Significance of the Renormalization Constant of the Color Gauge Field

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Abstract

It is shown that a sufficient condition for color confinement is given by $Z_3^{-1} = 0$, where $Z_3$ denotes the renormalization constant of the color gauge field.

I. INTRODUCTION

The color degree of freedom has been introduced into the quark model in order to save the connection between spin and statistics for quarks and strong interactions are considered to be invariant under the color $SU(3)$ group. This new degree of freedom is hardly recognizable, however, since all the familiar hadrons belong to the singlet representations of the color $SU(3)$ group as far as we are aware. Hadrons belonging to non-singlet representations of this group, if any, would be realized only at very high energies. In fact, quarks belonging to the triplet representation have never been observed to date, thereby suggesting that an isolated quark is, in principle, not subject to observation. This is the hypothesis of quark confinement. It indicates one of the most characteristic features of strong interactions to favor neutralization of color by realizing only color singlet states and suggests a strong resemblance to electromagnetic interactions that also favor neutralization, though not completely, because of repulsion between like charges and attraction between unlike charges. This forms a marked contrast with gravitational interactions which are always attractive and do not lead to neutralization.

This analogy led Nambu [1,2] to suspect that strong interactions are mediated by a gauge field coupled to the color charge. The quanta of the color gauge field are the color octet gluons. By now it is believed that no isolated colored particles are observable and we have promoted the hypothesis of quark confinement to that of color confinement. In other words, color confinement has guided us to the color gauge theory of strong interactions or QCD.
There is a considerable difference, however, in the degree of neutralization between QED and QCD. First, let us consider the multipole interactions between two electrically neutral systems, which are given by the van der Waals potential

\[ V_{vdW}(r) \propto r^{-6}. \] (1)

This shows that the electric field exerted by a neutral system can penetrate into the vacuum without any sharp cut-off.

The situation is completely different in QCD. Let us consider the interaction between two color singlet hadrons, for instance, nucleon-nucleon scattering. In the language of dispersion relations, the potential between them is given by the pole contributions in the crossed channels. The least massive color singlet particle that can be exchanged between them is the pion and the resulting nuclear forces are represented by the Yukawa potential

\[ V_Y(r) \propto \frac{e^{-\mu r}}{r}, \] (2)

where \( \mu \) denotes the pion mass. This is a consequence of color confinement in that isolated quarks and gluons are excluded in the crossed channels.

By comparing these two potentials we recognize that the flux of the color gauge field exerted by color singlet nucleons cannot penetrate into the confining vacuum as demonstrated by the presence of a sharp cut-off at the pion Compton wave length as the penetration depth. In this way we realize that the Yukawa mechanism of generating nuclear forces leaves a strong resemblance to the Meissner effect in the type II superconductor.

The hypothesis of color confinement also implies that the color \( SU(3) \) invariance be exact. Otherwise, an originally color singlet ground state would induce colored states through the symmetry-breaking perturbation resulting in the leakage of the unconfined color. Thus unbroken color gauge symmetry is an important condition for confinement and we shall take it for granted in what follows.

So far we have been guided by color confinement to reach the concept of color gauge field, but we shall also prove the converse that color confinement follows from QCD in return. For this purpose we first show that a sufficient condition for color confinement is given by

\[ Z_3^{-1} = 0, \] (3)

where \( Z_3 \) denotes the renormalization constant of the color gauge field. Then we prove that this condition is actually satisfied in a certain class of gauges provided that the color gauge symmetry is unbroken and asymptotic freedom is respected.

**II. FORMULATION OF QCD**

Let us start from the familiar Lagrangian density for QCD,

\[ L = L_{inv} + L_{gf} + L_{FP}, \] (4)

where
\[ L_{\text{inv}} = -\frac{1}{4} F_{\mu\nu} \cdot F_{\mu\nu} - \overline{\psi} (\gamma_\mu D_\mu + m) \psi, \]  
\[ L_{gf} = A_\mu \cdot \partial_\mu B + \frac{\alpha}{2} B \cdot B, \]  
\[ L_{\text{FP}} = i \partial_\mu \overline{\psi} \cdot D_\mu \psi. \] 

We have suppressed the color and flavor indices above. The first term in (4) is the gauge-invariant piece, the second one the gauge-fixing term and the last one the Faddeev-Popov ghost term expressed in the conventional notation. The Faddeev-Popov ghost fields \( c \) and \( \overline{c} \) are anticommuting quantities.

