NOETHER-SEVERI INEQUALITY AND EQUALITY FOR IRREGULAR THREEFOLDS OF GENERAL TYPE

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Dedicated to Professor Zhijie Chen on the occasion of his 80th birthday

Abstract. We prove the optimal Noether-Severi inequality that \( \text{vol}(X) \geq \frac{4}{3} \chi(\omega_X) \) for all smooth and irregular 3-folds \( X \) of general type over \( \mathbb{C} \). For those 3-folds \( X \) attaining the equality, we completely describe their canonical models and show that the topological fundamental group \( \pi_1(X) \cong \mathbb{Z}^2 \). As a corollary, we obtain for the same \( X \) another optimal inequality that \( \text{vol}(X) \geq \frac{4}{3} h^0_1(X, K_X) \) where \( h^0_1(X, K_X) \) stands for the continuous rank of \( K_X \), and we show that \( X \) attains this equality if and only if \( \text{vol}(X) = \frac{4}{3} \chi(\omega_X) \).

1. Introduction

Throughout this paper, we work over the complex numbers \( \mathbb{C} \), and all varieties are projective.

1.1. Motivation. A major problem in algebraic geometry is to classify algebraic varieties. Concerning this problem for varieties of general type, one well-known strategy is the geographical approach. That is, to find explicit relations among birational invariants of varieties first, and then to establish the fine classification based on these relations.

Such a strategy has been proved to be a great success for algebraic surfaces. For smooth surfaces \( S \) of general type, Noether [28] proved that

\[(1.1) \quad \text{vol}(S) \geq 2p_g(S) - 4,\]

which is now known as the Noether inequality. The classification of surfaces attaining the equality was sketched in Enriques’ book [12] and accomplished in detail afterwards by Horikawa [15]. However, the three dimensional analogue of the problem seems to be more challenging. By a recent result of J. Chen, M. Chen and C. Jiang [9], the Noether inequality

\[(1.2) \quad \text{vol}(X) \geq \frac{4}{3} p_g(X) - \frac{10}{3}\]

holds for “almost all” smooth 3-folds \( X \) of general type, but it is not known by far if (1.2) holds in general, and the classification of 3-folds attaining the equality is still open [9, Question 1.4, 1.5].

For an irregular variety \( X \), the inequality between \( \text{vol}(X) \) and \( \chi(\omega_X) \) attracts more attention. A notable feature is that the ratio \( \frac{\text{vol}(X)}{\chi(\omega_X)} \) is invariant.
under étale covers, thus it carries many geometric properties of $X$ itself. This type of inequalities plays the role of the Noether inequality as in the regular case, and is often referred as Severi type inequalities in the literature [32, 30, 34, 1, 25, 21]. Over the last decades, a number of fundamental Severi type inequalities have been established. For example, if $X$ is of maximal Albanese dimension, then the optimal inequality

$$\text{vol}(X) \geq 2(\dim X)! \chi(\omega_X)$$

holds. This inequality was originally stated for surfaces by Severi [32] and finally proved by Pardini [30]. Later, it was generalized to arbitrary dimension by Barja [1] and independently by the second named author [34]. Under some extra assumptions on $X$, this inequality has been improved by Lu and Zuo [25] for surfaces, and later by Barja, Pardini and Stoppino [4] in higher dimensions. Moreover, surfaces of maximal Albanese dimension attaining the equality in (1.3) have been explicitly characterized by Barja, Pardini and Stoppino [2], and independently by Lu and Zuo [25].

For general irregular varieties, the optimal Severi type inequality is known only for surfaces. Let $S$ be a smooth and irregular surface of general type. Bombieri [7] showed that the inequality

$$\text{vol}(S) \geq 2\chi(\omega_S)$$

holds. In [17], Horikawa completely described the structure of all irregular surfaces $S$ of general type attaining the equality. More concretely, the Albanese map of $S$ induces a fibration by curves of genus two, the canonical model of $S$ is a flat double cover over an elliptic ruled surface via its relative canonical map with respect to the Albanese fibration, and $\pi_1(S) \simeq \mathbb{Z}^2$.

By virtue of the importance of the geographical classification and also parallel to the Noether inequality problem, the following two questions arise naturally:

(Q1) What is the analogue of (1.4) for general irregular varieties of general type of dimension $n \geq 3$ (See also [21, §1, Question (1)])?

(Q2) Once we obtain the optimal $n$-dimensional inequality in (Q1), can we describe the geometry of varieties for which the equality hold?

1.2. Main results. The first main result of this paper is to establish an optimal three dimensional version of (1.4), thus answers Question (Q1) in dimension three.

**Theorem 1.1** (Theorem 3.4, 4.2 and 4.7). Let $X$ be a smooth and irregular 3-fold of general type. Then we have the following optimal inequality

$$\text{vol}(X) \geq \frac{4}{3} \chi(\omega_X).$$

If the equality holds, then

1. $q(X) = 1$, $h^2(X, \mathcal{O}_X) = 0$, and the general Albanese fiber of $X$ is a smooth surface $F$ with $\text{vol}(F) = 1$ and $p_g(F) = 2$;
(2) all minimal models of $X$ are Gorenstein;

(3) the topological fundamental group $\pi_1(X) \simeq \mathbb{Z}^2$.

As the title suggests, we call (1.5) the Noether-Severi inequality, because it is an analogue of the Noether inequality (1.2) and also of Severi type. Note that by [20, Remark 3.6], for any integer $e > 0$, there exists a smooth and irregular 3-fold $X$ of general type with $\text{vol}(X) = 4e$ and $\chi(\omega_X) = 3e$.

Previously, (1.5) was proved by the first named author [18] under the extra assumption that $X$ has a Gorenstein minimal model. Unfortunately, the method therein does not work in the general setting. Here in Theorem 1.1, we not only establish (1.5) in general, but also provide basic properties of the equality case in (1.5). An upshot in the characterization is that, all minimal models of $X$ are Gorenstein. This is a bit unexpected, at least to us. It suggests that the relation among birational invariants of a 3-fold may put constraints on singularities on its minimal models, which is not seen in the surface case. It is also a crucial ingredient for us to obtain the explicit structure of $X$ attaining the equality in (1.5).

**Definition 1.2.** A minimal and irregular 3-fold $V$ of general type is on the Noether-Severi line, if $K^3_V = \frac{3}{4} \chi(\omega_V)$.

As the second main result of this paper, we give a complete description of canonical models of all irregular 3-folds of general type on the Noether-Severi line, thus answer Question (Q2) in dimension three.

**Theorem 1.3** (Theorem 5.10). Let $X$ be the canonical model of a minimal and irregular 3-fold of general type on the Noether-Severi line, with its Albanese fibration $a : X \to B$, where $B$ is a smooth curve of genus one (see Theorem 1.1 (1)). Let $X'$ be the blow-up of $X$ along the base-locus section $\Gamma$ of $a$ (see Definition 5.2). Then the Albanese fibration $a' : X' \to B$ of $X'$ is factorized as

$$a' : X' \xrightarrow{f} Y \xrightarrow{q} S \xrightarrow{p} B$$

with the following properties:

1. $S = \mathbb{P}_B(a_*\omega_X)$ is a $\mathbb{P}^1$-bundle over $B$ with $p$ the projection.
2. $Y = \mathbb{P}_S(\mathcal{O}_S \oplus (\mathcal{O}_S(-2) \otimes K^*_1))$ is a $\mathbb{P}^1$-bundle over $S$ with $q$ the projection. Here $\gamma : B \to X$ denotes the section $\Gamma$, and $K_1 = p^*(\gamma^*\omega_X)$.
3. $f : X' \to Y$ is a flat double cover with the branch locus

$$D = D_1 + D_2,$$

where $D_1 \in |O_Y(1)|$, $D_2 \in |O_Y(5) \otimes q^*(O_S(10) \otimes K^{-4}_1 \otimes K^{-2}_2)|$ and $D_1 \cap D_2 = \emptyset$. Here $K_2 = p^*(\text{det} a_*\omega_X)$.

In one word, $X$ is a divisorial contraction of a double cover over a two-tower $\mathbb{P}^1$-bundle over a smooth curve of genus one.

In contrast with the rich understanding of the surface classification, to the best of our knowledge, classifications of irregular 3-folds $X$ of general
type with prescribed birational invariants are quite rare. The only known results are about those of maximal Albanese dimension with small \( \chi(\omega_X) \), in which the generic vanishing theory is essential:

- J. Chen, Debarre and Z. Jiang [10] have obtained, up to étale covers, an explicit description of 3-folds \( X \) of maximal Albanese dimension and of general type with \( \chi(\omega_X) = 0 \).
- Based on the work of Barja, Pardini and Stoppino [3], Z. Jiang [21] completely described 3-folds \( X \) of maximal Albanese dimension with \( \chi(\omega_X) = 1 \) and (the smallest volume) \( \text{vol}(X) = 12 \).

Theorem 1.3 is the first explicit description so far of an unbounded family of irregular 3-folds \( X \) of general type with \( \chi(\omega_X) > 0 \), and it is based on a very detailed study of the relative canonical map of \( X \) with respect to its Albanese fibration.

In a forthcoming paper [19], we will adopt the idea in this paper to study 3-folds on the Noether line, i.e., those attaining the equality in (1.2).

1.3. A corollary. Let \( X \) be a smooth and irregular variety. Let \( a : X \to A \) be the Albanese map of \( X \), with \( A \) the Albanese variety of \( X \). For a divisor \( L \) on \( X \), its continuous rank \( h^0_a(X, L) \) is defined as

\[
h^0_a(X, L) := \min \{ h^0(X, \mathcal{O}_X(L) \otimes a^*\alpha) | \alpha \in \text{Pic}^0(A) \}.
\]

It has been discovered in the work of Barja, Pardini, Stoppino and Z. Jiang [1, 3, 4, 21] that when \( h^0_a(X, L) > 0 \), the slope \( \lambda(L) := \frac{\text{vol}(L)}{h^0_a(X, L)} \) is closely related to the geometry of \( X \), especially when \( L = K_X \). From this perspective, finding the optimal lower or upper bound of \( \lambda(K_X) \) becomes an important problem.

When \( X \) is of maximal Albanese dimension, by the Severi inequality (1.3), we have the optimal lower bound \( \lambda(K_X) \geq 2(\dim X)! \). For general irregular varieties, the corresponding optimal lower bound is known only in dimension two. More precisely, let \( S \) be a smooth and irregular surface of general type. Then we have the following optimal lower bound \( \lambda(K_S) \geq 2 \), i.e.,

\[
(1.6) \quad \text{vol}(S) \geq 2h^0_a(S, K_S),
\]

which is an easy consequence of the Noether inequality (1.1). Note that \( h^0_a(S, K_S) \geq \chi(\omega_S) \) and that \( \chi(\omega_S) = p_g(S) \geq h^0_a(S, K_S) \) when \( q(S) = 1 \). It follows that \( \text{vol}(S) = 2h^0_a(S, K_S) \) if and only if \( \text{vol}(S) = 2\chi(\omega_S) \). Thus by [17], surfaces attaining the equality in (1.6) can be fully characterized.

Based on Theorem 1.1 and 1.3, we establish the optimal lower bound of \( \lambda(K_X) \) in dimension three and completely describe the equality case.

**Corollary 1.4** (Corollary 3.5). Let \( X \) be a minimal and irregular 3-fold of general type. Then we have the following optimal inequality

\[
(1.7) \quad \text{vol}(X) \geq \frac{4}{3} h^0_a(X, K_X).
\]
Moreover, the equality holds if and only if $X$ is on the Noether-Severi line.

Prior to (1.7), the best unconditional result was proved by Barja [1] stating that $\text{vol}(X) \geq h^0_n(X, K_X)$ for irregular 3-folds $X$ of general type. Thus (1.7) improves Barja’s result. Note that Corollary 1.4 offers more. Notably, it shows that Theorem 1.3 also gives a complete description of canonical models of those $X$ attaining the equality in (1.7).

1.4. Outline. The paper is organized as follows. In §2, we study the relative canonical map of minimal 3-folds fibered by (1, 2)-surfaces. The key results in this part are Proposition 2.5 and 2.6 which not only establish some basic numerical inequalities for these 3-folds but also describe the equality case. In §3, based on Proposition 2.5, we prove a part of Theorem 1.1, from which it is sufficient to deduce Corollary 1.4. In §4, based on Proposition 2.6, we complete the proof of Theorem 1.1. Moreover, we obtain some very explicit properties about the relative canonical map of minimal 3-folds on the Noether-Severi line (see Theorem 4.6). Finally, using these properties, we prove Theorem 1.3 in §5.

1.5. Notations and Preliminary. In this paper, we adopt the following notation and definitions.

Varieties and divisors. Let $V$ be a normal variety of dimension $d$. The geometric genus $p_g(V)$ of $V$ is defined as

$$p_g(V) := h^0(V, K_V).$$

For a divisor $L$ on $V$, the volume $\text{vol}(L)$ of $L$ is defined as

$$\text{vol}(L) := \limsup_{n \to \infty} \frac{h^0(V, nL)}{n^d/d!},$$

The volume $\text{vol}(K_V)$ is called the canonical volume of $V$, and is denoted by $\text{vol}(V)$. We say that $V$ is minimal, if $V$ has at worst $\mathbb{Q}$-factorial terminal singularity and $K_V$ is nef. If $V$ has at worst canonical singularities and $\text{vol}(V) > 0$, we say that $V$ is of general type. If $V$ is of general type and $K_V$ is ample, we say that $V$ is a canonical model. Note that if $V$ is minimal or a canonical model, then $\text{vol}(V) = K_V^d$.

For a linear system $\Lambda$, we denote by $\text{Mov}\Lambda$ and $\text{Bs}\Lambda$ the movable part and the base locus of $\Lambda$, respectively.

By an $(a, b)$-surface, we mean a normal surface $S$ having at worst canonical singularities with $\text{vol}(S) = a$ and $p_g(S) = b$.

Irregular varieties. Let $V$ be a normal variety with at worst rational singularities. We say that $V$ is irregular, if $q(V) := h^1(V, \mathcal{O}_V) > 0$. Note that $V$ has a well-defined Albanese map

$$a : V \to \text{Alb}(V),$$

where $A := \text{Alb}(V)$ is an abelian variety referred as the Albanese variety of $V$. Let $V \xrightarrow{f} W \xrightarrow{g} A$ be the Stein factorization of $a$. Then $f$ is called the
Albanese fibration of $V$, and a general fiber of $f$ is called an Albanese fiber. If $\phi : V' \to V$ is a resolution of singularities of $V$ and $a' : V' \to \text{Alb}(V')$ is the Albanese map of $V'$, then $\text{Alb}(V') \cong \text{Alb}(V)$ and $a' = a \circ \phi$. We refer the reader to [6, §2.4] for details.

