A 2-categorical state sum model

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It has long been argued that higher categories provide the proper algebraic structure underlying state sum invariants of 4-manifolds. This idea has been refined recently, by proposing to use 2-groups and their representations as specific examples of 2-categories. The challenge has been to make these proposals fully explicit. Here we give a concrete realization of this program. Building upon our earlier work with Baez and Wise on the representation theory of 2-groups, we construct a four-dimensional state sum model based on a categorified version of the Euclidean group. We define and explicitly compute the simplex weights, which may be viewed a categorified analogue of Racah-Wigner 6j-symbols. These weights solve an hexagon equation that encodes the formal invariance of the state sum under the Pachner moves of the triangulation. This result unravels the combinatorial formulation of the Feynman amplitudes of quantum field theory on flat spacetime proposed in [1], which was shown to lead after gauge-fixing to Korepanov’s invariant of 4-manifolds.

I. INTRODUCTION

This paper results from the convergence of two lines of investigation in state sum models: state sums can be viewed either as topological objects [1] or categorical constructions [2]. Well understood in dimension two [2] and three [3, 4], this convergence is established here in dimension four.

State sum models provide a powerful technical tool for the combinatorial construction of manifold invariants and topological quantum field theories. The idea is to rewrite a path integral as a sum of local weights defined using a triangulation of the manifold, and to reformulate topological invariance as a set of algebraic equations for the weights. These equations encode the invariance of the state sum under elementary re-buildings of the triangulation, the so-called Pachner moves, which are known to relate any topologically equivalent configurations. This procedure is the core of the lattice definition of two-dimensional topological field theory by Fukuma, Hosono and Kawai [3]. Notorious examples of state sum models in three dimensions are the Ponzano-Regge and Turaev-Viro models based on the representation category of a (quantum) group, leading to a state sum formulation of quantum gravity in three-dimensional spacetime [4, 5].

State sum models are also at the root of the spin foam approach to quantum gravity in four dimensions [6]. Stemming from a formulation of gravity as a constrained topological theory, the main strategy in this approach has been to quantize the topological theory using a state sum model and to impose the constraints in the resulting quantum theory. Specific realizations of this idea led to a background independent formulation of the gravity path integral as a sum over geometries displaying a fundamental discreteness at the Planck scale. Remarkably, this formulation enables one to define transition amplitudes between the states of the gravitational field in loop quantum gravity [7, 8]. The close relation to topological field theory is one of the striking features of this approach: not only is it sufficient to determine the form of the boundary states, but it may arguably enable one to keep the diffeomorphism symmetry, and with it the low energy behavior of the theory, under control.

Category theory appears to be a natural arena for constructing state sum models and generalizing the Fukuma, Hosono and Kawai’s procedure to dimensions higher than two [3, 11]. This is due to a remarkable correspondence between the combinatorics of Pachner moves and the coherence laws of (higher) categories. Well understood in three dimensions [12], this correspondence is however much more difficult to exploit in higher dimensions, due to the complexity of higher algebraic structures. Formalisms have been proposed for the construction of four-dimensional models using 2-categories [13], but the only known examples of such models use a very restricted class of 2-categories, ones with a single object [14, 15].

As an attempt to find analogues of three-dimensional models built from representations of a group, Barrett and Mackaay proposed in [2] to build four-dimensional state sum models starting with a categorical group, or 2-group. 2-groups play the same role in higher gauge theory as groups do in gauge theory: just as groups can be used to describe connections defining parallel transport along curves, 2-groups can be used to describe ‘2-connections’ defining parallel transport along both curves and surfaces [16]. A 2-group can be defined as a ‘crossed module’, which is a pair of groups related by an homomorphism \( H \rightarrow G \) and an action of \( G \) on \( H \) satisfying some compatibility conditions. One of the simplest examples of (Lie) 2-group is the Poincaré 2-group [17], determined by the Lorentz group \( G \) acting on the translation group \( H \) of Minkowski space, taken with the trivial homomorphism \( H \rightarrow G \). The idea of [2], also expressed and further ex-

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explored by Crane, Sheppeard and Yetter in [18–20], was to try using the representations of the Poincaré 2-group to construct a four-dimensional analogue of the Ponzano-Regge model for three-dimensional quantum gravity.

This is the first of the two lines of investigation leading to the results presented here. We explicitly construct the model outlined in these works, using a 2-group closely related to the Poincaré 2-group, the Euclidean 2-group. The core of our construction is the complete calculation of the weight for the 4-simplex, which can be seen as a categorified analogue of Racah-Wigner 6j-symbols. The Euclidean 2-group is a categorified version of the Euclidean group: it differentiates the roles of rotations and translations by treating the former as objects in a category and the latter as morphisms. Consequently its representation theory looks quite different from that of the group [21]. In fact, just as the representations of a group can be viewed as objects in a category, the representations of a 2-group can be viewed as objects in a 2-category. The model developed in this paper thus gives an explicit realization of the 2-categorical approach to state sum models in four dimensions.

The second line of investigation concerns the combinatorial reformulation of Feynman amplitudes in quantum field theory on flat spacetime proposed in [1]. The goal of that work, motivated by earlier results in the state sum model was shown in [1] to reproduce the 4-manifold invariant previously constructed by Korepanov [23–25].

The original motivation for the present paper was to unravel the algebraic nature of the state sum model discovered in [1]. This is where the two lines of investigation meet: as we show here, the relevant structure is the 2-category of representations of the Euclidean 2-group. The fact that such a 2-categorical model shows up naturally in ordinary quantum field theory is an intriguing and exciting outcome of our work.

Explicitly, given a triangulated closed 4-manifold $\Delta$, the state sum model of [1] is characterized by weights which are (real) functions of a set of positive numbers $l_e \in \mathbb{R}_+$ labeling the edges $e$ and a set of integer spins $s_t \in \mathbb{Z}$ labeling the triangles $t$. These weights are given by the formula:

$$W_{\Delta}(l_e, s_t) = \prod_{t \in \Delta} 2A_t(l_e) \prod_{\sigma \in \Delta} \frac{\cos \left[ \sum_{t \in \sigma} s_t \phi_t^\sigma(l_e) \right]}{V_{\sigma}(l_e)}$$

where the products are over all triangles $t$ and all 4-simplices $\sigma$. There is a factor $2A_t$ for each triangle $t$, which depends on the three labels $l_e \in \mathbb{R}_+$ on the edges of $t$. Whenever these numbers satisfy the triangle inequality, i.e if they define a Euclidean geometry for $t$, $A_t$ is equal to the area of the triangle; otherwise, it is equal to zero. This means the weight $W_{\Delta}$ is zero unless the set of labels $l_e$ is consistent with all triangle inequalities. There is a factor for each 4-simplex $\sigma$, which depends on ten labels $l_e \in \mathbb{R}_+$, one for each edge of $\sigma$; and ten labels $s_t \in \mathbb{Z}$, one for each of triangle of $\sigma$. $V_{\sigma}(l_e)$ is equal to $4!$ times the volume of the 4-simplex with edge lengths $l_e$; the sum in the argument of the cosine is over all ten triangles of $\sigma$, and $\phi_t^\sigma$ denotes the dihedral angle between the two tetrahedra of $\sigma$ that meet at the triangle $t$. The partition function, or state sum, is formally obtained by summing the weights (1) over all values of the labels, using the Lebesgue measures $dl_e^2$ for the real variables and the counting measure $\pi^2 \sum_{s_t}^{\infty}$ for the integer variables.

The conjectured 2-categorical nature of the weights (1) prompted the in-depth study [21] of the infinite dimensional representation theory of Lie 2-groups introduced in [19]. The case of the Euclidean 2-group, determined by the pair of groups $G = SO(4)$ and $H = \mathbb{R}^4$, was described in [21]. In Section II we begin by reviewing this description. We explain how the variables $l_e \in \mathbb{R}_+$ and $s_t \in \mathbb{Z}$ can be understood as labeling ‘irreducible representations’ of the Euclidean 2-group and ‘irreducible 1-intertwiners’ between representations. A key aspect of this theory is that any such 1-intertwiner defines an ordinary representation of the rotation group $SO(4)$. There are also ‘2-intertwiners’ between 1-intertwiners, which define ordinary intertwiners between $SO(4)$ representations. So, just like models built from $SO(4)$ representation theory, the model based on the representation theory of the Euclidean 2-group starts with an assignment of a representation of $SO(4)$ on each triangle and an $SO(4)$ intertwiner on each tetrahedron. The crucial difference, however, is that in the latter case the representation on each triangle is infinite-dimensional and depends on the data labeling the bounding edges.

Section III presents our main result: the definition and computation of the weights of the state sum. The weight for each 4-simplex, coined ‘10j 2-symbol’ in this paper, gives an analogue of Racah-Wigner 6j symbols in the 2-categorical context. We explain the construction in explicit detail and perform a full calculation of the weights, which leads precisely to the formula (1).

We conclude with an outlook in Section IV.

II. LABELED TRIANGULATIONS

In this section we review the key ingredients from the representation theory of the Euclidean 2-group required for the construction of the state sum model.

Given an triangulated 4-manifold $\Delta$, we will:

- assign an irreducible representation of the Euclidean 2-group to each edge of $\Delta$
• assign an irreducible 1-intertwiner to each triangle of Δ, and
• assign a 2-intertwiner to each tetrahedron.

The state sum model gives a way to compute an amplitude for any such assignment. We begin with a geometrical description of what the notions listed above amount to. We refer the reader to [21] for the full details of the representation theory.

