Darboux transformations for supersymmetric two-boson equation

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Abstract

In this paper we construct Darboux transformations for the supersymmetric Two-boson equation. Two Darboux transformations and associated Bäcklund transformations are presented. For one of them, we also obtain the corresponding the nonlinear superposition formula.

Keywords: Darboux transformation, Bäcklund transformation, nonlinear superposition formula, supersymmetric integrable systems.

1 Introduction

Broer-Kaup (BK) system or classical Boussinesq equation

\[
\begin{align*}
    u_t &= (2h + u^2 - u_x)_x, \\
    h_t &= (2uh + h_x)_x,
\end{align*}
\]  

(1)

is a very important integrable system. The system was introduced by Kaup when he was extending the newly established method of inverse scattering transform and studying an energy-dependent Schrödinger spectral problem \[^\text{[1]}\]. Kaup further showed that the BK system may be derived from the water-wave equations if one more order of nonlinearity is included in the derivation of the Boussinesq equation \[^\text{[3]}\]. Independently, Broer obtained the BK system as the equation approximating Boussinesq type equations \[^\text{[2]}\]. As a typical integrable system, BK system has been studied extensively and a large number of results have been accumulated. From the viewpoint of solutions, BK system possesses various types of solutions such as the standard
solitons [3][4], rational solutions [5][6] and fission-fusion solutions [7][8]. From the algebraic viewpoint, Kupershmidt, coined it as a dispersive water wave equation, showed that it is a tri-Hamiltonian system, namely it has three local Hamiltonian structures which are compatible [9]. Leo et al, within the framework of prolongation theory, worked out its Bäcklund transformation [24] and the related nonlinear superposition formula is derived by Gordoa and Conde [25]. The constructions of Darboux transformations for BK system were considered by Leble and Ustinov [11] and Li et al [10]. Besides, we mention that BK system also plays a role in the matrix models as explained in [12][13] and thus the name two-boson system.

A supersymmetric extension of BK system, named as supersymmetric two-boson system, was constructed by Brunelli and Das [14]. It reads as

\[\alpha_t = 2\beta_x + 2\alpha' \alpha_x - \alpha_{xx}, \quad \beta_t = (2\alpha' \beta + \beta_x)_x,\]

where \(\alpha\) and \(\beta\) are fermionic or Grassmann odd variables depending on super-spatial variables \((x, \theta)\) and temporal variable \(t\). \(D = \partial_\theta + \theta \partial_x\) is the usual super derivative and for a given function \(f\), \(f' = (Df)\). Brunelli and Das further demonstrated that the supersymmetric two-boson system is a bi-Hamiltonian system and closely associated with other known integrable systems [15]. With Yang, one of the authors succeeded in converting the supersymmetric two-boson system into Hirota form [16]. Recently, by means of super-Bell polynomials, Fan obtained the bilinear Bäcklund transformation for it [18]. Very recently, Xue and one of the authors examined Fan’s Bäcklund transformation and worked out its related nonlinear superposition formula [19].

The purpose of this paper is to construct Darboux transformations for the supersymmetric two-boson system. In general, Darboux transformations have been playing vital roles in the study of integrable systems [20]. In particular, a DT may provide a convenient tool to construct particular solutions of a (integrable) nonlinear differential equation. To the best of our knowledge, any form of Darboux transformations for the system (2) has not been constructed. We also notice that the Darboux transformations and nonlinear superposition formulae are valid for the \(N = 2, a = 4\) SUSY KdV equation [21][22], since it shares the same hierarchy with the SUSY Two-Boson equation [17][23].

The paper is organised as follows. Next section, we will construct two Darboux transformations and the associated Bäcklund transformations for the supersymmetric two-boson system (2). We will also demonstrate the relationship between our Darboux transformations and those for classical Boussinesq equation found in [10]. In Section 3, we will build the nonlinear supersymmetric formula for one of the Bäcklund transformations obtained in Section 2.
2 Darboux-Bäcklund transformation

It is known that (2) possesses the following spectral problem

\[ L \psi = \lambda \psi, \quad \psi_t = P \psi, \]

where

\[ L = \partial_x - \alpha' + D^{-1} \beta, \quad P = -\partial_x^2 + 2\alpha' \partial_x + 2\beta D. \]

