Stochastic differential equations with multiplicative noise: the Markov solution

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Abstract
Fokker-Planck equations are decisive for the Markov property. With multiplicative noise they have non-Gaussian solutions in short times. Corresponding new path increments agree with a FPE by both their expectation and their mode. The mean implies the “anti-Ito” sense, and the mode follows the noiseless motion. The paths are given by the anti-Ito integral, except for smallest integration times.

If the noiseless drift has an attractor, a possible steady density has a maximum there. Such a density is determined by the Freidlin equation for the quasipotential. The global existence depends on a new condition which is related with “detailed balance”, but does not reverse the time.

Key words: Stochastic differential equations; multiplicative noise; random paths; Fokker-Planck equation; steady states.
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I. Introduction

A state-dependent coupling with the noise sources ("multiplicative noise") entails a diffusion depending on the (random) state. Solutions of FPEs are therefore not Gaussian in short time steps. Paths of a Markov solution must thus have non-Gaussian increments. These are readily found by expanding the coupling coefficients with the noise: their first derivatives introduce the squared increments \((dW)^2\) of the driving Wiener process. The mean of these path increments entails the “anti-Ito” case – as well as the corresponding FPE - while their mode coincides with the noiseless motion. In the stochastic integral for the paths (in \([0, t]\)) one may replace \((dW)^2\) by \(dt\), and the result is the anti-Ito integral. The asymptotics for small \(t\) is however not unique, and the replacement is thus not possible for smallest \(t\). Nevertheless the standard methods apply for the numerical computation of the paths in any given \([0, t]\).

The FPE can be cast into a form that merely involves the noiseless drift and the diffusion matrix; the latter is only differentiated once.

Focusing on the Markov property saves the well-known problems of modelling [1,2]. It further provides a new analysis of possible steady densities. These must have a maximum on the attractor of the noiseless motion. Their local shape is determined by the Freidlin equation [3] (with a new result for the second derivatives), and the global validity of that equation will be shown to depend on a new explicit condition that entails a main feature of “detailed balance” [4], but without reversing the time.

II. Background

The continuous Markov process \(\tilde{X}(t)\) is assumed to fulfill the stochastic differential equation (SDE) [3,5-8]

\[
dX^i = a^i(\tilde{X})dt + b^{ik}(\tilde{X})dW_k \quad \text{or} \quad d\tilde{X} = \tilde{a}(\tilde{X})dt + \tilde{B}(\tilde{X})d\tilde{W}
\]

(2.1)

with smooth functions \(a^i(\tilde{x}), b^{ik}(\tilde{x})\). The drift \(\tilde{a}\) is supposed to be independent of the
noise. As usual, (2.1) denotes an integral equation, with the “integration sense” specified by $\alpha$ ($0 \leq \alpha \leq 1$; “Ito” for $\alpha = 0$, “Stratonovich” for $\alpha = 1/2$ and “anti-Ito” for $\alpha = 1$).

The Wiener processes $W_k(t)$ are Gaussian distributed, with $<W_k(t) - W_k(0)> = 0$ and $<[W_k(t) - W_k(0)]^2> = t$; they are independent of each other.

The standard path increments, with given $\mathbf{X}(t)$, are given by

$$
\mathbf{X}(t + dt) - \mathbf{X} = \tilde{a}(\mathbf{x})dt + B(\mathbf{x})d\mathbf{W} + \alpha \tilde{a}_{sp}(\mathbf{x})dt + o(dt),
$$

(2.2)

where $d\mathbf{W} = \mathbf{W}(t + dt) - \mathbf{W}(t)$, and with the “spurious” drift

$$
a^i_{sp}(\mathbf{x}) = b^{ij, k}(\mathbf{x})b^{kj}(\mathbf{x}) = (B_k B^T)^{ik}.
$$

(2.3)

Note that (2.2) is Gaussian distributed, since the coefficients are taken at the initial $\mathbf{x}$.

The “diffusion matrix”

$$
D(\mathbf{x}) = B(\mathbf{x})B^T(\mathbf{x})
$$

(2.4)

is symmetric and nonnegative.

