We introduce the pseudohole and heavy-pseudoparticle operator algebra that generates all Hubbard-chain eigenstates from a single reference vacuum. In addition to the pseudoholes already introduced for the description of the low-energy physics, this involves the heavy pseudoparticles associated with Hamiltonian eigenstates whose energy spectrum has a gap relatively to the many-electron ground state. We introduce a generalized pseudoparticle perturbation theory which describes the relevant finite-energy ground state transitions. In the present basis these excitations refer to a small density of excited pseudoparticles. Our operator basis goes beyond the Bethe-ansatz solution and it is the suitable and correct starting point for the study of the finite-frequency properties, which are of great relevance for the understanding of the unusual spectral properties detected in low-dimensional novel materials.

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I. INTRODUCTION

The unusual spectral properties of one-dimensional electronic quantum liquids [1,2] imply that they cannot be described by one-electron, Fermi-liquid-like models [3,4]. Further, descriptions of these quantum systems in terms of exotic excitations such as holons and spinons [5,6], or pseudoparticles [7,8,9], have been limited to the low-energy Hilbert subspace, and both bosonization [1,2] and conformal-field theory techniques [10,11,12,13] also apply only in that limit. However, low-energy studies cannot describe the finite-frequency properties, which are of great interest for the understanding of the unusual properties detected in real low-dimensional novel materials. Thus developing a microscopic operator description of these systems for arbitrary energies remains an important theoretical challenge.

For more than sixty years the “Bethe ansatz” (BA) [14] has played a central role in the analytic solution of a variety of “integrable” many-body problems in condensed matter physics and quantum field theory [14,15,16,17,18]. Techniques based on the exact BA solutions were used to describe excited states, thermodynamics, and low-energy correlation functions [13,14,17,19] and remain an active subject of study today. However, the BA solution only provides limited information on the many-particle problem. For instance, it does neither provide off-diagonal operator expressions nor matrix elements.

In this paper we introduce a suitable pseudohole and heavy-pseudoparticle operator algebra which generates all Hubbard-chain eigenstates from a single pseudohole and heavy-pseudoparticle vacuum. This is the pseudohole vacuum already introduced in Ref. [20] for the low-energy Hilbert subspace spanned by the lowest-weight states (LWS’s) and highest-weight states (HWS’s) of the SU(2) η-spin and spin algebras [21,22,23,24] of type I. (See below the definition of types I and II LWS’s and HWS’s.) That vacuum is the half-filling and zero-magnetic-field ground state (GS).

Recent investigations have established that although not themselves complete, the Hamiltonian eigenstates which refer to the BA solution can be extended to form a complete set of states for the one-dimensional Hubbard model [2]. However, the vital and to date open issue of how to describe specifically those excitations that dominate response and transport at finite energies in integrable quantum liquids and related models remains an open problem. Our generalized operator algebra is complete and allows the construction of a pseudoparticle perturbation theory which is the correct starting point to solve this problem. The main point is that the GS transitions which determine and dominate response and transport at finite energies involve a small density of excited pseudoparticles only. Therefore, the description of these relevant transitions is within the range of our perturbation theory. In this paper we introduce the generalized operator representation. The application of our approach to study the finite-energy spectral and transport properties of the Hubbard chain will be presented elsewhere [25].

In contrast to three-dimensional Fermi liquids [1,2], the low-energy excitations of one-dimensional many-electron quantum problems are at zero magnetic field characterized by charge - spin separation [1,4,26]. This can be interpre-
tated in terms of holon and spinon modes [18,20]. The BA solution of the Hubbard chain [18] at zero magnetic field and chemical potential has allowed the identification and study of the holon and spinon excitations and corresponding symmetry transformations [18,20]. (The spinon excitations of the Hubbard chain are similar to the corresponding spinon excitations of the spin 1/2 isotropic Heisenberg chain [27].)

On the other hand, at finite magnetic fields the charge and spin separation is replaced by a more exotic c and s separation [1]. Here c and s refer to orthogonal small-momentum and low-energy modes which couple both to the charge and spin channels. These can be described in terms of pseudoparticle and pseudohole excitations [20]. In Ref. [20] it was shown that the zero-magnetic field holon and spinon excitations are limiting cases of the general pseudohole excitations. This has allowed to obtain the exact expression of the electron operator in terms of holon and spinon operators [20]. This study has also provided the relation between Hamiltonian symmetry and the transformation of the elementary excitations at finite magnetic fields. (At finite magnetic fields the usual holon and spinon picture breaks down [20].)

The present pseudohole and heavy-pseudoparticle operator algebra is a generalization of the low-energy pseudoparticle operator representation of Refs. [5,6,18,20] and pseudohole representation of Ref. [20]. For the general case of the Hubbard chain at finite magnetic field and chemical potential both the non-LWS’s and non-HWS’s of the η-spin and spin algebras have energy gaps relative to the corresponding canonical-ensemble GS [20,25]. (The expressions of these gaps are determined in this paper.) The LWS’s and HWS’s of these algebras can be classified into two types, the states I (or LWS’s I and HWS’s I) and the states II (or LWS’s II and HWS’s II). While the Hamiltonian eigenstates I are described only by real BA rapidities, all or some of the rapidities associated with the eigenstates II are complex and non real. Since for finite magnetic field and chemical potential the states II have energy gaps relative to the corresponding canonical-ensemble GS, in that case the low-energy physics is exclusively determined by the states I [20]. Our algebra refers to the whole parameter space and is expressed most naturally in terms of operators describing the pseudohole excitations introduced in Ref. [20], the new heavy-pseudoparticle excitations, and the associate topological momentum-shift excitations.

We demonstrate that not only the familiar BA solution of the Hubbard chain [18,19,23] can be interpreted naturally at all energy scales in terms of the pseudohole and heavy-pseudoparticle basis, but that such basis goes beyond that solution. We then establish that the generalized pseudoparticle perturbation theory, which is also naturally described in terms of the new operator algebra, can be used to study the low (ω − ω₀) energy excitations. Here ω₀ represents the energy gaps of particular finite-energy states II and non-LWS’s and non-HWS’s which proven to dominate the finite-energy response functions [20,24].

In contrast to the states I which at finite values of the chemical potential and magnetic field span the low-energy Hilbert subspace [20], the LWS’s and HWS’s II are characterized by finite occupancies of the heavy-pseudoparticle bands, whose energy spectrum has a finite gap. In addition, our pseudohole and heavy-pseudoparticle operator basis generates the non-LWS’s and non-HWS’s of the η-spin and spin algebras absent in the BA solution. It goes beyond the BA solution [18,19,23] because it i)- generates from a single vacuum all Hamiltonian eigenstates (including the above η-spin and spin non-LWS’s and non-HWS’s); ii)- provides relevant information on non-diagonal operators [20,24]; and iii)- since the relevant finite-energy transitions involve a small density of pseudoparticles, it allows the direct study of the finite-energy transport properties [20,24].

Importantly, we find that at all energy scales there is only zero-momentum pseudoparticle forward scattering in the Hubbard chain. Although this exact property holds only for integrable systems, we believe that our pseudohole and heavy-pseudoparticle operator description will provide the most suitable starting point for description of the unusual finite-frequency properties of both integrable and non-integrable electronic Luttinger liquids. In the case of the latter liquids this could be achieved by means of the bosonization of the pseudoparticle theory [1].

We fully define the pseudohole and heavy-pseudoparticle statistics by evaluating their infinite-dimensional statistical-interaction matrix [20]. They are neither fermions nor bosons, yet they obey an anticommuting algebra. Furthermore, the introduction of topological-momentum-shift operators [20,24] allows the use of a suitable second-quantization formalism.

Since the Hamiltonian-eigenstate transitions involve only our pseudohole, heavy-pseudoparticle, and topological-momentum-shift operators, all operators acting onto the Hilbert space can be written in terms of the former operators. However, the BA alone does not provide directly the expressions of off-diagonal [in the energy eigenstate basis] part of the physical operators. Among the specific results we quote in this paper are the expressions in terms of pseudohole and heavy-pseudoparticle operators for the Hamiltonian, momentum operator, and six generators of the η-spin and spin SO(4) algebra (yet four of these generators are off diagonal in the energy representation). Further, the algebra can provide diagonal part of all operators acting in the Hilbert space. On the other hand, combining BA with the pseudoparticle operator basis and the invariances of the quantum liquid around particular values of momentum and energy provides important information on correlation functions [20,24]. (This goes beyond BA because these functions are determined by the off-diagonal part of the physical operators.) Our pseudohole and heavy-pseudoparticle operator representation is the natural generalization of the low-energy pseudoparticle description which was studied in detail.
in Refs. \[31,32,33,34,35,36\].

In Section II we introduce the pseudohole and heavy-pseudoparticle operator description. In this operator representation all Hamiltonian eigenstates have a simple Slater-determinant like form of pseudohole and heavy-pseudoparticle levels.

In Section III we introduce the key concept of generalized ground state (GGS) of which the GS is a particular case. We also define the pseudohole and heavy-pseudoparticle statistics \[29\].

In Section IV we write the Hamiltonian in the pseudohole and heavy-pseudoparticle operator basis. For this we have in each subspace a simple form. This permits the introduction of a systematic, non-singular pseudoparticle operators. We generalize the rapidity operators for states out of the BA solution and show that the same rapidity eigenvalue describes a \(SU(2)\) \(\eta\)-spin tower or a \(SU(2)\) spin tower of Hamiltonian eigenstates.

In Sec. V we introduce the generalized pseudoparticle perturbation theory for Hilbert subspaces of arbitrary energy. This includes evaluation of the heavy-pseudoparticle energy gaps. We discuss the effects of normal-ordering (relative to the suitable GGS or GS) and show that, in terms of the pseudoparticle operators, the normal-ordered Hamiltonian in each subspace a simple form. This permits the introduction of a systematic, non-singular pseudoparticle perturbation theory valid at low \((\omega - \omega_0)\) energy and establishes directly the universal form of the Hamiltonian for integrable electronic quantum liquids.

Finally, in Sec. VI we present the discussion and concluding remarks.

II. THE GENERALIZED PSEUDOHOLE AND HEAVY-PSEUDOHOLE OPERATOR BASIS

We consider the Hubbard chain \[18,19\] in a magnetic field \(H\) and chemical potential \(\mu\)

\[
\hat{H} = \hat{H}_{SO(4)} + 2\mu \hat{S}_z^c + 2\mu_0 \hat{S}_z^s,
\]

(1)

where

\[
\hat{H}_{SO(4)} = -t \sum_{j,\sigma} \left[ c_{j+1,\sigma}^\dagger c_{j,\sigma} + c_{j,\sigma}^\dagger c_{j+1,\sigma} \right] + U \sum_{j} \left[ c_{j,\uparrow}^\dagger c_{j,\uparrow} - \frac{1}{2} \right] \left[ c_{j,\downarrow}^\dagger c_{j,\downarrow} - \frac{1}{2} \right].
\]

(2)

In the absence of the chemical-potential and magnetic-field terms the Hamiltonian (1) reduces to (2) and has \(SO(4) = SU(2) \otimes SU(2) / Z_2\) symmetry \[21,22,23\]. In equation (1) we used the notations \(\eta = S^c\) and \(\hat{S} = \hat{S}^s\) for \(\eta\)-spin and spin, respectively. With this notation the generators of the \(\eta\)-spin and spin \(SO(4)\) algebra are \[21,22,23\],

\[
\hat{S}_z^c = -\frac{1}{2} \left[ N_a - \sum_{\sigma} \hat{N}_\sigma \right], \quad \hat{S}_z^s = \sum_{j} (-1)^j c_{j,\uparrow}^\dagger c_{j,\downarrow}, \quad \hat{S}_z^c = \sum_{j} (-1)^j c_{j,\downarrow} c_{j,\uparrow}^\dagger,
\]

(3)

for \(\eta\)-spin and

\[
\hat{S}_z^s = \sum_{j} \sigma \hat{N}_\sigma, \quad \hat{S}_s^c = \sum_{j} c_{j,\uparrow}^\dagger c_{j,\downarrow}, \quad \hat{S}_s^c = \sum_{j} c_{j,\downarrow} c_{j,\uparrow}^\dagger,
\]

(4)

for spin. Since \(N_a\) is even, the operator \(\hat{S}_z^c + \hat{S}_z^s\) has only integer eigenvalues and all half-odd integer representations of \(SU(2) \otimes SU(2)\) are projected out \[21,22,23\]. This selection rule excludes transitions with \(\Delta \hat{S}_z^c + \Delta \hat{S}_z^s\) half-odd integer.

In equations (2) – (4) the operator \(c_{j,\sigma}^\dagger\) and \(c_{j,\sigma}\) creates and annihilates, respectively, one electron of spin projection \(\sigma\) (\(\sigma\) refers to the spin projections \(\sigma = \uparrow, \downarrow\) when used as an operator or function index and is given by \(\sigma = \pm 1\) otherwise) at the site \(j\),

\[
\hat{N}_\sigma = \sum_{j} c_{j,\sigma}^\dagger c_{j,\sigma},
\]

(5)

is the number operator for \(\sigma\) spin-projection electrons, and \(t, U, \text{ and } \mu_0\) are the first-neighbor transfer integral, the onsite Coulomb repulsion, and the Bohr magneton, respectively.

There are \(N_u\) up-spin electrons and \(N_d\) down-spin electrons in the chain of \(N_a\) sites and with lattice constant \(a\) associated with the model (1). We use periodic boundary conditions and consider \(N_a\) to be even and when \(N = N_a\)
(half filling) both $N_\uparrow$ and $N_\downarrow$ to be odd and employ units such that $a = t = \mu_0 = \hbar = 1$. The dimensionless onsite repulsion is $u = U/4t$. When $N_\sigma$ is odd the Fermi momenta are given by $k_{F\sigma}^+ = \pm \left[ k_{F\sigma} - \frac{\pi n^\sigma}{N_\sigma} \right]$ where

$$k_{F\sigma} = \frac{\pi N_\sigma}{N_a}.$$  

When $N_\sigma$ is even the Fermi momenta are given by $k_{F\sigma}^+ = k_{F\sigma}$ and $k_{F\sigma}^- = - \left[ k_{F\sigma} - \frac{\pi n^\sigma}{N_\sigma} \right]$ or by $k_{F\sigma}^+ = \left[ k_{F\sigma} - \frac{\pi n^\sigma}{N_\sigma} \right]$ and $k_{F\sigma}^- = - k_{F\sigma}$. Often we can ignore the $\frac{1}{N_\sigma}$ corrections of these expressions and consider $k_{F\sigma}^\pm \simeq \pm k_{F\sigma} = \pm \pi n^\sigma$ and $k_{F} = (k_{F\uparrow} + k_{F\downarrow})/2 = \pi n/2$, where $n^\sigma = N_\sigma/N_n$ and $n = N/N_a$. We consider the electronic density $n = n_\uparrow + n_\downarrow$ and the spin density $m = n_\uparrow - n_\downarrow$.

For finite values of both the magnetic field and chemical potential the symmetry of the quantum problem is reduced to $U(1) \otimes U(1)$, with $S^c_\sigma$ and $S^s_\sigma$ commuting with $\hat{H}$. The eigenvalues $S^c_\sigma$ and $S^s_\sigma$ determine the different symmetries of the Hamiltonian (1). When $S^c_\sigma \neq 0$ and $S^s_\sigma \neq 0$ the symmetry is $U(1) \otimes U(1)$, for $S^c_\sigma = 0$ and $S^s_\sigma \neq 0$ (and $\mu = 0$) it is $SU(2) \otimes U(1)$, when $S^c_\sigma \neq 0$ and $S^s_\sigma = 0$ it is $U(1) \otimes SU(2)$, and at $S^c_\sigma = 0$ and $S^s_\sigma = 0$ (and $\mu = 0$) the Hamiltonian symmetry is $SO(4)$.

There are four $U(1) \otimes U(1)$ sectors of parameter space which correspond to $S^c_\sigma < 0$ and $S^s_\sigma < 0$; $S^c_\sigma < 0$ and $S^s_\sigma > 0$; $S^c_\sigma > 0$ and $S^s_\sigma < 0$; and $S^c_\sigma > 0$ and $S^s_\sigma > 0$. We follow Ref. [25] and refer these sectors by $(l_c, l_s)$ where

$$l_\alpha = \frac{S^\alpha}{|S^\alpha|}.$$  

The sectors $(-1, 1); (-1, -1); (1, -1); (1, 1)$ refer to electronic densities and spin densities $0 < n < 1$ and $0 < m < n$; $0 < n < 1$ and $-n < m < 0$; $1 < n < 2$ and $0 < m < (2 - n)$; and $1 < n < 2$ and $-(2 - n) < m < 0$, respectively.

There are two $(l_c)$ sectors of $SU(2) \otimes U(1)$ Hamiltonian symmetry [and two $(l_c)$ sectors of $U(1) \otimes SU(2)$ Hamiltonian symmetry] which correspond to $S^c_\sigma < 0$ and $S^s_\sigma > 0$ for $l_c = -1$ and $l_c = 1$, respectively, (and to $S^c_\sigma < 0$ and $S^s_\sigma > 0$ for $l_c = -1$ and $l_c = 1$, respectively). There is one $SO(4)$ sector of parameter space [which is constituted only by the $S^c_\sigma = 0$ (and $\mu = 0$) and $S^s_\sigma = 0$ canonical ensemble].

The related holon and spinon study of Ref. [1] did not introduce the excitation operator generators (some of these were studied in Ref. [20]) and only refers and applies to states with eigenvalues $\frac{S^c}{N_a} \sim 0$ and $\frac{S^s}{N_a} \sim 0$. In this paper we generate all Hamiltonian eigenstates from the pseudohole vacuum $|V\rangle$ introduced in Ref. [24], which is the half-filled and zero-magnetic-field GS. It is convenient for our complete operator description of the quantum problem to consider explicitly all the above sectors of parameter space.

In the case of integrable models of simple Abelian $U(1)$ symmetry the elementary excitations are generated by a single type of pseudoparticles (pseudoholes) [27]. On the other hand, in the present case of the Hubbard chain we have shown [20] that in terms of pseudoholes the description of the states I involves four branches of $\alpha, \beta$ pseudoholes, where $\alpha = c, s$ and $\beta = \pm \frac{1}{2}$. In the particular case of LWS’s I and (or) HWS’s I we have that the $\alpha, \beta$ pseudoholes are such that $\alpha = c, s$ and $\beta = \pm \frac{1}{2}$. Therefore, in each $(l_c, l_s)$ sector of parameter space of Hamiltonian symmetry $U(1) \otimes U(1)$ only the $c, +\frac{1}{2}$ and $s, \pm \frac{1}{2}$ pseudohole branches are involved in the description of the corresponding states I. However, the relation $\beta = \pm \frac{1}{2}$ is not valid for the general case considered in this paper, as we find below, yet the general description also involves the $\alpha, \beta$ pseudoparticles introduced in Ref. [24], which have creation operators $a_{q,\alpha,\beta}^\dagger$. In addition, it involves an infinite number of $\alpha, \gamma$ heavy-pseudoparticle branches, with creation operators $b_{q,\alpha,\gamma}^\dagger$. Here the quantum numbers $\gamma$ are the positive integers ($\gamma = 1, 2, \ldots, \infty$).

