Entropy and a generalisation of “Poincare’s Observation”

Oliver Johnson*

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Abstract

Consider a sphere of radius $\sqrt{n}$ in $n$ dimensions, and consider $X$, a random variable uniformly distributed on its surface. Poincaré’s Observation states that for large $n$, the distribution of the first $k$ coordinates of $X$ is close in total variation distance to the standard normal $N(0, I_k)$. In this paper, we consider a larger family of manifolds, and $X$ taking a more general distribution on the surfaces. We establish a bound in the stronger Kullback–Leibler sense of relative entropy, and discuss its sharpness, providing a necessary condition for convergence in this sense. We show how our results imply the equivalence of ensembles for a wider class of test functions than is standard. We also deduce results of de Finetti type, concerning a generalisation of the idea of orthogonal invariance.

1 Notation and Definitions

Diaconis and Freedman [6] consider $X$, a random variable uniformly distributed on the surface of a sphere of radius $\sqrt{n}$ in $n$ dimensions. They show that the distribution of $\pi_{n,k}X$, the first $k$ coordinates of $X$, is close in total variation distance to a Gaussian for large $n$. They indicate a natural connection between this problem and the equivalence of ensembles for the free Hamiltonian, where a sphere corresponds to a surface of constant kinetic energy. In this paper, we consider a more general family of manifolds defined by symmetric, additive Hamiltonians. Unlike Diaconis and Freedman we shall not assume a uniform distribution on these surfaces. We prove convergence in the stronger Kullback–Leibler sense of relative entropy distance.

*Statistical Laboratory, CMS, Wilberforce Road, Cambridge, CB3 0WB, UK. Email: otj1000@cam.ac.uk. Fax: +44 1223 337956

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First, we introduce notation. Given measurable \( f : \mathbb{R} \to \mathbb{R} \cup \{ \infty \} \) we define \( F \) to be the set on which \( f \) is finite, and let \( R_n = R_n(f) \) be given by \( R_n(x) = \sum_{i=1}^{n} f(x_i) \). We define the surface \( S_n(t) = \{ x : R_n(x) = nt \} \subseteq F^n \). Given \( f \) and \( c > 0 \), define \( g_{n,c} \) for the density of a Gibbs distribution: \( g_{n,c}(x) = \exp(-cR_n(x))/Z^n_c \). We assume that \( f \) has the property that \( Z_c = \int \exp(-cf(x))dx \) is finite for all \( c > 0 \). Define the projection \( \pi_{n,k} : \mathbb{R}^n \to \mathbb{R}^k \) restricting to the first \( k \) coordinates: \( \pi_{n,k}(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_k) \). For any probability density \( q \) on \( \mathbb{R}^n \) write the energy \( \varepsilon(q) = \int q(x)R_n(x)dx \) and the entropy \( h(q) = \int -q(x) \log q(x)dx \).

For a given \( f \), we know that Gibbs densities maximise the entropy for a given energy. (This characterisation of Gibbs measures via a variational principle is discussed in Chapter 15 of Georgii [7]). This is done by considering the Kullback–Leibler distance \( D(p||g_{n,c}) \), where \( D(f||g) = \int f(x) \log(f(x)/g(x))dx \), for \( f \) and \( g \) probability densities on \( \mathbb{R}^k \).

**Example 1.1** Two cases in particular are significant here:

1. If \( f(x) = x^2 \), then \( Z_c = \sqrt{\pi/c} \), and \( g_{n,c}(x) = \prod_{i=1}^{n} g_{1,c}(x_i) \), where \( g_{1,c} \) is a \( N(0, (2c)^{-1}) \) density, the density which maximises entropy subject to a variance constraint.

2. If \( f(x) = x \) for \( x \geq 0 \) and \( f(x) = \infty \) for \( x < 0 \), then \( Z_c = 1/c \), and \( g_{n,c}(x) = \prod_{i=1}^{n} g_{1,c}(x_i) \), where \( g_{1,c} \) is an \( \text{Exp}(c) \) density, the density which maximises entropy on the positive half-line subject to a mean constraint.