A. Representations on edges

A representation of the Euclidean 2-group is given by an SO(4)-equivariant map χ: X → ℝ⁴, where X is some space on which the rotation group acts, (g, x) → g ⋅ x. ‘Equivariant’ means that χ commutes with the group action:

\[ χ(g \cdot x) = g \chi(x) \quad (2) \]

where \( g \chi(x) \) is the image of the ℝ⁴-vector \( χ(x) \) by the rotation \( g \in \text{SO}(4) \). Irreducible representations are the ones for which the action on \( X \) is transitive and the map \( χ \) is one-to-one: in this case \( X \) is identified to a single SO(4)-orbit in \( \mathbb{R}^4 \), a 3-sphere of given radius \( l \in \mathbb{R}_+ \):

\[ X_l = \{ x \in \mathbb{R}^4, |x| = l \} \quad (3) \]

So, irreducible representations are effectively labeled by positive numbers, the radii of spheres. There is a notion of tensor product of representations [21]: the tensor product \( l \otimes m \) of two irreducible representations corresponds to the map \( X_l \times X_m \to \mathbb{R}^4 \) given by \((x, x') \mapsto x + x'\).

In what follows, we will work with the spherical measures \( d^3_l x \) induced on \( X_l \) by the Lebesgue measure on \( \mathbb{R}^4 \) and normalized as:

\[ d^3_l x := \frac{1}{\pi} d^3 x \delta(|x|^2 - l^2), \quad (4) \]

giving a total volume of \( \pi l^3 \) for the 3-sphere \( X_l \).

Other notations are as follows. We assume \( \mathbb{R}^4 \) is equipped with its standard basis of unit vectors \( e_1, e_2, e_3 \) and \( e_4 \). We identify \( U(1) \) with the subgroup of \( \text{SO}(4) \) rotations that leave the whole plane \( \{e_1, e_2\} \) fixed; we denote by \( h_o \) the \( U(1) \) element rotating the plane \( \{e_3, e_4\} \) by the angle \( φ \). In Section [11] below we will consider the \( \text{SO}(3) \)-subgroup of rotations that leave \( e_4 \) fixed; abusing notation, we will refer to this subgroup simply as \( \text{SO}(3) \).

B. 1-Intertwiners on triangles

Consider three irreducible representations labeled by \( l, m, n \in \mathbb{R}_+ \) satisfying the (strict) triangle inequality. A 1-intertwiner between \( l \otimes m \) and \( n \), drawn as

\[
\begin{array}{c}
|l| \\
\downarrow \\
|m| \\
\downarrow \\
n
\end{array}
\]

is described as follows. Let \( T \) be the set of triples of \( \mathbb{R}^3 \)-vectors \((x, y, z)\) forming a triangle of lengths \( l, m, n \):

\[ |x| = l, |y| = m, |z| = n, \quad x + y = z \quad (5) \]

It will be convenient to use the symbolic notation \( \triangle \) for a generic point \((x, y, z)\) in \( T \). The natural action of \( \text{SO}(4) \) on \( \mathbb{R}^4 \) induces a transitive action on \( T \), which we write \((g, \triangle) \mapsto g \triangle \). A 1-intertwiner amounts to a \( \text{SO}(4) \) vector bundle over \( T \), that is, a vector bundle \( V \) with a fiber-preserving \( \text{SO}(4) \) action. More precisely, for \( \triangle \in T, \varphi \in V_\triangle, \) and \( g \in \text{SO}(4) \), we can write the action as:

\[ g(\triangle, \varphi) = (g \triangle, \Phi^g_\triangle(\varphi)), \quad (6) \]

where \( \Phi^g_\triangle : V_\triangle \to V_{g \triangle} \) are invertible linear maps satisfying the rule:

\[ \Phi^g_\triangle = \Phi^g_{g \triangle} \Phi^g_\triangle \quad (7) \]

This rule says that \( \triangle \) defines a representation of \( \text{SO}(4) \) on \textit{sections} of the vector bundle \( V \). Alternatively, if we fix a point \( \triangle \in T \) and restrict the action to its stabilizer

\[ G_\triangle := \{ h \in \text{SO}(4) : h \triangle = \triangle \} \quad (8) \]

then \( \Phi^g_\triangle \) is just the equation for a representation of \( G_\triangle \) on \( V_\triangle \). As shown in [21], two such 1-intertwiners defining equivalent \( G_\triangle \) representations at the same point \( \triangle \in T \) belong to the same equivalence class\(^1\). In the language of Mackey [30], the full \( \text{SO}(4) \) representation is the one \textit{induced} by the \( G_\triangle \) representation. Irreducible 1-intertwiners are the ones for which the \( G_\triangle \) representation is irreducible. Here the \( \text{SO}(4) \) action on \( T \) has a \( U(1) \) subgroup as stabilizer group: its irreducible representations are one-dimensional, so that the bundle \( V_\triangle \) is a line bundle \( V_\triangle \simeq \mathbb{C} \). Irreducible 1-intertwiners are labeled by elements of Irrep(\( U(1) \)) \( \simeq \mathbb{Z} \).

Since \( \text{SO}(4) \) acts transitively on \( T \), the rule \( \Phi^g_\triangle \) determines all maps \( \Phi^g_\triangle \) in terms of the map at a given point \( \triangle \in T \). For practical calculations, which will require to pick a representative in each equivalence class of 1-intertwiners, it will be convenient to single out a ‘reference’ triangle in \( T \). For each set of values for the edge labels, we denote by \( \triangle^0 \) the unique triangle \((x_0, y_0, z_0) \in T \) such that:

\[ x_0 = le_1, \quad z_0 = n(\sin \gamma e_1 + \cos \gamma e_2) \quad (9) \]

for some \( \gamma \in [0, \pi] \), where \( e_i \) are the basis vectors. Thus, the pair of vectors \((x_0, z_0)\) defines a triangle with positive orientation in the oriented plane \( \{e_1, e_2\} \), with \( U(1) \) as stabilizer group \( G_{\triangle^0} \). The maps \( \Phi^g_{\triangle^0} \) at any point \( \triangle = k\triangle^0 \), where \( k \in \text{SO}(4) \), can then be expressed as:

\[ \Phi^{g k}_{k \triangle^0} = \Phi^{g k}_{k \triangle^0} \Phi^{k}_{\triangle^0} \quad (10) \]

\(^1\) Two 1-intertwiners are equivalent when there is an invertible 2-intertwiner between them.
For each $s \in \mathbb{Z}$, let us fix once and for all a nowhere vanishing complex function $g \mapsto \Phi^s(g)$ on $SO(4)$ such that $\Phi^s(1) = 1$ and
\begin{equation}
\Phi^s(gh_\theta) = e^{i s \theta} \Phi^s(g),
\end{equation}
for all $g \in SO(4)$ and all $U(1)$ rotation $h_\theta$ of angle $\theta$. A representative in the equivalence class of irreducible 1-intertwiners labeled by $s$ is obtained by choosing $\Phi^s_\nu : V^\nu \to \mathbb{C}$, which is a map $\mathbb{C} \to \mathbb{C}$, to act by multiplication by the complex number $\Phi^s(g)$.

Using an $SO(4)$-invariant measure $\mu_T$ on $T$, we may also view the $SO(4)$ representation $\bigoplus$ as acting on the space
\begin{equation}
\int_T d\mu_T(\Delta) V_\Delta
\end{equation}
of $L^2$ sections\footnote{The set of such functions is not empty: consider for example the product $\Phi^s(g) := D^{j_+}_{\frac{\pi}{2}}(g_L) D^{j_+}_{\frac{\pi}{2}}(g_R)$ of two SU(2) Wigner $D$-matrices diagonal elements in a given irreducible SU(2) representation $j$, where $g_L, g_R$ are the left and right SU(2) components of $g$ and $U(1)$ is identified to the subgroup of diagonal elements $g_L = g_R = h_\theta$.} of the bundle $V$. When the 1-intertwiner is irreducible, $V$ is a line bundle; in this case the direct integral is just the space $L^2(T, \mu_T)$ of complex functions $f$ for which
\begin{equation}
\int_T d\mu_T(\Delta) |f(\Delta)|^2 < \infty.
\end{equation}
Since $SO(4)$ acts transitively on $T$, all invariant measures coincide up to a multiplicative constant. We will use the following measure on $T$:
\begin{equation}
d\mu_T = d^3x d^3y d^3z \delta^4(x + y - z)
\end{equation}
expressed in terms of the spherical measures $\bigoplus$. With the chosen normalization, the total volume of $T$ gives the area $A$ of a triangle of lengths $l, m, n$:
\begin{equation}
\mu_T(T) = 2A(l,m,n).
\end{equation}

The composition of two 1-intertwiners
\begin{equation}
\begin{array}{c}
 l \\
 \downarrow \bullet \\
 m \\
 \downarrow \\
 n \\
 \downarrow \\
 r
\end{array}
\end{equation}
drawn as:
\begin{equation}
\begin{array}{c}
 l \\
 \downarrow \quad \bullet \\
 m \\
 \downarrow \quad \\
 n \\
 \downarrow \quad \\
 r
\end{array}
\end{equation}
is described as follows. Consider the set $Q$ of quadruples of $\mathbb{R}^4$-vectors $(x, y, v, w)$ forming a quadrangle of lengths $l, m, q, r$:
\begin{equation}
|x| = l, |y| = m, |v| = q, |w| = r, \quad x + y + v + w = w
\end{equation}
Let $Q^-_n \subset Q$ be the subset of such quadrangles with the additional condition that
\begin{equation}
|x + y| = n
\end{equation}
It will be convenient to use the symbolic notation $\bigoplus$ for an element $(x, y, v, w)$ of $Q^-_n$; and $\bigoplus'$ and $\bigoplus''$ for the corresponding triangles $(x, y, z := x + y)$ and $(z := x + y, v, w)$. We also write $\bigoplus$ for a generic quadrangle in $Q$. Denote by $(V^\nu, \Phi^\nu_\nu)$ and $(V^\nu, \Phi^\nu_\nu')$ the two 1-intertwiners $l \otimes m \otimes n$ and $n \otimes q \otimes r$. The composite 1-intertwiner
\begin{equation}
l \otimes m \otimes n \otimes q \otimes r
\end{equation}
amounts to a $SO(4)$ vector bundle $\bigoplus$ over $Q^-_n$, with fibers $\bigoplus$ and maps $\Psi^\nu_\nu : \bigoplus \to \bigoplus$ given by the tensor products:
\begin{equation}
\bigoplus = \bigoplus' \otimes \bigoplus,
\quad \Psi^\nu_\nu' = \Phi^\nu_\nu' \otimes \Phi^\nu_\nu''
\end{equation}
If the two 1-intertwiners are irreducible, this is a line bundle: $\bigoplus \simeq \mathbb{C}$; and each of the maps $\Psi^\nu_\nu$ acts by multiplication by a complex number.

