The Asymptotic Behaviour of the Sum of Negative Eigenvalues of a Self-Adjoint Operator Given in Semi-Axis

Özlem Bakși

Department of Mathematics,
Faculty of Arts and Science, Yıldız Technical University
(34210), Davutpaşa, İstanbul, Turkey

e-mail: baksi@yildiz.edu.tr

Abstract

In this work, we find the asymptotic formulas for the sum of the negative eigenvalues smaller than $-\varepsilon$ ($\varepsilon > 0$) of a self-adjoint operator $L$ which is defined by the following differential expression

$$\ell(y) = -(p(x)y'(x))' - Q(x)y(x)$$

with the boundary condition

$$y(0) = 0$$

in the space $L_2(0, \infty; H)$.

AMS Subj. Classification: 34B24, 47A10

Keywords: Self-adjoint operator, Sturm-Liouville operator, spectrum, negative eigenvalues, asymptotic behaviour.
1 Introduction

Let $H$ be an infinite dimensional separable Hilbert space. Let us consider the operator $L$ in the Hilbert space $L^2(0, \infty; H)$ defined by the differential equation

$$\ell(y) = -(p(x)y'(x))' - Q(x)y(x)$$  \hspace{1cm} (1)

and with the boundary condition $y(0) = 0$.

Let us assume the scalar function $p(x)$ and the operator function $Q(x)$ satisfy the following conditions:

p1) For every $x \in [0, \infty)$, there are positive constants $c_1, c_2$ such that

$$c_1 \leq p(x) \leq c_2.$$

p2) The function $p(x)$ has continuous and bounded derivative.

p3) The function $p(x)$ is not decreasing in the interval $[0, \infty)$.

Q1) For every $x \in [0, \infty)$ the operator $Q(x) : H \rightarrow H$ is self-adjoint, compact and positive.

Q2) The operator $Q(x)$ is monotone decreasing.

Q3) $Q(x)$ is a continuous operator function with respect to the norm in $B(H)$ and

$$\lim_{x \rightarrow \infty} \|Q(x)\| = 0.$$

$D(L)$ denotes the set of all functions $y(x) \in L^2(0, \infty; H)$ satisfying the following conditions:

y1) $y(x)$ and $y'(x)$ are absolute continuous with respect to the norm in the space $H$ in every finite interval $[0, a]$.

y2) $l(y) = -(p(x)y'(x))' - Q(x)y(x) \in L^2(0, \infty; H)$.

y3) $y(0) = 0$ and,

$$(Ly)(x) = -(p(x)y'(x))' - Q(x)y(x).$$

It is proved that the operator $L : D(L) \rightarrow L^2(0, \infty; H)$ is self-adjoint, semi bounded-below and the negative part of the spectrum of the operator $L$ is discrete [1]. Let $-\lambda_1 \leq -\lambda_2 \leq \cdots \leq -\lambda_n \leq \cdots$ be negative eigenvalues of the operator $L$. In this work we find an asymptotic formula for the sum

$$\sum_{-\lambda_i < -\varepsilon} \lambda_i \hspace{1cm} (\varepsilon > 0),$$

as $\varepsilon \rightarrow +0$.

In [2] and [3], the asymptotic formulas for the sum of the negative eigenvalues of second order differential operator with scalar coefficient are calculated. In [1], [4], [5], [6], [7] the asymptotic behaviour of the number of the negative eigenvalues are investigated.
2 Some Inequalities For the Sum of the Eigenvalues

Let $\alpha_1(x) \geq \alpha_2(x) \geq \cdots \geq \alpha_j(x) \geq \cdots$ be the eigenvalues of the operator $Q(x) : H \to H$. Since the operator function $Q(x)$ is monotone decreasing, the functions $\alpha_1(x), \alpha_2(x), \cdots, \alpha_j(x), \cdots$ are also monotone decreasing, [5]. Moreover, since

$$\alpha_1(x) = \sup_{\|f\|=1} (Q(x)f, f),$$

[8] and

$$\|Q(x)\| = \sup_{\|f\|=1} \|Q(x)f, f\| = \sup_{\|f\|=1} (Q(x)f, f),$$

[9] then $\alpha_1(x) = \|Q(x)\|$.

On the other hand, since $\lim_{x \to \infty} \alpha_1(x) = 0$, then the function $\alpha_1$ has a continuous inverse function defined in the interval $(0, \alpha_1(0)]$. Let

$$\psi_j(\varepsilon) = \sup\{x \in [0, \infty); \alpha_j(x) \geq \varepsilon\} \quad (j = 1, 2, \cdots)$$

(2)

and $\psi_1$ denote the inverse function of $\alpha_1$. We consider the following operators:

1) Let $L^0$ and $L'$ be operators in the space $L_2(0, \psi_1(\varepsilon); H)$, which are formed by expression (1) and with the boundary conditions

$$y(0) = y(\psi_1(\varepsilon)) = 0$$
$$y'(0) = y'(\psi_1(\varepsilon)) = 0,$$

respectively. Here, $\varepsilon \in (0, \alpha_1(0)]$.

2) $L_i$ and $L'_i$ be operators in the space $L_2(x_{i-1}, x_i; H)$ which are formed by expression (1) and with the boundary conditions

$$y(x_{i-1}) = y(x_i) = 0$$
$$y'(x_{i-1}) = y'(x_i) = 0,$$

respectively.

3) $L_{i(1)}$ be operator in the space $L_2(x_{i-1}, x_i; H)$ which is formed by the differential equation

$$-p(x_i)y''(x) - Q(x_i)y(x)$$

and with the boundary conditions $y(x_{i-1}) = y(x_i) = 0$.

4) Let $L'_{i(1)}$ be operator in the space $L_2(x_{i-1}, x_i; H)$ which is formed by the differential equation

$$-p(x_{i-1})y''(x) - Q(x_{i-1})y(x)$$
and with boundary conditions \( y'(x_{i-1}) = y'(x_i) = 0 \).

Let us divide the interval \([0, \psi_1(\varepsilon)]\) by the intervals at the length

\[
\delta = \frac{\psi_1(\varepsilon)}{||\psi_1'(\varepsilon)||} + 1 \quad (3)
\]

Here, \( a \in (0, 1) \) is a constant number and \( \varepsilon \) is any positive number satisfying the inequality \( \psi_1'(\varepsilon) \geq 2 \). And also \( ||\psi_1'(\varepsilon)|| \) shows exact part of \( \psi_1'(\varepsilon) \).

Let the partition points of the interval \([0, \psi_1(\varepsilon)]\) be

\[ 0 = x_0 < x_1 < \cdots < x_M = \psi_1(\varepsilon). \]

Let \( N(\lambda), N^0(\lambda), N'(\lambda), n_i(\lambda) \) and \( n_{i(1)}(\lambda) \) be numbers of eigenvalues smaller than \(-\lambda \) \( (\lambda > 0) \) of the operators \( L, L^0, L', L_i \) and \( L_{i(1)} \), respectively. Let us write \( n_i, n_{i(1)} \) instead of \( n_i(\varepsilon), n_{i(1)}(\varepsilon) \), respectively.

Şengül [1] proved that the inequalities

\[
N^0(\varepsilon) \leq N(\varepsilon) \leq N'(\varepsilon) \quad (4)
\]

are satisfied, if \( Q(x) \) satisfies the conditions \( Q1), Q2), Q3) \) and \( p(x) \) satisfies the conditions \( p1), p3).\)

We want to show that the inequalities

\[
N^0(\lambda) \leq N(\lambda) \leq N'(\lambda) \quad (\forall \lambda \in [\varepsilon, \infty)) \quad (5)
\]

are satisfied. Let \( u_1, u_2, \cdots, u_n, \cdots \) be orthonormal eigenvectors corresponding to the eigenvalues \(-\lambda_1, -\lambda_2, \cdots, -\lambda_n, \cdots \). Let us consider the following operators:

\[
S = L + \lambda I \quad (6)
\]

\[
S^0 = L^0 + \lambda I, \quad S' = L' + \lambda I \quad (7)
\]

Here \( I \) in (6) is identity operator in the space \( L_2(0, \infty; H) \); \( I \) in (7) is identity operator in the space \( L_2(0, \psi_1(\varepsilon); H) \). We have

\[
-\lambda_1 \leq -\lambda_2 \leq \cdots \leq -\lambda_{N(\lambda)} < -\lambda, \quad \lambda_{N(\lambda)+1} \geq -\lambda. \quad (8)
\]

