Linear constrained Cosserat-shell models including terms up to $O(h^5)$: conditional and unconditional existence and uniqueness

Ionel-Dumitrel Ghiba and Patrizio Neff

Abstract. In this paper, we linearise the recently introduced geometrically nonlinear constrained Cosserat-shell model. In the framework of the linear constrained Cosserat-shell model, we provide a comparison of our linear models with the classical linear Koiter shell model and the “best” first-order shell model. For all proposed linear models, we show existence and uniqueness based on a Korn’s inequality for surfaces.

Mathematics Subject Classification. 4A05, 74A60, 74B20, 74G65, 74K20, 74K25, 74Q05.

Keywords. Cosserat shell, 6-Parameter resultant shell, In-plane drill rotations, Constrained Cosserat elasticity, Isotropy, Existence of minimisers, Linear theories, Deformation measures, Change of metric, Bending measures, Change of curvature measures.

Contents

1. Introduction 2

2. Short overview of the geometrically nonlinear constrained Cosserat-shell model including terms up to $O(h^5)$ 2
   2.1. Notations 2
   2.2. Shell geometry and kinematics 3
   2.3. Variational problem for the constrained nonlinear Cosserat-shell model 5

3. The classical linear (first) Koiter model and the corresponding existence results 8

4. The linear constrained Cosserat-shell model 10
   4.1. The deformation measures in the linear constrained Cosserat-shell model 10
   4.2. The constrained linear $O(h^5)$-Cosserat-shell model: conditional existence 14
   4.3. The constrained linear $O(h^3)$-Cosserat-shell model. Conditional existence 22

5. A modified constrained linear $O(h^5)$-Cosserat-shell model. Unconditional existence 23
   5.1. Variational problem for the modified constrained nonlinear Cosserat-shell model 23
   5.2. The modified constrained linear $O(h^5)$-Cosserat-shell model 25
   5.3. Unconditional existence results 25

6. Conclusion 27

Acknowledgements 27

References 27
1. Introduction

The aim of this paper is to provide the linearised formulation of the constrained geometrically nonlinear Cosserat-shell model including terms up to order $O(h^5)$ in the shell thickness $h$ proposed in [33]. The usage of the Cosserat directors [25,26] is very important in the study of thin bodies like rods, plates and shells, since they are able to capture three-dimensional effects in one- or two-dimensional models, each particle being endowed with a triad of directors. The Cosserat approach to shell theory (also called micropolar shell theory) was initiated by the Cosserat brothers, who were the first to elaborate a rigorous study on directed media. Some theories of Cosserat surfaces have been presented in the monograph [38,42] and the linearised theory has been investigated in a large number of papers, see for instance [9–11,27]. The theory of simple shells (models which describe the shell-like body as a deformable surface endowed with an independent triad of orthonormal vectors connected to each point of the surface) has been given by Zhilin and Altenbach in [2,3,44,45] and a mathematical study of the linearised equations for this model has been presented in the papers [12,13]. We refer to the review paper [4] for a detailed presentation of various approaches and developments concerning Cosserat-type shell theories and to the books [21–23] for linear shell theories in the classical elasticity framework.

The constrained geometrically nonlinear Cosserat-shell model including terms up to order $O(h^5)$ in the shell thickness $h$ proposed in [33] is obtained as limit case of the novel geometrically nonlinear Cosserat introduced in [30,39], i.e., letting the Cosserat couple modulus $\mu_c \to \infty$, the independent Cosserat microrotation must coincide with the continuum macrorotation locally. In the constrained elastic model, the parental three-dimensional model turns into the Toupin couple stress model [43, Eq. 11.8]. The original dimensional descent was obtained starting with a 3D-parent Cosserat model and we obtained a kinematical model which is equivalent to the kinematical model of 6-parameter shells. Nevertheless, the theory of 6-parameter shells was developed for shell-like bodies made of Cauchy materials, see the monographs [20,37] or [29,41]. Here, the constrained geometrically nonlinear Cosserat-shell model is considered in order to be able to compare the strain measures and the energies involved in our model with those used in other theories which do not consider independent Cosserat directors. In the present paper, we give the linearised form of the constrained Cosserat-shell model. We establish that the kinematics is consistent with both the classical linear Koiter–Sanders–Budianski [18] “best” first-order shell model and the Anicic-Léger model [5–7].

For all the proposed models, we identify the set of admissible solutions and we show existence of solution. Comparing the conditions imposed on the shell thickness versus the initial curvature for the $O(h^5)$-models vs. the $O(h^3)$-models, we see that the existence result for the $O(h^5)$-model needs less stringent assumptions. While in the unconstrained linear Cosserat-shell model there is no need for the use of Korn-type inequalities, in the constrained linear shell models the Korn-type inequalities must be applied.

2. Short overview of the geometrically nonlinear constrained Cosserat-shell model including terms up to $O(h^5)$

2.1. Notations

In this paper, for $a, b \in \mathbb{R}^n$ we let $\langle a, b \rangle_{\mathbb{R}^n}$ denote the scalar product on $\mathbb{R}^n$ with associated vector norm $\|a\|_{\mathbb{R}^n}^2 = \langle a, a \rangle_{\mathbb{R}^n}$. The standard Euclidean scalar product on the set of real $n \times m$ second-order tensors $\mathbb{R}^{n \times m}$ is given by $\langle X, Y \rangle_{\mathbb{R}^{n \times m}} = \text{tr}(XY^T)$, and thus the (squared) Frobenius tensor norm is $\|X\|_{\mathbb{R}^{n \times m}}^2 = \langle X, X \rangle_{\mathbb{R}^{n \times m}}$. The identity tensor on $\mathbb{R}^{n \times n}$ will be denoted by $\mathbb{I}_n$, so that $\text{tr}(X) = \langle X, \mathbb{I}_n \rangle$, and the zero matrix is denoted by $0_n$. We let $\text{Sym}(n)$ and $\text{Sym}^+(n)$ denote the symmetric and positive definite symmetric tensors, respectively. We adopt the usual abbreviations of Lie-group theory, i.e., $\text{GL}(n) =$...
\(\{X \in \mathbb{R}^{n \times n} \mid \det(X) \neq 0\} \) the general linear group \(\text{SO}(n) = \{X \in \text{GL}(n) \mid X^T X = I_n, \det(X) = 1\}\) with corresponding Lie-algebras \(\mathfrak{s}o(n) = \{X \in \mathbb{R}^{n \times n} \mid X^T = -X\}\) of skew-symmetric tensors and \(\mathfrak{s}l(n) = \{X \in \mathbb{R}^{n \times n} \mid \text{tr}(X) = 0\}\) of traceless tensors. For all \(X \in \mathbb{R}^{n \times n}\) we set \(\text{sym} X = \frac{1}{2}(X + X^T) \in \mathfrak{s}o(n)\), skew \(X = \frac{1}{2}(X - X^T) \in \mathfrak{s}o(n)\) and the deviatoric part \(\text{dev} X = X - \frac{1}{n} \text{tr}(X) \cdot I_n \in \mathfrak{s}o(n)\) and we have the orthogonal Cartan-decomposition of the Lie-algebra \(\mathfrak{g}l(n) = \{\mathfrak{s}l(n) \cap \text{Sym}(n)\} \oplus \mathfrak{s}o(n) \oplus \mathbb{R} \cdot I_n\), \(X = \text{dev} X + \text{skew} X + \frac{1}{n} \text{tr}(X) \cdot I_n\). A matrix having the three column vectors \(A_1, A_2, A_3\) will be written as \((A_1 \mid A_2 \mid A_3)\). We make use of the operator \(\text{axl} : \mathfrak{s}o(3) \rightarrow \mathbb{R}^3\) associating with a skew-symmetric matrix \(A \in \mathfrak{s}o(3)\) the vector \(\text{axl}(A) := (-A_{23}, A_{13}, -A_{12})^T\). The inverse operator will be denoted by \(\text{Anti} : \mathbb{R}^3 \rightarrow \mathfrak{s}o(3)\).

For an open domain \(\Omega \subseteq \mathbb{R}^3\), the usual Lebesgue spaces of square integrable functions, vector or tensor fields on \(\Omega\) with values in \(\mathbb{R}, \mathbb{R}^3, \mathbb{R}^{3 \times 3}\) or \(\text{SO}(3)\), respectively, will be denoted by \(L^2(\Omega; \mathbb{R}), L^2(\Omega; \mathbb{R}^3), L^2(\Omega; \mathbb{R}^{3 \times 3})\) and \(L^2(\Omega; \text{SO}(3))\), respectively. Moreover, we use the standard Sobolev spaces \(H^1(\Omega; \mathbb{R})\) and \(H^{2, \infty}(\Omega; \mathbb{R})\) \([1, 34, 36]\) of functions \(u\). For vector fields \(u = (u_1, u_2, u_3)^T\) with \(u_i \in H^1(\Omega), i = 1, 2, 3\), we define \(\nabla u := (\nabla u_1 \mid \nabla u_2 \mid \nabla u_3)^T\). The corresponding Sobolev space will be denoted by \(H^1(\Omega; \mathbb{R}^3)\). A tensor \(Q : \Omega \rightarrow \text{SO}(3)\) having the components in \(H^1(\Omega; \mathbb{R})\) belongs to \(H^1(\Omega; \text{SO}(3))\). In writing the norm in the corresponding Sobolev space, we will specify the space. The space will be omitted only when the Frobenius norm or scalar product is considered.

In the formulation of the minimization problem, for a \(C^2(\omega)\) function \(y_0 : \omega \rightarrow \mathbb{R}^3\) such that \(\|\partial_x y_0 \times \partial_x y_0\| \neq 0\) we consider the Weingarten map (or shape operator) defined by \(L_{y_0} = \Gamma_{y_0}^{-1} \Pi_{y_0} \in \mathbb{R}^{2 \times 2}\), where \(\Pi_{y_0} := \nabla y_0^T \nabla y_0 \in \mathbb{R}^{2 \times 2}\) and \(\Gamma_{y_0} := -\nabla y_0^T \nabla n_0 \in \mathbb{R}^{2 \times 2}\) are the matrix representations of the first fundamental form (metric) and the second fundamental form of the surface, respectively, and \(n_0 = \frac{\partial_x y_0 \times \partial_x y_0}{\|\partial_x y_0 \times \partial_x y_0\|}\) is the unit normal to the curved reference surface. Then, the Gauß curvature \(K\) of the surface is determined by \(K := \det(L_{y_0})\) and the mean curvature \(H\) through \(2H := \text{tr}(L_{y_0})\). We also use the tensors defined by

\[
A_{y_0} := (\nabla y_0^T | \nabla y_0)^{-1} \in \mathbb{R}^{3 \times 3}, \quad B_{y_0} := -(\nabla n_0^T | \nabla n_0)^{-1} \in \mathbb{R}^{3 \times 3},
\]

and the so-called alternator tensor \(C_{y_0}\) of the surface \([45]\)

\[
C_{y_0} := \det \nabla \Theta | \nabla \Theta)^{-T} \left( \begin{array}{ccc} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) | \nabla \Theta)^{-1}.
\]

We define the lifted quantities \(\tilde{y}_0 \in \mathbb{R}^{3 \times 3}\) by \(\tilde{y}_0 = (\nabla y_0^T | n_0) \nabla y_0 | n_0) = \tilde{I}_{y_0} + \tilde{\Theta}_3\) and \(\tilde{\Pi}_{y_0} \in \mathbb{R}^{3 \times 3}\) by \(\tilde{\Pi}_{y_0} := (\nabla y_0 | n_0)^T (\nabla n_0^T | n_0) = \tilde{\Pi}_{y_0} - \tilde{\Theta}_3\), where for a given matrix \(M \in \mathbb{R}^{2 \times 2}\) we employ the notations

\[
M^p = \left( \begin{array}{cc} M_{11} & M_{12} \\ M_{21} & M_{22} \end{array} \right) \in \mathbb{R}^{3 \times 3} \quad \text{and} \quad \tilde{M} = \left( \begin{array}{cc} M_{11} & M_{12} \\ M_{21} & M_{22} \\ 0 & 0 \end{array} \right) \in \mathbb{R}^{3 \times 3}.
\]

Corresponding quantities are definite for any \(C^2\)-functions \(m : \omega \rightarrow \mathbb{R}^3\) such that \(\|\partial_x m \times \partial_x m\| \neq 0\).

2.2. Shell geometry and kinematics

Let \(\Omega_\xi \subset \mathbb{R}^3\) be a three-dimensional curved shell-like thin domain. Here, the domains \(\Omega_\xi\) and \(\Omega_\xi\) are referred to a fixed right Cartesian coordinate frame with unit vectors \(e_i\) along the axes \(Ox_i\). A generic point of \(\Omega_\xi\) will be denoted by \((\xi_1, \xi_2, \xi_3)\). The elastic material constituting the shell is assumed to be homogeneous and isotropic and the reference configuration \(\Omega_\xi\) is assumed to be a natural state. The deformation of the body occupying the domain \(\Omega_\xi\) is described by a vector map \(\varphi_\xi : \Omega_\xi \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3\) (called deformation) and by a microrotation tensor \(\overline{R}_\xi : \Omega_\xi \subset \mathbb{R}^3 \rightarrow \text{SO}(3)\) attached at each point. We denote the current configuration (deformed configuration) by \(\Omega_\eta := \varphi_\xi(\Omega_\xi) \subset \mathbb{R}^3\). We consider a fictitious
Cartesian (planar) configuration of the body \( \Omega_h \). This parameter domain \( \Omega_h \subset \mathbb{R}^3 \) is a right cylinder of the form

\[ \Omega_h = \left\{ (x_1, x_2, x_3) \mid (x_1, x_2) \in \omega, \; -\frac{h}{2} < x_3 < \frac{h}{2} \right\} = \omega \times \left( -\frac{h}{2}, \frac{h}{2} \right), \tag{2.3} \]

where \( \omega \subset \mathbb{R}^2 \) is a bounded domain with Lipschitz boundary \( \partial \omega \) and the constant length \( h > 0 \) is the thickness of the shell. For shell–like bodies, we consider the domain \( \Omega_h \) to be thin, i.e., the thickness \( h \) is small.

The diffeomorphism \( \Theta : \mathbb{R}^3 \to \mathbb{R}^3 \) describing the reference configuration (the curved surface of the shell) will be chosen in the specific form

\[ \Theta(x_1, x_2, x_3) = y_0(x_1, x_2) + x_3 n_0(x_1, x_2), \quad n_0 = \frac{\partial x_1 y_0 \times \partial x_2 y_0}{\| \partial x_1 y_0 \times \partial x_2 y_0 \|}, \tag{2.4} \]

where \( y_0 : \omega \to \mathbb{R}^3 \) is a function of class \( C^2(\omega) \). If not otherwise indicated, by \( \nabla \Theta \) we denote \( \nabla \Theta(x_1, x_2, 0) \).

Now, let us define the map \( \varphi : \Omega_h \to \Omega_c, \; \varphi(x_1, x_2, x_3) = \varphi_\xi(\Theta(x_1, x_2, x_3)) \). We view \( \varphi \) as a function which maps the fictitious planar reference configuration \( \Omega_h \) into the deformed configuration \( \Omega_c \) and it is the only one unknown of the model, contrary to the unconstrained Cosserat-shell model where the microrotation tensor field \( \mathcal{Q}_c : \Omega_h \to \text{SO}(3) \), is \( \mathcal{Q}_c(x_1, x_2, x_3) = \mathcal{R}_c(\Theta(x_1, x_2, x_3)) \) is an independent unknown, too.