The composition of 1-intertwiners tells us also how to obtain an $SO(4)$-invariant measure $\mu_{Q^-_n}$ on $Q^-_n$ from the measures $\mu_T$ and $\nu_T$ on the two sets of triangles $(x, y, z)$ and $(z, v, w)$ given by (14). It is defined by the formula:
\begin{equation}
\mu_{Q^-_n} = \pi \int d^3z \mu_T \otimes \nu_T,
\end{equation}
where $d^3z$ is the spherical measure $\bigoplus$ and $\mu_T$ denotes the disintegration of $\mu_T$ with respect to $d^3z$, that is, $d\mu_T = d^3z d\mu_T$. The factor $\pi$ was inserted to simplify the formulas. Explicitly, $d\mu_{Q^-_n}$ reads:
\begin{equation}
d^3x d^3y d^3v d^3w \delta(|x| + |y| - n^2) \delta^4(x + y + v + w)
\end{equation}
The data [15] gives a representation of $SO(4)$ on the space
\begin{equation}
\int d\mu_{Q^-_n} \bigoplus V^\nu \otimes \bigoplus
\end{equation}
of $L^2$ sections of the bundle $\bigoplus$. If the two 1-intertwiners are irreducible, $\bigoplus$ is line bundle; in this case the direct integral is just the space $L^2(Q^-_n, \mu_{Q^-_n})$ of square integrable functions on $Q^-_n$.

Letting the label $n \in \mathbb{R}_+$ run within the range of values allowed by triangular inequalities, the union of the bundles over all subsets $Q^-_n \subset Q$ gives an $SO(4)$ bundle over $Q$. We may consider the resulting $SO(4)$ representation on $L^2$ sections with respect to the measure:
\begin{equation}
\mu_Q = \int d\mu^2_{Q^-_n}
\end{equation}
where \(dn^2 = 2dn)\) is the Lebesgue measure on \(\mathbb{R}_+\). Explicitly:

\[
d\mu_Q = d^2_x d^3_m y d^4_v d^5_w \delta^4(x + y + v - w)
\]  

(23)

This representation is the direct integral of the representations \(\mathcal{V}^l\) labeled by \(\mathcal{P}\). For \(\mathcal{P} \in Q\) and \(\varphi \in V_g \otimes V_d\), the action of \(g \in SO(4)\) can be written as:

\[
g(\mathcal{P}, \varphi) = (g \mathcal{P}, (\Phi^g_q \otimes \Phi^g_q)(\varphi))
\]  

(24)

In just the same way, two 1-intertwiners

\[
e \quad m \quad q \quad p \quad l \quad r
\]

are composed as drawn:

\[
\begin{array}{ccc}
m & q & p \\
l & m & q \\
r & l & p
\end{array}
\]

Using an obvious extension of the above notations, we denote by \((V^l, \Phi^l_q)\) and \((V^q, \Phi^q_q)\) the two 1-intertwiners \(l \otimes p \rightarrow r\) and \(m \otimes q \rightarrow r\). The composite 1-intertwiner \(l \otimes m \otimes q \rightarrow r\) amounts to a \(SO(4)\) vector bundle \((W^l, \Psi^l_q)\) over the set \(Q^l \subset Q\) of quadrangles \((10)\) such that \(|y + v| = p\), where fibers \(W^l\) and maps \(\Phi^l_q, W^l \rightarrow W^l\) are given by the tensor products:

\[
W^l = V^l \otimes V^q, \quad \Psi^l_q = \Phi^l_q \otimes \Phi^q_q
\]  

(25)

This data gives a representation of \(SO(4)\) on the space

\[
\int d\mu_{Q^l_+}(\mathcal{P}) V^l \otimes V^q
\]  

(26)

of \(L^2\) sections on the bundle \(W^l\), where the measure \(\mu_{Q^l_+}\) is defined by a formula analogous to \((19)\). Explicitly, \(d\mu_{Q^l_+}\) reads:

\[
d^2_x d^3_m y d^4_v d^5_w \delta(|y + v| - p) \delta^4(x + y + v - w)
\]  

(27)

Letting the label \(p \in \mathbb{R}_+\) run within the range of values allowed by triangular inequalities, the direct integral of all representations \((26)\) gives a \(SO(4)\) representation on \(L^2\) sections of a bundle over \(Q\), with respect to the measure:

\[
\mu_Q = \int dp^2 d\mu_{Q^l_+}
\]  

(28)

also given by \((29)\). For \(\mathcal{P} \in Q\) and \(\psi \in V^l \otimes V^q\), the action of \(g \in SO(4)\) can be written as:

\[
g(\mathcal{P}, \psi) = (g \mathcal{P}, (\Phi^g_q \otimes \Phi^g_q)(\psi))
\]  

(29)

The above constructions can be extended to more general triangulated surfaces with boundaries. For example, by composing three 1-intertwiners as:

\[
\begin{array}{c}
m & q & p \\
l & m & q \\
r & l & p
\end{array}
\]

and by letting the labels \(n, r \in \mathbb{R}_+\) on the dashed edges run within the range of values allowed by triangular inequalities, we obtain a representation of \(SO(4)\) on \(L^2\) sections

\[
\int d\mu_{P}(\mathcal{P}) V^q \otimes V^l \otimes V^q
\]  

(30)

of a vector bundle over a set \(P\) of pentagons \((x, y, v, w, z)\) of lengths \(l, m, q, s, t \in \mathbb{R}^4\), endowed with the measure:

\[
d\mu_P = d^2_x d^3_m y d^4_v d^5_w v d^6_z \delta^4(x + y + v + w - z)
\]  

(31)

This representation is the direct integral of representations on \(L^2\) sections of bundles over sets \(P_{nr}\) of pentagons with two fixed diagonal lengths, with respect to the measures \(\mu_{P_{nr}}\), showing up in the decomposition:

\[
\mu_{P} = \int dn^2 dq^2 d\mu_{P_{nr}}
\]  

(32)

When the three 1-intertwiners are irreducible, the direct integral \((30)\) is just the space \(L^2(P, \mu_P)\) of square integrable functions on \(P\).

C. 2-Intertwiners on tetrahedra

Given two composite 1-intertwiners

\[
\begin{array}{c}
m & q & p \\
l & m & q \\
r & l & p
\end{array}
\]

constructed as above, a 2-intertwiner between these is drawn as:

\[
\begin{array}{c}
m & q \\
l & m \\
r & r
\end{array}
\]

The two sides of this diagram correspond to the splitting of the boundary of a tetrahedron with edge lengths \(l, m, n, p, q, r\) into two pairs of triangles sharing an edge.

Using the notations of the previous section, let \((V^l, \Phi^l_q), (V^r, \Phi^r_q), (V^m, \Phi^m_q)\) and \(V^q, \Phi^q_q\) be the 1-intertwiners that label the four triangles. The two composite intertwiners \(l \otimes m \otimes q \rightarrow r\) give two \(SO(4)\) vector
bundles \((V_{\mathcal{P}} \otimes V_{\mathcal{O}} \otimes \Phi_{\mathcal{P}}^i \otimes \Phi_{\mathcal{O}}^k)\) and \((V_{\mathcal{K}} \otimes V_{\mathcal{S}} \otimes \Phi_{\mathcal{K}}^i \otimes \Phi_{\mathcal{S}}^k)\) over the sets \(Q_n^\pm\) and \(Q_p^+\), respectively. We also introduce the set

\[ Q_{np} = Q_n^- \cap Q_p^+ \]

of quadrangles \((x, y, v, w)\) with fixed diagonal lengths \(|x + y| = n\) and \(|y + v| = p\). Note that a generic element \(n \in Q_{np}\) defines a tetrahedron of lengths \(l, m, n, p, q, r\) embedded in \(\mathbb{R}^4\). The natural action of \(SO(4)\) on \(\mathbb{R}^4\) induces a transitive action on \(Q_{np}\), which we write \((g, \mathfrak{m}) \mapsto g\mathfrak{m}\).

A 2-intertwiner amounts to a family of linear maps

\[ m_{\mathfrak{m}}: V_{\mathcal{P}} \otimes V_{\mathcal{O}} \rightarrow V_{\mathcal{K}} \otimes V_{\mathcal{S}} \tag{33} \]

indexed by elements \(\mathfrak{m} \in Q_{np}\), satisfying the intertwining rule:

\[ (\Phi_{\mathcal{P}}^i \otimes \Phi_{\mathcal{O}}^j) m_{\mathfrak{m}} = m_{\mathfrak{m}^*} (\Phi_{\mathcal{P}}^i \otimes \Phi_{\mathcal{O}}^j) \tag{34} \]

for all \(g \in SO(4)\).

