DYNAMICAL SYSTEMS METHOD FOR SOLVING NONLINEAR EQUATIONS WITH LOCALLY HÖLDER CONTINUOUS MONOTONE OPERATORS

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Abstract. A version of the Dynamical Systems Method for solving ill-posed nonlinear equations with monotone and locally Hölder continuous operators is studied in this paper. A discrepancy principle is proposed and justified under natural and weak assumptions. The only smoothness assumption on F is the local Hölder continuity of order $\alpha > 1/2$.

1. Introduction

In this paper we study a version of the Dynamical Systems Method (DSM) for solving the equation

$$F(u) = f,$$

where $F$ is a nonlinear monotone operator in a real Hilbert space $H$, and equation \((1.1)\) is assumed solvable, possibly nonuniquely. An operator $F$ is called monotone if

$$\langle F(u) - F(v), u - v \rangle \geq 0, \quad \forall u, v \in H.$$  \hspace{1cm} (1.2)

Here, $\langle \cdot, \cdot \rangle$ denotes the inner product in $H$. It is known (see, e.g., [14]), that the set $\mathcal{N} := \{ u : F(u) = f \}$ is closed and convex if $F$ is monotone and continuous. A closed and convex set in a Hilbert space has a unique minimal-norm element. This element in $\mathcal{N}$ we denote by $y$, $F(y) = f$, and call it the minimal-norm solution to equation (1). We assume in addition that $F$ is locally Hölder continuous of order $\alpha > 1/2$, i.e.,

$$\| F(u) - F(v) \| \leq C_R \| u - v \|^\alpha, \quad \forall u, v \in B(y, R).$$  \hspace{1cm} (1.3)

Assume that $f = F(y)$ is not known but $f_\delta$, the noisy data, are known, and $\| f_\delta - f \| \leq \delta$. If $F'(u)$ is not boundedly invertible then solving equation \((1.1)\) for $u$ given noisy data $f_\delta$ is often (but not always) an ill-posed problem. When $F$ is a linear bounded operator many methods for stable solution of \((1.1)\) were proposed (see [8]–[14] and the references therein). When $F$ is nonlinear several methods have been proposed and studied (see, e.g., [8], [10], [11], [9], [13], [20], [21] and references therein). The most frequently used and studied methods are regularized Newton-type and gradient-type methods. These methods requires the knowledge of the Fréchet derivative of $F$. Therefore, they are not applicable if $F$ is not Fréchet.
differentiable. Our goal in this paper is to study a method for a stable solution to problem (1.1) when \( F \) is not Fréchet differentiable.

In this paper we study a version of the Dynamical Systems Method (DSM) for solving (1.1). In the formulation given in [14], the DSM consists of finding a nonlinear map \( \Phi(t,u) \) such that the Cauchy problem

\[
\dot{u} = \Phi(t,u), \quad u(0) = u_0,
\]

has a unique solution for all \( t \geq 0 \), there exists \( \lim_{t \to \infty} u(t) := u(\infty) \), and \( F(u(\infty)) = f \).

Various choices of \( \Phi \) satisfying (1.4) were proposed and justified in [14]. Each such choice yields a version of the DSM.

The DSM for solving equation (1.1) was extensively studied in [14]–[19]. In [14], the following version of the DSM was investigated for monotone operators \( F \):

\[
\dot{u}_\delta = -(F(u_\delta) + a(t)I)^{-1}(F(u_\delta) + a(t)u_\delta - f_\delta), \quad u_\delta(0) = u_0.
\]

The convergence of this method was justified with an \( a \) priori choice of stopping rule in [14]. An \( a \) posteriori choice of stopping rule for this method was proposed and justified in [7]. Another version of the DSM with an \( a \) posteriori choice of stopping rule was formulated and justified in [5].

In this paper we consider the following version of the DSM for a stable solution to equation (1.1):

\[
\dot{u}_\delta = -(F(u_\delta) + a(t)u_\delta - f_\delta), \quad u_\delta(0) = u_0,
\]

where \( F \) is a monotone continuous operator and \( u_0 \in H \). It is known that a local solution of (1.6) exists under the assumption that \( F \) is monotone continuous and \( a(t) > 0 \) (see, e.g., [1, p.99] and [14, p.165]). When \( \delta = 0 \) and \( a(t) \) satisfies some conditions then it is known that the solution to (1.6) exists globally (see, e.g., [14, p.170]).

The advantage of the method in (1.6) compared with the one in (1.5) is the absence of the inverse operator in the algorithm, which makes the algorithm (1.6) less expensive than (1.5). On the other hand, algorithm (1.5) converges faster than (1.6) in many cases. Another advantage of the DSM (1.6) is the applicability when \( F \) is locally Hölder continuous of order \( \alpha > 0 \) but not Fréchet differentiable as shown in this paper.

The convergence of the method (1.6) for any initial value \( u_0 \) with an \( a \) priori choice of stopping rule was justified in [14, p.170]. In [6] the DSM (1.6) with a stopping rule of Discrepancy Principle type was proposed and justified under the assumption that \( F \) is Fréchet differentiable. There, convergence of \( u_\delta(t_\delta) \), chosen by a stopping rule of Discrepancy Principle type, is proved for the regularizing function \( a(t) = d/(c + t)^b \) where \( c \geq 1 \), \( b \in (0, 1/2) \) and \( d \) is sufficiently large. However, how large one should choose the parameter \( d \) is not quantified in [6].

In this paper we study the DSM (1.6) with the stopping rule proposed in [6] under weaker assumption on \( F \) and for a larger class of regularizing function \( a(t) \). The novel results in this paper include a justification of the DSM (1.6) with our stopping rule for a stable solution to (1.1) under the assumption that \( F \) is locally Hölder continuous of order \( \alpha > 1/2 \). This condition is much weaker than the Fréchet differentiability of \( F \) which was used in [6]. Moreover, our results are justified for
Lemma 2.1. Assume that equation \( F(\delta) + a\delta - f_\delta = 0, \quad a > 0, \)
where \( a = \text{const}. \) It is known (see, e.g., [14], [22]) that equation (2.1) with monotone continuous operator \( F \) has a unique solution for any \( f_\delta \in H. \)

Let us recall the following result from [14]:

\[ \lim_{a \to 0} ||V_a - y|| = 0, \]

where \( V_a \) solves (2.1) with \( \delta = 0. \)

Let \( a = a(t) \) be a strictly monotonically decaying continuous positive function on \([0, \infty)\), \( 0 < a(t) \searrow 0, \) and assume \( a \in C^1[0, \infty). \) These assumptions hold throughout the paper and often are not repeated. Then the solution \( V_\delta \) of (2.1) is a function of \( t, V_\delta = V_\delta(t). \)

Below the words decreasing and increasing mean strictly decreasing and strictly increasing.

