TWISTING OF QUANTUM SPACES AND TWISTED COHOM OBJECTS

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Abstract. Twisting process for quantum linear spaces is defined. It consists in a particular kind of globally defined deformations on finitely generated algebras. Given a quantum space \( A = (A_1, A) \), a multiplicative cosimplicial quasicomplex \( C^\bullet [V] \) in the category Grp is associated to \( A_1 \), in such a way that for every \( n \in \mathbb{N} \) a subclass of linear automorphisms of \( A^\otimes n \) is obtained from the groups \( C^n [A_1] \). Among the elements of this subclass, the comitial 2-cocycles are those which define the twist transformations. In these terms, the twisted internal coHom objects \( \text{hom}^T [B, A] \), constructed in \([4]\), can be described as twisting of the proper coHom objects \( \text{hom} [B, A] \). Moreover, twisted tensor products \( A \circ B \), in terms of which above objects were built up, can be seen as particular 2-cocycle twisting of \( A \circ B \), enabling us to generalize the mentioned constructions. The quasicomplexes \( C^T [V] \) are further studied, showing for instance that, when \( V \) is a coalgebra the quasicomplexes related to Drinfeld twisting, corresponding to bialgebras generated by \( V \), are subobjects of \( C^\bullet [V] \).

Introduction

In a previous paper \([5]\), combining the ideas of twisted tensor products (TTP) \([1]\) and internal coHom objects, we have built up on the monoidal category \((CA, \circ, K)\) of conic algebras or conic quantum spaces, i.e. finitely generated graded algebras (which constitutes a full subcategory of the -general- quantum linear spaces \([12, 13]\)), functions \( (B, A) \mapsto \text{hom}^T [B, A] \in CA \), one for each collection of automorphisms \( \Upsilon = \{ \sigma_A : A \subset A \}_{A \in CA} \), defining \( CA^{op}\)-based categories distinct from the one related to its proper internal coHom objects \( \text{hom} [B, A] \). The opposite objects \( \text{hom}^T [B, A]^{op} \neq \text{hom}^T [B^{op}, A^{op}] \in CA^{op} \)

represent ‘spaces of morphisms’ whose coordinate rings, given precisely by the algebras \( \text{hom}^T [B, A] \), do not commute with the ones of their respective domain \( B^{op} \), in the sense that they give rise to twisted coevaluation arrows \( A \rightarrow \text{hom}^T [B, A] \circ B \), where \( \circ \) is a TTP constructed in terms of \( \sigma_A, \sigma_B \in \Upsilon \). This is why we have called them twisted coHom objects.

We have seen in that paper the objects \( \text{end}^T [A] = \text{hom}^T [A, A] \), which are endowed with a bialgebra structure, are comitial 2-cocycle twisting \([4]\) of the coEnd objects \( \text{end} [A] = A^{\triangleright} A \). In other words, those objects are related by a (non formal) bialgebra deformation. However, we have not been able to make an analogous claim relating the objects \( \text{hom}^T [B, A] \) and \( \text{hom} [B, A] \) for \( B \neq A \). To this end, we will develop in this paper a suitable kind of globally defined deformations of quantum spaces, which we shall call twisting of quantum spaces, in such a way that a multiplicative cosimplicial quasicomplex structure \( C^T \) can be related to them, being 2-cocycles the well-behaved deformations. We shall see the objects \( \text{hom}^T [B, A] \) are twisting by 2-cocycles of \( \text{hom} [B, A] \). In particular, for the coEnd objects, those transformations define the previously cited 2-cocycle bialgebra twisting.

The main aim of our work is to construct and analyze the mentioned cosimplicial quasicomplex structure and related algebra deformation process. We invoke the concept of twisted coHom objects just because they were our main motivation to this paper.

Although our twist transformation will be defined on the hole class of quantum spaces, i.e. the category FGA of finitely generated algebras, they are mainly designed to be applied on the conic ones. This is why, among other things, the first chapter is dedicated to them. Thus, in \( \S 1 \) we make a brief review about conic quantum spaces, recalling their monoidal structures and related internal (co)Hom objects. For a more extended treatment, reference \([4]\) can be consulted.

In order to introduce the twisting process in a (quasi)cohomological framework, in \( \S 2 \) we endow every linear space \( V \) (or every tensor algebra \( V^\otimes \)) with a multiplicative cosimplicial quasicomplex structure \( (C^\bullet [V], \partial) \). Recall that formal deformations \([3]\) of an associative algebra \( A \) are controlled by the Hochschild complex \([3]\) of \( A \). Given a pair \( A = (A_1, A) \) and a (comitial) 2-cocycle \( \psi \) inside certain subclass of \( C^2 [A_1] \), the admissible 2-cochains, we define in \( \S 3 \) a deformation of \( A \) as a new quantum space \( A_\psi \), the twisting of \( A \) by \( \psi \). The mentioned subclass defines a subgroup of linear automorphisms of \( A \otimes A \). Deformed quantum spaces \( A_\psi \) and

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$A_\varphi$ are isomorphic \textit{iff} the cocycles $\psi$ and $\varphi$ are cohomologous through some admissible 1-cochain. We show that twist transformation can be composed and have inverse. Moreover, an equivalence relation between quantum spaces can be defined, namely: $A$ and $B$ are \textit{twist or gauge related} when there exists $\psi$ such that $A_\varphi \simeq B$ (as in the case of bialgebra twisting). We also show the twisting of bialgebras in the category $FGA$ are particular twisting of quantum spaces.

Finally, we investigate in §4 the behavior of twist transformations in relation to certain functorial structures in $FGA$. We see, for example, that products $\otimes$ give linear maps $A \rightarrow B$ such that restricted to particular kind of twist transformations, for functors $\cdot$, $\circ$, and products $\otimes\circ$, $\circ\cdot\otimes$ and $\otimes\cdot\circ\cdot\otimes$ defined in [6]. This study lead us to a better understanding of the construction of twisted coHom objects and, in turn, a way to generalize it.

We often adopt definitions and notation extracted form Mac Lane's book [10]. $k$ indicates some of the numeric fields, $\mathbb{R}$ or $\mathbb{C}$. The usual tensor product on $k$–$\text{Alg} \equiv \text{Alg}$ and $\text{Vct}_k \equiv \text{Vct}$ (the categories of unital associative $k$-algebras and of $k$-vector spaces, respectively) is denoted by $\otimes$. Grp denotes the category of groups and their homomorphisms.

1. Conic quantum spaces

In this chapter we recall definition, functorial structures and internal coHom objects (standard and twisted ones) of a subclass of quantum spaces that we have called \textit{conic quantum spaces} [6], introducing the needed notation for the rest of the paper. We suppose the reader is familiar with the concepts of quadratic and general quantum spaces [12][13].

Conic quantum spaces define a category $CA$ such that the full inclusions $QA \subset CA \subset FGA$ holds, being $FGA$ and $QA$ the categories of finitely generated algebras and of quadratic algebras, respectively. Remember that $FGA$ is formed out by pairs $(A_1, A)$ where $A$ is a unital algebra generated by a finite linear subspace $A_1 \subset A$, and its morphisms are algebra morphisms $A \rightarrow B$ such that restricted to $A_1 \subset A$ give linear maps $A_1 \rightarrow B_1$ ($\subset B$). The objects of $CA$ are pairs $A = (A_1, A) \in FGA$ such that $A$ is a graded algebra with grading given by

$$A = \bigoplus_{n \in \mathbb{N}_0} A_n; \quad A_n = \Pi \left( A_1^{\otimes n} \right); \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\},$$

where $\Pi = \Pi_A : A_1^\otimes \rightarrow A$ is the canonical epimorphism of algebras such that restricted to $A_1$ gives the inclusion $A_1 \hookrightarrow A$. We are denoting by $A_1^\otimes$ the tensor algebra of $A_1$, i.e. $A_1^\otimes = \bigoplus_{n \in \mathbb{N}_0} A_1^{\otimes n}$. In the general case the related algebras are just filtered, more precisely

$$A = \bigcup_{n \in \mathbb{N}_0} F_n; \quad F_n = \Pi \left( \bigoplus_{i=0}^{n} A_1^{\otimes i} \right).$$

Alternatively, the objects of $CA$ can be described as those pairs $A$ in $FGA$ such that $\ker \Pi$ is a graded bilateral ideal $\ker \Pi_A = \bigoplus_{n \in \mathbb{N}_0} I_n$ with $I_0 \subset A_1^{\otimes n}$. In these terms, given another pair $(B_1, B)$ with $\ker \Pi_B = \bigoplus_{n \in \mathbb{N}_0} J_n$, the arrows $(A_1, A) \rightarrow (B_1, B)$ in $CA$ are characterized by linear maps $\alpha_1 : A_1 \rightarrow B_1$ such that $\alpha_1^{\otimes n}(I_n) \subset J_n$, being

$$\alpha_1^\otimes = \bigoplus_{n \in \mathbb{N}_0} \alpha_1^{\otimes n} : A_1^\otimes \rightarrow B_1^\otimes$$

the unique extension of $\alpha_1$ to $A_1^\otimes$ as a morphism of algebras. The algebra map $\alpha : A \rightarrow B$ that defines any morphism in $FGA$ is the unique algebra homomorphism such that $\alpha \Pi_A = \Pi_B \alpha_1^\otimes$. In particular, $\alpha_1 = \alpha|_{A_1}$. Returning to the ideal of a conic quantum space, since the restriction of $\Pi_A$ to $A_1$ is the identity, we always have $I_1 = \{0\}$. In addition, $I_0 \neq \{0\} \iff A = \{0\}$. Thus, generically $\ker \Pi_A = \bigoplus_{n \geq 2} I_n$, unless $A = \{0\}$.

Examples of conic quantum spaces are, beside quadratics, those with ker $\Pi = I[Y]$, where $Y \subset A_1^{\otimes m}$ for some $m > 2$, indicating by $I[X]$ the bilateral ideal generated algebraically by $X \subset A_1^\otimes$, i.e. $I[X] = A_1^\otimes \otimes X \otimes A_1^\otimes$. We have called them \textit{m-th quantum spaces}, and denoted $CA^m$ the full subcategory of $FGA$ which has these pairs as objects. Thus $QA = CA^2$.

We recall that from the geometric point of view, the quantum spaces actually are the opposite objects $A^{op}$ to the pairs $A = (A_1, A) \in FGA$, being the latter their (generically) noncommutative coordinate rings. That is to say $FGA^{op}$, which we indicate $QLS$, is the category of quantum spaces, and $FGA$ the category of their coordinate rings. Nevertheless, since the duality between $QLS$ and $FGA$, for quantum spaces we understand the objects of each one of these categories. Categories $CA$ and $CA^m$ give rise to full subcategories of $QLS$. 


1.1. Functorial structures and internal coHom objects. The bifunctor \( \circ \) defines an (strict and symmetric) monoidal structure with unit \( K = (k, k^\otimes) \) in CA. Recall that \( A \circ B = (A_1 \otimes B_1, A \circ B) \), being \( A \circ B \) the subalgebra of \( A \otimes B \) generated by \( A_1 \otimes B_1 \), and that the unit for \( \circ \) is \( I \simeq (k, k) \) in FGA. If \( A, B \in CA \), the algebra \( A \circ B \) effectively defines an object of CA due to the kernel of \( \Pi : [A_1 \otimes B_1] \otimes \to A \circ B \), using the (canonical) homogeneous isomorphism of graded algebras

\[
[A_1 \otimes B_1] \otimes = [A_1 \otimes B_1] \simeq \bigoplus_{n \in \mathbb{N}_0} (A_1^{\otimes n} \otimes B_1^{\otimes n}),
\]

being the latter the subalgebra of \( A_1^{\otimes n} \otimes B_1^{\otimes n} \) generated by \( A_1 \otimes B_1 \), is a graded bilateral ideal isomorphic to

\[
\bigoplus_{n \in \mathbb{N}_0} (I_n \otimes B_1^{\otimes n} + A_1^{\otimes n} \otimes J_n) \left( \subset [A_1 \otimes B_1] \right).
\]

Note that \( [A_1 \otimes B_1] \simeq A_1^{\otimes} \circ B_1^{\otimes} \). We frequently identify the latter algebra and \( [A_1 \otimes B_1] \). In particular, \( \Pi \) will also be understood as a map \( A_1^{\otimes} \circ B_1^{\otimes} \to A \circ B \).

For quadratic algebras there is another monoidal structure \( \bullet \) and a functor \( ! \) such that

\[
A^{\bullet} \simeq A, \quad (A \circ B)^{\bullet} \simeq A^\bullet \bullet B^\bullet, \quad (A \bullet B)^{\bullet} \simeq A^\bullet \circ B^\bullet, \quad K^{\bullet} \simeq U,
\]

being \( U = (k, U) \), \( U = k[e]/I [e^2] \), a unit object for \( \bullet \). Analogous functors to \( \bullet \) and \( ! \) can be given in CA, namely

\[
A \odot B = A^{\circ} \circ B^{\circ} / \bigoplus_{n \in \mathbb{N}_0} I_n \otimes J_n
\]

and

\[
A^\dagger = A^{\circ} / I^\dagger; \quad I^\dagger = \bigoplus_{n \in \mathbb{N}_0} I_n \simeq I \left[ \bigoplus_{n \geq 2} I_n^+ \right],
\]

with \( I_n^+ = \{ x \in A_1^{\otimes n} : \langle x, y \rangle = 0, \forall y \in I_n \} \), for \( n \geq 2 \). We shall take \( I_{0,1}^+ = \{ 0 \} \). A unit element for \( (CA, \circ) \) is \( U \) as for QA, but the restriction of \( \circ \) to QA does not coincide with the functor \( \bullet \). On the other hand, the functor \( ! \) does coincide with the corresponding to the quadratic cases, although \( I^\dagger \not\cong id_{CA} \). The preserved properties are, with respect to \( \bullet \) and \( ! \),

\[
(A \circ B)^{\bullet} \simeq A^\bullet \circ B^\bullet, \quad K^\bullet \simeq U, \quad K \simeq U^\dagger,
\]

while \( (A \circ B)^{!} \not\cong A^! \circ B^! \) and \( A^{\bullet} \not\cong A \). Some kind of covariant mixing of \( \circ \) and \( ! \) can be defined, namely

\[
\triangleright : CA^{op} \times CA \to CA, \quad \triangleright : CA \times CA^{op} \to CA,
\]

\[
\circ : CA^{op} \times CA^{op} \to CA,
\]

satisfying

\[
\triangleright = \bullet (I \times id), \quad \triangleright = \bullet (id \times !), \quad \circ = \bullet (I \times !)
\]

when restricted to QA. They are given on objects by (identifying each object with its opposite)

\[
A \triangleright B = A^{\circ} \circ B^{\circ} / I \left[ \bigoplus_{n \in \mathbb{N}_0} I_n^+ \otimes J_n \right],
\]

\[
A \triangleright B = A^{\circ} \circ B^{\circ} / I \left[ \bigoplus_{n \in \mathbb{N}_0} I_n \otimes J_n \right],
\]

\[
A \triangleright B = A^{\circ} \circ B^{\circ} / I \left[ \bigoplus_{n \in \mathbb{N}_0} I_n^+ \otimes J_n \right].
\]

The functors \( \triangleright \) and \( \triangleright \) have \( K \) as left and as a right unit, respectively, in the sense that \( K \triangleright A \simeq A \) and \( A \triangleright K \simeq A \) for any \( A \) in CA (on the other hand, \( A \triangleright U \simeq A^! \), \( U \triangleright A \simeq A^! \) and \( K \triangleright A \simeq A \circ K \simeq A^! \)).

While the internal coHom objects of QA are given by \( \hom[BA] = B \triangleright A \) with coevaluation arrow \( A \to (B \triangleright A) \circ B \) defined by the map \( a_i \to [b^2 \otimes a_i] \otimes b_j \) (sum over repeated indices is understood), where \( \{a_i\}, \{b_j\} \) and \( \{b^i\} \) are bases of \( A_1, B_1 \) and \( B_1^* \), respectively.

Theorem 1. The category \((CA, \circ, K)\) has internal coHom objects given by \( \hom[BA] = B \triangleright A \) with coevaluation arrow \( A \to (B \triangleright A) \circ B \) defined by the map \( a_i \to [b^2 \otimes a_i] \otimes b_j \) (sum over repeated indices is understood), where \( \{a_i\}, \{b_j\} \) and \( \{b^i\} \) are bases of \( A_1, B_1 \) and \( B_1^* \), respectively. \( \blacksquare \)

The proof is given in [6].
1.2. The twisted internal coHom objects. An internal coHom object in FGA is an initial object of the comma category \((A \downarrow \text{FGA} \circ B)\). The objects of each \((A \downarrow \text{FGA} \circ B)\), diagrams in the Manin terminology, are pairs \(\langle \varphi, \mathcal{H} \rangle_{A,B} = \langle \varphi, \mathcal{H} \rangle\), with \(\mathcal{H} \in \text{FGA}\) and \(\varphi\) a morphism \(A \to \mathcal{H} \circ B\); and its arrows \(\langle \varphi, \mathcal{H} \rangle \to \langle \varphi', \mathcal{H}' \rangle\) are given by morphisms \(\alpha : \mathcal{H} \to \mathcal{H}'\) satisfying \(\varphi' = (\alpha \circ I_B) \varphi\). For every such category \(\langle \varphi, \mathcal{H} \rangle \to \mathcal{H}\) defines an embedding \(U : (A \downarrow \text{FGA} \circ B) \hookrightarrow \text{FGA}\). The disjoint union of the family \(\{(A \downarrow \text{FGA} \circ B)\}_{A,B \in \text{FGA}}\), namely FGA*, has a semigroup structure given by the partial product functor \(\delta\). U extends to an obvious embedding FGA° \to \text{FGA}, which we shall also call U, and that satisfies

\[
(1.6) \quad U \circ = \circ (U \times U) \quad \text{and} \quad U (\ell_A, I) = I.
\]

So U : FGA° \to FGA is an embedding of categories with unital associative partial products. The same is true for CA and every CA°, but changing \(I\) by \(\mathcal{K}\) in Eq. (1.6). We have seen in E that it is possible to arrive at the notions of cocomposition and coidentity from the semigroup structure in FGA°, the initiality of each \(\text{hom}_{\mathcal{K}}[B, A] \in (A \downarrow \text{FGA} \circ B)\), and the existence of a functor U satisfying Eq. (1.6). Then, the function \((B, A) \mapsto \text{hom}_{\mathcal{K}}[B, A]\), together with the corresponding cocomposition and coidentity arrows, defines an FGA-cubed (or dually FGA°-based) category with objects also in FGA. Such a cobased category has in addition the usual notion of coevaluation \(A \to \text{hom}_{\mathcal{K}}[B, A] \circ B\).

Based on these ideas, we construct in that paper a family of categories \(Y^{A,B}\) with initial objects, denoted \(\text{hom}^T_{\mathcal{K}}[B, A]\), and with a family of embedding \(\text{hom}^T_{\mathcal{K}}[A,B] \to \text{FGA}\). The objects of each \(Y^{A,B}\) are essentially arrows \(A \to \mathcal{H} \circ B\), where \(\circ\) denotes a twisted tensor product between \(\mathcal{H}\) and \(B\). Each category is defined by a linear isomorphism \(\hat{\tau}_{A,B} : B_1 \otimes B_2 \otimes A_1 \cong B_1 \otimes B_2 \otimes A_1\), from which the twisted products \(\circ\) are built up. Restricting \(A, B\) to CA and considering \(\hat{\tau}_{A,B} = id \otimes \sigma_{S^{-1}} \otimes \sigma_A\), the disjoint union \(Y\) of these categories has a semigroupoid structure such that an equation like (1.6) holds for the extended functor \(Y \to CA\). Thus, \((B, A) \mapsto \text{hom}^T_{\mathcal{K}}[B, A]\) gives rise to an CA-cubed category with the additional notion of coevaluation \(A \to \text{hom}^T_{\mathcal{K}}[B, A] \circ_\sigma B\). We will not give the details of this construction, referring the interested reader to E. Nevertheless, in the rest of this section we review the main properties of the objects \(\text{hom}^T_{\mathcal{K}}[B, A]\) and its relationship to the proper coHom objects \(B \triangleright A\) of CA.

Let \(\{\sigma_A : A_1 \simeq A_1\}_{A \in \text{CA}}\) be a collection defining \(Y\), such that each map \(\sigma_A\) can be extended to an automorphism \(A \simeq A\). Consider a couple \(A\) and \(B\) of quantum spaces given by algebras \(A \simeq A_1 \otimes I\) and \(B \simeq B_1 \otimes J\), being \(I\) and \(J\) the ideals linearly generated by

\[
(1.7) \quad R_{\lambda_n}^{k_1...k_n} a_{k_1...a_{k_n}} \in I_n, \quad S_{\mu_n}^{k_1...k_n} b_{k_1...b_{k_n}} \in J_n,
\]

with \(\lambda_n\) and \(\mu_n\) in index sets \(\Lambda_n\) and \(\Phi_n\), respectively. Writing \(\phi = \sigma_A\) and \(\rho = \sigma_B^{s-1}\), the initial objects of each category \(Y^{A,B}\) can be defined by a conic quantum space

\[
\text{hom}^T_{\mathcal{K}}[B, A] = B^T \triangleright A^T, \quad A^T = (A_1, A_1 \otimes I_\sigma), \quad B^T = (B_1, B_1 \otimes J_\sigma),
\]

with \(I_\sigma\) and \(J_\sigma\) linearly generated by

\[
\left\{ R_{\lambda_n}^{k_1...k_n} a_{k_1...a_{k_n}} \right\}_{\lambda_n \in \Lambda_n}, \quad \left\{ S_{\mu_n}^{k_1...k_n} b_{k_1...b_{k_n}} \right\}_{\mu_n \in \Phi_n}
\]

where

\[
(1.8) \quad \sigma R_{\lambda_n}^{k_1...k_n} \equiv R_{\lambda_n}^{k_1...k_n} \phi_{j_2}^{k_2} \phi_{j_3}^{k_3} \cdots (\phi_n^{1-n})^{k_n},
\]

\[
(1.9) \quad \sigma S_{\mu_n}^{k_1...k_n} \equiv S_{\mu_n}^{k_1...k_n} (\rho^{1-n})^{k_2} \rho^{1-n})^{k_3} \cdots (\rho^{1-n})^{k_n}.
\]

On the other hand, if each \(J_\sigma = B_1 \otimes \sigma_n\) is the span of

\[
(1.10) \quad \left\{ b_{k_1...b_{k_n}} (S_{\omega_n}^{1})_{k_1...k_n} \right\}_{\omega_n \in \Omega_n},
\]

\(J_\sigma\) is spanned by the set

\[
(1.11) \quad \left\{ b_{k_1...b_{k_n}} (S_{\omega_n}^{1})_{k_1...k_n} \right\}_{\omega_n \in \Omega_n},
\]

with

\[
(1.12) \quad \left\{ S_{\omega_n}^{1} \right\}_{k_1...k_n} \equiv (\sigma_S^{1})_{k_1...k_n} (S_{\omega_n}^{1})_{k_1...k_n}.
\]

Therefore, \(\text{hom}^T_{\mathcal{K}}[B, A]\) will be the algebra generated by \(z_i^j = b_i \otimes a_j\) and quotient by the ideal algebraically generated by

\[
\left\{ \sigma R_{\lambda_n}^{k_1...k_n} z_{k_1}^{j_1} z_{k_2}^{j_2} \cdots z_{k_n}^{j_n} (\sigma_S^{1})_{j_1...j_n} \right\}_{\lambda_n \in \Lambda_n, n \in \mathbb{N}_0}.
\]
From the semigroupoid structure of $\mathcal{T}$ and the embedding $\mathcal{T}^A \ni \mathcal{T}^A \mapsto \mathcal{T}^B \ni \mathcal{T}^B$, there exist arrows

$$\text{hom}_{\mathcal{T}}^A [\mathcal{B}, \mathcal{A}] \to \text{hom}_{\mathcal{T}}^C [\mathcal{A}, \mathcal{C}] \circ \text{hom}_{\mathcal{T}}^A [\mathcal{C}, \mathcal{B}], \quad \text{end}_{\mathcal{T}}^A [\mathcal{A}] \to \mathcal{K},$$

giving us the notions of cocomposition and coidentity we have just mentioned. Of course, these arrows define a bialgebra structure on $\text{end}_{\mathcal{T}}^A [\mathcal{A}]$. Moreover, there exists a counital 2-cocycle $\chi = \chi$, such that $\text{end}_{\mathcal{T}}^A [\mathcal{A}] \simeq (\mathcal{A} \triangleright \mathcal{A})$. That means $\text{end}_{\mathcal{T}}^A [\mathcal{A}]$ is a twisting by $\chi$ of $\text{end}_{\mathcal{T}}^A [\mathcal{A}]$. Moreover, from the actions $\mathcal{A} \rightarrow \text{end}_{\mathcal{T}}^A [\mathcal{A}] \circ \mathcal{A}$ (the standard coevaluation maps for $\mathcal{A}$), the above twisting can be translated to each $\mathcal{A}$ defining a new quantum space $\mathcal{A}_r$ isomorphic to $\mathcal{T}^r$ (c.f. [11], page 54). Finally,

$$\text{end}_{\mathcal{T}}^A [\mathcal{A}] \simeq \mathcal{A}_r \triangleright \mathcal{A}_r = \text{end} [\mathcal{A}_r] \simeq \text{end} [\mathcal{A}]_r.$$

This paper was mainly motivated to make valid the last equation for every coHom object. Note that for $\text{hom}_{\mathcal{T}}^A [\mathcal{B}, \mathcal{A}]$, with $\mathcal{B} \neq \mathcal{A}$, only a part of this equation is valid, i.e.

$$\text{hom}_{\mathcal{T}}^A [\mathcal{B}, \mathcal{A}] \simeq \mathcal{B}_r \triangleright \mathcal{A}_r = \text{hom} [\mathcal{B}_r, \mathcal{A}_r],$$

since the last part has no sense for a quantum space which has not a bialgebra structure. To do that we shall define along the next chapters a twisting process for all quantum spaces.

2. Cosimplicial quasicomplexes for tensor algebras

Let us consider a (general) quantum space $(\mathcal{A}_1, \mathcal{A})$ with related algebraic structure $(m, \eta)$. Any linear endomorphism $\Xi \in \text{End}_{\text{Vct}}[\mathcal{A} \otimes \mathcal{A}]$ defines a new product $m_{\Xi} = m \Xi$ on $\mathcal{A}$, which can be called twisting or deformation of $m$ by $\Xi$. Let us suppose $\Xi$ is such that $m_{\Xi}$ is associative. If in addition $\Xi$ satisfies

$$(2.1) \quad \Xi (\eta \otimes I) = \eta \otimes I \quad \text{and} \quad \Xi (I \otimes \eta) = I \otimes \eta,$$

then $\eta$ is also a unit for $m_{\Xi}$ and we have a new algebraic structure $(m_{\Xi}, \eta)$ on $\mathcal{A}$, namely the twisting or deformation $\mathcal{A}_{\Xi}$ of $\mathcal{A}$ by $\Xi$. In general, the pair $(\mathcal{A}_1, \mathcal{A}_{\Xi})$ is not a quantum space, because the vector space generated algebraically by $\mathcal{A}_1$ through the product $m_{\Xi}$ is not all of $\mathcal{A} = \mathcal{A}_{\Xi}$ (the equality is for the underlying vector spaces). Just take $\Xi = 0$. Things change when we restrict ourself to the set $\text{Aut}_{\text{Vct}}[\mathcal{A} \otimes \mathcal{A}]$, as we shall see in §3.1. On the other hand, if $(\mathcal{A}_1, \mathcal{A})$ is a conic quantum space with $\mathcal{A} = \bigoplus_{n \in \mathbb{N}_0} \mathcal{A}_n$, we can ask $\Xi$ to be a bihomogeneous of degree zero, i.e. $\Xi (\mathcal{A}_r \otimes \mathcal{A}_s) \subset \mathcal{A}_r \otimes \mathcal{A}_s$, $\forall r, s \in \mathbb{N}_0$. In such a case $m_{\Xi}$ gives rise clearly to the same gradation as $m$. Thus, our objects of interest are homogeneous linear automorphisms satisfying (2.1).

