Separation of spin and charge in the continuum
Schrodinger equation

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Abstract

I describe here the attempt to introduce spin-charge separation in Schrodinger equation. The construction we present here gives a decomposed Schrodinger spinor that has one problem: Its absolute value can only have value between 0 and $\frac{1}{2}$. The problem we solve is to expand and generalize this construction so that one can have a Schrodinger spinor with absolute value that are arbitrary non-negative numbers.

It may be that one has to introduce a set of different decompositions to cover all nonnegative values, that is to introduce patches over $\mathbb{R}_3^+$ so that in each patch one has a different representation.

It seems that the decomposition has a direct relation to so called entangled states that have been discussed very much in connection of e.g. quantum computing, and we would like to find this relation and discuss it in detail.

1 Introduction

Separation of spin and charge is an behavior of electrons in some materials in which they split into three independent particles (spinon, orbiton and chargon). It is one of most manifestations of quasiparticles, although that spinon and chargon are note gauge invariant quantity. A spin-charge separation could have far-reaching practical consequences to spintronics [1] that develops devices which are driven by the spin properties of electrons. In a wider context [2], the spin-charge separation could possibly explain the behavior of elementary particles in dense environments such as Early Universe and the interior of compact stars. It might even become visible in the LHC-ALICE experiment at CERN. In this paper I extend the result on [3] and provide new class of spin and charge decomposition.

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Plan of paper

The work is organized as follows. In section 2 we discuss the separation of spin and charge in spin chains and lattice electrons in the case of non-relativistic context of lattice condensed matter electrons. In section 3 we show how the previous lattice decomposition is realized in the context of the nonrelativistic Schrödinger quantum mechanics in combination with Maxwell’s electrodynamics. We come to the problem with this decomposition and show that the decomposition is not complete because $0 \leq |\psi| \leq \frac{1}{2}$. For the Schrödinger equation we demand that the integral of $|\psi|^2 = 1$ but at a given point $x$ the absolute value of the wavefunction can be any real number. In section 4 we propose the generalization of the decomposition and solve the problem of the limit of absolute value of $\psi$. This generalization shows that $a \leq |\psi| < \infty$, where $a$ is a real positive value.

2 Spin chains and lattice electrons

In this section we describe spin-charge separation in the non-relativistic context of lattice condensed matter electrons.

More extensive discussions on spin-charge separation and strongly correlated lattice electrons can be found e.g. in [4], [5], [6].

2.1 Lattice fermions

We now describe the quantitative aspects of the spin-charge separation in a fermionic spin one-half system such as (condensed matter) electrons.

We consider non-relativistic fermions (electrons) that we describe by anti-commutating creation and annihilation operators $c_{l\alpha}^\dagger$ and $c_{l\alpha}$, respectively. Here $\alpha$ with the two value $\alpha = \uparrow, \downarrow$ is an index for the spin-up and spin-down states at the site $l$ of a lattice: We do not specify the dimensionality of the lattice, in particular it can now be more than one dimensional. The operators subject to the anti commutation relations

$$\{c_{l\alpha}, c_{l'\beta}^\dagger\} = \{c_{l\alpha}^\dagger, c_{l'\beta}\} = 0, \quad \{c_{l\alpha}, c_{l'\beta}^\dagger\} = \delta_{ll'} \delta_{\alpha\beta}. \tag{2.1}$$

At each site of the lattice these operators span a Hilbert space with four states. These are the (Fock) vacuum $|0\rangle$, the state with spin-up ($\uparrow$) and the state with spin-down ($\downarrow$),

$$c_{\uparrow}^\dagger |0\rangle = |\uparrow\rangle, \quad c_{\downarrow}^\dagger |0\rangle = |\downarrow\rangle \tag{2.2}$$

and the doubly occupied state

$$c_{\uparrow}^\dagger c_{\downarrow}^\dagger |0\rangle = |\uparrow\downarrow\rangle. \tag{2.3}$$
The Fock vacuum and the doubly occupied state are bosonic, the other two are fermionic. We introduce a spin-charge decomposition in these creation and annihilation operators. It is defined by the following transformation

\[ c_{l\alpha}^{\dagger} = b_{l}.s_{l\alpha}^{\dagger} + d_{l}^{\dagger}.\epsilon_{\alpha\beta}s_{l\beta}, \quad c_{l\alpha} = b_{l}^{\dagger}.s_{l\alpha} + d_{l}.\epsilon_{\alpha\beta}s_{l\beta}^{\dagger} \] (2.4)

with \( \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} \) and \( \epsilon_{12} = 1 \). Here \( s_{l\alpha}^{\dagger} \) and \( s_{l\alpha} \) are fermionic creation and annihilation operators. They are called spinon operators. The spinons clearly carry the spin degree of freedom in the decomposition. The two bosonic operators \( b_{l}^{\dagger} \) and \( b_{l} \) correspond to states that are called holons. The two bosonic operators \( d_{l}^{\dagger} \) and \( d_{l} \) correspond to states that are called doublons.

By construction, the bosonic operators have no spin. But they do carry the electric charge which becomes apparent when we consider the symmetry properties of the decomposition. For this we introduce a local Maxwell \( U(1) \) gauge transformation. It acts on the electron operators in the standard fashion as follows,

\[ c_{l\alpha}^{\dagger} \rightarrow e^{i\phi_{l}}.c_{l\alpha}^{\dagger} \] (2.5)

and

\[ c_{l\alpha} \rightarrow e^{-i\phi_{l}}.c_{l\alpha}. \] (2.6)

From (2.4) we conclude immediately that we can take

\[ b_{l} \rightarrow e^{i\phi_{l}}.b_{l}, \quad d_{l} \rightarrow e^{-i\phi_{l}}.d_{l}, \quad s_{l\alpha} \rightarrow s_{l\alpha}. \] (2.7)

This implies that the holons and the doublons carry an electric charge which is equal to that of an electron, while the spinons are charge neutral. As a consequence (2.4) indeed decomposes the electron creation and annihilation operators in terms of independent spin and charge carriers.

If we substitute the decomposition (2.4) in the anticommutation relations (2.1) of \( c_{l\alpha} \) and \( c_{l\alpha}^{\dagger} \) and assume that the holon, doublon and spinon operators each obey standard canonical anti commutation relations of bosons and fermions respectively, we find that the decomposed electron operator is also subject to the canonical anti commutators but with the following modification

\[ \{c_{l\alpha},c_{l'\beta}^{\dagger}\} = \left(b_{l}^{\dagger}.b_{l'} + d_{l}^{\dagger}.d_{l'} + s_{l\downarrow,\uparrow}^{\dagger}.s_{l'\downarrow,\uparrow} + s_{l\downarrow,\downarrow}^{\dagger}.s_{l'\downarrow,\downarrow}\right)\delta_{ll'}\delta_{\alpha\beta}. \] (2.8)

Here

\[ \hat{N}_{l} = b_{l}^{\dagger}.b_{l} + d_{l}^{\dagger}.d_{l} + s_{l\downarrow,\uparrow}^{\dagger}.s_{l\downarrow,\uparrow} + s_{l\downarrow,\downarrow}^{\dagger}.s_{l\downarrow,\downarrow} \] (2.9)

is the total number operator at the site \( l \). In the subspace of states \(|\text{phys}>\) that obey

\[ \hat{N}_{l}|\text{phys}> = |\text{phys}> \] (2.10)
for every \( l \), the decomposed electron creation and annihilation operators then reproduce the canonical anti commutators (2.1).

The constraint (2.10) states that exactly one particle occupies each lattice site. In this sense the liquid composed of spinons and chargons is incompressible.

We also note that the constraint (2.10) resembles the Gaub law constraint of a gauge theory, with \( \hat{N}_l \) the (abelian) charge operator.

