FABER POLYNOMIAL COEFFICIENT ESTIMATES FOR A CLASS OF ANALYTIC BI-UNIVALENT FUNCTIONS

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ABSTRACT. In the present paper, we were mainly concerned with obtaining estimates for the general Taylor-Maclaurin coefficients for functions in a certain general subclass of analytic bi-univalent functions. For this purpose, we used the Faber polynomial expansions. Several connections to some of the earlier known results are also pointed out.

1. Introduction

Let \( A \) denote the class of all analytic functions \( f \) defined in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) and normalized by the conditions \( f(0) = 0 \) and \( f'(0) = 1 \). Thus each \( f \in A \) has a Taylor-Maclaurin series expansion of the form:

\[
  f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U).
\]  

(1.1)

Further, let \( S \) denote the class of all functions \( f \in A \) which are univalent in \( U \) (for details, see [9]; see also some of the recent investigations [3, 4, 5, 21, 23]).

Two of the important and well-investigated subclasses of the analytic and univalent function class \( S \) are the class \( S^*(\alpha) \) of starlike functions of order \( \alpha \) in \( U \) and the class \( K(\alpha) \) of convex functions of order \( \alpha \) in \( U \). By definition, we have

\[
  S^*(\alpha) := \left\{ f : f \in S \text{ and } \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad (z \in U; 0 \leq \alpha < 1) \right\}, \quad (1.2)
\]

and

\[
  K(\alpha) := \left\{ f : f \in S \text{ and } \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad (z \in U; 0 \leq \alpha < 1) \right\}. \quad (1.3)
\]

It is clear from the definitions (1.2) and (1.3) that \( K(\alpha) \subset S^*(\alpha) \). Also we have

\[
  f(z) \in K(\alpha) \iff zf'(z) \in S^*(\alpha),
\]

and

\[
  f(z) \in S^*(\alpha) \iff \int_{0}^{z} \frac{f(t)}{t} dt = F(z) \in K(\alpha).
\]

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It is well-known that, if \( f(z) \) is an univalent analytic function from a domain \( D_1 \) onto a domain \( D_2 \), then the inverse function \( g(z) \) defined by

\[
g(f(z)) = z, \quad (z \in D_1),
\]
is an analytic and univalent mapping from \( D_2 \) to \( D_1 \). Moreover, by the familiar Koebe one-quarter theorem (for details, see \([9]\)) we know that the image of \( U \) under every function \( f \in S \) contains a disk of radius \( \frac{1}{4} \).

According to this, every function \( f \in S \) has an inverse map \( f^{-1} \) that satisfies the following conditions:

\[
f^{-1}(f(z)) = z, \quad (z \in U),
\]
and

\[
f(f^{-1}(w)) = w, \quad (|w| < r_0(f); r_0(f) \geq \frac{1}{4}).
\]

In fact, the inverse function is given by

\[
f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \tag{1.4}
\]

A function \( f \in A \) is said to be bi-univalent in \( U \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( U \). Let \( \Sigma \) denote the class of bi-univalent functions in \( U \) given by (1.1). Examples of functions in the class \( \Sigma \) are

\[
z - z, -\log(1 - z), \frac{1}{2} \log\left(\frac{1 + z}{1 - z}\right), \cdots.
\]

It is worth noting that the familiar Koebe function is not a member of \( \Sigma \), since it maps the unit disk \( U \) univalently onto the entire complex plane except the part of the negative real axis from \(-1/4\) to \(-\infty\). Thus, clearly, the image of the domain does not contain the unit disk \( U \). For a brief history and some intriguing examples of functions and characterization of the class \( \Sigma \), see Srivastava et al. \([19]\), Frasin and Aouf \([11]\), and Yousef et al. \([24]\).

In 1967, Lewin \([17]\) investigated the bi-univalent function class \( \Sigma \) and showed that \( |a_2| < 1.51 \). Subsequently, Brannan and Clunie \([6]\) conjectured that \( |a_2| \leq \sqrt{2} \). On the other hand, Netanyahu \([18]\) showed that \( \max_{f \in \Sigma} |a_2| = \frac{1}{3} \). The best known estimate for functions in \( \Sigma \) has been obtained in 1984 by Tan \([20]\), that is, \( |a_2| < 1.485 \). The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients \( |a_n| \) \((n \in \mathbb{N} \setminus \{1, 2\})\) for each \( f \in \Sigma \) given by (1.1) is presumably still an open problem.

