COEFFICIENT ESTIMATES OF CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH HOHLOV OPERATOR

P. GOCHHAYAT, A. PRAJAPATI, AND A. K. SAHOO

Abstract. A typical quandary in geometric functions theory is to study a functional composed of amalgamations of the coefficients of the pristine function. Conventionally, there is a parameter over which the extremal value of the functional is needed. The present paper deals with consequential functional of this type. By making use of linear operator due to Hohlov [12], a new subclass $R_{a,b}^c$ of analytic functions defined in the open unit disk is introduced. For both real and complex parameter, the sharp bounds for the Fekete-Szegö problems are found. An attempt has also been taken to found the sharp upper bound to the second and third Hankel determinant for functions belonging to this class. All the extremal functions are express in term of Gauss hypergeometric function and convolution. Finally, the sufficient condition for functions to be in $R_{a,b}^c$ is derived. Relevant connections of the new results with well known ones are pointed out.

1. Introduction and Preliminaries

Let $A$ be the class of functions analytic in the open unit disk

$$U := \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \},$$

normalized by the condition $f(0) = 0$, $f'(0) = 1$ and has the Taylor-Maclaurin series of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in U).$$

Let $S$ be the subclass of $A$ consisting of univalent functions. Suppose that $f$ and $g$ are in $A$. We say that $f$ is subordinate to $g$, (or $g$ is superordinate to $f$), write as

$$f \prec g \text{ in } U \text{ or } f(z) \prec g(z) \quad (z \in U),$$

2010 Mathematics Subject Classification. Primary: 30C45; Secondary: 30C50.

Key words and phrases. Univalent function, Hohlov operator, Coefficient estimates, Fekete-Szegö problem, Hankel determinant, sufficient condition.

The present investigation of the second author is supported under the INSPIRE fellowship, Department of Science and Technology, New Delhi, Government of India, Sanction Letter No. REL1/2016/2/2015-16.
if there exists a function \( \omega \in \mathcal{A} \), satisfying the conditions of the Schwarz lemma (i.e. \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \)) such that
\[
    f(z) = g(\omega(z)) \quad (z \in \mathcal{U}).
\]

It follows that
\[
    f(z) \prec g(z) \quad (z \in \mathcal{U}) \implies f(0) = g(0) \quad \text{and} \quad f(\mathcal{U}) \subset g(\mathcal{U}).
\]

In particular, if \( g \) is univalent in \( \mathcal{U} \), then the reverse implication also holds (cf. [28]). Denote \( \mathcal{P} \), the class of functions \( \phi \) which is analytic in \( \mathcal{U} \) and is of the form
\[
    \phi(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \mathcal{U}),
\]
with \( \phi(0) = 1 \) and \( \Re(\phi(z)) > 0 \).

If \( f \) and \( g \) are functions in \( \mathcal{A} \) and given by the power series
\[
    f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (z \in \mathcal{U}),
\]
then the Hadamard product (or Convolution) of \( f \) and \( g \) denoted by \( f \ast g \), is defined by
\[
    (f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g \ast f)(z) \quad (z \in \mathcal{U}).
\]

Note that \( f \ast g \in \mathcal{A} \).

For the complex parameters \( a, b \) and \( c \) with \( c \neq 0, -1, -2, -3, \cdots \), the Gauss hypergeometric function denoted by \( _2F_1(a, b; c; z) \) and is defined by
\[
    _2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!} \quad (z \in \mathcal{U}),
\]
where \( (\alpha)_n \) denotes the Pochhammer symbol (or shifted factorial) given in terms of the Gamma function \( \Gamma \), by
\[
    (\alpha)_n = \frac{\Gamma(\alpha + n)}{\alpha} = \begin{cases} 1; & \text{if } n = 0, \\ \frac{\alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + n - 1)}{n!}; & \text{if } n \neq 0. \end{cases}
\]

In terms of Gauss hypergeometric function and convolution, Hohlov (cf. [12], [13]) introduced and studied a linear operator denoted by \( \mathcal{I}_{a,b}^c \) and defined by \( \mathcal{I}_{a,b}^c f : \mathcal{A} \to \mathcal{A} \), as
\[
    \mathcal{I}_{a,b}^c f(z) := z_2 F_1(a, b; c; z) \ast f(z) \quad (z \in \mathcal{U}).
\]
COEFFICIENT ESTIMATES OF CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH... 3

We note that upon suitable choice of the parameters, the above defined three-parameter family of operator unifies various other linear operators which are introduced and studied earlier. For example

(1) $I_{a,1}^c := \mathcal{L}(a, c)$, the well known Carlson-Shaffer operator (cf. [6]).
(2) $I_{1+1}^\lambda := \mathcal{D}^\lambda (\lambda > -1)$, is the Ruscheweyh derivative operator of order $\lambda$ (cf. [44]).
(3) $I_{1+\eta}^{2+} := I_B^\eta$, the well known Bernardi integral operator (cf. [4], also see [47]).
(4) $I_{2,1}^{-\alpha} := \Omega_z^\alpha$, the fractional differential operator (cf. [39]), also renamed as Owa-Srivastava fractional differential operator (cf. [29–31]).
(5) $I_{1+1}^n := I_n$, the Noor integral operator (cf. [38], also see [32]).
(6) $I_{1,\mu}^{\lambda+1} := \mathcal{J}_{\lambda,\mu}$, the well known Choi-Saigo-Srivastava operator (cf. [8]).
(7) $I_{1,2}^{1} := \mathcal{L}$, is the Libera integral operator (see [47]).
(8) $I_{1,2}^{1} := I_A$, is the Alexander transformation, where as $I_{1,1}^{z} = \int_0^z \frac{f(t) dt}{t}$ is its inverse transform (see [10]).

