Nucleon axial structure from lattice QCD

Gunnar S. Bali, Lorenzo Barca, Sara Collins, Michael Gruber, Marius Löffler, Andreas Schäfer, Wolfgang Söldner, Philipp Wein, Simon Weishäupl, and Thomas Wurm

Abstract: We present a new analysis method that allows one to understand and model excited state contributions in observables that are dominated by a pion pole. We apply this method to extract axial and (induced) pseudoscalar nucleon isovector form factors, which satisfy the constraints due to the partial conservation of the axial current up to expected discretization effects. Effective field theory predicts that the leading contribution to the (induced) pseudoscalar form factor originates from an exchange of a virtual pion, and thus exhibits pion pole dominance. Using our new method, we can recover this behavior directly from lattice data. The numerical analysis is based on a large set of ensembles generated by the CLS effort, including physical pion masses, large volumes, and lattice spacings down to 0.039 fm, which allows us to take all relevant limits. We find that some observables are much more sensitive to the choice of parametrization for the form factors than others. On the one hand, the $z$-expansion leads to significantly smaller values for the axial dipole mass than the dipole ansatz ($M_A^{z\text{-exp}} = 1.01(9)$ GeV versus $M_A^{\text{dipole}} = 1.30(7)$ GeV). On the other hand, we find that the result for the induced pseudoscalar coupling at the muon capture point is almost independent of the choice of parametrization ($g^{z\text{-exp}}_P = 8.69(49)$ and $g^{\text{dipole}}_P = 8.32(18)$), and is in good agreement with both, chiral perturbation theory predictions and experimental measurement via ordinary muon capture.

Keywords: Lattice QCD, Nonperturbative Effects
1 Introduction

The axial structure of the nucleon is relevant for the description of experiments that involve weak interactions. The most precisely known quantity in this context is the axial coupling constant $g_A$, which corresponds to the axial form factor at vanishing momentum transfer and can be determined experimentally from $\beta$ decay (see refs. [1–3]; cf. also ref. [4]). At finite momentum transfer $Q^2$, the axial and the induced pseudoscalar form factors are much less well known. They enter the description of exclusive pion electroproduction [5–8] (e.g., $e^-p \rightarrow \pi^-p\nu$), (quasi-)elastic neutrino-nucleon scattering [9–12], radiative muon capture [13–15], and ordinary muon capture [16–19]. Via weak muon capture in muonic hydrogen a combination of the Dirac, Pauli, axial, and induced pseudoscalar form factors
can be measured, constraining the latter at the muon capture point \([15, 18–21]\). The direct determination of the induced pseudoscalar coupling in refs. \([18, 19]\) shows that, at small momentum transfer, the induced pseudoscalar form factor is indeed well approximated by a pion pole dominance (PPD) ansatz.

From the theoretical side, one can gain insight into the form factors through various techniques. At small momentum transfer, chiral perturbation theory (ChPT) yields valuable low energy theorems \([8, 22–25]\) (motivating, e.g., the above mentioned PPD ansatz), while, at intermediate and large virtualities, the form factors can be determined (up to some systematic uncertainty of \(\sim 15\%\)) using light-cone sum rules \([26, 27]\). Another interesting approach is the application of functional renormalization group methods \([28]\).

In this work, we will use lattice QCD, which enables a determination of hadronic observables from first principles. Once all systematic uncertainties are under control, this method provides the cleanest and most direct access to hadron form factors. Many studies of nucleon couplings and form factors have been carried out in the past using a wide variety of lattice actions and analysis methods (see, e.g., refs. \([29–63]\)). Recent studies of form factors at finite virtualities with data close to physical pion masses have faced two problems: first of all, it is difficult to reconcile the data with the partial conservation of the axial current (PCAC). Even though PCAC is approximately fulfilled on the correlation function level, the corresponding relation between the ground state form factors, which are extracted using a spectral decomposition, is broken to a much larger extent. Secondly, the PPD ansatz for the induced pseudoscalar form factor fails to describe the data at small momentum transfer and small pion masses, which is the domain where one would expect this ansatz to give the best approximation. In both cases an explanation in terms of finite lattice spacing effects is unlikely, since the violation of PCAC is largest at small virtualities and masses. For nice presentations of these problems see, e.g., refs. \([55, 58]\). A prime suspect that may be responsible for both effects is a particularly large excited state contamination, albeit it was demonstrated in ref. \([57]\) that the problem persist if one uses a traditional fit ansatz with up to three free excited states. In ref. \([60]\) we have proposed a subtraction method that removes excited state contributions that violate the equations of motion for the nucleon. While this leads to a recovery of the PCAC relation on the form factor level, the PPD ansatz still remained strongly broken. While this is not impossible as such, the induced pseudoscalar charge at the muon capture point remained at variance with the experimental value \([15, 18–21]\).

A deeper insight into the excited state effects is possible using effective field theory (EFT). ChPT based analyses \([64–66]\) (along the lines of refs. \([67–70]\) using interpolating currents from ref. \([71]\)) indicate that the subtraction method mentioned above does not remove all excited states and that the violation of the PPD ansatz is due to additional, large contributions from \(N\pi\) excited states that predominantly affect the induced pseudoscalar and the pseudoscalar form factors. While an a posteriori subtraction of the effect performed in refs. \([64, 65]\) leads to satisfying results, such a procedure appears inadequate from the lattice QCD perspective as it introduces a dependence on ChPT input parameters and cannot be consistently combined with standard excited state fits. Moreover, an independence of the operator smearing is assumed, which may not be justified in lattice
simulations where spatially extended (smeared) nucleon interpolators with radii that are not very small compared to the inverse pion mass are employed.

The procedure advocated in this article makes use of the same effective field theory methods as refs. [64–66, 72] in order to calculate the leading excited state contribution to the correlation function explicitly for all axial and pseudoscalar channels with less assumptions. We will show that exploiting the EFT knowledge stabilizes excited state fits considerably and allows us to extract ground state form factors, which are found to obey both the PCAC relation on the form factor level and the PPD ansatz reasonably well. Recently an alternative analysis method has been proposed in ref. [73], which also allows an extraction of nucleon form factors that satisfy PCAC. We will discuss similarities and differences between our approach and this method in sections 3.3 and 4.1.

This article is structured as follows. In section 2 we will give a detailed description of the EFT calculation needed to determine the leading $N\pi$ contribution to the correlation function and how it can be combined with the usual excited state analysis. The lattice setup and the employed ratios of correlation functions are detailed in section 3. Section 4 contains the results for the form factors (using both, dipole fits and the $z$-expansion) and includes an analysis of the PCAC relation as well as of the PPD ansatz. We also explore parametrizations that are consistent with PCAC in the continuum. In the latter case the continuum limit is under much better control. We summarize our findings in section 5.

## 2 Correlation functions

### 2.1 Definitions

In order to study hadron structure using lattice QCD one has to calculate two- and three-point correlation functions, where hadron states with matching quantum numbers are created by a suitable interpolating current $\mathcal{N}$ at the source time $t_{src}$, and are destroyed by $\mathcal{N}$ at the sink time $t_{snk}$ (here, we will always set $t_{src} = 0$ and $t_{snk} = t$ without loss of generality). In the case of three-point correlation functions one inserts a local current $\mathcal{O}$ at some insertion time $\tau$ with $t > \tau > 0$ and, usually, one is interested in the ground state matrix element of this current insertion. The momenta can be fixed by appropriate Fourier transforms, in our case at the sink and the insertion, such that the initial state and final state momenta are $p$ and $p'$, respectively:

\[
C_{2pt,P_+}^\mathcal{O}(t) = P_+^{\alpha\beta} C_{2pt,\beta\alpha}^\mathcal{O}(t) = a^3 \sum_x e^{-ip_+ \cdot x} \langle \mathcal{N}^\beta(x, t) \mathcal{N}^{\alpha}(0, 0) \rangle ,
\]

\[
C_{3pt,\Gamma}^{\mathcal{O}}(t, \tau) = \Gamma^{\alpha\beta} C_{3pt,\beta\alpha}^{\mathcal{O}}(t, \tau) = a^6 \sum_{x,y} e^{-ip' \cdot x + i(p' - p) \cdot y} \Gamma^{\alpha\beta} \langle \mathcal{N}^\beta(x, t) \mathcal{O}(y, \tau) \mathcal{N}^{\alpha}(0, 0) \rangle ,
\]

where $C_{2pt, P_+}$, $C_{3pt, \Gamma}^{\mathcal{O}}$, $P_+$, and $\Gamma$ are matrices in Dirac space with the corresponding spin indices $\alpha$ and $\beta$. The three-quark nucleon interpolating current is defined via the usual quark-diquark structure with the charge conjugation matrix $C$,

\[
\mathcal{N}^{\alpha}(x, t) = (u(x, t)^T C \gamma_5 d(x, t)) u^\alpha(x, t) ,
\]
where each quark is smeared separately in the spatial directions using Wuppertal smearing [74] on spatially APE-smoothed links [75]. Note that Minkowski scalar products and gamma matrix conventions are used throughout this work. At zero three-momentum \( P_+ = (1 + \gamma_0)/2 \) annihilates the leading negative parity contribution. For the analysis of the pseudoscalar and axialvector form factors we choose \( \Gamma \) to be \( \tilde{P}_+ = P_+ \gamma^i \gamma_5, \ i = 1, 2, 3 \). In order to relate the correlation functions to matrix elements, one inserts identity operators (corresponding to sums over all hadronic states) and uses the translational properties of the Fourier transforms. When evaluating the result at large Euclidean times, \( t, \tau \), and \( t - \tau \), excited states are exponentially suppressed and the correlation functions can be approximated by the ground state contributions:

\[
C^P_{\text{2pt}, P_+}(t) \approx \sum_\sigma P^\alpha_+ \langle 0 | N^\sigma | \gamma^\beta | N_\sigma^P \rangle \langle N_\sigma^P | N^\alpha | 0 \rangle \frac{e^{-E_p t}}{2E_p}, \tag{2.4}
\]

\[
C^P_{\text{3pt}, \Gamma}(t, \tau) \approx \sum_{\sigma, \sigma'} \Gamma^{\alpha\beta} \langle 0 | N^\sigma | N_\sigma^P \rangle \langle N_\sigma^P | \gamma^\alpha \gamma^\beta | \gamma^\sigma \gamma^\tau | 0 \rangle \frac{e^{-E_p(t-\tau)} e^{-E_p \tau}}{2E_p 2E_p}, \tag{2.5}
\]

where all currents are located at the origin and \( | N_\sigma^P \rangle \) corresponds to a nucleon state with three-momentum \( p \) and spin-projection \( \sigma \). The parity projected overlap matrix elements can be parametrized as

\[
P^\alpha_+ \langle 0 | N^\sigma | \gamma^\beta | N_\sigma^P \rangle = P^\alpha_+ \sqrt{Z_p^\pm} u^\beta_{p, \sigma}, \tag{2.6}
\]

where \( u^\beta_{p, \sigma} \) is a nucleon spinor and \( \sqrt{Z_p^\pm} \) are momentum- and smearing-dependent overlap factors. For smeared currents \( \sqrt{Z_p^+} \) and \( \sqrt{Z_p^-} \) can differ from each other due to the explicit breaking of Lorentz invariance by the operator smearing, cf. refs. [76] and [77] for more details. Since in our analysis only positive parity projected overlap matrix elements occur, we define \( \sqrt{Z_p^\pm} \equiv \sqrt{Z_p^\pm} \). The form factor decomposition for the nucleon-nucleon matrix element of a generic current can be written as

\[
\langle N_\sigma^P | \gamma^\beta | N_\sigma^P \rangle = \bar{u}^\beta_{p', \sigma'} J[\gamma^\alpha] u^\alpha_{p, \sigma}, \tag{2.7}
\]

where \( J[\gamma^\alpha] \) is matrix valued and can be parametrized in terms of form factors, cf. eqs. (2.12) and (2.13) below. Using the spinor identity \( \sum_\sigma u_{p, \sigma} \bar{u}_{p, \sigma} = \bar{p} + m \), where \( m \) is the nucleon mass, one arrives at the ground state contribution

\[
C^P_{\text{2pt}}(t) \approx \frac{Z_p}{2E_p} e^{-E_p t} (\bar{p} + m), \tag{2.8}
\]

\[
C^P_{\text{3pt}, \Gamma}(t, \tau) \approx \frac{\sqrt{Z_p^+} \sqrt{Z_p^-}}{2E_p 2E_p} e^{-E_p(t-\tau)} e^{-E_p \tau} (\bar{p} + m) J[\gamma^\alpha] (\bar{p} + m). \tag{2.9}
\]

For the two-point function we can explicitly evaluate the trace with \( P_+ \) to find

\[
C^P_{\text{2pt}, P_+}(t) \approx \frac{Z_p}{2E_p} e^{-E_p t} |\text{tr}\{P_+(\bar{p} + m)\}| = \frac{Z_p}{E_p} \frac{E_p + m}{E_p} e^{-E_p t}. \tag{2.10}
\]

For the three-point functions the trace with \( \Gamma \) depends on the current-specific decomposition (2.7). In practice it turns out (in particular in case of the three-point functions) that,
at Euclidean time distances $t$ and $\tau$ with acceptable signal-to-noise ratio, not only the
ground state contributes. In most cases this problem can be treated by taking into account
generic excited state contributions in the fit functions (see section 2.3). However, there
are situations in which the excited states constitute (at the available temporal distances)
the dominant contribution. In the latter case the generic excited state parametrizations
fail to describe the data appropriately and further physical insight into the excited state
structure is needed, cf. section 2.2.

For the isovector pseudoscalar and axialvector currents used in this work,
\[ P = \bar{u}\gamma_5 u - \bar{d}\gamma_5 d, \quad A_\mu = \bar{u}\gamma_\mu \gamma_5 u - \bar{d}\gamma_\mu \gamma_5 d, \quad (2.11) \]
the explicit decompositions in terms of the pseudoscalar form factor, $G_P(Q^2)$, as well as
the axial and induced pseudoscalar form factors, $G_A(Q^2)$ and $\tilde{G}_P(Q^2)$, are
\[ J^P = \gamma_5 G_P(Q^2), \quad (2.12) \]
\[ J^A_\mu = \gamma_\mu \gamma_5 G_A(Q^2) + \frac{q_\mu}{2m} \gamma_5 \tilde{G}_P(Q^2), \quad (2.13) \]
where $q = p' - p$ is the momentum transfer and $Q^2 = -q^2$ is the virtuality. The three
form factors used above are not independent in the continuum theory, since the axial Ward
identity yields $\partial^\mu A_\mu = 2i m_\ell \mathcal{P}$ known as partial conservation of the axialvector current
(PCAC). Here $m_\ell$ is the light quark mass. On the lattice this relation can be broken by
discretization effects. For the nucleon matrix elements it implies that
\[ 2i m_\ell \langle N^P_p | \mathcal{P} | N^P_p \rangle = \langle N^P_p | \partial^\mu A_\mu | N^P_p \rangle + O(a^2), \quad (2.14) \]
where we can safely ignore discretization effects linear in the lattice spacing $a$, since our
analysis is fully order $a$ improved, cf. section 3.1. Using the definitions (2.12) and (2.13)
together with the equations of motion one can deduce the corresponding relation for the
form factors (called PCACFF in ref. [60]):
\[ \frac{m_\ell}{m} G_P(Q^2) = G_A(Q^2) - \frac{Q^2}{4m^2} \tilde{G}_P(Q^2) + O(a^2). \quad (2.15) \]
Eqs. (2.14) and (2.15) should both be satisfied, once the ground state matrix elements have
been extracted reliably.

2.2 EFT-based analysis

Employing a theory where hadrons are the effective degrees of freedom (like baryon chiral
perturbation theory) in order to elucidate the excited state structure in correlation func-
tions is appealing, in particular if multi-hadron states with additional pions are the relevant
excitations, see refs. [67–70, 78]. In many cases, however, these contributions are relatively
small and one can deal with them using standard methods like, e.g., source/sink-smearing
and multi-exponential fits that allow for generic excited state contributions. As will be
explained in detail in this section, the situation is different in the context of isovector axial
and pseudoscalar form factors, where $N\pi$ contributions can actually be a leading term
from the EFT point of view due to pion pole dominance (PPD). Especially for small pion masses this effect outweighs the exponential suppression at the currently available source-sink distances due to the small energy gap.\footnote{Note that, due to the exponential deterioration of the signal, one cannot expect the source-sink distances to become dramatically larger in future simulations.} In this situation multi-exponential fits with generic excited states become very unstable and usually fail to resolve the ground state.\footnote{An alternative method has been proposed in ref. [73], which appears to resolve the ground state contribution in this situation. We will comment on this method in some detail in sections 3.3 and 4.1.}

In refs. [65, 66] nucleon three-point functions with axialvector and pseudoscalar current insertions have been analyzed using ChPT and compelling qualitative evidence has been presented that the violations of the PCAC and PPD relations are indeed caused by $N\pi$ excited states. This is done as follows: first, one calculates the excited state contribution to the form factor using ChPT. The predicted, excited state contaminated form factor is found to agree quite well with recent data from the PACS collaboration [58], cf. refs. [66, 72]. In a second step, one may attempt to correct the error by subtracting the calculated excited state contaminations a posteriori (see, e.g., refs. [64, 65], where such a subtraction has been performed for the induced pseudoscalar form factor). While this method yields convincing qualitative results, there are some open questions and limitations that need to be addressed:

1. The EFT calculation disregards the effect of operator smearing. There are heuristic arguments that the effect of the smearing should be negligible as long as the smearing radii $r_{sm}$ are much smaller than the Compton wavelength of the pion $\lambda_\pi \approx 1.41$ fm, cf. refs. [67, 68, 70, 78]. However, these arguments are to some extent called into question by the observation that the operator smearing used in actual simulations has, in general, a strong impact on the signal of excited states. In refs. [38, 40] it has been found that smearing radii of roughly $r_{sm} \sim 0.5$ fm maximize the ground state overlap. In the lattice analysis performed in this article, the optimized smearing radii are on some ensembles even larger (up to 0.8 fm, cf. table 2), and it is questionable whether a dependence on the smearing can be completely excluded for such smearing radii.

2. So far, an a posteriori subtraction of the excited states has only been performed in combination with the ratio method on the lattice. It is unclear how one would avoid double counting, if one combines it with a standard excited state analysis, e.g., by using multi-exponential fits.

3. Estimating the systematic error tied to the ChPT based subtraction is challenging.

From a lattice QCD perspective the situation is in our opinion quite clear concerning point 2. If there is a large $N\pi$ excited state contribution, then it should be taken into account explicitly in the multi-state fits to the correlation functions.\footnote{One can also try to circumvent the problem entirely by either suppressing or subtracting the unwanted excited state contributions. In ref. [79] the pion pole contribution is suppressed by analyzing the matrix elements of currents with a Gaussian profile instead of local currents. Ref. [60] presents a method to subtract some of the excited state contributions.} In this approach point 1 can be addressed simultaneously by allowing for a smearing dependence of the $N\pi$
Figure 1. Feynman diagrams showing the most important (tree-level) contributions to the axial and pseudoscalar three-point functions. The squares correspond to explicitly inserted operators: the right and left ones correspond to smeared three-quark baryon interpolating currents at the source (at time 0) and the sink (at time $t$), respectively, while the ones in the middle depict a pseudoscalar or an axialvector operator insertion (at time $\tau$). The circles correspond to pion-nucleon interaction vertices, while the dashed and solid lines represent pion and nucleon propagators, respectively. The dotted red vertical lines indicate the sums over hadronic states one usually introduces to interpret correlation functions.

coupling to the interpolating currents. Furthermore, we can avoid systematic uncertainties (point 3) by relaxing ChPT constraints. In the following, we will describe in detail how this can be achieved.

