Test of quantum atmosphere in the dimensionally reduced Schwarzschild black hole

Myungseok Eune$^1$,∗ and Wontae Kim$^2$,†

$^1$Gyedang College of General Education, Sangmyung University, Cheonan, 31066, Republic of Korea
$^2$Department of Physics, Sogang University, Seoul, 04107, Republic of Korea

(Dated: November 9, 2021)

Abstract

It has been suggested by Giddings that the origin of Hawking radiation in black holes is a quantum atmosphere of near-horizon quantum region by investigating both the total emission rate and the stress tensor of Hawking radiation. Revisiting this issue in the exactly soluble model of a dimensionally reduced Schwarzschild black hole, we shall confirm that the dominant Hawking radiation in the Unruh vacuum indeed occurs at the quantum atmosphere, not just at the horizon by exactly calculating the out-temperature responsible for outgoing Hawking particle excitations. Consequently we show that the out-temperature vanishes at the horizon and has a peak at a scale whose radial extent is set by the horizon radius, and then decreases to the Hawking temperature at infinity. We also discuss bounds of location of the peak for the out-temperature in our model.

Keywords: Hawking temperature, Stefan-Boltzmann law, Tolman temperature, Hartle-Hawking vacuum, Unruh vacuum

∗ eunems@smu.ac.kr
† wtkim@sogang.ac.kr
I. INTRODUCTION

Hawking radiation as information carrier [1] leads to black hole complementarity in such a way that there would be no contradictory physical observations between static and freely falling observers [2]. In connection with black hole complementarity, one of the solutions to the firewall paradox [3] is that the infalling observer crossing the horizon could find the firewall of high frequency quanta after the Page time [4]. Thus, the Hawking radiation at the horizon should be highly excited beyond the Planckian scale. The existence of the firewall might also be explained by the Tolman temperature [5, 6] since the Hawking radiation at infinity is ascribed to the infinitely blue-shifted radiation at the horizon.

On the other hand, Unruh showed numerically that the process of thermal particle creation is low-energy behavior so that the highest frequency mode does not matter for the thermal emission [7]. It was also claimed that the Hawking radiation can be retrieved by an alternative Boulware accretion scenario without recourse to a pair creation scenario at the horizon [8]. Recently, Giddings raised a refined question regarding the origin of the Hawking radiation in the Unruh vacuum [9]. He investigated both the total emission rate and the stress tensor of Hawking radiation and then concluded that the origin of Hawking radiation is the near-horizon quantum region of the quantum atmosphere whose radial extent is set by the horizon radius scale. Subsequently, there have been some related works to the quantum atmosphere; analyses of the stress tensor and the effective temperature [10–12], and calculations of emission rate of Hawking radiation in arbitrary dimensions [13].

In particular, from the analysis of a stress tensor, it was claimed that in the Unruh vacuum the Tolman temperature near the horizon does not originate from the out-going particles but the in-going particles from the fact that the negative influx would transition to the positive outward flux over the quantum region outside the horizon [9]. It means that the out-going Hawking radiation originates from the quantum atmosphere. This claim was also discussed by employing the local temperature responsible for the out-going particles [10]; thus the local temperature related to the out-going particles must be finite over the whole region, in particular, it has a peak at a macroscopic distance outside the horizon. The crucial difference from conventional results comes from a modification of the Stefan-Boltzmann law for the out-going particles. In Ref. [11], the authors also advocated the quantum atmosphere with two different arguments. Heuristically, the first was based on
the gravitational Schwinger effect for particle production by the tidal force outside a black hole horizon. Next, the second argument of our concern made use of a calculation of the stress tensor to derive the energy density for an observer at a constant Kruskal position in order to investigate the quantum atmosphere. However, the second argument relied on the observer in Kruskal coordinates, which are not free-fall coordinates at a finite distance from the horizon. Subsequently, this issue was resolved by using a free-fall coordinate system without acceleration [12]. Furthermore, the adiabaticity of the test field modes, which allows us to test where the Wentzel-Kramers-Brillouin breaks down, was taken as an indicator of a particle creation process. The authors found a peak in the violation condition for the field modes; then the condition agrees to the peak of the discrepancy between the energy density and the effective temperature. This discrepancy was regarded as a signal for the location of quantum atmosphere where a particle creation process is taking place.

