STRICHARTZ ESTIMATES FOR THE ONE-DIMENSIONAL WAVE EQUATION

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ABSTRACT. We study the hyperboloidal initial value problem for the one-dimensional wave equation perturbed by a smooth potential. We show that the evolution decomposes into a finite-dimensional spectral part and an infinite-dimensional radiation part. For the radiation part we prove a set of Strichartz estimates. As an application we study the long-time asymptotics of Yang-Mills fields on a wormhole spacetime.

1. Introduction

Strichartz estimates were originally discovered in the context of the Fourier restriction problem [13] but only later their true power was exploited in the study of nonlinear wave equations [9]. To illustrate this point, consider for instance the Cauchy problem for the cubic wave equation in three spatial dimensions,

\[
\begin{cases}
(\partial_t^2 - \Delta_x)u(t, x) = u(t, x)^3 & (t, x) \in \mathbb{R} \times \mathbb{R}^3 \\
u(t, x) = f(x), & \partial_t u(t, x) = g(x) & (t, x) \in \{0\} \times \mathbb{R}^3,
\end{cases}
\]

for given initial data \(f, g \in S(\mathbb{R}^3)\), say. A weak formulation of Eq. (1.1) is provided by Duhamel’s formula

\[
u(t, \cdot) = \cos(t|\nabla|)f + \frac{\sin(t|\nabla|)}{|\nabla|} g + \int_0^t \frac{\sin((t-t')|\nabla|)}{|\nabla|} \left(u(t', \cdot)^3\right) dt'
\]

with the wave propagators \(\cos(t|\nabla|)\) and \(\frac{\sin(t|\nabla|)}{|\nabla|}\). The latter are the Fourier multipliers that yield the solution to the free wave equation \((\partial_t^2 - \Delta_x)u(t, x) = 0\). The point is that Eq. (1.2) is a reformulation of Eq. (1.1) as a fixed point problem. Proving the existence of solutions to Eq. (1.1) therefore amounts to showing that the operator on the right-hand side of Eq. (1.2) has a fixed point. The main issue then is to find suitable spaces that are compatible with the free evolution and that allow one to control the nonlinear term.

For the cubic equation (1.2) the Sobolev embedding \(\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)\) suffices but if one increases the power of the nonlinearity or the spatial dimension, a more sophisticated argument is required. The crucial tool is provided by the Strichartz estimates which are mixed spacetime bounds on the wave propagators of the form

\[
\|\cos(t|\nabla|)f\|_{L^p_t(L^q_x)} \lesssim \|f\|_{\dot{H}^s(\mathbb{R}^d)}
\]

for certain admissible values of \(p, q, s, \) and \(d\). For instance, the sine propagator satisfies the Strichartz estimate

\[
\left\| \frac{\sin(t|\nabla|)}{|\nabla|} \right\|_{L^p_t(L^q_x)} \lesssim \|g\|_{L^2(\mathbb{R}^3)}
\]

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which allows one to control a quintic nonlinearity in three dimensions. At the same time, the Strichartz estimates provide information on the long-time asymptotics which makes them crucial in proving scattering.

The physical effect that is responsible for the existence of Strichartz estimates is dispersion. The latter refers to the observation that waves of different frequencies travel at different speeds. In other words, a wave packet tends to spread out which leads to an averaged decay that is quantified by the Strichartz estimates. The strength of the dispersive decay depends strongly on the underlying spatial dimension: The higher the space dimension, the more room there is for the wave to spread out. On the other hand, in the one-dimensional case, there is no dispersion at all and the evolution is a pure transport phenomenon. This precludes the existence of Strichartz estimates as is easily seen by noting that $u(t, x) = \chi(t - x)$ for a $\chi \in C^\infty_c(\mathbb{R})$ solves $(\partial^2_t - \partial^2_x)u(t, x) = 0$. By translation invariance we have

$$\|u(t, \cdot)\|_{L^q(\mathbb{R})} = \|\chi\|_{L^q(\mathbb{R})}$$

and $\|u\|_{L^p(\mathbb{R})L^q(\mathbb{R})} = \infty$, unless $p = \infty$. The weak dispersion in low dimensions causes severe difficulties in understanding the asymptotics of many models in quantum field theory, see e.g. [8, 12, 7] for recent work.

In this paper we show that one can recover Strichartz estimates even in the one-dimensional case if one studies a hyperboloidal evolution problem instead of the standard Cauchy problem. The key observation is that the standard Cartesian coordinates are not very well suited for describing radiation processes. The foliation induced by the standard coordinates is singular at null infinity and therefore unnatural in this context, see e.g. [4] for a discussion on this. Consequently, as suggested in many physics papers, e.g. [6, 14, 15, 1], we choose a hyperboloidal foliation instead, where the leaves are asymptotic to translated forward lightcones. In this setup we study the evolution problem for the one-dimensional wave equation with an arbitrary potential added (to avoid technicalities we restrict ourselves to smooth potentials). We show that the solution decomposes into a finite dimensional part which is controlled by spectral theory and an infinite-dimensional “radiation” part which satisfies Strichartz estimates, provided a certain spectral assumption holds. We remark in passing that there are some technical similarities with Strichartz estimates in the context of self-similar blow up established in [2, 3].

As a first application we consider Yang-Mills fields on a wormhole geometry. Under a certain symmetry reduction, we study small-energy perturbations of an explicit Yang-Mills connection and prove its asymptotic stability in a Strichartz sense.

1.1. Main results. We use the hyperboloidal coordinates from [1] defined by $\Phi : \mathbb{R} \times (-1, 1) \to \mathbb{R}^2$, $\Phi(s, y) := (s - \log \sqrt{1 - y^2}, \text{artanh } y)$.

The map $\Phi$ is a diffeomorphism onto its image with inverse $\Phi^{-1}(t, x) = (t - \log \cosh x, \tanh x)$.

In these coordinates, the one-dimensional wave equation

$$(\partial^2_t - \partial^2_x)v(t, x) = 0 \quad (1.3)$$

reads

$$[\partial^2_s + 2y\partial_s\partial_y + \partial_s - (1 - y^2)\partial_y^2 + 2y\partial_y] u(s, y) = 0, \quad (1.4)$$
where \( v(t, x) = u(t - \log \cosh x, \tanh x) \). By testing with \( \partial_s u(s, y) \), we formally find the energy identity

\[
\frac{1}{2} \frac{d}{ds} \left[ \int_{-1}^{1} (1 - y^2) |\partial_y u(s, y)|^2 dy + \int_{-1}^{1} |\partial_s u(s, y)|^2 dy \right] = -|\partial_s u(s, -1)|^2 - |\partial_s u(s, 1)|^2
\]

and this motivates the introduction of the following energy norm.

**Definition 1.1.** For functions \((f, g) \in C^1(-1, 1) \times C(-1, 1)\), we define the energy norm \( \|(f, g)\|_H \) by

\[
\|(f, g)\|_H^2 := \int_{-1}^{1} (1 - y^2) |f'(y)|^2 dy + \int_{-1}^{1} |g(y)|^2 dy.
\]

Our main result is concerned with a more general class of wave equations, that is to say, we study the initial value problem

\[
\begin{align*}
[\partial^2_s + 2y \partial_y \partial_y + \partial_s - (1 - y^2) \partial^2_y + 2y \partial_y + V(y)]u(s, y) &= 0 \quad (s, y) \in (0, \infty) \times (-1, 1), \\
u(s, y) &= f(y), \\
\partial_s u(s, y) &= g(y) 
\end{align*}
\]

for an unknown \( u : [0, \infty) \times (-1, 1) \to \mathbb{C} \), prescribed initial data \( f, g : (-1, 1) \to \mathbb{C} \), and a given potential \( V : (-1, 1) \to \mathbb{C} \).

**Definition 1.2.** Let \( V \in C^\infty([-1, 1]) \) be even. We define a set \( \Sigma_V \subset \mathbb{C} \) by saying that \( \lambda \in \mathbb{C} \) belongs to \( \Sigma_V \) if \( \Re \lambda \geq 0 \) and there exists a nontrivial odd \( f_\lambda \in C^\infty([-1, 1]) \) that satisfies

\[-(1 - y^2)f''_\lambda(y) + 2(\lambda + 1)yf'_\lambda(y) + \lambda(\lambda + 1)f_\lambda(y) + V(y)f_\lambda(y) = 0\]

for all \( y \in (-1, 1) \).

**Theorem 1.3.** Let \( V : [-1, 1] \to \mathbb{C} \) be smooth and even, \( p \in [2, \infty) \), \( q \in [1, \infty) \), and \( \epsilon > 0 \). Then there exist constants \( C_{p, q}, C_\epsilon > 0 \) such that the following holds.

1. The set \( \Sigma^+_V := \Sigma_V \cap \{ z \in \mathbb{C} : \Re z > 0 \} \) consists of finitely many points.
2. For any given odd initial data \( f, g \in C^\infty([-1, 1]) \), there exists a unique solution \( u = u_{f, g} \in C^\infty([0, \infty) \times (-1, 1)) \) to the initial value problem (1.6) that satisfies, for each \( s \geq 0 \),

\[
\|(u(s, \cdot), \partial_s u(s, \cdot))\|_H < \infty.
\]

3. For each \( \lambda \in \Sigma^+_V \) there exists a number \( n(\lambda) \in \mathbb{N}_0 \) and a set \( \{\phi^{\lambda, k}_{f, g} \in C^\infty(-1, 1) : k \in \{0, 1, \ldots, n(\lambda)\}\} \) of odd functions satisfying \( \|\phi^{\lambda, 0}_{f, g}\|_H < \infty \) and such that the solution \( u_{f, g} \) decomposes according to

\[
u_{f, g}(s, y) = \sum_{\lambda \in \Sigma^+_V} e^{\lambda s} \sum_{k=0}^{n(\lambda)} s^k \phi^{\lambda, k}_{f, g}(y) + \bar{u}_{f, g}(s, y).
\]

The map \((f, g) \mapsto \phi^{\lambda, k}_{f, g}\) has finite rank and

\[
\|(\bar{u}_{f, g}(s, \cdot), \partial_s \bar{u}_{f, g}(s, \cdot))\|_H \leq C_\epsilon e^{\epsilon s}\|(f, g)\|_H
\]

for all \( s \geq 0 \).

4. If \( \Sigma_V \cap i\mathbb{R} = \emptyset \), we have the Strichartz estimates

\[
\|\bar{u}_{f, g}\|_{L^p(0, \infty)L^q(-1, 1)} \leq C_{p, q}\|(f, g)\|_H.
\]
Remark 1.4. With slightly more effort it is also possible to improve the energy bound to
\[\|(\tilde{u}_{f,g}(s,\cdot), \partial_s \tilde{u}_{f,g}(s,\cdot))\|_H \lesssim \|(f, g)\|_H\]
for all \(s \geq 0\), provided \(\Sigma_V \cap i\mathbb{R} = \emptyset\). To keep the paper at a reasonable length, however, we refrain from working out the details.

Remark 1.5. The smoothness assumptions are imposed for convenience and can of course be considerably weakened. This produces some inessential technicalities but no new insight.

2. Application: Asymptotics of Yang-Mills fields on wormholes

We give an application of Theorem 1.3 to Yang-Mills fields on wormholes studied in [1].

2.1. Setup. As in [1], we consider \(M^4 := \mathbb{R} \times \mathbb{R} \times (0, \pi) \times (0, 2\pi)\). Let \((t, r, \theta, \varphi) : M^4 \to \mathbb{R}^4\) be a chart on \(M^4\). We define a Lorentzian metric \(g\) on \(M^4\) by
\[g := -dt \otimes dt + dr \otimes dr + \cosh(r)^2(\sin(\theta) d\theta \otimes d\theta + \sin(\theta)^2 d\varphi \otimes d\varphi)\]
Then \((M^4, g)\) is a Lorentzian manifold with 2 asymptotic ends (as \(r \to \pm \infty\)), which physically represents a wormhole spacetime. We would like to study Yang-Mills connections on the principal bundle \(M^4 \times \text{SU}(2)\). That is to say, we are looking for \(\text{su}(2)\)-valued one-forms
\[A = A_0 dt + A_1 dr + A_2 d\theta + A_3 d\varphi\]
on \(M^4\) that formally\(^1\) extremize the Yang-Mills action
\[\int_{(M^4, g)} \text{tr}(F_{\mu\nu} F^{\mu\nu}),\]
where \(F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\) is the curvature two-form. The Euler-Lagrange equation associated to the Yang-Mills action reads
\[\frac{1}{\sqrt{|\det g|}} \partial_\mu \left( \sqrt{|\det g|} F^{\mu\nu} \right) + [A_\mu, F^{\mu\nu}] = 0 \quad (2.1)\]
and is called the Yang-Mills equation. Here, Greek indices run from 0 to 3 and we use Einstein’s summation convention. As usual, indices are raised and lowered by the metric, i.e., \(F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}\), where \(g_{\mu\nu} = g(\partial_\mu, \partial_\nu)\) with \(\partial_0 = \partial_t, \partial_1 = \partial_r, \partial_2 = \partial_\theta, \partial_3 = \partial_\varphi\), and \(g^{\mu\nu}\) is defined by the requirement that \(g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu\), where \(\delta^\mu_\nu\) is the Kronecker symbol. Furthermore, \(\det g = -\cosh(r)^4 \sin(\theta)^2\) is the determinant of the matrix \((g_{\mu\nu})\).

We choose the basis \(\{\tau_1, \tau_2, \tau_3\}\) for \(\text{su}(2)\), where
\[\tau_1 := -\frac{i}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 := -\frac{i}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 := -\frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\]
Then \(\cos \theta \tau_3 d\varphi\) solves the Yang-Mills equation, as is easily checked. We would like to study the stability of the explicit solution \(\cos \theta \tau_3 d\varphi\). Following [1], we consider the perturbation ansatz
\[A = \cos \theta \tau_3 d\varphi + W(t, r)(\tau_1 d\theta + \sin \theta \tau_2 d\varphi)\]
for a real-valued function \(W\). By noting the commutator relations \([\tau_1, \tau_2] = \tau_3, [\tau_1, \tau_3] = -\tau_2,\) and \([\tau_2, \tau_3] = \tau_1\), we readily compute the nonvanishing components of \(F_{\mu\nu}\), which

\(^1\)“Formally” here means that we are in fact looking for solutions of the Euler-Lagrange equation associated to the Yang-Mills action.
are given by \( F_{02} = \partial_0 W \tau_1 \), \( F_{03} = \partial_0 W \sin \theta \tau_2 \), \( F_{12} = \partial_1 W \tau_1 \), \( F_{13} = \partial_1 W \sin \theta \tau_2 \), and \( F_{23} = -(1 - W^2) \sin \theta \tau_3 \). Consequently,

\[
F_{02} = -\frac{\partial_0 W \sin \theta}{\sqrt{|\det g|}} \tau_1, \quad F_{03} = -\frac{\partial_0 W}{\sqrt{|\det g|}} \tau_2, \quad F_{12} = \frac{\partial_1 W}{\sqrt{|\det g|}} \tau_1, \quad F_{13} = -\frac{1 - W^2}{\cosh(r)^2 \sqrt{|\det g|}} \tau_3,
\]

and for \( \nu \in \{2, 3\} \), Eq. (2.1) reduces to

\[
(\partial_t^2 - \partial_r^2) W(t, r) = \frac{W(t, r)(1 - W(t, r)^2)}{\cosh(r)^2},
\]

whereas for \( \nu \in \{0, 1\} \), Eq. (2.1) is satisfied identically. In particular, we observe that \( W = 0 \) is a solution, showing that \( \cos \theta \tau_3 d\phi \) indeed solves the Yang-Mills equation. Consequently, under this particular symmetry reduction enforced by the perturbation ansatz, the study of the stability of \( \cos \theta \tau_3 d\phi \) as a solution to the Yang-Mills equation boils down to analyzing the stability of the trivial solution \( W = 0 \) of Eq. (2.2). Note that Eq. (2.2) is effectively a one-dimensional semilinear wave equation and studying its asymptotics might seem hard due to the lack of dispersion.

