1. Introduction

The study of systems of strongly correlated electrons has received an increasing attention in the last years, especially after the discoveries of the Fractional Quantum Hall Effect and the High Temperature Superconductivity. In this context, one of the most studied is the Hubbard model [1] which describes the dynamics of non-relativistic electrons moving on a lattice. Its Hamiltonian contains a kinetic term and a quartic potential term which effectively describes the Coulomb interactions among electrons.

Another lattice model that has been suggested as an appropriate starting point for the theory of the High Temperature Superconductivity [2,3], is the so-called $t$-$J$ model [4] which is characterized by the absence of doubly occupied electron states. Its Hamiltonian contains a kinetic term of strength $t$ responsible for the hopping of electrons from one lattice site to its nearest neighbors, and a potential term of strength $J$ which describes nearest neighbor spin exchange interactions. If $J << t$, the $t$-$J$ model can be mapped onto a Hubbard model with very strong Coulomb repulsion; for other values of the coupling constants instead, the two models exhibit a quite different behavior. A particularly interesting feature of the $t$-$J$ model is the presence of a supersymmetry [5]: In fact, when $J = \pm 2t$, the Hamiltonian of the $t$-$J$ model commutes with the generators of the superalgebra $SU(1|2)$ [6]. It is remarkable that precisely at the supersymmetric point $J = \pm 2t$, the $t$-$J$ model in one dimension is exactly solvable by Bethe Ansatz [7,8]. We recall that the supersymmetric $t$-$J$ model cannot be derived from the Hubbard model, but instead can be mapped onto a quantum lattice gas of hard core bosons and fermions. This latter system was formulated in the mid-seventies and solved exactly by Bethe Ansatz in one dimension [9,8].

The availability of a complete solution for the one-dimensional supersymmetric $t$-$J$ model allows a detailed analysis of its collective excitations [10]. In particular two kinds of low lying quasiparticle excitations have been identified above the antiferromagnetic ground state ($J > 0$): One carries charge and no spin and is called holon; the other carries spin and no charge and is called spinon. This remarkable phenomenon of spin-charge separation, which is rigorously proven only in one dimension, has been conjectured to occur also in two dimensions [3,11]. To implement this separation between spin and charge, one often uses the formalism of the slave operators [12] and makes the Ansatz in which the electron is a product of a charged spinless antiholon and a neutral spinon of spin $1/2$. Since the electron is a fermion, holons and spinons must have complementary statistics. For example, the holon may be fermionic and the spinon bosonic (slave fermion representation [13]), or vice-versa the holon may be bosonic and the spinon fermionic (slave boson representation [3,11]). Actually, since in two dimensions the statistics is arbitrary [14,15], there are in principle other possibilities; in particular it has been suggested [16] that holons and spinons are semions, i.e. anyons that are exactly half-way between bosons and fermions. In [17,18] it has been shown that these different statistics assignments can be interpreted in a unified way within the framework of the bosonization procedure of two-dimensional systems. More precisely, the slave boson and slave fermion representations arise as two different
gauge fixings in the abelian bosonization of the $t$-$J$ model, whereas the semion case naturally occurs in the non-abelian bosonization.

In this paper we present a generalization of the analysis of [17,18], and in particular we show that performing an abelian bosonization with two independent Chern-Simons fields, the holons and the spinons of the two-dimensional $t$-$J$ model may become anyons of arbitrary complementary statistics, in such a way that the product of one antiholon and one spinon (i.e. the electron) is a fermion. We call this the slave anyon representation, which interpolates continuously between the slave boson and the slave fermion representations; in this way the semionic statistics of [16] simply becomes a particular case. Therefore, we deduce that at the formal kinematical level, all consistent statistics assignments are possible and only a dynamical analysis can eventually select one among these many possibilities.

Another interesting fact is that the supersymmetry of the $t$-$J$ model at $J = \pm 2t$ becomes simpler and more transparent when formulated in terms of holon and spinon operators. In fact, since these have complementary statistics, it is natural to expect that the fermionic supercharges transform holons into spinons and vice-versa. Indeed this is what happens, because the superalgebra $SU(1|2)$ is linearly realized on holons and spinons in contrast to the non-linear realization in terms of the electron operators. In this paper we discuss in particular the supersymmetry of the $t$-$J$ model in the slave anyon representation and analyze the braiding properties of the supercharge densities with holons and spinons.

The paper is organized as follows: In Section 2 we review the main features of the $t$-$J$ model, write its Hamiltonian and present its bosonization with two independent Chern-Simons fields to yield holons and spinons with complementary statistics. In Section 3 we discuss the supersymmetry of the $t$-$J$ model in the slave fermion formalism, whereas in Section 4 we introduce slave holons and spinons in the anyon representation and discuss their braiding properties. Then we prove the invariance under $SU(1|2)$ of the Hamiltonian of the $t$-$J$ model at the supersymmetric point $J = 2t$ in the slave anyon representation. Finally, Section 5 presents our conclusions, and Appendix A contains an alternative proof of the supersymmetry of the Hamiltonian of the $t$-$J$ model when $J = 2t$.

2. The $t$-$J$ Model and its Bosonizations

The (two-dimensional) $t$-$J$ model describes a lattice system of strongly correlated non-relativistic electrons. It was proposed a few years ago by Zhang and Rice [4] as

\footnote{To avoid confusions, we stress that here the terms holons and spinons are used in a different way with respect to [10] where holons and spinons denote different supermultiplets of $SU(1|2)$.}
an effective model for the electronic dynamics in the copper-oxide planes of the high temperature superconductors. Given a two-dimensional square lattice $\Omega$ of spacing $a = 1$, one assumes that in each site at most one electron can be accommodated, i.e. doubly occupied electron states are excluded from the physical Hilbert space of the system; this is a way to simulate strong Coulomb repulsion. The $t$-$J$ model with chemical potential $\mu$ is described by the following Hamiltonian

$$H_{tJ} = \mathcal{P} \left[ -t \sum_{<i,j>} \sum_{\alpha = \pm 1} \left( c^\dagger_\alpha(i) c_\alpha(j) + \text{h. c.} \right) + J \sum_{<i,j>} \left( \mathbf{S}(i) \cdot \mathbf{S}(j) - \frac{1}{4} n(i) n(j) \right) + \mu \sum_{i \in \Omega} n(i) \right] \mathcal{P}$$

where $\sum_{<i,j>}$ denotes the sum over nearest-neighbor lattice sites; $c^\dagger_\alpha(i)$ and $c_\alpha(i)$ are respectively creation and annihilation operators for electrons with third component of the spin $\alpha/2$ at the site $i$. They satisfy the usual anticommutation relations

$$\{ c_\alpha(i), c^\dagger_\beta(j) \} = \delta_{\alpha\beta} \delta(i,j)$$

where $\delta(i,j)$ is the lattice delta function. The vector $\mathbf{S}(i)$ is the spin operator whose components are

$$S^a(i) = \sum_{\alpha, \beta = \pm 1} c^\dagger_\alpha(i) \left( \frac{\sigma^a}{2} \right)_{\alpha\beta} c_\beta(i) \quad a = 1, 2, 3, \quad (\sigma^a \text{ being the Pauli matrices}),$$

$n(i)$ is the number operator

$$n(i) = \sum_{\alpha = \pm 1} n_\alpha(i) = \sum_{\alpha = \pm 1} c^\dagger_\alpha(i) c_\alpha(i).$$

Finally, $\mathcal{P}$ denotes the Gutzwiller projection operator which ensures that doubly occupied electrons states are removed from the Hilbert space of the system [19]. The parameters $t$ and $J$ in (2.1) are coupling constants which we take to be positive: $t$ denotes the strength of the kinetic nearest-neighbor hopping term; $J$ denotes the strength of the spin exchange nearest-neighbor interaction. We recall that for $J \ll t$ the Hamiltonian (2.1) can be derived from the Hubbard model in the limit of (infinite) strong on-site repulsion and that for $t = 0$ it reduces to the antiferromagnetic Heisenberg XXX model [1].

In the following we will discuss the bosonization procedure with Chern-Simons fields and the presence of anyons, which are specific features of the two-dimensional $t$-$J$ model; however, most of the considerations that we present hereinafter are valid for lattices of any dimensions. As we mentioned, doubly occupied states are excluded from the Hilbert space of the $t$-$J$ model; therefore if we denote by $|0\rangle$ the Fock vacuum
which satisfies \( c_\alpha(i)|0\rangle = 0 \) for all \( i \in \Omega \) and \( \alpha = \pm 1 \), there are only three possible electronic states at any given lattice site \( i \), namely

\[
|0\rangle \ , \ \ c_+^\dagger(i)|0\rangle \ , \ c_-^\dagger(i)|0\rangle \ . \tag{2.4}
\]

The fourth state \( c_+^\dagger(i) c_+^\dagger(i)|0\rangle \), which would be allowed by Fermi statistics, is removed by the Gutzwiller projector \( P \). There is an equivalent way to realize this truncation in the Fock space: Instead of using the operators \( c_\alpha^\dagger \) to construct the physical states and then the Gutzwiller projector to remove those with double occupancy, one can use the \textit{projected} fermionic operators

\[
c'_\alpha(i) \equiv \left(1 - n_{-\alpha}(i)\right) c_\alpha(i) \ , \ c_{\alpha}^\dagger(i) \equiv c_{\alpha}^\dagger(i) \left(1 - n_{-\alpha}(i)\right) \tag{2.5}
\]

for \( \alpha = \pm 1 \). It is immediate to verify that these operators satisfy a hard-core constraint even when their spin index is different. Indeed one has

\[
c'_\alpha(i) \ c'_\beta(i) = c_{\alpha}^\dagger(i) \ c_{\beta}^\dagger(i) = 0 \tag{2.6}
\]

for all \( \alpha \) and \( \beta \). Furthermore, using (2.2) it is straightforward to verify that

\[
\left\{ c'_\alpha(i) , c'_\beta(j) \right\} = \left\{ c_{\alpha}^\dagger(i) , c_{\beta}^\dagger(j) \right\} = 0 \ , \\
\left\{ c'_\alpha(i) , c_{\beta}^\dagger(j) \right\} = \delta(i,j) \left[ \left(1 - \frac{1}{2} n(i)\right) \delta_{\alpha\beta} + \sum_{a=1}^{3} S^a(i) \sigma_{a\beta} \right] . \tag{2.7}
\]

The operators \( S^a(i) \) and \( n(i) \) are defined respectively in (2.3a) and (2.3b), but it is easy to realize that they can be written indifferently in terms of \( c_\alpha \) or \( c'_\alpha \). In fact, with simple algebra we see that

\[
S^a(i) \equiv \sum_{\alpha,\beta = \pm 1} c_{\alpha}^\dagger(i) \left(\frac{\sigma^a}{2}\right)_{\alpha\beta} c_{\beta}(i) = \sum_{\alpha,\beta = \pm 1} c_{\alpha}^\dagger(i) \left(\frac{\sigma^a}{2}\right)_{\alpha\beta} c_{\beta}(i) = S^a(i) \ , \tag{2.8a}
\]

and

\[
n'_\alpha(i) \equiv c_{\alpha}^\dagger(i) c'_{\alpha}(i) = c_{\alpha}^\dagger(i) c_{\alpha}(i) \left(1 - n_{-\alpha}(i)\right)^2 = n_\alpha(i) \ , \tag{2.8b}
\]

where the last equality follows from \( n_\alpha(i) n_{-\alpha}(i) = 0 \) which is a consequence of the absence of doubly occupied states. Using the projected operators (2.5), the Hamiltonian (2.1) becomes

\[
H_{tJ} = \left[ -t \sum_{<i,j>} \sum_{\alpha = \pm 1} \left( c_{\alpha}^\dagger(i) c'_{\alpha}(j) + \text{h.c.} \right) + J \sum_{<i,j>} \left( S(i) \cdot S(j) - \frac{1}{4} n(i) n(j) \right) + \mu \sum_{i \in \Omega} n(i) \right] . \tag{2.9}
\]
Since the Hilbert space of the $t$-$J$ model does not contain any doubly occupied states, the hopping term of the Hamiltonian can be dropped at half-filling, i.e. when the number of electrons is equal to the number of lattice sites. In this case the $t$-$J$ model becomes equivalent to the (antiferromagnetic) XXX model. The situation is clearly different below half-filling: In such a case there are some empty points and some electrons can hop into them from neighboring occupied sites with a rate that is proportional to the coupling constant $t$. A deviation from half-filling is usually called “doping” and can be realized by suitably choosing the value of the chemical potential $\mu$ in the Hamiltonian. Several authors suggested that, at small doping, the spectrum of the lowest lying excitations above the antiferromagnetic ground state consists of holons and spinons [2,3,11]. The holons carry charge and no spin, whilst the spinons carry spin and no charge. Only an accurate study of the dynamics could allow to check if this mechanism of spin-charge separation actually occurs. To this aim it may be useful to perform a change of variables and write the projected electron operators, which obviously have both charge and spin, as products of an antiholon and a spinon. More precisely one writes

$$c_i' = e_i^\dagger s_i^\alpha , \quad c_i'^\dagger = e_i s_i^\dagger$$  \hspace{1cm} (2.10)

where $e_i^\dagger(i)$ and $e(i)$ are respectively the creation and annihilation operators for charged spinless holons, and $s_i^\dagger(\alpha)$ and $s_i(\alpha)$ are respectively the creation and annihilation operators for neutral spinons of spin $\alpha$, and are such that

$$e_i^\dagger(i)|0\rangle = s_i(\alpha)|0\rangle = 0$$

where $|0\rangle$ is the Fock vacuum of Eq. (2.4) representing a hole. The requirement that there are no doubly occupied electron states, translates into the constraint

$$\phi(i) \equiv e_i^\dagger(i) e(i) + \sum_{\alpha=\pm 1} s_i^\dagger(\alpha) s_i(\alpha) - 1 = 0$$  \hspace{1cm} (2.11)

for all $i \in \Omega$. This constraint implies that in each lattice site one of the following three possibilities must occur: Either there is one hole or there is one electron with spin $+1$ or one electron with spin $-1$. There are no other possibilities, just like in the original electronic description where there were only three allowed configurations at each site.

