Optimal Robust Constraint-Following Control for Permanent Magnet Linear Motor: A Fuzzy approach

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Abstract—In this paper, robust constraint-following control (RCFC) with the optimal design is developed to handle the trajectory tracking control issues for permanent magnet linear motors, which control performance is deteriorated mainly by friction, ripple force, and external disturbance. Specifically, fuzzy description for the main nonlinearity of permanent magnet linear motor and its fuzzy dynamic model is formulated. Then, the tracking specification is modeled as a performance constraint, the RCFC algorithm following the Udwadia-Kalaba theory is designed to comply with this constraint, and possess strong robustness to uncertainties simultaneously. The resulting controller is demonstrated to be uniform bounded and uniform ultimate bounded with Lyapunov analysis. Furthermore, the optimal design issue via a fuzzy approach is investigated to achieve the optimal tradeoff between control effort and system performance. Finally, the rapid control prototype platform CSPACE is employed to implement the real-time control and debugging. Simulation and experiment results illustrate the actual effectiveness of the proposed algorithm.

Index Terms—Constraint-following, Fuzzy approach, Linear motor, Optimal design, Robust control.

I. INTRODUCTION

PERMANENT magnet linear motor (PMLM) eliminates the intermediate conversion mechanism on the basis of the law of magnetic induction, which exhibits the non-contact property that the electromagnetic energy is directly converted into linear motion [1]–[3]. Without the interference of mechanical transmission, the influence of contact-type disturbances and nonlinearities such as mechanical elements, gear backlash, and friction is greatly reduced [4], [5]. Due to these superior properties, PMLMs have been extensively applied on many occasions involving high-speed and/or high-precision linear motions, such as precision machine tools, scanning machines, medical operation, hard disk drive, etc [6]. Hence, research for PMLM has recently attracted conspicuous attention from academia. From the standpoint of intelligent control engineering, the principal limiting factors of PMLM performance are inevitable frictions, force ripple, external disturbances, and parameter uncertainties [7], [8]. The proportional-integral-derivative (PID) control is broadly applied in the engineering field, especially in industrial robots which can be quite accurately modeled. While for the PMLM system, it is difficult to obtain an accurate model of nonlinear effects, particularly the cogging effect in linear motion. Thus, it is very crucial to develop more advanced control algorithms to promote the development of the PMLM system towards ultra-precision tracking and positioning, because the conventional PID control usually does not exist alone.

Until now, energetical and continuous attempts have been devoted to boosting the control performance of the PMLM system. Various controllers have been implemented on the linear motor platform, such as sliding mode control [9], adaptive robust control [10], iterative learning control [11], backstepping control [12], fuzzy control [13], neural network control [14], fault-tolerant control [15], model predictive control [16], etc. The mentioned-above researches have played significant roles in improving the dynamic precision of the PMLM system. Different from the previous studies that mainly focus on motion control itself, the control issues in this paper are regarded as the servo constraint problems. According to the rigorous theoretic framework established by Udwadia and Kalaba [17], the system’s motion requirement is treated as a kind of constraint, which is termed as trajectory tracking constraint (also called performance constraint). From the perspective of inverse dynamics control, in order to obey the specified performance constraint, a certain servo constraint force can be imposed on the system as a control input. This servo constraint force can be obtained by the fundamental equation derived by Udwadia and Kalaba [18].

Begin from the concept of servo constraint [19], [20], the robust constraint following control (RCFC) is structured to render the PMLM system to follow the pre-specified performance constraints in case of nonlinearity and uncertainty. Specifically, the control design can fall into three components. Firstly, the constraint force is formulated as the $u_1$ component to cope with the dynamics in the system when uncertainty is not involved. Secondly, considering that the initial conditions may not strictly follow the constraints, the $u_2$ component of the controller is devised. The combination of $u_1$ and $u_2$ components is the nominal control of the system, namely Udwadia-Kalaba nominal control (UKNC), which is designed via Udwadia-Kalaba theory. Thirdly, the $u_3$ component is...
deduced as robust feedback terms to restrain the influence of model uncertainty and extrinsic disturbance. It is worth emphasizing that the initial condition shift issues are considered in our control design, which has practical significance. Because in practical terms, the PMLM system does not necessarily start from the setting conditions strictly due to positioning errors, output noises, and other factors, which would degrade the control performance [21].

In the aspect of trajectory tracking control, the design idea of constraint-following has been successfully developed in some mechanical systems, such as active suspension [22], electronic throttle [23], mobile robot [24], etc. In addition, robust control is also recognized as having inherent reliability in dealing with uncertainties. Inspired by the previous researches, this paper innovatively incorporates the idea of constraint-following and the design of robust feedback to settle the tracking control problems of PMLM system. The proposed RCFC strategy is concerned with the exogenous and internal uncertainties of the model, including mass, friction, ripple force, and load variation, etc. Nevertheless, obtaining a precise model is sometimes more complicated than the control design itself. A fuzzy approach is adopted to handle uncertainties. Specifically, assuming that the uncertainties in the PMLM system (probably fast time-varying) are unknown but bounded, fuzzy-set theory is invoked to describe those unknown bounds.

In view of the fuzzy information of the uncertainties, the fuzzy dynamic model is established at first, and then the RCFC strategy is devised to render the PMLM system control performance. Since robust feedback design generally considers the most unsatisfactory situation, the optimal gain of the controller is finally explored to avoid high control costs without sacrificing the control performance. Among them, the idea of optimal design is to minimize a comprehensive performance index that formulates the control cost, transient response performance, and steady-state tracking performance. As a result, the optimal RCFC (ORCFC) algorithm based on the fuzzy approach is validated by simulation and experiment.

