The decompositions of Werner and isotropic states

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Abstract
The decompositions of separable Werner states and isotropic states are well-known tough issues in quantum information theory. In this work, we investigate them in the Bloch vector representation, exploring the symmetric informationally complete positive operator-valued measure (SIC-POVM) in the Hilbert space. In terms of regular simplexes, we successfully get the decomposition for arbitrary Werner state in $\mathbb{C}^N \otimes \mathbb{C}^N$, and the explicit separable decompositions are constructed based on the SIC-POVM. Meanwhile, the decomposition of isotropic states is found related to the decomposition of Werner states via partial transposition. It is interesting to note that when dimension $N$ approaches to infinity, the Werner states are either separable or non-steerably entangled, and most of the isotropic states tend to be steerable.

Keywords Separable decomposition of Werner states · SIC-POVM · Isotropic states · Bloch vectors

1 Introduction
As one of the essential physical resources in quantum information processing, entanglement has been studied extensively, while we are still in the early stage of fully understanding it [1]. The essence of the quantum entanglement is the inherent non-local correlation that is fundamentally different from the classical situation. The quantum non-locality, which violates the local realistic theory, has experienced a large number of various experimental tests [2, 3] and is well confirmed.
Werner proposed an important class of states, the Werner states, for the study the separability of mixed states [4]. It was found that the non-separability of these states is not enough to guarantee the Bell non-locality. The Werner state bears high symmetry, and hence, its many properties can be conveniently investigated for the aim of quantum information theory, such as steering [5] and quantum discord [6]. Practically, the Werner states are also of fundamental importance in the studies of quantum error correction [7], entanglement with white noise, and the entanglement distillation [8,9].

The Werner states have been investigated with great effort. However, to find out the product decomposition for a separable Werner state in any dimension is still an open task [10]. Most of the decomposition methods in the literature are only suitable for qubit systems [11,12]. By means of the angular momentum approach, Ref. [10] realized the separable decomposition validated for a subset of separable Werner states when \( N > 2 \). It was shown that the conical designs may provide simple decompositions for some separable Werner and isotropic states [13]. In Ref. [14], people conjectured that the Werner states may be decomposed by the regular simplex with circumradius of \( \sqrt{\frac{2(N-1)}{N}} \) in the Bloch vector space of \( N \)-dimensional mixed states. We know that the existence of the regular simplex in this case is equivalent to the existence of the SIC-POVM, which has been extensively studied [15].

In this paper, we obtain explicit decompositions for arbitrary separable Werner and isotropic states by means of the SIC-POVM. In Sect. 2, a one-parameter representation of the Werner and isotropic states in the Bloch vector space is introduced. Then, in Sect. 3, we decompose the Werner and isotropic states in the one-parameter representation, where the general separable decompositions of the Werner and isotropic states are given, respectively. Section 4 remains for a summary.

### 2 The Werner and isotropic states in the Bloch vector space

An arbitrary \( N \)-dimensional density matrix \( \rho \) may be represented as follows:

\[
\rho = \frac{1}{N} \mathbb{I} + \frac{1}{2} \sum_{\mu=1}^{N^2-1} r_{\mu} \lambda_{\mu} = \frac{1}{N} \mathbb{I} + \frac{1}{2} \vec{r} \cdot \vec{\lambda}, \tag{1}
\]

where \( \lambda_{\mu} \) are the \( N^2 - 1 \) traceless generators of \( SU(N) \) group with \( \text{Tr}[\lambda_{\mu} \lambda_{\nu}] = 2 \delta_{\mu\nu} \) and \( \vec{r} \) is an \( (N^2 - 1) \)-dimensional Bloch vector with component \( r_{\mu} = \text{Tr}[\rho \lambda_{\mu}] \). The density matrix \( \rho \) needs to be positive semidefinite and trace one, which impose constraints on the Bloch vector \( \vec{r} \) [16,17]. For example, the norm of the vector \( |\vec{r}| \leq \sqrt{2(N-1)/N} \) since \( \text{Tr}[\rho^2] \leq 1 \). Consider the positivity of density matrix, we have the following lemma:

