Large 3-groups of automorphisms of algebraic curves in characteristic 3

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December 19, 2013

Abstract

Let $S$ be a $p$-subgroup of the $K$-automorphism group $\text{Aut}(\mathcal{X})$ of an algebraic curve $\mathcal{X}$ of genus $g \geq 2$ and $p$-rank $\gamma$ defined over an algebraically closed field $K$ of characteristic $p \geq 3$. In this paper we prove that if $|S| > 2(g - 1)$ then one of the following cases occurs.

(i) $\gamma = 0$ and the extension $K(\mathcal{X})/K(\mathcal{X})^S$ completely ramifies at a unique place, and does not ramify elsewhere.

(ii) $\gamma > 0$, $p = 3$, $\mathcal{X}$ is a general curve, $S$ attains the Nakajima’s upper bound $3(\gamma - 1)$ and $K(\mathcal{X})$ is an unramified Galois extension of the function field of a general curve of genus 2 with equation $Y^2 = cX^6 + X^4 + X^2 + 1$ where $c \in K^*$. 

Case (i) was investigated by Stichtenoth, Lehr, Matignon, and Rocher, see [18, 11, 12, 15, 16].

1 Introduction

In the present paper, $K$ is an algebraically closed field of characteristic $p \geq 3$, $\mathcal{X}$ is a (projective, non-singular, geometrically irreducible, algebraic) curve of genus $g(\mathcal{X}) \geq 2$ with function field $K(\mathcal{X})$, $\text{Aut}(\mathcal{X})$ is the $K$-automorphism group of $\mathcal{X}$, and $S$ is a (non-trivial) subgroup of $\text{Aut}(\mathcal{X})$ whose order is a power of $p$.

The earliest results on the maximum size of $S$ date back to the 1970’s and have played an important role in the study of curves with large automorphism groups exceeding the classical Hurwitz bound $84(g(\mathcal{X}) - 1)$. Stichtenoth proved that if $S$ fixes a place $\mathcal{P}$ of $K(\mathcal{X})$ then

$$|S| \leq \frac{p}{p-1} g(\mathcal{X})$$

(1)

unless the extension $K(\mathcal{X}) : K(\mathcal{X})^S$ completely ramifies at $\mathcal{P}$, and does not ramify elsewhere; in geometric terms, $S$ fixes a point $P$ of $\mathcal{X}$ and acts on $\mathcal{X} \setminus \{P\}$ as a semiregular permutation group; see [18] and also [10, Theorem 11.78]. In the latter case, the Stichtenoth bound is

$$|S| \leq \frac{4p}{p-1} g(\mathcal{X})^2.$$  

(2)

In his 1987 paper [13] Nakajima pointed out that the maximum size of $S$ is related to the Hasse-Witt invariant $\gamma(\mathcal{X})$ of $\mathcal{X}$. It is known that $\gamma(\mathcal{X})$ coincides with the $p$-rank of $\mathcal{X}$ defined to be the rank of the (elementary abelian) group of the $p$-torsion points in the Jacobian variety of $\mathcal{X}$; moreover, $\gamma(\mathcal{X}) \leq g(\mathcal{X})$.
and when equality holds then \( \mathcal{X} \) is called an ordinary (or general) curve; see \cite{10} Section 6.7. If \( S \) fixes a point and \((1) \) fails then \( \gamma(\mathcal{X}) = 0 \); conversely, if \( \gamma(\mathcal{X}) = 0 \), then \( S \) fixes a point, see \cite{10} Lemma 11.129. For \( \gamma(\mathcal{X}) > 0 \), Nakajima proved that \( |S| \) divides \( g(\mathcal{X}) - 1 \) when \( \gamma(\mathcal{X}) = 1 \), and \( |S| \leq p/(p - 2)(\gamma(\mathcal{X}) - 1) \) otherwise; see \cite{13} and also \cite{10} Theorem 11.84. Therefore, the Nakajima bound \cite{13} Theorem 1] is

\[
|S| \leq \begin{cases} 
\frac{p}{p - 2}(g(\mathcal{X}) - 1) & \text{for } \gamma(\mathcal{X}) > 1, \\
\frac{g(\mathcal{X}) - 1}{\gamma(\mathcal{X}) - 1} & \text{for } \gamma(\mathcal{X}) = 1.
\end{cases}
\]

(3)

In this context, a major issue is to determine the possibilities for \( \mathcal{X} \), \( g \) and \( S \) when either \( |S| \) is close to the Stichtenoth bound (2), or \( |S| \) is close to the Nakajima bound (3).

Lehr and Matignon \cite{11} investigated the case where \( S \) fixes a point and were able to determine all curves \( \mathcal{X} \) with

\[
|S| > \frac{4}{(p - 1)^2} g^2,
\]

proving that \((4) \) only occurs when the curve is birationally equivalent over \( K \) to an Artin-Schreier curve of equation \( Y^q - Y = f(X) \) such that \( f(X) = XS(X) + cX \) where \( S(X) \) is an additive polynomial of \( K[X] \). Later on, Matignon and Rocher \cite{12} showed that the action of a \( p \)-subgroup of \( K \)-automorphisms \( S \) satisfying

\[
|S| > \frac{4}{(p^2 - 1)^2} g^2,
\]

corresponds to the étale cover of the affine line with Galois group \( S \cong (\mathbb{Z}/p\mathbb{Z})^n \) for \( n \leq 3 \). These results have been refined by Rocher, see \cite{14} and \cite{16}. The essential tools used in the above mentioned papers are ramification theory and some structure theorems about finite \( p \)-groups.

Curves close to the Nakajima bound are investigated in this paper. The main result is stated in the following theorem.

**Theorem 1.1.** Let \( S \) be a \( p \)-subgroup of the \( K \)-automorphism group \( \text{Aut}(\mathcal{X}) \) of an algebraic curve \( \mathcal{X} \) of genus \( g(\mathcal{X}) \geq 2 \) defined over an algebraically closed field \( K \) of characteristic \( p \geq 3 \). If

\[
|S| > 2(g(\mathcal{X}) - 1)
\]

(5)

then one of the following cases occurs:

(i) \( \gamma = 0 \) and the extension \( K(\mathcal{X})/K(\mathcal{X})^S \) completely ramifies at a unique place, and does not ramify elsewhere.

