ISOMORPHISM AND BI-LIPSCHITZ EQUIVALENCE BETWEEN THE UNIVOQUE SETS

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(Communicated by Shaobo Gan)

Abstract. In this paper, we consider a class of self-similar sets, denoted by $A$, and investigate the set of points in the self-similar sets having unique codings. We call such set the univoque set and denote it by $U_1$. We analyze the isomorphism and bi-Lipschitz equivalence between the univoque sets. The main result of this paper, in terms of the dimension of $U_1$, is to give several equivalent conditions which describe that the closure of two univoque sets, under the lazy maps, are measure theoretically isomorphic with respect to the unique measure of maximal entropy. Moreover, we prove, under the condition $U_1$ is closed, that isomorphism and bi-Lipschitz equivalence between the univoque sets have resonant phenomenon.

1. Introduction. Let $\{g_j\}_{j=1}^{N}$ be an iterated function system (IFS) of similitudes defined on $\mathbb{R}$ by

$$g_j(x) = r_jx + a_j,$$

where the similarity ratios satisfy $0 < r_j < 1$, and $a_j \in \mathbb{R}$. Hutchinson [14] proved that there exists a unique non-empty compact set $K \subset \mathbb{R}$ such that

$$K = \bigcup_{j=1}^{N} g_j(K).$$

We call $K$ the self-similar set or attractor for the IFS $\{g_j\}_{j=1}^{N}$. For any $x \in K$, there at least exists a sequence $(i_k)_{k=1}^{\infty} \in \{1, \ldots, N\}^{\mathbb{N}}$ such that

$$x = \lim_{k \to \infty} g_{i_{1}} \circ \cdots \circ g_{i_{k}}(0).$$

We call such a sequence a coding of $x$. We can define a surjective projection map between the symbolic space $\{1, \ldots, N\}^{\mathbb{N}}$ and the self-similar set $K$ by

$$\pi((i_k)_{k=1}^{\infty}) := \lim_{n \to \infty} g_{i_{1}} \circ \cdots \circ g_{i_{k}}(0).$$

An $x \in K$ may have many different codings, if $(i_k)_{k=1}^{\infty}$ is unique then we call $x$ a univoque point. If $x$ has multiple codings, then usually the attractor $K$ is complicated, see [5, 6]. We say $\{g_j\}_{j=1}^{N}$ satisfies the open set condition (OSC) [14] if there exists a non-empty bounded open set $O \subseteq \mathbb{R}$ such that

$$g_i(O) \cap g_j(O) = \emptyset, \quad i \neq j.$$
and \( g_j(O) \subseteq O \) for all \( 1 \leq j \leq N \). Usually, self-similar sets with the open set condition is relatively easy to be analyzed. For instance, the Hausdorff dimension of \( K \) coincides with the similarity dimension which is the unique solution \( s \) of the equation \( \sum_{j=1}^N r_j^s = 1 \).

Self-similar sets without the open set condition were analyzed by many scholars, see [11, 13, 18, 22, 23] and references therein. The main concern of these references is to calculate the Hausdorff dimension of the attractors and some associated measures. However, for the fractal structure, the self-similar sets are far beyond understood. For instance, how can we classify different fractal sets from the dynamical and fractal perspective. In dynamical systems, usually we utilize the measure-theoretic isomorphism to classify dynamical systems. In fractal geometry, the bi-Lipschitz equivalence is viewed as an appropriate definition which allows us to distinguish two fractal sets. We will make use of these two definitions to classify some fractal sets.

In this paper, we shall consider the measure-theoretic isomorphism and bi-Lipschitz equivalence between two univoque sets. Partial motivation of this consideration is from the classical \( \beta \)-expansions. For this case, there are many results concerning the univoque sets [7, 8, 12]. However, to the best of our knowledge, there are few papers considering the classification of the univoque sets. For the classical \( \beta \)-expansions, for different \( \beta \)'s the Hausdorff dimension of the univoque sets may differ. Therefore, usually we cannot find a bi-Lipschitz map or an isomorphism between the univoque sets. In the setting of self-similar sets, generally, we can consider this problem. Another reason why we consider the univoque set is that it reflects, in some sense, the complexity of the structure of self-similar set. It has some intimate relation with the original self-similar set [4]. We want to find some classification results such that the original self-similar sets and their subsets have resonant phenomenon, namely, two self-similar sets are bi-Lipschitz equivalent (measure theoretically isomorphic) if and only if their associated univoque sets are bi-Lipschitz equivalent (measure theoretically isomorphic).

Many articles are devoted to the study of the bi-Lipschitz equivalence [3, 9, 10, 16, 17, 19, 21, 25, 27, 29, 30, 31, 32, 33, 34]. It is an important problem to construct a bi-Lipschitz map between two fractal sets. Usually, it is not easy to find such a map, in particular for the self-similar sets which are not totally disconnected [26]. Comparing with the self-similar sets, for the univoque sets it is much more difficult to construct a bi-Lipschitz map as generally the structure of the univoque sets is not clear. Moreover, the univoque sets could not be closed. This fact makes the bi-Lipschitz equivalence between the univoque sets difficult.

It is common knowledge that many fractal sets can be taken as some dynamical systems [2, Lemma 2.1]. As such we may simultaneously classify different fractal sets from the dynamical perspective as well as from the fractal point of view.

Generally, for the self-similar sets without the open set condition, it is difficult to analyze. In this paper, we shall consider a class of overlapping self-similar sets as follows. Fix \( \lambda \in (0, 1) \). Let \( K \) be the attractor of \( \{f_i(x) = \lambda x + b_i\}_{i=1}^n \), where \( n \geq 3 \) and \( b_i \in \mathbb{R} \) for all \( i \). Let \( \mathcal{A} \) be the collection of all the self-similar sets satisfying the following (1) – (4) conditions,

1. \( 0 = b_1 < b_2 < \cdots < b_n = 1 - \lambda \);
2. \( f_i([0, 1]) \cap f_j([0, 1]) = \emptyset \) for any \( |i - j| \geq 2 \);
3. There exist \( i, j \in \{1, \cdots, n - 1\} \) such that
Let some of these words will be clear, see Definition 3.10 and the example below. We use the stand for “left”, “right”, “middle” and “independent”, respectively. The meaning of $k$ of $f_i([0, 1]) \cap f_{i+1}([0, 1])$ is $\lambda^2$ whenever $f_i([0, 1]) \cap f_{i+1}([0, 1]) \neq \emptyset$, where $|J|$ denotes the length of $J$.

For convenience we give a finer classification of $\mathcal{A}$. Given the following two conditions

(5) $f_n([0, 1]) \cap f_{n-1}([0, 1]) \neq \emptyset$ or $f_1([0, 1]) \cap f_2([0, 1]) \neq \emptyset$;

(5') $f_n([0, 1]) \cap f_{n-1}([0, 1]) = \emptyset$ and $f_1([0, 1]) \cap f_2([0, 1]) = \emptyset$.

Denote by $\mathcal{A}_1$ all the self-similar sets satisfying conditions (1) – (5) and by $\mathcal{A}_2$ all the self-similar sets satisfying conditions (1) – (4) and (5'). It is easy to prove that by conditions (1) – (4) if

$$f_i([0, 1]) \cap f_{i+1}([0, 1]) \neq \emptyset,$$

then we necessary have $f_{i_n} = f_{i(i+1)}$, where $f_{ij} = f_i \circ f_j$. We call $f_i([0, 1]), 1 \leq i \leq n$, a basic interval of $[0, 1]$ with respect to the IFS $\{f_i\}_{i=1}^n$. Let

$$k_i = \sharp\{1 \leq p \leq n : f_p([0, 1]) \cap f_q([0, 1]) = \emptyset \text{ for any } q \neq p\},$$

and

$$n_K = \sharp\{1 \leq i \leq n-1 : |f_i([0, 1]) \cap f_{i+1}([0, 1])| = \lambda^2\},$$

where $\sharp$ means cardinality.

If there exist some $1 \leq i \leq p \leq n$ such that $\bigcup_{k=i}^p f_k([0, 1])$ is an interval and that $p - i$ is largest, then we call $\bigcup_{k=i}^p f_k([0, 1])$ a maximal connected component of $[0, 1]$. This definition is the same as defined in topology. Denote by $C_n$ the number of all the maximal connected components. Define $k_i = k_r = C_n - k_i$, and $k_m = n - k_i - 2k_i$. It is easy to find that $n_K = k_i + k_m$. The subscripts $l, r, m, i$ stand for “left”, “right”, “middle” and “independent”, respectively. The meaning of these words will be clear, see Definition 3.10 and the example below. We use the following example to illustrate the above definitions.

**Example 1.1.** Let $K$ be the attractor of the IFS

$$\{f_1(x) = lx, f_2(x) = \lambda x + \lambda - \lambda^2, f_3(x) = \lambda x + 2\lambda - 2\lambda^2, f_4(x) = \lambda x + 3\lambda - 3\lambda^2, f_5(x) = \lambda x + 4\lambda, f_6(x) = \lambda x + 3\lambda + 2\lambda^2, f_7(x) = \lambda x + 1 - 2\lambda + \lambda^2, f_8(x) = \lambda x + 1 - \lambda\},$$

where $0 < \lambda < \frac{1}{10}$.

The basic intervals in the first level for $K$ are as follows.

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0 ----- ----- ----- ----- 1
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**Figure 1.** First iteration of $K$

For $K$, there are 3 maximal connected components, i.e.

$$\bigcup_{k=1}^4 f_k([0, 1]), f_5([0, 1]), \bigcup_{k=6}^8 f_k([0, 1]).$$

In our definition we emphasize that $\bigcup_{k=2}^4 f_k([0, 1])$ is not a maximal connected component as for this set $p - i = 2$ is not the largest number. Therefore $C_n = 3$. Moreover, $n_K = 5$, which refers to the number of the following intersections

$$f_1([0, 1]) \cap f_2([0, 1]), f_2([0, 1]) \cap f_3([0, 1]), f_3([0, 1]) \cap f_4([0, 1]), f_4([0, 1]) \cap f_5([0, 1]), f_5([0, 1]) \cap f_6([0, 1]), f_6([0, 1]) \cap f_7([0, 1]), f_7([0, 1]) \cap f_8([0, 1]).$$
$k_i = 1$, namely, we only have that $f_5([0, 1])$ does not intersect with other basic intervals. Moreover,
\[ k_m = 3, k_l = k_r = 2. \]
The first result of this paper is from the ergodic perspective. We prove the existence of the unique measures of maximal entropy with respect to the lazy map for $U_1$ and $K$, where $\overline{U_1}$ is the closure of $U_1$. For the definition of lazy map, we refer to the next section.

**Theorem 1.2.** Let $K \in \mathcal{A}$. Then $\overline{U_1}$ has a unique measure of maximal entropy with respect to the lazy map. Moreover, the same statement is correct for $K_1$.

The next result is giving a uniform dimensional formula of $U_1$. The key idea, which we implement, is the configuration set with finite patterns [16].

**Theorem 1.3.** Let $K \in \mathcal{A}$ be the self-similar set with the IFS \( \{f_i\}_{i=1}^n \). Then $U_1 \setminus U_1$ is at most a countable set.

