The Dirac particle on de Sitter background

Ion I. Cotăescu  
_The West University of Timișoara,_
_V. Parvan Ave. 4, RO-1900 Timișoara_

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Abstract

We show that the Dirac equation on de Sitter background can be analytically solved in a special static frame where the energy eigen-spinors can be expressed in terms of usual angular spinors known from special relativity, and a pair of radial wave functions.

1 Introduction

One of the most important but difficult problems of the quantum field theory in curved spacetime is the problem of finding analytic solutions of Dirac equation on given backgrounds. Since there are no general methods to obtain such solutions, these must be derived in each particular case separately, starting with a suitable chart and tetrad gauge and by using appropriate calculation procedures. From the results reported up to now [1, 2, 3] we understand that the form of the solutions obtained through separation of variables as well as their physical significance are strongly dependent on the choice of all these ingredients. In these conditions it is helpful to exploit, in addition, the effects of the global symmetries of the background.

An important case is that of the Dirac equation in spherically symmetric (central) static charts which have the global symmetry of the group
\( T(1) \otimes SO(3) \), of time translations and rotations of the Cartesian space coordinates. Recently, we have proposed a Cartesian gauge \([4]\) in which the whole theory remains covariant under this group when the Cartesian holonomic coordinates are used. Then the energy and angular momentum are conserved like in special relativity from which we can take over the method of separation of variables in spherical coordinates. In fact, our gauge defines Cartesian local (unholonomic) frames which play here the same role as the Cartesian natural frame of the Minkowski spacetime, since their third axes are just those of projections of the whole angular momentum. This allowed us to separate the spherical variables in terms of usual angular spinors such that all the constants involved in separation of variables get physical meaning. Therefore we are sure that the local properties of the Dirac field can be correctly interpreted.

On this way we have found a complete formulation of the radial problem of the Dirac equation on central static backgrounds, obtaining the radial equations and the form of the radial scalar product in the most general case \([4]\). Based on these results, we have solved the Dirac equation on anti-de Sitter static background, giving the formula of the discrete energy levels and the form of the energy eigenspinors in the spherical coordinates of a special natural frame. Moreover, we have observed therein that the solutions of the Dirac equation on de Sitter static backgrounds could be derived in the same manner, following step by step the technique of the anti-de Sitter case. This is just what we would like to present here. Our aim is to study the radial problem and its supersymmetry which helps us to find the radial wave functions we need in order to write down the energy eigenspinors in spherical coordinates (and natural units, \( \hbar = c = 1 \)).

We start in the second section with a short review of our main results concerning the separation of spherical variables in the Dirac equation on central static charts. The third section is devoted to the hidden supersymmetry of the radial problem which can be pointed out by using an appropriate transformation. This simplifies the form of the radial Hamiltonian allowing us to completely solve the radial problem, as shown in Sec.4.

\section{Preliminaries}
The main point of our approach \cite{4} is the choice of tetrads in Cartesian gauge and Cartesian coordinates such that Dirac equation takes a simple form which transforms manifestly covariant under the rotations of the space Cartesian coordinates. Consequently, its theory in the spherical natural coordinates, \((t, r, \theta, \phi)\), can be done like in special relativity. Our tetrad fields depend on three arbitrary functions of \(r\), \(u\), \(v\) and \(w\), which give the line element

\[
ds^2 = w^2 \left[ dt^2 - \frac{dr^2}{u^2} - \frac{r^2}{v^2} (d\theta^2 + \sin^2 \theta d\phi^2) \right].
\]

(1)

and determine the form of the Dirac equation of the free field, \(\psi\), of mass \(M\).

We have shown that this equation has particular positive frequency solutions of energy \(E\),

\[
\psi_{E,j,\kappa,j,m_j}(t, r, \theta, \phi) = u_{E,j,\kappa,j,m_j}(r, \theta, \phi) e^{-iEt}
\]

(2)

\[
= \frac{v}{r w^{3/2}} \left[ f^+(r) \Phi^+_{m_j,\kappa,j}(\theta, \phi) + f^-(r) \Phi^-_{m_j,\kappa,j}(\theta, \phi) \right] e^{-iEt}.
\]

Here \(u_{E,j,\kappa,j,m_j}\) are the particle-like energy eigenspinors which depend on the radial wave functions \(f^{(\pm)}\) and on the four-component angular spinors, \(\Phi^{\pm}_{m_j,\kappa,j}\), known from special relativity \cite{5}. These are orthogonal to each other being completely determined by the quantum number \(j\), of the angular momentum, the quantum number \(m_j\), of its projection along the third axis of the local Cartesian frame, and the value of \(\kappa_j = \pm (j + 1/2)\).