Evidently, there are two more trivial solutions to Eq. (2.2), namely \( W = \pm 1 \). In [1] it is shown that these solutions are linearly stable under general perturbations, whereas the solution \( W = 0 \) has one unstable mode. In this paper, we restrict ourselves to odd solutions. Under this assumption, the solutions \( W = \pm 1 \) disappear and \( W = 0 \) becomes stable, as we will see. The restriction to odd solutions is important for our method of proof, where the free wave equation and the corresponding explicit solution formula is taken as a starting point. We believe that our results remain valid in the general case but this would require a different proof. We plan to address this issue in the future.

2.2. Hyperboloidal formulation. The hyperboloidal initial value problem for the Yang-Mills equation (2.2) takes the form

\[
\left\{ \begin{array}{ll}
\partial_s^2 u + 2y \partial_s \partial_y + \partial_y - (1 - y^2) \partial_y^2 + 2y \partial_y - 1)u(s, y) = -u(s, y)^3 & (s, y) \in (0, \infty) \times (-1, 1) \\
u(s, y) = f(y), & (s, y) \in \{0\} \times (-1, 1)
\end{array} \right.
\]

(2.3)

where \( W(t, r) = u(t - \log \cosh r, \tanh r) \). Note that the linear part in (2.3) is Eq. (1.6) with \( V(y) = -1 \). We compute \( \Sigma_V \).

Lemma 2.1. Let \( V(y) = -1 \) for all \( y \in [-1, 1] \). Then \( \Sigma_V = \emptyset \).

Proof. According to Definition [12], we have to solve the spectral problem

\[-(1 - y^2)f''(y) + 2(\lambda + 1)yf'(y) + \lambda(\lambda + 1)f(y) - f(y) = 0 \]

for \( f \in C^\infty([-1, 1]) \) odd and \( \text{Re} \lambda \geq 0 \). This ODE can be solved explicitly in terms of hypergeometric functions and it follows that no solution other than \( f = 0 \) exists. We refrain from giving the details here because this spectral problem was discussed at length in [1].

Definiton 2.2. Set \( V(y) := -1 \) for all \( y \in [-1, 1] \) and let \( u_{f,g} \) be the solution of Eq. (1.6) provided by Theorem [13]. We define the wave propagators by

\[
C(s)f := u_{f,0}(s, \cdot), \quad S(s)g := u_{0,g}(s, \cdot).
\]
By Theorem 1.3 and Lemma 2.1 we have the bound \( \|S(s)g\|_{L^6(-1,1)} \lesssim \|g\|_{L^2(-1,1)} \) for any \( s \geq 0 \) and thus, \( S(s) \) uniquely extends to a bounded operator \( S(s) : L^2_\text{odd}(-1,1) \to L^6_\text{odd}(-1,1) \), where \( L^6_\text{odd}(-1,1) \) denotes the completion of \\
\{ f \in C^\infty([-1,1]) : f \text{ is odd}\}

with respect to \( \| \cdot \|_{L^6(-1,1)} \). Then a weak formulation of Eq. (2.3) is given by 
\[
    u(s,\cdot) = C(s)f + S(s)g - \int_0^s S(s-s') (u(s',\cdot)^3) \, ds'.
\] (2.4)

**Theorem 2.3.** There exists a \( \delta > 0 \) such that for all odd functions \( f,g \in C^\infty([-1,1]) \) with \( \|(f,g)\|_H < \delta \), Eq. (2.4) has a unique solution \( u \) in \( C([0,\infty), L^6_\text{odd}(-1,1)) \). Furthermore, \( u \in L^p((0,\infty), L^6_\text{odd}(-1,1)) \) for any \( p \in [3,\infty] \).

**Remark 2.4.** Theorem 2.3 implies that the Yang-Mills connection \( \cos \theta \tau_3 d \varphi \) is asymptotically stable under odd small-energy perturbations on the hyperboloid
\[
\{ (\frac{1}{2} \log(1-y^2), \text{artanh} y) \in \mathbb{R}^{1,1} : y \in (-1,1) \}.
\]

### 2.3. Proof of Theorem 2.3

For \( R > 0 \) we define 
\[
    X_R := \{ u \in C([0,\infty), L^6_\text{odd}(-1,1)) : \|u\|_{L^3(0,\infty)L^6(-1,1)} + \|u\|_{L^\infty(0,\infty)L^6(-1,1)} \leq R \}.
\]

Note that \( u \in X_R \) implies \( u(s,\cdot) \in L^6_\text{odd}(-1,1) \) and thus, \( u(s,\cdot)^3 \in L^2_\text{odd}(-1,1) \) for any \( s \geq 0 \). Consequently, 
\[
    K_{f,g}(u)(s) := C(s)f + S(s)g - \int_0^s S(s-s') (u(s',\cdot)^3) \, ds'
\]

is well-defined as a map \( K_{f,g} : X_R \to C([0,\infty), L^6_\text{odd}(-1,1)) \) for any \( R > 0 \) by Theorem 1.3.

**Lemma 2.5.** There exist \( M, \delta_0 > 0 \) such that for all \( \delta \in (0, \delta_0] \) and any pair of odd functions \( f,g \in C^\infty([-1,1]) \) satisfying \( \|(f,g)\|_H < \delta \), \( K_{f,g} \) maps \( X_M \) to itself.

**Proof.** Let \( u \in X_R \) for some \( R > 0 \) and assume that \( \|(f,g)\|_H < \delta \). By Theorem 1.3 and Lemma 2.1 we have 
\[
    \|K_{f,g}(u)\|_{L^3(0,\infty)L^6(-1,1)}
    \lesssim \|(f,g)\|_H + \int_0^\infty \|1_{[0,s)}(s')S(s-s') (u(s',\cdot)^3)\|_{L^2(0,\infty)L^6(-1,1)} \, ds'
    = \|(f,g)\|_H + \int_0^\infty \|S(s-s') (u(s',\cdot)^3)\|_{L^2(s',\infty)L^6(-1,1)} \, ds'
    = \|(f,g)\|_H + \int_0^\infty \|S(s) (u(s',\cdot)^3)\|_{L^2(0,\infty)L^6(-1,1)} \, ds'
    \lesssim \|(f,g)\|_H + \int_0^\infty \|u(s',\cdot)^3\|_{L^2(-1,1)} \, ds'
    = \|(f,g)\|_H + \int_0^\infty \|u(s',\cdot)^3\|^3_{L^6(-1,1)} \, ds'
\]

Analogously, 
\[
    \|K_{f,g}(u)\|_{L^3(0,\infty)L^6(-1,1)} \lesssim \|(f,g)\|_H + \|u\|_{L^3(0,\infty)L^6(-1,1)}^3
\]

and we obtain 
\[
    \|K_{f,g}(u)\|_{L^3(0,\infty)L^6(-1,1)} + \|K_{f,g}(u)\|_{L^\infty(0,\infty)L^6(-1,1)} \leq C\delta + CR^3
\]
for some constant $C > 0$. Now we choose $\delta_0 = (8C^3)^{-\frac{1}{2}}$ and $R = 2C\delta$. Then we have

$$C\delta + CR^3 = C\delta + C(2C\delta)^3 \leq C\delta + 8C^3\delta^3 C\delta = 2C\delta$$

and the claim follows with $M = 2C$ since the wave propagators preserve oddness. □

Now we set up an iteration by $a_n := 0$ and $b_n := K_{f,g}(u_{n-1})$ for $n \in \mathbb{N}$. For brevity we define

$$\|u\|_X := \|u\|_{L^3(0,\infty)L^6(-1,1)} + \|u\|_{L^\infty(0,\infty)L^6(-1,1)}.$$

**Lemma 2.6.** There exist $M, \delta > 0$ such that $u_n \in X_M\delta$ for all $n \in \mathbb{N}$ and the sequence $(u_n)_{n \in \mathbb{N}}$ is Cauchy with respect to $\| \cdot \|_X$, provided that $\|(f, g)\|_H < \delta$.

**Proof.** The first statement follows from Lemma 2.5. The algebraic identity $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ and Hölder’s inequality yield

$$\|u_{n+1} - u_n\|_{L^3(0,\infty)L^6(-1,1)} = \|K_{f,g}(u_n) - K_{f,g}(u_{n-1})\|_{L^3(0,\infty)L^6(-1,1)}$$

$$\lesssim \int_0^\infty \|S(s)[u_n(s', \cdot)^3 - u_{n-1}(s', \cdot)^3]\|_{L^2(0,\infty)L^6(-1,1)} \, ds'$$

$$\lesssim \int_0^\infty \|u_n(s', \cdot)^3 - u_{n-1}(s', \cdot)^3\|_{L^2(-1,1)} \, ds'$$

$$\lesssim \int_0^\infty \|u_n(s', \cdot) - u_{n-1}(s', \cdot)\|_{L^6(-1,1)} \left(\|u_n(s', \cdot)\|_{L^6(-1,1)}^2 + \|u_{n-1}(s', \cdot)\|_{L^6(-1,1)}^2\right) \, ds'$$

$$\lesssim \|u_n - u_{n-1}\|_{L^3(0,\infty)L^6(-1,1)}^2 \left(\|u_n\|_{L^3(0,\infty)L^6(-1,1)}^2 + \|u_{n-1}\|_{L^3(0,\infty)L^6(-1,1)}^2\right).$$

Analogously, we obtain the bound

$$\|u_{n+1} - u_n\|_{L^\infty(0,\infty)L^6(-1,1)} \lesssim \|u_n - u_{n-1}\|_{L^3(0,\infty)L^6(-1,1)} \left(\|u_n\|_{L^3(0,\infty)L^6(-1,1)}^2 + \|u_{n-1}\|_{L^3(0,\infty)L^6(-1,1)}^2\right)$$

and in summary, $\|u_{n+1} - u_n\|_X \leq CM^2\delta^2\|u_n - u_{n-1}\|_X$ for some constant $C > 0$. Thus, by choosing $\delta$ sufficiently small, we find $\|u_{n+1} - u_n\|_X \leq \frac{1}{2}\|u_n - u_{n-1}\|_X$ for all $n \in \mathbb{N}$ and this implies the claim. □

As a consequence of Lemma 2.6, the sequence $(u_n)_{n \in \mathbb{N}}$ converges to an element

$$u \in C([0, \infty), L^6_{\text{odd}}(-1,1)) \cap L^3((0, \infty), L^6_{\text{odd}}(-1,1)) \cap L^\infty((0, \infty), L^6_{\text{odd}}(-1,1)),$$

which satisfies Eq. (2.4). It remains to prove the uniqueness.

**Lemma 2.7.** Let $f, g \in C^\infty([-1,1])$ be odd. Then there exists at most one function $u \in C([0, \infty), L^6_{\text{odd}}(-1,1))$ that satisfies Eq. (2.4).

**Proof.** Suppose $u, \bar{u} \in C([0, \infty), L^6_{\text{odd}}(-1,1))$ satisfy Eq. (2.4) and let $s_0 > 0$ be arbitrary. Then, for $s \in [0, s_0]$, we have

$$\|u(s, \cdot) - \bar{u}(s, \cdot)\|_{L^6(-1,1)} \leq \int_0^s \|S(s - s')[u(s', \cdot)^3 - \bar{u}(s', \cdot)^3]\|_{L^6(-1,1)} \, ds'$$

$$\lesssim \int_0^s \|u(s', \cdot)^3 - \bar{u}(s', \cdot)^3\|_{L^2(-1,1)} \, ds'$$

$$\lesssim \left(\|u\|_{L^\infty(0, s_0)L^6(-1,1)}^2 + \|\bar{u}\|_{L^\infty(0, s_0)L^6(-1,1)}^2\right) \times \int_0^s \|u(s', \cdot) - \bar{u}(s', \cdot)\|_{L^6(-1,1)} \, ds'$$
and Gronwall’s inequality implies that \( \|u(s, \cdot) - \tilde{u}(s, \cdot)\|_{L^2(-1, 1)} = 0 \) for all \( s \in [0, s_0] \). □

3. **The hyperboloidal initial value problem for the free wave equation**

Now we turn to the proof of Theorem 1.3 and start with the hyperboloidal initial value problem for the free wave equation, i.e., we study

\[
\begin{aligned}
\left\{ \begin{array}{l}
\partial^2_s + 2y\partial_y + \partial_s - (1 - y^2)\partial_y^2 + 2y\partial_y]u(s, y) = 0 \quad (s, y) \in (0, \infty) \times (-1, 1) \\
u(s, y) = f(y), \quad \partial_s u(s, y) = g(y) \\
\end{array} \right.
\end{aligned}
\tag{3.1}
\]

for an unknown \( u : [0, \infty) \times (-1, 1) \to \mathbb{R} \) and given data \( f, g : (-1, 1) \to \mathbb{R} \).

3.1. **Classical solution of the initial value problem.** The solution to (3.1) can be given explicitly. This is a straightforward consequence of the fact that the general solution of Eq. (1.3) is of the form \( v(t, x) = F(t - x) + G(t + x) \).

**Definition 3.1.** For \( f, g \in C^\infty(-1, 1) \) and \( (s, y) \in [0, \infty) \times (-1, 1) \) we set

\[
u_{f,g}(s, y) := f(0) - \frac{1}{2} \int_{-1+e^{-s}(1+y)}^0 (1 - x) f'(x) dx + \frac{1}{2} \int_0^{1-e^{-s}(1-y)} (1 + x) f'(x) dx
\]

\[ + \frac{1}{2} \int_{-1+e^{-s}(1+y)}^{1-e^{-s}(1-y)} g(x) dx.
\]

**Lemma 3.2 (Existence and uniqueness of smooth solutions).** Let \( f, g \in C^\infty(-1, 1) \). Then \( u_{f,g} \in C^\infty([0, \infty) \times (-1, 1)) \) and \( u = u_{f,g} \) is a solution to (3.1). Furthermore, this solution is unique in \( C^\infty([0, \infty) \times (-1, 1)) \).

**Proof.** Since \( -1 + e^{-s}(1 + y) \in (-1, 1) \) and \( 1 - e^{-s}(1 - y) \in (-1, 1) \) for all \( s \geq 0 \) and \( y \in (-1, 1) \), it is evident that \( u_{f,g} \in C^\infty([0, \infty) \times (-1, 1)) \) and a straightforward computation shows that \( u = u_{f,g} \) solves (3.1). In fact, the formula for \( u_{f,g} \) is derived from the general solution \( v(t, r) = F(t - r) + G(t + r) \) of Eq. (1.3) and thus, \( u_{f,g} \) is necessarily unique in \( C^\infty([0, \infty) \times (-1, 1)) \). □

**Lemma 3.3 (Boundedness of the energy).** Let \( f, g \in C^\infty(-1, 1) \) with \( \|(f, g)\|_H < \infty \). Then we have

\[
\|(u_{f,g}(s, \cdot), \partial_s u_{f,g}(s, \cdot))\|_H \lesssim \|(f, g)\|_H
\]

for all \( s \geq 0 \).

**Proof.** This is a simple exercise. □

3.2. **Solution for odd data and Strichartz estimates.** The existence of the constant finite-energy solution \( u(s, y) = 1 \) precludes the possibility of Strichartz estimates. Consequently, we restrict ourselves to odd data \( f, g \in C^\infty(-1, 1) \). Then the solution \( u_{f,g} \) is given by

\[
u_{f,g}(s, y) = \frac{1}{2} \int_{1-e^{-s}(1+y)}^{1-e^{-s}(1-y)} [(1 + x) f'(x) + g(x)] dx.
\]

The following simple Sobolev embedding shows that the energy is strong enough to control \( L^q \), provided \( q < \infty \).

**Lemma 3.4.** Let \( q \in [1, \infty) \). Then we have the bound

\[
\|f\|_{L^q(-1, 1)} \lesssim \left\| (1 - |\cdot|^2)^{\frac{q}{2}} f' \right\|_{L^2(-1, 1)}
\]

for all odd \( f \in C^1(-1, 1) \) such that the right-hand side is finite.

