Frustrated Bonds and Long Range Order in Quasi-2D Magnets.

I. Ya. Korenblit

School of Physics and Astronomy, Raymond and Beverly Sackler Faculty of Exact Sciences,
Tel Aviv University, Tel Aviv 69978, Israel

Abstract

We employ the Schwinger boson mean-field approach to study the effects of arbitrary frustrated bonds and plaquettes (formed from four frustrated bonds) in two-dimensional ferro- and antiferromagnets on the spin-wave spectrum and the correlation length at finite temperatures. We distinguish between strongly frustrated bonds (plaquettes), when the frustrated coupling $J'$ exceeds the spin canting threshold $J_c$, and weakly frustrated bonds (plaquettes), with $J' < J_c$, $(J_c - J')/J_c \sim 1$. It is shown that in antiferromagnets the amplitude of spin-wave scattering on strongly frustrated bonds or plaquettes grows with the decrease of the temperature. A small amount of such defects reduces significantly the spin-wave stiffness and the correlation length at low temperatures. As a result, the quasi-2D Néel temperature is sharply suppressed. Quantum fluctuations are also considered and their effect on the spin-wave spectrum is shown to be of the order of $(2S)^{-2}\ln^{-1}2S$ in the large spin limit. For weakly frustrated (nonfrustrated) defect bonds (plaquettes) the spin-wave stiffness renormalization is of the order of the dopant concentration and does not depend on the temperature. The results account for the observed properties of doped quasi-2D $La_2CuO_{4+x}$. The frustrated bonds in 2D ferromagnets act in the same way as in antiferromagnets, while the effect
of frustrated plaquettes is weak and temperature independent.

PACS numbers 74.72-h, 75.30Dx, 75.50Ee
I. INTRODUCTION

The discovery of high $T_c$ superconductivity in the doped cuprates $La_2CuO_4+x$ and $YBa_2Cu_3O_{6+x}$ enhanced the interest in the study of two-dimensional (2D) quantum antiferromagnets. It is now well established that the system of localized $Cu^{2+}$ spins in both families (near $x=0$) is described by the 2D Heisenberg Hamiltonian, and a small coupling between $Cu^{2+}$ layers leads to the formation of a three-dimensional Néel ordered state at temperatures much smaller than the intraplane exchange interaction energy. Above the Néel temperature $T_N$ one observes 2D antiferromagnetic correlations. To leading order, the correlation length $\xi$ in undoped samples follows the theoretically predicted law

$$\xi \propto \exp(A/T). \quad (1.1)$$

The calculation by Chakravarty et al. based on the quantum nonlinear sigma model ($QNL\sigma M$), showed that $A = 2\pi \rho_s$, where $\rho_s$ is the spin stiffness constant. The preexponential factor in the two-loop approximation does not depend on the temperature up to terms of the order of $T/2\pi \rho_s$ in antiferromagnets and scales as $T^{1/2}$ in ferromagnets. A very good quantitative agreement between the $QNL\sigma M$ correlation length and the results of a neutron scattering study of the model 2D, $S = 1/2$, square-lattice Heisenberg antiferromagnet $Sr_2CuO_2Cl_2$ was recently obtained in a surprisingly wide temperature range $T_N < T < 2\pi \rho_s$.

Another approach to low-dimensional ($D = 1, 2$) magnets has been proposed by Arovas and Auerbach. They employed the Schwinger boson representation for spins in a $SU(N)$ generalization of the Heisenberg model. Their expression for the correlation length corresponds to the one-loop approximation in the $QNL\sigma M$. The constant $A$ in Eq. (1.1) is $2\pi JS^2$ in the limit of large $S$, and $J$ stands for the intraplane coupling constant.

The doping has a drastic effect on the magnetic properties of the cuprates. Even a very small dopant concentration, which introduces holes residing on the oxygen orbitals in the $CuO_2$ planes, substantially reduces the Néel temperature. In $La_2CuO_4$, doped with
strontium or with excess oxygen, the long range order disappears at doping concentration as small as 2%. It has been shown that in samples of $La_2CuO_{4+x}$ with a small concentration of defects (of the order or less than 1%) the correlation length varies little with doping at very high temperatures 500 - 600 K, however the growth of the spin correlations is increasingly inhibited by the defects as the temperature is decreased.

In this paper we shall focus on the properties of doped 2D magnets. Doping with excess oxygen or with strontium creates holes on the oxygen sites in the CuO$_2$ planes. Since the Cu-O distance is half the Cu-Cu distance, the exchange interaction, $J_\sigma$, of the hole spin with the two nearest-neighbor Cu spins is much larger than the Cu-Cu exchange, $J$. For either sign of $J_\sigma$ the hole spin would thus like to be parallel (antiparallel) to the two neighboring Cu spins. Therefore the oxygen hole introduces an effective ferromagnetic coupling, $J'$, between its two Cu neighbors, with $J' = O(| J_\sigma |) \gg J$. This was the basic idea of the frustration model of Aharony et al.

Motivated by these arguments and the results of the experimental investigation of $La_2CuO_{4+x}$ and $La_{2-x}Sr_xCuO_4$, we shall consider two simple models of defects, namely, frustrated bonds and frustrated plaquettes. We hope that the main qualitative results, presented in this paper, survive in the case of more complicated frustrating defects.

The effect of randomly distributed frustrated bonds on the Heisenberg ferro- and antiferromagnets in the ordered state has been investigated by many authors. It has been shown that there is a threshold energy of the frustrated bond,

$$J_c = (z/2 - 1)J,$$

which determines the local stability of the system. Here $z$ is the number of the nearest neighbors. For a 2D square lattice, $z = 4$ and $J_c = J$. If the (positive) energy of the frustrated bond $J'$ exceeds $J_c$, the two spins connected by the frustrated bond cant. In the classical limit the canted spins act on the magnetic background as a dipole. As a result, the spins around the ferromagnetic bond are also canted, and the canting angle in a 2D Heisenberg magnet decays at large distances $r$ from the defect as $r^{-1}$.
The quantum fluctuations of the spins, connected by the frustrated bond, have been calculated for \( J' < J_c \) in the framework of the linear spin wave theory.\(^{17,18}\) These fluctuations diverge when \( J' \to J_c^- \), which reflects the breakdown of the linear spin wave approach.

The effect of localized holes on the properties of quasi-2D antiferromagnets at finite temperatures has been studied by Glazman and Ioselevich.\(^{20}\) A classical approach, based on the dipole model of Aharony et al.,\(^{12}\) led them to the conclusion that the renormalized stiffness is a function of \((x/T)\), where \(x\) is the concentration of localized holes.

The model of a single frustrated bond is reasonable for samples doped with excess oxygen. In contrast, Sr doping replaces a trivalent La\(^{3+}\) ion with a divalent Sr\(^{2+}\) ion in the plane above a CuO\(_2\) plane. The Coulomb potential that pins the hole in the CuO\(_2\) plane is then due to the Sr\(^{2+}\) ions, which project onto the centers of the copper-oxygen plaquettes. Therefore the holes are localized on small regions around the centers of the above plaquettes. The simplest defect model of this kind is a plaquette formed by four frustrated bonds.\(^{21,22}\) The canted spins at the corners of the plaquette act on the long-range order parameter (at \( T \to 0 \)) as a quadrupole rather than a dipole. The energy spectrum of the hole, localized on a plaquette, was calculated in Refs. \(^21\) and \(^22\). To our best knowledge neither the critical value \( J_c \) of the frustrated coupling (at \( T=0 \)), nor the effect of the frustrated plaquette at finite temperatures, has been considered till now.

In this paper we study the effects of defect (frustrated and nonfrustrated) bonds and plaquettes in 2D ferro- and antiferromagnets on the spin wave spectrum and on the correlation length at finite temperatures. The Schwinger boson mean-field approach (SBMFA) will be used. This method has been applied successfully to study a variety of properties of low dimensional antiferromagnets.\(^{13,23-31}\)

We show that there is a striking distinction between the effect of strongly and weakly frustrated defects on the behavior of doped antiferromagnets. For strongly frustrated defects, with \( t = (J' - J_c)/J_c > 0 \) \((J_c = J\) for frustrated bonds, \(J_c = 0.376J\) for frustrated plaquettes), the renormalization of the spin wave velocity \( c \) and the correlation length in the large spin limit (to leading linear order in \( x \)) is given by
\[
\frac{c(x) - c(0)}{c(0)} = -\alpha \frac{4JS^2t}{T(1 + t)} x, \tag{1.3}
\]
\[
\ln \frac{\xi(x, T)}{\xi(x, 0)} = -\alpha \frac{8\pi J^2 S^4 t}{T^2(1 + t)} x, \tag{1.4}
\]
where the multiplier \(\alpha\) depends on the type of the defect (\(\alpha \approx 1\) for frustrated bonds, \(\alpha \approx 3.5\) for frustrated plaquettes), and decreases slightly with the decrease of the temperature. Hence, a small amount of frustrated bonds or plaquettes reduces significantly the spin-wave velocity and the correlation length at low temperatures \(T \ll 4JS^2\), if \(t\) is of the order or larger than unity. The decrease of the 2D correlation length with doping causes the strong supression of the quasi-2D long range magnetic order. The Néel temperature changes with the dopant concentration as
\[
\frac{T_N(x) - T_N(0)}{T_N(0)} \approx -\alpha \frac{4JS^2}{T_N(0)} x. \tag{1.5}
\]
Because of the large factor \(4JS^2\alpha/T_N(0)\), the Néel temperature extrapolates to zero at a doping concentration which is much smaller than the percolation threshold.

The quantum fluctuations, associated with the frustrated bonds, are relatively large near the threshold \(J' = J_c\). However, when \(J' \gg J_c\), they are surprisingly small, of the order of \((2S)^{-2} \ln^{-1} 2S\). For weakly frustrated (nonfrustrated) defect bonds or plaquettes, with \((J_c - J')/J_c \approx 1\), the spin-wave stiffness renormalization is of the order of the dopant concentration and does not depend on the temperature.

The frustrated bonds in 2D ferromagnets act in the same way as in antiferromagnets. However the effect of frustrated plaquettes in ferromagnets is entirely different. The renormalization of the spin-wave stiffness and, hence, of the correlation length is not enhanced, and it is temperature independent at both \(J' < J_c\) and \(J' > J_c\).

The above results account for the observed properties of doped quasi-2D antiferromagnets.

The paper is organized as follows. In Sec. II the SBMFA is used to study the problem of a defect bond in a 2D Heisenberg ferromagnet. We derive the mean-field Hamiltonian and
find the temperature dependence of the spin-wave scattering amplitude on the defect bond. Then we treat the ferromagnet with a low concentration of bond defects, and obtain the renormalization of the spin-wave stiffness and the correlation length at small temperatures $T \ll 2\pi \rho_s$. In Sec. III we extend this method to treat 2D antiferromagnets with defect bonds, and study the effect of such defects on $T_N$. The plaquette type defects are considered in Sec. IV. In Sec. V we summarize our results, and compare them with the experimental findings. Some comments are made concerning the temperature and defect concentration dependence of the spin wave velocities in quasi-2D antiferromagnets at $T \ll T_N$. The canting of the spins at large distances from the strongly frustrated ($J' > J_c$) bond is addressed in Appendix A. In Appendix B we calculate the $T$-matrix for the scattering of the Holstein-Primakoff spin waves on frustrated plaquettes at $T = 0$ and $J' < J_c$.

II. FRUSTRATED BONDS IN A 2D FERROMAGNET

A. The Hamiltonian

We consider a Heisenberg ferromagnet on a square lattice with nearest neighbor (nn) interactions only. Thus, in the standard Heisenberg Hamiltonian

$$H = - \sum_{<ij>} J_{ij} \vec{S}_i \cdot \vec{S}_j,$$

the sum is over nn pairs, $<ij>$, and the exchange interactions $J_{ij}$ are equal to $J$ (for most bonds) or $-J'$ (for a small concentration of bonds).

In the Schwinger representation, each spin is replaced by two bosons:

$$S^+_i = a^\dagger_i b_i, \quad S^-_i = a_i b^\dagger_i, \quad S^z_i = \frac{1}{2}(a^\dagger_i a_i - b^\dagger_i b_i),$$

(2.2)

together with the constraints

$$\frac{1}{2}(a^\dagger_i a_i + b^\dagger_i b_i) = S.$$

(2.3)

After this transformation is performed, the Hamiltonian can be written as
\[ H = -2 \sum_{<ij>} J_{ij} : B_{ij}^\dagger B_{ij} : + \sum_i \lambda_i (a_i^\dagger a_i + b_i^\dagger b_i), \]  
\( (2.4) \)

where \( B_{ij}^\dagger = \frac{1}{2}(a_i^\dagger a_j + b_i^\dagger b_j) \), and : : denotes normal ordering. The second term in this expression is added to take into account the constraint \( (2.3) \), \( \lambda_i \) are the Lagrange multipliers.

An important approximation in the mean field theory of Arovas and Auerbach

is that the constraints \( (2.3) \) are imposed on the average, i.e. they are taken into account by introducing a single Lagrange multiplier \( \lambda \). This approximation is plausible in a spatially ordered magnet, but in a disordered one the dependence of \( \lambda_i \) on the position may be important. In what follows we introduce two Lagrange multipliers: \( \lambda_1 \) for the sites connected by the defect bond, and \( \lambda \) for all other sites.