For this paper, to summarize, the predominant objective is to pursue the optimal robust constraint-following control (ORCFC) scheme for the uncertain PMLM system via a fuzzy approach to obtain superior trajectory tracking performance. The resulting controller is demonstrated to be uniform bounded and uniform ultimate bounded with Lyapunov analysis. The primary contributions are stated as below:

i) Start from tracking requirements: From another novel point viewpoint, the concept of servo constraint is innovatively applied to redesign robust control of PMLM system. The developed RCFC algorithm starting from the tracking requirement can better guarantee the system performance.

ii) Converge under any initial condition: The uncertainty of the initial state is considered, and the UKNC and RCFC algorithms can still converge to the desired trajectory under the mismatched initial conditions. Therefore, the PMLM system can also meet the performance requirements without strictly starting from the set initial conditions.

iii) Optimal design by fuzzy approach: Through the fuzzy description of an uncertain PMLM system, the optimal gain of the RCFC controller is solved by fuzzy-set theory. The optimal RCFC (ORCFC) algorithm can avoid the control cost waste caused by robust design structure, and achieve the optimal tradeoff between control effort and system performance.

iv) Real-time experimental validation: Rapid control prototype platform CSPAC is employed to carry out the real-time control for the PMLM system while avoiding the time-consuming repetitive programming and debugging. Due to the convenience of practical operation, experiments have been implemented to verify the superior trajectory tracking property and strong robustness against disturbance.

II. UDWAIDIA-KALABA EQUATION AND SYSTEM DESCRIPTION

A. Fundamental equation of constrained system

Since the proposed control design is characterized by constraint-following, it is essential to introduce the fundamental equation and the concept of constraint. The fundamental equation introduced in this section is called Udwadia-Kalaba equation [17]. A systematic three-step procedure is summarized to derive the fundamental equation as follow:

Firstly, an unconstrained system with $n$ generalized coordinates $x := [x_1, x_2, \ldots, x_n]^T$ is considered. By applying Lagrangian or Newtonian mechanics, its equation of motion can be presented as

$$M(x, t)\ddot{x} = Q(x, \dot{x}, t).$$

where the inertia matrix $M(x, t)$ is positive definite, $\ddot{x}$ is the acceleration, $Q(x, \dot{x}, t)$ denotes the force impressed on the system in which the constraints are released. Thus, the acceleration of unconstrained system denoted by the vector $a(x, \dot{x}, t)$ is obtained

$$\ddot{x} = Q(x, \dot{x}, t)M^{-1}(x, t) = a(x, \dot{x}, t).$$

Secondly, when constraints exist, at arbitrary time $t$, the acceleration of constrained system recorded as the vector $\ddot{x}$ differs from $a(x, \dot{x}, t)$, because additional forces are generated to cater for constraints. Suppose that the system is governed by $m$ constraints, including $h$ holonomic constraints with the mathematical form of

$$\varphi_i(x, t) = 0, \quad i = 1, 2, \ldots, h.$$

and $m - h$ nonholonomic constraints with the mathematical form of

$$\varphi_i(x, \dot{x}, t) = 0, \quad i = h + 1, h + 2, \ldots, m.$$

Differentiating $h$ holonomic constraints twice and $m - h$ nonholonomic constraints once with regard to the time $t$, yields

$$A(x, \dot{x}, t)\ddot{x} = c(x, \dot{x}, t), \quad A(x, \dot{x}, t)\ddot{x} = b(x, \dot{x}, t),$$

where the $m \times n$ matrix $A(x, \dot{x}, t)$ and the $m$-vector $b(x, \dot{x}, t)$ are known functions.

Thirdly, the actual equations of motion with constraints can be written in the following form
\[ M(x, t) \ddot{x} = Q(\dot{x}, \ddot{x}, t) + Q^c(\dot{x}, \ddot{x}, t). \]  

(6)

where \( Q^c(\dot{x}, \ddot{x}, t) \) is a constraint force that is enabled to satisfy the constraint equation. By Gauss’s principle, Udwadia and Kalaba put forward the fundamental equation

\[ \ddot{x} = a + M^{-1/2} B^+(b - Aa). \]  

(7)

where \( B = AM^{-1/2} \), the symbol \(+\), called the Moore-Penrose (MP) generalized inverse on the condition of the following:

\[ BB^+ B = B, \]
\[ B^+ BB^+ = B^+, \]
\[ BB^+ = (BB^+)^T, \]
\[ B^+ B = (B^+ B)^+. \]  

(8)

From Eq.(6) and Eq.(7), the additional constraint force \( Q^c(\dot{x}, \ddot{x}, t) \), which caused the acceleration of the system at time \( t \) to change from its unconstrained value of \( a(t) \) to its constrained value of \( \ddot{x}(t) \), is given by

\[ Q^c = M^{1/2} B^+(b - AM^{-1} Q). \]  

(9)

B. Description of the PMLM system

According to the previous derivation [10], PMLM can be described as follows:

\[ M \ddot{x} = u - F, \]
\[ F = F_{fric} + F_{ripple} - F_d. \]  

(10)

where \( x \) represents the position, \( \dot{x} \) and \( \ddot{x} \) are the corresponding velocity and acceleration, \( u \) denotes the input voltage. Mass of the coil assembly plus the inertia load is denoted by \( M \). The lumped nonlinearities recorded as \( F \) consists of friction \( F_{fric} \), ripple force \( F_{ripple} \), and the other external disturbances \( F_d \). The friction is generally modeled as [1]:

\[ F_{fric}(\dot{x}) = [f_c + (f_s - f_c)e^{-(\dot{x}/x_s)^2} + f_v \dot{x}] \text{sign}(\dot{x}). \]  

(11)

where \( f_c \) denotes the viscous friction parameter, \( f_s \) denotes the static friction level, \( f_v \) denotes the minimum Coulomb friction level, and the lubricant coefficient generally determined by empirical experiments is denoted as \( \dot{x}_s \).