Elsewhere we will characterize the connection of the pseudohole and heavy-pseudoparticle quantum numbers to electronic quantities in some limiting cases of physical interest. For instance, in the $n \to 1$ and $S^c \to 0$ (or $m \to 0$ and $S^s \to 0$) limit, $\alpha = c$ (or $\alpha = s$) becomes charge (or spin) $\hat{c}$; and the $c, \beta$ (or $s, \beta$) pseudoholes are such that $\beta = S^c = \pm \frac{1}{2}$ (or $\beta = -S^c = \pm \frac{1}{2}$). We emphasize that in that limit and in the particular case we just considered $S^c = \pm \frac{1}{2}$ (or $S^c = \pm \frac{1}{2}$) denotes the $\eta$ spin (or spin) projection of the individual $c, \beta$ (or $s, \beta$) pseudohole. Therefore, it does not refer to the general definition associated with Eqs. (3) and (4) which defines the $\eta$ spin (or spin) projection of an Hamiltonian eigenstate. Moreover, for $U \to \infty$, the creation of one $c, \gamma$ pseudoparticle ($\gamma > 0$) is associated with the creation of $D = \gamma$ doubly occupied sites. Note that in that limit the Hamiltonian commutes with the doubly-occupancy operator, i.e. $[\hat{H}, \hat{D}] = 0$.

The above operators obey an anticommuting algebra such that

$$\{a_{q,\alpha,\beta}^\dagger, a_{q',\alpha',\beta'}\} = \delta_{q,q'}\delta_{\alpha,\alpha'}\delta_{\beta,\beta'},$$  

(8)
and

\[ \{b_{q,\alpha,\gamma}^\dagger, b_{q',\alpha',\gamma'}\} = \delta_{q,q'} \delta_{\alpha,\alpha'} \delta_{\gamma,\gamma'}, \]

and all remaining anticommutators vanish.

As was found in Ref. [24], the \( \alpha, \beta \) pseudoholes are the “holes” of the \( \alpha \) pseudoparticles considered in the low-energy representation introduced in Refs. [7,8,9,31] and studied in more detail in Refs. [32,33,34,35,36]. In contrast to their “holes”, the usual \( \alpha \) pseudoparticles (of number \( N_\alpha \)) are not labeled by the \( \beta \) pseudohole quantum number \( \theta \). Moreover, there are no \( \beta \) bands, i.e. \( \alpha, + \frac{1}{2} \) and \( \alpha, - \frac{1}{2} \) pseudoholes occupy different pseudomomentum values in the same \( \alpha \) band (as the right-hand-side (rhs) of Eq. (23) below confirms). For convenience we shall call the usual \( \alpha \) pseudoparticles \([8,9,31,32,33,34,35,36]\) and the corresponding \( \alpha \) bands the \( \alpha, 0 \) pseudoparticles and \( \alpha, 0 \) bands, respectively.

The description of the whole Hilbert space involves the set of \( \alpha, \gamma \) pseudomomentum bands which are occupied by the \( \alpha, \beta \) pseudoholes (or \( \alpha, 0 \) pseudoparticles) for \( \gamma = 0 \) and by the \( \alpha, \gamma \) heavy pseudoparticles for \( \gamma > 0 \). In particular, the description of the non-LWS’s and non-HWS’s out of the BA requires the use of the \( \alpha, \beta \) pseudoholes instead of the usual \( \alpha, 0 \) pseudoparticles. However, we find in futures sections that both the \( SO(4) \) Hamiltonian term (2) and the BA rapidities are \( \beta \) independent. Moreover, we find that the BA rapidities can be generalized to describe a whole \( SU(2) \) tower of \( \eta \) spin or spin Hamiltonian eigenstates. We emphasize that the BA solution refers directly only to the LWS’s (or HWS’s) of these towers \([8]\). In the case of such \( \beta \)-independent quantities it is often convenient to use the \( \alpha, 0 \)-pseudoparticle representation rather than the \( \alpha, \beta \)-pseudohole representation. The \( \alpha, 0 \) pseudoparticles have creation operators \( b_{q,\alpha,0}^\dagger \) and obey an anticommuting algebra such that

\[ \{b_{q,\alpha,\beta}^\dagger, b_{q',\alpha',\beta'}\} = \delta_{q,q'} \delta_{\alpha,\alpha'} \]

and all remaining anticommutators vanish (including the ones involving \( \alpha, 0 \) pseudoparticles and heavy pseudoparticles).

Often, when we refer \( \alpha, \gamma \) pseudoparticles (bands) we consider \( \gamma = 0, 1, 2, ... \) whereas the notation \( \alpha, \gamma \) heavy pseudoparticles refers only to \( \gamma = 1, 2, ... \). The use of \( \alpha, \beta \) pseudoholes and \( \alpha, 0 \) pseudoparticles is alternative and in general we do not use them at the same time. While in the characterization of all Hamiltonian eigenstates and in studies of \( \beta \)-dependent operators the use of the \( \alpha, \beta \)-pseudohole and \( \alpha, \gamma \)-heavy-pseudoparticle representation is required, in studies involving \( \beta \)-independent operators (as for instance in charge transport \([23]\)) it is often more convenient to use the \( \alpha, \gamma \)-pseudoparticle representation (with \( \gamma = 0, 1, 2, 3 ... \)).

A crucial point is that the pseudohole and heavy-pseudoparticle numbers are constant for each Hamiltonian eigenstate. The study of the non-LWS’s outside the BA solution \([24]\) requires the introduction of the generalized \( \alpha, \beta \) pseudohole numbers. These are given by

\[ N_{c,\beta}^h = \frac{N^h}{2} - \beta[N_\alpha - N], \quad N_{s,\beta}^h = \frac{N^h}{2} - \beta[N_\gamma - N_\delta], \]

where

\[ N_\alpha^h = N_\alpha - N_{c,0}, \quad N_\delta^h = N_{c,\beta} - 2 \sum_{\gamma=0}^{\infty} N_{s,\gamma}, \]

and the total number of \( \alpha \) pseudoholes is \( N_\alpha^h = \sum_\beta N_{c,\beta}^h \).

These important expressions are generalizations of the pseudohole expressions of Ref. [24], which refer only to the \( S^e \) and \( S^s \) LWS’s and HWS’s I (which we call simply states I). It is convenient to introduce the numbers \( N_\delta^h \) such that

\[ N_\alpha^h = S^\alpha - l_\alpha S_\zeta^\alpha = S^\alpha - |S_\zeta^\alpha|, \]

where for \( S_\zeta^\alpha \neq 0 \) the parameter \( l_\alpha = \pm 1 \) is given in Eq. (7). When dealing with energies and eigenvalues instead of Hamiltonians and operators the numbers of Eq. (13) together with the \( \alpha, \gamma \)-pseudoparticle quantities and numbers constitute an alternative representation for the \( \alpha, \beta \)-pseudohole and \( \alpha, \gamma \)-heavy-pseudoparticle quantities and numbers \([24]\). The knowledge of all the \( N_{c,\beta}^h \) pseudohole and \( N_{s,\alpha,\gamma}^h \) heavy-pseudoparticle (\( \gamma > 0 \)) numbers provides the knowledge of all pseudoparticle numbers \( N_{\alpha,\gamma} (\gamma = 0, 1, 2, 3 ... \) and \( N_\delta^h \) numbers, and the inverse is also true. While the set of \( N_{\alpha,\gamma} \) numbers are directly given by the BA solution \([13]\), the \( \beta \)-dependent pseudohole numbers \( N_{c,\beta}^h \) and the numbers \( N_{\delta}^h \) are not considered by that solution. In Appendix A we relate the pseudoparticle numbers \( N_{\alpha,\gamma} \) with the usual BA notation \([19]\). On the other hand, we find below that \( \beta \) is the quantum number needed to describe the Hamiltonian eigenstates out of the BA solution.
It follows from Eq. (11) that the electron numbers are exclusive functions of the pseudohole numbers and read
\[
N_\uparrow = \frac{N_\alpha}{2} + \sum_\beta \beta [N_{c,\beta}^h - N_{s,\beta}^h], \quad N_\downarrow = \frac{N_\alpha}{2} + \sum_\beta \beta [N_{c,\beta}^h + N_{s,\beta}^h]. \tag{14}
\]

As for the case of the Hilbert subspace spanned by the states I, the pseudomomentum-number operators
\[
\hat{N}_{\alpha,\beta}^h(q) = a_{q,\alpha,\beta}^\dagger a_{q,\alpha,\beta}, \quad \hat{N}_{\alpha,\gamma}(q) = b_{q,\alpha,\gamma}^\dagger b_{q,\alpha,\gamma}, \tag{15}
\]
(with \(\gamma = 1, 2, \ldots\)), and
\[
\hat{N}_{\alpha,0}(q) \equiv 1 - \sum_\beta \hat{N}_{\alpha,\beta}^h(q), \tag{16}
\]
play a central role in the generalized theory. The operator (16) has the following alternative representation in terms of \(\alpha, 0\) pseudoparticle operators
\[
\hat{N}_{\alpha,0}(q) = b_{q,\alpha,0}^\dagger b_{q,\alpha,0}. \tag{17}
\]
The number operators can be expressed in terms of the pseudomomentum distributions as follows
\[
\hat{N}_{\alpha,\beta}^h = \sum_q \hat{N}_{\alpha,\beta}^h(q), \quad \hat{N}_{\alpha,\gamma} = \sum_q \hat{N}_{\alpha,\gamma}(q). \tag{18}
\]

We emphasize that while all Hamiltonian eigenstates are also eigenstates of the operator \(\hat{N}_{\alpha,\beta}^h\) because these numbers are good quantum numbers, this does not hold true for the operator \(\hat{N}_{\alpha,\beta}^h(q)\). We find below that only the states I are eigenstates of the latter operator, yet all Hamiltonian eigenstates are also eigenstates of the pseudoparticle operators \(\hat{N}_{\alpha,\gamma}(q)\).

The \(SO(4)\) operator generators, \(\hat{S}_z^x\) and \(\hat{S}_z^y\) [see Eqs. (3) – (4)], and the \(\eta\)-spin and spin numbers \(S^\alpha\) and \(N^\alpha_z\) [see Eq. (13)], are given by
\[
\hat{S}_z^x = \sum_{q,\beta} \beta \hat{N}_{\alpha,\beta}^h(q); \quad \hat{S}_z^y = \sum_q a_{q,\alpha,\pm 1}^\dagger a_{q,\alpha,\pm 1}, \tag{19}
\]
and
\[
S^\alpha = \frac{1}{2} [N_a^h - \sum_{\gamma=1}^\infty 2\gamma N_{\alpha,\gamma}]; \quad N^\alpha_z = \frac{1}{2} \left[ \sum_\beta (1 - 2\beta l_\alpha) \hat{N}_{\alpha,\beta}^h - \sum_{\gamma=1}^\infty 2\gamma N_{\alpha,\gamma} \right], \tag{20}
\]
respectively. Note that the expressions for the off-diagonal \(SO(4)\) generators go beyond BA, which provides the expressions only for diagonal operators. The application a suitable number of times of these operators onto the LWS’s of the \(\eta\)-spin and spin algebras generates the non-LWS’s towers of Hamiltonian eigenstates outside the BA \[23\]. Therefore, their off-diagonal form in the quantum number \(\beta\), Eq. (19), reveals that in the present operator basis the construction of the states out of the BA requires the introduction of these pseudohole quantum numbers. This is confirmed by the form of the Hamiltonian-eigenstate pseudohole and heavy-pseudoparticle generators relative to the theory vacuum [see Eq. (23) below].

The generalization of the LWS’s BA total momentum expression \[19\], which we denote by \(P_{BA}\), to all Hamiltonian eigenstates, requires addition of an extra term associated with \(\eta\) pairing \[22,23\]. This leads to
\[
P = P_{BA} + \pi [S^x + S_z^x]. \tag{21}
\]

When this expression provides values of the momentum such that \(|P| > \pi\), the total momentum is defined as the corresponding value at the first Brillouin zone. In operator form and for sub-canonical ensembles such that \(S_z^x \neq 0\) and with \(l_\alpha\) given by Eq. (7) we have that \(N_{\alpha,\dagger}^h < N_{\alpha,\dagger}^h\) and the present representation leads to the following simple expressions for the momentum operator \(\hat{P}\) and its eigenvalue \(P\)
\[
\hat{P} = \sum_{q,\alpha,\gamma=0}^\infty qC_{\alpha,\gamma} \hat{N}_{\alpha,\gamma}(q) + \pi [N_{c,\dagger}^h + \sum_{\gamma=1}^\infty \hat{N}_{s,\gamma}];
\]
\[
P = \sum_{q,\alpha,\gamma=0}^\infty qC_{\alpha,\gamma} N_{\alpha,\gamma}(q) + \pi \sum_{\gamma=1}^\infty [(1 + \gamma)N_{c,\gamma} + N^\alpha_z], \tag{22}
\]
respectively, where \( C_{c,\gamma} = -1 \) for \( \gamma > 0 \) and \( C_{a,\gamma} = 1 \) otherwise. We emphasize that the momentum term, \( \pi \sum_{\gamma=1}^n [(1+\gamma)N_{c,\gamma} + N_{c,\gamma}] \), is always a multiple of \( \pm \pi \).

The pseudoparticle perturbation theory introduced in Refs. [8,12,31,32,34] and developed in a suitable operator basis in Refs. [21,23] refers to the Hilbert subspace spanned by the Hamiltonian eigenstates \( I \). Rather than holons and spinons [12,21,27], at finite values of the magnetic field and chemical potential and at constant electronic numbers the low-energy excitations \( I \) are described by pseudoparticle-pseudohole processes relative to the canonical-ensemble GS. The latter state, as well as all excited states \( I \) with the same electron numbers, are simple Slater determinants of pseudoparticle levels [12,23,24,28,36].

Although we find in Sec. III that the pseudoholes and heavy pseudoparticles obey generalized [23] rather than Fermi or Bose statistics, one can use their formal anticommutation relations (together with suitable topological-momentum-shift operators introduced in Refs. [20,30] and generalized for the heavy pseudoparticles in Sec. III) to develop an extremely simple second-quantized representation of all \( 4^N \) orthonormal Hamiltonian eigenstates [23]. As for the above states \( I \), in our generalized operator basis all these states are simple Slater determinants of pseudohole and heavy-pseudoparticle levels. They are of the form

\[
|\psi; \{N_{h,\alpha,\beta}\}, \{N_{a,\alpha}\}\rangle = \frac{1}{\sqrt{C}} \prod_{\alpha} A_\alpha \prod_{\gamma=1}^\infty \left[ [\hat{S}_\gamma^a]^N_{h,\alpha} \prod_{q,q'} a_{\eta,\alpha,-\frac{1}{2}}^\dagger b_{\eta',\alpha,\gamma}^\dagger \right] |V\rangle,
\]

where the generator \( \hat{S}_\gamma^a \) is given in Eq. (19), \( \{N_{h,\alpha,\beta}\} \) and \( \{N_{a,\alpha}\} \) are abbreviations for the sets

\[
\{N_{c,\frac{1}{2}}, N_{c,-\frac{1}{2}}, N_{h,\alpha,\gamma}, N_{h,\alpha,-\frac{1}{2}}\},
\]

and

\[
\{N_{c,1}, ..., N_{c,\infty}, N_{s,1}, ..., N_{s,\infty}\},
\]

respectively,

\[
A_\alpha = \prod_\beta \Theta(N_{h,\alpha,\beta} - \sum_{\gamma=1}^\infty \gamma N_{a,\alpha,\gamma}),
\]

and \( \Theta(x) = 1 \) for \( x \geq 0 \) and \( \Theta(x) = 0 \) for \( x < 0 \). Note that \( A_\alpha \) equals either 1 or 0. This is because in the present representation the Hamiltonian eigenstates are characterized by pseudohole and heavy-pseudoparticle numbers such that \( N_{h,\alpha,\beta} \geq \sum_{\gamma=1}^\infty \gamma N_{a,\alpha,\gamma} \). We note that application of the non-diagonal operators of Eq. (19) onto \( \eta \) spin or spin LWS’s or HWS’s generates often states with numbers such that \( N_{h,\alpha,\beta} < \sum_{\gamma=1}^\infty \gamma N_{a,\alpha,\gamma} \). The presence of \( A_\alpha \) (see Eq. (26)) in the rhs of Eq. (23) (which is to be changed according to the changes of the numbers \( N_{h,\alpha,\beta} \)) then assures that application of the above generators in these LWS’s or HWS’s gives 0.

The existence of the half-filling Mott-Hubbard gap [32,33], which we denote here by \( \Delta_{MH} \), implies that \( S^c = S^c_z = 0 \) for chemical potentials \( \mu \) such that \( -\frac{\Delta_{MH}}{2} < \mu < \frac{\Delta_{MH}}{2} \). Therefore, the \( S^c = S^c_z = 0 \) and \( S^s = S^s_z = 0 \) vacuum \( |V\rangle \) of the rhs of Eq. (23) can be associated with different chemical-potential values within the range \( -\frac{\Delta_{MH}}{2} < \mu < \frac{\Delta_{MH}}{2} \). The vacuum chemical-potential value is chosen according to the suitable reference level from which the energy of the pseudohole and pseudoparticle seas of the states (23) is to be measured. Therefore, we assume in the definition of the vacuum of Eq. (23) that \( |V\rangle \) is such that \( \mu = 0 \) for \( S^c_z = 0 \) states but \( \mu \to \pm \frac{\Delta_{MH}}{2} \) for \( S^c_z \to 0^\mp \).

The possible pseudohole and heavy-pseudoparticle numbers are constrained by Eqs. (19) and (20) and together with the number of discrete pseudomomentum values in each band leads to \( 4^N \) possible orthonormal Hamiltonian eigenstates of form (23), in agreement with Ref. [23]. The number of discrete pseudomomentum values in each \( \alpha, \gamma \) band (\( \alpha, \gamma \)-band Fock-space dimension) is given by

\[
d_{F,\alpha,\gamma} = N_{h,\alpha} + N_{a,\alpha,\gamma} - \sum_{\gamma' = 1}^{\infty} \left( |\gamma + \gamma'\rangle - |\gamma - \gamma'\rangle \right) N_{a,\gamma'}. \]

In the above summations and products, the pseudomomentum takes on values

\[
q_j = \frac{2\pi}{N_{F}} f_j^{\alpha,\gamma},
\]
where in contrast to the usual momentum, $I_j^{α,γ}$ are consecutive integers or half-odd integers for $\bar{N}_{α,γ}$ odd and even, respectively. Here

\[
\bar{N}_{c,0} = \sum_α \frac{N^h_α}{2} - \sum_γ ^∞ N_{c,γ} = \frac{N_a}{2} - N_{s,0} - \sum_α,γ=1 ^∞ N_{α,γ},
\]

(29)

and

\[
\bar{N}_{α,γ} = d_{F_{α,γ}},
\]

(30)

for values of $α, γ$ other than $c, 0$. It follows that for each $α, γ$ band $q_{α,γ}^{(−)} ≤ q ≤ q_{α,γ}^{(+)}$, with the limits of the pseudo-Brillouin zones given by

\[
q_{c,0}^{(±)} = \pm π[1 - \frac{1}{N_a}],
\]

(31)

for $\bar{N}_{c,0}$ even and

\[
q_{c,0}^{(+)} = \pi, \quad q_{c,0}^{(−)} = -π[1 - \frac{2}{N_a}],
\]

(32)

or

\[
q_{c,0}^{(+)} = \pi[1 - \frac{2}{N_a}], \quad q_{c,0}^{(−)} = -π,
\]

(33)

for $\bar{N}_{c,0}$ odd. On the other hand, the limits of the pseudo-Brillouin zones are simply given by

\[
q_{α,γ}^{(±)} = ±q_{α,γ} = ± \frac{π}{N_a}[d_{F_{α,γ}} - 1],
\]

(34)

for all the remaining $α, γ$ bands (other than the $c, 0$ band).