Since \(SO(4)\) acts transitively on \(Q_{np}\), this rule determines all maps \(m_{\mathfrak{m}}\) in terms of the map at a given point of \(Q_{np}\). To fix the normalization of our 2-intertwiners, it will be convenient to single out a ‘reference’ quadrangle in \(Q_{np}\). For each set of values for the edge labels, we denote by \(\mathfrak{s}\) be the unique quadrangle \((x_0, y_0, v_0, w_0)\) in \(Q_{np}\) such that:

\[ x_0 = le_1, \]
\[ z_0 := x_0 + y_0 = n(\sin \gamma e_1 + \cos \gamma e_2), \]
\[ w_0 = r \cos \gamma' e_1 + r \sin \gamma' (\cos \theta e_2 + \sin \theta e_3) \tag{35} \]

for some angles \(\gamma, \gamma', \theta \in [0, \pi]\), where \(e_i\) are the basis vectors. Thus, the triple of vectors \((x_0, y_0, w_0)\) defines a tetrahedron with positive orientation in the 3-dimensional space \((e_1, e_2, e_3)\). We also introduce the four triangles \((x_0, y_0, v_0, w_0), (x_0 + y_0, y_0, v_0, w_0), (y_0, v_0, y_0 + v_0)\) and \((x_0, y_0 + v_0, w_0)\) induced by the reference tetrahedron; we use the symbols \(\gamma, \gamma', \gamma''\) for these. Each of these triangles is the image by a unique \(SO(3)\) rotation in the space \((e_1, e_2, e_3)\) of one of the reference triangles \(\rho, \alpha, \sigma\), and \(\gamma''\) lying in the plane \((e_1, e_2)\) and specified by \(\mathfrak{m}\). We denote as \(k_{\rho}, k_{\alpha}, k_{\sigma}\) such \(SO(3)\) rotations:

\[ k_{\rho}^{\mathfrak{m}} = \rho(\mathfrak{m}), \quad k_{\alpha}^{\mathfrak{m}} = \alpha(\mathfrak{m}), \quad k_{\sigma}^{\mathfrak{m}} = \sigma(\mathfrak{m}) \tag{36} \]

As a consequence of the rules \((34)\) and \((38)\), the map \(m_{\mathfrak{m}}\) at any point \(\mathfrak{m} = g\mathfrak{m}'\) of \(Q_{np}\) can always be expressed as:

\[ m_{g\mathfrak{m}} = (\Phi_{\mathcal{P}}^{k_{\rho}^{\mathfrak{m}}} \otimes \Phi_{\mathcal{O}}^{k_{\alpha}^{\mathfrak{m}}} m_0 (\Phi_{\mathcal{P}}^{k_{\rho}^{g^{-1}}} \otimes \Phi_{\mathcal{O}}^{k_{\alpha}^{g^{-1}}})^{-1} \tag{37} \]

for some linear map \(m_0: V_{\mathcal{P}} \otimes V_{\mathcal{O}} \rightarrow V_{\mathcal{K}} \otimes V_{\mathcal{S}}\) depending on the labels. Note that the map \(m_0\) is independent of our choice \((35)\) of positively oriented tetrahedron in the space \((e_1, e_2, e_3)\). In fact, upon a rotation \(\varphi \rightarrow u\varphi\) of the reference tetrahedron by \(u \in SO(3)\), the \(SO(3)\) rotations \(k_\varphi\) defined in \((30)\) simply change as \(k_\varphi \rightarrow k_\varphi^u = uk_\varphi\). Hence such a rotation of the reference tetrahedron simply corresponds to a shift \(g \rightarrow gu\) in \((37)\).

When the 1-intertwiners on the four triangles are all irreducible, labeled by \(s_i \in \mathbb{Z}, i = 1, \ldots, 4\), all bundles are line bundles and the maps \((38)\) act by multiplication by a complex number \(m_\mathfrak{m} \in \mathbb{C}\). In this case, the map \(m_0\) showing up in \((37)\) only contributes to a normalization factor, which we set to one:

\[ m_0 = 1 \tag{38} \]

Given such choices of a reference tetrahedron and normalization condition, the set of labels on the edges and triangles of a tetrahedron uniquely specifies a 2-intertwiner \((39)\) via the formula \((37)\).

Note that our normalized 2-intertwiner depends on the orientation (here chosen to be positive) of the reference tetrahedron in the space \((e_1, e_2, e_3)\). In fact, a flip of the orientation by means of the \(U(1)\) rotation \(\mathfrak{s} \rightarrow h_\mathfrak{s} \mathfrak{s}'\) of angle \(\pi\) around the plane \((e_1, e_2)\), induces the change \(k_\mathfrak{s} \rightarrow h_\mathfrak{k} k_\mathfrak{s} h_\mathfrak{k}\) of the rotations \((36)\). Using the rule \((10)\), this in turn leads to the rescaling \(m_{\mathfrak{m}} \rightarrow (-1)^{\Sigma_i s_i} m_{\mathfrak{m}}\) of the 2-intertwiner.

We also introduce the dual 2-intertwiner between the two composite 1-intertwiners:

\[ l \quad m \]
\[ r \quad q \quad \text{and} \quad l \quad m \]
\[ r \quad q \]

\[ \downarrow \]
\[ \downarrow \]

\[ \downarrow \]
\[ \downarrow \]

\[ \downarrow \]
\[ \downarrow \]

This is the unique 2-intertwiner whose maps

\[ \tilde{m}_{\mathfrak{m}}: V_{\mathcal{K}} \otimes V_{\mathcal{S}} \rightarrow V_{\mathcal{P}} \otimes V_{\mathcal{O}} \tag{39} \]

satisfy the normalization condition \((38)\) for the flipped reference tetrahedron \(\mathfrak{s}' = h_\mathfrak{s} \mathfrak{s}'\). The normalization is chosen in such a way that the maps \(m_{\mathfrak{m}}\) of the 2-intertwiner and those \(\tilde{m}_{\mathfrak{m}}\) of its dual satisfy the identity:

\[ (-1)^{\Sigma_i s_i} \text{Tr}[\tilde{m}_{\mathfrak{m}} m_{\mathfrak{m}}] = 1 \tag{40} \]

where \(\text{Tr}\) is the trace in \(V_{\mathcal{K}} \otimes V_{\mathcal{S}}\) and the sum is over the four spins \(s_i \in \mathbb{Z}\) labeling the irreducible 1-intertwiners.

Just as 1-intertwiners define \(SO(4)\) representations, 2-intertwiners define \(SO(4)\) intertwining operators. To see this, let the labels \(n, p \in \mathbb{R}_+\) on the dashed edges of the above diagrams run in the range of values allowed by the triangular inequalities. The union of the bundles over all subsets \(Q_n^-\) and \(Q_p^+ \subset Q\) give two \(SO(4)\) bundles over \(Q\). Consider the map

\[ (\varphi) \mapsto (m_{\mathfrak{m}}(\varphi)) \tag{41} \]
from sections of one bundle to sections of the other. In this formula, if $\square$ is the quadrangle $(x,y,v,w)$, $\star$ symbolizes the corresponding element in the subset $Q_{np}$ with $n = |x + y|$ and $p = |y + v|$. The rule (34) says that defines an intertwiner between the two SO(4) representations $(23)$ and $(29)$. Restricting to $L^2$ sections with respect to the measure (33), we obtain a map

$$M : \int d\mu_Q(\square) V_\square \otimes V_\Delta \to \int d\mu_Q(\square) V_\triangle \otimes V_\gamma$$

(42)

This map is the direct integral of the maps (33). When the 1-intertwiners on the four triangles are all irreducibles, $M$ is the diagonal operator on the space $L^2(Q, \mu_Q)$ of square integrable functions on $Q$ acting by multiplication by the function $\square \mapsto m_\square$.

These constructions can be extended to 2-intertwiners between more general composite 1-intertwiners. For example, if we supplement the six representations and the four 1-intertwiners considered above with two irreducible representations $s,t \in \mathbb{R}_+$ and a 1-intertwiner $(V_\Psi, \Phi_\Psi)$ between $r \otimes s$ and $t$, we may consider the 2-intertwiner drawn as:

\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\node (s) at (0,0) {$s$};
\node (t) at (1,0) {$t$};
\node (l) at (0.5,1) {$l$};
\node (m) at (0.5,0.5) {$m$};
\node (n) at (0.5,-0.5) {$n$};
\node (p) at (0.5,-1) {$p$};
\node (q) at (1,1) {$q$};
\draw (s) -- (t);
\draw (l) -- (m);
\draw (m) -- (p);
\draw (p) -- (q);
\end{tikzpicture}
\end{array}
\end{align*}

\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\node (s) at (0,0) {$s$};
\node (t) at (1,0) {$t$};
\node (l) at (0.5,1) {$l$};
\node (m) at (0.5,0.5) {$m$};
\node (n) at (0.5,-0.5) {$n$};
\node (p) at (0.5,-1) {$p$};
\node (q) at (1,1) {$q$};
\draw[dashed] (s) -- (t);
\draw[dashed] (l) -- (m);
\draw[dashed] (m) -- (p);
\draw[dashed] (p) -- (q);
\end{tikzpicture}
\end{array}
\end{align*}

defined by taking the tensor product of the maps (33) with identity maps $1_\Psi : V_\Psi \to V_\Psi$:

$$m_\square \otimes 1_\Psi : (V_\Psi \otimes V_\Delta) \otimes V_\Psi \to (V_\triangle \otimes V_\gamma) \otimes V_\Psi$$

(43)

Letting the labels on the dashed lines edges run within the range of values allowed by the triangular inequalities, we obtain a map

$$\int \mu_{\mathcal{P}}(\square) V_\Psi \otimes V_\Delta \otimes V_\Psi \to \int \mu_{\mathcal{P}}(\square) V_\triangle \otimes V_\gamma$$

that intertwines two SO(4) representations on $L^2$ sections of bundles over the set $\mathcal{P}$ of pentagons of lengths $l,m,q,s,t$ in $\mathbb{R}^4$, endowed with the measure (31). We used our symbolic notation where $\square$ denotes a generic pentagon $(x,y,v,w,z)$ in $\mathcal{P}$; and $\star$ denotes the corresponding quadrangle $(x,y,v,x+y+v)$ in $Q_{np}$ with $|x+y|=n$ and $|y+v|=p$. When the 1-intertwiners are all irreducibles, the vectors spaces are all one dimensional and the maps (13) act by multiplication by a complex number $m_\square \in \mathbb{C}$. In this case, the SO(4) intertwiner is the diagonal operator on the space $L^2(\mathcal{P}, \mu_{\mathcal{P}})$ of square integrable functions on $\mathcal{P}$ acting by multiplication by the function $\square \mapsto m_\square$.