Besides friction, ripple force is another major factor affecting the tracking control performance, especially in the low-speed motor. The ripple force is caused by the cogging forces and the magnetic resistance in the actuator structure. It is mainly related to the displacement and is generally modeled as:

\[ F_{ripple}(x) = \sum_{n=1}^{\infty} a_n \sin \left( \frac{2n\pi}{t} x + \varphi_n \right) + \sum_{n=1}^{\infty} b_n \sin \left( \frac{2n\pi}{t} x + \sigma_n \right). \]  

(12)

High order harmonics in (12) are often ignored in control design, and the following simplified ripple force model is usually applied.

\[ F_{ripple}(x) = S_1 \sin(\omega x) + S_2 \sin(3\omega x) + S_3 \sin(5\omega x). \]  

(13)

where \( \omega, S_1, S_2, S_3 \) are constants.

Aiming at the main uncertainties of PMLM system, we adopt a fuzzy description method and combine fuzzy theory with system theory to re-describe the dynamic model as follows:

\[ M(x, \delta, t) \ddot{x} + F_{fric}(\dot{x}, \ddot{x}, t) + F_{ripple}(x, \delta, t) - F_d(x, \dot{x}, \ddot{x}, t) = u. \]  

(14)

where \( u \in \mathbb{R}^n \) is the control input voltage, and \( \delta \) denotes the uncertain parameter. Assumed that \( \delta \in \sum \subset \mathbb{R}^n \), which represents the bound of \( \delta \) is known and compact.

In order to facilitate the nominal control design based on Udwadia-Kalaba theory, the following definition is necessary. Let

\[ M(x, \delta, t) = \bar{M}(x, \delta, t) + \Delta M(x, \delta, t), \]
\[ F_{fric}(\dot{x}, \ddot{x}, t) = \bar{F}_{fric}(\dot{x}, t) + \Delta F_{fric}(\dot{x}, \ddot{x}, t), \]
\[ F_{ripple}(x, \delta, t) = \bar{F}_{ripple}(x, t) + \Delta F_{ripple}(x, \delta, t), \]
\[ F_d(x, \dot{x}, \ddot{x}, t) = \bar{F}_d(x, \dot{x}, \ddot{x}, t) + \Delta F_d(x, \dot{x}, \ddot{x}, t). \]  

(15)

where \( \bar{M}, \bar{F}_{fric}, \bar{F}_{ripple}, \) and \( \bar{F}_d \) denote the nominal portions, \( \Delta M, \Delta F_{fric}, \Delta F_{ripple}, \) and \( \Delta F_d \) denote the uncertain portions, which depend on \( \delta \).

III. ROBUST CONSTRAINT-FOLLOWING CONTROL DESIGN

In this paper, we consider the PMLM system as a constrained system, in which the performance requirement is modeled as a servo constraint (we call it a performance constraint). Then the performance constraint can be described by Eq.(5) of second-order form.

A. Constraint force under the known uncertainty

Firstly, we present the constraint force if the uncertainty is not considered in PMLM system (14). Here, \( \delta \) is considered known.

**Theorem 1.** (Udwadia and Kalaba [25]): Consider the PMLM system (14) and the constraint (5), the expression for the force of constraint, which complies with Gauss’s principle and Lagrange’s form of d’Alembert’s principle, is given by

\[ Q^c(x, \dot{x}, \ddot{x}, t) = \begin{cases} M^{1/2}(x, \dot{x}, \ddot{x}, t)(A(t)M^{-1}(x, \dot{x}, \ddot{x}, t)) & \text{sign}(\dot{x}) \end{cases} \]  

(16)

where \( a(x, \dot{x}, t) = M^{-1}(x, \dot{x}, \ddot{x}, t)(F_{fric}(\dot{x}, \ddot{x}, t) + F_{ripple}(x, \delta, t) - F_d(x, \dot{x}, \ddot{x}, t)) \) for PMLM system (14), such that the system exactly satisfies all the constraint.

**Remark 1.** Theorem 1 suggests that one can employ the control input \( u = Q^c \) to drive PMLM system to meet Eq. (5) if there is no uncertainty. However, in the practical PMLM system, the uncertainty is widespread and unknown. Therefore, robust constraint-following design to deal with uncertainty for position tracking control need to be investigated.

B. Robust constraint-following control

Based on the actual situation, we consider the uncertainty while designing the controller. The matrices/vectors \( M, F_{fric}, F_{ripple}, F_d \) can be decomposed as shown in Eq. (15). Let \( C(x, t) := M^{-1}(x, \dot{x}, \ddot{x}, t), \Delta C(x, \delta, t) := M^{-1}(x, \dot{x}, \ddot{x}, t) - M^{-1}(x, \dot{x}, \ddot{x}, t), D(x, \delta, t) := M(x, t)M^{-1}(x, \dot{x}, \ddot{x}, t) - I. \) Hence, \( \Delta C(x, \delta, t) = C(x, t)D(x, \delta, t) \) can be obtained.

**Assumption 1.** For each \( (x, t) \in \mathbb{R}^n \times \mathbb{R} \), \( A(x, t) \) is full rank. That means \( A(x, t)A^T(x, t) \) is invertible.