The usual first-quantized BA wave function expression for the Hamiltonian eigenstates has an involved form which includes many permutations \[18,19\]. That spatial wave function depends on the quantum numbers $I_j^{α,γ}$ of Eq. (28) through infinite sets of complex numbers, which many authors call rapidities. The expression of the spatial wave functions in terms of the quantum numbers $I_j^{α,γ}$ requires the solution of an infinite system of algebraic equations which we present in Sec. IV. These equations define the real part of the BA rapidities as functions of the quantum numbers $I_j^{α,γ}$ \[18,19\]. Although the expression of the spatial wave function for the Hamiltonian eigenstates in terms of the quantum numbers $I_j^{α,γ}$ requires the solution of the above systems of equations, which constitutes a problem of considerable complexity, the description of these eigenstates in the basis of the above BA quantum numbers $I_j^{α,γ}$ that diagonalize the quantum liquid, Eq. (23), is much simpler. Historically, the Hamiltonian eigenstates were introduced in terms of the spatial BA wave functions \[13,14\]. The diagonalization of the problem leads then to infinite systems of algebraic equations. These equations introduce the integer or half-odd integer quantum numbers $I_j^{α,γ}$ of Eq. (28) which describe the Hamiltonian eigenstates.

One of the principal advantages of the representation (23) is that it permits the description of the Hamiltonian eigenstates in terms of the quantum numbers $I_j^{α,γ}$ and does not require the spatial wave-function representation. In the basis associated with these quantum numbers the description of these states does not involve the rapidity numbers. Moreover, our algebraic representation (23) refers also to the Hamiltonian eigenstates out of the BA solution.

A $S^α$ LWS is such that

\[
N_{α,\downarrow}^h = \sum_γ ^∞ γN_{α,γ},
\]

(35)

whereas for a $S^α$ HWS we have that

\[
N_{α,\uparrow}^h = \sum_γ ^∞ γN_{α,γ},
\]

(36)

with $N_{α,\downarrow}^h ≥ N_{α,\uparrow}^h$ for a LWS and $N_{α,\uparrow}^h ≥ N_{α,\downarrow}^h$ for a HWS.
For the particular case of states I, we have that \( N_{\alpha,\gamma} = 0 \) for all \( \gamma = 1, 2, \ldots \) branches in the former equations. Therefore, the states I are characterized by zero occupancies of the heavy-pseudoparticle bands \( \alpha, \gamma \) (with \( \gamma > 0 \)). It then follows from the general Eqs. (35) and (36) that a \( S^\alpha \) LWS I is such that
\[
N_{\alpha,\frac{1}{2}}^h = 0 ,
\]
whereas for a \( S^\alpha \) HWS I we have that
\[
N_{\alpha,-\frac{1}{2}}^h = 0 ,
\]
with \( N_{\alpha,-\frac{1}{2}}^h \geq 0 \) for a LWS I and \( N_{\alpha,\frac{1}{2}}^h \geq 0 \) for a HWS I. This together with Eq. (23) leads to the simple Slater-determinant form found for the states I in Ref. [20]. These results also confirm that in the case of states I the \( \alpha, \beta \) pseudoholes are such that \( \beta = \frac{1}{2} \) but that this equality does not hold true in the general case. While in the case of the states I the Slater determinant (23) either involves \( \alpha, \frac{1}{2} \) or \( \alpha, -\frac{1}{2} \) pseudoholes only, in the case of the non-LWS’s and non-HWS’s the states I the Slater determinant (23) either involves \( \alpha, \frac{1}{2} \) or \( \alpha, -\frac{1}{2} \) pseudohole occupancies.

In the case of the non-LWS’s and non-HWS’s out of BA the relative number of \( \alpha, \frac{1}{2} \) and \( \alpha, -\frac{1}{2} \) pseudoholes is determined by the values of \( S^\alpha \) and \( S^\gamma \) through Eqs. (19), (20), and (39). The pseudohole numbers which characterize these states are such that
\[
N_{\alpha,\gamma}^h - \sum_{\gamma=1}^{\infty} 2\gamma N_{\alpha,\gamma} > |N_{\alpha,-\frac{1}{2}}^h - N_{\alpha,\frac{1}{2}}^h| .
\]
They can be generated from LWS’s (or HWS’s) by acting onto the latter the off-diagonal generator \( \hat{S}^\alpha_2 \) (or \( \hat{S}^\gamma_2 \)) the required number of times. According to Eq. (19), in the present basis these off-diagonal generators just produce \( \beta \)-flip processes. In the case of the LWS’s II the \( \beta \) flips generated by these operators create a number \( \gamma \) of \( \alpha, \frac{1}{2} \) pseudoholes for each \( \alpha, \gamma \) heavy pseudoparticle, which implies these states are inside BA. Following Eqs. (19) – (20) this is required for the LWS condition \( S^\alpha = -S^\gamma \). Note that both for the non-LWS’s and non-HWS’s out of the BA and for the LWS’s II and HWS’s II there is no relation between the quantity \( l_{\alpha} \) defined in Eq. (7) and the pseudohole quantum number \( \beta \).

In the case of non-LWS’s and non-HWS’s the wave function (23) uses as starting state a LWS. We then call it for LWS general representation. The general state (23) can be rewritten, alternatively, in terms of a starting HWS (HWS general representation). This just requires replacing in the rhs of Eq. (23) \( \pm \frac{1}{2} \) by \( \mp \frac{1}{2} \) and the generator \( \hat{S}_{\alpha}^\gamma \) by \( \hat{S}_{\alpha}^\gamma \). In the corresponding HWS general representation the off-diagonal generators create new states from a reference HWS. One can also use a representation where the starting state is a \( S^\alpha \) (or \( S^\gamma \)) HWS and a \( S^\alpha \) (or \( S^\gamma \)) LHS. However, the four possible general representations describe the same states.

The form of the pseudohole and heavy-pseudoparticle generators of the rhs of Eq. (23) confirms that the corresponding Hamiltonian eigenstates are not in general eigenstates of the pseudohole operator \( \hat{N}^h_{\alpha,\beta}(q) \). This is because due to the transformation laws of the off-diagonal \( \eta \)-spin and spin generators present in the rhs of Eq. (23) a Hamiltonian eigenstate is, in general, a mixture of different \( \beta = \frac{1}{2} \) and \( \beta = \frac{3}{2} \) pseudomomentum distributions. The exception are the states I which have only either \( \alpha, \frac{1}{2} \) or \( \alpha, -\frac{1}{2} \) pseudohole occupancy.

### III. GENERALIZED GROUND STATES AND STATISTICS

The conservation of the pseudohole and heavy-pseudoparticle numbers permits dividing the Hilbert space into subspaces spanned by the set of Hamiltonian eigenstates (23) with the same \( \{N_{\alpha,\beta}^h\} \) and \( \{N_{\alpha,\gamma}\} \) numbers, which correspond to the same sub-canonical ensemble. Obviously, a canonical ensemble (with constant electron numbers and thus with constant values of \( N_{\alpha,\frac{1}{2}}^h - N_{\alpha,-\frac{1}{2}}^h \) for both \( \alpha = c, s \) is usually realized by several \( \{N_{\alpha,\beta}^h\} \), \( \{N_{\alpha,\gamma}\} \) sub-canonical ensembles. Let us introduce the generalized ground state (GGS) as the Hamiltonian eigenstate(s) (23) of lowest energy in each Hilbert subspace. We find it is (they are) of the form
\[
|GGS; \{N_{\alpha,\beta}^h\}, \{N_{\alpha,\gamma}\}\rangle = \frac{1}{\sqrt{C}} \prod_{\alpha} A_{\alpha} \prod_{\gamma=1}^{\infty} [\hat{S}_{\alpha}^\gamma]^{-\frac{1}{2}} \prod_{q=a_{\alpha}}^{-\frac{1}{2}} \prod_{q'=a_{\alpha}}^{\frac{1}{2}} \prod_{\gamma_{\alpha,\beta}^{(\pm)}} \prod_{\gamma_{\alpha,\gamma}^{(\pm)}} \prod_{\gamma_{\beta,\gamma}^{(\pm)}} \prod_{\gamma_{\beta,\beta}^{(\pm)}} a_{q,\alpha,-\frac{1}{2}}^\dagger b_{q',\gamma,\alpha,\beta}^\dagger b_{q',\gamma,\alpha,\beta} |V\rangle ,
\]
(40)
where

$$q_{F\alpha,\gamma}^{(\pm)} = q_{F\alpha,\gamma}^{(\pm)} \pm C_{\alpha,\gamma} \frac{2\pi}{N_a},$$  \hfill (41)$$

and when (i) \(\alpha = c, \gamma > 0\), and \(N_{c,\gamma}\) is even or (ii) \(\alpha = s\) or \(\alpha = c\) and \(\gamma = 0\) and \(N_{c,\gamma}\) is odd (even) and \(I_j^{\alpha,\gamma}\) are integers (half-odd integers) the pseudo-Fermi points are symmetric and given by

$$q_{F\alpha,\gamma}^{(\pm)} = \pm [q_{F\alpha,\gamma} - C_{\alpha,\gamma} \frac{2\pi}{N_a}],$$  \hfill (42)$$

where

$$q_{Fc,\gamma} = \frac{\pi [d_{c,\gamma} - N_{c,\gamma}]}{N_a}, \quad \gamma > 0,$$
$$q_{Fc,0} = \frac{\pi N_{c,0}}{N_a},$$
$$q_{Fs,\gamma} = \frac{\pi N_{s,\gamma}}{N_a}. \quad (43)$$

On the other hand, when (i) \(\alpha = c, \gamma > 0\), and \(N_{c,\gamma}\) is odd or (ii) \(\alpha = s\) or \(\alpha = c\) and \(\gamma = 0\) and \(N_{c,\gamma}\) is odd (even) and \(I_j^{\alpha,\gamma}\) are half-odd integers (integers) we have that

$$q_{F\alpha,\gamma}^{(+)} = q_{F\alpha,\gamma}, \quad q_{F\alpha,\gamma}^{(-)} = -[q_{F\alpha,\gamma} - C_{\alpha,\gamma} \frac{2\pi}{N_a}],$$  \hfill (44)$$

or

$$q_{F\alpha,\gamma}^{(+)} = q_{F\alpha,\gamma} - C_{\alpha,\gamma} \frac{2\pi}{N_a}, \quad q_{F\alpha,\gamma}^{(-)} = -q_{F\alpha,\gamma}. \quad (45)$$

The GS associated with a given canonical ensemble is always a state I of the form presented in Ref. [20], which is a particular case of the general GGS expression (40). It is useful to denote the GS’s by \(|GS; S_z^c, S_z^s\rangle\), where \(S_z^c\) and \(S_z^s\) are the eigenvalues of the corresponding canonical ensemble. The GS expression associated with the \((l_c, l_s)\) sector of Hamiltonian symmetry \(U(1) \otimes U(1)\) reads

$$|GS; S_z^c, S_z^s\rangle = \prod_{q=q_{c,0}^{(-)}}^{q_{c,0}^{(+)}} \prod_{q=q_{s,0}^{(-)}}^{q_{s,0}^{(+)}} a_{q,\pm}^\dagger \prod_{q=q_{c,0}^{(-)}}^{q_{c,0}^{(+)}} a_{q,\pm}^\dagger |V\rangle. \quad (46)$$

In all sectors of Hamiltonian symmetry there are states I. In the particular case of the \(SO(4)\) zero-chemical potential and zero-magnetic field canonical ensemble there is only one state I, which is the pseudohole and heavy-pseudoparticle vacuum, \(|V\rangle\), which is nothing but the \(SO(4)\) GS [20]. The same applies to the sectors of Hamiltonian symmetry \(SU(2) \otimes U(1)\) and \(U(1) \otimes SU(2)\), the GS being always a state I. (In addition, in these sectors there is a large number of excited states I.)

In the case of the two \((l_s)\) \(SU(2) \otimes U(1)\) sectors the GS is both a LWS and HWS of the \(\eta\)-spin algebra. Therefore, it is empty of c pseudoholes and reads

$$|GS; 0, S_z^s\rangle = \prod_{q=q_{c,0}^{(-)}}^{q_{c,0}^{(+)}} \prod_{q=q_{s,0}^{(-)}}^{q_{s,0}^{(+)}} a_{q,\pm}^\dagger |V\rangle. \quad (47)$$

In the case of the \((l_c)\) \(U(1) \otimes SU(2)\) sector the GS is both a LWS and a HWS of the spin algebra and is empty of \(s\) pseudoholes. It reads

$$|GS; S_z^c, 0\rangle = \prod_{q=q_{c,0}^{(-)}}^{q_{c,0}^{(+)}} \prod_{q=q_{s,0}^{(-)}}^{q_{s,0}^{(+)}} a_{q,\pm}^\dagger |V\rangle. \quad (48)$$
Finally, the $S^c = S^h = 0$ (and $\mu = 0$) and $S^s = S^p = 0$ SO(4) ground state is, at the same time, a LWS and HWS of both the $\eta$-spin and spin algebras, ie following the notation of Refs. [20,23] it is a LWS,LWS, a LWS,HWS, a [HWS,LWS], and a [HWS,HWS]. Therefore, it is empty of both $c$ and $s$ pseudoholes and is the vacuum of the pseudohole and heavy-pseudoparticle theory. All the remaining states I can be described by particular cases of the general expression (23) which are Slater determinants of $\alpha, \beta$ pseudoholes and $\alpha, \gamma$ heavy pseudoparticles [20].

The number of Hamiltonian eigenstates associated with a sub-canonical ensemble and thus characterized by the same $\{N^h_{\alpha,\beta}\}$ and $\{N_{\alpha,\gamma}\}$ numbers is given by

$$n (\{N^h_{\alpha,\beta}\}, \{N_{\alpha,\gamma}\}) = \prod_{\alpha=c,h} \left[ N^h_{\alpha} + \frac{d_{F,0}}{N_{\alpha}} \right] \prod_{\gamma=1}^{\infty} \left[ N_{\alpha,\gamma} \right] = \prod_{\alpha=c,h} \left[ d_{F,0} N^h_{\alpha} \right] \prod_{\gamma=1}^{\infty} \left[ d_{F,\alpha,\gamma} \right],$$

(49)

which in terms of pseudoparticle numbers only can be rewritten simply as

$$\prod_{\alpha=c,s,\gamma=0}^{\infty} \left[ d_{F,\alpha,\gamma} \right].$$

(50)

The square brackets in the above equations refer to the usual combinatoric coefficients.

To classify the statistics of our pseudoholes and heavy pseudoparticles according to the generalized Pauli principle of Ref. [24], we introduce the pseudohole and heavy-pseudoparticle dimensions

$$d^h_{\alpha} \equiv 1 + d_{F,\alpha,0} - N^h_{\alpha}, \quad d_{\alpha,\gamma} \equiv 1 + d_{F,\alpha,\gamma} - N_{\alpha,\gamma},$$

(51)

respectively. An Hamiltonian eigenstate transition producing changes $\Delta N^h_{\alpha}$ and $\Delta N_{\alpha,\gamma}$ in the pseudohole and heavy pseudoparticle numbers, respectively, leads then to the following changes in the corresponding dimensions

$$\Delta d^h_{\alpha} = - \sum_{\alpha'} g_{\alpha,\alpha'} \Delta N^h_{\alpha'} - \sum_{\alpha', \gamma'=1}^{\infty} g_{\alpha;\alpha',\gamma'} \Delta N_{\alpha',\gamma'},$$

(52)

and

$$\Delta d_{\alpha,\gamma} = - \sum_{\alpha'} g_{\alpha;\gamma;\alpha'} \Delta N^h_{\alpha'} - \sum_{\alpha', \gamma'=1}^{\infty} g_{\alpha;\gamma;\alpha',\gamma'} \Delta N_{\alpha',\gamma'},$$

(53)

where the statistical-interaction matrix has infinite dimension and reads

$$g_{c,\alpha} = \delta_{c,\alpha}, \quad g_{c;\alpha,\gamma} = 0,$$

(54)

$$g_{s,\alpha} = \frac{1}{2}, \quad g_{s;\alpha,\gamma} = \delta_{s,\alpha},$$

(55)

$$g_{\alpha;\gamma;\alpha'} = - \delta_{\alpha,\alpha'}, \quad g_{\alpha;\gamma;\alpha',\gamma'} = \delta_{\alpha,\alpha'} (\gamma + \gamma' - |\gamma - \gamma'|).$$

(56)

This fully defines the statistics, the pseudoholes and heavy pseudoparticles being neither fermions nor bosons [29].

The simple form of the GGS expression (40), GS expressions (46) – (48), and general-Hamiltonian eigenstates (23) has a deep physical meaning. It reveals that in the present basis these eigenstates of the many-electron quantum problem are “non-interacting” states of simple Slater-determinant form. However, that the numbers $I^h_\gamma$ of the rhs of Eq. (28) can be integers or half-odd integers for different Hamiltonian eigenstates, makes the problem much more involved than a simple non-interacting case. This change in the integer or half-odd integer character of some of the numbers $I^h_\gamma$ of two states, shifts all the occupied pseudomomenta and leads to the orthogonal catastrophes [5]. These shifts are associated with topological excitations which are generated by the topological-momentum shift operators introduced in Refs. [21,23] for the particular case of the states I.

Let us introduce the $\alpha, 0$ and $\alpha, \gamma$ topological-momentum-shift operators (for the states I see Refs. [20,23]) which generate topological excitations associated with a shift of all the pseudomomenta of the corresponding band by $\pm \frac{\alpha \gamma}{N_{\gamma}}$. Creation or annihilation of odd numbers of pseudoholes or heavy-pseudoparticles always requires the occurrence of such topological-momentum shifts. Therefore, in addition to the pseudohole and heavy-pseudoparticle operators our second-quantized algebra also includes topological-momentum shift operators. Their unitary generators are such that
\[ U_{\alpha,\beta}^{\pm1} a_{q,\alpha,\beta}^\dagger = a_{q,\alpha,\beta}^\dagger U_{\alpha,\beta}^{\pm1}, \]  
and
\[ U_{\alpha,\gamma}^{\pm1} b_{q,\alpha,\gamma}^\dagger = b_{q,\alpha,\gamma}^\dagger U_{\alpha,\gamma}^{\pm1}, \]  
and read
\[ U_{\alpha,0}^{\pm1} = \exp \left[ -\sum_{q,\beta} a_{q,\alpha,\beta}^\dagger a_{q,\alpha,\beta} \right], \]  
and
\[ U_{\alpha,\gamma}^{\pm1} = \exp \left[ \sum_{q} b_{q,\alpha,\gamma}^\dagger b_{q,\alpha,\gamma} \right]. \]

As we discuss in future sections, a remarkable property is that the transition between a GS (46) and any eigenstate (23) can be separated into two types of excitations: (a) a topological GS – GGS transition which involves the creation or annihilation of pseudoholes and (or) heavy pseudoparticles as well as the occurrence of topological momentum shifts and (b) a Landau-liquid excitation \[ GGS \] associated with pseudoparticle - pseudohole processes relative to the GGS.

The topological transitions (a) are basically superpositions of three kinds of elementary transitions: (i) GS - GS transitions involving changes in the \( \sigma \) electron numbers by one, (ii) single \( \beta \) flips of \( \alpha, \beta \) pseudoholes which lead to non-LWS's and non-HWS's outside the BA, and (iii) creation of single \( \alpha, \gamma \) pseudoparticles at constant values of \( S^c, S^z, S^x, \) and \( S^y \). While the transitions (i) are gapless, the elementary excitations (ii) and (iii) require a finite amount of energy.

