General relativity can be recast as a theory of connections by performing a canonical transformation on its phase space. In this form, its (kinematical) structure is closely related to that of Yang-Mills theory and topological field theories. Over the past few years, a variety of techniques have been developed to quantize all these theories non-perturbatively. These developments are summarized with special emphasis on loop space methods and their applications to quantum gravity.

1. Introduction

In the first conference in this series, held at Goa in 1988, I presented some results on a new Hamiltonian formulation of general relativity and outlined how these results could be used in the construction of a non-perturbative quantum theory of gravity. During the last three years, this general program has been pursued vigorously by a number of groups. The key idea in the initial stage was to exploit the fact that, in its new version, the phase space of general relativity is the same as that of theories of connections –gauge theories and topological field theories– to import into quantum gravity ideas and techniques from these better understood theories. It soon turned out, however, that the methods initially developed for quantum gravity could in turn be applied to other theories of connections. Thus, there has been a synergetic exchange of ideas between general relativity and gauge theories and a variety of new results have been obtained. Indeed, in the space allotted to me it would be impossible to do justice to all the developments that have occurred. Fortunately, there already exist in the literature a number of detailed reviews: there is a monograph [1] addressed to research students which treats all the basic issues with due care to mathematical subtleties; there is a more compact review [2] which appeared in a journal, addressed to standard physics audiences; and, there is a summer school report [3], geared to particle physicists and field theorists, which emphasizes the more recent work in this area.

In this article, therefore, I shall not attempt to present a comprehensive summary. Rather, my aim is to outline the main directions in which the work has proceeded since the Goa conference and to supply references where details can be found. When one tries to squeeze diverse themes along which research has naturally progressed into a few categories, there are inevitable omissions. I apologize in advance to my colleagues if they find that their favorite ideas have unfortunately been skipped.

I have divided the material according to themes. Consequently, individual sections vary a great deal in their length. Section 2 is devoted to the developments in classical general relativity and geometry. Section 3 addresses the general quantization program, emphasizing the points at which it goes beyond Dirac’s treatment of quantization of constrained systems. Section 4 is devoted to the quantum theory. Open problems and directions of current work are summarized in section 5.

2. Hamiltonian general relativity and geometry:

In the new Hamiltonian framework, it is convenient to begin with complex general relativity. The phase space then is the same as that of $SO(3)$ Yang-Mills (or, topological field) theory: the configuration variable is a complex-valued, $SO(3)$ connection, $A_i^a(x)$, and the canonically conjugate momentum is a vector density $E^a_i(x)$ of weight 1, where $a$ is a (tangent space) vector index while $i$ is an internal, $SO(3)$ index. In Yang-Mills theory, $E^a_i$ has the interpretation of the electric field. In general relativity, it represents a (density weighted) triad. There are seven first class constraints:

\[ \mathcal{D}_a E^a_i = 0; \quad E^a_i F_{ab} = 0; \quad \text{and} \quad \epsilon^{ijk} E^a_i E^b_j F_{abk} = 0; \]

where $\mathcal{D}$ is the gauge covariant derivative operator defined by $A_i^a$ and $F_{ab}^k := 2 \partial (A_i^a) B^i + G \epsilon^{ijk} A_i^a A_j A_k$ is the field strength of $A_i^a$. (G is Newton’s constant. Note that $GA_i^a$ has dimensions of inverse length, the usual physical dimensions of a connection.) Since the first three of these constitute precisely the Gauss
constraint of gauge theories, we see that the constraint surface of general relativity is in fact embedded in that of SO(3)-gauge theories. The remaining four constraints are the “genuine relativity constraints”. The first three of these –called collectively the vector constraint– generate spatial diffeomorphisms while the last one –the scalar constraint– generates “pure” time evolution. Apart from a surface term, the Hamiltonian is a linear combination of these seven constraints. Like the constraint functionals, the evolution equations are low order polynomials in the basic canonical variables. For example, setting the Lagrange multipliers of the Gauss and the vector constraints equal to zero, for “pure” time evolution, we have the following equations of motion:

\[ \dot{A}_i^a = iN E_i^b F_{b a k} \epsilon^{i j k} \quad \text{and} \quad \dot{E}_i^a = i D_b (N E_i^a E_b^k) \epsilon^{j k} \]

where \( N \) is the lapse, i.e., the Lagrange multiplier of the scalar constraint. All these equations are significantly simpler than those encountered in geometrodynamics, where the 3-metric, rather than a connection, is the configuration variable. In addition, the Hamiltonian description of general relativity is now closely related to that of gauge theories. As we will see, these features have been exploited in a number of ways in the recent developments.

Note however that so far we have considered complex relativity. To recover the real theory, we have to restrict ourselves to the appropriate “real section” of the complex phase space. In the early papers there was some confusion on the expression and implementation of the reality conditions. This issue has been clarified now: the reality conditions again involve only low order polynomials in the basic canonical variables. (See [4] and chapters 8, 3.3 and 4.3 of [1]. While I will restrict myself to the source-free theory in this article, all results mentioned above have been extended in [4,5] to include matter fields and the cosmological constant.)

In Euclidean general relativity, these conditions require simply that \( A_i^a \) and \( E_i^a \) be real. In the Lorentzian theory, on the other hand, although they involve expressions which are (at worst) only quartic in \( A_i^a \) and \( E_i^a \), their implementation is not so straightforward. Thus, it does appear that the simplicity of the field equations has been attained, to some extent, at the cost of having to deal with the reality conditions. For example, the dynamical trajectories in the space of connections can be interpreted as “null geodesics” of a super-metric. This interpretation provides a useful technique in Bianchi models to find the qualitative behavior of solutions and, in many cases, complete solutions. However, in general, it is hard to pick out the null geodesics which would correspond to real, Lorentzian solutions. More generally, the implementation of the Lorentzian reality conditions remains one of the most important open problems in the program. However, some work has been done to make these conditions more manageable and I will return to this point in sections 3-5.