**Assumption 2.** For given \( P \in \mathbb{R}^{n \times n}, P > 0 \), let

\[ W(x, t) := PA(x, t)C(x, t)A^T(x, t)P. \]  

(17)

Define that \( \Lambda \) is a positive scalar constant, have

\[ \Lambda \leq \inf_{(x, \dot{x}) \in \mathbb{R}^n \times \mathbb{R}} \lambda_m(W(x, t)). \]  

(18)
Next, we consider the constraint-following problem. That is, \( Ax \neq c \) or \( Ax \neq b \) results in the constraint tracking error. Let

\[
V(t) = \frac{1}{2} \sum_{(x, t) \in R} \lambda_m(D(x, \delta, t) + D^T(x, \delta, t)) \geq \rho_D(x, t).
\]

(21)

Now, the RCFC design is provided as follows:

\[
u = u_1(x, x, t) + u_2(x, x, t) + u_3(x, x, t)
\]

(22)

with

\[
u_3(x, x, t) := -\gamma(x, x, t)\mu(x), \dot{x}(x, x, t), t)
\]

(23)

where

\[
\gamma(x, x, t) = \begin{cases}
\frac{1}{\mu(x, x, t) + \mu(x, x, t)} & \text{if } ||\mu(x, x, t)|| > \varepsilon, \\
\frac{1}{\mu(x, x, t) + \mu(x, x, t)} & \text{if } ||\mu(x, x, t)|| \leq \varepsilon.
\end{cases}
\]

(24)

\[
\mu(x, x, t) = \eta(x, x, t)\rho(x, x, t),
\]

\[
\eta(x, x, t) = \bar{\mu}(x, x, t)\eta(x, x, t),
\]

\[
\bar{\mu}(x, x, t) = \bar{M}^{-1}(x, x)A^T(x, t)P,
\]

(25)

where \( \rho(x, x, t) \) represents the upper bound of uncertainties is defined as follows:

\[
\rho(x, x, t) \geq \max_{\delta \in \Sigma} \|PAC\Delta C(F_d - F_{fric} - F_{ripple} + u_1 + u_2) + PAC(\Delta F_d - \Delta F_{fric} - \Delta F_{ripple})\|.
\]

(26)

Remark 2. From the structure of the controller, it consists of the nominal term based on Udvardi-Kalaba theory and the robust feedback term characterized by constraint following. Essentially, the control in Eq. (22) with \( u_3 = 0 \) is the nominal control based on U-K theory (UKNC), and \( u_3 \) is the robust feedback design used to restrain the influence of model uncertainty and extrinsic disturbance.

Remark 3. The parameter \( \varepsilon \) can be chosen arbitrarily small to achieve a high-performance constraint following. Nevertheless, \( \varepsilon \) can not be unlimited small, otherwise, the corresponding control input may be severely chattering. Consequently, there should be an optimal tradeoff between control effort and system performance.

C. Theoretical proof of the controller

Theorem 2. Consider the PMLM system (14), subject to Assumptions 1, 2, and 3, is under the control (22). Then, the following performances can be guaranteed:

1) Uniform boundedness: For any \( r > 0 \), there exists a \( d(r) < \infty \) such that \( ||y(x(t), \dot{x}(t), t)|| \leq d(r) \) for all \( t \geq t_0 \).

2) Uniform ultimate boundedness: For any \( r > 0 \), there exists a \( d > 0 \) such that \( ||y(x(t), \dot{x}(t), t)|| \leq d \) for all \( d > d \) as \( t \geq t_0 + T(\bar{d}, r) \).

Proof: Consider the Lyapunov function \( V(\eta) = \eta^TP\eta \). The time derivative of \( V(\cdot) \) is given by

\[
\dot{V}(\eta) = 2\eta^TP(Ax - b)
\]

(27)

Decompose \( M^{-1}, F_{fric}, F_{ripple} \) and \( F_d \), have

\[
A\left[ M^{-1}(F_d - F_{fric} - F_{ripple}) + M^{-1}(u_1 + u_2 + u_3) \right] - b
\]

(28)

Recall Theorem 1, when there is no uncertainty, we can apply \( u_1 \) as the control input, and \( Ax - b = 0 \),

\[
\bar{M}\dot{\bar{x}} + \bar{F}_{fric} + \bar{F}_{ripple} = \bar{F}_d = u_1
\]

(29)

Thus

\[
A\left[ \bar{M}^{-1}(\bar{F}_d - \bar{F}_{fric} - \bar{F}_{ripple} + u_1) \right] - b = 0
\]

(30)

Next, by (26),

\[
2\eta^TPA\Delta C(\Delta F_d - \Delta F_{fric} - \Delta F_{ripple}) + 2\eta^TPA\Delta C(F_d - F_{fric} - F_{ripple} + u_1 + u_2) \leq 2\eta^TPA\Delta C(F_d - F_{fric} - F_{ripple} + u_1 + u_2) \leq 2\eta^TPA\Delta C(F_d - F_{fric} - F_{ripple} + u_1 + u_2) \leq 2\eta^TPA\Delta C(F_d - F_{fric} - F_{ripple} + u_1 + u_2) \leq 2\eta^TPA\Delta C(F_d - F_{fric} - F_{ripple} + u_1 + u_2)
\]

(31)

By applying \( \Delta C = CD, \bar{M}^{-1} = C, (23), (25) \), and (21), we can obtain

\[
2\eta^TPA\Delta C(F_d - F_{fric} - F_{ripple} + u_1 + u_2)
\]

(32)

If \( ||\mu|| > \varepsilon \), according to (24) and (25),

\[
-2\gamma||\mu||^2(1 + \rho_D) \leq -2\gamma||\mu||^2 - 2\gamma\mu^T D\mu \leq -2\gamma||\mu||^2 - 2\gamma\mu^T D\mu \leq -2\gamma||\mu||^2(1 + \rho_D).
\]

(33)

If \( ||\mu|| \leq \varepsilon \), according to (24),

\[
-2\gamma||\mu||^2(1 + \rho_D) \leq -2\gamma||\mu||^2(1 + \rho_D)
\]

(34)

From (30)-(35), when \( ||\mu|| > \varepsilon \), there is

\[
\dot{V} \leq 2\eta^T||\rho - 2\varepsilon||^2 - 2\eta^T||\rho - 2\varepsilon||^2.
\]

