LOSS OF MEMORY OF RANDOM FUNCTIONS OF MARKOV
CHAINS AND LYAPUNOV EXPONENTS

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Abstract. In this paper we prove that the asymptotic rate of exponential loss of
memory of a random function of a Markov chain \((Z_t)_{t \in \mathbb{Z}}\) is bounded above by the
difference of the first two Lyapunov exponents of a certain product of matrices. We
also show that this bound is in fact realized, namely for almost all realization of the
process \((Z_t)_{t \in \mathbb{Z}}\), we can find symbols where the asymptotic exponential rate of loss
of memory attains the difference of the first two Lyapunov exponents. This shows
that the process has infinite memory and leads to a lower bound on the asymptotic
exponential loss of memory which is saturated (and equal to the upper bound for an
adequate choice of the symbols) on a set of full measure.

1. Introduction

Let \((X_t)_{t \in \mathbb{Z}}\) be a Markov chain over a finite alphabet \(\mathcal{A}\). We consider a probabilistic
function \((Z_t)_{t \in \mathbb{Z}}\) of this chain, a model introduced by Petrie (1969). More precisely,
there is another alphabet \(\mathcal{B}\) and for any \(X_t\) we choose at random a \(Z_t\) in \(\mathcal{B}\). The
random choice of \(Z_t\) depends only on the value \(X_t\) of the original process at time \(t\).

We are interested in the asymptotic loss of memory of the process \((Z_t)_{t \in \mathbb{Z}}\). For
example, if the conditional probability of \(Z_t\) given \(X_t\) does not depend on \(X_t\), the
process \((Z_t)_{t \in \mathbb{Z}}\) is an independent process. Another trivial example is when there is no
random choice, namely \(Z_t = X_t\), in this case the process \((Z_t)_{t \in \mathbb{Z}}\) is Markovian. However
as we will see, under natural assumptions, the process \((Z_t)_{t \in \mathbb{Z}}\) has infinite memory. Our
goal is to investigate how fast this process looses memory.

Exponential upper bounds for this asymptotic loss of memory have been obtained
in various papers, see for example Douc et al. (2009a,b), and references therein. For

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the case of projections of Markov chains and the relation with Gibbs measures, see Chazottes & Ugalde (2009) and references therein.

In the present paper, under generic assumptions, we prove that the asymptotic rate of exponential loss of memory is bounded above by the difference of the first two Lyapunov exponents of a certain product of matrices. We also show that this bound is in fact realized, namely for almost all realization of the process \((Z_t)_{t \in \mathbb{Z}}\), we can find symbols where the asymptotic exponential rate of loss of memory attains the difference of the first two Lyapunov exponents. This shows that the process has infinite memory and leads to a lower bound on the asymptotic exponential loss of memory which is saturated (and equal to the upper bound for an adequate choice of the symbols) on a set of full measure.

As an application, we consider the case of a randomly perturbed Markov chain with two symbols. We show that the asymptotic rate of loss of memory can be expanded in powers of the perturbation with a logarithmic singularity. This was our original motivation coming from our previous work with A. Galves (Collet et al., 2008).

The content of the paper is as follows. In Section 2 we give a precise definition of the asymptotic exponential rate of loss of memory and state the main results about the relation of this rate with the first two Lyapunov exponents. Proofs are given in Section 3. In Section 4 we give the application to the random perturbation of a two states Markov chain.

2. Definitions and main results

Let \((X_t)_{t \in \mathbb{Z}}\) be an irreducible aperiodic Markov chain over a finite alphabet \(\mathcal{A}\) with transition probability matrix \(p(\cdot|\cdot)\) and unique invariant measure \(\pi\). Without loss of generality we will assume \(\mathcal{A} = \{1,2,\ldots,k\}\). Consider another finite alphabet \(\mathcal{B} = \{1,2\ldots,\ell\}\), with \(\ell \geq k\), and a process \((Z_t)_{t \in \mathbb{Z}}\), a probabilistic function of the Markov chain \((X_t)_{t \in \mathbb{Z}}\) over \(\mathcal{B}\). That is, there exists a matrix \(q(\cdot|\cdot) \in \mathbb{R}^{k \times \ell}\) such that
for any $n \geq 0$, any $z^n_0 \in B^{n+1}$ and any $x^n_0 \in A^{n+1}$ we have

$$\mathbb{P}(Z^n_0 = z^n_0 | X^n_0 = x^n_0) = \prod_{i=0}^{n} \mathbb{P}(Z_i = z_i | X_i = x_i) = \prod_{i=0}^{n} q(z_i | x_i). \tag{2.1}$$

From now on, the symbol $z$ will represent an element in $B^\mathbb{Z}$. Define the shift-operator $\mathcal{S}: B^\mathbb{Z} \to B^\mathbb{Z}$, by

$$(\mathcal{S}z)_n = z_{n+1}.$$ 

The shift is invertible and its inverse is given by

$$(\mathcal{S}^{-1}z)_n = z_{n-1}.$$ 

To state our results we will need the following hypothesis.

(H1) $\min_{i,j} p(j|i) > 0$, $\min_{i,m} q(m|i) > 0$.

(H2) $\det(p) \neq 0$, $\text{rank}(q) = k$.

For the convenience of the reader we recall Oseledec’s theorem in finite dimension, see for example Ledrappier (1984). As usual, we denote by $\log^+(x) = \max(\log(x), 0)$.

**Oseledec’s theorem.** Let $(\Omega, \mu)$ be a probability space and let $T$ be a measurable transformation of $\Omega$ such that $\mu$ is $T$-ergodic. Let $L_\omega$ be a measurable function from $\Omega$ to $\mathcal{L}(\mathbb{R}^k)$ (the space of linear operators of $\mathbb{R}^k$ into itself). Assume the function $L_\omega$ verifies

$$\int \log^+ \| L_\omega \| d\mu(\omega) < +\infty.$$ 

Then, there exist $\lambda_1 > \lambda_2 > \ldots > \lambda_s$, with $s \leq k$ and there exists an invariant set $\tilde{\Omega} \subset \Omega$ of full measure ($\mu(\Omega \setminus \tilde{\Omega}) = 0$) such that for all $\omega \in \tilde{\Omega}$ there exist $s+1$ sub-vector spaces

$$\mathbb{R}^k = V^{(1)}_\omega \supset V^{(2)}_\omega \supset \ldots \supset V^{(s+1)}_\omega = \{\vec{0}\}$$

such that for any $\vec{v} \in V^{(j)}_\omega \setminus V^{(j+1)}_\omega$ ($1 \leq j \leq s$) we have

$$\lim_{n \to +\infty} \frac{1}{n} \log \| L^{[n]}_\omega \vec{v} \| = \lambda_j,$$
where \( L_\omega^{[n]} = L_{T^{n-1}(\omega)} \ldots L_\omega \). Moreover, the subspaces satisfy the relation

\[
L_\omega V^{(j)}_\omega \subseteq V^{(j)}_{T_\omega}.
\]

The numbers \( \lambda_1, \lambda_2, \ldots, \lambda_s \) are called the Lyapunov exponents.