Another important issue is the statistics of holons and spinons. Since the electron operators must be fermionic, $e(i)$ and $s_i(\alpha)$ cannot have the same statistics. In principle we have two possibilities: The holon is a fermion and the spinon is a boson, or vice-versa, the holon is a boson and the spinon is a fermion. These two scenarios are usually called the slave-fermion and the slave-boson descriptions respectively, and both of them have been proposed and studied in the literature (the slave-fermion approach for the $t$-$J$ model has been considered in [13,20], while the slave-boson approach has been considered in [3,11,21]). However, for two-dimensional systems there are other possibilities, since in two dimensions the statistics is not necessarily bosonic.
or fermionic but can be anyonic [22] (for a review see for instance [14,15]). Indeed, in [16] Laughlin conjectured that the two-dimensional holons and spinons are semions, i.e. anyons that are exactly half-way between bosons and fermions. Actually, as we will show later, holons and spinons for the $t$-$J$ model in two dimensions can in principle be anyons of arbitrary statistics with the only constraint that the composite object formed by one holon and one spinon be a fermion.

A few remarks are in order at this point. First of all, we should keep in mind that the replacement of the electron operators with products of holons and spinons according to (2.10) is a purely formal operation. Differently from the electron operators $c_{\alpha}(i)$ and $c_{\alpha}^{\dagger}(i)$, the holon and spinon oscillators do not commute with the constraint operator $\phi(i)$; therefore the spin-charge separation, i.e. the excitation of a single holon or spinon, can occur only if the constraint (2.11) is somehow released. Secondly, the issue of the statistics of holons and spinons cannot be resolved at this formal stage, since all the above mentioned assignments of statistics are formally correct. In fact, to make any progress it would be necessary to determine what is the dynamics of holons and spinons and to find their Hamiltonian. The first step towards this aim is simply to substitute (2.10) into the Hamiltonian of the $t$-$J$ model. For definiteness we choose the slave fermion description, but similar considerations would apply equally well to the slave boson formalism. In the slave fermion description, the fermionic holon operators obey

$$\{e(i), e^{\dagger}(j)\} = \delta(i,j), \quad (2.12a)$$

and the bosonic spinon operators obey

$$[s_\alpha(i), s_\beta^{\dagger}(j)] = \delta(i,j) \delta_{\alpha\beta}, \quad (2.12b)$$

with all other possible (anti)commutators vanishing. A straightforward substitution of (2.10) into the Hamiltonian (2.9) leads to

$$H_{t,J} = \left\{ + t \sum_{<i,j>} \left( e^{\dagger}(j)e(i)s_\alpha^{\dagger}(i)s_\alpha(j) + \text{h.c.} \right) + \frac{J}{4} \sum_{<i,j>} e(i)e^{\dagger}(i)e(j)e^{\dagger}(j) \times \right.$$ 

$$\times \left[ \sum_{a=1}^{3} \left( s_\alpha^{\dagger}(i) \sigma_{\alpha\beta}^a s_\beta(i) s_\alpha^{\dagger}(j) \sigma_{\alpha'\beta'}^a s_{\beta'}(j) \right) - s_\alpha^{\dagger}(i) s_\alpha(i) s_\alpha^{\dagger}(j) s_\alpha(j) \right]$$

$$\left. + \sum_{i \in \Omega} \left[ \mu e(i)e^{\dagger}(i)s_\alpha^{\dagger}(i)s_\alpha(i) + i \Lambda(i) \phi(i) \right] \right\}$$

(2.13)

where we have introduced the Lagrange multiplier $\Lambda(i)$ to enforce the constraint (2.11).

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2 To simplify the notation, here and in the following we understand the summation symbols over repeated spin indices.
Using the anticommutation relations of the holons, the $J$-term of (2.13) can be rewritten as follows

\[
\frac{J}{4} \sum_{<i,j>} \left[ 1 - e^{\dagger}(i) e(i) - e^{\dagger}(j) e(j) + e^{\dagger}(i) e^{\dagger}(j) e(j) \right] \times \\
\times \left[ \sum_{a=1}^{3} \left( s_{\alpha}^{\dagger}(i) \sigma_{\alpha\beta}^{a} s_{\beta}(i) s_{\alpha'}^{\dagger}(j) \sigma_{\alpha'\beta'}^{a} s_{\beta'}(j) \right) - s_{\alpha}^{\dagger}(i) s_{\alpha}(i) s_{\alpha}^{\dagger}(j) s_{\alpha'}(j) \right] .
\]

(2.14)

Due to the constraint (2.11), it is easy to realize that only the 1 of the first square bracket of (2.14) yields a non vanishing contribution. In fact the other three terms, when multiplied by the second square bracket, give rise to expressions containing annihilation operators in the same point both for holons and spinons, and hence vanish when applied to physical states satisfying the constraint (2.11). With a similar argument, we can also rewrite the chemical potential term simply as

\[
\mu_s s_{\alpha}(i) s_{\alpha}(i).
\]

Thus, the Hamiltonian (2.13) can be drastically simplified and one gets

\[
H_{tJ} = \sum_{<i,j>} \left\{ + t \left( e^{\dagger}(j) e(i) s_{\alpha}^{\dagger}(i) s_{\alpha}(j) + \text{h.c.} \right) \right.
\]

\[
+ \frac{J}{4} \left[ \sum_{a=1}^{3} \left( s_{\alpha}^{\dagger}(i) \sigma_{\alpha\beta}^{a} s_{\beta}(i) s_{\alpha'}^{\dagger}(j) \sigma_{\alpha'\beta'}^{a} s_{\beta'}(j) \right) - s_{\alpha}^{\dagger}(i) s_{\alpha}(i) s_{\alpha}^{\dagger}(j) s_{\alpha'}(j) \right] \\
\left. + \sum_{i \in \Omega} \left[ \mu_s s_{\alpha}(i) s_{\alpha}(i) + i \Lambda(i) \phi(i) \right] \right\} .
\]

(2.15)

We stress that the formal manipulations that yielded the Hamiltonian (2.15) are valid in any dimensions. In the particular case of two dimensional lattices, it is possible to obtain (2.15) in an alternative way based on the abelian bosonization, as shown in [17,18]. The two-dimensional abelian bosonization consists of replacing the original fermionic fields by new bosonic degrees of freedom interacting with an abelian Chern-Simons gauge field [23]. In this framework, holons and spinons arise, roughly speaking, as the modulus and phase of the field which bosonizes the electron operators, and thus at first are both bosonic \(^3\). The final statistics assignments are dictated by the choice of the coupling constants between the bosonic holons and spinons on the one hand, and the abelian Chern-Simons field on the other.

\(^3\) Actually the correspondence is more involved than this, and moreover one of the two fields must be a hard-core boson. For details we refer to the original papers [17,18].
We now give some details of this construction following [18], and later we present its generalization. In the formalism of second quantization, the grand canonical partition function for the $t$-$J$ model at temperature $\beta$ and chemical potential $\mu$ is written as an Euclidean functional integral over a complex Grassmann field $\psi_\alpha$ with antiperiodic boundary conditions, according to

$$Q(\beta, \mu) = \int D\psi \ e^{-S(\psi)}$$

where

$$S(\psi) = \int_0^\beta d\tau \left[ \sum_{i \in \Omega} \psi^*_\alpha(i, \tau) \frac{\partial}{\partial \tau} \psi_\alpha(i, \tau) + \mathcal{H}_{tJ}(\psi) \right].$$

In this formula $\mathcal{H}_{tJ}(\psi)$ denotes the functional obtained from the Hamiltonian (2.9) by replacing the fermionic operators $c'_\alpha(i)$ and $c'^\dagger_\alpha(i)$ with the Grassmann fields $\psi_\alpha(i, \tau)$ and $\psi^*_\alpha(i, \tau)$ respectively. This functional is explicitly given by

$$\mathcal{H}_{tJ}(\psi) = -t \sum_{<i,j>} \left( \psi^*_\alpha(i, \tau) \psi_\alpha(j, \tau) + \text{h. c.} \right) + \mathcal{H}_J(\psi)$$

where

$$\mathcal{H}_J(\psi) = \frac{J}{4} \left[ \sum_{a=1}^3 \left( \psi^*_\alpha(i, \tau) \sigma^a_{\alpha\beta} \psi_\beta(i, \tau) \psi^*_\alpha'(j, \tau) \sigma^a_{\alpha'\beta'} \psi_{\beta'}(j, \tau) - \psi^*_\alpha(i, \tau) \psi_\alpha(i, \tau) \psi^*_\alpha'(j, \tau) \psi_{\alpha'}(j, \tau) \right) \right].$$

As is well known, the path integral over anticommuting Grassmann fields automatically takes into account the sign factors due to the fermionic statistics. An equivalent expression for $Q(\beta, \mu)$ can be obtained by using, in place of $\psi_\alpha(i, \tau)$ and $\psi^*_\alpha(i, \tau)$, complex commuting hard core bosonic fields $\phi_\alpha(i, \tau)$ and $\phi^*_\alpha(i, \tau)$ together with a suitable prescription to implement the original fermionic statistics. For two dimensional systems, this prescription is quite simple and elegant: One couples these bosonic fields to an abelian Chern-Simons gauge field $A_\lambda (\lambda = 0, 1, 2)$ with level $k = 1$, and averages over all its possible configurations [24]. More precisely, the following identity holds

$$Q(\beta, \mu) = \int D\psi \ e^{-S(\psi)} = \left\langle \int D\varphi \ e^{-\tilde{S}(\varphi; A)} \right\rangle_{k=1}$$

where $\tilde{S}(\varphi; A)$ is the bosonized action obtained from $S(\psi)$ by replacing $\psi_\alpha$ with $\varphi_\alpha$ minimally coupled to $A_\lambda$. The symbol $\langle \ldots \rangle_k$ denotes the expectation value over the Chern-Simons field, namely

$$\langle \mathcal{O}(A) \rangle_k = \frac{\int DA \mathcal{O}(A) e^{-S_{CS}^k(A)}}{\int DA e^{-S_{CS}^k(A)}}$$

8
where $O(A)$ is any functional of $A$ and

$$S_{CS}^k(A) = \frac{k}{4\pi i} \int_0^\beta d\tau \int d^2x \, e^{\lambda\mu\nu} A_\lambda(\tau, x) \partial_\mu A_\nu(\tau, x)$$  \hspace{1cm} (2.21)$$
is the abelian Chern-Simons action of level $k$.

The bosonization identity (2.19) is essentially based on the fundamental property that the expectation values of Wilson loops in Chern-Simons theories are topological invariants [24,25]. To see the implications of this fact, let us consider an arbitrary braid $b$ of $N$ objects corresponding to the evolution of $N$ particles between their configurations at Euclidean times $\tau = 0$ and $\tau = \beta$. Due the periodic boundary conditions on $\tau$, the braid $b$ actually forms a closed link $L(b)$, and an abelian Wilson loop can be defined as follows

$$W\left(L(b), A\right) = e^{i \int_{L(b)} \left(A_0(x, \tau) d\tau + \sum_{i=1}^2 A_i(x, \tau) dx^i\right)}$$  \hspace{1cm} (2.22)$$
where the line integral in the exponent is computed along the link $L(b)$ with a suitable choice of framing [25]. If $A_\lambda$ is a Chern-Simons gauge field with action (2.21), one can prove that

$$\left\langle W\left(L(b), A\right)\right\rangle_k = e^{i \frac{2\pi}{k} n(b)}$$  \hspace{1cm} (2.23)$$
where

$$n(b) = \frac{1}{2} \left[ n_+(b) - n_-(b) \right]$$  \hspace{1cm} (2.24)$$
is the the winding number of the braid $b$, i.e. the difference between the number of overcrossings $n_+(b)$ and the number of undercrossings $n_-(b)$. If $k = 1$, we have

$$\left\langle W\left(L(b), A\right)\right\rangle_{k=1} = e^{i \pi \left(n_+(b) - n_-(b)\right)} = e^{i \pi \left(n_+(b) + n_-(b)\right)} \equiv (-1)^{\sigma(\pi(b))}$$  \hspace{1cm} (2.25)$$
where $\sigma(\pi(b))$ is the signature of the permutation $\pi(b)$ associated to the braid $b$. From (2.25) we see that the average over an abelian Chern-Simons field with level 1 reproduces precisely the right sign factors needed to implement the fermionic statistics in terms of purely bosonic variables. Inserting this result into the grand canonical partition function and using a path integral notation, one can finally prove the bosonization identity (2.19).

