Cosmological evolution in \( f(R) \) gravity and a logarithmic model

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In this paper, we revisit the conditions for cosmological viability of \( f(R) \) gravity and formulate them in terms of constraints on the potential of the scalaron field \( f' \). When the scalaron is heavy, \( f(R) \) goes to general relativity, and the matter-domination epoch could last long enough to ensure the formation of the large scale structure. The idea of having the Newton’s constant change its value depending on the scalar curvature leads to a class of \( f(R) \) models. In particular, one obtains an \( R \ln R \) model when the running of a dimensionless coupling constant is described by a beta-function similar to the one in quantum chromodynamics. In this \( R \ln R \) model, the effective cosmological constant is exponentially suppressed compared to a reference scale. We explore the phase space dynamics and the cosmological evolution of the model in details.

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I. INTRODUCTION

The measurements of Type Ia supernova luminosity-distance indicate that the current universe is undergoing an accelerated expansion \([1, 2, 3, 4, 5]\). The simplest approach to address this issue is to introduce the \( \Lambda \text{CDM} \) model, in which 31.7% of the mass-energy density of the universe is made up by ordinary matter and dark matter, and the rest is constituted by the cosmological constant, \( \Lambda \) \([5]\). The cosmological constant has large negative pressure, and the equation of state \( w = \frac{P}{\rho} \) is equal to \(-1\), where \( P \) and \( \rho \) are the pressure and the energy density of the cosmological constant, respectively. It is the large negative pressure that functions as the repulsive force field against the regular gravity, thus drives the cosmic acceleration. However, the value of the observed cosmological constant is smaller than the theoretical one by a factor of 120 orders of magnitude \([6]\).

Another possibility is that this cosmic speed-up might be caused within general relativity by a mysterious cosmic fluid with negative pressure, which is usually called “dark energy”. However, the nature of dark energy is still unknown. Alternatively, the acceleration could be due to purely gravitational effects, i.e. one may consider modifying the current gravitational theory to produce an effective dark energy \([7, 8, 9, 10]\). A natural approach is to replace the Ricci curvature scalar in the Einstein-Hilbert action with an arbitrary function of the scalar,

\[
S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} f(R) + S_m, \tag{1}
\]

where \( G \) is the Newtonian constant, \( S_m \) is the matter term in the action.

Any modified gravity model should fit the conventional standard cosmology as well as explain the current cosmic speed-up issue. Specifically, in a viable model, the universe should have had a matter-domination epoch in the early universe era to enable the formation of the large-scale structure, and it should have transited from a matter-domination epoch into the current dark-energy-domination one. Moreover, in order to be able to drive the cosmic acceleration, the effective dark energy in this model should have sufficiently large negative pressure, and the effective equation of state \( w_{\text{eff}} \) should be smaller than \(-1/3\).

The currently observed value of the cosmological constant presents a hierarchy problem between the cosmological acceleration scale and the Planck scale \([25, 26]\). The running of the (classical) gravitational coupling with the curvature scale is explored in our previous work \([27]\). If the running is defined by the same beta-function as the (quantum) function in quantum chromodynamics, one is ultimately led to an \( R \ln R \) model. In this model, the cosmological constant can be exponentially suppressed compared to some reference scale. We will explore the dynamical analysis and the cosmological evolution of this model in this study.

The paper is organized as follows. In Sec. II, we construct the dynamical system for \( f(R) \) cosmology. In Sec. III, the conditions for the cosmological viability of the \( f(R) \) gravity are explored. Sec. IV introduces the \( R \ln R \) model. In Secs. V and VI, the dynamical analysis and the cosmological evolution of the \( R \ln R \) model are studied, respectively. Lastly Sec. VII summarizes our
II. THE DYNAMICAL SYSTEM IN $f(R)$ COSMOLOGY