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Proof. By the fundamental theorem of calculus and the oddness of \( f \), we infer

\[ f(y) = \int_0^y f'(x)dx \]

for all \( y \in (-1, 1) \) and Cauchy-Schwarz yields

\[
|f(y)| \leq \int_0^{|y|} |f'(x)|dx = \int_0^{|y|} (1 - x^2)^{-\frac{1}{2}}(1 - x^2)^{\frac{1}{2}}|f'(x)|dx \\
\leq \left( \int_0^{|y|} (1 - x^2)^{-1}dx \right)^{\frac{1}{2}} \left\| (1 - |x|^2)^{\frac{1}{2}}f' \right\|_{L^2(-1,1)}.
\]

Since

\[
\int_0^{|y|} (1 - x^2)^{-1}dx \lesssim |\log(1 - y^2)| + 1
\]

and the square root of the latter function belongs to \( L^q(-1,1) \) for any \( q \in [1,\infty) \), the stated bound follows. \( \square \)

**Proposition 3.5** (Strichartz estimates for the free equation). Let \( p \in [2,\infty] \) and \( q \in [1,\infty) \). Then we have the Strichartz estimates

\[
\|u_{f,g}\|_{L^p(0,\infty)L^q(-1,1)} \lesssim \|(f,g)\|_H
\]

for all odd \( f,g \in C^\infty(-1,1) \) with \( \|(f,g)\|_H < \infty \).

Proof. The case \( p = \infty \) is a consequence of Lemmas 3.3 and 3.4. Thus, it suffices to prove the bound

\[
\|u_{f,g}\|_{L^2(1,\infty)L^q(-1,1)} \lesssim \|(f,g)\|_H.
\]

We first consider the case \( g = 0 \). Then we have

\[
|u_{f,0}(s,y)| \lesssim \int_{1-e^{-s}(1-y)}^{1-e^{-s}(1+y)} |f'(x)|dx = e^{-s} \int_{1-y}^{1+y} |f'(1 - e^{-s}x)|dx \\
= \int_0^2 1_{[1-y,1+y]}(x) e^{-s} f'(1 - e^{-s}x)|dx
\]

for all \( y \in [0,1) \) and thus, by Minkowski’s inequality and the oddness of \( u_{f,0}(s,\cdot) \),

\[
\|u_{f,0}(s,\cdot)\|_{L^q(-1,1)} \lesssim \left\| \int_0^2 1_{[1-y,1+y]}(x) e^{-s} f'(1 - e^{-s}x)|dx \right\|_{L^q_0(0,1)} \\
\lesssim \int_0^2 \|1_{[1-y,1+y]}(x)\|_{L^q_0(0,1)} e^{-s} f'(1 - e^{-s}x)|dx.
\]

Now note that \( 1_{[1-y,1+y]}(x) \leq 1_{[1-x,1]}(y) \) for all \( x \in [0,2] \) and \( y \in [0,1] \). This yields \( \|1_{[1-y,1+y]}(x)\|_{L^q_0(0,1)} \lesssim x^{\frac{1}{q}} \) and thus,

\[
\|u_{f,0}(s,\cdot)\|_{L^q(-1,1)} \lesssim \int_0^2 x^{\frac{1}{q}} |e^{-s} f'(1 - e^{-s}x)|dx.
\]

Consequently,

\[
\|u_{f,0}\|_{L^2(1,\infty)L^q(-1,1)} \lesssim \left\| \int_0^2 x^{\frac{1}{q}} |e^{-s} f'(1 - e^{-s}x)|dx \right\|_{L^2(1,\infty)} \\
\lesssim \int_0^2 x^{\frac{1}{q}} |e^{-s} f'(1 - e^{-s}x)|dx
\]
again by Minkowski’s inequality. Now we have
\[
\|e^{-s}f'(1 - e^{-s}x)\|_{L^2(1, \infty)}^2 = \int_1^\infty |f'(1 - e^{-s}x)|^2 e^{-2s} ds \\
= x^{-2} \int_{1-e^{-1}x}^1 (1 - \eta)|f'(\eta)|^2 d\eta \\
\lesssim x^{-2} \|(f, 0)\|_H^2
\]
for all \( x \in (0, 2) \) and in summary, we obtain
\[
\|u_{f,0}\|_{L^2(1, \infty)L^2(-1,1)} \lesssim \|(f, 0)\|_H \int_0^2 x^{1/2 - 1} dx \lesssim \|(f, 0)\|_H.
\]

The case \( f = 0 \) is much simpler and it suffices to note that
\[
|u_{0,g}(s, y)| \leq \int_{1-e^{-1}(1-y)}^{1-e^{-1}(1+y)} |g(x)| dx \lesssim e^{-\frac{s}{2}} \|g\|_{L^2(-1, 1)} \lesssim e^{-\frac{s}{2}} \|(0, g)\|_H
\]
for all \( s \geq 0 \) and \( y \in [0, 1] \).

In particular, Proposition \ref{prop:asymptotic} shows that the zero solution is asymptotically stable under odd perturbations in the energy space.

3.3. Semigroup formulation. For later purposes it is desirable to translate the results obtained so far into semigroup language. First, we need to define proper function spaces and operators.

Definition 3.6. We set
\[
\tilde{\mathcal{H}} := \{ f = (f_1, f_2) \in C^\infty([-1, 1]) \times C^\infty([-1, 1]) : f \text{ is odd} \}.
\]
The vector space \( \tilde{\mathcal{H}} \) equipped with the inner product
\[
(f \mid g)_H := \int_{-1}^1 (1 - y^2)f_1'(y)\overline{g_1(y)} dy + \int_{-1}^1 f_2(y)\overline{g_2(y)} dy
\]
is a pre-Hilbert space and we denote by \( \mathcal{H} \) its completion. Furthermore, we consider the formal differential expression
\[
\mathcal{L}_0 f(y) := \begin{pmatrix} (1 - y^2)f_1''(y) - 2yf_1'(y) - 2yf_2'(y) - f_2(y) \\ f_2(y) \end{pmatrix}
\]
and define the operator \( \tilde{\mathcal{L}}_0 : \mathcal{D}(\tilde{\mathcal{L}}_0) \subset \mathcal{H} \to \mathcal{H} \) by \( \mathcal{D}(\tilde{\mathcal{L}}_0) := \tilde{\mathcal{H}} \) and \( \tilde{\mathcal{L}}_0 f := \mathcal{L}_0 f \).

By construction, \( \tilde{\mathcal{L}}_0 \) is a densely-defined operator on \( \mathcal{H} \). With these definitions at hand, the initial value problem \eqref{eq:ivp} can be written as
\[
\begin{cases}
\partial_s \Phi(s) = \tilde{\mathcal{L}}_0 \Phi(s) & \text{for all } s > 0 \\
\Phi(0) = f
\end{cases}
\]
for \( \Phi(s) = (u(s, \cdot), \partial_s u(s, \cdot)) \) and \( f = (f, g) \). The well-posedness of this initial value problem now means that (the closure of) \( \tilde{\mathcal{L}}_0 \) generates a semigroup.

Lemma 3.7. The operator \( \tilde{\mathcal{L}}_0 : \mathcal{D}(\tilde{\mathcal{L}}_0) \subset \mathcal{H} \to \mathcal{H} \) is closable and its closure \( \mathcal{L}_0 \) generates a strongly continuous one-parameter semigroup \( \{S_0(s) \in \mathcal{B} \mathcal{H} : s \geq 0 \} \). Furthermore, we have the estimate \( \|S_0(s)f\|_H \leq \|f\|_H \) for all \( s \geq 0 \) and all \( f \in \mathcal{H} \).
Proof. A straightforward computation shows that
\[ \text{Re}(\mathcal{L}_0 f | f)_{\mathcal{H}} = -|f_2(-1)|^2 - |f_2(1)|^2 \leq 0 \]
for all \( f = (f_1, f_2) \in \mathcal{D}(\mathcal{L}_0) \) (this is just an instance of the energy identity Eq. (1)). Furthermore, for \( g = (g_1, g_2) \in \mathcal{H} \) we set \( g(y) := yg'_1(y) + g_1(y) + \frac{1}{2}g_2(y) \) and
\[ f_1(y) := \frac{1}{1+y} \int_{-1}^{y} (1+x)g(x)dx + \frac{1}{1-y} \int_{y}^{1} (1-x)g(x)dx. \]
Note that \( f_1 \) is odd and belongs to \( C^\infty([-1,1]) \). We set \( f := (f_1, f_1 - g_1) \). Then we have \( f \in \mathcal{D}(\mathcal{L}_0) \) and a straightforward computation shows that \( (1-\mathcal{L}_0)f = g \). Since \( g \in \mathcal{H} \) was arbitrary, we see that the range of \( 1-\mathcal{L}_0 \) is dense in \( \mathcal{H} \) and an application of the Lumer-Phillips theorem (see e.g. [5], p. 83, Theorem 3.15) completes the proof. \( \square \)

Corollary 3.8. We have \( \sigma(\mathcal{L}_0) \subset \{ z \in \mathbb{C} : \text{Re} \, z \leq 0 \} \).

Proof. The statement is a consequence of the growth bound in Lemma 3.7 and [5], p. 55, Theorem 1.10. \( \square \)

Remark 3.9. In fact, we have \( \sigma_p(\mathcal{L}_0) = \{ z \in \mathbb{C} : \text{Re} \, z < 0 \} \) (and hence \( \sigma(\mathcal{L}_0) = \{ z \in \mathbb{C} : \text{Re} \, z \leq 0 \} \)). This follows easily by noting that for any \( \lambda \in \mathbb{C} \), the function \( f = (f_1, \lambda f_1) \) with
\[ f_1(y) := (1+y)^{-\lambda} - (1-y)^{-\lambda} \]
satisfies \( (\lambda - \mathcal{L}_0)f = 0 \). However, we omit a formal proof of this result since it is not needed in the following.

4. The Wave Equation with a Potential

Now we move on to the main problem and add a potential \( V \in C^\infty([-1,1]) \). In order to retain the parity symmetry, we require \( V \) to be even. That is to say, we study the initial value problem
\[
\begin{align*}
\left[ \partial^2_x + 2y\partial_x \partial_y + \partial_y - (1-y^2) \partial_y^2 + 2y\partial_y + V(y) \right] u(s,y) &= 0 \quad (s,y) \in (0,\infty) \times (-1,1) \\
u(s,y) &= f(y) \quad (s,y) \in \{ 0 \} \times (-1,1) \\
\partial_x u(s,y) &= g(y) \quad (s,y) \in (0,\infty) \times (-1,1)
\end{align*}
\]  
(4.1)

4.1. Semigroup formulation. We immediately switch to the semigroup picture. Note that by Lemma 3.4, the operator \((f_1, f_2) \mapsto (0, -Vf_1)\) is bounded on \( \mathcal{H} \).

Definition 4.1. Let \( V \in C^\infty([-1,1]) \) be even. Then we define the bounded operator \( \mathcal{L}_V : \mathcal{H} \to \mathcal{H} \) by
\[ \mathcal{L}_V f := \begin{pmatrix} 0 \\ -Vf_1 \end{pmatrix}. \]
Furthermore, we set \( \mathcal{L}_V := \mathcal{L}_0 + \mathcal{L}_V \), where \( \mathcal{L}_0 \) is the closure of \( \mathcal{L}_0 \), see Lemma 3.7.

Eq. (4.1) can be written as
\[
\begin{align*}
\partial_s \Phi(s) &= \mathcal{L}_V \Phi(s) \quad \text{for all } s > 0 \\
\Phi(0) &= (f,g)
\end{align*}
\]
and the abstract theory immediately tells us that this initial value problem is well-posed.
Lemma 4.2. The operator \( L_V : D(L_0) \subset \mathcal{H} \rightarrow \mathcal{H} \) generates a strongly continuous one-parameter semigroup \( \{ S_V(s) \in \mathcal{B}(\mathcal{H}) : s \geq 0 \} \) and we have the bound
\[
\| S_V(s)f \|_\mathcal{H} \leq e^{\| L_V \| s} \| f \|_\mathcal{H}
\]
for all \( s \geq 0 \) and all \( f \in \mathcal{H} \).

Proof. The statement is a consequence of the bounded perturbation theorem, see e.g. [3], p. 158, Theorem 1.3.

4.2. Analysis of the generator. In order to relate the semigroup formulation to the classical picture, we need some technical results on the generator \( L_V \). The point is that the latter is only abstractly defined as the closure of \( \tilde{L}_V := \tilde{L}_0 + L'_V \).

Lemma 4.3. Let \( n \in \mathbb{N} \), \( \delta \in (0, 1) \), and \( I_\delta := (-1 + \delta, 1 - \delta) \). Then we have the bound
\[
\| f_1 \|_{W^n,\infty(I_\delta)} + \| f_2 \|_{W^{n-1,\infty}(I_\delta)} \lesssim \| f \|_\mathcal{H} + \| \tilde{L}_V^n f \|_\mathcal{H}
\]
for all \( f = (f_1, f_2) \in \tilde{\mathcal{H}} \).

Proof. Since \( f_1 \) and \( f_2 \) are odd, we have \( f_j(y) = \int_0^y f_j(x) dx, \ j \in \{1, 2\} \), and Cauchy-Schwarz yields
\[
\| f_1 \|_{L^\infty(I_\delta)} \lesssim \| f'_1 \|_{L^2(I_\delta)} \lesssim \| f \|_\mathcal{H}
\]
\[
\| f_2 \|_{L^\infty(I_\delta)} \lesssim \| f'_2 \|_{L^2(I_\delta)} = \| [\tilde{L}_V f']_1 \|_{L^2(I_\delta)} \lesssim \| \tilde{L}_V f \|_\mathcal{H}.
\]
Furthermore,
\[
f''_1(y) = \frac{1}{1 - y^2} \left[ [\tilde{L}_V f]_2(y) + 2y f'_1(y) + 2y f'_2(y) + f_2(y) + V(y) f_1(y) \right]
\]
and thus, by the one-dimensional Sobolev embedding,
\[
\| f'_1 \|_{L^\infty(I_\delta)} \lesssim \| f'_1 \|_{L^2(I_\delta)} + \| f''_1 \|_{L^2(I_\delta)} \lesssim \| f \|_\mathcal{H} + \| \tilde{L}_V f \|_\mathcal{H}.
\]
This settles the case \( n = 1 \) and from here we proceed inductively.

Corollary 4.4. Let \( n \in \mathbb{N} \). If \( f \in D(L^n_V) \) then \( f \) can be identified with an odd function in \( C^n(-1, 1) \times C^{n-1}(-1, 1) \).

Proof. Fix \( \delta \in (0, 1) \), set \( I_\delta := (-1 + \delta, 1 - \delta) \), and let \( f \in D(L^n_V) \). Then there exists a sequence \( (f_k)_{k \in \mathbb{N}} \subset \mathcal{H} \) such that \( f_k \rightarrow f \) and \( \tilde{L}_V^n f_k \rightarrow L^n_V f \). In particular, \((f_k)_{k \in \mathbb{N}} \) and \((\tilde{L}_V^n f_k)_{k \in \mathbb{N}} \) are Cauchy sequences with respect to \( \| \cdot \|_\mathcal{H} \). By Lemma 4.3, we see that \( f_k \) converges to an odd function in \( C^n(I_\delta) \times C^{n-1}(I_\delta) \). Since \( \delta \in (0, 1) \) was arbitrary, \((f_k)_{k \in \mathbb{N}} \) converges pointwise on \((-1, 1)\) to an odd function in \( C^1(-1, 1) \times C(-1, 1) \), which may be identified with \( f \).

Remark 4.5. From now on we will implicitly make the identification suggested in Corollary 4.4. Consequently, any \( f \in D(L^n_V) \) is an odd function in \( C^n(-1, 1) \times C^{n-1}(-1, 1) \) and we have the inclusion \( D(L^n_V) \subset C^{n-1}(-1, 1) \times C^{n-1}(-1, 1) \).

Corollary 4.6. On \( D(L^n_0) \), \( L_0 \) acts as a classical differential operator, i.e., if \( f \in D(L^n_0) \subset C^2(-1, 1) \times C^1(-1, 1) \), we have \( L_0 f = L^n_0 f \) on \((-1, 1)\).