In this work we would like to revisit the spatial origin of Hawking radiation by investigating the local temperature responsible for the outgoing Hawking particles in a dimensionally reduced Schwarzchild black hole which is more or less realistic and exactly soluble model. The organization of this paper is as follows. In Sec. II we study expectation values of the stress tensor for scalar fields from the one-loop effective action in the dimensionally reduced model based on Refs. [14–17]. Then in Sec. III the stress tensor in thermodynamic equilibrium states proposed by Balbinot and Fabbri [18] will be extended to the stress tensor in the Unruh vacuum of nonequilibrium state. In Sec. IV we first calculate the local temperature in the Hartle-Hawking vacuum and then find it is finite everywhere. Next we show that the local temperature in the Unruh vacuum is just the Tolman temperature, but it consists of the influx and outward flux. Since the only outgoing modes contribute to the Hawking temperature at spatial infinity, we should discard the influx from the Tolman temperature and then justify the out-temperature purely characterized by the outgoing Hawking radiation. Eventually the out-temperature turns out to be the same as the local temperature in the Hartle-Hawking vacuum; it vanishes at the horizon and has a peak at a scale whose radial extent is set by the horizon radius in the quantum atmosphere, and then approaches the Hawking temperature at infinity. Furthermore we find the lower bound and the upper bound of the peak. Finally conclusion will be given in Sec. V.
II. DIMENSIONAL REDUCTION OF THE EINSTEIN-HILBERT ACTION

Let us start with the Einstein-Hilbert action and the matter action defined by

\[ I_{\text{EH}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g^{(4D)}} R^{(4D)}, \]  
(1)

\[ I_m^{(4D)} = -\frac{1}{8\pi} \int d^4x \sqrt{-g^{(4D)}} \sum_{i=1}^{N} (\nabla f_i)^2, \]  
(2)

where \( f_i \) is a scalar field and \( N \) is the number of the scalar fields. In the spherically symmetric space, the four-dimensional line element can be written as

\[ ds^2_{(4D)} = ds^2 + \frac{1}{\lambda^2} e^{-2\phi} d\Omega^2, \]  

with \( ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu \) and \( \phi = \phi(x) \), where \( \lambda^2 = \pi/(2G) \), \( e^{-\phi} \) is the radius, and \( d\Omega_2 \) is the two-dimensional solid angle. Then, the four-dimensional actions (1) and (2) reduce to

\[ I = \frac{1}{4G} \int d^2x \sqrt{-g} e^{-2\phi} \left[ R + 2(\nabla \phi)^2 + 2\lambda^2 e^{2\phi} \right], \]  
(3)

\[ I_m = -\frac{1}{2} \int d^2x \sqrt{-g} e^{-2\phi} \sum_{i=1}^{N} (\nabla f_i)^2. \]  
(4)

Solving the equations of motion from the actions (3) and (4), one can obtain the two-dimensional vacuum solution for a static black hole described by

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)}, \]  
(5)

where \( f(r) = 1 - (2GM)/r, \phi = -\ln(\lambda r) \), and \( f_i = 0 \). The event horizon is located at \( r_h = 2GM \). In terms of the light-cone coordinates, the line element (5) is also written as

\[ ds^2 = -e^{2\rho}d\sigma^+d\sigma^-, \]  
(6)

where \( \rho = (1/2)\ln f \) and \( \sigma^\pm = t \pm r^* \). The tortoise coordinate is defined by \( r^* = r + r_h \ln (r f/r_h) \), and its inverse relation is given by \( r = r_h[1 + W(e^{r^*/r_h-1})] \), where the function \( W(z) \) is the Lambert \( W \) function satisfying \( z = We^W \).

On the other hand, the one-loop effective action for the scalar fields in Eq. (4) is obtained as

\[ \bar{\Gamma} = -\frac{N}{96\pi} \int d^2x \sqrt{-g} \left[ R^{(2D)} - 12R^{(2D)}(\nabla \phi)^2 + 12\phi R^{(2D)} \right], \]  
(7)

and the trace anomaly of the semi-classically quantized stress tensor reads as \[14–17\]

\[ \langle T_{\mu}^{\mu} \rangle = \frac{N}{24\pi} \left[ R - 6(\nabla \phi)^2 + 6\Box \phi \right]. \]  
(8)
However, the flux in the Hartle-Hawking vacuum is unfortunately negative at spatial infinity. In order to evade the negative flux at infinity [19, 20], Balbinot and Fabbri [18] considered the Weyl invariant action not affecting the trace anomaly such as

$$\frac{N}{96\pi} \left( b^2 - 36 \right) \int d^2x \sqrt{-g} (\nabla \phi)^2 \frac{1}{\Box} (\nabla \phi)^2$$

which is arbitrary but phenomenologically sensible in that Eq. (9) will provide the desired expression of the Hawking radiation at infinity. The constant $b$ was determined as $b = 2\sqrt{3}$, which can be identified with $b = 4\sqrt{3}\pi \ell_1$ in Ref. [18]. Combining Eq. (7) with Eq. (9), we will take the one-loop effective action as

$$\Gamma = -\frac{N}{48\pi} \int d^2x \sqrt{-g} \left[ \psi (R - 6(\nabla \phi)^2) + \frac{1}{2}(\nabla \psi)^2 + 6\phi R - \frac{1}{2}(\nabla \chi)^2 - 2\sqrt{3\chi}(\nabla \phi)^2 \right],$$

where the two auxiliary scalar fields $\psi$ and $\chi$ satisfy

$$\Box \psi = R - 6(\nabla \phi)^2, \quad \Box \chi = 2\sqrt{3}(\nabla \phi)^2.$$  