**Lemma 1** For trace-one Hermitian matrices \( \rho = \frac{1}{N} \mathbb{I} + \frac{1}{2} \hat{r} \cdot \vec{\lambda} \) and \( \rho' = \frac{1}{N} \mathbb{I} + \frac{1}{2} s \hat{r} \cdot \vec{\lambda} \), where \( \hat{r} \) is a unit vector, the following statements are true:
(1) The positive semidefiniteness of $\rho$ and $\rho'$ gives
\[ rs \geq -\frac{2}{N}. \] (2)

(2) If $\rho = \frac{1}{N} \mathbb{1} + \frac{1}{2} r \hat{r} \cdot \hat{\lambda}$ is a pure state with $r = \sqrt{2(N-1)/N}$, then $\rho' = \frac{1}{N} \mathbb{1} + \frac{1}{2} s \hat{r} \cdot \hat{\lambda}$ is positive semidefinite if and only if
\[ -\sqrt{\frac{2}{N(N-1)}} \leq s \leq \sqrt{\frac{2(N-1)}{N}}. \] (3)

**Proof** (1). Considering that $\rho$ is a positive semidefinite Hermitian matrix, we have the spectrum decomposition of $\rho$, namely $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$, where $p_i$ and $|\psi_i\rangle$ are eigenvalues and eigenvectors of $\rho$, respectively. Thus, $\text{Tr}[\rho \rho'] = \sum_i p_i \langle \psi_i | \rho' | \psi_i \rangle \geq 0$ due to the facts $p_i \geq 0$ and $\rho' \geq 0$. Then, we have
\[ 0 \leq \text{Tr}[\rho \rho'] = \frac{1}{N} + \frac{rs}{4} \text{Tr} \left[ \left( \hat{r} \cdot \hat{\lambda} \right) \left( \hat{r} \cdot \hat{\lambda} \right) \right] = \frac{1}{N} + \frac{1}{2} rs. \] (4)

This gives Eq. (2) where we have used $\text{Tr}[\lambda_{\mu\nu}] = 0$ and $\text{Tr}[\lambda_{\mu\nu}\lambda_{\mu\nu}] = 2\delta_{\mu\nu}$.

(2). First, suppose $\rho'$ is positive semidefinite, then Eq. (2) gives
\[ s \geq -\sqrt{\frac{2}{N(N-1)}}. \] (5)

Because $\text{Tr}[\rho'] = 1$, the positive semidefiniteness of $\rho'$ means that it is a density matrix and therefore $\text{Tr}[\rho'^2] \leq 1$, from which we can get
\[ s \leq \sqrt{\frac{2(N-1)}{N}}. \] (6)

Equations (5) and (6) give Eq. (3).

Second, if $0 \leq s \leq \sqrt{\frac{2(N-1)}{N}}$, then $\rho' = \frac{1}{N} \mathbb{1} + \frac{1}{2} s \hat{r} \cdot \hat{\lambda}$ is a density matrix. This is because we can always write
\[ \rho' = p \rho + (1-p) \frac{1}{N}, \] (7)

with $0 \leq p \leq 1$ and $s = pr$. On the other hand, from the Observation 4 of Ref. [14] we have that $\rho'$ is a density matrix whenever $|s|^2 \leq \frac{2}{N(N-1)}$, i.e. $-\sqrt{\frac{2}{N(N-1)}} \leq s \leq \sqrt{\frac{2}{N(N-1)}}$. See also Eq. (11) in Ref. [18]. Therefore, we have that if Eq. (3) is satisfied, $\rho'$ is a density matrix and thus positive semidefinite. \qed
By definition, the Werner states satisfy

$$\rho_W = \left( \frac{N - \phi}{N^3 - N} \right) \mathbb{1} \otimes \mathbb{1} + \left( \frac{N \phi - 1}{N^3 - N} \right) V ,$$