In the sequel we will use this theorem with \( \Omega = \mathcal{B}^\mathbb{Z} \), \( \mu \) the stationary ergodic measure of the process \( (Z_t)_{t \in \mathbb{Z}} \) (Cappé et al., 2005), \( T = \mathcal{S}^{-1} \) and \( L_z \) the linear operator in \( \mathbb{R}^k \) with matrix given by

\[
(L_z)_{i,j} = q(z_0|j)p(j|i).
\]

From now on we will use the \( \ell^2 \) norm \( \| \cdot \| \) and the corresponding scalar product on \( \mathbb{R}^k \). Note that from our definition of \( L_z \) we have

\[
\sup_{z} \| L_z \| < +\infty.
\]

Therefore we can apply Oseledec’s theorem to get the existence of the Lyapunov exponents.

For any \( \tilde{z} \in \mathcal{B}^\mathbb{Z} \), \( a \in \mathcal{A} \) and \( b, c \in \mathcal{B} \) define

\[
\Delta_a^{[n]}(\tilde{z}) = \mathbb{P}(X_0 = a \mid Z_{-n+1} = z_{-n+1}, Z_{-n} = b) - \mathbb{P}(X_0 = a \mid Z_{-n+1} = z_{-n+1}, Z_{-n} = c)
\]

and the asymptotic exponential rate \( \tau_{\tilde{z}}(a, b, c) \) given by

\[
\tau_{\tilde{z}}(a, b, c) = \limsup_{n \to +\infty} \frac{1}{n} \log |\Delta^{[n]}_{a,b,c}(\tilde{z})|.
\]

We can state now our main results.

**Theorem 2.2.** Under the hypothesis (H1), for each \( a \in \mathcal{A} \), \( b, c \in \mathcal{B} \),

\[
\tau_{\tilde{z}}(a, b, c) \leq \lambda_2 - \lambda_1,
\]

\( \mu \)-almost surely.
Remark. When \( \mathcal{A} = \mathcal{B} \) and \( q \) is the identity matrix, \((Z_t) = (X_t)\) is a Markov chain. The second part of hypothesis (H1) does not hold, but it is easy to adapt the proof of Theorem 2.2 for this particular case. It is easy to verify recursively that the matrices \( L^{[n]} \) are of rank one. The Lyapunov exponents can be computed explicitly. One gets \( \lambda_1 = -H(p) \) (the entropy of the Markov chain with transition probability \( p \)) from the ergodic Theorem, and \( \lambda_2 = -\infty \) with multiplicity \( k - 1 \).

**Theorem 2.3.** Under hypothesis (H1-H2), for \( \mu \) almost all \( z \) there exists \( a \in \mathcal{A} \), \( b, c \in \mathcal{B} \) (which may depend on \( z \)) such that
\[
\tau_z(a, b, c) = \lambda_2 - \lambda_1.
\]

As a corollary, we derive equivalent results for the loss of memory of the process \((Z_t)_{t \in \mathbb{Z}}\), which was our main goal. Define for any \( e, b, c \in \mathcal{B} \) and any \( z \in \Omega \) the value
\[
\Delta^{[n]}_{e,b,c}(z) = \mathbb{P}(Z_0 = e \mid Z_{-n+1}^{-1} = z_{-n+1}^{-1}, Z_{-n} = b) - \mathbb{P}(Z_0 = e \mid Z_{-n+1}^{-1} = z_{-n+1}^{-1}, Z_{-n} = c)
\]
and the exponential rate of loss of memory of the process \((Z_t)_{t \in \mathbb{Z}}\) given by
\[
\hat{\tau}_z(e, b, c) = \limsup_{n \to +\infty} \frac{1}{n} \log |\Delta^{[n]}_{e,b,c}(z)|.
\]

**Corollary 2.4.** Under the hypothesis (H1), for each \( e, b, c \in \mathcal{B} \),
\[
\hat{\tau}_z(e, b, c) \leq \lambda_2 - \lambda_1,
\]
m\( \mu \)-almost surely. Moreover, under hypothesis (H1-H2), for \( \mu \) almost all \( z \) there exists \( e, b, c \in \mathcal{B} \) (which may depend on \( z \)) such that
\[
\hat{\tau}_z(e, b, c) = \lambda_2 - \lambda_1.
\]

From a practical point of view, one can prove various lower bounds for the quantity \( \lambda_2 - \lambda_1 \). As an example we give the following result.

Let
\[
R = \frac{1}{\min_{m,i} \{q(m|i)\}}.
\]
Proposition 2.5. Under hypothesis (H1-H2) we have
\[ \lambda_2 - \lambda_1 \geq \frac{1}{k-1} \log | \det(p) | - \frac{k}{k-1} \log R. \]

3. Proofs

We begin by proving some lemmas which will be useful later. We introduce the order 
\((\mathbb{R}^k, \leq)\) given by \(\vec{v} \leq \vec{w}\) if and only if \(v_i \leq w_i\) for all \(i = 1, \ldots, k\). When needed, we will also make use of the symbols \(<, >\) and \(\geq\), defined in an analogous way. Note that since the matrices \(L_z\) have strictly positive entries, if \(\vec{v} \leq \vec{w}\) then \(L_z \vec{v} \leq L_z \vec{w}\). We will use the notation \(\vec{1} \in \mathbb{R}^k\) for the vector with components \((\vec{1})_i = 1\) for each \(i = 1, \ldots, k\).

Lemma 3.1. Under hypothesis (H1), if \(\vec{\xi} \in V^{(2)}_z \setminus \{\vec{0}\}\) then \(\vec{\xi}\) has two non-zero components of opposite signs, \(\mu\)-almost surely.

Proof. Assume there exits \(\vec{\xi} \in V^{(2)}_z \setminus \{\vec{0}\}\) with \(\vec{\xi}_i \geq 0\) for all \(i = 1, \ldots, k\). Then, from hypothesis (H1) it follows that there exists \(\alpha > 0\) such that, for all \(z\),

\[ L_z \vec{\xi} \geq \alpha \|\vec{\xi}\| \vec{1}. \]

One may take, for example,

\[ \alpha = \frac{1}{\sqrt{k}} \inf_{z_0 \in (i,j)} q(z_0 | j)p(j | i) = \frac{1}{\sqrt{k}} \inf_{z_0 \in (i,j)} (L_z)_{i,j}. \]

We can apply \(L_{[n-1]}^{[n]}\) to both sides, use monotonicity and take norms, to obtain

\[ \| L_z^{[n]} \vec{\xi} \| \geq \alpha \|\vec{\xi}\| \| L_{[n-1]}^{[n]} \vec{1} \|. \]

Let \(\vec{w} \in V^{(1)}_{[n-1]} \setminus V^{(2)}_{[n-1]}\). Then

\[ \| L_{[n-1]}^{[n]} \vec{w} \| \leq \| L_{[n-1]}^{[n]} \vec{w} \| \leq \| \vec{w} \| \| L_{[n-1]}^{[n]} \vec{1} \| \leq \frac{\| \vec{w} \| \| L_{z}^{[n]} \vec{\xi} \|}{\alpha \|\vec{\xi}\|}. \]