One can show that in the subspace of states \(|\text{phys}\rangle\) the holon operators can be interpreted as creation and annihilation operators corresponding to the bosonic and spinless Fock vacuum state \(|0\rangle\) of the electron, the doublons as creation and annihilation operators for the spinless and bosonic doubly occupied state \(|\uparrow\downarrow\rangle\), and the spinons as creation and annihilation operators corresponding to the fermionic spin \(\pm 1/2\) states \(|\uparrow\rangle\) and \(|\downarrow\rangle\).

Beside the action of the Maxwellian \(U(1)\) phase rotation, we also identify in the decomposition (2.4) the following local internal \(U_{\text{int}}(1)\) symmetry: if we send

\[
 b_l \rightarrow e^{i\eta_l} b_l, \quad d_l \rightarrow e^{i\eta_l} d_l, \quad s_{l\alpha} \rightarrow e^{i\eta_l} s_{l\alpha}
\]

the electron operators \(c_{l\alpha}\) and \(c_{l\alpha}^\dagger\) remain intact. Since the decomposition should not introduce any (vortex-like) line singularities in the operators, we define the angle \(\eta_l\) modulo \(2\pi\). This implies that the internal \(U_{\text{int}}(1)\) gauge group is compact. This internal \(U(1)\) symmetry is presumed to have important physical consequences. In particular, since \(U_{\text{int}}(1)\) is a compact group we expect \([7]\) that there is a first-order phase transition between a strong coupling confined phase and a week coupling deconfined phase. Since there is a priori no natural kinetic term for any dynamically independent gauge field of the internal group, we are formally in the infinite coupling limit with respect to a coupling of the internal group. This means that under normal circumstances we are in the confinement phase of the compact gauge theory. From the point of view of physical applications this could be the explanation why under these normal circumstances the spinon and the chargon are tightly confined to each other into a pointlike electron.

In the minimal version of the compact \(U(1)\) theory the coupling does not run. But it is conceivable that in the presence of additional fields such as the spin and charge variables, the \(\beta\)--function for the coupling becomes nontrivial. Since the gauge group is Abelian, we expect that the strength of the coupling increases at high energies. This leads us to a picture which is consistent with the known pointlike structure of an individual \(S\)--matrix electron in the high energy limit. The shorter the distance, the tighter the chargon and holon become confined to each other. But at low energies and in a proper quantum protectorate where the coupling of the internal gauge group becomes weak the spinon and chargons may become deconfined. A spin-charge decomposition can then take place and instead of pointlike electrons, fundamental constituents are deconfined spinons and chargons.
Suppose that the internal gauge symmetry corresponds to a fundamental interaction akin Maxwell’s electromagnetism, that there is indeed a yet unseen fundamental force which under normal material conditions is so strong that it confines the spinons and chargons into pointlike electrons. The condensed matter electron would like the Cooper pair of spinon and chargon, and the different quantum protectorates would describe different phases in the conventional sense, separated from each other by conventional phase transitions of the internal gauge group. The concept of a quantum protectorate would then reduce to the standard concept of a phase. But until now, there is no experimental evidence of a new fundamental force corresponding to the compact internal $U(1)$. Thus this internal symmetry remains a mathematical construct and the notion of a quantum protectorate endures as conceptually distinct from the notion of a phase.

The decomposition (2.14) is not unique. For example, we could also introduce the truncated decomposition

$$c_{l\alpha}^\dagger = b_{l\alpha}^\dagger s_{l\alpha}, \quad c_{l\alpha} = b_{l\alpha}^\dagger s_{l\alpha}$$ (2.12)

Now the canonical fermionic anticommutation relations lead to the following structure (we suppress the lattice site index $l$)

$$\{c_{\uparrow}^\dagger, c_{\uparrow}\} = b^\dagger b + s_{\uparrow}^\dagger s_{\uparrow} =: \hat{N}_{\uparrow}$$ (2.13)

$$\{c_{\downarrow}^\dagger, c_{\downarrow}\} = b^\dagger b + s_{\downarrow}^\dagger s_{\downarrow} =: \hat{N}_{\downarrow}$$ (2.14)

$$\{c_{\uparrow}^\dagger, c_{\downarrow}\} = s_{\uparrow}^\dagger s_{\downarrow} =: \hat{C}^-$$ (2.15)

$$\{c_{\downarrow}^\dagger, c_{\uparrow}\} = s_{\downarrow}^\dagger s_{\uparrow} =: \hat{C}^+$$ (2.16)

with all other anticommutators vanishing. The relation (2.13), (2.14) (2.15) (2.16) can be generalized as

$$\{c_{\alpha}^\dagger, c_{\beta}\} = s_{\alpha}^\dagger s_{\beta} + b^\dagger b \delta_{\alpha\beta}$$ (2.17)

As a consequence the decomposed operators (2.12) reproduce the canonical fermionic anticommutation relations in the subspace of states $|\text{phys}\rangle$ that are subject to the conditions

$$\hat{N}_{\downarrow}|\text{phys}\rangle = \hat{N}_{\uparrow}|\text{phys}\rangle = |\text{phys}\rangle$$ (2.18)

and

$$\hat{C}^\pm|\text{phys}\rangle = 0$$ (2.19)

and in addition the (anti)commutators of these four operators must vanish weakly, in the subspace of states $|\text{phys}\rangle$. 

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We first observe that

\[ [\hat{C}^-, \hat{C}^+] = s^\dagger_\uparrow s^\uparrow_\uparrow - s^\dagger_\downarrow s^\downarrow_\downarrow =: \hat{C}^0 \]  

(2.20)

and

\[ [\hat{C}^+, \hat{C}^0] = s^\dagger_\uparrow s^\uparrow_\uparrow (2s^\dagger_\uparrow s^\uparrow_\uparrow - 2s^\dagger_\downarrow s^\downarrow_\uparrow) = 2\hat{C}^+ \hat{C}^0 \]  

(2.21)

\[ [\hat{C}^-, \hat{C}^0] = s^\dagger_\uparrow s^\downarrow_\downarrow (2s^\dagger_\uparrow s^\uparrow_\uparrow - 2s^\dagger_\downarrow s^\downarrow_\downarrow) = 2\hat{C}^- \hat{C}^0 \]  

(2.22)

consequently in the constraint surface \((\hat{C}^0, \hat{C}^\pm)\) defines a \(SU(2)\) algebra and consistency of our construction demands that the states \(|\text{phys} \rangle\) are subject to the (first class) \(SU(2)\) constraint algebra (Gaub law).

\[ \hat{C}^\pm |\text{phys} > = \hat{C}^0 |\text{phys} > = 0. \]  

(2.23)

Furthermore, since we clearly have

\[ [\hat{C}^\pm, \hat{N}_\uparrow] = [\hat{C}^\pm, \hat{N}_\downarrow] = [\hat{C}^0, \hat{N}_\uparrow] = [\hat{C}^0, \hat{N}_\downarrow] = 0 \]  

(2.24)

the constraint algebra is also consistent with the conditions \((2.18)\). This ensures that in the subspace \(|\text{phys} >\) the decomposed operators do realize the canonical fermionic anti commutation relations.

Finally, the states \(|\text{phys} >\) can be constructed as follows: We start from the following Fock ground state \(|0 >\),

\[ b|0 >= s^\dagger_\uparrow |0 >= s^\dagger_\downarrow |0 >= 0. \]  

(2.25)

Clearly, this state also obeys

\[ \hat{C}^\pm |0 >= \hat{C}^0 |0 >= 0. \]  

(2.26)

We then introduce the following two states

\[ |b >= b^\dagger |0 >, \quad |\uparrow\downarrow >= s^\dagger_\uparrow s^\dagger_\downarrow |0 >. \]  

(2.27)

One can easily verify that these two states satisfy the conditions we have imposed on \(|\text{phys} >\). These two states span the Hilbert space where the decomposed operators \((2.12)\) realize the fermionic anti commutation relation.