But, what kind of condition(s) must be imposed on $\Xi$ to insure $m_{\Xi}$ is associative (when $m$ is). We shall show immediately that there is a cosimplicial quasicomplex structure related to every tensor algebra of a given vector space. Since every linear map in $\mathcal{A}$ can be defined by one in $\mathcal{A}_r$ through the epimorphism $\mathcal{A}_r \ni \mathcal{A}$, we will study the maps on $\mathcal{A} \otimes \mathcal{A}$ in terms of those on $\mathcal{A}_r \otimes \mathcal{A}_s$, and the mentioned quasicomplex structure. As one might expect, 2-cocycles will define maps whose related twisted products are associative.

Our construction of the cosimplicial object and the associated cochain quasicomplex $\mathcal{C}$ was inspired by the one developed in [11] and related to twisting of bialgebras,\textsuperscript{2} which we shall call $\mathcal{G}$. At the end of this chapter the relationship between these quasicomplexes will be analyzed.

2.1. The cosimplicial object. Consider $\mathcal{V} \in \text{Vct}$ and its associated tensor algebra $\mathcal{V}^\otimes = \bigoplus_{n \in \mathbb{N}_0} \mathcal{V}^\otimes_n$, $\mathcal{V}^\otimes_0 = k$. We shall call

$$\text{C}^n [\mathcal{V}] \subset \text{Aut}_{\text{Vct}} \left( (\mathcal{V}^\otimes)^\otimes_n \right), \quad n \in \mathbb{N}_0,$$

the subgroup of $n$-homogeneous of degree zero, or simply homogeneous, linear automorphisms of $(\mathcal{V}^\otimes)^\otimes_n$, i.e. those linear maps $\psi : (\mathcal{V}^\otimes)^\otimes_n \simeq (\mathcal{V}^\otimes)^\otimes_n$ such that the equalities of sets

$$(2.2) \quad \psi (V^\otimes r_1 \otimes \cdots \otimes V^\otimes r_n) = V^\otimes r_1 \otimes \cdots \otimes V^\otimes r_n,$$

hold for every $r_k \in \mathbb{N}_0$ and $k = 1\ldots n$. In the finite dimensional case, such conditions are equivalent to the inclusions

$$\psi (V^\otimes r_1 \otimes \cdots \otimes V^\otimes r_n) \subset V^\otimes r_1 \otimes \cdots \otimes V^\otimes r_n.$$

The unit element of $\text{C}^n [\mathcal{V}]$ is $\mathbb{I}^\otimes_n$, being $\mathbb{I}$ the identity map of $\text{Aut}_{\text{Vct}} [\mathcal{V}^\otimes]$. In terms of $n$-fold multi-index $R = (r_1, \ldots, r_n) \in \mathbb{N}^n_0$, we can write

$$(2.3) \quad (\mathcal{V}^\otimes)^\otimes_n = \bigoplus_{R \in \mathbb{N}^n_0} \mathcal{V}^\otimes_R; \quad \mathcal{V}^\otimes_R = V^\otimes r_1 \otimes \cdots \otimes V^\otimes r_n,$$

and

$$\psi = \bigoplus_{R \in \mathbb{N}^n_0} \psi_R; \quad \psi_R : V^\otimes_R \simeq V^\otimes R.$$

\textsuperscript{1}Since $m$ is unital, and therefore a surjective linear map, any product $m'$ on $\mathcal{A}$ can be obtained in that way. Indeed, $m' = m_{\Xi}$ with $\Xi = \omega m'$, being $\omega$ some right inverse of $m$.

\textsuperscript{2}A similar construction can be also found in the work of Davidov [11], on twisting of monoidal structures.
By $\mathbb{N}_0^\times$ we understand the singleton set. For the units we write $\mathbb{I}^\otimes_n = \bigoplus_{R \subseteq \mathbb{N}_0^\times} \mathbb{I}_R$ and $\mathbb{I}_1 = \mathbb{I}$. The composition of two maps, namely $\psi$ and $\varphi$, takes the form

$$\psi \varphi = \bigoplus_{R \subseteq \mathbb{N}_0^\times} [\psi \varphi]_R = \bigoplus_{R \subseteq \mathbb{N}_0^\times} \psi_R \varphi_R.$$

This notation lead us to the identification

$$C^n [V] = \times_{R \subseteq \mathbb{N}_0^\times} Aut_{Vct} [V^\otimes R].$$

In other terms, each element of $C^n [V]$ can be seen as a section of a fiber bundle with base $\mathbb{N}_0^\times$ and fibers $Aut_{Vct} [V^\otimes R]$. From now on, whenever we are considering a fixed generic vector space $V$, we shall write $C^n [V] = C^n$, just for brevity. Based on groups $C^n$, we now define a cosimplicial object in the category Grp as follows.

The multiplication $m_0$ of $V^\otimes$ restricted to each subspace $V^\otimes r \otimes V^\otimes s \subseteq V^\otimes \otimes V^\otimes$ is an injective map which has as image the vector subspace $V^\otimes r+s \subseteq V^\otimes$. Then, $m_0$ defines canonical isomorphisms $V^\otimes r \otimes V^\otimes s \simeq V^\otimes r+s$. On the other hand, consider the functions

$$D_i^n = I \times \ldots \times D \times \ldots \times I : \mathbb{N}_0^{\times n+1} \rightarrow \mathbb{N}_0^{\times n}; \quad i \in \{1, \ldots, n\},$$

where the sum on integers $D : (n, m) \mapsto n + m$ acts on the $i$-th and $i+1$-th factors of $\mathbb{N}_0^{\times n+1}$. In terms of these functions, given $R = (r_1, \ldots, r_{n+1})$ with $n > 0$, the restrictions to $V^\otimes R$ of the maps

$$m_i^n = I \otimes \ldots \otimes m_0 \otimes I \otimes \ldots \otimes I : \left( V^\otimes \right)^{\otimes n+1} \rightarrow \left( V^\otimes \right)^{\otimes n},$$

$i \in \{1, \ldots, n\}$, define canonical bijections $m_i^n : V^\otimes R \simeq V^\otimes D_i^n(R)$. The latter give rise to group homomorphisms $\delta_i^n : C^n \rightarrow C^{n+1}$ by the assignment

$$\psi \mapsto \delta_i^n \psi = \bigoplus_{R \subseteq \mathbb{N}_0^{\times n+1}} [\delta_i^n \psi]_R; \quad [\delta_i^n \psi]_R = (m_i^n)^{-1} \psi_{D_i^n(R)} m_i^n.$$

The elements $\delta_i^n \psi$ are uniquely determined by

$$m_i^n \delta_i^n \psi = \psi m_i^n,$$

since the restriction of above equation to $V^\otimes R$ gives precisely the last part of Eq. (2.6). Using the identification $V^\otimes R \simeq V^\otimes D_i^n(R)$, we can write in compact form

$$\delta_i^n \psi \cong \bigoplus_{R \subseteq \mathbb{N}_0^{\times n+1}} \psi_{D_i^n(R)}.$$

Let us also define the homomorphisms $\delta_{0,n+1}^n : C^n \rightarrow C^{n+1}, n \neq 0$, as $\delta_{0,n}^n \psi = \mathbb{I} \otimes \psi$ and $\delta_{n+1,n}^n \psi = \psi \otimes \mathbb{I}$. For $n = 0$ any element $\chi \in C^0$ is a map $1 \mapsto \lambda \in k^\times \equiv k - \{0\}$. In other words $C^0 \simeq k^\times$, the multiplicative group of the field $k$. So we can define $\delta_{0,1}^0 \chi$ by the assignment $v \in V^\otimes \mapsto \lambda \cdot v \in V^\otimes$. In resume, we have built up a set of group homomorphisms $\delta_i^n$, $i \in \{0, 1, \ldots, n+1\}, n \in \mathbb{N}_0$. We shall call them cofaces operators.

Consider now, for $i \in \{0, 1, \ldots, n\}$, the linear transformations

$$\eta_i^n = I \otimes \ldots \otimes I \otimes \eta_0 \otimes \ldots \otimes I : \left( V^\otimes \right)^{\otimes n} \rightarrow \left( V^\otimes \right)^{\otimes n+1},$$

related to the unit map $\eta_0 : k \rightarrow V^\otimes$, and the functions

$$S_i^n : \mathbb{N}_0^{\times n} \rightarrow \mathbb{N}_0^{\times n+1} / \ S_i^n (r_1, \ldots, r_i, r_{i+1}, \ldots, r_n) = (r_1, \ldots, r_i, 0, r_{i+1}, \ldots, r_n).$$

Since $\eta_0$ as a map over $V^\otimes 0$ is the identity, given $R \subseteq \mathbb{N}_0^{\times n}$ the map $\eta_i^n$ restricted to $V^\otimes R$ defines a canonical bijection $\eta_i^n : V^\otimes R \simeq V^\otimes S_i^n(R)$. From $\eta_i^n$ and $S_i^n$, the group homomorphisms $\sigma_i^n : C^{n+1} \rightarrow C^n$, with

$$\psi \mapsto \sigma_i^n \psi = \bigoplus_{R \subseteq \mathbb{N}_0^{\times n}} [\sigma_i^n \psi]_R; \quad [\sigma_i^n \psi]_R = (\eta_i^n)^{-1} \psi_{S_i^n(R)} \eta_i^n,$$

can be defined. The equation $\eta_i^n \sigma_i^n \psi = \psi \eta_i^n$ determines $\sigma_i^n \psi$ completely. From the identification $V^\otimes R \simeq V^\otimes S_i^n(R)$ we can write

$$\sigma_i^n \psi \cong \bigoplus_{R \subseteq \mathbb{N}_0^{\times n}} \psi_{S_i^n(R)}.$$

They will be called codegeneracies.

**Theorem 2.** The map $[n+1] \mapsto C^n [V]$ defines a cosimplicial object in Grp; i.e. it can be extended to a functor $\Delta \rightarrow$ Grp, being $\Delta$ the simplicial category.
We define the multplicative quasicomplex structure, in a very close fashion to the cochain quasicomplex for twisting of bialgebras. (2.13)

\[m_j \circ_i m_i = m_i \circ_{i+1} m_{i+1}\]

These are direct consequences of the associative monoidal structures defined by \((D, S)\) in \(N_0\) and by \((m, \eta)\) in \(V\). Using this fact, let us show the first one of the above equalities. Taking \(\psi \in C^n\) we have for all \(\psi \in C^{n+2}\)

\[\psi \circ_i \psi |_R = (m_i^n)^{-1} \psi |_{D^n_i R} m_{i+1}^{D^n_i R} m_i R\]

Note that, for instance, \(m_j^{D^n_j R} \) is \(m_j^{n+1}\) restricted to \(V^\otimes D^{n+1}_j R\). Restricting Eq. (2.12) to \(V^\otimes R\) and taking into account the domains where \(m_i^n\)'s are applied, we arrive at

\[\psi \circ_i \psi |_{R} = (m_j^{D^n_j R} m_i R)^{-1} \psi |_{D^n_j R} m_{i+1}^{D^n_j R} m_i R\]

The others equalities can be shown in an analogous way.

Now, we are going to endow the family of groups \(\{C^n[V]\}_{n \in \mathbb{N}_0}\) with a multiplicative quasicomplex structure, in a very close fashion to the cochain quasicomplex for twisting of bialgebras.

2.2. The multiplicative quasicomplex \(C^*\). Unless a confusion may arise, the index \(n\) on maps \(\delta^n\) will be omitted. Consider the map \(\partial\) that assigns to each \(\psi \in C^n\), for every \(n \in \mathbb{N}_0\), an element of \(C^{n+1}\) given by the equation

\[\partial \psi = \left( \prod_{i \text{ odd}} \delta_i \psi \right) \left( \prod_{i \text{ even}} \delta_i \psi \right)^{-1},\]

where \(i \in \{0, \ldots, n+1\}\) and being

\[\prod_{\{m_1 < \ldots < m_k\}} \varphi_i \overset{=} {\circ} \varphi_{m_1} \varphi_{m_2} \ldots \varphi_{m_k}, \quad \prod_{\{m_1 < \ldots < m_k\}} \varphi_i \overset{=} {\circ} \varphi_{m_k} \varphi_{m_{k-1}} \ldots \varphi_{m_1}.\]

The notation \(\partial \psi = \partial_+ \psi (\partial_- \psi)^{-1}\),

\[\partial_+ \psi = \prod_{i \text{ odd}} \delta_i \psi, \quad \partial_- \psi = \prod_{i \text{ even}} \delta_i \psi,\]

will be very useful. Note that \(\partial : C^n \to C^{n+1}\) is not a group homomorphism for every \(n\), but satisfies \(\partial^\otimes n = \otimes^{n+1}\). The same holds for maps \(\partial_{\pm}\).

**Definition 1.** We define the multiplicative cosimplicial quasicomplex of \(V\) (or its tensor algebra), denoted by the pair \((C^* [V], \partial)\), as the sequence

\[C^0 [V] \xrightarrow{\partial} C^1 [V] \xrightarrow{\partial} C^2 [V] \xrightarrow{\partial} C^3 [V] \xrightarrow{\partial} \ldots .\]

The prefix *quasi* is used because in general \( \partial^2 (C^n [V]) \neq \{I^{\otimes n+2}\} \). However, we shall see for \( n = 0, 1 \) the equality holds. Sometimes we shall omit for brevity this prefix.

Let \( \text{Grp}_n \) be the full subcategory of Set* whose objects are groups based on their respective unit elements (i.e. its morphisms are unit preserving functions), and denote by \( \text{Grp}_{eq} \) the category of quasiobjects in \( \text{Grp}_n \subset \text{Set}^* \). Its arrows \( \mathfrak{f} : (C^*, \partial_c) \rightarrow (D^*, \partial_0) \) are units of unit preserving functions \( f^n : C^n \rightarrow D^n \) such that equality \( f^{n+1} \partial_c = \partial_0 f^n \) holds for all \( n \). Then, the pair \( (C^* [V], \partial) \) is an object of \( \text{Grp}_{eq} \), and the triple \((C^* [V], \partial_+ , \partial_-)\) is a unipotent multiplication *quasi*complex in \( \text{Grp}_n \) (see [2] and references therein).

The notions of cocycles, coboundaries, cohomology relation and cocontinuity for \((C^*, \partial)\) are the following.

- **An n-cocycle** is a cochain \( \chi \in C^n \) satisfying \( \partial \chi = I^{\otimes n+1} \). We indicate \( Z^n = Z^n [V] \) the set of n-cocycles in \( C^n \).

- Two n-cochains \( \chi \) and \( \chi' \) are called **cohomologous** if there exists \( \theta \in C^{n-1}, n \in \mathbb{N} \), such that the equation \( \partial \theta \chi (\partial \theta) \chi' = \chi' \) holds. We denote \( \chi \sim \theta \chi' \) if \( \chi \) and \( \chi' \) are cohomologous through \( \theta \), or simply \( \chi \sim \chi' \), and call \( \text{Coh}^n = \text{Coh}^n [V] \) the subset of \( C^n \times C^n \) of cohomologous pairs.

- **An n-coboundary** is an element \( \omega \in C^n \) with \( \Gamma^{\otimes n} \sim \omega \), i.e. such that \( \exists \theta \in C^{n-1} \) satisfying \( \partial \theta (\partial \theta)^{-1} = \omega \). We denote \( B^n = B^n [V] \) the n-coboundaries of \( C^n \). In particular, \( \{ \Gamma^{\otimes n} \} \times B^n \subset \text{Coh}^n \), for all \( n \in \mathbb{N} \).

- **An n-counital** cochain is a map \( \chi \in C^n \) satisfying \( 0 \leq i \leq n-1, \chi \eta_i^{n-1} = \eta_i^{n-1} \), i.e. \( \sigma^n_{i-1} \chi = I^{\otimes n-1} \). Each \( C^n = C^n [V] \) indicates the subgroup of counital cocycles in \( C^n \).

We must mention that the cohomology relation is not an equivalence relation for all \( n \). In general it is only reflexive, because \( \chi \sim \chi \) through \( \theta = I^{\otimes n} \).

As we have said at the beginning of this chapter, we are interested in cochains satisfying Eq. (2.11), i.e. \( \chi (\eta \otimes I) = (\eta \otimes I) \) and \( \chi (I \otimes \eta) = (I \otimes \eta) \). Since \( \eta_i^1 \) and \( \eta_i^1 \) both can be rewritten \( \beta_i^1 = \eta_i^1 \), \( i = 0, 1 \). This are exactly the counital cochains. It follows from (2.11) that, if \( \chi \in C^n \), then \( \delta^n \chi \) is not necessarily in \( C^{n+1} \). For example, provided \( \sigma^n_i \delta_i^n = id \), then \( \sigma^n_i \delta_i^n \chi = \chi \). Hence, the groups \( C^n \) do not define a cosimplicial subobject of \( C^* \). Nevertheless, for the quasicomplex structure we have the following result.

**Proposition 1.** The restriction \( \partial_c \) of \( \partial \) to the groups \( C^n \) defines a quasicomplex \((C^*, \partial_c) \in \text{Grp}_{eq} \), such that the inclusions \( u : C^n \hookrightarrow C^n \) make \( C^* \) a subobject of \( C^* \).

**Proof.** We must show that \( \partial (C^n) \subset C^{n+1} \). So, let us calculate \( \sigma_j^n \partial \psi \) for \( \psi \) such that \( \sigma_j^{n-1} \psi = I^{\otimes n-1} \), with \( j = 0, ..., n-1 \). Since \( \sigma_j^n \) is a group homomorphism, \( \sigma_j^n \partial \psi = \left( \prod_{i \text{ odd}} \sigma_j^n \delta_i^n \psi \right) \left( \prod_{i \text{ even}} \sigma_j^n \delta_i^n \psi \right)^{-1} \). On the other hand, from (2.11) (remember that \( i = 0, 1, ..., n+1 \))

\[
\sigma_j^n \delta_i^n \psi = \delta_i^{n-1} \sigma_{j-i}^{n-1} \psi = \delta_i^{n-1} I^{\otimes n-1} = I^{\otimes n}, \quad \text{if } i < j;
\]

\[
\sigma_j^n \delta_i^n \psi = \delta_i^{n-1} \sigma_{j-i}^{n-1} \psi = \delta_i^{n-1} I^{\otimes n-1} = I^{\otimes n}, \quad \text{if } i > j+1;
\]

\[
\sigma_j^n \delta_i^n \psi = \psi, \quad \text{if } i = j, i = j+1;\]

hence, if \( j \) is odd, \( \sigma_j^n \partial \psi = \sigma_j^n \delta_j^n \psi \left( \sigma_j^{n} \delta_{j+1}^{n} \psi \right)^{-1} = \psi \psi^{-1} = I^{\otimes n} \). The same can be done for \( j \) even. Therefore \( \sigma_j^n \partial \psi = I^{\otimes n} \) for \( j = 0, ..., n-1 \); that is to say, \( \partial \psi \in C^{n+1} \) if \( \psi \in C^n \).

From now on, \( \partial \) will denote the coboundary operator for both \( C^* \) and \( C^n \). It is worth mentioning \((C^*, \partial_+, \partial_-)\) is not a parait subquasicomplex of \((C^*, \partial_+, \partial_-)\).

The subsets of cocycles, coboundaries and cohomologous cochains in \( C^* \) will be indicated by \( Z^n \subset Z^n \), \( B^n \subset B^n \) and \( \text{Coh}^n \subset \text{Coh}^n \), respectively. Note that \( \mathfrak{f} = Z^n \cap C^n \), but for \( B^n \) and \( \text{Coh}^n \), in general, (2.15) \( B^n \subset B^n \cap C^n \) and \( \text{Coh}^n \subset \text{Coh}^n \cap C^n \times C^n \).

2.2.1. **Elementary calculations in low geometric dimensions.**

**n = 0:** Given \( \chi : k \rightarrow k : \lambda \rightarrow \lambda \), then \( \chi^{-1} : 1 \rightarrow \lambda^{-1} \), it follows that \( \delta_{0,1} \chi : v \rightarrow \lambda : v \), and

\[
\partial \chi = \delta \chi \left( \delta_0 \chi \right)^{-1} = I : V^\otimes \rightarrow V^\otimes.
\]

Hence \( B^1 = \{I\} \), and as a consequence \( B^1 = B^1 \). In particular, \( \partial \) restricted to \( C^0 \) is the trivial group morphism, and accordingly \( Z^0 = \{ \chi \in C^0 : \partial \chi = I \} = C^0 = k^\times \). In particular, \( \partial^2 \chi = I^\otimes 2 \) for every \( \chi \in C^0 \). On the other hand, the unique counital 0-cochain is the identity map \( I_k = \mathbb{I}_0 \). Thus, \( C^0 = \{\mathbb{I}_0\} \subset C^0 \).
\[ (2.16) \quad \partial \alpha = \delta_1 \alpha \ (\delta_0 \alpha)^{-1} \ (\delta_2 \alpha)^{-1} = (\partial \alpha) \ (\alpha \otimes \alpha)^{-1}. \]

Then, a 1-cocycle is an element \( \alpha \in C^1 \) such that \( \delta_1 \alpha = \alpha \otimes \alpha \), or \( \alpha_{r+s} \approx \alpha_r \otimes \alpha_s \). In particular, \( \alpha_s \approx \alpha_0 \otimes \alpha_s \), so \( \alpha_0 = I_k \). This means that \( \alpha \) must be counital, i.e. the inclusion \( Z^1 \subset C^1 \) holds. Moreover, from Eq. \( (2.16) \) the cocycle condition is equivalent to \( \alpha m_\otimes = m_\otimes \delta_1 \alpha = m_\otimes (\alpha \otimes \alpha) \), hence \( \alpha \) is a 1-cocycle iff it is an algebra automorphism of \( V^\otimes \) obeying \( \alpha(V) \subset V \). Then, such algebra automorphisms are in bijection with linear automorphisms \( Aut_{VC}[V] = \text{GL}(n) \), \( n = \dim V \), i.e. \( \text{GL}(n) \approx Z^1 = C^1 \). With respect to the square of \( \partial \) on \( C^1 \), we have
\[ \partial^2 \alpha = \delta_1 (\partial \alpha) \delta_2 (\partial \alpha)^{-1}. \]
Further, restricting \( \delta_{1,2} (\alpha) \) to some \( V^\otimes \otimes V^\otimes \otimes V^\otimes \), and using the identifications \( \lambda_3 = \lambda_1 \), we have
\[ \partial^2 \alpha = \delta_1 (\partial \alpha) \delta_2 (\partial \alpha)^{-1} = (\partial \alpha) \ (\alpha \otimes \alpha)^{-1} \ (\partial \alpha)^{-1}, \]
and using that \( \delta_1 (\alpha \otimes \alpha) = \delta_1 \alpha \otimes \alpha, \delta_2 (\alpha \otimes \alpha) = \alpha \otimes \delta_1 \alpha, \) and that the \( \delta_i \)'s are group homomorphisms, it follows from Eq. \( (2.16) \)
\[ \partial^2 \alpha = \delta_1 (\partial \alpha) \delta_2 (\partial \alpha)^{-1}. \]
\[ (2.17) \]
\[ \partial^2 \alpha = \delta_1 (\partial \alpha) \delta_2 (\partial \alpha)^{-1}. \]

But \( \partial \) is not a morphism of groups for \( n = 1 \).

Given \( \alpha, \beta \in C^1 \), they are cohomological iff there exists \( \lambda \in C^0 = k^\times \) such that \( \lambda \cdot \alpha \cdot \lambda^{-1} = \beta \) iff \( \alpha = \beta \). In particular \( B^1 = \{ I \} = B^1 \), as we have previously seen. Restricting ourself to counital cochains we have \( (\alpha, \beta) \in \text{Coh}^1 \) iff there exists \( \lambda \in C^0 = k^\times \) such that \( \lambda \cdot \alpha \cdot \lambda^{-1} = \beta \). But \( C^0 = \{ I_0 \} \), so \( (\alpha, \beta) \in \text{Coh}^1 \) iff \( \alpha = \beta \), and consequently \( \text{Coh}^1 = \text{Coh}^1 \cap C^1 \cap C^1 \).

\( n = 2 \) : A 2-cocycle is an element \( \psi \in C^2 \) satisfying \( \delta_1 \psi (\psi \otimes I) = \delta_2 \psi (I \otimes \psi) \), or
\[ (2.19) \quad (\delta_1 \psi) (\psi_{r,s} \otimes I_t) (I_r \otimes \psi_{s,t}) = (\delta_2 \psi) (\psi_{r,s} \otimes I_t) (I_r \otimes \psi_{s,t}). \]
A counital 2-cochain fulfills the equations \( \psi (1 \otimes a) = 1 \otimes a \) and \( \psi (a \otimes 1) = a \otimes 1 \), that is to say, \( \psi_{0,s} \approx I_s \) and \( \psi_{r,0} \approx I_r \). From Eq. \( (2.19) \) is easy to see that
\[ (2.20) \quad \psi_{0,s} \approx \psi_{0,0} \cdot I_s \quad \text{and} \quad \psi_{r,0} \approx \psi_{0,0} \cdot I_r, \]
regarding \( \psi_{0,0} \) as an element of \( k^\times \). That means,
\[ (2.21) \quad \psi \in Z^2 \quad \text{iff} \quad \psi \in Z^2 \text{ and } \psi_{0,0} = 1. \]

Suppose \( (\psi, \varphi) \in \text{Coh}^2 \), i.e. \( \partial_\theta \psi = \varphi \partial_\theta \psi \) for some \( \theta \in C^1 \), or in other terms, \( \theta_{r+s} \psi_{r,s} \approx \varphi_{r,s} (\theta_r \otimes \theta_s) \). If \( (\psi, \varphi) \in \text{Coh}^2 \cap C^2 \times C^2 \), then \( \psi_{0,0} = \varphi_{0,0} = 1 \) and \( \theta_0 \approx \theta_0 \psi_{0,0} \approx \varphi_{0,0} (\theta_0 \otimes \theta_0) \approx (\theta_0 \otimes \theta_0) \), and in consequence \( \theta_0 = 1 \), i.e. \( \theta \in C^1 \). Therefore, \( (\psi, \varphi) \in \text{Coh}^2 \). Summing up (compare with Eq. \( (2.19) \)),
\[ (2.22) \quad \text{Coh}^i = \text{Coh}^i \cap C^i \times C^i \quad \text{and} \quad B^i = B^i \cap C^i, \quad \text{for } i = 1, 2. \]

2.2.2. (Anti)bicharacter like cochains. Using the terminology of linear forms over a bialgebra, we shall say \( \psi \in C^2 \) is a bicharacter if
\[ (2.23) \quad \delta_1 \psi = \psi_{23} \psi_{13}, \quad \delta_2 \psi = \psi_{12} \psi_{13}, \]
and anti-bicharacter (see [2] for examples in the bialgebra case) if
\[ (2.24) \quad \delta_1 \psi = \psi_{13} \psi_{23}, \quad \delta_2 \psi = \psi_{13} \psi_{12}, \]
where \( \psi_{12} = \psi \otimes I, \psi_{23} = I \otimes \psi \) and \( \psi_{13} = (I \otimes f_{0,1}) \psi_{12} (I \otimes f_{0,0}) \). We are denoting by \( f_{0} \) the flipping operator on \( V^\otimes \otimes V^\otimes \). Using the identifications \( \lambda_3 = \lambda_1 \) and following obvious notation, Eq. \( (2.23) \) can be written
\[ (2.25) \quad \psi_{r,s,t} \approx (\psi_{r,s} \otimes I_t) (I_r \otimes f_{s,t}^{-1}) (\psi_{r,t} \otimes I_s) (I_r \otimes f_{s,t}), \]
Note that every (anti)bicharacter \( \psi \) is always counital; in fact,
\[ (2.26) \quad \psi_{0,t} \approx \psi_{0,t} \approx (I_0 \otimes \psi_{0,t}) (I_0 \otimes f_{0,t}^{-1}) (\psi_{0,t} \otimes I_0) (I_0 \otimes f_{0,t}) \approx \psi_{0,t}^2, \]
thus \( \psi_{0,t} \approx I_t \). Also, \( \psi \) is completely define by \( \psi_{1,1} \).