The radial wave functions are solutions of a pair of radial equations which can be written in compact form as the eigenvalues problem

\[
H \mathcal{F} = E \mathcal{F}
\]

(3)

of the radial Hamiltonian

\[
H = \begin{pmatrix}
Mw & -u \frac{d}{dr} + \kappa_j \frac{v}{r} \\
\frac{d}{dr} + \kappa_j \frac{v}{r} & -Mw
\end{pmatrix},
\]

(4)

in the space of the two-component vectors \(\mathcal{F} = |f^{(+)} , f^{(-)}|^T\) where the radial scalar product is

\[
(\mathcal{F}_1, \mathcal{F}_2) = \int_{D_r} \frac{dr}{u(r)} \mathcal{F}^+_1 \mathcal{F}_2.
\]

(5)
This selects the "good" radial wave functions (i.e. square integrable functions or tempered distributions) which enter in the structure of the particle-like energy eigenspinors of (2). Furthermore, the negative frequency (antiparticle-like) eigenspinors can be obtained directly by using the charge conjugation 4.

Here we use three arbitrary functions \((u, v, w)\) which define the metric and implicitly the form of the radial Hamiltonian. However, it is known that, in general, for any central static metric two such functions are enough. This means that we have a supplementary degree of freedom allowing us to choose suitable radial coordinate. The most convenient is to work in the special natural frame 4 where, by definition, \(u = 1\). According to (2) its radial coordinate is

\[
    r_s(r) = \int \frac{dr}{u(r)} + \text{const.}
\]

Here the constant assures the condition \(r_s(0) = 0\).

3 The supersymmetry of the radial problem

Let us consider now the problem of the massive Dirac particle on de Sitter background. There exists a static chart with usual spherical coordinates, \((t, \hat{r}, \theta, \phi)\), where the line element is

\[
ds^2 = (1 - \omega^2 \hat{r}^2) dt^2 - \frac{d\hat{r}^2}{1 - \omega^2 \hat{r}^2} - \hat{r}^2 (d\theta^2 + \sin^2 \theta d\phi^2).
\]

In the corresponding special frame, denoted from now by \((t, r, \theta, \phi)\), the radial coordinate is

\[
r = \frac{1}{\omega} \arctanh \omega \hat{r},
\]

as it is obtained from (3) and (7). Here, the radial domain is \(D_r = [0, \infty)\) and the line element has the form

\[
ds^2 = \frac{1}{\cosh^2 \omega r} \left[ dt^2 - dr^2 - \frac{1}{\omega^2} \sinh^2 \omega r \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right],
\]
from which we can identify the functions \( v \) and \( w \) we need to write the radial Hamiltonian

\[
H = \begin{bmatrix}
\frac{\omega k}{\cosh \omega r} & -d \frac{d}{dr} + \frac{\omega \kappa_j}{\sinh \omega r} \\
\frac{d}{dr} + \frac{\omega \kappa_j}{\sinh \omega r} & -\frac{\omega k}{\cosh \omega r}
\end{bmatrix},
\]

with the notation \( k = M/\omega \) (i.e. \( Mc^2/h\omega \) in usual units).

Our radial problem has hidden supersymmetry like in the anti-de Sitter case [4]. This can be easily pointed out with the help of the transformation \( F \rightarrow \hat{F} = U(r)F \) where

\[
U(r) = \begin{bmatrix}
\cosh \frac{\omega r}{2} & -i \sinh \frac{\omega r}{2} \\
i \sinh \frac{\omega r}{2} & \cosh \frac{\omega r}{2}
\end{bmatrix}.
\]

A little calculation shows us that the transformed Hamiltonian,

\[
\hat{H} = U(r)HU^{-1}(r) - \frac{i}{2} I_{2 \times 2},
\]

which gives the new eigenvalue problem

\[
\hat{H}\hat{F} = \left( E - \frac{i}{2} \right) \hat{F},
\]

has supersymmetry since it has the requested specific form,

\[
\hat{H} = \begin{bmatrix}
\nu & -d \frac{d}{dr} + W \\
\frac{d}{dr} + W & -\nu
\end{bmatrix}.
\]