When this technique is applied to the $t$-$J$ model, the electron is described by a hard-core boson $\varphi_\alpha$ coupled to an abelian Chern-Simons field. The modulus and phase of $\varphi_\alpha$ are then interpreted as holon and spinon fields respectively. More precisely, one writes

$$\varphi_\alpha(i, \tau) = H^*(i, \tau) \Sigma_\alpha(i, \tau)$$  \hspace{1cm} (2.26)$$
with
\[ \Sigma_\alpha^*(i, \tau) \Sigma_\alpha(i, \tau) = 1 \quad (2.27) \]
Clearly, there is an arbitrariness in the choice of the phases of \( H^* \) and \( \Sigma_\alpha \); in fact if
\[ H^*(i, \tau) \rightarrow H^*(i, \tau) e^{-i\theta(i, \tau)}, \]
\[ \Sigma_\alpha(i, \tau) \rightarrow e^{i\theta(i, \tau)} \Sigma_\alpha(i, \tau), \quad (2.28) \]
the field \( \varphi_\alpha \) remains unchanged. However, modulo this \( U(1) \) gauge invariance, one can still consider \( H^* \) and \( \Sigma_\alpha \) as the generalized “modulus” and “phase” of \( \varphi_\alpha \). Indeed, the constraint (2.27) expresses the fact that \( \Sigma_\alpha \) is confined on the unit circle like any phase, whereas \( H^* \) must be a hard-core field such that
\[ H^*(i, \tau) H^*(i, \tau) = 0 \]
in order for \( \varphi_\alpha \) to be a hard-core complex bosonic field as required by the bosonization procedure. \( H^* \) and \( \Sigma_\alpha \) are not yet the most convenient holon and spinon fields, but these can be obtained with a further change of variables. In fact, as shown in [18], there are some simplifications if one introduces new fields \( \tilde{E}(i, \tau) \) and \( S_\alpha(i, \tau) \) according to
\[ H^*(i, \tau) = \left( 1 - \tilde{E}^*(i, \tau) \tilde{E}(i, \tau) \right)^{-1/2} \tilde{E}(i, \tau), \]
\[ \Sigma_\alpha(i, \tau) = \left( 1 - \tilde{E}^*(i, \tau) \tilde{E}(i, \tau) \right)^{-1/2} S_\alpha(i, \tau). \quad (2.29) \]
In terms of these new fields the constraint (2.27) becomes
\[ \tilde{\Phi}(i, \tau) \equiv \tilde{E}^*(i, \tau) \tilde{E}(i, \tau) + S_\alpha^*(i, \tau) S_\alpha(i, \tau) - 1 = 0 \quad (2.30) \]
which is formally similar to (2.11).

In the abelian bosonization of the \( t-J \) model proposed in [17,18], the Chern-Simons field \( A_\lambda \) is coupled either only to the holons \( \tilde{E} \) or only to the spinons \( S_\alpha \). In the first case the bosonic holons are transmuted into fermions after integrating out \( A_\lambda \), while the spinons remain bosonic realizing, in this way, the slave fermion representation. To see this, let us recall that the bosonized action of the \( t-J \) model, written in terms \( \tilde{E} \) and \( S_\alpha \), is
\[ S(\tilde{E}, S; A) = \int_0^\beta d\tau \left\{ \sum_{i \in \Omega} \left[ \tilde{E}^*(i, \tau) \frac{\partial}{\partial \tau} \tilde{E}(i, \tau) + i A_0(i, \tau) \left( 1 - \tilde{E}^*(i, \tau) \tilde{E}(i, \tau) \right) \right] + S_\alpha^*(i, \tau) \frac{\partial}{\partial \tau} S_\alpha(i, \tau) + \mu S_\alpha^*(i, \tau) S_\alpha(i, \tau) \right) \]
\[ - t \sum_{<i,j>} \left[ \tilde{E}^*(i, \tau) e^{i \int_{<i,j>} A_\lambda(x) dx} \tilde{E}(j, \tau) S_\alpha^*(i, \tau) S_\alpha(j, \tau) + h.c. \right] \]
\[ + \mathcal{H}_J(S) + \sum_{i \in \Omega} i \Lambda(i, \tau) \tilde{\Phi}(i, \tau) \right\} \quad (2.31) \]
where $\Lambda(i, \tau)$ is the Lagrange multiplier enforcing the constraint (2.30), and $\mathcal{H}_J(S)$ is given by (2.18b) with $\psi_\alpha(i, \tau)$ and $\psi^*_\alpha(i, \tau)$ replaced by $S_\alpha(i, \tau)$ and $S^*_\alpha(i, \tau)$ respectively. As mentioned before, in (2.31) the Chern-Simons field is coupled (in the standard minimal way) only to the holon fields $\tilde{E}$ and $\tilde{E}^*$. Using this action and (2.19), the partition function of the $t$-$J$ model is formally given by

$$Q(\beta, \mu) = \left\langle \int \mathcal{D}\tilde{E} \mathcal{D}S \ e^{-\hat{S}(\tilde{E}, S; A)} \right\rangle_{k=1}.$$  \hfill (2.32)

Finally, after computing the expectation value over the Chern-Simons field (or equivalently after eliminating $A_\lambda$ through its field equations) one gets the effective action of $t$-$J$ model in the slave fermion representation, i.e.

$$S_{\text{eff}}(E, S) = \int_0^\beta d\tau \left\{ \sum_{i \in \Omega} \left[ E^*(i, \tau) \frac{\partial}{\partial \tau} E(i, \tau) + S^*_\alpha(i, \tau) \frac{\partial}{\partial \tau} S_\alpha(i, \tau) \right] - t \sum_{\langle i, j \rangle} \left[ E^*(i, \tau) E(j, \tau) S^*_\alpha(i, \tau) S_\alpha(j, \tau) + \text{h.c.} \right] \right\} + \mathcal{H}_J(S) + \sum_{i \in \Omega} \left[ \mu \ S^*_\alpha(i, \tau) S_\alpha(i, \tau) + i \Lambda(i, \tau) \Phi(i, \tau) \right].$$ \hfill (2.33)

Here $E$ and $E^*$ denote the hard-core holon fields dressed by the effective Chern-Simons contribution. They are defined as follows

$$E(i, \tau) \equiv e^{\sum_{j \in \Omega} \Theta(i, j) \left( \tilde{E}^*(j, \tau) \tilde{E}(j, \tau) - 1 \right)} \tilde{E}(i, \tau)$$ \hfill (2.34)

where $\Theta(i, j)$ is the lattice angle function (see [26,27,28] and Section 4 for a detailed discussion of its properties). Using the canonical commutation relations of $\tilde{E}$ and $\tilde{E}^*$, it is easy to show that $E$ and $E^*$ are fermions satisfying the equal time anticommutation relations

$$\left\{ E(i, \tau), E^*(j, \tau) \right\} = \delta(i, j).$$ \hfill (2.35)

Furthermore, from (2.34) it is clear that

$$E^*(i, \tau) E(i, \tau) = \tilde{E}^*(i, \tau) \tilde{E}(i, \tau),$$

and hence

$$\Phi(i, \tau) \equiv E^*(i, \tau) E(i, \tau) + S^*_\alpha(i, \tau) S_\alpha(i, \tau) - 1 = \tilde{\Phi}(i, \tau).$$

Using these results, it becomes evident that the gran canonical partition function can be written as a path integral over both fermionic and bosonic fields according to

$$Q(\beta, \mu) = \int \mathcal{D}E \mathcal{D}S \ e^{-S_{\text{eff}}(E, S)}.$$ \hfill (2.36)
We observe that this path integral implements correctly the fermionic statistics of the original electronic problem. From the effective action (2.33) one can easily derive the Hamiltonian of the $t$-$J$ model in the slave fermion description; when written in the operator formalism, such Hamiltonian is exactly the same as the one given in (2.15), which we obtained by simply inserting (2.10) into (2.9).

As remarked in [18], also the slave boson description of the $t$-$J$ model can be realized via abelian bosonization. In fact, if we start from an action similar to (2.31) but with the Chern-Simons field $A_\lambda$ coupled only to the spinons $S_\alpha$, upon elimination of $A_\lambda$, these are transmuted into fermions while the holons remain bosons. Using the bosonization formalism, it becomes apparent that the slave fermion and the slave boson descriptions are deeply related to each other and can be viewed as two different gauge fixings of an abelian gauge invariance exhibited by the bosonized Hamiltonian of the $t$-$J$ model.

The authors of [18] considered also a non abelian variant of this bosonization technique by introducing two Chern-Simons gauge fields: One abelian, and one in the fundamental representation of $SU(2)$. In order to represent correctly the fermionic statistics of the electronic problem, the coefficients of the two Chern-Simons terms must be judiciously chosen (see e.g. Eq. (3.10) of [18]). Furthermore, in order to describe only abelian representations of the braid group (i.e. fields whose statistics is simply described by a phase and not by a braiding matrix), the level $k$ of the $SU(2)$ Chern-Simons term must be related to the spin $s$ of the $SU(2)$ representation according to $k = \pm 2s$. As mentioned, the $SU(2)$ field is taken in the fundamental representation with $s = 1/2$ (the non-relativistic electrons are indeed a doublet of $SU(2)$), and thus $k$ must be either $+1$ or $-1$. In this non abelian bosonization, the abelian field is coupled to the (bosonic) holons while the $SU(2)$ field is coupled to the (bosonic) spinons; integrating out these Chern-Simons fields, holons and spinons are transmuted into semions.

Now we are going to generalize (and to simplify) the construction of [18] by showing that the semionic statistics of holons and spinons can be realized within the context of the abelian bosonization without using non abelian Chern-Simons fields. More generally, we will show that holons and spinons in the two-dimensional $t$-$J$ model can be arbitrary anyons subject only to the constraint that the composite object made of one holon and one spinon be a fermion. We call this the slave anyon representation of the $t$-$J$ model which, as we will show later, smoothly interpolates between the slave fermion and the slave boson descriptions that we mentioned earlier. Once again we stress that this “anyonization” is a purely formal operation based on mathematical manipulations on the partition function; to see which statistics actually occurs, requires a serious investigation of the dynamical aspects of the model.

Let us now give some details of our construction. We start from the effective action (2.33) in the slave fermion picture and, as in [18], we introduce two independent Chern-Simons fields, $B_\lambda$ and $V_\lambda$, but, contrary to [18], we take both of them abelian \(^4\). Then we minimally couple the fermionic holon field $E$ only to $B_\lambda$ and the bosonic...

\(^4\) One could equivalently consider the action (2.31) and introduce only a new Chern-Simons field $V_\lambda$ coupled to $S_\alpha$ and suitably change the coupling constant in $S_{CS}(A)$. 

spinon field $S_{\alpha}$ only to $V_{\lambda}$. The resulting action is

$$\tilde{S}(E, S; B, V) = \int_{0}^{\beta} d\tau \left\{ \sum_{i \in \Omega} \left[ E^{*}(i, \tau) \frac{\partial}{\partial \tau} E(i, \tau) + i B_{0}(i, \tau) \left( 1 - E^{*}(i, \tau) E(i, \tau) \right) \right] \\
+ S_{\alpha}^{*}(i, \tau) \frac{\partial}{\partial \tau} S_{\alpha}(i, \tau) + i V_{0}(i, \tau) S_{\alpha}^{*}(i, \tau) S_{\alpha}(i, \tau) \right\} \right. \\
- t \sum_{<i,j>} \left[ E^{*}(i, \tau) e^{i \int_{<i,j>} B_{\lambda}(x) dx} E(j, \tau) \times \\
\times S_{\alpha}^{*}(i, \tau) e^{i \int_{<i,j>} V_{\lambda}(x) dx} S_{\alpha}(j, \tau) + h.c. \right] \\
+ \mathcal{H}_{J}(S) + \sum_{i \in \Omega} \left[ \mu S_{\alpha}^{*}(i, \tau) S_{\alpha}(i, \tau) + i \Lambda(i, \tau) \Phi(i, \tau) \right] \right\} . \quad (2.37)$$

Notice that the time component $B_{0}$ is coupled to $(1 - E^{*} E)$ and not simply to the density $E^{*} E$: This is not an artifact but is a direct consequence of expressing the original action of the $t$-$J$ model in terms of $E$ and $S_{\alpha}$ (cf Eq. (2.31)). The other couplings with the gauge fields are no surprise, and one easily sees that

$$\left. \tilde{S}(E, S; B, V) \right|_{B = V = 0} = S_{\text{eff}}(E, S) .$$

In the spirit of the bosonization procedure, there should be no difference even when the two Chern-Simons fields are non vanishing. In particular, the introduction of $B_{\lambda}$ and $V_{\lambda}$, with their corresponding actions at level $k_{B}$ and $k_{V}$ respectively (see Eq. (2.21)), should not spoil the overall fermionic statistics that is already implemented in a correct way by the effective action (2.33). Therefore the levels of the two Chern-Simons terms must be carefully chosen. Using (2.23), it is not difficult to realize that the correct statistics is maintained if

$$k_{B} = -k_{V} \mod 2 . \quad (2.38)$$

The minimal choice is

$$k_{B} = -k_{V} = \frac{1}{\nu} \quad (2.39)$$

where the parameter $\nu$ is introduced for later convenience. Contrarily to the non abelian bosonization of [18] where the levels of the Chern-Simons terms are both fixed, in our approach there is a free parameter which, as we will see in a moment, allows to realize anyons of arbitrary statistics and not only semions.