From (1.3) it is clear that

$$z(I_{a,b}^c f(z))' = a(I_{a+1,b}^c f(z)) - (a - 1)I_{a,b}^c f(z).$$

By using Hohlov operator, we now defined a new subclass of $\mathcal{A}$ as follows:

**Definition 1.1.** A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}_{a,b}^c$, if and only if $\left(\frac{I_{a,b}^c f(z)}{z}\right)$ takes all values to the bounded region by the right half plane of the lemniscate of Bernoulli given by,

$$\{w \in \mathbb{C} : |w^2 - 1| < 1\} = \{u + iv : (u^2 + v^2)^2 = 2(u^2 - v^2)\}.$$

In terms of subordination, we have $f \in \mathcal{R}_{a,b}^c$ if it satisfies

$$\frac{(I_{a,b}^c f(z))}{z} \prec \sqrt{(1 + z)}, \quad (z \in \mathcal{U}).$$

**Remark 1.2.** Taking $b = 1$, the class $\mathcal{R}_{a,1}^c := \mathcal{R}(a, c)$, recently introduced and studied by Patel and Sahoo [40].

**Remark 1.3.** Taking $a = 2$, $b = 1$ and $c = 1$, we say a function $f$ given by (1.1) is in the class $\mathcal{R}_{2,1}^1$ if it satisfies the subordination relation

$$f'(z) \prec \sqrt{(1 + z)}, \quad (z \in \mathcal{U}).$$

The family $\mathcal{R}_{2,1}^1$ is recently studied by Sahoo and Patel [45] which is close-to-convex and hence univalent.
Remark 1.4. Taking $a = 1$, $b = 1$ and $c = 1$, we say a function $f$ given by (1.1) is in the class $\mathcal{R}_{1,1}$ if it satisfies the subordination relation

$$
\frac{f(z)}{z} \prec \sqrt{1+z}, \quad (z \in \mathcal{U}).
$$

Note that, the family $\mathcal{R}_{1,1}$ contain univalent as well as non univalent functions (cf. [10]).

It is well known that the $n^{th}$ coefficient of function belonging to the class $\mathcal{S}$ is bounded by $n$ and the bounds for the coefficients gives information about the geometric properties of the functions. For example, the $n^{th}$ coefficient gives information about the area whereas the second coefficient of functions in the family $\mathcal{S}$ yields the growth and distortion properties of function. A typical problem in geometric function theory is to study a functional made up of combinations of the coefficients of the original function. Usually, there is a parameter over which the extremal value of the functional is needed. Some of our results deal with one important functional of this type: the Fekete-Szegö functional. The classical problem settled by Fekete-Szegö [11] is to find for each $\lambda \in [0,1]$ the maximum value of the coefficient functional is defined by

$$
\Phi_\lambda(f) := |a_3 - \lambda a_2^2|
$$

over the class $\mathcal{S}$ and was proved by using Loewner method. Several researchers solved the Fekete-Szegö problem for various subclasses of the class of $\mathcal{S}$ and related subclasses of functions in $\mathcal{A}$. For instance see [3, 9, 17, 21, 30, 33, etc.]. For a systematic survey on Fekete-Szegö problem of classical subclasses of $\mathcal{S}$ we refer [46]. In [46], Srivastava et al. held that the inequality was sharp, however recently Peng (cf. [41]) showed that the extremal function given there for the case of $\mu \in (2/3, 1]$ is not sharp. Cho et al. [7] obtained Fekete-Szegö inequalities for close-to-convex function with respect to a certain convex function which improve the bound studied in [46].

Another way to investigate the sharp bound for the non linear functional is by using Hankel or Toeplitz determinant. Recalling the $q^{th}$ Hankel determinant of $f$ for $q \geq 1$ and $n \geq 1$ which is introduced and studied by Noonan and Thomas [36] as

$$(1.5) \quad H_q(n) = \begin{vmatrix}
    a_n & a_{n+1} & \cdots & a_{n+q} \\
    a_{n+1} & a_{n+2} & \cdots & a_{n+q+1} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n+q} & a_{n+q+1} & \cdots & a_{n+2q-2}
\end{vmatrix}, \quad (q, n \in \mathbb{N}).$$

This determinant has been studied by several authors including Noor [37] with the subject of inquiry ranging from the rate of growth of $H_q(n)$ (as $n \to \infty$) to the determinant of precise bounds with specific values of $n$ and $q$ for certain subclasses of analytic functions in the unit disk $\mathcal{U}$. For $q = 2$, $n = 1$, $a_1 = 1$, then the Hankel determinant simplifies to
In our present investigation, following the techniques adopted by Libera and Zlotkiewicz (cf. [24], [25]), for functions belongs to the family $R^{c}_{a,b}$, the Fekete-Szegö problem is completely solved for both real and complex parameter. All the extremal functions are presented in terms of Gauss Hypergeometric functions and convolution. Secondly, using the techniques of Hankel determinant, the sharp upper bound for the non linear functional $|a_2a_4 - a_3^2|$ is derived. Motivated by the work of Babalola [2] we found the sharp upper bound to the $|H_3(1)|$ for the function belonging to the class $R^{c}_{a,b}$ related with lemniscate of Bernoulli. Sufficient condition for functions to be in $R^{c}_{a,b}$ is also presented.

To establish our main results, we need the following lemmas:
Lemma 1.5. (cf. [16], [24–26]) Let the function $\phi \in P$, given by (1.2). Then

\begin{align*}
|p_k| &\leq 2 \quad (k \geq 1), \\
|p_2 - \nu p_1^2| &\leq 2 \max\{1, |2\nu - 1|\}, \quad (\nu \in \mathbb{C}), \\
p_2 &= \frac{1}{2} \{p_1^2 + (4 - p_1^2)x\}, \\
p_3 &= \frac{1}{4} \{p_1^3 + 2(4 - p_1^2)p_1x - (4 - p_1^2)p_1x^2 + 2(4 - p_1^2)(1 - |x|^2)z\},
\end{align*}

for some complex numbers $x, z$ satisfying $|x| \leq 1$ and $|z| \leq 1$. The estimates in (1.7) and (1.8) are sharp for the functions given by

\[ f(z) = \frac{1 + z}{1 - z}, \quad g(z) = \frac{1 + z^2}{1 - z^2} \quad (z \in \mathcal{U}). \]

Lemma 1.6. (cf. [26]) Let $\phi \in P$ and of the form (1.2), then

\[ |p_2 - \nu p_1^2| \leq \begin{cases} 
-4\nu + 2 & \text{; if } \nu < 0, \\
2 & \text{; if } 0 \leq \nu \leq 1, \\
4\nu - 2 & \text{; if } \nu > 1.
\end{cases} \]

For $\nu < 0$ or $\nu > 1$, equality holds if and only if $\phi(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. For $0 < \nu < 1$, the equality holds if and only if $\phi(z) = \frac{1+z^2}{1-z^2}$ or one of its rotations. For $\nu = 0$, the equality holds if and only if $\phi(z) = \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{2}\right) \frac{1}{1+z^2} \left(0 \leq \eta \leq 1\right)$ or one of its rotations. For $\nu = 1$, the equality holds if and only if $\phi$ is the reciprocal of one of the functions such that the equality holds in the case $\nu = 0$.