The first and second rows of figure 1 show the tree-level Feynman diagrams that contribute to the correlation functions. As discussed in ref. [65], these yield the most important contribution to the correlation function. The squares on the right and left depict the smeared source and sink currents, while the one in the middle corresponds to the inserted local quark bilinears (axialvector or pseudoscalar currents in our case). The dashed and solid lines depict pion and nucleon propagators, while the circle stands for a pion-nucleon interaction vertex. The dotted red lines are for illustration only and indicate the identity operators (i.e., the sum over all hadronic states) that are usually inserted between source and current as well as between current and sink, cf. eq. (2.5). This elucidates that the diagram in the first row yields a contribution to the ground state, while the diagrams on the left- and right-hand sides in the second row give rise to a nucleon-pion excitation in the final and initial state, respectively. For the diagram in the middle of the second row, however, the situation is not that simple, since the nucleon-pion interaction is not restricted to a specific time-slice. As a consequence, the diagram contributes to both the ground state and the excited states, as shown in the bottom row of figure 1. This follows from an explicit calculation of the diagrams (see below). We emphasize that there is no one-to-one correspondence between the individual contributions in the spectral decomposition and the diagrams. For example, both the diagram in the first row and the
diagram in the middle of the second row contribute to the ground state and, actually, an infinite number of diagrams will contribute to each state if one takes into account higher orders in ChPT (see ref. [65] for a list of one-loop diagrams). Finally, a single diagram can contribute to multiple states in the spectral decomposition, cf. the bottom row of figure 1. We will exploit the fact that the pion pole contribution to the ground state automatically gives rise to an associated excited state.

Before addressing the details, let us note that the following calculation is in large parts already contained in refs. [65, 66], where also one-loop diagrams are taken into account. Also the presentation in ref. [79] is based on similar considerations (cf. also ref. [80]). However, we will present the result in a more general way (without using a particular spin projection or fixing initial and final state momenta to a predefined configuration) such that it can be used in a variety of simulation setups. The first ingredient we need in order to evaluate the diagrams in figure 1 are the corresponding Feynman rules. Here we follow the conventions of ref. [81], but adapt them to our choices for the currents (see eq. (2.11)) and convert them to position space. We work in two-flavor baryon ChPT here. However, since we only consider the nucleon sector and are only working at tree-level accuracy, a three-flavor calculation would give exactly the same result. Note that in this section all time variables are in Minkowski time and will be rotated to imaginary times only at the very end. The pion and nucleon propagators read

\[
S_N(x) = i \int \frac{d^4 q}{(2\pi)^4} \frac{e^{-iq \cdot x}}{q^2 - m_N^2 + i\epsilon},
\]

\[
S^{ab}_\pi(x) = \delta^{ab} \frac{1}{q^2 - m_\pi^2 + i\epsilon}.
\]

For the vertices of the current insertions we have

\[
\begin{align*}
\mathcal{A}^\mu &= g_A \gamma^\mu \gamma_5 \sigma^3, \\
P &= 0, \\
\mathcal{A}^\mu &= -2 F_\pi \partial^\mu \delta^{a3}, \\
P &= -2i F_\pi B \delta^{a3}, \quad B \equiv \frac{m_\pi^2}{2m_\ell},
\end{align*}
\]

where we only take into account the leading contribution in the chiral counting and all derivatives are understood to act on the pion propagator. Here, \( F_\pi \) and \( g_A \) correspond to the pion decay constant and the axial coupling in the chiral limit, respectively, while \( B \) is the condensate parameter and \( \sigma^a \) are Pauli matrices. For the leading \( N\pi \) interaction vertex we have

\[
\begin{align*}
\mathcal{V} &= -\frac{i g_A}{2 F_\pi} \phi \gamma_5 \sigma^a.
\end{align*}
\]

The vertices for local three-quark currents have been derived in ref. [71]. We adapt these to the smeared interpolating currents used here by allowing for momentum- and smearing-dependent couplings. With the nucleon isospinor \( \Psi_N \), where \( \Psi_p = (1, 0)^T \) and \( \Psi_n = (0, 1)^T \),
the leading order vertices read

\[
\begin{align*}
\begin{array}{c}
\text{vertex} = \sqrt{Z_p} \Psi_N, \\
\text{vertex} = \sqrt{Z_{p, q}} \frac{i}{2F_\pi} \gamma_5 \sigma^a \Psi_N,
\end{array}
\end{align*}
\]

(2.23)

where one can actually assume \( Z_p = Z_p(p^2) \) and \( Z_{p, q} = Z_{p, q}(p^2, p \cdot q, q^2) \) up to lattice artifacts (obviously, the couplings will also depend on the masses, the smearing method and the smearing radii). We will use \( Z = Z_p, Z' = Z_p', \tilde{Z} = \tilde{Z}_{p', q}, \) and \( \tilde{Z}' = \tilde{Z}_{p, q} \) as shorthand notations. In the following we always consider protons, i.e., \( \bar{\Psi}\sigma^a\Psi_p = 1 \). We will not assume

\[
\sqrt{Z_{p, q}} = \sqrt{Z_{p'}} + \text{higher order}, \quad \sqrt{Z_{p', q}} = \sqrt{Z_p} + \text{higher order},
\]

(2.25)

which should hold at least approximately for small smearing radii, as discussed above. Instead, we will test the validity of this assumption by comparing it to our data, cf. figure 5 in section 3.2. We complete the setup with the definition of the following energies and four-momenta

\[
\begin{align*}
E & = \sqrt{p^2 + m^2}, \quad E' = \sqrt{p'^2 + m^2}, \quad E_\pi = \sqrt{(p' - p)^2 + m_\pi^2}, \\
p & = \left( \frac{E}{p} \right), \quad p' = \left( \frac{E'}{p'} \right), \quad q = \left( \frac{E' - E}{p' - p} \right), \quad r_\pm = \left( \frac{E_\pi}{\pm(p' - p)} \right).
\end{align*}
\]

(2.26)

(2.27)

We will now consider one example for each type of diagram in figure 1 with an axial-vector current insertion, starting with the purely nucleonic diagram (in the first row of figure 1). Defining the four-vectors \( x = (t, x), \ y = (\tau, y) \) and the energies \( E_i = q_i^0 \), we obtain

\[
\sqrt{Z} \int d^3x e^{-ip \cdot x} \int d^3 y e^{-i(p - p') \cdot y} S_N(x - y)g_A \gamma^\mu \gamma_5 S_N(y) = \\
= -\sqrt{Z} \int \frac{dE_2}{2\pi} e^{-iE_2(t - \tau)} \int \frac{dE_1}{2\pi} e^{-iE_1(\tau - t)} \gamma_0 E_2 - \gamma \cdot p' + m)g_A \gamma^\mu \gamma_5(\gamma_0 E_1 - \gamma \cdot p + m) \frac{(E_2^2 - p'^2 - m^2 + ie)(E_1^2 - p^2 - m^2 + ie)}{(E_2^2 - p'^2 - m^2 + ie)(E_1^2 - p^2 - m^2 + ie)} \\
= \frac{\sqrt{Z} \sqrt{Z}}{2E'2E} e^{-iE'(t - \tau)} e^{-iE\tau(\mathbf{p}' + m)g_A \gamma^\mu \gamma_5(\mathbf{p} + m)}.
\]

(2.28)

In the first step, one integrates over the positions which gives delta distributions in momentum space, which in turn eliminate the integrals over the three-momenta from the propagators. Then, we close both integration contours in the lower half of the complex plane and use Cauchy’s residue theorem twice. Rotating to imaginary times (\( t \to -it \) and \( \tau \to -i\tau \)) one obtains the axial part of eq. (2.9) to zeroth order accuracy in ChPT, exactly as expected.

Next, we consider the left diagram in the second row of figure 1, where the current insertion couples to a pion that directly connects to the sink, while the nucleon propagates
directly from source to sink. We find

\[
\sqrt{Z'} \sqrt{Z} \int d^3x e^{-i p' \cdot x} \int d^3y e^{-i (p-p') \cdot y} \left( \frac{i}{2F_\pi} \frac{\partial}{\partial y^\mu} \right) \left( -2F_\pi \frac{\partial}{\partial y^\mu} \right) S_\pi(x - y) S_N(x) =
\]

\[
= -\sqrt{Z'} \sqrt{Z} \int \frac{dE_2}{2\pi} e^{-i E_2(t-\tau)} \int dE_1 \frac{dE_1}{2\pi} e^{-i E_1 t} \frac{(E_2^\mu)}{E_2^2 - q^2 - m_\pi^2 + i\epsilon} \frac{(E_1^\mu)}{E_1^2 - p^2 - m_\pi^2 + i\epsilon} \gamma_5 (\gamma_0 E_1 - \gamma \cdot p + m),
\]

\[\tag{2.29}\]

where we have introduced the notation \((E_2^\mu)/(E_1^\mu)\), etc., to list the components of a 4-vector. The pion carries the three-momentum \(q\), while the nucleon propagates with momentum \(p\). As in the first diagram, the integrals over the energies can be calculated independently. The diagram yields an \(N\pi\) excitation in the final state with the energy \(E + E_\pi\). In general this will not be the excited state with the smallest possible energy. For example, if the final state momentum \(p'\) is fixed to zero, the excited state with the smallest possible energy gap would be the one where both hadron momenta vanish, and \(E + E_\pi = \sqrt{m^2 + p'^2} + \sqrt{m_\pi^2 + q^2} \geq m + m_\pi\). For the diagram where the pion propagates from the source to the insertion (cf. the right diagram in the second row of figure 1) one obtains, carrying out an analogous calculation,

\[
-\frac{\sqrt{Z'} \sqrt{Z}}{2E' 2E_\pi} e^{-i E't} e^{-i E_\pi t} \gamma_5 (\gamma_0 E_1 - \gamma \cdot p + m),
\]

\[\tag{2.30}\]

which yields an \(N\pi\) excitation in the initial state.

Finally, the diagram where the nucleon-pion interaction happens dynamically (the middle diagram in the second row of figure 1) gives

\[
\sqrt{Z'} \sqrt{Z} \int d^3x e^{-i p' \cdot x} \int d^3y e^{-i (p-p') \cdot y} \int d^4z \times S_N(x - z) \left[ -i \frac{g_\pi}{2F_\pi} \gamma_\mu \gamma_5 \frac{\partial}{\partial y^\mu} \right] \left( -2F_\pi \frac{\partial}{\partial y^\mu} \right) S_\pi(z - y) S_N(z) =
\]

\[
g_A \sqrt{Z'} \sqrt{Z} \int \frac{dE_2}{2\pi} e^{-i E_2(t-\tau)} \int dE_1 \frac{dE_1}{2\pi} e^{-i E_1 t} \]

\[
\times \left( \frac{(E_2^\mu)}{(E_2 - E_1)^2 - q^2 - m_\pi^2 + i\epsilon} \right)^\nu \left( \frac{(E_2 - E_1)^\mu}{(E_2 - E_1)^2 - q^2 - m_\pi^2 + i\epsilon} \right)^\nu \lambda (\gamma_0 E_2 - \gamma \cdot p' + m) \gamma_\nu \gamma_5 (\gamma_0 E_1 - \gamma \cdot p + m).\]

\[\tag{2.31}\]

In this case, where the virtual pion has the three-momentum \(q\) and the energy \(E_2 - E_1\), the remaining integrations over \(E_1\) and \(E_2\) are not independent of each other. We will perform them consecutively starting with \(E_1\). Similarly to the procedure for the other diagrams, both integration contours can be closed in the lower half of the complex plane. There, the integrand has two single poles, which collapse to a double pole, if \(E_2 = E - E_\pi\). The latter case has to be treated separately. The result after the first integration is

\[
g_A \sqrt{Z'} \sqrt{Z} \int \frac{dE_2}{2\pi} f(E_2),
\]

\[\tag{2.32}\]
where, for $E_2 \neq E - E_\pi$,
\[
 f(E_2) = e^{-iE_2(t-\tau)} e^{-iE_\tau(q - E)^\mu(q - E)^\nu} \frac{(\gamma_0 E_2 - \gamma \cdot p + m) \gamma_\nu \gamma_5 (\gamma_0 + m)}{2E((E_2 - E)^2 - E_\pi^2 + i\epsilon)(E_2^2 - E^2 + i\epsilon)}
\]
\[
 + e^{-iE_2(t-\tau)} e^{-iE_\tau(q - E)^\mu(q - E)^\nu} \frac{(\gamma_0 E_2 - \gamma \cdot p + m) \gamma_\nu \gamma_5 (\gamma_0(E_2 + E_\pi) - \gamma \cdot p + m)}{2E_\pi((E_2 + E_\pi)^2 - E^2 + i\epsilon)}. \tag{2.33}
\]

For $E_2 = E - E_\pi$, one can check that $f(E_2)$ is finite, which is the only relevant information since it means that there is no pole at this point when using the residue theorem for $E_2$ later on. Thus, one finds that $f(E_2)$ has three poles in the lower half of the complex plane. The first term in eq. (2.33) has two single poles, while the second term in eq. (2.33) has only one single pole. Its second, seeming pole is at $E_2 = E - E_\pi$, where eq. (2.33) is not evaluated. One obtains three contributions that correspond to the diagrams in the bottom row of figure 1:

\[
 \begin{align*}
 &- \frac{g_A \sqrt{Z} \sqrt{Z}}{2E' 2E} e^{-iE'(t-\tau)} e^{-iE_\tau} q^\mu q^\nu \frac{(\gamma' + m) \gamma_\nu \gamma_5 (\gamma_0 + m)}{q^2 - m_\pi^2}, \\
 &- \frac{g_A \sqrt{Z} \sqrt{Z}}{2E 2E_\pi} e^{-iE_\tau} e^{-iE r^\mu r^\nu} \frac{(\gamma_0 + m) \gamma_\nu \gamma_5 (\gamma_0 + m)}{(p + r_+)^2 - m^2}, \\
 &- \frac{g_A \sqrt{Z} \sqrt{Z}}{2E' 2E_\pi} e^{-iE' r^\mu r^\nu} \frac{(\gamma' + m) \gamma_\nu \gamma_5 (\gamma_0 + m)}{(p' + r_-)^2 - m^2},
\end{align*} \tag{2.34}
\]

where we have written the result in terms of the four-vectors defined in eqs. (2.27). The first term yields a contribution to the ground state. It is responsible for the leading, pole dominant contribution to the induced pseudoscalar form factor. The second and the third term contribute to the same $N\pi$ excitations in the final and initial states as those in eqs. (2.29) and (2.30), respectively.

This concludes our calculation of the tree-level diagrams shown in figure 1 for the axialvector current insertion. For the pseudoscalar current the calculation is analogous and we will not repeat it here. By matching the result obtained for the ground state with the usual form factor decompositions (using eq. (2.9) in combination with eqs. (2.12) and (2.13) after rotating to Euclidean times) one finds

\[
 G_A = g_A + \text{higher order}, \tag{2.35}
\]
\[
 G_\rho = g_A \frac{4m^2}{Q^2 + m_\pi^2} + \text{higher order}, \tag{2.36}
\]
\[
 G_P = g_A \frac{m^2}{m_\ell Q^2 + m_\pi^2} + \text{higher order}. \tag{2.37}
\]

We emphasize that we will not enforce these results for the ground state contribution. In eq. (2.35) this corresponds to augmenting the axial coupling in the chiral limit to the full axial form factor, which is justified at leading order accuracy. In the same spirit, we have already tacitly used the actual nucleon mass in the propagator instead of its chiral limit value, which is also consistent to leading order accuracy in ChPT.
perform the same replacement \( g_A \mapsto G_A \) in the complete calculation. (We will show that this choice is in much better agreement with the data at nonzero \( Q^2 \), cf. section 3.2 and, in particular, figure 5.) After doing so, eq. (2.36) yields the PPD assumption for the induced pseudoscalar form factor, as expected.

It turns out to be convenient to define the ratios

\[
a = \frac{\sqrt{Z}}{\sqrt{Z}}, \quad a' = \frac{\sqrt{Z'}}{\sqrt{Z'}},
\]

where \( a = a' = 1 \) would correspond to the assumption that the smearing does not affect the overlap of the interpolating currents with the \( N\pi \) excited states (compared to the ground state). Note that in general \( a \) and \( a' \) are functions of the momenta. Putting everything together and rotating to Euclidean time \( (t \to -it \text{ and } \tau \to -i\tau) \) we find

\[
C^{p,p,A}_{3pt} = + \frac{\sqrt{Z}/\sqrt{Z}}{Z} e^{-E'(t-\tau)} e^{-E\tau} (\gamma + m) \left[ G_A \gamma^\mu \gamma_5 + G_P \frac{q^\mu}{2m} \gamma_5 \right] (\gamma + m)
\]

\[
- \frac{\sqrt{Z}/\sqrt{Z}}{Z} e^{-(E+E_a)(t-\tau)} e^{-E\tau} \gamma^\mu \left( b' \gamma_5 (\gamma + m) + G_A \frac{(\gamma + m) \gamma_5 (\gamma + m)}{(p + r_+)^2 - m^2} \right)
\]

\[
+ \frac{\sqrt{Z}/\sqrt{Z}}{Z} e^{-E'(t-\tau)} e^{-(E'+E_a)\tau} \gamma^\mu \left( b (\gamma + m) \gamma_5 - G_A \frac{(\gamma + m) \gamma_5 (\gamma + m)}{(p' + r_-)^2 - m^2} \right)
\]

\[
+ \ldots,
\]

\[
C^{p,p,P}_{3pt} = + \frac{\sqrt{Z}/\sqrt{Z}}{Z} e^{-E'(t-\tau)} e^{-E\tau} (\gamma + m) \gamma_5 (\gamma + m)
\]

\[
- \frac{\sqrt{Z}/\sqrt{Z}}{Z} e^{-(E+E_a)(t-\tau)} e^{-E\tau} B \left( b' \gamma_5 (\gamma + m) + G_A \frac{(\gamma + m) \gamma_5 (\gamma + m)}{(p + r_+)^2 - m^2} \right)
\]

\[
- \frac{\sqrt{Z}/\sqrt{Z}}{Z} e^{-E'(t-\tau)} e^{-(E'+E_a)\tau} B \left( b (\gamma + m) \gamma_5 - G_A \frac{(\gamma + m) \gamma_5 (\gamma + m)}{(p' + r_-)^2 - m^2} \right)
\]

\[
+ \ldots,
\]

where

\[
b = -a + G_A \frac{m^2}{(p + r_+)^2 - m^2} \quad \text{and} \quad b' = -a' + G_A \frac{m^2}{(p + r_+)^2 - m^2}.
\]

and the dots represent additional excited state contributions. These results can be used for all momentum configurations and with arbitrary spin projections. After taking the trace with the specific matrices \( P^i_+ \) that we use here, the result can be further simplified, see below.