In this paper, we use the Faber polynomial expansions for a general subclass of analytic bi-univalent functions to determine estimates for the general coefficient bounds \( |a_n| \).

The Faber polynomials introduced by Faber \([10]\) play an important role in various areas of mathematical sciences, especially in geometric function theory. The recent publications \([12]\) and \([13]\) applying the Faber polynomial expansions to meromorphic bi-univalent functions motivated us to apply this technique to classes of analytic bi-univalent functions. In the literature, there are only a few works determining the general coefficient bounds \( |a_n| \) for the analytic bi-univalent functions given by (1.1) using Faber polynomial expansions (see for example, \([14]\) \([15]\) \([16]\)). Hamidi and Jahangiri \([14]\) considered the class of analytic bi-close-to-convex functions. Jahangiri and Hamidi \([16]\) considered the class defined by
Frasin and Aouf [11], and Jahangiri et al. [15] considered the class of analytic bi-univalent functions with positive real-part derivatives.

2. The class $B_{\Sigma}^{\mu}(\alpha, \lambda, \delta)$

Firstly, we consider a comprehensive class of analytic bi-univalent functions introduced and studied by Yousef et al. [25] defined as follows:

**Definition 2.1.** (See [25]) For $\lambda \geq 1$, $\mu \geq 0$, $\delta \geq 0$ and $0 \leq \alpha < 1$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $B_{\Sigma}^{\mu}(\alpha, \lambda, \delta)$ if the following conditions hold for all $z, w \in \mathbb{U}$:

\[
\text{Re} \left( (1 - \lambda) \left( \frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} + \xi \delta z f''(z) \right) > \alpha \tag{2.1}
\]

and

\[
\text{Re} \left( (1 - \lambda) \left( \frac{g(w)}{w} \right)^\mu + \lambda g'(w) \left( \frac{g(w)}{w} \right)^{\mu-1} + \xi \delta w g''(w) \right) > \alpha, \tag{2.2}
\]

where the function $g(w) = f^{-1}(w)$ is defined by (1.4) and $\xi = \frac{2\lambda + \mu}{2\lambda + 1}$.

**Remark 2.2.** In the following special cases of Definition 2.1 we show how the class of analytic bi-univalent functions $B_{\Sigma}^{\mu}(\alpha, \lambda, \delta)$ for suitable choices of $\lambda$, $\mu$ and $\delta$ lead to certain new as well as known classes of analytic bi-univalent functions studied earlier in the literature.

(i) For $\delta = 0$, we obtain the bi-univalent function class $B_{\Sigma}^{\mu}(\alpha, \lambda, 0) := B_{\Sigma}^{\mu}(\alpha, \lambda)$ introduced by Çağlar et al. [8].

(ii) For $\delta = 0$ and $\mu = 1$, we obtain the bi-univalent function class $B_{\Sigma}^{1}(\alpha, \lambda, 0) := B_{\Sigma}(\alpha, \lambda)$ introduced by Frasin and Aouf [11].

(iii) For $\delta = 0$, $\mu = 1$, and $\lambda = 1$, we obtain the bi-univalent function class $B_{\Sigma}^{1}(\alpha, 1, 0) := B_{\Sigma}(\alpha)$ introduced by Srivastava et al. [19].

(iv) For $\delta = 0$, $\mu = 0$, and $\lambda = 1$, we obtain the well-known class $B_{\Sigma}^{0}(\alpha, 1, 0) := S_{\Sigma}(\alpha)$ of bi-starlike functions of order $\alpha$.

