Continuum directed random polymers on disordered hierarchical diamond lattices

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Abstract

I discuss a family of models for a continuum directed random polymer in a disordered environment in which the polymer lives on a fractal, $D^{b,s}$, called the diamond hierarchical lattice. The diamond hierarchical lattice is a compact, self-similar metric space forming a network of intertwining pathways (continuum polymers) connecting a beginning node, $A$, to a termination node, $B$. This fractal depends on a branching parameter $b \in \{2, 3, \cdots\}$ and a segmenting number $s \in \{2, 3, \cdots\}$, and there is a canonical uniform probability measure $\mu$ on the collection of directed paths, $\Gamma^{b,s}$, for which the intersection set of two randomly chosen paths is almost surely either finite or of Hausdorff dimension $(\log s - \log b)/\log s$ when $s \geq b$. In the case $s > b$, my focus is on random measures on the set of directed paths that can be formulated as a subcritical Gaussian multiplicative chaos measure with expectation $\mu$. When normalized, this random path measure is analogous to the continuum directed random polymer (CDRP) introduced by Alberts, Khanin, Quastel [Journal of Statistical Physics 154, 305-326 (2014)], which is formally related to the stochastic heat equation for a $(1+1)$-dimension polymer.

Keywords: Gaussian multiplicative chaos, diamond hierarchical lattice, random branching graphs

1 Introduction

Alberts, Khanin, and Quastel [2, 3] introduced a continuum directed random polymer (CDRP) model for a one-dimensional Wiener motion (the polymer) over a time interval $[0, 1]$ whose law is randomly transformed through a field of impurities spread throughout the medium of the polymer. The polymer’s disordered environment is generated by a time-space Gaussian white noise $\{W(x)\}_{x \in D}$ where $D := [0, 1] \times \mathbb{R}$ (in other terms, $W$ is a $\delta$-correlated Gaussian field). For an inverse temperature parameter $\beta > 0$, the CDRP is a random probability measure $Q^W_\beta$ on the set of trajectories $\Gamma := C([0, 1])$ that is formally expressed as

$$Q^W_\beta(dp) = \frac{1}{M(\Gamma)} M(dp) \quad \text{for} \quad M(dp) = e^{\beta W_p - \frac{\beta^2}{2} E[W_p^2]} \mu(dp), \quad (1.1)$$

where $\mu$ refers to the standard Wiener measure on $\Gamma$ and $\{W_p\}_{p \in \Gamma}$ is a Gaussian field formally defined by integrating the white noise over a Brownian trajectory: $W_p = \int_0^1 W(p(r)) dr$. The random measure $M \equiv M(W)$ is a function of the field such that $E[M] = \mu$ and yet $M$ is a.s. singular with respect to $\mu$.

The rigorous mathematical meaning of the random measure $M$ in (1.1) requires special consideration since exponentials of the field $W_p$ do not have an immediately clear meaning, and, indeed, if the measure $M$ is singular with respect to $\mu$ the expression $\exp\{\beta W_p - (\beta^2/2)E[W_p^2]\}$ cannot define a Radon-Nikodym derivative $dM/d\mu$ anyway. The construction approach of $M$ in [3] involves an analysis of the finite-dimensional distributions through Wiener chaos expansions. Another point-of-view

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is that the random measure $M$ has the form of a Gaussian multiplicative chaos (GMC) measure over the Gaussian field $\{W_p\}_{p \in \Gamma}$. GMC theory began with an article by Kahane [21] and much of the progress on this topic has been motivated by the demands of quantum gravity theory [4, 11, 12, 25] although GMC theory arises in many other fields, including random matrix theory [29] and number theory [27]. A GMC is classified as subcritical or critical, respectively, depending on whether the expectation measure, $\mathbb{E}[M]$, is $\sigma$-finite or not. Because of its relevance to quantum gravity, the relatively unwieldy case of critical GMC has attracted the most attention, with recent results in [5, 20, 24]. The random measure $M$ in (1.1) is subcritical since $\mathbb{E}[M] = \mu$ is a probability measure. Shamov [28] has formulated subcritical GMC measure theory in a particularly complete and accessible way.

In this article, I will study a GMC measure analogous to (1.1) for a CDRP living on a fractal $D^{b,s}$ referred to as the diamond hierarchical lattice. Given a branching parameter $b \in \{2, 3, \cdots \}$ and a segmenting parameter $s \in \{2, 3, \cdots \}$, the diamond hierarchical lattice is a compact metric space that embeds $bs$ shrunken copies of itself, which are arranged through $b$ branches that each have $s$ copies running in series; see the construction outline below in (A) and (B) of Section 1.1. Diamond hierarchical lattices provide a useful setting for formulating toy statistical mechanical models; for example, [14, 15, 16, 17, 18, 22, 26]. Lacoin and Moreno [23] studied (discrete) directed polymers on disordered diamond hierarchical lattices, classifying the disorder behavior based on the cases $b < s$, $b = s$, and $b > s$, which are combinatorially analogous, respectively, to the $d = 1$, $d = 2$, and $d > 3$ cases of directed polymers on the $(1 + d)$-rectangular lattice. In [1], we studied a functional limit theorem for the partition function of the $b < s$ diamond lattice polymer in a scaling limit in which the temperature grows as a power law of the length of the polymer. This limit result is analogous to the intermediate disorder regime in [2] for directed polymers on the $(1 + 1)$-rectangular lattice. My focus here will be developing theory for a CDRP corresponding to the limiting partition function obtained in [1].

A similar CDRP model on the $b = s$ diamond lattice, if it exists, would likely require a different approach for its construction; see [6] for computations relevant to the continuum limit of discrete polymers. It is interesting to compare this question with results [7, 8] by Caravenna, Sun, and Zygouras on scaling limits for $(1 + 2)$-rectangular lattices polymers.

### 1.1 Overview of the continuum directed random polymer on the diamond lattice

In this section I will sketch the construction of the continuum directed random polymer on the diamond hierarchical lattice and explore some of its properties. I discuss the diamond lattice fractal and its relevant substructures in more detail in Section 2. Proof of propositions are placed in Section 3.

**A.** The sequence of diamond hierarchical graphs

For a branching number $b \in \{2, 3, 4, \cdots \}$ and a segmenting number $s \in \{2, 3, 4, \cdots \}$, the first diamond graph $D^{b,s}_1$ is defined by $b$ parallel branches connecting two root nodes $A$ and $B$ wherein each branch is formed by $s$ bonds running in series. The graphs $D^{b,s}_{n+1}$, $n \geq 1$ are then constructed inductively by replacing each bond on $D^{b,s}_1$ by a nested copy of $D^{b,s}_n$; see the illustration below of the $(b, s) = (2, 3)$ case.

![Diagram](image-url)
The set of bonds (edges) on the graph $D_{n}^{b,s}$ is denoted by $E_{n}^{b,s}$. A directed path is a one-to-one function $p: \{1, \cdots, s^n\} \to E_{n}^{b,s}$ such that the bonds $p(j)$, $p(j+1)$ are adjacent for $1 \leq j \leq s^n - 1$ and the bonds $p(1)$ and $p(s^n)$ connect to $A$ and $B$, respectively. I will use the following notations:

\[
\begin{align*}
V_{n}^{b,s} &= \text{Set of vertex points on } D_{n}^{b,s} \\
E_{n}^{b,s} &= \text{Set of bonds on the graph } D_{n}^{b,s} \\
\Gamma_{n}^{b,s} &= \text{Set of directed paths on } D_{n}^{b,s} \\
[p]_{n} &= \text{The path in } \Gamma_{n}^{b,s} \text{ determined by } p \in \Gamma_{N}^{b,s} \text{ for } N > n
\end{align*}
\]

**Remark 1.1.** The hierarchical structure of the sequence of diamond graphs implies that $V_{n}^{b,s}$ is canonically embedded in $V_{N}^{b,s}$ for $N > n$. The vertices in $V_{n}^{b,s} \setminus V_{n-1}^{b,s}$ are referred to as the $n^{th}$ generation vertices. In the same vein, $E_{n}^{b,s}$ and $\Gamma_{n}^{b,s}$ define equivalence relations on $E_{N}^{b,s}$ and $\Gamma_{N}^{b,s}$, respectively. For instance, $p, q \in \Gamma_{N}^{b,s}$ are equivalent up to generation $n$ if $[p]_{n} = [q]_{n}$.

**Remark 1.2.** $V_{n}^{b,s}$ is a countable, dense subset of $D_{n}^{b,s}$.

**Remark 1.3.** In analogy to Remark 1.1, $V_{n}^{b,s}$ is canonically identifiable with $\cup_{n=1}^{\infty} V_{n}^{b,s}$. Also, $E_{n}^{b,s}$ and $\Gamma_{n}^{b,s}$ define equivalence relations on $E_{n}^{b,s}$ and $\Gamma_{n}^{b,s}$ for each $n \in \mathbb{N}$.

**Remark 1.4.** The measure $\nu$ is defined such that $\nu(V_{n}^{b,s}) = 0$ and, under the interpretation of Remark 1.3, $\nu(e) = 1/|E_{n}^{b,s}|$ for each $n \in \mathbb{N}$ and $e \in E_{n}^{b,s}$. Similarly $\mu$ is defined so that $\mu(p) = 1/|\Gamma_{n}^{b,s}|$ for any $p \in \Gamma_{n}^{b,s}$.

**Remark 1.5.** Let $(\Gamma_{i,j}^{b,s}, \mu^{(i,j)})$ be copies of $(\Gamma_{N}^{b,s}, \mu)$ corresponding to the embedded subcopies, $D_{i,j}^{b,s}$, of $D_{b,s}$. The path space $(\Gamma_{b,s}, \mu)$ can be decomposed as

\[
\Gamma_{b,s} = \bigcup_{i=1}^{b} \bigtimes_{j=1}^{s} \Gamma_{i,j}^{b,s} \quad \text{and} \quad \mu = \frac{1}{b} \sum_{i=1}^{b} \prod_{j=1}^{s} \mu^{(i,j)} \quad (1.2)
\]

by way of $s$-fold concatenation of the paths.

**Proposition 1.6.** Fix some $p \in \Gamma_{b,s}$ and let $q \in \Gamma_{b,s}$ be chosen uniformly at random, i.e., according to the measure $\mu(dq)$. Define the set of intersection times $I_{p,q} = \{ r \in [0,1] | p(r) = q(r) \}$, and define $N_{p,q}^{(n)} := \sum_{k=1}^{s^n} 1_{[p]_{n}(k) = [q]_{n}(k)}$, in other terms, as the number of bonds shared by $[p]_{n}$, $[q]_{n} \in \Gamma_{n}^{b,s}$.

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(i). As \( n \to \infty \) the sequence of random variables \( T_{p,q}^{(n)} = (\frac{1}{sn})^b N_{p,q}^{(n)} \) converges almost surely to a limit \( T_{p,q} \). The moment generating function \( \varphi_n(t) = \mathbb{E}\exp\{tT_{p,q}^{(n)}\} \) converges pointwise to the moment generating function of \( T_{p,q} \), and, in particular, the second moment of \( T_{p,q}^{(n)} \) converges to the second moment of \( T_{p,q} \).

(ii). \( I_{p,q} \) is a finite set with probability \( 1 - p_{b,s} \) for \( p_{b,s} \in (0,1) \) satisfying \( p_{b,s} = \frac{1}{2}[1 - (1 - p_{b,s})^s] \). In this case, the intersections occur only at vertex points.

(iii). In the event that \( I_{p,q} \) is infinite, the Hausdorff dimension of \( I_{p,q} \) is almost surely \( h := \frac{\log s - \log b}{\log s} \).

**Definition 1.7.** The intersection time \( T_{p,q} \) of two paths \( p, q \in \Gamma_{b,s} \) is defined as

\[
T_{p,q} := \lim_{n \to \infty} \left( \frac{1}{sn} \right)^h N_{p,q}^{(n)} .
\]

**Remark 1.8.** The intersection time satisfies the formal identity

\[
\int_0^1 \int_0^1 \delta_D(p(r), q(t)) \, dr \, dt = T_{p,q} ,
\]

where \( \delta_D \) is the \( \delta \)-distribution on \( D_{b,s} \) satisfying \( f(x) = \int_{D_{b,s}} \delta_D(x,y) f(y) \nu(y) \) for a test function \( f : D_{b,s} \to \mathbb{R} \). The above identity can be understood in terms of the discrete graphs \( D_n^{b,s} \) for which any two directed paths \( p, q : \{1, \cdots, s^n\} \to E_n^{b,s} \) satisfy

\[
\frac{1}{sn} \sum_{1 \leq j \leq s^n} \frac{1}{sn} \sum_{1 \leq k \leq s^n} 1_{p_j = q_k} \cdot (\frac{1}{sn})^h \sum_{1 \leq j \leq s^n} 1_{p_j = q_j} = (\frac{1}{sn})^h N_{p,q}^{(n)} .
\]

(C). A Gaussian field on directed paths

Let \( W \) denote a Gaussian white noise on \( (D_{b,s}, \nu) \) defined within some probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), i.e., a linear map from \( \mathcal{H} := L^2(D_{b,s}, \nu) \) into a Gaussian subspace of \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \) such that

\[
\psi \in \mathcal{H} \quad \mapsto \quad W(\psi) \sim \mathcal{N}(0, \|\psi\|^2_{\mathcal{H}}) .
\]

I also use the alternative notations

\[
W(\psi) \equiv (W, \psi) \equiv \int_{D_{b,s}} W(x) \psi(x) \nu(dx) ,
\]

where the field \( W(x) \), \( x \in D_{b,s} \) formally satisfies the \( \delta \)-correlation \( \mathbb{E}[W(x)W(y)] = \delta_D(x,y) \).

