GENERALIZED JACOBIAN RINGS FOR OPEN COMPLETE INTERSECTIONS

MASANORI ASAKURA AND SHUJI SAITO

Abstract

In this paper, we develop the theory of Jacobian rings of open complete intersections, which mean a pair \((X, Z)\) where \(X\) is a smooth complete intersection in the projective space and and \(Z\) is a simple normal crossing divisor in \(X\) whose irreducible components are smooth hypersurface sections on \(X\). Our Jacobian rings give an algebraic description of the cohomology of the open complement \(X - Z\) and it is a natural generalization of the Poincaré residue representation of the cohomology of a hypersurface originally invented by Griffiths. The main results generalize the Macaulay’s duality theorem and the Donagi’s symmetrizer lemma for usual Jacobian rings for hypersurfaces. A feature that distinguishes our generalized Jacobian rings from usual ones is that there are instances where duality fails to be perfect while the defect can be controlled explicitly by using the defining equations of \(Z\) in \(X\). Two applications of the main results are given: One is the infinitesimal Torelli problem for open complete intersections. Another is an explicit bound for Nori’s connectivity in case of complete intersections. The results have been applied also to study of algebraic cycles in several other works.

Contents

§0 Introduction
§1 Jacobian rings for open complete intersections
§2 Green’s Jacobian rings
§3 Proof of Theorem(I)
§4 A vanishing lemma
§5 Proof of Theorem(II)
§6 Proof of Theorem(II’)
§7 Proof of Theorem(III)
§8 Infinitesimal Torelli for open complete intersections
§9 Explicit bound for Nori’s connectivity
References

§0. Introduction.

The purpose of this paper is to develope the theory of Jacobian rings of open complete intersections. Here, by “open complete intersection” we mean a pair \((X, Z = \bigcup_{1 \leq j \leq s} Z_j)\) where \(X\) is a smooth complete intersection in \(\mathbb{P}^n\) and \(Z_j \subset X\) is a smooth hypersurface section such that \(Z\) is a simple normal crossing divisor in \(X\). Our Jacobian rings give an algebraic description of the cohomology of \(X - Z\) and it is a natural generalization of the Poincaré residue representation of the cohomology of a hypersurface that played a significant role in the work of Griffiths [Gri]. The fundamental results on the generalized Jacobian rings have been stated without proof in [AS] where it is applied to the Beilinson’s Hodge and Tate conjecture for open complete intersections (see [A], [MSS] and [SaS] for other applications of the generalized Jacobian rings). The main results are stated in §1. The proofs occupying §2 through §7 are
based on the basic techniques in computations of Koszul cohomology developed by M. Green ([G1] and [G2]). Two applications of the results in §1 are given in §§8 and §9.

In §8 we study the infinitesimal Torelli problem for open complete intersection \((X, Z)\) as an application of the duality theorem for the generalized Jacobian rings. It concerns the injectivity of the following map

\[ H^1(X, T_X(-\log Z)) \to \bigoplus_{1 \leq q \leq m} \text{Hom}(H^{m-q}(X, \Omega_X^q(\log Z)), H^{m-q+1}(X, \Omega_X^{q-1}(\log Z))) \]

induced by the cup product and the contraction \(\Omega_X^q(\log Z) \otimes T_X(-\log Z) \to \Omega_X^{q-1}(\log Z)\). Here \(\Omega_X^q(\log Z)\) is the sheaf of differential \(q\)-forms on \(X\) with logarithmic poles along \(Z\) and \(T_X(-\log Z)\) is the \(\Omega_X\)-dual of \(\Omega_X^1(\log Z)\). We show that the map is injective under a mild numerical assumption. Since the above map is interpreted as the derivative of the period map from an appropriate moduli space of isomorphism classes of pairs \((X, Z)\) to the period domain (cf. [U2]), it implies that the mixed Hodge structure on \(H^m(X \setminus Z, \mathbb{Q})\) determines \((X, Z)\) up to isomorphisms locally on the moduli space. It is a generalization of the infinitesimal Torelli for hypersurfaces due to Griffiths [Gri] and for complete intersections due to Peters [P] and Usui [U1].

In §9 we prove the following result as an application of the symmetrizer lemma for the generalized Jacobian rings. We fix integers \(r, s \geq 1\) and \(d_1, \ldots, d_r, e_1, \ldots, e_s \geq 1\). Let \(S\) be a non-singular affine variety over \(\mathbb{C}\) and assume that we are given schemes over \(S\)

\[(0-1)\quad \mathbb{P}^n_S \leftrightarrow \mathcal{X} \leftrightarrow Z = \bigcup_{1 \leq j \leq s} Z_j,\]

whose fibers are open complete intersections. Assume that the fibers of \(\mathcal{X}/S\) are smooth complete intersection of multi-degree \((d_1, \ldots, d_r)\) in \(\mathbb{P}^n\) and that those of \(Z_j \subset \mathcal{X}\) are smooth hypersurface section of degree \(e_j\). Write \(\mathcal{U} = \mathcal{X} \setminus Z\). We will introduce an invariant \(c_S(\mathcal{X}, Z)\) that measures the “generality” of the family \((0-1)\), or how many independent parameters \(S\) contains.

**Theorem(0-1). Assuming \(n - r \geq 2\), we have**

\[
\begin{align*}
F^{t-n+r+1}H^t(\mathcal{U}, \mathcal{C}) &= 0 \quad \text{if } s \leq n - r + 2 \text{ and } \delta_{\min r} \geq t + r + 1 + c_S(\mathcal{X}, Z), \\
F^{t-n+r+1}H^t(\mathcal{X}, \mathcal{C}) &= 0 \quad \text{if } s = 1 \text{ and } \delta_{\min r} + e_1 \geq t + r + 1 + c_S(\mathcal{X}, Z),
\end{align*}
\]

where \(\delta_{\min} = \min \{d_i, e_j\}_{1 \leq i \leq r, 1 \leq j \leq s}\) and \(F^*\) denotes the Hodge filtration defined in [D1] and [D2].

The second vanishing of Th.(0-1) gives an explicit bound for Nori’s connectivity [N] in case of complete intersections in the projective space. Nagel [Na2] has obtain similar degree bounds for complete intersections in a general projective smooth variety.

**§1. Jacobian rings for open complete intersections.**

Throughout the whole paper, we fix integers \(r, s \geq 0\) with \(r+s \geq 1\), \(n \geq 2\) and \(d_1, \ldots, d_r, e_1, \ldots, e_s \geq 1\). We put

\[
d = \sum_{i=1}^r d_i, \quad e = \sum_{j=1}^s e_j, \quad \delta_{\min} = \min \{d_i, e_j\}_{1 \leq i \leq r, 1 \leq j \leq s}, \quad d_{\max} = \max \{d_i\}_{1 \leq i \leq r}, \quad e_{\max} = \max \{e_j\}_{1 \leq j \leq s}.
\]

We also fix a field \(k\) of characteristic zero. Let \(P = k[X_0, \ldots, X_n]\) be the polynomial ring over \(k\) in \(n+1\) variables. Denote by \(P^l \subset P\) the subspace of the homogeneous polynomials of degree \(l\). Let \(A\) be a polynomial ring over \(P\) with indeterminants \(\mu_1, \ldots, \mu_r, \lambda_1, \ldots, \lambda_s\). We use the multi-index notation

\[
\mu^{\underline{a}} = \mu_1^{a_1} \cdots \mu_r^{a_r} \quad \text{and} \quad \lambda^{\underline{b}} = \lambda_1^{b_1} \cdots \lambda_s^{b_s} \quad \text{for } \underline{a} = (a_1, \ldots, a_r) \in \mathbb{Z}_{\geq 0}^r, \quad \underline{b} = (b_1, \ldots, b_s) \in \mathbb{Z}_{\geq 0}^s.
\]
For $q \in \mathbb{Z}$ and $\ell \in \mathbb{Z}$, we write

$$A_q(\ell) = \bigoplus_{a+b=q} P^{ad+bc+\ell} \cdot \mu^a \lambda^b \quad (a = \sum_{i=1}^{r} a_i, b = \sum_{j=1}^{s} b_j, \mu = \sum_{j=1}^{r} a_i d_i, \lambda = \sum_{j=1}^{r} b_j \varepsilon_j)$$

By convention $A_q(\ell) = 0$ if $q < 0$.

**Definition (1-1).** For $\mathcal{F} = (F_1, \cdots, F_r), \mathcal{G} = (G_1, \cdots, G_s)$ with $F_i \in P^{d_i}, G_j \in P^{e_j}$, we define the Jacobian ideal $J(\mathcal{F}, \mathcal{G})$ to be the ideal of $A$ generated by

$$\sum_{1 \leq i \leq r} \frac{\partial F_i}{\partial x_k} \mu_i + \sum_{1 \leq j \leq s} \frac{\partial G_j}{\partial x_k} \lambda_j, \quad F_i, \quad G_j \lambda_j \quad (1 \leq i \leq r, 1 \leq j \leq s, 0 \leq k \leq n).$$

The quotient ring $B = B(\mathcal{F}, \mathcal{G}) = A/J(\mathcal{F}, \mathcal{G})$ is called the Jacobian ring of $(\mathcal{F}, \mathcal{G})$. We denote

$$B_q(\ell) = B_q(\ell)(\mathcal{F}, \mathcal{G}) = A_q(\ell)/A_q(\ell) \cap J(\mathcal{F}, \mathcal{G}).$$

**Definition (1-2).** Suppose $n \geq r + 1$. Let $\mathbb{P}^n = \text{Proj} \ P$ be the projective space over $k$. Let $X \subset \mathbb{P}^n$ be defined by $F_1 = \cdots = F_r = 0$ and let $Z_j \subset X$ be defined by $G_j = F_1 = \cdots = F_r = 0$ for $1 \leq j \leq s$. We also call $B(\mathcal{F}, \mathcal{G})$ the Jacobian ring of the pair $(X, Z = \cup_{1 \leq j \leq s} Z_j)$ and denote $B(\mathcal{F}, \mathcal{G}) = B(X, Z)$ and $J(\mathcal{F}, \mathcal{G}) = J(X, Z)$.

In what follows we fix $\mathcal{F}$ and $\mathcal{G}$ as Def.(1-1) and assume the condition

$$(1 - 1) \quad F_i = 0 \ (1 \leq i \leq r) \quad \text{and} \quad G_j = 0 \ (1 \leq j \leq s) \quad \text{intersect transversally in} \ \mathbb{P}^n.$$

We mention three main theorems. The first main theorem concerns with the geometric meaning of Jacobian rings.

**Theorem (1).** Suppose $n \geq r + 1$. Let $X$ and $\mathcal{Z}$ be as Definition (1-2).

1. For integers $0 \leq q \leq n - r$ and $\ell \geq 0$ there is a natural isomorphism

$$\phi_{X, \mathcal{Z}}^q : B_q(d + e - n - 1 + \ell) \xrightarrow{\sim} H^q(X, \Omega_{X}^{n-r-q}(\log Z)(\ell))_{\text{prim}}.$$

Here $\Omega_{X}^p(\log Z)$ is the sheaf of algebraic differential $q$-forms on $X$ with logarithmic poles along $Z$ and `prim' means the primitive part:

$$H^q(X, \Omega_{X}^p(\log Z)(\ell))_{\text{prim}} = \begin{cases} \text{Coker}(H^q(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^p) \rightarrow H^q(X, \Omega_X^p)) & \text{if } q = p \text{ and } s = \ell = 0, \\ H^q(X, \Omega_{X}^p(\log Z)(\ell)) & \text{otherwise}. \end{cases}$$

2. There is a natural map

$$\psi_{X, \mathcal{Z}} : B_1(0) \rightarrow H^1(X, T_X(-\log Z))_{\text{alg}} \subset H^1(X, T_X(-\log Z))$$

which is an isomorphism if $\dim(X) \geq 2$. Here $T_X(-\log Z)$ is the $\mathcal{O}_X$-dual of $\Omega_X^{1}(\log Z)$ and the group on the right hand side is defined in Def.(1-3) below. The following map

$$H^1(X, T_X(-\log Z)) \otimes H^q(X, \Omega_X^p(\log Z)) \rightarrow H^{q+1}(X, \Omega_X^{p-1}(\log Z)).$$

induced by the cup-product and the contraction $T_X(-\log Z) \otimes \Omega_X^p(\log Z) \rightarrow \Omega_X^{p-1}(\log Z)$ is compatible through $\psi_{X, \mathcal{Z}}$ with the ring multiplication up to scalar.

Roughly speaking, the generalized Jacobian rings describe the infinitesimal part of the Hodge structures of open variety $X \setminus Z$, and the cup-product with Kodaira-Spencer class coincides with the ring
The map $h$ is a natural map (called the trace map)

$$h : B_{n-r}(2(d - n - 1) + e) \rightarrow k.$$ 

Let $h_p(\ell) : B_p(d - n - 1 + \ell) \rightarrow B_{n-r-p}(d + e - n - 1 - \ell)^*$ be the map induced by the following pairing induced by the multiplication

$$B_p(d - n - 1 + \ell) \otimes B_{n-r-p}(d + e - n - 1 - \ell) \rightarrow B_{n-r}(2(d - n - 1) + e) \rightarrow k.$$ 

When $r > n$ we define $h_p(\ell)$ to be the zero map by convention.

(2) The map $h_p(\ell)$ is an isomorphism in either of the following cases.

(i) $s \geq 1$ and $p < n - r$ and $\ell < e_{\text{max}}$.

(ii) $s \geq 1$ and $0 \leq \ell \leq e_{\text{max}}$ and $r + s \leq n$.

(iii) $s = \ell = 0$ and either $n - r \geq 1$ or $n - r = p = 0$.

(3) The map $h_{n-r}(\ell)$ is injective if $s \geq 1$ and $\ell < e_{\text{max}}$.

We have the following auxiliary result on the duality.

Theorem(II'). Assume $n - r \geq 1$ and consider the composite map

$$\eta_{X,Z} : H^0(X, \Omega_X^{n-r}(\log Z)) \xrightarrow{(\phi_{X,Z})^{-1}} B_0(d + e - n - 1) \xrightarrow{h_{n-r}(0)^*} B_{n-r}(d - n - 1)^*$$

where the second map is the dual of $h_{n-r}(0)$. Then $\eta_{X,Z}$ is surjective and we have (cf. Def.(1-4) below)

$$\text{Ker}(\eta_{X,Z}) = \bigwedge_X^{n-r}(G_1, \ldots, G_s).$$

Definition(1-4). Let $G_1, \ldots, G_s$ be as in Def.(1-1) and let $Y_j \subset \mathbb{P}^n$ be the smooth hypersurface defined by $G_j = 0$. Let $X \subset \mathbb{P}^n$ be a smooth projective variety such that $Y_j$ ($1 \leq j \leq s$) and $X$ intersect transversally. Put $Z_j = X \cap Y_j$. Take an integer $q$ with $0 \leq q \leq s - 1$. For integers $1 \leq j_1 < \cdots < j_{q+1} \leq s$, let

$$\omega_X(j_1, \ldots, j_{q+1}) \in H^0(X, \Omega_X^q(\log Z)) \quad (Z = \sum_{1 \leq j \leq s} Z_j)$$

be the restriction of

$$\sum_{\nu=1}^{q+1} (-1)^{\nu-1} e_{j\nu} \frac{dG_{j\nu}}{G_{j\nu}} \wedge \cdots \wedge \frac{dG_{j_{\nu+1}}}{G_{j_{\nu+1}}} \wedge \cdots \wedge \frac{dG_{j_{q+1}}}{G_{j_{q+1}}} \in H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^q(\log Y))$$

This result was originally invented by P. Griffiths in case of hypersurfaces and generalized to complete intersections by Konno [K]. Our result is a further generalization.
Further we fix a global section be the global section associated respectively to the effective divisors is exact if one of the following conditions is satisfied.

We let where we denote where 

Our last main theorem is the generalization of Donagi's symmetrizer lemma [Do] (see also [DG], [Na1] and [N]) to the case of open complete intersections at higher degrees.

**Theorem (III).** Assume \( s \geq 1 \). Let \( V \subset B_1(0) \) is a subspace of codimension \( c \geq 0 \). Then the Koszul complex 

is exact if one of the following conditions is satisfied.

(i) \( p \geq 0, \ q = 0 \) and \( \delta_{\min p} + \ell \geq c \).

(ii) \( p \geq 0, \ q = 1 \) and \( \delta_{\min p} + \ell \geq 1 + c \) and \( \delta_{\min (p + 1)} + \ell \geq d_{\max} + c \).

(iii) \( p \geq 0, \ \delta_{\min (r + p)} + \ell \geq d + q + c, \ d + e_{\max} - n - 1 > \ell \geq d - n - 1 \) and either \( r + s \leq n + 2 \) or \( p \leq n - r - [q/2] \), where \([q/2]\) denotes the Gaussian symbol.

**Remark (1-1).** In case \( q \geq 2, \ \ell = d - n - 1, \ r + s > n + 2 \) and \( p = n - r - 1 \), the complex in Th. (III) is not injective but the cohomology is controlled by motivic elements. We shall study it in a future paper.