(iv) For $\mu = 1$, we obtain the well-known class $B_{\Sigma}^{1}(\alpha, \lambda, \delta) := B_{\Sigma}(\alpha, \lambda, \delta)$ of bi-univalent functions.
3. COEFFICIENT ESTIMATES

Using the Faber polynomial expansion of functions \( f \in A \) of the form \([1,1]\), the coefficients of its inverse map \( g = f^{-1} \) may be expressed as in \([1]\):

\[
g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}(a_2, a_3, \ldots) w^n,
\]

where

\[
K_{n-1} = \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))(n-3)!} a_2^{n-3} a_3 + \frac{(-n)!}{(-2n+3)(n-4)!} a_2^{n-4} a_4 \\
+ \frac{(-n)!}{(2(-n+2))(n-5)!} a_2^{n-5} [a_5 + (-n+2) a_3^2] + \frac{(-n)!}{(-2n+5)(n-6)!} a_2^{n-6}
\]

\[
[a_6 + (-2n+5) a_3 a_4] + \sum_{j \geq 7} a_2^{n-j} V_j,
\]

such that \( V_j \) with \( 7 \leq j \leq n \) is a homogeneous polynomial in the variables \( a_2, a_3, \ldots, a_n \) \([2]\).

In particular, the first three terms of \( K_{n-1}^{-1} \) are

\[
K_1^{-2} = -2a_2, \quad K_2^{-3} = 3 (2a_2^2 - a_3), \quad K_3^{-4} = -4 (5a_2^3 - 5a_2 a_3 + a_4).
\]

In general, for any \( p \in \mathbb{N} := \{1, 2, 3, \ldots\} \), an expansion of \( K_n^p \) is as in \([1]\),

\[
K_n^p = p a_n + \frac{p(p-1)}{2} D_2 + \frac{p!}{(p-3)!} D_3 + \cdots + \frac{p!}{(p-n)!} D_n,
\]

where \( D_n^p = D_n^p(a_2, a_3, \ldots) \), and by \([22]\), \( D_n^m(a_1, a_2, \ldots, a_n) = \sum_{i_1 \ldots i_n} a_1^{i_1} \cdots a_n^{i_n} \) while \( a_1 = 1 \), and the sum is taken over all non-negative integers \( i_1, \ldots, i_n \) satisfying \( i_1 + i_2 + \cdots + i_n = m \), \( i_1 + 2i_2 + \cdots + ni_n = n \), it is clear that \( D_n^m(a_1, a_2, \ldots, a_n) = a_1^m \).

Consequently, for functions \( f \in \mathcal{B}_\Sigma^\mu(\alpha, \lambda, \delta) \) of the form \([1,1]\), we can write:

\[
(1 - \lambda) \left( \frac{f(z)}{z} \right)^\mu + \lambda f'(z) \left( \frac{f(z)}{z} \right)^{\mu-1} + \xi \delta z f''(z) = 1 + \sum_{n=2}^{\infty} F_{n-1} (a_2, a_3, \ldots, a_n) z^{n-1},
\]

where

\[
F_1 = (\mu + \lambda + 2\xi\delta) a_2, \quad F_2 = (\mu + 2\lambda + 6\xi\delta) \left[ \frac{\mu-1}{2} a_2^2 + \left( 1 + \frac{6\delta}{2\lambda+1} \right) \right] a_3.
\]

In general,

\[
F_{n-1} (a_2, a_3, \ldots, a_n) = [\mu + (n-1) \lambda + n(n-1) \xi \delta] \times [(\mu - 1)!] \times \left[ \sum_{\substack{i_1 \cdots i_n \geq 2 \atop i_1 \geq 1}} \frac{a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n}}{i_1! i_2! \cdots i_n! [\mu - (i_1 + i_2 + \cdots + i_n-1)]!} \right]
\]

is a Faber polynomial of degree \((n-1)\).

Our first theorem introduces an upper bound for the coefficients \(|a_n|\) of analytic bi-univalent functions in the class \( \mathcal{B}_\Sigma^\mu(\alpha, \lambda, \delta) \).
Theorem 3.1. For $\lambda \geq 1$, $\mu \geq 0$, $\delta \geq 0$ and $0 \leq \alpha < 1$, let the function $f \in \mathcal{B}_{\Sigma}^{\mu}(\alpha, \lambda, \delta)$ be given by (1.1). If $a_k = 0$ ($2 \leq k \leq n - 1$), then

$$|a_n| \leq \frac{2 (1 - \alpha)}{\mu + (n - 1) \lambda + n (n - 1) \xi \delta} \quad (n \geq 4).$$