Next I discuss a field \( W_p \), \( p \in \Gamma_{b,s} \) formally defined by integrating the white noise along paths:

\[
W_p = \int_0^1 W(p(r)) \, dr .
\]

The kernel \( K_{\Gamma}(p,q) \) is given by the intersection time by the identity [1.3]

\[
K_{\Gamma}(p,q) = \mathbb{E}[W_p W_q] = \int_0^1 \int_0^1 \delta_D(p(r), q(t)) \, dr \, dt = T_{p,q} .
\]

Define \( Y : \mathcal{H} \to L^2(\Gamma_{b,s}, \mu) \) as

\[
(Y\psi)(p) = \int_0^1 \psi(p(r)) \, dr .
\]
Theorem 1.13. Let $\beta Y$ be a Gaussian multiplicative chaos on paths.

Remark 1.10. I prove Proposition 1.9 by constructing a sequence of GMC measures $\{M^{(n)}_\beta\}_{n \in \mathbb{N}}$ that form a martingale and have well-defined Radon-Nikodým derivatives $dM^{(n)}_\beta/d\mu$.

Remark 1.11. I prove Proposition 1.9 by constructing a sequence of GMC measures $\{M^{(n)}_\beta\}_{n \in \mathbb{N}}$ that form a martingale and have well-defined Radon-Nikodým derivatives $dM^{(n)}_\beta/d\mu$.

By \cite{28} the existence and uniqueness of the subcritical GMC measure $M_\beta$ in Proposition 1.9 is equivalent to the operator $\beta Y$ defining a random shift of the field $W$. In other terms, the law $\mathbb{P}_\beta$ determined by $L_{\mathbb{P}_\beta}[W] := \mathbb{P}_{\beta \times \mu}[W + \beta Y]$ is absolutely continuous with respect to the law $\mathbb{P}$.

Theorem 1.12. $\beta Y$ defines a random shift of the field $W$.

Theorem 1.13. For $\beta > 0$ let the random measure $M_\beta(dp)$ be defined as in Proposition 1.9.

(i) $M_\beta$ is a.s. mutually singular to $\mu$.

(ii) The product measure $M_\beta \times M_\beta$ is a.s. supported on pairs $(p,q) \in \Gamma^{b,s} \times \Gamma^{b,s}$ such that the intersection set $I_{p,q} = \{r \in [0,1] | p(r) = q(r)\}$ is either finite or has Hausdorff dimension $\delta = (\log s - \log b)/\log s$. 

To see that $Y$ is a bounded operator, notice that by Jensen’s inequality

$$
\int_{\Gamma^{b,s}} |(Y\psi)(p)|^2 \mu(dp) \leq \int_{\Gamma^{b,s}} \int_0^1 |\psi(p(r))|^2 dr \mu(dp) = \int_{\Gamma^{b,s}} |\psi(x)|^2 \nu(dx) = \|\psi\|_{\mathcal{H}}^2.
$$

In particular, $Y$ is continuous when the topology of the codomain is identified with $L^0(\Gamma^{b,s}, \mu)$, i.e., convergence in measure with respect to $\mu$. In the terminology of \cite{28}, a continuous function $Y$ from $\mathcal{H}$ to $L^0(\Gamma^{b,s}, \mu)$ is referred to as a generalized $\mathcal{H}$-valued function over the measure space $(\Gamma^{b,s}, \mu)$. The pair $(W,Y)$ encodes the Gaussian field $W$ on $\Gamma^{b,s}$ by defining

$$
\int_{\Gamma^{b,s}} W_p f(p) \mu(dp) =: \langle W, Y^* f \rangle \text{ for a test function } f \in L^2(\Gamma^{b,s}, \mu).
$$

The adjoint $Y^*: L^2(\Gamma^{b,s}, \mu) \to \mathcal{H}$ can be expressed in the form $(Y^*f)(x) = \int_{\Gamma^{b,s}} f(p) \int_0^1 \delta_D(p(r), x)$. I will use the notation $(Y\psi)(p) \equiv \langle Y_p, \psi \rangle$ and summarize (1.5) as $W_p = (Y_p, W)$.

**Proposition 1.9.** There exists a unique random measure, $M_\beta(dp)$, on $(\Gamma^{b,s}, \mu)$ satisfying the properties (I)-(III) below.

(I). $E[M_\beta] = \mu$

(II). $M_\beta$ is adapted to the white noise $W$. Thus I can write $M_\beta(dp) \equiv M_\beta(W, dp)$.

(III). For $\psi \in \mathcal{H}$ and a.e. realization of the field $W$,

$$
M_\beta(W + \psi, dp) = e^{\beta(Y\psi)(p)} M_\beta(W, dp).
$$

**Remark 1.10.** $M_\beta$ is the subcritical Gaussian multiplicative chaos on $(\Gamma^{b,s}, \mu)$ over the field $(W,Y)$ with expectation $\mu$.

**Remark 1.11.** I prove Proposition 1.9 by constructing a sequence of GMC measures $\{M^{(n)}_\beta\}_{n \in \mathbb{N}}$ that form a martingale and have well-defined Radon-Nikodým derivatives $dM^{(n)}_\beta/d\mu$.
(iii). Let $(\Gamma^{b,s}_{i,j}, M^{(i,j)}_\beta)$ be independent copies of $(\Gamma^{b,s}, M_\beta)$ corresponding to the first-generation embedded copies, $D^{b,s}_{i,j}$, of $D^{b,s}$. Then there is equality in distribution of random measures

$$M^{\sqrt{\beta}} = \frac{1}{b} \sum_{i=1}^{b} \prod_{j=1}^{s} M^{(i,j)}_\beta$$

under the identification $\Gamma^{b,s} = \bigcup_{i=1}^{b} \times \Gamma^{b,s}_{i,j}$.

**Remark 1.14.** Part (ii) of the Theorem 1.13 implies that the random measure $M_\beta$ almost surely has no atoms.

The next theorem states two strong disorder properties in the $\beta \gg 1$ regime. Analogous results were obtained in [23] for discrete polymers on diamond graphs.

**Theorem 1.15.** Let the random measure $M_\beta(dp)$ be defined as in Proposition 1.9 and define the random probability measure $Q_\beta(dp) = M_\beta(dp)/M_\beta(\Gamma^{b,s})$. As $\beta \to \infty$,

(i). the random variable $M_\beta(\Gamma^{b,s})$ converges in probability to 0, and

(ii). the random variable $\max_{p \in \Gamma^{b,s}} Q_\beta(p)$ converges in probability to 1.

**Remark 1.16.** In particular, part (ii) implies that when $\beta \gg 1$ most of the weight of the measure $M_\beta(dp)$ is concentrated on a single course-grained path $p \in \Gamma^{b,s}$.

(E). A Gaussian multiplicative chaos martingale

I will expand on the structure and properties of the map $Y : \mathcal{H} \to L^2(\Gamma^{b,s}, \mu)$.

**Proposition 1.17.** The linear operator $Y$ is compact and has the following properties:

(i). $Y$ can be decomposed as $Y = UD$ where $U : \mathcal{H} \to L^2(\Gamma^{b,s}, \mu)$ is an isometry and $D : \mathcal{H} \to \mathcal{H}$ is a self-adjoint operator with eigenvalues $\lambda_n = s^{-n+1}$ for $n \in \mathbb{N} \cup \{\infty\}$.

(ii). The null space of $Y$, which I denote by $\mathcal{H}_\infty$, is infinite dimensional. For $2 \leq n < \infty$, the eigenspace $\mathcal{H}_n$ corresponding to the eigenvalue $\lambda_n$ has dimension $(bs)^{n-1}(b-1)$. The eigenspace corresponding to $\lambda_0 = 1$ has dimension $b$, which I decompose into the one-dimensional space of constant functions, $\mathcal{H}_0$, and an orthogonal complement, $\mathcal{H}_1$.

(iii). For $1 \leq n < \infty$ the space $\mathcal{H}_n$ has an orthogonal basis $f_{(e,\ell)}$ labeled by $(e, \ell) \in E^{b,s}_{n-1} \times \{1, \cdots, b-1\}$, where the function $f_{(e,\ell)}$ is supported on $e \subset D^{b,s}$.

(iv). $YY^* : L^2(\Gamma^{b,s}, \mu) \to L^2(\Gamma^{b,s}, \mu)$ is a self-adjoint operator with eigenvalues $\tilde{\lambda}_n = s^{-n+1}$ for $n \in \mathbb{N} \cup \{\infty\}$ with eigenspaces having dimension $b$ for $n = 1$ and $(bs)^{n-1}(b-1)$ for $n \geq 2$. Hence $YY^*$ has Hilbert-Schmidt norm $\|YY^*\|_{HS} = (b^{n-1}+1)^{1/2}$ but is not traceclass.

**Remark 1.18.** $\oplus_{k=0}^{\infty} \mathcal{H}_k$ is the orthogonal complement to the space of all $\psi \in \mathcal{H} = L^2(D^{b,s}, \nu)$ such that $\int_0^1 \psi(p(r)) \, dr$ is zero for every path $p \in \Gamma^{b,s}$.

**Definition 1.19.** Define $Y^{(n)} : \mathcal{H} \to L^2(\Gamma^{b,s}, \mu)$ to act as $Y^{(n)} \psi = E[Y \psi \mid F_n]$, where $F_n$ is the $\sigma$-algebra on $\Gamma^{b,s}$ generated the map $p \mapsto [p]_n$.

**Remark 1.20.** $Y^{(n)}$ can also be written in the following forms:

- $(Y^{(n)} \psi)(p) = \int_{\tilde{p} \in [p]_n} (Y \psi)(\tilde{p}) \mu(d\tilde{p})$, in other terms, the average of $(Y \psi)(\tilde{p})$ over all $\tilde{p} \in \Gamma^{b,s}$ in the same generation-$n$ equivalence class of $p$. 

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• \( (Y^{(n)} \psi)(p) = \langle Y_p^{(n)}, \psi \rangle \) for \( Y_p^{(n)} \in H \) defined as \( Y_p^{(n)} := \chi_{T_p^{(n)}} / \nu(T_p^{(n)}) \), where \( T_p^{(n)} = \cup_{k=1}^n [p]_n(k) \), i.e., the generation-\( n \) course-grained trace of the path \( p \) through the space \( D^{b,s} \).

Proposition 1.21. The maps \( Y^{(n)} : H \rightarrow L^2(\Gamma^{b,s}, \mu) \) satisfy the properties below.

(i). Let \( P_n : H \rightarrow H \) be the orthogonal projection onto \( \oplus_{k=0}^n \mathcal{H}_k \) for \( \mathcal{H}_k \) defined as in part (ii) of Proposition 1.17. For any \( \psi \in H \),

\[
Y^{(n)} \psi = Y P_n \psi.
\]

(ii). As \( n \rightarrow \infty \), the map \( Y^{(n)} \) converges in operator norm to \( Y \).

(iii). As \( n \rightarrow \infty \), \( Y^{(n)}(Y^{(n)})^* \) converges in Hilbert-Schmidt norm to \( YY^* \), which has integral kernel \( K_\Gamma(p,q) = T_{p,q} \), i.e., the intersection time of the paths.

(iv). For any \( k \in \mathbb{N} \) and \( p \in \Gamma^{b,s} \), \( Y_p^{(k)} - Y_p^{(k-1)} \in \mathcal{H}_k \). In particular, the following sequence of vectors in \( H \) are orthogonal:

\[
Y_p^{(0)}, Y_p^{(1)} - Y_p^{(0)}, Y_p^{(2)} - Y_p^{(1)}, \ldots
\]

Proposition 1.22. Define \( \mathcal{F}_n \) to be the \( \sigma \)-algebra on \( \Omega \) generated by the field variables \( \langle W, \psi \rangle \) for \( \psi \in \oplus_{k=0}^n \mathcal{H}_k \). Let \( M^{(n)}_\beta \) be the GMC measure over the finite-dimensional field \( (W, \beta Y^{(n)}) \), i.e., with Radon-Nikodym derivative

\[
\frac{dM^{(n)}_\beta}{d\mu} = \exp \left\{ \beta \langle W, Y^{(n)} \rangle - \frac{\beta^2}{2} \| Y^{(n)} \|^2_{L^2} \right\}.
\]

The sequence of measures \( \{M^{(n)}_\beta\}_{n \in \mathbb{N}} \) forms a martingale with respect to the filtration \( \mathcal{F}_n \) and a.s. converges vaguely to the GMC measure \( M_\beta \).

(F). Chaos expansion construction of the GMC measure

The Gaussian multiplicative chaos \( M_\beta \) can also be constructed through chaos expansions analogous to those in [3]. The chaos decomposition generated by the field \( W \) is

\[
L^2(\Omega, \mathcal{F}(W), \mathbb{P}) = \bigoplus_{k=0}^\infty \mathcal{H}_W^{(k)},
\]

where \( \mathcal{H}_W^{(k)} \) is the orthogonal complement of the set \( \overline{P}_{k-1}(H) \) within \( \overline{P}_k(H) \) for

\[
\mathcal{P}_k(H) := \{ p(\langle W, \psi_1 \rangle, \ldots, \langle W, \psi_k \rangle) \mid p : \mathbb{R}^k \rightarrow \mathbb{R} \text{ is a degree-}k \text{ polynomial and } \psi_1, \ldots, \psi_k \in \mathcal{H} \}.
\]

There is a canonical isometry between \( \mathcal{H}_W^{(k)} \) and the \( k \)-fold symmetric tensor, \( \mathcal{H}^{\otimes k} \), of \( H \). This isometry can be expressed as a map from symmetric functions \( f(x_1, \ldots, x_k) \) in \( L^2((D^{b,s})^k, 1/m^k) \) to elements of \( \mathcal{H}_W^{(k)} \) expressed as stochastic integrals:

\[
\frac{1}{k!} \int_{(D^{b,s})^k} f(x_1, \ldots, x_k) W(x_1) \cdots W(x_k) \nu(dx_1) \cdots \nu(dx_k).
\]

Intuitively, the integral above is to be understood as over points \( (x_1, \ldots, x_k) \in (D^{b,s})^k \) for which the components \( x_j \) are distinct. See [19] Section 7.2 for the general theory of stochastic integrals.

