Convex integer optimization with Frank-Wolfe methods

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Abstract
Mixed-integer nonlinear optimization is a broad class of problems that feature combinatorial structures and nonlinearities. Typical exact methods combine a branch-and-bound scheme with relaxation and separation subroutines. We investigate the properties and advantages of error-adaptive first-order methods based on the Frank-Wolfe algorithm for this setting, requiring only a gradient oracle for the objective function and linear optimization over the feasible set. In particular, we will study the algorithmic consequences of optimizing with a branch-and-bound approach where the subproblem is solved over the convex hull of the mixed-integer feasible set thanks to linear oracle calls, compared to solving the subproblems over the continuous relaxation of the same set. This novel approach computes feasible solutions while working on a single representation of the polyhedral constraints, leveraging the full extent of Mixed-Integer Programming (MIP) solvers without an outer approximation scheme.

1. Introduction
Mixed-integer nonlinear optimization problems (MINLP) is a challenging class of problems combining both combinatorial structures and nonlinearities which can model a broad range of problems arising in engineering, transportation, and more generally operations and other application contexts. In Machine Learning, combinatorial constraints can capture rich properties demanded of a solution, e.g. solutions that must be a path, cycle, or tour in a graph, or solutions with maximum support or guaranteed sparsity. The dominant algorithmic frameworks for solving such problems are combinations of BnB with spatial and integer branching and cutting planes tightening local linear relaxations.

We focus in this paper on mixed-integer convex problems in which the nonlinear constraints and objectives are convex and present a new algorithmic framework for solving these problems that exploit recent advances in so-called Frank-Wolfe (FW) methods (also called: conditional gradient (CG) methods). The problem class we consider is of the type:

\[
\min_{x,y} f(x, y) \quad (1a)
\]

\[
\text{s.t. } (x, y) \in X \times Y \quad (1b)
\]

\[
x_j \in \mathbb{Z} \ \forall j \in J, \quad (1c)
\]

where \(X\) is a compact subset formed by polyhedral constraints and possibly contains combinatorial structures, \(Y\) is a (potentially non-polyhedral) compact convex set over which optimizing a linear function is computationally tractable and less demanding than the original problem, \(f : X \times Y \to \mathbb{R}\) is a differentiable convex function with a Lipschitz-smooth gradient accessed as an oracle, we do not require an expression graph.
Several families of methods have been developed over the last decades, rooted in both nonlinear and discrete optimization approaches; see Kronqvist et al. (2019) for a recent review of algorithms and solvers for convex MINLPs. Some solvers implement a branch-and-bound algorithm that solves a continuous relaxation including the nonlinearities at each node of the branch-and-bound tree; see e.g., Knitro (Byrd et al., 2006) or Couenne (Belotti, 2009). Solving nonlinear problems at each node is however expensive: the typical algorithms used are based on interior-point methods (almost always relying on expensive second-order information), which yield high-precision results at the cost of more expensive iterations. Another approach, introduced by Quesada & Grossmann (1992), solves Linear Programs (LPs) at each node and nonlinear problems only to compute feasible solutions and additional cuts when integer solutions are found. This method is also implemented in SCIP (Bestuzheva et al., 2021) for convex or nonconvex nonlinearities. The solvers SHOT (Lundell et al., 2022) and Pavito.jl (JuMP-dev, 2018) use single-tree polyhedral outer approximation approaches, relying on the Mixed-Integer Programming (MIP) solver to build the main relaxation of the problem. Bonmin (Bonami et al., 2008) uses a hybrid approach with both outer polyhedral approximations and branch-and-bound. In Chen & Goulart (2022), mixed-integer conic problems are tackled with ADMM using inexact termination and a safe dual bound recovery in a BnB framework, similarly to our approach for the inexact subproblem exploitation. However, in our case, the FW gap computed at each iteration directly produces a dual bound that can be used to terminate node processing and obtain a global tree bound. Furthermore, our approach avoids projections onto cones which can be expensive (requiring eigendecompositions) or numerically unstable (for exponential and positive semidefinite cones for instance). It is important to note that in many MINLP solution approaches, nonlinearities are handled through separation (via gradient cuts or supporting hyperplane cuts). Separation is not possible if the nonlinearity is in the objective and most models therefore only allow for nonlinearities in the constraints requiring an epigraph formulation for the objective function. By doing so, the initial structure of Problem (1) is lost. Some lines of work focused on branch-and-bound approaches for specialized sparse regression problems, also referred to as best subset selection (Bertsimas et al., 2016; Hazimeh & Mazumder, 2020; Gómez & Prokopyev, 2021; Moreira Costa et al., 2022). These approaches however rely on the properties of the quadratic loss and/or of the constraint set and would not extend to other loss functions (e.g. logistic or Poisson regression) or additional constraints (arbitrary linear constraints on the features, grouped subset selection).