(ii) \( \gamma > 0 \), \( p = 3 \), \( \mathcal{X} \) is a general curve, \( S \) attains the Nakajima’s upper bound \( 3(\gamma - 1) \) and \( K(\mathcal{X}) \) is an unramified Galois extension of the function field of a general curve of genus 2 with equation \( Y^2 = cX^6 + X^4 + X^2 + 1 \) where \( c \in K^* \).

The analogous problem for 2-groups of automorphisms \( S \) in characteristic \( p = 2 \) was investigated in \cite{4}.

One may also ask how the above results may be refined when \( \text{Aut}(\mathcal{X}) \) is larger than \( S \). So far, this problem has been investigated for zero \( p \)-rank curves \( \mathcal{X} \) such that \( \text{Aut}(\mathcal{X}) \) fixes no point of \( \mathcal{X} \); see \cite{5, 6, 7}.

2 **Background and Preliminary Results**

Let \( \tilde{\mathcal{X}} \) be a non-singular model of \( K(\mathcal{X})^S \), that is, a projective non-singular geometrically irreducible algebraic curve with function field \( K(\mathcal{X})^S \), where \( K(\mathcal{X})^S \) consists of all elements of \( K(\mathcal{X}) \) fixed by every element in \( S \).
Usually, \( \tilde{X} \) is called the quotient curve of \( X \) by \( S \) and denoted by \( X/S \). The field extension \( \mathbb{K}(X)/\mathbb{K}(\tilde{X}) \) is Galois of degree \(|S|\).

Let \( \bar{P}_1, \ldots, \bar{P}_k \) be the points of the quotient curve \( \tilde{X} = X/S \) where the cover \( X \mapsto \tilde{X} \) ramifies. For \( 1 \leq i \leq k \), let \( L_i \) denote the set of points of \( X \) which lie over \( \bar{P}_i \). In other words, \( L_1, \ldots, L_k \) are the short orbits of \( S \) on its faithful action on \( X \). Here the orbit of \( P \in X \)

\[ o(P) = \{ Q \mid Q = P^g, \ g \in S \} \]

is long if \(|o(P)| = |S|\), otherwise \( o(P) \) is short. It may be that \( S \) has no short orbits. This is the case if and only if every non-trivial element in \( S \) is fixed–point-free on \( X \). On the other hand, \( S \) has a finite number of short orbits.

If \( P \) is a point of \( X \), the stabilizer \( S_P \) of \( P \) in \( S \) is the subgroup of \( S \) consisting of all elements fixing \( P \). For a non-negative integer \( i \), the \( i \)-th ramification group of \( X \) at \( P \) is denoted by \( S_P^{(i)} \) (or \( S_i(P) \) as in [19, Chapter IV]) and defined to be

\[ S_P^{(i)} = \{ g \mid \text{ord}_P(g(t) - t) \geq i + 1, \ g \in S_P \}, \]

where \( t \) is a uniformizing element (local parameter) at \( P \). Here \( S_P^{(0)} = S_P^{(1)} = S_P \).

Let \( \bar{g} \) be the genus of the quotient curve \( \tilde{X} = X/S \). The Hurwitz genus formula gives the following equation

\[ 2\bar{g} - 2 = |S|(2\bar{g} - 2) + \sum_{P \in X} d_P. \tag{6} \]

where

\[ d_P = \sum_{i \geq 0} (|S_P^{(i)}| - 1). \tag{7} \]

Let \( \gamma \) be the \( p \)-rank of \( X \), and let \( \bar{\gamma} \) be the \( p \)-rank of the quotient curve \( \tilde{X} = X/S \). The Deuring-Shafarevich formula, see [23] or [19, Theorem 11.62], states that

\[ \gamma - 1 = |S|(|\bar{\gamma} - 1| + \sum_{i=1}^k (|S| - \ell_i) \tag{8} \]

where \( \ell_1, \ldots, \ell_k \) are the sizes of the short orbits of \( S \). If \( G \) has no short orbits, that is, the Galois extension \( \mathbb{K}(X) \) of \( \mathbb{K}(\tilde{X}) \) is unramified, then \( G \) can be generated by \( \bar{\gamma} \) elements by Shafarevich’s theorem [20, Theorem 2], whereas the largest elementary abelian subgroup of \( G \) has rank at most \( \bar{\gamma} \) see [14, Section 4.7].

The following result is a special case of the classification of automorphism groups of genus 2 curves due to Igusa [9, Section 8] and refined in [2, Section 1]; see also [21, Lemma 1], and [3].

**Proposition 2.1.** Let \( G \) be the automorphism group of a genus 2 curve \( C \) in characteristic 3. If \( 3 \) divides \(|G|\), then either \( G \cong D_{12} \), or \( G \cong GL(2,3) \), and \( C \) is a non-singular model of an irreducible plane curve of equation \( Y^2 = cX^6 + X^4 + X^2 + 1 \) where \( c \in K^* \) with \( c = 1 \) when \( G \cong GL(2,3) \).

**Remark 2.2.** An equivalent equation for \( C \) in Proposition 2.1 is

\[ X(Y^3 - Y) - X^2 + c = 0, \tag{9} \]

and the linear map \((X, Y) \mapsto (X, Y + 1)\) is an automorphism of \( C \).
From group theory we use the following results, see for instance [24].

**Proposition 2.3.** Let $G$ be a non-abelian group of order 27. Then either $G \cong C_9 \times C_3$ or $G \cong UT(3,3)$ where $UT(r,3)$ denotes the group of upper-triangular unipotent $r \times r$ matrices over the field with three elements.

From Projective geometry, the following known result is used. For the sake of completeness we include a proof.

**Lemma 2.4.** In the $r$-dimensional projective space $PG(r, \mathbb{K})$ over an algebraically closed field $\mathbb{K}$ of characteristic $p$, let $S$ be a finite $p$-subgroup of $PGL(r+1, \mathbb{K})$. If $r \geq 2$ then $S$ preserves a flag

$$\Pi_0 \subset \Pi_1 \subset \ldots \subset \Pi_{r-1}$$

where $\Pi_i$ is an $i$-dimensional projective subspace of $PG(r, \mathbb{K})$.

**Proof.** In $PG(2, \mathbb{K})$, any $p$-subgroup of $PGL(3, \mathbb{K})$ fixes an incident point-line pair. Hence, the claim holds for $r = 2$. We prove it for $r \geq 3$ by induction on $r$.