Lemma 2.2. Assume \( ||F(0) - f_\delta|| > 0. \) Let \( 0 < a(t) \searrow 0, \) and \( F \) be monotone. Denote

\[ \psi(t) := ||V_\delta(t)||, \quad \phi(t) := a(t)\psi(t) = ||F(V_\delta(t)) - f_\delta||, \]

where \( V_\delta(t) \) solves (2.1) with \( a = a(t). \) Then \( \phi(t) \) is decreasing, and \( \psi(t) \) is increasing.

Proof. Since \( ||F(0) - f_\delta|| > 0, \) one has \( \psi(t) \neq 0, \forall t \geq 0. \) Indeed, if \( \psi(t)|_{t=\tau} = 0, \) then \( V_\delta(\tau) = 0, \) and equation (2.1) implies \( ||F(0) - f_\delta|| = 0, \) which is a contradiction.

Note that \( \phi(t) = a(t)||V_\delta(t)||. \) One has

\[ 0 \leq \langle F(V_\delta(t_1)) - F(V_\delta(t_2)), V_\delta(t_1) - V_\delta(t_2) \rangle \]
\[ = -(a(t_1)||V_\delta(t_1)|| + a(t_2)||V_\delta(t_2)||, V_\delta(t_1) - V_\delta(t_2)) \]
\[ = (a(t_1) + a(t_2))(V_\delta(t_1) - V_\delta(t_2)) - a(t_1)||V_\delta(t_1)||^2 - a(t_2)||V_\delta(t_2)||^2. \]

Thus,

\[ 0 \leq (a(t_1) + a(t_2))(V_\delta(t_1) - V_\delta(t_2)) - a(t_1)||V_\delta(t_1)||^2 - a(t_2)||V_\delta(t_2)||^2 \]
\[ \leq (a(t_1) + a(t_2)||V_\delta(t_1)|| ||V_\delta(t_2)|| - a(t_1)||V_\delta(t_1)||^2 - a(t_2)||V_\delta(t_2)||^2 \]
\[ = (a(t_1)||V_\delta(t_1)|| - a(t_2)||V_\delta(t_2)||)(||V_\delta(t_2)|| - ||V_\delta(t_1)||) \]
\[ = (\phi(t_1) - \phi(t_2))(\psi(t_2) - \psi(t_1)). \]

If \( \psi(t_2) > \psi(t_1), \) then (2.3) implies \( \phi(t_1) \geq \phi(t_2), \) so

\[ a(t_1)\psi(t_1) \geq a(t_2)\psi(t_2) > a(t_2)\psi(t_1). \]

Thus, if \( \psi(t_2) > \psi(t_1), \) then \( a(t_2) < a(t_1) \) and, therefore, \( t_2 > t_1, \) because \( a(t) \) is strictly decreasing.
Similarly, if $\psi(t_2) < \psi(t_1)$, then $\phi(t_1) \leq \phi(t_2)$. This implies $a(t_2) > a(t_1)$, so $t_2 < t_1$.

Suppose $\psi(t_1) = \psi(t_2)$, i.e., $\|V_0(t_1)\| = \|V_0(t_2)\|$. From (2.2), one has

$$\|V_0(t_1)\| = \|V_0(t_2)\| = \|V_0(t_1)\|,$$

This implies $V_0(t_1) = V_0(t_2)$, and then equation (2.1) implies $a(t_1) = a(t_2)$. Hence, $t_1 = t_2$, because $a(t)$ is strictly decreasing.

Therefore, $\phi(t)$ is decreasing and $\psi(t)$ is increasing. \hfill \Box

**Lemma 2.3.** Let $F$ be a monotone continuous operator. Then,

(2.4) \[ \lim_{t \to \infty} \|F(V_0(t)) - f_\delta\| \leq \delta. \]

**Proof.** We have $F(y) = f$, and

\[ 0 = (F(V_0) + aV_0 - f_\delta, F(V_0) - f_\delta) \]

\[ = \|F(V_0) - f_\delta\|^2 + a\langle V_0 - y, F(V_0) - f_\delta \rangle + a\langle y, F(V_0) - f_\delta \rangle \]

\[ = \|F(V_0) - f_\delta\|^2 + a\langle V_0 - y, F(V_0) - F(y) + a\langle y, f - f_\delta \rangle \]

\[ + a\langle y, F(V_0) - f_\delta \rangle \]

\[ \geq \|F(V_0) - f_\delta\|^2 + a\langle V_0 - y, f - f_\delta \rangle \]

\[ \geq a\|V_0 - y\|\|f - f_\delta\| + a\|y\|\|F(V_0) - f_\delta\| \]

\[ \geq a\|V_0 - y\| + a\|y\|\|F(V_0) - f_\delta\|. \]

On the other hand, we have

\[ 0 = (F(V_0) - F(y) + aV_0 + f - f_\delta, V_0 - y) \]

\[ = \langle F(V_0) - F(y), V_0 - y \rangle + a\|V_0 - y\|^2 + a\langle y, V_0 - y \rangle + \langle f - f_\delta, V_0 - y \rangle \]

\[ \geq a\|V_0 - y\|^2 + a\langle y, V_0 - y \rangle + \langle f - f_\delta, V_0 - y \rangle, \]

where the inequality $\langle V_0 - y, F(V_0) - F(y) \rangle \geq 0$ was used. Therefore,

\[ a\|V_0 - y\|^2 \leq a\|y\|\|V_0 - y\| + \delta\|V_0 - y\|. \]

This implies

(2.6) \[ a\|V_0 - y\| \leq a\|y\| + \delta. \]

From (2.6) and (2.6), and an elementary inequality $ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}, \forall \epsilon > 0$, one gets:

\[ \|F(V_0) - f_\delta\|^2 \leq \delta^2 + a\|y\|\delta + a\|y\|\|F(V_0) - f_\delta\| \]

\[ \leq \delta^2 + a\|y\|\delta + \epsilon\|F(V_0) - f_\delta\|^2 + \frac{1}{4\epsilon}a^2\|y\|^2, \]

where $\epsilon > 0$ is fixed, independent of $t$, and can be chosen arbitrary small. Let $t \to \infty$ and $a = a(t)$. Then (2.7) implies

\[ \lim_{t \to \infty} (1 - \epsilon)|F(V_0(t)) - f_\delta|^2 \leq \delta^2, \]

for any fixed $\epsilon > 0$ arbitrarily small. This implies (2.4). Lemma 2.3 is proved. \hfill \Box
Remark 2.4. Let \( V := V_\delta(t)|_{\delta=0} \), so
\[
F(V) + a(t)V - f = 0.
\]
Let \( y \) be the minimal-norm solution to equation (1.1). We claim that
\[
\|V_\delta - V\| \leq \frac{\delta}{a}.
\]
Indeed, from (2.1) one gets
\[
F(V_\delta) = F(V) + a(V_\delta - V) = f - f.
\]
Multiply this equality with \((V_\delta - V)\) and use the monotonicity of \( F \) to get
\[
a\|V_\delta - V\|^2 \leq \delta\|V_\delta - V\|.
\]
This implies (2.8).