(8)

where $\phi$ is an arbitrary number $\in [-1, 1]$, and $V$ is defined by $V |\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle$, $\forall |\psi\rangle, |\phi\rangle \in \mathcal{H}$, the $N$-dimensional Hilbert space. It is known that $\rho_W$ is separable if and only if $\phi \geq 0$ [4]. In the study of entanglement distillation, the Werner states are usually parameterized as [20]

$$\rho_W = \frac{1}{N^2 + \alpha N} \left( \mathbb{1} \otimes \mathbb{1} + \alpha V \right) ,$$

(9)

where $\alpha = \frac{N \phi - 1}{N - \phi} \in [-1, 1]$. For steerability consideration, it is convenient to express Werner states in the following form [5]

$$\rho_W = \left( \frac{N - 1 + \beta}{N^3 - N^2} \right) \mathbb{1} \otimes \mathbb{1} - \left( \frac{\beta}{N^2 - N} \right) V ,$$

(10)

where $\beta = \frac{1 - N \phi}{N + 1} \in \left[ \frac{1 - N}{N + 1}, 1 \right]$. By definition, the isotropic states are those states satisfying $\rho_I = (u \otimes u^*) \rho_I (u^\dagger \otimes u^{*\dagger})$ where $u$ is an arbitrary unitary matrix and the asterisk denotes the complex conjugate [19]. When parameterized in a symmetric form, $\rho_I$ then writes

$$\rho_I = \frac{1 - \eta}{N^2} \mathbb{1} \otimes \mathbb{1} + \eta P_+ .$$

(11)

Here, $\frac{1}{N^2 - 1} \leq \eta \leq 1$ and $P_+ = |\psi_+\rangle \langle \psi_+|$ with $|\psi_+\rangle = \frac{1}{\sqrt{N}} \sum_i |ii\rangle$. The isotropic states turn out to be entangled if and only if $\frac{1}{N + 1} < \eta$ [19], while it will be steerable if and only if $\frac{H_N - 1}{N - 1} < \eta \leq 1$ with $H_N = \sum_{n=1}^N \frac{1}{n}$ the Harmonic number [5].

In Bloch representation, Werner states and isotropic states can be reformulated as [18]

$$\rho_W = \frac{1}{N^2} \mathbb{1} \otimes \mathbb{1} + \frac{1}{4} \sum_{\mu=1}^{N^2 - 1} 2(N\phi - 1) \lambda_\mu \otimes \lambda_\mu ,$$

(12)

and

$$\rho_I = \frac{1}{N^2} \mathbb{1} \otimes \mathbb{1} + \frac{1}{4} \sum_{\mu=1}^{N^2 - 1} 2\eta \lambda_\mu \otimes \lambda_\mu^T ,$$

(13)

respectively, where superscript $T$ signifies the transpose operation of a matrix. To investigate the non-local nature of these two classes of quantum states uniformly, we...
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Fig. 1 The non-localities of Werner and isotropic states. Different types of non-localities emerge with different values of $\tau$ for Werner states $\rho_W$ and isotropic states $\rho_I$, where the solid and hollow circles mean the closed and open interval, respectively, and $H_N$ denotes the $N$-th Harmonic number.

reparameterize the Werner states and isotropic states as

$$\rho_W = \frac{1}{N^2} \mathbb{1} \otimes \mathbb{1} + \frac{1}{4} \sum_{\mu=1}^{N^2-1} \left( \frac{\tau}{N^2-1} \right) \lambda_\mu \otimes \lambda_\mu, \quad (14)$$

$$\rho_I = \frac{1}{N^2} \mathbb{1} \otimes \mathbb{1} + \frac{1}{4} \sum_{\nu=1}^{N^2-1} \left( \frac{\tau}{N^2-1} \right) \lambda_\nu \otimes \lambda_\nu^T. \quad (15)$$

