A Reverse Jensen Inequality Result with Application to Mutual Information Estimation

Gerhard Wunder*, Benedikt Groß*, Rick Fritschek*, and Rafael F. Schaefer†

* Cybersecurity and AI Group
Freie Universität Berlin
Takustr. 9, 14195 Berlin, Germany
{g.wunder, benedikt.gross, rick.fritschek}@fu-berlin.de

† Chair of Communications Engineering and Security
University of Siegen
Hölderlinstr. 3, 57068 Siegen, Germany
rafael.schaefer@uni-siegen.de

Abstract—The Jensen inequality is a widely used tool in a multitude of fields, such as for example information theory and machine learning. It can be also used to derive other standard inequalities such as the inequality of arithmetic and geometric means or the Hölder inequality. In a probabilistic setting, the Jensen inequality describes the relationship between a convex function and the expected value. In this work, we want to look at the probabilistic setting from the reverse direction of the inequality. We show that under minimal constraints and with a proper scaling, the Jensen inequality can be reversed. We believe that the resulting tool can be helpful for many applications and provide a variational estimation of mutual information, where the reverse inequality leads to a new estimator with superior training behavior compared to current estimators.

I. INTRODUCTION

The Jensen inequality is an important tool in many fields of pure and applied mathematics, such as convex analysis, information theory, and machine learning. For an integrable, real-valued random variable $X$ and a convex function $\varphi$ it states that

$$\varphi(\mathbb{E}[X]) \leq \mathbb{E}[\varphi(X)].$$

(1)

It can be used to derive other standard inequalities such as the inequality of arithmetic and geometric means or the Hölder inequality. In this paper, we proof a reversion of Jensen’s inequality for a certain class of measures. Several reversion results for specific functions or classes of distributions are known in the literature. In [1], a reverse Jensen inequality for a family of exponential distributions is used to obtain tractable bounds for expectation maximisation (EM) for conditional and discriminative learning of certain latent variable models. In [2], several reverse Jensen results for discrete variables are formulated with applications in information theory. Further results can be found for example in [3], [4]. In [5], a converse of the Jensen inequality is derived and used to compute upper bounds on some divergences.

This paper is structured as follows. In Section II we derive our reverse inequalities and give concrete examples for its application. In Section III, the reverse inequality is used to obtain a variational lower bound on the mutual information (MI). Mutual information is a central quantity in information theory that measures the amount of information one random variable carries about another. Recently, MI estimators gained much interest in the field of machine learning, e.g., in the context of representation learning [6], generative models and deep information bottleneck but also in wireless communications [7]–[9]. In Section III an overview over recent developments in the use of MI for several machine learning applications and estimators of MI from finite samples is given. Section IV validates our MI estimator on synthetic data. The paper concludes with a discussion on limitations of MI estimation from finite samples and future research directions in Section V.

II. REVERSE JENSEN INEQUALITIES

A. Concave Functions

Our initial lemma is as follows. Let $b = b(p) := E[X^p]/E[X] \geq 1$ be the ratio of first and $p$-th non-centralized moment.

Lemma 1. Let $f : \mathbb{R}_0^+ \rightarrow \mathbb{R}_0^+$ be concave and $f(0) = 0$ and set

$$\zeta_b(a) := \sup_{\frac{1}{p} + \frac{1}{q} = 1} \left[ \frac{1 - b(p)^{\frac{1}{p}}a^{-\frac{1}{q}}}{a} \right]^+.$$

Then for any random variable $X$, we have

$$f(\mathbb{E}[X]) \geq \mathbb{E}[f(X)] \geq \sup_{a > b} f(a \mathbb{E}[X]) \zeta_b(a).$$

Remark 1. The case $b = \infty$ with vanishing first or second moment yields a trivial lower bound.

Remark 2. It can be easily checked that in the deterministic case where $b(p) = 1$ and $\zeta_b(a) = \max_{a \geq 1}(1 - a^{-1})/a$. In all cases, the inequality appears to be not tight.