The spectral problem (3) may be reformulated as matrix form. To this end, we introduce the new variable as \( \Psi = (\psi, (D^{-1}\beta \psi), \psi')^T \), and obtain

\[ \Psi' = U \Psi, \quad U = \begin{pmatrix} 0 & 0 & 1 \\ \beta & 0 & 0 \\ \lambda + \alpha' & -1 & 0 \end{pmatrix} \]

and

\[ \Psi_t = V \Psi, \quad V = - \begin{pmatrix} \lambda^2 + \alpha_x' - \alpha'^2 - \beta' & \alpha' - \lambda & -\beta \\ (\lambda - \alpha')\beta' - \beta_x' + \beta \alpha_x & -\beta' & (\alpha' - \lambda)\beta + \beta_x' \\ (\alpha_x - \beta_x) - 2\alpha'(\alpha_x - \beta) & \alpha_x - \beta & \lambda^2 + \alpha_x' - \alpha'^2 - 2\beta' \end{pmatrix}. \]

To construct a Darboux transformation for (4)-(5), we assume that there exists a gauge transformation \( \tilde{\Psi} \equiv W \Psi \) such that \( \tilde{\Psi} \) solves

\[ \tilde{\Psi}' = \tilde{U} \tilde{\Psi}, \quad \tilde{\Psi}_t = \tilde{V} \tilde{\Psi}, \]

where \( \tilde{U}, \tilde{V} \) are the matrices \( U, V \) but with \( \alpha, \beta \) replaced by the new field variables \( \tilde{\alpha}, \tilde{\beta} \). To be qualified as a Darboux transformation, the matrix \( W \) has to satisfy

\[ W' + W^\dagger U - \tilde{U} W = 0, \quad W_t + WV - \tilde{V} W = 0. \]

It is remarked that to be consistent the gauge matrix \( W \) has to be structured as \( \begin{pmatrix} \text{even even odd} \\ \text{even even odd} \\ \text{odd odd even} \end{pmatrix} \).

Also for a super matrix \( A = (a_{ij})_{m \times n} \), we define \( A^\dagger = (a_{ij}^\dagger)_{m \times n} \), and \( a_{ij}^\dagger = (-1)^{p(a_{ij})}a_{ij} \), with \( p(a_{ij}) \) denoting the parity of entry \( a_{ij} \).

To achieve a meaningful result, we make the simplest ansatz \( W = \lambda \mathcal{M} + \mathcal{N}, \quad \mathcal{M} = (m_{ij})_{3 \times 3}, \quad \mathcal{N} = (n_{ij})_{3 \times 3} \). It follows from the first equation of (8) that the matrices \( \mathcal{M} \) and \( \mathcal{N} \) should take the following forms:

\[ \mathcal{M} = \begin{pmatrix} m_{11} & 0 & 0 \\ 0 & m_{22} & 0 \\ m_{31} & 0 & m_{11} \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} n_{11} & m_{22} - m_{11} & m_{11}' - m_{31} \\ n_{21} & n_{22} & m_{22} \beta - m_{11} \tilde{\beta} \\ n_{31} & -m_{31} & n_{33} \end{pmatrix}, \]
where the entries are determined by the following equations

\[ m'_{22} = 0, \]  
\[ n'_{22} + m_{22} (\beta - \tilde{\beta}) = 0, \]  
\[ n_{11} - n_{33} - m'_{31} + m_{11,x} = 0, \]  
\[ n_{33} - n_{11} + m'_{31} + m_{11} (\alpha' - \tilde{\alpha}') = 0, \]  
\[ n_{22} - n_{33} - m'_{31} - (m_{22} - m_{11})\tilde{\alpha}' = 0, \]  
\[ n_{21} + (m_{22}\beta - m_{11}\tilde{\beta})' - \tilde{\beta}(m'_{11} - m_{31}) = 0, \]  
\[ n_{31} - n'_{11} - (m_{22} - m_{11})\beta + (m'_{11} - m_{31})\alpha' = 0, \]  
\[ n_{33} - n_{31} - (m_{11} - m_{31})\tilde{\alpha}' + m_{22}\beta - m_{11}\tilde{\beta} = 0, \]  
\[ n'_{31} + n_{21} + n_{33}\alpha' - n_{11}\tilde{\alpha}' + m_{31}\beta = 0, \]  
\[ n_{21} + m_{22}\beta - n_{11}\tilde{\beta} - (m_{22}\beta - m_{11}\tilde{\beta})\alpha' = 0. \]

It follows from (9) that \( m_{22} \) is a constant. Without loss of generality, one may consider \( m_{22} = 0 \) or \( m_{22} = 1 \). Also, (12)-(15) yield

\[ m_{11,x} + m_{11}(\alpha' - \tilde{\alpha}') = 0. \]  

Now, we consider the following cases:

**Case I:** We assume \( m_{11} = 0, \ m_{22} = 1 \) and replace \( n_{11} \) by \(-r\), for simplicity. Then the equations (12)-(15) lead to

\[ n_{33} = -r - m_{31}', \quad n_{21} = -\beta' - \tilde{\beta} m_{31}, \]
\[ n_{22} = -r + \tilde{\alpha}', \quad n_{31} = -r' + \beta + m_{31}\alpha'. \]

Above equations together (16) produce

\[ m_{31}(\tilde{\alpha}' - \alpha') - m_{31,x} = 0, \]

which holds if we choose \( m_{31} = 0 \). Now (17) gives

\[ r_x = r(\tilde{\alpha} - \alpha)' \]

Next supposing \( r \neq 0 \), integrating above equation once, and setting the integral constant to be zero, without loss of generality, one has

\[ \tilde{\alpha} = \alpha + (\ln r)' \]

Then (10) supplies us with

\[ \tilde{\beta} = \beta + \alpha_x - r' + (\ln r)'_x. \]
Finally, (18) gives the equation
\[ \alpha_x - \frac{(\beta / r)_x}{r'} + (\ln r)'_x = 0. \]
which may be integrated once and yields
\[ r_x = r^2 - \lambda_1 r - r\alpha + r(\beta / r)'. \] (21)
where \( \lambda_1 \) is an integral constant. Above equation constitutes a (spatial part) BT for the SUSY Two-Boson equation. The associated temporal part of BT may be obtained as follows
\[ r_t = (r_x + 2r\alpha' - r^2)_x. \] (22)

To sum up, Darboux transformation for the field variables reads as
\[ \tilde{\alpha} = \alpha + (\ln r)', \quad \tilde{\beta} = \beta + (\beta / r)_x, \] (23)
and Darboux transformation for the eigenfunctions reads as
\[ \tilde{\Psi} = \mathcal{W}_I \Psi, \quad \mathcal{W}_I = \begin{pmatrix} -r & 1 & 0 \\ -\beta' & \lambda - \lambda_1 + (\beta / r)' & \beta \\ \beta - r' & 0 & -r \end{pmatrix}, \] (24)
together with (21)-(22), which define the Bäcklund transformations for the field variables.

**Remark 1:** On the level of eigenfunctions, the Darboux transformation in scalar form is easily found to be:
\[ \tilde{\psi} = D^{-1}(\beta \psi) - r\psi. \]

**Case II:** As we know that the inverse of a Darboux matrix is also a Darboux matrix, thus we may construct the second Darboux transformation from the case I. In fact, we consider the following replacement:
\[ (\lambda_1, \alpha, \beta, \tilde{\alpha}, \tilde{\beta}, r) \rightarrow (\lambda_2, \tilde{\alpha}, \tilde{\beta}, \alpha, \beta, s), \]
then it follows from (21) and (23) that
\[ s_x = s^2 - \lambda_2 s - s\tilde{\alpha} + s(\tilde{\beta} / s)', \] (25)
\[ \tilde{\alpha} = \alpha - (\ln s)', \] (26)
\[ \tilde{\beta} = \beta - \left(\frac{\tilde{\beta} / s}{x}\right). \] (27)
Substituting (26) into (25), one has
\[ (\tilde{\beta} / s)' = \alpha' + \lambda_2 - s. \] (28)
From above equation together with (27), one obtains

\[ \tilde{\beta} = \beta - \alpha x + s' \tag{29} \]

and

\[ s_x = s(\lambda_2 + \alpha') - s^2 + s\left(s^{-1}(\alpha x - \beta)\right)' \tag{30} \]

Thus (26) and (29) are the Darboux transformation for the field variables, (30) serves as the spatial part of Bäcklund transformation. By direct calculation, we find the temporal part of BT as follows

\[ s_t = -s_{xx} - (s^2 - 2s\alpha')_x. \]

Now we construct the corresponding Darboux matrix. Considering the matrix \((\lambda - \lambda_1)W_I^{-1}\), replacing \(\lambda_1, \beta, r\) by \(\lambda_2, \tilde{\beta}, s\), together with (28) and (29), one obtains the second Darboux transformation for the wave function

\[
\tilde{\Psi} = W_{II} \Psi, \quad W_{II} = -\frac{1}{s} \begin{pmatrix}
\lambda - s + \alpha' & -1 & (\zeta - s')/s \\
-s^2 + s(\lambda_2 + \alpha') & -s & \zeta - s' \\
\zeta - (\lambda + \alpha')\zeta/s & \zeta/s & \lambda - \lambda_2 - s'\zeta/s^2
\end{pmatrix},
\]

where

\[ \alpha x - \beta = \zeta. \]

**Remark 1**: Up to a simple change of variables, the BT in the Case II coincides with the one appeared in the framework of super-Bell polynomials (see [18] or [19]).