**Our approach and contribution**

Our approach for Problem (1) is markedly different from these previous approaches: the branch-and-bound process is used to enforce integrality constraints (as expected). In each branch-and-bound node, we have to solve a nonlinear program (NLP) based on the original problem, the current variable bounds arising from the BnB process, and relaxed integrality constraints. The NLP itself is solved using a particularly well-suited variant of the Frank-Wolfe algorithm, the so-called Blended Pairwise Conditional Gradient (BPCG) algorithm (Tsuij et al., 2021) that requires a linear minimization oracle (here given by the MIP solver) which is called repeatedly in the solution process and access to gradients from the objective by means of a first-order oracle; see also Braun et al. (2022) for a general overview of conditional gradients and Frank-Wolfe methods. No other access to the feasible region (e.g., complete description) or the objective function (e.g., Hessian information) is required and in particular we do not require epigraph formulations and we retain the original polyhedral structure (except for bound updates); see also Figure 1 for a schematic overview. In terms of key differences this results in:

1. Instead of solving weaker but expensive convex relaxations at each node of the branch-and-bound tree, we optimize over the convex hull of integer-feasible solutions. This lets us compute much stronger relaxations and automatically derive several feasible solutions at each node,

2. We replace exact convex solvers with a Frank-Wolfe based error-adaptive solution process, i.e. a solution process for which the amount of computations performed is gradually increasing with the specified precision,

3. We leverage the active set representation of the solution at each node to warm-start the children iterates and reduce the number of calls to the Mixed-Integer Programming (MIP) solver.
4. We develop new lazification techniques and strong branching strategies for our framework.

We will now discuss the above in more detail. The main motivation for outer-approximation frameworks is the ability to leverage MIP solvers as the core of their procedure through single-tree approaches i.e. the MIP solver is called only once with separation callbacks. This class of approaches however carries several issues: only near-feasible and near-optimal solutions are obtained in the limit towards the end of the solution process and rely on MIP heuristics for primal feasible solutions. Furthermore, the separation constraints for the nonlinear feasible set, often linear cutting planes, are in general dense in the variables involved in the nonlinear structures. These high-density constraints slow down the MIP solving process and yield numerically ill-conditioned LP relaxations. We face none of these issues due to not relying on outer approximations at all but rather relying on nonlinear relaxations. While this may seem inefficient at first and naive realizations would require a very large number of MIP subproblems to be solved, our framework allows utilizing recent advances in Frank-Wolf methods and MIP methodology to reduce the number MIP subproblems dramatically.

Within the context of FW, we utilize (a) blending (Braun et al., 2019) and lazification (Braun et al., 2017) to aggressively reuse previously discovered vertices and in fact we extend the lazification approach from a single subproblem to the whole tree for huge additional gains in performance, (b) adaptive error guarantees and dual bounds to allow for inexact and incomplete resolution of nodes in the tree, interrupting the solving process as soon as the dual bound is satisfying for the current node, (c) warm-starting BPCG, using the vertices and weights of the parent node’s active set, creating high-quality warm-start solutions and significantly reducing the number of required FW iterations. These are combined with modern MIP solution techniques (see Koch et al. (2022) for an overview) that we can fully leverage as the subproblems solved by BPCG are standard MIPs: (a) primal heuristics provide feasible solutions (also valid for Problem (1) and not just the subproblem) along the way that can be reused across all other MIP calls via active set and shadow set representations (b) cutting planes over standard MIPs can be fully leveraged for the resolution of the MIP subproblems occurring in BPCG. These techniques are combined in one overarching framework and we develop techniques such as strong branching and hybrid branching for our setup here, further improving the solution process.