Proof. Let \( f \in D(L^n_0) \). Then there exists a sequence \( (f_k)_{k \in \mathbb{N}} \) with \( \tilde{L}_0^n f_k \rightarrow L^n_0 f \) for \( n \in \{0, 1, 2\} \). By the definition of \( \tilde{L}_0 \) and Lemma 4.3, we see that \((L^n_0 f_k)_{k \in \mathbb{N}} \) converges pointwise on \((-1, 1)\) to \( L^n_0 f \in C^1(-1, 1) \times C(-1, 1) \), and the latter function is identified with \( L^n_0 f \).

Corollary 4.7. Let \( f = (f, g) \in \tilde{\mathcal{H}} \) and set \( u(s, \cdot) := [S_0(s)f]_1 \). Then we have \( u = u_{f,g} \).
Proof. We have \( f \in D(L_0^n) \) for any \( n \in \mathbb{N} \). Thus, by [5], p. 124, Proposition 5.2, we obtain \( S_0(s)f \in D(L_0^n) \) for all \( s \geq 0 \) and any \( n \in \mathbb{N} \). Furthermore, since \( \partial^\alpha_+ S_0(s)f = S_0(s)L_0^n f \), it follows that \( u \in C^\infty([0, \infty) \times (-1, 1)) \). Thus, by Corollary 4.6, \( u \) is a smooth finite-energy solution of Eq. (4.11) and by Lemma 5.2 we must have \( u = u_{f,g} \). \( \square \)

4.3. Spectral properties. The special structure of the operator \( \mathbf{L}_t^0 \) allows us to obtain important spectral information, even at this level of generality. First, we need a simple compactness result.

**Lemma 4.8.** Let \((f_n)_{n \in \mathbb{N}} \subset C^1(-1,1)\) be a sequence of odd functions that satisfy

\[
\left\| (1 - | \cdot |^2) \frac{1}{2} f_n' \right\|_{L^2(-1,1)} \lesssim 1
\]

for all \( n \in \mathbb{N} \). Then there exists a subsequence of \((f_n)_{n \in \mathbb{N}}\) that is Cauchy in \( L^2(-1,1) \).

**Proof.** We mimic the classical proof of the Arzelà-Ascoli theorem. The set \((-1,1) \cap \mathbb{Q}\) is countable and dense in \((-1,1)\) and we write \((-1,1) \cap \mathbb{Q} = \{ y_j : j \in \mathbb{N} \} \). By the fundamental theorem of calculus and the oddness of \( f_n \), we have

\[
f_n(y) = \int_0^y f_n'(x) dx
\]

and thus, by Cauchy-Schwarz,

\[
|f_n(y)| \leq \int_0^y |f_n'(x)| dx = \int_0^y (1 - x^2)^{-\frac{1}{2}}(1 - x^2)^{\frac{1}{2}} |f_n'(x)| dx
\]

\[
\leq \left( \int_0^y (1 - x^2)^{-1} dx \right)^{1/2} \left\| (1 - | \cdot |^2) \frac{1}{2} f_n' \right\|_{L^2(-1,1)}
\]

\[
\lesssim |\log(1-y^2)|^{\frac{1}{2}} + 1
\]

for all \( y \in (-1,1) \) and all \( n \in \mathbb{N} \). Since \( y_j \in (-1,1) \), this estimate shows that for each \( j \in \mathbb{N} \), the sequence \((f_n(y_j))_{n \in \mathbb{N}} \subset \mathbb{C}\) is bounded. By Cantor’s classical diagonal argument we extract a subsequence \((f_{n_k})_{k \in \mathbb{N}}\) of \((f_n)_{n \in \mathbb{N}}\) such that for each \( j \in \mathbb{N} \), \((f_{n_k}(y_j))_{k \in \mathbb{N}}\) is Cauchy in \( \mathbb{C} \).

Now note that for any \( \delta \in (0,1) \), we have the bound

\[
|f_n(x) - f_n(y)| \lesssim \delta^{-\frac{1}{2}} |x - y|^{\frac{1}{2}}
\]

for all \( x, y \in [-1+\delta, 1-\delta] \) and \( n \in \mathbb{N} \). Indeed,

\[
f_n(x) - f_n(y) = \int_y^x f_n'(t) dt
\]

and thus, by Cauchy-Schwarz,

\[
|f_n(x) - f_n(y)| \leq |x - y|^{\frac{1}{2}} \left( \int_{y-\delta}^{y+\delta} |f_n'(t)|^2 dt \right)^{1/2} \lesssim \delta^{-\frac{1}{2}} |x - y|^{\frac{1}{2}} \left\| (1 - | \cdot |^2) \frac{1}{2} f_n' \right\|_{L^2(-1,1)}
\]

\[
\lesssim \delta^{-\frac{1}{2}} |x - y|^{\frac{1}{2}},
\]

as claimed. As a consequence of this estimate, \((f_n)_{n \in \mathbb{N}}\) is equicontinuous on \([-1+\delta, 1-\delta]\) and the density of \( \{ y_j : j \in \mathbb{N} \} \) implies that \((f_{n_k})_{k \in \mathbb{N}}\) is Cauchy in \( L^\infty(-1,1) \).

Now let \( \epsilon \in (0,1) \). Then there exists an \( N_\epsilon \in \mathbb{N} \) such that

\[
\| f_{n_k} - f_{n_l} \|_{L^\infty(-1,1+\epsilon,1-\epsilon)} \leq \epsilon
\]
for all $k, \ell \geq N_\varepsilon$. Consequently,
\[
\|f_{n_k} - f_{n_\ell}\|_{L^2(-1,1)}^2 \leq \|f_{n_k} - f_{n_\ell}\|_{L^2(-1+\epsilon,1-\epsilon)}^2 + \|f_{n_k} - f_{n_\ell}\|_{L^2(-1-1+\epsilon)}^2 + \|f_{n_k} - f_{n_\ell}\|_{L^2(1-\epsilon,1)}^2 \lesssim \varepsilon^\frac{1}{2}
\]
for all $k, \ell \geq N_\varepsilon$ since
\[
\|f_n\|_{L^2(-1,1+\epsilon)}^2 \lesssim \int_{-1}^{-1+\epsilon} |\log(1-y^2)| dy \lesssim \varepsilon^\frac{1}{2}
\]
for all $n \in \mathbb{N}$ and analogously for $\|f_n\|_{L^2(1-\epsilon,1)}^2$. \hfill \Box

We continue with a simple resolvent bound. Note that this bound is just a consequence of the fact that the operator $L_V'$ maps the first component to the second component.

**Lemma 4.9.** Let $\varepsilon > 0$. Then we have the bound
\[
\|L_V'R_{L_0}(\lambda)f\|_{\mathcal{H}} \lesssim \frac{1}{|\lambda|} \|f\|_{\mathcal{H}}
\]
for all $\lambda \in \mathbb{C}$ with $\text{Re}\; \lambda \geq \varepsilon$ and all $f \in \mathcal{H}$.

**Proof.** To begin with, let $f = (f_1, f_2) \in D(L_0)$ and set $u := R_{L_0}(\lambda)f$. Then $u = (u_1, u_2) \in D(L_0)$ and $(\lambda - L_0)u = f$. By Corollary 4.4 $u \in C^2(-1,1) \times C^1(-1,1)$ and Corollary 4.6 yields $(\lambda - L_0)u = f$. The first component of this equation reads $\lambda u_1 - u_2 = f_1$ or, equivalently,
\[
[R_{L_0}(\lambda)f]_1 = \frac{1}{\lambda} ((R_{L_0}(\lambda)f)_2 + f_1).
\]
Consequently,
\[
\|L'_V R_{L_0}(\lambda)f\|_{\mathcal{H}} = \|V[R_{L_0}(\lambda)f]_1\|_{L^2(-1,1)} \lesssim \frac{1}{|\lambda|} \left( \|R_{L_0}(\lambda)f\|_{L^2(-1,1)} + \|f_1\|_{L^2(-1,1)} \right)
\lesssim \frac{1}{|\lambda|} \left( \|R_{L_0}(\lambda)f\|_{\mathcal{H}} + \|f\|_{\mathcal{H}} \right) \lesssim \frac{1}{|\lambda|} \|f\|_{\mathcal{H}}
\]
by Lemma 3.4 and 5, p. 55, Theorem 1.10. Thus, the claim follows by density. \hfill \Box

**Lemma 4.10.** The operator $L'_V : \mathcal{H} \to \mathcal{H}$ is compact. As a consequence, the set
\[
\sigma(L_V') \cap \{z \in \mathbb{C} : \text{Re} \; z > 0\}
\]
consists of finitely many eigenvalues of finite algebraic multiplicity.

**Proof.** Let $(f_n)_{n \in \mathbb{N}} \subset \mathcal{H}$ be a bounded sequence and write $f_n = (f_{n,1}, f_{n,2})$. Then we have
\[
\left\| \left(1 - |\cdot|^2 \right)^{\frac{3}{2}} f_{n,1} \right\|_{L^2(-1,1)} \lesssim 1
\]
for all $n \in \mathbb{N}$ and Lemma 1.5 implies that $(f_{n,1})_{n \in \mathbb{N}}$ has a subsequence, again denoted by $(f_{n,1})_{n \in \mathbb{N}}$, that is Cauchy in $L^2(-1,1)$. We have
\[
\|L'_V f_m - L'_V f_n\|_{\mathcal{H}} = \|V(f_{m,1} - f_{n,1})\|_{L^2(-1,1)} \lesssim \|f_{m,1} - f_{n,1}\|_{L^2(-1,1)}
\]
and thus, $(L'_V f_n)_{n \in \mathbb{N}}$ has a convergent subsequence. This shows that $L'_V$ is compact. By Corollary 3.8 $R_{L_0}$ is holomorphic on the open right half-plane $H^+ := \{z \in \mathbb{C} : \text{Re} \; z > 0\}$ and the obvious identity $\lambda - L_V = (I-L'_V R_{L_0}(\lambda))(\lambda - L_0)$ shows that $\lambda \in H^+$ belongs to $\rho(L_V')$ if and only if $I-L'_V R_{L_0}(\lambda)$ is bounded invertible. By Lemmas 4.2 and 4.9...
we immediately see that \(\sigma(L_V) \cap H^+\) is bounded. Furthermore, the map \(\lambda \mapsto L_V' R_{L_0}(\lambda)\) is holomorphic on \(H^+\) and has values in the set of compact operators on the Hilbert space \(\mathcal{H}\). Consequently, the analytic Fredholm theorem (see e.g. [11], p. 194, Theorem 3.14.3) implies that the inverse \(\lambda \mapsto [I - L_V' R_{L_0}(\lambda)]^{-1}\) has finitely many poles of finite order with finite rank residues. For every \(\lambda \in \sigma(L_V) \cap H^+\) we therefore have \(1 \in \sigma(L_V' R_{L_0}(\lambda))\) and thus, \(1 \in \sigma_p(L_V' R_{L_0}(\lambda))\). Let \(f_\lambda \in \mathcal{H}\) be an associated eigenfunction. Then \(R_{L_0}(\lambda)f_\lambda \in \mathcal{D}(L_V)\) is an eigenfunction of \(L_V\) and we see that every \(\lambda \in \sigma(L_V) \cap H^+\) is an eigenvalue of \(L_V\) and the corresponding spectral projection has finite rank. \(\square\)

Lemma 4.10 allows us to remove the unstable part of the spectrum by a finite-rank projection.

**Definition 4.11.** Let \(\gamma : [0, 2\pi] \to \rho(L_V)\) be a positively oriented, regular, smooth, simple closed curve that encircles the set \(\sigma(L_V) \cap \{z \in \mathbb{C} : \text{Re } z > 0\}\) (the existence of such a curve is guaranteed by Lemma 4.10). Then we define

\[
P V := \frac{1}{2\pi i} \int_\gamma R_{L_V}(\lambda) d\lambda.
\]

Our goal now is to prove a set of Strichartz estimates for the *reduced semigroup* \(S_V(s)(I - P_V)\) under a suitable *spectral assumption* on \(L_V\). To this end, we first need to clarify the relation between the abstract Hilbert space \(\mathcal{H}\) and the standard Lebesgue spaces.

**Definition 4.12.** For \(q \in [1, \infty)\), we define the Banach space \(L^q_{\text{odd}}(-1, 1)\) as the completion of \(\{f \in C^\infty([-1, 1]) : f \text{ odd} \}\) with respect to \(\|\cdot\|_{L^q(-1,1)}\).

**Lemma 4.13.** Let \(q \in [1, \infty)\). Then there exists a linear, bounded, and injective map \(i : \mathcal{H} \to L^q_{\text{odd}}(-1, 1) \times L^2_{\text{odd}}(-1, 1)\).

**Proof.** For \(f \in \tilde{\mathcal{H}}\) we set \(i(f) := f\). By Lemma 3.3 we obtain the bound

\[
\|i(f)\|_{L^q(-1,1) \times L^2(-1,1)} \lesssim \|f\|_{\mathcal{H}}
\]

for all \(f \in \tilde{\mathcal{H}}\) and by density, \(i\) extends to a linear and bounded map \(i : \mathcal{H} \to L^q_{\text{odd}}(-1, 1) \times L^2_{\text{odd}}(-1, 1)\). To show the injectivity, suppose that \(i(f) = 0\) for \(f \neq 0\) in \(\mathcal{H}\). Then there exists a sequence \((f_n)_{n \in \mathbb{N}} \in \tilde{\mathcal{H}}\) such that \(f_n \to f\) in \(\mathcal{H}\) and \(i(f_n) \to i(f) = 0\) in \(L^q_{\text{odd}}(-1, 1) \times L^2_{\text{odd}}(-1, 1)\), as \(n \to \infty\). In particular, \(f_n \to f\) in \(\mathcal{H}\). Furthermore, since

\[
\int_{-1}^{1} (1 - y^2)f'(y)g'(y)dy = -\int_{-1}^{1} (1 - y^2)f(y)g''(y)dy + 2\int_{-1}^{1} yf(y)g'(y)dy
\]

for all \(f, g \in C^\infty([-1, 1])\), we see that \(f_n = i(f_n) \to 0\) in \(L^q_{\text{odd}}(-1, 1) \times L^2_{\text{odd}}(-1, 1)\) implies \(f_n \to 0\) in \(\mathcal{H}\) and the uniqueness of weak limits yields \(f = 0\). \(\square\)

**Remark 4.14.** By Lemma 4.13 we may identify \(f \in \mathcal{H}\) with \(i(f) \in L^q_{\text{odd}}(-1, 1) \times L^2_{\text{odd}}(-1, 1)\) and this yields the continuous embedding \(\mathcal{H} \hookrightarrow L^q_{\text{odd}}(-1, 1) \times L^2_{\text{odd}}(-1, 1)\).

Now we aim for proving the following Strichartz estimates for the reduced semigroup.

**Theorem 4.15.** Let \(V \in C^\infty([-1, 1])\) be even, \(p \in [2, \infty]\), and \(q \in [1, \infty)\). Furthermore, suppose that the operator \(L_V\) has no eigenvalues on the imaginary axis. Then we have the Strichartz estimates

\[
\| S_V(s)(I - P_V)f \|_{L^q_t(0,\infty)L^p_x(-1,1)} \lesssim \|(I - P_V)f\|_{\mathcal{H}}
\]

for all \(f \in \mathcal{H}\).
4.4. **Explicit representation of the semigroup.** First, we show that the reduced semigroup $S_V(s)(I - P_V)$ inherits the decay from the free semigroup $S_0$, up to an $\epsilon$-loss. This follows from the celebrated Gearhart-Prüss theorem and the simple resolvent bound from Lemma 4.16.

**Lemma 4.16.** Let $\epsilon > 0$. Then there exists a $C_\epsilon > 0$ such that

$$\|S_V(s)(I - P_V)f\|_H \leq C_\epsilon e^{Cs}\|(I - P_V)f\|_H$$

for all $s \geq 0$ and $f \in H$.