Let \( S \) be a finite subset of \( E^{b,s} \) and define \( \Gamma^{b,s}_S \) as the collection of paths \( p \in \Gamma^{b,s} \) such that \( S \subset \text{Range}(p) \). If \( \Gamma^{b,s}_S \) is nonempty, define \( \mu_S \) as the probability measure uniformly distributed over paths in \( \Gamma^{b,s}_S \) (and thus supported on \( \Gamma^{b,s}_S \)). If \( \Gamma^{b,s}_S = \emptyset \), i.e., there is no path containing all the points in \( S \), then \( \mu_S \) is defined as zero.
Definition 1.23. For a Borel set $A \subset \Gamma^{b,s}$, define $\rho_n(x_1, \ldots, x_k; A)$ as a map from $(D^{b,s})^k$ to $[0, \infty)$ with

$$
\rho_n(x_1, \ldots, x_k; A) := \begin{cases} 
\beta^n(x_1, \ldots, x_k) \mu_{\{x_1, \ldots, x_k\}}(A) & x_1, \ldots, x_k \in E^{b,s} \\
0 & \text{otherwise,}
\end{cases}
$$

where $\gamma(S)$ is defined for a finite subset of $E^{b,s}$ as

$$
\gamma(S) := \sum_{k=0}^{\infty} \left( |S| - \left| \{e \in E^{b,s}_k \mid e \cap S \neq \emptyset \} \right| \right).
$$

In the above formula for $\gamma(S)$, elements of $E^{b,s}_k$ are to be understood as subsets of $E^{b,s}$, and the $k = 0$ term of the sum is always interpreted as $|S| - 1$.

Remark 1.24. The term $\left| \{e \in E^{b,s}_k \mid e \cap S \neq \emptyset \} \right|$ counts the number of distinct equivalence classes from $E^{b,s}_k$ corresponding to elements in $S$.

Theorem 1.25. For any Borel set $A \subset D^{b,s}$, the random variable $M_\beta(W, A)$ is equal to the chaos expansion

$$
\mu(A) + \sum_{k=1}^{\infty} \frac{\beta^{n_k}}{k!} \int_{(D^{b,s})^k} \rho_k(x_1, \ldots, x_k; A) W(x_1) \cdots W(x_k) \nu(dx_1) \cdots \nu(dx_k).
$$

The sequence of symmetric functions $\{\rho_k(x_1, \ldots, x_k; A)\}_{k \in \mathbb{N}}$ satisfy

(i). $\int_{D^{b,s}} \rho_k(x_1, \ldots, x_k; A) \nu(dx_k) = \rho_{k-1}(x_1, \ldots, x_{k-1}; A)$ and

(ii). $\int_{(D^{b,s})^k} \rho_k(x_1, \ldots, x_k; A) \nu(dx_1) \cdots \nu(dx_k) = \mu(A)$.

(G). Scaling limits from non Gaussian variables

For each $n \in \mathbb{N}$ let $\{\omega^{(n)}_a\}_{a \in E^{b,s}_n}$ be a family of i.i.d. random variables indexed by the edge set of the diamond lattice $D^{b,s}$. Assume that the variables have mean zero, variance one, and finite exponential moments: $E[\exp(\beta \omega^{(n)}_a)] < \infty$ for $\beta \in \mathbb{R}$. Define a random measure $M^{(n)}_{\beta}$ on $\Gamma^{b,s}_n$ as follows:

$$
M^{(n)}_{\beta}(A) = \frac{1}{|\Gamma_n^{b,s}|} \sum_{p \in A} \prod_{k=1}^{s^n} \left[ \frac{\exp \left\{ \beta \omega^{(n)}_{p(k)} \right\}}{\exp \left\{ \beta \omega^{(n)}_{p(k)} \right\}} \right] \quad \text{for } A \subset \Gamma^{b,s}_n. \tag{1.10}
$$

The theorem below follows as a consequence of Theorem 4.6 of [1].

Theorem 1.26. Fix $N \in \mathbb{N}$ and let subsets of $\Gamma_N^{b,s}$ be identified with the canonically corresponding subsets of $\Gamma^{b,s}$ and of $\Gamma_N^{b,s}$ for $n > N$. For $\beta := (\beta(b/s))^{n/2}$, the family of random variables $M^{(n)}_{\beta_n}(A)$ labeled by $A \subset \Gamma^{b,s}_N$ has joint convergence in law as $n \to \infty$ given by

$$
\{M^{(n)}_{\beta_n}(A)\}_{A \subset \Gamma^{b,s}_N} \Rightarrow \{M_\beta(A)\}_{A \subset \Gamma^{b,s}_N}. \tag{1.11}
$$

2 Diamond hierarchical lattice

In this section I will provide a path-based construction of the diamond hierarchical lattice as a compact metric space. The proofs of propositions in this section are in the appendix.
2.1 Construction of the diamond lattice as a metric space

The hierarchical formulation of the diamond graph $D_{b,s}^n$ in terms of $b \cdot s$ embedded copies of $D_{n-1}^{b,s}$ carries with it a canonical one-to-one correspondence between $\{\{1, \cdots, b\} \times \{1, \cdots, s\}\}^n$ and the set of bonds, $E_{n}^{b,s}$. I will construct the diamond hierarchical lattice, $D_{b,s}^n$, as an equivalence class on the set of sequences $D_{b,s}^n := \{\{1, \cdots, b\} \times \{1, \cdots, s\}\}^\infty$ determined by a semi-metric $d_D : D_{b,s}^n \times D_{b,s}^n \rightarrow [0,1]$ defined below.

Let $\tilde{\pi} : D_{b,s}^n \rightarrow [0,1]$ be the “projective” map sending a sequence $x = \{(b_{k}^{x}, s_{k}^{x})\}_{k \in \mathbb{N}}$ to

$$\tilde{\pi}(x) := \sum_{k=1}^{\infty} \frac{s_{k}^{x} - 1}{s_{k}^{x}}.$$ 

Of course, the right side above is the base $s$ generalized decimal expansion of the number $\tilde{\pi}(x) \in [0,1]$ having $k^{th}$ digit $s_{k}^{x} - 1$. The root vertices of the continuum lattice will be identified with the sets $A := \{x \in D_{b,s}^n | \tilde{\pi}(x) = 0\}$ and $B := \{x \in D_{b,s}^n | \tilde{\pi}(x) = 1\}$.

For two points $x = \{(b_{k}^{x}, s_{k}^{x})\}_{k \in \mathbb{N}}$ and $y = \{(b_{k}^{y}, s_{k}^{y})\}_{k \in \mathbb{N}}$ in $D_{b,s}^n$, I write $x \downarrow y$ if $x$ or $y$ is contained in $A \cup B$ or for some $n \in \mathbb{N}$

$$(b_{k}^{x}, s_{k}^{x}) = (b_{k}^{y}, s_{k}^{y}) \quad \text{for} \quad 1 \leq k < n-1 \quad \text{and} \quad b_{n}^{x} = b_{n}^{y} \quad \text{but} \quad s_{n}^{x} \neq s_{n}^{y}.$$ 

In other terms the sequence of pairs defining $x$ and $y$ disagree for the first time at an $s$-component value. Intuitively, this means that there exists a directed path going through both $x$ and $y$. We then define the semi-metric $d_D$ on $D_{b,s}^n$ as the traveling distance

$$d_D(x,y) := \begin{cases} |\tilde{\pi}(x) - \tilde{\pi}(y)| & \text{if } x \downarrow y, \\ \inf_{z \in D_{b,s}^n, z \downarrow x, z \downarrow y} (d_D(x,z) + d_D(z,y)) & \text{otherwise}. \end{cases}$$

The semi-metric is bounded by 1 since by choosing appropriate $z \in A$ or $z \in B$ in the infimum above, I can conclude that $d_D(x,y) \leq \min (\tilde{\pi}(x) + \tilde{\pi}(y), 2 - \tilde{\pi}(x) - \tilde{\pi}(y)).$

**Definition 2.1.** The diamond hierarchical lattice is defined as $$D_{b,s}^n := D_{b,s}^n / \{(x,y) \in D_{b,s}^n | d_D(x,y) = 0\}.$$ 

In future, I will treat the metric $d_D(x,y)$ and the map $\tilde{\pi}$ as acting on $D_{b,s}^n$.

**Remark 2.2.** The vertex set, $V_{b,s}^n$, on the diamond graph $D_{b,s}^n$ is canonically embedded on $D_{b,s}^n$; see Appendix A. The representation of elements in $D_{b,s}^n$ by sequences $\{(b_{j}, s_{j})\}_{j \in \mathbb{N}}$ is unique except for the countable collection of vertices $V_{b,s}^n := \bigcup_{n} V_{b,s}^n$.

**Remark 2.3.** The self-similar structure of the fractal $D_{b,s}^n$ can be understood through a family of contractive shift maps $S_{i,j} : D_{b,s}^n \rightarrow D_{b,s}^n$ for $(i,j) \in \{1, \cdots, b\} \times \{1, \cdots, s\}$ that send $x = \{(b_{k}^{x}, s_{k}^{x})\}_{k \in \mathbb{N}}$ to $S_{i,j}(x) = y = \{(b_{k}^{y}, s_{k}^{y})\}_{k \in \mathbb{N}}$ with $(b_{k}^{y}, s_{k}^{y}) = (i,j)$ and $(b_{k}^{y}, s_{k}^{y}) = (b_{k-1}, s_{k-1})$ for $k \geq 2$. The $S_{i,j}$’s are well-defined as functions on $D_{b,s}^n$, and map $D_{b,s}^n$ onto the shrunken subcopies $D_{b,s}^{n}$ with

$$d_D(S_{i,j}(x), S_{i,j}(y)) = \frac{1}{s} d_D(x,y), \quad x,y \in D_{b,s}^n.$$

**Proposition 2.4.** $(D_{b,s}^n, d_D)$ is a compact metric space with Hausdorff dimension $1 + \frac{\log b}{\log s}$. The vertex set $V_{b,s}^n$ is dense in $D_{b,s}^n$.

The import of the next proposition is that a probability measure $\nu$ can be placed on $D_{b,s}^n$ such that subsets identifiable with elements of $E_{b,s}^n$ are assigned measure $(bs)^{-n}$. 

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Proposition 2.5. Let $\mathcal{B}_D$ be the Borel $\sigma$-algebra on $\mathbb{D}^{b,s}$ generated by the metric $d_D$. There is a unique measure $\nu$ on $(\mathbb{D}^{b,s}, \mathcal{B}_D)$ such that $\nu(\mathbb{D}^{b,s}) = 0$ and for $(b_j, s_j) \in \{1, \ldots, b\} \times \{1, \ldots, s\}$ the cylinder sets
\[
C(b_1, s_1) \times \cdots \times (b_n, s_n) := \left\{ x \in \mathbb{D}^{b,s} \mid x = \{(b_j^x, s_j^x)\}_{j \in \mathbb{N}} \text{ with } b_j^x = b_j \text{ and } s_j^x = s_j \text{ for } 1 \leq j \leq n \right\}
\]
(identified with elements in $\mathbb{D}^{b,s}_n$) have measure $\nu(C(b_1, s_1) \times \cdots \times (b_n, s_n)) = |\mathbb{D}^{b,s}_n|^{-1} = (bs)^{-n}$.

2.2 Directed paths on the diamond hierarchical lattice

I define a directed path on $\mathbb{D}^{b,s}$ to be a continuous function $p : [0, 1] \to \mathbb{D}^{b,s}$ such that $\hat{\pi}(p(r)) = r$ for all $r \in [0, 1]$. I will use the uniform metric on the set of directed paths:

\[
d_F(p_1, p_2) = \max_{0 \leq r \leq 1} d_D(p_1(r), p_2(r)) \quad p_1, p_2 \in \mathbb{D}^{b,s}.
\]

Remark 2.6. Note that $d_F(p_1, p_2) = s^{-(n-1)}$, where $n \in \mathbb{N}$ is the first generation such that there is a vertex $v \in V^{b,s}_n$ in the range of $p_1$ but not of $p_2$.

Remark 2.7. $\mathbb{D}^{b,s}_n$ is canonically identified with an equivalence relation of $\mathbb{D}^{b,s}$ in which $q \equiv_n p$ iff $[p]_n = [q]_n$, or, equivalently, $d_F(p, q) \leq s^{-n}$.

Remark 2.8. The metric $d_D$ on $\mathbb{D}^{b,s}$ can be reformulated in terms of the space of directed paths, $\mathbb{D}^{b,s}$, as

\[
d_D(x, y) = \inf_{p, q \in \mathbb{D}^{b,s}, x \in \mathbb{D}^{b,s}, z \in \text{Range}(p) \cap \text{Range}(q)} \left( |\hat{\pi}(x) - \hat{\pi}(z)| + |\hat{\pi}(z) - \hat{\pi}(y)| \right).
\]

The uniform measure on $\mathbb{D}^{b,s}$ refers to the triple $(\mathbb{D}^{b,s}, \mathcal{B}_\mathbb{D}, \mu)$ in the proposition below.