1. If \( f_n([0, 1]) \cap f_{n-1}([0, 1]) \neq \emptyset, f_1([0, 1]) \cap f_2([0, 1]) = \emptyset \), or \( f_n([0, 1]) \cap f_{n-1}([0, 1]) = \emptyset, f_1([0, 1]) \cap f_2([0, 1]) \neq \emptyset \), then
   \[ \dim_H(U_1) = \frac{\log \gamma_1}{-\log \lambda}, \]
   where $\gamma_1$ is a Pisot number of the following equation:
   \[ x^3 - nx^2 + (n - k_i + k_m)x - (k_m + k_l) = 0. \]
2. If \( f_1([0, 1]) \cap f_2([0, 1]) \neq \emptyset \) and \( f_n([0, 1]) \cap f_{n-1}([0, 1]) \neq \emptyset \), then
   \[ \dim_H(U_1) = \frac{\log \gamma_2}{-\log \lambda}, \]
   where $\gamma_2$ is a Pisot number of the following equation:
   \[ x^2 - (n - 1)x - k_i + k_m = 0. \]
3. If \( f_1([0, 1]) \cap f_2([0, 1]) = \emptyset \) and \( f_{n-1}([0, 1]) \cap f_n([0, 1]) = \emptyset \), then
   \[ \dim_H(U_1) = \frac{\log \gamma_3}{-\log \lambda}, \]
   where $\gamma_3$ is the appropriate root of the following equation:
   \[ x^2 - nx + (n - k_i + k_m) = 0. \]

Moreover, $\gamma_3$ is a Pisot number if and only if $k_m + 1 < k_i$.

We give some remarks upon this result. Firstly, the Pisot numbers play a crucial role in proving all the following theorems. Secondly, the significance of this result is that we give a uniform formula of the Hausdorff dimension of $U_1$. Without this formula, we cannot analyze the relation between the bi-Lipschitz equivalence and measure-theoretic isomorphism for the univoque sets. A useful corollary of Theorem 1.3 is the following result.

**Corollary 1.4.** Given $i = 1$ or 2. Let $K_1, K_2 \in \mathcal{A}_i$ be two self-similar sets with the IFS’s \( \{f_i\}_{i=1}^n \) and \( \{g_j\}_{j=1}^m \). Suppose that \( \dim_H(U_1) = \dim_H(U_2) \). Then the following conditions are equivalent.

1. $k_l^{(1)} = k_l^{(2)}$;
Theorem 1.5. Let $K \in \mathcal{A}$ with the IFS’s $\{f_i\}_{i=1}^n$. Then
\[
\dim_H(K) = \frac{\log \gamma}{\log \lambda},
\]
where $\gamma$ is a Pisot number of the following polynomial
\[x^2 - nx + n_K = 0.\]

The following corollary follows from Theorems 1.5, 1.3 and Corollary 1.4.

Corollary 1.6. Given $i = 1$ or 2. For any $K_1, K_2 \in \mathcal{A}_i$. Then
\[
\dim_H(U_i^{(1)}) = \dim_H(U_i^{(2)})
\]
if and only if
\[
\dim_H(K_1) = \dim_H(K_2).
\]

We may find some counterexamples in the class $\mathcal{A}$ such that $\dim_H(U_i^{(1)}) = \dim_H(U_i^{(2)})$ but $\dim_H(K_1) \neq \dim_H(K_2)$, see one example in Section 4. The next theorem is one of the main results of this paper.

Theorem 1.7. Given $i = 1$ or 2. For any $K_1, K_2 \in \mathcal{A}_i$. Then the following conditions are equivalent.

1. $\dim_H(U_i^{(1)}) = \dim_H(U_i^{(2)})$, and $k_i^{(1)} = k_i^{(2)}$;
2. $\dim_H(K_1) = \dim_H(K_2)$, and $k_i^{(1)} = k_i^{(2)}$;
3. $K_1$ and $K_2$ are quasi-Lipschitz equivalent, and $k_i^{(1)} = k_i^{(2)}$;
4. $K_1$ and $K_2$ are bi-Lipschitz equivalent, and $k_i^{(1)} = k_i^{(2)}$;
5. $g_{K_1} = n_{K_2}$, and $k_i^{(1)} = k_i^{(2)}$;
6. $U_i^{(1)}$ and $U_i^{(2)}$ are measure theoretically isomorphic with respect to the unique measures of maximal entropy, and $k_i^{(1)} = k_i^{(2)}$.
7. $K_1$ and $K_2$ are measure theoretically isomorphic with respect to the unique measures of maximal entropy, and $k_i^{(1)} = k_i^{(2)}$.

The definition of bi(qusi)-Lipschitz equivalence is available in the next section. We emphasize that $\dim_H(U_i^{(1)}) = \dim_H(U_i^{(2)})$ cannot imply $k_i^{(1)} = k_i^{(2)}$, and vice versa, see Example 4.1. It is natural to consider the bi-Lipschitz equivalence between $U_i^{(1)}$ and $U_i^{(2)}$. We can find some counterexample such that $\dim_H(U_i^{(1)}) = \dim_H(U_i^{(2)})$ cannot imply that $U_i^{(1)}$ and $U_i^{(2)}$ are bi-Lipschitz equivalent. More precisely, we may find some example such that $U_i^{(1)}$ is closed but $U_i^{(2)}$ is not closed, see Example 4.1. However, if we suppose that $U_i^{(1)}$ and $U_i^{(2)}$ are closed, then the following result holds.

Theorem 1.8. For any $K_1, K_2 \in \mathcal{A}$ with IFS’s $\{f_i\}_{i=1}^n$ and $\{g_i\}_{i=1}^m$, respectively. Suppose that $U_i^{(1)}$ and $U_i^{(2)}$ are closed. Then the following conditions are equivalent.

\[k_i^{(1)} = k_i^{(2)};\]
\[k_i^{(1)} = k_i^{(2)};\]
\[k_i^{(1)} = k_i^{(2)};\]
\[k_i^{(1)} = k_i^{(2)};\]
(1) \( \dim_H(U_1^{(1)}) = \dim_H(U_1^{(2)}) \);
(2) \( U_1^{(1)} \) and \( U_1^{(2)} \) are bi-Lipschitz equivalent;

With a little effort, Theorems 1.7, 1.8 and Corollary 1.6 yield the following resonant result.

**Corollary 1.9.** For any \( U_1, U_2 \in A \). Suppose that \( U_1^{(1)} \) and \( U_1^{(2)} \) are closed. Then the following conditions are equivalent.

1. \( U_1^{(1)} \) and \( U_1^{(2)} \) are bi-Lipschitz equivalent, and \( k_m^{(1)} = k_m^{(2)} \);
2. \( K_1 \) and \( K_2 \) are bi-Lipschitz equivalent, and \( k_m^{(1)} = k_m^{(2)} \);
3. \( U_1^{(1)} \) and \( U_1^{(2)} \) are measure theoretically isomorphic with respect to the unique measures of maximal entropy, and \( k_m^{(1)} = k_m^{(2)} \);
4. \( K_1 \) and \( K_2 \) are measure theoretically isomorphic with respect to the unique measures of maximal entropy, and \( k_m^{(1)} = k_m^{(2)} \).

**Remark 1.10.** When \( U_1 \) is not closed, we may still prove some resonant result.

For the class \( A \), we may further classify two sub-classes, i.e. \( A_1 = A'_1 \cup A''_1 \) such that we may prove the following result.

Let \( K_1, K_2 \in \mathbb{A}^* \), where \( A_1^* = A'_1 \) or \( A''_1 \). Then the following conditions are equivalent.

1. \( \dim_H(U_1^{(1)}) = \dim_H(U_1^{(2)}) \);
2. \( U_1^{(1)} \) and \( U_1^{(2)} \) are quasi-Lipschitz equivalent;
3. \( U_1^{(1)} \) and \( U_1^{(2)} \) are Lipschitz equivalent;
4. \( n_{K_1} = n_{K_2} \);
5. \( K_1 \) and \( K_2 \) are Lipschitz equivalent;
6. \( K_1 \) and \( K_2 \) are quasi-Lipschitz equivalent;
7. \( \dim_H(K_1) = \dim_H(K_2) \).

However, the proof of this result is more complicated than the method in our paper, although the main idea is similar. We shall prove this result in another paper.

This paper is arranged as follows. In section 2, we give some basic definitions which will appear in the paper. In section 3, we give the proofs of the main theorems. In section 4, we give one example. Finally, we give some remarks and open problems.

2. **Some basic definitions and lemmas.** Given \( K \in A \) with the IFS \( \{f_i\}_{i=1}^n \), we say \( f_{i_1 \cdots i_n}(x) \) is an exact overlap if there exist two different \( i_1 \cdots i_k \in \{1, \cdots, n\}^k \) and \( j_1 \cdots j_m \in \{1, \cdots, n\}^m \) such that

\[
  f_{i_1 \cdots i_k} = f_{j_1 \cdots j_m}.
\]

For any \( k \geq 1 \), let \( (i_1 \cdots i_k) \in \{1, \cdots, n\}^k \) we call \( f_{i_1 \cdots i_k}(\{0,1\}) \) the \( k \)-th level basic interval. It is well known that any self-similar set can be taken as a topological dynamical system [2]. Let \( K \) be a self-similar set from the class \( A \) with IFS \( \{f_i\}_{i=1}^n \).

Define

\[
  T_j(x) := f_j^{-1}(x) = (x - b_j)\lambda^{-1}
\]

for \( x \in f_j(K) \) and \( 1 \leq j \leq n \). We denote the concatenation \( T_{i_n} \circ \cdots \circ T_{i_1}(x) \) by \( T_{i_1 \cdots i_n}(x) \). The following lemma was proved in [2, Lemma 2.1].

**Lemma 2.1.** Let \( x \in K \). Then \( (i_k)_{k=1}^\infty \in \{1, \cdots, n\}^\mathbb{N} \) is a coding for \( x \) if and only if \( T_{i_1 \cdots i_k}(x) \in K \) for all \( k \in \mathbb{N} \).
Motivated by this lemma, we define the orbits.

**Definition 2.2.** Let \( x \in K \) with a coding \((i_k)_{k=1}^{\infty}\), we call the set
\[ \{T_{i_1\ldots i_k}(x) : k \geq 0\} \]
an orbit set of \( x \).

Clearly, for a generic point \( x \in K \), it may have multiple codings. In other words, it has multiple orbits. The following definition is to define a special orbit, namely, the lazy orbit.

**Definition 2.3.** Let \( x \in K \) with a coding \((i_k)_{k=1}^{\infty}\), we call \((i_k)_{k=1}^{\infty}\) the lazy coding of \( x \) if whenever the associated orbit of \( x \), denoted by \( \{T_{i_1\ldots i_k}(x) : k \geq 1\} \), hits some switch region \( f_i(K) \cap f_j(K) \neq \emptyset \), where \( i < j \), i.e. there exists some \( l \) such that \( T_{i_1\ldots i_l}(x) \in f_i(K) \cap f_j(K) \), then we let \( i_{l+1} = i \). We call such map the lazy map, and the orbit \( \{T_{i_1\ldots i_k}(x) : k \geq 0\} \) the lazy orbit. We denote the lazy orbit of \( x \) by \( \{L^k(x) : k \geq 0\} \), where \( L^0(x) = x \).

Let \( K \in \mathcal{A} \). Denote \( V = \{f_i(K) \cap f_j(K) : f_i(K) \cap f_j(K) \neq \emptyset \text{ for some } i,j\} \). Evidently, we have a dynamical description of \( U_1 \).

\[
U_1 = \left\{ x \in K : \text{the lazy orbit of } x \text{ does not hit } V \right\}. \tag{1}
\]

Now we introduce some definitions related with the bi-Lipschitz equivalence.

Two metric spaces \((X_1, d_1)\) and \((X_2, d_2)\) are said to be **bi-Lipschitz equivalent** if there exists a bijection \( \phi : X_1 \rightarrow X_2 \) and a constant \( c > 0 \) such that
\[
c^{-1} d_1(x, x') \leq d_2(\phi(x), \phi(x')) \leq c d_1(x, x') \]
for all \( x, x' \in X_1 \).