1.6. Birational modification with respect to linear systems. Let $X$ be a minimal 3-fold of general type. Let

$$\alpha : X_0 \to X$$

be a resolution of singularities of $X$, where $X_0$ is smooth. We may further assume that $\alpha$ is an isomorphism over the smooth locus of $X$. Let $L$ be a Weil divisor on $X$ with $h^0(X,L) \geq 2$. Since $L$ is $\mathbb{Q}$-Cartier, we have $h^0(X_0, [\alpha^*L]) = h^0(X,L) \geq 2$. Let $|M_0| = \text{Mov}|[\alpha^*L]|$ be the movable part of $|[\alpha^*L]|$. By Hironaka’s big theorem, we may resolve the base locus of $|M_0|$ by taking successive blow-ups as follows:

$$\beta : X' = X_{n+1} \xrightarrow{\pi_{n+1}} X_{n} \to \cdots \to X_{i+1} \xrightarrow{\pi_{i+1}} X_i \to \cdots \to X_{1} \xrightarrow{\pi_{1}} X_0.$$  

Here each $\pi_i$ is a blow-up along a nonsingular center $W_i$ contained in the base locus of $\text{Mov}|(\pi_0 \circ \cdots \circ \pi_{i-1})^*M_0|$. As a result, the linear system $|M| = \text{Mov}|\beta^*M_0|$ is base point free. Set

$$\pi := \alpha \circ \beta : X' \to X.$$  

Such a birational modification will be used throughout this paper.

Acknowledgement. Y.H. would like to thank Professors Meng Chen and Yongnam Lee for their hospitality and support. Both authors are grateful to Professors Jin-Xing Cai, Jungkai Alfred Chen and Jun Lu for stimulating questions and fruitful discussions. Both authors also would like to thank the anonymous referee sincerely for valuable comments which helped to improve the presentation of the paper dramatically.

Y.H. is supported by a KIAS Individual Grant (MP 062501) at Korea Institute for Advanced Study and the Shanghai Pujiang Program Grant No. 21PJ1405200. T.Z. is supported by the Science and Technology Commission of Shanghai Municipality (STCSM) Grant No. 18dz2271000 and the National Natural Science Foundation of China (NSFC) General Grant No. 12071139.

2. Geometry of 3-folds fibered by $(1,2)$-surfaces

Throughout this section, we always assume that $X$ is a minimal 3-fold of general type with $p_g(X) \geq 4$ and with a fibration

$$f : X \to B$$

over a smooth curve $B$ such that the general fiber $F$ of $f$ is a $(1,2)$-surface.
2.1. Setting. By the assumption, $F$ is a minimal $(1,2)$-surface. Thus $|K_F|$ has a unique base point. Therefore, the horizontal part of the base locus of the relative canonical map of $X$ with respect to $f$ is just the base point of $|K_F|$. Since $\Gamma$ is a section of $f$, we have $\Gamma \cong B$. Thus $\Gamma$ is smooth.

Let $T$ be an effective divisor on $B$ with $t = \deg T \geq 0$. Taking a birational modification $\pi : X' \to X$ as in §1.6 with respect to the linear system $|K_X + f^*T|$, we may write

$$\pi^*(K_X + f^*T) = M + Z,$$

where $|M| = \text{Mov}||\pi^*(K_X + f^*T)||$ is base point free and $Z$ is an effective $\mathbb{Q}$-divisor. Note that $X$ has at worst terminal singularities. Thus

$$K_{X'} = \pi^*K_X + E_\pi,$$

where $E_\pi$ is an effective $\pi$-exceptional $\mathbb{Q}$-divisor.

Let $\phi_{K_X + f^*T} : X \dasharrow \mathbb{P}^{h^0(X,K_X + f^*T) - 1}$ be the rational map of $X$ induced by $|K_X + f^*T|$, with the image $\Sigma$. Then we have the following commutative diagram

$$\begin{array}{ccc}
X' & \stackrel{\psi}{\longrightarrow} & \Sigma' \\
\downarrow \phi_M & & \downarrow \tau \\
X & \stackrel{\phi_{K_X + f^*T}}{\longrightarrow} & \Sigma \\
\downarrow \pi & & \downarrow \\
B & \xleftarrow{f} & X
\end{array}$$

where $\phi_M : X' \to \Sigma$ is the morphism induced by $|M|$, $X' \xrightarrow{\psi} \Sigma' \xrightarrow{\tau} \Sigma$ is the Stein factorization of $\phi_M$, and $f' = f \circ \pi$ is the fibration of $X'$. Denote by $F'$ a general fiber of $f'$. Since $p_g(F) = 2$, we have $\dim \phi_M(F') \leq 1$. Thus $\dim \Sigma \leq 2$.

Remark 2.1. We would like to point out that the varieties as well as the maps in above diagram may vary when the divisor $T$ varies. However, for simplicity, we will use the same set of notation as above as a universal one. For the convenience of the reader, each time we will specify clearly which $T$ we use in the corresponding setting. In particular, by $T$ being sufficiently ample, we mean that $h^0(X,K_X + f^*T) \geq g(B) + 2$ and $\phi_{K_X + f^*T}$ is the relative canonical map $X \dasharrow \mathbb{P}^{h^0(X,K_X + f^*T)}$ of $X$ with respect to $f$.

2.2. The case when $\dim \Sigma = 2$. In this subsection, we always assume that $\dim \Sigma = 2$, and we have no a priori restriction on $T$. In fact, when $T$ is sufficiently ample, we always have $\dim \Sigma = 2$.

Let $C$ be a general fiber of $\psi$. Then we have

$$M^2 \equiv dC,$$

where $d := (\deg \tau) \cdot (\deg \Sigma)$. Since $\Sigma \subset \mathbb{P}^{h^0(X,K_X + f^*T) - 1}$ is a non-degenerate surface, we have $\deg \Sigma \geq h^0(X,K_X + f^*T) - 2$. 
Lemma 2.2. We have $|M||_{F'} = \text{Mov}|(\pi|_{F'})^*K_F|$, where $\pi|_{F'} : F' \to F$ coincides with the blow-up of the unique base point of $|K_F|$. In particular, $g(C) = 2$ and $f'$ factors through $\psi$ birationally (i.e., there is a rational map $p_{\Sigma'} : \Sigma' \dasharrow B$ such that $f' = p_{\Sigma'} \circ \psi$).

Proof. Since $p_g(F) = 2$ and $\phi_M(F')$ is a curve, the restriction map

$$H^0(X, K_X + f^*T) \to H^0(F, K_F)$$

is surjective. Thus the horizontal part of $B_s|K_X + f^*T|$ with respect to $f$ is just the section $\Gamma$. From the construction of $\pi$, we see that $\pi|_{F'} : F' \to F$ is just the blow-up of the unique base point of $|K_F|$. It follows that $|M||_{F'} = \text{Mov}|(\pi|_{F'})^*K_F|$. Note that $\text{Mov}|(\pi|_{F'})^*K_F|$ is a rational pencil of curves of genus two. We deduce that $g(C) = 2$. Moreover, $(F' \cdot C) = 0$. Thus $f'$ factors birationally through $\psi$.

The following estimate will be used when we treat the canonical map or the relative canonical map of $X$.

Lemma 2.3. The following statements hold:

1. If $T = 0$, then we have
   $$d \geq \min\{2p_g(X) - 4, \, p_g(X) + g(B) - 2\} \geq p_g(X) - 2.$$

2. If $g(B) > 0$ and $T$ is sufficiently ample as in Remark 2.1, then we have
   $$\deg \Sigma \geq \max\{h^0(X, K_X + f^*T), h^0(X, K_X + f^*T) + g(B) - 2\}.$$

Proof. For (1), the second inequality is obvious. Thus we only need to prove the first one. Take a general member $S \in |M|$. By Bertini’s theorem, $S$ is smooth. Moreover, by Lemma 2.2, $S \cap F' = C$. Now a general fiber of the induced morphism $f'|_S : S \to B$ is $S \cap F' = C$, and it is just a general fiber of $\psi|_S$. We conclude that $f'|_S$ and $\psi|_S$ are identical to each other.

Since $M|_S \equiv dC$, we may write $M|_S = (\psi|_S)^*D$, where $D$ is an effective divisor on $B$ of degree $d$. Note that $h^0(B, D) = h^0(S, M|_S) \geq p_g(X) - 1$. If $h^1(B, D) > 0$, by Clifford’s inequality,

$$\deg D \geq 2h^0(B, D) - 2 \geq 2p_g(X) - 4.$$

If $h^1(B, D) = 0$, by the Riemann-Roch theorem, we have

$$\deg D = h^0(B, D) + g(B) - 1 \geq p_g(X) + g(B) - 2.$$

This proves (1).

For (2), by the assumption on $T$, the map $\phi_{K_X + f^*T}$ is the relative canonical map $X \dasharrow \mathbb{P}_B(f_*\omega_{X/B})$ of $X$ with respect to $f$. In particular, $\Sigma = \mathbb{P}_B(f_*\omega_{X/B}) \subset \mathbb{P}^{h^0(X, K_X + f^*T) - 1}$ is a smooth $\mathbb{P}^1$-bundle over $B$. Since $g(B) > 0$, $\Sigma$ is irregular. Thus by [27, §10 and Theorem 8], we deduce that

$$\deg \Sigma \geq h^0(X, K_X + f^*T).$$
For the rest part of (2), let $H$ be the restriction on $\Sigma$ of a general hyperplane of $\mathbb{P}^{h^0(X,K_X+f^*T)-1}$. By Bertini’s theorem, $H$ is smooth. Moreover, $\deg \Sigma = \deg \mathcal{O}_H(H)$. Since $H$ has a natural cover to $B$ induced by the projection, we have $g(H) \geq g(B)$. If $h^1(H,\mathcal{O}_H(H)) > 0$, by Clifford’s inequality,

$$\deg \Sigma \geq 2h^0(H,\mathcal{O}_H(H)) - 2 \geq 2h^0(X,K_X + f^*T) - 4.$$  

If $h^1(H,\mathcal{O}_H(H)) = 0$, by the Riemann-Roch theorem, we have

$$\deg \Sigma = h^0(H,\mathcal{O}_H(H)) + g(H) - 1 \geq h^0(X,K_X + f^*T) + g(B) - 2.$$  

By Remark 2.1, we have $h^0(X,K_X + f^*T) \geq g(B) + 2$. It follows that

$$\deg \Sigma \geq h^0(X,K_X + f^*T) + g(B) - 2.$$  

Thus the proof is completed. □

**Lemma 2.4.** There exists a unique $\pi$-exceptional prime divisor $E_0$ on $X'$ such that

1. $\text{coeff}_{E_0}(Z) = \text{coeff}_{E_0}(E_\pi) = 1$;
2. $\pi(E_0) = \Gamma$, $\phi_M(E_0) = \Sigma$;
3. $(E_0 \cdot C) = (Z \cdot C) = (E_{\pi} \cdot C) = ((\pi^*K_X) \cdot C) = 1$.

**Proof.** By Lemma 2.2, we may assume that $C$ is a general member of $\text{Mov}(\pi|_{F'})^*K_F|$. Then we have

$$(\pi^*K_X \cdot C) = ((\pi^*K_X)|_{F'} \cdot C) = (((\pi|_{F'})^*K_F) \cdot C) = 1.$$  

Since $(M \cdot C) = 0$, it follows that

$$(Z|_{F'} \cdot C) = (Z \cdot C) = ((\pi^*(K_X + f^*T)) \cdot C) = ((\pi^*K_X) \cdot C) = 1.$$  

By Lemma 2.2, $Z|_{F'}$ is a $(-1)$-curve on $F'$. Thus there exists a unique prime divisor $E_0 \subseteq Z$ with $\text{coeff}_{E_0}(Z) = 1$ such that $E_0|_{F'} = Z|_{F'}$. In particular, $\pi(E_0) = \Gamma$. Moreover,

$$(E_0 \cdot C) = (E_0|_{F'} \cdot C) = (Z|_{F'} \cdot C) = 1.$$  

Thus $\phi_M(E_0) = \Sigma$.

By the adjunction formula,

$$K_{F'} = K_X|_{F'} = (\pi^*K_X)|_{F'} + E_\pi|_{F'} = (\pi|_{F'})^*K_F + E_\pi|_{F'}.$$  

By Lemma 2.2, we deduce that $E_\pi|_{F'}$ is just the $(-1)$-curve $Z|_{F'}$. Using the same argument as for $Z$, we deduce that $\text{coeff}_{E_0}(E_\pi) = 1$ and $(E_\pi \cdot C) = 1$.

The proof is completed. □

**Proposition 2.5.** Suppose either $p_g(X) \geq 4$ or $T$ is sufficiently ample as in Remark 2.1. Then the following inequalities hold:

1. $$(K_X \cdot \Gamma) \geq \frac{1}{3}d - \frac{2}{3}t + \frac{2}{3}(g(B) - 1);$$
2. $$K_X^3 \geq \frac{4}{3}d - \frac{8}{3}t + \frac{2}{3}(g(B) - 1).$$
Proof. Let $E_0$ be the unique $\pi$-exceptional prime divisor as in Lemma 2.4. Similarly as in the proof of Lemma 2.3, take a general member $S \in |M|$. By Bertini’s theorem and Lemma 2.4 (2), $S$ is smooth and $E_0|_S$ is irreducible. Then we obtain the fibration $f'|_S = \psi|_S : S \to B$.

Denote $\Gamma_S = E_0|_S$. Then $\Gamma_S$ is a section of $\psi|_S$. By Lemma 2.4 (1), we may write

$$E_\pi|_S = \Gamma_S + E_V, \quad Z|_S = \Gamma_S + Z_V. \quad (2.3)$$

Here $E_V$ and $Z_V$ are effective $\mathbb{Q}$-divisors on $S$. By Lemma 2.4 (3),

$$(E_V \cdot C) = ((E_\pi - E_0) \cdot C) = 0, \quad (Z_V \cdot C) = ((Z - E_0) \cdot C) = 0. \quad (2.4)$$

We deduce that both $E_V$ and $Z_V$ are vertical with respect to $\psi|_S$.

Note that $(f^*T)|_S \equiv tF'|_S \equiv tC$. By the adjunction formula, (2.1), (2.2) and (2.3),

$$K_S = (K_{X'} + S)|_S \equiv (2d - t)C + 2\Gamma_S + E_V + Z_V. \quad (2.4)$$

Moreover, since $p_g(X) > 0$, we have $M \geq f^*T$. Thus $dC \equiv M|_S \geq (f^*T)|_S$. We deduce that $d \geq t$. Thus

$$2d - t \geq d \geq h^0(S, M|_S) - 1 \geq h^0(X, K_X + f^*T) - 2.$$ 

Therefore, if $p_g(X) \geq 4$ or $T$ is sufficiently ample, we always have $2d - t \geq 2$.