Since the eigenvalues smaller than \( \lambda \) are \( \mu_i = \lambda_i + \lambda \) \( (i = 1, 2, \cdots) \) from (8)

\[
-\mu_1 \leq -\mu_2 \leq \cdots \leq -\mu_{N(\lambda)} < 0, \quad -\mu_{N(\lambda)+1} \geq 0 \quad (9)
\]

is obtained. By the similar way we can show that the number of negative eigenvalues of the operators \( S^0 \) and \( S' \) are \( N^0(\lambda) \) and \( N'(\lambda) \), respectively. Let

\[
-\mu_{(1)1} \leq -\mu_{(1)2} \leq \cdots \leq -\mu_{(1)N^0(\lambda)}, \quad -\mu_{(2)1} \leq -\mu_{(2)2} \leq \cdots \leq -\mu_{(2)N'(\lambda)} \quad (10)
\]

be negative eigenvalues of the operators \( S^0 \) and \( S' \) respectively. Let the orthonormal eigenvectors corresponding these eigenvalues be \( \varphi_1, \varphi_2, \cdots, \varphi_{N^0(\lambda)} \) and \( \psi_1, \psi_2, \cdots, \psi_{N'(\lambda)} \) respectively.
Lemma 2.1 If the operator function $Q(x)$ satisfies the conditions $Q1), Q2), Q3$ and the function $p(x)$ satisfies the conditions $p1), p2$ then

$$N(\lambda) \geq N^0(\lambda) \quad (\forall \lambda \in (0, \infty))$$

(11)

Proof: To obtain a contradiction, we suppose that

$$N(\lambda) < N^0(\lambda).$$

Then, there is a non-zero linear combination

$$\varphi = \sum_{i=1}^{N^0(\lambda)} \beta_i \varphi_i$$

of the functions $\varphi_1, \varphi_2, \cdots, \varphi_{N^0(\lambda)}$ such that

$$\left( u_i, \varphi \right)_{(0, \psi_1(\varepsilon))} = \int_0^{\psi_1(\varepsilon)} \left( u_i(x), \varphi(x) \right) dx = 0 \quad (i = 1, 2, \cdots, N(\lambda))$$

By using (12)

$$\left( S^0 \varphi, \varphi \right)_{(0, \psi_1(\varepsilon))} = \left( S^0 \left( \sum_{i=1}^{N_1(\lambda)} \beta_i \varphi_i \right), \sum_{i=1}^{N_1(\lambda)} \beta_i \varphi_i \right)_{(0, \psi_1(\varepsilon))}$$

$$= \left( \sum_{i=1}^{N_1(\lambda)} \beta_i \mu_{(1)i} \varphi_i, \sum_{i=1}^{N_1(\lambda)} \beta_i \varphi_i \right)_{(0, \psi_1(\varepsilon))}$$

$$> \sum_{i=1}^{N_1(\lambda)} \mu_{(1)i} |\beta_i|^2 = \alpha < 0$$

(13)

In the similar way as proved in Glazman [10] there exists a vector function $\tilde{\varphi}$ which has the following properties:

$\tilde{\varphi}1)$ The vector function $\tilde{\varphi} = \tilde{\varphi}(x)$ has second second order continuous derivative respect to the norm in the space $H$ in the interval $[0, \psi_1(\varepsilon)]$.

$\tilde{\varphi}2)$ $\tilde{\varphi}(x)$ is equal to zero outside of the interval $[a, b] \subset (0, \psi_1(\varepsilon))$.

$\tilde{\varphi}3)$ $\left| \left( S^0 \tilde{\varphi}, \tilde{\varphi} \right)_{(0, \psi_1(\varepsilon))} - \left( S^0 \varphi, \varphi \right)_{(0, \psi_1(\varepsilon))} \right| < \frac{\alpha}{2}$

$\tilde{\varphi}4)$ $\left( u_i, \tilde{\varphi} \right)_{(0, \psi_1(\varepsilon))} = 0 \quad (i = 1, 2, \cdots, N(\lambda))$.

As it is known,

$$\inf_{\substack{y \in D(S), \|y\|_{(0, \infty)} = 1 \\ y \perp u_i \ (i = 1, 2, \cdots, N(\lambda))}} \left( Sy, y \right)_{(0, \infty)} = \mu_{N(\lambda) + 1}$$. 
Therefore

\[
(S^0(\tilde{\varphi}, \varphi)_{(0,\psi_1(\varepsilon)}) = (S(\tilde{\varphi}, \varphi)_{(0,\psi_1(\varepsilon)}) (0,\infty) \\
\geq \mu_{N(\lambda)+1} \geq 0.
\]

By the last inequality,

\[
(S^0_\varphi, \varphi)_{(0,\psi_1(\varepsilon))} \geq 0 \quad (14)
\]

is obtained. By (13) and (14)

\[
(S^0\tilde{\varphi}, \tilde{\varphi})_{(0,\psi_1(\varepsilon))} - (S^0\varphi, \varphi)_{(0,\psi_1(\varepsilon))} = (S^0\tilde{\varphi}, \tilde{\varphi})_{(0,\psi_1(\varepsilon))} - \alpha \geq -\alpha \quad (15)
\]

is found. On the other hand this result in (15) contradicts with the property \(\tilde{\varphi}^3\). Hence

\[
N(\lambda) \geq N^0(\lambda).
\]

**Lemma 2.2** If the operator function \(Q(x)\) satisfies the conditions \(Q1), Q2), Q3) and function \(p(x)\) satisfies the conditions \(p1), p2) then \(N(\lambda) \leq N'(\lambda) for all \(\lambda \in [\varepsilon, \infty)).

**Proof:** Suppose for contradiction that \(N(\lambda) > N'(\lambda).\) Then, there is a non-zero linear combination

\[
u = \sum_{i=1}^{N(\lambda)} d_i u_i \quad (16)
\]

of the vector functions \(u_1, u_2, \cdots, u_{N(\lambda)}\) such that

\[
(\psi_i, u)_{(0,\psi_1(\varepsilon))} = \int_0^\psi_1(\lambda) (\psi_i(x), u(x))dx = 0 \quad (i = 1, 2, \cdots, N'(\lambda))
\]

By using (16)

\[
(Su, u)_{(0,\infty)} = (S(\sum_{i=1}^{N(\lambda)} d_i u_i), \sum_{i=1}^{N(\lambda)} d_i u_i)_{(0,\infty)} \\
= (\sum_{i=1}^{N(\lambda)} d_i \mu_i u_i, \sum_{i=1}^{N(\lambda)} d_i u_i)_{(0,\infty)} = \sum_{i=1}^{N(\lambda)} \mu_i |d_i|^2 < 0 \quad (17)
\]

is obtained. We can write the equation (17) as

\[
(Su, u)_{(0,\infty)} = \int_0^{\psi_1(\varepsilon)} (S(u(x)), u(x))dx + \int_\psi_1(\varepsilon) \infty (S(u(x)), u(x))dx < 0 \quad (18)
\]
Since
\[ \int_{\psi_1(\varepsilon)}^{\infty} \left( S(u(x)), u(x) \right) dx \geq 0 \]
then we have
\[ \int_{0}^{\psi_1(\varepsilon)} \left( S(u(x)), u(x) \right) dx < 0. \quad (19) \]

If we consider the equality
\[ (u, \psi_1(0, \psi_1(\varepsilon))) = \int_{0}^{\psi_1(\varepsilon)} \left( u(x), \psi_1(x) \right) (0, \infty) \, dx = 0 \quad (i = 1, 2, \ldots, N'(\lambda)) \]
from (19)
\[ \inf_{y \in D(S), \|y\|_{(0, \psi_1(\varepsilon))}=1, y \perp \psi_i \, (i=1, 2, \ldots, N'(\lambda))} \psi_1(\varepsilon) \int_{0}^{\psi_1(\varepsilon)} \left( S(y(x)), y(x) \right) dx < 0 \quad (20) \]
is obtained. From (20)
\[ \inf_{y' \in D(S'), \|y'\|_{(0, \psi_1(\varepsilon))}=1, y'(0)=y'(\psi_1(\varepsilon))=0, y \perp \psi_i \, (i=1, 2, \ldots, N'(\lambda))} \psi_1(\varepsilon) \int_{0}^{\psi_1(\varepsilon)} \left( S(y(x)), y(x) \right) dx < 0 \quad (21) \]
is found. By (21)
\[ \inf_{y' \in D(S'), \|y'\|_{(0, \psi_1(\varepsilon))}=1, y' \parallel \psi_i \, (i=1, 2, \ldots, N'(\lambda))} \psi_1(\varepsilon) \int_{0}^{\psi_1(\varepsilon)} \left( S'(y(x)), y(x) \right) dx < 0 \quad (22) \]
is obtained. On the other hand, we have
\[ \inf_{y' \in D(S'), \|y'\|_{(0, \psi_1(\varepsilon))}=1, y \perp \psi_i \, (i=1, 2, \ldots, N'(\lambda))} \psi_1(\varepsilon) \int_{0}^{\psi_1(\varepsilon)} \left( S'(y(x)), y(x) \right) dx = \mu_{2(N'(\lambda)+1)} \geq 0 \quad (23) \]
This result contradicts with (22). Therefore \( N(\lambda) \leq N'(\lambda) \). \( \square \)