We first show that $S$ fixes a point of $PG(r, \mathbb{K})$. Take a non-trivial element $s$ in the center $Z(S)$ of $S$. Since $\mathbb{K}$ is algebraically closed, $s$ fixes some points of $PG(r, \mathbb{K})$. If $s$ has a unique fixed point, then $S$ itself fix that point. Therefore, we may assume that $s$ has more than one fixed point. Since $s$ has order a power of $p$, the set of its fixed points is a proper projective subspace $\Pi$ of $PG(r, \mathbb{K})$. From $s \in Z(S)$ it follows that $S$ preserves $\Pi$. By the inductive hypothesis, $S$ fixes a point in $\Pi$, and hence in $PG(r, \mathbb{K})$.

Now, let $P \in PG(r, \mathbb{K})$ be a fixed point of $S$. The linear system of hyperplanes through $P$ is a projective space $PG(r - 1, \mathbb{K})$ over $\mathbb{K}$, and $S$ acts on $PG(r - 1, \mathbb{K})$, not necessarily faithfully, as a $p$-subgroup $S_1$ of $PGL(r, \mathbb{K})$. From what we have already proven, $S_1$ fixes a point $Q$ of $PG(r - 1, \mathbb{K})$. Therefore, $Q$ regarded as a hyperplane $\Pi_{r-1}$ of $PG(r, \mathbb{K})$ through $P$ is preserved by $S$. From the inductive hypothesis on $r$, $S$ has an invariant flag $\Pi_0 \subset \Pi_1 \subset \ldots \subset \Pi_{r-2}$. This together with $\Pi_{r-1}$ gives an $S$-invariant flag in $PG(r, \mathbb{K})$. □

### 3 The action of $S$ on $\mathcal{X}$

In this section, $\mathcal{X}$ stands for a curve which satisfies the hypotheses of Theorem 1.1 but does not have the property given in (i) of Theorem 1.1.

**Proposition 3.1.** The following results hold:

(I) $p = 3$,

(II) $S$ fixes no point in $\mathcal{X}$,

(III) $\gamma(\mathcal{X}) > 0$.

**Proof.** (II) holds, otherwise comparing (2) with (1) would yield (i) of Theorem 1.1. From this (III) follows by [10, Lemma 11.129]. In particular, (3) applies, and comparing (5) with (3) gives (I). □

**Proposition 3.2.** $\mathcal{X}$ is an ordinary curve with $g(\mathcal{X}) - 1 = \frac{1}{2}|S|$. Moreover, $S$ has exactly two short orbits on $\mathcal{X}$, both of length $\frac{1}{2}|S|$, and if $|S| > 3$ then the identity is the unique element in $S$ fixing every point of $\Omega_1 \cup \Omega_2$. 

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Proof. Let \( g = g(\mathcal{X}) \) and \( \gamma = \gamma(\mathcal{X}) \). Our hypothesis \((\text{E})\) yields \( \gamma \geq 2 \) by \((\text{B})\). Let \( \bar{\gamma} \) be the 3-rank of the quotient curve \( \mathcal{X} = \mathcal{X}/S \). From \((\text{B})\),

\[
\gamma - 1 = \bar{\gamma}|S| - |S| + k \sum_{i=1}^k (|S| - \ell_i) = (\bar{\gamma} + k - 1)|S| - k \sum_{i=1}^k \ell_i \geq (\bar{\gamma} + \frac{2}{3}k - 1)|S|, \tag{10}
\]

where \( \ell_1, \ldots, \ell_k \) are the sizes of the short orbits of \( S \).

If no such short orbits exist, then \( \gamma - 1 = |S|/\bar{\gamma} - 1 \) whence \( \bar{\gamma} > 1 \) by \( \gamma \geq 2 \). Therefore, \( |S| \leq \gamma - 1 \leq g - 1 \) contradicting \((\text{B})\).

Hence \( k \geq 1 \), and if \( \bar{\gamma} \geq 1 \) then \((\text{10})\) yields that \( |S| \leq \frac{3}{2}(\gamma - 1) \leq \frac{3}{2}(g - 1) \) contradicting \((\text{B})\). So, \( \bar{\gamma} = 0 \), and \((\text{10})\) together with \((\text{B})\) imply that \( k < \frac{g}{2} \) whence \( 1 \leq k \leq 2 \). The case \( k = 1 \) cannot actually occur by \((\text{10})\).

Therefore, \( \bar{\gamma} = 0 \) and \( k = 2 \). Let \( \Omega_1 \) and \( \Omega_2 \) be the short orbits of \( S \), and let \( \ell_i = |\Omega_i| \) for \( i = 1, 2 \). Then \((\text{10})\) reads

\[
\gamma - 1 = |S| - (\ell_1 + \ell_2) \tag{11}
\]

whence \( |S| > 2(\gamma - 1) = 2|S| - 2(\ell_1 + \ell_2) \). Also, \( \ell_1 + \ell_2 < |S| \). Write \( |S| = 3^h, \ell_1 = 3^m, \ell_2 = 3^n \) with \( h > m \geq r \). Then \( 3^m + 3^n < 3h < 2(3^m + 3^n) \). Here \( r > 0 \) by (II) of Proposition 3.1. Therefore \( 3^{m-r} + 1 < 3^{h-r} < 2(3^{m-r} + 1) \). Since \( h > m \geq r \), this yields \( m = r \), that is, \( \ell_1 = \ell_2 \), and \( h = r + 1 \). In other words, \( \ell_1 = \ell_2 = \frac{1}{3}|S| \). From \((\text{11})\), \( \gamma - 1 = \frac{1}{3}|S| \).

Let \( \bar{g} \) be the genus of the quotient curve \( \bar{\mathcal{X}} = \mathcal{X}/S \). The Hurwitz genus formula, \((\text{E})\) and \((\text{F})\), applied to \( S \) gives

\[
2g - 2 = |S|(2\bar{g} - 2) + \frac{2}{3}|S|(4 + k_1 + k_2) \tag{12}
\]

where, for a point \( P_i \in \Omega_i \), \( k_i \) is the smallest non-negative integer such that \( |S|^{g_i} \equiv 1 \). Suppose on the contrary that \( \mathcal{X} \) is not an ordinary curve, that is, \( g > \gamma \). Then \( k_1 + k_2 \geq 1 \). From \((\text{12})\), \( 2g - 2 \geq -2|S| + \frac{10}{3}|S| = \frac{4}{3}|S| \) contradicting \((\text{B})\).