Moreover, when $0 < \nu < 1$, although the above upper bound is sharp, it can also be improved as follows:

\[ |p_2 - \nu p_1^2| + \nu |p_1|^2 \leq 2, \quad (0 < \nu \leq \frac{1}{2}), \]

and

\[ |p_2 - \nu p_1^2| + (1 - \nu) |p_1|^2 \leq 2, \quad (\frac{1}{2} < \nu \leq 1). \]
2. Main Results

Unless otherwise mentioned, throughout this sequel we assume that both \( a, b \geq c > 0 \). We begin with the proof of Fekete-Szegö problem for the class \( R^c_{a,b} \).

**Theorem 2.1.** If the function \( f \) given by (1.1) belongs to the class \( R^c_{a,b} \), then for any \( \mu \in \mathbb{C} \),

\[
|a_3 - \mu a_2^2| \leq \frac{(c)^2}{(a)^2(b)^2} \max \left\{ 1, \frac{ab(c+1) + \mu c(a+1)(b+1)}{4ab(c+1)} \right\}.
\]

The estimate (2.1) is sharp.

**Proof.** If \( f \in R^c_{a,b} \) then by the definition of \( R^c_{a,b} \), satisfies the condition

\[
\left| \left( \frac{T^c_{a,b} f(z)}{z} \right)^2 - 1 \right| < 1, \quad (z \in \mathcal{U}).
\]

So now using (2.2) and definition of subordination that satisfies, the relation

\[
T^c_{a,b} f(z) = \sqrt{1 + w(z)},
\]

where \( w \) is analytic in \( \mathcal{U} \) and satisfies the conditions of Schwarz lemma \( w(0) = 0 \) and \( |w(z)| < 1 \). Setting

\[
\phi(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + p_1z + p_2z^2 + \cdots, \quad (z \in \mathcal{U}),
\]

implies that \( \phi \in \mathcal{P} \). From the above expression, we get

\[
w(z) = \frac{\phi(z) - 1}{\phi(z) + 1} \quad (z \in \mathcal{U}).
\]

Therefore, (2.3) gives

\[
\left( \frac{T^c_{a,b} f(z)}{z} \right)^2 = \left( \frac{2\phi(z)}{1 + \phi(z)} \right)^{1/2} \quad (z \in \mathcal{U}).
\]

Which upon simplification and comparing the co-efficient of \( z, z^2, z^3 \) both side of (2.4) yields.

\[
a_2 = \frac{c}{4ab} p_1,
\]

\[
a_3 = \frac{2(c)^2}{(a)(b)^2} \left( \frac{1}{4} p_2 - \frac{5}{32} p_1^2 \right),
\]

and

\[
a_4 = \frac{6(c)^3}{(a)(b)^3} \left( \frac{1}{4} p^3 - \frac{5}{16} p_1 p_2 + \frac{13}{128} p_1^3 \right).
\]
Thus, by using (2.5) and (2.6), we get

\begin{equation}
|a_3 - \mu a_2^2| \leq \frac{1}{2} \left( \frac{c_2}{(a_2(b_2)} \right) \left| p_2 - \frac{5ab(c + 1) + \mu c(a + 1)(b + 1)}{8ab(c + 1)} p_1^2 \right|.
\end{equation}

Now using (1.8) in (2.8), we get

\begin{equation}
|a_3 - \mu a_2^2| \leq \left( \frac{c_2}{(a_2(b_2)} \max \left\{ 1, \frac{|ab(c + 1) + \mu c(a + 1)(b + 1)|}{4ab(c + 1)} \right\} \right).
\end{equation}

The estimate (2.1) is sharp for the function \( f \in A \) defined in \( U \) by

\[ f(z) = \begin{cases} 
  z_1 F_2(c, b, a; z) \ast 4z \sqrt{1 + z^2}; & \frac{|ab(c + 1) + \mu c(a + 1)(b + 1)|}{4ab(c + 1)} \leq 1, \\
  z_1 F_2(c, b, a; z) \ast \frac{1}{2}(1 + z)^{-1}; & \frac{|ab(c + 1) + \mu c(a + 1)(b + 1)|}{4ab(c + 1)} > 1.
\end{cases} \]

This completes the proof of Theorem 2.1.

For \( \mu = 1 \), the bound \(|H_2(1)|\) directly follows from Theorem 2.1.

**Corollary 2.2.** If the function \( f \) given by (1.1) belongs to the class \( R_{a,b}^c \), then

\[ |H_2(1)| = |a_3 - \mu a_2^2| \leq \frac{(c_2}{(a_2(b_2)} \max \left\{ 1, \frac{|ab(c + 1) + \mu c(a + 1)(b + 1)|}{4ab(c + 1)} \right\} \right).
\]

**Remark 2.3.** Putting \( b = 1 \) in Theorem 2.1, we get the sharp bound for the function belonging to the subclass of \( A \) associated with the Carlson-Shaffer operator (cf. [40], Theorem 3).