Since the basic variable is a connection, it is tempting to look for phase space variables which are manifestly gauge invariant and base quantization on them. For configuration variables, there is an obvious, natural choice: traces of holonomies around closed loops. More precisely, given any closed loop \( \gamma \) (in the “spatial” 3-manifold on which \( A_i^a \) are defined), we can set:

\[ Q_{\gamma} (A) := \frac{1}{2} \text{tr} \mathcal{P} G \oint_{\gamma} d^{\sigma} A_{\alpha} , \]

where trace is taken in the spin \( \frac{1}{2} \)-representation of SO(3), \( \mathcal{P} \) stands for “path-ordered” and \( G \), as before, is Newton’s constant. The vector space generated by these \( Q_{\gamma} \) is closed under the product because of the Madelstam identities [6,7]. To get momentum variables –i.e., functions on the phase space which are linear in momenta– we introduce 2-dimensional (closed) strips. A strip \( S \) is a mapping from \( S^1 \times (-\epsilon, \epsilon) \) to the 3-manifold. Thus, a strip is coordinated by two parameters, \( \sigma \in [0, 2\pi] \) and \( \tau \in (-\epsilon, \epsilon) \). (These coordinates can be changed by a reparametrization. The invariant structure is the topology \( S^1 \times \mathbb{R} \) and the foliation of \( S \) by a fixed family of circles.) With each strip \( S \), we associate a momentum variable

\[ P_S (A, E) := \int_S dS^{ab} \eta_{abc} \text{tr} E^c U_{\tau} (\sigma, \tau) , \]

where \( U_{\tau} \) is the group element representing the (untraced) holonomy of the connection around the loop \( \tau = \text{const} \). Clearly, these functions are also gauge invariant. Now, the surprising result is that the loop-strip (i.e these configuration and momentum) variables are in fact closed under the Poisson bracket! Furthermore, the structure constants involved can be constructed from simple geometric properties of loops and strips in
three-dimensional space [1,2,7]. Finally, these variables are (over)complete almost everywhere on the phase space. An exhaustive characterization of regions (sets of “measure zero”) of the phase space where they fail to be complete and the structure of the intersection of these regions with the constraint surface and the real section of the phase space is available. However, results of [8] are based on the “small $T$-algebra” of Rovelli and Smolin [6] which is closely related to but not the same as the loop-strip algebra discussed above. Therefore, strictly speaking, some further work is needed to show that the last set of results holds also for the loop-strip algebra [8]. It is therefore natural to use the loop-strip algebra as the starting point in quantum theory.

The simple form of the constraints has led to two ingenious methods of solving them. Capovilla, Jacobson and Dell (CDJ) [9] have pointed out that the scalar and the vector constraint can be solved simply by making an algebraic ansatz for $E^i_k$ in terms of $A^i_k$. Furthermore, the solution is generic. This is a striking result. However, one is still left with the Gauss constraint and a useful technique to impose it on the CDJ “free data” has not yet emerged. More recently, Newman and Rovelli [10] have used Hamilton-Jacobi techniques to address a more important problem: that of finding gauge invariant free data. They have been able to treat in this way the Gauss and the vector constraints. Thus their analysis provides elegant geometric coordinates on the phase space reduced with respect to these six constraints. Work is in progress on the scalar constraint. If this program is carried to completion, one would have available a complete set of Dirac-observables for full general relativity. These would play a key role in quantization. Note, however, that both the CDJ and the Newman-Rovelli techniques have so far been applied to the complex theory and imposition of reality conditions does not appear straightforward.

The choice of connections $A^i_k$ as the configuration variables is also natural from the viewpoint of dynamics of the theory. First, there is a striking geometric result [8]: the holonomy group of $A^i_k$ is the same on any Cauchy slice of the real, Lorentzian space-time; it is a “constant of motion”. Second, because the scalar constraint is purely quadratic in momenta—in contrast to geometrodynamics, there is no “super-potential” in connection-dynamics—the dynamical trajectories in the connection superspace can be interpreted as the null geodesics of the “super-metric” $\epsilon^{ijk} F_{abk}$. There are thus numerous advantages in regarding general relativity as a theory of dynamics of (self-dual) 3-connections rather than of 3-metrics. This viewpoint was pushed to its logical extreme by Capovilla, Dell and Jacobson [9]. They begin with an action which is built out only of self-dual connections and which, in its Hamiltonian version, reproduces the description given above. In this description then, the “triads” $E^i_k$ appear only as momenta canonically conjugate to the 3-connection. The action, or indeed the entire framework, knows nothing about a space-time metric which can, if one wishes, be introduced as a secondary field constructed from the curvature of the connection. This analysis has had a number of interesting off-shoots. For example, it led Capovilla [11] to construct a family of “neighbors of general relativity”. In the Hamiltonian version of these theories, the configuration variable is again a connection, the Gauss and the vector constraints have the same form as in Eq. (1) but the scalar constraint has additional terms. Recently, Peldán [12] has taken this work further by showing that one can replace the internal group $SO(3)$ by any Lie group of dim $\geq 3$; the theory then represents gravity coupled to matter although the physical meaning of these coupleings is still unclear.

Finally, there are the applications of the framework to geometry. A number of interesting results about half-flat metrics as well as Einstein spaces with self-dual Weyl curvature have been obtained in the Riemannian (i.e., $++++$) regime [13]. Perhaps the most striking of these is the exhaustive analysis of the structure of the moduli space of gravitational instantons with self dual Weyl curvature and positive cosmological constant due to Torre [14]. This imaginative analysis is an illustration of how the relation between the Yang-Mills theory and general relativity discussed above can be exploited to obtain rigorous results of interest to differential geometry.