In the constrained Cosserat-shell model, the reconstructed total deformation \( \varphi_s : \Omega_h \subset \mathbb{R}^3 \to \mathbb{R}^3 \) of the shell-like body is assumed to be

\[ \varphi_s(x_1, x_2, x_3) = m(x_1, x_2) + \left( x_3 b_m(x_1, x_2) + \frac{x_3^2}{2} b_b(x_1, x_2) \right) \mathcal{Q}_\infty(x_1, x_2) \nabla \Theta e_3, \tag{2.5} \]

where \( m : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3 \) represents the total deformation of the midsurface, \( \mathcal{Q}_\infty : \Omega_h \to \text{SO}(3) \) is the constrained rotation field and which depends on the deformation of the midsurface by extracting the appropriate continuum rotation of the midsurface through the polar decomposition \([40]\) in the sense that

\[ \mathcal{Q}_\infty = \text{polar}(\langle \nabla m | n \rangle \langle \nabla \Theta \rangle^{-1}) = (\nabla m | n) \langle \nabla \Theta \rangle^{-1} \sqrt{\langle \nabla \Theta \rangle \hat{\imath}_m^{-1} \langle \nabla \Theta \rangle^T}, \tag{2.6} \]

with the lifted quantity \( \hat{\imath}_m \in \mathbb{R}^{3 \times 3} \) given by \( \hat{\imath}_m := (\nabla m | n)^T (\nabla m | n), \; n = \frac{\partial x_1 m \times \partial x_2 m}{\| \partial x_1 m \times \partial x_2 m \|} \), and \( b_m, b_b : \omega \subset \mathbb{R}^2 \to \mathbb{R} \) are introduced to allow in principal for symmetric thickness stretch (\( b_m \neq 1 \)) and asymmetric thickness stretch (\( b_b \neq 0 \)) about the midsurface and they are given by

\[ b_m = 1 - \frac{\lambda}{\lambda + 2\mu} \langle \mathcal{Q}_\infty^T (\nabla m | 0) \langle \nabla \Theta \rangle^{-1} e_3, 1_3 \rangle - 2, \]

\[ b_b = -\frac{\lambda}{\lambda + 2\mu} \langle \mathcal{Q}_\infty^T (\nabla \mathcal{Q}_\infty \nabla \Theta e_3 | 0) \langle \nabla \Theta \rangle^{-1}, 1_3 \rangle \]

\[ + \frac{\lambda}{\lambda + 2\mu} \langle \mathcal{Q}_\infty^T (\nabla m | 0) \langle \nabla \Theta \rangle^{-1}(\nabla n_0 | 0) \langle \nabla \Theta \rangle^{-1} e_3, 1_3 \rangle. \tag{2.7} \]

In this fashion, we obtain a fully two-dimensional minimization problem in which the energy density is expressed in terms of the following tensor fields (the same strain measures are also considered in \([14,15,20,29,37]\) but with other motivations of their importance) on the surface \( \omega \):

- the symmetric elastic shell strain tensor \( \mathcal{E}_\infty \in \text{Sym}(3) \):

\[ \mathcal{E}_\infty := \mathcal{Q}_\infty^T (\nabla m | \mathcal{Q}_\infty \nabla \Theta e_3) \langle \nabla \Theta \rangle^{-1} - 1_3 = [\text{polar}(\langle \nabla m | n \rangle \langle \nabla \Theta \rangle^{-1})]^T (\nabla m | n) \langle \nabla \Theta \rangle^{-1} - 1_3 \]

\[ = \sqrt{\langle \nabla \Theta \rangle^{-T} \hat{\imath}_m 1_3^T \hat{\imath}_m \langle \nabla \Theta \rangle^{-1} - \sqrt{\langle \nabla \Theta \rangle^{-T} \hat{\imath}_y 0_2 \langle \nabla \Theta \rangle^{-1}}; \tag{2.8} \]
the non-symmetric elastic shell bending–curvature tensor $\mathcal{K}_\infty \notin \text{Sym}(3)$:

\[
\mathcal{K}_\infty := \left( \text{axl}(Q^T_{\infty} \partial_{x_1} Q_{\infty}) \right) \text{axl}(Q^T_{\infty} \partial_{x_2} Q_{\infty}) [\nabla \Theta]^{-1} \\
= \left( \text{axl}(\sqrt{[\nabla \Theta] \hat{T}_m [\nabla \Theta]^{-T} [\nabla \Theta]^{-1} \sqrt{[\nabla \Theta]} \hat{T}_m^{-1} [\nabla \Theta]^T}) \right) \\
\text{axl}(\sqrt{[\nabla \Theta] \hat{T}_m^{-T} [\nabla \Theta]^{-T} [\nabla \Theta]^{-1} \sqrt{[\nabla \Theta]} \hat{T}_m^{-1} [\nabla \Theta]^T}) [0] [\nabla \Theta]^{-1}.
\]

(2.9)

Other consequences, besides the continuum rotation constraint $Q_\infty = \text{pol}((\nabla m|n)[\nabla \Theta]^{-1})$ of the constrained Cosserat-shell model are

\[
\mathcal{E}_\infty B_{y_0} + C_{y_0} \mathcal{K}_\infty \in \text{Sym}(3) \Leftrightarrow \sqrt{[\nabla \Theta]^{-T} \hat{T}_m [\nabla \Theta]^{-1} [\nabla \Theta] (L^b_{y_0} \hat{L}_m^b) [\nabla \Theta]^{-1} \\
\overset{!}{=} [\nabla \Theta]^{-T} \left( (L^b_{y_0})^T - (L^b_m)^T \right) [\nabla \Theta]^T \sqrt{[\nabla \Theta]^{-T} \hat{T}_m [\nabla \Theta]^{-1}} - (2.10)
\]

and

\[
(\mathcal{E}_\infty B_{y_0} + C_{y_0} \mathcal{K}_\infty) B_{y_0} \in \text{Sym}(3) \Leftrightarrow \sqrt{[\nabla \Theta]^{-T} \hat{T}_m [\nabla \Theta]^{-1} [\nabla \Theta] (L^b_{y_0} - L^b_m) L^b_{y_0} [\nabla \Theta]^{-1} \\
\overset{!}{=} [\nabla \Theta]^{-T} (L^b_{y_0})^T \left( (L^b_m)^T - (L^b_m)^T \right) [\nabla \Theta]^T \sqrt{[\nabla \Theta]^{-T} \hat{T}_m [\nabla \Theta]^{-1}}. - (2.11)
\]

These restrictions remain additional constraints in the minimization problem and they assure\(^1\) that the (through the thickness reconstructed) strain tensor

\[
\tilde{\mathcal{E}}_s = 1 \left[ \mathcal{E}_\infty - \frac{\lambda}{\lambda + 2\mu} \text{tr}(\mathcal{E}_\infty) \left( 0 | 0 | n_0 \right) (0 | 0 | n_0)^T \right] \\
+ x_3 \left[ (\mathcal{E}_\infty B_{y_0} + C_{y_0} \mathcal{K}_\infty) - \frac{\lambda}{\lambda + 2\mu} \text{tr}(\mathcal{E}_\infty B_{y_0} + C_{y_0} \mathcal{K}_\infty) \left( 0 | 0 | n_0 \right) (0 | 0 | n_0)^T \right] \\
+ x_3 \left[ (\mathcal{E}_\infty B_{y_0} + C_{y_0} \mathcal{K}_\infty) B_{y_0} \right] + O(x_3^3) - (2.12)
\]

remains symmetric.

### 2.3. Variational problem for the constrained nonlinear Cosserat-shell model

The variational problem for the constrained Cosserat $O(h^5)$-shell model (see [33]) is to find a deformation of the midsurface $m : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ minimizing on $\omega$ the functional

\[
I = \int_{\omega} \left[ \left( h + \frac{h^3}{12} \right) W^\infty_{\text{shell}} (\sqrt{[\nabla \Theta]^{-T} \hat{T}_m 1^b_2 [\nabla \Theta]^{-1} - \sqrt{[\nabla \Theta]^{-T} \hat{T}_{y_0} 1^b_2 [\nabla \Theta]^{-1}}) \\
+ \left( \frac{h^3}{12} - \frac{h^5}{80} \right) W^\infty_{\text{shell}} \sqrt{[\nabla \Theta]^{-T} \hat{T}_m [\nabla \Theta]^{-1} [\nabla \Theta] (L^b_{y_0} - L^b_m) [\nabla \Theta]^{-1} \hat{T}_m^{-1} [\nabla \Theta]^{-1}} \\
- \frac{h^3}{3} H W^\infty_{\text{shell}} (\sqrt{[\nabla \Theta]^{-T} \hat{T}_m 1^b_2 [\nabla \Theta]^{-1} - \sqrt{[\nabla \Theta]^{-T} \hat{T}_{y_0} 1^b_2 [\nabla \Theta]^{-1}}) \\
\sqrt{[\nabla \Theta]^{-T} \hat{T}_m [\nabla \Theta]^{-1} [\nabla \Theta] (L^b_{y_0} - L^b_m) [\nabla \Theta]^{-1}} \right)
\]

\(^1\) The terms of order $O(x_3^3)$ are not relevant here, since we have taken a quadratic ansatz for the deformation.
such that

\[
\varepsilon_\infty B_{\text{y}_0} + C_{y_0} K_\infty \in \text{Sym}(3)
\]

\[
\Leftrightarrow \sqrt{|\nabla \Theta| - T \hat{I}_m |\nabla \Theta|^{-1} |\nabla \Theta| \left( L_{\text{y}_0}^b - L_m^b \right) |\nabla \Theta|^{-1}}
\]

\[
\varepsilon_\infty B_{\text{y}_0} + C_{y_0} K_\infty \in \text{Sym}(3)
\]

\[
\Leftrightarrow \sqrt{|\nabla \Theta| - T \hat{I}_m |\nabla \Theta|^{-1} |\nabla \Theta| \left( L_{\text{y}_0}^b - L_m^b \right) L_{\text{y}_0}^b |\nabla \Theta|^{-1}}
\]

\[
(2.13)
\]

where

\[
K_\infty = \left( \text{axl}(Q^T \partial_{x_1} Q^T) | \text{axl}(Q^T \partial_{x_2} Q^T) | 0 \right) |\nabla \Theta|^{-1},
\]

\[
Q_\infty = \text{polar}( (\nabla m |n |\nabla \Theta|^{-1}) = (\nabla m |n |\nabla \Theta|^{-1} \sqrt{|\nabla \Theta| \hat{I}_m |\nabla \Theta|},
\]

\[
W^\infty_{\text{shell}}(S) = \mu \| S \|^2 + \frac{\lambda \mu}{\lambda + 2 \mu} \left( \text{tr}(S) \right)^2, \quad W^\infty_{\text{shell}}(S, T) = \mu \langle S, T \rangle + \frac{\lambda \mu}{\lambda + 2 \mu} \text{tr}(S) \text{tr}(T),
\]

\[
(2.14)
\]

\[
W^\infty_{\text{mp}}(S) = \mu \| S \|^2 + \frac{\lambda}{2} \left( \text{tr}(S) \right)^2 \quad \forall S, T \in \text{Sym}(3),
\]

\[
W_{\text{curv}}(X) = \mu L^2_c \left( b_1 \| \text{dev sym} X \|^2 + b_2 \| \text{skew} X \|^2 + b_3 \| \text{tr}(X) \|^2 \right) \quad \forall X \in \mathbb{R}^{3 \times 3}.
\]

The parameters \( \mu \) and \( \lambda \) are the usual Lamé constants of classical isotropic elasticity, \( \kappa = \frac{2 \mu + 3 \lambda}{3} \) is the infinitesimal (elastic) bulk modulus, \( b_1, b_2, b_3 \) are non-dimensional constitutive curvature coefficients (weights) and \( L_c > 0 \) introduces an internal length which is characteristic for the material, e.g., related to the grain size in a polycrystal. The internal length \( L_c > 0 \) is responsible for size effects in the sense that smaller samples are relatively stiffer than larger samples. If not stated otherwise, we assume that \( \mu > 0, \kappa > 0, b_1 > 0, b_2 > 0, b_3 > 0 \). All the constitutive coefficients are coming from the three-dimensional Cosserat formulation (in the constrained model from the three-dimensional Toupin couple stress model, too), without using any a posteriori fitting of some two-dimensional constitutive coefficients.

The potential of applied external loads \( \Pi(m, Q_\infty) \) appearing in the variational problem is expressed by

\[
\Pi(m, Q_\infty) = \Pi_\omega(m, Q_\infty) + \Pi_{\gamma_t}(m, Q_\infty), \quad \text{with}
\]

\[
\Pi_\omega(m, Q_\infty) = \int_\omega \langle f, u \rangle \, da + \Lambda_\omega(Q_\infty) \quad \text{and} \quad \Pi_{\gamma_t}(m, Q_\infty) = \int_{\gamma_t} \langle t, u \rangle \, ds + \Lambda_{\gamma_t}(Q_\infty), \quad (2.16)
\]
where \( u(x_1, x_2) = m(x_1, x_2) - y_0(x_1, x_2) \) is the displacement vector of the midsurface, \( \Pi_\omega(m, Q_\infty) \) is the potential of the external surface loads \( f \), while \( \Pi_\gamma(m, Q_\infty) \) is the potential of the external boundary loads \( t \). The functions \( \Lambda_\omega, \Lambda_\gamma : L^2(\omega, SO(3)) \rightarrow \mathbb{R} \) are expressed in terms of the loads from the three-dimensional parental variational problem, see [30], and they are assumed to be continuous and bounded operators. Here, \( \gamma_t \) and \( \gamma_d \) are non-empty subsets of the boundary of \( \omega \) such that \( \gamma_t \cup \gamma_d = \partial \omega \) and \( \gamma_t \cap \gamma_d = \emptyset \). On \( \gamma_t \) we have considered traction boundary conditions, while on \( \gamma_d \) we have the Dirichlet-type boundary conditions:

\[
\left. m \right|_{\gamma_d} = m^*, \quad Q_\infty Q_0 e_3 \big|_{\gamma_d} = \frac{\partial x_1 m^* \times \partial x_2 m^*}{||\partial x_1 m^* \times \partial x_2 m^*||},
\]

(2.17)

where the boundary conditions are to be understood in the sense of traces.

Since we would like to obtain a compatible form of our linear model with the classical linear Koiter model presented in Sect. 3, before performing the linearisation, let us remark that we can express the nonlinear strain tensors in the form

\[
\begin{align*}
\mathcal{E}_\infty &= \left[ \nabla \Theta \right]^{-T} \left[ (\widehat{Q}_\infty \nabla y_0)^T \nabla m - I_{y_0} \right]^b \left[ \nabla \Theta \right]^{-1} = \left[ \nabla \Theta \right]^{-T} \mathcal{G}_\infty^b \left[ \nabla \Theta \right]^{-1}, \\
C_{y_0} \mathcal{K}_\infty &= \left[ \nabla \Theta \right]^{-T} \left[ (\widehat{Q}_\infty \nabla y_0)^T \nabla (\widehat{Q}_\infty n_0) + \Pi_{y_0} \right]^b \left[ \nabla \Theta \right]^{-1} = -\left[ \nabla \Theta \right]^{-T} \mathcal{R}_\infty^b \left[ \nabla \Theta \right]^{-1}, \\
C_{y_0} \mathcal{K}_\infty B_{y_0} &= - \left[ \nabla \Theta \right]^{-T} \left[ \mathcal{R}_\infty L_{y_0} \right]^b \left[ \nabla \Theta \right]^{-1}, \\
C_{y_0} \mathcal{K}_{e,s} B_{y_0}^2 &= - \left[ \nabla \Theta \right]^{-T} \left[ \mathcal{R}_\infty L_{y_0}^2 \right]^b \left[ \nabla \Theta \right]^{-1}, \\
\mathcal{E}_\infty B_{y_0} + C_{y_0} \mathcal{K}_{e,s} B_{y_0} &= - \left[ \nabla \Theta \right]^{-T} \left[ \mathcal{R}_\infty - \mathcal{G}_\infty L_{y_0} \right]^b \left[ \nabla \Theta \right]^{-1}, \\
\mathcal{E}_\infty^2 B_{y_0} + C_{y_0} \mathcal{K}_\infty B_{y_0} &= - \left[ \nabla \Theta \right]^{-T} \left[ \mathcal{R}_\infty - \mathcal{G}_\infty L_{y_0} \right] L_{y_0}^b \left[ \nabla \Theta \right]^{-1},
\end{align*}
\]

(2.18)

where

\[
\begin{align*}
\mathcal{G}_\infty := (\widehat{Q}_\infty \nabla y_0)^T \nabla m - I_{y_0} \in \text{Sym}(2) & \quad \text{the change of metric tensor,} \\
\mathcal{R}_\infty := -(\widehat{Q}_\infty \nabla y_0)^T \nabla (\widehat{Q}_\infty n_0) - \Pi_{y_0} \notin \text{Sym}(2) & \quad \text{the bending strain tensor.}
\end{align*}
\]

(2.19)

The non-symmetric quantity \( \mathcal{R}_\infty - \mathcal{G}_\infty L_{y_0} \) represents the change of curvature tensor. The choice of this name will be justified in a forthcoming paper [32]. For now, we just mention that the definition of \( \mathcal{G}_\infty \) is related to the classical change of metric tensor in the Koiter model

\[
\mathcal{G}_{\text{Koiter}} := \frac{1}{2} \left[ (\nabla m)^T \nabla m - I_{y_0} \right] = \frac{1}{2} \left( I_m - I_{y_0} \right) \in \text{Sym}(2),
\]

(2.20)

while the bending strain tensor \( \mathcal{R}_\infty \) may be compared with the classical bending strain tensor in the Koiter model

\[
\mathcal{R}_{\text{Koiter}} := -(\nabla m)^T \nabla n - \Pi_{y_0} = \Pi_m - \Pi_{y_0} \in \text{Sym}(2).
\]

(2.21)

We also note that the constrained Cosserat-shell model is not able to reflect the effect of the transverse shear vector \( \mathcal{T}_\infty := (\widehat{Q}_\infty n_0)^T (\nabla m) \), since it vanishes as is intended.

In the constrained nonlinear Cosserat-shell model, the tensor \( \mathcal{G}_\infty \) is symmetric (since \( \mathcal{E}_\infty \) is symmetric). The same will be true in the linear model, too.

In our model, the total energy is not simply the sum of energies coupling the membrane and the change of curvature effects, respectively. Two further coupling energies are still present, see the lines 3–6 from (2.13) and (2.18), and they result directly from the dimensional reduction of the variational problem from geometrically nonlinear three-dimensional Cosserat elasticity. Beside this, the bending and the drilling effects are present in the model, see the 8th line from (2.13) and (2.18), due to the parental three-dimensional curvature energy, and not derived to the parental three-dimensional quadratic energy depending on the Biot-strain tensor.
Our model is constructed in [30] under the following assumptions upon the thickness
\[
h \max \left\{ \sup_{(x_1, x_2) \in \omega} |\kappa_1|, \sup_{(x_1, x_2) \in \omega} |\kappa_2| \right\} < 2 \tag{2.22}
\]
where \( \kappa_1 \) and \( \kappa_2 \) denote the principal curvatures of the surface. The hypothesis (2.22) which guarantees the local injectivity of the parametrization of the shell represents the key point of the works [5–7] on strongly curved shells.

