Generalized stability of Heisenberg coefficients

LI YING

Abstract. Stembridge introduced the notion of stability for Kronecker triples, which generalizes Murnaghan's classical stability result for Kronecker coefficients. Sam and Snowden proved a conjecture of Stembridge concerning stable Kronecker triples, and they also showed an analogous result for Littlewood–Richardson coefficients. Heisenberg coefficients are Schur structure constants of the Heisenberg product which generalize both Littlewood–Richardson coefficients and Kronecker coefficients. We show that any stable triple for Kronecker coefficients or Littlewood–Richardson coefficients also stabilizes Heisenberg coefficients, and we classify the triples stabilizing Heisenberg coefficients. We also follow Vallejo's idea of using matrix additivity to generate Heisenberg stable triples.

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1. Introduction

We assume familiarity with the basic results in the (complex) representation theory of symmetric groups and symmetric functions (see [6,15]). There are two famous structure constants, Kronecker coefficients and Littlewood–Richardson coefficients, which can be defined in terms of representations of symmetric groups. Given a partition \( \lambda \) of \( n \) (written as \( \lambda \vdash n \), or \( \lambda \) has size \( n \)), let \( V_\lambda \) be the associated irreducible representation of the symmetric group \( S_n \). The Kronecker coefficient \( g^\lambda_{\mu,\nu} \) is the multiplicity of \( V_\lambda \) in the irreducible decomposition of \( \text{Res}^{S_n \times S_n}_{S_n}(V_\mu \otimes V_\nu) \) (viewing \( S_n \) as a subgroup of \( S_n \times S_n \) via the canonical diagonal embedding), the Kronecker product of \( V_\mu \) and \( V_\nu \). That is, \( g^\lambda_{\mu,\nu} = \langle \text{Res}^{S_n \times S_n}_{S_n}(V_\mu \otimes V_\nu), V_\lambda \rangle \), where \( \lambda, \mu, \) and \( \nu \) are partitions of \( n \), and \( \langle , \rangle \) denotes the scalar product that makes the irreducible representations orthonormal. The Littlewood–Richardson product...
coefficient $c_{\mu,\nu}^\lambda$ is the multiplicity of $V_\lambda$ in the irreducible decomposition of $\text{Ind}_{S_n \times S_m}^{S_{n+m}} (V_\mu \otimes V_\nu)$, the *induction product* of $V_\mu$ and $V_\nu$. That is,

$$c_{\mu,\nu}^\lambda = \langle \text{Ind}_{S_n \times S_m}^{S_{n+m}} (V_\mu \otimes V_\nu), V_\lambda \rangle,$$

where $\lambda \vdash n + m$, $\mu \vdash n$, and $\nu \vdash m$ for some positive integers $n$ and $m$.

We view partitions as vectors so we define addition, subtraction, and scalar multiplication on them. While the Littlewood–Richardson coefficient is well-studied and has several beautiful combinatorial interpretations (see [4,6,12]), no explicit combinatorial description for the Kronecker coefficient is known. In 1938 Murnaghan [10] discovered a remarkable stability property for Kronecker coefficients. He stated without proof that for any partitions $\lambda, \mu, \nu$ of the same size, the sequence $\{g_{\mu+\alpha}^{\lambda+(n)}\}$ is eventually constant. There are many proofs with different flavours for this fact, see [2,5,19]. Stembridge [18] vastly generalized this result by introducing the concept of a stable triple.

**Definition 1.1.** A triple $(\alpha, \beta, \gamma)$ of partitions of the same size with $g_{\beta,\gamma}^\alpha > 0$ is a K-triple. It is K-stable if, for any other triple of partitions $(\lambda, \mu, \nu)$ with $|\lambda| = |\mu| = |\nu|$, the sequence $\{g_{\mu+n\beta}^{\lambda+n\alpha} \}$ is eventually constant.

Thus, Murnaghan observed that $((1),(1),(1))$ is K-stable. Stembridge conjectured a characterization for K-stability, and he proved its necessity. Sam and Snowden [16] proved the sufficiency.

**Proposition 1.2.** A K-triple $(\alpha, \beta, \gamma)$ is K-stable if and only if $g_{n\beta,n\gamma}^{n\alpha} = 1$ for all $n > 0$.

Sam and Snowden [16] also proved an analogous result for Littlewood–Richardson coefficients, which can also be deduced from some earlier work (see [16, Remark 4.7]).

**Definition 1.3.** A triple $(\alpha, \beta, \gamma)$ of partitions with $|\alpha| = |\beta| + |\gamma|$ and $c_{\beta,\gamma}^\alpha > 0$ is an LR-triple. It is LR-stable if, for any other triple of partitions $(\lambda, \mu, \nu)$ with $|\lambda| = |\mu| + |\nu|$, the sequence $\{c_{\mu+n\beta,\nu+n\gamma}^{\lambda+n\alpha} \}$ is eventually constant.

**Proposition 1.4.** ([16] Theorem 4.6) The following are equivalent for an LR-triple $(\alpha, \beta, \gamma)$:

(a) $(\alpha, \beta, \gamma)$ is LR-stable;
(b) $c_{\beta,\gamma}^\alpha = 1$;
(c) $c_{n\beta,n\gamma}^{n\alpha} = 1$ for all $n > 0$.

**Remark 1.5.** Sam and Snowden [16] did not require $c_{\beta,\gamma}^\alpha > 0$, which should be added. For example, when $\beta$ is not contained in $\alpha$, we have that $c_{\beta,\gamma}^\alpha = 0$ and $\{c_{\mu+n\beta,\nu+n\gamma}^{\lambda+n\alpha} \}$ is eventually zero.
Aguiar, Ferrer Santos, and Moreira introduced a new (commutative and associative) product, the Heisenberg product (denoted #), on representations of symmetric groups in [1,9]. This product interpolates between the induction product and the Kronecker product (see Definition 2.1 for details), its structure constants are the Heisenberg coefficients, which are a common generalization of the Littlewood–Richardson coefficient and the Kronecker coefficient. We show that K-stable triples and LR-stable triples also stabilize Heisenberg coefficients, and we characterize the triples which do this.