Now we can state Diaconis and Freedman’s results on the normal and exponential case in the form:

**Theorem 1.2** For \( f \) as in Example 1.1.1 or 1.1.2 and for a given \( t \in (0, \infty) \), take \( c \) such that \( \varepsilon(g_{1,c}) = t \) and let \( X \) be distributed uniformly on the surface \( S_n(t) \subseteq F^n \). Writing \( p_{n,k,t}(y) \) for the density of \( \pi_{n,k}X \):

For \( f(x) = x^2 \), as in Example 1.1.1, \( d_{TV}(p_{n,k,t}, g_{k,c}) \leq 2(k+3)/(n-k-3). \)

For \( f(x) = x \), as in Example 1.1.2, \( d_{TV}(p_{n,k,t}, g_{k,c}) \leq 2(k+1)/(n-k-1). \)

The following theorem is the main result of this paper. It involves a class \( \mathcal{F} \) of functions (we postpone the precise definition to Definition 2.2 but roughly speaking we want \( f \) to be strictly increasing and well-behaved at zero). Let \( h_{n,t} \) be the density corresponding to uniform distribution on \( S_n(t) \).

**Theorem 1.3** Assuming \( f \in \mathcal{F} \), for any \( t \in (0, \infty) \) take \( c \) such that \( \varepsilon(g_{1,c}) = t \) and consider \( X \), a random variable with density \( p \) on \( S_n(t) \). Writing \( p_{n,k,t}(y) \) for the density of \( \pi_{n,k}X \), for some constant \( C = C(f) \):

\[
D(p_{n,k,t}||g_{k,c}) \leq D(p||h_{n,t}) + \log \left( \frac{n}{n-k} \right) + \frac{2}{\sqrt{n/C - 1}}.
\]
A major motivation for this work, beyond the intrinsic interest of generalising Diaconis and Freedman’s work \[6\], comes from the question of so-called ‘equivalence of ensembles’. Given a Hamiltonian \( H(x) \), for \( x \in \mathbb{R}^n \) we can consider the microcanonical ensemble (uniform measure with density \( h_{n,t} \) on the surface \( \{x : H(x) = nt\} \)) and grand canonical ensemble (Gibbs measure with density \( g_{n,c} \) proportional to \( \exp(-cH(x)) \) for \( x \in \mathbb{R}^n \)). The principle of equivalence of ensembles suggests that in some sense \( h_{n,t} \) and \( g_{n,c} \) are close together, that is

\[
\int p(x)h_{n,t}(dx) \simeq \int p(x)g_{n,c}(x)
\]

for some class of test functions \( p \). Convergence in total variation distance (as established by Diaconis and Freedman) implies that Equation (1) holds for \( p \) bounded and continuous and depending only on \( k = o(n) \) coordinates. Theorem 1.3 implies this for a wider class of test functions, including the Hamiltonian itself.

**Corollary 1.4** Assuming \( f \in \mathcal{F} \), for any \( t \in (0, \infty) \)

\[
\lim_{n \to \infty} \left( \int p(x)h_{n,t}(dx) - \int p(x)g_{n,c}(x) \right) = 0,
\]

for \( p \) depending only on \( k \) coordinates, if \( p(x_1, \ldots, x_k) \) is bounded above by a multiple of \( 1 + \sum_{i=1}^k f(x_i) \). Here, \( k \) need not be fixed, and can grow as \( o(n) \).

**Proof** This is a consequence of Theorem 1.3 in conjunction with Lemma 3.1 of Csiszár \[3\]. The latter states that \( D(u_n \parallel v) \to 0 \) implies \( \int p(x)u_n(x)dx \to \int p(x)v(x)dx \), for any \( p \) such that \( \int \exp(tp(x))v(x)dx < \infty \) for small \( |t| \). This integral is seen to be finite by definition of the partition function.

