AN M-FUNCTION ASSOCIATED WITH GOLDBACH’S PROBLEM

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Abstract. We prove the existence of the $M$-function, by which we can state the limit theorem for the value-distribution of the main term in the asymptotic formula for the summatory function of the Goldbach generating function.

1. The Goldbach generating function

One of the most famous unsolved problems in number theory is Goldbach’s conjecture, which asserts that all even integer $\geq 6$ can be written as a sum of two odd primes.

Let

$$r_2(n) = \sum_{l+m=n} \Lambda(l)\Lambda(m),$$

where $\Lambda(\cdot)$ denotes the von Mangoldt function. This may be regarded as the Goldbach generating function. In fact, Goldbach’s conjecture would imply $r_2(n) > 0$ for all even $n \geq 6$. Hardy and Littlewood [15] conjectured that $r_2(n) \sim nS_2(n)$ for even $n$ as $n \to \infty$, where

$$S_2(n) = \prod_{p | n} \left(1 + \frac{1}{p-1}\right) \prod_{p \nmid n} \left(1 - \frac{1}{(p-1)^2}\right)$$

($p$ denotes the primes). In view of this conjecture, it is interesting to evaluate the sum

$$A_2(x) = \sum_{n \leq x} (r_2(n) - nS_2(n)) \quad (x > 0).$$

It is known that the estimate $A_2(x) = O(x^{3/2+\varepsilon})$ (where, and in what follows, $\varepsilon$ is an arbitrarily small positive number) is equivalent to the Riemann hypothesis (RH) for the Riemann zeta-function $\zeta(s)$ (see Granville [14], Bhowmik and Ruzsa [4], Bhowmik et al. [3]).

2010 Mathematics Subject Classification. Primary 11M41, Secondary 11P32, 11M26, 11M99.

Key words and phrases. Goldbach’s problem, $M$-function, Riemann zeta-function.
The unconditional estimate \( A_2(x) = O(x^2 \log x)^{-A} \) \((A > 0)\) was classically known. In 1991, Fujii published a series of papers [11] [12] [13], in which he refined this classical estimate under the RH. Fujii first proved \( A_2(x) = O(x^{3/2}) \) in [11], and then in [12], he gave the following asymptotic formula

\[
A_2(x) = -4x^{3/2} \cdot \Re \Psi(x) + R(x),
\]

where \( R(x) \) is the error term, and

\[
\Psi(x) = \sum_{\gamma > 0} \frac{x^{i\gamma}}{(1/2 + i\gamma)(3/2 + i\gamma)} = \sum_{m=1}^{\infty} \frac{x^{i\gamma_m}}{(1/2 + i\gamma_m)(3/2 + i\gamma_m)},
\]

with \( \gamma \) running over all imaginary parts of non-trivial zeros of \( \zeta(s) \) which are positive. We number those imaginary parts as \( 0 < \gamma_1 < \gamma_2 < \cdots < \gamma_m < \cdots \).

Concerning the error term \( R(x) \), Fujii [12] showed the estimate \( R(x) = O(x^{4/3}(\log x)^{4/3}) \). Egami and the author [10] raised the conjecture

\[
R(x) = O(x^{1+\varepsilon}), \quad R(x) = \Omega(x).
\]

This conjecture was settled by Bhowmik and Schlage-Puchta [5] in the form

\[
R(x) = O(x(\log x)^5), \quad R(x) = \Omega(x \log \log x).
\]

The best upper-bound estimate at present is \( O(x(\log x)^3) \). As for the more detailed history, see [2].

Properties of the main term on the right-hand side of (1.1) was first considered by Fujii [13]. Let

\[
f(\alpha) = \Psi(e^{\alpha}) \quad (\alpha \in \mathbb{R}).
\]

In [13], Fujii studied the value-distribution of \( f(\alpha) \), and proved the following limit theorem. Assume that \( \gamma \)'s are linearly independent over \( \mathbb{Q} \) (which we call the LIC). Then Fujii stated the existence of the “density function” \( F(z) \) \((z = x + iy \in \mathbb{C})\) for which

\[
\lim_{X \to \infty} \frac{1}{X} \mu\{0 \leq \alpha \leq X \mid f(\alpha) \in R\} = \iint_{R} F(x + iy)dx\,dy
\]

holds for any rectangle \( R \) in \( \mathbb{C} \), where \( \mu\{\cdot\} \) means the one-dimensional Lebesgue measure. This is an analogue of the following result of Bohr and Jessen [8] [9] for the value-distribution of \( \zeta(s) \). Let \( \sigma > 1/2 \). Bohr and Jessen proved the existence of a continuous function \( F_\sigma(z) \) for
which

$$\lim_{T \to \infty} \frac{1}{2T} \mu\{-T \leq t \leq T \mid \log \zeta(\sigma + it) \in R\} = \iint_R F_\sigma(x + iy) \, dx \, dy$$

holds for any rectangle $R$.

Fujii gave a sketch of the proof, which is along the same line as in [8]. In particular, Fujii indicated explicitly how to construct $F(x + iy)$, following the method of Bohr and Jessen [7].