From the localized action (10), the quantum-mechanical stress tensor is easily obtained as

$$\langle T_{\mu\nu} \rangle = \frac{N}{24\pi} \left[ -\nabla_\mu \nabla_\nu \psi + \frac{1}{2}\nabla_\mu \psi \nabla_\nu \psi - 6\psi \nabla_\mu \phi \nabla_\nu \phi - 6(\nabla_\mu \nabla_\nu \phi - g_{\mu\nu}\Box \phi) - \frac{1}{2}\nabla_\mu \chi \nabla_\nu \chi \\
- 2\sqrt{3}\chi \nabla_\mu \phi \nabla_\nu \phi + g_{\mu\nu} \left( \Box \psi - \frac{1}{4}(\nabla \psi)^2 + 3\psi(\nabla \phi)^2 + \frac{1}{4}(\nabla \chi)^2 + \sqrt{3\chi}(\nabla \phi)^2 \right) \right].$$

In the conformal gauge (6), Eq. (11) is written as

$$\partial_+ \partial_- \psi = -2\partial_+ \partial_- \rho - 6\partial_+ \phi \partial_- \phi, \quad \partial_+ \partial_- \chi = 2\sqrt{3}\partial_+ \phi \partial_- \phi,$$

and the general solutions for $\psi$ and $\chi$ are easily obtained as

$$\psi = -2\rho + 6\xi + \omega_+ (\sigma^+) + \omega_- (\sigma^-),$$

$$\chi = -2\sqrt{3}\xi + \eta_+ (\sigma^+) + \eta_- (\sigma^-),$$

where $\xi = \ln \left( (r/r_h) \sqrt{T} \right)$ and $\omega_\pm (\sigma^\pm)$ and $\eta_\pm (\sigma^\pm)$ are arbitrary holomorphic/antiholomorphic functions determined by some boundary conditions of vacuum states. Plugging Eqs. (14)
and (15) into Eq. (12), the components of the stress tensor (12) are expressed as
\[
\langle T_{+-} \rangle = \frac{N}{12\pi} \left[ -\partial_+ \partial_- \rho - 3\partial_+ \phi \partial_- \phi + 3\partial_+ \partial_- \phi \right],
\]
\[
\langle T_{\pm \pm} \rangle = \frac{N}{12\pi} \left[ \partial_\pm^2 \rho - (\partial_\pm \rho)^2 + 6\rho (\partial_\pm \phi)^2 - 3(\partial_\pm^2 \phi - 2\partial_\pm \rho \partial_\pm \phi) + 3\partial_\pm^2 \xi \\
+ 6(\partial_\pm \xi)^2 - 288\xi (\partial_\pm \phi)^2 + \partial_\pm \xi (3\partial_\pm \omega_\pm + \sqrt{3}\partial_\pm \eta_\pm) \\
- (\partial_\pm \phi)^2 \left( (3\omega_+ + \sqrt{3}\eta_+) + (3\omega_- + \sqrt{3}\eta_-) \right) - t_\pm \right],
\]
where we defined \( t_\pm = (1/2)\partial_\pm^2 \omega_\pm - (1/4)(\partial_\pm \omega_\pm)^2 + (1/4)(\partial_\pm \eta_\pm)^2 \). Here, in order for the stress tensor (17) to be static, one of the simplest solutions in Eqs. (14) and (15) might be chosen as \( \omega_\pm = \pm(1/2)c_1 \sigma^\pm + d_1 \) and \( \eta_\pm = \pm(1/2)c_2 \sigma^\pm + d_2 \) [18], where \( c_1, d_1, c_2, d_2 \) are constants. The two constants of \( d_1 \) and \( d_2 \) can be written as one constant without affecting the stress tensor due to the translation symmetry in it; however, this choice of \( \omega_\pm \) and \( \eta_\pm \) always results in \( t_+ = t_- \) in Eq. (17), which describes only equilibrium states such as the Hartle-Hawking and the Boulware vacuum. In order to incorporate non-equilibrium vacuum, we impose a less strong condition for the auxiliary fields as
\[
3\omega_\pm + \sqrt{3}\eta_\pm = \pm \frac{C}{2} \sigma^\pm - \frac{1}{2} D,
\]
which also successfully makes Eqs. (16) and (17) static, where \( C \) and \( D \) are constants. Note that arbitrary holomorphic/antiholomorphic integration functions in Eqs. (14) and (15) can be written as \( \omega_\pm = \sum_{n=0}^{\infty} a_n^\pm (\sigma^\pm)^n \) and \( \eta_\pm = \sum_{n=0}^{\infty} b_n^\pm (\sigma^\pm)^n \) with the constants \( a_n \) and \( b_n \), and thus Eq. (18) can be obtained by choosing \( 3a_0^\pm + \sqrt{3}b_0^\pm = -(1/2)D \) and \( 3a_1^\pm + \sqrt{3}b_1^\pm = \pm C/2 \). Eventually, the integration constant \( D \) comes from the zero modes of \( \omega_\pm \) and \( \eta_\pm \).