Let \(-\mu_{i(1)} \leq -\mu_{i(2)} \leq -\mu_{i(3)} \leq \cdots \) be eigenvalues of the operator \( L_i(1) \) and let we have the following equalities
\[ a_j(x, t) = \alpha_j(x) - p(x) \left( \frac{\pi t}{\delta} \right)^2 \quad (j = 1, 2, \ldots) \quad (24) \]
\[ b_j(\varepsilon, x) = \frac{\delta}{\pi} \sqrt{\frac{\alpha_j(x) - \varepsilon}{p(x)}} \quad (j = 1, 2, \ldots) \quad (25) \]
\[ \beta_j(\varepsilon, x) = \int_0^{b_j(\varepsilon, x)} a_j(x, t) \, dt \quad (j = 1, 2, \cdots) \quad (26) \]

\[ \varphi_{i,j}(\varepsilon) = \min\{x_{i+1}, \psi_j(\varepsilon)\} \quad (i = 1, 2, \cdots, M - 1). \quad (27) \]

**Theorem 2.3** If the operator function \( Q(x) \) and the scalar function \( p(x) \) satisfy the conditions \( Q1) - Q3) \) and \( p1) - p3) \), then we have

\[ \sum_{m=1}^{n_i(1)} \mu_{i(1)m} > \frac{1}{\delta} \sum_{\alpha_j(x_i) > \varepsilon} \varphi_{i,j}(\varepsilon) \int_{x_i}^{x_{i+1}} \beta_j(\varepsilon, x) \, dx - 3 \sum_{\alpha_j(0) > \varepsilon} \alpha_j(0) \]

for small positive values of \( \varepsilon \).

**Proof:** Let us consider the operator \( L_{i(1)} \) which is formed by the differential expression

\[-p(x_i)y''(x) - Q(x_i)y(x)\]

with the boundary conditions \( y(x_{i-1}) = y(x_i) = 0 \).

We wish to obtain the eigenvalues of the operator \( L_{i(1)} \). In order to find the eigenvalues, we will solve the eigenvalues problem

\[-du'' = \lambda u \]
\[ u(a) = u(b) = 0 \quad (28) \]

in the space \( L_2(a,b) \). Here, \( a = x_{i-1}, \ b = x_i \) and \( d = p(x_i) \). Moreover, \( \gamma \) is an eigenvalue of the operator \( Q(b) : H \rightarrow H \). The eigenvalues of boundary-value problem (28) are in the form

\[ \lambda_n = d\left(\frac{n\pi}{b-a}\right)^2, \quad (n \in \mathbb{N}). \]

So, the eigenvalues of the operator \( L_{i(1)} \) are of the form

\[ \lambda_n - \gamma = p(x_i)\left(\frac{n\pi}{x_i - x_{i-1}}\right)^2 - \gamma. \]

Since the eigenvalues of the operator \( Q(x) : H \rightarrow H \) are \( \alpha_1(x) \geq \alpha_2(x) \geq \cdots \geq \alpha_j(x) \geq \cdots \) then the eigenvalues of the operator \( L_{i(1)} \) are

\[ p(x_i)\left(\frac{m\pi}{x_i - x_{i-1}}\right)^2 - \alpha_j(x_i) \quad (m = 1, 2, \cdots; j = 1, 2, \cdots), \]

therefore \( n_{i(1)} \) is the number of pairs \( (m, j) \quad (m, j \geq 1) \) satisfying the inequality

\[ p(x_i)(\frac{m\pi}{\delta})^2 - \alpha_j(x_i) < -\varepsilon \quad (\delta = x_i - x_{i-1}). \quad (29) \]
By using (24), (25) and (29), we obtain

\[ \sum_{m=1}^{n_{i(1)}} \mu_{i(1)m} = \sum_{\alpha_j(x_i) > \varepsilon}^j \sum_{a_j(x_i,m) > \varepsilon}^m a_j(x_i,m) \]

\[ \geq \sum_{\alpha_j(x_i) > \varepsilon}^j \sum_{m=1}^{[b_j(\varepsilon,x_i)]-1} a_j(x_i,m) \]

(30)

For the sum \[ \sum_{m=1}^{[b_j(\varepsilon,x_i)]-1} a_j(x_i,m) \] in (30)

\[ \frac{b_j(\varepsilon,x_i)-2}{b_j(\varepsilon,x_i)} \]

\[ \leq \int_1^{b_j(\varepsilon,x_i)} a_j(x_i,t)dt \]

\[ = \int_0^{b_j(\varepsilon,x_i)} a_j(x_i,t)dt - \int_0^{b_j(\varepsilon,x_i)-2} a_j(x_i,t)dt \]

\[ = \beta_j(\varepsilon,x_i) - 3\alpha_j(x_i) \]

(31)

is obtained. If we consider that the functions \( \beta_j(\varepsilon,x) \) \( (j = 1,2,\cdots) \) are decreasing, from (27), (30) and (31)

\[ \sum_{m=1}^{n_{i(1)}} \mu_{i(1)m} > \frac{1}{\delta} \sum_{\alpha_j(x_i) > \varepsilon}^j \int_{x_i}^{x_i+1} \beta_j(\varepsilon,x_i)dx - 3 \sum_{\alpha_j(0) > \varepsilon}^j \alpha_j(0) \]

\[ \geq \frac{1}{\delta} \sum_{\alpha_j(x_i) > \varepsilon}^j \int_{x_i}^{x_i+1} \beta_j(\varepsilon,x)dx - 3 \sum_{\alpha_j(0) > \varepsilon}^j \alpha_j(0) \]

(32)

is obtained. □

**Theorem 2.4** If the operator function \( Q(x) \) and the scalar function \( p(x) \)
satisfy the conditions $Q1) - Q3), p1) - p3)$, then we have

$$N(\varepsilon) \sum_{i=1}^{\lambda_i} > \frac{1}{\delta} \sum_{j=1}^{\varepsilon} \int_{0}^{\beta_j(\varepsilon, x)dx - \text{const.}} \int_{0}^{\delta_j} \alpha_j^{j}(x)dx - \text{const.} \psi^{n}_1(\varepsilon) \sum_{j=1}^{\alpha_j(0)}$$

for small positive values of $\varepsilon$.

Here, $l_\varepsilon = \sum_{\alpha_j(0) \geq \varepsilon} 1$.

**Proof:** We can easily show that $L_i < L_{i(1)}$. In the case, it is known that

$$n_i(\lambda) \geq n_{i(1)}(\lambda)$$

[11]. On the other hand, from variation principles of R. Courant [12], we have

$$N^0(\lambda) \geq \sum_{i=1}^{M} n_i(\lambda).$$

From (32) and (33)

$$N^0(\lambda) \geq \sum_{i=1}^{M} n_i(\lambda(1)) (\lambda \geq \varepsilon)$$

is obtained. From (5) and (34)

$$N(\lambda) \geq \sum_{i=1}^{M} n_i(1)(\lambda) (\forall \lambda \geq \varepsilon)$$

is found. By using (35), we can show that the inequality

$$N(\varepsilon) \sum_{i=1}^{\lambda_i} \geq \sum_{i=1}^{M} \sum_{m=1}^{n_{i(1)}} \mu_{i(1)m}$$

is satisfied. By the Theorem 2.1 and (36)

$$N(\varepsilon) = \sum_{i=1}^{\lambda_i} \geq \sum_{i=1}^{M-1} \left\{ \frac{1}{\delta} \sum_{j=1}^{\delta_i} \int_{0}^{\beta_j(\varepsilon, x)dx - \text{const.}} \int_{0}^{\delta_j} \alpha_j^{j}(x)dx - \text{const.} \psi^{n}_1(\varepsilon) \sum_{j=1}^{\alpha_j(0)} \right\}$$

$$= \frac{1}{\delta} \sum_{j(x) \geq \varepsilon}^{\varphi_{i,j}(\varepsilon)} \int_{x_i}^{\beta_j(\varepsilon, x)dx - \text{const.}} \int_{0}^{\delta_j} \alpha_j^{j}(x)dx - \text{const.} \sum_{j=1}^{\alpha_j(0)}$$