If \( \psi \) is a bicharacter, it will be a 2-cocycle iff
\[ (2.27) \quad \psi_{12} \psi_{13} \psi_{23} = \psi_{23} \psi_{13} \psi_{12}, \]
i.e. $\psi$ satisfies the Yang-Baxter (YB) equation. For an anti-bicharacter, $\psi$ is a 2-cocycle iff

\begin{equation}
\psi_{12}\psi_{23} = \psi_{23}\psi_{12}.
\end{equation}

It can be shown it is enough for $\psi_{1,1}$ to satisfy some of the above equations (depending on the case) in order to ensure $\psi$ is a 2-cocycle. Then, given $M \in GL(n^2)$ satisfying \[\psi_{1,1}(a \otimes a) = M_{ij}^k a_i \otimes a_j\] (resp. \[\psi_{1,2}(a \otimes a) = M_{ij}^k a_i \otimes a_j\]), we can define a counital 2-cocycle $\psi$ related to an $n$-dimensional vector space $V$, just taking $\psi_{1,1}(a \otimes a) = M_{ij}^k a_i \otimes a_j$ and extending $\psi$ to all of $V^\otimes \otimes V^\otimes$ using Eq. \[\psi_{1,2}(a \otimes a) = M_{ij}^k a_i \otimes a_j\] (resp. \[\psi_{1,2}(a \otimes a) = M_{ij}^k a_i \otimes a_j\]).

2.2.3. First and second cohomology spaces. We have just seen that the cohomology relation in $C^1[V]$ is the identity relation, so is an equivalence one. In particular, it can be used to define the first cohomology space $H^1$ as the quotient of $Z^1$, i.e. $H^1 \cong Z^1 \cong GL[V]$. The same can be done for $n = 2$.

Proposition 2. The cohomology relation for $n = 2$ is an equivalence relation, and every equivalence class with some element in $Z^2$ is contained there. This is also valid for $C^\bullet$.

Proof. We need to show that the relation is symmetric and transitive. Suppose $(\varphi, \psi) \in \text{Coh}^2$, i.e. there exists $\alpha \in C^1$ such that $\partial_+ \varphi (\partial_+ \alpha)^{-1} = \psi$. Let us first note that $\partial_\pm$ are group homomorphisms when restricted to $C^1$. Indeed, $\partial_- (\alpha) = \delta_1 (\alpha)$ and $\partial_+ (\alpha) = \alpha \otimes \alpha$ for all $\alpha \in C^1$. In particular $\partial_- (\alpha^{-1}) = \delta_1 (\alpha^{-1}) = (\delta_1 \alpha)^{-1}$ and $\partial_+ (\alpha^{-1}) = \alpha^{-1} \otimes \alpha^{-1} = (\partial_+ \alpha)^{-1}$. Thus

\[
\partial_- (\alpha^{-1}) \psi \partial_+ (\alpha^{-1})^{-1} = \varphi, \quad i.e. \quad (\psi, \varphi) \in \text{Coh}^2,
\]

and accordingly the relation is symmetric. To show the transitivity, consider again the cohomologous maps $\varphi$ and $\psi$, and a 2-cochain $\phi \in C^2$ such that there exists $\beta$ satisfying $\partial_- \beta \psi (\partial_+ \beta)^{-1} = \phi$, i.e. $\psi \sim \phi$. Then

\[
\partial_- (\beta \alpha) \varphi (\partial_+ (\beta \alpha))^{-1} = \partial_- \beta \left( \partial_- \alpha \varphi \partial_+ (\alpha^{-1}) \right) (\partial_+ \beta)^{-1} = \phi.
\]

Accordingly, $\varphi \sim \phi$ through $\beta \alpha$. It rests to prove each equivalence class containing an element of $Z^2$ is a subset of $Z^2$, that is to say, given a 2-cocycle $\varphi \in C^2$ and a 1-cochain $\alpha \in C^1$, $\psi = \delta_1 \alpha \varphi (\alpha \otimes \alpha)^{-1}$ is a 2-cocycle too. But,

\[
\partial_- \psi = \delta_1 (\delta_1 \alpha) \delta_1 \varphi \left( \delta_1 (\alpha)^{-1} \otimes \alpha^{-1} \right) \left( \delta_1 \alpha \varphi (\alpha \otimes \alpha)^{-1} \otimes \mathbb{I} \right)
\]

and

\[
(\partial_+ \psi)^{-1} = \left( \mathbb{I} \otimes (\alpha \otimes \alpha) \varphi^{-1} \delta_1 (\alpha)^{-1} \right) (\alpha \otimes \delta_1 \alpha) \delta_2 (\varphi)^{-1} \delta_2 (\delta_1 \alpha)^{-1}
\]

Hence, from the last equations and Eq. \[\psi_{1,2}(a \otimes a) = M_{ij}^k a_i \otimes a_j\], as we wanted to show.

For $C^\bullet$ we just have to use the monic $u : C^\bullet \hookrightarrow C^\bullet$, which is given by group monomorphisms. $lacksquare$

Then we can define the second cohomology space $H^2[V]$ as the set of cohomologous classes in $Z^2$, i.e. $H^2 = Z^2 / \sim_{\text{Coh}}$. We are going to see in the next chapter that any 2-cocycle is a 2-coboundary. Therefore, every $\psi \in Z^2$ is cohomologous to $\mathbb{I}^\otimes 2$, and accordingly $H^2[V] = \{ \mathbb{I}^\otimes 2 \}$. Nevertheless, this does not mean all deformations on a given quantum space generated by $V$ are trivial, because the maps that define twisted algebras isomorphic to the original one are those cohomologous to the identity through certain class of 1-cochains (what we shall call admissible), and not through any of them.

2.3. Functorial properties of $C^\bullet[V]$. Let us consider the cosimplicial objects $C^\bullet_V : \Delta \rightarrow \text{Grp}$ and $C^\bullet_W : \Delta \rightarrow \text{Grp}$ with $V$ and $W$ in Vct.

Theorem 3. The functors $C^\bullet_V$ and $C^\bullet_W$ are naturally equivalent if $V$ and $W$ are isomorphic vector spaces.

Proof. We must show that there exists a family \( \{ \alpha_{[n+1]} \}_{n \in \mathbb{N}_0} \) of group isomorphisms

\[
C^\bullet_V [n+1] = C^\bullet [V] \cong C^\bullet [W] = C^\bullet_W [n+1]
\]

such that

\[
\alpha_{[n+1]} \delta^n_{i, V} = \delta^n_{i, W} \alpha_{[n+1]} \quad \text{and} \quad \alpha_{[n+1]} \sigma^n_{i, V} = \sigma^n_{i, W} \alpha_{[n+1]}.
\]
Let \( f \) be an isomorphism between \( V \) and \( W \). Then, the map \( f^\circ : V^\circ \to W^\circ \), the unique extension of \( f \) to \( V^\circ \) as an algebra homomorphism, defines for every \( n \in \mathbb{N}_0 \) a group isomorphism \( C^n[V] \simeq C^n[W] \) given by
\[
\psi \mapsto \psi^f = (f^\circ)^{\otimes n} \psi \left( (f^\circ)^{\otimes n} \right)^{-1}.
\]
In particular, since
\[
(f^\circ)^{\otimes n} (V^\circ \otimes \ldots \otimes V^\circ) \subset W^\circ \otimes \ldots \otimes W^\circ,
\]
or in compact form \((f^\circ)^{\otimes n} (V^\circ)^{\otimes R} \subset W^\circ^{\otimes R}\), we can write
\[
(f^\circ)^{\otimes n} = \bigoplus_{R \in \mathbb{N}_0^+} f^\circ_R \quad \text{and} \quad \psi^f = \bigoplus_{R \in \mathbb{N}_0^+} f^\circ_R \psi_R \left( f^\circ_R \right)^{-1}.
\]
Let us call \( f^n \) such group isomorphisms, and show that
\[
(f^\circ)^{\otimes n} m_{i,V} = m_{i,W} \quad \text{and} \quad (f^\circ)^{\otimes n+1} \eta_{i,V} = \eta_{i,W}.
\]
and analogous equations to its inverse, hold. Restricting to some \( R \in \mathbb{N}_0^{n+1} \) they translate into
\[
(f^\circ)^{\otimes D^n(R)} m_{i,V} = m_{i,W} \quad \text{and} \quad (f^\circ)^{\otimes S^{n+1}(R)} \eta_{i,V} = \eta_{i,W}.
\]
For the first equality of (2.29), given \( \psi \in C^n[V] \) and \( R \in \mathbb{N}_0^{n+1} \), we have
\[
[f^{n+1} \psi_R]_R = f^{\otimes R} \left[ \delta^n_{i,W} \psi_R \left( f^\circ_R \right)^{-1} = f^{\otimes R} \left( \eta_{i,V} \psi_R \left( f^\circ_R \right)^{-1}
\]
and using (2.30),
\[
[f^{n+1} \psi_R]_R = (f^{\otimes D^n(R)} \psi_R \left( f^\circ D^n(R) \right)^{-1} m_{i,W}^R
\]
and taking \( \alpha_{n+1} = f^n \) we have the natural equivalence \( C^n[V] \simeq C^n[W] \) we are looking for. 

This equivalence can be translated to the corresponding multiplicative quasicomplex structures, as follows. Let us call \( G[Vct] \subset Vct \) the groupoid associated to \( Vct \), i.e. the subcategory of \( Vct \) whose morphisms are isomorphisms.

**Theorem 4.** The map \( V \mapsto (C^* [V], \partial_V) \) defines a functor \( G[Vct] \to \text{Grp}_{eq} \).

**Proof.** Let us show the map \( f \mapsto f^*, f : V \simeq W \) and \( f^n \psi = \psi^f \) for all \( \psi \in C^n[V] \) (see Eq. (2.28)), extends \( V \mapsto (C^* [V], \partial_V) \) to a functor. That is to say, \( f^* \) is such that \( f^{n+1} \partial_V = \partial_W \). Using Eq. (2.29)
\[
f^{n+1} \partial_V = \left( \prod_{i \text{ odd}} \left( f^{n+1} \delta_{i,V} \psi \right) \right) \left( \prod_{i \text{ even}} \left( f^{n+1} \delta_{i,V} \psi \right) \right)^{-1}
\]
(2.31)
\[
= \left( \prod_{i \text{ odd}} \delta_{i,W} \left( f^n \psi \right) \right) \left( \prod_{i \text{ even}} \delta_{i,W} \left( f^n \psi \right) \right)^{-1} = \partial_W f^n \psi,
\]
since each \( f^n \) is a group homomorphism. Hence, the claim of the proposition follows. 

Note the functor \( G[Vct] \to \text{Grp}_{eq} \) defines an homomorphism of groupoids
\[
G[Vct] \to G[\text{Grp}_{eq}]
\]
that assigns to each linear isomorphism \( f : V \simeq W \) a quasicomplex isomorphism (see (2.28))
\[
f^* : (C^* [V], \partial_V) \simeq (C^* [W], \partial_W) : \psi \mapsto \psi^f.
\]
From this, we have the bijections
\[
Z^n [V] \simeq Z^n [W], B^n [V] \simeq B^n [W] \quad \text{and} \quad \text{Coh}^n [V] \simeq \text{Coh}^n [W]
\]
for all \( n \in \mathbb{N}_0 \). Clearly, all these results are also true for the subquasicomplexes \( C^* [V] \) of countial cochains.

In the following subsections we present some structural results. They relate the quasicomplex of a vector space \( W \) with one of its subspaces \( V \subset W \), and to its dual \( W^* \). Also relate \( C^* [U] \) and \( C^* [V] \) with the
quasicomplexes $C^\bullet [U \otimes V]$. From these results we are able to define a full subcategory of $\text{Grp}_{eq}$, with monoidal structure given by the direct product $\times$ of groups.

All constructions and results will be based on $C^\bullet$, but are also valid, at the quasicomplex level, for the related subquasicomplex $C^\bullet$.

2.3.1. Linear subspaces and subobjects in $\text{Grp}_{eq}$. Consider the inclusion of vector spaces $V \subset W$. The cochains $\psi \in C^n[W]$ such that $\psi(V^\otimes R) \subset V^\otimes R$ holds $\forall R \in \mathbb{N}_0^n$, form a subgroup of $C^n[W]$ that we shall denote $C^n[V \subset W]$. Given $\psi \in C^n[V \subset W]$, let us indicate by $\psi|_V$ and $[\psi|_V]_R$ the restrictions of $\psi$ and $\psi_R$ to $(V^\otimes)^\otimes_R$ and $V^\otimes R$, respectively. Of course, $\psi|_V$ is an element of $C^n[V]$ and any element there can be obtained in that way. We are interested on the relation between the cohomological properties of the elements $\psi \in C^n[V \subset W]$, and the ones of their associated elements $\psi|_V$ in $C^n[V]$. To this end, consider the following result.

**Proposition 3.** The subgroups $C^n[V \subset W]$ define a multiplicative cosimplicial quasicomplex $C^\bullet [V \subset W]$, such that the canonical inclusions and projections

$$
C^n[V \subset W] \hookrightarrow C^n[W]; \quad C^n[V \subset W] \twoheadrightarrow C^n[V] : \psi \mapsto \psi|_V,
$$

give rise to natural transformations of the respective cosimplicial objects, and to monic and epimorphic arrows, respectively, in $\text{Grp}_{eq}$.

**Proof.** Note that if $V \subset W$ as vector spaces, then $V^\otimes \subset W^\otimes$ as algebras. Hence (see Eq. (2.6)), for every $\psi \in C^n[V \subset W]$ and $R \in \mathbb{N}_0^n$, we have

$$
(\delta_i,\psi)|_{V^\otimes R} = (m^R_{i,W})^{-1} \psi|_{D^R_i(R)} m^R_{i,W} \big|_{V^\otimes R} = (\delta_i,\psi|_V)|_R.
$$

In particular,

$$
\delta_i,\psi|_V (V^\otimes R) \subset V^\otimes R, \quad \forall R \in \mathbb{N}_0^n,
$$

i.e. $\delta_i,\psi \in C^n[V \subset W]$ if $\psi \in C^n[V \subset W]$, and consequently Eq. (2.32) can be written

$$
(\delta_i,\psi)|_V = \delta_i,\psi|_V.
$$

Furthermore, the restriction of $\delta_i,\psi$ to $C^n[V \subset W]$ gives a map

$$
\delta_i,\psi|_V : C^n[V \subset W] \rightarrow C^n[V \subset W] + 1.
$$

such that

$$
\delta_i,\psi|_V \psi = \delta_i,\psi
$$

and (using Eq. (2.33))

$$
(\delta_i,\psi|_V \psi)|_V = \delta_i,\psi|_V.
$$

The same is true for the maps $\sigma_i,\psi$. In consequence, from $\delta_i,\psi|_V$ and $\sigma_i,\psi|_V$ a cosimplicial object

$$
C^\bullet[V \subset W] : \Delta \rightarrow \text{Grp} : [n+1] \rightarrow C^n[V \subset W]
$$

is defined and, since Eqs. (2.34) and (2.35), the proposition follows.

We must mention the group injections $C^n[V] \rightarrow C^n[V \subset W]$ given by the map $\phi \mapsto \phi \otimes 1_K$, where we are decomposing the algebra $(W^\otimes)^\otimes_n$ in the subalgebra $(V^\otimes)^\otimes_n$ and a bilateral ideal $K$, are not morphisms of quasicomplexes (the problem is with the coface operators $\delta^R_{i,n+1}$). Hence, the resulting group monomorphisms $C^n[V] \rightarrow C^n[W]$ do not define $C^\bullet[V]$ as a subobject of $C^\bullet[W]$.

Now, to the wanted result. The cohomological properties of the elements $\psi \in C^\bullet[V \subset W]$, determine completely the ones of the restrictions $\psi|_V \in C^\bullet[V]$. More precisely,

**Proposition 4.** Let $\psi, \varphi$ be elements of $C^n[V \subset W]$.

a) $\psi \in Z^n[V \subset W]$ if and only if $\psi|_V \in Z^n[V]$.

b) If $(\psi, \varphi) \in \text{Coh}^n[V \subset W]$, then $(\psi, \varphi) \in \text{Coh}^n[W]$ and $(\psi|_V, \varphi|_V) \in \text{Coh}^n[V]$.

**Proof.** a) The first part follows from the facts that $\partial V \subset W \psi = \partial W \psi$, and

$$
(\partial V \subset W \psi)|_V = \partial W \psi|_V = \partial V (\psi|_V)
$$

(direct consequences of Eqs. (2.34) and (2.35), respectively).

b) $(\psi, \varphi) \in \text{Coh}^n[V \subset W]$ means that there exists $\theta \in C^{n-1}[V \subset W]$ such that

$$
[\partial V \subset W]_\theta \psi = \varphi [\partial V \subset W]_\theta.
$$
It is clear this implies \((\psi, \varphi) \in \text{Coh}^n [W]\). In addition, \(\theta|_V, [\partial_{C,W}] \ldots \theta|_V\) and \([\partial_{C,W}]_+ \theta|_V\) define elements of \(C^* [V]\), thus
\[
\partial_{V,-} (\theta|_V) (\psi|_V) = ([\partial_{C,W}] \ldots \theta|_V) \psi|_V = [\partial_{C,W}]_+ \theta|_V (\varphi|_V) \partial_{V,+} (\theta|_V),
\]
i.e. \(\psi|_V \sim \varphi|_V\) through \(\theta|_V\).

2.3.2. 

**Coadjoints and products.** Let us restrict ourself to the category \(Vct_f \subset \text{Vct}\) of finite dimensional \(k\)-vector spaces. In such a case, for each isomorphism \(V \cong V^*\) there is a related isomorphism of quasicomplexes \((C^* [V], \partial_V) \cong (C^* [V^*], \partial_{V^*})\) and, in particular, group isomorphisms \(C^n [V] \cong C^n [V^*]\) for every \(n\). On the other hand, the natural pairing between \((V^n)^{\otimes n}\) and \((V^n)^{\otimes n}\), namely
\[
\langle \cdot, \cdot \rangle : (V^n)^{\otimes n} \times (V^n)^{\otimes n} \rightarrow k,
\]
gives to group anti-homomorphisms (the transposition maps)
\[
* : C^n [V] \rightarrow C^n [V^*] : \psi \mapsto \psi^*
\]
with \((\psi^* v, w) = (v, \psi w)\), and
\[
* : C^n [V^*] \rightarrow C^n [V] : \psi \mapsto \psi^*,
\]
being \((v, \psi w) = (\psi v, w)\). In both cases we have \((\psi^*)_R = (\psi_R)^*, \forall R\), and consequently,
\[
(2.36) \quad \delta_{V^*, i} \psi^* = (\delta_i, V \psi)^* \quad \text{and} \quad \sigma_{V^*, i} \psi^* = (\sigma_i, V \psi)^*.
\]
Of course, \(*_* = *_* = \text{id}\); that is to say, \(*_* \text{ and } *_\text{ are mutually inverse natural equivalences of the corresponding cosimplicial objects. But these maps do not define morphisms between quasicomplexes } (C^* [V], \partial_V) \text{ and } (C^* [V^*], \partial_{V^*})\), unless they are composed with the inversion \(\psi \mapsto \psi^{-1}\).

**Proposition 5.** The maps \(\psi \in C^n [V] \mapsto \psi^{-1} = \psi^{-1*}, n \in \mathbb{N}_0\), define an isomorphism in \(\text{Grp}_{\text{ab}}\). The same is true for \(\psi \in C^n [V^*] \mapsto \psi^{-1*}\).

**Proof.** Since they are clearly bijective maps, we just must prove that
\[
(\partial \psi)^{-1} = \partial \psi^{-1}, \forall \psi \in C^n [V], n \in \mathbb{N}_0.
\]
Using Eq. (2.36), we have
\[
(\partial \psi)^{-1} = \prod_{i \text{ odd}} (\delta_{i, V} \psi)^{-1} \prod_{i \text{ even}} (\delta_{i, V} \psi)^{-1} = \prod_{i \text{ even}} (\delta_{i, V} \psi)^{^{-1}} \prod_{i \text{ odd}} (\delta_{i, V} \psi)^{-1} = \prod_{i \text{ odd}} (\delta_{i, V} \psi)^{-1} \prod_{i \text{ even}} (\delta_{i, V} \psi)^{-1},
\]
where in the last step we use \(\delta\)’s are homomorphisms of groups. Inverting the last member we arrive at the wanted result, i.e.
\[
(\partial \psi)^{-1} = \prod_{i \text{ odd}} (\delta_{i, V} \psi)^{-1} \prod_{i \text{ even}} (\delta_{i, V} \psi)^{-1} = \partial \psi^{-1}.
\]
In the same way the another claim can be shown.

For reason that will become clear later, we indicate \(\psi^{-1} \) by \(\psi^i\). Following this notation, we can call \(C^* [V^*]\) the coadjoint quasicomplex to \(C^* [V]\), and denote it \(C^* [V]^i\). Its coboundary is \(\partial^i \psi = \partial \psi^i = (\partial \psi^i)^*\).

Given \(g : C^* [V] \rightarrow C^* [W]\), its coadjoint arrow \(g^i : C^* [V]^i \rightarrow C^* [W]^i\) can be defined as \(g^i (\psi) = (g (\psi^i))^*\). Of course, \(g^i\) is (by composition) a morphism in \(\text{Grp}_{\text{ab}}\) and, \(C^* [V]^i = C^* [V], g^0 = g\).

In these terms, if \(\psi \in Z^n [V]\), we have as a consequence of the last proposition that \(\psi^i \in Z^n [V]^i\). In addition, if \(\psi\) is an (anti)character in \(C^2 [V]\), then \(\psi^i\) is an (anti)character in \(C^2 [V]^i\).

We also can define a product between our quasicomplexes. Indeed, for every \(n \in \mathbb{N}_0\), and \(V, W \in \text{Vct}\), consider the groups \(C^n [V] \times C^n [W]\). We shall indicate \(C^* [V] \times C^* [W]\) the product cosimplex of \(C^* [V]\) and \(C^* [W]\), with object function \([n + 1] \rightarrow C^n [V] \times C^n [W]\) and coface operators and codegeneracies
\[
(2.37) \quad \delta^n \phi = (\delta^n, V \phi, \delta^n, W \phi), \quad \sigma^n \phi = (\sigma^n, V \phi, \sigma^n, W \phi).
\]
Naturally, the associated product quasicomplex will have a coboundary map such that \(\delta (\phi, \varphi) = (\partial \phi, \partial \varphi)\).

It is clear that given a pair of morphisms \(f^* : C^* [V] \rightarrow C^* [V']\) and \(g^* : C^* [W] \rightarrow C^* [W']\), the related maps
\[
f^n \times g^n : C^n [V] \times C^n [W] \rightarrow C^n [V'] \times C^n [W']
\]
define an arrow $C^\bullet [V] \times C^\bullet [W] \to C^\bullet [V'] \times C^\bullet [W']$.

**Proposition 6.** $C^\bullet [V] \times C^\bullet [W]$ is a subobject of $C^\bullet [V \otimes W]$ in $\text{Grp}_{q}^\bullet$.

**Proof.** Consider the canonical isomorphisms $t_R : (V \otimes W)^{\otimes R} \simeq V^{\otimes R} \otimes W^{\otimes R}$. We shall show the group monomorphisms $j : C^n [V] \times C^n [W] \to C^n [V \otimes W]$,

$$j (\phi, \varphi) = \bigoplus_{R \in \mathbb{N}_0^{n}} [j (\phi, \varphi)]_R = \bigoplus_{R \in \mathbb{N}_0^{n}} t_R^{-1} (\phi_R \otimes \varphi_R) t_R,$$

define a morphism in $\text{Grp}_{q}^\bullet$. To see that, it is enough to show $j \delta_i^n (\phi, \varphi) = \delta_i^n |_{V \otimes W} j (\phi, \varphi)$. Given $R \in \mathbb{N}_0^{n+1}$, by definition of $j$,

$$[j \delta_i^n (\phi, \varphi)]_R = t_R^{-1} \left( [\delta_i^n |_{V \otimes W} \otimes [\delta_i^n |_{W} \varphi]_R \right) t_R$$

$$= t_R^{-1} \left( (m^n_{i,V})^{-1} \phi_{D^n_i (R)} m^n_{i,V} \otimes (m^n_{i,W})^{-1} \varphi_{D^n_i (R)} m^n_{i,W} \right) t_R$$

$$= t_R^{-1} (m^n_{i,V} \otimes m^n_{i,W})^{-1} \left( \phi_{D^n_i (R)} \otimes \varphi_{D^n_i (R)} \right) (m^n_{i,V} \otimes m^n_{i,W}) t_R.$$ 

Noting that $m^n_{i,V} \otimes W = t_{D^n_i (R)}^{-1} (m^n_{i,V} \otimes m^n_{i,W}) t_R$, the last member is precisely

$$(m^n_{i,V} \otimes W)^{-1} [j (\phi, \varphi)]_{D^n_i (R)} m^n_{i,V} \otimes W = [\delta_i^n |_{V \otimes W} j (\phi, \varphi)]_R.$$

The same can be done for $\delta_{i,n+1}^n$. This concludes our proof. \[\blacksquare\]

In forthcoming sections, we shall identify $(V \otimes W)^{\otimes R}$ with $V^{\otimes R} \otimes W^{\otimes R}$, enabling us to describe each $j (\phi, \varphi)$ as a map

$$(2.38) \quad j (\phi, \varphi) = \bigoplus_{R \in \mathbb{N}_0^{n}} [j (\phi, \varphi)]_R = \bigoplus_{R \in \mathbb{N}_0^{n}} \phi_R \otimes \varphi_R,$$

or equivalently, as the restriction of $\phi \otimes \varphi$ to

$$\bigoplus_{R \in \mathbb{N}_0^{n}} V^{\otimes R} \otimes W^{\otimes R} \subset (V^{\otimes})^{\otimes n} \otimes (W^{\otimes})^{\otimes n},$$

i.e. $j (\phi, \varphi) \subset \phi \otimes \varphi$.