Proposition 2.9. Let $\mathcal{B}_\mathbb{D}$ be the Borel $\sigma$-algebra generated by the metric $d_F$. There is a unique measure $\mu$ on $(\mathbb{D}^{b,s}, \mathcal{B}_\mathbb{D})$ satisfying $\mu([p]_n) = |\mathbb{D}^{b,s}_n|^{-1} = b^{-\sum_{j=1}^{n} s_j} - 1$ for all $n \in \mathbb{N}$ and $p \in \mathbb{D}^{b,s}_n$.

Remark 2.10. One relationship between $\mu$ and $\nu$ is that for any $R \in \mathcal{B}_\mathbb{D}$

\[
\nu(R) = \int_{\mathbb{D}^{b,s}} \int_{[0,1]} 1_{p(r) \in R} \, d\mu(dp).
\]

3 Proofs

3.1 Intersection time between random directed paths

The intersections between randomly chosen paths $p, q \in \mathbb{D}^{b,s}$ can be encoded into realizations of a discrete-time branching process that begins with a single node and for which each generation $n$ node has exactly $s$ children independently with probability $1/b$, or has no children at all.

Given $p, q \in \mathbb{D}^{b,s}$ recall that $N^{(n)}_{p, q}$ is the number of bonds shared by the course-grained paths $[p]_n, [q]_n \in \mathbb{D}^{b,s}_n$. For $q \in \mathbb{D}^{b,s}$ chosen at random (i.e., according to $\mu$), let $F^{(q)}_n$ be the $\sigma$-algebra of subsets of $\mathbb{D}^{b,s}$ generated by $[q]_n$.

Proof of Proposition 2.6. We can write $I_{p,q}$ as

\[
I_{p,q} = \bigcap_{n=1}^{\infty} I_{p,q}^{(n)} \text{ for } I_{p,q}^{(n)} = [0, 1] - \bigcup_{1 \leq k \leq s^n, [p]_n(k) \neq [q]_n(k)} \left( \frac{k-1}{s^n}, \frac{k}{s^n} \right).
\]
Let $p_{b,s}^{(n)}$ be the probability that the number, $N_{p,q}^{(n)}$, of bonds shared by $[p]_n$ and $[q]_n$ is not zero. Then the probability that $N_{p,q}^{(n)}$ never becomes zero is the limit $p_{b,s}^{(n)} \wedge p_{b,s}$. The probabilities $p_{b,s}^{(n)}$ satisfy the recursive relation
\[
p_{b,s}^{(n+1)} = G_{b,s}(p_{b,s}^{(n)}) \quad \text{for} \quad G_{b,s}(x) := \frac{1}{b} \left[ 1 - (1 - x)^{s} \right]
\]
and have initial value $p_{b,s}^{(0)} = 1$. When $s$ is larger than $b$, the probability $p_{b,s} \in (0, 1)$ is the unique attractive fixed point of the map $G_{b,s} : [0, 1] \to [0, 1]$.

Part (i): The variables $m_n := (\frac{1}{s})^n N_{p,q}^{(n)}$ form a nonzero, mean-one martingale with respect to the filtration, $F_n^{(q)}$, generated by $[q]_n$. Hence, there is an a.s. limit $m_{\infty} = \lim_{n \to \infty} m_n$. The moment generating functions $\varphi_n(t) := \mathbb{E}[e^{tm_n}]$ satisfy the recursive relation
\[
\varphi_{n+1}^{b,s}(t) = \frac{b - 1}{b} + \frac{1}{b} \left( \varphi_{n}^{b,s}(\frac{b}{s} t) \right)^s \quad \text{with} \quad \varphi_0^{b,s}(t) = e^t. \quad (3.1)
\]
The sequence $\varphi_{n}^{b,s}(t)$ converges pointwise to a nontrivial limit $\varphi_{\infty}^{b,s}(t)$, which is the moment generating function of $m_{\infty}$, satisfying $\varphi_{\infty}^{b,s}(t) = \frac{b - 1}{b} + \frac{1}{b} (\varphi_{\infty}^{b,s}(\frac{b}{s} t))^s$. Note that the limit of $\varphi_{\infty}^{b,s}(t)$ as $t \to -\infty$ solves $x = \frac{b - 1}{b} + \frac{1}{b} x^s$, and thus $\mathbb{P}[m_{\infty} = 0] = 1 - p_{b,s}$.

Parts (ii) and (iii): In the event that $N_{p,q}^{(n)}$ is zero for some $n \in \mathbb{N}$, $\mathcal{I}_{p,q}$ is finite and $p(t) = q(t) \in V^{b,s}$ for $t \in \mathcal{I}_{p,q}$. Conversely, I will show below that if $N_{p,q}^{(n)}$ is never zero, then the set $\mathcal{I}_{p,q}$ is a.s. infinite since its dimension-$h$ Hausdorff measure is infinite for any $0 < h < \mathfrak{h}$. This would suffice to prove the proposition since the above remarks show that $N_{p,q}^{(n)}$ becomes zero for large enough $n \in \mathbb{N}$ with probability $1 - p_{b,s}$.

I will split up the analysis between proving $\dim_H(\mathcal{I}_{p,q}) \leq \mathfrak{h}$ and $\dim_H(\mathcal{I}_{p,q}) \geq \mathfrak{h}$. To show that $\dim_H(\mathcal{I}_{p,q}) \leq \mathfrak{h}$, I will argue that the Hausdorff measure $H_\mathfrak{h}(\mathcal{I}_{p,q})$ is a.s. finite. For a given $\delta > 0$ pick $n$ with $(\frac{1}{s})^n < \delta$. Since $\mathcal{I}_{p,q} \subset \mathcal{I}_{p,q}^{(n)}$ and $\mathcal{I}_{p,q}^{(n)}$ is covered by $N_{p,q}^{(n)}$ intervals of length $1/s^n$
\[
H_{\mathfrak{h},\delta}(\mathcal{I}_{p,q}) = \inf_{\mathcal{I}_{p,q} \subset \bigcup_k \mathcal{I}_k, |I_k| \leq \delta} \sum_k |I_k|^\mathfrak{h} \leq N_{p,q}^{(n)} \left( \frac{1}{s^n} \right)^{b_n} = m_n.
\]
Thus,
\[
H_\mathfrak{h}(\mathcal{I}_{p,q}) = \lim_{\delta \to 0} H_{\mathfrak{h},\delta}(\mathcal{I}_{p,q}) \leq \lim_{n \to \infty} m_n = m_{\infty}.
\]
Therefore $H_\mathfrak{h}(\mathcal{I}_{p,q})$ is a.s. finite and $\dim_H(\mathcal{I}_{p,q}) \leq \mathfrak{h}$.

Next I will condition on the event that $N_{p,q}^{(n)}$ is not zero for any $n \in \mathbb{N}$ and show that $\dim_H(\mathcal{I}_{p,q}) \geq \mathfrak{h}$ almost surely. It suffices to show that $H_\mathfrak{h}(\mathcal{I}_{p,q}) > 0$ for any $0 < h < \mathfrak{h}$. Let $\mathcal{S}_{p,q}^{(n)}$ be the collection of intervals $[\frac{k-1}{s^n}, \frac{k}{s^n}] \subset \mathcal{I}_{p,q}$ such that $[\frac{k-1}{s^n}, \frac{k}{s^n}] \cap \mathcal{I}_{p,q}^{(N)}$ is not finite for any $N > n$ (in other terms, ancestors of the interval do not go extinct). For a Borel set $A \subset [0, 1]$, let $\mathcal{C}(A)$ be the collection of coverings of $A$ by elements in $\bigcup_{n=1}^{\infty} \mathcal{S}_{p,q}^{(n)}$. Define the Hausdorff-like measure $H_\mathfrak{h}$ as
\[
\tilde{H}_\mathfrak{h}(A) = \lim_{n \to \infty} \tilde{H}_\mathfrak{h}^{\frac{1}{s^n}}(A) \quad \text{for} \quad \tilde{H}_\mathfrak{h}^{\frac{1}{s^n}}(A) = \inf_{\{I_k\} \in \mathcal{C}(A)} \sum_k |I_k|^\mathfrak{h}. \quad (3.2)
\]
For any Borel $A \subset [0, 1]$ we have that
\[
\frac{1}{2s^n} \tilde{H}_\mathfrak{h}(A) \leq H_\mathfrak{h}(A) \leq \tilde{H}_\mathfrak{h}(A). \quad (3.3)
\]
The second inequality above holds since \( \tilde{H}_{h,1/s^n}(A) \) is defined as an infimum over a smaller collection of coverings than \( H_{h,1/s^n}(A) \). The first inequality holds since any interval \( I \subset [0,1] \) is covered by two adjacent intervals of the form \( \left[ \frac{k-1}{s^n}, \frac{k}{s^n} \right] \) for \( n := \lfloor \log_{1/s} |I| \rfloor \). Thus if \( \tilde{H}_{h,1}(I_{p,q}) > 0 \) almost surely then \( H_{h}(I_{p,q}) > 0 \) almost surely.

Let \( \tilde{N}_{p,q}^{(n)} \) be the number elements in \( S_{p,q}^{(n)} \). Conditioned on the event that \( N_{p,q}^{(n)} \) is never zero, \( \tilde{N}_{p,q}^{(n)} \) forms a Markov chain taking values in \( \mathbb{N} \) with initial value \( \tilde{N}_{p,q}^{(0)} = 1 \) and satisfying the distributional equality

\[
\tilde{N}_{p,q}^{(n+1)} \overset{d}{=} \sum_{j=1}^{\tilde{N}_{p,q}^{(n)}} n_j \quad \text{for i.i.d. variables } n_j \in \{1, \cdots, s\} \text{ with } \mathbb{P}[n_j = \ell] = \left( \frac{s}{\ell} \right) \frac{p_{b,s}^{\ell}(1-p_{b,s})^{s-\ell}}{1-(1-p_{b,s})^{s}}.
\]

Fix some \( 0 < h < \hbar \). Define the variables

\[
L_{p,q,n} := \inf_{\{I_k\} \in C(I_{p,q})} \sum_{k} |I_k|^h,
\]

which have the a.s. convergence \( L_{p,q,n} \searrow L_{p,q,\infty} := \tilde{H}_{h,1}(I_{p,q}) \). The hierarchical symmetry of the model implies that the \( L_{p,q,n} \)'s satisfy the distributional recursion relation

\[
L_{p,q,n+1} \overset{d}{=} \min \left( 1, \sum_{j=1}^{n} \left( \frac{1}{s} \right)^h L_{p,q,n}^{(j)} \right),
\]

where the \( L_{p,q,n}^{(j)} \)'s are independent copies of \( L_{p,q,n} \) and \( n \in \{1, \cdots, s\} \) is independent of the \( L_{p,q,n}^{(j)} \)'s with \( \mathbb{P}[n = \ell] = \left( \frac{s}{\ell} \right) \frac{p_{b,s}^{\ell}(1-p_{b,s})^{s-\ell}}{1-(1-p_{b,s})^{s}} \). The distribution of \( L_{p,q,\infty} \) is a fixed point of (3.5). The probability \( x = \mathbb{P}[L_{p,q,\infty} = 0] \) satisfies

\[
x = \frac{(xp_{b,s} + 1 - p_{b,s})^s - (1-p_{b,s})^s}{1-(1-p_{b,s})^s},
\]

which has solutions only for \( x = 0 \) and \( x = 1 \). However, \( x = 1 \) is not possible since a.s. convergence \( L_{p,q,n} \searrow 0 \) as \( n \to \infty \) contradicts (3.5). To see the rough idea for this, notice that if \( 0 < L_{p,q,n} \ll 1 \) with high probability when \( n \gg 1 \) then the expectation of (3.5) yields

\[
\mathbb{E}[L_{p,q,n+1}] \approx \mathbb{E} \left[ \sum_{j=1}^{n} \left( \frac{1}{s} \right)^h L_{p,q,n}^{(j)} \right] = \left( \frac{1}{s} \right)^h \mathbb{E}[n] \mathbb{E}[L_{p,q,n}] = s^{h-h} \mathbb{E}[L_{p,q,n}]
\]

because \( \mathbb{E}[n] = \frac{s}{b} = s^h \). The above shows that the expectations of \( \mathbb{E}[L_{p,q,n}] \) will contract away from 0 since \( h-h > 0 \).

\[
\square
\]

### 3.2 The compact operator \( Y \)

In this section I will prove Propositions 1.17 and 1.21.

**Definition 3.1.** For \( \ell \in \{1, \cdots, b\} \), let \( v^{(\ell)} = (v_1^{(\ell)}, \cdots, v_b^{(\ell)}) \) be orthonormal vectors in \( \mathbb{R}^b \) where \( v^{(1)} = \frac{1}{\sqrt{b}}(1, \cdots, 1) \). Let \( p_1, \cdots, p_b \) be an enumeration of the elements in \( \Gamma_1^{b,s} \), i.e., the branches of \( D_1^{b,s} \).