Note that on the remaining two states at each lattice site,

\[ |\uparrow >= s^\dagger_\uparrow |0 >, \quad |\downarrow >= s^\dagger_\downarrow |0 > \]  

(2.28)

the action of the constraint operators \((\hat{C}^0, \hat{C}^\pm)\) becomes realized in terms of conventional Pauli matrices.
Clearly, the physical state $|0>$ can be identified as the holon, and the physical state $|\uparrow\downarrow>$ can be identified as the doublon. Consequently (2.12) is a projection of the electron creation and annihilation operators to the subspace of chargons.

Finally, there is also the following decomposition [6].

$$c_{l\alpha}^{\dagger} = \frac{b_{l}^{\dagger}}{\sqrt{b_{l}^{\dagger}b_{l}}} s_{l\alpha}^{\dagger}, \quad c_{l\alpha} = \frac{b_{l}}{\sqrt{b_{l}^{\dagger}b_{l}}} s_{l\alpha}. \quad (2.29)$$

These decomposed operators satisfy the fermionic anti commutation relations without any additional constraints.

A decomposition such as (2.4), (2.12) and (2.29) admits a well-defined interpretation in terms of group representation theory: the original creation and annihilation operators realize a tensor product representation of the Maxwellian gauge group and the spatial rotation group. The spin-charge operators can be interpreted in terms of a Clebsch-Gordan expansion that decomposes this tensor product into irreducible representations of the two groups.

Usually in Physics, we have learned to expect that the irreducible components in a Clebsch-Gordan expansion have more fundamental physical value than their reducible tensor products. But in the present case the physical relevance of the decomposition derives from two added dynamical criteria:

- The decomposition must be consistent with the particle interpretation of the operators involved. This means that (anti)commutation relations of all operators involved must have the canonical structure which is consistent with the creation and annihilation of individual particle states.

- When the Hamiltonian is realized in terms of the decomposed operators, it must have a physically meaningful form.

Only when these two added criteria are fulfilled, we can expect that a spin-charge separation becomes realized in the Nature. From our schematic examples we expect that this will be the case at least when we implement a spin-charge decomposition in a proper Hamiltonian realization of the one dimensional Neel antiferromagnet.

### 2.2 Hubbard Model

Nowadays, there is little doubt about the microscopic theory that describes cuprate superconductors [4]. These materials are modeled by a (single band) Hubbard-model [8] with Hamiltonian of the form

$$H = H_{t} + H_{U} = - \sum_{l,l',\alpha} t_{l-l'} c_{l\alpha}^{\dagger} c_{l\alpha} + U \sum_{l} n_{l\uparrow} n_{l\downarrow}. \quad (2.30)$$
Here the summations extend over all \((2d)\) lattice sites \(l\) and over the spin degrees of freedom \(\alpha = \uparrow, \downarrow\) and

\[
n_{l\alpha} = c_{l\alpha}^\dagger c_{l\alpha} \tag{2.31}
\]

is the number operator and \(t_{l-l'}\) and \(U\) are phenomenological parameters.

The properties of the Hubbard model including spin-charge separation have been discussed widely in the literature, and we refer to [4], [5], [8] for details.

When we introduce the spin operator

\[
S_l = \frac{1}{2} \sum_{\alpha\beta} c_{l\alpha}^\dagger \hat{\sigma}_{\alpha\beta} c_{l\beta} \tag{2.32}
\]

where

\[
\hat{\sigma} = (\sigma^1, \sigma^2, \sigma^3) \tag{2.33}
\]

are the three Pauli matrices, we can write

\[
H_U = \frac{U}{2} \sum_l (n_{l\uparrow} + n_{l\downarrow}) - \frac{2U}{3} \sum_l S_l^2 \tag{2.34}
\]

and if we demand the strong repulsion constraint

\[
\sum_{\alpha} c_{l\alpha}^\dagger c_{l\alpha} = 1 \tag{2.35}
\]

the components of the spin operator \((2.32)\) satisfy the spin commutation relations of the Pauli matrices.

The condition \((2.35)\) imposes the constraint that exactly half of the band is filled. If we remove electrons from the half-filled state the system becomes underdoped and the constraint \((2.35)\) is replaced by the non-holonomic

\[
\sum_{\alpha} c_{l\alpha}^\dagger c_{l\alpha} \leq 1. \tag{2.36}
\]

There are now holes in the otherwise uniform charge distribution. As in our schematic example, these holes which correspond to sites that are unoccupied by electrons propagate along the lattice when electrons jump from an occupied site to an unoccupied site. This leads to the propagation of charge but with no transport of spin. As a consequence in an underdoped system we expect that the spin and the charge can be transported independently. Instead of electrons the fundamental constituents are now quasiparticles that are either chargeless fermionic spinons or spinless bosonic holons, which in general travel with different velocities.

A cuprate superconductor is supposedly described by such an underdoped system. The constraint \((2.36)\) is non-holonomic but it becomes elegantly resolved in terms of
the decomposed variables (2.4). For this we simply project the decomposition (2.4) to
the subspace where

\[ d_l|\text{phys}> = d^\dagger_l|\text{phys}> = 0. \quad (2.37) \]

This clearly removes the doubly occupancy of the states. For simplicity one may wish
to use the truncated decomposition (2.12) in an effective manner, but with proper
care since the constraint structures are different: A naive use of (2.12) implements a
projection to the subspace of holons and doublons instead.

In a mean field approximation where we integrate over the fermions \( s_{l\alpha} \) and \( s^\dagger_{l\alpha} \), we now
expect to get (d-wave) superconductivity when the holon fields \( b_l \) condense,

\[ <b^\dagger_l b_l> = \Delta_b \neq 0 \quad (2.38) \]

which corresponds to the relation (2.36) in the present case.

The ensuing phase diagram is quite elaborate, there are several different phase regions
[4]. Besides (2.38), of particular interest in the pseudo-gap phase with the characteristic
property that even though the underlying symmetry is broken the effective bosonic order
parameter \( \Delta_b \) vanishes due to quantum fluctuations.

Finally, our one dimensional example (figures 3-9) reveals that there is an apparent
symmetry between the transport of holons and doublons. Obviously, the condition for
overdoping can be introduced in manner which is parallel to (2.37), by projecting to
the subspace where

\[ b_l|\text{phys}> = b^\dagger_l|\text{phys}> = 0 \quad (2.39) \]

and assuming that there is a doublon condensation.

### 3 Spin charge separation and the Schrodinger equation

In the continuum limit the interactive dynamics of condensed matter electrons is gov-
erned by nonrelativistic Schrodinger quantum mechanics in combination with Maxwell’s
electrodynamics. In this section we study how the previous lattice decompositions be-
come realized in that context. We find that the spin-charge decomposed two-component
Pauli spinor has a structure which is very similar to that of the lattice fermion, including
the appearance of a \( SU(2) \) constraint algebra.

#### 3.1 Decomposing the Pauli spinor

In obvious notation the classical Schrodinger Lagrangian including the Maxwellian con-
tribution is

\[ L[\psi, \psi^\dagger, A_\mu] = \psi^\dagger \left( i\partial_0 - eA_0 \right) \psi + \frac{1}{2m} \left| \left( i\partial_k - eA_k \right) \psi \right|^2 - \frac{1}{4} F^2_{\mu\nu}. \quad (3.1) \]
Here $\psi$ is the two-component commuting (Pauli) spinor. We interpret it as a Hartree-type many-body wavefunction that describes the nonrelativistic dynamics of an ensemble of interacting electrons in its totally antisymmetric subspace \[ \wedge \].

From (3.1) we infer the following familiar canonical Poisson brackets for the Pauli spinor

\[
\{ \psi_\alpha^*(x), \psi_\beta(x') \} = \delta_{\alpha\beta} (x - x'); \quad \{ \psi_\alpha^*(x), \psi_\beta^*(x') \} = \{ \psi_\alpha(x), \psi_\beta(x') \} = 0. \tag{3.2}
\]

The structure and normalization of these brackets ensure that at the level of the quantum theory where the brackets become replaced by (anti)commutators, the Pauli spinors acquire the standard interpretation in terms of proper fermion creation and annihilation operators.

