On finite $P\sigma T$-groups

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Abstract

Let $\sigma = \{\sigma_i | i \in I\}$ be some partition of the set of all primes $\mathbb{P}$ and $G$ a finite group. $G$ is said to be $\sigma$-soluble if every chief factor $H/K$ of $G$ is a $\sigma_i$-group for some $i = i(H/K)$.

A set $\mathcal{H}$ of subgroups of $G$ is said to be a complete $\sigma$-Hall set of $G$ if every member $\neq 1$ of $\mathcal{H}$ is a Hall $\sigma_i$-subgroup of $G$ for some $\sigma_i \in \sigma$ and $\mathcal{H}$ contains exactly one Hall $\sigma_i$-subgroup of $G$ for every $i \in I$ such that $\sigma_i \cap \pi(G) \neq \emptyset$. A subgroup $A$ of $G$ is said to be $\sigma$-permutable or $\sigma$-quasinormal in $G$ if $G$ has a complete $\sigma$-Hall set $\mathcal{H}$ such that $AH^x = H^xA$ for all $x \in G$ and all $H \in \mathcal{H}$.

We obtain a characterization of finite $\sigma$-soluble groups $G$ in which $\sigma$-quasinormality is a transitive relation in $G$.

1 Introduction

Throughout this paper, all groups are finite and $G$ always denotes a finite group. Moreover, $\mathbb{P}$ is the set of all primes, $\pi \subseteq \mathbb{P}$ and $\pi' = \mathbb{P} \setminus \pi$. If $n$ is an integer, the symbol $\pi(n)$ denotes the set of all primes dividing $n$; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of $G$. $G$ is said to be a $D_\pi$-group if $G$ possesses a Hall $\pi$-subgroup $E$ and every $\pi$-subgroup of $G$ is contained in some conjugate of $E$.

In what follows, $\sigma$ is some partition of $\mathbb{P}$, that is, $\sigma = \{\sigma_i | i \in I\}$, where $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$; $\Pi$ is always supposed to be a subset of the set $\sigma$ and $\Pi' = \sigma \setminus \Pi$.

By the analogy with the notation $\pi(n)$, we write $\sigma(n)$ to denote the set $\{\sigma_i | \sigma_i \cap \pi(n) \neq \emptyset\}$; $\sigma(G) = \sigma(|G|)$. $G$ is said to be: $\sigma$-primary [1] if $|\sigma(G)| \leq 1$; $\sigma$-decomposable (Shemetkov [2]) or $\sigma$-nilpotent (Guo and Skiba [3]) if $G = G_1 \times \cdots \times G_n$ for some $\sigma$-primary groups $G_1, \ldots, G_n$; $\sigma$-soluble [1] if every chief factor of $G$ is $\sigma$-primary; a $\sigma$-full group of Sylow type [1] if every subgroup $E$ of $G$ is a $D_{\sigma_i}$-group for every $\sigma_i \in \sigma(E)$.

A natural number $n$ is said to be a $\Pi$-number if $\sigma(n) \subseteq \Pi$. A subgroup $A$ of $G$ is said to be: a Hall $\Pi$-subgroup of $G$ [1] if $|A|$ is a $\Pi$-number and $|G : A|$ is a $\Pi'$-number; a $\sigma$-Hall subgroup of $G$ if $A$ is a Hall $\Pi$-subgroup of $G$ for some $\Pi \subseteq \sigma$.

Keywords: finite group, $\sigma$-quasinormal subgroup, $P\sigma T$-group, $\sigma$-soluble group, $\sigma$-nilpotent group.

Mathematics Subject Classification (2010): 20D10, 20D15, 20D30
A set \( \mathcal{H} \) of subgroups of \( G \) is a **complete Hall \( \sigma \)-set** of \( G \) [1, 5] if every member \( \neq 1 \) of \( \mathcal{H} \) is a Hall \( \sigma_i \)-subgroup of \( G \) for some \( \sigma_i \in \sigma \) and \( \mathcal{H} \) contains exact one Hall \( \sigma_i \)-subgroup of \( G \) for every \( \sigma_i \in \sigma(G) \).

Recall that a subgroup \( A \) of \( G \) is said to be: **\( \sigma \)-permutable** or **\( \sigma \)-quasinormal** in \( G \) [1] if \( G \) possesses a complete Hall \( \sigma \)-set \( \mathcal{H} \) such that \( AH^x = H^x A \) for all \( H \in \mathcal{H} \) and all \( x \in G \); **\( \sigma \)-subnormal** in \( G \) [1] if there is a subgroup chain \( A = A_0 \leq A_1 \leq \cdots \leq A_t = G \) such that either \( A_{i-1} \leq A_i \) or \( A_i/(A_{i-1})_{A_i} \) is \( \sigma \)-primary for all \( i = 1, \ldots, t \).

