Entanglement in model independent cosmological scenario

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We propose a \textit{model-independent} approach to study entanglement creation due to the dynamics between two asymptotic quantum regimes in the framework of homogeneous and isotropic universe. We realize it by Padé expansion to reconstruct the functional form of scale factor, rather than postulating it a priori. This amounts to fix the Padé approximants constraining the free parameters in terms of current cosmic observations. Assuming fermions, we solve the Dirac equation for massive particles and we investigate entanglement entropy in terms of modes $k$ and mass $m$. We consider two rational approximations of (1,1) and (1,2) orders, which turn out to be the most suitable choices for guaranteeing cosmic bounds. Our results show qualitative agreement with those known in literature and arising from toy models, but with sensible quantitative discrepancies. Moreover, our outcomes are model independent reconstructions which show that any higher orders departing from (1,1) do not significantly modify particle-antiparticle production (hence entanglement), if cosmic bounds are taken into account.

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I. INTRODUCTION

Quantum correlations, also known as \textit{entanglement}, captured renewed attention after the development of quantum theory of information \textsuperscript{[1]}. Recently it has been realized that entanglement also arises in cosmological scenarios \textsuperscript{[2]}. It is actually related to the mechanism of particle-antiparticle production during cosmic evolution, a phenomenon pointed out some time ago \textsuperscript{[3]}. To ascertain that, given the difficulties in solving the dynamics, especially for what concern the Dirac field, it is often resorted to cosmological toy models.

Widely employed models are due to Duncan \textsuperscript{[4]} and Birrell and Davies \textsuperscript{[5]}, where the universe is viewed as a dynamical system obeying the hypothesis of homogeneity and isotropy, i.e. satisfying the Friedmann-Robertson-Walker (FRW) line element $ds^2 = d\tau^2 - a(\tau)^2 \left[dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)\right]$. The scale factor $a(t)$ was devised to simply show an accelerated expansion phase of the universe between two stationary asymptotic regions, corresponding to the early and the late universe.

Nowadays the most accredited paradigm is the cosmological concordance model, named $\Lambda$CDM \textsuperscript{[6]}. There, the component responsible for the cosmic speed up is the cosmological constant $\Lambda$ and manifests a negative equation of state \textsuperscript{[7–9]} providing gravitational repulsive effects\textsuperscript{1}. Thus in order to study entanglement there is a need to elaborate a \textit{model-independent} approach that formulate the scale factor evolution within the concordance paradigm, since the term pushing the universe up is still undetermined. Any model-independent treatment needs to leave unset the free parameters that can be bounded from experimental observations to agree with both late and early dynamics \textsuperscript{[11]}.

In this work, we propose the Padé rational expansions \textsuperscript{[12] [13]} as \textit{model-independent} reconstruction for the scale factor $a(t)$ in order to analyze the entanglement production. The use of Padé approximations is motivated by the fact that they represent the most suitable way to build up a high order convergent series on $a(t)$ which converges at both late and early stages \textsuperscript{[14] [18]}. We show that the involved polynomials might be rational. They well adapt to describe the universe without assuming the functional form of the Hubble parameter at arbitrary times with two peculiar expansion orders, namely (1,1) and (1,2). We fix the free parameters in terms of modern observations. We thus use the Padé polynomials to study the entanglement production.

Actually we consider quantum states associated to matter at two epochs: the first concerning earliest phases, i.e. before expansion, whereas the second as expansion is over, i.e. at the future. Then we introduce the dynamical process

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\textsuperscript{1}Alternatives to the concordance model are commonly named \textit{dark energy} scenarios, in which the equation of state is a varying function.
in a way that the functional form of \( a(t) \) is reconstructed from recent kinematic data coming from cosmography, without postulating the model \textit{a priori}. We solve the Dirac equation for massive particles and we compute the Bogoliubov transformations between the two involved epochs. We evaluate the particle-antiparticle entanglement as subsystem (particle) entropy in terms of mode \( k \) and mass \( m \).

Particularly, models of Refs. \[4,5\] are featured by introducing \textit{ad hoc} scale factors which do not match any cosmological requirements. The models depend upon free constants, which have been introduced to stress the expansion rate and the volume variation. The Padé expansions do not fix \textit{a priori} the number of free parameters. Although the set of free parameters may be in principle larger, each term simply corresponds to a series coefficient and is not imposed by hand.