Manivel [7] and Vallejo [21] independently generated K-stable triples using additive matrices. Manivel also showed that the set of stable triples is the intersection of the Kronecker semigroup with a union of faces of the Kronecker cone [8, Proposition 2]. Later, Pelletier [14] produced particular faces of the Kronecker cone containing only stable triples, which generalized Manivel and Vallejo’s work. We will use the idea of additive matrices to generate triples of partitions which stabilize the Heisenberg coefficient.

This paper is organized as follows. In the second section, we give the definition of the Heisenberg product and some related results, and define the H-stable triple for Heisenberg coefficients. Section 3 shows that K-stable triples and LR-stable triples are H-stable, and gives a necessary and sufficient condition for a triple to be H-stable. In Sect. 4, we define H-additive matrices which generalize additive matrices, and prove some results on H-additive matrices which are analogous to those on additive matrices. In the last section, we prove that each H-additive matrix gives an H-stable triple.

2. Heisenberg product

**Definition 2.1.** Let $V$ and $W$ be representations of $S_n$ and $S_m$ respectively. Fix an integer $l$ (weakly) between $\max\{m, n\}$ and $m + n$, and let $p = l - m$, $q = n + m - l$, and $r = l - n$. The following diagram of inclusions (solid arrows) commutes:

\[
\begin{array}{c}
S_p \times S_q \times S_q \times S_r & \longrightarrow & S_{p+q} \times S_{q+r} = S_n \times S_m \\
\text{Res} & & \\
\text{Ind} & & \\
S_p \times S_q \times S_r & \longrightarrow & S_{p+q+r} = S_l \\
\end{array}
\]

where $\Delta_{S_q} : S_q \hookrightarrow S_q \times S_q$ is the canonical diagonal embedding. The **Heisenberg product** (denoted \#) of $V$ and $W$ is

\[
V \# W = \bigoplus_{l = \max\{n, m\}}^{n+m} (V \# W)_l,
\]
where the degree $l$ component is defined using the dashed arrows in (2.1):

\[(V \# W)_l = \text{Ind}_{S_p \times S_q \times S_r}^{S_{n+m} \times S_{m+n} \times S_m} (V \otimes W). \tag{2.3}\]

The Heisenberg product connects the induction product and the Kronecker product. When $l = m + n$, $(V \# W)_l$ is the induction product $\text{Ind}_{S_p \times S_q \times S_r}^{S_{n+m} \times S_{m+n} \times S_m} (V \otimes W)$. When $l = n = m$, $(V \# W)_l$ is the Kronecker product $\text{Res}_{S_p \times S_q \times S_r}^{S_{n+m} \times S_{m+n} \times S_m} (V \otimes W)$.

Remarkably, this product is associative [1, Theorem 2.3, Theorem 2.4, Theorem 2.6]. The Heisenberg coefficient $h^\lambda_{\mu, \nu}$ is the multiplicity of $V_\lambda$ in $V_\mu \# V_\nu$,

\[h^\lambda_{\mu, \nu} = \langle V_\mu \# V_\nu, V_\lambda \rangle. \tag{2.4}\]

The Heisenberg coefficient generalizes the Littlewood-Richardson coefficient and the Kronecker coefficient. In [22], we gave a formula for the Heisenberg coefficient.

**Proposition 2.2.** Given partitions $\lambda \vdash l$, $\mu \vdash m$, and $\nu \vdash n$,

\[h^\lambda_{\mu, \nu} = \sum_{\alpha \vdash p, \beta \vdash r, \tau \vdash q} c^\mu_{\alpha, \beta} c^\nu_{\eta, \rho} g^\delta_{\beta, \eta} c^\tau_{\alpha, \delta} c^\lambda_{\tau, \rho} \tag{2.4}\]

where $\max\{m, n\} \leq l \leq m + n$, $p = l - n$, $q = m + n - l$, and $r = l - m$.

**Remark 2.3.** The Heisenberg product can be described in terms of the Kronecker product of Thibon characters [3, Section 4.4]. The calculation of the Heisenberg coefficient (2.4) can be obtained from [19, Theorem 2.1] and [3, Theorem 4.9].

In [22], we showed that Heisenberg coefficients stabilize in low degrees, which is analogous to Murnaghan’s stability result. It is also worth trying to generalize stability for Heisenberg coefficients.

**Definition 2.4.** A triple $(\alpha, \beta, \gamma)$ of partitions with $\max\{ |\beta|, |\gamma| \} \leq |\alpha| \leq |\beta| + |\gamma|$ and $h^\alpha_{\beta, \gamma} > 0$ is an H-triple. It is H-stable if, for any other triple of partitions $(\lambda, \mu, \nu)$ with $\max\{ |\mu|, |\nu| \} \leq |\lambda| \leq |\mu| + |\nu|$, the sequence $\{ h^\lambda_{\mu+n\beta, \nu+n\gamma} \}$ is eventually constant.

In [22, Theorem 2.5], we proved the following result.

**Proposition 2.5.** $((1), (1), (1))$ is an H-stable triple

3. **H-stable triples**

We show that K-stable triples and LR-stable triples are H-stable. As in [22], we begin with a stability result for Littlewood-Richardson coefficients.

**Lemma 3.1.** Given partitions $\lambda$, $\mu$, $\nu$, $\alpha$, and a positive integer $n \geq |\mu|$ with $|\lambda + n\alpha| = |\mu| + |\nu|$ and $\nu \subset \lambda + n\alpha$, 

(1) there exists a partition \( \nu^0 \) such that \( \nu^0 = \nu - (n - |\mu|)\alpha \);
(2) \( c_{\mu,\nu}^{\lambda+m\alpha} = c_{\mu,\nu+(m-n)\alpha}^{\lambda} \) for all \( m \geq n \).