In this section, we prepare for the dynamical analysis of $f(R)$ cosmology. The equivalent of the Einstein equation in $f(R)$ gravity reads,

$$f'R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - [\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box] f' = 8\pi G T_{\mu\nu},$$  \hspace{1cm} (2)

where $f'$ denotes the derivative of the function $f$ with respect to its argument $R$, and $\Box$ is the usual notation for the covariant D'Alembert operator $\Box \equiv \nabla_\alpha \nabla^\alpha$. Compared to general relativity, $f(R)$ gravity has one extra scalar degree of freedom, $f'$. The dynamics of this degree of freedom are determined by the trace of Eq. (2)

$$\Box f' = \frac{1}{3} (2f' - f'R) + \frac{8\pi G}{3} T,$$  \hspace{1cm} (3)

where $T$ is the trace of the stress-energy tensor $T_{\mu\nu}$. Identifying $f'$ by

$$\phi \equiv \frac{df}{dR},$$  \hspace{1cm} (4)

one can rewrite Eq. (3) as

$$\Box \phi = V'(\phi) + \frac{8\pi G}{3} T,$$  \hspace{1cm} (5)

where

$$V'(\phi) \equiv \frac{dV}{d\phi} - \frac{1}{3} [2f - \phi R].$$  \hspace{1cm} (6)

We consider the homogeneous universe in the flat Friedmann-Robertson-Walker metric, $ds^2 = -dt^2 + a^2(t)dx^2$. In this case, the universe can be modeled by a four-dimensional dynamical system of $\{\phi, \pi, H, a\}$, where

$$\pi \equiv \dot{\phi},$$  \hspace{1cm} (7)

$H$ is Hubble parameter, and the dot (\cdot) denotes the derivative with respect to time. Equation (5) provides the equation of motion for $\pi$

$$\ddot{\pi} = -3H \pi - V'(\phi) + \frac{8\pi G}{3} \rho_m.$$  \hspace{1cm} (8)

The equation of motion for $H$ is

$$\dot{H} = \frac{R}{6} - 2H^2.$$  \hspace{1cm} (9)

The definition of the Hubble parameter implies

$$\dot{a} = aH.$$  \hspace{1cm} (10)

The system is constrained by

$$H^2 + \frac{\pi}{\phi} \dot{H} + \frac{1}{6} \frac{f - \phi R}{\phi} - \frac{8\pi G}{3\phi} (\rho_m + \rho_r) = 0,$$  \hspace{1cm} (11)

where $\rho_m$ and $\rho_r$ are the density of matter and the one of radiation, respectively. Equations (7)-(11) provide a closed description of the dynamical system of $\{\phi, \pi, H, a\}$.

In order to explore how $f(R)$ gravity causes cosmic speed-up, it is convenient to cast the formulation of $f(R)$ gravity in a format similar to that of general relativity. We rewrite Eq. (2) as

$$G_{\mu\nu} = 8\pi G \left[T_{\mu\nu} + \tilde{T}^{(\text{eff})}_{\mu\nu}\right],$$  \hspace{1cm} (12)

where

$$8\pi G \tilde{T}^{(\text{eff})}_{\mu\nu} = \frac{f - f'R}{2} g_{\mu\nu} + \nabla_\mu \nabla_\nu f' - g_{\mu\nu} \Box f' + (1 - f') G_{\mu\nu}.$$  \hspace{1cm} (13)

$\tilde{T}^{(\text{eff})}_{\mu\nu}$ is the energy-momentum tensor of the effective dark energy, and it is guaranteed to be conserved, $\tilde{T}^{(\text{eff})}_{\mu\nu} = 0$. Equation (13) reveals the definition of the equation of state of the effective dark energy

$$w_{\text{eff}} \equiv \frac{p_{\text{eff}}}{\rho_{\text{eff}}},$$  \hspace{1cm} (14)

where

$$8\pi G \bar{\rho}_{\text{eff}} = 3H^2 - 8\pi G (\rho_m + \rho_r) = \frac{f - f'R}{2} - 3H \dot{f}' + 3H^2 (1 - f'),$$  \hspace{1cm} (15)

$$8\pi G \bar{p}_{\text{eff}} = H^2 - R/3 - 8\pi G \rho_r = \dot{f}' + 2H \dot{f}' + \frac{f - f'R}{2} + (H^2 - R/3)(1 - f').$$  \hspace{1cm} (16)

In order for an $f(R)$ model to account for the cosmic speed-up, $w_{\text{eff}}$ should be less than $-1/3$.