(37)
is obtained. Since the functions $\alpha_j(x) \ (j = 1, 2, \cdots)$ are decreasing, then we have

$$\sum_{\alpha_j(x_i) > \varepsilon} \sum_i \int_{x_i} \beta_j(\varepsilon, x) dx = \sum_{\alpha_j(x_1) > \varepsilon} \sum_{\alpha_j(x_i) > \varepsilon} \int_{x_i} \beta_j(\varepsilon, x) dx. \quad (38)$$

From (37) and (38)

$$\sum_{i=1}^{N(\varepsilon)} \lambda_i \geq \frac{1}{\delta} \sum_{\alpha_j(x_1) > \varepsilon} \sum_{\alpha_j(x_i) > \varepsilon} \int_{x_i} \beta_j(\varepsilon, x) dx - 3M \sum_{j=1}^{l_\varepsilon} \alpha_j(0) \quad (39)$$

is obtained. By using (27) on the right-hand side of inequality (39)

$$\sum_{i=1}^{N(\varepsilon)} \lambda_i \geq \frac{1}{\delta} \sum_{\alpha_j(x_1) > \varepsilon} \int_{x_1}^{x_2} \beta_j(\varepsilon, x) dx + \int_{x_2}^{x_3} \beta_j(\varepsilon, x) dx + \cdots + \int_{x_{i_0}} \beta_j(\varepsilon, x) dx$$

$$- 3M \sum_{j=1}^{l_\varepsilon} \alpha_j(0) \quad (40)$$

is found. Here, $i_0$ is a natural number satisfying the following condition:

$$x_{i_0} < \psi_j(\varepsilon) \leq x_{i_0+1}.$$ 

By using (27) and (40)

$$\sum_{i=1}^{N(\varepsilon)} \lambda_i \geq \frac{1}{\delta} \sum_{\psi_j(\varepsilon) > x_1} \int_{x_1}^{x_j} \beta_j(\varepsilon, x) dx - 3M \sum_{j=1}^{l_\varepsilon} \alpha_j(0)$$

$$= \frac{1}{\delta} \sum_{j=1}^{l_\varepsilon} \int_{0}^{x_j} \beta_j(\varepsilon, x) dx - \frac{1}{\delta} \sum_{\psi_j(\varepsilon) < x_1} \int_{0}^{\psi_j(\varepsilon)} \beta_j(\varepsilon, x) dx$$

$$- \frac{1}{\delta} \sum_{\psi_j(\varepsilon) \geq x_1} \int_{0}^{x_j} \beta_j(\varepsilon, x) dx - 3M \sum_{j=1}^{l_\varepsilon} \alpha_j(0)$$

$$= \frac{1}{\delta} \sum_{j=1}^{l_\varepsilon} \int_{0}^{x_j} \beta_j(\varepsilon, x) dx - \frac{1}{\delta} \sum_{j=1}^{l_\varepsilon} \int_{0}^{\psi_j(\varepsilon)} \beta_j(\varepsilon, x) dx$$

$$- 3M \sum_{j=1}^{l_\varepsilon} \alpha_j(0) \quad (41)$$
is obtained. From (24), (25) and (26)

\[
\frac{1}{\delta} \beta_j(\varepsilon, x) = \frac{1}{\delta} \int_0^{b_j(\varepsilon, x)} \left[ \alpha_j(x) - p(x) \left( \frac{\pi t}{\delta} \right)^2 \right] dt
\]

\[
= \frac{1}{\delta} \alpha_j(x) b_j(\varepsilon, x) - \pi^2 \frac{p(x)}{3\delta^3} b_j^3(\varepsilon, x)
\]

\[
= \frac{1}{\delta} b_j(\varepsilon, x) \left[ \alpha_j(x) - \pi^2 \frac{p(x)}{3\delta^2} b_j^2(\varepsilon, x) \right]
\]

\[
= \frac{1}{\pi} \sqrt{\frac{\alpha_j(x) - \varepsilon}{p(x)}} \left[ \alpha_j(x) - \pi^2 \frac{p(x)}{3\delta^2} \frac{\delta^2 (\alpha_j(x) - \varepsilon)}{\pi^2 p(x)} \right]
\]

\[
= \frac{1}{\pi} \sqrt{\frac{\alpha_j(x) - \varepsilon}{p(x)}} \left[ \frac{2}{3} \alpha_j(x) + \frac{\varepsilon}{3} \right] < \text{const.} \alpha_j^\frac{4}{3} (x) \tag{42}
\]

is found for the expression \( \frac{1}{\delta} \beta_j(\varepsilon, x) \). From (27) and (42),

\[
\frac{1}{\delta} \sum_{j=1}^{l_e} \int_0^{\varphi_0,j(\varepsilon)} \beta_j(\varepsilon, x) dx < \text{const.} \sum_{j=1}^{l_e} \int_0^{\delta} \alpha_j^\frac{3}{2} (x) dx \tag{43}
\]

is obtained. From (3), (41) and (43)

\[
\sum_{i=1}^{N(\varepsilon)} \lambda_i > \frac{1}{\delta} \sum_{j=1}^{l_e} \int_0^{\psi_j(\varepsilon)} \beta_j(\varepsilon, x) dx - \text{const.} \sum_{j=1}^{l_e} \int_0^{\delta} \alpha_j^\frac{3}{2} (x) dx - \text{const.} \psi_1^\frac{4}{3} (\varepsilon) \sum_{j=1}^{l_e} \alpha_j(0)
\]

is found. □

Let \( -\mu_{i(1)}'1 \leq -\mu_{i(1)}'2 \leq -\mu_{i(1)}'3 \leq \cdots \) be eigenvalues of the operator \( L_{i(1)}' \) and \( n_{i(1)}'(\lambda) \) be number of the eigenvalues smaller than \( -\lambda \) \( (\lambda > 0) \) of the operator \( L_{i(1)}' \). Moreover, we will simply write \( n_{i(1)}' \) instead of \( n_{i(1)}'(\varepsilon) \).

**Theorem 2.5** If the operator function \( Q(x) \) and the scalar function \( p(x) \) satisfy the conditions \( Q1) - Q3), p1) - p3) \) then the inequality

\[
\sum_{m=1}^{n_{i(1)}'} \mu_{i(1)}'m \leq \frac{1}{\delta} \sum_{j=1}^{l_e} \int_0^{\frac{x_{i-1}}{\alpha_j(\varepsilon, x)}} \beta_j(\varepsilon, x) dx + \sum_{j=1}^{l_e} \alpha_j(0) \quad (i = 2, 3, \cdots)
\]

is satisfied for the small positive values of \( \varepsilon \).
**Proof:** The eigenvalues of the operator \( L_{i(1)}' \) are in the form

\[
p(x_{i-1}) \left[ \frac{(m-1)\pi}{(x_i - x_{i-1})} \right]^2 - \alpha_j(x_{i-1}) \quad (m = 1, 2, \ldots; j = 1, 2, \ldots).
\]

Therefore \( n_{i(1)}' \) is the number of the pairs \( (m, j) \quad (m, j \geq 1) \) satisfying the inequality

\[
p(x_{i-1}) \left[ \frac{(m-1)\pi}{(x_i - x_{i-1})} \right]^2 - \alpha_j(x_{i-1}) < -\varepsilon. \tag{44}
\]

From (24), (25), and (44)

\[
\sum_{m=1}^{\infty} \mu_{i(1)m}' = \sum_{\alpha_j(x_{i-1}) > \varepsilon} \sum_{a_j(x_{i-1}, m-1) > \varepsilon} a_j(x_{i-1}, m-1)
\]

\[
= \sum_{\alpha_j(x_{i-1}) > \varepsilon} \sum_{m=1}^{[b_j(\varepsilon, x_{i-1})] + 1} a_j(x_{i-1}, m-1) \tag{45}
\]

is found. It is easy to see that

\[
\sum_{m=1}^{[b_j(\varepsilon, x_{i-1})] + 1} a_j(x_{i-1}, m-1) \leq \alpha_j(x_{i-1}) + \int_0^{b_j(\varepsilon, x_{i-1})} a_j(x_{i-1}, t)dt
\]

\[
= \alpha_j(x_{i-1}) + \beta_j(\varepsilon, x_{i-1}). \tag{46}
\]

