge e_j & \text{if } p \leq j \leq n - 1.
\end{cases}
\]
Therefore we have
\[
\dot{\pi}(I_{nj})P_1\dot{\pi}(I_{nj})(e_1 \wedge \cdots \wedge e_{p-1} \wedge e_n) = \begin{cases} 
0 & \text{if } 1 \leq j \leq p-1, \\
-e_1 \wedge \cdots \wedge e_{p-1} \wedge e_n & \text{if } p \leq j \leq n-1.
\end{cases}
\]

Hence \(\lambda(0, 1) = -(n - p)\).

Similarly, to get \(\lambda(1, 0)\) it is enough to compute
\[
\sum_{j=1}^{n-1} \dot{\pi}(I_{nj})P_0\dot{\pi}(I_{nj})(e_1 \wedge \cdots \wedge e_p).
\]

We have
\[
\dot{\pi}(I_{nj})(e_1 \wedge \cdots \wedge e_p) = \begin{cases} 
-e_1 \wedge \cdots \wedge e_n \wedge \cdots \wedge e_p & \text{if } 1 \leq j \leq p, \\
0 & \text{if } p+1 \leq j \leq n-1.
\end{cases}
\]

where \(e_n\) appears in the \(j\)-place. Therefore
\[
\dot{\pi}(I_{nj})P_0\dot{\pi}(I_{nj})(e_1 \wedge \cdots \wedge e_p) = \begin{cases} 
-e_1 \wedge \cdots \wedge e_p & \text{if } 1 \leq j \leq p, \\
0 & \text{if } p+1 \leq j \leq n-1.
\end{cases}
\]

Hence \(\lambda(1, 0) = -p\).

Also it is clear now that \(\sum_{j=1}^{n-1} \dot{\pi}(I_{nj})P_1\dot{\pi}(I_{nj})(e_1 \wedge \cdots \wedge e_p) = 0\), hence \(\lambda(1, 1) = 0\).

Therefore, when \(\pi\) is the standard representation of \(K\) in \(\Lambda^p(V), 1 \leq p \leq \ell - 1\), we have
\[
(\lambda(r, s))_{0 \leq r, s \leq 1} = \begin{pmatrix} 0 & p - n \\ -p & 0 \end{pmatrix}.
\]

Therefore, we obtain a more explicit version of Corollary 3.6 using (3.2) and Remark 3.8.

**Corollary 5.1.** Let \(\Phi\) be an irreducible spherical function on \(G\) of type \(\pi \in \text{SO}(n), n = 2\ell\). If the highest weight of \(\pi\) is of the form \((1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^\ell\), with \(p\) ones, \(1 \leq p \leq \ell - 1\), then the function \(H : (0, 1) \to \text{End}(\mathbb{C}^2)\) associated with \(\Phi\) satisfies
\[
y(1 - y)H''(y) + \frac{1}{2}n(1 - 2y)H'(y) + \frac{1 + (1 - 2y)^2}{4y(1 - y)} \begin{pmatrix} p - n & 0 \\ -p & 0 \end{pmatrix} H(y)
\]
\[
+ \frac{(1 - 2y)}{2y(1 - y)} \begin{pmatrix} 0 & p - n \\ -p & 0 \end{pmatrix} H(y) = \lambda H(y),
\]

for some \(\lambda \in \mathbb{C}\).

**5.2 The odd case: \(K = \text{SO}(2\ell + 1)\)**

We now study the case \(n = 2\ell + 1\), with \(\ell \geq 1\). The fundamental weights of \(\text{so}(2\ell + 1, \mathbb{C})\) are
\[
\lambda_p = \epsilon_1 + \cdots + \epsilon_p, \quad 1 \leq p \leq \ell - 1,
\]
\[
\lambda_\ell = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_\ell).
\]

Here we will consider the fundamental \(K\)-modules
\[
\Lambda^1(\mathbb{C}^n), \quad \Lambda^2(\mathbb{C}^n), \quad \ldots, \quad \Lambda^\ell(\mathbb{C}^n).
\]
Spherical Functions of Fundamental $K$-Types Associated with the $n$-Dimensional Sphere

We will show that the highest weight of $\Lambda^p(\mathbb{C}^n)$ is $\epsilon_1 + \cdots + \epsilon_p$ for $1 \leq p \leq \ell$. Also we will establish that $\Lambda^p(\mathbb{C}^n)$ splits into the direct sum of two $M$-submodules for $1 \leq p \leq \ell - 1$, while $\Lambda^\ell(\mathbb{C}^n)$ splits into the sum of three $M$-submodules; for this reason it will be treated separately in Section 8.

Observe that $\lambda_\ell$ is not analytically integral and therefore it will not be considered, although we will consider the $K$-module with highest weight $2\lambda_\ell$.

As in the even case we are interested in computing

$$\sum_{j=1}^{n-1} \hat{\pi}(I_{nj}) P_0 \hat{\pi}(I_{nj}) |_{V_r} = \lambda(r, s) I_{V_r},$$

with $r, s = 0, 1$ corresponding to the two $M$-submodules $V_0$ and $V_1$ of the representation $\pi$, corresponding to $m_{n-1} = (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^\ell$ with $p - 1$ and $p$ ones respectively (see the betweenness conditions in Section 2.5). Being $(\pi, \circ)$, respectively. It is easy to see that $(\pi, \circ)$, $\pi$, $(\pi, \circ)$. Then we have the irreducible $K$-module $\Lambda^p(V)$ for $1 \leq p \leq \ell - 1$. The vector $(\epsilon_1 - i\epsilon_2) \wedge (\epsilon_3 - i\epsilon_4) \wedge \cdots \wedge (\epsilon_{2p-1} - i\epsilon_{2p})$ is the unique, up to a scalar, dominant vector and its weight is $(1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^\ell$ with $p$ ones. Then, if $V'$ is the subspace generated by $\{\epsilon_1, \ldots, \epsilon_{n-1}\}$, $\Lambda^p(V)$ is the direct sum of two irreducible $M$-submodules, namely

$$\Lambda^p(V) = V_0 \oplus V_1 = \Lambda^p(V') \wedge e_{n-p} \oplus \Lambda^p(V')$$

of highest weights $(1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^\ell$ with $p - 1$ ones, and $(1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^\ell$ with $p$ ones, respectively. It is easy to see that $(\epsilon_1 - i\epsilon_2) \wedge (\epsilon_3 - i\epsilon_4) \wedge \cdots \wedge (\epsilon_{2p-3} - i\epsilon_{2p-2}) \wedge e_n$ is an $M$-highest weight vector in $\Lambda^{p-1}(V') \wedge e_n$ and that $(\epsilon_1 - i\epsilon_2) \wedge (\epsilon_3 - i\epsilon_4) \wedge \cdots \wedge (\epsilon_{2p-1} - i\epsilon_{2p})$ is an $M$ highest weight vector in $\Lambda^p(V')$.

To get $\lambda(0, 0)$ it is enough to compute

$$\sum_{j=1}^{n-1} \hat{\pi}(I_{nj}) P_0 \hat{\pi}(I_{nj}) (\epsilon_1 \wedge \cdots \wedge \epsilon_{p-1} \wedge e_n).$$

Since we have that $\hat{\pi}(I_{nj}) (\epsilon_1 \wedge \cdots \wedge \epsilon_{p-1} \wedge e_n) = e_1 \wedge \cdots \wedge e_{p-1} \wedge e_j$, we obtain $P_0 \hat{\pi}(I_{nj}) (\epsilon_1 \wedge \cdots \wedge e_{p-1} \wedge e_n) = 0$ and $\lambda(0, 0) = 0$.

To get $\lambda(0, 1)$ it is enough to compute

$$\sum_{j=1}^{n-1} \hat{\pi}(I_{nj}) P_1 \hat{\pi}(I_{nj}) (\epsilon_1 \wedge \cdots \wedge \epsilon_{p-1} \wedge e_n).$$

We have

$$P_1 \hat{\pi}(I_{nj}) (\epsilon_1 \wedge \cdots \wedge \epsilon_{p-1} \wedge e_n) = \begin{cases} 0 & \text{if } 1 \leq j \leq p - 1, \\ \epsilon_1 \wedge \cdots \wedge \epsilon_{p-1} \wedge e_j & \text{if } p \leq j \leq n - 1. \end{cases}$$

Therefore

$$\hat{\pi}(I_{nj}) P_1 \hat{\pi}(I_{nj}) (\epsilon_1 \wedge \cdots \wedge \epsilon_{p-1} \wedge e_n) = \begin{cases} 0 & \text{if } 1 \leq j \leq p - 1, \\ -\epsilon_1 \wedge \cdots \wedge \epsilon_{p-1} \wedge e_n & \text{if } p \leq j \leq n - 1. \end{cases}$$

Hence $\lambda(0, 1) = -(n - p)$.

Similarly, to get $\lambda(1, 0)$ it is enough to compute

$$\sum_{j=1}^{n-1} \hat{\pi}(I_{nj}) P_0 \hat{\pi}(I_{nj}) (\epsilon_1 \wedge \cdots \wedge e_p).$$
We have that
\[
\hat{\pi}(I_{nj})(e_1 \wedge \cdots \wedge e_p) = \begin{cases} 
-e_1 \wedge \cdots \wedge e_n \wedge \cdots \wedge e_p & \text{if } 1 \leq j \leq p, \\
0 & \text{if } p + 1 \leq j \leq n - 1,
\end{cases}
\]
where \(e_n\) appears in the \(j\)-place. Therefore
\[
\hat{\pi}(I_{nj})P_0\hat{\pi}(I_{nj})(e_1 \wedge \cdots \wedge e_p) = \begin{cases} 
-e_1 \wedge \cdots \wedge e_p & \text{if } 1 \leq j \leq p, \\
0 & \text{if } p + 1 \leq j \leq n - 1.
\end{cases}
\]
Hence \(\lambda(1,0) = -p\).

Also it is clear now that \(\sum_{j=1}^{n-1} \hat{\pi}(I_{nj})P_0\hat{\pi}(I_{nj})(e_1 \wedge \cdots \wedge e_p) = 0\), hence \(\lambda(1,1) = 0\).

Therefore, when \(\pi\) is the standard representation of \(K\) in \(\Lambda^p(V), 1 \leq p \leq \ell - 1\), we have
\[
(\lambda(r,s))_{0 \leq r,s \leq 1} = \begin{pmatrix} 
0 & p - n \\
-p & 0
\end{pmatrix}.
\]

Therefore, we obtain a more explicit version of Corollary 3.6 using (3.2) and Remark 3.8.

**Corollary 5.2.** Let \(\Phi\) be an irreducible spherical function on \(G\) of type \(\pi \in \hat{SO}(n), n = 2\ell + 1\). If the highest weight of \(\pi\) is of the form \((1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^\ell\), with \(p\) ones, \(1 \leq p \leq \ell - 1\), then the function \(H : (0, 1) \to \text{End}(\mathbb{C}^2)\) associated with \(\Phi\) satisfies
\[
y(1-y)H''(y) + \frac{1}{2}n(1-2y)H'(y) + \frac{1 + (1-2y)^2}{4y(1-y)} \begin{pmatrix} 
p - n & 0 \\
0 & -p
\end{pmatrix} H(y) + \frac{(1-2y)}{2y(1-y)} \begin{pmatrix} 
0 & p - n \\
-p & 0
\end{pmatrix} H(y) = \lambda H(y),
\]
for some \(\lambda \in \mathbb{C}\).