Further, by specializing the parameters \( a, b \) and \( c \) we have the following sharp bounds:

**Remark 2.4.** Putting \( a = \lambda + 1, b = 1, c = 1 \) in the Theorem 2.1, the required sharp bound for the function belonging to the subclass of \( A \) associated with Ruscheweyh operator is given by

\[ |a_3 - \mu a_2^2| \leq \frac{1}{(\lambda + 1)_2} \max \left\{ 1, \frac{|(\lambda + 1) + \mu (\lambda + 2)|}{4(\lambda + 1)} \right\} \right).
\]

**Remark 2.5.** Putting \( a = 1, b = 1 + \eta, c = 2 + \eta \) in the Theorem 2.1, the required sharp bound for the function belonging to the subclass of \( A \) associated with Bernardi operator is given by

\[ |a_3 - \mu a_2^2| \leq \frac{3 + \eta}{2(1 + \eta)} \max \left\{ 1, \frac{|(1 + \eta)(3 + \eta) + 2\mu(2 + \eta)^2|}{4(1 + \eta)(3 + \eta)} \right\} \right).
\]

**Remark 2.6.** Putting \( a = 2, b = 1 c = 1 \) in the Theorem 2.1, the required sharp bound for the function belonging to the subclass of \( A \) associated with Alexander differential operator (cf. [45], Theorem 2.1) is given by

\[ |a_3 - \mu a_2^2| \leq \frac{1}{6} \max \left\{ 1, \frac{|2 + 3\mu|}{8} \right\} \right).
\]
Corollary 2.7. If the function $f$ given by (1.1) belongs to the class $\mathcal{R}_{a,b}^c$, then it follows from (2.3) that $|a_2| \leq \frac{2}{a_2(b+1)}$ and Theorem 2.7 gives $|a_3| \leq \frac{(c_2)}{(a_2(b+1))}$. The estimate for $|a_2|$ is sharp if $f$ is defined by

$$f(z) = z_1 F_2(c, b, a; z) \ast \frac{-z}{2}(1 + z)^{-1} \quad (z \in \mathcal{U}),$$

and the estimate for $|a_3|$ is sharp for the function $g$ defined by

$$g(z) = z_1 F_2(c, b, a; z) \ast 4z \sqrt{1 + z^2} \quad (z \in \mathcal{U}).$$

We will proceed this Theorem in the case of $\mu \in \mathbb{R}$.

Theorem 2.8. Let $\mu \in \mathbb{R}$. If the function $f$ given by (1.1) belongs to the class $\mathcal{R}_{a,b}^c$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(c_2)}{2(a_2(b+1))} \left( -\frac{(c+1)ab + \mu c(a+1)+1)}{2ab(c+1)} \right); & \mu < \frac{-5(c+1)ab}{c(a+1)(b+1)}; \\ \frac{(c_2)}{2(a_2(b+1))} \left( \frac{(c+1)ab + \mu c(a+1)+1)}{2ab(c+1)} \right); & \mu > \frac{3(c+1)ab}{c(a+1)(b+1)}. \end{cases}$$

The estimate is sharp for the functions $f$ defined in $\mathcal{U}$ by

$$f(z) = \begin{cases} z_1 F_2(c, b, a; z) \ast 4z \sqrt{1 + z^2}; & -\frac{5(c+1)ab}{c(a+1)(b+1)} \leq \mu \leq \frac{3(c+1)ab}{c(a+1)(b+1)}; \\ z_1 F_2(c, b, a; z) \ast \frac{-z}{2}(1 + z)^{-1}; & \mu < \frac{-5(c+1)ab}{c(a+1)(b+1)} \text{ or } \mu > \frac{3(c+1)ab}{c(a+1)(b+1)}. \end{cases}$$

Proof. Since from (2.8), we have

$$|a_3 - \mu a_2^2| = \frac{1}{2(a_2(b+1))} \left| p_2 \left( \frac{5ab(c+1) + \mu c(a+1)+1}{8ab(c+1)} \right) p_1 \right|^2.$$ 

The result follows upon applications of Lemma 1.6 in (2.8). This completes the proof of Theorem 2.8.

Remark 2.9. Putting $b = 1$ in the Theorem 2.8 we obtained the recent result, due to Patel and Sahoo (cf. [14], Corollary 5).

In the following theorem, we find sharp upper bound to the second Hankel determinant for the class $\mathcal{R}_{a,b}^c$.

Theorem 2.10. If $a \geq c \geq 1/2$ and the function $f$ given by (1.1) belongs to the class $\mathcal{R}_{a,b}^c$, then

$$|a_2a_4 - a_3^2| \leq \left( \frac{c_2}{a_2(b+1)} \right)^2.$$ 

The estimate in (2.9) is sharp for the functions $g$, given by

$$g(z) = z_1 F_2(c, b, a; z) \ast 4z \sqrt{1 + z^2} \quad (z \in \mathcal{U}).$$
Proof. Let the function $f \in \mathcal{R}_{a,b}$. From (2.5), (2.6) and (2.7), we get

$$|a_2a_4 - a_3^2| = \left| \frac{3}{8} \frac{c(c_3)}{a_3b(b_3)} \left( p_1p_3 - \frac{5}{4} p_1^2p_2 + \frac{13}{32} p_1^4 \right) - \left\{ \frac{c(c_2)}{(a_2)^2b_2} \right\}^2 \frac{1}{4} \left( p_2^2 - \frac{5}{4} p_2^2p_2 + \frac{25}{64} p_1^4 \right) \right|.$$

Since the function $\phi(z) \in \mathcal{P}$. We assume that $p_1 > 0$ and $p_1 = p$ ($0 \leq p \leq 2$). Now by using (1.9) and (1.10) in (2.10), we get

$$|a_2a_4 - a_3^2| = \left| \frac{3}{8} \frac{c(c_3)}{a_3b(b_3)} \left\{ p^4 + 2(4 - p^2)p^2x - (4 - p^2)p^2x^2 + 2(4 - p^2)(1 - |x|^2)p^2 \right\} 
+ \frac{5c(c_2)}{32a(a_2)b(b_2)} \left\{ -3(c + 2)(a + 1)(b + 1) + 2(c + 1)(a + 2)(b + 2) \right\} \right| 
\frac{p^4 + (4 - p^2)p^2x}{256(a_3)(a_2)b_2(b_3)} \frac{2}{16} \left\{ \frac{c(c_2)}{(a_2)^2b_2} \right\}^2 \frac{p^4 + (4 - p^2)p^2x}{256(a_3)(a_2)(b_3)^2} 
\frac{(39(c + 2)(a + 1)(b + 1) - 25(c + 1)(a + 2)(b + 2))c(c_2)}{256(a_3)(a_2)(b_3)^2} p^4 \right|,$$