2. The theory of $M$-functions and the statement of the main result

The result (1.5) of Bohr and Jessen has been generalized to a wider class of zeta-functions. The existence of the limit on the left-hand side of (1.5) is now generalized to a fairly general class (see [25]).

It is more difficult to prove the integral expression like the right-hand side of (1.5). The case of Dirichlet $L$-functions $L(s, \chi)$ is essentially the same as in the case of $\zeta(s)$ (see Joyner [22]). The case of Dedekind zeta-functions of algebraic number fields was studied by the author [26] [27] [28]. The case of automorphic $L$-functions attached to $\text{SL}(2, \mathbb{Z})$ or its congruence subgroups was established recently in [30] [31].

All of those generalizations consider the situation when $t = \Im(s)$ varies (like the left-hand side of (1.5)). When we treat more general $L$-functions, various other aspects can be considered. In 2008, Ihara [16] studied the $\chi$-aspect for $L$-functions defined on number fields or function fields. His study was then further refined in a series of papers of Ihara and the author [17] [18] [19] [20]. Let us quote a result proved in [18].

**Theorem 2.1.** Let $s = \sigma + it \in \mathbb{C}$ with $\sigma > 1/2$. There exists an explicitly constructable density function $M_\sigma(w)$, continuous and non-negative, for which

$$\text{Avg}_\chi \Phi(\log L(s, \chi)) = \int_\mathbb{C} M_\sigma(w) \Phi(w) |dw|$$

holds, where $\text{Avg}_\chi$ stands for some average with respect to characters, $|dw| = du \, dv / (2\pi)$ (for $w = u + iv$), and $\Phi$ is the test function which is either (i) some continuous function, or (ii) the characteristic function of a compact subset of $\mathbb{C}$ or its complement.

The density function $M_\sigma$ is called an $M$-function. Here we do not give the details how to define $\text{Avg}_\chi$, but in [18], two types of averages were considered. One of them is a certain average with respect to Dirichlet
characters, and the other is essentially the same as the average in $t$-aspect like (1.5). In this sense, $F_\sigma$ in (1.5) may be regarded as an example of $M$-functions.

Since then, various analogues of Theorem 2.1 were discovered by Mourtada and Murty [35], Akbary and Hamieh [1], Lebacque and Zykin [24], Matsumoto and Umegaki [29], Mine [32, 33, 34], and so on.

The aim of the present article is to show the following “limit theorem”, which is a generalization of Fujii’s (1.4) in the framework of the theory of $M$-functions.

**Theorem 2.2.** We assume the LIC. There exists an explicitly constructable density function ($M$-function) $M : \mathbb{C} \to \mathbb{R}_{\geq 0}$, for which

$$
\lim_{X \to \infty} \frac{1}{X} \int_0^X \Phi(f(\alpha))d\alpha = \int_\mathbb{C} M(w)\Phi(w)|dw|
$$

holds for any test function $\Phi : \mathbb{C} \to \mathbb{C}$ which is continuous, or which is the characteristic function of either a compact subset of $\mathbb{C}$ or the complement of such a subset. The function $M(w)$ is continuous, tends to 0 when $|w| \to \infty$, $M(\overline{w}) = M(w)$, and

$$
\int_\mathbb{C} M(w)|dw| = 1.
$$

**Remark 2.3.** Choosing $\Phi = 1_R$, we recover Fujii’s result (1.4).

The above theorem is an analogue of the absolutely convergent case in the theory of $M$-functions (that is, an analogue of [18, Theorem 4.2]). In this sense, our theorem is a rather simple example of $M$-functions. In particular, complicated mean-value arguments (such as [18, Sections 5–8]) are not necessary. Still, however, our theorem gives a new evidence of the ubiquity of $M$-functions.

3. The finite truncation

The rest of the present paper is devoted to the proof of Theorem 2.2.

We first define the finite truncation of $f(\alpha)$. Let $b_m = (1/2 + i\gamma_m)(3/2 + i\gamma_m)$, $c_m = 1/|b_m|$, and $\beta_m = \arg b_m$. Then

$$
f(\alpha) = \sum_{m=1}^{\infty} \frac{e^{i\alpha\gamma_m}}{b_m} = \sum_{m=1}^{\infty} c_m e^{i(\alpha\gamma_m - \beta_m)}.
$$

It is to be noted that

$$
c_m = \frac{1}{\sqrt{\frac{1}{4} + \gamma_m^2} \sqrt{\frac{9}{4} + \gamma_m^2}} \sim \frac{1}{\gamma_m^2} \sim \left(\frac{\log m}{2\pi m}\right)^2
$$
as $m \to \infty$, hence the above series expression of $f(\alpha)$ is absolutely convergent.

We first consider the finite truncation

$$f_N(\alpha) = \sum_{m=1}^{N} c_m e^{i(\alpha \gamma_m - \beta_m)}. \quad (3.3)$$

Let $T$ be the unit circle on $\mathbb{C}$, and $T_N = \prod_{m \leq N} T$. Define

$$S_N(t_N) = \sum_{m \leq N} c_m t_m, \quad (3.4)$$

where $t_N = (t_1, \ldots, t_N) \in T_N$. Then obviously

$$f_N(\alpha) = S_N(e^{i(\alpha \gamma_1 - \beta_1)}, \ldots, e^{i(\alpha \gamma_N - \beta_N)}). \quad (3.5)$$

The idea of attaching the mapping $S_N : T_N \to \mathbb{C}$ to $f_N$ goes back to the work of Bohr [6]. We denote by $d^*t_N$ the normalized Haar measure on $T_N$, that is the product measure of $d^*t = (2\pi)^{-1}d\theta$ for $t = e^{i\theta} \in T$. The following is an analogue of [16, Theorem 1].