### 6 The spherical functions of fundamental \(K\)-types

Let \(n = 2\ell\), the irreducible spherical functions of \(K\)-type
\[
m_n = (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^\ell,
\]
with \(p\) ones, \(1 \leq p \leq \ell - 1\), are those associated with the irreducible representations of \(G\) of highest weights of the form \(m_{n+1} = (w + 1, 1, \ldots, 1, \delta, 0, \ldots, 0) \in \mathbb{C}^\ell\) that interlaces \(m_n\),
\[
\begin{array}{cccccccc}
w + 1 & 1 & \ldots & 1 & \delta & 0 & \ldots & 0 \\
1 & \ldots & \ldots & 1 & 0 & \ldots & \ldots & 0
\end{array}
\]

We now consider the \(K\)-module \(\Lambda^p(\mathbb{C}^n)\) which has highest weight \(m_n\).

For \(w = 0\) and \(\delta = 0\) we consider the \(G\)-module \(\Lambda^p(\mathbb{C}^{n+1})\) whose highest weight is \(m_{n+1}\), and we have the following \(K\)-module decomposition
\[
\Lambda^p(\mathbb{C}^{n+1}) = \Lambda^p(\mathbb{C}^n) \oplus \Lambda^{p-1}(\mathbb{C}^n) \wedge e_{n+1},
\]
where \(\Lambda^p(\mathbb{C}^n)\) is the sum of two \(SO(n-1)\)-modules:
\[
\Lambda^p(\mathbb{C}^n) = \Lambda^p(\mathbb{C}^{n-1}) \oplus \Lambda^{p-1}(\mathbb{C}^{n-1}) \wedge e_n.
\]
We observe that
\[
a(s)(e_1 \wedge \cdots \wedge e_{p-1} \wedge e_n) = e_1 \wedge \cdots \wedge e_{p-1} \wedge (\cos s e_n - \sin s e_{n+1}) \\
= \cos s(e_1 \wedge \cdots \wedge e_{p-1} \wedge e_n) - \sin s(e_1 \wedge \cdots \wedge e_{p-1} \wedge e_{n+1}).
\]

Hence, if \(\Phi_0\) is the spherical function associated with the irreducible representation of \(G\) of highest weight \(m_{n+1} = (1,1,\ldots,1,\delta,0,\ldots,0) \in \mathbb{C}^\ell\) with \(\delta = 0\), we have that
\[
\Phi_0(a(s))(e_1 \wedge \cdots \wedge e_{p-1} \wedge e_n) = \cos s(e_1 \wedge \cdots \wedge e_{p-1} \wedge e_n).
\]

Also we have that \(a(s)(e_1 \wedge \cdots \wedge e_p) = e_1 \wedge \cdots \wedge e_p\). Thus the vector valued function \(F_0(s)\) given by the irreducible spherical function \(\Phi_0(a(s))\) is
\[
F_0(s) = \begin{pmatrix} \cos s \\ 1 \end{pmatrix}.
\]

For \(w = 0\) and \(\delta = 1\) we consider the \(G\)-module \(\Lambda^{p+1}(\mathbb{C}^{n+1})\) whose highest weight \(m_{n+1}\), and for \(1 \leq p \leq \ell - 1\) we have the following \(K\)-module decomposition
\[
\Lambda^{p+1}(\mathbb{C}^{n+1}) = \Lambda^{p+1}(\mathbb{C}^n) \oplus \Lambda^p(\mathbb{C}^n) \wedge e_{n+1},
\]
where \(\Lambda^p(\mathbb{C}^n) \wedge e_{n+1}\) is the sum of two \(\text{SO}(n-1)\)-modules:
\[
\Lambda^p(\mathbb{C}^n) \wedge e_{n+1} = \Lambda^p(\mathbb{C}^{n-1}) \wedge e_{n+1} \oplus \Lambda^{p-1}(\mathbb{C}^{n-1}) \wedge e_n \wedge e_{n+1}.
\]

We observe that
\[
a(s)(e_1 \wedge \cdots \wedge e_{p-1} \wedge e_n \wedge e_{n+1}) = e_1 \wedge \cdots \wedge e_{p-1} \wedge (\sin s e_n + \cos s e_{n+1}) \\
= \sin s(e_1 \wedge \cdots \wedge e_{p-1} \wedge e_n) + \cos s(e_1 \wedge \cdots \wedge e_{p-1} \wedge e_{n+1}).
\]

Hence, if \(\Phi_1\) is the spherical function associated with the irreducible representation of \(G\) of highest weight \(m_{n+1} = (1,1,\ldots,1,\delta,0,\ldots,0) \in \mathbb{C}^\ell\) with \(\delta = 1\), we have that \(\Phi_1(a(s))(e_1 \wedge \cdots \wedge e_{p-1} \wedge e_n \wedge e_{n+1}) = \cos s(e_1 \wedge \cdots \wedge e_{p-1} \wedge e_n \wedge e_{n+1})\). Also we have that
\[
a(s)(e_1 \wedge \cdots \wedge e_p \wedge e_{n+1}) = e_1 \wedge \cdots \wedge e_{p-1} \wedge e_n \wedge e_{n+1}.
\]

Thus the vector valued function \(F_1(s)\) given by the irreducible spherical function \(\Phi_1(a(s))\) is
\[
F_1(s) = \begin{pmatrix} 1 \\ \cos s \end{pmatrix}.
\]

**Definition 6.1.** We shall consider the \(2 \times 2\) matrix-valued function \(\Psi = \Psi(y)\), for \(0 < y < 1\), whose columns are given by the functions \(H_0(y) = F_0(s)\) and \(H_1(y) = F_1(s)\), with \(\cos s = 2y - 1\):
\[
\Psi(y) = \begin{pmatrix} 2y - 1 & 1 \\ 1 & 2y - 1 \end{pmatrix}.
\]

Since the functions \(H_0(y)\) and \(H_1(y)\) are associated with irreducible spherical functions, they satisfy the differential equation given in **Corollary 5.1**; moreover, the respective eigenvalues are \(\lambda = -p\) and \(\lambda = p - n\). Therefore, we have
\[
y(1 - y)\Psi'' + \frac{1}{2}n(1 - 2y)\Psi' + \frac{1}{4y(1 - y)} \begin{pmatrix} p - n & 0 \\ 0 & -p \end{pmatrix} \Psi \\
+ \frac{(1 - 2y)}{2y(1 - y)} \begin{pmatrix} 0 & p - n \\ -p & 0 \end{pmatrix} \Psi = \Psi \begin{pmatrix} -p & 0 \\ 0 & p - n \end{pmatrix}.
\]

Furthermore, it is easy to check that the function \(\Psi(y)\) also satisfy the equation above even when \(n\) is odd.
Theorem 6.2. The function $\Psi$ can be used to obtain a hypergeometric differential equation from the one given in Corollaries 5.1 and 5.2. Precisely, if $H$ is a vector-valued solution of the differential equation in Corollaries 5.1 or 5.2, with eigenvalue $\lambda$, then $P = \Psi^{-1}H$ is a solution of $DP = \lambda P$, where $D$ is the hypergeometric differential operator given by

$$DP = y(1 - y)P'' - \left(\frac{n}{2} + 1\right)(2y - 1) - 1 \left(\frac{n}{2} + 1\right)(2y - 1) P' - \left(\begin{array}{cc} p & 0 \\ 0 & n - p \end{array}\right) P.$$

Proof. By hypothesis we have that

$$y(1 - y)H''(y) + \frac{1}{2}n(1 - 2y)H'(y) + \frac{1 + (1 - 2y)^2}{4y(1 - y)} \left(\begin{array}{cc} p - n & 0 \\ 0 & -p \end{array}\right) H(y) + \frac{(1 - 2y)}{2y(1 - y)} \left(\begin{array}{cc} 0 & p - n \\ -p & 0 \end{array}\right) H(y) = \lambda H(y),$$

Then, writing $H = \Psi P$, we have

$$y(1 - y)P'' + (2y(1 - y)\Psi^{-1}\Psi' + \frac{n}{2}(1 - 2y)I)P'$$

$$+ \Psi^{-1} \left( y(1 - y)\Psi'' + \frac{n}{2}(1 - 2y)\Psi' + \frac{1 + (1 - 2y)^2}{4y(1 - y)} \left(\begin{array}{cc} p - n & 0 \\ 0 & -p \end{array}\right) \Psi' \right.$$

$$+ \frac{(1 - 2y)}{2y(1 - y)} \left(\begin{array}{cc} 0 & p - n \\ -p & 0 \end{array}\right) \Psi \right) P = \lambda P.$$

Now we compute

$$2y(1 - y)\Psi^{-1}\Psi' = \frac{4y(1 - y)}{4y(y - 1)} \left(\begin{array}{cc} 2y - 1 & -1 \\ -1 & 2y - 1 \end{array}\right) = - \left(\begin{array}{cc} 2y - 1 & -1 \\ -1 & 2y - 1 \end{array}\right).$$

Therefore

$$y(1 - y)P'' - \left(\frac{n}{2} + 1\right)(2y - 1) - 1 \left(\frac{n}{2} + 1\right)(2y - 1) P' - \left(\begin{array}{cc} \lambda + p & 0 \\ 0 & \lambda + n - p \end{array}\right) P = 0.$$

This completes the proof of the theorem. ■

6.1 $\Delta$-eigenvalues of spherical functions

As we said, when $n = 2\ell$ the irreducible spherical functions of the pair $(SO(n + 1), SO(n))$, of type $m_n = (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^{\ell}$ with $p$ ones, $1 \leq p \leq \ell - 1$ are those associated with the irreducible representations $\tau$ of $G$ of highest weights of the form $m_{n+1} = (w + 1, 1, \ldots, 1, \delta, 0, \ldots, 0) \in \mathbb{C}^{\ell}$ with $p - 1$ ones, such that the following pattern holds

$$w + 1 \quad 1 \quad \ldots \quad 1 \quad \delta \quad 0 \quad \ldots \quad 0,$$

$$1 \quad \ldots \quad \ldots \quad 1 \quad 0 \quad \ldots \quad \ldots \quad 0.$$ 

Let $\Phi_{w,\delta}$ be the corresponding spherical function. Then $\Delta \Phi_{w,\delta} = \lambda \Phi_{w,\delta}$, where the eigenvalue $\lambda = \lambda_n(w, \delta)$ can be computed from the expression $\Delta = Q_{n+1} - Q_n$. If $v \in V_{m_{n+1}}$ is a highest weight vector from (2.6) we have

$$\hat{\tau}(Q_{2\ell+1})v = -((w + 1)^2 + (2\ell - 1)(w + 1) + (2\ell - p)(p - 1) + 2\delta(\ell - p))v.$$

If $v \in V_{m_{2\ell}}$ is a highest weight vector, then from (2.5) we have

$$\hat{\tau}(Q_n)v = -p(2\ell - p)v.$$
Since \( \Delta = Q_{n+1} - Q_n \) it follows that
\[
\lambda_{2\ell}(w, \delta) = -(w + 1)^2 - (2\ell - 1)(w + 1) + (2\ell - p) - 2\delta(\ell - p)
\]
Analogously, we obtain that the eigenvalues of the spherical functions \( \Phi_{w,\delta} \) of the pair \((\text{SO}(2\ell + 2), \text{SO}(2\ell + 1))\) are of the form
\[
\lambda_{2\ell+1}(w, \delta) = -(w + 1)(w + 2\ell + 1) + 2\ell - p + 1 - \delta(\ell - p) - \delta^2,
\]
here \( \delta \) is 0 or 1 when we are in the cases \( 1 \leq p < \ell \) but \( \delta \) could also be \(-1\) in the particular case \( p = \ell \).