Denote by $\sigma : S \to S_0$ the contraction onto its minimal model $S_0$. By [9, Corollary 2.3 for $\lambda = 1$ and $D = K_X + f^*T$], we have

$$\left(\pi^* \left( K_X + \frac{1}{2}f^*T \right) \right)|_S \sim \sigma^* \frac{1}{2} \sigma^* K_{S_0} + H, \quad (2.5)$$

where $H$ is an effective $\mathbb{Q}$-divisor. Therefore, by Lemma 2.4 (3), $((\sigma^* K_{S_0}) \cdot C) \leq ((\pi^* (2K_X + f^*T)) \cdot C) = 2$. Let $C_0 = \sigma_* C$ and $\Gamma_{S_0} = \sigma_* \Gamma_S$. Then we have $(K_{S_0} \cdot C_0) \leq 2$. On the other hand, by (2.4),

$$K_{S_0} \equiv (2d - t)C_0 + 2\Gamma_{S_0} + \sigma_*(E_V + Z_V).$$

We deduce that $(K_{S_0} \cdot C_0) \geq (2d - t)C_0^2 \geq 2C_0^2$. By parity, it follows that $(K_{S_0} \cdot C_0) = 2$ and $C_0^2 = 0$. In particular, the fibration $\psi|_S$ descends to a fibration $S_0 \to B$ whose general fiber is $C_0$ with $g(C_0) = 2$. Moreover,

$$(H \cdot C) = \left( \left( \pi^* \left( K_X + \frac{1}{2}f^*T \right) \right) \cdot C \right) - \frac{1}{2}(K_{S_0} \cdot C_0) = 0.$$

This implies that $H$ is vertical with respect to $\psi|_S$.

---

\[^{1}\text{Here } D \text{ is semi-ample by [24, Theorem 3.3].}\]
Since $\Gamma_{S_0}$ is a section of the fibration $S_0 \to B$, $\Gamma_{S_0} \simeq B$. Thus $g(\Gamma_{S_0}) = g(B)$. By the adjunction formula on $S_0$, we have

$$2g(B) - 2 = (K_{S_0} \cdot \Gamma_{S_0}) + \Gamma_{S_0}^2$$

$$= (K_{S_0} \cdot \Gamma_{S_0}) + \frac{1}{2} ((K_{S_0} - (2d - t)C_0 - \sigma^*(E_V + Z_V)) \cdot \Gamma_{S_0})$$

(2.6) $$= \frac{3}{2} (K_{S_0} \cdot \Gamma_{S_0}) - \left( d - \frac{1}{2} t \right) - \frac{1}{2} (\sigma^*(E_V + Z_V) \cdot \Gamma_{S_0})$$

$$\leq \frac{3}{2} (K_{S_0} \cdot \Gamma_{S_0}) - \left( d - \frac{1}{2} t \right),$$

where the last inequality follows from the fact that $\sigma^*(E_V + Z_V)$ is vertical with respect to the fibration $S_0 \to B$. Together with (2.5) and the fact that $(H \cdot \Gamma_S) \geq 0$, we deduce that

$$(K_X \cdot \Gamma) = ((\pi^*K_X)|_S \cdot \Gamma_S) \geq \frac{1}{2} ((\sigma^*K_{S_0}) \cdot \Gamma_S) - \frac{1}{2} t$$

$$\geq \frac{1}{3} d - \frac{2}{3} t + \frac{2}{3} (g(B) - 1).$$

This proves (1).

For (2), note that

$$K_X^3 = (K_X \cdot (K_X + f^*T)^2) - 2tK_X^2$$

(2.7) $$\geq ((\pi^*K_X) \cdot M^2) + ((\pi^*K_X)|_S \cdot Z_S) - 2t.$$

By Lemma 2.4 (3),

$$((\pi^*K_X) \cdot M^2) = d((\pi^*K_X) \cdot C) = d.$$

By (1), we have

$$((\pi^*K_X)|_S \cdot Z_S) \geq ((\pi^*K_X)|_S \cdot \Gamma_S) = (K_X \cdot \Gamma) \geq \frac{1}{3} d - \frac{2}{3} t + \frac{2}{3} (g(B) - 1).$$

Combine the above inequalities together. We deduce that

$$K_X^3 \geq \frac{4}{3} d - \frac{8}{3} t + \frac{2}{3} (g(B) - 1).$$

This proves (2). \qed

The following is a crucial proposition that we will frequently use in our later argument.

**Proposition 2.6.** Keep the same notation as in the proof of Proposition 2.5. Suppose the equality in Proposition 2.5 (2) holds. Then we have

1. $\sigma^*(E_V + Z_V) = 0$ and $K_{S_0} \equiv (2d - t)C_0 + 2\Gamma_{S_0}$;
2. $K_{S_0}^2 = \frac{2}{3} (2d - t + g(B) - 1)$;
3. $(\pi^*(2K_X + f^*T)|_S \sim_\Theta \sigma^*K_{S_0}$.

**Proof.** Since the equality in Proposition 2.5 (2) holds, all inequalities in the above proof become equalities. On the one hand, (2.6) becomes an equality, and we have $(\sigma^*(E_V + Z_V) \cdot \Gamma_{S_0}) = 0$. Thus $(\sigma^*(E_V + Z_V) \cdot (2\Gamma_{S_0} + (2d -
is just induced by the base-point-free linear system $|\omega|$. We deduce that 

$$K_{S_0} = (2d - t)C_0 + 2\Gamma_{S_0}.$$ 

Thus (1) is proved. Also by (2.6), we have 

$$\Gamma_{S_0}^2 = -(K_{S_0} \cdot \Gamma_{S_0}) + 2g(B) - 2 = -\frac{1}{3}(2d - t) + \frac{2}{3}(g(B) - 1).$$ 

Together with (1), it follows that 

$$K_{S_0}^2 = ((2d - t)C_0 + 2\Gamma_{S_0})^2 = \frac{8}{3}(2d - t + g(B) - 1),$$ 

and (2) is proved. Now (2.7) also becomes an equality. This implies that 

$$(\pi^*(2K_X + f^*T))|_S)^2 = 4(\pi^*K_X)|_S \cdot (\pi^*(K_X + f^*T)|_S) = 4((\pi^*K_X) \cdot M^2) + 4((\pi^*K_X)|_S \cdot Z|_S) = \frac{8}{3}(2d - t + g(B) - 1).$$ 

By (2.5), we have 

$$(\pi^*(2K_X + f^*T))|_S)^2 \geq (\pi^*(2K_X + f^*T))|_S \cdot \sigma^*K_{S_0} \geq K_{S_0}^2.$$ 

Combine the above equalities and inequalities with (2.5) and apply the Hodge index theorem. We deduce that 

$$(\pi^*(2K_X + f^*T))|_S \equiv \sigma^*K_{S_0},$$ 

i.e., $H \equiv 0$. Since $H$ is effective, we conclude that $H = 0$ and 

$$(\pi^*(2K_X + f^*T))|_S \sim q \sigma^*K_{S_0}.$$ 

This proves (3). \hfill \Box

### 2.3. A particular case when $T = 0$, $g(B) > 0$ and $\dim \Sigma = 2$

In this subsection, we still assume that $\dim \Sigma = 2$, but we further assume that $T = 0$, $g(B) > 0$ and $p_g(X) \geq 4$. That is, we consider the canonical map of $X$. By Lemma 2.3 (1), we have 

$$d = (\deg \tau) \cdot (\deg \Sigma) \geq p_g(X) - 1.$$ 

In the following, we focus on the specific case when 

$$d = p_g(X) - 1.$$ 

In this case, since $\deg \Sigma \geq p_g(X) - 2$ and $p_g(X) \geq 4$, we deduce that $\deg \Sigma = p_g(X) - 1$ and $\deg \tau = 1$. Since $f'$ factors birationally through $\psi$ by Lemma 2.2, we further deduce that $\Sigma$ is birationally fibered over $B$. Note that $g(B) > 0$. By [27, Theorem 8], $g(B) = 1$ and $\Sigma$ is a cone over $B$. In particular, $\Sigma$ is normal. Thus $\Sigma' = \Sigma$. Moreover, $\phi_M(F')$ is a line on $\Sigma$.

Let $\Sigma_0$ be the blow-up of $\Sigma$ at the unique cone singularity $v$. Then we obtain a ruled surface $p : \Sigma_0 \rightarrow B$ over $B$. Let $s$ be the exceptional curve on $\Sigma_0$, and let $l$ be a ruling on $\Sigma_0$. Then the morphism $\Sigma_0 \rightarrow \Sigma \subset \mathbb{P}^{p_g(X) - 1}$ is just induced by the base-point-free linear system $|s + p^*e_B|$ on $\Sigma_0$, where
$e_B$ is a divisor on $B$ of degree $d = p_g(X) - 1$. Since $s$ is contracted via this morphism, we deduce that $s^2 = -(p_g(X) - 1) \leq -3$. By [14, V, Theorem 2.15], $p_*O_{\Sigma_0}(s)$ is decomposable. Thus

$$p_*O_{\Sigma_0}(s) = O_B \oplus O_B(-e_B).$$

Therefore, we have the following commutative diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{\psi_0} & \Sigma_0 \\
\psi = \phi_M & \downarrow & \dashedrightarrow \\
\Sigma & \xrightarrow{p} & B
\end{array}
$$

where the rational map $\psi_0$ is induced by the blow-up of $\Sigma$.

**Lemma 2.7.** The rational map $\psi_0$ is a morphism. Moreover, $p \circ \psi_0 = f'$ and

$$
|M| = \psi_0^*|s + p^*e_B|.
$$

**Proof.** Take a birational modification $\nu : X'' \to X'$ such that:

1. $X''$ is smooth and $\nu$ is an isomorphism over $X'' - \psi^{-1}(v)$;
2. the rational map $\mu = \psi_0 \circ \nu : X'' \dasharrow \Sigma_0$ is a morphism.

Let $f'' : X'' \to B$ be the Albanese fibration of $X''$. Then $f'' = f' \circ \nu$. By Lemma 2.2, $f'$ birationally factors through $\psi$. It follows that $f'' = p \circ \mu$.

Suppose $\psi_0$ is not a morphism. By Zariski’s main theorem, there is an integral curve $A \subset X''$ such that $\nu(A) \in \psi^{-1}(v)$ is a point and $\mu(A) \subset \Sigma_0$ is a curve. Since $\psi(\nu(A)) = v$, we deduce that $\mu(A) = s$. Thus $f''(A) = p(s) = B$. On the other hand, $f''(A) = f'(\nu(A))$ is a point. This is a contradiction. Thus $\psi_0$ is a morphism.

Now $\psi_0^*l = F'$. Thus it follows that $p \circ \psi_0 = f'$, and (2.8) simply follows from the above commutative diagram.

In the following, we adopt the idea in [9] to prove the “weak pseudo-effectivity” of $3\pi^*K_X - dF'$ (Lemma 2.10). Recall that by Lemma 2.4, there is a unique $\pi$-exceptional prime divisor $E_0$ satisfying the condition therein.

**Lemma 2.8.** There exists a unique prime divisor $D_0$ such that

1. $\text{coeff}_{D_0}(\psi_0^*s) = 1$;
2. $D_0 \cdot E_0 \cdot F') = 1$ and $(\pi^*K_X \cdot D_0 \cdot F') = 1$.

**Proof.** By the abuse of notation, we still denote by $C$ the general fiber of $\psi_0$. By Lemma 2.7, we have

$$
M|_{F'} = (\psi_0^*|s + p^*e_B|)|_{F'} \equiv (\psi_0^*s)|_{F'}.
$$

By Lemma 2.2, $M|_{F'} \equiv C$. Thus we have $(\psi_0^*s)|_{F'} \equiv C$. By Lemma 2.4 (3), $(\psi_0^*s) \cdot E_0 \cdot F') = (E_0 \cdot C) = 1$. Since $\psi_0^*s$ is Cartier and $E_0 \notin \text{Supp}(\psi_0^*s)$, for any prime divisor $D$ with $\text{coeff}_D(\psi_0^*s) > 0$, we have $\text{coeff}_D(\psi_0^*s) \geq 1$.
and \((D \cdot E_0 \cdot F')\) is a non-negative integer. Thus there exists a unique prime divisor \(D_0\) with \(\text{coeff}_{D_0}(\psi_0^*s) = 1\) such that 
\[
(D_0 \cdot E_0 \cdot F') = 1.
\]
Note that we have \((\pi^*K_X)|_{F'} = (\pi|_{F'})^*K_F \equiv C + E_0|_{F'}\) by Lemma 2.2. Therefore, 
\[
((\pi^*K_X) \cdot D_0 \cdot F') = (E_0 \cdot D_0 \cdot F') = 1.
\]
The proof is completed. \(\square\)

**Lemma 2.9.** Let \(A\) be an ample Cartier divisor on \(\Sigma_0\). For any integer \(m > 0\), there exists an integer \(c > 0\) and an effective divisor \(H_m \sim cm(K_{X'}/\Sigma_0 + E_0) + cv_0^*A\) such that \(E_0 \not\subseteq \text{Supp}(H_m)\), where \(K_{X'}/\Sigma_0 = K_{X'} - \psi_0^*K_{\Sigma_0}\) and \(E_0\) is the \(\pi\)-exceptional divisor as in Lemma 2.4.

**Proof.** The proof of [9, Claim 4.9] works verbatim in our setting, and we only need to replace \(W, g, F_a\) and \(E_0\) therein by \(X', \psi_0, \Sigma_0\) and \(E_0\) in our context. We leave the detailed proof to the interested reader. \(\square\)

**Lemma 2.10.** For any nef \(\mathbb{Q}\)-divisor \(L\) on \(X'\), we have 
\[
((3\pi^*K_X - dF') \cdot D_0 \cdot L) \geq 0,
\]
where \(D_0\) is the divisor as in Lemma 2.8.

**Proof.** The proof is just a slight modification of that of [9, Claim 4.10]. For the convenience of the reader, we give a detailed proof here.

By (2.1), (2.2) and Lemma 2.7, we have 
\[
K_{X'}/\Sigma_0 + E_0 = (\pi^*K_X + E_\pi) + \psi_0^*(2s + p^*eB) + E_0 \\
= \pi^*K_X + 2M - \psi_0^*(p^*eB) + E_\pi + E_0 \\
= 3\pi^*K_X - \psi_0^*(p^*eB) + E_\pi + E_0 - 2Z.
\]
Write \(E_\pi + E_0 - 2Z = N_+ - N_-\), where \(N_+\) and \(N_-\) are both effective \(\mathbb{Q}\)-divisors with no common components. By Lemma 2.4 (1), we deduce that \(E_0 \not\subseteq \text{Supp}(N_+)\) and \(E_0 \not\subseteq \text{Supp}(N_-)\).