Proof. For the function $f \in \mathcal{B}_{\Sigma}^{\mu}(\alpha, \lambda, \delta)$ of the form (1.1), we have the expansion (3.5) and for the inverse map $g = f^{-1}$, we obtain

$$(1 - \lambda) \left( \frac{g(w)}{w} \right)^{\mu} + \lambda g'(w) \left( \frac{g(w)}{w} \right)^{\mu - 1} + \xi \delta wg''(w) = 1 + \sum_{n=2}^{\infty} F_{n-1} (A_2, A_3, \ldots, A_n) w^{n-1}, \quad (3.7)$$

with

$$A_n = \frac{1}{n} K_{n-1}^{-\mu} (a_2, a_3, \ldots). \quad (3.8)$$

On the other hand, since $f \in \mathcal{B}_{\Sigma}^{\mu}(\alpha, \lambda, \delta)$ and $g = f^{-1} \in \mathcal{B}_{\Sigma}^{\mu}(\alpha, \lambda, \delta)$, by definition, there exist two positive real-part functions $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{A}$ and $q(w) = 1 + \sum_{n=1}^{\infty} d_n w^n \in \mathcal{A}$, where Re $[p(z)] > 0$ and Re $[q(w)] > 0$ in $\mathbb{U}$ so that

$$(1 - \lambda) \left( \frac{f(z)}{z} \right)^{\mu} + \lambda f'(z) \left( \frac{f(z)}{z} \right)^{\mu - 1} + \xi \delta z f''(z) = \alpha + (1 - \alpha) p(z) = 1 + (1 - \alpha) \sum_{n=1}^{\infty} K_n^1 (c_1, c_2, \ldots, c_n) z^n \quad (3.9)$$

and

$$(1 - \lambda) \left( \frac{g(w)}{w} \right)^{\mu} + \lambda g'(w) \left( \frac{g(w)}{w} \right)^{\mu - 1} + \xi \delta wg''(w) = \alpha + (1 - \alpha) q(w) = 1 + (1 - \alpha) \sum_{n=1}^{\infty} K_n^1 (d_1, d_2, \ldots, d_n) w^n \quad (3.10)$$

Note that, by the Caratheodory lemma (e.g., [9]), $|c_n| \leq 2$ and $|d_n| \leq 2$ ($n \in \mathbb{N}$). Comparing the corresponding coefficients of (3.5) and (3.9), for any $n \geq 2$, yields

$$F_{n-1} (a_2, a_3, \ldots, a_n) = (1 - \alpha) \sum_{n=1}^{\infty} K_n^1 (c_1, c_2, \ldots, c_{n-1}), \quad (3.11)$$

and similarly, from (3.7) and (3.10) we find

$$F_{n-1} (A_2, A_3, \ldots, A_n) = (1 - \alpha) \sum_{n=1}^{\infty} K_n^1 (d_1, d_2, \ldots, d_{n-1}). \quad (3.12)$$

Note that for $a_k = 0$ ($2 \leq k \leq n - 1$), we have

$$A_n = - a_n$$

and so $[\mu + (n - 1) \lambda] a_n = (1 - \alpha) c_{n-1}$ and $- [\mu + (n - 1) \lambda] a_n = (1 - \alpha) d_{n-1}$.
Taking the absolute values of the above equalities, we obtain
\[
|a_n| = \frac{(1-\alpha)|c_{n-1}|}{\mu + (n-1)\lambda + n(n-1)\xi\delta}
\]
\[
= \frac{(1-\alpha)|d_{n-1}|}{\mu + (n-1)\lambda + n(n-1)\xi\delta} \leq \frac{2(1-\alpha)}{\mu + (n-1)\lambda + n(n-1)\xi\delta},
\]
which completes the proof of (3.1).

The following corollary is an immediate consequence of the above theorem.

**Corollary 3.2.** For \(\lambda \geq 1, \delta \geq 0\) and \(0 \leq \alpha < 1\), let the function \(f \in \mathcal{B}_\Sigma(\alpha, \lambda, \delta)\) be given by (1.1). If \(a_k = 0\) \((2 \leq k \leq n-1)\), then
\[
|a_n| \leq \frac{2(1-\alpha)}{1 + (n-1)\lambda + n(n-1)\xi\delta} \quad (n \geq 4).
\]