We simply say \( X_1 \) and \( X_2 \) are Lipschitz equivalent if they are bi-Lipschitz equivalent. For simplicity, we denoted by \( A_1 \simeq A_2 \) when they are Lipschitz equivalent. We say that \( X_1 \) and \( X_2 \) are **quasi-Hölder equivalent** [29] if there exists a bijection \( \phi : X_1 \rightarrow X_2 \) and \( t > 0 \) such that for all different \( x, x' \in X_1 \),
\[
\frac{\log d_2(\phi(x), \phi(x'))}{\log d_1(x, x')} \rightarrow t \quad \text{uniformly as } d_1(x, x') \rightarrow 0.
\]

In particular, when \( t = 1 \), we say that \( X_1 \) and \( X_2 \) are **quasi-Lipschitz equivalent** [29]. Clearly, if \( X_1 \) and \( X_2 \) are bi-Lipschitz equivalent, then they are quasi-Lipschitz equivalent. Moreover, \( \dim_H(X_1) = \dim_H(X_2) \), provided that \( X_1 \) and \( X_2 \) are quasi-Lipschitz equivalent.

An algebraic number is called a Pisot number if all its conjugates have modules strictly smaller than 1. The Pisot number plays a pivotal role in this paper, it has many useful properties, see [1] and references therein. There are many definitions regarding the ergodic theory, for instance, the unique measure of maximal entropy, isomorphism. We do not introduce them in details. The reader can find these definitions in the book [28].

3. **Proofs of main theorems.**

3.1. **Proof of Theorem 1.2.** In this section, we shall give the proofs of the main results introduced in Section 1. We first prove Theorem 1.2. In [15], Jiang and Dajani proved that for the self-similar sets with the exact overlaps they gave an algorithm which can calculate the dimension of \( U_1 \), see [15, Theorem 2.18] or [4]. The key idea of [15] is partitioning the self-similar sets and constructing a graph-directed self-similar set with the open set condition. Then the univoque set \( U_1 \) can
be identified with a graph-directed self-similar set. This idea enables us to find the individual example rather than a class of examples. Nevertheless, the partition idea allows us to find the unique measure of maximal entropy for $U_1$. For convenience, we introduce how to give a partition for any $K \in \mathcal{A}$. Let

$$
\mathcal{P} = \bigcup_{i=1}^{2n} \{ f_i(0), f_i(1) \} = \{ 0 = s_1 < \cdots < s_{2n} = 1 \}.
$$

For two consecutive members $s_i$ and $s_{i+1}$ of $\mathcal{P}$, we call $\{ s_i, s_{i+1} \}$ an admissible pair if there exists a $j$ such that

$$
s_i, s_{i+1} \in f_j([0, 1]).
$$

Define

$$
\mathcal{Q} = \{ 1 \leq i < 2n : \{ s_i, s_{i+1} \} \text{ is an admissible pair} \}.
$$

For an admissible pair $\{ s_i, s_{i+1} \}$, there exist at most two $j$’s satisfying (2) and we denote by $\alpha(i)$ the smaller $j$. Dynamically, we choose the lazy algorithm. It is not difficult to check that $f_{\alpha(i)}^{-1}(s_i), f_{\alpha(i)}^1(s_{i+1}) \in \mathcal{P}$. For $s, t \in \mathcal{P}$ with $s < t$ define

$$
\mathcal{V}[s, t] = \{ \{ s_j, s_{j+1} \} : j \in \mathcal{Q} \text{ and } [s_j, s_{j+1}] \subseteq [s, t] \}
$$

For an admissible pair $\{ s_i, s_{i+1} \}$ let

$$
\mathcal{A}\{ s_i, s_{i+1} \} = \mathcal{V}[f_{\alpha(i)}^{-1}(s_i), f_{\alpha(i)}^1(s_{i+1})] \text{ and } [s_i, s_{i+1}]_K = [s_i, s_{i+1}] \cap K.
$$

Note that for some $[s_i, s_{i+1}]_K$, $s_i$ or $s_{i+1}$ could be an isolated point in $[s_i, s_{i+1}]_K$, i.e. there is no sequence $x_k \in [s_i, s_{i+1}]_K$ which is different from $s_i$ or $s_{i+1}$ such that

$$
\lim_{k \to \infty} x_k = s_i \text{ or } s_{i+1}.
$$

For this case, we delete the isolated point from $[s_i, s_{i+1}]_K$. Since $[s_i, s_{i+1}]_K$ is a closed set, it follows that after removing the isolated point of $[s_i, s_{i+1}]_K$, $[s_i, s_{i+1}]_K \setminus \{ s_i \}$ or $[s_i, s_{i+1}]_K \setminus \{ s_{i+1} \}$ is still closed. For simplicity, we still utilize $[s_i, s_{i+1}]_K$ to replace $[s_i, s_{i+1}]_K \setminus \{ s_i \}$ or $[s_i, s_{i+1}]_K \setminus \{ s_{i+1} \}$ if there is no confusion. In what follows, we always obey this rule. To illustrate this modification, we give the following example.

**Example 3.1.** Let $K$ be the attractor of the IFS

$$
\begin{align*}
\left\{ f_1(x) = \frac{x}{3}, f_2(x) = \frac{x}{3} + \frac{2}{9}, f_3(x) = \frac{x + 2}{3} \right\},
\end{align*}
$$

$s_1 = 0, s_2 = 2/9, s_3 = 1/3, s_4 = 5/9, s_5 = 2/3, s_6 = 1$. Note that

$$
[s_1, s_2]_K = [0, 2/9] \cap K = f_{11}(K) \cup f_{12}(K) \cup \{ 2/9 \}.
$$

The point $2/9$ is an isolated point in $[s_1, s_2]_K$. However, $2/9$ is an accumulation point in $[s_2, s_3]_K = [2/9, 1/3] \cap K$.

**Lemma 3.2.**

1. $K = \bigcup_{i \in \mathcal{Q}} [s_i, s_{i+1}]_K$.

2. The compact sets $\{ [s_i, s_{i+1}]_K, i \in \mathcal{Q} \}$ have the graph-directed structure:

$$
[s_i, s_{i+1}]_K = \bigcup_{\{s_j, s_{j+1}\} \in \mathcal{A}\{s_i, s_{i+1}\}} f_{\alpha(i)}([s_j, s_{j+1}]_K).
$$

Moreover, the graph-directed structure above satisfies the open set condition with the open sets $\{ (s_i, s_{i+1}) : i \in \mathcal{Q} \}$.
Proof. The first statement follows from the definitions of partition $P$ and $[s_i, s_{i+1}]_K$. For the second statement, we only need to consider the image of $[s_i, s_{i+1}]_K$ under the expanding map $f_{\alpha(i)}^{-1}$, i.e., it suffices to prove that

$$f_{\alpha(i)}^{-1}([s_i, s_{i+1}]_K) = \bigcup_{\{s_i, s_{j+1}\} \in A(s_i, s_{i+1})} [s_j, s_{j+1}]_K.$$ 

However, this is a directed consequence of the definitions of $A(s_i, s_{i+1})$ and $f_{\alpha(i)}$. For the open set condition [22], we can check it directly from the equation in the second statement.

Lemma 3.2 is essentially giving a Markov partition [20] for $K$. Note that by the reformulation (1), for the univoque point, its lazy orbit never hits the set $V$. Roughly speaking, if we delete these $[s_i, s_{i+1}]_K, i \in \mathcal{Q}$ which are associated with $V$ in the above graph-directed self-similar set, then we obtain a subset that is equal to the univoque set (up to a countable set). Therefore, we have the following construction. Let $\mathcal{Q}^*$ be the subset of $\mathcal{Q}$ defined in (3) by deleting those $j$ such that

$$\{s_j, s_{j+1}\} = \{f_{\ell+1}(0), f_{\ell}(1)\}$$

for some $\ell$. For this $\mathcal{Q}^*$, we are allowed to define a new Markov chain as follows. Let $i \in \mathcal{Q}^*$, we define the image of $[s_i, s_{i+1}]_K$, under the expanding map $f_{\alpha(i)}^{-1}$, by

$$\bigcup_{\{s_i, s_{j+1}\} \in A(s_i, s_{i+1}), j \in \mathcal{Q}^*} [s_j, s_{j+1}]_K.$$ 

In terms of this rule, we define a new directed graph with vertex set $\{[s_i, s_{i+1}]_K, i \in \mathcal{Q}^*\}$. For any two different $i, j \in \mathcal{Q}^*$, if

$$f_{\alpha(i)}^{-1}[s_i, s_{i+1}]_K \supset [s_j, s_{j+1}]_K,$$

then we label an edge from $[s_i, s_{i+1}]_K$ to $[s_j, s_{j+1}]_K$. The corresponding similitude between these two vertices is $f_{\alpha(i)}$. Therefore, we have constructed a graph-directed self-similar set $K^*$ satisfying the open set condition [22].

Lemma 3.3. Let $K \in \mathcal{A}$. Then $\dim_H(U_1) = \dim_H(K^*)$. Moreover, $U_1 = K^*$ except for a countable set, and $K^* = \overline{U_1} \cup C$, where $C$ is a countable set.

Proof. First, by the reformulation (1), we have $U_1 \subset K^*$. Conversely, take $x \in K^*$. Then we can find a lazy coding $(i_n) \in \{1, 2, \ldots, n\}^\mathbb{N}$ (we choose the lazy algorithm) such that

$$f_{i_n}^{-1} \circ f_{i_{n-1}}^{-1} \circ \ldots \circ f_{i_1}^{-1}(x) \in \mathcal{V}^*[0, 1]$$

for any $n \geq 1$, where

$$\mathcal{V}^*[0, 1] = \{[s_j, s_{j+1}]_K : j \in \mathcal{Q}^*\}.$$ 

If the orbit of $x$ never hits the endpoints of every $[s_j, s_{j+1}]_K, j \in \mathcal{Q} \setminus \mathcal{Q}^*$, denoted the set of these endpoints by $W$ (clearly, we have $W \subset \mathcal{P}$), then $x \in U_1$. Here the set $W$ can be defined as follows:

$$W = \begin{cases} 
\mathcal{V}^*[0, 1] \cup f_{i_1}([0, 1]) & \text{for } i_1 \neq 0, \\
\mathcal{V}^*[0, 1] \cup f_{i_1}([0, 1]) & \text{for } i_1 = 0, \\
\emptyset & \text{for } i_1 = 1.
\end{cases}$$

If there exists some smallest $n_0$ such that

$$f_{i_n}^{-1} \circ f_{i_{n-1}}^{-1} \circ \ldots \circ f_{i_1}^{-1}(x) \in W \subset \mathcal{P}.$$
Lemma 3.4. Let $C := \{ x \in K^* :$ there exists a sequence $(i_n)_{n=1}^\infty \in \{1, \ldots, n\}^\mathbb{N}$ and $n_0 \geq 0$

such that $f_{n_0}^{-1} \circ f_{n_0-1}^{-1} \circ \ldots \circ f_1^{-1}(x) \in W \}$,

where $f_{n_0}(x) = x$. Therefore, we have proved that $C$ is at most a countable set as

$C \subset \bigcup_{n=1}^{\infty} \cup (i_1, i_2, \ldots, i_n) = f_{i_1 i_2 \ldots i_n}(P)$.

Therefore, we have proved that $U_1 \subset K^* \subset U_1 \cup C, C \subset K^*$. Subsequently

$K^* = \overline{U_1} \cup \overline{C}$.

\[ \square \]

Lemma 3.4. $C \subset \overline{U_1}, \overline{U_1} = K^*$.

**Proof.** By Lemma 3.3, it suffices to prove that $C \subset \overline{U_1}$. We suppose without loss of generality that

$f_n([0, 1]) \cap f_{n-1}([0, 1]) = \emptyset, f_1([0, 1]) \cap f_2([0, 1]) \neq \emptyset$.

Let

$W = \bigcup_{f_i([0, 1]) \cap f_{i+1}([0, 1]) \neq \emptyset} \{f_i(1)\}$.