Choose an ample divisor \(A = t_1p^*eB + t_2s\) on \(\Sigma_0\), where \(t_1 > t_2\) are two positive integers such that \(t_2K_X\) is Cartier. Let \(m\) be a positive integer such that \(mK_X\) is Cartier. By Lemma 2.9, there exists an integer \(c > 0\) and an effective divisor \(H_m \sim cm(K_{X'}/\Sigma_0 + E_0) + cv_0^*A\) such that \(E_0 \not\subseteq \text{Supp}(H_m)\). Thus we have 
\[
H_m + cmN_- + ct_2Z \\
\sim cm(K_{X'}/\Sigma_0 + E_0) + cv_0^*A + cmN_- + ct_2Z \\
= cm(3\pi^*K_X - \psi_0^*(p^*eB)) + E_\pi + E_0 - 2Z + N_- + cv_0^*A + ct_2Z \\
= cm(3\pi^*K_X - \psi_0^*(p^*eB)) + c((t_1\psi_0^*(p^*eB) + t_2(\psi_0^*s + Z)) + cmN_+ \\
= cm(3\pi^*K_X - \psi_0^*(p^*eB)) + c((t_1 - t_2)\psi_0^*(p^*eB) + t_2(M + Z) + cmN_+ \\
= cm(3\pi^*K_X - \psi_0^*(p^*eB)) + c((t_1 - t_2)\psi_0^*(p^*eB) + t_2\pi^*K_X) + cmN_+.
\]
Here the first equality is by (2.9), and the last two equalities are by (2.8) and (2.1), respectively. By Lemma 2.7, \( p \circ \psi_0 = f' = f \circ \pi \). Thus \( \psi_0(p' e_B) = \pi^*(f^* e_B) \). This implies

\[
H_m + cmN_- + ct_2 Z - cmN_+ \\
\sim cm\pi^*(3K_X - f^* e_B) + cm\pi^*(t_2K_X + (t_1 - t_2)f^* e_B) .
\]

Note that \( N_+ \) is \( \pi \)-exceptional. We deduce that \( cmN_+ \) is contained in the fixed part of \( |H_m + cmN_- + ct_2 Z| \). In particular, \( H_m + cmN_- + ct_2 Z - cmN_+ \) is effective.

Let \( G_m = \frac{1}{cm}(H_m + cmN_- + ct_2 Z - cmN_+) \). Since \( E_0 \not\subseteq \text{Supp}(H_m) \cup \text{Supp}(N_+) \cup \text{Supp}(N_-) \), by Lemma 2.4 (1), \( \text{coeff}_E E_0(G_m) = \frac{t_2}{m} \text{coeff}_E (Z) = \frac{t_2}{m} \).

By Lemma 2.8 (2),

\[
\left( G_m - \frac{t_2}{m} E_0 \right) \cdot E_0 \cdot F' \geq \mu_m (D_0 \cdot E_0 \cdot F') = \mu_m ,
\]

where \( \mu_m = \text{coeff}_D E_0(G_m) \). Since \( E_0 \) is \( \pi \)-exceptional and both \( G_m \) and \( F' \) are \( \pi \)-trivial, \( (G_m \cdot E_0 \cdot F') = 0 \). It follows that

\[
- \frac{t_2}{m} (E_0^2 \cdot F') \geq \mu_m \geq 0 .
\]

In particular, \( \lim_{m \to \infty} \mu_m = 0 \). Thus for any nef \( \mathbb{Q} \)-divisor \( L \) on \( X' \), we have

\[
\lim_{m \to \infty} (G_m \cdot D_0 \cdot L) = \lim_{m \to \infty} ((G_m - \mu_m D_0) \cdot D_0 \cdot L) \geq 0 .
\]

By the definition of \( G_m \), the above inequality just implies that

\[
\left( (3\pi^* K_X - dF') \cdot D_0 \cdot L \right) = \lim_{m \to \infty} (G_m \cdot D_0 \cdot L) \geq 0 .
\]

The proof is completed.

\[ \square \]

**Proposition 2.11.** Suppose \( g(B) > 0 \) and \( p_g(X) \geq 4 \). If \( T = 0 \), \( \deg \Sigma = 2 \) and \( d = p_g(X) - 1 \), then we have

\[ K_X^3 > \frac{4}{3} d = \frac{4}{3} (p_g(X) - 1) . \]

**Proof.** By (2.1) and (2.8), \( \pi^* K_X \equiv dF' + \psi_0^* s + Z \). Thus

\[
K_X^3 \geq d \left( (\pi^* K_X)^2 \cdot F' \right) + \left( (\pi^* K_X^2) \cdot (\psi_0^* s) \right) \geq d + \left( (\pi^* K_X)^2 \cdot D_0 \right) ,
\]

where \( D_0 \) is the unique divisor as in Lemma 2.8. Note that we have \( g(B) = 1 \), we deduce that \( K_{X/B} = K_X \). Thus \( K_X - \epsilon F \) is nef for some \( \epsilon > 0 \) by [29, Theorem 1.4 and Lemma 1.6]. By Lemma 2.8 (2) and Lemma 2.10,

\[
0 \leq \left( (3\pi^* K_X - dF') \cdot D_0 \cdot (\pi^* K_X - \epsilon F') \right) = 3 \left( (\pi^* K_X)^2 \cdot D_0 \right) - (3\epsilon + d) \left( (\pi^* K_X) \cdot D_0 \cdot F' \right) = 3 \left( (\pi^* K_X)^2 \cdot D_0 \right) - (3\epsilon + d) .
\]

That is, \( (\pi^* K_X)^2 \cdot D_0 \geq \frac{d}{3} + \epsilon \). It follows that

\[
K_X^3 \geq d + \frac{d}{3} + \epsilon > \frac{4}{3} d = \frac{4}{3} (p_g(X) - 1) .
\]

The proof is completed. 
\[ \square \]
Remark 2.12. Keep the same assumption as in Proposition 2.11. If we further assume that $K_X - cF$ is nef for some $c > 0$, then the same proof leads to the inequality that

$$K_X^3 \geq \frac{4}{3}(p_g(X) - 1) + c.$$ 

2.4. The case when $T = 0$ and $\dim \Sigma = 1$. We have the following result.

Proposition 2.13. Suppose that $T = 0$ and $\dim \Sigma = 1$. Then $\psi = f'$. Moreover, if $X$ is irregular, then $q(X) = 1$ and $h^2(X, \mathcal{O}_X) = 0$.

Proof. By the assumption, $p_g(X) \geq 4$. Note that a general fiber $F'$ cannot be contained in the base locus $Z$ of $|\pi^*K_X + f^*T|$. Thus

$$h^0(X', M - F') = h^0(X', K_{X'} - F') \geq p_g(X) - p_g(F) \geq 1.$$ 

Since $\dim \Sigma = 1$, $\psi$ contracts every element in $|M|$ to points. Note that $h^0(X', M - F') > 0$. Thus $\psi$ contracts general fibers of $f'$ to points. Since $f'$ has connected fibers, we conclude that $\psi$ contracts every fiber of $f'$ by [24, Lemma 1.6]. It follows that $\psi = f'$ and the general fiber of $\psi$ is $F'$. If $X$ is irregular, by [11, Lemma 4.5 (i)], we have $q(X) = 1$ and $h^2(X, \mathcal{O}_X) = 0$. The proof is completed. \hfill $\Box$

3. Noether-Severi inequality for irregular 3-folds

The goal of this section is to prove the Noether-Severi inequality (1.5).

3.1. Noether-Severi inequality for 3-folds with (1, 2)-surface Albanese fibers. Throughout this subsection, we always assume that $X$ is a minimal and irregular 3-fold of general type with

$$f : X \to B$$

the Albanese fibration of $X$ such that the general fiber $F$ of $f$ is a minimal (1, 2)-surface. Here $B$ is a smooth curve. By the same argument as in the proof of [5, Proposition V.15], we deduce that $B$ is of genus $g(B) = q(X)$. In the following, we fix a sufficiently ample divisor $T$ on $B$ as in Remark 2.1 with $\deg T = t$.

Lemma 3.1. We have

$$h^0(X, K_X + f^*T) = \chi(\omega_X) + 2t - (g(B) - 1).$$

Proof. By [23, Theorem 2.1], $R^1f_*\omega_X$ and $R^2f_*\omega_X$ are both torsion free sheaves. Now $q(F) = 0$. We deduce that $R^1f_*\omega_X = 0$. Moreover, $R^2f_*\omega_X = \omega_B$. Since $T$ is sufficiently ample, we have

$$h^1(X, K_X + f^*T) = h^1(B, f_*\omega_X \otimes \mathcal{O}_B(T)) = 0,$$

and

$$h^2(X, K_X + f^*T) = h^0(B, R^2f_*\omega_X \otimes \mathcal{O}_B(T)) = h^0(B, K_B + T) = t + g(B) - 1.$$
Note that \( \chi(\omega_F) = p_g(F) + 1 = 3 \). Combine these results together, and it follows that
\[
h^0(X, K_X + f^*T) = \chi(O_X(K_X + f^*T)) - t - (g(B) - 1) \\
= \chi(\omega_X) + 2t - (g(B) - 1).
\]

Thus the proof is completed. \( \square \)

**Lemma 3.2.** We have
\[
K_X^3 \geq \frac{4}{3}\chi(\omega_X) + \frac{2}{3}(q(X) - 1) - \frac{4}{3}.
\]

**Proof.** Consider the map
\[
\phi_{K_X + f^*T} : X \rightarrow \Sigma \subset \mathbb{P}^{h^0(X, K_X + f^*T) - 1}
\]
induced by \( |K_X + f^*T| \). By the assumption on \( T \), we have \( \dim \Sigma = 2 \). Combine Proposition 2.5 (2) and Lemma 2.3 (2) together. It follows that
\[
K_X^3 \geq \frac{4}{3}h^0(X, K_X + f^*T) - \frac{8}{3}t + 2(g(B) - 1) - \frac{4}{3}.
\]
Then the result just follows from Lemma 3.1. \( \square \)

**Proposition 3.3.** We have
\[
K_X^3 \geq \frac{4}{3}\chi(\omega_X) + \frac{2}{3}(q(X) - 1).
\]

**Proof.** Let \( \pi_k : B_k \rightarrow B \) be any étale cover of degree \( k > 1 \) and let \( X_k = X \times_{\pi_k} B_k \). Then we have the following commutative diagram:
\[
\begin{array}{ccc}
X_k & \longrightarrow & X \\
\downarrow f_k & & \downarrow f \\
B_k & \longrightarrow & B
\end{array}
\]
It is easy to see that the induced fibration \( f_k : X_k \rightarrow B_k \) via \( \pi_k \) is exactly the Albanese fibration of \( X_k \), and the Albanese fiber of \( X_k \) is a minimal \((1,2)\)-surface. By the Hurwitz formula, \( g(B_k) - 1 = k(g(B) - 1) \). By Lemma 3.2, we have
\[
K_{X_k}^3 \geq \frac{4}{3}\chi(\omega_{X_k}) + \frac{2}{3}(g(B_k) - 1) - \frac{4}{3}.
\]
Since \( X_k \rightarrow X \) is étale of degree \( k \), the above inequality is equivalent to
\[
K_X^3 \geq \frac{4}{3}\chi(\omega_X) + \frac{2}{3}(g(B) - 1) - \frac{4}{3k}.
\]
Thus the proof is completed by taking \( k \rightarrow \infty \). \( \square \)
3.2. Main theorem. Now we prove the main result in this section.

**Theorem 3.4.** Let $X$ be a minimal and irregular 3-fold of general type. Then we have the following optimal inequality:

\[(3.1) \quad K_X^3 \geq \frac{4}{3} \chi(\omega_X). \]

If the equality holds, then $q(X) = 1$, $h^2(X, \mathcal{O}_X) = 0$, and the general Albanese fiber of $X$ is a minimal $(1,2)$-surface.

**Proof.** If the general Albanese fiber of $X$ is a $(1,2)$-surface, then (3.1) follows from Proposition 3.3. Otherwise, by [20, Theorem 1.8], we have a stronger inequality $K_X^3 \geq 2\chi(\omega_X)$. Therefore, (3.1) always holds. By the example constructed in [20, Section 3], (3.1) is optimal.

From now on, we assume that $K_X^3 = \frac{4}{3} \chi(\omega_X)$. Then $K_X^3 < 2\chi(\omega_X)$. By [20, Theorem 1.8] again, the general Albanese fiber of $X$ is a $(1,2)$-surface. Then by Proposition 3.3, $q(X) = 1$.

Since $\chi(\omega_X) = \frac{4}{3} K_X^3 > 0$, we have $\chi(\omega_X) \geq 1$. Let $\pi'_k : X_k \to X$ be an étale cover of $X$ of degree $k \geq 4$. Then we still have $K_X^3 = \frac{4}{3} \chi(\omega_{X_k})$. By Proposition 3.3 again, $q(X_k) = 1$. Since $\chi(\omega_{X_k}) = k\chi(\omega_X) \geq 4$, we deduce that $p_g(X_k) \geq \chi(\omega_{X_k}) - q(X_k) + 1 \geq 4$. Consider the canonical map of $X_k$ (i.e., taking $T = 0$ in §2). If the canonical image of $X_k$ is a curve, by Proposition 2.13, $h^2(X_k, \mathcal{O}_{X_k}) = 0$. If the canonical image is a surface, by Lemma 2.3 (1), Proposition 2.5 (2) and Proposition 2.11, we have

\[
\frac{4}{3} \chi(\omega_{X_k}) = K_X^3 \geq \frac{4}{3} (p_g(X_k) - 1) = \frac{4}{3} (\chi(\omega_{X_k}) + h^2(X_k, \mathcal{O}_{X_k}) - 1).
\]

Thus we still have $h^2(X_k, \mathcal{O}_{X_k}) = 0$. Since $\mathcal{O}_X$ is a direct summand of $\pi'_k^* \mathcal{O}_{X_k}$, we deduce that $h^2(X, \mathcal{O}_X) = 0$. Thus the proof is completed. \(\square\)

Theorem 3.4 has the following consequence.

**Corollary 3.5.** Let $X$ be a minimal and irregular 3-fold of general type. Then we have the following optimal inequality

\[K_X^3 \geq \frac{4}{3} h^0(X, K_X). \]

Moreover, the equality holds if and only if $K_X^3 = \frac{4}{3} \chi(\omega_X)$.

**Proof.** When the Albanese dimension of $X$ is not one, by [1, Corollary B], $K_X^3 \geq 4 h^0(X, K_X)$. Thus we only need to treat the case when $X$ is of Albanese dimension one.