We consider that the functions \( \beta_j(\varepsilon, x) \quad (j = 1, 2, \ldots) \) are monotone decreasing, by (45) and (46),

\[
\sum_{m=1}^{n_{i(1)}'} \mu_{i(1)m}' \leq \sum_{j=1}^{l_x} \alpha_j(0) + \frac{1}{\delta} \sum_{\alpha_j(x_{i-1}) > \varepsilon} \int_{x_{i-1}}^{x_{i-2}} \beta_j(\varepsilon, x)dx
\]

\[
< \frac{1}{\delta} \sum_{\alpha_j(x_{i-1}) > \varepsilon} \int_{x_{i-2}}^{x_{i-1}} \beta_j(\varepsilon, x)dx + \sum_{j=1}^{l_x} \alpha_j(0) \quad (i = 1, 2, \ldots)
\]

is obtained. \( \square \)

Let \( n_{i}'(\lambda) \) be number of the eigenvalues smaller than \(-\lambda \quad (\lambda > 0)\) of the operator \( L_i' \), \(-\mu_1' \leq -\mu_2' \leq -\mu_3' \leq \cdots\) be eigenvalues of the operator \( L_i' \) and \( n_{i}'(\varepsilon) = n_i' \).
**Theorem 2.6** If the operator function $Q(x)$ and the scalar function $p(x)$ satisfy the conditions $Q(1) - Q(3)$, and $p(1) - p(3)$, then we have

$$\sum_{i=1}^{N(\varepsilon)} \lambda_i < \sum_{m=1}^{n'_i} \mu'_m + \frac{1}{\delta} \sum_{j=1}^{l_\varepsilon} \int_0^{x_{i+1}} \beta_j(\varepsilon, x)dx + \frac{\psi_1(\varepsilon)}{\delta} \sum_{j=1}^{l_\varepsilon} \alpha_j(0)$$

for the small values of $\varepsilon$.

**Proof:** We can easily show that $L'_i > L'_{i(1)}$. In this case we have

$$N'_i(\lambda) \leq n'_{i(1)}(\lambda)$$

(47) [11]. On the other hand, from variation principles of R. Courant [12], we have

$$N'(\lambda) \leq \sum_{i=1}^{M} n'_i(\lambda).$$

(48)

From (47) and (48),

$$N'(\lambda) \leq \sum_{i=2}^{M} n'_{i(1)}(\lambda) + n'_1(\lambda)$$

(49)

is obtained. From (5) and (49)

$$N(\lambda) \leq \sum_{i=2}^{M} n'_{i(1)}(\lambda) + n'_1(\lambda) \quad (\forall \lambda \geq \varepsilon)$$

(50)

is found. By using (50), we have

$$\sum_{i=1}^{N(\varepsilon)} \lambda_i \leq \sum_{i=2}^{M} \sum_{m=1}^{n'_{i(1)}} \mu'_i(\lambda)m + \sum_{m=1}^{n'_1} \mu'_m.$$  

(51)

By using Theorem 2.3 and (51)

$$\sum_{i=1}^{N(\varepsilon)} \lambda_i \leq \sum_{m=1}^{n'_1} \mu'_m + \frac{1}{\delta} \sum_{j=1}^{l_\varepsilon} \int_{\alpha_j(x_{i-1})}^{x_{i+1}} \beta_j(\varepsilon, x)dx + M \sum_{j=1}^{l_\varepsilon} \alpha_j(0)$$

(52)

is found. Since the functions $\alpha_j(x)$ $(j = 1, 2, \cdots)$ are monotone decreasing, then we have

$$\sum_{j, \alpha_j(x_{i-1}) > \varepsilon}^{\int \beta_j(\varepsilon, x)dx} = \sum_{j, \alpha_j(x_{i-1}) > \varepsilon}^{\int \beta_j(\varepsilon, x)dx}$$

(53)
From (52) and (53)

\[
\sum_{i=1}^{N(\varepsilon)} \lambda_i < \sum_{m=1}^{n'_1} \mu'_m + \frac{1}{\delta} \sum_{j \in J(\varepsilon)} \sum_{i \in J(\varepsilon-1)} x_{i-1}^i \int \beta_j(\varepsilon, x) dx + M \sum_{j=1}^{l_\varepsilon} \alpha_j(0)
\]

\[
= \sum_{m=1}^{n'_1} \mu'_m + \frac{1}{\delta} \sum_{j \in J(\varepsilon)} \left[ \int \beta_j(\varepsilon, x) dx + \int_{x_1}^{x_2} \beta_j(\varepsilon, x) dx \right]_{x_1}^{x_2}
\]

\[
+ \cdots + \int_{x_{i_0-1}}^{x_{i_0}} \beta_j(\varepsilon, x) dx + M \sum_{j=1}^{l_\varepsilon} \alpha_j(0)
\]

is obtained. Here, \(i_0\) is a natural number satisfying the conditions

\[
\alpha_j(x_{i_0}) > \varepsilon, \quad \alpha_j(x_{i_0+1}) \leq \varepsilon. \tag{54}
\]

From (2)

\[
x_{i_0} \leq \psi_j(\varepsilon). \tag{55}
\]

From (54) and (55)

\[
\sum_{i=1}^{N(\varepsilon)} \lambda_i < \sum_{m=1}^{n'_1} \mu'_m + \frac{1}{\delta} \sum_{j=1}^{l_\varepsilon} \psi_j(\varepsilon) \int \beta_j(\varepsilon, x) dx + \frac{\psi_j(\varepsilon)}{\delta} \sum_{j=1}^{l_\varepsilon} \alpha_j(0)
\]

is found. \(\square\)

Let

\[
\delta_i = \frac{\delta_{i-1}}{|\delta_{i-1} \psi_1^{(i+1)(\delta-1)}(\varepsilon)|} + 1, \quad (i = 1, 2, \cdots; \delta_0 = \delta) \tag{56}
\]

\[
a_{j(i)}(x, t) = \alpha_j(x) - p(x) \left( \frac{\pi t}{\delta_i} \right)^2,
\]

\[
b_{j(i)}(\varepsilon, x) = \frac{\delta_i}{\pi} \sqrt{\frac{\alpha_j(x) - \varepsilon}{p(x)}},
\]

\[
\beta_{j(i)}(\varepsilon, x) = \int_0^{b_{j(i)}(\varepsilon, x)} a_{j(i)}(x, t) dt,
\]

\[
\varphi_j(\delta_i, \varepsilon) = \min \{ \delta_i, \psi_j(\varepsilon) \} \quad (i = 0, 1, 2, \cdots). \tag{57}
\]

Let \(L(i)\) be operator in the space \(L_2(0, \delta_i; H)\) which is formed by the expression (1) and with the boundary condition

\[
y'(0) = y'(\delta_i) = 0. \tag{58}
\]
Moreover, let \( L^{(0)}_{(i)} \) be the operator which is formed by the expression
\[-p(0) y''(x) - Q(0) y(x)\]
and with the boundary condition (58).

Let \(-\mu_{(i)1} \leq -\mu_{(i)2} \leq \cdots \) and \(-\mu_{(i)1}^{(0)} \leq -\mu_{(i)2}^{(0)} \leq \cdots \) be eigenvalues smaller than \(-\lambda, (\lambda > 0)\) of the operators \( L_{(i)} \) and \( L^{(0)}_{(i)} \), respectively.

Moreover, let \( n_{(i)}(\lambda) \) and \( n_{(i)}^{(0)}(\lambda) \) be numbers of the eigenvalues smaller than \(-\lambda, (\lambda > 0)\) of the operators \( L_{(i)} \) and \( L^{(0)}_{(i)} \), respectively.

Since \( L_{(i)} \geq L^{(0)}_{(i)} \), then we have
\[ n_{(i)}(\lambda) \leq n_{(i)}^{(0)}(\lambda), \] (59)
[11]. By using (59), we can show that
\[ \sum_{m=1}^{n_{(i)}} \mu_{(i)m} \leq \sum_{m=1}^{n_{(i)}^{(0)}} \mu_{(i)m}^{(0)}. \] (60)

Here, \( n_{(i)} = n_{(i)}(\varepsilon), \quad n_{(i)}^{(0)} = n_{(i)}^{(0)}(\varepsilon). \) \( \delta_{-1} = \psi_1(\varepsilon) \) and from the formula (56)
\[ \frac{\delta_{i-1}}{\delta_i} = \frac{\delta_{i-2}}{[\delta_{i-1} \psi_1^{(i+1)a-1}(\varepsilon)]} \geq 1 \leq \delta_{i-1} \psi_1^{(i+1)a-1}(\varepsilon) + 1 \]
\[ = \frac{\delta_{i-2}}{[\delta_{i-1} \psi_1^{(i+1)a-1}(\varepsilon)]} \psi_1^{(i+1)a-1}(\varepsilon) + 1 \]
\[ < \frac{\delta_{i-2}}{\delta_{i-1} \psi_1^{(i+1)a-1}(\varepsilon)} \psi_1^{(i+1)a-1}(\varepsilon) + 1 \]
\[ = \psi_1^a(\varepsilon) + 1 \quad (i = 1, 2, \cdots) \]
is obtained. From the last relation, we find
\[ \frac{\delta_{i-1}}{\delta_i} < 2 \psi_1^a(\varepsilon), \quad (i = 1, 2, \cdots) \] (61)
for the values of \( \varepsilon \) satisfying the inequality \( \psi_1^a(\varepsilon) > 2. \)