III. COSMOLOGICAL VIABILITY

Many $f(R)$ models have been presented to address the current cosmic speed-up problem. It is necessary to check whether these models agree with the observations for both early and late universe. The conditions for cosmological viability for the $f(R)$ theory were discussed via dynamical analysis in phase space in [12]. With this approach one could find the conditions for the existence of a viable matter-domination epoch prior to a late-time acceleration, which can be expressed as

$$m(r) \approx 0^+ \text{ and } \frac{dm}{dr} > -1, \text{ at } r \approx -1,$$  \hspace{1cm} (17)
where \( m \equiv f''R/f' \) and \( r \equiv -f'R/f \). Actually, \( r \) and \( m \) are closely related to \( V'(\phi) \) and \( V''(\phi) \), respectively, with \( V''(\phi) \) being defined by Eq. (6) and

\[
V''(\phi) = \frac{f' - f''R}{3f''}. \tag{18}
\]

Then, in this section we will revisit these cosmological viability conditions by considering how the potential \( V(\phi) \) determines the dynamics of \( f(R) \) cosmology.

In the standard cosmology based on general relativity, a matter-domination epoch is ensured in the early universe. Therefore in order to obtain the conditions for this matter-domination epoch in \( f(R) \) gravity, one may need to consider how \( f(R) \) gravity could be reduced to general relativity. This implies that

\[
f(R) \approx f'R, \text{ with } f' \sim 1 \text{ i.e. } |\Delta f'| \ll 1, \tag{19}
\]

which results in \( r \approx -1 \), as shown in Eq. (17). Combine Eqs. (7) and (8) as

\[
\ddot{\phi} = -3H\dot{\phi} - V'(\phi) + \frac{8\pi G}{3} \rho_m. \tag{20}
\]

We require \( \phi \) to evolve very slowly, then we have

\[
|\ddot{\phi}| \sim 3H \dot{\phi} \ll V'(\phi) \approx \frac{8\pi G}{3} \rho_m. \tag{21}
\]

Note that \( \rho_m = \rho_{m0}/a^3 \) and \( a = aH \), where \( \rho_{m0} \) is the matter density of the present universe. These, together with \( V'(\phi) \approx 8\pi G \rho_m/3 \), lead to

\[
\dot{\phi} \approx -3H \frac{V'}{V''}, \tag{22}
\]

\[
|3\dot{\phi}H| \approx 9H^2 \frac{V'}{V''} \ll V'(\phi) \approx \frac{8\pi G}{3} \rho_m, \tag{23}
\]

and

\[
V''(\phi) \gg 9H^2 \approx 3 \cdot 8\pi G(\rho_m + \rho_r). \tag{24}
\]

Combination of Eqs. (18) and (24) yields

\[
f' \gg f'' \cdot 8\pi G(10\rho_m + \rho_r). \tag{25}
\]

Equation (24) or (25) is the condition for the recovery of general relativity. In the matter-domination case, Eq. (25) becomes

\[
f' \gg f'' R. \tag{26}
\]

This is just the condition of \( m(r) \approx 0^+ \) in Eq. (17).

Moreover, Eq. (25) also shows the requirement on \( f(R) \) gravity to have a radiation-domination epoch. Take the \( R \ln R \) as an example which we will introduce in Sec. IV.

For this model, \( f(R) = R \left(1 + \alpha_0 \ln \frac{R}{R_0} \right) \), and Eq. (25) implies that

\[
f' = 1 + \alpha + \alpha \ln R/R_0 \gg \alpha \left(9 \frac{\Omega_r}{\Omega_m} + 10 \right). \tag{27}
\]

Since at general relativity limit, \( f' \approx 1 \), then we have

\[
\alpha \ll \frac{1}{9 \frac{\Omega_r}{\Omega_m} + 10}. \tag{28}
\]

If we require general relativity to be recovered to have a matter domination, \( \alpha > 0.1 \) should be much less than 0.1. However, if we require general relativity to be recovered to have a radiation domination, \( \alpha \) should be much less than \( \Omega_m/(9\Omega_r) \). This is a very stringent constraint.

