Stability analysis and absorption cross-section in
three-dimensional black string

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Abstract

It is shown that all string fields except dilaton are non-propagating in the
(2+1)-dimensional black string. One finds that the perturbation around the
black string reveals a mixing between the dilaton and other fields. Under
the new gauge(dilaton gauge), we disentangle this mixing and obtain one
decoupled dilaton equation. It turns out that this black string is stable.
From the scattering of dilaton off the neutral black string(\(N = 0\)), we find the
absorption cross-section. Further the absorption cross-section for minimally
coupled scalar is obtained and we compared it with that of dilaton.
I. INTRODUCTION

Black strings play an important role in understanding the microscopic origin for the black hole entropy. The statistical interpretation of the Bekenstein-Hawking entropy was made first in the extremal or near-extremal five-dimensional (5D) black holes \([1]\). The microstates of 5D extremal black hole arise mainly from the fields moving around a circle in the internal dimension. In order to understand this situation, it is useful to take this internal direction as a spacetime direction explicitly. This is the six-dimensional (6D) black string \([2]\). The entropy calculation of 6D black string gives us the same result as in the 5D black hole.

On the other hand, to calculate the entropy of Schwarzschild black hole, we need a nonperturbative formulation of string theory, M–theory. This is because the Schwarzschild black hole belongs to the non–extremal one. Recently, there have been many attempts along this direction \([3]\). The idea is to map the Schwarzschild black hole to extremal or near-extremal configurations, where the entropy counting might be possible \([4]\). It is shown that black strings give rise to Schwarzschild black hole with the compactification radius \((R_1)\) or charged dilaton black hole with the radius \((R_2)\), when compactified to lower dimensions before or after boosting\((\alpha)\) along uncompactified direction. One finds \(R_1 = R_2 \cosh \alpha\) and the Newton constant \(G_1^N = G_2^N / \cosh \alpha\). Thus the entropy of two holes are the same. Further a 7D Schwarzschild black hole is obtained as a compactification of a three brane in 11D supergravity \([5]\). And we relate it to a charged black hole with the same statistical entropy. The charged black hole will be found from subjecting the three brane to a boost in uncompactified space–time, followed by Kaluza-Klein compactification. A near–extremal charged hole is defined such that the Schwarzschild radius remains arbitrarily large at infinite boost\((\alpha \to \infty)\). This case is used to derive the entropy of neutral Schwarzschild hole from the charged ones.

In this paper, we study the propagation of string fields in the three–dimensional(3D) black string background with the new(dilaton) gauge. The 3D black string is used as a toy model for investigating higher–dimensional black strings \([6]\). Especially, the s–mode
approximation of higher-dimensional black string is exactly the same as in the 3D black string \[7\]. Here we are interested in the stability of 3D black string. The transition between Schwarzschild black holes to black branes in M–theory corresponds to the black hole–black string transition in a boosted frame \[3,4\]. The latter is related to the instability of a black string \[7\]. In this sense, it is very important to analyze the stability of a black string. Further we investigate the dynamical behavior (absorption cross-section=greybody factor) of 3D black string rather than the static behavior (entropy) \[8,9\]. Apart from counting the microstates of black holes, the dynamical behavior is also an important issue. This is so because the greybody factor for the black hole arises as a consequence of the gravitational potential barrier surrounding the horizon. That is, this is an effect of space–time curvature. Together with the Bekenstein–Hawking entropy, this seems to be the strong hint of a deep and mysterious connection between curvature and statistical mechanics.

The organization of this paper is as follows. In Sec.II we set up the equations of motion for the 3D black string and linearize these equations around the background solution. And we introduce both the transverse gauge and the new (dilaton) gauge. In Sec.III, we study the propagation of string fields with the gauge conditions. It turns out that a physically propagating mode is just the dilaton. Further we show that the 3D black string is stable against the external perturbation. Section IV is concerned with the dynamical behavior of the neutral (N=0) black string. We derive the absorption cross section from the scattering process of dilaton off the 3D black string. Finally we discuss our results in Sec.V.

**II. FORMALISM**

Let us start with the σ-model action of string theory \[10\]

\[
S_\sigma = -\frac{1}{4\pi \alpha'} \int d^2 \xi \left( \sqrt{\gamma} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu g_{\mu \nu} + \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu \nu} - \frac{1}{2} \alpha' \sqrt{\gamma} R^{(2)} \Phi \right),
\]

where \(R^{(2)}\) is the Ricci curvature of the world sheet, \(B_{\mu \nu}\) two–form field, and \(\Phi\) dilaton. The conformal invariance requires the \(\beta\)-function equations.
\[ R_{\mu\nu} - \nabla_\mu \nabla_\nu \Phi - \frac{1}{4} H_{\mu\rho\sigma} H^{\rho\sigma}_\nu = 0, \quad (2) \]

\[ \nabla^2 \Phi + (\nabla \Phi)^2 - \frac{8}{k} - \frac{1}{6} H^2 = 0, \quad (3) \]

\[ \nabla_\mu H^{\mu\nu\rho} + (\nabla_\mu \Phi) H^{\mu\nu\rho} = 0, \quad (4) \]

where \( H_{\mu\nu\rho} = 3 \partial_{[\mu} B_{\nu\rho]} \) is the Kalb–Ramond field corresponding to \( B_{\nu\rho} \). The above equations are also derived from the requirement that the fields must be an extremum of the low-energy string action in string frame,

\[ S_{l-e} = \int d^3 x \sqrt{-g} e^\Phi \{ R + (\nabla \Phi)^2 + \frac{8}{k} - \frac{1}{12} H^2 \}. \quad (5) \]