**Theorem 3.3.** For \(\lambda \geq 1, \mu \geq 0\), \(\delta \geq 0\) and \(0 \leq \alpha < 1\), let the function \(f \in \mathcal{B}^\mu(\alpha, \lambda, \delta)\) be given by (1.1). Then one has the following
\[
|a_2| \leq \left\{ \begin{array}{ll}
\sqrt{\frac{4(1-\alpha)}{2(1-\alpha)}} & \text{if } 0 \leq \alpha \leq \frac{\mu + 2\lambda - \lambda^2}{(\mu + 2\lambda + 6\xi\delta)(\mu + 1)}, \\
\mu + 2\lambda + 2\xi\delta & \text{if } \frac{\mu + 2\lambda - \lambda^2}{(\mu + 2\lambda + 6\xi\delta)(\mu + 1)} \leq \alpha < 1.
\end{array} \right.
\]

(3.13)
\[
|a_3| \leq \left\{ \begin{array}{ll}
\min \left\{ \frac{4(1-\alpha)^2}{(\mu + \lambda + 2\xi\delta)^2} + \frac{2(1-\alpha)}{\mu + 2\lambda + 6\xi\delta}, \frac{4(1-\alpha)}{(\mu + 2\lambda + 6\xi\delta)(\mu + 1)} \right\} & \text{if } 0 \leq \mu < 1, \\
\mu + 3 & \text{if } \mu \geq 1
\end{array} \right.
\]

(3.14)
and
\[
\left|a_3 - \frac{\mu + 3}{2}a_2^2\right| \leq \frac{2(1-\alpha)}{\mu + 2\lambda + 6\xi\delta}.
\]

**Proof.** If we set \(n = 2\) and \(n = 3\) in (3.11) and (3.12), respectively, we get
\[
(\mu + \lambda + 2\xi\delta) a_2 = (1-\alpha) c_1,
\]

(3.15)
\[
(\mu + 2\lambda + 6\xi\delta) \left[ \left(\frac{\mu - 1}{2}\right) a_2^2 + \left(1 + \frac{6\delta}{2\lambda + 1}\right) a_3 \right] = (1-\alpha) c_2,
\]

(3.16)
\[
- (\mu + \lambda) a_2 = (1-\alpha) d_1,
\]

(3.17)
\[
(\mu + 2\lambda + 6\xi\delta) \left[ \frac{\mu + 3}{2} a_2^2 - \left(1 + \frac{6\delta}{2\lambda + 1}\right) a_3 \right] = (1-\alpha) d_2.
\]

(3.18)
From (3.15) and (3.17), we find (by the Carathéodory lemma)
\[
|a_2| = \frac{(1-\alpha)|c_1|}{\mu + \lambda + 2\xi\delta} = \frac{(1-\alpha)|d_1|}{\mu + \lambda + 2\xi\delta} \leq \frac{2(1-\alpha)}{\mu + \lambda + 2\xi\delta}.
\]

(3.19)
Also from (3.16) and (3.18), we obtain
\[
(\mu + 2\lambda + 6\xi\delta) (\mu + 1) a_2^2 = (1-\alpha) (c_2 + d_2).
\]

(3.20)
Using the Carathéodory lemma, we get \(|a_2| \leq \sqrt{\frac{4(1-\alpha)}{(\mu + 2\lambda + 6\xi\delta)(\mu + 1)}}\), and combining this with inequality (3.19), we obtain the desired estimate on the coefficient \(|a_2|\) as asserted in (3.13).

Next, in order to find the bound on the coefficient \(|a_3|\), we subtract (3.18) from (3.16).
We thus get
\[
(\mu + 2\lambda + 6\xi\delta) \left( -2a_2^2 + 2 \left( 1 + \frac{6\delta}{2\lambda + 1} \right) a_3 \right) = (1 - \alpha) (c_2 - d_2).
\]
or
\[
a_3 = a_2^2 + \frac{(1 - \alpha) (c_2 - d_2)}{2 (\mu + 2\lambda + 6\xi\delta)}.
\] (3.21)

Upon substituting the value of \(a_2^2\) from (3.15) into (3.21), it follows that
\[
a_3 = \frac{(1 - \alpha)^2 c_1^2}{(\mu + \lambda + 2\xi\delta)^2} + \frac{(1 - \alpha) (c_2 - d_2)}{2 (\mu + 2\lambda + 6\xi\delta)}.
\]