In the classical case, when \( \sigma = \sigma^0 = \{\{2\}, \{3\}, \ldots\} \), \( \sigma \)-quasinormal subgroups are also called **\( S \)-quasinormal** or **\( S \)-permutable** [6, 7], and a subgroup \( A \) of \( G \) is subnormal in \( G \) if and only if it is \( \sigma^0 \)-subnormal in \( G \).

We say that \( G \) is a **\( P \sigma T \)-group** [1] if \( \sigma \)-quasinormality is a transitive relation in \( G \), that is, if \( K \) is a \( \sigma \)-quasinormal subgroup of \( H \) and \( H \) is a \( \sigma \)-quasinormal subgroup of \( G \), then \( K \) is a \( \sigma \)-quasinormal subgroup of \( G \). In the case, when \( \sigma = \{\{2\}, \{3\}, \ldots\} \), \( P \sigma T \)-groups are called **\( P ST \)-groups** [6].

In view of Theorem B in [1], \( P \sigma T \)-groups can be characterized as the groups in which every \( \sigma \)-subnormal subgroup is \( \sigma \)-quasinormal in \( G \).

Our first observation is the following fact, which generalizes the sufficiency condition in Theorem A of the paper [1].

**Theorem A.** Let \( G \) have a normal \( \sigma \)-Hall subgroup \( D \) such that:

(i) \( G/D \) is a \( P \sigma T \)-group, and

(ii) every \( \sigma \)-subnormal subgroup of \( D \) is normal in \( G \).

If \( G \) is a \( \sigma \)-full group of Sylow type, then \( G \) is a \( P \sigma T \)-group.

**Corollary 1.1** (See Theorem A in [1]). Let \( G \) have a normal \( \sigma \)-Hall subgroup \( D \) such that:

(i) \( G/D \) is \( \sigma \)-nilpotent, and

(ii) every subgroup of \( D \) is normal in \( G \).

Then \( G \) is a \( P \sigma T \)-group.

In the case when \( \sigma = \{\{2\}, \{3\}, \ldots\} \), we get from Theorem A the following

**Corollary 1.2** (See Theorem 2.4 in [5]). Let \( G \) have a normal Hall subgroup \( D \) such that:

(i) \( G/D \) is a \( PST \)-group, and

(ii) every subnormal subgroup of \( D \) is normal in \( G \).

Then \( G \) is a \( PST \)-group.

Recall that \( G^{\sigma_{nil}} \) denotes the **\( \sigma \)-nilpotent residual** of \( G \), that is, the intersection of all normal subgroups \( N \) of \( G \) with \( \sigma \)-nilpotent quotient \( G/N \); \( G^{\mathfrak{n}} \) denotes the **nilpotent residual** of \( G \) [9].
Definition 1.3. We say that $G$ is a special $P\sigma T$-group provided the $\sigma$-nilpotent residual $D = G^{\sigma_1}$ of $G$ is contained in a Hall $\sigma_i$-subgroup $E$ of $G$ for some $i$ and the following conditions hold:

(i) $D$ is a Hall subgroup of $G$ and every element of $G$ induces a power automorphism in $D$;
(ii) $D$ has a normal complement $S$ in $E$.

Note that if $G = C_5 \times (C_3 \rtimes C_2)$, where $C_3 \rtimes C_2 \simeq S_3$ and $\sigma = \{\{3, 5\}, \{3, 5\}'\}$, then $G$ is a special $P\sigma T$-group with $C_3 = G^{\sigma_1}$.

The following theorem shows that every special $P\sigma T$-group is a $P\sigma T$-group.

**Theorem B.** Suppose that $G$ has a $\sigma$-nilpotent normal Hall subgroup $D$ with $\sigma$-nilpotent quotient $G/D$ such that $G/O^{\sigma_i}(D)$ is a special $P\sigma T$-group for each $\sigma_i \in \sigma(D)$. Then $G$ is a $P\sigma T$-group.

Generalizing the concept of complete Wielandt $\sigma$-set of a group in [3], we say that a complete Hall $\sigma$-set $H$ of $G$ is a generalized Wielandt $\sigma$-set of $G$ if every member $H$ of $H$ is $\pi(G^{\sigma_1})$-supersoluble.

Using Theorem B, we prove also the following revised version of Theorem A in [1].

**Theorem C.** Let $G$ be $\sigma$-soluble and $D = G^{\sigma_1}$. Suppose that $G$ has a generalized Wielandt $\sigma$-set. Then $G$ is a $P\sigma T$-group if and only if the following conditions hold:

(i) $D$ is an abelian Hall subgroup of $G$ of odd order and every element of $G$ induces a power automorphism in $D$;
(ii) $G/O^{\sigma_i}(D)$ is a special $P\sigma T$-group for each $\sigma_i \in \sigma(D)$.

**Corollary 1.4** (See Theorem 2.3 in [3]). Let $G$ be a soluble and $D = G^{\sigma_1}$. If $G$ is a $PST$-group, then $D$ is an abelian Hall subgroup of $G$ of odd order and every element of $G$ induces a power automorphism in $D$.

2 Some preliminary results

In view of Theorems A and B in [4], the following fact is true.