The static black string solution to (2)–(4) is found to be

\[ \bar{H}_{rtx} = \frac{Q}{r^2}, \quad \bar{\Phi} = \ln r - \frac{1}{2} \ln \left( \frac{k}{2} \right), \]

\[ \bar{g}_{\mu\nu} = \begin{pmatrix} -(1 - \frac{M}{r}) & 0 & 0 \\ 0 & (1 - \frac{N}{r}) & 0 \\ 0 & 0 & \frac{k}{8\pi^2} (1 - \frac{M}{r})^{-1} (1 - \frac{N}{r})^{-1} \end{pmatrix}, \quad (6) \]

with \( N \equiv Q^2/M(M > N) \). The Christoffel symbols are given by

\[ \Gamma^t_{tr} = \frac{1}{2} \left( \frac{1}{r-M} - \frac{1}{r} \right), \]

\[ \Gamma^x_{tx} = \frac{1}{2} \left( \frac{1}{r-N} - \frac{1}{r} \right), \]

\[ \Gamma^r_{tt} = \frac{4}{k} M \left( 1 - \frac{M}{r} \right) \left( 1 - \frac{N}{r} \right), \]

\[ \Gamma^r_{xx} = -\frac{4}{k} N \left( 1 - \frac{M}{r} \right) \left( 1 - \frac{N}{r} \right), \]

\[ \Gamma^r_{rr} = -\frac{1}{2} \left( \frac{1}{r-M} + \frac{1}{r-N} \right). \quad (7) \]

A simple extension of Witten’s construction for a gauged WZW model also yields the 3D charged black string [11]. The solution (6) is characterized by three parameters: \( M \) (mass), \( Q \) (axion charge per unit length), and \( k \) (cosmological constant). For \( 0 < |Q| < M \), black string is similar to the 4D Reissner-Nordström solution. In addition to the event (outer) horizon \( (r_{EH} = M) \), there exist an inner horizon \( (r_{IH} = N) \). When \( |Q| = M \), this becomes
the extremal black string. Finally, when $|Q| > M$, the spacetime has neither a horizon nor a curvature singularity and this case is not relevant to us.

To study the propagation of fields specifically, we introduce the small perturbation fields around the background solution as \[12,13\]

\[H_{rtx} = \bar{H}_{rtx}, \]
\[\Phi = \bar{\Phi} + \phi, \]
\[g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \tag{8}\]

In order to obtain the equations governing the perturbations, we introduce the notation

\[\hat{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} h. \tag{9}\]

And then one needs to linearize (2)–(4) to obtain

\[
\delta R_{\mu\nu}(h) = - \frac{1}{2} \bar{\nabla}_{\nu} \bar{\nabla}_{\rho} \phi + \delta \Gamma_{\nu}^{\rho}(h) \nabla_{\rho} \bar{\Phi} - \frac{1}{2} \bar{H}_{\mu\rho\sigma} \bar{H}_{\nu}^{\rho\sigma} + \frac{1}{2} \bar{H}_{\mu\rho\sigma} \bar{H}_{\nu}^{\rho\sigma} h^{\rho\sigma} = 0, \tag{10}\n
\]

\[
h_{\mu\nu} \bar{\nabla}_{\mu} \bar{\nabla}_{\nu} \phi + h_{\mu\nu} \partial_{\mu} \bar{\Phi} \partial_{\nu} \phi + \bar{\nabla}_{\mu} \bar{H}_{\nu}^{\rho\sigma} \partial_{\nu} \phi - \bar{\nabla}_{\nu} \phi - 2 \bar{g}_{\nu}^{\mu\rho} \partial_{\nu} \bar{\Phi} \partial_{\nu} \phi + 1/2 \{2 \bar{H}_{\mu\rho\sigma} \bar{H}_{\nu}^{\rho\sigma} - 3 \bar{H}_{\mu\rho\sigma} \bar{H}_{\alpha}^{\rho\sigma} h_{\alpha}^{\mu} \} = 0, \tag{11}\n
\]

\[
(\bar{\nabla}_{\mu} + \partial_{\mu} \bar{\Phi}) \bar{H}_{\nu}^{\mu\rho} - (\bar{\nabla}_{\mu} h_{\beta}^{\nu}) \bar{H}_{\beta}^{\mu\nu} + (\bar{\nabla}_{\mu} h_{\beta}^{\nu}) \bar{H}_{\beta}^{\mu\nu} - (\bar{\nabla}_{\mu} \bar{h}_{\alpha}^{\mu}) \bar{H}_{\nu}^{\alpha\rho} + (\partial_{\mu} \phi) \bar{H}_{\nu}^{\mu\rho} = 0, \tag{12}\n
\]

where

\[
\delta R_{\mu\nu}(h) = - \frac{1}{2} \bar{\nabla}_{\nu} \bar{h}_{\mu\nu} - \frac{1}{2} \bar{\nabla}_{\mu} \bar{h}_{\nu\rho} + \frac{1}{2} \bar{\nabla}_{\rho} \bar{h}_{\nu\rho} + \frac{1}{2} \bar{\nabla}_{\rho} \bar{h}_{\mu\rho}, \tag{13}\n
\]

\[
\delta \Gamma_{\mu\nu}(h) = \frac{1}{2} \bar{g}_{\rho\sigma} (\bar{\nabla}_{\nu} h_{\mu\sigma} + \bar{\nabla}_{\mu} h_{\nu\sigma} - \bar{\nabla}_{\sigma} h_{\mu\nu}). \tag{14}\n
\]