Within our BnB tree, the MIP solver serves multiple functions: (a) the Linear Minimization Oracle (LMO) calls a MIP solver, leveraging it to its full extent with cutting planes, conflict analysis, restarts, (b) the MIP model can integrate combinatorial constraints that are not directly representable with linear inequalities (SOS1, indicator constraints), (c) some structures like cutting planes, conflicts, variable bounds computed for a MIP call can be reused by following subproblems within a node and its children, (d) primal heuristics called by the MIP solver collect multiple solutions that can all be evaluated for our original problem.

Our framework is in particular the first one, to the best of our knowledge, that can exploit the flexibility and advances of modern MIP solvers outside of a polyhedral outer approximation scheme, staying feasible throughout the process. Ideas of warm-starting Frank-Wolfe variants based on active sets have been leveraged in the past. In Moondra et al. (2021), an away-step Frank-Wolfe variant leverages active sets from previous iterations in an online optimization setting. In Joulin et al. (2014), Frank-Wolfe or variants thereof are called over the integer hull of combinatorial problems, calling a MIP solver as LMO and additional roundings to compute feasible solutions of good quality. The hardness of optimization over the integer hull motivated lazified conditional gradient approaches in Braun et al. (2017), opening a stream of literature. We however generalize lazification techniques to consider the whole tree. Branch-and-bound with Away-step Frank-Wolfe (AFW) subproblems has been explored in Buchheim et al. (2018) for a portfolio problem. Each subproblem is partly solved to compute lower bounds from the Frank-Wolfe dual gap and each branching performs fixing to all possible values in the case of integer variables. We leverage BPCG (Tsuji et al., 2021) which enables us to aggressively reuse information and whose convergence speed is typically higher than that of AFW, while maintaining a higher sparsity which greatly benefits our branching as the fractionality of solutions is lower in most cases.

We complement the above with extensive computational results to demonstrate the effectiveness of our approach. The code will be made available as a Julia package Boscia.jl under the MIT license.
2. Nonlinear branch-and-bound over integer hulls with linear oracles

In this section, we present the main paradigm for our solver framework. Our approach is primarily based on branch-and-bound, the feasible region handled at each node is identical except for updated fixings and bounds on the variables that must take integer values. At each node, we solve a convex relaxation with updated local bounds for the integer variables using a FW-based algorithm. This family of algorithms only requires a gradient oracle for the objective function \( f \), i.e. given a point \( x \) return \( \nabla f(x) \), and not a full analytical expression nor Hessian information. It accesses the feasible region \( P \) though a Linear Minimization Oracle (LMO), which, given a direction \( d \), solves \( \min_{x \in P} d \cdot (x) \), where \( v \) is an extreme point or vertex of the feasible set. This means in particular that the feasible region does not have to be given in closed form, e.g., with constraint matrix and right-hand side, as long as the LMO is well-defined and tractable. Instead of optimizing over the continuous relaxation of Problem (1) with updated bounds at each node, we optimize over the convex hull of integer feasible solutions at each node. For that purpose, the LMO called within the FW procedure is the MIP solver with bounds updated for the node and objective given by the gradient information at the current solution. This design choice is illustrated in Figure 2. Optimizing over the convex hull at each node means that all vertices computed across FW iterations are feasible for the original problem, the feasible region itself is tightened and the tree height is greatly reduced for problems that have a loose continuous relaxation.

**Blended Pairwise Conditional Gradient (BPCG).** The classic FW algorithm calls the LMO once per iteration. Even though this is acceptable in contexts where the LMO is inexpensive compared to gradient evaluations or in large dimensions in which storage of vertices is an issue, the plain FW algorithm is not the best fit when optimizing over polytopes, not only because the descent direction can be significantly misaligned with the negative of the gradient, but also and especially because the LMO is particularly expensive (consisting of solving a MIP) in our context. We need methods that exploit already-computed extreme points of the feasible set before making new calls to the LMO, which requires a representation of the iterate as an active set, an ordered set of vertices \( S_t := \{v_1, \ldots, v_k\} \) with associated weights \( \lambda \geq 0, \sum \lambda = 1 \) such that the iterate is a convex combination \( x_t := \sum_k \lambda_k v_k \). Typical algorithms leveraging this representation are the
Away Frank-Wolfe (AFW) (Guélat & Marcotte, 1986), Pairwise Frank-Wolfe (PFW) (Lacoste-Julien & Jaggi, 2015) and Blended Conditional Gradient (BCG) (Braun et al., 2019). The main algorithm we focus on is the recently-proposed Blended Pairwise Conditional Gradient (BPCG) algorithm (Tsuji et al., 2021) which blends Frank-Wolfe steps in which a vertex is added to the active set with weight transfer steps which do not require any LMO call and only perform inner product between vertices and the gradient. Experimentally, BPCG produces iterates with a smaller active set across all iterations than other FW algorithms. We use more specifically a slight modification of the lazified version of BPCG from (Tsuji et al., 2021) as implemented in FrankWolfe.jl. We present the pseudo-code for the variant that we are using in the Technical Appendix.