**Remark 2**: One may rewrite the transformation as a scaler form:

\[ \tilde{\psi} = \psi - s^{-1}\psi_x + s^{-2}(s' - \alpha x + \beta)\psi'. \]

So far, we have constructed two different Darboux transformations for (4) and the related Bäcklund transformations for the supersymmetric two-boson system. As usual, it is interesting to work out the Darboux matrices in more explicit forms, namely, to represent the entries of Darboux matrices in terms of the solutions of the linear spectral system or its adjoint.

In **Case I**, a direct calculation shows that the Berezin or super-determinant of \(W_I\) reads as

\[ Ber(W_I) = \lambda - \lambda_1, \]

thus take a particular solution \(\Psi_1 = (f_1, g_1, \sigma_1)^T\) of the spectral problem (4) at \(\lambda = \lambda_1\). Here, \(f_1\) and \(g_1\) are bosonic, \(\sigma_1\) is fermionic. Imposing \(W_I \Psi_1|_{\lambda=\lambda_1} = 0\), we have

\[ r = g_1/f_1. \]

In this way, the deserved explicit form of the Darboux matrix \(W_I\) is found.
For the Case II, we need the adjoint linear problem of (4), which is given by

\[ - (\chi^\dagger)' = \chi U. \quad (31) \]

Now since each Darboux transformation for the linear spectral problem induces a Darboux transformation for the adjoint linear spectral problem, we have

\[ \tilde{\chi} = \chi T \]

with

\[
T = (\lambda - \lambda_2)(W_{II}^{-1})^\dagger = \begin{pmatrix}
-s & 1 & 0 \\
-s^2 - s(\lambda_2 + \alpha') + s' s^{-1} \zeta & \lambda - s + \alpha' & \zeta - s' \\
\zeta & 0 & -s
\end{pmatrix}. 
\]

Note that

\[ \text{Ber}(T) = \lambda - \lambda_2. \]

Thus, we take a particular solution \( \chi_2 = (g_2, f_2, \sigma_2) \) of (31) at \( \lambda = \lambda_2 \), where \( f_2 \) and \( g_2 \) are bosonic, \( \sigma_2 \) is fermionic. By imposing \( \chi_2 T|_{\lambda=\lambda_2} = 0 \), we obtain

\[ s = \lambda_2 + \alpha' + g_2/f_2. \]

With this result, the explicit form of Darboux matrix \( W_{II} \) is achieved.

As a final part of the section, we relate our Darboux transformations to those for the classical Boussinesq equation obtained in [10]. Then letting \( \alpha = \xi + \theta u \) and \( \beta = \eta + \theta h \), from (4) we find we easily obtain the bosonic limit

\[ \varphi_x = U_1 \varphi, \quad U_1 = \begin{pmatrix} \lambda + u & -1 \\ h & 0 \end{pmatrix}. \quad (32) \]

From (5), we have

\[ \varphi_t = V_1 \varphi, \quad V_1 = -\begin{pmatrix} \lambda^2 - u^2 - h + u_x & u - \lambda \\ \lambda h - uh - h_x & -h \end{pmatrix}. \quad (33) \]

The compatibility of (32) and (33) gives (1). By taking the gauge transformation

\[ \varphi = G \phi, \quad G = g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

where \( g \) satisfies the following equation

\[ g_x = \frac{1}{2}(\lambda + u)g, \]
the spectral problem (32) becomes
\[ \phi_x = M \phi, \quad M = \begin{pmatrix} (u - \lambda)/2 & 1 \\ -h & -(u + \lambda)/2 \end{pmatrix} \]
which is essentially the one considered in [10] (cf. (4) of [10]). The bosonic limit of the Darboux matrix \( W_I \) reads as
\[ W_1 = \begin{pmatrix} -a & 1 \\ -h & \lambda - \lambda_1 + \frac{h}{a} \end{pmatrix} \]
with
\[ \tilde{u} = u + (\ln a)_x, \quad \tilde{h} = h + (h/a)_x, \]
where \( a \) is defined by
\[ a_x = a^2 - \lambda_1 a - au + h. \]
Then, direct calculation shows that the Darboux transformation
\[ \tilde{\phi} = T \phi, \quad T = \tilde{G}^{-1} W_1 G = d \begin{pmatrix} -a & -1 \\ h & \lambda - \lambda_1 + \frac{h}{a} \end{pmatrix}, \quad ad^2 = -1 \]
which is depicted by
\[ \begin{array}{ccc}
\phi & \xrightarrow{G} & \tilde{\phi} \\
W_1 & \xrightarrow{T} & \tilde{W}_1 \\
\phi & \xrightarrow{G^{-1}} & \tilde{\phi}
\end{array} \]
is exactly the Darboux transformation described by the Proposition 2 of [10].