Assume that a non-trivial element \( s \) of \( S \) fixes \( \Omega_1 \cup \Omega_2 \) pointwise. From the Deuring-Shafarevic formula \((\text{S})\) applied to \((s)\),

\[
|s| = \frac{1}{3}|S|(2|s| - 3),
\]

which is only possible for \(|s| = |S| = 3\). \(\square\)

From now on, \( \Omega_1 \) and \( \Omega_2 \) denote the short orbits of \( S \) on \( \mathcal{X} \). By the second claim of Proposition 3.2,

\[
|S_P| = 3 \text{ for } P \in \Omega_1 \cup \Omega_2. \tag{13}
\]

Proposition 3.3. If \( S \) is abelian then either \( |S| = 3 \), or \( |S| = 9 \) and \( S \) is elementary abelian.

Proof. For a point \( P \in \Omega_1 \), the stabilizer \( S_P \) of \( P \) in \( S \) is a subgroup of order 3. Since \( S \) is abelian, \( S_P \) fixes every point in \( \Omega_1 \). Let \( \gamma^* \) be the 3-rank of the quotient curve \( \mathcal{X}/S_P \). From \((\text{S})\) and Proposition 3.2,

\[
\frac{1}{3}|S| = \gamma - 1 \geq 3(\gamma^* - 1) + \frac{2}{3}|S| \geq \frac{2}{3}|S| - 3
\]

whence \( |S| = 9 \) and \( g = \gamma = 4 \). Assume on the contrary that \( S \) is cyclic. For a point \( Q \in \Omega_2 \) the stabilizer \( S_Q \) is a subgroup of \( S \) of order 3. Since \( S \) is cyclic, it has only one subgroup of order 3. Therefore \( S_P = S_Q \), and

\[
\frac{1}{3}|S| = \gamma - 1 \geq 3(\gamma^* - 1) + \frac{2}{3}|S| + \frac{2}{3}|S| \geq \frac{4}{3}|S| - 3,
\]

which implies \(|S| = 1\), a contradiction. \(\square\)
Proposition 3.4. Assume that \(|S| \geq 9\). Let \(N\) be a non-trivial normal subgroup of \(S\). Then either \(N\) is semiregular on \(X\), or \(S = N \rtimes \langle \varepsilon \rangle\) with \(\varepsilon^3 = 1\).

Proof. The assertion trivially holds for \(|S| = 9\). Assume that some non-trivial element in \(N\) fixes point \(P\). From the Hurwitz genus formula applied to \(N\), we have \(\frac{1}{2}|S| > |N|/(\bar{g} - 1)\) where \(\bar{g}\) is the 3-rank of the quotient curve \(\bar{X} = X/N\). Let \(S\) be the automorphism group of \(\bar{X}\) induced by \(S\). Then \(|\bar{S}|/|N| = |S|\) and hence \(\frac{1}{2}|S| > \bar{g} - 1\). Assume that \(\bar{g} \geq 2\). Then (I) holds for \(\bar{S}\). If (II) were also true for \(\bar{S}\), then Proposition 3.2 would imply \(|\bar{S}| = 3(\bar{g} - 1)\). So, \(\bar{S}\) fixes a point in \(\bar{X}\). This remains true when \(\bar{g} = 0\). Let \(\bar{Q} \in \bar{X}\) be such a fixed point. Then the orbit \(O\) of \(N\) consisting of all points of \(\bar{X}\) lying over \(\bar{Q}\) is also an orbit of \(S\) for \(\Omega_1\) and \(\Omega_2\) are the only short orbits of \(S\), this verifies that either \(\Omega\) coincides with one of them, say \(\Omega_1\). Therefore, \(|N| = \frac{1}{3}|S|\). The stabilizer \(\varepsilon\) of a point \(R \in \Omega_2\) on \(S\) has order 3 and \(\varepsilon \notin N\). Therefore \(S = N \rtimes \langle \varepsilon \rangle\). We are left with the case \(\bar{g} = \bar{\gamma} = 1\). Let \(\Omega_1, \ldots, \Omega_m\) be the short orbits of \(N\). Since the stabilizer \(N_Q\) of any point \(Q \in \Omega_i\) has order 3, Proposition 3.3 applied to \(\bar{N}\) gives

\[
\frac{1}{2}|S| = |\gamma - 1| = \frac{2}{3}|N|m
\]

whence \(|S| = 2|N|m\). But this is impossible as \(|S|\) is a power of 3.

Proposition 3.5. If \(|S| > 9\) then the center \(Z(S)\) of \(S\) is semiregular on \(X\).

Proof. Since \(Z(S)\) is a normal subgroup of \(S\), Proposition 3.3 applies to \(Z(S)\). The case \(S = Z(S) \rtimes \langle \varepsilon \rangle\) cannot actually occur since this semidirect product would be direct and \(S\) would be abelian contradicting Proposition 3.3.

Proposition 3.6. Let \(N\) be a normal subgroup of \(G\) such that \(|N| \leq \frac{1}{3}|S|\). Then the quotient curve \(\bar{X} = X/N\) with \(S = S/N\) and \(g(\bar{X}) = 1\) satisfies the hypotheses of Theorem 1.1 but does not have the property given in (i) of Theorem 1.1.

Proof. By Proposition 3.3, the Galois \(p\)-extension of \(\bar{X}\) with Galois group \(\bar{G}\) is unramified. Therefore, the Deuring-Shafarevich formula applied to \(\bar{G}\) gives \(|g - 1| = |\bar{G}|(\bar{g} - 1)\).

Since the center of any \(p\)-group is non-trivial, a straightforward inductive argument on \(|S|\) depending on Proposition 3.6 gives the following result.

Proposition 3.7. If there exists a curve which satisfies the hypothesis of Theorem 1.1 for \(|S| = 3^k\) but does not have the property (i), then for any \(1 \leq j < k\) there also exists a curve which satisfies the hypothesis of Theorem 1.1 for \(|S| = 3^j\) but does not have the property (i).

Proposition 3.8. Let \(N\) be a non-trivial normal subgroup of \(S\). If the factor group \(S/N\) is a abelian then either \(|N| = \frac{1}{3}|S|\) or \(|N| = \frac{1}{9}|S|\), and in the latter case, \(S/N\) is an abelian group.

