EXEMPLARY DECAY OF RÉNYI DIVERGENCE UNDER FOKKER-PLANCK EQUATIONS

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ABSTRACT. We prove the exponential convergence to the equilibrium, quantified by Rényi divergence, of the solution of the Fokker-Planck equation with drift given by the gradient of a strictly convex potential. This extends the classical exponential decay result on the relative entropy for the same equation.

1. INTRODUCTION AND MAIN RESULTS

We consider the long time behavior of the following Fokker-Planck equation on \( \mathbb{R}^d \)

\[
\partial_t p_t(x) = \text{div} \left( p_t(x) \nabla V(x) \right) + \Delta p_t(x),
\]

where \( V \) is a smooth potential function on \( \mathbb{R}^d \), and the initial datum \( p_0 \) is smooth and decays sufficiently fast at infinity. It is well-known \[16, 10\] that if \( V \) satisfies the uniform convexity (or Bakry-Émery \[2\]) condition:

\[
D^2 V(x) \geq K \cdot I_d \quad \text{for every } x \in \mathbb{R}^d
\]

with some constant \( K > 0 \), then the solution \( p_t \) of the Fokker-Planck equation dissipates the relative entropy (or Kullback-Leibler divergence) exponentially fast towards the Gibbs equilibrium distribution

\[
p_{\infty}(x) = e^{-V(x)},
\]

where we assume that the normalization constant is one without loss of generality. More precisely,

\[
D(p_t \| p_{\infty}) \leq e^{-2Kt} D(p_0 \| p_{\infty}).
\]

Recall that the relative entropy \( D(p \| q) \) is defined by

\[
D(p \| q) := \begin{cases} \int p \ln \left( \frac{p}{q} \right) dq, & p \ll q, \\ \infty, & \text{otherwise}. \end{cases}
\]

For convenience, we will abuse notation and use symbols \( p, q, \) etc., to represent probability measures as well as the density functions associated with them. Whether a symbol refers to a probability measure or a density should be clear from the context. In addition, to avoid technicalities, all probability density functions of consideration will be assumed to be smooth.

The exponential decay \[1.3\] can be established by the entropy dissipation method, which usually relies on the validity of a log-Sobolev inequality with respect to \( p_{\infty} \).
In fact, the entropy production (time-derivative of entropy) is
\[
\frac{d}{dt} D(p_t \parallel p_\infty) = -I(p_t \parallel p_\infty),
\]
where \( I \) is the relative Fisher information defined by
\[
I(p \parallel q) := \int \left| \nabla \ln \left( \frac{p}{q} \right) \right|^2 \, dp,
\]
if probability distribution \( q > 0 \). We say that the measure \( p_\infty = e^{-V} \) satisfies the log-Sobolev inequality (LSI) with constant \( K > 0 \) if for all probability measures \( p \) (absolutely continuous w.r.t. \( p_\infty \)),
\[
D(p \parallel p_\infty) \leq \frac{1}{2K} I(p \parallel p_\infty).
\]
Then (1.3) follows directly from (1.4) and (1.6) and the Grönwall’s inequality. The entropy dissipation method exemplified as above has become an important tool to quantitatively study solutions of partial differential equations. We refer interested readers to the review paper [19] and the books [18, 20] for more extensive discussion on this method.

The decay of the solution of Fokker-Planck equation in relative entropy can be viewed from a different, yet much deeper perspective. This dates back to the celebrated work by Jordan, Kinderlehrer and Otto [8], where the Fokker-Planck equation is regarded as the gradient flow of the relative entropy with respect to the 2-Wasserstein distance in the space of probability measures. Based on identifying a Riemannian structure on the Wasserstein space of probability measures, Otto [14] showed that a large number of evolution equations could also be viewed as the gradient flow in the 2-Wasserstein metric for certain functionals. Moreover, in this geometric setting, the strong geodesic convexity of the functionals gives rise to a number of functional inequalities, including the LSI; see e.g. [1, 13]. By now, similar results in this direction have been obtained in various settings, such as finite Markov chains [9], discrete porous medium equation [6], and quantum Lindblad equation [5], just to name a few.

In this paper, we study the dissipation behavior of the solution of Fokker-Planck equation with respect to Rényi divergence, including the relative entropy as a special instance. The precise definition of Rényi divergence is given as follows.

**Definition 1.1 (Rényi divergence).** For two probability distributions \( p \ll q \), Rényi divergence is defined as
\[
D_\alpha(p \parallel q) = \begin{cases} 
\frac{1}{\alpha - 1} \ln \left( \int \left( \frac{p}{q} \right)^\alpha \, dq \right), & 0 < \alpha < \infty, \alpha \neq 1; \\
\int \frac{p}{q} \ln \left( \frac{p}{q} \right) \, dq, & \alpha = 1.
\end{cases}
\]
If \( p \) is not absolutely continuous with respect to \( q \), set \( D_\alpha(p \parallel q) = \infty \).

Our interest in Rényi divergence is motivated by the recent work on the second laws of quantum thermodynamics [3], which states that a family of free energies – quantum Rényi divergence, never increase during state transitions in a microscopic quantum system. It has been proved rigorously by Frank and Lieb [7] that for order \( \alpha \) larger or equal to \( \frac{1}{2} \), the quantum Rényi divergence is monotonically decreasing under all completely positive trace preserving maps. In particular, this implies...
that the quantum Rényi divergence decreases under Lindblad equation, which is generally viewed as the quantum analogy of Fokker-Planck equation. There are some attempts to characterize the convergence rate for quantum Rényi divergence under Lindblad equations [11]. One important issue at the quantum level is that there is no consensus about how Rényi divergence should be defined due to the non-commutative nature of quantum system. Two incompatible versions of quantum Rényi divergence can be found in [12] and [3]. Motivated by this, we pull ourselves back from microscopic dynamics (quantum) to macroscopic system (classical), and examine the decay rate of the classical Rényi divergence, defined in (1.7), under the Fokker-Planck equation. To the best of our knowledge, this topic has not been presented in literature. We also refer to our forthcoming paper [4] for a similar study on the Lindblad equation involving a specific quantum Rényi divergence.