In this approach the grand canonical partition function is

$$Q(\beta, \mu) = \left\langle \left\langle \int DEDS \; e^{-\tilde{S}(E, S; B, V)} \right\rangle \right\rangle_{\nu} . \quad (2.40)$$
where the symbol $\langle \langle \cdot \rangle \rangle_\nu$ denotes the expectation value with respect to the Chern-Simons fields $B_\lambda$ and $V_\lambda$ whose levels are given as in (2.39). Eliminating $B_\lambda$ and $V_\lambda$ through their field equations, the resulting action can be entirely expressed in terms of new fields, $\hat{E}$ and $\hat{S}_\alpha$ and their complex conjugates, defined as follows

$$
\hat{E}(i, \tau) \equiv e^{i\nu \sum_{j \in \Omega} \Theta(i,j) \left( E^*(j,\tau) E(j,\tau) - 1 \right)} E(i, \tau),
$$

$$
\hat{S}_\alpha(i, \tau) \equiv e^{-i\nu \sum_{j \in \Omega} \Theta(i,j) S^*_\alpha(j,\tau) S_\alpha(j,\tau)} S_\alpha(i, \tau).
$$

These are anyonic fields of statistics $\nu$ and $-\nu$ respectively, and, as such, satisfy braiding relations among themselves. We postpone the detailed discussion of their properties to Section 4; here we simply notice that $\hat{E}$ is a fermion based anyon whilst $\hat{S}_\alpha$ is boson based. The effective action resulting from (2.40) realizes the slave anyon representation of the $t$-$J$ model and is essentially given by (2.33) with $E$ and $S_\alpha$ replaced by $\hat{E}$ and $\hat{S}_\alpha$. However, this representation is purely formal because the anyonic fields (2.41) are non local. In fact, the angle function appearing in the exponents of (2.41) is multivalued and, to remove all ambiguities, a cut and a fundamental domain must be chosen implying that the resulting fields become non-local [29]. Of course, if $\nu = 1$ or $\nu = 0$ these problems cease to exist since in these circumstances the anyons becomes ordinary bosons or fermions. These two cases are clearly in direct correspondence with the slave fermion and slave boson representations that we discussed earlier, and thus our slave anyon representation smoothly interpolates between them. It is worth to point out that the effective action $S_{\text{eff}}(\hat{E}, \hat{S}_\alpha)$ is well-defined and unambiguous for any value of $\nu$ and does not depend on the detailed definition of the angle function; however due to the non locality of $\hat{E}$ and $\hat{S}_\alpha$, it is problematic to path-integrate over them. Therefore, in the functional formalism the slave anyon representation must be written in terms of bosonic and fermionic fields coupled to two abelian Chern-Simons fields as in (2.37), with the grand canonical partition function given by (2.40). Things are different in the operator formalism where one can define generalized commutation relations for anyonic operators. These are actually braiding relations which depend crucially on how the angle function is defined, as we will discuss in Section 4.
3. The Supersymmetric $t$-$J$ Model

When the coupling constants and the chemical potential of the $t$-$J$ model are

$$J = 2t \quad , \quad \mu = \gamma(d) t$$

(3.1)

where $\gamma(d)$ is the coordination number of the $d$-dimensional lattice $\Omega$, the Hamiltonian (2.1) enjoys very special properties: It can be written as a graded permutation operator (up to an irrelevant constant term proportional to the number of lattice points) and it commutes with the generators of a superalgebra $SU(1|2)$ [6]. In $d = 1$ the $t$-$J$ model at the supersymmetric point (3.1) can be exactly solved by Bethe Ansatz [7,8], and it is tempting to speculate that also in higher dimensions the supersymmetry has some non trivial implications.

Let us first observe that there is a very simple realization of the superalgebra $SU(1|2)$ in terms of the projected fermionic operators (2.6) introduced in Section 2. In fact, defining

$$J^+(i) = c'^\dagger_+(i) c'_-(i) \quad , \quad J^-(i) = c'^\dagger_-(i) c'_+(i) \quad ,$$

$$J^0(i) = \frac{1}{2} (n'_+(i) - n'_-(i)) \quad ,$$

$$T(i) = 1 - \frac{1}{2} (n'_+(i) + n'_-(i)) \quad ,$$

$$Q_\alpha(i) = c'_\alpha(i) \quad , \quad Q^{\dagger}_\alpha(i) = c'^\dagger_\alpha(i) \quad (\alpha = \pm 1) \quad ,$$

(3.2)

and using (2.7), it is immediate to verify that

$$[J^+, J^-] = 2 J^0 \quad , \quad [J^0, J^\pm] = \pm J^\pm \quad,$$

$$\{Q^{\dagger}_\alpha, Q_{-\alpha}\} = J^\alpha \quad , \quad \{Q^{\dagger}_\alpha, Q_{\alpha}\} = \alpha J^0 + T \quad ,$$

$$[J^0, Q_\alpha] = -\frac{\alpha}{2} Q_\alpha \quad , \quad [J^0, Q^{\dagger}_\alpha] = \frac{\alpha}{2} Q^{\dagger}_\alpha \quad ,$$

$$[T, Q_\alpha] = \frac{1}{2} Q_\alpha \quad , \quad [T, Q^{\dagger}_\alpha] = -\frac{1}{2} Q^{\dagger}_\alpha \quad ,$$

(3.3)

where $J^+ = \sum_{i \in \Omega} J^+(i)$, and so on. These are precisely the graded commutation relations of the superalgebra $SU(1|2)$, which in some circles is also denoted by spl$(2,1)$. Notice that $Q^{\dagger}_\alpha$ and $Q^{\dagger}_\alpha$ for $\alpha = \pm 1$ form two doublets of fermionic generators; and hence one can say that (3.3) is a $N = 2$ supersymmetry algebra [31].

---

5 Given any lattice site $i \in \Omega$, the coordination number is defined as the number of points which are nearest neighbors to $i$. For example, for a square lattice one has $\gamma(d) = 2d$. 

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Choosing the coupling constants of $H_{tJ}$ as in (3.1), it is easy to check that
\[
[T^A, H_{tJ}] = 0
\]  \hspace{1cm} (3.4)
where $T^A$ is any one of the eight generators of $SU(1|2)$. Thus the $t$-$J$ model with $J = 2t$ and $\mu = \gamma(d) t$ is manifestly $N = 2$ supersymmetric in any dimension $d$. We denote the corresponding Hamiltonian by $H_{tJ}^{\text{SUSY}}$. It is interesting to observe that the supersymmetry is non-linearly realized in $H_{tJ}^{\text{SUSY}}$; in fact the action of the generators $T^A$ on the projected electronic operators yields expressions which, in general, are non-linear. For example one finds
\[
\{Q_+, c^\dagger_- (i)\} = c'^\dagger_-(i) c'_+(i) \ .
\]  \hspace{1cm} (3.5)
We now show that there is a simpler way to describe this supersymmetry by using the slave fermion or the slave boson representations with which it can be realized in a linear way. To avoid repetitions, here we limit our discussion to the slave fermion case, while in the next section we will discuss in detail the supersymmetry in the slave anyon representation.

Let us then consider the fermionic holon operators $e(i)$ and $e^\dagger(i)$, and the bosonic spinon operators $s_\alpha(i)$ and $s^\dagger_\alpha(i)$ which satisfy the (anti)commutation relations (2.12), and let us define
\[
J^+ = \sum_{i \in \Omega} s^\dagger_+(i) s_-(i) \ , \quad J^- = \sum_{i \in \Omega} s^\dagger_-(i) s_+(i) \ ,
\]  \hspace{1cm} (3.6a)
\[
J^0 = \sum_{i \in \Omega} \frac{1}{2} \left( s^\dagger_+(i) s_+(i) - s^\dagger_-(i) s_-(i) \right) \ ,
\]  \hspace{1cm} (3.6b)
\[
T = \sum_{i \in \Omega} \left( e^\dagger(i) e(i) + \frac{1}{2} s^\dagger_\alpha(i) s_\alpha(i) \right) \ ,
\]  \hspace{1cm} (3.6c)
\[
Q_\alpha = \sum_{i \in \Omega} e^\dagger(i) s_\alpha(i) \ , \quad Q^\dagger_\alpha = \sum_{i \in \Omega} e(i) s^\dagger_\alpha(i) \quad (\alpha = \pm 1) \ .
\]
Using (2.12), it is very easy to check that these eight operators close the superalgebra $SU(1|2)$ given in (3.3). Furthermore, the action of these generators on holons and spinons is always linear. In particular, the $SU(2)$ generators $J^\pm$ and $J^0$ commute with $e(i)$ and $e^\dagger(i)$, which are indeed scalars under rotations, while on $s_\alpha(i)$ and $s^\dagger_\alpha(i)$ they act as follows
\[
\left[ J^0, s^\dagger_\alpha(i) \right] = \frac{\alpha}{2} s^\dagger_\alpha(i) \ , \quad \left[ J^0, s_\alpha(i) \right] = -\frac{\alpha}{2} s_\alpha(i) \ ,
\]  \hspace{1cm} (3.7a)
\[
\left[ J^\alpha, s^\dagger_\alpha(i) \right] = s^\dagger_\alpha(i) \ , \quad \left[ J^\alpha, s_\alpha(i) \right] = -s_\alpha(i) \ .
\]  \hspace{1cm} (3.7a)
(Of course, in the last two commutators there is no sum over $\alpha$.) These commutation relations simply express the fact that $s_\alpha(i)$ and $s^\dagger_\alpha(i)$ for $\alpha = \pm 1$ are two spin $1/2$
doublets of $SU(2)$. The abelian generator $T$ measures the $U(1)$ charge of holons and spinons according to

\[ [T, s_{\alpha}(i)] = -\frac{1}{2}s_{\alpha}(i) \quad , \quad [T, s^\dagger_{\alpha}(i)] = \frac{1}{2}s^\dagger_{\alpha}(i) , \]
\[ [T, e(i)] = -e(i) \quad , \quad [T, e^\dagger(i)] = e^\dagger(i) . \]  

(3.7b)

Finally, the supersymmetry generators $Q_\alpha$ and $Q^\dagger_\alpha$ transform the fermionic holons into the bosonic spinons and vice-versa. More precisely one finds

\[ [Q_\alpha, s^\dagger_\beta(i)] = e^\dagger(i) \delta_{\alpha\beta} \quad , \quad [Q^\dagger_\alpha, s_\beta(i)] = -e(i) \delta_{\alpha\beta} \quad , \]
\[ \{Q_\alpha, e(i)\} = s_\alpha(i) \quad , \quad \{Q^\dagger_\alpha, e^\dagger(i)\} = s^\dagger_\alpha(i) \]  

with the other possible (anti)commutators vanishing.

Before looking at the invariance properties of the Hamiltonian, let us observe that the constraint $\phi(i)$ in (2.11) is invariant under $SU(1|2)$, i.e.

\[ [T^A, \phi(i)] = 0 . \]  

(3.8)

This property is certainly welcome and will have very important implications in the following. Let us now turn to the Hamiltonian $H_{tJ}$ in the slave fermion representation given by (2.15), and chose the coupling constants as in (3.1). A simple calculation shows that this Hamiltonian commutes with the generators of $SU(1|2)$ only if the Lagrange multiplier $\Lambda(i)$ transforms in a non trivial way under the supersymmetries. To be precise one finds that

\[ [Q_\alpha, H_{tJ}] = 0 \]

only if

\[ [Q_\alpha, \Lambda(i)] = -i \frac{J}{2} \left( \Delta + \gamma(d) \right) e^\dagger(i) s_\alpha(i) \]  

(3.9)

where $\Delta$ is the lattice Laplace operator. Even though acceptable on general grounds, this transformation property is rather complicated, especially when compared with the ones of the other operators entering the Hamiltonian (see (3.7)). Moreover it poses serious problems in setting up a mean field theory without breaking the supersymmetry. Indeed, the mean field approximation (MFA) instructs us to replace the operator $\Lambda(i)$ with its mean value $\lambda(i)$ which is a number and, as such, commutes with the supersymmetry generators. Therefore, if $\Lambda(i)$ has non trivial transformation properties like (3.9), the MFA certainly breaks the supersymmetry. This is very unsatisfactory because the mean field approximation is one of the most viable methods to analyze the theory in dimensions bigger than one, and if (3.9) were unavoidable, it would seem that the role of the supersymmetry of the $t-J$ model can not be investigated in this approach.
However, this problem can be overcome. Indeed, since the Hilbert space of the \( t-J \) model is characterized by \( \phi(i) = 0 \), we can add to the Hamiltonian (2.15) any expression proportional to \( \phi(i) \) without changing the physical dynamics of the system. Thus we can try to modify the Hamiltonian \( H_{tJ} \) in such a way that the supersymmetry be realized without transforming the Lagrange multiplier. This is possible if we consider the following operator

\[
H'_{tJ} = H_{tJ} + \frac{J}{4} \sum_{<i,j>} \left[ \left( s^\dagger_\alpha(i) s_\alpha(i) - e^\dagger(i) e(i) \right) \phi(j) + \left( i \leftrightarrow j \right) \right]
\]  

(3.10)

where \( H_{tJ} \) is the Hamiltonian (2.15) with the coupling constants chosen according to (3.1). Because of our previous observation, \( H'_{tJ} \) is equivalent to \( H_{tJ} \), since their difference vanishes on the physical states. With a straightforward calculation, one can easily prove that the new Hamiltonian \( H'_{tJ} \) commutes with all generators of \( SU(1|2) \) without requiring any transformation property of the Lagrange multiplier \( \Lambda(i) \), namely

\[
\left[ T^A, H'_{tJ} \right] = 0 \quad \text{and} \quad \left[ T^A, \Lambda(i) \right] = 0 .
\]  

(3.11)

Now the mean field approximation can be used on \( H'_{tJ} \) without breaking the supersymmetry. Actually, we can go even further. With simple algebra, the Hamiltonian (3.10) can be rewritten as follows \(^6\)

\[
H'_{tJ} = \frac{J}{2} \sum_{<i,j>} \left| s^\dagger_\alpha(i) s_\alpha(j) + e^\dagger(i) e(j) \right|^2 + \sum_{i \in \Omega} \left[ \frac{J}{4} \gamma(d) \left( s^\dagger_\alpha(i) s_\alpha(i) + e^\dagger(i) e(i) \right) + i \Lambda(i) \phi(i) \right] .
\]  

(3.12)

Using (2.11), we then have

\[
H'_{tJ} = \frac{J}{2} \sum_{<i,j>} \left| s^\dagger_\alpha(i) s_\alpha(j) + e^\dagger(i) e(j) \right|^2 + \sum_{i \in \Omega} i \Lambda'(i) \phi(i) - \frac{J}{4} \gamma(d) V
\]  

(3.13)

where

\[
\Lambda'(i) = \Lambda(i) - i \frac{J}{4} \gamma(d) ,
\]

and \( V \) is the volume of \( \Omega \), \( i.e. \) the number of lattice points.