Active and shadow set branching. We accelerate the solving process of individual nodes through warm starts. At the end of each BPCG call, the optimal solution is given as a convex combination of vertices in an active set. All these vertices are obtained by calls to the MIP LMO and are thus extreme points of the convex hull of the integer feasible set. When branching at a given node, the active set vertices can be partitioned to form active sets for the left and right children. Furthermore, as the relaxed solution is fractional in the variable branched on, it must contain at least a vertex for each of the children. The weights are also transferred to each child for the corresponding vertex and renormalized. This ensures a high-quality starting point at each child node in the tree. When weights are adjusted via pairwise steps of BPCG these steps do not require an LMO call and are thus computationally rather cheap. The LMO is only called when performing FW steps that add a new vertex to the active set. BPCG also drops vertices from the active set when the pairwise step is performed completely (i.e. $\gamma_i = 1$). In a single run of the BPCG algorithm, a dropped vertex is usually not part of the optimal active set. However, we perform many runs for many subproblems arising across different nodes. As such a dropped vertex might very well be relevant for another subproblem further down the tree when the bounds have been updated, adding computational burden by redundant calls to the LMO. We therefore maintain a shadow set, a set of discarded vertices collected within each node when they are
Figure 3: Comparison of branching strategies on an integer sparse regression example with dimension 40. Hybrid branching performs partial strong branching up to the switching point corresponding to a certain depth in the tree. Partial strong branching uses a FW gap of $10^{-3}$ and at most 10 FW iterations. Hybrid branching operates partial strong branching up to depth 5, the “switching” limit corresponds to the last call to strong branching by the hybrid strategy.

removed from the active set. The shadow set can be partitioned in a fashion similar to the active set. At all non-root nodes, an extra-lazification layer is added to BPCG: if no suitable pair of vertices can be found in the pairwise step, the algorithm searches the shadow set for a vertex with a suitable alignment with the gradient, using identical criteria as in the lazy BPCG algorithm. Only if this additional step fails to find a good pair does the algorithm perform an LMO call computing a new vertex. By propagating both the active and shadow set, we significantly reduce the number of calls to the LMO, ensuring we never recompute a given vertex twice within a run for our algorithm.

3. Branch-and-bound techniques for error-adaptive convex relaxations

In this section, we present the techniques derived from branch-and-bound techniques that can be used in our framework. In particular, we highlight the advantages from the error-adaptivity of BPCG, i.e. the property of the computational cost decreasing gradually when the error tolerance increases, and how the various components of modern MIP solvers can be used to enhance the higher-level branch-and-bound process.
MIP solvers are implementations of complex algorithms with several key components including presolving, heuristics, bound tightening, and conflict analysis. This is usually also a main motivation for single-tree MINLP solvers handling nonlinearities as a special case of separation within a MIP solving process. We can however also leverage this additional information about the feasible set within our framework by transferring it across nodes, and potentially exploiting a reoptimization emphasis within a given node that solvers like SCIP support (Gamrath et al., 2015). This avoids the recomputation of already fixed terms across successive LMO calls.

**FW gap-based termination.** Frank-Wolfe methods produce at each iteration a primal feasible iterate and a dual gap upper-bounding the optimality gap. As a first-order class of methods, they typically feature many inexpensive iterations (unlike e.g. interior point methods which require few expensive iterations), with a dual bound gradually increasing with iterations. We leverage this fact to adapt the solving process at each node. Whenever the dual bound reaches the objective value of the best incumbent (or becomes close to the gap tolerance), the node can be terminated early. A significant part of the solving process can be avoided in this manner which is not necessarily the case when nodes are processed with other non-convex or convex solvers. Pushing this further, the solution process of a node can be stopped at any point to produce a dual bound, even if the solution is not an exact but approximate optimum. The only strict requirement is that the dual bound after solving the current node becomes high enough to yield progress in the overall tree dual bound.