The obvious Ansatz for a spin-charge decomposition is

\[
\psi_\alpha = \phi_\alpha S_\alpha \tag{3.3}
\]

where $\phi(x)$ is to be viewed as the chargon and $S_\alpha$ as the spinon. However, despite its naturalness this decomposition leads to difficulties: Even though (3.3) can be viewed as a change of variables in the Lagrangian once we first remove the degeneracy in the right hand side, somewhat unexpectedly in turns out that (3.3) is not consistent with the interpretation of the chargon and the spinon in terms of particle states in the second quantized quantum theory. This can be seen either by substituting (3.3) in (3.1) and carefully analysing the properties of the ensuing decomposed Lagrangian, or more directly by inspecting the canonical structure of (3.3). Here we follow the latter route as it is an independent model and thus it has more generality.

In order that $\phi$ and $S_\alpha$ acquire a proper particle interpretation at the quantum level, their classical Poisson brackets should have the following structure:

\[
\{ \phi^*(x), \phi(x') \} = \delta(x - x'); \quad \{ S_\alpha^*(x), S_\beta(x') \} = \delta_{\alpha\beta} (x - x') \tag{3.4}
\]

with all other brackets vanishing. These Poisson brackets ensure that at the second quantized level we can consistently expand the fields $\phi(x)$ and $S_\alpha$ in terms of particle creation and annihilation operators, in the usual manner.

If we substitute the decomposition (3.3) in (3.2), we arrive at the following non-vanishing brackets for the decomposed Pauli spinor,

\[
\{ \psi_\alpha^*(x), \psi_\alpha(x') \} = \left( \phi^*(x)\phi(x) + S_\alpha^*(x)S_\alpha(x) \right) \delta(x - x') \\
= \delta_\alpha(x) \delta(x - x'), \quad (\alpha = 1, 2) \tag{3.5}
\]

and

\[
\{ \psi_1^*(x), \psi_2(x') \} = S_1^*(x)S_2(x) \delta(x - x') =: C^- (x) \delta(x - x') \tag{3.6}
\]

\[
\{ \psi_2^*(x), \psi_1(x') \} = S_2^*(x)S_1(x) \delta(x - x') =: C^+ (x) \delta(x - x'). \tag{3.7}
\]
These brackets reproduce \(^{3,2}\) provided we can consistently impose the constraints,

\[
g_\alpha(x) \approx 1, \quad (\alpha = 1, 2), \quad \text{and} \quad C^\pm(x) \approx 0
\]  

These constraint functionals have the following non-vanishing Poisson bracket relations

\[
\{C^-(x), g_1(x')\} = S_1^*(x)S_2(x)\delta(x - x') = C^-(x)\delta(x - x') \quad (3.9)
\]

\[
\{C^+(x), g_1(x')\} = -\delta(x - x')S_2^*(x)S_1(x) = -C^+(x)\delta(x - x') \quad (3.10)
\]

and

\[
\{C^-(x), g_2(x')\} = -S_1^*(x)S_2(x)\delta(x - x') = -C^-(x)\delta(x - x') \quad (3.11)
\]

\[
\{C^+(x), g_2(x')\} = S_2^*(x)S_1(x)\delta(x - x') = C^+(x)\delta(x - x'). \quad (3.12)
\]

Then

\[
\{C^\pm(x), g_1(x')\} = \mp C^\pm(x)\delta(x - x'), \quad \{C^\pm(x), g_2(x')\} = \pm C^\pm(x)\delta(x - x'). \quad (3.13)
\]

We have also

\[
\{C^-(x), C^+(x')\} = \left(S_2^*(x)S_2(x) - S_1^*(x)S_1(x)\right)\delta(x - x') =: C^0(x)\delta(x - x'). \quad (3.14)
\]

and

\[
\{C^+(x), C^0(x')\} = 2S_2^*(x')S_1(x)\delta(x - x') = 2C^+(x)\delta(x - x') \quad (3.15)
\]

\[
\{C^-(x), C^0(x')\} = -2S_2^*(x)S_2(x)\delta(x - x') = -2C^-(x)\delta(x - x'). \quad (3.16)
\]

We conclude that this gives us the additional non-vanishing brackets

\[
\{C^\pm(x), C^0(x')\} = \pm 2C^\pm(x). \quad (3.17)
\]

In particular, \((C^\pm(x), C^0(x))\) determines an \(SU(2)\) algebra.

Combining these Poisson brackets we conclude that we have a first-class constraint algebra in the sense of Dirac \(^{12}\), provided we impose the additional constraint

\[
C^0(x) \approx 0. \quad (3.18)
\]

We now resolve this constraint algebra. We first note that \(^{3,18}\) gives

\[
S_1^*(x)S_1(x) \approx S_2^*(x)S_2(x) \quad (3.19)
\]
on the constraint surface. When we combine this with (3.8) we conclude that on the
constraint surface

$$S_1(x) \approx S_2(x) \approx 0.$$  \hspace{1cm} (3.20)

Thus the spinon is weakly zero on the constraint surface, and only the phase of the
chargon survives. In particular, the entire Pauli spinor vanishes weakly on the constraint
surface.

We conclude that even though (3.3) is an appealing and natural choice for the spin-
charge decomposition Pauli spinor, it does not lead to a consistent particle interpretation
of the spinon and chargon when we replace the Poisson brackets with (anti)commutators.

We now show how the spin-charge separation can be imposed in a manner which is
consistent with the particle interpretation. For this we recall (2.4) and introduce the
following decomposed Pauli spinor \[3\] (we denote $\alpha, \beta = 1, 2$)

$$\psi_\alpha(x) = \phi_+(x)S_+\alpha(x) + \phi_-(x)S_-\alpha(x).$$  \hspace{1cm} (3.21)

Here $\phi_\pm$ are two complex functions and $S_\pm$ are two a priori linearly independent complex
spinors.

We first argue that (3.21) can be viewed as a decomposition of the Schrodinger wave
function into its independent spin and charge constituents. For this we implement a
Maxwellian $U(1)$ gauge transformation that sends the Pauli spinor into

$$\psi(x) \rightarrow e^{i\eta}\psi(x).$$  \hspace{1cm} (3.22)

This suggests \[3\] that for the decomposed fields we take

$$\phi_\pm \rightarrow e^{i\eta}\phi_\pm, \quad S_\pm \rightarrow S_\pm.$$  \hspace{1cm} (3.23)

Consequently $\phi_\pm$ become chargons that carry the electric charge of the Pauli spinor.
The $S_\pm$ are charge neutral. But since $S_\pm$ are spinors, they are the spinons that carry
the spin of the electron [?].

In addition of (3.23), we have the following two internal compact $U^\pm_{int}(1)$ gauge symme-
tries in the decomposition:

$$\phi_\pm \rightarrow e^{i\gamma_\pm}\phi_\pm, \quad S_\pm \rightarrow e^{i\gamma_\pm}S_\pm, \quad \psi \rightarrow \psi$$  \hspace{1cm} (3.24)

These internal rotations are akin the internal gauge symmetry (2.11). We propose that
in analogy with (2.11) the internal symmetry transformation (3.24) ensures that under
normal circumstances the spinon and the chargon are confined into the Pauli spinor $\psi$.