The generators of the excitations (i) were studied in Ref. [20] and the ones of (ii) involve only topological momentum shifts and \( \beta \) flips (which describe either creation of electron pairs or spin flip processes). Consider as an example of a transition (iii) the creation of one \( \alpha, \gamma \) pseudoparticle at the pseudo-Fermi points, \( q = q_{F,\alpha,\gamma}^{(\pm)} \). (We examine here explicitly topological momentum shifts of occupied bands only, i.e. these which generate momentum.) Let us introduce the notation \(-\alpha\) such that \(-c = s\) and \(-s = c\). When \( \gamma > 0 \) is even (or odd) the generator reads \( U_{\alpha,0}^{\pm1} G_{\alpha,\gamma} \) (or \( G_{\alpha,\gamma} U_{-\alpha,0}^{\pm1} \)), where the topological-momentum-shift operator was defined above and
\[ G_{\alpha,\gamma} = \left[ \hat{S}_+^\alpha \right]^\gamma \prod_{q = q_{F,\alpha,\gamma}^{(\pm)}} a_{q,\alpha,\beta}^\dagger \prod_{q = q_{F,\alpha,\gamma}^{(\pm)}} b_{q,\alpha,\gamma}^\dagger. \]

Therefore, such transitions involve one \( \alpha, 0 \) or \(-\alpha, 0\) topological momentum shift, the creation of a number \( \gamma \) of \( \alpha, \frac{1}{2} \) pseudoholes and \( \gamma \) of \( \alpha, -\frac{1}{2} \) pseudoholes, and the creation of one \( \alpha, \gamma \) pseudoparticle at one of the pseudo-Fermi points. (This can be considered as the antiparticle of the “cloud” of \( 2\gamma \) associated pseudoholes.)

IV. PSEUDOPARTICLE HAMILTONIAN AND BA RAPIDITIES

For simplicity let us denote the general Hamiltonian eigenstates \( |\psi; \{ N_{\alpha,\beta}^h \}, \{ N_{\alpha,\gamma} \} \rangle \) by \( |\psi \rangle \) [and the GGS’s (40) by \( GGS \)] . These states are also eigenstates of the \( \alpha, \gamma \) pseudomomentum-distribution operators (15) such that
\[ \hat{N}_{\alpha,\gamma}(q)|\psi \rangle = N_{\alpha,\gamma}(q)|\psi \rangle, \]
where \( N_{\alpha,\gamma}(q) \) represents the eigenvalue of the operator (15), which is given by 1 and 0 for occupied and empty values of \( q \), respectively.

It follows from the form of the state-generator of the rhs of Eq. (23) that all \( 2S^\alpha + 1 \) Hamiltonian eigenstates constructed from the same \( S^\alpha = -S^z \) LWS by applying onto it \( 1, 2, \ldots, 2S^\alpha \) number of times the off-diagonal operator \( \hat{S}_+^\alpha \) (19) have the same \( N_{\alpha,\gamma}(q) \) pseudomomentum distribution. On the other hand, the pseudohole numbers \( N_{\alpha,-\frac{1}{2}}^h \) and \( N_{\alpha,+\frac{1}{2}}^h \) are different for each of the \( 2S^\alpha + 1 \) states of such \( S^\alpha \) tower. These numbers change from \( N_{\alpha,-\frac{1}{2}}^h = N_{\alpha,-\frac{1}{2}}^h \) to
\[ \sum_{\gamma=1}^{\infty} \gamma N_{\alpha,\gamma} \text{ and } N_{\alpha,\gamma}^h = \sum_{\gamma=1}^{\infty} \gamma N_{\alpha,\gamma} \text{ for the initial LWS to } N_{\alpha,\gamma}^h = \sum_{\gamma=1}^{\infty} \gamma N_{\alpha,\gamma} \text{ and } N_{\alpha,\gamma}^h = N_{\alpha,\gamma}^h - \sum_{\gamma=1}^{\infty} \gamma N_{\alpha,\gamma} \text{ for the corresponding HWS, where } N_{\alpha}^h \text{ is given in Eq. (12).} \]

As in the case of the spatial wave functions, the expression of the energy in terms of the quantum numbers \( I_j^{\alpha,\gamma} \) of Eq. (28) involves the BA rapidities. These are complex functions of the numbers \( I_j^{\alpha,\gamma} \). Following our pseudomomentum definition, Eq. (28), in our description they are functions of the pseudomomentum \( q \). The real part of the BA rapidities, which we denote by \( R_{\alpha,\gamma}(q) \), are in the present basis eigenvalues of corresponding rapidity operators, \( \hat{R}_{\alpha,\gamma}(q) \). If we combine our operator representation with the properties of the BA solution, it is straightforward to confirm from the relation between the rapidity real part, \( R_{\alpha,\gamma}(q) \), and the pseudomomentum distributions of Eq. (62) that these real rapidity-function parts are eigenvalues of the rapidity operators \( \hat{R}_{\alpha}(q) \). These rapidity operators are exclusive functions of the pseudoparticle operators \( \hat{N}_{\alpha,\gamma}(q) \). It follows that all Hamiltonian eigenstates are also eigenstates of the rapidity operators \( \hat{R}_{\alpha,\gamma}(q) \) such that

\[ \hat{R}_{\alpha,\gamma}(q) | \psi \rangle = R_{\alpha,\gamma}(q) | \psi \rangle, \]  

where for the particular case of \( R_{c,0}(q) \) we also consider the associate rapidity \( K(q) \) which is defined by the following equation

\[ R_{c,0}(q) = \frac{\sin K(q)}{u}. \]  

This notation refers to the spectral parameter \( \frac{\sin K(q)}{u} \) which appears often in operator form. The rapidity \( K(q) \) is the eigenvalue of the rapidity operator \( \hat{K}(q) \) such that

\[ \hat{K}(q) | \psi \rangle = K(q) | \psi \rangle. \]  

In Appendix A we present the relation of our rapidity eigenvalues to the real part of the BA rapidity numbers \[ [9]. \]

The real part of the BA rapidities, which in our basis are the eigenvalues \( R_{\alpha,\gamma}(q) \) [and \( K(q) \)], are fully determined by the pseudoparticle distributions \( N_{\alpha,\gamma}(q) \) through a system of coupled integral equations. The standard treatment of the BA solutions is very lengthy, e.g., for each state we have to rewrite a new set of equations. This is because the eigenvalues \( R_{\alpha,\gamma}(q) \) of the rapidity operators \( R_{\alpha,\gamma}(q) \) are different for each state \( | \psi \rangle \) (23).

However, within the present operator description, we can introduce a single set of infinite general equations which apply to all LWS’s. In addition, our operator representation allows the generalization of the BA rapidities to the non-LWS’s out of BA. It follows that the above general equations can be generalized to all Hamiltonian eigenstates. Each of these equations defines one of the rapidity operators \( \hat{R}_{\alpha,\gamma}(q) \) in terms of a \( q \), \( \alpha \), and \( \gamma \) summation containing functionals of the rapidity operators and pseudomomentum-distribution operators. This system of coupled equations has a unique solution which defines the expressions for the rapidity operators in terms of the pseudomomentum-distribution operators. In terms of the corresponding eigenvalues the system of coupled integral equation reads

\[
K(q) = q - \frac{2}{N_a} \sum_{\gamma=0}^{\infty} \sum_{q'} N_{s,\gamma}(q') \tan^{-1} \left( \frac{R_{c,0}(q) - R_{s,\gamma}(q')} {1 + \gamma} \right) \\
- \frac{2}{N_a} \sum_{\gamma=1}^{\infty} \sum_{q'} N_{c,\gamma}(q') \tan^{-1} \left( \frac{R_{c,0}(q) - R_{c,\gamma}(q')} {\gamma} \right),
\]

\[ 2\text{Re} \sin^{-1} \left( u(R_{c,\gamma}(q) - i\gamma) \right) = q + \frac{2}{N_a} \sum_{q'} N_{c,0}(q') \tan^{-1} \left( \frac{R_{c,\gamma}(q) - R_{c,0}(q')} {\gamma} \right) \\
+ \frac{1}{N_a} \sum_{\gamma'=1}^{\infty} \sum_{q'} N_{c,\gamma'}(q') \Theta_{\gamma,\gamma'} \left( R_{c,\gamma}(q) - R_{c,\gamma'}(q') \right),
\]

and

\[
0 = q - \frac{2}{N_a} \sum_{q'} N_{c,0}(q') \tan^{-1} \left( \frac{R_{s,\gamma}(q) - R_{c,0}(q')} {1 + \gamma} \right) \\
+ \frac{1}{N_a} \sum_{\gamma=0}^{\infty} \sum_{q'} N_{s,\gamma}(q') \Theta_{1+\gamma,1+\gamma'} \left( R_{s,\gamma}(q) - R_{s,\gamma'}(q') \right),
\]

\[ 13 \]
where
\[
\Theta_{\gamma,\gamma'}(x) = \Theta_{\gamma',\gamma}(x) = \delta_{\gamma,\gamma'} \left\{ 2 \tan^{-1} \left( \frac{x}{2\gamma} \right) + \sum_{l=1}^{\gamma-1} 4 \tan^{-1} \left( \frac{x}{2l} \right) \right\} 
+ (1 - \delta_{\gamma,\gamma'}) \left\{ 2 \tan^{-1} \left( \frac{x}{|\gamma - \gamma'|} \right) + 2 \tan^{-1} \left( \frac{x}{|\gamma + \gamma'|} \right) \right\} + \frac{x + \gamma' - |\gamma - \gamma'| - 1}{4} \left\{ 1 - 2 \tan^{-1} \left( \frac{x}{|\gamma - \gamma'| + 2} \right) \right\} .
\]

The limits of the pseudo-Brillouin zones, \( q_{\alpha,\gamma}^{(\pm)} \), associated with the pseudomomentum summations are given in Eqs. (31) – (34). Although the integral Eqs. (66) – (68) are coupled note that each of these equations refers to one \( \alpha, \gamma \) channel and defines the corresponding \( \alpha, \gamma \) rapidity in terms of other rapidities.

The key point is that since the eigenvalue \( N_{\alpha,\gamma}(q) \) (here \( \gamma = 0, 1, 2, 3, \ldots \)) is common to the whole tower of \( 2S^\alpha + 1 \) Hamiltonian eigenstates constructed from the same \( S^\alpha \) LWS, also the associate rapidity eigenvalue \( R_{\alpha,\gamma}(q) \) of Eq. (63) is common to these \( 2S^\alpha + 1 \) Hamiltonian eigenstates. This follows from the fact that rapidity solutions \( R_{\alpha,\gamma}(q) \) of Eqs. (66) – (68) are functionals of the \( N_{\alpha,\gamma}(q) \) distributions and \( \beta \) independent. We thus conclude that the LWS rapidities extracted from the BA provide full information on the non-LWS’s rapidity operators.

The rapidity eigenvalues of Eqs. (63) and (65) are independent of the pseudohole numbers \( N_{h,\beta}^h \) and only involve the pseudoparticle distributions \( N_{\alpha,\gamma}(q) \). The same holds true for the associate rapidity, pseudohole, and pseudoparticle operators. Since the \( SO(4) \) Hamiltonian (2) and other physical quantities turn out to be exclusive functionals of the rapidity operators, in studies involving such quantities it is often more convenient to use the \( \alpha, \gamma \) pseudoparticle representation (with \( \gamma = 0, 1, 2, 3, \ldots \)) than the \( \alpha, \beta \) pseudohole and \( \alpha, \gamma \) heavy-pseudoparticle representation (with \( \gamma = 1, 2, 3, \ldots \)). We remind that the set of all \( N_{h,\beta}^h \) and \( N_{\alpha,\gamma} \) (with \( \gamma = 1, 2, 3, \ldots \)) numbers of a given Hamiltonian eigenstate (23) have the same information as the corresponding set of all \( N_{\alpha}^\alpha \) and \( N_{\alpha,\gamma} \) (with \( \gamma = 0, 1, 2, 3, \ldots \)) numbers which characterize the same state.

By direct insertion in each of Eqs. (66) – (68) of the corresponding \( q_{\alpha,\gamma}^{(\pm)} \) pseudomomenta we find that the functions \( R_{\alpha,\gamma}(q) \) have for all eigenstates the following boundary values at the limits of the pseudo-Brillouin zones
\[
K(q_{c,0}^{(\pm)}) = \pm \pi, \quad R_{c,0}(q_{c,0}^{(\pm)}) = 0, \quad (70)
\]
for \( \alpha = c \) and \( \gamma = 0 \) and
\[
R_{\alpha,\gamma}(q_{\alpha,\gamma}^{(\pm)}) = \pm \infty, \quad (71)
\]
for the remaining choices of the quantum numbers \( \alpha, \gamma \) other than \( c, 0 \).

It is useful to consider the rapidity functions \( K^{(0)}(q) \) and \( R_{\alpha,\gamma}^{(0)}(q) \) which are the GGS eigenvalues (or GS eigenvalues – a GS is a particular case of a GGS) such that
\[
\hat{K}(q)|GGS\rangle = K^{(0)}(q)|GGS\rangle, \quad (72)
\]
and
\[
\hat{R}_{\alpha,\gamma}(q)|GGS\rangle = R_{\alpha,\gamma}^{(0)}(q)|GGS\rangle. \quad (73)
\]
These are defined by Eqs. (66) – (68) with the pseudomomentum distribution given by its GGS value which we denote by \( N_{\alpha,\gamma}^{(0)}(q) \) and is such that
\[
\hat{N}_{\alpha,\gamma}(q)|GGS\rangle = N_{\alpha,\gamma}^{(0)}(q)|GGS\rangle. \quad (74)
\]

Following the GGS expression (40), \( N_{\alpha,\gamma}^{(0)}(q) \) has the simple free Fermi-like form
\[
N_{\alpha,\gamma}^{(0)}(q) = \Theta \left( C_{\alpha,\gamma}^{(0)} \left[ q_{\alpha,\gamma}^{(\pm)} - q \right] \right), \quad 0 < q < q_{\alpha,\gamma}^{(\pm)} = \Theta \left( C_{\alpha,\gamma}^{(0)} \left[ q - q_{\alpha,\gamma}^{(\pm)} \right] \right), \quad q_{\alpha,\gamma}^{(\pm)} < q < 0. \quad (75)
\]
For the particular case of a GS these distributions read
\[ N^{(0)}_{\alpha,0}(q) = \Theta\left(q^{(+)\alpha}_{F\alpha,0} - q\right), \quad 0 < q < q^{(+)\alpha}_{\alpha,0} \]
\[ = \Theta\left(q - q^{(-)\alpha}_{F\alpha,0}\right), \quad q^{(-)\alpha}_{\alpha,0} < q < 0, \]
\[ N^{(0)}_{\alpha,\gamma}(q) = 0, \quad \gamma > 0. \quad (76) \]

By inserting in Eqs. (66) – (68) the GGS distribution (75) we define the GGS rapidities \( K^{(0)}(q) \) and \( R^{(0)}_{\alpha,\gamma}(q) \) in terms of their inverse functions with the result
\[ q = \int_0^{K^{(0)}(q)} dk 2\pi \rho_{\alpha,0}(k), \quad (77) \]
\[ q = \int_0^{R^{(0)}_{\alpha,\gamma}(q)} dr 2\pi \rho_{\alpha,\gamma}(r), \quad (78) \]

where the functions \( 2\pi \rho_{\alpha,0}(k) \) and \( 2\pi \rho_{\alpha,\gamma}(r) \) are the solutions of the integral equations (B4)-(B6) of Appendix B. In that Appendix we study the rapidity solutions of Eqs. (66) – (68) both for GGS’s and for Hamiltonian eigenstates differing from a GGS by a small density of excited pseudoparticles. In equations (B4)-(B6) the parameters \( Q^{(\pm)} \) and \( r^{(\pm)}_{\alpha,\gamma} \) are defined combining the following equations
\[ Q^{(\pm)} = K^{(0)}(q^{(\pm)}_{F\alpha,0}), \quad r^{(\pm)}_{\alpha,\gamma} = R^{(0)}_{\alpha,\gamma}(q^{(\pm)}_{F\alpha,0}), \quad (79) \]

with equations (77) and (78). We also introduce the associate parameters
\[ Q = K^{(0)}(q_{F\alpha,0}), \quad r_{\alpha,\gamma} = R^{(0)}_{\alpha,\gamma}(q_{F\alpha,\gamma}). \quad (80) \]

The system of coupled algebraic equations (66) – (68) is obtained from the BA solution (see Ref. 14 and Appendix A which connects our rapidity representation to the notation used in that reference), yet also describes states out of that solution. Given the configuration of pseudoparticle quantum numbers that describes each Hamiltonian eigenstate of form (23), these equations fully define the corresponding rapidities. In the thermodynamic limit, we can take the pseudomomentum continuum limit \( q_j \to q \) and the real part of the rapidities become functions of \( q \) which we have called here \( R_{\alpha,\gamma}(q) \). In that limit the BA system of algebraic equations are replaced by the system of infinite coupled integral equations presented in Appendix B.

We have mentioned that the BA solution is most naturally expressed in the pseudoparticle basis. One reflection of this is the simple expression for the Hamiltonian (1) in that basis. The BA energy expression given in Refs. 2, 13 for the LWS’s of the \( \eta \)-spin and spin algebras, which we denote by \( E_{BA} \), can be generalized to all Hamiltonian eigenstates. For that we replace the \(-S^\alpha_z\) dependence of \( E_{BA} \) by a \( S^\alpha \)
\[ E \equiv E_{BA}(-S^\alpha_z \to S^\alpha). \quad (81) \]

Combining Eq. (19) with the energy expression (81) leads then to the following expression for the Hamiltonian (1)
\[ \hat{H} = \hat{H}_{SO}(4) + 2\mu \sum_\beta \beta \hat{\chi}^h_{c,\beta} + 2\mu_0 \hat{H} \sum_\beta \beta \hat{\chi}^h_{s,\beta}, \quad (82) \]

where
\[ \hat{H}_{SO}(4) = -2t \sum_q \hat{\chi}^h_{c,0}(q) \cos[\hat{K}(q)] + 2t \sum_{q,\gamma=1} \hat{N}_{c,\gamma}(q) \sum_{j=\pm 1} \sqrt{1 - |u(R_{c,\gamma}(q) - ji\gamma)|^2} \]
\[ + U[\frac{N_a}{4} - \frac{\hat{\chi}^h_{c,0}}{2} - \sum_{\gamma=1}^\alpha q_j \hat{N}_{c,\gamma}]. \quad (83) \]

This is the rigorous pseudoparticle expression of the Hamiltonian (2) at all energy scales. It gives the exact expression of that Hamiltonian in the full Hilbert space. Despite its simple appearance, the Hamiltonian (83) describes a many-pseudoparticle problem. The reason is that the expression of the rapidity operator in terms of the operators \( \hat{N}_{\alpha,\gamma}(q) \) contains many-pseudoparticle interacting terms.
To specify the Hamiltonian (83) completely, we must indicate the equations that define the rapidity operator eigenvalues in terms of the pseudoparticle momentum distribution operators. In the case of the present model these are Eqs. (66) – (68).