Using Eqs. (6) and (18), one can write Eqs. (16) and (17) as
\[
\langle T_{+-} \rangle = \frac{N f}{48\pi} \left[ \frac{1}{2} f'' + 3\frac{f'}{r} \right],
\]
\[
\langle T_{\pm \pm} \rangle = \frac{N f}{48\pi} \left[ \frac{1}{2} f'' - \frac{1}{4}(f')^2 + \frac{3}{2r^2}(1 + f)^2 + \frac{C}{2r}(1 + f) - 4t_\pm \\
+ \frac{f^2}{2r^2} \left( -(6 + 2Crh) \ln f - (24 + 2Crh) \ln \frac{r}{rh} - 2Cr + 2D \right) \right],
\]
where the integration function is written as \( t_\pm = (1/2)\partial_\pm^2 \omega_\pm + (1/2)(\partial_\pm \omega_\pm)^2 \mp (\sqrt{3}C/4)\partial_\pm \omega_\pm + C^2/48 \).
III. THREE VACUUM STATES

We consider three vacuum states; the Boulware [21], the Hartle-Hawking [22, 23], and the Unruh vacuum [24]; thus $C = \{C_B, C_{\text{HH}}, C_U\}$, $D = \{D_B, D_{\text{HH}}, D_U\}$, and $t^{\text{B,H,U}}_{\pm}$ will be determined for a given vacuum state. Firstly in the Boulware vacuum there are no influx and outward flux at spatial infinity, so that we can choose $t^{B}_{\pm} = 0$ in Eq. (20). Furthermore in the limit of the Minkowski spacetime of $M = 0$ [18], Eq. (20) should vanish as

$$\langle T^{\pm\pm}\rangle_B \to \lim_{r_h \to 0} \frac{N}{48\pi} \left[ \frac{12 + 3C_B r_h + 2D_B}{2r^2} - \frac{12 + C_B r_h}{r^2} \ln \frac{r}{r_h} \right] = 0,$$

which determines $C_B = -12/r_h$ and $D_B = 12$. Consequently, we get

$$\langle T^{\pm\pm}\rangle_B = \frac{N x^2}{192\pi r_h^2} \left[ 19x(3x - 4) + 36f^2 \ln f \right],$$

where $f = 1 - x$ with $x = r_h/r$. All constants can be completely fixed in the Boulware vacuum thanks to the additional condition (21).

Secondly in the Hartle-Hawking vacuum both the influx and outward flux are finite at the horizon, in other words, $\langle T^{\pm\pm}\rangle_{\text{HH}}$ is regular. It requires that $\langle T^{\pm\pm}\rangle_{\text{HH}}$ should vanish necessarily at the horizon since $\langle T^{\pm\pm}\rangle_{\text{HH}} = (T^{\pm\pm})_{\text{HH}}/f^2$, and thus we can fix the boundary conditions as $t^{\text{HH}}_{\pm} = -1/(4r_h)^2$ and $C_{\text{HH}} = -3/r_h$. Plugging these conditions into Eq. (20), we can obtain

$$\langle T^{\pm\pm}\rangle_{\text{HH}} = \frac{N f^2}{192\pi r_h^2} \left[ 1 + 2x + x^2(9 + 4D_{\text{HH}} + 36 \ln x) \right],$$

where $D_{\text{HH}}$ is not fixed by the Hartle-Hawking boundary condition. Note that one cannot require the condition like Eq. (21) in the Hartle-Hawking vacuum since there does not exist the limit of the Minkowski spacetime of $M = 0$ unlike the case of the Boulware vacuum. Note that Eqs. (22) and (23) are exactly the same as the results in Ref. [18]. As expected, the flux at infinity is well-defined as $\langle T^{\pm\pm}\rangle_{\text{HH}} \to N(\pi/12)T_H^2$, where $T_H = 1/(4\pi r_h)$.