**Proof.** For part (1), since for any two partitions, the conjugate of their sum is equal to the union of their conjugates, we consider \( \lambda' = (1^{a_1}2^{a_2} \ldots) \) and \( \alpha' = (1^{b_1}2^{b_2} \ldots) \) in exponential notations, then \( (\lambda + n\alpha)' = (1^{a_1+nb_1}2^{a_2+nb_2} \ldots) \).

Since \( \nu' \subseteq (\lambda + n\alpha)' \), the diagram of \( \nu' \) can be obtained from the diagram of \( (\lambda + n\alpha)' \) by removing \( |(\lambda + n\alpha)'| - |\nu'| = |\mu| \leq n \) boxes. For each \( i \) with \( b_i > 0 \), there are \( a_i + nb_i \geq n \) columns in the diagram \( (\lambda + n\alpha)' \) which each contains exactly \( i \) boxes. Among those columns, only in the rightmost columns might boxes be removed and suppose that in \( m_i < |\mu| \) columns there are some boxes removed. So in the leftmost \( a_i + nb_i - m_i \geq (n - |\mu|)b_i \) columns there are no boxes removed, hence, they are columns of the diagram of \( \nu' \). So \( \nu' \) is the union of \((n - |\mu|)\alpha\)' and some other partition, call it \( \nu^0' \). Then \( \nu^0 = \nu - (n - |\mu|)\alpha \) is a partition (contained in \( \lambda + |\mu|\alpha \)).

For part (2), the connected components of the skew shapes \( (\lambda + n\alpha)/\nu \) and \( (\lambda + |\mu|\alpha)/\nu^0 \) are the same except for some horizontal shifts, hence, by the Littlewood–Richardson rule, \( c_{\mu,\nu}^{\lambda+n\alpha} = c_{\mu,\nu^0}^{\lambda+|\mu|\alpha} \). Following the same arguments, we have \( c_{\mu,\nu+(m-n)\alpha}^{\lambda+m\alpha} = c_{\mu,\nu^0}^{\lambda+|\mu|\alpha} = c_{\mu,\nu}^{\lambda+n\alpha} \) for all \( m \geq n \). \( \square \)

Similarly, we have the following result.

**Lemma 3.2.** Given partitions \( \lambda, \mu, \nu, \alpha \), and a positive integer \( n \geq |\mu| \), with \( |\lambda| = |\mu| + |\nu + n\alpha| \) and \( \nu + n\alpha \subseteq \lambda \),
(1) \( \lambda - (n - |\mu|)\alpha \) is a partition;
(2) \( c_{\mu,\nu+n\alpha}^{\lambda} = c_{\mu,\nu+(m-n)\alpha}^{\lambda+\alpha} \) for all \( m \geq n \).

**Remark 3.3.** The fact that \( c_{\mu,\nu+(m-n)\alpha}^{\lambda+\alpha} \) (resp. \( c_{\mu,\nu+n\alpha}^{\lambda+\alpha} \)) in Lemma 3.1(2) (resp. Lemma 3.2(2)) is a constant when \( m \) is large also follows from Proposition 1.4 as \((\alpha,(0),\alpha)\) is LR-stable.

The next theorem generalizes a result in [22].

**Theorem 3.4.** A \( K \)-stable triple is \( H \)-stable.

Let \((\alpha,\beta,\gamma)\) with \( \alpha, \beta, \gamma \vdash s > 0 \) be \( K \)-stable. Suppose \( \lambda, \mu, \) and \( \nu \) are partitions with \( \lambda \vdash p, \mu \vdash q, \) and \( \nu \vdash r \) and \( \max\{q,r\} \leq p \leq q + r \). Theorem 3.4 states that the sequence \( \{h_{\mu+n\beta,\nu+n\gamma}^{\lambda+n\alpha}\} \) is eventually constant. According to Proposition 2.2, we have

\[
h_{\mu+n\beta,\nu+n\gamma}^{\lambda+n\alpha} = \sum_{K_n} c_{\xi,\theta}^{\mu+n\beta} c_{\eta,\rho}^{\nu+n\gamma} g_{\theta,\eta}^{\delta} c_{\xi,\delta}^{\tau} c_{\tau,\rho}^{\lambda+n\alpha},
\]

where \( K_n = \{((\xi, \theta, \eta, \rho, \delta, \tau) | \theta, \eta, \delta \vdash (q + r - p) + ns, \xi \vdash p - r, \\
\rho \vdash p - q, \tau \vdash q + ns\} \).
Define $f_n : K_n \rightarrow \mathbb{Z}_{\geq 0}$ as
\[
f_n(\xi, \theta, \eta, \rho, \delta, \tau) = \xi^\mu \eta^\nu \rho^\lambda \delta^\gamma \tau^\alpha \text{ (the summands in (3.1)).}
\]
(3.2)

Some terms in the sum of (3.1) vanish. Let us consider the nonvanishing terms only. Let $K^0_n = K_n \setminus f_n^{-1}(0)$. To prove Theorem 3.4, it is enough to prove
\[
\sum_{u \in K^0_n} f_n(u) = \sum_{u \in K^0_{n+1}} f_{n+1}(u)
\]
(3.3)
for $n$ sufficiently large.

We have a natural embedding $\varphi_n : K_n \hookrightarrow K_{n+1}$,
\[
\varphi_n(\xi, \theta, \eta, \rho, \delta, \tau) = (\xi, \theta + \beta, \eta + \gamma, \rho, \delta + \alpha, \tau + \alpha).
\]
We show that when $n$ is large, $\varphi_n$ induces a bijection between $K^0_n$ and $K^0_{n+1}$ with $f_n = f_{n+1} \circ \varphi_n$. From the definition of K-stability, we know that there exists a positive integer $N$, such that for all $n \geq N$, we have
\[
g_{\xi+n\beta, \eta+n\gamma}^{\epsilon+n\alpha} = g_{\xi+(n+1)\beta, \eta+(n+1)\gamma}^{\epsilon+(n+1)\alpha},
\]
(3.4)
for all $\epsilon, \xi, \eta, \rho, \delta, \tau$.