The main result of the present paper is the following theorem.

**Theorem 1.2.** Assume that \( V \) satisfies (1.2). Fix \( \alpha \in (0, \infty) \), \( \delta \in (0, 2K) \), and a smooth initial probability distribution \( p_0 \) which decays sufficiently fast at infinity. Let \( p_t \) be the solution of Fokker-Planck equation given in (1.1). Then there exists \( t^* \geq 0 \), \( C > 0 \) and \( \kappa_\alpha > 0 \) such that

\[
D_\alpha(p_t \parallel p_\infty) \leq CD_\alpha(p_0 \parallel p_\infty)e^{-\kappa_\alpha t}
\]

for any \( t \geq t^* \), where

\[
\kappa_\alpha = \begin{cases} 
2K & \text{if } \alpha \in (0, 1], \\
2K - \delta & \text{if } \alpha \in (1, \infty),
\end{cases}
\]

and an upper bound of \( t^* \) is given by

\[
t^* = \begin{cases} 
0, & \alpha \in (0, 1]; \\
\frac{(\alpha - 1)D_\alpha(p_0 \parallel p_\infty)}{\delta}, & \alpha \in (1, 2]; \\
\frac{D_2(p_0 \parallel p_\infty)}{\delta} + \frac{1}{2K} \ln(\alpha - 1), & \alpha \in (2, \infty).
\end{cases}
\]

and the constant \( C \) can be bounded from above as

\[
C = \begin{cases} 
D_1(p_0 \parallel p_\infty)/D_\alpha(p_0 \parallel p_\infty), & \alpha \in (0, 1]; \\
\exp((2K - \delta)(\alpha - 1)D_\alpha(p_0 \parallel p_\infty)/\delta), & \alpha \in (1, 2].
\end{cases}
\]

**Remark 1.3.** Observe that when \( \alpha = 1 \), Theorem 1.2 recovers the classical dissipation estimate (1.3) for relative entropy under Fokker-Planck equation.

When \( \alpha > 1 \), the exponential decay of the Rényi divergence \( D_\alpha(p_t \parallel p_\infty) \) with a rate close to \( 2K \) only kicks in after some waiting period \( t^* \); see Example 1.4 below. The upper bound of the waiting time \( t^* \) given in (1.10) may not be sharp.

Compared to (1.3), the exponential decay rate in Theorem 1.2 seems sub-optimal when \( \alpha \in (1, \infty) \). This is due to the necessity of introducing small \( \delta \) in generalized log-Sobolev inequality (see Proposition 3.4 below). It remains an open question whether/how the sharp exponential convergence rate \( 2K \) can be obtained.

**Example 1.4.** Consider a one dimensional Ornstein-Uhlenbeck process: dimension \( d = 1 \) and \( V(x) = \frac{x^2}{2} + \frac{1}{2} \ln(2\pi) \); then \( K = 1 \) and \( p_\infty = N(0, 1) \). Choose the initial distribution \( p_0 \) to be \( N\left(1, \frac{1}{2}\right) \). Then at time \( t \), \( p_t = N\left(e^{-t}, 1 - \frac{1}{2}e^{-2t}\right) \). A direct
calculation leads to
\[
D_\alpha(p_t \parallel p_\infty) = -\frac{1}{2} \ln \left(1 - \frac{1}{2} e^{-2t}\right) - \frac{1}{2(\alpha - 1)} \ln \left(1 + \frac{1}{2} e^{-2t} (\alpha - 1)\right) + \frac{\alpha}{(\alpha - 1) + 2e^{2t}}.
\]
Hence as \( t \to \infty \), \( D_\alpha(p_t \parallel p_\infty) \approx \frac{\alpha e^{-2t}}{2} \), which decays exponentially fast with rate \( 2K \) (\( 2K = 2 \) in the current case) for any \( \alpha \in (0, \infty) \). However, when \( t \sim O(1) \), this is not the case: in the third term above, for large \( \alpha \gg 1 \), the additive term \( \alpha - 1 \) might dominate the term \( 2e^{2t} \) in the denominator; thus we cannot observe exponential decay with rate close to \( 2K \). Therefore, in general one needs a waiting period \( t^* \) as in Theorem 1.2 before achieving the asymptotic exponential decay.

The proof of Theorem 1.2 is presented in Section 3. Let us explain here briefly the key idea of our proof. In the case that \( \alpha \in (0, 1] \), the estimate (1.8) simply follows from (1.3) and the monotonicity of Rényi divergence with respect to order \( \alpha \) (c.f. Lemma 3.1). The majority of the effort is devoted to the case \( \alpha \in (1, 2] \). In this regime, we prove that \( p_\infty \) satisfies a generalized log-Sobolev inequality (c.f. Proposition 3.4), which combining with the standard entropy production argument leads to (1.8). The generalized log-Sobolev inequality could be better understood via variational formalism of Fokker-Planck equation, with Rényi divergence as the energy functional (c.f. Section 2), which is a new result, to the best of our knowledge. Finally, the result in the regime \( \alpha \in (2, \infty) \) follows from the previous result in the case \( \alpha \in (1, 2] \) and a comparison lemma of Rényi divergence between different orders \( \alpha \), under Fokker-Planck equation (c.f. Lemma 3.6). We want to emphasize that most proofs presented in this paper are established by assuming the validity of various integration by parts formulas to simplify the technicality of the presentation.

The rest of the paper is organized as follows. In Section 2 we first show that the Fokker-Planck equation can be viewed as the gradient flow of Rényi divergence under a certain metric tensor. Section 3 is devoted into the proof of Theorem 1.2 (some lengthy calculations and proofs are postponed into appendix).