We now proceed to inspect the Poisson bracket structure of the decomposed spinor.
For the chargons $\phi_{\pm}$, the consistency with the particle interpretation at the second quantized level proposes that we postulate the non-vanishing Poisson brackets

$$\{ \phi_{\pm}^*(x), \phi_{\pm}(x') \} = \delta(x - x'). \quad (3.25)$$

Similarly, we assume that the spinon components are subject to the non-vanishing brackets

$$\{ S_{\pm\alpha}^*, S_{\pm\beta} \} = \delta_{\alpha\beta}(x - x'). \quad (3.26)$$

These Poisson bracket relations ensure that the decomposed spinor (3.21) satisfies

$$\{ \psi_\alpha(x), \psi_\beta(x') \} = 0, \quad \{ \psi_\alpha^*(x), \psi_\beta^*(x') \} = 0. \quad (3.27)$$

When we compute the remaining Poisson brackets of the decomposed spinors we arrive at the following non-vanishing brackets,

$$\{ \psi_1^*(x), \psi_1(x') \} = \left( \phi^*_+ \phi_+ + \phi^*_- \phi_- + S_{1+1}^* S_{1+1} + S_{1-1}^* S_{1-1} \right) \delta(x - x') =: \varrho_1(x) \delta(x - x') \quad (3.28)$$

$$\{ \psi_2^*(x), \psi_2(x') \} = \left( \phi^*_+ \phi_+ + \phi^*_- \phi_- + S_{2+2}^* S_{2+2} + S_{2-2}^* S_{2-2} \right) \delta(x - x') =: \varrho_2(x) \delta(x - x') \quad (3.29)$$

$$\{ \psi_1^*(x), \psi_2(x') \} = \left( S_{1+2}^* S_{1+2} + S_{1-2}^* S_{1-2} \right) \delta(x - x') =: C^- (x) \delta(x - x') \quad (3.30)$$

$$\{ \psi_2^*(x), \psi_1(x') \} = \left( S_{2+1}^* S_{2+1} + S_{2-1}^* S_{2-1} \right) \delta(x - x') =: C^+ (x) \delta(x - x'). \quad (3.31)$$

The Poisson brackets of $C^+$ and $C^-$ give

$$\{ C^+(x), C^-(x') \} = \left( \varrho_1(x) - \varrho_2(x) \right) \delta(x - x') \quad (3.32)$$

and since

$$\{ C^-(x), C^+(x') \} = \left( \varrho_2(x) - \varrho_1(x) \right) \delta(x - x') =: C^0(x) \delta(x - x') \quad (3.33)$$

and

$$\{ C^+(x), C^0(x') \} = 2C^+(x) \delta(x - x') \quad (3.34)$$

$$\{ C^-(x), C^0(x') \} = -2C^-(x) \delta(x - x') \quad (3.35)$$

$$\{ C^\pm(x), C^0(x') \} = \pm 2C^\pm(x) \delta(x - x') \quad (3.36)$$
the constraint functionals \( (C_0(x), C^\pm(x)) \) determine a \( SU(2) \) algebra. Finally, since
\[
\{C_0(x), \varrho_1(x')\} = C_0(x) \delta(x - x') \tag{3.37}
\]
\[
\{C^\pm(x), \varrho_1(x')\} = C^\pm(x) \delta(x - x'), \tag{3.38}
\]
we conclude that if we define the total density operator
\[
\varrho = \frac{1}{2}(\varrho_1 + \varrho_2) = \phi^*_+ \phi_+ + \phi^*_- \phi_- + \frac{1}{2}\left( S^*_+ S_+ + S^*_- S_- + S^*_+ S^- + S^*_- S^+ \right) \tag{3.39}
\]
we have
\[
\{C^0(x), \varrho(x')\} = \{C^\pm(x), \varrho(x')\} = 0 \tag{3.40}
\]
and the following set of first class constraints
\[
C^0(x) \approx 0 \tag{3.41}
\]
\[
C^\pm(x) \approx 0 \tag{3.42}
\]
\[
\varrho(x) \approx 1 \tag{3.43}
\]
ensures that on the constraint surface the decomposed Pauli spinor \( (3.21) \) reproduces the Poisson brackets \( (3.2) \) of the Pauli spinor.

Clearly, the condition \( (3.43) \) is a generalization of \( (2.10) \) to the present case. It can be interpreted as a statement that the decomposed liquid of spinons and chargons is incompressible. There is also an obvious analogy between \( (3.43) \) and the Gauss law constraint in a \( U(1) \) gauge theory.

Notice that for the bosonic fields the number operator \( (3.39) \) has eigenvalue one
\[
\{\varrho(x), \phi_\pm(x')\} = \phi_\pm(x) \delta(x - x') \tag{3.44}
\]
but for the spinor components we get
\[
\{\varrho(x), S_{\pm\alpha}(x')\} = \frac{1}{2} S_{\pm\alpha}(x) \delta(x - x'). \tag{3.45}
\]
This is since according to our normalization of \( \varrho(x) \), the two components of \( S_{\pm\alpha} \) together form one spinor with particle number one. The condition \( (3.43) \) states that at each point we have one complete particle, either a chargon or a spinon with its two components.

We also note that since \( (3.41)-(3.43) \) is a first-class algebra, once we introduce the subsidiary i.e. gauge fixing conditions \[12\], each constraint leads to the elimination of a pair of canonical variables. We then have four complex conditions and as a consequence both sides of \( (3.21) \) describe an equal number of four independent field degrees of freedom:
The decomposition (3.21) is complete and fully consistent with the canonical structure of the original Pauli spinor.

When we compute the Poisson bracket of the constraint functionals $C^0, C^\pm$ with the decomposed Pauli spinor, we find that they act like the Pauli matrices,

$$\{C^0(x), \psi_1(x')\} = -\psi_1(x)\delta(x - x') \quad (3.46)$$

$$\{C^0(x), \psi_2(x')\} = \psi_2(x)\delta(x - x'). \quad (3.47)$$

Let us now defined the Pauli spinor by $\psi = (\psi_1, \psi_2)$ and the third component of Pauli matrix by $\sigma^3$, then

$$\{C^0(x), \psi(x')\} = -\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right) \left(\begin{array}{c} \psi_1(x) \\ \psi_2(x) \end{array}\right) \delta(x - x')$$

$$\{C^0(x), \psi(x')\} = -\sigma^3 \left(\begin{array}{c} \psi_1(x) \\ \psi_2(x) \end{array}\right) \delta(x - x') \quad (3.48)$$

$$\{C^+(x), \psi_1(x')\} = 0 \quad (3.49)$$

$$\{C^+(x), \psi_2(x')\} = \psi_1(x)\delta(x - x') \quad (3.50)$$

$$\{C^-(x), \psi_1(x')\} = \psi_2(x)\delta(x - x') \quad (3.51)$$

$$\{C^-(x), \psi_2(x')\} = 0 \quad (3.52)$$

Note that $\frac{i}{2}(\sigma^1 + i\sigma^2) = \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right)$ and $\frac{i}{2}(\sigma^1 - i\sigma^2) = \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right)$. Then

$$\{C^+(x), \psi(x')\} = \frac{1}{2} \left(\sigma^1 - i\sigma^2\right) \left(\begin{array}{c} \psi_1(x) \\ \psi_2(x) \end{array}\right) \delta(x - x') \quad (3.53)$$

$$\{C^-(x), \psi(x')\} = \frac{1}{2} \left(\sigma^1 + i\sigma^2\right) \left(\begin{array}{c} \psi_1(x) \\ \psi_2(x) \end{array}\right) \delta(x - x'). \quad (3.54)$$

We conclude that

$$\{C^\pm(x), \psi(x')\} = \frac{1}{2} \left(\sigma^1 \mp i\sigma^2\right) \left(\begin{array}{c} \psi_1(x) \\ \psi_2(x) \end{array}\right) \delta(x - x'). \quad (3.55)$$

For the density operator we get

$$\{\rho(x), \psi_1(x')\} = \frac{3}{2} \psi_1(x)\delta(x - x') \quad (3.56)$$
\[ \{ \varrho(x), \psi_2(x') \} = \frac{3}{2} \psi_2(x) \delta(x - x') \] (3.57)

Then
\[ \{ \varrho(x), \psi(x') \} = \frac{3}{2} f \left( \begin{array}{c} \psi_1(x) \\ \psi_2(x) \end{array} \right) \delta(x - x'), \] (3.58)

where the eigenvalue 3/2 is consistent with our interpretation of the condition (3.43) with each component of the Pauli spinor containing one full bosonic chargon and one of the two components of a full spinon.