for some $x$ ($|x| \leq 1$) and for some $z$ ($|z| \leq 1$). Applying the triangle inequality in (2.11) and replacing $|x|$ by $y$ in the equation, we get

$$|a_2a_4 - a_3^2| \leq \frac{c(c_2)}{16a(a_2)b(b_2)} \left\{ \frac{2abc + ac + bc - c + 5ab + 4a + 4b + 2}{16(a + 1)(b + 1)} \right\} p^4 + \frac{c(c_2)}{64(a_3)(a_2)(b_3)^2} p^4$$

$$\frac{(4 - p^2)^2y\{abc - ac - bc - 5c + 4ab + 2a + 2b - 2\}}{32(a_2)^2(b_3)^2} y^2$$

$$\left\{ \frac{(4 - p^2)^2c(c_2)}{32(a_2)^2(b_3)^2} + \frac{32(3c + 2)(a + 1)(b + 1)3p(p - 2) + 2(4 - p^2)(c + 1)(a + 2)(b + 2)}{16(a_3)(a_2)(b_3)^2} \right\} \leq \frac{16}{16(a + 1)(b + 1)} + \frac{16}{16(a + 1)(b + 1)}$$

$$\{3(c + 2)(a + 1)(b + 1)3p(p - 2) + 2(4 - p^2)(c + 1)(a + 2)(b + 2)\} + \frac{32(3c + 2)(a + 1)(b + 1)3p(p - 2) + 2(4 - p^2)(c + 1)(a + 2)(b + 2)}{16(a_3)(a_2)(b_3)^2} \} > 0.$$

Next we maximize the function $G(p, y)$ on the closed rectangle $[0, 2] \times [0, 1]$. Indeed, for $0 \leq p \leq 1$ and $0 \leq y \leq 1$, we have

$$\frac{\partial G}{\partial y} = \frac{c(c_2)(4 - p^2)}{8(a_2)(a_3)(b_2)(b_3)} \left\{ \frac{p^2}{8} (abc - ac - bc - 5c + 4ab + 2a + 2b - 2) \right. + \left. y \left( 3(c + 2)(a + 1)(b + 1)3p(p - 2) + 2(4 - p^2)(c + 1)(a + 2)(b + 2) \right) \right\} > 0.$$

Which clearly shows that $G(p, y)$ cannot attain maximum in the interior of the closed rectangle $[0, 2] \times [0, 1]$.

$$\max_{0 \leq y \leq 1} G(p, y) = G(p, 1) = F(p) \text{ (say)},$$
Therefore maximum must be attain on the boundary. Thus for fixed \( p, \ (0 \leq p \leq 2) \), we have

\[
F(p) = \frac{c(c_2)p^4}{16a(a_2)b(b_2)} \left\{ \frac{-10abc + 13ac + 13bc + 59c - 43ab - 20a - 20b + 26}{16(a + 1)(a + 2)(b + 1)(b + 2)} \right\} 
\]

(2.13)

\[
+ \frac{c(c_2)p^2}{2a(a_2)b(b_2)} \left\{ \frac{-3abc - 9ac - 11bc - 31c + 6a - 6b - 18}{8(a + 1)(a + 2)(b + 1)(b + 2)} \right\} + \frac{c(c_2)(c + 1)}{(a_2)(b_2)(b_2)}.
\]

Differentiating (2.13) partially w.r.t. \( p \) and equating to zero yields,

\[
\frac{\partial F}{\partial p} = \frac{c(c_2)}{a(a_2)b(b_2)} \left\{ \frac{-10abc + 13ac + 13bc + 59c - 43ab - 20a - 20b + 26}{64(a + 1)(a + 2)(b + 1)} \right\} p^2 
\]

\[
+ \left\{ \frac{-3ab - 9ac - 11bc - 31c + 6a - 6b - 18}{8(a + 1)(b + 1)} \right\} \right\} = 0,
\]

which implies that either \( p = 0 \) or \( p^2 = \frac{8(-3abc - 9ac - 11bc - 31c + 6a - 6b - 18)}{(-10abc + 13ac + 13bc + 59c - 43ab - 20a - 20b + 26)}. \)

Further, we have \( F''(0) < 0 \). Thus the maximum value of \( F \) is attained at \( p = 0 \). Therefore, the upper bound in (2.12) corresponds to \( p = 0 \) and \( y = 1 \) becomes

\[
|a_2a_4 - a_3^2| \leq \left( \frac{(c_2)}{(a_2)(b_2)} \right)^2.
\]

This completes the proof of Theorem 2.10.

Remark 2.11. Putting \( b = 1 \) in the Theorem 2.10 we obtained the recent result, due to Patel and Sahoo (cf. [40], Theorem 7).

Next we find the sharp upper bound for the fourth co-efficient of functions belonging to the class \( R_{a,b}^c \).

Theorem 2.12. If the function \( f \) given by (1.1) belongs to the class \( R_{a,b}^c \), then

(2.14)

\[
|a_4| \leq \frac{3(c_3)}{(a_3)(b_3)}.
\]

The estimate (2.14) is sharp.

Proof. Using (1.10) in (2.7), we assume that \( p_1 > 0 \) and write \( p_1 = p \ (0 \leq p \leq 2) \), then we deduce that

(2.15)

\[
|a_4| = \frac{6(c_3)}{(a_3)(b_3)} \left\{ \frac{p^3}{128} - \frac{(4 - p^2)p}{32}x - \frac{1}{16} (4 - p^2)x^2 + \frac{1}{8} (4 - p^2)(1 - |x|^2)^2 \right\}.
\]
for some $x$ ($|x| \leq 1$) and for some $z$ ($|z| \leq 1$). Applying the triangle inequality and replace $|x|$ by $y$ in (2.15), we get

$$
(2.16) \quad |a_4| \leq \frac{6(c)_3}{(a)_3(b)_3} \left( \frac{p^3}{128} + \frac{(4 - p^2)py}{32} + \frac{1}{16} (4 - p^2)py^2 + \frac{1}{8} (4 - p^2)(1 - |y|^2)z \right) = G(p, y), \quad (0 \leq y \leq 1; 0 \leq p \leq 2) \quad (say).
$$