Let $f : X \to B$ be the Albanese fibration of $X$, where $B$ is a smooth projective curve of genus $g(B) = q(X) \geq 1$. Denote by $F$ a general fiber of $f$. By [13], $f_* \omega_X$ is a generic vanishing sheaf on $B$. We deduce that

\[(3.2) \quad h^0(X, K_X) = \chi(f_* \omega_X). \]

Suppose first that $F$ is not a $(1,2)$-surface. By [20, Theorem 1.6],

\[K_{X/B}^3 \geq 2 \deg f_* \omega_{X/B}. \]
Note that 
\[ K^3_X = K^3_{X/B} + 6(g(B) - 1)K^2_F = K^3_{X/B} + 6\chi(\omega_B)K^2_F. \]

By the Riemann-Roch theorem,
\[ \chi(f_*\omega_X) = \deg f_*\omega_X - p_g(F)\chi(\omega_B) = \deg f_*\omega_{X/B} + p_g(F)\chi(\omega_B). \]

By the Noether inequality, it is easy to check that \( 3K^2_F \geq p_g(F) \). Combine the above results with (3.2). It follows that
\[ K^3_X \geq 2\chi(f_*\omega_X) = 2h^0_a(X, K_X). \]

Now suppose \( F \) is a \((1, 2)\)-surface. As is shown in the proof of Lemma 3.1, \( R^1f_*\omega_X = 0 \) and \( R^2f_*\omega_X = \omega_B \). By Theorem 3.4, we deduce that
\[ (3.3) \quad K^3_X \geq \frac{4}{3}\chi(\omega_X) = \frac{4}{3}\chi(f_*\omega_X) + \frac{4}{3}\chi(\omega_B) \geq \frac{4}{3}\chi(f_*\omega_X). \]

Thus by (3.2), we always have
\[ (3.4) \quad K^3_X \geq \frac{4}{3}h^0_a(X, K_X). \]

If the equality in (3.4) holds, then (3.3) becomes an equality. Thus \( K^3_X = \frac{4}{3}\chi(\omega_X) \). On the other hand, if \( K^3_X = \frac{4}{3}\chi(\omega_X) \), by Theorem 3.4, the general Albanese fiber \( F \) is a \((1, 2)\)-surface and \( q(X) = 1 \). Thus (3.3) becomes an equality, so does (3.4). The proof is completed. \( \Box \)

4. 3-folds on the Noether-Severi line: more properties

Throughout this section, we assume that \( X \) is a minimal and irregular 3-fold of general type with \( K^3_X = \frac{4}{3}\chi(\omega_X) \). Let
\[ a : X \to \text{Alb}(X) \]
be the Albanese morphism of \( X \). By Theorem 3.4, \( B := \text{Alb}(X) \) is a smooth curve of genus one, and \( a \) is the Albanese fibration of \( X \). We will also fix a sufficiently ample divisor \( T \) on \( B \) as in Remark 2.1, and denote \( t = \deg T \).

Consider the map \( \phi_{K_X + a^*T} \) induced by \( |K_X + a^*T| \). We will keep on using the same notation as in §2.1. Recall the following commutative diagram:

Here all the notation are the same as in §2.1, except that we replace \( f \) and \( f' \) therein by \( a \) and \( a' \). Let \( F \) be a general fiber of \( a \). Since \( p_g(F) = 2 \), by the choice of \( T \), we know that \( \phi_{K_X + a^*T} \) is the relative canonical map of \( X \) with respect to \( a \). Moreover, \( \Sigma \simeq \mathbb{P}_B(a_*\omega_X) \) is a smooth elliptic ruled surface contained in \( \mathbb{P}^{h^0(X, K_X + a^*T) - 1} \). We still write \( d = (\deg \tau) \cdot (\deg \Sigma) \).

All the above notation and facts will be used throughout this section.
4.1. Relative canonical image of \( X \). We have the following lemma.

**Lemma 4.1.** Keep the notation as above. Then the morphism \( \tau : \Sigma' \to \Sigma \) is an isomorphism, and

\[
d = \deg \Sigma = h^0(X, K_X + a^*T) = \chi(\omega_X) + 2t.
\]

**Proof.** Since now \( g(B) = 1 \), we have

\[
K_X^3 \geq \frac{4}{3} d - \frac{8}{3} t \geq \frac{4}{3} h^0(X, K_X + a^*T) - \frac{8}{3} t = \frac{4}{3} \chi(\omega_X).
\]

Here the first inequality is from Proposition 2.5 (2), the second is from Lemma 2.3, and the third equality is by Lemma 3.1. As \( K_X^3 = \frac{4}{3} \chi(\omega_X) \), the above inequalities must be equalities. Thus \( \deg \tau = 1 \) and \( d = \deg \Sigma = h^0(X, K_X + a^*T) \). The proof is completed.

\[\square\]

4.2. Cartier index of \( X \). The main result here is that \( X \) is Gorenstein.

**Theorem 4.2.** Let \( X \) be a minimal and irregular 3-fold of general type with \( K_X^3 = \frac{4}{3} \chi(\omega_X) \). Then \( X \) is Gorenstein. It follows that \( X \) is factorial.

**Proof.** Suppose \( X \) is Gorenstein. Then by [22, Lemma 5.1], \( X \) is factorial. Thus to prove the theorem, it suffices to prove that \( X \) is Gorenstein. Recall the following Riemann-Roch formula

\[
\chi(\omega_X^{[2]}) = h^0(X, 2K_X) = \frac{1}{2} K_X^3 + 3 \chi(\omega_X) + l_2(X)^2
\]

in [31, Corollary 10.3]. Here the correction term \( l_2(X) = 0 \) if and only if \( X \) is Gorenstein. By the Kawamata-Viehweg vanishing theorem, we have

\[
h^0(X, 2K_X + 2a^*T) = \chi(\mathcal{O}_X(2K_X + 2a^*T)) = \chi(\omega_X^{[2]}) + 2t \chi(\omega_F^{[2]}).
\]

Since \( F \) is a (1,2)-surface, \( \chi(\omega_F^{[2]}) = K_F^2 + \chi(\mathcal{O}_F) = 4 \). Thus we deduce that

\[
(4.1) \quad h^0(X, 2K_X + 2a^*T) = \frac{1}{2} K_X^3 + 3 \chi(\omega_X) + 8t + l_2(X).
\]

In the following, we prove in steps that \( l_2(X) > 0 \) would lead to a contradiction.

**Step 0.** From now on, suppose \( l_2(X) > 0 \). For any \( k \in \mathbb{Z}_{>0} \), let \( \mu_k : B \to B \) be the multiplication map by \( k \) on \( B \), and let \( X_k = X \times_{\mu_k} B \). We have

\[
h^0(X_k, 2K_{X_k}) = k^2 h^0(X, 2K_X), \quad K_{X_k}^3 = k^2 K_X^3, \quad \chi(\omega_{X_k}) = k^2 \chi(\omega_X).
\]

This implies that \( l_2(X_k) = k^2 l_2(X) \). Thus replacing \( X \) by \( X_k \) for a sufficiently large \( k \), we may assume that \( l_2(X) \geq 6 \).

Recall that \( |M| = \text{Mov}[\pi^*(K_X + a^*T)] \). Denote by \( C \) a general fiber of \( \psi = \phi_M \). Set \( |M_0| = \text{Mov}[2K_X + 2a^*T], \ |M_1| = \text{Mov}[2K_X + 2a^*T - M] \) and \( |M_2| = \text{Mov}[2K_X + 2a^*T - 2M] \). Replacing \( X \) by a further blow-up, we may assume that \( |M_0|, \ |M_1| \) and \( |M_2| \) are all base point free. By

\[\text{In the Riemann-Roch formula, } \omega_X^{[2]} \text{ means the reflexive sheaf } \mathcal{O}_X(2K_X).\]
Bertini’s theorem, we may take a smooth general member \( S \in |M| \). Then \( (a^*T)|_S \equiv tC \). Moreover, it is easy to see that

\[
h^0(X, 2K_X + 2a^*T) = h^0(X', 2K_{X'} + 2a^*T)
\]

(4.2)

\[
u_0 + u_1 + h^0(X', 2K_{X'} + 2a^*T - 2M),
\]

where

\[
u_i = \dim \text{Im} \left( H^0(X', M_i) \to H^0(S, M|_S) \right) \quad (i = 0, 1).
\]

Note that now we are in the equality case of Proposition 2.5 (2). By Proposition 2.6 (2), (3) and Lemma 4.1, we have

\[
\left( (\pi^*K_X)|_S \right)^2 = \left( \pi^* \left( K_X + \frac{1}{2}a^*T - \frac{1}{2}a^*T \right) \right)|_S^2
\]

\[
= \left( \left( \pi^* \left( K_X + \frac{1}{2}a^*T \right) \right)|_S^2 - t \left( (\pi^*K_X) \cdot C \right)
\]

(4.3)

\[
= \frac{2}{3} (2 \deg \Sigma - t) - t
\]

\[
= \frac{4}{3} \chi(\omega_X) + t.
\]

**Step 1.** In this step, we prove that

\[
u_0 \leq \frac{8}{3} \chi(\omega_X) + 6t + 2.
\]

Consider the complete linear system \(|M_0|_S\|. By our assumption, it is base point free, thus induces a morphism \( \phi_0 : S \to \mathbb{P}^{h^0(S, M_0|_S) - 1} \). If \( \dim \phi_0(S) = 2 \), by [29, Lemma 1.8],

\[
4 \left( (\pi^*K_X)|_S + tC \right)^2 \geq (M_0|_S)^2 \geq 2h^0(S, M_0|_S) - 4.
\]

If \( \dim \phi_0(S) = 1 \), since \( M_0|_S \geq M|_S \), the general fiber of \( \phi_0 \) is identical to \( C \). Now \( M_0|_S \equiv bC \), where \( b \geq h^0(S, M_0|_S) - 1 \). Thus by Lemma 2.4 (3),

\[
2 \left( (\pi^*K_X)|_S + tC \right)^2 \geq b \left( (\pi^*K_X) \cdot C \right) \geq h^0(S, M_0|_S) - 1.
\]

Thus in both cases, we always have

\[
2 \left( (\pi^*K_X)|_S + tC \right)^2 \geq h^0(S, M_0|_S) - 2.
\]

Together with (4.3), it follows that

\[
u_0 \leq h^0(S, M_0|_S) \leq 2 \left( (\pi^*K_X)|_S + tC \right)^2 + 2 = \frac{8}{3} \chi(\omega_X) + 6t + 2.
\]

**Step 2.** In this step, we prove that

\[
u_1 \leq \chi(\omega_X) + 2t + 2.
\]

Similarly as in Step 1, the complete linear system \(|M_1|_S\| induces a morphism \( \phi_1 : S \to \mathbb{P}^{h^0(S, M_1|_S) - 1} \). If \( \dim \phi_1(S) = 2 \), by [29, Lemma 1.8],

\[
(M_1|_S)^2 \geq 2h^0(S, M_1|_S) - 4.
\]

It follows that

\[
4 \left( (\pi^*K_X)|_S + tC \right)^2 \geq (M_1|_S + M|_S)^2 \geq 2h^0(S, M_1|_S) - 4 + 2(M_1|_S \cdot M|_S).
\]
Note that \(g(C) = 2\). Since \(\phi_1\) does not contract \(C\), the linear system \(|M_1|_S|_C\) induces a finite morphism on \(C\). Note that \(g(C) = 2\) by Lemma 2.2. We deduce that \((M_1|_S \cdot C) \geq 2\). Moreover, as in \(\S 2.2\) and by Lemma 4.1, we have \(M|_S \equiv dC = (\chi(\omega_X) + 2t)C\). Thus
\[
(M_1|_S \cdot M|_S) \geq 2\chi(\omega_X) + 4t.
\]
Combine the above inequalities with (4.3) together, we deduce that
\[
h^0(S, M_1|_S) \leq 2(\langle\pi^*K_X|_S + tC\rangle^2 + 2 - 2\chi(\omega_X) - 4t = \frac{2}{3} \chi(\omega_X) + 2t + 2.
\]
Since \(\chi(\omega_X) > 0\), we deduce that \(u_1 \leq \chi(\omega_X) + 2t + 2\).

If dim \(\phi_1(S) = 1\), since \(M_1|_S \geq M|_S\), the general fiber of \(\phi_1\) is just \(C\). Thus \(M_1|_S \equiv bC\), where \(b \geq h^0(S, M_1|_S) - 1\). Note that \(M|_S \equiv (\chi(\omega_X) + 2t)C\) as before. Therefore, the divisor \(2(\pi^*K_X)|_S - (b + \chi(\omega_X))C\) on \(S\) is pseudo-effective. Let \(\sigma : S \to S_0\) be the contraction morphism onto the minimal model of \(S\). By Proposition 2.6 (3), we deduce that \(K_{S_0} - (t + b + \chi(\omega_X))C_0\) is pseudo-effective, where \(C_0 = \sigma_*C\). In the meantime, by Proposition 2.6 (1) and Lemma 4.1, \(K_{S_0} = (2\chi(\omega_X) + 3t)C_0 + 2\Gamma_{S_0}\), where \(\Gamma_{S_0}\) is a section of the fibration \(S_0 \to B\) with \(g(\Gamma_{S_0}) = g(B) = 1\). Then
\[
3\Gamma_{S_0}^2 = ((K_{S_0} + \Gamma_{S_0}) \cdot \Gamma_{S_0}) - (2\chi(\omega_X) + 3t)(\Gamma_{S_0} \cdot C_0) = -(2\chi(\omega_X) + 3t).
\]
This implies that the divisor
\[
K_{S_0} - \frac{1}{3}(2\chi(\omega_X) + 3t)C_0 \equiv \frac{2}{3}(2\chi(\omega_X) + 3t)C_0 + 3\Gamma_{S_0}
\]
is nef. Therefore, it follows that
\[
(K_{S_0} - \frac{1}{3}(2\chi(\omega_X) + 3t)C_0) \cdot (K_{S_0} - (t + b + \chi(\omega_X))C_0) \geq 0,
\]
i.e.,
\[
K_{S_0}^2 \geq \left(\frac{5}{3}\chi(\omega_X) + 2t + b\right)(K_{S_0} \cdot C_0).
\]
Note that \((K_{S_0} \cdot C_0) = 2\) and \(K_{S_0}^2 = \frac{16}{3}\chi(\omega_X) + 8t\) by Proposition 2.6 (2) and Lemma 4.1. We deduce that \(b \leq \chi(\omega_X) + 2t\). It follows that
\[
u_1 \leq h^0(S, M_1|_S) \leq b + 1 \leq \chi(\omega_X) + 2t + 1.
\]

**Step 3.** In this step, we prove that
\[
h^0(X', 2K_{X'} + 2a'^*T - 2M) = 1.
\]
Suppose on the contrary that \(h^0(X', 2K_{X'} + 2a'^*T - 2M) \geq 2\). Then \(|M_2|\) is base point free. Since \(K_X\) is semi-ample and big, we have \((\pi^*K_X)^2 \cdot M_2 > 0\). Otherwise, any effective divisor linear equivalent to \(M_2\) would be contracted by the pluricanonical morphism of \(X\), which is a contradiction. Thus by (4.3), we deduce that
\[
K_X^3 = ((\pi^*K_X)^2 \cdot (K_{X'} + a'^*T)) - t > ((\pi^*K_X)|_S)^2 - t = \frac{4}{3}\chi(\omega_X).
\]
This is a contradiction. Thus \(h^0(X', 2K_{X'} + 2a'^*T - 2M) = 1\).
Step 4. Now we can finish the whole proof. By (4.1), we have
\[ h^0(X, 2K_X + 2a^*T) = \frac{1}{2}K_X^3 + 3\chi(\omega_X) + l_2(X) + 8t \geq \frac{11}{3} \chi(\omega_X) + 8t + 6. \]

On the other hand, by (4.2), (4.4), (4.5) and (4.6),
\[ h^0(X, 2K_X + 2a^*T) \leq \frac{11}{3} \chi(\omega_X) + 8t + 5. \]

This is a contradiction. Thus the whole proof is completed. □

4.3. Relative canonical linear system of \( X \). Recall in §2.1 that there
is a canonical section \( \Gamma \) of the fibration \( a : X \to B \) whose intersection \( \Gamma \cap F \) with \( F \) is just the unique base point of \( |K_F| \). Since \( \Sigma \) is a surface, the
restriction morphism
\[ H^0(X, K_X + a^*T) \to H^0(F, K_F) \]
is surjective, which implies that \( \Gamma \) is the only horizontal base locus of \( |K_X + a^*T| \) with respect to \( a \).