As in the case of the lattice fermion we again conclude that the decomposition (3.21) is not unique. For example, we may introduce

\[ \psi_\alpha = f(\phi, \phi^*), \mathcal{S}_\alpha \] (3.59)

and if we e.g. select \( f \) to have the functional form

\[ f(\phi, \phi^*) = f(\frac{\phi}{\phi^*}); \quad f^*(\phi, \phi^*) = f(\phi^*, \phi); \quad |f| = 1 \] (3.60)

which is a generalization of (2.29), the decomposed spinor obeys the Poisson brackets (3.2). In the slave-fermion approach, when we project the field \( \psi(x) \) to a many-particle subspace so that the chargon contribution is totally antisymmetric under exchange of the particle coordinates and the spinon is symmetric, this decomposition incorporates the (normalized) Laughlin wavefunction [13], [11] of fractional quantum Hall effect.

### 3.2 Resolving the constraints

We now proceed to explicitly resolve the constraints (3.41), (3.42), (3.43). For this we introduce a tree component unit vector \( \vec{s} = (s_1, s_2, s_3) \) i.e. \( s_1^2 + s_2^2 + s_3^2 = 1 \). We denote

\[ s_\pm = s_1 \pm is_2 \] (3.61)

and define two spinors \( \chi^\uparrow \) and \( \chi^\downarrow \) as follows,

\[ \chi^\uparrow = \left( \begin{array}{c} \chi_{\uparrow 1} \\ \chi_{\uparrow 2} \end{array} \right) = \frac{1}{\sqrt{2(1 - s_3)}} \left( \begin{array}{c} 1 - s_3 \\ -s_+ \end{array} \right) \] (3.62)

\[ \chi^\downarrow = \left( \begin{array}{c} \chi_{\downarrow 1} \\ \chi_{\downarrow 2} \end{array} \right) = \frac{1}{\sqrt{2(1 - s_3)}} \left( \begin{array}{c} s_- \\ 1 - s_3 \end{array} \right). \] (3.63)

This defines an orthonormal basis of complex spinors,

\[ \chi^\dagger_{\uparrow} \chi^\uparrow = \frac{1}{2(1 - s_3)} \left( \begin{array}{c} 1 - s_3, -s_- \\ s_- \end{array} \right) \left( \begin{array}{c} 1 - s_3 \\ -s_+ \end{array} \right) = 1 \] (3.64)
\[ \chi_\uparrow^\dagger \cdot \chi_\downarrow = \frac{1}{2(1-s_3)} \begin{pmatrix} s_+ & 1-s_3 \end{pmatrix} \begin{pmatrix} s_- \\ 1-s_3 \end{pmatrix} = 1. \] (3.65)

Then

\[ \chi_\uparrow^\dagger \cdot \chi_\uparrow = \chi_\downarrow^\dagger \cdot \chi_\downarrow = 1. \] (3.66)

In the same case

\[ \chi_\uparrow^\dagger \cdot \chi_\downarrow = \frac{1}{2(1-s_3)} \begin{pmatrix} 1-s_3, -s_- \end{pmatrix} \begin{pmatrix} s_- \\ 1-s_3 \end{pmatrix} = 0. \] (3.67)

We then can resume (3.66) and (3.67) as

\[ \chi_\uparrow^\dagger \chi_{\alpha}^\uparrow + \chi_\downarrow^\dagger \chi_{\alpha}^\downarrow = \delta_{\alpha \beta}. \] (3.68)

These two spinors are related to each other by a charge conjugation

\[ -i\sigma^2 \chi_\uparrow^* = -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \frac{1}{\sqrt{2(1-s_3)}} \begin{pmatrix} 1-s_3 \\ -s_- \end{pmatrix} = \chi_\downarrow. \] (3.69)

This relation is equivalent to

\[ \begin{pmatrix} \chi_{\uparrow 1} \\ \chi_{\uparrow 2} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \chi_{\uparrow 1}^* \\ \chi_{\uparrow 2}^* \end{pmatrix} \iff (\chi_{\uparrow 1} = -\chi_{\uparrow 2}^*, \ \chi_{\uparrow 2} = \chi_{\uparrow 1}^*). \] (3.70)

Note that if we introduce the spin projection operator

\[ \hat{s} = -\frac{1}{2} \vec{s} \cdot \vec{\sigma} \] (3.71)

where \( \vec{\sigma} \) is the vector with Pauli matrices as components. Then

\[ \hat{s}\chi_\uparrow = -\frac{1}{2} \left( s_1\sigma_1 + s_2\sigma_2 + s_3\sigma_3 \right) \begin{pmatrix} \chi_{\uparrow 1} \\ \chi_{\uparrow 2} \end{pmatrix} = \frac{1}{2} \chi_\uparrow. \] (3.72)

In the same computation

\[ \hat{s}\chi_\downarrow = \frac{1}{2} \frac{1}{\sqrt{2(1-s_3)}} \begin{pmatrix} s_3 & -s_- \\ s_+ & -s_3 \end{pmatrix} \begin{pmatrix} s_- \\ 1-s_3 \end{pmatrix} = -\frac{1}{2} \chi_\downarrow. \] (3.73)

We conclude that the orthonormal spinors are the \( \pm \frac{1}{2} \) eigenstates of \( \hat{s} \)

\[ \hat{s}\chi_\uparrow = \frac{1}{2} \chi_\uparrow, \ \ \hat{s}\chi_\downarrow = -\frac{1}{2} \chi_\downarrow. \] (3.74)

These relations can be inverted to represent \( \vec{s} \) in terms of the spinors,

\[ \begin{pmatrix} s_+ \\ s_3 \end{pmatrix} = -\begin{pmatrix} 2\chi_{\uparrow 1}^* \chi_{\uparrow 2} \\ \chi_{\uparrow 1}^* \chi_{\uparrow 1} - \chi_{\uparrow 2}^* \chi_{\uparrow 2} \end{pmatrix} = \begin{pmatrix} 2\chi_{\downarrow 1}^* \chi_{\downarrow 2} \\ \chi_{\downarrow 1}^* \chi_{\downarrow 1} - \chi_{\downarrow 2}^* \chi_{\downarrow 2} \end{pmatrix}. \] (3.75)
This relation can be compressed in the form

$$\vec{s} = \begin{pmatrix} s_+ \\ s_3 \end{pmatrix} = -\chi^\dagger_\uparrow \hat{\sigma} \chi^\uparrow = +\chi_\downarrow \hat{\sigma} \chi^\downarrow$$

$$= -\begin{pmatrix} \chi^\dagger_\uparrow \sigma^+ \chi^\uparrow \\ \chi^\dagger_\downarrow \sigma^+ \chi^\downarrow \end{pmatrix} = \begin{pmatrix} \chi^\dagger_\uparrow \sigma^+ \chi^\uparrow \\ \chi^\dagger_\uparrow \sigma^3 \chi^\uparrow \end{pmatrix}$$

(3.76)

where $\sigma^+ = \sigma_1 + i\sigma_2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$.