**Lemma 2.1.** If $G$ is $\sigma$-soluble, then $G$ is a $\sigma$-full group of Sylow type.

We use $\mathfrak{N}_\sigma$ to denote the class of all $\sigma$-nilpotent groups.

**Lemma 2.2** (See Corollary 2.4 and Lemma 2.5 in [1]). The class $\mathfrak{N}_\sigma$ is closed under taking products of normal subgroups, homomorphic images and subgroups. Moreover, if $E$ is a normal subgroup of $G$ and $E/E \cap \Phi(G)$ is $\sigma$-nilpotent, then $E$ is $\sigma$-nilpotent.

In view of Proposition 2.2.8 in [9], we get from Lemma 2.2 the following

**Lemma 2.3.** If $N$ is a normal subgroup of $G$, then

$$(G/N)^{\mathfrak{N}_\sigma} = G^{\mathfrak{N}_\sigma} N/N.$$
Lemma 2.4 (See Knyagina and Monakhov [10]). Let $H, K$ and $N$ be pairwise permutable subgroups of $G$ and $H$ be a Hall subgroup of $G$. Then

\[ N \cap HK = (N \cap H)(N \cap K). \]

Lemma 2.5. The following statements hold:

(i) $G$ is a $P\sigma T$-group if and only if every $\sigma$-subnormal subgroup of $G$ is $\sigma$-quasinormal in $G$.

(ii) If $G$ is a $P\sigma T$-group, then every quotient $G/N$ of $G$ is also a $P\sigma T$-group.

(iii) If $G$ is a special $P\sigma T$-group, then every quotient $G/N$ of $G$ is also a special $P\sigma T$-group.

Proof. (i) This follows from the fact (see Theorem B in [1]) that every $\sigma$-quasinormal subgroup of $G$ is $\sigma$-subnormal in $G$.

(ii) Let $H/N$ be a $\sigma$-subnormal subgroup of $G/N$. Then $H$ is a $\sigma$-subnormal subgroup of $G$ by Lemma 2.6(5) in [1], so $H$ is $\sigma$-quasinormal in $G$ by hypothesis and Part (i). Hence $H/N$ is $\sigma$-quasinormal in $G/N$ by Lemma 2.8(2) in [1]. Hence $G/N$ is a $P\sigma T$-group by Part (i).

(iii) Suppose that $D = G^{\sigma_{T}}$ is a Hall subgroup of $G$ and $D \leq E$, where $E = D \times S$ is a Hall $\sigma_{T}$-subgroup $E$ of $G$, and every element of $G$ induces a power automorphism in $D$. Then $EN/N$ is a Hall $\sigma_{T}$-subgroup of $G/N$ and $DN/N = (G/N)^{\sigma_{T}}$ is a Hall subgroup of $G/N$ by Lemma 2.3. Moreover, $EN/N = (DN/N)(SN/N)$ and, by Lemma 2.4,

\[ DN \cap SN = N(D \cap SN) = N(D \cap S)(D \cap N) = N(D \cap N) = N, \]

which implies that $(DN/N) \cap (SN/N) = 1$. Hence $EN/N = (DN/N) \times (SN/N)$.

Finally, let $H/N \leq DN/N$. Then $H = N(H \cap D)$, where $H \cap D$ is normal in $G$ by hypothesis. But then $H/N = N(H \cap D)/N$ is normal in $G/N$, so every element of $G/N$ induces a power automorphism on $DN/D$. Hence $G/N$ is a special $P\sigma T$-group.

The lemma is proved.

3 Proofs of the results

Proof of Theorem A. Since $G$ is a $\sigma$-full group of Sylow type by hypothesis, it possesses a complete Hall $\sigma$-set $\mathcal{H} = \{H_1, \ldots, H_t\}$, and a subgroup $H$ of $G$ is $\sigma$-quasinormal in $G$ if and only if $HH_i^x = H_i^x H$ for all $H_i \in \mathcal{H}$ and $x \in G$. We can assume without loss of generality that $H_i$ is a $\sigma_i$-group for all $i = 1, \ldots, t$.

Assume that this theorem is false and let $G$ be a counterexample of minimal order. Then $D \neq 1$ and for some $\sigma$-subnormal subgroup $H$ of $G$ and for some $x \in G$ and $k \in I$ we have $HH_k^x \neq H_k^x H$ by Lemma 2.5(i). Let $E = H_k^x$. 

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(1) The hypothesis holds for every quotient $G/N$ of $G$.

It is clear that $G/N$ is a $\sigma$-full group of Sylow type and $DN/N$ is a normal $\sigma$-Hall subgroup of $G/N$. On the other hand,

$$(G/N)/(DN/N) \simeq G/DN \simeq (G/D)/(DN/D),$$

so $(G/N)/(DN/N)$ is a $P\sigma T$-group by Lemma 2.5(ii). Finally, let $H/N$ be a $\sigma$-subnormal subgroup of $DN/N$. Then $H = N(H \cap D)$ and, by Lemma 2.6(5) in [1], $H$ is $\sigma$-subnormal in $G$. Hence $H \cap D$ is $\sigma$-subnormal in $D$ by Lemma 2.6(1) in [1], so $H \cap D$ is normal in $G$ by hypothesis. Thus $H/N = N(H \cap D)/N$ is normal in $G/N$. Therefore the hypothesis holds on $G/N$.

