Extended framework of Hamilton’s principle for single-degree-of-freedom nonlinear damping systems

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Abstract
The paper explores application of the variational formalism called extended framework of Hamilton’s principle to nonlinear damping systems. Single-degree-of-freedom systems with dominant source of nonlinearity from polynomial powers of the velocity are initially considered. Appropriate variational formulation is provided, and then the corresponding weak form is discretized to produce a novel computational method. The resulting low-order temporal finite element method utilizes non-iterative algorithm, and some examples are provided to verify its performance. The present temporal finite element method using small time step is equivalent to the adaptive Runge–Kutta–Fehlberg method with default error tolerances in MATLAB, and additional simulation shows its good convergence characteristics.

Keywords
Extended framework of Hamilton’s principle, nonlinear damping system, non-iterative algorithm, variational formalism, temporal finite element method

Introduction
In variational principle, state or transient response of physical system is determined from the extremal property (i.e., minimum, maximum, or stationarity) of functional. Particularly, Hamilton’s principle\(^1,2\) provides a general method for dynamics with the concept of stationary action. Nonlinear oscillations can also be encapsulated in this variational framework.\(^3–11\) For examples, from the Hamiltonian, He\(^3–7\) provided analytical methods to calculate angular frequencies of oscillators where nonlinear source came from the function of displacement. Also, with semi-inverse method, He\(^8\) found functional actions for generalized Korteweg–de Vrie (KdV) equation and nonlinear Schrödinger equation. Despite theoretical and analytical significance, Hamilton’s principle has critical weakness. More specifically, it cannot properly consider initial conditions, rather, it uses end-point constraints (or temporal boundary conditions) implying that positions of a system at the beginning and end of the time interval are known (interested reader is referred to chapter 5 of Ref.\(^12\)). Since one of the main objectives in dynamic analysis is to find the way of system evolution, one does not know the final position of the system and this may indicate physical inconsistency in Hamilton’s principle.

Previously, to overcome such end-point constraints in Hamilton’s principle, the extended framework of Hamilton’s principle (EHP) was established for linear elasticity.\(^13–15\) EHP resolves this issue with using mixed variables and external specification of initial values in the action variation. In addition, dissipative process can be integrated in this framework by introducing Rayleigh’s dissipation.\(^16\) This EHP formalism was successfully applied to dynamics of viscoplastic material, heat diffusion, and thermoelasticity.\(^17–20\)
The current work applies EHP formalism to nonlinear damping systems. The focus is on a single-degree-of-freedom (SDOF) system with nonlinear damping force proportional to even or/and odd orders of velocity. Such systems have been widely used for analysis of rolling ship subjected to beam waves,21–24 and they are adopted to investigate potential benefits in vibration control 25–27 and energy harvesting, 28–30 where iterative algorithm is essentially required in a numerical method. As shall be shown in this paper, the present temporal finite element method (TFEM) stemming from EHP formalism adopts non-iterative algorithm and this method with small size of time step is equivalent to the Runge–Kutta–Fehlberg (RKF45) method with default error criteria. The paper is structured as follows: The next section presents variational formulation of a nonlinear damping system from the perspective of EHP formalism. Based upon this formulation, a representative TFEM is developed in the subsequent section. This is followed by the section in which several computational examples are provided to validate the present approach. Finally, the work is summarized and general conclusions are drawn.

Variational formulation

This section is devoted to an overview of EHP formalism and extension of this EHP formalism to dynamics of nonlinear damping systems. It begins with a harmonic oscillation, where the equation of motion in a standard displacement-based approach is given by

\[ m \frac{d^2 u(t)}{dt^2} + k u(t) = \tilde{f}(t) \]  

(1)

with specified initial conditions

\[ u(0) = \dot{u}_0; \quad \dot{u}(0) = \ddot{u}_0 \]  

(2)

In equations (1) and (2), \( m \) and \( k \) represent mass and linear elastic stiffness. Also, \( u(t) \) and \( \tilde{f}(t) \) are the displacement of the mass and the known applied force at time \( t \).