## §2. Green's Jacobian rings

Let the notation be as §1. In this section we study our Jacobian rings by using the method in [G2] and [G3]. We write

\[ E = E_0 \oplus E_1 \]  

with \( E_0 = \bigoplus_{i=1}^r \mathcal{O}(d_i) \) and \( E_1 = \bigoplus_{j=1}^s \mathcal{O}(e_j) \)

where we denote \( \mathcal{O} = \mathcal{O}_{\mathbb{P}^n} \). We consider the projective space bundle

\[ \pi : \mathbb{P} := \mathbb{P}(E) \longrightarrow \mathbb{P}^n. \]

We let \( \mathcal{L} = \mathcal{O}_{\mathbb{P}(E)}(1) \), the tautological line bundle. We let

\[ \mu_i \in H^0(\mathbb{P}, \mathcal{L} \otimes \pi^* \mathcal{O}(-d_i)) \quad \text{and} \quad \lambda_j \in H^0(\mathbb{P}, \mathcal{L} \otimes \pi^* \mathcal{O}(-e_j)) \]

be the global section associated respectively to the effective divisors

\[ \mathbb{P}(\bigoplus_{\alpha \neq i} \mathcal{O}(d_\alpha) \oplus E_1) \hookrightarrow \mathbb{P}(E) \quad \text{and} \quad \mathbb{P}(E_0 \oplus \bigoplus_{\beta \neq j} \mathcal{O}(e_\beta)) \hookrightarrow \mathbb{P}(E). \]

Further we fix a global section

\[ (2 - 1) \]

\[ \sigma = \sum_{i=1}^r F_i \mu_i + \sum_{j=1}^s G_j \lambda_j \in H^0(\mathbb{P}, \mathcal{L}), \]

and put

\[ Q_i : \mu_i = 0 \subset \mathbb{P}, \quad P_j : \lambda_j = 0 \subset \mathbb{P}, \quad Z : \sigma = 0 \subset \mathbb{P}, \]

\[ X_i : F_i = 0 \subset \mathbb{P}^n, \quad Y_j : G_j = 0 \subset \mathbb{P}^n. \]
We assume that
\[(2-2) \quad \bigcup_{1 \leq i \leq r} X_i \cup \bigcup_{1 \leq j \leq s} Y_j \subset \mathbb{P}^n \text{ is a simple normal crossing divisor}\]
that implies that \(Z\) is a nonsingular divisor in \(\mathbb{P}\). We will use the following divisors on \(\mathbb{P}\)

\[Q_* = \sum_{1 \leq i \leq r} Q_i \quad \text{and} \quad P_* = \sum_{1 \leq j \leq s} P_j\]

The following facts are well-known.

**Lemma (2-1).** Put \(t = r + s\).

1. We have the isomorphisms

\[R^i \pi_* \mathcal{L}^\nu \simeq \begin{cases} S^\nu(\mathcal{E}) & \text{if } \nu \geq 0, \quad i = 0 \\ \det \mathcal{E}^* \otimes S^{-\nu-t}(\mathcal{E}^*) & \text{if } \nu \leq -t, \quad i = t - 1 \\ 0 & \text{otherwise} \end{cases}\]

2. We have the isomorphisms

\[H^q(\mathbb{P}, \mathcal{L}^\nu \otimes \pi^* \mathcal{V}) \simeq \begin{cases} H^q(\mathbb{P}, S^\nu(\mathcal{E}) \otimes \mathcal{V}) & \text{if } \nu \geq 0 \\ H^q(\mathbb{P}, S^{-\nu-t}(\mathcal{E}^*) \otimes \det \mathcal{E}^* \otimes \mathcal{V}) & \text{if } \nu \leq -t \\ 0 & \text{if } -t + 1 \leq \nu \leq -1 \end{cases}\]

where \(\mathcal{V}\) is a vector bundle on \(\mathbb{P}^n\).

3. We have the commutative diagram with the exact horizontal sequences (called the Euler sequences)

\[0 \longrightarrow \mathcal{O}_\mathbb{P} \longrightarrow \pi^* \mathcal{E}^* \otimes \mathcal{L} \longrightarrow T_{\mathbb{P}/\mathbb{P}^n} \longrightarrow 0\]

where the middle vertical maps are given by the global sections

\[\mu_i \in H^0(\mathbb{P}, \mathcal{L} \otimes \pi^* \mathcal{O}(-d_i)) \quad \text{and} \quad \lambda_j \in H^0(\mathbb{P}, \mathcal{L} \otimes \pi^* \mathcal{O}(-e_j)).\]

We introduce the sheaf of differential operators of \(\mathcal{L}\) of order \(\leq 1\) as follows:

\[\Sigma_\mathcal{L} := \mathcal{D}iff^{\leq 1}(\mathcal{L}) = \{ P \in \mathcal{E}nd_{\mathbb{P}}(\mathcal{L}) : Pf - fP \text{ is } \mathcal{O}_\mathbb{P}\text{-linear (}f \in \mathcal{O}_\mathbb{P}) \}\]

\[\simeq \mathcal{L} \otimes D^\leq_{\mathbb{P}} \otimes \mathcal{L}^*.\]

By definition it admits the following exact sequence

\[(2-3) \quad 0 \longrightarrow \mathcal{O}_\mathbb{P} \longrightarrow \Sigma_\mathcal{L} \longrightarrow T_{\mathbb{P}} \longrightarrow 0\]

with the extension class

\[-c_1(\mathcal{L}) \in \text{Ext}^1(T_{\mathbb{P}}, \mathcal{O}_\mathbb{P}) \simeq \text{Ext}^1(\mathcal{O}_\mathbb{P}, \Omega^1_{\mathbb{P}} \otimes \mathcal{O}_\mathbb{P}) \simeq H^1(\mathbb{P}, \Omega^1_{\mathbb{P}}).\]
Letting \( U \subset \mathbb{P}^n \) be an affine subspace and \( x_1, \cdots, x_n \) be its coordinate, \( \Gamma(\pi^{-1}(U), \Sigma_L) \) is generated by the following sections

\[
(2-4) \quad \frac{\partial}{\partial x_i}, \lambda_i \frac{\partial}{\partial \lambda_j}, \mu_i \frac{\partial}{\partial \mu_j}, \quad \text{\( O_\Sigma \)-linear maps.}
\]

The section \( \sigma \) defines a map

\[
j(\sigma) : \Sigma_L \rightarrow \mathcal{L}, \quad P \mapsto P(\sigma),
\]

which is surjective by the assumption (2-2) and it gives rise to the exact sequence

\[
(2-5) \quad 0 \rightarrow T_\mathbb{P}(- \log \mathcal{Z}) \rightarrow \Sigma_L \xrightarrow{j(\sigma)} \mathcal{L} \rightarrow 0.
\]

**Definition (2-1).** We define

\[
\Sigma_L(- \log \mathbb{P}_x) \subset \Sigma_L \quad \text{and} \quad \Sigma_L(- \log \mathbb{P}_x + \mathbb{Q}_x) \subset \Sigma_L
\]

to be the inverse image of \( T_\mathbb{P}(- \log \mathbb{P}_x) \) and \( T_\mathbb{P}(- \log \mathbb{P}_x + \mathbb{Q}_x) \) respectively via the map in (2-3).

By the assumption (2-2) \( j(\sigma) \) restricted on \( \Sigma_L(- \log \mathbb{P}_x + \mathbb{Q}_x) \) is surjective so that (2-5) gives rise to the following commutative diagram of the exact sequences

\[
\begin{array}{c}
\begin{array}{c}
0 \rightarrow T_\mathbb{P}(- \log \mathcal{Z}) \rightarrow \Sigma_L \rightarrow \mathcal{L} \rightarrow 0 \\
\uparrow \quad \uparrow \quad \uparrow j(\sigma) \\
0 \rightarrow T_\mathbb{P}(- \log \mathcal{Z} + \mathbb{P}_x) \rightarrow \Sigma_L(- \log \mathbb{P}_x) \rightarrow \mathcal{L} \rightarrow 0 \\
\uparrow \quad \uparrow \quad \uparrow j(\sigma) \\
0 \rightarrow T_\mathbb{P}(- \log \mathcal{Z} + \mathbb{P}_x + \mathbb{Q}_x) \rightarrow \Sigma_L(- \log \mathbb{P}_x + \mathbb{Q}_x) \rightarrow \mathcal{L} \rightarrow 0.
\end{array}
\end{array}
\]

On the other hand (2-3) and the Euler sequences give rise to the following commutative diagram of the exact sequences

\[
\begin{array}{c}
\begin{array}{c}
0 \rightarrow \pi^* \mathcal{E}^* \otimes \mathcal{L} \rightarrow \Sigma_L \rightarrow \pi^* T_\mathbb{P}^n \rightarrow 0 \\
\uparrow \quad \uparrow \quad \uparrow \iota \\
0 \rightarrow \pi^* \mathcal{E}_0^* \otimes \mathcal{L} \oplus \mathcal{O}_\mathbb{P}^{\mathbb{Q}_x} \rightarrow \Sigma_L(- \log \mathbb{P}_x) \rightarrow \pi^* T_\mathbb{P}^n \rightarrow 0 \\
\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\
0 \rightarrow \mathcal{O}_\mathbb{P}^{\mathbb{Q}_x + \mathbb{P}_x} \rightarrow \Sigma_L(- \log \mathbb{P}_x + \mathbb{Q}_x) \rightarrow \pi^* T_\mathbb{P}^n \rightarrow 0.
\end{array}
\end{array}
\]

Here \( \iota \) is the sum of the maps \( \mathcal{L} \otimes \pi^* \mathcal{O}(-d_i) \rightarrow \Sigma_L \) and \( \mathcal{L} \otimes \pi^* \mathcal{O}(-e_j) \rightarrow \Sigma_L \) given by

\[
\frac{\partial}{\partial \mu_i} \in H^0(\mathbb{P}, \mathcal{L}^{-1} \otimes \pi^* \mathcal{O}(d_i) \otimes \Sigma_L) \quad \text{and} \quad \frac{\partial}{\partial \lambda_j} \in H^0(\mathbb{P}, \mathcal{L}^{-1} \otimes \pi^* \mathcal{O}(e_j) \otimes \Sigma_L)
\]

respectively. The left vertical maps are given by the global sections

\[
\mu_i \in H^0(\mathbb{P}, \mathcal{L} \otimes \pi^* \mathcal{O}(-d_i)) \quad \text{and} \quad \lambda_j \in H^0(\mathbb{P}, \mathcal{L} \otimes \pi^* \mathcal{O}(-e_j)).
\]

The following lemma is straightforward from the definition.

**Lemma (2-2).** For integers \( \ell \) and \( q \), we have

\[
A_q(\ell) = H^0(\mathcal{L}^\ell \otimes \pi^* \mathcal{O}(\ell)) \quad \text{and} \quad B_q(\ell) = A_q(\ell)/J(\Sigma_L(- \log \mathbb{P}_x)).
\]
where \( J(\Sigma E(- \log \mathbb{P}_s)) \subset A_q(\ell) \) is the image of the map
\[
j(\sigma) \otimes 1 : H^0(\Sigma E(- \log \mathbb{P}_s) \otimes \mathcal{L}^{r-1} \otimes \pi^* \mathcal{O}(\ell)) \to H^0(\mathcal{L}^s \otimes \pi^* \mathcal{O}(\ell)).
\]

We define the sheaf of differential operators of \( E \) of order \( \leq 1 \) as follows:
\[
\Sigma_E := \text{Diff}^{\leq 1}(E) = \{ P \in \mathcal{E} \text{End}_k(E) ; \ P f - f P \text{ is } \mathcal{O}_{\mathbb{P}^n}\text{-linear} \ (f \in \mathcal{O}_{\mathbb{P}^n}) \}
\]
which admits the exact sequence
\[
0 \to E \otimes E^* \to \Sigma'_E \to E \otimes E^* \otimes T_{\mathbb{P}^n} \to 0.
\]

We define \( \Sigma_E \) to be the inverse image of \( T_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n} \to E \otimes E^* \otimes T_{\mathbb{P}^n} \) where \( \mathcal{O}_{\mathbb{P}^n} \to E \otimes E^* \) is the diagonal embedding. By definition we have the exact sequence
\[
0 \to E \otimes E^* \to \Sigma_E \to T_{\mathbb{P}^n} \to 0.
\]

**Lemma 2.3.** We have the isomorphism \( \pi_* \Sigma_{\mathcal{L}} \xrightarrow{\sim} \Sigma_E \).

**Proof.** It is easy to see that the image of the natural map \( \pi_* \Sigma_{\mathcal{L}} \to \Sigma'_E \) is contained in the sheaf \( \Sigma_E \). Since the sheaf \( \pi_* \Sigma_{\mathcal{L}} \) is generated by the sections (2-4), it is surjective. Due to the exact sequences
\[
0 \to \mathcal{O}_{\mathbb{P}^n} \to \pi_* \Sigma_{\mathcal{L}} \to \pi_* T_{\mathbb{P}^n} \to 0,
\]
\[
0 \to \pi_* T_{\mathbb{P}^n/\mathbb{P}^n} \to \pi_* T_{\mathbb{P}^n} \to T_{\mathbb{P}^n} \to 0,
\]
we can see that \( \pi_* \Sigma_{\mathcal{L}} \) is a locally free sheaf of rank \( n + (r + s)^2 \), which is the same as the one of \( \Sigma_E \). Thus the surjective map \( \pi_* \Sigma_{\mathcal{L}} \to \Sigma_E \) is also injective. \( \Box \)

**Definition 3.3.** Define
\[
\Sigma^0_E = \pi_* \Sigma_{\mathcal{L}}(- \log \mathbb{P}_s) \subset \Sigma_E, \quad \text{and} \quad \Sigma^{00}_E = \pi_* \Sigma_{\mathcal{L}}(- \log \mathbb{P}_s + \mathbb{Q}_s) \subset \Sigma_E.
\]

By (2-7) we have the commutative diagram of the exact sequences
\[
\begin{array}{c}
0 \to E \otimes E^* \to \Sigma_E \to T_{\mathbb{P}^n} \to 0 \\
\uparrow \quad \uparrow \quad \uparrow = \\
(2-8) \quad 0 \to E \otimes E_0^s \oplus \mathcal{O}_{\mathbb{P}^n}^{\oplus s} \to \Sigma^0_E \to T_{\mathbb{P}^n} \to 0 \\
\uparrow \quad \uparrow \quad \uparrow = \\
0 \to \mathcal{O}_{\mathbb{P}^n}^{\oplus r+s} \to \Sigma^{00}_E \to T_{\mathbb{P}^n} \to 0,
\end{array}
\]

We have the global section
\[
\sigma' = \pi_* \sigma = (F_i, G_j)_{1 \leq i \leq r, 1 \leq j \leq s} \in H^0(E) = \bigoplus_{1 \leq i \leq r} H^0(\mathcal{O}(d_i)) \oplus \bigoplus_{1 \leq j \leq s} H^0(\mathcal{O}(e_j)).
\]

It induces the surjective map
\[
(2-9) \quad j(\sigma') : \Sigma^0_E \to E, \quad P \mapsto P(\sigma')
\]
that by (2-8) induces the exact sequence
\[
(2-10) \quad 0 \to T_{\mathbb{P}^n}(- \log X_s + Y_s) \to \Sigma^{00}_E \xrightarrow{j(\sigma')} E \to 0.
\]
where \( X_s = \sum_{1 \leq j \leq s} X_i \) and \( Y_s = \sum_{1 \leq j \leq s} Y_j \) with \( X_i : F_i = 0 \subset \mathbb{P}^n \) and \( Y_j : G_j = 0 \subset \mathbb{P}^n \).
In this section we complete the proof of Theorem(I). First we prove the following.

**Theorem (3-1).** Let the assumption be as in Theorem(I). For integers \(q, \ell\) with \(0 \leq q \leq n - r\) and \(\ell \geq 0\) there is a natural isomorphism

\[
B_q(d + e - n - 1 + \ell) \xrightarrow{\sim} H^q(X, \Omega^{n-r-q}_X(\log Z)(\ell))_{prim}.
\]

We start with the following lemma.

**Lemma (3-1).** Let \(q \geq 0\) and \(\ell\) be integers. Assuming \(\ell \geq -d - e\), there is a natural isomorphism

\[
\phi_q(\ell) : A_q(d + e - n - 1 + \ell)/J(\Sigma^{(0)}_E) \xrightarrow{\sim} H^q(\mathbb{P}^n, \Omega^{n-q}_p(\log X_\ast + Y_\ast)(\ell)).
\]

where \(J(\Sigma^{(0)}_E) \subset A_q(d + e - n - 1 + \ell)\) is the image of the map (cf. (2-9))

\[
\chi : H^q(\Sigma^{(0)}_E \otimes S^{q-1}(E) \otimes O(\ell)) \to H^q(S^q(E) \otimes O(\ell)) = A_q(d + e - n - 1 + \ell).
\]

**Proof.** (2-10) gives rise to the following exact sequence

\[
0 \to \Omega^{n-q}_p(\log X_\ast + Y_\ast)(\ell) \to \Omega^{q-1}_E \otimes O(d + e - n - 1 + \ell) \to \cdots
\]

Thus we have the natural map

\[
A_q(d + e - n - 1 + \ell)/J(\Sigma^{(0)}_E) \rightarrow H^q(\Omega^{n-q}_p(\log X_\ast + Y_\ast)(\ell)).
\]

In order to show that this is an isomorphism, it suffices to show the following:

\[
H^b(\Sigma^{(0)}_E \otimes S^{q-b}(E) \otimes O(d + e - n - 1 + \ell)) = 0, \quad \text{for } 1 \leq b \leq q
\]

\[
H^{b-1}(\Sigma^{(0)}_E \otimes S^{q-b}(E) \otimes O(d + e - n - 1 + \ell)) = 0, \quad \text{for } 2 \leq b \leq q
\]

By the bottom sequence of (2-8) we have a decreasing filtration \(F\) on \(\Sigma^{(0)}_E\) such that

\[
Gr^p_{\ell}(\Sigma^{(0)}_E) = \bigoplus_{(\nu, \ell) \in T_{\mathbb{P}^n}} \Omega^{n-b+\nu}_p(\mathbb{P}^n, \Omega^{n-q}_p(\log X_\ast + Y_\ast)(\ell)).
\]

Thus the above vanishing follows from the Bott vanishing theorem. \(\square\)

Now Th.(3-1) follows from the following two lemmas. Recall the assumption (2-2).

**Lemma (3-2).** Assume \(\ell \geq 0\). Let \(J(\Sigma_E(- \log \mathbb{P}^n)) \subset A_q(d + e - n - 1 + \ell)\) be defined as Lemma (2-2). Its image via \(\phi_q(\ell)\) coincides with the image of

\[
\bigoplus_{1 \leq \alpha \leq r} H^q(\mathbb{P}^n, \Omega^{n-q}_p(\log X^{(\alpha)}_\ast + Y_\ast)(\ell)) \to H^q(\mathbb{P}^n, \Omega^{n-q}_p(\log X_\ast + Y_\ast)(\ell)).
\]

where \(X^{(\alpha)}_\ast = \sum_{1 \leq \alpha \leq r} X_i\).