A second application is to results of de Finetti type, as described in \[6\]. For example, define an infinite sequence \( X_1, X_2, \ldots \) of random variables to be orthogonally invariant if for any \( n \), the law of \( (X_1, \ldots, X_n) \) is invariant under orthogonal transformations of \( \mathbb{R}^n \). Schoenberg \[12\] states that all orthogonally invariant distributions are mixtures of normals. Diaconis and Freedman \[6\] prove this by showing that the first \( k \) of \( n \) orthogonally invariant variables are within \( 2k/n \) of a mixture of normals, so a passage to the limit provides the infinite result. In a similar way we can consider \( f \)-invariant measures; that is, sequences of random variables such that for any \( n \), the law of \( (X_1, \ldots, X_n) \) is invariant under continuous transformations of \( \mathbb{R}^n \) which preserve \( R_n(x_1, \ldots, x_n) \). We show:

**Corollary 1.5** For \( f \in \mathcal{F} \), the only \( f \)-invariant measures \( P \) are mixtures of Gibbs measures. That is, if we write \( G_{\infty,c} \) for the distribution of \( Y_1, Y_2, \ldots \), where \( Y_i \) are independent with density \( g_{1,c} \), there exists a measure \( \lambda \), valued on \( c > 0 \), such that

\[
P = \int G_{\infty,c} \lambda(c) dc.
\]
Proof First we consider finite subsequences of $X_1, \ldots$. If measure $P_n$ is invariant under $R_n$-preserving transformations then (as in [6]), $P_n$ is constant on each manifold $S_n(t)$, so:

$$P_n = \int h_{n,t} \mu_n(t) \, dt,$$

for some measure probability measure $\mu_n$. Now projecting down to the first $k$ coordinates:

$$Q_k = \pi_{n,k} P_n = \int (\pi_{n,k} h_{n,t}) \mu_n(t) \, dt.$$

Writing $c(t)$ for the unique $c$ such that $\varepsilon(g_{1,c}) = t$, and defining $R_k = \int g_{k,c(t)} \mu_n(t) \, dt$, then:

$$\|R_k - Q_k\|_{TV} \leq \int \mu_n(t) \| (\pi_{n,k} h_{n,t}) - g_{k,c(t)} \| \, dt \leq \sqrt{\frac{2k}{n-k}},$$

by Theorem 1.3.

Duplicating Diaconis and Freedman’s tightness argument we can show that as $n \to \infty$, $\mu_n$ must have a convergent subsequence, with limit $\mu$. Mapping from $t$ to $c(t)$, we find a $\lambda$ with the required properties.

The fact that projecting a uniform distribution on a sphere approximately gives a Gaussian is often referred to as Poincaré’s Observation (see for example [10]). However, in Section 6 of their paper, Diaconis and Freedman suggest that this attribution is wrong, and that the earliest reference to it in the probability literature comes in the work of Borel. Nonetheless, it appears that the observation is even older than this, and can be traced back to Mehler [11].

Csiszár [4] uses entropy-theoretic methods to consider the distribution of a random variable $X_1$, conditional on the vector $X = (X_1, \ldots, X_n) \in A$ for some set $A$. However, these results rely on $A$ being a ‘thick set’ of positive measure, whereas in the present paper we consider the so-called ‘thin shell’ case. Dembo and Zeitouni [3] extended Csiszár’s results to the distribution of the first $k$ coordinates, where $k = k(n)$ can vary with $n$, and (as in this paper) discovered that a sufficient condition for convergence is that $k(n) = o(n)$. Schroeder [13] generalised this result to Markov processes, and Comets and Zeitouni [2] even to mean–field perturbations of Markov processes. However, all these papers use the assumption that $A$ has non-empty interior. This corresponds to a weak limit theorem, bounding the probability of a particular set, whereas our methods require a local limit theorem, bounding densities.