The BRS transformations are introduced for the quark and gauge fields by replacing the infinitesimal gauge function by either \( c \) or \( \overline{c} \) in their gauge transformation, and they are denoted by \( \delta \) or \( \overline{\delta} \), correspondingly. The BRS transformations for other auxiliary fields \( B, c \) and \( \overline{c} \) are defined so as to leave the total Lagrangian invariant. The conserved BRS charges denoted by \( Q_B \) and \( \overline{Q}_B \) are related to the BRS transformations of an operator \( F \) through

\[ \epsilon \delta F = i[\epsilon Q_B, F], \quad \epsilon \overline{\delta} F = i[\epsilon \overline{Q}_B, F], \]

where \( \epsilon \) is a Grassmann variable which anticommutes with \( c \) and \( \overline{c} \). We shall skip the BRS transformations of individual fields since they are well-known.

In terms of the BRS transformations the equation for the gauge field is given by

\[ \partial_\mu F_{\mu\nu} + g J_\nu = i \delta A_\nu, \]  

where \( J_\nu \) denotes the color current density and \( g \) the gauge coupling constant.

The propagation function of the color gauge field is given by the vacuum expectation value of the time-ordered product of two gauge fields,

\[ \langle A^a_\mu(x), A^b_\nu(y) \rangle = \delta_{ab} \frac{-i}{(2\pi)^4} \int d^4 k \ e^{ik(x-y)} D_{\mu\nu}(k), \]  

where \( a \) and \( b \) are color indices, and

\[ D_{\mu\nu}(k) = \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2 - i\epsilon} \right) D(k^2) + \frac{\alpha k_\mu k_\nu}{(k^2 - i\epsilon)^2}. \]

The function \( D(k^2) \) obeys the Lehmann representation [3],

\[ D(k^2) = \int dm^2 \frac{\rho(m^2)}{k^2 + m^2 - i\epsilon}. \]

The renormalization constant of the gauge field \( Z_3 \) is given by Lehmann’s theorem [3] by

\[ Z_3^{-1} = \int dm^2 \rho(m^2). \]
III. INDEFINITE METRIC AND SUBSIDIARY CONDITIONS

When a gauge field is quantized in a covariant gauge, introduction of indefinite metric is inevitable since it is inherited from the Minkowski metric through the gauge field that transforms as a four-vector. In order to eliminate indefinite metric from physical interpretations of the gauge theory, subsidiary conditions are introduced in order to select observable physical states.

In QED the subsidiary condition called the Lorentz condition is given by

$$B^{(+)}(x)|\alpha\rangle = 0,$$

(14)

where $|\alpha\rangle$ denotes a physical state and the superscript $(+)$ refers to the positive frequency part. In QED described by the Lagrangian (4) the Faddeev-Popov ghost fields $c$ and $\overline{c}$ turn out to be free fields and we may also require their absence to define a physical state $|\alpha\rangle$ by

$$c^{(+)}(x)|\alpha\rangle = \overline{c}^{(+)}(x)|\alpha\rangle = 0.$$

(15)

Thus the subsidiary conditions in QED are given by (14) and (15). It is not difficult, however, to verify that the above conditions are equivalent to those given by (15) and (16):

$$Q_B|\alpha\rangle = 0.$$

(16)

We are aware of the fact that (16) is the only possible form of the subsidiary condition that can be employed in non-Abelian gauge theories since the auxiliary fields $B$, $c$ and $\overline{c}$ are no longer free fields in this case.

4. Interpretation of Confinement

In discussing confinement there are two important questions to be answered. First, we have to answer the question of what confinement means. Once this question is settled we have to prove that it is a consequence of QCD in the second stage. In this section we shall answer the first question.

In order to give an interpretation of confinement, we shall look for a known example of confinement within the framework of known theories. We easily find a simple example in QED. Namely, longitudinal and scalar photons are never subject to observation and they provide simple examples of confined particles. They are confined since they fail to satisfy the Lorentz condition or they belong to zero-norm states. In other words these unphysical photons are confined by metric cancellation.