**Tree state-dependent termination and evolving error.** We implement additional termination criteria for node processing which do not guarantee the node should be pruned but that save on the total number of iterations. One of them is the number of open nodes with a lower bound (obtained from the parent) that is lower than the dual bound of the node being processed during the BPCG run. This set of lower nodes will be processed before the children of the current node are. If the number of these more promising nodes is high enough, there is a higher probability for the children of the current node not to be processed at all, for instance if a better incumbent is found elsewhere. We also add an adaptive Frank-Wolfe gap criterion, increasing the precision with the depth in the BnB tree:

\[ \varepsilon_n = \varepsilon_0 \rho^n \]

with \( \rho \in [0, 1] \) where \( n \) is the depth of the current node, starting from the root at depth 0. With some convex solvers, using a reduced precision could slow down the search, as approximate solutions might exhibit a much higher fractionality when a weaker stopping criterion is used, requiring more branching. In the case of BPCG however, the iterates are a convex combination of few integer extreme points. Furthermore, if the optimal solution at a node is an extreme point itself, the process converges fast by dropping the other extreme points from the active set once the optimal one has been added. As such solutions obtained from low precision runs do not necessarily exhibit a higher fractionality which in turn might require more branching.

**Strong branching.** When tackling large discrete problems, the choice of variables to branch on can yield drastic differences in how fast the lower and upper bounds evolve and the overall size of the branch-and-bound tree. A powerful technique to estimate the lower bound increase when branching on a given variable is the so-called *strong branching* which solves the children subproblems of a given node for all candidate variables, selecting the variable to branch on that improves the lowest lower bound across children. Other techniques like pseudo-cost branching or recent machine learning approaches try to construct surrogate models to avoid solving the multiple convex subproblems induced by strong branching (Nair et al., 2020).

We propose a new family of branching techniques that leverage the properties of FW algorithms to obtain a partial estimate of the lower bound improvement while greatly reducing the cost. In our context, strong branching with nonlinear subproblems over the integer hull would be too costly to perform variable selection. However, we can (a) relax the strong branching over the integer hull to the continuous relaxation (solving LPs for the LMO instead of MIPs), (b) run few iterations of the subproblem (and/or setting a very high FW gap tolerance). In the limit case, on the one extreme end this corresponds to performing a single FW iteration for the strong branching estimation and the optimal value of a single linear problem is used to select the branching variable. On the other extreme end the complete high-accuracy solve would correspond to using
the continuous relaxation of the node. By carefully limiting the precision we can interpolate between these two regimes.

4. Computational experiments

We leverage the FrankWolfe.jl framework (Besançon et al., 2022) as a subsolver. All features specific to our branch-and-bound framework are implemented in the Julia package Boscia.jl which will be available in open-source under MIT license for the community to test, use, and build upon. The BnB core structures are implemented in Bonobo.jl, the underlying MIP solver is SCIP (Bestuzheva et al., 2021). All experiments were carried out on an 8-core compute node equipped with an Intel Xeon E3-1245v5 3.50GHz CPU and 32GB RAM. We use Julia 1.7.0. The package versions used are FrankWolfe.jl v.0.2.11, Bonobo.jl v0.1.2, SCIP.jl v0.11.3 with SCIP v8.0.0.

4.0.1 Problem classes.

We considered the following problem classes.

Sparse regression problems. Sparsity is a desirable property for prediction models for reasons of robustness, explainability, computational efficiency or other underlying motivation. Our framework allows for solving all cardinality-constrained regression models including linear regression, sparse Poisson regression, and logistic regression, as long as the loss function is convex and loss gradients are available. We also experiment with regression models where the predictor coefficients themselves are constrained to take integer values.

Grouped sparse regression. In addition to a simple cardinality constraint, our framework can represent richer generic constraints in regression models, for example adding group sparsity, where variables are partitioned into subsets and only a few of those subsets can be in the support of the predictor. This is a cardinality-constrained version of the grouped lasso model (Lounici et al., 2011; Friedman et al., 2010). Similarly to simple sparse regression, our framework can accommodate all these formulations for all differentiable convex losses including Poisson, logistic, and linear regression with or without $\ell_2$ regularization.