Let us denote $C^\bullet [\text{Vct}]$ and $C^\bullet [\text{Vct}_f]$ the full subcategories of $\text{Grp}_{q}^\bullet$ whose objects are of the form

$$C^\bullet \{V_i\} \simeq C^\bullet [V_1] \times ... \times C^\bullet [V_m] : V_i \in \text{Vct},$$

and define $C^\bullet \{V_i\} \uparrow \simeq C^\bullet \{V_1\} \uparrow ... \times C^\bullet \{V_m\}$ for $V_i \in \text{Vct}_f$. The following result resumes all we have discussed.

**Theorem 5.** Each quasicomplex $C^\bullet \{V_i\}$ is a subobject of $C^\bullet \{\otimes_i V_i\}$. The map

$$\times : (C^\bullet \{V_i\}, C^\bullet \{W_i\}) \to C^\bullet \{V_i\} \times C^\bullet \{W_i\}$$

defines a monoidal structure in $C^\bullet [\text{Vct}]$ with unit $C^\bullet [k]$; while $!: C^\bullet \{V_i\} \to C^\bullet \{V_i\} \uparrow$ gives rise to a covariant monoidal functor in $C^\bullet [\text{Vct}_f]$. \[\blacksquare\]

It is important to note, among other things, the set inclusions (under natural identifications)

$$(2.39) \quad Z^n [V_1] \times ... \times Z^n [V_m] \subset Z^n [V_1 \otimes ... \otimes V_m]$$

hold. This is a consequence of the following more general fact. Since the image of $G [\text{Vct}] \to \text{Grp}_{q}^\bullet$ (see §2.3), is contained in $C^\bullet [\text{Vct}]$, it can be regarded as a functor $C^\bullet : G [\text{Vct}] \to C^\bullet [\text{Vct}]$. Then, the monics $j : C^\bullet [V] \times C^\bullet [W] \to C^\bullet [V \otimes W]$ define natural transformations $C^\bullet \times C^\bullet \to C^\bullet (\otimes \cdot)$. The same is true for the full subfunctor $C^\bullet [V] \subset C^\bullet [V \otimes W]$. On the other hand, for each $V \in \text{Vct}_f$, $C^\bullet [V] \simeq C^\bullet [V^\star]$ is a functorial isomorphism for the natural equivalences $C^\bullet \simeq C^\bullet [\cdot]$ and $C^\bullet \simeq C^\bullet [\cdot]$.
2.4. Comparison with the quasicomplex for bialgebra twisting. Let \( A \) be a bialgebra, i.e. \( A \in k - \text{Bialg} = \text{Bialg} \), with coalgebra structure \((\Delta, \varepsilon)\). Consider the groups \( G^n[A] \), \( n \in \mathbb{N}_0 \), formed out by the invertible elements of \( \text{Lin} \left[ A^{\otimes n}, k \right] \) under the convolution product \(*\). Such a product is given by
\[
\psi * \varphi = (\psi \otimes \varphi) \Delta^{(n)}, \quad \psi, \varphi \in \text{Lin} \left[ A^{\otimes n}, k \right],
\]
being \( \Delta^{(n)} \) the usual coproduct on \( A^{\otimes n} \) (in particular, \( \Delta^{(1)} = \Delta \) and \( \Delta^{(0)} = I_k \)). The unit of \(*\) is \( \varepsilon^{\otimes n} \), the usual counit of \( A^{\otimes n} \).

Denote by the pair \((m, \eta)\) the algebraic structure of \( A \), and define (as we have done for a tensor algebra \( V^{\otimes} \)) the maps \( m^n \) and \( \eta^n \) as in Eqs. 2.36 and 2.39. Then, the group homomorphisms \( d_i^n : G^n[A] \to G^{n+1}[A] \) and \( s_i^n : G^{n+1}[A] \to G^n[A], \) with
\[
d_i^n : \psi \mapsto \begin{cases} 
\varepsilon \otimes \psi, & \text{if } i = 0, \\
\psi m_i^n, & \text{if } i \in \{1, ..., n\}, \\
\psi \otimes \varepsilon, & \text{if } i = n + 1,
\end{cases}
\]
and
\[
s_i^n : \psi \mapsto \psi \eta_i^n, \quad i \in \{0, 1, ..., n\},
\]
define a cosimplicial object \( G^\bullet_A : [n+1] \to G^n[A] \) in Grp \([3,11]\). Moreover, using the convolution product, the map
\[
d : \psi \mapsto \left( \prod_{i \text{ even}} d_i \psi \right)^{-1} \left( \prod_{i \text{ odd}} d_i \psi \right),
\]
where \( i \in \{0, ..., n+1\} \), supplies the groups \( G^n[A] \) with a (cosimplicial multiplicative) cochain quasicomplex structure.\(^3\) Note that the order is reversed w.r.t. the coboundary \( \partial \) of the quasicomplexes \( C^\bullet \).

We recall that \( \psi \in G^n[A] \) is a counital \( n \)-cochain if satisfies \( s_i^n \psi = \psi \eta_i^n = \varepsilon^{\otimes n} \) for all \( i \). Also, recall, a bicharacter is a map \( \psi \in G^2[A] \) such that
\[
d_1 \psi = \psi (m \otimes I) = \psi_{13} \ast \psi_{23}, \quad d_2 \psi = \psi (I \otimes m) = \psi_{13} \ast \psi_{12},
\]
while an anti-bicharacter\([2]\) fulfill the opposite equations: \( d_1 \psi = \psi_{23} \ast \psi_{13} \) and \( d_2 \psi = \psi_{12} \ast \psi_{13} \).

Since \( d \) is a unit preserving map, the pair \((G^\bullet[A], d)\) is a quasicomplex in Grp\(_q\), and the factorization of \( d \) in \( d_+, d_- \), namely \( d \psi = (d_+ \psi) - d_- \psi \), makes the triple \((G^\bullet[A], d_+, d_-)\) a parity quasicomplex there. For later convenience, we mention the following known facts without proof.

**Proposition 7.** If a pair of bialgebras \( A \) and \( B \) are homomorphic by means of a map \( \alpha : A \to B \), then there exists a natural transformation \( G_B^\bullet \to G_A^\bullet \) of cosimplicial objects given by functions
\[
\psi \in G^n[B] \mapsto \psi \alpha^{\otimes n} \in G^n[A].
\]
Moreover, the same functions define a morphism \((G^\bullet[B], d_B) \to (G^\bullet[A], d_A)\) of quasicomplexes, becoming the assignment \( A \mapsto G^\bullet[A] \) into a contravariant functor Bialg \( \to \text{Grp}_q\). \( \blacksquare \)

When the bialgebra \( A \) is a tensor algebra \( C^{\otimes} \),\(^4\) we can ask which is the relation between \( G^\bullet[C^{\otimes}] \) and \( C^\bullet[C] \). In order to find this relation, given \( \psi \in G^n[C^{\otimes}] \), consider the endomorphism of \( (C^{\otimes})^{\otimes n} \) of the form\(^5\)
\[
\psi \ast \varepsilon^{\otimes n} \ast \psi^{-1} = (\psi \otimes \varepsilon^{\otimes n} \otimes \psi^{-1}) \left( \varepsilon^{\otimes n} \otimes \Delta^{(n)} \right) \Delta^{(n)}.
\]
Given another \( \varphi \in G^n[C^{\otimes}] \), we have
\[
(\psi \ast \varphi) \ast \varepsilon^{\otimes n} \ast (\psi \ast \varphi) = (\varphi \ast \psi) \ast \varepsilon^{\otimes n} \ast \varphi^{-1} \ast \psi^{-1} = (\varphi \ast \varepsilon^{\otimes n} \ast \varphi^{-1}) \left( \psi \ast \varepsilon^{\otimes n} \ast \psi^{-1} \right),
\]
and obviously for \( \varepsilon^{\otimes n}, \varepsilon^{\otimes n} \ast \varepsilon^{\otimes n} = \varepsilon^{\otimes n} = \varepsilon^{\otimes n} \). It follows that \( \psi^{-1} \ast \varepsilon^{\otimes n} \ast \psi \) is the inverse of \( \psi \ast \varepsilon^{\otimes n} \ast \psi^{-1} \), hence the latter is a linear automorphism of \( (C^{\otimes})^{\otimes n} \). Therefore, the function \( f^n : \psi \mapsto \psi^{-1} \ast \varepsilon^{\otimes n} \ast \psi \) is a group antihomomorphism, i.e. \( f^n(\varepsilon^{\otimes n}) = I_{\otimes n} \) and \( f^n(\psi \ast \varphi) = f^n(\varphi) f^n(\psi) \). Suppose now from on the comultiplication on \( C^{\otimes} \) is such that the inclusion \( \Delta(C) \subset C \otimes C \) holds. In other words, we are saying \( C \) is a coalgebra. Then, since \( \Delta^{(n)} \) is an algebra morphism, for every \( R \in \mathbb{N}_0^{\otimes n} \)
\[
\text{Im} \Delta_R \subset C^{\otimes R} \otimes C^{\otimes R}, \quad \Delta_R \overset{\Delta^{(n)}}{\to} (C^{\otimes R})^{\otimes R},
\]
\[3\]We reorder the factor, with respect to [11], in a way that is convenient for our work.

[4]Every tensor algebra \( C^{\otimes} \) has, fixing a basis \( \{v_i\} \) on \( C \), a bialgebra structure given by the extensions of \( \Delta : v_i \mapsto v_i \otimes v_i \) and \( \varepsilon : v_i \mapsto 1 \) to algebra homomorphisms. This is cocommutative.

[5]Given a bialgebra \( A \) and two linear maps \( \alpha, \beta : A \to B_{\alpha, \beta} \), \( \alpha \ast \beta \) will always mean \((\alpha \otimes \beta) \Delta : A \to B_{\alpha} \otimes B_{\beta} \).
and accordingly each map $F^n \psi$ is homogeneous, thus an $n$-cochain in $C^n [C]$. Moreover,

**Theorem 6.** Let $C$ be a coalgebra. Then there exist group anti-homomorphisms $F^n : G^n [C^{\otimes}] \to C^n [C]$, $n \in \mathbb{N}_0$, that define:

- a natural transformation of the functors $G^n [C^{\otimes}]$ and $C^n [C]$, and
- a morphism of cochain quasicomplexes $(G^* [C^{\otimes}], d) \to (C^* [C], \partial)$ in $\text{Grp}_{eq}$.

**Proof.** For the first statement we must show that $F^{n+1} \delta_i^n = \delta_i^n F^n$ and $F^{n+1} \sigma_i^n = \sigma_i^n F^{n+2}$. Let us apply the first member of the first equality to $\psi \in G^n [C^{\otimes}]$. We have, for $i \neq 0, n+1$, the element of $C^{n+1} [C]$

$$F^{n+1} \delta_i^n \psi = F^{n+1} (\psi m_i^n) = (\psi m_i^n) * \big(I^{n+1} * (\psi^{-1} m_i^n)\big)$$

which restricted to $C^{\otimes R}$, for some $R \in \mathbb{N}_0^{n+1}$, gives

$$[F^{n+1} \delta_i^n \psi]_R = (\psi m_i^n \otimes I_R \otimes \psi^{-1} m_i^n) (I_R \otimes \Delta_R) \Delta_R.$$

Writing $I_R = (m_i^n)^{-1} m_i^n$ and using

$$(m_i^n \otimes m_i^n) \Delta_R = \Delta_{D_i^{n+1}(R)} m_i^n,$$

it follows that $[F^{n+1} \delta_i^n \psi]_R$ is equal to

$$F^{n+1} (\varepsilon \otimes \psi) = (\varepsilon \otimes \psi) * I^{n+1} * (\varepsilon \otimes \psi^{-1}) = (\varepsilon \otimes \psi) * (I \otimes I^{n}) * (\varepsilon \otimes \psi^{-1})$$

what follows from the algebra map character of $\varepsilon$.

Let us prove the second statement. $F^* : G^* [C^{\otimes}] \to C^* [C]$ is a morphism in $\text{Grp}_{eq}$ if and only if equation $F^{n+1} d = \partial F^n$ is satisfied. Using $F^{n+1} \delta_i^n = \delta_i^n F^n$ and the fact that each $F^n$ is an anti-homomorphism of groups, the last equation is immediately fulfilled (compare Eqs. (2.13) with (2.40)). That concludes our proof. [1]

The above theorem describes completely the relation between $G^* [C^{\otimes}]$ and $C^* [C]$ (provided $C$ is a coalgebra) as multiplicative cosimplicial quasicomplexes. Nevertheless, let us look a little closer at the maps $F^n$. Observe that

$$F^n \psi = I^{\otimes n} \iff \psi * \varphi = \varphi * \psi \text{ for all } \varphi \in G^n [C^{\otimes}].$$

In particular, if the coproduct $\Delta$ of $C^{\otimes}$ is cocommutative, then $F^n \psi = I^{\otimes n}$ for all $\psi$ in $G^n [C^{\otimes}]$, for all $n$. In any case, we may consider the quotient groups $G_2^n [C^{\otimes}] \cong G^n [C^{\otimes}] / \mathbb{Z}^n$ and the corresponding injections $i^n : G_2^n [C^{\otimes}] \hookrightarrow C^n [C]$ induced by each $F^n$. Thus, roughly speaking, the elements of $G^n$ form, up to the center, a subset of $C^n$. Moreover, we have:

**Theorem 7.** The groups $G_2^n [C^{\otimes}]$ define a multiplicative cosimplicial quasicomplex in such a way that the commutative diagrams of group (anti)homomorphisms

---

6We are using that given a pair of coalgebras $(A, \Delta_A)$ and $(B, \Delta_B)$, and linear maps $\alpha, \beta$ and $\gamma, \delta$ with domain $A$ and $B$, resp., the convolution product related to the usual coalgebra in $A \otimes B$ satisfies $(\alpha \otimes \gamma) * (\beta \otimes \delta) = (\alpha * \beta) \otimes (\gamma * \delta)$.

7As we will see later, from the point of view of twisting of bialgebras, the groups of interest are precisely $G_2^n$, instead of $G^n$. 
where \( p^n : G^n[\mathcal{C}^\otimes] \to G^n_2[\mathcal{C}^\otimes] \) is the canonical projection, give rise to a commutative diagram of natural transformations among their related cosimplicial objects, and a commutative diagram of morphisms in \( \text{Grp}_{eq} \) of their corresponding cochain quasicomplex.

The theorem follows from the lemmas below. To enunciate the first one, let us note that given \( R \in N_0^{\times n} \), the pair \( (\Delta_R, \varepsilon_R) \) defines a coalgebra structure on \( \mathcal{C}^\otimes_R \). Then, the set \( \text{Lin}[\mathcal{C}^\otimes] \) is supplied with a convolution product \(*\) which enable us to define the groups \( G^R = G^R[\mathcal{C}^\otimes] \), given by the invertible linear forms \( \mathcal{C}^\otimes_R \to k \). Of course, the unit is \( \varepsilon_R \). On the other hand, because of Eq. (2.23), every element \( \psi \) in \( G^n = G^n[\mathcal{C}^\otimes] \) is defined by a family of linear forms \( \psi_R \in G^R \), one for each \( R \in N_0^{\times n} \), being \( \psi_R = \psi|_{G^R} \). In other words, the elements of \( G^n \) can be characterized as functions \( R \in N_0^{\times n} \to \psi_R \in G^R \). Furthermore, the restrictions \( \psi \to \psi_R \) are group epimorphisms, since \( [\psi * \varphi]_R = \psi_R * \varphi_R \).

**Lemma 1.** The following statements are equivalent:

i) \( \psi \in Z^n \subset G^n \).

ii) The equation \( \psi_R * \gamma = \gamma * \psi_R \) holds \( \forall \gamma \in G^R, \forall R \in N_0^{\times n} \).

**Proof.** A linear form \( \psi \) in the center of \( G^n \) iff \( \psi_R * \varphi_R = \varphi_R * \psi_R \) for all \( R \in N_0^{\times n} \), for all \( \varphi \in G^n \). But, to give an arbitrary element \( \varphi \in G^n \) the same as giving for each \( R \) an arbitrary element of \( G^R \), by the characterization above. Then, \( \psi \in Z^n \iff \psi_R * \gamma = \gamma * \psi_R \) for all \( \gamma \in G^R \) and \( R \in N_0^{\times n} \).

**Lemma 2.** For all \( n \in \mathbb{N}_0 \), \( d^n_i (Z^n) \subset Z^{n+1} \) and \( s^n_i (Z^{n+1}) \subset Z^n \).

**Proof.** We shall only prove the inclusions \( d^n_i (Z^n) \subset Z^{n+1} \), since the another ones can be proven in a similar way. Let \( \psi \) and \( \varphi \) be arbitrary linear forms in \( G^n \) and \( G^{n+1} \), respectively. Then, for \( i \neq 0, n + 1 \),

\[
(d^n_i \psi) * \varphi = (\psi m^n_i \otimes \varphi) \Delta^{(n+1)}
\]

restricted to some \( \mathcal{C}^\otimes_R (R \in N_0^{\times n+1}) \) is equal to

\[
(\psi m^n_i \otimes \varphi) \Delta_R = \left( \psi \otimes \varphi (m^n_i)^{-1} \right) (m^n_i \otimes m^n_R) \Delta_R
\]

\[
= \left( \psi \otimes \gamma \right) \Delta_{D^{n+1}_R} (m^n_R) = \left( \psi \otimes \gamma \right) m^n_R,
\]

where \( \gamma \equiv \varphi (m^n_R)^{-1} \) is a linear form \( \mathcal{C}^\otimes_{D^{n+1}_R} \to k \). This linear form is invertible w.r.t. the convolution product, and \( \gamma^{-1} \equiv \varphi^{-1} (m^n_R)^{-1} \), because (see Eqs. (2.24) and (2.13))

\[
\gamma^{-1} * \varphi = (\varphi^{-1} (m^n_R)^{-1} \otimes \varphi (m^n_i)^{-1}) \Delta_{D^{n+1}_R} (m^n_R)^{-1} \Delta_R (m^n_i)^{-1} = \varepsilon_R (m^n_i)^{-1} = \varepsilon_{D^{n+1}_R}.
\]

Therefore, \( \gamma \in \mathcal{G}^{D^{n+1}_R} \). Now, if \( \psi \in Z^n \), from the last lemma

\[
\left( \psi \otimes \gamma \right) m^n_R = \left( \gamma * \psi \right) m^n_R = (\varphi \otimes \psi m^n_i) \Delta_R,
\]

for all \( R \), hence \( (d^n_i \psi) * \varphi = \varphi * (d^n_i \psi) \) for all \( \varphi \in G^{n+1} \). In the \( i = 0 \) case,

\[
(\varepsilon * \varphi) = \varphi \left( \varepsilon \otimes \psi \right) \left( \mathbb{I} \otimes \mathbb{I}^\otimes \right) = \varphi \left( \varepsilon \otimes \mathbb{I} \right) \left( \psi \otimes \mathbb{I}^\otimes \right) = \varphi \left( \mathbb{I} \otimes \left( \psi \otimes \mathbb{I}^\otimes \right) \right),
\]

and similarly

\[
\varphi * (\varepsilon \otimes \psi) = \varphi \left( \mathbb{I} \otimes \varepsilon \otimes \left( \mathbb{I}^\otimes \otimes \psi \right) \right) = \varphi \left( \mathbb{I} \otimes \left( \mathbb{I}^\otimes \otimes \psi \right) \right).
\]

In addition, we have supposed \( \psi \in Z^n \), then \( \psi \otimes \mathbb{I}^\otimes = \mathbb{I}^\otimes \otimes \psi \) and \( d^n_i \psi * \varphi = \varphi * d^n_i \psi \) holds. And analogously for \( i = n + 1 \). Hence, \( d^n_i (Z^n) \subset Z^{n+1} \) for all \( n, i \).

As a direct consequence of the last theorem, the cocycles, (anti)bicharacters, coboundaries and every cohomological class of \( G^n_2 \) can be seen as subsets of the cocycles, the coboundaries and corresponding cohomological class of \( \mathcal{C}^* \), respectively. Furthermore, the counital cochains of \( G^n_2 \) define another quasicomplex, namely \( \mathcal{G}^{*}_2 \), such that there exists a monic \( \mathcal{G}^{*}_2 \to \mathcal{C}^* \) given by group anti-monomorphisms.

By last, consider a vector space \( V \) and the free bialgebra generated by a multiplicative matrix \( t^i_j = v^j \otimes v_i \in V^* \otimes V \). That is to say, the algebra is \( [V^* \otimes V]^\otimes \) and the coalgebra structure is given by assignments \( \Delta : t^i_j \to t^k_l \otimes t^k_j \) and \( \varepsilon : t^i_j \to \delta^i_j \). It can be shown the bijection \( End[V] = \text{Lin}[V, V] \simeq \text{Lin}[V^* \otimes V, k] \) gives rise to an isomorphism of quasicomplexes

\[
(2.44) \quad C^* [V] \simeq G^* \left[ [V^* \otimes V]^\otimes \right].
\]
3. Twisting of quantum linear spaces

As a previous step toward the quantum space twist transformations we need the concept of admissible cochain. Consider a pair $A = (A_1, A)$ and a 2-cochain $\psi \in C^2[A_1]$. The map $\psi$ defines a linear map in $A \otimes A$ iff

$$\psi (A_1^\otimes \otimes \ker \Pi + \ker \Pi \otimes A_1^\otimes) \subset A_1^\otimes \otimes \ker \Pi + \ker \Pi \otimes A_1^\otimes,$$

where $\Pi$ is the canonical epimorphism $A_1^\otimes \to A$. Such a map, namely $\Psi: A \otimes A \to A \otimes A$, would be given by the equation $\Psi \Pi^{\otimes 2} = \Pi^{\otimes 2} \psi$.

**Definition 2.** We shall call $A$-admissible, or admissible for $A$, the $n$-cochains satisfying

$$\psi (\ker \Pi^{\otimes n}) \subset \ker \Pi^{\otimes n} \text{ and } \psi^{-1} (\ker \Pi^{\otimes n}) \subset \ker \Pi^{\otimes n},$$

or equivalently, $\psi (\ker \Pi^{\otimes n}) = \ker \Pi^{\otimes n}$. ■

If $A$ is conic, admissibility condition for $\psi \in C^2[A_1]$ reduces to

$$\psi (A_1^\otimes \otimes I_s + I_r \otimes A_1^\otimes) \subset A_1^\otimes \otimes I_s + I_r \otimes A_1^\otimes,$$

for all $r, s \in \mathbb{N}_0$, where $\ker \Pi = \bigoplus_{n \geq 2} I_n$. The other inclusions (and therefore the equalities) follow from the fact that the involved vector spaces are finite dimensional.

The $A$-admissible $n$-cochains form a subgroup of $C^n[A_1]$. It can be shown the quotient of this subgroup by the identification

$$\psi \sim \psi' \iff \Pi^{\otimes n} (\psi - \psi') = 0,$$

is such that the map $[\psi] \mapsto \Psi$, from admissible equivalence classes to $Aut_{\mathcal{V}^n} [A^{\otimes n}]$, is an injective group homomorphism.

It is worth remarking admissibility of $\psi$ does not imply the one of $\partial \psi$. For instance, consider the algebra generated by $A_1 = \text{span} [a, b]$ and quotient by the relation $ab = 0$. Define the 1-cochain $\theta: A_1^\otimes \simeq A_1^\otimes$ such that $\theta_a$ is the identity unless for the elements $axb$ and $xab$, where $\theta (abx) = xab$ and $\theta (xab) = abx$, being $x$ equal to $a$ or $b$. Hence, $\theta$ is clearly admissible. Then, for instance,

$$\delta_1 \theta (ab \otimes b) = (m_1^R)^{-1} \theta_3 m_1^R (ab \otimes b) = (m_1^R)^{-1} (bab) = ba \otimes b,$$

with $R = (2, 1)$, what implies $\delta_1 \theta (\ker \Pi^{\otimes 2}) \neq \ker \Pi^{\otimes 2}$, i.e. $\partial \theta$ is not admissible. Therefore, unfortunately the subgroups of admissible cochains do not define a subcomplex of $C^* [A_1]$. Nevertheless, in §4.1 we consider for each conic quantum space $A = (A_1, A)$ a stronger condition than admissibility, which enable us to define a subcomplex $C^* [A] \subset C^* [A_1]$.

3.1. Counital 2-cocycle twisting. Consider a quantum space $A = (A_1, A)$ with related unital associative algebra structure $(m, \eta)$, and an admissible counital 2-cochain $\psi \in C^2[A_1]$. Since $\psi$ gives rise to a unique automorphism $\Psi$ of $A \otimes A$ by the equation $\Psi \Pi^{\otimes 2} = \Pi^{\otimes 2} \psi$, we can define, as at the beginning of §2, a new algebra structure over $A$, namely $A_\psi = (A, m_\psi, \eta)$, with $m_\psi = m \Psi$. In these terms we have the following notion of twisting process on quantum spaces.

**Definition 3.** Given a quantum space $A = (A_1, A) \in \text{FGA}$ (resp. CA) and a related admissible $\psi \in C^2[A_1]$, we define the twisting of $A$ by $\psi$ as the pair $A_\psi = (A_1, A_\psi) \in \text{FGA}$ (resp. CA). ■

The following theorem justifies above definition.

**Theorem 8.** Let $\Psi$ be a map defined by an $A$-admissible counital 2-cocycle $\psi \in C^2[A_1]$. Then $m_\psi = m \Psi$ is an associative product with unit $\eta$, and the associative unital algebra $A_\psi = (A, m_\psi, \eta)$ is also generated by $A_1$. Moreover, $m_\psi$ defines a filtration that coincides with the one associated to $m$ (see Eq. [17]). In particular, for the conic case, $m_\psi$ defines the same gradation as $m$.