**Proposition 3.1.** We may construct a function $M_N : \mathbb{C} \to \mathbb{R}_{\geq 0}$, for which

$$\int_{\mathbb{C}} M_N(w)|\Phi(w)|dw = \int_{T_N} \Phi(S_N(t_N))d^*t_N \quad (3.6)$$

holds for any continuous function $\Phi$ on $\mathbb{C}$. In particular, choosing $\Phi \equiv 1$ we obtain

$$\int_{\mathbb{C}} M_N(w)|dw| = 1. \quad (3.7)$$

Also for $N \geq 2$ the function $M_N(w)$ is compactly supported, non-negative and $M_N(\overline{w}) = M_N(w)$.

**Proof.** First consider the case $N = 1$. Let $s_n(t_n) = c_n t_n$. For $w = re^{i\theta} \in \mathbb{C}$ ($r = |w|, \theta = \arg w$), define

$$m_n(w) = \frac{1}{r} \delta(r - c_n), \quad (3.8)$$

where $\delta$ is the Dirac delta function.
where $\delta(\cdot)$ stands for the usual Dirac delta distribution. We have

\[
\int_{\mathbb{C}} m_n(w)\Phi(w)|dw| = \int_{0}^{2\pi} \int_{0}^{\infty} m_n(re^{i\theta})\Phi(re^{i\theta}) \frac{1}{2\pi} r dr d\theta
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} d\theta \int_{0}^{\infty} \delta(r - c_n)\Phi(re^{i\theta}) dr
\]

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} \Phi(c_ne^{i\theta}) d\theta
\]

\[
= \int_{T} \Phi(s_n(t_n))d^*t_n.
\]

In particular, putting $n = 1$ in (3.9), we find

\[
\int_{\mathbb{C}} m_1(w)\Phi(w)|dw| = \int_{T} \Phi(s_1(t_1))d^*t_1,
\]

which implies that the case $N = 1$ of Proposition 3.1 is valid with $M_1 = m_1$.

Now we prove the general case by induction on $N$. Define

\[
M_N(w) = \int_{\mathbb{C}} M_{N-1}(w')m_N(w - w')|dw'|
\]

for $N \geq 2$. This is compactly supported, and

\[
\int_{\mathbb{C}} M_N(w)\Phi(w)|dw| = \int_{\mathbb{C}} \int_{\mathbb{C}} M_{N-1}(w')m_N(w - w')|dw'|\Phi(w)|dw|
\]

\[
= \int_{\mathbb{C}} M_{N-1}(w')|dw'| \int_{\mathbb{C}} m_N(w - w')\Phi(w)|dw|.
\]

The exchange of the integrations is verified because $M_N$ is compactly supported. Putting $w'' = w - w'$ we see that the inner integral is

\[
= \int_{\mathbb{C}} m_N(w'')\Phi_{w'}(w'')|dw''| \quad \text{ (where } \Phi_{w'}(w'') = \Phi(w'' + w'),)\]

which is, by (3.9),

\[
= \int_{T} \Phi_{w'}(s_N(t_N))d^*t_N.
\]
Therefore
\[ \int_{C} M_N(w) \Phi(w) \, dw = \int_{C} M_{N-1}(w') \, dw' \int_{T} \Phi_{w'}(s_N(t_N)) \, d^*t_N \]
\[ = \int_{T} d^*t_N \int_{C} M_{N-1}(w') \Phi_{w'}(s_N(t_N)) \, dw' \]
\[ = \int_{T} d^*t_N \int_{C} M_{N-1}(w') \Phi_{s_N}(w') \, dw', \]
where \( \Phi_{s_N}(w') = \Phi(s_N(t_N) + w') = \Phi_{w'}(s_N(t_N)) \). Using the induction assumption we see that the right-hand side is
\[ = \int_{T} d^*t_N \int_{T} M_{N-1}(s_N-1(t_N-1)) \, d^*t_N = \int_{T} M_{N-1}(s_N-1(t_N-1)) \, d^*t_N. \]

Since
\[ \Phi_{s_N}(s_{N-1}(t_N-1)) = \Phi(s_{N-1}(t_N-1) + s_N(t_N)) = \Phi(s_N(t_N)), \]
we obtain the assertion of the proposition.

The following two propositions are analogues of [18, Remark 3.2 and Remark 3.3]. For any \( A \subset \mathbb{C} \), by \( 1_A \) we denote the characteristic function of \( A \). By \( \text{Supp}(\phi) \) we mean the support of a function \( \phi \).

**Proposition 3.2.** The formula (3.6) is valid when \( \Phi = 1_A \), where \( A \) is either a compact subset of \( \mathbb{C} \) or the complement of such a subset.