**Theorem 2.7** If the operator function \( Q(x) \) and the scalar function \( p(x) \) satisfy the conditions \( Q1) - Q3), and p1) - p3), then we have
\[ \sum_{i=1}^{N} \lambda_i < \frac{1}{\delta} \sum_{j=1}^{l} \beta_j(\varepsilon, x) dx + \text{const.} \sum_{j=1}^{l} \int_0^\delta \alpha_j^2(x) dx + \text{const.} \psi_1^a(\varepsilon) \sum_{j=1}^{l} \alpha_j(0) \]
for small positive values of \( \varepsilon. \)
\textbf{Proof :} By the similar way to the proof of Theorem 2.6, the following inequality
\begin{equation}
\sum_{m=1}^{n_1} \mu'_m < \sum_{m=1}^{n_1} \mu(1)m + \frac{1}{\delta_1} \sum_{\psi_j(\varepsilon) < \delta_0} \int_0^1 \beta_j(1)(\varepsilon, x)dx + \frac{\delta_0}{\delta_1} \sum_{j=1}^{l_\varepsilon} \alpha_j(0)
\end{equation}
(62)
can be proved. If we replace the equation (57) in (62), then we have
\begin{equation}
\sum_{m=1}^{n_1} \mu'_m < \sum_{m=1}^{n_1} \mu(1)m + \frac{1}{\delta_1} \sum_{j=1}^{l_\varepsilon} \int_0^\delta_0 \psi_j(\varepsilon) \int_0^1 \beta_j(1)(\varepsilon, x)dx + \frac{\delta_0}{\delta_1} \sum_{j=1}^{l_\varepsilon} \alpha_j(0).
\end{equation}
(63)
If we apply the inequality (63) for the eigenvalues of the operator $L(i)$, then
\begin{equation}
\sum_{m=1}^{n(i)} \mu(i)m < \sum_{m=1}^{n(i+1)} \mu(i+1)m + \frac{1}{\delta_{i+1}} \sum_{j=1}^{l_\varepsilon} \int_0^1 \beta_j(i+1)(\varepsilon, x)dx + \frac{\delta_{i+1}}{\delta_{i+1}} \sum_{j=1}^{l_\varepsilon} \alpha_j(0)
\end{equation}
(64)
is obtained. From (61) and (64)
\begin{equation}
\sum_{m=1}^{n(i)} \mu(i)m < \sum_{m=1}^{n(i+1)} \mu(i+1)m + \frac{1}{\delta_{i+1}} \sum_{j=1}^{l_\varepsilon} \int_0^1 \beta_j(i+1)(\varepsilon, x)dx + \sum_{j=1}^{l_\varepsilon} \alpha_j(0)
\end{equation}
(65)
is found. By using (45) and (46)
\begin{equation}
\sum_{m=1}^{n(i+1)} \mu(i+1)m \leq \sum_{j=1}^{l_\varepsilon} \left(\alpha_j(0) + \beta_j(i+1)(\varepsilon, 0)\right)
\end{equation}
(66)
is obtained. Moreover, if we use the equation (42), then we get
\begin{equation}
\beta_j(i+1)(\varepsilon, x) \leq \text{const.} \delta_{i+1}^3 \beta_j^3(x).
\end{equation}
(67)
From (60), (66) and (67),
\begin{equation}
\sum_{m=1}^{n(i+1)} \mu(i+1)m \leq \sum_{j=1}^{l_\varepsilon} \alpha_j(0) + \text{const.} \delta_{i+1}^3 \sum_{j=1}^{l_\varepsilon} \alpha_j^3(0)
\end{equation}
(68)
is obtained. By using inequality (56), we find
\begin{equation}
\delta_{i+1} \leq 1.
\end{equation}
(69)
Here, \( i_0 \in \mathbb{N} \) is a constant satisfying the condition
\[
i_0 \geq \frac{1}{a} - 2
\]
From (68) and (69), we get
\[
\sum_{m=1}^{n(i_0+1)} \mu_{(i_0+1)m} \leq \text{const.} \sum_{j=1}^{l_\varepsilon} \alpha_j(0).
\]
(70)
From (61), (63), (65) and (70),
\[
\sum_{m=1}^{n'_i} \mu'_m \leq \text{const.} \sum_{j=1}^{l_\varepsilon} \alpha_j(0) + \sum_{i=0}^{i_0} \frac{1}{\delta_i+1} \int_0^\varepsilon \beta_j(i+1)(\varepsilon, x) dx
\]
\[
+ 2(i_0 + 1)\psi_1^q(\varepsilon) \sum_{j=1}^{l_\varepsilon} \alpha_j(0)
\]
(71)
is found. From (57), (67) and (71),
\[
\sum_{m=1}^{n'_i} \mu'_m < \text{const.} \sum_{j=1}^{l_\varepsilon} \int_0^\delta \alpha_j^q(x) dx + \text{const.} \psi_1^q(\varepsilon) \sum_{j=1}^{l_\varepsilon} \alpha_j(0)
\]
(72)
is obtained. By the Theorem 2.6 and (72), we have
\[
\sum_{i=1}^{N(\varepsilon)} \lambda_i < \frac{1}{\delta} \sum_{j=1}^{l_\varepsilon} \int_0^\delta \beta_j(\varepsilon, x) dx + \text{const.} \sum_{j=1}^{l_\varepsilon} \int_0^\delta \alpha_j^q(x) dx + \text{const.} \psi_1^q(\varepsilon) \sum_{j=1}^{l_\varepsilon} \alpha_j(0).
\]
is obtained. □

3 Asymptotic Formulas For The Sum Of Negative Eigenvalues

In this section, we find asymptotic formulas for the sum \( \sum_{\lambda_i < -\varepsilon} \lambda_i \) as \( \varepsilon \to +0 \).

Let us denote the functions of the form \( \ln_0 x = x, \ ln_n x = \ln(\ln_{n-1} x) \) by \( \ln_n x \quad (n = 0, 1, 2, \cdots) \) and we suppose that the function \( \alpha_1(x) = \|Q(x)\| \) satisfies the following condition:

\( \alpha_1 \) There are a number \( \xi > 0 \) and a natural number \( n \geq 1 \) such that the function \( \alpha_1(x) - (\ln_n x)^{-\xi} \) is neither negative nor monotone increasing in the interval \([b, \infty) \quad (b > 0)\).
Theorem 3.1 If the conditions $Q_1 - Q_3$, $p_1 - p_3$ and $\alpha_1$ are satisfied and the series $\sum_{j=1}^{\infty} [\alpha_j(0)]^m$ is convergent for a constant $m \in (0, \infty)$, then the asymptotic formula

$$\sum_{-\lambda_i < -\varepsilon} \lambda_i = \frac{1}{3\pi} \left[ 1 + O(e^{-\varepsilon}) \right] \sum_{j} \int_{\alpha_j(x) \geq \varepsilon} \frac{\alpha_j(x) - \varepsilon}{p(x)} \left( 2\alpha_j(x) + \varepsilon \right) dx$$

is satisfied as $\varepsilon \to +0$. Here, $\beta$ is a positive constant.