The relevant $\tilde{a}_{sp}$ is determined by the $\mathbf{x}$- dependence of $D(\mathbf{x})$:

$$
a^i_{sp} = D^{ik, k}/2.
$$

(2.5)

This is evident when $B(\mathbf{x})$ is diagonal (thus in one dimension), and it generally holds by stochastic equivalence, see [9] Appendix B; an independent argument will be given below.

The density $w(\mathbf{x}, t)$ of $\mathbf{X}(t)$ (i.e. of the leading points of the random paths) is determined by the FPE

$$
w_t = -(a^i + \alpha a^i_{sp})w + (1/2)(D^{ik}w)_k,l.
$$

(2.6)

By $(D^{ik}w)_k = D^{ik, k}w + D^{ik}w_k$ and by (2.5) it can be rewritten as

$$
w_t = \nabla \cdot \{-[\tilde{a} + (\alpha - 1)\tilde{a}_{sp}]w + D\nabla w/2\}.
$$

(2.7)

With the probability current

$$
\tilde{f}(\mathbf{x}, t) = [\tilde{a} + (\alpha - 1)\tilde{a}_{sp}]w - D\nabla w/2
$$

(2.8)
it amounts to the continuity equation $w_x + \nabla \cdot \vec{j} = 0$.

The “propagator” is defined as the solution of (2.7) with an initial deltafunction, asymptotically for small time steps. It is only a Gaussian when $D(\vec{x})$ is constant [every $D(\vec{x})$ with a constant rank becomes constant under a nonlinear transform of the variables $\vec{x}$ [9], and the resulting Gaussian propagator is distorted by the inverse transform].

III. The new approach for multiplicative noise $(a_{Sp} \neq \vec{0})$

3.1 Preliminary remarks

The SDE is understood as an integral equation and can thus have non-Markovian solutions. Constructing the paths by conditional increments in consecutive short time steps is only meaningful in the Markov case, since otherwise an increment would stochastically depend on some earlier information than its initial value. Each continuous Markov process is associated with a FPE, and the propagator of that FPE is the probability density of the path increments. That correspondence is incomplete in the existing theory. New path increments are required; their agreement with the FPE includes the mode, in addition to the mean.

3.2 The path increments

For simplicity this is first explained in one dimension and without the drift $a(x)$. Letting $b(X) \approx b(x) + b'(x) b(x) dW$ (and observing the order $O(\sqrt{dt})$ of $dW$) results in

$$X(dt) - x = b(x) dW + a_{Sp}(x) (dW)^2 + o(dt). \quad (3.1)$$

While the mean of (3.1) agrees with the standard increment for $\alpha = 1$, its most probable value vanishes, since the density of $Y := (dW)^2$ is given by

$$(8\pi dt y)^{-1/2} \exp(-y/2dt),$$

which even diverges at $y = 0$.

In higher dimensions consider $b^{\mu}(\vec{x} + d\vec{x}) = b^{\mu}(\vec{x}) + b^{\mu,\lambda}(\vec{x}) \, d\vec{x}^\lambda$ and insert $d\vec{x}^\lambda \equiv b^{\lambda \nu} dW_\nu$ to obtain the extra noise term
\[ b^{\mu \nu} b^\lambda \lambda dW_\mu dW_\nu := Q^i. \] (3.2)

**Properties:** \( \langle \tilde{Q} \rangle = \tilde{a}_{sp} dt \) by (2.3), and the most probable \( \tilde{Q} \) is zero.

The new (non-Gaussian) path increment is thus given by
\[
\tilde{X}(dt) - \tilde{x} = \tilde{a}(\tilde{x}) dt + \tilde{B}(\tilde{x}) d\tilde{W} + \tilde{Q}(\tilde{x}) + o(dt).
\] (3.3)

Comparing the expectations of (3.3) and (2.2) shows that \( \alpha = 1 \).

### 3.3 The Fokker-Planck equation

The FPE with \( \alpha = 1 \) is given by
\[
w_t = \nabla \cdot (-\tilde{a} w + D \nabla w/2),
\] (3.4)
with the FP-operator \( L := -\frac{\partial}{\partial x^i} a^i + \frac{\partial}{\partial x^i} D^{ik} \frac{\partial}{\partial x^k} \), and with the current
\[
\tilde{j} = \tilde{a} w - D \nabla w/2.
\] (3.5)

Note that (3.4) and (3.5) do not involve \( \tilde{a}_{sp} \). It is interesting to consider the mean increment from \( \tilde{x} \). Its time-derivative is given by \( \int (\tilde{\xi} - \tilde{x}) L(\tilde{\xi}) \delta(\tilde{\xi} - \tilde{x}) d\tilde{\xi} \).