2. Fokker-Planck equation as the gradient flow of Rényi divergence

This section is devoted to identifying Fokker-Planck equation (1.1) as the gradient flow of Rényi divergence for any order \( \alpha \in (0, \infty) \), with respect to a certain metric tensor (2.6) in the space of probability measures, which generalizes the well-known fact that Fokker-Planck equation is \( L^2 \)-Wasserstein gradient flow of the relative entropy [8]. Interested readers may refer to e.g., [1, 13, 14] for an extensive treatment of gradient flows in the space of probability measures.

We want to define a Riemannian structure on a space of probability measures under which the gradient flow of \( D_\alpha(\cdot \parallel p_\infty) \) gives the Fokker-Planck equation (1.1). By Riemannian structure, we mean a manifold (denoted by \( M_\alpha \)) and a metric tensor, denoted by \( g_{\alpha,p}(\cdot, \cdot) \), defined on the tangent space \( \mathcal{T}_p M_\alpha \). The dependence of the metric tensor on \( \alpha \) and \( p \) will be clear in the sequential. For a fixed a Riemannian structure (\( M_\alpha, g_{\alpha,p}(\cdot, \cdot) \)), the gradient of the energy functional \( D_\alpha(\cdot \parallel p_\infty) \) at \( p \in M_\alpha \) is defined as the element in \( \mathcal{T}_p M_\alpha \), denoted by \( \nabla D_\alpha|_p \).
or simply $\text{grad} D_\alpha$ (when no confuse arises for $p$), such that

\begin{equation}
\tag{2.1}
g_{\alpha,p}(\text{grad} D_\alpha, \nu) = \frac{d}{d\epsilon} D_\alpha(p + \epsilon \nu \parallel p_\infty) \bigg|_{\epsilon = 0}, \quad \forall \nu \in \mathcal{T}_p M_\alpha.
\end{equation}

The corresponding gradient flow dynamics (of the Rényi divergence) is given by

\begin{equation}
\tag{2.2}
\partial_t p_t = -\text{grad} D_\alpha|_{p_t}.
\end{equation}

Below we define the space $M_\alpha$ and metric tensor $g_{\alpha,p}(\cdot, \cdot)$. Let $M_\alpha$ be the space of smooth probability distributions, which have finite Rényi divergence with respect to $p_\infty$, i.e.

\[ M_\alpha := \{ \text{smooth } p : D_\alpha(p \parallel p_\infty) < \infty \}. \]

We will not go into technical details of the differential structure of the manifold and think of the tangent space $\mathcal{T}_p M_\alpha$ at $p \in M_\alpha$ as

\[ \mathcal{T}_p M_\alpha = \{ \text{signed functions on } \mathbb{R}^d \text{ with } \int \nu(x) \, dx = 0 \}. \]

For any $\nu \in \mathcal{T}_p M_\alpha$, let $\Psi_\nu$ be a weak solution to the equation

\begin{equation}
\tag{2.3}
\nu + \text{div}(p \nabla \Psi_\nu) = 0.
\end{equation}

Namely, for all smooth and compactly supported test functions $f$, we have

\[ \int f \nu \, dx = \int \nabla f \cdot \nabla \Psi_\nu \, dp. \]

Note that $\Psi_\nu$ is defined uniquely up to some additive constant. Then whenever dealing with an element $\nu \in \mathcal{T}_p M_\alpha$, it is equivalent to consider its associated $\Psi_\nu$.

In order to define the metric tensor $g_{\alpha,p}(\cdot, \cdot)$, we also need to introduce an inner product $\langle \cdot, \cdot \rangle_{\alpha,p}$ on the space of vector fields. To be more precise, we define, for vector fields $U = [u_1 \ u_2 \ \cdots \ u_d]$ and $V = [v_1 \ v_2 \ \cdots \ v_d]$ where $u_j$ and $v_j$ are functions on $\mathbb{R}^d$, for all $1 \leq j \leq d$, the inner product $\langle \cdot, \cdot \rangle_{\alpha,p}$ by

\begin{equation}
\tag{2.4}
\langle U, V \rangle_{\alpha,p} := \sum_{j=1}^d \alpha \int u_j v_j \, d\mu_{\alpha,p},
\end{equation}

where $\mu_{\alpha,p}$ is a probability distribution defined by

\begin{equation}
\tag{2.5}
\mu_{\alpha,p} := \frac{\left( \frac{p}{p_\infty} \right)^\alpha p_\infty}{\int \left( \frac{p}{p_\infty} \right)^\alpha \, dp_\infty}.
\end{equation}

With this inner product, we define the metric tensor $g_{\alpha,p}(\cdot, \cdot)$ by

\begin{equation}
\tag{2.6}
g_{\alpha,p}(\nu_1, \nu_2) := \langle \nabla \Psi_{\nu_1}, \nabla \Psi_{\nu_2} \rangle_{\alpha,p},
\end{equation}

for $\nu_k \in \mathcal{T}_p M_\alpha$, where $\Psi_{\nu_k}$ are related to $\nu_k$ via the equation

\[ \nu_k + \text{div}(p \nabla \Psi_{\nu_k}) = 0 \quad k = 1, 2. \]

