Spectra of Quantized Walks and a $\sqrt{\delta \varepsilon}$-Rule

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Abstract

We introduce quantized bipartite walks, compute their spectra, generalize the algorithms of Grover [G96] and Ambainis [Amb03] and interpret them as quantum walks with memory. We compare the performance of walk based classical and quantum algorithms and show that the latter run much quicker in general. Let $P$ be a symmetric Markov chain with transition probabilities $P[i, j]$, $(i, j \in [n])$. Some elements of the state space are marked. We are promised that the set of marked elements has size either zero or at least $\varepsilon n$. The goal is to find out with great certainty which of the above two cases holds. Our model is a black box that can answer certain yes/no questions and can generate random elements picked from certain distributions. More specifically, by request the black box can give us a uniformly distributed random element for the cost of $\mathcal{P}_0$. Also, when “inserting” an element $i$ into the black box we can obtain a random element $j$, where $j$ is distributed according to $P[i, j]$. The cost of the latter operation is $\mathcal{P}_1$. Finally, we can use the black box to test if an element $i$ is marked, and this costs us $\mathcal{P}_2$. If $\delta$ is the eigenvalue gap of $P$, there is a simple classical algorithm with cost $O(\mathcal{P}_0 + (\mathcal{P}_1 + \mathcal{P}_2)/\delta \varepsilon)$ that solves the above promise problem. (The algorithm is efficient if $\mathcal{P}_0$ is much larger than $\mathcal{P}_1 + \mathcal{P}_2$.) In contrast, we show that for the “quantized” version of the algorithm it costs only $O(\mathcal{P}_0 + (\mathcal{P}_1 + \mathcal{P}_2)/\sqrt{\delta \varepsilon})$ to solve the problem. We refer to this as the $\sqrt{\delta \varepsilon}$ rule. Among the technical contributions we give a formula for the spectrum of the product of two general reflections.

1 Introduction

The recent algorithm of Ambainis [Amb03] which exploits properties of quantum walks on versions of Johnson graphs opens up a new avenue for walk based algorithm designs. It claims as its direct predecessor the algorithm of Grover [G96] for database search. Ambainis’s algorithm, like [G96], is a search algorithm, but for many problems to which it applies, such as element distinctness, it runs quicker than any application of [G96]. In the present article we give a thorough analysis of the type of walks that power the construction in [Amb03], and use them in a very general algorithmic scheme. In our efforts to simplify the proofs we modify the paradigm that Grover has set up.

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Ambainis’s idea that a quantum walk can accelerate black box search for pairs of colliding elements demonstrates that diffusion-based quantum walks are more versatile than it had been thought previously. The consequences are numerous. Based on it Magniez, Santha and Szegedy [MSS] give a quicker algorithm for the oracle version of the triangle finding problem. Childs and Eisenberg [CE] streamline the analysis of [Amb03], and give further examples to its use. These papers, even the latter one do not change the proof structure of [Amb03] significantly.

In contrast, our discussion departs from that of [Amb03] at several points: 1. We treat not only the Johnson graphs but all Markov chains; 2. We show how to eliminate the restriction Ambainis imposes on the state of the machine: we no longer need to request that it stays in a constant dimensional subspace of the entire state space. 3. We circumvent relevant parts of the proof by introducing “memory” for walks. As a consequence of 1., 2., and 3. we obtain:

Let $P$ be a symmetric Markov chain with transition probabilities $P[i, j]$, $(i, j \in [n])$. Some elements of the state space are marked. We are promised that the set of marked elements has size either zero or at least $\varepsilon n$. The goal is to find out with great certainty which of the above two cases holds. Our model is a black box that can answer certain yes/no questions and can generate random elements picked from certain distributions. More specifically, by request the black box can give us a uniformly distributed random element for the cost of $\varphi_0$. Also, when “inserting” an element $i$ into the black box we can obtain a random element $j$, where $j$ is distributed according to $P[i, j]$. The cost of the latter operation is $\varphi_1$. Finally, we can use the black box to test if an element $i$ is marked, and this costs us $\varphi_2$. If $\delta$ is the eigenvalue gap of $P$, there is a simple classical algorithm with cost $O(\varphi_0 + (\varphi_1 + \varphi_2)/\delta \varepsilon)$ that solves the above promise problem. (The algorithm is efficient if $\varphi_0$ is much larger than $\varphi_1 + \varphi_2$.) In contrast, we show that for the “quantized” version of the algorithm it costs only $O(\varphi_0 + (\varphi_1 + \varphi_2)/\sqrt{\delta \varepsilon})$ to solve the problem. We refer to this as the $\sqrt{\delta \varepsilon}$ rule. Among the technical contributions we give a formula for the spectrum of the product of two general reflections.

We settle at a walk model, directly derived from [Amb03], that we call “bipartite.” We focus only on walks that do diffusion for coin flip. Our formulas for the spectra and the eigenvalues of these walks should be useful in other contexts as well.

In the quantum walk literature one can find two separate directions. Discrete time walks were introduced by Y. Aharonov, L. Davidovich, and N. Zagury [ADZ] and re-introduced by D. A. Meyer [Mey]. The properties of these walks were studied in one dimension by Ambainis, E. Bach, A. Nayak, A. Vishwanath, and J. Watrous: [ABNV], and in general by D. Aharonov, A. Ambainis, J. Kempe, U. Vazirani [AAKV].

Continuous time walks were introduced by E. Farhi, S. Gutmann [FG], and they were studied by J. Roland, N. Cerf [RC], by Wim van Dam, Michele Mosca, Umesh V. Vazirani [DMV], by Childs and Goldstone [CG] and many others. Andrew M. Childs, Richard Cleve, Enrico Deotto, Edward Farhi, Sam Gutmann, Daniel A. Spielman [CCDF] show an example where a continuous quantum walk exponentially quicker traverses a graph than its deterministic counterpart. Continuous walks are intimately related to the paradigm of adiabatic computation.

In this article we are concerned about special discrete time walks. The results are self-contained.
2 Notations

We develop notations for bipartite objects, both classical and quantum. The symbol $[n]$ denotes a left-set of size $n$ and symbol $|m|$ denotes a right-set of size $m$. A classical bipartite walk takes place on the node set $[n] \cup |m|$:

| Left | Right |
|------|-------|
| Set: $[n]$ | $|m|$ |
| Vector: $a = (\alpha_1, \ldots, \alpha_n)$ | $b = (\beta_1, \ldots, \beta_m)$ |

We can concatenate left and right vectors into vectors of length $n + m$:

$$(a, b) = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m).$$

Left and right vectors and their concatenations are classical objects. To quantize them we form tensor products from them that lie in the Hilbert space $\mathbb{C}[n] \otimes \mathbb{C}[m]$. Among the other not entirely standard notations we need are $\circ$, which means point-wise product of vectors and matrices and the square root of a vector or a matrix, which means point-wise square root:

$$\sqrt{M[i, j]} = \sqrt{M[i,j]}.$$

We apply the latter operation only if the elements of the matrix or vector are non-negative. Let $v_1, \ldots, v_k$ be vectors in the same Hilbert space. We denote the Gram matrix of $v_1, \ldots, v_k$ (the matrix made of the inner products $\langle v_s, v_t \rangle$) by $\text{Gram}(\{v_i\}_{i=1}^k)$.

Linear operators will be denoted by any of lower case, upper case or Greek letters. The matrix of transition probabilities of a Markov chain will usually be denoted by $P$. The typical state space for our Markov chains is $[n] = \{1, \ldots, n\}$. Although we avoided using $i$ as a running index in formulas containing the complex root of $-1$, the reader needs to use judgment about the meaning of $i$ in each formula.

3 Bipartite Walks

Let $[n]$ and $|m|$ be two disjoint sets of size $n$ and $m$, respectively. A bipartite walk on $[n] \cup |m|$ is a probabilistic map with domain $[n]$ and range $|m|$, and another probabilistic map with domain $|m|$ and range $[n]$. A probabilistic map is a stochastic matrix with rows and columns indexed by the elements of the domain and range of the map, respectively. Thus a walk is described by a pair of stochastic matrices $(c, r)$ of dimensions $(n, m)$ and $(m, n)$, respectively. Since $c$ and $r$ are stochastic, we have that $c[i, j] \geq 0$ and $r[j, i] \geq 0$ for all $1 \leq i \leq n, 1 \leq j \leq m$, $\sum_j c[i, j] = 1$ for all $1 \leq i \leq n$, $\sum_i r[j, i] = 1$. for all $1 \leq j \leq m$.

To implement the walk we need registers Left and Right, that hold values from $[n]$ and $|m|$, respectively. An instantiation of the walk starts at an initial value $a \in [n]$. We alternately set Right to $j$ with probability $c[\text{Left}, j]$ and then Left to $i$ with probability $r[i, \text{Right}]$, etc. If
Walk

Prob = 1/2 x 1 x 1/2 x 1 x 1/2
Instantiation of length 5

Figure 1: Example to a bipartite walk, and one of its instantiations

α comes from an initial distribution on \([n]\), then executing the walk for an even number of steps results in a distribution on \([n]\), and executing the walk for an odd number of steps results in a distribution on \([m]\).