Here \( \nu = \omega(k - i\kappa_j) \) is a constant and

\[
W(r) = \omega(ik \tanh \omega r + \kappa_j \coth \omega r)
\]

is the superpotential of the radial problem [4]. The transformed radial wave functions \( \hat{f}^{(\pm)} \) (which are the components of \( \hat{F} \)) satisfy the second order equations resulted from the square of (13). These are

\[
\left( -\frac{d^2}{dr^2} - \omega^2 \frac{ik(ik \pm 1)}{\cosh^2 \omega r} + \omega^2 \frac{2\kappa_j(k_j \pm 1)}{\sinh^2 \omega r} \right) \hat{f}^{(\pm)}(r) = \omega^2 \epsilon^2 \hat{f}^{(\pm)}(r),
\]

where we have denoted \( \epsilon = E/\omega - i/2. \)
4 Solutions

The solutions of Eqs. (16) are well-known to be expressed in terms of Gauss hypergeometric functions \( F \), \( F_\pm(y) \equiv F(\alpha\pm, \beta\pm, \gamma\pm, y) \), depending on the new variable \( y = -\sinh^2 \omega r \), as

\[
\hat{f}^{(\pm)}(y) = N_\pm (1 - y)^{p\pm} y^{s\pm} F_\pm(y)
\]

(17)

where

\[
\alpha_\pm = s_\pm + p_\pm + \frac{i\epsilon}{2}, \quad \beta_\pm = s_\pm + p_\pm - \frac{i\epsilon}{2}, \quad \gamma_\pm = 2s_\pm + \frac{1}{2},
\]

(18)

\( N_\pm \) are normalization factors while the parameters \( p_\pm \) and \( s_\pm \) are related with \( k \) and \( \kappa_j \) through

\[
2s_\pm (2s_\pm - 1) = \kappa_j(\kappa_j \pm 1),
\]

(19)

\[
2p_\pm (2p_\pm - 1) = ik (ik \pm 1).
\]

(20)

Furthermore, we have to find the suitable values of these parameters such that the functions (17) should be solutions of the transformed radial problem (13). If we replace (17) in (13), after a few manipulation, we obtain

\[
y(1 - y) \frac{dF_\pm(y)}{dy} - y \left( p_\pm + \frac{ik}{2} \right) F_\pm(y) + (1 - y) \left( s_\pm + \frac{\kappa_j}{2} \right) F_\pm(y)
\]

\[
= \frac{\eta}{2} \frac{N_\pm}{N_\mp} \left( \kappa_j \pm \frac{1}{2} \pm \frac{iE \pm M}{\omega} \right) y^{s_\pm - s_\mp + 1/2} (1 - y)^{p_\mp - p_\pm + 1/2} F_\pm(y),
\]

(21)

where \( \eta = \pm 1 \). These equations are nothing else than the usual identities of hypergeometric functions if the values of \( s_\pm, p_\pm \) and \( N_+/N_- \) are correctly matched. First we observe that the differences \( s_+ - s_- \) and \( p_+ - p_- \) must be half-integer since we work with analytic functions of \( y \). This means that the allowed groups of solutions of (19) are

\[
2s_+^1 = -\kappa_j, \quad 2s_+^2 = \kappa_j + 1,
\]

\[
2s_-^1 = -\kappa_j + 1, \quad 2s_-^2 = \kappa_j,
\]

(22)

while Eq. (20) gives us

\[
2p_+^1 = -ik, \quad 2p_+^2 = ik + 1,
\]

\[
2p_-^1 = -ik + 1, \quad 2p_-^2 = ik.
\]

(23)
On the other hand, it is known that the hypergeometric functions of (17) are analytical on the domain $D_y = (-\infty, 0]$, corresponding to $D_r$, only if $\Re(\gamma_\pm) > \Re(\beta_\pm) > 0$ [8]. Moreover, their factors must be regular on this domain including $y = 0$. Obviously, both these conditions are accomplished if we take

$$s_\pm > 0.$$  \hspace{1cm} (24)

We specify that there are no restrictions upon the values of the parameters $p_\pm$.