**Proof.** Since $\Pi$ is a morphism of unital algebras $\Pi m_\otimes = m \Pi$ and $\Pi (\lambda) = \eta (\lambda), \forall \lambda \in A_1^{\otimes 0} = k$. Then, if $\psi$ is counital we have

$$\Psi (\eta (\lambda) \otimes [a]) = \Psi \Pi^{\otimes 2} (\lambda \otimes a) = \Pi^{\otimes 2} \psi (\lambda \otimes a) = \eta (\lambda) \otimes [a],$$

being $[a] = \Pi (a)$. So, $\Psi (\eta \otimes I) = (\eta \otimes I)$ and

$$m_\psi (\eta \otimes I) = m \Psi (\eta \otimes I) = m (\eta \otimes I) = I.$$

The same holds for $I \otimes \eta$; hence $\eta$ is a unit map for the product $m_\psi$. Let us see that $m_\psi$ is associative. It will be associative iff

$$m_\psi (m_\psi \otimes I) \Pi^{\otimes 3} = m_\psi (I \otimes m_\psi) \Pi^{\otimes 3},$$

iff

$$\Pi m_\otimes \psi (m_\otimes \otimes I) (\psi \otimes I) = \Pi m_\otimes \psi (I \otimes m_\otimes) (I \otimes \psi).$$
From Equation 2.7 for \( n = 2 \), \( \psi (m_\otimes I) = (m_\otimes I) \delta_1 \psi \) and \( \psi (I \otimes m_\otimes) = (I \otimes m_\otimes) \delta_2 \psi \), so \( m_\psi \) is associative iff
\[
\Pi m_\otimes (I \otimes m_\otimes) (\partial \psi - \Pi^{\otimes 3}) = 0,
\]
where we have used the associativity of \( m_\otimes \). But the last eq. holds trivially since \( \psi \) is a 2-cocycle.

It rests to show that \( A_1 \) generates \( A \) through \( m_\psi \). We know that, given a basis \( \{ a_i \} \) of \( A_1 \), it is possible to express any element of \( A \) as a linear combination of words written with the letters \( a_i \)'s and glued with the product \( m \). Let us show the same is true for \( m_\psi \). We will proceed by induction on the length of the words.

Denoting \( m \) by a blank space and \( m_\psi \) by \( " \cdot " \), the map \( \Psi \) on \( r, s \in N_0 \) is given by a \((\dim A_1)^{r+s}\) square invertible matrix representing \( \psi_{r,s} \), namely (remember that \( \Pi (a_i) = a_i \))
\[
\Psi (a_{i_1} \cdots a_{i_r} \otimes a_{j_1} \cdots a_{j_s}) = \psi_{i_1,j_1}^{k_1} \cdots \psi_{i_r,j_s}^{k_s} a_{k_1} \cdots a_{k_r} \otimes a_{i_1} \cdots a_{i_r}.
\]
Since
\[
a_{k_r} a_l = (\psi^{-1})^{i,l}_k a_l a_{j_r},
\]
then, for words of length 2 there exists an invertible linear map \( \theta_2 : A_1^{\otimes 2} \cong A_1^{\otimes 2}, \) with matrix coefficients \( (\theta_2)_{ij} = \psi_{ij}^{kl} \), such that
\[
a_{k_l} a_i = (\theta^{-1}_2)_{ij} a_i a_{j_l}.
\]
Suppose that for words of \( n \) letters there exists an invertible linear map \( \theta_2 \) such that
\[
(\theta^{-1})_{ij} = (\theta^{-1}_2)_{ij} a_{i_1} \cdots a_{i_n}.
\]
Then, for \( n + 1 \),
\[
(a_{j_1} \cdots a_{j_n} \cdot a_{j_{n+1}}) = (\theta^1_{n+1})_{i_1 j_1} \cdots (\theta^1_{n+1})_{i_n j_n} a_{i_1} \cdots a_{i_n} a_{j_{n+1}},
\]
\[
= (\theta^1_{n+1})_{i_1 j_1} a_{j_1} \cdots a_{j_{n+1}} = (\theta^1_{n+1})_{i_1 j_1} \cdots (\theta^1_{n+1})_{i_n j_n} a_{j_1} \cdots a_{j_{n+1}},
\]
i.e.
\[
a_{k_1} \cdots a_{k_{n+1}} = \theta^1_{n+1} a_{j_1} \cdots a_{j_{n+1}},
\]
being
\[
(\theta^{-1})^1_{n+1} a_{j_1} \cdots a_{j_{n+1}} = (\theta^{-1})^1_{n+1} a_{j_1} \cdots a_{j_{n+1}}.
\]
Hence, by induction, any element of \( A \) can be expressed as linear combinations of words constructed with elements of \( A_1 \) and glued with \( m_\psi \). Both words have the same length, in the sense that can be obtained as the image by \( \Pi \) of homogeneous elements with same number of factors. From that it is clear \( m_\psi \) defines the same filtration as \( m \), and for the conic case the same gradation, as we wanted to show.

In other words, the last theorem says the pair \((A_1, A_\psi)\) is a quantum space, and if \((A_1, A)\) is conic, so is its twisting. In the following, we shall describe some interesting examples.

3.1.1. The quantum plane as a 2-cocycle twisting. Let \( A^{2|0} = (A_1, A) \) be the quantum space with \( A_1 = \text{span} \{ a, b \} \) and \( A = k[a,b] \), the commutative algebra freely generated by the symbols \( a \) and \( b \). In other words, \( A^{2|0} \) gives the coordinate ring of the two dimensional affine space. Consider the basis
\[
\{ a^f b^g \}_{f, g \in N_0, \ k \in N} \}
\]
of \( A_1^{\otimes} \), and for every element \( v \) in that basis the positive numbers \( m^a(v) = \sum_{i=1}^k f_i \) and \( m^b(v) = \sum_{i=1}^k g_i \). Now, given \( \hbar \in k \) let us define the 2-cocochain
\[
\psi_\hbar : v \otimes w \mapsto \exp [m^a(v) m^b(w) \hbar] v \otimes w.
\]
The counitality is immediate, since taking \( v = w = 1 \), equation \( m^a(v) m^b(w) = 0 \) holds. This cochain is \( A \)-admissible, in fact, \( \psi_\hbar \) gives rise to a linear automorphism \( \Psi_\hbar \) of \( A^{\otimes 2} \) such that
\[
\Psi_\hbar (a^m b^n \otimes a^s b^t) = \exp (n\hbar) a^n b^m \otimes a^s b^t.
\]
The formula above defines \( \Psi_\hbar \) completely, because the set \( \{ a^m b^n \}_{m, n \in N_0} \) is a basis for \( A \). Let us see that \( \psi_\hbar \) is a 2-cocycle. Straightforward calculations show that
\[
\delta_1 \psi_\hbar (u \otimes v \otimes w) = \exp [m^a(u) m^a(v)] m^b(w) \hbar] u \otimes v \otimes w,
\]
\[
\delta_2 \psi_\hbar (u \otimes v \otimes w) = \exp [m^a(u) m^b(v) + m^b(w)] \hbar] u \otimes v \otimes w,
\]
for any elements \( u, v \) and \( w \) of the basis of \( A \otimes \mathcal{B} \). Now, applying consecutively \( \psi_h \otimes \mathcal{I} \) and \( \delta_1 \psi_h \) on such elements we have
\[
\begin{align*}
  u \otimes v \otimes w &\mapsto \exp \left[ m^a(u) \, m^b(v) \, \hbar \right] u \otimes v \otimes w \\
  &\mapsto \exp \left[ (m^a(u) + m^a(v)) \, m^b(w) \, \hbar \right] \exp \left[ m^a(u) \, m^b(v) \, \hbar \right] u \otimes v \otimes w \\
  &\mapsto \exp \left[ (m^a(u) \, m^b(w) + m^a(v) \, m^b(w) + m^a(u) \, m^b(v)) \, \hbar \right] u \otimes v \otimes w,
\end{align*}
\]
and doing the same for \( \mathcal{I} \otimes \psi_h \) and \( \delta_2 \psi_h \),
\[
\begin{align*}
  u \otimes v \otimes w &\mapsto \exp \left[ m^a(u) \, m^b(v) \, \hbar \right] u \otimes v \otimes w \\
  &\mapsto \exp \left[ m^a(u) \, (m^b(v) + m^b(w)) \, \hbar \right] \exp \left[ m^a(u) \, m^b(w) \, \hbar \right] u \otimes v \otimes w \\
  &\mapsto \exp \left[ (m^a(u) \, m^b(v) + m^a(v) \, m^b(w) + m^a(u) \, m^b(w)) \, \hbar \right] u \otimes v \otimes w.
\end{align*}
\]
Thus \( \delta_1 \psi_h \) (\( \psi_h \otimes \mathcal{I} \) = \( \delta_2 \psi_h \) (\( \mathcal{I} \otimes \psi_h \)). Moreover, it can be seen that \( \psi_h \) is a bicharacter and also an anti-bicharacter. Accordingly, the product \( m_{\psi_h} = m \Psi_h = * \) is associative, and from Eq. (3.6) for \( n = s = 0 \) and \( m = r = 1 \), \( a * b = \exp (\hbar \, b * a) \) follows. Thus, \( A^{n|0}_{\psi_h} \) is isomorphic to the quantum plane
\[
A^{n|0}_{\psi_h} = \mathbb{K} \left\{ \langle x, y \rangle \mid \{ xy - \exp (\hbar \, xy) \} \right\}
\]
under the extension of \( x \mapsto a, y \mapsto b \) to an algebra map. The map \( \psi_h \) also defines an admissible 2-cocycle for the superplane \( A^{n|2} \), given by a Grassmann algebra in two variables. Writing again \( m_{\psi_h} = * \), we have the relation \( a * b = - \exp (\hbar \, b * a) \), thus \( A^{n|2}_{\psi_h} \) is isomorphic to the quantum 0|2-dimensional superplane
\[
A^{0|2}_{\psi_h} = \mathbb{K} \left\{ \langle \eta, \xi \rangle \mid \{ \eta \xi + \exp (\hbar \, \xi \eta) \, \eta^2; \, \xi^2 \} \right\}.
\]
This result can easily be generalized to any quantum space \( A^{n|0}_{h} \) and \( A^{0|n}_{h} \). In general, given \( M \in GL (n^2) \) fulfilling the YB equation or such that \( M_{12} \, M_{23} = M_{23} \, M_{12} \), we can define, as at the end of §2.2.2, a bicharacter or an anti-bicharacter \( \psi \) related to an \( n \)-dimensional vector space \( A \). Such a cochain is always admissible for commutative and anticommutative quantum spaces \( A \) generated by \( A \). The ideal related to \( A \) is generated by
\[
( N^{kl}_{ij} + N^{kl}_{ji} ) \, a_k \otimes a_l; \quad N = M^{-1},
\]
the \( \mp \) corresponding to the \( A \) commutative or \( A \) anticommutative cases, respectively. (We need the results of §3.2.2 to show that.) Note relations in (3.7) can be written \( ( R^{kl}_{ij} + P^{kl}_{ij} ) \, a_k \otimes a_l \), being \( R = M_{21} \) and \( P \) the permutation matrix. It can be shown \( R \) is a YB operator, in fact a triangular one, i.e. \( R_{21} \, R = I \). Thus, twisting \( A \) by (anti)bicharacters defines quantum spaces with relations given by triangular YB operators of the form \( M_{21} \, M^{-1} \). As a particular case, to obtain \( A^{n|0}_{h} \) or \( A^{0|n}_{h} \) from \( A \), \( \psi \) can be defined by
\[
M^{kl}_{ij} = \exp \left\{ [1 - \exp (\hbar / 2)] \, \hbar \right\} \delta^k_j \delta^l_i; \quad i, j, k, l = 1 \ldots n;
\]
being \( \exp (0) = 1 \). The extension of \( M \) to a bicharacter and to an anti-bicharacter gives rise to the same map. Since \( M \) is a YB operator, the resulting cochain \( \psi \) is effectively a 2-cocycle.

3.1.2. Symmetric twisted tensor products. Given a couple of quantum spaces \( A \) and \( B \), we recall that a symmetric twisted tensor product (STTP) of them is (essentially) a quantum space \( A \circ \tau B = (A \otimes B) \circ \tau (A \otimes B) \), being \( A \circ \tau B \) the subalgebra of \( A \otimes \mathcal{B} \) \( \mathbb{I} \) generated by \( A \otimes B \), and where the related (symmetric) twisting map \( \tau \) is a linear bijection \( B \otimes A \subset A \otimes B \) that defines, by restriction, an isomorphism \( B_1 \otimes A_1 \cong A_1 \otimes B_1 \). Thus, \( \tau \) is completely defined by the last isomorphism and by the properties of a twisting map, namely
\[
\begin{align*}
  \tau (m_B \otimes I_A) &= (I_A \otimes m_B) \, (\tau \otimes I_B) \, (I_B \otimes \tau) \, (I_A \otimes \tau), \\
  \tau (I_B \otimes m_A) &= (m_A \otimes I_B) \, (I_A \otimes \tau) \, (\tau \otimes I_A),
\end{align*}
\]
and
\[
\begin{align*}
  \tau (I_B \otimes \eta_A) &= \eta_A \otimes I_B; \quad \tau (\eta_B \otimes I_A) = I_A \otimes \eta_B.
\end{align*}
\]
Remember that the algebra structure of \( A \otimes \mathcal{B} \) is given by the maps
\[
\begin{align*}
  m^\tau_{AB} &= (m_A \otimes m_B) \, (I_A \otimes \tau) \, (I_B), \quad \text{and} \quad \eta_{A\otimes B} = \eta_A \otimes \eta_B.
\end{align*}
\]
We will show that any STTP can be seen as a twisting of \( A \circ B \) by an element of \( 3^2 [A_1 \otimes B_1] \).
The usual product in $A \otimes B$ is $m_{A \otimes B} = (m_A \otimes m_B)$, having the canonical flipping map. Then,
given $\tau$ we have that $m_{A \otimes B}^\tau = m_{A \otimes B} \Omega$, where
$$
\Omega \equiv (I \otimes f^{-1} \otimes I) : (A \otimes B)^{\otimes^2} \to (A \otimes B)^{\otimes^2}.
$$
And since $\Omega \left((A \circ B)^{\otimes^2}\right) \subseteq (A \circ B)^{\otimes^2}$ (which follows from property (3.8)),
we also have $m_{A \otimes B}^\tau = m_{A \circ B} \Omega$, by making the corresponding restriction. Now,
let us extend the isomorphism $	au_{A \circ B}$ to all of $B_1 \otimes \cdots \otimes 1$ using Eq. (3.10),
and call $\tau_\otimes$ the resulting map. By construction, $\tau_\otimes$ defines isomorphisms
\begin{equation}
\tau_\otimes : B_1 \otimes A_1 \cong A_1 \otimes B_1.
\end{equation}
to all of $B_1 \otimes A_1^{\otimes}$ using Eq. (3.10) and (3.10),
and call $\tau_\otimes$ the resulting map. By construction, $\tau_\otimes$ defines isomorphisms
\begin{equation}
\tau_\otimes : B_1 \otimes A_1 \cong A_1 \otimes B_1.
\end{equation}
That is, $\tau_\otimes$ gives rise (by restriction) to a 2-cocycle in $C_2^2[A_1 \otimes B_1]$,
which is automatically $A \circ B$-admissible. Let us call $\omega$ such a 2-cocycle. Because
$\tau_\otimes$ satisfies Eq. (3.3), $\omega$ is counital, i.e. $\omega \in C_2^2[A_1 \otimes B_1]$. It remains to see that $\omega$ is a 2-cocycle
in order to prove our claim. Eq. (3.3) for $\omega$ translates into equations $\delta_1 \omega = \omega_{13} \omega_{23}$ and $\delta_2 \omega = \omega_{13} \omega_{12}$,
with $\omega_{12} = \omega \otimes 1$, $\omega_{23} = 1 \otimes \omega$, and $\omega_{13} = (\omega \otimes f_0^{-1}) \omega_{12} (1 \otimes f_0)$.
That is, $\omega$ is an anti-bicharacter. Then, since clearly $\omega_{23} \omega_{12} = \omega_{12} \omega_{23} = \delta_2 \omega$,
$\omega$ is effectively a 2-cocycle; explicitly
$$
\delta_1 \omega (\omega \otimes 1) = \omega_{13} \omega_{23} \omega_{12} = \omega_{13} \omega_{12} \omega_{23} = \delta_2 \omega (1 \otimes \omega).
$$

Let us recall that the twisting $\omega$ of a bialgebra $A \circ B$ is by $\omega$ the bialgebra $A \circ B$ such that $A \circ B = (A \circ B)\omega$.
Moreover, $\omega$ is an anti-bicharacter. $\blacksquare$

### 3.1.3. Twisting of bialgebras

Suppose a quantum space $A$ is a bialgebra in FGA (resp. CA), i.e. the coalgebra structure is
given by arrows $\Delta : A \to A \otimes A$ and $\varepsilon : A \to k$ (resp. $\varepsilon : A \to k$) there. In particular, $A_1$
is a coalgebra with coproduct $\Delta_1 = \Delta|A_1$, and counit $\varepsilon_1 = \varepsilon|A_1$. That is to say, a bialgebra
in FGA is a bialgebra algebraically generated by a finite dimensional coalgebra. The pair $(A_1^{\otimes}, \varepsilon_1)$
defines a bialgebra structure on $A_1^{\otimes}$, and the canonical epimorphism $\Pi$ is a bialgebra homomorphism w.r.t. this structure.

Let $\chi : A^{\otimes^2} \to k$ be a 2-cocycle of $G^2[A]$ (see §2.4). From the morphism of
quasicomplexes $G^* \vert[A] \to G^* \vert[A_1^{\otimes}]$ induced by the bialgebra map $\Pi : A_1^{\otimes} \to A$ (see Eq. (2.11) in Prop. 7),
the map $\chi \Pi^{\otimes^2} : A_1^{\otimes} \otimes A_1^{\otimes} \to k$ is also a 2-cocycle. Since $\Pi$ is surjective, the maps $\chi \to \chi \Pi^{\otimes^2}$,
$n \in \mathbb{N}_0$, give rise to a monic $G^* \vert[A] \to G^* \vert[A_1^{\otimes}]$. Now, from $f^* : G^* \vert[A_1^{\otimes}] \to C^* \vert[A_1]$, we have in addition the automorphism
$$
F_\chi \equiv f^2 \left(\chi \Pi^{\otimes^2}\right) = \left(\chi \Pi^{\otimes^2}\right) \ast \Pi^{\otimes^2} \ast \left(\chi \Pi^{\otimes^2}\right)^{-1},
$$
which, by Theor. 6, defines a 2-cocycle in $C_2[A_1]$. The composition of $f^*$
with the above monic, quotient out the corresponding centers $Z$, is another monic $G_2^* \vert[A] \to C^* \vert[A_1]$
(see Theor. 7), which enables us to think of $G^* \vert[A]$, as a subcoalgebra of $C^* \vert[A_1]$. Of course, all that is also true if $G^*$
and $C^*$ are replaced by their counital subgroups $G^{\otimes}$ and $C^{\otimes}$.

We recall that the twisting maps $\Pi \Pi$ of a bialgebra $A$ by $\chi$ is the bialgebra $A_\chi$ with the same coalgebra structure,
and associative product $m_\chi = \chi \ast m \ast \chi^{-1}$ with the same unit.\footnote{If a couple of elements $\chi, \vartheta \in G^{\otimes}$ differ by an element of the center $Z$, both linear forms define the same twisted bialgebra. This is why the interesting objects are the quotients $G_2^{\otimes}$ instead of entire groups $G^n$.}

$$
F_\chi \text{ is admissible, in fact}
$$
\begin{equation}
\Pi^{\otimes^2} F_\chi = F_\chi \Pi^{\otimes^2} \left(\chi \Pi^{\otimes^2}\right) = \left(\chi \ast \Pi^{\otimes^2} \ast \chi^{-1}\right) \Pi^{\otimes^2}.
\end{equation}

Hence, given a counital 2-cocycle $\chi \in G_2^{\otimes} \vert[A]$, we have an admissible cocycle $f_\chi$ in $Z^3[A_1]$. Moreover,
the twisting $A_\chi = (A_1, A_\chi)$ by $\chi \in G_2^{\otimes} \vert[A]$ of the bialgebra $A$, coincides with the twisting $A_{f_\chi}$
of $A$ as a quantum space, because from Eq. (3.11), $m_{A_{f_\chi}} = \chi \ast m \ast \chi^{-1} = m_{A_\chi}$.

Let us consider the twisted internal coEnd objects $\text{end}_\chi^\Pi \vert[A]$, related to a symmetric twisting map defined by
$\hat{\tau}_A = id \otimes \sigma \otimes \sigma$,
where $\sigma = \sigma_A : A_1 \subset A_1$ and $\sigma' = \sigma'^{-1}$, such that $\sigma_A$ can be extended to quantum space
automorphism $A \simeq A$. We mentioned in §1.2 that each one of these bialgebras are isomorphic to the twisting $(A \triangleright A)_\chi$ (as bialgebras). Such $\chi$ is a 2-cocycle in $\mathfrak{G}^2[A \triangleright A]$ defined by

$$\chi\left(z_{k_1}^{j_1} \cdots z_{k_r}^{j_r} \otimes z_{k_{r+1}}^{j_{r+1}} \cdots z_{k_{r+s}}^{j_{r+s}}\right) = \delta_{k_1}^{j_1} \cdots \delta_{k_r}^{j_r} \left(\sigma^{\rho_{k_{r+1}}} \cdots \sigma^{\rho_{k_{r+s}}}\right),$$

being $z_i^j = a_i \otimes a_i$. Then, via $F$, $\text{end}^T[A]$ can be seen as a twisting of $\text{end}[A]$ by $F \chi \in \mathfrak{G}^2[A_1^1 \otimes A_1^1]$, where

$$F \chi\left(z_{k_1}^{j_1} \cdots z_{k_r}^{j_r} \otimes z_{k_{r+1}}^{j_{r+1}} \cdots z_{k_{r+s}}^{j_{r+s}}\right) =$$

$$= z_{k_1}^{j_1} \cdots z_{k_r}^{j_r} \otimes \left(\left(\sigma^{\rho_{k_{r+1}}} \cdots \sigma^{\rho_{k_{r+s}}}\right) z_{a_1}^{b_1} \cdots z_{a_s}^{b_s} \left(\left(\sigma^{\rho_{k_{r+1}}} \cdots \sigma^{\rho_{k_{r+s}}}\right)\right)\right).$$

One can show $\chi$, and consequently $F \chi$, is bicharacter and anti-bicharacter.

In the following we generalize this correspondence between $\text{end}^T[A]$ and $\text{end}[A]$ to all of twisted internal coHom objects.

3.1.4. The twisted coHom objects. Consider the twisted coHom objects $\text{hom}^T[B, A]$ related to a twisting map defined by

$$\hat{\tau}_{A,B} = id \otimes \rho \otimes \phi, \quad \phi = \sigma_A, \quad \rho^{-1} = \sigma_B,$$

where $\sigma_A$ and $\sigma_B$ can be extended to automorphisms $A \simeq A$ and $B \simeq B$. Then, we know that $\text{hom}^T[B, A]$ is the algebra generated by $z_i^j = b_i \otimes a_i \in B_i^j \otimes A_1^1$ and quotient by the ideal algebraically generated by (see Eqs. 4.7 to 4.12)

$$\mathfrak{S} \left(\mathfrak{S}^{\mathfrak{n}}_n z_{a_1}^{j_1} \cdots z_{k_n}^{j_n} \left(\sigma S^1\right)_{\omega_n} \right)_{\omega_n}, n \in \mathbb{N}_0.$$ 

Now, let us define for $\text{hom}[B, A] = B \triangleright A$ the counital 2-cochain $\varsigma \in \mathfrak{C}^2[B_1^1 \otimes A_1^1]$ given by

$$\varsigma\left(z_{k_1}^{j_1} \cdots z_{k_r}^{j_r} \otimes z_{k_{r+1}}^{j_{r+1}} \cdots z_{k_{r+s}}^{j_{r+s}}\right) = z_{k_1}^{j_1} \cdots z_{k_r}^{j_r} \otimes \xi^{[r]} \left(z_{k_{r+1}}^{j_{r+1}} \cdots z_{k_{r+s}}^{j_{r+s}}\right),$$

being $\xi^{[r]} \in \mathfrak{C}^1[B_1^1 \otimes A_1^1]$ such that

$$\xi^{[r]} \left(z_{k_1}^{j_1} \cdots z_{k_r}^{j_r}\right) = \left(\phi \sigma^{\rho_{k_{r+1}}} \cdots \phi \sigma^{\rho_{k_{r+s}}}\right) \left(\sigma^{\rho_{k_{r+1}}} \cdots \sigma^{\rho_{k_{r+s}}}\right) \left(\sigma^{\rho_{k_{r+1}}} \cdots \sigma^{\rho_{k_{r+s}}}\right) \left(\sigma^{\rho_{k_{r+1}}} \cdots \sigma^{\rho_{k_{r+s}}}\right).$$

Defining $\xi^{[0]} = 1$, we can write $\varsigma_{r,s} = \iota_r \otimes \xi^{[r]} \otimes \iota_s$ for $r, s \in \mathbb{N}_0$. From the fact that $\phi$ and $\rho$ define algebra automorphisms, it follows that $\varsigma$ is $B \triangleright A$-admissible. Let us see that $\varsigma$ is a 2-cocycle. Straightforward calculations show that

$$\varsigma^{[r+s]} = \varsigma^{[r]} \otimes \varsigma^{[s]} \quad \text{and} \quad \varsigma^{[s]} = \varsigma^{[s]} \otimes 1.$$ 

Then, comparing

$$\varsigma_{r+s,t} \left(\iota_r \otimes \iota_t\right) = \left(\iota_{r+s} \otimes \varsigma^{[r+s]}_t\right) \left(\iota_r \otimes \varsigma^{[r]}_t \otimes \iota_s\right) = \iota_r \otimes \varsigma^{[r]}_t \otimes \varsigma^{[s]}_t,$$

and

$$\varsigma_{r+s,t} \left(\iota_r \otimes \varsigma^{[s]}_t \otimes \iota_s\right) = \left(\iota_r \otimes \varsigma^{[r]}_s \otimes \iota_t\right) \left(\iota_r \otimes \iota_s \otimes \varsigma^{[s]}_t\right) = \iota_r \otimes \varsigma^{[r]}_s \otimes \varsigma^{[s]}_t,$$

we have from Eq. 5.10 that $\varsigma_{r+s,t} \left(\iota_r \otimes \iota_s\right) \approx \varsigma_{r,s} \left(\iota_r \otimes \iota_s\right)$, i.e. $\varsigma$ is an admissible element of $\mathfrak{G}^2[B_1^1 \otimes A_1^1]$. Moreover, using again Eq. 5.10, we can see $\varsigma$ is a bicharacter and anti-bicharacter. The twisting of $B \triangleright A$ by $\varsigma$ defines on the vector space $B \triangleright A$ the product $m_\varsigma = m_\mathbb{E}$, with $\Xi \Pi^{\mathfrak{G}^2} = \Pi^{\mathfrak{G}^2} \varsigma$. Then, the underlying algebra of $(B \triangleright A)_\varsigma$ is the one generated by $z_i^j = b_i \otimes a_i$ and quotient by the ideal algebraically generated by

$$\left\{ R^n_{\lambda_n} z_{k_1}^{j_1} \cdots z_{k_n}^{j_n} \left(\sigma S^1\right)_{\omega_n} \right\}_{\lambda_n, n \in \mathbb{N}_0},$$

denoting by $\varsigma$ the product $m_\varsigma$, and the isomorphism $\text{hom}^T[B, A] / \simeq \text{hom}[B, A]_\varsigma$ follows. Therefore,

**Proposition 9.** The quantum spaces $\text{hom}^T[B, A]$ and $\text{hom}[B, A]$ are related by a twist transformation, i.e.

$$\text{hom}^T[B, A] / \simeq \text{hom}[B, A]_\varsigma,$$

being $\varsigma$ a bicharacter in $\mathfrak{G}^2[B_1^1 \otimes A_1^1]$. ■
For the coEnd objects the equation
\[
\text{end} [A]_\zeta \cong \text{end}^T [A] \cong \text{end} [A]
\]
holds, provided \( \zeta = f \chi \) (see Eq. 3.1.2). Moreover, \( A_\chi \) and \( B_\chi \) can alternatively be seen as twisting of the quantum spaces \( A = \text{hom} [\mathcal{K}, A] \) and \( B = \text{hom} [\mathcal{K}, B] \) by cocycles
\[
\begin{align*}
  a_{k_1 \ldots k_r} \otimes a_{k_{r+1} \ldots k_{r+s}} & \mapsto a_{k_1 \ldots k_r} \otimes (\phi^{-r})_{k_{r+1}}^{j_1} \cdots (\phi^{-r})_{k_{r+s}}^{j_s} a_{j_1 \ldots j_s},
  b_{k_1 \ldots k_r} \otimes b_{k_{r+1} \ldots k_{r+s}} & \mapsto b_{k_1 \ldots k_r} \otimes (\rho^{-r})_{k_{r+1}}^{j_1} \cdots (\rho^{-r})_{k_{r+s}}^{j_s} b_{j_1 \ldots j_s},
\end{align*}
\]
in \( \mathbb{Z}^2 [A_1] \) and \( \mathbb{Z}^2 [B_1] \), respectively, which we also denote \( \zeta \). In resume,
\[
\text{hom}^T [B, A] \cong B \triangleright A_\zeta = \text{hom} [B, A_\zeta] \cong \text{hom} [B, A],
\]
unifying in this way the correspondence between the objects \( \text{hom}^T [B, A] \) and \( \text{hom} [B, A] \) (in particular, \( A^T = \text{hom}^T [\mathcal{K}, A] \) and \( A = \text{hom} [\mathcal{K}, A] \)), just in terms of twisting of quantum spaces.