2 Class of Functions Considered

It is natural to ask for the widest possible class of functions $f$ such that results such as Theorem 1.2 hold. For technical reasons, we will need to control $w_n$, the density of $R_n(X)$, when $X$ has Gibbs density $g_{n,c}$. Specifically we need an upper
bound on \(\log(w_{n-k}(s)/w_n(t))\) for certain \(s, t\). Theorem 1.2 holds because \(w_n\) is known exactly (in Example 1.1.1 \(R_n(X)\) is the sum of squares of normals, with the \(\chi^2_n\) distribution, and in Example 1.1.2 \(R_n(X)\) is the sum of exponentials, with the \(\Gamma(n, c)\) distribution). A later paper by Borovkov [1] extends Diaconis and Freedman’s work to the case \(f(x) = |x|^p\), again using exact calculations of the density. Whilst his bounds are tighter, the method will not extend to the general case – it gives no information about Hamiltonians of the form \(f(x) = x^2 + \epsilon x^4\).

However, we can make progress in other cases too. The key observation is that \(X_i\) are IID (with marginal density \(g_{1,c}\)), so \(R_n(X) = \sum_{i=1}^n f(X_i)\) is a sum of IID random variables. Using a local version of the Central Limit Theorem, we will be able to show that the densities are sufficiently close in supremum norm to a Gaussian density for our result to go through. We obtain a proposition reminiscent of Equation (2.9) of [6].

**Proposition 2.1** Consider \(X\) with density \(g_{n,c}\) and let \(Y_j = f(X_j)\), with \(EY_j = \mu, \text{Var}(Y_j) = \sigma^2\), and characteristic function \(\phi(u) = E\exp(iuY_j)\). Assume that \(I = \int |\phi(u)|^r du\) is finite for some \(r \geq 1\) and \(m = E|Y - \mu|^3\) is finite. If \(w_k\) is the density of \(R_k(X) = \sum_{i=1}^k Y_i\) then there exists a constant \(C = C(Y)\) such that for \(n - k \geq r\):}

\[
\log \left( \frac{w_{n-k}(z)}{w_n(n\mu)} \right) \leq \log \left( \frac{n}{n-k} \right) + \frac{2}{\sqrt{n/C}-1} \quad \text{for all } z \in \mathbb{R}.
\]

**Proof** The local limit theorem tells us that there exists a constant \(C(Y)\) such that if \(f_n\) is the density of \(\sum(Y_i - \mu)/\sqrt{n\sigma^2}\) then for \(n \geq r\):

\[
\sup_x \left| f_n(x) - \frac{1}{\sqrt{2\pi}} \exp(-x^2/2) \right| \leq \frac{C(Y)}{\sqrt{2\pi n}}.
\]

A careful reading of (for example) Section 46 of Gnedenko and Kolmogorov [8] shows that the dependence of \(C(Y)\) on \(Y\) comes through \(m, \sigma^2, I\) and \(\nu = \sup_{t > \sigma^2/m} |\phi(t)| < 1\).

Rescaling this, we know that for any \(z\):

\[
w_{n-k}(z) \leq \frac{\sqrt{n-k} + C}{(n-k)\sqrt{2\pi \sigma^2}} \quad \text{and} \quad w_n(n\mu) \geq \frac{\sqrt{n} - C}{n\sqrt{2\pi \sigma^2}}.
\]

Taking the ratio of these terms we deduce the result, since:

\[
\log \left( \frac{\sqrt{n-k} + C}{\sqrt{n} - C} \right) \leq \log \left( 1 + \frac{2C}{\sqrt{n-C}} \right).
\]

\(\Box\)

We can now describe the class of surfaces that our Theorem will cover. These conditions on function \(f\) are chosen so that they imply that the local limit theorem
will hold in Proposition 2.1. We confirm this in Lemmas 2.3 and 2.4 below. Roughly speaking, we require the function $f$ to grow faster than a linear function on most of the domain, and to be well-behaved at 0.

**Definition 2.2** Define $\mathcal{F}$ to be the class of functions $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ with:

1. $f(0) = 0$, $f(x)$ right continuous at 0.

2. $f(x)$ is differentiable for $x > 0$ with $f’(x) > 0$ and:
   
   (a) There exist $a_1, a_2$ such that $f’(x) \geq a_1 > 0$ for $x \in (a_2, \infty)$.
   