Therefore, we have that the eigenvalues of the spherical functions \( \Phi_{w,\delta} \) of the pair \((\text{SO}(n + 1), \text{SO}(n))\) are of the form
\[
\lambda_n(w, \delta) = \begin{cases} 
-w(w + n + 1) - p & \text{if } \delta = 0, \\
-w(w + n + 1) - n + p & \text{if } \delta = \pm 1.
\end{cases} \tag{6.2}
\]

### 6.2 Polynomial eigenfunctions of the hypergeometric operator \( D \)

Let \( D \) be the differential operator on the real line introduced in Theorem 6.2:
\[
DP = y(1 - y)P'' + (C - yU)P' - VP,
\tag{6.3}
\]
with
\[
C = \begin{pmatrix}
(n/2 + 1) & 1 \\
1 & (n/2 + 1)
\end{pmatrix}, \quad U = (n + 2)I, \quad V = \begin{pmatrix}
p & 0 \\
0 & n - p
\end{pmatrix},
\]
where \( n \) is of the form \( 2\ell \) or \( 2\ell + 1 \) for \( \ell \in \mathbb{N} \) and \( 1 \leq p < \ell \).

We will study the \( \mathbb{C}^2 \)-vector valued polynomial eigenfunctions of \( D \).

The equation \( DP = \lambda P \) is an instance of a matrix hypergeometric differential equation studied in [22]. Since the eigenvalues of \( C, n/2 \) and \( n/2 + 2 \), are not in \( -\mathbb{N}_0 \) the function \( P \) is determined by \( P_0 = P(0) \). For \( |y| < 1 \) it is given by
\[
P(y) = 2H_1 \left( \begin{array}{c} U, V + \lambda \end{array} ; y \right) P_0 = \sum_{j=0}^{\infty} \frac{y^j}{j!} [C; U; V + \lambda]_j P_0, \quad P_0 \in \mathbb{C}^2,
\]
where the symbol \( [C; U; V + \lambda]_j \) is inductively defined by
\[
[C; U; V + \lambda]_0 = 1,
\]
\[
[C; U; V + \lambda]_{j+1} = (C + j)^{-1} (j(U + j - 1) + V + \lambda) [C; U; V + \lambda]_j,
\]
for all \( j \geq 0 \).

Therefore, we have that there exists a polynomial solution if and only if the coefficient \( [C; U; V + \lambda]_{j+1} \) is a singular matrix for some \( j \in \mathbb{Z} \). Since the matrix \( C + j \) is invertible for all \( j \in \mathbb{N}_0 \), we have that there is a polynomial solution of degree \( j \) for \( DP = \lambda P \) if and only if there exists \( P_0 \in \mathbb{C}^2 \) such that \( [C; U; V + \lambda]_j P_0 \neq 0 \) and \( (j(U + j - 1) + V + \lambda) [C; U; V + \lambda]_j P_0 = 0 \).

Now we easily observe that the only possible values for \( \lambda \) such that \( j(U + j - 1) + V + \lambda \) has non trivial kernel are those given in (6.2). Then, if \( \lambda = -w(2n + 1) - p \), it is easy to check that the first and only \( j \) for which \( j(U + j - 1) + V + \lambda \) is singular is \( j = w \), and its kernel (of dimension 1) is the subspace generated by \( (\frac{1}{2}) \). Analogously, if \( \lambda = -w(2n + 1) - n + p \), it is easy to check that the first and only \( j \) for which \( j(U + j - 1) + V + \lambda \) is singular is \( j = w \), and its kernel (of dimension 1) is the subspace generated by \( (\frac{1}{2}) \) respectively. Therefore we have the following result.
Theorem 6.3. For a given $\ell \in \mathbb{N}$ take $n = 2\ell$ or $2\ell + 1$ and $1 \leq p \leq \ell - 1$, then the polynomial eigenfunctions of
\[ DP = y(1-y)P'' + (C - yU)P' - VP, \]
with
\[ C = \begin{pmatrix} (n/2 + 1) & 1 \\ 1 & (n/2 + 1) \end{pmatrix}, \quad U = (n + 2)I, \quad V = \begin{pmatrix} p & 0 \\ 0 & n - p \end{pmatrix} \]
have eigenvalues $-w(w + n + 1) - p$ or $-w(w + n + 1) - n + p$, with $w \in \mathbb{N}_0$; in both cases the degree of the polynomial is $w$ with leading coefficient a multiple of $\left(\frac{1}{2}\right)$ or $\left(\frac{n}{2}\right)$, respectively.

7 The inner product

Given a finite dimensional irreducible representation $\pi$ of $K$ in the vector space $V_\pi$ let $(C(G) \otimes \text{End}(V_\pi))^{K \times K}$ be the space of all continuous functions $\Phi : G \to \text{End}(V_\pi)$ such that $\Phi(k_1gk_2) = \pi(k_1)\Phi(g)\pi(k_2)$ for all $g \in G$, $k_1, k_2 \in K$. Let us equip $V_\pi$ with an inner product such that $\pi(k)$ becomes unitary for all $k \in K$. Then we introduce an inner product in the vector space $(C(G) \otimes \text{End}(V_\pi))^{K \times K}$ by defining
\[ \langle \Phi_1, \Phi_2 \rangle = \int_G \text{tr}(\Phi_1(g)\Phi_2(g)^*) dg, \]
where $dg$ denote the Haar measure on $G$ normalized by $\int_G dg = 1$, and where $\Phi_2(g)^*$ denotes the adjoint of $\Phi_2(g)$ with respect to the inner product in $V_\pi$.

By using Schur's orthogonality relations for the unitary irreducible representations of $G$, it follows that if $\Phi_1$ and $\Phi_2$ are non equivalent irreducible spherical functions, then they are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$, i.e.
\[ \langle \Phi_1, \Phi_2 \rangle = 0. \]

Recall that, given an irreducible spherical function $\Phi$ of type $\pi$ of the pair $(G, K)$, the function $\Phi(a(s))$ is scalar valued when restricted to any $\text{SO}(n-1)$-module (see (3.1) for $a(s)$). We shall denote by $m$ the number of $\text{SO}(n-1)$-submodules of $\pi$, and by $d_1, d_2, \ldots, d_m$ the respective dimensions of each one of those submodules.

In particular, if $\Phi_1$ and $\Phi_2$ are two irreducible spherical functions of type $\pi \in \tilde{K}$, we consider the vector valued functions $H_1(y)$ and $H_2(y)$ given by the diagonal matrix valued functions $\Phi_1(a(s))$ and $\Phi_2(a(s))$ (see Remark 3.7), with $y = (\cos s + 1)/2$, respectively, denoting
\[ H_1(y) = (h_1(y), \ldots, h_m(y))^t, \quad H_2(y) = (f_1(y), \ldots, f_m(y))^t. \]

Proposition 7.1. If $\Phi_1$, $\Phi_2$ are two irreducible spherical functions of type $\pi \in \tilde{K}$ then
\[ \langle \Phi_1, \Phi_2 \rangle = \frac{(n-1)!!}{(n-2)!!} \frac{2}{\omega_s} \sum_{i=1}^m d_i \int_0^1 (y(1-y))^{n/2-1} h_i(y)f_i(y) dy, \]
with $\omega_s = \pi$ if $n$ is even and $\omega_s = 2$ if $n$ is odd.

Proof. Let $A = \exp \mathbb{R}I_{n+1,n}$ be the Lie subgroup of $G$ of all elements of the form
\[ a(s) = \exp sI_{n+1,n} = \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & \cos s & \sin s \\ 0 & -\sin s & \cos s \end{pmatrix}, \quad s \in \mathbb{R}, \]
where $I_{n-1}$ denotes the identity matrix of size $n - 1$. 

Now [12, Theorem 5.10, p. 190] establishes that for every \( f \in C(G/K) \) and a suitable constant \( c_\ast \)
\[
\int_{G/K} f(gK) dg_K = c_\ast \int_{K/M} \left( \int_{-\pi}^\pi \delta_\ast(a(s)) f(ka(s)K) ds \right) dk_M,
\]
where \( dg_K \) and \( dk_M \) are respectively the invariant measures on \( G/K \) and \( K/M \) normalized by \( \int_{G/K} dg_K = \int_{K/M} dk_M = 1 \) and the function \( \delta_\ast : A \to \mathbb{R} \) is defined by
\[
\delta_\ast(a(s)) = \prod_{\nu \in \Sigma^+} \sin i\nu(I_{n+1,n}),
\]
with \( \Sigma^+ \) the set of those positive roots whose restrictions to \( a \), the Lie algebra of \( A \), are not zero. In our case we have \( \delta_\ast(a(s)) = |\sin^{n-1}s| \).

To find the value of \( c_\ast \) we consider the function \( f \equiv 1 \), having then
\[
1 = 2c_\ast \int_0^\pi \sin^{n-1}s ds.
\]

Since
\[
\int \sin^{n-1}s ds = -\frac{1}{n-1} \sin^{n-2}s \cos s + \frac{n-2}{n-1} \int \sin^{n-3}s ds,
\]
we obtain that, for \( n = 2\ell \) or \( 2\ell + 1 \),
\[
\int_0^\pi \sin^{n-1}s ds = \frac{n-2}{n-1} \frac{n-4}{n-3} \cdots \frac{n-2\ell+1}{n-2\ell+2} \int_0^\pi \sin^{n-2\ell}s ds.
\]

Therefore
\[
c_\ast = \frac{(n-1)!!}{(n-2)!!} \frac{1}{2\omega_\ast},
\]
with \( \omega_\ast = \pi \) for \( n = 2\ell \) and \( \omega_\ast = 2 \) for \( 2\ell + 1 \).

Since the function \( g \mapsto \text{tr}(\Phi_1(g)\Phi_2(g)^*) \) is invariant under left and right multiplication by elements in \( K \), we have
\[
\langle \Phi_1, \Phi_2 \rangle = \int_G \text{tr}(\Phi_1(g)\Phi_2(g)^*) dg = 2c_\ast \int_0^\pi \sin^{n-1}s \text{tr} (\Phi_1(a(s)\Phi_2(a(s))^*)) ds.
\]

If we put \( y = \frac{1}{2}(\cos s + 1) \) for \( 0 < s < \pi \) we have
\[
\text{tr} (\Phi_1(a(s)\Phi_2(a(s))^*)) = \sum_{i=1}^m d_i h_i(y) f_i(y).
\]

Then
\[
\langle \Phi_1, \Phi_2 \rangle = 4c_\ast \sum_{i=1}^m d_i \int_0^1 (4y(1-y))^{(n-2)/2} h_i(y) f_i(y) dy,
\]
and the proposition follows.