Proof. The assertion is a corollary of Propositions 3.4 and 3.6.

Proposition 3.9. If \(|S| \geq 9\) then the following hold.

(i) \(\Phi(S) = S'\).

(ii) \(|\Phi(S)| = \frac{1}{9}|S|\).

(iii) \(S\) contains exactly four maximal subgroups, each being a normal subgroup of \(S\) of index 3.
(iv) Exactly two of the four maximal subgroups of $S$ are semiregular on $X$.

(v) $S$ can be generated by two elements.

Proof. From Proposition 3.10 either $|S'| = \frac{1}{4} |S|$, or $|S'| = \frac{1}{9} |S|$. In the former case, $S$ is cyclic by Hilfssatz 7.1.b] but this contradicts Proposition 3.3. Therefore, $|S'| = \frac{1}{9} |S|$. Since $S/S'$ is elementary abelian by Proposition 3.8 (i) holds. From this (ii) follows. Let $\varphi$ be the natural homomorphism $S \rightarrow S/\Phi(S)$. Since every maximal subgroup of $S$ contains $\Phi(S)$, there is a one-to-one correspondence between the maximal subgroups of $S$ and the subgroups of $S/\Phi(S)$. By (ii), $S/\Phi(S)$ is an elementary abelian group of order 9 with exactly four proper subgroups. Therefore there are exactly four maximal subgroups in $S$. Also, the subgroups of $S/\Phi(S)$ are normal, and hence each of the four maximal subgroups of $S$ is normal, as well. Furthermore, the four maximal subgroups of $S/\Phi(S)$ partition the set of non-trivial elements of $S/\Phi(S)$. Hence every element of $S \setminus \Phi(S)$ belongs to exactly one of the four maximal subgroups of $S$. Take a point $P \in \Omega_1$, and let $M_1$ be the maximal subgroup of $S$ containing $S_P$. Since $M$ is a normal subgroup of $S$ and $\Omega_1$ is an $S$-orbit, this yields that $M$ contains $S_Q$ for every $Q \in \Omega_1$. Repeating the above argument for a point in $\Omega_2$ shows that a maximal normal subgroup contains the stabilizer of each point in $\Omega_2$. From the last claim of Proposition 3.2 these two maximal subgroups are distinct. Therefore, the remaining two maximal subgroups are both semiregular on $X$.

Finally, (i) together with the Burnside fundamental theorem, Chapter III, Satz 3.15] imply that $S$ can be generated by two elements.

From now on, the following notation is used: For $i = 1, 2, M_i$ denotes the maximal normal subgroup of $S$ containing the stabilizer of a point of $\Omega_i$ while $M_3$ and $M_4$ stand for the semiregular maximal subgroups of $S$, respectively. Proposition 3.4 has the following corollary.

Proposition 3.10. Neither $M_1$ nor $M_2$ is cyclic.

Proposition 3.11. Every normal subgroup of $S$ whose order is at most $\frac{1}{9} |S|$ is contained in $\Phi(S)$.

Proof. Let $N$ be a normal subgroup of $S$. From Chapter III, Hilfssatz 3.4.a], $\Phi(S)N/N$ is a subgroup of $\Phi(S/N)$. From Propositions 3.10 and Proposition 3.9 applied to $\chi = \chi/N$, we have $|\Phi(S/N)| = \frac{1}{9} |S|/|N|$. Since $\Phi(S)(\Phi(S) \cap N) \cong \Phi(S)N/N$, this yields $|N| \leq |\Phi(S) \cap N|$. Therefore, if $|N| \leq |\Phi(S)|$ then $N$ is contained in $\Phi(S)$.

Proposition 3.12. If $M_i$ is abelian for some $1 \leq i \leq 4$, then $S$ has maximal class.

Proof. By (i) of Proposition 3.9 the assertion follows from an elementary result, see for instance Theorem 5.2.

It should be noted that $p$-groups of maximal class have intensively investigated, see and 11.

Proposition 3.13. If $|S| > 3$ then $\chi$ is not hyperelliptic.

Proof. Since the length of any $S$-orbit in $\chi$ is divisible by 3, the number of distinct Weierstrass points of $\chi$ is also divisible by 3. On the other hand, a hyperelliptic curve of genus $g$ defined over a field of zero or odd characteristic has as many as $2g + 2$ Weierstrass points, see Theorem 7.103]. Therefore, if $\chi$ were hyperelliptic, both numbers $2g + 2$ and $2g - 2 - \frac{2}{3} |S|$ would be divisible by 3, a contradiction.

Proposition 3.14. If $|S| = 27$ then $S \cong UT(3,3)$.
Proof. If $|S| = 27$ then either $S = C_9 \times C_3$, or $S$ is isomorphic to the group $UT(3, 3)$ of all upper-triangular unipotent $3 \times 3$ matrices over the field with three elements. Since the group $C_9 \times C_3$ has only one non-cyclic maximal subgroup, the assertion follows from Proposition 3.10.

Proposition 3.15. If $|S| \geq 81$ then $S$ is not metacyclic.

Proof. We show that $S$ has a normal subgroup of index 27. This is certainly true when $M_3$ is cyclic. Otherwise, $M_3$ can be generated by two elements, and its Frattini subgroup $\Phi(M_3)$ has index 27 in $S$. Since $\Phi(M_3)$ is a characteristic subgroup of $M_3$ and $M_3$ is a normal subgroup of $S$, $\Phi(M_3)$ has the required property. From Proposition 3.6 applied to $N = \Phi(M_3)$ shows that $\bar{S} = S/\Phi(M_3)$ is a subgroup of $\text{Aut}(\bar{X})$ with $\bar{X}$ the quotient curve of $\bar{X} = X/\Phi(M_3)$. Since $\Phi(M_3)$ is semiregular on $X$ by Proposition 3.1, $\bar{X}$ has genus 10. As $|S| = 27$, Proposition 3.14 implies that $\bar{S} \cong UT(3, 3)$. In particular, $\bar{S}$ is not metacyclic. Since every factor group of a metaacyclic group is still metacyclic, the assertion follows.

4 Small genera

4.1 Cases $|S| = 3, 9$

We prove that if $X$ satisfies the hypotheses of Theorem 1.1 for $|S| = 3$ then (ii) holds. For this case, our hypothesis (i) yields $g = 2$. From (i), every automorphism of $\text{Aut}(X)$ of order 3 has two fixed points on $X$. Therefore, (i) of Theorem 1.1 cannot occur, and the assertion follows from Proposition 2.1.