**Proof.** It is enough to prove the case when \( A \) is compact. Let \( \phi_1, \phi_2 \) be continuous non-negative functions, defined on \( \mathbb{C} \), compactly supported, satisfying \( 0 \leq \phi_1 \leq 1_A \leq \phi_2 \leq 1 \) and \( \text{Vol}(\text{Supp}(\phi_2 - \phi_1)) < \varepsilon \) (where “Vol” denotes the volume measured by \( d^*t_N \)). Then
\[ \int_{C} M_N(w)(1_A - \phi_1)(w) \, dw < C_N \varepsilon, \quad \int_{C} M_N(w)(\phi_2 - 1_A)(w) \, dw < C_N \varepsilon, \]
where \( C_N = \sup\{M_N(w)\} \). Therefore, using Proposition 3.1 we have
\[ \int_{C} M_N(w)1_A(w) \, dw - C_N \varepsilon \leq \int_{C} M_N(w)\phi_1(w) \, dw \]
\[ = \int_{T_N} \phi_1(s_N(t_N)) \, d^*t_N \leq \int_{T_N} 1_A(s_N(t_N)) \, d^*t_N \]
\[ \leq \int_{T_N} \phi_2(s_N(t_N)) \, d^*t_N = \int_{C} M_N(w)\phi_2(w) \, dw \]
\[ \leq \int_{C} M_N(w)1_A(w) \, dw + C_N \varepsilon, \]
from which the desired assertion follows. \( \square \)
In the proof of Proposition 3.1 we have shown that $M_N$ is compactly supported. Now we show more explicitly what is the support.

**Proposition 3.3.** The support of $M_N$ is the image of the mapping $S_N$.

**Proof.** Let $A$ be a compact subset of $\mathbb{C}$. We can use (3.6) with $\Phi = 1_A$ because of Proposition 3.2. Then

$$\int_A M_N(w) |dw| = \int_{T_N} 1_A(S_N(t_N)) d^* t_N = \text{Vol}(S_N^{-1}(A)),$$

which implies the proposition. \qed

4. THE FINITE-TRUNCATION VERSION OF THE THEOREM

The aim of this section is to prove

**Proposition 4.1.** Under the assumption of the LIC, we have

$$\lim_{X \to \infty} \frac{1}{X} \int_0^X \Phi(f_N(\alpha)) d\alpha = \int_{T_N} \Phi(S_N(t_N)) d^* t_N$$

for any continuous function $\Phi$ on $\mathbb{C}$.

Then, combining this with Proposition 3.1, we have

$$\lim_{X \to \infty} \frac{1}{X} \int_0^X \Phi(f_N(\alpha)) d\alpha = \int_{\mathbb{C}} M_N(w) \Phi(w) |dw|$$

for any continuous $\Phi$, which is the “finite-truncation” analogue of our main theorem.

In view of (3.5), in order to prove Proposition 4.1, it is enough to prove the following

**Proposition 4.2.** Under the assumption of the LIC, we have

$$\lim_{X \to \infty} \frac{1}{X} \int_0^X \Psi(e^{i(\alpha \gamma - \beta_1)}, \ldots, e^{i(\alpha \gamma - \beta_N)}) d\alpha = \int_{T_N} \Psi(t_N) d^* t_N$$

holds for any continuous $\Psi : T_N \to \mathbb{C}$.

This is an analogue of [16, Lemma 4.3.1].

**Proof.** Write $t_N = (e^{i\theta_1}, \ldots, e^{i\theta_N})$. Then the right-hand side of Proposition 4.2 is

$$= \frac{1}{(2\pi)^N} \int_0^{2\pi} \ldots \int_0^{2\pi} \Psi(e^{i\theta_1}, \ldots, e^{i\theta_N}) d\theta_1 \cdots d\theta_N.$$

To show that this is equal to the left-hand side, by Weyl’s criterion (see [23, Chapter 1, Theorem 9.9]), it is enough to show the equality when $\Psi = t_1^{n_1} \cdots t_N^{n_N}$ for any $(n_1, \ldots, n_N) \in \mathbb{Z}^N \setminus \{(0, \ldots, 0)\}$. But in
this case, since $\Psi(e^{i\theta_1}, \ldots, e^{i\theta_N}) = e^{i(n_1\theta_1 + \cdots + n_N\theta_N)}$, the right-hand side is clearly equal to 0. The left-hand side is
\[
\lim_{X \to \infty} \frac{1}{X} \int_{0}^{X} e^{in_1(\alpha\gamma_1 - \beta_1) + \cdots + in_N(\alpha\gamma_N - \beta_N)} d\alpha
\]
\[
= \lim_{X \to \infty} \frac{1}{X} e^{-i(n_1\beta_1 + \cdots + n_N\beta_N)} \int_{0}^{X} e^{i\alpha(n_1\gamma_1 + \cdots + n_N\gamma_N)} d\alpha.
\]
Since we assume the LIC, $n_1\gamma_1 + \cdots + n_N\gamma_N \neq 0$ because $(n_1, \ldots, n_N) \neq (0, \ldots, 0)$. Therefore the above is
\[
= \lim_{X \to \infty} \frac{1}{X} e^{-i(n_1\beta_1 + \cdots + n_N\beta_N)} \frac{e^{iX(n_1\beta_1 + \cdots + n_N\beta_N)} - 1}{i(n_1\beta_1 + \cdots + n_N\beta_N)}
\]
which is also equal to 0. The proposition is proved. $\square$

5. The existence of the $M$-function

In this section we prove the existence of the limit function
\[
M(w) = \lim_{N \to \infty} M_N(w).
\]

For this purpose we consider the Fourier transform. We follow the argument on pp. 644-647 in [18], which is based on the ideas of Ihara [16] and of the author [27].