**Lemma (3-3).** Suppose \(n \geq r + 1\) and put \(X = X_1 \cap \cdots \cap X_r\) and \(Z_j = X \cap Y_j\). By the assumption (2-2) \(X \subset \mathbb{P}^n\) is a nonsingular complete intersection of codimension \(r\) and \(Z = \Sigma_{1 \leq j \leq s} Z_j\) is a normal crossing divisor in \(X\). Let \(r \leq a \leq n\) and \(\ell \geq 0\). Then the sequence

\[
\bigoplus_{1 \leq \alpha \leq r} H^{n-a}(\Omega^{a}_p(\log X^{(\alpha)}_\ast + Y_\ast)(\ell)) \to H^{n-a}(\Omega^{a}_p(\log X_\ast + Y_\ast)(\ell)) \to H^{n-a}(\Omega^{n-r}_X(\log Z)(\ell))_{prim} \to 0
\]
is exact where the last map is the composite of the successive residue maps along $X_i$ ($1 \leq i \leq r$).

Proof of Lem.(3-2) First we claim that we may show Lem.(3-2) replacing $H^q(\Omega_{\mathbb{P}^n}^{n-q}(\log X^* + Y_{\ast})(\ell))$ with $H^q(\Omega_{\mathbb{P}^n}^{n-q}(\log X_{\ast} + Y_{\ast})(-X_{\ast})(\ell))$. The claim follows from the general lemma (3-4) below. The exact sequence

$$0 \to \Omega_{\mathbb{P}^n}^{n-q}(\log X_{\ast} + Y_{\ast})(-X_{\ast})(\ell) \to \bigoplus_{1 \leq \alpha < \beta \leq r} \Omega_{\mathbb{P}^n}^{\alpha\beta}(\log X^{(\alpha\beta)}_{\ast}) + Y_{\ast} \to$$

$$\to \bigoplus_{1 \leq \alpha \leq r} \Omega_{\mathbb{P}^n}^{\alpha}(\log X^{(\alpha)}_{\ast}) + Y_{\ast} \to \Omega_{\mathbb{P}^n}^q(\log X_{\ast} + Y_{\ast}) \to \Omega_{X_{\ast}}^{n-r}(\log Z) \to 0,$$

where $X^*_{(\alpha\beta)} = \sum_{1 \leq i \leq s} X_i$ and so on. Thus the desired assertion follows from the following general lemma.

Lemma (3-4). Let the notation and the assumption be as Def.(1-2) and let $d = \dim(X)$.

(1) If $a + b \neq \dim X$ and $a \geq 1$ and $\ell \geq 0$, $H^a(X, \Omega_{X_{\ast}}^d(\log Z)(\ell))_{\text{prim}} = 0$.

(2) For $1 \leq \alpha \leq r$ and $a \geq 1$ the natural map

$$H^a(X, \Omega_{X_{\ast}}^{d-a}(\log Z)(-Z_{\ast})(\ell)) \to H^a(X, \Omega_{X_{\ast}}^{d-a}(\log Z^{(\alpha)}_{\ast})(\ell))_{\text{prim}}$$

is surjective where $Z^{(\alpha)}_{\ast} = \sum_{1 \leq j \leq s} Z_j$.
We have the following commutative diagram (cf. (2-8) and (2-9)).

\[ \sum \text{image of } \]

By induction on \( s \) and \( b \) we are reduced to the case \( \dim(X) = 1 \). In this case we have only to consider \( H^1(X, \Omega_X^1(\log Z)(\ell))_{prim} \) which we easily see vanishes. This completes the proof of Lem.(3-4)(1). Lem.(3-4)(2) follows from (1) in view of the exact sequence

\[ 0 \to \Omega_X^1(\log Z)(-Z_a) \to \Omega_X^1(\log Z(\alpha)) \to \Omega_{Z_a}^1(\log W(\alpha)) \to 0 \quad (W(\alpha) = \sum_{1 \leq j \leq s} Z_a \cap Z_j) \]

Next we show Theorem(I)(2), namely the following statement.

**Lemma(3-5).** Let the assumption be as Lem.(3-2). There is a natural map

\[ B_1(0) \to H^1(X, T_X(-\log Z))_{alg} \]

which is an isomorphism if \( \dim(X) \geq 2 \).

**Proof.** The section \((F_i, G_j)_{1 \leq i \leq r, 1 \leq j \leq s} \in H^0(\mathcal{E})\) (cf. (2-1)) defines the surjective map

\[ j_1 : \mathcal{E}_0^* \otimes \mathcal{E} \to I_X \otimes \mathcal{E}, \quad \xi_i^* \otimes \cdot \mapsto f_i \otimes \cdot \]

\((I_X \text{ denotes the ideal sheaf of } X)\) and the map (We denote \( O = O_{\mathbb{P}^n} \))

\[ j_2 : O^{\oplus s} \to \mathcal{E}, \quad e_j = (0, \cdots, 1, \cdots, 0) \to g_j \eta_j \quad (1 \leq j \leq s) \]

Here \( \xi_i \) (resp. \( \eta_j \)) is a local frame of \( O(d_i) \) (resp. \( O(e_j) \)) and \( \Sigma f_i \xi_i + \Sigma g_j \eta_j \) is the local description of the image of \( \Sigma_{i=1}^r F_i \mu_i + \Sigma_{j=1}^s G_j \lambda_j \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{L}) \) under the isomorphism

\[ H^0(\mathbb{P}, \mathcal{L}) \simeq H^0(\mathbb{P}^n, \mathcal{E}) = \bigoplus_{i=1}^r H^0(\mathbb{P}^n, O(d_i)) \oplus \bigoplus_{j=1}^s H^0(\mathbb{P}^n, O(e_j)). \]

Put

\[ I = \text{Im}(j_1 + j_2 : (\mathcal{E}_0^* \otimes \mathcal{E}) \oplus O^{\oplus s} \to \mathcal{E}), \]

which is generated by local sections

\[ f_i \xi_i, \quad f_i \eta_j, \quad g_j \eta_j \quad (1 \leq i, i' \leq r, 1 \leq j \leq s). \]

We have the following commutative diagram (cf. (2-8) and (2-9))

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & L & K & T & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{E}_0^* \otimes \mathcal{E} \oplus O^{\oplus s} & \Sigma_{E}^0 & T_{\mathbb{P}^n} & 0 \\
\downarrow j_1 + j_2 & \downarrow j_3 (\sigma') & \downarrow j_3 & 0 \\
0 & I & \mathcal{E} & \mathcal{E}/I & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 \\
\end{array}
\]
The map $j_3$ in the above diagram can be written as follows:

$$j_3 : \frac{\partial}{\partial x} \mapsto \sum_{i=1}^{r} \frac{\partial f_i}{\partial x} \xi_i + \sum_{j=1}^{s} \frac{\partial g_j}{\partial x} \eta_j \mod I$$

and it is easy to see that this implies the following exact sequence

$$(3-3) \quad 0 \to I_X \otimes T_{\mathbb{P}^n} \to T \to T_X (-\log Z) \to 0.$$  

We get the map

$$B_1(0) = \text{Coker}(H^0(\mathbb{P}, \Sigma_L(- \log \mathbb{P})) \to H^0(\mathbb{P}, \mathcal{L}))$$

$$\simeq \text{Coker}(H^0(\mathbb{P}^n, \Sigma_L) \to H^0(\mathbb{P}^n, \mathcal{E}))$$

$$\xrightarrow{\alpha} H^1(\mathbb{P}^n, K) \text{ (from the middle vertical sequence in (3-2))}$$

$$\xrightarrow{\beta} H^1(\mathbb{P}^n, T) \text{ (from the top horizontal sequence in (3-2))}$$

$$\xrightarrow{\gamma} H^1(X, T_X (-\log Z)) \text{ (from (3-3))}$$

Thus Lem.(3-5) follows from the following.

**Lemma(3-6).** Assume $\dim(X) = n - r \geq 2$ and $n \geq 3$.

(1) $H^1(\Sigma_L) = 0$.

(2) $H^1(L) = H^2(L) = 0$.

(3) $H^1(I_X \otimes T_{\mathbb{P}^n}) = 0$.

(4) $\text{Ker}(H^1(T_X (-\log Z)) \xrightarrow{\delta} H^1(I_X \otimes T_{\mathbb{P}^n})) = H^1(T_X (-\log Z))_{\text{alg}}$ where $\delta$ is induced by (3-3).

**Proof.** (1) follows from (2-8) and the Bott vanishing. To show (2) we consider the following commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & L_1 & \to & L & \to & L_2 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{E}_0^{*} \otimes \mathcal{E} & \to & \mathcal{E}_0^{*} \otimes \mathcal{E} \oplus \mathcal{O}^{\oplus s} & \to & \mathcal{O}^{\oplus s} & \to & 0 \\
\downarrow j_1 & & \downarrow j_1 + j_2 & & \downarrow j_2' & & \\
0 & \to & I_X \otimes \mathcal{E} & \to & \mathcal{E} & \to & \mathcal{E} \oplus \mathcal{O}_X & \to & 0 \\
0 & \to & 0 & & 0 & & 0 & & 
\end{array}
$$

where $j_2' : e_k \mapsto g_k \eta_k \mod I_X (1 \leq k \leq s)$. Therefore we have $L_2 = \text{Ker}(j_2') = I_X^{\oplus s}$. From the Koszul exact sequence

$$0 \to \wedge \mathcal{E}_0^{*} \to \cdots \to \wedge^s \mathcal{E}_0^{*} \to \mathcal{E}_0^{*} \to I_X \to 0,$$

we can see that $L_i$ has the following resolution.

$$0 \to (\wedge \mathcal{E}_0^{*})^{\oplus s} \to \cdots \to \mathcal{E}_0^{*} \otimes \mathcal{E} \to L_1 \to 0.$$

Therefore (2) follows from the Bott vanishing. (3) is an easy consequence of the Euler exact sequence

$$(3-4) \quad 0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \to T_{\mathbb{P}^n} \to 0.$$  

Finally we see easily that (4) is reduced to prove

$$(*) \quad \text{Ker}(H^1(X, T_X) \xrightarrow{\delta_1} H^2(I_X \otimes T_{\mathbb{P}^n})) = \text{Ker}(H^1(X, T_X) \xrightarrow{\delta_2} H^2(X, \mathcal{O}_X)),$$
where \( \delta_1 \) is induced by the exact sequence

\[
0 \to I_X \otimes T_{\mathbb{P}^n} \to T' \to T_X \to 0 \quad \text{with} \quad T' = \text{Ker}(T_{\mathbb{P}^n} \to \mathcal{E}_0 \otimes \mathcal{O}_X)
\]

and \( \delta_2 \) is the map in Def.(1-3). By the commutative diagram

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & I_X \otimes T_{\mathbb{P}^n} \to T' \to T_X \to 0 \\
\downarrow & & \downarrow \iota \\
0 & \to & I_X \otimes T_{\mathbb{P}^n} \to T_{\mathbb{P}^n} \to T_{\mathbb{P}^n} \otimes \mathcal{O}_X \to 0 \\
\downarrow & & \downarrow \\
\mathcal{E}_0 \otimes \mathcal{O}_X & = & \mathcal{E}_0 \otimes \mathcal{O}_X \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

and the fact \( H^2(T_{\mathbb{P}^n}) = 0 \), we may prove (*) after replacing the left hand side with the kernel of \( H^1(X,T_X) \to H^1(T_{\mathbb{P}^n} \otimes \mathcal{O}_X) \). (3-4) induces the boundary map \( H^1(T_{\mathbb{P}^n} \otimes \mathcal{O}_X) \to H^2(\mathcal{O}_X) \) which is injective. We see that the composite of \( \iota \) and \( \delta_3 \) coincides with the map in Def.(1-3) by noting that the extension class of (3-4) is given by \( c_1(\mathcal{O}_{\mathbb{P}^n}(1)) \in H^1(\Omega^1_{\mathbb{P}^n}) \). This completes the proof. \( \square \)

§4. A VANISHING LEMMA

The following result is the technical heart of the proof of Theorem(II) and (III). For a vector bundle \( \mathcal{F}, \mathcal{F}^* \) denote its dual.

**Theorem(4-1)(vanishing lemma).** Assume \( s \geq 1 \). Let \( p, w, \nu, \ell \) be integers. Then

\[
H^w(\mathbb{P}^r, \Sigma_{\mathcal{L}}(-\log \mathbb{P}^*_{\nu}) \otimes \mathcal{L}^\nu \otimes \pi^* \mathcal{O}(\ell)) = 0
\]

if one of the following conditions is satisfied. We put \( m = n + r + s - 1 = \dim \mathbb{P} \).

1. \( w > 0, \nu \geq -s + 1, \ell \geq 0 \) and \((\nu, \ell) \neq (0, 0)\)
2. \( p - \nu \leq w < m \) and \( \nu \leq -1 \)
3. \( e_1 = e_2 = \cdots = e_s \) and \( 0 < w \neq n \) and \( \nu \geq -s + 1 \)
4. \( e_1 = e_2 = \cdots = e_s, 0 \leq w, \nu = \ell = 0 \) and \( p \leq n \)
5. \( p - \nu \leq r + s - 1 \) and \( \nu \leq -1 \).

\( 1^* \) \( w < m, \nu \leq -1, \ell \leq \mathbf{e} \) and \((\nu, \ell) \neq (-s, \mathbf{e})\)

\( 2^* \) \( 0 < w < p - \nu - s \) and \( \nu \geq -s + 1 \)

\( 3^* \) \( e_1 = e_2 = \cdots = e_s \) and \( m > w \neq r + s - 1 \) and \( \nu < 0 \)

\( 4^* \) \( e_1 = e_2 = \cdots = e_s, m > w, \nu = -s, \ell = \mathbf{e} \) and \( p \geq r + s \)

\( 5^* \) \( p - \nu \geq r + 1 \) and \( \nu \geq -s + 1 \).

**Proof.** For simplicity we put \( \Sigma = \Sigma_{\mathcal{L}}(-\log \mathbb{P}^*_{\nu}) \). By the Euler exact sequence (cf. Lem.(2-1)(3)) we have the isomorphism

\[
\Omega^1_{\mathbb{P}^r/\mathbb{P}^n}(\log \mathbb{P}^*) \simeq (\pi^* \mathcal{E}_0 \otimes \mathcal{L}^{-1}) \oplus \mathcal{O}_{\mathbb{P}^r}^{\oplus s-1}.
\]

and we have

\[
\Omega^1_{\mathbb{P}^r/\mathbb{P}^n}(\log \mathbb{P}^*) \to \mathcal{L}^{-r} \otimes \pi^* \mathcal{O}(d - n - 1).
\]

Noting

\[
K_{\mathbb{P}} = \det \Omega^1_{\mathbb{P}} = \pi^*(K_{\mathbb{P}^n} \otimes \det \mathcal{E}) \otimes \mathcal{L}^{-r-s} = \mathcal{L}^{-r-s} \otimes \pi^* \mathcal{O}(d + e - n - 1),
\]
we have the Serre duality:
\[ H^w(P, \Lambda \Sigma^* \otimes L^r \otimes \pi^* O(\ell))^* \simeq H^{n-w}(P, \Lambda \Sigma^* \otimes L^{-r-s} \otimes \pi^* O(e - \ell)). \]

Therefore the assertion (n) \((1 \leq n \leq 5)\) is equivalent to \((n)^*\) and we only need to show (1), (2), (3), (4) and (5). By Def.(2-1) we have the exact sequence
\[ 0 \to O_p \to \Sigma \to T_p(-\log P_*) \to 0 \]
that induces the exact sequence
\[ (4-4) \quad 0 \to \Omega^p_0(log P_*) \to \lambda \Sigma^* \to \Omega^p_1(log P_*) \to 0. \]
Moreover the exact sequence
\[ 0 \to \pi^* \Omega^p_{2^n} \to \Omega^p_{2^n} \to \Omega^p_{2^n/p^n}(log P_*) \to 0 \]
gives rise to a finite decreasing filtration \(F^*\) on \(\Omega^p_0(log P_*)\) such that
\[ \text{Gr}^i_p(\Omega^p_0(log P_*)) = \pi^* \Omega^p_{2^n} \otimes \Omega^p_{2^n/p^n}(log P_*) \simeq \bigoplus_{i=0}^{q-a} \pi^* \Omega^p_{2^n} \otimes [\lambda \pi^* \xi^0 \otimes L^{-i}]^{(s-r-1-i)} \]
where the second isomorphism follows from (4-1). Hence we obtain the spectral sequence
\[ (4-5) \quad qE_1^{a,b} = \bigoplus_{i=0}^{q-a} H^{a+b}(P, L^{r-i} \otimes \pi^* (\Omega^p_{2^n}(\ell) \otimes \lambda \xi^0))^{(s-r-1-i)} \Rightarrow H^{a+b}(P, \Omega^p_0(log P_*) \otimes L^r \otimes \pi^* O(\ell)) \]
Noting that \(\lambda \xi^0 = 0\) for \(i > r\), Lem.(2-1)(2) implies
\[ H^w(P, L^{r-i} \otimes \pi^* (\Omega^p_{2^n}(\ell) \otimes \lambda \xi^0)) \]
\[ (4-7) \begin{cases} H^w(P^n, S^{r-i}(E) \otimes \Omega^p_{2^n}(\ell) \otimes \lambda \xi^0) & \text{if } \nu \geq i, \\ H^{w-1}(P^n, S^{r-i}(E^*) \otimes \Omega^p_{2^n} \otimes \lambda \xi^0(\ell - d - e)) & \text{if } \nu \leq i - t, \\ 0 & \text{if } i > r \text{ or } i - t < \nu < i. \end{cases} \]

Here we put \(t = r + s\). By (4-6) and (4-4) the desired vanishing in cases (1), (2) and (5) follows from the following.

Claim. (i) Under the assumption (1), \(qE_1^{a,b} = 0\) if \(w = a + b > 0\).

(ii) Under the assumption (2), \(qE_1^{a,b} = 0\) if \(w = a + b > 0\) and \(q \leq p\).

(iii) Under the assumption (5), \(qE_1^{a,b} = 0\) if \(q \leq p\).