Therefore

\[ \| L_z^{[n]} \vec{\xi} \| \geq \frac{\alpha \|\vec{\xi}\|}{\| \vec{w} \|} \| L_{[n-1]}^{[n]} \vec{w} \|. \]

and using Oseledece’s theorem we have \(\mu\) almost surely that

\[ \lim_{n \to +\infty} \frac{1}{n} \log \| L_z^{[n]} \vec{\xi} \| \geq \lim_{n \to +\infty} \frac{1}{n} \log \| L_{[n-1]}^{[n]} \vec{w} \| = \lambda_1, \]
which contradicts the fact that $\vec{\xi} \in V^{(2)}_\zeta \setminus \{\vec{0}\}$. \qed

**Lemma 3.2.** Under hypothesis (H1) we have $\text{Codim}(V^{(2)}_\zeta) = 1$, $\mu$-almost surely.

**Proof.** Assume $\text{Codim}(V^{(2)}_\zeta) \geq 2$. Since any vector $\vec{w}_1$ of norm one in the cone $\mathcal{C}_k = \{\vec{w}: \vec{w} > 0\}$ does not belong to $V^{(2)}_\zeta$ (by Lemma 3.1), the vector space $V^{(2)}_\zeta \oplus \mathbb{R} \vec{w}_1$ is of codimension one, $\mu$-almost surely. Therefore we can find a vector $\vec{w}_2$ of norm one in $\mathcal{C}_k \setminus (V^{(2)}_\zeta \oplus \mathbb{R} \vec{w}_1)$. Note that

$$\inf_{\vec{y} \in V^{(2)}_\zeta, \gamma} \|\vec{w}_1 - \gamma\vec{w}_2 - \vec{y}\| > 0 \quad (3.3)$$

since otherwise, the minimum is reached at a finite non zero pair $(\gamma, \vec{y})$ which would contradict $\vec{w}_2 \in \mathcal{C}_k \setminus (V^{(2)}_\zeta \oplus \mathbb{R} \vec{w}_1)$. Let $\zeta$ be a fixed element in $\mathcal{B}^\mathbb{Z}$.

Define

$$\gamma_n = \max_i \frac{(L^{[n]}_\zeta \vec{w}_1)_i}{(L^{[n]}_\zeta \vec{w}_2)_i} \quad \text{and} \quad \delta_n = \min_i \frac{(L^{[n]}_\zeta \vec{w}_1)_i}{(L^{[n]}_\zeta \vec{w}_2)_i}.$$

Let

$$\phi = \inf \min_{\zeta, \gamma, r, s} \frac{(L^{[r]}_\zeta \vec{w}_2)_s}{(L^{[s]}_\zeta \vec{w}_2)_r},$$

it follows from hypothesis (H1) that $\phi > 0$ and it turns out that this number is independent of $\zeta$. Let

$$\alpha = \frac{1 - \sqrt{\phi}}{1 + \sqrt{\phi}} < 1.$$

From the Birkhoff-Hopf theorem, see for example Cavazos-Cadena (2003), there exists a constant $\beta > 0$ such that for all $\zeta \in \mathcal{B}^\mathbb{Z}$ and all $n$

$$1 \leq \frac{\gamma_n}{\delta_n} \leq 1 + \beta \alpha^n. \quad (3.4)$$

We now prove that

$$\frac{\gamma_n}{1 + \beta \alpha^n} \leq \delta_{n+1} \leq \gamma_{n+1} \leq \gamma_n.$$

To see this observe that $\delta_{n+1} \leq \gamma_{n+1}$ by definition. We also have by monotonicity of $L^{[n]}_\zeta$

$$\gamma_{n+1} = \max_i \frac{(L^{[n+1]}_\zeta \vec{w}_1)_i}{(L^{[n+1]}_\zeta \vec{w}_2)_i} = \max_i \frac{(L^{[n]}_\zeta L^{[n]}_\zeta \vec{w}_1)_i}{(L^{[n+1]}_\zeta \vec{w}_2)_i} \leq \max_i \frac{(L^{[n]}_\zeta L^{[n]}_\zeta \vec{w}_1)_i}{(L^{[n]}_\zeta \vec{w}_2)_i} = \frac{\gamma_n}{\delta_n}.$$
and similarly
\[ \delta_{n+1} \geq \delta_n = \gamma_n \frac{\delta_n}{\gamma_n} \geq \frac{\gamma_n}{1 + \beta \alpha^n}. \]

Since the sequence \( (\gamma_n) \) is decreasing, there exists \( \gamma^* \) and \( \beta' > 0 \) such that
\[ |\gamma_n - \gamma^*| \leq \beta' \alpha^n. \]

On the other hand, it follows immediately from (3.4) that for any \( i = 1, \ldots, k \), we have
\[ -\gamma_n \beta \alpha^n (L_z^{[n]} \vec{w}_2)_i \leq (L_z^{[n]} \vec{w}_1)_i - \gamma_n (L_z^{[n]} \vec{w}_2)_i \leq 0. \]

Then there exists \( \beta'' > 0 \) such that
\[ \frac{\|L_z^{[n]} \vec{w}_1 - \gamma_n L_z^{[n]} \vec{w}_2\|}{\|L_z^{[n]} \vec{w}_2\|} \leq \beta'' \alpha^n. \]

This implies
\[ \frac{\|L_z^{[n]} \vec{w}_1 - \gamma^* L_z^{[n]} \vec{w}_2\|}{\|L_z^{[n]} \vec{w}_2\|} \leq \left( \beta' + \beta'' \right) \alpha^n. \]

Since \( \vec{w}_1 \) and \( \vec{w}_2 \) are linearly independent we have \( \vec{w}_1 - \gamma^* \vec{w}_2 \neq \vec{0} \). This and the previous inequality imply that
\[ \lim_{n \to +\infty} \frac{1}{n} \log \|L_z^{[n]}(\vec{w}_1 - \gamma^* \vec{w}_2)\| \leq \lambda_1 + \log \alpha < \lambda_1, \]

then \( \vec{w}_1 - \gamma^* \vec{w}_2 \in V_{z}^{(2)} \setminus \{0\} \) and this contradicts (3.3). \( \square \)

**Proof of Theorem 2.2.** First observe that
\[ \mathbb{P}(X_0 = a, Z_{-n+1}^{-1} = z_{-n+1}^{-1}, Z_{-n} = b) = \]
\[ \sum_{x_{-l} \in \mathcal{X}^n} \pi(x_{-n}) q(b|x_{-n}) p(a|x_{-1}) \prod_{l=1}^{n-1} q(z_{-l}|x_{-l}) p(x_{-l}|x_{-l-1}). \quad (3.5) \]