Let $\psi_z(w) = \exp(i \Re(\overline{z}w))$, and define the Fourier transform of $m_n$ as
\[
\tilde{m}_n(z) = \int_{C} m_n(w) \psi_z(w) |dw|.
\]

Applying (3.9) with $\Phi = \psi_z$, we see that the right-hand side of the above is
\[
= \int_{\mathbb{T}} \psi_z(s_n(t_n)) d^* t_n = \frac{1}{2\pi} \int_{0}^{2\pi} \psi_z(c_n e^{i\theta_n}) d\theta_n
\]
\[
= \frac{1}{2\pi} \int_{0}^{2\pi} \exp(i \Re(\overline{z} \cdot c_n e^{i\theta_n})) d\theta_n.
\]
Writing $\overline{z} \cdot c_n e^{i\theta_n} = c_n |z| e^{i(\theta_n - \tau)}$ ($\tau = \arg z$), we have
\[
\Re(\overline{z} \cdot c_n e^{i\theta_n}) = c_n |z| \cos(\theta_n - \tau) = c_n |z| (\cos \theta_n \cos \tau + \sin \theta_n \sin \tau)
\]
and so
\[
\tilde{m}_n(z) = \frac{1}{2\pi} \int_{0}^{2\pi} \exp(i c_n |z| (\cos \theta_n \cos \tau + \sin \theta_n \sin \tau)) d\theta_n.
\]

Now quote:
Lemma 5.1. (Jessen and Wintner [21, Theorem 12]) Let $C$ be a closed convex curve in $\mathbb{C}$ parametrized by $x(\theta) = (\xi_1(\theta), \xi_2(\theta))$, $z = |z|e^{i\tau} \in \mathbb{C}$, and let $g_\tau(\theta) = \xi_1(\theta) \cos \tau + \xi_2(\theta) \sin \tau$. Assume that $\xi_1, \xi_2 \in C^2$ and $g''_\tau(\theta)$ has (for each fixed $\tau$) exactly two zeros on $C$. Then

\begin{equation}
\int_C \exp(i|z|g_\tau(\theta))d\theta = O(|z|^{-1/2}),
\end{equation}

where the implied constant depends on $C$.

In the present case $\xi_1(\theta) = c_n \cos \theta$, $x_2(\theta) = c_n \sin \theta$, and $C$ is the circle of radius $c_n$. Since

\begin{equation}
g''_\tau(\theta) = -c_n (\cos \theta \cos \tau + \sin \theta \sin \tau) = -c_n \cos(\theta - \tau),
\end{equation}

the assumption of the lemma is clearly satisfied, and hence by the lemma we have

\begin{equation}
\tilde{m}_n(z) = O_n(|z|^{-1/2}).
\end{equation}

Now define

\begin{equation}
\tilde{M}_N(z) = \prod_{n \leq N} \tilde{m}_n(z).
\end{equation}

Then from (5.6) and the obvious inequality $|\tilde{m}_n(z)| \leq 1$ (which immediately follows from (5.4)), we have

\begin{equation}
\tilde{M}_N(z) = O_N(|z|^{-N/2})
\end{equation}

and

\begin{equation}
|\tilde{M}_N(z)| \leq 1.
\end{equation}

From these inequalities we obtain (i) and (ii) of the following

Proposition 5.2. Let $N_0 \geq 5$.

(i) $\tilde{M}_{N_0} \in L^t$ for any $t \in [1, +\infty]$,

(ii) $|\tilde{M}_N(z)| \leq |\tilde{M}_{N_0}(z)|$ for all $N \geq N_0$,

(iii) $\tilde{M}_N(z)$ converges to a certain function $\tilde{M}(z)$ uniformly in any compact subset when $N \to \infty$.

Proof of (iii). It is clear from (5.3) that

\begin{equation}
\frac{1}{2\pi} \int_0^{2\pi} \Re(z \cdot c_ne^{i\theta_n})d\theta_n = 0.
\end{equation}

Therefore we can write

\begin{equation}
\tilde{m}_n(z) - 1 = \frac{1}{2\pi} \int_0^{2\pi} \left(\exp(i\Re(z \cdot c_ne^{i\theta_n})) - 1 - \Re(z \cdot c_n e^{i\theta_n})\right)d\theta_n.
\end{equation}
Since $|e^{ix} - 1 - ix| \ll x^2$ for any real $x$ (by the Taylor expansion for small $|x|$, and by the fact $|e^{ix}| = 1$ for large $|x|$), we obtain

\begin{equation}
|\tilde{m}_n(z) - 1| \ll \int_0^{2\pi} |\mathcal{R}(\mathcal{E} \cdot c_n e^{i\theta_n})|^2 d\theta_n \ll |z|^2 c_n^2.
\end{equation}