### 3. Extension of the Dirac program

Attempts at canonical quantization of general relativity have traditionally followed the general program introduced by Dirac for quantization of constrained systems [15]. The Dirac program, however, is incomplete in one important respect: while it tells us that constraints should be incorporated in the quantum theory as conditions which select the physical states, it provides no guidelines for introducing the appropriate inner product on the space of these physical states. In particle mechanics or Minkowskian field theories this does
not pose problems in practice because one can use the available symmetries to select a preferred inner-product. For examples, in Minkowskian field theories, one selects the vacuum state by invoking Poincaré invariance and then uses the vacuum expectation values of physical operators to obtain the required Hilbert space structure. In quantum gravity, on the other hand, such a space-time group of symmetries is simply not available. (One might imagine using the spatial diffeomorphism group to select the vacuum. However, the strategy fails because every physical state is invariant under this group.) Therefore, Dirac’s program has to be supplemented with a new guiding principle.

While this problem exists for any canonical approach to quantum general relativity, there are additional features which are peculiar to our specific approach which also require an extension of the Dirac program. First, for real Lorentzian relativity, the “canonical” variables constitute a hybrid pair: the Lorentzian reality conditions require that $E_i^a$ be real, while $A_i^a$ is allowed to be complex (but such that the time-derivative of $E_i^a$ is again real). Consequently, the canonical quantization procedure itself is at first somewhat obscure. For example, in the $A_i^a$ representation, the states $\Psi(A)$ are now holomorphic functionals of connections $A_i^a$. Can one still represent the $E_i^a$ operator by the functional derivative with respect to $A_i^a$? A second problem is associated with the loop-strip variables discussed in section 2. Because they are manifestly gauge invariant and closed under the Poisson brackets, it is attractive to use their Poisson algebra as the starting point in canonical quantization. However, unlike the habitual canonical pairs such as the metric and the extrinsic curvature, the loop-strip variables are overcomplete almost everywhere on the phase space. How does one handle this overcompleteness in the quantum theory? Further, there are regions (although of “measure zero”) of the classical phase space, where these variables fail to be complete. Do we now have a problem just of the opposite sort in quantum theory?

Since the Goa conference, these issues have been analysed in detail and an appropriate extension of the Dirac program has been constructed. (See chapter 10 of [1].) The idea is to use an algebraic approach. One first constructs an abstract algebra of quantum operators based on a given Poisson algebra, and then looks for its representations. The possible relations that may exist due to overcompleteness of the basic classical variables—such as the loop-strip functionals—are now built in to the very structure of the quantum algebra. Similarly, the reality conditions in the classical theory are to be coded in the $*$-relations of the quantum algebra. These $*$-relations in turn are to determine the inner-product: one seeks that inner-product with respect to which the $*$-relations, defined abstractly on the algebra, are realized as the Hermitian adjoint relations on the Hilbert space. Thus, far from being a nuisance, the classical reality conditions may ultimately determine the quantum inner-product. This specific idea has been tested in source-free Maxwell theory [16,17], linearized gravity [18] and 2+1-dimensional gravity ([19] and chapter 17 of [1]). The first two examples are especially instructive because in these Minkowskian field theories, the new strategy leads one to the correct inner product without having to appeal to the Poincaré group. Finally, in this program, the issue of time is decoupled from that of finding the inner-product. In the 2+1-theory, for example, the issue of singling out a time variable is almost as hard as in the 3+1 theory: both theories are diffeomorphism invariant and therefore devoid of a background metric. Using the general program sketched above, we are able to find the required inner product on physical states without having to first isolate the time variable.

This extension of the Dirac program is meant to provide general guidelines; it is not a rigid set of rules. It has been carried out to completion in a number of examples which mimic various features of the new Hamiltonian formulation of general relativity [20] and this detailed analysis has provided confidence in the underlying ideas. It has also given rise to a number of conjectures. For example, we suspect that if there exists an inner-product which can implement the $*$-relations faithfully and if the resulting representation of the algebra is irreducible, then the inner product is unique (up to an overall constant factor). Such conjectures remain unproven however and further work is clearly needed. Much of this work would rely only on general techniques from mathematical physics; these results should therefore have validity well beyond quantum general relativity.

4. Quantum general relativity

In quantum general relativity, several developments have occurred. The results obtained so far clarify a number of issues and provide considerable support for the strategies which were outlined at Goa. Furthermore, as we shall see, some of these results are quite striking and extremely encouraging. Nonetheless, the program as a whole is still far from being complete. For example, while a number of new solutions to all quantum constraints have been found, the set of solutions obtained so far is still quite incomplete.
Perhaps more importantly, there remain several conceptual problems. Some of these are common to any non-perturbative approach to quantum gravity. Examples are: singling out useful observables, resolving the issue of time and interpreting the framework as a whole. There are others which are specific to the approach being pursued. By now, there does exist a large body of results that has been obtained using the loop variables and the general quantization techniques summarized in section 3. However, when looked at in detail, one finds that these results do not quite fit together in to a precise and uniform mathematical framework. There are several gaps that need to be filled before we have “global understanding”. (One should perhaps emphasize, however, that this issue has become relevant precisely because the program has reached a certain level of maturity, not enjoyed by other non-perturbative approaches to quantum general relativity.) In this section, I will summarize the results obtained so far as well as the puzzles that still remain.

4.1 Regularization and Weaves

As explained in the Goa conference, Rovelli and Smolin [6] introduced a new representation for quantum general relativity in which states are certain functionals of closed loops. This provides a representation of the loop-strip algebra on a vector space – the space of loop states One can equip the space of loop states with an Hermitian inner product (using the Gel’fand-Naimark-Segal construction) in which, moreover, the diffeomorphism group of the underlying 3-manifold acts unitarily. However, it is not obvious that this is the “correct” inner-product especially because the resulting Hilbert space is non-separable. Therefore, most results have been obtained using just the vector space structure of the space of loop functionals. The action of the loop-strip operators involves simple geometric operations such as breaking, re-routing and gluing loops. This is the starting point for quantization. Perhaps the most interesting of the “kinematic” results so obtained are the following: Regularization of operators respecting the (3-dimensional) diffeomorphism invariance of the theory and existence of states that approximate a flat metric on a large scale but necessarily exhibit a discrete structure on the Planck scale. These states are called weaves; they represent how a macroscopic, classical geometry can be “woven” using excitations along loops as “quantum threads”.