(2) $H_G = 1$.

Assume that $H_G \neq 1$. Clearly, $H/H_G$ is $\sigma$-subnormal in $G/H_G$. Claim (1) implies that the hypothesis holds for $G/H_G$, so the choice of $G$ implies that $G/H_G$ is a $P\sigma T$-group. Hence

$$(H/H_G)(EH_G/H_G) = (EH_G/H_G)(H/H_G).$$

by Lemma 2.5(i). Therefore $EH = EH_HG$ is a subgroup of $G$ and so $HE = EH$, a contradiction. Hence $H_G = 1$.

(3) $DH = D \times H$.

By Lemma 2.6(1) in [1], $H \cap D$ is $\sigma$-subnormal in $D$. Hence $H \cap D$ is normal in $G$ by hypothesis, which implies that $H \cap D = 1$ by Claim (2). Lemma 2.6(1) in [1] implies also that $H$ is $\sigma$-subnormal in $DH$. But $H$ is a $\sigma$-Hall subgroup of $DH$ since $D$ is a $\sigma$-Hall subgroup of $G$ and $H \cap D = 1$. Therefore $H$ is normal in $DH$ by Lemma 2.6(10) in [1], so $DH = D \times H$.

Final contradiction. Since $D$ is a $\sigma$-Hall subgroup of $G$, then either $E \leq D$ or $E \cap D = 1$. But the former case is impossible by Claim (3) since $HE \neq EH$, so $E \cap D = 1$. Therefore $E$ is a $\Pi'$-subgroup of $G$, where $\Pi = \sigma(D)$. By the Schur-Zassenhaus theorem, $D$ has a complement $M$ in $G$. Then $M$ is a Hall $\Pi'$-subgroup of $G$ and so for some $x \in G$ we have $E \leq M^x$ since $G$ is a $\sigma$-full group of Sylow type. On the other hand, $H \cap M^x$ is a Hall $\Pi'$-subgroup of $H$ by Lemma 2.6(7) in [1] and hence $H \cap M^x = H \leq M^x$ since $H \cap D = 1$ by Claim (3). Lemma 2.6(1) in [1] implies that $H$ is $\sigma$-subnormal in $M^x$. But $M^x \simeq G/D$ is a $P\sigma T$-group by hypothesis, so $HE = EH$ by Lemma 2.5(i). This contradiction completes the proof of the theorem.

**Lemma 3.1.** If $G$ is a special $P\sigma T$-group, then it is a $P\sigma T$-group.

**Proof.** Let $D = G^{\sigma_\mathcal{H}}$ and $E$ be a normal Hall $\sigma_i$-subgroup of $G$ such that $E = D \times S$. Since $G/D$ is $\sigma$-nilpotent, $G$ is $\sigma$-soluble. Hence $G$ is a $\sigma$-full group of Sylow type by Lemma 2.1. Therefore $G$ possesses a complete Hall $\sigma$-set $\mathcal{H} = \{H_1, \ldots, H_t\}$, and a subgroup $H$ of $G$ is $\sigma$-quasinormal in $G$ if and only if $HH_j = H_j H$ for all $H_j \in \mathcal{H}$ and $x \in G$. We can assume without loss of generality that $H_j$ is a $\sigma_j$-group for all $j = 1, \ldots, t$.

Assume that this lemma is false and let $G$ be a counterexample of minimal order. Then $G$ is not $\sigma$-nilpotent, and for some $\sigma$-subnormal subgroup $H$ of $G$ and for some $x \in G$ and $k \in I$ we
have $HH_k^x \neq H_k^x H$ by Lemma 2.5(i). Let $E = H_k^x$. The subgroup $S$ is normal in $G$ since it is characteristic in $E$. Since $G$ is not $\sigma$-nilpotent, $D \neq 1$. On the other hand, Theorem A and the choice of $G$ imply that $S \neq 1$ since every subgroup of $D$ is normal in $G$ by hypothesis. Let $R$ and $N$ be minimal normal subgroups of $G$ such that $R \leq D$ and $N \leq S$. Then $R$ is a group of order $p$ for some prime $p$. Hence $R \cap HN \leq O_p(HN) \leq P$, where $P$ is a Sylow $p$-subgroup of $H$ since $\pi(D) \cap \pi(S) = \emptyset$, so $R \cap HN = R \cap H$.