3.2. Twisting and the cohomology relation. We have seen that cocycle condition for a 2-cocycle insure the associativity of its related deformed product. But, when can it be insured these products define a twisted quantum space isomorphic to the original one, or when two twisted quantum spaces are isomorphic? The following theorem gives the answer to these questions.

**Theorem 9.** Given an \( A \)-admissible cochain \( \psi \in \mathbb{Z}^2, A \cong A_\psi \) iff \( \psi = \Theta \) being \( \theta \) an \( A \)-admissible counital 1-cochain. More generally, consider a pair of \( A \)-admissible counital 2-cocycles \( \varphi \) and \( \psi \). Then, \( A_\varphi \cong A_\psi \) iff \( \varphi \) and \( \psi \) are cohomologous through an \( A \)-admissible counital 1-cochain \( \theta \). In particular, if \( \varphi \) and \( \psi \) are related as in Eq. 3.2.1, then \( \varphi \varphi \psi \psi \) through an admissible \( \theta \).

We show below one of the implications. The proof of the other will be given at the end of §3.2.2.

**Proof.** (Part 1) Suppose \( \psi = \delta \Theta (\Theta \otimes \Theta)^{-1} = \Theta \theta \) with
\[
\psi (\text{ker} \Pi^{\otimes 2}) = \text{ker} \Pi^{\otimes 2} \quad \text{and} \quad \theta (\text{ker} \Pi) = \text{ker} \Pi.
\]
(Since \( \partial^2 \Theta = 1 \) for all \( \Theta \in C^1 \), \( \psi \) is a 2-cocycle.) In particular \( \psi \) defines a linear automorphism \( \Psi : A^{\otimes 2} \to A^{\otimes 2} \) such that \( \Pi^{\otimes 2} = \Pi^{\otimes 2} \psi \), which gives rise to the twisted product \( m_\psi = m \Psi \). And \( \theta \) defines, by the equation \( \Theta \Pi = \Pi \Theta \), a linear automorphism \( \Theta : A \to A \) satisfying \( \Theta (A_1) \subset A_1 \). Then,
\[
m_\psi \Pi^{\otimes 2} = m \Pi^{\otimes 2} \psi = m \Pi^{\otimes 2} \delta \Theta (\Theta \otimes \Theta)^{-1} = \Pi m_\Theta \delta \Theta (\Theta \otimes \Theta)^{-1} = \Pi \Theta m_\Theta (\Theta \otimes \Theta)^{-1},
\]
where Eq. 2.7 was used in the last equality. Multiplying to the right by \( \Theta \otimes \Theta \) and using the relation between \( \theta \) and \( \Theta \), we arrive at \( m_\phi (\Theta \otimes \Theta) \Pi^{\otimes 2} = \Theta m \Pi^{\otimes 2} \), and since \( \Pi \) is right invertible, equality \( m_\phi (\Theta \otimes \Theta) = \Theta m \) follows. This means the linear automorphism \( \Theta \) defines an algebra map \( \Theta : A \to A_\psi \) such that \( \Theta (A_1) \subset A_1 \), therefore it defines a quantum space isomorphism \( A \cong A_\psi \).

Now, consider a pair of admissible 2-cochains \( \varphi \) and \( \psi \) such that \( \varphi \varphi \psi \psi \), being \( \theta \) admissible. By definition of cohomologous elements \( \delta \Theta \varphi (\Theta \otimes \Theta)^{-1} = \psi \). Hence, reproducing the calculations above, \( m_\psi (\Theta \otimes \Theta) = \Theta m_\varphi \), being \( \Theta \) the linear automorphism defined by \( \theta \) and \( \Pi \). That is to say \( A_\varphi \cong A_\psi \).

Consider on the set of \( A \)-admissible counital 2-cocycles, the cohomology relation through \( A \)-admissible 1-cochains; i.e. the \( A \)-admissible cochains \( \varphi, \psi \in \mathbb{Z}^2 \) are cohomologous if \( \varphi \varphi \psi \psi \) being \( \theta \) admissible. It is an equivalence relation since proposition of §2.2.3 and the fact that admissible n-cochains form a subgroup of \( C^n [A_1] \) for all \( n \). Then, by theorem above, the resulting quotient set, namely \( H^2_A [A_1] \), characterize completely the isomorphism classes of twisted of \( A \). Since the last claim of theorem above (related to Eq. 3.3.3), the set \( H^2_A [A_1] \) can be regarded as a quotient of \( \text{Aut}_{\mathcal{K}C} [A \otimes A] \).

Following an analogous reasoning, one can define a space \( H^3_A [A_1] \), given by \( A \)-admissible 1-cocycles (ipsos facto counital) quotient by the cohomology relation through admissible cochains (which gives the equality relation, as follows from the general case analyzed in §2.2.1). This space is in bijection with the group of automorphisms of \( A \), i.e. \( H^3_A [A_1] \cong \text{Aut}_{\mathcal{K}C} [A_1] \).

It is worth mentioning that non every admissible 2-cocycle is a coboundary of an admissible 1-cochain. If this were the case, the commutative algebra \( k[a, b] \) of §3.1.1 would be isomorphic to the quantum plane \( \mathbb{A}_k^{20} = \mathbb{K}[a, b] \). In particular, we can affirm that there do exist non trivial twisting of quantum spaces, i.e. \( H^2_A [A_1] \neq \{1^{\otimes 2}\} \). Nevertheless, for quantum spaces formed out by free algebras, every counital 2-cocycle gives

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9To give an automorphism of \( A = (A_1, \mathcal{A}) \) is the same as to give an element \( \alpha \in \text{Aut} [A_1] \) such that \( \alpha, \alpha^{-1} \) can be extended to algebra maps on \( A \). But these are precisely the admissible 1-cocycles in \( C^1 [A_1] \). Since \( Z^1 [A_1] = Z^3 [A_1] \subset \text{Aut} [A_1] \), we have the mentioned bijection.
rise to an isomorphic twisted quantum space. This is a consequence of the fact that every counital 2-cocycle is a counital 2-coboundary (and the admissibility condition is immediate), as we show below.

3.2.1. The equality $Z^2 = B^2$. We have seen that $B^2 \subseteq Z^2$ for every vector space. In this subsection we show the other inclusion and some related consequences.

**Theorem 10.** Every $\psi \in Z^2$ is cohomologous to the identity, i.e. $\psi \in B^2$. Moreover, for every linear automorphism $\varpi : A_1 \cong A_1$, there exists a unique $\theta \in C^1$ such that $\psi = \partial \theta$ and $\theta_1 = \varpi$.

**Proof.** Let us come back to the proof of Theor. 8. Given a cochain $\psi \in C^2$ (non necessarily a counital 2-cocycle) we have defined inductively, from Eq. (3.5), a family of linear isomorphisms $\theta_n : A_1 \cong A_1, n \geq 2$. In an analogous way, take a map $\varpi : A_1 \cong A_1$ and consider $\theta \in C^1$ such that $\theta_0 = 1/\psi_0, \theta_1 = \varpi$, and 

$$\theta_{n+1} \equiv \psi_{n+1} (\theta_0 \otimes \varpi^\theta), \quad n \in \mathbb{N}.$$ 

Let us show that if $\psi$ is a 2-cocycle, then, using the identification given by Eq. (2.8),

$$\theta_{r+s} \equiv \psi_{r,s} (\theta_r \otimes \theta_s),$$

or equivalently $\psi = \delta_1 \theta (\theta \otimes \theta)^{-1} = \partial \theta$. We will make induction on $s$. Recall that, since Eq. (2.22), $\psi_{r,0} \equiv \psi_{0,0} \cdot 1_r$ for all 2-cocycle. Then for $s = 0$, by definition of $\theta_0$ and $\theta_1$, Eq. (3.17) holds. For $s + 1$,

$$\psi_{r,s+1}(\theta_r \otimes \theta_{s+1}) \equiv \psi_{r,s+1}(\theta_r \otimes \psi_{s,1}(\theta_s \otimes \varpi)) = \psi_{r,s+1}(\theta_r \otimes \theta_s \otimes \varpi).$$

Using Eq. (2.19) for $t = 1$,

$$\psi_{r,s+1}(\theta_r \otimes \theta_{s+1}) \equiv \psi_{r+1,s}(\theta_r \otimes \theta_s \otimes \varpi) = \psi_{r,s+1}(\theta_r \otimes \theta_s \otimes \varpi),$$

and from the inductive hypothesis (i.e. validity of Eq. (3.17))

$$\psi_{r,s+1}(\theta_r \otimes \theta_{s+1}) \equiv \psi_{r,s+1}(\theta_r \otimes \theta_{s+1}) = \theta_{r+s+1},$$

as we wanted to show.

Now, consider another $\chi$ such that $\psi = \partial \chi$ and $\chi_1 = \varpi$. From Eq. (3.17), $\chi_0 = 1/\psi_0, \varpi$ and the rest of $\chi$ is given inductively by the equation $\chi_{n+1} = \psi_{n+1} (\chi_n \otimes \varpi)$, concluding in this way the proof of the our claim. \[\square\]

**Corollary 1.** If $\psi \in Z^2$, then $\psi \in B^2$. In addition, fixing an automorphism $\varpi$ of $A_1$, there exists a unique $\theta \in C^1$ such that $\psi = \partial \theta$ and $\theta_1 = \varpi$.

**Proof.** From previous theorem, if $\psi \in Z^2 = Z^2 \cap C^2$, then $\psi \in B^2 \cap C^2$ and there exists a unique $\theta \in C^1$ such that $\psi = \partial \theta$ and $\theta_1 = \varpi$. But, as we have shown in §2.2.1 (see Eq. (2.22)), $B^2 \cap C^2 = B^2$. Thus $\psi \in B^2$ and accordingly $\theta \in C^1$. \[\square\]

This corollary can also be proven by noting that $\psi$ is counital if and only if $\psi_{0,0} = 1$ (see Eq. (2.21)). This makes $\theta_0 = 0 = 1$, i.e. $\theta$ is counital.

**Corollary 2.** Eventy twisting of the quantum space $A = (A_1, A_1^\otimes)$ is isomorphic to $A$.

**Proof.** Every $n$-cochain in $C^n(A_1)$ is obviously admissible for the quantum space $A = (A_1, A_1^\otimes)$ (since $\ker \Pi = 0$). Then, from above results, given a counital 2-cocycle $\psi$ there exists a counital 1-cochain $\theta$ (admissible for $A$) such that $\psi = \partial \theta$. In this situation, Theor. 9 insures $A \cong A_\theta$. \[\square\]

These results motive the definition of a subgroup $B_1^1 |A_1| \subset C^1 |A_1|$ constituted by counital 1-cochains $\theta$ such that $\theta_1 = \Pi$. We shall call primitive the elements of this subgroup. From now on, when we write $\psi = \partial \theta$, we are supposing $\theta$ is in $B^1$, unless we say the contrary.

As an example, the cochain

$$\varsigma_A : a_{k_1} \ldots a_{k_r} \otimes a_{k_{r+1}} \ldots a_{k_{r+s}} \rightarrow a_{k_1} \ldots a_{k_r} \otimes (\sigma_{\mathcal{A}}^{r \ast})_{k_{r+1}} \ldots (\sigma_{\mathcal{A}}^{r \ast})_{k_{r+s}} a_{j_1} \ldots a_{j_s}$$

related to the twisted internal coHom objects $\mathcal{B} \otimes \mathcal{A}$, has primitive

$$\theta_A : a_{k_1} \ldots a_{k_r} \rightarrow \delta_{k_1}^{j_1} (\sigma_{\mathcal{A}}^{r \ast})_{k_2} (\sigma_{\mathcal{A}}^{r \ast})_{k_3} \ldots (\sigma_{\mathcal{A}}^{r \ast})_{k_{r+s}} a_{j_1} \ldots a_{j_s}.$$ 

Since we have shown in §2.2.3 that cohomology relation is an equivalence relation in $C^2 |A_1|$, previous results imply, in particular, that every pair of 2-cocycles are cohomologous. The following proposition tell us which class of 1-cochains implement such a relation.

\[10\text{If the twisted space is not also free, then it would have a non cero related ideal, which would imply the underlying vector space is smaller than the untwisted one.}\]
Proposition 10. Consider a pair of cochains \( \psi, \varphi \in \mathbb{Z}^2 \) with primitive \( \lambda, \chi \in \mathfrak{P}^1 \), respectively.

a) They are cohomologous through \( \theta \) iff \( \theta = \lambda \omega ^{-1} \), with \( \omega \in \mathbb{Z}^1 \).

b) They are cohomologous through an admissible cochain iff there exists \( \omega \in \mathbb{Z}^1 \) such that \( \lambda \omega ^{-1} \) is admissible.

Proof. a) If \( \psi = \partial \lambda, \varphi = \partial \chi, \theta = \lambda \omega ^{-1} \) and \( \partial \omega = \mathbb{I} \circ \mathbb{I} \) (i.e. \( \omega \) is a 1-cocycle), then

\[
\delta_1 \theta \varphi (\theta \otimes \theta)^{-1} = \delta_1 (\lambda \omega ^{-1}) \left( \delta_1 \chi (\chi \otimes \chi)^{-1} \right) (\lambda \omega ^{-1} \otimes \lambda \omega ^{-1})^{-1} = \delta_1 \lambda \left( \delta_1 \omega (\omega \otimes \omega )^{-1} \right) (\lambda \otimes \lambda )^{-1} = \delta_1 \lambda (\lambda \otimes \lambda )^{-1} = \psi.
\]

That means \( \varphi \sim_\theta \psi \), and one implication follows. Now suppose \( \varphi \sim_\theta \psi \). We always can write \( \theta = \lambda \omega ^{-1} \), for every \( \theta \in \mathbb{C}^1 \). It is enough to take \( \omega = \lambda ^{-1} \theta \chi \). Let us see that \( \omega \in \mathbb{Z}^1 \). From equation above if \( \varphi \sim_\theta \psi \) we have that

\[
\delta_1 \lambda \left( \delta_1 \omega (\omega \otimes \omega )^{-1} \right) (\lambda \otimes \lambda )^{-1} = \delta_1 \lambda (\lambda \otimes \lambda )^{-1} ,
\]

which is fulfilled if and only if \( \delta_1 \omega = \omega \otimes \omega \), i.e. \( \omega \in \mathbb{Z}^1 \), as we wanted to show.

b) This part is an immediate consequence of the first one. \( \blacksquare \)

Recall that to give an element \( \omega \in \mathbb{Z}^1 [A_1] \) is the same as giving one in \( \varpi \in Aut [A_1] \), since every 1-cocycle is of the form \( \omega = \varpi \circ \omega \). This connects above proposition with Theor. 9.

A brief comment about bialgebras is in order. The quasicomplex \( \mathfrak{G}^* [A^1 \mathfrak{P}] \) also satisfies that every 2-cocycle is a 2-coboundary. In fact, given a cocycle \( \chi \in \mathfrak{G}^2 \), a cochain \( \lambda \in \mathfrak{G}^1 \), such that \( d\lambda = \chi \), can be inductively defined by the formula

\[
\lambda_0 = 1 / \chi_{0,0}, \lambda_{n+1} = (\lambda_n \otimes \varepsilon) * \chi_{n,1}.
\]

Moreover, every pair of 2-cocycles in \( \mathfrak{G}^2 [A^1 \mathfrak{P}] \) are cohomologous.\(^{11}\) But for \( \mathfrak{G}^2 [A] \), given a 2-cocycle \( \phi \) there, we can just say \( \phi \Pi \mathbb{I} = d\theta ; \) and \( \phi \) will be a coboundary iff \( \theta = \theta \Pi \).

3.2.2. Role of primitive 1-cochains. The role of the 1-cochains \( \theta \) defining a given 2-cocycle \( \psi \), for a generic quantum space \( A \), is expressed by the following theorem.

Theorem 11. Let \( A = (A_1, A) \) be a quantum space and consider an admissible \( \psi \in \mathbb{Z}^2 [A_1] \). The canonical epimorphism \( A^1 \otimes A \psi \rightarrow A \psi \) associated to \( A \psi = (A_1, A_\psi) \) is given by \( \Pi \theta \), being \( \theta \in \mathfrak{P}^1 [A_1] \) the primitive of \( \psi \). Furthermore, if \( I \) is the ideal related to \( A \), then \( I \psi \), the one related to \( A \psi \), is equal to \( \theta^{-1} (I) \).

Proof. Since \( \theta_{0,1} = I_{0,1} \) (which means \( \theta (A_1) = A_1 \) and \( \theta (\lambda) = \eta (\lambda) \)), \( \Pi \theta \) defines the inclusion \( A_1 \hookrightarrow A \psi \) and a unit preserving map. If we show that

\[
(3.19)
\]

what implies (multiplying by \( \Pi \) to the left) \( m_\psi (\Pi \theta \otimes \Pi \theta) = \Pi \theta m_\psi \), we are proving \( \Pi \theta \) is an algebra homomorphism \( A^1 \otimes A \psi \rightarrow A \psi \).

Eq. (3.19) restricted to a subspace \( A^1 \otimes A \psi \) means \( \psi_{r,s} (\theta_r \otimes \theta_s) = [\delta_1 \theta]_{r,s} \), or equivalently, \( \psi_{r,s} (\theta_r \otimes \theta_s) \approx \theta_{r,s} \). But this is true since \( \partial \theta = \psi \), hence Eq. (3.19) holds.

Finally, we must show \( \theta^{-1} (I) \) is equal to \( I \psi \), the ideal related to \( A \psi \). By definition, \( I = \ker \Pi \) and, from the last result, \( I \psi = \ker \Pi \theta \). In addition, it is well known that \( \ker \Pi \theta = \theta^{-1} (\ker \Pi) \), and consequently our claim follows. \( \blacksquare \)

Thus we can write (compare to Eq. (3.3))

\[
a_{i_1} \cdot \psi \ldots \cdot \psi a_{i_n} = \Pi \theta (a_{i_1} \otimes \ldots \otimes a_{i_n}) ,
\]

being \( " \cdot \psi " = m_\psi \). On the other hand, note that if \( \theta \) is \( A \)-admissible, then \( I \psi = \theta^{-1} (I) = I \) and, therefore, \( A \cong A \psi \), as we proved before.

As an immediate corollary of last theorem we have:

Corollary 3. Consider a conic quantum space \( A \) with \( I = \bigoplus_{n \geq 2} I_n \), and a counital \( A \)-admissible 2-cocycle \( \psi = \partial \theta \in \mathbb{Z}^2 [A_1] \). The ideal related to the twisted quantum space \( A \psi \) is

\[
I \psi = \bigoplus_{n \geq 2} I_{\psi,n} = \bigoplus_{n \geq 2} \theta^{-1} (I_n) . \]

\(^{11}\)The cochains that implement the cohomology relation have a form analogous to the ones given in last proposition for the complex \( \mathfrak{C}^* [A_1] \).
In coordinates, if $I$ is linearly generated by
\[ \left\{ R^{k_1 \ldots k_n}_{\lambda_1 \ldots \lambda_n} \right\}_{\lambda_n \in \Lambda_n} \subset I_n, \]
then $I_\psi$ is linearly generated by the elements
\[ R^{k_1 \ldots k_n}_{\lambda_1 \ldots \lambda_n} \theta^{-1}(\alpha_{k_1} \ldots \alpha_{k_n}) = R^{k_1 \ldots k_n}_{\lambda_1 \ldots \lambda_n} (\theta^{-1})^{j_1 \ldots j_n}_{\kappa_1 \ldots \kappa_n} a_{j_1} \ldots a_{j_n}. \]

Last corollary gives us another way to see that twisting of conic quantum spaces are also conic. Further work shows the same for $m$-th quantum spaces. In what follows, and to avoid any confusion, we shall write $X \cdot Y \subset A_1^\otimes$ for the image under $m_\otimes$ of a couple of subspaces $X, Y \subset A_1^\otimes$.

Proposition 11. If $A \in CA^n$ and $I = I[Y]$, the ideal generated by $Y \subset A_1^{\otimes m}$, then $A_\psi \in CA^n$ and $I_\psi = I[\theta^{-1}(Y)]$.

Thus, twist transformations of $A \in CA^n$ give objects of $CA^n$. Moreover, if $A \in CA$ has an ideal $I$ generated by a graded vector subspace of $A_1^\otimes$
\[ S = \bigoplus_{n \in \mathbb{N}_0} S_n; \quad S_{0,1} \doteq \{0\}, \]
i.e. $I = I[S] = A_1^\otimes \cdot S \cdot A_1^\otimes$, then the ideal related to $A_\psi$ is
\[ I_\psi = \theta^{-1}(I[S]) = I[\theta^{-1}(S)] = A_1^\otimes \cdot \theta^{-1}(S) \cdot A_1^\otimes. \]

Before going to the proof, let us show the lemma below.

Lemma 3. Let $A$ be a conic quantum space with ideal $I = \bigoplus_{n \geq 2} I_n$. $\psi = \partial \theta$ is admissible iff
\[ \theta^{-1}(A_1^\otimes \cdot I_r + I_r \cdot A_1^\otimes) = A_r^\otimes \cdot \theta^{-1}(I_r) + \theta^{-1}(I_r) \cdot A_r^\otimes. \]

Proof. If $\psi$ is admissible, then (see Eq. (3.2))
\[ \psi_{r,s}(A_1^\otimes \cdot I_s + I_r \cdot A_1^\otimes) = A_{r,s}^\otimes \cdot I_s + I_r \cdot A_{r,s}^\otimes. \]

Since $\psi = \delta_1 \theta (\theta \otimes \theta)^{-1}$, equation above says
\[ [\delta_1 \theta^{-1}]_{r,s}(A_1^\otimes \cdot I_s + I_r \cdot A_1^\otimes) = (\theta_r \otimes \theta_s)^{-1}(A_{r,s}^\otimes \cdot I_s + I_r \cdot A_{r,s}^\otimes). \]

Applying to the left $m_\otimes$, using $m_\otimes \delta_1 \theta = m_\otimes \theta$ and $\theta(A_1^\otimes) = A_1^\otimes \cdot I_r$, we arrive precisely at Eq. (3.20).

Reciprocally, if $\theta$ satisfies (3.20), Eq. (3.21) follows immediately and with it the admissibility of $\partial \theta$. 

Proof. (of proposition) Let us consider the more general case. Suppose we have $A \in CA$ with
\[ I[S] = \bigoplus_{n \geq 2} I_n; \quad I_n = \sum_{r=2}^n \sum_{i=0}^{n-r} A_i^\otimes \cdot I_r \cdot A_r^\otimes, \]
and an $A$-admissible 2-cocycle $\psi = \partial \theta$. From lemma above, since $I_1 = \{0\}$, we have
\[ \theta^{-1}(A_1 \cdot I_r) = A_1 \cdot \theta^{-1}(I_r) \quad \text{and} \quad \theta^{-1}(I_r \cdot A_1) = \theta^{-1}(I_r) \cdot A_1, \]
for all $r$, and in particular
\[ \theta^{-1}(A_1 \cdot S_2) = A_1 \cdot \theta^{-1}(S_2) \quad \text{and} \quad \theta^{-1}(S_2 \cdot A_1) = \theta^{-1}(S_2) \cdot A_1. \]

Consequently, $\theta^{-1}(I_3) = A_1 \cdot \theta^{-1}(S_2) + \theta^{-1}(S_2) \cdot A_1 + \theta^{-1}(S_3)$. We can show by induction on $n \geq 3$ that
\[ \theta^{-1}(I_n) = \sum_{r=2}^n \sum_{i=0}^{n-r} A_i^\otimes \cdot \theta^{-1}(I_r) \cdot A_r^\otimes. \]

In fact, because $I_{n+1} = A_1 \cdot I_n + I_n \cdot A_1 + S_{n+1}$, and using (3.22)
\[ \theta^{-1}(I_{n+1}) = \theta^{-1}(A_1 \cdot I_n + I_n \cdot A_1 + S_{n+1}) = A_1 \cdot \theta^{-1}(I_n) + \theta^{-1}(I_n) \cdot A_1 + \theta^{-1}(S_{n+1}), \]
we have from inductive hypothesis
\[ \theta^{-1}(I_{n+1}) = \sum_{r=2}^n \sum_{i=0}^{n-r} A_i^\otimes \cdot \theta^{-1}(I_r) \cdot A^\otimes_i + \sum_{r=2}^n \sum_{i=0}^{n-r} A_i^\otimes \cdot \theta^{-1}(S_r) \cdot A_i^\otimes_i + \theta^{-1}(S_{n+1}) = \sum_{r=2}^n \sum_{i=0}^{n-r} A_i^\otimes \cdot \theta^{-1}(S_r) \cdot A^\otimes_i. \]
Thus, $\theta^{-1}(I[S]) = I[\theta^{-1}(S)]$, as we wanted to see. 

\[ ^{12} \text{This is the case of quantum spaces of the form } A \otimes B \text{ with } \otimes = \bullet, \cdot, , \circ \text{ or } \odot, \text{ and } A'. \]
Example: In the quadratic case $I = I \{Y\}$ is an ideal generated by $Y \subset A_1^{\otimes 2}$. Then, since $\theta_2 \approx \psi_{1,1}$, we have that $I_\psi = I \{\psi_{1,1}^{-1}(Y)\}$. Let us suppose $Y$ is given by a set $Y^{ki}_\lambda a_k \otimes a_i$, $\lambda \in \Lambda$. If $\psi_{1,1} \approx M \in GL (n^2)$, $I_\psi$ is algebraically generated by the elements $Y^{ki}_\lambda A^{kj}_\lambda a_i \otimes a_j$, with $N = M^{-1}$. For a freely commutative algebra, we have $\Lambda = [n] \times [n]$ and $Y^{ki}_\lambda = Y^{ij}_\lambda = \delta_k^i \delta_j^\lambda - P^{ki}_\lambda$. From all that Equation (3.23) follows.