**Proposition 7.2.** If \( \Phi_1, \Phi_2 \in (C^\infty(G) \otimes \text{End}(V_\pi))^K \times K \) then
\[
\langle \Delta \Phi_1, \Phi_2 \rangle = \langle \Phi_1, \Delta \Phi_2 \rangle.
\]
Proof. If we apply a left invariant vector field $X \in \mathfrak{g}$, to the function on $G$ given by $g \mapsto \text{tr}(\Phi_1(g)\Phi_2(g)^*)$, and then we integrate over $G$ we obtain

$$0 = \int_G \text{tr} ((X\Phi_1)(g)\Phi_2(g)^*)\,dg + \int_G \text{tr} (\Phi_1(g)(X\Phi_2)(g)^*)\,dg.$$ 

Therefore $\langle X\Phi_1, \Phi_2 \rangle = -\langle \Phi_1, X\Phi_2 \rangle$. Now let $\tau : \mathfrak{g}_C \to \mathfrak{g}_C$ be the conjugation of $\mathfrak{g}_C$ with respect to the real linear form $\mathfrak{g}$. Then $-\tau$ extends to a unique antilinear involutive * operator on $D(G)$ such that $(D_1D_2)^* = D_2^*D_1^*$ for all $D_1, D_2 \in D(G)$. This follows easily from the fact that the universal enveloping algebra over $\mathbb{C}$ of $\mathfrak{g}$ is canonically isomorphic to $D(G)$. Then it follows that $\langle D\Phi_1, \Phi_2 \rangle = \langle \Phi_1, D^*\Phi_2 \rangle$.

Finally, it is easy to verify that $\Delta^* = \Delta$. 

\begin{section}{Spherical functions as polynomial solutions of $DP = \lambda P$}

Let us consider $\tilde{D}$, the differential operator on $(0,1)$ introduced in Corollaries 5.1 and 5.2:

$$y(1-y)H''(y) + \frac{1}{2}n(1-2y)H'(y) + \frac{1+(1-2y)^2}{4y(1-y)} \begin{pmatrix} p-n & 0 \\ 0 & -p \end{pmatrix} H(y)$$

$$+ \frac{(1-2y)}{2y(1-y)} \begin{pmatrix} 0 & p-n \\ -p & 0 \end{pmatrix} H(y) = \lambda H(y), \quad (7.1)$$

Recall that the operator $D$ that appears in (6.3) extends the differential operator $D = \Psi \tilde{D} \Psi^{-1}$ to the whole real line, where

$$\Psi(y) = \begin{pmatrix} 2y - 1 & 1 \\ 1 & 2y - 1 \end{pmatrix}$$

is the matrix function given in (6.1) and used in Theorem 6.2.

We want to focus our attention on the following vector spaces of $\mathbb{C}^2$-valued analytic functions on $(0,1)$:

$$S_\lambda = \{H = H(y) : \tilde{D}H = \lambda H, \ H(\cos \frac{s+1}{2}) \text{analytic at } s = 0\},$$

$$W_\lambda = \{P = P(y) : DP = \lambda P, \text{analytic on } [0,1]\}.$$ 

From Theorem 6.2 we know that the correspondence $P \mapsto \Psi P$ is an injective linear map from $W_\lambda$ into $S_\lambda$. Now we want to prove that this map is bijective.

\begin{theorem}

The linear map $P \mapsto \Psi P$ is an isomorphism from $W_\lambda$ onto $S_\lambda$.

\end{theorem}

Proof. A vector valued function $P \in W_\lambda$ is an eigenfunction of the hypergeometric operator $D$. Since it is analytic at $y = 1$ it is determined by $P(1)$, therefore $\dim(W_\lambda) = 2$.

On the other hand, if $H \in S_\lambda$ then there is a function $F(s)$ analytic at $s = 0$, such that it extends the function $H(\cos \frac{s+1}{2})$ defined on $(0, \pi)$. Then, $F$ satisfies the following differential equation

$$F''(s) + (n-1)\frac{\cos s}{\sin s} F'(s) + \frac{1+\cos^2 s}{\sin^2 s} \begin{pmatrix} p-n & 0 \\ 0 & -p \end{pmatrix} F(s)$$

$$- \frac{\cos s}{\sin^2 s} \begin{pmatrix} 0 & p-n \\ -p & 0 \end{pmatrix} F(s) = \lambda F(s),$$

or equivalently

$$\sin^2 s F''(s) + \frac{n-1}{2} \sin(2s) F'(s) + (2 - \sin^2 s) \begin{pmatrix} p-n & 0 \\ 0 & -p \end{pmatrix} F(s)$$

$$= \lambda F(s).$$
\[-2 \cos s \begin{pmatrix} 0 & p-n \\ -p & 0 \end{pmatrix} F(s) = \lambda \sin^2 s F(s), \]

(7.2)

Let \( a_j \in \mathbb{C}^2 \) and \( \alpha_j, \beta_j, \gamma_j \in \mathbb{C} \), for \( j \geq 0 \), be the Taylor coefficients of \( F, \sin, \sin^2 \) and \( \cos \) at \( s = 0 \):

\[
F(s) = \sum_{j \geq 0} a_j s^j, \quad \sin s = \sum_{j \geq 1} \alpha_j s^j,
\]

\[
F'(s) = \sum_{j \geq 0} a_{j+1} (j+1) s^j, \quad \sin^2 s = \sum_{j \geq 2} \beta_j s^j,
\]

\[
F''(s) = \sum_{j \geq 0} a_{j+2} (j+1)(j+2) s^j, \quad \cos s = \sum_{j \geq 0} \gamma_j s^j.
\]

Therefore, from (7.2) we have

\[
\sum_{j \geq 0} \sum_{k=0}^{j-2} \beta_{j-k} a_{k+2}(k+2)(k+1) + \frac{n-1}{2} \sum_{k=0}^{j-1} 2^j \alpha_{j-k} a_{k+1}(k+1) + \begin{pmatrix} p-n & 0 \\ -p & 0 \end{pmatrix} \sum_{k=0}^{j} \gamma_{j-k} a_k
\]

\[
\times \left( 2a_j - \sum_{k=0}^{j-2} \beta_{j-k} a_k \right) + 2 \begin{pmatrix} 0 & p-n \\ -p & 0 \end{pmatrix} \sum_{k=0}^{j} \gamma_{j-k} a_k \right] s^j.
\]

Hence, since \( \beta_2 = \alpha_1 = \gamma_0 = 1 \), we have that

\[
\left[ j(j-1) + (n-1)j + 2 \begin{pmatrix} p-n & -p+n \\ p & -p \end{pmatrix} \right] a_j
\]

is a linear combination with matrix coefficients of \( \{a_0, a_1, \ldots, a_{j-1}\} \); it is clear that for \( j = 1 \) and \( j > 2 \) the matrix above is non singular, therefore \( \{a_0, a_2\} \) determine completely the sequence \( \{a_j\}_{j \geq 0} \). Also it is clear that when \( j = 0 \) or \( 2 \), that matrix has nullity 1. Therefore we can conclude that \( \dim(S_\lambda) = 2 \). The theorem follows. \( \blacksquare \)

**Theorem 7.4.** Let \( H \) be the \( \mathbb{C}^2 \)-valued analytic function on \((0,1)\) given by an irreducible spherical function \( \Phi \) on \( G \) of fundamental \( K \)-type \((1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^\ell \), with \( p \) ones, \( 0 < p < \ell \). If \( P = \Psi^{-1} H \), then \( P \) is polynomial.

**Proof.** We know that the function \( H \) is analytic in \((0,1)\), and from Corollary 5.1 we know that it is an eigenfunction of the operator \( \bar{D} \) (see (7.1)). Also we know that the function \( H(\frac{1+\cos s}{2}) \) is analytic at \( s = 0 \), since \( \Phi(a(s)) \) it is. Therefore from Theorem 7.3 the function \( P = \Psi^{-1} H \) is an analytic eigenfunction of \( \bar{D} \) on the closed interval \([0,1]\).

If we introduce the following matrix-weight function \( V = V(y) \) supported on the interval \([0,1]\)

\[
V(y) = \frac{(n-1)!!}{(n-2)!!} \frac{2}{\omega_\pi (y(1-y))^{n/2-1}} \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},
\]

with \( \omega_\pi = \pi \) if \( n \) is even and \( \omega_\pi = 2 \) if \( n \) is odd, then from Proposition 7.1 we have

\[
\langle \Phi_0, \Phi_1 \rangle = \int_0^1 H_2^*(y) V(y) H_1^*(y) dy.
\]

It follows from Propositions 7.1 and 7.2 that \( \bar{D} \) is a symmetric operator with respect to the inner product defined among continuous vector-valued functions on \([0,1]\) by

\[
\langle H_1, H_2 \rangle_V = \int_0^1 H_2^*(y) V(y) H_1(y) dy.
\]
Then, since \( D = \Psi^{-1} \tilde{D} \Psi \), we have that \( D \) is a symmetric operator with respect to the inner product defined among continuous vector-valued functions on \([0, 1]\) by
\[
\langle P_1, P_2 \rangle_W = \int_0^1 P_2^*(y)W(y)P_1(y)dy,
\]
where
\[
W = \Psi^* V \Psi.
\]
Actually, we have that \((W, D)\) is a classical pair in the sense of [7], see also [2]. As the weight \( W \) has finite moments there exists a sequence \( \{Q_r\}_{r \geq 0} \) of \( 2 \times 2 \) matrix-valued orthonormal polynomials, such that \( DQ_r = Q_r \Lambda_r \) where \( \Lambda_r \) is a real diagonal matrix (for precise definitions and general facts on matrix-valued orthogonal polynomials see [5] and [2]).

Let \( \{e_1, e_2\} \) be the canonical basis of \( \mathbb{C}^2 \). Then
\[
\langle Q_r e_j, Q_s e_i \rangle_W = e_i^* \left( \int_0^1 Q_s^*(y)W(y)Q_r^*(y)dy \right) e_j = e_i^* \delta_{si} I e_j = \delta_{r,s} \delta_{i,j}.
\]
Therefore, for \( r \geq 0, \ j = 1, 2, \ \{Q_r e_j\} \) is a family of \( \mathbb{C}^2 \)-valued orthonormal polynomials such that
\[
D(Q_r e_j) = (DQ_r)e_j = (Q_r \Lambda_r)e_j = Q_r(\Lambda_r e_j) = \lambda_r^j(Q_r e_j),
\]
where \( \Lambda_r = \text{diag}(\lambda_r^1, \lambda_r^2) \).

Now we write our function \( P = \Psi^{-1} H \) as \( P = \sum_{r,j} a_{r,j} Q_r e_j \), where \( a_{r,j} = \langle P, Q_r e_j \rangle_W \). Since \( P \) is analytic on \([0, 1]\) the sum converges not only in the \( L^2 \)-norm but also in the topology based on uniform convergence of sequences of functions and their successive derivatives.

Therefore,
\[
\lambda P = DP = \sum_{r,j} a_{r,j} \lambda_r^j Q_r e_j.
\]
Then \( a_{r,j} = 0 \) if \( \lambda_r^j \neq \lambda \). Since \( \dim W_\lambda = 2 \) it follows that \( P \) is a polynomial. \( \blacksquare \)

**Remark 7.5.** It is easy to see from (5.1) and (5.2) that the dimensions of the \( M \)-submodules of the fundamental representation of \( K \) with highest weight of the form \((1, \ldots, 1, 0, \ldots, 0)\), with \( p \) ones, are given by
\[
d_1 = \frac{(n - 1)!}{(p - 1)!(n - p)!}, \quad d_2 = \frac{(n - 1)!}{p!(n - 1 - p)!},
\]
therefore the weight \( W \) is given by
\[
W = \frac{(n - 1)!!}{(n - 2)!!} \omega_s p!(n - p)! (y(1 - y))^{n/2 - 1} \Psi^* \begin{pmatrix} p & 0 \\ 0 & n - p \end{pmatrix} \Psi,
\]
with \( \omega_s = \pi \) if \( n \) is even and \( \omega_s = 2 \) if \( n \) is odd. Then, \( W \) is a scalar multiple of
\[
\begin{pmatrix} p(2y - 1)^2 + n - p & n(2y - 1) \\ n(2y - 1) & (n - p)(2y - 1)^2 + p \end{pmatrix}.
\]
Even more, since \( 0 < p < \ell \) and \( n = 2\ell, 2\ell + 1 \) it follows that \( p \neq n - p \). Then it can be proved that the weight \( W \) does not reduce to a smaller size, i.e., there is not any invertible matrix \( M \) such that \( M^* W(y) M \) is diagonal for all \( y \in [0, 1] \).
For a given fundamental $K$ type $\pi \in \hat{\text{SO}}(n)$, $n = 2\ell$ or $2\ell + 1$, with highest weight of the form $(1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^\ell$ with $p$ ones ($0 < p < \ell$), let $\Phi_{w,\delta}$ denote the irreducible spherical function of the pair $(\text{SO}(n+1), \text{SO}(n))$ given by $\tau \in \hat{\text{SO}}(n+1)$ with highest weight of the form $(w+1, 1, \ldots, 1, \delta, 0, \ldots, 0)$ with $p-1$ ones.