The geometrically nonlinear model admits global minimizers for materials with the Poisson ratio \( \nu = \frac{\lambda}{2(\lambda+\mu)} \) and Young’s modulus \( E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} \) such that \(-\frac{1}{2} < \nu < \frac{1}{2} \) and \( E > 0 \) [31]. Under these assumptions on the constitutive coefficients, together with the positivity of \( b_1, b_2 \) and \( b_3 \), and the orthogonal Cartan-decomposition of the Lie-algebra \( \mathfrak{gl}(3) \) it follows that there exists positive constants \( c_1^+, c_2^+, C_1^+ \) and \( C_2^+ \) such that for all \( X \in \mathbb{R}^{3\times 3} \) the following inequalities hold
\[
\begin{align*}
C_1^+ \| S \|^2 &\geq W_{\text{shell}}^\infty(S) \geq c_1^+ \| S \|^2 \quad \forall S \in \text{Sym}(3), \\
C_2^+ \| X \|^2 &\geq W_{\text{curv}}(X) \geq c_2^+ \| X \|^2 \quad \forall X \in \mathbb{R}^{3\times 3}.
\end{align*}
\tag{2.23}
\]

Step by step, in the next sections we will see that our models generalize a whole family of models. In the end, the linearization of the deformation measures of our constrained Cosserat-shell model are those preferred in the later works by Sanders and Budiansky [18,19] and by Koiter and Simmonds [35], who called the obtained theory the “best first-order linear elastic shell theory” but our change of curvature measure will be that proposed by Anicic and Léger [5–7].

3. The classical linear (first) Koiter model and the corresponding existence results

According to [22, page 344], [24, page 154] in the linear (first) Koiter model, the variational problem is to find a midsurface displacement vector field \( v : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3 \) minimizing
\[
\int_\omega \left\{ h \left( \mu \| \nabla \Theta \|^2 + \frac{\lambda}{\lambda + 2\mu} \text{tr} \left( [\nabla \Theta]^{-1} [\mathcal{G}^{\text{lin}}_{\text{Koiter}}] [\nabla \Theta]^{-1} \right)^2 \right) \right\} \det(\nabla y_0|n_0) \, da,
\tag{3.1}
\]
where the strain measures [22] are given by
\[
\mathcal{G}^{\text{lin}}_{\text{Koiter}} := \frac{1}{2} \left[ I_m - I_{y_0} \right] \quad \text{lin} = \frac{1}{2} \left[ (\nabla y_0)^T v + (v)^T (\nabla y_0) \right] = \text{sym}((\nabla y_0)^T v) \in \text{Sym}(2) \tag{3.2}
\]
and [16]
\[
\mathcal{R}^{\text{lin}}_{\text{Koiter}} := \left[ \Pi_m - \Pi_{y_0} \right] \quad \text{lin} = \left( (n_0, \partial_{x\alpha} y_\alpha v - \sum_{\gamma=1,2} \Gamma^\gamma_{\alpha\beta} \partial_{x\gamma} v) \right)_{\alpha\beta} \in \text{Sym}(2). \tag{3.3}
\]
Here, and in the rest of the paper, \( a_1, a_2, a_3 \) denote the columns of \( \nabla \Theta \), while \( a^1, a^2, a^3 \) denote the rows of \( [\nabla \Theta]^{-1} \), i.e.,
\[
\nabla \Theta = (\nabla y_0|n_0) = (a_1| a_2| a_3), \quad [\nabla \Theta]^{-1} = (a^1| a^2| a^3)^T. \tag{3.4}
\]
In fact, \( a_1, a_2 \) are the covariant base vectors and \( a^1, a^2 \) are the contravariant base vectors in the tangent plane given by
\[
a_\alpha := \partial_{x\alpha} y_0, \quad \langle a^\beta, a_\alpha \rangle = \delta^\beta_\alpha, \quad \alpha, \beta = 1, 2, \quad \text{and} \quad a_3 = a^3 = n_0. \tag{3.5}
\]
\[2\text{Observe that } \det(\nabla y_0|n_0) = \sqrt{\det([\nabla y_0]^T v \nabla y_0)} \text{ is independent of } n_0.\]
The following relations hold [22, page 95]:

\[\|a_1 \times a_2\| = \sqrt{\det I_{y_0}}, \quad a_3 \times a_1 = \sqrt{\det I_{y_0}} a_2, \quad a_2 \times a_3 = \sqrt{\det I_{y_0}} a_1. \tag{3.6}\]

The expression of \( R_{\text{Koiter}}^{\text{lin}} \) involves \( \Gamma_{\alpha\beta}^{\gamma} \) on the surface given by \( \Gamma_{\alpha\beta}^{\gamma} = \langle a_{\gamma}, \partial_{x_{\alpha}} a_{\beta} \rangle = -\langle \partial_{x_{\alpha}} a_{\gamma}, a_{\beta} \rangle = \Gamma_{\beta\alpha}^{\gamma}. \)

Note that, using the expansion \( m = y_0 + v \) and \( (\nabla m)^T \nabla m = (\nabla y_0)^T \nabla y_0 + (\nabla y_0)^T \nabla v + (\nabla v)^T \nabla y_0 + \text{h.o.t} \in \mathbb{R}^{2 \times 2}, \) the linear approximation of the difference \( \frac{1}{2}[I_m - I_{y_0}]^{\text{lin}} \) appearing in the Koiter model is easy to be obtained [22, page 92], the linear approximation of the difference \( [I_m - I_{y_0}]^{\text{lin}} \) needs some more insights from differential geometry [22, page 95] and it is based on formulas of Gauß \( \partial_{x_{\alpha}} a_{\beta} = \sum_{\gamma=1,2} \Gamma_{\alpha\beta}^{\gamma} a_{\gamma} + b_{\alpha\beta} a_3 \) and \( \partial_{x_{\alpha}} a_{\alpha} = -\sum_{\gamma=1,2} \Gamma_{\alpha\beta}^{\gamma} a_{\gamma} + b_{\alpha\beta} a_0 \) and the formulas of Weingarten \( \partial_{x_{\alpha}} a_3 = \partial_{x_{\alpha}} a_3 = -\sum_{\beta=1,2} b_{\alpha\beta} a_{\beta} = -\sum_{\beta=1,2} b_{\alpha\beta} a_{\gamma} \) together with the relations [22, page 76] \( b_{\alpha\beta}(m) = -\langle \partial_{\alpha} a_{\beta}(m), a_{\beta}(m) \rangle = \langle \partial_{\alpha} a_{\beta}(m), a_{\beta}(m) \rangle = b_{\alpha\beta}(m) \), where \( b_{\alpha\beta}(m) \) are the components of the second fundamental form corresponding to the map \( m, b_{\alpha\beta}(m) \) are the components of the matrix associated to the Weingarten map (shape operator), and on the following sets of linear approximations of the normal to the surface

\[ n = \frac{\partial_{x_1} m \times \partial_{x_2} m}{\sqrt{\det((\nabla m)^T \nabla m)}} = \frac{1}{\sqrt{\det((\nabla m)^T \nabla m)}} (\partial_{x_1} y_0 \times \partial_{x_2} y_0 + \partial_{x_1} y_0 \times \partial_{x_2} v + \partial_{x_1} v \times \partial_{x_2} y_0 + \text{h.o.t}), \]

\[
\det((\nabla m)^T \nabla m) = \det((\nabla y_0)^T \nabla y_0) (1 + \text{tr}(((\nabla y_0)^T \nabla y_0)^{-1} \text{sym}((\nabla y_0)^T \nabla v)) + \text{h.o.t})
\]
\[
= \det((\nabla y_0)^T \nabla y_0) (1 + 2 \sum_{\alpha=1,2} \langle \partial_{x_{\alpha}} v, a_{\alpha} \rangle + \text{h.o.t})
\]

\[ n = \frac{\partial_{x_1} m \times \partial_{x_2} m}{\sqrt{\det((\nabla m)^T \nabla m)}} = n_0 + \frac{1}{\sqrt{\det((\nabla y_0)^T \nabla y_0)}} (\partial_{x_1} y_0 \times \partial_{x_2} v + \partial_{x_1} v \times \partial_{x_2} y_0 + \text{h.o.t})
\]
\[
- \text{tr}(((\nabla y_0)^T \nabla y_0)^{-1} \text{sym}((\nabla y_0)^T \nabla v)) n_0.
\tag{3.7}
\]

Here, “h.o.t” stands for terms of order higher than linear with respect to \( v \).

We also note that other alternative (but equivalent) forms of the change of metric tensor and the change of curvature tensor given in [22, page 181] are

\[ \varphi_{\text{Koiter}}^{\text{lin}} = \left( \frac{1}{2} (\partial_{\beta} v_{\alpha} + \partial_{\alpha} v_{\beta}) - \sum_{\gamma=1,2} \Gamma_{\alpha\beta}^{\gamma} v_{\gamma} - b_{\alpha\beta} v_3 \right)_{\alpha\beta} \in \text{Sym}(2), \tag{3.8} \]

and

\[ R_{\text{Koiter}}^{\text{lin}} = \left( \partial_{x_{\alpha} x_{\beta}} v_3 - \sum_{\gamma=1,2} \Gamma_{\alpha\beta}^{\gamma} \partial_{x_{\gamma}} v_3 - \sum_{\gamma=1,2} b_{\alpha\beta} b_{\gamma\beta} v_3 + \sum_{\gamma=1,2} b_{\alpha\beta} \partial_{x_{\beta}} v_{\gamma} - \sum_{\tau=1,2} \Gamma_{\beta\gamma}^{\tau} v_{\tau} \right.
\]
\[ + \sum_{\gamma=1,2} b_{\beta}^\gamma (\partial_{x_{\alpha}} v_{\gamma} - \sum_{\tau=1,2} \Gamma_{\alpha\tau}^{\beta} v_{\gamma}) + \sum_{\tau=1,2} (\partial_{x_{\alpha}} b_{\beta}^\gamma + \sum_{\gamma=1,2} \Gamma_{\alpha\gamma}^{\tau} b_{\beta}^\gamma - \sum_{\gamma=1,2} \Gamma_{\alpha\beta} b_{\gamma}^\tau \partial_{x_{\gamma}} v_{\tau}) \right)_{\alpha\beta} \in \text{Sym}(2), \tag{3.9} \]

respectively. Actually, on the one hand, the last form (3.9) of the curvature tensor will be considered when the admissible set of solutions of the variational problem will be defined. On the other hand, as noticed in [23, page 175] by considering the form (3.3) of the change of metric tensor, we can impose substantially weaker regularity assumptions on the mapping \( y_0 \).

Regarding the existence of the solution, the following results are known, see [17] and [22].
Theorem 3.1. (Existence results for the linear Koiter model on a general surface) [22, Theorem 7.1.-1] Let there be given a domain \( \omega \subset \mathbb{R}^2 \) and an injective mapping \( y_0 \in C^3(\omega, \mathbb{R}^3) \) such that the two vectors \( \alpha = \partial_{x \alpha} y_0, \alpha = 1, 2, \) are linear independent at all points of \( \overline{\omega} \). Then, the variational problem (3.1) has one and only one solution in the admissible set of solutions \[ \mathcal{A}_1 := \{ v = (v_1, v_2, v_3) \in H^1(\omega, \mathbb{R}) \times H^1(\omega, \mathbb{R}) \times H^2(\omega, \mathbb{R}) \mid v_1 = v_2 = v_3 = \langle \nabla v_3, \nu \rangle = 0 \quad \text{on} \quad \partial \omega, \] (3.10) where \( \nu \) is the outer unit normal to \( \partial \omega \).

Theorem 3.2. (Existence results for the linear Koiter model for shells whose middle surfaces have little regularity) [17, 22, Theorem 7.1.-2] Let there be given a domain \( \omega \subset \mathbb{R}^2 \) and an injective mapping \( y_0 \in H^{2,\infty}(\omega, \mathbb{R}^3) \) such that the two vectors \( \alpha = \partial_{x \alpha} y_0, \alpha = 1, 2, \) are linear independent at all points of \( \overline{\omega} \). Then, the variational problem (3.1) has one and only one solution in the admissible set of solutions \[ \mathcal{A}_2 := \{ v \in H^1_0(\omega, \mathbb{R}^3) \mid \langle \partial_{x \alpha} x \beta v, n_0 \rangle \in L^2(\omega) \} \] (3.11)

The proof of Theorem 3.1 is based on the next inequality of Korn’s type on a general surface.

Theorem 3.3. (Korn-type inequality on a general surface) [22, Theorem 2.6.-4] Let there be given a domain \( \omega \subset \mathbb{R}^2 \) and an injective mapping \( y_0 \in C^3(\omega, \mathbb{R}^3) \) such that the two vectors \( \alpha = \partial_{x \alpha} y_0, \alpha = 1, 2, \) are linear independent at all points of \( \overline{\omega} \). Let the change of metric and change of curvature be given by (3.8) and (3.9), respectively. Then, there exists a constant \( c = c(\omega, \partial \omega, y_0) > 0 \) such that
\[ \|v_1\|_{H^1(\omega; \mathbb{R})}^2 + \|v_2\|_{H^1(\omega; \mathbb{R})}^2 + \|v_3\|_{H^2(\omega; \mathbb{R})}^2 \leq c \left( \|G_{\text{Koiter}}^\text{lin}\|_{L^2(\omega)}^2 + \|R_{\text{Koiter}}^\text{lin}\|_{L^2(\omega)}^2 \right) \quad \forall \ v \in \mathcal{A}_1. \] (3.12)

The existence result given by Theorem 3.2 is based on another Korn-type inequality which is valid for shells whose middle surfaces have little regularity.

Theorem 3.4. (Korn inequality on a general surface for shells whose middle surfaces have little regularity) [17, 22, Theorem 2.6.-6] Let there be given a domain \( \omega \subset \mathbb{R}^2 \) and an injective mapping \( y_0 \in H^{2,\infty}(\omega, \mathbb{R}^3) \) such that the two vectors \( \alpha = \partial_{x \alpha} y_0, \alpha = 1, 2, \) are linear independent at all points of \( \overline{\omega} \). Then, there exists a constant \( c > 0 \) such that
\[ \|v\|_{H^1(\omega; \mathbb{R}^3)}^2 + \sum_{\alpha, \beta = 1, 2} \|\partial_{x \alpha} x \beta v, n_0\|_{L^2(\omega)}^2 \leq c \left( \|G_{\text{Koiter}}^\text{lin}\|_{L^2(\omega)}^2 + \|R_{\text{Koiter}}^\text{lin}\|_{L^2(\omega)}^2 \right) \quad \forall \ v \in \mathcal{A}_2, \] (3.13)

where the change of metric and the change of curvature are given by (3.2) and (3.3), respectively (note that \( G_{\text{Koiter}}^\text{lin} \in L^2(\omega) \) and \( R_{\text{Koiter}}^\text{lin} \in L^2(\omega) \), \( \forall \ v \in \mathcal{A}_2 \)).

4. The linear constrained Cosserat-shell model

4.1. The deformation measures in the linear constrained Cosserat-shell model

We express the total midsurface deformation as
\[ m(x_1, x_2) = y_0(x_1, x_2) + v(x_1, x_2), \] (4.1)

with \( v : \omega \to \mathbb{R}^3 \), the infinitesimal shell-midsurface displacement. For the elastic rotation tensor \( \overline{Q}_{e,s} \in \text{SO}(3) \), there is a skew-symmetric matrix
\[ \overline{A}_\theta := \text{Anti}(\vartheta_1, \vartheta_2, \vartheta_3) := \begin{pmatrix} 0 & -\vartheta_3 & \vartheta_2 \\ \vartheta_3 & 0 & -\vartheta_1 \\ -\vartheta_2 & \vartheta_1 & 0 \end{pmatrix} \in \text{so}(3), \quad \text{Anti} : \mathbb{R}^3 \to \text{so}(3), \] (4.2)

where \( \vartheta = \text{axl}(\overline{A}_\theta) \) denotes the axial vector of \( \overline{A}_\theta \), such that \( \overline{Q}_{e,s} := \exp(\overline{A}_\theta) = \sum_{k=0}^{\infty} \frac{1}{k!} \overline{A}_\theta^k = \mathbb{I}_3 + \overline{A}_\theta + \text{h.o.t.} \).
The tensor field $\mathbf{A}_\vartheta$ is the infinitesimal elastic microrotation. Here, “h.o.t” stands for terms of order higher than linear with respect to $v$ and $\mathbf{A}_\vartheta$.