(36)

and when \( ||\mu|| \leq \varepsilon \), there is
\begin{align}
\dot{V} &\leq 2\|\eta\|^2 - 2\|\eta\|^2 - 2\|\eta\|^2 \leq \varepsilon/2 - 2\kappa\|\eta\|^2. \tag{37}
\end{align}

With (36) and (37), we can obtain
\begin{align}
\dot{V} &\leq \varepsilon/2 - 2\kappa\|\eta\|^2. \tag{38}
\end{align}

Apply the Rayleigh’s principle [26] and Assumption 2, have
\begin{align}
\|\eta\|^2 = \eta^T P ACCA T P \eta \geq \lambda_m(P)\|\eta\|^2 \geq \Delta\|\eta\|^2. \tag{39}
\end{align}

Thus,
\begin{align}
\dot{V} &\leq \varepsilon/2 - 2\kappa\|\eta\|^2. \tag{40}
\end{align}

Upon invoking arguments in [27], [28], the uniform boundedness is guaranteed with
\begin{align}
d(r) = \begin{cases}
r \sqrt{\frac{\lambda_m(P)}{\lambda_m(P)}} & \text{if } r > R, \\
\frac{\varepsilon}{4\kappa^2} & \text{if } r \leq R.
\end{cases} \tag{41}
\end{align}

Moreover, uniform ultimate boundedness is also guaranteed with
\begin{align}
\bar{d} = R \sqrt{\frac{\lambda_m(P)}{\lambda_m(P)}} = \sqrt{\frac{\varepsilon\lambda_m(P)}{4\kappa^2\lambda_m(P)}}, \tag{42}
\end{align}

yielding
\begin{align}
T(\bar{d}, r) = \begin{cases}
0 & \text{if } r \leq \bar{d} \sqrt{\frac{\lambda_m(P)}{\lambda_m(P)}}, \\
\frac{\lambda_m(P)}{\lambda_m(P)} - 2\varepsilon r^2 \frac{\lambda_m(P)}{\lambda_m(P)} & \text{otherwise.}
\end{cases} \tag{43}
\end{align}

IV. OPTIMAL GAIN DESIGN

From the above Lyapunov analysis, system performance can be guaranteed by the proposed RCFP scheme. The size of uniform ultimate bounded region decreases with the increase of \(\kappa\). We can choose an infinitesimal \(\kappa\) to achieve excellent performance, but it will be accompanied by high control cost. Therefore, it is of practical engineering significance to seek an optimal gain for \(\kappa\) to achieve the best tradeoff between control effort and system performance. Here, the optimal design for RCFP is conducted by a fuzzy approach.

System performance is the major factor of concern. According to the Rayleigh’s principle [26], we know
\begin{align}
-2\kappa\|\eta\|^2 \leq -\frac{2\kappa}{\lambda_m(P)} \eta^T P \eta. \tag{45}
\end{align}

Substituting (45) into (40), we then get
\begin{align}
\dot{V} \leq \Delta(-2\kappa\|\eta\|^2) + \varepsilon/2 \leq \frac{2\kappa}{\lambda_m(P)} \eta^T P \eta + \varepsilon/2.
\end{align}

with \(V_0 = V(t_0) = V(\xi(t_0))\). From the analogous property of differential inequality, have
\begin{align}
V(t) \leq (V_0 - \frac{\varepsilon}{2\chi})e^{-\chi(t-t_0)} + \frac{\varepsilon}{2\chi}, \tag{47}
\end{align}

for all \(t \geq t_0\). We can also deduce that, for arbitrary \(t_i\) or arbitrary \(\tau \geq t_i\),
\begin{align}
V(\tau) \leq (V_i - \frac{\varepsilon}{2\chi})e^{-\chi(\tau-t_i)} + \frac{\varepsilon}{2\chi}, \tag{48}
\end{align}

with \(V_i = V(t_i) = V(\xi(t_i))\). The time \(t_i\) refers to the initial execution time of the control algorithm.

As \(V(\varepsilon) \geq \lambda_m(P)\|\xi\|^2\), combine (48), we can obtain an upper bound of \(\|\xi\|^2\). For any \(\tau \geq t_i\), let
\begin{align}
\Theta(\kappa, \tau, t_i) = (V_i - \frac{\varepsilon}{2\chi})e^{-\chi(\tau-t_i)}.
\end{align}

The components \(\Theta(\kappa, \tau, t_i)\) and \(\Theta_{\infty}(\kappa)\) depending on \(\varepsilon\) can be treated as transient-state response performance and steady-state tracking performance, respectively. Although their exact values are unknown, their possible values can be delivered by known membership functions. Next, a fuzzy performance index is formulated.

\begin{align}
J(\kappa, t_i) := D \int_{t_i}^{\infty} \Theta^2(\kappa, \tau, t_i) \, d\tau + \theta_1 D \Theta_{\infty}(\kappa) + \theta_2 \kappa^2
\end{align}

the weighting factors \(\theta_1 > 0, \theta_2 > 0\), while \(J_1(\kappa, t_i), J_2(\kappa)\), \(J_3(\kappa)\) can be viewed as the average of overall transient performance, steady-state performance, and control cost (via a fuzzy D-operation).

There exist a balance and trade-off between system performance and control effort. Motivated by this, the goal of the optimal design is to pursue \(\kappa > 0\) such that the performance index (51) is minimized. From (49), it can be shown that
\begin{align}
\int_{t_i}^{\infty} \Theta^2(\kappa, \tau, t_i) \, d\tau = (V_i - \frac{\varepsilon}{2\chi})^2 \int_{t_i}^{\infty} e^{-2\chi(\tau-t_i)} \, d\tau
\end{align}

Applying the D-operation, and recalling that \(\chi = \frac{2\kappa}{\lambda_m(P)}\) in (46), the performance index can be rewritten as
\begin{align}
J(\kappa, t_i) &= D \left[ \frac{V_i^2}{2\chi} - V_i \frac{\varepsilon}{2\chi} + \frac{\varepsilon^2}{8\chi^3} \right] + \theta_1 D \left[ \frac{\varepsilon^2}{2\chi} \right] + \theta_2 \kappa^2
\end{align}

where \(\chi_1 = \frac{\lambda_m(P)}{4\kappa^2}, \chi_2 = \frac{\lambda_m(P)}{8\kappa^2}, \chi_3 = \frac{\lambda_m(P)}{16\kappa^2}\).