For any $x \in C \setminus W \subset K^*$, there exists a smallest $k \geq 1$ and $i_1 \cdots i_k \in \{1, \ldots, n\}^k$

such that $T_{i_1 \cdots i_k}(x)$ hits $W$ for the first time (here we emphasize that we choose the lazy orbit of $x$), i.e. $T_{i_1 \cdots i_k}(x) \notin W$ for any $1 \leq j \leq k-1$ and $T_{i_1 \cdots i_k}(x) \in W$, or $x \notin W$ and $T_{i_1}(x) \in W$. In other words, $f_{i_1 \cdots i_k}(x)$ is not an exact overlap. We may assume that $T_{i_1 \cdots i_k}(x) = f_i(1)$ for some $1 \leq i \leq n$, where $f_i([0, 1]) \cap f_{i+1}([0, 1]) \neq \emptyset$.

By the fourth condition of \( \mathcal{A} \), it follows that $f_{in}(x) = f_{i(i+1)}(x)$. Since $f_1([0, 1]) \cap f_2([0, 1]) \neq \emptyset$, we suppose that $t_0 \geq 2$ is the largest integer such that

$f_j([0, 1]) \cap f_{j+1}([0, 1]) \neq \emptyset, 1 \leq j \leq t_0 - 1$

and $f_{t_0}([0, 1]) \cap f_{t_0+1}([0, 1]) = \emptyset$. Define

$x_l = \pi(i_1 i_2 \cdots i_k (i+1)2^l n^\infty)$.

Clearly,

$\lim_{l \to \infty} x_l = f_{i_1 \cdots i_k (i+1)}(f_1(1)) = f_{i_1 \cdots i_k} (f_{in}(1)) = f_{i_1 \cdots i_k} (f_1(1)) = x$ as $T_{i_1 \cdots i_k}(x) = f_i(1)$.

Here, we use a fact $f_1(1) = \pi(2^\infty)$. Hence, it suffices to prove that $x_l \in U_1$ for any $l \geq 1$. Evidently,

$x_l = f_{i_1 \cdots i_k (i+1)2^l n^\infty} \subset f_{i_1 \cdots i_k (i+1)2^l t_0}([0, 1])$.

By the assumption, $f_n([0, 1]) \cap f_{n-1}([0, 1]) = \emptyset$, it follows that $i + 1 \leq n - 1$ and that $t_0 \leq n - 1$. Therefore, $f_{i_1 \cdots i_k (i+1)2^l t_0}(x)$ is not an exact overlap (if $f_{i_1 \cdots i_k}(x)$ is an exact overlap and $f_{i_1 \cdots i_k}(x)$ is not an exact overlap, then $t_p$ should be 1 or $n$). Moreover,

$f_{i_1 \cdots i_k (i+1)2^l t_0}([0, 1]) \cap f_{i_1 \cdots i_k (i+1)2^l t_0+1}([0, 1]) = \emptyset$

by the maximum of $t_0$. Thus, $x_l = f_{i_1 \cdots i_k (i+1)2^l t_0}(1)$ is a univoque point for any $l \geq 1$. With a slight change, we can prove $W \subset \overline{U_1}$. Now $\overline{U_1} = K^*$ follows from $K^* = \overline{U_1} \cup \overline{C}$.

\[ \square \]
The graph-directed construction in Lemma 3.2 has an associated matrix which is defined as follows: $S = (s_{i,j})_{i \times l}$, where

$$s_{i,j} = \begin{cases} 0 & \text{if } [s_i, s_{i+1}] \not\supset f_{\alpha(i)}([s_j, s_{j+1}]_K) \\ 1 & \text{if } [s_i, s_{i+1}]_K \supset f_{\alpha(i)}([s_j, s_{j+1}]_K). \end{cases}$$

The corresponding subshift of finite type generated by $S$ is defined as follows:

$$\Sigma = \{(i_k)_{k=1}^\infty \in \{1, 2, 3, \ldots, l\}^\infty : s_{i_k, i_{k+1}} = 1 \text{ for any } k\}.$$ Similarly, we may define the adjacent matrix with the directed graph of $K^*$, denoted by $S^*$. The subshift of finite type generated by this matrix is denoted by $\Sigma^*$.

The following lemma can be checked directly.

**Lemma 3.5.** Let $K \in A$. Then the associated graph-directed construction of $K$ and $K^*$ have irreducible matrices $S$ and $S^*$, respectively.

**Proof.** Without loss of generality, we prove only one case, i.e.

$$f_n([0,1]) \cap f_{n-1}([0,1]) = \emptyset, f_1([0,1]) \cap f_2([0,1]) \neq \emptyset.$$

First, we prove that $S$ is irreducible. It suffices to show that the directed graph of $K$ is strongly connected. For any vertex $[s_i, s_{i+1}]_K$, if $i \in Q^*$, then there exists some $1 \leq j \leq n - 1$ such that

$$f_{\alpha(i)}^{-1}([s_i, s_{i+1}]_K) \supset f_j(K).$$

If $f_j(K) \cap (\cup_{k \neq j} f_k(K)) = \emptyset$, then

$$f_{\alpha(i)}^{-1}(f_j(K)) = K = \cup_{i \in Q}[s_i, s_{i+1}]_K.$$

Therefore, the vertex $[s_i, s_{i+1}]_K$ can reach any other vertices. If

$$f_j([0,1]) \cap f_{j+1}([0,1]) \neq \emptyset,$$

then

$$f_j(K) \supset f_{j+1}(K) = f_{(j+1)1}(K).$$

Therefore,

$$f_{\alpha(j)}^{-1}(f_j(K)) \supset f_{\alpha(j)}^{-1}(f_{j+1}(K)) = f_n(K).$$

Since

$$f_{n-1}([0,1]) \cap f_n([0,1]) = \emptyset,$$

it follows that

$$f_{\alpha(i)}^{-1}(f_n(K)) = K = \cup_{i \in Q}[s_i, s_{i+1}]_K.$$

Hence, the vertex $[s_i, s_{i+1}]_K$ can reach any other vertices. If $i \in Q \setminus Q^*$, then we may repeat the above discussion (the same as the case $f_j(K) \supset f_{j+1}(K)$). Now we have proved that the directed graph of $S$ is strongly connected. With analogous discussion, we are able to prove that the graph of $S^*$ is also strongly connected. 

By Lemma 3.5, it follows that the associated subshift of finite type generated by the matrix $S$, i.e. $\Sigma$, has a unique measure of maximal entropy (Parry measure [24]), denoted by $\mu$. On the other hand, we may construct a natural map $\phi$ between $\Sigma$ and $K$ by

$$\phi(t_n) = x = \lim_{n \to \infty} f_{j_n} \circ \cdots \circ f_{j_1}(0),$$

where for each $[s_{i_n}, s_{i_n+1}]_K$, there is a unique $f_{j_n}$ (we choose the lazy algorithm) such that

$$[s_{i_n}, s_{i_n+1}]_K \subset f_{j_n}(K).$$
Lemma 3.6. \((\Sigma, \sigma, \mu)\) and \((K, L, \mu \circ \phi^{-1})\) are measure theoretically isomorphic, where \(\mu\) is the Parry measure, and \(\sigma\) is a left shift. 

Proof. We first prove that \(\phi\) is a bijection. Note that we chose the lazy orbit when we give a Markov partition of \(K\), therefore, \(\phi\) is well-defined and it is one-to-one. Moreover, it is also onto. Therefore, \(\phi\) is a bijection. It is easy to check that \(L \circ \phi = \phi \circ \sigma\), where \(L\) is the lazy map. Hence, we finish the proof. \(\square\)

Similarly, for the graph-directed self-similar set \(K^* = U_1\), we can obtain the following lemma.

Lemma 3.7. Let \(K \in \mathcal{A}\). Then \(K^* = U_1\) has, under the lazy map, a unique measure of maximal entropy.

Proof of Theorem 1.2. Theorem 1.2 follows from Lemmas 3.6 and 3.7. \(\square\)

3.2. Proof of Theorem 1.3. Now we define a set which offers another method that can calculate the Hausdorff dimension of \(U_1\). Before we introduce this approach, we recall some definitions. Let \(K \in \mathcal{A}\) with the IFS \(\{f_i\}_{i=1}^n\). For any \(k \geq 1\), let \((i_1 \cdots i_k) \in \{1, 2, \cdots, n\}^k\), we call \(f_{i_1} \cdots f_{i_k}(x)\) a \(k\)-level basic interval. If \(f_{i_1} \cdots f_{i_k}(x)\) is an exact overlap, then any univoque point of \(U_1\) cannot be in \(f_{i_1} \cdots f_{i_k}(K)\). Therefore, if we need to find all the univoque points, we should delete all possible exact overlaps in every \(k\)-th level. This simple observation leads to the following construction. Let \(k \in \mathbb{N} \geq 2\). Define

\[E_k = \{f_{j_1} \cdots f_{j_k}(x) : j_1 \cdots j_k \in \{1, \cdots, n\}^k, f_{j_1} \cdots f_{j_k}(x)\text{ is an exact overlap}\}.\]

Roughly speaking, \(E_k\) is a family of similitudes which are exact overlaps. Define

\[E_1 = (\bigcup_{i=1}^n f_i([0, 1])) \setminus H_1,\]

where

\[H_1 = \bigcup_{i_1, i_2 \in \{1, 2, \cdots, n\}^2} f_{i_1} \circ f_{i_2}((0, 1)).\]

Generally for \(k \geq 2\), we let

\[E_k = (E_{k-1} \cap (\bigcup_{i_1, i_2, \cdots, i_k} f_{i_1} \cdots f_{i_k}([0, 1]))) \setminus H_k,\]

where

\[H_k = \bigcup_{i_1, \cdots, i_k+1 \in \{1, 2, \cdots, n\}^{k+1}} f_{i_1} \cdots f_{i_k+1}((0, 1)).\]

Note that for any \(k \geq 1\), \(E_k\) is a union of some basic intervals with length \(\lambda^k\) and some basic intervals deleting some appropriate sets. These appropriate sets are associated with the exact overlaps (see the following example). We denote the cardinality of all the closed intervals in \(E_k\) by \(\sharp(E_k)\), \(k \geq 1\), i.e. if \(I\) is a basic interval in \(E_k\) with length \(k\), then we count this interval for one time. If \(I\) is a closed interval generated by a basic interval deleting some appropriate intervals (these intervals are associated with the exact overlaps, moreover, we delete at most two appropriate intervals), then we also count this closed interval for one time. We shall use the following example to illustrate the above definitions.
Let \( E = \left(0, 1\right) \) be the interval \((f, \lambda) = \left[0, 1\right] \cup [1-\lambda, 1]\) \( \lambda \) type if \( \lambda(x) \leq \lambda(x) \). Therefore, \( \mathcal{P}(E_1) = 3 \). Here \( [0, \lambda(1-\lambda)] \) is exactly the basic interval \( f_1([0, 1]) \) deleting the interval \( \lambda - \lambda^2, \lambda \) = \( f_1([0, 1]) \) (this interval is associated with the exact overlap \( f_1(x) = f_2(1) \)). The second interval is obtained in a similar way.

\[
E_2 = \left( E_1 \cap \left( \bigcup_{i \in \{1, 2, 3\}} f_3 \left( [0, 1] \right) \right) \right) \setminus \left( f_1 \left( [0, 1] \right) \cup f_2 \left( [0, 1] \right) \cup f_3 \left( [0, 1] \right) \right) \\
= \left( f_1([0, 1]) \setminus f_3([0, 1]) \right) \cup \left( f_2([0, 1]) \setminus f_3([0, 1]) \right) \\
\cup \left( f_3([0, 1]) \setminus f_3([0, 1]) \right) \\
\cup \left( f_3([0, 1]) \setminus f_3([0, 1]) \right) \\
\cup \left( f_3([0, 1]) \setminus f_3([0, 1]) \right).
\]

Therefore, \( \mathcal{P}(E_2) = 7 \). Generally, we can count the number of closed intervals of \( E_k, k \geq 3 \) in this way.