Let $N < N'$. Then

\begin{align*}
|\tilde{M}_{N'}(z) - \tilde{M}_N(z)| &\leq \sum_{j=1}^{N'-N} |\tilde{M}_{N+j}(z) - \tilde{M}_{N+j-1}(z)| \\
&= \sum_{j=1}^{N'-N} |\tilde{M}_{N+j-1}(z)| \cdot |\tilde{m}_{N+j}(z) - 1| \\
&\ll |z|^2 \sum_{j=1}^{N'-N} c_{N+j}^2
\end{align*}

by (5.9) and (5.12). Because of (5.2) we see that the series on the right-hand side converges as $N, N' \to \infty$. Therefore by Cauchy’s criterion we obtain the assertion (iii). \qed

Now we prove the following result, which is an analogue of [18, Proposition 3.4].

**Proposition 5.3.** $\tilde{M}_N(z)$ converges to $\tilde{M}(z)$ uniformly in $\mathbb{C}$ when $N \to \infty$. The limit function $\tilde{M}(z)$ is continuous and belongs to $L^t$ (for any $t \in [1, \infty]$), and the above convergence is also $L^t$-convergence.

**Proof.** Let $0 < \varepsilon < 1$. By Proposition 5.2 (i) we can find $R = R(N_0) > 1$ for which

\begin{equation}
\int_{|z| \geq R} |\tilde{M}_{N_0}(z)|^t |dz| < \varepsilon
\end{equation}

for any $1 \leq t < \infty$ and (noting (5.8))

\begin{equation}
\sup_{|z| \geq R} |\tilde{M}_{N_0}(z)| < \varepsilon.
\end{equation}

(Here $R$ is independent of $t$, because by (5.9) the inequality (5.13) for $t = 1$ implies (5.13) for other finite values of $t$.) Because of Proposition 5.2 (ii), the above inequalities are valid also for $\tilde{M}_N(z)$ for all $N \geq N_0$.

Taking $N \to \infty$ in the above inequalities, we find that $\tilde{M} \in L^t$ ($1 \leq t \leq \infty$).
Let \( N' > N \). Then

\[
|\tilde{M}_{N'}(z) - \tilde{M}_N(z)|^t = \left| \prod_{N < n \leq N'} m_n(z) - 1 \right|^t \cdot |\tilde{M}_N(z)|^t \leq 2^t |\tilde{M}_N(z)|^t
\]

for any \( z \in \mathbb{C} \), so taking the limit \( N' \to \infty \) we have

\[
(5.15) \quad |\tilde{M}(z) - \tilde{M}_N(z)|^t \leq 2^t |\tilde{M}_N(z)|^t.
\]

Therefore from \((5.13)\) and \((5.14)\) we obtain

\[
(5.16) \quad \int_{|z| \geq R} \left| |\tilde{M}(z) - \tilde{M}_N(z)|^t \right| dz < 2^t \varepsilon
\]

and (using the case \( t = 1 \) of \((5.15)\))

\[
(5.17) \quad \sup_{|z| \geq R} |\tilde{M}(z) - \tilde{M}_N(z)| < 2 \varepsilon
\]

for all \( N \geq N_0 \).

Now we apply Proposition 5.2 (iii) for the compact subset \( \{|z| \leq R\} \) to obtain that if \( N = N(R, \varepsilon) \geq N_0 \) is sufficiently large, then

\[
(5.18) \quad |\tilde{M}(z) - \tilde{M}_N(z)| \leq \varepsilon / R^2
\]

for all \( z \) satisfying \( |z| \leq R \). Therefore

\[
(5.19) \quad \int_{|z| \leq R} |\tilde{M}(z) - \tilde{M}_N(z)|^t |dz| < \pi R^2 \left( \frac{\varepsilon}{R^2} \right)^t \leq \pi R^2 \frac{\varepsilon}{R^2} \leq \pi \varepsilon
\]

and

\[
(5.20) \quad \sup_{|z| \leq R} |\tilde{M}(z) - \tilde{M}_N(z)| < \frac{\varepsilon}{R^2} \leq \varepsilon.
\]

Now we arrive at

\[
(5.21) \quad \int_{\mathbb{C}} |\tilde{M}(z) - \tilde{M}_N(z)|^t |dz| < (2^t + \pi) \varepsilon
\]

and

\[
(5.22) \quad \sup_{z \in \mathbb{C}} |\tilde{M}(z) - \tilde{M}_N(z)| < 3 \varepsilon.
\]

Therefore we obtain the assertions of the proposition. \( \square \)

Since \( M_N \) is given by the convolution product of \( m_1, \ldots, m_N \) (see \((3.11)\)), by the definition \((5.7)\), \( \tilde{M}_N(z) \) is the Fourier transform of \( M_N(w) \). Therefore we can write

\[
(5.23) \quad M_N(w) = \int_{\mathbb{C}} \tilde{M}_N(z) \psi_w(z) |dz|.
\]
Define
\begin{equation}
M(w) = \int_{\mathbb{C}} \tilde{M}(z)\psi_{-w}(z)|dz|.
\end{equation}

Then we obtain

**Proposition 5.4.** When $N \to \infty$, $M_N(w)$ converges to $M(w)$ uniformly in $w \in \mathbb{C}$. The limit function $M(w)$ is continuous, non-negative, tends to 0 when $|w| \to \infty$, $M(\overline{w}) = M(w)$, and

\begin{equation}
\int_{\mathbb{C}} M(w)|dw| = 1.
\end{equation}

The functions $M$ and $\tilde{M}$ are Fourier duals of each other.