Similarly, it can be shown that the bosonic limit of our Darboux transformation presented in Case III relates to the Darboux transformation presented by Proposition 1 of [10].

## 3 Nonlinear Superposition Formula

For a given a Bäcklund transformation, it is interesting to find the corresponding nonlinear superposition formula. For the Case II of the last section, such formula has been worked out already in [19]. In this section, we consider the nonlinear superposition formula for the Bäcklund transformation given in the Case I. Thus, we start with a solution \((\alpha, \beta)\) of (2) and take two solutions \(\Psi_1 = (f_1, g_1, \sigma_1)^T\) and \(\Psi_2 = (f_2, g_2, \sigma_2)^T\) corresponding to arbitrary constants \(\lambda_1\) and \(\lambda_2\). Then we consider two Darboux transformations
\[
\Psi_{[k]} = W_k \Psi, \quad W_k \equiv W_{II}|_{\lambda = \lambda_k} = \begin{pmatrix} -r_k & 1 \\ -\beta' & \lambda - r_k + \alpha' + \frac{r_{k,k}}{r_k} \beta \\ \beta - r_k' & 0 \\ 0 & -r_k \end{pmatrix}, \quad (34)
\]
\[ \alpha_{[k]} = \alpha + (\ln r_k)' \quad \beta_{[k]} = \beta + \left( \frac{\beta}{r_k} \right)_x, \quad r_k = \frac{g_k}{f_k}. \]

Then from the compatibility of the two equations, or the Bianchi identity

\[
\begin{align*}
(\alpha, \beta; \Psi) & \xrightarrow{\lambda_1} (\alpha_1, \beta_1; \Psi_1) \quad (\alpha_2, \beta_2; \Psi_2) \\
& \xrightarrow{\lambda_2} (\alpha_2, \beta_2; \Psi_2) \quad (\alpha_{[1]}, \beta_{[1]}; \Psi_{[1]}) \\
& \xrightarrow{\lambda_1} (\alpha_{[1]}, \beta_{[1]}; \Psi_{[1]}) \quad (\alpha_{[2]}, \beta_{[2]}; \Psi_{[2]} = (\alpha_{[21]}, \beta_{[21]}; \Psi_{[21]})
\end{align*}
\]

we have

\[ W_{2[1]} W_1 = W_{1[1]} W_2, \]

where

\[
W_{1[1]} = W_{11,\alpha=a_{[2]}, \beta=\beta_{[2]}, r_1=r_{21}}, \quad W_{2[1]} = W_{21, \alpha=\alpha_{[1]}, \beta=\beta_{[1]}, r_2=r_{12}},
\]

\[ \alpha_{[21]} = \alpha_{[2]} + (\ln r_{21})', \quad \alpha_{[12]} = \alpha_{[1]} + (\ln r_{12})', \quad \beta_{[21]} = \beta_{[2]} + \left( \frac{\beta_{[2]}}{r_{21}} \right)_x, \quad \beta_{[12]} = \beta_{[1]} + \left( \frac{\beta_{[1]}}{r_{12}} \right)_x. \]

From \( \alpha_{[12]} = \alpha_{[21]}, \beta_{[12]} = \beta_{[21]}, \Psi_{[12]} = \Psi_{[21]} \), we can get

\[ \alpha_{[12]} = \alpha + (\ln r_1 r_{12})' \quad \beta_{[12]} = \beta + \left( \frac{\beta}{r_1} + \frac{\beta + (\frac{\beta}{r_1})_x}{r_{12}} \right)_x, \]

where

\[ r_{12} = \frac{(\lambda_2 - \lambda_1) r_2}{r_2 - r_1} = \frac{\beta'}{r_1} + \frac{\beta r_2}{r_2 - r_1} \left( \frac{r_1'}{r_1^2} - \frac{r_2'}{r_2^2} \right). \]

As a final remark, we notice that our results and Darboux transformations and nonlinear superposition formulae are valid for the \( \mathcal{N} = 2, a = 4 \) SUSY KdV equation \([21, 22]\), since it shares the same hierarchy with the SUSY Two-Boson equation \([17, 23]\).

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