Denoting by \( S_{a,b}^{[n]}(\vec{z}) \) the sum in (3.5) we have that
\[ S_{a,b}^{[n]}(\vec{z}) = < \vec{\theta}_b, L_{\vec{z}_{-1}^{[n-1]}} \vec{\psi}_a >, \]
where $\bar{\theta}_b, \bar{\psi}_a \in \mathbb{R}^k$ are given by

$$(\bar{\theta}_b)_i = \pi(i) q(b|i),$$

$$(\bar{\psi}_a)_i = p(a|i).$$

Observe that

$$\sum_{a \in \mathcal{A}} S_{a,b}^{[n]}(z) = \langle \bar{\theta}_b, L_{\mathcal{Y}^{-1}_z}^{[n-1]} \rangle.$$  

By Lemma 3.1, as $(\bar{\psi}_a)_i > 0$ for any $i = 1, \ldots, k$, then $\bar{\psi}_a \notin V_2^{(2)}$, $\mu$-almost surely. From Lemma 3.1 we have $\mathcal{I} \in V_2^{(1)} \setminus V_2^{(2)}$. Therefore, by Lemma 3.2 we have that for any $a \in \mathcal{A}$,

$$(\bar{\psi}_a)_i = u_a \mathcal{I} + \xi_a,$$  \hspace{1cm} (3.6)

where $\xi_a \in V_2^{(2)}$, $u_a \neq 0$, $\psi_a \neq 0$ by hypothesis (H2), and this decomposition is unique.

Then

$$S_{a,b}^{[n]}(z) = u_a \langle \bar{\theta}_b, L_{\mathcal{Y}^{-1}_z}^{[n-1]} \rangle + \langle \bar{\theta}_b, L_{\mathcal{Y}^{-1}_z}^{[n-1]} \xi_a \rangle$$

and

$$\frac{S_{a,b}^{[n]}(z)}{\sum_{a \in \mathcal{A}} S_{a,b}^{[n]}(z)} = u_a + \frac{\langle \bar{\theta}_b, L_{\mathcal{Y}^{-1}_z}^{[n-1]} \xi_a \rangle}{\langle \bar{\theta}_b, L_{\mathcal{Y}^{-1}_z}^{[n-1]} \rangle}.$$  

Therefore,

$$\Delta_{a,b,c}^{[n]}(z) = \frac{\langle \bar{\theta}_b, L_{\mathcal{Y}^{-1}_z}^{[n-1]} \xi_a \rangle}{\langle \bar{\theta}_b, L_{\mathcal{Y}^{-1}_z}^{[n-1]} \rangle} - \frac{\langle \bar{\theta}_c, L_{\mathcal{Y}^{-1}_z}^{[n-1]} \xi_a \rangle}{\langle \bar{\theta}_c, L_{\mathcal{Y}^{-1}_z}^{[n-1]} \rangle}.$$  

Fix $b_0 \in \mathcal{B}$. Define for any $b$, $n$ and $z$

$$\gamma_b(n,z) = \frac{\langle \bar{\theta}_{b_0}, L_{\mathcal{Y}^{-1}_z}^{[n-1]} \rangle}{\langle \bar{\theta}_b, L_{\mathcal{Y}^{-1}_z}^{[n-1]} \rangle}.$$  \hspace{1cm} (3.7)

and

$$\bar{\theta}_b(n,z) = \gamma_b(n,z) \bar{\theta}_b.$$  \hspace{1cm} (3.8)

We have

$$\langle \bar{\theta}_{b_0}, L_{\mathcal{Y}^{-1}_z}^{[n-1]} \rangle = \sum_i (\bar{\theta}_{b_0})_i (L_{\mathcal{Y}^{-1}_z}^{[n-1]})_i \leq R \sum_i (\bar{\theta})_i (L_{\mathcal{Y}^{-1}_z}^{[n-1]})_i$$

$$= R \langle \bar{\theta}_b, L_{\mathcal{Y}^{-1}_z}^{[n-1]} \rangle.$$
In other words for any $b$, $n$ and $z$

$$R^{-1} \leq \gamma_b(n, z) \leq R. \quad (3.9)$$

Then

$$\Delta_{a,b,c}^{[n]}(z) = \left( <\tilde{\theta}_b, L_{\gamma_{n-1}^{-1}}^{[n-1]} \tilde{I} > \right)^{-1} <\tilde{\theta}_b(n, z) - \tilde{\theta}_c(n, z), L_{\gamma_{n-1}^{-1}}^{[n-1]} \tilde{\xi}_a >.$$ 

Note that

$$| <\tilde{\theta}_b, L_{\gamma_{n-1}^{-1}}^{[n-1]} \tilde{I} > | \geq \frac{1}{\sqrt{k}} \| L_{\gamma_{n-1}^{-1}}^{[n-1]} \tilde{I} \| \inf_i \{ \langle \tilde{\theta}_b, i \rangle \}$$

and

$$| <\tilde{\theta}_b(n, z) - \tilde{\theta}_c(n, z), L_{\gamma_{n-1}^{-1}}^{[n-1]} \tilde{\xi}_a > | \leq \| \tilde{\theta}_b(n, z) - \tilde{\theta}_c(n, z) \| \cdot \| L_{\gamma_{n-1}^{-1}}^{[n-1]} \tilde{\xi}_a \| \leq 2 R \sup_b \| \tilde{\theta}_b \| \| L_{\gamma_{n-1}^{-1}}^{[n-1]} \tilde{\xi}_a \|.$$ 

Then, using Oseledec’s theorem we have

$$\tau_{a,b,c}(z) = \limsup_{n \to +\infty} \frac{1}{n} \log | \Delta_{a,b,c}^{[n]}(z) | \leq \lambda_2 - \lambda_1. \quad \square$$

Before proceeding with the proof of Theorem 2.3 we will prove a useful lemma.

**Lemma 3.10.** Under hypotheses (H1-H2), let $\bar{\Omega}$ be a set of full $\mu$ measure where the Oseledec Theorem holds. Then for any $z \in \bar{\Omega}$, there exists a symbol $a = a(z) \in \mathcal{A}$ such that $\tilde{\xi}_a \in V_{\gamma}^{(2)} \setminus V_{\gamma}^{(3)}$, where $\tilde{\xi}_a$ is the vector in decomposition (3.6).