Here, the parameter $\tau$ relates to $\phi$ and $\eta$ in Werner and isotropic states as

$$\rho_W : \quad \tau = \frac{2(N\phi - 1)}{N}; \quad \rho_I : \quad \tau = \frac{2\eta(N^2 - 1)}{N}. \quad (16)$$

According to the existing non-local criteria for $\rho_W$ and $\rho_I$, one may readily find the corresponding conditions for parameter $\tau$. For example, from Eq. (8) we know that for $\rho_W$, the $\phi \in [-1, 1]$ and it is separable if and only $\phi \geq 0$. According to Eq. (16), therefore $\tau \in \left[ -\frac{2(N+1)}{N}, \frac{2(N-1)}{N} \right]$ and $\rho_W$ is separable if and only $\tau \geq -\frac{2}{N}$, see Fig. 1. One may also translate the steerable condition for parameter $\beta$ of Eq. (10) in Ref. [5] to the condition of $\tau$. Similar procedure applies to isotropic states as well, and we will eventually have Fig. 1.
3 The separable decomposition of Werner and isotropic states

3.1 General decomposition in regular simplex

A set of \((N^2 - 1)\)-dimensional unit real vectors \(\mathcal{A} = \{\vec{a}_i|i = 1, \cdots, N^2; |\vec{a}_i| = 1\}\) form a regular simplex, \(N^2\)-simplex, if

\[
\vec{a}_i \cdot \vec{a}_j = -\frac{1}{N^2 - 1}, \forall i \neq j.
\]  

For Werner and isotropic states, the following fact holds:

**Lemma 2** For arbitrary \(N^2\)-simplex \(\mathcal{A}\) in dimension \(N^2 - 1\), the Werner state \(\rho_W\) can always be decomposed as

\[
\rho_W = \sum_{i=1}^{N^2} \frac{1}{N^2} R_i(r) \otimes S_i(s) \tag{18}
\]

with \(rs = \tau\), and

\[
R_i(r) = \frac{1}{N} \mathbb{1} + \frac{r}{2} \vec{a}_i \cdot \vec{a}_i, \quad S_i(s) = \frac{1}{N} \mathbb{1} + \frac{s}{2} \vec{a}_i \cdot \vec{a}_i, \tag{19}
\]

are two trace-one Hermitian matrices whose Bloch vectors form two \(N^2\)-simplexes of size \(r\) and \(s\), respectively. For isotropic states, the similar decomposition exists:

\[
\rho_I = \sum_{i=1}^{N^2} \frac{1}{N^2} R_i(r) \otimes S_i^T(s). \tag{20}
\]

Note here \(R_i(r)\) and \(S_i(s)\) in Eqs. (18) and (20) may not be positive semidefinite.

**Proof** For \(\vec{a}_i \in \mathcal{A}\), we may construct an \(N^2 \times N^2\) matrix

\[
O = \sqrt{\frac{N^2 - 1}{N^2}} \begin{pmatrix}
\vec{a}_1^T \\
\vec{a}_2^T \\
\vdots \\
\vec{a}_{N^2}^T
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\sqrt{N^2-1}} \\
\frac{\vec{a}_1}{\sqrt{N^2-1}} \\
\vdots \\
\frac{\vec{a}_{N^2}}{\sqrt{N^2-1}}
\end{pmatrix}, \tag{21}
\]

where \(\vec{a}_i\) are \((N^2 - 1)\)-dimensional column vectors. It is easy to verify \(OO^T = \mathbb{1}\) considering of the condition Eq. (17):