Let us begin with regularization. Since a basic phase space variable is a (density weight one) triad \(E_i^a(x)\), the spatial metric (of density weight two) is a “composite” field given by \(q^{ab}(x) = E_i^a(x)E_i^b(x)\). In the quantum theory, therefore, this operator must be regulated. The obvious possibility is point splitting. One might set \(q^{ab}(x) = \lim_{y \to x} E^{ai}(x)E^b_i(y)\). However, the procedure violates gauge invariance since the internal indices at two different points have been contracted. A gauge invariant prescription is to use the Rovelli-Smolin loop variable \(T^{ab}([\gamma])(x,y)\) defined in the classical theory by

\[
T^{aa'}([\gamma])(y, y') := \text{tr} \left[ (\mathcal{P} \exp G \int_{y'} y A_a dl^a) E^{a'}(y') (\mathcal{P} \exp G \int_{y} y' A_a dl^a) E^a(y) \right],
\]

where \(y\) and \(y'\) are any two points on the loop \(\gamma\), and note that in the limit \(\gamma\) shrinks to zero, \(T^{aa'}([\gamma])(y, y')\) tends to \(q^{aa'}\). In quantum theory, one can formally define the action of the operator \(T^{aa'}([\gamma])(y, y')\) directly on the loop states. As with other loop operators, it acts by breaking and re-routing the loops that appear in the argument of the quantum state. One may therefore try to define a quantum operator \(\hat{q}^{aa'}\) as a limit of \(\hat{T}^{aa'}([\gamma])\) as \(\gamma\) shrinks to zero. The resulting operator does exist after suitable regularization and renormalization. However, because of the density weights involved, the operator necessarily carries memory of the background metric used in regularization. Roughly, this comes about as follows. The operator in question is analogous to the product \(\delta^3(x) \cdot \delta^3(x)\) of distributions at the same point. To regulate it, we have to introduce a background metric. The final result is a distribution of the form \(N(x)\delta^3(x)\) where, because \(\delta^3(x)\) is a density of weight one, the renormalization parameter \(N\) is now a density of weight one, proportional to the determinant of the background metric. Since the final answer carries a memory of the particular metric used to regulate the operator, we have violated diffeomorphism invariance. Although there is no definitive proof, there do exist arguments which suggest that any local operator carrying the information about geometry will face the same problem.

There do exist, however, non-local operators which can be regulated in a way that respects diffeomorphism invariance [21,3]. Furthermore, the resulting operators are finite without the need of any renormalization. Thus, there are no free renormalization constants in the final expressions. As the first example, of loop functionals.
consider the function $q(\omega)$—representing the smeared 3-metric—on the classical phase space, defined by

$$q(\omega) := \int d^3 x \left( q^{ab} \omega_a \omega_b \right)^{1/2},$$

where $\omega_a$ is any smooth 3-form of compact support. (Note that the integral is well-defined without the need of a background volume element because $q^{ab}$ is a density of weight two.) The corresponding operator is defined as follows: First re-express $q(\omega)$ using the loop variable $T^{ab}$ of Eq. (5), then replace $T^{ab}$ by a regulated operator $\bar{T}^{ab}$ on the loop states and finally take the limit as the regulator $\epsilon$ goes to zero. The result, $\bar{q}(\omega)$, is a well-defined operator on the loop states; it carries no imprint of background structures used in regularization. For example, if $\gamma$ is a smooth loop without self-intersections, we have a rather simple action:

$$\bar{q}(\omega) \cdot \Psi(\gamma) = l_P^2 \int_\gamma ds |\gamma^a \omega_a| \cdot \Psi(\gamma),$$

where $l_P = \sqrt{\bar{\gamma}}\bar{h}$ is the Planck length, $s$, a parameter along the loop and $\gamma^a$ the tangent vector to the loop. (The $G$ in $l_P$ comes from the fact that $G\bar{A}^a_\mu$ has the usual dimensions of a connection (see Eq. (3)) and $\bar{h}$ comes from the fact that $\bar{E}^a_\mu$ is $\bar{h}$ times a functional derivative.) One can similarly define the operator corresponding to the area $\mathcal{A}_S$ of a smooth 2-surface $S$. The result is:

$$\mathcal{A}_S \cdot \Psi(\gamma) = l_P^2 I(\gamma, S) \cdot \Psi(\gamma),$$

where $I(\gamma, S)$ is the unoriented intersection number between the loop $\gamma$ and the surface $S$. Thus, the final result is simple and geometrical: each intersection of the loop with the surface contributes a Planck area to the surface. If the state has a support on a single loop, it would clearly resemble a classical geometry; most surfaces would have no intersection with that loop whence the area of most surfaces would be simply zero. Such a state would represent “an elementary” excitation of geometry; like a 1-photon state in the classical geometry, it wold not have a classical analog. To approximate a classical geometry, the state must involve many loops so that given any smooth surface $S$ with area $A$ in the classical geometry, there are approximately $A/l_P^2$ intersections between $S$ and the loops.

With these operators at hand, we can now introduce weaves. Fix on $\mathbb{R}^3$ a flat metric $h_{ab}$ and let us ask if we can construct states which approximate $h_{ab}$ on a scale $L$ which large compared to $l_P$. Note the logic of the argument: we first fix a flat metric we want to approximate and then use it repeatedly to define the length scales need in the argument. This procedure is necessary because in non-perturbative quantum gravity there is no background metric and therefore no a physical notion of length. In particular, although we can set $l_P = \sqrt{\bar{\gamma}}\bar{h}$, this quantity does not represent a physical length unless we have a metric which enables one to measure lengths. His question can be phrased without reference to a specific inner-product because the operators $\bar{q}(\omega)$ and $\mathcal{A}_S$ carrying information about geometry are “diagonal” in the loop representation. That is, using the form of the operators (7) and (8) we can ask: Are there states $\Psi(\gamma)$ for which

$$\bar{q}(\omega) \cdot \Psi(\gamma) = [h(\omega) + O(\tfrac{l_P}{L})] \cdot \Psi(\gamma),$$