- Define \( f^{(\ell)} \in \mathcal{H} \) and \( \hat{f}^{(\ell)} \in L^2(\Gamma_1^{b,s}, \mu) \) for \( \ell \in \{1, \cdots, b\} \) as

\[
f^{(\ell)}(x) = \sqrt{b} \sum_{i=1}^{b} v_i^{(\ell)} \chi_{\cup_{k=1}^{b}[p_i](k)}(x) \quad \text{and} \quad \hat{f}^{(\ell)}(p) = \sqrt{b} \sum_{i=1}^{b} v_i^{(\ell)} \chi_{p_i}(p).
\]

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For \((e, \ell) \in \cup_{n=0}^{\infty} E_n^{b,s} \times \{1, \ldots, b\}\), define \(f_{(e, \ell)} \in \mathcal{H}\) as
\[
 f_{(e, \ell)}(x) = (sb)^{\frac{n}{2}} \chi_e(x) f^{(\ell)}(x_e),
\]
where for \(x \in e\) the point \(x_e \in D^{b,s}\) refers to the position of \(x\) in the shrunken copy of \(D^{b,s}\) corresponding to \(e\).

For \((e, \ell) \in \cup_{n=0}^{\infty} E_n^{b,s} \times \{1, \ldots, b\}\), define \(\tilde{f}_{(e, \ell)} \in L^2(\Gamma^{b,s}, \mu)\) as
\[
 \tilde{f}_{(e, \ell)}(p) = b^{\frac{n}{2}} \chi_{e \cap \text{Range}(p) \neq \emptyset} f^{(\ell)}(p_e),
\]
where if \(e \cap \text{Range}(p) \neq \emptyset\) the path \(p_e \in \Gamma^{b,s}\) refers to a magnification of the portion of the path \(p\) in the shrunken copy of \(D^{b,s}\) corresponding to \(e\).

**Proof of Proposition 1.17** It suffices to show that the operator \(Y : \mathcal{H} \to L^2(\Gamma^{b,s}, \mu)\) has the form
\[
 Y = |\tilde{f}_{(D^{b,s}, 1)}\rangle \langle \tilde{f}_{(D^{b,s}, 1)}| + \sum_{k=0}^{\infty} \sum_{e \in E_k^{b,s}} \sum_{\ell \in \{2, \ldots, b\}} s^{-\frac{k}{2}} |\tilde{f}_{(e, \ell)}\rangle \langle f_{(e, \ell)}|.
\] (3.6)

Clearly \(Y\) maps \(f_{(D^{b,s}, 1)} = 1_{D^{b,s}}\) to \(\tilde{f}_{(D^{b,s}, 1)} = 1_{\Gamma^{b,s}}\). Pick \(e \in E_k^{b,s}\) and \(\ell \in \{2, \ldots, b\}\). For \(p \in \Gamma^{b,s}\),
\[
 (Y f_{(e, \ell)})(p) = \int_0^1 f_{(e, \ell)}(p(r)) dr = \int_0^1 (sb)^{\frac{k}{2}} \chi_e(p(r)) f^{(\ell)}((p(r))_e) dr = \frac{1}{s^k} (sb)^{\frac{k}{2}} \chi_{e \cap \text{Range}(p) \neq \emptyset} f^{(\ell)}(p_e) = s^{-\frac{k}{2}} \tilde{f}_{(e, \ell)}(p).
\]
The third equality holds since the path \(p(r)\) is in \(e\) for a time interval of length \(1/s^k\) in the event that \(e \cap \text{Range}(p)\) is nonempty.

The orthogonal complement in \(\mathcal{H}\) of the space spanned by the vectors \(\tilde{f}_{(D^{b,s}, 1)}\) and \(f_{(e, \ell)}\) with \((e, \ell) \in E_k^{b,s} \times \{2, \ldots, b\}\) is comprised of all vectors \(\psi\) for which \(0 = \int_0^1 \psi(p(r)) dr\) for all \(p \in \Gamma^{b,s}\), which is, by definition, the null space of \(Y\).

**Proof of Proposition 1.21** Part (i): The conditional expectation \(E[ \cdot \mid F_n]\) satisfies
\[
 E[\tilde{f}_{(e, \ell)} \mid F_n] = \begin{cases} 
 \tilde{f}_{(e, \ell)} & e \in \bigcup_{k=0}^{n-1} E_k^{b,s}, \\
 0 & e \in \bigcup_{k=n}^{\infty} E_k^{b,s} \text{ and } \ell \in \{2, \ldots, b\}.
\end{cases}
\]
The result then follows from the form (3.6) of \(Y\) since \(Y^{(n)}\) has the form
\[
 Y^{(n)} = |\tilde{f}_{(D^{b,s}, 1)}\rangle \langle f_{(D^{b,s}, 1)}| + \sum_{k=0}^{n-1} \sum_{e \in E_k^{b,s}} \sum_{\ell \in \{2, \ldots, b\}} s^{-\frac{k}{2}} |\tilde{f}_{(e, \ell)}\rangle \langle f_{(e, \ell)}| = Y P_n.
\]

Part (ii): As a consequence of (i), the operator norm of the difference between \(Y^{(n)}\) and \(Y\) has the form \(\|Y^{(n)} - Y\|_\infty = s^{-n/2}\).
Part (iii): The vectors $Y^{(n)}(Y^{(n)})^*$ can be written in the form

$$ Y^{(n)}(Y^{(n)})^* = |\hat{f}_{(D^{b,s},1)}(\hat{f}_{(D^{b,s},1)})| + \sum_{k=0}^{n-1} \sum_{e \in \mathcal{E}^{b,s}} s^{-k}|\hat{f}_{e,\ell}|^2 \hat{f}_{e,\ell} \rangle \langle \hat{f}_{e,\ell} |. $$

It follows that the Hilbert-Schmidt norm of the difference between $Y^{(n)}(Y^{(n)})^*$ and $YY^*$ is $(\frac{k}{2})^{n/2} \sqrt{s^{-1} b}$. Thus, we can write $YY^*$ has integral kernel $T(p,q)$. The map $(Y^{(n)})^*: L^2(\Gamma^{b,s}, \mu) \to \mathcal{H}$ sends $f \in L^2(\Gamma^{b,s}, \mu)$ to

$$(Y^{(n)})^* f = \int_{\Gamma^{b,s}} f(p)Y^{(n)}_p(p) \, dp = b^n \int_{\Gamma^{b,s}} f(p)\chi_{T^{(n)}}(dp).$$

Thus, $Y^{(n)}(Y^{(n)})^*: L^2(\Gamma^{b,s}, \mu) \to L^2(\Gamma^{b,s}, \mu)$ has kernel $K^{(n)}(p,q)$

$$K^{(n)}(p,q) = \langle Y^{(n)}_p, Y^{(n)}_q \rangle = \frac{b^n}{s^n} \sum_{k=1}^{s^n} 1_{[p,q]}(k)=q|n(k) := m_n.$$

However, $m_n \equiv m_n(p,q)$ converges to $m_\infty(p,q) = T_{p,q}$ in $L^2(\Gamma^{b,s} \times \Gamma^{b,s}, \mu \times \mu)$ by part (i) of Proposition 1.6

Part (iv): The vectors $Y^{(n)}_p \in \mathcal{H}$ satisfy $Y^{(k)}_p = P_k Y^{(n)}_p$ for $k \leq n$. Thus

$$Y^{(n)}_p - Y^{(n-1)}_p = (P_n - P_{n-1})Y^{(n)}_p \in \mathcal{H}_n. \square$$

3.3 Existence and uniqueness of the GMC measure for the CDRP

3.3.1 GMC martingale

**Proposition 3.2.** Define $\mathcal{F}_n$ to be the $\sigma$-algebra on $\Omega$ generated by variables $(W, \psi)$ for $\psi \in \oplus_{k=0}^{n} \mathcal{H}_k$ where $\mathcal{H}_k$ is defined as in part (ii) of Proposition 1.17

(i). The sequence of random variables $\{e^{\beta (W,Y^{(n)}_p) - \frac{\beta^2}{2} ||Y^{(n)}_p||^2_{\mathcal{H}}}\}_{n \in \mathbb{N}}$ forms a mean-one martingale with respect to $\mathcal{F}_n$.

(ii). For any Borel set $A \subset \Gamma^{b,s}$, the sequence of random variables $\left\{ \int_A e^{\beta (W,Y^{(n)}_p) - \frac{\beta^2}{2} ||Y^{(n)}_p||^2_{\mathcal{H}}} \mu(dp) \right\}_{n \in \mathbb{N}}$ is a mean-$\mu(A)$, square-integrable martingale with respect to $\mathcal{F}_n$ that converges a.s. to a nonzero limit.

**Proof.** Part (i): If $N > n$, then $\langle W, Y^{(N)}_p - Y^{(n)}_p \rangle$ and $\langle W, Y^{(n)}_p \rangle$ are independent random normal variables since $Y^{(N)}_p - Y^{(n)}_p$ and $Y^{(n)}_p$ are orthogonal elements in $\mathcal{H}$ by part (iv) of Proposition 1.21. Thus, we can write

$$e^{\beta (W,Y^{(N)}_p) - \frac{\beta^2}{2} ||Y^{(n)}_p||^2_{\mathcal{H}}} = e^{\beta (W,Y^{(N)}_p - Y^{(n)}_p) - \frac{\beta^2}{2} ||Y^{(n)}_p - Y^{(n)}_p||^2_{\mathcal{H}}} e^{\beta (W,Y^{(n)}_p) - \frac{\beta^2}{2} ||Y^{(n)}_p||^2_{\mathcal{H}}}.$$

The conditional expectation with respect to $\mathcal{F}_n$ is $e^{\beta (W,Y^{(n)}_p) - \frac{\beta^2}{2} ||Y^{(n)}_p||^2_{\mathcal{H}}}.$

Part (ii): The sequence $\left\{ \int_A e^{\beta (W,Y^{(n)}_p) - \frac{\beta^2}{2} ||Y^{(n)}_p||^2_{\mathcal{H}}} \mu(dp) \right\}_{n \in \mathbb{N}}$ is a mean-$\mu(A)$ martingale. We can write

$$\int_A e^{\beta (W,Y^{(n)}_p) - \frac{\beta^2}{2} ||Y^{(n)}_p||^2_{\mathcal{H}}} \mu(dp) = \int_A \prod_{k=1}^{n} e^{\beta (W,Y^{(n)}_p) - \frac{\beta^2}{2} ||Y^{(n)}_p||^2_{\mathcal{H}}} \mu(dp)$$

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Since the $(\mathbf{W}, b^n X_{[p,n,κ(k)]})$’s are independent, centered normal variables with variance $(b/s)^n$, the second moment of the above is equal to

\[
\int_{A \times A} e^{\beta T_{p}^n \psi_{p}} \mu(dp) \mu(dq), \tag{3.7}
\]

where $N_{p,q}^{(n)}$ is the number of bonds shared by the course-grained paths $[p]_n$ and $[q]_n$. By part (i) of Proposition 1.6, the sequence $(\frac{1}{n} N_{p,q}^{(n)})$ converges $\mu \times \mu$-a.e. as $n \to \infty$ to the intersection time $T_{p,q}$, and when $A = \Gamma^{b,s}$ the expression (3.7) converges to the moment generating function, $\mathbb{E}[\exp(\beta T_{p,q})]$. It follows that (3.7) converges to $\int_{A \times A} e^{\beta T_{p}^n \psi_{p}} \mu(dp) \mu(dq)$ for an arbitrary Borel set $A \in \mathcal{B}_\Gamma$.

\[\square\]

### 3.3.2 Martingale limit construction of the GMC measure

**Proof of Propositions 1.9 and 1.22** Recall that $M_{\beta}^{(n)}$ is defined as the GMC measure on $(\Gamma^{b,s}, \mu)$ over the finite-dimensional field $(\mathbf{W}, \beta Y^{(n)})$:

\[
M_{\beta}^{(n)}(A) = \int_A e^{\beta (\mathbf{W}, Y^{(n)}_{p}) - \frac{\beta^2}{2} \|Y^{(n)}_{p}\|^2} \mu(dp), \quad A \in \mathcal{B}_\Gamma. \tag{3.8}
\]

The space $\mathcal{H}_\mu = \oplus_{k=0}^\infty \mathcal{H}_k$ is the orthogonal complement of the null space of $\mu$. Define $\mathcal{F}_\infty$ as the $\sigma$-algebra generated by the variables $\langle \mathbf{W}, \psi \rangle$ for $\psi \in \mathcal{H}_\mu$.

**Existence:** Let $D$ be a countable subcollection of continuous functions on $\Gamma^{b,s}$ that are dense with respect to the norm $d_\Gamma$. For each $\psi \in D$, the sequence $\{ \int_{\Gamma^{b,s}} \psi(p) M_{\beta}^{(n)}(dp) \}_{n \in \mathbb{N}}$ is a martingale w.r.t. the filtration $\mathcal{F}_n$ having uniformly bounded second moments as a consequence of Proposition 3.2. Consequently, $\{ \int_{\Gamma^{b,s}} \psi(p) M_{\beta}^{(n)}(dp) \}_{n \in \mathbb{N}}$ converges a.s. to a limit in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Thus the measures $M_{\beta}^{(n)}$ a.s. converge vaguely to a limit measure $M_{\beta}$ adapted to the field $\mathbf{W}$, i.e., $M_{\beta} \equiv M_{\beta}(\mathbf{W})$.