Therefore, combining (6.2), Theorems 6.3 and 7.4 we have the following statement.

**Theorem 7.6.** Given $w \in \mathbb{N}_0$, every irreducible spherical function $\Phi_{w,\delta}$ of the pair $(\text{SO}(n+1), \text{SO}(n))$, with $n = 2\ell$ or $2\ell + 1$, of type $m_n = (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^\ell$ with $p$ ones ($0 < p < \ell$), corresponds to a vector valued function $P_{w,\delta}$, which is a polynomial of degree $w$; and the leading coefficients of $P_{w,0}$ and $P_{w,1}$ are multiples of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively. Precisely

$$P_{w,\delta}(y) = \sum_{j=0}^{w} \frac{y^j}{j!} [C; U + V + \lambda]_j P_{w,\delta}(0),$$

with

$$C = \begin{pmatrix} (n/2 + 1) \\ 1 \end{pmatrix}, \quad U = (n+2)I, \quad V = \begin{pmatrix} p & 0 \\ 0 & n-p \end{pmatrix},$$

$$\lambda = \lambda_n(w, \delta) = \begin{cases} -w(w + n + 1) - p & \text{if } \delta = 0, \\ -w(w + n + 1) - n + p & \text{if } \delta = 1. \end{cases}$$

Even more, the value of $P_{w,\delta}(0)$ can be computed.

**Proof.** It only remains to prove that $P_{w,\delta}(0)$ can be computed.

Let us consider the case $\delta = 0$. We know from (6.2) and Theorem 6.3 that there is some $c \in \mathbb{C}$ such that

$$[C; U + V + \lambda]_w P_{w,0}(0) = c \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$ 

Since $[C; U + V + \lambda]_w$ is invertible, this $c$ is univocally determined by the condition $\Phi(e) = I$, which implies

$$\Psi(1) \sum_{j=0}^{w} \frac{1}{j!} [C; U + V + \lambda]_j P_{w,0}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$ 

Similarly, we can prove the same for $P_{w,1}(0).$ \hfill \blacksquare

**Remark 7.7.** It is worth to observe that for $w, w' \geq 0$ and $\delta, \delta' = 0, 1$, since $\langle P_{w,\delta}, P_{w',\delta'} \rangle_W = \langle \Phi_{w,\delta}, \Phi_{w',\delta'} \rangle_W$, we have that if $(w, \delta) \neq (w', \delta')$ then

$$\langle P_{w,\delta}, P_{w',\delta'} \rangle_W = 0.$$ 

Therefore, our construction encodes all equivalent classes of irreducible spherical functions of a fundamental $K$-type of highest weight $\lambda_p$, $0 < p < \ell$, in the orthogonal set of $\mathbb{C}^2$-valued polynomials $\{P_{w,0}, P_{w,1}\}$. The degree of $P_{w,0}$ and $P_{w,1}$ is $w$, and the leading coefficient is a multiple of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, respectively.
8 Matrix valued orthogonal polynomials

8.1 Matrix valued orthogonal polynomials

In this subsection, given \( n \) of the form \( 2\ell \) or \( 2\ell+1 \) with \( \ell \in \mathbb{N} \), for a fixed \( 0 < p < \ell \) we shall construct a sequence of matrix-valued polynomials \( \{ P_w \}_{w \geq 0} \) directly related to irreducible spherical functions of type \( \pi \in \text{SO}(n) \) of highest weight \( m_\pi = (1, \ldots, 1, 0, \ldots, 0) \in \mathbb{C}^\ell \), with \( p \) ones.

Given a nonnegative integer \( w \) and \( \delta = 0, 1 \), we can consider \( \Phi_{w,\delta} \), the irreducible spherical function of type \( \pi \) associated with the irreducible representation \( \tau \in \widehat{\text{SO}(n+1)} \) of highest weight \( m_\tau = (w+1, 1, 1, \ldots, 1, \delta, 0, \ldots, 0) \) with \( p-1 \) ones.

We insist on recalling that, since \( \pi \) has only two \( \text{SO}(n-1) \)-submodules, we can interpret the diagonal matrix-valued function \( \Phi_{w,\delta}(a(s)), s \in (0, \pi) \), as a 2 column vector function.

Now we consider the vector-valued function \( P_{w,\delta}: (0, 1) \rightarrow \mathbb{C}^2 \) given by the vector function

\[
P_{w,\delta}(y) = \Psi^{-1}(y) \Phi_{w,\delta}(a(s)), \quad \text{with } \cos(s) = 2y - 1.
\]

Then, we define the matrix-valued function

\[
P_w = P_w(y),
\]

whose \( \delta \)-th column \( (\delta = 0, 1) \) is given by the \( \mathbb{C}^2 \)-valued polynomial \( P_{w,\delta}(y) \).

Let consider the matrix-valued skew symmetric bilinear form defined among \( C^\infty \mathbb{R} \times \mathbb{R} \) matrix-valued functions on \([0, 1]\) by

\[
\langle P, Q \rangle_W = \int_0^1 Q^*(y) W(y) P(y) dy,
\]

where

\[
W = \frac{(n-1)!! 2}{(n-2)!!} \frac{\omega_s}{p! (n-p)!} \frac{(1-y)^{n/2-1}}{(n/2)!} (p^2 y - n - p) \frac{n^2}{(2y-1)^2 + p}.
\]

See Remark 7.5. Then we state the following theorem.

**Theorem 8.1.** The matrix-valued polynomial functions \( P_w, w \geq 0 \), form a sequence of orthogonal polynomials with respect to \( W \), which are eigenfunctions of the symmetric differential operator \( D \) in (6.3). Moreover,

\[
DP_w = P_w \left( \begin{array}{cc} \lambda(w,0) & 0 \\ 0 & \lambda(w,1) \end{array} \right),
\]

where

\[
\lambda(w, \delta) = \begin{cases} -w(w+n+1) - p & \text{if } \delta = 0, \\ -w(w+n+1) - n + p & \text{if } \delta = 1. \end{cases}
\]

**Proof.** From Theorem 6.2 we have that the \( \delta \)-th column of \( P_w \) is an eigenfunction of the operator \( D \) with eigenvalue \( \lambda(w, \delta) \), see (6.2) and (6.3). Therefore we have

\[
DP_w = P_w \Lambda_w,
\]

with

\[
\Lambda_w = \left( \begin{array}{cc} \lambda(w,0) & 0 \\ 0 & \lambda(w,1) \end{array} \right).
\]
From Theorem 7.6 we know that each column of $P_w$ is a polynomial function of degree $w$ and, even more, that $P_w$ is a polynomial whose leading coefficient is a nonsingular diagonal matrix. Given $w$ and $w'$, non negative integers, by using Remark 7.7 we have

$$
\langle P_{w'}, P_w \rangle_W = \int_0^1 P_w(y)^* W(y) P_{w'}(y) du = \sum_{\delta,\delta'} \int_0^1 (P_{w,\delta}(y)^* W(y) P_{w',\delta'}(y) du) E_{\delta,\delta'}
$$

$$
= \sum_{\delta,\delta'=0} \delta_{w,w'} \delta_{\delta,\delta'} \left( \int_0^1 P_{w,\delta}(y)^* W(y) P_{w',\delta'}(y) du \right) E_{\delta,\delta'}
$$

$$
= \delta_{w,w'} \sum_{\delta=0} \int_0^1 (P_{w,\delta}(y)^* W(y) P_{w',\delta}(y) du) E_{\delta,\delta},
$$

which proves the orthogonality. Even more, it also shows us that $\langle P_w, P_w \rangle_W$ is a diagonal matrix. Also, making a few simple computations we have that

$$
\langle DP_w, P_{w'} \rangle = \delta_{w,w'} \langle P_w, P_{w'} \rangle \Lambda_w = \delta_{w,w} \Lambda_w^* \langle P_w, P_{w'} \rangle = \langle P_w, DP_{w'} \rangle,
$$

for every $w, w' \in \mathbb{N}_0$, since $\Lambda_w$ is real and diagonal. This concludes the proof of the theorem. ■

9 The SO($2\ell + 1$)-type with highest weight $2\lambda_\ell$

In this section $K = \text{SO}(2\ell + 1)$. We will focus on the particular case when the $K$-type is given by an irreducible representation $\pi$ with highest weight $2\lambda_\ell = (1, 1, \ldots, 1)$. We will first see that such $K$-module is the direct sum of three $M$-modules, and we will find similar results to those obtained for the fundamental $K$-types $\lambda_1, \ldots, \lambda_{\ell-1}$ that are direct sum of two $M$-submodules.

Let us consider the irreducible $K$-module $\Lambda^\ell(V)$, with $V = \mathbb{C}^n$, $n = 2\ell + 1$. The vector $v = (e_1 - ie_2) \wedge (e_3 - ie_4) \wedge \cdots \wedge (e_{2\ell-1} - ie_{2\ell})$ is the unique, up to a scalar, dominant vector and its weight is $2\lambda_\ell = (1, 1, \ldots, 1)$.

It is not difficult to see that $\Lambda^\ell(V)$ is the sum of three $M$-irreducible submodules, namely

$$
\Lambda^\ell(V) = V_1 \oplus V_0 \oplus V_{-1}
$$

with respective highest weights $(1, \ldots, 1), (1, \ldots, 1, 0), (1, \ldots, 1, -1) \in \mathbb{C}^\ell$ and having $V_0 = \Lambda^{\ell-1}(V) \Lambda e_n$ and $V_1 \oplus V_{-1} \simeq \Lambda^\ell(\mathbb{C}^{n-1})$.

The vectors

$$
v_1 = (e_1 - ie_2) \wedge (e_3 - ie_4) \wedge \cdots \wedge (e_{2\ell-1} - ie_{2\ell}),
$$

$$
v_0 = -(e_1 - ie_2) \wedge (e_3 - ie_4) \wedge \cdots \wedge (e_{2\ell-3} - ie_{2\ell-2}) \wedge e_n,
$$

$$
v_{-1} = (e_1 - ie_2) \wedge (e_3 - ie_4) \wedge \cdots \wedge (e_{2\ell-1} + ie_{2\ell})
$$

are $M$-highest weight vectors in $V_1$, $V_0$ and $V_{-1}$, respectively. Also let us call $P_1$, $P_0$ and $P_{-1}$ the respective projections on $V_1$, $V_0$ and $V_{-1}$, according to the decomposition (9.1).