where $h(\omega) := \int d^3 x (h^{ab} \omega_a \omega_b)^{1/2}$ is the value that $q(\omega)$ assumes at the given flat metric $h_{ab}$, and for which

$$\mathcal{A}_S \cdot \Psi(\gamma) = [\mathcal{A}_S(h) + O(\tfrac{l_P}{L})] \cdot \Psi(\gamma),$$

where $\mathcal{A}_S(h)$ is the area of $S$ measured by the flat metric $h_{ab}$. It turns out that such states do exist but they display a discrete structure of a specific type at the Planck length. Roughly, the situation is as follows. To recover $h_{ab}$, the state must have excitations along loops which are separated by the Planck length as measured by $h_{ab}$. If the loop separation is large, the metric represented by the state goes to zero. In the continuum limit, on the other hand, where the loop separation goes to zero, the metric represented by the state diverges.
It is precisely when the loop separation is \( l_P \) that one recovers \( h_{ab} \). I should emphasize, however, that this is a simplified, qualitative picture and the condition on the loop separation is only necessary and not sufficient. The detailed calculation is quite involved and uses the “uniformity” of the flat metric \( h_{ab} \). In particular, although qualitative ideas are available, detailed constructions of weaves approximating general 3-metrics have not yet been attempted.

That something unusual should happen at the Planck length has been anticipated for a long time. Indeed, a number of approaches to quantum gravity begin by assuming some discrete structure at the Planck scale and then attempt to recover familiar macroscopic physics from it. The situation in the present treatment is quite different. Here, one begins with well-established principles of general relativity and quantum mechanics and combines them using loop variables. The framework then predicts that there should be a discrete structure, and indeed of a specific type, at the Planck scale.

Note finally that although I have used the loop representation here for concreteness, the same result could have been obtained using the connection representation where quantum states are holomorphic functionals \( \Psi(A) \) of the connection \( A^i_a \). It is the use of loop variables in the regularization process that is critical to the argument.

4.2 Knots and links.

In the canonical approach to quantum general relativity, dynamics is governed by constraints. Consequently, a key step in the program is to impose quantum constraints to single out physical states. It is here that the simplicity of the expression (1) plays an important role: In sharp contrast to the Wheeler-DeWitt equation of quantum geometrodynamics, infinitely many solutions to the full set of quantum constraints has been available in quantum connection-dynamics.

Almost all known solutions have been obtained using the loop representation. Furthermore, we have not been able to translate the answer back into the connection (or the metric) representation in most cases. Thus, while the loop representation is not essential in the discussion of kinematic issues of section 4.1, it does seem to play a fundamental role in the discussion of dynamics.

In broad terms, so far, two avenues have been followed to solve the constraints. The first, initiated by Rovelli and Smolin [6] and adopted with some variations by Blencowe [22] and others [23], deals directly with suitable functionals of loops. The second approach, initiated by Gambini and his collaborators [24], begins by introducing what one may call “SU(2)-form factors” of loops. That is, given any loop, one introduces certain fields on the spatial 3-manifold which capture precisely that information about the loop which needed to construct traces of holonomies of arbitrary SU(2) connections. (These are non-trivial generalizations of the “U(1)-form factors” introduced in [17] to discuss quantum Maxwell fields in the loop representation.) Quantum states are then represented as suitable functionals of the form factors. An advantage of this approach is that the states have a form which is familiar from other field theories: they are functionals of fields on 3-manifold rather than on the loop space. In both approaches, one formulates the quantum constraints as linear operators on the wave functionals one is dealing with and looks for their kernel. The detailed techniques used are, however, different and the two approaches complement each other rather well.

The results may be summarized as follows. A key feature of the loop representation is that the Gauss constraint is automatically satisfied; everything one does is manifestly gauge invariant. It therefore remains to impose the vector and the scalar constraints. The vector constraint implies that the physical states are functionals only of diffeomorphism equivalence classes of loops, i.e., of knot classes of individual smooth loops, or link classes of smooth multi-loops. In the Gambini approach, these knot invariants are constructed out of the “form factors”; one is led to use (infinite-dimensional) differential geometrical methods to construct these invariants. For example, the Gauss linking number arises from a metric on the infinite dimensional space of divergence free vector fields which constitute the simplest of the “form factors”. This is an interesting development in mathematical physics, and may play a role also outside quantum gravity. In general, however, the states may have support on loops with self-intersections or corners. One is therefore led to consider “generalized” knot and link classes. Thus, any functional on the space of loops (possibly with self-intersections and corners) which assumes the same value on loops related by smooth diffeomorphisms satisfies the vector constraint. Furthermore, this is the general solution! This result is probably the most striking and definitive application of the loop representation to quantization of diffeomorphism invariant theories. The fact that the space of knot (or link) classes —now the domain space of quantum states— is discrete is expected to simplify a number of mathematical problems in the remainder of the program.
The requirement that the norm be finite can in fact carry the crucial physical information. For example, if one solves the eigenvalue equation for the harmonic oscillator in the position representation, one finds that there exist eigenstates $\Psi(x)$ for any value of energy. It is the requirement that the norm be finite that enforces both positivity and quantization of energy. Note however that whether all solutions to the eigenvalue equation are normalizable depends on the choice of representation. For example, if states are taken to be holomorphic functions of $z = q - ip$, every eigenstate $\Psi(z)$ turns out to be normalizable whence the conclusion that energy is positive and quantized can be arrived at simply by solving the eigenvalue equation. Of course, a priori it is not clear whether the loop representation is analogous to the $x$ representation of the $z$ representation in this respect. Finally, the regularization procedure used for finding these solutions is not as clear-cut as that discussed in section 4.1.

### 4.3 Global picture

The results discussed in the previous two sections are based on the general framework introduced by Rovelli and Smolin [6] in quantum general relativity. However, at certain steps, their construction is only formal. It is therefore appropriate to ask if there is a precise mathematical structure underlying the loop representation and if one can use the loop space methods in familiar theories where we already knows what the “correct answer” is.