Proof. The left inequality is the standard Jensen inequality. For any $X^* > 0$ we have

$$\mathbb{E}[f(X)] \geq \mathbb{E}[f(X) 1\{X \leq X^*\}]$$

$$= \mathbb{E}\left[ \frac{f(X)}{X^*} X 1\{X \leq X^*\} \right]$$

$$\geq \frac{f(X^*)}{X^*} \mathbb{E}[X 1\{X \leq X^*\}].$$
Then, for any random variable fulfilling condition (3)

$$E[f(X)] \geq 
\frac{f(X^*)}{X^*} E[X(1-I\{X > X^*\})]
$$

$$= 
\frac{f(X^*)}{X^*} \left[ E[X] - E[I\{X > X^*\}] \right],
$$

where the first inequality follows from the positivity of $X$, the second from concavity and the intercept theorem. For $0 < p, q < \infty$ with $p^{-1} + q^{-1} = 1$, we have:

$$E[f(X)] \geq \frac{f(X^*)}{X^*} \left[ E[X] - E[I\{X > X^*\}] \right] 
= \frac{f(X^*)}{X^*} E[X] \left[ 1 - \left( \frac{E[X^p]}{E^p[X]} \right) \right] 
\geq \frac{f(X^*)}{X^*} E[X] \left[ 1 - \left( \frac{E[X^p]}{E^p[X]} \right) \right],
$$

where we applied Hölder’s inequality in the first step and Markov’s inequality in the second step. If we choose $X^* = aE[X]$ we get

$$E[f(X)] \geq \frac{f(aE[X])}{a} \left[ 1 - b^{\frac{1}{\beta}} (p) \right],$$

and since the inequality is true for any sequence $p_n$, we obtain

$$E[f(X)] \geq \limsup_{n \to \infty} \frac{f(aE[X])}{a} \left[ 1 - b^{\frac{1}{\beta}} (p) \right],$$

and the claim follows.

This recovers the corresponding lemma in [10] in case of $p = q = 2$. Obviously, a non-trivial lower bound exists if

$$E[X^p] < \infty \text{ for some } p \geq 1. \quad (2)$$

Its actual tightness depends on the growth of the higher moments. Condition (2) is certainly true for any positive random variable with finite support, i.e.: $X = 0$ for $X > aE[X]$ (3)

by the Lebesgue space inclusion theorem. Then, Lemma (1) can be strengthened.

**Example 1.** We can use this result for the special case

$$g(z_1, \ldots, z_n) = \sum_{i=1}^n \log (1 + z_i), \quad z_i \geq 0.$$ 

Setting $f(z) = \log(1 + z)$ and letting $Z$ be a discrete random variable in $\mathbb{R}_+^n$, taking on values $z_1, \ldots, z_n$, each with probabilities $p_z(z_i) = 1/n$ the above result yields

$$\log \left( 1 + \frac{1}{n} \sum_{i=1}^n z_i \right) \geq \frac{1}{n} \sum_{i=1}^n \log (1 + z_i) \geq \sup_{a > b} \log \left( 1 + \frac{n}{a} \sum_{i=1}^n z_i \right) \zeta_b (a).$$

Another often used example includes $f(z) = \log(1 + \frac{z}{1+z})$ where similar inequalities can be obtained.

**B. Convex Functions**

We have the following lemma for convex functions.

**Lemma 3.** Let $f : \mathbb{R}_+^+ \to \mathbb{R}_+^+$ be convex, increasing and $f(0) = 0$. Then for any random variable $X \geq 0$

$$f \left( \zeta^{-1}_b (a) E[X] \right) \geq aE[f(X)].$$

**Proof.** We have from Lemma (1) with concave $g$

$$E_X [g(X)] \geq g(aE_X [X]) \zeta_b (a)$$

where we indicate with $E_X$ that the expectation is with respect to the distribution of $X$. Setting $Y = g(X)$ and using change of measures in the expectation yields:

$$E_Y [Y] \geq g(aE [g^{-1}(Y)]) \zeta_b (a)$$

and eventually:

$$g^{-1} \left( \zeta^{-1}_b (a) E_Y [Y] \right) \geq E_Y [g^{-1} (Y)]$$

Since $X$ and thus $Y$ and $g$ were arbitrary (under the posed constraints) the result follows by replacing the convex $g^{-1}$ with $f$.