This is an analogue of [18, Proposition 3.5], and the proof is exactly the same.

6. **Completion of the proof**

Now we finish the proof of our main Theorem 2.2. Among the statement of Theorem 2.2, the properties of $M(w)$ is already shown in the above Proposition 5.4. Therefore the only remaining task is to prove (2.2).

First consider the case when $\Phi$ is continuous. We have already shown the “finite-truncation” version of (2.2) as (4.2). We will prove that it is possible to take the limit $N \to \infty$ on the both sides of (4.2).

From (3.4) we see that the image of the mapping $S_N$ is included in the disc of radius $\sum_{m=1}^{\infty} c_m$ for any $N$. Therefore by Proposition 3.3 we find that the support of $M_N$ for any $N$ is also included in the same disc, hence is the support of $M$. The image of $f$ is clearly also bounded. Therefore, to prove (2.2), we may assume that $\Phi$ is compactly supported, hence is uniformly continuous.

Then, as $N \to \infty$, $\Phi(f_N(\alpha))$ tends to $\Phi(f(\alpha))$ uniformly in $\alpha$. Also, $M_N(w)\Phi(w)$ tends to $M(w)\Phi(w)$ uniformly in $w$, because of Proposition 5.4. This yields that, when we take the limit $N \to \infty$ on (1.2), we may change the integration and this limit. Therefore we obtain (2.2) for continuous $\Phi$.

Finally, similarly to the proof of Proposition 3.2, we can deduce the assertion in the case when $\Phi$ is a characteristic function of a compact subset or its complement. This completes the proof of Theorem 2.2.

**Remark 6.1.** Consider the Dirichlet series
\begin{equation}
\Psi(s,x) = \sum_{\gamma>0} \frac{x^{i\gamma}}{(1/2+i\gamma)^s(3/2+i\gamma)^s}.
\end{equation}
where \( s \in \mathbb{C} \). Obviously \( \Psi(1, x) = \Psi(x) \). Because of (3.2), the series (6.1) is absolutely convergent when \( \Re s > 1/2 \). It is easy to see that we can extend Theorem 2.2 to \( \Psi(s, x) \) in this domain of absolute convergence.

**Remark 6.2.** A generalization of the theory of the Goldbach generating function to the case with congruence conditions was first considered by Rüppel [36], and the generalized form of \( \Psi(x) \) in this case (written in terms of the zeros of Dirichlet \( L \)-functions) was determined by Suzuki [37]. (See also [2] [3] [4].) It is desirable to generalize our result in the present paper to Suzuki’s generalized \( \Psi \). Probably more interesting is to consider the \( \chi \)-analogue; that is instead of the average with respect to \( \alpha \) as in our Theorem 2.2 consider some analogue with respect to \( \chi \) (cf. [10], [18]).