Integration by parts leads to \( a^i + D^{ik} k/2 \) [the same follows by \( L^+(x) \) is the backward operator, i.e. the adjoint of \( L \)]. With the expectation of (3.3) this restates (2.5), besides confirming (3.4) for the new paths.

**Further properties:**

(i) Its propagator has its maximum at the most probable value of (3.3), i.e. at \( \tilde{a}(\tilde{x})dt \).

It is not a Gaussian, in view of the mean (\( \tilde{a} + \tilde{a}_{sp} \))dt.

(ii) Wherever \( \nabla w(\tilde{x}, t) = \tilde{0} \) the velocity of the probability current (3.5) just equals \( \tilde{a}(\tilde{x}) \)

(the noise interferes by \( \nabla w \)). That evident feature holds by \( \alpha = 1 \).

**An explicit example in one dimension:** Assume \( a(x) = -x \), as well as \( 0 < b(x) < \infty \), \( b'(x) > 0 \), whence \( a_{sp} > 0 \). The steady density follows by \( J = -xw - b^2 w'/2 \equiv 0 \). Its maximum is at \( x = 0 \). There the most probable increment \( a(0)dt \) vanishes, while the mean one \( [a(0) + a_{sp}(0)]dt \) is positive (in accordance with the positive mean of the steady
density). This agrees with (3.1), while no (Gaussian) increment (2.2) distinguishes these cases. The adequate path increment is indeed given by (3.1).

3.4 The most probable path

The most probable path (starting from some $\vec{x}_0$) is given by adding up the most probable increments $\vec{a} \, dt$ in consecutive time steps. This yields the noiseless path $\vec{x}_\ell(t)$ given by $\vec{x}_\ell = \vec{a}(\vec{x}_\ell)$, with $\vec{x}_\ell(0) = \vec{x}_0$. It follows that the density (starting with a deltafunction at $\vec{x}_0$) has its maximum moving on $\vec{x}_\ell(t)$; recall the property (ii) above which shows that $\vec{J}$ parallels the path. When $\vec{a}(\vec{x})$ has an attractor, $\vec{x}_\ell(t)$ describes the most probable approach to it, and the steady density thus has a maximum there. This is essential for the new analysis of the steady states in the following Chap. IV. [A superposition of densities with different starting points (or times) is not admitted].

3.5 The integral for the paths

The integral for the resolving paths is obtained by partitioning a given interval $[0, t]$ into $m$ steps, by adding up the respective conditional increments (3.3) (with the initial value given by the $\vec{X}$-value reached after the preceding step), and by letting $m \to \infty$; $dt = t / m$. It is a basic fact of stochastic analysis that in an integrand $(dW)^2$ can be replaced by $dt$, as well as $dW_i \, dW_k$ ($i \neq k$) by zero. Since $\vec{Q}$ of (3.2) then reduces to $\vec{a}_{sp} dt$, the above Riemannian sum converges to the anti-Ito integral. This holds for a given $t > 0$. It would not be meaningful to let $t \to 0$ a posteriori (the derivative of a stochastic integral with respect to the upper limit does not exist). Path increments in infinitesimal times are given by the exact $\vec{Q}$. Numerical computations of the paths in a given $[0, t]$ can still be carried out by the methods of [10].