**Remark 3.** The result gives a nice moment bound. Let $X = 0$

$$\text{for } X \geq aE_X [X] \text{ and } f(X) = Y = X^\frac{1}{c} \text{ then } f^{-1}(Y) = X = Y^c \text{ for some } c > 1. \text{ By Lemma (2) i.e. } \zeta^{-1}_b (a) = a, \text{ and Lemma (3) i.e. } aE_X [Y] \geq aE_Y [Y^c], \text{ we have }$$

$$E_Y [Y^c] \leq a^{c-1} E_Y [Y] \text{.}$$

A result for decreasing $f$ is as follows.

**Lemma 4.** Suppose $f : \mathbb{R}_+^+ \to \mathbb{R}_+^+$ is some decreasing convex function with $f(x) = c$. Then, we have for any random variable $Z \geq x$

$$E[f(Z + x)] \leq \inf_{a > b} \left\{ f(E[aZ + x]) + [c - f(E[aZ + x])] (1 - \zeta_b (a)) \right\}.$$

**Proof.** The trick is to transform the problem into an equivalent concave problem. Set $F(z) := -f(z + x) + c$ so that $F$ is concave and $F(0) = 0, F(z) \geq 0$. By Lemma (1) we have
\[ \mathbb{E} [ F(Z) ] \geq F(\alpha ) \mathbb{E} [ Z ] \zeta_b (a) \] for \( a > b \). Substituting this, we get
\[ \mathbb{E} [ \aleph (Z+x) + c ] \geq (\aleph (\mathbb{E} [ Z ] + x) + c ) \zeta_b (a) > 0 \]
so that
\[ -\mathbb{E} [ \aleph (Z+x) ] \geq -\aleph (\mathbb{E} [ aZ+x ] ) \zeta_b (a) + c \zeta_b (a) - c. \]

Multiplying by \(-1\) yields the result.

**Example 2.** Let \( f \) be the function \( f(z) := 1/z \) and let \( Z \) be a discrete random variable with \( \mathbb{E} [ Z ] := z \) in \( \mathbb{R}^n \) taking on \( z_1, ..., z_n \) with \( p_Z (z_i) = 1/n \). Then, we have by Lemma \( \ref{lem:1} \)
\[ \frac{1}{z} \leq \frac{1}{\mathbb{E} [ Z ]} \leq \frac{1}{a (z - z_1) + z_1} + \frac{(1 - \zeta_b (a))}{z_1}. \]

**C. Special Functions**

1) The log-function: The usefulness is shown in the next theorem.

**Theorem 1.** For any random variable \( X > 0 \) with \( \zeta_b (a), b \) (defined above), any \( a > b \), we have the lower and upper bound
\[
\log (\mathbb{E} [ X ] ) \leq \frac{1}{\zeta_b (a)} \mathbb{E} \left[ \log \left( \frac{1 + X}{\mathbb{E} [ X ]} \right) \right]
\leq \frac{1}{\zeta_b (a)} \log (1 + \mathbb{E} [ X ] ) + \log \left( \frac{\mathbb{E} [ X ]}{1 + a \mathbb{E} [ X ]} \right),
\]
of which the difference is uniformly bounded in \( \mathbb{E} [ X ] \).

**Proof.** For the left-hand side, fix some \( c > 0, a > b \), then
\[
\mathbb{E} \left[ \log \left( \frac{1 + X}{c} \right) \right]
\geq \zeta_b (a) \log (1 + a \mathbb{E} [ X ] ) + \log \left( \frac{1}{c} \right)
= \zeta_b (a) \log \left( \frac{1 + a \mathbb{E} [ X ]}{\mathbb{E} [ X ]} \right)
\]
using the above lemma. For any (appropriate) \( c (a, \mathbb{E} [ X ] ) > 0 \) we obtain
\[
\log (\mathbb{E} [ X ] ) \leq \log \left( \frac{1 + a \mathbb{E} [ X ]}{\mathbb{E} [ X ]} \right)
\leq \frac{1}{\zeta_b (a)} \mathbb{E} \left[ \log \left( \frac{1 + X}{c} \right) \right].
\]

Selecting the optimal parameter \( c (a, \mathbb{E} [ X ] ) \) as
\[
c (a, \mathbb{E} [ X ] ) = \left( \frac{1}{\mathbb{E} [ X ]} + a \right) \zeta_b (a)
\]
yields the desired result.