**Lemma 3.5.** When $n \geq N + 2p - q - r$, $\varphi_n|_{K^0_n} : K^0_n \rightarrow K^0_{n+1}$ is a well-defined bijection. Moreover, $f_n|_{K^0_n} = f_{n+1} \circ \varphi_n|_{K^0_n}$.

**Proof.** We just describe the main idea of the proof, and leave the details to the readers. The partitions $\xi$ and $\rho$ have fixed sizes (independent of $n$) and as a consequence, Lemmas 3.1 and 3.2 imply that the sum on the left hand side of (3.3) (also the right hand side) can be reduced into a sum where the partitions have the special form, $\theta = \theta_0 + (n-a)\beta$, $\eta = \eta_0 + (n-b)\gamma$, $\tau = \tau_0 + (n-c)\alpha$, $\delta = \delta_0 + (n-d)\alpha$, for partitions $\theta_0, \eta_0, \tau_0, \delta_0$ of fixed sizes (the other terms having a 0 contribution in the summation). The sum writes then as a sum over a finite set of tuples of partitions $(\xi, \theta_0, \eta_0, \rho_0, \delta_0, \tau_0)$, where each summand is a product of coefficients that stabilize.

More precisely, $n \geq N + 2p - q - r$ is large enough so we can apply Lemmas 3.1 and 3.2 and show that $\theta - (n-p+r)\beta, \eta - (n-p+q)\gamma, \tau - (n-p+q)\alpha, \delta - (n-2p+q+r)\alpha$ are partitions. Then we can write $\theta = \theta_0 + (n-2p+q+r)\beta, \eta = \eta_0 + (n-2p+q+r)\gamma, \delta = \delta_0 + (n-2p+q+r)\alpha$.

From (3.4), we have that
\[
g_{\eta,\rho}^\delta = g_{\eta+\beta,\rho+\gamma}^{\delta+\alpha}.
\]

**Lemma 3.1** and **3.2** also tell us
\[
c_{\xi,\theta}^{\mu+n\beta} = c_{\xi,\theta+\beta}^{\mu+(n+1)\beta}, \quad c_{\eta,\rho}^{\nu+n\gamma} = c_{\eta,\rho+\gamma}^{\nu+(n+1)\gamma}, \quad c_{\tau,\rho}^{\lambda+n\alpha} = c_{\tau,\rho+\alpha}^{\lambda+(n+1)\alpha}, \quad c_{\xi,\delta}^{\tau+\alpha} = c_{\xi,\delta+\alpha}^{\tau+\alpha}.
\]
Using the above results one can see $\varphi_n|_{K^0_n}$ is a bijection and $f_n|_{K^0_n} = f_{n+1} \circ \varphi_n|_{K^0_n}$. □
**Proof of Theorem 3.4.** Applying Lemma 3.5, we prove (3.3), hence Theorem 3.4.

Theorem 3.4 shows that some Heisenberg coefficients in low degree components stabilize. Our next result gives a stability result for the relatively high degree components.

**Theorem 3.6.** LR-stable triples are H-stable.

The idea of the proof is the same (without using the stability of Kronecker coefficients) as the proof of Theorem 3.4. Given an LR-stable triple \((\alpha, \beta, \gamma)\) with \(\alpha \vdash a + b\), \(\beta \vdash a\), and \(\gamma \vdash b\), and partitions \(\lambda \vdash p\), \(\mu \vdash q\), and \(\nu \vdash r\) with \(\max\{q, r\} \leq p \leq q + r\), we define \(f'_n : LR_n \mapsto \mathbb{Z}_{\geq 0}\) as

\[
f'_n(\xi, \theta, \eta, \rho, \delta, \tau) = c^{\mu+n\beta}_{\xi,\theta} c^{\nu+n\gamma}_{\eta,\rho} g^\delta_{\theta,\eta} c^\tau_{\xi,\delta} c^{\lambda+n\alpha}_{\tau,\rho},
\]

where

\[
LR_n = \{(\xi, \theta, \eta, \rho, \delta, \tau) \mid \theta, \eta, \delta \vdash (q + r - p), \\
\xi \vdash p - r + na, \ \rho \vdash p - q + nb, \ \tau \vdash q + na\}.
\]

Applying Proposition 2.2, Theorem 3.6 states that

\[
\sum_{u \in LR^0_n} f'_n(u) = \sum_{u \in LR^0_{n+1}} f'_n(u)
\]

for all large \(n\), where \(LR^0_n = LR_n \setminus f'^{-1}_n(0)\).

**Proof of Theorem 3.6.** Consider the map \(\phi_n : LR_n \hookrightarrow LR_{n+1}\),

\[
\phi_n(\xi, \theta, \eta, \rho, \delta, \tau) = (\xi + \beta, \theta, \eta + \gamma, \delta, \tau + \beta).
\]

Using Lemmas 3.1, 3.2, and the same idea as in the proof of Lemma 3.5, we get \(f'_n = f'_{n+1} \circ \phi_n\) on \(LR^0_n\) when \(n\) is large, and it is not hard to see that \(\phi_n\) is a bijection between \(LR^0_n\) and \(LR^0_{n+1}\). So (3.6) is true, and hence we have proved the theorem.

Propositions 1.2 and 1.4 (3) have a similar form. A natural question is whether the necessary and sufficient condition for being an H-stable triple has the same form. The answer is yes, and Pelletier [13, Theorem 3.6] used (2.4) and proved the following

**Proposition 3.7.** An H-triple \((\alpha, \beta, \gamma)\) is H-stable if \(h^{\alpha}_{n\beta, n\gamma} = 1\) for all \(n > 0\).

**Remark 3.8.** By Propositions 1.2 and 1.4, Proposition 3.7 also shows that K-stable triples and LR-stable triples are H-stable, but our proofs of Theorem 3.4 and Theorem 3.6 are direct, without reference to the nontrivial Proposition 1.2.