Similarly, multiplying the equation
\[
F(V) + aV - F(y) = 0,
\]
by \( V - y \) one derives the inequality:
\[
\|V\| \leq \|y\|.
\]
Similar arguments one can find in [14].

From (2.8) and (2.9), one gets the following estimate:
\[
\|V_\delta\| \leq \|V\| + \frac{\delta}{a} \leq \|y\| + \frac{\delta}{a}.
\]

From the monotonicity of \( F \) and (2.1) one gets
\[
0 \leq \langle F(V_\delta(t)) - F(V_\delta(t')), V_\delta(t) - V_\delta(t') \rangle
\]
\[
\leq \langle a(t')V_\delta(t') - a(t)V_\delta(t), V_\delta(t) - V_\delta(t') \rangle
\]
\[
= -a(t')\|V_\delta(t) - V_\delta(t')\|^2 + (a(t') - a(t))\langle V_\delta(t), V_\delta(t) - V_\delta(t') \rangle
\]
\[
\leq -a(t')\|V_\delta(t) - V_\delta(t')\|^2 + |a(t') - a(t)|\|V_\delta(t)\|\|V_\delta(t) - V_\delta(t')\|,
\]
for all \( t, t' > 0 \). This implies:
\[
\limsup_{\xi \to 0} \frac{\|V_\delta(t + \xi) - V_\delta(t)\|}{|\xi|} \leq \frac{\|V_\delta(t)\|}{a(t)} \|V_\delta(t)\|, \quad t > 0.
\]

Let us formulate and prove a version of the Gronwall’s inequality for continuous functions.

Lemma 2.5. Let \( \alpha(t) \) and \( \beta(t) \) be continuous nonnegative functions on \([0, \infty)\). Let \( 0 \leq g(t) \) be a continuous function on \([0, \infty)\) satisfying the following condition:
\[
\limsup_{\xi \to 0} \frac{g^2(t + \xi) - g^2(t)}{\xi} \leq -2\alpha(t)g^2(t) + 2\beta(t)g(t), \quad \forall t \geq 0.
\]
Then
\[
g(t) \leq g(0)e^{-\tilde{\varphi}(t)} + e^{-\tilde{\varphi}(t)}\int_0^t e^{\tilde{\varphi}(s)}\beta(s)ds, \quad \tilde{\varphi}(t) := \int_0^t \alpha(s)ds.
\]
Proof. Let
\[ g_\epsilon(t) := \left( g^2(t) + \epsilon e^{-2\int_0^t \alpha(\xi)d\xi} \right)^\frac{1}{2}, \quad t \geq 0, \quad \epsilon > 0. \]
From (2.13) one obtains
\[ \limsup_{\xi \to 0} \frac{g^2_\epsilon(t + \xi) - g^2_\epsilon(t)}{\xi} \leq \frac{\epsilon}{\xi} e^{-2\int_0^t \alpha(\xi)d\xi} \]
(2.15)
\[ \leq -2\alpha(t)g^2(t) + 2\beta(t)g(t) - 2\epsilon \alpha(t)e^{-2\int_0^t \alpha(\xi)d\xi} \]
\[ \leq -2\alpha(t)g^2(t) + 2\beta(t)g_\epsilon(t), \quad \forall t \geq 0. \]
Since \( g_\epsilon(t) > 0 \), it follows from (2.13) and the continuity of \( g_\epsilon(t) \) that
\[ \limsup_{\xi \to 0} \frac{g_\epsilon(t + \xi) - g_\epsilon(t)}{\xi} \leq -\alpha(t)g_\epsilon(t) + \beta(t). \]
(2.16)
From the Taylor expansion of \( e^{\int_0^t \epsilon a(s)ds} \), we have
\[ e^{\varphi(t + \xi)} = e^{\varphi(t) + \int_0^{t+\xi} \alpha(s)ds} = e^{\varphi(t)} \left( 1 + \int_t^{t+\xi} a(s)ds + O(\xi^2) \right), \quad \xi \to 0. \]
This, (2.16), the mean value theorem for integration, and the continuity of \( g_\epsilon(t) \) imply
\[ \limsup_{\xi \to 0} \frac{e^{\varphi(t+\xi)}g_\epsilon(t + \xi) - e^{\varphi(t)}g_\epsilon(t)}{\xi} \leq e^\varphi(t)\beta(t). \]
(2.17)
From (2.17) one obtains
\[ e^{\varphi(t)}g_\epsilon(t) - e^{\varphi(0)}g_\epsilon(0) \leq \int_0^t e^{\varphi(s)}\beta(s)ds, \quad t \geq 0. \]
(2.18)
This implies
\[ g(t) < g_\epsilon(t) \leq \left( g^2(0) + \epsilon \right)^\frac{1}{2} e^{-\varphi(t)} + e^{-\varphi(t)} \int_0^t e^{\varphi(s)}\beta(s)ds, \quad \forall t \geq 0. \]
(2.19)
Letting \( \epsilon \to 0 \) in (2.19) one obtains (2.14). Lemma 2.3 is proved. \( \square \)

**Lemma 2.6.** Let \( a(t) \in C^1[0, \infty) \) satisfy the following conditions (see also (3.2))
\[ 0 < a(t) \to 0, \quad 0 < \frac{|\dot{a}(t)|}{a^2(t)} \to 0. \]
(2.20)
Let \( \phi(t) := \int_0^t a(s)ds \) and \( V_0(t) \) be the solution to (2.1) with \( a = a(t) \). Then the following relations hold:
\[ \lim_{t \to \infty} \phi(t) = \infty, \]
(2.21)
\[ \lim_{t \to \infty} e^{r\phi(t)}a(t) = \infty, \quad r = const > 0, \]
(2.22)
\[ \lim_{t \to \infty} \frac{\int_0^t e^{\phi(s)}\|\dot{a}(s)\|V_0(s)\|ds}{e^{\phi(t)}} = 0, \]
(2.23)
\[ M := \lim_{t \to \infty} \frac{\int_0^t e^{\phi(s)}\|\dot{a}(s)\|ds}{e^{\phi(t)}a(t)} = 0. \]
(2.24)
Proof. Let us first prove (2.21). It follows from (2.20) that there exists $t_1 \geq 0$ such that
\[ a(t) \geq \frac{\dot{a}(t)}{a(t)}, \quad \forall t \geq t_1. \]
This implies
\[ \phi(t) \geq \int_{t_0}^{t} a(s)ds \geq \int_{t_0}^{t} \frac{-\dot{a}(s)}{a(s)} ds = -\ln a(s) \bigg|_{t_0}^{t} = \ln a(t_0) - \ln a(t). \]
Relation (2.21) follows from the relation $\lim_{t \to \infty} a(t) = 0$ and (2.25).