### 2.3 Spectral decomposition

In this section we will provide the explicit expressions for the correlation functions that are used in our analysis, including our parametrization of additional generic excited states. For the latter we will assume that they occur with the same energies in both, two- and three-point functions. State-of-the-art lattice analyses of form factors take into account
up to three excited states in the two-point and up to two excited states in the three-point functions, see, e.g., ref. [82]. Whether this is necessary depends on the available statistics and on the applied source/sink smearing. In our simulation a relatively large number of smearing steps was performed, leading to large smearing radii, cf. table 2. In this situation, we find it sufficient to add only one generic excited state to the two- and three-point correlators on top of the pion pole enhanced state that we have calculated in the last section. Including the additional generic excited state term, we obtain for the two-point function

$$C^p_{2pt,P_\pi}(t) = Z_p \frac{E_p + m}{E_p} e^{-E_p t} (1 + A_pe^{-\Delta E_p t}). \quad (2.42)$$

In the following we will abbreviate $\Delta E = \Delta E_p$ and $\Delta E' = \Delta E_{p'}$. Note that we do not assume any dispersion relation for the excited state energies, nor do we assume that these are single hadron states. Instead, we treat them as free fit parameters. We define the trace occurring in the ground state contribution to the three-point function as

$$B_{\Gamma,\mathcal{O}}^{p'p} = \text{Tr} \{ \Gamma(p' + m)J[\mathcal{O}](p + m) \}. \quad (2.43)$$

The explicit results can be found in appendix A, together with the remaining traces needed to evaluate eqs. (2.39) and (2.40). For the three-point functions we obtain the parametrization

$$C^p_{3pt,P_\pi} = \sqrt{Z} \sqrt{Z} \frac{2}{2E' 2E} e^{-E'(t-\tau) - E\tau}$$

$$\times \left[ B^p_{P_\pi A} \left( 1 + B_{01} e^{-\Delta E'/(t-\tau)} + B_{11} e^{-\Delta E\tau} \right) + e^{-\Delta E\tau} \frac{E'}{E_p} r^{\mu} \left( c' p^i + d' q^i \right) \right], \quad (2.44)$$

$$C^{p'p}_{3pt,P_\pi} = \sqrt{Z} \sqrt{Z} \frac{2}{2E' 2E} e^{-E'(t-\tau) - E\tau}$$

$$\times \left[ B^p_{P_\pi A} \left( 1 + B_{01} e^{-\Delta E'/(t-\tau)} + B_{11} e^{-\Delta E\tau} \right) + e^{-\Delta E\tau} \frac{E'}{E_p} m^2 \left( c' p^i + d' q^i \right) \right], \quad (2.45)$$

where we have suppressed the dependence of the excited state parameters on the momenta, the spin-projection and the current insertion: $B_{ij} = B_{ij}(p', p, \Gamma, \mathcal{O})$. We have defined $\Delta E_{\pi} = E_{\pi}$, $\Delta E_{N_{\pi}} = E_{\pi} - (E' - E)$ and

$$c = -2b - 4G_A \frac{m_{E\pi} + p \cdot r_-}{(p' + r_-)^2 - m^2}, \quad c' = -2b' - 4G_A \frac{m_{E\pi} + p \cdot r_+}{(p' + r_+)^2 - m^2}, \quad (2.46)$$

$$d = -G_A \frac{4m^2(m + E)}{(p' + r_-)^2 - m^2}, \quad d' = G_A \frac{4m^2(m + E)}{(p + r_+)^2 - m^2}. \quad (2.47)$$

Equations (2.46) and (2.47) are only valid up to higher order corrections in ChPT. For instance, one could replace $G_A$ by $(Q^2 + m_{\pi}^2)G_P/(4m^2)$ or by $(Q^2 + m_{\pi}^2)m_{\pi}G_P/(mm_{\pi}^2)$.
in the $N\pi$ excited state contributions (cf. eqs. (2.35), (2.36) and (2.37)) and the result would still be valid at leading order. From a plain vanilla ChPT power-counting point of view one could even replace $G_A$ by $g_A$. Therefore, in anticipation of possible higher order corrections, we may relax the assumptions even further by using $c, c', d,$ and $d'$ as free fit parameters, which minimizes the ChPT bias. This has the additional advantage, that it does not allow the excited state signal to have a direct influence on the result for the ground state form factors. Naturally, one has to pay for the increased number of fit parameters with a slightly larger statistical error for the ground state result – a small price considering that one gets rid of one entire source of systematic uncertainty. In section 3.2 we will assess the validity of the ChPT predictions by comparing them to the results obtained from the fits. In particular we will be able to check whether the data is consistent with the parameter-free ChPT prediction for $d$ and whether the direct coupling of the smeared three-quark interpolating currents to the $N\pi$ state differs from the leading order ChPT prediction calculated for local currents.

Note the elegance of the parametrization given in eqs. (2.44) and (2.45). Even after relaxing the conditions (2.46) and (2.47), it encodes the relative strength of the $N\pi$ excited state contribution in the different channels. The importance of this knowledge must not be underestimated. For instance, one can directly see that any determination of the axial form factor using solely the $A_1, A_2,$ and $A_3$ channels is not affected by these excited states at all.

Finally, let us note that for the kinematics we use in the numerical analysis, setting the final state momentum to zero, $p' = 0$, such that $p = -q$ (this setup is used in many lattice simulations), the parametrization becomes even simpler since one can replace $c'p^i + d'q^i = e'q^i$ (with $e' = d' - c'$) and $cp^i + dq^i = dq^i$. In this kinematic situation, the $N\pi$ excited state energy corresponds to $E_N(0) + E_\pi(-q)$ in the initial state, and $E_N(p) + E_\pi(q)$ in the final state.

3 Data analysis

3.1 Lattice setup

In order to determine the axial and (induced) pseudoscalar form factors using the correlation functions described in section 2, we have analyzed a large set of lattice ensembles generated within the CLS effort [83]. The ensembles have been generated using a tree-level Symanzik improved gauge action and $N_f = 2 + 1$ flavors of nonperturbatively order $a$ improved Wilson (clover) fermions. An efficient and stable hybrid Monte Carlo sampling is achieved by applying twisted-mass determinant reweighting [85], which avoids near-zero modes of the Wilson Dirac operator. The individual quarks in the nucleon interpolators at the source and the sink are Wuppertal-smeared [74], employing spatially APE-smoothed [75] gauge links.

Some of the CLS ensembles (cf. table 2 for a full list of the ensembles used in this work) have been simulated employing very fine lattices down to $a = 0.039$ fm. For these

\footnote{The ensembles rqcd021, and rqcd030 have been generated using the BQCD code [84].}
Table 1. Lattice spacings $a$, corresponding to the five different inverse couplings $\beta$ used in this study. The lattice spacings have been obtained by determining the Wilson flow time at the SU(3) symmetric point in lattice units $t_0^*/a^2$ and equating $t_0^*$ with the result $\mu_{\text{ref}}^*/(8t_0^*)^{-1/2} \approx 478$ MeV of ref. [87].

| $\beta$ | 3.40 | 3.46 | 3.55 | 3.70 | 3.85 |
|---------|------|------|------|------|------|
| $a$ [fm] | 0.086 | 0.076 | 0.064 | 0.050 | 0.039 |

Figure 2. Schematic visualization of the analyzed CLS ensembles in the space spanned by the lattice spacing and quark masses. On the flavor symmetric plane (blue), where $m_\ell = m_s$, flavor multiplets of hadrons have degenerate masses (e.g., $m_K^2 = m_\pi^2$ and $m_N = m_{\Sigma} = m_{\Xi} = m_{\Lambda}$). The green lines are defined to have physical average quadratic meson mass ($2m_K^2 + m_\pi^2 = \text{phys.}$). This corresponds to an approximately physical mean quark mass ($2m_\ell + m_s \approx \text{phys.}$). The red lines are defined by $2m_K^2 - m_\pi^2 = \text{phys.}$ and indicate an almost physical strange quark mass ($m_s \approx \text{phys.}$). Physical masses are reached at the intersections of green and red lines.

In practice the ensembles do not always lie exactly on top of the green and red trajectories shown in figure 2. lattices we avoid large autocorrelation times by using open boundary conditions in the time direction [85, 86]. The latter allow the topological charge to flow into and out of the simulation volume through the temporal boundaries and thus topological freezing is avoided. While employing open boundary conditions is crucial for fine lattice spacings, we use lattices with both open and periodic boundary conditions for the coarser spacings. In total we have five different lattice spacings ranging from $a = 0.039$ fm to $a = 0.086$ fm, see table 1.

As illustrated in figure 2, the available ensembles have been generated along three different trajectories in the quark mass plane. This strategy is explained in ref. [88]. The ensembles cover a range of volumes with $3.5 \leq m_\pi L \leq 6.4$ allowing us to investigate and control finite volume effects. The majority of the ensembles has $m_\pi L > 4$. Having multiple quark mass trajectories with a wide range of lattice spacings and volumes enables us to simultaneously extrapolate to physical masses, to infinite volume, and to the continuum limit by means of a global fit to 37 ensembles. Our extrapolation strategy is explained in detail in section 4.2.

5In practice the ensembles do not always lie exactly on top of the green and red trajectories shown in figure 2.
Table 2. List of the ensembles used in this work, labeled by their identifier and sorted by the inverse coupling $\beta$ and the pion masses. We specify the geometries $N_t \times N_s$ as well as the boundary conditions in time (periodic (p) or open (o)). The light and strange hopping parameters $\kappa_\ell$ and $\kappa_s$ used in the simulation and the resulting approximate meson masses are given in MeV. Where available, we also provide the root mean squared smearing radii $r_{sm}$ for the light quark sources in fm. #conf. gives the number of configurations analyzed. The column $t/a$ lists the source-sink distances in lattice units that have been analyzed on this lattice. The subscript #meas. specifies how many measurements have been performed for the respective source-sink distance. In physical units these distances roughly correspond to 0.7 fm, 0.9 fm, 1.0 fm, and 1.2 fm. The last column specifies on which trajectories in the quark mass plane the ensemble lies, cf. figure 2. An in-depth description of the ensemble generation can be found in ref. [83]. Note that ensemble D201 was only used for the test with nonzero final momentum shown in figure 7.

| Ens. | $\beta$ | $N_t \times N_s$ | #conf. | $m_\pi$ | $f_{\pi}$ | $r_{sm}$ | $t/a$ | #meas. | traj. |
|------|---------|-----------------|--------|---------|---------|---------|-------|--------|-------|
| U103 | 3.4     | 24 128 o       | 417    | 0.13675962 | 0.13675962 | 417 | 4.4 | 0.638 | 2473 8 |
| H101 | 3.4     | 32 96 o        | 420    | 0.136865 | 0.13654933 | 354 | 4.9 | 0.669 | 1997 8 |
| H102 | 3.4     | 32 96 o        | 279    | 0.13634079 | 0.13634079 | 279 | 5.8 | 0.722 | 320 8 |
| H105 | 3.4     | 32 96 o        | 279    | 0.13634079 | 0.13634079 | 279 | 5.8 | 0.722 | 320 8 |
| N101 | 3.4     | 48 128 o       | 220    | 0.13697 | 0.13634079 | 220 | 4.6 | 0.772 | 243 8 |
| C101 | 3.4     | 48 96 o        | 220    | 0.13697 | 0.13634079 | 220 | 4.6 | 0.772 | 243 8 |
| D101 | 3.4     | 64 128 o       | 220    | 0.13697 | 0.13634079 | 220 | 4.6 | 0.772 | 243 8 |
| H107 | 3.4     | 32 96 p        | 126    | 0.13634079 | 0.13634079 | 126 | 4.9 | 0.742 | 1997 8 |
| H106 | 3.4     | 32 96 p        | 126    | 0.13634079 | 0.13634079 | 126 | 4.9 | 0.742 | 1997 8 |
| C102 | 3.4     | 48 96 o        | 222    | 0.13697 | 0.13634079 | 222 | 4.6 | 0.772 | 243 8 |
| D102 | 3.4     | 64 128 o       | 222    | 0.13697 | 0.13634079 | 222 | 4.6 | 0.772 | 243 8 |
| B450 | 3.46    | 32 64 p        | 126    | 0.13697 | 0.13634079 | 126 | 4.9 | 0.742 | 1997 8 |
| S400 | 3.46    | 32 128 o       | 126    | 0.13697 | 0.13634079 | 126 | 4.9 | 0.742 | 1997 8 |
| N401 | 3.46    | 48 128 o       | 286    | 0.13697 | 0.13634079 | 286 | 5.8 | 0.722 | 1997 8 |
| D450 | 3.46    | 64 128 p        | 286    | 0.13697 | 0.13634079 | 286 | 5.8 | 0.722 | 1997 8 |

Continued on next page
| Ens.  | β    | $N_s$ | $N_t$ | bc | $\kappa_t$ | $\kappa_s$ | $m_\pi$ | $m_K$ | $m_\pi L$ | $r_{sm}$ | #conf. | $t/a$#meas. | traj. |
|-------|------|-------|-------|----|-----------|-----------|--------|------|----------|--------|-------|----------------|-------|
| B452  | 3.46 | 32    | 64    | p  | 0.1370455 | 0.136378044 | 350    | 545  | 4.3      | 0.650  | 1944  | 9, 11, 13, 13, 16 | msc |
| N450  | 3.46 | 48    | 128   | p  | 0.1370986 | 0.136352601 | 285    | 524  | 5.3      | 0.706  | 1132  | 9, 11, 13, 16 | msc |
| D451  | 3.46 | 64    | 128   | p  | 0.13714   | 0.136337761 | 200    | 503  | 4.9      | 0.784  | 532   | 9, 11, 13, 16 | msc |
| rqcd030 | 3.46 | 32    | 64    | p  | 0.1369587 | 0.1369587 | 319    | 319  | 3.9      | —      | 1224  | 9, 11, 13, 16 | sym |
| X450  | 3.46 | 48    | 64    | p  | 0.136994  | 0.136994  | 263    | 263  | 4.9      | 0.739  | 400   | 9, 11, 13, 16 | sym |
| N450  | 3.55 | 48    | 128   | o  | 0.137     | 0.137     | 411    | 411  | 6.4      | 0.610  | 884   | 11, 14, 16, 19 | trm, sym |
| N203  | 3.55 | 48    | 128   | o  | 0.13708   | 0.136840284 | 345    | 442  | 5.4      | 0.660  | 1543  | 11, 14, 16, 19 | trm |
| N200  | 3.55 | 48    | 128   | o  | 0.13714   | 0.13672086 | 284    | 462  | 4.4      | 0.696  | 1712  | 11, 14, 16, 19 | trm |
| D200  | 3.55 | 64    | 128   | o  | 0.1372    | 0.136601748 | 201    | 481  | 4.2      | 0.786  | 2001  | 11, 14, 16, 19 | trm |
| E250  | 3.55 | 96    | 192   | p  | 0.137232867 | 0.136536633 | 130    | 489  | 4.0      | 0.829  | 490   | 11, 14, 16, 19 | trm, msc |
| N204  | 3.55 | 48    | 128   | o  | 0.137112  | 0.136575049 | 351    | 545  | 5.5      | 0.661  | 1500  | 11, 14, 16, 19 | msc |
| N201  | 3.55 | 48    | 128   | o  | 0.13715968 | 0.136561319 | 285    | 523  | 4.5      | 0.727  | 1522  | 11, 14, 16, 19 | msc |
| D201  | 3.55 | 64    | 128   | o  | 0.1372067 | 0.136546844 | 199    | 501  | 4.1      | 0.778  | 1078  | 11, 14, 16, 19 | msc |
| X250  | 3.55 | 48    | 64    | p  | 0.13705   | 0.13705   | 348    | 348  | 5.4      | 0.655  | 345   | 11, 14, 16, 19 | sym |
| X251  | 3.55 | 48    | 64    | p  | 0.1371    | 0.1371    | 268    | 268  | 4.2      | 0.719  | 436   | 11, 14, 16, 19 | sym |
| N300  | 3.7  | 48    | 128   | o  | 0.137     | 0.137     | 420    | 420  | 5.1      | 0.591  | 1014  | 14, 17, 21, 24 | trm, sym |
| N302  | 3.7  | 48    | 128   | o  | 0.137064  | 0.1368721791358 | 346    | 451  | 4.2      | 0.644  | 1383  | 14, 17, 21, 24 | trm |
| J303  | 3.7  | 64    | 192   | o  | 0.137123  | 0.1367546608 | 258    | 476  | 4.2      | 0.705  | 634   | 14, 17, 21, 24 | trm |
| X304  | 3.7  | 48    | 128   | o  | 0.137079325093654 | 0.136665430105663 | 352    | 554  | 4.3      | 0.620  | 1482  | 14, 17, 21, 24 | msc |
| J304  | 3.7  | 64    | 192   | o  | 0.137123  | 0.1367546608 | 260    | 523  | 4.2      | 0.708  | 1525  | 14, 17, 21, 24 | msc |
| J500  | 3.85 | 64    | 192   | o  | 0.136852  | 0.136852  | 410    | 410  | 5.2      | 0.579  | 750   | 17, 22, 27, 32 | trm, sym |
| J501  | 3.85 | 64    | 192   | o  | 0.1369032 | 0.136749715 | 333    | 445  | 4.2      | 0.613  | 1507  | 17, 22, 27, 32 | trm |
The local axial and pseudoscalar currents in our calculation have to be renormalized. We use the renormalization factors $Z_A$ from ref. [89] (as recommended in this reference, we use the values $Z_{A,sub}$ from their table 7), which have been determined using a new method based on the chirally rotated Schrödinger functional [90]. In addition, we use the nonperturbative quark mass-dependent order $a$ improvement coefficients described in ref. [91] (but with updated values from ref. [92]). The isovector currents are multiplicatively renormalized using

$$A^\text{ren} = Z_A(\beta) \left[ 1 + 2am_\ell^\text{bare} b_A(\beta) + 2a(2m_\ell^\text{bare} + m_s^\text{bare}) \tilde{b}_A(\beta) \right] A^\text{imp},$$

$$m_\ell^\text{ren} = Z_A(\beta) \left[ 1 + 2am_\ell^\text{bare} b_A(\beta) + 2a(2m_\ell^\text{bare} + m_s^\text{bare}) \tilde{b}_A(\beta) \right] m_\ell^\text{imp} P^\text{imp},$$

where $m_\ell^\text{imp}$ is the PCAC light quark mass obtained from improved currents,

$$m_\ell^\text{imp} = \frac{\langle 0 | \partial_\mu A^\text{imp}_\mu | \pi \rangle}{2i \langle 0 | P^\text{imp} | \pi \rangle}.$$  

(3.3)

The bare quark mass

$$m_q^\text{bare} = \frac{1}{2a} \left( \frac{1}{\kappa_q} - \frac{1}{\kappa_{\text{crit}}} \right).$$

(3.4)

is calculated using the hopping parameter $\kappa_q$ (cf. table 2) and its critical value $\kappa_{\text{crit}}$ [93]. We exploit the fact that the product of quark mass and pseudoscalar current renormalizes in exactly the same way as the axialvector current. $\tilde{b}_A$ has been found to be zero within errors smaller than 0.1 [92]. This corresponds to shifts of at most 4%, depending on the ensemble, that decrease towards the continuum limit. We neglect this effect, which is small compared to the other sources of error, and proceed with continuum limit extrapolations that are quadratic in $a$. Within the Symanzik improvement program [94, 95] also the currents themselves have to be $O(a)$-improved. For the axialvector current this yields

$$A^\mu_\text{imp} = A^\mu + c_A a \partial_\mu P,$$

(3.5)

where we use the improvement coefficient $c_A$, nonperturbatively determined in ref. [96], and $\partial_\mu$ denotes the symmetrically discretized derivative. For the pseudoscalar current $P^\text{imp} = P$.

### 3.2 Fits to the correlation functions

To calculate the three-point functions (2.2) one has to evaluate all possible contractions. Disconnected diagrams do not contribute in our case, since we only consider isovector currents. The connected diagrams can be evaluated using sequential sources [97]. Since each sink momentum requires new inversions, we restrict the numerical analysis to the case in which the final state three-momentum is set to zero ($p' = 0$). Note, however, that the parametrizations provided in section 2.3 are applicable to all possible kinematic situations.

On each ensemble we have analyzed 4 source-sink separations that have been chosen such that they correspond roughly to the physical distances 0.7 fm, 0.9 fm, 1.0 fm, and
Figure 3. Nucleon dispersion relation for the ensembles listed in table 2. The data points show the squared ground state energies obtained from fits to two-point functions using the ansatz (2.42) and treating the energies as free fit parameters. The lines correspond to $E^2 = m^2 + p^2$ using the nucleon mass $m$ determined at zero momentum.

1.2 fm. The source-sink distance in lattice units and the corresponding number of measurement per configuration are specified in table 2. On some ensembles we have reduced the computational cost by applying the coherent sink technique [33], where one inverts on multiple, temporally separated sequential sources simultaneously. For the statistical analysis we generate 500 bootstrap samples per ensemble using a bin size of 20 molecular dynamics units to eliminate autocorrelations.

The nucleon energies determined from fits to two-point functions using a spectral decomposition with one generic excited state (2.42) agree with the continuum dispersion relation, see figure 3. With this justification, we employ the continuum dispersion relation for single nucleon energies in the subsequent analysis. The nucleon isovector form factors are obtained by a simultaneous fit to two-point functions and to the ratio

$$R_{\Gamma,\mathcal{O}}^{P,P}(t, \tau) = \frac{C_{3pt,\Gamma}^{P,\mathcal{O}}(t, \tau)}{C_{2pt,\mathcal{O}}^{P}(t)}.$$  

using the parametrizations given in section 2.3. In the literature also the ratio

$$\frac{C_{3pt,\Gamma}^{P,\mathcal{O}}(t, \tau)}{C_{2pt,\mathcal{O}}^{P}(t)} = \sqrt{\frac{C_{2pt,\mathcal{O}}^{P}(t)C_{2pt,\mathcal{O}}^{P}(t)}{C_{2pt,\mathcal{O}}^{P}(t)C_{2pt,\mathcal{O}}^{P}(t)}}$$  

(3.7)
is found, which is constructed such that the overlap factors drop out and the ground state contribution is time-independent. This is not the case for the ratio (3.6), where the ground state contribution is $\propto e^{-(E-E')\tau}$. Nevertheless, we find it to be advantageous for various reasons:

1. It allows for a maximal cancellation of correlations, since the interpolating currents at the source and the sink occur at exactly the same spacetime positions with exactly the same phase factors in two- and three-point functions, cf. eqs. (2.1) and (2.2).