**Proof.** We denote by $L^s$ the part of $L$ in $\ker P$. Then $R_{L^s}(\lambda)$ is the part of $R_{L^s}(\lambda)$ in $\ker P$. By construction, $\sigma(L^s) \cap \{z \in \mathbb{C} : \Re z > 0\} = \emptyset$ and Lemma 4.19 together with the identity $\lambda - L_V = [I - L^s V]\lambda_0(\lambda - \lambda_0)$ shows that

$$\sup\{\|R_{L^s}(\lambda)\|_H : \Re \lambda \geq \epsilon\} < \infty.$$ 

Consequently, the Gearhart-Prüss Theorem, see e.g. [5], p. 302, Theorem 1.11, implies the claim. \hfill $\square$

In the following, we denote by $S^s_V : [0, \infty) \to \mathcal{B}(\ker P_V)$ the reduced semigroup, i.e., $S^s_V(s)f := S_V(s)f$ for all $f \in \ker P_V$. The generator of the semigroup $S^s_V$ is $L^s$, the part of $L$ in $\ker P$. From Lemma 4.16 and [5], p. 234, Corollary 5.15, we obtain the representation

$$S^s_V(s)f = \frac{1}{2\pi i} \lim_{\nu \to \infty} \int_{c-i\nu}^{c+i\nu} e^{\lambda s} R_L^s(\lambda)f d\lambda$$

for any $\epsilon > 0$ and $f \in D(L^s)$. If we set $\nu := (\lambda - L^s)^{-1}f$, we obtain $(\lambda - L^s)\nu = f$. Formally at least, this equation is equivalent to

$$\lambda u_1(y) - u_2(y) = f_1(y)$$

$$\lambda u_2(y) - (1 - y^2) u''_1(y) + 2y u'_1(y) + 2 y u''_2(y) + u_2(y) + V(y) u_1(y) = f_2(y)$$

and inserting the first equation into the second one yields

$$- (1 - y^2) u''_1(y) + 2(\lambda + 1)y u'_1(y) + \lambda(\lambda + 1) u_1(y) + V(y) u_1(y) = F_\lambda(y)$$

with $F_\lambda(y) := 2y f'_1(y) + (\lambda + 1) f_1(y) + f_2(y)$. Consequently, our next goal is to solve Eq. (4.2).

5. **The Green function**

In order to solve Eq. (4.2), we need to first construct a suitable fundamental system for the homogeneous equation

$$- (1 - y^2) u''(y) + 2(\lambda + 1)y u'(y) + \lambda(\lambda + 1) u(y) + V(y) u(y) = 0.$$ (5.1)

5.1. **Construction of a fundamental system.** In terms of $v(y) := (1 - y^2)^{\frac{1}{2}(\lambda + 1)} u(y)$, Eq. (5.1) reads

$$v''(y) + \frac{1 - \lambda^2}{(1 - y^2)^2} v(y) = \frac{V(y)}{1 - y^2} v(y).$$ (5.2)

**Definition 5.1.** For $y \in (-1, 1)$ and $\lambda \in \mathbb{C}$ we set

$$\psi_1(y, \lambda) := (1 - y)^{\frac{1}{2}(1+\lambda)}(1 + y)^{\frac{1}{2}(1-\lambda)}.$$
Note that
\[
\partial_y^2 \psi_1(y, \lambda) + \frac{1 - \lambda^2}{1 - y^2} \psi_1(y, \lambda) = 0 \tag{5.3}
\]
for all \( y \in (-1, 1) \) and \( \lambda \in \mathbb{C} \). We construct a perturbative solution to Eq. (5.2) with good control of the error near the singularity at \( y = 1 \).

**Proposition 5.2.** There exists a solution \( v(y) = v_1(y, \lambda) \) to Eq. (5.2) of the form
\[
v_1(y, \lambda) = \psi_1(y, \lambda) [1 + a_1(y, \lambda)],
\]
where the function \( a_1 \) satisfies \(|a_1(y, \lambda)| \lesssim (1 - y)^{\frac{3}{4}} |\lambda|^{-1} \) for all \( y \in [0, 1) \) and \( \lambda \in \mathbb{C} \) with \( \text{Re} \lambda \geq -\frac{1}{4} \). Furthermore, for all \( k, \ell, m \in \mathbb{N}_0 \), there exists a constant \( C_{k, \ell, m} > 0 \) such that
\[
|\partial_\kappa \partial_\omega^\ell \partial_y^m a_1(y, \kappa + i \omega)| \leq C_{k, \ell, m} (1 - y)^{\frac{3}{4} - k} |\omega|^{-1 - \ell}
\]
for all \( y \in [0, 1) \), \( \omega \in \mathbb{R} \), and \( \kappa \in [-\frac{1}{4}, \frac{1}{4}] \).

**Proof.** To begin with, we assume \( \lambda \neq 0 \) and define
\[
\psi_0(y, \lambda) := \psi_1(y, \lambda) - \psi_1(y, -\lambda).
\]
Note that \( W(\psi_0(\cdot, \lambda), \psi_1(\cdot, \lambda)) = 2\lambda \). Consequently, by the variation of parameters formula and Eq. (5.3), \( v_1 \) has to satisfy the integral equation
\[
v_1(y, \lambda) = \psi_1(y, \lambda) + \int_y^1 \psi_0(y, \lambda) \psi_1(x, \lambda) \psi_0(x, \lambda) \psi_1(x, \lambda) \frac{V(x)}{1 - x^2} v_1(x, \lambda) dx \tag{5.4}
\]
for all \( y \in [0, 1) \). Conversely, any continuous solution to Eq. (5.4) belongs to \( C^2([0, 1)) \) and solves Eq. (5.2). We write \( v_1 = \psi_1 h_1 \). Then, Eq. (5.4) is equivalent to the Volterra equation
\[
h_1(y, \lambda) = 1 + \int_y^1 K(y, x, \lambda) h_1(x, \lambda) dx \tag{5.5}
\]
with the kernel
\[
K(y, x, \lambda) = \frac{1}{2\lambda} \left[ \frac{\psi_0(y, \lambda)}{\psi_1(y, \lambda)} \psi_1(x, \lambda)^2 - \psi_0(x, \lambda) \psi_1(x, \lambda) \right] \frac{V(x)}{1 - x^2} = \frac{1}{2\lambda} \left[ 1 - \left( \frac{1 - y}{1 + y} \right)^{-\lambda} \left( \frac{1 - x}{1 + x} \right)^{\lambda} \right] V(x). \tag{5.6}
\]
We have the bound \(|K(y, x, \lambda)| \lesssim (1 - x)^{\frac{3}{4}} |\lambda|^{-1} \) for all \( 0 \leq y \leq x < 1 \) and all \( \lambda \in \mathbb{C} \setminus \{0\} \) with \( \text{Re} \lambda \geq -\frac{1}{4} \). If, in addition, \( |\lambda| \geq 1 \), this yields
\[
\int_0^1 \sup_{y \in (0, x)} |K(y, x, \lambda)| dx \lesssim |\lambda|^{-1} \lesssim 1
\]
and the Volterra existence theorem (see e.g. [10], Lemma 2.4) shows that Eq. (5.5) has a solution \( h_1(\cdot, \lambda) \in L^\infty(0, 1) \) satisfying \( \|h_1(\cdot, \lambda)\|_{L^\infty(0, 1)} \lesssim 1 \) for all \( \lambda \in \mathbb{C} \) with \( \text{Re} \lambda \geq -\frac{1}{4} \) and \( |\lambda| \geq 1 \). It follows that \( h_1(\cdot, \lambda) \in C([0, 1]) \) and
\[
|h_1(y, \lambda) - 1| \lesssim \int_y^1 |K(x, y, \lambda)| h_1(x, \lambda) dx \lesssim |\lambda|^{-1} \|h_1(\cdot, \lambda)\|_{L^\infty(0, 1)} \int_y^1 (1 - x)^{-\frac{1}{4}} dx \lesssim (1 - y)^{\frac{3}{4}} |\lambda|^{-1} \lesssim (1 - y)^{\frac{3}{4}} (\lambda^{-1}),
\]
which implies the claimed estimate on \( a_1 \).

The difficulty in proving the derivative bounds in the regime \( |\lambda| \geq 1 \) lies with the fact that \( \lambda = \kappa + i \omega \) appears in the exponent in Eq. (5.6). Thus, it seems that differentiating
with respect to $\omega$ does not improve the decay in $\omega$. This problem can be dealt with by a suitable change of variables. More precisely, we consider the diffeomorphism $\varphi : (0, \infty) \to (0, 1)$, $\varphi(\xi) := \frac{1 - e^{-\xi}}{1 + e^{-\xi}}$, with inverse $\varphi^{-1}(x) = -\log \frac{1 - x}{1 + x}$. We write $\lambda = \kappa + i\omega$ and it suffices to consider the case $\omega \geq 1$. Then we may rewrite Eq. (5.5) as

$$h_1(\varphi(\eta), \lambda) = \int_{\eta}^{\infty} K(\varphi(\eta), \varphi(\xi), \lambda)h_1(\varphi(\xi), \lambda)\varphi'(\xi)d\xi$$

$$= \frac{1}{\omega} \int_{\eta}^{\infty} K(\varphi(\eta), \varphi(\omega^{-1}\xi + \eta), \lambda)\varphi'(\omega^{-1}\xi + \eta)h_1(\varphi(\omega^{-1}\xi + \eta), \lambda)d\xi$$

$$= \frac{1}{2\lambda\omega} \int_{\eta}^{\infty} \left(1 - e^{- (\kappa\omega^{-1} + i)\xi}\right) V(\varphi(\omega^{-1}\xi + \eta))\varphi'(\omega^{-1}\xi + \eta)$$

$$\times h_1(\varphi(\omega^{-1}\xi + \eta), \lambda)d\xi$$

and from this representation the derivative bounds follow inductively.

In the case $|\lambda| \leq 1$ we need to argue differently due to the apparent singularity of $K(y, x, \lambda)$ at $\lambda = 0$. In fact, this singularity is removable because $\psi_0(y, 0) = 0$. In order to exploit this, we first note that

$$\partial_t \psi_1(y, t\lambda) = \frac{\lambda}{2} \log \left(1 + \frac{1 - y}{1 + y}\right) \psi_1(y, t\lambda)$$

for $t \in \mathbb{R}$ and then we use the fundamental theorem of calculus to write

$$\psi_0(y, \lambda) = \int_{0}^{1} \partial_t \psi_0(y, t\lambda)dt = \lambda \tilde{\psi}_0(y, \lambda)$$

with

$$\tilde{\psi}_0(y, \lambda) := \frac{1}{2} \log \left(1 + \frac{1 - y}{1 + y}\right) \int_{0}^{1} [\psi_1(y, t\lambda) + \psi_1(y, -t\lambda)] dt.$$ We have the bound

$$|\tilde{\psi}_0(y, \lambda)| \lesssim |\log(1 - y)| \int_{0}^{1} \left[(1 - y)^{1/2(1 + t\Re \lambda)} + (1 - y)^{1/2(1 - t\Re \lambda)}\right] dt$$

$$\lesssim |\log(1 - y)| \left[(1 - y)^{1/2(1 + \Re \lambda)} + (1 - y)^{1/2(1 - \Re \lambda)}\right]$$

and thus,

$$|K(y, x, \lambda)| \lesssim |\log(1 - x)|(1 - x)^{-\frac{1}{4}} \lesssim (1 - x)^{-\frac{1}{2}}$$

for all $0 \leq y \leq x < 1$ and $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$, $\Re \lambda \geq -\frac{1}{4}$. Consequently, a Volterra iteration yields the stated estimate for $a_1$. For the derivative bounds it suffices to note that each derivative with respect to $\omega$ or $\kappa$ produces a singular term $\log(1 - x)$ which, however, is harmless since $|\log(1 - x)|^n(1 - x)^{-\frac{1}{4}} \lesssim C_n(1 - x)^{-\frac{1}{2}}$ for any $n \in \mathbb{N}_0$. Consequently, the derivative bounds follow inductively. \hfill $\square$

**Definition 5.3.** In the following, $v_1$ always refers to the solution constructed in Proposition 5.2.

Proposition 5.2 shows that the error $a_1$ improves upon differentiation with respect to $\omega$. On the other hand, when differentiating with respect to $y$, the bounds get worse. Both operations have in common that taking a derivative results in the loss of one power of the respective variable in the estimate. This is a crucial property and we introduce a more economical notation to keep track of this behavior.
Definition 5.4. For $\alpha, \beta \in \mathbb{R}$, we write $f(y, \omega) = \mathcal{O}((1 - y)^\alpha \langle \omega \rangle^\beta)$ if for all $k, \ell \in \mathbb{N}_0$ there exist constants $C_{k, \ell} > 0$ such that

$$|\partial_y^k \partial_{\omega}^\ell f(y, \omega)| \leq C_{k, \ell} (1 - y)^{\alpha - k} \langle \omega \rangle^{\beta - \ell}$$

in a range of the variables $y$ and $\omega$ that is specified explicitly or follows from the context. In other words, the $\mathcal{O}$-terms may be formally differentiated. Such functions are said to be of symbol type. We also use self-explanatory variants of this notation.

Remark 5.5. By Proposition 5.2 we have, with $\omega = \text{Im} \lambda$,

$$W(v_1(\cdot, \lambda), v_1(\cdot, -\lambda)) = W(\psi_1(\cdot, \lambda), \psi_1(\cdot, -\lambda))[1 + \mathcal{O}((1 - y)^{\frac{1}{2}} \langle \omega \rangle^{-1})] + \psi_1(y, \lambda)\psi_1(y, -\lambda)\mathcal{O}((1 - y)^{-\frac{1}{2}} \langle \omega \rangle^{-1})$$

$$= 2\lambda[1 + \mathcal{O}((1 - y)^{\frac{1}{2}} \langle \omega \rangle^{-1})] + \mathcal{O}((1 - y)^{\frac{1}{2}} \langle \omega \rangle^{-1})$$

for all $y \in [0, 1)$ and $\lambda \in \mathbb{C}$ with $|\text{Re} \lambda| \leq \frac{1}{4}$. This expression is in fact independent of $y$ and thus, we may evaluate it at $y = 1$ which yields

$$W(v_1(\cdot, \lambda), v_1(\cdot, -\lambda)) = 2\lambda.$$

The bounds on the derivatives of $v_1$ are sufficient for our purposes and easy to work with but certainly not optimal, as the following result shows.

Lemma 5.6. Let $\text{Re} \lambda \geq -\frac{1}{4}$. Then we have

$$\frac{v_1(\cdot, \lambda)}{\psi_1(\cdot, \lambda)} \in C^\infty([0, 1]).$$

Proof. As in the proof of Proposition 5.2 we write $v_1 = \psi_1h_1$ and from Eqs. (5.5) and (5.6), we obtain

$$\partial_y h_1(y, \lambda) = -\frac{1}{1 - y^2} \left(\frac{1 - y}{1 + y}\right)^{-\lambda} \int_0^1 \left(\frac{1 - x}{1 + x}\right)^\lambda V(x)h_1(x, \lambda)dx.$$ 

The change of variables $x = y + t(1 - y)$ yields

$$\partial_y h_1(y, \lambda) = -(1 + y)^{\lambda - 1} \int_0^1 \left(\frac{1 - t}{1 + y + t(1 - y)}\right)^\lambda V(y + t(1 - y))h_1(y + t(1 - y), \lambda)dt$$

and from this expression the statement follows inductively. □

The solution $v_1$ is sufficient to construct the Green function for Eq. (4.2).

Definition 5.7. For $y \in [0, 1)$ and $\lambda \in \mathbb{C}$ with $\text{Re} \lambda \geq -\frac{1}{4}$ we set

$$u_1(y, \lambda) := (1 - y^2)^{-\frac{1}{2}(1 + \lambda)}v_1(y, \lambda).$$

Furthermore, for $|\text{Re} \lambda| \leq \frac{1}{4}$, we define

$$u_0(y, \lambda) := (1 - y^2)^{-\frac{1}{2}(1 + \lambda)}[v_1(0, -\lambda)v_1(y, \lambda) - v_1(0, \lambda)v_1(y, -\lambda)].$$

5.2. Regularity theory. We take up the opportunity to establish the link between $\Sigma_V$, see Definition 1.2 and the spectrum of $L_V$. The key observation in this respect is a regularity result for the operator $L_V$.

Lemma 5.8. For any $\lambda \in \mathbb{C}$ with $\text{Re} \lambda \geq 0$, we have $\ker(\lambda - L_V) \subset \tilde{H}$. 