The optimal design problem can be transformed into a constrained optimization issue. For any \(t_i\),
\begin{align}
\min_{\kappa} J(\kappa, t_i), \quad \text{subject to } \kappa > 0. \tag{53}
\end{align}

The first-order derivative of \(J\) with respect to \(\kappa\) is given by
\begin{align}
\frac{\partial J}{\partial \kappa} = -\frac{\chi_1}{\kappa^2} + 2\frac{\chi_2}{\kappa^2} - 3\frac{\chi_3}{\kappa^3} - 2\theta_1 \frac{\chi_4}{\kappa^3} + 2\theta_2 \kappa.
\end{align}

The fact that \(\frac{\partial J}{\partial \kappa} = 0\) leads to
Displacement (mm) following control (ORCFC) for PMLM is shown in Figure 1.

The major advantages of fuzzy optimal design for constraint-following control (ORCFC) can be summarized as threefold. First, the ORCFC scheme can formulate analytically, which can be reflected in the form of analytic expressions of the optimal gain, the system performance, and the minimum cost. It can provide guidance for engineering designers in practice. Second, due to the small amount of calculation and the minimum cost, it can provide guidance for engineering designers in practice. Third, the ORCFC scheme does not need initial on-line training or learning, which is easier to implement in practical engineering applications.

V. NUMERICAL SIMULATIONS AND EXPERIMENTAL RESULTS

A. Parameters selection and simulation results

The PMLM system’s parameters are listed in Table I.

\[
2\theta_4 \kappa^3 + 2\chi_3 \kappa = \chi_1 \kappa^2 + 2\theta_1 \chi_4 \kappa + 3\chi_3.
\]

which is a quintic polynomial equation. Through mathematical analysis, we can get that the solution \( \kappa > 0 \) to (56) always exists and is unique.

The well organized optimal design procedure is concluded as follows:

- **Step 1**: According to the desirable performance of PMLM system, convert it to the form of (5).
- **Step 2**: Choose design parameters \( P \) and \( \rho \) to meet (26), choose parameter \( \rho_D(x,t) \) to meet (21), and choose parameter \( \varepsilon \) small enough.
- **Step 3**: Construct robust constraint-following controller (22) on account of (19), (20), and (23).
- **Step 4**: Choose parameter \( \lambda \) based on (17) and (18), determine parameter \( \lambda_3 \) (\( P \)). Then, \( \chi \) is given in (46).
- **Step 5**: Calculate \( \chi_1 \), \( \chi_2 \), \( \chi_3 \), \( \chi_4 \) in (53) through D-operation.
- **Step 6**: Set weighting factors \( \theta_1 \) and \( \theta_2 \), then the solution of the optimal gain \( \kappa \) in (56) is obtained.
- **Step 7**: The optimal robust constraint-following control (ORCFC) is expressed in (22) using the optimized gain \( \kappa \) in (56).

The control design flow chart of the optimal robust constraint-following control (ORCFC) for PMLM is shown in Figure 1.

**Remark 4.** The major advantages of fuzzy optimal design for RCFC (ORCFC) can be summarized as threefold. First, the ORCFC scheme can formulate analytically, which can be reflected in the form of analytic expressions of the optimal gain, the system performance, and the minimum cost. It can provide guidance for engineering designers in practice. Second, due to the small amount of calculation and simple implementation of the ORCFC scheme, it is applicable for control and analysis of other mechatronic systems which can be modeled as second-order form. Third, the ORCFC scheme does not need initial on-line training or learning, which is easier to implement in practical engineering applications.

**TABLE I: The PMLM system’s parameters.**

| Symbol | Descriptions | Value | Unit |
|--------|--------------|-------|------|
| \( m \) | Mass of the moving thrust block | 1.4 | Kg |
| \( R \) | Resistance between any two phases | 5.6 | Ohms |
| \( k_f \) | Force constant | 290 | N/Amp |
| \( f_c \) | Level of static friction | 6 | N |
| \( f_e \) | Coulomb friction’s minimum level | 2 | N |
| \( \dot{x}_s \) | Lubricant parameter | 0.5 | m/s |
| \( f_v \) | Viscous friction parameter | 2 | N |
| \( S_1 \) | Correction constant of amplitude | 3 | |
| \( S_2 \) | Correction constant of amplitude | 2 | |
| \( S_3 \) | Correction constant of amplitude | 1 | |
| \( \omega \) | Periodic constant | 314 | rad/s |

Considering a sinusoidal signal, which frequency and amplitude are 1 rad/s and 100 mm, i.e., \( x^d = 0.1 \sin(t) \), as the reference trajectory. Based on the Udwadia-Kalaba theory, the constraint of PMLM system should be satisfied as to the desired trajectory, which can be written in the form of (5)

\[
\dot{x}^d = 0.1 \cos(t), \quad \dot{x}^d = -0.1 \sin(t).
\]

that is,

\[
A = 1, \quad c = 0.1 \cos(t), \quad b = -0.1 \sin(t).
\]

We define \( \tilde{F}_{\text{ric}} = (f_c + f_v \dot{x}) \text{sign}(\dot{x}) \), \( F_{\text{ripple}} = F_{\text{ripple}}, \dot{F}_d = 0 \), so \( \Delta F_{\text{ric}} = \left( f_c - f_v \right) e^{-t/(\dot{x}_s^2)} \text{sign}(\dot{x}) \), \( \Delta F_{\text{ripple}} = 0 \), \( \Delta F_d = 0 \).
The optimal parameters are set as \( (k, \theta) \), and then calculate the optimal gain \( \kappa \) and the control input.