Let us prove (2.22). We claim that, for sufficiently large $t > 0$, the following inequality holds:
\[ \phi(t) = \int_{0}^{t} a(s)ds > \frac{1}{r} \ln \frac{1}{a^2(t)}. \]
Indeed, by L’Hospital’s rule and (2.20), one gets
\[ \lim_{t \to \infty} \frac{\phi(t)}{\ln a(t)} = \lim_{t \to \infty} \frac{a^2(t)}{2\dot{a}(t)} = \infty. \]
This implies that (2.26) holds for all $t \geq \tilde{T}$ provided that $\tilde{T} > 0$ is sufficiently large. It follows from inequality (2.26) that
\[ \lim_{t \to \infty} a(t)e^{\phi(t)} \geq \lim_{t \to \infty} a(t)e^{\ln \frac{1}{a(t)}} = \lim_{t \to \infty} \frac{1}{a(t)} = \infty. \]
Thus, equality (2.22) is proved.

Let us prove (2.23). Since $a(t)\|V_\delta(t)\|$ is a decreasing function of $t$ (cf. Lemma 2.2) one gets
\[ \lim_{t \to \infty} \int_{0}^{t} \frac{e^{\phi(s)}|\dot{a}(s)|\|V_\delta(s)\|ds}{e^{\phi(t)}} \leq \lim_{t \to \infty} \int_{0}^{t} \frac{e^{\phi(s)}|\dot{a}(s)|a(0)\|V_\delta(0)\|ds}{e^{\phi(t)}}. \]
We claim that
\[ \lim_{t \to \infty} \int_{0}^{t} \frac{e^{\phi(s)}|\dot{a}(s)|\|V_\delta(s)\|ds}{e^{\phi(t)}} = 0. \]
Indeed, if $\int_{0}^{t} e^{\phi(s)}|\dot{a}(s)|\|V_\delta(s)\|ds < \infty$, then (2.30) follows from (2.21). Otherwise, relation (2.30) follows from L’Hospital’s rule and the relation $\lim_{t \to \infty} \frac{|\dot{a}(t)|}{a^2(t)} = 0$.

From (2.29) and (2.30) one gets (2.23).

Let us prove (2.24). Since (2.21) holds and $a(t)e^{\phi(t)} \to 0$ as $t \to \infty$, by (2.22) with $r = 1$, relation (2.24) holds if the numerator of (2.24) is bounded. Otherwise, L’Hospital’s rule yields
\[ M = \lim_{t \to \infty} \frac{e^{\phi(t)}|\dot{a}(t)|}{e^{\phi(t)}a^2(t) + e^{\phi(t)}\dot{a}(t)} = 0. \]
Here we have used the relation $\lim_{t \to \infty} \frac{|\dot{a}(t)|}{a^2(t)} = 0$.

Lemma 2.4 is proved. \(\square\)

Remark 2.7. From (2.23) and the inequality $\|V_\delta(t)\| \geq \|V_\delta(0)\| > 0$, $\forall t \geq 0$, (see Lemma 2.2), one gets the following relation
\[ \lim_{t \to \infty} \int_{0}^{t} \frac{e^{\phi(s)}|\dot{a}(s)|ds}{e^{\phi(t)}} = 0. \]
Let $\epsilon > 0$ be arbitrary. It follows from (2.24) that there exists $t_\epsilon > 0$ such that the following inequality holds:

\begin{equation}
0 < a(t) \leq 0, \quad \frac{1}{2} > q > \frac{\|a(t)\|}{a^2(t)} \gg 0. \tag{2.34}
\end{equation}

Lemma 2.8. Let $a(t)$ satisfy (2.34) and $\varphi(t) := (1 - q) \int_0^t a(s)ds$, $q \in (0, 1/2)$.
Then one has

\begin{equation}
e^{-\varphi(t)} \int_0^t e^{\varphi(s)}|\dot{a}(s)|\|V_\delta(s)\|ds \leq \frac{q}{1 - 2q}a(t)\|V_\delta(t)\|, \quad t \geq 0. \tag{2.35}
\end{equation}

Proof. Let us prove that

\begin{equation}
e^{\varphi(t)}|\dot{a}(t)| \leq \frac{q}{1 - 2q}(a(t)e^{\varphi(t)})^r, \quad \forall t \geq 0. \tag{2.36}
\end{equation}

Inequality (2.36) is equivalent to

\begin{equation} \left(1 - \frac{2}{2q}\right) e^{\varphi(t)}|\dot{a}(t)| \leq \dot{a}(t)e^{\varphi(t)} + (1 - q)a^2(t)e^{\varphi(t)}, \quad t \geq 0. \tag{2.37}
\end{equation}

Note that $\dot{a} = -|\dot{a}|$. Inequality (2.37) holds because from (2.34) one obtains

\begin{equation} \left(1 - \frac{2}{2q}\right) |\dot{a}(t)| \leq |\dot{a}(t)| + (1 - q)a^2(t), \quad t \geq 0. \tag{2.38}
\end{equation}

Thus, inequality (2.36) holds. Integrate (2.36) from 0 to $t$ and get

\begin{equation} \int_0^t e^{\varphi(s)}|\dot{a}(s)|ds \leq \frac{q}{1 - 2q} \left( a(t)e^{\varphi(t)} - a(0)e^0 \right) < \frac{q}{1 - 2q}a(t)e^{\varphi(t)}, \quad \forall t \geq 0. \tag{2.39}
\end{equation}

Multiplying (2.39) by $e^{-\varphi(t)}\|V_\delta(t)\|$ and using the fact that $\|V_\delta(t)\|$ is increasing, one gets inequality (2.35). Lemma 2.8 is proved. \hfill \Box

3. Main results

3.1. Dynamical Systems Method. Let $u_\delta(t)$ solve the following Cauchy problem:

\begin{equation} \dot{u}_\delta = -[F(u_\delta) + a(t)u_\delta - f_\delta], \quad u_\delta(0) = u_0. \tag{3.1}
\end{equation}

Assume

\begin{equation} 0 < a(t) \gg 0, \quad 0 < \frac{\|\dot{a}(t)/a^2(t)\|}{a^2(t)} \gg 0, \quad t \geq 0. \tag{3.2}
\end{equation}

Remark 3.1. Let $a(t) = \frac{d}{(e + t)^b}$, where $b \in (0, 1)$, $c > 0$ and $d > 0$. Then this $a(t)$ satisfies (3.2).