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By definition, an odd solution of Eq. (5.1) on \((0,\infty)\) and Corollary 4.6 shows that \((\lambda - \lambda_0 - L'_V)f = 0\). As a consequence, \(f_2 = \lambda f_1\) and \(f_1\) is an odd solution of Eq. (5.1) on \((-1,1)\). Hence, it remains to show that \(f_1 \in C^\infty([-1,1])\).

\[ u_1(y,\lambda) = (1 - y^2)^{-\frac{1}{2}(1+\lambda)}\psi(y,\lambda) \frac{v_1(y,\lambda)}{\psi(y,\lambda)} = (1 + y)^{-\lambda} \frac{v_1(y,\lambda)}{\psi(y,\lambda)} \]

and Lemma 5.6 shows that \(u_1(\cdot,\lambda) \in C^\infty([0,1])\). Furthermore, by Proposition 5.2 \(u_1(y,\lambda) = (1 + y)^{-\lambda}[1 + O((1 - y)^2(\lambda - 1)^{-1})]\). In particular, there exists a \(c \in (0,1)\) such that \(|u_1(y,\lambda)| > 0\) for all \(y \in [c,1]\) and we set

\[ \tilde{u}_1(y,\lambda) := u_1(y,\lambda) \int_c^y \frac{(1 - x^2)^{-1-\lambda}}{u_1(x,\lambda)^2} dx. \]

Then \(\{u_1(\cdot,\lambda), \tilde{u}_1(\cdot,\lambda)\}\) is a fundamental system for Eq. (5.1) on \([c,1]\). As a consequence, there exist constants \(a, b \in \mathbb{C}\) such that

\[ f_1(y) = au_1(y,\lambda) + b\tilde{u}_1(y,\lambda) \]

for \(y \in [c,1]\). We have \(|\partial_y \tilde{u}_1(y,\lambda)| \gtrsim (1 - y)^{-1-\Re\lambda}\) for \(y \in [c,1]\) and thus,

\[ \int_c^1 (1 - y^2)|\partial_y \tilde{u}_1(y,\lambda)|^2 dy \gtrsim \int_c^1 (1 - y)^{-2\Re\lambda} dy = \infty. \]

Consequently, since \(\|f\|_H < \infty\), we must have \(b = 0\) and therefore, \(f_1 \in C^\infty([-1,1])\).

**Lemma 5.9.** We have

\[ \Sigma_V = \sigma_p(L_V) \cap \{z \in \mathbb{C} : \Re z \geq 0\}. \]

**Proof.** Let \(\lambda \in \Sigma_V\). Then \(\Re\lambda \geq 0\) and there exists a nontrivial, odd \(f_\lambda \in C^\infty([-1,1])\) that satisfies Eq. (5.1) for all \(y \in (-1,1)\). We set \(f := (f_\lambda, \lambda f_\lambda)\). Then \(f \in \tilde{H}\) and \((\lambda - \lambda_0 - L'_V)f = 0\), which implies that \(\lambda \in \sigma_p(L_V)\).

Conversely, if \(\Re\lambda \geq 0\) and \(\lambda \in \sigma_p(L_V)\), Lemma 5.8 implies that there exists a nontrivial \(f = (f_1, f_2) \in \tilde{H}\) such that \((\lambda - \lambda_0 - L'_V)f = 0\). In other words, \(f_2 = \lambda f_1\) and \(f_1\) is a nontrivial solution of Eq. (5.1). Consequently, \(\lambda \in \Sigma_V\).

Next, we relate the point spectrum of \(L_V\) to the value of \(u_1(y,\lambda)\) at \(y = 0\).

**Lemma 5.10.** Let \(\lambda \in \mathbb{C}\) with \(\Re\lambda \geq 0\). If \(u_1(0,\lambda) = 0\) then \(\lambda \in \sigma_p(L_V)\).

**Proof.** The function \(u_1(\cdot,\lambda)\) satisfies Eq. (5.1) for all \(y \in [0,1]\) and by evaluation at \(y = 0\), we find inductively that \(\partial_y^{2k}u_1(y,\lambda)|_{y=0} = 0\) for all \(k \in \mathbb{N}_0\) (here the assumption \(u_1(0,\lambda) = 0\) enters). We extend \(u_1(\cdot,\lambda)\) to \([-1,1]\) by setting \(u_1(-y,\lambda) := -u_1(y,\lambda)\) for \(y \in [0,1]\). Then \(u_1(\cdot,\lambda) \in C^\infty([-1,1])\), \(u_1(\cdot,\lambda)\) is odd and satisfies Eq. (5.1) for all \(y \in (-1,1)\). This means that \(\lambda \in \Sigma_V\) and Lemma 5.9 finishes the proof.

### 5.3. Construction of the Green function.

In order to construct the Green function, we need a more explicit expression for the Wronskian of \(u_0\) and \(u_1\). Note carefully that this is the place where the spectral assumption enters.

**Lemma 5.11.** We have

\[ W(u_0(\cdot,\lambda), u_1(\cdot,\lambda))(y) = 2\lambda u_1(0,\lambda)(1 - y^2)^{-1-\lambda} \]

for all \(\lambda \in \mathbb{C}\) with \(|\Re\lambda| \leq \frac{1}{2}\). Furthermore, if \(L_V\) has no eigenvalues on the imaginary axis, there exists an \(\epsilon_0 > 0\) such that \(|u_1(0,\lambda)| \gtrsim 1\) for all \(\lambda \in \mathbb{C}\) with \(\Re\lambda \in [0,\epsilon_0]\).
Proof. By definition and Remark \ref{205}, we have

\[
W(u_0(\cdot, \lambda), u_1(\cdot, \lambda))(y) = (1 - y^2)^{-1-\lambda}W(v_1(0, -\lambda)v_1(\cdot, -\lambda) - v_1(0, \lambda)v_1(\cdot, -\lambda), v_1(\cdot, \lambda))
\]

\[
= -v_1(0, \lambda)W(v_1(\cdot, -\lambda), v_1(\cdot, \lambda))(1 - y^2)^{-1-\lambda}
\]

\[
= 2\lambda v_1(0, \lambda)(1 - y^2)^{-1-\lambda}
\]

\[
= 2\lambda u_1(0, \lambda)(1 - y^2)^{-1-\lambda}
\]

for all \(\lambda \in \mathbb{C}\) with \(|\text{Re}\lambda| \leq \frac{1}{4}\).

By assumption and Lemma \ref{209}, there exists an \(\epsilon_0 > 0\) such that there are no eigenvalues of \(L_V\) in the strip \(\{ z \in \mathbb{C} : \text{Re} \ z \in [0, \epsilon_0] \}\). Consequently, by Lemma \ref{209} \(u_1(0, \lambda) \neq 0\) for all \(\lambda \in \mathbb{C}\) with \(\text{Re} \ \lambda \in [0, \epsilon_0]\) and, since \(u_1(0, \lambda) = 1 + O((\lambda)^{-1})\) by Proposition \ref{202}, the claim follows.

**Definition 5.12.** For any \(\lambda \in \mathbb{C}\) with \(\text{Re} \ \lambda \in (0, \frac{1}{4}]\) and \(\lambda \not\in \sigma_p(L_V)\), we set

\[
G_V(y, x, \lambda) := -\frac{1}{(1 - x^2)W(u_0(\cdot, \lambda), u_1(\cdot, \lambda))(x)} \begin{cases} u_0(y, \lambda)u_1(x, \lambda) & 0 \leq y \leq x < 1 \\ u_1(y, \lambda)u_0(x, \lambda) & 0 \leq x \leq y < 1 \end{cases}
\]

**Lemma 5.13.** There exists an \(\epsilon_0 > 0\) such that any \(\lambda \in \mathbb{C}\) with \(\text{Re} \ \lambda \in (0, \epsilon_0]\) belongs to \(\rho(L_V)\) and for any \(f = (f_1, f_2) \in \mathcal{H}\) we have

\[
\mathcal{R}_{L_V}(\lambda)f(y) = \left( \int_0^1 G_V(y, x, \lambda)[2xf'_1(x) + (\lambda + 1)f_1(x) + f_2(x)]dx \right)
\]

for \(y \in [0, 1]\).

Proof. By Lemma \ref{209} it follows that there exists an \(\epsilon_0 > 0\) such that \(\text{Re} \ \lambda \in (0, \epsilon_0]\) implies \(\lambda \in \rho(L_V)\). Recall from the proof of Lemma \ref{208} that \(u_1(\cdot, \lambda) \in C^\infty([0, 1])\). Furthermore,

\[
(1 - y^2)^{-\frac{1}{2}(1+\lambda)}v_1(y, -\lambda) = (1 - y^2)^{-\frac{1}{2}(1+\lambda)}\psi_1(y, -\lambda) \frac{v_1(y, -\lambda)}{\psi_1(y, -\lambda)} = (1 - y)^{-\lambda} \frac{v_1(y, -\lambda)}{\psi_1(y, -\lambda)}
\]

and by Lemma \ref{210} we see that

\[
u_0(y, \lambda) = v_1(0, -\lambda)u_1(y, \lambda)+(1-y)^{-\lambda}h(y, \lambda),
\]

where \(h(\cdot, \lambda) \in C^\infty([0, 1])\). For brevity we set \(F_\lambda(x) := 2xf'_1(x) + (\lambda + 1)f_1(x) + f_2(x)\). Then, by Lemma \ref{211}, we have

\[
\int_0^1 G_V(y, x, \lambda)F_\lambda(x)dx = -\frac{1}{2\lambda u_1(0, \lambda)} \sum_{k=1}^4 I_k(y)
\]

with

\[
I_1(y) := v_1(0, -\lambda)u_1(y, \lambda) \int_y^1 (1 - x^2)^{\lambda}u_1(x, \lambda)F_\lambda(x)dx
\]

\[
I_2(y) := h(y, \lambda)(1 - y)^{-\lambda} \int_y^1 (1 - x^2)^{\lambda}u_1(x, \lambda)F_\lambda(x)dx
\]

\[
I_3(y) := v_1(0, -\lambda)u_1(y, \lambda) \int_0^y (1 - x^2)^{\lambda}u_1(x, \lambda)F_\lambda(x)dx
\]

\[
I_4(y) := u_1(y, \lambda) \int_0^y (1 + x)^{\lambda}h(x, \lambda)F_\lambda(x)dx.
\]
By assumption, $F_{\lambda} \in C^\infty([0,1])$ and therefore, $I_4 \in C^\infty([0,1])$. Furthermore,

$$I_1(y) + I_3(y) = v_1(0,-\lambda)u_1(y,\lambda) \int_0^1 (1 - x^2)^\lambda u_1(x,\lambda)F_{\lambda}(x)dx$$

and thus, $I_1 + I_3 \in C^\infty([0,1])$. Finally, the change of variables $x = y + t(1 - y)$ yields

$$I_2(y) = h(y,\lambda)(1 - y) \int_0^1 (1 - t)^\lambda[1 + y + t(1 - y)]^\lambda u_1(y + t(1 - y),\lambda)F_{\lambda}(y + t(1 - y))dt$$

and thus, $I_2 \in C^\infty([0,1])$. In summary, the function $w_{\lambda}(y) := \int_0^1 G(y, x, \lambda)F_{\lambda}(x)dx$ belongs to $C^\infty([0,1])$ and by construction, $w_{\lambda}$ satisfies Eq. (4.2) for $y \in [0,1]$. Furthermore, $w_{\lambda}(0) = 0$ and by the oddness of $F_{\lambda}$, we find inductively from Eq. (4.2) that $w_{\lambda}^{(2k)}(0) = 0$ for all $k \in \mathbb{N}_0$. This means that $w_{\lambda}$ extends to an odd function in $C^\infty([-1,1])$ and $w_{\lambda}$ satisfies Eq. (4.2) for all $y \in [-1,1]$. As a consequence, $u := (w_{\lambda}, \lambda w_{\lambda} - f_1)$ belongs to $\tilde{H}$ and satisfies $(\lambda - L_V)u = f$, which means that $u = R_{L_V}(\lambda)f$. \hfill $\Box$

5.4. Time evolution on the unstable subspace. By now we have collected enough information so that we can prove the first part of Theorem 1.3 which is a consequence of the following result combined with Lemmas 4.16 and 5.9.

**Lemma 5.14.** We have $\text{rg}
\bar{P}_V \subset C^\infty(-1,1) \times C^\infty(-1,1)$ and for every $\lambda \in \sigma(L_V) \cap \{z \in \mathbb{C} : \text{Re } z > 0\} =: \sigma_u(L_V)$ there exists a number $n(\lambda) \in \mathbb{N}_0$ such that

$$S_V(s)f = \sum_{\lambda \in \sigma_u(L_V)} e^{\lambda s} \sum_{k=0}^{n(\lambda)} \frac{s^k}{k!}(L_V - \lambda)^k f$$

for all $f \in \text{rg}
\bar{P}_V$ and all $s \geq 0$.

**Proof.** By Lemma 4.10, $\sigma_u(L_V)$ is finite and consists of eigenvalues with finite algebraic multiplicities. For each $\lambda \in \sigma_u(L_V)$, let $\bar{P}_{V,\lambda}$ be the corresponding spectral projection. Then

$$\text{rg}
\bar{P}_V = \bigoplus_{\lambda \in \sigma_u(L_V)} \text{rg}
\bar{P}_{V,\lambda}.$$

Denote by $L_{V,\lambda}$ the part of $L_V$ in the finite-dimensional subspace $\text{rg}
\bar{P}_{V,\lambda}$. Clearly, $\text{rg}
\bar{P}_{V,\lambda} \subset D(L_V)$ and for any $f \in \text{rg}
\bar{P}_{V,\lambda}$, we have $L_{V,\lambda}f = L_Vf \in \text{rg}
\bar{P}_{V,\lambda} \subset D(L_V)$. Inductively, this implies $\text{rg}
\bar{P}_{V,\lambda} \subset D(L_{V,n})$ for any $n \in \mathbb{N}$ and Corollary 4.4 shows that $\text{rg}
\bar{P}_{V,\lambda} \subset C^\infty(-1,1) \times C^\infty(-1,1)$.

Let $S_{V,\lambda}(s)$ be the part of $S_V(s)$ in $\text{rg}
\bar{P}_{V,\lambda}$ and set $\tilde{S}_{V,\lambda}(s) := e^{-\lambda s}S_{V,\lambda}(s)$. Then $\tilde{S}_{V,\lambda}(s)$ is a semigroup on $\text{rg}
\bar{P}_{V,\lambda}$ with generator $L_{V,\lambda} - \lambda$. Since $\sigma(L_{V,\lambda} - \lambda) = \{0\}$ and $\dim \text{rg}
\bar{P}_{V,\lambda} < \infty$, it follows that $L_{V,\lambda} - \lambda$ is nilpotent and there exists an $n(\lambda) \in \mathbb{N}$ such that $(L_{V,\lambda} - \lambda)^{n(\lambda)} = 0$. Note that $\partial_s^n\tilde{S}_{V,\lambda}(s)f = \tilde{S}_{V,\lambda}(s)(L_{V,\lambda} - \lambda)^nf$ for all $n \in \mathbb{N}_0$ and $f \in \text{rg}
\bar{P}_{V,\lambda}$. Consequently,

$$\partial_s^n\tilde{S}_{V,\lambda}(s)f = \tilde{S}_{V,\lambda}(s)(L_{V,\lambda} - \lambda)^nf = 0$$

and integrating this equation yields

$$\tilde{S}_{V,\lambda}(s)f = \sum_{k=0}^{n(\lambda)} \frac{s^k}{k!}(L_{V,\lambda} - \lambda)^kf = \sum_{k=0}^{n(\lambda)} \frac{s^k}{k!}(L_V - \lambda)^kf.$$
In order to separate the free evolution from the effect of the potential, we introduce suitable operators that account for the difference.