Our aim now is to express all the deformation measures and the linearised models in terms of $v$ and $\mathbf{A}_\vartheta$, as well as in terms of its axial vector $\vartheta$. The following definitions are used to express these quantities in terms of $\vartheta$. For any column vector $q \in \mathbb{R}^3$ and any matrix $M = (M_1 | M_2 | M_3) \in \mathbb{R}^{3 \times 3}$, we define the cross-product [8]

$$q \times M := (q \times M_1 | q \times M_2 | q \times M_3)$$

(operates on columns) and

$$M^T \times q^T := -(q \times M)^T$$

(operates on rows). \hfill (4.3)

Note that $M$ can also be a $3 \times 2$ matrix, the definition remains the same. We note some properties of these operations: for any column vectors $q_1, q_2 \in \mathbb{R}^3$ and any matrices $M, N \in \mathbb{R}^{3 \times 3}$ (or $\mathbb{R}^{3 \times 2}$) we have

$$(q_1 \times M) q_2 = q_1 \times (M q_2), \quad q_1^T (q_2 \times M) = (q_1 \times q_2)^T M = (q_1^T \times q_2^T) M = -q_2^T (q_1 \times M) \hfill (4.4)$$

and, more general

$$(q_1 \times M) N = q_1 \times (M N), \quad N^T (q_2 \times M) = -(q_2 \times N)^T M = (N^T \times q_2^T) M. \hfill (4.5)$$

With these relations, the infinitesimal microrotation $\mathbf{A}_\vartheta$ can be expressed as

$$\mathbf{A}_\vartheta := \vartheta \times \mathbf{I}_3 = \mathbf{I}_3 \times \vartheta^T \in \mathbb{R}^{3 \times 3}. \hfill (4.6)$$

For $m = y_0 + v$, starting from

$$\mathcal{E}_\infty = \sqrt{[\nabla \Theta]^{-T} \mathbf{P}_m [\nabla \Theta]^{-1} - \sqrt{[\nabla \Theta]^{-T} \mathbf{P}_{y_0} [\nabla \Theta]^{-1}} = \mathcal{E}_\infty^{\text{lin}} + \text{h.o.t.}, \hfill (4.7)$$

we find the expression of the linear approximation of $\mathcal{E}_\infty$, given by

$$[\nabla \Theta]^{-T} \left( G_{\infty}^{\text{lin}} \right)^0 \left[ \nabla \Theta \right]^{-1} = G_{\infty}^{\text{lin}} = \frac{1}{2} \sqrt{[\nabla \Theta]^{-T} \mathbf{P}_{y_0} [\nabla \Theta]^{-1} \left[ \nabla \Theta \right]^{-T} ((\nabla y_0)^T \nabla v + (\nabla v)^T \nabla y_0)^3 \left[ \nabla \Theta \right]^{-1}} \hfill (4.8)$$

where $G_{\infty}^{\text{lin}}$ denotes the linear approximation of $\mathcal{G}_\infty$ and we have used the expansion $\sqrt{\frac{x + \delta x}{2 \delta x}} = \sqrt{x} + O(x^2) \quad \forall \ x \geq 0, \delta x > 0$.

While we have the identity

$$\mathcal{G}_\infty = \mathcal{G}_\text{Koiter} \in \text{Sym}(2) \quad \text{and} \quad G_{\infty}^{\text{lin}} = G_{\text{Koiter}}^{\text{lin}} \in \text{Sym}(2), \hfill (4.9)$$

we are only able to find the relation

$$\mathcal{R}_\infty^{\text{lin}} = \mathcal{R}_\text{Koiter}^{\text{lin}} - G_{\text{Koiter}}^{\text{lin}} L_{y_0} \notin \text{Sym}(2) \hfill (4.10)$$

between $\mathcal{R}_\infty^{\text{lin}}$ and $\mathcal{R}_\text{Koiter}^{\text{lin}}$ and not between $\mathcal{R}_\infty$ and $\mathcal{R}_\text{Koiter}$. This is not surprising, since only symmetric stress tensors are taken into account in the classical linear Koiter model, i.e., the internal strain energy does not depend on the skew-symmetric part of the considered strain measures (since it is work conjugate to the skew-symmetric part of the stress tensor). In addition, the linear Koiter model does not consider extra degrees of freedom.

In order to see how the rotation vector $\vartheta$ depends on $v$ in the linear constrained Cosserat-shell model, let us do something different, and linearise the condition $\mathcal{G}_\infty := (Q_\infty \nabla y_0)^T \nabla m - I_{y_0} \in \text{Sym}(2)$ to obtain

$$(\nabla y_0)^T (1 - \mathbf{A}_\vartheta) (\nabla y_0 + \nabla v) + \text{h.o.t.} = (\nabla y_0 + \nabla v)^T (1 + \mathbf{A}_\vartheta) (\nabla y_0) + \text{h.o.t.} \quad \text{and} \quad (1 + \mathbf{A}_\vartheta) n_0 + \text{h.o.t.} = n, \hfill (4.11)$$
i.e.,
\[(\nabla y_0)^T (\nabla v) - (\nabla y_0)^T \mathcal{A}_{\partial \infty} \nabla y_0 + \text{h.o.t.} = (\nabla v)^T (\nabla y_0) + (\nabla y_0)^T \mathcal{A}_{\partial \infty} (\nabla y_0) + \text{h.o.t.} \quad \text{and} \quad n = n_0 + \partial_{\infty} \times n_0 + \text{h.o.t.} \] (4.12)
or alternatively
\[
\text{skew}[(\nabla y_0)^T (\nabla v)] = (\nabla y_0)^T [\partial_{\infty} \times (\nabla y_0)] + \text{h.o.t.} \quad \text{and} \quad n = n_0 + \partial_{\infty} \times n_0 + \text{h.o.t.} \] (4.13)

Using that $T^{\text{lin}}_\infty + \text{h.o.t.} = T_\infty = 0$, i.e., $(Q_\infty n_0)^T (\nabla m) = 0$, and that the linear approximation of the transverse shear vector $T_\infty = (Q_\infty n_0)^T (\nabla m)$ is given by [8]
\[
T^{\text{lin}}_\infty = n_0^T (\nabla v - A_{\partial \infty} \nabla y_0) \quad \iff \quad T^{\text{lin}}_\infty = n_0^T (\nabla v - \partial_{\infty} \times \nabla y_0)
\]
\[
\iff \quad T^{\text{lin}}_\infty = n_0^T \nabla v + \partial_{\infty}^T (n_0 \times \nabla y_0) = n_0^T \nabla v + (\partial_{\infty} \times n_0)^T \nabla y_0,
\] (4.14)
we find
\[
(\partial_{\infty} \times n_0)^T \nabla y_0 = -n_0^T \nabla v \quad \iff \quad \langle a_\alpha, \partial_{\infty} \times n_0 \rangle = -\langle n_0, \partial_{\infty} v \rangle \quad \iff \quad \partial_{\infty} \times n_0 = -\sum_{\alpha=1,2} \langle n_0, \partial_{\alpha} v \rangle a_\alpha.
\] (4.15)

Hence, we obtain finally
\[
n = n_0 - \sum_{\alpha=1,2} \langle n_0, \partial_{\alpha} v \rangle a_\alpha + \text{h.o.t.}
\] (4.16)

We can also express the $2 \times 2$ skew-symmetric matrix $C = \sqrt{\det I_{y_0}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ using the cross-product. Similar as in [8], we have
\[
(a^1 | a^2)^T C = -n_0 \times \nabla y_0.
\] (4.17)

Thus, we deduce
\[
C = -(\nabla y_0)^T (n_0 \times \nabla y_0) = (n_0 \times \nabla y_0)^T (\nabla y_0),
\] (4.18)
and it holds
\[
\langle \partial_{\infty}, n_0 \rangle C = -(\nabla y_0)^T (\partial_{\infty} \times \nabla y_0) = (\partial_{\infty} \times \nabla y_0)^T (\nabla y_0).
\] (4.19)

Let us now obtain an alternative form of $G^{\text{lin}}_\infty$. We have
\[
G^{\text{lin}}_\infty = (\nabla y_0)^T (\nabla v - A_{\partial \infty} \nabla y_0) = (\nabla y_0)^T (\partial_{\alpha_1} u + a_1 \times \partial_{\infty} | \partial_{\alpha_2} u + a_2 \times \partial_{\infty})
\]
\[
= (\nabla y_0)^T (\nabla v) + (a_1 | a_2)^T (a_1 \times \partial_{\infty} | a_2 \times \partial_{\infty}) = (\nabla y_0)^T (\nabla v) + \left( \begin{array}{c} 0 \\ \langle \partial_{\infty}, a_1 \times a_2 \rangle \end{array} \right)
\]
\[
= (\nabla y_0)^T (\nabla v) + \ partial_{\infty}^T n_0 \left( -\frac{0}{\sqrt{\det I_{y_0}}} \sqrt{\det I_{y_0}} \right) = (\nabla y_0)^T (\nabla v) + \partial_{\infty}^T n_0 C.
\] (4.20)

Remember that $C$ is a $2 \times 2$ skew-symmetric matrix. Using in the following (4.20) and the condition $G^{\text{lin}}_\infty \in \text{Sym}(2)$ (i.e., skew($G^{\text{lin}}_\infty$) = 0), we have
\[
\langle \partial_{\infty}, n_0 \rangle C = -\text{skew}[(\nabla y_0)^T (\nabla v)] \quad \iff \quad \langle \partial_{\infty}, n_0 \rangle \mathbb{I}_2 = -\text{skew}[(\nabla y_0)^T (\nabla v)] C^{-1}
\]
\[
\iff \quad 2 \langle \partial_{\infty}, n_0 \rangle = -\text{tr}(\text{skew}[(\nabla y_0)^T (\nabla v)] C^{-1}) \quad \iff \quad \langle \partial_{\infty}, n_0 \rangle = -\frac{1}{2} \text{tr}(\text{skew}[(\nabla y_0)^T (\nabla v)] C^{-1}).
\] (4.21)
Thus, the rotation vector $\vartheta_\infty$ is completely determined by the relations (4.15) and (4.21) as a function of the midsurface displacement $v$, by knowing its component in the direction of $n_0$ and its components in the tangent plane.

Further, we can write

$$R^{\text{lin}}_\infty = (\nabla y_0)^T (\vartheta_\infty \times \partial_{x_1} n_0 \mid \vartheta_\infty \times \partial_{x_1} n_0) - (\nabla y_0)^T (\partial_{x_1} (\vartheta_\infty \times n_0) \mid \partial_{x_2} (\vartheta_\infty \times n_0)).$$

To compute the last term in (4.22), we use (4.15) and write

$$-\partial_{x_\beta} (\vartheta_\infty \times n_0) = \partial_{x_\beta} \sum_{\alpha=1,2} \langle n_0, \partial_{x_\alpha} v \rangle a_\alpha = \sum_{\alpha=1,2} \partial_{x_\beta} \left[ \langle n_0, \partial_{x_\alpha} v \rangle a_\alpha \right] + \sum_{\alpha=1,2} \left[ \langle n_0, \partial_{x_\alpha} v \rangle \partial_{x_\beta} a_\alpha \right]$$

$$= \sum_{\alpha=1,2} \left[ \langle n_0, \partial_{x_\alpha, x_\beta} v \rangle a_\alpha \right] + \sum_{\alpha=1,2} \left[ \langle \partial_{x_\beta} n_0, \partial_{x_\alpha} v \rangle a_\alpha \right]$$

$$+ \sum_{\alpha=1,2} \left[ \langle n_0, \partial_{x_\alpha} v \rangle \left( - \sum_{\gamma=1,2} \Gamma^\alpha_{\gamma\beta} a_\gamma + b_3^\alpha n_0 \right) \right]$$

$$= \sum_{\alpha=1,2} \left[ \langle n_0, \partial_{x_\alpha, x_\beta} v \rangle - \sum_{\gamma=1,2} \Gamma^\alpha_{\gamma\beta} \partial_{x_\gamma} v \right] a_\alpha$$

$$- \sum_{\alpha,\gamma=1,2} \left[ \langle a_\gamma, \partial_{x_\alpha} v \rangle b_3^\alpha a_\alpha \right]$$

$$+ \sum_{\alpha=1,2} \left[ \langle n_0, \partial_{x_\alpha} v \rangle b_3^\alpha n_0 \right],$$

(4.23)
due to Gauß and Weingarten formulas [28]. Therefore, we obtain

$$-(\nabla y_0)^T (\vartheta_\infty \times n_0) \mid \partial_{x_2} (\vartheta_\infty \times n_0) = R^{\text{lin}}_\text{Koiter} - (\partial_{x_1} v \mid \partial_{x_2} v)^T (a_1 \mid a_2) \begin{pmatrix} b_1^1 & b_1^2 \\ b_1^2 & b_2^2 \end{pmatrix}$$

$$= R^{\text{lin}}_\text{Koiter} - \left[ (\nabla y_0)^T (\nabla v) \right]^T L_{y_0}.$$  

(4.24)

To compute the first term in (4.22), we decompose $\vartheta_\infty = \langle \vartheta_\infty, n_0 \rangle n_0 + \sum_{\alpha=1,2} \left[ \langle \vartheta_\infty, a_\alpha \rangle a_\alpha \right]$ and write

$$\vartheta_\infty \times \partial_{x_\beta} n_0 = \langle \vartheta_\infty, n_0 \rangle n_0 \times \partial_{x_\beta} n_0 + \sum_{\alpha=1,2} [\langle \vartheta_\infty, a_\alpha \rangle a_\alpha \times \partial_{x_\beta} n_0].$$

(4.25)

Using

$$n_0 \times \partial_{x_\beta} n_0 = n_0 \times \left( - \sum_{\alpha=1,2} b_3^\alpha a_\alpha \right) = \sqrt{\text{det} \Gamma_{y_0}} \left( -b_3^1 a_2^1 + b_3^2 a_1^1 \right)$$

(4.26)

and

$$a_\alpha \times \partial_{x_\beta} n_0 = a_\alpha \times \left( - \sum_{\gamma=1,2} b_3^\gamma a_\gamma \right) = - \sum_{\gamma=1,2} b_3^\gamma \epsilon_{\alpha\gamma} n_0,$$

(4.27)

with $\epsilon_{12} = -\epsilon_{21} = 1$, this yields

$$\langle \nabla y_0 \rangle ^T (\vartheta_\infty \times \partial_{x_1} n_0 \mid \vartheta_\infty \times \partial_{x_2} n_0) = \sqrt{\text{det} \Gamma_{y_0}} \langle \vartheta_\infty, n_0 \rangle (\nabla y_0)^T (\langle b_1^1 a_2^1 + b_1^2 a_1^1 \rangle - b_2^1 a_2^1 + b_2^2 a_1^1)$$

$$= \sqrt{\text{det} \Gamma_{y_0}} \langle \vartheta_\infty, n_0 \rangle (\nabla y_0)^T \begin{pmatrix} b_1^1 & b_1^2 \\ b_1^2 & b_2^2 \end{pmatrix}$$

(4.28)

and inserting here (4.21) we obtain

$$\langle \nabla y_0 \rangle ^T (\vartheta_\infty \times \partial_{x_1} n_0 \mid \vartheta_\infty \times \partial_{x_2} n_0) = -\text{skew} \left[ (\nabla y_0)^T (\nabla v) \right] L_{y_0}.$$  

(4.29)

Substituting (4.24) and (4.29) into (4.22), we are left with

$$R^{\text{lin}}_\infty = R^{\text{lin}}_\text{Koiter} - \text{sym} \left[ (\nabla y_0)^T (\nabla v) \right] L_{y_0},$$

$$\vartheta^{\text{lin}}_\infty = \text{sym} \left[ (\nabla y_0)^T (\nabla v) \right] = G^{\text{lin}}_\text{Koiter},$$

(4.30)
which can be written by virtue of (4.9) in the form
\[ R_{\text{lin}}^\infty = R_{\text{Koiter}}^\infty - \mathcal{G}_{\text{Koiter}}^\infty L_{y_0}. \] (4.31)

Some direct calculations are also possible, see also (4.12), since
\[
\mathcal{Q}_\infty = \text{polar}((\nabla m | n)[\nabla \Theta]^{-1}) = \text{polar}(((\nabla y_0 + \nabla v | n_0 + \partial_\infty \times n_0)[\nabla \Theta]^{-1})
= \text{polar}((\nabla \Theta + (\nabla v | n_0)[\nabla \Theta]^{-1}) = \text{polar}(1 + (\nabla v | \partial_\infty \times n_0)[\nabla \Theta]^{-1})
= 1 + \text{skew}((\nabla v | \partial_\infty \times n_0)[\nabla \Theta]^{-1}) + \text{h.o.t.},
\]
which leads to
\[
\bar{A}_{\phi_\infty} = -\text{skew}((\nabla v | \sum_{\alpha=1,2} \langle n_0, \partial_{x_\alpha} v \rangle a^\alpha)[\nabla \Theta]^{-1}).
\] (4.33)

Summarizing, we have the linear approximations of the deformations measures
\[
\mathcal{E}_{\text{lin}}^\infty = [\nabla \Theta]^{-T}[\mathcal{G}_{\text{Koiter}}^\text{lin}]^b[\nabla \Theta]^{-1},
\]
\[
C_{y_0}^\infty \mathcal{K}_{\text{lin}} = -[\nabla \Theta]^{-T}[R_{\text{Koiter}}^\infty - \mathcal{G}_{\text{Koiter}}^\infty L_{y_0}]^b[\nabla \Theta]^{-1}
\]
\[
C_{y_0}^\infty B_{y_0} = -[\nabla \Theta]^{-T}[[R_{\text{Koiter}}^\infty - \mathcal{G}_{\text{Koiter}}^\infty L_{y_0}]L_{y_0}]^b[\nabla \Theta]^{-1},
\]
\[
\mathcal{E}_{\text{lin}}^\infty B_{y_0} + C_{y_0}^\infty B_{y_0} = -[\nabla \Theta]^{-T}[[R_{\text{Koiter}}^\infty - 2 \mathcal{G}_{\text{Koiter}}^\infty L_{y_0}]L_{y_0}]^b[\nabla \Theta]^{-1},
\]
\[
\bar{A}_{\phi_\infty} \equiv \text{Anti}\phi_\infty = -\text{skew}((\nabla v | \sum_{\alpha=1,2} \langle n_0, \partial_{x_\alpha} v \rangle a^\alpha)[\nabla \Theta]^{-1}),
\]
\[
\phi_\infty = -\frac{1}{2} \text{tr} (\text{skew} \left( [(\nabla y_0)^T(\nabla v)] C^{-1} \right) n_0 - \sum_{\alpha=1,2} \langle n_0, \partial_{x_\alpha} v \rangle a^\alpha \in \mathbb{R}^3,
\]
\[
\mathcal{K}_{\text{lin}}^\infty = (\nabla \phi_\infty | 0)[\nabla \Theta]^{-1},
\]
\[
\mathcal{K}_{\text{lin}}^\infty B_{y_0} = (\nabla \phi_\infty | 0) L_{y_0}^b[\nabla \Theta]^{-1},
\]
\[
\mathcal{K}_{\text{lin}}^\infty B_{y_0}^2 = (\nabla \phi_\infty | 0) (L_{y_0}^b)^2[\nabla \Theta]^{-1}.
\] (4.34)