We prove the reverse direction and complete the characterization of H-stability using the monotonicity of Heisenberg coefficients. This is deduced from the monotonicity of Littlewood–Richardson coefficients and Kronecker coefficients. We start with the monotonicity of Kronecker coefficients. Stembridge [19] proved the following for Kronecker coefficients.
Proposition 3.9. Let $(\alpha, \beta, \gamma)$ be a K-triple. Then

(1) the sequence $\{g_{\lambda + n\alpha, \mu + n\beta, \nu + n\gamma}\}$ is weakly increasing for any partitions $\lambda$, $\mu$, and $\nu$ with the same size;

(2) if $g_{\beta, \gamma}^{\alpha} \geq 2$, then $g_{n\beta, n\gamma}^{\alpha} \geq n + 1$.

Using the hive model of Littlewood–Richardson coefficients (see [4,12]), we prove an analogous result for Littlewood–Richardson coefficients.

Proposition 3.10. Let $(\alpha, \beta, \gamma)$ be an LR-triple. Then

(1) the sequence $\{c_{\lambda + n\alpha, \mu + n\beta, \nu + n\gamma}\}$ is weakly increasing for any partitions $\lambda$, $\mu$, and $\nu$ with $|\lambda| = |\mu| + |\nu|$;

(2) if $c_{\beta, \gamma}^{\alpha} \geq 2$, then $c_{n\beta, n\gamma}^{\alpha} \geq n + 1$.

Proof. We follow the notation used for hives in [12, Section 4]. Let $k$ be a positive integer larger than the lengths of $\lambda$, $\mu$, $\nu$, $\alpha$, $\beta$, and $\gamma$. We define (coordinatewise) addition and scalar multiplication on hives (as for vectors and matrices).

For (1), it suffices to show $c_{\lambda}^{\mu, \nu} \leq c_{\lambda + n\alpha}^{\mu + n\beta, \nu + n\gamma}$. Since $c_{\beta, \gamma}^{\alpha} \geq 1$, there exists a hive $\Delta \in H_k(\alpha, \beta, \gamma)$. Then the map $\iota : H_k(\lambda, \mu, \nu) \rightarrow H_k(\lambda + n\alpha, \mu + n\beta, \nu + n\gamma)$

$$\iota(\Theta) = \Theta + \Delta$$

where $\Theta \in H_k(\lambda, \mu, \nu)$, gives a well-defined injection. So (1) is proved.

For (2), we have two different hives $\Delta_1$ and $\Delta_2$ in $H_k(\alpha, \beta, \gamma)$ as $c_{\beta, \gamma}^{\alpha} \geq 2$. Then $i\Delta_1 + (n - i)\Delta_2$ ($0 \leq i \leq n$) gives $n + 1$ different hives in $H_k(n\alpha, n\beta, n\gamma)$, so $c_{n\beta, n\gamma}^{\alpha} \geq n + 1$.

Propositions 2.2, 3.9, and 3.10 together imply the following

Proposition 3.11. Let $(\alpha, \beta, \gamma)$ be a H-triple. Then

(1) the sequence $\{h_{\lambda + n\alpha, \mu + n\beta, \nu + n\gamma}\}$ is weakly increasing for any partitions $\lambda$, $\mu$, and $\nu$ with $\max\{|\mu|, |\nu|\} \leq |\lambda| \leq |\mu| + |\nu|$;

(2) if $h_{\beta, \gamma}^{\alpha} \geq 2$, then $h_{n\beta, n\gamma}^{\alpha} \geq n + 1$.

Proof. For (1), it is enough to show $h_{\lambda}^{\mu, \nu} \leq h_{\lambda + n\alpha}^{\mu + n\beta, \nu + n\gamma}$. Since $h_{\beta, \gamma}^{\alpha} > 0$, by formula (2.4), there exists a sextuple $(\xi, \theta, \eta, \rho, \delta, \tau)$ of partitions with appropriate sizes such that $c_{\beta, \gamma}^{\delta} c_{\eta, \rho}^{\theta} g_{\delta, \eta}^{\theta} c_{\xi, \delta}^{\alpha} c_{\tau, \rho}^{\alpha} > 0$. The triples appearing in the coefficients on the left hand side are LR-triples or K-triples. As in the proofs of Theorems 3.4 and 3.6, applying Proposition 2.2, we write

$$h_{\lambda}^{\mu, \nu} = \sum_{u \in \Lambda} f_u \quad \text{and} \quad h_{\lambda + n\alpha}^{\mu + n\beta, \nu + n\gamma} = \sum_{u' \in \Lambda'} f'_{u'},$$

where $\Lambda$ and $\Lambda'$ are sets of sextuples of partitions with appropriate sizes, $f_u$ and $f'_{u'}$ are the summands given by the sextuples $u$ and $u'$. We view the sextuples as vectors whose coordinates are partitions, so we may define addition and
scalar multiplication on them. The map $u \rightarrow u + (\xi, \theta, \eta, \rho, \delta, \tau) =: u'$ embeds $\Lambda$ into $\Lambda'$. From Proposition 3.9 and Proposition 3.10, we know that $f_u \leq f_{u'}$, so (1) is proved.

For (2), if $h_{\beta,\gamma}^\delta \geq 2$, then there are two possibilities.

Case 1. There exists a sextuple $(\xi, \theta, \eta, \rho, \delta, \tau)$ of partitions with appropriate sizes such that $c_{\xi,\theta}^{\beta,\rho} c_{\eta,\eta}^{\gamma} g_{\theta,\theta}^{\delta} c_{\xi,\delta}^{\epsilon} c_{\tau,\rho}^{\sigma} \geq 2$. So all the five coefficients on the left hand side are positive and at least one of them is at least 2. From Propositions 3.9 and 3.10, we have $h_{n\beta,n\gamma}^{n\alpha} \geq n + 1$.