The gain confidence as well as physical insight into these methods, therefore, (source-free) Maxwell theory and linearized gravity were considered in detail. In the Maxwell case [17], the starting point is the Bargmann representation in which states are holomorphic functionals of positive frequency Maxwell fields and one arrives at the loop representation by performing a transform along the lines suggested by Rovelli and Smolin in [6]. However, now one can show that this loop transform exists rigorously. Since in full general relativity there is no notion of positive and negative frequency decomposition, in the case of linearized gravity we used variables which are the analogs of the $A_\alpha^i$ and $E_\alpha^i$ used in the full theory. Quantization was completed in the connection representation (see chapter 11 in [1] and also [16]) as well as the loop representation [18] following the general program outlined in section 2. In particular, the correct inner-product on the physical states is recovered without having to make explicit use of the underlying Poincaré group. However, now, certain new features are encountered. In the connection representation, while the quantum states are holomorphic functionals of the (self dual, linearized) connections of one helicity as anticipated, they are “holomorphic distributions” of the (self dual, linearized) connections of the other helicity. While this causes no problem in the connection representation itself, now the loop transform fails to be well-defined. The loop representation is constructed directly [18] starting from the linearized loop variables following a second strategy suggested by Rovelli and Smolin [6]. However, we find that states are now functionals of “thickened” loops. Somewhat surprisingly, for any (non-zero) value of the “thickening parameter”, the resulting loop representation is isomorphic to the Fock representation (where, however, the isomorphism does depend on the value of the parameter). Thus, both the connection and the loop representations based on self-dual $A_\alpha^i$ and real $E_\alpha^i$ are somewhat different from those based on negative frequency $A_\alpha^i$ and positive frequency $E_\alpha^i$. To summarize, it is reassuring that the general ideas underlying loop quantization successfully reproduce known physics in the case of the Maxwell theory and linearized gravity. However, when looked at in detail, one finds that the techniques used here do differ from those employed in full quantum general relativity. Furthermore,
because of the absence of a background metric, it does not seem possible to change the strategy and use, e.g., thickened loops in the full theory.

Another development [7] addresses the issue of precision. The goal here was to sharpen some of the heuristic notions used by Rovelli and Smolin [6] by constructing a precise algebra of loop operators and a proper representation theory for it. This was achieved by using the Gel’fand spectral theory of $C^\ast$-algebras (see e.g. [26]), the $C^\ast$-algebra in question being built directly out of quantum holonomy operators $\hat{Q}_\gamma$. These are the quantum analogs of the holonomy functionals $Q_\gamma$ of Eq.(3) where, however, no assumption is made on the existence of the operator-valued distribution $A^i_\gamma(x)$. Thus, $\hat{Q}_\gamma$ are the primary objects; in the quantum theory, $A^i_\gamma$ may not even exist. The representation theory leads us directly to the connection representation. However, the quantum states are functionals not on the space $A/G$ of smooth connections modulo gauge transformations but on a space $\Delta$ which is a “completion” of $A/G$. This is analogous to the quantum theory of scalar fields where the states are functionals not on the space of scalar fields but on a completion thereof containing distributions. Indeed, the structures involved in the present case closely parallel those involved in the case of a scalar field. Finally, in this framework, the loop transform is rigorously defined and hence one has fuller control on and a deeper understanding of the mathematical structures involved. Unfortunately, however, as it stands the framework is not directly applicable to quantum general relativity. The problem is with the non-triviality of the Lorentzian reality conditions. Had the connection been real—as in Yang-Mills or topological theories, or Euclidean general relativity—everything would have been straightforward. With the Lorentzian reality conditions, however, the $\star$-relations in the required $C^\ast$-algebra are hard to introduce whence, as the matters stand, one cannot even get started. To summarize, once again, these results give us confidence in the general ideas surrounding the loop representation. They have proven to be especially useful in certain topological field theories, including 2+1-dimensional general relativity, where they provide a unifying and precise mathematical framework. However, a significant new input is needed to make them directly applicable to full quantum general relativity. As of now, the treatment of the 3+1-dimensional theory remains heuristic.

5. Discussion

In the last two sections, I sketched the new developments that have occurred in the quantization program since the Goa conference. In this section, I will summarize the open problems and directions of current research.

At this stage of the program, I believe that the major mathematical problems are the following: i)Understanding the structure of the space of physical states found so far and addressing the issue of completeness; ii)Interpreting operators that act on the physical states, e.g., by finding their classical analogs. These would be the “Dirac observables”, i.e. functions on the phase space which weakly commute with constraints; and, iii)Finding ways to impose the reality conditions to select the inner product on physical states. These mathematical issues are related to the key physical questions facing the program: i)Resolving the issue of time; ii)Developing approximation methods to help with the interpretation of states and observables; iii)Dealing squarely with the Planck regime where the approximations break down and new concepts are needed. In particular, one would have to learn how to interpret quantum mechanics in absence of a background space-time geometry.

There is, of course, no a guarantee that these steps can be satisfactorily completed. Yet, there do exist a number of model systems in which the mathematical problems have been fully resolved and significant progress has been made on the physical issues. These models have been obtained by truncating general relativity in various ways and therefore share several key features with the full theory. I will first outline these developments and then indicate the strategies they suggest to address the problems listed above.