The main novelty for applications is the fact that $\alpha = 1$. It is also worth noting that the FPE (3.4) merely involves the noiseless drift and the diffusion matrix; the latter is only differentiated once.
IV. Steady states

4.1 The quasipotential

It is understood that a steady solution $w(\vec{x})$ of (3.4) can be written as $N \exp[-\Phi(\vec{x})]$ with the “quasipotential” $\Phi$. Note that $\nabla w = -w \nabla \Phi$. The current $\vec{j} = \vec{\alpha} w - D \nabla w / 2$ thus takes the form

$$\vec{j} = w \vec{\alpha}_c$$

with the “conservative drift” $\vec{\alpha}_c := \vec{\alpha} + D \nabla \Phi / 2$, \hfill (4.1)

where $\vec{\alpha}_c$ is the velocity of the steady current. The FPE (3.4) becomes $0 = \nabla \cdot (w \vec{\alpha}_c)$ which yields the general equation

$$\nabla \cdot \vec{\alpha}_c - \vec{\alpha}_c \cdot \nabla \Phi = 0$$

for $\Phi(\vec{x})$.

In one dimension the current $\vec{j} = aw - Dw'/2 = (a + D\Phi'/2)$ vanishes when $0 = a + D\Phi'/2 = \vec{\alpha}_c$, whence $\Phi' = 2a/D$. In higher dimensions $\vec{j}$ satisfies $\nabla \cdot \vec{j} = 0$, and it only must vanish where $\vec{\alpha} = 0$. The straightforward extension of $a + D\Phi'/2 = 0$ is

$$\vec{\alpha} = (-D/2 + A) \nabla \Phi$$

with an antisymmetric (and nonsingular) $A(\vec{x})$, to be determined below. Note that

$$\nabla \Phi = 0$$

where $\vec{\alpha} = 0$ and that $\vec{j} = 0$ there by (4.1). It further follows by (4.1) that

$$\vec{\alpha}_c = A \nabla \Phi ,$$

so that $\vec{\alpha}_c \perp \nabla \Phi$, which amounts to the Freidlin equation [3]

$$\vec{\alpha} + D \nabla \Phi / 2 \cdot \nabla \Phi = 0 .$$

Any $\Phi$ given by

$$\nabla \Phi = (-D/2 + A)^{-1} \vec{\alpha}$$

thus satisfies (4.5). By (4.2) it then follows that $\nabla \cdot \vec{\alpha}_c = 0$, and (4.4) then implies that $A$ is constant (since $\nabla \Phi$ is a gradient).

Clearly (4.6) must fulfill $\Phi_{,ik} = \Phi_{,ki}$, i.e. as many conditions as independent elements of $A$. 

Before specifying \( A \), it is useful to insert \( \nabla \Phi = A^{-1} \vec{a}_c \) into \( \vec{a} = (-D/2 + A) \nabla \Phi \) and to take the divergence. With the dissipation \( -\nabla \cdot \vec{a} := \rho \) this yields
\[
2\rho = \nabla \cdot D A^{-1} \vec{a}_c = \nabla \cdot D A^{-1}[\vec{a} + (D/2) \nabla \Phi].
\]
That relation must not really involve \( \Phi \) or a functional of it (this would create an extra condition for \( \Phi \)). This leaves the necessary condition
\[
\nabla \cdot D A^{-1} \vec{a} = 2\rho .
\] (4.7)
Since \( D A^{-1}D \) is antisymmetric, (4.7) follows when \( D \) is nonsingular and constant; this excludes multiplicative noise. A nonconstant \( D \) is however admitted where \( \nabla \Phi = \vec{0} \).

*Note:* when \( D \) is singular, \( D A^{-1}D \) may vanish, so that (4.7) also applies with a nonconstant \( D \), see the Klein-Kramers example below.

The relation (4.7) provides the evaluation of \( A \). Since \( A \) is constant, this can be carried out where \( \vec{a} = \vec{0} \) (and \( \nabla \Phi = \vec{0} \) so that (4.7) applies). With the matrix \( M \) of the elements \( a^i_k \), (4.7) reduces to
\[
D^{ik} A^{-1}_{ki} M^{li} = tr D A^{-1} M = tr A^{-1} M D = 2\rho .
\]
Splitting \( M D \) into its symmetric and antisymmetric parts shows that only the latter contributes, and it follows that
\[
A = (MD)_a/2\rho \quad \text{where} \quad (MD)_a := [MD - (MD)^T]/2 .
\] (4.8)
This is a *new* and *explicit* result for \( A \).