The right-hand side is
\[
\frac{1}{\zeta_b (a)} \mathbb{E} \left[ \log \left( \frac{1 + X}{\mathbb{E} [ X ]} \right) \right]
= \frac{1}{\zeta_b (a)} \mathbb{E} \left[ \log (1 + X ) \right] + \mathbb{E} \left[ \log \left( \frac{\mathbb{E} [ X ]}{1 + \mathbb{E} [ X ]} \right) \right]
\leq \frac{1}{\zeta_b (a)} \log (1 + \mathbb{E} [ X ] ) + \log \left( \frac{\mathbb{E} [ X ]}{1 + \mathbb{E} [ X ]} \right)
\]
due to Jensen’s inequality. Eventually, it is easy to see that the difference is uniformly bounded.

2) **Product-type functions:** Suppose we have real numbers \( a_1, ..., a_n \geq \alpha \) and \( b_1, ..., b_n \geq \beta > 0 \) (denominator shall be non-zero). Set without loss of generality \( \alpha = \beta = 1 \). Further, suppose \( a_i (\phi) \) and \( b_i (\phi) \) both depend on some parameter set \( \phi \subset D \) such that any \( \phi \subset D \) guarantees the lower bounds for \( a_i, b_i \). Consider the metric to be maximized over \( D \) as
\[
f (\phi ) = \prod_{i=1}^{n} a_i (\phi) \]

Multiply by \( \beta / \alpha \) if the lower does not hold. Let \( C (s_\alpha) \) and \( C (s_\beta) \) be the spread functions that depend on the spreads
\[
s_\alpha := \max_{a} a_i \geq 1, \quad s_\beta := \max_{b} b_i \geq 1
\]
and
\[
C (s_\alpha / s_\beta ) := \max_{c_1} \frac{1}{c_1} \left( 1 - \frac{1 + s_\alpha / s_\beta}{c_1} \right).
\]

Then using above lemmas we can prove the following theorem.

**Theorem 2.** Suppose we have real numbers \( a_i (\phi) \geq 1 \) and \( b_i (\phi) \geq 1 \) depending on some parameter set \( \phi \subset D \). We have
\[
\left( \frac{1}{n} \sum a_i (\phi) \right)^{C (s_\alpha)} \right) \leq \left( \prod_{i=1}^{n} a_i (\phi) \right)^{\frac{1}{n}} \leq \left( \frac{1}{n} \sum b_i (\phi) \right)^{C (s_\beta)},
\]
where \( C (s_\alpha), C (s_\beta) \) are the spread parameters.

Obviously, depending on the spread of \( a_1, ..., a_n \) and \( b_1, ..., b_n \) we can replace product terms with sums. Applications of such estimations can be applied in waveform design, see e.g. \( \ref{17} \).

The proof is omitted due to lack of space.

**III. Neural Estimation of Mutual Information**

Precise estimation of mutual information is a long standing problem due to its dependence on the underlying probabilities, which are normally unknown in most applications, and only observable through samples. Classical approaches tackled this problem based on a binning of the probability space \( \ref{12}, \ref{13} \), \( k \)-nearest neighbor statistics \( \ref{14} - \ref{16} \), maximum likelihood estimation \( \ref{17} \), and variational lower bounds \( \ref{18} \). However, most of these techniques are limited to very low dimensional problems. However, due to its use in the field of deep learning, for example as a metric in representation learning, it received renewed interest. In particular, \( \ref{19} \), showed a promising new direction in combining deep learning methods with variational bounds which resulted in a surge of papers building upon this new direction. It also presented the first estimator within this framework, the mutual information neural estimator (MINE), which is based on the Donsker-Varadhan lower bound of the Kullback-Leibler divergence but is biased, see \( \ref{20} \). Another closely related estimator is the Nguyen-Wainwright-Jordan (NWJ) estimator, based on the lower bound \( \ref{21} \), also known...
as f-GAN [22]. This estimator is unbiased and can be derived through the Fenchel duality. Both estimators are limited by their variance which scales exponentially with the mutual information. The info noise contrastive estimator (InfoNCE) estimator [23] was derived in the context of representation learning, and gives a bound with low variance, but high bias, dependent on the sample size and the mutual information. Recent works have combined the high variance, low bias estimators (NWJ, MINE) with the high bias, low variance NCE method in [20], [24]. Note that the marginal term is the culprit of high variance fluctuations due to the exponential function inside the expected value. This part of the estimator can be dominated by extremely rare events, which are unlikely to be sampled from, see [25].