Properties (I)-(III) follow easily from this construction. For instance, to verify property (III) fix some $\phi \in \bigcup_{N=1}^\infty \oplus_{k=0}^N \mathcal{H}_k$. Notice that since $M_{\beta}^{(n)}(\mathbf{W}, dp)$ a.s. converges vaguely to $M_{\beta}(\mathbf{W}, dp)$, I have the a.s. equality

\[
M_{\beta}(\mathbf{W} + \phi, dp) = \lim_{n \to \infty} M_{\beta}^{(n)}(\mathbf{W} + \phi, dp)
= \lim_{n \to \infty} e^{\beta \langle Y^{(n)}_{p}, \phi \rangle} M_{\beta}^{(n)}(\mathbf{W}, dp).
\]

However, since $\psi \in \oplus_{k=0}^N \mathcal{H}_k$ for some $N$, I have $\langle Y^{(n)}_{p}, \phi \rangle = \langle Y_{p}, \phi \rangle$ when $n \geq N$, and thus

\[
e^{\beta \langle Y_{p}, \phi \rangle} \lim_{n \to \infty} M_{\beta}^{(n)}(\mathbf{W}, dp)
= e^{\beta \langle Y_{p}, \phi \rangle} M_{\beta}(\mathbf{W}, dp). \tag{3.9}
\]

The space $\bigcup_{N=1}^\infty \oplus_{k=0}^N \mathcal{H}_k$ is dense in the complement of the null space of $\mathcal{H}$, and it follows that the above extends to all values in $\mathcal{H}$.

**Uniqueness:** Next I argue that $M_{\beta}$ is the unique GMC measure over the field $(\mathbf{W}, \beta Y)$. I will reduce the uniqueness of $M_{\beta}$ to the uniqueness of the the GMC measures $M_{\beta}^{(n)}$ over the finite-dimensional fields $(\mathbf{W}, \beta Y^{(n)})$. Let $M_{\beta}$ be a random measure satisfying (I)-(III), and define $\tilde{M}_{\beta}^{(n)}$ as the conditional expectation of $M_{\beta}$ w.r.t. $\mathcal{F}_n$

\[
\tilde{M}_{\beta}^{(n)}(\mathbf{W}^{(n)}, A) = \mathbb{E}[\tilde{M}_{\beta}(\mathbf{W}, A) | \mathcal{F}_n], \tag{3.10}
\]
where \( W^{(n)} \) refers to the finite-dimensional field of variables \( \langle W, \psi \rangle \) for \( \psi \in \otimes_{k=0}^n \mathcal{H}_k \). Since \( \mathcal{H} = \oplus_{k=0}^\infty \mathcal{H}_k \), the random variable \( \tilde{M}_\beta(W, A) \) is recovered as the a.s. limit

\[
\tilde{M}_\beta(W, A) = \lim_{n \to \infty} \mathbb{E}[\tilde{M}_\beta(W, A) | F_n].
\]

(3.11)

Obviously \( (3.10) \) implies that \( \mathbb{E}[\tilde{M}_\beta^{(n)}] = \mathbb{E}[\tilde{M}\beta] = \mu \). For \( \phi \in \otimes_{k=0}^n \mathcal{H}_k \),

\[
\tilde{M}_\beta^{(n)}(W^{(n)} + \phi, A) = \mathbb{E}[\tilde{M}_\beta(W + \phi, A) | F_n] = e^{\beta Y_{p,\phi}} \mathbb{E}[\tilde{M}_\beta(W, A) | F_n] = e^{\beta Y_{p,\phi}} \tilde{M}_\beta^{(n)}(W^{(n)}, A).
\]

If \( \tilde{M}_\beta^{(n)} = \tilde{M}_\beta^{(n)}(W) \) is viewed as a function of the entire field \( W \), then

\[
\tilde{M}_\beta^{(n)}(W + \phi, dp) = e^{\beta Y_{p,\phi}} \tilde{M}_\beta^{(n)}(W, dp)
\]

since \( Y_{p}^{(n)} = P_n Y_{p} \) for the projection \( P_n : \mathcal{H} \to \oplus_{k=0}^n \mathcal{H}_k \). Thus \( \tilde{M}_\beta^{(n)} \) is a GMC measure over the trivial field \( (W, \beta Y^{(n)}) \) and must be equal to \( M_\beta^{(n)} \). From the construction analysis above, \( M_\beta^{(n)} \) a.s. converges vaguely to a GMC measure \( M_\beta \) over the field \( (W, \beta Y) \). From \( (3.11) \) it follows that \( M_\beta = \tilde{M}_\beta \).

\[ \square \]

3.4 Properties of the GMC measure

In this section I will prove Theorem 1.13

Proof of Theorem 1.13 Part (i): First I will show that for \( \beta > 0 \) the GMC measure \( M_\beta \) is mutually

singular to \( \mu \) with probability 0 or 1. Let \( M_\beta^{(c)} \) and \( M_\beta^{(s)} \) be the continuous and singular components in the Hahn decomposition of \( M_\beta \) w.r.t. \( \mu \). Since the law of the random measure \( M_\beta \) is uniform over the path space \( \Gamma_{b,s} \), \( \mathbb{E}[M_\beta^{(c)}] = \lambda \mu \) and \( \mathbb{E}[M_\beta^{(s)}] = (1 - \lambda)\mu \) for some value \( \lambda \in [0, 1] \). However, if \( \lambda \in (0, 1) \) then the random measures \( \lambda^{-1}M_\beta^{(c)} \) and \( (1 - \lambda)^{-1}M_\beta^{(s)} \) must both be GMC measures

satisfying property (I)-(III) of Proposition 1.9. This, however, contradicts uniqueness of that GMC measure. Therefore, \( \lambda \in \{0, 1\} \). Suppose to reach a contradiction that \( \lambda = 1 \), i.e., \( M_\beta = M_\beta^{(c)} \). Let \( G_\beta(W, p) \) be the Radon-Nikodým derivative of \( M_\beta \) with respect to \( \mu \). For almost every \( p \), \( G_\beta(W, p) \) is a random variable with finite second moment and for all \( \phi \in \mathcal{H} \)

\[
G_\beta(W + \phi, p) = e^{\beta \langle Y(p), \phi \rangle} G_\beta(W, p).
\]

(3.12)

However, the above is only possible if \( Y(p) \in \mathcal{H} \), which is a contradiction.

Part (ii): The intersection time, \( T_{p,q} \), of two random paths \( p, q \in \Gamma_{b,s} \) chosen independently according to the measure \( M_\beta \) has moment generating function \( \int_{\Gamma_{b,s} \times \Gamma_{b,s}} e^{\alpha T_{p,q}} M_\beta(dp) M_\beta(dq) \), which has expectation

\[
\mathbb{E} \left[ \int_{\Gamma_{b,s} \times \Gamma_{b,s}} e^{\alpha T_{p,q}} M_\beta(dp) M_\beta(dq) \right] = \int_{\Gamma_{b,s} \times \Gamma_{b,s}} e^{(\alpha + \beta^2/2)} \mu(dp) \mu(dq) = \varphi^{b,s}(\alpha + \beta^2/2) < \infty.
\]

(3.13)

It follows that the set of pairs \( (p, q) \) for which the intersection set \( I_{p,q} \) has Hausdorff dimension > \( h \), and thus for which \( T_{p,q} = \infty \), is almost surely of measure zero with respect to \( M_\beta \times M_\beta \). The set of pairs \( (p, q) \) for which the intersection set \( I_{p,q} \) has Hausdorff dimension < \( h \) satisfy \( T_{p,q} = 0 \), and

\[
\lim_{\alpha \to -\infty} \int_{\Gamma_{b,s} \times \Gamma_{b,s}} e^{\alpha T_{p,q}} M_\beta(dp) M_\beta(dq) = M_\beta \times M_\beta \left( \left\{ (p, q) \mid T_{p,q} = 0 \right\} \right).
\]
Taking the expectation
\[
\mathbb{E}\left[M_\beta \times M_\beta \left( \{(p, q) \mid T_{p,q} = 0 \} \right) \right] = \lim_{\gamma \to -\infty} \varphi^{b,s}_\gamma = p_{b,s}. \tag{3.14}
\]
However, \( \{(p, q) \mid T_{p,q} = 0 \} \) contains the set of pairs such that the intersection set \( I_{p,q} \) is finite and
\[
\mathbb{E}\left[M_\beta \times M_\beta \left( \{(p, q) \mid |I_{p,q}| < \infty \} \right) \right] = \mu \times \mu \left( \{(p, q) \mid |I_{p,q}| < \infty \} \right) = p_{b,s}. \tag{3.15}
\]
The second equality above holds by part (ii) of Proposition 1.6. To see the first equality in (3.15), notice that \( \{(p, q) \mid |I_{p,q}| < \infty \} \) can be written as the limit as the \( n \to \infty \) limit
\[
\bigcup_{p,q \in \Gamma^{b,s}_n, \forall (k) p(k) \neq q(k)} p \times q \supset \{(p, q) \mid |I_{p,q}| < \infty \}.
\]
Moreover the random measure \( M_\beta \) is independent over sets \( p, q \in \Gamma^{b,s}_n \) for which \( p(k) \neq q(k) \) for \( 1 \leq k \leq s^n \).

It follows from (3.14) and (3.15) that
\[
\mathbb{E}\left[M_\beta \times M_\beta \left( \{(p, q) \mid T_{p,q} = 0 \text{ and } |I_{p,q}| = \infty \} \right) \right] = 0. \tag{3.16}
\]
Therefore, \( M_\beta \times M_\beta \) is supported on pairs \( (p, q) \) for which either \( I_{p,q} \) is finite or has Hausdorff dimension \( b \).

Part (iii): Let \( M^{(i,j)}_\beta \) be measurable with respect to independent copies \( W^{(i,j)} \) of the field \( W \), and define the field \( \hat{W} \) such that
\[
\langle \hat{W}, \phi \rangle := \frac{1}{\sqrt{bs}} \sum_{i=1}^b \sum_{j=1}^s \langle W^{(i,j)}, \phi^{(i,j)} \rangle \quad \text{for} \quad \phi \in \hat{H} = \bigoplus_{i=1}^b \bigoplus_{j=1}^s \mathcal{H}^{(i,j)},
\]
where \( \mathcal{H}^{(i,j)} \) are copies of the Hilbert space \( \mathcal{H} = L^2(D^{b,s}, \mathcal{B}_D, \nu) \). Define \( (\hat{\Gamma}^{b,s}, \hat{\mu}) \) for
\[
\hat{\Gamma}^{b,s} := \bigcup_{i=1}^b \bigtimes_{j=1}^s \Gamma^{b,s}_{i,j} \quad \text{and} \quad \hat{\mu} := \frac{1}{b} \sum_{i=1}^b \prod_{j=1}^s \mu^{(i,j)}. \tag{3.17}
\]
Finally, define \( \hat{Y} : \mathcal{H} \to L^2(\hat{\Gamma}^{b,s}, \hat{\mu}) \) as
\[
\langle \hat{Y}, \phi \rangle (p) = \frac{1}{s} \sum_{i=1}^b \chi \left( p \in \times_{j=1}^s \Gamma^{b,s}_{i,j} \right) \sum_{j=1}^s \langle Y^{(i,j)}, \phi^{(i,j)} \rangle \left( p^{(j)} \right).
\]
In the above \( p_j \in \Gamma^{b,s}_{i,j} \) are components of \( p \in \times_{j=1}^s \Gamma^{b,s}_{i,j} \).

The computation below shows that \( \hat{M} = \frac{1}{b} \sum_{i=1}^b \Pi_{j=1}^s M^{(i,j)}_\beta \) defines a GMC measure over the field \( (\hat{W}, \sqrt{\hat{\mu}} \hat{Y}) \):
\[
\hat{M}(\hat{W} + \phi, dp) = \frac{1}{b} \sum_{i=1}^b \chi \left( p \in \times_{j=1}^s \Gamma^{b,s}_{i,j} \right) \prod_{j=1}^s M^{(i,j)}_\beta \left( W^{(i,j)} + \frac{\phi^{(i,j)}}{\sqrt{bs}}, dp^{(j)} \right)
\]
\[
= \frac{1}{b} \sum_{i=1}^b \chi \left( p \in \times_{j=1}^s \Gamma^{b,s}_{i,j} \right) e^{\beta \frac{1}{\sqrt{bs}} \sum_{j=1}^s \langle Y^{(i,j)}, \phi^{(i,j)} \rangle \left( p^{(j)} \right)} \prod_{j=1}^s M^{(i,j)}_\beta \left( W^{(i,j)}, dp^{(j)} \right)
\]
\[
= \frac{1}{b} \sum_{i=1}^b \chi \left( p \in \times_{j=1}^s \Gamma^{b,s}_{i,j} \right) e^{b \beta \langle \hat{Y}, \phi \rangle (p)} \prod_{j=1}^s M^{(i,j)}_\beta \left( W^{(i,j)}, dp^{(j)} \right)
\]
\[
= e^{b \beta \langle \hat{Y}, \phi \rangle (p)} \hat{M}(\hat{W}, dp)
\]
Since the terms \( P \) the analysis below shows that there exist \( c, C > 0 \) such that for all \( \beta > 1 \)

\[
P \left[ \frac{\nu_\beta(W, p)}{\nu_\beta(W, q)} \in (\lambda^{-1}, \lambda) \right] \leq \min_{|m| \leq c \log(\beta)} C \sqrt{P \left[ \frac{\nu_\beta(W, p)}{\nu_\beta(W, q)} \in (\lambda^{2m-1}, \lambda^{2m+1}) \right]}. \tag{3.20}
\]

Since the terms \( P \left[ \frac{\nu_\beta(W, p)}{\nu_\beta(W, q)} \in (\lambda^{2m-1}, \lambda^{2m+1}) \right] \) sum to 1 over \( m \in \mathbb{Z} \), the above must be smaller than \( C(1/|c \log \beta|)^{\frac{1}{2}} \), thus implying (3.19).

\[ \square \]

### 3.5 Strong disorder behavior as \( \beta \to \infty \)

In this section I will prove Theorem 1.15. The proof below that \( M_\beta(W, \Gamma^{b,s}) \) converges in probability to zero is a straightforward adaption of the argument of Lacoin and Moreno for discrete polymers on diamond lattices in [23].