Remark 3.2. It is known that there exists a unique local solution to problem (3.1) for any initial data $u_0$ if $F$ is monotone continuous and $0 < a(t)$ is a continuous function. Proofs for this are often based on Peano approximations (see, e.g., [11], p.99 and [14], p.165). When $F$ is monotone and hemicontinuous then equation (3.1) is understood in the weak sense. When $F$ is monotone and continuous it is
It follows from (2.12) and (3.6) that

\[ t \quad \phi_K \quad (3.5) \]

for all \( \phi \in K \) this and Lemma 2.5 imply (3.4).

It follows from (2.30) that

\[ p.167 \]

known that equation (3.1) can be understood in the strong sense (see, e.g., [14, p.167]).

The following lemma guarantees the global existence of a unique solution to (3.1).

**Lemma 3.3.** Let \( F \) be monotone and continuous. Let \( 0 < a(t) \) be a continuous function satisfying (3.2). Then the unique solution to (3.1) exists globally.

**Proof.** Assume the contrary, i.e., that \( u_\delta(t) \) exists on interval \([0, T]\) but does not exist on \([T, T + d]\), where \( d > 0 \) is arbitrary small. Let us prove that the following limit

\[ \lim_{t \to T} u_\delta(t) = u_\delta(T) \]

exists and is finite. This contradicts the definition of \( T \) since one can consider \( u_\delta(T) \) as an initial data and construct the solution \( u_\delta(t) \) on interval \([T, T + d]\), for sufficiently small \( d > 0 \), by using the local existence of \( u_\delta(t) \).

Let us first prove that

\[ \| u_\delta(t) - V_\delta(t) \| \leq e^{-\phi(t)} \| w(0) \| + e^{-\phi(t)} \int_0^t e^{\phi(s)} \frac{|\dot{a}(s)|}{a(s)} \| V_\delta(s) \| ds, \]

for all \( t \in [0, T) \) where

\[ \phi(t) := \int_0^t a(s) ds, \quad w(t) := u_\delta(t) - V_\delta(t). \]

From (3.1), (2.1), and the monotonicity of \( F \) one gets

\[ \langle \dot{u}_\delta(t), w(t) \rangle = -\langle F(u_\delta(t)) + a(t)u_\delta(t) - F(V_\delta(t)) - a(t)V_\delta(t), w(t) \rangle \]

\[ \leq -a(t)\| w(t) \|^2. \]

It follows from (2.12) and (3.6) that

\[ \lim_{\xi \to 0} \sup_{\xi \to 0} \frac{\| w(t + \xi) \|^2 - \| w(t) \|^2}{\xi} = \lim_{\xi \to 0} \sup_{\xi \to 0} \frac{\langle w(t + \xi) - w(t), w(t + \xi) + w(t) \rangle}{\xi} \]

\[ \leq 2\langle \dot{u}_\delta(t), w(t) \rangle + \lim_{\xi \to 0} \sup_{\xi \to 0} \frac{\langle V_\delta(t + \xi) - V_\delta(t), w(t + \xi) + w(t) \rangle}{\xi} \]

\[ \leq -2a(t)\| w(t) \|^2 + 2\| w(t) \| \frac{|\dot{a}(t)|}{a(t)} \| V_\delta(t) \|. \]

This and Lemma 2.5 imply (3.3).

Let

\[ K = 1 + \sup_{t \geq 0} e^{-\phi(t)} \int_0^t e^{\phi(s)} \frac{|\dot{a}(s)|}{a(s)} ds. \]

It follows from (2.30) that \( K \) is bounded. From (3.4), (3.8), and the fact that the function \( \| V_\delta(t) \| \) is increasing, one obtains

\[ \| u_\delta(t) \| \leq e^{-\phi(t)} \| w(0) \| + K \| V_\delta(t) \| \]

\[ \leq K_T := \| w(0) \| + K \| V_\delta(T) \|, \quad \forall t \in [0, T). \]

Let \( z_h(t) := u_\delta(t + h) - u_\delta(t) \). It follows from (3.1) that

\[ \dot{z}_h(t) = -\langle F(u_\delta(t + h)) - F(u_\delta(t)) + a(t)z_h(t) \]

\[ + (a(t) - a(t + h))u_\delta(t + h), \quad 0 < t < t + h < T. \]
Multiply (3.10) by \( z_h(t) \) and use the monotonicity of \( F \) to get

\[
(3.11) \quad \|z_h(t)\| \frac{d}{dt} \|z_h(t)\| \leq -a(t)\|z_h(t)\|^2 + (a(t) - a(t + h))(u_\delta(t + h), z_h(t)).
\]

This and (3.9) imply

\[
(3.12) \quad \frac{d}{dt} \|z_h(t)\| \leq -a(t)\|z_h(t)\| + (a(t) - a(t + h))\|u_\delta(t + h)\|
\]

\[
\leq -a(t)\|z_h(t)\| + (a(t) - a(t + h))K_T, \quad 0 < t < t + h < T.
\]

From (3.12) and the Gronwall’s inequality one obtains

\[
(3.13) \quad \|z_h(t)\| \leq e^{-\phi(t)}\|z_h(0)\| + e^{-\phi(t)}K_T \int_0^t e^{\phi(s)}(a(s) - a(s + h))ds,
\]

for \( 0 < t < t + h < T \). It follows from (3.13) and the uniform continuity of \( a(t) \) on \([0, T]\) that

\[
(3.14) \quad \lim_{h \to 0} \|u_\delta(t + h) - u_\delta(t)\| \leq \lim_{h \to 0} \|u_\delta(0 + h) - u_\delta(0)\| = 0,
\]

and this relation holds uniformly with respect to \( t \) and \( t + h \) such that \( t < t + h < T \). Here, the last equality in (3.14) follows from the fact that \( u_\delta(t) \) solves (3.1) on \([0, T]\). Relation (3.14) and the Cauchy criterion for convergence imply the existence of the finite limit in (3.3).