Here \(\delta R_{\mu\nu}\) can be transformed to the Lichnerowicz operator \[7\]

\[
\delta R_{\mu\nu} = - \frac{1}{2} \bar{\nabla}_{\nu} \hat{h}_{\mu\nu} + \bar{\nabla}_{\nu} (h \bar{h}_{\mu}) - \bar{R}_{\nu\alpha} h_{\alpha}^{\mu} + \bar{\nabla}_{\nu} (h \bar{h}_{\mu}). \tag{15}\n
\]

These are the bare perturbation equations. We have to examine whether there exist any choice of gauge which can simplify (10)–(12). Conventionally, we choose the harmonic(transverse) gauge (\(\bar{\nabla}_{\mu} \hat{h}_{\nu}^{\mu} = 0\)) to describe the propagation of gravitons. Further
one requires the traceless condition \( h = 0 \), which corresponds to de Donder gauge. However this does not eliminate all of the gauge degrees of freedom for the metric. There still exist the residual gauge transformations,

\[
\delta_\xi h_{\mu\nu} = \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu
\]

with \( \bar{\nabla}_\mu \xi^\mu = 0 \) and \( \bar{\nabla}^2 \xi^\mu = 0 \). The counting of gravitational degrees of freedom is as follows. A symmetric traceless tensor has \( D(D + 1)/2 - 1 \) in \( D \)-dimensions. \( D \) of them are eliminated by the harmonic gauge condition. Also \( D - 1 \) are eliminated from our freedom to take further residual gauge transformations. Thus gravitational degrees of freedom are \( D(D + 1)/2 - 1 - D - (D - 1) = D(D - 3)/2 \). We have no gravitational degrees of freedom in \( D = 3 \). Without loss of generality, the Kalb–Ramond perturbation \( \mathcal{H}_{rtx} \) may be taken to be zero in the black string background \[7\]. Hence the physical degree of freedom in the 3D black string turns out to be the dilaton field. As we will see later, however, the conventional (harmonic) gauge is not appropriate for studying the propagation of dilaton.

Here we introduce a new gauge (dilaton gauge) to investigate the propagation of dilaton. The harmonic gauge is originated from the harmonic coordinates which satisfy

\[
g^{\mu\nu} \Gamma^\lambda_{\mu\nu} = 0.
\]

Linearization of \([17]\) leads to

\[
h^{\mu\nu} \Gamma^\lambda_{\mu\nu} - \bar{g}^{\mu\nu} \delta \Gamma^\lambda_{\mu\nu} = 0
\]

with \( \bar{g}^{\mu\nu} \delta \Gamma^\lambda_{\mu\nu} = \bar{\nabla}_\mu \hat{h}^{\mu\lambda} \). For the Minkowski background (\( \bar{g}^{\mu\nu} = \eta^{\mu\nu} \)), \([18]\) becomes the harmonic gauge (\( \bar{\nabla}_\mu \hat{h}^{\mu\lambda} = 0 \)) \([14]\]. But we may choose a new gauge \( h^{\mu\nu} \Gamma^\lambda_{\mu\nu} = \bar{\nabla}_\mu \hat{h}^{\mu\lambda} \) when \( \bar{g}^{\mu\nu} \neq \eta^{\mu\nu} \), instead of the harmonic gauge. In our coordinate system, one finds

\[
\bar{g}^{\mu\nu} \Gamma^t_{\mu\nu} = 0, \quad \bar{g}^{\mu\nu} \Gamma^x_{\mu\nu} = 0, \quad \bar{g}^{\mu\nu} \Gamma^r_{\mu\nu} = -\frac{8}{kr} (r^2 - MN),
\]

which means that \( (t, x, r) \) do not belong to the harmonic coordinates. Requiring that \([19]\) remain unchanged at the linearized level, one always choose the new gauge (\( h^{\mu\nu} \Gamma^\lambda_{\mu\nu} = \bar{\nabla}_\mu \hat{h}^{\mu\lambda} \)). This gauge is suitable for studying the propagation of dilaton.
III. PROPAGATION OF STRING FIELDS

First we consider the symmetries of the background space–time in (3). In our case, the $t$ and $x$–translational symmetries means that we can decompose $h_{\mu\nu}$ into frequency modes in these variables [7]. Hence we take the form for $h_{\mu\nu}$,

$$h_{\mu\nu}(t, x, r) = e^{i\omega t} e^{i\mu x} \left[ H_{tt}(r) \begin{array}{ccc} H_{tx}(r) & H_{tx}(r) & H_{tx}(r) \\ H_{xt}(r) & H_{xx}(r) & H_{xx}(r) \\ H_{rt}(r) & H_{rx}(r) & H_{rx}(r) \end{array} \right].$$

Similarly, one chooses the perturbations for Kalb–Ramond field and dilaton as

$$\mathcal{H}_{rtx}(t, x, r) = \bar{\mathcal{H}}_{rtx}(r) e^{i\omega t} e^{i\mu x} \bar{\mathcal{H}}(r),$$

$$\phi(t, x, r) = e^{i\omega t} e^{i\mu x} \bar{\phi}(r).$$

Since we start with full degrees of freedom (20), we should choose a gauge to study the propagation of string fields.