In particular, we recover the Duncan’s model [4] from (1,1) Padé approximant. Further, the entanglement obtainable from it (see e.g. [19]) is in good agreement with current cosmic observations only for small and high values of \( k \). For intermediate and large values of \( k \), higher order Padé approximants do not significantly modify the entanglement production got from (1,1) approximation.

The paper is structured as follows. In Sec. II, we present our model-independent approach based on Padé approximations for the universe dynamics. Actually we show how to construct a viable approximation to \( a(t) \) motivating our choice through the use of recent developments in the literature. In Sec. III, we describe the simplest solution to \( a(t) \) as function of Padé polynomials. Using it we derive analytical solutions of Dirac equation in terms of Hypergeometric functions of second kinds. In turn this allows us to compute the entanglement production. In Sec. IV we consider higher order solution to \( a(t) \) as function of Padé polynomials and compute numerically the entanglement production. We give also a detailed interpretation of the so-obtained results. Finally, we report conclusions and perspectives in Sec. V.

II. THE PADÉ APPROXIMATION AS MODEL-INDEPENDENT TECHNIQUE

The simplest approach in defining rational approximations to the scale factor is offered by the Padé series [12,13]. The basic demand over the Padé construction is to overcome the model-dependence issue on \( a(\tau) \) as one postulates the functional time-dependence of \( a(\tau) \). Indeed, expanding in Taylor series around a precise time, say \( \tau_* \), leads to powers of \( \tau \) which are defined only around \( \tau \approx \tau_* \). In other words, the \( a(\tau) \) Taylor series, taking \( \tau_* \) as reference time, gives:

\[
a(\tau) \equiv 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{d^k a}{dt^k}|_{\tau=\tau_*} (\tau - \tau_*)^k,
\]

where the coefficients are named \textit{cosmographic parameters}. These coefficients are well defined as they are fixed at \( \tau = \tau_* \) and give information on the universe dynamics at large scales. Indeed, the advantage of expanding \( a(\tau) \) is that the cosmographic parameters may be directly measured with cosmic data, i.e. without fixing \textit{a priori} the cosmological model. Approaches toward the definition of the cosmographic series and bounds over it have been extensively investigated in the literature [20,21].

From a pure experimental point of view, this procedure if compared directly with data, introduces errors into the analysis since the employed formulae only represent approximations to the true expressions. Moderating the caveat may involve higher orders of the Taylor series, but this comes at the expense of introducing more fitting parameters. Furthermore the flat-to-flat behavior is not preserved. This would considerably complicate the corresponding statistical analysis [22].

To go beyond the expected convergence range of Taylor series\(^3\), one can shift to the Padé series. Let us recall that for a generic function \( f(z) \) the Padé approximant with fixed orders \((m,n)\) is defined as:

\[
P_{mn}(z) = \frac{a_0 + a_1 z + a_2 z^2 + \ldots + a_m z^m}{1 + b_1 z + b_2 z^2 + \ldots + b_n z^n}.
\]

A more general \((M,N)\) Padé approximant may be written under the form:

\[
P_{M,N}(z) = \frac{\sum_{m=0}^{M} a_m Z^m}{1 + \sum_{n=1}^{N} b_n Z^n},
\]

\(^2\) In observational cosmology, it is customary to take \( \tau_* \) as our time.

\(^3\) This problem is known in the literature as convergence problem [23].
where \( Z = Z(z) \) is an arbitrary function of the independent variable (for example the redshift \( z \)).

The aforementioned definition of Padé series is here used to match cosmic rulers, assuming that Padé expansions are equivalent to the standard Taylor series up to the highest possible orders. To do so, we require

\[
P_{M,N}(0) = f(0), \quad P'_{M,N}(0) = f'(0), \quad \vdots \quad P^{(M+N)}_{M,N}(0) = f^{(M+N)}(0).
\]

In principle the net number of total unknown terms is \( M + N + 1 \), defined as the sum between the \( M + 1 \) coefficients of the numerator and \( N \) terms of the denominator. In turn, we have:

\[
\sum_{i=0}^{\infty} c_i z^i = \sum_{m=0}^{M} a_m z^m + O(z^{M+N+1})
\]

and then

\[
(1 + b_1 z + \ldots + b_N z^N)(c_0 + c_1 z + \ldots) = a_0 + a_1 z + \ldots + a_M z^M + O(z^{M+N+1}).
\]

Plugging together the terms with the same power, one gets a set of \( M + N + 1 \) equations for any \( M + N + 1 \) unknown coefficients \( a_i \) and \( b_i \). For \( z \gg 1 \) the series converges and thus it candidates to overcome Taylor’ series divergencies.