When $\alpha = 1$, it is easy to see that $\mu_{\alpha,p} = p$ and that the resulting metric tensor reduces to the one that has been defined in [14].
Finally, we check that the Fokker-Planck equation is indeed the gradient flow of $D_\alpha(p \parallel p_\infty)$ with respect to the Riemannian structure defined above. In fact, by definition of the gradient in (2.1), we have from direct computations that

$$g_{\alpha,p} (\text{grad} D_\alpha, \nu) = \frac{d}{d\epsilon} D_\alpha(p + \epsilon \nu \parallel p_\infty) \bigg|_{\epsilon = 0}$$

$$= \alpha \int \left( \frac{p}{p_\infty} \right)^{\alpha - 1} \nu \, dp_\infty$$

$$= -\alpha \int \left( \frac{p}{p_\infty} \right)^{\alpha - 1} \text{div}(p \nabla \Psi) \, dx$$

$$= \alpha \int \left( \frac{p}{p_\infty} \right)^{\alpha - 1} \nabla \left( \frac{p}{p_\infty} \right) \cdot \nabla \Psi \, dp_\infty$$

$$= \langle -\nabla \phi, \nabla \Psi \rangle_{\alpha, p},$$

where

$$\phi := -\ln(p/p_\infty) = -\ln(p) - V.$$  

In view of the definition of metric tensor (2.6), this implies that

$$\text{grad} D_\alpha + \text{div}(p(-\nabla \phi)) = 0,$$

As a consequence, the corresponding gradient flow dynamics is

$$\partial_t p_t = -\text{grad} D_\alpha|_{p_t} = -\text{div}(p_t \nabla \phi_t),$$

where $\phi_t = -\ln(p_t/p_\infty)$. This recovers exactly the Fokker-Planck equation (1.1).

3. Proof of Theorem 1.2

The proof of Theorem 1.2 is divided into three subsections according to three regimes of $\alpha$.

3.1. Case (I): $\alpha \in (0, 1]$. We first recall a useful lemma on the monotonicity of Rényi divergence in the order $\alpha$.

Lemma 3.1. [17, Theorem 3] Let $p, q$ be two probability distributions. Then $D_\alpha(p \parallel q)$ is non-decreasing in order $\alpha$ for any $\alpha \in (0, \infty)$.

Proof of Theorem 1.2 in Case (I). Thanks to Lemma 3.1 and (1.3), we immediately have that

$$D_\alpha(p_t \parallel p_\infty) \leq D_1(p_t \parallel p_\infty) \leq e^{-2Kt} D_1(p_0 \parallel p_\infty) \leq Ce^{-2Kt} D_\alpha(p_0 \parallel p_\infty),$$

where $C = D_1(p_0 \parallel p_\infty)/D_\alpha(p_0 \parallel p_\infty)$. This proves (1.8). Apparently, waiting period $t^* = 0$. $\square$
3.2. Case (II): $\alpha \in (1, 2]$. In this case, we will prove Theorem 1.2 by applying the entropy production method. Recall the space $\mathcal{M}_\alpha$ of probability measures and its metric tensor $\langle \cdot, \cdot \rangle_{\alpha, p}$ defined by (2.6). If $p_t$ solves the Fokker-Planck equation (i.e., $p_t$ is the gradient flow), then

$$
\frac{d}{dt} D_\alpha(p_t \parallel p_\infty) = -g_{\alpha, p_t}(\partial_t p_t, \partial_t p_t)
$$

$$
= -\alpha \int |\nabla \phi_t|^2 \, d\mu_{\alpha, p_t}
$$

$$
= -\alpha \int \left| \nabla \ln \left( \frac{p_t}{p_\infty} \right) \right|^2 \, d\mu_{\alpha, p_t}.
$$

Recall that $\phi_t := -\ln(p_t/p_\infty)$ (2.7). As a remark, the above equality can also be obtained via direct calculation. Then define

$$
\Phi_t := -\ln \left( \frac{\mu_{\alpha, p_t}}{p_\infty} \right) = -\ln \left( \frac{p_t}{p_\infty} \right)^\alpha + (\alpha - 1) D_\alpha(p_t \parallel p_\infty)
$$

$$
= \alpha \phi_t + (\alpha - 1) D_\alpha(p_t \parallel p_\infty).
$$

Then the time derivative of Rényi divergence is linked to the relative Fisher information via

$$
\frac{d}{dt} D_\alpha(p_t \parallel p_\infty) = -\frac{1}{\alpha} \int |\nabla \Phi_t|^2 \, d\mu_{\alpha, p_t} = -\frac{1}{\alpha} I(\mu_{\alpha, p_t} \parallel p_\infty).
$$

To distinguish this relative Fisher information $I(\mu_{\alpha, p_t} \parallel p_\infty)$ and the classical one $I(p_t \parallel p_\infty)$, we shall call $I(\mu_{\alpha, p_t} \parallel p_\infty)$ relative $\alpha$-Fisher information here and below.

In order for the entropy production method to work, we need to bound the relative $\alpha$-Fisher information from below by Rényi divergence. This is formalized as a generalized log-Sobolev inequality below. It turns out this inequality is only valid for a restrictive class of probability measures $\mathcal{M}_{\alpha, \delta}$ that is close to the invariant measure $p_\infty$. To be more precise, fixing $\delta \in (0, 2K)$ and $\alpha \in [1, 2]$, we define

$$
\mathcal{M}_{\alpha, \delta} := \left\{ p \in \mathcal{M}_\alpha \mid \frac{\alpha - 1}{\alpha} I(\mu_{\alpha, p} \parallel p_\infty) \leq \delta \right\}.
$$

When $\alpha = 1$, the inequality in above definition holds trivially and hence $\mathcal{M}_{\alpha, \delta} = \mathcal{M}_\alpha$; when $\alpha \in (1, 2]$, the above constraint is non-trivial and we have that $\mathcal{M}_{\alpha, \delta} \subsetneq \mathcal{M}_\alpha$.

The following lemma characterizes the exponential decay of relative $\alpha$-Fisher information on the set $\mathcal{M}_{\alpha, \delta}$, and it plays a key role in proving both generalized log-Sobolev inequality and case (II) of Theorem 1.2. The proof of this lemma is postponed to Section 4.1.