**Proof.** By hypothesis (H2), the set of vectors $\{ \psi_a : a \in \mathcal{A} \}$ forms a basis of $\mathbb{R}^k$. Assume $\tilde{\xi}_a \in V_{\gamma}^{(3)}$ for all $a$. Then, as $\text{Codim}(V_{\gamma}^{(3)}) \geq 2$, the set $\{ \psi_a \}$ generates a sub-space of co-dimension 1, which contradicts (H2). \hfill $\square$

**Proof of Theorem 2.3.** Let $\bar{\Omega}$ be a set of full $\mu$ measure where the Oseledec’s Theorem holds. Let $z \in \bar{\Omega}$, and let $a = a(z) \in \mathcal{A}$ such that $\tilde{\xi}_a \in V_{\gamma}^{(2)} \setminus V_{\gamma}^{(3)}$. The existence of such and element is guaranteed by Lemma 3.10. Let

$$\tilde{\xi}_a(n, z) = \frac{L_{\gamma_{n-1}^{-1}}^{[n-1]} \tilde{\xi}_a}{\| L_{\gamma_{n-1}^{-1}}^{[n-1]} \tilde{\xi}_a \|} \in V_{\gamma_{n-1}}^{(2)}.$$
We now show that there exists $b$ and $c$ such that
\[
\limsup_{n \to \infty} \left| \left< \tilde{\theta}_b(n, z) - \tilde{\theta}_c(n, z), \tilde{\xi}_a(n, z) \right> \right| > 0,
\]
where the vectors $\tilde{\theta}_b(n, z)$ and $\tilde{\theta}_c(n, z)$ are defined by (3.8). Assume this is not the case, namely for any $b$ and $c$
\[
\lim_{n \to \infty} \left| \left< \tilde{\theta}_b(n, z) - \tilde{\theta}_c(n, z), \tilde{\xi}_a(n, z) \right> \right| = 0. \tag{3.11}
\]
Choose for any $n$ (and $z$ fixed) a normalized vector $\vec{f}(n, z)$ orthogonal to $V^{(2)}_{\mathcal{S} - n z}$. Such a vector exists by Lemma 3.2. Note that for any $b$, $n$ and $z$, we have from the bounds in (3.9) and the hypothesis $(H1)$
\[
0 < R^{-1} \min_c \| \tilde{\theta}_c \| \leq \| \tilde{\theta}_b(n, z) \| \leq R \max_c \| \tilde{\theta}_c \| .
\]
This implies that the vectors $(\tilde{f}(n, z), \tilde{\xi}_a(n, z), \tilde{\theta}_1(n, z), \ldots, \tilde{\theta}_\ell(n, z))$ belongs to a compact subset of $\mathbb{R}^{k(\ell+2)}$. Therefore, we can find a subsequence $(n_j)$ of integers such that
\[
\lim_{j \to \infty} (\tilde{f}(n_j, z), \tilde{\xi}_a(n_j, z), \tilde{\theta}_1(n_j, z), \ldots, \tilde{\theta}_\ell(n_j, z)) = (\tilde{f}(z), \tilde{\xi}_a(z), \tilde{\theta}_1(z), \ldots, \tilde{\theta}_\ell(z)).
\]
The vectors $\tilde{f}(z)$ and $\tilde{\xi}_a(z)$ have norm one, and the vectors $\tilde{\theta}_b(z)$ satisfy
\[
0 < R^{-1} \min_c \| \tilde{\theta}_c \| \leq \| \tilde{\theta}_b(z) \| \leq R \max_c \| \tilde{\theta}_c \| .
\]
We have also for any $b$ and $c$ that
\[
< \tilde{\theta}_b(z) - \tilde{\theta}_c(z), \tilde{\xi}_a(z) > = 0 .
\]
We now show that the set of vectors $\{\tilde{\theta}_b(z)\}$ contains a basis of $\mathbb{R}^k$. From hypothesis $(H2)$, there exists $(b_1, \ldots, b_k) \in \{1, \ldots, \ell\}^k$ such that
\[
\det (\tilde{\theta}_{b_1}, \ldots, \tilde{\theta}_{b_k}) \neq 0 .
\]
Then
\[
\left| \det (\bar{\theta}_b(z), \ldots, \bar{\theta}_b(z)) \right| = \lim_{j \to \infty} \prod_{m=1}^{k} \left| \gamma_{b_m}(n_j, z) \right| \left| \det (\bar{\theta}_b, \ldots, \bar{\theta}_b) \right|
\geq R^{-k} \left| \det (\bar{\theta}_b, \ldots, \bar{\theta}_b) \right| > 0.
\]

Let
\[
\zeta(n, z) = \frac{1}{k} \sum_{m=1}^{k} \bar{\theta}_{b_m}(n, z),
\]
and
\[
\bar{\zeta}(z) = \lim_{j \to \infty} \zeta(n_j, z) = \frac{1}{k} \sum_{m=1}^{k} \bar{\theta}_{b_m}(z).
\]

We now observe that since all the components of the vector \( \zeta(n, z) \) are strictly positive, and since by Lemma 3.1 any vector in \( V^{(2)}_{\gamma-n} \) has two components of opposite sign, we get
\[
\frac{|\langle \bar{f}(n, z), \zeta(n, z) \rangle|}{\|\zeta(n, z)\|} = \inf_{\bar{y} \in V^{(2)}_{\gamma-n}} \left\| \zeta(n, z) - \bar{y} \right\| \\
\geq \min_i \left\{ \frac{|\langle \zeta(n, z) \rangle_i|}{\|\zeta(n, z)\|} \right\} \geq \frac{1}{k R^2} \frac{\min_i (\bar{\theta}_b)_i}{\max_i (\bar{\theta}_b)_i} > 0.
\]

Taking the limit we get
\[
\frac{|\langle \bar{f}(z), \bar{\zeta}(z) \rangle|}{\|\bar{\zeta}(z)\|} \geq \frac{1}{k R^2} \frac{\min_i (\bar{\theta}_b)_i}{\max_i (\bar{\theta}_b)_i} > 0.
\]

We now define the orthogonal projection \( \mathcal{P} \) on the orthogonal \( \bar{f}^\perp \) of \( \bar{f} \) parallel to \( \bar{\zeta} \), namely for any vector \( v \)
\[
\mathcal{P}v = v - \bar{\zeta} \left( \frac{\langle \bar{f}, v \rangle}{\|\bar{\zeta}\|} \right).
\]

We claim that the vectors \( (\mathcal{P}(\bar{\theta}_{b_m}(z) - \bar{\theta}_{b_{m+1}}(z)))_{m=1}^{k-1} \) form a basis of \( \bar{f}^\perp \). Indeed, if this is not true, there exists real numbers, \( \alpha_1, \ldots, \alpha_{k-1} \), with at least one nonzero such that
\[
\sum_{m=1}^{k-1} \alpha_m \mathcal{P}(\bar{\theta}_{b_m}(z) - \bar{\theta}_{b_{m+1}}(z)) = 0.
\]
In other words, there exists a number \( \alpha \) such that
\[
\sum_{m=1}^{k-1} \alpha_m (\bar{b}_m(z) - \bar{b}_{m+1}(z)) = \alpha \zeta.
\]
But this is impossible since the vectors \((\bar{b}_m(z) - \bar{b}_{m+1}(z))_{m=1,\ldots,k-1}\) and \(\zeta\) form a basis of \(\mathbb{R}^k\). Since
\[
\langle \bar{f}(z), \xi_a(z) \rangle = \lim_{j \to \infty} \langle f(n_j, z), \bar{\xi}_a(n_j, z) \rangle = 0
\]
we obtain that the normalized vector \(\bar{\xi}_a(z)\) would be orthogonal to the basis \((\bar{\theta}_b^m(z) - \bar{\theta}_{b+1}^m(z))_{m=1,\ldots,k-1}\) of \(\bar{f}^\perp\) which is a contradiction with (3.11). In other words, there exists \(a = a(z), b = b(z)\) and \(c = c(z)\) such that
\[
\lim_{n \to \infty} \left| < \bar{\theta}_b(n, z) - \bar{\theta}_c(n, z), \bar{\xi}_a(n, z) > \right| > 0.
\]
Using the notations of the proof of Theorem 2.2 we have by the Schwarz’s inequality
\[
| \Delta_{a,b,c}^n(z) | = \frac{1}{| \bar{\theta}_b^m, L_{n-1}^m \bar{\zeta} |} \left| < \bar{\theta}_b(n, z) - \bar{\theta}_c(n, z), \bar{\xi}_a(n, z) > \right|
\leq \frac{\| L_{n-1}^m \bar{\zeta} \|}{\| \bar{\theta}_b^m \| \| L_{n-1}^m \bar{\zeta} \|} \left| < \bar{\theta}_b(n, z) - \bar{\theta}_c(n, z), \bar{\xi}_a(n, z) > \right|. 
\]
Therefore, for this choice of \(a(z) \in A\) and \(b(z), c(z) \in B\) we have
\[
\tau_{z}(a, b, c) = \lambda_2 - \lambda_1. 
\]