First assume \(\nu \geq -s + 1\). For \(i \leq r, \nu \geq i - t + 1\). Hence the first assertion of the claim follows immediately from (4-7) and the Bott vanishing. Next assume (2) and \(q \leq p\). We have \(\nu - i < \nu < s - 1\) so that by (4-7) we may assume \(\nu - i \leq -t\). Hence \(a \leq q - i \leq p - i \leq p - \nu - t\). By (4-7) and the Bott vanishing we get \(qE_1^{a,b} = 0\) if \(p - \nu - t \leq w - t + 1 < n\), that is, \(p - \nu \leq w < m\). This completes the proof in case (2). Finally the assertion in case (5) follows from (4-7) by noting \(0 \leq i < q - a\) in (4-6).

Next we treat the (3). Assume \(\nu \geq -s + 1\). By (4-7)
\[ (4-8) \quad qE_1^{a,b} = \bigoplus_{i=0}^{q-a} H^{a+b}(P^n, S^{r-i}(E) \otimes \Omega^p_{2^n}(\ell) \otimes \lambda \xi^0)^{(s-r-1-i)} \]
By the Bott vanishing, \(E_{1}^{a,0}, E_{1}^{a,-a}, E_{1}^{a,n-a}\) with \(0 \leq a \leq \min(n,q)\) are the only terms which are possibly non-zero. Therefore we have

(i) \(E_{1}^{a,b} = qE_{\infty}^{a,b}\) for any \(a,b\).

(ii) \(E_{1}^{a,-a} = E_{\infty}^{a,-a}\) and \(E_{1}^{a,n-a} = E_{\infty}^{a,n-a}\) for \(a \neq 0, n\).

(iii) \(E_{1}^{a,0} = H^0(P, \Omega^p_0(log P_*) \otimes L^r \otimes \pi^* O(\ell))\) for \(a \neq 0, n\).
Note that the boundary map coming from (4-4)
\[ \partial_a : H^{a-1}(P, \Omega^{a-1}_P(\log P_*)) \otimes \mathcal{L}' \otimes \pi^* \mathcal{O}(\ell) \to H^a(P, \Omega^a_P(\log P_*) \otimes \mathcal{L}' \otimes \pi^* \mathcal{O}(\ell)) \]
is induced by the cup product with the class
\[ \hat{c} := c_1(L)|_{P | P_*} \in H^1(P, \Omega^1_P(\log P_*)). \]

**Claim.** Assume \( e_1 = e_2 = \cdots = e_s = e \). Then the natural map
\[ \pi^* : H^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}) \to H^1(P, \Omega^1_P(\log P_*)) \]
is injective and we have \( \hat{c} = \pi^*(c_1(O_{\mathbb{P}^n}(e))). \)

We know that \( H^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}) \) is the direct sum of \( \pi^*(H^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n})) \) and the subspace spanned by \( c_1(L) \). The kernel of \( H^1(\mathbb{P}^n, \Omega^1_{\mathbb{P}^n}) \to H^1(P, \Omega^1_P(\log P_*)) \) is generated by \( c_1(O_{\mathbb{P}^n}(P_j)) \) for \( 1 \leq j \leq s \). The claim follows from the fact \( L \otimes \pi^* \mathcal{O}(-e_j) = O_{\mathbb{P}^n}(P_j) \).

From the claim the map \( \partial_a \) and the maps \( \delta : p_{-1}E_{1}^{a-1,b} \xrightarrow{\cup c} pE_{1}^{a,b} \) with \( c \) a non-zero multiple of \( c_1(O_{\mathbb{P}^n}(e)) \), are compatible with respect to the spectral sequence (4-6). Thus we get the commutative diagram
\[
\begin{array}{cccccccc}
p_{-1}E_{1}^{0,0} & \to & p_{-1}E_{1}^{1,0} & \to & \cdots & \to & p_{-1}E_{1}^{n-2,0} & \to & p_{-1}E_{1}^{n-1,0} & \to & p_{-1}E_{1}^{n,0} \\
pE_{1}^{0,0} & \to & pE_{1}^{1,0} & \to & pE_{1}^{2,0} & \to & \cdots & \to & pE_{1}^{n-1,0} & \to & pE_{1}^{n,0}
\end{array}
\]

where the horizontal arrows are the differentials in (4-6) and the slanting arrows are the above map \( \delta \). To show the desired vanishing in case (3), we need show that if \( w \neq 0, n \) then \( \partial_w \) is surjective and \( \partial_{w+1} \) is injective. Therfor it suffices to show the following

**Claim.** The map \( \delta : p_{-1}E_{1}^{a-1,0} \xrightarrow{\cup c} pE_{1}^{a,0} \) is an isomorphism for \( 2 \leq a \leq n-1 \) and surjective for \( a = 1 \) and injective for \( a = n \).

By (4-8), for \( a \neq 0, n \), we have the isomorphism
\[ qE_{1}^{a,0} \simeq H^a(\mathbb{P}^n, \Omega^a_{\mathbb{P}^n}) \oplus \phi. \]

Here \( \phi \) is determined as follows. If \( q < a, \phi = 0 \). If \( q \geq a \), writing
\[ q = a \oplus \bigoplus_{i=0}^{r-1} [S^{r-i}(E) \otimes \mathcal{O}(\ell)] \otimes \mathcal{O}(\phi) \]
we put \( \phi = \# \{ k \mid \ell_k = \emptyset \} \). Thus, fixing \( \nu \) and \( \ell, \phi = \phi(q-a) \) is a function of \( q-a \). Note that even for \( a = 0 \) or \( n \), we have the injection \( H^a(\mathbb{P}^n, \Omega^a_{\mathbb{P}^n}) \oplus \phi \to qE_{1}^{a,0} \). Since \( H^{a-1}(\mathbb{P}^n, \Omega^{a-1}_{\mathbb{P}^n}) \xrightarrow{\cup c} H^n(\mathbb{P}^n, \Omega^a_{\mathbb{P}^n}) \) is clearly an isomorphism, the claim is proven.

Finally we treat the case (4). By the vanishing lemma(3) it suffices to show \( H^n(\mathbb{P}^n, \ell \Sigma^*) = 0 \). By (4-4) this is reduced to proving the following.

(i) \( H^n(\mathbb{P}^n, \Omega^{n-1}_P(\log P_*)) = 0 \).

(ii) \( H^{n-1}(\mathbb{P}^n, \Omega^{n-1}_P(\log P_*)) = 0 \xrightarrow{\cup c} H^n(\mathbb{P}^n, \Omega^a_P(\log P_*)) \) is surjective.

As before we have the spectral sequence
\[ qE_{1}^{a,b} \Rightarrow H^{a+b}(\mathbb{P}^n, \Omega^a_P(\log P_*)) \]
where
\[ qE_{1}^{a,b} \simeq \begin{cases} H^{a+b}(\mathbb{P}^n, \Omega^a_P(\log P_*)) \oplus \mathcal{O}(\phi) & \text{if } 0 \leq a \leq \min\{q, n\}, \\ 0 & \text{otherwise}. \end{cases} \]

Thus \( qE_{1}^{a,b} = 0 \) unless \( b = 0 \) and the spectral sequence degenerates at \( E_2 \). Now the assertion (i) follows from the fact that \( p_{-1}E_{1}^{n,0} = 0 \) by the assumption \( p \leq n \). The assertion (ii) can be shown by the same argument as the proof in case (3). \( \square \)
§5. Proof of Theorem(II)

In this section we prove Theorem(II). We deduce it from the Serre duality theorem and Lem.(4-1). The exact sequence (cf.(2-6))

\[ 0 \to T_p(- \log Z + \mathbb{P}_x) \to \Sigma_L(- \log \mathbb{P}_x) \xrightarrow{j(p)} L \to 0, \]

and (4-2) induce the Koszul exact sequence

\[ (5-1) \quad 0 \to L^{-m-1} \to \Sigma^* \otimes L^{-m} \to \cdots \to \wedge^{m+1} \Sigma^* \to 0. \quad (\Sigma = \Sigma_L(- \log \mathbb{P}_x)) \]

Tensoring with \( L^{r+q} \otimes \pi^* \mathcal{O}(\ell) \), we get the exact sequence where we denote \( \mathcal{O} = \mathcal{O}_{\mathbb{P}_x} \)

\[ (5-2) \quad 0 \to L^{-m+r+q-1} \otimes \pi^* \mathcal{O}(\ell) \to \cdots \to \wedge^m \Sigma^* \otimes L^{r+q-1} \otimes \pi^* \mathcal{O}(\ell) \xrightarrow{\delta} \wedge^{m+1} \Sigma^* \otimes L^{r+q} \otimes \pi^* \mathcal{O}(\ell) \to 0. \]

By (4-2) we have

\[ \wedge^m \Sigma^* \otimes L^{r+q-1} \otimes \pi^* \mathcal{O}(\ell) = \Sigma \otimes L^{r-1} \otimes \pi^* \mathcal{O}(d - n - 1 + \ell), \]

\[ \wedge^{m+1} \Sigma^* \otimes L^{r+q} \otimes \pi^* \mathcal{O}(\ell) = \mathcal{O} \otimes \pi^* \mathcal{O}(d - n - 1 + \ell) \]

and the map \( \delta \) in (5-2) is nothing but \( j(\sigma) \otimes 1 \). Therefore we have the canonical map

\[ B_q(d - n - 1 + \ell) \to \ker[H^m(L^{-m+r+q-1} \otimes \pi^* \mathcal{O}(\ell)) \to H^m(\Sigma^* \otimes L^{-m+r+q} \otimes \pi^* \mathcal{O}(\ell))] \]

By (4-3) and the Serre duality the right hand side is isomorphic to the dual of

\[ \text{Coker}[H^0(\Sigma \otimes L^{n-r-q-1} \otimes \pi^* \mathcal{O}(d + e - n - 1 - \ell)) \to H^0(L^{n-r-q} \otimes \pi^* \mathcal{O}(d + e - n - 1 - \ell))] \]

\[ = B_{n-r-q}(d + e - n - 1 - \ell). \]

Thus we get the canonical maps

\[ h_q(\ell) : B_q(d - n - 1 + \ell) \to B_{n-r-q}(d + e - n - 1 - \ell)^*, \]

\[ h_q^*(\ell) : B_{n-r-q}(d + e - n - 1 - \ell) \to B_q(d - n - 1 + \ell)^*, \]

where \( h_q^*(\ell) \) is the dual of \( h_q(\ell) \). In particular we get the trace map

\[ (5-3) \quad \tau := h_{n-r}(d + e - n - 1) : B_{n-r}(2(d - n - 1) + e) \to B_0(0)^* = k. \]

For \( x \in B_q(d - n - 1 + \ell) \) and \( y \in B_{n-r-q}(d + e - n - 1 - \ell) \) we let \( < x, y > \in k \) be the evaluation of \( h_q(\ell)(x) \) at \( y \). This gives us a bilinear pairing

\[ < , > : B_q(d - n - 1 + \ell) \otimes B_{n-r-q}(d + e - n - 1 - \ell) \to k; (x, y) \to < x, y >. \]

Lemma(5-1). (1) \( h_q^*(\ell) \) coincides with

\[ h_{n-r-q}(e - \ell) : B_{n-r-q}(d + e - n - 1 - \ell) \to B_q(d - n - 1 + \ell)^*. \]

(2) We have

\[ < x, y > = \tau(xy) \quad \text{for} \quad x \in B_q(d - n - 1 + \ell) \quad \text{and} \quad y \in B_{n-r-q}(d + e - n - 1 - \ell), \]

where \( xy \in B_{n-r}(2(d - n - 1) + e) \) is the multiplication of \( x \) and \( y \).
Proof. The first assertion follows by taking the dual of \((5-2)\) in view of the isomorphism
\[
\left( \bigwedge^{n} \Sigma \otimes \mathcal{L}^n \otimes \pi^* \mathcal{O}(\ell) \right)^* \otimes K_{P} \cong \bigwedge^{m+1-n} \Sigma^* \otimes \mathcal{L}^{-r-s} \otimes \pi^* \mathcal{O}(e - \ell).
\]
The second assertion follows from the commutative diagram
\[
0 \to \mathcal{L}^{-r-q-1} \otimes \pi^* \mathcal{O}(\ell) \to \cdots \to m^{1} \Sigma^* \otimes \mathcal{L}^{r+q-1} \otimes \pi^* \mathcal{O}(\ell) \to 0
\]
where the cup product with an element of $H^0(\mathcal{L}^{n-\ell} \otimes \pi^* \mathcal{O}(d + e - n - 1 - \ell))$ gives the vertical maps.

Lemma(5-2). Assume $s \geq 1$ and $e_1 = \cdots = e_s$.

(1) $h_\ell(\ell)$ is injective under one of the following conditions.
   (i) $n - r \geq q$ and $\ell < e$.
   (ii) $n - r \geq q + \frac{r - 1}{2}$.

(2) $h_\ell(\ell)$ is an isomorphism under one of the following conditions.
   (i) $0 < q < n - r$ and $\ell < e$.
   (ii) $0 \leq \ell \leq e$ and $r + s \leq n$.

(3) Assuming $n - r \geq 1$, $\tau$ is an isomorphism.

Proof. By the exact sequence \((5-2)\), $h_\ell(\ell)$ is surjective if
\[
(a) \quad H^a(\bigwedge^{m+1-a} \Sigma^* \otimes \mathcal{L}^{r+q-a} \otimes \pi^* \mathcal{O}(\ell)) = 0 \text{ for } 1 \leq a \leq m - 1,
\]
and is injective if
\[
(b) \quad H^b(\bigwedge^{m-b} \Sigma^* \otimes \mathcal{L}^{r+q-b-1} \otimes \pi^* \mathcal{O}(\ell)) = 0 \text{ for } 1 \leq b \leq m - 1.
\]

To show the injectivity, we show (b). First we assume (1)(i). By the assumption $\ell < e$ and Lem.(4-1)(1)* we may assume $r + q - b - 1 \geq 0$, namely $b \leq r + q - 1$. By the assumption $q \leq n - r$ this implies $b \leq n$. Hence Lem.(4-1)(3) completes the proof. Next we show (1)(ii). By what we have shown, we may assume $\ell \geq e$. By Lem.(4-1)(1) we may suppose $r + q - b - 1 \leq -s$, namely $b \geq r + s + q - 1$. The assumption $q \geq \frac{r - 1}{2}$ implies $r + s + q - 1 \geq (m - b) - (r + q - b - 1) = n + s - q$. Hence Lem.(4-1)(2) completes the proof.

To show that $h_\ell(\ell)$ is an isomorphism we show (a) and (b). First assume (2)(i). The assertion (b) has been shown in this case. To show (a), by Lem.(4-1)(1)* we may assume $r + q - a \geq 0$, namely $a \leq r + q$. By the assumption $q \leq n - r$ this implies $a \leq b$. Hence Lem.(4-1)(3) completes the proof. (2)(ii) follows from (2)(i) and Lem.(5-1)(1) by replacing $q$ by $n - r - q$ and $\ell$ by $e - \ell$. Next assume (2)(iii). We only show (b). The proof of (a) is similar. By Lem.(4-1)(1), (2), (3) and (1)*, (2)*, (3)*, we have only to consider either of the case $\ell = 0$, $r + q - b - 1 = 0$ and $b = n$ or the case $\ell = e$, $r + q - b - 1 = -s$ and $b = r + s - 1$. In the former case we have $m - b = m - n = r + s - 1 < n$. Hence Lem.(4-1)(4) completes the proof. In the latter case we have $m - b = m - (r + s - 1) = n \geq r + s$. Hence Lem.(4-1)(4)* completes the proof.

Finally we show (3). By (1)(ii) $\tau$ is injective. By (1)(i) and Lem.(5-2)(2), $\tau$ cannot be the zero map. Thus $\tau$ is always an isomorphism. \(\square\)

By Lem.(5-2) Theorem(II)(2) and (3) holds true in case $s = 1$. The case $s \geq 2$ is reduced to the special case by the induction on $s$ due to the following lemma(5-3). Let the notation be as §2. We put
\[
\Sigma' = \Sigma_{\mathcal{L}}(- \log \sum_{j=2}^{S} \mathbb{P}_{j}) \quad \text{and} \quad \Sigma = \Sigma_{\mathcal{P}}(- \log \sum_{j=2}^{S} \mathbb{P}_{j}),
\]
where $\overline{\mathcal{C}} = \mathcal{L}|_{\mathbb{P}_1}$ and $\overline{\mathbb{P}}_j = \mathbb{P}_j \cap \mathbb{P}_1$ for $2 \leq j \leq s$. We also define the Jacobian rings
\[
B_q' = \text{Coker} \left( H^0(\Sigma' \otimes \mathcal{L}^{q-1} \otimes \pi^* \mathcal{O}(\ell)) \right) ^{j(\pi)} \to H^0(\mathcal{L}^q \otimes \pi^* \mathcal{O}(\ell))
\]
\[
\overline{B}_q = \text{Coker} \left( H^0(\Sigma \otimes \overline{\mathcal{L}}^{q-1} \otimes \pi^* \mathcal{O}(\ell)) \right) ^{j(\pi)} \to H^0(\overline{\mathcal{L}}^q \otimes \pi^* \mathcal{O}(\ell))
\]
where $\pi = \sum_{i=1}^{r} F_i \mu_i + \sum_{j=2}^{s} G_j \lambda_j \in H^0(\mathcal{L})$. Put $d' = d + e_1$ and $e' = e - e_1 = \overline{e}$.

**Lemma (5-3).** (1) We have the exact sequence
\[
B_{q-1}(d' - n - 1 + \ell) \xrightarrow{\phi} B_q(d - n - 1 + \ell) \xrightarrow{\psi} \overline{B}_q(d - n - 1 + \ell) \to 0,
\]
where $\rho$ is the reduction modulo $\lambda_1 \in A_q(-e_1)$ and $\phi$ is the multiplication by $\lambda_1$.