III. 10j 2-SYMBOLS

We are now in a position to define and compute the weight that the state sum associates to each 4-simplex of the triangulation. This weight, which we refer to as ‘10j 2-symbol’, is a function of ten positive numbers labeling the irreducible representations of the Euclidean 2-group, and ten integer spins labeling irreducible 1-intertwiners. As we have seen in the previous section, each set of such labels determines one normalized 2-intertwiner for each of the five boundary tetrahedra of the 4-simplex.

A. Definition

10j 2-symbols will be defined by taking an appropriate trace of the product of five normalized 2-intertwiners, depicted by the following diagram:

\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\node (s) at (0,0) {$s$};
\node (t) at (1,0) {$t$};
\node (l) at (0.5,1) {$l$};
\node (m) at (0.5,0.5) {$m$};
\node (n) at (0.5,-0.5) {$n$};
\node (p) at (0.5,-1) {$p$};
\node (q) at (1,1) {$q$};
\draw (s) -- (t);
\draw (l) -- (m);
\draw (m) -- (p);
\draw (p) -- (q);
\end{tikzpicture}
\end{array}
\end{align*}

\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\node (s) at (0,0) {$s$};
\node (t) at (1,0) {$t$};
\node (l) at (0.5,1) {$l$};
\node (m) at (0.5,0.5) {$m$};
\node (n) at (0.5,-0.5) {$n$};
\node (p) at (0.5,-1) {$p$};
\node (q) at (1,1) {$q$};
\draw[dashed] (s) -- (t);
\draw[dashed] (l) -- (m);
\draw[dashed] (m) -- (p);
\draw[dashed] (p) -- (q);
\end{tikzpicture}
\end{array}
\end{align*}

The details of the construction are as follows. Consider the standard 4-simplex [12345], endowed with the standard orientation defined by the ordering $\{1, \cdots, 5\}$ of its vertices. The 4-simplex has boundary

$$[2345] - [1345] + [1245] - [1235] + [1234]$$

(44)

where $[jklm]$ is the tetrahedron with vertices $\{j,k,l,m\}$. The sign indicates the induced orientation of the 4-simplex boundary: for the tetrahedron $i := [jklm]$ opposite to the vertex $i$, the boundary orientation is defined as the ordering of its vertices in an even permutation of $(12345)$ where $i$ appears first. Each boundary tetrahedron shows up in the sequence (11) with a (+) or a (-) sign depending on whether or not the numerical order agrees with the boundary orientation.

There is an irreducible representation for each edge $(ij)$, labeled by $l_{ij} \in \mathbb{R}_+$. The edges drawn as dashed lines on the diagram are coined ‘interior edges’, those drawn as plain lines are coined ‘exterior edges’. For each triangle $ijk$ with vertices $i < j < k$, there is an irreducible 1-intertwiner $l_{ij} \otimes l_{jk} \to l_{ik}$, labeled by a spin $s_{ijk} \in \mathbb{Z}$. For each tetrahedron $i := [jklm]$ with vertices $j < k < l < m$, there is a normalized 2-intertwiner $m_{ijklm}$ indexed by the
Recalling the definitions of the previous sections, \( P \) is the set of pentagons \((x_{ij})\) of lengths \( l_{ij} \) in \( \mathbb{R}^4 \), where \((ij) \in \{(12), (23), (34), (45), (15)\} \) runs over the five exterior edges. \( P \) is endowed with the measure \( d\mu_P \) given by the formula (51):

\[
d\mu_P = \prod_{\text{ext.}(ij)} d^3\ell_{ij} x_{ij} \delta^4(x_{12} + \ldots + x_{45} - x_{15})
\]

where \( d^3\ell_{ij} x_{ij} \) is the spherical measure \( [4] \) on the 3-sphere of radius \( l_{ij} \); the delta function imposes the closure of the pentagons. Letting the labels on the interior edges run within the range of values allowed by the triangular inequalities, each of the five pentagonal figures of the diagram gives an SO(4) representation on \( L^2 \) sections of a vector bundle over \( P \), defined as in \([30]\); and each double arrow gives a SO(4) intertwiner between these, defined as in \([13]\). There is thus one such SO(4) intertwiner \( M_i \) for each tetrahedron \( i := [ijklm] \). We also denote by \( M_i \) the SO(4) intertwiner determined by the dual 2-intertwiner.

We consider the product

\[
M = M_2 M_3 M_4 M_5 M_6
\]

of the five SO(4) intertwiners. The use of dual 2-intertwiners is dictated by the orientation of the tetrahedron: for each \( i \), the product involves either \( M_i \) or the dual \( \bar{M}_i \), whether the tetrahedron shows up in \([13]\) with a (+) or a (-) sign, that is, whether \((ijklm)\) where \( j<k<l<m \) is an even or an odd permutation of (12345). By construction, \( M \) is the direct integral over \( P \) of maps \( M_{Q} \) indexed by elements \( Q \in P \) and given by:

\[
(1 \otimes \bar{m}_2^2, m_3^2 \otimes 1)(1 \otimes m_1^1, \bar{m}_4^3 \otimes 1)(1 \otimes m_5^5, \bar{m}_6^6 \otimes 1)
\]

where \( \bar{m}_2^2 \) denotes the unique quadrangle (tetrahedron) in \( \mathbb{R}^4 \) that the point \( Q \in P \) associates to \( i = [ijklm] \).

10j 2-symbols are defined by means of the identity:

\[
\text{Tr} M = \kappa \int \prod_{\text{int.}(ij)} d^2\ell_{ij} \left\{ \ell_{e_1} \ldots \ell_{e_{10}} \right\} \delta_{+} s_{t_1} \ldots \delta_{+} s_{t_{10}}
\]

where the integral is over the labels on the interior edges and \( \kappa \) is an overall constant, which will later be chosen to be \( \kappa = \pi^{4/26} \) for practical convenience. The 2-symbol, which we wrote here using brackets, depends on the labels \( e_j \) on the ten edges \( e_j \) and the labels \( s_{t_j} \) on the ten triangles \( t_j \) of the 4-simplex. \( \text{Tr} M \) denotes the trace of \( M \), where the trace of a direct integral is defined as the integral of the trace:

\[
\text{Tr} M := \int_P d\mu_P(Q) / \text{Tr} M_Q
\]

The formula (18) should be understood as an identity of measures, as follows. Writing \( P = \bigsqcup P_t \) as the disjoint union of subsets \( P_t \) of pentagons with given interior edge lengths \( \ell = (l_{ij}) \), the identity (18) says that upon the decomposition

\[
\mu_P = \kappa \int \prod_{\text{int.}(ij)} d^2\ell_{ij} \mu_{P_t}
\]

of the measure \( \mu_P \) into measures \( \mu_{P_t} \) on \( P_t \), the trace (49) decomposes into 10j 2-symbols. Hence these symbols are explicitly given by the formula:

\[
\left\{ \ell_{e_1} \ldots \ell_{e_{10}} \right\} := \int_{P_t} d\mu_{P_t}(Q) \bar{m}_2^2 m_3^2 m_4^3 m_5^5 m_6^6
\]

where the diagram is the graphical representation of the trace \( \text{Tr} M_Q \) of the map (47). The integration is over the subset \( P_t \subset P \) of pentagons whose edge lengths match with the labels \( l_{ij} \). Note that each of such pentagons defines a Euclidean 4-simplex embedded in \( \mathbb{R}^4 \).

### B. Explicit computation

The goal of the remainder of this section is to evaluate the integral in the right-hand side of (51). This will enable us to write the 10j 2-symbol as an explicit function of the labels. The result is the formula (71) below.

1. The measure

First, combining Equ. (45), (50) and (11) gives the expression of the measure on \( P_t \) in terms of the Lebesgue measure on \( \mathbb{R}^4 \). Solving the delta function in (15) by integrating over \( x_{15} \) yields

\[
d\mu_{P_t} = \frac{1}{\kappa \pi^{2}} \prod_{i=1}^{4} d^2 x_{i+1} \prod_{i<j} \delta(|x_{ij}|^2 - l_{ij}^2)
\]

where we introduced the vectors \( x_{ij} := x_{i+1} + \ldots + x_{j-1} \). Each point of \( P_t \) corresponds to a 4-simplex embedded in \( \mathbb{R}^4 \), with edge-vectors \( x_{ij} \). Upon the action of SO(4), \( P_t \) has two orbits, labeled by the orientation \( \eta = \pm 1 \) of the 4-simplices in \( \mathbb{R}^4 \). The measure \( d\mu_{P_t} \) being SO(4) invariant, it is thus equivalent to the product \( d\bar{g} \sum_{\eta = \pm 1} \) of a Haar measure on SO(4) and the counting measure on the set of orbits. The resulting Jacobian has been computed explicitly in [22]. For the value \( \kappa = \pi^{8/26} \), it leads to the identity:

\[
d\mu_{P_t} = \frac{1}{V(l_{ij})} d\bar{g} \sum_{\eta = \pm 1}
\]

where \( d\bar{g} \) is the normalized Haar measure on SO(4) and \( V(l_{ij}) \) is \( 4! \) times the volume of a Euclidean 4-simplex with edge lengths \( l_{ij} \).
2. The trace

The integrand in \( [51] \) is the trace \( \text{Tr} M_\mathcal{O} \) of the map \( [47] \). By writing each of the five 2-intertwiners in the form \( [57] \), we obtain the formula:

\[
\text{Tr} M_\mathcal{O} = \langle \bigotimes_{i} m_{i}^{\perp} | \bigotimes_{\Delta=\langle ij \rangle} \left( \Phi_{g_{i}k_{i}^{\triangle}} \right)^{-1} \Phi_{g_{j}k_{j}^{\triangle}} \rangle \tag{54}
\]
whose right-hand side is the complex number obtained by tracing out the 1-intertwiner maps \( \Phi_{\mathcal{O}}^{\triangle} \) with the maps \( m_{i}^{\perp} \) showing up in the decomposition \( [57] \) of the 2-intertwiners. Our notation is as follows: given distinct \( i, j, k, l, m \) with \( k < l < m \), \( \Delta = \langle ij \rangle \) represents the triangle \( \{k | l | m \} \) common to the two tetrahedra \( i \) and \( j \); moreover the pair \( \langle ij \rangle \) is ordered by requiring that \( \langle ijklm \rangle \) is an even permutation of \( (12345) \). The group elements \( g_{i} \) and \( k_{i}^{\triangle} \) are respectively the SO(4) and SO(3) rotations defined by:

\[
a_{i} = g_{i}a_{i}^{\prime}, \quad k_{i}^{\Delta}a_{i}^{\prime} = \delta(\mathcal{O}) \tag{55}
\]
when the tetrahedron \( i \) shows up in \( [44] \) with a (+) sign \((i = 1, 3, 5)\), and

\[
a_{i} = g_{i}a_{i}^{\prime}, \quad k_{i}^{\Delta}a_{i}^{\prime} = -\delta(\mathcal{O}) \tag{56}
\]
when \( i \) shows up in \( [44] \) with a (-) sign \((i = 2, 4)\).