One can now also specify the second derivatives of \( \Phi \) where \( \vec{a} = \vec{0} \): their matrix \( S \) is given by (4.6) as
\[
S = (-D/2 + A)^{-1} M .
\] (4.9)
This includes multiplicative noise. (These second derivatives provide the starting values of the characteristics of (4.5) [11], but a smooth \( \Phi(\vec{x}) \) is thereby only obtained when (4.7) is fulfilled).

As an example with a singular \( D \), consider the Klein-Kramers equation [4] with the variables \( x, v : v w_x + (\gamma v - U')w + \gamma T w_v = 0 \). Clearly
\[ \vec{a} = (-\gamma v - U') \quad \text{and} \quad D = \begin{pmatrix} 0 & 0 \\ 0 & \gamma T \end{pmatrix}, \] and (4.8) yields \[ A = T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \] Mind that the temperature \( T \) must be constant, while the friction parameter \( \gamma \) is not involved in \( A \) and can thus be any \( \gamma(x, v) > 0 \); (4.7) is therefore fulfilled with a nonconstant \( D \).

**Discussion**

- The Freidlin equation plays a central role (without referring to weak noise). According to the Sect. 3.4 \( \nabla \Phi = \vec{0} \) where \( \vec{a} = \vec{0} \). The local quadratic approximation of \( \Phi \) always holds, but the further extension requires the new explicit condition (4.7). That extension applies in Euclidian spaces and with natural boundaries, escapes to infinity are prevented by \( \vec{a}(\vec{x}) \).

- The expression (4.3) agrees with detailed balance: the drift \( A \nabla \Phi = \vec{a}_c \) is reversible, since a change of its sign is immaterial for (4.5), and \( \vec{a} - \vec{a}_c = -D \nabla \Phi / 2 \) is irreversible. Time is however not reversed in the new approach. This means that the individual \( \vec{x}^i \) need not have any pertinent symmetries, which is a substantial generalization.

- Smooth solutions of the Freidlin equation also exist with a \( \vec{x} \)-dependent \( A \) [12], but they do not satisfy (4.2) since \( \nabla \cdot \vec{a}_c \neq 0 \).

- For the second derivatives of \( \Phi \) where \( \nabla \Phi = \vec{0} \) see also [13,14].

**References**

[1] van Kampen N.G. *Stochastic Processes in Physics and Chemistry* (Elsevier/North-Holland, Amsterdam, 3rd ed. 2008).

[2] Mannella R., McClintock P.V.E. *Itô versus Stratonovich: 30 years later*, Fluctuation and Noise Letters, 11 (1) pp. 1240010-1240019, 2012.

[3] Freidlin M.I., Wentzell A.D. *Random Perturbations of Dynamical Systems*, Springer, Berlin, 1984.
[4] Risken H. *The Fokker-Planck Equation*, Springer, Berlin, 1989 2nd ed.

[5] Arnold L. *Stochastische Differentialgleichungen*, Oldenbourg, München, 1973, and
*Stochastic Differential Equations: Theory and Applications*, Wiley, New York, 1974

[6] Gichman I.I., Skorochod A.W. “Stochastische Differentialgleichungen”, Akademie-
Verlag, Berlin, 1971

[7] Oksendal B. *Stochastic differential equations*, Springer, Berlin, Heidelberg, 1992

[8] Gardiner C.W. *Handbook of Stochastic Methods*, Springer, Berlin, Heidelberg 1994

[9] Ryter D. *The intrinsic "sense" of stochastic differential equations*, arXiv 1605.02897
(v3)

[10] Kloeden P., Platen E. *Numerical Solution of Stochastic Differential Equations*,
Springer 1992

[11] Ludwig D. *Persistence of dynamical systems under random perturbations*, SIAM Rev.
vol.17, no.4, pp.605-640,1975.

[12] Ryter D. *The exit problem at weak noise, the two-variable quasipotential, and the
Kramers problem* J.Stat.Phys.149, pp. 1069–1085, 2012

[13] Kwon C., Ao P., Thouless D.J. *Structure of stochastic dynamics near fixed points*
Proc.Natl.Acad.Sci.USA, vol. 102, pp.13029-13033, 2005

[14] Ryter D. *Weak-noise solution of Fokker–Planck equations with n (> 2) variables: A
simpler evaluation near equilibrium points* Phys.Lett.A, vol. 377, pp. 2280-2283,
2013