Sparse coding with permutation matrices. We consider the problem of finding a convex combination of $K$ permutation matrices that approximate an $n \times n$ doubly-stochastic matrix $\hat{X}$:

$$
\min_{X \in P_n, \theta \in \Delta_k} \left\| \sum_{i \in [k]} \theta_i X_i - \hat{X} \right\|^2,
$$

where $\Delta_k$ is the $k$-dimensional probability simplex, $[k]$ represents the set of integers from 1 to $k$, and $P_n$ is the set of $n \times n$ permutation matrices, which we can convexify with auxiliary variables using the bounds on $\theta$ and $X$, and the fact that one of the variables of the bilinear terms is always binary:

$$
\min_{X \in P_n, \theta \in \Delta_k, Y \in \mathbb{R}^n} \left\| \sum_{i \in [k]} Y_i - \hat{X} \right\|^2
$$

s.t. $0 \leq Y_i \leq X_i, 0 \leq \theta_i - Y_i \leq 1 - X_i$.

This problem or variants have been considered in Valls et al. (2021); Dufossé & Uçar (2016). It can be viewed as a cardinality-constrained version of sparse coding over the Birkhoff polytope, the convex hull of permutation matrices.

Portfolio optimization. We revisit the example of Buchheim et al. (2018), selecting a portfolio with budget $b$, integrality requirements on shares for some assets, and with a generic convex differentiable risk penalty term $h(\cdot)$:

$$
\min_x h(x)^T M x - \langle r, x \rangle
$$

s.t. $\langle c, x \rangle \leq b, x_j \in \mathbb{Z} \forall j \in J$.

MIPLIB instance. Our framework reads standard instance formats automatically thanks to the MathOptInterface backend (Legat et al., in press) in Julia. As an arbitrary example, we use the (bounded) instance 22433.mps
from the MIPLIB 2017 (Gleixner et al., 2021) and as a simple experiment, we compute vertices for random directions and then minimize the sum of squared distances to these vertices, therefore promoting interior integral solutions.

### 4.0.2 Results

In Figure 5, we test the effect of the lazification and warm-starting techniques added to BPCG, the root node requires the most LMO calls of all layers and populates the active and shadow sets. Depending on the instances, the cardinality of the active set and shadow sets evolve differently, depending on whether the last layers of nodes densify or sparsify the solution.

An example of primal-dual convergence of the branch-and-bound tree for a sparse regression example is shown in Figure 4. As typical for our framework, the optimal primal is found early in the search process thanks to the MIP heuristics and vertices, the process could be stopped much earlier with a non-zero dual gap but a high-quality (most of the time optimal) primal solution. Figure 6 shows the same primal-dual evolution with the LMO calls per node. Figure 8 shows the primal-dual convergence for an integer regression problem with cardinality constraints. On all instances, the optimal primal solution is found early in the process, typically at the root node thanks to the optimization performed over the integer hull, the following nodes are used only to prove (near-)optimality.

Figure 7 shows the effect of tuning the FW gap tolerance at each node with the layer-dependent error described in Equation (2).

A comparison of branching strategies on a sparse regression problem is presented in Figure 3. On some problem types, strong and hybrid strong branching provide a significant speed-up, especially early on in the branching process since superior variable branching strategies impact the rest of the process. We also observed that the FW gap used for strong branching does not have a significant impact in terms of progress per node or time. This is likely due to the highly lazified nature of BPCG which performs few LMO calls mostly at the beginning of the solving process. Higher precision for the continuous relaxation used for strong branching typically does not require more LMO calls but also provides little gain in terms of prediction for the variable to branch on. This limited effect can be explained by the fact that the individual solutions or bounds are not used directly but only rank the different variables, high precision is therefore useless as long as the ranking is not changing when accuracy increases. On the 22433.mps MIPLIB instance, our algorithm requires between one and three nodes to solve the problem depending on the quadratic objective, despite about 231 integer variables. This is due to the number of fractional variables remaining small when optimizing over the integer hull.
Figure 5: Average cardinality of the active & shadow sets and number of LMO calls over nodes at each depth level of the tree for the permutation matrices example with $n, k = 3$. The number of vertices in the active and shadow sets increases moderately while the number of LMO calls drops quickly after the root node.