A. Harmonic Gauge

With the harmonic gauge condition ($\bar{\nabla}_\mu \hat{h}^\mu = 0$), the equations (10)–(12) become

$$\nabla^2 h_{\mu\nu} - \bar{R}_{\sigma\nu} h^\sigma_{\mu} - \bar{R}_{\sigma\mu} h^\sigma_{\nu} + 2 \bar{R}_{\sigma\rho\nu} h^{\sigma\rho} + 2 \nabla_\mu \nabla_\nu \phi$$

$$- 2 \delta \Gamma^\rho_{\mu\nu}(r) \nabla_\rho \bar{\Phi} + \bar{H}_{\mu\rho\sigma} H^{\rho\sigma} - \bar{H}_{\mu\rho\sigma} H^{\rho\sigma} = 0,$$

$$h^{\mu\nu} \nabla_\mu \nabla_\nu \bar{\Phi} + h^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \Phi - \nabla^2 \phi - 2 g^{\mu\nu} \nabla_\mu \Phi \nabla_\nu \phi$$

$$+ \frac{1}{6} \left\{ 2 \bar{H}_{\mu\rho\sigma} H^{\mu\rho\sigma} - 3 \bar{H}_{\mu\rho\sigma} H^{\rho\sigma} h^{\mu}_{\alpha} \right\} = 0,$$

$$\left( \nabla_\mu + \partial_\mu \bar{\Phi} \right) \mathcal{H}^{\mu\nu} - \left( \nabla_\mu h^{\nu}_{\beta} \right) \bar{H}^{\mu\beta\rho} + \left( \nabla_\nu h^{\rho}_{\beta} \right) \bar{H}^{\mu\beta\nu} + \left( \partial_\nu \phi \right) \bar{H}^{\mu\nu} = 0,$$

Hereafter we are interested only in the dilaton propagation. From (24) one finds

$$\phi'' + \left( \frac{1}{r} + \frac{1}{r - M} + \frac{1}{r - N} \right) \phi' + \frac{kr}{8(r - M)(r - N)} \left( \frac{\omega^2}{r - M} - \frac{\mu^2}{r - N} \right) \phi$$

$$- \frac{1}{2r^2(r - M)(r - N)} \left[ M(r + N) h^t_t + N(r + M) h^x_x \right.$$

$$\left. + \left\{ 2r^2 - (M + N)r + 2MN \right\} h^r_r - 4MN \mathcal{H} \right] = 0,$$
where the prime(′) means the derivative with respect to \( r \). We note that the analysis of dilaton fluctuation around the black string reveals a surprising mixing between the dilaton and other fields. We have to disentangle this mixing and obtain one decoupled dilaton equation by using the harmonic gauge(\( \nabla_\mu \hat{h}^{\mu \rho} = 0 \)) and Kalb–Ramond equation (25). However we fail to obtain a decoupled dilaton equation.

**B. Dilaton Gauge**

We recognize that it is not easy to decouple the dilaton equation with the harmonic gauge condition. But if we choose the dilaton gauge(\( h_{\mu \nu} \Gamma^\sigma_{\mu \nu} \)) the dilaton equation can be diagonalized easily. Under this gauge the equations (10)–(12) are given by

\[
\nabla^2 h_{\mu \nu} - \bar{R}_{\sigma \nu} h_{\mu}^{\sigma} - \bar{R}_{\sigma \mu} h_{\nu}^{\sigma} + 2 \bar{R}_{\sigma \mu \nu} h^{\sigma \rho} + 2 \nabla_\mu \nabla_\nu \phi - g_{\mu \nu} \nabla_\rho (h^{\alpha \beta} \Gamma^\sigma_{\alpha \beta}) - g_{\nu \phi} \nabla_\mu (h^{\alpha \beta} \Gamma^\sigma_{\alpha \beta}) - 2 \delta \Gamma^\rho_{\mu \nu} (h^{\sigma \rho}) = 0, \quad (27)
\]

\[
h^{\mu \nu} \nabla_\mu \nabla_\nu \phi + h^{\mu \nu} \nabla_\mu \Phi \nabla_\nu \phi + \bar{g}^{\mu \nu} \delta \Gamma^\rho_{\mu \nu} \partial_\rho \phi - \nabla^2 \phi - 2 \bar{g}^{\mu \nu} \nabla_\mu \Phi \nabla_\nu \phi + \frac{1}{6} \{ 2 \bar{H}^{\mu \rho \sigma} \bar{H}^{\mu \rho \sigma} - 3 \bar{H}^{\mu \rho \sigma} \bar{H}^{\alpha \rho \sigma} h^\alpha \} = 0, \quad (28)
\]

\[
(\nabla_\mu + \partial_\mu \bar{\phi}) \bar{H}^{\mu \rho \sigma} - \bar{\nabla}_\mu h^{\beta \rho} \bar{H}^{\mu \beta \rho} + \bar{\nabla}_\mu h^{\beta \rho} \bar{H}^{\mu \beta \rho} + \partial_\mu \phi \bar{H}^{\mu \rho \sigma} - h^\delta \Gamma^\alpha_{\delta \eta} \bar{H}^{\rho \sigma} h^\alpha = 0. \quad (29)
\]