We next maximize the function $G(p, y)$ on the closed rectangle $[0, 2] \times [0, 1]$. Since

$$
G'(y) = \frac{6(c)_3}{(a)_3(b)_3} \left( \frac{(4 - p^2)p}{32} + \frac{1}{8} (4 - p^2)py - \frac{1}{4} (4 - p^2)y \right) < 0,
$$

for $0 < p < 2$ and $0 < y < 1$; it follows that $G(p, y)$ can’t have a maximum value in the interior of the closed rectangle $[0, 2] \times [0, 1]$. Thus, for fixed $p \in [0, 2],$

$$
\max_{0 \leq y \leq 1} G(p, y) = G(p, 0) = F(p),
$$

where

$$
F(p) = \frac{6(c)_3}{(a)_3(b)_3} \left( \frac{p^3}{128} + \frac{1}{2} - \frac{p^2}{8} \right).
$$

We further note that,

$$
(2.17) \quad F'(p) = \frac{6(c)_3}{(a)_3(b)_3} \left( \frac{3p}{128} - \frac{1}{4} \right) p,
$$

for $p = 0$ or $p = 32/3$. Since

$$
F''(0) = \frac{-3(c)_3}{2(a)_3(b)_3} < 0,
$$

the function $F$ attains maximum value at $p = 0$. Thus, the upper bound of the function $G$ corresponding to $p = y = 0$. Therefore, putting $p = y = 0$ in (2.16), we get

$$
(2.18) \quad |a_4| \leq \frac{3(c)_3}{(a)_3(b)_3}.
$$

The estimate in (2.18) is sharp for the function $f$ defined by,

$$
f(z) = z_1 F_2(c, b, a; z) * 36z \sqrt{1 + z^3}.
$$

This completes the proof of Theorem 2.12. \qed

**Remark 2.13.** Putting $b = 1$ in the Theorem 2.12, we obtained the recent result due to Patel and Sahoo (cf. [40], Theorem 9).

**Theorem 2.14.** If the function $f$ given by (1.1) belongs to the class $R^c_{a,b}$, then

$$
|a_5| \leq \frac{15}{16} \frac{(c)_4}{(a)_4(b)_4}.
$$
COEFFICIENT ESTIMATES OF CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED

Putting $b = 1$ in the Theorem 2.14 we get following:

**Corollary 2.15.** If the function $f$ given by (1.1) belongs to the class $\mathcal{R}(a, b)$, then

$$|a_5| \leq \frac{15 (c)_4}{384 (a)_4}.$$  

The following Theorems are straight forward verification on applying the same procedure as described in Theorem 2.10.

**Theorem 2.16.** If the function $f$ given by (1.1) belongs to the class $\mathcal{R}_{a,b}$, then

$$|a_2a_3 - a_4| \leq \frac{13(c)_2}{64a(a)_3b(b)_3} \left( \frac{c(a + 2)(b + 2) + 9ab(c + 2)}{c(a + 2)(b + 2) + 11ab(c + 2)} \right)^{\frac{1}{2}} (c(a + 2)(b + 2) + 9ab(c + 2)).$$

Putting $b = 1$ in the Theorem 2.16 we get following result for the function class $\mathcal{R}(a, c)$.

**Corollary 2.17.** If the function $f$ given by (1.1) belongs to the class $\mathcal{R}(a, c)$, then

$$|a_2a_3 - a_4| \leq \frac{13(c)_2}{384a(a)_3} \left( \frac{3c(a + 2) + 9a(c + 2)}{3c(a + 2) + 11a(c + 2)} \right)^{\frac{1}{2}} (3c(a + 2) + 9a(c + 2)).$$

Next we find the sharp upper bound for third Hankel determinant of functions belonging to the class $\mathcal{R}_{a,b}$.

**Theorem 2.18.** If the function $f$ given by (1.1) belongs to the class $\mathcal{R}_{a,b}$, then

$$|H_3(1)| \leq \left( \frac{(c)_2}{(a)_2(b)_2} \right)^3 + \frac{39(c)_2(c)_3}{64((a)_3)^2(b)_3^2} \left( \frac{c(a + 2)(b + 2) + 9ab(c + 2)}{c(a + 2)(b + 2) + 11ab(c + 2)} \right)^{\frac{1}{2}} (c(a + 2)(b + 2) + 9ab(c + 2)) + \frac{15(c)_4(c)_2}{64(a)_2b(b)_3}. $$

**Proof.** Suitable applications of Theorems 2.10, 2.12, 2.16, 2.2, 2.14 and Corollary 2.7 in equation (1.6), the result follows. This completes the proof of Theorem 2.18.  □

Putting $b = 1$ in the Theorem 2.18 we obtained the following third Hankel determinant for the function class $\mathcal{R}(a, c)$.

**Corollary 2.19.** If the function $f$ given by (1.1) belongs to the class $\mathcal{R}(a, c)$ then

$$|H_3(1)| \leq \left( \frac{(c)_2}{2(a)_2} \right)^3 + \frac{39(c)_2(c)_3}{2304((a)_3)^2} \left( \frac{3c(a + 2) + 9a(c + 2)}{3c(a + 2) + 11a(c + 2)} \right)^{\frac{1}{2}} (3c(a + 2) + 9a(c + 2)) + \frac{15(c)_4(c)_2}{3072(a)_2(a)_4}.$$ 

Finally, we have following sufficient condition for a function in $\mathcal{A}$ to be in the class $\mathcal{R}_{a,b}^c$:
Theorem 2.20. Let $\gamma > 0$. If $f \in A$ satisfies
\[
\Re \left\{ \frac{I_{a+1,b}^c f(z)}{I_{a,b}^c f(z)} \right\} < 1 + \frac{1}{2a\gamma} \quad (z \in \mathcal{U}),
\]
then
\[
\frac{I_{a,b}^c f(z)}{z} < (1 + z)^{1/\gamma} \quad (z \in \mathcal{U})
\]
and the result is the best possible.