Unsurprisingly, it is difficult to solve the BA operator equations (66) – (68) directly and to obtain the explicit expression for the rapidity operators in terms of the pseudomomentum distribution operators (15). In contrast, it is easier to calculate their normal-ordered expression in terms of the normal-ordered operators. The rapidity operators \( \hat{R}_{\alpha,\gamma}(q) \) contain all information about the many-pseudoparticle interactions of the quantum-liquid Hamiltonian. There are two fundamental properties which imply the central role that the rapidity operators of Eqs. (72) and (73) have in the present quantum problem:

(a) As we find in Sec. V, each of the normal-ordered rapidity operator :\( \hat{R}_\alpha(q) : \) (relative to the suitable GGS or GS) can be written exclusively in terms of the pseudomomentum distribution operators (15);

(b) The normal-ordered version of the \( SO(4) \) Hamiltonian (83) can be written, exclusively, in terms of the pseudomomentum distribution operators (15), but all the corresponding many-pseudoparticle interaction terms can be written in terms of the rapidity operators :\( \hat{R}_{\alpha,\gamma}(q) : \). It follows that the rapidity operators commute with the Hamiltonian.

In the ensuing section, we introduce the pseudoparticle perturbation theory which leads to the normal-ordered expressions for the Hamiltonian and rapidity operators. Despite the non-interacting form of the Hamiltonian eigenstates (23), the normal-ordered Hamiltonian includes pseudoparticle interaction terms and is, therefore, a many-pseudoparticle operator. However, we find in Sec. V that these pseudoparticle interactions have a pure forward-scattering, zero-momentum transfer, character.

V. PSEUDOPARTICLE HAMILTONIAN AND PERTURBATION THEORY

We will be mostly interested in GS - GGS transitions followed by pseudoparticle - pseudohole excitations involving a small density of \( \alpha, \gamma \) pseudoparticles relative to both the GS and GGS distributions. Let us then introduce the normal-ordered pseudomomentum distribution operator

\[
: \hat{N}_{\alpha,\gamma}(q) : = \hat{N}_{\alpha,\gamma}(q) - N^{(0)}_{\alpha,\gamma}(q).
\]

We emphasize that when we choose \( N^{(0)}_{\alpha,\gamma}(q) \) to be the GS pseudomomentum distribution (76) we have that

\[
: \hat{N}_{\alpha,\gamma}(q) : = \hat{N}_{\alpha,\gamma}(q), \quad \gamma > 0.
\]

The normal-ordered distribution (84) obeys the eigenvalue equation

\[
: \hat{N}_{\alpha,\gamma}(q) : |\psi\rangle = \delta N_{\alpha,\gamma}(q) |\psi\rangle.
\]

We also consider the normal-ordered rapidity operator

\[
: \hat{R}_{\alpha,\gamma}(q) : = \hat{R}_{\alpha,\gamma}(q) - R^{(0)}_{\alpha,\gamma}(q),
\]

where \( R^{(0)}_{\alpha,\gamma}(q) \) is the GGS eigenvalue of Eq. (73).

In the pseudoparticle basis the normal-ordered rapidity operators :\( \hat{R}_{\alpha,\gamma}(q) : \) contain an infinite number of terms, as we shall demonstrate below. The first of these terms is linear in the pseudomomentum distribution operator :\( \hat{N}_{\alpha,\gamma}(q) : \), (84), whereas the remaining terms consist of products of two, three, ......, until infinity, of these operators. The number of :\( \hat{N}_{\alpha,\gamma}(q) : \) operators which appears in these products equals the order of the scattering in the corresponding rapidity term.

A remarkable property is that in the pseudoparticle basis the seemingly “non-perturbative” quantum liquids become perturbative: while the two-electron forward scattering amplitudes and vertices diverge, the two-pseudoparticle \( f \) functions [given by Eq. (117) below] are finite.

By “perturbative” we also mean here the following: at each point of parameter space (canonical ensemble) the excited eigenstates are of form (23) and correspond to quantum-number configurations involving a density of excited pseudoholes and heavy pseudoparticles relative to the GS configuration (46) and (76). We introduce the following density

\[
n'_{ex} = \sum_{\alpha,\beta} n^{\alpha,\beta}_{ex} + \sum_{\alpha,\gamma=1} n^{\alpha,\gamma}_{ex},
\]

(88)
which is kept small. Here

\[ n_{ex}^{\alpha,\beta} = \frac{1}{N_a} \sum_q [1 - N_{\alpha,\beta}^{h,(0)}(q)] \delta N_{\alpha,\beta}(q), \]  

(89)

and

\[ n_{ex}^{\alpha,\gamma} = \frac{1}{N_a} \sum_q N_{\alpha,\gamma}(q), \]  

(90)

define the densities of excited \( \alpha,\beta \) pseudoholes and \( \alpha,\gamma \) heavy pseudoparticles, respectively, associated with the Hamiltonian eigenstate \( |\psi\rangle \).

In Eq. (89) \( N_{\alpha,\beta}^{h,(0)}(q) \) is the GS pseudohole distribution which reads

\[
\begin{align*}
N_{\alpha,-\frac{1}{2}}^{h,(0)}(q) &= 1 - N_{\alpha,0}^{(0)}(q), \\
N_{\alpha,\frac{1}{2}}^{h,(0)}(q) &= 0,
\end{align*}
\]

(91)

if the GS is a \( S^\alpha \) LWS and

\[
\begin{align*}
N_{\alpha,-\frac{1}{2}}^{h,(0)}(q) &= 0, \\
N_{\alpha,\frac{1}{2}}^{h,(0)}(q) &= 1 - N_{\alpha,0}^{(0)}(q),
\end{align*}
\]

(92)

if the GS is a \( S^\alpha \) HWS, where \( N_{\alpha,0}^{(0)}(q) \) is the \( \alpha,0 \) pseudoparticle GS distribution (76). Our perturbation theory refers to general GS transitions to Hamiltonian eigenstates \( |\psi\rangle \) involving (a) a GS - GGS transition which costs the energy \( \omega_0 \) [given by Eq. (124) below] and (b) a Landau-liquid pseudoparticle-pseudohole excitation around the GGS. The pseudoparticle perturbation theory corresponds to such GS transitions with excitation energy \( \omega \) such that the energy \( \omega - \omega_0 \) is small. Since the initial state is a GS we have assumed the initial GS distribution (76) for the \( \alpha,\gamma \) heavy pseudoparticles. Following Eq. (76) we thus have that \( N_{\alpha,\gamma}(q) = \delta N_{\alpha,\gamma}(q) \) in the rhs of Eq. (90).

The expectation values of the \( SO(4) \) Hamiltonian (83) in the final states \( |\psi\rangle \) are functions of the density of excited pseudoparticles only. This density is given by

\[ n_{ex} = \sum_{\alpha,\gamma=0} n_{ex}^{\alpha,\gamma}, \]  

(93)

where \( n_{ex}^{\alpha,\gamma} \) is given by

\[ n_{ex}^{\alpha,0} = \frac{1}{N_a} \sum_q [1 - N_{\alpha,0}^{(0)}(q)] \delta N_{\alpha,0}(q), \]  

(94)

for \( \gamma = 0 \) and by (90) otherwise. When both the pseudoparticle density (93) and \( n^z = \sum_{\alpha} \frac{N_{\alpha}^{(0)}}{N_a} \) [see Eq. (13)] are small implies that the density (88) is also small. On the other hand, the pseudoparticle density (93) is small provided the density (88) is also small. This then implies that all the densities \( n_{ex}^{\alpha,\gamma} \) are small and that we can expand the expectation values in these densities. The perturbative character of the quantum liquid rests on the fact that the evaluation of the expectation values up to the \( j \)th order in the densities (93) requires considering only the corresponding operator terms of scattering orders less than or equal to \( j \). This follows from the linearity of the density of excited \( \alpha,\gamma \) pseudoparticles, which are the elementary “particles” of the quantum liquid, in

\[ \delta N_{\alpha,\gamma}(q) = \langle \psi | : \hat{N}_{\alpha,\gamma}(q) : |\psi\rangle, \]  

(95)

and from the form of (90) and (94). The perturbative character of the quantum liquid implies, for example, that, to second order in the density of excited pseudoparticles, the energy involves only one- and two-pseudoparticle Hamiltonian terms, as in the case of the quasiparticle terms of a Fermi-liquid energy functional [3,4]. In this case the corresponding energy \( \omega - \omega_0 \) is small.

It follows from the eigenfunction Eq. (86) that the problem of using the rapidity Eqs. (66) – (68) to derive the expression of the operators : \( \hat{R}_{\alpha,\gamma}(q) : \) in terms of the operators : \( \hat{N}_{\alpha,\gamma}(q) : \) is equivalent to the problem of evaluating the corresponding expansion of the rapidity eigenvalues \( \delta R_{\alpha,\gamma}(q) \) in terms of the pseudoparticle deviations \( \delta N_{\alpha,\gamma}(q) \)
where $\alpha,\gamma$ pseudoparticle phase shifts $\Phi_{\alpha,\gamma}$ 

(66) and expand in the scattering order. This leads to

\[
R_{\alpha,\gamma}(q) = \Phi_{\alpha,\gamma;(\alpha',\gamma')}(q,q') : N_{\alpha'}(q') :. 
\]

where the phase-shift expressions are given below. We emphasize that while the expressions for the phase shifts $\Phi_{\alpha,\gamma;(\alpha',\gamma')}(q,q')$ are specific to each model because they involve the spectral parameters, the form of the operator term $\hat{X}_{\alpha,\gamma}^{(1)}(q)$ (98) is universal and refers to all the solvable electronic multicomponent quantum liquids.

It is useful to introduce the phase shifts $\tilde{\Phi}_{\alpha,\gamma;(\alpha',\gamma')}$ such that

\[
\tilde{\Phi}_{\alpha,\gamma;(\alpha',\gamma')}(k,k') = \tilde{\Phi}_{\alpha,\gamma;(\alpha',\gamma')}(\sin k, \sin k') , \tag{99}
\]

\[
\tilde{\Phi}_{\alpha,\gamma;0,c}(k,r') = \tilde{\Phi}_{\alpha,\gamma;0,c}(\sin k, r') , \tag{100}
\]

\[
\tilde{\Phi}_{\alpha,\gamma;0,0}(k',r') = \tilde{\Phi}_{\alpha,\gamma;0,0}(r, \sin k') , \tag{101}
\]

\[
\tilde{\Phi}_{\alpha,\gamma;0}(r,k') = \tilde{\Phi}_{\alpha,\gamma;0}(r, \text{Re} k') , \tag{102}
\]

where $\gamma > 0$ and the phase shifts $\tilde{\Phi}_{\alpha,\gamma;(\alpha',\gamma')}$ are defined by the integral equations (B30)-(B40) of Appendix B.

The two-pseudoparticle phase shifts can be defined in terms of the phase shifts $\tilde{\Phi}_{\alpha,\gamma;0}(q,q')$ as follows

\[
\Phi_{\alpha,\gamma;0}(q,q') = \tilde{\Phi}_{\alpha,\gamma;0}(q,q') \left( R_{\alpha,\gamma}(q), R_{\alpha,\gamma}(q') \right) . \tag{103}
\]

The quantity $\Phi_{\alpha,\gamma;0}(q,q')$ represents the shift in the phase of the $\alpha',\gamma'$ pseudoparticle of pseudomomentum $q'$ due to a zero-momentum forward-scattering collision with the $\alpha,\gamma$ pseudoparticle of pseudomomentum $q$.

In Appendix C we use the Hamiltonian expression (83) in terms of the rapidity operators to derive the expression for the normal-ordered Hamiltonian. We find that in normal order relative to the suitable GGS (or GS) the Hamiltonian $\hat{H}_{SO(4)}$, Eq. (83), can be written as
\[ \hat{H}_{SO(4)} := \sum_{j=1}^{\infty} \hat{H}^{(j)}. \]  

(104)

For example, the first and second pseudoparticle-scattering order terms read

\[ \hat{H}^{(1)} = \sum_{q, \alpha, \gamma} \epsilon_{\alpha, \gamma}^0(q) : \hat{N}_{\alpha, \gamma}(q) :, \]

(105)

and

\[ \hat{H}^{(2)} = \frac{1}{N_\alpha} \sum_{q, \alpha, \gamma} \sum_{q', \alpha', \gamma'} \frac{1}{2} f_{\alpha, \gamma; \alpha', \gamma'}(q, q') : \hat{N}_{\alpha, \gamma}(q) :: \hat{N}_{\alpha', \gamma'}(q') :, \]

(106)

respectively. All the remaining higher-order operator terms of expressions (97) and (104) can be obtained by combining the rapidity equations (66) – (68) and the Hamiltonian expression (83).

In Appendix C it is shown that the pseudoparticle bands \( \epsilon_{\alpha, \gamma}^0(q) \) can either be expressed in terms of the phase shifts (103), with the result

\[
\begin{align*}
\epsilon_{c,0}^0(q) &= -\frac{U}{2} - 2t \cos K^{(0)}(q) + 2t \int_{Q(-)}^{Q(+) \pi} dk \Phi_{c,0}(k) \sin k, \\
\epsilon_{c,\gamma}^0(q) &= -\gamma U + 4t \text{Re} \sqrt{1 - u^2 |R_{c,\gamma}^{(0)}(q) - i\gamma|^2} + 2t \int_{Q(-)}^{Q(+) \pi} dk \Phi_{c,0}(k) \sin k, \\
\epsilon_{s,\gamma}^0(q) &= 2t \int_{Q(-)}^{Q(+) \pi} dk \Phi_{c,0}(k) \sin k,
\end{align*}
\]

(107)

for \( \gamma > 0 \) and

\[
\begin{align*}
\epsilon_{c,0}^0(q) &= \int_{0}^{\pi} K^{(0)}(q) dk, \\
\epsilon_{c,\gamma}^0(q) &= \int_{0}^{\pi} R_{c,\gamma}^{(0)}(q) dk, \\
\epsilon_{s,\gamma}^0(q) &= \int_{0}^{\pi} \eta_{s,\gamma}(k) dk,
\end{align*}
\]

(108)

(110)

for \( \gamma > 0 \) and

\[
\begin{align*}
\epsilon_{c,\gamma}^0(q) &= \int_{0}^{\pi} K^{(0)}(q) dk, \\
\epsilon_{s,\gamma}^0(q) &= \int_{0}^{\pi} R_{s,\gamma}^{(0)}(q) dk, \\
\epsilon_{s,\gamma}^0(q) &= \int_{0}^{\pi} \eta_{s,\gamma}(k) dk,
\end{align*}
\]

(111)

(112)

for \( \gamma = 0, 1, 2, ..., \) where the functions \( 2t\eta_{c,0}(k) \) and \( 2t\eta_{c,\gamma}(r) \) are the solutions of the integral equations (C7)-(C9) of Appendix C. The GGS (or GS) rapidity functions \( K^{(0)}(q) \) and \( R_{c,\gamma}^{(0)}(q) \) are defined by Eqs. (77), (78) combined with Eqs. (B4)-(B6) of Appendix B. Combining Eqs. (70) and (71) with the band expressions we find that the bands (108), (111) and (109), (112) vanish at the limits of the pseudo-Brillouin zones, i.e.

\[ \epsilon_{\alpha, \gamma}^0(q_{\alpha, \gamma}) = 0. \]

(113)

The associate group velocities, \( v_{\alpha, \gamma}(q) \), are given by

\[ v_{\alpha, \gamma}(q) = \frac{d\epsilon_{\alpha, \gamma}^0(q)}{dq}. \]

(114)

The two-pseudoparticle forward-scattering phase shifts \( \Phi_{\alpha, \gamma; \alpha', \gamma'}(q, q') \) defined by Eq. (103) and velocities
play an important role in the physical quantities when criticality is approached \[23,26\).

All \( \hat{X}^{(j)}_{a,\gamma}(q) \) terms of the rhs of Eq. (97) are such that both the \( f \) functions of the rhs of Eq. (106) and all the remaining higher order coefficients associated with the operators \( \hat{H}^{(j)} \) of order \( j > 1 \) have universal forms in terms of the two-pseudoparticle phase shifts and pseudomomentum derivatives of the bands and coefficients of order \( < j \). This follows from the fact that the \( S \)-matrix for \( j \)-pseudoparticle scattering factorizes into two-pseudoparticle scattering matrices, as in the case of the usual BA \( S \)-matrix \[23,24,26\]. For example, we show in Appendix C that although the second-order term \( \hat{H}^{(2)} \) of Eq. (106) involves an integral over the second-order function \( \hat{X}^{(2)}_{a,\gamma}(q) \) [see Eq. (14) of Appendix C], this function is such that \( \hat{H}^{(2)} \) can be written exclusively in terms of the first-order functions (98),

\[
\hat{H}^{(2)} = \sum_{q,a,\gamma} v_{a,\gamma}(q) \hat{X}^{(1)}_{a,\gamma}(q) : \hat{N}_{a,\gamma}(q) : + \frac{N_a}{2\pi} \sum_{a,\gamma} \theta(N_{a,\gamma}) \frac{|v_{a,\gamma}|}{2} \sum_{j=\pm 1} [\hat{X}^{(1)}_{a,\gamma}(jq_{F,a,\gamma})]^2 ,
\]

where \( |v_{a,\gamma}| = C_{a,\gamma} v_{a,\gamma} \), \( \theta(x) = 1 \) for \( x > 0 \), and \( \theta(x) = 0 \) for \( x \leq 0 \) and the “Landau” \( f \) functions, \( f_{a,\gamma;\alpha'\gamma'}(q,q') \), are found in that Appendix to have universal form which reads

\[
f_{a,\gamma;\alpha'\gamma'}(q,q') = 2\pi v_{a,\gamma}(q) \Phi_{a,\gamma;\alpha'\gamma'}(q,q') + 2\pi v_{a',\gamma'}(q') \Phi_{a',\gamma';a'\gamma'}(q',q) + 2\pi \sum_{j=\pm 1} \sum_{\alpha''} \sum_{\gamma''} \theta(N_{a'',\gamma''}) C_{a'',\gamma''} v_{a'',\gamma''} \Phi_{a'',\gamma'';a\gamma}(jq_{F\alpha''\gamma''},q) \Phi_{a'',\gamma'';a\gamma}(jq_{F\alpha''\gamma''},q') ,
\]

where the pseudoparticle group velocities are given by Eqs. (114) and (115).

We can also write the expression of \( \hat{H}^{(j)} \) for higher scattering orders \( j > 2 \). The main feature is that for all sub-canonical ensembles, energy scales, and pseudoparticle scattering orders, the Landau-liquid terms of that Hamiltonian involve only zero-momentum forward-scattering.

Let us consider that the normal-ordering of the Hamiltonian (104) refers to the initial GS. In this case, and in contrast to the low-energy Landau theory studied in Refs. \[23,24,26\], the Hamiltonian (104) describes finite-energy transitions involving a small density of excited pseudoparticles. Furthermore, a finite-size analysis \[26\] confirms that the excitations which control the response and transport properties at finite energies involve a small density of excited pseudoparticles. Fortunately, their description is thus within the range of our generalized pseudoparticle-perturbation theory. It is convenient to define the normal ordering relative to the initial GS but with energy shifted by

\[
\omega_0 = \lim_{N_a \to \infty} \left[ E_{GGS} - E_{GS} \right] .
\]

This leads to a small \( \omega - \omega_0 \) energy description. Fortunately, the relevant GS - GGS transitions correspond to energy values \( \omega_0 \) associated with small-pseudoparticle-density excitations \[23,26\]. For each of these \( \omega_0 \) values there is a low-energy \( \omega - \omega_0 \) pseudoparticle-perturbation theory. The associate Hilbert subspace is spanned by the set of GSS’s of energy \( \omega_0 \) and by states originated from these by pseudoparticle-pseudohole processes. The latter states have small momentum and low \( \omega - \omega_0 \) energy relatively to the initial GS (we have shifted its energy by \( \omega_0 \)). Normal-ordering relative to the energy-\( \omega_0 \) shifted initial GS is equivalent to separate the problem (104) into (a) the GS - GGS transition of finite-energy (118) and (b) a low \( \omega - \omega_0 \) energy theory associated with both the corrections of order \( 1/N_a \) to the GS - GGS-transition energy (118) and the Landau-liquid-pseudoparticle-pseudohole processes around the final GGS. In most cases the excitation energy (118) associated with the GS - GGS transitions is finite. The finite energies are the energy gaps of the states II and (or) non-LWS’s and non-HWS’s relative to the initial GS.