Lemma 3.3 can be expressed in the following way.

**Lemma 3.9.** Let \( K \in \mathcal{A} \). Then \( U_1 \setminus U_1 \) is at most a countable set. \( K^* = U_1 = \cap_{k=1}^{\infty} \). Moreover,

\[
\dim_H(U_1) = \dim_H(U_1) = \lim_{k \to \infty} \frac{\log \mathcal{P}(E_k)}{-k \log \lambda}.
\]

Lemma 3.3 gives an algorithm which may calculate the Hausdorff dimension of \( U_1 \). However, we cannot directly obtain Theorem 1.3 as for different IFS’s, the matrices may be distinct. Hence, we need to find a uniform approach which can calculate the Hausdorff dimension of \( U_1 \). Our idea of solving this problem is motivated by the Ngai and Wang’s finite type condition [23]. First, we give the following definition of types.

**Definition 3.10.** Given any \( k \geq 1 \), let \([a, b] \in E_k \). We call \([a, b] \) an independent type if \( \lambda^{-k}(b-a) = 1 \). We call \([a, b] \) a left type if there exists some \( (f_{i_1}, \ldots, f_{i_k}) \in \{f_1, \ldots, f_n\}^k \) such that \( \lambda^{-1} \circ \cdots \circ \lambda^{-1}([a, b]) = [0, 1-\lambda] \). We call \([a, b] \) a right type if there exists some \( (f_{i_1}, \ldots, f_{i_k}) \in \{f_1, \ldots, f_n\}^k \) such that \( \lambda^{-1} \circ \cdots \circ \lambda^{-1}([a, b]) = [\lambda, 1] \). We call \([a, b] \) a middle type if there exists some \( (f_{i_1}, \ldots, f_{i_k}) \in \{f_1, \ldots, f_n\}^k \) such that

\[
\lambda^{-1} \circ \cdots \circ \lambda^{-1}([a, b]) = [\lambda, 1-\lambda] .
\]

In Section 1, we define some numbers, namely \( k_i, k_l, k_r, k_m \). Now we use the above definition to give an explanation to these numbers.

**Example 3.11.** Let \( K \) be the attractor of the IFS

\[
\{ f_1(x) = \lambda x, f_2(x) = \lambda x + \lambda - \lambda^2, f_3(x) = \lambda x + 2\lambda - 2\lambda^2, f_4(x) = \lambda x + 3\lambda - 3\lambda^2, f_5(x) = \lambda x + 4\lambda, f_6(x) = \lambda x + 1 - 3\lambda + 2\lambda^2, f_7(x) = \lambda x + 1 - 2\lambda + 2\lambda^2, f_8(x) = \lambda x + 1 - \lambda \},
\]

where \( 0 < \lambda < \frac{1}{10} \).

The basic intervals in the first level for \( K \) is as follows.

From the figure, we know that \( f_5([0, 1]) \) is an independent type as \( f_5^{-1}(f_5([0, 1])) = [0, 1] \). Similarly,

\[
f_1([0, 1]) \setminus f_2([0, 1]) = [0, \lambda - \lambda^2], f_6([0, 1]) \setminus f_7([0, 1]) = [1 - 3\lambda + 2\lambda^2, 1 - 3\lambda + \lambda^2 + \lambda]
\]
are the left types.
\[ f_4([0, 1]) \setminus f_3([0, 1]) = [3\lambda - 2\lambda^2, 4\lambda - 3\lambda^2], f_5([0, 1]) \setminus f_7([0, 1]) = [1 - \lambda + \lambda^2, 1] \]
are the right types.
\[ f_6([0, 1]) \setminus (f_1([0, 1]) \cup f_3([0, 1])) = [\lambda, 2\lambda - 2\lambda^2], \]
\[ f_7([0, 1]) \setminus (f_2([0, 1]) \cup f_4([0, 1])) = [2\lambda - \lambda^2, 3\lambda - 3\lambda^2], \]
and
\[ f_7([0, 1]) \setminus (f_6([0, 1]) \cup f_8([0, 1])) = [1 - 2\lambda + 2\lambda^2, 1 - \lambda] \]
are the middle types.

Now, it is easy to see that \( k_i, k_l, k_r, k_m \) refer to the numbers of all possible independent, left, right and middle types in \( E_1 \), respectively. In terms of Definition 3.10, given \( K \in A \) with the IFS \( \{f_i\}_{i=1}^{n} \), suppose there are \( k_i \) independent types, \( k_l \) left types, \( k_r \) right types, \( k_m \) middle types in \( E_1 \). Clearly,
\[ n = k_l + k_r + k_i + k_m, k_r = k_l, n_k = k_m + k_i. \]

To prove Theorem 1.3, we need the idea of configuration which was the main tool of [16].

**Definition 3.12.** Suppose \((X, d)\) is a compact metric space. Let \(|A|\) be the diameter of \( A \subset X \), and \( \text{dist}(A, B) = \inf_{x \in A, y \in B} d(x, y) \). We say that \((X, d, \{D^k\}_k, \{\delta_k\}_k)\) (for simplicity we may replace \((X, d, \{D^k\}_k, \{\delta_k\}_k)\) by \(X\)) is a configuration set if there exists a constant \( c > 0 \) such that \( \{\delta_k\}_k \) is a decreasing sequence with \( \lim_{k \to \infty} \delta_k = 0 \) and \( \delta_{k+1} \geq c^{-1} \delta_k \) for all \( k \), \( D^k \) consists of finitely many compact subsets of \( X \) for any \( i \geq 0 \) with \( D^0 = \{X\} \), and for any \( A \in D^k \),
\[ c^{-1} \delta_k \leq |A| \leq c \delta_k, \]
and there exists some \( F(A) \subset D^{k+1} \) satisfying
\[ A = \cup_{B \in F(A)} B \text{ and } \text{dist}(B, B') \geq c^{-1} \delta_k \forall B \neq B' \in F(A). \]

**Definition 3.13.** Let \((X, d, \{D^k\}_k, \{\delta_k\}_k)\) be a configuration set. We say that \( X \) is a configuration set of finite pattern if the following conditions are satisfied:

1. \( \delta_k = \lambda_k \) for some \( \lambda \in (0, 1) \);
2. there is a surjective label mapping \( l : \cup_{k=0}^\infty D^k \to \{1, 2, \ldots, m\} \) and a transition matrix \( M = (a_{ij})_{m \times m} \) such that for any \( 1 \leq i, j \leq m \), any \( k \geq 0 \) and any \( A \in D^k \) with \( l(A) = i \),
\[ \sharp\{B \in F(A) : l(B) = j\} = a_{ij}. \]

The following result was proved in [16].

**Theorem 3.14.** Suppose that \( X \) is a configuration set of finite patterns. Let \( \rho \) be the spectral radius of \( M \). Then
\[ \dim_H(X) = \dim_B(X) = s = \frac{\log \rho}{-\log \lambda}, \]
and \( \mathcal{H}^s(X) > 0 \). Moreover, if the matrix \( M \) is irreducible, then
\[ 0 < \mathcal{H}^s(X) < \infty. \]
Lemma 3.15. Let $K \in A$, then $\overline{U_1}$ is a configuration set with finite patterns.

Proof. First, we prove that for any interval $A \subset E_k$, $c_1 \lambda^k \leq |A| \leq c_2 \lambda^k$. The right inequality is clear if we take $c_2 = 1$. For the left inequality, $|A| \geq \lambda^k - 2\lambda^{k+1}$. By the condition of $A$, it follows that $1 > 2\lambda$, therefore, there is some $c_0 > 0$ such that $1 > (2 + c_0)\lambda$. Thus, $|A| \geq \lambda^k - 2\lambda^{k+1} \geq c_0 \lambda^{k+1}$. Now we take $c_1 = c_0 \lambda$ and prove the left inequality. Next, we show that if $A = \bigcup_{B \in F(A)} B$, where $F(A) \subset E_{k+1}$, then

$$\text{dist}(B, B') \geq c_3 \lambda^k, \forall B \neq B' \in F(A),$$

for some $c_3 > 0$. By the construction of the univoque set and Lemma 3.9, we take $c_3 = \min\{\lambda^2, c_4\}$, where $c_4 = \min_{x \neq j}\{\text{dist}(D_i, D_j)\}$, and $\{D_i\}$ are the maximal connected components of $[0, 1]$ with respect to the IFS (see the definition in the first section). Hence, we have proved that $\overline{U_1}$ is a configuration set. Now, we prove that $\overline{U_1}$ is a configuration set with finite patterns. By the definition of $A$, there are only 4 types for the univoque sets. Given one type from $E_k, k \geq 1$, without loss of generality, we may assume that it is an independent type. In $E_{k+1}$ the independent type generates (under the IFS \{$f_i$\}) $k_i$ independent types, $k_l$ left types, $k_r$ right types, and $k_m$ middle types. For other three types, the discussion is similar, see Lemma 3.19, 3.20, 3.21.

The following lemma was proved by Akiyama [1], which is crucial for the proof of Theorems 1.7 and 1.3.

Lemma 3.16. Let $x_1, x_2, x_3$ be roots of the polynomial $g(x) = x^3 - ax^2 - bx - c \in \mathbb{Z}[x]$. If $g(1) < 0, g(-1) < 0$ and $g(|c|) < 0$, then

$$x_1 > 1, |x_i| < 1, i = 2, 3.$$

Lemma 3.17. Let

$$f(x) = x^n + a_nx^{n-1} + \cdots + a_0 \in \mathbb{Z}[x], a_0 \neq 0.$$

Suppose $\{x_i\}_{i=1}^{n-1}$ is the first $n-1$ roots of $f(x)$ satisfying $|x_i| < 1, 1 \leq i \leq n-1$, and the last real root of $f(x)$ is $\beta > 1$. Then $f(x)$ is irreducible over $\mathbb{Z}[x]$.

Proof. Let $\{x_i\}_{i=1}^{n-1}$ be the first $n-1$ roots lying inside the unit circle, and $\beta > 1$ be the last root of $f(x)$. If

$$f(x) = Q_1(x)Q_2(x) = \left(\prod_{i=1}^{n-1}(x - x_i)\right)(x - \beta),$$

where $Q_1(x), Q_2(x) \in \mathbb{Z}[x]$ are monic polynomials with $\deg(Q_1), \deg(Q_2) \geq 1$, without loss of generality, there exist $1 \leq i_1 < i_2 < \cdots < i_t \leq n - 1$ such that $Q_1(x) = \prod_{j=1}^{i_t}(x - x_{ij})$. Subsequently $|Q_1(0)| = |\prod_{j=1}^{i_t}x_{ij}| \in \mathbb{Z}$ which implies

$$|Q_1(0)| < 1$$

since $|x_{ij}| < 1$. On the other hand, we notice that $|Q_1(0)| \cdot |Q_2(0)| = f(0) = a_0 \neq 0$, hence

$$0 < |Q_1(0)| < 1$$

and $|Q_1(0)| \in \mathbb{Z}$.