By generalizing this argument to QCD we may assume that quarks and gluons or generally colored particles are confined just because they fail to satisfy the subsidiary condition (16),

$$Q_B|\text{quark}\rangle \neq 0, \quad Q_B|\text{gluon}\rangle \neq 0.$$

(17)

From the definition of the physical states (16) in QCD we expect

$$\langle \beta|\delta \overline{A}_\nu(x)|\alpha\rangle = 0,$$

(18)
when both $|\alpha\rangle$ and $|\beta\rangle$ are physical. So we shall show

$$\langle\text{quark}|\delta A_\nu(x)|\text{quark}\rangle \neq 0,$$

$$\langle\text{gluon}|\delta A_\nu(x)|\text{gluon}\rangle \neq 0,$$

for the purpose of proving (17).

By making use of the field equation (9) and the resulting Ward-Takahashi identities, we can verify that Eqs. (19) follows from

$$\partial_\mu \langle \delta A^a_\mu(x), A^b_\nu(y) \rangle = 0.$$  (20)

This is a sufficient condition for color confinement, but it is violated when the color symmetry is spontaneously broken as suggested in the introduction by an intuitive argument. Also, in QED Eq. (20) is not satisfied but it is replaced by

$$\partial_\mu \langle \delta A^a_\mu(x), A^b_\nu(y) \rangle = \partial_\nu \delta^4(x - y),$$  (21)

so that there is no charge confinement.

Furthermore, it can be shown on the basis of renormalization group and an analysis of the Goto-Imamura-Schwinger term that Eq. (3) is a sufficient condition for Eq. (20). Thus the problem of color confinement reduces to that of evaluating the renormalization constant of the color gauge field [4].

**IV. EVALUATION OF $Z_3$**

It is worth emphasizing that $Z_3$ can be evaluated exactly in QCD by means of renormalization group. In the renormalization group approach we study the dependence of the parameters characterizing the theory, such as the gauge coupling constant $g$ and the gauge parameter $\alpha$, on the scale change of the renormalization point $\mu$. These parameters are called the running coupling constant and the running gauge parameter as functions of $\mu$ and are denoted by $\bar{g}$ and $\bar{\alpha}$. The asymptotic values of these running parameters in the limit of $\mu \to \infty$ are denoted by $g_\infty$ and $\alpha_\infty$, respectively. It should be stressed that these asymptotic values can be identified with their unrenormalized ones.

In QED, Gell-Mann and Low [5] reached the conclusion that the unrenormalized coupling constant $e^2/4\pi$ may behave in either of two ways:

(a) It may really be infinite as perturbation indicates;

(b) It may be a finite number independent of $e^2/4\pi$.

In gauge theories such as QED and QCD, we generally have the relation [6]

$$Z_3^{-1} = \frac{\alpha}{\alpha_\infty}.$$  (22)

Now asymptotic freedom in QCD is represented by

$$g_\infty = 0,$$  (23)
and in this case the asymptotic limit of the gauge parameter can assume one of the following three alternative values:

\[ \alpha_\infty = -\infty, 0, \alpha_0, \]  
\[ \text{(24)} \]

where

\[ \alpha_0 = \frac{1}{3} \left( 13 - \frac{4}{3} N_f \right) \]  
\[ \text{(25)} \]

for the system consisting of gluons and \( N_f \) flavors of quarks.

In QED the renormalized coupling constant \( e^2/4\pi \) is fixed and only one of the two alternative cases (a) and (b) is realized. In QCD, however, which one of the three possible values in (24) is realized depends on the initial choice of the two parameters \( g \) and \( \alpha \). Although \( g \) is fixed as \( e \) is, the choice of \( \alpha \) is quite arbitrary. We can always find a domain of \( \alpha \) for a given value of \( g \) in which \( \alpha_\infty = -\infty \) is realized and consequently the sufficient condition for confinement (3) is satisfied. Indeed, for small values of \( g^2 \) this domain is given by [4,6]

\[ \alpha < \text{Min}(\alpha_0, 0). \]  
\[ \text{(26)} \]
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