\[
OO^T = \frac{N^2 - 1}{N^2} \begin{pmatrix}
\vec{a}_1^T & \vec{a}_2^T & \cdots & \vec{a}_{N^2}^T
\end{pmatrix} \begin{pmatrix}
\frac{1}{\sqrt{N^2-1}} \\
\frac{\vec{a}_1}{\sqrt{N^2-1}} \\
\vdots \\
\frac{\vec{a}_{N^2}}{\sqrt{N^2-1}}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\sqrt{N^2-1}} \\
\frac{\vec{a}_1}{\sqrt{N^2-1}} \\
\vdots \\
\frac{\vec{a}_{N^2}}{\sqrt{N^2-1}}
\end{pmatrix}.
\[
\begin{align*}
N^2 - 1 & = \frac{N^2 - 1}{N^2} \left( \sum_{i=1}^{N^2} \bar{a}_i \bar{a}_i^T \left| \sum_{j=1}^{N^2} \bar{a}_j \right| \sqrt{N^2 - 1} \right) \\
& \quad \left( \sum_{k=1}^{N^2} \bar{a}_k \right) \sqrt{N^2 - 1} \\
& = 1.
\end{align*}
\] (22)

Here, \( N^2 - 1 \sum_{i=1}^{N^2} \bar{a}_i \bar{a}_i^T = 1 \) is an \((N^2 - 1) \times (N^2 - 1)\) identity matrix and \( \sum_{i=1}^{N^2} \bar{a}_i = 0 \) \( (\sum_{i=1}^{N^2} \bar{a}_i^T = 0) \). We notice that Eq. (18) can be expressed as

\[
\rho_W = \frac{1}{N^2} \mathbb{1} \otimes \mathbb{1} + \frac{r + s}{2N} \left( \sum_{i=1}^{N^2} \bar{a}_i \right) \cdot \bar{\lambda} + \frac{r s}{4N^2} \sum_{i=1}^{N^2} \bar{a}_i \cdot \bar{\lambda} \otimes \bar{\lambda} \\
= \frac{1}{N^2} \mathbb{1} \otimes \mathbb{1} + \frac{r s}{4N^2} \sum_{v, \mu=1}^{N^2-1} \sum_{i=1}^{N^2} a_{i \mu} a_{i \nu} \lambda_{\mu} \otimes \lambda_{\nu},
\] (23)

where \( a_{i \mu} \) are the components of \( \bar{a}_i \). Define a matrix \( T \) whose elements \( T_{\mu \nu} = \sum_{i=1}^{N^2} a_{i \mu} a_{i \nu} \), then the matrix \( T = \sum_{i=1}^{N^2} \bar{a}_i \bar{a}_i^T = \frac{N^2}{N^2 - 1} \mathbb{1} \) according to Eq. (22). Furthermore, Eq. (23) can be reformulated as

\[
\rho_W = \frac{1}{N^2} \mathbb{1} \otimes \mathbb{1} + \frac{r s}{4N^2} \sum_{v, \mu=1}^{N^2-1} T_{\mu \nu} \lambda_{\mu} \otimes \lambda_{\nu} \\
= \frac{1}{N^2} \mathbb{1} \otimes \mathbb{1} + \frac{r s}{4} \sum_{\mu=1}^{N^2-1} \left( \frac{r s}{N^2 - 1} \right) \lambda_{\mu} \otimes \lambda_{\mu}.
\] (24)

Here, \( r s \) is identified as \( \tau \), and we have used \( T_{\mu \nu} = \frac{N^2}{N^2 - 1} \delta_{\mu \nu} \) for diagonal matrix \( T \). Equation (24) tells that Eq. (18) is indeed a decomposition of Eq. (14).