The quantized version of the walk takes place in the \(n \times m\) dimensional Hilbert space of quantum registers \(|\text{Left}\rangle|\text{Right}\rangle\). The roles of \(c\) and \(r\) are taken up by diffusion operators \(2C - I\) and \(2R - I\). Operator \(2C - I\) is controlled by the first register and acts on the second register. Let \(c_i\), \((1 \leq i \leq n)\) be the vector of probabilities, where \(c_i[j]\) is the probability that \(c\) takes the \(i^{th}\) element of \([n]\) into the \(j^{th}\) element of \([m]\), and let \(r_j\), \((1 \leq j \leq m)\) be the vector of probabilities, where \(r_j[i]\) is the probability that \(r\) takes the \(j^{th}\) element of \([m]\) into the \(i^{th}\) element of \([n]\). When \(|\text{Left}\rangle\) is set to basis vector \(|i\rangle\) we define the controlled diffusion operator on register \(|\text{Right}\rangle\) as

\[
2\sqrt{c_i}\sqrt{c_i^T} - I_{[m]}.
\]

The controlled diffusion operator of the \(|\text{Left}\rangle\) register, when \(j\) is the value of the \(|\text{Right}\rangle\) register is similarly defined by

\[
2\sqrt{r_j}\sqrt{r_j^T} - I_{[n]}.
\]

Our focus of interest will be the operator, which does a controlled diffusion on \(|\text{Right}\rangle\) and then another one on \(|\text{Left}\rangle\). We redefine the above using the ket formalism. Let

\[
c_i = \sum_{j=1}^{m} c[i, j]|i\rangle|j\rangle, \quad (1 \leq i \leq n);
\]

\[
r_j = \sum_{i=1}^{n} r[j, i]|i\rangle|j\rangle, \quad (1 \leq j \leq m).
\]

Writing all vectors in the basis \(|i\rangle|j\rangle\) \((1 \leq i \leq n, 1 \leq j \leq m)\) we define the projection operators

\[
C = \sum_{i=1}^{n} \sqrt{c_i}\sqrt{c_i^T}; \quad (1)
\]

\[
R = \sum_{j=1}^{m} \sqrt{r_j}\sqrt{r_j^T}. \quad (2)
\]
Definition 1. The quantized version of the bipartite walk \((c, r)\) is the pair \((2C - I, 2R - I)\) of diffusion operators on \(C^{[n] \times [m]}\). The two-step walk operator is \(\mu = (2R - I)(2C - I)\).

In Section 12 we shall compute the eigenvalues and eigenvectors of \(\mu\) in a more general setting. Equipped with these expressions in Section 8 we analyze the running time of our generalization of Grover/Ambainis type algorithms.

4 The Discriminant Matrix

The spectrum of \(\mu\) of the previous Section is completely determined by matrices \(c\) and \(r\). But how? The following definition takes us one step closer to answering this question.

Definition 2 (Discriminant Matrix). We define the discriminant matrix of a bipartite walk \((c, r)\) as:

\[
M = \begin{pmatrix}
0 & \sqrt{c \circ r^T} \\
\sqrt{r \circ c^T} & 0
\end{pmatrix}.
\]

(3)

The significance of the discriminant matrix will become clear from our Spectral Theorem. Below we give an equivalent definition of it, that will yield itself to a natural generalization. Notice that:

\[
\langle \sqrt{c_i} | \sqrt{c_{i'}} \rangle = \delta_{i,i'} \quad \text{for all } 1 \leq i \leq i' \leq n; \quad (4)
\]

\[
\langle \sqrt{r_j} | \sqrt{r_{j'}} \rangle = \delta_{j,j'} \quad \text{for all } 1 \leq j \leq j' \leq m. \quad (5)
\]

Furthermore:

\[
\langle \sqrt{c_i} | \sqrt{r_j} \rangle = \sqrt{c[i,j]} \sqrt{r[j,i]} = M[i,j].
\]

Thus we can also express \(M\) as:

\[
M = \text{Gram}(\sqrt{c_1}, \ldots, \sqrt{c_n}, \sqrt{r_1}, \ldots, \sqrt{r_m}) - I_{n+m}.
\]

5 A Spectral Theorem

Let \(H\) be a Hilbert space and \(\mathcal{A}\) be a subspace of \(H\). The unitary operator that that leaves \(\mathcal{A}\) invariant and takes all vectors in \(\mathcal{A}^\perp\) to their opposite is called a general reflection, and it is denoted by \(\text{ref}_\mathcal{A}\). Let \(\mathcal{A}, \mathcal{B} \leq H\) be defined via two separate orthogonal bases of unit vectors:

\[
\mathcal{A} = \langle v_1, \ldots, v_n \rangle; \\
\mathcal{B} = \langle w_1, \ldots, w_m \rangle.
\]
In Section 12 we show the easy fact that for $C = \sum_{i=1}^{n} v_i v_i^*$ and $R = \sum_{j=1}^{m} w_j w_j^*$ the operators $2C - I$ and $2R - I$ are exactly $\text{ref}_A$ and $\text{ref}_B$. Expressions (1), (2) and relations (4) and (5) tie the problem of computing the spectrum and eigenvectors of $\mu$ in Definition 11 to the problem of computing the spectrum of operator $\mu = \text{ref}_B \text{ref}_A$.

**Definition 3.** We call $A^\perp \cap B^\perp$ the idle subspace, and its orthogonal complement the busy subspace.

The justification for the above definition is that the idle subspace lies in the kernel of operators $C$, $R$, and hence $\mu$ acts in the idle subspace as the identity. Thus in order to get the spectral decomposition of $\mu$ it is enough to compute it restricted on the busy subspace.

**Definition 4.** For $a = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^{|n|}$ and $b = (\beta_1, \ldots, \beta_m) \in \mathbb{C}^{|m|}$ we define

\[
\tilde{a} = \sum_{i=1}^{n} \alpha_i v_i \\
\tilde{b} = \sum_{j=1}^{m} \beta_j w_j \\
(a, b)^\sim = \tilde{a} + \tilde{b} = \sum_{i=1}^{n} \alpha_i v_i + \sum_{j=1}^{m} \beta_j w_j.
\]

Every vector in the busy subspace has a convenient (albeit not unique) expression using the tilde:

\[
(A^\perp \cap B^\perp)^\perp = \langle A, B \rangle = A + B = \{ \tilde{a} + \tilde{b} \mid a \in \mathbb{C}^{|n|}, b \in \mathbb{C}^{|m|} \}.
\]

We also need:

**Definition 5 (Discriminant Matrix (generalized)).** The Discriminant Matrix of an ordered pair $((v_1, \ldots, v_n), (w_1, \ldots, w_m))$ of orthonormal systems is:

\[
M = \text{Gram}(v_1, \ldots, v_n, w_1, \ldots, w_m) - I
\]

Our Spectral Theorem relates the spectrum and eigenvectors of $\mu$ on the busy subspace to the spectrum and eigenvectors of $M$. The eigenvalues of $M$ are symmetric to 0 and are in the $[-1, 1]$ range.

**Theorem 1 (Spectral Theorem).** Let $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ be two orthogonal systems of unit vectors spanning spaces $A$ and $B$, respectively. The eigenvectors and eigenvalues of the operator $\mu = \text{ref}_B \text{ref}_A$ on $A + B$ are derived from those of the discriminant matrix $M$ of the pair $((v_i)_{i \in [n]}, (w_j)_{j \in [m]})$ as follows:
Figure 2: The transformation that takes the eigenvalues of $M$ into the eigenvalues of $\mu$ folds the $[-1, 1]$ interval and then expands it to the complex unit circle (dotted circle).

| Eigenvalue | Eigenvector/Space |
|------------|--------------------|
| 1          | All vectors in $\mathcal{A} \cap \mathcal{B}$. This space has dimension $d_1$, where $d_1$ is dimension of the eigen-space of $M$ associated with eigenvalue 1. |
| $2\lambda^2 - 1 - 2i\lambda \sqrt{1 - \lambda^2}$ | All vectors of the form $\tilde{a} - \lambda \tilde{b} + i \sqrt{1 - \lambda^2} \tilde{b}$, where $(a, b)$ is an eigenvector of $M$ with eigenvalue $1 > \lambda > 0$. |
| $2\lambda^2 - 1 + 2i\lambda \sqrt{1 - \lambda^2}$ | All vectors of the form $\tilde{a} - \lambda \tilde{b} - i \sqrt{1 - \lambda^2} \tilde{b}$, where $(a, b)$ is an eigenvector of $M$ with eigenvalue $1 > \lambda > 0$. |
| $-1$       | All vectors of the form $\tilde{a}$ or $\tilde{b}$, where $(a, b)$ is an eigenvector of $M$ with eigenvalue 0. (This implies that $(a, 0)$ and $(0, b)$ are eigenvectors of $M$ with eigenvalue 0 too.) |

We prove the Spectral Theorem and other unproven claims of this section in Section 12. Let $(a, b) \in \mathbb{C}^{[n] \cup [m]}$. When $\{v_i\}_{i \in [n]}$ and $\{w_j\}_{j \in [m]}$ specialize to $\{\sqrt{c_i}\}$ and $\{\sqrt{\tau_j}\}_j$, the tilde operator takes the following form:

$$
(a, b)^\sim = \sum_{i=1}^{n} \alpha_i \left( \sum_{j=1}^{m} M[i, j] |j\rangle \right) + \sum_{j=1}^{m} \beta_j \left( \sum_{i=1}^{n} N[j, i] |i\rangle \right) |j\rangle = \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i M[i, j] + \beta_j N[j, i]) |i\rangle |j\rangle.
$$
The busy subspace is this case is the set of all vectors that can be expressed as above, and the discriminant matrix specializes to \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