### 4.2. The constrained linear $O(h^5)$-Cosserat-shell model: conditional existence

We can now easily obtain the linearised constrained Cosserat $O(h^5)$-shell model by inserting the linearised deformation measures in the quadratic variational problem of the nonlinear constrained Cosserat model and the obtained minimization problem is to find the midsurface displacement vector field $v : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ minimizing on $\omega$:

\[
I = \int_\omega \left[ \left( h + \frac{h^3}{12} \right) W_{\text{shell}}^\infty ((\nabla \Theta) - T[\mathcal{G}_{\text{Koiter}}^\text{lin}]^b[\nabla \Theta]^{-1}) 
+ \left( \frac{h^3}{12} - \frac{h^5}{80} \right) W_{\text{shell}}^\infty ((\nabla \Theta) - T[R_{\text{Koiter}}^\text{lin} - 2 \mathcal{G}_{\text{Koiter}}^\text{lin} L_{y_0}]^b[\nabla \Theta]^{-1}) 
+ \frac{h^3}{3} W_{\text{shell}}^\infty ((\nabla \Theta) - T[\mathcal{G}_{\text{Koiter}}^\text{lin}]^b[\nabla \Theta]^{-1}, [\nabla \Theta] - T[R_{\text{Koiter}}^\text{lin} - 2 \mathcal{G}_{\text{Koiter}}^\text{lin} L_{y_0}]^b[\nabla \Theta]^{-1}) 
- \frac{h^3}{6} W_{\text{shell}}^\infty ((\nabla \Theta) - T[\mathcal{G}_{\text{Koiter}}^\text{lin}]^b[\nabla \Theta]^{-1}, [\nabla \Theta] - T[R_{\text{Koiter}}^\text{lin} - 2 \mathcal{G}_{\text{Koiter}}^\text{lin} L_{y_0}]^b[\nabla \Theta]^{-1}) \right]
\]
Moreover, we define two sets of admissible functions

\[ \mathcal{A}_\text{lin}^\infty = \left\{ v = (v_1, v_2, v_3) \in H^1(\omega, \mathbb{R}) \times H^1(\omega, \mathbb{R}) \times H^2(\omega, \mathbb{R}) \mid v_1 = v_2 = v_3 = \langle \nabla v_3, v \rangle = 0 \quad \text{on} \quad \partial \omega, \right. \]

\[
\nabla \left[ \text{skew} \left( \sum_{\alpha=1}^{3} \langle n_0, \partial_{x_\alpha} v \rangle a^\alpha \right) [\nabla \Theta]^{-1} \right] \in L^2(\omega),
\]

\[ = \delta_\infty \]

such that

\[ \mathcal{G}_{\text{Koiter}}^{\text{lin}} L_{y_0} \in \text{Sym}(2) \quad \text{and} \quad (\mathcal{R}_{\text{Koiter}}^{\text{lin}} - 2 \mathcal{G}_{\text{Koiter}}^{\text{lin}} L_{y_0}) L_{y_0} \in \text{Sym}(2), \]
\[ G_{\text{Koiter}}^\text{lin}(v) L_{y_0} \in \text{Sym}(2) \quad \text{and} \quad (R_{\text{Koiter}}^\text{lin}(v) - 2 G_{\text{Koiter}}^\text{lin}(v) L_{y_0}) L_{y_0} \in \text{Sym}(2) \],

where \( \nu \) is the outer unit normal to \( \partial \omega \), and

\[
\tilde{A}_\text{lin}^\infty = \left\{ v \in H_0^1(\omega, \mathbb{R}^3) \mid \langle \partial_{x_\alpha} v, n_0 \rangle \in L^2(\omega), \quad \nabla[\text{skew}((\nabla v) \left( \sum_{\alpha=1,2} \langle n_0, \partial_{x_\alpha} v \rangle a^\alpha \rangle [\nabla \Theta]^{-1})] \in L^2(\omega), \right. \\
\left. G_{\text{Koiter}}^\text{lin}(v) L_{y_0} \in \text{Sym}(2) \quad \text{and} \quad (R_{\text{Koiter}}^\text{lin}(v) - 2 G_{\text{Koiter}}^\text{lin}(v) L_{y_0}) L_{y_0} \in \text{Sym}(2) \right\},
\]

\[ (4.39) \]

depending on the expressions of \( G_{\text{Koiter}}^\text{lin} \) and \( R_{\text{Koiter}}^\text{lin} \) which we consider, i.e., \( (3.8) \) and \( (3.9) \) or \( (3.2) \) and \( (3.3) \), respectively. Both admissible sets of functions incorporate a weak reformulation of the symmetry constraint in \((4.36)\), in the sense that all the derivatives are considered now in the sense of distributions, and the boundary conditions are to be understood in the sense of traces. The admissible set \( A_\text{lin}^\infty \) defined by \((4.39)\) is a closed subset of the admissible set \( A_1 \) given by \((3.10)\) and usually considered in the linear Koiter theory, while \( \tilde{A}_\text{lin}^\infty \) defined by \((4.40)\) is a closed subset of the admissible set \( A_2 \) given by \((3.11)\).

However, it is not clear a priori if the sets \( A_\text{lin}^\infty \) and \( \tilde{A}_\text{lin}^\infty \) are non-empty, since we still do not know if there exists \( v \in A_1 \) or \( v \in A_2 \) such that \( G_{\text{Koiter}}^\text{lin} L_{y_0} \in \text{Sym}(2) \) and \( (R_{\text{Koiter}} - 2 G_{\text{Koiter}} L_{y_0}) L_{y_0} \in \text{Sym}(2) \). The situation is similar to pure bending type models applied on curved surfaces. In general, the set of admissible deformations leaving the first fundamental form invariant may be empty.

For the moment, let us consider the admissible set \( A_\text{lin}^\infty \). Similarly as in the linear unconstrained Cosserat-shell theory, we rewrite the minimization problem in a weak form. To this aim, we define the following operators (we use the same notations, but we point out the dependence on \( v \) of all involved quantities)

\[ \mathcal{C}_\infty^\text{lin} : A_\text{lin}^\infty \to \mathbb{R}^{3 \times 3}, \quad \mathcal{C}_\infty^\text{lin}(v) = \langle \nabla \Theta \rangle^{-T} [G_{\text{Koiter}}^\text{lin}(v)]^\flat \langle \nabla \Theta \rangle^{-1} = \langle \nabla \Theta \rangle^{-T} (\text{sym} [(\nabla v)^T (\nabla v)])^\flat \langle \nabla \Theta \rangle^{-1}, \]

\[ A_\infty : A_\text{lin}^\infty \to \mathbb{R}^{3 \times 3}, \quad A_\infty(v) = -\text{skew}((\nabla v) \left( \sum_{\alpha=1,2} \langle n_0, \partial_{x_\alpha} v \rangle a^\alpha \rangle [\nabla \Theta]^{-1}), \]

\[ K_\infty : A_\text{lin}^\infty \to \mathbb{R}^{3 \times 3}, \quad K_\infty(v) = \left( \text{axl}(\partial_{x_1} A_\infty(v)) \mid \text{axl}(\partial_{x_2} A_\infty(v)) \mid 0 \right) \langle \nabla \Theta \rangle^{-1}, \]

\[ (4.41) \]

the bilinear form

\[ B_\infty^\text{lin} : A_\infty \times A_\infty \to \mathbb{R}, \]

\[ B_\infty^\text{lin}(v, \tilde{v}) := \int_\omega \left[ \left( h + \frac{\hbar^5}{12} \right) W_\text{shell}(\mathcal{C}_\infty^\text{lin}(v), \mathcal{C}_\infty^\text{lin}(\tilde{v})) \\
+ \left( \frac{\hbar^3}{12} - K \frac{\hbar^5}{80} \right) W_\text{shell}(\mathcal{C}_\infty^\text{lin}(v) B_{y_0} + C_{y_0} \mathcal{K}_\infty^\text{lin}(v), \mathcal{C}_\infty^\text{lin}(\tilde{v}) B_{y_0} + C_{y_0} \mathcal{K}_\infty^\text{lin}(\tilde{v})) \\
- \frac{\hbar^3}{6} H W_\text{shell}(\mathcal{C}_\infty^\text{lin}(v), \mathcal{C}_\infty^\text{lin}(\tilde{v}) B_{y_0} + C_{y_0} \mathcal{K}_\infty^\text{lin}(v)) \\
- \frac{\hbar^3}{6} H W_\text{shell}(\mathcal{C}_\infty^\text{lin}(\tilde{v}), \mathcal{C}_\infty^\text{lin}(v) B_{y_0} + C_{y_0} \mathcal{K}_\infty^\text{lin}(v)) \\
+ \frac{\hbar^3}{12} W_\text{shell}(\mathcal{C}_\infty^\text{lin}(v), (\mathcal{C}_\infty^\text{lin}(\tilde{v}) B_{y_0} + C_{y_0} \mathcal{K}_\infty^\text{lin}(v)) B_{y_0}) \\
+ \frac{\hbar^3}{12} W_\text{shell}(\mathcal{C}_\infty^\text{lin}(\tilde{v}), (\mathcal{C}_\infty^\text{lin}(v) B_{y_0} + C_{y_0} \mathcal{K}_\infty^\text{lin}(v)) B_{y_0}) \right] \]

\[ (4.43) \]
\begin{align*}
+ \frac{h^5}{80} & \mathcal{W}_{\text{lin}}^{\infty} (\mathcal{E}_{\delta_{\infty}}(v) B_{y_0} + C_{y_0} \kappa_{\infty}(v)) B_{y_0}, (\mathcal{E}_{\delta_{\infty}}(\bar{v}) B_{y_0} + C_{y_0} \kappa_{\infty}(\bar{v})) B_{y_0}) \\
+ \left( h - K \frac{h^3}{12} \right) & \mathcal{W}_{\text{curv}}(\kappa_{\infty}(v), \kappa_{\infty}(\bar{v})) \\
+ \left( \frac{h^5}{12} - K \frac{h^3}{80} \right) & \mathcal{W}_{\text{curv}}(\kappa_{\infty}(v) B_{y_0}, \kappa_{\infty}(\bar{v}) B_{y_0}) \\
+ \frac{h^5}{80} & \mathcal{W}_{\text{curv}}(\kappa_{\infty}(v) B_{y_0}^2, \kappa_{\infty}(\bar{v}) B_{y_0}^2) \mid \det(\nabla y_0 | n_0)\right) \ R_{\omega}, 
\end{align*}

and the linear operator

\[
\Pi_{\infty}^{\text{lin}} : \mathcal{A}_{\infty}^{\text{lin}} \rightarrow \mathbb{R}, \quad \Pi_{\infty}^{\text{lin}}(\bar{v}) = \Pi_{\infty}(\bar{v}).
\]

Therefore, the weak form of the equilibrium problem of the linear theory of constrained Cosserat shells including terms up to order \(O(h^5)\) is to find \(v \in \mathcal{A}_{\infty}^{\text{lin}}\) satisfying

\[
B_{\infty}^{\text{lin}}(v, \bar{v}) = \Pi_{\infty}^{\text{lin}}(\bar{v}) \quad \forall \bar{v} \in \mathcal{A}_{\infty}^{\text{lin}}.
\]

We endow \( \mathcal{A}_{\infty}^{\text{lin}} \) with the norm

\[
v = (v_1, v_2, v_3) \mapsto \left( ||v_1||_{\mathcal{H}^1(\omega; \mathbb{R})}^2 + ||v_2||_{\mathcal{H}^1(\omega; \mathbb{R})}^2 + ||v_3||_{\mathcal{H}^2(\omega; \mathbb{R})}^2 + ||\nabla[\text{skew}((\nabla v | \sum_{\alpha=1,2} (n_0, \partial_{x,\alpha} v) a^\alpha)(\nabla \Theta^{-1})])||_{L^2(\omega)}^2 \right)^{1/2}.
\]

Let us recall that one decisive point in the proof of the existence result is the coercivity of the internal energy density, see [33]:

**Lemma 4.1.** (Coercivity in the theory including terms up to order \(O(h^5)\)) For sufficiently small values of the thickness \(h\) such that

\[
h \max\{\sup_{x \in \omega} |\kappa_1|, \sup_{x \in \omega} |\kappa_2|\} < \alpha \quad \text{with} \quad \alpha < \sqrt{\frac{2}{3}} (29 - \sqrt{761}) \simeq 0.97083
\]

and for constitutive coefficients satisfying \(\mu > 0, 2\lambda + \mu > 0, b_1 > 0, b_2 > 0\) and \(b_3 > 0\), the energy density

\[
W(\mathcal{E}_{m,s}, \kappa_{e,s}) = W_{\text{memb}}(\mathcal{E}_{m,s}) + W_{\text{memb,bend}}(\mathcal{E}_{m,s}, \kappa_{e,s}) + W_{\text{bend,curv}}(\kappa_{e,s})
\]

is coercive in the sense that there exists a constant \(a_1^+ > 0\) such that \(W(\mathcal{E}_{m,s}, \kappa_{e,s}) \geq a_1^+ (||\mathcal{E}_{m,s}||^2 + ||\kappa_{e,s}||^2)\), where \(a_1^+\) depends on the constitutive coefficients. In fact, the following inequality holds true

\[
W(\mathcal{E}_{m,s}, \kappa_{e,s}) \geq a_1^+ (||\mathcal{E}_{m,s}||^2 + ||\mathcal{E}_{m,s} B_{y_0} + C_{y_0} \kappa_{e,s}||^2 + ||\kappa_{e,s}||^2).
\]

For the proof of the existence of the solution, we need the next estimate given by the following result.

**Lemma 4.2.** Let there be given a domain \(\omega \subset \mathbb{R}^2\) and assume that the initial configuration of the curved shell is defined by a continuous injective mapping \(y_0 \in C^3(\bar{\omega}, \mathbb{R}^3)\) such that the two vectors \(a_\alpha = \partial_{x,\alpha} y_0, \alpha = 1, 2\) are linear independent at all points of \(\bar{\omega}\). There exists a constant \(c > 0\) such that

\[
||G_{\text{Koiter}}(v)||^2 + ||R_{\text{Koiter}}(v) - 2 G_{\text{Koiter}}(v) L_{y_0}||^2 \geq c (||G_{\text{Koiter}}(v)||^2 + ||R_{\text{Koiter}}(v)||^2),
\]

everywhere in \(\omega\).
Proof. We split our discussion. On the one hand, in all the points of \( \Omega \) for which \( L_{y_0} = 0 \) it follows that
\[
\| \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) \|^2 + \| \mathcal{R}_{\text{Koiter}}^{\text{lin}}(v) - 2 \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) L_{y_0} \|^2 = \| \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) \|^2 + \| \mathcal{R}_{\text{Koiter}}^{\text{lin}}(v) \|^2. \tag{4.50}
\]

On the other hand, due to the properties of the Frobenius norm (including sub-multiplicity), in all the points of \( \Omega \) for which \( L_{y_0} \neq 0 \) we deduce
\[
\begin{align*}
\| \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) \|^2 + \| \mathcal{R}_{\text{Koiter}}^{\text{lin}}(v) - 2 \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) L_{y_0} \|^2 & = \frac{1}{2} \| \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) \|^2 + \frac{1}{2} \| \mathcal{R}_{\text{Koiter}}^{\text{lin}}(v) \|^2 + \| \mathcal{R}_{\text{Koiter}}^{\text{lin}}(v) - 2 \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) L_{y_0} \|^2 \\
& = \frac{1}{2} \| \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) \|^2 + \frac{1}{2} \| 2 L_{y_0} \|^2 \| \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) \|^2 + \| \mathcal{R}_{\text{Koiter}}^{\text{lin}}(v) - 2 \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) L_{y_0} \|^2 \\
& \geq \frac{1}{2} \| \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) \|^2 + \frac{1}{2} \| 2 L_{y_0} \|^2 \| \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) \|^2 + \| \mathcal{R}_{\text{Koiter}}^{\text{lin}}(v) - 2 \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) L_{y_0} \|^2 \\
& \geq \frac{1}{2} \| \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) \|^2 + \frac{1}{2} \| 2 L_{y_0} \|^2 \| \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) \|^2 + \| \mathcal{R}_{\text{Koiter}}^{\text{lin}}(v) - 2 \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) L_{y_0} \|^2 \\
& \geq \frac{1}{2} \frac{\| \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) \|^2}{\| 2 L_{y_0} \|^2} + \frac{1}{2} \| 2 L_{y_0} \|^2 \| \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) \|^2 + \| \mathcal{R}_{\text{Koiter}}^{\text{lin}}(v) - 2 \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) L_{y_0} \|^2 \\
& = \frac{1}{2} \min \left\{ \frac{\| \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) \|^2}{\| 2 L_{y_0} \|^2}, 1 \right\} \left( \| \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) \|^2 + \| 2 \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) L_{y_0} \|^2 + \| \mathcal{R}_{\text{Koiter}}^{\text{lin}}(v) - 2 \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) L_{y_0} \|^2 \right). \tag{4.51}
\end{align*}
\]