Performing the mean-field decoupling\(^{32}\) in Eq. \((2.4)\), one obtains

\[ H = H_0 + H_{int}. \]  
\( (2.5) \)

Here \( H_0 \) is the Hamiltonian of the undoped ferromagnet,

\[ H_0 = -2Q_f J \sum_{<ij>} (B_{ij} + B_{ij}^\dagger) + \lambda \sum_i (a_i^\dagger a_i + b_i^\dagger b_i), \]  
\( (2.6) \)

where the mean-field amplitude \( Q_f = \langle B_{ij}^\dagger \rangle = \langle B_{ij} \rangle \) describes the short-ranged ferromagnetic correlations.\(^{32,33}\) The amplitude \( Q_f \), like \( \lambda \), is supposed to be positionally independent in the ordered crystal.\(^6\)

The term \( H_{int} \) in Eq. \((2.5)\) gives the interaction of the spin waves with the defect bonds, \( \langle lm \rangle \),

\[ H_{int} = 2W_f \sum_{<lm>} (B_{lm} + B_{lm}^\dagger) + (\lambda_1 - \lambda) \sum_{<lm>} (a_l^\dagger a_l + a_m^\dagger a_m + b_l^\dagger b_l + b_m^\dagger b_m), \]  
\( (2.7) \)

where

\[ W_f = Q_f J + Q_1 J'. \]  
\( (2.8) \)

The mean-field amplitude \( Q_1 = \langle B_{\vec{r},\vec{r}+\vec{a}} \rangle \) describes the ferromagnetic correlations of the spins at the ends of the frustrated bond \( \vec{a} \). It differs from the amplitude \( Q_f \) for the unfrustrated bonds, and this difference is crucial for the properties of the doped magnets.
Transforming Eq. (2.6) to the momentum space, we find

$$H_0 = \sum_{\vec{q}} \omega_{\vec{q}}(a_{\vec{q}}^\dagger a_{\vec{q}} + b_{\vec{q}}^\dagger b_{\vec{q}}).$$  \hspace{1cm} (2.9)

The excitation spectrum $\omega_{\vec{q}}$ has the form

$$\omega_{\vec{q}} = \Delta + JQ_f z(1 - \gamma_{\vec{q}}),$$  \hspace{1cm} (2.10)

where $\gamma_{\vec{q}} = z^{-1} \sum_{\vec{\delta}} \exp(i\vec{q} \cdot \vec{\delta}) = (\cos q_x + \cos q_y)/2$, $z = 4$ is the number of nearest neighbors, and $\vec{\delta}$ is summed over nn vectors. The spectrum is characterized by two temperature dependent parameters, namely, the mean-field amplitude $Q_f$ and the gap $\Delta = \lambda - J z Q_f$. They are governed by the equations

$$\frac{1}{N} \sum_{\vec{q}} n(\omega_{\vec{q}}) = S \hspace{1cm} (2.11)$$

$$Q_f = S - \frac{1}{N} \sum_{\vec{q}} n(\omega_{\vec{q}})(1 - \gamma_{\vec{q}}), \hspace{1cm} (2.12)$$

which follow from the constraint (2.3) and the above definition of $Q_f$. Here $n(\omega_{\vec{q}}) = [\exp(\omega_{\vec{q}}/T) - 1]^{-1}$ is the Bose distribution function, $N$ is the total number of spins.

It is seen from the last equation that $Q_f$ is equal to $S$ up to corrections of order $(T/4JS)^2$. Then Eq. (2.11) yields that the gap $\Delta$ is finite and exponentially small at low temperatures,

$$\Delta \propto T \exp(-4\pi JS^2/T). \hspace{1cm} (2.13)$$

The formula (1.1) for the correlation length $\xi(T)$ follows immediately from this expression and Eq. (2.10).

The $2 \times 2$ interaction matrix for each defect bond in Eq. (2.7) can be easily diagonalized. The interaction Hamiltonian (2.7) is then rewritten as

$$H_{int} = (2JQ_f \mu_f - W_f) \sum_{<lm>} [(a_l^\dagger - a_m^\dagger)(a_l - a_m) + (b_l^\dagger - b_m^\dagger)(b_l - b_m)]$$

$$+ 2JQ_f \mu_f \sum_{<lm>} [(a_l^\dagger + a_m^\dagger)(a_l + a_m) + (b_l^\dagger + b_m^\dagger)(b_l + b_m)], \hspace{1cm} (2.14)$$

9
where \( \mu_f = (\lambda_1 - \lambda + W_f)/4JQ_f \). Performing the Fourier-transformation of this Hamiltonian we find

\[
H_{int} = \sum_{\nu=1}^{2} v_\nu \sum_{l} \frac{1}{N} \sum_{\vec{q}_1, \vec{q}_2} e^{i(\vec{q}_1 - \vec{q}_2) \cdot \vec{r}_l} f'_\nu(\vec{q}_1) f_{\nu}^*(\vec{q}_2) (a^\dagger_{\vec{q}_1} a_{\vec{q}_2} + b^\dagger_{\vec{q}_1} b_{\vec{q}_2}).
\]

(2.15)

Here

\[
f_1(\vec{q}) = 1 - e^{-i\vec{q} \cdot \vec{a}}, \quad f_2(\vec{q}) = 1 + e^{-i\vec{q} \cdot \vec{a}},
\]

(2.16)

\[v_1 = 2JQ_f \mu_f - W_f, \quad v_2 = 2JQ_f \mu_f, \text{ and } l \text{ is summed over the defect bonds.}\]

**B. The single defect problem**

In this subsection we use the Green’s function method to solve the problem of spin wave scattering on a single defect bond. Let us define the retarded Green’s function by the relation

\[
G(\vec{q}, \vec{q}_1, t - t_1) = -i\Theta(t - t_1) < [a_{\vec{q}}(t), a^\dagger_{\vec{q}_1}(t_1)] >_T
\]

(2.17)

Here \( \Theta(t) \) is the step-function and \( < \cdots >_T \) denotes the thermodynamic average.

In a ferromagnet with one defect bond, the Fourier-transform \( G(\vec{q}, \vec{q}_1, \omega) \) is linked to the \( T \)-matrix by the relation

\[
G(\vec{q}, \vec{q}_1, \omega) = G^0(\vec{q}, \omega)\delta_{\vec{q},\vec{q}_1} + G^0(\vec{q}, \omega)T(\vec{q}, \vec{q}_1, \omega)G^0(\vec{q}_1, \omega),
\]

(2.18)

where \( G^0(\vec{q}, \omega) \) is the Green’s function for the pure ferromagnet,

\[
G^0(\vec{q}, \omega) = (\omega - \omega_{\vec{q}})^{-1},
\]

(2.19)

and the \( T \)-matrix satisfies the equation

\[
T(\vec{q}, \vec{q}_1, \omega) = V(\vec{q}, \vec{q}_1) + \frac{1}{N} \sum_{\vec{q}_2} V(\vec{q}, \vec{q}_2)G^0(\vec{q}_2, \omega)T(\vec{q}_2, \vec{q}_1, \omega),
\]

(2.20)

with the interaction energy

\[
V(\vec{q}, \vec{q}_1) = \sum_{\nu=1}^{2} v_\nu f'_\nu(\vec{q}) f_{\nu}^*(\vec{q}_1).
\]

(2.21)
The Green’s function for the $b$-operators obeys the same equations (2.18) and (2.20).

The constraint (2.3) for the spins, at the ends of the frustrated bond, can be written in the form

$$<a_l^\dagger a_l>_T = -\frac{1}{\pi} \int_{-\infty}^{+\infty} n(\omega) \frac{1}{N} \sum_{\vec{q},\vec{q}_1} \text{Im} G(\vec{q}, \vec{q}_1, \omega) d\omega = S. \quad (2.22)$$

We put here for sake of simplicity $\vec{r}_l = 0$. Substituting into this equation the Green’s function from Eq. (2.18), and taking into account that

$$-\frac{1}{\pi} \int_{-\infty}^{+\infty} n(\omega) \frac{1}{N} \sum_{\vec{q}} G^0(\vec{q}, \omega) = \frac{1}{N} \sum_{\vec{q}} n(\omega_{\vec{q}}) = S, \quad (2.23)$$

we find

$$\int_{-\infty}^{+\infty} d\omega n(\omega) \sum_{\vec{q},\vec{q}_1} \text{Im} G^0(\vec{q}, \omega)T(\vec{q}, \vec{q}_1, \omega)G^0(\vec{q}_1, \omega) = 0. \quad (2.24)$$

In the same way, from the definition of $Q_1$ we get the second equation

$$Q_1 = <a_l^\dagger a_m> = Q_f$$

$$+ \frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega n(\omega) \frac{1}{N^2} \sum_{\vec{q},\vec{q}_1} \text{Im} G^0(\vec{q}, \omega)T(\vec{q}, \vec{q}_1; \omega)G^0(\vec{q}_1, \omega)(1 - e^{i\vec{q} \cdot \vec{a}}). \quad (2.25)$$

When deriving this equation the constraint (2.24) has been taken into account.

We now solve the set of Eqs. (2.20), (2.24) and (2.25). The perturbation energy $V(\vec{q}, \vec{q}_1)$ is the sum of separable terms, and therefore the integral equation (2.20) is solved easily. The result is

$$T(\vec{q}, \vec{q}_1) = \sum_{\nu} \frac{\nu \nu f_\nu(\vec{q}) f_\nu^*(\vec{q}_1)}{D_\nu(\omega)}. \quad (2.26)$$

Here

$$D_\nu(\omega) = 1 - \nu \phi_\nu(\omega), \quad (2.27)$$

where

$$\phi_\nu(\omega) = \frac{1}{N} \sum_{\vec{q}} f_\nu^*(\vec{q}) G^0(\vec{q}, \omega) f_\nu(\vec{q}). \quad (2.28)$$
The functions $\phi_\nu(\omega)$ can be rewritten with the aid of Eqs. (2.16), (2.19) in the form

$$
\phi_1(\omega) = \frac{1}{2JS}[ -1 + \omega \varphi(\omega)],
$$

(2.29)

$$
\phi_2(\omega) = \frac{1}{2JS} + (4 - \frac{\omega}{2JS})\varphi(\omega).
$$

Here

$$
\varphi(\omega) = \frac{1}{N} \sum_{\vec{q}} (\omega - \omega_{\vec{q}})^{-1}
$$

(2.30)
is the local Green’s function for the undoped ferromagnet. We find from (2.30) that at small $\omega$ ($\omega \ll 4JS$)

$$
Re \varphi(\omega) = -\frac{1}{4\pi JS} \ln \left| \frac{4JS}{\omega - \Delta} \right|, \quad Im \varphi(\omega) = -\frac{1}{4JS} \Theta(\omega - \Delta).
$$

(2.31)

Eqs. (2.27) - (2.30)) yield

$$
D_1(\omega) = \zeta_f(T) + \mu_f + [1 - \zeta_f(T) - \mu_f] \omega \varphi(\omega),
$$

(2.32)

where

$$
\zeta_f(T) = 1 - \frac{W_f(T)}{2JQ}.
$$

(2.33)

We show next that $\mu_f(T)$ is always small at small temperatures $T \ll 4JS$, and the function $\zeta_f(T)$ is small, if the frustration is sufficiently strong. Thus, the function $D_1(\omega)$ is small too, when the defect bond is frustrated. This means, that the $T$-matrix and, hence, the spin wave scattering amplitude are enhanced strongly in this case. This is the reason of the unusually large effect of frustrated bonds on the properties of the doped magnets.

We substitute now the function $T(\vec{q}_1, \vec{q}_2)$ from (2.26) into Eqs. (2.24) and (2.25). Taking into account that

$$
\frac{1}{N} \sum_{\vec{q}} G^0(\vec{q}, \omega) f_n(\vec{q}) = \frac{1}{2} \phi_n(\omega),
$$

(2.34)

we rewrite these equations in terms of the functions $\phi_\nu(\omega)$
\[ v_1 \int_{-\infty}^{+\infty} d\omega n(\omega) \text{Im} \frac{\phi_1^2(\omega)}{D_1(\omega)} = -v_2 \int_{-\infty}^{+\infty} d\omega n(\omega) \text{Im} \frac{\phi_2^2(\omega)}{D_2(\omega)}. \quad (2.35) \]

\[ Q_1 = S + \frac{v_1}{2\pi} \int_{-\infty}^{+\infty} d\omega n(\omega) \text{Im} \frac{\phi_1^2(\omega)}{D_1(\omega)}. \quad (2.36) \]

When writing Eq. (2.36) we have replaced the parameter \( Q_f \) for the pure crystal by \( S \), since, as we shall see later, the difference \( S - Q_f \sim (T/4JS)^2 \) is always smaller than the second term in the r.h.s. of this equation.

It follows from Eq. (2.33) and the definition (2.8) of \( W_f \) that

\[ Q_1 = S \frac{1 - 2\zeta_f}{t + 1}, \quad (2.37) \]

where \( t = (J' - J)/J \). Eqs. (2.36) and (2.37) yield

\[ \frac{t + 2\zeta_f}{t + 1} = -\frac{v_1}{2\pi S} \int_{-\infty}^{+\infty} d\omega n(\omega) \text{Im} \frac{\phi_1^2(\omega)}{D_1(\omega)}. \quad (2.38) \]

After the expressions (2.29) - (2.31) for the functions \( \phi_1(\omega) \) and \( \phi_2(\omega) \) are inserted into Eqs. (2.35) and (2.38), one finds

\[ \frac{2JS^2\mu_f}{T + 8JS^2\mu_f} = \frac{1}{8S}K(T) \quad (2.39a) \]

\[ \frac{t + 2\zeta_f}{t + 1} = \frac{1}{S}K(T), \quad (2.39b) \]

where

\[ K(T) = \frac{1}{\pi} \int_0^{\infty} dx x n(4JSx)[(\zeta + \mu_f - \frac{x}{\pi} \ln \frac{1}{x})^2 + x^2]^{-1}. \quad (2.40) \]

When deriving these equations we neglected terms of higher order in the small quantities \( \zeta_f \) and \( \mu_f \).

In the quasiclassical limit, \( S \gg 1 \), we find that with logarithmic accuracy

\[ K(T) = \frac{Tl(T)}{8JS \mid \zeta_f(T) + \mu_f(T) \mid}, \quad (2.41) \]

where
\[ l(T) = 1 + \frac{2}{\pi} \arctan \left[ \frac{1}{\pi} \ln \frac{1}{\zeta_f + \mu_f} \right], \quad 1 < l(T) \leq 2, \] (2.42)

if \( \zeta_f + \mu_f > 0 \), and

\[ l(T) = 1 - \frac{2}{\pi} \arctan \left[ \frac{1}{\pi} \ln \frac{1}{|\zeta_f + \mu_f|} \right], \] (2.43)

if \( \zeta_f + \mu_f < 0 \).