From now on, $|S| = 9$ and $X$ is a curve satisfying the hypotheses of Theorem 1.1 but does not have the property given in (i) of Theorem 1.1. We prove that then (iii) holds. From Propositions 3.2 and 3.3, $X$ is an ordinary curve of genus $g = 4$ with an elementary abelian subgroup $S$ of $\text{Aut}(X)$ of order 9.

Let $N$ be the kernel of the permutation representation of $S$ on $\Omega_1 \cup \Omega_2$. If $N$ is not trivial then it has order 3, and the Hurwitz genus formula (6) and (7) applied to $N = \Phi(M_3)$ gives $6 = 2(g-1) \geq 6(\bar{g}-1) + 24$. Therefore $S$ acts on $\Omega_1 \cup \Omega_2$ faithfully.

By Proposition 3.10, $X$ is assumed to be a canonical curve embedded in $PG(3, K)$. Then $S$ extends to a subgroup of $PG(3, K)$ which preserves $X$ and acts on $X$ faithfully.

According to Lemma 2.4, choose the projective coordinate system $(X_0 : X_1 : X_2 : X_3)$ in $PG(3, K)$ in such a way that $S$ preserves the canonical flag

$$P_0 \subset \Pi_1 \subset \Pi_2$$

where $P_0 = (1 : 0 : 0 : 0)$, $\Pi_1$ is the line through $P_0$ and $P_1 = (0 : 1 : 0 : 0)$ while $\Pi_2$ is the plane of equation $X_3 = 0$. Here $P_0 \not\in X$, since $S$ fixes no point in $X$. Moreover, $\Pi_2 \cap X = \Omega_1 \cup \Omega_2$. In fact, for any point $R \in \Pi_2 \cap X$, Proposition 3.2 implies that the $S$-orbit of $R$ has size 9 unless $R \in \Omega_1 \cup \Omega_2$. On the other hand $S$ preserves $\Pi_2 \cap X$, and this implies that the $S$-orbit of $R$ cannot exceed 6.

Lemma 4.1. Both $\Omega_1$ and $\Omega_2$ consist of three collinear points.

Proof. Assume on the contrary that $\Omega_1$ is a triangle. Take $g \in S$ such that $g$ fixes each vertex of $\Omega_1$. Since $g$ is a projectivity of $PG(3, K)$ it fixes $\Pi_2$ pointwise. As $\Pi_2$ also contains $\Omega_2$, $g$ must fix $\Omega_2$ pointwise. But this is impossible as $S$ acts on $X$ faithfully.

As a corollary, $I(R, X \cap \Pi_2) = 1$ for every point $R \in \Omega_1 \cup \Omega_2$. For $i = 1, 2$, let $r_i$ denote the line containing $\Omega_i$. Their common point is fixed by $S$, and may be chosen for $P_0$. Let $M$ be the subgroup of $S$ which preserves every line through $P_0$. Since $\deg X = 6$, no line meets $X$ in more than six distinct points.
Therefore, either $|M| = 1$ or $|M| = 3$. In the latter case, $M$ is an elation group of order 3 with center $P_0$. If $\Delta$ is its axis then every point in $\Delta \cap X$ is fixed by $M$. Therefore $\Delta$ is not $\Omega_2$ and contains either $r_1$ or $r_2$. Since $S$ is abelian, it preserves $\Delta$ and hence every plane through $r_1$. But then every $S$-orbit has length at most 3. A contradiction with Proposition 3.1. Hence $M$ is trivial.

Since $P_0 \notin X$, the linear system $\Sigma$ of all planes through $P_0$ cuts out on $X$ a linear series without fixed point. Therefore this effective linear series has dimension 2 and degree 6, and is denoted by $g_2^6$.

It might happen that $g_2^6$ is composed of an involution, and we investigate such a possibility. From Section 7.4, there is a curve $Z$ whose function field $\mathbb{K}(Z)$ is an $S$-invariant proper subfield of $\mathbb{K}(X)$. Since no non-trivial element in $S$ fixes every line through $P_0$ and hence every plane through $P_0$, $S$ acts on $Z$ faithfully.

As the genus of $Z$ is less than 4, applying (3) to $Z$ gives $\gamma(Z) = 0$. Therefore, every non-trivial element in $S$ has a unique fixed point $\bar{T}$, see [10] Lemma 11.129]. From this, the support of the divisor of $K(Z)$ lying over $\bar{T}$ contains the points in $\Omega_1 \cup \Omega_2$. Therefore, the line through $P_0$ and a point in $\Omega_1 \cup \Omega_2$ must contain all the points in $\Omega_1 \cup \Omega_2$. But this would imply that $r_1 = r_2$, a contradiction.

Therefore, $g_2^6$ is simple and without fixed point. The projection of $X$ from $P_0$ is an irreducible plane curve $C$ of degree 6 and genus 4 with two triple points $R_1$ and $R_2$ arising from $\Omega_1$ and $\Omega_2$, respectively. Here $C$ and $X$ are birationally equivalent, and $S$ is a subgroup of $PGL(3, \mathbb{K})$ preserving $C$. For $i = 1, 2$, a non-trivial projectivity $s_i \in S$ fixing $\Omega_i$ pointwise acts on $C$ fixing the point $R_i$.