**Lemma 3.2.** Fix $\delta \in (0, 2K)$ and $\alpha \in [1, 2]$. Let $p_t$ be the solution to (1.1) with initial condition $p_0 \in \mathcal{M}_{\alpha, \delta}$. Then $I(\mu_{\alpha, p_t} \parallel p_\infty)$ is monotonically decreasing with respect to time $t$ and moreover, it decays exponentially fast with rate $2K - \delta$, i.e.,

$$
I(\mu_{\alpha, p_t} \parallel p_\infty) \leq I(\mu_{\alpha, p_0} \parallel p_\infty)e^{-(2K-\delta)t}.
$$

**Remark 3.3.** The monotonicity of relative $\alpha$-Fisher information implies that the space $\mathcal{M}_{\alpha, \delta}$ is an invariant set under the Fokker-Planck dynamics.
**Proposition 3.4** (Generalized log-Sobolev inequality). Consider fixed \( \delta \in (0, 2K) \) and \( \alpha \in [1, 2] \). Then \( p_\infty \) satisfies a generalized log-Sobolev inequality in the following sense:

\[
D_\alpha(q \parallel p_\infty) \leq \frac{I(\mu_{\alpha, q} \parallel p_\infty)}{\alpha(2K - \delta)}, \quad \forall q \in \mathcal{M}_{\alpha, \delta}.
\]

**Proof of Proposition 3.4.** Let \( p_t \) be the solution of the Fokker-Planck equation whose initial condition \( p_0 = q \in \mathcal{M}_{\alpha, \delta} \). By (3.2) and Lemma 3.2,

\[
D_\alpha(q \parallel p_\infty) \equiv D_\alpha(p_0 \parallel p_\infty) = \frac{1}{\alpha} \int_0^\infty I(\mu_{\alpha, p_t} \parallel p_\infty) \, dt \leq \frac{I(\mu_{\alpha, p_0} \parallel p_\infty)}{\alpha} \int_0^\infty e^{-(2K - \delta)t} \, dt = \frac{I(\mu_{\alpha, q} \parallel p_\infty)}{\alpha(2K - \delta)},
\]

which proves (3.5). \( \square \)

For \( \alpha = 1, \delta = 0 \), then \( \mathcal{M}_{\alpha, \delta} = \mathcal{M}_\alpha \) and the above inequality (3.5) reduces to the classical log-Sobolev inequality in (1.6).

**Remark 3.5.** We remark that a linearization of the generalized log-Sobolev inequality above gives Poincaré inequality. Similar results hold for the (classical) log-Sobolev inequality in (1.6) and have been proved in [13]. In fact, assuming that the \( q \) is a small perturbation of equilibrium, i.e. \( q = (1 + \epsilon f)p_\infty \), where \( \epsilon \ll 1 \) and \( f \) is a function satisfying \( \int f \, dp_\infty = 0 \), one can derive an asymptotic expansion of the generalized LSI (3.5) as \( \epsilon \to 0 \). The leading order term will lead to Poincaré inequality \( \int f^2 \, dp_\infty \leq \frac{1}{\alpha} \int |\nabla f|^2 \, dp_\infty \). See [13] Section 7] for more details.

**Proof of Theorem 1.2 in Case (II).** Recall that for case (II), we consider \( \alpha \in (1, 2] \), though in this subsection, many concepts are introduced for \( \alpha \in [1, 2] \). Let \( t^* \) be the first passage time into set \( \mathcal{M}_{\alpha, \delta} \) for Fokker-Planck dynamics \( p_t \). We first prove that \( t^* < \infty \) and provide an upper bound estimation for it. By the definition of \( \mathcal{M}_{\alpha, \delta} \) and \( t^* \), \( I(\mu_{\alpha, p_t} \parallel p_\infty) \geq \frac{\alpha}{\alpha - 1} \delta \) for \( t \leq t^* \). Then by (3.2),

\[
\frac{d}{dt} D_\alpha(p_t \parallel p_\infty) \leq -\frac{\delta}{\alpha - 1} \quad \forall t \leq t^*.
\]

This leads to

\[
0 - D_\alpha(p_0 \parallel p_\infty) \leq D_\alpha(p_t \parallel p_\infty) - D_\alpha(p_0 \parallel p_\infty) \leq -\frac{\delta}{\alpha - 1} t^*,
\]

from which we have

\[
t^* \leq \frac{(\alpha - 1)D_\alpha(p_0 \parallel p_\infty)}{\delta}.
\]

We know that \( p_{t^*} \in \mathcal{M}_{\alpha, \delta} \). Then by Lemma 3.2, \( p_t \in \mathcal{M}_{\alpha, \delta} \) for all \( t \geq t^* \). By the generalized LSI (see Proposition 3.4 and 3.2) we have

\[
\frac{d}{dt} D_\alpha(p_t \parallel p_\infty) = -\frac{1}{\alpha} I(\mu_{\alpha, p_t} \parallel p_\infty) \leq -(2K - \delta) D_\alpha(p_t \parallel p_\infty), \quad \forall t \geq t^*.
\]

By Grönwall’s inequality, for \( t \geq t^* \),

\[
D_\alpha(p_t \parallel p_\infty) \leq e^{-(2K - \delta)(t - t^*)} D_\alpha(p_{t^*} \parallel p_\infty),
\]

where \( C = e^{t^*(2K - \delta)} D_\alpha(p_{t^*} \parallel p_\infty) \leq e^{t^*(2K - \delta)} \leq \exp((2K - \delta)(\alpha - 1)D_\alpha(p_0 \parallel p_\infty)/\delta).\) This finishes the proof of Theorem 1.2 for case (II). \( \square \)
3.3. Case (III): $\alpha \in (2, \infty)$. In this case, we would like to prove Theorem 1.2 by making use of the results proved in the previous case (II). To this end, we first state a useful comparison principle for the family of Rényi divergences $\{D_{\alpha}(p_t \parallel p_\infty)\}_{\alpha>1}$ when $p_t$ solves the Fokker-Planck equation (1.1).