With these spinors we can resolve the constraints (3.41), (3.42), (3.43) as follows: We set

$$S_+ = \begin{pmatrix} S_{+1} \\ S_{+2} \end{pmatrix} =: \rho_. \chi^\uparrow$$

and

$$S_- = \begin{pmatrix} S_{-1} \\ S_{-2} \end{pmatrix} =: \rho_. \chi^\downarrow$$

(3.77)

where

$$|\rho_\pm|^2 = S_\pm^\dagger S_\pm.$$  (3.79)

When we substitute these in the constraints (3.41) and (3.42) we get

$$C^0 = \varphi_1 - \varphi_2 = s_3 \left( |\rho_-|^2 - |\rho_+|^2 \right) \approx 0$$

(3.80)

$$C^- = S_{+1}^* S_{+2} + S_{-1}^* S_{-2} = \frac{1}{2} s_+ \left( |\rho_-|^2 - |\rho_+|^2 \right) \approx 0$$

(3.81)

$$C^+ = S_{+2}^* S_{+1} + S_{-2}^* S_{-1} = \frac{1}{2} s_- \left( |\rho_-|^2 - |\rho_+|^2 \right) \approx 0.$$  (3.82)

Then

$$C^{\pm} = \frac{1}{2} s_{\mp} \left( |\rho_-|^2 - |\rho_\pm|^2 \right) \approx 0.$$  (3.83)

Consequently, these constraints become resolved if we set

$$\rho_\pm = re^{i\omega_{\pm}}, \ r \in \mathbb{R}.$$  (3.84)

When we substitute the following decomposed spinor

$$\psi = \phi_+ r \chi^\uparrow (\vec{s}) e^{i\omega_+} + \phi_- r \chi^\downarrow (\vec{s}) e^{i\omega_-}$$

(3.85)

in the remaining constraint (3.43), we get

$$\varrho = \frac{1}{2} (\varrho_1 + \varrho_2) = \phi_+^* \phi_+ + \phi_-^* \phi_- + \frac{1}{2} \left( S_{+1}^* S_{+1} + S_{-1}^* S_{-1} + S_{+2}^* S_{+2} + S_{-2}^* S_{-2} \right)$$
\[ \phi^+ \phi_+ + \phi^- \phi_- + r^2 \approx 1 \]
\[ \Rightarrow |\phi_+|^2 + |\phi_-|^2 \approx 1 - r^2. \]  
(3.86)

This defines a sphere \( S^2 \), which we parametrize by setting
\[
\begin{pmatrix}
|\phi_+| \\
|\phi_-| \\
r
\end{pmatrix} =
\begin{pmatrix}
\cos \alpha \sin \beta \\
\sin \alpha \sin \beta \\
\cos \beta
\end{pmatrix}
\]  
(3.87)

and we define
\[ \phi_{\pm} = |\phi_{\pm}| e^{i \eta_{\pm}}. \]  
(3.88)

For the Pauli spinor (3.85), we then arrive at the following explicit parametrization
\[
\psi =
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}
= \phi_+ r \chi_\uparrow(\vec{s}) e^{i \omega_+} + \phi_- r \chi_\downarrow(\vec{s}) e^{i \omega_-}
\]
\[ = \cos \beta \left( \cos \alpha \sin \beta \chi_\uparrow(\vec{s}) e^{i (\omega_+ + \eta_+)} + \sin \alpha \sin \beta \chi_\downarrow(\vec{s}) e^{i (\omega_- + \eta_-)} \right) \]
\[ = \frac{1}{2} \sin 2 \beta \left( \cos \alpha \sin \chi_\uparrow(\vec{s}) e^{i (\omega_+ + \eta_+)} + \sin \alpha \chi_\downarrow(\vec{s}) e^{i (\omega_- + \eta_-)} \right). \]  
(3.89)

Here the phase combinations \( \omega_\pm + \eta_\pm \) parametrize the internal \( U(1) \) symmetries. If we choose the relative phases of the spinons and chargons to cancel each other,
\[ \omega_\pm + \eta_\pm = \]  
(3.90)

we obtain
\[ \psi =: \phi_+ \chi_\uparrow + \phi_- \chi_\downarrow \]  
(3.91)

with only four field degrees of freedom in (3.91) that is consistent with the constraint structure. However, here we prefer to leave the relative phases between the spinons and chargons as unspecified. We wish to treat the spinon fields and the chargon fields as independent dynamical degrees of freedom.

We now come to the problem with such a decomposition, and the question is how to resolve it: For the Pauli spinor the condition (3.43) yields
\[
|\psi|^2 = \left( \phi_+ r \chi_\uparrow(\vec{s}) e^{i \omega_+} + \phi_- r \chi_\downarrow(\vec{s}) e^{i \omega_-} \right) \left( \phi_+ r \chi_\uparrow(\vec{s}) e^{i \omega_+} + \phi_- r \chi_\downarrow(\vec{s}) e^{i \omega_-} \right)^\dagger \]
\[ = r^2 (|\phi_+|^2 + |\phi_-|^2) = r^2 (1 - r^2). \]  
(3.92)

Note that this vanishes when either the density of spinons \( (r = 0) \) or the density of chargons \( (r = 1) \) vanishes, and that the maximum value is obtained at \( r = 1/\sqrt{2} \) which gives for the maximum value of the absolute value of spinor
\[ 0 \leq |\psi| \leq \frac{1}{2}. \]  
(3.93)

This relation shows that the construction is not complete: For a Schrodinger equation we demand that the integral of \( |\psi|^2 \) is equal to \( = 1 \), but at a given point \( x \) the absolute value of the wavefunction can be any real number.
4 New class of decomposition: resolution of the constraint equations

In this section, I show that not all decompositions are possible.

- Let us define the spin-charge decomposition as follows:
  \[
  \psi_\alpha(x) = \phi_+ S_+ \alpha + \epsilon_{\alpha\beta} \phi_- S_- \beta
  \]  
  (4.1)
  with the Poisson brackets of the fields \(\phi_\pm\) and \(S_{\ell\alpha}\) as
  \[
  \{\phi_+^*(x), \phi_\pm(x')\} = \pm \delta(x - x'), \quad \{S_{\ell\alpha}^*(x), S_{\ell'\beta}(x')\} = \pm \delta_{\alpha\beta}(x - x')
  \]  
  (4.2)
  and
  \[
  \{\phi_\pm^*(x), \phi_\mp(x')\} = 0, \quad \{S_{\ell\alpha}^*(x), S_{\ell'\beta}(x')\} = 0.
  \]  
  (4.3)
  There result the Poisson brackets
  \[
  \{\psi_1^*(x), \psi_1(x')\} = \left(\phi_+^* \phi_+ - \phi_-^* \phi_- + S_{+1}^* S_{+1} - S_{-2}^* S_{-2}\right) \delta(x - x')
  \]  
  \[
  =: \varrho_1(x) \delta(x - x')
  \]  
  (4.4)
  \[
  \{\psi_2^*(x), \psi_2(x')\} = \left(\phi_+^* \phi_+ - \phi_-^* \phi_- + S_{+2}^* S_{+2} - S_{-1}^* S_{-1}\right) \delta(x - x')
  \]  
  \[
  =: \varrho_2(x) \delta(x - x')
  \]  
  (4.5)
  \[
  \{\psi_1^*(x), \psi_2(x')\} = \left(S_{+1}^* S_{+2} + S_{-2}^* S_{-1}\right) \delta(x - x') = C^-(x) \delta(x - x')
  \]  
  (4.6)
  \[
  \{\psi_2^*(x), \psi_1(x')\} = \left(S_{+2}^* S_{+1} + S_{-1}^* S_{-2}\right) \delta(x - x') = C^+(x) \delta(x - x')
  \]  
  (4.7)
  Then
  \[
  \{C^-(x), C^+(x')\} = \left(\varrho_2 - \varrho_1\right)(x) \delta(x - x') =: C^0(x) \delta(x - x').
  \]  
  (4.8)
  and
  \[
  \{C^+(x), C^0(x')\} = 2C^+(x) \delta(x - x'), \quad \{C^-(x), C^0(x')\} = -2C^-(x) \delta(x - x').
  \]  
  (4.9)
  showing that \((C^\pm, C^0)\) determine an \(SU(2)\) algebra. Besides,
  \[
  \{C^+(x), \varrho_1(x')\} = -\{C^+(x), \varrho_2(x')\} = -C^+(x) \delta(x - x')
  \]  
  (4.10)
  \[
  \{C^-(x), \varrho_1(x')\} = -\{C^-(x), \varrho_2(x')\} = C^-(x) \delta(x - x').
  \]  
  (4.11)
  If we define the total density operator as
  \[
  \rho = \frac{1}{2}(\varrho_1 + \varrho_2) = \phi_+^* \phi_+ - \phi_-^* \phi_- + \frac{1}{2}\left(S_{+1}^* S_{+1} - S_{-2}^* S_{-2} + S_{+2}^* S_{+2} - S_{-1}^* S_{-1}\right)
  \]  
  (4.12)
we have

\[ \{C^0(x), \varrho(x')\} = \{C^\pm(x), \varrho(x')\} = 0 \]  