Lemma 3.4 is proved.

\[\square\]

**Theorem 3.4.** Let \( a(t) \) satisfy (3.2). Assume that \( F: H \to H \) is a monotone operator satisfying condition (1.3), and \( u_0 \) is an element of \( H \), satisfying inequality

\[
(3.15) \quad \|F(u_0) - f_\delta\| > C\delta^\zeta > 0,
\]

where \( C > 0 \) and \( 0 < \zeta \leq 1 \) are constants. Assume that equation \( F(u) = f \) has a solution, \( f \) is unknown but \( f_\delta \) is given, \( \|f_\delta - f\| \leq \delta \). Let \( y \) be the minimal-norm solution to (1.1). Then the solution \( u_\delta(t) \) to problem (3.1) exists globally and there exists a unique \( t_\delta \) such that

\[
(3.16) \quad \|F(u_\delta(t_\delta)) - f_\delta\| = C\delta^\zeta, \quad \|F(u_\delta(t)) - f_\delta\| > C\delta^\zeta, \quad \forall t \in [0, t_\delta).
\]

If \( \zeta \in (0, 1) \) and

\[
(3.17) \quad \lim_{\delta \to 0} t_\delta = \infty,
\]

then

\[
(3.18) \quad \lim_{\delta \to 0} \|u_\delta(t_\delta) - y\| = 0.
\]

**Remark 3.5.** Inequality (3.15) is not a restrictive assumption. Indeed, if it does not hold and \( \|u_0\| \) is not too large, then \( u_0 \) can be considered as an approximate solution to (1.1).

**Proof.** The uniqueness of \( t_\delta \) follows from (3.16). Indeed, if \( t_\delta \) and \( \tau_\delta > t_\delta \) both satisfy (3.16), then the second inequality in (3.16) does not hold on the interval \([0, \tau_\delta]\).

Let us verify the existence of \( t_\delta \).
Denote
\[(3.19) \quad v := F(u_\delta) + au_\delta - f_\delta, \quad h = \|v\|.
\]

We have
\[(3.20) \quad \limsup_{\xi \to 0} \frac{h^2(t + \xi) - h^2(t)}{\xi} = \limsup_{\xi \to 0} \frac{\langle v(t + \xi) - v(t), v(t + \xi) + v(t) \rangle}{\xi}
\]
\[
\leq \limsup_{\xi \to 0} \frac{(F(u_\delta(t + \xi)) - F(u_\delta(t)), v(t + \xi) + v(t))}{\xi}
+ 2(\alpha_\delta(t), v(t)) + 2(\alpha(t)u_\delta(t), v(t)).
\]

From (3.11) and (3.19) one gets \(u_\delta(t + \xi) - u_\delta(t) = -\int_t^{t+\xi} v(s)ds\). This and the monotonicity of \(F\) imply
\[(3.21) \quad \left\langle F(u_\delta(t + \xi)) - F(u_\delta(t)), \int_t^{t+\xi} v(s)ds \right\rangle \leq 0.
\]

Since \(F\) is Hölder continuous of order \(\alpha\) and \(u_\delta(t)\) is differentiable one obtains
\[(3.22) \quad \|F(u_\delta(t + \xi)) - F(u_\delta(t))\| = O(\|\xi\|^{\alpha}),
\]
and
\[(3.23) \quad \left\| 2 \int_t^{t+\xi} v(s)ds - \xi [v(t + \xi) + v(t)] \right\| = O(\|\xi\|^{1+\alpha}).
\]

Relations (3.22), (3.23) and the inequality \(\alpha > 1/2\) imply
\[(3.24) \quad \lim_{\xi \to 0} \frac{\langle F(u_\delta(t + \xi)) - F(u_\delta(t)), v(t + \xi) + v(t) - \frac{\xi}{2} \int_t^{t+\xi} v(s)ds \rangle}{\xi} = 0.
\]

From (3.21) and (3.24) we get
\[(3.25) \quad \limsup_{\xi \to 0} \frac{\langle F(u_\delta(t + \xi)) - F(u_\delta(t)), v(t + \xi) + v(t) \rangle}{\xi} \leq 0.
\]

This, the relation \(\dot{u}_\delta = -v\) (see (3.1)), and (3.20) imply
\[(3.26) \quad \limsup_{\xi \to 0} \frac{h^2(t + \xi) - h^2(t)}{\xi} \leq -2a(t)h^2(t) + 2|\alpha(t)||u_\delta(t)||h(t)|.
\]

This, Lemma 2.3\(\alpha\) and (3.3) imply
\[(3.27) \quad h(t) \leq e^{-\phi(t)}h(0) + e^{-\phi(t)} \int_0^t e^{\phi(s)}|\dot{u}(s)| \left(\|w(0)\| + K\|V_\delta(s)\|\right)ds.
\]

Since \(\langle F(u_\delta) - F(V_\delta), u_\delta - V_\delta \rangle \geq 0\), one obtains two inequalities
\[(3.28) \quad a\|u_\delta - V_\delta\|^2 \leq \langle v, u_\delta - V_\delta \rangle \leq \|u_\delta - V_\delta\|h,
\]
and
\[(3.29) \quad \|F(u_\delta) - F(V_\delta)\|^2 \leq \langle v, F(u_\delta) - F(V_\delta) \rangle \leq h\|F(u_\delta) - F(V_\delta)\|.
\]

Inequalities (3.28) and (3.29) imply:
\[(3.30) \quad a\|u_\delta - V_\delta\| \leq h, \quad \|F(u_\delta) - F(V_\delta)\| \leq h.
\]
The triangle inequality, the second inequality in (3.30), and (3.27) imply
\[
\|F(u_\delta(t)) - f_\delta\| \leq \|F(V_\delta(t)) - f_\delta\| + \|F(u_\delta) - F(V_\delta)\|
\leq \|F(V_\delta(t)) - f_\delta\| + h
\]
(3.31)
\[\leq \|F(V_\delta(t)) - f_\delta\| + h(0)e^{-\phi(t)} + e^{-\phi(t)} \int_0^t e^{\phi(s)}|\dot{a}(s)|\left(\|w(0)\| + K\|V_\delta(s)\|\right)ds.\]
This, (2.23), (2.21), Lemma 2.3, and (2.32) imply
(3.32)
\[\lim_{t \to \infty} \|F(u_\delta(t)) - f_\delta\| \leq \lim_{t \to \infty} \|F(V_\delta(t)) - f_\delta\| \leq \delta.\]
The existence of \(t_0\) satisfying (3.16) follows from (3.32) and the continuity of the function \(\|F(u_\delta(t)) - f_\delta\|\).

Let us prove (3.18) given that (3.17) holds.