**Definition 6.1.** For any \( \epsilon > 0 \) and \( f \in C([0, 1]) \), we set

\[
[T_\epsilon(s)f](y) := \frac{1}{2\pi i} \lim_{N \to \infty} \int_{-\epsilon-iN}^{\epsilon+iN} e^{\lambda s} \int_0^1 [G_V(y, x, \lambda) - G_0(y, x, \lambda)] f(x) dx d\lambda
\]

\[
[\tilde{T}_\epsilon(s)f](y) := \frac{1}{2\pi i} \lim_{N \to \infty} \int_{-\epsilon-iN}^{\epsilon+iN} \lambda e^{\lambda s} \int_0^1 [G_V(y, x, \lambda) - G_0(y, x, \lambda)] f(x) dx d\lambda
\]

The key result for the Strichartz estimates are the following bounds on \( T_\epsilon \) and \( \tilde{T}_\epsilon \).

**Theorem 6.2.** Let \( p \in [2, \infty] \) and \( q \in [1, \infty) \). Then there exists an \( \epsilon_0 > 0 \) such that

\[
\|e^{-\epsilon T_\epsilon(s)f}\|_{L^p_t(0,\infty)L^q(0,1)} \lesssim \left\| (1 - |\cdot|^2)^{\frac{1}{2}} f \right\|_{L^2(0,1)}
\]

\[
\|e^{-\epsilon \tilde{T}_\epsilon(s)f}\|_{L^p_t(0,\infty)L^q(0,1)} \lesssim \left\| (1 - |\cdot|^2)^{\frac{1}{2}} f \right\|_{L^2(0,1)} + \|f\|_{L^2(0,1)}
\]

for all \( \epsilon \in (0, \epsilon_0] \) and \( f \in C^1([0, 1]) \).

We now reduce the proof of Theorem 4.15 to Theorem 6.2. The rest of this section is then devoted to the proof of Theorem 6.2.

**Lemma 6.3.** Assume that Theorem 6.2 holds. Then Theorem 4.15 follows.

**Proof.** Let \( f \in D(L_V) \). Then \((I - P_V)f \in D(L_V)\) and by [3], p. 234, Corollary 5.15, we obtain

\[
S_V(s)(I - P_V)f = S_V^\epsilon(s)(I - P_V)f = \frac{1}{2\pi i} \lim_{N \to \infty} \int_{-\epsilon-iN}^{\epsilon+iN} e^{\lambda s} R_{L_V}(\lambda)(I - P_V)f d\lambda
\]

for all \( \epsilon > 0 \). By Lemma 5.13 there exists an \( \epsilon_0 > 0 \) such that for all \( \epsilon \in (0, \epsilon_0] \),

\[
S_V(s)(I - P_V)f = \frac{1}{2\pi i} \lim_{N \to \infty} \int_{-\epsilon-iN}^{\epsilon+iN} e^{\lambda s} R_{L_V}(\lambda)(I - P_V)f d\lambda.
\]

Now we set \( \mathcal{H}_0 := (\ker P_V \cap D(L_V)) + \mathcal{H}, \mathcal{Y} := L_0^p((0, \infty), L^q(0,1)) \), and for \( \epsilon \in (0, \epsilon_0] \) and \( f \in \mathcal{H}_0 \) we define

\[
\Phi_\epsilon(f)(s) := \frac{1}{2\pi i} \lim_{N \to \infty} \int_{-\epsilon-iN}^{\epsilon+iN} e^{\lambda s} [R_{L_V}(\lambda)f]_1 d\lambda - [S_V(s)f]_1.
\]

We claim that \( \Phi_\epsilon \) maps \( \mathcal{H}_0 \) into \( \mathcal{Y} \). Indeed, for \( f \in \ker P_V \cap D(L_V) \), we have \( \Phi_\epsilon(f)(s) = [S_V(s)f]_1 - [S_0(s)f]_1 \) and thus, \( \Phi_\epsilon(f) \in \mathcal{Y} \) by Remark 4.14. Furthermore, for \( f = (f_1, f_2) \in \mathcal{H}, \) Lemma 5.13 shows that

\[
\Phi_\epsilon(f)(s) = T_\epsilon(s)(2 \cdot |f_1 + f_2| + \tilde{T}_\epsilon(s)f_1
\]

and Theorem 6.2 yields the bound

\[
\|e^{-\epsilon \Phi_\epsilon(f)(s)}\|_{L^p_t(0,\infty)L^q(0,1)} \lesssim \|f\|_{\mathcal{H}}.
\]  (6.1)

Consequently, \( \Phi_\epsilon(f) \in \mathcal{Y} \) for all \( f \in \mathcal{H}_0 \), as claimed. By density, \( \Phi_\epsilon \) uniquely extends to a map \( \Phi_\epsilon : \mathcal{H} \to \mathcal{Y} \) and the bound (6.1) holds for all \( f \in \mathcal{H} \). For \( f \in \ker P_V \cap D(L_V) \) we obtain

\[
\|e^{-\epsilon S_V(s)f}1\|_{L^p_t(0,\infty)L^q(0,1)} \lesssim \|e^{-\epsilon \Phi_\epsilon(f)(s)}\|_{L^p_t(0,\infty)L^q(0,1)} + \|e^{-\epsilon [S_0(s)f]_1}\|_{L^p_t(0,\infty)L^q(0,1)} \lesssim \|f\|_{\mathcal{H}}
\]

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by Proposition \ref{prop:smoothness} and monotone convergence yields
\[
\|\mathbb{S}_V(f)\|_{L^p(0,\infty)L^q(0,1)} \lesssim \|f\|_H
\]
which, by density, extends to all \( f \in \ker P_V \).

\[\square\]

6.1. Analysis of the operator \( T_e \). First, we identify the integral kernel of \( T_e \).

Lemma 6.4. Let \( \lambda \in \mathbb{C} \setminus \{0\} \) with \( |\Re \lambda| \leq \frac{1}{4} \) and \( \omega = \Re \lambda \). Then we have
\[
G_V(y, x, \lambda) - G_0(y, x, \lambda) = \sum_{j=1}^{4} G_{V,j}(y, x, \lambda) = \sum_{j=1}^{4} \tilde{G}_{V,j}(y, x, \lambda),
\]
where
\[
G_{V,1}(y, x, \lambda) = \int_{0}^{1} (1 + y)^{-\lambda} (1 - x)^{\lambda} \mathcal{O}((1 - y)^0(1 - x)^0(\omega)^{-1})
\]
\[
G_{V,2}(y, x, \lambda) = \int_{0}^{1} (1 + y)^{-\lambda} (1 - x)^{\lambda} \mathcal{O}((1 - y)^0(1 - x)^0(\omega)^{-1})
\]
\[
G_{V,3}(y, x, \lambda) = \int_{0}^{1} (1 + y)^{-\lambda} (1 - x)^{\lambda} \mathcal{O}((1 - y)^0(1 - x)^0(\omega)^{-1})
\]
\[
G_{V,4}(y, x, \lambda) = \int_{0}^{1} (1 + y)^{-\lambda} (1 - x)^{\lambda} \mathcal{O}((1 - y)^0(1 - x)^0(\omega)^{-1})
\]
as well as
\[
\tilde{G}_{V,1}(y, x, \lambda) = \int_{0}^{1} (1 - y)^{-\lambda} (1 - x)^{\lambda} \mathcal{O}((1 - y)^0(1 - x)^0(\omega)^{00}) dt
\]
\[
\tilde{G}_{V,2}(y, x, \lambda) = \int_{0}^{1} (1 + y)^{-\lambda} (1 - x)^{\lambda} \mathcal{O}((1 - y)^0(1 - x)^0(\omega)^{00}) dt
\]
\[
\tilde{G}_{V,3}(y, x, \lambda) = \int_{0}^{1} (1 + y)^{-\lambda} (1 - x)^{\lambda} \mathcal{O}((1 - y)^0(1 - x)^0(\omega)^{00}) dt
\]
\[
\tilde{G}_{V,4}(y, x, \lambda) = \int_{0}^{1} (1 + y)^{-\lambda} (1 - x)^{\lambda} \mathcal{O}((1 - y)^0(1 - x)^0(\omega)^{00}) dt
\]
for all \( y, x \in [0, 1) \).

Proof. By definition and Proposition \ref{prop:asymptotics}.

\[
u_1(y, \lambda) = (1 - y^2)^{-\frac{1}{4}(1 + \lambda)} v_1(y, \lambda) = (1 - y^2)^{-\frac{1}{4}(1 + \lambda)} \psi_1(y, \lambda)[1 + \mathcal{O}((1 - y)^{\frac{1}{2}}(\omega)^{-1})]
\]
\[
= (1 + y)^{-\lambda}[1 + \mathcal{O}((1 - y)^{\frac{1}{2}}(\omega)^{-1})]
\]
as well as
\[
u_0(y, \lambda) = (1 - y^2)^{-\frac{1}{4}(1 + \lambda)} [v_1(0, -\lambda)v_1(y, \lambda) - v_1(0, \lambda)v_1(y, -\lambda)]
\]
\[
= (1 - y^2)^{-\frac{1}{4}(1 + \lambda)} \psi_1(y, \lambda)[1 + \mathcal{O}((1 - y)^0(\omega)^{-1})]
\]
\[
= (1 + y)^{-\lambda}[1 + \mathcal{O}((1 - y)^0(\omega)^{-1})]
\]
\[
= (1 + y)^{-\lambda}[1 + \mathcal{O}((1 - y)^0(\omega)^{-1})]
\]
Finally, by Lemma \ref{lem:asymptotics}.

\[
\frac{1}{(1 - x^2)W(u_0(\cdot, \lambda), u_1(\cdot, \lambda))(x)} = \frac{(1 - x^2)^{\lambda}}{2\lambda u_1(0, \lambda)} = \frac{(1 - x^2)^{\lambda}}{2\lambda}[1 + \mathcal{O}((\omega)^{-1})]
\]
and the first representation follows.

For the second representation, we need to exploit the fact that \( u_0(y, 0) = 0 \) to get rid of the apparent singularity of the Green function at \( \lambda = 0 \). By the fundamental theorem of calculus we obtain

\[
  u_0(y, \lambda) = \int_0^1 \partial_t u_0(y, t\lambda) dt = \lambda \int_0^1 (1 + y)^{-t\lambda} O((1 - y)^0(t\omega)^0) dt
  + \lambda \langle \log(1 - y) \rangle \int_0^1 (1 - y)^{-t\lambda} O((1 - y)^0(t\omega)^0) dt.
\]

Inserting this expression for \( u_0 \) in the definition of the Green function yields the second representation. \( \square \)

In order to estimate the kernel of the operators \( T_\epsilon \) and \( \hat{T}_\epsilon \), we make frequent use of the following elementary bound.

**Lemma 6.5.** We have \( \langle a - b \rangle \gtrsim \langle a \rangle^{-1} \langle b \rangle \) for all \( a, b \in \mathbb{R} \).

**Proof.** If \( |b| \leq 2|a| \) we have \( \langle a \rangle^{-1} \langle b \rangle \lesssim 1 \lesssim \langle a - b \rangle \) and if \( |b| \geq 2|a| \),

\[
  \langle a - b \rangle \simeq 1 + |a - b| \geq 1 + |b| - |a| \geq 1 + \frac{1}{2}|b| \simeq \langle b \rangle \gtrsim \langle a \rangle^{-1} \langle b \rangle.
\]

\( \square \)

**Proposition 6.6.** There exists an \( \epsilon_0 > 0 \) such that

\[
  K_\epsilon(s, y, x) := \frac{1}{2\pi i} \lim_{N \to \infty} \int_{\epsilon-iN}^{\epsilon+iN} e^{\lambda s} [G_V(y, x, \lambda) - G_0(y, x, \lambda)] d\lambda
\]

exists for any \( (s, y, x) \in [0, \infty) \times [0, 1] \times [0, 1] \) and \( \epsilon \in (0, \epsilon_0] \) and we have

\[
  T_\epsilon(s)f(y) = \int_0^1 K_\epsilon(s, y, x) f(x) dx.
\]

Furthermore,

\[
  |K_\epsilon(s, y, x)| \lesssim e^{\epsilon s} (\log(1 - y))^3 (s + \log(1 - x))^{-2}
\]

for all \( (s, y, x) \in [0, \infty) \times [0, 1] \times [0, 1] \) and all \( \epsilon \in (0, \epsilon_0] \).

**Proof.** Lemma 6.4 yields the rough bound

\[
  |G_V(y, x, \lambda) - G_0(y, x, \lambda)| \lesssim (1 - y)^{-\frac{1}{2}}(1 - x)^{-\frac{1}{2}} (\lambda)^{-2}
\]

for all \( x, y \in [0, 1] \) and \( \lambda \in \mathbb{C} \) with \( \text{Re} \lambda \in (0, \epsilon_0] \), provided \( \epsilon_0 > 0 \) is sufficiently small. As a consequence, the existence of \( K_\epsilon(s, y, x) \) follows and Fubini’s theorem yields the stated expression for \( T_\epsilon(s)f \).

To prove the bound on \( K_\epsilon \), we need to distinguish between \( \lambda \) small and \( \lambda \) large. To this end, we use a standard cut-off \( \chi : \mathbb{R} \to [0, 1] \) that satisfies \( \chi(t) = 1 \) if \( |t| \leq 1 \) and \( \chi(t) = 0 \) if \( |t| \geq 2 \). Then we split

\[
  K_\epsilon(s, y, x) = \frac{e^{\epsilon s}}{2\pi} \int_{\mathbb{R}} e^{i\omega s} [G_V(y, x, \epsilon + i\omega) - G_0(y, x, \epsilon + i\omega)] d\omega
  =: \frac{e^{\epsilon s}}{2\pi} [I_\epsilon(s, y, x) + J_\epsilon(s, y, x)]
\]
and use Lemma 6.4 to decompose
\[
I_\epsilon(s, y, x) = \int \chi(\omega) e^{i\omega s} [G_V(y, x, \epsilon + i\omega) - G_0(y, x, \epsilon + i\omega)] d\omega
\]
\[
= \sum_{j=1}^{4} \int \chi(\omega) e^{i\omega s} \tilde{G}_{V,j}(y, x, \epsilon + i\omega) d\omega
=: I_{\epsilon,j}(s, y, x)
\]
and
\[
J_\epsilon(s, y, x) = \int [1 - \chi(\omega)] e^{i\omega s} [G_V(y, x, \epsilon + i\omega) - G_0(y, x, \epsilon + i\omega)] d\omega
\]
\[
= \sum_{j=0}^{4} \int [1 - \chi(\omega)] e^{i\omega s} G_{V,j}(y, x, \epsilon + i\omega) d\omega
=: J_{\epsilon,j}(s, y, x)
\]
By Lemma 6.4 we have
\[
[1 - \chi(\omega)] G_{V,1}(y, x, \epsilon + i\omega) = [1 - \chi(\omega)](1 + y)^{-\epsilon}(1 + y)^{-i\omega}(1 - x)^{i\omega}
\times (\epsilon + i\omega)^{-1} \mathcal{O}((1 - y)^0(1 - x)^0(\omega)^{-1})
= e^{i\omega(-\log(1+y) + \log(1-x))} \mathcal{O}((1 - y)^0(1 - x)^0(\omega)^{-2})
\]
and thus,
\[
|J_{\epsilon,1}(s, y, x)| \lesssim \left| \int e^{i\omega (s - \log(1+y) + \log(1-x))} \mathcal{O}((1 - y)^0(1 - x)^0(\omega)^{-2}) d\omega \right|
\lesssim (s - \log(1+y) + \log(1-x))^{-2}
\lesssim (s + \log(1-x))^{-2}
\]
by means of two integrations by parts. For \( J_{\epsilon,2} \) we note that
\[
[1 - \chi(\omega)] G_{V,2}(y, x, \epsilon + i\omega) = \frac{1}{[0,1]}(y - x)[1 - \chi(\omega)](1 + y)^{-\epsilon}(1 + y)^{-i\omega}(1 - x)^{i\omega}
\times (\epsilon + i\omega)^{-1} \mathcal{O}((1 - y)^0(1 - x)^0(\omega)^{-1})
= e^{i\omega(-\log(1+y) + \log(1-x))} \mathcal{O}((1 - y)^0(1 - x)^0(\omega)^{-2})
\]
and we obtain
\[
|J_{\epsilon,2}(s, y, x)| \lesssim (s - \log(1-y) + \log(1-x))^{-2} \lesssim (\log(1-y))^2(s + \log(1-x))^{-2}.
\]
The terms \( J_{\epsilon,3} \) and \( J_{\epsilon,4} \) are handled analogously and in summary, we obtain
\[
|J_{\epsilon}(s, y, x)| \lesssim (\log(1-y))^2(s + \log(1-x))^{-2}.
\]
Now we turn to the low-frequency part \( I_\epsilon \). By Lemma 6.3 we have
\[
\chi(\omega) \tilde{G}_{V,1}(y, x, \epsilon + i\omega) = \int_{0}^{1} (1 + y)^{-i\omega}(1 - x)^{i\omega} \mathcal{O}((1 - y)^0(1 - x)^0(\omega)^{-2}t^0) dt
\]
\[
= \int_{0}^{1} e^{i\omega(-t \log(1+y) + \log(1-x))} \mathcal{O}((1 - y)^0(1 - x)^0(\omega)^{-2}t^0) dt
\]
and thus, by Fubini and two integrations by parts,
\[
|I_{r,1}(s, y, x)| \leq \int_0^1 \left| \int_\mathbb{R} e^{i\omega (s - t \log(1+y) + \log(1-x))} O((1-y)^0(1-x)^0\langle \omega \rangle^{-2}t^0) d\omega \right| dt \\
\lesssim \int_0^1 \langle s - t \log(1+y) + \log(1-x) \rangle^{-2} dt \\
\lesssim \int_0^1 \langle t \log(1+y) \rangle^2 \langle s + \log(1-x) \rangle^{-2} dt \\
\lesssim \langle s + \log(1-x) \rangle^{-2}.
\]