A. Mutual Information Estimators from Unnormalized Gibbs Measures

Let us identify \( P(X, Y) \) with the ground truth joint probability measure \( P \) and \( Q := P_X \times P_Y \) with the product of the marginals \( P(X) \) and \( P(Y) \). We use the notation \( P \ll Q \), to denote that \( P \) is absolutely continuous w.r.t. \( Q \). Further, let \( P^n \) and \( Q^n \) denote the empirical measures from a set of i.i.d. samples. Moreover, let \( \mathcal{F} \) be a family of functions \( T_\theta : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \) parametrized by the weights \( \theta \in \Theta \) which can be a neural network.

To derive the Donsker-Varadhan representation of the Kullback-Leibler divergence, i.e., the starting point of MINE

\[
D_{KL}(P \| Q) = \sup_{T: \Theta} \mathbb{E}_P[T] - \log \mathbb{E}_Q[e^{T}],
\]

one can utilize Gibbs measures. Let \( G \) be any positive measure with total variation \(|G| = \int dG\). One can now identify a Gibbs’s measure \( dG = \exp(\theta T) dQ/dQ \) for a given critic \( T \), which needs to be optimized. One can see that by construction, \( Q \) dominates \( G \), i.e., \( G \ll Q \) and \( G \) is a probability measure.

Hence, we have

\[
\mathbb{E}_P \log \frac{dG}{dQ} = \mathbb{E}_P[T] - \log \mathbb{E}_Q[e^T] \leq \mathbb{E}_P \log \frac{dP}{dQ},
\]

where the inequality is due to the positiveness of the Kullback-Leibler divergence \( D_{KL}(P \| G) \), cf. [19] Th. 4]. Moreover, a non-unique optimum is \( T^* = \log \frac{dP}{dQ} + c \), i.e., the estimate of MINE can be unnormalized. However, as briefly discussed above, the MINE estimator suffers from a bias because of the \( \log \) in the marginal term, when using Monte Carlo sampling, i.e. \( \mathbb{E}_Q[\log \mathbb{E}_Q[e^{T}]] \leq \log \mathbb{E}_Q[e^T] \), which leads to instability.

An interesting new direction is obtained when we consider the family of unnormalized Gibbs measures.

**Definition 1.** An unnormalized Gibbs measure is defined through

\[
dG = \frac{e^T}{Z(T)} dQ, \quad \exists c \in \mathbb{R} : \int \frac{e^{T+c}}{Z(T+c)} dQ = 1
\]

where \( Z \) is some normalization function.

For the NWJ estimator, where \( dG = \frac{e^T}{\exp(e^{-T} \mathbb{E}_Q|c_T|)} dQ \), we have that

\[
\mathbb{E}_P \log \frac{dG}{dQ} = \mathbb{E}_P[T] - e^{-1} \mathbb{E}_Q[e^T] \leq \mathbb{E}_P \log \frac{dP}{dQ}
\]

by the simple inequality \( \log(x) \leq \frac{x}{x} \), cf. [20]. Notably, the measures identified with \( T \) in the first and second line are actually different, but all inequalities become tight for \( T^* = \log \frac{dP}{dQ} + 1 \). As discussed earlier, the resulting NWJ estimator is unbiased, due to \( \mathbb{E}_Q[\log(1 + 1 - e^{T})] = \mathbb{E}_Q[e^{T-1}] \).

B. Bounds on Variance

Lower bounds for the marginal terms of the MINE and NWJ estimator follow from the analysis in [26, Th. 2]. For MINE, the variance \( V_{G,Q} \) can be given as

\[
\lim_{n} V_{G,Q} \mathbb{E}_Q[\log \mathbb{E}_Q[e^T]] \geq e^{D_{KL}(G \| Q)} - 1,
\]

due to the independence of \( Q^n \) and \( P^n \). Thus, it can be seen that the variance of MINE and NWJ scales exponentially with the estimated mutual information. Notably, [26] has addressed this issue with the smoothed mutual information lower bound estimator (SMILE) which simply clips \( T \) in the marginal term by \( \tau \) and thereby bounds the variance of the marginal term but introduces a bias. However, since the clipping is on the marginal only the estimator is in general instable.