## 6 Bipartite Walks from Ordinary Markov Chains

A Markov chain with state set \([n]\) is an \(n\) by \(n\) stochastic matrix \(P\). We associate the (classical) bipartite walk \((P, P)\) with chain \(P\). The discriminant matrix of the walk is

\[
M = \begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix},
\]

where \(D = \sqrt{P \circ P^T}\). We call \(D\) the half discriminant matrix. Since \(M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes D\), its eigenvectors are of the form \((a, a)\) and \((a, -a)\), where \(a\) is an eigenvector of \(D\). Matrix \(D\) in general is not stochastic, and it can be the zero matrix, for instance when \(P\) is the matrix of a cyclic permutation. A stationary distribution of a Markov chain \(P\) is a probability distribution \(\pi\) on \([n]\) such that \(\pi P = \pi\). It is an eigenvector of \(P\) with all non-negative components. A Markov chain \(P\) is symmetric if \(P = P^T\). In this case \(D = P\) and its matrix is doubly stochastic. The uniform distribution on \([n]\) is stationary for symmetric Markov chains. In the quantized case:

**Lemma 1.** Let \(P\) be a symmetric Markov chain on \([n]\) and let \(\mu\) be the quantized walk operator associated with \(P\). Let \(a = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right), \ b = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)\). Then \(u = \hat{a} = \hat{b}\) is a unit vector and \(\mu u = u\).

**Proof.** We have

\[
u = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sqrt{c_i} = \sum_{i=1}^{n} \frac{1}{\sqrt{n}} \sum_{j=1}^{m} \sqrt{P[i, j]} \, |i\rangle \langle j| = \sum_{1 \leq i, j \leq n} \frac{P[i, j]}{n} \, |i\rangle \langle j| = \sum_{j=1}^{m} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sqrt{P[j, i]} \, |i\rangle \langle j| = \frac{1}{\sqrt{n}} \sum_{j=1}^{m} \sqrt{c_j}.
\]

From Lemma 8 it follows that \(u\) is an eigenvector of \(\mu\) with eigenvalue 1. Direct calculation or Theorem 5 gives \(|u| = 1\). \(\square\)

## 7 Walks with “Memory”

We first study the classical setting. Let \(P\) be a symmetric Markov chain with state set \([n]\). Assume furthermore that elements of a subset \(G\) of \([n]\) are marked, where \(G\) is either the empty set or \(|G| \geq \varepsilon n\). In this section we consider the task of designing an efficient algorithm that differentiates
in between the above two cases. At our disposal there are three subroutines, each with different associated costs:

| Name of the routine | Description                                               | Cost  |
|---------------------|-----------------------------------------------------------|-------|
| PickUniform()       | Picks a random \( i \in [n] \) distributed according to \( \mathcal{W} \) | \( \mathcal{W}_0 \) |
| ApplyChain(i)       | Applies the randomized map \( P \) on input \( i \in [n] \) | \( \mathcal{W}_1 \) |
| IsMarked(i)         | Returns 0 if \( i \in [n] \setminus G \), returns 1 if \( i \in G \) | \( \mathcal{W}_2 \) |

Under the above conditions what is the price of completing the task? The optimal algorithm depends on the ratios of \( \mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2 \). If \( \mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2 \) have the same magnitude we just repeatedly pick random elements of \([n]\) using \texttt{PickUniform()} and test them with \texttt{IsMarked(i)} until we either find a marked element or declare that \( G \) is empty. We need to do this \( O(1/\varepsilon) \) times to achieve small constant error probability. However, if \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \) are much smaller than \( \mathcal{W}_0 \), then the following algorithm will perform better:

**Algorithm \texttt{FindMarked}(K) (Classical):**

```
Set \( i = \texttt{PickUniform}() \);
Do \( K \) times {
    If ( \texttt{IsMarked}(i) == 0 ) \( i = \texttt{ApplyChain}(i) \);
}
Output \texttt{IsMarked}(i);
```

If \( G = \emptyset \), the output of the above algorithm is always 0. On the other hand, if the eigenvalue gap of \( P \) is \( \delta, |G| \geq \varepsilon n \) and \( K \) is 1000/\( \delta \varepsilon \), it is easy to see that \( i \) will almost certainly "converge" to an element of \( G \) making the output 1 with high probability. The associated cost is:

\[
\text{Classical cost of detecting large } G = \mathcal{W}_0 + 1000(\mathcal{W}_1 + \mathcal{W}_2)/(\delta \varepsilon). \tag{10}
\]

We shall "quantize" Algorithm \texttt{FindMarked}(\( K \)) and show that in the quantum case the right hand term of Expression (10) is replaced with 1000(\( \mathcal{W}_1 + \mathcal{W}_2 \))/\( \sqrt{\delta \varepsilon} \). The merit of the result is that both \( \varepsilon \) and \( \delta \) get square rooted, while what (almost) trivially follows from Grover is only that \( \varepsilon \) gets square rooted. Our quantum machine will have registers:

- **Control register:** \( |b\rangle \), where \( b \in \{0, 1\} \);
- **Walk registers:** \( |i\rangle|j\rangle \), where \( i, j \in [n] \).

We explain the role of the control register later. The \( |i\rangle|j\rangle \) register pair is used to perform a quantized bipartite walk with \( |n| = |m| = |n| \). Our new twist is that instead of quantizing \( P \), we quantize:

\[
P' \overset{\text{def}}{=} \begin{cases} 
\text{if } i \notin G \text{ use map } P \text{ on } i; \\
\text{if } i \in G \text{ then map } i \text{ into itself with probability 1.}
\end{cases}
\]
We might say that $P'$ “remembers” if it ever sees a marked element. If $G = \emptyset$ then $P' = P$. For the general case, express:

$$ P = \begin{pmatrix} P_1 & P_2 \\ P'^T & P_3 \end{pmatrix} \quad \text{with coordinate-division} \quad \begin{bmatrix} n \setminus G \\ G \end{bmatrix} = \begin{bmatrix} P_1 & P_2 \\ P'^T & P_3 \end{bmatrix} $$

Then the matrix form of $P'$ is:

$$ P' = \begin{pmatrix} P_1 & P_2 \\ 0 & I \end{pmatrix}. $$

Above $I$ is the identity matrix. The half discriminant matrix of the associated bipartite walk is:

$$ D = \begin{pmatrix} P_1 & 0 \\ 0 & I \end{pmatrix}. $$

Let us denote by $\nu$ the quantum walk associated with $P'$ and by $\mu$ the quantum walk associated with $P$. Define

$$ u = \sum_{1 \leq i,j \leq n} \sqrt{P[i,j]/n} |i\rangle|j\rangle. $$

By Lemma 1 and (8) we have that $\mu u = u$. Therefore $\mu^K u = u$ for every integer $K$. Our algorithm will utilize:

1. If $G = \emptyset$ then $\nu = \mu$, so $\left| \frac{u + \nu^K u}{2} \right| = \left| (u + u)/2 \right| = 1$.

2. If $|G| \geq \varepsilon n$ and $K \geq 1000/\sqrt{\delta \varepsilon}$ then $\left| \frac{u + \nu^K u}{2} \right| \leq 3/4$.

The above formula explains the role of the control register. At the start we split the computation into two branches: in one branch we leave $u$ untouched while in the other we apply $\nu^K$ on it. The splitting operator is $H = (= H^{-1}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. The algorithm:

**Algorithm FindMarked($K$) (Quantum):**

1. Put the pair of walk registers into the state $u$;
2. Apply $H$ on the Control register; /* This puts the system into state $\frac{1}{\sqrt{2}}|0\rangle u + \frac{1}{\sqrt{2}}|1\rangle u$; */
3. Do $K$ times {
   1. If ( the Control register is $|1\rangle$ ) {
      1. apply $\nu$ on the pair of walk registers;
   }
   } /* Now the machine will be at state $\frac{1}{\sqrt{2}}|0\rangle u + \frac{1}{\sqrt{2}}|1\rangle \nu^K u$ */
4. Apply $H^{-1}$ on the Control register;
5. Measure the final state: $|b\rangle|i\rangle|j\rangle$;
6. If ( $b = 1$ or $i \in G$ ) output 1, else output 0.
Next we give a pair of subroutines that put the walk register $|i⟩|j⟩$ into the state $\nu|i⟩|j⟩$. These routines implement the operators $2C' - I$ and $2R' - I$, where $(2C' - I, 2R' - I)$ is the quantization of Markov chain $P'$. At our disposal, like in the classical analogue, we only have operators $2C - I$ and $2R - I$, where $(2C - I, 2R - I)$ is the quantization of Markov chain $P$. While $P$ does not depend on the marked subset, $P'$ does. Thus our subroutines will need to use subroutine IsMarked(i). They will also use an ancilla register $|a⟩$ named Marked, where $a \in \{0, 1\}$. This register stays $|0⟩$ in between applications of the subroutines and is not to be confused with the Control register.