Choose the projective coordinate system $(X_0 : X_1 : X_2)$ in $PG(2, \mathbb{K})$ so that $R_1 = (0 : 0 : 1)$ and $R_2 = (0 : 1 : 0)$. In affine coordinates $(X, Y)$ with $X = X_1/X_0$, $Y = X_2/X_0$, an equation of $C$ is $f = 0$ with an irreducible polynomial $f \in \mathbb{K}[X, Y]$ of degree six. W.l.o.g. the origin $O = (0, 0)$ is the common point of two tangents to $C$, say $t_1$ at $R_1$ and $t_2$ at $R_2$. Furthermore, $s_1(O) = (\lambda, 0), s_2(O) = (0, \mu)$ with $\lambda, \mu \in \mathbb{K}^*$, and $\lambda = \mu = 1$ may be assumed. Thus $s_1 : (X, Y) \mapsto (X + 1, Y)$ and $s_2 : (X, Y) \mapsto (X, Y + 1)$. Hence

\[ f(X + 1, Y) = f(X, Y), \quad f(X, Y + 1) = f(X, Y). \]

Since $R_1$ is a triple point of $C$, there exist $h_0, h_1, h_2, h_3 \in \mathbb{K}[Y]$ such that

\[ f(X, Y) = h_3X^3 + h_2X^2 + h_1X + h_0 = 0, \]

where $\deg h_0 \leq 2$ by the particular choice of $t_2$. From this and (14), the polynomial

\[ f(X + 1, Y) - f(X, Y) = h_3 - h_2X + h_2 + h_1 \]

vanishes at every affine point of $C$. Since $C$ is not rational, this is only possible when [10] is the zero polynomial, that is, $h_2 = h_3 + h_1 = 0$. Thus $f(X, Y) = h_3(X^3 - X) + h_0$. The second mixed partial derivate is $f_{XY} = -dh_3/dY$. Similarly, as $R_2$ is a triple point of $C$ there exist $k_0, k_3 \in \mathbb{K}[X]$ with $\deg k_0 \leq 2$ such that $f(Y, X) = k_3(Y^3 - Y) + k_0$. Since $f_{XY} = f_{YX}$, this yields $dh_3/dY = dk_3/dX$, whence $dh_3/dY$ and $dk_3/dX$ both have degree 0. Thus

\[ h_3 = c_3Y^3 + c_1Y + c_0, \quad k_3 = d_3X^3 + d_1X + d_0 \]

where $c_0, c_1, c_3, d_0, d_1, d_3 \in \mathbb{K}$. Therefore

\[ (c_3Y^3 + c_1Y + c_0)(X^3 - X) + h_0 = (d_3X^3 + d_1X + d_0)(Y^3 - Y) + k_0. \]

Comparison of the coefficients of $X^3$ shows that $c_3 = -c_1, c_0 = 0$. Similarly, $d_3 = -d_1, d_0 = 0$. Thus

\[ f(X, Y) = c(X^3 - X)(Y^3 - Y) + h_0 = d(Y^3 - Y)(X^3 - X) + k_0 \]


where \( c, d \in \mathbb{K} \). From this \((c - d)(X^3 - X)(Y^3 - Y) = k_0 - h_0 \) whence \( c = d \) and \( k_0 = h_0 = u \) with \( u \in \mathbb{K}^* \). Therefore
\[
f(X, Y) = (X^3 - X)(Y^3 - Y) + c = 0
\]
where \( c \in \mathbb{K}^* \). This ends the proof of Theorem 3.11 for \(|S| = 9\).

### 4.2 Case \(|S| = 27\)

We exhibit an explicit example. Let \( \mathcal{X} \) be a non-singular model of the irreducible curve \( \mathcal{Y} \) embedded in \( PG(3, \mathbb{K}) \) defined by the affine equations

1. \((X^3 - X)(Y^3 - Y) - 1 = 0\);
2. \(Z^3 - Z + X^3Y - XY^3 = 0\).

A straightforward Magma computation shows that \( g(\mathcal{X}) = \gamma(\mathcal{X}) = 10\). Moreover, both maps

\[g: (X, Y, Z) \mapsto (X + 1, Y, Z + Y) \quad h: (X, Y, Z) \mapsto (X, Y - 1, Z + X)\]

are in \( \text{Aut}(\mathcal{X}) \). They generate a non-abelian group \( S \) of order 27 and exponent 3. Therefore \( S \cong UT(3, 3) \). Actually, \( \text{Aut}(\mathcal{X}) \) is larger than \( S \) since it also contains \( r: (X, Y, Z) \mapsto (Y, X, Z) \). This shows that \( \mathcal{X} \) satisfies the hypotheses of Theorem 3.11 with \( S \cong UT(3, 3) \).

### 4.3 Case \(|S| = 81\)

**Proposition 4.2.** For \(|S| = 81\) there are only two possibilities for \( S \), namely

(a) \( S \cong S(81, 7) \) where \( S(81, 7) = C_3 \wr C_3 \) is the Sylow 3-subgroup of the symmetric group of degree 9, moreover \( M_1 \cong C_3 \times C_3 \times C_3 \), \( M_2 \cong UT(3, 3) \), \( M_3 \cong M_4 \cong C_9 \times C_3 \).

(b) \( S \cong S(81, 9) = \langle a, b, c | a^9 = b^3 = c^3 = 1, ab = ba, cac^{-1} = ab^{-1}, cbc^{-1} = a^3b \rangle \) with exactly 62 elements of order 3; moreover \( M_1 \cong M_2 \cong M_3 \cong UT(3, 3) \), \( M_4 \cong C_9 \times C_3 \).

**Proof.** There exist exactly seven groups of order 81 generated by two elements, namely \( S(81, i) \) with \( i = 1, \ldots, 7 \), and each of them has an abelian normal subgroup of index 3. By Proposition 4.2, \( S \) is of maximal class. There are four pairwise non-isomorphic groups of order 81 and maximal class, namely (a), (b) and

(c) \( S(81, 8) \cong \langle a, b, c | a^9 = b^3 = c^3 = 1, ab = ba, cac^{-1} = ab, cbc^{-1} = a^3b \rangle \) with 26 elements of order 3;

(d) \( S(81, 10) \cong (C_9 \times C_3) \times C_3 \) with 8 elements of order 3.

One of the four maximal normal subgroups of \( S(81, 8) \) is isomorphic to \( U(3, 3) \) and hence it contains all elements of order 3. On the other hand, (iv) of Proposition 3.9 yields that two of the maximal normal subgroups of \( S \), namely \( M_1 \) and \( M_2 \), have non-trivial 1-point stabilizer in \( \Omega_1 \) and \( \Omega_2 \), respectively. Hence, both must have an element of order 3 not contained in \( \Phi(S) \). Since \( M_1 \cap M_2 = \Phi(S) \), these elements are not in the same maximal normal subgroup. This contradiction shows that (c) cannot actually occur in our situation. Regarding \( S(81, 10) \), all elements of order 3 lie in \( \Phi(S) \) as \( \Phi(S) \) is an elementary abelian group of order 9. But this is impossible in our situation since \( M_1 \) must have an element of order 3 not in \( \Phi(S) \) by Propositions 3.9 and 3.11.