   (b) There exist $1 < q < 2$ and $a_3 > 0$ such that:
   $$\liminf_{x \searrow 0} \frac{f’(x)^q}{f(x)} \geq a_3.$$

3. Either (a) $f(x) \equiv \infty$ for $x < 0$ or (b) $f(x) = f(-x)$ for all $x$. As before, we write $F$ for the interval on which $f$ is finite.

Notice that for any $p \geq 1$, picking $f(x) = x^p$ on $x \geq 0$ and infinity elsewhere, or $f(x) = |x|^p$ everywhere ensure that $f \in \mathcal{F}$. Thus the cases considered by Diaconis and Freedman [6] and Borovkov [1] are included in our theorems.

Conditions 1 and 2 ensure that $Z_c$ is finite and non-zero for $c \in (0, \infty)$, since then

$$\int \exp(-cf(x))\,dx \leq a_2 + \int_{a_2}^\infty f’(x) \exp(-cf(x))/a_1 \leq a_2 + \exp(-cf(a_2))/ca_1,$$

and continuity provides boundedness away from zero.

**Lemma 2.3** Assume $f \in \mathcal{F}$. If $X$ has density $g_{1,c}(x) = \exp(-cf(x))/Z_c$ and $Y = f(X)$, then $m = \mathbb{E}|Y - \mathbb{E}Y|^3$ is finite.

**Proof** In case (a) and (b) of Definition 2.2 $f(X)$ will have the same density, hence we need only consider case (a). Further, note that in this case, since $Y \geq 0$, $m = \mathbb{E}(Y - \mu)^3$, the 3rd cumulant. Since the moment generating function of $Y$ is $M(t) = Z_{c-t}/Z_c$, the cumulant generating function $\log M(t) = \log Z_{c-t} - \log Z_c$, and the result follows.

**Lemma 2.4** If $f \in \mathcal{F}$, then if $X$ has density $g_{1,c}(x) = \exp(-cf(x))/Z_c$ and $Y = f(X)$, with characteristic function $\phi$, then there exists $r$ such that $I = \int |\phi(u)|^r\,du$ is finite.

**Proof** The density $g(y)$ of $Y$ is $Z_c^{-1}\exp(-cy)/f’(f^{-1}(y))$. As described in for example Theorem 74 of [14], if $g \in L^p$, for some $1 < p \leq 2$ then $I$ is finite for $r \geq p/(p - 1)$. By Condition (a) of Definition 2.2, for $y$ close to zero $g(y) \leq \text{const.} \exp(-cy)y^{1/q}$, so picking $p > q$ ensures the finiteness of the integral at 0. Condition 2a gives us control as $y \to \infty$.  

\[ \square \]
3 Nesting and Projection of Surfaces

Recall that we want to project a density \( p \) from a manifold \( S_n(t) \) onto its first \( k \) coordinates. Given \( p \), we will first create \((\theta p)\), a density on \( F^n \), by placing weighted copies of \( p \) on each manifold \( S_n(u) \). One has a free choice of how to weight the individual manifolds. The picture is that of fitting together an infinite set of ‘Russian dolls’, each of different sizes, but each with the same pattern of densities on their surface. We then consider the projection of \( p \) by considering the projection of \((\theta p)\) conditional on being on \( S_n(t) \).

By a suitable choice of weighting, we can make the projection well-behaved. For example, in the case of the uniform density on the sphere, we weight concentric spheres by the \( \chi^2 \) distribution to produce the normal density on \( \mathbb{R}^n \).

We need to develop a coordinate system that allows us to describe a point \( \mathbf{x} \) in space by giving its distance from the origin \( \mathbf{0} \) and the point where the line from \( \mathbf{0} \) to \( \mathbf{x} \) crosses \( S_n(t) \). In the case of the sphere \( f(x) = x^2 \), this corresponds to transforming between rectangular and polar coordinates. We shall require one further technical lemma, not proved here:

**Lemma 3.1** For any \( t \in (0, \infty) \), the equation \( \varepsilon(g_{1,c}) = t \) has a unique solution as an equation in \( c \), if \( f(0) = 0 \), \( f \) is right continuous at zero, increasing on \( x > 0 \), \( f(x) \to \infty \) as \( x \to \infty \), and \( f(x) = \infty \) for \( x < 0 \).