Algorithm PerturbedWalk, diffusion on $j$:

$|i⟩|j⟩|0⟩ \rightarrow |i⟩|j⟩|g(i)⟩$, where $g(i) = \text{IsMarked}(i)$;
If $g(i) = 0$ apply $(2C - I)$ on the pair of walk registers;
/* This takes $|i⟩|j⟩|g(i)⟩$ for $g(i)=0$ into the state
$|i⟩\left(\sum_{j'\in[n]}2\sqrt{P'[i,j']}|j'⟩\right)|g(i)⟩ - |i⟩|j⟩|g(i)⟩ */
$|i⟩|j⟩|a⟩ \rightarrow |i⟩|j⟩|a \oplus g(i)⟩$, where $g(i) = \text{IsMarked}(i)$;
/* This step ‘‘forgets’’ $g(i)$, since $a=g(i)$ at this stage */

Similarly:

Algorithm PerturbedWalk, diffusion on $i$:

$|i⟩|j⟩|0⟩ \rightarrow |i⟩|j⟩|g(j)⟩$, where $g(j) = \text{IsMarked}(j)$;
If $g(j) = 0$ apply $(2R - I)$ on the pair of walk registers;
/* This takes $|i⟩|j⟩|g(j)⟩$ for $g(j)=0$ into the state
$\left(\sum_{i'\in[n]}2\sqrt{P[j,i']}|i'⟩\right)|j⟩|g(j)⟩ - |i⟩|j⟩|g(j)⟩ */
$|i⟩|j⟩|a⟩ \rightarrow |i⟩|j⟩|a \oplus g(j)⟩$, where $g(j) = \text{IsMarked}(j)$;
/* This step ‘‘forgets’’ $g(j)$, since $a=g(j)$ at this stage */

In the quantized setting we need to reinterpret costs $\wp_0$ and $\wp_1$: The cost of putting the machine into state $u$ is $\wp_0$; The cost of applying operators $2C - I$ and $2R - I$ on the pair of walk registers is $\wp_1$. With this pricing the total cost of the operation FindMarked($K$) is

$$O(\wp_0 + K(\wp_1 + \wp_2)).$$

8 The $\sqrt{\delta\varepsilon}$ rule

Theorem 2. Let $P$ be a symmetric Markov chain with state set [n], and let $G$ be a subset of [n] marked via operator IsMarked(i). Let $\delta$ be the eigenvalue gap of $P$. Then for a randomly picked $K \in [1,1000/\sqrt{\delta\varepsilon}]$:

1. If $G = \emptyset$ FindMarked($K$) outputs 0 with probability 1;
2. If $|G| \geq \varepsilon n$ FindMarked($K$) outputs 1 with probability at least 1/1000.
(The probabilities are both over \( K \) and the output of \( \text{FindMarked}(K) \).)

**Proof.** Recall that state of the machine during the execution of \( \text{FindMarked}(K) \) evolves as:

\[
|0\rangle_0 \rightarrow |0\rangle u \rightarrow \frac{1}{\sqrt{2}}|0\rangle u + \frac{1}{\sqrt{2}}|1\rangle u \rightarrow \frac{1}{\sqrt{2}}|0\rangle u + \frac{1}{\sqrt{2}}|1\rangle \nu^K u \rightarrow |0\rangle \frac{u + \nu^K u}{2} + |1\rangle \frac{u - \nu^K u}{2}.
\]

The proof of the last transition:

\[
\left( \frac{1}{\sqrt{2}}|0\rangle u + \frac{1}{\sqrt{2}}|1\rangle \nu^K u \right) \left( \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right) \otimes I_{C_{[n] \times n}} = \\
\frac{1}{2} |0\rangle (u + \nu^K u) + \frac{1}{2} |1\rangle (u - \nu^K u).
\]

The last arrow of (11) represents a transition (actually the only one) that depends on the input, since \( \nu \), the quantized version of the modified walk, \( P' \), depends on what \( G \) is. If \( G = \emptyset \) then \( \nu = \mu \) (see previous section), and since \( u \) is left invariant under \( \mu \), the final state is \( |0\rangle u \), and the output is 0 with probability 1. This proves the first part of theorem.

In order to prove the second part we show that if \(|G| = \varepsilon n\), then

\[
\left| \frac{u + \nu^K u}{2} \right| \leq \frac{7}{8}
\]

or the left walk register contains an element of \( G \) with probability 1/16. In fact, if \( \varepsilon > 1/8 \) the latter is the case. To see this notice that the last operation does not effect the walk register, thus we get the same measurement for the walk register as if we measured it in the previous step. Taking a look now at Formula (14), and the fact that \( u \) us “uniform,” we can easily see the claim. Therefore in the sequel we shall assume that \( \varepsilon \leq 1/8 \).

We use Theorem 1 to compute the spectral decomposition of \( \nu \) and in turn to compute the effect of \( \nu^K \) on \( u \). Recall that \( \nu \) is the quantization of the chain \( P' \), so in order to apply Theorem 1 we first need to determine the eigenvalues and eigenvectors of the half discriminant matrix

\[
D = \sqrt{P' \circ P'^T} = \begin{pmatrix} P_1 & 0 \\ 0 & I \end{pmatrix}.
\]

For \( i \in [n] \) let \( e_i \) be the unit vector that takes 1 on \( i \) and 0 elsewhere. Because of the block structure of \( D \), for all \( i \in G \) the vector \( e_i \) is an eigenvector of \( D \). The remaining eigenvectors are those that are eigenvectors of \( P_1 \) augmented with zeros on the coordinates corresponding to the elements of \( G \).

**Lemma 2.** The spectral radius of \( P_1 \) is at most \( 1 - \delta \varepsilon / 2 \).
Proof. It is an easy fact that there exists a non-negative unit vector \( \rho' \) such that \( \rho' P_1 = \lambda' \rho' \) and \( \lambda' \) is the spectral radius of \( P_1 \). Define \( \rho \) to be the augmentation of \( \rho' \) with zeros on the coordinates corresponding to the elements of \( G \), and define \( a = \sum_{i=1}^{n} \frac{1}{\sqrt{n}} e_i \). Since \( P \) is doubly stochastic we have \( a P = a \). Consider the spectral decomposition of \( \rho \) in the basis formed by the eigenvectors of \( P \):

\[
\rho = \alpha a + \sum_k \alpha_k a_k,
\]

where \( \{ a_k \} \cup \{ a \} \) is a complete set of eigenvectors for \( P \). Since \( P \) is symmetric, its eigenvectors are orthogonal and we have \( \alpha^2 + \sum_k \alpha_k^2 = 1 \). For a vector \( v \in \mathbb{C}^n \) we denote by \( v_G \) and \( v_{\overline{G}} \) its \( G \) and \( [n] \setminus G \) components. \( |G| \geq \varepsilon n \) is equivalent to:

\[
\langle a_{\overline{G}}, a_{\overline{G}} \rangle = \langle a_{\overline{G}}, a \rangle \leq 1 - \varepsilon.
\]

Since \( \rho_G = 0 \):

\[
\alpha = \langle \rho, a \rangle = \langle \rho, a_G + a_{\overline{G}} \rangle = \langle \rho, a_{\overline{G}} \rangle \leq |\rho||a_{\overline{G}}| \leq \sqrt{1 - \varepsilon}.
\]

Hence, \( \sum_k \alpha_k^2 = 1 - \alpha^2 \geq \varepsilon \). Let \( \lambda_k \) be the eigenvalue of \( P \) associated with \( a_k \). Then \( \rho P = \alpha a + \sum_k \alpha_k \lambda_k a_k \). Since the eigenvalue gap of \( P \) is at least \( \delta \), we have

\[
|\rho P|^2 = \alpha^2 + \sum_k \alpha_k^2 \lambda_k^2 \leq \alpha^2 + (1 - \delta)^2 \sum_k \alpha_k^2 \leq \alpha^2 + (1 - \delta) \sum_k \alpha_k^2 = \alpha^2 + \sum_k \alpha_k^2 - \delta \sum_k \alpha_k^2 \leq 1 - \delta \varepsilon.
\]

On the other hand

\[
|\rho P|^2 \geq |\rho D|^2 = |\rho' P_1|^2 = \lambda'^2
\]

which implies \( \lambda'^2 \leq 1 - \delta \varepsilon \), \( \lambda' \leq 1 - \delta \varepsilon / 2 \), as needed. \( \square \)

From the above lemma one can anticipate the proof of the theorem: Any eigenvector of \( \nu \) that comes from the "quantization" of a shrinking eigenvector of \( P' \) has eigenvalue \( e^{i\theta} \), where \( \theta \) is separated away from 0 by at least a constant factor times \( \sqrt{\delta \varepsilon} \). This we deduce from the previous lemma and Theorem 1. Hence, if \( K \) is a random number in the range \([1, 1000/\sqrt{\delta \varepsilon}]\), the phase shift, when applying \( \nu^K \) on any of the above eigenvectors is some non-zero constant times \( \pi \) on expectation. Among the technical details we need to work out the most compelling (although not very hard) is that the projection of \( u \) on the space spanned by the above eigenvectors is large. Define

\[
u' = \sum_{i \in [n] \setminus G; j \in [n]} \sqrt{\frac{P[i,j]}{P}} |i\rangle |j\rangle.
\]
From our assumption that $|G|/n \leq 1/8$ we obtain:

$$\langle u, u' \rangle = \sum_{i \in [n] \setminus G, j \in [n]} P[i, j] \frac{n - |G|}{n} \geq \frac{7}{8}. $$

Since $\nu$ is unitary so is $\nu^K$, therefore

$$|\nu^K u - \nu^K u'|^2 = |\nu^K (u - u')|^2 = |u - u'|^2 = |u|^2 + |u'|^2 - \langle u, u' \rangle - \langle u', u \rangle \leq \frac{1}{4}. $$

From this

$$\frac{1}{4} |u' + \nu^K u'|^2 + \frac{1}{4} |u - u'|^2 + \frac{1}{4} |\mu^K (u - u')|^2 \leq \frac{1}{4} |u' + \nu^K u'|^2 + \frac{1}{8}. $$

Set $$\text{amp}_K \overset{\text{def}}{=} |u' + \nu^K u'|^2.$$ We are done if we show that

**Lemma 3.** The probability that for a random $K \in [1, 1000/\sqrt{\delta \varepsilon}]$ the value of $\text{amp}_K$ is at most 3 is at least $1/6$.