The hypothesis holds for $G/R$ and $G/N$ by Lemma 2.5(iii). Hence the choice of $G$ and Lemma 2.5(i) imply that

$$EHR/R = (ER/R)(HR/R) = (HR/R)(EHR/R)$$

and so $EHR$ is a subgroup of $G$. Similarly we get that $EHN$ is a subgroup of $G$. Since $|R| = p$ and $EH$ is not a subgroup of $G$, $R \cap E = 1$. Therefore from Lemma 2.4 we get that $R \cap EHN = R \cap E(HN) = (R \cap E)(R \cap HN) = R \cap HN$. Hence

$$EHR \cap EHN = E(ER \cap EHN) = EH(R \cap EHN) = EHR \cap HN = EHR \cap H = EH$$

is a subgroup of $G$. Hence $HE = EH$, a contradiction. The lemma is proved.

**Lemma 3.2.** If $\mathcal{H} = \{H_1, \ldots, H_t\}$ is a generalized Wielandt $\sigma$-set of $G$, then

$$\mathcal{H}_0 = \{H_1N/N, \ldots, H_tN/N\}$$

is a generalized Wielandt $\sigma$-set of $G/N$.

**Proof.** It is clear that $\mathcal{H}_0$ is a complete Hall $\sigma$-set of $G/N$. Now let $D = G^{\sigma_0}$ and $\pi = \pi(G^{\sigma_0})$. Then $(G/N)^{\sigma_0} = DN/N$ by Lemma 2.3, so

$$\pi_0 = \pi((G/N)^{\sigma_0}) = \pi(DN/N) \subseteq \pi(D) = \pi.$$  

Hence every member $H_i$ of $\mathcal{H}$ is $\pi_0$-supersoluble, so $H_iN/N$ is $\pi_0$-supersoluble. Hence $\mathcal{H}_0$ is a generalized Wielandt $\sigma$-set of $G/N$. The lemma is proved.

**Proof of Theorem B.** Clearly, $G$ is $\sigma$-soluble, so $G$ is a $\sigma$-full group of Sylow type by Lemma 2.1. Therefore $G$ possesses a complete Hall $\sigma$-set $\mathcal{H} = \{H_1, \ldots, H_t\}$, and a subgroup $H$ of $G$ is $\sigma$-quasinormal in $G$ if and only if $HH_i^x = H_i^x H$ for all $H_i \in \mathcal{H}$ and $x \in G$. We can assume without loss of generality that $H_i$ is a $\sigma_x$-group for all $i = 1, \ldots, t$.

Assume that this theorem is false and let $G$ be a counterexample of minimal order. Then $D \neq 1$ and for some $\sigma$-subnormal subgroup $H$ of $G$ and for some $x \in G$ and $k \in I$ we have $HH_k^x \neq H_k^x H$ by Lemma 2.5(i). Let $E = H_k^x$.

(1) $G$ is not a special $P\sigma T$-group (This follows from Lemma 3.1 and the choice of $G$).

(2) $|\sigma(D)| > 1$.  

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Indeed, suppose that \( \sigma(D) = \{\sigma_i\} \). Then \( O^{\sigma_i}(D) = 1 \), so \( G \simeq G/O^{\sigma_i}(D) \) is a special \( P\sigma T \)-group by hypothesis, contrary to Claim (1).

(3) The hypothesis holds for every quotient \( G/N \) of \( G \), where \( N \leq D \).

First we show that \( (G/N)/O^{\sigma_i}(D/N) \) is a special \( P\sigma T \)-group for each \( \sigma_i \in \sigma(D/N) \). Note that \( \sigma_i \in \sigma(D/N) = \sigma(D/(D \cap N)) \subseteq \sigma(D) \), so \( G/O^{\sigma_i}(D) \) is a special \( P\sigma T \)-group by hypothesis. It is not difficult to show that

\[
O^{\sigma_i}(D)N/N = O^{\sigma_i}(D/N).
\]

Hence

\[
(G/N)/(O^{\sigma_i}(D/N)) = (G/N)/(O^{\sigma_i}(D)N/N) \simeq G/NO^{\sigma_i}(D) \simeq (G/O^{\sigma_i}(D))/O^{\sigma_i}(D)N/O^{\sigma_i}(D)
\]

is a special \( P\sigma T \)-group by Lemma 2.5(iii).

It is clear also that \( DN/N \simeq D/D \cap N \) is a \( \sigma \)-nilpotent normal Hall subgroup of \( G/N \) with \( \sigma \)-nilpotent quotient

\[
(G/N)/(DN/N) \simeq G/DN \simeq (G/D)/(DN/D)
\]

by Lemma 2.2. Hence we have (3).

(4) If \( N \) is a minimal normal subgroup of \( G \) contained in \( D \), then \( EHN \) is a subgroup of \( G \).

Claim (3) and the choice of \( G \) implies that the conclusion of the theorem holds for \( G/N \). On the other hand, \( EN/E \) is a Hall \( \sigma_k \)-subgroup of \( G/N \) and, by Lemma 2.6(4) in [1], \( HN/N \) is a \( \sigma \)-subnormal subgroup of \( G \). Note also that \( G/N \) is \( \sigma \)-soluble, so every two Hall \( \sigma_k \)-subgroups of \( G/N \) are conjugate by Lemma 2.1. Thus,

\[
(HN/N)(EN/N) = (EN/N)(HN/N) = EHN/N
\]

by Lemma 2.5(i). Hence \( EHN \) is a subgroup of \( G \).