Following the standard recipe of Padé polynomials the most suitable set of exponents requires that:

1. Padé series behaves smoothly with high redshift cosmic data;
2. Padé series minimize systematics in approximating the universe expansion history;
3. orders \( M, N \) might be comparable to converge to constants asymptotically;
4. convergence is calibrated \textit{a posteriori} by means of data surveys;
5. coefficients are fixed to avoid possible poles;
6. Padé series might agree with previous approaches\(^4\).

We are interested in approximating \textit{directly} the scale factor \( a(\tau) \) using the Padé series, in agreement with current cosmological bounds. Thus, the simplest orders that seem to be viable for approximating \( a(\tau) \) are \((1, 1)\) and \((1, 2)\) Padé series respectively. These two orders well adapt to the six conditions above described.

In our scenario, the Padé approximation will be built up in terms of a positive-definite functional \( t(\tau) = -\ln \tau \). The functional dependence is chosen to match time \( t \in (-\infty, +\infty) \) with time \( \tau \in [0, +\infty) \), so to respect the fact that \( \tau_0 = 0 \) is current time, while \( \tau = +\infty \) is the remote past. Moreover, one can notice that \( \tau_* \) corresponds to the cosmographic time at which the scale factor is \textit{conventionally} normalized to \( a(\tau_*) \neq 1 \). In addition, it is also demanded the stationarity of the spacetime for \( t \to \pm \infty \) as in Refs. \cite{4} and \cite{5}. This enables to have two natural quantizations of the field associated with two Fock spaces \cite{24}.

\section*{III. ENTANGLEMENT FROM THE SCALE FACTOR (1, 1) PADÉ APPROXIMANT}

In the previous section, we introduced the main reasons behind the choice of Padé expansions instead of using the standard Taylor approach. We highlight that the smallest orders of Padé polynomials, namely the \((1, 1)\) and \((1, 2)\), as applied to luminosity distance \( d_L \) significantly improve the convergence radius. Hence, we can start adopting the expansion

\[
a_{(1,1)}(\tau) = \frac{\beta_0 + \beta_1 \tau}{1 + \beta_2 \tau}.
\]

\(^4\) This gives a robust physical explanation to the toy model discussed in \cite{4}.
In terms of $t$ we have

\begin{equation}
\begin{aligned}
a_{(1,1)}(t = -\infty) &= \frac{\beta_1}{\beta_2}, \\
a_{(1,1)}(t = +\infty) &= \beta_0,
\end{aligned}
\end{equation}

as well as $da_{(1,1)}/dt|_{t=-\infty} = da_{(1,1)}/dt|_{t=+\infty} = 0$.

Physically speaking one can assume the Big Bang time, as the one at which quantum effects are remarkable. However, the choice of this value does not influence our approach. Indeed, it only gives a shift to the weight and strength of all curves and functions that we are working with, leaving unaltered the physical properties behind the choice of our Padé formalism. As a matter of fact, we note that Eqs. (10) represent conditions which work well only in certain epochs of universe’s evolution. They cannot be used for the whole large scale dynamics.

Below we shall consider matter field $\psi$ (of mass $m$ and spin $\frac{1}{2}$) in 1+1 spacetime. In the far future we assume a Dirac field, with spins $\frac{1}{2}$ associated to each particle. A simple and immediate request is the existence of a Hilbert space both in the past flat region ($t = -\infty$) and in the future flat epoch ($t = +\infty$), with a suitable choice concerning basis vectors for each era.

A. Limits over (1,1) Padé expansion of $a(t)$

We need to fix the parameters in (9) in the most suitable way, i.e. to agree with observational properties associated to scale factor evolution at late and early times. In particular, computing the kinematics of our model, one immediately finds that $H$ and $q$ read:

\begin{equation}
\begin{aligned}
H &= \frac{\dot{a}}{a} \quad q = -1 - \frac{\dot{H}}{H^2},
\end{aligned}
\end{equation}

one immediately finds that $H$ and $q$ read:

\begin{equation}
\begin{aligned}
H_0 &= \frac{\beta_0}{\beta_0 + \beta_1} - \frac{1}{1 + \beta_2},
q_0 &= -\left(\frac{(\beta_0 + \beta_1)(\beta_2 - 1)}{\beta_0 \beta_2 - \beta_1}\right),
\end{aligned}
\end{equation}

where the subscript 0 refers to $H$ and $q$ as computed at current time. By virtue of Eqs. (12), one can remove some arbitrariness on the constants, requiring that

\begin{equation}
\begin{aligned}
\beta_0 &> 0, \\
\beta_1 &\neq \beta_2.
\end{aligned}
\end{equation}