From Eqs. (2.39) - (2.41) we finally obtain

\[ \mu_f \gtrless |\zeta_f + \mu_f| = \frac{l(T)}{2} \left( \frac{T}{8JS^2} \right)^2 (1 + \frac{8JS^2 \mu_f}{T}), \] (2.44a)

\[ \frac{t + 2\zeta_f}{t + 1} \gtrless |\zeta_f + \mu_f| = \frac{Tl(T)}{8JS^2}. \] (2.44b)

Eqs. (2.44) have a solution with \( \zeta_f + \mu_f > 0 \) at both \( t > 0 \) and \( t < 0 \):

\[ \zeta_f(T) = \frac{1}{4} \left\{ -t[1 + \frac{T}{2JS^2(t + 2)}] + [t^2 + \frac{Tl(T)(t + 1)}{JS^2}]^{1/2} \right\}, \] (2.45a)

\[ \mu_f = \frac{T}{8JS^2} \frac{t + 2\zeta_f}{t + 2}, \] (2.45b)

and

\[ D_1(0) = \zeta_f + \mu_f = \frac{1}{4} \left\{ -t + [t^2 + \frac{Tl(T)(t + 1)}{JS^2}]^{1/2} \right\}. \] (2.46)

If \( J' < J \) \( (t < 0) \) Eqs. (2.37) and (2.45) yield

1) \( T/JS^2 \gg t^2 \)

\[ \zeta_f = \left( \frac{Tl(T)}{16JS^2} \right)^{1/2} + O[t^2 \left( \frac{JS^2}{T} \right)^{1/2}], \quad \mu_f = \frac{T\zeta_f}{4JS^2} \ll \zeta_f \] (2.47a)

\[ Q_1 = S[1 - \left( \frac{Tl(T)}{4JS^2} \right)^{1/2}] \] (2.47b)

2) \( T/JS^2 \ll t^2 \ll 1 \)

\[ \zeta_f = \frac{|t|}{2} + \frac{Tl(T)}{8JS^2 \|t\|} + O\left( \frac{T^2}{J^2 S^4 \|t\|^3} \right), \quad \mu_f = \frac{T^2 l(T)}{(8JS^2)^2 \|t\|} \ll \zeta_f \] (2.48a)
The function $Q_1$ decreases with the increase of $T$ much faster than $Q$. When $T \to 0$, $D_1(0) = \zeta_f = |t|/2$, and $Q_1$, like $Q$, tends to $S$. Thus, the correlations of the spins, connected by the frustrated bond, are purely ferromagnetic, when the frustrated exchange $J'$ is lower than the threshold $J_c = J$.

The properties of the defect change drastically, when $J' > J$ ($t > 0$). If $T/JS^2 \ll t^2$, we obtain

$$\zeta_f = \frac{T}{8JS^2} \left[ l(T) \frac{t+1}{t} - \frac{t}{t+2} \right] + O\left( \frac{T^2}{J^2S^4t^3} \right),$$

$$D_1(0) = \frac{Tl(T)(t+1)}{8JS^2t}.$$ (2.49a, 2.49b)

It follows from Eqs. (2.37) and (2.49) that $Q_1(T \to 0) = S/(t+1) < S$. Thus, when $J' > J$, the ferromagnetic correlations decrease with the increase of $J'$. At large $J' \gg J$, the correlation amplitude $Q_1(0) \ll S$, i.e. the spins are almost antiparallel. The correlation function $Q_{\vec{r},\vec{r}+\delta}$ of two neighboring spins at large distances $r$ from the frustrated bond is also less than $S$, and approaches $S$ with the increase of $r$ as $r^{-4}$ (See Appendix A). Thus, the spins at the ends of the frustrated bond act on the ferromagnetic background like a dipole, in agreement with the classical picture.

The negative solution of Eqs. (2.44)

$$\zeta_f = D_1(0) = -t/2 + O(T/4JS^2), \quad t \ll 1,$$ (2.50)

which appears, when $t > 0$, resembles the bound state solution, obtained in Ref. 14 for a frustrated bond in a 3D ferromagnet. Indeed, the $T$-matrix (2.26) in this case diverges at the negative energy $\omega = -2\pi JS t/\ln(1/t)$, which is the condition of a bound state formation. At $t \approx 1$ this state lies far from the bottom of the spin-wave band. Hence, it almost does not alter the spin-wave spectrum. In contrast, the dipole-type state affects strongly the properties of the ferromagnet. Therefore we consider in what follows only the dipole-type states.
Eqs. (2.39) can be solved also at arbitrary spins. If $|t| \ll 1$, the function $\mu_f \ll \zeta_f$, and $\zeta_f = D_1(0)$ is governed by the equation:

$$
\zeta_f^2 (2\zeta_f + t) = 0.524 S \left( \frac{T}{4J S^2} \right)^2.
$$

(2.51)

This equation yields

1) $J' < J, \ |t| \gg (T/4JS)^{2/3}$

$$
\zeta_f(T) = \frac{|t|}{2} + 1.05 \frac{1}{St^2} \left( \frac{T}{4JS} \right)^2 + O[\left( \frac{T}{4JS^2} \right)^4 \left( \frac{1}{|t|^5} \right)],
$$

(2.52)

2) $|t| \ll (T/4JS)^{2/3}$

$$
\zeta_f(T) = \frac{0.645}{S^{1/3}} \left( \frac{T}{4JS^2} \right)^{2/3} + O(|t|),
$$

(2.53)

3) $J' > J, \ 1 \gg t \gg (T/4JS)^{2/3}$

$$
\zeta_f(T) = \frac{0.724S^{1/2}}{t^{1/2}} \frac{T}{4JS^2} + O\left( \frac{T}{4JS^{2T}} \right)^2.
$$

(2.54)

Finally, if $t \geq 1$, the function $D_1$ is given by Eq. (2.49b) with $l(T) = 2$, up to small terms of the order of $\ln^{-1} 4JS/T$. Thus, at $t > 0$ the main features of the functions $\zeta_f(T)$ and $Q_1(T)$ are the same, as for large spins. In other words, the dipole-like picture of the frustrated bond defect is valid not only in the classical limit, but rather at arbitrary spins.

C. The renormalization of the spin-wave spectrum

Averaging Eq. (2.18) over the distribution of the defects and the orientations of the defect bonds, we find the renormalized spin wave spectrum $\epsilon_{\vec{q}}$. To first order in the defect concentration $x$ the configurationally averaged Green’s function is given as

$$
G^{-1}(\vec{q}, \omega) = \omega - \omega_{\vec{q}} - \Sigma(\vec{q}, \omega), \quad \epsilon_{\vec{q}}.
$$

(2.55)

where the self-energy is given by $\Sigma(\vec{q}, \omega) = xT(\vec{q}, \vec{q}; \omega)$. Thus,

$$
\epsilon_{\vec{q}} = \omega_{\vec{q}} + xT(\vec{q}, \vec{q}; \omega_{\vec{q}}).
$$

(2.56)
In the long-wave limit \( \epsilon \sim T / \ln(4JS/\epsilon) \), we obtain for \( \epsilon \)

\[
\epsilon = \Omega |1 - \frac{x}{D_1(0)}| + \Delta_1(T),
\]

where \( \Omega = 4JS(1 - \gamma \epsilon) \). The properties of the renormalized gap \( \Delta_1 \) will be discussed later.

When the frustration is weak, we obtain \( D_1(0) \approx \ell \approx 1 \). The renormalization of the spin-wave stiffness is of the order of \( x \) and temperature independent. At strong frustrations one has \( D_1(0) \approx \zeta_f(T) \ll 1 \). The renormalization is enhanced, and the enhancement increases with the decrease of the temperature (see Eqs. (2.47) - (2.49), (2.52) - (2.54)).

It follows from Eqs. (2.57) and (2.49a) that for strongly frustrated bonds \( J' > J \) the renormalization of the stiffness \( \rho_s \) can be written as

\[
\frac{\rho_s(0) - \rho_s(x)}{\rho_s(0)} = \frac{U_f}{T} x,
\]

where \( U_f = 8JS^2 t/l(T)(t + 1) \). Even a small defect concentration \( x < T/U_f \ll 1 \) reduces significantly the stiffness.

**D. The correlation length**

The gap in the spin wave spectrum of a pure ferromagnet is governed, as mentioned above, by the constraint (2.3). A self-consistent way to obtain the gap in a doped configurationally averaged crystal is to impose on the spectrum (2.57) the constraint (2.3), averaged over the defect distribution. Since the defects under consideration preserve the value of the spins, we obtain the same Eqs. (2.11), (2.12) as for the undoped crystal, with \( \omega \) replaced by \( \epsilon \). The gap \( \Delta_1 \) is then given by Eq (2.13) with the exchange energy \( J \) replaced by the renormalized value \( J(1 - x/D_1(0)) \). Thus, the correlation length in the doped crystal is given by

\[
\xi(T, x) \propto \exp\left[ \frac{2\pi JS^2 T}{T}(1 - \frac{x}{D_1(0)}) \right].
\]

At a given \( x \) the ratio \( \xi(T, x)/\xi(T, 0) \) decreases rapidly with the decrease of the temperature. Even though the expression (2.57) is valid only at small concentrations \( x \ll D_1(0) \), the renormalization of the correlation length may be exponentially large, if \( x \gg D_1(0)(T/2\pi JS^2) \).
III. DEFECT BONDS IN ANTIFERROMAGNETS

A. The Hamiltonian

We shall consider a bipartite antiferromagnet on a square lattice. Only nearest neighbor interactions are included in the Hamiltonian, and it is supposed, as before, that there is a small number of defect bonds with exchange energy \( J' \neq J \), where \( J \) is the host exchange energy. The Hamiltonian is given by

\[
H = \sum_{<ij>} J_{ij} \vec{S}_i \cdot \vec{S}_j,
\]

where the couplings \( J_{ij} \) are equal to \( J \) or \( -J' \). The transformation to Schwinger bosons for the spins on the sublattices \( A \) and \( B \) is as follows:

\[
S^+_A = a^\dagger b, \quad S^-_A = ab^\dagger, \quad S^z_A = \frac{1}{2}(a^\dagger a - b^\dagger b)
\]

\[
S^+_B = -ab^\dagger, \quad S^-_B = -a^\dagger b, \quad S^z_B = \frac{1}{2}(b^\dagger b - a^\dagger a)
\]

with the constraint \( \langle 2.3 \rangle \). After the mean-field decoupling is performed, the pure antiferromagnet Hamiltonian \( H_0 \) and the interaction Hamiltonian \( H_{int} \) transform to

\[
H_0 = -\frac{JQ}{2} \sum_{<ij>} (a_i a_j + a_i^\dagger a_j^\dagger + b_i b_j + b_i^\dagger b_j^\dagger)
\]

\[
+ \lambda \sum_i (a_i^\dagger a_i + b_i^\dagger b_i), \quad (3.3)
\]

\[
H_{int} = (W - 2JQ\mu) \sum_{<lm>} [(a_i - a_i^\dagger)(a_m - a_m^\dagger) + (b_i - b_i^\dagger)(b_m - b_m^\dagger)]
\]

\[
+ 2JQ\mu \sum_{<lm>} [(a_i^\dagger + a_m)(a_m^\dagger + a_i) + (b_i^\dagger + b_m)(b_m^\dagger + b_i)]. \quad (3.4)
\]

Here \( W = JQ + J'Q' \), \( \mu = (1/4JQ)(\lambda_1 - \lambda + W) \),

\[
Q = \frac{1}{2} < a_i a_j + b_i b_j > = \frac{1}{2} < a_i^\dagger a_j^\dagger + b_i^\dagger b_j^\dagger >, \quad (3.5)
\]

and \( Q' \) is given by Eq. \( \text{(3.3)} \) with the bonds \( < ij > \) replaced by the defect bond \( < lm > \). The sums in \( \text{(3.4)} \) are over all defect bonds \( < lm > \).
In momentum space the Hamiltonian $H_0$ can be diagonalized by the standard Bogoliubov transformation

$$a_\vec{q} = u_\vec{q} \alpha_\vec{q} + v_\vec{q} \alpha^\dagger_{-\vec{q}},$$  \hspace{1cm} (3.6a)$$

$$b_\vec{q} = u_\vec{q} \beta_\vec{q} + v_\vec{q} \beta^\dagger_{-\vec{q}},$$  \hspace{1cm} (3.6b)$$

where

$$u_\vec{q}^2 = \frac{JQz + \lambda + \omega_\vec{q}}{2\omega_\vec{q}}, \quad v_\vec{q}^2 = \frac{JQz + \lambda - \omega_\vec{q}}{2\omega_\vec{q}}.$$  \hspace{1cm} (3.7)$$

The quasiparticle energy has the form

$$\omega_\vec{q}^2 = (JzQ)^2 (1 - \gamma_\vec{q}^2) + \Delta^2,$$  \hspace{1cm} (3.8)$$

where the gap is given by $\Delta^2 = (\lambda + JQz)^2 - (JQz)^2$. From equations (2.3) and (3.5)-(3.8) the self-consistent equations, obeyed by the mean-field amplitude $Q$ and the gap $\Delta$, can be obtained:

$$(JQz + \lambda) \frac{1}{N} \sum_\vec{q} \omega_\vec{q}^{-1} (n_\vec{q} + \frac{1}{2}) = S + \frac{1}{2},$$  \hspace{1cm} (3.9)$$

$$\frac{Jz}{N} \sum_\vec{q} \gamma_\vec{q}^2 \omega_\vec{q}^{-1} (n_\vec{q} + \frac{1}{2}) = 1.$$  \hspace{1cm} (3.10)$$

In the limit of large spin $S \gg 1$, one has the amplitude $Q = S(1 + 0.158/2S)$, the gap

$$\Delta \propto T \exp(-2\pi JQS^*/T),$$  \hspace{1cm} (3.11)$$

and the correlation length

$$\xi(T) \propto \exp(2\pi JQS^*/T),$$  \hspace{1cm} (3.12)$$

where $S^* = S(1 - 0.197/S)$.  