(2) We have the exact sequence
\[
\overline{B}_{n-r-q}(\ell' - e_1) \xrightarrow{\psi} B_{n-r-q}(\ell') \xrightarrow{\pi} B_{n-r-q}(\ell') \to 0,
\]
where $\pi$ is the natural projection arising from the natural injection $\Sigma' \subset \Sigma$ and $\psi$ is the unique map which fits into the commutative diagram
\[
\begin{array}{ccc}
B_{n-r-q}(\ell' - e_1) & \to & 0 \\
\downarrow \rho & & \downarrow \\
\overline{B}_{n-r-q}(\ell' - e_1) & \xrightarrow{\psi} & B_{n-r-q}(\ell')
\end{array}
\]
where $\psi$ is the multiplication by $G_1 \in A_0(e_1)$.

(3) The following diagram is commutative.
\[
\begin{array}{ccc}
B_{q-1}(d' - n - 1 + \ell) & \xrightarrow{h_{q-1}(\ell)} & B'_{n-(r+1)-(q-1)}(d' + e' - n - 1 - \ell) \\
\downarrow \phi & & \downarrow \pi^* \\
B_q(d - n - 1 + \ell) & \xrightarrow{h_{q}(\ell)} & B_{n-r-q}(d + e - n - 1 - \ell) \\
\downarrow \rho & & \downarrow \psi^* \\
\overline{B}_q(d - n - 1 + \ell) & \xrightarrow{\tau_{q}(\ell)} & \overline{B}_{n-r-q}(d + \overline{e} - n - 1 - \ell) \\
\downarrow 0 & & \\
0 & & 0
\end{array}
\]
where the horizontal arrows are the duality maps defined before.

**Proof.** The exactness of the sequences together with the well-definedness of $\psi$ is seen immediately from the explicit description of the Jacobian rings (cf. Lem.(2-2)). The commutativity of the upper square of the diagram in (3) is an easy consequence of the commutative diagram (cf. (5-2))
\[
\begin{array}{ccc}
\mathcal{L}^{-m+r+q-1}(\ell) & \to & \Sigma^* \otimes \mathcal{L}^{-m+r+q}(\ell) \\
\downarrow & & \downarrow \\
\mathcal{L}^{-m+r+q-1}(\ell) & \to & \Sigma^* \otimes \mathcal{L}^{-m+r+q}(\ell)
\end{array}
\]
where we put $\mathcal{L}(\ell) = \mathcal{L} \otimes \pi^* \mathcal{O}(\ell)$. The vertical maps are the dual of the natural embedding $\Sigma \hookrightarrow \Sigma'$.

Next we show the commutativity of the lower square. By (5-1)(2) it suffices to show
\[
(5-3-1) \quad \overline{\tau}(x) = \tau(\overline{\psi}(x)) \quad \text{for} \quad x \in \overline{B}_{n-r}(2(d - n - 1) + \overline{e}),
\]
where $\tau$ and $\tau$ are the trace maps (cf. (5-3)). We consider the following commutative diagram

\[
\begin{array}{cccc}
0 & \rightarrow & \mathcal{L}^{n-r}(\ell) & \rightarrow 0 \\
\uparrow & & \uparrow j' & \\
\mathcal{L}^{n-r}(\ell - e_1) & \leftarrow & \mathcal{L}^{n-r-1}(\ell) \otimes \Sigma' & \leftarrow \mathcal{L}^{n-r-1}(\ell) \otimes \Sigma \\
\uparrow & & \uparrow j & \\
(\ell - e_1) & \leftarrow & \mathcal{L}^{n-r-1}(\ell) \otimes 2\Sigma' & \leftarrow \mathcal{L}^{n-r-1}(\ell) \otimes 2\Sigma \\
\uparrow \vdots & & \uparrow \vdots & \\
0 & \leftarrow & \mathcal{L}^{n-2r-s+1}(\ell - e_1) \otimes \Sigma & \leftarrow \mathcal{L}^{n-2r-s}(\ell) \otimes \Sigma \\
\end{array}
\]

with $\ell = 2(d - n - 1) + e$. Here the vertical exact sequences come from (5-2) and the horizontal exact sequences come from the exact sequence

\[
0 \rightarrow \Sigma \rightarrow \Sigma' \rightarrow N_{P_1/P} \rightarrow 0
\]

(coming from the exact sequence $0 \rightarrow T_p(- \log \sum_{j=1}^s \bar{p}_j) \rightarrow T_p(- \log \sum_{j=2}^s \bar{p}_j) \rightarrow N_{P_1/P} \rightarrow 0$) and the isomorphism

\[
N_{P_1/P} \simeq \mathcal{O}_P(\mathcal{P}) \otimes \mathcal{O}_{P_1} \simeq \mathcal{L}(-e_1) \otimes \mathcal{O}_{P_1}.
\]

We note that the bottom row is isomorphic to the adjunction sequence

\[
0 \leftarrow K_{\mathcal{P}_1} \leftarrow K_{\mathcal{P}} \otimes \mathcal{O}_P(\mathcal{P}_1) \leftarrow K_{\mathcal{P}} \leftarrow 0
\]

and the left and right vertical sequences induce the maps

\[
\text{Coker}(H^0(\mathcal{L}^{n-r-1}(\ell) \otimes \Sigma)) \xrightarrow{j} H^0(\mathcal{L}^{n-r}(\ell)) \rightarrow H^m(\mathcal{P}, K_{\mathcal{P}}) \simeq k,
\]

\[
\text{Coker}(H^0(\mathcal{L}^{n-r-1}(\ell - e_1) \otimes \Sigma)) \xrightarrow{j} H^0(\mathcal{L}^{n-r}(\ell - e_1)) \rightarrow H^{m-1}(\mathcal{P}_1, K_{\mathcal{P}_1}) \simeq k,
\]

which are nothing but the trace maps $\tau$ and $\tau$ respectively. On the other hand we note that the map

\[
\alpha : H^0(\mathcal{L}^{n-r-1}(\ell) \otimes \Sigma') \rightarrow H^0(\mathcal{L}^{n-r}(\ell - e_1))
\]

is surjective. Using this we define the map

\[
\delta : H^0(\mathcal{L}^{n-r}(\ell - e_1)) \rightarrow \text{Coker}(H^0(\mathcal{L}^{n-r-1}(\ell) \otimes \Sigma)) \xrightarrow{j} H^0(\mathcal{L}^{n-r}(\ell))
\]

by $\delta(x) = j'(\tilde{x}) \mod \text{Im}(j)$ for $x \in H^0(\mathcal{L}^{n-r}(\ell - e_1))$ and $\tilde{x} \in H^0(\mathcal{L}^{n-r-1}(\ell) \otimes \Sigma')$ with $\alpha(\tilde{x}) = x$. It is easily seen that $\delta$ coincides with the multiplication by $G_1 \in H^0(\mathcal{O}_{\mathcal{P}_1}(e_1))$. Thus, to show (5-3-1) it suffices to prove $\partial_1(x) = \partial_2(j'(\tilde{x}))$, where

\[
\partial_1 : H^0(\mathcal{L}^{n-r}(\ell - e_1)) \rightarrow H^1(\text{Ker}(j)) \quad \text{and} \quad \partial_2 : H^0(\mathcal{L}^{n-r}(\ell)) \rightarrow H^1(\text{Ker}(j))
\]

are the boundary maps coming from the exact sequences

\[
0 \rightarrow \text{Ker}(j) \xrightarrow{j} \text{Ker}(j') \xrightarrow{\alpha} \mathcal{L}^{n-r}(\ell - e_1) \rightarrow 0, \quad 0 \rightarrow \text{Ker}(j) \rightarrow \mathcal{L}^{n-r-1}(\ell) \otimes \Sigma \xrightarrow{j} \mathcal{L}^{n-r}(\ell) \rightarrow 0.
\]
Take an open covering \( \mathbb{P} = \bigcup_{i \in I} U_i \) and \( \{ \eta_i \}_{i \in I} \in \prod_{i \in I} H^0(U_i, \mathcal{L}^n-r-1(\ell) \otimes \Sigma) \) such that \( j(\eta_i) = j'(\tilde{x})|_{U_i} \). Putting \( \xi_i = \tilde{x}|_{U_i} - \iota(\eta_i) \), we see \( \eta_i \in H^0(U_i, \text{Ker}(j')) \) and \( \alpha(\xi_i) = x_i|_{U_i} \). Thus

\[
\eta_i|_{U_i \cap U_j} - \eta_j|_{U_i \cap U_j} = \xi_i|_{U_i \cap U_j} - \xi_j|_{U_i \cap U_j} \in H^0(U_i \cap U_j, \text{Ker}(j))
\]

is a Cech cocycle representing both \( \partial_1(x) \) and \( \partial_2(j'(\tilde{x})) \). This completes the proof. \( \square \)

Finally we prove Theorem(II)(2)(iii). We reduce it to the case \( s = 1 \). For this we consider the diagram of Lem.(5-3) in case \( s = 1 \) and \( \ell = 0 \).

\[
\begin{array}{ccc}
B'_{q-1}(d' - n - 1) & \xrightarrow{h'} & B'_{n-(r+1)-(q+1)}(d' - n - 1) \\
\downarrow \phi & & \downarrow \pi^* \\
B_q(d - n - 1) & \xrightarrow{h} & B_{n-r-q}(d + e_1 - n - 1) \\
\downarrow \rho & & \downarrow \psi^* \\
\overline{B}_q(d - n - 1) & \xrightarrow{\overline{\kappa}} & \overline{B}_{n-r-q}(d - n - 1) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

We want to show \( \overline{\kappa} \) is an isomorphism. By Theorem(I) \( B_q(d - n - 1) \) and \( B_{n-r-q}(d - n - 1) \) have the same dimension. Hence it suffices to show the injectivity of \( \overline{\kappa} \). First we have the following.

**Claim.** \( \overline{B}_q(d - n - 1) = 0 \) if \( q > n - r \geq 1 \).

**Proof.** By Lem.(5-2)(2)(iii), \( B_q(d - n - 1) = 0 \) if \( q > n - r \geq 1 \). Hence the claim follows from the surjectivity of \( \rho \). \( \square \)

Assume \( n - r \geq 1 \). By Lem.(5-2) \( h \) is an isomorphism so that \( \overline{h} \) is injective if \( h' \) is surjective. By the claim and by the induction on \( r \) we are reduced to show the injectivity of \( \overline{h} \) in case \( n - r = 1 \) and \( q = 1 \) in which case \( \psi^* \) is surjective. For its dual \( \psi : \overline{B}_0(d - n - 1) \rightarrow B_0(d + e_1 - n - 1) \) that is the multiplication by \( G_1 \), is injective since \( (F_1, \ldots, F_r, G_1) \) is a regular sequence in \( k[X_0, \ldots, X_n] \). Since \( h \) is an isomorphism by Lem.(5-2), the diagram shows that \( \overline{h} \) is surjective so that injective by the reason of dimension. This completes the proof of Theorem(II)(2)(iii) in case \( n - r \geq 1 \). The diagram implies further that \( h' \) is an isomorphism in case \( n - r = 1 \) and \( q = 1 \) so that Theorem(II)(2)(iii) in case \( n - r = 0 \) is also proved. \( \square \)

§6. **Proof of Theorem(II').**

In this section we prove Theorem(II'). First the surjectivity of \( \eta_{X,Z} \) follows from Theorem(II)(3). As for Ker(\( \eta_{X,Z} \)) we first show the following.

**Proposition(6-1).** \( \wedge_X^{n-r}(G_1, \ldots, G_s) \subset \text{Ker}(\eta_{X,Z}) \).

Note that \( B_0(d + e - n - 1) = P^d+e-n-1/(F_1, \ldots, F_r) \) where \( P^\ell \subset k[X_0, \ldots, X_n] \) is the subspace of homogeneous polynomials of degree \( \ell \) and \( (F_1, \ldots, F_r) \subset P^\ell \) is the subspace generated by the multiples of \( F_i \). By Theorem(I) we have the isomorphisms

\[
B_0(d + e - n - 1) \xrightarrow{\sim} H^0(X, \Omega_{X}^{n-r}(\log Z)); \quad A \rightarrow \text{Res}X_{F_1, \ldots, F_r, G_1, \ldots, G_s} \Omega,
\]

where

\[
\Omega := \sum_{i=0}^n (-1)^i X_i dX_0 \wedge \cdots \wedge dX_i \wedge \cdots \wedge dX_n,
\]
and

\[ \text{Res}_X : \Omega^n_{\mathbb{P}^n}(\log X_\ast + Y_\ast) \rightarrow \Omega^n_X(\log Z) \]

is the composite of the residue maps along \( F_1 = F_2 = \cdots = F_r = 0. \)

**Definition (6-1).** Assume \( s \geq n - r + 1. \) For integers \( 1 \leq j_1 < \cdots < j_{n-r+1} \leq s, \) we write

\[ A(j_1, \ldots, j_{n-r+1}) := \sum_{\sigma \in \mathcal{S}_{n+1}} \text{sign}(\sigma) \frac{\partial F_1}{\partial X_{\sigma(0)}} \cdots \frac{\partial F_r}{\partial X_{\sigma(r-1)}} \frac{\partial G_1}{\partial X_{\sigma(r)}} \cdots \frac{\partial G_{j_{n-r+1}}}{\partial X_{\sigma(n)}}. \]

\[ A'(j_1, \ldots, j_{n-r+1}) := A(j_1, \ldots, j_{n-r+1}) : \frac{G_1 \cdots G_s}{G_{j_1} \cdots G_{j_{n-r+1}}} \in \mathcal{S}_{d+e-n-1}. \]

**Lemma (6-1).** We have (cf. Def. (1-3))

\[ \text{Res}_X \frac{A'(j_1, \ldots, j_{n-r+1})}{F_1 \cdots F_r G_1 \cdots G_s} \Omega = (-1)^{r+1} \omega_X(j_1, \ldots, j_{n-r+1}). \]

**Lemma (6-2).** Write \( A' = A'(j_1, \ldots, j_{n-r+1}). \) Then we have (cf. Def. (1-1))

\[ A' \mu_i, A' \lambda_j \in J(F, G) \quad \text{for} \quad 1 \leq i \leq r, \ 1 \leq j \leq s. \]

Pr. (6-1) follows from the above lemmas: By Lem. (6-1) it suffices to show

\[ A'(j_1, \ldots, j_{n-r+1}) \in \text{Ker}(B_0(d + e - n - 1) \xrightarrow{h_{n-r}(0)^*} B_{n-r}(d - n - 1)^*). \]

Since \( h_{n-r}(0)^* \) is given by the pairing

\[ B_0(d + e - n - 1) \otimes B_{n-r}(d - n - 1) \rightarrow B_{n-r}(2(d - n - 1) + e) \xrightarrow{v} k, \]

Pr. (6-1) follows from Lem. (6-2).

**Proof of Lem. (6-1)** We may suppose \( j_1 = 1, \ldots, j_{n-r+1} = n - r + 1. \) We may prove the formula on the affine subspace \( \{X_0 \neq 0\}. \) Let

\[ A = \sum_{\sigma \in \mathcal{S}_{n+1}} \text{sign}(\sigma) \frac{\partial F_1}{\partial X_{\sigma(0)}} \cdots \frac{\partial F_r}{\partial X_{\sigma(r-1)}} \frac{\partial G_1}{\partial X_{\sigma(r)}} \cdots \frac{\partial G_{n-r+1}}{\partial X_{\sigma(n)}}. \]

For polynomials \( h_1, \ldots, h_n \in k[X_0, \ldots, X_n] \) write

\[ J(h_1, \ldots, h_n) = \det \begin{pmatrix} \frac{\partial h_1}{\partial X_1} & \cdots & \frac{\partial h_1}{\partial X_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_n}{\partial X_1} & \cdots & \frac{\partial h_n}{\partial X_n} \end{pmatrix}. \]

Writing \( G_j = F_{r+j} \) and \( e_j = d_{r+j} \) for \( 1 \leq j \leq s, \) we claim

\[ (*) \quad X_0 A = \sum_{\nu=1}^n (-1)^{\nu-1} (d_\nu \cdot F_\nu) J(F_1, \ldots, \widehat{F_\nu}, \ldots, F_{n+1}) \]

that implies that we have on \( \{X_0 \neq 0\} \)

\[ \text{Res}_X \frac{A'(j_1, \ldots, j_{n-r+1})}{F_1 \cdots F_r G_1 \cdots G_s} \Omega = \text{Res}_X \sum_{\nu=1}^r (-1)^{\nu+1} d_\nu \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_\nu}{f_\nu} \wedge \cdots \wedge \frac{df_r}{f_r} \wedge \frac{dg_1}{g_1} \wedge \cdots \wedge \frac{dg_{n-r+1}}{g_{n-r+1}} \]

\[ + \text{Res}_X \sum_{\mu=1}^{n-r+1} (-1)^{r+1+\mu} e_\mu \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_r}{f_r} \wedge \frac{dg_1}{g_1} \wedge \cdots \wedge \frac{dg_{r+1}}{g_{r+1}} \wedge \cdots \wedge \frac{dg_{n-r+1}}{g_{n-r+1}}. \]
where \( f_i = F_i/X_0^{d_i} \) and \( g_j = G_j/X_0^{e_j} \). Since the first term vanishes this completes the proof of Lem.(6-1). To show the formula (*) we note

\[
X_0 \frac{\partial F_\nu}{\partial X_0} = d_\nu \cdot F_\nu - \sum_{i=1}^{n} X_i \frac{\partial F_\nu}{\partial X_i}.
\]