More formally, we give a \( C^1 \)-foliation of \( F^n \setminus \{0\} \) by compact \((n-1)\)-dimensional manifolds \( S_n(u), u > 0 \). Each manifold is endowed with a standard Riemannian metric and diffeomorphic, by a central projection, to either a unit sphere centered at the origin or to its intersection with a non-negative orthant.

**Proposition 3.2** For given \( t \), under Conditions 1 and 2 of Definition 2.2, there exists a bijection \( \Phi_t \) between \( \{x : x > 0\} \times S_n(t) \) and \( F^n \setminus \{0\} \). The bijection \( \Phi_t \) has a Jacobian \( A_{n,t} \) which is positive everywhere.

**Proof** For given \( u \) and for any \( \mathbf{x} \in F^n \setminus \{0\} \), under Conditions 1 and 2 of Definition 2.2, the equation \( R_n(k\mathbf{x}) = u \) has a unique solution in \( k > 0 \), by the Intermediate Value Theorem. Hence given \( (r,s) \), where \( r > 0 \) and \( s \in S_n(t) \), we can find a unique \( \mathbf{x} = ks \), where \( k \) is chosen such that \( R_n(ks) = nr \). Conversely, given \( \mathbf{x} \), we can define the central projection \( S_{n,t}(\mathbf{x}) = k\mathbf{x} \), where \( k \) is chosen such that \( R_n(k\mathbf{x}) = nt \), which is equivalent to saying that \( k\mathbf{x} \in S_n(t) \). We take the pair \( (r,s) = (R_n(\mathbf{x}), S_{n,t}(\mathbf{x})) \).

This foliation induces a coordinate system on \( F^n \setminus \{0\} \) via the bijection \( \Phi_t \). Let \( \sigma_{n,t} \) denote the induced Riemannian volume on \( S_n(t) \). The Jacobian \( A_{n,t}(\mathbf{x}) \) is determined since for any measurable function \( f \):

\[
\int_{\mathbb{R}_+ \times S_n(t)} f(t,s)\Phi_t(dt,ds) = \int_{\mathbb{R}^n} f(R_n(\mathbf{x}), S_{n,t}(\mathbf{x}))d\mathbf{x}
\]

\[
= \int_{\mathbb{R}_+ \times S_n(t)} f(t,s)A_{n,t}(t,s)dt\sigma_{n,t}(ds).
\]
The positivity of $f'$ ensures the positivity of $A_{n,t}$, since the local structure of the above foliation (and the induced map $\Phi_t$) may be also described as follows. Given a point $x$, such that $\Phi_t(x) = (R_n(x), S_n(x))$, for small $\delta x$, we calculate $\Phi_t(x + \delta x)$. Now $R_n(x + \delta x) = R_n(x) + \sum_i \delta x_i f'(x_i) + O(\delta x^2)$, which is not identically equal to $R_n(x)$. Similarly, if $S_n(x) = kx \in S_n(t)$, then $S_n(x + \delta x)$ is $k(1 - \epsilon)(x + \delta x)$, where $\epsilon$ is chosen such that this lies on $S_n(t)$. The choice of $\epsilon$ that achieves this is $\epsilon = \left(\sum f'(x_i)\delta x_i\right)/\left(\sum f'(x_i)\right)$, which again ensures that $S_n(x + \delta x)$ is not identically equal to $S_n(x)$. 

Now having developed our coordinate system, we can describe the map $\theta$ which takes a density on $S_n(t)$ and gives a density on $F^n$. The motivation for this definition is that it gives an isometry between densities (see Lemma 3.4).

**Definition 3.3** Given a probability density $p$ on $S_n(t)$, we can define the product density $(w_n \times p)$ on $\mathbb{R}^+ \times S_n(t)$, where $w_n$ is the density of $R_n(X)$ when $X$ has density $g_{n,c}$. We can thus define the density $(\theta p)$ induced by $\Phi_t$ on $F^n \setminus \{0\}$, since transforming to Cartesian coordinates, we know that:

$$
(\theta p)(x) = \frac{w_n(R_n(x))p(S_{n,t}(x))}{A_{n,t}(R_n(x), S_{n,t}(x))}, \text{ for } x \neq 0,
$$

which is by construction a probability density on $F^n \setminus \{0\}$.