As in §2.1, we have
\[ \pi^*(K_X + a^*T) = M + Z, \quad K_X^V = \pi^*K_X + E_\pi. \]
Let \( S \in |M| \) be a general member. Then \( S \) is smooth, and \( a'|_S : S \to B \) is a fibration with a general fiber \( C \). As in (2.3), we may write
\[ E_\pi|_S = \Gamma_S + E_V, \quad Z|_S = \Gamma_S + Z_V. \]

Here \( \Gamma_S \) is a section of \( a'|_S \) with \( \pi(\Gamma_S) = \Gamma \), and \( E_V \) and \( Z_V \) are the vertical parts of \( E_\pi|_S \) and \( Z|_S \) with respect to \( a'|_S \), respectively. Denote by \( \sigma : S \to S_0 \) the contraction onto the minimal model of \( S \). Note that the fibration \( a'|_S \) descends to a fibration \( S_0 \to B \) with a general fiber \( C_0 = \sigma_C \).

Lemma 4.3. The divisor \( E_\pi|_S - Z|_S = E_V - Z_V \) is effective.

Proof. By (2.1) and (2.2), we have
\[ E_\pi|_S - Z|_S = (K_X^V + S)|_S - (\pi^*(2K_X + a^*T))|_S = K_S - (\pi^*(2K_X + a^*T))|_S. \]
Since \( K_S \geq \sigma^*K_{S_0} \), by Proposition 2.6 (3), \( E_V - Z_V = E_\pi|_S - Z|_S \) is \( \mathbb{Q} \)-linearly equivalent to an effective \( \mathbb{Q} \)-divisor. In particular, there exist an integer \( n > 0 \) and an effective divisor \( D \) on \( S \) such that
\[ nE_V \sim D + nZ_V. \]
On the other hand, Proposition 2.6 (1) also tells us that \( \sigma_*E_V = 0 \). It implies that \( h^0(S, nE_V) = h^0(S_0, \mathcal{O}_{S_0}) = 1 \). Thus
\[ nE_V = D + nZ_V. \]
It follows that \( E_V - Z_V = \frac{1}{n}D \) is effective. □

Lemma 4.4. The linear system \( |K_X + a^*T| \) has no fixed part.
Proof. Suppose $|K_X + a^*T|$ has nonzero fixed part $Z_X$. Then $Z_X$ is vertical respect to $a$. Let $S_X = \pi(S)$. By Theorem 4.2, both $S_X$ and $Z_X$ are Cartier divisors on $X$. By the Kawamata-Viehweg vanishing theorem, we have $H^1(X, \mathcal{O}_X(-S_X - Z_X)) = 0$. As a result, the natural restriction map

$$H^0(X, \mathcal{O}_X) \to H^0(S_X + Z_X, \mathcal{O}_{S_X + Z_X})$$

is surjective. Thus $h^0(S_X + Z_X, \mathcal{O}_{S_X + Z_X}) = 1$, which implies that $S_X + Z_X$ is connected. In particular, we have $S_X \cap Z_X \neq \emptyset$. Let $A$ be an integral curve supported on $Z_X|\mathcal{O}_X$. By Lemma 4.3, $E_\pi|\mathcal{O}_X - (\pi^*Z_X)|_{\mathcal{O}_X} = (E_\pi|\mathcal{O}_X - Z|_{\mathcal{O}_X})$ is effective. In particular, $A \subseteq \text{Supp}(Z_X|\mathcal{O}_X) \subseteq \pi(E_\pi)$. We deduce that $A \subseteq \text{Bs}|\mathcal{O}_X|$.

By the construction of $\pi$ in §1.6 and by induction, there is a $\beta$-exceptional prime divisor $E_A$ such that $E_A|\mathcal{O}_X \neq 0$, $\pi(E_A) = A$ and $\text{coeff}_{E_A}(E_\pi - Z) = \text{coeff}_{E_A}(E_\pi - \pi^*(S_X + Z_X)) < 0$. We deduce that $E_\pi|\mathcal{O}_X - Z|_{\mathcal{O}_X}$ is not effective, which contradicts Lemma 4.3. As a result, $|K_X + a^*T|$ has no fixed part. □

Lemma 4.5. A general member $S_X \in |K_X + a^*T|$ is normal with at worst canonical singularities. Moreover, $\pi|\mathcal{O}_X : S \to S_X$ factors through $\sigma : S \to S_0$.

Proof. By Bertini’s theorem, $S_X$ is integral. Moreover, since $X$ is Gorenstein, $S_X$ is Cohen-Macaulay. To prove that $S_X$ is normal, we only need to show that $S_X$ is smooth in codimension one.

Suppose the singular locus of $S_X$ contains an integral curve $A$. Then $S_X$ has multiplicity at least two at a general point of $A$. Since $S$ is smooth, similarly to the proof of Lemma 4.4, there is a $\beta$-exceptional divisor $E_A'$ on $X'$ such that $\pi(E_A') = A$, $E_A'|\mathcal{O}_X \neq 0$, and $\text{coeff}_{E_A'}(E_\pi' - Z) < 0$. Thus $E_\pi'|\mathcal{O}_X - Z|_{\mathcal{O}_X}$ is not effective. However, it is impossible by Lemma 4.3. Therefore, $S_X$ is normal. By the adjunction formula, $K_{S_X} = (2K_X + a^*T)|_{S_X}$. Then

$$K_S - (\pi|\mathcal{O}_X)^*K_{S_X} = (K_X' + S)|_{S} - (\pi^*(2K_X + a^*T))|_{S} = E_\pi|\mathcal{O}_X - Z|_{S},$$

which is effective by Lemma 4.3. Thus $S_X$ has at worst canonical singularities. By the uniqueness of the minimal model, the minimal resolution of $S_X$ is just $S_0$. The proof is completed. □

Theorem 4.6. The following statements hold:

1. $\text{Bs}|K_X + a^*T| = \Gamma$, and $\Gamma$ lies in the smooth locus of $X$.
2. A general member $S_X \in |K_X + a^*T|$ is smooth.
3. Let $\pi_\Gamma : Y \to X$ be the blow-up of $X$ along $\Gamma$. Then the linear system $|\pi_\Gamma^*(K_X + a^*T) - E_\Gamma|$ is base point free, where $E_\Gamma$ is the $\pi_\Gamma$-exceptional divisor.

Proof. By Lemma 4.4, $|K_X + a^*T|$ has no fixed part. Thus $\text{Bs}|K_X + a^*T| = \pi(Z \cap S)$. Thus to prove $\text{Bs}|K_X + a^*T| = \Gamma$, we only need to prove that $\Gamma = \pi(Z|S)$. Since $Z|S = \Gamma|S + Z_V$ and $\pi(\Gamma|S) = \Gamma$, it suffices to show that $\pi(Z_V) \subseteq \Gamma$.

By Proposition 2.6 (1), $\sigma(Z_V)$ consists of finitely many points on $S_0$. Thus by Lemma 4.5, $\pi(Z_V)$ also consists of finitely many points on $\pi(S)$. Suppose
there is a point \( p \in \sigma(Z_V) \) but \( p \notin \Gamma \). We may write

\[ Z_V = Z_1 + Z_2, \]

where \( Z_1 \) and \( Z_2 \) are effective, \( \sigma(Z_1) = p \), and \( p \notin \sigma(Z_2) \). By the argument in §2.2, now we have

\[ (\pi^*(K_X + a^*T))|_S = M|_S + Z|_S \equiv dC + \Gamma_S + Z_1 + Z_2. \]

However, since \( Z_1 \) does not intersect \( Z_2 \) or \( \Gamma_S \), we deduce that

\[ (Z_1 \cdot (dC + \Gamma_S + Z_2)) = 0. \]

This is a contradiction, because \( (\pi^*(K_X + a^*T))|_S \) is nef and big, thus \( 1 \)-connected. As a result, \( \sigma(Z_V) \subseteq \Gamma \) and \( Bs(K_X + a^*T) = \Gamma \).

By Theorem 4.2, every Weil divisor on \( X \) is Cartier. Take any fiber \( F_1 \) of \( a \). Since \( (\Gamma : F_1) = 1 \), there exists exactly one irreducible component \( F_{1,0} \) of \( F_1 \) such that \( F_{1,0} \cap \Gamma \neq \emptyset \) is a point. Moreover, \( (\Gamma \cdot F_{1,0}) = 1 \) and coeff\( F_{1,0}(F_1) = 1 \). It implies that \( F_1 \) is smooth at this point, so is \( X \). Thus \( (1) \) is proved.

To prove \( (2) \), suppose \( S_X \) is singular. By Bertini’s theorem and \( (1) \), the singular locus of \( S_X \) is contained in \( \Gamma \). Let \( p \in S_X \) be a singularity. By Lemma 4.5, we have the minimal resolution \( \sigma_0 : S_0 \rightarrow S_X \) such that \( K_{S_0} = \sigma_0^* K_{S_X} \). Let \( E_p \) be the exceptional divisor on \( S_0 \) lying over \( p \). Since \( p \) is a canonical singularity, every irreducible component of \( E_p \) is a \((-2)\)-curve. In particular, \((K_{S_0} \cdot E_p) = 0 \). Let \( \Gamma_{S_0} \subset S_0 \) be the strict transform of \( \Gamma \subset S_X \) under \( \sigma_0 \). Then we have \((\Gamma_{S_0} \cdot E_p) \geq 0 \). On the other hand, by Proposition 2.6 \((1) \), \( K_{S_0} \equiv 2\Gamma_{S_0} + (2d - t)C_0 \). Thus

\[ (K_{S_0} \cdot E_p) = 2(\Gamma_{S_0} \cdot E_p) + (2d - t)(C_0 \cdot E_p) \geq 2(\Gamma_{S_0} \cdot E_p). \]

This is a contradiction. As a result, \( S_X = S_0 \) is smooth, and \( (2) \) is proved.

To prove \( (3) \), first note that \( \Gamma \) is smooth. By \( (1) \) and \( (2) \), we have

\[ K_Y = \pi_Y^* K_X + E_{\Gamma}, \quad \pi_Y^*(K_X + a^*T) = M_{\Gamma} + E_{\Gamma}. \]

Let \( S_Y \in |M_{\Gamma}| \) be a general member. To prove that \( |M_{\Gamma}| \) is base point free, it suffices to show that the restricted linear system \( |M_{\Gamma}||S_Y \) is base point free.

Suppose \( Bs(|M_{\Gamma}||S_Y|) \neq \emptyset \). Since \( S_X \) is smooth and \( S_Y \) is the blow-up of \( S_X \) along the curve \( \Gamma \), the morphism \( \pi_Y|_{S_Y} : S_Y \rightarrow S_X \) is actually an isomorphism. Moreover, \( (\pi_Y|_{S_Y})^* \Gamma = E_{\Gamma}|_{S_Y} \). We also have a fibration \( b|_{S_Y} : S_Y \rightarrow B \), where \( b = a \circ \pi_Y \). Now \( S_X = S_0 \). By Proposition 2.6 \((1) \) and \( (3) \), we have \((K_X + a^*T)|_{S_X} \sim_{\mathbb{Q}} \frac{1}{2}(K_{S_X} + (a^*T)|_{S_X}) \equiv \Gamma + dC \). Note that \( \Gamma \) lies in the base locus of \( |K_X + a^*T||S_X| \). Therefore, we may write \((K_X + a^*T)|_{S_X} = \Gamma + (a|_{S_X})^{*} D \), where \( |D| \) is base point free on \( B \). Thus we deduce that

\[ M_{\Gamma}|_{S_Y} = (\pi_Y^*(K_X + a^*T))|_{S_Y} - E_{\Gamma}|_{S_Y} = (b|_{S_Y})^{*} D. \]

This implies that \( Bs(|M_{\Gamma}||S_Y|) \) must be vertical curves with respect to \( b|_{S_Y} \).

On the other hand, note that by \( (1) \), \( Bs|M_{\Gamma}| \subseteq E_{\Gamma} \). This implies that
Bs(|MΓ||Sγ) ⊆ EΓ|Sγ, which is horizontal with respect to b|Sγ. This is a contradiction. Thus |MΓ||Sγ| base point free, and (3) is proved. □

4.4. Fundamental group. In this subsection, we compute the topological fundamental group of smooth models of X following the idea of Xiao in [33].

Theorem 4.7. Let X1 be a smooth model of X. Then the topological fundamental group

\[ \pi_1(X_1) \simeq \pi_1(B) \simeq \mathbb{Z}^2. \]

Proof. By the assumption, \( B = \text{Alb}(X_1) \). Let

\[ a_1 : X_1 \to B \]

be the Albanese fibration of \( X_1 \). Then a general fiber \( F_1 \) of \( a_1 \) is a smooth (1,2)-surface. By Theorem 4.2, \( X \) is factorial. Note that the Albanese fibration \( a \) has a section \( \Gamma \). We deduce that \( a \) does not have multiple fibers. Neither does \( a_1 \).

Let \( V \) be the image of \( \pi_1(F_1) \) into \( \pi_1(X_1) \). By a similar argument as in the proof of [33, Lemma 1], we deduce that \( V \) is a normal subgroup of \( \pi_1(X_1) \).