We used the symbolic notation introduced in the previous section: \( \mathcal{O} \) denotes the unique tetrahedron in \( \mathbb{R}^{4} \) that is the point \( \mathcal{O} \in \mathcal{P} \) associated to \( i = [jk|lm] \); \( a_{i}^{\prime} \) is the reference tetrahedron \( [53] \) used to normalize 2-intertwiners; \( a_{i}^{\prime} = h_{\mathcal{O}}a_{i} \), where \( h_{\mathcal{O}} \) is the rotation of angle \( \pi \) around the plane \( \langle e_{1}, e_{2} \rangle \), is the (flipped) reference tetrahedron used to normalize dual 2-intertwiners. Given a triangle \( \Delta \) and a tetrahedron \( i \) adjacent to it, \( \Delta^{\prime}, \delta(\mathcal{O}) \) (or \( \delta(\mathcal{O}) \)) denote respectively the reference triangle \( [9] \) and the triangle that \( a_{i}^{\prime} \) (or \( a_{i}^{\prime} \)) associated to \( \Delta \) in \( \mathbb{R}^{4} \). Note that, written in the form \( [54] \), the trace of \( M_\mathcal{O} \) depends on the point \( \mathcal{O} \in \mathcal{P} \) only through \( g_{i} := g_{i}(\mathcal{O}) \).

Since in our case the 1-intertwiners are irreducible, all vectors space are one-dimensional and all maps act by multiplication by a complex number. In this case, using the normalization \( [58] \) for the 2-intertwiners, the trace reduces to the following product:

\[
\text{Tr} M_\mathcal{O} = \prod_{\Delta=\langle ij \rangle} \left( \Phi_{g_{i}k_{i}^{\triangle}} \right)^{-1} \Phi_{g_{j}k_{j}^{\triangle}} \tag{57}
\]
over the ten triangles \( \Delta = \langle ij \rangle \) of the 4-simplex.

The next step is to observe that the contribution of the 1-intertwiner map corresponding to each triangle \( \Delta = \langle ij \rangle \) reduces to a phase \( e^{i\mathcal{O}}\mathcal{O}_{\mathcal{O}} \), where \( \mathcal{O}_{\mathcal{O}} \in \mathbb{Z} \) is the spin label on \( \Delta \), and \( \mathcal{O}_{\mathcal{O}} \) is some angle in \([0, 2\pi]\). Indeed, it is clear from the definitions \( [55] \) and \( [50] \) that the image of the reference triangle \( \Delta^{\prime} \) by the rotation \( g_{i}k_{i}^{\triangle} \) coincides with the triangle \( \delta(\mathcal{O}) \) that \( g_{i} \) associates to \( \Delta \) in \( \mathbb{R}^{4} \). Since this triangle is common to \( i \) and \( j \), we have that \( \Delta(\mathcal{O}) = \Delta(\mathcal{O}) \).

This shows that the rotation \( (g_{j}k_{j}^{\triangle})^{-1}g_{i}k_{i}^{\triangle} \) stabilizes \( \mathcal{O} \), hence belongs to \( U(1) \):

\[
(g_{j}k_{j}^{\triangle})^{-1}g_{i}k_{i}^{\triangle} := h_{\mathcal{O}} \in U(1) \tag{58}
\]
for some \( \mathcal{O}_{\mathcal{O}} \in [0, 2\pi] \). Together with the relations \( [7] \) and \( [11] \), it yields the formula:

\[
\left( \Phi_{g_{i}k_{i}^{\triangle}} \right)^{-1} \Phi_{g_{j}k_{j}^{\triangle}} = \Phi_{(g_{i}k_{i}^{\triangle})^{-1}g_{i}k_{i}^{\triangle}} = e^{i\mathcal{O}}\mathcal{O}_{\mathcal{O}} \tag{59}
\]

Note also that the rotations \( h_{\mathcal{O}} := h_{\mathcal{O}}(\mathcal{O}) \) are all invariant under the SO(4) action \( (g, \mathcal{O}) \rightarrow g\mathcal{O} \) on \( \mathcal{P}_{\mathcal{O}} \). This means that, upon integration over \( \mathcal{P}_{\mathcal{O}} \) with the measure \( [54] \), the angles \( \mathcal{O}_{\mathcal{O}} \), and hence the integrand \( [51] \), depend on the point \( \mathcal{O} \in \mathcal{P}_{\mathcal{O}} \) only through the orientation \( \eta = \pm 1 \) of the corresponding 4-simplex in \( \mathbb{R}^{4} \). Hence, performing the integral of \( [57] \) with respect to the measure \( [53] \) gives the quantity:

\[
\frac{1}{V(i_{ij})} \sum_{\mathcal{O} = \pm 1} \prod_{\Delta=\langle ij \rangle} e^{i\mathcal{O}_{\mathcal{O}}(\mathcal{O})} \tag{60}
\]

3. Relation to dihedral angles

The last step is to relate the angles \( \mathcal{O}_{\mathcal{O}} \) to the dihedral angles of the 4-simplex. The (interior) dihedral angle \( \phi_{ij} \in [0, \pi] \) between the two tetrahedra sharing the triangle \( \Delta = \langle ij \rangle \) is defined as \( (\pi \text{ minus}) \) the angle between their outwards unit normal vectors \( n_{i} \) and \( n_{j} \):

\[
\cos \phi_{ij} = -n_{i} \cdot n_{j} \tag{61}
\]

It is clear that the two angles \( \mathcal{O}_{\mathcal{O}} \) and \( \phi_{ij} \) are closely related. For example we may note that, since by construction \( k_{i}^{\triangle} \) leaves \( e_{4} \) invariant and \( g_{i} \) maps it to a vector normal to the tetrahedron \( i \), the image of \( e_{4} \) by \( g_{i}k_{i}^{\triangle} \) must coincide with \( \pm n_{i} \). This means the scalar product \( e_{4} \cdot h_{\mathcal{O}} e_{4} \) equals \( n_{i} \cdot n_{j} \) modulo a sign, and thus

\[
|\cos \mathcal{O}_{\mathcal{O}}| = |\cos \phi_{ij}| \tag{62}
\]
which says \( \mathcal{O}_{\mathcal{O}} = \pm \phi_{ij} \) or \( \pi \pm \phi_{ij} \).

The exact relation is the following: let \( \Delta = \langle ij \rangle \) represent the triangle \( \{k | l | m \} \) common to the tetrahedra \( i \) and \( j \), where \( k < l < m \) and \( \langle ijk \rangle \) is an even permutation of \( (12345) \). The angle \( \mathcal{O}_{\mathcal{O}} \) of the rotation \( [58] \) is given by

\[
\mathcal{O}_{\mathcal{O}} = \pi + \eta \phi_{ij} \tag{63}
\]
where \( \eta \) is the orientation of the 4-simplex in \( \mathbb{R}^{4} \), that is, if \( (x_{ij})_{i<j} \) are the edge vectors, the sign of the determinant \( \det(x_{12}, \cdots, x_{15}) \).

There are various ways to prove the relation \( [63] \). An elegant algebraic proof relies on the following lemma. Let \( \mathcal{O}_{ij} \) be the rotation of angle \( \theta_{ij} \) around the plane \( \langle e_{1}, e_{4} \rangle \), where \( \theta_{ij} \in [0, \pi] \) is the 3-dimensional dihedral angle between the faces \( [ijk] \) and \( [ijl] \) in the tetrahedron \( \tilde{m} \).
Lemma 1: Given any permutation \((ijklm)\) of \((12345)\), the following equation for the triple of angles \((\xi_{ijk}, \xi_{ijl}, \xi_{ijm})\) in \([0, 2\pi]^3\):

\[
h_{\xi_{ijk}} a_{ijkl} h_{\xi_{ijl}} a_{ijlm} h_{\xi_{ijm}} a_{ijkl} = 1
\]

has exactly two solutions \(\xi_{ijk}^\pm\) given by:

\[
\xi_{ijk}^\pm = \pi \pm \phi_{ijk}
\]

where \(\phi_{ijk}\) is the dihedral angle between the two tetrahedra sharing the triangle \([ijk]\).

There is one such identity satisfied by the dihedral angles \(\xi_{ijk}\) for each edge \((ij)\) of the 4-simplex. These identities can be understood as vanishing curvature conditions around the edges of the 4-simplex.