C. Extension with the Reverse Jensen Inequality

Our new approach is based on the partial converse of Jensen’s inequality for the \( \log(1 + X) \) function above in Theorem [1] With that result, we can now identify with any \( T \) the measure

\[
dG = \max_{a \gg b} \frac{e^T}{\exp \left( \frac{1}{\zeta(a)} \mathbb{E} \left[ \log \left( 1 + e^{T} \right) \right] \right)} dQ,
\]

where \( c(a, \mathbb{E} [e^T]) = \left( \frac{1}{\zeta(a)} + a \right)^{-\zeta(a)} \).

Hence, we can write the inequality chain

\[
\mathbb{E}_P \log \frac{dG}{dQ} = \mathbb{E}_P[T] - \min_{a \gg b} \left( \frac{1}{\zeta_b(a)} \mathbb{E} \left[ \log \left( 1 + e^{T} \right) \right] \right)
\]

\[
\leq \mathbb{E}_P[T] - \log \mathbb{E}_Q[e^T] \leq \mathbb{E}_P \log \frac{dP}{dQ}.
\]

This leads to the following estimator, which is due to its construction, a lower bound on the mutual information:

**Definition 2** (Reverse-Jensen-Estimator (RJE)).

\[
I_{RJE} := \sup_{T \in \mathcal{X}} \mathbb{E}_P[T] - \frac{1}{\zeta_b(a)} \mathbb{E}_Q \left[ \log \left( 1 + \frac{e^T}{c} \right) \right].
\]

This time (proof deferred) \( \frac{1}{\zeta_b(a)} \mathbb{E}_Q \left[ \log \left( 1 + \frac{e^T}{c} \right) \right] \geq \log \mathbb{E}_Q[e^T] \), provided \( \mathbb{E}_Q[e^T] \) is large, so that the estimator is stable. This can be also now seen from the numerical results.
IV. NUMERICAL EXPERIMENTS

We implemented two versions of our MI estimator: a) one which contains the moments ratio \( b \) defined in Lemma 1 and b) in which \( b \) is treated as a predefined parameter to be set for the problem at hand. Hence, for a batch of \( n \) pairs \((x,y)\) drawn from the joint distribution \( p_{XY} \) and \( n \) pairs \( \tilde{x}, \tilde{y} \) from the product of the marginal distributions \( p_X p_Y \) and a critic \( f : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \), an estimate of the mutual information \( I(X; Y) \) is calculated as follows:

- a) We set the parameter \( a \) in Theorem 1 to \( a = cr \) for a scalar parameter \( c > 0 \) and moments ratio

\[
r = \frac{1}{n} \sum_{i=1}^{n} e^{2f(x_i, y_i)} \left( \frac{1}{n} \sum_{i=1}^{n} e^{f(\tilde{x}_i, \tilde{y}_i)} \right)^{-2}.
\]

The MI estimate is then computed as

\[
\hat{I}(X; Y) = \frac{1}{n} \sum_{i=1}^{n} f(x_i, y_i) - \log \left( \frac{\frac{1}{n} \sum_{i=1}^{n} e^{f(\tilde{x}_i, \tilde{y}_i)}}{1 - cr \frac{1}{n} \sum_{i=1}^{n} e^{f(\tilde{x}_i, \tilde{y}_i)}} \right) - \frac{c}{1 - \sqrt[1/c]{n}} \sum_{i=1}^{n} \log \left( 1 + e^{f(\tilde{x}_i, \tilde{y}_i)} \right).
\]

- b) For scalar parameters \( 0 < b < a \), we estimate the MI as

\[
\hat{I}(X; Y) = \frac{1}{n} \sum_{i=1}^{n} f(x_i, y_i) + \log \left( \frac{1}{n} \sum_{i=1}^{n} e^{f(\tilde{x}_i, \tilde{y}_i)} \right) + a
\]

\[
- \frac{a}{1 - \sqrt{b/a}} \sum_{i=1}^{n} \log \left( 1 + e^{f(\tilde{x}_i, \tilde{y}_i)} \right).
\]