**Lemma 3.6** (Comparison of orders). Let $1 < \alpha_0 < \alpha_1 < \infty$. If $p_t$ solves the Fokker-Planck equation with initial condition $p_0$, then

$$D_{\alpha_1}(p_t \parallel p_\infty) \leq \frac{\alpha_1(\alpha_0 - 1)}{\alpha_0(\alpha_1 - 1)}D_{\alpha_0}(p_0 \parallel p_\infty),$$

where $t_1 = \frac{1}{2K} \ln \left(\frac{\alpha_1 - 1}{\alpha_0 - 1}\right)$.

Lemma 3.6 states that the Rényi divergence $D_{\alpha}(p_t \parallel p_\infty)$ can be bounded from above by a Rényi divergence with a smaller order than $\alpha$ at the expense of longer marching time. A simpler version of Lemma 3.6 for the Ornstein-Uhlenbeck process ($V = \frac{|x|^2}{2} + \frac{d}{2} \ln(2\pi)$) was proved in [15, Theorem 3.2.3]. Since we are unaware of the proof of this lemma for Fokker-Planck dynamics with a general potential $V$ in the literature, we include a proof in the appendix for completeness.

**Proof of Theorem 1.2 in Case (III).** Fix $\alpha > 2$. The proof in this case needs to use the result from Case (II). In fact, combing (3.6) with order 2, and Lemma 3.6 with $\alpha_0 = 2, \alpha_1 = \alpha > 2$, we can obtain that for $t \geq t^* + t_1$ (Here $t^*$ is the waiting time for $\alpha_0 = 2$ in case (II))

$$D_{\alpha}(p_t \parallel p_\infty) \leq \frac{\alpha}{2(\alpha - 1)}D_2(p_{t-t_1} \parallel p_\infty) \leq \frac{\alpha}{2(\alpha - 1)}C_2D_2(p_0 \parallel p_\infty)e^{-\left(2K-\delta\right)(t-t_1)} = Ce^{-\left(2K-\delta\right)t}D_{\alpha}(p_0 \parallel p_\infty),$$

where $C_2$ is the prefactor for $\alpha = 2$ case, and

$$C = \frac{\alpha}{2(\alpha - 1)}C_2D_2(p_0 \parallel p_\infty)e^{\left(2K-\delta\right)t_1}.$$ 

Notice that $t_1 = \frac{1}{2K} \ln(\alpha - 1)$, thus $t^* + t_1 \leq D_2(p_0 \parallel p_\infty) + \frac{1}{2K} \ln(\alpha - 1)$, which becomes the upper bound for the waiting time in the case $\alpha \in (2, \infty)$. This completes the proof of Theorem 1.2.

4. Appendix

4.1. Proof of Lemma 3.2. Recall the definition

$$\Phi_t := -\ln \left(\frac{\mu_{\alpha,p_t}}{p_\infty}\right)$$

from (3.1). First, we need to compute the time derivative of $I(\mu_{\alpha,p_t} \parallel p_\infty)$, given in the following lemma.
Lemma 4.1. The time derivative of $I(\mu_{\alpha,p} \parallel p_\infty)$ is given by

$$
\frac{d}{dt}I(\mu_{\alpha,p} \parallel p_\infty) = -2 \int \text{tr}(D^2 \Phi_t^T D^2 \Phi_t) + \nabla \Phi_t \cdot D^2 V \nabla \Phi_t \, d\mu_{\alpha,p_t}
$$

$$
+ \frac{4}{\alpha} (\alpha - 1) \sum_{i,j} \int \partial_i \Phi_t \partial_j \Phi_t \partial_{i,j} \Phi_t \, d\mu_{\alpha,p_t}
$$

$$
- \frac{\alpha - 1}{\alpha} \int |\nabla \Phi_t|^4 \, d\mu_{\alpha,p_t} + \frac{\alpha - 1}{\alpha} I(\mu_{\alpha,p} \parallel p_\infty)^2.
$$

(4.1)

The terms on the second and third lines vanish when $\alpha = 1$, so they are essentially the correction terms for Rényi divergence (when $\alpha \neq 1$). Here, $\partial_i$ means partial derivative with respect to $x_i$, and similar rule applies to notations $\partial_j$ and $\partial_{i,j}$.

Proof. Step (I): Recall $\phi_t := -\log(p_t/p_\infty)$ from (2.7), where $p_\infty = e^{-V}$. It is straightforward to show

$$
\partial_t \phi_t = p_t^{-1} \text{div} (p_t \nabla \phi_t) = \Delta \phi_t + \nabla \phi_t \cdot \nabla p_t
$$

$$
\nabla \phi_t = -\frac{\nabla p_t}{p_t} - \nabla V.
$$

Hence, $\partial_t \phi_t = \Delta \phi_t - |\nabla \phi_t|^2 - \nabla \phi_t \cdot \nabla V$. Then by (3.1) and (3.2),

$$
\partial_t \Phi_t = \alpha \partial_t \phi_t - \frac{\alpha - 1}{\alpha} I(\mu_{\alpha,p} \parallel p_\infty)
$$

$$
= \alpha \left( \Delta \phi_t - |\nabla \phi_t|^2 - \nabla \phi_t \cdot \nabla V \right) - \frac{\alpha - 1}{\alpha} I(\mu_{\alpha,p} \parallel p_\infty).
$$

Also from (3.1), $\nabla \Phi_t = \alpha \nabla \phi_t$, then

$$
\partial_t \Phi_t = \Delta \Phi_t - \frac{1}{\alpha} |\nabla \Phi_t|^2 - \nabla \Phi_t \cdot \nabla V - \frac{\alpha - 1}{\alpha} I(\mu_{\alpha,p} \parallel p_\infty).
$$