Thanks to the dilaton gauge, the first three terms in (28) cancel out. Then the dilaton equation (28) leads to

\[
\phi'' + \left( \frac{1}{r} + \frac{1}{r - M} + \frac{1}{r - N} \right) \phi' + \frac{k r}{8 (r - M) (r - N)} \left( \frac{\omega^2}{r - M} - \frac{\mu^2}{r - N} \right) \phi - \frac{M N}{r^2 (r - M) (r - N)} (h - 2 \bar{H}) = 0. \quad (30)
\]

Now we attempt to disentangle the final term in (30) by using the dilaton gauge (18) and Kalb–Ramond equation (29). Each component of dilaton gauge condition gives rise to

\[
t : (\partial_r - \frac{1}{r}) h^{tr} + i \omega h^{tt} + i \mu h^{tx} = 0, \quad (31)
\]

\[
x : (\partial_r - \frac{1}{r}) h^{xr} + i \omega h^{xt} + i \mu h^{xx} = 0, \quad (32)
\]

\[
r : (\partial_r - \frac{1}{r}) h^{rr} + i \omega h^{rt} + i \mu h^{rx} = 0. \quad (33)
\]
And the Kalb–Ramond equation (29) leads to

\[ tx : r(\phi' + \mathcal{H}' - h_k' - h_x') + \frac{8}{k}(r^2 - MN)h_{rr} + i\omega h_x' + i\mu h_r' = 0, \]  
(34)

\[ tr : i\mu(\phi + \mathcal{H} - h_k' - h_r') - \frac{8}{k}(r - M)h_{xx} + i\omega h_t' + h_x = 0, \]  
(35)

\[ xr : i\omega(\phi + \mathcal{H} - h_x' - h_r') - \frac{8}{k}(r - N)h_{tr} + i\mu h_t' + h_r' = 0. \]  
(36)

Solving six equations (34)–(36), one finds an important equation

\[ \partial_\mu(\phi + \mathcal{H} - h) = 0. \]  
(37)

The solution to this is given by

\[ h = \phi + \mathcal{H}. \]  
(38)

This means that the trace of \( h_{\mu\nu}(h) \) is a redundant field. On the other hand, the Einstein equations from (27) are given by

\[ tt : \frac{8}{k}(r - M)(r - N)h''_{tt} + \frac{8}{k}(2r^2 - (3M + N)r + 2MN)h'_{tt} + r\left(\frac{\omega^2}{r - M} - \frac{\mu^2}{r - N}\right)h_{tt} \]
\[ + \quad \frac{8}{k}M^2(r - N)h_{tt} - \frac{8}{k}MN(r - M)h_{xx} - \frac{64}{k}M(r - M)(r - N)(r^2 - MN)h_{rr} \]
\[ - \quad \frac{16}{k}i\omega(r - M)(r - N)h_{tr} \]
\[ - \quad \frac{8}{k}M(r - M)(r - N)\phi' - 2\omega^2\phi + 16\frac{MN(r - M)}{kr^3}\mathcal{H} = 0, \]  
(39)

\[ xx : \frac{8}{k}(r - M)(r - N)h''_{xx} + \frac{8}{k}(2r^2 - (M + 3N)r + 2MN)h'_{xx} + r\left(\frac{\omega^2}{r - M} - \frac{\mu^2}{r - N}\right)h_{xx} \]
\[ - \quad \frac{8}{k}MN(r - N)h_{tt} + \frac{8}{k}N^2(r - M)h_{xx} + \frac{64}{k}N(r - M)(r - N)(r^2 - MN)h_{rr} \]
\[ - \quad \frac{16}{k}i\mu(r - M)(r - N)h_{tr} \]
\[ + \quad \frac{8}{k}N(r - M)(r - N)\phi' - 2\mu^2\phi + 16\frac{MN(r - N)}{kr^3}\mathcal{H} = 0, \]  
(40)

\[ rr : \frac{8}{k}(r - M)(r - N)h''_{rr} + \frac{16}{k}(3r^2 - (M + N)r - MN)h'_{rr} + r\left(\frac{\omega^2}{r - M} - \frac{\mu^2}{r - N}\right)h_{rr} \]
\[ - \quad \frac{M}{(r - M)^2}h_{tt}' + \frac{N}{(r - N)^2}h_{xx}' - \frac{M(M - N)}{(r - M)^3(r - N)}h_{tt} - \frac{N(M - N)}{(r - M)(r - N)^3}h_{xx} \]
\[ + \quad \frac{8}{k}(6r^4 - 5(M + N)r + 4MN^2 - MN(M + N)r + 2M^2N^2)h_{rr} \]
\[ H = 0, \quad (41) \]

\[ tx : \frac{8}{k} (r - M)(r - N) h''_{tx} + \frac{16(r - M)(r - N)}{r} h'_{tx} + r \left( \frac{\omega^2}{r - M} - \frac{\mu^2}{r - N} \right) h_{tx} \\
- \frac{8}{k} i\mu \frac{(r^2 - 2Nr + MN)}{r} h_{tx} + \frac{8}{k} i\omega \frac{(r^2 - 2Mr + MN)}{r} h_{tx} - 2\omega \mu \phi = 0, \quad (42) \]