Final contradiction. Since \( |\sigma(D)| > 1 \) by Claim (2) and \( D \) is \( \sigma \)-nilpotent, \( G \) has at least two \( \sigma \)-primary minimal normal subgroups \( R \) and \( N \) such that \( R, N \leq D \) and \( \sigma(R) \neq \sigma(N) \). Then at least one of the subgroups \( R \) or \( N \), \( R \) say, is a \( \sigma_i \)-group for some \( i \neq k \). Then \( R \cap HN \leq O_{\sigma_i}(HN) \leq V \), where \( V \) is a Hall \( \sigma_i \)-subgroup of \( H \), since \( N \) is a \( \sigma_i \)-group and \( G \) is a \( \sigma \)-full group of Sylow type. Hence \( R \cap HN = R \cap H \). Claim (4) implies that \( EHR \) and \( EHN \) are subgroups of \( G \). Now, arguing similarly as in the proof of Lemma 3.1, one can show that \( EHR \cap EHN = EH \) is a subgroup of \( G \), so \( HE = EH \). This contradiction completes the proof of the result.

Proof of Theorem C. Let \( \pi = \pi(D) \) and \( \mathfrak{H} = \{H_1, \ldots, H_t\} \) be a generalized Wielandt \( \sigma \)-set of \( G \). We can assume without loss of generality that \( H_i \) is a \( \sigma_i \)-group for all \( i = 1, \ldots, t \). Since \( G \) is \( \sigma \)-soluble by hypothesis, \( G \) is a \( \sigma \)-full group of Sylow type by Lemma 2.1.

Necessity. Assume that this is false and let \( G \) be a counterexample of minimal order. Then \( D \neq 1 \).
(1) If $R$ is a non-identity normal subgroup of $G$, then the hypothesis holds for $G/R$. Hence the
necessity condition of the theorem holds for $G/R$ (Since the hypothesis holds for $G/R$ by Lemmas
2.5(ii) and 3.2, this follows from the choice of $G$).

(2) If $E$ is a proper $\sigma$-subnormal subgroup of $G$, then $E^{\Omega_\sigma} \leq D$ and the necessity condition
of the theorem holds for $E$.

Every $\sigma$-subnormal subgroup $H$ of $E$ is $\sigma$-subnormal in $G$ by Lemma 2.6(2) in [1] and hence $H$
is $\sigma$-quasinormal in $G$ by hypothesis and Lemma 2.5(i). Thus $H$ is $\sigma$-quasinormal in $E$ by Lemma
2.8(1) in [1] since $G$ is a $\sigma$-full group of Sylow type. Thus, $E$ is a $\sigma$-soluble $P\sigma T$-group. It is clear
that $E$ possesses a complete Hall $\sigma$-set $H_0 = \{E_1, \ldots, E_n\}$ such that $E_i \leq H_i^{x_i}$ for some $x_i \in G$
for all $i = 1, \ldots, n$. Hence every member of $H_0$ is $\pi$-supersoluble. Moreover, since

$$E/E \cap D \simeq ED/D \in \mathfrak{R}_\sigma$$

and $\mathfrak{R}_\sigma$ is a hereditary class by Lemma 2.2, we have $E/E \cap D \in \mathfrak{R}_\sigma$. Hence $E^{\Omega_\sigma} \leq E \cap D$. Therefore,

$$\pi_0 = \pi(E^{\Omega_\sigma}) \subseteq \pi.$$ Hence every member of $H_0$ is $\pi_0$-supersoluble. Hence $H_0$ is a generalized Wielandt
$\sigma$-set of $E$.

Therefore the hypothesis holds for $E$, so the necessity condition of the theorem holds for $E$ by
the choice of $G$.

(3) $D$ is nilpotent.

Assume that this is false and let $R$ be a minimal normal subgroup of $G$. Then $RD/R = (G/R)^{\Omega_\sigma}$
is abelian by Lemma 2.3 and Claim (1). Therefore $R \leq D$, $R$ is the unique minimal normal subgroup
of $G$ and $R \not\leq \Phi(G)$ by Lemma 2.2. Let $V$ be a maximal subgroup of $R$. Since $G$ is $\sigma$-soluble by
hypothesis, $R$ is a $\sigma_i$-group for some $i$. Hence $V$ is $\sigma$-subnormal in $G$ by Lemma 2.6(6) in [1], so $V$
is $\sigma$-quasinormal in $G$ by hypothesis and Lemma 2.5(i). Then $R \leq D \leq O^{\sigma_i}(G) \leq N_G(V)$ by Lemma
3.1 in [1]. Hence $R$ is abelian, so $R = C_G(R)$ is a $p$-group for some prime $p$ by [11] A, 15.2]

It is clear that $R \leq H_i \cap D$ for some $i$. Then $H_i$ is $p$-supersoluble by hypothesis, so some subgroup
$L$ of $R$ of order $p$ is normal in $H_i$. On the other hand, $L$ is clearly $\sigma$-quasinormal in $G$ and hence
$G = H_i O^{\sigma_i}(G) \leq N_G(L)$ by Lemma 3.1 in [1], so $R = L$. Therefore $G/C_G(R) = G/R$ is a cyclic
group. Hence $G$ is supersoluble and therefore $D$ is nilpotent.