For \(\widetilde{G}_{V,2}\) we have
\[
\chi(\omega)\widetilde{G}_{V,2}(y, x, \epsilon + i\omega) = \int_0^1 \langle \log(1-y) \rangle (1-y)^{-i\omega} (1-x)^i\omega O((1-y)^0(1-x)^0\langle \omega \rangle^{-2}t^0) dt
\]
and thus,
\[
\frac{|I_{r,2}(s, y, x)|}{\langle \log(1-y) \rangle} \leq \int_0^1 \left| \int_\mathbb{R} e^{i\omega (s - t \log(1+y) + \log(1-x))} O((1-y)^0(1-x)^0\langle \omega \rangle^{-2}t^0) d\omega \right| dt \\
\lesssim \int_0^1 \langle s - t \log(1+y) + \log(1-x) \rangle^{-2} dt \\
\lesssim \int_0^1 \langle t \log(1-y) \rangle^2 \langle s + \log(1-x) \rangle^{-2} dt \\
\lesssim \langle \log(1-y) \rangle^2 \langle s + \log(1-x) \rangle^{-2}.
\]

The corresponding bound for \(I_{r,3}\) follows analogously and for \(I_{r,4}\) we note that
\[
\chi(\omega)\widetilde{G}_{V,4}(y, x, \epsilon + i\omega) = \int_0^1 1_{[0,1]}(y-x) \langle \log(1-x) \rangle (1+y)^{-i\omega} (1+x)^i\omega (1-x)^{(1-t)\omega} \\
\quad \times O((1-y)^0(1-x)^0\langle \omega \rangle^{-2}t^0) dt
\]
\[
= \int_0^1 1_{[0,1]}(y-x) \langle \log(1-y) \rangle e^{i\omega (-1+y) + \log(1+y) + (1-t) \log(1-x)} \\
\quad \times O((1-y)^0(1-x)^0\langle \omega \rangle^{-2}t^0) dt
\]
and thus,
\[
\frac{|I_{r,4}(s, y, x)|}{\langle \log(1-y) \rangle} \leq \int_0^1 \left| \int_\mathbb{R} 1_{[0,1]}(y-x) e^{i\omega (s - \log(1+y) + \log(1+x) + (1-t) \log(1-x))} \\
\quad \times O((1-y)^0(1-x)^0\langle \omega \rangle^{-2}t^0) d\omega \right| dt \\
\lesssim \int_0^1 1_{[0,1]}(y-x) \langle s - \log(1+y) + \log(1+x) + (1-t) \log(1-x) \rangle^{-2} dt \\
\lesssim \int_0^1 1_{[0,1]}(y-x) \langle - \log(1+y) + \log(1+x) - t \log(1-x) \rangle^2 \\
\quad \times \langle s + \log(1-x) \rangle^{-2} dt \\
\lesssim \langle \log(1-y) \rangle^2 \langle s + \log(1-x) \rangle^{-2}.
\]

Now we can conclude the desired bound for the operator \(T_{\epsilon}\).
Lemma 6.7. Let $p \in [2, \infty]$ and $q \in [1, \infty)$. Then there exists an $\epsilon_0 > 0$ such that
\[
\|e^{-\epsilon s}T_\epsilon(s)f\|_{L^p(0,\infty)L^q(0,1)} \lesssim \left\| (1 - |\cdot|^2)^{\frac{1}{2}} f \right\|_{L^2(0,1)}
\]
for all $f \in C([0,1])$ and $\epsilon \in (0, \epsilon_0]$.

Proof. By Proposition 6.6 we have
\[
|e^{-\epsilon s}T_\epsilon(s)f(y)| \leq \int_0^1 e^{-\epsilon s}|K_\epsilon(s,y,x)||f(x)|dx
\]
\[
= \int_0^\infty e^{-\epsilon s}|K_\epsilon(s,y,1 - e^{-\eta})||f(1 - e^{-\eta})|e^{-\eta}d\eta
\]
\[
\lesssim (\log(1 - y))^3 \int_\mathbb{R} (s - \eta)^{-2}1_{[0,\infty)}(\eta)|f(1 - e^{-\eta})|e^{-\eta}d\eta
\]
and thus,
\[
\|e^{-\epsilon s}T_\epsilon(s)f\|_{L^q(0,1)} \lesssim \int_\mathbb{R} (s - \eta)^{-2}1_{[0,\infty)}(\eta)f(1 - e^{-\eta})|e^{-\eta}d\eta.
\]
Consequently, Young’s inequality yields
\[
\|e^{-\epsilon s}T_\epsilon(s)f\|_{L^2(0,1)} \lesssim \| (\cdot )^{-2} \|_{L^1(\mathbb{R})} \int_0^\infty |f(1 - e^{-\eta})|^2e^{-2\eta}d\eta
\]
\[
\lesssim \int_0^1 (1 - x)|f(x)|^2dx.
\]
On the other hand, by Cauchy-Schwarz, we also have
\[
\|e^{-\epsilon s}T_\epsilon(s)f\|_{L^2(0,1)} \lesssim \int_0^1 (1 - x)|f(x)|^2dx
\]
for all $s \geq 0$ and this yields
\[
\|e^{-\epsilon s}T_\epsilon(s)f\|_{L^\infty(0,\infty)L^q(0,1)} \lesssim \left\| (1 - |\cdot|^2)^{\frac{1}{2}} f \right\|_{L^2(0,1)} + \|f\|_{L^2(0,1)}.
\]

6.2. Analysis of the operator $\hat{T}_\epsilon$. The treatment of the operator $\hat{T}_\epsilon$ is very similar.

Proposition 6.8. Let $p \in [2, \infty]$ and $q \in [1, \infty)$. Then there exists an $\epsilon_0 > 0$ such that
\[
\|e^{-\epsilon s}\hat{T}_\epsilon(s)f\|_{L^p(0,\infty)L^q(0,1)} \lesssim \left\| (1 - |\cdot|^2)^{\frac{1}{2}} f \right\|_{L^2(0,1)} + \|f\|_{L^2(0,1)}
\]
for all $f \in C^1([0,1])$ and $\epsilon \in (0, \epsilon_0]$.

Proof. Let $\tilde{\chi} : \mathbb{R}^2 \to [0,1]$ be a smooth cut-off that satisfies $\tilde{\chi}(x) = 1$ if $|x| \leq 1$ and $\tilde{\chi}(x) = 0$ if $|x| \geq 2$. We define $\chi : \mathbb{C} \to [0,1]$ by $\chi(z) := \tilde{\chi}(\text{Re } z, \text{Im } z)$. We split
\[
\hat{T}_\epsilon(s) = \hat{T}_\epsilon^\oplus(s) + \hat{T}_\epsilon^\perp(s),
\]
where
\[
\hat{T}_\epsilon^\oplus(s)f(y) := \frac{1}{2\pi i} \lim_{N \to \infty} \int_{-iN}^{iN} \chi(\lambda)\lambda e^{\lambda s}[G_V(y,x,\lambda) - G_0(y,x,\lambda)]f(x)dxd\lambda
\]
\[
\hat{T}_\epsilon^\perp(s)f(y) := \frac{1}{2\pi i} \lim_{N \to \infty} \int_{-iN}^{iN} [1 - \chi(\lambda)]\lambda e^{\lambda s}[G_V(y,x,\lambda) - G_0(y,x,\lambda)]f(x)dxd\lambda.
\]
For the low-frequency part $\hat{T}_\varepsilon^b(s)f$, the additional factor of $\lambda$ (compared to $T_\varepsilon(s)f$) is helpful as it cancels the singularity of the Green function at $\lambda = 0$. Consequently, we immediately infer the bound

$$\|e^{-\varepsilon \hat{T}_\varepsilon^b(s)f}\|_{L^2(0,\infty)L^3(0,1)} \lesssim \|(1 - |\cdot|^2)^{\frac{1}{2}} f\|_{L^2(0,1)} \lesssim \|f\|_{L^2(0,1)}$$

by proceeding as in the proofs of Proposition 6.6 and Lemma 6.7.

The high-frequency part $\hat{T}_\varepsilon^c(s)f$ is more delicate as the additional $\lambda$ destroys the inverse square decay of the Green function as $|\text{Im} \lambda| \to \infty$. Consequently, we need to perform an integration by parts with respect to $x$ in order to recover the decay. More precisely, by Lemma 6.4, we have

$$\hat{T}_\varepsilon^c(s)f(y) = \frac{1}{2\pi i} \lim_{N \to \infty} \sum_{j=1}^{4} \int_{\varepsilon-iN}^{\varepsilon+iN} [1 - \chi(\lambda)]e^{\lambda s} \int_0^1 \lambda G_{V,j}(y, x, \lambda)f(x)dx d\lambda$$

and an integration by parts yields

$$\int_0^1 \lambda G_{V,1}(y, x, \lambda)f(x)dx$$

$$= (1 + y)^{-\lambda} \int_y^1 (1 - x)^{\lambda} O((1 - y)^0(1 - x)^0 \langle \omega \rangle^{-1})f(x)dx$$

$$= (1 + y)^{-\lambda} (1 - y)^{\lambda} O((1 - y)\langle \omega \rangle^{-2})f(y)$$

$$+ (1 + y)^{-\lambda} \int_y^1 (1 - x)^{\lambda} O((1 - y)^0(1 - x)^0 \langle \omega \rangle^{-2})f(x)dx$$

$$+ (1 + y)^{-\lambda} \int_y^1 (1 - x)^{\lambda} O((1 - y)^0(1 - x)\langle \omega \rangle^{-2})f'(x)dx$$

$$=: (1 + y)^{-\lambda} (1 - y)^{\lambda} O((1 - y)\langle \omega \rangle^{-2})f(y) + \int_0^1 H_{V,1}(y, x, \lambda)f(x)dx$$

$$+ \int_0^1 H'_{V,1}(y, x, \lambda)f'(x)dx.$$
and analogously for $G_{V,4}$. This means that
\[
e^{-\varepsilon s T^2_{\varepsilon}(s)} f(y) = f(0) \int_{\mathbb{R}} e^{i\omega(s-\log(1+y))} O(\langle \omega \rangle^{-2}) d\omega \\
+ f(y) \int_{\mathbb{R}} e^{i\omega(s-\log(1+y)+\log(1-y))} O((1-y)\langle \omega \rangle^{-2}) d\omega \\
+ f(y) \int_{\mathbb{R}} e^{i\omega s} O((1-y)\langle \omega \rangle^{-2}) d\omega + [A_\varepsilon(s)f](y) + [B_\varepsilon(s)f'](y) \\
= O((s)^{-1})f(0) + O((s)^{-1}(1-y)^{3})f(y) + [A_\varepsilon(s)f](y) + [B_\varepsilon(s)f'](y),
\]
where the operators $A_\varepsilon(s)$ and $B_\varepsilon(s)$ satisfy the bound for $T_\varepsilon(s)$ from Lemma 6.7. Consequently, we find
\[
\|e^{-\varepsilon s T^2_{\varepsilon}(s)} f\|_{L^p_t(0,\infty) L^q_y(0,1)} \lesssim |f(0)| + \| (1 - | \cdot |)^{\frac{3}{4}} f \|_{L^q_y(0,1)} + \| (1 - | \cdot |)^{\frac{3}{4}} f' \|_{L^q_y(0,1)} \\
\lesssim \| (1 - | \cdot |)^{\frac{3}{4}} f \|_{L^p_y(\infty,1)} + \| (1 - | \cdot |)^{\frac{3}{4}} f' \|_{L^p_y(0,1)} + \| f \|_{L^r_y(0,1)}
\]
and the simple estimate
\[
(1 - y)^{\frac{3}{4}} | f(y) | \lesssim \int_y^1 (1 - x)^{\frac{3}{4}} | f'(x) | dx + \int_y^1 (1 - x)^{-\frac{3}{4}} | f(x) | dx \\
\lesssim \| (1 - | \cdot |)^{\frac{3}{4}} f' \|_{L^p_y(0,1)} + \| f \|_{L^r_y(0,1)}
\]
for all $y \in [0,1]$ finishes the proof. \qed

References

[1] Piotr Bizoń and Patryk Mach. Global dynamics of a Yang-Mills field on an asymptotically hyperbolic space. *Trans. Amer. Math. Soc.*, 369(3):2029–2048, 2017.

[2] Roland Donninger. Strichartz estimates in similarity coordinates and stable blowup for the critical wave equation. *Duke Math. J.*, 166(9):1627–1683, 2013.

[3] Roland Donninger and Ziping Rao. Blowup stability at optimal regularity for the critical wave equation. Preprint arXiv:1811.08130, 2018.

[4] Roland Donninger and Anıl Zenginoğlu. Nondispersive decay for the cubic wave equation. *Anal. PDE*, 7(2):461–495, 2014.

[5] Klaus-Jochen Engel and Rainer Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[6] Helmut Friedrich. Cauchy problems for the conformal vacuum field equations in general relativity. *Comm. Math. Phys.*, 91(4):445–472, 1983.

[7] Michał Kowalczyk, Yvan Martel, and Claudio Muñoz. Kink dynamics in the $\phi^4$ model: asymptotic stability for odd perturbations in the energy space. *J. Amer. Math. Soc.*, 30(3):769–798, 2017.

[8] Hans Lindblad and Avy Soffer. Scattering for the Klein-Gordon equation with quadratic and variable coefficient cubic nonlinearities. *Trans. Amer. Math. Soc.*, 367(12):8861–8909, 2015.

[9] Hans Lindblad and Christopher D. Sogge. On existence and scattering with minimal regularity for semilinear wave equations. *J. Funct. Anal.*, 130(2):357–426, 1995.

[10] Wilhelm Schlag, Avy Soffer, and Wolfgang Staubach. Decay for the wave and Schrödinger evolutions on manifolds with conical ends. *Trans. Amer. Math. Soc.*, 362(1):19–52, 2010.

[11] Barry Simon. *Operator theory. A Comprehensive Course in Analysis*, Part 4. American Mathematical Society, Providence, RI, 2015.
[12] Jacob Sterbenz. Dispersive decay for the 1D Klein-Gordon equation with variable coefficient nonlinearities. *Trans. Amer. Math. Soc.*, 368(3):2081–2113, 2016.

[13] Robert S. Strichartz. Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.*, 44(3):705–714, 1977.

[14] Anıl Zenginoğlu. Hyperboloidal foliations and scri-fixing. *Classical Quantum Gravity*, 25(14):145002, 19, 2008.

[15] Anıl Zenginoğlu. Hyperboloidal layers for hyperbolic equations on unbounded domains. *J. Comput. Phys.*, 230(6):2286–2302, 2011.

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