The condition expressed by Eq. (24) has an intuitive interpretation. Note that the potential \( V(\phi) \) should have a minimum so that there could be a dark-energy domination in the late universe. The field \( \phi \) and the matter density \( \rho_m \) are coupled in the early universe, from this coupling the field \( \phi \) acquires mass. A heavy mass of \( \phi \) (large \( V''(\phi) \) ) makes for a slow-roll evolution of \( \phi \) and thus a long-term matter-domination epoch. In the late universe where the scalar curvature is around the cosmological constant scale, the field \( \phi \) will be released from the coupling and approach to the de Sitter minimum of the potential \( V(\phi) \), then the dark energy will drive the universe accelerating efficiently.

The process of the field \( \phi \) obtaining mass from its coupling to the matter density is very close to the chameleon mechanism employed in the study of the solar system tests of \( f(R) \) gravity \([3, 28, 29, 30, 31, 32, 33, 34, 35, 36]\). In this chameleon mechanism, the field \( f' \) is coupled to the matter densities of the Sun and of the background inside and outside the Sun, respectively. The field \( f' \) acquires mass from this coupling and thus the \( f(R) \) gravity could evade the solar system tests.

\[\text{IV. INTRODUCTION TO THE } R \ln R \text{ MODEL}\]

In the rest of this paper, we present the results on the cosmological dynamics of an \( R \ln R \) model. First, we will introduce this model using the idea of the running coupling proposed in our previous work \([27]\). When the running of the gravitational coupling is defined by the same beta-function as the (quantum) function in quantum chromodynamics, one is led to the \( R \ln R \) model, \( f(R) = \frac{R}{\alpha_0} \left(1 + \alpha_0 \ln \frac{R}{R_0} \right) \). For convenience, we rewrite it as

\[
f(R) = \frac{R}{\alpha_0} \left(1 + \alpha_0 \ln \frac{R}{R_0} \right), \tag{29}
\]

where \( \alpha_0 \) and \( R_0 \) are positive. \( \alpha_0 \) is a dimensionless parameter. However, \( R_0 \) is not really a parameter in the action, but merely the scale at which the action is evaluated. In this study, we let \( R_0 \) be dimensionless and equal to 1. The de Sitter curvature \( \Lambda = R_0^{-1/\alpha_0 + 1} \) is exponentially suppressed as being compared to \( R_0 \). We require \( f' \) to be positive to avoid the anti-gravity \([37]\), and \( f'' \) positive to avoid the Dolgov-Kawasaki instability \([38]\). In this model, \( V'(\phi) = \frac{1}{3} \alpha e^{\phi/\alpha_0 - 2}(\phi - 2\alpha_0) \).
V. DYNAMICAL ANALYSIS IN PHASE SPACE

For the $R \ln R$ model reads

$$\dot{\phi} = -3H \pi - \frac{1}{3} \Lambda e^{\frac{\phi}{\alpha_0}} - 2(\phi - 2\alpha_0) + \frac{8\pi G}{3} \rho_m,$$  \hspace{1cm} (35)

$$\dot{H} = \frac{1}{6} \Lambda e^{\frac{\phi}{\alpha_0}} - 2 - 2H^2,$$ \hspace{1cm} (36)

$$\dot{a} = aH.$$ \hspace{1cm} (37)

The constraint equation is

$$H^2 + \frac{\pi}{\phi} H - \frac{\Lambda}{6\phi} e^{\frac{\phi}{\alpha_0}} - 2 - \frac{8\pi G}{3\phi} (\rho_m + \rho_r) = 0.$$ \hspace{1cm} (38)

A. Dynamics in the vacuum

For simplicity, we first consider the dynamics in the vacuum, where both $\rho_m$ and $\rho_r$ are equal to zero. In this case, the solutions to the constraint Eq. (38) are

$$H = \frac{1}{2} \left[ -\frac{\pi}{\phi} \pm \sqrt{\left(\frac{\pi}{\phi}\right)^2 + \frac{2\Lambda}{3\phi} e^{\frac{\phi}{\alpha_0}} - 2} \right].$$ \hspace{1cm} (39)