Case 2. Two distinct sextuples $u = (\xi, \theta, \eta, \rho, \delta, \tau)$ and $u' = (\xi', \theta', \eta', \rho', \delta', \tau')$ give positive summands for $h_{\beta,\gamma}^\delta$. Then $iu + (n - i)u' (0 \leq i \leq n)$ gives $n + 1$ different sextuples, and due to Propositions 3.9 and 3.10, they all give positive summands for $h_{n\beta,n\gamma}^{n\alpha}$, so $h_{n\beta,n\gamma}^{n\alpha} \geq n + 1$. Hence, we have proved the proposition.

Combining Propositions 3.7 and 3.11, we arrive at the main theorem of this paper.

\textbf{Theorem 3.12.} An $H$-triple $(\alpha, \beta, \gamma)$ is $H$-stable if and only if $h_{n\beta,n\gamma}^{n\alpha} = 1$ for all $n > 0$.

4. Additive Matrices

Manivel [7] and Vallejo [21] used additive matrices to produce examples of $K$-stable triples. We first recall some definitions and results concerning additive matrices, then we give an analogous result for $H$-stable triples.

For a positive integer $n$, let $[n] := \{1, 2, \ldots, n\}$. Given a matrix $A$, we arrange its entries in weakly decreasing order. The resulting sequence is called the $\pi$-sequence of $A$, and denoted by $\pi(A)$. Let $M(\alpha, \beta)$ denote the set of matrices with nonnegative integer entries, row-sum vector $\alpha$ and column-sum vector $\beta$, and $M(\alpha, \beta)_\gamma$ be the subset of $M(\alpha, \beta)$ whose elements have $\pi$-sequence $\gamma$.

\textbf{Definition 4.1.} A $p \times q$ matrix $A = (a_{i,j})$ with nonnegative integer entries is called $K$-additive if there exist real numbers $x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q$, such that

$$a_{i,j} > a_{k,l} \implies x_i + y_j > x_k + y_l$$

for all $i, k \in [p]$ and $j, l \in [q]$.

\textbf{Proposition 4.2.} ([21] Theorem 1.1) Let $\alpha$, $\beta$, and $\gamma$ be partitions of the same size. If there is a matrix $A \in M(\beta, \gamma)$ which is $K$-additive, then $(\alpha, \beta, \gamma)$ is $K$-stable.
Moreover, Manivel [7, Section 5.3] showed that each K-additive matrix defines a regular face of the corresponding Kronecker polyhedron of minimal dimension.

For any (weak) composition \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r) \) of \( n \) (written as \( \alpha \vdash n \)), let \( h_{\alpha} \) be the induced representation from the trivial representation of the Young subgroup \( S_\alpha = S_{\alpha_1} \times S_{\alpha_2} \times \cdots \times S_{\alpha_r} \) to \( S_n \). For partitions \( \lambda \) and \( \mu \) with the same size, the multiplicity of \( V_\lambda \) in the irreducible decomposition of \( h_\mu \) is \( K_{\lambda, \mu} \), the Kostka number. This counts the number of semistandard Young tableaux of shape \( \lambda \) and content \( \mu \). It is well-known that \( K_{\lambda, \mu} > 0 \) if and only if \( \lambda \succeq \mu \), where \( \succeq \) is the dominance order on partitions. In particular, \( K_{\lambda, \lambda} = 1 \). So

\[
\begin{equation}
\label{eq:4.1}
\begin{split}
    h_{\alpha} &= \bigoplus_{\lambda \succeq \mu} K_{\lambda, \mu} V_\lambda = V_\mu \oplus \left( \bigoplus_{\lambda \succ \mu} K_{\lambda, \mu} V_\lambda \right).
\end{split}
\end{equation}
\]

For a (weak) composition \( \alpha \) of \( |\lambda| \), \( h_\alpha \) is isomorphic to \( h_{\pi(\alpha)} \) as representations of \( S_{|\lambda|} \), so we define \( K_{\lambda, \alpha} = K_{\lambda, \pi(\alpha)} \). One of the most important steps in Vallejo’s proof of Proposition 4.2 is the following formula which computes the Kronecker product of two \( h \)'s, see [17, Exercise 7.84b].

**Proposition 4.3.** Let \( \beta \) and \( \gamma \) be (weak) compositions of \( n \), then the Kronecker product of \( h_\beta \) and \( h_\gamma \) is

\[
\begin{equation}
\label{eq:4.1}
\begin{split}
    \text{Res}_{S_n \times S_n} (h_\beta \otimes h_\gamma) &= \bigoplus_{A \in M(\beta, \gamma)} h_{\pi(A)} = \bigoplus_{\alpha \vdash n} \bigoplus_{A \in M(\beta, \gamma) \alpha} h_\alpha.
\end{split}
\end{equation}
\]

Aguiar et al. [1] provided a similar formula for the Heisenberg product. To express the formula, we introduce some notation. Given three finite sequences of real numbers \( \alpha, \beta, \gamma \), let \( F(\alpha, \beta) \) be the set of matrices with real entries, zero at the top left corner, row-sum vector (ignoring the first row) \( \alpha \) and column-sum vector (ignoring the first column) \( \beta \). We denote by \( H(\alpha, \beta, \gamma) \) the subset of \( H(\alpha, \beta) \) whose elements have \( \pi \)-sequence \( \gamma \).

**Example 4.4.** The following matrix is in \( H(18,10,12,18,3) \)

\[
\begin{pmatrix}
0 & 4 & 6 & 1 \\
4 & 5 & 7 & 2 \\
2 & 3 & 5 & 0
\end{pmatrix}
\]

**Proposition 4.5.** ([1] Theorem 3.1) Let \( \beta \) and \( \gamma \) be two (weak) compositions, then the Heisenberg product of \( h_\beta \) and \( h_\gamma \) is

\[
\begin{equation}
\label{eq:4.1}
\begin{split}
    h_\beta \# h_\gamma &= \bigoplus_{A \in H(\beta, \gamma)} h_{\pi(A)}.
\end{split}
\end{equation}
\]

We introduce \( H \)-additive matrices and use Proposition 4.5 to show that each \( H \)-additive matrix gives an \( H \)-stable triple.
Definition 4.6. A \((p + 1) \times (q + 1)\) matrix \(A = (a_{i,j})\) with nonnegative integer entries and \(a_{1,1} = 0\) is called \(H\)-additive if there exist real numbers \(x_1 = 0, x_2, \ldots, x_{p+1}, y_1 = 0, y_2, \ldots, y_{q+1}\), such that
\[
a_{i,j} > a_{k,l} \implies x_i + y_j > x_k + y_l
\]
for all \((i, j), (k, l) \in [p + 1] \times [q + 1] \setminus \{(1, 1)\}.