The interaction Hamiltonian (3.4) can be written in the momentum representation in a simple form,
\[ H_{\text{int}} = \frac{1}{N} \sum_{\nu=1}^{2} V_{\nu} \sum_{\vec{q}, \vec{q}_1} \langle A_{\vec{q}} | \eta_{\nu}(\vec{q}) \rangle \langle \eta_{\nu}(\vec{q}_1) | A_{\vec{q}_1} \rangle. \]  

(3.12)

Here we introduced the two-component column vectors \(| A_{\vec{q}} \rangle, | \eta_{\nu}(\vec{q}) \rangle\), and row vectors \langle A_{\vec{q}} |, \langle \eta_{\nu}(\vec{q}) |\) as:

\[
| A_{\vec{q}} \rangle = \begin{pmatrix} a_{\vec{q}} \\ a_{\vec{q}}^\dagger \end{pmatrix}, \quad \langle A_{\vec{q}} | = (a_{\vec{q}}^\dagger, a_{\vec{q}}^-) ;
\]

(3.13)

\[
| \eta_{\nu}(\vec{q}) \rangle = \begin{pmatrix} 1 \\ (-1)^{\nu} e^{-i\vec{q} \cdot \vec{a}} \end{pmatrix}, \quad \langle \eta_{\nu}(\vec{q}) | = \begin{pmatrix} 1 \\ (-1)^{\nu} e^{i\vec{q} \cdot \vec{a}} \end{pmatrix},
\]

(3.14)

and \(V_1 = 2JQ\mu - W, V_2 = 2JQ\mu.\)

**B. The one-bond problem**

Let us define the Green’s function matrix by the relation

\[
G_{\alpha\beta}(\vec{q}, \vec{q}_1; t - t_1) = -i\Theta(t - t_1) [A_{\alpha\vec{q}}(t), A_{\beta\vec{q}_1}(t_1)]^T,
\]

(3.15)

where \(A_{1\vec{q}} = a_{\vec{q}}, A_{2\vec{q}} = a_{\vec{q}}^\dagger\). The unperturbed Green’s function matrix is as follows:

\[
\hat{G}^0(\vec{q}, \omega) = \frac{1}{\omega^2 - \omega_{\vec{q}}^2} \begin{pmatrix} 4JQ + \omega & 4JQ\gamma_{\vec{q}} \\ 4JQ\gamma_{\vec{q}} & 4JQ - \omega \end{pmatrix}.
\]

(3.16)

The one-defect problem is solved in the same way, as for the ferromagnet. The Green’s function matrix and the \(T\)-matrix obey Eqs. (2.18) and (2.20), with the functions \(G\) and \(T\) replaced by the \(2 \times 2\) matrices \(\hat{G}\) and \(\hat{T}\). The solution for \(\hat{T}(\vec{q}, \vec{q}_1; \omega)\) is

\[
\hat{T}(\vec{q}, \vec{q}_1; \omega) = \sum_{\nu=1}^{2} \frac{V_{\nu} \langle \eta_{\nu}(\vec{q}) \rangle \langle \eta_{\nu}(\vec{q}_1) |}{D_{\nu}(\omega)}.
\]

(3.17)

Here

\[
D_{\nu}(\omega) = 1 - V_{\nu} \psi_{\nu}(\omega),
\]

(3.18)

and
\( \psi_{\nu} = \frac{1}{N} \sum_{\vec{q}} < \eta_{\nu}(\vec{q}) | \hat{G}^0(\vec{q}; \omega) | \eta_{\nu}(\vec{q}) > . \) (3.19)

Using Eqs. (3.14) and (3.16) we find

\( \psi_1(\omega) = -\frac{1}{2JQ} [1 - (\omega^2 - \Delta^2)g(\omega)], \) (3.20a)

\( \psi_2(\omega) = \frac{1}{2JQ} + (16JQ - \frac{\omega^2 - \Delta^2}{2JQ})g(\omega), \) (3.20b)

where the local unperturbed Green’s function \( g(\omega) \) is

\( g(\omega) = \frac{1}{N} \sum_{\vec{q}} (\omega^2 - \omega_{\vec{q}}^2)^{-1}. \) (3.21)

The function \( g(\omega) \) is easily calculated at small frequencies \( \omega \ll 4JQ: \)

\( Reg(\omega) = -\frac{1}{8\pi J^2 Q^2} [\ln \frac{8JQ}{|\omega^2 - \Delta^2|^{1/2}} - \frac{1}{2\pi} + O(\frac{\omega^2}{J^2 Q^2})], \) (3.22a)

\( Img(\omega) = -\frac{1}{(4JQ)^2} [\Theta(\omega - \Delta) - \Theta(-\omega - \Delta)]. \) (3.22b)

From Eqs. (3.20) and (3.21) we find the function \( D_1(\omega), \)

\( D_1(\omega) = \zeta(T) + \mu + (1 - \zeta - \mu)\omega^2 g(\omega), \) (3.23)

where

\( \zeta(T) = 1 - \frac{W(T)}{2JQ}. \) (3.24)

We see that the spin-wave scattering on frustrated bonds is enhanced strongly, since, as we show below, \( \zeta(T) \) and \( \mu(T) \) are small.

Let us proceed to the derivation of the equations, which govern the function \( Q' \) and the local Lagrange multiplier \( \mu. \) In terms of the Green’s function matrix, the local constraint in the momentum representation can be written as

\[
\int_{-\infty}^{+\infty} d\omega \ n(\omega) Im \sum_{\vec{q}\vec{q}_1} < \chi_\uparrow | \hat{G}(\vec{q}, \vec{q}_1; \omega) - \hat{G}^0(\vec{q}, \omega) \delta_{\vec{q}\vec{q}_1} | \chi_\uparrow > = 0,
\] (3.25)
where \( | \chi_\uparrow > \) is the spin 1/2 spinor with spin up.

Equations (2.18), (3.17) and (3.25) yield

\[
\int_{-\infty}^{+\infty} d\omega n(\omega) \Im \sum_{\nu=1}^{2} V_{\nu} \frac{L_{\nu}(\omega)L_{\nu}\uparrow(\omega)}{D_{\nu}(\omega)} = 0.
\] (3.26)

Here

\[
L_{\nu}(\omega) = L_{\nu}\uparrow(\omega) = \frac{1}{N} \sum_{\vec{q}} < \chi_{\downarrow} | \hat{G}^{0}(\vec{q},\omega) | \eta_{\nu}(\vec{q}) >.
\] (3.27)

The equation for \( Q' \) can be derived in the same way. The first step is to transform the initial expression

\[
Q' = \frac{1}{N} \sum_{\vec{q}\vec{n}} < a_{\vec{q}}^\dagger a_{-\vec{q}}^\dagger > e^{i\vec{q}\cdot\vec{a}}
\] (3.28)

with the aid of Eq. (3.17), and the constraint (3.25). We find

\[
Q' = Q - \frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega n(\omega)\ Im \sum_{\nu} \frac{V_{\nu}L_{\nu}\uparrow(\omega)}{D_{\nu}(\omega)} \frac{1}{N} \sum_{\vec{q}} < \chi_{\downarrow} e^{i\vec{q}\cdot\vec{a}} - \chi_{\uparrow} | \hat{G}^{0}(\vec{q},\omega) | \eta_{\nu}(\vec{q}) >,
\] (3.29)

where \( < \chi_{\downarrow} \mid \) is the spinor with spin down. Taking into account the relation

\[
< \chi_{\downarrow} e^{i\vec{q}\cdot\vec{a}} - \chi_{\uparrow} | = - < \eta_{1}(\vec{q}) |,
\]

we find

\[
Q' = Q + \frac{1}{\pi} \int_{-\infty}^{+\infty} d\omega n(\omega) \ Im \frac{V_{1}\psi_{1}(\omega)}{D_{1}(\omega)} L_{1}\uparrow(\omega).
\] (3.30)

The functions \( L_{\nu\uparrow}(\omega) \) can be written in terms of the functions \( \psi_{\nu}(\omega) \) and \( g(\omega) \), as

\[
L_{1}\uparrow(\omega) = \frac{\psi_{1}(\omega)}{2} + \omega g(\omega),
\] (3.31a)

\[
L_{2}\uparrow(\omega) = (\omega + 8JQ)g(\omega) - \frac{1}{2}\psi_{1}(\omega).
\] (3.31b)

Using the relation between \( Q' \) and \( \zeta \)

\[
Q' = Q \frac{1 - 2\zeta}{1 + t}
\] (3.32)

and Eqs. (3.11), (3.20), (3.26), and (3.30) we finally obtain
\[ \mu = \frac{T}{16JQ^2S}\left(1 + \frac{8JS^*\mu}{T}\right)I(T), \]  

(3.33a)

\[ \frac{t + 2\zeta}{t + 1} = \frac{1}{Q}I(T), \]  

(3.33b)

where

\[ I(T) = \frac{1}{\pi} \int_0^\infty dx \frac{x^2 \coth \frac{2JQx}{T}}{x^2} \left[ (\zeta + \mu - \frac{x^2}{\pi} \ln \frac{1}{x^2})^2 + x^4 \right]^{-1}. \]  

(3.34)

In the large spin limit the integral \( I(T) \) is given by

\[ I(T) = \frac{Tl(T)}{8JQ(\zeta + \mu)} + \frac{1}{2} \left[ \frac{\pi}{2(\zeta + \mu) \ln(\zeta + \mu)^{-1}} \right]^{1/2} + O(1) + O(\ln^{-2} \frac{1}{\zeta}), \zeta + \mu > 0. \]  

(3.35)

The second term in the r.h.s. of this equation takes into account the effect of quantum fluctuations. The slowly varying function \( l(T) \) is given by Eq. (2.42) with \( \zeta_f, \mu_f \) replaced by \( \zeta, \mu \).

Several results follow immediately. First, we see that \( \mu \), like \( \mu_f \), tends to 0, when \( T \to 0 \).

Near the threshold, when \( |t| \ll 1 \), the quantum correction to the mean-field amplitude \( Q' \) exceeds the correction (of the order of 1/S) to the function \( Q \) in the undoped samples. Indeed, if \( S^{-2/3} \ll |t| \ll 1 \), and \( T = 0 \), the mean-field amplitude becomes

\[ Q' = Q - \left( \frac{\pi}{|t| \ln(1/|t|)} \right)^{1/2}. \]  

(3.36)

The quantum correction increases with the decrease of \( |t| \) as \( (|t| \ln |t|)^{-1/2} \), in agreement with the result obtained in the linear spin wave theory.\(^{17}\) However, while in the linear spin-wave theory the quantum corrections diverge when \( |t| \to 0 \), we find a finite value in this case

\[ Q' = Q[1 - \left( \frac{\pi}{(2Q)^2 \ln 2Q} \right)^{1/3}]. \]  

(3.37)

It follows from Eqs. (3.33b) and (3.33) that the quantum corrections, associated with the spin-wave scattering on strongly frustrated bonds \( (t \gg S^{-2/3}) \), are of the order of \( S^{-2} \ln^{-1} S \), i.e. smaller than the usual 1/S corrections:
When deriving these equations we neglected terms of order unity in Eq. (3.35). The effect of these terms is to replace $2Q$ in Eqs. (3.38), (3.39) by $2Q + \kappa$, with $\kappa \approx 1$. Thus $\zeta \ll 1$ for any value of the spin. This implies that the quantum effects are not very important even for real spins $S = 1/2$.

Neglecting the quantum fluctuations, we find that at $T = 0$, $t > 0$ one has $\zeta(T) = 0$, and $Q' = S/(1 + t) < S$. Like in the ferromagnet, the function $S - Q_{\vec{r}, \vec{r} + \delta}$ decays as $r^{-4}$ at large distances $r$ from the frustrated bond (see Appendix A). Hence, the spins at the end of the frustrated bond act as a dipole, when $J' > J$.

At finite temperatures Eqs. (3.33) and (3.35) yield in the large spin limit that at $t \gg \zeta$ one has

$$D_1(0) = \zeta + \mu = \frac{Tl(T)(t + 1)}{8J^2 t}.$$  

The function $D_1(0)$ coincides, up to terms of the order of $1/S$, with Eq. (2.49) for $D_1(0)$ in a ferromagnet. The solution of Eqs. (3.33)-(3.35) in the quasiclassical limit at small $t$ also coincides with the corresponding solutions [Eqs. (2.47), (2.48)] in a ferromagnet, and will not be given here.

So far, we discussed only the positive solution $D_1 > 0$ of Eqs. (3.33). As for the ferromagnet, these equations also have a negative solution at $J' > J$. However, unlike the ferromagnet, this solution does not correspond to a bound state, since the zero temperature $T$-matrix in the antiferromagnet has no pole at negative $\omega$.

C. The spin-wave spectrum and the correlation length

To leading order in $x$, the configurationally averaged Green’s function is given by
\[ [1 - \tilde{G}_0(\tilde{q}; \omega) \tilde{T}(\tilde{q}, \tilde{q}; \omega)x] \tilde{G}(\tilde{q}, \omega) = \tilde{G}_0(\tilde{q}; \omega). \] (3.41)

The renormalized spin-wave spectrum, which follows from the solution of this equation, can be written in terms of the \( T(\tilde{q}, \tilde{q}) \)-matrix elements as

\[ \epsilon^2_{\tilde{q}} = \omega^2_{\tilde{q}} + x [4JQ(T_{11} + T_{22}) + \omega_q(T_{11} - T_{22}) + 4JQ\gamma_{\tilde{q}}(T_{12} + T_{21})]. \] (3.42)

We have at small frequencies, \( \omega_q \ll 4JQ \),

\[ \epsilon^2_{\tilde{q}} = \omega^2_{\tilde{q}}[1 - \text{Re} \frac{x}{D_1(\omega_q)}]. \] (3.43)

In the small \( q \) limit, when \( D_1(0) \gg (\omega^2_q/8\pi J^2 Q^2) \ln(4JQ/\omega_q) \), the spin-wave spectrum is

\[ \epsilon^2_{\tilde{q}} = E^2_{\tilde{q}}[(1 - \frac{x}{D_1(0)}) + \Delta^2(T)], \] (3.44)

where \( E^2_{\tilde{q}} = (4JQ)^2(1 - \gamma^2_{\tilde{q}}) \). At higher frequencies the function \( D_1(0) \) can be dropped in Eq. (3.43), and we find

\[ \epsilon^2_{\tilde{q}} = E^2_{\tilde{q}} + 8\pi J^2 Q^2 x \ln^{-1} \frac{4JQ}{E_{\tilde{q}}}. \] (3.45)

The wave-vector dependence of the last term in the r.h.s. of this equation is subtle. The spin-wave spectrum, hence, acquires a concentration dependent gap, the spin-wave velocity being the same as in the undoped antiferromagnet.