This last form of the Hamiltonian is particularly simple and suggestive. In fact, using (3.7) it is easy to see that the expression inside the absolute value sign,

\[
G(i, j) = s^\dagger_\alpha(i) s_\alpha(j) + e^\dagger(i) e(j)
\]  

(3.14)

\(^6\) Here and in the following, for any operator \( O \), the symbol \( |O|^2 \) means \( O^\dagger O \).
is invariant under $SU(1|2)$. Then, since the constraint $\phi(i)$ is also invariant, $H'_{tJ}$ commutes with the generators of $SU(1|2)$ only if the Lagrange multiplier $\Lambda'(i)$ does not transform, as desired. Moreover, when written as in (3.13), the Hamiltonian of the $t$-$J$ model appears as the natural generalization of the Heisenberg XXX model. Indeed, if we delete all $e$ and $e^\dagger$ operators from (3.13), we recover precisely the antiferromagnetic Heisenberg model which, as well known, is invariant under $SU(2)$. Thus, we could say that the supersymmetric $t$-$J$ model is the natural generalization of the Heisenberg model, when the symmetry algebra $SU(2)$ is generalized to $SU(1|2)$. A further generalization of the $t$-$J$ model in this respect, is given by the model recently proposed in [31], which is invariant under the superalgebra $SU(2|2)$. In fact, when written in the slave fermion representation, the Hamiltonian of [31] is like (3.13) but with two sets of fermionic oscillators, corresponding respectively to holons and localons. The latter are possible because in this model the double electronic occupancy on the same site is allowed and local pairs of electrons (localons) can arise in the spectrum.

The supersymmetric Hamiltonian (3.13) is particularly suited for a further simplification: We can perform on it the Hubbard-Stratonovich transformation [32] to simplify the $J$-term, which is quartic in the operators, without breaking the supersymmetry. In fact, let us introduce a complex bosonic Hubbard-Stratonovich operator $F_{ij}$ defined on the link between the nearest neighbor points $i$ and $j$, and consider the Hamiltonian

$$
\tilde{H}'_{tJ} = \sum_{<i,j>} \left[ -\frac{2}{J} F_{ij}^\dagger F_{ij} + F_{ij}^\dagger \left( s^\dagger_{\alpha}(i) s_{\alpha}(j) + e^\dagger(i) e(j) \right) + \left( s^\dagger_{\alpha}(j) s_{\alpha}(i) + e^\dagger(j) e(i) \right) F_{ij} \right] + \sum_{i \in \Omega} i \Lambda'(i) \phi(i) - \frac{J}{4} \gamma(d) V .
$$

(3.15)

Upon elimination of $F_{ij}$ through its “equation of motion”, $\tilde{H}'_{tJ}$ becomes equivalent to $H'_{tJ}$ and thus it describes the same dynamics. The advantage of writing the Hamiltonian as in (3.15) is that the coupling among the operators is much simpler than in (3.13): There are no more quartic interactions. Moreover, since the quantity in (3.14) is invariant under $SU(1|2)$ as we have previously remarked, $\tilde{H}'_{tJ}$ is invariant only if the Hubbard-Stratonovich operator does not transform, $i.e.$

$$
\left[ T^A, F_{ij} \right] = 0 .
$$

(3.16)

Again this property is welcome, because it allows to make the MFA on $\tilde{H}'_{tJ}$ without breaking supersymmetry. In fact, we can replace the operators $F_{ij}$ and $\Lambda'(i)$ with their corresponding mean values $f_{ij}$ and $\lambda'(i)$, determined in a self-consistent way, and get the mean field Hamiltonian

$$
\tilde{H}_{MF} = \sum_{<i,j>} \left[ -\frac{2}{J} f_{ij}^\dagger f_{ij} + f_{ij}^\dagger \left( s^\dagger_{\alpha}(i) s_{\alpha}(j) + e^\dagger(i) e(j) \right) + \left( s^\dagger_{\alpha}(j) s_{\alpha}(i) + e^\dagger(j) e(i) \right) f_{ij} \right] + \sum_{i \in \Omega} i \lambda'(i) \phi(i) - \frac{J}{4} \gamma(d) V
$$

(3.17)
which is invariant under $SU(1|2)$.

A few remarks are in order. First of all, once the MFA is made and in particular $\Lambda'(i)$ is replaced by $\lambda'(i)$, the constraint $\phi(i) = 0$ is not enforced any more. Then, the condition that only the states without double electronic occupancy are permitted is released, and in principle new states and excitations, where the number of holons and spinons is not restricted as in (2.11), become allowed. Secondly, one can improve the MFA without breaking supersymmetry if one considers the phase of the complex field $f_{ij}$ and the amplitude of $\lambda'(i)$ as fluctuating. In other words, one writes

$$f_{ij} = \tilde{f}_{ij} e^{i A_{ij}}, \quad \lambda'(i) = \tilde{\lambda}(i) + A(i)$$

(3.18)

where $\tilde{f}_{ij}$ and $\tilde{\lambda}(i)$ are fixed complex numbers while $A_{ij}$ and $A(i)$ are dynamical variables. In particular, if one chooses $\tilde{f}_{ij} = \tilde{f}$ and $\lambda(i) = \lambda$, where $\tilde{f}$ and $\tilde{\lambda}$ are real constants, one selects the so-called uniform phase of the slave fermion representation [13,20], whose Hamiltonian is

$$\tilde{H} = \tilde{f} \sum_{<i,j>} \left( s_{\alpha}^\dagger(i) e^{i A_{ij}} s_{\alpha}(j) + e^{i A_{ij}} e^{i A_{ij}} e(i(j) + h.c. \right)$$

$$+ \sum_{i \in \Omega} i A(i) \left( s_{\alpha}^\dagger(i) s_{\alpha}(i) + e^{i A_{ij}} e(i) \right)$$

$$+ i \tilde{\lambda} \sum_{i \in \Omega} \left( s_{\alpha}^\dagger(i) s_{\alpha}(i) + e^{i A_{ij}} e(i) \right).$$

(3.19)

The first two lines can be interpreted as the Hamiltonian of a non relativistic fermion (holon) and a pair of non relativistic spin 1/2 bosons (spinons) minimally coupled to an Abelian gauge field whose components $A_\mu(i)$ are

$$A_0(i) = A(i), \quad A_1(i) = A_{i,i+\hat{1}}, \quad A_2(i) = A_{i,i+\hat{2}}$$

(3.20)

where $\hat{1}$ and $\hat{2}$ denote the unit lattice vectors along the $x$- and $y$-axis respectively. This gauge field can be identified with the resonating valence bond (RVB) field introduced in [2,3,11]. A system similar to that described by (3.19) was also considered and analyzed in [33], but there the masses of the fermions and bosons were taken to be different, thus breaking explicitly the supersymmetry.

We conclude this section with a few comments. First of all, the discussion we have presented using the operators in the slave fermion representation can be easily repeated in the slave boson formalism, where the holons are bosonic and the spinons fermionic. In this case most of the formulas, in particular the Hamiltonians (3.13), (3.15), (3.17) and (3.19), remain unchanged. The only formal difference is in the action of the supersymmetry generators on holons and spinons, since, changing the
statistics, the commutators must be replaced by anticommutators and vice-versa. Indeed, Eqs. (3.7c) are replaced by

\[ \{ Q_\alpha , s_\beta ^\dagger (i) \} = e^\dagger (i) \delta _{\alpha \beta } , \quad \{ Q_\alpha ^\dagger , s_\beta (i) \} = e(i) \delta _{\alpha \beta } , \]

\[ [ Q_\alpha , e(i) ] = -s_\alpha (i) , \quad [ Q_\alpha ^\dagger , e^\dagger (i) ] = s_\alpha ^\dagger (i) . \]  

(3.21)

On the contrary, (3.7a-b) remain valid also in the slave boson formalism, because \( J^\pm , J^0 \) and \( T \) are bosonic generators which always close commutators independently of the statistics of the operators on which they act.

Finally, we observe that the analysis of the supersymmetry properties of the \( t-J \) model as presented in this section, can be equivalently formulated in the functional approach, using fields instead of operators. In this case, the action of the generators of the superalgebra \( SU(1|2) \) on holons and spinons is expressed as transformation laws of the corresponding fields, instead of (anti)commutators. For example, using the notations of Section 2, the infinitesimal variations of the holon and spinon fields turn out to be

\[ \delta E(i, \tau) = -\eta E(i, \tau) + \epsilon_+ S_+(i, \tau) + \epsilon_- S_-(i, \tau) , \]

\[ \delta E^*(i, \tau) = +\eta E^*(i, \tau) + \epsilon_+^* S^*_+(i, \tau) + \epsilon_-^* S^*_-(i, \tau) , \]

\[ \delta S_+(i, \tau) = -\frac{\theta^0}{2} S_+(i, \tau) - \theta^+ S_-(i, \tau) - \frac{\eta}{2} S_+(i, \tau) - \epsilon_+^* E(i, \tau) , \]

\[ \delta S_-(i, \tau) = +\frac{\theta^0}{2} S_-(i, \tau) - \theta^- S_+(i, \tau) - \frac{\eta}{2} S_-(i, \tau) - \epsilon_-^* E(i, \tau) , \]

\[ \delta S^*_+(i, \tau) = +\frac{\theta^0}{2} S^*_+(i, \tau) + \theta^- S^*_-(i, \tau) + \frac{\eta}{2} S^*_+(i, \tau) + \epsilon_+ E^*(i, \tau) , \]

\[ \delta S^*_-(i, \tau) = -\frac{\theta^0}{2} S^*_-(i, \tau) + \theta^+ S^*_+(i, \tau) + \frac{\eta}{2} S^*_-(i, \tau) + \epsilon_- E^*(i, \tau) . \]

(3.22)

Here \( \theta^0 , \theta^\pm \) and \( \eta \) are infinitesimal real parameters associated to the bosonic generators \( J^0 , J^\pm \) and \( T \) respectively, whereas \( \epsilon_\alpha \) and \( \epsilon^*_\alpha \) are infinitesimal grassmann parameters associated to the fermionic generators \( Q_\alpha \) and \( Q_\alpha ^\dagger \). It is straightforward to verify that the action (2.33) is invariant under the transformations (3.22) if the coupling constants are chosen as in (3.1) and the constraint \( \Phi(i, \tau) = 0 \) is enforced. Notice that (3.22) are valid both in the slave fermion and in the slave boson representations, the only difference being the statistics of the fields. On the contrary, (3.22) are not suited for a discussion of the supersymmetry of the \( t-J \) model in the slave anyon representation, since in this case holons and spinons are anyonic non-local objects for which a functional approach is not well-defined (cf (2.41) and following comments). Therefore, the analysis of the invariance of the \( t-J \) model under the superalgebra \( SU(1|2) \) in the slave anyon representation must be performed in the operator approach, and to this task we turn in the following section.
4. The Slave Anyon Representation of the Supersymmetric \( t-J \) Model

In Section 2 we discussed the slave anyon representation of the two-dimensional \( t-J \) model, and showed that holons and spinons become anyons of opposite statistics if one introduces two independent Chern-Simons fields in the bosonization procedure. In this section we want to extend the analysis of the supersymmetry properties presented in Section 3 to the slave anyon representation of the \( t-J \) model. To this aim, we first introduce anyonic operators by means of a generalized Jordan-Wigner transformation [34,35]. This is given by Eq. (2.41), which, when written in terms of oscillators (denoted by lower case letters), becomes

\[
\hat{e}(i) \equiv e^{i \nu \sum_{j \in \Omega} \Theta(i,j) \left( e^\dagger(j) e(j) - 1 \right)} e(i),
\]

\[
\hat{s}_\alpha(i) \equiv e^{-i \nu \sum_{j \in \Omega} \Theta(i,j) s^\dagger_\alpha(j) s_\alpha(j)} s_\alpha(i).
\]

Here \( e(i) \) and \( s_\alpha(i) \) are the holon and spinon oscillators in the slave fermion description, \( \nu \) is the statistical parameter related to the Chern-Simons actions as in (2.39), and \( \Theta(i,j) \) is the lattice angle function. The exponential factors in (4.1) are sometimes called disorder operators [35]. One can prove that \( \hat{e}(i) \) and \( \hat{s}_\alpha(i) \) are anyons of statistics \( \nu \) and \( -\nu \) respectively, and satisfy braiding relations among themselves. To give a rigorous discussion of these braiding relations, a detailed analysis of the lattice angle function is necessary. We recall that \( \Theta(i,j) \) naively denotes the angle from a base point \( B \) (eventually taken to infinity) to \( i \) as seen by \( j \). However, some care must be used to define this angle on a lattice. In fact, to avoid ambiguities, the center of the angle \( \Theta(i,j) \) must be a point of the dual lattice near \( j \) and a path from \( B \) to \( i \) must be chosen [26,27,28]. Once these two specifications are given, the lattice angle function is unambiguously defined. Different choices of the center of \( \Theta(i,j) \) or of the path along which \( \Theta(i,j) \) is measured, lead to different angle functions and hence, through (4.1), to different anyonic oscillators characterized by different braiding properties. This point was discussed in great detail in [28,36] where the relation between anyons and quantum groups was established. Here, instead, we are interested in a different perspective, and will choose the angle \( \Theta(i,j) \) to define holon and spinon operators in such a way that the supersymmetry properties of the \( t-J \) model in the slave anyon representation become as simple as possible.