\[ tr : \frac{8}{k} (r - M)(r - N) h''_{tr} + \frac{8(r - M)(3r - N)}{r} h'_{tr} + r \left( \frac{\omega^2}{r - M} - \frac{\mu^2}{r - N} \right) h_{tr} \\
+ \frac{8}{k} (r - M) h_{tr} - i\mu \frac{N}{(r - N)} h_{tx} \\
+ \frac{i\omega}{2} \frac{M}{(r - M)^2} h_{tt} + \frac{i\omega}{2} \frac{N}{(r - N)^2} h_{xx} + \frac{4i\omega}{k} \frac{(3M + N)r - 4MN}{r} h_{rr} \\
+ 2i\omega \phi' - i\omega \frac{M}{r(r - M)} \phi = 0, \quad (43) \]

\[ xr : \frac{8}{k} (r - M)(r - N) h''_{xr} + \frac{8(3r - M)(r - N)}{r} h'_{xr} + r \left( \frac{\omega^2}{r - M} - \frac{\mu^2}{r - N} \right) h_{xr} \\
+ \frac{8}{k} \frac{r}{r} h_{xr} + i\omega \frac{M}{(r - M)^2} h_{tx} \\
- \frac{i\mu}{2} \frac{M}{(r - M)^2} h_{tt} - \frac{i\mu}{2} \frac{N}{(r - N)^2} h_{xx} + \frac{4i\mu}{k} \frac{(M + 3N)r - 4MN}{r} h_{rr} \\
+ 2i\mu \phi' - i\mu \frac{N}{r(r - N)} \phi = 0. \quad (44) \]

From (34)–(36) and (39)–(41) we may set \( H = 0 \) without loss of generality. Also this is confirmed from Ref. [7]. With \( H = 0 \), (30) becomes a decoupled dilaton equation

\[ \tilde{\phi}'' + \left( \frac{1}{r} + \frac{1}{r - M} + \frac{1}{r - N} \right) \tilde{\phi}' \\
+ \frac{kr}{8(r - M)(r - N)} \left( \frac{\omega^2}{r - M} - \frac{\mu^2}{r - N} \right) \tilde{\phi} - \frac{MN}{r^2(r - M)(r - N)} \tilde{\phi} = 0. \quad (45) \]

### C. Stability Analysis

Since the physical field in the 3D black string is dilaton, we will consider the stability of dilaton. We remind the reader that the stability analysis should be based on the physical fields. In order to check whether there exists an exponentially growing mode, we replace \( \omega \)
by $\omega = i\Omega$ in (45) [15]. Now we investigate the behavior of $\tilde{\phi}$ as $r \to \infty$ and as $r \to M$. In the asymptotically flat region ($r \to \infty, \ k \to \infty$), one finds the relevant equation

$$\frac{d^2\tilde{\phi}}{d\rho^2} - (\Omega^2 + \mu^2)\tilde{\phi} = 0$$

with the new coordinate $\rho = \sqrt{\frac{k}{8}}\ln \left( r\sqrt{\frac{2}{k}} \right)$. The regular solution to (46) is given by

$$\tilde{\phi}_\infty \sim e^{-\sqrt{\Omega^2 + \mu^2}\rho}.$$  

(47)

On the other hand, near the horizon ($r \to M$) we obtain

$$\tilde{\phi}'_M - \frac{kM\Omega^2}{8(M - N)(r - M)^2} \tilde{\phi} = 0.$$  

(48)

Here one finds the regular solution

$$\tilde{\phi}_M \sim (r - M)^{\sqrt{\frac{kM}{8(M - N)}}}\Omega.$$  

(49)

It turns out that a necessary condition to obtain a regular solution in the whole region is to have a change in sign in the coefficient of the undifferentiated $\phi$ in (45) [7,15]. It is obvious that this coefficient does not change sign when one moves from $r = M$ to $r = \infty$. Therefore we conclude that the 3D black string is stable. Also $N = 0$ case (neutral black string) cannot lead to an instability.

**IV. ABSORPTION CROSS–SECTION**

In this section we will calculate the absorption cross–section to obtain the dynamic behavior of the black string. The low energy condition ($\omega \ll \frac{1}{M}$) is assumed, which implies that the Compton wavelength of the particle is much larger than the gravitational radius of black string. Since it is hard to find a solution to (15), we consider the case of $N = 0$(neutral black string). We also assume $\mu \leq \omega$ and use a matching procedure. The space–time is divided into two regions: the near region ($r \sim M$) and far region ($r \gg \frac{1}{\omega}$) [16]. We now study each region in turn.
A. Near–Region Solution

In this case, (45) leads to
\[
\ddot{\tilde{\phi}} + \left( \frac{2}{r} + \frac{1}{(r-M)} \right) \frac{\dot{\tilde{\phi}}}{r} + \frac{k}{8} \left( \frac{\omega^2}{r-M} - \frac{\mu^2}{r} \right) \tilde{\phi} = 0.
\] (50)

In order to solve the above equation, we introduce a new variable
\[
z = 1 - \frac{M}{r}, \quad 0 \leq z \leq 1.
\] (51)

The horizon is located at \( z = 0 \) and the asymptotically flat region is at \( z = 1 \). Then (50) is given by
\[
z(1-z) \partial_z^2 \tilde{\phi} + \partial_z \tilde{\phi} + \frac{k}{8} \left( \frac{\omega^2}{z} + \frac{\omega^2 - \mu^2}{1-z} \right) \tilde{\phi} = 0.
\] (52)

This can be transformed into the hypergeometric form by defining
\[
\tilde{\phi} = z^\alpha (1-z)^\beta \Psi.
\] (53)