As to the uncertainties in (15), we choose \( \Delta M, \Delta F_{\text{fric}}, \Delta F_{\text{ripple}} \) to be "close to 0, 0.01, 0.002, 0" and associated with the following membership functions

\[
\mu_{\Delta M, \Delta F_{\text{fric}}} = \begin{cases} 
1 + 100\epsilon, & -0.01 \leq \epsilon \leq 0, \\
1 - 100\epsilon, & 0 \leq \epsilon \leq 0.01.
\end{cases}
\]

\[
\mu_{\Delta F_{\text{fric}}} = \begin{cases} 
100\epsilon, & 0 \leq \epsilon \leq 0.01, \\
-100\epsilon + 2, & 0.01 \leq \epsilon \leq 0.02.
\end{cases}
\]

\[
\mu_{\Delta F_{\text{ripple}}} = \begin{cases} 
500\epsilon, & 0 \leq \epsilon \leq 0.002, \\
-500\epsilon + 2, & 0.002 \leq \epsilon \leq 0.004.
\end{cases}
\]

Based on the design Step 4 in Section 4, we obtain \( \lambda = 0.01, \lambda_M(P) = 1 \). According to the design Step 5, we select five sets of weight coefficients \( \theta_1 \) and \( \theta_2 \), the optimal gains \( \kappa_{\text{opt}} \) and the corresponding minimum performance indexes \( J_{\text{min}} \) are listed in Table II.

The sinusoidal signal with the function \( x(t) = 0.1\sin(t) \) is considered as the desired trajectory. Under the three control algorithms \( u_1, \text{UKNC} \ (u_1 + u_2), \text{RCFC} \ (u_1 + u_2 + u_3) \), the tracking curves are plotted in Figure.2. Note that, we set the initial displacement of the three tracking curves not at the original point. It can also be observed that the curves of UNKC and RCFC can gradually reach the desired trajectory, while \( u_1 \) can't. In fact, in our control design, the initial position of the tracking curve can be at any position in the presence of \( u_2 \) and the tracking performance can be guaranteed.

The tracking performance curves for sinusoidal signal under the RCFC algorithms with different \( \kappa_{\text{opt}} \) in Table II and the UKNC algorithm are provided in Figure.3. The results show that compared with the UKNC algorithm, the RCFC algorithm with \( \kappa_{\text{opt}} \) can achieve a smaller steady-state displacement error. For RCFC algorithm, when \( \kappa_{\text{opt}} \) increases, both the displacement error and the control input decreases. Therefore, the RCFC algorithm with \( \kappa_{\text{opt}} = 7.59 \) (ORCFC) is chosen as the optimal design, which achieves the tradeoff between control cost and performance.

### B. Experimental platform and experimental results

To further illustrate the practical effectiveness of the proposed ORCFC algorithm, experiments are conducted on the real-time PMLM position tracking control system. Experimental setup is depicted in Figure 4, which is composed of CSPACE control box platform, PMLM, industrial PC installed MATLAB/Simulink, linear motor driver, grating displacement sensor, and other platform configurations. The CSPACE control box platform, combined with MATLAB/Simulink real-time workshop, seamlessly integrates the total development lifecycle into an independent environment. In this way, each development phase between testing and simulation can be done in a real-time environment. The experiments are conducted as follows.

- **i)** Program the control algorithm in MATLAB/Simulink software and directly convert it into C code in the automatic code generation software via the CSPACE control box platform.
- **ii)** Compile the C code in the Compose Code Studio (CCS) software, and then download it into the digital control system processor (DSP, TMS320F28335) via the emulator.

### Table II: Weighting \( \theta_1/\theta_2 \)/Optimal gain\( \kappa_{\text{opt}} \)/Minimum cost \( J_{\text{min}} \)

| \( (\theta_1/\theta_2) \) | \( \theta_1/\theta_2 \) | \( \kappa_{\text{opt}} \) | \( J_{\text{min}} \) |
|----------------|----------------|----------------|----------------|
| (1,100)       | 0.01           | 2.65           | 1038.01        |
| (1,10)        | 0.1            | 5.37           | 456.72         |
| (1,1)         | 1              | 6.48           | 181.87         |
| (10,1)        | 10             | 6.91           | 179.04         |
| (100,1)       | 100            | 7.59           | 177.37         |
Displacement Error (mm)

Control (V)

sinusoidal signal (i.e., x performance.
between 6 and 10 seconds is estimated to reflect the steady-state
indexes, i.e., rise time and stability time. Meanwhile, the error range
of the proposed algorithm, Table III provides specific performance
superior. To intuitively exhibit the excellent transient performance
them, the dynamic performance of the ORCFC algorithm is the most
faster than PID control. On the other side, the steady-state error
side, the response speed of both UKNC and ORCFC algorithms is
are depicted in Figure 5. From the experimental results, on one
results of displacement, error, and the control input for step signal
chosen as the desired reference trajectory. The experimental response
carried out under the following three conditions, i.e., without load and
mass and friction of the PMLM system. To further observe the impact
parameter variations on the system performance, experiments are
added to the moving thrust block, which would lead to the change of
mass and friction of the PMLM system. To further observe the impact
of parameter variations on the system performance, experiments are
carried out under the following three conditions, i.e., without load and
with 4kg, 8kg load. The experimental results under different payloads
are presented in Figure 7. Furthermore, Table V lists the comparison
of specific data reflecting the steady-state tracking performance under
the ORCFC, UKNC, and PID control. Through comparison study, it
can be summed up that the ORCFC strategy demonstrates excellent
robustness against parameter variations.