Finally in the Unruh vacuum there is no influx at spatial infinity and the outward flux is finite at horizon, specifically, $\langle T^{--}\rangle_U$ vanishes at infinity and $\langle T^{++}\rangle_U$ is regular on the horizon, so that the boundary conditions are chosen as $t^{U}_{+} = 0$ and $t^{U}_{-} = -(4r_h)^{-2}$ with $C_U = -3/r_h$. Then the resulting stress tensor can be newly obtained as

$$\langle T^{++}\rangle_U = \frac{N x^2}{192\pi r_h^2} \left[ 6 + 4D_U - 8(2 + D_U)x + (9 + 4D_U)x^2 + 36f^2 \ln x \right],$$

$$\langle T^{--}\rangle_U = \frac{N f^2}{192\pi r_h^2} \left[ 1 + 2x + x^2(9 + 4D_U + 36 \ln x) \right].$$
Near the horizon the influx is negative finite as $\langle T_{++}\rangle_U \rightarrow -N/(192\pi r_0^2)$, while $\langle T_{--}\rangle_U \rightarrow 0$. At spatial infinity $\langle T_{++}\rangle_U \rightarrow 0$, while $\langle T_{--}\rangle_U \rightarrow N(\pi/12)T_\text{H}^2$.

IV. LOCAL TEMPERATURES IN EQUILIBRIUM AND NON-EQUILIBRIUM STATE

In this section, we calculate the local temperature in order to study the origin of Hawking radiation in the evaporating black hole. For this purpose, the proper velocity of radiation flow can be found by solving the geodesic equation of $u^\mu \nabla_\mu u^\nu = 0$, where the proper velocity $u^\mu$ is defined by $u^\mu = dx^\mu/d\tau$ with a proper time $\tau$. Then the proper velocity can be obtained as

$$u^\mu = \left(\frac{\sqrt{f(r_0)}}{f(r)}, \pm \sqrt{f(r_0) - f(r)}\right).$$

If the frame is released from rest at an arbitrary point of $r = r_0$, then Eq. (26) reduces to $u^\mu = (1/\sqrt{f}, 0)$. Next, we define the local quantities related to the stress tensor as

$$\varepsilon = \langle T_{\mu\nu}\rangle u^\mu u^\nu, \quad p = \langle T_{\mu\nu}\rangle n^\mu n^\nu, \quad \mathcal{F} = -\langle T_{\mu\nu}\rangle u^\mu n^\nu,$$

where $\varepsilon$, $p$, and $\mathcal{F}$ are the proper energy density, pressure, and flux, respectively. Note that $n^\mu$ is a spacelike unit vector normal to $u^\mu$ given by $n^\mu = (0, \sqrt{f})$. In the light-cone coordinates, the energy density and flux in Eq. (27) can also be expressed as

$$\varepsilon = \frac{1}{f}(\langle T_{++}\rangle + \langle T_{--}\rangle + 2\langle T_{+-}\rangle),$$

$$\mathcal{F} = -\frac{1}{f}(\langle T_{++}\rangle - \langle T_{--}\rangle),$$

where we used $u^\pm = 1/\sqrt{f}$ and $n^\pm = \pm 1/\sqrt{f}$ in the light-cone coordinates.

Before studying the local temperature in the Unruh vacuum, we first investigate the Stefan-Boltzmann law in the Hartle-Hawking vacuum in order to relate the proper energy density (28) to the local temperature along the line of Refs. [5, 6]; however, we will take into account the trace anomaly in deriving the Stefan-Boltzmann law as compared to the conventional procedure. Let us start with the first law of thermodynamics written as $dU = TdS - pdV$ where $U$, $S$, $V$, and $T$, and $p$ are the internal energy, entropy, volume, temperature, and pressure of a system. At a fixed temperature, the first law of thermodynamics
can be rewritten as
\[
\left( \frac{\partial U}{\partial V} \right)_T = T \left( \frac{\partial S}{\partial V} \right)_T - p, \tag{30}
\]
where the left-hand side is the energy density, \(\varepsilon = (\partial U/\partial V)_T\). Using the Maxwell relation of \((\partial S/\partial V)_T = (\partial p/\partial T)_V\), one can see that Eq. (30) becomes
\[
\varepsilon = T \left( \frac{\partial p}{\partial T} \right)_V - p. \tag{31}
\]
Assuming the trace of the stress tensor does not vanish as seen from Eq. (8), we can write
\[
\langle T^\mu_\mu \rangle = -\varepsilon + p, \tag{32}
\]
by using Eq. (27). From Eqs. (31) and (32), one can eliminate the pressure term and thus obtain the first order differential equation as
\[
T \left( \frac{\partial \varepsilon}{\partial T} \right)_V - 2\varepsilon = \langle T^\mu_\mu \rangle, \tag{33}
\]
where we used the fact that the trace anomaly is independent of temperature [25]. Then, the energy density is solved as [26]
\[
\varepsilon = \gamma T^2 - \frac{1}{2} \langle T^\mu_\mu \rangle, \tag{34}
\]
where \(\gamma\) is an integration constant which turns out to be the Stefan-Boltzmann constant. Note that the Stefan-Boltzmann law (34) modified by the trace anomaly naturally reduces to the well-known Stefan-Boltzmann law in Refs. [5, 6] if \(\langle T^\mu_\mu \rangle = 0\). The second term in Eq. (34) will modify the behavior of the local temperature significantly near the horizon.