As shown in Fig. 2, the surface can be clearly viewed via the Poincaré compactification in cylindrical coordinate system which transforms $\phi$, $\pi$, and $H$ to $\phi_P \equiv \phi/(12 + \phi^2 + \pi^2)^{1/2}$, $\pi_P \equiv \pi/(12 + \phi^2 + \pi^2)^{1/2}$, and $H_P \equiv H/(12 + H^2)^{1/2}$, respectively. The Hubble parameter in the upper surface is positive, corresponding to an expanding universe; whereas the lower surface corresponds to a contracting one. The constraint surfaces of the expanding and contracting universes are disconnected, despite that they both asymptotically approach $(\phi_P \to -1, \pi_P \to 0, H_P \to 0)$. The zero value of the square root term in Eq. (39) makes a cutting edge in the projection of the constraint surface onto the plane $(\phi_P, \pi_P)$. The edge is described by

$$\pi_P = \pm \sqrt{\frac{2}{3} \Lambda e^{\frac{\phi}{\alpha_0}} - 2} \phi.$$ \hspace{1cm} (40)

When $\phi \to -\infty$, one has $\pi_P \to 0$ and the cutting edge is almost closed, as shown in Fig. 3.

In the vacuum evolution, the phase space flows will stay on the constraint surface. Some of them are plotted in Figs. 2 and 5. There are four special points in the figures: $A(\phi_P = 0^+, \pi_P = 1, H_P = 1)$, $B(\phi_P = -1, \pi_P = 0, H_P = 0)$, $C(\phi_P = 0^+, \pi_P = -1, H_P = 1)$, and $D(\phi_P = 0.5, \pi_P = 0, H_P = \sqrt{\frac{12}{12 + \phi^2}})$. At Point A, $\phi_P = 0$ and $\pi_P = H_P = 1$. All of the phase currents flow out of Point A and move to Point D. Therefore, loosely speaking, Point A is an repeller. At Point B, $\phi_P = \pi_P = H_P = 0$. Moreover, near Point B the currents slowly approach and then move away from B. So B is a saddle point. Similarly, C is also a saddle point. When $\rho_m$ is equal to zero, Eq. (5) for this $R \ln R$ model reads

$$\ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0.$$ \hspace{1cm} (41)
FIG. 2: The constraint surface for the $R \ln R$ model with $\alpha_0 = R_0 = 1$.

FIG. 3: Top view of the constraint surface for the $R \ln R$ model with $\alpha_0 = R_0 = 1$.

On the upper constraint surface with $H > 0$, due to the friction force $-3H\dot{\phi}$, the field $\phi$ will eventually arrive and stay at the minimum of the potential, where $V'(\phi) = 0$, $\phi = 2\alpha_0$, $\pi = 0$ and $H = \sqrt{\Lambda/12}$. This minimum corresponds to Point $D$ in Fig. 5, which is an attractor and is also called the de Sitter point.

We project the phase diagrams onto the plane $(\phi_P, \pi_P, H_P = \text{constant})$. Subsequently, near the cutting edge the direction of the flow can be derived to be

$$\left.\frac{d\pi}{d\phi}\right|_{\text{flow}} = \frac{\dot{\pi}}{\dot{\phi}} = -(\phi + 1) \sqrt{-\frac{\Lambda}{6\phi} e^{\frac{\alpha_0}{2}} - 2}.$$  \hspace{1cm} (42)

On the other hand, with Eq. (40) the slope of the tangent to the edge yields the same value. To conclude, the phase space flows are tangential to the cutting edge and will not enter the forbidden area enclosed by the edge. In other words, the constraint equation forces the currents to stay on the surface.

The corresponding behaviors of the phase currents in the contracting branch of the constraint surface can be analyzed in a similar way and are omitted.

B. Dynamics in the presence of matter

The constraint surface described by Eq. (38) is three dimensional when the matter/radiation density is not zero. For ease of visualization, we explore the dynam-
FIG. 6: The vector fields of $\{\dot{\phi}_P, \dot{\pi}_P, \dot{H}_P\}$ on the slice of $H_P \to 1$ for the $R \ln R$ model with $\alpha_0 = R_0 = 1$. The blue arrows denote that $H_P < 0$ at the positions of the arrows, and the red arrows for $H_P > 0$. The black line are the intersection between the 2D constraint surface of (39) and the slice of $H_P = \text{Constant}$. The cyan line is the trace of $\dot{\pi}_P = 0$, and Point C is one end of this trace. Point A is a repeller, and Point C is a saddle point.

FIG. 7: The vector fields of $\{\dot{\phi}_P, \dot{\pi}_P, \dot{H}_P\}$ on the slice of $H_P = 0.986$ for the $R \ln R$ model with $\alpha_0 = R_0 = 1$. Point B is a saddle point.