Proof of Corollary 2.4. The upper bound follows by noting that for all \(z \in \mathcal{B^Z}\) and \(e, b, c \in B\)
\[
\Delta_{e,b,c}^n(z) = \sum_{a \in \mathcal{A}} q(e|a) \Delta_{a,b,c}^n(z) \quad (3.12)
\]
and then applying Theorem 2.2. We now prove that the upper bound is reached for almost all \(z \in \mathcal{B^Z}\). Let's suppose that for all \(z\) on a set of positive measure and for all \(e, b, c \in B\) we have
\[
\tau_{z}(e, b, c) < \lambda_2 - \lambda_1. 
\]
By (H2), as \(\text{rank}(q) = k\) there exists symbols \(e_1, \ldots, e_k \in B\) such that the matrix \(M \in \mathbb{R}^{k \times k}\) with elements \(M_{i,j} = q(e_i|j)\) is invertible. Denote by \(U_{b,c,z}^n\) and \(V_{b,c,z}^n\) the
vectors in $\mathbb{R}^k$ with elements $(U_{b,c,z}^{[n]})_i = \tilde{\Delta}_{e_i,b,c,\bar{z}}^{[n]}(\bar{z})$ and $(V_{b,c,z}^{[n]})_i = \Delta_{e_i,b,c,\bar{z}}^{[n]}(\bar{z})$. By (3.12) we have

$$U_{b,c,z}^{[n]} = MV_{b,c,z}^{[n]}$$

and as $M$ is invertible

$$V_{b,c,z}^{[n]} = M^{-1} U_{b,c,z}^{[n]}.$$ 

Then, for all $a \in \mathcal{A}$

$$|\Delta_{a,b,c}^{[n]}(\bar{z})| \leq \|V_{b,c,z}^{[n]}\| \leq \|M^{-1}\| \|U_{b,c,z}^{[n]}\| \leq \sqrt{k} \|M^{-1}\| \max_e \{|\tilde{\Delta}_{e,b,c}^{[n]}(\bar{z})|\}.$$ 

Applying the logarithm on both sides, dividing by $n$ and taking limits we have that for all $\bar{z}$ on a set of positive measure, for all $a \in \mathcal{A}$ and for all $b, c \in \mathcal{B}$

$$\tau_{\bar{z}}(a, b, c) < \lambda_2 - \lambda_1$$

which contradicts Theorem 2.3. \qed

**Proof of Proposition 2.5.** It is well known that the sequence of Lyapunov exponents $\lambda_1, \ldots, \lambda_s$ satisfy

$$\lambda_1 + m_2 \lambda_2 + \ldots + m_s \lambda_s = \mathbb{E}_\mu[\log |\det L_{\bar{z}}|],$$

where the numbers $m_i$ denote the multiplicity of $\lambda_i$, namely $\dim(V_{\bar{z}}^{(j)}) = m_j + \ldots + m_s$ (see Ledrappier (1984)). In particular, $1 + m_2 + \ldots + m_s = k$. Let $E = \mathbb{E}_\mu[\log |\det L_{\bar{z}}|]$. Then we have

$$E \leq \lambda_1 + (k - 1) \lambda_2$$

and

$$\lambda_2 - \lambda_1 \geq \frac{E}{k - 1} - \frac{k}{k - 1} \lambda_1.$$ 

Note that by Lemma 3.1, for almost all $\bar{z}$ we have

$$\lambda_1 = \lim_{n \to \infty} \frac{1}{n} \log \|L_{\bar{z}}^{[n]}\| \leq \lim_{n \to \infty} \frac{1}{n} \log \|I\| = 0.$$ 

Moreover,

$$\det L_{\bar{z}} = \left( \prod_{i=1}^k q(z_0|i) \right) \det(p).$$
Therefore,
\[
\lambda_2 - \lambda_1 \geq \frac{1}{k - 1} \log |\det(p)| + \frac{1}{k - 1} \sum_{i=1}^{k} E\mu[\log q(\cdot |i)] \\
\geq \frac{1}{k - 1} \log |\det(p)| - \frac{k}{k - 1} \log R.
\]

\[\square\]

4. Perturbed processes over a binary alphabet

Consider the chain \((X_t)_{t \in \mathbb{Z}}\) over the alphabet \(\mathcal{A} = \{0, 1\}\) with matrix of transition probabilities given by
\[
P = \begin{pmatrix}
p_0 & 1 - p_0 \\p_1 & 1 - p_1
\end{pmatrix}
\]
where we assume \(p_0 \neq p_1\) and
\[
0 < \beta = \min\{p_0, p_1, 1 - p_0, 1 - p_1\}.
\]
The quantities \(p(j|i)\) are given by
\[
p(j|i) = P_{i,j}.
\]
Consider also the process \((Z_t)_{t \in \mathbb{Z}}\) over the alphabet \(\mathcal{B} = \{0, 1\}\) with output matrix
\[
Q = \begin{pmatrix}
1 - \epsilon & \epsilon \\
\epsilon & 1 - \epsilon
\end{pmatrix}
\]
such that \(Q_{i,j} = \mathbb{P}(Z_0 = j \mid X_0 = i)\). From now on we will assume \(\epsilon \in (0, 1) \setminus \{1/2\}\). Then
\[
L_{Z,\epsilon} = \begin{pmatrix}
\eta_0 p_0 & (1 - \eta_0)(1 - p_0) \\
\eta_0 p_1 & (1 - \eta_0)(1 - p_1)
\end{pmatrix}
\]
where \(\eta_0 = 1 - \epsilon\) and \(\eta_1 = \epsilon\). We have the following equality
\[
\lambda_1 + \lambda_2 = E\mu[\log |\det L_{.,\epsilon}|]
\]
see for example Ledrappier (1984) for a proof. Therefore

\[ \lambda_1 + \lambda_2 = \mathbb{P}(Z_0 = 0) \log(\eta_0(1 - \eta_0)|p_0(1 - p_1) - p_1(1 - p_0)|) \]

\[ + \mathbb{P}(Z_0 = 1) \log(\eta_1(1 - \eta_1)|p_0(1 - p_1) - p_1(1 - p_0)|) \]

\[ = \log \epsilon + \log(1 - \epsilon) + \log |\det(P)|. \tag{4.1} \]