To end this subsection, let us prove the other implication of Theor. 9.

Proof. (of Theor. 9, part 2) Suppose $A_\varphi \preceq A_\psi$. That is to say, there exists a linear automorphism $\Theta : A \preceq A$ such that $\Theta (1) = 1$ and, in a basis $\{a_i\}$ of $A_1$, $\Theta (a_i) = \varpi^i_1 a_j$ and

$$\Theta (a_{i_1} \otimes \cdots \otimes a_{i_k}) = \varpi^{i_1}_1 \cdots \varpi^{i_k}_1 a_{j_1} \cdot \cdots \cdot a_{j_k},$$

being $\varpi^i_1$ the matrix elements of an automorphism $\varpi : A_1 \preceq A_1$. If $\chi$ and $\lambda$ are the primitive of $\varphi$ and $\psi$, resp., the last equation can be written $\Theta \Pi \chi_k = \Pi \lambda_k \varpi^\otimes k$, $k \in N_0$. Thus, the 1-cocycle $\theta$ given by $\theta_k = \lambda_k \varpi^\otimes k \chi_k^{-1}$ is $A$-admissible. Calling $\omega$ the 1-cocycle such that $\omega \varphi \sim \omega \psi$, then $\theta = \lambda \omega \chi^{-1}$. From Prop. 10 it follows that $\varphi \sim \theta \psi$. In particular, if $\Pi^\otimes 2 (\varphi - \psi) = 0$ (see Eq. (3.3)), then $m_\varphi = m_\psi$ and, in consequence, $A_\varphi = A_\psi$. That implies $\varphi$ and $\psi$ are cohomologous through the admissible 1-cocycle $\lambda \chi^{-1}$ (since $\varpi = 1$).

3.3. The gauge equivalence.

3.3.1. Composition and inversion of twist transformations. In this subsection we show that consecutive applications of twist transformations is again a twist transformation, and that twist transformations have inverse. To start with, the following results will be crucial.

Proposition 12. The map $\partial$ defines a surjection $C^1 \to \hat{3}^2$ which restricted to the subgroup $\hat{3}^1 \subset C^1$ became a bijection $\hat{3}^1 \approx \hat{3}^2$.

This is an immediate corollary of Theor. 10.

Proposition 13. The set $\hat{3}^2$ is a group under the product

$$(\psi, \varphi) \mapsto \psi \ast \varphi \equiv \psi (\theta \otimes \theta) \varphi (\theta \otimes \theta)^{-1},$$

being $\theta \in \hat{3}^1$ the primitive of $\psi$. With respect to this group structure the bijection $\hat{3}^1 \approx \hat{3}^2$ is a group isomorphism.

Proof. From $\partial : \hat{3}^1 \approx \hat{3}^2$ we have that $\psi \ast \varphi = \partial \theta \ast \partial \chi$ is equal to

$$\partial \theta (\theta \otimes \theta) \partial \chi (\theta \otimes \theta)^{-1} = \delta_1 \theta \delta_1 \chi (\theta \otimes \theta)^{-1} (\chi \otimes \chi)^{-1} = \delta_1 \theta (\theta \chi \otimes \theta \chi)^{-1} = \partial (\theta \chi),$$

thus $\theta \chi \in \hat{3}^1$ is sent to $\psi \ast \varphi$ via $\partial$. In other words, $\partial$ restricted to $\hat{3}^1$ translates the product on its domain into the map $\ast$. Hence, $\ast$ is an associative product with unit $\Pi^\otimes 2 = \partial I$, and each $\psi \in \hat{3}^2$ has inverse

$$i\psi = \partial (\theta^{-1}) = (\delta_1 \theta^{-1}) (\theta \otimes \theta) = (\theta \otimes \theta)^{-1} (\theta \otimes \theta)^{-1} \psi^{-1} (\theta \otimes \theta).$$

This concludes the proof. 

Now we are ready to prove our claim.

Proposition 14. Given a quantum space $A = (A_1, A)$ and a couple of counital 2-cocycles $\psi$ and $\varphi$, admissible for $A$ and $A_\psi$, respectively, then $\psi \ast \varphi$ is $A$-admissible and $(A_\psi)_{\varphi} = A_\psi \ast \varphi$. In addition $i\psi$, the inverse of $\psi$ under the product $\ast$, is admissible for $A_\psi$, and accordingly $(A_\psi)_{i\psi} = A$.

Proof. Let $\Psi$ and $\Phi$ be the automorphisms of $A^\otimes 2$ related to $\psi$ and $\varphi$, respectively, i.e.

$$\Psi \Pi^\otimes 2 = \Pi^\otimes 2 \psi \quad \text{and} \quad \Phi (\Pi \theta)^\otimes 2 = (\Pi \theta)^\otimes 2 \varphi,$$

if $\psi = \partial \theta$. The last equality implies $\Phi \Pi^\otimes 2 = \Pi^\otimes 2 (\theta \otimes \theta) \varphi (\theta \otimes \theta)^{-1}$ and

$$\Psi \Phi \Pi^\otimes 2 = \Pi^\otimes 2 \psi (\theta \otimes \theta) \varphi (\theta \otimes \theta)^{-1} = \Pi^\otimes 2 \psi \ast \varphi.$$

Thus $\psi \ast \varphi$ is $A$-admissible, and since the product of $(A_\psi)_{\varphi}$ is $(m \Psi) \Phi = m (\Psi \Phi)$, the first claim of the proposition follows. Now, we must show that the 2-cocycle $i\psi = \partial (\theta^{-1}) = (\theta \otimes \theta)^{-1} \psi^{-1} (\theta \otimes \theta)$ is admissible for $A_\psi$. But

$$(\Pi \theta)^\otimes 2 i\psi = \Pi^\otimes 2 \psi^{-1} (\theta \otimes \theta) = \Psi^{-1} (\Pi \theta)^\otimes 2,$$

hence $i\psi (\ker (\Pi \theta)^\otimes 2) \subset \ker (\Pi \theta)^\otimes 2$. Then, the twisting of $A_\psi$ by $i\psi$ can be defined, and from the above result $(A_\psi)_{i\psi} = A_{\psi \ast i\psi} = A$. In particular, the multiplication of $(A_\psi)_{i\psi}$ will be, since Eq. (3.23), $(m \Psi) \Psi^{-1} = m (\Psi \Psi^{-1}) = m$. 

Let us recall that something similar happens in the bialgebra case. It is well-known that if \( \chi, \zeta \) are 2-cocycles in \( G^2[A] \) and \( G^2[A_\chi] \), respectively, then \( \zeta \ast \chi \) is a 2-cocycle in \( G^2[A] \), and \((A_\chi)_\zeta = A_{\chi \ast \chi} \). In addition, \( \chi \) is a 2-cocycle in \( G^2[A_\chi] \), therefore \((A_\chi)_{\chi^{-1}} = A\). The main difference is that in \( G^2[A] \) the convolution does not define a subgroup structure for its 2-cocycles.

3.3.2. Gauge transformations. Let us call \( T_\psi \) the twist transformation related to \( \psi \in \mathbb{Z}^2 \) over a given quantum space. The results of the previous subsection imply the twist transformations, associated to an inclusion of vector spaces \( A_1 \subset A \), define a groupoid with (partial) composition \( T_\psi T_\varphi = T_{\psi \ast \varphi} \) and inversion \( T_\psi^{-1} = T_{\psi^{-1}} \). As a generalization of this fact, let us consider the following definition.

**Definition 4.** We shall say a couple of quantum spaces \( A \) and \( B \) are **twist or gauge related**, namely \( A \sim B \), if there exists an \( A \)-admissible \( \psi \in \mathbb{Z}^2[A_1] \) and an isomorphism of quantum spaces \( \alpha : A_\psi \cong B \). The pairs \((\alpha, \psi)\) will be called **gauge transformations** between \( A \) and \( B \).

**Proposition 15.** The twist relation is an equivalence relation.

**Proof.**

*Reflexivity:* Since \( \mathbb{I}^\otimes 2 \in \mathbb{Z}^2[A_1] \) is trivially \( A \)-admissible, the identity morphism insures \( A \sim A \).

*Symmetry:* Suppose \( A \sim B \) via a cochain \( \psi \) and an isomorphism \( \alpha \). Let \( \theta \) be the primitive of \( \psi \), and consider the 2-cocycle \( \psi (\theta \otimes \theta)^{-1} \psi^{-1} (\theta \otimes \theta) \in \mathbb{Z}^2[A_1] \). Since the restriction \( \alpha_1 = \alpha|_A \) defines an isomorphism \( A_1 \cong B_1 \), through the functor given in \( \mathbb{2.3} \) we have a bijection \( \mathbb{2.3} \) \( [A_1] \cong [B_1] \), such that

\[
\psi \mapsto \psi \circ_1 = (\alpha_1 \otimes \alpha_1^\otimes) \psi (\alpha_1 \otimes \alpha_1^\otimes)^{-1}.
\]

In particular, \( \psi \circ_1 = (\alpha_1 \otimes \alpha_1^\otimes) \psi (\alpha_1 \otimes \alpha_1^\otimes)^{-1} \in \mathbb{2.3} \). Using that \( \alpha_1 \Pi_\psi = \Pi_1 \alpha_1^\otimes \) and \( \psi_1 \Pi_\psi = \Pi_1 \psi \), we have

\[
\Pi_1 \alpha_1 \psi \circ_1 = (\alpha_1 \otimes \alpha_1) \psi^{-1} (\alpha_1 \otimes \alpha_1)^{-1} \Pi_1 \psi.
\]

Thus, \( \psi \circ_1 \) is \( B \)-admissible and defines the twisted product

\[
m_B (\alpha \otimes \alpha) \Psi^{-1} (\alpha \otimes \alpha)^{-1}.
\]

On the other hand, since \( m_B (\alpha \otimes \alpha) = \alpha m_A \Psi \),

\[
a_1 [m_B (\alpha \otimes \alpha) \Psi^{-1} (\alpha \otimes \alpha)^{-1}] = m_A (\alpha \otimes \alpha)^{-1},
\]

therefore \( a_1 \) defines a quantum space isomorphism \( B_\psi \circ_1 \cong A \), and consequently \( B \sim A \).

*Transitivity:* Consider another quantum space \( C \) such that \( B \sim C \) via a cochain \( \varphi \) and an isomorphism \( \beta \). Using again \( \alpha_1 \Pi_\psi = \Pi_1 \alpha_1^\otimes \) and \( \psi_1 \Pi_\psi = \Pi_1 \psi \), it can see that

\[
\psi \circ_1 = (\alpha_1 \otimes \alpha_1^\otimes) \psi (\alpha_1 \otimes \alpha_1^\otimes)^{-1} \varphi (\alpha_1 \otimes \alpha_1^\otimes)^{-1} (\theta \otimes \theta)^{-1}
\]

is \( A \)-admissible and has a related automorphism \( \Psi (\alpha_1 \otimes \alpha_1)^{-1} \Phi (\alpha \otimes \alpha) \), if \( \Phi \Pi_B = \Pi_B \varphi \). It defines on \( A \) the twisted product \( m_A \Psi (\alpha \otimes \alpha)^{-1} \Phi (\alpha \otimes \alpha) \).

\[
\beta \alpha \left[ m_A \Psi (\alpha \otimes \alpha)^{-1} \Phi (\alpha \otimes \alpha) \right] = \beta m_B \Phi (\alpha \otimes \alpha) = m_C (\beta \otimes \alpha \otimes \beta \alpha),
\]

where we have used \( \beta m_B \Phi = m_C (\beta \otimes \beta \otimes \beta) \). It follows that \( A \sim C \).

Denoting by \( T_{(\alpha, \psi)} \) the gauge transformation defined by \( (\alpha, \psi) \), the theorem above says those transformations form a groupoid (or a category based on quantum spaces) with composition and inverse

\[
T_{(\beta, \psi)} T_{(\alpha, \psi)} = T_{(\beta, \psi) \ast (\alpha, \psi)}, \quad T_{(\alpha, \psi)}^{-1} = T_{(\alpha, \psi)},
\]

being

\[
(\beta, \varphi) \ast (\alpha, \psi) = (\beta \alpha, \psi \ast \varphi)^{-1} \quad \text{and} \quad i (\alpha, \psi) = (\alpha^{-1}, \psi \circ_1).
\]

As examples of gauge equivalence we have \( hom^X[B, A] \sim hom^X[B, A] \) for every pair \( B, A \) in \( CA \), and \( A \circ_r B \sim A \circ B \) for every STTP of \( A \) and \( B \).

Geometrically, we are defining an equivalence relation among non commutative algebraic varieties in QLS.

A characterization of gauge equivalence between conic quantum spaces can be given in terms of their corresponding ideals. More precisely,

**Theorem 12.** Let \( A, B \) be objects of \( CA \), with related ideals \( I \) and \( J \), respectively. \( A \sim B \) iff there exists an homogeneous (of degree zero) linear isomorphism \( \vartheta : A^\otimes 1 \cong B^\otimes 1 \) such that

\[
\vartheta (A^\otimes r \cdot I_s + I_r \cdot A^\otimes s) = B^\otimes r \cdot J_s + J_r \cdot B^\otimes s, \quad r, s \in \mathbb{N}.
\]
Proof. Suppose $A$ is gauge equivalent to $B$, i.e. there exists an $A$-admissible $\psi \in Z^2[A_1]$ and an isomorphism of quantum spaces $\alpha : A_\psi \simeq B$. If $\psi = \partial \theta$, from Lemma 3, admissibility condition is equivalent to
\begin{equation}
(3.25)
\theta^{-1} \left( A_1^{\otimes r} \cdot I_s + I_r \cdot A_1^{\otimes s} \right) = A_1^{\otimes r} \cdot \theta^{-1}(I_s) + \theta^{-1}(I_r) \cdot A_1^{\otimes s}.
\end{equation}
Because $\alpha$ defines an algebra homomorphism $A_\psi \simeq B$, that is to say
\begin{equation}
(3.26)
\alpha_1^{\otimes} (I_{\psi,r}) = \alpha_1^{\otimes} \theta^{-1}(I_r) = J_r,
\end{equation}
we have
\begin{align*}
\alpha_1^{\otimes} (I_{\psi,r}) & = \alpha_1^{\otimes} (A_1^{\otimes r}) \cdot \alpha_1^{\otimes} \theta^{-1}(I_s) + \alpha_1^{\otimes} \theta^{-1}(I_r) \cdot \alpha_1^{\otimes} (A_1^{\otimes s}) \\
& = B_1^{\otimes r} \cdot J_r + J_r \cdot B_1^{\otimes s},
\end{align*}
The linear map $\theta = \alpha_1^{\otimes} \theta^{-1} : A_1^{\otimes} \rightarrow B_1^{\otimes}$ is obviously the map we are looking for.

Reciprocally, consider an homogeneous linear bijection $\vartheta : A_1^{\otimes} \simeq B_1^{\otimes}$ satisfying (3.24). Given a basis $\{a_i\}$ of $A_1$, if $\vartheta(a_i) = b_i$, then $\{b_i\}$ is a basis of $B_1$, and we can write
\begin{equation}
\vartheta (a_{i_1} \ldots a_{i_n}) = \vartheta r^n_{j_1 \ldots j_n} b_{j_1} \ldots b_{j_n} \vartheta (1) = 1.
\end{equation}
The assignment $\alpha_1 = \vartheta|_{A_1} : a_i \mapsto b_i$ defines a linear bijection, and $\vartheta$ can be decomposed as $\vartheta = \alpha_1^{\otimes} \theta^{-1}$ with
\begin{equation}
\alpha_1^{\otimes} \theta^{-1}(a_{i_1} \ldots a_{i_n}) = (\alpha_1^{\otimes} \theta^{-1} n_{j_1 \ldots j_n} a_{j_1} \ldots a_{j_n} \vartheta (1) = 1).
\end{equation}
In particular, since $(\vartheta 1)^j_1 = 1^j_1$ and $\vartheta (1) = 1$, then $\theta_{0,1} = I_{0,1}$ and, as a consequence, $\theta \in \mathcal{P}^1[A_1]$. Let us consider the counital 2-cocycle $\psi = \partial \theta$. We shall see $\psi$ is $A$-admissible and $\alpha_1^{\otimes}$ defines an isomorphism $A_\psi \simeq B$.

Multiplying Eq. (3.24) by $(\alpha_1^{\otimes})^{-1}$ to the left, we obtain precisely Eq. (3.26). But this is the admissibility condition for $\psi$, provided $\psi_{r,s} \approx r_{t+s} (\theta_r \otimes \theta_s)^{-1}$. Now, using again (3.24), but in the form (3.26) and fixing $s = 0$, equation $\alpha_1^{\otimes} (I_{\psi,r}) = J_r$ follows.

In the general case we can say,

**Proposition 16.** Given $A, B \in FGA$, with related ideals $I$ and $J$, if $A \simeq B$, then there exists an homogeneous (of degree zero) linear bijection $\vartheta : A_1^{\otimes} \simeq B_1^{\otimes}$ such that $\vartheta(I) = J$.

**Proof.** If $A \simeq B$ through $\psi = \partial \theta \in Z^2[A_1]$ and an isomorphism $\alpha : A_\psi \simeq B$, then $\vartheta = \alpha_1^{\otimes} \theta^{-1}$ is an homogeneous linear isomorphism such that $\vartheta(I) = J$.

Thus, gauge equivalent non commutative spaces in QLS have isomorphic defining ideals.

## 4. Twisting and functors on CA

In this chapter we mainly deal with conic quantum spaces, analyzing the relationship between certain functors defined on their category, and quasicomplexes naturally related to them. For instance, $c^* [A_1]$ and $c^* [B_1]$ are quasicomplexes naturally related to the functors $! : A \rightarrow A$, while $c^* [A_1], c^* [B_1]$ and $c^* [A_1 \otimes B_1]$ are related to bifunctors $\circ, \bullet$ and $\otimes$. We shall study the following problems. Suppose a twisting $A_\varphi$ of $A$ is given (of course, $\varphi \in Z^2[A_1]$): when can we insure there exists $\chi \in Z^2[A_1]$ such that $(A_\varphi)^* = (A)^* \chi$? On the other hand, given another twisted quantum space, namely $B_\phi$ (now $\phi \in Z^2[B_1]$): can we write
\begin{equation}
A_\varphi \otimes B_\phi = (A \otimes B)\kappa, \quad \kappa = \circ, \bullet \text{ or } \otimes,
\end{equation}
for some $\kappa \in Z^2[A_1 \otimes B_1]$?

In the following, we answer these questions and other corresponding to the functors $\prec, \succ$ and $\circ$. In relation to latter functors, we study the internal coHom objects of twisted quantum spaces, comparing them with the untwisted ones.

### 4.1. 2nd admissibility condition

For conic quantum spaces, a stronger condition than admissibility is defined. Its consequences are analyzed along all subsequent subsections.

**Definition 5.** Let $A$ be a conic quantum space with related ideal $I = \bigoplus_{n \geq 2} I_n$. We shall call **2nd $A$-admissible**, or **2nd admissible** for $A$, the $n$-cochains satisfying
\begin{equation}
(4.1)
\psi \left( A_1^{\otimes p_1} \otimes \ldots \otimes A_1^{\otimes p_k} \cdot I_q \cdot A_1^{\otimes r} \otimes \ldots \otimes A_1^{\otimes p_n} \right) = A_1^{\otimes p_1} \otimes \ldots \otimes \left( A_1^{\otimes p_k} \cdot I_q \cdot A_1^{\otimes r} \right) \otimes \ldots \otimes A_1^{\otimes p_n},
\end{equation}
for all $p_k, q, r \in \mathbb{N}_0, k, i = 1 \ldots n$. ■
For 2-cochains, 2nd admissibility is equivalent to the inclusions
\[
\psi \left( A_1^{\otimes s} \otimes (A_1^{\otimes p} \cdot I_q \cdot A_1^{\otimes r}) \right) \subset A_1^{\otimes s} \otimes (A_1^{\otimes p} \cdot I_q \cdot A_1^{\otimes s}) ,
\]
\[
\psi \left( (A_1^{\otimes p} \cdot I_q \cdot A_1^{\otimes r}) \otimes A_1^{\otimes s} \right) \subset (A_1^{\otimes p} \cdot I_q \cdot A_1^{\otimes r}) \otimes A_1^{\otimes s} ;
\]
and Lemma 3 translates into the next result.

Lemma 4. Under the conditions of above definition, \( \psi = \partial \theta \) is 2nd admissible iff
\[
(4.3) \quad \theta^{-1} \left( A_1^{\otimes p} \cdot I_q \cdot A_1^{\otimes r} \right) = A_1^{\otimes p} \cdot \theta^{-1} (I_q) \cdot A_1^{\otimes r}.
\]

Proof. If \( (4.3) \) holds for the primitive \( \theta \) of \( \psi \), we have, putting \( p = x + y \),
\[
\theta^{-1} \left( A_1^{\otimes x} \cdot A_1^{\otimes y} \cdot I_q \cdot A_1^{\otimes r} \right) = A_1^{\otimes x} \cdot A_1^{\otimes y} \cdot \theta^{-1} (I_q) \cdot A_1^{\otimes r}.
\]
But \( A_1^{\otimes y} \cdot \theta^{-1} (I_q) \cdot A_1^{\otimes r} = \theta^{-1} \left( A_1^{\otimes y} \cdot I_q \cdot A_1^{\otimes r} \right) \) and \( A_1^{\otimes x} = \theta^{-1} (A_1^{\otimes x}) \), thus
\[
\theta^{-1} \left( A_1^{\otimes x} \cdot (A_1^{\otimes y} \cdot I_q \cdot A_1^{\otimes r}) \right) = \theta^{-1} (A_1^{\otimes x}) \cdot \theta^{-1} (A_1^{\otimes y} \cdot I_q \cdot A_1^{\otimes r} ),
\]
and using the fact that \( \psi = \partial \theta \), the first part of \( (4.3) \) follows. In a similar way the second part can be shown, and 2nd admissibility is fulfilled. The converse follows in a similar way. \( \blacksquare \)

Clearly, if \( \psi \) is 2nd admissible, then is admissible. Furthermore, the 2nd admissible cochains form a subgroup of the admissible ones.

4.1.1. Restricted gauge equivalence. For 2-cocycles we have, in relation to the product \( * \) in \( \mathfrak{Z}^2 \), that:

Proposition 17. Let \( \psi \) and \( \varphi \) be a couple of counital 2-cocycles, 2nd admissible for \( A \) and \( A_\psi \), respectively. Then \( \psi * \varphi \) is 2nd \( A \)-admissible. Furthermore \( \psi \), the inverse of \( \psi \) under the product \( * \), is 2nd admissible for \( A_\psi \).

Proof. Suppose \( \psi = \partial \lambda \) and \( \varphi = \partial \chi \). From lemma above, \( \lambda \) and \( \chi \) satisfies
\[
(4.4) \quad \lambda^{-1} \left( A_1^{\otimes p} \cdot I_q \cdot A_1^{\otimes r} \right) = A_1^{\otimes p} \cdot \lambda^{-1} (I_q) \cdot A_1^{\otimes r}
\]
and
\[
\chi^{-1} \left( A_1^{\otimes p} \cdot \lambda^{-1} (I_q) \cdot A_1^{\otimes r} \right) = A_1^{\otimes p} \cdot (\lambda \chi)^{-1} (I_q) \cdot A_1^{\otimes r},
\]
since \( \lambda^{-1} (I) \) is the ideal related to \( A_\psi \). In consequence,
\[
(\lambda \chi)^{-1} \left( A_1^{\otimes p} \cdot I_q \cdot A_1^{\otimes r} \right) = A_1^{\otimes p} \cdot (\lambda \chi)^{-1} (I_q) \cdot A_1^{\otimes r}.
\]
On the other hand, \( \psi * \varphi = \partial \lambda * \partial \chi = \partial (\lambda \chi) \), hence \( \psi * \varphi \) has a primitive satisfying \( (4.3) \). Therefore, \( \psi * \varphi \) is 2nd admissible for \( A \).

For the second claim of our proposition, using that \( i_\psi = \partial \left( \lambda^{-1} \right) \) and (from Eq. \( (4.4) \))
\[
\lambda \left( A_1^{\otimes p} \cdot \lambda^{-1} (I_q) \cdot A_1^{\otimes r} \right) = \lambda \left( \lambda^{-1} \left( A_1^{\otimes p} \cdot I_q \cdot A_1^{\otimes r} \right) \right)
\]
\[
= A_1^{\otimes p} \cdot I_q \cdot A_1^{\otimes r} = A_1^{\otimes p} \cdot \lambda \left( \lambda^{-1} (I_q) \right) \cdot A_1^{\otimes r},
\]
the 2nd \( A_\psi \)-admissibility of \( i_\psi \) is immediate. \( \blacksquare \)

Therefore, in terms of 2nd admissible cocycles a restricted twist equivalence can be defined. A characterization of equivalent quantum spaces through this gauge transformations follow from the proof of Theor. 12, and says:

Theorem 13. Let \( A, B \) be objects of \( CA \), with related ideals \( I \) and \( J \), respectively. \( A \sim B \) iff there exists an homogeneous (of degree cero) linear isomorphism \( \vartheta : A_1^{\otimes} \simeq B_1^{\otimes} \) such that
\[
\vartheta \left( A_1^{\otimes r} \cdot I_s \cdot A_1^{\otimes t} \right) = B_1^{\otimes r} \cdot J_s \cdot B_1^{\otimes t} ; \quad r, s, t \in \mathbb{N}_0. \quad \blacksquare
\]

By last, consider the following property of 2nd admissible 2-cocycles:

Proposition 18. If a pair of 2nd admissible 2-cocycles are cohomologous through an admissible 1-cochain \( \theta \), then \( \theta \) is 2nd admissible.
Proof. Let \( \psi = \partial \lambda \) and \( \varphi = \partial \chi \) be cohomologous 2-cocycles through an admissible cochain \( \theta \). We know from Prop. 11 that \( \theta = \lambda \omega^{-1} \), with \( \omega \) a 1-cocycle. Using Lemma 4 for \( \chi \) and then the admissibility of \( \theta \) (and the fact that \( \omega \) is a 1-cocycle),

\[
\lambda \omega^{-1} (A_1^{\otimes p} \cdot I_\theta \cdot A_1^{\otimes r}) = \lambda (A_1^{\otimes p} \cdot \omega \chi^{-1} (I_\theta) \cdot A_1^{\otimes r}) = (A_1^{\otimes p} \cdot \chi^{-1} (I_\theta) \cdot A_1^{\otimes r}).
\]

Using same lemma for \( \lambda \) we have \( \theta (A_1^{\otimes p} \cdot I_\theta \cdot A_1^{\otimes r}) = A_1^{\otimes p} \cdot I_\theta \cdot A_1^{\otimes r} \), as we have claimed. \[\blacksquare\]

From that, Theor. 9 can be rephrased in terms of 2nd admissibility too.