Plugging Eq. (28) into Eq. (34), we obtain the local temperature in the Hartle-Hawking vacuum as
\[
\gamma T_{\text{HH}}^2 = \frac{1}{f} (\langle T_{++} \rangle_{\text{HH}} + \langle T_{--} \rangle_{\text{HH}}) = \frac{2}{f} \langle T_{\pm \pm} \rangle_{\text{HH}}, \tag{35}
\]
where \(\langle T_{++} \rangle_{\text{HH}} = \langle T_{--} \rangle_{\text{HH}}\) from Eq. (23), and requiring \(T_{\text{HH}} \to T_H\) at infinity determines \(\gamma = \pi N/6\). The local temperature \(T_{\text{HH}}\) in the Hartle-Hawking vacuum (35) can be explicitly calculated as
\[
T_{\text{HH}}(r) = T_H \sqrt{1 - \frac{r_h}{r}} \sqrt{1 + \frac{2r_h}{r} + \left( \frac{r_h}{r} \right)^2 \left( 9 + 4D_{\text{HH}} + 36 \ln \frac{r_h}{r} \right)}, \tag{36}
\]
FIG. 1. The local temperatures $T_{\text{HH}}$ in the Hartle-Hawking vacuum are plotted for $D_{\text{HH}} \geq D_c \approx 23.03$, where the temperatures are real and monotonically decreasing after each peaks. In case of $D_{\text{HH}} = D_c$, the ratio of the maximum local temperature to the Hawking temperature is $T_{\text{HH}}/T_H \approx 3.70$ and it occurs at $r_c/r_h \approx 1.43$. The location of the peak temperature is shifted to the right from $r_c$ as $D_{\text{HH}}$ is getting large.

by using Eq. (23). It is interesting to note that the local temperature vanishes at the horizon. Of course, it approaches the expected Hawking temperature at infinity. The near horizon limit of the local temperature is very different from that of the conventional Tolman temperature defined by $T = T_H/\sqrt{f(r)}$ which is divergent at the horizon. The zero temperature at the horizon is compatible with the fact that there exist neither influx nor outward flux at the horizon in thermal equilibrium.

The constant $D_{\text{HH}}$ in Eq. (36) is still arbitrary because it is not enough to fix all constants just by using the Hartle-Hawking boundary condition imposed on the stress tensor in contrast to the case of the Boulware vacuum. Thus, it should be fixed by an additional condition. If $D_{\text{HH}} < D_0$, the temperature (36) is imaginary in a certain region outside the horizon, where $D_0 = -(53 + \sqrt{73})/8 + 9 \ln \left[(\sqrt{73} - 1)/2\right] \approx 4.26$. If $D_0 \leq D_{\text{HH}} < D_c$ where $D_c \approx 23.03$, the temperature is real in the whole region, but unfortunately it is not decreasing monotonically. Finally if we take $D_{\text{HH}} \geq D_c$, then the temperature is not only
real in the whole region but also decreasing monotonically as \( r \) increases after it reaches a maximum value at \( r_c \approx 1.43 r_h \) (see Fig. 1). In this respect we will take \( D_{HH} \geq D_c \) from now on.

Let us now find the locations of peaks from \( \partial_r T_{HH} \big|_{r=r_{\text{peak}}} = 0 \) with assuming \( D_{HH} \geq D_c \), where \( r_{\text{peak}} \) is a position at which the maximum temperature occurs, then we obtain the relation between \( D_{HH} \) and \( r_{\text{peak}} \) as

\[
D_{HH} = \frac{63 - 50 r_{\text{peak}}/r_h - (r_{\text{peak}}/r_h)^2}{8(r_{\text{peak}}/r_h - 3/2)} + 18 \left( \frac{r_{\text{peak}}}{r_h} - \frac{3}{2} \right) \ln \frac{r_{\text{peak}}}{r_h}. \tag{37}
\]

If \( r_{\text{peak}} \) occurs at \( r_h \), then \( D_{HH} = -3 \), which is prohibited by the assumption that \( D_{HH} \geq D_c \approx 23.03 \). Hence, for \( D_{HH} \geq D_c \), \( r_{\text{peak}} \) should be at a finite distance from the horizon and increases to \((3/2)r_h\) monotonically as \( D_{HH} \) increases. As a result, the peaks of the local temperatures in equilibrium should lie in

\[ 1.43 r_h \lesssim r_{\text{peak}} < 1.5 r_h. \tag{38} \]