In order to obtain the explicit expression of $E$ in (3.2) we are interested to compute

$$
\sum_{j=1}^{n-1} \hat{\pi}(I_{n_j}) P_s \hat{\pi}(I_{n_j}) \big|_{V_r} = \lambda(r,s)I_{V_r},
$$

with $r, s = 1, 0, -1$ corresponding to the three $M$-submodules $V_1$, $V_0$ and $V_{-1}$ of the representation $\pi$. 

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If $1 \leq j \leq \ell$, then
\[
\hat{\pi}(I_{n,2j-1})(e_{2k-1} - ie_{2k}) = \begin{cases} 
0 & \text{if } k \neq j, \\
e_n & \text{if } k = j, 
\end{cases}
\]
\[
\hat{\pi}(I_{n,2j})(e_{2k-1} - ie_{2k}) = \begin{cases} 
0 & \text{if } k \neq j, \\
i e_n & \text{if } k = j, 
\end{cases}
\]
therefore, it is easy to see that $P_0 \hat{\pi}(I_{n,2j-1})v_0 = P_0 \hat{\pi}(I_{n,2j})v_0 = 0$ and that $P_r \hat{\pi}(I_{n,2j-1})v_s = P_r \hat{\pi}(I_{n,2j})v_s = 0$ when $s \pm 1$ and $r \pm 1$; i.e.
\[
\lambda(0,0) = \lambda(-1,-1) = \lambda(1,-1) = \lambda(-1,1) = \lambda(1,1) = 0.
\]
Furthermore, it is easy to see that, for $1 \leq j \leq \ell$ and $r$ equal to $1$ or $-1$, we have
\[
\hat{\pi}(I_{n,2j-1})P_0 \hat{\pi}(I_{n,2j-1})v_r + \hat{\pi}(I_{n,2j})P_0 \hat{\pi}(I_{n,2j})v_r = -v_r,
\]
then $\lambda(-1,0) = \lambda(1,0) = -\ell$. Therefore, it only remains to compute
\[
\sum_{j=1}^{\ell} (\hat{\pi}(I_{n,2j-1})P_s \hat{\pi}(I_{n,2j-1})v_0 + \hat{\pi}(I_{n,2j})P_s \hat{\pi}(I_{n,2j})v_0),
\]
for $s = \pm 1$.
To obtain $P_s \hat{\pi}(I_{n,k})v_0$ it is necessary to decompose $\hat{\pi}(I_{n,k})v_0$ according to the direct sum (9.1).
We know that $\hat{\pi}(X_{-\epsilon_j-\epsilon_{\ell}})v_1 \in V_1$ and $\hat{\pi}(X_{-\epsilon_j+\epsilon_{\ell}})v_{-1} \in V_{-1}$; recall that
\[
X_{-\epsilon_j-\epsilon_{\ell}} = I_{2\ell-1,2j-1} - I_{2\ell,2j} + i(I_{2\ell-1,2j} + I_{2\ell,2j-1}),
\]
\[
X_{-\epsilon_j+\epsilon_{\ell}} = I_{2\ell-1,2j-1} + I_{2\ell,2j} + i(I_{2\ell-1,2j} - I_{2\ell,2j-1}),
\]
see (2.2). We have
\[
\hat{\pi}(X_{-\epsilon_j-\epsilon_{\ell}})(e_{2j-1} - ie_{2j}) = -2(e_{2\ell-1} + ie_{2\ell}),
\]
\[
\hat{\pi}(X_{-\epsilon_j-\epsilon_{\ell}})(e_{2\ell-1} - ie_{2\ell}) = 2(e_{2j-1} + ie_{2j}),
\]
\[
\hat{\pi}(X_{-\epsilon_j-\epsilon_{\ell}})(e_{2k-1} - ie_{2k}) = 0, \quad \text{for } k \neq s, \ell.
\]
Therefore, for $1 \leq j \leq \ell$,
\[
\hat{\pi}(X_{-\epsilon_j-\epsilon_{\ell}})v_1 = 2(e_1 - ie_2) \land \cdots \land (e_{2(\ell-1)-1} - ie_{2(\ell-1)}) \land (e_{2j-1} + ie_{2j})
\]
\[
-2(e_1 - ie_2) \land \cdots \land (e_{2j-3} - ie_{2j-2}) \land (e_{2\ell-1} + ie_{2\ell}) \land (e_{2j+1} - ie_{2j+2}) \land 
\]
\[
\cdots \land (e_{2\ell-1} - ie_{2\ell}).
\]
Similarly, for $1 \leq j \leq \ell$,
\[
\hat{\pi}(X_{-\epsilon_j+\epsilon_{\ell}})v_1 = 2(e_1 - ie_2) \land \cdots \land (e_{2(\ell-1)-1} - ie_{2(\ell-1)}) \land (e_{2j-1} + ie_{2j})
\]
\[
+2(e_1 - ie_2) \land \cdots \land (e_{2j-3} - ie_{2j-2}) \land (e_{2\ell-1} + ie_{2\ell}) \land (e_{2j+1} - ie_{2j+2}) \land 
\]
\[
\cdots \land (e_{2\ell-1} - ie_{2\ell}).
\]
Hence, for $1 \leq j \leq \ell$, we have
\[
\frac{1}{8}(\hat{\pi}(X_{-\epsilon_j-\epsilon_{\ell}})v_1 + \hat{\pi}(X_{-\epsilon_j+\epsilon_{\ell}})v_{-1})
\]
\[
= (e_1 - ie_2) \land \cdots \land (e_{2(\ell-1)-1} - ie_{2(\ell-1)}) \land e_{2j} = \hat{\pi}(I_{n,2j})v_0,
\]
\[
\frac{1}{8}(\hat{\pi}(X_{-\epsilon_j-\epsilon_{\ell}})v_1 + \hat{\pi}(X_{-\epsilon_j+\epsilon_{\ell}})v_{-1})
\]
\[ = (e_1 - ie_2) \land \cdots \land (e_{2(\ell-1)} - ie_{2(\ell-1)}) \land e_{2j-1} = \hat{\pi}(I_{n,2j-1})v_0. \]

Then, for \( 1 \leq j < \ell, \)
\[
\hat{\pi}(I_{n,2j-1})P_1\hat{\pi}(I_{n,2j-1})v_0 = \frac{1}{\ell} \hat{\pi}(I_{n,2j-1}) \hat{\pi}(X_{-\varepsilon_j-\varepsilon_\ell})v_1 \\
= \frac{i}{\ell}(e_1 - ie_2) \land \cdots \land e_{2j} \land \cdots \land (e_{2(\ell-1)} - ie_{2(\ell-1)}) \land e_n,
\]
\[
\hat{\pi}(I_{n,2j})P_1\hat{\pi}(I_{n,2j})v_0 = -\hat{\pi}(I_{n,2j}) \hat{\pi}(X_{-\varepsilon_j-\varepsilon_\ell})v_1 \\
= -\frac{1}{\ell}(e_1 - ie_2) \land \cdots \land e_{2j-1} \land \cdots \land (e_{2(\ell-1)} - ie_{2(\ell-1)}) \land e_n.
\]

Therefore, for \( 1 \leq j < \ell, \)
\[
\hat{\pi}(I_{n,2j-1})P_1\hat{\pi}(I_{n,2j-1})v_0 + \hat{\pi}(I_{n,2j})v_0P_1\hat{\pi}(I_{n,2j})v_0 = -\frac{1}{\ell}v_0.
\]

Besides, for \( j = \ell \) we have
\[
\hat{\pi}(I_{n,2\ell})v_0 = \frac{1}{2\ell}(-v_1 + v_{-1}) \quad \text{and} \quad \hat{\pi}(I_{n,2\ell-1})v_0 = \frac{1}{2}(v_1 + v_{-1}).
\]

Therefore, since
\[
\hat{\pi}(I_{n,2\ell})P_1\hat{\pi}(I_{n,2\ell})v_0 = -\frac{1}{2\ell}\hat{\pi}(I_{n,2\ell})v_1 = -\frac{1}{2}v_0,
\]
\[
\hat{\pi}(I_{n,2\ell-1})P_1\hat{\pi}(I_{n,2\ell-1})v_0 = \frac{1}{\ell}\hat{\pi}(I_{n,2\ell-1})v_1 = -\frac{1}{2}v_0,
\]
we have that
\[
\sum_{j=0}^{n-1} \hat{\pi}(I_{n,j})P_1\hat{\pi}(I_{n,j})v_0 = -\frac{\ell + 1}{2}v_0,
\]
i.e.
\[
\lambda(0,1) = -\frac{\ell + 1}{2}.
\]

Analogously we obtain
\[
\lambda(0,-1) = -\frac{\ell + 1}{2}.
\]

Hence
\[
(\lambda(r,s))_{-\ell \leq r,s \leq \ell} = \begin{pmatrix}
0 & -\ell & 0 \\
-\ell & 0 & -\frac{\ell+1}{2} \\
0 & -\ell & 0
\end{pmatrix}.
\]

Therefore, we obtain a more explicit version of Corollary 3.6 using (3.2) and Remark 3.8.

Confront Corollary 5.2.

**Corollary 9.1.** Let \( \Phi \) be an irreducible spherical function on \( G \) of type \( \pi \in SO(n), n = 2\ell + 1. \)
If the highest weight of \( \pi \) is of the form \( (1, \ldots, 1) \in \mathbb{C}^\ell \), then the function \( H : (0,1) \rightarrow \mathbb{C}^3 \)
associated with \( \Phi \) satisfies \( \tilde{D}H = \lambda H \), for some \( \lambda \in \mathbb{C} \) with
\[
\tilde{D}H = y(1 - y)H''(y) + \frac{1}{2}n(1 - 2y)H'(y) + \frac{(1 - 2y)^2 + 1}{4y(1 - y)} \begin{pmatrix}
-\ell & 0 & 0 \\
0 & -\ell - 1 & 0 \\
0 & 0 & -\ell
\end{pmatrix} H(y)
\]
\[+ \frac{(1 - 2y)}{2y(1 - y)} \begin{pmatrix}
0 & -\ell & 0 \\
-\ell & 0 & -\frac{\ell+1}{2} \\
0 & -\ell & 0
\end{pmatrix} H(y).\]
9.1 Spherical functions of SO(2ℓ + 1)-type 2λℓ

Let \( n = 2\ell + 1 \), we now focus on the spherical functions \( \Phi_{w,\delta} \) of type \( \mathfrak{m}_n = (1, \ldots, 1) \in \mathbb{C}^\ell \), which are associated with the irreducible representations of SO\((n + 1)\) of highest weights of the form \( \mathfrak{m}_{n+1} = (w + 1, 1, \ldots, 1, \delta) \in \mathbb{C}^{\ell+1} \) such that the following pattern holds

\[
\begin{pmatrix}
w + 1 & 1 & \cdots & 1 & \delta \\
1 & \cdots & \cdots & 1 & -1
\end{pmatrix}
\]

As before we make the function \( \Psi \) whose columns are given by the spherical functions \( \Phi_{0,\delta} \), \( \delta = -1, 0, 1 \). When \( w = 0 \), this is calculable using \([24, \text{p. 364, equation (8)}]\) or alternatively by considering the \( G \)-modules \( \Lambda^{\ell+1}(\mathbb{C}^{n+1}) = V_1 \oplus V_{-1} \) and \( \Lambda^\ell(\mathbb{C}^{n+1}) = V_0 \) and working in the same way that we already did in the beginning of Section 6 for the \( 2 \times 2 \) cases (here \( V_t \), for \( t = 1, 0, -1 \), are the irreducible \( G \)-modules with highest weights \( (1, \ldots, 1, t) \in \mathbb{C}^{\ell+1} \)).