Like for ferromagnets, the renormalization in the small \( q \) region is of the order of \( x \) for weakly frustrated bonds, and is enhanced and temperature dependent, when \( t < 0, |t| \ll 1 \) or \( t > 0 \). For strongly frustrated bonds \( (t > 0) \) we have

\[ \epsilon^2_{\tilde{q}} = E^2_{\tilde{q}}(1 - \frac{2Ux}{T}) + \Delta^2_1(T), \] (3.46)

where

\[ U = \frac{4JQ^2 t}{l(T)(t + 1)}. \] (3.47)

The renormalized spin wave velocity is, hence, given by
\[ c(x) = 2^{3/2} J Q \left( 1 - \frac{U x}{T} \right). \]  

(3.48)

The *averaged* constraint (2.3) governs, like for ferromagnets, the renormalized spin-wave gap \( \Delta_1(x) \). For the correlation length at \( S \gg 1 \) we find Eq. (3.11), with the exchange coupling renormalized according to Eq. (3.44):

\[ \xi(T) = C \exp \left[ \frac{2 \pi J Q S^*}{T} \left( 1 - \frac{x}{2 D_1(0)} \right) \right]. \]  

(3.49)

As noted above, the SBMFA gives correctly only the exponent. The two-loop calculation of Chakravarty *et al.*\(^5\) as well as Monte Carlo simulations (Ref. 36 and references therein) and experimental data\(^2,3\) show that the prefactor \( C \) in this equation does not depend on the temperature up to small terms of the order of \( T/2 \pi \rho_s \). At \( J' > J \), we have

\[ \xi(T) = C \exp \left[ \frac{2 \pi J Q S^*}{T} \left( 1 - \frac{U x}{T} \right) \right]. \]  

(3.50)

At sufficiently low temperatures the ratio \( \xi(x, T)/\xi(0, T) \) is small even at small dopant concentration \( x \ll T/U \ll 1 \).

**D. The phase transition in the quasi-2D antiferromagnets**

The decrease of the 2D correlation length with doping causes the rapid reduction of the Néel temperature in the quasi-2D antiferromagnets. Starting from the relation \(^1\)

\[ T_N(x) \approx J_\perp \xi^2(T_N, x), \]  

(3.51)

Eq. (3.49) yields

\[ T_N(x) = T_N(0) \left[ 1 - \frac{x}{D_1(T = T_N(0))} \right], \]  

(3.52)

where the Néel temperature for the undoped antiferromagnet is given by

\[ T_N(0) \approx 4 \pi J S^2 / \ln(J/J_\perp). \]  

(3.53)
Since \( x \) in Eq. (3.52) is multiplied by a large factor \( 1/D_1(T_N(0)) \), the Néel temperature is suppressed rapidly, when the dopant concentration increases. When \( J' > J \), the Néel temperature decreases with the increase of doping as

\[
T_N(x) = T_N(0)(1 - \frac{UX}{T_N(0)}),
\]

(3.54)

At a sufficiently large ratio \( J/J_\perp \), the Néel temperature extrapolates to zero at a doping concentration \( x \) which is much smaller than the percolation threshold.

**IV. FRUSTRATED PLAQUETTES**

In this section we consider the effect of a more complicated defect, i.e. a frustrated plaquette, on the properties of a 2D antiferromagnet. The plaquette is formed from 4 frustrated bonds, which connect 4 neighboring spins. To begin with, let us summarize the results obtained in Appendix B in the harmonic spin-wave approximation. The \( T \)-matrix has a pole, when the frustrated coupling, \( J' \), reaches the critical value \( J_c = 0.376 J \). Like in the case of a single frustrated bond, the divergence of the \( T \)-matrix signals a local instability of the system. When \( J' > J_c \), the defect gives rise to a canted ground state. The threshold for instability of the perfectly aligned antiferromagnetic ground state is shifted to a weaker value, when the extra hole frustrates four bonds in a plaquette, rather than one bond. The reason is that in a plaquette two of the four bonds, connecting each spin with its neighbors, are frustrated.

There is a remarkable difference between the effect of frustrated plaquettes on the spin-wave stiffness in the ferro- and antiferromagnets. In antiferromagnets the stiffness diverges when \( J' \rightarrow J_c^- \). In ferromagnets the divergent term in the \( T \)-matrix scales as \( q^4 \), and hence the stiffness passes smoothly the singular point \( J' = J_c \). Therefore we employ the SBMFA to consider frustrated plaquettes only in antiferromagnets.

The interaction Hamiltonian we wish to treat is the following

\[
H_{int} = W_{pl} \sum_{\langle lm \rangle} (a_l a_m + a_l^\dagger a_m^\dagger) + (\lambda_1 - \lambda) \sum_{l=1}^4 a_l^\dagger a_l.
\]

(4.1)
Here the first sum < \l m > runs over the frustrated bonds < 12 >, < 13 >, < 24 >, < 34 >; λ₁ is the Lagrange multiplier for the spins 1-4 at the corners of the plaquette. The energy \( W_{pl} = J'Q_{pl} + JQ \), and \( Q_{pl} \) is the correlation amplitude of the frustrated bonds. Evidently, \( Q_{pl} \) is the same for all 4 frustrated bonds in the plaquette.

The interaction energy matrix

\[
\hat{V} = \begin{pmatrix}
\lambda_1 - \lambda & W_{pl} & W_{pl} & 0 \\
W_{pl} & \lambda_1 - \lambda & 0 & W_{pl} \\
W_{pl} & 0 & \lambda_1 - \lambda & W_{pl} \\
0 & W_{pl} & W_{pl} & \lambda_1 - \lambda
\end{pmatrix}
\]  

(4.2)

is diagonalized by the unitary transformation (B3). The Fourier-transformed interaction Hamiltonian (4.1) becomes

\[
H_{int} = \sum_{i=1}^{4} w_i \sum_{\vec{q}, \vec{q}_1} \langle A_{\vec{q}} | y_i(\vec{q}) \rangle \langle y_i(\vec{q}_1) | A_{\vec{q}} \rangle .
\]  

(4.3)

Here the two-component column (row) vector \( | A_{\vec{q}} \rangle \) (\( < A_{\vec{q}} | \)) is given by Eq. (3.13),

\[
w_1 = w_3 = JQ\mu_{pl} - W_{pl}/2, \quad w_2 = JQ\mu_{pl} - W_{pl}, \quad w_4 = JQ\mu_{pl} = (\lambda_1 - \lambda + 2W_{pl})/4,
\]

\[
| y_i(\vec{q}) \rangle = \begin{pmatrix} l_i(\vec{q}) \\ m_i(\vec{q}) \end{pmatrix}, \quad < y_i(\vec{q}) | = (l_i^*(\vec{q}), \quad m_i^*(\vec{q})) ,
\]  

(4.4)

where

\[
l_1(\vec{q}) = l_3(\vec{q}) = 1 - e^{-i(q_x + q_y)} , \quad m_1(\vec{q}) = -m_3(\vec{q}) = e^{-iq_x} - e^{-iq_y} \]  

(4.5a)

\[
l_2(\vec{q}) = l_4(\vec{q}) = 1 + e^{-i(q_x + q_y)} , \quad m_2(\vec{q}) = -m_4(\vec{q}) = -e^{-iq_x} - e^{-iq_y} .
\]  

(4.5b)

The equation for the one-defect \( T \)-matrix can be solved as before. The result is

\[
\hat{T}(\vec{q}, \vec{q}_1; \omega) = \sum_{i=1}^{4} w_i \left| \frac{y_i(\vec{q}) \langle y_i(\vec{q}_1) |}{\Lambda_i(\omega)} \right|.
\]  

(4.6)

Here

\[
\Lambda_i(\omega) = 1 - w_i\Psi_i(\omega),
\]  

(4.7)
where
\[ \Psi_i(\omega) = \frac{1}{N} \sum_{\vec{q}} < y_i(\vec{q}) | \hat{G}^0(\vec{q}, \omega) | y_i(\vec{q}) >. \] (4.8)

It is straightforward to calculate the functions \( \Psi_i(\omega) \) substituting expressions (3.16), (4.4), (4.5) into Eq. (4.8). This yields
\[ \Psi_1(\omega) = \Psi_3(\omega) = \frac{16JQ}{N} \sum_{\vec{q}} \frac{1 - \cos q_x \cos q_y}{\omega^2 - \omega^2_q}, \] (4.9a)
\[ \Psi_2(\omega) = \frac{8JQ}{N} \sum_{\vec{q}} \frac{\sin q_x + \sin q_y}{\omega^2 - \omega^2_q}, \] (4.9b)
\[ \Psi_4(\omega) = \frac{8JQ}{N} \sum_{\vec{q}} \frac{2 + \cos^2 q_x + \cos^2 q_y + 4 \cos q_x \cos q_y}{\omega^2 - \omega^2_q}. \] (4.9c)

Setting \( \mu_{pl} = 0 \) in Eq. (4.6) recovers in the quasiclassical limit the \( T \)-matrix expression derived in Appendix B, for the plaquette defect in the long range ordered antiferromagnet at zero temperature.

Converting the sums over \( \vec{q} \) into integrals over the first Brillouin zone in the reciprocal lattice, one finds numerically the values
\[ \Psi_1(0) = -1.273/JQ; \quad \Psi_2(0) = -0.727/JQ. \] (4.10)

Hence,
\[ \Lambda_1(0) = \Lambda_3(0) = 0.124 + 0.876\zeta_{pl} + 1.273\mu_{pl}, \] (4.11a)
\[ \Lambda_2(0) = \zeta_{pl} + 0.727\mu_{pl}, \] (4.11b)

where
\[ \zeta_{pl} = 1 - 0.727 \frac{W_{pl}}{JQ} \] (4.12)

Next we show that \( \zeta_{pl} \) and \( \mu_{pl} \), like \( \zeta \) and \( \mu \) in the previous section, are small at low temperatures, if the frustration of the bonds in the defect plaquette is sufficiently strong,
i.e. if \(J' > J_c\), or \(|J' - J_c|/J_c \ll 1\). Hence, \(\Lambda_1(0) \gg \Lambda_2(0)\), i.e. the first and third modes in Eq. (4.10) may be neglected when considering the effect of the frustrated plaquette.

Given the expressions for the \(T\)-matrix and the functions \(\Psi_i(\omega)\), we can derive, as before, the equations for the functions \(\zeta_{pl}(T)\) and \(\mu_{pl}(T)\). We find

\[
\int_{-\infty}^{+\infty} d\omega n(\omega) \left\{ w_2 Im \frac{[\Psi_2(\omega) + 4\omega g(\omega)]^2}{\Lambda_2(\omega)} + w_4 Im \frac{[\Psi_4(\omega) + 8\omega g(\omega)]^2}{\Lambda_4(\omega)} \right\} = 0, \tag{4.13}
\]

\[
Q_{pl} = Q + \frac{w_2}{8\pi} \int_{-\infty}^{+\infty} d\omega n(\omega) Im \frac{\Psi_2(\omega)[\Psi_2(\omega) + 4\omega g(\omega)]}{\Lambda_2(\omega)}. \tag{4.14}
\]

Taking into account that at low frequencies, \(\omega \ll 4JS\), the functions \(\Psi_2(\omega)\) and \(\Psi_4(\omega)\) can be simplified as

\[
\Psi_2(\omega) = \Psi_2(0) + \frac{\omega^2}{JQ} g(\omega), \quad \Psi_4(\omega) = 64JQ g(\omega), \tag{4.15}
\]

we find from Eqs. (4.13) and (4.14):

\[
\mu_{pl} = \frac{T}{16JS^2} (1 + \frac{8JS^2}{T}) I_1(T), \tag{4.16a}
\]

\[
\frac{t + \zeta_{pl}(j_c^{-1} + 1)}{t + 1} = \frac{1}{Q} I_1(T), \tag{4.16b}
\]

\[
Q_{pl}(T) = Q \frac{j_c - \zeta_{pl}(T)(1 + j_c)}{j_c(t + 1)}. \tag{4.16c}
\]

Here \(t = (J' - J_c)/J_c, j_c = J_c/J = 0.376\), and

\[
I_1(T) = \frac{1}{\pi} \int_0^\infty dx x^2 \times \coth \frac{2JQx}{T} \{[\zeta_{pl} + \frac{\mu_{pl}}{1 + j_c} - \frac{1 + j_c}{\pi} x^2 \ln \frac{1}{x^2}]^2 + (1 + j_c)^2 x^4\}^{-1}. \tag{4.17}
\]

Reasoning analogous to that given in the preceding section shows that the quantum corrections in Eqs. (4.16) are of the order of \((2S)^{-2}\), when \(t > 0\). In the large spin limit the functions \(\mu_{pl}(T)\) and \(\zeta_{pl}(T)\) are governed by equations, similar to Eqs. (3.33) for one frustrated bond in an antiferromagnet,
\[
\mu_{pl}(\zeta_{pl}) + \frac{\mu_{pl}}{1 + j_c} = \frac{l(T)S^*}{2Q}(\frac{T}{16JS^*2})^2(1 + \frac{16JS^*}{T}),
\]  
(4.18a)

\[
\frac{t + \zeta_{pl}(j_c^{-1} + 1)}{t + 1} = \frac{Tl(T)}{16JQ^2(1 + j_c)}.
\]  
(4.18b)

When \( t < 0 \) and \( T/4JQ^2 \ll t^2 \ll 1 \) the solution of these equations is as follows:

\[
\zeta_{pl} = \frac{j_c}{1 + j_c} |t| + \frac{Tl(T)}{16JQ^2(1 + j_c) |t|}, \quad \mu_{pl} \sim \frac{T^2}{(16JS^*)^2 |t|} \ll \zeta_{pl},
\]  
(4.19)

\[
Q_{pl} = Q(1 - \frac{Tl(T)}{16j_c |t| JQ^2}).
\]  
(4.20)

At positive \( t \) and small temperatures \( T/4JQ^2 \ll t^2 \), we find

\[
\zeta_{pl} = \frac{T}{16JQ^2}\left[\frac{(t + 1)l(T)}{t(1 + j_c)} - \frac{Q^3t}{2S^*3(1 + t) - S^*Q^2t}\right],
\]  
(4.21a)

\[
\mu_{pl} = \frac{QT}{16JS^*} \frac{t}{2S^*3(1 + t) - Q^2t},
\]  
(4.21b)

\[
\Lambda_2(0) = \zeta_{pl} + \frac{\mu_{pl}}{1 + j_c} = \frac{t + 1}{t(1 + j_c)} \frac{Tl(T)}{16JQ^2},
\]  
(4.21c)

\[
Q_{pl} = \frac{Q}{t + 1}\left[1 - \frac{T}{16j_cQ^2}l(T)\left(\frac{1}{t} + 1\right) - \frac{tQ^3}{2S^*3(1 + t) - Q^2tS^*}\right].
\]  
(4.21d)

In the zero temperature limit, \( Q_{pl} = Q \), if \( t < 0 \), and \( Q_{pl} = Q/(t + 1) < Q \), if \( t > 0 \). Thus, at \( t = 0 \) or \( J' = J_c = 0.376J \), the ground state changes from collinear to canted. It is argued in Appendix A that at large distances \( r \) from the defect plaquette, the correlation amplitude \( Q_{\vec{r},\vec{r}+\vec{d}} \) approaches to \( S \) as \( r^{-6} \), i.e. the frustrated plaquette acts on the antiferromagnetic background at large distances as a quadrupole.