At finite values of \( S_z^\gamma \) and \( S_z^\alpha \) the low-energy Hilbert space is entirely spanned by states I \[23,26\]. We note that for those states fixing the electron numbers fixes the pseudohole numbers \[26\] and the \( \alpha,\gamma \) bands are empty for \( \gamma > 0 \). Therefore, if we fix the electron numbers we have that at energy scales smaller than the gaps (118) for the states II and non-LWS’s and non-HWS’s only Landau-liquid excitations (b) are allowed. This justifies the Landau-liquid character of the problem at low energies \[7,23,26\]. Obviously, if we change the electron numbers, there occur at low energy GS - GS transitions which are particular cases of the above GS - GGS transitions (a). Understanding these GS - GS transitions permits one to express the electron in terms of pseudoholes and topological momentum shifts \[23,26\].

Let us evaluate the general \( \omega_0 \) expression for GS - GGS transitions such that the final sub-canonical ensemble is characterized by vanishing or small values of \( N_e^a \to 0 \). Here \( l_\alpha = \pm 1 \) is defined by Eq. (7). We also assume that both the initial GS and final GGS correspond to canonical ensembles such that \( S_z^\alpha \neq 0 \) and belonging the same sector.
of parameter space, i.e. with the same \( l_s \) numbers. We note that since GS’s are always \( S^\alpha \) LWS’s I or HWS’s I there are neither \( \alpha, \frac{\pi}{4} \) pseudoholes [see Eqs. (37) and (38)] nor \( \alpha, \gamma \) heavy pseudoparticles \((\gamma > 0)\) in the initial GS and that \( N^{z}_{\alpha} = 0 \) [see Eq. (13)] for that state. From Eqs. (104) – (106), we find for the GS - GGS gap

\[
\omega_0 = \sum_{\alpha} \epsilon_{\alpha,0}^{0}(q_{Fc,0}) \Delta N_{\alpha,0} + \sum_{\gamma=1} \epsilon_{s,\gamma}^{0}(0) N_{s,\gamma} + 2\mu \Delta S_z^c + 2\mu_0 H \Delta S_z^c ,
\]

(119)

where the band expressions at the pseudo-Fermi points are given in Eq. (C12) of Appendix C. Inserting in Eqs. (11) both the pseudohole expressions (12) and the expressions \( S_z^c = -\frac{1}{2} [N_a - N] \) and \( S_z^c = -\frac{1}{2} [N_\uparrow - N_\downarrow] \) we find after use of Eqs. (13) and (20) the following general expressions for \( \Delta N_{c,0} \) and \( \Delta N_{s,0} \)

\[
\Delta N_{c,0} = -2|\epsilon_{c,0}|^2 \Delta S_z^c - 2N_{c} - \sum_{\gamma=1}^{2} \gamma N_{c,\gamma} = -[N_{c,\uparrow}^h - \Delta N_{c,\uparrow}] - 2N_{c} - \sum_{\gamma=1}^{2} \gamma N_{c,\gamma} ,
\]

(120)

and

\[
\Delta N_{s,0} = - \sum_{\alpha} [\epsilon_{\alpha,0}^{0}(q_{Fc,0}) \Delta N_{\alpha}^{h} + N_{\alpha}^{z}] - \sum_{\gamma=1}^{2} \gamma N_{c,\gamma} - \sum_{\gamma=1}^{2} (1 + \gamma) N_{s,\gamma} \\
\]

\[
= \frac{1}{2} \sum_{\alpha} [\epsilon_{\alpha,0}^{0}(q_{Fc,0}) \Delta N_{\alpha}^{h} + 2N_{\alpha}^{z}] - \sum_{\gamma=1}^{2} \gamma N_{c,\gamma} - \sum_{\gamma=1}^{2} (1 + \gamma) N_{s,\gamma} .
\]

(121)

Generalization of the chemical-potential and magnetic-field expressions (47) of Ref. [33] to all sectors of parameter space leads to

\[
|\mu| = -\epsilon_{c,0}^{0}(q_{Fc,0}) - \frac{\epsilon_{s,0}^{0}(q_{Fs,0})}{2} , \quad 2\mu_0 |H| = -\epsilon_{s,0}^{0}(q_{Fs,0}) .
\]

(122)

From the use of Eqs. (20) and (120) – (122) in Eq. (119) leads to the following gap expression

\[
\omega_0 = 2|\mu| N_{c,\uparrow}^{h} + 2\mu_0 |H| \sum_{\gamma=1}^{\infty} N_{s,\gamma} \right] + \sum_{\gamma=1}^{\infty} \epsilon_{s,\gamma}^{0}(0) N_{s,\gamma} ,
\]

(123)

which is the general expression for the gap for GS - GGS transitions. It follows from Eqs. (13), (19), and (20) that the gap expression (123) can be rewritten as

\[
\omega_0 = 2|\mu| \left( \sum_{\gamma=1}^{\infty} \gamma N_{c,\gamma} + N_{c}^{z} \right) + 2\mu_0 |H| \sum_{\gamma=1}^{\infty} (1 + \gamma) N_{s,\gamma} + N_{s}^{z} + \sum_{\gamma=1}^{\infty} \epsilon_{s,\gamma}^{0}(0) N_{s,\gamma} .
\]

(124)

In reference [20] the concept of GGS is generalized to \( \alpha, \gamma > 0 \) pseudoparticle filled seas with pseudo-Fermi points either of the form (a) \( q_{F\alpha,\gamma} = \frac{\pi d_{\alpha,\gamma} - N_{\alpha}}{N_{\alpha}} \) or (b) \( q_{F\alpha,\gamma} = \frac{\pi N_{\alpha}}{N_{\alpha}} \). We note that our definition (43) corresponds to the case (a) for \( \alpha = c \) and (b) for \( \alpha = s \). While the first two terms of the rhs of Eq. (124) are the same for either case, the third term vanishes for the case (a) [this follows from Eq. (113)] and reads \( \epsilon_{\alpha,\gamma}^{0}(0) \) for \( \alpha, \gamma \) seas corresponding to the case (b). This explains the form of the latter term in Eq. (124) which refers only to the \( s, \gamma \) pseudoparticles.

For GS - GGS transitions to pure LWS’s or HWS’s II the use of Eq. (13) leads to

\[
N^{z}_{\alpha} = 0 ,
\]

(125)

for both \( \alpha = c, s \). It follows that for such transitions the gap is given by expression (124) with \( N^{z}_{c} = 0 \) and \( N^{z}_{s} = 0 \). Therefore, the single \( (\gamma > 0) \) \( \alpha, \gamma \)-heavy-pseudoparticle gap is \( 2\gamma |\mu| \) and \( 2(1 + \gamma) \mu_0 |H| + \epsilon_{\alpha,\gamma}^{0}(0) \) for \( c \) and \( s \), respectively.

The gap expression (123) – (124) involves the chemical-potential \( \mu \), magnetic field \( H \), and the zero-momentum \( \alpha, \gamma \) (with \( \gamma > 0 \)) band values. For constant values of \( U \) the above quantities change as follows: For fixed values of \( S^{z}_{\alpha} \), \( |\mu| \) changes between \( |\mu| = \Delta M_H \) for \( \frac{N_{\alpha}}{N_{\alpha}} \rightarrow 0 \) (and \( n \rightarrow 1 \)) and a maximum value, \( |\mu| = \Delta M_H + 4t \), for \( S^{z}_{\alpha} \rightarrow \pm \frac{N_{\alpha}}{2} \) (and both \( n \rightarrow 0 \) and \( n \rightarrow 2 \)), where \( \Delta M_H \) is the half-filled Mott-Hubbard gap [18, 22]. The gap \( \Delta M_H \) is both an increasing function of \( U \) and of \( |S^{z}_{\alpha}| \) [22]. On the other hand, for fixed values of \( S^{z}_{\alpha} \), \( |H| \) changes between \( |H| = 0 \) for \( \frac{N_{\alpha}}{N_{\alpha}} \rightarrow 0 \) (and \( m \rightarrow 0 \)) and a maximum value, \( |H| = H_c \), for \( S^{z}_{\alpha} \rightarrow \pm \frac{N_{\alpha}}{2} \) (and \( m \rightarrow \pm n \) and \( n \rightarrow 2 \)), where \( H_c \) is the critical field for onset of fully polarized ferromagnetism defined in Refs. [18, 22]. Its expression presented in these papers refers to the parameter-space sectors where the electronic density is such that \( 0 < n < 1 \). The generalization
of that expression to the sectors where the density is such that $1 < n < 2$ is obtained by replacing in it $n$ by $2 - n$.

Finally, for fixed values of $S^z$ the band $\epsilon_{\alpha,\gamma}^0(0)$ changes between $\epsilon_{\alpha,\gamma}^0(0) = 0$ for $\frac{S^z}{N_0} \to 0$ and a minimum negative value. A more detailed study of the $U$, density, and magnetization dependence of the gap expression (123) – (124) will be presented elsewhere.

Given an initial GS and a final sub-canonical ensemble we can write

$$\hat{H} := \hat{H}_0 + \hat{H}_{\text{Landau}},$$  \hspace{1cm} (126)

where $\hat{H}_0$ has eigenvalue $\omega_0$ and corresponds to the GS - GGS energy (a) and $\hat{H}_{\text{Landau}}$ is normal-ordered relative to the initial GS (with energy shifted by $\omega_0$) and is of the form

$$\hat{H}_{\text{Landau}} = \hat{H}_L^{(1)} + \hat{H}_L^{(2)},$$

where

$$\hat{H}_L^{(1)} = \sum_{q,\alpha,\gamma} \epsilon_{\alpha,\gamma}(q) : \hat{N}_{\alpha,\gamma}(q) :,$$

$$\epsilon_{\alpha,0}(q) = \epsilon_{\alpha,0}^0(q) - \delta_{\alpha,s} \epsilon_{\alpha,\gamma}^0(0),$$

and $\hat{H}_L^{(2)}$ is given in Eq. (106). The Hamiltonian (127) describes both the GS - GGS transition and Landau-liquid pseudoparticle - pseudohole excitations relative to the GGS. (The $c,0$ and $s,0$ pseudoparticle bands are shown in Figs. 7 and 8, respectively, of Ref. [32].) Here (127) are the Landau-liquid Hamiltonian terms which are relevant at low energy ($\omega - \omega_0$). Therefore, the corresponding second-order Hamiltonian is suitable to study the physics at low positive energy above the gap, i.e., small $(\omega - \omega_0)$ [25,26].

In general, the different final GGS’s of the Hilbert subspace where a given initial GS is transformed upon excitations involving a small density of pseudoparticles have different energies (123) – (124). This implies that the study of the quantum-liquid physics at energy-scale $\omega_0$ involves, in general, transitions to one sub-cannonical ensemble only. When two or several possible final GGS’s have the same energy gap ($\omega_0$) the physics involves the Hilbert subspace spanned by all Hamiltonian eigenstates associated with the corresponding different final sub-canonical ensembles.

For sub-canonical ensembles such that $S^z_0 \neq 0$ and characterized by small values of $N^h_{\alpha,\omega} / N^h_{\alpha}$ [where $l_\alpha$ is given in Eq. (7)], the gap expression (123) leads directly to

$$\hat{H}_0 = 2|\mu|\hat{N}_{c,\omega}^h + 2\rho_0 H [\hat{N}_{s,\omega}^h + \sum_{\gamma=1}^\infty \hat{N}_{s,\gamma}] + \sum_{\gamma=1}^\infty \epsilon_{\alpha,\gamma}(0) \hat{N}_{\alpha,\gamma}.$$  \hspace{1cm} (130)

There is a remarkable similarity between the general Hamiltonian (126) and its version in the Hilbert subspace spanned by the states $|l_\alpha\rangle$ [23,24,26]. Moreover, in the case that the GGS is the GS itself or a GS differing from it by one or a few electrons the gap (123) – (124) vanishes and the Hamiltonian (126) reduces to the above low-energy Hamiltonian. In this case it refers to gapless GS - GS transitions which change the electron and $\alpha,0$ pseudoparticle numbers and to Landau-liquid $\alpha,0$ pseudoparticle-pseudohole excitations around the final GS.

The two-pseudoparticle forward-scattering phase shifts $\Phi_{\alpha,\gamma;\alpha',\gamma'}(q, q')$ defined by Eqs. (99) – (103) and (B30)-(B40) and the velocities (115) play an important role for the physics at energy scales around $\omega_0$ [23,26]. [In the case of $\gamma = 0$, the velocities (115) are plotted in Fig. 9 of Ref. [23].]

In this section we have presented the expression of the normal-ordered Hamiltonian and rapidity operators in the pseudoparticle basis. This confirms the consistency of the one-dimensional Landau-liquid theory which was shown to refer to this basis. The advantage of using the pseudoparticle basis is that (i) the problem becomes perturbative, i.e. from the point of view of the pseudoparticle interactions it is possible to classify which scatterings are relevant and (ii) we can describe the relevant high-energy GS transitions because these refer to a small density of excited pseudoparticles and are, therefore, within the range of the perturbation theory.

For energies $\omega$ just above the set of energy values $\omega_0$ of the form given in Eqs. (123) – (124) only the two-pseudoparticle interactions are relevant, as can be confirmed by finite-size studies at the corresponding critical point [24]. Therefore, and for simplicity, we have presented in this paper only the two first Hamiltonian terms of expression (104). However, Eqs. (66) – (68) and (83) contain full information about all the remaining terms of higher-scattering order. All these many-pseudoparticle terms only describe zero-momentum forward-scattering interactions.
VI. CONCLUDING REMARKS

In this paper we have introduced a pseudohole and heavy-pseudoparticle representation for all the Hamiltonian eigenstates of the Hubbard chain in a magnetic field and chemical potential which is valid for all sectors of Hamiltonian symmetry. In this picture all Hamiltonian eigenstates can be generated from a single reference vacuum, the half-filling and zero-magnetic-field GS.

Our algebraic approach permits an operator analysis of the BA solution and goes beyond that solution by generating the non-LWS’s and non-HWS’s of the η-spin and spin algebras. We find that all GS excitations can be divided into (a) a GS - GGS topological transition which changes the α, β pseudohole and α, γ heavy-pseudoparticle numbers and (b) one pseudoparticle-pseudohole excitation around the final GGS. It will be shown elsewhere [23,24] that the quantum-liquid physically important transitions are the GS - GGS transitions which occur at energies ω_0 of form (124). Further, for low (ω − ω_0) energies our algebraic approach leads to the construction of a GS normal-ordered Hamiltonian which in the pseudoparticle basis a universal form involving only k = 0, forward-scattering pseudoparticle interaction terms.

A central result of our paper is that the perturbative character of the pseudoparticle basis also refers to high-energy states provided that the density of excited α, γ pseudoparticles is small. This perturbative character implies that the two-pseudoparticle Landau f functions and forward-scattering amplitudes are finite, in contrast to the non-perturbative electronic representation, in which the two-electron forward-scattering amplitudes and vertices diverge.

The pseudoparticle algebra allows us to see immediately the origin of the “universal” character of the one-dimensional integrable quantum liquids at all energy scales, which can be understood as a straightforward generalization to pseudoparticles of Wilson’s low-energy renormalization group arguments: the pseudo-Fermi points of the pseudoparticle GS’s and GGS’s replace the particle Fermi points, and close to the pseudo-Fermi points only few types of two-pseudoparticle scattering processes are relevant for the low (ω − ω_0) energy physics.

Concerning applications of our ideas to other theoretical models and to problems in real materials, we note that it is of considerable interest to determine the extent to which the algebraic structure and pseudoparticle perturbation theory remain valid in systems which are not integrable but which behave as electronic Luttinger liquids; an example of such a system is the one-dimensional extended Hubbard model. In terms of real materials, it is known that the present one-dimensional quantum systems provide useful (albeit idealized) descriptions of the physics of quasi-one-dimensional solids.

Finally, we turn to directions for further research. In addition to some specific calculations mentioned above, our immediate aims are to apply the concepts and techniques developed here to study the finite-frequency spectral and transport properties of the Hubbard chain. The occurrence of only zero-momentum pseudoparticle forward scattering combined with the infinite pseudohole- and pseudoparticle-number conservations laws can be used to extract important information on the finite-frequency behavior of correlation functions [20]. We will show elsewhere [25] that the α, γ pseudoparticles are the transport carriers at all energy scales and couple to external potentials and that the topological GS - GGS transitions introduced in this paper give rise to the finite-frequency-conductivity absorption edges. We emphasize that our pseudoparticle operator basis provides a more suitable starting point for future investigations on that open problem. This is a subject of considerable interest for the understanding of the mechanisms behind the unusual spectral and transport properties of the novel low-dimensional materials.

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APPENDIX A: RELATION TO THE USUAL BA NOTATION

Our pseudoparticle operator representation required a suitable choice of notation which is directly related to the BA notation used by Takahashi [19]. In this Appendix we relate the pseudoparticle notation to his notation. In our $c, \gamma$ and $s, \gamma$ parameters below we consider that $\gamma > 0$ and $\gamma = 0, 1, 2, \ldots$, respectively.

The $N_{\alpha,\gamma}$ pseudoparticle numbers are related to the numbers $N, M', M'_n$, and $M_n$ of Ref. [19] as

\[ N_{c,0} \equiv N - 2M', \quad N_{c,\gamma} \equiv M'_\gamma, \quad N_{s,\gamma} \equiv M_{1+\gamma}. \]  
(A1)

Moreover, our onsite parameter $u = \frac{U}{4t}$ equals the interaction $U$ of that reference. Considering the discrete pseudo-momentum $q_j$ of Eq. (28), our rapidity eigenvalues are related to the rapidity-real parts $k_j$, $\Lambda^\gamma_{\alpha_n}$, and $\Lambda_{\alpha_n}$ of Ref. [19] as follows

\[ K(q_j) \equiv k_j, \quad R_{c,\gamma}(q_j) \equiv \frac{\Lambda^\gamma_{\alpha_n}}{u}, \quad R_{s,\gamma}(q_j) \equiv \frac{\Lambda^{1+\gamma}_{\alpha_n}}{u}. \]  
(A2)

Finally, the relation of the pseudoparticle quantum numbers of Eq. (28) to the Takahashi quantum numbers $I_j$, $J^{\alpha_n}_n$, and $J^{\gamma}_{\alpha_n}$ is

\[ I^{c,0}_j \equiv I_j, \quad I^{\gamma}_{j,\gamma} \equiv J^{\alpha_n}_{j,\gamma}, \quad I^{s,\gamma}_j \equiv J^{1+\gamma}_{j}. \]  
(A3)
APPENDIX B: NORMAL-ORDERED RAPIDITY OPERATOR EXPRESSIONS

Following the discussion of Sec. V, the perturbative character of the system implies the equivalence between expanding in the pseudoparticle scattering order and/or in the pseudomomentum deviations (95). In this Appendix we give a short description of the calculation of the normal-ordered operator expansion for the pseudoparticle rapidities (97). We focus our study on the evaluation of the functions $X_{\alpha,\gamma}^{(1)}(q)$ and $X_{\alpha,\gamma}^{(2)}(q)$ and associate functions $Q_{\alpha,\gamma}^{(1)}(q)$ and $Q_{\alpha,\gamma}^{(2)}(q)$ (eigenvalues of the operators $X_{\alpha,\gamma}^{(1)}(q)$ and $X_{\alpha,\gamma}^{(2)}(q)$ and $Q_{\alpha,\gamma}^{(1)}(q)$ and $Q_{\alpha,\gamma}^{(2)}(q)$, respectively).