Therefore, if \( \cos s = 2y - 1 \) we have

\[
\Psi(y) = \begin{pmatrix}
e^{is} & 1 & e^{-is} \\
1 & \frac{1}{2}(e^{is} + e^{-is}) & 1 \\
e^{-is} & 1 & e^{is}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
2y - 1 + 2i\sqrt{y - y^2} & 1 & 2y - 1 - 2i\sqrt{y - y^2} \\
1 & 2y - 1 & 1 \\
2y - 1 - 2i\sqrt{y - y^2} & 1 & 2y - 1 + 2i\sqrt{y - y^2}
\end{pmatrix}
\]

Each column of \( \Psi \) satisfies the differential equation given in Corollary 9.1. And it is easy to check that we have

\[
y(1 - y)H''(y) + \frac{1}{2}n(1 - 2y)H'(y) + \frac{(1 - 2y)^2 + 1}{4y(1 - y)} \begin{pmatrix}
-\ell & 0 & 0 \\
0 & -\ell - 1 & 0 \\
0 & 0 & -\ell
\end{pmatrix} \Psi(y) \\
+ \frac{(1 - 2y)}{2y(1 - y)} \begin{pmatrix}
0 & -\ell & 0 \\
-\ell & 0 & -\ell + 1/2 \\
0 & -\ell & 0
\end{pmatrix} \Psi(y) = \Psi(y) \begin{pmatrix}
-\ell - 1 & 0 & 0 \\
0 & -\ell & 0 \\
0 & 0 & -\ell - 1
\end{pmatrix}.
\]

**Theorem 9.2.** The function \( \Psi \) can be used to obtain a hypergeometric differential equation from the one given in Corollary 9.1. Precisely, if \( H \) is a vector-valued solution of the differential equation in Corollary 9.1, with eigenvalue \( \lambda \), then \( P = \Psi^{-1}H \) is a solution of \( DP = \lambda P \), where \( D \) is the hypergeometric differential operator given by

\[
DP = y(1 - y)P'' + (C - yU)P' - VP,
\]

with

\[
C = \begin{pmatrix}
(n + 2)/2 & 1/2 & 0 \\
1 & (n + 2)/2 & 1 \\
0 & 1/2 & (n + 2)/2
\end{pmatrix}, \quad U = (n + 2)I,
\]

\[
V = \begin{pmatrix}
-\ell - 1 & 0 & 0 \\
0 & -\ell & 0 \\
0 & 0 & -\ell - 1
\end{pmatrix}.
\]

**Proof.** Let us write \( H = \Psi P \). Then

\[
y(1 - y)P'' + \left(2y(1 - y)\Psi^{-1}\Psi' + \frac{n}{2}(1 - 2y)I\right)P'
\]
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\[ + \Psi^{-1}\left( y(1-y)\Psi'' + \frac{n}{2}(1-2y)\Psi' + \frac{1+(1-2y)^2}{4y(1-y)} \begin{pmatrix} -\ell & 0 & 0 \\ 0 & -\ell - 1 & 0 \\ 0 & 0 & -\ell \end{pmatrix} \Psi \right) \\
+ \frac{(1-2y)}{2y(1-y)} \begin{pmatrix} 0 & -\ell & 0 \\ 0 & 0 & -\ell - 1 \\ -\ell & 0 & 0 \end{pmatrix} \Psi \right) P = \lambda P. \]

Now we compute

\[ 2y(1-y)\Psi^{-1}\Psi' + \frac{n}{2}(1-2y)I = -(n+2)yI + \begin{pmatrix} (n+2)/2 & 1/2 & 0 \\ 1 & (n+2)/2 & 1 \\ 0 & 1/2 & (n+2)/2 \end{pmatrix}. \]

Therefore

\[ y(1-y)P'' + (\begin{pmatrix} (n+2)/2 & 1/2 & 0 \\ 1 & (n+2)/2 & 1 \\ 0 & 1/2 & (n+2)/2 \end{pmatrix}) P' \]
\[ + \begin{pmatrix} -\ell - 1 & 0 & 0 \\ 0 & -\ell & 0 \\ 0 & 0 & -\ell - 1 \end{pmatrix} - \lambda I \right) P = 0. \]

This completes the proof of the theorem. \qed

We obtain a similar result to Theorem 6.3, with an analogous proof:

**Theorem 9.3.** For a given $\ell \in \mathbb{N}$ let $n = 2\ell + 1$, then the nonzero polynomial eigenfunctions of

\[ DP = y(1-y)P'' + (C - yU)P' - VP, \]

with

\[ C = \begin{pmatrix} (n+2)/2 & 1/2 & 0 \\ 1 & (n+2)/2 & 1 \\ 0 & 1/2 & (n+2)/2 \end{pmatrix}, \quad U = (n+2)I, \]
\[ V = \begin{pmatrix} -\ell - 1 & 0 & 0 \\ 0 & -\ell & 0 \\ 0 & 0 & -\ell - 1 \end{pmatrix}, \]

have eigenvalues $-w(w + n + 1) - \ell$ or $-w(w + n + 1) - \ell - 1$, with $w \in \mathbb{N}_0$. In both cases the degree of the polynomial is $w$ and the leading coefficient can be any multiple of \begin{pmatrix} 1 \\ 0 \end{pmatrix} or any linear combination of \begin{pmatrix} 1 \\ 0 \end{pmatrix} and \begin{pmatrix} 0 \\ 1 \end{pmatrix}, respectively.

Let us consider $\widetilde{D}$, the differential operator on $(0,1)$ introduced in Corollary 9.1:

\[ \widetilde{D}H = y(1-y)H''(y) + \frac{1}{2}n(1-2y)H'(y) \]
\[ + \frac{(1-2y)^2+1}{4y(1-y)} \begin{pmatrix} -\ell & 0 & 0 \\ 0 & -\ell - 1 & 0 \\ 0 & 0 & -\ell \end{pmatrix} H(y) + \frac{(1-2y)}{2y(1-y)} \begin{pmatrix} 0 & -\ell & 0 \\ -\ell & 0 & -\ell - 1 \end{pmatrix} H(y). \]

Recall that the operator $D$ that appears in Theorem 9.3 extends the differential operator $D = \Psi \widetilde{D} \Psi^{-1}$ to the whole real line.
We want to focus our attention on the following vector spaces of \( C^3 \)-valued analytic functions on \((0,1)\):

\[
S_\lambda = \{ H = H(y) : \tilde{D}H = \lambda H, \ H(\cos s + 1/2) \text{ analytic at } s = 0 \},
\]

\[
W_\lambda = \{ P = P(y) : DP = \lambda P, \text{ analytic on } [0,1] \}.
\]

From Theorem 9.2 we know that the correspondence \( P \mapsto \Psi P \) is an injective linear map from \( W_\lambda \) into \( S_\lambda \). In fact, \( \Psi((\cos s + 1)/2) \) is analytic as a function of \( s \) and \( P \) is analytic at \( y = 1 \), hence \( H((\cos s + 1)/2) = (\Psi P)((\cos s + 1)/2) \) is analytic at \( s = 0 \).

Then, we have an analogous result to Theorem 7.3, whose proof is quite similar and therefore we will omit it.

**Theorem 9.4.** The linear map \( P \mapsto \Psi P \) is an isomorphism from \( W_\lambda \) onto \( S_\lambda \).

Now, we can easily make a proof similar to that one of Theorem 7.4 in order to obtain next theorem.

**Theorem 9.5.** Let \( H \) be the \( C^3 \)-valued analytic function on \((0,1)\) given by an irreducible spherical function \( \Phi \) on \( SO(2\ell + 2) \) of fundamental \( SO(2\ell + 1) \)-type \((1,\ldots,1) \in \mathbb{C}^\ell \). If \( P = \Psi^{-1}H \), then \( P \) is polynomial.

For a given fundamental \( K \)-type \( \pi \in \hat{SO}(n) \), \( n = 2\ell + 1 \), with highest weight \((1,\ldots,1) \in \mathbb{C}^\ell \), let \( \Phi_{w,\delta} \) denote the irreducible spherical function of the pair \((SO(n+1),SO(n))\) given by \( \tau \in \hat{SO}(n+1) \) with highest weight of the form \((w+1,1,\ldots,1,\delta) \in \mathbb{C}^{\ell+1} \), \( \delta = -1,0,1 \).

Now, combining Theorems 9.3, 9.5 and the expression of the eigenvalue \( \lambda_n(w,\delta) \) given in (6.2) we have the following statement.

**Theorem 9.6.** Given \( w \in \mathbb{N} \), every irreducible spherical function \( \Phi_{w,\delta} \) of the pair \((SO(n+1),SO(n))\) with \( n = 2\ell + 1 \), of type \( m_n = (1,\ldots,1) \in \mathbb{C}^\ell \), corresponds to a vector-valued function \( P_{w,\delta} \) \((w \geq 0, \delta = -1,0,1)\), which is a polynomial of degree \( w \). The leading coefficients of \( P_{w,0} \) is a multiple of \( \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \) and the leading coefficients of \( P_{w,-1} \) and \( P_{w,1} \) are both linear combinations of \( \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) \) and \( \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \). Precisely

\[
P_{w,\delta}(y) = \sum_{j=0}^{w} \frac{y^j}{j!} [C;U;V + \lambda]_j P_{w,\delta}(0),
\]

with

\[
C = \begin{pmatrix}
\frac{(n+2)/2}{1} & \frac{1/2}{1} & 0 \\
1 & \frac{(n+2)/2}{1} & 1 \\
0 & \frac{1/2}{1} & \frac{(n+2)/2}{1}
\end{pmatrix},
\]

\[
U = (n+2)I, \quad V = \begin{pmatrix}
-\ell & 1 & 0 & 0 \\
0 & -\ell & 0 & 0 \\
0 & 0 & -\ell - 1
\end{pmatrix},
\]

\[
\lambda = \lambda_n(w,\delta) = \begin{cases}
-w(w + n + 1) - \ell & \text{if } \delta = 0, \\
-w(w + n + 1) - \ell - 1 & \text{if } \delta = \pm 1.
\end{cases}
\]

Even more, the value of \( P_{w,\delta}(0) \) can be computed.
Proof. It only remains to prove that $P_{w,\delta}(0)$ can be computed.

Let us consider the case $\delta = 0$. We know from (6.2) and Theorem 9.3 that there is some $c \in \mathbb{C}$ such that

$$[C; U; V + \lambda]_w P_{w,0}(0) = c \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$ 

Since $[C; U; V + \lambda]_w$ is invertible this $c$ is univocally determined by the condition $\Phi(e) = I$ which implies

$$\Psi(1) \sum_{j=0}^{w} \frac{1}{j!} [C; U; V + \lambda]_j P_{w,0}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$ 

Now let us consider the cases $\delta = \pm 1$. We know from (6.2) and Theorem 9.3 that

$$[C; U; V + \lambda]_w P_{w,\delta}(0) \in \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle;$$

since $[C; U; V + \lambda]_w$ is invertible, this condition tells us that $P_{w,\delta}(0)$ belongs to a plane which contains the origin and does not depend on $\delta$.