FIG. 8: The vector fields of $\{\dot{\phi}_P, \dot{\pi}_P, \dot{H}_P\}$ on the slice of $H_P = \sqrt{\Lambda/\sqrt{12}}/\sqrt{\sqrt{12} + \Lambda/12} = 0.083$ for the $R \ln R$ model with $\alpha_0 = R_0 = 1$. Point D is an attractor.

FIG. 9: The vector fields of $\{\dot{\phi}_P, \dot{\pi}_P, \dot{H}_P\}$ on the slice of $H_P = 0$ for the $R \ln R$ model with $\alpha_0 = R_0 = 1$. Point B is a saddle point.

The 2D constraint surface described by Eq. (39) is the separation surface for the signs of the density term. The matter density is positive in the space enclosed by the constraint surface, and is negative outside of the surface.

The fact that Point D is still an attractor in the presence of matter is related to the dynamics of $\dot{a}$. Inside the upper branch of the 2D constraint surface the Hubble parameter is positive. Therefore, Eq. (10) implies that $\dot{a}$ is positive, then the density term keeps decreasing in the evolution and asymptotically comes to zero at Point D.

The dynamics for the negative $H_P$ can be analyzed in a similar way as in the case of positive $H_P$, and are not
FIG. 10: Some phase space flows in the nonvacuum case for the $R \ln R$ model with $\alpha_0 = R_0 = 1$. Compared to the vacuum case, in the presence of matter the phase space flows still originate at Point A and terminate at Point D. The part between A and C for the trajectory in blue is not plotted due to the difficulty obtaining accurate numerical integration near the boundary.

FIG. 11: Top view of some phase space flows in the nonvacuum case for the $R \ln R$ model with $\alpha_0 = R_0 = 1$.

FIG. 12: The cosmological evolution for the $R \ln R$ model with $\alpha_0 = 0.002$ and $R_0 = 1$. A tiny value of $\alpha_0$ will generate an extreme small value of $\Lambda (= R_0 e^{-1/\alpha_0+1})$, then a super long matter-domination stage.

FIG. 13: The cosmological evolution for the $R \ln R$ model with $\alpha_0 = 0.04$ and $R_0 = 1$. A large $\alpha_0$ will make a small $V''(\phi)$, then a fast evolution of $\phi$. Consequently, the effective dark energy density also changes dramatically as implied in Eq. (15), a matter-domination stage even does not exist.

FIG. 14: The cosmological evolution for the $R \ln R$ model with $\alpha_0 = 0.02$ and $R_0 = 1$.

VI. THE COSMOLOGICAL EVOLUTION OF THE $R \ln R$ MODEL

We generally studied the cosmological dynamics in Sec. V. In this section we will explore the most important solution which trajectory tracks the diluting matter included.
energy density,
\[
\frac{8\pi G}{3} (\rho_m + \rho_r) = \left( \frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{r,0}}{a^4} \right) H_0^2, \tag{43}
\]
where the 0 in the indices denotes that the quantities are measured at the present universe, \( z = 0 \). The \( \Omega_i \)'s are defined as \( \Omega_i = 8\pi G \rho_i / (3H^2) \), where \( i \) refers to the indexing of radiation, matter or effective dark energy. In the past the field \( \phi \) of this solution closely follows the minimum of the effective potential \( V_{\text{eff}} \) defined by \( V_{\text{eff}} = V'(\phi) - \frac{8\pi G}{3} \rho_m \) shown in Eq. (3), until the field \( \phi \) becomes very light and “releases”, approaching the de Sitter minimum of the potential \( V(\phi) \).