2. In contrast to eq. (3.7) it does not introduce additional excited states from two-point functions at small separations $\tau$ or $t - \tau$.

3. One avoids a technical problem of eq. (3.7): in the course of the error analysis one can encounter negative values for single bootstrap samples due to statistical fluctuations such that the argument of the square root is negative.

Results of the simultaneous fits using the ratio (3.6) are shown in figure 4, where we have selected cases in which the effect due to the pion pole enhanced excited states is large, i.e., ensembles with small pion masses at small (but nonzero) momentum transfer. Note that for our kinematics the parametrization (2.44) and (2.45) only includes two additional fit parameters ($d$ and $e'$) in addition to the usual excited state parametrization. These two parameters describe the $N\pi$ related excited state contributions for the axialvector and pseudoscalar channels simultaneously, for all spin-projections. That this is even possible strongly indicates that the results given in section 2.3 are a very good approximation of the underlying physics.

In order to take into account systematic uncertainties of our excited state analysis, we perform a fit range variation, where the minimal distance between the operators is varied between $2a$ and $4a$ in the ratios, and between $2a$ and $3a$ in the two-point functions. In figures 4 and 6 full circles (dots) correspond to data points that are always (never) part of the fitted window, while the open symbols indicate data points that are used only in some of the fits. The error bands of the extracted ground state contributions contain both the statistical error and the error related to the choice of the fit range.

In figure 4 the yellow bands correspond to the ground state contributions extracted from the EFT-inspired ansatz for the three-point function (eqs. (2.44) and (2.45)), while the gray band is the ground state signal obtained from a traditional multistate fit ansatz (also using eqs. (2.44) and (2.45), but without the explicit $N\pi$ contribution, i.e., setting $c = c' = d = d' = 0$). As one can see, the ground state contribution can be disentangled from the huge signal of the $N\pi$ state (which fails to be resolved using the traditional ansatz with generic excited state contributions). Here, it is particularly advantageous that the coefficients of the $N\pi$ contributions are constrained for various channels and spin projections in our fit. This results in a crosstalk between the different rows in figure 4 and allows us to pin down the corresponding excited state parameters very precisely. To this end, the seemingly strange linear behavior in $A_0$ (i.e., row 3 in figure 4, where the spin projection is aligned with the momentum) is actually helpful and it is noteworthy that the data can be described almost perfectly by our fit ansatz. The ratio shown in the top panels...
Figure 4. Fits to the ratio $R_{Q}^{0,\rho}$ (defined in eq. (3.6)) at a momentum transfer $\mathbf{q} = -\mathbf{p} = \frac{2\pi}{L}(0,0,1)^T$ for ensemble D200 (left side) and C101 (right side) for various channels and spin projections, where we have exploited rotational symmetry to average over equivalent directions. The solid lines correspond to a simultaneous fit to all channels taking into account the leading $N\pi$ contribution using eqs. (2.44) and (2.45). The yellow band corresponds to the ground state. The gray band (dashed lines) shows the ground state extracted from a traditional fit using one generic excited state. The bands include the statistical error and an error due to a variation of the fit range.
Figure 5. The plot on the left shows how well the parameter free tree-level ChPT prediction $d_{\text{ChPT}}$ (circles; see eq. (2.47)) describes the data obtained from the fit ($d_{\text{fit}}$). As anticipated in section 2.2, the estimate using $g_A$ instead of $G_A$ (crosses) is not as good. This simply means that at nonzero momentum transfer the coupling of the pion to the nucleon is given by $G_A(Q^2)$ instead of $g_A$, as expected. In the plot on the right we show $a'$ (cf. eq. (2.38)) obtained from our fit to the data. A value of $a' = 1$ would imply that the leading order ChPT estimate for the coupling of $N\pi$ to the three-quark operators is exact and that the operator smearing does not affect the coupling at all. As one can see, the data is not very sensitive to the value of $a'$. We do not see any significant momentum dependence and no strong smearing effect.

(which is sensitive to $G_A$ but independent of $G_{\bar{P}}$) is not affected by the $N\pi$ excited state contribution at all. This explains why past lattice determinations of the axial form factor using this channel in combination with traditional excited state fits were able to extract a correct, reliable result.

The ansatz including the $N\pi$ excited states explicitly allows for a much better description of the data. In the case of D200 for instance, fits using block-correlated covariance matrices yield $\chi^2/$d.o.f. $\approx 1.05$ (including $N\pi$) versus $\chi^2/$d.o.f. $\approx 5.59$ (excluding $N\pi$). Note, however, that we have decided to use uncorrelated fits to extract the results. This avoids instabilities in the covariance matrix and prevents an underestimation of the statistical errors.

We find that almost the complete excited state contamination can be attributed to this $N\pi$ state, and that there are only very mild additional contributions in the final state (where $p' = 0$). Nevertheless, we refrain from removing the additional generic excited states from the parametrization, in order to exclude an underestimation of the error in the extracted ground state contribution. Actually, one can also obtain a very good description of the data with even smaller statistical errors if one would use the ChPT-biased parametrizations discussed in section 2.2, which may indicate that possible higher order corrections are small. Nevertheless, the latter would entail a systematic uncertainty that we intend to avoid.

However, we can confront the results of our fits with the corresponding ChPT prediction, see figure 5. In particular for the parameters $d$ and $d'$ in eqs. (2.44) and (2.45) ChPT yields a parameter free prediction, see eq. (2.47). Since $d$ corresponds to one of our fit parameters, a direct comparison is possible (left plot in figure 5). As anticipated in section 2.2, the prediction using $G_A(Q^2)$ (circles) as the pion-nucleon coupling instead
of \( g_A = G_A(0) \) (crosses) is in almost perfect agreement with our data, even at large \( Q^2 \), where one would usually not expect ChPT to work. For our kinematics, the \( N\pi \) excitation in the final state can also couple directly to the three-quark operator (this corresponds to the diagrams on the left and right in the second row of figure 1). Therefore, we can try to determine \( a' \) (defined in eq. (2.38)) directly from the data. A value \( a' = 1 \) means that the leading order ChPT estimate for the coupling of \( N\pi \) to the three-quark operators calculated for local currents is exact in spite of the smearing. As one can see from the large statistical errors in the right plot of figure 5, our data is not very sensitive to \( a' \). We neither see a significant momentum dependence nor a strong smearing effect. If anything, the direct coupling of the three-quark operators to \( N\pi \) seems to be slightly enhanced by the smearing.

In figure 6 we reinvestigate the subtraction method that some of us have proposed in ref. [60]. As one can clearly see in the upper panels of figure 6, it removes the strangely linear behavior in the \( A_0 \) channel almost entirely. We find that the results for the ground state obtained from fits to the unsubtracted (solid lines; ground state yellow) and the subtracted (dashed lines; ground state red) data are mutually compatible, once we take
Figure 7. Fits to the ratio $R_{p,\Gamma}^{p,p}$ (as defined in eq. (3.6), but rescaled such that the ground state contribution in all channels corresponds to $g_A$) at momentum transfer $q = 0$, with $p' = p = 2\pi L (0, 0, 1)^T$ and with $p' = p = 0$ for the contributing axial channels. This analysis has been performed on ensemble D201. The solid lines correspond to a simultaneous fit to all channels taking into account the leading $N\pi$ contribution using eqs. (2.44) and (2.45), where the yellow band corresponds to the ground state. The bands include the statistical error and an error due to a variation of the fit range.

into account the leading $N\pi$ contribution.\textsuperscript{6} For the subtracted correlation functions, the fit ansatz given in section 2.3 has to be adapted appropriately, cf. appendix B. However, the ground state extracted from the subtracted data has a much larger statistical uncertainty. A closer look shows that the subtraction method here has fallen victim to its own success: since the largest and clearest excited state contaminations (in $A_0$) have been subtracted successfully, the corresponding parameters can be determined less reliably, which in turn leads to a large error in the ground state. One can conclude that a combination of the analysis method proposed here (taking into account the relevant $N\pi$ excitation explicitly in the fit to the correlation function) and the subtraction method proposed in ref. [60] is not advantageous.

As a consistency check, we have also considered the case $q = 0$ with $p' = p \neq 0$ on one of our ensembles (D201). In this situation eq. (2.44) predicts that the correlation functions of $A_1$, $A_2$, and $A_3$ are not affected by the $N\pi$ excited state, while $A_0$ gets a contribution

\textsuperscript{6}Note that the subtraction method in combination with traditional excited state fits (as used in ref. [60]) does not yield the correct ground state. In particular in the pseudoscalar channel the correction overshoots and yields too large values. This has strong effects on $G_P$ and $G_R$, while $G_A$ is unaffected.
Figure 8. Energy gaps to the excited states on the ensemble D200. The crosses have been obtained from a fit using the ansatz from eqs. (2.44) and (2.45) but taking $\Delta E_{3\text{pt}} = \Delta E_{N\pi}$ and $\Delta E'_{3\text{pt}} = \Delta E'_{N\pi}$ as free fit parameters, while $\Delta E_{2\text{pt}} = \Delta E$ corresponds to the energy of the generic excited state determined from two- and three-point functions. The dots have been obtained from a fit without an explicit $N\pi$ state (i.e., $e = e' = 0$ in eqs. (2.44) and (2.45)) but relaxing the condition that the excited state energies in two- and three-point function have to match (i.e., $\Delta E_{3\text{pt}} = \Delta E$ and $\Delta E'_{3\text{pt}} = \Delta E'$ from the three-point function and $\Delta E_{2\text{pt}} = \Delta E$ from the two-point function). The orange, dotted line and the green, dashed line show the energy gaps for a noninteraction nucleon-pion system in the initial and the final state, respectively, as obtained from the diagrams in the left and the right column of figure 1. For our kinematics the energys are $E_{N\pi} = E_\pi(q) + E_N(0)$ and $E'_{N\pi} = E_\pi(q) + E_N(-q)$.

$\propto \exp\left(-\left(E_N + m_\pi/2\right)t\right) \cosh\left(m_\pi(\tau - t/2)\right)$ in the three-point function. In figure 7 we show that this is indeed the case and that a simultaneous fit using eq. (2.44) yields a consistent description of the data for all channels.

3.3 Excited state energies

In ref. [73] it has been proposed to use the signal of the timelike axialvector channel to determine the energy of the low-lying $N\pi$ excitation. The main difference to the traditional excited state fit method is that one does not impose that the leading excited states in the two- and three-point functions have the same energy. In figure 8 (which roughly reproduces Fig. 3 of ref. [73]) we show the energy gaps to the various excited states obtained from two different fits to the correlation functions on ensemble D200. The dots (fit 1) have been obtained using the method proposed in ref. [73] (with the slight difference that we perform a simultaneous fit to all channels instead of the two-step method presented in ref. [73]), while the crosses (fit 2) have been obtained using our fit ansatz from eqs. (2.44) and (2.45) but taking $\Delta E_{N\pi}$ and $\Delta E'_{N\pi}$ as free fit parameters. In contrast to fit 1, fit 2 contains the additional excited states known from the two-point function, which leads to larger statistical uncertainties in particular when the energy levels of the $N\pi$ state and the

\footnote{Figure number from the arXiv v2 version.}
excited state from the two point function (blue data points) get close to each other. Both kinds of fits lead to energies for the nucleon-pion states that approximately correspond to those of a noninteracting system (cf. the diagrams in the left and the right column of figure 1), which for our kinematics means $E_{N\pi} = E_\pi(q) + E_N(0)$ in the initial state (orange, dotted line) and $E_{N'\pi} = E_\pi(q) + E_N(-q)$ in the final state (green, dashed line). That both methods lead to compatible values for the $N\pi$ excited state energies is encouraging and suggests that the physical interpretation obtained using EFT (cf. section 2.2) is correct.

In particular for the low-lying $N\pi$ state (which for our kinematics occurs in the initial state) at intermediate $Q^2$ one can see that the energies obtained from the fits slightly undershoot those of the noninteracting system. This effect is found to be a bit more significant in ref. [73]. One may speculate that this small deviation is due to an interaction between the nucleon and the pion. For the time being we have chosen to ignore these small deviations in our fits in favor of a better stability.

We comment that the pole enhanced $N\pi$ excited state contribution occurs either in the initial state or in the final state, but certainly not in both at once. Therefore, it is not necessary to allow for an excited-state to excited-state contribution in this case.

4 Form factors

4.1 Approximate restoration of PCAC and PPD

As mentioned in the introduction, form factors extracted from data using a traditional fit ansatz (with same excited state energies in the two- and the three-point function) show strong violations of PCAC and PPD. In particular in the case of PCAC this result was puzzling since the latter is fulfilled at the correlation function level (up to some small, expected discretization effects). In order to quantify the violation of the PCAC relation at the form factor level (cf. eq. (2.15)), we define the ratio (cf. also ref. [55])

$$r_{PCAC} = \frac{m \ell m}{G_P(Q^2) + \frac{Q^2}{4m^2} G_{\tilde{P}}(Q^2)} G_A(Q^2), \quad (4.1)$$

where $r_{PCAC} = 1$ if PCAC is fulfilled exactly. As the panel on the left-hand side of figure 9 demonstrates, using the parametrization of excited state contributions described in section 2.3 the PCAC relation is now fulfilled reasonably well on all ensembles, in particular on the ensembles with small pion masses, which previously exhibited the largest deviations. We emphasize that our fit ansatz does not impose PCAC on the ground state. While we see a significant improvement for all ensembles, small deviations of $\sim 5\%$ remain in some cases.

The induced pseudoscalar form factor is often estimated by

$$G_{\tilde{P}} \approx \frac{4m^2 G_A}{m^2_\pi + Q^2} \Rightarrow r_{PPD} = \frac{(m^2_\pi + Q^2) G_{\tilde{P}}(Q^2)}{4m^2 G_A(Q^2)} = 1, \quad (4.2)$$

which is usually referred to as the pion pole dominance (PPD) assumption. Note that this relation does not have to hold exactly, even in the continuum. However, one would expect
it to be fulfilled at least approximately for small pion mass. The panel on the right-hand side of figure 9 shows that this is indeed the case if one explicitly takes into account the pion pole enhanced excited states in the spectral decomposition of the correlation function.

As reported in ref. [73] the problem can also be resolved (though within larger statistical uncertainties), if one uses a traditional multi state fit ansatz, but relaxes the condition that the excited state energies in the two- and three-point function have to match. One exploits the huge excited state signal in the timelike axialvector channel to determine the energy gaps quite precisely (cf. also section 3.3). This can be seen as further confirmation that the previously observed large deviations from PCAC and PPD were indeed caused by unresolved, pion pole enhanced excited states. Note, however, that our ansatz (shown in eqs. (2.44) and (2.45)) conveys a deeper insight into the structure of the excited state contribution compared to a generic parametrization. For instance, it is clear that, for $A_{\mu}$ with $\mu = 1, 2, 3$, the result for $G_A$ cannot be affected by the $N\pi$ excited state at all. Heuristically speaking, this is because $G_A$ is not subject to pion pole dominance. Therefore, the deviation in $G_A$ reported in ref. [73] is clearly an artifact.

### 4.2 Parametrization and extrapolation

In this section we will explore two common form factor parametrizations: the traditional dipole ansatz and the $z$-expansion, which has become fashionable lately. In both cases we also consider parametrizations that are consistent with PCAC in the continuum (section 4.2.3) and we will use a generic ansatz for the combined continuum, chiral and volume extrapolation explained in section 4.2.4.

#### 4.2.1 Dipole ansatz

Motivated by eqs. (2.35), (2.36), and (2.37), we rewrite the form factors as

$$G_A \equiv A(Q), \quad G_{\tilde{P}} \equiv \frac{4m^2}{Q^2 + m_{\pi}^2} \tilde{P}(Q), \quad G_P \equiv \frac{m}{m_{\ell} Q^2 + m_{\pi}^2} P(Q), \quad (4.3)$$
where the pion pole is isolated (cf. also ref. [52]) such that one can use similar parametrizations for the residual form factors $X(Q)$, $X \in \{A, \tilde{P}, P\}$. The prefactors not only ensure that all $X(Q)$ have the same mass dimension, but also allow us to obtain the correct chiral behavior of the form factors at small $Q^2$ despite using the same generic ansatz for all form factors, see section 4.2.4 below.

One can consider various parametrizations for the residual form factors. For instance, one can use a dipole ansatz

$$X(Q) = \frac{g_X}{\left(1 + Q^2/M_X^2\right)^2}, \quad (4.4)$$

which reproduces the traditional dipole form for the axial form factor with the axial coupling $g_A$ and the axial dipole mass $M_A$. This parametrization not only yields the correct low-energy behavior (if one uses a generic parametrization for the pion mass, volume and lattice spacing dependence of $g_X$ and $M_X$, cf. section 4.2.4 below), but also yields the correct asymptotic limit $G_A \propto 1/Q^4$, $G_{\tilde{P}} \propto 1/Q^6$, and $G_P \propto 1/Q^6$ [98], at large momentum transfer.

### 4.2.2 $z$-expansion

One may also parametrize the residual form factors using the $z$-expansion [99, 100], which automatically imposes analyticity constraints. This corresponds to an expansion of the form factors in the variable

$$z = \frac{\sqrt{t_{\text{cut}} + Q^2} - \sqrt{t_{\text{cut}} - t_0}}{\sqrt{t_{\text{cut}} + Q^2} + \sqrt{t_{\text{cut}} - t_0}}, \quad (4.5)$$

where $t_{\text{cut}} = 9 m_{\pi}^2$ is the particle production threshold and $t_0$ is a tunable parameter.\footnote{We have set $t_0$ to $-t_{\text{phys}} = -9 m_{\pi, \text{phys}}^2$ in our analysis. By choosing a negative value one can avoid the erratic behavior at $t_{\text{cut}} = t_0$, while approaching the chiral limit.} We then parametrize

$$X(Q) = \sum_{n=0}^{N} a_n^X z^n(Q), \quad (4.6)$$

where the $X(Q)$ are defined as in section 4.2.1. Without additional constraints this parametrization has $N + 1$ free parameters and is usually called a $z^{(N+1)}$ ansatz. Again, the generic parametrization discussed in section 4.2.4 will yield the correct chiral behavior. However, eq. (4.6) does not incorporate any constraints at large momentum transfer. In order to reproduce the correct asymptotic behavior one has to enforce constraints of the type

$$\lim_{Q^2 \to \infty} Q^k X(Q) = 0, \quad \text{for } 0 \leq k \leq n, \quad (4.7)$$

which can be implemented (as long as $n < N$) by demanding

$$0 = \sum_{l=0}^{N} l^k a_l^X, \quad \text{for } 0 \leq k \leq n. \quad (4.8)$$
These can be incorporated, e.g., by fixing

\[ a_k^X = \frac{(-1)^{k+n+1}}{k!(n-k)!} \sum_{l=n+1}^{N} \frac{l!}{(l-(n+1))!(l-k)} a_l^X, \quad \text{for } 0 \leq k \leq n. \]  

(4.9)

Alternatively, one can solve the problem recursively by setting

\[ a_k^X = \frac{(-1)^{2k+1}}{k!} \sum_{l=k+1}^{N} \frac{l!}{(l-(k+1))!(l-k)} a_l^X, \quad \text{for } 0 \leq k \leq n. \]  

(4.10)

To enforce the correct scaling in the asymptotic limit, \( G_A \propto 1/Q^4 \), \( G_\rho \propto 1/Q^6 \), and \( G_P \propto 1/Q^8 \) [98], we have to apply the formulas above for \( n = 3 \), thereby fixing \( a_k^X \) for \( k = 0, 1, 2, 3 \), such that 4 coefficients are fixed and only \( N - 3 \) coefficients are free parameters.\(^9\) This parametrization with correct asymptotic behavior is usually referred to as the \( z^{4+(N-3)} \) ansatz.