The performance of the RJE is assessed through a setup with two multivariate Gaussian random variables \( X, Y \in \mathbb{R}^n \) with means 0 and covariance matrices \( C_X = C_Y = I_n \) and \( C_{XY} = \nu I_n \) for \( \nu \in (0, 1) \). Hence,

\[
\begin{bmatrix} X \\ Y \end{bmatrix} \sim \mathcal{N}(0, C), \quad C = \begin{bmatrix} I_n & \nu I_n \\ \nu I_n & I_n \end{bmatrix},
\]

and the mutual information is given by \( I(X; Y) = -\frac{1}{2} \log \det C \). The critic \( f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) computes scores from samples drawn from the joint distribution \( p_{XY} \), and from the product of marginal distributions \( p_X p_Y \), obtained by pairing the samples of one batch from \( X \) with shuffled samples from \( Y \), which are then fed into the estimator to produce the MI estimates. The architecture of the critic network is as follows: The samples from \( X \) and \( Y \) are concatenated and then forwarded through two fully connected layers with 100 neurons and relu activations. The final layer is a fully connected layer with linear activation and 1 output neuron. For dimension 50 and varying parameter \( \nu \), estimator a) is trained for 400 epochs with a batch size of 256. In the estimator, we set \( c = 2 \). The estimated MI and its variance over 1000 batches is shown in Fig. 1 and 2. It is worth noting that the RJE is stable without any clipping whatsoever as suggested.

V. DISCUSSION AND CONCLUSIONS

We proved a reversion of the Jensen inequality for a class of functions with relevance in statistics and machine learning. From these results, we derived an estimator for mutual information from finite samples and evaluated its performance on synthetic data. These types of estimators have gained huge interest in the field of machine learning due to the use of MI as regularization for generative models [27], [28] and as objective for representation learning [6], among others. Black box estimators of MI such as the one derived here suffer from fundamental limitations [25], and many of the estimators found in the literature either fail to capture high MI, or exhibit huge variance, stemming from the marginal term in the variational formulation. By applying our reverse Jensen result, the expectation over the product of marginal distributions on \( X \) and \( Y \) is pulled out of the logarithm, thereby reducing the variance of the estimation with only a small bias, making the estimator very stable. The proper tuning of the free parameters in our MI estimators, as well as a thorough investigation of its performance in the machine learning applications mentioned above are a topic of future research.
REFERENCES

[1] T. Jehara and A. Pentland, “On reversing Jensen’s inequality,” 2001, pp. 231–237.

[2] I. Budimir, S. S. Dragomir, and J. Pecaric, “Further reverse results for Jensen’s discrete inequality and applications in information theory,” Journal of Inequalities in Pure and Applied Mathematics, vol. 2, no. 1, p. 5, 2001.

[3] S. S. Dragomir and M. Ionescu, “Some converse of Jensen’s inequality and applications,” Revue d’Analyse Numérique et de Théorie de l’Approximation, vol. 23, no. 1, pp. 71–78, 1994.

[4] S. S. Dragomir, “Some reverses of the Jensen inequality with applications,” Bulletin of the Australian Mathematical Society, vol. 87, no. 2, pp. 177–194, 2013.

[5] S. Khan, M. Adil Khan, and Y.-M. Chu, “Converses of the Jensen inequality derived from the green functions with applications in information theory,” Mathematical Methods in the Applied Sciences, vol. 43, no. 5, pp. 2577–2587, 2020.

[6] R. D. Hjelm, A. Fedorov, S. Lavoie-Marchildon, K. Grewal, P. Bachman, A. Trischler, and Y. Bengio, “Learning deep representations by mutual information estimation and maximization,” in Proc. 7th Int. Conf. Learning Representations, New Orleans, LA, USA, May 2019.

[7] R. Fritsche, R. F. Schafer, and G. Wunder, “Deep learning for channel coding via neural mutual information estimation,” in Proc. 26th IEEE Int. Workshop Signal Process. Adv. Wireless Commun., Cannes, France, Jul. 2019, pp. 1–5.

[8] ——, “Deep learning based wiretap coding via mutual information estimation,” in Proc. 2nd ACM Workshop Wireless Security and Machine Learning, Linz, Austria, Jul. 2020, pp. 74–79.

[9] ——, “Neural mutual information estimation for channel coding: State-of-the-art estimators, analysis, and performance comparison,” in Proc. 21th IEEE Int. Workshop Signal Process. Adv. Wireless Commun., Atlanta, GA, USA, May 2020, pp. 1–5.