We begin by constructing the Hamiltonian of the \( t-J \) model using the anyonic oscillators (4.1). We just sketch its derivation without entering the details since only elementary manipulations are involved. We start from the action (2.37) and fix the coupling constants as in (3.1). Then, we compute the corresponding Hamiltonian, rewrite it in the operator formalism and modify it by adding the constraint as we did in (3.10). Finally, after removing the Chern-Simons fields, we can introduce the anyonic operators (4.1) and obtain the following Hamiltonian

\[
H'_{tJ} = \frac{J}{2} \sum_{\langle i,j \rangle} \left| \hat{s}^\dagger_\alpha(i) \hat{s}_\alpha(j) - \hat{e}(j) \hat{e}^\dagger(i) \right|^2 + \sum_{i \in \Omega} i \Lambda'(i) \phi(i)
\]
where $\Lambda'(i')$ is the Lagrange multiplier enforcing the constraint $\phi(i) = 0$ to select the physical states of the $t$-$J$ model. This constraint can be written equivalently either with the anyonic operators (4.1) or with the oscillators in the slave fermion representation; in fact

$$
\phi(i) = \hat{s}_{\alpha}^\dagger(i) \hat{s}_{\alpha}(i) + \hat{e}_{\alpha}^\dagger(i) \hat{e}(i) - 1
= \hat{s}_{\alpha}^\dagger(i) \hat{s}_{\alpha}(i) + e_{\alpha}^\dagger(i) e(i) - 1 .
$$

(4.3)

This equality follows from the fact that the disorder operators cancel exactly in the combinations $\hat{s}_{\alpha}^\dagger(i) \hat{s}_{\alpha}(i)$ and $\hat{e}_{\alpha}^\dagger(i) \hat{e}(i)$, and hence any dependence on the statistical parameter $\nu$ and on the angle function drops out from $\phi(i)$.

For later convenience, and in analogy with (3.14), we introduce the operator

$$
\hat{G}(i, j) = \hat{s}_{\alpha}^\dagger(i) \hat{s}_{\alpha}(j) - \hat{e}(j) \hat{e}_{\alpha}^\dagger(i) \quad \text{for} \ i \neq j ,
$$

(4.4)

which appears inside the absolute value of (4.2) and is such that

$$
\left[ \phi(k), |\hat{G}(i, j)|^2 \right] = 0 .
$$

(4.5)

for all $k$, as one can easily check with some straightforward algebra. Notice that the holon term of $\hat{G}(i, j)$ has a relative minus sign with respect to the spinon term, and that the creation operator $\hat{e}_{\alpha}^\dagger(i)$ stands on the right of the annihilation operator $\hat{e}(j)$. Obviously the standard normal order can be restored by exchanging $\hat{e}(j)$ and $\hat{e}_{\alpha}^\dagger(i)$, but then, due the braiding properties of these oscillators, a phase depending on the relative position of $i$ and $j$ is produced (see for example [28]). Of course, when $\nu = 0$, i.e. when the holons are fermionic and the spinons bosonic, this phase factor is simply a minus sign and (4.4) reduces exactly to (3.14). Moreover, in this case the Hamiltonian (4.2) becomes equal to that of the slave fermion representation given in (3.13), apart from the irrelevant constant proportional to the volume of the lattice that we have dropped.

The anyonic oscillators (4.1) and their braiding properties crucially depend on how one chooses the angle function in the disorder operators [28]; on the contrary the Hamiltonian (4.2) is independent of such a choice. To prove this, let us consider the operator

$$
\hat{G}'(i, j) = \hat{s}_{\alpha}^\dagger(i) \hat{s}_{\alpha}(j) - \hat{e}'(j) \hat{e}'_{\alpha}(i)
$$

(4.6)

where the primed anyonic oscillators are defined as in (4.1) but with a different angle function denoted by $\Theta'(i, j)$. Expressing these primed oscillators in terms of the unprimed ones, (4.6) becomes

$$
\hat{G}'(i, j) = \hat{s}_{\alpha}^\dagger(i) q \sum_{k \in \Omega} \left[ \chi(i, k) - \chi(j, k) \right] s_{\beta}(k) s_{\beta}(k)
- q \sum_{k \in \Omega} \chi(j, k) (e_{\alpha}(k) e(k)-1) \hat{e}(j) \hat{e}_{\alpha}(i) q
- \sum_{k \in \Omega} \chi(i, k) (e_{\alpha}(k) e(k)-1)
$$

(4.7)
where
\[ q = e^{i \pi \nu}, \]  
and
\[ \chi(i, j) = \frac{1}{\pi} \left[ \Theta'(i, j) - \Theta(i, j) \right]. \]  

If we rewrite the exponents in the holon term using (4.3), and then rearrange the factors, we get
\[ \hat{G}'(i, j) = q \left\{ \chi(j, i) + \sum_{k \in \Omega} \left[ \chi(i, k) - \chi(j, k) \right] s^+_{\beta}(k) s_{\beta}(k) \right\} \left\{ q^{-\chi(i, j)} \hat{s}_{\alpha}(i) \hat{s}_{\alpha}(j) \right\} \right. \]  
\[ \left. - q^{-\chi(j, i)} \hat{e}(j) \hat{e}^+ (i) q^{-\sum_{k \in \Omega} \left[ \chi(i, k) - \chi(j, k) \right] \phi(k)} \right\}. \]  

From the definition (4.9), it is clear that
\[ \chi(i, i) = \chi(j, j) = \chi. \]  

If we take into account the constraint \( \phi(i) = 0 \) for all \( i \in \Omega \), the right hand side of (4.10) simplifies considerably and one gets
\[ \hat{G}'(i, j) = q \left\{ -\chi + \chi(j, i) + \sum_{k \in \Omega} \left[ \chi(i, k) - \chi(j, k) \right] s^+_{\beta}(k) s_{\beta}(k) \right\} \hat{G}(i, j). \]  

Therefore, on the physical states of the \( t-J \) model, \( \hat{G}'(i, j) \) and \( \hat{G}(i, j) \) simply differ by a unitary operator, and thus
\[ \left| \hat{G}'(i, j) \right|^2 = \hat{G}'(i, j) \hat{G}'(i, j) = \hat{G}(i, j) \hat{G}(i, j) = \left| \hat{G}(i, j) \right|^2 \]  
for all \( i \) and \( j \). This equality implies that the Hamiltonian (4.2) does not depend on the details of the angle function. This property should not be totally unexpected, since different choices of the angle function simply correspond to different gauges for the Chern-Simons fields used in the bosonization procedure.

We now come to the slave anyon representation of the superalgebra \( SU(1|2) \). As far as the generators \( J^\pm, J^0 \) and \( T \) are concerned, there is no difference with respect to the case discussed in Section 3, and Eqs. (3.6a-b) and (3.7a-b) hold also in the anyonic representation, apart from some obvious notational changes. Things are different instead for the supersymmetry generators \( Q_\alpha \) and \( Q^\dagger_\alpha \). In fact, since these are fermionic, they are “sensitive” to the statistics of the operators on which they act. For example, in the slave fermion representation \( Q_\alpha \) and \( Q^\dagger_\alpha \) close anticommutators with \( e \) and \( e^\dagger \), but in the slave boson representation they close commutators, as one can see from Eqs. (3.7c) and (3.21). Therefore, for intermediate statistics it is natural to expect a generalization of these equations.
The slave anyon representation of the supercharges can be written in analogy with (3.6c); in fact one has

\[ Q_\alpha = \sum_{i \in \Omega} \hat{Q}_\alpha(i) = \sum_{i \in \Omega} \hat{e}_i^\dagger \hat{s}_\alpha(i), \]
\[ Q_\alpha^\dagger = \sum_{i \in \Omega} \hat{Q}_\alpha^\dagger(i) = \sum_{i \in \Omega} \hat{e}(i) \hat{s}_\alpha^\dagger(i). \]

(4.14)

With simple algebra using (4.1) and (2.12), one can check that \( Q_\alpha \) and \( Q_\alpha^\dagger \) in (4.14) close the proper anticommutators of \( SU(1|2) \), and also that the densities \( \hat{Q}_\alpha(i) \) and \( \hat{Q}_\alpha^\dagger(i) \), while being fermions with respect to each other, are anyons with respect to the constituent holons and spinons. Indeed, \( \hat{Q}_\alpha(i) \) and \( \hat{Q}_\alpha^\dagger(i) \) close anticommutators among themselves, but obey braiding relations with \( \hat{e}(i) \) and \( \hat{s}_\alpha(i) \). Let us now analyze if the supercharges (4.14) depend on the choice of the angle function in the anyonic oscillators. To this purpose, we consider the density

\[ \hat{Q}_\alpha'(i) = \hat{e}_i^\dagger \hat{s}_\alpha'(i) \]

(4.15)

where the primed oscillators are those introduced earlier. Expressing the latter ones in terms of the unprimed oscillators (4.1), we get

\[ \hat{Q}_\alpha'(i) = \hat{e}_i^\dagger(i) q \sum_{j \in \Omega} \chi(i,j) \phi(j) \hat{s}_\alpha(i). \]

(4.16)

Bringing the \( q \)-factor to the far right and taking into account (4.11), we finally obtain

\[ \hat{Q}_\alpha'(i) = q^\chi \hat{Q}_\alpha(i) q \sum_{j \in \Omega} \chi(i,j) \phi(j). \]

(4.17)

This equation implies that on the physical states of the \( t-J \) model, which satisfy \( \phi(i) = 0 \) for all \( i \in \Omega \), the two supersymmetry densities \( \hat{Q}_\alpha'(i) \) and \( \hat{Q}_\alpha(i) \) differ only by a phase factor independent of \( i \). Thus, on the physical states also the corresponding supersymmetry generators simply differ by a constant phase:

\[ Q'_\alpha = \sum_{i \in \Omega} \hat{Q}_\alpha'(i) = q^\chi \sum_{i \in \Omega} \hat{Q}_\alpha(i) = q^\chi Q_\alpha, \]

(4.18)

and hence they close the same superalgebra (3.3). In fact, the (anti)commutation relations of \( SU(1|2) \) remain unchanged if the supersymmetry charges are rescaled by a phase factor. Moreover, it is clear that if \( Q_\alpha \) commutes with the Hamiltonian, so does \( Q'_\alpha \).

The result contained in Eqs. (4.13) and (4.18) is quite important: All possible choices of angle function for the anyonic oscillators are equivalent both in the Hamiltonian and in the supersymmetry charges. Therefore, we are free to choose those angle functions that are most convenient for our purposes and that simplify most
our calculations. Actually, in discussing the supersymmetry properties of the slave anyon representation of the $t$-$J$ model we find very convenient to introduce two angle functions that we call horizontal and vertical, and correspondingly define two kinds of anyonic oscillators that we also call horizontal (or of type $\mathcal{H}$) and vertical (or of type $\mathcal{V}$). The reason for this terminology will be evident in a moment.

For the type $\mathcal{H}$ anyons, the angle function $\Theta_{\mathcal{H}}(i,j)$ is defined as follows: First, for any point $i \in \Omega$ we consider a straight lattice path $\mathcal{C}_{\mathcal{H}}(i)$ parallel to the $x$-axis from the base point $B$ (taken to infinity in the direction of the positive $x$-axis) to the point $i$; then we take the dual point

$$j^* \equiv \left( j_x + \frac{\varepsilon}{2}, j_y + \frac{\varepsilon}{2} \right)$$

(4.19)

where $j_x$ and $j_y$ are the coordinates of $j \in \Omega$, and $\varepsilon \leq 1$ is the spacing of a new lattice $\Omega'$ in which $\Omega$ is embedded (eventually $\varepsilon$ can be taken to zero, see [28]). The point $j^*$ belongs to the lattice dual to $\Omega'$, and can also be viewed as the center of the elementary plaquette of $\Omega'$ whose lower left corner is $j$. Finally, we define $\Theta_{\mathcal{H}}(i,j)$ as the angle between $B$ and $i$ measured from $j^*$ along the path $\mathcal{C}_{\mathcal{H}}(i)$. This is represented in Fig. 1.