(4.13)

and the following set of first class constraints

\[ C^0(x) \approx 0, \ C^\pm(x) \approx 0, \ \varrho(x) \approx 1 \]  

(4.14)

ensuring that on the constraint surface the decomposed Pauli spinor reproduces the Poisson brackets (3.2).

We have to solve the constraint equation (4.14). By expansion we get

\[ C^0 = S^\dagger_1 S^*_1 - S^\dagger_2 S^*_2 + S^*_1 S^+_1 - S^*_2 S^+_2 \approx 0. \]  

(4.15)

Recall that

\[ S^\dagger\sigma^3 S^+ = S^*_1 S^+ + S^*_2 S^+ + S^\dagger\sigma^3 S^- = S^*_1 S^- + S^*_2 S^- \]  

(4.16)

Then

\[ C^0 = -S^\dagger_1 \sigma^3 S^+ - S^\dagger_2 \sigma^3 S^- = s_3(|\rho_+|^2 - |\rho_-|^2) \approx 0 \]  

(4.17)

\[ C^- = S^*_1 S^+ + S^*_2 S^- \quad C^+ = S^*_2 S^+ + S^*_1 S^- \]  

(4.18)

Besides,

\[ S^\dagger_1 \sigma^+ S^+ = 2S^*_1 S^+ + 2S^*_1 S^- \quad S^\dagger_2 \sigma^- S^+ = 2S^*_2 S^+ + 2S^*_2 S^- \]  

(4.19)

\[ S^\dagger_1 \sigma^- S^+ = 2S^*_2 S^+ + 2S^*_2 S^- \quad S^\dagger_2 \sigma^- S^- = 2S^*_2 S^- + 2S^*_2 S^- \]  

(4.20)

yielding

\[ \chi^\dagger_\uparrow \sigma^- \chi_\downarrow = s_\downarrow = -\chi^\dagger_\downarrow \sigma^- \chi_\uparrow. \]  

(4.21)

Then we get the following results

\[ C^- = \frac{1}{2} \left( -|\rho_+|^2 s_+ + |\rho_-|^2 s_- \right) \approx 0, \]  

(4.22)

\[ C^+ = \frac{1}{2} \left( -|\rho_+|^2 s_- + |\rho_-|^2 s_+ \right) \approx 0, \]  

(4.23)

and

\[ \varrho = |\phi_+|^2 - |\phi_-|^2 + \frac{1}{2} \left( S^\dagger_1 S_+ - S^\dagger_2 S_- \right) \]  

\[ = |\phi_+|^2 - |\phi_-|^2 + \frac{1}{2} \left( |\rho_+|^2 - |\rho_-|^2 \right) \approx 1. \]  

(4.24)

The equations (4.17) and (4.24) show that

\[ |\phi_+|^2 - |\phi_-|^2 \approx 1 \]  

(4.25)
which can be solved in the constraint surface to give

$$\phi_+ = e^{i\nu_+} \cosh \alpha, \quad \phi_- = e^{i\nu_-} \sinh \alpha$$  \hspace{1cm} (4.26)

where $\alpha$ and $\nu_\pm$ are real parameters.

If we choose $\rho_\pm = re^{i\omega_\pm}$, (4.17) can be easily solved and the relations (4.22) give

$$C^\pm = \pm \frac{1}{2} r^2 (s_+ - s_-) \approx 0 \Rightarrow s_+ \approx s_- \Rightarrow s_2 \approx 0$$  \hspace{1cm} (4.27)

The absolute value of field $\psi$ is then given by

$$|\psi| = r, \quad 0 \leq |\psi| < \infty.$$  \hspace{1cm} (4.28)

• We adapt here the following spin charge decomposition

$$\psi = \frac{1}{2} \left( \phi_+ S_{+\alpha} + \phi_- S_{-\alpha} + \epsilon_{\alpha\beta} \phi_- S_{-\beta} + \epsilon_{\alpha\beta} \phi_+ S_{+\beta} \right).$$  \hspace{1cm} (4.29)

One can show that

$$\{ \psi^*_1(x), \psi_1(x') \} = \left( \mathcal{R}_e^2(\phi_-) - \mathcal{R}_e^2(\phi_+) \right)(x) \delta(x - x') = \{ \psi_2^*(x), \psi_2(x') \}$$  \hspace{1cm} (4.30)

and

$$\varrho_1 = \varrho_2 = \mathcal{R}_e^2(\phi_+) - \mathcal{R}_e^2(\phi_-) \approx 1.$$  \hspace{1cm} (4.31)

Beside

$$\{ \psi_1^*(x), \psi_2(x') \} = \{ \psi_2^*(x), \psi_1(x') \} = 0.$$  \hspace{1cm} (4.32)

We now assumed that $\mathcal{R}_e^2 \phi_- = \lambda$. Then (4.31) give $\mathcal{R}_e^2 \phi_+ \approx 1 + \lambda$. the function $\rho_-$ and $\rho_+$ defined in (3.77) and (3.78) are expressed as:

$$\rho_+ = re^{i\theta_+}, \quad \rho_- = (1 - r)e^{i\theta_-}.$$  \hspace{1cm} (4.33)

finally we conclude that

$$|\psi|^2 = r^2 \mathcal{R}_e^2(\phi_+) + (1 - r)^2 \mathcal{R}_e^2(\phi_-) = r^2 + \left( 2r^2 - 2r + 1 \right) \lambda$$  \hspace{1cm} (4.34)

and

$$|\psi| = \sqrt{r^2 + \left( 2r^2 - 2r + 1 \right) \lambda}.$$  \hspace{1cm} (4.35)

We arrived to the bound

$$\sqrt{\frac{\lambda(\lambda + 1)}{2\lambda + 1}} \leq |\psi| < \infty.$$  \hspace{1cm} (4.36)
Let us remark that in general case we can choose arbitrary function $f(r)$ and $g(r)$ such that

$$\rho_+ = f(r)e^{i\theta_+}, \quad \rho_- = g(r)e^{i\theta_-}$$  \hspace{1cm} (4.37)

and then

$$|\psi| = \sqrt{f(r)^2 + (f(r)^2 + g(r)^2)\lambda}. \hspace{1cm} (4.38)$$

We defined $r_0^i, i = 1, 2 \cdots$ as the solution of $(1 + \lambda)f'(r)f(r) + \lambda g'(r)g(r) = 0$ such that

$$\sqrt{f(r_0^i)^2 + (f(r_0^i)^2 + g(r_0^i)^2)\lambda} \leq |\psi| < \infty. \hspace{1cm} (4.39)$$

For example, setting

$$f(r) = e^{kr}, \quad g(r) = e^{-\frac{1+\lambda}{\lambda}kr}; \quad n \in \mathbb{N} \hspace{1cm} (4.40)$$

and

$$\forall r > 0, \sqrt{f(r)^2 + (f(r)^2 + g(r)^2)\lambda} \leq |\psi| < \infty. \hspace{1cm} (4.41)$$

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