The one significant difference, as mentioned by Borovkov, is that we will no longer consider the uniform distribution on the surface but rather consider $h_{n,t}(v) = A_{n,t}(nt, v)/\left(\int A_{n,t}(nt, s)\sigma_{n,t}(ds)\right)$ for $v \in S_n(t)$.

**Lemma 3.4** With the definitions above: $D(\theta p\|g_{n,c}) = D(p\|h_{n,t})$.

**Proof** Note that $w_n$ is characterized by:

$$
\int_0^r w_n(t)dt = \int g_{n,c}(x)\mathbb{I}(R_n(x) \leq r)dx = \int \exp(-ct)Z_c^{-n}\mathbb{I}(t \leq r) \left(\int A_{n,t}(nt, s)\sigma_{n,t}(ds)\right) dt.
$$

We deduce that for any $u \in F^n$, $g_{n,c}(u) = w_n(R_n(u))/\left(\int A_{n,t}(nt, s)\sigma_{n,t}(ds)\right)$. Hence by definition, $(\theta h_{n,t})(u) = w_n(R_n(u))/\left(\int A_{n,t}(nt, s)\sigma_{n,t}(ds)\right) = g_{n,c}(u)$. This means that

$$
D(\theta p\|g_{n,c}) = D(\theta p\|\theta h_{n,t}) = \int \theta p(x) \log \left(\frac{p(S_{n,t}(x))}{h_{n,t}(S_{n,t}(x))}\right) dx
$$

$$
= \int p(s) \log \left(\frac{p(s)}{h_{n,t}(s)}\right) ds
$$

as required.
Definition 3.5 Given \( t > 0 \) and a density \( p \) on \( S_n(t) \), write \( p_{n,k,t}(y) \) for the density of \( p \) projected by \( \pi_{n,k} \) onto \( \mathbb{R}^k \).

The key observation is that \( p_{n,k,t}(y) = q_{n,k}(y, nt)/w_n(nt) \), which is the projection of \((\theta p)\) on \( \mathbb{R}^k \) conditioned on \( R_n = nt \). Here \( q_{n,k}(y, nt) \) may be treated as the joint density of \( \pi_{n,k}X = y \) and \( R_n(X) = nt \), where \( X \) is a random point of \( F^n \) with density \( \theta p \). We thus establish the principal result of this paper:

**Proof of Theorem 1.3** We can write:

\[
D(p_{n,k,t}(y, nt) \log \left( \frac{q_{n,k}(y, nt)}{w_n(nt)} \right) dy \leq D(p\|h_n).
\]

**Proof** Given integrable functions \( g(x), h(x) \), normalising to get probability densities \( p(x) = g(x)/\int g(x)dx \), \( q(x) = h(x)/\int h(x)dx \), the Gibbs inequality gives:

\[
0 \leq D(p\|q) \int g(x)dx = \int g(x) \log \left( \frac{g(x)}{h(x)} \right) dx - \left( \int g(x)dx \right) \log \left( \frac{\int g(x)dx}{\int h(x)dx} \right).
\]

Now, writing \( x = (y, u) \), where \( y \in \mathbb{R}^k \), \( u \in \mathbb{R}^{n-k} \), notice that:

\[
q_{n,k}(y, nt) = \int (\theta p)(y, u)\mathbb{I}(R_k(y) + R_{n-k}(u) = nt)du
\]

\[
g_{k,c}(y)w_{n-k}(nt - R_k(y)) = \int (\theta h_{n,t})(y, u)\mathbb{I}(R_k(y) + R_{n-k}(u) = nt)du
\]