In EHP formalism, Lagrangian function \( L \) for the harmonic oscillation can be written as

\[ L(u, \dot{u}, J; \tau) = \frac{1}{2} m \dot{u}^2 + \frac{1}{2} a J^2 - J u + \tilde{f} u \]  

(3)

In equation (3), a superposed dot represents a derivative with respect to \( \tau \), while \( a \) is the flexibility of linear elastic spring \( (a = 1/k) \). Also, \( J \) denotes the impulse of internal spring force \( F_s \), thus, the relation becomes

\[ J(\tau) = F_s(\tau) = k u(\tau) \text{ or } a \dot{J}(\tau) - J(\tau) - u(\tau) = 0 \]  

(4)

The action functional \( I \) for the fixed time interval from 0 to \( t \) is written as

\[ I = \int_0^t L(u, \dot{u}, J; \tau) d\tau \]  

(5)

The stationarity of action is newly defined in EHP as

\[ \delta I_{NEW} = -\delta \int_0^t L(u, \dot{u}, J; \tau) d\tau + [m \dot{u} \delta \dot{u}]_0^t = 0 \]  

(6)

and this shows main difference between EHP and Hamilton’s principle. Thus, in EHP, the underlined terms in equation (6) are externally added to \( \delta I \) in the original Hamilton’s principle where \( \delta I = -\delta \int_0^t L(u, \dot{u}, J; \tau) d\tau \). Such terms have confining effects to have unique dynamic evolution with unspecified pairs of displacement and velocity at the ends of time interval such as \( (\dot{u}(0), \ddot{u}(0)) \) and \( (\dot{u}(t), \ddot{u}(t)) \). Among unspecified pairs, sequentially assigning given initial values such as \( \ddot{u}_0 \) and \( \dot{u}_0 \) to unspecified initial values \( (\ddot{u}(0), \dot{u}(0)) \) completes this framework.
In detail, equation (6) can be explicitly rewritten as
\[
\delta I_{\text{NEW}} = - \int_0^t \left[ m \ddot{u} \delta \dot{u} + a \dot{J} \delta \dot{J} - u \delta \dot{J} - \dot{J} \delta u + \dot{J} \delta \dot{u} \right] dt + [m \dddot{u} \delta \dot{u}]_0^t = 0 \tag{7}
\]

After performing integration-by-parts on the first two terms, equation (7) becomes
\[
\delta I_{\text{NEW}} = \int_0^t \left( m \ddot{u} + \dot{J} - f \right) \delta \dot{u} dt + \int_0^t (a \dot{J} - \dddot{u}) \delta J dt + \left[ m \dddot{u} \delta \dot{u} \right]_0^t - [m \dddot{u} \delta \dot{u}]_0^t + [u \delta \dddot{u} + \dddot{u} \delta \dddot{u}]_0^t = 0 \tag{8}
\]

By allowing arbitrary variation for \( \delta u \) and \( \delta J \) over the interval, the first line of equation (8) produces equation of motion and rate-constitutive relation
\[
m \dddot{u}(\tau) + \dot{J}(\tau) - f(\tau) = 0 \tag{9}
\]
\[
a \dot{J}(\tau) - \dddot{u}(\tau) = 0 \tag{10}
\]
for \( \tau \in (0, t) \). Also, the last term in equation (8) becomes zero because of the constitutive relation in equation (4).

More importantly, in EHP, initial conditions are properly considered through subsequently assigning given initial values. First, by enforcing the following equations,
\[
\dddot{u}(0) = \dot{u}(0); \quad \delta \dot{u}(0) = \delta \dddot{u}(0); \quad \dddot{u}(t) = \dot{u}(t); \quad \delta u(t) = \delta \dddot{u}(t) \tag{11}
\]
only unique dynamic evolution is considered with unspecified values: \( \dot{u}(0), \delta \dddot{u}(0), \dot{u}(t), \) and \( \delta \dddot{u}(t) \). The given initial velocity \( \dot{u}_0 \) is firstly assigned to \( \dot{u}(0) \) as
\[
\dot{u}(0) = \dot{u}_0 \tag{12}
\]

Then, the given initial displacement \( \dddot{u}_0 \) is subsequently assigned to \( \dddot{u}(0) \) so that the variation of \( \dddot{u}(0) \) becomes
\[
\delta \dddot{u}(0) = 0 \tag{13}
\]

Equation (6) does not necessarily have equation (13) explicitly as far as this sequentially assigning process is kept in mind.\textsuperscript{13,15} Thus, a harmonic oscillation is encapsulated within EHP variational formalism with properly accounting for initial conditions.