**Proof.** We need to set up the stage to use Theorem 1 for the operator $\nu$. Since $\nu$ is the quantization of $P'$, the busy subspace will be $A' + B'$, where $A'$ is generated by

$$v_i = \sum_{j \in [n]} \sqrt{P[i, j]} \ |i\rangle \langle j| \quad \text{for } i \in [n] \setminus G;$$

$$|i\rangle \langle i| \quad \text{for } i \in G.$$ and $B'$ is generated by

$$w_j = \sum_{i \in [n]} \sqrt{P[j, i]} \ |i\rangle \langle j| \quad \text{for } j \in [n] \setminus G;$$

$$|i\rangle \langle i| \quad \text{for } i \in G.$$ The space spanned by the vectors $|i\rangle \langle i|$ ($i \in G$) is invariant under $\nu$ (from Lemma 8 for instance) and it is orthogonal to $u' = \sum_{i \in [n] \setminus G} \frac{1}{\sqrt{n}} v_i$. Define $A'' = \langle v_i \mid 1 \leq i \leq n \rangle$ and $B'' = \langle w_j \mid 1 \leq j \leq n \rangle$. Let $Z$ be the orthogonal complement of the subspace generated by $|i\rangle \langle i|$ ($i \in G$). We have:

$$u' \in Z \quad (16)$$

$$\nu|_Z = (\text{ref}_{B''} \text{ref}_{A''})|_Z. \quad (17)$$

The discriminant matrix associated with the $\{(v_i)_{i \in [n] \setminus G}, \{w_j\}_{j \in [n] \setminus G}\}$ pair is

$$M_1 = \begin{pmatrix} 0 & P_1 \\ P_1 & 0 \end{pmatrix}. $$
It follows from Lemma 2 that all eigenvalues of $M_1$ are less than $1 - \delta\varepsilon/2$ in absolute value (the eigenvalues of $M_1$ and those of $P_1$ coincide up to a sign). Since the action of $\nu$ on $Z$ is the same as that of $\text{ref}_B \text{ref}_A$ and $u' \in Z$, we decompose $u'$ according to orthogonal unit eigenvectors of $\text{ref}_B \text{ref}_A$:

$$u' = \sum_{k=1}^{\ell} \gamma_k z_k.$$ 

Since $u' \in \text{A}'$, it lies in the busy subspace of operators $\text{ref}_A$ and $\text{ref}_B$. We use Theorem 1 to claim that each $z_k$ on the busy subspace has eigenvalue $e^{i\theta_k}$ def $2\omega_k^2 - 1 + 2i\omega_k\sqrt{1 - \omega_k^2}$, where $\omega_k$ is some eigenvalue of $M_1$. Consequentially $|\omega_k| \leq 1 - \delta\varepsilon/2$. Hence

$$|\cos \theta_k| = 2\omega_k^2 - 1 < 2(1 - \delta\varepsilon/2) - 1 = 1 - \delta\varepsilon.$$ 

Therefore when representing angles in $[-\pi, \pi]$:

$$|\theta_k| \geq \sqrt{\delta\varepsilon} \quad \text{for } 1 \leq k \leq \ell. \quad (18)$$

We have

$$\nu^K u' = \sum_{k=1}^{\ell} \gamma_k e^{i\theta_k K} z_k$$

$$u' + \nu^K u' = \sum_{k=1}^{\ell} \gamma_k (1 + e^{i\theta_k K}) z_k$$

$$\text{amp}_K = |u' + \nu^K u'|^2 = \sum_{k=1}^{\ell} |\gamma_k|^2 |1 + e^{i\theta_k K}|^2. \quad (19)$$

In Equation 19 we used the fact that unitary operators have orthogonal eigenvector-systems. We show that for an individual $k$ the expected value of $|1 + e^{i\theta_k K}|^2$ for a random $K \in [0, 1000/\sqrt{\delta\varepsilon}]$ is close to 2. Let us denote $1000/\sqrt{\delta\varepsilon}$ by $N$. Then:

$$\frac{1}{N} \sum_{K=1}^{N} |1 + e^{i\theta_k K}|^2 =$$

$$\frac{1}{N} \sum_{K=1}^{N} 2 + e^{i\theta_k K} + e^{-i\theta_k K} =$$

$$2 + \frac{1}{N} \sum_{K=1}^{N} e^{i\theta_k K} + \frac{1}{N} \sum_{K=1}^{N} e^{-i\theta_k K} =$$

$$2 + \frac{e^{i\theta_k (N+1)} - 1}{N(e^{i\theta_k} - 1)} + \frac{e^{-i\theta_k (N+1)} - 1}{N(e^{-i\theta_k} - 1)} \leq$$

$$2 + \frac{2}{N(e^{i\theta_k} - 1)} + \frac{2}{N(e^{-i\theta_k} - 1)} \leq 2.5,$$
when $N$ is $1000/\sqrt{\delta \varepsilon}$. The last inequality comes from Inequality (18). Let $1 - p$ be the probability that $\text{amp}_K > 3$ for a random $K$. Then its expectation is lower bounded by $3 - 3p$. Since $2.5 \sum_{k=1}^{\ell} a_k^2 = 2.5$ is an upper bound on the expectation, we have $p \geq 1/6$ as needed.

9 Consequences

As a first consequence we reprove the result of Ambainis [Amb03] with a stronger implication:

**Theorem 3.** Let $X, Y$ be finite sets, $f : X \to Y$ be an oracle function and let $\mathcal{R} \subseteq Y \times Y$ be a binary relation known to us. For $H \subseteq X$ we define $f(H) = \{ f(i) \mid i \in H \}$. Define

$$p(f, \alpha) = \text{The probability that } \mathcal{R} \cap (f(H) \times f(H)) \neq \emptyset \text{ for a random set } H \text{ with size } |X|^\alpha.$$

Then there is a quantum query machine with oracle $f$ that can differentiate in between the cases when $p(f, \alpha) = 0$ and $p(f, \alpha) \geq \varepsilon$ that runs in time $O(|X|^\alpha + 1000\sqrt{|X|^\alpha / \varepsilon})$.

**Proof.** Let $k = \lceil |X|^\alpha \rceil$, $n = \binom{|X|}{k}$, and $P$ be the Markov chain on all $k$ subsets of $X$ with transitions: the probability that $H$ goes to $H'$ is zero if the the symmetric difference of $H$ and $H'$ is not two, and $\frac{1}{k(|X| - k)}$ otherwise. We quantize this chain, but with the caveat that with $H$ we keep track of $f(H)$. A set $H$ is marked if $\mathcal{R} \cap (H \times H) \neq \emptyset$. Since at all times we update $f(H)$, it does not cost us queries to find out if $H$ is marked. Hence $\varphi_2 = 0$. The cost to perform $2C - I$ or $2R - I$ is 2. Hence $\varphi_1 = 2$. To put the walk registers into the position

$$u = \frac{1}{\sqrt{n k(|X| - k)}} \sum_{|H \Delta H'| = 2} |H, f(H) \rangle |H', f(H') \rangle$$

costs $O(|X|^\alpha)$ queries. Hence $\varphi_0 = |X|^\alpha$. The probability that an item is marked is $p(f, \alpha)$, which is either zero or at least $\varepsilon$. Finally, the eigenvalues of the above Markov chain (derived from the Johnson graph) are well known to be $\frac{(k-j)(|X| - k - j)}{k(|X| - k)}$ for $j = 0, 1, \ldots, n$. Thus its eigenvalue gap is

$$\delta = 1 - \frac{(k-1)(|X| - k - 1) - 1}{k(|X| - k)} = \frac{n}{k(|X| - k)} > 1/k.$$ 

The theorem now follows from Theorem 2.

**Corollary 1.** There is a quantum query machine running in time $O(|X|^{2/3})$ that differentiates in between the cases when $f(i) \neq f(i')$ for all $i \neq i' \in X$ and when there are $i \neq i' \in X$ such that $f(i) = f(i')$.

**Proof.** Let $\mathcal{R}$ be the equality relation and set $\alpha = |X|^{2/3}$.

The second consequence of Theorem 2 is immediate:

**Theorem 4.** Let $G$ be $d$-regular expander on $\{1, \ldots, n\}$. Then there is a version of Grover search that runs in $\sqrt{n}$ steps and all transitions are done along the edges of $G$.