With this definition, the matrix in Example 4.4 is \(H\)-additive (consider setting \(x_0 = y_0 = 0, x_1 = 1, x_2 = -1, y_1 = 1, y_2 = 3, y_3 = -2\)).

Theorem 4.7. Let \(\alpha, \beta, \text{ and } \gamma\) be partitions with \(\max\{|\beta|, |\gamma|\} \leq |\alpha| \leq |\beta| + |\gamma|\). If there is a matrix \(A \in \mathcal{H}(\beta, \gamma)_\alpha\) which is \(H\)-additive, then \((\alpha, \beta, \gamma)\) is \(H\)-stable.

Remark 4.8. Theorem 4.7 is equivalent to Proposition 4.2 if \(|\alpha| = |\beta| = |\gamma|\). The only LR-stable triples it can produce are of the form \((\beta \cup \gamma, \beta, \gamma)\) where \(\beta \cup \gamma\) is the partition whose parts are those of \(\beta\) and \(\gamma\), arranged in decreasing order. It is not hard to see \(c_{\beta \cup \gamma} = 1\).

Omn and Vallejo’s proof [11,21] for Proposition 4.2 can be applied to prove Theorem 4.7 with some small changes. We first recall some basic notions, many introduced in [11,21]. We move away from integers for a while and work with real numbers. For a vector \(a = (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m\), we denote by \(\pi(a)\) the vector formed by the entries of \(a\) arranged in weakly decreasing order. We say that \(a\) is dominated by \(b\) (both are vectors in \(\mathbb{R}^m\)), written as \(a \preceq b\), if
\[
\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_i \quad \text{and} \quad \sum_{i=1}^{k} \pi(a)_i \leq \sum_{i=1}^{k} \pi(b)_i, \quad \text{for all } k \in [m].
\]
If \(a \preceq b\) and \(\pi(a) \neq \pi(b)\), then we write \(a \prec b\). In particular, when \(a\) and \(b\) are partitions of some integer \(n\), then \(\preceq\) coincides with the dominance order for partitions. A matrix \(A \in \mathcal{F}(\alpha, \beta)\) is real-minimal if there is no other matrix \(B \in \mathcal{F}(\alpha, \beta)\) such that \(\pi(B) \prec \pi(A)\). Omn and Vallejo’s [11] results can be rephrased and generalized as follows.

Let \(E\) be an affine subspace of \(\mathbb{R}^m\). Define \(a \in \mathbb{R}^m\) to be real-minimal in \(E\) if \(E\) does not contain any \(b\) such that \(\pi(b) \prec \pi(a)\). Let \(L : \mathbb{R}^m \rightarrow \mathbb{R}^k\) be a linear map such that \(E\) is a level set of \(L\) (i.e. \(E = L^{-1}(v)\) for some \(v \in \mathbb{R}^k\)). Then \(a\) is real-minimal in \(E\) if and only if there exists \(z \in \mathbb{R}^k\) such that, for all \(i, j\) with \(a_i > a_j\), we have \(\langle z, L(e_i) \rangle_{\mathbb{R}^k} > \langle z, L(e_j) \rangle_{\mathbb{R}^k}\), where \(\langle \cdot, \cdot \rangle_{\mathbb{R}^k}\) is the standard scalar product on \(\mathbb{R}^k\) and \(e_i\) are the vectors of the canonical basis of \(\mathbb{R}^m\).

Suppose \(\alpha\) and \(\beta\) are two finite sequences of real numbers whose lengths are \(p\) and \(q\), respectively. We consider the linear map \(\Phi : \mathcal{F}(\alpha, \beta) \rightarrow \mathbb{R}^{pq+p+q}\),
\[
\Phi(A) = (a_{1,2}, a_{1,3}, \ldots, a_{1,q+1}, a_{2,1}, a_{2,2}, \ldots, a_{2,q+1}, \ldots, a_{p+1,1}, a_{p+1,2}, a_{p+1,q+1}),
\]
where \(A = (a_{i,j}) \in \mathcal{F}(\alpha, \beta)\). Let \(E := \Phi(\mathcal{F}(\alpha, \beta))\) be an affine subspace of \(\mathbb{R}^{pq+p+q}\), and \(L : \mathbb{R}^{pq+p+q} \rightarrow \mathbb{R}^{p+q}\) be the linear map which sends a matrix.
(we view a vector in \( \mathbb{R}^{pq+p+q} \) as a \((p+1) \times (q+1)\) matrix with 0 at the top left corner) to its marginals, i.e. \( L(A) = (\text{row}(A), \text{col}(A)) \), where \( \text{row}(A) \in \mathbb{R}^p \) (resp. \( \text{col}(A) \in \mathbb{R}^q \)) is the vector whose \( i \)-th entry is the sum of entries in the \((i+1)\)-st row (resp. column) of \( A \). In particular, \( L(A) = (\alpha, \beta) \) if and only if \( A \in \mathcal{F}(\alpha, \beta) \), so \( E = L^{-1}(\alpha, \beta) \) is a level of \( L \).

**Theorem 4.9.** Let \( A \in \mathcal{F}(\alpha, \beta) \). Then \( A \) is real-minimal if and only if \( A \) is H-additive.