(52) leads to
\[
z(1-z) \partial_z^2 \Psi + \{1 + 2\alpha - 2(\alpha + \beta)z\} \partial_z \Psi - (\alpha + \beta)(\alpha + \beta - 1)\Psi = 0
\] (54)

where
\[
\alpha = i\sqrt{\frac{k}{8}\omega}, \quad \beta = 1 + i\sqrt{\frac{k}{8}(\omega^2 - \mu^2)} - 1.
\] (55)

Comparing (54) with the standard hypergeometric equation, one finds the solution
\[
\tilde{\phi}(z) = z^\alpha (1-z)^\beta \left[ C_1 F(\alpha + \beta, \alpha + \beta - 1, 1 + 2\alpha; z) 
+ C_2 z^{-2\alpha} F(-\alpha + \beta, -\alpha + \beta - 1, 1 - 2\alpha; z) \right]
\] (56)

where \( C_1 \) and \( C_2 \) are to–be–determined constants. Near the horizon \((z \to 0, r \to M)\), one gets
\[
\tilde{\phi}_M(z) = \frac{C_1}{M^\alpha} e^{i\sqrt{\mu}\omega \ln(r-M)} + C_2 M^\alpha e^{-i\sqrt{\mu}\omega \ln(r-M)},
\] (57)
where the first term (last term) correspond to the ingoing(outgoing) wave. We impose the condition that there be only ingoing flux at the horizon to obtain the absorption cross-section. This implies \( C_2 = 0 \) and thus the solution around \( z = 0 \) is

\[
\tilde{\phi}_{\text{near}}(z) = C_1 z^\alpha (1 - z)^\beta F(\alpha + \beta, \alpha + \beta - 1, 1 + 2\alpha; z). \tag{58}
\]

**B. Far–Region Solution**

In this region (50) reduces to

\[
\tilde{\phi}'' + \frac{3}{r} \tilde{\phi}' + \frac{k \omega^2 - \mu^2}{8} \frac{1}{r^2} \tilde{\phi} = 0. \tag{59}
\]

The solution of this equation is

\[
\tilde{\phi}_{\text{far}}(r) = A \left( \frac{M}{r} \right)^{2-\beta} + B \left( \frac{M}{r} \right)^{\beta} = \frac{1}{r} \left[ A M^{2-\beta} e^{i \sqrt{\frac{k}{8} (\omega^2 - \mu^2)} - 1 \ln r} + B M^\beta e^{-i \sqrt{\frac{k}{8} (\omega^2 - \mu^2)} - 1 \ln r} \right], \tag{60}
\]

where \( A \) and \( B \) are the normalization constants. Here the first (last) term correspond to the ingoing (outgoing) wave in the asymptotically flat region.

**C. Matching the far and near solutions**

Now we need to match the far–region solution (50) to the large \( r(z \to 1) \) limit of near–region solution (58) in the overlapping region. The \( z \to 1 \) behavior of (58) follows from the \( z \to 1 - z \) transformation rule for hypergeometric functions. This takes the form

\[
\tilde{\phi}_{n \to f}(z) = C_1 z^\alpha (1 - z)^\beta \left[ \frac{\Gamma(1 + 2\alpha)\Gamma(2 - 2\beta)}{\Gamma(1 + \alpha - \beta)\Gamma(2 + \alpha - \beta)} F(\alpha + \beta, \alpha + \beta - 1, 2\beta - 1; 1 - z) \\
+ (1 - z)^{2-2\beta} \frac{\Gamma(1 + 2\alpha)\Gamma(-2 + 2\beta)}{\Gamma(\alpha + \beta)\Gamma(\alpha + \beta - 1)} F(\alpha - \beta + 1, \alpha - \beta + 2, 3 - 2\beta; 1 - z) \right]. \tag{61}
\]

Using \( 1 - z = \frac{M}{r} \), one obtains the explicit form

\[
\tilde{\phi}_{n \to f}(r) = C_1 \left( \frac{M}{r} \right)^{\beta} \frac{\Gamma(1 + 2\alpha)\Gamma(2 - 2\beta)}{\Gamma(1 + \alpha - \beta)\Gamma(2 + \alpha - \beta)} + C_1 \left( \frac{M}{r} \right)^{2-\beta} \frac{\Gamma(1 + 2\alpha)\Gamma(-2 + 2\beta)}{\Gamma(\alpha + \beta)\Gamma(\alpha + \beta - 1)}. \tag{62}
\]
Matching (60) to (62), we find

\[ A = \frac{\Gamma(1 + 2\alpha)\Gamma(-2 + 2\beta)}{\Gamma(\alpha + \beta)\Gamma(\alpha + \beta - 1)} C_1, \]

\[ B = \frac{\Gamma(1 + 2\alpha)\Gamma(2 - 2\beta)}{\Gamma(1 + \alpha - \beta)\Gamma(2 + \alpha - \beta)} C_1. \]  

(63)

The reflection coefficient \( R \) is then given by

\[ R = \left| \frac{B}{A} \right|^2 = \left| \frac{\Gamma(\alpha + \beta)\Gamma(\alpha + \beta - 1)}{\Gamma(1 + \alpha - \beta)\Gamma(2 + \alpha - \beta)} \right|^2. \]  

(64)