The first model is obtained by a weak-field truncation of general relativity [27]. The idea is to choose a background point in the phase space, $A^i_a = 0$ and $E^i_a = h^a_i (\equiv \text{flat})$, corresponding to flat space-time, consider the deviations $\Delta A^i_a = A^i_a - 0$ and $\Delta E^i_a = E^i_a - h^a_i$ and keep in the expressions of operator constraints terms which are at most quadratic in these deviations. These truncated constraints are then imposed on wave functions $\Psi[A]$ in the connection representation. To see the result, let us first use the (c-number) flat background triad $h^a_i$ to convert internal indices to space indices and define operators $\Delta A_{ab}$ and $\Delta E_{ab}$. Next, let us decompose these operators into anti-symmetric and symmetric parts and the symmetric parts into trace, longitudinal and transverse-traceless parts. The result of imposition of the constraints is then as
follows. The Gauss constraint is essentially a functional differential equation which tells us how a physical state \( \Psi(A) \) changes when we change \( A^a_i \) by a purely anti-symmetric term. Similarly, the vector constraint provides a functional differential equation determining the dependence of \( \Psi(A) \) on the longitudinal part of \( A^a_i \) while the scalar constraint determines the dependence on the trace-part. Thus, the dependence of the wave function on the (symmetric) transverse, traceless part is arbitrary (although it does of course have to be holomorphic) and the constraints determine its dependence on the remaining parts of \( A^a_i \). Thus, as expected, the constraints ensure that the true degrees of freedom lie in the transverse trace-less part of \( A^a_i \).

However, this is not all: the specific form of the quantum constraints contains more information. Let us focus on the scalar constraint. Its precise form can be reduced to:

\[
i\hbar \frac{\delta}{\delta A^{STT}_{\text{im}}(\vec{x})} \Psi(A) = (\hat{STT}\hat{A}^{*}_{ab}(\vec{x}) \hat{STT}\hat{A}^{ab}(\vec{x})) \circ \Psi(A),
\]

where \( A^T \) and \( A^{STT}_{ab} \) denote the trace and the symmetric, transverse, traceless parts of \( A^a_i(\vec{x}) \), the subscript “im” stands for “imaginary part” and \( \ast \) denotes the Hermitian conjugate. The operator on right is precisely the Hamiltonian density in the weak field limit. Therefore, the equation can be re-interpreted as a (bubble-)time evolution equation, where the role of “local” time is played by the imaginary part of \( A^T(\vec{x}) \). If we integrate this equation on \( \mathbb{R}^3 \), we obtain the Schrödinger evolution equation. Thus, in the truncated model, one can single out, from various components of the connection, an appropriate time variable. When this is done, the familiar Schrödinger evolution emerges simply as one of the “components” of the quantum scalar constraint. Note the dual role of constraints. On the one hand, they simply constrain the form of the allowable wave functionals: One begins with arbitrary holomorphic functionals \( \Psi(A) \) and finds that their dependence on the anti-symmetric part, the longitudinal part and the trace is completely fixed by their dependence on the transverse trace-less part of \( A^a_i \). In this picture, nothing “happens”. Physical states are simply certain type of functionals on the connection superspace. On the other hand, if we now foliate the infinite dimensional connection superspace by level surfaces of \( A^T \), we discover that the physical states “evolve” from one level surface to another in a specific way and that the operator governing this evolution is simply the Hamiltonian density. If one adds matter sources, the right hand side of Eq.(11) is augmented precisely by the Hamiltonian density of matter fields. Thus, in the connection representation, we see in detail how familiar Minkowskian physics can emerge, in the appropriate approximation, from the constraint equations of non-perturbative quantum general relativity. Finally, note that such a “deparametrization” of the truncated quantum theory is not possible in the metric representation of geometrodynamics [28].

The remaining truncated models which have been studied in detail have only a finite number of degrees of freedom: type I and II Bianchi models [29], spherically symmetric gravitational fields [30] and 2+1 dimensional general relativity [1,19,31]. However, they capture some of the genuinely non-linear features of full general relativity. In all these cases, the quantization program was completed in the connection representation: there is a complete set of solutions to all quantum constraints, a complete set of operators corresponding to the classical Dirac observables is known explicitly, the reality conditions are well-understood and an inner product can be introduced on physical states unambiguously. Furthermore, in the Bianchi models and 2+1 gravity [31], the theory can be “deparametrized” and Schrödinger evolution can be recovered naturally. The work on spherically symmetric models is still unpublished; I have not seen proofs and am not sure of the status of the issue of time. In the case of 2+1 gravity, the program is completed also in the loop representation. Now, the (2-dimensional) diffeomorphism invariance requires that physical states can only depend on homotopy classes of closed loops and the inner product is introduced by exploiting the fact that the space of homotopy classes —like the space of knot classes in the 3+1-dimensional case— is discrete and therefore admits a natural measure.

These results provide a number of lessons for the full 3+1 theory. First, in all these cases, the reality conditions turn out to be rather simple and determine the inner-product on the space of physical states. This is done prior to deparametrization: The mathematical structure can be introduced before the variable corresponding to time is isolated. This is significant because, in the full theory, it is still unclear if there exists an exact decomposition of the components of the connection in to the “time part” and the “dynamical part”. As noted above, however, such a decomposition does exist in the weak field truncation and it enables of the issue of time. .
one to recover familiar physics. It is conceivable that, in the full theory, the notion of time can only be introduced in an approximate sense. This may suffice for the purpose of interpretation. In regimes in which the approximation is good, one could interpret the theory using space-time concepts we are used to, while in other regimes, we may have to get accustomed to doing physics using genuinely new concepts, perhaps suggested by the mathematical framework. On the other hand, had the deparametrization of the theory been essential for introduction of the inner-product, as was widely assumed in quantum geometrodynamics, the approximate notion of time would not have sufficed: it is hard to do mathematical physics with approximate inner-products! It is fortunate therefore that the reality conditions provide us with a well-defined principle to select the inner product and the potential fuzziness in the notion of time affects the only the physical interpretation, where one can afford to be somewhat imprecise.