From (3.33)–(3.35), (2.10), and (3.31) with \(r = 1\), and the inequality \(\|V_\delta(t)\| \geq \|V_\delta(0)\| > 0, t \geq 0,\) one gets, for all sufficiently small \(\delta > 0\), the following inequality:
(3.33)
\[h(0)e^{-\phi(t_\delta)} \leq a(t_\delta)\|V_\delta(0)\| \leq a(t_\delta)\|V_\delta(t_\delta)\|.\]
From the fact that \(\|V_\delta(t)\|\) is a nondecreasing function of \(t\), (2.33), and (3.17) one obtains
(3.34)
\[Ke^{-\phi(t_\delta)} \int_0^{t_\delta} e^{\phi(s)}|\dot{a}(s)||V_\delta(s)||ds \leq K\|V_\delta(t_\delta)\|e^{-\phi(t_\delta)} \int_0^{t_\delta} e^{\phi(s)}|\dot{a}(s)||ds \leq a(t_\delta)\|V_\delta(t_\delta)\|,\]
for all sufficiently small \(\delta > 0\). From (2.33) and (3.17) one gets, for all sufficiently small \(\delta > 0\), the following inequality
(3.35)
\[\|w(0)||e^{-\phi(t_\delta)} \int_0^{t_\delta} e^{\phi(s)}|\dot{a}(s)||ds \leq a(t_\delta)\|V_\delta(0)\| \leq a(t_\delta)\|V_\delta(t_\delta)\|.\]
From (3.33–3.35), (2.10), and (3.31) with \(t = t_\delta\), one obtains
(3.36)
\[C\delta^\zeta \leq 4a(t_\delta)\|V_\delta(t_\delta)\| \leq 4\left(\frac{a(t_\delta)}{\zeta} + \delta\right).\]
This and the relation \(\lim_{\delta \to 0} \frac{\delta^\zeta}{\zeta} = \infty\) for a fixed \(\zeta \in (0, 1)\) imply
(3.37)
\[\lim_{\delta \to 0} \frac{\delta^\zeta}{\zeta} \leq \frac{\|y\|}{C}.\]
Relation (3.37) and the first inequality in (2.10) imply, for sufficiently small \(\delta > 0\), the following inequality
(3.38)
\[\|V_\delta(t)\| \leq \|y\| + \frac{\delta}{a(t_\delta)} < \|y\| + \frac{C\delta^\zeta}{a(t_\delta)} < 5\|y\|, \quad 0 \leq t \leq t_\delta.\]
This implies
(3.39)
\[\lim_{\delta \to 0} \int_0^{t_\delta} e^{\phi(s)}|\dot{a}(s)||V_\delta(s)||ds \leq 5\|y\| \lim_{\delta \to 0} \int_0^{t_\delta} e^{\phi(s)}|\dot{a}(s)||ds \leq \frac{\int_0^{t_\delta} e^{\phi(s)}|\dot{a}(s)||V_\delta(s)||ds}{e^{\phi(t_\delta)}a(t_\delta)}.\]
It follows from (2.24) and (3.39) that
(3.40)
\[\lim_{\delta \to 0} \int_0^{t_\delta} e^{\phi(s)}|\dot{a}(s)||V_\delta(s)||ds = 0.\]
It follows from (3.30) and (3.27) that

\[ \|u_\delta(t) - V_\delta(t)\| \leq h(0) \frac{e^{-\phi(t)}}{a(t)} + \frac{e^{-\phi(t)}}{a(t)} \int_{0}^{t} e^{\phi(s)} |\dot{a}(s)| \left( \|w(0)\| + K\|V_\delta(s)\| \right) ds. \]

This, (2.24), (2.22) with \( r = 1 \), and (3.40) imply that

\[ \lim_{\delta \to 0} \|u_\delta(t_\delta) - V_\delta(t_\delta)\| = 0. \]

From (3.41) one gets

\[ \lim_{\delta \to 0} \frac{\delta}{a(t_\delta)} = 0. \]

Now let us finish the proof of Theorem 3.4.

From the triangle inequality and inequality (3.28) one obtains:

\[ \|u_\delta(t_\delta) - y\| \leq \|u_\delta(t_\delta) - V_\delta(t_\delta)\| + \|V(t_\delta) - V_\delta(t_\delta)\| + \|V(t_\delta) - y\| \]

(3.43)

\[ \leq \|u_\delta(t_\delta) - V_\delta(t_\delta)\| + \delta \frac{\delta}{a(t_\delta)} + \|V(t_\delta) - y\|. \]

From (3.41), (3.43), (3.17), and Lemma 2.1 one obtains (3.18). Theorem 3.4 is proved.

Assume that \( a(t) \) satisfies the following conditions

(3.44) \[ 0 < a(t) \searrow 0, \quad \frac{1}{3} > q > \frac{\phi(t)}{a^2(t)} \searrow 0. \]

**Remark 3.6.** Let \( a(t) = \frac{d}{c + t} \), where \( c \in (0, 1) \), \( d > 0 \) and \( d > bq^{-1}c^{b-1} \). Then this \( a(t) \) satisfies (3.44).

**Theorem 3.7.** Let \( a(t) \) satisfy (3.44). Let \( F, f, f_\delta \) be as in Theorem 3.4. Assume that \( u_0 \in H \) satisfies either

(3.45) \[ \|F(u_0) + a(0)u_0 - f_\delta\| \leq pa(0)\|V_\delta(0)\|, \quad 0 < p < 1 - \frac{q}{1 - 2q}, \]

or

(3.46) \[ \|F(u_0) + a(0)u_0 - f_\delta\| \leq \theta \delta^c, \quad 0 \leq \theta < C, \]

where \( C > 0 \) is the constant from Theorem 3.4. Let \( t_\delta \) be defined by (3.16). Then

(3.47) \[ \lim_{\delta \to 0} t_\delta = \infty. \]

**Remark 3.8.** One can easily choose \( u_0 \) satisfying inequality (3.45). Indeed, (3.45) holds if \( u_0 \) is sufficiently close to \( V_\delta(0) \). Note that inequality (3.45) is a sufficient condition for (3.58), i.e.,

(3.48) \[ e^{-\varphi(t)}h(0) \leq pa(t)\|V_\delta(t)\|, \quad t \geq 0, \]

to hold. In our proof inequality (3.18) (or (3.58)) is used at \( t = t_\delta \). The stopping time \( t_\delta \) is often sufficiently large for the quantity \( e^{\varphi(t_\delta)}a(t_\delta) \) to be large. In this case inequality (3.18) with \( t = t_\delta \) is satisfied for a wide range of \( u_0 \). Note that by (2.22) one gets \( \lim_{t \to \infty} e^{\varphi(t)}a(t) = \infty \). Here \( \varphi(t) = (1 - q)\phi(t) \) (see also (3.9) and (3.12)).
Proof of Theorem 3.7. Let us prove (3.47) assuming that (3.45) holds. The proof goes similarly when (3.46) holds instead of (3.45).