From the above expression for $L_{z, \epsilon}$ we have

\[ L_{z, \epsilon} = M_{z_0} + \epsilon A_{z_0}, \]

where

\[ M_{z_0} = \begin{pmatrix}
(1 - z_0)p_0 & z_0(1 - p_0) \\
(1 - z_0)p_1 & z_0(1 - p_1)
\end{pmatrix} \]

and

\[ A_{z_0} = (2z_0 - 1) \begin{pmatrix}
p_0 & -(1 - p_0) \\
p_1 & -(1 - p_1)
\end{pmatrix}. \]

Note that

\[ \sup_{z_0} \|M_{z_0}\| \leq 2 \quad \text{and} \quad \sup_{z_0} \|A_{z_0}\| \leq 2. \]

For $b \in \{0, 1\}$ define the vectors

\[ \vec{e}_b = \begin{pmatrix}
p_0(1 - b) + (1 - p_0)b \\
p_1(1 - b) + (1 - p_1)b
\end{pmatrix}, \]

and

\[ \vec{f}_b = \begin{pmatrix}
-p_1(1 - b) - (1 - p_1)b \\
p_0(1 - b) + (1 - p_0)b
\end{pmatrix}. \]

An easy computation shows that for any $b$ we have

\[ \beta \leq \|e_b\| \leq \sqrt{2} \quad \text{and} \quad \beta \leq \|f_b\| \leq \sqrt{2}. \]

We recall that a distance $d$ can be defined on $\Omega$ as follows. For $\underline{z}$ and $\underline{z}'$ in $\Omega$, let

\[ \tilde{d}(\underline{z}, \underline{z}') = \inf\{|i|, z_i \neq z'_i\}. \]
Then
\[ d(\mathbf{z}, \mathbf{z}') = e^{-\tilde{d}(\mathbf{z}, \mathbf{z}')} . \]

We refer to Bowen (2008) for details, in particular \( \Omega \) equipped with this distance is a compact metric space. We now prove the following result.

**Lemma 4.2.** There exists two constants \( \epsilon_0 > 0 \) and \( D > 0 \) and two continuous functions \( \rho(\epsilon, \mathbf{z}) \) and \( h(\epsilon, \mathbf{z}) \) such that for any \( \epsilon \in [0, \epsilon_0] \) the vectors
\[ \mathbf{g}(\epsilon, \mathbf{z}) = \mathbf{e}_1 + \epsilon h(\epsilon, \mathbf{z}) \mathbf{f}_1 \]
satisfy
\[ L_{\mathbf{z}, \epsilon} \mathbf{g}(\epsilon, \mathbf{z}) = \rho(\epsilon, \mathbf{z}) \mathbf{g}(\epsilon, \mathbf{z}). \]

Moreover, there is a constant \( U > 1 \) such that for any \( \epsilon \in [0, \epsilon_0] \), any \( n \) and any \( \mathbf{z} \in \Omega \)
\[ \| \mathbf{g}(\epsilon, \mathbf{z}) - \mathbf{e}_1 \| \leq U \epsilon , \quad \left| \rho(\epsilon, \mathbf{z}) - \frac{\langle M_{\mathbf{z}_1} \mathbf{e}_{21}, \mathbf{e}_{11} \rangle}{\| \mathbf{e}_{21} \|^2} \right| \leq U \epsilon \quad \text{and} \quad U^{-1} \mathbf{1} \leq \mathbf{g}(\epsilon, \mathbf{z}) \leq U \mathbf{1} . \]

**Proof.** The equation for \( \mathbf{g} \) is equivalent to
\[ L_{\mathbf{x}, \epsilon} \mathbf{g}(\epsilon, \mathbf{x}) = \rho(\epsilon, \mathbf{x}) \mathbf{g}(\epsilon, \mathbf{x}). \]

Note that
\[ \mathbf{g}(\epsilon, \mathbf{x}) = \mathbf{e}_{21} + \epsilon h(\epsilon, \mathbf{x}) \mathbf{f}_{21} \quad \text{and} \quad L_{\mathbf{x}, \epsilon} = M_{\mathbf{x}_1} + \epsilon A_{\mathbf{x}_1} . \]

Taking the scalar product of both terms in equation (4.3) with \( \mathbf{e}_{21} \) and \( \mathbf{f}_{21} \) we get
\[ \rho(\epsilon, \mathbf{z}) = \frac{1}{\| \mathbf{e}_{21} \|^2} \left[ \langle M_{\mathbf{z}_1} \mathbf{e}_{22}, \mathbf{e}_{21} \rangle + \epsilon h(\epsilon, \mathbf{z}) \langle M_{\mathbf{z}_1} \mathbf{f}_{22}, \mathbf{e}_{21} \rangle + \epsilon \langle A_{\mathbf{z}_1} \mathbf{e}_{22}, \mathbf{e}_{21} \rangle + \epsilon^2 \langle A_{\mathbf{z}_1} \mathbf{f}_{22}, \mathbf{e}_{21} \rangle \right] \quad (4.4) \]

and since \( M_{\mathbf{z}_1} \mathbf{f}_{21} = 0 \)
\[ \rho(\epsilon, \mathbf{z}) h(\epsilon, \mathbf{z}) = \frac{1}{\| \mathbf{f}_{21} \|^2} \left[ \langle A_{\mathbf{z}_1} \mathbf{e}_{22}, \mathbf{f}_{21} \rangle + \epsilon h(\epsilon, \mathbf{z}) \langle A_{\mathbf{z}_1} \mathbf{f}_{22}, \mathbf{f}_{21} \rangle \right] . \]
We denote by \( D \) the Banach space of continuous functions on \([0, \epsilon_0] \times \Omega\) equipped with the sup norm. On the ball \( B_D \) of radius \( D = 16\beta^{-5} \) centered at the origin in \( D \) we define a transformation \( \mathcal{T} \) given by

\[
\mathcal{T}(h)(\epsilon, \bar{z}) = \frac{u_1(\epsilon, \bar{z}) + \epsilon u_2(\epsilon, \bar{z})h(\epsilon, \mathcal{T} \bar{z})}{u_3(\epsilon, \bar{z}) + \epsilon u_4(\epsilon, \bar{z})h(\epsilon, \mathcal{T} \bar{z})}
\]

where

\[
u_1(\epsilon, \bar{z}) = \frac{\|\vec{e}_{z_1}\|^2}{\|\vec{f}_{z_1}\|^2} \langle A_{z_1} \vec{e}_{z_2}, \vec{f}_{z_1} \rangle, \quad \nu_2(\epsilon, \bar{z}) = \frac{\|\vec{e}_{z_1}\|^2}{\|\vec{f}_{z_1}\|^2} \langle A_{z_1} \vec{f}_{z_2}, \vec{f}_{z_1} \rangle, \quad \nu_3(\epsilon, \bar{z}) = \langle M_{z_1} \vec{e}_{z_2}, \vec{e}_{z_1} \rangle + \epsilon \langle A_{z_1} \vec{e}_{z_2}, \vec{e}_{z_1} \rangle + \epsilon^2 \langle A_{z_1} \vec{f}_{z_2}, \vec{e}_{z_1} \rangle \quad \text{and} \quad \nu_4(\epsilon, \bar{z}) = \langle M_{z_1} \vec{f}_{z_2}, \vec{e}_{z_1} \rangle.
\]