Proof. Setting
\[
(2.19) \quad \frac{I_{a,b}^c f(z)}{z} = (1 + w(z))^{1/\gamma} \quad (z \in \mathcal{U}).
\]
Choosing the principal branch in (2.19), we see that $w$ is analytic in $\mathcal{U}$ with $w(0) = 0$. Taking the logarithmic differentiation in (2.19) and using the identity (1.4) in the resulting equation, we deduce that
\[
(2.20) \quad \frac{(I_{a+1,b}^c f(z))}{(I_{a,b}^c f(z))} = 1 + \frac{zw'(z)}{a\gamma(1 + w(z))} \quad (z \in \mathcal{U}).
\]
Next to claim that $|w(z)| < 1, z \in \mathcal{U}$. \exists a $z_0 \in \mathcal{U}$ such that
\[
\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1 \quad (w(z_0) \neq 1).
\]
Letting $w(z_0) = e^{i\theta} (-\pi < \theta \leq \pi)$ and applying Jack’s lemma [14], we have
\[
(2.21) \quad z_0w'(z_0) = kw(z_0) \quad (k \geq 1).
\]
Using (2.21) in (2.20) then, we get
\[
\Re \left( \frac{I_{a+1,b}^c f(z)}{I_{a,b}^c f(z)} \right) = 1 + \frac{1}{a\gamma} \Re \left\{ \frac{zw'(z_0)}{1 + w(z_0)} \right\}
\]
\[
= 1 + \frac{k}{a\gamma} \Re \left( \frac{e^{i\theta}}{1 + e^{i\theta}} \right)
\]
\[
\geq 1 + \frac{k}{2a\gamma}.
\]
Thus we conclude that $|w(z)| < 1$ for $z \in \mathcal{U}$ and the theorem follows from (2.19). For sharpness we consider the principal branch of the function $f_0$ defined as,
\[
(2.23) \quad f_0(z) = z_1 F_2(c, b, a; z) \ast z(1 + z)^{\frac{1}{\gamma}} \quad (z \in \mathcal{U}).
\]
Therefore (2.23), yields
\[
\frac{I_{a,b}^c f_0(z)}{z} = (1 + z)^{1/\gamma}.
\]
Taking logarithmic differentiation and suitable application of (1.3), gives
\[ \Re \left( \frac{T_{a+1,b}^c f_0(z)}{T_{a,b}^c f_0(z)} \right) = 1 + \frac{1}{a\gamma} \frac{z}{1+z} \]
\[ \rightarrow 1 + \frac{1}{2a\gamma} \text{ as } z \rightarrow 1^-. \]
This completes the proof of Theorem 2.20.

\[ \square \]

**Remark 2.21.** Putting \( b = 1 \) in the theorem 2.20, we get the result due to Patel and Sahoo (cf. [40], Theorem 11).

Putting \( \gamma = 2 \) in Theorem 2.20, we have the following

**Corollary 2.22.** If \( f \in A \) satisfies
\[ \Re \left\{ \frac{T_{a+1,b}^c f(z)}{T_{a,b}^c f(z)} \right\} < 1 + \frac{1}{4a} \quad (z \in U), \]
then \( f \in R_{a,b}^c \). The result is the best possible.

**References**

[1] A. Abubaker and M. Darus, Hankel determinant for a class of analytic functions involving a generalized linear differential operator, *Int. J. Pure Appl. Math.*, 69, (2011), 429-435.

[2] K. O. Babalola, On \( H_3(1) \) Hankel determinant for some classes of univalent functions, *Inequal. Theory Appl.*, 6, (2007), 1–7.

[3] D. Bansal, S. Maharana and J. K. Prajapat, Third order Hankel determinant for certain univalent functions, *J. Korean Math. Soc.*, 52 (6), (2015), 1139-1148.

[4] S. D. Bernardi, Convex and Starlike univalent functions, *Trans. Amer. Math. Soc.*, 135, (1969), 429–446.

[5] B. Bhowmik, S. Ponnusamy and K. -J. Wirths, On the Fekete- Szegö problem for close-to-convex univalent functions, *J. Math. Anal. Appl.*, 373(2), (2011), 432 438.

[6] B. C. Carlson and D. B. Shaffer, Starlike and prestarlike hypergeometric functions, *SIAM Journal Anal.*, 15, (1984), 737–745.

[7] N. E. Cho, B. Kowalczyk and A. Lecko, Fekete-Szegö problem for close-to-convex functions with respect to a certain convex function depend on a real parameter, *Front. Math. China*, 11(6) (2016), 1471–1500.

[8] J. H. Choi, M. Saigo and H. M. Srivastava, Some inclusion properties of a certain family of integral operators, *J. Math. Anal. Appl.*, (2002), 432–445.
[9] J. Dziok, A general solution of Fekete-Szegö problem, *Bound. Value Probl.*, **1(98)**, (2013), 1–13.

[10] P. L. Duren, Univalent functions, *Grundlehrender Mathematischer Wissenschaffer*, **259**, Springer, New york, (1983).

[11] M. Fekete and G. Szego, Eine bemerkung uber ungerade schlichte funktionen, *J. Lond. Math. Soc.*, **8**, (1933), 85–89.

[12] Y. E. Hohlov, Hadamard convolution, hypergeometric functions and linear operators in the class of univalent functions, *Dokl. Akad. Nauk Ukr. SSR, Ser., A(7)*, (1984), 25–27.

[13] Y. E. Hohlov, Convolution operators preserving univalent functions, *Ukrainian Math. J.*, **37**, (1985), 220–226.

[14] I. S. Jack, Functions starlike and convex of order $\alpha$, *J. Lond. Math. Soc.*, **3**, (1971), 469–474.

[15] A. Janteng, S. A. Halim and M. Darus, Coefficient inequality for a function whose derivative has positive real part, *J. Inequal. Pure Appl. Math.*, **7(2)**, (2006).

[16] A. Janteng, S. A. Halim and M. Darus, Hankel determinant for starlike and convex functions, *Int. J. Math. Anal.*, **1(13)**, (2007), 619–625.

[17] S. Kanas and H. E. Darwish, Fekete-Szegö problem for starlike and convex functions of complex order, *Appl. Math. Letters*, **23**, (2010), 777–782.