### 4.2.3 Consistency with PCAC in the continuum

Let us assume the following ansatz for the extrapolation to the physical point \((m_\pi \to m_\pi^{\text{phys}}, a \to 0, L \to \infty)\),

\[ x = x^a(m_\pi, m_K, L)x^a(a, m_\pi, m_K), \]  

(4.11)

where we have factorized the dependence on the lattice spacing with

\[ x^a(0, m_\pi, m_K) = 1, \]  

(4.12)

for all parameters in the form factor decompositions, i.e., \( x \in \{g_A, M_A, g_\rho, M_\rho, g_P, M_P\} \) for the dipole ansatz, and \( x \in \{a_n^A, a_n^\rho, a_n^P\}, n = 4, 5, \ldots, N \) for the \( z \)-expansion. This allows us to perform a combined fit to all ensembles for each form factor. The expressions used for \( x^a \) and \( x^a \) will be given below in section 4.2.4.

Since we know that the partial conservation of the axial current has to be satisfied exactly in the continuum limit, we can use eq. (2.15) to obtain \( G_P \) from \( G_A \) and \( G_\rho \):

\[ \frac{m_\ell}{m} G_P(Q^2) = G_A(Q^2) - \frac{Q^2}{4m^2} G_\rho(Q^2) + O(a^2). \]  

(4.13)

However, one then has to impose the additional constraints

\[ \lim_{Q \to \infty} Q^n \left( G_A - \frac{Q^2}{4m^2} G_\rho \right) \Bigg|_{a=0}^\dagger = 0 \iff \lim_{Q \to \infty} Q^n \left( A(Q) - \hat{P}(Q) \right) \Bigg|_{a=0}^\dagger = 0, \quad \text{for } n = 4, 5, \]  

(4.14)

in order to preserve the correct asymptotic behavior of \( G_P \), cf. also eq. (4.3). For the dipole parametrizations one gets

\[ g_AM_A^0 \Bigg|_{a=0}^\dagger = g_\rho M_\rho^0 \Bigg|_{a=0}^\dagger. \]  

(4.15)

\(^9\)We neglect possible \( O(Q^2a^2) \) lattice artifacts since we only have lattice data with \( Q^2 \ll a^{-2} \). Such effects could be implemented by relaxing the constraint (4.7) at nonzero lattice spacing.
The equivalent constraints for the \(z\)-expansion can be obtained using eq. (4.9) and read
\[
(a_k^A - a_k^P) \bigg|_{a=0} = \frac{(-1)^k}{k!(5-k)!} \sum_{l=6}^{N} \frac{l!}{(l-6)!(l-k)!} (a_l^A - a_l^P) \bigg|_{a=0}, \text{ for } k = 4, 5. \tag{4.16}
\]

Let us now parametrize the pseudoscalar form factor using
\[
P(Q) = \left(1 + \frac{Q^2}{m_\pi^2}\right) P_1(Q) - \frac{Q^2}{m_\pi^2} P_2(Q). \tag{4.17}
\]
The ansatz (4.17) becomes consistent with PCAC in the continuum limit once we demand that
\[
P_1(Q) \bigg|_{a=0} = A(Q) \bigg|_{a=0}, \quad P_2(Q) \bigg|_{a=0} = \tilde{P}(Q) \bigg|_{a=0}. \tag{4.18}
\]
Unfortunately, PCAC is broken on the lattice by discretization effects, such that \(P_1(Q)\) and \(P_2(Q)\) differ from \(A(Q)\) and \(\tilde{P}(Q)\) at nonzero lattice spacing. Hence, we use the same ansatz for both (e.g., the dipole form (4.4) or the \(z\)-expansion (4.6)), but we start with independent parameters. Here, the asymptotic constraints yield
\[
\lim_{Q \to \infty} Q^n \left( P_1(Q) - P_2(Q) \right)^\frac{1}{4} = 0, \text{ for } n < 6, \tag{4.19}
\]
independent of \(a\). Note, that eq. (4.18) and (4.19) can only be fulfilled simultaneously if the axial and induced pseudoscalar form factors meet the requirement (4.14). For the two parametrizations (cf. sections 4.2.1 and 4.2.2) that we consider, the constraints for \(n < 4\) are fulfilled automatically. Similar to the above, the remaining two constraints can be satisfied by
\[
g_{P_1} M_{P_1}^\frac{1}{4} = g_{P_2} M_{P_2}^\frac{1}{4}. \tag{4.20}
\]
when using the dipole ansatz, and by
\[
(a_k^{P_1} - a_k^{P_2}) = \frac{(-1)^k}{k!(5-k)!} \sum_{l=6}^{N} \frac{l!}{(l-6)!(l-k)!} (a_l^{P_1} - a_l^{P_2}), \text{ for } k = 4, 5. \tag{4.21}
\]
when using the \(z\)-expansion.

To summarize, if we want our form factor parametrizations to obey PCAC in the continuum limit, we start by parametrizing \(P(Q)\) as in eq. (4.17), thereby introducing more parameters at first. However, as discussed above, these parameters are highly constrained such that the ansatz enforcing PCAC will have less free fit parameters in the end. Using the dipole ansatz, we have \(g_A, M_A, g_{P_1}, M_{P_1}, g_{P_2}, M_{P_2}\), which can be factorized in a lattice spacing dependent and a lattice spacing independent part as shown in eq. (4.11). The constraints discussed above can be incorporated by setting
\[
g_{P_2} = g_{P_1} \left( \frac{M_{P_1}}{M_{P_2}} \right)^{\frac{1}{4}}, \quad g_{P_1}^{\tilde{A}} = g_A^{\tilde{P}}, \quad \left[ g_{P_2}^{\tilde{A}} = g_{P_2}^{\tilde{P}} \right]. \tag{4.22}
\]
The green line with physical average masses is defined by \( \bar{m} \) such that \( \delta m \bar{m} \). To parametrize the quark mass plane we have defined the linear combinations the leading contribution found in ChPT calculations of the axial coupling, cf. refs. [101, 102].

In our combined analysis of all ensembles we will consider four kinds of fits: the dipole expansion with correct asymptotic behavior (\( z^{4+(N-3)} \)), and the two corresponding parametrizations where PCAC holds automatically in the continuum (\!2P and \!z^{4+(N-3)}, respectively). They are listed in table 3. We have factorized the occurring parameters \( x = x^a x^\alpha \) (see eq. (4.11)) into a continuum limit part \( x^\alpha \), and a part which describes discretization effects \( x^a \). In particular, in the parametrizations that respect PCAC, the number of parameters is reduced due to the constraints derived in section 4.2.3 (see also table 3). We perform a combined continuum, chiral and volume extrapolation using the generic ansatz

\[
x^\alpha(m_\pi, m_K, L) = c_1^\alpha + c_2^\alpha \bar{m}^2 + c_3^\alpha \delta m^2 + c_4^\alpha \frac{m_\pi^2}{\sqrt{m_\pi L}} e^{-m_\pi L} + c_5^\alpha \frac{m_K^2}{\sqrt{m_K L}} e^{-m_K L} + c_6^\alpha \frac{m_\eta^2}{\sqrt{m_\eta L}} e^{-m_\eta L},
\]

\[
x^a(a, m_\pi, m_K) = 1 + a^2 (d_1^a \bar{m}^2 + d_2^a \delta m^2 + d_3^a \delta m^2),
\]

(4.28) (4.29)

where \( m_\eta^2 = (4m_K^2 - m_\pi^2)/3 \). The functional form of the finite volume terms is motivated by the leading contribution found in ChPT calculations of the axial coupling, cf. refs. [101, 102].

To parametrize the quark mass plane we have defined the linear combinations

\[
\delta m^2 = m_K^2 - m_\pi^2 \approx B(m_s - m_\ell),
\]

\[
\bar{m}^2 = (2m_K^2 + m_\pi^2)/3 \approx 2B(m_s + 2m_\ell)/3,
\]

(4.30)

such that \( \delta m = 0 \) corresponds to exact flavor symmetry, i.e., the blue line in figure 2, while the green line with physical average masses is defined by \( \bar{m} = \text{phys.} \approx 411 \text{ MeV} \). Along

\[
g_\rho^\alpha = g_\alpha^\alpha \left( \frac{M_\rho^2}{M_\rho^2} \right)^4, \quad M_{P_1}^2 = M_{P_1}^2, \quad M_{P_2}^2 = M_{P_2}^2,
\]

(4.23)

where the constraint in brackets is not independent of the others. If one uses the \( z \)-expansion, one starts with \( a_n^A, a_n^P, a_n^P, a_n^P, n = 4, 5, \ldots, N \). Again, we assume these coefficients to be factorized as in eq. (4.11). Here, the constraints discussed above can be implemented by setting

\[
a_k^P = a_k^P + \frac{(-1)^k}{k!(5-k)!} \sum_{l=6}^N \frac{l!}{(l-6)!(l-k)!} (a_l^P - a_l^P), \quad \text{for } k = 4, 5,
\]

(4.24)

\[
a_k^{A\alpha} = a_k^{A\alpha} + \frac{(-1)^k}{k!(5-k)!} \sum_{l=6}^N \frac{l!}{(l-6)!(l-k)!} (a_l^{A\alpha} - a_l^{A\alpha}), \quad \text{for } k = 4, 5,
\]

(4.25)

\[
a_k^{A\alpha} = a_k^{A\alpha}, \quad \text{for } k = 4, 5, \ldots, N
\]

(4.26)

\[
a_k^{P\alpha} = a_k^{P\alpha}, \quad \text{for } k = 4, 5, 6, \ldots, N
\]

(4.27)

As above, the constraints in brackets are not independent of the others.

### 4.2.4 Continuum, chiral, and volume extrapolation

In our combined analysis of all ensembles we will consider four kinds of fits: the dipole ansatz (\( 2P \)), the \( z \)-expansion with correct asymptotic behavior (\( z^{4+(N-3)} \)), and the two corresponding parametrizations where PCAC holds automatically in the continuum (\!2P and \!z^{4+(N-3)}, respectively). They are listed in table 3. We have factorized the occurring parameters \( x = x^a x^\alpha \) (see eq. (4.11)) into a continuum limit part \( x^\alpha \), and a part which describes discretization effects \( x^a \). In particular, in the parametrizations that respect PCAC, the number of parameters is reduced due to the constraints derived in section 4.2.3 (see also table 3). We perform a combined continuum, chiral and volume extrapolation using the generic ansatz

\[
x^\alpha(m_\pi, m_K, L) = c_1^\alpha + c_2^\alpha \bar{m}^2 + c_3^\alpha \delta m^2 + c_4^\alpha \frac{m_\pi^2}{\sqrt{m_\pi L}} e^{-m_\pi L} + c_5^\alpha \frac{m_K^2}{\sqrt{m_K L}} e^{-m_K L} + c_6^\alpha \frac{m_\eta^2}{\sqrt{m_\eta L}} e^{-m_\eta L},
\]

\[
x^a(a, m_\pi, m_K) = 1 + a^2 (d_1^a \bar{m}^2 + d_2^a \delta m^2 + d_3^a \delta m^2),
\]

(4.28) (4.29)
do not consider terms with chiral logarithms ($m$ average mass varies; all ensembles used in this study have $\bar{m} <$ the line of an approximately physical strange quark mass, i.e., the red line in figure 2, the average mass varies; all ensembles used in this study have $\bar{m} <$ 500 MeV. Note that we do not consider terms with chiral logarithms ($m^2_{\pi} \ln(m^2_{\pi}/m^2)$, etc.) in eq. (4.28) since they are of the same order in chiral counting as $m^2_{\pi}$. Therefore, their inclusion does not really make sense as long as not all of them are known explicitly from a calculation within chiral perturbation theory.

### 4.3 Results

Figure 10 provides a compilation of (continuum, chiral and finite volume extrapolated) form factors that have been obtained from the parametrizations discussed in the previous sections. The parameters producing the central values can be taken from table 4. Surprisingly, even the fits using a dipole ansatz (2P) give a reasonable description of the data (actually, it has in most cases the smallest $\chi^2$/d.o.f. of all fits, cf. table 4), despite the fact that the functional form is very constrained. However, the latter may lead to an underestimation of the error, and it may also induce a smaller slope at zero momentum transfer. In order to reduce this bias one may relax the constraints due to the choice of parametrization. The currently most popular and probably best suited ansatz for this task is the $z$-expansion described in section 4.2.2, which allows us to increase the number of parameters seamlessly. To this end, we have performed $z^{4+3}$, and $z^{4+4}$ fits (and the corresponding fits that are constrained to be consistent with PCAC in the continuum limit, see below). While the
Figure 10. Comparison of continuum results at the physical point for the residual form factors obtained using the different fits, cf. table 3. The fits enforcing PCAC in the continuum (lower panels) yield significantly smaller statistical errors. The mean values of the plotted curves can be reproduced using the parameters provided in table 4.

The $z^{4+3}$ fit is almost as restrictive as the dipole ansatz, expansions with a larger number of parameters ($z^{4+4}$, $z^{4+5}$, etc.) introduce less and less parametrization bias. In practice, however, the choice will always be a balancing act between reducing parametrization bias and being able to control the systematics of all occurring parameters (due to discretization effects, volume effects and the necessary chiral extrapolation, cf. section 4.2.4). Therefore, we have decided to perform the analysis using various parametrizations.

We emphasize that PCAC was not enforced when extracting the form factors from fits to the correlators. Nevertheless, due to the advances in the understanding of excited state contaminations in the correlation functions, we are now able to resolve the ground state contributions such that the obtained form factors agree with PCAC (and also PPD) reasonably well. This enables us to perform combined fits to all form factors using parametrizations that automatically obey PCAC in the continuum limit. As one can easily see in table 3, the resulting parametrizations are much more restrictive in the continuum than their counterparts. For example, the dipole fit (!2P) has in total three free parameters (at the physical point in the continuum limit) for all form factors. However, in contrast to the parametrization bias discussed above, the PCAC constraints do not evoke any kind of systematic uncertainty, since they only reflect an exactly known symmetry. Unsurprisingly, we find that the continuum extrapolation is more stable when using these PCAC-consistent parametrizations. Overall, we find that both the !2P and the $!z^{4+3}$ fit yield a very good
Table 4. Results for the parameters at the physical point in the continuum occurring in the dipole ansatz (4.4) and the z-expansion (4.6) together with the uncorrelated $\chi^2$ per degree of freedom of the corresponding fit. For convenience, we also provide the values for the parameters, which are entirely fixed by constraints.

| id   | $X$  | $\chi^2$/d.o.f. | $g_X$ | $M_X$ [GeV] |
|------|------|-----------------|-------|-------------|
| 2P A |      | 0.78            | 1.226 | 1.310       |
|      | $\tilde{P}$ | 0.61          | 1.310 | 1.188       |
|      | $P$   | 0.68            | 1.258 | 1.482       |
| !2P A | $P_1$ | 0.69            | 1.231 | 1.304       |
|      | $\tilde{P} = P_2$ | 1.225 | 1.305 |

| id   | $X$  | $\chi^2$/d.o.f. | $a_0^X$ | $a_1^X$ | $a_2^X$ | $a_3^X$ | $a_4^X$ | $a_5^X$ | $a_6^X$ | $a_7^X$ |
|------|------|-----------------|---------|---------|---------|---------|---------|---------|---------|---------|
| $z^{4+3}$ | $A$ | 0.94            | 1.009   | -1.765 | -1.039 | 1.627   | 3.869   | -5.691  | 1.990   |
|         | $\tilde{P}$ | 0.60          | 1.036   | -1.710 | -2.489 | 6.342   | -2.528  | -1.680  | 1.028   |
|         | $P$ | 0.69            | 1.072   | -1.683 | 0.313  | -5.860  | 16.453  | -14.595 | 4.300   |
| !$z^{4+3}$ | $A = P_1$ | 0.81           | 1.014   | -1.721 | -0.559 | -0.827  | 7.842   | -8.442  | 2.694   |
|         | $\tilde{P} = P_2$ | 1.008 | -1.683 | 0.654  | -0.700 | 7.747   | -8.404  | 2.688   |
| $z^{4+4}$ | $A$ | 0.96            | 1.011   | -1.822 | -0.840 | 1.356   | 4.026   | -5.715  | 1.985   | -0.00004 |
|         | $\tilde{P}$ | 0.58          | 1.052   | -1.826 | -2.198 | 6.011   | -2.363  | -1.695  | 1.020   | -0.00025 |
|         | $P$ | 0.68            | 1.124   | -1.977 | 0.952  | -6.513  | 16.740  | -14.607 | 4.282   | -0.00022 |
| !$z^{4+4}$ | $A = P_1$ | 0.79           | 1.022   | -1.761 | -0.485 | -0.893  | 7.867   | -8.442  | 2.692   | 0.00000  |
|         | $\tilde{P} = P_2$ | 1.020 | -1.751 | -0.505 | -0.872 | 7.858   | -8.445  | 2.696   | -0.00088 |

description of the data ($\chi^2$/d.o.f. = 0.69 and $\chi^2$/d.o.f. = 0.81, respectively), while still allowing for a controlled extrapolation to the physical point. Our final results will therefore be obtained from these fits. The !$z^{4+4}$ fit also provides a very good description of our data ($\chi^2$/d.o.f. = 0.79). However, it is less trustworthy since it relies on an excessive number of parameters, which leads to larger systematic uncertainties in the combined continuum, chiral, and volume extrapolation.

In figures 11, 12 and 13, we show how well the !$z^{4+3}$ fit actually describes the data. (For the !2P fit such plots look similarly convincing.) The 6 rows in each figure correspond to the five available lattice spacings and to the continuum limit, while the columns correspond to the different quark mass trajectories. The yellow band corresponds to the extrapolated result at physical masses and in infinite volume. The data show that the form factors exhibit an increasing slope (in $Q^2$) for decreasing pion masses and lattice spacings. It is particularly encouraging that the data for our physical meson mass ensemble (E250) nicely reproduces the expected pion pole structure in the (induced) pseudoscalar form factor (cf. eq. (4.3)).

Above, in figure 9, we have demonstrated that the nucleon form factor data extracted from the correlation functions using the results presented in section 2.3 agrees reasonably
Figure 11. The axial form factor $G_A(Q^2)$ obtained using the $!z^{4+3}$ ansatz fitted to all available ensembles. This is a combined fit to all form factors with $\chi^2$/d.o.f. = 0.81. The panels correspond to different lattice spacings and quark mass trajectories, where the yellow band corresponds to the form factor at physical masses and at infinite volume.
Figure 12. The induced pseudoscalar form factor $G_P(Q^2)$ obtained using the $!z^4$ ansatz fitted to all available ensembles. This is a combined fit to all form factors with $\chi^2$/d.o.f. = 0.81. The panels correspond to different lattice spacings and quark mass trajectories, where the yellow band corresponds to the form factor at physical masses and at infinite volume.
Figure 13. The pseudoscalar form factor $\frac{m^2}{2m}G_P(Q^2)$ obtained using the $\bar{z}^{4+3}$ ansatz fitted to all available ensembles. This is a combined fit to all form factors with $\chi^2/\text{d.o.f.} = 0.81$. The panels correspond to different lattice spacings and quark mass trajectories, where the yellow band corresponds to the form factor at physical masses and at infinite volume.
Figure 14. The $r_{PCAC}$ (left panel) and $r_{PPD}$ (center panel) ratios (defined in eqs. (4.1) and (4.2)) at the physical point in the continuum limit. They are obtained using a 2P (dotted, blue), $z^{4+3}$ (solid, green) or a $z^{4+4}$ (dashed, red) fit ansatz. In the case of $r_{PPD}$, we also show results obtained from the corresponding fits that are constrained to be consistent with PCAC in the continuum limit (right panel), cf. table 3.