The absorption coefficient \( A \) (partial wave absorption cross-section) is calculated as

\[ A = 1 - R = 1 - \frac{\sinh 2\pi \sqrt{k_8\omega} \sinh 2\pi \sqrt{k_8(\omega^2 - \mu^2) - 1}}{\sinh^2 \pi \left( \sqrt{k_8\omega} + \sqrt{k_8(\omega^2 - \mu^2) - 1} \right)}. \]  

(65)

This result can also be obtained from the flux calculation. The conserved flux due to (50) is defined as

\[ \mathcal{F} = 2\pi i \left( \tilde{\phi}^* \Delta \partial_r \tilde{\phi} - \tilde{\phi} \Delta \partial_r \tilde{\phi}^* \right). \]  

(66)

Using (58) and \( \Delta = r^2(r - M) \), the incoming flux across the horizon is given by

\[ \mathcal{F}(0) = 4\pi M^2 \sqrt{k_8} |C_1|^2 \omega. \]  

(67)

From (60) the incoming flux at infinity is found to be

\[ \mathcal{F}_\infty = 4\pi |A|^2 M^2 \sqrt{k_8}(\omega^2 - \mu^2) - 1. \]  

(68)

Hence the partial wave absorption cross-section is

\[ \sigma_{\phi}^{\text{BS}} = \frac{\mathcal{F}(0)}{\mathcal{F}_\infty} \]

\[ = \sqrt{k_8} \frac{\omega}{\sqrt{k_8(\omega^2 - \mu^2) - 1}} \left| \frac{\Gamma(\alpha + \beta)\Gamma(\alpha + \beta - 1)}{\Gamma(1 + 2\alpha)\Gamma(2\beta - 2)} \right|^2 \]

\[ = \frac{\sinh 2\pi \sqrt{k_8\omega} \sinh 2\pi \sqrt{k_8(\omega^2 - \mu^2) - 1}}{\sinh^2 \pi \left( \sqrt{k_8\omega} + \sqrt{k_8(\omega^2 - \mu^2) - 1} \right)}. \]  

(69)

We note that in the case of 3D black string the absorption coefficient \( A = \sigma_{\phi}^{\text{BS}} \) is just the plane-wave absorption cross-section.
V. DISCUSSIONS

First, let us discuss the importance of our new gauge (dilaton gauge). It turns out that the conventional (harmonic) gauge is not appropriate for investigating the propagation of dilaton. Instead, here we introduce the dilaton gauge. This gauge transforms the graviton part into complicated form, while it simplifies the mixing between the dilaton and the other fields. Further, the dilaton gauge condition itself provides us with the simple relations (31)–(33). This is so because $h_{\mu\nu}\Gamma^\rho_{\mu\nu} = \hat{\nabla}_\mu \hat{h}^{\mu\rho}$ is reduced to $\partial_\mu \hat{h}^{\mu\rho} = \frac{1}{r} h^{\mu\rho}$. And the Kalb–Ramond equation is also simplified as in (34)–(36) by this gauge. We note that the trace of $h_{\mu\nu}(h)$ is not zero but a redundant one ($h = \phi + \mathcal{H}$) under this gauge. Thanks to the dilaton gauge, we obtain the decoupled dilaton equation. The stability analysis based on this equation shows that the 3D black string is stable.

Now we turn our attention to the dynamic behavior of neutral black string. The relevant quantity is the absorption cross–section. In the case of charged black string ($N \neq 0$), we cannot obtain this quantity. For the low energy ($\omega \ll \frac{1}{M}$) and $\omega \geq \mu$, one finds the absorption cross–section using a matching procedure. Since the dilaton belongs to the fixed scalar, it is useful to compare the absorption cross–section with that of minimally coupled scalar ($\psi$). This field satisfies the equation: $\nabla^2 \psi = 0$. By similar procedure, one can obtain its cross–section as

$$\sigma^\psi_{BS} = \frac{\sinh 2\pi \sqrt{\frac{k}{8}} \omega \sinh 2\pi \sqrt{\frac{k}{8}} (\omega^2 - \mu^2)}{\sinh^2 \pi \left( \sqrt{\frac{k}{8}} \omega + \sqrt{\frac{k}{8}} (\omega^2 - \mu^2) \right)}.$$  \hspace{1cm} (70)

If $\frac{k}{8}(\omega^2 - \mu^2) \gg 1$, there is essentially no difference between (69) and (70). This means that to extract the dynamic behavior of the neutral ($N = 0$) black string, one can use either the minimally coupled scalar or the dilaton (fixed scalar). Further let us compare (70) with the absorption cross–section of minimally coupled scalar in the BTZ black hole. In the BTZ black hole, one finds $\sigma^\psi_{BTZ} \rightarrow A_H (= 2\pi r_+)$ when $m = 0$, $\omega \rightarrow 0$. In our case, we find the similar result ($\sigma^\phi_{BS} \rightarrow 1$: total absorption) for $\mu = 0$, $\omega \rightarrow 0$.

Finally we comment on the 3D extremal ($M = N$) black string. This was discussed in
Ref. [13]. In this case the metric is not only translationally invariant, but boost invariant. In other words, the space–time of 3D extremal black string has a null Killing symmetry. In this case, it turns out that the graviton become a propagating mode, whereas the dilaton is non–propagating. This case contrast well with the present case. However, the graviton may become a propagating mode by the transmutation of the degrees of freedom with other field (here, dilaton) in the extremal balck string space–time. This is very similar to the Higgs mechanism for gauge fields in the Minkowski space–time.

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