(4) $D$ is a Hall subgroup of $G$.

Suppose that this is false and let $P$ be a Sylow $p$-subgroup of $D$ such that $1 < P < G_p$, where
$G_p \in \text{Syl}_p(G)$. We can assume without loss of generality that $G_p \leq H_1$.

(a) $D = P$ is a minimal normal subgroup of $G$.

Let $R$ be a minimal normal subgroup of $G$ contained in $D$. Since $D$ is nilpotent by Claim (3), $R$
is a $q$-group for some prime $q$. Moreover, $D/R = (G/R)^{\Omega_\sigma}$ is a Hall subgroup of $G/R$ by Claim (1)
and Lemma 2.3. Suppose that $PR/R \neq 1$. Then $PR/R \in \text{Syl}_p(G/R)$. If $q \neq p$, then $P \in \text{Syl}_p(G)$.
This contradicts the fact that $P < G_p$. Hence $q = p$, so $R \leq P$ and therefore $P/R \in \text{Syl}_p(G/R)$ and
we again get that \( P \in \text{Syl}_p(G) \). This contradiction shows that \( PR/R = 1 \), which implies that \( R = P \) is the unique minimal normal subgroup of \( G \) contained in \( D \). Since \( D \) is nilpotent, a \( p' \)-complement \( E \) of \( D \) is characteristic in \( D \) and so it is normal in \( G \). Hence \( E = 1 \), which implies that \( R = D = P \).

(b) \( D \not\leq \Phi(G) \). Hence for some maximal subgroup \( M \) of \( G \) we have \( G = D \times M \) (This follows from Lemma 2.2 since \( G \) is not \( \sigma \)-nilpotent).

(c) If \( G \) has a minimal normal subgroup \( L \neq D \), then \( G_p = D \times (L \cap G_p) \). Hence \( O_{p'}(G) = 1 \).

Indeed, \( DL/L \simeq D \) is a Hall subgroup of \( G/L \) by Claim (1). Hence \( G_pL/L = RL/L \), so \( G_p = D \times (L \cap G_p) \). Thus \( O_{p'}(G) = 1 \) since \( D < G_p \) by Claim (a).

(d) \( V = C_G(D) \cap M \) is a normal subgroup of \( G \) and \( C_G(D) = D \times V \leq H_1 \).

In view of Claim (b), \( C_G(D) = D \times V \), where \( V = C_G(D) \cap M \) is a normal subgroup of \( G \). By Claim (a), \( V \cap D = 1 \) and hence \( V \simeq DV/D \) is \( \sigma \)-nilpotent by Lemma 2.2. Let \( W \) be a \( \sigma_1 \)-complement of \( V \). Then \( W \) is characteristic in \( V \) and so it is normal in \( G \). Therefore we have (d) by Claim (c).

(e) \( G_p \neq H_1 \).

Assume that \( G_p = H_1 \). Let \( Z \) be a subgroup of order \( p \) in \( Z(G_p) \cap D \). Then, since \( D \leq O^{\sigma_1}(G) = O^p(G) \), \( Z \) is normal in \( G \) by Lemma 3.1 in [1]. Hence \( D = Z < G_p \) and so \( D < C_G(D) \). Then \( V = C_G(D) \cap M \neq 1 \) is a normal subgroup of \( G \) and \( V \leq H_1 = G_p \) by Claim (d). Let \( L \) be a minimal normal subgroup of \( G \) contained in \( V \). Then \( G_p = D \times L \) is a normal elementary abelian subgroup of \( G \). Therefore every subgroup of \( G_p \) is normal in \( G \) by Lemma 3.1 in [1]. Hence \( |D| = |L| = p \). Let \( D = \langle a \rangle \), \( L = \langle b \rangle \) and \( N = \langle ab \rangle \). Then \( N \not\leq D \), so in view of the \( G \)-isomorphisms

\[
DN/D \simeq N \simeq NL/L = G_p/L = DL/L \simeq D
\]

we get that \( G/C_G(D) = G/C_G(N) \) is a \( p \)-group since \( G/D \) is \( \sigma \)-nilpotent by Lemma 2.2. But then Claim (d) implies that \( G \) is a \( p \)-group. This contradiction shows that we have (e).