Of course \( \| 2 L_{y_0} \|^2 \) depends on the points in \( \Omega \). Since \( y_0 \in C^3(\Omega, \mathbb{R}^3) \) is such that the two vectors \( a_\alpha = \partial_{x_\alpha} y_0, \alpha = 1, 2 \) are linear independent at all points of \( \Omega \) it follows that \( \det \left[ \nabla_x \Theta(0) \right] \geq c_0 > 0 \) at all points of \( \Omega \). Moreover, we have
\[
\| L_{y_0} \| = \| I_{y_0}^{-1} \Pi_{y_0} \| \leq \| I_{y_0}^{-1} \| \| \Pi_{y_0} \| = \frac{1}{\det L_{y_0}} \| \text{Cof} L_{y_0} \| \| \Pi_{y_0} \| \tag{4.52}
\]
and since \( y_0 \in C^2(\Omega; \mathbb{R}^3) \) implies \( \text{Cof} L_{y_0} \in C^1(\Omega; \mathbb{R}^{2 \times 2}) \) and \( \Pi_{y_0} \in C(\Omega; \mathbb{R}^{2 \times 2}) \), together with \( \det \left[ \nabla_x \Theta(0) \right] \geq c_0 > 0 \) it follows that there is \( c > 0 \) independent of \( x \in \Omega \) such that
\[
\| L_{y_0} \| \leq \frac{1}{\det L_{y_0}} \| \text{Cof} L_{y_0} \| \| \Pi_{y_0} \| < c, \tag{4.53}
\]
i.e., there exists another constant \( c > 0 \) such that
\[
\frac{1}{\| 2 L_{y_0} \|^2} > c. \tag{4.54}
\]

Therefore, using (4.51) and (4.54) we deduce that there exists a constant \( c > 0 \) such that
\[
\| \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) \|^2 + \| \mathcal{R}_{\text{Koiter}}^{\text{lin}}(v) - 2 \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) L_{y_0} \|^2 \geq \frac{1}{2} \frac{\| \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) \|^2}{\| 2 L_{y_0} \|^2} + \frac{1}{2} \| 2 L_{y_0} \|^2 \| \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) \|^2 + \| \mathcal{R}_{\text{Koiter}}^{\text{lin}}(v) - 2 \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) L_{y_0} \|^2
\geq c \left( \| \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) \|^2 + \| 2 \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) L_{y_0} \|^2 + \| \mathcal{R}_{\text{Koiter}}^{\text{lin}}(v) - 2 \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) L_{y_0} \|^2 \right). \tag{4.55}
\]

Now we apply the reverse triangle inequality to see that
\[
\| \mathcal{R}_{\text{Koiter}}^{\text{lin}}(v) - 2 \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) L_{y_0} \| \geq \| \mathcal{R}_{\text{Koiter}}^{\text{lin}}(v) \| - \| 2 \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) L_{y_0} \|, \tag{4.56}
\]
hence
\[
\| 2 \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) L_{y_0} \| + \| \mathcal{R}_{\text{Koiter}}^{\text{lin}}(v) - 2 \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) L_{y_0} \| \geq \| \mathcal{R}_{\text{Koiter}}^{\text{lin}}(v) \|. \tag{4.57}
\]

Inserting this in (4.55), it follows that there exists a positive constant \( c > 0 \) such that
\[
\| \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) \|^2 + \| \mathcal{R}_{\text{Koiter}}^{\text{lin}}(v) - 2 \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) L_{y_0} \|^2 \geq c \left( \| \mathcal{G}_{\text{Koiter}}^{\text{lin}}(v) \|^2 + \| \mathcal{R}_{\text{Koiter}}^{\text{lin}}(v) \|^2 \right), \tag{4.58}
\]
everywhere in \( \omega \), and the proof is complete. \( \square \)

After having Lemma 4.2, other similar arguments as in the unconstrained linear Cosserat-shell model allow us to formulate the following existence result.
Theorem 4.3. (A conditional existence result for the theory including terms up to order $O(h^5)$ on a general surface) Let there be given a domain $\omega \subset \mathbb{R}^2$ and an injective mapping $y_0 \in C^3(\omega, \mathbb{R}^3)$ such that the two vectors $a_\alpha = \partial_x \omega, \alpha = 1, 2$, are linear independent at all points of $\omega$. Assume that the admissible set $A_{x, y_0}^\infty$ is non-empty and that the linear operator $\Pi_{x, y_0}^\infty$ is bounded. Then, for sufficiently small values of the thickness $h$ such that condition (4.46) is satisfied and for constitutive coefficients satisfying $\mu > 0, 2 \lambda + \mu > 0, b_1 > 0, b_2 > 0$ and $b_3 > 0$, the problem (4.44) admits a unique solution $v \in A_{x, y_0}^\infty$.

Proof. First, we show that for sufficiently small values of the thickness $h$ such that (4.46) is satisfied and for constitutive coefficients satisfying $\mu > 0, 2 \lambda + \mu > 0, b_1 > 0, b_2 > 0$ and $b_3 > 0$, the bilinear form $B_{x, y_0}^\infty$ is coercive. According to Proposition 3.3. from [33], see also Lemma 4.1, and considering that energy terms containing $\mu_c$ are absent everywhere, there exists a constant $a_1^+ > 0$ such that

$$W^{\infty}(E_{\omega_0}^\infty, \kappa_{x, y_0}^{\infty}(v)) \geq a_1^+ (\|E_{\omega_0}^\infty(v)\|^2 + \|E_{\omega_0}^\infty(v)y_0 + C_{y_0}^\infty\kappa_{x, y_0}^{\infty}(v)\|^2 + \|\kappa_{x, y_0}^{\infty}(v)\|^2) \quad \forall v \in A_{x, y_0}^\infty,$$

where $a_1^+$ depends on the constitutive coefficients (but not on the Cosserat couple modulus $\mu_c$). Therefore, it follows that there exists a constant $a_1^+ > 0$ such that

$$B_{x, y_0}^\infty(v, v) = \int_{\omega} W^{\infty}(E_{\omega_0}^\infty(v), \kappa_{x, y_0}^{\infty}(v)) \det(\nabla y_0|n_0) da \geq a_1^+ (\|E_{\omega_0}^\infty(v)\|^2_{L^2(\omega)} + \|\kappa_{x, y_0}^{\infty}(v)\|^2_{L^2(\omega)}) \quad \forall v \in A_{x, y_0}^\infty.$$

Since

$$E_{\omega_0}^\infty(v) = [\nabla \Theta]^{-T}[\gamma_{\text{Koiter}}^\infty(v)]^b[\nabla \Theta]^{-1},$$

we obtain

$$\|E_{\omega_0}^\infty(v)\|^2 = \langle [\nabla \Theta]^{-T}[\gamma_{\text{Koiter}}^\infty(v)]^b[\nabla \Theta]^{-1}, [\nabla \Theta]^{-T}[\gamma_{\text{Koiter}}^\infty(v)]^b[\nabla \Theta]^{-1} \rangle \geq \langle [\nabla \Theta]^{-T}[\gamma_{\text{Koiter}}^\infty(v)]^b[\nabla \Theta]^{-1}, [\gamma_{\text{Koiter}}^\infty(v)]^b[\nabla \Theta]^{-1} \rangle \geq \lambda_{\min}(\tilde{I}_{y_0}^{-1})\langle [\gamma_{\text{Koiter}}^\infty(v)]^b, [\gamma_{\text{Koiter}}^\infty(v)]^b[\nabla \Theta]^{-1}\rangle \geq \lambda_{\min}(\tilde{I}_{y_0}^{-1})\|\gamma_{\text{Koiter}}^\infty(v)\|^2,$$

where $0 < \lambda_{\min}(\tilde{I}_{y_0}^{-1}) = \min\{\lambda_{\min}(I_{y_0}^{-1}), 1\} \leq 1$ is the smallest eigenvalue of the positive definite matrix $\tilde{I}_{y_0}^{-1}$.

Similarly, from

$$E_{\omega_0}^\infty(v)y_0 + C_{y_0}^\infty\kappa_{x, y_0}^{\infty}(v) = [\nabla \Theta]^{-T}[\gamma_{\text{Koiter}}^\infty(v) - 2 \gamma_{\text{Koiter}}^\infty(v) L_{y_0}]^b[\nabla \Theta]^{-1},$$

we obtain

$$\|E_{\omega_0}^\infty(v)y_0 + C_{y_0}^\infty\kappa_{x, y_0}^{\infty}(v)\|^2 \geq \lambda_{\min}(\tilde{I}_{y_0}^{-1})\|\gamma_{\text{Koiter}}^\infty(v) - 2 \gamma_{\text{Koiter}}^\infty(v) L_{y_0}\|^2,$$

while

$$\|\kappa_{x, y_0}^{\infty}(v)\|^2 \geq \lambda_{\min}(\tilde{I}_{y_0}^{-1})(\|\text{axl}(\partial_{x_1} A_{\omega_0}(v))\|^2 + \|\text{axl}(\partial_{x_2} A_{\omega_0}(v))\|^2) = \frac{\lambda_{\min}(\tilde{I}_{y_0}^{-1})}{2} (\|\partial_{x_1} A_{\omega_0}(v)\|^2 + \|\partial_{x_2} A_{\omega_0}(v)\|^2) = \frac{\lambda_{\min}(\tilde{I}_{y_0}^{-1})}{2} \|\nabla A_{\omega_0}(v)\|^2.$$
where $\lambda_{\text{max}}(I_{y_0}^{-1})$ is the largest eigenvalue of the positive definite matrix $\hat{I}_{y_0}^{-1}$, and

$$
\lambda_{\text{min}}(I_{y_0}^{-1}) = \frac{\det I_{y_0}^{-1}}{\lambda_{\text{max}}(I_{y_0}^{-1})} = \frac{1}{\lambda_{\text{max}}(I_{y_0}^{-1})} \geq \frac{1}{\sqrt{2 \left\| I_{y_0}^{-1} \right\| \det I_{y_0}^{\text{lin}}}} = \frac{1}{\sqrt{2 \left\| \text{Cof} I_{y_0} \right\|}}.
$$

(4.67)

Since $y_0 \in C^2(\overline{\omega}; \mathbb{R}^3)$ implies $\text{Cof} I_{y_0} \in C^1(\overline{\omega}; \mathbb{R}^{2\times2})$, we have that $\text{Cof} I_{y_0}$ is bounded above. Hence, there exists a positive constant $c > 0$ such that

$$
\lambda_{\text{min}}(I_{y_0}^{-1}) = \frac{\det I_{y_0}^{-1}}{\lambda_{\text{max}}(I_{y_0}^{-1})} \geq \frac{1}{\sqrt{2 \left\| \text{Cof} I_{y_0} \right\|}} \geq c > 0,
$$

(4.68)
i.e., that $\lambda_{\text{min}}(I_{y_0}^{-1}) > 0$ is bounded below over $\overline{\omega}$ by a positive constant.

Thus, from (4.62), (4.64) and (4.65), we deduce that there exists a positive constant $c > 0$ such that

$$
\mathcal{B}^{\text{lin}}(v, v) \geq c \left( \| \mathcal{G}^{\text{lin}}_{\text{Koiter}}(v) \|^2_{L^2(\omega)} + \| \mathcal{R}^{\text{lin}}_{\text{Koiter}}(v) \|^2_{L^2(\omega)} + \| \nabla A_\infty(v) \|^2_{L^2(\omega)} \right) \quad \forall v \in \mathcal{A}_{\infty}^{\text{lin}}.
$$

(4.69)

The estimate (4.49) together with (4.60) and (4.65) leads us to the existence of a positive constant $c > 0$ such that

$$
\mathcal{B}^{\text{lin}}(v, v) \geq c \left( \| \mathcal{G}^{\text{lin}}_{\text{Koiter}}(v) \|^2_{L^2(\omega)} + \| \mathcal{R}^{\text{lin}}_{\text{Koiter}}(v) \|^2_{L^2(\omega)} \right) \quad \forall v \in \mathcal{A}_{\infty}^{\text{lin}}.
$$

(4.70)

Using the Korn inequality given by Theorem 3.3, we have that there exists a constant $c > 0$ such that

$$
\| \mathcal{G}^{\text{lin}}_{\text{Koiter}} \|^2_{L^2(\omega)} + \| \mathcal{R}^{\text{lin}}_{\text{Koiter}} \|^2_{L^2(\omega)} \geq c \left( \| v_1 \|^2_{H^1(\omega; \mathbb{R}^3)} + \| v_2 \|^2_{H^1(\omega; \mathbb{R}^3)} + \| v_3 \|^2_{H^2(\omega; \mathbb{R}^3)} \right) \quad \forall v \in \mathcal{A}_{\infty}^{\text{lin}} \subset \mathcal{A}_1.
$$

(4.71)

As a conclusion, we have that there exists a constant $c > 0$ such that

$$
\mathcal{B}^{\text{lin}}(v, v) \geq c \left( \| v_1 \|^2_{H^1(\omega; \mathbb{R}^3)} + \| v_2 \|^2_{H^1(\omega; \mathbb{R}^3)} + \| v_3 \|^2_{H^2(\omega; \mathbb{R}^3)} + \| \nabla A_\infty(v) \|^2_{L^2(\omega)} \right) \quad \forall v \in \mathcal{A}_{\infty}^{\text{lin}}.
$$

(4.72)

In consequence, the bilinear form is coercive on $\mathcal{A}_{\infty}^{\text{lin}}$ and the Lax–Milgram theorem leads us to the conclusion of the theorem. \hfill \Box 

**Theorem 4.4.** (A conditional existence result for the theory including terms up to order $O(h^5)$ for shells whose middle surface has little regularity) Let there be given a domain $\omega \subset \mathbb{R}^2$ and an injective mapping $y_0 \in H^{2,\infty}(\omega, \mathbb{R}^3)$ such that the two vectors $a_\alpha = \partial_{x_\alpha} y_0$, $\alpha = 1, 2$ are linear independent at all points of $\overline{\omega}$. Assume that the admissible set $\mathcal{A}_{\infty}^{\text{lin}}$ is non-empty and that the linear operator $\Pi_{\infty}^{\text{lin}}$ is bounded. Then, for sufficiently small values of the thickness $h$ such that condition (4.46) is satisfied and for constitutive coefficients such that $\mu > 0$, $2 \lambda + \mu > 0$, $b_1 > 0$, $b_2 > 0$ and $b_3 > 0$, the problem (4.44) admits a unique solution $v \in \mathcal{A}_{\infty}^{\text{lin}}$.

**Proof.** The proof is similar with that of the first existence result Theorem 4.3. There are only three important differences. Before proceeding to discuss these differences, let us notice that since $y_0 \in H^{2,\infty}(\omega, \mathbb{R}^3)$ is such that the two vectors $a_\alpha = \partial_{x_\alpha} y_0$, $\alpha = 1, 2$ are linear independent at all points of $\overline{\omega}$ it follows that $\det[\nabla_x \Theta(0)] \geq c_0 > 0$ at all points of $\overline{\omega}$.

The first difference is regarding the proof of the estimates

$$
\| \mathcal{E}^{\infty}_d(v) B_{y_0} + C_{y_0} \mathcal{K}^{\infty}(v) \|^2_{L^2(\omega)} \geq c \left( \| \mathcal{R}^{\text{lin}}_{\text{Koiter}}(v) \|^2_{L^2(\omega)} + 2 \mathcal{G}^{\text{lin}}_{\text{Koiter}}(v) L_{y_0} \right) \| \mathcal{R}^{\text{lin}}_{\text{Koiter}}(v) - 2 \mathcal{G}^{\text{lin}}_{\text{Koiter}}(v) L_{y_0} \|_{L^2(\omega)},
$$

(4.73)

with $c > 0$ a positive constant over $\overline{\omega}$, since in the proof of Lemma 4.2 which was used in the proof of Theorem 4.3 we have used that $y_0 \in C^3(\overline{\omega}, \mathbb{R}^3)$, while in the hypothesis of the theorem under discussion we have $y_0 \in H^{2,\infty}(\omega, \mathbb{R}^3)$.