For the isotropic states, Eq. (20) gives

\[
\rho_I = \frac{1}{N^2} \mathbb{1} + \frac{r s}{4N^2} \sum_{i=1}^{N^2} \bar{a}_i \cdot \bar{\lambda} \otimes \bar{a}_i \cdot \bar{\lambda}^T \\
= \frac{1}{N^2} \mathbb{1} \otimes \mathbb{1} + \frac{1}{4} \sum_{\mu=1}^{N^2-1} \left( \frac{r s}{N^2 - 1} \right) \lambda_{\mu} \otimes \lambda_{\mu}^T,
\] (25)

where we have used the same matrix \( T \) as that in Eq. (24), and \( r s \) is identified as \( \tau \). Hence, Eq. (25) tells that Eq. (20) is a decomposition of Eq. (15).
Fig. 2 The separable decomposition of Werner states. An infinite number of decompositions for each \( \tau \in [- \frac{2}{N}, \frac{2(N-1)}{N}] \) exist, except the endpoints of \(- \frac{2}{N}\) and \(\frac{2(N-1)}{N}\). The decompositions are achieved by the values of \(r\) and \(s\) fitting the contour \(rs = \tau\).

### 3.2 Separable decomposition in SIC-POVM

Above, the general separable decompositions of the Werner and isotropic states are given; however, they are not all semipositive. To get explicit separable decompositions, we employ the SIC-POVM formalism.

A SIC-POVM in \(N\)-dimensional Hilbert space is represented by a set of \(N^2\) vectors \(|\psi_i\rangle\) satisfying [15]

\[
|\langle \psi_i | \psi_j \rangle|^2 = \frac{N \delta_{ij} + 1}{N + 1}, \quad i, j \in \{1, \cdots, N^2\}.
\]  

(26)

Let \(\vec{r}_i\) and \(\vec{r}_j\) be Bloch vectors of pure states \(\rho_i = |\psi_i\rangle\langle \psi_i|\) and \(\rho_j = |\psi_j\rangle\langle \psi_j|\), respectively, we have

\[
|\langle \psi_i | \psi_j \rangle|^2 = \text{Tr}[\rho_i \rho_j] = \text{Tr} \left[ \left( \frac{1}{N} \mathbb{1} + \frac{1}{2} \vec{r}_i \cdot \vec{\lambda} \right) \left( \frac{1}{N} \mathbb{1} + \frac{1}{2} \vec{r}_j \cdot \vec{\lambda} \right) \right] = \frac{1}{N} + \frac{\vec{r}_i \cdot \vec{r}_j}{N + 1}, \quad \forall i \neq j.
\]  

(27)

For pure states with \(|\vec{r}_i| = |\vec{r}_j| = \sqrt{2(N-1)/N}\), Eq. (27) predicts

\[
\frac{\vec{r}_i \cdot \vec{r}_j}{|\vec{r}_i||\vec{r}_j|} = -\frac{1}{N^2 - 1}, \quad \forall i \neq j.
\]  

(28)
Comparing with Eq. (17), we have that the \( N^2 \) unit vectors \( \vec{r}_i/|\vec{r}_i| \) from SIC-POVM configure an \( N^2 \)-simplex in the \((N^2 - 1)\)-dimensional Bloch vector space. Therefore, we have the following theorem

**Theorem 1** If there exists a SIC-POVM in the \( N \)-dimensional Hilbert space, the separable Werner states, i.e. \(-\frac{2}{N} \leq \tau \leq \frac{2(N-1)}{N}\), can always be decomposed as

\[
\rho_W = \sum_{i=1}^{N^2} \frac{1}{N^2} R_i(r) \otimes S_i(s)
\]

with

\[
R_i(r) = \frac{1}{N} \mathbb{1} + \frac{r}{2|\vec{r}_i|} \vec{r}_i \cdot \vec{\lambda}, \quad S_i(s) = \frac{1}{N} \mathbb{1} + \frac{s}{2|\vec{r}_i|} \vec{r}_i \cdot \vec{\lambda}.
\]

Here, \( \{\vec{r}_i\} \) are Bloch vectors, \( \tau = rs \in \left[ -\frac{2}{N(N-1)} , \frac{2(N-1)}{N} \right] \). 