From (3.26), the triangle inequality one gets

\[
\lim_{\xi \to 0} \frac{h^2(t + \xi) - h^2(t)}{\xi} \leq -2a(t)h^2(t) + 2|\dot{a}(t)||V_\delta(t)||h(t) + 2|\dot{a}(t)||u_\delta(t) - V_\delta(t)||h(t)
\]

This and the first inequality in (3.30) imply

\[
\lim_{\xi \to 0} \frac{h^2(t + \xi) - h^2(t)}{\xi} \leq -2\left(a(t) - \frac{|\dot{a}(t)|}{a(t)}\right)h^2(t) + 2|\dot{a}(t)||V_\delta(t)||h(t)
\]

Since \( a - \frac{|\dot{a}|}{a} \geq (1 - q)a \), by (3.44), it follows from (3.50) and Lemma 2.8 that

\[
h(t) \leq h(0)e^{-\varphi(t)} + e^{-\varphi(t)} \int_0^t e^{\varphi(s)}|\dot{a}(s)||V_\delta(s)||ds,
\]

where

\[
\varphi(t) := \int_0^t (1 - q)a(s)ds = (1 - q)\phi(t), \quad t > 0.
\]

From (3.51) and (3.30), one gets

\[
\|F(u_\delta(t)) - F(V_\delta(t))\| \leq h(0)e^{-\varphi(t)} + e^{-\varphi(t)} \int_0^t e^{\varphi(s)}|\dot{a}(s)||V_\delta(s)||ds.
\]

It follows from inequality (3.53) and the triangle inequality that

\[
\|F(u_\delta(t)) - f_\delta\| \geq \|F(V_\delta(t)) - f_\delta\| - \|F(V_\delta(t)) - F(u_\delta(t))\|
\]

\[
\geq a(t)||V_\delta(t)|| - h(0)e^{-\varphi(t)} - e^{-\varphi(t)} \int_0^t e^{\varphi(s)}|\dot{a}\||V_\delta(s)||ds.
\]

Since \( a(t) \) satisfies (3.44) one gets by Lemma 2.8 the following inequality

\[
\frac{q}{1-2q}a(t)||V_\delta(t)|| \geq e^{-\varphi(t)} \int_0^t e^{\varphi(s)}|\dot{a}\||V_\delta(s)||ds.
\]

From the relation \( h(t) = \|F(u_\delta(t)) + a(t)u_\delta(t) - f_\delta\| \) (cf. (3.19)) and inequality (3.45) one gets

\[
h(0)e^{-\varphi(t)} \leq pa(0)||V_\delta(0)||e^{-\varphi(t)}, \quad t \geq 0.
\]

It follows from (3.2) that

\[
e^{-\varphi(t)}a(0) \leq a(t).
\]

Indeed, inequality \( a(0) \leq a(t)e^{\varphi(t)} \) is obviously true for \( t = 0 \), and

\[
(a(t)e^{\varphi(t)})' = a^2(t)e^{\varphi(t)} \left(1 - q - \frac{|\dot{a}(t)|}{a^2(t)}\right) \geq 0,
\]

by (3.2). Here, we have used the relation \( \dot{a} = -|\dot{a}| \) and the inequality \( 1 - q > q \).

Inequalities (3.56) and (3.57) imply

\[
e^{-\varphi(t)}h(0) \leq pa(t)||V_\delta(0)|| \leq pa(t)||V_\delta(t)||, \quad t \geq 0,
\]
where we have used the inequality $\|V_\delta(t)\| \leq \|V_\delta(t')\|$ for $t \leq t'$, established in Lemma 2.2. From (3.54) and (3.55), one gets
\[ C\delta^\zeta = \|F(u_\delta(t)) - f_\delta\| \geq (1 - p - \frac{q}{1 - 2q})a(t_\delta)\|V_\delta(t_\delta)\|. \]
Thus,
\[ \lim_{\delta \to 0} a(t_\delta)\|V_\delta(t_\delta)\| = 0. \]
From (2.8) and the triangle inequality we obtain
\[ a(t_\delta)\|V(t_\delta)\| \leq a(t_\delta)\|V_\delta(t_\delta)\| + a(t_\delta)\|V(t_\delta) - V_\delta(t_\delta)\| \leq a(t_\delta)\|V_\delta(t_\delta)\| + \delta. \]
This and (3.59) imply
\[ \lim_{\delta \to 0} a(t_\delta)\|V_\delta(t_\delta)\| = 0. \]
Since $\|V(t)\|$ is increasing and $\|V(0)\| > 0$, relation (3.61) implies $\lim_{\delta \to 0} a(t_\delta) = 0$. Since $0 < a(t) \to 0$, it follows that (3.47) holds.

Theorem 3.7 is proved.

If $F$ is a monotone operator then $F_1(u) = F(u + \bar{u})$, where $\bar{u} \in H$, is also a monotone operator. Consider the following Cauchy problem
\[ \dot{u} = -(F(u) + a(t)(u - \bar{u}) - f_\delta), \quad n \geq 0. \]
Applying Theorem 3.4 and Theorem 3.7 for $F_1$ one gets the following corollaries:

**Corollary 3.9.** Let $\bar{u} \in H$ be arbitrary and $y^*$ be the solution to (1.1) with minimal distance to $\bar{u}$. Let $a(t)$ satisfy (3.2). Assume that $F : H \to H$ is a monotone operator satisfying condition (1.3), and $u_0$ is an element of $H$, satisfying the inequality:
\[ \|F(u_0) - f_\delta\| > C\delta^\zeta > \delta, \]
where $C > 0$ and $0 < \zeta \leq 1$ are constants.

Then the solution $u_\delta(t)$ to problem (3.62) exists globally, and there exists a unique $t_\delta > 0$ such that
\[ \|F(u_\delta(t)) - f_\delta\| = C\delta^\zeta, \quad \|F(u_\delta(t)) - f_\delta\| > C\delta^\zeta, \quad \forall t \in [0, t_\delta). \]

If $\zeta \in (0, 1)$ and
\[ \lim_{\delta \to 0} t_\delta = \infty, \]
then
\[ \lim_{\delta \to 0} \|u_\delta(t) - y^*\| = 0. \]

**Corollary 3.10.** Let $a(t)$ satisfy (3.34). Let $F$ and $f_\delta$ be as in Corollary 3.9. Assume that $u_0$ be an element of $H$ such that $\|F(u_0) - f_\delta\| > C\delta^\zeta > \delta$. Assume in addition that $u_0$ satisfies either
\[ \|F(u_0) + a(0)(u_0 - \bar{u}) - f_\delta\| \leq p\|V_\delta(0)\|, \quad 0 < p < 1 - \frac{q}{1 - 2q}, \]
or
\[ \|F(u_0) + a(0)(u_0 - \bar{u}) - f_\delta\| \leq \theta\delta^\zeta, \quad 0 < \theta < C, \]
where $C > 0$ and $0 < \zeta \leq 1$ are constants from Corollary 3.10. Let $t_\delta$ be defined by (3.64). Then
\[ \lim_{\delta \to 0} t_\delta = \infty. \]
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