We still know from (4.64) and (4.65) that

$$
\| \mathcal{E}^{\infty}_d(v) B_{y_0} + C_{y_0} \mathcal{K}^{\infty}(v) \|^2 \geq \lambda_{\text{min}}(I_{y_0}^{-1}) \| \mathcal{R}^{\text{lin}}_{\text{Koiter}}(v) - 2 \mathcal{G}^{\text{lin}}_{\text{Koiter}}(v) L_{y_0} \|^2,
$$

(4.74)
and
\[
\|K_{\infty}^{\text{lin}}(v)\|^2 \geq \frac{\lambda_{\text{min}}(I_{y_0}) - 1}{2} \|\nabla A_{\infty}(v)\|^2,
\] (4.75)
since they are independent by the assumption \(y_0 \in C^3(\overline{\omega}, \mathbb{R}^3)\) and \(\lambda_{\text{min}}(I_{y_0}) \leq 1\). In addition, for \(\det[\nabla \Theta(0)] \geq c_0 > 0\) it is true that \(\lambda_{\text{min}}(I_{y_0}) \geq \frac{\sqrt{2} \|\text{Cof } I_{y_0}\|}{\sqrt{2} \|\text{Cof } I_{y_0}\|}\). Therefore, \(y_0 \in H^{2, \infty}(\omega, \mathbb{R}^3)\) implies \(\text{Cof } I_{y_0} \in L^\infty(\omega)\) and
\[
\int_\omega \|E_{\infty}^{\text{lin}}(v)B_{y_0} + C_{y_0}K_{\infty}^{\text{lin}}(v)\|^2 \, \text{d}a \geq \int_\omega \frac{1}{\sqrt{2} \|\text{Cof } I_{y_0}\|} \|R_{\text{Koiter}}^{\text{lin}}(v) - 2 G_{\text{Koiter}}^{\text{lin}}(v) L_{y_0}\|^2 \, \text{d}a
\]
\[
\geq \frac{1}{\sqrt{2} \|\text{Cof } I_{y_0}\|} \int_\omega \|R_{\text{Koiter}}^{\text{lin}}(v) - 2 G_{\text{Koiter}}^{\text{lin}}(v) L_{y_0}\|^2 \, \text{d}a
\] (4.76)
i.e., there exists a constant \(c > 0\) such that
\[
\|E_{\infty}^{\text{lin}}(v)B_{y_0} + C_{y_0}K_{\infty}^{\text{lin}}(v)\|^2_{L^2(\omega)} \geq c \|R_{\text{Koiter}}^{\text{lin}}(v) - 2 G_{\text{Koiter}}^{\text{lin}}(v) L_{y_0}\|^2_{L^2(\omega)}.
\] (4.77)
In a similar way, it follows that there exists a constant \(c > 0\) such that
\[
\|K_{\infty}^{\text{lin}}(v)\|^2_{L^2(\omega)} \geq c \|\nabla A_{\infty}(v)\|^2_{L^2(\omega)}.
\] (4.78)
The second difference is regarding the needed estimate
\[
\|G_{\text{Koiter}}^{\text{lin}}(v)\|^2_{L^2(\omega)} + \|R_{\text{Koiter}}^{\text{lin}}(v) - 2 G_{\text{Koiter}}^{\text{lin}}(v) L_{y_0}\|^2_{L^2(\omega)} \geq c \left(\|G_{\text{Koiter}}^{\text{lin}}(v)\|^2_{L^2(\omega)} + \|R_{\text{Koiter}}^{\text{lin}}(v)\|^2_{L^2(\omega)}\right),
\] (4.79)
where \(c\) is a positive constant. However, the estimates (4.50) and (4.51) are true for \(y_0 \in H^{2, \infty}(\omega, \mathbb{R}^3)\), too. Moreover, since
\[
\|L_{y_0}\| \leq \frac{1}{\det I_{y_0}} \|\text{Cof } I_{y_0}\| \|\Pi_{y_0}\|
\] (4.80)
and \(\det[\nabla \Theta(0)] \geq c_0 > 0\) at all points of \(\overline{\omega}\), from (4.50) and (4.51) we deduce the estimate
\[
\|G_{\text{Koiter}}^{\text{lin}}(v)\|^2 + \|R_{\text{Koiter}}^{\text{lin}}(v) - 2 G_{\text{Koiter}}^{\text{lin}}(v) L_{y_0}\|^2 \geq \left\{\begin{array}{ll}
\left(\frac{1}{2} \min_{I_{y_0}} \left\{\frac{1}{4} \|\text{Cof } I_{y_0}\|^2 \|\Pi_{y_0}\|^2, 1\right\} \left(\|G_{\text{Koiter}}^{\text{lin}}(v)\|^2 + \|R_{\text{Koiter}}^{\text{lin}}(v)\|^2\right)\right) & \text{if } L_{y_0} = 0, \\
\frac{1}{2} \min_{I_{y_0}} \left\{\frac{1}{4} \|\text{Cof } I_{y_0}\|^2 \|\Pi_{y_0}\|^2, 1\right\} \left(\|G_{\text{Koiter}}^{\text{lin}}(v)\|^2 + \|R_{\text{Koiter}}^{\text{lin}}(v)\|^2\right) & \text{if } L_{y_0} \neq 0,
\end{array}\right.
\] (4.81)
where \(c > 0\) is a positive constant. Hence, since \(y_0 \in H^{2, \infty}(\omega, \mathbb{R}^3)\) implies \(\text{Cof } I_{y_0} \in L^\infty(\omega)\) and \(\Pi_{y_0} \in L^\infty(\omega)\) we deduce
\[
\int_\omega \left(\|G_{\text{Koiter}}^{\text{lin}}(v)\|^2 + \|R_{\text{Koiter}}^{\text{lin}}(v) - 2 G_{\text{Koiter}}^{\text{lin}}(v) L_{y_0}\|^2\right) \, \text{d}a
\]
\[
\geq \min_{I_{y_0}} \left\{\frac{c}{8 \|\text{Cof } I_{y_0}\|^2 \|\Pi_{y_0}\|^2}, \frac{1}{2} \right\} \int_\omega \left(\|G_{\text{Koiter}}^{\text{lin}}(v)\|^2 + \|R_{\text{Koiter}}^{\text{lin}}(v)\|^2\right) \, \text{d}a.
\] (4.82)
Thus, there exists the positive constant $c > 0$ such that
\[
\|G_{\text{Koiter}}^{\text{lin}}(v)\|_{L^2(\omega)}^2 + \|R_{\text{Koiter}}^{\text{lin}}(v) - 2G_{\text{Koiter}}^{\text{lin}}(v) I_{y_0}\|_{L^2(\omega)}^2 \geq c \left( \|G_{\text{Koiter}}^{\text{lin}}(v)\|_{L^2(\omega)}^2 + \|R_{\text{Koiter}}^{\text{lin}}(v)\|_{L^2(\omega)}^2 \right).
\]

The third difference is that the Korn inequality for shells whose middle surface has little regularity given by Theorem 3.4 is now applicable, instead of the Korn inequality on general surfaces. \hfill \Box

### 4.3. The constrained linear $O(h^3)$-Cosserat-shell model. Conditional existence

By ignoring the $O(h^5)$ terms from the functional defining the variational problem of the constrained linear $O(h^3)$-Cosserat shell model, we obtain the linearised constrained Cosserat $O(h^3)$-shell model which is to find the midsurface displacement vector field $v : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ minimizing on $\omega$:

\[
I = \int_{\omega} \left[ \left( h + K \frac{h^3}{12} \right) W_{\text{shell}}^{\infty}(\nabla \Theta)^{-T}[G_{\text{Koiter}}^{\text{lin}}]^{\flat}[\nabla \Theta]^{-1}ight. \\
+ \frac{h^3}{12} W_{\text{shell}}^{\infty}(\nabla \Theta)^{-T}[R_{\text{Koiter}}^{\text{lin}} - 2G_{\text{Koiter}}^{\text{lin}} I_{y_0}]^{\flat}[\nabla \Theta]^{-1}ight. \\
+ \frac{h^3}{3} H W_{\text{shell}}^{\infty}(\nabla \Theta)^{-T}[G_{\text{Koiter}}^{\text{lin}}]^{\flat}[\nabla \Theta]^{-1}, (\nabla \Theta)^{-T}[R_{\text{Koiter}}^{\text{lin}} - 2G_{\text{Koiter}}^{\text{lin}} I_{y_0}]^{\flat}[\nabla \Theta]^{-1}ight. \\
- \frac{h^3}{6} W_{\text{shell}}^{\infty}(\nabla \Theta)^{-T}[G_{\text{Koiter}}^{\text{lin}}]^{\flat}[\nabla \Theta]^{-1}, (\nabla \Theta)^{-T}[R_{\text{Koiter}}^{\text{lin}} - 2G_{\text{Koiter}}^{\text{lin}} I_{y_0}]^{\flat}[\nabla \Theta]^{-1}ight. \\
+ \left( h - K \frac{h^3}{12} \right) W_{\text{curv}}((\nabla \theta_\omega | 0) (\nabla \Theta)^{-1}) \\
+ \frac{h^3}{12} W_{\text{curv}}((\nabla \theta_\omega | 0) I_{y_0}^{\flat}(\nabla \Theta)^{-1}) \right] \det((\nabla y_0 | n_0) da - \Pi_{\infty}(v), \tag{4.84}
\]

such that

\[
G_{\text{Koiter}}^{\text{lin}} I_{y_0} \in \text{Sym}(2) \quad \text{and} \quad (R_{\text{Koiter}}^{\text{lin}} - 2G_{\text{Koiter}}^{\text{lin}} I_{y_0}) I_{y_0} \in \text{Sym}(2). \tag{4.85}
\]

The coercivity result given by the following lemma, see [33], and similar arguments as in proving the existence results in the theory including terms up to order $O(h^5)$ lead to corresponding existence results.

In the constrained nonlinear Cosserat-shell model up to $O(h^3)$ the shell energy density $W(h^3)(\varepsilon_\infty, \kappa_\infty)$ is given by

\[
W(h^3)(\varepsilon_\infty, \kappa_\infty) = \left( h + K \frac{h^3}{12} \right) W_{\text{shell}}^{\infty}(\varepsilon_\infty) + \frac{h^3}{12} W_{\text{shell}}^{\infty}(\varepsilon_\infty B_{y_0} + C_{y_0} \kappa_\infty) \\
- \frac{h^3}{3} H W_{\text{shell}}^{\infty}(\varepsilon_\infty, \varepsilon_\infty B_{y_0} + C_{y_0} \kappa_\infty) + \frac{h^3}{6} W_{\text{shell}}^{\infty}(\varepsilon_\infty, (\varepsilon_\infty B_{y_0} + C_{y_0} \kappa_\infty)B_{y_0}) \\
+ \left( h - K \frac{h^3}{12} \right) W_{\text{curv}}(\kappa_\infty) + \frac{h^3}{12} W_{\text{curv}}(\kappa_\infty B_{y_0}). \tag{4.86}
\]

**Lemma 4.5.** (Coercivity in the theory including terms up to order $O(h^3)$) Assume that the constitutive coefficients are such that $\mu > 0$, $2 \lambda + \mu > 0$, $b_1 > 0$, $B_2 > 0$, $b_3 > 0$ and $L_c > 0$ and let $c_2^+$ denote the smallest eigenvalue of $W_{\text{curv}}(S)$, and $c_1^+$ and $C_1^+$ denote the smallest and the largest eigenvalues of the quadratic form $W_{\text{shell}}^{\infty}(S)$, If the thickness $h$ satisfies one of the following conditions:

i) $h \max\{\sup_{x \in \omega} |\kappa_1|, \sup_{x \in \omega} |\kappa_2|\} < \alpha$ \quad and \quad $h^2 < \frac{(5 - 2\sqrt{6})(\alpha^2 - 12)^2}{4 \alpha^2 C_1^+}$ with $0 < \alpha < 2$;
then the total energy density \( W^{(h^3)}(\mathcal{E}_\infty, \mathcal{K}_\infty) \) is coercive, in the sense that there exists a constant \( a_1^+ > 0 \) such that
\[
W^{(h^3)}(\mathcal{E}_\infty, \mathcal{K}_\infty) \geq a_1^+ (\|\mathcal{E}_\infty\|^2 + \|\mathcal{K}_\infty\|^2),
\]
where \( a_1^+ \) depends on the constitutive coefficients. In fact, the following inequality holds true
\[
W^{(h^3)}(\mathcal{E}_\infty, \mathcal{K}_\infty) \geq a_1^+ (\|\mathcal{E}_\infty\|^2 + \|\mathcal{E}_\infty B_{y_0} + C_{y_0} \mathcal{K}_\infty\|^2 + \|\mathcal{K}_\infty\|^2).
\]

Indeed, we have:

**Theorem 4.6.** (A conditional existence result for the theory including terms up to order \( O(h^3) \) on a general surface) Let there be given a domain \( \omega \subset \mathbb{R}^2 \) and an injective mapping \( y_0 \in C^3(\overline{\omega}, \mathbb{R}^3) \) such that the two vectors \( a_\alpha = \partial_{x_\alpha} y_0, \alpha = 1, 2 \), are linear independent at all points of \( \overline{\omega} \). Assume that the admissible set \( \mathcal{A}_\infty^{\text{lin}} \) is non-empty and that the linear operator \( \Pi^{\text{lin}}_\infty \) is bounded. Assume that the constitutive coefficients are such that \( \mu > 0, 2 \lambda + \mu > 0, b_1 > 0, b_2 > 0, b_3 > 0 \) and \( L_c > 0 \) and let \( c_2^+ \) denote the smallest eigenvalue of \( W_{\text{curv}}(S) \), and \( c_1^+ \) and \( C_1^+ > 0 \) denote the smallest and the largest eigenvalues of the quadratic form \( W_{\text{shell}}^\infty(S) \). If the thickness \( h \) satisfies one of the conditions (4.87), then the problem (4.84) admits a unique solution \( v \in \mathcal{A}_\infty^{\text{lin}} \).

**Theorem 4.7.** (A conditional existence result for the theory including terms up to order \( O(h^3) \) for shells whose middle surface has little regularity) Let there be given a domain \( \omega \subset \mathbb{R}^2 \) and an injective mapping \( y_0 \in H^2(\omega, \mathbb{R}^3) \) such that the two vectors \( a_\alpha = \partial_{x_\alpha} y_0, \alpha = 1, 2 \), are linear independent at all points of \( \omega \). Assume that the admissible set \( \tilde{\mathcal{A}}_\infty^{\text{lin}} \) is non-empty and that the linear operator \( \tilde{\Pi}^{\text{lin}}_\infty \) is bounded. Assume that the constitutive coefficients are such that \( \mu > 0, 2 \lambda + \mu > 0, b_1 > 0, b_2 > 0, b_3 > 0 \) and \( L_c > 0 \) and let \( c_2^+ \) denote the smallest eigenvalue of \( W_{\text{curv}}(S) \), and \( c_1^+ \) and \( C_1^+ > 0 \) denote the smallest and the largest eigenvalues of the quadratic form \( W_{\text{shell}}^\infty(S) \). If the thickness \( h \) satisfies one of the conditions (4.87), then the problem (4.84) admits a unique solution \( v \in \tilde{\mathcal{A}}_\infty^{\text{lin}} \).

**Remark 4.8.** As we have already mentioned for the nonlinear Cosserat-shell models [31,33], a large value for \( \alpha \) will relax the first condition (4.87)\_1 while the other condition (4.87)\_2 on the thickness will remain more restrictive. While for the \( O(h^5) \)-model the conditions imposed on the thickness do not depend on the constitutive parameters, in \( O(h^5) \)-model the conditions (4.87) are expressed in terms of all constitutive parameters, through \( c_1^+, c_2^+ \) and \( C_1^+ \).

5. A modified constrained linear \( O(h^5) \)-Cosserat-shell model. Unconditional existence

In order to present an unconditional existence result, independent of whether the admissible set is non-empty, we need to modify the model slightly. We have already done so in the geometrically nonlinear case [33] and here we just adapt that procedure properly to the linearised case.

5.1. Variational problem for the modified constrained nonlinear Cosserat-shell model

In the modified constrained Cosserat-shell model, the (through the thickness reconstructed) strain tensor is considered to be
\[
\tilde{\mathcal{E}}_s = \frac{1}{\lambda + 2\mu} \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \frac{\lambda}{\lambda + 2\mu} \text{tr}(\mathcal{E}_\infty) (0 | 0 | n_0) (0 | 0 | n_0)^T
\]
\[ + x_3 \left[ \text{sym}(\mathcal{E}_\infty B_{y_0} + C_{y_0} \mathcal{K}_\infty) - \frac{\lambda}{(\lambda + 2\mu)} \text{tr} \left[ \text{sym}(\mathcal{E}_\infty B_{y_0} + C_{y_0} \mathcal{K}_\infty) \right] (0|0)n_0 (0|0)n_0^T \right] \\
+ x_3^2 \text{sym} \left[ (\mathcal{E}_\infty B_{y_0} + C_{y_0} \mathcal{K}_\infty) B_{y_0} \right] + O(x_3^3). \]

which is now symmetric by definition through the additional by applied symmetrization.

This ansatz of the (through the thickness reconstructed) strain tensor leads to a model in which only the symmetric parts of \( \mathcal{E}_\infty B_{y_0} + C_{y_0} \mathcal{K}_\infty \) and \( (\mathcal{E}_\infty B_{y_0} + C_{y_0} \mathcal{K}_\infty) B_{y_0} \) are involved and there will be no need to assume a priori that they have to be symmetric.