A special case, where we perform Grover search along the edges of the hypercube was studied by Julia Kempe [K].
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11 Operator Notation in Linear Algebra

This part of the paper builds on ordinary linear algebra. Correspondingly, every vector (unless transposed) is a row vector. Operators (as opposed to the first part) are acting on the right, and in general we have to reverse the direction of all formulas of the first part involving vectors, stars and operators. For inner product we keep the angular notation, but we separate with commas rather than bars. In particular, if \( v \) is a vector and \( M \) and \( N \) are linear operators then \( vMN \) is a row vector that we obtain by applying \( M \) and \( N \) on \( v \) in this order. Also:

\[
\langle v, v \rangle = vv^* \quad \text{is a scalar;}
\]
\[
v^*v \quad \text{is a dim } v \text{ by dim } v \text{ matrix.}
\]

In order to represent elements of \( \mathbb{C}^{[n]} \otimes \mathbb{C}^{[m]} \) graphically we introduce Rectangular vectors. These are vectors with index set \([n] \times [m]\), drawn in an array format and delimited by double bars in order to differentiate them from operators (see figure above). Regardless of their rectangular shape, they are row vectors in the sense that operators act on them on the right. If \( v \) and \( w \) are two rectangular vectors of the same dimensions, we can take their scalar product or we can create an operator by writing \( v^*w \) that acts on \( \mathbb{C}^{[n]} \otimes \mathbb{C}^{[m]} \). A nice thing about rectangular vectors is that we can conveniently express the tilde operator with them. Let \( M \) be an \( n \) by \( m \) matrix and \( N \) be an \( m \) by \( n \) matrix. We obtain the rectangular vector \( M_i \) by replacing all entries of \( M \) with zero, except those ones in the \( i^{th} \) row. Similarly, we obtain \( N_j \) by replacing all entries of \( N^T \) with zero, except those ones in the \( j^{th} \) column. When \( N \) and \( M \) are given and fixed, and \( a = (\alpha_1, \ldots, \alpha_n), \ b = (\beta_1, \ldots, \beta_m) \), then

\[
\tilde{a} = \sum_{i=1}^{n} \alpha_i M_i; \quad (20)
\]
\[
\tilde{b} = \sum_{j=1}^{m} \beta_j N_j; \quad (21)
\]
\[
(a, b)^\sim = \tilde{a} + \tilde{b}. \quad (22)
\]

We use the rectangular vector notation only in Section [14], where where we work out a specific example.
12 Spectra of Products of two General Reflections

In this section we prove the claims of Section 5. We need the definitions of that section with the modification that in this linear algebra inspired part we write operators on the right. Recall that subspaces $A, B \leq H$ are defined via two separate orthogonal bases of unit vectors:

$$A = \langle v_1, \ldots, v_n \rangle,$$
$$B = \langle w_1, \ldots, w_m \rangle,$$

and

$$C = \sum_{i=1}^{n} v_i^* v_i,$$
$$R = \sum_{j=1}^{m} w_j^* w_i.$$

Lemma 4. $C$ is an orthogonal projection to $A$ and $R$ is an orthogonal projection to $B$. Also:

$$\text{ref}_A = 2C - I; \quad (23)$$
$$\text{ref}_B = 2R - I. \quad (24)$$

Proof. We prove that $C$ is an orthogonal projection to $A$ and (23). The other claims are analogous. For $v_l (1 \leq l \leq n)$ we have

$$v_l C = \sum_{i=1}^{n} v_l v_i^* v_i = \sum_{i=1}^{n} \langle v_l, v_i \rangle v_i = v_l. \quad (25)$$

For any $u$ which is orthogonal to all $v_i$s we have:

$$u C = \sum_{i=1}^{n} u v_i^* v_i = \sum_{i=1}^{n} \langle u, v_i \rangle v_i = 0. \quad (26)$$

Equations (25) and (26) prove that $C$ is an orthogonal projection to $A$. Also from (25) and (26): $v_l(2C - I) = 2v_l - v_l = v_l$ and $u(2C - I) = -u$, and (23) follows. 

Before computing the eigenvalues/vectors of $\mu$ we study the discriminant matrix of Definition 5. $M$ has a blocked structure corresponding to the subdivision of its rows and columns to vs and ws. The two diagonal blocks are 0 and the two off-diagonal blocks are transposed conjugates of each other.

Lemma 5. The spectral norm of $M$ is at most 1. Furthermore, if $(a, b) = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$ is an eigenvector of $M$ with eigenvalue 1 then

$$\sum_{i=1}^{n} \alpha_i v_i = \sum_{j=1}^{m} \beta_j w_j.$$
Proof. Since $M$ is hermitian, all its eigenvalues are real. Let $(a, b) = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$ be any unit. Then

$$(a, b)M(a, b)^* = 2 \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j \langle v_i, w_j \rangle = 2 \left\langle \sum_{i=1}^{n} \alpha_i v_i, \sum_{j=1}^{m} \beta_j w_j \right\rangle.$$  \hspace{1cm} (27)

Let $\sum_{i=1}^{n} |\alpha_i|^2 = q$, $\sum_{j=1}^{m} |\beta_j|^2 = 1 - q$. Since $|\sum_{i=1}^{n} \alpha_i v_i| = \sqrt{q}$ and $|\sum_{j=1}^{m} \beta_j w_j| = \sqrt{1-q}$, the right hand side of (27) is at most $2\sqrt{q(1-q)} \leq 1$ with equality only if $q = 1 - q = \frac{1}{2}$. This implies the first part of the lemma. From the above it also follows that the right hand side of Equation (27) is one iff $\sum_{i=1}^{n} \alpha_i v_i = \sum_{j=1}^{m} \beta_j w_j$. \hfill \Box

We also show that the eigenvalues of $M$ are distributed symmetrically to zero. The following observation will prove useful in many contexts:

**Lemma 6.** If $(a, b)M = (a', b')$ for some $a, a' \in C^n$, $b, b' \in C^m$, then

$$(a, 0)M = (0, b'), \quad (b, 0)M = (a', 0).$$

Proof. Since the diagonal blocks of $M$ are zero, $(a, 0)M$ is of the form $(0, b'')$ and $(0, b)M$ is of the form $(a'', 0)$, which implies $(a, b)M = (a'', b'')$. But $(a, b)M = (a', b')$, which gives $a'' = a'$, $b'' = b'$, as needed. \hfill \Box

**Lemma 7.** If $(a, b), a \in C^n, b \in C^m$ is an eigenvector of $M$ with eigenvalue $\lambda$ then $(a, -b)$ is an eigenvector of $M$ with eigenvalue $-\lambda$.

Proof. From $(a, b)M = (\lambda a, \lambda b)$ Lemma gives that $(a, 0)M = (0, \lambda b)$ and $(0, b)M = (\lambda a, 0)$. Then $(a, -b)M = (-\lambda a, \lambda b) = -\lambda(a, -b)$. \hfill \Box

We denote the eigenvalues of $M$ by $1 \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{m+n}$ and with $d_\lambda$ the dimension of the eigen-space associated with eigenvalue $\lambda$.

Let us now undertake the task of computing the spectrum and eigenvalues of

$$\mu = \text{ref}_A\text{ref}_B = (2C - I)(2R - I).$$

Instead of $\mu$ it will be slightly more convenient to analyze the operator $\kappa = \frac{1}{2}(\mu - I) = 2CR - C - R$ and its action on the busy subspace. Recall the definition of the tilde operation from Section S

From Lemma \ref{lem:tilde}

$$\tilde{a}C = \tilde{a}; \quad \tilde{b}R = \tilde{b}. \hspace{1cm} (28)$$

$$\tilde{a}C = (a, 0)M; \quad \tilde{b}R = (0, b)M. \hspace{1cm} (30)$$

The proof of the theorem is powered by the following relations:
Proof. We prove only Equation (30), the proof of (31) is analogous. Because of linearity it is enough to prove (30) for the basis vectors. Let $1 \leq i \leq n$ be arbitrary and let $a = (0, \ldots, 0, 1, 0, \ldots, 0)$ be the unit vector, with a 1 in the $i^{th}$ position. Then $\tilde{a} = v_i$. We have:

$$\tilde{a}R = \sum_{j=1}^{m} v_i w_j^* w_j = \sum_{j=1}^{m} (v_i, w_j) w_j = \sum_{j=1}^{m} M[v_i, w_j] w_j.$$ 

Above $M[v_i, w_j]$ means the entry of $M$ indexed by the row associated with $v_i$ and by the column associated with $w_j$. Thus $\tilde{a}R = \tilde{b}$, where

$$b = |M[v_i, w_1], \ldots, M[v_i, w_m]|.$$ 

But this $b$ is exactly $(a, 0)M$. (More precisely, $(a, 0)M = (0, b)$.)

First we look at the action of $\kappa$ on $A \cap B$:

Lemma 8. We characterize $A \cap B$ different ways:

1. A vector is in $A \cap B$ if and only if it can be written both as $\tilde{a}$ and $\tilde{b}$.
2. Every vector in $A \cap B$ is an eigenvector of $\kappa$ with eigenvalue 0. (Or, equivalently, an eigenvector of $\mu$ with eigenvalue 1.)
3. $\tilde{a} = \tilde{b}$ iff $(a, b)M = (a, b)$, i.e. $(a, b)$ is an eigenvector of $M$ with eigenvalue 1.