**Proof** In Lemma 2, we have proved that the Werner and isotropic states can be decomposed by \( R_i(r) \) and \( S_i(s) \), whose Bloch vectors correspond to a regular simplex. Thus, here we only need to prove the existence of some semidefinite positive matrices \( R_i(r) \) and \( S_i(s) \) for the separable Werner and isotropic states. This in fact is obvious by noticing that \( \{\vec{r}_i\} \) are Bloch vectors corresponding to a SIC-POVM. That is, \( \{\vec{r}_i\} \) are Bloch vectors of pure states. In light of Lemma 1, \( r, s \in \left[ -\frac{2}{N(N-1)} , \frac{2(N-1)}{N} \right] \) correspond to positive semidefinite density matrices.  

The result of Theorem 1 is exhibited in Fig. 2, where every point on the contour \( rs = \tau \) provides a decomposition form for the state with parameter \( \tau \). Analogously, considering the parameter space for isotropic states in Fig. 1 and the relation between Eqs. (14) and (15), the separable decomposition for isotropic states would be

\[
\rho_I = \sum_{i=1}^{N^2} \frac{1}{N^2} \left( \frac{1}{N} \mathbb{1} + \frac{r}{2|\vec{r}_i|} \vec{r}_i \cdot \vec{\lambda} \right) \otimes \left( \frac{1}{N} \mathbb{1} + \frac{s}{2|\vec{r}_i|} \vec{r}_i \cdot \vec{\lambda}^T \right).
\]

Consequently, in this way all the separable Werner and isotropic states are decomposed into products of local density matrices. To write out the decompositions, we need to know the \( N^2 \) vectors \(|\psi_i\rangle\) in SIC-POVM. The explicit construction of SIC-POVM in lower-dimensional space can be found in Refs. [21,22].

As mentioned above, the decomposition is generally not unique, i.e. for a given \( \tau \) every point on the contour \( rs = \tau \) in Fig. 2 corresponds to a separable decomposition. Every Bloch vector of the decomposed local density matrix matches to a SIC-POVM in \( N \)-dimensional Hilbert space. Note that though the existence of SIC-POVM in arbitrary dimension is still an open question, we find the following dimensions may yield definite solutions

\[ N = 2, 3, \ldots, 24, 28, 30, 31, 35, 37, 39, 43, 48, 124, \]
Fig. 3  The non-localities of Werner and isotropic states in large $N$ limit. Asymptotically, when dimension $N$ approaches to infinite, the Werner states $\rho_W$ tend to be either separable or non-steerably entangled, while nearly all isotropic states $\rho_I$ are steerable.

Readers may refer to Ref. [23,24] for a recent review on this point.

The parameterization of Werner and isotropic states with single parameter $\tau$ enables us to study the asymptotic behaviour in high dimension $N$, see Fig. 3. When dimension $N$ goes to infinity, in the parameter region of the Werner states, there would be one half of separable states (the upper green region in Fig. 3a), and one half of non-steerable entangled states (the lower yellow region in Fig. 3a), while for isotropic states, the separable and non-steerable entangled states (green and yellow regions in Fig. 3b) both become negligible in comparison with the steerable states (dark yellow regions in Fig. 3b). The relative amount of the different types of non-locality may be well understood through the parametrization shown in Fig. 1.

4 Summary

In this paper, we propose a novel scheme for the decomposition of all separable Werner and isotropic states in any dimension. To this aim, we first associate the decomposition of the separable Werner states with the existence of SIC-POVM. And then by taking advantage of the largest regular simplex in the Bloch vector space, the separable Werner states and isotropic states are decomposed. With the unified parametrization scheme in Bloch vector representation, the Werner states are shown to be relatively either separable or non-steerably entangled in large dimensions, asymptotically, while the isotropic states tend considerably to steerable states. As the Werner states are important models in the studies of quantum non-locality and quantum information processing [5–9], the general and separable decompositions of them may provide useful tools to analyse how the entangled states perform in different quantum information tasks.

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