The resulting variational problem for the modified constrained Cosserat \( O(h^5) \)-shell model [33] is to find a deformation of the midsurface \( m : \omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) minimizing on \( \omega \):

\[
I = \int_\omega \left[ (h + K\frac{h^3}{12}) W_{\text{shell}}(\sqrt{[\nabla \Theta]^{-T} \hat{T}_m \mathbb{1}_2 [\nabla \Theta]^{-1} - \sqrt{[\nabla \Theta]^{-T} \hat{I}_{y_0} \mathbb{1}_2 [\nabla \Theta]^{-1}}) \\
+ \left( \frac{h^3}{12} - K\frac{h^5}{80} \right) W_{\text{shell}}(\text{sym} \left[ \sqrt{[\nabla \Theta]^{-T} \hat{T}_m [\nabla \Theta]^{-1} [\nabla \Theta] \left( L_{y_0}^b - L_m^b \right) [\nabla \Theta]^{-1} \right]) \\
- \frac{h^3}{3} H W_{\text{shell}}(\text{sym} \left[ \sqrt{[\nabla \Theta]^{-T} \hat{T}_m [\nabla \Theta]^{-1} [\nabla \Theta] \left( L_{y_0}^b - L_m^b \right) [\nabla \Theta]^{-1} \right]) \\
+ \frac{h^3}{6} W_{\text{shell}}(\text{sym} \left[ \sqrt{[\nabla \Theta]^{-T} \hat{T}_m [\nabla \Theta]^{-1} [\nabla \Theta] \left( L_{y_0}^b - L_m^b \right) [\nabla \Theta]^{-1} \right]) \\
+ \frac{h^5}{80} W_{\text{curv}}(\text{sym} \left[ \sqrt{[\nabla \Theta]^{-T} \hat{T}_m [\nabla \Theta]^{-1} [\nabla \Theta] \left( L_{y_0}^b - L_m^b \right) [\nabla \Theta]^{-1} \right]) \\
+ \left( h - K\frac{h^3}{12} \right) W_{\text{curv}}(\mathcal{K}_\infty) + \left( \frac{h^3}{12} - K\frac{h^5}{80} \right) W_{\text{curv}}(\mathcal{K}_\infty B_{y_0}) + \frac{h^5}{80} W_{\text{curv}}(\mathcal{K}_\infty B_{y_0}^2) \right] \text{det} \nabla \Theta \text{da} \\
- \Pi(m, \mathcal{Q}_\infty). \]

The admissible set \( \mathcal{A}_{\text{mod}} \) is defined by\(^3\)

\[
\mathcal{A}_{\text{mod}} = \left\{ (m, \mathcal{Q}_\infty) \in H^1(\omega, \mathbb{R}^3) \times H^1(\omega, \text{SO}(3)) \mid m|_{\gamma_d} = m^*, \mathcal{Q}_\infty Q_0 e_3|_{\gamma_d} = \frac{\partial x_1 m^* \times \partial x_2 m^*}{\|\partial x_1 m^* \times \partial x_2 m^*\|} \right\}
\]

where

\[
\mathcal{K}_\infty = \left( \text{axl}(Q_\infty^T \partial x_1 Q_\infty) \mid \text{axl}(Q_\infty^T \partial x_2 Q_\infty) \mid 0 \right) [\nabla \Theta]^{-1},
\]

\[
Q_\infty = \text{polar}(([\nabla m[n]][\nabla \Theta]^{-1}) = ([\nabla m[n]][\nabla \Theta]^{-1} \sqrt{[\nabla \Theta]^{-1} \hat{T}_m [\nabla \Theta]^{-1}}),
\]

\(^3\)The definition of the admissible set \( \mathcal{A}_{\text{mod}} \) incorporates a weak reformulation of the imposed symmetry constraint \( \mathcal{E}_\infty \in \text{Sym}(3) \). We notice that the constraints \( U := Q_\infty^T (\nabla m[Q_\infty Q_0 e_3])[\nabla \Theta]^{-1} \in L^2(\omega, \text{Sym}^+(3)) \) together with the compatibility conditions between \( \mathcal{Q}_\infty \) and the values of \( m \) on \( \gamma_d \) will imply that \( \mathcal{Q}_\infty \) and \( m \) are not independent variables and \( \mathcal{Q}_\infty = \text{polar}([\nabla m[n]][\nabla \Theta]^{-1}) \in \text{SO}(3) \), where \( n = \frac{\partial x_1 m \times \partial x_2 m}{\|\partial x_1 m \times \partial x_2 m\|} \) is the unit normal vector to the deformed midsurface. Assuming that the boundary data satisfy the conditions \( m^* \in H^1(\omega, \mathbb{R}^3) \) and \( \text{polar}(\nabla m^* \mid n^*) \in H^1(\omega, \text{SO}(3)) \), it follows that the admissible set is not empty.
Taking into account the modified constitutive nonlinear $O(h^5)$-Cosserat-shell model, the variational problem for the constrained Cosserat $O(h^5)$-shell linear model is now to find a deformation of the midsurface $v: \omega \subset \mathbb{R}^2 \to \mathbb{R}^3$ minimizing on $\omega$:

\[
I = \int_\omega \left[ \left( h + K \frac{h^3}{12} \right) W_{\text{shell}}^\infty (\nabla \Theta)^{-T} G_{Koiter}^{\text{lin}} [\nabla \Theta]^{-1} \right. \\
+ \left( \frac{h^3}{12} - K \frac{h^5}{80} \right) W_{\text{shell}}^\infty (\nabla \Theta)^{-T} \text{sym} \left[ (\nabla \Theta)^{-1} \right] \\
+ \frac{h^3}{3} \frac{1}{W_{\text{shell}}^\infty (\nabla \Theta)^{-T} \text{sym} \left[ (\nabla \Theta)^{-1} \right]} \\
- \frac{h^3}{6} W_{\text{shell}}^\infty (\nabla \Theta)^{-T} \text{sym} \left[ (\nabla \Theta)^{-1} \right] \\
+ \frac{h^5}{80} W_{\text{mp}}^\infty (\nabla \Theta)^{-T} \text{sym} \left[ (\nabla \Theta)^{-1} \right] \\
+ \left( h - K \frac{h^3}{12} \right) W_{\text{curv}}((\nabla \Theta)) \\
+ \left( \frac{h^3}{12} - K \frac{h^5}{80} \right) W_{\text{curv}}((\nabla \Theta)) \\
+ \frac{h^5}{80} W_{\text{curv}}((\nabla \Theta)) \\
\right] \det(\nabla y_0) da - \Pi(u),
\]

where

\[
W_{\text{shell}}^\infty (S) = \mu \| S \|^2 + \frac{\lambda \mu}{\lambda + 2\mu} [\text{tr}(S)]^2, \quad W_{\text{shell}}^\infty (S, T) = \mu \langle S, T \rangle + \frac{\lambda \mu}{\lambda + 2\mu} \text{tr}(S) \text{tr}(T),
\]

\[
W_{\text{mp}}^\infty (S) = \mu \| S \|^2 + \frac{\lambda}{2} [\text{tr}(S)]^2 \quad \forall S, T \in \text{Sym}(3),
\]

\[
W_{\text{curv}}(X) = \mu L_2^\infty (b_1 \| \text{dev sym} X \|^2 + b_2 \| \text{skew} X \|^2 + b_3 [\text{tr}(X)]^2) \quad \forall X \in \mathbb{R}^{3 \times 3}.
\]

### 5.3. Unconditional existence results

The sets of admissible functions are accordingly defined by

\[
\mathcal{A}_{\text{lin}}^\text{mod} = \left\{ v = (v_1, v_2, v_3) \in H^1(\omega, \mathbb{R}) \times H^1(\omega, \mathbb{R}) \times H^2(\omega, \mathbb{R}) \mid v_1 = v_2 = v_3 = \langle \nabla v_3, \nu \rangle = 0 \text{ on } \partial \omega, \right. \\
\left. \nabla \text{skew}((\nabla v | \sum_{\alpha=1,2} \langle n_0, \partial_{x_\alpha} v \rangle a^\alpha) \nabla \Theta |^{-1}) \rangle \in L^2(\omega), \right\}
\]
and
\[ \mathcal{A}_{\text{lin}}^{\text{mod}} = \left\{ v \in H_0^1(\omega, \mathbb{R}^3) : \langle \partial_{x_\alpha x_\beta} v, n_0 \rangle \in L^2(\omega), \nabla[\text{skew}((\nabla v)^T \sum_{\alpha=1,2} \langle n_0, \partial_{x_\alpha} v \rangle a^\alpha)(\nabla \Theta)^{-1}] \in L^2(\omega) \right\}, \]
(5.6)
depending on the expressions of \( G_{\text{Koiter}}^{\text{lin}} \) and \( R_{\text{Koiter}}^{\text{lin}} \) which we consider, i.e., (3.8) and (3.9) or (3.2) and (3.3), respectively.

Hence, we have avoided the problem that the new set of admissible functions may be non-empty and, since all our inequalities from the proof of the conditional existence result involve only symmetric matrices, we have the following unconditional existence result:

**Theorem 5.1.** (Unconditional existence result for the theory including terms up to order \( O(h^5) \) on a general surface) Let there be given a domain \( \omega \subset \mathbb{R}^2 \) and an injective mapping \( y_0 \in C^3(\overline{\omega}, \mathbb{R}^3) \) such that the two vectors \( a_\alpha = \partial_{x_\alpha} y_0 \), \( \alpha = 1,2 \), are linear independent at all points of \( \overline{\omega} \). Assume that the linear operator \( \Pi_{\text{lin}}^{\text{mod}} \) is bounded. Then, for sufficiently small values of the thickness \( h \) such that such that condition (4.46) is satisfied and for constitutive coefficients such that \( \mu > 0, 2 \lambda + \mu > 0, b_1 > 0, b_2 > 0 \) and \( b_3 > 0, 0 > 0 \), the problem (5.4) admits a unique solution \( v \in \mathcal{A}_{\text{lin}}^{\text{mod}} \).

**Theorem 5.2.** (Unconditional existence result for the theory including terms up to order \( O(h^5) \) for shells whose middle surface has little regularity) Let there be given a domain \( \omega \subset \mathbb{R}^2 \) and an injective mapping \( y_0 \in C^2(\overline{\omega}, \mathbb{R}^3) \) such that the two vectors \( a_\alpha = \partial_{x_\alpha} y_0 \), \( \alpha = 1,2 \), are linear independent at all points of \( \overline{\omega} \). Assume that the linear operator \( \Pi_{\text{lin}}^{\text{inf}} \) is bounded. Then, for sufficiently small values of the thickness \( h \) such that condition (4.46) is satisfied and for constitutive coefficients such that \( \mu > 0, 2 \lambda + \mu > 0, b_1 > 0, b_2 > 0 \) and \( b_3 > 0, \) the problem (5.4) admits a unique solution \( v \in \mathcal{A}_{\text{lin}}^{\text{mod}} \).

By ignoring the \( O(h^5) \)-terms from the functional defining the variational problem, we obtain the linearised constrained Cosserat \( O(h^3) \)-shell model which is to find the midsurface displacement vector field \( v : \omega \subset \mathbb{R}^2 \to \mathbb{R}^3 \) minimizing on \( \omega \):

\[
I = \int_{\omega} \left[ \left( h + K \frac{h^3}{12} \right) W_{\text{shell}}^\infty ([\nabla \Theta]^{-1} g^{\text{lin}}_{\text{Koiter}})^b \right] \nabla \Theta^{-1} + \frac{h^3}{12} W_{\text{shell}}^\infty ([\nabla \Theta]^{-1} T \text{sym} R_{\text{Koiter}}^{\text{lin}} - 2 g^{\text{lin}}_{\text{Koiter}} L_{y_0})^b \nabla \Theta^{-1} + \frac{h^3}{3} H W_{\text{shell}}^\infty ([\nabla \Theta]^{-1} T g^{\text{lin}}_{\text{Koiter}}^b \nabla \Theta^{-1}, [\nabla \Theta]^{-1}) \nabla \Theta^{-1} + \frac{h^3}{6} W_{\text{shell}}^\infty ([\nabla \Theta]^{-1} T g^{\text{lin}}_{\text{Koiter}}^b \nabla \Theta^{-1}, [\nabla \Theta]^{-1} T \text{sym} R_{\text{Koiter}}^{\text{lin}} - 2 g^{\text{lin}}_{\text{Koiter}} L_{y_0}) L_{y_0} \nabla \Theta^{-1} \right. \\
- \left. \left( h - K \frac{h^3}{12} \right) W_{\text{curv}}((\nabla \partial_\infty | 0) \nabla \Theta^{-1}) + \left( h^2 - K \frac{h^5}{80} \right) W_{\text{curv}}((\nabla \partial_\infty | 0) L_{y_0}^b \nabla \Theta^{-1}) \right] \det(\nabla y_0|n_0) \, da - \tilde{\Pi}(u). \quad (5.7)
\]

Accordingly, the following results are true:

**Theorem 5.3.** (Unconditional existence result for the theory including terms up to order \( O(h^3) \) on a general surface) Let there be given a domain \( \omega \subset \mathbb{R}^2 \) and an injective mapping \( y_0 \in C^3(\overline{\omega}, \mathbb{R}^3) \) such that the two vectors \( a_\alpha = \partial_{x_\alpha} y_0 \), \( \alpha = 1,2 \), are linear independent at all points of \( \overline{\omega} \). Assume that the constitutive coefficients are such that \( \mu > 0, 2 \lambda + \mu > 0, b_1 > 0, b_2 > 0, b_3 > 0 \) and \( L_c > 0 \) and let \( c_2^+ \) denote the smallest eigenvalue of \( W_{\text{curv}}(S) \), and \( c_1^+ > 0, c_4^+ > 0 \) denote the smallest and the largest eigenvalues of the quadratic form \( W_{\text{shell}}^\infty(S) \). If the thickness \( h \) satisfies one of the conditions (4.87), then the problem (5.7) admits a unique solution \( v \in \mathcal{A}_{\text{mod}}^{\text{lin}} \).
Theorem 5.4. (Unconditional existence result for the theory including terms up to order $O(h^3)$ for shells whose middle surface has little regularity) Let there be given a domain $\omega \subset \mathbb{R}^2$ and an injective mapping $y_0 \in H^{2,\infty}(\omega, \mathbb{R}^3)$ such that the two vectors $a_\alpha = \partial_x a_\alpha y_0$, $\alpha = 1, 2$, are linear independent at all points of $\overline{\omega}$. Assume that the linear operator $\Pi_{\text{lin}}^{\infty}$ is bounded. Assume that the constitutive coefficients are such that $\mu > 0$, $2\lambda + \mu > 0$, $b_1 > 0$, $b_2 > 0$, $b_3 > 0$ and $L_{\text{c}} > 0$ and let $c_2^+$ denote the smallest eigenvalue of $W_{\text{curv}}(S)$, and $c_1^+$ and $C_1^+$ denote the smallest and the largest eigenvalues of the quadratic form $W_{\text{shell}}^\infty(S)$. If the thickness $h$ satisfies one of the conditions (4.87), then the problem (5.7) admits a unique solution $v \in \hat{A}_{\text{mod}}^{\text{lin}}$.

6. Conclusion

We have considered the geometrically nonlinear constrained Cosserat shell model and we deduced it’s linearization. The constrained model is interesting in it’s own right since it provides a natural bridge to more classical shell models, in which no independent triad of directors is considered. We were able to show existence and uniqueness, using precisely those tools that are usually employed: the classical Korn’s inequality on a surface. However, one important caveat remained: the results provide only a conditional existence, since the requirements on the admissible set are too strong. Therefore, we were led to modify the variational problem. More precisely, we only consider the symmetric parts of some strain measures in the elastic energy and this allowed us to prove unconditional existence results. Since shell theory is an approximation anyway, we believe that the modified approach is vastly superior and merits further investigation if one is interested in remaining close to more classical shell models.

Acknowledgements

This research has been funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation)—Project No. 415894848: NE 902/8-1 (P. Neff).

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

References

[1] Adams, R.A.: Sobolev Spaces, Volume 65 of Pure and Applied Mathematics, 1st edn. Academic Press, London (1975)
[2] Altenbach, H., Zhilin, P.A.: A general theory of elastic simple shells (in Russian). Uspekhi Mekhaniki 11, 107–148 (1988)
[3] Altenbach, H., Zhilin, P.A.: The theory of simple elastic shells. In: Kienzler, R., Altenbach, H., Ott, I. (eds.) Theories of Plates and Shells. Critical Review and New Applications, Euromech Colloquium, vol. 444, pp. 1–12. Springer, Heidelberg (2004)
[4] Altenbach, J., Altenbach, H., Eremeyev, V.A.: On generalized Cosserat-type theories of plates and shells: a short review and bibliography. Arch. Appl. Mech. 80, 73–92 (2010)
[5] Anicic, S.: A shell model allowing folds. In: Numerical Mathematics and Advanced Applications, pp. 317–326. Springer (2003)
[6] Anicic, S.: Du modèle de Kirchhoff-Love exact à un modèle de coque mince et à un modèle de coque pliée. PhD thesis, Université Joseph Fourier (Grenoble 1) (2001)
[7] Anicic, S., Léger, A.: Formulation bidimensionnelle exacte du modèle de coque 3D de Kirchhoff-Love. C. R. Acad. Sci. Paris Ser. Math. 329(8), 741–746 (1999)
[45] Zhilin, P.A.: Applied Mechanics—Foundations of Shell Theory. State Polytechnical University Publisher, Sankt Petersburg (2006). (in Russian)

Ionel-Dumitrel Ghiba
Department of Mathematics
Alexandru Ioan Cuza University of Iaşi
Blvd. Carol I, no. 11
700506 Iasi
Romania
e-mail: dumitrel.ghiba@uaic.ro

Ionel-Dumitrel Ghiba
Octav Mayer Institute of Mathematics of the Romanian Academy
Iaşi Branch
700505 Iasi
Romania

Patrizio Neff
Head of Lehrstuhl für Nichtlineare Analysis und Modellierung, Fakultät für Mathematik
Universität Duisburg-Essen
Thea-Leymann Str. 9
45127 Essen
Germany
e-mail: patrizio.neff@uni-due.de

(Received: November 22, 2022; revised: November 22, 2022; accepted: January 2, 2023)