The first condition, i.e. $\beta_0 > 0$, states that the universe’s volume increases as byproduct of the standard Big Bang model. The second request over $\beta_1$ and $\beta_2$ is less stringent and represents our arbitrariness to frame how cosmic expansion rate behaves far from asymptotic regimes. In fact, asymptotically it is licit to relax the second condition, having a non-accelerating (but expanding) universe in the limits $\tau \to 0$ and $\tau \to \infty$, while it is not licit to take $\beta_0 < 1$ even asymptotically. Bearing in mind these bounds, without losing generality we require $\beta_1$ and $\beta_2$ to be positive-definite in general and $\beta_0 > 1$. Moreover to guaranteeing the physical robustness of Eqs. (12) we take the range of values satisfying the condition

\begin{equation}
\beta_0 \beta_2 > \beta_1.
\end{equation}

The request (15) over $\beta_0, \beta_1$ and $\beta_2$ implies a degeneracy among the coefficients. Thus a mathematical trick useful to characterize the scale factor evolution at large scales can be based on recasting the constants in Eq. (9). In particular, if we consider

\begin{equation}
\beta_2 e^{-t} \to e^{-T},
\end{equation}

we have $t = T + \ln \beta_2$ and so $a_{1,1}(t)$ transforms as

\begin{equation}
a_{1,1}(T) = \frac{\beta_0 + \beta'e^{-T}}{1 + e^{-T}}.
\end{equation}

This choice disguises $\beta_1$ and $\beta_2$ through a single constant, namely $\beta' \equiv \beta_1/\beta_2$. Notice that the derivatives with respect to $t$ are equivalent to the derivatives with respect to $T$. Furthermore, the asymptotic limits on $t$ are the same on $T$, i.e. as $t \to \infty, T \to \infty$. Hence below we can interchange $T$ with $t$ without affecting the final outcomes.
B. Getting solutions for Dirac’s field

On curved space-time, the Dirac equation for the field $\psi$ of mass $m$ reads:

$$\left[\tilde{\gamma}^{\mu} D_{\mu} + m\right] \psi = 0,$$

(18)

where, working on a FRW space-time, we defined $\tilde{\gamma}^{\mu} \equiv [a(t)]^{-1} \gamma^{\mu}$ that are the re-scaled $2 \times 2$ spinor matrices and $D_{\mu}$ the covariant derivative. Employing the auxiliary field $\varphi$, we may look for solutions under the form [27]:

$$\psi = a^{-1/2}(\gamma^\nu \partial_\nu - M)\varphi,$$

(19)

with effective mass given by: $M = ma(t)$. So that, recasting the Dirac equation (18) by

$$\Box \varphi - \gamma^0 M \varphi - M^2 \varphi = 0,$$

(20)

where $\Box \equiv g_{\mu\nu} \partial^\mu \partial^\nu$, we get the corresponding solutions as:

$$\varphi(-) \equiv N(-) f(-)(t) e^{ikx},$$

(21)

$$\varphi(+) \equiv N(+) f(+) (t) e^{ikx},$$

(22)

with $u, v$ the flat spinors and $k$ the momentum. After some algebra, one can get the differential equation for the time dependent functions $f^{(\pm)}$:

$$\ddot{f}^{(\pm)} + \left( k^2 + M^2 \pm i\dot{M} \right) f^{(\pm)} = 0.$$

(23)

Let us now define $f^{(\pm)}_{in/out}$ and $f^{(\pm)*}_{in/out}$ the solutions behaving as positive and negative frequency modes with respect to time $t$ near the asymptotic past/future, i.e.

$$f^{(\pm)}_{in/out}(t) \approx -iE_{in/out} f^{(\pm)}_{in/out}(t),$$

(24)

where

$$E_{in/out} \equiv \sqrt{k^2 + M^2_{in/out}},$$

(25a)

$$M_{in/out} \equiv ma(t \to -\infty/ + \infty).$$

(25b)

Then, from Eq. (24), we have

$$f^{(\pm)}_{in}(t) = e^{i\delta t} \mathcal{F}_1 \left[ \theta^\pm_1, \theta^\pm_2, \theta^\pm_3, \ell_1(t) \right],$$