Figure 6: Primal-dual progress on the permutation matrices example with $n = 3, k = 2$. Unlike many examples, improving primal solutions are found late in the search process. The non-cumulative number of LMO calls per node is displayed to highlight the decrease throughout the tree.
Figure 7: Effect of the FW gap tolerance parameters on the total solving time on integer sparse regression. Solving the subproblems with a too high accuracy is much costlier for the runtime. Furthermore, lower decay rates, that imply nodes lower in the tree are solved with an accuracy that does not increase a lot seem to accelerate the solving process.

Figure 8: Convergence plot for an integer regression example in dimension 40 with a cardinality constraint on the support of the predictor. On most instances, the primal optimum is found at the root node, the remaining nodes are spent proving near-optimality with the specified relative gap tolerance of 1%.
5. Conclusion

In this paper, we proposed a novel algorithm for mixed-integer convex optimization relying only on gradient and function evaluations of the objective. By embedding a FW-based subsolver within a BnB framework, our method does not rely on outer approximation & separation nor on projection subproblems. Since FW algorithms rely on LMO calls to handle the constraint set, we can significantly strengthen convex relaxations by optimizing at each node over the convex hull of mixed-integer feasible solutions, therefore leveraging the capabilities of modern MIP solvers. Lazification techniques within and across nodes also avoid expensive MIP solves by exploiting all vertices that have been discovered and further MIP information. Future work will include exploiting MIP reoptimization techniques and transferring additional MIP information across LMO calls within and across nodes, and designing adaptive rules for early termination criteria.

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**Appendix A. Blended Pairwise Conditional Gradient**

We present below the modified Lazy Blended Pairwise Conditional Gradient (L-BPCG) from Tsuji et al. (2021) to solve relaxations at each node. The procedure populates the shadow set with dropped vertices and starts from the warm-started active set. The convergence follows from that of BPCG, we use the same progress measure on whether a shadow vertex offers sufficient decrease as FW direction (vertex the iterates moves towards) as we do for a classic pairwise step. The additional shadow vertex selection can be viewed as a special case of a pairwise step where the initial weight in the active set is zero.

**Appendix B. Branching strategies**

We present additional results comparing the strong and hybrid branching strategies to most infeasible (most fractional) branching. Strong branching is not beneficial on all problems. In particular, strong and hybrid branching provides a strong advantage for the dual bound progress when some branching decisions significantly tighten the problem. In some cases such as the mixed-binary quadratic problem shown in Figure 9, the most infeasible branching strategy is less efficient both against the number of nodes and time despite being a much cheaper variable selection heuristic.

Figure 11 shows the evolution of the dual bound when using different maximum depth levels for hybrid strong branching. The most expensive strategies using strong branching further down the tree are are favourable even against time. This opens a range of algorithmic questions on adaptive transitions from strong branching to less expensive heuristics.

Figure 12 shows the evolution of the number of LMO calls and cardinality of the average number of active and shadow set for each node depth. We systematically observe a decrease in the number of LMO calls across layers thanks to warm starts and lazification. For some instances, the active set cardinality drops in the last layer, i.e. final solutions are sparser than the ones from previous layers. In some other instances, the leaf nodes produce denser solutions once all integers are fixed.

Figure 14 shows the primal dual convergence on the sparse poisson regression example in dimension 70. Unlike most other examples we covered, the gap is closed by a sharp drop of the incumbent value and not by an increase of the dual bound. This suggests a potential for improvement from designing and adding more advanced objective-aware heuristics (e.g. a nonlinear feasibility pump) or adjusting the heuristic collection from the MIP solver.
Algorithm A.1 Lazy Blended Pairwise Conditional Gradient with Shadow Set

Require: Starting active set $\mathcal{A}_0$ and weights $\lambda$, shadow set $S$, function $f$, feasible set of current node $P$, $\epsilon_{tol}$, accuracy $K \geq 1$