Proof. 1. is true by definition. For 2. assume that $\tilde{a} = \tilde{b}$. Then $\tilde{b}C = \tilde{a}C = \tilde{a}$ and $\tilde{a}R = \tilde{b}R = \tilde{b}$. Hence

$$2\tilde{a}CR - \tilde{a}C - \tilde{a}R = 2\tilde{a}R - \tilde{a} - \tilde{b} = 2\tilde{b} - \tilde{a} - \tilde{b} = 0,$$

since $\tilde{a} = \tilde{b}$. Next we prove item 3. If $\tilde{a} = \tilde{b}$ then $\tilde{b}C = \tilde{a}C = \tilde{a}$, so by (30) also $(0, b)M = (a, 0)$. Similarly, $\tilde{a}R = \tilde{b}$, so by (31) $(a, 0)M = (0, b)$. Therefore $(a, b)M = (a, b)$. Conversely, if $(a, b) = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$ is an eigenvector of $M$ with eigenvalue 1, then by Lemma 5

$$\tilde{a} = \tilde{b}.$$ 

Next we shall create eigenvectors for $\kappa$ from the eigenvectors for $M$ with eigenvalue less than 1. Let $(a, b), a \in C^{[n]}, b \in C^{[m]}$ be an eigenvector of $M$ with eigenvalue $\lambda \neq 0$, i.e.

$$(a, b)M = (\lambda a, \lambda b).$$

Then Lemma 6 gives:

$$(a, 0)M = (0, \lambda b); \quad (32)$$

$$(0, b)M = (\lambda a, 0). \quad (33)$$

Combining these with (30) and (31) we get:
\[ \tilde{a} R = \lambda \tilde{b}; \quad (34) \]
\[ \tilde{b} C = \lambda \tilde{a}; \quad (35) \]

We would like to find a \( \beta \) such that \( v = \tilde{a} + \beta \tilde{b} \) is an eigenvector of \( \kappa \). We use (34)-(35) to compute \( v\kappa \):

\[ v\kappa = (\tilde{a} + \beta \tilde{b})(2CR - C - R) = \]
\[ 2\tilde{a}CR - \tilde{a}C - \tilde{a}R + 2\beta \tilde{b}CR - \beta \tilde{b}C - \beta \tilde{b}R = \]
\[ 2\lambda \tilde{b} - \tilde{a} - \lambda \tilde{b} + 2\beta \lambda^2 \tilde{b} - \beta \lambda \tilde{a} - \beta \tilde{b} = \]
\[ (-1 - \beta \lambda)\tilde{a} + (\lambda + 2\beta \lambda^2 - \beta)\tilde{b}. \quad (36) \]

We conclude that as long as
\[ \lambda + 2\beta \lambda^2 - \beta = \beta(-1 - \beta \lambda), \quad (40) \]
v is an eigenvector of \( \kappa \) with eigenvalue \(-1 - \beta \lambda \). Let us express \( \beta \) from (40):
\[ \beta^2 \lambda + 2\beta \lambda^2 + \lambda = 0. \quad (41) \]

Solving the equation gives \( \beta = -\lambda \pm \sqrt{\lambda^2 - 1} \). Considering that \( \mu = 2\kappa + I \) we can now write down the eigenvectors and eigenvalues of \( \kappa \) and \( \mu = 2\kappa + I \) that we obtain from eigenvectors and eigenvalues of \( M \). We summarize the formulas in the following two tables. The first table refers to the case of \( \lambda \in (0, 1) \):

| Eigenvector | Eigenvalue |
|-------------|------------|
| \( M \) \( (a, b) \) | \( \lambda \) |
| \( (a, -b) \) | \(-\lambda \) |
| \( \kappa \) \( \tilde{a} - \lambda \tilde{b} + i \sqrt{1 - \lambda^2} \tilde{b} \) | \( \lambda^2 - 1 - i \lambda \sqrt{1 - \lambda^2} \) |
| \( \tilde{a} - \lambda \tilde{b} - i \sqrt{1 - \lambda^2} \tilde{b} \) | \( \lambda^2 - 1 + i \lambda \sqrt{1 - \lambda^2} \) |
| \( \mu \) \( \tilde{a} - \lambda \tilde{b} + i \sqrt{1 - \lambda^2} \tilde{b} \) | \( 2\lambda^2 - 1 - 2i \lambda \sqrt{1 - \lambda^2} \) |
| \( \tilde{a} - \lambda \tilde{b} - i \sqrt{1 - \lambda^2} \tilde{b} \) | \( 2\lambda^2 - 1 + 2i \lambda \sqrt{1 - \lambda^2} \) |

The second table refers to the case when \( \lambda = 0 \). In this case \( (a, b)M = (0, 0) \) implies that \( (a, 0)M = (0, 0) \) and \( (0, b)M = (0, 0) \). Therefore the zero subspace of \( M \) decomposes into the direct sum of the (possibly 0-dimensional) subspaces \{\( (a, 0) \mid (a, 0)M = (0, 0) \}\) and \{\( (0, b) \mid (0, b)M = (0, 0) \}\}. Since Equation (41) in this special case holds with every \( \beta \), we obtain:

| Eigenvector | Eigenvalue |
|-------------|------------|
| \( M \) \( (a, 0) \) | 0 |
| \( (0, b) \) | 0 |
| \( \kappa \) \( \tilde{a} \) | \(-1 \) |
| \( \tilde{b} \) | \(-1 \) |
| \( \mu \) \( \tilde{a} \) | \(-1 \) |
| \( \tilde{b} \) | \(-1 \) |
We are left to show that we have found all \( n + m - \dim(A \cap B) \) orthogonal eigenvectors of the busy subspace. In the above tables we lined up the eigenvectors of \( \kappa \) with the eigenvectors of \( M \) from which they originate to suggest a one-one correspondence. Observe that among the eigenvalues the correspondence is established by

\[
(M\text{-side}) \quad \pm \lambda \quad \longleftrightarrow \quad \lambda^2 - 1 \pm i \lambda \sqrt{1 - \lambda^2} \quad (\kappa\text{-side})
\]

The numbers seem to match, since \( M \) has \( n + m - d_1 - d_{-1} = n + m - 2d_1 \) eigenvectors with eigenvalues in the range \((-1, 1)\). These correspond to the the same number of eigenvectors for \( \kappa \) in the busy subspace with non-zero eigenvalues. In addition, the busy subspace contains \( d_1 \) independent eigenvectors with eigenvalue zero. We get a total of \( n + m - d_1 \) eigenvectors. By \( \dim(A + B) = \dim A + \dim B - \dim(A \cap B) \) and and Lemma 8 the dimension of the busy subspace is also \( n + m - d_1 \). We cannot walk away from the task, however, of showing that no dependencies occur among the eigenvectors we constructed. Since eigen-spaces associated with different eigenvalues are orthogonal, is sufficient to show that

**Lemma 9.** Let \( |\lambda| < 1 \). Then the dimensions of the eigen-spaces of \( \kappa \) associated with eigenvalues \( \lambda^2 - 1 \pm i \lambda \sqrt{1 - \lambda^2} \) are \( d_\lambda \) (each). Also, the dimension of the \(-1\)-eigen-space of \( \kappa \) is \( d_0 \).

**Proof.** Let \( \lambda \in (0, 1) \) and

\[
S_\lambda = \{(a, b) \mid a \in \mathbb{C}^n, b \in \mathbb{C}^m, (a, b)M = \lambda(a, b)\}.
\]

Let \( \tau_+ \) and \( \tau_- \) be the operators from \( S_\lambda \) to \( \mathbb{C}^{[n] \times [m]} \) defined by

\[
\tau_+ : \quad (a, b) \rightarrow \bar{a} - \lambda \bar{b} + i \sqrt{1 - \lambda^2} \bar{b},
\]

\[
\tau_- : \quad (a, b) \rightarrow \bar{a} - \lambda \bar{b} - i \sqrt{1 - \lambda^2} \bar{b}.
\]

We need to show that the images of \( \tau_+ \) and \( \tau_- \) have dimension \( d_\lambda = \dim S_\lambda \). We prove that the kernel of both \( \tau_+ \) and \( \tau_- \) are trivial. We show this only for \( \tau_+ \), since the proof for \( \tau_- \) goes in the same way. Let us assume, contrary to the lemma, that for some \( a \in \mathbb{C}^n \), \( b \in \mathbb{C}^m \), \( (a, b) \in S_\lambda \), \( (a, b) \neq (0, 0) \) we have \( (a, b)\tau_+ = \bar{a} - (\lambda - i \sqrt{1 - \lambda^2}) \bar{b} = 0 \). In fact it is enough to show that the assumption implies \( a = 0 \) or \( b = 0 \), since both imply the other. By Lemma 8, \( (a, (\lambda - i \sqrt{1 - \lambda^2}) b) \) is an eigenvector of \( M \) with eigenvalue 1, which implies \( (a, 0)M = (0, (\lambda - i \sqrt{1 - \lambda^2}) b) \). Since \( (a, b) \in S_\lambda \), we also have \( (a, 0)M = (0, \lambda b) \). Hence

\[
\lambda b = \left(\lambda - i \sqrt{1 - \lambda^2}\right) b,
\]

which, since \( |\lambda| < 1 \) can happen only if \( b = 0 \), a contradiction. Note that the proof works for the \( \lambda = 0 \) case too.

\[\square\]