(26a)

$$f^{(\pm)}_{out}(t) = e^{i\delta t} \mathcal{F}_1 \left[ \theta^\pm_1, \theta^\pm_2, \theta^\pm_3, \ell_2(t) \right].$$

(26b)

The coefficients $\theta^\pm_1, \theta^\pm_2$ and $\theta^\pm_3$ are defined as

$$\theta^\pm_1 \equiv 1 + i \left[ 2E_{\pm} \pm m(\beta_0 - \beta') \right],$$

(27a)

$$\theta^\pm_2 \equiv \left[ 2E_{\pm} \pm m(\beta_0 - \beta') \right],$$

(27b)

$$\theta^\pm_3 \equiv 1 \pm 2i E_{out/in},$$

(27c)

with

$$E_{\pm} \equiv \frac{1}{2} (E_{out} \pm E_{in}).$$

(28)

Furthermore it is $\ell_2 = 1 - \ell_1$ with

$$\ell_1 = \frac{\exp \left[ \frac{t}{2} \right]}{\exp \left[ \frac{t}{2} \right] + \exp \left[ -\frac{t}{2} \right]},$$

(29)
Going back to (26), the function $\delta$ depends in principle upon the whole set of parameters defined in Eqs. (9) and (25), i.e. $\delta = \delta(E_{\text{out}}, E_{\text{in}}, \beta_0, \beta')$. Cumbersome algebra shows it does not depend on $\beta_0$ and $\beta'$ since it turns out to be a phase factor. Indeed it reads

$$
\delta \equiv E_+ t + 2E_- \ln \left(2 \cosh \left(t/2 \right) \right).
$$

Our solutions, $f_{\text{in}}^{(\pm)}(t)$ and $f_{\text{out}}^{(\pm)}(t)$, are written in terms of hypergeometric functions of the second kind, i.e. $2F_1$, that satisfy

$$
2F_1(\theta_1^\pm, \theta_2^\pm, \theta_3^\pm, \ell(t)) \approx 1 + F_{1,0}\ell(t) + F_{2,0}\ell(t)^2,
$$

with $F_{1,0} \equiv \frac{\theta_3^+\theta_3^-}{\theta_2^+}$ and $F_{2,0} \equiv \frac{\theta_3^+\theta_3^-\theta_2^+\theta_2^-}{2\theta_3^+\theta_3^-\theta_2^+\theta_2^-}$, as $\ell(t) \ll 1$.

### C. Particle-antiparticle production

As one expands the field $\psi$ over spinors in input and output regions, it is possible to relate the coefficients of such expansions, namely the in-out ladder operators for particles and antiparticles (denoted by $a$, $a^\dagger$ and $b$, $b^\dagger$ respectively), by Bogolubov transformations [4]:

$$
a_{\text{out}}(k) = \alpha(k)a_{\text{in}}(k) - \beta(k)b_{\text{in}}^\dagger(-k),
$$

$$
b_{\text{out}}^\dagger(-k) = \beta^*(k)a_{\text{in}}(k) + \alpha^*(k)b_{\text{in}}^\dagger(-k),
$$

with coefficients satisfying $|\alpha|^2 + |\beta|^2 = 1$ and $\alpha \beta^* - \alpha^* \beta = 0$.

Analogously, Bogolubov transformations interconnect the solutions $f_{\text{in/out}}^{(\pm)}$ giving

$$
f_{\text{in}}^{(\pm)}(t) = A^{(\pm)}(k)f_{\text{out}}^{(\pm)}(t) + B^{(\pm)}(k)f_{\text{out}}^{(\mp)*}(t).
$$

Clearly the coefficients $A^{(\pm)}$, $B^{(\pm)}$ are related to $\alpha$, $\beta$, in particular it results [28]

$$
|\alpha(k)|^2 = \frac{E_{\text{out}}}{E_{\text{in}}} \left(\frac{E_{\text{in}} - M_{\text{in}}}{E_{\text{out}} - M_{\text{out}}} \right) |A^{(-)}(k)|^2.
$$