Lemma 3.18. If $f$ and $g$ are two irreducible polynomials over $\mathbb{Z}[x]$, then $(f, g) = 1$ over $\mathbb{C}$. In other words, $f, g$ do not have a common root over the field $\mathbb{C}$.
Proof. Since $f$ and $g$ are two irreducible polynomials over $\mathbb{Z}[x]$, we have $(f, g) = 1$ over $\mathbb{Z}[x]$. We may implement the Euclidean algorithm for $f$ and $g$. There exist some $r_i(x) \in \mathbb{Q}[x], 0 \leq i \leq 2n + 5$ such that
\[
\begin{align*}
f(x) &= r_0(x)g(x) + r_1(x) \\
g(x) &= r_2(x)r_1(x) + r_3(x) \\
r_1(x) &= r_4(x)r_3(x) + r_5(x) \\
r_3(x) &= r_6(x)r_5(x) + r_7(x) \\
&\vdots \\
r_{2n+1}(x) &= r_{2n+4}(x)r_{2n+3}(x) + r_{2n+5}(x),
\end{align*}
\]
where
\[
\partial(g(x)) > \partial(r_1(x)) > \partial(r_2(x)) > \cdots > \partial(r_{2n+5}(x)) = 0,
\]
$\partial(h(x))$ denotes the degree of $h(x)$. Note that $r_{2n+5}(x)$ is a constant. Since $(f, g) = 1$, it follows that $r_{2n+5}(x) \neq 0$. If $f$ and $g$ have a common root over $\mathbb{C}$, denoted by $x_0$, then by the above equations, we have $r_{2n+5}(x_0) = 0$, leading to a contradiction. \hfill \Box

Lemma 3.19. Let $K \in A$ be the self-similar set with the IFS $\{ f_i \}_{i=1}^n$. If
\[
f_n([0, 1]) \cap f_{n-1}([0, 1]) = \emptyset, f_1([0, 1]) \cap f_2([0, 1]) \neq \emptyset,
\]
or
\[
f_n([0, 1]) \cap f_{n-1}([0, 1]) \neq \emptyset, f_1([0, 1]) \cap f_2([0, 1]) = \emptyset,
\]
then
\[
\dim_H(U_1) = \frac{\log \gamma_1}{-\log \lambda},
\]
where $\gamma_1$ is a Pisot number of the following equation:
\[
x^3 - nx^2 + (n - k_i + k_m)x - (k_m + k_l) = 0.
\]

Proof. By Lemma 3.15, $U_1$ is a configuration set with finite patterns. Therefore, we only need to find the transition matrix $M$. We suppose that
\[
f_n([0, 1]) \cap f_{n-1}([0, 1]) = \emptyset, f_1([0, 1]) \cap f_2([0, 1]) \neq \emptyset.
\]
For the other case, the proof is similar. We consider the relation between 4 different types. First, we consider the offspring of independent type. Clearly, an independent type can generate $k_i$ independent types, $k_l$ left types, $k_r$ right types, and $k_m$ middle types. A left type can generate $k_i - 1$ independent types, $k_l$ left types, $k_r$ right types, and $k_m$ middle types. A right type can generate $k_i$ independent types, $k_l - 1$ left types, $k_r$ right types, and $k_m$ middle types. Finally, a middle type can generate $k_i - 1$ independent types, $k_l - 1$ left types, $k_r$ right types, and $k_m$ middle types. We may use the following matrix to describe this process.
\[
M = \begin{pmatrix}
k_i & k_l & k_r & k_m \\
k_i - 1 & k_l & k_r & k_m \\
k_i & k_l - 1 & k_r & k_m \\
k_i - 1 & k_l - 1 & k_r & k_m
\end{pmatrix}.
\]
Then we can calculate the Hausdorff dimension of $U_1$ in terms of the spectral radius of the above matrix.
\[
\dim_H(U_1) = \frac{\log \gamma_1}{-\log \lambda},
\]
where $\gamma_1$ is largest real root of the following equation:
\[ x^3 - nx^2 + (n - k_i + k_{m})x - (k_{m} + k_i) = 0. \]

Note that $n_K = k_m + k_l, n = k_m + k_l + k_i + k_r$. Therefore, the above equation is exactly
\[ x^3 - nx^2 + 2n_Kx - n_K = 0. \]

In the remaining of the proof, we prove that $\gamma_1$ is a Pisot number. Let
\[ f(x) = x^3 - nx^2 + 2n_Kx - n_K. \]

Note that $f(n) = (2n - 1)n_K > 0$ and
\[ f(n - 2) = -2(n - 2)^2 - n_K + 2n_K(n - 2). \]

Since $n - 2 \geq n_K$, we have
\[ 2(n - 2)^2 \geq 2n_K(n - 2). \]

Therefore, $f(n - 2) < 0$. Hence, by the intermediate value theorem, we have
\[ \gamma_1 \in (n - 2, n). \]

Now we claim that $\gamma_1 \neq n - 1$. If not, we have $f(n - 1) = 0$, i.e.
\[ 2n_K(n - 1) = n_K + (n - 1)^2. \]

Therefore,
\[ n - 1 \mid n_K + (n - 1)^2. \]

In other words, there exist some $t \in N$ such that
\[ (n - 1)t = (n - 1)^2 + n_K. \]

Note that $t \geq n$. Otherwise, $(n - 1)^2 < (n - 1)^2 + n_K = (n - 1)t \leq (n - 1)^2$, leading to a contradiction. Equivalently, we have
\[ (n - 1)(t - n + 1) = n_K. \]

However, this is also impossible as
\[ (n - 1)(t - n + 1) \geq (n - 1) > n_K. \]

It is easy to check by the fact $n - 2 \geq n_K$ that
\[ f(1) < 0, f(-1) < 0, f(n_K) < 0. \]

Therefore, by Lemma 3.16 and 3.17, we have that $f(x)$ is an irreducible polynomial. Now, we prove that $f(x)$ is the minimal polynomial of $\gamma_1$. If the degree of $\gamma_1$ is strictly smaller than 3, then by the conclusion obtained above, we have that $\gamma_1$ is not an integer, therefore the degree of the minimal polynomial of $\gamma_1$ must be 2. We let this polynomial be
\[ g(x) = ax^2 + bx + c. \]

Namely, $g(\gamma_1) = 0$. Using the Euclidean algorithm again, we may find some
\[ p(x), r(x), r_1(x), r_2(x) \in \mathbb{Q}[x] \]

such that
\[
\begin{align*}
  f(x) &= p(x)g(x) + r(x) \\
  g(x) &= r_2(x)r(x) + r_2(x),
\end{align*}
\]

where $\partial(r(x)) = 1, \partial(r_2(x)) = 0$, i.e. $r_2(x)$ is a constant. Since $f$ is irreducible over $\mathbb{Z}[x]$, we have $(f, g) = 1$. Thus, $(f, g) = 1$ over $\mathbb{C}[x]$. However, $f(x)$ and $g(x)$ have
a common root $\gamma_1$. Therefore, we conclude that $f(x)$ is the minimal polynomial of $\gamma_1$. Now, we have proved that $\gamma_1$ is a Pisot number. 

Lemma 3.20. Let $K \in \mathcal{A}$ be the self-similar set with the IFS $\{f_i\}_{i=1}^n$. If 
\[ f_1([0,1]) \cap f_2([0,1]) \neq \emptyset, f_n([0,1]) \cap f_{n-1}([0,1]) \neq \emptyset, \]
then 
\[ \dim_H(U_1) = \frac{\log \gamma_2}{-\log \lambda}, \]
where $\gamma_2$ is a Pisot number of the following equation: 
\[ x^2 - (n-1)x + (k_m - k_i) = 0; \]

Proof. We consider the relation between 4 different types. The proof is similar as Lemma 3.19. The matrix $M$ is 
\[
\begin{pmatrix}
    k_i & k_l & k_r & k_m \\
    k_i & k_l & k_r - 1 & k_m \\
    k_i & k_l - 1 & k_r & k_m \\
    k_i & k_l - 1 & k_r - 1 & k_m
\end{pmatrix}
\]
Then we can calculate the Hausdorff dimension of $U_1$ in terms of the spectral radius of the above matrix. Note that 0 and 1 are also the roots of the characteristic polynomial of $M$. The Pisot property is trivial as $n > k_m + 2$. 

Lemma 3.21. Let $K \in \mathcal{A}$ be the self-similar set with the IFS $\{f_i\}_{i=1}^n$. If 
\[ f_1([0,1]) \cap f_2([0,1]) = \emptyset \]
and $f_{n-1}([0,1]) \cap f_n([0,1]) = \emptyset$, then 
\[ \dim_H(U_1) = \frac{\log \gamma_3}{-\log \lambda}, \]
where $\gamma_3$ is the appropriate root of the following equation: 
\[ x^2 - nx + (n - k_i + k_m) = 0. \]
Moreover, $\gamma_3$ is a Pisot number if and only if $k_m + 1 < k_i$. 

Proof. The matrix $M$ is 
\[
\begin{pmatrix}
    k_l & k_l & k_r & k_m \\
    k_i - 1 & k_l & k_r & k_m \\
    k_i - 1 & k_l & k_r & k_m \\
    k_i - 2 & k_l & k_r & k_m
\end{pmatrix}
\]
Then we can calculate the Hausdorff dimension of $U_1$ in terms of the spectral radius of the above matrix. The remaining proof is trivial. 

Proof of Theorem 1.3. Theorem 1.3 follows from Lemmas 3.19, 3.20, 3.21.
3.3. Proof of Theorem 1.7. The following results were proved in [16].

Lemma 3.22. Let \( K \in \mathcal{A} \) with the IFS’s \( \{ f_i \}_{i=1}^n \). Then
\[
\dim_H(K) = \frac{\log \gamma}{-\log \lambda},
\]
where \( \gamma \) is a Pisot number of the following polynomial
\[
x^2 - nx + nK = 0
\]

Lemma 3.23. If \( K_1, K_2 \in \mathcal{A} \), then the following four conditions are equivalent:
1. \( \dim_H K_1 = \dim_H K_2 \);
2. \( K_1 \) and \( K_2 \) are quasi-Lipschitz equivalent;
3. \( K_1 \) and \( K_2 \) are Lipschitz equivalent;
4. \( n_{K_1} = n_{K_2} \).

Now, we prove some lemmas.

Lemma 3.24. Let \( p, q \in \mathbb{N}^+ \) be two integers. If \( p \geq q \) and \( \sqrt{p} - \sqrt{q} = r \in \mathbb{N} \), then \( \sqrt{p}, \sqrt{q} \in \mathbb{N} \).

Proof. Note that \( p = r^2 + q + 2r\sqrt{q}, q = r^2 + p - 2r\sqrt{p}, \) then
\[
\sqrt{q} = \frac{b - q - r^2}{2r}, \sqrt{p} = \frac{b - q + r^2}{2r} \in \mathbb{Q}.
\]
We let \( \sqrt{q} = \frac{a}{b} \in \mathbb{Q}, a, b \in \mathbb{N} \) and \( (a, b) = 1 \), then \( a^2 = qb^2 \). Clearly, \( b \mid qb^2 \). Therefore, \( b \mid a^2 \). By the assumption \( (a, b) = 1 \), we have \( b = 1 \). With a similar discussion, we may show that \( \sqrt{p} \in \mathbb{N} \).

Lemma 3.25. Let \( n, l \in \mathbb{N}, n \geq 4 \) and \( l \leq n - 3 \). If \( \sqrt{n^2 - 8l} = n - t \) for some \( t \in \mathbb{N} \), then \( t = 2 \).

Proof. Note that \( t^2 = 2(nt - 4l) \). Therefore, \( t \) is an even number. Since \( l \leq n - 3 \), then
\[
\sqrt{n^2 - 8l} \geq \sqrt{(n - 4)^2 + 8} \geq 2\sqrt{2}.
\]
Therefore, \( t \leq n - 3 \). If \( t = 4 \) then \( 8l = 8n - 16 \) and \( l = n - 2 \) which contradicts to \( l \leq n - 3 \). Assume that \( t = 2k \) for some \( k \geq 3 \). Then we have
\[
8l = 2nt - t^2 = 4nk - 4k^2.
\]
Therefore, \( 2l = nk - k^2 \leq 2n - 6 \), i.e.
\[
n(k - 2) \leq k^2 - 6 \quad (4)
\]
On the other hand, we also have \( n \geq t + 3 = 2k + 3 \), then \( n(k - 2) \geq 2k^2 - k - 6 > k^2 \) for any \( k \geq 3 \). This contradicts to the inequality (4).