We now average over the random distribution of the frustrated plaquettes, and obtain the renormalized spin-wave energy

\[
\epsilon_q^2 = \omega_q^2[1 - x(1 + j_c)Re\frac{1}{\Lambda_2(\omega_q)}],
\]  
(4.22)
In the small $q$ limit this becomes

$$
\epsilon_q^2 = 8J^2Q^2[1 - x\frac{1 + j_c}{\Lambda_2(0)}] + \Delta_{pl}^2,
$$

(4.23)

where the gap $\Delta_{pl}$ should be calculated self-consistently, as explained in Sec. [II]. Eqs. (4.23) and (4.21c) yield the renormalized spin wave velocity for strongly frustrated plaquettes with $t > 0$,

$$
c(x) = 2^{3/2}(1 - \frac{U_{pl}x}{T}),
$$

(4.24)

with

$$
U_{pl} = \frac{8JQ^2l(1 + j_c)^2}{T(t + 1)}.
$$

(4.25)

Note that the energy $U_{pl}$ is larger than the corresponding energy $U$ (Eq. (3.47)) for frustrated bonds. For strongly frustrated plaquettes, $t > 0$, at low temperatures $T/4JQ^2 \ll t^2$, the renormalized correlation length $\xi$ and Néel temperature $T_N(x)$ are thus given by

$$
\xi(T, x) = C \exp\left[\frac{2\pi JS^*Q}{T}(1 - \frac{U_{pl}x}{T})\right],
$$

(4.26a)

$$
T_N(x) = T_N(0)[1 - \frac{U_{pl}x}{T_N(0)}].
$$

(4.26b)

V. DISCUSSION. THEORY VS EXPERIMENT

We have generalized the SBMFA for 2D doped magnets, which allowed us to study the effect of noninteracting arbitrary frustrated bonds on the spin-wave spectrum and on the 2D magnetic correlation length at finite temperatures. We describe the defect bond by two local parameters, namely, the mean-field amplitude and the Lagrange multiplier. In a more rigorous treatment local mean-field parameters should be introduced also for the near-by bonds. However, even our simplified consideration is sufficient to describe correctly many peculiar properties of the doped 2D and quasi-2D antiferromagnets.
As noted in the Introduction, it is expected that in real cuprates the effective ferromagnetic coupling $J'$ generated by the localized holes is larger than the coupling $J_{12}$ i.e. $J'$ exceeds the spin canting threshold $J_c$ (for frustrated bonds $J_c = J$, while for plaquettes $J_c = 0.376J$, in agreement with the results of the linear spin-wave theory, see Refs. 16, 17 and Appendix B). In this case the renormalization of the antiferromagnet spin-wave velocity in the large spin limit is of the order of $Ex/T$, with $E = U$ for frustrated bonds, and $E = U_{pl}$ for frustrated plaquettes (see Eqs. (3.48), (3.47), (4.24) and (4.25)). Because of the large factor $E/T$ the renormalization at low temperatures is large even at small doping level.

It follows from Eqs. (3.50) and (4.26a) that the effect of doping upon the correlation length, like upon the spin-wave velocity, increases with the decrease of the temperature. The ratio $\xi(x,T)/\xi(0,T)$ is of the order of unity if $2\pi\rho_s E/T^2 \ll x^{-1}$, and is exponentially small when $(2\pi\rho_s Ex)^{1/2} \gg T \gg Ex$. The rapid decrease of the 2D correlation length strongly reduces the quasi-2D Néel temperature (see Eqs. (3.54) and (4.26b)). The above results hold also for frustrated bonds in ferromagnets (Eqs. (2.58), (2.59), (2.49b)). In contrast, frustrated plaquettes have only weak effects on the properties of the ferromagnet, the renormalization of the spin wave stiffness being of the order of $x$ and temperature independent (Appendix B).

The effect of frustrated bonds is large also at $J' < J_c$, provided $|t| = (J_c - J')/J_c \ll 1$. However, the temperature dependence is weaker than at $J' > J_c$. The renormalization of the spin wave spectrum in the large spin limit is proportional to $x/T^{1/2}$ if $T \gg 4JS^{2t^2}$, and it tends to a constant value $x/|t| \gg x$ when $T \rightarrow 0$ (Eqs. (2.57), (2.46), (2.47a), (2.48a)).

The strong effect of frustrating defects on the spin-wave stiffness and on the correlation length is related to the local instability leading to the spin canting. Indeed, the spinwave scattering amplitude at $T = \omega = 0$ diverges at the local stability threshold $J' = J_c$. At $J' > J_c$ the defects act as dipoles (bonds) or quadrupoles (plaquettes). Hence, the disturbance of the background decays as a power of the distance from the defect rather than exponentially. This means that the scattering amplitude has a zero frequency pole at any $J' > J_c$. At finite temperatures the scattering amplitude is finite but large and increases.
with the decrease of the temperature. We believe that these arguments hold for any type of frustrating defects, and therefore the above results are valid qualitatively not only for frustrated bonds and plaquettes.

We also calculated the quantum corrections to the local mean-field amplitude in antiferromagnets in the large spin limit. The corrections are relatively large, \( \sim (S^2 \ln S)^{-1/3} \), near the threshold (Eq. (3.37)), however at stronger frustration \( (J' > J_c) \) the corrections are unusually small, \( \sim (2S)^{-2} \ln^{-1} 2S \) (Eqs. (3.38)) and (3.39)). This result gives evidence in favour of the classical model for strongly frustrated bonds in 2D antiferromagnets, proposed in Ref. 12.

It follows from our theory that weakly frustrating and nonfrustrating impurities act quite differently. The renormalization of the spin stiffness is of the order of \( x \ll 1 \), and does not depend on the temperature. Thus, the correlation length is given by the same Eq. (1.1), as for undoped samples, with a concentration dependent constant \( A \). Similar results have been obtained in Ref. 37 by means of the quantum Monte-Carlo method.

Let us compare our results with the experimental findings\(^{2,9,10,12,38,39}\) for the temperature and doping concentration dependences of the correlation length in quasi-2D antiferromagnets. It has been shown in these papers that one should distinguish between two kinds of impurities. Dopants which introduce holes and, presumably, frustrate the Cu-Cu bonds, suppress strongly the correlation length at low temperatures and reduce rapidly the Néel temperature. In contrast, dopants which introduce vacancies or excess electrons do not change the temperature dependence of the correlation length and simply reduce the stiffness in accord with percolation. The reduction of \( T_N \) is much more gradual than that in hole doped samples. The results, presented here, account qualitatively for all these properties of doped antiferromagnets.\(^{40}\)

A remarkable difference between the magnetic properties of the electron- and hole-doped materials was observed in Ref. 39. In electron doped \( Nd_{2-x}Ce_xCuO_4 \) the reduction of \( T_N \) and \( \rho_s \) is of the same order of magnitude. However, in the hole doped \( La_2CuO_4+x \) the correlation length (and, hence, \( \rho_s \)) at high temperatures (\( T \approx 500 K \gg T_N(x) \)) is almost
unchanged by doping, while \( T_N \) essentially decreases. Our results permit to explain this puzzle. In electron doped samples the effect of doping does not depend on the temperature, and the decrease of \( \rho_s \), as well as of \( \xi \) and \( T_N \), is of the order of \( x \). In hole doped samples the change of \( T_N \) is larger than that of \( \xi (T \gg T_N) \), since the effect of doping grows with the decrease of the temperature.

We now show that the experimental results confirm the temperature dependence of the renormalized correlation length, derived in the preceding sections. It follows from Eqs. (3.50) and (4.26a) that in antiferromagnets, doped with strongly frustrating impurities, the quantity \( T \ln(\xi/C) \) should be a linear function of \( x/T \):

\[
T \ln \left( \frac{\xi(T,x)}{C} \right) = 2\pi \rho_s (1 - \frac{E_x}{T}),
\]

where \( E \), as before, stands for \( U \) or \( U_{pl} \). In Refs. 2,9,10 the correlation length was measured for three doped samples \( La_2CuO_{4+x} \), with \( T_N(x) = 245 \) K, 190 K and 90 K. The precise hole concentrations for these samples are not given in the references cited above. It is known, however,2,10 that the hole concentration varies approximately linearly with \( 325 - T_N(x) \). Therefore one has an estimate for their ratios, \( x(90) : x(190) : x(245) = 1 : 0.59 : 0.34 \).

The quantity \( C \) can be determined from the measurements of \( \xi(T,0) \) in an undoped sample2 (wee neglect its week temperature dependence, see Sec. III C), and the stiffness constant is \( 2\pi \rho_s \simeq 150 \) meV.\(^2\) Using these data, we plotted in Fig. 1 the experimental values of \( T \ln(\xi/C) \) vs \( x/T \). It is seen that the experimental points for all three samples fall on a single straight line, as expected from Eq. (5.1).

According to Ref. 2, the linear function \( T_N(x) \) extrapolates to zero at \( x \approx 2\% \). Thus, we can estimate the hole concentration \( x(90) \approx 1.4\% \). Then, from the slope of the straight line in Fig. 1 we estimate the energy \( E \approx 500 \) meV. The theoretical values, which follow at \( t \gg 1 \) from Eqs. (3.47), (4.27) and the relation\(^2\) \( 2\pi \rho_s = 1.15 J \), are: \( U = 110 \) meV, \( U_{pl} = 410 \) meV. The last number is close to the above experimental value. Note that the extra hole in lanthanum cuprate is localized on a region of the order of two lattice constants,\(^2\) and hence the real defect is more complicated than the above models. What is more, our
study is based on the assumption that the impurity holes are localized, which is, apparently, violated at high temperatures. Nevertheless, the above analysis shows that the theory of strongly frustrating defects explains the experimental findings in samples with Néel order at least semiquantitatively.

Keimer et al. also measured the temperature dependence of the correlation length $\xi(T, x)$ of Sr doped samples ($x=0.02; 0.03; 0.04$), without Néel order. Eq. (5.1) fails to account for these experimental data even at high temperatures, when $Ex/T < 1$. Keimer et al. showed that in this case the correlation length obeys the empirical relation

$$\xi^{-1}(x, T) = \xi^{-1}(x, 0) + \xi^{-1}(0, T),$$

(5.2)

where $\xi(0, T)$ is the correlation length of the carrier-free sample, and $\xi(x, 0)$ is the measured correlation length at low temperatures ($\xi(x, 0) = 150, 65$ and $45 \text{ Å}$ for $x = 0.02, 0.03$ and $0.04$ respectively). They also attempted to fit Eq. (5.2) to the data for the above less doped samples, which exhibit 3D long-range order, $\xi(x, 0)$ being a fitting parameter ($\xi(x, 0) = 140$ and $275 \text{ Å}$ for the $T_N = 90$ and $190$ K samples respectively). The fit is good only at relatively high temperatures. Figure 1 demonstrates that Eq. (5.1) describes the experimental data better than Eq. (5.2), especially for the $T_N = 90$ K sample, although one needs only one parameter ($E$) to fit the data for all samples. This implies, first, that $\xi(x, 0)$ for the less doped samples is larger than the measured $\xi(x, T)$, namely, $\xi(x, 0) > 400 \text{ Å}$ for the $T_N = 90$ K sample. One cannot rule out the possibility that $\xi(x, 0)$ is infinite for the concentrations $x$ smaller than a certain (small) critical concentration $x_c$. This standpoint is somewhat supported by the numerical simulations in the zero-temperature limit. It has been shown that the frustrated bonds destroy the 2D long-range order in the Heisenberg antiferromagnet ($\xi(x, 0)$ is, perhaps, finite at any $x$), while in the XY model the correlations do not decay exponentially, but rather as a power of the distance. One can, thus, suggest that the small XY-anisotropy, which exists in the real cuprates, stabilizes the 2D, $T = 0$ long-range order in doped samples at some small, but finite doping concentration.

The finite value of $\xi(x, 0)$ in the samples with high doping level also affects the renor-
malization of the spin-stiffness. We showed in Sections III C and IV that the stiffness renormalization given by Eqs. (3.46) and (4.23) holds only in the small $q$ limit, when

\[ q^2 \ll \frac{T}{E \ln(4JS/\omega q)}, \quad (5.3) \]

while at larger wave vectors the spin-stiffness is the same as in the undoped samples. On the other hand, the spin waves exist only within a region of size $\xi(x, T)$, i.e.

\[ q \gg \xi^{-1}(x, T) \quad (5.4) \]

(See Ref. 36 and references therein). In slightly doped samples, the inequalities (5.3) and (5.4) are always fulfilled, since the inverse correlation length is exponentially small at low temperatures. In strongly doped samples, a new temperature independent length scale $\xi(x, 0)$ appears. Thus, the region of well-defined spin waves is restricted to distances smaller than $\xi(x, 0)$, while according to Eq. (5.3) only large wavelength spin waves are renormalized. At sufficiently high doping level, i.e. small $\xi(x, 0)$, the inequalities (5.3) and (5.4) are violated, and hence the spin-stiffness is not renormalized. This reasoning explains why the correlation length for the $x = 0.02$ sample is larger than for the $T_N = 90$ K sample (compare Figs. 8b and 10 in Ref. 2).