Additionally the Bogolubov coefficients can be related to particle-antiparticle production by

$$
|\beta(k)|^2 = 1 - |\alpha(k)|^2 \equiv \frac{n(k)}{2},
$$

where $n$ is the density of particles per mode at the output ($0 \leq n \leq 2$ with the maximum accounting for the two possible spin orientations). We rewrite Eq. (33) by

$$
2F_1(\theta_1^+, \theta_2^+, \theta_3^+, \ell_1) = A^{(-)}2F_1(\theta_1^-, \theta_2^+, \theta_3^+, \ell_2) + B^{(-)}e^{2i\delta}2F_1(\theta_1^+, \theta_2^-, \theta_3^+, \ell_2)^*.
$$

and furthermore we have

$$
2F_1(\theta_1^-, \theta_2^+, \theta_3^-, \ell_1) = A^{(-)}2F_1(\theta_1^-, \theta_2^+, \theta_2^+ + \theta_3^- + 1, \ell_2) + B^{(-)}\ell_2^{\theta_2^+ - \theta_2^-}\ell_1^{\theta_1^- - \theta_2^-}2F_1(1 + \theta_2^+, \theta_2^+, 1, \theta_1^- + \theta_2^+ - \theta_3^- + 1, \ell_2)^*.
$$

Hence, given that $\ell_1 \in [0, 1]$, we can use the property of hypergeometric functions of second kind

$$
2F_1(\theta_1^+, \theta_2^+, \theta_3^+, \ell_1) = A^{(-)}2F_1(\theta_1^+, \theta_2^+, \theta_1^- + \theta_2^+ - \theta_3^- + 1, \ell_2) + B^{(-)}\ell_2^{\theta_2^+ - \theta_2^-}\ell_1^{\theta_1^- - \theta_2^-}2F_1\left[1 + \theta_2^+, \theta_1^+, \theta_1^- + \theta_2^+ - \theta_3^- + 1, \ell_2(t) \right],
$$

which is compatible with Eq. (31) and comes from the fact that $2F_1(\theta_1^+, \theta_2^+, \theta_1^-, \ell_1)^* = 2F_1(\theta_1^+, \theta_2^+, \theta_3^-, \ell_1)$ in the interval $\ell_1 \in [0, 1]$. 

Adopting the symmetry of $2\mathcal{F}_1$ with respect to the exchange of the first two arguments we get

$$2\mathcal{F}_1(\theta_1^-, \theta_2^+, \theta_3^-, \ell_1) = A(-) 2\mathcal{F}_1(\theta_1^+, \theta_2^+, \theta_3^-, \theta_1^- + 1, \ell_2) + B(-) \ell_1^{\theta_2^- - \theta_1^-} \ell_1^{-\theta_2^+} 2\mathcal{F}_1(1 - \theta_1^-, 1 - \theta_2^+, \theta_3^-, \theta_1^- - \theta_2^+ + 1, \ell_2).$$

Moreover since

$$\ell_1^{-\theta_2^-} 2\mathcal{F}_1(1 - \theta_1^-, 1 - \theta_2^+, \theta_3^-, \theta_1^- - \theta_2^+ + 1, \ell_2) = 2\mathcal{F}_1(\theta_3^-, \theta_3^-, \theta_2^+, \theta_3^-, \theta_1^- - \theta_2^+ + 1, \ell_2),$$

it is licit to write

$$2\mathcal{F}_1(\theta_1^-, \theta_2^+, \theta_3^-, \ell_1) = A(-) F(\theta_1^+, \theta_2^+, \theta_1^- + 1, \ell_2) + B(-) \ell_1^{\theta_2^- - \theta_1^-} 2\mathcal{F}_1(\theta_3^-, \theta_3^-, \theta_2^+ - \theta_1^- - \theta_2^+ + 1, \ell_2).$$

We need $A(-)$ to compute $n(k)$ as shown in Eqs. (34)-(35). Hence, using Eqs. (37)-(38) and the property (A1) of Hypergeometric function reported in Appendix A, we get

$$A(-) = \frac{\Gamma(1 - 2iE_{\text{in}})\Gamma(-2iE_{\text{out}})}{\Gamma(1 - i2E_+ - im(\beta_0 - \beta'))\Gamma(-2iE_+ + im(\beta_0 - \beta'))},$$

and

$$B(-) = \frac{\Gamma(1 - 2iE_{\text{in}})\Gamma(2iE_{\text{out}})}{\Gamma(1 + 2iE_+ - im(\beta_0 - \beta'))\Gamma(2iE_+ + im(\beta_0 - \beta'))}.$$ 