Fig. 5: The experimental response results to step signal under the ORCFC, UKNC, and PID control, (a) Displacement, (b) Control input.

Fig. 6: The experimental tracking results of the sinusoidal signal under the ORCFC, UKNC, and PID control, (a) Displacement error, (b) Control input.

1) Transient response performance: Step signal \( x^d = 0.1 \) is chosen as the desired reference trajectory. The experimental response
results of displacement, error, and the control input for step signal
are depicted in Figure 5. From the experimental results, on one
side, the response speed of both UKNC and ORCFC algorithms is
faster than PID control. On the other side, the steady-state error
of both UKNC and ORCFC algorithms is also smaller. Among
them, the dynamic performance of the ORCFC algorithm is the most
superior. To intuitively exhibit the excellent transient performance
of the proposed algorithm, Table III provides specific performance
indexes, i.e., rise time and stability time. Meanwhile, the error range
between 6 and 10 seconds is estimated to reflect the steady-state
performance.

2) Steady-state tracking performance: Same as the simulation, the
sinusoidal signal (i.e., \( x^d = 0.1 \sin(t) \)) is chosen as the desired
reference trajectory. Similarly, the corresponding experimental tracking
results of the displacement error and control input are shown in
Figure 6. Obviously, the proposed ORCFC and UKNC algorithms can
achieve smaller errors than PID control. Simultaneously, the control
input of PID control is higher than that of the ORCFC and UKNC
algorithms. The ORCFC algorithm can achieve the best steady-state
tracking performance.

For quantitative comparison, Table IV gives the concrete data about
the average error (AVE), the maximum error (MAEx), and the error’s
standard deviation (STD), which are mathematically described as follows:

\[
AVE = \frac{1}{n - i + 1} \sum_{i} |e(i)|
\]
\[
MAXE = \max \{|e(i)|\}
\]
\[
STD = \sqrt{\frac{1}{n - i + 1} \sum_{i} (e(i) - MAXE)^2}
\]

where \( i \) and \( n \) denote the sampling points for displacement error,
and \( e(i) \) is the \( i \)-th sampling error. For sinusoidal signal, \( i \) and \( n \)
are set to 1 and 5000, which correspond to the time from 0 to 25
seconds, because the sample period is 0.005 (sec). Similarly, for step
signal, we calculate the steady-state error from 3 to 10 seconds, so \( i \)
and \( n \) are set to 601 and 2000, respectively.

3) Robustness against parameter variations: Different payloads are
added to the moving thrust block, which would lead to the change of
mass and friction of the PMLM system. To further observe the impact
of parameter variations on the system performance, experiments are
carried out under the following three conditions, i.e., without load and
with 4kg, 8kg load. The experimental results under different payloads
are presented in Figure 7. Furthermore, Table V lists the comparison
of specific data reflecting the steady-state tracking performance under
the ORCFC, UKNC, and PID control. Through comparison study, it
can be summed up that the ORCFC strategy demonstrates excellent
robustness against parameter variations.
4) Robustness against a sudden disturbance: The purpose of this experiment is to verify the anti-interference ability of the control system. When tracking a sinusoidal signal, the sudden disturbance load is applied to the moving thrust block of the PMLM system. Under the ORCFC, UKNC, and PID control, the experimental curves for the control input, displacement error are depicted in Figure 8. It serves to show that the ORCFC algorithm demonstrates a better capacity for resisting disturbance.

VI. CONCLUSIONS

In this paper, optimal robust constraint-following control is proposed to achieve high-performance position tracking control of the PMLM. The PMLM system’s control performance is deteriorated mainly by friction, ripple force, and external disturbance. Those uncertainties are described via a fuzzy number, and then system’s fuzzy dynamic model is generated. The RCFC algorithm (theoretic framework established by Udwadia and Kalaba) is devised to obey system’s position tracking performance constraint. The resulting controller is demonstrated to be uniform bounded and uniform ultimate bounded with Lyapunov analysis. Next, the optimal design for RCFC (ORCFC) is formulated as a comprehensive performance index minimization problem. On the rapid control prototype platform CSPACE, the real-time experiments are implemented to demonstrate that the proposed ORCFC algorithm can reach superior performance.

CONFLICTS OF INTEREST

The authors declare no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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DATA AVAILABILITY STATEMENT

The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

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TABLE III: Dynamic performance comparison of PMLM system under step response

| Dynamic performance | ORCF | UKNC | PID |
|---------------------|------|------|-----|
| Transient           | 0.245| 0.390| 1.175|
| Rise time (sec)     |      |      |     |
| Settling time (sec) | 0.325| 0.535| 2.065|
| Steady-state        | -0.0005 ~ 0.0005 | -0.004 ~ 0.002 | -0.1 ~ 0.1 |
| Estimated error range (mm) | | | |

TABLE IV: Comparisons of the steady-state tracking performance of the PMLM system.

| Algorithms          | Step | Sinusoid |
|---------------------|------|----------|
| Desired reference signals |      |          |
| Average error AVE (mm) | 0.0005 | 0.0533 |
| Maximum error MAXE (mm) | 0.0016 | 0.1138 |
| Standard deviation STD (mm) | 0.0005 | 0.0585 |

TABLE V: Comparisons of the steady-state tracking performance of the PMLM system in case of the variations of payload.

| Additional payloads | 0kg | 4kg | 8kg |
|---------------------|-----|-----|-----|
| Average error AVE (mm) | 0.0533 | 0.0356 | 0.0238 |
| Maximum error MAXE (mm) | 0.1138 | 0.1027 | 0.0894 |
| Standard deviation STD (mm) | 0.0585 | 0.0453 | 0.0315 |
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