Using this horizontal angle in the Jordan-Wigner transformation (4.1) we get the anyons of type $\mathcal{H}$, which from now on will be simply denoted by the symbols $\hat{e}(i)$, $\hat{e}^\dagger(i)$, $\hat{s}_\alpha(i)$ and $\hat{s}_\beta^\dagger(i)$ with no further specifications. These operators satisfy braiding relations among themselves that can be easily obtained using the (anti)commutators (2.12) and the properties of the angle function $\Theta_{\mathcal{H}}(i,j)$. For example, if $i$ and $j$ are two horizontal nearest neighbor points, viz. $j = i + \hat{1}$, one can show that in the limit $\varepsilon \to 0$

\begin{align}
\hat{e}(i) \hat{e}(j) + q \hat{e}(j) \hat{e}(i) &= 0 \\
\hat{e}(j) \hat{e}^\dagger(i) + q \hat{e}^\dagger(i) \hat{e}(j) &= 0 \\
\hat{s}_\alpha(i) \hat{s}_\beta(j) - q^{-1} \hat{s}_\beta(j) \hat{s}_\alpha(i) &= 0 \\
\hat{s}_\alpha(j) \hat{s}_\beta^\dagger(i) - q^{-1} \hat{s}_\beta^\dagger(i) \hat{s}_\alpha(j) &= 0
\end{align}

(4.20a-d)

where $q$ is given in (4.8). We refer the reader to [28] where these braiding relations are discussed and derived in full generality. Here we simply point out that there are no phases in the relations between operators in the same point which indeed are

\begin{align}
\hat{e}(i) \hat{e}^\dagger(i) + \hat{e}^\dagger(i) \hat{e}(i) &= 1 \\
\hat{s}_\alpha(i) \hat{s}_\beta^\dagger(i) - \hat{s}_\beta^\dagger(i) \hat{s}_\alpha(i) &= \delta_{\alpha\beta}
\end{align}

(4.21a-b)

It is also important to realize that $\hat{e}$ and $\hat{s}_\alpha$ commute among themselves, for example

$$\hat{e}(i) \hat{s}_\alpha(j) - \hat{s}_\alpha(j) \hat{e}(i) = 0$$

(4.22)

for all $i$ and $j$. From Eqs. (4.20), (4.21) and (4.22), it is straightforward to obtain the braiding relations between the supercharge densities $\hat{Q}_\alpha(i)$ and $\hat{Q}_\alpha^\dagger(i)$ and the
anyonic holon and spinon operators. These braiding relations generalize Eqs. (3.7c) and (3.21) to the case of arbitrary statistics, and provide an anyonic linear realization of the $N = 2$ supersymmetry.

Let us now come to the anyons of type $V$, which are characterized by the vertical angle function $\Theta_V(i,j)$. The latter is defined as $\Theta_H(i,j)$ but with a rotation of $\pi/2$ in a counterclockwise way. More precisely, for any point $i \in \Omega$ we consider a straight lattice path $C_V(i)$ parallel to the $y$-axis from the base point $B'$ (taken to infinity in the direction of the positive $y$-axis) to $i$; then given $j \in \Omega$, we take the dual point $\ast j \equiv (j_x - \frac{\epsilon}{2}, j_y + \frac{\epsilon}{2})$ (4.23)

which is the center of the elementary plaquette of $\Omega'$ whose lower right corner is $j$, and define $\Theta_V(i,j)$ as the angle between $B$ and $i$ measured from $\ast j$ along the path $C_V(i)$. This is represented in Fig. 2.

The anyonic oscillators of type $V$ are obtained by using the vertical angle function $\Theta_V(i,j)$ in (4.1); they satisfy braiding relations among themselves that can be easily determined from the oscillator algebra (2.12), and are formally the same as those satisfied by the horizontal anyons (indeed, Eqs. (4.20) still hold with $j = i + \hat{2}$).

These two types of anyonic oscillators are very helpful in simplifying our calculations. In fact, if we distinguish between horizontal and vertical lattice bonds and correspondingly introduce horizontal and vertical anyons, the Hamiltonian (4.2) can be conveniently rewritten as follows

$$H'_{tJ} = H_H + H_V + \sum_{i \in \Omega} i \Lambda'(i) \phi(i)$$

$$= \frac{J}{2} \sum_{<i,j>_{H}} \left| G^H(i,j) \right|^2 + \frac{J}{2} \sum_{<i,j>_{V}} \left| G^V(i,j) \right|^2 + \sum_{i \in \Omega} i \Lambda'(i) \phi(i)$$

where

$$G^H(i,j) = \hat{s}_\alpha(i) \hat{s}_\alpha(j) - \hat{e}(j) \hat{e}^\dagger(i)$$

and $G^V(i,j)$ is the same but with vertical anyons. The horizontal Hamiltonian $H_H$ contains only the interactions between horizontal nearest-neighbor points denoted by $<i,j>_H$, whereas the vertical Hamiltonian $H_V$ contains only the contributions from vertical nearest-neighbor points denoted by $<i,j>_V$. In some extended sense, $H_H$ and $H_V$ can be considered as one-dimensional Hamiltonians corresponding respectively to the rows and the columns of the original two-dimensional square lattice.

Actually, in writing the Hamiltonian (4.24), we could have used only one kind of anyonic oscillators, for instance only horizontal ones. But in such a case, the term corresponding to vertical nearest neighbor interactions, $G^V(i,j)$, would have been more complicated and would have contained extra non-local operators, as shown in [27] for a different model. However, as we discussed before, these extra non-local operators can be written simply as $q$ raised to suitable combinations of the holon and spinon densities, and, after using the constraint $\phi(i) = 0$, they can be reabsorbed into
an overall unitary operator which drops out in taking the absolute value (see (4.12) and (4.13)). Finally, we remark that when $\nu = 0$ holons and spinons become fermions and bosons respectively, and the distinction between horizontal and vertical operators ceases to exist. In this case the Hamiltonians $H_{\mathcal{H}}$ and $H_{\mathcal{V}}$ in (4.24) can be combined, and together they reproduce exactly the Hamiltonian (3.13) of the supersymmetric $t$-$J$ model in the slave fermion representation, up to an irrelevant additive constant that we have dropped.

Let us now discuss the supersymmetry properties the $t$-$J$ model in the slave anyon representation. To this purpose, it is convenient to start from the form (4.24) of the Hamiltonian and assume that the supersymmetry generators $Q_\alpha$ and $Q^\dagger_\alpha$ are made of anyonic oscillators of type $\mathcal{H}$. Since we are free to choose any angle function due to the relation (4.18), there is no loss of generality in making this assumption. Moreover, to avoid ambiguities we denote the supercharge densities by the symbols $\hat{Q}_\alpha^\mathcal{H}(i)$ and $\hat{Q}^\dagger_\alpha^\mathcal{H}(i)$. From the braiding relations (4.20) and the (anti)commutators (4.21) and (4.22), it follows that

$$\left[\hat{Q}^\mathcal{H}_\alpha(k), \hat{G}^\mathcal{H}(i,j)\right] = 0 \quad \text{for} \quad k \neq i,j \quad , \quad (4.26)$$

and

$$\left(\hat{Q}^\mathcal{H}_\alpha(i) + \hat{Q}^\mathcal{H}_\alpha(j)\right)\hat{G}^\mathcal{H}(i,j) = q^{-1}\hat{G}^\mathcal{H}(i,j)\left(\hat{Q}^\mathcal{H}_\alpha(i) + \hat{Q}^\mathcal{H}_\alpha(j)\right) ,$$

$$\left(\hat{Q}^\mathcal{H}_\alpha(i) + \hat{Q}^\mathcal{H}_\alpha(j)\right)\hat{G}^\dagger\mathcal{H}(i,j) = q\hat{G}^\dagger\mathcal{H}(i,j)\left(\hat{Q}^\mathcal{H}_\alpha(i) + \hat{Q}^\mathcal{H}_\alpha(j)\right) . \quad (4.27)$$

Of course the corresponding equations involving the density $\hat{Q}_\alpha^\dagger\mathcal{H}(i)$ can be simply obtained from (4.26) and (4.27) by hermitean conjugation. These equations imply that

$$\left[Q_\alpha, \left|\hat{G}^\mathcal{H}(i,j)\right|^2\right] = \left[Q^\dagger_\alpha, \left|\hat{G}^\mathcal{H}(i,j)\right|^2\right] = 0 \quad , \quad (4.28)$$

and hence the horizontal Hamiltonian $H_{\mathcal{H}}$ is invariant under supersymmetry.

To prove that also the vertical Hamiltonian commutes with $Q_\alpha$ and $Q^\dagger_\alpha$, we first observe that Eqs. (4.26) and (4.27) can be translated into the corresponding ones for vertical anyons by simply replacing everywhere the superscript $\mathcal{H}$ with $\mathcal{V}$. Then we make use of (4.18) and its hermitean conjugate to rewrite $\hat{Q}_\alpha$ and $\hat{Q}^\dagger_\alpha$ in terms of vertical anyons, and conclude that

$$\left[Q_\alpha, \left|\hat{G}^\mathcal{V}(i,j)\right|^2\right] = \left[Q^\dagger_\alpha, \left|\hat{G}^\mathcal{V}(i,j)\right|^2\right] = 0 \quad . \quad (4.29)$$

Finally, it is straightforward to check that the supercharges commute with the constraint operator $\phi(i)$. This completes the proof of the invariance of the whole Hamiltonian (4.24) under the $N = 2$ supersymmetry.

In Appendix A we will show that this same result can be obtained by a different definition of the angle function in the anyonic oscillators forming the supersymmetry generators $\hat{Q}_\alpha$ and $\hat{Q}^\dagger_\alpha$. This alternative method is based on a more intensive use of
the constraint \( \phi(i) = 0 \), but has the advantage of making \( \hat{G}(i, j) \), and not only its modulus square, invariant under \( SU(1|2) \).

5. Conclusions

In this paper we have studied the formal properties of the \( t-J \) model in two dimensions using the formalism of the slave operators, and showed that, by means of a generalized abelian bosonization with two independent Chern-Simons fields, holons and spinons may acquire arbitrary complementary statistics in such a way that the electron remains a fermion. We would like to stress that the factorization of the electron in an antiholon and a spinon is a formal \textit{Ansatz}, and only a detailed analysis at the dynamical level could shed some light on its correctness.

The \( t-J \) model at \( J = 2t \) is characterized by the invariance of its Hamiltonian under the superalgebra \( SU(1|2) \), which is linearly realized on holons and spinons. In the case of anyonic statistics, the supercharge densities of \( SU(1|2) \) have braiding properties with the holon and spinon operators, which generalize the standard (anti)commutation relations with fermions and bosons. These supersymmetry properties of the \( t-J \) model at \( J = 2t \) could play an important role in the analysis of the dynamical aspects of the spin-charge separation, which certainly deserves to be investigated further.

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Appendix A

In this Appendix we propose two new definitions for the angle function in the anyonic oscillators forming the supercharges $\hat{Q}_\alpha$ and $\hat{Q}^\dagger_\alpha$ that simplify considerably the discussion of the supersymmetry properties of the $t$-$J$ model in the slave anyon representation. In fact with this choice the operator $\hat{G}(i, j)$, and not only $|\hat{G}(i, j)|^2$, will be supersymmetric. We denote these new angle functions by $\tilde{\Theta}_H(i, j)$ and $\tilde{\Theta}_V(i, j)$ corresponding respectively to the horizontal and vertical anyons. The function $\tilde{\Theta}_H(i, j)$ is defined as follows: For any point $i \in \Omega$, we choose the horizontal path $C_H(i)$ from the base point $B$ to $i$, and consider the dual point $\ast j \equiv (j_x - \frac{\varepsilon}{2}, j_y - \frac{\varepsilon}{2})$ (A.1) which is the center of the elementary plaquette of $\Omega'$ whose upper right corner is $j$. Then we define $\tilde{\Theta}_H(i, j)$ as the angle between $B$ and $i$ measured from $\ast j$ along the path $C_H(i)$. This is represented in Fig. 3.

Using this twiddled horizontal angle in the Jordan-Wigner transformation (4.1) we get twiddled anyons of type $H$, which, from now on, we simply denote by the symbols $\tilde{\hat{e}}(i)$, $\tilde{\hat{e}}^\dagger(i)$, $\tilde{\hat{s}}_\alpha(i)$ and $\tilde{\hat{s}}^\dagger_\alpha(i)$ with no further specifications. These twiddled horizontal oscillators are used to define a new anyonic representation of the supersymmetry generators. In fact, we can posit

$$Q_\alpha = \sum_{i \in \Omega} \tilde{\hat{Q}}_\alpha(i) = \sum_{i \in \Omega} \tilde{\hat{e}}^\dagger(i) \tilde{\hat{s}}_\alpha(i) \ ,$$

$$Q^{\dagger}_\alpha = \sum_{i \in \Omega} \tilde{\hat{Q}}^\dagger_\alpha(i) = \sum_{i \in \Omega} \tilde{\hat{e}}(i) \tilde{\hat{s}}^\dagger_\alpha(i) \ .$$

As discussed in Section 4, these supercharges are equivalent to those defined in (4.14), since they simply differ in the definition of the angle function in the anyonic oscillators (cf (4.18)).

We now study the braiding relations between the supersymmetry generators (A.2) and the Hamiltonian (4.24) which is made out of anyons of type $H$ and type $V$. To this aim it is therefore necessary to establish the precise connection between $\tilde{\Theta}_H(i, j)$ and the horizontal and vertical angle functions defined in Section 4. To avoid repetitions, we limit our discussion to the horizontal case, but of course similar considerations can be made also in the vertical case.

From the definitions of $\Theta_H(i, j)$ and $\tilde{\Theta}_H(i, j)$, one can derive the following identity

$$\Theta_H(i, j) - \tilde{\Theta}_H(j, i) = \begin{cases} \pi & \text{for } i_y > j_y \\ -\pi & \text{for } i_y \leq j_y \end{cases} \ ,$$

which holds for any value of $\varepsilon$ and not only for $\varepsilon \to 0$. The derivation of this identity is completely similar to that discussed in detail in [27,28], and so we do not reproduce it here. Notice that the case $i = j$ is included in this equation since

$$\Theta_H(i, i) = -\frac{3\pi}{4} \ , \quad \tilde{\Theta}_H(i, i) = \frac{\pi}{4} \ .$$
for any $i \in \Omega$.