Finally we arrive at

$$n(k) = 2 \left[ 1 - \frac{E_{\text{out}}}{E_{\text{in}}} \frac{(E_{\text{in}} - M_{\text{in}})}{(E_{\text{out}} - M_{\text{out}})} \right] \left[ \frac{\Gamma(1 - 2iE_{\text{in}})\Gamma(-2iE_{\text{out}})}{\Gamma(1 - i2E_+ - im(\beta_0 - \beta'))\Gamma(-2iE_+ + im(\beta_0 - \beta'))} \right]^2.$$

### D. Entanglement entropy

We now have all the ingredients to compute the particle antiparticle entanglement. Recall, e.g. from [28, 29], that it can be quantified by the subsystem (particle) entropy given by

$$S_{\text{out}} = -\frac{n}{2} \log_2 \frac{n}{2} - \left(1 - \frac{n}{2}\right) \log_2 \left(1 - \frac{n}{2}\right).$$

The behavior of $S$ in terms of $k$ is reported in Figs. [1].

We can see that the entropy, as the momentum increases from zero on, reaches a maximum and then decreases. Since we are interested in asymptotic regimes, we take $\beta' = 1$ and note the curves are broadened (and with higher maxima) as far as the free parameter of the scale factor fulfills the condition: $\beta_0 > 1$.

Additionally if the mass increases the entropy production gradually tends to be suppressed. This behavior is depending also on $\beta_0$, as above discussed and degenerates with $m$. In fact, as $m$ tends to be smaller, the entropy is suppressed away at smaller $k$.

### IV. ENTANGLEMENT FROM THE SCALE FACTOR (1,2) PADÉ APPROXIMANT

By referring to Eq. [3] we can consider a second case in which the expansion converges faster than $a_{(1,1)}$, that is

$$a_{(1,2)}(\tau) = \frac{\beta_0}{1 + \tau + \beta_3 \tau^2}.$$ 

The picture of Eq. [12] is motivated by the fact that the scale factor should reproduce cosmic data at high redshift. Among all possible choices, the aforementioned one corresponds to another suitable landscape in depicting the evolution of the luminosity distance at redshift much larger than $z > 1$. 

FIG. 1: Behavior of $S$ vs momentum $k$. Here, we consider $m = 0.01$ (left) and $m = 0.001$ (right). Curves from top to bottom correspond to values: $\beta_0 \in [1, 10]$ with step 1 and $\beta' = 1$.

In the case of (42), the input and output limits read

$$a_{(1,2)}(t = -\infty) = 0,$$

$$a_{(1,2)}(t = +\infty) = \beta_0,$$

as well as $da_{(1,2)}/dt|_{t=-\infty} = da_{(1,2)}/dt|_{t=+\infty} = 0$.

Unfortunately the Dirac equation (18) with $a_{(1,2)}(t)$ cannot be solved analytically. Before resorting to numerics, we have to notice that here we have an additional new free term, namely $\beta_3$. However, we require that $\beta_3$ is fixed to guarantee that the present limits over $H$ and $q$ are still valid. For being compatible with current observations and in particular to have that the present value of the Hubble parameter, i.e. $H_0$, lies in the domain predicted by the Planck satellite [30], it is possible to have the tight interval: $\beta_3 \in [10^{-2}; 10^{-1}]$. This slightly modifies the evolution of $a(t)$ in (42) with respect to (9), as shown in Fig. (2).

FIG. 2: Behavior of $a_{(1,1)}$ (dashed black curve) and $a_{(1,2)}$ (thick gray curve) as function of the cosmic time $t$. Here we adopted the indicative values $\beta_0 = 5$ and $\beta' = 1$. With the here-adopted constraint $\beta_3 = 0.05$, the behavior of the two scale factors is slightly different only at small times. This guarantees the Hubble rate is preserved as a genuine de-Sitter expansion rate at small redshift.

In the case of the Padé $(1, 2)$ expansion, the main deviations from the $(1, 1)$ case are essentially depending upon the values of $k$, although the functional forms of $S$ is unaltered. The discrepancy between entanglement entropies of the two cases has been portrayed in Fig. (3). The discrepancy is extremely small even for different choices of $m$. Precisely, by shifting up the mass magnitude, discrepancies become smaller for fixed $k$. 
FIG. 3: Difference between entanglement entropy computed by the Padé expansions (1,1) (dashed line) and (1,2) (thick line). The parameters here used are $\beta_0 = 5$, $\beta' = 1$ and $\beta_3 = 0.05$, with $m = 10^{-2}$.