Table 5. Results for the form factors $G_X(0)$ at zero momentum transfer and for the mean squared radii $r_X^2 = -6G_X'(0)/G_X(0)$ obtained from fits using various form factor parametrizations. We also provide results for the pion-nucleon coupling $g_{\pi NN}$ and for the induced pseudoscalar coupling at the muon capture point $g_\pi^\star$, which can be directly compared to the experimental value $g_\pi^\star = 8.01(55)$ from muon capture [18, 19].

| id   | $G_A(0)$  | $r_A^2$ [fm$^2$] | $G_P(0)$  | $r_P^2$ [fm$^2$] | $m_\pi G_P(0)$ | $r_P^2$ [fm$^2$] | $g_P$       | $g_{\pi NN}$ |
|------|-----------|-----------------|-----------|-----------------|----------------|-----------------|-------------|-------------|
| 2P   | 1.226(23) | 0.272(22)       | 243(21)   | 11.96(11)       | 1.258(87)      | 11.85(7)        | 8.88(74)    | 14.71(2.82) |
| !2P  | 1.231(21) | 0.275(18)       | 227(5)    | 11.91(2)        | 1.231(21)      | 11.85(11)       | 8.32(17)    | 12.72(55)   |
| $z^{4+3}$ | 1.277(43) | 0.354(60)       | 226(27)   | 11.72(25)       | 1.416(190)     | 12.25(23)       | 8.36(95)    | 12.16(3.51) |
| !$z^{4+3}$ | 1.305(36) | 0.454(36)       | 238(8)    | 12.07(4)        | 1.305(36)      | 11.94(18)       | 8.69(29)    | 14.54(1.08) |
| $z^{4+4}$ | 1.297(55) | 0.387(47)       | 235(30)   | 11.78(18)       | 1.540(184)     | 12.31(13)       | 8.66(1.05)  | 13.26(3.77) |
| !$z^{4+4}$ | 1.323(42) | 0.466(22)       | 244(10)   | 12.09(3)        | 1.323(42)      | 12.06(26)       | 8.89(34)    | 15.29(1.29) |

well with PCAC and PPD. In figure 14 we show the result for the ratios $r_{PCAC}$ (left panel) and $r_{PPD}$ (center panel) after the extrapolation using the previously discussed form factor parametrizations that do not enforce PCAC. We find that both PCAC and PPD are fulfilled within very large statistical errors. This observation is true for dipole fits as well as fits using the $z$-expansion. For the parametrizations that enforce PCAC in the continuum $r_{PCAC} = 1$ by construction. As one can see by comparing the center and the right panel (note the difference in the scale between the two plots), the PCAC-consistent fits allow for a much better resolution of possible deviations from the pion pole dominance assumption for the induced pseudoscalar form factor. We find an almost perfect realization of the PPD assumption (at a 1%–2% level) for all values of momentum transfer independent of the choice of parametrization.
The results for the form factors at zero momentum transfer and for the mean squared radii are given in table 5, where we also provide the results for the induced pseudoscalar coupling at the muon capture point

\[ g_P^* = \frac{m_\mu}{2m} G_P(0.88m_\mu^2), \]  

(4.31)

with the muon mass \( m_\mu = 105.6 \text{ MeV}, \) and for the pion-nucleon coupling constant

\[ g_{\pi NN} = \lim_{Q^2 \to -m_\pi^2} \frac{m_\pi^2 + Q^2}{4m F_\pi} G_P(Q^2) = \frac{m}{F_\pi} \tilde{P}(-m_\pi^2), \]  

(4.32)

where we use the PDG value of \( F_\pi = 92.07 \text{ MeV} \) \[103\]. As a general trend we find that the fits which ensure that PCAC is fulfilled in the continuum limit yield a smaller statistical uncertainty. A main achievement is that we now have control over the pion pole enhanced excited states occurring in the pseudoscalar channels. As a consequence, we find reasonable values for \( g_P^* \) that are in agreement with an approximate realization of PPD in nature.\(^{10}\)

From table 5 one can actually read off that the different parametrizations yield compatible results, with the exception of the axial radius, where the dipole fits give significantly smaller values, and the pion-nucleon coupling, where the \(!z^{4+4}\) fit seems to be an outlier.

In our opinion, the \(!2P\) and the \(!z^{4+3}\) yield the most trustable results (for the fits with more free parameters the chiral and continuum extrapolation is less stable). However, given our set of available data, we cannot decide whether the \(!2P\) or the \(!z^{4+3}\) fit is better. We have therefore decided to perform an analysis of systematic uncertainties for both of these fits. In table 6 we provide, in addition to the statistical error \((\sigma)\), estimates for the systematic uncertainties due to the chiral extrapolation \((\delta m)\) and the continuum extrapolation \((\delta a)\). To this end, we have perform additional fits with cuts in the fit ranges \((\bar{m} < 450 \text{ MeV} \text{ and } a < 0.08 \text{ fm}, \text{ respectively})\). We then take the difference between the results from these fits and our main result as an estimate for the corresponding systematic uncertainty.

4.4 Discussion

As one can see in table 6, the \(!2P\) and the \(!z^{4+3}\) fit yield compatible results for almost all observables. For definiteness we choose to quote the values from the \(!z^{4+3}\) fit as our final result in these cases, merely because it might have less parametrization bias and because the slightly larger statistical uncertainty is more conservative. In the case of the axial radius, which is directly linked to the axial dipole mass, however, we find that the dipole fit and the \(z\)-expansion yield significantly different results. Our main conclusion here has to be that \(r_A\) is highly parametrization dependent – a nuisance which also plagues determinations from experiment.

In figure 15 we show a compilation of experimental data and lattice data for the axial dipole mass. While the 20\(^{th}\) century world average (cf. ref. [24]) supports a value around 1 GeV, newer experiments by K2K [108], MINOS [109], and, in particular, MiniBooNE [110] yield larger values. This has fueled some discussions lately. One possible

\(^{10}\)In earlier work [60] we found much smaller values that would have suggested a strong violation of the PPD assumption.
Table 6. Final results obtained from the !2P and the !z4+3 fit including the statistical error ()s and estimates for the systematic uncertainties due to chiral extrapolation ()m and due to the continuum extrapolation ()a. Since both fits satisfy PCAC in the continuum, $G_A(0) = \frac{m}{m} G_P(0)$ is fulfilled automatically.

|                | !2P                      | !z4+3                    |
|----------------|--------------------------|--------------------------|
| $G_A(0) = \frac{m}{m} G_P(0)$ | 1.231(21)s(5)m(17)a      | 1.305(36)s(62)m(8)a      |
| $G_P(0)$       | 227(5)s(1)m(2)a          | 238(8)s(12)m(1)a         |
| $r_A^2$ [fm^2] | 0.275(18)s(8)m(25)a      | 0.454(36)s(86)m(8)a      |
| $M_A$ [GeV]    | 1.304(43)s(18)m(55)a     | 1.014(41)s(85)m(9)a      |
| $r_P^2$ [fm^2] | 11.91(2)s(1)m(2)a       | 12.07(4)s(9)m(1)a       |
| $g_T^P$        | 8.32(17)s(2)m(5)a       | 8.69(29)s(40)m(1)a      |
| $g_{\pi NN}$   | 12.98(57)s(9)m(23)a     | 14.84(1.10)s(1.78)m(4)a |
| $\Delta_{GT}$  | 1.06(2.82)s(24)m(22)a%   | 6.65(4.69)s(5.41)m(38)a% |

Another line of inquiry is followed, e.g., by refs. [100] and [12], and is based on the suspicion that the dipole ansatz might be too restrictive. Using the $z$-expansion one finds smaller values and much larger errors for $M_A$. In ref. [100] it is shown that the MiniBooNE data is consistent with old $\pi$ electroproduction data under these circumstances. Our analysis supports this picture. The results for the axial radii obtained from the dipole fit (!2P) and the $z$-expansion (！z4+3) correspond to the axial pole masses of $M_A = 1.30(7)$ GeV (dipole) and $M_A = 1.01(9)$ GeV ($z$-exp). The situation we find is thus very similar to the one reported in ref. [100], where extractions using a dipole ansatz yield $M_A = 1.29(5)$ GeV (dipole, [100]), while the $z$-expansion yields a smaller value $M_A = (0.85^{+0.22}_{-0.07} \pm 0.09)$ GeV ($z$-exp, [100]). It is notable that the $z$-expansion coefficients we obtain from our fits (see table 4) roughly fulfill the constraints that are imposed in ref. [100].

For the dipole ansatz our result is in good agreement with previous lattice determinations. In particular the agreement with the continuum extrapolated value from [55] is encouraging. For the $z$-expansion the situation is not so clear, since the lattice results scatter over a wide range. In part this may be caused by the use of different variants of the $z$-expansion (number of parameters, use of priors, choice of $t_0$, implementation of constraints, etc.).

In figure 16 we have compiled results for the induced pseudoscalar coupling at the muon capture point $g^{\star}_P$ from experiment, ChPT, and lattice QCD. The ChPT predictions\textsuperscript{11}

\textsuperscript{11}Heavy baryon ChPT actually reproduces the Adler–Dothan–Wolfenstein formula [145, 146], cf. ref. [24].
| id | ref. | description |
|----|------|-------------|
| A  | [24] reanalysis of experimental data (year \(\leq 1999\)) |
| A1 | \(\nu\) scattering; various targets; world avg. year \(\leq 1990\) |
| A2 | \(\pi\) electroproduction; world avg. year \(\leq 1999\); H1ChPT corrected |
| A3 | various targets; RFG model; dipole ansatz |
| B  | [104] \(\nu\) scattering; reanalysis of ANL, BNL, FNAL, CERN, and IHEP data; B2 ANL data; dipole ansatz |
| B1 | BNL data; dipole ansatz |
| B2 | ANL data; dipole ansatz |
| B3 | FNAL data; dipole ansatz |
| B4 | combined analysis of BNL, ANL, and FNAL data; \(z\)-exp |
| C  | [12] reanalysis of \(\nu\) scattering data (from BNL, [105], ANL [106], FNAL [107]) |
| C1 | BNL data; dipole ansatz |
| C2 | ANL data; dipole ansatz |
| C3 | FNAL data; dipole ansatz |
| C4 | combined analysis of BNL, ANL, and FNAL data; \(z\)-exp |
| D  | [108] \(\nu\) scattering, K2K (SciFi); oxygen target; dipole ansatz |
| E  | [109] \(\nu\) scattering; MINOS; iron target; dipole ansatz |
| F  | [110] \(\nu\) scattering; MiniBooNE; carbon target; assuming RFG model; dipole ansatz |
| G  | [111] reanalysis of [110]; RFG model and spectral function model; dipole ansatz |
| H  | [100] reanalysis of MiniBooNE [110] and \(\pi\) electroproduction data |
| H1 | MiniBooNE [110] data; dipole ansatz |
| H2 | \(\pi\) electroproduction data (from refs. [112–116]); dipole ansatz |
| H3 | MiniBooNE [110] data; \(z\)-exp |
| H4 | \(\pi\) electroproduction data (from refs. [112–116]); \(z\)-exp |
| I  | [117] reanalysis of MiniBooNE data [110] |
| I1 | LFG model; dipole ansatz |
| I2 | LFG model + multi-nucleon reactions + RPA, etc., see [118] |

**Figure 15.** Compilation of results for the axial dipole mass \(M_A\) from experiment (A-I) and lattice simulations (J-Q). Extractions based on a dipole ansatz are colored red, while those using any variant of the \(z\)-expansion are colored blue. The dashed lines show the mean result of the \(1^2P\) (red) and the \(1^4P_3\) (blue) fit.  

**Symbols:** crosses: \(\nu\) scattering; circles: \(\pi\) electroproduction; line: not continuum extrapolated; dot: single ensemble; square: continuum extrapolated.  

**Abbreviations:** RFG: relativistic Fermi gas [119]; LFG: local Fermi gas; RPA: random phase approximation [120–122]; DWF: domain wall fermions; HISQ: highly improved staggered quarks; CE: continuum extrapolated.

are based on measurements of the axial radius and experimental data for \(g_{\pi NN}\). They persistantly call for a value slightly above 8. While older measurements of ordinary muon capture (OMC) were in agreement with this prediction (within large errors), the TRIUMF measurement [14, 15] lies significantly higher. It has to be seen as a success of BChPT that the new OMC measurement by MuCap [18, 19] is spot on with a small error. Independent of the choice of parametrization our results are in perfect agreement with both the ChPT prediction and the MuCap result. In particular recent lattice results that include a chiral and a continuum extrapolation using ensembles with close to physical pion masses have yielded much smaller values. In retrospect, it is clear that these findings were caused by
Figure 16. Compilation of data for the pseudoscalar coupling at the muon capture point $g_P^*$ from experiment (A-E), BChPT (F-I), and lattice simulations (J-S). Extractions based on a dipole ansatz are colored red, while those using any variant of the $z$-expansion are colored blue. Some lattice calculations use an EFT ansatz colored green (pion pole term combined with Taylor expansion). The dashed lines show the mean result of the $2\pi$ (red) and the $4\pi^3$ (blue) fit. The lattice results in parentheses are outdated, since they are strongly affected by the pion pole enhanced excited states treated in this article, cf. also the discussion in ref. [73].

**Symbols:** circle: radiative muon capture; triangle: ordinary muon capture; line: not continuum extrapolated; dot: single ensemble; square: continuum extrapolated.

**Abbreviations:** RMC: radiative muon capture; OMC: ordinary muon capture; HBChPT: heavy baryon ChPT; EOMS: extended on-mass-shell scheme; DWF: domain wall fermions; HISQ: highly improved staggered quarks; CE: continuum extrapolated.

The pion pole enhanced $N\pi$ excited state contribution, which was not fully under control. See also ref. [73], where the same conclusion has been drawn.

Results for the pion-nucleon coupling constant $g_{\pi NN}$ are composed in figure 17. The experimental results from $\pi N$ scattering, $NN$ scattering, and pionic atoms have reached a high precision, and in particular recent determinations are in quite good agreement with each other and with the ChPT analysis of refs. [140, 141]. The discussion is now centering on the understanding of charge and isospin breaking effects — a question that is out of reach of current lattice QCD analyses of nuclear structure, which usually ignore QED effects and use degenerate light quark masses. Also the experimental precision is not yet within reach.\(^\text{12}\) However, a comparison of the lattice values with the experimental

\(^\text{12}\)There are a number of indirect estimates based on the Goldberger–Treiman relation, see, e.g., refs. [30, 32, 40, 53]. While such estimates can have quite small statistical errors and may serve as consistency checks, they should not be considered as independent measurements of $g_{\pi NN}$.
### Table

| id | ref. | description |
|----|------|-------------|
| A  | [126] | $\pi N$ scattering; PWA |
| B  | [127–129] | np, pp scattering; PWA |
| C  | [130, 131] | $\pi N$ scattering; PWA |
| D  | [132] | $\pi N$ scattering; PWA; GMO |
| E  | [133] | np backward cross section |
| F  | [134] | $\pi N$ scattering; PWA; DR |
| G  | [135] | $\pi^- p$ and $\pi^- d$ pionic atoms; GMO |
| H  | [136] | $\pi^- p$ and $\pi^- d$ pionic atoms; GMO |
| I  | [137] | $\pi N$ scattering; DR; J1 CERN data |
| J  | [138] | np, pp scattering; PWA |
| K  | [139] | $\pi N$ scattering; PWA; DR |
| L  | [139] | np, pp scattering; PWA |
| M  | [140, 141] | ChPT analysis of $\pi N$, np scattering length, and GMO sum rule |
| N  | [32] | $N_f = 2 + 1$ DWF; RBC/UKQCD; $a = 0.114$ fm; dipole ansatz |
| O  | [52] | $N_f = 2 + 1$ Wilson (clover) fermions; $a = 0.114$ fm; z-exp |
| P  | [55] | $N_f = 2 + 1 + 1$ Wilson (clover-on-HISQ) fermions; PNDME; CE; EFT ansatz |
| Q  | This work | $N_f = 2 + 1$ Wilson (clover) fermions; RQCD; full resolution of $N\pi$ state; CE |
| Q1 | dipole ansatz |
| Q2 | z-exp |

### Figure 17

Compilation of data for the pion-nucleon coupling constant $g_{\pi NN}$ from experiment (A-L), ChPT (M), and lattice simulations (N-Q). We do not discriminate between charged and neutral pion-nucleon couplings here, which can be slightly different. In the lattice section we have only listed direct determinations, ignoring all results that are merely based on the Goldberger–Treiman relation [142]. Extractions based on a dipole ansatz are colored red, while those using any variant of the z-expansion are colored blue. Some lattice calculations use an EFT ansatz colored green (a pion pole term combined with a Taylor expansion). The dashed lines show the mean result of the $|2P$ (red) and the $|z^{4+3}$ (blue) fit. For a recent review, see ref. [143].

**Symbols:** circle: $N\pi$ scattering; triangle (up): $NN$ scattering; triangle (down): pionic atoms; line: not continuum extrapolated; dot: single ensemble; square: continuum extrapolated.

**Abbreviations:** PWA: partial wave analysis; GMO: Goldberger-Miyazawa-Oehme sum rule [144]; DWF: domain wall fermions; HISQ: highly improved staggered quarks; CE: continuum extrapolated.

Results and with the ChPT analysis of refs. [140, 141], which also includes an estimate of systematic uncertainties, can serve as a consistency check. It is thus quite encouraging that our results for $g_{\pi NN}$ from both, the $|2P$ and the $|z^{4+3}$ fit, are in agreement with these determinations. As one can see in table 6, a meaningful determination of the Goldberger–Treiman discrepancy $\Delta_{GT} = 1 - \frac{m_A}{f_{\pi} g_{\pi NN}}$ is not possible with our current accuracy.

### 5 Summary

In this article we have presented a method that can control pion pole enhanced excited state contributions that occur in axial and pseudoscalar channels, which previously have not been resolved to a satisfactory degree. The technique is based on similar EFT considerations as refs. [64–70, 78], but simultaneously minimizes the ChPT bias. The EFT analysis presented in section 2.2 is mainly used to understand the general structure of the pole enhanced $N\pi$ contribution, which then can be taken into account explicitly in the spectral decomposition.
of the three-point functions, see section 2.3. Our numerical analysis presented in section 3 demonstrates that, using our new technique, the ground state can be extracted reliably, even at small pion masses where the pole enhanced excited state constitutes (at currently available source-sink distances) the largest contribution in some channels.

We find that the extracted nucleon form factors satisfy constraints from PCAC up to small deviations of roughly 5%, which can be attributed to discretization effects. We find the PPD assumption to be fulfilled to the same degree. Note, however, that the pion pole dominance assumption for the pseudoscalar form factors is only a (seemingly very good) estimate and is not expected to be fulfilled exactly, even in the continuum. PCAC, however, has to hold exactly in the continuum. We leverage the latter information in our form factor analysis: in addition to the usual dipole ansatz and the \( z \)-expansion, we have derived (for both cases) parametrizations that are consistent with PCAC in the continuum, cf. section 4.2.3. The latter stabilize the continuum extrapolation considerably, without adding any parametrization bias.

Using a large set of available CLS ensembles, we are able to take all relevant limits (continuum limit, infinite volume limit, chiral extrapolation to physical masses) in a controlled fashion. To this end, we use generic extrapolation formulas (see section 4.2.4) for the parameters occurring in the form factor parametrization. The results at the physical point (in the continuum and for infinite volume) obtained from various form factor parametrizations are given in tables 4 and 5. For the presently available data we find that both dipole and \( z \)-expansion fits (in both cases we use that PCAC is exact in the continuum) yield a very good description of our data. The final numbers, including estimates for systematic uncertainties due to the chiral and the continuum extrapolation, can be taken from table 6. It is particularly interesting that our results for the axial mass are in agreement with the findings obtained from quasi-elastic neutrino nucleon scattering data (MiniBooNE, [100]) and that we find the same parametrization bias, i.e., larger values of the axial pole mass for dipole fits and smaller values for the \( z \)-expansion.

In figure 14, we plot the ratios \( r_{\text{PCAC}} \) and \( r_{\text{PPD}} \) at the physical point, where deviations from one correspond to a violation of PCAC and deviations from the PPD assumption, respectively. In particular the fits with exact PCAC in the continuum (i.e., \( r_{\text{PCAC}} = 1 \) automatically) allow us to draw conclusions with respect to the pion pole dominance ansatz for the pseudoscalar form factors. We find that our results are consistent with the PPD ansatz independent of the choice of parametrization for the form factor. The values we extract for the induced pseudoscalar coupling at the muon capture point are in good agreement with the experimental value obtained from muon capture [18, 19].

Acknowledgments

We are grateful to O. Bär, R. Gupta, and K.-F. Liu for fruitful discussions. The authors also would like to express their gratitude towards B. Gläße, P. Georg, D. Richtmann, and J. Simeth for various support. This work was supported by the Deutsche Forschungsgemeinschaft (collaborative research centre SFB/TRR-55), and the European Union’s Horizon 2020 Research and Innovation programme under the Marie Skłodowska-Curie grant.
agreement no. 813942 (ITN EuroPLEx).

We used a modified version of the CHROMA [147] software package along with the LIBHADRONANALYSIS library [148] and improved inverters [85, 149–151]. The configurations were generated as part of the CLS effort [83, 88] using openQCD (https://luscher.web.cern.ch/luscher/openQCD/) [85]. We thank all our CLS colleagues for the joint generation of the gauge ensembles. Additional \( m_\ell = m_s \) ensembles were generated with openQCD by members of the Mainz group on the Wilson and Clover HPC Clusters of IKP Mainz as well as by RQCD on the QPACE computer using the BQCD code [84].