**Proof of part (i) of Theorem 1.15.** It suffices to show that the fractional moment \( E\left[ \sqrt{M_\beta(W, \Gamma^{b,s})} \right] \) converges to zero as \( n \to \infty \).

Let \( h : D^{b,s} \to \mathbb{R} \) be the constant function \( h(x) = \lambda \) for some \( \lambda \in \mathbb{R} \), and \( \tilde{P}_\lambda \) be the measure on \( W \) with derivative

\[
\frac{d\tilde{P}_\lambda}{dP} = e^{(W, h) - \frac{1}{2} \|h\|^2} = e^{\lambda W(D^{b,s}) - \frac{1}{2} \lambda^2}.
\]

Let \( \tilde{E}_\lambda \) denotes the expectation with respect to \( \tilde{P}_\lambda \). The Cauchy-Schwarz inequality yields that

\[
E \left[ \left( M_\beta(W, \Gamma^{b,s}) \right)^\frac{1}{2} \right]^2 = \tilde{E}_\lambda \left[ \left( M_\beta(W, \Gamma^{b,s}) \right)^\frac{1}{2} e^{-\frac{1}{2} \lambda W(D^{b,s}) + \frac{1}{2} \lambda^2} \right] \leq \tilde{E}_\lambda \left( M_\beta(W, \Gamma^{b,s}) \right)^\frac{1}{2} \tilde{E} \left[ e^{-\frac{1}{2} \lambda W(D^{b,s}) + \frac{1}{2} \lambda^2} \right]^{\frac{1}{2}}.
\]

Since \( \tilde{E}_\lambda[F(W)] = E[F(W + h)] \) for any integrable function \( F \) of the field and \( M_\beta(W + h, d\nu) = e^{\beta(Y_h|\nu)} M_\beta(W, d\nu) \) where \( \langle Y_h | \nu \rangle = \lambda \), the above is equal to

\[
= E \left[ e^{\beta Y_h} M_\beta(W, \Gamma^{b,s}) \right]^{\frac{1}{2}} E \left[ e^{-\frac{1}{2} \lambda W(D^{b,s}) + \frac{1}{2} \lambda^2} \right]^{\frac{1}{2}}
\]

\[
= e^{\frac{1}{2} \lambda \beta + \frac{1}{2} \lambda^2}. \tag{3.18}
\]

The above is minimized as \( \exp\left\{ -\frac{1}{8} \beta^2 \right\} \) when \( \lambda = -\frac{1}{2} \beta \), and thus tends to zero as \( \beta \) grows.

\[ \square \]

The proof below also borrows ideas from [23].

**Proof of part (ii) of Theorem 1.15.** Fix \( n \in \mathbb{N}, q, p \in \Gamma_n^{b,s} \) with \( q \neq p \), and \( \alpha > 1 \). It suffices to show that as \( n \to \infty \)

\[
P \left[ \frac{\nu_\beta(W, p)}{\nu_\beta(W, q)} \in (\lambda^{-1}, \lambda) \right] \to 0. \tag{3.19}
\]

The analysis below shows that there exist \( c, C > 0 \) such that for all \( \beta > 1 \)

\[
P \left[ \frac{\nu_\beta(W, p)}{\nu_\beta(W, q)} \in (\lambda^{-1}, \lambda) \right] \leq \min_{|m| \leq c \log(\beta)} C \sqrt{P \left[ \frac{\nu_\beta(W, p)}{\nu_\beta(W, q)} \in (\lambda^{2m-1}, \lambda^{2m+1}) \right]}. \tag{3.20}
\]
Next I will show (3.20). Define \( h \in L^2(D^{b,s}, \nu) \) as 
\[
h = \alpha \chi(\cup_{p(k) \neq q(k)} P(k)) - \alpha \chi(\cup_{p(k) \neq q(k)} Q(k))
\]
for a parameter \( \alpha \in [-1, 1] \). Then for any \( p \in \mathfrak{p} \) and \( q \in \mathfrak{p} \)
\[
\langle h, Y_p \rangle = \frac{\alpha n}{s^n} \quad \text{and} \quad \langle h, Y_q \rangle = -\frac{\alpha n}{s^n},
\]
where \( 1 \leq n < s^n \) is the number of edges not shared by the paths \( p, q \in \Gamma_{n}^{b,s} \). Notice that
\[
\frac{\nu_\beta(W+h,p)}{\nu_\beta(W+h,q)} = \frac{M_\beta(W+h,p)}{M_\beta(W+h,q)} = e^{\beta(h,Y_p)-\beta(h,Y_q)} \frac{M_\beta(W,p)}{M_\beta(W,q)} = e^{2\beta \alpha n} \frac{\nu_\beta(W,p)}{\nu_\beta(W,q)}.
\]
Define \( \hat{P} \) to have derivative \( \frac{d\hat{P}}{dx} = \exp\{\langle W, h \rangle - \frac{1}{2} \|h\|_h^2 \} \). Applying the Cauchy-Schwarz inequaly,
\[
P\left[ \frac{\nu_\beta(W,p)}{\nu_\beta(W,q)} \in [\lambda^{-1}, \lambda] \right] = \hat{P}\left[ e^{-(\langle W, h \rangle + \frac{1}{2} \|h\|_h^2)} \chi\left( \frac{\nu_\beta(W,p)}{\nu_\beta(W,q)} \in [\lambda^{-1}, \lambda] \right) \right]
\leq \hat{P}\left[ (e^{-(\langle W, h \rangle + \frac{1}{2} \|h\|_h^2)} \right)^{\frac{1}{2}} P\left[ \frac{\nu_\beta(W,p)}{\nu_\beta(W,q)} \in [\lambda^{-1}, \lambda] \right]^{\frac{1}{2}}.
\]
Since the law \( \hat{P} \) is a shift of \( P \) by \( h \), the above is equal to
\[
= e^{\frac{1}{2} \|h\|_h^2} P\left[ \frac{\nu_\beta(W+h,p)}{\nu_\beta(W+h,q)} \in [\lambda^{-1}, \lambda] \right]^{\frac{1}{2}}
\leq e^{\alpha^2 n} P\left[ e^{2\beta \alpha n} \frac{\nu_\beta(W,p)}{\nu_\beta(W,q)} \in [\lambda^{-1}, \lambda] \right]^{\frac{1}{2}},
\]
where the second inequality is by (3.22). With \( \alpha \) ranging over \([-1, 1]\), the above implies (3.20) with \( C := \exp\{1/b^3\} \) and \( c := \log(2n)/\log(\lambda) \).

\[
\square
\]

### 3.6 Chaos expansion

The proof of the proposition below is in the Appendix.

**Proposition 3.3.** Let \( S \subset E^{b,s} \) be finite and \( \Gamma_{S}^{b,s} \) be nonempty. Define the measure \( \mu_S^{(n)} \) such that
\[
\mu_S^{(n)}(A) = \frac{\mu(A \cap G_S^{(n)})}{\mu(G_S^{(n)})},
\]
where \( G_S^{(n)} \) is the set of \( p \in \Gamma_{S}^{b,s} \) such that \( S \subset \cup_{k=1}^{s^n} [p]_n(k) \). Then the sequence \( \{\mu_S^{(n)}\}_{n \in \mathbb{N}} \) converges vaguely to a limiting probability measure \( \mu_S \).

The following defines a generalization of the measure \( \rho_k(x_1, \cdots, x_k; dp) \) on \( \Gamma_{b,s} \); see Definition 1.23.

**Definition 3.4.** Let \( x_1, \cdots, x_k \) be distinct elements in \( E^{b,s} \). If \( \Gamma_{x_1,\cdots,x_k}^{b,s} \) is nonempty, define the measure \( \rho_k^{(n)}(x_1, \cdots, x_k; ds) \) such that for a Borel set \( A \subset \Gamma_{x_1,\cdots,x_k}^{b,s} \)
\[
\rho_k^{(n)}(x_1, \cdots, x_k; A) = b^{(n)}(\{x_1, \cdots, x_k\}) \mu_k^{(n)}(x_1, \cdots, x_k; A),
\]
where \( \gamma^{(n)}(S) \) is defined for \( n \in \mathbb{N} \) and a finite set \( S \subset E^{b,s} \)
\[
\gamma^{(n)}(S) = \sum_{k=0}^{n-1} \left( |S| - |\{e \in E^{b,s}_k \mid e \cap S \neq \emptyset\}| \right).
\]
If \( \Gamma_{x_1,\cdots,x_k}^{b,s} \) is empty, I define \( \rho_k^{(n)}(x_1, \cdots, x_k; dp) = 0 \).
Corollary 3.5. For fixed, distinct \( x_1, \ldots, x_k \in E^{b,s} \), the sequence of measures \( \{\rho_k^{(n)}(x_1, \ldots, x_k; dp)\}_{n \in \mathbb{N}} \) converges vaguely to \( \rho_k(x_1, \ldots, x_k; dp) \). Moreover, if \( A \subset \Gamma^{b,s} \), then \( \rho_k^{(n)}(x_1, \ldots, x_k; p) = \rho_k(x_1, \ldots, x_k; p) \) for all \( n > N \).

Remark 3.6. I drop superscripts and subscripts in the following cases: \( \gamma \equiv \gamma(\infty), \rho_k \equiv \rho_k(\infty) \), and \( \rho_k(x_1, \ldots, x_k) \equiv \rho_k(x_1, \ldots, x_k; \Gamma^{b,s}) \).

Remark 3.7. The measure \( \rho_k^{(n)}(x_1, \ldots, x_k; dp) \) is equal to \( \tau_k^{(n)}(x_1, \ldots, x_k; p) \mu(dp) \) where the density can be written as

\[
\tau_k^{(n)}(x_1, \ldots, x_k; p) = b^{n^2} \frac{\chi_{G_{(x_1, \ldots, x_k)}^{(n)}}(p)}{\mu(G_{(x_1, \ldots, x_k)}^{(n)})} = b^{nk} \chi\left\{ x_1, \ldots, x_k \right\} \subset \sum_{\ell=1}^{b^n} [p]_n(\ell) = \prod_{\ell=1}^{k} Y_p^{(n)}(x_\ell).
\]

Proposition 3.8. Let \( M_{(n)}^{(n)}(x_1, \ldots, x_k; A) \) be the GMC measure over the field \( \left( W, \beta Y^((n)) \right) \). For a Borel set \( A \subset \Gamma^{b,s} \), the random variable \( M_{\beta}^{(n)}(A) \) has the chaos expansion

\[
\mu(A) + \sum_{k=1}^{\infty} \frac{\beta_k}{k!} \int_{(D^{b,s})^k} \rho_k^{(n)}(x_1, \ldots, x_k; A) W(x_1) \cdots W(x_k) \nu(dx_1) \cdots \nu(dx_k). \tag{3.25}
\]

The hierarchy of functions \( \{\rho_k^{(n)}(x_1, \ldots, x_k; A)\}_{k \in \mathbb{N}} \) satisfies

(I). \( \int_{D^{b,s}} \rho_k^{(n)}(x_1, \ldots, x_k; A) \nu(dx_k) = \rho_k^{(n)}(A) \) and

(II). \( \int_{(D^{b,s})^k} \rho_k^{(n)}(x_1, \ldots, x_k; A) \nu(dx_1) \cdots \nu(dx_k) = \mu(A) \).

Proof. Recall that

\[
M_{\beta}^{(n)}(A) = \int_{A} e^{\beta(W, Y_p^{(n)} - \frac{\beta}{2} \|Y_p^{(n)}\|_H^2)} \mu(dp), \tag{3.26}
\]

where \( Y_p^{(n)} \in \mathcal{H} \) is equal to \( Y_p^{(n)} = b^n \chi(\bigcup_{k=1}^{b^n} [p]_n(k)) \). The integrand above has the chaos expansion

\[
e^{\beta(W, Y_p^{(n)} - \frac{\beta}{2} \|Y_p^{(n)}\|_H^2)} = 1 + \sum_{k=1}^{\infty} \frac{\beta_k}{k!} \int_{(D^{b,s})^k} \prod_{\ell=1}^{k} Y_p^{(n)}(x_\ell) W(x_1) \cdots W(x_k) \nu(dx_1) \cdots \nu(dx_k).
\]

By Remark 3.7, \( \prod_{\ell=1}^{k} Y_p^{(n)}(x_\ell) \) is equal to the density, \( \tau_k^{(n)}(x_1, \ldots, x_k; p) \), of the measure \( \rho_k^{(n)}(x_1, \ldots, x_k; dp) \). Thus the expressions (3.25) and (3.26) are equal.