Denote \( H = \pi_1(X_1)/V \). Then we have the following exact sequence

\[ 1 \to V \to \pi_1(X_1) \to H \to 1 \]

as in [33, §1]. By [16, Theorem 4.8], \( F_1 \) is simply connected. Thus \( V = 1 \).

Let \( b_1, \ldots, b_n \) be all points on \( B \) lying under singular fibers of \( a_1 \), and let \( B_0 = B \setminus \{b_1, \ldots, b_n\} \). Using exactly the same proof as [33, Lemma 2] for the fibration \( a_1 \) together with the fact that \( a_1 \) has no multiple fibers, we deduce that \( H \) is the quotient of \( \pi_1(B_0) \) by the normal subgroup generated by the homotopy classes of \( \gamma_1, \ldots, \gamma_n \) in \( \pi_1(B_0) \), where each \( \gamma_i \) is a small loop in \( B \) around \( b_i \). Note that this quotient is exactly \( \pi_1(B) \). Thus \( \pi_1(X_1) \simeq \pi_1(B) \simeq \mathbb{Z}^2 \), and the proof is completed. □

5. 3-folds on the Noether-Severi line: fine classification

In the last section, we give an explicit description of the canonical model of irregular 3-folds on the Noether-Severi line.

Let \( X \) be a minimal and irregular 3-fold of general type with \( K_X^3 = 4 \chi(\omega_X) \). Let \( a : X \to B \) be the Albanese fibration of \( X \). By Theorem 3.4, \( B \) is a smooth curve of genus \( g(B) = q(X) = 1 \). Let \( T \) be a sufficiently ample divisor on \( B \) as in Remark 2.1. Denote \( t = \text{deg}T \).

Let

\[ \epsilon : X \to X_0 := X_{\text{can}} \]

be the contraction from \( X \) onto its canonical model \( X_{\text{can}} \). Let \( \Gamma_0 = \epsilon(\Gamma) \), where \( \Gamma \) is the section of \( a \) as is described in §2.1. Thus \( \Gamma_0 \) is a section of the Albanese fibration \( a_0 : X_0 \to B \) of \( X_0 \). In particular, \( \Gamma_0 \) is smooth.

Proposition 5.1. We have

1. \( \text{Bs} |K_{X_0} + a_0^*T| = \Gamma_0 \), and \( \Gamma_0 \) lies in the smooth locus of \( X_0 \).
(2) Let \( \pi_0 : X_1 \to X_0 \) be the blow-up along \( \Gamma_0 \) with \( E_0 \) the exceptional divisor. Then \( E_0 \) is a \( \mathbb{P}^1 \)-bundle over \( \Gamma_0 \), and \( |M_0| := |\pi_0^*(K_{X_0} + a_0^*T) - E_0| \) is base point free.

Proof. For (1), note that \( K_X + a^*T = e^*(K_{X_0} + a_0^*T) \). By Theorem 4.6 (1),

\[
\epsilon^{-1}(\text{Bs}[K_{X_0} + a_0^*T]) = \text{Bs}[K_X + a^*T] = \Gamma.
\]

Thus \( \text{Bs}[K_{X_0} + a_0^*T] = \Gamma_0 \). Since \( \epsilon|_\Gamma : \Gamma \to \Gamma_0 \) is an isomorphism and \( \epsilon^{-1}(\Gamma_0) = \Gamma \), by Zariski’s main theorem, \( \epsilon \) is an isomorphism over a neighbourhood of \( \Gamma_0 \). By Theorem 4.6 (1) again, we deduce that \( \Gamma_0 \) lies in the smooth locus of \( X_0 \). For (2), note that \( \Gamma_0 \) is smooth. Thus \( E_0 \) is a \( \mathbb{P}^1 \)-bundle over \( \Gamma_0 \). The rest part of (2) just follows from Theorem 4.6 (3). \( \square \)

**Definition 5.2.** We call \( \Gamma_0 \) the base-locus section of \( a_0 \).

By Proposition 5.1, we have the following commutative diagram

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\psi} & \Sigma \\
\downarrow{\pi_0} & & \downarrow{p} \\
X_0 & \underset{\alpha_0}{\xrightarrow{\phi}} & B
\end{array}
\]

where \( \phi := \phi_{K_{X_0} + a_0^*T} \) is the relative canonical map of \( X_0 \) with respect to \( a_0 \),

\( \Sigma = \mathbb{P}_B((a_0)\omega_{X_0}) \) is a \( \mathbb{P}^1 \)-bundle over \( B \) with the natural projection \( p \), and \( \psi \) is the morphism induced by \( |M_0| \) which, by Lemma 4.1, has connected fibers.

Denote by \( C \) a general fiber of \( \psi \). By Lemma 2.4 (3), we have \( ((\pi_0^*K_{X_0}) \cdot C) = 1 \). By Lemma 2.2, \( (K_{X_1} \cdot C) = 2g(C) - 2 = 2 \). Thus we deduce that

\[
(E_0 \cdot C) = ((K_{X_1} - \pi_0^*K_{X_0}) \cdot C) = 1.
\]

In particular, \( \psi|_{E_0} : E_0 \to \Sigma \) is birational. By Proposition 5.1 (2), \( E_0 \) itself is a \( \mathbb{P}^1 \)-bundle. We deduce that \( \psi|_{E_0} \) is an isomorphism.

**Lemma 5.3.** The morphism \( \psi \) is flat, and every fiber of \( \psi \) is integral.

Proof. Let \( W \) be any fiber of \( \psi \). Since \( \psi|_{E_0} \) is an isomorphism, \( E_0 \) is a section of \( \psi \). Thus \( W \cap E_0 \) is a point. Suppose \( W \) contains an irreducible component \( W_0 \) of dimension two. Then \( W_0 \cap E_0 = \emptyset \). Thus

\[
((K_{X_0} + a_0^*T)^2 \cdot ((\pi_0)_*W_0)) = ((M_0 + E_0)^2 \cdot W_0) = (M_0^2 \cdot W_0) = 0.
\]

Since \( K_{X_0} \) is ample, the above equality implies that \( \dim((\pi_0)_*W_0) \leq 1 \). This is impossible, because the only \( \pi_0 \)-exceptional divisor is \( E_0 \). As a result, \( \dim W = 1 \). By [26, Theorem 23.1], \( \psi \) is flat.

Note that \( K_{X_0} \) is Cartier and \((\pi_0^*K_{X_0} \cdot W) = 1 \). If \( W \) is reducible, then there is an irreducible component \( W_1 \) of \( W \) such that \((\pi_0^*K_{X_0} \cdot W_1) = 0 \), i.e., \((K_{X_0} \cdot ((\pi_0)_*W_1)) = 0 \). Thus \((\pi_0)_*W_1 \) is a point, so \( W_1 \subset E_0 \). This is a contradiction. Therefore, \( W \) is irreducible. Since \( K_{X_0} \) is Cartier, \((\pi_0^*K_{X_0} \cdot W) = 1 \) and \( W \) is irreducible, we deduce that \( W \) is reduced. \( \square \)

**Lemma 5.4.** Let \( \mathcal{E} = \psi_*\mathcal{O}_{X_1}(2E_0) \). Then \( \mathcal{E} \) is locally free of rank two.
\textbf{Proof.} Take any fiber $C$ of $\psi$. Since a general fiber of $\psi$ is of genus $2$, by Lemma 5.3, it follows that $C$ is an integral curve of arithmetic genus $2$. We deduce that $h^4(C, O_C) = 2$. By Theorem 4.2 and Proposition 5.1, $X_1$ is Cohen-Macaulay and the dualizing sheaf $\omega_{X_1}$ is invertible. Since $C$ is a fiber of $\psi$ and $\Sigma$ is smooth, $C$ is Cohen-Macaulay and the dualizing sheaf $\omega_C = \omega_{X_1}|_C$ is invertible. We deduce that

$$h^0(C, K_C) = h^0(C, \omega_C) = h^1(C, O_C) = 2,$$

where the last equality follows from the Serre duality. On the other hand, note that $K_{X_1} = \pi_0^*K_X + E_0 = M_0 - \pi_0^*(a_0T) + 2E_0$. Thus $h^0(C, 2E_0|_C) = h^0(C, K_{X_1}|_C) = h^0(C, K_C) = 2$. By Grauert’s theorem [14, III, Corollary 12.9], the result follows. \hfill \square

Let $q : Y := \mathbb{P}_\Sigma(E) \to \Sigma$ denote the $\mathbb{P}^1$-bundle over $\Sigma$. By Lemma 5.3, every fiber $C$ of $\psi$ is integral. By [8, Theorem 3.3], we deduce that $|K_C| = |2E_0|_C$ is base point free. Thus we obtain a morphism

$$f : X_1 \to Y,$$

which is just the relative canonical map of $X_1$ with respect to $\psi$. Moreover, $f$ is a finite morphism of degree two. By [26, Theorem 23.1], $f$ is flat. Let $E_Y = f(E_0)$. It is easy to see that $E_Y$ is a section of $q$. Let $j : \Sigma \to Y$ denote this section. Thus we have the following commutative diagram

$$\begin{array}{c}
X_1 \\
\downarrow f \\
\Sigma \\
\downarrow q \\
Y \\
\downarrow j \\
B
\end{array}$$

For any fiber $C$ of $\psi$, $f|_C : C \to f(C) \cong \mathbb{P}^1$ is just the canonical map of $C$. Note that $2E_0|_C \in |K_C|$. Thus we have $(f^*E_Y)|_C = 2E_0|_C$. We conclude that $f^*E_Y = 2E_0$. Since $f^*E_Y = 2E_0$, we have $\mathcal{E} = q_*\mathcal{O}_Y(E_Y)$. Consider the short exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(E_Y) \to \mathcal{O}_{E_Y}(E_Y) \to 0.$$ 

Note that $R^1q_*\mathcal{O}_Y = 0$ and $q_*\mathcal{O}_{E_Y}(E_Y) = \mathcal{O}_{\Sigma}(j^*E_Y)$. Pushing forward by $q$ on the above short exact sequence, we obtain

$$0 \to \mathcal{O}_{\Sigma} \to \mathcal{E} \to \mathcal{O}_{\Sigma}(j^*E_Y) \to 0. \quad (5.1)$$

In the following, we will compute $j^*E_Y$. Now $a_1 := p \circ \psi : X_1 \to B$ is the Albanese fibration of $X_1$. Then

$$p_0 := a_1|_{E_0} : E_0 \to B$$

induces the $\mathbb{P}^1$-bundle structure on $E_0$. Denote $(a_0)_{0} \omega_{X_0}$ by $\mathcal{E}_B$. By the isomorphism $\psi|_{E_0}$, we may identify $E_0$ with $\mathbb{P}_B(\mathcal{E}_B)$. Denote by $F_1$ a general fiber of $a_1$. Then $F_1|_{E_0}$ is just a fiber of $p_0$. Since $M_0 \cdot F_1 \equiv C$, we have $(M_0|_{E_0}, F_1|_{E_0}) = (M_0 \cdot F_1 \cdot E_0) = (E_0 \cdot C) = 1$. Thus we may write

$$M_0|_{E_0} = s + p_0^*A_1. \quad (5.2)$$
where \( s \) is a relative hyperplane section of \( p_0 \), i.e., \( \mathcal{O}_{E_0}(s) = \mathcal{O}_{E_0}(1) \). Note that \( E_0 \) is horizontal with respect to \( \psi \) but \( M_0 \) is vertical. We deduce that \( h^0(X_1, M_0 - E_0) = 0 \). Therefore, the map \( \psi|_{E_0} : E_0 \to \Sigma \) is just induced by \( |M_0|_{E_0} \). By Lemma 4.1, we have

\[
(5.3) \quad \deg \mathcal{E}_B + 2 \deg A_1 = (s + p_0^*A_1)^2 = \deg \Sigma = \chi(\omega_X) + 2t.
\]

Recall that now the inequality in Proposition 2.5 (1) becomes an equality. Thus we deduce from Lemma 4.1 that \( (K_{X_0} \cdot \Gamma_0) = (K_X \cdot \Gamma) = \frac{1}{3} \chi(\omega_X) \). Let \( \gamma_0 : B \to X_0 \) represent the base-locus section \( \Gamma_0 \). Then we have

\[
(5.4) \quad (\pi_0^*K_{X_0})|_{E_0} = p_0^*A_0,
\]

where \( A_0 = \gamma_0^*K_{X_0} \) with \( \deg A_0 = \frac{1}{3} \chi(\omega_X) \). Set \( A_2 = A_0 + T - A_1 \). Then we have

\[
(5.5) \quad \deg A_2 = \frac{1}{3} \chi(\omega_X) + t - \deg A_1
\]

and

\[
(5.6) \quad E_0|_{E_0} = (\pi_0^*K_{X_0} + a_1^*T - M_0)|_{E_0} = -s + p_0^*A_2.
\]

Remark 5.5. In fact, from (5.3) and (5.5), we deduce that

\[
\deg A_1 + 3 \deg A_2 - \deg \mathcal{E}_B - t = 0.
\]

From now on, we identify \( E_0 \) and \( p_0 \) with \( \Sigma \) and \( p \) under the isomorphism \( \psi|_{E_0} \). Under this identification and by (5.6), we have

\[
(5.7) \quad f^*E_Y = (\psi|_{E_0})_*((f|_{E_0})^*E_Y) = (\psi|_{E_0})_* (2E_0|_{E_0}) = -2s + 2p^*A_2.
\]

Thus it follows from (5.1) that

\[
K_Y = -2E_Y + q^*(K_{\Sigma} - 2s + 2p^*A_2)
\]

\[
= -2E_Y + q^*(-4s + p^* \det \mathcal{E}_B + 2p^*A_2).
\]

Let \( D \) be the branch locus of \( f : X_1 \to Y \). Since \( X_1 \) has at worst canonical singularities, \( D \) must be reduced. Since \( E_Y \) is contained in \( D \), we may write

\[
D = E_Y + D',
\]

where \( D' \) does not contain \( E_Y \) as an irreducible component. Furthermore, there exists a divisor \( L \) on \( Y \) such that \( D \sim 2L \) and \( K_{X_1} = f^*(K_Y + L) \).

Lemma 5.6. We have

1. \( L \sim 3E_Y + q^*(5s + p^*(A_1 - 2A_2 - \det \mathcal{E}_B - T)) \);
2. \( D' \sim 5E_Y + 2q^*(5s + p^*(A_1 - 2A_2 - \det \mathcal{E}_B - T)) \).