4.1.2. The (anti)bicharacter case. Examples of 2nd admissible cochains are the admissible (anti)bicharacters.

Proposition 19. Let \( \psi \in \mathfrak{e}^2 [A_1] \) be an (anti)bicharacter. \( \psi \) is admissible iff is 2nd admissible. \[\blacksquare\]

That follows from proposition below.

Proposition 20. Let \( \psi \in \mathfrak{e}^2 [A_1] \) be an (anti)bicharacter for which there exists a graded vector space

\[ S = \bigoplus_{n \in \mathbb{N}_0} S_n \subset A_1^{\otimes n} = \bigoplus_{n \in \mathbb{N}_0} A_1^{\otimes n}; \ S_{0,1} = \{0\}, \]

such that

\[
(4.5) \quad \psi (A_1^{\otimes m} \otimes S_n + S_m \otimes A_1^{\otimes n}) = A_1^{\otimes m} \otimes S_n + S_m \otimes A_1^{\otimes n}.
\]

Then, for all \( p, q, r, u \in \mathbb{N}_0 \), we have

\[
(4.6) \quad \psi ((A_1^{\otimes p} \cdot S_q \cdot A_1^{\otimes r}) \otimes A_1^{\otimes u}) = (A_1^{\otimes p} \cdot S_q \cdot A_1^{\otimes r}) \otimes A_1^{\otimes u},
\]

and if \( \psi \) is a 2-cocycle with primitive \( \theta \),

\[
(4.7) \quad \theta^{-1} (A_1^{\otimes p} \cdot S_q \cdot A_1^{\otimes r}) = A_1^{\otimes p} \cdot \theta^{-1} (S_q) \cdot A_1^{\otimes r}.
\]

Proof. Consider an element \( a \in A_1 \) and \( s \in S_n \). Since Eq. (4.5),

\[
\psi (a \otimes s) = \psi_{1,n} (a \otimes s) \in A_1 \otimes S_n + S_1 \otimes A_1^{\otimes n}.
\]

But \( S_1 = \{0\} \), hence

\[
(4.8) \quad \psi_{1,n} (A_1 \otimes S_n) = A_1 \otimes S_n \quad \text{or} \quad \psi_{1,n} (A_1 \otimes S_n) = A_1 \otimes S_n.
\]

Now, consider an element \( a \otimes b = a \cdot b \in A_1^{\otimes 2} \). Suppose \( \psi \) is a bicharacter. From the first part of Eq. (4.8) we have

\[
\psi_{2,n} ((a \cdot b) \otimes s) = \psi_{1+1,n} ((a \cdot b) \otimes s) = (1 \otimes \psi_{1,n}) (1 \otimes f_{1,n}^{-1}) \psi_{1,n} (a \otimes s) \otimes b,
\]

and from Eq. (4.6) it follows that \( \psi (A_1^{\otimes 2} \otimes S_n) = A_1^{\otimes 2} \otimes S_n \). An inductive reasoning shows that

\[
\psi (A_1^{\otimes m} \otimes S_n) = \psi_{m,n} (A_1^{\otimes m} \otimes S_n) = A_1^{\otimes m} \otimes S_n
\]

for all \( m \). Now, consider an element \( c \otimes (a \cdot s) \in A_1^{\otimes m} \otimes A_1 \cdot S_n \). Using the second part of Eq. (4.8),

\[
\psi_{m,1+n} (c \otimes (a \cdot s)) = (\psi_{m,1} \otimes I_n) (I_{m} \otimes f_{1,n}^{-1}) (\psi_{m,n} (c \otimes s) \otimes a),
\]

hence \( \psi (A_1^{\otimes m} \otimes A_1 \cdot S_n) = A_1^{\otimes m} \otimes A_1 \cdot S_n \). Analogous arguments, followed by inductive reasoning, lead us finally to the equalities (4.6).

To prove the second part of the lemma, i.e. Eq. (4.7), recall that a 2-cocycle \( \psi \) and its primitive \( \theta \) are related by the equation

\[
\theta_{m+n}^{-1} \approx (\theta_{m}^{-1} \otimes \theta_{n}^{-1}) \psi_{m,n}^{-1},
\]

Thus, the previous result implies \( \theta^{-1} (S_m \cdot A_1^{\otimes n}) = \theta^{-1} (S_m) \cdot \theta^{-1} (A_1^{\otimes n}) \), and the other equalities follow in a similar way. This ends our proof. \[\blacksquare\]

Each bicharacter \( \varsigma_A \) appearing in twisted coHom objects (see 3.1.5) is admissible, ipso facto 2nd admissible, thanks to \( \sigma_A \) defines an automorphism. In fact,

\[
\varsigma_A (A_1^{\otimes s} \otimes (A_1^{\otimes p} \cdot I_\theta \cdot A_1^{\otimes r})) = A_1^{\otimes s} \otimes [\sigma_A^{-s}]^\otimes (A_1^{\otimes p} \cdot I_\theta \cdot A_1^{\otimes r}) = A_1^{\otimes s} \otimes [\sigma_A^{-s}]^\otimes (A_1^{\otimes p}) \cdot [\sigma_A^{-s}]^\otimes (I_\theta) \cdot [\sigma_A^{-s}]^\otimes (A_1^{\otimes r}) = A_1^{\otimes s} \otimes A_1^{\otimes p} \cdot I_\theta \cdot A_1^{\otimes r}.\]
4.1.3. Quasicomplexes of 2nd admissible cochains. From Eq. (44), easy calculations show that the coface operators and codegeneracies on $C^*$ send the subgroup of 2nd admissible cochains to itself. Thus, we can define for each conic quantum space $A = (A_1, A)$ a subquasicomplex $C^*[A]$ of $C^*[A_1]$ formed out by 2nd $A$-admissible cochains. We are not going to employ explicitly this quasicomplex in any calculation. However, for the sake of completeness, we enumerate some of its properties below (without proof).

- As in §2.3, the cosimplicial objects $[n + 1] \mapsto C^n[A]$ and $[n + 1] \mapsto C^n[B]$ are naturally equivalent when $A \simeq B$.
- The map $A \mapsto C^*[A]$ defines a functor $C^* : G[CA] \to Grp_{eq}$ and a morphism between the groupoids $G[CA]$ and $G[Grp_{eq}]$ such that to every isomorphism $\alpha : A \simeq B$, it assigns the isomorphism of quasicomplexes $\alpha^*$ defined as in Eq. (2.22), replacing $f$ by $\alpha_1$.
- The subgroups of counital cochains give rise to a subquasicomplex $C^*[A]$ satisfying Eq. (2.22).

The first and second cohomology spaces, namely $H^{1,2}[A]$, can be defined as in §3.2. Now, they are usual cohomological spaces in the sense that are given by quotients $Z^{1,2}[A]/\sim_{coh}$. It follows that

$$H^1[A] = Z^1[A] \simeq Aut_CA[A] \simeq H^1_\text{A}[A_1]$$

and from Prop. 18 the inclusion $H^2[A] \subset H^2_\text{A}[A_1]$ is immediate.

- From Eq. (2.44) (putting $V = A_1$), it follows by restriction the isomorphism

$$(4.9) \quad C^*[A] \simeq G^*[\text{end}[A]] \simeq G^*[A \triangleright A].$$

In particular, suppose $A$ is quadric and consider the algebras $C[A]$ and $A(R)$ defined in [12] and [13], respectively. The well known associated epimorphisms $A(R) \hookrightarrow \text{end}[A] \to C[A]$, which follows from initiality of $\text{end}[A]$, gives rise to monics

$$G^*[A(R)] \hookrightarrow G^*[\text{end}[A]] \hookrightarrow G^*[C[A]].$$

Thus, the bialgebra twist of $A(R)$ and $C[A]$ are given by a particular subgroup of 2nd admissible twisting of the quantum space $A$.

Another properties of $C^*[A]$, in relation with the involutio and the product between quasicomplexes in $C^*[Vct]$, will be briefly commented in the following section.

4.2. Coadjoint products and internal coHom objects.

4.2.1. Dual quantum spaces and coadjoint cochains. Let us first enunciate the following well-known result (without proof).

**Lemma 5.** Consider a pair of finite dimensional $k$-vector spaces $U$ and $V$, supplied with a non degenerated pairing $U \times V \to k$. Let $\alpha$ be an automorphism in $V$. Then, for every subset $S \subset V$ we have the equality of vector spaces

$$(4.10) \quad \alpha(S^\perp) = \alpha^{-1}(S^\perp),$$

where $\alpha^*$ denotes the (transpose) automorphism induced in $U$ by the mentioned pairing. □

Consider a conic quantum space $A$ and a counital $A$-admissible 2-cocycle $\psi = \partial \theta \in 3^2[A_1]$. The ideal $I_\psi$ related to $A_\psi$ is determined by $I_{\psi,n} = \theta^{-1}(I_n)$, hence (from lemma above)

$$(I_{\psi,n})^\perp = \theta^{-1}(I_n)^\perp = \theta^*(I_n^\perp),$$

and recalling Eq. (1.13), $(I_\psi)^\perp = I\left(\bigoplus_{n \geq 2} (I_{\psi,n})^\perp\right) = I\left[\bigoplus_{n \geq 2} \theta^*(I_n^\perp)\right]$. On the other hand, consider the 2-cocycle

$$\psi^{*,-1} = \psi^1 = (\partial \theta^1) = \partial^1(\theta^1) \in 3^2[A_1] = 3^2[A_1]^\perp.$$

If $\psi^1$ were $A^1$-admissible, it follows from Prop. 12 (for $S = \bigoplus_{n \geq 2} I_n^\perp$) that the ideal related to $(A^1)_{\psi^1}$ would be

$$(I_1^\perp)_{\psi^1} = \theta^*(I_1^\perp) = I\left[\bigoplus_{n \geq 2} \theta^*(I_n^\perp)\right].$$

That is to say, $(A_\psi)^1$ and $(A^1)_{\psi^1}$ coincide. The problem is to insure the admissibility of $\psi^1$. The things change when we suppose $\psi$ is 2nd admissible or, in particular, if $\psi$ is an admissible (anti)bicharacter.

**Proposition 21.** If $\psi \in C^n[A_1]$ is 2nd $A$-admissible, then $\psi^1 \in C^n[A_1]^\perp$ is 2nd $A^1$-admissible. For $A \in CA^n$, the converse is also valid.

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13If the ideal related to $A$ is generated by elements $R_{ij}^l a_k a_l$, being $R_{ij}^l$ the coefficients of a map $R : A_{1}^{\text{op}2} \to A_{1}^{\text{op}2}$, then the ideal related to $frt[A,R]$ is generated by $R_{ij}^l z_k^m z_l^m - z_k^l z_l^m R_{ij}^m$.
14As usual, given a subset $S \subset V$, $S^\perp$ is the vector space $S^\perp = \{x \in U : \langle x, y \rangle = 0, \forall y \in S\}$.  
15Recall the isomorphism of quasicomplexes $C^*[V] \simeq C^*[V]^\perp = C^*[V^\ast]$. 
Proof. We just consider $n = 2$, since the other cases can be shown in a similar way. From Lemma 5,
\[
\psi^j(A_1^{\otimes u} \otimes (A_1^{\otimes p} \cdot I_q^+ \cdot A_1^{\otimes r})) = \psi^j \left( (A_1^{\otimes u} \otimes (A_1^{\otimes p} \cdot I_q \cdot A_1^{\otimes r}))^\perp \right)
\]
= $\psi(A_1^{\otimes u} \otimes (A_1^{\otimes p} \cdot I_q \cdot A_1^{\otimes r}))^\perp$,
and from 2nd $A$-admissibility of $\psi$, i.e.
\[
\psi(A_1^{\otimes u} \otimes (A_1^{\otimes p} \cdot I_q \cdot A_1^{\otimes r})) = A_1^{\otimes u} \otimes (A_1^{\otimes p} \cdot I_q \cdot A_1^{\otimes r}),
\]
we have
\[
\psi^j(A_1^{\otimes u} \otimes (A_1^{\otimes p} \cdot I_q^+ \cdot A_1^{\otimes r})) = A_1^{\otimes u} \otimes (A_1^{\otimes p} \cdot I_q^+ \cdot A_1^{\otimes r}).
\]
Straightforwardly, because $I_q^n = \sum_{r=2}^n \sum_{j=0}^n A_1^{\otimes n-r-1} \cdot I_q^r \cdot A_1^{\otimes 1}$, $\psi^j$ is 2nd $A^j$-admissible. Now, let us show the other claim. Suppose $A \in CA^m$. From the last result, if $\psi^j \in C^n [A_1]^{\perp}$ is 2nd $A^j$-admissible, then $\psi^{j!}$ is 2nd $A^{j!}$-admissible. But $\psi^{j!} = \psi$ and the ideal related to $A^{j!}$ is the same as the one related to $A$. So, the proposition have been proven.

In terms of complexes $C^* [A]$, last proposition says isomorphism $C^* [A_1] \cong C^* [A_1]^{\perp}$ gives rise to a monic $C^* [A] \rightarrow C^* [A^j]$ which define a natural transformation $C^* \rightarrow C^* !$. In addition, when $C^*$ is restricted to $CA^m$, the isomorphisms $C^* [A] \cong C^* [A_1]^{\perp}$ and the related natural equivalence $C^* \cong C^* !$ follow.

In resume, we have shown:

Theorem 14. Let $A$ be an object in $CA$. For every counital 2nd $A$-admissible $\psi \in Z^2 [A_1]$, $\psi^j$ is a 2nd $A^j$-admissible element of $Z^2 [A_1]$ and $(A^j)^\perp = (A^j)^\perp$. ■

4.2.2. Compositions of quantum spaces and quasi-complexes. Given $A, B \in FGA$, consider a couple of related admissible cochains $\psi \in Z^2 [A_1]$ and $\varphi \in Z^2 [B_1]$. From the discussion we made in §2.3.2, they give rise to a 2-cocycle $(\psi, \varphi)$ in the quasi-complex $C^* [A_1] \times C^* [B_1]$ which, through the monic $j : C^* [A_1] \times C^* [B_1] \rightarrow C^* [A_1 \otimes B_1]$, can be identified with a map (recall Eq. (268))
\[
j(\psi, \varphi) = \bigoplus_{r,s} \psi_{r,s} \otimes \varphi_{r,s} \in Z^2 [A_1 \otimes B_1].
\]
Also, they can be seen as a restriction of $\psi$ and $\varphi$. In these terms, admissibility condition for $\psi$ and $\varphi$ lead us immediately to $A \circ B$-admissibility of $j(\psi, \varphi) \subset \psi \otimes \varphi$. If $\psi = \partial \theta$ and $\varphi = \partial \chi$, then the primitive of $j(\psi, \varphi)$ is $j(\theta, \chi) = \bigoplus_{r,s} \theta_r \otimes \chi_s$, and consequently $j(\theta, \chi) \subset \theta \otimes \chi$. From that, it follows the ideal related to $(A \circ B)_j(\psi, \varphi)$ is precisely the one related to $A_{\psi} \circ B_{\varphi}$. So, we have

Theorem 15. Given $A, B \in FGA$ and a couple of counital admissible cochains $\psi \in Z^2 [A_1]$ and $\varphi \in Z^2 [B_1]$, then $j(\psi, \varphi) \in Z^2 [A_1 \otimes B_1]$ is $A \circ B$-admissible and $A_{\psi} \circ B_{\varphi} \simeq (A \circ B)_j(\psi, \varphi)$. ■

For other composition of quantum spaces, it is not enough for $\psi$ and $\varphi$ to be admissible. For instance, the ideal related to $A \bowtie B$, $\bigoplus_{n \in \mathbb{N}} I_n \otimes J_n$, is not preserved by $j(\psi, \varphi)$. Nevertheless, if $\psi$ and $\varphi$ are 2nd admissible we can show, using similar techniques to the ones developed in the previous subsection, that:

- $j(\psi, \varphi)$ is 2nd $A \bowtie B$-admissible and the related ideal of $(A \bowtie B)_j(\psi, \varphi)$ is
\[
(\bigoplus_{n \in \mathbb{N}} I_n \otimes J_n)^{-1} = (\bigoplus_{n \in \mathbb{N}} \theta^{-1} (I_n) \otimes \chi^{-1} (J_n));
\]

- $j(\psi, \varphi)^!$ is 2nd $A \circ B$-admissible and the related ideal of $(A \circ B)_j(\psi, \varphi)^!$ is
\[
(\bigoplus_{n \in \mathbb{N}} \theta^* (I_n^+) \otimes \chi^{-1} (J_n));
\]

and analogous results for the functors $\circ$ and $\bowtie$. The same can be done if $\psi$ and $\varphi$ are admissible bicharacters, since, for instance, cochains like $\psi^j$ and $j(\psi, \varphi)$ will also have that property. Note that 4.11 and 4.12 are precisely the ideals related to $A_{\psi} \bowtie B_{\varphi}$ and $A_{\psi} \circ B_{\varphi}$, respectively. Summing up,

Theorem 16. Let $\psi \in Z^2 [A_1]$ and $\varphi \in Z^2 [B_1]$ be 2nd $A$ and $B$-admissible, respectively. Then,
\[
A_{\psi} \bowtie B_{\varphi} = (A \circ B)_j(\psi, \varphi); \quad A_{\psi} \circ B_{\varphi} = (A \circ B)_j(\psi, \varphi)^!;
\]
\[
A_{\psi} \bowtie B_{\varphi} = (A \circ B)_j(\psi, \varphi); \quad A_{\psi} \circ B_{\varphi} = (A \circ B)_j(\psi, \varphi)^!.
\]
And for quadratic and $m$-th quantum spaces, $A_{\psi} \bowtie B_{\varphi} = (A \bullet B)_j(\psi, \varphi)$. ■
Among other things, results above imply the existence of functorial injections
\[ C^\bullet [A] \times C^\bullet [B] \to C^\bullet [A \odot B] \], \quad \odot = \circ, \odot, \triangleleft, \triangleright, \text{and} \oplus. \]

4.2.3. **Twist transformations and the coHom objects.** As an application of formulae above, given \( A \) and \( B \) in CA and a pair of 2nd admissible cocycles \( \psi \in S^2[A] \) and \( \varphi \in S^2[B] \), \( \text{hom} [B, A_\psi] = \text{hom} [B, A_i(\psi, \varphi)] \). Then, for twisted coHom objects we have
\[ \text{hom}^\tau [B_\psi, A_i] = \text{hom} [B_\psi, A_i] \circ \text{hom} [B, A_i] \circ \psi \circ \varphi. \]

On the other hand, from \( \text{end} [A_i] \) and \( \psi_h \) (see §3.1.1) we can compute the bialgebra \( \text{end} [A_i] \) by making a twist transformation \( j (\psi_h, \psi_h) \) (recall \( \psi_h \) is a bicharacter). In general, as a direct consequence of **Theor. 16** for \( \triangleright \): \[ \text{Proposition 22.} \quad \text{If} \ A \triangleleft C \text{ and} B \triangleleft D \text{ through 2nd admissible cocycles, then} \ \text{hom} [B, A] \triangleleft \text{hom} [D, C]. \]

Now, we shall study twisted coHom objects under the perspective of twist transformation. From §3.1.4, we know that there exist admissible bicharacters (see Eqs. (3.14) and (3.15)), \( \psi \) and \( \varsigma \), and so \( \text{ipso facto} \) 2nd admissible cochains, \( \varsigma_A \in S^2[A] \), \( \varsigma_B \in S^2[B] \) and \( \text{hom} [\varsigma_A, A] \in S^2[B_\varsigma \otimes A_1] \) such that
\[ \text{hom}^\tau [B, A] \equiv B_\triangleright A_\varsigma = \text{hom} [B, A_\varsigma] \setminus \text{hom} [B, A]. \]
By direct calculations, it can be seen that \( \text{hom} [B, A] \equiv j (\varsigma_A, \varsigma_A) \). On the other hand, the product \( \text{hom}^\tau [B, A] \circ \triangleright B \) appearing in twisted coevaluation map \( A \to \text{hom}^\tau [B, A] \circ \triangleright B \), can be regarded as a twisting of the quantum space \( \text{hom}^\tau [B, A] \circ \triangleright B \) by a cochain \( \omega \in S^2[B_\omega \otimes A_1 \otimes B_1] \). The latter is an admissible anti-bicharacter defined by \( \omega_{1,1} \) as (recall Eq. (3.13))
\[ \omega (z^*_i \otimes z^*_k \otimes b_k \otimes b_i) = \phi_i^*_k \phi_k^* \rho_i^* \rho_k^* \otimes b_k \otimes b_i, \]
writing \( z^*_i = b^i \otimes a_i \) and making usual identifications. Using \( (2.22) \) we find that \( \omega = j (\varsigma_B, \varsigma_A, \omega_{1,1} \otimes \omega_{1,1}, \omega_{1,1} \otimes \omega_{1,1}, \omega_{1,1} \otimes \omega_{1,1}) \). Since for all 2-cocycles \( \psi \) and \( \varphi \),
\[ j (\psi, \varphi) = j (\theta^{-1}, \chi^{-1}) = j (\theta, \chi^{-1}) = j (\psi, \varphi), \]
then \( \omega = j (\varsigma_B, \varsigma_A, \omega_{1,1} \otimes \omega_{1,1}, \omega_{1,1} \otimes \omega_{1,1}, \omega_{1,1} \otimes \omega_{1,1}) \), and consequently
\[ \text{hom}^\tau [B, A] \circ \triangleright B = \left( \text{hom} [B, A] \circ \triangleright B \right) \circ \omega = \left( \text{hom} [B, A] \circ \triangleright B \right) \omega_{1,1} = \text{hom} [B, A] \circ \triangleright B. \]

That is to say, the initiality of \( \text{hom}^\tau [B, A] \) follows from that of \( \text{hom} [B, A] \), because the related coevaluation maps are given by a same arrow. This way we see the construction of objects \( \text{hom}^\tau [B, A] \) enable us to generalize it to more general twist transformations. In fact, instead of maps \( A \to H \circ \triangleright B \) and twisting \( \tau_{A,B} \), we can study a class of arrows \( A \to (H \circ \triangleright B) \omega \), where \( \omega \) defines an element of \( S^2[H_1 \otimes B_1] \). In terms of them, for each pair \( A, B \in \text{CA} \), a full subcategory of \( \Omega (A) \downarrow \Omega (CA \circ \triangleright B) \), being \( \Omega : \text{CA} \to \text{GrVct} \) the forgetful functor to graded vector spaces, can be defined as in \( [3] \). If such a category, namely \( \Omega^{A,B} \), is related to a cochain of the form \( \omega_{A,B} = j (\varsigma_A^1, \varsigma_A, \omega_{1,1} \otimes \omega_{1,1}, \omega_{1,1} \otimes \omega_{1,1} \otimes \omega_{1,1} \otimes \omega_{1,1}) \) (in a similar way that \( \Upsilon^{A,B} \) is related to \( \tau_{A,B} \)), \( \varsigma \in \chi \) denotes a 2nd \( \chi \)-admissible 2-cocycle, then it will have initial object
\[ \text{hom}^\Omega [B, A] = \text{hom} [B, A] \circ \chi = B_\triangleright A_\varsigma. \]
Also, the disjoint union \( \Omega = \bigcup_{A,B} \Omega^{A,B} \) has a semigroupoid structure, and there exists a related embedding \( \Omega \hookrightarrow \text{CA} \) that preserves the corresponding (partial) products. Accordingly, the assignment \( (B, A) \mapsto \text{hom}^\Omega [B, A] \) defines a CA-cobased and (by duality) a QLS-category based for each collection of cocycles \( \Omega_{A,B} \). These results are developed in \( [7] \).

Finally, let us say the bialgebras \( \text{end}^\Omega [A] \) are twisting of \( \text{end} [A] \) also as bialgebras (i.e., in the Drinfeld’s sense). In fact, the isomorphism given by Eq. (1.10) ensure the twisting \( j (\varsigma^1 \varsigma, \varsigma) \) on each bialgebra \( \text{end} [A] \) is in the image of the map \( F \), i.e. there exists \( \chi \in G^\star [A \circ \triangleright A] \) such that \( F \chi = j \varsigma \varsigma \). Explicitly,
\[ \chi_A (z_{i_1} \cdots z_{i_r} \otimes k_{m_1} \cdots k_{m_s}) = (\varsigma_A)_{i_1 \cdots i_r, m_1 \cdots m_s}, \]
The fact that \( \varsigma_A \) is 2nd admissible is crucial for \( \chi_A \) to be well defined on \( (A \circ \triangleright A)^{\otimes 2} \).

\(^{16}\)To define \( \omega_{A,B} \) in a diagram \( (\varsigma, H) \), **Prop. 4** must be taken into account for the inclusions \( H_1 \otimes B_1 \subset B_\varsigma \otimes A_1 \otimes B_1 \).
Conclusions

Motivated by the idea of twisted internal coHom objects, we have defined a non formal algebra deformation process over the category of quantum spaces. Such deformations, or twisting, are controlled by a cosimplicial multiplicative quasicomplex structure $C^*$ in the category $\text{Grp}^*$ of groups and unit preserving maps. Their counital 2-cocycles are the elements implementing the mentioned deformations. Also, given a quantum space $A$, the cohomology classes of 2-cocycles, through the so called $A$-admissible cochains, defines the isomorphisms classes of twisting of $A$.

We have shown that for each bialgebra in the category of quantum spaces, the corresponding $C^*$ has the quasicomplex of bialgebra twist transformations as a subobject. More precisely, if $V$ is a coalgebra, then there exists a monic $G^*_V [V] \rightarrow C^* [V]$ in the category of group quasicomplexes, where $V$ is a bialgebra generated by $V$.

The twist transformations define a gauge equivalence between quantum spaces. For instance, every $n$-dimensional quantum plane $A^n_{\hbar} | 0$ is gauge equivalent to the affine space $A^n | 0$. Also, the twisted coHom objects are equivalent, in this sense, to the untwisted ones.

The quasicomplexes $C^* [V], V \in \text{Vct}_f$, generate a monoidal category with involution. We have shown that, for a particular class of twist transformations, the monoidal structure is compatible with products $\circ, \bullet, \odot, \triangleright, \triangleleft$ and $\triangleright$ between quantum spaces, and the involution with the functor $!$, in the sense that these functors (acting on objects) ‘commute’ with the twisting process.

Now, it is natural to ask what happen if, instead of groups $C^n [V]$, we consider the algebras of degree zero $n$-homogeneous linear endomorphisms $E^n [V]$. Straightforwardly, the function $[n + 1] \mapsto E^n [V]$ together with Eqs. (2.6) and (2.10) define a cosimplicial object in Alg. We just have to repeat the proof of Theor. 2. Moreover, maps $\partial_\pm$ given in Eq. (2.14), supply $E^*$ with a structure of multiplicative cosimplicial parity quasicomplex in Alg, (the category of algebras and unit preserving functions). From the algebra deformation process given at the beginning of §2, new twisting on quantum spaces can be constructed within the cohomological framework defined by $(E^*, \partial_+ , \partial_-)$. These new transformations will be investigated elsewhere.

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