The dark energy is a low-curvature-scale problem. Consequently, the \( f(R) \) gravity should reduce to general relativity at high curvature scale, and only deviate from general relativity at the low curvature scale. However, in the \( R \ln R \) model the modification term matters is not negligible at both high and low curvature scales. In order to reduce this model to general relativity at high curvature scale, the parameter \( \alpha_0 \) should be much less than 1, as discussed in Sec. IV. However, \( \alpha_0 \) can not be too small because of the connection between the de Sitter curvature and \( \alpha_0 \), \( \Lambda = R_0 e^{-1/\alpha_0 + 1} \). Otherwise, the matter-domination stage would last too long due to the extreme small value of \( \Lambda \), and the evolution of \( \phi \) would be very slow, as shown in Figs. 12 and 15. The parameter \( \alpha_0 \) can not be too large either. In this model \( V''(\phi) = \frac{1}{4} \lambda e^{\phi/\alpha_0 - 2}(\phi/\alpha_0 - 1) \). Therefore, a large \( \alpha_0 \) will make a small \( V''(\phi) \), which results in a fast evolution of \( \phi \) and a short matter-domination epoch (if such an epoch exists). These are illustrated in Figs. 13 and 15. Consequently, one needs to choose an intermediate value for \( \alpha_0 \). Let \( \alpha_0 = 0.02 \) and \( R_0 = 1 \), we obtain the evolutions of \( \Omega_i \)'s, \( \omega_{\text{eff}} \) and \( \phi \) plotted in Figs. 14, 15 respectively. In this solution, matter-radiation equality takes place around redshift of \( z = 3250 \) [40], and \( \Omega_m = 0.32 \) at \( z = 0 \). The field \( \phi \) runs significantly depending on the matter density as shown in Eq. (32). Consequently, the effective dark energy density also changes dramatically as implied in Eq. (15). As a result, there is no ideal matter-domination epoch at high redshift. Moreover, \( \omega_{\text{eff}} \) is far away from the expected value \(-1\) in the late universe for this model. The trajectory of this evolution in the phase space is plotted in Figs. 16, 17.
VII. CONCLUSIONS

In this article, we studied the cosmological evolution in $f(R)$ gravity, and obtained the conditions for cosmological viably by exploring the geometric features of the potential $V(\phi)$ in the equation of motion for $\phi$. In the early universe the field $\phi$ is coupled to the matter density, and acquires mass from this coupling, thus has a slow-roll evolution. Consequently, general relativity is recovered and a matter-domination stage is ensured in the early universe. In the late universe when the scalar curvature is around the cosmological constant scale, the field $\phi$ will be released from its coupling to the matter density and approach to the de Sitter minimum of the potential $V(\phi)$. Then, the dark energy will be dominate and drive the cosmic speed-up. The process that in the early universe the field $\phi$ evolves as a slow-roll due to a heavy mass obtained from the coupling between the field $\phi$ and the matter density is very close to the chameleon mechanism in the solar system tests of $f(R)$ gravity.

The dynamics in phase space for the $R \ln R$ model were studied in details. In the presence of matter, for ease of visualization we projected the 4D phase space onto the 3D phase space of $\{\phi, \pi, H, a\}$ by taking the scale factor $a$ as an implicit variable, and explored the vector fields of the phase flows on the slices of $H_p = \text{constant}$. This method is generic and could be applied to other modified gravity models. In the $R \ln R$ model, general relativity is recovered only for some scales due to the logarithmical running of $f'$ with respect to the matter density. This makes it hard for the $R \ln R$ model to have a sensible cosmological evolution. Actually, this problem becomes alleviated in the following modified logarithmic model [27]

$$f(R) = \frac{1 + \ln R}{1 + \frac{3}{4} \ln R} R, \quad (44)$$

in which $f'$ asymptotes to a finite value at high curvature. However, the $f(R)$ model should be very close to the $\Lambda$CDM model so as to fit the cosmological observations in both early and late universe. In this class of $f(R)$ models, the scalar field $f'$ is almost frozen at curvature higher than the cosmological constant scale and becomes activated near the cosmological constant scale.

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Appendix: Lambert $W$ function

The Lambert $W$ function is defined \cite{11} by

$$W = W(Y)e^{W(Y)}. \quad (A.1)$$

$W$ can be a negative or complex number. In this study, we only consider the case of $Y > 0$. When $0 < Y \ll 1$, $W(Y) \ll 1$, $e^{W(Y)} \approx 1$, then $W(Y) \approx Y$. When $Y \gg 1$, $W(Y) \gg 1$, then $\ln Y \approx W$. Concisely,

$$W(Y) = \begin{cases} Y & \text{if } 0 < Y \ll 1; \\ \ln Y & \text{if } Y \gg 1. \end{cases} \quad (A.2)$$

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