Ensure: final iterate $x_T$ such that $f(x_T) \leq f^* + \epsilon_{tol}$

1: $x_0 \leftarrow \sum_{k=1}^{\left|\mathcal{A}_0\right|} \lambda_k v_k$
2: $\Phi_0 \leftarrow \max_{v \in P} \langle \nabla f(x_0), v \rangle / 2$
3: $t = 0$
4: while dual_gap $> \epsilon_{tol}$ do
5:   $a_t \leftarrow \arg\max_{v \in \mathcal{A}_t} \langle \nabla f(x_t), v \rangle$ \hspace{1cm} \text{\textsuperscript{*} away vertex}
6:   $s_t \leftarrow \arg\min_{v \in \mathcal{A}_t} \langle \nabla f(x_t), v \rangle$ \hspace{1cm} \text{\textsuperscript{*} local forward vertex}
7:   if $\langle \nabla f(x_t), a_t - s_t \rangle \geq \Phi_t$ then
8:     $d_t \leftarrow a_t - s_t$
9:     $\gamma_t \leftarrow \text{weightOf}(\mathcal{A}_t, a_t)$
10:    $\gamma_t \leftarrow \arg\min_{\gamma \in [0, \lambda_{\max}]} f(x - \gamma d_t)$
11:    $x_{t+1} \leftarrow x_t - \gamma_t d_t$
12:    $\Phi_{t+1} \leftarrow \Phi_t$
13:    if $\gamma_t < \gamma_{\max}$ then \hspace{1cm} \text{\textsuperscript{*} descent step}
14:       $\mathcal{A}_{t+1} \leftarrow \mathcal{A}_t$
15:    else \hspace{1cm} \text{\textsuperscript{*} drop step}
16:       $\mathcal{A}_{t+1} \leftarrow \mathcal{A}_t \setminus \{a_t\}$
17:       $S_{t+1} \leftarrow S_t \cup \{a_t\}$
18:    end if
19:   else
20:     $s_t \leftarrow \arg\min_{v \in S_t} \langle \nabla f(x_t), v \rangle$ \hspace{1cm} \text{\textsuperscript{*} forward vertex from dropped vertices}
21:     if $\langle \nabla f(x_t), a_t - s_t \rangle \geq \Phi_t$ then
22:       $\mathcal{A}_{t+1} \leftarrow \mathcal{A}_t \cup \{a_t\}$
23:       $S_{t+1} \leftarrow S_t \setminus \{s_t\}$
24:       descent or drop step
25:     else
26:       $w_t \leftarrow \arg\min_{v \in P} \langle \nabla f(x_t), v \rangle$ \hspace{1cm} \text{\textsuperscript{*} Global LMO}
27:       if $\langle \nabla f(x_t), x_t - w_t \rangle \geq \Phi_t / K$ then
28:         $d_t = x_t - w_t$
29:         $\gamma_t \leftarrow \arg\min_{\gamma \in [0, \lambda_{\max}]} f(x - \gamma d_t)$
30:         $x_{t+1} \leftarrow x_t - \gamma_t d_t$
31:         $\Phi_{t+1} \leftarrow \Phi_t$
32:         $\mathcal{A}_{t+1} \leftarrow \mathcal{A}_t \cup \{w_t\}$
33:       else
34:         $x_{t+1} \leftarrow x_t$
35:         $\Phi_{t+1} \leftarrow \Phi_t / 2$
36:         $\mathcal{A}_{t+1} \leftarrow \mathcal{A}_t$
37:       end if
38:     end if
39:   end if
40:   $t \leftarrow t + 1$
41: end while
Figure 9: Primal-dual bound convergence for a low-dimensional quadratic mixed-binary problem (12) with a high inner dimension (500) taken from Hunkenschröder et al. (2022).
Figure 10: Dual bound convergence for grouped sparse regression against time and number of nodes. The switching limit corresponds to the last node where the hybrid branching applies strong branching.
Figure 11: Depth adjustment for hybrid branching i.e. maximum depth up to which partial strong branching is applied before switching to most fractional branching on an integer sparse regression example. Num_int corresponds to a maximum depth equal to the number of integer variables. Since this instance has general integer and not only binary variables, this is not the maximum depth of the tree. The lowest numbers correspond to fewer strong branching calls. The maximum number of FW iterations was 10 and the gap tolerance $10^{-3}$. 
Figure 12: Evolution of the LMO call, active and shadow set cardinalities for the permutation matrices example.
Figure 13: Primal-dual convergence on the permutation matrices example. The number of LMO calls is displayed for each node and decreases with the layers.
Figure 14: Primal-dual convergence on the sparse Poisson regression example.