Final contradiction for (4). In view of Theorem A in [1], \( G \) has a \( \sigma_1 \)-complement \( E \) such that \( EG_p = G_pE \). Let \( V = (EG_p)^{G_p} \). By Claim (e), \( EG_p \neq G \). On the other hand, since \( D \leq EG_p \) by Claim (a), \( EG_p \) is \( \sigma \)-subnormal in \( G \) by Lemma 2.6(5) in [1]. Therefore the necessity condition of the theorem holds for \( EG_p \) by Claim (2). Hence \( V \) is a Hall subgroup of \( EG_p \). Moreover, by Claim (2) we have \( V \leq D \), so for a Sylow \( p \)-subgroup \( V_p \) of \( V \) we have \( |V_p| \leq |P| < |G_p| \). Hence \( V \) is a \( p' \)-group and so \( V \leq C_G(D) \leq H_1 = G_p \). Thus \( V = 1 \). Therefore \( EG_p = E \times G_p \) is \( \sigma \)-nilpotent and so \( E \leq C_G(D) \leq H_1 \) by Claim (d). Hence \( E = 1 \) and so \( D = 1 \), a contradiction. Thus, \( D \) is a Hall subgroup of \( G \).

(5) \( G/O^{\sigma_i}(D) \) is a special \( P\sigma T \)-group for each \( \sigma_i \in \sigma(D) \).

First assume that \( O^{\sigma_i}(D) \neq 1 \) and let \( N \) be a minimal normal subgroup of \( G \) contained in \( O^{\sigma_i}(D) \). Then \( G/N \) is a \( P\sigma T \)-group by Lemma 2.5(ii), so the choice of \( G \) implies that

\[
(G/N)/O^{\sigma_i}(D)/N = (G/N)/(O^{\sigma_i}(D)/N) \simeq G/O^{\sigma_i}(D)
\]
is a special $P\sigma T$-group. Now assume that $O^\sigma_i(D) = 1$, that is, $D$ is a $\sigma_i$-group. Since $G/D$ is
$\sigma$-nilpotent by Lemma 2.2, $H_i/D$ is normal in $G/D$ and hence $H_i$ is normal in $G$. Therefore all
subgroups of $D$ are $\sigma$-permutable in $G$ by Lemma 2.3(2)(3) and hypothesis. Since $D$ is a normal
Hall subgroup of $H_i$, it has a complement $S$ in $H_i$ by the Schur-Zassenhaus theorem. Lemma 3.1 in
[1] implies that $D \leq O^\sigma_i(G) \leq N_G(S)$. Hence $H_i = D \times S$. Therefore
$$G = H_i O^\sigma_i(G) = SO^\sigma_i(G) \leq N_G(L)$$
for every subgroup $L$ of $D$. Hence every element of $G$ induces a power automorphism in $D$. Hence
$G$ is a special $P\sigma T$-group.

(6) Every subgroup $H$ of $D$ is normal in $G$. Hence every element of $G$ induces a power automor-
phism in $D$.

Since $D$ is nilpotent by Claim (3), it is enough to consider the case when $H$ is a subgroup of the
Sylow $p$-subgroup $P$ of $D$ for some prime $p$. For some $i$ we have $P \leq O_{\sigma_i}(D) = H_i \cap D$. On the
other hand, we have
$$D = O_{\sigma_i}(D) \times O^\sigma_i(D)$$
since $D$ is nilpotent. Then
$$H O^\sigma_i(D)/O^\sigma_i(D) \leq D/O^\sigma_i(D) = (G/O^\sigma_i(D))^{\mathfrak{m}_p},$$
so $H O^\sigma_i(D)/O^\sigma_i(D)$ is normal in $G/O^\sigma_i(D)$ by Claim (5). Hence $H O^\sigma_i(D)$ is normal in $G$, which
implies that
$$H = H(O^\sigma_i(D) \cap O_{\sigma_i}(D)) = H O^\sigma_i(D) \cap O_{\sigma_i}(D)$$
is normal in $G$.

(7) If $p$ is a prime such that $(p - 1, |G|) = 1$, then $p$ does not divide $|D|$. In particular, $|D|$ is
odd.

Assume that this is false. Then, by Claim (6), $D$ has a maximal subgroup $E$ such that $|D : E| = p$
and $E$ is normal in $G$. It follows that $C_G(D/E) = G$ since $(p - 1, |G|) = 1$. Since $D$ is a Hall subgroup
of $G$, it has a complement $M$ in $G$. Hence $G/E = (D/E) \times (ME/E)$, where $ME/E \cong M \cong G/D$ is
$\sigma$-nilpotent. Therefore $G/E$ is $\sigma$-nilpotent by Lemma 2.2. But then $D \leq E$, a contradiction. Hence
$p$ does not divide $|D|$. In particular, $|D|$ is odd.

(8) $D$ is abelian.

In view of Claim (6), $D$ is a Dedekind group. Hence $D$ is abelian since $|D|$ is odd by Claim (7).

From Claims (4)–(8) we get that the necessity condition of the theorem holds for $G$.

Sufficiency. This directly follows from Theorem B.

The theorem is proved.
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