Besides, the condition $\Phi_{w,\delta}(e) = I$, for $\delta = \pm 1$, tells us

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sum_{j=0}^{w} \frac{1}{j!} [C; U; V + \lambda]_j P_{w,\delta}(0).$$

Then, $P_{w,\delta}(0)$ belongs to a plane, parallel to the kernel of

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sum_{j=0}^{w} \frac{1}{j!} [C; U; V + \lambda]_j,$$

which does not contain the origin and does not depend on $\delta$. Therefore we know that both $P_{w,1}(0)$ and $P_{w,-1}(0)$ are in the same straight line.

On the other hand, recall that we have

$$\Phi_{w,\delta}(a(s)) = \Psi \left( \frac{\cos s + 1}{2} \right) P_{w,\delta} \left( \frac{\cos s + 1}{2} \right),$$

where

$$a(s) = \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & \cos s & \sin s \\ 0 & -\sin s & \cos s \end{pmatrix},$$

then

$$\left. \frac{d}{ds} \Phi(a(s)) \right|_{s=0} = \begin{pmatrix} i & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & i \end{pmatrix} P_{w,\delta}(1).$$
From [24, p. 364, equation (8)] we can easily compute \( \frac{d}{ds} \Phi_{w,\delta}(a(s)) \) at \( s = 0 \), which is obtained by looking at the action of \( \dot{\tau}(I_{n+1,n}) \) and considering the corresponding projection, see (2.1); having then

\[
\frac{i(w + \ell + 1)}{1 + \ell} \begin{pmatrix} -1 & 0 & -i \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix} = \sum_{j=0}^{w} \frac{1}{j!} [C; U; V + \lambda]_{j} P_{w,\delta}(0).
\]

This last condition establishes that \( P_{w,1}(0) \) and \( P_{w,-1}(0) \) are in two different and parallel planes, and the line mentioned above does not belong to any of them since each plane has to intersect it. Therefore the values of \( P_{w,1}(0) \) and \( P_{w,-1}(0) \) are univocally determined.

9.2 Matrix-valued orthogonal polynomials of size 3

In this subsection, given \( n \) of the form \( 2\ell + 1 \) with \( \ell \in \mathbb{N} \), we shall construct a sequence of matrix-valued polynomials \( \{P_w\}_{w \geq 0} \) directly related to irreducible spherical functions of type \( \pi \in \hat{\text{SO}}(n) \) of highest weight \( m_\pi = (1, \ldots, 1) \in \mathbb{C}^\ell \).

Given a nonnegative integer \( w \) and \( \delta = -1, 0, 1 \), we can consider \( \Phi_{w,\delta} \), the irreducible spherical function of type \( \pi \) associated with the irreducible representation \( \tau \in \hat{\text{SO}}(n+1) \) of highest weight of the form \( m_\tau = (w + 1, 1, \ldots, 1, \delta) \).

We insist on recalling that, since \( \pi \) has only three \( \text{SO}(2\ell) \)-submodules, we can interpret the diagonal matrix-valued function \( \Phi_{w,\delta}(a(s)), s \in (0, \pi) \), as a 3 column vector function.

Now we consider the vector-valued function

\[
P_{w,\delta} : (0, 1) \to \mathbb{C}^3
\]

given by the vector function \( P_{w,\delta}(y) = \Psi^{-1}(y) \Phi_{w,\delta}(a(s)) \), with \( \cos(s) = 2y - 1 \). Then, we define the matrix-valued function

\[
P_w = P_w(y),
\]

whose \( \delta \)-th column \( (\delta = -1, 0, 1) \) is given by the \( \mathbb{C}^3 \)-valued polynomial \( P_{w,\delta}(y) \).

Let consider the matrix-valued skew symmetric bilinear form defined among continuous \( 3 \times 3 \) matrix-valued functions on \([0, 1]\) by

\[
\langle P, Q \rangle_W = \int_0^1 Q^*(y)W(y)P(y)dy,
\]

where the \( 3 \times 3 \) weight-matrix \( W \) is given by

\[
W(y) = \frac{(n-1)!!}{(n-2)!!}(y(1-y))^{n/2-1}\Psi^*(y)\begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \Psi(y)
\]

with

\[
d_1 = d_3 = \frac{(2\ell + 1)!}{\ell!(\ell + 2)!}, \quad d_2 = \frac{(2\ell + 1)!}{\ell!(\ell + 1)!},
\]

and

\[
\Psi(y) = \begin{pmatrix} 2y - 1 + 2i\sqrt{y - y^2} & 1 & 2y - 1 - 2i\sqrt{y - y^2} \\ 1 & 2y - 1 & 1 \\ 2y - 1 - 2i\sqrt{y - y^2} & 1 & 2y - 1 + 2i\sqrt{y - y^2} \end{pmatrix}
\]

Let us recall that, from Proposition 7.1, we have

\[
\langle \Phi_{w,\delta}, \Phi_{w',\delta} \rangle = \int_0^1 P_{w,\delta}^* W(y)P_{w',\delta} dy.
\]
Remark 9.7. Notice that $W$ reduces to a smaller size: if $M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}$, we have

$W(y)M^* = \frac{(n - 1)!!}{(n - 2)!!}(y(1 - y))^{n/2 - 1/4}
\times \begin{pmatrix}
2d_1(2y - 1)^2 + d_2 & d_1(2y - 1) + d_2(2y - 1)/\sqrt{2} & 0 \\
d_1(2y - 1)/\sqrt{2} + d_2(2y - 1)/\sqrt{2} & d_1 + d_2(2y - 1)^2/2 & 0 \\
0 & 0 & d_18(y - y^2)
\end{pmatrix}.$

Then we state the following theorem.

Theorem 9.8. The matrix-valued polynomial functions $P_w$, $w \geq 0$, form a sequence of orthogonal polynomials with respect to $W$, which are eigenfunctions of the symmetric differential operator $D$ from Theorem 9.2. Moreover,

$$DP_w = \left(\lambda(w_{-1}) \ 0 \ \ 0 \\
0 \ \ \lambda(w, 0) \ \ 0 \\
0 \ \ 0 \ \ \lambda(w, 1)\right),$$

where

$$\lambda(w, \delta) = \begin{cases}
-w(w + n + 1) - p & \text{if } \delta = 0, \\
-w(w + n + 1) - n + p & \text{if } \delta = \pm 1.
\end{cases}$$

Proof. The proof is completely analogous to the proof of Theorem 8.1 □

Appendix

Proof of Proposition 3.2. For $|t|$ sufficiently small $A(s, t)$ is close to the identity of $K$, i.e. to the identity matrix $I_n$. So we can consider the function

$$X(s, t) = \log(A(s, t)) = B(s, t) - \frac{B(s, t)^2}{2} + \frac{B(s, t)^3}{3} - \cdots, \quad (9.2)$$

where $B(s, t) = A(s, t) - I_n$. Then

$$\pi(A(s, t)) = \pi(\exp X(s, t)) = \exp \hat{\pi}(X(s, t)) = \sum_{j \geq 0} \frac{\hat{\pi}(X(s, t))^j}{j!}.$$

Now we differentiate with respect to $t$ to obtain

$$\frac{\partial (\pi \circ A)}{\partial t} = \hat{\pi} \left( \frac{\partial X}{\partial t} \right) + \frac{1}{2!} \hat{\pi} \left( \frac{\partial X}{\partial t} \right) \hat{\pi}(X) + \frac{1}{3!} \hat{\pi}(X) \hat{\pi} \left( \frac{\partial X}{\partial t} \right) + \frac{1}{4!} \hat{\pi}(X)^2 \hat{\pi} \left( \frac{\partial X}{\partial t} \right) + \cdots. \quad (9.3)$$

Since $X(s, 0) = 0$, if we differentiate (9.2) with respect to $t$ and evaluate at $(s, 0)$ we obtain

$$\left. \frac{\partial^2 (\pi \circ A)}{\partial t^2} \right|_{t=0} = \hat{\pi} \left( \frac{\partial^2 X}{\partial t^2} \bigg|_{t=0} \right) + \frac{1}{2} \hat{\pi} \left( \frac{\partial X}{\partial t} \bigg|_{t=0} \right)^2.$$

To compute $\frac{\partial X}{\partial t} \bigg|_{t=0}$ and $\frac{\partial^2 X}{\partial t^2} \bigg|_{t=0}$ we differentiate (9.2) and we get

$$\frac{\partial X}{\partial t} = \frac{\partial B}{\partial t} - \frac{1}{2} \left( \frac{\partial B}{\partial t} \right) B - \frac{1}{2} B \left( \frac{\partial B}{\partial t} \right) + \frac{1}{2} \left( \frac{\partial B}{\partial t} \right) B^2 + \frac{1}{3} B \left( \frac{\partial B}{\partial t} \right) B + \frac{1}{4} B^2 \left( \frac{\partial B}{\partial t} \right) + \cdots.$$
Since \( B(s, 0) = 0 \) we have
\[
\frac{\partial X}{\partial t} \bigg|_{t=0} = \frac{\partial B}{\partial t} \bigg|_{t=0} = \frac{\partial A}{\partial t} \bigg|_{t=0}.
\]

We also get
\[
\frac{\partial^2 X}{\partial t^2} \bigg|_{t=0} = \frac{\partial^2 A}{\partial t^2} \bigg|_{t=0} - \left( \frac{\partial A}{\partial t} \bigg|_{t=0} \right)^2.
\]

Now we will first consider the case \( A(s, t) = k(s, t) \). A direct computation yields to
\[
\frac{\partial k}{\partial t} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
-\sin s \sin t & 0 & 0 & 0 & 0 \\
\frac{\sin^2 s \cos t}{(1-\cos^2 s \cos^2 t)^{3/2}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
in particular \( \frac{\partial k}{\partial t} \bigg|_{t=0} = \frac{1}{\sin^2 s} I_{n,j} \). Differentiating once more with respect to \( t \) and evaluating at \( t = 0 \) we obtain
\[
\frac{\partial^2 k}{\partial t^2} \bigg|_{t=0} = -\frac{1}{\sin^2 s} \left( E_{jj} + E_{n,n} \right). \quad \text{Then we get}
\]
\[
\frac{\partial^2 A}{\partial t^2} \bigg|_{t=0} - \left( \frac{\partial A}{\partial t} \bigg|_{t=0} \right)^2 = -\frac{1}{\sin^2 s} \left( E_{jj} + E_{n,n} \right) - \frac{1}{\sin^2 s} I_{n,j}^2 = 0.
\]

Similarly when \( A(s, t) = h(s, t) \) we obtain
\[
\frac{\partial h}{\partial t} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\frac{-\cos s \cos t \sin t}{(1-\cos^2 s \cos^2 t)^{3/2}} & 0 & 0 & 0 & 0 \\
\frac{\cos s \cos t \sin^2 s}{(1-\cos^2 s \cos^2 t)^{3/2}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
in particular \( \frac{\partial h}{\partial t} \bigg|_{t=0} = -\frac{\cos^2 s}{\sin^2 s} I_{n,j} \). Differentiating once more with respect to \( t \) and evaluating at \( t = 0 \) we obtain
\[
\frac{\partial^2 h}{\partial t^2} \bigg|_{t=0} = -\frac{\cos^2 s}{\sin^2 s} \left( E_{jj} + E_{n,n} \right). \quad \text{Then we get}
\]
\[
\frac{\partial^2 A}{\partial t^2} \bigg|_{t=0} - \left( \frac{\partial A}{\partial t} \bigg|_{t=0} \right)^2 = -\frac{\cos^2 s}{\sin^2 s} \left( E_{jj} + E_{n,n} \right) - \frac{\cos^2 s}{\sin^2 s} I_{n,j}^2 = 0.
\]

Proposition follows.

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