Now, in the Unruh vacuum, one can calculate the local temperature by substituting Eqs. (24) and (25) into Eq. (29) as [9, 10, 27]

\[
\mathcal{F}_U = -\frac{1}{f} (\langle T_{++} \rangle_U - \langle T_{--} \rangle_U) = \sigma \left( \frac{T_H}{\sqrt{f}} \right)^2 = \sigma T_U^2, \tag{39}
\]

where \( \sigma = \gamma/2 \). The local temperature in the Unruh vacuum takes exactly the Tolman’s form, which is not new; however, it is worth noting that the local temperature (39) consists of the negative influx and the positive outward flux. Defining

\[
\sigma T_{in}^2 = -\frac{1}{f} \langle T_{++} \rangle_U, \quad \sigma T_{out}^2 = \frac{1}{f} \langle T_{--} \rangle_U, \tag{40}
\]

we obtain

\[
T_{in}^2 + T_{out}^2 = T_U^2, \tag{41}
\]

where \( T_{in} \) and \( T_{out} \) are related to the influx and outward flux, respectively. Note that the divergence of the Tolman temperature at the horizon comes from \( T_{in} \) since the influx (24) is negative finite at the horizon but it is infinitely blue-shifted there. Moreover, it turns out that \( T_{out} \) is exactly the same as the local temperature (35) in the Hartle-Hawking vacuum as seen from Eq. (23) and Eq. (25) if \( D_U = D_{HH} \), which results in

\[ T_{out} = T_{HH}. \tag{42} \]
Therefore, the local temperature associated with the outgoing Hawking radiation can be finite everywhere, and its peak occurs at a macroscopic distance outside the horizon, which means that the main excitations occur not at the horizon but at the peak in the quantum atmosphere.

V. CONCLUSION

In the dimensionally reduced Schwarzschild black hole, we found that the divergence of the temperature at the horizon comes from the infinite blue-shift of the negative influx, and the out-temperature responsible for the Hawking radiation is always finite, more importantly, its peak occurs in the quantum atmosphere bounded by $1.43 r_h \lesssim r_{\text{peak}} < 1.5 r_h$, which is indeed the size of the black hole. It means that the main excitations of Hawking particles dominantly happens at the peak but it spreads throughout the whole region well outside the horizon. Consequently, in the spherically reduced Schwarzschild black hole, we confirmed that the origin of the Hawking radiation is the quantum atmosphere not just at the horizon from the viewpoint of the local temperature in the semi-classical regime.

As a first comment, we discuss what it happens when the trace anomaly (8) is ignored in our calculations. In Eq. (33), if we replace $\langle T^\mu_\mu \rangle$ by $q \langle T^\mu_\mu \rangle$ where $q$ is 0 or 1, the temperature is obtained as $T_{\text{HH}}^2 = T_H^2 \left[ f \left( 1 + 2x + x^2(9 + 4D_{\text{HH}} + 36 \ln x) \right) + 8(1-q)x^2 \right]$. At the horizon, $T_{\text{HH}} = 2\sqrt{2}T_H \sqrt{1-q}$ so that in the usual case of $q = 0$ the temperature does not vanish. In our case of $q = 1$ the temperature vanishes at the horizon, which means that the modified Stefan-Boltzmann law (34) induced by the trace anomaly is essential in our atmosphere argument.

Finally, one might wonder what the physical meaning of $D$ is, in consideration of the fact that three regimes are identified in Sec. IV and thus only $D \geq D_c$ seems to be physically relevant. Unfortunately, we are not well aware of the meaning of the constant $D$ generated from the semiclassical treatment although $D$ plays the important role in our calculations. Instead, we notice some intriguing points for further study: (i) $D$ arises from the solution of the auxiliary scalar fields to localize the non-local effective actions (7) and (9). What needs to be answered is that it can be a quantum-mechanical hair or not. Otherwise, we must find a way to fix the constant like the case of Boulware state. (ii) For $D \neq D'$, the Hartle-Hawking states such as $|\text{HH}; D\rangle$ and $|\text{HH}; D'\rangle$ are degenerated at the horizon, and
thus the local temperatures are coincident at the horizon; however, they depend on $D$ in the bulk region. We hope these issues will be resolved elsewhere.

ACKNOWLEDGMENTS

We would like to thank D. V. Vassilevich for valuable comments on our previous version of the manuscript, and Hwajin Um for exciting discussions. Eune was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2018R1D1A1B05050636). Kim was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIP) (2017R1A2B2006159).