The computation of observables was carried out on the QPACE 2 and QPACE 3 systems of the SFB/TRR-55, on the Regensburg QPACE B machine, the Regensburg HPC-cluster ATHENE 2, and at various supercomputer centers. In particular, the authors gratefully acknowledge computing time granted by the John von Neumann Institute for Computing (NIC), provided on the Booster partition of the supercomputer JURECA [152] at Jülich Supercomputing Centre (JSC, http://www.fz-juelich.de/ias/jsc/).

Regarding the generation of recent gauge ensembles, the authors gratefully acknowledge the Gauss Centre for Supercomputing (GCS) for providing computing time for GCS Large-Scale Projects on the GCS share of the two supercomputers JUQUEEN [153] and JUWELS [154] at JSC as well as on SuperMUC at Leibniz Supercomputing Centre (LRZ, https://www.lrz.de). GCS is the alliance of the three national supercomputing centres HLRS (Universität Stuttgart), JSC (Forschungszentrum Jülich), and LRZ (Bayerische Akademie der Wissenschaften), funded by the German Federal Ministry of Education and Research (BMBF) and the German State Ministries for Research of Baden-Württemberg (MWK), Bayern (StMWFK) and Nordrhein-Westfalen (MIWF).

A Traces

For the ground state contributions defined in eq. (2.43) one finds

\[
B_{P^\mu P^\nu}^{P^\mu P^\nu} = 2G_A \left( p^\mu p^\mu + p^i p^i + m(p^i + p^i) \gamma^0 - g^i \left( m^2 + mE + mE + p \cdot p \right) \right) 
+ 2G_P \frac{q^0}{2m} \left( (m + E^i) p^i - (m + E) p^0 \right), \tag{A.1}
\]

\[
B_{P^\mu P^\nu}^{P^\mu P^\nu} = 2G_P \left( (m + E^i) p^i - (m + E) p^0 \right). \tag{A.2}
\]

For the remaining traces that are needed for the determination of our final parametrizations given in section 2.3 one gets

\[
\text{Tr} \{ P^i (\bar{p} + m) \gamma^5 (\bar{p} + m) \} = 4(p^i (mE + p \cdot r) - mr^i (m + E)), \tag{A.3}
\]

\[
\text{Tr} \{ P^i (\bar{p} + m) \gamma^5 (\bar{p} + m) \} = 4(p^i (mE + p \cdot r) - mr^i (m + E')), \tag{A.4}
\]

\[
\text{Tr} \{ P^i \gamma^5 (\bar{p} + m) \} = +2p^i, \tag{A.5}
\]

\[
\text{Tr} \{ P^i (\bar{p} + m) \gamma_5 \} = -2p^i. \tag{A.6}
\]
B  Fit ansatz for the subtracted currents

For the subtracted correlation functions defined in ref. [60] one inserts

\[ A^\mu_\perp = \left( g^{\mu\nu} - \frac{\vec{p}' \cdot \vec{p}}{p^2} \right) A^\nu, \quad P_\perp = P - \frac{1}{2i m_\ell} \frac{\vec{p}' \cdot \vec{p}}{p^2} \partial_\mu A^\mu, \]  

(B.1)

instead of the usual currents. Here \( \vec{p} = (p' + p)/2 \). By construction, this does not change the ground state contribution at all. In contrast, the excited state contributions are affected very strongly. Therefore, the fit ansatz given in eqs. (2.44) and (2.45) has to be adapted to this case. Following the same steps as discussed in detail for the standard currents in section 2.2, we find

\[ C_{3pt.P_\perp}^p A^\mu_\perp = + \sqrt{\frac{Z}{Z'}} \left( e^{-E'(t-\tau)} e^{-E_{\tau}} \right) \]

\[ \times \left[ D_{P_\perp,3pt}^p A^\mu \left( 1 + B_{10} e^{-\Delta E'(t-\tau)} + B_{01} e^{-\Delta E_{\tau}} + B_{11} e^{-\Delta E'(t-\tau)} e^{-\Delta E_{\tau}} \right) \right. \]

\[ + e^{-\Delta E_{\pi}(t-\tau)} \frac{E'}{E_{\pi}} \left( \gamma^\mu - p^\mu \frac{\vec{p} \cdot r_\perp}{p^2} \right) \left( c' p^i + d' q^i \right) \]

\[ \left. + e^{-\Delta E_{\pi}(t-\tau)} \frac{E}{E_{\pi}} \left( \gamma^\mu - p^\mu \frac{\vec{p} \cdot r_\perp}{p^2} \right) \left( c p^i + d q^i \right) \right], \quad (B.2) \]

\[ C_{3pt.P_\perp}^p P_\perp = + \sqrt{\frac{Z}{Z'}} e^{-E'(t-\tau)} e^{-E_{\tau}} \]

\[ \times \left[ D_{P_\perp,3pt}^p \left( 1 + B_{10} e^{-\Delta E'(t-\tau)} + B_{01} e^{-\Delta E_{\tau}} + B_{11} e^{-\Delta E'(t-\tau)} e^{-\Delta E_{\tau}} \right) \right. \]

\[ + e^{-\Delta E_{\pi}(t-\tau)} \frac{1}{E_{\pi}} \left( m_{\pi}^2 - \frac{\vec{p} \cdot r_\perp}{p^2} \right) \left( c' p^i + d' q^i \right) \]

\[ - e^{-\Delta E_{\pi}(t-\tau)} \frac{1}{E_{\pi}} \left( m_{\pi}^2 - \frac{\vec{p} \cdot r_\perp}{p^2} \right) \left( c p^i + d q^i \right) \right]. \quad (B.3) \]

Similar to the situation with unsubtracted correlation functions, the parametrization simplifies for the particular kinematics we are using in our numerical analysis (\( \vec{p}' = 0 \) such that \( \vec{q} = -\vec{p} \)).

References

[1] UCNA collaboration, M. A.-P. Brown et al., New result for the neutron \( \beta \)-asymmetry parameter \( A_0 \) from UCNA, Phys. Rev. C97 (2018) 035505 [1712.00884].

[2] B. Märkisch et al., Measurement of the Weak Axial-Vector Coupling Constant in the Decay of Free Neutrons Using a Pulsed Cold Neutron Beam, Phys. Rev. Lett. 122 (2019) 242501 [1812.04666].

[3] M. González-Alonso, O. Naviliat-Cuncic and N. Severijns, New physics searches in nuclear and neutron \( \beta \) decay, Prog. Part. Nucl. Phys. 104 (2019) 165 [1803.08732].

[4] L. Hayen and N. Severijns, Radiative corrections to Gamow-Teller decays, 1906.09870.

[5] S. Choi et al., Axial and Pseudoscalar Nucleon Form Factors from Low Energy Pion Electroproduction, Phys. Rev. Lett. 71 (1993) 3927.
[6] V. Bernard, U.-G. Meißner and N. Kaiser, "Comment on ‘Axial and Pseudoscalar Nucleon Form Factors from Low Energy Pion Electroproduction’", Phys. Rev. Lett. 72 (1994) 2810.

[7] A. Liesenfeld et al., "A measurement of the axial form factor of the nucleon by the \( p(e,e'\pi^+)n \) reaction at \( W = 1125 \text{ MeV} \), Phys. Lett. B468 (1999) 20 [nucl-ex/9911003].

[8] T. Fuchs and S. Scherer, "Pion electroproduction, partially conserved axial-vector current, chiral Ward identities, and the axial form factor revisited", Phys. Rev. C68 (2003) 055501 [nucl-th/0303002].

[9] L. A. Ahrens et al., "A study of the axial-vector form factor and second-class currents in antineutrino quasielastic scattering", Phys. Lett. B202 (1988) 284.

[10] T. Kitagaki et al., "Study of \( \nu d \rightarrow \mu^- p p_s \) and \( \nu d \rightarrow \mu^- \Delta^{++}(1232) n_s \) using the BNL 7-foot deuterium-filled bubble chamber", Phys. Rev. D42 (1990) 1331.

[11] A. Bodek, S. Avvakumov, R. Bradford and H. Budd, "Extraction of the Axial Nucleon Form Factor from Neutrino Experiments on Deuterium", J. Phys. Conf. Ser. 110 (2008) 082004 [0709.3538].

[12] A. S. Meyer, M. Betancourt, R. Gran and R. J. Hill, "Deuterium target data for precision neutrino-nucleus cross sections", Phys. Rev. D93 (2016) 113015 [1603.03048].

[13] R. D. Hart, C. R. Cox, G. W. Dodson, M. Eckhause, J. R. Kane, M. S. Pandey, A. M. Rushton, R. T. Siegel and R. E. Welsh, "Radiative Muon Capture in Calcium", Phys. Rev. Lett. 39 (1977) 399.

[14] G. Jonkmans et al., "Radiative Muon Capture on Hydrogen and the Induced Pseudoscalar Coupling", Phys. Rev. Lett. 77 (1996) 4512 [nucl-ex/9608005].

[15] D. H. Wright et al., "Measurement of the induced pseudoscalar coupling using radiative muon capture on hydrogen", Phys. Rev. C57 (1998) 373.

[16] T. Gorringe and H. W. Fearing, "Induced pseudoscalar coupling of the proton weak interaction", Rev. Mod. Phys. 76 (2004) 31 [nucl-th/0206039].

[17] G. Bardin, J. Duclos, A. Magnon, J. Martino, A. Richter, E. Zavattini, A. Bertin, M. Piccinini, A. Vitale and D. F. Measday, "A novel measurement of the muon capture rate in liquid hydrogen by the lifetime technique", Nucl. Phys. A352 (1981) 365.

[18] MuCap collaboration, V. A. Andreev et al., "Measurement of Muon Capture on the Proton to 1% Precision and Determination of the Pseudoscalar Coupling gp", Phys. Rev. Lett. 110 (2013) 012504 [1210.6545].

[19] MuCap collaboration, V. A. Andreev et al., "Measurement of the formation rate of muonic hydrogen molecules", Phys. Rev. C91 (2015) 055502 [1502.00913].

[20] P. Winter, "Muon capture on the proton", AIP Conf. Proc. 1441 (2012) 537 [1110.5090].

[21] R. J. Hill, P. Kammel, W. J. Marciano and A. Sirlin, "Nucleon axial radius and muonic hydrogen — a new analysis and review", Rep. Prog. Phys. 81 (2018) 096301 [1708.08462].

[22] V. Bernard, N. Kaiser and U.-G. Meißner, "QCD accurately predicts the induced pseudoscalar coupling constant", Phys. Rev. D50 (1994) 6899 [hep-ph/9403351].

[23] H. W. Fearing, R. Lewis, N. Mobed and S. Scherer, "Muon capture by a proton in heavy baryon chiral perturbation theory", Phys. Rev. D56 (1997) 1783 [hep-ph/9702394].

[24] V. Bernard, L. Elouadrhiri and U.-G. Meißner, "Axial structure of the nucleon", J. Phys. G28 (2002) R1 [hep-ph/0107088].
[25] M. R. Schindler, T. Fuchs, J. Gegelia and S. Scherer, \textit{Axial, induced pseudoscalar, and pion-nucleon form factors in manifestly Lorentz-invariant chiral perturbation theory}, \textit{Phys. Rev.} \textbf{C75} (2007) 025202 [nucl-th/0611083].

[26] V. M. Braun, A. Lenz and M. Wittmann, \textit{Nucleon form factors in QCD}, \textit{Phys. Rev.} \textbf{D73} (2006) 094019 [hep-ph/0604050].

[27] I. V. Anikin, V. M. Braun and N. Offen, \textit{Axial form factor of the nucleon at large momentum transfers}, \textit{Phys. Rev.} \textbf{D94} (2016) 034011 [1607.01504].

[28] G. Eichmann and C. S. Fischer, \textit{Nucleon axial and pseudoscalar form factors from the covariant Faddeev equation}, \textit{Eur. Phys. J.} \textbf{A48} (2012) 9 [1111.2614].

[29] G. Martinelli and C. T. Sachrajda, \textit{A lattice study of nucleon structure}, \textit{Nucl. Phys.} \textbf{B316} (1989) 355.

[30] H.-W. Lin, T. Blum, S. Ohta, S. Sasaki and T. Yamazaki, \textit{Nucleon structure with two flavors of dynamical domain-wall fermions}, \textit{Phys. Rev.} \textbf{D78} (2008) 014505 [0802.0863].

[31] RBC/UKQCD collaboration, T. Yamazaki, Y. Aoki, T. Blum, H.-W. Lin, M. F. Lin, S. Ohta, S. Sasaki, R. J. Tweedie and J. M. Zanotti, \textit{Nucleon Axial Charge in (2+1)-Flavor Dynamical-Lattice QCD with Domain-Wall Fermions}, \textit{Phys. Rev. Lett.} \textbf{100} (2008) 171602 [0801.4016].

[32] T. Yamazaki, Y. Aoki, T. Blum, H.-W. Lin, S. Ohta, S. Sasaki, R. Tweedie and J. Zanotti, \textit{Nucleon form factors with 2+1 flavor dynamical domain-wall fermions}, \textit{Phys. Rev.} \textbf{D79} (2009) 114505 [0904.2039].

[33] LHPC collaboration, J. D. Bratt et al., \textit{Nucleon structure from mixed action calculations using 2+1 flavors of asqtad sea and domain wall valence fermions}, \textit{Phys. Rev.} \textbf{D82} (2010) 094502 [1001.3620].

[34] C. Alexandrou, M. Brinet, J. Carbonell, M. Constantinou, P. A. Harraud, P. Guichon, K. Jansen, T. Korzec and M. Papinutto, \textit{Axial nucleon form factors from lattice QCD}, \textit{Phys. Rev.} \textbf{D83} (2011) 045010 [1012.0857].

[35] S. Capitani, M. Della Morte, G. von Hippel, B. Jäger, A. Jüttner, B. Knippschild, H. B. Meyer and H. Wittig, \textit{Nucleon axial charge from lattice QCD with controlled errors}, \textit{Phys. Rev.} \textbf{D86} (2012) 074502 [1205.0180].

[36] J. R. Green, M. Engelhardt, S. Krieg, J. W. Negele, A. V. Pochinsky and S. N. Syritsyn, \textit{Nucleon structure from Lattice QCD using a nearly physical pion mass}, \textit{Phys. Lett.} \textbf{B734} (2014) 290 [1209.1687].

[37] R. Horsley, Y. Nakamura, A. Nobile, P. E. L. Rakow, G. Schierholz and J. M. Zanotti, \textit{Nucleon axial charge and pion decay constant from two-flavor lattice QCD}, \textit{Phys. Lett.} \textbf{B732} (2014) 41 [1302.2233].

[38] PNDME collaboration, T. Bhattacharya, S. D. Cohen, R. Gupta, A. Joseph, H.-W. Lin and B. Yoon, \textit{Nucleon charges and electromagnetic form factors from 2+1+1-flavor lattice QCD}, \textit{Phys. Rev.} \textbf{D89} (2014) 094502 [1306.5435].

[39] CSSM/QCDSF/UKQCD collaboration, A. J. Chambers et al., \textit{Feynman-Hellmann approach to the spin structure of hadrons}, \textit{Phys. Rev.} \textbf{D90} (2014) 014510 [1405.3019].

[40] G. S. Bali, S. Collins, B. Gläfle, M. Göckeler, J. Najjar, R. H. Rödl, A. Schäfer, R. W. Schiel, W. Söldner and A. Sternbeck, \textit{Nucleon isovector couplings from N_{f} = 2 lattice QCD}, \textit{Phys. Rev.} \textbf{D91} (2015) 054501 [1412.7336].
[41] G. von Hippel, T. D. Rae, E. Shintani and H. Wittig, *Nucleon matrix elements from lattice QCD with all-mode-averaging and a domain-decomposed solver: An exploratory study*, Nucl. Phys. **B914** (2017) 138 [1605.00564].

[42] PNDME collaboration, T. Bhattacharya, V. Cirigliano, S. D. Cohen, R. Gupta, H.-W. Lin and B. Yoon, *Axial, scalar, and tensor charges of the nucleon from 2+1+1-flavor Lattice QCD*, Phys. Rev. **D94** (2016) 054508 [1606.07049].

[43] A. S. Meyer, R. J. Hill, A. S. Kronfeld, R. Li and J. N. Simone, *Calculation of the Nucleon Axial Form Factor Using Staggered Lattice QCD*, PoS **LATTICE2016** (2017) 179 [1610.04593].

[44] B. Yoon et al., *Isovector charges of the nucleon from 2+1-flavor QCD with clover fermions*, Phys. Rev. **D95** (2017) 074508 [1611.07452].

[45] J. Liang, Y.-B. Yang, K.-F. Liu, A. Alexandru, T. Draper and R. S. Sufian, *Lattice Calculation of Nucleon Isovector Axial Charge with Improved Currents*, Phys. Rev. **D96** (2017) 034519 [1612.04388].

[46] C. Bouchard, C. C. Chang, T. Kurth, K. Orginos and A. Walker-Loud, *On the Feynman-Hellmann theorem in quantum field theory and the calculation of matrix elements*, Phys. Rev. **D96** (2017) 014504 [1612.06963].

[47] C. Alexandrou, M. Constantinou, K. Hadjiyiannakou, K. Jansen, C. Kallidonis, G. Koutsou, K. Ottnad and A. Vaquero, *Nucleon electromagnetic and axial form factors with \( N_f = 2 \) twisted mass fermions at the physical point*, PoS **LATTICE2016** (2017) 154 [1702.00984].

[48] E. Berkowitz et al., *An accurate calculation of the nucleon axial charge with lattice QCD*, 1704.01114.

[49] D.-L. Yao, L. Alvarez-Ruso and M. J. Vicente-Vacas, *Extraction of nucleon axial charge and radius from lattice QCD results using baryon chiral perturbation theory*, Phys. Rev. **D96** (2017) 116022 [1708.08776].

[50] C. Chang et al., *A per-cent-level determination of the nucleon axial coupling from quantum chromodynamics*, Nature **558** (2018) 91 [1805.12130].

[51] C. Alexandrou, M. Constantinou, K. Hadjiyiannakou, K. Jansen, C. Kallidonis, G. Koutsou and A. Vaquero Avilés-Casco, *Connected and disconnected contributions to nucleon axial form factors using \( N_f = 2 \) twisted mass fermions at the physical point*, EPJ Web Conf. **175** (2018) 06003 [1807.11203].

[52] J. Green, N. Hasan, S. Meinel, M. Engelhardt, S. Krieg, J. Laechli, J. Negele, K. Orginos, A. Pochinsky and S. Syritsyn, *Up, down, and strange nucleon axial form factors from lattice QCD*, Phys. Rev. **D95** (2017) 114502 [1703.06703].

[53] C. Alexandrou, M. Constantinou, K. Hadjiyiannakou, K. Jansen, C. Kallidonis, G. Koutsou and A. Vaquero Avilés-Casco, *Nucleon axial form factors using \( N_f = 2 \) twisted mass fermions with a physical value of the pion mass*, Phys. Rev. **D96** (2017) 054507 [1705.03399].

[54] S. Capitani, M. Della Morte, D. Djukanovic, G. M. von Hippel, J. Hua, B. Jäger, P. M. Jumnikar, H. B. Meyer, T. D. Rae and H. Wittig, *Iso-vector axial form factors of the nucleon in two-flavour lattice QCD*, Int. J. Mod. Phys. **A34** (2019) 1950009 [1705.06186].

[55] PNDME collaboration, R. Gupta, Y.-C. Jang, H.-W. Lin, B. Yoon and T. Bhattacharya, *Axial-vector form factors of the nucleon from lattice QCD*, Phys. Rev. **D96** (2017) 114503
1705.06834).

[56] PACS collaboration, N. Tsukamoto, K.-I. Ishikawa, Y. Kuramashi, S. Sasaki and T. Yamazaki, Nucleon structure from 2+1 flavor lattice QCD near the physical point, *EPJ Web Conf.* **175** (2018) 06007 [1710.10782].

[57] Y.-C. Jang, T. Bhattacharya, R. Gupta, H.-W. Lin and B. Yoon, Nucleon Axial and Electromagnetic Form Factors, *EPJ Web Conf.* **175** (2018) 06033 [1801.01635].