[18] F. R. Keogh and E. P. Merkes, A co-efficient inequality for certain classes of analytic functions, *Proc. Amer. Math. Soc.*, **20**, (1969), 8–12.

[19] W. Koepf, On the Fekete-Szegö problem for close to convex functions, *Proc. Amer. Math. Soc.*, **101(1)**, (1987), 89–95.

[20] W. Koepf, On the Fekete-Szegö problem for close to convex functions II, *Archiv der Mathematik*, **49(5)**, (1987), 420–433.

[21] B. Kowalczyk and A. Lecko, Fekete-Szegö inequality for close-to-convex functions with respect to a certain starlike function depend on a real parameter, *J. Inequal. Appl.*, **1(65)**, (2014), 1–16.

[22] D. V. Krishna and T. Ram Reddy, Coefficient inequality for certain subclasses of analytic functions associated with Hankel determinant, *Indian J. Pure Appl. Math.*, **46(1)**, (2015), 91–106.

[23] S. K. Lee, V. Ravichandran and S. Supramaniam, Bounds for the second Hankel determinant of certain univalent functions, *J. Inequal. Appl.*, **2013**, (2013), 281.

[24] R. J. Libera and E. J. Zlotkiewicz, Early coefficients of the inverse of a regular convex functions, *Proc. Amer. Math. Soc.*, **85(2)**, (1982), 225–230.
COEFFICIENT ESTIMATES OF CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED

[25] R. J. Libera and E. J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivative in $P$, Proc. Amer. Math. Soc., 87(2), (1983), 251–257.

[26] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, In Proc. Conference on Complex Analysis (Tianjin, 1992), Z. Li, F. Ren, L. Yang and S. Zhang, Eds., 157–169, International Press, Cambridge, Mass, USA, (1994).

[27] T. H. MacGregor, Functions whose derivative have a positive real part, Trans. Amer. Math. Soc., 104(3), (1962), 532–537.

[28] S. S. Miller and P. T. Mocanu, Differential Subordinations: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, 225, Marcel Dekker, New York, (2000).

[29] A. K. Mishra and P. Gochhayat, Second Hankel determinant for a class of analytic functions defined by fractional derivative, Int. J. Math. Math. Sci., (2008).

[30] A. K. Mishra and P. Gochhayat, Applications of the Owa-Srivastava operator to the class of k-uniformly convex functions, Fract. Calc. Appl. Anal., 9(4), (2006), 323–331.

[31] A. K. Mishra and P. Gochhayat, The Fekete-Szegő problem for k-uniformly convex functions and for a class defined by the Owa-Srivastava operator, J. Math. Anal. Appl., 397(9), (2008), 563–572.

[32] A. K. Mishra and P. Gochhayat, The Fekete-Szegő problem for a class defined by an integral operator, Kodai Math. J., 33, (2010), 310–328.

[33] A. K. Mishra and P. Gochhayat, A coefficient inequality for a subclass of the Carathéodory functions defined by conical domains, Comput. Math. Appl., 61(9), (2011), 2816–2820.

[34] A. K. Mishra and S. N. Kund, The second Hankel determinant for a class of analytic functions associated with the Carlson-Shaffer operator, Tamkang J. Math., 44(1), (2013), 73-82.

[35] G. Murugusundaramoorthy and K. Vijaya, Second Hankel determinant for bi-univalent analytic functions associated with Hohlov operator, Int. J. Anal. Appl., 8(1), (2015), 22–29

[36] J. W. Noonan and D. K. Thomas, On the second Hankel determinant of areally mean p-valent functions, Trans. Amer. Math. Soc., 223 (1976), 337–346.

[37] K. I. Noor, Hankel determinant problem for the class of functions with bounded boundary rotation, Rev. Roumaine Math. Pures Appl., 28(8), (1983), 731–739.

[38] K. I. Noor and M. A. Noor, On integral operators, J. Anal. Appl., 238, (1999), 341–352.
[39] S. Owa and H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, *Canad. J. Math.*, 39(5), (1987), 1057–1077.

[40] J. Patel and A. K Sahoo, On certain subclasses of analytic functions involving Carlson-Shaffer operator and related to Lemniscate of bernoulli, *J. Complex Anal.*, 2014, (2014), 1–7.

[41] Z. Peng, On the Fekete-Szegő problem for a class of analytic functions, *ISRN Math. Anal.*, 2014, (2014), 1–4.

[42] M. Raza, S. N. Malik, Upper bound of the third Hankel determinant for a class of analytic functions related with Lemniscate of Bernoulli, *Int. J. Inequal. Appl.*, 2013, (2013), Article 412.

[43] T. R. Reddy and D. Vamshee Krishna, Hankel determinant for starlike and convex functions with respect to symmetric points, *J. Indian Math. Soc.*, 79, (2012), 161-171.

[44] St. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math. Soc.*, 49, (1975), 109–115.

[45] A. K. Sahoo and J. Patel, Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli, *Int. J. Anal. Appl.*, 6 (2), (2014), 170–177.

[46] H. M. Srivastava, A. K. Mishra and M. K. Das The Fekete-Szegő problem for a subclasses of close to convex functions, *Complex Var. Elliptic Equ.*, 44(2), (2001), 145-163.

[47] H. M. Srivastava and S. Owa (Editors), Current Topics in Analytic Function Theory, *World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong* (1992).

[48] T. Yavuz, Second Hankel determinant problem for a certain subclass of univalent functions, *Int. J. Math. Anal.*, 9(10), (2015), 493 – 498.

[49] T. Yavuz, Second Hankel determinant for analytic functions defined by Ruscheweyh derivative, *Int. J. Anal. Appl.*, 8(1), (2015), 63–68.

Department of Mathematics, Sambalpur University, Jyoti Vihar 768019, Burla, Sambalpur, Odisha, India

E-mail address: pgochhayat@gmail.com

E-mail address: anujaprajapati49@gmail.com

Department of Mathematics, Veer Surendra Sai University of Technology, Sidhi Vihar 768018, Burla, Sambalpur, Odisha, India

E-mail address: ashokuumt@gmail.com