We evaluate here the first-order and second-order terms of the eigenvalues of the operator (97). Equation (86) then allows the straightforward calculation of the corresponding operator expressions.

In the thermodynamic limit, Eqs. (66) − (68) lead to the following equations

$$K(q) = q - \frac{1}{\pi} \sum_{\gamma=0}^{\infty} \int_{q_{\epsilon,\gamma}^{(-)}}^{q_{\epsilon,\gamma}^{(+)}} dq' N_{s,\gamma}(q') \tan^{-1} \left( \frac{R_{c,0}(q) - R_{s,\gamma}(q')}{1 + \gamma} \right)$$

$$- \frac{1}{\pi} \sum_{\gamma=1}^{\infty} \int_{q_{\epsilon,\gamma}^{(-)}}^{q_{\epsilon,\gamma}^{(+)}} dq' N_{c,\gamma}(q') \tan^{-1} \left( \frac{R_{c,0}(q) - R_{c,\gamma}(q')}{\gamma} \right), \quad (B1)$$

$$2R e \sin^{-1} \left( u(R_{c,\gamma}(q) - i \gamma) \right) = q + \frac{1}{\pi} \int_{q_{c,0}^{(-)}}^{q_{c,0}^{(+)}} dq' N_{c,0}(q') \tan^{-1} \left( \frac{R_{c,\gamma}(q) - R_{c,0}(q')}{\gamma} \right)$$

$$+ \frac{1}{2\pi} \sum_{\gamma'=1}^{\infty} \int_{q_{c,\gamma'}^{(-)}}^{q_{c,\gamma'}^{(+)}} dq' N_{c,\gamma'}(q') \Theta_{\gamma,\gamma'} \left( R_{c,\gamma}(q) - R_{c,\gamma'}(q') \right), \quad (B2)$$

and

$$0 = q - \frac{1}{\pi} \int_{q_{c,0}^{(-)}}^{q_{c,0}^{(+)}} dq' N_{c,0}(q') \tan^{-1} \left( \frac{R_{s,\gamma}(q) - R_{c,0}(q')}{1 + \gamma} \right)$$

$$+ \frac{1}{2\pi} \sum_{\gamma'=1}^{\infty} \int_{q_{c,\gamma'}^{(-)}}^{q_{c,\gamma'}^{(+)}} dq' N_{s,\gamma'}(q') \Theta_{\gamma,\gamma'} \left( R_{s,\gamma}(q) - R_{s,\gamma'}(q') \right). \quad (B3)$$

We start by considering the GGS eigenstate rapidities of Eqs. (72) and (73). If we insert in Eqs. (66) − (68) the GGS distribution (75) after some algebra we find that the functions of the rhs of Eqs. (77) − (78) defining the inverse of these GGS rapidities are solutions of the following integral equations

$$2\pi \rho_{c,0}(k) = 1 + \frac{\cos k}{u} \int_{r_{c,0}^{(-)}}^{r_{c,0}^{(+)}} dr \frac{2\pi \rho_{s,0}(r)}{\pi \left[ 1 + (r - \frac{\sin k}{u})^2 \right]}, \quad (B4)$$

$$2\pi \rho_{c,\gamma}(r) = 2R e \left( \frac{u}{\sqrt{1 - u^2(r - i \gamma)^2}} \right) - \int_{Q(\gamma)}^{Q(\gamma)} dk \frac{2\pi \rho_{c,0}(k)}{\pi \gamma \left[ 1 + \left( \frac{\sin r}{\gamma} \right)^2 \right]}, \quad (B5)$$

and

$$2\pi \rho_{s,\gamma}(r) = \int_{Q(\gamma)}^{Q(\gamma)} dk \frac{2\pi \rho_{c,0}(k)}{\pi(1 + \gamma) \left[ 1 + (\frac{\sin k}{1 + \gamma})^2 \right]}$$

$$- \int_{r_{s,0}^{(-)}}^{r_{s,0}^{(+)}} dr' \frac{\Theta_{1,1+\gamma}(r - r')}{2\pi} 2\pi \rho_{s,0}(r'), \quad (B6)$$

where
\[ \Theta^{[1]}_{\gamma', \gamma'}(x) = \frac{d\Theta_{\gamma', \gamma'}(x)}{dx} \]
\[ = \delta_{\gamma', \gamma'} \left\{ \frac{1}{\gamma [1 + (\frac{x}{\gamma})^2]} + \sum_{l=1}^{\gamma-1} \frac{2}{[1 + (\frac{x}{\gamma})^2]} \right\} \\
+ (1 - \delta_{\gamma', \gamma'}) \left\{ \frac{2}{[\gamma - \gamma''][1 + (\frac{x}{\gamma})^2]} + \frac{2}{(\gamma + \gamma'')^2} \right\} \\
+ \sum_{l=1}^{\gamma + \gamma' - |\gamma' - \gamma| - 1} \left\{ \frac{4}{[\gamma - \gamma'][1 + (\frac{x}{\gamma})^2]} \right\}, \tag{B7} \]

is the derivative of the function (69) and the parameters \( Q^{(\pm)} \) and \( r_{\alpha, \gamma}^{(\pm)} \) are defined combining Eqs. (77) – (78) and (79). We have omitted in the rhs of Eqs. (B4)-(B6) terms associated with contributions of order \( j > 2 \) in the density of excited pseudoparticles. These terms do not contribute to the quantities to be evaluated in this paper, as we discuss below.

Let us now consider small deviations from a GGS or GS. The eigenvalue form of Eq. (96) is
\[ \delta K(q) = K^{(0)}(q) + \delta X_{c, 0}(q) - K^{(0)}(q), \tag{B8} \]
for \( \alpha = c \) and \( \gamma = 0 \) and
\[ \delta R_{\alpha, \gamma}(q) = R_{\alpha, \gamma}^{(0)}(q) + \delta X_{\alpha, \gamma}(q) - R_{\alpha, \gamma}^{(0)}(q), \tag{B9} \]
for all remaining values of the quantum numbers \( \alpha \) and \( \gamma \). Here \( \delta K(q), \delta R_{\alpha, \gamma}(q), \) and \( \delta X_{\alpha, \gamma}(q) \) are the eigenvalues of the operators \( \hat{K}(q) ; \hat{R}_{\alpha, \gamma}(q) ; \) and \( \hat{X}_{\alpha, \gamma}(q) ; \) respectively. From Eq. (97) \( \delta X_{\alpha, \gamma}(q) \) can be written as
\[ \delta X_{\alpha, \gamma}(q) = X_{\alpha, \gamma}^{(1)}(q) + X_{\alpha, \gamma}^{(2)}(q) + \ldots, \tag{B10} \]
where \( X_{\alpha, \gamma}^{(i)}(q) \) is the eigenvalue of the operator \( \hat{X}_{\alpha, \gamma}^{(i)}(q) \). Expanding the \( \delta R_{\alpha, \gamma}(q) \) expressions (B8) and (B9) we find
\[ K(q) = \sum_{i=0}^{\infty} K^{(i)}(q), \tag{B11} \]
and
\[ R_{\alpha, \gamma}(q) = \sum_{i=0}^{\infty} R_{\alpha, \gamma}^{(i)}(q), \tag{B12} \]
respectively, \( \delta K(q) = \sum_{i=1}^{\infty} K^{(i)}(q) \) and \( \delta R_{\alpha, \gamma}(q) = \sum_{i=1}^{\infty} R_{\alpha, \gamma}^{(i)}(q) \] where the zero-order GGS functions \( K^{(0)}(q) \) and \( R_{\alpha, \gamma}^{(0)}(q) \) are defined by Eqs. (77) – (78). From the resulting equations we can obtain all derivatives of the GGS functions \( K^{(0)}(q) \) and \( R_{\alpha, \gamma}^{(0)}(q) \) with respect to \( q \). The terms of the rhs of Eqs. (B11) and (B12) involve these derivatives. For instance, the first-order and second-order terms read
\[ K^{(1)}(q) = \frac{dK^{(0)}(q)}{dq} X_{c, 0}^{(1)}(q), \tag{B13} \]
\[ R_{\alpha, \gamma}^{(1)}(q) = \frac{dR_{\alpha, \gamma}^{(0)}(q)}{dq} X_{\alpha, \gamma}^{(1)}(q), \tag{B14} \]
and
\[ K^{(2)}(q) = \frac{dK^{(0)}(q)}{dq} X_{c, 0}^{(2)}(q) + \frac{1}{2} \frac{d^2 K^{(0)}(q)}{dq^2} [X_{c, 0}^{(1)}(q)]^2, \tag{B15} \]
\[ R_{\alpha, \gamma}^{(2)}(q) = \frac{dR_{\alpha, \gamma}^{(0)}(q)}{dq} X_{\alpha, \gamma}^{(2)}(q) + \frac{1}{2} \frac{d^2 R_{\alpha, \gamma}^{(0)}(q)}{dq^2} [X_{\alpha, \gamma}^{(1)}(q)]^2, \tag{B16} \]
respectively, and involve the first and second derivatives.

From Eqs. (B1)-(B3) [with \( N_{\alpha,\gamma}(q') \) given by the GGS distribution (75)] we find that the first derivatives \( \frac{dK^{(0)}(q)}{dq} \) and \( \frac{dR^{(0)}_{\alpha,\gamma}(q)}{dq} \) can be expressed in terms of the functions (B4)-(B6) as follows

\[
\frac{dK^{(0)}(q)}{dq} = \frac{1}{2\pi \rho_{c,0}(K^{(0)}(q))},
\]

(B17)

and

\[
\frac{dR^{(0)}_{\alpha,\gamma}(q)}{dq} = \frac{1}{\pi \rho_{c,\gamma}(R^{(0)}_{\alpha,\gamma}(q))}.
\]

(B18)

The second derivatives \( \frac{d^2K^{(0)}(q)}{dq^2} \) and \( \frac{d^2R^{(0)}_{\alpha,\gamma}(q)}{dq^2} \) then read

\[
\frac{d^2K^{(0)}(q)}{dq^2} = -\frac{1}{\left[2\pi \rho_{c,0}(K^{(0)}(q))\right]^2} \cdot \frac{\frac{d\rho_{c,0}(k)}{dk}}{\left|k = K^{(0)}(q)\right|},
\]

(B19)

and

\[
\frac{d^2R^{(0)}_{\alpha,\gamma}(q)}{dq^2} = -\frac{1}{\left[2\pi \rho_{c,\gamma}(R^{(0)}_{\alpha,\gamma}(q))\right]^3} \cdot \frac{\frac{d\rho_{c,\gamma}(r)}{dr}}{\left|r = R^{(0)}_{\alpha,\gamma}(q)\right|}.
\]

(B20)

By introducing both the distributions

\[
N_{\alpha,\gamma}(q) = N^{(0)}_{\alpha,\gamma}(q) + \delta N_{\alpha,\gamma}(q),
\]

(B21)

and the first-order and second-order functions (B13)-(B16) into Eqs. (B1)-(B3), we find after expanding to second order that for \( j = 1 \) and \( j = 2 \) the functions \( \hat{X}^{(j)}_{\alpha,\gamma}(q) \) can be written as

\[
\hat{X}^{(j)}_{\alpha,\gamma}(q) = \hat{Q}^{(j)}_{\alpha,\gamma}(q) + \hat{Y}^{(j)}_{\alpha,\gamma}(q),
\]

(B22)

where \( \hat{Y}^{(j)}_{\alpha,\gamma}(q) \) is even in \( q \)

\[
\hat{Y}^{(j)}_{\alpha,\gamma}(q) = \hat{Y}^{(j)}_{\alpha,\gamma}(-q),
\]

(B23)

and does not contribute to the physical quantities to second scattering order (and to second order in the density of excited pseudoparticles). Up to this order only \( \hat{Q}^{(j)}_{\alpha,\gamma}(q) \) contributes.

To derive this result we have expanded the expression for the even functions \( Y^{(j)}_{\alpha,\gamma}(q) \) for \( j = 1 \) and \( j = 2 \) to second order in the density of excited pseudoparticles. Following the perturbative character of the quantum liquid in the pseudoparticle basis, the obtained expression is exact up to \( j = 2 \) pseudoparticle scattering order. Moreover, we find that the even function \( Y^{(1)}_{\alpha,\gamma}(q) \) does not contribute to the physical quantities up to that order and can, therefore, be omitted.

Let us introduce the functions \( \tilde{Q}^{(1)}_{\alpha,\gamma}(r) \) such that

\[
Q^{(1)}_{\alpha,\gamma}(q) = \tilde{Q}^{(1)}_{\alpha,\gamma}(R^{(0)}_{\alpha,\gamma}(q)),
\]

(B24)

for all values of \( \alpha \) and \( \gamma \). Note that following Eq. (64)

\[
R^{(0)}_{c,0}(q) = \sin K^{(0)}(q) u.
\]

(B25)

It is also useful to define the function \( \tilde{Q}^{(1)}(k) \) such that

\[
\tilde{Q}^{(1)}(k) = \hat{Q}^{(1)}_{c,0}(\frac{\sin k}{u}).
\]

(B26)
By introducing both the distributions (B21) and the first-order functions defined in Eq. (B24) into Eqs. (B1)-(B3), we find after expanding to first order that the functions \( \bar{Q}_{c,\alpha,\gamma}^{(1)}(r) \) are defined by the following system of coupled integral equations

\[
\bar{Q}_{c,\alpha,\gamma}^{(1)}(r) = -\sum_{\gamma=0}^{\infty} \int_{\gamma_0^{-}(q,\gamma)}^{\infty} dq \delta N_{c,\gamma}(q) \frac{1}{\pi} \tan^{-1} \left( \frac{r - R_{c,\alpha,\gamma}^{(0)}(q)}{1 + \gamma} \right) \]

\[- \sum_{\gamma=1}^{\infty} \int_{\gamma_0^{-}(q,\gamma)}^{\infty} dq \delta N_{c,\gamma}(q) \frac{1}{\pi} \tan^{-1} \left( \frac{r - R_{c,\alpha,\gamma}^{(0)}(q)}{\gamma} \right) + \int_{r_{c,0}^{-}(+)_{\gamma}}^{\infty} dr' \frac{\bar{Q}_{c,0}^{(1)}(r')}{\pi \gamma [1 + (r - r')^2]}, \quad (B27)
\]

\[
\bar{Q}_{c,\alpha,\gamma}^{(1)}(r) = \int_{q_{c,0}^{-}(q)}^{q_{c,0}^{(+)(q)}} dq \delta N_{c,\gamma}(q) \frac{1}{\pi} \tan^{-1} \left( \frac{r - R_{c,0}^{(0)}(q)}{1 + \gamma} \right)
\]

\[+ \sum_{\gamma'=1}^{\infty} \int_{q_{c,\gamma'}^{-}(q,\gamma')}^{q_{c,\gamma'}^{(+)(q,\gamma')}} dq \delta N_{c,\gamma'}(q) \frac{1}{2\pi} \Theta_{\gamma',\gamma'}(r - R_{c,\gamma'}^{(0)}(q)) - \int_{r_{c,0}^{-}(+)_{\gamma}}^{r_{c,0}^{(+)_{\gamma}}} dr' \frac{\bar{Q}_{c,0}^{(1)}(r')}{\pi \gamma [1 + (r - r')^2]}, \quad (B28)
\]

and

\[
\bar{Q}_{s,\alpha,\gamma}^{(1)}(r) = -\int_{q_{c,0}^{-}(q)}^{q_{c,0}^{(+)(q)}} dq \delta N_{c,\gamma}(q) \frac{1}{\pi} \tan^{-1} \left( \frac{r - R_{c,0}^{(0)}(q)}{1 + \gamma} \right)
\]

\[+ \sum_{\gamma'=1}^{\infty} \int_{q_{c,\gamma'}^{-}(q,\gamma')}^{q_{c,\gamma'}^{(+)(q,\gamma')}} dq \delta N_{s,\gamma'}(q) \frac{1}{2\pi} \Theta_{1 + \gamma,1 + \gamma'}(r - R_{c,\gamma'}^{(0)}(q)) + \int_{r_{c,0}^{-}(+)_{\gamma}}^{r_{c,0}^{(+)_{\gamma}}} dr' \frac{\bar{Q}_{s,0}^{(1)}(r')}{\pi (1 + \gamma) [1 + (r - r')^2]} - \int_{r_{c,0}^{-}(+)_{\gamma}}^{r_{c,0}^{(+)_{\gamma}}} dr' \bar{Q}_{s,0}^{(1)}(r') \frac{1}{1 + \gamma,1 + \gamma'} (r - r'). \quad (B29)
\]