We first prove that \( \mathcal{T} \) maps \( B_D \) into itself. Indeed for \( h \in B_D \), since \( D = 16\beta^{-5} \) there exists \( \epsilon'_0 > 0 \) small enough such that for any \( \epsilon \in [0, \epsilon'_0] \)

\[
|\mathcal{T}(h)(\epsilon, \bar{z})| \leq \frac{2}{\beta^2} \frac{4 + 4\epsilon D}{\beta^3 - 4\epsilon D - 4\epsilon - 4\epsilon^2} \leq D.
\]

We leave to the reader the proof that \( \mathcal{T}(h) \) is a continuous function of \( \epsilon \) and \( \bar{z} \). We now prove that \( \mathcal{T} \) is a contraction on \( B_D \). For \( h \) and \( h' \) in \( B_D \), since \( D = 16\beta^{-5} \) there exists \( \epsilon_0 > 0 \) small enough, and smaller than \( \epsilon'_0 \), such that for any \( \epsilon \in [0, \epsilon_0] \) we have

\[
|\mathcal{T}(h)(\epsilon, \bar{z}) - \mathcal{T}(h')(\epsilon, \bar{z})| = \epsilon \left| \frac{u_1(\epsilon, \bar{z})u_4(\epsilon, \bar{z}) - u_2(\epsilon, \bar{z})u_3(\epsilon, \bar{z})}{(u_3(\epsilon, \bar{z}) + \epsilon u_4(\epsilon, \bar{z})h(\epsilon, \mathcal{T} \bar{z}))(u_3(\epsilon, \bar{z}) + \epsilon u_4(\epsilon, \bar{z})h(\epsilon, \mathcal{T} \bar{z}))} \right| |h(\epsilon, \bar{z}) - h'(\epsilon, \bar{z})| \\
\leq \epsilon \frac{2}{\beta^2} \frac{16}{(\beta^3 - 4\epsilon D - 4\epsilon - 4\epsilon^2)^2} \left| h(\epsilon, \bar{z}) - h'(\epsilon, \bar{z}) \right| \leq \frac{1}{2} \left| h(\epsilon, \bar{z}) - h'(\epsilon, \bar{z}) \right|.
\]

By the contraction mapping principle (see for example Dieudonné (1969)), the map \( \mathcal{T} \) has a unique fixed point \( h \) in \( B_D \). It follows at once that the vectors

\[
\bar{g}(\epsilon, \bar{z}) = \vec{e}_{z_1} + \epsilon h(\epsilon, \bar{z}) \vec{f}_{z_1}
\]

satisfy equation (4.3). The estimate on \( \bar{g}(\epsilon, \bar{z}) \) follows immediately from the fact that \( h \in B_D \), and the estimate on \( \rho(\epsilon, \bar{z}) \) follows from (4.4).

\[\square\]

Remark. An easy improvement of the above proof allows to show that \( \rho \) and \( h \) depend analytically on \( \epsilon \) in a small (complex) neighborhood of 0.
By the estimate on $\bar{g}(\epsilon, z)$ of the previous lemma and Lemma 3.1 applied to the vector $\bar{1}$, we have $\mu$ almost surely
\[
\lim_{n \to \infty} \frac{1}{n} \log \| L^{n-1}_{\mathcal{Y}^{-1}_z} \bar{g}(\epsilon, z) \| = \lim_{n \to \infty} \frac{1}{n} \log \| L^{n-1}_{\mathcal{Y}^{-1}_z} \bar{1} \| = \lambda_1 .
\]
On the other hand from Lemma 4.2 it follows that
\[
\log \| L^{n-1}_{\mathcal{Y}^{-1}_z} \bar{g}(\epsilon, z) \| = \sum_{j=0}^{n-1} \log \rho(\epsilon, \mathcal{Y}^{-j} z) + \| \bar{g}(\epsilon, \mathcal{Y}^{-n} z) \|.
\]
Using again the estimate on $\bar{g}(\epsilon, z)$ from Lemma 4.2, the Birkhoff ergodic theorem (Krengel, 1985) and the ergodicity of $\mu$, we have $\mu$-almost surely
\[
\lambda_1 = \int \log \rho(\epsilon, \mathcal{Y}^{-j} z) d\mu(z).
\]
Since
\[
\inf_{z_1, z_2} \frac{\langle M_{z_1} \bar{e}_{z_2}, \bar{e}_{z_1} \rangle}{\| \bar{e}_{z_1} \|^2} \geq \frac{\beta^3}{2},
\]
it follows from Lemma 4.2 and a direct computation that
\[
\lambda_1 = \int \log \left( \frac{\langle M_{z_1} \bar{e}_{z_2}, \bar{e}_{z_1} \rangle}{\| \bar{e}_{z_1} \|^2} \right) d\mu(z) + O(\epsilon)
\]
\[
= \sum_{z_1, z_2, x_1, x_2} \log \left( \frac{\langle M_{z_1} \bar{e}_{z_2}, \bar{e}_{z_1} \rangle}{\| \bar{e}_{z_1} \|^2} \right) q(z_1 | x_1) q(z_2 | x_2) p(x_2 | x_1) \pi(x_1) + O(\epsilon)
\]
\[
= \sum_{x_1, x_2} \log \left( \frac{\langle M_{x_1} \bar{e}_{x_2}, \bar{e}_{x_1} \rangle}{\| \bar{e}_{x_1} \|^2} \right) p(x_2 | x_1) \pi(x_1) + O(\epsilon)
\]
\[
= \pi(0) H(p_0) + \pi(1) H(p_1) + O(\epsilon),
\]
where for a number $x \in (0, 1)$, $H(x) = x \log x + (1 - x) \log(1 - x)$. The following theorem is an immediate consequence of the above estimates.

**Theorem 4.5.** If $p_0 \neq p_1$, $\min\{p_0, p_1, 1 - p_0, 1 - p_1\} > 0$ and $\epsilon > 0$ is small enough we have $\mu$-almost surely
\[
\tau_{\bar{z}}(a, b, c) \leq \log \epsilon + \log |\det(P)| - 2 [\pi(0) H(p_0) + \pi(1) H(p_1)] + O(\epsilon).
\]
Moreover, for $\mu$-almost all $\bar{z}$ there is a triplet $(a, b, c)$ (which may depend on $\bar{z}$) where the equality holds.
Proof. It is easy to verify that hypothesis (H1-H2) are satisfied. We therefore apply Theorems 2.2 and 2.3. The result follows from (4.1) and the above estimate on $\lambda_1$. □

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