Proof: By using Theorem 2.4 and Theorem 2.5, we have

$$\left| \sum_{i=1}^{N(\varepsilon)} \lambda_i - \frac{1}{\delta} \sum_{j=1}^{l_\varepsilon} \int_0^{\beta_j(\varepsilon, x)} \beta_j(\varepsilon, x) dx \right| < \text{const.} l_\varepsilon (\delta + \psi_1^a(\varepsilon))$$

for the small positive values of $\varepsilon$. If we take $a = \frac{1}{2}$ and consider (3)

$$\left| \sum_{i=1}^{N(\varepsilon)} \lambda_i - \frac{1}{\delta} \sum_{j=1}^{l_\varepsilon} \int_0^{\beta_j(\varepsilon, x)} \beta_j(\varepsilon, x) dx \right| < \text{const.} l_\varepsilon \psi_1^\frac{1}{2}(\varepsilon)$$

(73) is found. Let us take $f(\varepsilon) = \psi_1(\varepsilon) [\ln \psi_1(\varepsilon)]^{-1}$. By using the function $p(x)$ which satisfies the condition (p1) and the inequality (42)

$$\frac{1}{\delta} \sum_{j=1}^{l_\varepsilon} \int_0^{\beta_j(\varepsilon, x)} \beta_j(\varepsilon, x) dx > \frac{1}{\delta} \int_0^{\beta_1(\varepsilon, x)} \beta_1(\varepsilon, x) dx$$

$$= \frac{1}{3\pi} \int_0^{\frac{\psi_1}{\psi_1}} \left[ \frac{\alpha_1(x) - \varepsilon}{p(x)} \right] \left( 2\alpha_1(x) + \varepsilon \right) dx$$

$$> \frac{1}{3\pi} \int_{\frac{f}{2f(\varepsilon)}}^{\frac{f}{f(\varepsilon)}} \left[ \frac{\alpha_1(x) - \varepsilon}{p(x)} \right] \left( 2\alpha_1(x) + \varepsilon \right) dx$$

$$> \text{const.} f(\varepsilon) \left( \alpha_1(f(\varepsilon)) - \varepsilon \right)^\frac{3}{2}$$

(74) is obtained. Şengül showed

$$\alpha_1(f(\varepsilon)) - \varepsilon > \left( \ln \psi_1(\varepsilon) \right)^{-((\xi+1)(n+1))}$$

(75)

for the small values of $\varepsilon > 0$, [1]. From (74) and (75)
\[
\frac{1}{\delta} \sum_{j=1}^{l_\varepsilon} \int_0^{\psi_j(\varepsilon)} \beta_j(\varepsilon, x) dx > \text{const.} \frac{\psi_1(\varepsilon)}{\ln \psi_1(\varepsilon)} (\ln \psi_1(\varepsilon))^{(\varepsilon+1)(n+1)}
\]

\[
> \text{const.} \psi_1^{\frac{3}{4}}(\varepsilon)
\]  

(76)

is found. From (73) and (76)

\[
\left| \frac{1}{\delta} \sum_{j=1}^{l_\varepsilon} \frac{\sum_{i=1}^{N(\varepsilon)} \lambda_i}{\psi_j(\varepsilon)} \psi_j(\varepsilon) - 1 \right| < \text{const.} l_\varepsilon \psi_1^{\frac{1}{4}}(\varepsilon)
\]

(77)

is obtained. Since the series \( \sum_{m=1}^{\infty} [\alpha_j(0)]^m \) is convergent then we have

\[
\text{const} > \sum_{\alpha_j(0) \geq \varepsilon} [\alpha_j(0)]^m \geq \sum_{\alpha_j(0) \geq \varepsilon} \varepsilon^m = \varepsilon^m l_\varepsilon.
\]

From last inequality

\[
l_\varepsilon < \text{const.} \varepsilon^{-m}
\]

(78)

is found. Since the function \( \alpha_1(x) \) satisfy the condition \( \alpha_1 \), we have

\[
\varepsilon = \alpha_1(\psi_1(\varepsilon)) \geq (\ln \psi_1(\varepsilon))^{-\xi} \geq (\ln \psi_1(\varepsilon))^{-\xi}
\]

for the small values of \( \varepsilon > 0 \). From the last inequality above,

\[
\psi_1(\varepsilon) > e^{\varepsilon^{-\frac{1}{\xi}}}
\]

(79)

is obtained. From (77), (78) and (79)

\[
\left| \frac{1}{\delta} \sum_{j=1}^{l_\varepsilon} \frac{\sum_{i=1}^{N(\varepsilon)} \lambda_i}{\psi_j(\varepsilon)} \psi_j(\varepsilon) - 1 \right| < \text{const.} \varepsilon^{-m} e^{\frac{\varepsilon^{-1} \psi_1^{\frac{1}{4}}}{\psi_1^{\frac{1}{4}}} - \varepsilon^{-\beta}}
\]

(80)

is found. We can rewrite inequality (80)

\[
\frac{1}{\delta} \sum_{j=1}^{l_\varepsilon} \frac{\sum_{i=1}^{N(\varepsilon)} \lambda_i}{\psi_j(\varepsilon)} \psi_j(\varepsilon) - 1 = O(e^{-\varepsilon^{-\beta}})
\]

(81)
as \( \varepsilon \to 0 \). From (2), (42) and (81)

\[
\sum_{-\lambda_i < -\varepsilon} \lambda_i = \frac{1}{3\pi} [1 + O(e^{-\varepsilon^2})] \sum_j \int_{\alpha_j(x) \geq \varepsilon} \sqrt{\frac{\alpha_j(x) - \varepsilon}{p(x)}} (2\alpha_j(x) + \varepsilon) dx
\]
as \( \varepsilon \to 0 \), is obtained. \( \blacksquare \)

Let us assume that the function \( \alpha_1(x) \) satisfies the following condition:

\(\alpha_2\) For every \( \eta > 0 \)

\[
\lim_{x \to \infty} \alpha_1(x)x^{a_0 - \eta} = \lim_{x \to \infty} [\alpha_1(x)x^{a_0 + \eta}]^{-1} = 0
\]

Here, \( a_0 \) is a constant in the interval \((0, \frac{2}{3})\).

**Theorem 3.2** We suppose that the operator function \( Q(x) \), the scalar function \( p(x) \) satisfy the condition \( Q1) - Q3) \), \( p1) - p3) \) and \( \alpha_1(x) \) also satisfies the condition \( \alpha_2 \). In addition the series \( \sum_{j=1}^{\infty} [\alpha_j(0)]^m \) is convergent for a constant \( m \) satisfying the condition

\[
0 < m < \frac{(2 - 3a_0)^2}{2a_0(4 - 3a_0)}
\]

then the asymptotic formula

\[
\sum_{i=1}^{N(\varepsilon)} \lambda_i = \frac{1}{3\pi} [1 + O(\varepsilon^2)] \sum_j \int_{\alpha_j(x) \geq \varepsilon} \sqrt{\frac{\alpha_j(x) - \varepsilon}{p(x)}} (2\alpha_j(x) + \varepsilon) dx
\]
is satisfied as \( \varepsilon \to 0 \). Where \( t_0 \) is a positive constant.

**Proof**: By Theorem 2.4 and Theorem 2.5, we have

\[
\left| \sum_{i=1}^{N(\varepsilon)} \lambda_i - \delta^{-1} \sum_{j=1}^{\delta} \int_0^\psi_j(\varepsilon) \beta_j(\varepsilon, x) dx \right| < \text{const.} l_\delta \left( \int_0^\delta \alpha_1^2(x) dx + \psi_1^2(\varepsilon) \right)
\]

for the small values of \( \varepsilon > 0 \). Since the function \( \alpha_1(x) \) is decreasing,

\[
\alpha_1(x) \geq \alpha_1(\psi_1(2\varepsilon)) = 2\varepsilon
\]
in the interval \([0, \psi_1(2\varepsilon)]\). Since the function \( p(x) \) satisfies the condition \( p1) \) and (42), (84) then we find
\[ \delta^{-1} \sum_{j=1}^{\ell_x} \int_0^{\psi_j(\varepsilon)} \beta_j(\varepsilon, x) \, dx > \frac{1}{3\pi} \int_0^{\psi_1(\varepsilon)} \sqrt{\frac{\alpha_1(x) - \varepsilon}{p(x)}} \left(2\alpha_1(x) + \varepsilon\right) \, dx \]

\[ > \text{const.} \varepsilon^{\frac{3}{2}} \psi_1(2\varepsilon) \]  \hfill (85)

If we consider that the function \( \alpha_1(x) \) satisfies the condition \( \alpha_2 \) and \( \lim_{\varepsilon \to 0} \psi_1(\varepsilon) = \infty \), then we have

\[ \lim_{\varepsilon \to \infty} \frac{\alpha_1(\varepsilon)}{\psi_1(2\varepsilon)(\psi_1(2\varepsilon))^{a_0+\eta}} = 0 \]

From the last equality above, we obtain

\[ \psi_1(2\varepsilon) > (\varepsilon)^{a_0+\eta} \]  \hfill (86)

for the small value of \( \varepsilon > 0 \). From (85) and (86)

\[ \delta^{-1} \sum_{j=1}^{\ell_x} \int_0^{\psi_j(\varepsilon)} \beta_j(\varepsilon, x) \, dx > \text{const.} \varepsilon^{\frac{3a_0+3\eta}{2(a_0+\eta)}} \]  \hfill (87)

is found. We limit the integral \( \int_0^{\delta} \alpha_1^\frac{3}{2}(x) \, dx \) at the right hand side of the inequality (83). Since the function \( \alpha_1(x) \) satisfies the condition \( \alpha_2 \), then we have

\[ \alpha_1(x) \leq \text{const.} x^{\eta-a_0} \quad (\eta < a_0). \]  \hfill (88)