We have

\[
X_0 A = \sum_{\sigma \in S_{n+1}} \text{sign}(\sigma) X_0 \frac{\partial F_1}{\partial X_\sigma(0)} \cdots \frac{\partial F_{n+1}}{\partial X_\sigma(n)}
\]

\[
= \sum_{\sigma(0)=0} (\star) + \sum_{\sigma(1)=0} (\star) + \cdots + \sum_{\sigma(n)=0} (\star)
\]

\[
= \sum_{\sigma \in S_n} \text{sign}(\sigma)(d_1 \cdot F_1 - \sum_{i=1}^{n} X_i \frac{\partial F_1}{\partial X_i}) \cdot \frac{\partial F_2}{\partial X_\sigma(1)} \cdots \frac{\partial F_{n+1}}{\partial X_\sigma(n)}
\]

\[
- \sum_{\sigma \in S_n} \text{sign}(\sigma)(d_2 \cdot F_2 - \sum_{i=1}^{n} X_i \frac{\partial F_2}{\partial X_i}) \cdot \frac{\partial F_1}{\partial X_\sigma(1)} \cdots \frac{\partial F_{n+1}}{\partial X_\sigma(n)} + \cdots
\]

\[
+ (-1)^{n+1} \sum_{\sigma \in S_n} \text{sign}(\sigma)(d_{n+1} \cdot F_{n+1} - \sum_{i=1}^{n} X_i \frac{\partial F_{n+1}}{\partial X_i}) \cdot \frac{\partial F_1}{\partial X_\sigma(1)} \cdots \frac{\partial F_n}{\partial X_\sigma(n)}
\]

\[
= \sum_{\nu=1}^{n+1} (-1)^{\nu-1} (d_\nu \cdot F_\nu) J(F_1, \ldots, \widehat{F_\nu}, \ldots, F_{n+1}) \cdot \partial F_\nu
\]

where

\[
P_i = \sum_{\nu=1}^{n+1} (-1)^{\nu-1} \frac{\partial F_\nu}{\partial X_i} \cdot J(F_1, \ldots, \widehat{F_\nu}, \ldots, F_{n+1}) = \det \begin{pmatrix}
\frac{\partial F_1}{\partial X_1} & \frac{\partial F_2}{\partial X_1} & \cdots & \frac{\partial F_{n+1}}{\partial X_1} \\
\frac{\partial F_1}{\partial X_2} & \frac{\partial F_2}{\partial X_2} & \cdots & \frac{\partial F_{n+1}}{\partial X_2} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F_1}{\partial X_i} & \frac{\partial F_2}{\partial X_i} & \cdots & \frac{\partial F_{n+1}}{\partial X_i} 
\end{pmatrix} = 0.
\]

This completes the proof. □

**Proof of Lem.(6-2)** We may suppose \( j_1 = 1, \ldots, j_{n-r+1} = n - r + 1 \). Modulo \( J(\mathcal{F}, Q) \) we have

\[
A'\mu_1 = \sum_{\sigma \in S_{n+1}} \text{sign}(\sigma) \frac{\partial F_1}{\partial X_\sigma(0)} \mu_1 \cdot \frac{\partial F_2}{\partial X_\sigma(1)} \cdots \frac{\partial F_r}{\partial X_\sigma(r-1)} \cdot \frac{\partial G_1}{\partial X_\sigma(n)} \cdots \frac{\partial G_{n-r+1}}{\partial X_\sigma(n)} \cdot G_{n-r+2} \cdots G_s
\]

\[
= - \sum_{\sigma \in S_{n+1}} \text{sign}(\sigma) \left( \frac{\partial F_2}{\partial X_\sigma(0)} \mu_2 + \cdots + \frac{\partial G_s}{\partial X_\sigma(0)} \lambda_s \right) \cdot \frac{\partial F_1}{\partial X_\sigma(1)} \cdots \frac{\partial G_{n-r+1}}{\partial X_\sigma(n)} \cdots G_{n-r+2} \cdots G_s
\]

\[
= - \sum_{\sigma \in S_{n+1}} \text{sign}(\sigma) \left( \frac{\partial F_2}{\partial X_\sigma(0)} \mu_2 + \cdots + \frac{G_{n-r+1}}{\partial X_\sigma(0)} \lambda_{n-r+1} \right) \cdot \frac{\partial F_1}{\partial X_\sigma(1)} \cdots \frac{\partial G_{n-r+1}}{\partial X_\sigma(n)} \cdots G_{n-r+2} \cdots G_s
\]

The coefficient of \( \mu_i \) (\( 2 \leq i \leq r \)) in the above is

\[
-(G_{n-r+2} \cdots G_s) \sum_{\sigma \in S_{n+1}} \text{sign}(\sigma) \frac{\partial F_1}{\partial X_\sigma(0)} \cdot \frac{\partial F_2}{\partial X_\sigma(1)} \cdots \frac{\partial G_{n-r+1}}{\partial X_\sigma(n)} = 0
\]

Similarly the coefficient of \( \lambda_j \) (\( 1 \leq j \leq n - r + 1 \)) vanishes. This proves \( A'\mu_1 \equiv 0 \mod J(\mathcal{F}, Q) \). The rest of the assertion is proven in the same manner. □

Due to Pr.(6-1) Th.(II') now follows from

\[
(*) \quad \dim_k (\text{Ker}(\eta_{X,Z})) = \dim_k (h_{n-r}(0)^*) \leq \dim_k \wedge_{X}^{n-r} (G_1, \ldots, G_s).
\]
Note that by Th.(II)(2), \( \dim_k (h_{n-r}(0)^*) = 0 \) if \( s \leq n-r \) and \( n-r \geq 1 \). Consider the following commutative diagram that is the dual of the diagram in Lem.(5-3)

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
\overline{B}_0(d + e - n - 1) & \xrightarrow{\overline{\eta}_{n-r}(0)^*} & \overline{B}_{n-r}(d - n - 1)^* \\
\downarrow \psi & & \downarrow \rho^* \\
B_0(d + e - n - 1) & \xrightarrow{h_{n-r}(0)^*} & B_{n-r}(d - n - 1)^* \\
\downarrow \pi & & \downarrow \phi^* \\
B'_0(d' + e' - n - 1) & \xrightarrow{h'_{n-(r+1)}(0)^*} & B'_{n-(r+1)}(d' - n - 1)^* \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

(6-1)

where the horizontal maps are surjective by Theorem(II)(3). Combining this with the exact sequence

\[
0 \rightarrow \wedge^{n-r}_X(G_2, \ldots, G_s) \rightarrow \wedge^{n-r}_X(G_1, \ldots, G_s) \xrightarrow{\text{Res}_{Z_1}} \wedge^{n-r-1}_Z(G_2, \ldots, G_s) \rightarrow 0,
\]

the assertion (\( \ast \)) is reduced by induction on \( s \) to the case \( n-r = 1 \). Since \( \dim_k \wedge^1_X (G_1, \ldots, G_s) = s - 1 \) as is easily seen, it follows by induction on \( s \) from (6-1) and the following.

**Lemma(6-3).** Assuming \( n = r \) and \( s \geq 1 \), \( \dim_k \ker(h_0(0)^*) = 1 \) where

\[
h_0(0)^* : B_0(d + e - n - 1) \rightarrow B_0(d - n - 1)^*.
\]

**Proof.** We have the following commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
S^{d+e-n-1}/(F_1, \ldots, F_n) & \xrightarrow{\eta_{n-r}(0)^*} & S^{d-e-n-1}/(F_1, \ldots, F_n) \\
\downarrow G_1 & \xrightarrow{\eta_{n-r}(0)^*} & \downarrow \pi \\
S^{d+e-n-1}/(F_1, \ldots, F_n, G_1) & \xrightarrow{h_{n-r}(0)^*} & S^{d-e-n-1}/(F_1, \ldots, F_n, G_1) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

(6-2)

The left vertical sequence is exact due to the assumption (1-1) in §1. The induction hypothesis and Theorem(II)(2)(ii) imply

\[
\dim_k(\eta_{n-r}(0)^*) = \begin{cases} 0 & \text{if } s = 1, \\ 1 & \text{if } s \geq 2. \end{cases}
\]

Thus the lemma follows by induction on \( s \) from (6-2) and the following fact that is a consequence of Macaulay’s theorem (cf. [Do, Th.2.5])

\[
\dim_k S^{d+e-n-1}/(F_1, \ldots, F_n, G_1) = \begin{cases} 1 & \text{if } s = 1, \\ 0 & \text{if } s \geq 2. \end{cases}
\]

\(\square\)
§7. Proof of Theorem (III)

In this section we complete the proof of Th. (III). We deduce it from the following.

Theorem (7-1). Assume \( s \geq 1 \). Let \( W \subset A_1(0) \) is a base point free subspace of codimension \( c \) (i.e. for any \( x \in \mathbb{P}^n(\mathbb{C}) \), the evaluation map \( W \subset A_1(0) \to \bigoplus_i \mathbb{C}_x + \bigoplus_j \mathbb{C}_j \) at \( x \) is surjective). Then the Koszul complex

\[
B_p(\ell) \otimes \wedge^{q+1} W \to B_{p+1}(\ell) \otimes \wedge^q W \to B_{p+2}(\ell) \otimes \wedge^{q-1} W
\]

is exact if one of the following conditions is satisfied.

(i) \( p \geq 0 \), \( q = 0 \) and \( \delta_{\min} p + \ell \geq c \).

(ii) \( p \geq 0 \), \( q = 1 \) and \( \delta_{\min} p + \ell \geq 1 + c \) and \( \delta_{\min} (p+1) + \ell \geq d_{\max} + c \).

(iii) \( p \geq 0 \), \( \delta_{\min} (r+p) + \ell \geq q + c \), \( \ell \geq d - n - 1 \), \( e_1 = \cdots = e_s \) and either \( r + s \leq n + 2 \) or \( p \neq -n - r - 1 \).

(iv) \( p \geq 0 \), \( \delta_{\min} (r+p) + \ell \geq q + c \), \( d + e_{\max} - n - 1 > \ell \geq d - n - 1 \) and either \( r + s \leq n + 2 \) or \( p \neq -n - r - 1 \).

First we deduce Th. (III) from Th. (7-1). Let \( W := \text{Ker}(A_1(0) \to B_1(0)/V) \). Since \( W \) contains \( J := J(X, Z) \cap A_1(0) \) (cf. Def. (1-2)), it is a base point free subspace of codimension \( c \). We have the Koszul exact sequence

\[
0 \to S(J) \to W \otimes S^{-1}(J) \to \cdots \to \wedge^i W \otimes J \to \wedge W \to \wedge V \to 0.
\]

This complex tensored with \( B_s(\ell) \) induces the following diagram.

\[
\begin{array}{cccccccc}
\cdots & B_p(\ell) \otimes \wedge^{q+1-i} W \otimes S^i(J) & \cdots & B_p(\ell) \otimes \wedge^{q+1} W & B_p(\ell) \otimes \wedge^q V & 0 \\
\downarrow & \downarrow & & \downarrow & & \\
\cdots & B_{p+1}(\ell) \otimes \wedge^{q-1} W \otimes S^i(J) & \cdots & B_{p+1}(\ell) \otimes \wedge^{q} W & B_{p+1}(\ell) \otimes \wedge^q V & 0 \\
\downarrow & \downarrow & & \downarrow & & \\
\cdots & B_{p+2}(\ell) \otimes \wedge^{q-2} W \otimes S^i(J) & \cdots & B_{p+2}(\ell) \otimes \wedge^{q-1} W & B_{p+2}(\ell) \otimes \wedge^{q-1} V & 0 \\
\downarrow & \downarrow & & \downarrow & & \\
\vdots & \vdots & & \vdots & & \\
\end{array}
\]

where the vertical sequences are the Koszul complexes tensored with \( S(J) \). Since \( J \) annihilates \( B_p(\ell) \), the diagram is commutative. Therefore to show the exactness of the complex in Th. (III), it suffices to show that

\[
B_{p+i}(\ell) \otimes \wedge^{q-i} W \to B_{p+i+1}(\ell) \otimes \wedge^{q-i} W \to B_{p+i+2}(\ell) \otimes \wedge^{q-i} W
\]

is exact for \( \forall i \geq 0 \) and it follows from Th. (7-1) under the assumptions of Th. (III).

For the proof of Th. (7-1) we recall the regularity of sheaves ([G2]). A coherent sheaf \( \mathcal{F} \) on \( \mathbb{P}^n \) is called \( m \)-regular if

\[
H^i(\mathbb{P}^n, \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(m - i)) = 0 \quad \text{for} \quad \forall i > 0.
\]

We use the following properties of the regularity of sheaves, whose proof can be found in [G2].

1. If \( \mathcal{F} \) is \( m \)-regular, then also \( (m+1) \)-regular.

2. If \( \mathcal{F} \) and \( \mathcal{F}' \) are \( m \)-regular and \( m' \)-regular respectively, then \( \mathcal{F} \otimes \mathcal{F}' \) is \( (m+m') \)-regular.

In particular, if \( E \) is a \( m \)-regular locally free sheaf on \( \mathbb{P}^n \), then \( \wedge E \) is \( (mp) \)-regular since it is a direct summand of \( E^\otimes p \). Let \( \ell \geq 0 \) be an integer, and define a locally free sheaf \( E \) on a projective space \( \mathbb{P}^n \) by the exact sequence

\[
0 \to E \to H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\ell)) \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(\ell) \to 0.
\]

Then clearly \( E \) is \( 1 \)-regular, therefore \( \wedge E \) is \( p \)-regular. In [G2], there is a further result: We replace \( H^0(\mathcal{O}_{\mathbb{P}^n}(\ell)) \) by \( V \) a base point free linear subspace of \( H^0(\mathcal{O}_{\mathbb{P}^n}(\ell)) \) of codimension \( c \) and define \( E' \) by

\[
0 \to E' \to V \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(\ell) \to 0,
\]
then $\hat{\nu} E'$ is $(p + c)$-regular. This argument is applicable not only to $\mathcal{O}_{\mathbb{P}^n}(\ell)$ but also to any locally free sheaf satisfying certain conditions. We will need it later.

**Lemma (7-1).** Let $\mathcal{N}$ be a locally free sheaf on $\mathbb{P}^n$ generated by global sections. We assume that $\mathcal{N}$ satisfies $H^p(\mathcal{N}(-p)) = 0$ for $0 < p < n$ (e.g. $\mathcal{N} = \mathcal{E}$). Let $\mathcal{V}$ be a linear subspace of $H^0(\mathcal{N})$ of codimension $c$, such that $\mathcal{V} \otimes_k \mathcal{O}_{\mathbb{P}^n} \to \mathcal{N}$ is surjective (i.e. base point free). Define the locally free sheaf $\mathcal{N}$ by the exact sequence

$$0 \to \mathcal{N} \to \mathcal{V} \otimes_k \mathcal{O}_{\mathbb{P}^n} \to \mathcal{N} \to 0.$$ 

Then $\hat{\nu} \mathcal{N}$ is $(p + c)$-regular.

Now we go back to the proof of Th.(7-1). The following lemma is a generalization of [G2, Th.4.1].

**Lemma (7-2).** Let $q \geq 0$, $\nu \geq 0$, $\ell$ integers. Then the Koszul complex

$$A_\nu(\ell) \otimes \Lambda^{q+1} W \to A_{\nu+1}(\ell) \otimes \Lambda^q W \to A_{\nu+2}(\ell) \otimes \Lambda^{q-1} W$$

is exact if $\delta_{min} \nu + \ell \geq c + q$.

**Proof.** We define a locally free sheaf $M$ on $\mathbb{P}$ by the exact sequence

$$0 \to M \to W \otimes_k \mathcal{O}_\mathbb{P} \to \mathcal{L} \to 0.$$ 

where the first map comes from the identification $H^0(\mathbb{P}, \mathcal{L}) = A_1(0) \supset W$ (cf. Lem.(2-2)). Then we obtain a Koszul exact sequence

$$0 \to \Lambda^{q+1} M \to \Lambda^q W \otimes_k \mathcal{O}_\mathbb{P} \to \Lambda^{q-1} W \otimes_k \mathcal{L} \to \cdots \to W \otimes_k \mathcal{L}^q \to \mathcal{L}^{q+1} \to 0.$$ 

Tensoring with $\mathcal{L}^\nu \otimes \pi^* \mathcal{O}_{\mathbb{P}^n}(\ell)$, this gives an acyclic resolution of $\Lambda^{q+1} M \otimes \mathcal{L}^\nu \otimes \pi^* \mathcal{O}_{\mathbb{P}^n}(\ell)$. By Lem.(2-1)

$$H^i(\mathbb{P}, \mathcal{L}^\nu \otimes \pi^* \mathcal{O}_{\mathbb{P}^n}(\ell)) \simeq H^i(\mathbb{P}^n, S^\nu(\mathcal{E}) \otimes \mathcal{O}_{\mathbb{P}^n}(\ell)) = \oplus \ H^j(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\alpha))$$

for all $\nu \geq \nu$, with $\alpha \geq \delta_{min} \nu + \ell \geq \delta_{min} \nu + \ell \geq c + q \geq 0$ so that it vanishes if $i > 0$. Therefore the cohomology group in Lem.(7-2) is isomorphic to

$$H^1(\Lambda^{q+1} M \otimes \mathcal{L}^\nu \otimes \pi^* \mathcal{O}_{\mathbb{P}^n}(\ell))$$

which we shall prove vanishes. Let $\pi : \mathbb{P} \to \mathbb{P}^n$ be the projection. We apply $\pi_*$ to (*) and get the exact sequence

$$0 \to \pi_* M \to W' \otimes_k \mathcal{O}_{\mathbb{P}^n} \to \mathcal{E} \to 0$$ 

with $W' = \pi_* W \subset H^0(\mathbb{P}^n, \mathcal{E})$. The surjectivity of the right map is due to the base point freeness of $W$. Put $N = \pi_* M$. Then by Lem.(7-1), $\hat{\nu} N$ is $(c + i)$-regular. On the other hand we have the commutative diagram:

$$\begin{array}{ccc}
0 & \to & \pi^* N \\
\downarrow g & & \downarrow \nu \\
0 & \to & M \\
\end{array}$$

$$\begin{array}{ccc}
0 & \to & \mathcal{W} \otimes_k \mathcal{O}_\mathbb{P} \\
\downarrow g' & = & \downarrow \\
\mathcal{L} & \to & 0.
\end{array}$$

where the vertical maps are induced by the adjunction for $\pi$. By the snake lemma, $g$ is injective and $\text{Coker}(g) \simeq \text{Ker}(g')$. By the exact sequence (cf. Lem.(2-1)(3))

$$0 \to \Omega_{\mathbb{P}/\mathbb{P}^n} \to \pi^* \mathcal{E} \otimes \mathcal{L}^{-1} \to \mathcal{O}_\mathbb{P} \to 0,$$

we get $\text{Ker}(g') \simeq \Omega^1_{\mathbb{P}/\mathbb{P}^n} \otimes \mathcal{L}$. Hence we have the exact sequence