[1] S. W. Hawking, Particle Creation by Black Holes, Commun. Math. Phys. 43 (1975) 199–220.
[2] L. Susskind, L. Thorlacius and J. Uglum, The Stretched horizon and black hole complementarity, Phys. Rev. D48 (1993) 3743–3761, [hep-th/9306069].
[3] A. Almheiri, D. Marolf, J. Polchinski and J. Sully, Black Holes: Complementarity or Firewalls?, JHEP 02 (2013) 062, [1207.3123].
[4] D. N. Page, Information in black hole radiation, Phys. Rev. Lett. 71 (1993) 3743–3746, [hep-th/9306083].
[5] R. C. Tolman, On the Weight of Heat and Thermal Equilibrium in General Relativity, Phys. Rev. 35 (1930) 904–924.
[6] R. Tolman and P. Ehrenfest, Temperature Equilibrium in a Static Gravitational Field, Phys. Rev. 36 (1930) 1791–1798.
[7] W. G. Unruh, Dumb holes and the effects of high frequencies on black hole evaporation, gr-qc/9409008.
[8] W. Israel, Shenanigans at the black hole horizon: pair creation or Boulware accretion?, 1504.02419.
[9] S. B. Giddings, Hawking radiation, the StefanBoltzmann law, and unitarization, Phys. Lett. B754 (2016) 39–42, [1511.08221].
[10] W. Kim, *Origin of Hawking Radiation: Firewall or Atmosphere?,* *Gen. Rel. Grav.* **49** (2017) 15, [1604.00465].

[11] R. Dey, S. Liberati and D. Pranzetti, *The black hole quantum atmosphere,* *Phys. Lett.* **B774** (2017) 308–316, [1701.06161].

[12] R. Dey, S. Liberati, Z. Mirzaiyan and D. Pranzetti, *Black hole quantum atmosphere for freely falling observers,* *Phys. Lett.* **B797** (2019) 134828, [1906.02958].

[13] S. Hod, *Hawking radiation and the Stefan-Boltzmann law: The effective radius of the black-hole quantum atmosphere,* *Phys. Lett.* **B757** (2016) 121–124, [1607.02510].

[14] S. Nojiri and S. D. Odintsov, *Trace anomaly induced effective action for 2-D and 4-D dilaton coupled scalars,* *Phys. Rev.* **D57** (1998) 2363–2371, [hep-th/9706143].

[15] W. Kummer, H. Liebl and D. V. Vassilevich, *Hawking radiation for nonminimally coupled matter from generalized 2-D black hole models,* *Mod. Phys. Lett.* **A12** (1997) 2683–2690, [hep-th/9707041].

[16] W. Kummer, H. Liebl and D. V. Vassilevich, *Comment on: ‘Trace anomaly of dilaton coupled scalars in two-dimensions’,* *Phys. Rev.* **D58** (1998) 108501, [hep-th/9801122].

[17] J. S. Dowker, *Conformal anomaly in 2-d dilaton scalar theory,* *Class. Quant. Grav.* **15** (1998) 1881–1884, [hep-th/9802029].

[18] R. Balbinot and A. Fabbri, *Two-dimensional black holes and effective actions,* *Class. Quant. Grav.* **20** (2003) 5439–5454.

[19] V. F. Mukhanov, A. Wipf and A. Zelnikov, *On 4-D Hawking radiation from effective action,* *Phys. Lett.* **B332** (1994) 283–291, [hep-th/9403018].

[20] W. Kummer and D. V. Vassilevich, *Effective action and Hawking radiation for dilaton coupled scalars in two-dimensions,* *Phys. Rev.* **D60** (1999) 084021, [hep-th/9811092].

[21] D. G. Boulware, *Quantum Field Theory in Schwarzschild and Rindler Spaces,* *Phys. Rev.* **D11** (1975) 1404.

[22] W. Israel, *Thermo field dynamics of black holes,* *Phys. Lett.* **A57** (1976) 107–110.

[23] J. B. Hartle and S. W. Hawking, *Path Integral Derivation of Black Hole Radiance,* *Phys. Rev.* **D13** (1976) 2188–2203.

[24] W. G. Unruh, *Notes on black hole evaporation,* *Phys. Rev.* **D14** (1976) 870.

[25] H. Boschi-Filho and C. P. Natividade, *Anomalies in curved space-time at finite temperature,* *Phys. Rev.* **D46** (1992) 5458–5466.
[26] Y. Gim and W. Kim, *A Quantal Tolman Temperature*, *Eur. Phys. J. C*75* (2015) 549, [1508.00312].

[27] W. Kim, *The effective Tolman temperature in curved spacetimes*, *Int. J. Mod. Phys. D*26* (2017) 1730025, [1709.02537].