[10] G. Wunder, J. Schreck, and P. Jung, “Nearly doubling the throughput of multiuser MIMO systems using codebook tailored limited feedback protocol,” IEEE Trans. Wireless Commun., vol. 11, no. 11, pp. 3921–3931, Nov. 2012.

[11] W. Shin, G. Park, G. Wunder, S. Baek, and J. Kang, “Generalized subband-filtered and pulse-shaped multicarrier for quasi-synchronous uplink access,” in 28th Annu. IEEE Int. Symp. Personal, Indoor, and Mobile Radio Commun., Montreal, QC, Canada, Oct. 2017, pp. 1–7.

[12] A. M. Fraser and H. L. Swinney, “Independent coordinates for strange attractors from mutual information,” Phys. Rev. A, vol. 33, pp. 1134–1140, Feb. 1986.

[13] G. A. Darbellay and I. Vajda, “Estimation of the information by an adaptive partitioning of the observation space,” IEEE Trans. Inf. Theory, vol. 45, no. 4, pp. 1315–1321, May 1999.

[14] A. Kraskov, H. Stögbauer, and P. Grassberger, “Estimating mutual information,” Phys. Rev. E, vol. 69, p. 066138, Jun 2004.

[15] S. Gao, G. Ver Steeg, and A. Galstyan, “Efficient estimation of mutual information for strongly dependent variables,” in Proc. 18th Int. Conf. Artificial Intelligence and Statistics, San Diego, CA, USA, May 2015, pp. 277–286.

[16] W. Gao, S. Oh, and P. Viswanath, “Demystifying fixed k-nearest neighbor information estimators,” IEEE Trans. Inf. Theory, vol. 64, no. 8, pp. 5629–5661, Aug. 2018.

[17] T. Suzuki, M. Sugiyama, J. Sese, and T. Kanamori, “Approximating mutual information by maximum likelihood density ratio estimation,” in New challenges for feature selection in data mining and knowledge discovery, 2008, pp. 5–20.

[18] D. Barber and F. Agakov, “The IM algorithm: A variational approach to information maximization,” in Proc. 16th Int. Conf. Neural Information Processing Systems . MIT Press, 2003, pp. 201–208.

[19] I. Belghazi, S. Rajeswar, A. Baratin, R. D. Hjelm, and A. Courville, “MINE: Mutual information neural estimation,” in Proc. 35th Int. Conf. Machine Learning, Stockholm, Sweden, Jul. 2018.

[20] B. Poole, S. Ozair, A. van den Oord, A. A. Alemi, and G. Tucker, “On variational lower bounds of mutual information,” in NeurIPS Workshop on Bayesian Deep Learning, 2018.

[21] X. Nguyen, M. J. Wainwright, and M. I. Jordan, “Estimating divergence functionals and the likelihood ratio by convex risk minimization,” IEEE Trans. Inf. Theory, vol. 56, no. 11, pp. 5847–5861, Nov. 2010.

[22] S. Nowozin, B. Cseke, and R. Tomioka, “f-GAN: Training generative neural samplers using variational divergence minimization,” in Advances in Neural Information Processing Systems, 2016, pp. 271–279.

[23] A. van den Oord, Y. Li, and O. Vinyals, “Representation learning with contrastive predictive coding,” arXiv preprint arXiv:1807.03748, 2018.

[24] A. Sinha, J. Song, and S. Ermon, “Hybrid mutual information lower-bound estimators for representation learning,” in Neural Compression: From Information Theory to Applications–Workshop @ ICLR 2021, 2021.

[25] D. McAllester and K. Stratos, “Formal limitations on the measurement of mutual information,” in Proc. 23rd Int. Conf. Artificial Intelligence and Statistics, Aug. 2020, pp. 875–884.

[26] J. Song and S. Ermon, “Understanding the limitations of variational mutual information estimators,” in Proc. 8th Int. Conf. Learning Representations, Addis Ababa, Ethiopia, Apr. 2020.

[27] X. Chen, Y. Duan, R. Houthooft, J. Schulman, I. Sutskever, and P. Abbeel, “Infogan: Interpretable representation learning by information maximizing generative adversarial nets,” in Advances in Neural Information Processing Systems, 2016, pp. 2172–2180.

[28] D. P. Kingma and M. Welling, “Auto-encoding variational bayes,” arXiv preprint arXiv:1312.6114, 2013.