Therefore we have

\[ \int_0^{\delta} \alpha_1^\frac{3}{2}(x) \, dx \leq \text{const.} \int_0^{\delta} x^{\frac{3}{2}(\eta-a_0)} \, dx < \text{const.} \delta^{\frac{3}{2}(2-3a_0+3\eta)}. \]  \hfill (89)

On the other hand, from (3)

\[ \delta < \psi_1^{1-a}(\varepsilon) \]  \hfill (90)

is obtained. If we take \( x = \psi_1(\varepsilon) \) in the inequality (88), then we find

\[ \alpha_1(\psi_1(\varepsilon)) \leq \text{const.} \psi_1^{\eta-a_0}(\varepsilon) \quad (\eta < a_0) \]

or

\[ \psi_1(\varepsilon) \leq \text{const.} \varepsilon^{\frac{1}{a_0-\eta}} \]  \hfill (91)
From (89), (90) and (91), we have

\[ \frac{\delta}{\int_0^{\alpha} \frac{3}{2} (x) dx} \leq \text{const.} \varepsilon^{-(1-a)^2 (2-3a_0+3\eta) / 2(a_0-\eta)} \]  \hspace{1cm} (92)

From (78), (91) and (92)

\[ l_\varepsilon \frac{\delta}{\int_0^{\alpha} \frac{3}{2} (x) dx} < \text{const.} \varepsilon^{-m - (1-a)^2 (2-3a_0+3\eta) / 2(a_0-\eta)} \]  \hspace{1cm} (93)

\[ l_\varepsilon \psi_1^a(\varepsilon) < \text{const.} \varepsilon^{-m(a_0-\eta)/ (a_0-\eta)} \]  \hspace{1cm} (94)

are found. From (87), (93) and (94) we obtain

\[ \frac{l_\varepsilon \int_0^{\alpha} \frac{3}{2} (x) dx}{\delta^{-1} \sum_{j=1}^{l_\varepsilon} \int_0^{\beta_j(\varepsilon)} \psi_j(\varepsilon) dx} < \text{const.} \varepsilon^{F_1(\eta)} \]  \hspace{1cm} (95)

and

\[ \frac{l_\varepsilon \psi_1^a(\varepsilon) dx}{\delta^{-1} \sum_{j=1}^{l_\varepsilon} \int_0^{\beta_j(\varepsilon)} \psi_j(\varepsilon) dx} < \text{const.} \varepsilon^{F_2(\eta)} \]  \hspace{1cm} (96)

Here,

\[ F_1(\eta) = -m - \frac{(1-a) (2-3a_0+3\eta)}{2(a_0-\eta)} - \frac{3a_0 + 3\eta - 2}{2(a_0+\eta)} \]

\[ F_2(\eta) = -m(a_0-\eta)/ (a_0-\eta) - \frac{3a_0 + 3\eta - 2}{2(a_0+\eta)} \]

There is a number \( \omega = \omega(t) > 0 \) \( (0 < \eta < \omega) \) such that

\[ F_1(\eta) > \frac{2a - 2a_0m - 3aa_0}{2a_0} - t \]  \hspace{1cm} (97)

\[ F_2(\eta) > \frac{2 - 3a_0 - 2a_0m - 2a}{2a_0} - t \]  \hspace{1cm} (98)

for every \( t > 0 \). If we take

\[ a = \frac{(2-3a_0)^2 + 6a_0^2m}{4(2-3a_0)} \]

\[ t = t_0 = \frac{1}{16a_0} \left( (2-3a_0)^2 + 6a_0^2m - 8a_0m \right) \]
in the inequalities (97) and (98), then we have
\[ F_1(\eta) > t_0 \quad ; \quad F_2(\eta) > t_0. \]  
(99)

Since the number \( m \) satisfies the condition (82), we have \( a \in (0, 1) \) and \( t_0 > 0 \).
From (83), (95), (96) and (99) we obtain
\[ \left| \frac{N(\varepsilon)}{\delta^{-1} \sum_{j=1}^{N(\varepsilon)} \int_0^1 \beta_j(\varepsilon, x)dx} \sum_{i=1}^{N(\varepsilon)} \lambda_i \frac{\psi_j(\varepsilon)}{\psi_j(\varepsilon)} - 1 \right| < \text{const.} \varepsilon^{t_0}. \]  
(100)

By (42), (97) and (100) we have the asymptotic formula
\[ N(\varepsilon) = \frac{1}{3\pi} \left[ 1 + O(\varepsilon^{t_0}) \right] \sum_{j} \int_{\alpha_j(x) \geq \varepsilon} \sqrt{\frac{\alpha_j(x) - \varepsilon}{p(x)}} (2\alpha_j(x) + \varepsilon)dx \]
as \( \varepsilon \to 0. \) □

**Example 3.3** Let \( H = L^2[0, \pi] \) be a separable Hilbert space and \( e_i = \sqrt{\frac{2}{\pi}} \sin ix \quad (i = 1, 2, \cdots) \) be a standard basis in \( H \). Let \( Q(x) : H \to H \)
\[ Q(x)f = \sum_{i=1}^{\infty} \alpha(x)i^{-2}(f, e_i)e_i \quad (f \in H) \]
for all \( x \in [0, \infty) \). \( Q(x) \) is a self adjoint, completely continuous and positive operator function. The eigenvalues of \( Q(x) \) are in the form
\[ \alpha(x) = \begin{cases} \frac{2}{\ln \ln b} - \frac{x}{b \ln \ln b} & 0 \leq x \leq b \\ \frac{1}{\ln \ln x} & b \leq x < \infty \end{cases} \]
Here \( b > e^3 \) is a constant such that \( \ln x > (\ln \ln x)^2 \).

**References**

[1] Şengül, S. The asymptotic behaviour of the spectrum of negative part of Sturm-Liouville problem with operator coefficient, PhD thesis in YTU FBE (2006). (In Turkish).

https://tez.yok.gov.tr/Ulusaltetzgokmerkezi/TezGoster?key= -L8ilcw9ZRRcY-MKxXW1u5yjU0ABL-nqSBJoTERxJektuxRXLjFdBQGCdCtgfjhR
[2] Adıgüzelo, E.E., Oer, Z. Asymptotic Expansion for the sum of negative Eigenvalues of Sturm-Liouville operator given in Semi-axis, YTÜD, (2002), Vol 1,26-35.

[3] Bakși, Ö., Ismayılov, S. An asymptotic formula for the sum of negative eigenvalues of second order differential operator given in infinite interval, Sigma Mühendislik ve Fen Bilimleri Dergisi 2005-4, 87-98. (In Turkish)

[4] Skac.ek B.Y. Asymptod of Negative Part of Spectrum of One Dimensioned Differential Operators, Pribl. metodi res.eniya differn. uraveniy, Kiev, 1963”, Pribl. Metod reseniya differens, unavneniy, Kiev, (1963).

[5] Adıgüzelo, E.E. The asymptotic behaviour of the spectrum’s negative part of Sturm-Liouville problem with operator coefficient, Izv. AN Az.SSR, Seriya fiz.-tekn.i mat. nauk, No:6, 8-12, (1980). (In Russian)

[6] Maksudov F.G., Bayramoğlu M., Adıgüzelo E.,On asymptotics of spectrum and trace of high order differential operator with operator coefficients, Doğu-Turkish journal of Mathematics, (1993), vol.17.

[7] Adıgüzelo, E.E., Bakși, Ö., Bayramov, A.M. The Asymptotic Behaviour of the Negative Part of the Spectrum of Sturm-Liouville Operator with the Operator Coefficient which Has Singularity, International Journal of Differential Equations and Applications, Vol.6, No.3, 315-329, (2002).

[8] Gohberg, I.C. and Krein, M.G., Introduction to the Theory of Linear Non-self Adjoint Operators in Hilbert Space, Translation of Mathematical Monographs, Vol.18 (AMS, Providence, R.I.,1969).

[9] Lysternik, L.A. and Sobolev, V.I. Elements of Functional Analysis, (English translation ), John Willey Sons, New York, page 229, (1974).

[10] Glazman, I.M. Direct methods qualitative spectral analysis of singular differential operators, Jerusalem, pages 34-44, (1965).

[11] Smirnov,V.I., A Course of Higher Mathematics, vol.5, Pergamon Pres, New York, page 623, (1964).

[12] Courant, R. and Hilbert, D., Methods of Mathematical Physics, vol.1, New York, page 408, (1966).