**References**

[1] A. Akbary and A. Hamieh, Value-distribution of cubic Hecke \( L \)-functions, J. Number Theory 206 (2020), 81–122.
[2] G. Bhowmik and K. Halupczok, Asymptotics of Goldbach representations, in “Various Aspects of Multiple Zeta Functions”, H. Mishou et al. (eds.), Adv. Stud. Pure Math. 84, Math. Soc. Japan, 2020, pp. 1–21.
[3] G. Bhowmik, K. Halupczok, K. Matsumoto and Y. Suzuki, Goldbach representations in arithmetic progressions and zeros of Dirichlet \( L \)-functions, Mathematika 65 (2019), 57–97.
[4] G. Bhowmik and I. Z. Ruzsa, Average Goldbach and the quasi-Riemann hypothesis, Anal. Math. 44 (2018), 51–56.
[5] G. Bhowmik and J.-C. Schlage-Puchta, Mean representation number of integers as the sum of primes, Nagoya Math. J. 200 (2010), 27–33.
[6] H. Bohr, Zur Theorie der Riemann’schen Zetafunktion im kritischen Streifen, Acta Math. 40 (1915), 67–100.
[7] H. Bohr and B. Jessen, Om Sandsynlighedssfordelinger ved Addition af konvexe Kurver, Den. Vid. Selsk. Skr. Nat. Math. Afd. (8) 12 (1929), 1–82.
[8] H. Bohr and B. Jessen, Über die Werteverteilung der Riemannschen Zetafunktion, Erste Mitteilung, Acta Math. 54 (1930), 1–35.
[9] H. Bohr and B. Jessen, Über die Werteverteilung der Riemannschen Zetafunktion, Zweite Mitteilung, Acta Math. 58 (1932), 1–55.
[10] S. Egami and K. Matsumoto, Convolutions of the von Mangoldt function and related Dirichlet series, in “Number Theory: Sailing on the Sea of Number Theory”, S. Kanemitsu and J. Liu (eds.), Ser. Number Theory Appl. 2, World Scientific, 2007, pp. 1–23.
[11] A. Fujii, An additive problem of prime numbers, Acta Arith. 58 (1991), 173–179.
[12] A. Fujii, An additive problem of prime numbers II, Proc. Japan Acad. 67A (1991), 248–252.
[13] A. Fujii, An additive problem of prime numbers III, Proc. Japan Acad. 67A (1991), 278–283.
[14] A. Granville, Refinements of Goldbach’s conjecture, and the generalized Riemann hypothesis, Funct. Approx. Comment. Math. 37 (2007), 159–173; Corrigendum, ibid. 38 (2008), 235–237.
[15] G. H. Hardy and J. E. Littlewood, Some problems of “partitio numerorum” (V): A further contribution to the study of Goldbach’s problem, Proc. London Math. Soc. (2) 22 (1924), 46–56.
[16] Y. Ihara, On “M-functions” closely related to the distribution of $L'/L$-values, Publ. RIMS Kyoto Univ. 44 (2008), 893–954.
[17] Y. Ihara and K. Matsumoto, On $L$-functions over function fields: Power-means of error-terms and distribution of $L'/L$-values, in “Algebraic Number Theory and Related Topics 2008”, H. Nakamura et al. (eds.), RIMS Kôkyûroku Bessatsu B19, RIMS Kyoto Univ., 2010, pp. 221–247.
[18] Y. Ihara and K. Matsumoto, On certain mean values and the value-distribution of logarithms of Dirichlet $L$-functions, Quart. J. Math. (Oxford) 62 (2011), 637–677.
[19] Y. Ihara and K. Matsumoto, On log $L$ and $L'/L$ for $L$-functions and the associated “M-functions”: Connections in optimal cases, Moscow Math. J. 11 (2011), 73–111.
[20] Y. Ihara and K. Matsumoto, On the value-distribution of logarithmic derivatives of Dirichlet $L$-functions, in “Analytic Number Theory, Approximation Theory and Special Functions”, G. V. Milovanović and M. Th. Rassias (eds.), Springer, 2014, pp. 79–91.
[21] B. Jessen and A. Wintner, Distribution functions and the Riemann zeta function, Trans. Amer. Math. Soc. 38 (1935), 48–88.
[22] D. Joyner, Distribution Theorems of $L$-functions, Longman Scientific & Technical, 1986.
[23] L. Kuipers and H. Niederreiter, Uniform Distribution of Sequences, Wiley-Interscience, 1974.
[24] P. Lebacque and A. Zykin, On $M$-functions associated with modular forms, Moscow Math. J. 18 (2018), 437–472.
[25] K. Matsumoto, Value-distribution of zeta-functions, in “Analytic Number Theory”, K. Nagasaka and E. Fouvy (eds.), Lecture Notes in Math. 1434, Springer, 1990, pp. 178–187.
[26] K. Matsumoto, On the magnitude of asymptotic probability measures of Dedekind zeta-functions and other Euler products, Acta Arith. 60 (1991), 125–147.
[27] K. Matsumoto, Asymptotic probability measures of zeta-functions of algebraic number fields, J. Number Theory 40 (1992), 187–210.
[28] K. Matsumoto, On the speed of convergence to limit distributions for Dedekind zeta-functions of non-Galois number fields, in “Probability and Number Theory — Kanazawa 2005”, S. Akiyama et al. (eds.), Adv. Stud. Pure Math. 49, Math. Soc. Japan, 2007, pp. 199–218.
[29] K. Matsumoto and Y. Umegaki, On the value-distribution of the difference between logarithms of two symmetric power $L$-functions, Intern. J. Number Theory 14 (2018), 2045–2081.
[30] K. Matsumoto and Y. Umegaki, On the density function for the value-distribution of automorphic $L$-functions, J. Number Theory 198 (2019), 176–199.

[31] K. Matsumoto and Y. Umegaki. On the value-distribution of symmetric power $L$-functions, in “Topics in Number Theory”, T. Chatterjee and S. Gun (eds.), Ramanujan Math. Soc. Lecture Notes Ser. 26, 2020, pp. 147–167.

[32] M. Mine, On $M$-functions for the value-distribution of $L$-functions, Lith. Math. J. 59 (2019), 96–110.

[33] M. Mine, On certain mean values of logarithmic derivatives of $L$-functions and the related density functions, Funct. Approx. Comment. Math. 61 (2019), 179–199.

[34] M. Mine, The density function for the value-distribution of the Lerch zeta-function and its applications, Michigan Math. J. 69 (2020), 849–889.

[35] M. Mourtada and V. K. Murty, Distribution of values of $L'/L(\sigma, \chi_D)$, Moscow Math. J. 15 (2015), 497–509.

[36] F. Rüppel, Convolutions of the von Mangoldt function over residue classes, Šiauliai Math. Semin. 7(15) (2012), 135-156.

[37] Y. Suzuki, A mean value of the representation function for the sum of two primes in arithmetic progressions, Intern. J. Number Theory 13 (2017), 977–990.

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