The equalities (I) and (II) also follow from Remark 3.7 since, for instance,

\[
\int_{D^{b,s}} \rho_k^{(n)}(x_1, \ldots, x_k; A) \nu(dx_k) = \int_{D^{b,s}} \int_{A} \tau_k^{(n)}(x_1, \ldots, x_k; p) \mu(dp) \nu(dx_k)
\]

\[
= \int_{D^{b,s}} \int_{A} \prod_{\ell=1}^{k} Y_p^{(n)}(x_\ell) \mu(dp) \nu(dx_k),
\]

and switching the order of integration and using that \( \int_{D^{b,s}} Y_p^{(n)}(x) \nu(dx) = 1 \) yields

\[
= \int_{A} \prod_{\ell=1}^{k-1} Y_p^{(n)}(x_\ell) \mu(dp) = \rho_k^{(n)}(x_1, \ldots, x_{k-1}; A).
\]

\( \square \)
Proof of Theorem 1.25. By Proposition 3.2 the sequence \( \{M_\beta^{(n)}(A)\}_{n \in \mathbb{N}} \) converges in \( L^2(\Gamma^{b,s}, \mathcal{B}_\Gamma, \mu) \) to \( M_\beta(A) \). Thus

\[
\langle \rho_k^{(n)}(\cdot, A) \rangle \equiv (\mu(A), \rho_1^{(n)}(x_1; A), \rho_2^{(n)}(x_1, x_2; A), \cdots),
\]

viewed as an element of

\[
L^2\left( \bigcup_{k=0}^{\infty} \left( (D_{b,s}^k)^k, \bigoplus_{k=0}^{\infty} B_D^{k, k}, \bigoplus_{k=0}^{\infty} \frac{\beta^{2k}}{k!} \nu^k \right) \right),
\]

converges as \( n \to \infty \) to a limit \( \langle \tilde{\rho}_k(\cdot, A) \rangle \). However, if \( A \in \mathcal{B}_\Gamma^{(n)} := \mathcal{P}(\Gamma^{b,s}_n) \), Corollary 3.5 implies that \( \langle \tilde{\rho}_k(\cdot, A) \rangle = \langle \rho_k(\cdot, A) \rangle \)

\[ \square \]

A  Further discussion of the diamond hierarchical lattice

A.1  The vertex set

I will show how the set of vertices, \( V^{b,s}_n \), on the graph \( D^{b,s}_n \) are embedded in \( D^{b,s} \). For \( n \in \mathbb{N} \), I can label elements in \( V^{b,s}_n \setminus V^{b,s}_{n-1} \) by

\[
V^{b,s}_n \setminus V^{b,s}_{n-1} \equiv \left( \{1, \cdots, b\} \times \{1, \cdots, s\} \right)^{n-1} \times \left( \{1, \cdots, b\} \times \{1, \cdots, s-1\} \right).
\]

Given an element \( v = (b_1, s_1) \times \cdots \times (b_n, s_n) \in V^{b,s}_n \setminus V^{b,s}_{n-1} \), define \( U_v = L_v \cup R_v \subset D^{b,s} \) for

\[
L_v := \left\{ (b_1, s_1) \times \cdots \times (b_n, s_n) \times \prod_{j=1}^{\infty} (\hat{b}_j, s) \left| \hat{b}_j \in \{1, \cdots, b\} \right\} \subset D^{b,s}
\]

and

\[
R_v := \left\{ (b_1, s_1) \times \cdots \times (b_n, s_n + 1) \times \prod_{j=1}^{\infty} (\hat{b}_j, 1) \left| \hat{b}_j \in \{1, \cdots, b\} \right\} \subset D^{b,s}.
\]

Pairs \( x, y \in U_v \) satisfy \( d_D(x, y) = 0 \), and \( U_v \) is the maximal equivalence class with that property. Thus \( v \) is canonically identified with an element in \( D^{b,s} \). The root vertices \( V^{b,s}_0 = \{A, B\} \) of the graph are identified with the subsets of \( D^{b,s} \) given by

\[
A := \left\{ \prod_{j=1}^{\infty} (\hat{b}_j, 1) \left| \hat{b}_j \in \{1, \cdots, b\} \right\} \quad \text{and} \quad B := \left\{ \prod_{j=1}^{\infty} (\hat{b}_j, s) \left| \hat{b}_j \in \{1, \cdots, b\} \right\} \right.
\]

A.2  The metric space \( D^{b,s} \)

Next I prove the points in Proposition 2.4. Note that each element of \( E^{b,s}_n \) is equivalent to a nested sequence \( e_n \in E^{b,s}_n \).

Completeness: Let \( \{x_k\}_{k \in \mathbb{N}} \) be a Cauchy sequence in \( D^{b,s} \). The sequence \( \bar{\pi}(x_k) \in [0, 1] \) must be Cauchy and thus convergent to a limit \( \lambda \in [0, 1] \).

If \( \lambda \) is a multiple of \( b^{-N} \) for some nonnegative integer \( N \in \{0, 1, 2, \cdots\} \), let \( N \) be the smallest value. For large \( k \), the terms \( x_k \) must become arbitrarily close to generation \( N \) vertices. Since the generation \( N \) vertices are at least a distance \( 1/s^N \) apart, the terms must be close to the same generation vertex and thus convergent.
If \( \lambda \) is not a multiple of \( b^{-N} \) for any \( N \in \mathbb{N} \), then there must exist a nested sequence of edge sets \( e_n \in E_n^{b,s} \) such that each closure \( \bar{e}_n \) contains a tail of the sequence \( \{x_k\}_{k \in \mathbb{N}} \). The sequence \( \{x_k\}_{k \in \mathbb{N}} \) converges to the unique element in \( \cap_{n=1}^{\infty} \bar{e}_n \).

**Compactness:** Let \( \{x_k\}_{k \in \mathbb{N}} \) be a sequence in \( D^{b,s} \). Since \( D^{b,s} \) is covered by sets \( \bar{e} \) for \( e \in E_n^{b,s} \), the pigeonhole principle implies that there must exist a nested sequence of edge sets \( e_n \in E_n^{b,s} \) such that each \( \bar{e}_n \) contains \( x_k \) for infinitely many \( k \in \mathbb{N} \). Thus there is a subsequence of \( \{x_k\}_{k \in \mathbb{N}} \) converging to the unique element in \( \cap_{n=1}^{\infty} \bar{e}_n \).

**Hausdorff Dimension:** The contractive maps \( S_{i,j} \)’s defined in Remark 2.3 are the similitudes of the fractal \( D^{b,s} \). The open set \( O := D^{b,s} \setminus \{A, B\} \) is a separating set since
\[
\bigcup_{i,j} S_{i,j}(O) \subset O \quad \text{and} \quad S_{i,j}(O) \cap S_{k,l}(O) = \emptyset
\]
for \( (i, j) \neq (k, l) \). Since the \( S_{i,j} \)’s have contraction constant \( 1/s \) and there are \( bs \) maps, the Hausdorff dimension of \( D^{b,s} \) is \( \log(bs)/\log s \).

### A.3 The measures

Finally, I prove Propositions 2.5, 2.9, and 3.3.

**Uniform measure on the diamond lattice:** Let \( \mathcal{B}_E := \{A \cap E_n^{b,s} : |A| \in \mathcal{B}_D\} \) be the restriction of \( \mathcal{B}_D \) to \( E_n^{b,s} = D^{b,s} \setminus V^{b,s} \). Every element \( A \in \mathcal{B}_D \) can be decomposed as \( A = A_1 \cup A_2 \) for \( A_1 \subset V^{b,s} \) and \( A_2 \in \mathcal{B}_E \). I define \( \nu \) as zero on \( V^{b,s} \), and it remains to define \( \nu \) on \( \mathcal{B}_E \). The Borel \( \sigma \)-algebra \( \mathcal{B}_E \) is generated by the semi-algebra, \( \mathcal{C}_E = \cup_{n=0}^{\infty} E_n^{b,s} \), i.e., the collection of cylinder sets \( C((b_1,s_1) \times \cdots \times (b_n,s_n)) \). This follows since for any open set \( O \subset D^{b,s} \) we can write
\[
O \setminus V^{b,s} = \bigcup_{x \in O \setminus V^{b,s}} e_x,
\]
where \( e_x \in \mathcal{C}_E \) is the biggest set in \( \mathcal{C}_E \) satisfying \( x \in e_x \subset O \). A premeasure \( \hat{\nu} \) can be placed on \( \mathcal{C}_E \) by assigning \( \hat{\nu}(e) = |E_n^{b,s}|^{-1} = ((bs)^{-n} \) to each \( e \in E_n^{b,s} \). The finite premeasure \( (E^{b,s}, \mathcal{C}_E, \hat{\nu}) \) extends uniquely to a measure \( (\Gamma^{b,s}, \mathcal{B}_E, \nu) \) through the Carathéodory procedure.

**Uniform measure on paths:** Consider the semi-algebra \( \mathcal{C}_\Gamma := \cup_{n=0}^{\infty} \Gamma_n^{b,s} \) of subsets of \( \Gamma^{b,s} \). An arbitrary open set \( O \subset \Gamma^{b,s} \) be written as a disjoint union of elements in \( \mathcal{C}_\Gamma \) in analogy to \( \cup_{A \in \mathcal{B}_D} A \). Indeed, each element in \( \mathcal{C}_\Gamma \) is an open ball with respect to the metric \( d_\Gamma \). A finite premeasure \( \hat{\mu} \) is defined on \( \mathcal{C}_\Gamma \) by assigning each \( q \in \Gamma_n^{b,s} \) the value \( \hat{\mu}(q) = |\Gamma_n^{b,s}|^{-1} \). Again, by Carathéodory’s technique, the measure \( \hat{\mu} \) extends to a measure \( (\Gamma^{b,s}, \mathcal{B}_\Gamma, \mu) \).

**Uniform measure on paths through a finite subset of \( \Gamma^{b,s} \):** Let \( S \subset E_n^{b,s} \) be finite and \( \Gamma_n^{b,s} \) be nonempty. There exists an \( N \in \mathbb{N} \) such that no two elements in \( S \) fall into the same equivalence class \( e \in E_N^{b,s} \). For \( n > N \),
\[
\frac{d\mu_S^{(n)}}{d\mu} = J_S^{(n)} \quad \text{where} \quad J_S^{(n)}(p) := b^n|S|^{-\gamma(S)} \chi([p]_n \cap \Gamma_n^{b,s} \neq \emptyset).
\]
Moreover, \( J_S^{(n)} \) forms a nonnegative, mean-one martingale with respect to the filtration \( F_n = \Gamma_n^{b,s} \). If \( g : \Gamma^{b,s} \to \mathbb{R} \) is measurable with respect to \( F_m \) for some \( m \in \mathbb{N} \), then the sequence \( \int_{\Gamma_n^{b,s}} g(p) \mu^{(n)}(dp) \) is constant for \( n \geq m \) and thus convergent. A continuous function \( h : \Gamma^{b,s} \to \mathbb{R} \) must be uniformly
continuous since $\Gamma^{b,s}$ is a compact metric space. Thus given $\epsilon > 0$, there exists an $n \in \mathbb{N}$ such that $|g(p) - g(q)| < \epsilon$ when $[p]_n = [q]_n$. It follows that $Y_n g = \int_{\Gamma^{b,s}} g(p) \mu^{(n)}(dp)$ converges to a limit $Y_\infty g$ as $n \to \infty$. Thus $\mu^{(n)}_S$ converges vaguely to a limiting probability measure $\mu_S$ on $\Gamma^{b,s}$.

B Random Shift

The following argument from [28] shows that $\beta Y$ defines a random shift.

Proof of Theorem 1.12. Let $\mathbb{E}_\beta$ refer to the expectation with respect to $\widetilde{P}_\beta$. The calculation below shows that $\mathbb{E}_\beta [e^{\langle W, \psi \rangle}] = \mathbb{E}_\beta [M_\beta (W, \Gamma^{b,s}) e^{\langle W, \psi \rangle}]$ for any $\psi \in H$, and thus that $M_\beta (W, \Gamma^{b,s})$ is the Radon-Nikodym derivative of $\widetilde{P}_\beta$ with respect to $P$.

\[
\mathbb{E}_\beta [e^{\langle W, \psi \rangle}] := \int_{\Gamma^{b,s}} \mathbb{E} [e^{\langle W + \beta Y_p, \psi \rangle}] \mu(dp) \\
= \mathbb{E} [e^{\langle W, \psi \rangle}] \int_{\Gamma^{b,s}} e^{\beta \langle Y_p, \psi \rangle} \mu(dp).
\]

Since $Y_n$ converges strongly to $Y$ by part (i) of Proposition 1.21, the above is equal to

\[
e^{\frac{1}{2} \|\psi\|_H^2} \lim_{n \to \infty} \int_{\Gamma^{b,s}} e^{\beta \langle Y_p^{(n)}, \psi \rangle} \mu(dp) = \lim_{n \to \infty} \int_{\Gamma^{b,s}} e^{\frac{1}{2} \|\psi + \beta Y_p^{(n)}\|_H^2 - \frac{\beta^2}{2} \|Y_p^{(n)}\|_H^2} \mu(dp).
\]

Since $\mathbb{E} [\exp \{ \langle W, \phi \rangle \}] = \exp \{ \frac{1}{2} \|\phi\|_H^2 \}$, the above is equal to

\[
= \lim_{n \to \infty} \int_{\Gamma^{b,s}} \mathbb{E} [e^{\langle W, \psi + \beta Y_p^{(n)} \rangle} - \frac{\beta^2}{2} \|Y_p^{(n)}\|_H^2 \mu(dp)] \\
= \lim_{n \to \infty} \mathbb{E} \left[ \mathbb{E} \left[ e^{\langle W + \beta Y_p^{(n)} - \frac{\beta^2}{2} \|Y_p^{(n)}\|_H^2, \psi \rangle} \right] \right] = \lim_{n \to \infty} \mathbb{E} \left[ M_\beta^{(n)} (W, \Gamma^{b,s}) e^{\langle W, \psi \rangle} \right].
\]

Finally, the limit can be brought inside the expectation as a consequence of part (ii) of Proposition 3.2

\[
= \mathbb{E} \left[ M_\beta (W, \Gamma^{b,s}) e^{\langle W, \psi \rangle} \right].
\]

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