Substituting an estimated value of the energy $E \approx 500$ meV into Eq. (3.54) for the Néel temperature, we find $T_N(x) = T_N(0)(1 - bx)$, where $b \approx 20$ per 1%. The coefficient $b$ is smaller than the experimental value by a factor $2.5 - 3$. There are several reasons for this discrepancy. First, Eq. (3.54) holds only at small $x$ when $bx \ll 1$. Second, all our numerical estimates are based on the above approximate linear relation between the function $325 - T_N(x)$ and $x$. Finally, the mean-field equation (3.51) which relates the Néel temperature to the 2D correlation length should be improved.

As noted above, the holes in $La_2CuO_{4+x}$ are delocalized at temperatures $T > T_N$. The overall agreement of our theory, based on the frustration model, with the experimental results for hole doped lanthanum cuprate supports the standpoint that even delocalized holes frustrate the antiferromagnet. It would be difficult to understand the striking difference
between hole and electron (vacancy) doping, as well as the temperature dependence of the spin stiffness in hole doped samples, without this assumption. This conclusion agrees with the results of our analysis of the reentrant temperature dependence of the sublattice magnetization in oxygen doped cuprates. We gave evidence there that delocalized holes frustrate the antiferromagnet, though less than localized holes. New efforts are needed in order to elucidate this question.

Finally, we would like to note that, as has been shown in Sections [II] and [IV], the spin-wave velocity $c$ should also change drastically with hole doping. Even though we considered only 2D antiferromagnets, we expect that the same renormalization takes place in quasi-2D antiferromagnets at $T < T_N$ for spin waves with wave vectors in the range $J_\perp / J \ll q^2 \ll T / E \ln(4JS/\omega_\perp)$. Here the first inequality guarantees that the interplane coupling does not affect the spectrum, while the second one coincides with Eq. (5.3). Thus, we may conjecture that the renormalization of $c$ in hole doped quasi-2D antiferromagnets is of the order of $JS^2 / T$ in the large spin limit. The same conclusion follows from a direct calculation of the spin wave scattering on free dipoles at $T \ll T_N$. Unfortunately, the experimental information on this subject is very poor. We are aware only of the Ref. [45,46, where it is shown that a small amount of holes in the $CuO_2$ planes of $YBa_2Cu_3O_{6+x}$ renormalizes strongly the spin-wave velocity. Neither the temperature nor the concentration dependences of the spin-wave velocity has been studied. We hope that our results will stimulate new experiments in this field.

ACKNOWLEDGMENTS

I acknowledge A. Aharony for reading the manuscript and for many helpful discussions and comments. I would also like to thank A. B. Harris and A. Auerbach for stimulating discussions. This work was supported by the Wolfson Foundation and the U.S.-Israel Binational Science Foundation.
APPENDIX A: THE CORRELATION FUNCTION $Q_{\vec{r},\vec{r}+\vec{\delta}}$

In this appendix we calculate the correlation of the spins at the ends of a bond, $\vec{\delta}$, at a large distance from the frustrated bond, $\vec{a}$.

1. Ferromagnet

We start from the expression

$$Q_{\vec{r},\vec{r}+\vec{\delta}} = <a^\dagger_{\vec{r}} a_{\vec{r}+\vec{\delta}}>,\quad (A1)$$

and subtract the constraint equations for the spins at the ends of the bond $\vec{\delta}$,

$$\int_{-\infty}^{+\infty} d\omega n(\omega) \sum_{\vec{q}\vec{q}_1} e^{i(\vec{q}\cdot\vec{r}+i\vec{q}\cdot\vec{\delta})} Im\left[G(\vec{q},\vec{q}_1;\omega) - G^0(\vec{q},\omega)\delta_{\vec{q}\vec{q}_1}\right] = 0.\quad (A2)$$

We find

$$Q_{\vec{r},\vec{r}+\vec{\delta}} = S - \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega n(\omega) \frac{1}{N^2} \sum_{\vec{q}\vec{q}_1} e^{i(\vec{q}\cdot\vec{r})} \frac{1}{D_1(\omega)} \sum_{\vec{q}} G^0(\vec{q},\vec{q}_1;\omega) (\vec{q}\cdot\vec{\delta}) e^{i\vec{q}\cdot\vec{\delta}}.\quad (A3)$$

Eqs. (A3), (2.18), and (2.26) yield

$$Q_{\vec{r},\vec{r}+\vec{\delta}} = S + \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega n(\omega) \frac{1}{D_1(\omega)} \sum_{\vec{q}} G^0(\vec{q},\omega) (\vec{q}\cdot\vec{\delta}) e^{i\vec{q}\cdot\vec{\delta}}.\quad (A4)$$

where we have taken into account that at large $r$ only small wave-vectors are important, and expand the exponents in powers of $(\vec{q}\cdot\vec{a})$, $(\vec{q}\cdot\vec{\delta})$. The term proportional to $v_2 = \mu_f$ gives at low temperatures a negligible small contribution to $Q_{\vec{r},\vec{r}+\vec{\delta}}$. At small frequencies $\omega \ll JS/r^2$, the integration over $\vec{q}$ in Eq. (A4) is easily performed, and we find

$$Q_{\vec{r},\vec{r}+\vec{\delta}} = S - \frac{W_f}{(2JS)^2} F(\vec{r}) Z_f(T).\quad (A5)$$
Here

\[ F(\vec{r}) = \frac{1}{\pi^2 r^4} \left[ \vec{a} \cdot \vec{\delta} - \frac{2}{r^2} (\vec{r} \cdot \vec{a}) (\vec{r} \cdot \vec{\delta}) \right] ^2, \]  \hspace{1cm} (A6) 

\[ Z_f(T) = \frac{1}{2\pi} \int _{-\infty}^{+\infty} d\omega \, n(\omega) \text{Im} \frac{1}{D_1(\omega)}. \]  \hspace{1cm} (A7) 

It follows from Eqs. (2.48a), (2.49b), and (A7) that in the zero temperature limit

\[ Z_f(T) = 0 \quad \text{for} \quad t < 0, \]

\[ Z_f(T) = \frac{2JS^2 t}{t + 1} \quad \text{for} \quad t > 0. \]  \hspace{1cm} (A8) 

Thus, we finally obtain that \( Q_{\vec{r},\vec{r}+\vec{\delta}} = Q \), when \( t < 0 \), and

\[ Q_{\vec{r},\vec{r}+\vec{\delta}} = S \left[ 1 - \frac{t}{t + 1} F(\vec{r}) \right], \]  \hspace{1cm} (A9) 

when \( t > 0 \). The \( r \) dependence of \( Q_{\vec{r},\vec{r}+\vec{\delta}} \) given by Eqs. (A9) and (A6) holds in the region \( 1 \ll r \ll (4JS^2/T)^{1/2} \).

2. Antiferromagnet

We first consider a frustrated bond. In this case the function \( Q_{\vec{r},\vec{r}+\vec{\delta}} \) can be written as

\[ Q_{\vec{r},\vec{r}+\vec{\delta}} = Q - \frac{1}{\pi} \int _{-\infty}^{+\infty} d\omega \, n(\omega) \text{Im} \frac{1}{N^2} \sum _{\vec{q},\vec{q}_1} < \chi_\downarrow | \left[ \tilde{G}(\vec{q},\vec{q}_1; \omega) - \hat{G}^0(\vec{q},\omega) \delta_{\vec{q},\vec{q}_1} \right] | \chi_\uparrow > \times e^{i(\vec{q} - \vec{q}_1) \cdot \vec{r} + i\delta}. \]  \hspace{1cm} (A10) 

Taking into account the constraints for the spins at the ends of the frustrated bonds, we rewrite this equation as

\[ Q_{\vec{r},\vec{r}+\vec{\delta}} = Q + \frac{W}{2\pi} \int _{-\infty}^{+\infty} d\omega \, n(\omega) \text{Im} \frac{1}{D_1(\omega)} \]

\[ \times \frac{1}{N^2} \sum _{\vec{q},\vec{q}_1} [2e^{i\delta} < \chi_\downarrow | \hat{G}^0(\vec{q},\omega) | \eta_1(\vec{q}) > < \eta_1(\vec{q}_1) | \hat{G}^0(\vec{q}_1,\omega) | \chi_\uparrow > - < \chi_\uparrow | \hat{G}^0(\vec{q},\omega) | \eta_1(\vec{q}) > < \eta_1(\vec{q}_1) | \hat{G}^0(\vec{q}_1,\omega) | \chi_\uparrow > (1 + e^{i(\vec{q} - \vec{q}_1) \cdot \vec{r}})]e^{i(\vec{q} - \vec{q}_1) \cdot \vec{r}}. \]  \hspace{1cm} (A11)
We expand the expression within the square brackets in Eq. (A11) in powers of \((\vec{q} \cdot \vec{a})\) and \((\vec{q} \cdot \vec{\delta})\), and transform it into a sum of three terms

\[
\begin{align*}
-\frac{1}{N} & \sum_{\vec{q}} e^{i\vec{q} \cdot \vec{r}} G_{11}^0(\vec{q}, \omega)(\vec{q} \cdot \vec{a})(\vec{q} \cdot \vec{\delta})^2 + \left\{ \frac{2}{N} \sum_{\vec{q}} e^{i\vec{q} \cdot \vec{r}} [G_{11}^0(\vec{q}, \omega) - G_{12}^0(\vec{q}, \omega)] \right\}^2 \\
+ \frac{4i}{N^2} & \sum_{\vec{q}} [G_{11}^0(\vec{q}, \omega) - G_{12}^0(\vec{q}, \omega)]e^{i\vec{q} \cdot \vec{r}} \sum_{\vec{q}_1} (\vec{q}_1 \cdot \vec{a})G_{11}^0(\vec{q}_1, \omega)e^{-i\vec{q}_1 \cdot \vec{r}}.
\end{align*}
\]  

(A12)

The first of these terms, when integrated over \(\vec{q}\), transforms in the zero frequency limit into the above function (A6). The last two terms have no analogy in the case of ferromagnets. Their contribution to \(Q_{\vec{r}, \vec{r}+\vec{\delta}}\) can be important at small \(r\). However, at large distances, one has

\[
\int d\vec{q} [G_{11}^0(\vec{q}) - G_{12}^0(\vec{q})]e^{i\vec{q} \cdot \vec{r}} \sim \int d\vec{q} \frac{1 - \gamma \vec{q}}{\omega^2} e^{i\vec{q} \cdot \vec{r}} = 0.
\]  

(A13)

Thus the behavior of the function \(Q_{\vec{r}, \vec{r}+\vec{\delta}}\) at large distances in antiferromagnets is the same as in ferromagnets: \(S - Q_{\vec{r}, \vec{r}+\vec{\delta}} \propto r^{-4}\) in the region \(1 \ll r \ll (4JS^2/T)^{1/2}\). Analogous reasoning holds also in the case of frustrated plaquettes. Then, instead of the first term in Eq. (A12) we find

\[
\sum_{\vec{q}} e^{i\vec{q} \cdot \vec{r}} G_{11}^0(\vec{q})(\vec{q} \cdot \vec{a})(\vec{q} \cdot \vec{b})(\vec{q} \cdot \vec{\delta})^2,
\]

where \(\vec{a}\) and \(\vec{b}\) are unit vectors along the plaquette sides. Hence, at large distances the function \(S - Q_{\vec{r}, \vec{r}+\vec{\delta}}\) decays as \(r^{-6}\).

**APPENDIX B: THE FRUSTRATED PLAQUETTE AT ZERO TEMPERATURE.**

**HOLSTEIN-PRIMAKOFF REPRESENTATION**

In this Appendix we show that the effect of the frustrated plaquettes on the spin-wave spectrum is strikingly different in ferro- and antiferromagnets. While in antiferromagnets the renormalization diverges when the frustrated coupling \(J'\) tends to the local stability threshold, \(J_c\), in ferromagnets the renormalization is finite in this limit, and, hence, is qualitatively the same at \(J' < J_c\) and \(J' > J_c\).
1. Ferromagnet

We suppose that the strength of the frustrated bond $J'$ is smaller than the threshold value $J_c$ for local instability (the value of $J_c$ is obtained below). Then all the spins are parallel, and the Hamiltonian, which describes the interaction of the spin waves with the spins at the corners of the frustrated plaquette, is

$$H_{int} = (J + J') S \sum_{<lm>} (a_i^\dagger - a_m^\dagger)(a_m - a_i), \quad (B1)$$

where $<lm>$ is summed over the bonds in the plaquette. It follows from (B1) that the interaction matrix is

$$\hat{V} = (J + J') S \begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{pmatrix}, \quad (B2)$$

It is easy to verify that the matrix $\hat{P} = \hat{U}\hat{V}\hat{U}^{-1}$, where

$$\hat{U} = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad (B3)$$

is diagonal, and its elements are $P_1 = P_3 = -(J + J')S/2, P_2 = -(J + J')S, P_4 = 0$. The Fourier transform of the Hamiltonian (B1) is given by

$$H_{int} = \sum_{i=1}^{3} P_i \frac{1}{N} \sum_{\vec{q}\vec{q}_i} x_i(\vec{q}) x_i^*(\vec{q}_i) a_i^\dagger a_{\vec{q}_i}, \quad (B4)$$

with

$$x_1(\vec{q}) = (1 - e^{-i\vec{q} \cdot \vec{a}})(1 + e^{-i\vec{q} \cdot \vec{a}}), \quad (B5a)$$

$$x_2(\vec{q}) = (1 - e^{-i\vec{q} \cdot \vec{a}})(1 - e^{-i\vec{q} \cdot \vec{b}}), \quad (B5b)$$
\( x_3(\vec{q}) = (1 + e^{-i\vec{q} \cdot \vec{a}})(1 - e^{-i\vec{q} \cdot \vec{b}}). \) \hfill (B5c)

The solution of the \( T \)-matrix equation can be obtained in the usual way,

\[
T(\vec{q}, \vec{q}_1; \omega) = \sum_{i=1}^{3} P_i \frac{x_i(\vec{q})x_i^*(\vec{q}_1)}{1 - P_i \Phi_i(\omega)}, \tag{B6}
\]

where

\[
\Phi_i(\omega) = \frac{1}{N} \sum_{\vec{q}} x_i(\vec{q})G^0(\vec{q}, \omega)x_i^*(\vec{q}). \tag{B7}
\]

It follows from Eqs. (B7) and (B5) that

\[
\Phi_1(\omega) = \Phi_3(\omega) = 4 \int \frac{d^2 q}{(2\pi)^2} \frac{(1 + \cos q_x)(1 - \cos q_y)}{\omega - \omega_{\vec{q}}}, \tag{B8a}
\]

\[
\Phi_2(0) = 4 \int \frac{d^2 q}{(2\pi)^2} \frac{(1 - \cos q_x)(1 - \cos q_y)}{\omega - \omega_{\vec{q}}}. \tag{B8b}
\]

Thus, the function \( T(\vec{q}, \vec{q}_1; \omega) \) is given in the limit of small \( q \) and \( \omega \) by

\[
T(\vec{q}, \vec{q}_1; 0) = \frac{4P_1q^2}{1 - P_1 \Phi_1(0)} - \frac{P_2q_x^2q_y^2}{1 - P_2 \Phi_2(0)}. \tag{B9}
\]

The second term in the r.h.s. of this equation is of higher order in small \( q \), and can be neglected at small \( P_2 \). However, it appears that with the increase of \( P_2 \) it is the denominator of this term which approaches to zero first, and, hence, determines the threshold for the local instability.