□
We exhibit an explicit example. Let \( X \) be a non-singular model of the irreducible curve \( Y \) embedded in \( PG(4, K) \) defined by the affine equations

(i) \((X^3 - X)(Y^3 - Y) - 1 = 0;\)

(ii) \(X(U^3 - U) - 1 = 0;\)

(iii) \((X + 1)(V^3 - V) - 1 = 0.\)

A straightforward Magma computation shows that \( g(X) = \gamma(X) = 28.\) Furthermore, each of the following four maps

\[
g_1 : (X, Y, U, V) \mapsto (X, Y + 1, U, V), \quad g_2 : (X, Y, U, V) \mapsto (X, Y, U + 1, V),
\]

\[
g_3 : (X, Y, U, V) \mapsto (X, Y, U, V + 1), \quad g_4 : (X, Y, U, V) \mapsto (X + 1, Y, V, -Y - U - V),
\]

are in \( \text{Aut}(X) \). They generate a non-abelian group \( S \) of order 81 isomorphic to \( S(81, 7) \).

4.4 Case \( |S| = 243 \)

**Proposition 4.3.** If \( |S| = 243 \) and \( S \) has a maximal abelian subgroup, then there are only two possibilities for \( S \), namely

(i) \( S \) is isomorphic to the unique group \( S(243, 26) \) of order 243 with 170 elements of order 3, moreover \( M_2 \cong M_3 \cong M_4 \cong S(81, 9) \), and \( M_1 \cong C_9 \times C_9 \),

(ii) \( S \) is isomorphic to the unique group \( S(243, 28) \) of order 243 with 116 elements of order 3, moreover \( M_1 \cong M_2 \cong S(81, 9) \) while \( M_3 \cong S(81, 4) \) and \( M_4 \cong S(81, 10) \).

**Proof.** There exist exactly six pairwise non-isomorphic groups of order 81 and maximal class, namely (i), (ii) and \( S(243, 25) \) with 62 elements of order 3; \( S(243, 27) \) with 8 elements of order 3; \( S(243, 29) \) with 8 elements of order 3; \( S(243, 30) \) with 62 elements of order 3.

One of the four maximal normal subgroups of \( S(243, 28) \) (and of \( S(243, 30) \)) is isomorphic to \( S(81, 8) \) and hence it contains all elements of order 3. The argument in the proof of Proposition 4.2 ruling out possibility (c) also works in this case. Therefore, neither \( S \cong S(243, 25) \) nor \( S \cong S(243, 28) \) is possible. Regarding \( S(243, 27) \) and \( S(243, 29) \), we may use the argument from the proof of Proposition 4.2 that ruled out possibility (d). Therefore, \( S \cong S(243, 25) \) and \( S \cong S(243, 28) \) cannot occur in our situation.

5 Infinite Family of Examples

Let \( C \) be a general curve of genus 2 as given in Remark 2.1 with function field \( F = K(X) = K(x, y) \) where

\[
x(y^3 - y) - x^2 + c = 0. \tag{16}
\]

For a positive integer \( N \), let \( F_N \) be the largest unramified abelian extension of \( F \) of exponent \( N \); that is, \( F_N|F \) has the following three properties:

(i) \( F_N|F \) is an unramified Galois extension;

(ii) \( F_N \) is generated by all function fields which are cyclic unramified extensions of \( F \) of degree \( p^N \),
(iii) $\text{Gal}(F_N|F)$ is abelian and $u^{3^N} = 1$ for every element $u \in \text{Gal}(F_N|F)$.

From classical results due to Schmid and Witt [17], $\deg(F_N|F) = 3^N$ and $\text{Gal}(F_N|F)$ is the direct product of two copies of the cyclic group of order $3^N$. Let $X$ be the curve such that $F_N = \mathbb{K}(X)$. Since $F_N$ is an unramified extension of $F$, from the Deuring-Shafarevic formula [5] we have that $\gamma(X) - 1 = 3^{2N}(2 - 1) = 3^N$.

Our aim is to prove that $\text{Aut}(X)$ contains a $3$-group of order $3^{2N+1}$.

Let $\mathbb{K}(x)$ be rational the subfield of $F$ generated by $x$. Obviously, $\mathbb{K}(x)$ is a subfield of $F_N$ and we are going to consider the Galois closure $M$ of $F_N|\mathbb{K}(x)$. Let $M = \mathbb{K}(\mathcal{Y})$ where $\mathcal{Y}$ is an algebraic curve defined over $\mathbb{K}$. Take any $\mu \in \text{Gal}(M|\mathbb{K}(x))$. Then $\mu$ is a $\mathbb{K}$-automorphism of $\mathcal{Y}$ fixing $x$. Let $v = \mu(y)$. Since $\mu(x(y^3 - y) - x^2 + c) = x(v^3 - v) - x^2 + c$, from [10]

\[ x(v^3 - v) - x^2 + c = 0. \]

This together with [10] yield that either $v = y$ or $v = y \pm 1$. In both cases $v \in F$. Therefore, $\text{Gal}(M|\mathbb{K}(x))$ viewed as a subgroup $S$ of $\text{Aut}(\mathcal{Y})$ preserves $F$. From the definition of $F_N$, this implies that $S$ also preserves $F_N$, that is $S$ is a subgroup of $\text{Aut}(X)$. Since $F_N \subseteq M$, we have that

\[ [F_N : \mathbb{K}(x)] \leq [M : \mathbb{K}(x)] = |S|. \]

Furthermore,

\[ [F_N : \mathbb{K}(x)] = [F_N : F][F : \mathbb{K}(x)] = 3^{2N}3 = 3^{2N+1}. \]

On the other hand $|S| > 3^{2N+1}$ is impossible by the Nakajima’s bound.

As a corollary, $\text{Aut}(X)$ has a subgroup $S$ so that the Hasse-Witt invariant $\gamma(X)$ is equal to $|S|/3$.

Our construction together with Proposition 3.7 provides a curve of type (ii) in Theorem 1.1, for every power of $3$.

In particular, if $N = 1$ then the resulting curve is isomorphic to that given in Subsection 3.7. For $N = 2$, we have $S \cong S(243,26)$ and case (i) in Proposition 4.3 occurs; furthermore, $S/Z(S)$ is isomorphic to $S(81,9)$, and the quotient curve $X/Z(S)$ provides an example for type (b) in Proposition 4.2.

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