Step (II): We need to compute the time derivative of relative $\alpha$-Fisher information

$$
\frac{d}{dt}I(\mu_{\alpha,p} \parallel p_\infty) = 2 \int \nabla \Phi_t \cdot \nabla (\partial_t \Phi_t) \, d\mu_{\alpha,p_t} + \int |\nabla \Phi_t|^2 (-\partial_t \Phi_t) \, d\mu_{\alpha,p_t}
$$

$$
= 2 \int \nabla \Phi_t \cdot \nabla \left( \Delta \Phi_t - \frac{1}{\alpha} |\nabla \Phi_t|^2 - \nabla \Phi_t \cdot \nabla V \right) \, d\mu_{\alpha,p_t}
$$

$$
- \int |\nabla \Phi_t|^2 \left( \Delta \Phi_t - \frac{1}{\alpha} |\nabla \Phi_t|^2 - \nabla \Phi_t \cdot \nabla V \right) \, d\mu_{\alpha,p_t}
$$

$$
+ \frac{\alpha - 1}{\alpha} I(\mu_{\alpha,p} \parallel p_\infty)^2.
$$

$$
=: T_1 - T_2 + \frac{\alpha - 1}{\alpha} I(\mu_{\alpha,p} \parallel p_\infty)^2.
$$
In the first line, we have used the fact that $\partial_t \Phi_t = -\partial_t \mu_{\alpha,p_t}/\mu_{\alpha,p_t}$ from \eqref{eq:Fisher_info}. The term $T_1$ is
\[
T_1 = 2 \int \nabla \Phi_t \cdot \nabla \left( \Delta \Phi_t - \frac{1}{\alpha} |\nabla \Phi_t|^2 - \nabla \Phi_t \cdot \nabla V \right) d\mu_{\alpha,p_t}
\]
\[
= \sum_{i,j} 2 \int \left( \partial_j \Phi_t \partial_{i,j} \Phi_t - \frac{2}{\alpha} \partial_j \Phi_t \partial_i \Phi_t \partial_{i,j} \Phi_t 
- \partial_j \Phi_t \partial_i \Phi_t \partial_{i,j} V \right) d\mu_{\alpha,p_t}
- \partial_j \Phi_t \partial_i \Phi_t \partial_{i,j} \Phi_t \partial_i V \right) d\mu_{\alpha,p_t}
\]
\[
= -2 \int \text{tr}(D^2 \Phi_t^T D^2 \Phi_t) + \nabla \Phi_t \cdot D^2 V \nabla \Phi_t d\mu_{\alpha,p_t}
+ (2 - \frac{4}{\alpha}) \sum_{i,j} \int \partial_i \Phi_t \partial_j \Phi_t \partial_{i,j} \Phi_t d\mu_{\alpha,p_t}.
\]

From the second line to the third line, integration by parts to $\partial_t \Phi_t \partial_{i,j} \Phi_t \mu_{\alpha,p_t} = (\partial_j \Phi_t \mu_{\alpha,p_t}) \partial_t (\partial_j \Phi_t)$ has been used, as well as the definition \eqref{eq:Fisher_info}, i.e., $\mu_{\alpha,p_t} = e^{-\Phi_t - V}$.

Similarly, the term $T_2$ is
\[
T_2 = \int \nabla \Phi_t^2 \left( \Delta \Phi_t - \frac{1}{\alpha} |\nabla \Phi_t|^2 - \nabla \Phi_t \cdot \nabla V \right) d\mu_{\alpha,p_t}
\]
\[
= \sum_{i,j} \int (\partial_i \Phi_t)^2 \left( \partial_{i,j} \Phi_t - \frac{1}{\alpha} (\partial_j \Phi_t)^2 - \partial_j \Phi_t \partial_i V \right) d\mu_{\alpha,p_t}
\]
\[
= -\sum_{i,j} \int \partial_j ((\partial_i \Phi_t)^2 \mu_{\alpha,p_t}) \partial_i \Phi_t dx
+ \sum_{i,j} \int (\partial_i \Phi_t)^2 \left( \frac{1}{\alpha} (\partial_j \Phi_t)^2 - \partial_j \Phi_t \partial_j V \right) d\mu_{\alpha,p_t}
\]
\[
= -2 \sum_{i,j} \int \partial_i \Phi_t \partial_j \Phi_t \partial_{i,j} \Phi_t \ d\mu_{\alpha,p_t} + (1 - \frac{1}{\alpha}) \int \nabla \Phi_t^4 \ d\mu_{\alpha,p_t}.
\]

Then \eqref{eq:Fisher_info} can be obtained by replacing $T_1$ and $T_2$ in the time derivative of relative $\alpha$-Fisher information by the above two equations.

\[ \square \]

**Proof of Lemma \ref{lem:Fisher_info}** When $\alpha = 1$, this result is classical and has been well-presented in literatures. One could also easily modify the proof below to fit the case $\alpha = 1$; thus this case is omitted here and we will only consider $\alpha \in (1, 2]$ below. Fix coordinates $i, j$,
\[
\frac{4}{\alpha} (\alpha - 1) \int \partial_i \Phi_t \partial_j \Phi_t \partial_{i,j} \Phi_t d\mu_{\alpha,p_t}
\leq \frac{2(\alpha - 1)}{\alpha} \left( \int \left( \frac{\partial_i \Phi_t \partial_j \Phi_t}{\sqrt{2}} \right)^2 d\mu_{\alpha,p_t} + \int \left( \sqrt{2} \partial_{i,j} \Phi_t \right)^2 d\mu_{\alpha,p_t} \right)
\]
\[
= \alpha - 1 \int (\partial_i \Phi_t)^2 (\partial_j \Phi_t)^2 d\mu_{\alpha,p_t} + \frac{4(\alpha - 1)}{\alpha} \int (\partial_{i,j} \Phi_t)^2 d\mu_{\alpha,p_t}.
\]
Applying the above inequality to (4.1) and using \( \sum_{i,j} (\partial_i \Phi_t)^2 (\partial_j \Phi_t)^2 \equiv |\nabla \Phi_t|^4 \) and \( \sum_{i,j} (\partial_i \Phi_t)^2 = \text{tr}(D^2 \Phi_t^T D^2 \Phi_t) \), we have
\[
\frac{d}{dt} I(\mu_{\alpha,p_t} \parallel p_\infty) \leq -2 \int (D^2 \Phi_t^T D^2 \Phi_t) + |\nabla \Phi_t| \cdot D^2 V \nabla \Phi_t \, d\mu_{\alpha,p_t} + \frac{4(\alpha - 1)}{\alpha} \sum_{i,j} (\partial_i \Phi_t)^2 \, d\mu_{\alpha,p_t} + \frac{\alpha - 1}{\alpha} I(\mu_{\alpha,p_t} \parallel p_\infty).
\]