The second lesson drawn from the work on 2+1-gravity concerns the the solutions to the constraints in the connection and the loop representations. As in the 3+1 theory, the quantum constraints are easier to solve in connection dynamics than in geometrodynamics. However, initially this method of solving the constraints was criticized on the grounds that it did not provide insight into the nature of quantum geometry. If one can not address questions about geometry, why are these solutions of any use? Why are they physically interesting? Isn’t the ease of solving quantum constraints an illusion? These concerns are quite legitimate. However, they can in fact be addressed fully. For this, it is convenient to present an analogy. Consider the problem of finding the spectrum of the hydrogen atom. It is now well-known that the group theoretic methods based on the $SO(4)$-symmetry are well-tailored to solve this problem. They tell us that the eigenstates of the Hamiltonian can be labelled by kets $|n, l, m\rangle$, and that the eigenvalues of the Hamiltonian, total angular momentum and the z-component of angular momentum are given by $-13.6eV/n^2$, $(l+1)\hbar^2$ and $m\hbar$ respectively. However, this solution of the problem itself tells us nothing about the electron orbits or position or momentum distributions of electrons in various stationary states. Coming from the standard version of the classical theory based on position and momentum of the electron, therefore, one might conclude at first that the $|n, l, m\rangle$-description is no solution to the problem at all: It tells us nothing about the “physically interesting questions” pertaining to the position and momentum of the electron and their time developments. It is obvious, in the case of the Hydrogen atom, that this criticism is ill-placed first because there is indeed very interesting quantum theoretic information in the $|n, l, m\rangle$-description and second because, with an appropriate change of basis, one can recover the wave functions $\Psi_{n, l, m}(\vec{x})$ from the $|n, l, m\rangle$. I would like to suggest that the situation is rather similar in the case of 2+1 gravity. Just as the $|n, l, m\rangle$-basis is especially well-suited to the quantum dynamics of the Hydrogen atom, the connection and the loop basis are well-suited to the quantum constraints of the 2+1 theory. These solutions do carry, directly, information about certain loop operators which happen to be the quantum versions of classical Dirac observables, i.e., constants of motion. In geometrodynamics, it just happens that one is not used to think in terms of these observables. The geometrical variables one normally uses are analogous to the position and momenta of the electron. Therefore, it is not surprising that one cannot extract information about quantum geometry directly from the solutions. However, a change of basis would enable one to extract geometrodynamical information from the exact solutions. In fact, this transformation has been carried out explicitly for the case when the spatial topology is that of a 2-torus [31]. As in the case of the hydrogen atom the form of the transformation involved is somewhat complicated.

The situation with the 3+1 theory is likely to be the same: to extract geometrodynamical information from the solutions to quantum constraints, substantial amount of work is needed. The situation is further complicated by a number of factors which did not arise in the 2+1 theory. First, as the weak field truncation shows, the loop representation is not well-suited for extraction of time and deparametrizing the theory. In this respect, it is rather analogous to the metric representation. In the 2+1 theory, the loop transform is well-understood and to recover Schrödinger evolution, one can go back to the connection representation which is well-suited for deparametrization. In the full 3+1 theory, on the other hand, as we saw in Section 4.3, the complicated nature of the reality conditions has made it difficult to give a precise meaning to the transform, whence, as the matter stands, we cannot transform (most of) the known solutions to quantum constraints to the connection representation. Finally, unlike in the 2+1 theory, the classical Dirac observables are not known explicitly whence we cannot easily interpret the known solutions. A satisfactory completion of the Newman-Rovelli program, outlined in section 2, will be of tremendous help in addressing this issue.

I believe that there is an intermediate case whose analysis will clarify these issues significantly: 3+1
dimensional gravitational fields with one spatial Killing field. If the norm of the Killing field is constant and the twist is zero, this truncated theory is completely equivalent to 2+1 gravity discussed above. If the norm is arbitrary but the twist is zero, the theory is equivalent to 2+1 gravity interacting with a zero rest mass scalar field obeying the wave equation. If the twist is non-zero, the theory is equivalent to 2+1 gravity coupled to a doublet of scalar fields constituting a non-linear sigma model. These theories are interesting precisely because they are intermediate between the well understood vacuum 2+1 theory and the full 3+1 dimensional theory. The reality conditions are exactly the same as in the vacuum 2+1 theory: everything is real. Therefore, many of the (at least apparent) obstacles faced by the full, 3+1 program are absent. On the other hand, these are field theories with an infinite number of degrees of freedom. In fact, as 2+1 dimensional theories with matter, they have been studied in detail perturbatively [32] and appear to be finite! (There is a curious gap in the literature, however, because while people working on classical aspects of the theory are well aware of the relation between 3+1 and 2+1 theories mentioned above, those working in quantum theory [32, 33] are not. As a result, conceptually important connections have not been made.) These models are now being studied extensively. We expect the loop transform to exist rigorously: because all fields are real, the analysis of [7] should be applicable with only small modifications. Therefore, one should be able to solve the quantum constraints in the loop representation, use the reality conditions to select an inner-product and address the issue of time in the connection representation. Finally, since perturbation theory is available, one should be able to compare the exact results with those obtained perturbatively. Note that, in the 3+1 interpretation, these theories do have gravitons interacting non-linearly (with however only one polarization if the twist is set to zero); the only restriction is that the fields depend only on two spatial dimensions. In this sense, they capture a significant portion of the dynamics of the full 3+1 theory.

To conclude, I will point out another direction along which work is in progress: approximation methods. The idea [21] here is to analyse perturbations of the weave states. The focus is on understanding the effect of the intrinsic discreteness of weaves on perturbative results. Is there, in particular, an effective cut-off which makes the new perturbation theory automatically finite? These approximation methods would resemble the standard perturbation theory for long wave lengths but differ both conceptually and technically at short wave lengths. The general strategy can be illustrated by an analogy. In classical general relativity, a non-perturbative treatment is essential to obtain black hole solutions. However, once the essential physics of these solutions is understood, one can often ignore full dynamics and use perturbative calculations to obtain physical results, provided, of course, the boundary conditions at the horizons are correctly incorporated. For example, in astrophysical calculations of stellar distributions around black holes, it suffices to use Newtonian methods augmented only by the modified boundary conditions. I would like to suggest that the situation may be similar with the weave states of quantum gravity. As we saw in section 4.1, genuinely non-perturbative methods are essential to arrive at these states. However, once their structure is well-understood, one should be able to extract non-trivial physical information by studying perturbations around these states, possibly ignoring full dynamics but paying careful attention to the inherent discreteness.

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