Lemma 3.26. Given \( i = 1 \) or 2. Let \( K_1 \) and \( K_2 \) be two self-similar sets from \( \mathcal{A}_i \) with the IFS’s \( \{ f_i \}_{i=1}^n \) and \( \{ g_i \}_{i=1}^m \), respectively. If \( \dim_H(U_1^{(i)}) = \dim_H(U_2^{(i)}) \), then \( n = m, \dim_H(K_1) = \dim_H(K_2), n_{K_1} = n_{K_2} \).

Proof. Firstly, we suppose that \( K_1, K_2 \in \mathcal{A}_1 \). By Lemmas 3.19, 3.20 and 3.21, it follows that
\[
\dim_H(U_1^{(i)}) = \frac{\log \gamma_i}{-\log \lambda}, \dim_H(U_2^{(2)}) = \frac{\log \gamma_j}{-\log \lambda}.
\]
for some $1 \leq i, j \leq 2$. If $\dim_H(U^{(1)}_i) = \dim_H(U^{(2)}_j)$, then $\gamma_i = \gamma_j$, where $\gamma_i = \gamma_j$ is a Pisot number. By Lemmas 3.19, 3.20 and 3.21, $\gamma_i$ is the root of
\[ x^3 - nx^2 + (n - k^{(1)}_i + k^{(1)}_m)x - (k^{(1)}_m + k^{(1)}_i) = 0, \]
or
\[ x^2 - (n - 1)x - k^{(1)}_i + k^{(1)}_m = 0. \]
Similarly, $\gamma_j$ is the root of
\[ x^3 - mx^2 + (m - k^{(2)}_i + k^{(2)}_m)x - (k^{(2)}_m + k^{(2)}_i) = 0, \]
or
\[ x^2 - (m - 1)x - k^{(2)}_i + k^{(2)}_m = 0. \]
Hence, in terms of Lemmas 3.17 and 3.18 and the relation $n_K = k_m + k_l$, it follows that $n = m, n_{K_1} = n_{K_2}$. Therefore, using Lemma 3.23, we conclude $\dim_H(K_1) = \dim_H(K_2)$.

Secondly, suppose that $K_1, K_2 \in \mathcal{A}_2$. Then $\dim_H(K_1) = \dim_H(K_2)$ implies that $\gamma_i = \gamma_j$ is a Pisot number or an algebraic number. If $\gamma_i = \gamma_j$ is a Pisot number, then the proof is similar as the previous cases. If $\gamma_i = \gamma_j$ is an algebraic number, then
\[ \gamma_i = \frac{n + \sqrt{n^2 - 8nK_1}}{2} = \gamma_j = \frac{m + \sqrt{m^2 - 8nK_2}}{2}, \]
which implies by Lemmas 3.24 and 3.25 that
\[ n = m, n_{K_1} = n_{K_2}, \text{ and } \dim_H(K_1) = \dim_H(K_2). \]

\[ \square \]

**Proof of Corollary 1.4.** Proof of Corollary 1.4 follows from Lemma 3.26 and Theorem 1.3.

The following lemma is indeed Corollary 1.6.

**Lemma 3.27.** Given $i = 1$ or $2$. Let $K_1$ and $K_2$ be two self-similar sets from $\mathcal{A}_i$ with the IFS’s $\{f_i\}_{i=1}^n$ and $\{g_i\}_{i=1}^m$, respectively. Then $\dim_H(K_1) = \dim_H(K_2)$ if and only if $\dim_H(U^{(1)}_i) = \dim_H(U^{(2)}_j)$.

**Proof.** By Theorems 1.5, 1.3 and Corollary 1.4, it follows that the Hausdorff dimension of $K$ and $U_i$ are uniquely determined by $n$ and $n_K = k_m + k_l$.

We finish this subsection by giving a proof of Theorem 1.7.

**Proof of Theorem 1.7.** By Lemmas 3.26, 3.23, Corollaries 1.4 and 1.6, the first 5 conditions are equivalent. First, we prove that if $K_1$ and $K_2$ are measure theoretically isomorphic with respect to the unique measures of maximal entropy, then $\dim_H(K_1) = \dim_H(K_2)$. Since $K_1, K_2 \in \mathcal{A}_i, i = 1$ or $2$, we may give the Markov partitions for $K_1$ and $K_2$, respectively. The associated matrices of the Markov partitions are irreducible, and therefore we may find the unique measures of maximal entropy for the subshifts of finite type generated by the irreducible matrices, respectively. By Lemma 3.6, $K_1, K_2$ have unique measures of maximal entropy. By the assumption, i.e. $K_1$ and $K_2$ are measure theoretically isomorphic with respect to the unique measures of maximal entropy, it follows that the corresponding subshifts of finite type of the matrices are measure theoretically isomorphic. Therefore, the spectral radii of these two matrices are equal. By Theorem 1.5, it follows that $\dim_H(K_1) = \dim_H(K_2)$. Similar statement is
still correct for $\overline{U_1^{(1)}}$ and $\overline{U_1^{(2)}}$, that is, if $\overline{U_1^{(1)}}$ and $\overline{U_1^{(2)}}$ are measure theoretically isomorphic with respect to the unique measures of maximal entropy, then $\dim_H(U_1^{(1)}) = \dim_H(U_1^{(2)}) = \dim_H(U_1^{(1)}) = \dim_H(U_1^{(2)})$. Therefore, in terms of Corollary 1.4 and the discussion above, we prove one direction.

Conversely, if $\dim_H(U_1^{(1)}) = \dim_H(U_1^{(2)})$ or $\dim_H(K_1) = \dim_H(K_2)$, and $k_i^{(1)} = k_i^{(2)}$, by Corollary 1.4, it follows that $k_i^{(1)} = k_i^{(2)}$, $k_i^{(1)} = k_i^{(2)}$, and $k_i^{(1)} = k_i^{(2)}$. Thus, the graph-directed self-similar sets for $\overline{U_1^{(1)}}$ and $\overline{U_1^{(2)}}$, i.e. $K_1^*$ and $K_2^*$, have the same associated matrices $S_1^* = S_2^*$. Hence, by Lemma 3.7, $\overline{U_1^{(1)}}$ and $\overline{U_1^{(2)}}$ are measure theoretically isomorphic with respect to the unique measures of maximal entropy. Analogous proof is still correct for $K$.

3.4. Proof of Theorem 1.8. It is natural to give a checkable condition under which $U_1$ is closed. By virtue of some discussion from the symbolic space, we have the following result.

Theorem 3.28. Let $K \in \mathcal{A}$. Then $U_1$ is closed if and only if $f_1([0,1]) \cap f_2([0,1]) = \emptyset$ and $f_{n-1}([0,1]) \cap f_n([0,1]) = \emptyset$.

We first prove that the following lemma.

Lemma 3.29. Let $K \in \mathcal{A}$. Suppose $f_1(I) \cap f_2(I) \neq \emptyset$, $f_{n-1}(I) \cap f_n(I) = \emptyset$, $I = [0,1]$. Then $U_1$ is not closed.

Proof. Since $f_1(I) \cap f_2(I) \neq \emptyset$, by the fourth condition of $\mathcal{A}$, $f_{1n} = f_{21}$. We prove this lemma via some cases.

Case 1. If $f_2([0,1]) \cap f_i([0,1]) = \emptyset$, $i \geq 3$, then let $x_k = \pi(2^k n^\infty)$. Note that

$$\lim_{k \to \infty} x_k = \pi(2^\infty) = f_1(1) \in f_1(K) \cap f_2(K) = f_{1n}(K) = f_{21}(K).$$

Therefore, it suffices to prove that $x_k$ is a univoque point. However, since $x_k$ is the right endpoint of $f_{2i}([0,1])$ and $f_{2i}([0,1]) \cap f_i([0,1]) = \emptyset$, $i \geq 3$, it follows that $x_k$ is a univoque point.

Case 2. If $f_2([0,1]) \cap f_3([0,1]) \neq \emptyset$, then we suppose without loss of generality that there are some $3 \leq p \leq n - 1$ such that

$$f_i([0,1]) \cap f_{i+1}([0,1]) \neq \emptyset, 1 \leq i \leq p - 1.$$ 

For this case, we let $x_k = \pi(2^k n^\infty)$. Clearly, $x_k$ is a univoque point but

$$\lim_{k \to \infty} x_k = \pi(2^\infty) \in f_1(K) \cap f_2(K)$$

is not a univoque point. \hfill \Box

The following two cases can be proved in a similar way.

Lemma 3.30. Let $K \in \mathcal{A}$. Suppose $f_1(I) \cap f_2(I) = \emptyset$, $f_{n-1}(I) \cap f_n(I) = \emptyset$. Then $U_1$ is not closed.

Lemma 3.31. Let $K \in \mathcal{A}$. Suppose $f_1(I) \cap f_2(I) \neq \emptyset$, $f_{n-1}(I) \cap f_n(I) \neq \emptyset$. Then $U_1$ is not closed.

For the final case we have the following result.

Lemma 3.32. Let $K \in \mathcal{A}$. Suppose $f_1(I) \cap f_2(I) = \emptyset$, $f_{n-1}(I) \cap f_n(I) = \emptyset$. Then $U_1$ is closed.
Definition 3.34. Let another decomposition of $U$ for some $1 \leq i \leq n$ be hit the switch region for the first time.

Lemma 3.33. Suppose $f_1([0,1]) \cap f_2([0,1]) = \emptyset$ and $f_n([0,1]) \cap f_{n-1}([0,1]) = \emptyset$. Given $K_1, K_2 \in \mathcal{A}$, if $n_{K_1} = n_{K_2}$, then $U_1^{(1)} \simeq U_1^{(2)}$.

The proof of this result also needs the tool, i.e. configuration set. We give another decomposition of $U$.

Definition 3.34. Let $K \in \mathcal{A}$ with the IFS $\{f_i\}_{i=1}^n$. Suppose $f_1([0,1]) \cap f_2([0,1]) = \emptyset$ and $f_n([0,1]) \cap f_{n-1}([0,1]) = \emptyset$. Let $\mathcal{L} = f_1(K)$, $\mathcal{R} = f_n(K)$, and $\mathcal{M} = U_{i=2}^{n-1} f_i(K)$. We call $\mathcal{L}$ the left part of $K$, $\mathcal{M}$ the middle part of $K$, and $\mathcal{R}$ the right part of $K$.

In the $m$-th level, if a basic interval $J = f_{i_1 \cdots i_m}([0,1])$ does not intersect with other $m$-th basic intervals, then we call $J \cap K = f_{i_1 \cdots i_m}(K)$ the I pattern. Suppose that $J_1, J_2, \ldots, J_p$, arranged from left to right with $2 \leq p \leq n$, are $m$-th basic intervals such that $J_i \cap J_j = \emptyset$ whenever $|i - j| \geq 2$ and

$$\frac{|J_t \cap J_{t+1}|}{|J_t|} = \lambda, 1 \leq t \leq p - 1.$$ 

Assume that $u_1, \ldots, u_p$ are words such that $J_i = f_{u_i}([0,1])$ for $i = 1, \ldots, p$. 

Proof. It suffices to prove that $U^c_1$ is open. For any $x \in U^c_1 \cap K$, there is some $(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k$ such that

$$T_{i_1 \cdots i_k}(x) \in f_i(K) \cap f_{i+1}(K) \neq \emptyset$$

for some $1 \leq i \leq n$. We may assume that $k$ is smallest, i.e. the orbit of $x$ hit the switch region for the first time.