### 13 Norms and Inner Products

In this section we show how to compute norms of vectors in \( A + B \), and in particular we compute the norms of the eigenvectors we obtained in the previous section. We show that \( \tau_+ \) and \( \tau_- \) are scalar product preserving up to a constant scaling factor, and determine this constant. This gives an alternative proof to Lemma 9.
Lemma 10. For any $a, a' \in C^{[n]}$, $b, b' \in C^{[m]}$ it holds that

$$\langle \tilde{a}, \tilde{a}' \rangle = \langle a, a' \rangle;$$  
$$\langle \tilde{b}, \tilde{b}' \rangle = \langle b, b' \rangle;$$  
$$\langle \tilde{a}, \tilde{b} \rangle = (a, 0)M(0, b)^*.$$  

Furthermore, if $(a, b), (a', b') \in S_{\lambda}$:

$$\langle a, a' \rangle = \langle b, b' \rangle$$  
$$\langle \tilde{a}, \tilde{b}' \rangle = \lambda \langle a, a' \rangle$$  

Proof. Indeed, for $a = (\alpha_i), a' = (\alpha'_i), b = |\beta_i), b' = |\beta'_i)$

$$\langle \tilde{a}, \tilde{a}' \rangle = \sum_{1 \leq i \leq n} \alpha_i \bar{v}_i \bar{v}'_i \alpha_i = \sum_{1 \leq i \leq n} \alpha_i \bar{\alpha}'_i = \langle a, a' \rangle;$$  
$$\langle \tilde{b}, \tilde{b}' \rangle = \sum_{1 \leq j \leq m} \beta_j \bar{w}_j \bar{w}'_j \beta_j = \sum_{1 \leq i \leq n} \beta_j \bar{\beta}'_j = \langle b, b' \rangle;$$  
$$\langle \tilde{a}, \tilde{b} \rangle = \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq m} \alpha_i v_i \bar{w}_j \beta_j \bar{\alpha}'_j = \alpha_i \langle v_i, w_j \rangle \bar{\beta}_j = (a, 0)M(0, b)^*.$$  

Consider now any $(a, b), (a', b') \in S_{\lambda}$. We have $(a, 0)M(0, b)^* = (0, \lambda b)(0, b')^* = \lambda(b', b')$. But also, $(a, 0)M(0, b)^* = (a, 0)((0, b')M^*)^* = (a, 0)((0, b')M)^* = (a, 0)(\lambda a', 0)^* = \lambda(a, a')$. By the virtue of (44) the above shows not only (45) but also (46). \qed

Using Lemma 10

$$\langle \tilde{a} - \lambda \tilde{b} + i \sqrt{1 - \lambda^2} \tilde{b}, \tilde{a}' - \lambda \tilde{b}' + i \sqrt{1 - \lambda^2} \tilde{b}' \rangle = \langle \tilde{a}, \tilde{a}' \rangle + \langle \tilde{b}, \tilde{b}' \rangle - \lambda \langle \tilde{b}, \tilde{a}' \rangle - \lambda \langle \tilde{a}, \tilde{b}' \rangle.$$  

From Lemma 10 $\langle \tilde{b}, \tilde{a}' \rangle = \overline{\langle \tilde{a}', \tilde{b} \rangle} = \lambda \langle \tilde{a}', \tilde{a} \rangle = \lambda \langle \tilde{a}, \tilde{a}' \rangle$. By introducing $\gamma = \langle \tilde{a}, \tilde{a}' \rangle$ we can write Expression (47) as $\gamma(2 - 2\lambda^2)$.

On the other hand

$$\langle (a, b), (a', b') \rangle = \langle a, a' \rangle + \langle b, b' \rangle = 2\gamma.$$  

We conclude that the scaling factor is $1 - \gamma^2$ i.e. for every $v, w \in S_{\lambda}$:

$$\langle \tau_+(v), \tau_+(w) \rangle = (1 - \lambda^2) \langle v, w \rangle.$$  

Similarly we obtain that for every $v, w \in S_{\lambda}$:

$$\langle \tau_-(v), \tau_-(w) \rangle = (1 - \lambda^2) \langle v, w \rangle.$$  

In particular:

Theorem 5. The eigenvectors of $\mu$ in Theorem 7 have the norm

$$\sqrt{1 - \lambda^2}, \text{ if } |(a, b)| = 1 \text{ and } 0 \leq \lambda < 1;$$  
$$1, \text{ if } \tilde{a} \in A \cap B \text{ and } |a| = 1.$$  

25
14 An Example

In this section we give an example to the use of Theorem 1 for a Markov chain associated with Grover’s algorithm. Here we present the “concise version” of the chain which we call the Grover Chain. The Grover chain has two states: marked and unmarked. The “full version,” where different items correspond to different states has similar analysis. (We do not give a precise mathematical justification of the fact that clumping together all marked items and all unmarked items in the way we do gives formulas similar to those coming from the analysis of Algorithm FindMarked(K) for the chain $P = \frac{1}{n} E$, where $E$ is the all one matrix. Our example is interesting on its own right even without this connection.)

Assume that the probability that an item is marked is $p$. The chain corresponds to the classical (non-quantum) algorithm, where at each step we move to a random item, but when we find a marked item we never move away from it. The transition of this chain takes an unmarked item to an unmarked item with probability $1 - p$ and to a marked item with probability $p$. On the other hand marked items alway go into marked items with probability 1. Figure 14 shows the Markov chain and its associated bipartite maps, $c$ and $r$. The pair describing the walk is $(c, r)$, where

$$c = r = \begin{pmatrix} 1 - p & p \\ 0 & 1 \end{pmatrix}.$$ 

The half discriminant matrix of the walk is

$$D = \begin{pmatrix} 1 - p & 0 \\ 0 & 1 \end{pmatrix}.$$ 

With eigenvectors $a_1 = (1, 0)$ and $a_2 = (0, 1)$, and eigenvalues $1 - p$ and 1 respectively. Let $b_1 = (1, 0), b_2 = (0, 1)$. Then the eigenvectors of the discriminant matrix $M = \begin{pmatrix} 0 & D \\ D^T & 0 \end{pmatrix}$ are

$(a_1, b_1)$ with eigenvalue $1 - p$; $(a_1, -b_1)$ with eigenvalue $p - 1$; $(a_2, b_2)$ with eigenvalue 1; $(a_2, -b_2)$ with eigenvalue $-1$. 

Figure 4: The Markov chain associated with Grover’s algorithm, where items are marked with probability $p$
Let $\nu$ be the quantized version of this bipartite chain. By Theorem 11 the busy subspace of $\nu$ has eigenvectors: $\tilde{a}_1 - \tilde{b}_1 + p \tilde{b}_1 \pm i\sqrt{2p - p^2} \tilde{b}_1$ and $\tilde{a}_2$. The latter coincides with $\tilde{b}_2$. Representing these as rectangular vectors:

\[ v_1 = \begin{pmatrix} \sqrt{1 - p} (p + i\sqrt{2p - p^2}) & \sqrt{p} \\ \sqrt{p} (-1 + p + i\sqrt{2p - p^2}) & 0 \end{pmatrix}; \quad v_2 = \begin{pmatrix} \sqrt{1 - p} (p - i\sqrt{2p - p^2}) & \sqrt{p} \\ \sqrt{p} (-1 + p - i\sqrt{2p - p^2}) & 0 \end{pmatrix}; \]

\[ v_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]

One can compute that $|v_1|^2 = |v_2|^2 = 4p - 2p^2$. This is consistent with Theorem 5 since the squared norm of $(a_1, b_1)$ is 2, and Theorem 5 implies that the squared norms of $v_1$ and $v_3$ can be obtained from the norm of $(a_1, b_1)$ by scaling it with the factor $1 - (1 - p)^2$. Theorem 11 gives that the eigenvalues associated with $v_1$, $v_2$ and $v_3$ are $1 - 4p + 2p^2 - 2i(1 - p)\sqrt{2p - p^2}$, $1 - 4p + 2p^2 + 2i(1 - p)\sqrt{2p - p^2}$ and 1. What makes Grover’s algorithm work is that $1 - 4p + 2p^2 \pm 2i(1 - p)\sqrt{2p - p^2} = e^{\pm i\theta}$, where $\theta \in \Theta(\sqrt{p})$. Our version of Grover’s algorithm (Section 7) uses the initial state

\[ u = \begin{pmatrix} 1 - p \\ \sqrt{(1 - p)p} \end{pmatrix}. \]

This is the state we can easily produce (in the query model without cost, in the circuit model with small cost). Notice that $u$ has a large component in the space spanned by $v_1$ and $v_2$. Define

\[ u' = \frac{-i}{\sqrt{2 |v_1|}} v_1 + \frac{i}{\sqrt{2 |v_2|}} v_2 = \begin{pmatrix} \sqrt{1 - p} & 0 \\ 0 & 0 \end{pmatrix}. \]

Then

\[ \langle u , u' \rangle = \sqrt{1 - p}. \]

This again, should not surprise us because of Equation (16). We have

\[ u' u^K = \frac{-ie^{i\theta}}{\sqrt{2 |v_1|}} v_1 + \frac{ie^{-i\theta}}{\sqrt{2 |v_2|}} v_2, \]

and our analysis of Algorithm FindMarked($K$) can go on like in the previous section. The goal of the present article is exactly to show how to shortcut much of the calculations we have made in this section.