With Rayleigh’s dissipation function approach,\textsuperscript{16} EHP can embrace dissipative damping force. For the linearly damped system with damping coefficient \( c_1 \), Rayleigh’s dissipation \( \varphi_1 \) is written as
\[
\varphi_1(\dddot{u}; \tau) = \frac{1}{2} c_1 [\dddot{u}(\tau)]^2 \tag{14}
\]

The variation of Rayleigh’s dissipation (equation (14)) is added to the action variation \( \delta I \) in the form of
\[
\int_0^t \frac{d\varphi_1}{d\dot{u}} \delta \dot{u} dt = \int_0^t c_1 \dot{u} \delta \dot{u} dt \tag{15}
\]

Using such Rayleigh’s dissipation function has critical weakness because it does not strictly follow variational rule. The correct variation of equation (14) must be \( \delta \varphi_1(\dddot{u}; \tau) = c_1 \dddot{u} \delta \dot{u}, \) not \( c_1 \dot{u} \delta \dot{u} \) appeared in equation (15). However, this approach is simple and can embrace various dissipative systems easily.
Overall, along with equations (8) and (15), the action variation for the linearly damped oscillation in EHP formalism is written as

$$\delta I_{\text{NEW}} = \int_0^t (\ddot{u} + c_1 \dot{u} + J - f) \delta u \, dt + \int_0^t (\ddot{u} - \dot{u}) \delta J \, dt + \left[ m \ddot{u} \delta \ddot{u} \right]_0^t - \left[ m \dot{u} \delta \dot{u} \right]_0^t - \left[ u - a \cdot J \right] \delta J \, dt = 0 \quad (16)$$

From the first line of equation (16) and arbitrary variation for $\delta u$ and $\delta J$, the following equation of motion along with rate-constitutive equation (equation (10)) is recovered for $\tau \in (0, t)$, and initial conditions are properly taken into account through equations (12) and (13).

For a nonlinear damping force that is proportional to the $n$th power of velocity, the governing differential equation in displacement-based approach becomes

$$m \ddot{u} + c_1 \dot{u} + F_N + k u - f = 0 \quad (18)$$

with $F_N$ representing a nonlinear damping force as

$$F_N = c_n \dot{u}^n \quad \text{if } n \text{ is odd} \quad (19)$$

$$F_N = c_n \dot{u}^n \text{sgn}(\dot{u}) \quad \text{if } n \text{ is even} \quad (20)$$

In equations (19) and (20), $|x|$ and sgn represent the absolute value of $x$ and the signum function, respectively. Also, $c_n$ is a coefficient of the $n$th power of velocity.

By introducing the following Rayleigh’s dissipation function $\varphi_N$, dynamics of such nonlinear damping system can be encapsulated in EHP formalism.

$$\varphi_N = \begin{cases} 
\varphi_{\text{Odd}} = \frac{1}{n + 1} c_n \left( \dot{u}^{n+1} \right) & \text{if } n \text{ is odd} \\
\varphi_{\text{Even}} = \frac{1}{n + 1} c_n \left| \dot{u}^{n+1} \right| & \text{if } n \text{ is even} 
\end{cases} \quad (21)$$

That is, the variation of Rayleigh’s dissipation is added to the action variation in the form of

$$\int_0^t \frac{d\varphi_N}{du} \delta u \, dt = \left\{ \begin{array}{ll}
\int_0^t \frac{d\varphi_{\text{Odd}}}{du} \delta u \, dt = \int_0^t c_n \dot{u}^n \delta u \, dt \\
\int_0^t \frac{d\varphi_{\text{Even}}}{du} \delta u \, dt = \int_0^t c_n \dot{u}^n \text{sgn}(\dot{u}) \delta u \, dt 
\end{array} \right. \quad (22)$$