**Proof of Theorem 4.9.** Following Onn and Vallejo’s results we have that \( A \) is real-minimal if and only if there exists \( z = (x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q) \in \mathbb{R}^{p+q} \) such that, for all \((i, j), (k, l) \in [p+1] \times [q+1] \setminus \{(1, 1)\} \) with \( a_{i,j} > a_{k,l} \), we have

\[
\langle z, L(\Phi(E_{ij})) \rangle > \langle z, L(\Phi(E_{kl})) \rangle,
\]

where \( E_{ij} \) and \( E_{kl} \) are \((p+1) \times (q+1)\) matrices with only one nonzero entry 1 at position \((i, j)\) and \((k, l)\) respectively. Since \( \langle z, L(\Phi(E_{ij})) \rangle = x_{i-1} + y_{j-1} \) and \( \langle z, L(\Phi(E_{kl})) \rangle = x_{k-1} + y_{l-1} \), where we set \( x_0 = y_0 = 0 \), (4.2) is equivalent to

\[
x_{i-1} + y_{j-1} > x_{k-1} + y_{l-1}.
\]

So \( A \) is real-minimal if and only if \( A \) is H-additive.

An immediate corollary of Theorem 4.9 is the following.

**Corollary 4.10.** Let \( A \in \mathcal{F}(\alpha, \beta) \). Then \( A \) is H-additive if and only if there is no matrix \( B \in \mathcal{F}(\alpha, \beta) \) such that \( \pi(B) \preceq \pi(A) \), other than \( A \).

5. **Proof of Theorem 4.7**

Vallejo showed that the Kronecker coefficient indexed by the K-triple produced by a K-additive matrix is 1.

**Lemma 5.1.** ([20, Corollary 4.2]) Let \( A \in \mathcal{M}(\beta, \gamma)_{\alpha} \) be K-additive where \( \alpha, \beta, \) and \( \gamma \) are partitions with the same size, then \( g_{\beta,\gamma}^\alpha = 1 \).

The same is true for Heisenberg coefficients and H-additive matrices.

**Lemma 5.2.** Let \( A \in \mathcal{H}(\beta, \gamma)_{\alpha} \) be H-additive, where \( \alpha, \beta, \) and \( \gamma \) are partitions with \( \max\{|\beta|, |\gamma|\} \leq |\alpha| \leq |\beta| + |\gamma| \), then \( h_{\beta,\gamma}^\alpha = 1 \).

**Proof.** We first show that \( h_{\beta,\gamma}^\alpha \geq 1 \). Set \( \beta = (\beta_1, \beta_2, \ldots, \beta_p), \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_q) \). Since \( A = (a_{ij}) \) is additive, there exist real numbers \( x_i \) and \( y_j \) \((i \in [p+1] \) and \( j \in [q+1] \) satisfying the condition in Definition 4.6. After permuting rows and columns if necessary, we may assume \( x_2 \geq x_3 \geq \cdots \geq x_{p+1} \) and \( y_2 \geq y_3 \geq \cdots \geq y_{q+1} \). By H-additivity, this assumption implies that \( a_{i,j} \geq a_{i,j+1} \) for all \( 1 \leq i \leq p+1, \; 2 \leq j \leq q \), and \( a_{i,j} \geq a_{i+1,j} \) for all \( 2 \leq i \leq p, \; 1 \leq j \leq q+1 \). Set
\( \beta^{(1)} = (a_{2,1}, a_{3,1}, \ldots, a_{p+1,1}) \), \( \gamma^{(1)} = (a_{1,2}, a_{1,3}, \ldots, a_{1,q+1}) \), \( \beta^{(2)} = \beta - \beta^{(1)} \), and \( \gamma^{(2)} = \gamma - \gamma^{(1)} \). They are all partitions and, by the Littlewood–Richardson rule, we have

\[
\langle c_{\beta^{(1)}, \gamma^{(1)}, \gamma^{(2)}}^{\beta} \rangle = 1.
\]

Let \( A^{(1)} \) be the submatrix of \( A \) obtained by removing the first row, and \( A^{(2)} \) be the submatrix of \( A^{(1)} \) obtained by removing the first column. We set \( \alpha^{(1)} = \pi(A^{(1)}) \) and \( \alpha^{(2)} = \pi(A^{(2)}) \). From Remark 4.8, we have

\[
\langle c_{\alpha^{(1)}, \alpha^{(1)}}, \alpha^{(2)} \rangle = 1.
\]

Using Proposition 2.2 and Eqs. (5.1), (5.2), and (5.3), we have

\[
\langle h_{\alpha, \beta, \gamma} \rangle \leq 1.
\]

On the other hand, using Eq. (4.1) and properties of Kostka numbers, we have

\[
\langle h_{\alpha, \beta, \gamma} \rangle = \langle \bigoplus A \in H(\beta, \gamma) h_{\pi(A), \alpha} \rangle = \langle \bigoplus A \in H(\beta, \gamma) h_{\pi(A), \alpha} \rangle = \langle \bigoplus A \in H(\beta, \gamma) h_{\pi(A), \alpha} \rangle.
\]

Since \( A \in H(\beta, \gamma) \) is H-additive, according to Corollary 4.10, we have

\[
\langle |H(\beta, \gamma)|_\delta \rangle = 0
\]

for all \( \delta \prec \alpha \), and \( |H(\beta, \gamma)|_\alpha = 1 \). Equation (5.4) shows that \( h_{\beta, \gamma} \leq 1 \), and proves the lemma.

**Proof of Theorem 4.7.** If a matrix \( A \) is H-additive, then \( nA \) is H-additive. Consequently, by Lemma 5.2, \( h_{n\beta, n\gamma}^{n\alpha} = 1 \) for all \( n > 0 \). By Proposition 3.7, \( (\alpha, \beta, \gamma) \) is H-stable.

**Remark 5.9.** One may prove Theorem 4.7 without using Proposition 3.7. See [21, Section 5], and the proof there applies here. Also, as in [11, Theorem 7.1], given a rational matrix \( A \) with zero at the top left corner, it can be decided in polynomial time whether \( A \) is H-additive.
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Li Ying
Department of Applied and Computational Mathematics and Statistics
University of Notre Dame
Notre Dame IN 46556
USA
e-mail: 98yingli@gmail.com

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