We thus find (by the Caratheodory lemma) that
\[
|a_3| \leq \frac{4 (1 - \alpha)^2}{(\mu + \lambda + 2\xi\delta)^2} + \frac{2 (1 - \alpha)}{\mu + 2\lambda + 6\xi\delta}.
\] (3.22)

On the other hand, upon substituting the value of \(a_2^2\) from (3.20) into (3.21), it follows that
\[
a_3 = \frac{(1 - \alpha)}{2 (\mu + 2\lambda + 6\xi\delta) (\mu + 1)} [(\mu + 3) c_2 + (1 - \mu) d_2].
\]

Consequently (by the Caratheodory lemma), we have
\[
|a_3| \leq \frac{1 - \alpha}{(\mu + 2\lambda + 6\xi\delta)(\mu + 1)} [(\mu + 3) + |1 - \mu|].
\] (3.23)

Combining (3.22) and (3.23), we get the desired estimate on the coefficient \(|a_3|\) as asserted in (3.14). Finally, from (3.18), we deduce (by the Caratheodory lemma) that
\[
\left| a_3 - \frac{\mu + 3}{2} a_2 \right| = \frac{(1 - \alpha) |d_2|}{\mu + 2\lambda + 6\xi\delta} \leq \frac{2 (1 - \alpha)}{\mu + 2\lambda + 6\xi\delta}.
\]

This evidently completes the proof of 3.3. \(\Box\)

By setting \(\mu = 1\) in 3.3, we obtain the following consequence.

**Corollary 3.4.** For \(\lambda \geq 1, \delta \geq 0\) and \(0 \leq \alpha < 1\), let the function \(f \in \mathfrak{B}_\Sigma(\alpha, \lambda, \delta)\) be given by (1.1). Then one has the following
\[
|a_2| \leq \left\{ \begin{array}{ll}
\sqrt{\frac{2(1 - \alpha)}{(1 + 2\lambda + 6\xi\delta)}} , & 0 \leq \alpha \leq \frac{1 + 2\lambda - \lambda^2}{2(1 + 2\lambda + 6\xi\delta)} \\
1 + 2\lambda + 6\xi\delta , & \frac{1 + 2\lambda - \lambda^2}{2(1 + 2\lambda + 6\xi\delta)} \leq \alpha \leq 1
\end{array} \right.
\]
\[
|a_3| \leq \frac{2 (1 - \alpha)}{1 + 2\lambda + 6\xi\delta},
\]
and
\[
|a_3 - 2a_2^2| \leq \frac{2 (1 - \alpha)}{1 + 2\lambda + 6\xi\delta}.
\]
By setting $\lambda = 1$ in Theorem 3.3, we obtain the following consequence.

**Corollary 3.5.** For $\mu \geq 0$, $\delta \geq 0$ and $0 \leq \alpha < 1$, let the function $f \in \mathcal{B}_2^\mu(\alpha, \delta)$ be given by (1.1). Then one has the following

$$|a_2| \leq \left\{ \begin{array}{ll}
\frac{\sqrt{4(1-\alpha)(\mu+6\delta+2)(\mu+1)}}{\mu+2\delta+1}, & 0 \leq \alpha \leq \frac{1}{\mu+6\delta+2} \\
\frac{1}{\mu+6\delta+2} \leq \alpha \leq 1
\end{array} \right.$$

$$|a_3| \leq \left\{ \begin{array}{ll}
\min \left\{ \frac{4(1-\alpha)^2}{\mu+2\delta+1}, \frac{2(1-\alpha)}{\mu+6\delta+2}, \frac{4(1-\alpha)}{(\mu+6\delta+2)(\mu+1)} \right\}, & 0 \leq \mu < 1 \\
\frac{2(1-\alpha)}{\mu+6\delta+2}, & \mu \geq 1
\end{array} \right.$$

**Remark 3.6.** As a final remark, for $\delta = 0$ in

(i) Theorem 3.1 we obtain Theorem 1 in \cite{7}.

(ii) Theorem 3.3 we obtain Theorem 2 in \cite{7}.

(iii) Corollary 3.2 we obtain Theorem 1 in \cite{16}.

(iv) Corollary 3.4 we obtain Theorem 2 in \cite{16}.

(v) Corollary 3.5 we obtain Corollary 3 in \cite{7}.

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