Since \( \alpha \leq 2 \), then \( \frac{4(\alpha - 1)}{\alpha} \leq 2 \leq 0 \). By the fact that \( D^2 \Phi_t^T D^2 \Phi_t \geq 0 \),
\[
\frac{d}{dt} I(\mu_{\alpha,p_t} \parallel p_\infty) \leq -2 \int |\nabla \Phi_t| \cdot D^2 V \nabla \Phi_t \, d\mu_{\alpha,p_t} + \frac{\alpha - 1}{\alpha} I(\mu_{\alpha,p_t} \parallel p_\infty).
\]
Furthermore, by the assumption that \( D^2 V \) is strictly convex (used in the first line below), and that \( p_t \in M_{\alpha,\delta} \) (used in the third line below) for a particular time \( t \),
\[
\frac{d}{dt} I(\mu_{\alpha,p_t} \parallel p_\infty) \leq -2K \int |\nabla \Phi_t|^2 \, d\mu_{\alpha,p_t} + \frac{\alpha - 1}{\alpha} I(\mu_{\alpha,p_t} \parallel p_\infty)^2
\]
\[
= -2KI(\mu_{\alpha,p_t} \parallel p_\infty) + \frac{\alpha - 1}{\alpha} I(\mu_{\alpha,p_t} \parallel p_\infty)^2
\]
\[
\leq -(2K - \delta)I(\mu_{\alpha,p_t} \parallel p_\infty) \leq 0.
\]
Then \( I(\mu_{\alpha,p_t} \parallel p_\infty) \) is non-increasing in time and thus \( p_t \) will not leave \( M_{\alpha,\delta} \). Since we have assumed \( p_0 \in M_{\alpha,\delta} \), the above argument implies that \( I(\mu_{\alpha,p_t} \parallel p_\infty) \) is monotonically decreasing and \( p_t \in M_{\alpha,\delta} \) for any time \( t \geq 0 \). Finally, by Grönwall’s inequality, (4.4) easily follows.

\[
\square
\]

4.2. Proof of Lemma 3.6

Proof of Lemma 3.6. First, we need a variant of log-Sobolev inequality (1.6). Let \( p \) in (1.6) be \( p = \frac{f \, dp_\infty}{\int f \, dp_\infty} \) where \( f \) is a smooth, strictly positive function with \( \int f \, dp_\infty < \infty \). Then (1.6) can be re-written as
\[
\left( \int f \, dp_\infty \right) \ln \left( \int f \, dp_\infty \right) \leq \frac{1}{2K} \int \left| \nabla f \right|^2 \, dp_\infty.
\]

Then, we follow the proof of [15] Theorem 3.2.3. Let \( \alpha_t = 1 + (\alpha_0 - 1) e^{2Kt} \) and define
\[
F_t = \ln \left( \int h_t^{\alpha_t} \, dp_\infty \right)^{\frac{1}{\alpha_t}},
\]
where \( h_t := p_t/p_\infty \). It should be emphasized that both \( \alpha_t \) and \( h_t \) are changing during time evolution: the order \( \alpha_t \) is changing according to the above choice and the probability distribution \( p_t \) is evolving following the Fokker-Planck equation.

We shall show that \( F_t \) is non-increasing in time. In fact,
\[
\frac{d}{dt} F_t = \frac{1}{\alpha_t} \left[ \alpha_t \frac{d}{dt} \int h_t^{\alpha_t} \, dp_\infty - \frac{d\alpha_t}{dt} \ln \left( \int h_t^{\alpha_t} \, dp_\infty \right) \right].
\]
To simplify the notation, denote $Z_t := \int h_t^{\alpha_t} \, dp_\infty$. Multiplying both sides of last equation by $\alpha_t^2 Z_t$ and rearrange a few terms

$$\frac{d}{dt} F_t = \frac{\alpha_t}{\alpha_t - 1} \frac{\alpha_t}{\alpha_t - 1} - 1 \frac{\alpha_t}{\alpha_t - 1} \frac{\alpha_t}{\alpha_t - 1} - 1 = 0.$$ 

In the second line, log-Sobolev inequality (4.2) with $f = h_t^{\alpha_t}$ has been used.

Because $\alpha_t > 0$ and $Z_t > 0$, $F_t$ is non-increasing. Therefore, $F_t \leq F_0$, i.e.,

$$D_\alpha (p_t \parallel p_\infty) \leq \frac{\alpha_t}{\alpha_t - 1} \frac{\alpha_t}{\alpha_t - 1} - 1 \frac{\alpha_t}{\alpha_t - 1} \frac{\alpha_t}{\alpha_t - 1} - 1 = 0.$$

Then the lemma is proved by choosing time $t_1$ such that $\alpha_{t_1} = 1$, whence $t_1 = \frac{1}{2K} \ln \left( \frac{\alpha_1 - 1}{\alpha_0 - 1} \right).$ 

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