[58] PACS collaboration, K.-I. Ishikawa, Y. Kuramashi, S. Sasaki, N. Tsukamoto, A. Ukawa and T. Yamazaki, Nucleon form factors on a large volume lattice near the physical point in 2+1 flavor QCD, *Phys. Rev.* **D98** (2018) 074510 [1807.03974].

[59] χQCD collaboration, J. Liang, Y.-B. Yang, T. Draper, M. Gong and K.-F. Liu, Quark spins and anomalous Ward identity, *Phys. Rev.* **D98** (2018) 074505 [1806.08366].

[60] G. S. Bali, S. Collins, M. Gruber, A. Schäfer, P. Wein and T. Wurm, Solving the PCAC puzzle for nucleon axial and pseudoscalar form factors, *Phys. Lett.* **B789** (2019) 666 [1810.05569].

[61] E. Shintani, K.-I. Ishikawa, Y. Kuramashi, S. Sasaki and T. Yamazaki, Nucleon form factors and root-mean-square radii on a (10.8 fm)$^4$ lattice at the physical point, *Phys. Rev.* **D99** (2019) 014510 [1811.07292].

[62] PNDME collaboration, Y.-C. Jang, T. Bhattacharya, R. Gupta, H.-W. Lin and B. Yoon, Updates on Nucleon Form Factors from Clover-on-HISQ Lattice Formulation, *PoS LATTICE2018* (2019) 123 [1901.00060].

[63] J. R. Green, M. Engelhardt, N. Hasan, S. Krieg, S. Meinel, J. W. Negele, A. V. Pochinsky and S. N. Syritsyn, Excited-state effects in nucleon structure on the lattice using hybrid interpolators, *Phys. Rev.* **D100** (2019) 074510 [1907.11950].

[64] O. Bär, Nucleon-pion-state contamination in lattice calculations of the axial form factors of the nucleon, *PoS LATTICE2018* (2019) 105 [1808.08738].

[65] O. Bär, $N\pi$-state contamination in lattice calculations of the nucleon axial form factors, *Phys. Rev.* **D99** (2019) 054506 [1812.09191].

[66] O. Bär, $N\pi$-state contamination in lattice calculations of the nucleon pseudoscalar form factor, *Phys. Rev.* **D100** (2019) 054507 [1906.03652].

[67] O. Bär, Chiral perturbation theory and nucleon-pion-state contaminations in lattice QCD, *Int. J. Mod. Phys.* **A32** (2017) 1730011 [1705.02806].

[68] O. Bär, Multi-hadron-state contamination in nucleon observables from chiral perturbation theory, *EPJ Web Conf.* **175** (2018) 01007 [1708.00380].

[69] B. C. Tiburzi, Chiral corrections to nucleon two- and three-point correlation functions, *Phys. Rev.* **D91** (2015) 094510 [1503.06329].

[70] B. C. Tiburzi, Excited-state contamination in nucleon correlators from chiral perturbation theory, *PoS CD15* (2016) 087 [1508.00163].

[71] P. Wein, P. C. Bruns, T. R. Hemmert and A. Schäfer, Chiral extrapolation of nucleon wave function normalization constants, *Eur. Phys. J.* **A47** (2011) 149 [1106.3440].

[72] O. Bär, $N\pi$-excited state contamination in nucleon 3-point functions using ChPT, *PoS LATTICE 2019* (2019) 090 [1907.03284].
[73] Y.-C. Jang, R. Gupta, B. Yoon and T. Bhattacharya, Axial vector form factors from lattice QCD that satisfy the PCAC relation, 1905.06470.

[74] S. G"usken, A study of smearing techniques for hadron correlation functions, Nucl. Phys. B (Proc. Suppl.) 17 (1990) 361.

[75] M. Falcioni, M. L. Paciello, G. Parisi and B. Taglienti, Again on SU(3) glueball mass, Nucl. Phys. B251 (1985) 624.

[76] UKQCD collaboration, K. C. Bowler, R. D. Kenway, L. Lellouch, J. Nieves, O. Oliveira, D. G. Richards, C. T. Sachrajda, N. Stella and P. Ueberholz, First lattice study of semileptonic decays of $\Lambda_b$ and $\Xi_b$ baryons, Phys. Rev. D57 (1998) 6948 [hep-lat/9709028].

[77] F. M. Stokes, W. Kamleh and D. B. Leinweber, Opposite-parity contaminations in lattice nucleon form factors, Phys. Rev. D99 (2019) 074506 [1809.11002].

[78] O. B"ar, Nucleon-pion-state contribution to nucleon two-point correlation functions, Phys. Rev. D92 (2015) 074504 [1503.03649].

[79] H. B. Meyer, K. Ottnad and T. Schulz, A new method for suppressing excited-state contaminations on the nucleon form factors, PoS LATTICE2018 (2019) 062 [1811.03360].

[80] M. T. Hansen and H. B. Meyer, On the effect of excited states in lattice calculations of the nucleon axial charge, Nucl. Phys. B923 (2017) 558 [1610.03843].

[81] S. Scherer and M. R. Schindler, A Primer for Chiral Perturbation Theory, Lect. Notes Phys. 830 (2012) 1.

[82] Y.-C. Jang, R. Gupta, H.-W. Lin, B. Yoon and T. Bhattacharya, Nucleon Electromagnetic Form Factors in the Continuum Limit from 2+1+1-flavor Lattice QCD, 1906.07217.

[83] M. Bruno et al., Simulation of QCD with $N_f = 2 + 1$ flavors of non-perturbatively improved Wilson fermions, JHEP 02 (2015) 043 [1411.3982].

[84] Y. Nakamura and H. St"uben, BQCD - Berlin quantum chromodynamics program, PoS Lattice 2010 (2011) 040 [1011.0199].

[85] M. L"uscher and S. Schaefer, Lattice QCD with open boundary conditions and twisted-mass reweighting, Comput. Phys. Commun. 184 (2013) 519 [1206.2809].

[86] M. L"uscher and S. Schaefer, Lattice QCD without topology barriers, JHEP 07 (2011) 036 [1105.4749].

[87] ALPHA collaboration, M. Bruno et al., QCD Coupling from a Nonperturbative Determination of the Three-Flavor $\Lambda$ Parameter, Phys. Rev. Lett. 119 (2017) 102001 [1706.03821].

[88] RQCD collaboration, G. S. Bali, E. E. Scholz, J. Simeth and W. Söldner, Lattice simulations with $N_f = 2 + 1$ improved Wilson fermions at a fixed strange quark mass, Phys. Rev. D94 (2016) 074501 [1606.09039].

[89] M. Dalla Brida, T. Korzec, S. Sint and P. Vilaseca, High precision renormalization of the flavour non-singlet Noether currents in lattice QCD with Wilson quarks, Eur. Phys. J. C79 (2019) 23 [1808.09236].

[90] S. Sint, The chirally rotated Schrödinger functional with Wilson fermions and automatic O(a) improvement, Nucl. Phys. B847 (2011) 491 [1008.4857].
[91] P. Korcyl and G. S. Bali, Nonperturbative determination of improvement coefficients using coordinate space correlators in $N_f = 2 + 1$ lattice QCD, *Phys. Rev.* **D95** (2017) 014505 [1607.07090].

[92] G. S. Bali, K. G. Chetyrkin, P. Korcyl and J. Simeth, in preparation, .

[93] G. S. Bali et al., in preparation, .

[94] K. Symanzik, *Continuum Limit and Improved Action in Lattice Theories. 1. Principles and $\phi^4$ theory*, *Nucl. Phys.* **B226** (1983) 187.

[95] K. Symanzik, *Continuum Limit and Improved Action in Lattice Theories. 2. O(N) Non-linear sigma model in perturbation theory*, *Nucl. Phys.* **B226** (1983) 205.

[96] ALPHA collaboration, J. Bulava, M. Della Morte, J. Heitger and C. Wittemeier, *Non-perturbative improvement of the axial current in $N_f = 3$ lattice QCD with Wilson fermions and tree-level improved gauge action*, *Nucl. Phys.* **B896** (2015) 555 [1502.04999].

[97] L. Maiani, G. Martinelli, M. L. Paciello and B. Taglienti, *Scalar Densities and Baryon Mass Differences in Lattice QCD With Wilson Fermions*, *Nucl. Phys.* **B293** (1987) 420.

[98] C. Alabiso and G. Schierholz, *Asymptotic behavior of form factors for two- and three-body bound states. II. Spin-$\frac{1}{2}$ constituents*, *Phys. Rev.* **D11** (1975) 1905.

[99] R. J. Hill and G. Paz, *Model-independent extraction of the proton charge radius from electron scattering*, *Phys. Rev.* **D82** (2010) 113005 [1008.4619].

[100] B. Bhattacharya, R. J. Hill and G. Paz, *Model-independent determination of the axial mass parameter in quasielastic neutrino-nucleon scattering*, *Phys. Rev.* **D84** (2011) 073006 [1108.0423].

[101] S. R. Beane and M. J. Savage, *Baryon axial charge in a finite volume*, *Phys. Rev.* **D70** (2004) 074029 [hep-ph/0404131].

[102] A. A. Khan et al., *Axial coupling constant of the nucleon for two flavors of dynamical quarks in finite and infinite volume*, *Phys. Rev.* **D74** (2006) 094508 [hep-lat/0603028].

[103] PARTICLE DATA GROUP collaboration, M. Tanabashi et al., *Review of Particle Physics*, *Phys. Rev.* **D98** (2018) 030001.

[104] K. S. Kuzmin, V. V. Lyubushkin and V. A. Naumov, *Quasielastic axial-vector mass from experiments on neutrino-nucleus scattering*, *Eur. Phys. J.* **C54** (2008) 517 [0712.4384].

[105] N. J. Baker, A. M. Cnops, P. L. Connolly, S. A. Kahn, H. G. Kirk, M. J. Murtagh, R. B. Palmer, N. P. Samios and M. Tanaka, *Quasielastic neutrino scattering: A measurement of the weak nucleon axial-vector form factor*, *Phys. Rev.* **D23** (1981) 2409.

[106] K. L. Miller et al., *Study of the reaction $\nu\mu d \rightarrow \mu^- pp_s$*, *Phys. Rev.* **D26** (1982) 537.

[107] T. Kitagaki et al., *High-energy quasielastic $\nu_\mu n \rightarrow \mu^- p$ scattering in deuterium*, *Phys. Rev.* **D28** (1983) 436.

[108] K2K collaboration, R. Gran et al., *Measurement of the quasielastic axial vector mass in neutrino interactions on oxygen*, *Phys. Rev.* **D74** (2006) 052002 [hep-ex/0603034].

[109] MINOS collaboration, P. Adamson et al., *Study of quasielastic scattering using charged-current $\nu_\mu$-iron interactions in the MINOS near detector*, *Phys. Rev.* **D91** (2015) 012005 [1410.8613].

[110] MINIBOONE collaboration, A. A. Aguilar-Arevalo et al., *First measurement of the muon
neutrino charged current quasielastic double differential cross section, Phys. Rev. D81 (2010) 092005 [1002.2680].

[111] C. Juszczak, J. T. Sobczyk and J. Żmuda, Extraction of the axial mass parameter from MiniBooNE neutrino quasielastic double differential cross-section data, Phys. Rev. C82 (2010) 045502 [1007.2195].

[112] E. Amaldi, M. Benevantano, B. Borgia, F. De Notaristefani, A. Frondaroli, P. Pistilli, I. Sestili and M. Severi, Axial-vector form-factor of the nucleon from a coincidence experiment on electroproduction at threshold, Phys. Lett. 41B (1972) 216.

[113] P. Brauel et al., $\pi^+$ electroproduction on hydrogen near threshold at four-momentum transfers of 0.2, 0.4 and 0.6 GeV$^2$, Phys. Lett. 45B (1973) 389.

[114] A. Del Guerra, A. Giazotto, M. A. Giorgi, A. Stefanini, D. R. Botterill, D. W. Braben, D. Clarke and P. R. Norton, Measurements of threshold $\pi^+$ electroproduction at low momentum transfer, Nucl. Phys. B99 (1975) 253.

[115] A. Del Guerra, A. Giazotto, M. A. Giorgi, A. Stefanini, D. R. Botterill, H. E. Montgomery, P. R. Norton and G. Matone, Threshold $\pi^+$ electroproduction at high-momentum transfer: a determination of the nucleon axial vector form factor, Nucl. Phys. B107 (1976) 65.

[116] A. S. Esaulov, A. M. Pilipenko and Yu. I. Titov, Longitudinal and transverse contributions to the threshold cross-section slope of single-pion electroproduction by a proton, Nucl. Phys. B136 (1978) 511.

[117] J. Nieves, I. Ruiz Simo and M. J. Vicente Vacas, The nucleon axial mass and the MiniBooNE quasielastic neutrino-nucleus scattering problem, Phys. Lett. B707 (2012) 72 [1106.5374].

[118] J. Nieves, I. Ruiz Simo and M. J. Vicente Vacas, Inclusive charged-current neutrino-nucleus reactions, Phys. Rev. C83 (2011) 045501 [1102.2777].

[119] R. A. Smith and E. J. Moniz, Neutrino reactions on nuclear targets, Nucl. Phys. B43 (1972) 605 [Erratum: Nucl. Phys. B101 (1975) 547].

[120] D. Bohm and D. Pines, A Collective Description of Electron Interactions: 1. Magnetic Interactions, Phys. Rev. 82 (1951) 625.

[121] D. Pines and D. Bohm, A Collective Description of Electron Interactions: 2. Collective vs Individual Particle Aspects of the Interactions, Phys. Rev. 85 (1952) 338.

[122] D. Bohm and D. Pines, A Collective Description of Electron Interactions: 3. Coulomb Interactions in a Degenerate Electron Gas, Phys. Rev. 92 (1953) 609.

[123] H.-Ch. Schröder et al., The pion–nucleon scattering lengths from pionic hydrogen and deuterium, Eur. Phys. J. C21 (2001) 473.

[124] A. V. Butkevich and D. Perevalov, Determination of the axial nucleon form factor from the MiniBooNE data, Phys. Rev. D89 (2014) 053014 [1311.3754].

[125] A. Bodek, H. S. Budd and M. E. Christy, Neutrino quasielastic scattering on nuclear targets: Parametrizing transverse enhancement (meson exchange currents), Eur. Phys. J. C71 (2011) 1726 [1106.0340].

[126] R. Koch and E. Pietarinen, Low-energy $\pi N$ partial wave analysis, Nucl. Phys. A336 (1980) 331.

[127] R. A. M. Klomp, V. G. J. Stoks and J. J. de Swart, Determination of the $NN\pi$ coupling
constants in NN partial wave analyses, *Phys. Rev.* **C44** (1991) R1258.

[128] V. Stoks, R. Timmermans and J. J. de Swart, Pion-nucleon coupling constant, *Phys. Rev.* **C47** (1993) 512 [nucl-th/9211007].

[129] J. J. de Swart, M. C. M. Rentmeester and R. G. E. Timmermans, The status of the pion-nucleon coupling constant, *PiN Newslett.* **13** (1997) 96 [nucl-th/9802084].

[130] R. A. Arndt, Z. Li, L. D. Roper and R. L. Workman, Pion-nucleon coupling constant, *Phys. Rev.* **D44** (1991) 289.

[131] R. A. Arndt, Z. Li, L. D. Roper, R. L. Workman and M. Ford, Pion-nucleon partial-wave analysis to 2 GeV, *Phys. Rev.* **D43** (1991) 2131.

[132] R. A. Arndt, R. L. Workman and M. M. Pavan, Pion-nucleon partial-wave analysis with fixed-\(t\) dispersion relation constraints, *Phys. Rev.* **C49** (1994) 2729.

[133] J. Rahm et al., np scattering measurements at 162 MeV and the \(\pi NN\) coupling constant, *Phys. Rev.* **C57** (1998) 1077.

[134] M. M. Pavan, R. A. Arndt, I. I. Strakovsky and R. L. Workman, Determination of the \(\pi NN\) Coupling Constant in the VPI/GWU \(\pi N \rightarrow \pi N\) Partial-Wave and Dispersion Relation Analysis, *PiN Newslett.* **15** (1999) 171 [nucl-th/9910040]; [Phys. Scripta **T87** (2000) 65].

[135] H.-Ch. Schröder et al., Determination of the \(\pi N\) scattering lengths from pionic hydrogen, *Phys. Lett.* **B469** (1999) 25.

[136] T. E. O. Ericson, B. Loiseau and A. W. Thomas, Determination of the pion-nucleon coupling constant and scattering lengths, *Phys. Rev.* **C66** (2002) 014005 [hep-ph/0009312].

[137] D. V. Bugg, The pion nucleon coupling constant, *Eur. Phys. J.* **C33** (2004) 505.

[138] R. A. Arndt, W. J. Briscoe, I. I. Strakovsky and R. L. Workman, Extended partial-wave analysis of \(\pi N\) scattering data, *Phys. Rev.* **C74** (2006) 045205 [nucl-th/0605082].

[139] R. Navarro Pérez, J. E. Amaro and E. Ruiz Arriola, Precise determination of charge-dependent pion-nucleon-nucleon coupling constants, *Phys. Rev.* **C95** (2017) 064001 [1606.00592].

[140] V. Baru, C. Hanhart, M. Hoferichter, B. Kubis, A. Nogga and D. R. Phillips, Precision calculation of the \(\pi - d\) scattering length and its impact on threshold \(\pi N\) scattering, *Phys. Lett.* **B694** (2011) 473 [1003.4444].

[141] V. Baru, C. Hanhart, M. Hoferichter, B. Kubis, A. Nogga and D. R. Phillips, Precision calculation of threshold \(\pi - d\) scattering, \(\pi N\) scattering lengths, and the GMO sum rule, *Nucl. Phys.* **A872** (2011) 69 [1107.5509].

[142] M. L. Goldberger and S. B. Treiman, Form Factors in \(\beta\) Decay and \(\mu\) Capture, *Phys. Rev.* **111** (1958) 354.

[143] E. Matsinos, A brief history of the pion-nucleon coupling constant, 1901.01204.

[144] M. L. Goldberger, H. Miyazawa and R. Oehme, Application of Dispersion Relations to Pion-Nucleon Scattering, *Phys. Rev.* **99** (1955) 986.

[145] S. L. Adler and Y. Dothan, Low-Energy Theorem for the Weak Axial-Vector Vertex, *Phys. Rev.* **151** (1966) 1267 [Erratum: *Phys. Rev.* **164** (1967) 2062].

[146] L. Wolfenstein, Weak interactions of pions and muons, *Conf. Proc.* **C6909081** (1969) 661.

[147] SciDAC/LHPC/UKQCD collaboration, R. G. Edwards and B. Joó, The Chroma software
system for lattice QCD, *Nucl. Phys. B (Proc. Suppl.)* **140** (2005) 832 [hep-lat/0409003].

[148] G. S. Bali, S. Collins, B. Glässlle, S. Heybrock, P. Korcyl, M. Löffler, R. Rödl and A. Schäfer, *Baryonic and mesonic 3-point functions with open spin indices*, *EPJ Web Conf.* **175** (2018) 06014 [1711.02384].

[149] A. Nobile, *Solving the Dirac equation on QPACE*, *PoS Lattice 2010* (2011) 034.

[150] A. Frommer, K. Kahl, S. Krieg, B. Leder and M. Rottmann, *Adaptive Aggregation-Based Domain Decomposition Multigrid for the Lattice Wilson-Dirac Operator*, *SIAM J. Sci. Comput.* **36** (2014) A1581 [1303.1377].

[151] S. Heybrock, M. Rottmann, P. Georg and T. Wettig, *Adaptive algebraic multigrid on SIMD architectures*, *PoS LATTICE 2015* (2016) 036 [1512.04506].

[152] Jülich Supercomputing Centre, *JURECA: Modular supercomputer at Jülich Supercomputing Centre*, *JLSRF* **4** (2018) A132.

[153] Jülich Supercomputing Centre, *JUQUEEN: IBM Blue Gene/Q® Supercomputer System at the Jülich Supercomputing Centre*, *JLSRF* **1** (2015) A1.

[154] Jülich Supercomputing Centre, *JUWELS: Modular Tier-0/1 Supercomputer at the Jülich Supercomputing Centre*, *JLSRF* **5** (2019) A135.