The integration over the first Brillouin zone of the reciprocal lattice yields

\[
\Phi_2(0) = -0.727(J + J')/JS, \quad \Phi_1(0) = -1.363/JS.
\]

Hence \( P_1 \Phi_1(0) = 0.682 (J + J')/J < P_2 \Phi_2(0) = 0.727 (J + J')/J \), and the threshold value of \( J' \) should be found from the equation

\[
1 - P_2 \Phi_2(0) = 0.
\]

Thus \( J_c = 0.376J \), which exactly coincides with the value of \( J_c \), found in Sec. IV in the Schwinger boson representation for the plaquette defect in an antiferromagnet.

43
Note that the renormalized spin-wave spectrum
\[ \epsilon_{\vec{q}} = \omega_{\vec{q}} + xT(q, q) = \omega_{\vec{q}}[1 - \frac{1.572J_c(J' + J)}{J(J_c - 0.806J')}x] \]  
(B10)
is nonsingular at \( J' = J_c \). Therefore the renormalization of the spin-stiffness in the ferromagnet with frustrated plaquettes, unlike ferromagnets with frustrated bonds, is small, of the order of \( x \), at both \( J' < J_c \) and \( J' > J_c \).

2. Antiferromagnet

The Hamiltonian, which describes the interaction of the spin waves with the spins at the corners of the frustrated plaquette, when \( J' < J_c \), is
\[ H_{\text{int}} = -(J + J')S \sum_{\langle lm \rangle} \left( a_l b_m + a_l^\dagger b_m^\dagger + a_l a_l^\dagger + b_m^\dagger b_m \right), \]
(B11)
where \( a_l(b_m) \) are the Holstein-Primakoff operators for the spins in the sublattice \( A \) (\( B \)). The interaction matrix, like in ferromagnets, can be diagonalized by the transformation \([\text{B3}].\)

Then the Fourier transform of the Hamiltonian is given by
\[ H_{\text{int}} = \sum_{i=1}^{3} P_i \frac{1}{N} \sum_{\vec{q}\vec{l}} <c_{\vec{q}} | Y_i(\vec{q}) > < Y_i(\vec{q}) | c_{\vec{q}}>. \]
(B12)

Here
\[ | c_{\vec{q}} > = \begin{pmatrix} a_{\vec{q}} \\ b_{-\vec{q}}^\dagger \end{pmatrix}, \quad < c_{\vec{q}} | = \begin{pmatrix} a_{\vec{q}}^\dagger \quad b_{-\vec{q}} \end{pmatrix}, \]
(B13)
\[ | Y_i(\vec{q}) > = \begin{pmatrix} \alpha_i(\vec{q}) \\ \beta_i(\vec{q}) \end{pmatrix}, \quad < Y_i(\vec{q}) | = \begin{pmatrix} \alpha_i^\dagger(\vec{q}) \quad \beta_i^\dagger(\vec{q}) \end{pmatrix}, \]
(B14)
where
\[ \alpha_1(\vec{q}) = \alpha_3(\vec{q}) = 1 - e^{-i(q_x + q_y)}, \quad \beta_1(\vec{q}) = -\beta_3(\vec{q}) = e^{-i q_x} - e^{i q_y}, \]
(B15a)
\[ \alpha_2(\vec{q}) = 1 + e^{i(q_x + q_y)}, \quad \beta_2(\vec{q}) = e^{-i q_x} + e^{-i q_y}. \]
(B15b)
The next calculations are straightforward, provided we introduce the Green's function matrix by Eq. (3.15) with the vectors $<c_q|(|c_q>)$ substituted for $<A_q|(|A_q>)$.

The unperturbed Green's functions are

$$\hat{G}^0(q,\omega) = \frac{1}{\omega^2 - \omega_q^2} \begin{pmatrix} 4JS + \omega & -4JS\gamma_q \\ -4JS\gamma_q & 4JS - \omega \end{pmatrix},$$

(B16)

Like in the ferromagnet, only the mode $i = 2$ is relevant at $J'$ near the threshold $J_c = 0.376J$. Therefore the single-defect $T$-matrix can be written as

$$\hat{T}(q, q_1; \omega) = \frac{-(J + J') | Y_2(q) > < Y_2(q_1) |}{1 - P_2R(\omega)},$$

(B17)

where

$$R(\omega) = \frac{8JS}{N} \sum_q \frac{\sin^2 q_x + \sin^2 q_y}{\omega^2 - \omega_q^2}.$$

(B18)

The function $R(\omega)$ coincides up to the quantum corrections with the function $\Psi_2(\omega)$ from Eq. (4.9b). Thus $R(0) = 0.727(1 + J/J')$. The Eqs. (B17), (B14), and (B15) together with the relation

$$\epsilon_q = \omega_q + \frac{x}{2\omega_q^2} [4JS(T_{11}(q, q) + T_{22}(q, q)) + \omega_q(T_{12}(q, q) - T_{22}(q, q)) - 4JS\gamma_q(T_{12}(q, q) + T_{21}(q, q))]$$

(B19)

yield the renormalized spin-wave spectrum,

$$\epsilon_q = \omega_q [1 - \frac{1.76(J + J')x}{J_c - J'}].$$

(B20)

The remarkable difference between the effect of the frustrated plaquettes on the spin-wave spectrum in the ferro- and antiferromagnets is now evident.
REFERENCES

1 R. J. Birgeneau and G. Shirane, in Physical Properties of High Temperature Superconductors, edited by D. M. Ginzberg (World Scientific, Singapore, 1989).

2 B. Keimer, N. Belk, R. J. Birgeneau, A. Cassanho, C. Y. Chen, M. Greven, M. A. Kastner, A. Aharony, Y. Endoh, R. W. Erwin, and G. Shirane, Phys. Rev. B 46, 14034 (1992).

3 M. Greven, R. J. Birgeneau, Y. Endoh, M. A. Kastner, B. Keimer, M. Matsuda, G. Shirane, and T. R. Thurston Phys. Rev. Lett. 72, 1096 (1994).

4 A. M. Polyakov, Phys. Lett. B 59, 79 (1975).

5 S. Chakravarty, B. I. Halperin, and D. Nelson, Phys. Rev. B 39, 2344 (1989).

6 D. P. Arovas and A. Auerbach, Phys. Rev. B 38, 316 (1988).

7 J. H. Cho, F. C. Chou, and D. C. Johnston, Phys. Rev. Lett. 70, 222 (1993).

8 F. C. Chou, F. Borsa, J. H. Cho, D. C. Johnston, A. Lascialfari, D. R. Torgeson, and J. Ziolo, Phys. Rev. Lett. 71, 2323 (1993).

9 Y. Endoh, K. Yamada, R. J. Birgeneau, D. R. Gabbe, H. P. Jenssen, M. A. Kastner, C. J. Peters, T. R. Thurston, J. M. Tranquada, G. Shirane, Y. Hidaka, M. Oda, Y. Enomoto, M. Suzuki, and T. Murakami, Phys. Rev. B 37, 7443 (1988).

10 K. Yamada, K. Kakurai, Y. Endoh, T. R. Thurston, M. A. Kastner, R. Y. Birgeneau, G. Shirane, Y. Hidaka, and T. Murakami, Phys. Rev. B 40, 4557 (1989).

11 V. J. Emery, Phys. Rev. Lett. 58, 2794 (1987).

12 A. Aharony, R. J. Birgeneau, A. Coniglio, M. A. Kastner, and H. E. Stanley, Phys. Rev. Lett. 60, 1330 (1988).

13 B. Keimer, A. Aharony, A. Auerbach, R. J. Birgeneau, A. Cassanho, Y. Endoh, R. W. Er-
14 S. L. Ginsburg, Sov. Phys. JETP 49, 1127 (1979).

15 M. V. Feigelman and A. M. Tsvelik, Sov. Phys. JETP 49, 1136 (1979).

16 J. Vannimenus, S. Kirkpatrick, F. D. M. Haldane, and C. Jayaprakash, Phys. Rev. B 39, 4634 (1988).

17 D. N. Aristov and S. V. Maleev, Zs. Phys. B - Condensed Matter 81, 433 (1990).

18 Kong-Ju-Bock Lee and P. Schlottman, Phys. Rev. B 42, 4426 (1990).

19 J. V. Villain, Zs. Phys. B - Condensed Matter 3, 31 (1979).

20 L. I. Glazman and A. S. Ioselevich, Zs. Phys. B - Condensed Matter 80, 133 (1990).

21 V. L. Pokrovsky and G. V. Uimin, Physica C 60, 323 (1989).

22 R. J. Gooding, Phys. Rev. Lett. 66, 2266 (1991).

23 A. Chubukov, Phys. Rev. B, 44, 12318 (1991).

24 P. Kopietz and G. Castilla, Phys. Rev. B 43, 11100 (1991).

25 F. Milla, D. Poiblane, C. Bruder, Phys. Rev. B 43, 7891 (1991).

26 P. Kopietz, Phys. Rev. Lett 68, 3480 (1992).

27 T. K. Ng, Phys. Rev. B 45, 8181 (1992).

28 M. Azzouz and B. Doucot, Phys. Rev. B 47, 8660 (1993).

29 Zhong-Yi Lu, Zhao-Biu Su, and Lu Yu, Phys. Rev. B 47, 12276 (1993).

30 A. J. Millis and H. Monien, Phys. Rev. Lett. 70, 2810 (1993).

31 A. Mattson, P. Fröjdh, and T. Einarsson, Phys. Rev. B 49, 3997 (1994).

32 S. Sarker, C. Jayaprakash, H. R. Krishnamurty, and M. Ma, Phys. Rev. B 40, 5028 (1989).
The canting angle of the spins around the dipole decay as \( r^{-1} \) (Ref. [12,13]). Since \( Q_{\vec{r},\vec{r}+\vec{\delta}} \sim (1/2) |\vec{S}_{\vec{r}} + \vec{S}_{\vec{r}+\vec{\delta}}| \) (Ref. [33]), the difference \( S - Q_{\vec{r},\vec{r}+\vec{\delta}} \) decays with the increase of \( r \) as \( r^{-4} \).

J. Callaway, *Quantum Theory of Solid State* (Academic Press, New York - London, 1974), Part B.

E. Manousakis, Rev. Mod. Phys. 63, 1 (1991).

E. Manousakis, Phys. Rev. B 45, 7570 (1992).

M. Matsuda, K. Yamada, H. Kadowaki, T. R. Tharston, Y. Endoh, Y. Hidaka, R. J. Birgeneau, M. A. Kastner, P. M. Gehring, A. H. Moudden, and G. Shirane, Phys. Rev. B 42, 10098 (1990).

M. Matsuda, Y. Endoh, K. Yamada, H. Kojima, I. Tanaka, R. J. Birgeneau, M. A. Kastner, and G. Shirane, Phys. Rev. B 45, 12548 (1992).

It should be emphasized, however, that in 2D antiferromagnets the bond defect renormalization of the spin stiffness differs in some important respects from the site defect renormalization. In the later case, e.g., a low-frequency logarithmic divergence in the spin-wave stiffness was identified.

C. Y. Chen, R. J. Birgeneau, M. A. Kastner, N. W. Preyer, and Tineke Thio, Phys. Rev. B 43, 392 (1991).

I. Ya. Korenblit and A. Aharony, Phys. Rev. B 49, 13291 (1994).

B. I. Shraiman and E. D. Siggia, Phys. Rev. Lett. 61, 467 (1988).

R. J. Gooding and A. Mailhot, Phys. Rev. B 44, 11852 (1991).

J. Rossat-Mignod, L. P. Regnault, M. J. Jurgens, C. Vettier, P. Burlet, J. Y. Henry, and G. Laperot, Physica B 163, 4 (1990).
46. J. Rossat-Mignod, L. P. Regnault, P. Bourges, P. Burlet, C. Vettier, and J. Y. Henry, in *Selected Topics in Superconductivity*, edited by L. C. Gupta and M. S. Multani (World Scientific, Singapore, 1993).

47. I. Ya. Korenblit and E. F. Shender, Phys. Rev. B 48, 9478 (1993).

48. A. B. Harris and S. Kirkpatrick, Phys. Rev. B 16, 542 (1977).
FIG. 1. $(T/2\pi \rho_s) \ln(\xi/C)$ as a function of $2\pi \rho_s x/x(90)T$ for three samples $La_2CuO_{4+x}$ with $T_N = 245$ K (×), 190 K (○), and 90 K (●). The points are obtained from the experimental data (Refs. 2,9,10), as discussed in the text. The straight solid line shows the fit to Eq. (5.1). The dotted and dashed lines represent fits to Eq. (5.2) for the $T_N = 90$ and 190 K samples ($\xi(x,0) = 140$ and 275 Å, Ref. 2).