Let us now consider two anyonic holons, one in the horizontal representation and one in the twiddled representation. With simple algebra one can prove that

$$\hat{e}(i) \tilde{e}(j) = -e^{-i\nu[\Theta_{\mathcal{H}}(i,j) - \tilde{\Theta}_{\mathcal{H}}(j,i)]} \tilde{e}(j) \hat{e}(i) \ .$$  \hspace{1cm} (A.5a)

Using the identity (A.3), this equation becomes

$$\hat{e}(i) \tilde{e}(j) = \begin{cases} -q^{-1} \tilde{e}(j) \hat{e}(i) & \text{for } i_y > j_y \ , \\ -q \tilde{e}(j) \hat{e}(i) & \text{for } i_y \leq j_y \ . \end{cases} \hspace{1cm} (A.5b)$$

It is convenient to introduce the function $\sigma$, defined on the integers as

$$\sigma(m) = \theta(m) - \theta(-m) + \delta_{m,0} = \begin{cases} 1 & \text{for } m \geq 0 \ , \\ -1 & \text{for } m < 0 \ . \end{cases} \hspace{1cm} (A.6b)$$

where $\theta(m)$ is the standard Heaviside step function

$$\theta(m) = \begin{cases} 1 & \text{for } m > 0 \ , \\ \frac{1}{2} & \text{for } m = 0 \ , \\ 0 & \text{for } m < 0 \ . \end{cases} \hspace{1cm} (A.6b)$$

Then, we can rewrite (A.5b) as

$$\hat{e}(i) \tilde{e}(j) + q\sigma(j_y - i_y) \tilde{e}(j) \hat{e}(i) = 0 \ ,$$  \hspace{1cm} (A.7)

which is an example of the braiding relations we are looking for. We now consider how creation and annihilation operators braid among themselves. For $i \neq j$ we get

$$\hat{e}(i) \tilde{e}^\dagger(j) = \begin{cases} -q \tilde{e}^\dagger(j) \hat{e}(i) & \text{for } i_y > j_y \\ -q^{-1} \tilde{e}^\dagger(j) \hat{e}(i) & \text{for } i_y \leq j_y \ . \end{cases} \hspace{1cm} (A.8)$$

where again we have exploited the identity (A.3). If $i = j$ instead, using (A.4) and taking into account the inhomogeneous term in the anticommutation relations (2.12a) at the same point, we obtain

$$\hat{e}(i) \tilde{e}^\dagger(i) + q^{-1} \tilde{e}^\dagger(i) \hat{e}(i) = q \sum_{j \in \Omega} \chi(i,j) \left( e^\dagger(j) e(j) - 1 \right) \hspace{1cm} (A.9)$$

where

$$\chi(i,j) = \frac{1}{\pi} \left[ \Theta_{\mathcal{H}}(i,j) - \tilde{\Theta}_{\mathcal{H}}(i,j) \right] \ . \hspace{1cm} (A.10)$$
Notice that there is a braiding phase even in the relation at the same point, in contrast to (4.21a) which is a standard anticommutator. We can easily combine Eqs. (A.8) and (A.9) into a single formula as follows

\[ \hat{e}(i) \tilde{e}^\dagger(j) + q^{-\sigma(j_y-i_y)} \tilde{e}^\dagger(j) \hat{e}(i) = \delta(i,j) q \sum_{k \in \Omega} \chi(i,k) (e^\dagger(k)e(k)-1). \]  

(EQ 11)

Eqs. (A.7), (A.11) and their hermitean conjugates form the complete set of braiding relations between two anyonic holon operators in the horizontal and twiddled representations.

It is interesting to consider these relations in the limit \( \varepsilon \to 0 \) which is part of the continuum limit (we recall that \( \varepsilon \) is the spacing of the lattice \( \Omega' \) where \( \Omega \) is embedded, and is used to determine the centers from which the angles are measured, see for instance (4.2) and (A.1)). The braiding relation (A.7) remains unchanged when \( \varepsilon \to 0 \), whereas (A.11) simplifies a little, since the function \( \chi(i,k) \) appearing in the right hand side can be computed explicitly. In fact, from the definitions of \( \Theta_H(i,j) \) and \( \tilde{\Theta}_H(i,j) \), it is not difficult to show that, when \( \varepsilon \to 0 \),

\[
\begin{align*}
\Theta_H(i,j) &= \tilde{\Theta}_H(i,j) \quad \text{for } i_y \neq j_y, \\
\Theta_H(i,j) &= -\pi \quad \text{, } \tilde{\Theta}_H(i,j) = \pi \quad \text{for } i_y = j_y, \ i_x < j_x, \\
\Theta_H(i,j) &= \tilde{\Theta}_H(i,j) = 0 \quad \text{for } i_y = j_y, \ i_x > j_x.
\end{align*}
\]  

(EQ 12)

Using these values and those in (A.4), which hold for any \( \varepsilon \), we get

\[ \chi(i,j) = -2 \delta_{i_y,j_y} \theta(j_x-i_x) \quad \text{for } \varepsilon \to 0 \]  

(EQ 13)

where \( \theta \) is the step function (A.6b). Then, when \( \varepsilon \to 0 \) we can rewrite the right hand side of (A.12) and get

\[ \hat{e}(i) \tilde{e}^\dagger(j) + q^{-\sigma(j_y-i_y)} \tilde{e}^\dagger(j) \hat{e}(i) = \delta(i,j) q^{-2 \sum_{k_x} \theta(k_x-i_x) (e^\dagger(k)e(k)-1)}. \]  

(EQ 14)

We remark that it is much easier to guess a continuum limit of Eqs. (A.7) and (A.14) than of the analogous braiding relations involving only anyons of one kind [28].

What we have done so far with the holons, can be repeated with the spinons. Of course, since in the slave anyon representation the spinons are boson based, while the holons are fermion based, there are some sign changes; moreover, since the statistics of the spinons is opposite to that of the holons, \( q \) has to be changed into \( q^{-1} \). Keeping in mind these simple modifications, we can immediately translate all braiding relations of the holons into those of the spinons. For example, the spinon version of (A.7) is

\[ \hat{s}_\alpha(i) \tilde{s}_\beta(j) - q^{-\sigma(j_y-i_y)} \tilde{s}_\beta(j) \hat{s}_\alpha(i) = 0. \]  

(EQ 15)

The other braiding relations of the spinon operators can be obtained in a similar way from (A.11). Moreover it is clear that holons and spinons commute with each other, being independent operators (cf (4.22)).
After this detailed analysis of the braiding relations, we discuss the supersymmetry properties of the Hamiltonian (4.24) using the representation (A.2) of the supercharges. Let us first consider \( \tilde{\mathcal{Q}}_\alpha(i) \) and the horizontal nearest neighbor interaction \( G^H(i,j) \) defined in (4.25) for \( j = i + 1 \) (which implies in particular that \( i_y = j_y \)).

Then, it is easy to prove that

\[
[\tilde{\mathcal{Q}}_\alpha(k), G^H(i,j)] = 0 \tag{A.16}
\]

for any \( k \neq i,j \). The commutation relations of \( G^H(i,j) \) with the supercharge densities in \( i \) and \( j \) are more subtle. Due to the hard-core condition obeyed by the holon and spinon operators in the slave anyon representation, one immediately proves that

\[
[\tilde{\mathcal{Q}}_\alpha(i), \hat{e}(j) \hat{e}^\dagger(i)] = [\tilde{\mathcal{Q}}_\alpha(j), \hat{s}^\dagger_\alpha(i) \hat{s}_\alpha(j)] = 0 \tag{A.17}
\]

On the contrary, after some algebra involving the use of (A.7) and (A.11), we get

\[
[\tilde{\mathcal{Q}}_\alpha(j), \hat{e}(j) \hat{e}^\dagger(i)] = q^{\chi(i,j)+1} \hat{e}^\dagger(i) \sum_{k \in \Omega} \chi(j,k) \left( e^\dagger(k) e(k) - 1 \right) \hat{s}_\alpha(j) \tag{A.18}
\]

where in the last line we have transformed the spinon oscillator from the twiddled representation to the horizontal one, and have combined all exponents to reproduce the constraint operator \( \phi(k) \). Similarly, one can prove that

\[
[\tilde{\mathcal{Q}}_\alpha(i), \hat{s}^\dagger_\alpha(i) \hat{s}_\alpha(j)] = \hat{e}^\dagger(i) \sum_{k \in \Omega} \sum_{\beta = \pm 1} \chi(i,k) s^\dagger_\beta(k) s_\beta(k) \hat{s}_\alpha(j) \tag{A.19}
\]

where in the last line the holon oscillator has been transformed from the twiddled to the horizontal representation, and again all exponents have been combined to yield the constraint operator. Putting together the last three equations and rearranging the right hand sides of (A.18) and (A.19) to bring the factors with the constraint operators to the far right, one gets

\[
[Q_\alpha, G^H(i,j)] = \sum_{k \in \Omega} [\tilde{\mathcal{Q}}_\alpha(k), G^H(i,j)]
\]

\[
= \hat{e}^\dagger(i) \hat{s}_\alpha(j) \left\{ -\chi(i,j)+ \sum_{k \in \Omega} \chi(i,k) \phi(k) \right. \\
- q^{-\chi(i,j)+1-\chi(j,i)+1} \sum_{k \in \Omega} \chi(j,k) \phi(k) \right\} . \tag{A.20}
\]
From the definition of the function $\chi$ in (A.10), it follows that for $j = i + \hat{1}$

$$\chi(j, i) + 1 - \chi(j, j) = -\chi(i, j) .$$

Therefore, the numerical factors in the two terms inside the curly brackets of (A.20) are the same and can be factored out, leaving only exponentials of the constraint operator. On the physical states of the $t$-$J$ model where the constraint $\phi(i) = 0$ is identically satisfied, each exponential becomes 1, and thus we can conclude that

$$\left[ Q_{\alpha}, G^H(i, j) \right] = 0 . \quad (A.21a)$$

Similarly, one can show that

$$\left[ Q_{\alpha}^\dagger, G^H(i, j) \right] = 0 . \quad (A.21b)$$

We now turn to the commutation relations of the supersymmetry generators (A.2) with the vertical nearest neighbor interactions $G^V(i, j)$. To simplify the calculations, we first notice that all the braiding properties discussed so far, can be translated into those appropriate for the vertical anyons by simply using the angles $\Theta^V(i, j)$ and $\tilde{\Theta}^V(i, j)$ in place of $\Theta^H(i, j)$ and $\tilde{\Theta}^H(i, j)$ respectively. Notice that $\Theta^V(i, j)$ is defined as $\Theta^H(i, j)$ but rotated counterclockwise of $\pi/2$ (compare Fig.1 and Fig.2). Similarly, $\tilde{\Theta}^V(i, j)$ is defined as $\tilde{\Theta}^H(i, j)$ but rotated of $\pi/2$. Then, it is clear that all the manipulations we did before naturally apply also to the vertical anyons (both with and without the tilde sign). Since on the physical states of the $t$-$J$ model the supercharges constructed by twiddled vertical operators simply differ from those in (A.2) by a global phase factor, we conclude that

$$\left[ Q_{\alpha}, G^V(i, j) \right] = \left[ Q_{\alpha}^\dagger, G^H(i, j) \right] = 0 . \quad (A.22)$$

Finally, since the constraint $\phi(i)$ commutes with the supercharges, from (A.21) and (A.22) we deduce that the whole Hamiltonian of the $t$-$J$ model in the slave anyon representation given in (4.24) is supersymmetric.

If we compare Eqs. (A.21) and (A.22) with Eqs. (4.28) and (4.29), we see that using the representation (A.2) for the supercharges, the quadratic interactions $\hat{G}^H(i, j)$ and $\hat{G}^V(i, j)$ are themselves supersymmetric, whereas if the supercharges are written in the representation (4.14) only the quartic interactions $|\hat{G}^H(i, j)|^2$ and $|\hat{G}^V(i, j)|^2$ are invariant. In both cases, however, it is crucial to use the constraint $\phi(i) = 0$ which selects the physical states of the $t$-$J$ model. Clearly, in order to check the invariance of the Hamiltonian (4.24), the conditions expressed by (A.21) and (A.22) are sufficient but not necessary. However, in view of a possible mean field analysis of the slave anyon representation of the $t$-$J$ model after a Hubbard-Stratonovich transformation, it is more convenient to focus on the quadratic interactions $\hat{G}^H(i, j)$ and $\hat{G}^V(i, j)$ instead of the quartic Hamiltonian which derives from them.

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Figure Captions

Fig. 1 Representation of the lattice angle $\Theta_H(i, j)$ between the base point $B$ and $i$ measured along the horizontal curve $C_H(i)$ from the point $j^\ast$.

Fig. 2 Representation of the lattice angle $\Theta_V(i, j)$ between the base point $B'$ and $i$ measured along the vertical curve $C_V(i)$ from the point $^\ast j$.

Fig. 3 Representation of the lattice angle $\tilde{\Theta}_H(i, j)$ between the base point $B$ and $i$ measured along the curve $C_H(i)$ from the point $^\ast j$. 
SLAVE ANYONS IN THE $t$-$J$ MODEL
AT THE SUPERSYMMETRIC POINT *

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Abstract

We discuss the properties of the supersymmetric $t$-$J$ model in the formalism of the slave operators. In particular we introduce a generalized abelian bosonization for the model in two dimensions, and show that holons and spinons can be anyons of arbitrary complementary statistics (slave anyon representation). The braiding properties of these anyonic operators are thoroughly analyzed, and are used to provide an explicit linear realization of the superalgebra $SU(1|2)$. Finally, we prove that the Hamiltonian of the $t$-$J$ model in the slave anyon representation is invariant under $SU(1|2)$ for $J = 2t$.

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