Hence we deduce that for each \( y \):

\[
\frac{q_{n,k}(y, nt)}{w_n(nt)} \log \left( \frac{q_{n,k}(y, nt)}{g_{k,c}(y)w_{n-k}(nt - R_k(y))} \right) \leq \int \frac{(\theta p)(y, u)}{w_n(nt)} \log \left( \frac{(\theta p)(y, u)}{(\theta h_{n,t})(y, u)} \right) \mathbb{I}(R_k(y) + R_{n-k}(u) = nt)du
\]

\[
= \int p(S_n(y, u)) \log \left( \frac{p(S_n(y, u))}{h_{n,t}(S_n(y, u))} \right) \mathbb{I}(R_k(y) + R_{n-k}(u) = nt)du
\]
Integrating with respect to \( y \), we obtain:

\[
\int \frac{q_{n,k}(y,nt)}{w_n(nt)} \log \left( \frac{q_{n,k}(y,nt)}{g_{k,c}(y)w_{n-k}(nt - R_k(y))} \right) dy
= \int p(S_n(y,u)) \log \left( \frac{p(S_n(y,u))}{h_{n,t}(S_n(y,u))} \right) \mathbb{1}(R_k(y) + R_{n-k}(u) = nt) du dy
= \int p(S_n(x)) \log \left( \frac{p(S_n(x))}{h_{n,t}(S_n(x))} \right) \mathbb{1}(R_n(x) = nt) dx
= D(p\|h_n),
\]

as required. \( \square \)

**Corollary 3.7** Assume \( f \in \mathcal{F} \) and taking \( t = \varepsilon(g_{1,c}) \), given the density \( p = h_{n,t} \) on \( S_n(t) \), then the projection \( p_{n,k,t} \) satisfies:

\[
D(p_{n,k,t}\|g_{k,c}) \leq \log \left( \frac{n}{n-k} \right) + \frac{2}{\sqrt{n}/C - 1}.
\]

Observe that \( d_{TV}(f,g) = \int |f(x) - g(x)| dx \leq \sqrt{2D(f\|g)} \) (see [9]), and hence in this case then the rate of convergence in total variation distance is \( O(1/\sqrt{n}) \), as opposed to the \( O(1/n) \) which Diaconis and Freedman establish. This difference can be attributed to the fact that we approximate the densities \( w_n \), rather than being able to obtain exact bounds on them.

### 4 A converse

Given the density \( p = h_{n,t} \) on \( S_n(t) \) we can see from Corollary 3.7 that as \( n \to \infty \), and \( k/n \to 0 \) then \( D(p_{n,k,t}\|g_{k,c}) \to 0 \). This is also Diaconis and Freedman’s necessary and sufficient condition for convergence in total variation distance in the spherical case. We show that this condition holds for more surfaces than that:

**Proposition 4.1** Given \( f \in \mathcal{F} \), consider the uniform probability density \( h_{n,t} \) on \( S_n(t) \), and consider \( p_{n,k,t} \), the distribution of its first \( k \) coordinates. If \( p_{n,k,t} \to g_{k,c} \) in total variation distance then \( k/n \to 0 \). Hence the stronger result of \( D(p_{n,k,t}\|g_{k,c}) \to 0 \) implies that \( k/n \to 0 \).

**Proof** Now considering \( Y \) with density \( g_{n,c} \), by conditioning and independence, the density \( r_k(y) \) of \( (R_k(\pi_{n,k}Y)|R_n(Y) = nt) \) satisfies:

\[
r_k(y) = \frac{w_k(y)w_{n-k}(nt - y)}{w_n(nt)}.
\]
Hence since total variation distance is reduced by projection, we deduce that for any set $L$:

$$d_{TV}(p_{n,k,t}, g_{k,c}) \geq d_{TV}(r_k, w_k) \geq 2 \int_L w_k(y) \left( \frac{w_{n-k}(nt-y)}{w_n(nt)} - 1 \right) dy.$$ 

Now, using the estimates of the previous section, we know that choosing the interval $L = (kt - \epsilon \sqrt{n-k}, kt + \epsilon \sqrt{n-k})$ will ensure that the term in brackets is close to $\sqrt{n/n-k-1}$. Furthermore for $n \gg k$, $w_k(L)$ will be close to 1, so we deduce the result.

\[\square\]

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