The use of Eq. (B27) in Eqs. (B28) and (B29) allows the expression of both \( \bar{Q}_{c,\alpha,\gamma}^{(1)}(r) \) and \( \bar{Q}_{s,\alpha,\gamma}^{(1)}(r) \) in terms of free terms and integrals involving \( \bar{Q}_{s,0}^{(1)}(r) \). Combining Eqs. (B27)-(B29) with Eqs. (B24) and (B26) leads to Eq. (98) with the phase shifts defined below. Note that at first order we can either consider the function \( \bar{X}_{\alpha,\gamma}^{(1)}(q) \) or the associate function \( \bar{Q}_{\alpha,\gamma}^{(1)}(q) \) of Eq. (B22). Both functions are of the form (98). However, the general expressions for the phase shifts associated with the function \( \bar{X}_{\alpha,\gamma}^{(1)}(q) \) of Eq. (98) have extra terms. These arise from the function \( \bar{Y}_{\alpha,\gamma}^{(1)}(q) \) of the rhs of Eq. (B22). If we expand the physical quantities involving the phase shifts to first (and second) order in the density of excited pseudoparticles these extra terms lead to vanishing contributions. For instance, expression (98) is identical if we use in it either choice for the phase shift expressions. For simplicity, we omit here the phase-shift extra terms associated with the function \( \bar{Y}_{\alpha,\gamma}^{(1)}(q) \). We find that the phase shifts associated with the function \( \bar{Q}_{\alpha,\gamma}^{(1)}(q) \) are defined by the following coupled integral equations

\[
\Phi_{c,0;c,0}(r,r') = \frac{1}{\pi} \int_{r_{c,0}^{-}(+)_{\gamma}}^{r_{c,0}^{(+)_{\gamma}}} dq \delta N_{c,0}(q) \frac{1}{1 + (r - r')^2}, \quad (B30)
\]

\[
\Phi_{c,0;c,\gamma}(r,r') = -\frac{1}{\pi} \tan^{-1} \left( \frac{r - r'}{\gamma} \right) + \frac{1}{\pi} \int_{r_{c,0}^{-}(+)_{\gamma}}^{r_{c,0}^{(+)_{\gamma}}} dq \delta N_{c,0}(q) \frac{1}{1 + (r - r')^2}, \quad (B31)
\]

\[
\Phi_{c,0;s,\gamma}(r,r') = -\frac{1}{\pi} \tan^{-1} \left( \frac{r - r'}{1 + \gamma} \right) + \frac{1}{\pi} \int_{r_{c,0}^{-}(+)_{\gamma}}^{r_{c,0}^{(+)_{\gamma}}} dq \delta N_{c,0}(q) \frac{1}{1 + (r - r')^2}, \quad (B32)
\]

\[
\Phi_{c,s;c,0}(r,r') = \frac{1}{\pi} \tan^{-1} \left( \frac{r - r'}{\gamma} \right) - \frac{1}{\pi} \int_{r_{c,0}^{-}(+)_{\gamma}}^{r_{c,0}^{(+)_{\gamma}}} dq \delta N_{c,0}(q) \frac{1}{1 + (r - r')^2}, \quad (B33)
\]
\[ \Phi_{c,\gamma;c,\gamma'}(r, r') = \frac{1}{2\pi} \Theta_{\gamma,\gamma'}(r - r') - \frac{1}{\pi} \int_{r_{c,0}^{(-)}}^{r_{c,0}^{(+)}} dr' \frac{\Phi_{c,0,0;c,\gamma'}(r'', r')}{\gamma[1 + (\frac{r'' - r'}{\gamma})^2]}, \]  
(B34)

\[ \Phi_{c,\gamma};c,\gamma'(r, r') = -\frac{1}{\pi} \int_{r_{c,0}^{(-)}}^{r_{c,0}^{(+)}} dr'' \frac{\Phi_{c,0,0;c,\gamma'}(r'', r')}{\gamma[1 + (\frac{r'' - r'}{\gamma})^2]}, \]  
(B35)

\[ \Phi_{s,\gamma;c,0}(r, r') = \frac{1}{\pi} \tan^{-1}(\frac{r - r'}{1 + \gamma}) + \frac{1}{\pi} \int_{r_{c,0}^{(-)}}^{r_{c,0}^{(+)}} dr'' \frac{\Phi_{c,0,0;c,\gamma'}(r'', r')}{(1 + \gamma)[1 + (\frac{r'' - r'}{1 + \gamma})^2]} - \int_{r_{s,0}^{(-)}}^{r_{s,0}^{(+)}} dr'' \Phi_{s,0;c,0}(r'', r') \frac{\Theta_{1+\gamma,1}(r - r'')}{2\pi}, \]  
(B36)

\[ \Phi_{s,\gamma;c,\gamma'}(r, r') = \frac{1}{\pi} \int_{r_{c,0}^{(-)}}^{r_{c,0}^{(+)}} dr'' \frac{\Phi_{c,0,0;c,\gamma'}(r'', r')}{(1 + \gamma)[1 + (\frac{r'' - r'}{1 + \gamma})^2]} - \int_{r_{s,0}^{(-)}}^{r_{s,0}^{(+)}} dr'' \Phi_{s,0;c,\gamma'}(r'', r') \frac{\Theta_{1+\gamma,1}(r - r'')}{2\pi}, \]  
(B37)

\[ \Phi_{s,\gamma;0}(r, r') = \frac{1}{2\pi} \Theta_{1+\gamma,1}(r - r') - \frac{1}{\pi} \int_{r_{c,0}^{(-)}}^{r_{c,0}^{(+)}} dr'' G(r, r'') \Phi_{s,0;c,0}(r'', r') , \]  
(B38)

For \( \gamma = 0 \) Eqs. (B36) and (B38) can be rewritten as

\[ \Phi_{s,0;c,0}(r, r') = -\frac{1}{\pi} \tan^{-1}(\frac{r - r'}{2}) + \int_{r_{s,0}^{(-)}}^{r_{s,0}^{(+)}} dr'' G(r, r'') \Phi_{s,0;c,0}(r'', r') , \]  
(B39)

and

\[ \Phi_{s,0;0}(r, r') = \frac{1}{\pi} \tan^{-1}(\frac{r - r'}{2}) - \frac{1}{\pi^2} \int_{r_{c,0}^{(-)}}^{r_{c,0}^{(+)}} dr'' \tan^{-1}(\frac{r'' - r'}{1 + (r'' - r')^2}) + \int_{r_{s,0}^{(-)}}^{r_{s,0}^{(+)}} dr'' G(r, r'') \Phi_{s,0;0}(r'', r') , \]  
(B40)

and the kernel \( G(r, r') \) reads

\[ G(r, r') = -\frac{1}{2\pi} \left[ \frac{1}{1 + ((r - r')/2)^2} \right] \left[ 1 - \frac{1}{2} \left( t(r) + t(r') + \frac{l(r) - l(r')}{r - r'} \right) \right], \]  
(B41)

with

\[ t(r) = \frac{1}{\pi} \left[ \tan^{-1}(r + r_{c,0}^{(+)}) - \tan^{-1}(r + r_{c,0}^{(-)}) \right], \]  
(B42)

and

\[ l(r) = \frac{1}{\pi} \left[ \ln(1 + (r + r_{c,0}^{(+)})) - \ln(1 + (r + r_{c,0}^{(-)})) \right]. \]  
(B43)
In order to evaluate the second-order functions $Q^{(2)}_{\alpha,\gamma}(q)$ of the rhs of Eq. (B22) for $j = 2$ [see also Eq. (B10)] we introduce the functions (B13)-(B16) and distributions (B21) in Eqs. (B1)-(B3). Expanding to second order we find after some algebra

$$Q^{(2)}_{\alpha,\gamma}(q) = Q^{(2,+)}_{\alpha,\gamma}(q) + \frac{1}{2} \frac{d}{dq} \left[ (Q^{(1)}_{\alpha,\gamma}(q))^2 \right], \quad (B44)$$

where

$$Q^{(2,+)}_{\alpha,\gamma}(q) = \phi^{(2,+)}_{\alpha,\gamma}(q). \quad (B45)$$

It is also useful to define the function $\phi^{(2,+)}_{\alpha,\gamma}(k)$ such that

$$\phi^{(2,+)}_{\alpha,\gamma}(k) = \phi^{(2,+)}_{c,0,\alpha,\gamma}(k). \quad (B46)$$

The functions $\phi^{(2,+)}_{\alpha,\gamma}(r)$ are defined by the following system of coupled integral equations

$$\phi^{(2,+)}_{c,0}(r) = \sum_{\gamma=0}^{\gamma} \int_{q_{\alpha,\gamma}}^{q_{\alpha,\gamma}} dq N_{s,\gamma}(q) \frac{Q^{(1)}_{s,\gamma}(q)}{2\pi \rho_{s,\gamma}} \frac{1}{1 + (r - R_{s,\gamma}(q))^2} \left[ 1 + \left( \frac{r - R_{s,\gamma}(q)}{1 + \gamma} \right)^2 \right]$$

$$+ \sum_{\gamma=1}^{\gamma} \int_{q_{\alpha,\gamma}}^{q_{\alpha,\gamma}} dq N_{c,\gamma}(q) \frac{Q^{(1)}_{c,\gamma}(q)}{2\pi \rho_{c,\gamma}} \frac{1}{1 + (r - R_{c,\gamma}(q))^2} \left[ 1 + \left( \frac{r - R_{c,\gamma}(q)}{1 + \gamma} \right)^2 \right]$$

$$+ \sum_{\gamma=0}^{\gamma} \theta(N_{s,\gamma}) \sum_{j=\pm 1}^{(1 + \gamma)} \frac{[Q^{(1)}_{s,\gamma}(jqFs,\gamma)]^2}{2\pi (1 + \gamma)} \left[ 1 + (\frac{r - R_{s,\gamma}(q)}{1 + \gamma})^2 \right]$$

$$- \sum_{\gamma=1}^{\gamma} \theta(N_{c,\gamma}) \sum_{j=\pm 1}^{(1 + \gamma)} \frac{[Q^{(1)}_{c,\gamma}(jqFc,\gamma)]^2}{2\pi (1 + \gamma)} \left[ 1 + (\frac{r - R_{c,\gamma}(q)}{1 + \gamma})^2 \right]$$

$$+ \int_{r_{s,0}}^{r_{s,0}} dr' \frac{\phi^{(2,+)}_{0,0}(r')}{\pi (1 + (r - r')^2)}, \quad (B47)$$

$$\phi^{(2,+)}_{c,0}(r) = \int_{q_{c,0}}^{q_{c,0}} dq N_{c,0}(q) \frac{\cos K^{(0)}(q)}{2\pi \rho_{c,0}(k^{(0)}(q))} \frac{1}{\pi \gamma} \frac{1}{1 + (r - R_{c,0}(q))^2}$$

$$- \sum_{\gamma=1}^{\gamma} \int_{q_{c,\gamma}}^{q_{c,\gamma}} dq N_{c,\gamma}(q) \frac{Q^{(1)}_{c,\gamma}(q)}{2\pi \rho_{c,\gamma}} \frac{1}{2\pi \theta^{(1)}_{\gamma,\gamma}} (r - R_{c,\gamma}(q))$$

$$- \sum_{\gamma=1}^{\gamma} \frac{\theta(N_{c,\gamma})}{2\pi \rho_{c,\gamma}} \sum_{j=\pm 1}^{(1 + \gamma)} \frac{[Q^{(1)}_{c,\gamma}(jqFc,\gamma)]^2}{2\pi \theta^{(1)}_{\gamma,\gamma}} (r - jr_{c,\gamma})$$

$$- \int_{r_{c,0}}^{r_{c,0}} dr' \frac{\phi^{(2,+)}_{c,0}(r')}{\pi \gamma} \left[ 1 + (\frac{r - r'}{1 + \gamma})^2 \right], \quad (B48)$$

and

$$\phi^{(2,+)}_{s,\gamma}(r) = \int_{q_{s,\gamma}}^{q_{s,\gamma}} dq N_{s,\gamma}(q) \frac{\cos K^{(0)}(q)}{2\pi \rho_{s,\gamma}(k^{(0)}(q))} \frac{1}{\pi (1 + \gamma)} \frac{1}{1 + (r - R_{s,\gamma}(q))^2}$$

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As for the first-order case, the use of either the function $Q^{(2)}_{\alpha,\gamma}(q)$ defined by Eqs. (B44)-(B49) or of the full function $X^{(2)}_{\alpha,\gamma}(q)$ of Eq. (B22) for $j = 2$ leads to the same results for the physical quantities up to second order in the density of excited pseudoparticles.

Note that the free terms of Eqs. (B47)-(B49) involve the first-order functions only. This implies that the unique solutions of these integral equations can be expressed in terms of the first-order functions. Therefore, following Eqs. (98) and (B22) the second-order functions can also be expressed in terms of the pseudoparticle phase shifts.
APPENDIX C: NORMAL-ORDERED HAMILTONIAN EXPRESSION

In order to derive the first-order and second-order Hamiltonian terms of Eqs. (105) and (106), we again consider eigenvalues and deviations. The energy associated with the Hamiltonian (83) reads

$$E_{SO(4)} = -\frac{2tN_a}{2\pi} \int_{q_{e,0}}^{(+)q_{e,0}} N_{e,0}(q) \cos K(q) + \frac{4tN_a}{2\pi} \sum_{\gamma=1}^{(+)q_{c,\gamma}} N_{c,\gamma}(q) Re \sqrt{1 - [u(R_{c,\gamma}(q) - i\gamma)]^2}$$

$$+ U \left[ \frac{N_a}{4} \right] - \frac{N_{e,0}}{2} - \sum_{\gamma=1} \gamma N_{c,\gamma} ] .$$

To calculate the bands (107) – (109) we introduce in this energy Eqs. (B13), (B14), and (B21) and expand the obtained expression to first order in deviations with the result

$$\Delta E_{SO(4)}^{(1)} = \frac{N_a}{2\pi} \left\{ \int_{q_{e,0}}^{(+)q_{e,0}} \delta N_{e,0}(q)[-2t \cos K(0)(q) - U] \right\}$$

$$+ \sum_{\gamma=1}^{(+)q_{c,\gamma}} \delta N_{c,\gamma}(q)[4t Re \sqrt{1 - [u(R_{c,\gamma}(q) - i\gamma)]^2} - \gamma U]$$

$$+ 2t \int_{Q^{(-)}}^{Q^{(+)}} dk \tilde{Q}^{(1)}(k) \sin k - 4t \sum_{\gamma=1} \theta(N_{c,\gamma})(\int_{r_{\gamma,n}^{(-)}}^{r_{\gamma,n}^{(+)}} dr \tilde{Q}_{e,\gamma}^{(1)}(r) Re \left( \frac{u^2[r - i\gamma]}{1 - u^2[r - i\gamma]^2} \right) \right) ,$$

where the functions $\tilde{Q}^{(1)}(k)$ and $\tilde{Q}_{e,\gamma}^{(1)}(r)$ are defined by Eqs (B26)-(B28). As in the case of the second term of the rhs of Eq. (B22), up to second scattering order (and to second order in the density of excited pseudoparticles) the last term of Eq. (C2) does not contribute to the energy expression. Therefore, it can be omitted. The use of Eqs. (B22), (B24), (B26), and (98) in (C2) then leads after some straightforward algebra to

$$\Delta E_{SO(4)}^{(1)} = \sum_{q,\alpha,\gamma} \epsilon_{\alpha,\gamma}^{(0)}(q) \delta \tilde{N}_{\alpha,\gamma}(q) ,$$

with the bands given by Eqs. (107) – (109) in terms of the phase shifts (B30)-(B40).

To obtain the equivalent band expressions (110) – (112) requires introducing in the energy expression (C2) the functions defined by Eqs. (B26)-(B29) and performing some integrations by using symmetry properties of the obtained integral-equation kernels. After some algebra we find again the energy expression (C3) with the pseudoparticle bands given by

$$\epsilon_{c,0}^{(0)}(q) = -\frac{U}{2} - 2t \cos K(0)(q) - \int_{r_{\gamma,n}^{(-)}}^{r_{\gamma,n}^{(+)}} dr 2t \eta_{\gamma,0}(r) \frac{1}{\pi} \tan^{-1} \left( r - R_{\gamma,0}^{(0)}(q) \right) ,$$

$$\epsilon_{c,\gamma}^{(0)}(q) = -\gamma U + 4t Re \sqrt{1 - u^2[R_{c,\gamma}^{(0)}(q) - i\gamma]^2} - \int_{Q^{(-)}}^{Q^{(+)}} dk \frac{1}{\pi} \tan^{-1} \left( \frac{\sin k}{u} \frac{R_{c,\gamma}^{(0)}(q)}{1 + \gamma} \right) 2t \eta_{c,\gamma}(k) ,$$

and

$$\epsilon_{\alpha,\gamma}^{(0)}(q) = -2t \int_{Q^{(-)}}^{Q^{(+)}} dk \frac{1}{\pi} \tan^{-1} \left( \frac{\sin k}{u} \frac{R_{\alpha,\gamma}^{(0)}(q)}{1 + \gamma} \right) \sin k$$

$$+ \int_{r_{\gamma,n}^{(-)}}^{r_{\gamma,n}^{(+)}} dr 2t \eta_{\alpha,\gamma}(r) \left[ \frac{1}{2} \Theta_{1,1+\gamma} \left( r - R_{\alpha,\gamma}^{(0)}(q) \right) - \int_{r_{\gamma,n}^{(-)}}^{r_{\gamma,n}^{(+)}} dr' \frac{1}{\pi^2} \tan^{-1} \left( \frac{r' - R_{\alpha,\gamma}^{(0)}(q)}{1 + \gamma} \right) \right] ,$$

where the functions $2t \eta_{c,\gamma}(k)$ and $2t \eta_{\alpha,\gamma}(r)$ are defined by the integral equations.
\[2t \eta_{e,0}(k) = 2t \sin k + \frac{\cos k}{u} \int_{r_{e,0}^{(-)}}^{r_{e,0}^{(+)}} dr \frac{2t \eta_{e,0}(r)}{\pi \left[1 + (r - \sin k)^2\right]}, \quad (C7)\]

\[2t \eta_{e,\gamma}(r) = -4t Re\left(\frac{u^2[r - i \gamma]}{\sqrt{1 - u^2[r - i \gamma]^2}}\right) + \int_{Q^{(-)}}^{Q^{(+)}} dk \frac{2t \eta_{e,0}(k)}{\pi \gamma \left[1 + \left(\frac{\sin \gamma - r}{\gamma}\right)^2\right]}, \quad (C8)\]

and

\[2t \eta_{s,\gamma}(r) = \int_{Q^{(-)}}^{Q^{(+)}} dk \frac{2t \eta_{e,0}(k)}{\pi (1 + \gamma) \left[1 + \left(\frac{\sin \gamma - r}{1+\gamma}\right)^2\right]} - \int_{r_{s,0}^{(-)}}^{r_{s,0}^{(+)}} dr' \frac{2t \eta_{s,0}(r')}{2\pi} \Theta_{1,1+\gamma}\left(r' - r\right). \quad (C9)\]

The use of these equations and comparison of the band expressions (C4)-(C6) with the derivatives of the functions (C7)-(C9) readily leads to the simple expressions (110) – (112).

In order to derive the expression for the second-order Hamiltonian (106) and associate \( f \) functions (117) we expand the energy (C1) to second-order with the result

\[\Delta F_{SO(4)}^{(2)} = \frac{N_a}{2\pi} \int_{\mathbf{q}} d\mathbf{q} N_{c,0}(q) \frac{2t \sin K^{(0)}(q)}{2\pi \rho_{c,0}(K^{(0)}(q))} \left[Q_{c,0}^{(1)}(q) - \frac{1}{2} \sum_{\gamma=\pm 1} \left|Q_{c,\gamma}^{(1)}(j q_{F,0})\right|^2\right] \right.\]

\[\left. + \frac{2t \sin Q}{2\pi \rho_{c,0}(Q)} \sum_{j=\pm 1} \left|Q_{c,0}^{(1)}(j q_{F,0})\right|^2\right] + \sum_{\gamma=\pm 1} \frac{\theta(N_{c,0})}{2} \left[Q_{c,\gamma}^{(1)}(j q_{F,0})\right]^2 4t \quad Re\left(\frac{u^2[R_{c,0}^{(0)}(q) - i \gamma]}{\sqrt{1 - u^2[R_{c,0}^{(0)}(q) - i \gamma]^2}}\right)\]

\[+ \int_{Q^{(-)}}^{Q^{(+)}} dk \frac{2t \sin k}{2\pi \rho_{c,0}(Q)} 4t \quad Re\left(\frac{u^2[r_{c,\gamma} - i \gamma]}{\sqrt{1 - u^2[r_{c,\gamma} - i \gamma]^2}}\right)\]

\[- \sum_{\gamma=\pm 1} \theta(N_{c,0}) \int_{-\infty}^{r_{c,\gamma}^{(-)}} dr' \frac{2t \sin k}{2\pi \rho_{c,0}(Q) 4t \quad Re\left(\frac{u^2[r - i \gamma]}{\sqrt{1 - u^2[r - i \gamma]^2}}\right)\right]. \quad (C10)\]

Again, the last term of the rhs of this equation does not contribute to the energy expression to second scattering order (and to second order in the density of excited pseudoparticles). Inserting the suitable functions in the rhs of Eq. (C14), performing some integrations by using symmetry properties of the kernels of the integral equation (B27)-(B29) and (B47)-(B49), and replacing deviations by pseudomomentum normal-ordered operators (84) we find after some algebra expression (116). Note that replacing in Eq. (116) the function \( X_{\alpha,\gamma}^{(1)}(q) \) by the associate function \( Q_{\alpha,\gamma}^{(1)}(q) \) leads to the same result. By the use of Eq. (98) expression (116) can be rewritten in terms of the \( f \) functions (117) as given in the rhs of Eq. (106).

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