FIG. 4: Behavior of the transmissivity $\eta$ as function of the momentum $k$. Here we consider $\beta' = 1$ and $\beta_0 \in [1,15]$ with step 1. The left is obtained with $m = 10^{-2}$, while the right with $m = 10^{-3}$.

V. FINAL OUTLOOK AND PERSPECTIVES

We dealt with entanglement production for Dirac field in a homogeneous and isotropic universe assuming a model-independent framework for the scale factor. To do so, we considered two phases of universe’s evolution corresponding to past and future respectively. We thus proposed to use the Padé approximation on $a(t)$ built up in terms of functions of cosmic time $t$, fulfilling the fewest number of basic requirements dictated by current observations over the variations of $a(t)$. We computed Padé expansions (1,1) and (1,2) of the scale factor and we evaluated the asymptotic solutions of the Bogolubov transformations. Since the free constants entering our models are essentially influenced by cosmic observations, we have thus evaluated the particle-antiparticle entanglement in the output region according to current observations.

The results might appear qualitatively similar to those found in Ref. [19], however we here gave a physical interpretation of $a(t)$ evolution, without considering any ad hoc toy models. Further, in our approach we are led to take a single free parameter, $(\beta_0)$, to vary, while in Ref. [19] one was forced to have two ($\rho$ and $\epsilon$). Going on we have considered the (1,2) order of the scale factor expansion through Padé approximants. In this case we numerically determined the solution of dynamical equation to arrive at particle-antiparticle entanglement in the output region and we gave a tight range of values for $\beta_3$. Higher orders than (1,1) did not change significantly the results. Hence, we conclude that any orders higher than (1,1) do not furnish relevant deviations on $S$, if one guarantees cosmic requirements to be valid. Thus, any toy models developed in the literature are forced to be built up in terms of functions quite similar to (1,1) Padé rational expansion.

The use of (1,1) order expansion is qualitatively justified for investigating quantum information processing tasks. For instance, following Ref. [31] we could have considered how well information is transferred from remote past to far
future. The noise imparted to Dirac particles by the evolution of the universe still results equivalent to an amplitude damping channel. The transmissivity parameter turns out to be \( \eta = 1 - n/2 \) and its behavior is reported in Fig. [4].

At the end it is also worth mentioning that the model of Ref.[15] can be recovered from (1,1) Padé approximant, but with parameters values far away from fitting the observations.

Looking ahead it could be interesting to extend the presented approach to entanglement creation in non-homogeneous spacetimes [32]. Above all it would be interesting to work directly on the universe dynamics, considering with parameters values far away from fitting the observations.

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Appendix A: Properties of our solutions

The Hypergeometric functions of second kind satisfy the following property:

\[
2F_1(\theta_1^-, \theta_2^+, \theta_3^-, \ell_1) = \frac{\Gamma(\theta_3^-)\Gamma(\theta_3^- - \theta_1^- - \theta_2^+)\Gamma(\theta_3^- - \theta_2^+)}{\Gamma(\theta_3^- - \theta_1^-)\Gamma(\theta_3^- - \theta_2^+)} 2F_1(\theta_1^-, \theta_2^+, \theta_1^- + \theta_2^+ - \theta_3^- + 1, \ell_2) + \frac{\Gamma(\theta_3^-)\Gamma(\theta_3^- + \theta_2^+ - \theta_3^-)}{\Gamma(\theta_3^- - \theta_1^-)\Gamma(\theta_3^- - \theta_2^+)} \ell_2^{-\theta_1^- - \theta_2^+} 2F_1(\theta_3^- - \theta_1^-, \theta_3^- - \theta_2^+, \theta_3^- - \theta_1^- - \theta_2^+ + 1, \ell_2).
\]

(A1)

In addition, for completeness the hypergeometric functions satisfy the relation

\[
2F_1(\theta_1^+, \theta_2^+, \theta_3^+, 1) = \frac{\Gamma(\theta_3^+)\Gamma(\theta_3^+ - \theta_1^+ - \theta_2^+)}{\Gamma(\theta_3^+ - \theta_1^+)\Gamma(\theta_3^+ - \theta_2^+)},
\]

(A2)

known as Gauss’ hypergeometric theorem [33]. Since Eq. (A2) is valid for arbitrary sets of coefficients \( \theta_1^+, \theta_2^+, \theta_3^+ \), it will be also valid at asymptotic regime. Hence solutions (38)-(39) can be in principle obtained through its use as well.

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