Case 1. If $T_{i_1 \cdots i_k}(x) \in f_i(K) \cap f_{i+1}(K) \setminus \{f_i(1), f_{i+1}(0)\}$, then by the continuity of $T_{ij}, 1 \leq j \leq k$, it follows that there are some $\delta_1 > 0$ such that for any $y \in (x - \delta_1, x + \delta_1)$

$$T_{i_1 \cdots i_k}(y) \in f_i(K) \cap f_{i+1}(K) \setminus \{f_i(1), f_{i+1}(0)\}$$

or $y \not\in K$ (which implies that $y \in K^c \subset U^c_1$). Therefore,

$$y \in f_{i_1 \cdots i_k} \cap (f_i(K) \setminus f_{i+1}(0)) \subset U^c_1$$

or $y \in U^c_1$.

Subsequently, we have

$$(x - \delta_1, x + \delta_1) \subset U^c_1.$$ 

Case 2. $T_{i_1 \cdots i_k}(x) \in \{f_i(1), f_{i+1}(0)\}$. We may assume that $T_{i_1 \cdots i_k}(x) = f_{i+1}(0)$ (another case is similar). By the assumption $f_1(I) \cap f_2(I) = \emptyset$, $f_{n-1}(I) \cap f_n(I) = \emptyset$, there are some $\delta_2 > 0$ such that $(T_{i_1 \cdots i_k}(x) - \delta_2, T_{i_1 \cdots i_k}(x) + \delta_2) \subset K^c \subset U^c_1$ and

$$(T_{i_1 \cdots i_k}(x), T_{i_1 \cdots i_k}(x) + \delta_2) \cap K \subset f_i(K) \cap f_{i+1}(K).$$

Therefore, by the continuity of $T_{ij}, 1 \leq j \leq k$, it follows that there exists some $\delta_3 > 0$ such that

$$(x, x + \delta_3) \subset f_{i_1 \cdots i_k} \cap (f_i(K) \cap f_{i+1}(K)) \subset U^c_1$$

and

$$(x - \delta_3, x) \subset K^c \subset U^c_1.$$ 

Now we prove Theorem 1.8. First, we prove the following lemma.

Lemma 3.33. Suppose $f_1([0,1]) \cap f_2([0,1]) = \emptyset$ and $f_n([0,1]) \cap f_{n-1}([0,1]) = \emptyset$. Given $K_1, K_2 \in \mathcal{A}$, if $n_{K_1} = n_{K_2}$, then $U_1^{(1)} \simeq U_1^{(2)}$.
Denote
\[ L_u = f_u(L), M_u = f_u(M), R_u = f_u(R) \text{ and } K_u = f_u(K). \]
It is easy to give the following decomposition
\[ \cup_{i=1}^{p} f_u(K) \setminus \{ \cup_{i=1}^{p-1} (f_u(K) \cap f_u(K)) \} = (L_u \cup M_u \cup R_u) \cup (\cup_{i=2}^{p} M_u) \]
We call \( L_u \cup M_u \cup R_u \) the I pattern and each \( M_u \), \( 2 \leq i \leq p \) the II pattern. With this decomposition, we have the following lemma. The proof is similar as Lemma 3.15.

**Lemma 3.35.** Let \( K \in A \) with the IFS \( \{ f_i \}_{i=1}^{n} \). Suppose \( f_1([0,1]) \cap f_2([0,1]) = \emptyset \) and \( f_n([0,1]) \cap f_{n-1}([0,1]) = \emptyset \). Then \( U_1 \) is a configuration set with two patterns, i.e. I pattern and II pattern. Moreover, the matrix \( M \) is
\[
\begin{pmatrix}
    n - n_K & n_K \\
    n - n_K - 2 & n_K
\end{pmatrix}.
\]

We prove Lemma 3.33 in terms of the configuration set with finite patterns. First, we introduce a proposition which is useful to find the bi-Lipschitz map.

**Proposition 3.36.** Let \( (X, d_x, \{ D^k \}_k, \{ \delta_k \}_k), (Y, d_y, \{ G^k \}_k, \{ \delta_k \}_k) \) be two configuration sets. If there exists a bijection
\[ \eta : \bigcup_{k=0}^{\infty} D^k \to \bigcup_{k=0}^{\infty} G^k \]
such that for all \( k \), \( A \in D^k \) if and only if \( \eta(A) \in G^k \), and that \( B \subset A \) with \( A \in D^k \) and \( B \in D^{k+1} \) implies that \( \eta(B) \subset \eta(A) \), then
\[ X \simeq Y. \]

**Proof of Proposition 3.36.** For any \( x \in X \), there exists a unique sequence \( A^1, A^2, A^3, \ldots \), with \( A^k \in D^k \) such that
\[ \{ x \} = \cap_{i=1}^{\infty} A^i. \]
Then a mapping \( \bar{\eta} : X \to Y \) is defined as
\[ \{ \bar{\eta}(x) \} = \cap_{i=1}^{\infty} \eta(A^i). \]

Now we shall show that the mapping is a bi-Lipschitz mapping.

For any \( x \neq x' \) there exists some \( k \) such that \( x, x' \in A \) with \( A \in D^k \), but \( x \in B, x' \in B' \) for distinct \( B, B' \in D^{k+1} \). It follows from the definition of configuration set that
\[ c^{-1} \delta_k \leq \text{dist}(B, B') \leq d_x(x, x') \leq |A| \leq c \delta_k. \]
In the same way, we obtain that
\[ c^{-1} \delta_k \leq d_y(\bar{\eta}(x), \bar{\eta}(x')) \leq c \delta_k, \]
which implies
\[ c^{-2} d_x(x, x') \leq d_y(\bar{\eta}(x), \bar{\eta}(x')) \leq c^2 d_x(x, x'), \]
as required. \( \Box \)

The following corollary is a consequence of Proposition 3.36.

**Corollary 3.37.** Let \( X_1 \) and \( X_2 \) be two configuration sets of finite patterns. If their transition matrices coincide and \( l_1(X_1) = l_2(X_2) \) for the corresponding label mappings \( l_1 \) and \( l_2 \), then
\[ X_1 \simeq X_2. \]
Proof of Lemma 3.33. Lemma 3.33 follows immediately from Corollary 3.37, Proposition 3.36 and Lemma 3.35.

Proof of Theorem 1.8. It suffices to prove that \( \dim_H(U_1^{(1)}) = \dim_H(U_1^{(2)}) \) implies \( U_1^{(1)} \) and \( U_1^{(2)} \) are bi-Lipschitz equivalent. First, by Lemma 3.26, it follows that
\[
\dim_H(U_1^{(1)}) = \dim_H(U_1^{(2)})
\]
yields \( n_{K_1} = n_{K_2} \). Therefore, \( U_1^{(1)} \simeq U_1^{(2)} \) follows from Lemmas 3.33 and 3.35.

4. One example. In this section, we give one example which illustrates that generally \( \dim_H(U_1^{(1)}) = \dim_H(U_1^{(2)}) \) cannot imply that \( \dim_H(K_1) = \dim_H(K_2) \). Moreover, \( \dim_H(U_1^{(1)}) = \dim_H(U_1^{(2)}) \) does not yield \( U_1^{(1)} \) and \( U_1^{(2)} \) are bi-Lipschitz equivalent.

Example 4.1. Let \( K_1 \) be the attractor of the IFS
\[
\{f_1(x) = \lambda x, f_2(x) = \lambda x + \lambda - \lambda^2, f_3(x) = \lambda x + 2\lambda - 2\lambda^2, f_4(x) = \lambda x + 3\lambda - 3\lambda^2, f_5(x) = \lambda x + 4\lambda, f_6(x) = \lambda x + 1 - 3\lambda + 2\lambda^2, f_7(x) = \lambda x + 1 - 2\lambda + \lambda^2, f_8(x) = \lambda x + 1 - \lambda \},
\]
where \( 0 < \lambda < \frac{1}{10} \).

Let \( K_2 \) be the attractor of the IFS
\[
\{f_1(x) = \lambda x, f_2(x) = \lambda x + 2\lambda, f_3(x) = \lambda x + 4\lambda, f_4(x) = \lambda x + 6\lambda, f_5(x) = \lambda x + 7\lambda - \lambda^2, f_6(x) = \lambda x + 8\lambda, f_7(x) = \lambda x + 1 - \lambda \},
\]
where \( 0 < \lambda < \frac{1}{10} \). Then by Theorem 1.3, \( \dim_H(U_1^{(1)}) = \dim_H(U_1^{(2)}) = \frac{\log \gamma}{-\log \lambda} \), where \( \gamma \) is the Pisot number of the polynomial \( x^2 - 7x + 2 = 0 \).

The basic intervals in the first level for \( K_1 \) and \( K_2 \) are as follows.

\[
\begin{array}{cccccccc}
\hline
\hline
& & & & & & & \\
\hline
\end{array}
\]

\textbf{Figure 3.} First iteration of \( K_1 \)

\[
\begin{array}{cccccccc}
& & & & & & & \\
\hline
\hline
\end{array}
\]

\textbf{Figure 4.} First iteration of \( K_2 \)

For \( K_1 \),
\[
n = 8, n_{K_1} = 5, k^{(1)}_i = 1, k^{(1)}_m = 3, k^{(1)}_l = k^{(1)}_r = 2.
\]

For \( K_2 \),
\[
m = 7, n_{K_2} = 1, k^{(2)}_i = 5, k^{(2)}_m = 0, k^{(2)}_l = k^{(2)}_r = 1.
\]

Since \( n \neq m, n_{K_1} \neq n_{K_2} \), by Theorem 1.5 it follows that \( \dim_H(K_1) \neq \dim_H(K_2) \). By Theorem 3.28, \( U_1^{(2)} \) is closed but \( U_1^{(1)} \) is not. Therefore, \( U_1^{(1)} \) and \( U_1^{(2)} \) are not bi-Lipschitz equivalent.
5. **Final remarks.** In this paper, we only consider the case that $|f_i([0, 1]) \cap f_{i+1}([0, 1])| = \lambda^2$ whenever 

$$f_i([0, 1]) \cap f_{i+1}([0, 1]) \neq \emptyset.$$ 

In fact, we may investigate the following case: $|f_i([0, 1]) \cap f_{i+1}([0, 1])| = \lambda^k$ whenever $f_i([0, 1]) \cap f_{i+1}([0, 1]) \neq \emptyset$, where $k \geq 2$ is a given integer. Then we can obtain the Hausdorff dimension of $U_1$. For instance, when $f_n([0, 1]) \cap f_i([0, 1]) = \emptyset$ for any $i \neq n$ or $f_1([0, 1]) \cap f_i([0, 1]) = \emptyset$ for any $i \neq 1$, then 

$$\dim_H(U_1) = \frac{\log \eta}{-\log \lambda},$$

where $\eta$ is the appropriate root of the following equation:

$$x^{2k+1} - nx^{2k} + 2(k_m + k_l)x^k - (k_m + k_l) = 0.$$ 

However, for any $n \geq 3$ and $k \geq 2$, we cannot prove that the algebraic number $\eta$ is also a Pisot number (Partial results can be obtained if the coefficients of the above polynomial satisfying certain conditions, for instance, we may use the Rouché theorem if $n > 3(k_m + k_l) + 1$). As we mentioned in introduction, the Pisot property (Corollary 3.18) is essential for the classification results (Theorem 1.7, Corollary 1.9).

**Acknowledgments.** The work is supported by National Natural Science Foundation of China (Nos. 11371329, 11471124, 11701302, 11671147) and Philosophical and Social Science Planning of Zhejiang Province (No. 17NDJC108YB). The work is also supported by K.C. Wong Magna Fund in Ningbo University. Kan Jiang was also supported by Zhejiang Provincial Natural Science Foundation of China with No.LY20A010009. The authors are grateful to the referees for many useful comments and suggestions.

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Received April 2018; 1st revision May 2020; final revision May 2020.

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