We have hence all the possible combinations of parameter values giving the solutions of the second order equations (16) which satisfy the transformed radial problem. These solutions will be denoted by $(a, b)$, $a, b = 1, 2$, understanding that the corresponding parameters are $s_\pm^a, p_\pm^b$, as given by (22) and (23), and

$$\alpha_\pm^{(a,b)} = s_\pm^a + p_\pm^b + \frac{i\epsilon}{2}, \quad \beta_\pm^{(a,b)} = s_\pm^a + p_\pm^b - \frac{i\epsilon}{2}, \quad \gamma_\pm^{(a)} = 2s_\pm^a + \frac{1}{2}.$$  \hspace{1cm} (25)

The condition (24) requires to chose $a = 1$ when $\kappa_j = -j - 1/2$, and $a = 2$ if $\kappa = j + 1/2$. Thus, for each given set $(E, j, \kappa_j)$ we have a pair of different radial solutions, with $b = 1, 2$. Therefore, it is convenient to denote the transformed radial wave functions (17) by $\hat{f}_E^{(\pm)} j, a, b$, bearing in mind that the value of $a$ determines that of $\kappa_j$. The last step is to calculate the values of $N_+/N_-$. From (21) it results

$$\kappa_j = -j - \frac{1}{2} : \quad \eta \frac{N_+^{(1)}}{N_-^{(1)}} = -\frac{\alpha_+^{(1)}}{\gamma_+^{(1)}}, \quad \eta \frac{N_+^{(1)}}{N_-^{(1)}} = \frac{\beta_+^{(1)}}{\gamma_+^{(1)}} - 1, \hspace{1cm} (26)$$

$$\kappa_j = j + \frac{1}{2} : \quad \eta \frac{N_+^{(2)}}{N_-^{(2)}} = 1 - \frac{\beta_-^{(2)}}{\gamma_-^{(2)}}, \quad \eta \frac{N_+^{(2)}}{N_-^{(2)}} = \frac{\alpha_-^{(2)}}{\gamma_-^{(2)}}.$$  \hspace{1cm} (27)

Notice that here $\gamma_+^{(1)} = \gamma_-^{(2)} = j + 1$.

Now we can restore the form of the original radial wave functions of (4) by using the inverse of (17). In our new notation these wave functions are

$$f_{E,j,a,b}(r) = \cosh \frac{\omega r}{2} \hat{f}_{E,j,a,b}(r) \pm i \sinh \frac{\omega r}{2} \hat{f}_{E,j,a,b}(r). \hspace{1cm} (28)$$

For very large $r$ (when $y \to -\infty$) the hypergeometric functions behave as $F_\pm(y) \sim (-y)^{-\alpha_\pm}$ [8]. Thereby we deduce that $\hat{f}_{E,j,a,b} \sim \exp(-i\epsilon \omega r)$ and

$$f_{E,j,a,b} \sim e^{-iEr}.$$  \hspace{1cm} (29)
This means that the functions (28) represent tempered distributions corresponding to a continuous energy spectrum. On the other hand, (29) indicates that this energy spectrum covers the whole real axis, as it seems to be natural since the metric is not asymptotically flat. However, in our opinion, it is premature to draw definitive conclusions before to carefully study the properties of the radial state space. Anyway, it is clear that the energy spectrum is continuous, without discrete part, while the energy levels are infinitely degenerated since there are no restrictions upon the values of $j$, which can be any positive half-integer.

5 Comments

Here we have derived the solutions of the Dirac equation on de Sitter background. This was possible grace of our general method based on the Cartesian tetrad gauge which preserves the maximal global symmetry of the central static charts. We must say that these solutions are different from the other ones obtained by using also Cartesian gauges but in the spherical coordinates of moving charts [3]. The argument is that our solutions rewritten in the comoving frame have no more separated variables.

From the technical point of view, we have used here the same procedure as in Ref.[4]. Thus it is clear that the Dirac equation on de-Sitter or anti-de Sitter backgrounds can be solved in the same manner. However, despite this fact, the results are different since the corresponding energy spectra are of different kinds. We believe that these are good examples of free Dirac fields with continuous and respectively discrete energy spectra which could help us to understand some sensitive aspects of the quantum theory in curved spacetime. We refer especially to the structure of the operator algebra and its dependence on the choice of the tetrad gauge in a given chart. A guide in this direction could be our recent study of the algebras of the one-dimensional relativistic oscillators [7] which have similar superpotentials as those of the transformed Hamiltonians of the de Sitter or anti-de Sitter radial problems.

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