Proof. Recall that by (5.2), we have

\[
K_{X_1} = M_0 - a_1^*T + 2E_0 = f^*(E_Y + q^*(s + p^*(A_1 - T))).
\]

Let \( L' = 3E_Y + q^*(5s + p^*(A_1 - 2A_2 - \det \mathcal{E}_B - T)) \). By (5.8), we have

\[
f^*L' = K_{X_1} - f^*K_Y = f^*L,
\]
i.e., \( f^* \mathcal{O}_Y(L - L') = \mathcal{O}_{X_1} \). By the projection formula, we deduce that
\[
\mathcal{O}_Y \oplus \mathcal{O}_Y(-L) = f_* \mathcal{O}_{X_1} = \mathcal{O}_Y(L - L') \oplus \mathcal{O}_Y(-L').
\]
Thus \( L \sim L' \). Thus (1) is proved. Since \( D' \sim 2L - E_Y \), (2) follows immediately from (1). \( \square \)

**Lemma 5.7.** Under the isomorphism \( q|_{E_Y} : E_Y \rightarrow \Sigma \), we have
\[
\mathcal{O}_{E_Y}(D') = \mathcal{O}_{E_Y}(2p^*(A_1 + 3A_2 - \det E_B - T)) = \mathcal{O}_{E_Y}.
\]

**Proof.** By (5.7) and Lemma 5.6 (2), we have
\[
D'|_{E_Y} \sim 5E_Y|_{E_Y} + 2(5s + p^*(A_1 - 2A_2 - \det E_B - T)) \\
\sim 2p^*(A_1 + 3A_2 - \det E_B - T).
\]

By Remark 5.5, \( \deg A_1 + 3 \deg A_2 - \deg E_B - t = 0 \). Since \( h^0(E_Y, D'|_{E_Y}) > 0 \), we deduce that \( 2A_1 + 6A_2 - 2 \det E_B - 2T \sim 0 \). Thus the result follows. \( \square \)

**Lemma 5.8.** The short exact sequence (5.1) splits. In particular,
\[
\mathcal{E} = \mathcal{O}_\Sigma \oplus \mathcal{O}_\Sigma(-2s + 2p^*A_2).
\]

**Proof.** Let \( h : (\text{Sym}^4 \mathcal{E}) \otimes \mathcal{E} \rightarrow \text{Sym}^5 \mathcal{E} \) be the symmetrizing morphism. Then we have the following commutative diagram:

\[
\begin{array}{c}
0 \longrightarrow \text{Sym}^4 \mathcal{E} \longrightarrow (\text{Sym}^4 \mathcal{E}) \otimes \mathcal{E} \longrightarrow (\text{Sym}^4 \mathcal{E}) \otimes \mathcal{O}_\Sigma(j^* E_Y) \longrightarrow 0 \\
\end{array}
\]

\[
\begin{array}{c}
0 \longrightarrow \text{Sym}^4 \mathcal{E} \longrightarrow \text{Sym}^5 \mathcal{E} \longrightarrow \mathcal{O}_\Sigma(5j^* E_Y) \longrightarrow 0
\end{array}
\]

Here \((*)\) is obtained by tensoring (5.1) with \( \text{Sym}^4 \mathcal{E} \), and \((**)\) is obtained by pushing forward by \( q \) on
\[
0 \rightarrow \mathcal{O}_Y(4E_Y) \rightarrow \mathcal{O}_Y(5E_Y) \rightarrow \mathcal{O}_{E_Y}(5E_Y) \rightarrow 0
\]
as well as \( R^1q_* \mathcal{O}_Y(4E_Y) = 0 \). The splitting of (5.1) is equivalent to the splitting of \((*)\). Thus to prove the lemma, it suffices to prove that \((**)\) splits, because by composing \( h \), a splitting morphism \( \text{Sym}^5 \mathcal{E} \rightarrow \text{Sym}^4 \mathcal{E} \) gives a splitting morphism \( (\text{Sym}^4 \mathcal{E}) \otimes \mathcal{E} \rightarrow \text{Sym}^4 \mathcal{E} \).

Suppose \((***)\) does not split. Then the extension class
\[
[\text{Sym}^5 \mathcal{E}] \in \text{Ext}^1(\mathcal{O}_\Sigma(5j^* E_Y), \text{Sym}^4 \mathcal{E}) = \text{Ext}^1(\mathcal{O}_\Sigma, \text{Sym}^4 \mathcal{E} \otimes \mathcal{O}_\Sigma(-5j^* E_Y))
\]
is nonzero. By [14, III. Ex. 6.1], the map
\[
H^0(\Sigma, \mathcal{O}_\Sigma) \rightarrow H^1(\Sigma, \text{Sym}^4 \mathcal{E} \otimes \mathcal{O}_\Sigma(-5j^* E_Y))
\]
is nonzero, thus injective. From \((***)\), we deduce that
\[
h^0(\Sigma, \text{Sym}^4 \mathcal{E} \otimes \mathcal{O}_\Sigma(-5j^* E_Y)) = h^0(\Sigma, \text{Sym}^5 \mathcal{E} \otimes \mathcal{O}_\Sigma(-5j^* E_Y)).
\]
Together with the projection formula and (5.7), we have
\[
h^0(Y, 4E_Y + 10q^*(s - p^*A_2)) = h^0(Y, 5E_Y + 10q^*(s - p^*A_2)).
\]
In particular, \( E_Y \subseteq Bs|5E_Y + 10q^*(s - p^*A_2)| \). On the other hand, by Lemma 5.6 (2) and Lemma 5.7,
\[
5E_Y + 10q^*(s - p^*A_2) \sim 5E_Y + 2q^*(5s + p^*(A_1 - 2A_2 - \det E_B - T)) \sim D'.
\]
Thus \( E_Y \) is an irreducible component of \( D' \). This is a contradiction. As a result, \((**)\) splits, so does \((5.1)\).

**Lemma 5.9.** We have \( A_1 = T \). Thus \( A_2 = A_0 \).

**Proof.** Recall that \( K_{X_1} = f^*(E_Y + q^*(s + p^*(A_1 - T))) \). By the projection formula, we have
\[
\psi_*\omega_{X_1} = \mathcal{O}_\Sigma(s + p^*(A_1 - T)) \otimes q_* (\mathcal{O}_Y(E_Y) \otimes \mathcal{O}_Y(E_Y - L)).
\]
By Lemma 5.6 (1), it is clear that \( q_*\mathcal{O}_Y(E_Y - L) = 0 \). Thus by Lemma 5.8, we have
\[
\psi_*\omega_{X_1} = \mathcal{O}_\Sigma(s + p^*(A_1 - T)) \otimes q_*\mathcal{O}_Y(E_Y) = \mathcal{O}_\Sigma(s + p^*(A_1 - T)) \otimes \mathcal{O}_\Sigma(-s + p^*(A_1 + 2A_2 - T)).
\]
Therefore, it follows from the projection formula again that
\[
(a_1)_*\omega_{X_1} = p_*(\psi_*\omega_{X_1}) = p_*\mathcal{O}_\Sigma(s + p^*(A_1 - T)) = (a_0)_*\omega_{X_0} \otimes \mathcal{O}_B(A_1 - T).
\]
As a result, we deduce that \( A_1 = T \), and thus \( A_2 = A_0 + T - A_1 = A_0 \). □

Now we are ready to state the main theorem of this section.

**Theorem 5.10.** Let \( X_0 \) be a canonical and irregular 3-fold of general type with \( K_{X_0}^3 = \frac{1}{2} \chi(\omega_{X_0}) \). Let \( a_0 : X_0 \to B \) be the Albanese fibration of \( X_0 \), where \( B \) is a smooth curve of genus one. Let \( X_1 \) be the blow-up of \( X_0 \) along the base-locus section \( \Gamma_0 \) of \( a_0 \). Then the Albanese fibration \( a_1 : X_1 \to B \) of \( X_1 \) is factorized as
\[
a_1 : X_1 \xrightarrow{f} Y \xrightarrow{q} \Sigma \xrightarrow{p} B
\]
with the following properties:

1. \( \Sigma = \mathbb{P}(\mathcal{E}_B) \) with \( p : \Sigma \to B \) the projection, where \( \mathcal{E}_B = (a_0)_*\omega_{X_0} \).
2. \( Y = \mathbb{P}(\mathcal{O}_\Sigma \oplus (\mathcal{O}_\Sigma(-2) \otimes K_1^2)) \) with \( q : Y \to \Sigma \) the projection, where \( \gamma_0 : B \to X_0 \) corresponds to the section \( \Gamma_0 \) and \( K_1 = p^*(\gamma_0^*\omega_{X_0}) \).
3. \( f : X_1 \to Y \) is a flat double cover with the branch locus
\[
D = D_1 + D_2,
\]
where \( D_1 \in |\mathcal{O}_Y(1)| \), \( D_2 \in |\mathcal{O}_Y(5) \otimes q^*(\mathcal{O}_\Sigma(10) \otimes K_1^{-4} \otimes K_2^{-2})| \), and \( D_1 \cap D_2 = \emptyset \). Here \( K_2 = p^*(\det \mathcal{E}_B) \).

**Proof.** The statement (1) is from the definition. By Lemma 5.9, \( A_2 = A_0 \). Thus (2) just follows from Lemma 5.8. Now \( \mathcal{O}_Y(1) = \mathcal{O}_Y(E_Y) \). Moreover, by Lemma 5.9,
\[
5s + p^*(A_1 - 2A_2 - \det \mathcal{E}_B - T) = 5s - p^*(2A_0 + \det \mathcal{E}_B).
\]
Thus (3) follows from Lemma 5.6 (2). □
References

[1] Miguel A. Barja, Generalized Clifford-Severi inequality and the volume of irregular varieties, Duke Math. J. 164 (2015), no. 3, 541–568. MR 3314480

[2] Miguel Ángel Barja, Rita Pardini, and Lidia Stoppino, Surfaces on the Severi line, J. Math. Pures Appl. (9) 105 (2016), no. 5, 734–743. MR 3479190

[3] , Higher-dimensional Clifford-Severi equalities, Commun. Contemp. Math. 2 (2020), no. 8, 1950079, 15. MR 4142335

[4] , Linear systems on irregular varieties, J. Inst. Math. Jussieu 19 (2020), no. 6, 2087–2125. MR 4167003

[5] Arnaud Beauville, Complex algebraic surfaces, London Mathematical Society Student Texts, vol. 34, Cambridge University Press, Cambridge, 1996, Translated from the 1978 French original by R. Barlow, with assistance from N. I. Shepherd-Barron and M. Reid. Second edition. MR 1406314

[6] Mauro C. Beltrametti and Andrew J. Sommese, The adjunction theory of complex projective varieties, De Gruyter Expositions in Mathematics, vol. 16, Walter de Gruyter & Co., Berlin, 1995. MR 1318687

[7] E. Bombieri, Canonical models of surfaces of general type, Inst. Hautes Études Sci. Publ. Math. (1973), no. 42, 171–219. MR 318163

[8] Fabrizio Catanese, Marco Franciosi, Klaus Hulek, and Miles Reid, Embeddings of curves and surfaces, Nagoya Math. J. 154 (1999), 185–220. MR 1689180

[9] Jungkai A. Chen, Meng Chen, and Chen Jiang, The Noether inequality for algebraic 3-folds, Duke Math. J. 169 (2020), no. 9, 1603–1645, With an appendix by János Kollár. MR 4105534

[10] Jungkai Alfred Chen, Olivier Debarre, and Zhi Jiang, Varieties with vanishing holomorphic Euler characteristic, J. Reine Angew. Math. 691 (2014), 203–227. MR 3215551

[11] Meng Chen, Inequalities of Noether type for 3-folds of general type, J. Math. Soc. Japan 56 (2004), no. 4, 1131–1155. MR 2092941

[12] Federigo Enriques, Le Superficie Algebriche, Nicola Zanichelli, Bologna, 1949. MR 0031770

[13] Christopher D. Hacon, A derived category approach to generic vanishing, J. Reine Angew. Math. 575 (2004), 173–187. MR 2097552

[14] Robin Hartshorne, Algebraic geometry, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157

[15] Eiichi Horikawa, Algebraic surfaces of general type with small $c_1^2$. I, Ann. of Math. (2) 104 (1976), no. 2, 357–387. MR 424831

[16] , Algebraic surfaces of general type with small $c_1^2$. II, Invent. Math. 37 (1976), no. 2, 121–155. MR 460340

[17] , Algebraic surfaces of general type with small $c_1^2$. V, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 28 (1981), no. 3, 745–755 (1982). MR 656051

[18] Yong Hu, Inequality for Gorenstein minimal 3-folds of general type, Comm. Anal. Geom. 26 (2018), no. 2, 347–359. MR 3805162

[19] Yong Hu and Tong Zhang, Algebraic threefolds of general type with small volume, preprint.

[20] , Fibered varieties over curves with low slope and sharp bounds in dimension three, J. Algebraic Geom. 30 (2021), no. 1, 57–95. MR 4233178

[21] Zhi Jiang, On Severi type inequalities, Math. Ann. 379 (2021), no. 1-2, 133–158. MR 4211084

[22] Yujiro Kawamata, Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces, Ann. of Math. 127 (1988), no. 2, 93–163. MR 924674
[23] János Kollár, Higher direct images of dualizing sheaves. i., vol. 123, 1986, pp. 11–42. MR 825838
[24] János Kollár and Shigefumi Mori, Birational geometry of algebraic varieties, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original. MR 1658959
[25] Xin Lu and Kang Zuo, On Severi type inequalities for irregular surfaces, Int. Math. Res. Not. IMRN (2019), no. 1, 231–248. MR 3897429
[26] Hideyuki Matsumura, Commutative ring theory, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989, Translated from the Japanese by M. Reid, Second edition. MR 1011461
[27] Masayoshi Nagata, On rational surfaces. I. Irreducible curves of arithmetic genus 0 or 1, Mem. Coll. Sci. Univ. Kyoto Ser. A. Math. 32 (1960), 351–370. MR 126443
[28] Max Noether, Zur Theorie des eindeutigen Entsprechens algebraischer Gebilde, Math. Ann. 8 (1875), no. 2, 495–533. MR 1509663
[29] Koji Ohno, Some inequalities for minimal fibrations of surfaces of general type over curves, J. Math. Soc. Japan 44 (1992), no. 4, 643–666. MR 1180441
[30] Rita Pardini, The Severi inequality $K^2 \geq 4\chi$ for surfaces of maximal Albanese dimension, Invent. Math. 159 (2005), no. 3, 669–672. MR 2125737
[31] Miles Reid, Young person’s guide to canonical singularities, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 345–414. MR 927963
[32] Francesco Severi, La serie canonica e la teoria delle serie principali di gruppi di punti sopra una superficie algebraica, Comment. Math. Helv. 4 (1932), no. 1, 268–326. MR 1509461
[33] Gang Xiao, $\pi_1$ of elliptic and hyperelliptic surfaces, Internat. J. Math. 2 (1991), no. 5, 599–615. MR 1124285
[34] Tong Zhang, Severi inequality for varieties of maximal Albanese dimension, Math. Ann. 359 (2014), no. 3-4, 1097–1114. MR 3231026

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