$$0 \to \pi^* N \to M \to \Omega^1_{\mathbb{P}/\mathbb{P}^n} \otimes \mathcal{L} \to 0.$$
which induces the filtration
\[ q^{+1}M = F^0 \supset F^1 \supset \cdots \supset F^{q+1} \supset F^{q+2} = 0 \]
such that \( \text{Gr}_p^{(q^{+1}M)} = F^i/F^{i+1} \cong \pi^* F^i \otimes \Omega_{\mathbb{P}^n/P^{n}}^{q^{+1}-i} \). So it suffices to show that
\[ H^j(\mathbb{P}, \mathcal{L}^{q^{+1}-i} \otimes \Omega_{\mathbb{P}^n/P^{n}}^{q^{+1}-i} \otimes \pi^* (\check{N} \otimes \mathcal{O}_{P^n}(\ell))) = 0 \quad \text{for } 0 \leq i \leq q + 1 \]
The exact sequence \((\ast\ast)\) induces the exact sequence
\[ 0 \rightarrow r^{+s} \otimes \pi^* \mathcal{E} \otimes \mathcal{L}^{-r-s} \rightarrow \cdots \rightarrow p^{+1} \otimes \pi^* \mathcal{E} \otimes \mathcal{L}^{-p-1} \rightarrow \Omega_{\mathbb{P}^n/P^n}^{p} \rightarrow 0. \]
Therefore it suffices to show that
\[ H^j(\mathbb{P}, \mathcal{L}^{q^{+1}-i} \otimes \pi^* (\check{N} \otimes \mathcal{O}_{P^n}(\ell))) = 0 \quad \text{for } 1 \leq j \leq r + s - (q + 1 - i) \text{ and } 0 \leq i \leq q + 1. \]
In case \( 1 \leq j \leq \nu \) the above cohomology is isomorphic to
\[ H^j(\mathbb{P}^n, S^r-j(\mathcal{E}) \otimes \pi^* (\check{N} \otimes \mathcal{O}_{P^n}(\ell))) \simeq \bigoplus H^j(\mathbb{P}^n, \mathcal{O}_{P^n}(\alpha) \otimes \check{N}) \]
with \( \alpha \geq \delta_{\min}(\nu-j) + \delta_{\min}(q+1-i+j) + \ell = \delta_{\min}(\nu + q + 1 - i) + \ell. \) Since \( \check{N} \) is \((i+1)\)-regular, this vanishes if \( \alpha + j \geq i + c \), which holds since
\[ \alpha + j - i - c \geq \delta_{\min}(\nu + q + 1 - i) + \ell + j - i - c \geq \delta_{\min}(\nu + \ell) = 0. \]
Next assume \( j > \nu \). If \( \nu-j > -r-s \), the cohomology vanishes by Lem.(2.1)(2). Hence we only consider the case \( \nu-j \leq r+s \), namely \( j \geq r+s+\nu \). Since \( j \leq r+s -(q+1-i) \) by the assumption, we only consider the case \( \nu = 0 \), \( j = r+s+i = q+1 \). Then the cohomology is isomorphic to \( H^1(\mathbb{P}^n, \mathcal{O}_{P^n}(\ell) \otimes \check{N}) \).
Since \( \check{N} \) is \((q+1)\)-regular and \( \ell + 1 = \delta_{\min}(\nu + \ell + 1) \geq q + 1 + c \), it vanishes. This completes the proof of Lem.(7.2).

Now we prove Th.(7.1). Write \( \Sigma = \Sigma_{\mathcal{L}(\log \mathbb{P}^n)} \) and put
\[ \mathcal{M}_{k,h}(\ell) = \mathcal{M}^{m+1-k}_{\mathcal{L}} \otimes \mathcal{L}^{r+k-h} \otimes \pi^* \mathcal{O}_{P^n}(\ell - d + n + 1) \quad \text{and} \quad C_{k,h}(\ell) = H^0(\mathbb{P}, \mathcal{M}_{k,h}(\ell)). \]
From the exact sequence \((5-1)\) we obtain the exact sequence
\[(7-1) \quad 0 \rightarrow \mathcal{M}_{p,m+1}(\ell) \rightarrow \cdots \rightarrow \mathcal{M}_{p,1}(\ell) \rightarrow \mathcal{M}_{p,0}(\ell) \rightarrow 0,\]
that induces the following complex
\[ 0 \rightarrow C_{p,m+1}(\ell) \rightarrow \cdots \rightarrow C_{p,1}(\ell) \rightarrow C_{p,0}(\ell) \rightarrow 0, \]
Note that \( \text{Coker}\phi = B_p(\ell) \) by Lem.(2.2) and (4.2). We have the following commutative diagram:
\[
\begin{array}{cccccccc}
\cdots & \rightarrow & C_{p,1}(\ell) \otimes \check{N} & \rightarrow & C_{p,0}(\ell) \otimes \check{N} & \rightarrow & B_p(\ell) \otimes \check{N} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \rightarrow & C_{p+1,1}(\ell) \otimes \check{N} & \rightarrow & C_{p+1,0}(\ell) \otimes \check{N} & \rightarrow & B_{p+1}(\ell) \otimes \check{N} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \rightarrow & C_{p+2,1}(\ell) \otimes \check{N} & \rightarrow & C_{p+2,0}(\ell) \otimes \check{N} & \rightarrow & B_{p+2}(\ell) \otimes \check{N} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & \vdots & & \vdots & & \vdots & & \\
\end{array}
\]
By an easy diagram chase, we see that the exactness of
\[ B_p(\ell) \otimes \check{N} \rightarrow B_{p+1}(\ell) \otimes \check{N} \rightarrow B_{p+2}(\ell) \otimes \check{N}, \]
follows from the following.
\[
(1) \quad C_{p+2+a,a+1}(\ell) \rightarrow C_{p+2+a,a}(\ell) \rightarrow C_{p+2+a,a-1}(\ell) \quad \text{is exact for } 1 \leq a \leq q - 1. \\
(2) \quad C_{p+b,b}(\ell) \otimes \check{N} \rightarrow C_{p+b+1,b}(\ell) \otimes \check{N} \rightarrow C_{p+b+2,b}(\ell) \otimes \check{N} \quad \text{is exact for } b \geq 0.
\]
Note that (1) holds by trivial reason if $q = 0$ or $q = 1$.

**Lemma (7-3).** Assume $s \geq 1$ and $e_1 = \cdots = e_s$ and $p \geq 0$. (1) holds in either of the following cases

(i) $\ell \geq d - n - 1$ and $r + s \leq n + 2$.
(ii) $\ell \geq d - n - 1$ and $p \neq n - r - 1$.

**Lemma (7-4).** Assume $s \geq 1$ and $p \geq 0$. (2) holds in either of the following cases

(i) $q = 0$ and $\delta_{\min}p + \ell \geq c$.
(ii) $q = 1$ and $\delta_{\min}p + \ell \geq 1 + c$ and $\delta_{\min}(p + 1) + \ell \geq d_{\max} + c$.
(iii) $\delta_{\min}(r + p) + \ell \geq d + q + c$ and $\ell \geq d - n - 1$.

Before proving the lemmas, we finish the proof of Th.(7-1). In case (i), (ii) and (iii) it is a direct consequence of Lem.(7-4) and (7-3). The case (iv) is reduced to the case (iii) by induction on $s$. For this we use the following commutative diagram

\[
\begin{array}{cccc}
0 & \to & 0 & \to 0 \\
\downarrow & & \downarrow & \downarrow \\
B_p'_{-1}(\ell + e_1) \otimes q^1 W & \to & B_p'_{-1}(\ell + e_1) \otimes \hat{W} & \to \to B_p'_{-1}(\ell + e_1) \otimes q^{-1} W \\
\downarrow & & \downarrow & \downarrow \\
B_p(\ell) \otimes q^1 W & \to & B_p(\ell) \otimes \hat{W} & \to \to B_p(\ell) \otimes q^{-1} W \\
\downarrow & & \downarrow & \downarrow \\
\overline{B}_p(\ell) \otimes q^1 W & \to & \overline{B}_p(\ell) \otimes \hat{W} & \to \to \overline{B}_p(\ell) \otimes q^{-1} W \\
\downarrow & & \downarrow & \downarrow \\
0 & \to & 0 & \to 0 \\
\end{array}
\]

where the notation is the same as in Lem.(5-3). The exactness of the vertical sequences is a consequence of Lem.(5-3) and Th.(II) (Here we use the additional assumption $d + e_{\max} - n - 1 > \ell$). By the induction hypothesis we may assume the upper horizontal sequence is exact. It remains to show the exactness of the lower horizontal sequence

\[
\overline{B}_p(\ell) \otimes q^1 W \to \overline{B}_p(\ell) \otimes \hat{W} \to \overline{B}_p(\ell) \otimes q^{-1} W
\]

Letting $\overline{W} = \text{Im}(W \to H^0(\mathcal{L}))$ and $I = \text{Ker}(W \to H^0(\mathcal{L}))$, we have the filtration $F' / F^{i+1} \simeq (\hat{W}^q \otimes (\hat{I}^i))$. Since $I$ annihilates $\overline{B}_p(\ell)$, the above complex is filtered by the above filtration and its graded quotients are the complexes

\[
\overline{B}_p(\ell) \otimes q^1 \otimes \hat{W} \otimes \hat{I} \to \overline{B}_p(\ell) \otimes q^1 \otimes \hat{W} \otimes \hat{I} \to \overline{B}_p(\ell) \otimes q^1 \otimes \hat{W} \otimes \hat{I}
\]

for $0 \leq i \leq q$.

These are exact by the induction hypothesis and this completes the proof. □

**Proof of Lem.(7-3).** The exact sequence (7-1) induces a spectral sequence

\[
E^{\alpha,\beta}_1 = H^\beta(M_{k,m+1-\alpha}(\ell)) \implies H^{\alpha+\beta} = 0.
\]

We want to show that $E^{\alpha,0}_2 = 0$ in case:

(*) $p + 3 \leq k \leq p + q + 1$ and $k - (m + 1 - \alpha) = p + 2$ ($\iff \alpha = p - k + m + 3$).

Since $E^{\alpha,0}_2 = 0$, in order to show $E^{\alpha,0}_2 = 0$, it suffices to show that

\[
E^{\alpha-h,1}_1 = H^h(M_{k,m+2+h-\alpha}(\ell)) = 0 \quad \text{for all } h \geq 1.
\]

In case (*), putting $\ell' = \ell - d + n + 1$ we have

\[
E^{\alpha-h,1}_1 = H^h(M_{k,k-p+h-1}(\ell)) = H^h(M_{k,k-p+h-1}(\ell)) = H^h(M_{k,k-p+h-1}(\ell)) = H^h(M_{k,k-p+h-1}(\ell)) = H^h(M_{k,k-p+h-1}(\ell)).
\]
We want to show that it vanishes assuming $k \geq p + 3$ and $h \geq 1$. We may suppose $h + k \leq m + 2 + p$ which implies $h \leq m - 1$.

Case $p + r - h + 1 \leq -s$

We have $m - 1 \geq h \geq p + r + s + 1 > r + s$. The desired vanishing follows from Lem(4-1)(3)*.

Case $p + r - h + 1 \geq -s + 1$

By the assumption $\ell' \geq 0$ and by Lem(4-1)(1) and (3), we have only to check the case $h = p + r + 1 = n$ and $\ell' = 0$ so that we are concerned with the vanishing of $H^n(\bigwedge^{n+k} \Sigma^*)$. By Lem.(4-1)(4) this vanishes if $n + s - k \leq n \iff s \leq k$. Since $k \geq p + 3 = n - r + 2$, this holds if $r + s \leq n + 2$.

This completes the proof of Lem.(7-3). \qed

Proof of Lem.(7-4). By (4-2) we have

(*) \quad C_{p,0} = H^0(\mathcal{L}^p \otimes \pi^* \mathcal{O}_p(\ell)) = A_p(\ell) \quad \text{and} \quad C_{p,1} = H^0(\mathcal{L}^{p-1} \otimes \Sigma \otimes \pi^* \mathcal{O}_p(\ell)).

Thus, Lem.(7-4) in case (i) follows from Lem.(7-2). In case (ii) we need show the exactness of

\[ C_{p,0}(\ell) \otimes \mathcal{L}^2 \rightarrow C_{p+1,0}(\ell) \otimes \mathcal{L} \rightarrow C_{p+2,0}(\ell) \]

and the surjectivity of $C_{p+1,1}(\ell) \otimes \mathcal{L} \rightarrow C_{p+2,1}(\ell)$. By (*) the first assertion follows from Lem.(7-2).

To show the second assertion we recall the exact sequences

\[ 0 \rightarrow \mathcal{O}_p \rightarrow \Sigma(\log \mathbb{P}_s) \rightarrow T_p(\log \mathbb{P}_s) \rightarrow 0, \]

\[ 0 \rightarrow T_{p/p_n}(\log \mathbb{P}_s) \rightarrow T_p(\log \mathbb{P}_s) \rightarrow \pi^* T_{p/n} \rightarrow 0, \]

\[ 0 \rightarrow \mathcal{O}_p \rightarrow \pi^* \mathcal{E}_0^* \otimes \mathcal{L} \otimes \mathcal{O}_p^\otimes_s \rightarrow T_{p/p_n}(\log \mathbb{P}_s) \rightarrow 0, \]

\[ 0 \rightarrow \mathcal{O}_p \rightarrow \mathcal{O}_p(1)^{\oplus n+1} \rightarrow T_{p/n} \rightarrow 0. \]

Noting $H^1(\mathcal{L}^\ell \otimes \pi^* \mathcal{O}_p(\ell)) = 0$ for $\forall \nu \geq 0$, the assertion follows from the surjectivity of

\[ H^0(\mathcal{L}^p \otimes \pi^* \mathcal{O}_p(\ell)) \otimes \mathcal{L} \rightarrow H^0(\mathcal{L}^{p+1} \otimes \pi^* \mathcal{O}_p(\ell)), \]

\[ H^0(\mathcal{L}^{p+1} \otimes \pi^* \mathcal{E}_0^0(\ell)) \otimes \mathcal{L} \rightarrow H^0(\mathcal{L}^{p+2} \otimes \pi^* \mathcal{E}_0^0(\ell)) \]

which is a consequence of Lem.(7-2) under the assumption of (ii).

Finally we show Lem.(7-4) in case (iii). We denote $\Omega^2(\log \mathbb{P}_s)$ by $\Omega$ simply. We want to show that

\[ H^0(\bigwedge^{m+1-b} \Sigma^* \otimes \mathcal{L}^{r+p} \otimes \pi^* \mathcal{O}_p(\ell')) \otimes \mathcal{L}^{q+1-b} \rightarrow H^0(\bigwedge^{m+1-b} \Sigma^* \otimes \mathcal{L}^{r+p+1} \otimes \pi^* \mathcal{O}_p(\ell')) \otimes \mathcal{L}^{q-b} \rightarrow H^0(\bigwedge^{m+1-b} \Sigma^* \otimes \mathcal{L}^{r+p+2} \otimes \pi^* \mathcal{O}_p(\ell')) \otimes \mathcal{L}^{q-1-b} \]

is exact for $0 \leq \forall b \leq q$, where $\ell' = \ell - d + n + 1$. If $\delta > 0$ and $\ell' \geq 0$, the following exact sequence

\[ 0 \rightarrow \Omega^t \otimes \mathcal{L}^\ell \otimes \pi^* \mathcal{O}_p(\ell') \rightarrow \bigwedge^t \Sigma \otimes \mathcal{L}^\ell \otimes \pi^* \mathcal{O}_p(\ell') \rightarrow \Omega^{-1} \otimes \mathcal{L}^\ell \otimes \pi^* \mathcal{O}_p(\ell') \rightarrow 0 \]

remains exact after taking $H^0(\ )$ since $H^1(\mathcal{L}^\ell \otimes \Omega \otimes \pi^* \mathcal{O}_p(\ell')) = 0$, which we can see from the proof of Lem.(4-1) (cf. Claim below (4-7)). Thus it suffices to show that the following sequence is exact for all $t, b$ such that $m - b \leq t \leq m - b + 1$ and $0 \leq b \leq q$:

\[ H^0(\Omega^t \otimes \mathcal{L}^{r+p} \otimes \pi^* \mathcal{O}_p(\ell')) \otimes \mathcal{L}^{q+1-b} \rightarrow H^0(\Omega^t \otimes \mathcal{L}^{r+p+1} \otimes \pi^* \mathcal{O}_p(\ell')) \otimes \mathcal{L}^{q-b} \rightarrow H^0(\Omega^t \otimes \mathcal{L}^{r+p+2} \otimes \pi^* \mathcal{O}_p(\ell')) \otimes \mathcal{L}^{q-1-b}. \]
By (4-5), there is a filtration $F$ of $\Omega^t$ such that

$$\text{Gr}_F^u(\Omega^t) = \pi^*\Omega_{\mathbb{P}^m}^u \otimes \left( \bigoplus_{i=0}^{t-n} \left[ \pi^*\mathcal{E}_0 \otimes L^{-i} \right] \right),$$

where $(u, i)$ runs over $0 \leq u \leq n$ and $0 \leq i \leq \min\{t - u, r\}$. Since $H^1(L^{r-i} \otimes \text{Gr}_F^u \Omega^t \otimes \pi^*\mathcal{O}_{\mathbb{P}^m}(\ell')) = 0$ for $i \geq 0$ and $\ell' \geq 0$, it suffices to show that

$$H^0(L^{r-p-i} \otimes \pi^*(\Omega_{\mathbb{P}^m}^u \otimes \mathcal{E}_0(\ell')) \otimes q^{1-b} \wedge W$$

$$\to H^0(L^{r-p-i+1} \otimes \pi^*(\Omega_{\mathbb{P}^m}^u \otimes \mathcal{E}_0(\ell')) \otimes q^{-b} \wedge W$$

$$\to H^0(L^{r-p-i+2} \otimes \pi^*(\Omega_{\mathbb{P}^m}^u \otimes \mathcal{E}_0(\ell')) \otimes q^{-1-b} \wedge W.$$

is exact for $\forall b, u, i$ such that

$$0 \leq b \leq q$$
$$0 \leq u \leq n$$
$$0 \leq i \leq \min\{t - u, r\}$$

Finally, by the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^m}(-n - 1) \to \mathcal{O}_{\mathbb{P}^m}(-n)^{\oplus n + 1} \to \cdots \to \mathcal{O}_{\mathbb{P}^m}(-u - 1)^{\oplus (n - 1)} \to \Omega_{\mathbb{P}^m}^n \to 0,$$

we can reduce the assertion to show that

$$H^0(L^{r-p-i+j} \otimes \pi^*(\mathcal{E}_0(\ell' - u - j - 1)) \otimes q^{1-b-j} \wedge W$$

$$\to H^0(L^{r-p-i+j+1} \otimes \pi^*(\mathcal{E}_0(\ell' - u - j - 1)) \otimes q^{-b-j} \wedge W$$

$$\to H^0(L^{r-p-i+j+2} \otimes \pi^*(\mathcal{E}_0(\ell' - u - j - 1)) \otimes q^{-1-b-j} \wedge W$$

is exact for $\forall b, u, i, j$ such that

$$0 \leq b \leq q$$
$$0 \leq u \leq n$$
$$0 \leq i \leq \min\{t - u, r\}$$
$$0 \leq j \leq u - n$$

Let $\delta_i$ be the minimal degree of line bundles which are direct summands of $\mathcal{E}_0$. Then by Lem.(7-2), the above holds if $p \geq 0$ and

$$\delta_{\min}(r + p - i + j) + (\ell' + \delta_i - u - j - 1) \geq q - b - j + c$$

for $\forall b, u, i, j$ as above. By noting $\delta_i \geq \delta_{\min}i$, it is easy to see that this holds under the assumption $\delta_{\min}(r + p) + \ell - d \geq q + c$. This completes the proof. \qed

\section{Infinitesimal Torelli for Open Complete Intersections}

Let the notation and the assumption be as in Def.(1-2). The main result in this section is the infinitesimal Torelli for the pair $(X, Z)$, which concerns the injectivity of the following map

$$d\rho_{X,Z}^q : H^1(X, T_X(-\log Z)) \to \text{Hom}(H^{n-r-q}(X, \Omega_X^q(\log Z)), H^{n-r-q+1}(X, \Omega_X^{q+1}(\log Z)))$$

where $1 \leq q \leq n - r$ and $\Omega_X^q(\log Z)$ is the sheaf of algebraic differential $q$-forms on $X$ with logarithmic poles along $Z$ and $T_X(-\log Z)$ is the $\mathcal{O}_X$-dual of $\Omega_X^1(\log Z)$. The above map is induced by the cup product and the contraction $\Omega_X^q(\log Z) \otimes T_X(-\log Z) \to \Omega_X^{q+1}(\log Z)$.  