We refer to the Appendix B of [22] for the proof of a 3-dimensional analogue of this Lemma. One dimension down, the analogues of the equations \((63)\) hold in SO(3) and are associated to the vertices of a tetrahedron \([ijklm]\) in \(\mathbb{R}^3\). In the usual basis \((e_1, e_2, e_3, e_4)\) of \(\mathbb{R}^3\), they take the form:

\[
h_{\xi_{ijkl}} a_{ijkl} h_{\xi_{ijlm}} a_{ijlm} h_{\xi_{ijm}} a_{ijkl} = 1
\]

where \(h_{\xi}\) is the rotation of axis \(e_2\) and angle \(\xi\), and \(a_{ijkl}^k\) is the rotation of axis \(e_4\) and angle \(\theta_{ijkl}^k\), where \(\theta_{ijkl}^k\) is the angle between the edges \((ik)\) and \((il)\). These equations have two sets of solutions \(\xi_{ik}^\pm \in [0, 2\pi]\) related to the 3-dimensional dihedral angles \(\phi_{ik}\) as:

\[
\xi_{ik}^\pm = \pi \pm \phi_{ik}
\]

We can summarize the correspondence between the 3d and 4d cases as follows:

| 4d | 3d |
|-----|-----|
| basis \((e_1, e_2, e_3, e_4)\) | basis \((e_2, e_3, e_4)\) |
| 4-simplex \([ijklm]\) | tetrahedron \([iklm]\) |
| \(a_0\) around \((e_1, e_4)\)-plane | \(a_0\) around axis \(e_4\) |
| \(h_{\phi}\) around \((e_1, e_2)\)-plane | \(h_{\phi}\) around axis \(e_2\) |
| 3d dihedral angles \(\theta_{ijkl}^k\) | 2d angles \(\theta_{ijkl}^k\) |
| 4d dihedral angles \(\phi_{ijkl}\) | 3d dihedral angles \(\phi_{ik}\) |
| SO(4) edge identities | SO(3) vertex identities |

The proof of Lemma 1 can be inferred from its 3-dimensional analogue, by considering the orthogonal projection \(P: \mathbb{R}^4 \rightarrow \mathbb{R}^3\) onto the 3d space \((e_2, e_3, e_4)\) orthogonal to the basis vector \(e_1\). Assuming the vertex \(i\) of the 4-simplex \([ijklm]\) is at the origin and the edge \((ij)\) is along \(e_1\), \(P\) maps the 4-simplex to a tetrahedron \([iP(k)P(l)P(m)]\). The key observation is that the 3d dihedral angle \(\theta_{ijkl}^k\) between the faces \([ijk]\) and \([ijl]\) of the 4-simplex equals the 2d angle \(\theta_{ijkl}^k\) between the edges \((iP(k))\) and \((iP(l))\) of the image tetrahedron; and the 4d dihedral angle \(\phi_{ijkl}\) equals the 3d dihedral angle \(\phi_{ik}\) between the faces \([iP(k)P(l)]\) and \([iP(k)P(m)]\) of the image tetrahedron. In short, the projection \(P\) maps the left column of the above table to the right one. Using this correspondence, the Lemma 1 can be immediately deduced from the 3-dimensional result \((67)\).

To see why the angles \(\xi_{ijk}\) satisfy the relations \((64)\) of the Lemma, pick an edge \((ij)\) of the 4-simplex and let \((ijklm)\) be an even permutation of \((12345)\). The triangle common to the tetrahedra \(\hat{l}\) and \(\hat{m}\) is represented by the symbol \(\Delta = \langle \hat{l}\hat{m}\rangle\) (resp. \(\Delta = \langle \hat{m}\hat{l}\rangle\)) if \(i, j, k\) are numerically ordered as \(\sigma(i) < \sigma(j) < \sigma(k)\) by an even (resp. odd) permutation \(\sigma\). The reference triangle \(\delta\), whose edge vector \(x_{\sigma(i)\sigma(j)\sigma(k)}\) is along \(e_1\) and whose edge vector \(x_{\sigma(i)\sigma(j)\sigma(k)}\) is in the plane \((e_1, e_2)\) and points in the direction of \(e_2\), is the image by some SO(3) rotation \(\sigma_{ijk}\) of the triangle in \(\mathbb{R}^3\) whose edge vector \(x_{ijk}\) is along \(e_1\) and whose edge vector \(x_{ijk}\) is in the plane \((e_1, e_2)\) and points in the direction of \(e_2\). For example if \(j < k < i\), i.e. \(i\) \(\sigma\) simply swaps \(i\) and \(j\), the action of \(\sigma_{ijk}\) can be drawn as:

Note that if \(\sigma\) is even, the two triangles have the same orientation in the plane \((e_1, e_2)\) and \(\sigma_{ijk}\) leaves the whole plane \((e_3, e_4)\) invariant. If \(\sigma\) is odd, the two triangles have opposite orientations and \(\sigma_{ijk} = \bar{\sigma}_{ijk}a_\pi\), where \(\bar{\sigma}_{ijk}\) leaves the whole plane \((e_3, e_4)\) invariant and \(a_\pi\) is the rotation of angle \(\pi\) around the plane \((e_1, e_4)\).

If we then let \(k_{ijk}^\pm \coloneqq k_{ijk}^\pm\) the rotations:

\[
h_{\xi_{ijk}} a_{ijkl}^{-1} g_{\hat{m}}^{-1} g_{\hat{n}}^{-1} g_{\hat{m}} g_{\hat{n}}^{-1} g_{\hat{n}}^{-1} g_{\hat{m}}^{-1} a_{ijkl} = 1
\]

coincide with those defined in \((68)\). This is clear when \(\sigma\) is even: in this case \(\Delta = \langle \hat{l}\hat{m}\rangle\) and \(\sigma_{ijk}\) commutes with all U(1) elements. When \(\sigma\) is odd, then \(\Delta = \langle \hat{m}\hat{l}\rangle\) and \(\sigma_{ijk} = \bar{\sigma}_{ijk}a_\pi\) where \(\bar{\sigma}_{ijk}\) commutes with using all U(1) elements, and we can conclude by using the equality \(a_\pi h_{\chi}^{-1} a_\pi = h_{\chi}\). Observe also that the rotations:

\[
a_{ijkl}^k = a_{ijkl}^{-1} g_{\hat{m}}^{-1} g_{\hat{n}}^{-1} g_{\hat{m}} g_{\hat{n}}^{-1} g_{\hat{n}}^{-1} g_{\hat{m}}^{-1} a_{ijkl}
\]

are those of the Lemma, of angle \(\theta_{ijkl}^k\) around the plane \((e_1, e_4)\). To see this, let the edge vector \(x_{ij}\) be along \(e_1\), the edge vector \(x_{ij}\) be in the plane \((e_1, e_2)\) and point in the direction of \(e_2\), and the tetrahedron \(\hat{n}\) be in the hyperplane \((e_1, e_2, e_3)\) with the orientation of its reference tetrahedron; since \((ijklm)\) is an even permutation of \((12345)\), this means that \(x_{jk}\) points in the direction opposite to \(e_3\). The action of \(a_{ijkl}^k\) places the edge vector
The rotations \( \mathbf{R} \) are thus solutions of the equation (68). Applying the lemma yields \( \xi_\Delta = \pi + \epsilon \omega_0 \) for some \( \epsilon = \pm 1 \) which does not depend on the triangle \( \Delta \). It is then straightforward to relate the sign \( \epsilon \) to the orientation \( \eta \) of the 4-simplex in \( \mathbb{R}^4 \) and reach the conclusion (63).

Plugging the result (63) into the formula (60) and summing over the orientation label gives the final expression for the 2-10j symbol:

\[
\left\{ \frac{1}{2} \sum_{s_t} \int_{\phi_t} \right\} = (-1)^{\sum_{s_t}^e} \frac{\cos \sum_{s_t}^e \phi_t(l_e)}{V(l_e)}
\]

(70)

where both sums on the right hand side are over the ten triangles of the 4-simplex; and \( \phi_t \) is the dihedral angle between the two tetrahedra sharing the triangle \( t \).

### C. State sum model

Let \( \Delta \) be a triangulated orientable 4-manifold. With any assignment of an irreducible representation \( l_e \in \mathbb{R}^+ \) of the Euclidean 2-group to each edge \( e \) and an irreducible 1-intertwiner \( s_t \in \mathbb{Z} \) to each triangle \( t \) of \( \Delta \), the model associates a weight \( W_\Delta(l_e, s_t) \in \mathbb{R} \) given by the formula:

\[
W_\Delta = \prod_t 2A_t(l_e) \prod_{\sigma} \left\{ \frac{l_{e_{i_1}} \cdots l_{e_{i_{10}}}}{s_t_{i_1} \cdots s_t_{i_{10}}} \right\}
\]

(71)

The products are over the triangles \( t \), the tetrahedra \( t \) and the 4-simplices \( \sigma \) of the manifold. In the case of a manifold with boundary, the triangles and tetrahedra are those in the interior, i.e. which do not lie on the boundary.

The weight for each triangle corresponds to the volume of the measures (44): it is equal to twice the area of the triangle with edge lengths \( l_e \) when the triangular inequality is satisfied, and zero otherwise. The weight for each tetrahedron corresponds to the normalization (10) of the 2-intertwiners: in general it is defined to be \( 1/\text{Tr}[\bar{\eta}_0 \epsilon^{1/2}] \), which ensures that \( W_\Delta \) is independent of the normalization choice for an orientable manifold. For the normalization (68), it reduces to a sign factor \( (-1)^{\sum_{s_t}^e} \), where the sum is over the four triangles of the tetrahedron. These signs can be absorbed into the 4-simplex weight, giving a factor \( i^2 \sum_{s_t}^e \) which compensates the signs showing up in (72). Hence the formula agrees with (11) for a closed manifold; in general the two formulas agree up to a global sign depending only on the boundary data.

The partition function is formally defined by summing up these weights over all values of the labels, using the Lebesgue measures \( dl_e^2 \) for the real variables \( l_e \in \mathbb{R}^+ \) and the counting measure \( \sum_{s_t} \) for the integer variables \( s_t \in \mathbb{Z} \). Thus the 2-categorical state sum model constructed here is the same as the model of (1).