**Theorem (8-1).** Assume $\delta_{\text{min}}(n - r - q) + d + e \geq n - 1$ and $\delta_{\text{min}}(q - 1) + d \geq n - 1$. Then $dp^q_{X,Z}$ is injective.

**Proof.** By noting $\Omega^r_X \simeq O_X(d - n - 1)$, the dual of $dp^q_{X,Z}$ is identified with the map

$$H^{n-r-q}(X, \Omega^q_X (\log Z)) \otimes H^{n-r-q+1}(X, \Omega^{q-1}_X (\log Z))^* \to H^{n-r-1}(X, \Omega^1_X (\log Z) \otimes O_X(d - n - 1)).$$

By Th.(I) and Th.(II) in §1 it is identified with the multiplication of Jacobian rings

$$B_{n-r-q}(d + e - n - 1) \otimes B_{q-1}(d - n - 1) \to B_{n-r-1}(2(d - n - 1) + e).$$

By definition the condition of Th.(8-1) implies that every (bi)homogeneous polynomial appearing in the Jacobian rings on the left hand side has a non-negative degree. Hence the above map is surjective under the assumption. □

§9. EXPLICIT BOUND FOR NORTI’S CONNECTIVITY

In this section we deduce Th.(0-1) from Th.(9-1) and Th.(9-3), the symmetrizer lemmas for open complete intersections. Let the assumption be as in §1. We fix a non-singular algebraic variety $S$ over $k$ and the following schemes over $S$

$$(9 - 1) \quad \mathbb{P}^n_S \to X \leftarrow Z = \bigcup_{1 \leq j \leq s} Z_j$$

whose fibers are as in Def.(1-2). Let $f : X \to S$ be the natural morphism and write $U = X \setminus Z$. For integers $p, q$ we introduce the following sheaf on $S_{zar}$

$$H^{p,q}(U/S) = R^p f_* \Omega^p_{X/S}(\log Z),$$

where $\Omega^p_{X/S}(\log Z) = \wedge^p \Omega^1_{X/S}(\log Z)$ with $\Omega^1_{X/S}(\log Z)$, the sheaf of relative differentials on $X$ over $S$ with logarithmic poles along $Z$. We assume $s \geq 1$. Then the Lefschetz theory implies $H^{p,q}(U/S) = 0$ if $p + q \neq n - r$. We consider the following Koszul complex

$$(9 - 2) \quad \Omega^q_S \otimes \mathcal{O} H^{a+2,b-2}(U/S) \xrightarrow{\nabla} \Omega^q_S \otimes \mathcal{O} H^{a+1,b-1}(U/S) \xrightarrow{\nabla} \Omega^{q+1}_S \otimes \mathcal{O} H^{a,b}(U/S).$$

Here $\nabla$ is induced by the Kodaira-Spencer map

$$\kappa_{(X,Z)} : \Theta_S \to R^1 f_* T_{X/S}(-\log Z),$$

with $\Theta_S = \text{Hom}_{O_S}(\Omega^1_S, O_S)$ and $T_{X/S}(-\log Z) = \text{Hom}_{O_X}(\Omega^1_{X/S}(\log Z), O_X)$, and the map

$$R^1 f_* T_{X/S}(-\log Z) \otimes R^{b-1} f_* \Omega^{a+1}_{X/S}(\log Z) \to R^b f_* \Omega^a_{X/S}(\log Z)$$

induced by the cup product and $T_{X/S}(-\log Z) \otimes \Omega^{a+1}_{X/S}(\log Z) \to \Omega^a_{X/S}(\log Z)$, the contraction.

**Theorem (9-1).** Let $c = c_S(X, Z)$ be as in Def.(9-1) below and assume $n - r \geq 2$. Assume also that $(*)$ either $a < n - r - 1$ or $r + s \leq n$. Then the complex $(9-2)$ with $a + b = n - r$ is exact under one of the following conditions

(i) $a \geq 0$, $q = 0$ and $\delta_{\text{min}} a + d \geq c + n + 1$.

(ii) $a \geq 0$, $q = 1$ and $\delta_{\text{min}} a + d \geq c + n + 2$ and $\delta_{\text{min}} (a + 1) + d \geq c + n + 1 + d_{\text{max}}$.

(iii) $a \geq 0$, $\delta_{\text{min}} (r + a) \geq q + c + n + 1$ and $r + s \leq n + 2$.

(iv) $a \geq 0$, $\delta_{\text{min}} (r + a) \geq q + c + n + 1$ and $a < n - r - \frac{r}{2}$.
Definition (9-1). For \( x \in S \) let \( U_x \subset X_x \supset Z_x \) denote the fibers of the family (9-1) and let
\[
\kappa_{x, z}^{\log} : T_x S \to H^1(X_x, T_{X_x}(- \log Z_x)) \quad \text{(resp. } \psi_{(x, z)} : B_1(0) \to H^1(X_x, T_{X_x}(- \log Z_x))\text{)}
\]
be the Kodaira-Spencer map (resp. the map in Th.(1)(2) for \((X_x, Z_x)\)). We define
\[
c_S(X, Z) = \max_{x \in S} \{ \dim_k(\text{Im}(\psi_{(x, z)}))/\text{Im}(\psi_{(x, z)}) \cap \text{Im}(\kappa_{x, z}^{\log}) \}.
\]

If \( n - r \geq 2 \) and \( X_x \) is not a K3 surface, \( \psi_{(x, z)} \) is surjective so that
\[
c_S(X, Z) = \max_{x \in S} \{ \dim_k(\text{Coker}(\kappa_{(x, z)})) \otimes \mathcal{O}_S k(x)) \}.
\]

Now we prove Th.(9-1). We fix \( 0 \in S \) and let \( X \supset Z \) be the fiber over \( 0 \) of \( X \supset Z \). By Th.(I) and (II) the assumption \((*)\) implies that the dual of the fiber over \( 0 \) of the complex (9-2) is identified with
\[
(*) \quad B_q(\ell_0) \otimes T_0 S \to B_{a+1}(\ell_0) \otimes T_0 S \to B_{a+2}(\ell_0) \otimes T_0 S \quad (\ell_0 := d - n - 1)
\]
where \( B_\ell(\ell) \) denotes the Jacobian ring for \( (X, Z) \) and the maps are induced by the composite map
\[
\rho : T_0 S \xrightarrow{\kappa_{0, Z}^{\log}} H^1(X, T_X(- \log Z))_{\text{alg}} \xrightarrow{\psi_{X, Z}} B_1(0) \quad \text{(cf. Th.(I))}
\]
and the multiplication on the Jacobian rings. Let \( V = \text{Im}(\rho) \subset B_1(0) \) and \( K = \text{Ker}(\rho) \). By definition \( V \) is of codimension \( \leq c \) in \( B_1(0) \). We have the filtration \( F^q(\wedge T_0 S) \subset \wedge T_0 S \) such that
\[
F^q(\wedge T_0 S)/F^{q-1}(\wedge T_0 S) \cong \wedge K \otimes \wedge V.
\]
The filtration induces a filtration of the complex \((*)\) so that it suffices to show the exactness of \((*)\) with \( T_0 S \) replaced by \( V \). Then it follows from Th.(III). \( \square \)

Theorem (9-2). Let \( c = c_S(X, Z) \) be as in Def.(9-1). Assume \( n - r \geq 2 \) and that \( S \) is affine. Then
\[
H^b(X, \Omega^a_{X/k}(\log Z)) = 0 \quad \text{if } s \leq n - r + 2, \ b \leq n - r - 1, \ \delta_{min}(n - b - 1) \geq a + b + 1 + r + c,
\]
where \( \Omega^a_{X/k}(\log Z) \) is the sheaf of differential forms of \( X \) over \( k \) with logarithmic poles along \( Z \).

Proof. Filter \( \Omega^a_{X/k}(\log Z) \) by the subsheaves
\[
F^q S \Omega^a_{X/k}(\log Z) = \text{Im}(f^* \Omega^a_{S} \otimes \Omega^{a-q}_{X/S}(\log Z) \to \Omega^a_{X/k}(\log Z))
\]
so that
\[
Gr^q F^q S \Omega^a_{X/k}(\log Z) = f^* \Omega^q_{S} \otimes \Omega^{a-q}_{X/S}(\log Z).
\]
The filtration gives rise to the spectral sequence
\[
E_{1}^{q, p} = H^{q+p}(Gr^q F^p S \Omega^a_{X/k}(\log Z)) = \Omega^q_{S} \otimes H^{n-q+a+p}(U/S) \Rightarrow H^{q+p}(X, \Omega^a_{X/k}(\log Z)).
\]
By the Lefschetz theory \( E_{1}^{q, b-q} = 0 \) unless \( a + b - q = n - r \) in which case \( E_{2}^{q, b-q} \) is computed as the cohomology of the complex (9-2). Th.(9-1) implies that \( E_{2}^{q, b-q} = 0 \) if \( s \leq n - r + 2 \) and \( a - q - 1 \geq 0 \) and \( \delta_{min}(r + a - q - 1) \geq q + c + n + 1 \), which is in case \( a + b - q = n - r \) equivalent to the assumption of Th.(9-2). \( \square \)

Now the first vanishing of Th.(0-1) is an easy consequence of Th.(9-2) since the vanishing of \( F^q H^1(U, \mathbb{C}) \) is reduced to that of \( H^{q}(X, \Omega^{2\ell}_{X/C}(\log Z)) \) by [D1, Pr.3.1.8].
In order to show the second vanishing of Th.(0-1) we consider the family (9-1) assuming \( s = 1 \). For integers \( a, b \) we write \( H^{a,b}_{O,c}(U/S) = R^bf_*\Omega^a_{(X,Z)/S} \) where \( \Omega^a_{(X,Z)/S} \) is defined by the exact sequence

\[
0 \to \Omega^a_{(X,Z)/S} \to \Omega^a_{X/S} \to i_*\Omega^a_Z \to 0.
\]

By the Lefschetz theory and the Serre duality

\[
H^{a,b}_{O,c}(U/S) = 0 \quad \text{if} \ a + b \neq n - r \quad \text{and} \quad H^{a,b}_{O,c}(U/S) = H^{b,a}(U/S)^*.
\]

We consider the complex

\[
(9-3) \quad \Omega^{a-1}_S \otimes O H^{a+2,b-2}_{O,c}(U/S) \overset{\nabla}{\longrightarrow} \Omega^a_S \otimes O H^{a+1,b-1}_{O,c}(U/S) \overset{\nabla}{\longrightarrow} \Omega^{a+1}_S \otimes O H^{a,b}_{O,c}(U/S)
\]

where the maps are induced by the Gauss-Manin connection. By the same argument as the proof of Th.(9-1) we can show the following.

**Theorem (9-3).** Assume \( n - r \geq 2 \) and \( s = 1 \) and write \( e = e_1 \). Let \( c = c_S(X,Z) \) be as in Def.(9-1). The complex (9-3) with \( a + b = n - r \) is exact under one of the following conditions

(i) \( a \geq 0, q = 0 \) and \( \delta_{\text{min}} a + d + e \geq c + n + 1 \).

(ii) \( a \geq 0, q = 1 \) and \( \delta_{\text{min}} a + d + e \geq c + n + 2 \) and \( \delta_{\text{min}}(a+1) + d + e \geq c + n + 1 + d_{\text{max}} \).

(iii) \( a \geq 0, \delta_{\text{min}}(r+a) + e \geq q + c + n + 1 \).

**Theorem (9-4).** Let \( c = c_S(X,Z) \) be as in Def.(9-1). Assume \( n - r \geq 2 \) and that \( S \) is affine. Then

\[
H^b(X,\Omega^a_{(X,Z)/k}) = 0 \quad \text{if} \ b \leq n - r - 1 \quad \text{and} \quad \delta_{\text{min}}(n-1-b) + e \geq a+b+1+r+c
\]

where \( \Omega^a_{(X,Z)/k} \) is defined by the exact sequence

\[
0 \to \Omega^a_{(X,Z)/k} \to \Omega^a_{X/k} \to i_*\Omega^a_Z \to 0.
\]

**Proof.** Filter \( \Omega^a_{(X,Z)/k} \) by the subsheaves

\[
F^q_S \Omega^a_{(X,Z)/k} = \text{Im}(f^*\Omega^a_S \otimes \Omega^{a-q}_{(X,Z)/k} \to \Omega^a_{(X,Z)/k})
\]

so that

\[
G^q_{F_S} \Omega^a_{(X,Z)/k} = f^*\Omega^a_S \otimes \Omega^{a-q}_{(X,Z)/k}/S.
\]

The rest of the argument is the same as the proof of Th.(9-2). \( \square \)

As is shown in [N], the vanishing of \( F^dH^l(X,Z,\mathbb{C}) \) is reduced to that of \( H^l(X,\Omega^{2d}_{(X,Z)/\mathbb{C}}) \). Thus the second vanishing of Th.(0-1) is an easy consequence of Th.(9-4).

**REFERENCES**

A1. M. Asakura, On the K1-groups of algebraic curves, preprint.

A2. M. Asakura and S. Saito, Beilinson’s Hodge and Tate conjectures for open complete intersections, preprint.

D1. P. Deligne, Théorie de Hodge II, Publ. Math. IHES 40 (1972), 5-57.

D2. Théorie de Hodge III, Publ. Math. IHES 44 (1974), 5-78.

D3. R. Donagi, Generic Torelli for projective hypersurfaces, Compositio Math. 50 (1983), 325-353.

D4. R. Donagi and M. Green, A new proof of the symmetrizer lemma and a stronger weak Torelli theorem for projective hypersurfaces, J.Diff. Geom. 20 (1984), 459-461.

E1. M. Green, The period map for hypersurface sections of high degree on an arbitrary variety, Compositio Math. 55 (1984), 135-156.

E2. M. Green, Koszul cohomology and hypersurface sections of high degree, Lectures on Riemann surfaces (Cornalba, Gomez-Mont, and Verjovsky, eds.), ICTP, Trieste, Italy, pp. 177-200.

Gri. P. Griffiths, Periods of certain rational integrals: I and II, Ann. of Math. 90 (1969), 460-541.

K. K. Konno, On the variational Torelli problem for complete intersections, Compositio Math. 78 (1991), 271-296.

[MS] S. Müller-Stach and S. Saito, On K2 of algebraic surfaces, preprint.
[Na1] J. Nagel, *The Abel-jacobi map for complete intersections*, Indag. Mathem. 8(1) (1997), 95–113.

[Na2] ——, *Effective bounds for Hodge-theoretic connectivity*, preprint.

[N] M. V. Nori, *Algebraic cycles and Hodge theoretic connectivity*, Invent. of Math. 111 (1993), 349–373.

[P] C. Peters, *The local Torelli theorem I: Complete intersections*, Math. Ann. 217 (1975), 1–16.

[SaS] S. Saito, *Higher normal functions and Griffiths groups*, to appear in J. of Algebraic Geometry.

[U1] S. Usui, *Local Torelli theorem for non-singular complete intersections*, Japan. J. Math. 2-2 (1976), 411–418.

[U2] ——, *Variation of mixed Hodge structure arising from family of logarithmic deformations II: Classifying space*, Duke Math. J. 51 (1984), 851–875.

Graduate School of Mathematics, Kyushu University 33 FUKUOKA 812-8581, JAPAN

E-mail: asakura@math.kyushu-u.ac.jp

Graduate School of Mathematics, Nagoya University, Chikusa-ku, NAGOYA, 464-8602, JAPAN

E-mail: sshuji@msb.biglobe.ne.jp