The physical interpretation of the model is best understood by writing the weight \( W_\Delta \) in terms of the exponential of a discrete classical action, using the expression (60) of the 10j symbols. This action, which depends on the labels \( l_e, s_t \) and an orientation \( \eta_\sigma = \pm 1 \) for each 4-simplex, reads

\[
S_\Delta(l_e, s_t; \eta_\sigma) = \sum_t s_t \omega_t(l_e, \eta_\sigma)
\]

(72)

where the sum is over all the triangles \( t \). If we regard the edge labels \( \{ l_e \} \) as defining a discrete geometry in the sense of Regge calculus, the functions \( \omega_t(l_e, \eta_\sigma) \), defined as \( \omega_t = \sum_{\eta_\sigma \in \eta} \eta_\sigma \phi_t^2 \), are the deficit angles associated to the triangles: they represent curvature in this geometry. The equations of motion obtained by varying the action (72) imply that the deficit angles are trivial, i.e. that the geometry is flat.

The topological invariance of the state sum has been discussed in detail in (1). The core of this discussion is an hexagonal identity satisfied by the weight (44), giving a four-dimensional analogue of the Biedenharn-Elliott identity of 6j symbols and an algebraic expression of the four-dimensional Pachner move invariance. Upon a regularization procedure resulting from the gauge-fixing of symmetries of the action (72), it was shown in (1) that the partition function of the model reproduces the formula for the 4-manifold invariant defined by Korepanov in (26–28).

### IV. Outlook

We have developed in explicit detail a state sum model starting from the 2-category of representations of a 2-group. We have defined and computed the 4-simplex weights, which may be viewed as a categorified analogue of Racah-Wigner 6j symbols. The 2-group we considered is a categorified version of the four-dimensional Euclidean group ISO(4): though it is built from the same ingredients as the Euclidean group, it differentiates the roles of rotations and translations by treating the former as objects in a category and the latter as morphisms. As anticipated in (2), the resulting model has a remarkable geometrical flavour, where each set of irreducible representations labeling the edges of the triangulations are interpreted as specifying a Euclidean geometry on the underlying manifold.

Our construction bridges results from several works in the recent literature. First, it gives an explicit realization of the proposal of (2) to use 2-group representations to define new state sum models in four dimensions. Second, as shown in (1), it provides a state sum formulation of the 4-manifold invariant constructed in (26–28). Third, it unravels the algebraic structure underlying the state
sum formulation of the Feynman amplitudes of quantum field theory on flat spacetime proposed in [1].

Our results suggest several interesting avenues for future work. To clarify the physical meaning of the model, an important step will be to identify the corresponding classical field theory (if any). A natural candidate is a higher gauge generalization of BF theory called ‘BFCG theory’, involving flat 2-connections [12]. We expect the model [22] to provide a state sum formulation of such a theory having the Euclidean 2-group as gauge 2-group.

State sum models built from group representations have well-known generalizations obtained by replacing the group by a quantum group. In the case of the Lie group SU(2), one such generalization based on the quantum deformation $U_q(sl_2)$ for $q$ root of unity leads to the Turaev-Viro and Crane-Yetter models, which are finite and produce genuine manifold invariants [3, 4]. It is worth investigating the construction of analogous models in the context of 2-groups, using a suitable notion of quantum 2-group [23]. In the case of the Euclidean 2-group, this may lead to a regularization of the state sum [71] alternative to the one provided by gauge fixing. Moreover, from the point of view of [1], this may enable one to propose and study possible dimension-full deformations of ordinary quantum field theory. This strategy has already revealed particularly useful to understand the coupling of matter fields to three-dimensional quantum gravity in the context of spin foam models [22, 23].

Finally, several works pointed out the possible relevance of higher algebraic structures, and in particular the Poincaré 2-group, for the formulation of a model of quantum gravity in four dimensions [2, 20, 21, 23, 33]. Clarifying this relationship is yet another area for future study.

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[1] A. Baratin, L. Freidel, Hidden quantum gravity in 4D Feynman diagrams: Emergence of spin foams, Class. Quant. Grav. 24 (2007), 2027-2060. arXiv:hep-th/0611042
[2] J. W. Barrett, M. Mackaay, Categorical representations of categorical groups, Th. Appl. Cat. 16 (2006), 529–557. arXiv:math/0407463
[3] M. Fukuma, S. Hosono, H. Kawai, Lattice topological field theory in two-dimensions, Commun. Math. Phys. 161, 157-176 (1994). arXiv:hep-th/9212154
[4] G. Ponzano and T. Regge, Semiclassical limits of Racah coefficients, in Spectroscopic and Group Theoretical Methods in Physics: Racah Memorial Volume, ed. F. Bloch, North-Holland, Amsterdam, 1968, pp. 75–103.
[5] V. Turaev and O. Viro, State sum invariants of 3-manifolds and quantum 6j-symbols, Topology 31 (1992), 865-902.
[6] A. Perez, The Spin Foam Approach to Quantum Gravity. Living Rev. Relativity 16, (2013) 3. arXiv:1205.2019/gr-qc
[7] L. Freidel, K. Krasnov, A new spin foam model for 4d gravity, Class. Quant. Grav. 25, 125018 (2008). arXiv:0708.1559/gr-qc
[8] J. Engle, E. Livine, R. Pereira, C. Rovelli, LQG vertex with finite Immirzi parameter, Nucl. Phys. B 799, 136 (2008). arXiv:0711.0146/gr-qc
[9] J. C. Baez, J. Dolan, Higher dimensional algebra and topological quantum field theory, J. Math. Phys. 36 (1995) 6073-6105. arXiv:q-alg/9503002
[10] J. S. Cartera, L. H. Kauffman, M. Saito, Structures and Diagrammatics of Four Dimensional Topological Lattice Field Theories, Adv. in Math. 146, 1 (1999), 39100. arXiv:math/9806023
[11] J. C. Baez and A. Lauda, A prehistory of n-categorical physics. arXiv:0908.2469
[12] J. W. Barrett, B. W. Westbury, Invariants of piecewise-linear 3-manifolds, Trans. Amer. Math. Soc. 348 (1996), 3997-4022. arXiv:hep-th/9311155
[13] M. Mackaay, Spherical 2-categories and 4-manifold invariants, Adv. Math. 143 (1999), 288-348. arXiv:math/9805030
[14] L. Crane, L. H. Kauffman, D. N. Yetter, State-sum invariants of 4-manifolds, J. Knot Theory Ram. 6 (1997), no. 2, 177234. arXiv:hep-th/9409167
[15] M. Mackaay, Finite groups, spherical 2-categories, and 4- manifold invariants, Adv. Math. 153 (2000), no. 2, 353-390. arXiv:math/9903003
[16] J. C. Baez, J. Huerta, An Invitation to Higher Gauge Theory. arXiv:1003.4485/hep-th
[17] J. C. Baez, Higher Yang-Mills Theory, arXiv:hep-th/0206130
[18] L. Crane, D. N. Yetter, A More Sensitive Lorentzian State Sum, Appl. Cat. Str. 13 (2005), 501-516. arXiv:math/0305176
[19] L. Crane, D. N. Yetter, Measurable categories and 2-groups, Appl. Cat. Struct. 13 (2005), 501-516. arXiv:math/0305176
[20] L. Crane, M. Sheppeard, 2-Categorical Poincaré representations and state sum applications, arXiv:math/0306440
[21] J. C. Baez, A. Baratin, L. Freidel, D. K. Wise, Infinite dimensional representations of 2-groups, Amer. Math. Soc. 219 (2012), No.1032. arXiv:0812.4969/math.QA
[22] A. Baratin, L. Freidel, Hidden Quantum Gravity in 3D Feynman diagrams, Class. Quant. Grav. 24 (2007), 1993-2026. arXiv:gr-qc/0604016
[23] J. W. Barrett, Feynman diagrams coupled to three dimensional quantum gravity, *Class. Quant. Grav.* **23**, 137 (2006). [arXiv:gr-qc/0502048]

[24] L. Freidel, E. Livine, Ponzano-Regge model revisited III: Feynman diagrams and Effective field theory, *Class. Quant. Grav.* **23** (2006), 2021. [arXiv:hep-th/0502106]

[25] L. Freidel, E. Livine, Effective 3D quantum gravity and non-commutative quantum field theory, *Phys. Rev. Lett.* **96**, 221301 (2006). http://arxiv.org/abs/hep-th/0512113

[26] I. G. Korepanov, Euclidean 4-simplices and invariants of four-dimensional manifolds: I. Moves 3-3, *Theor. Math. Phys.* **131** (2002) 765-774. [arXiv:math/0211165]

[27] I. G. Korepanov, Euclidean 4-simplices and invariants of four-dimensional manifolds: II. An algebraic complex and moves 2-4, *Theor. Math. Phys.* **133** (2002) 1338-1347. [arXiv:math/0211166]

[28] I. G. Korepanov, Euclidean 4-simplices and invariants of four-dimensional manifolds: III. Moves 1-5 and related structures, *Theor. Math. Phys.* **135** (2003) 601-613. [arXiv:math/0211167]

[29] A. Baratin, D. K. Wise, 2-group representations for spin foams, *AIP Conf. Proc.* 1196:28-35 (2009). [arXiv:0910.1542[hep-th]]

[30] G. W. Mackey, Unitary Group Representations in Physics, Probability and Number Theory, Benjamin–Cummings, New York, 1978.

[31] B. Dittrich, S. Speziale, Area-angle variables for general relativity, *New J. Phys.* **10** (2008) 083006. [arXiv:0802.0864[gr-qc]]

[32] F. Girelli, H. Pfeiffer, E. M. Popescu, Topological higher gauge theory - from BF to BFCG theory, *J. Math. Phys.* **49** (2008), 032503. [arXiv:0708.3051[hep-th]]

[33] S. Majid, Strict quantum 2-groups, [arXiv:1208.6265[math.QA]]

[34] A. Mikovic, M. Vojinovic, Poincaré 2-group and quantum gravity, *Class. Quant. Grav.* **29**, 165003 (2012). [arXiv:1110.4694[gr-qc]]

[35] J. C. Baez, D. K Wise, Teleparallel Gravity as a Higher Gauge Theory, [arXiv:1204.4339[gr-qc]]