From equation (22), one can see that the damping force $F_N$ in equations (19) and (20) is properly recovered. Overall, EHP formalism can embrace this system by defining the action variation as following form.

$$\delta I_{\text{NEW}} = \delta I + \left[ m \ddot{u} \right]_0^t + \int_0^t \frac{d\varphi_N}{du} \delta u \, dt + \int_0^t \frac{d\varphi_N}{du} \delta u \, dt = 0 \quad (23)$$

**Numerical implementation**

Based upon the weak formalism presented in “Variational formulation” section, various TFEMs can be developed. In order to show its viability, the TFEM for velocity-squared and/or cubic damping nonlinearities is
developed using low-order temporal shape functions. It starts from the following weak form that is the explicit expression of equation (23).

\[ \delta I_{[0, \Delta t]} = - \int_0^{\Delta t} (a \dot{J} - \dot{u}) \delta J \, dt + \left[ \hat{p} \delta u \right]_0^{\Delta t} - \int_0^{\Delta t} \left[ m \ddot{u} \delta u + \{ c_1 \dot{u} + c_2 \dot{u}^2 \text{sgn}(\dot{u}) + c_3 \dot{u}^3 \} \delta u - \dot{J} \delta u + \hat{f} \delta u \right] \, dt = 0 \]

(24)

Here, a single time-step variation of the action \( \delta I_{[0, \Delta t]} \) is considered with small time duration \([0, \Delta t]\). Also, in the discretization, the linear momentum \( \hat{p} (= m \ddot{u}) \) is introduced to use initial velocity, explicitly.

Equation (24) is the continuity-balanced equation, where the first temporal derivatives of \( \dot{u} \) and \( J \) as well as \( \delta u \) and \( \delta J \) appear. Thus, \( C^0 \) continuity is required to discretize primary variables and their corresponding variation. With introducing temporally linear shape functions

\[ L_0(\tau) = 1 - \tau / \Delta t; \quad L_{\Delta t}(\tau) = \tau / \Delta t \]

(25)

\( u \) and \( J \) can be approximated as

\[ u(\tau) = L_0(\tau) u_0 + L_{\Delta t}(\tau) u_{\Delta t}; \quad J(\tau) = L_0(\tau) J_0 + L_{\Delta t}(\tau) J_{\Delta t} \]

(26)

In equation (26), subscripts represent a discrete value of variable at time points. For example, \( u_{\Delta t} \) represents the displacement at \( \Delta t \).

Subsequently, \( \dot{u} \) and \( \ddot{u} \) are approximated as

\[ \dot{u}(\tau) = \dot{L}_0(\tau) u_0 + \dot{L}_{\Delta t}(\tau) u_{\Delta t}; \quad \ddot{u}(\tau) = \dot{L}_0(\tau) \dot{J}_0 + \dot{L}_{\Delta t}(\tau) \dot{J}_{\Delta t} \]

(27)

with first differentiation of shape functions

\[ \dot{L}_0(\tau) = -1 / \Delta t; \quad \dot{L}_{\Delta t}(\tau) = 1 / \Delta t \]

(28)

Similarly, \( \delta u \) and \( \delta J \) are approximated with temporal shape functions and discrete values.

With equations (25) to (28), each term in equation (24) can be temporally discretized. For example, the first term and the last term in equation (24) are discretized as

\[ - \int_0^{\Delta t} a \dot{J} \delta J \, dt = - \int_0^{\Delta t} a P \dot{J} \delta J \, dt = -a \int_0^{\Delta t} \delta J_0 \delta J_{\Delta t} \left\{ \frac{L_0}{L_{\Delta t}} \right\} \left\{ \frac{\dot{L}_0}{\dot{L}_{\Delta t}} \right\} \left\{ \frac{J_0}{J_{\Delta t}} \right\} \, dt \]

\[ = -a \int_0^{\Delta t} \delta J_0 \delta J_{\Delta t} \left[ \frac{1}{\Delta t} \frac{\dot{L}_0}{\dot{L}_{\Delta t}} \frac{\dot{L}_0}{\dot{L}_{\Delta t}} \right] \left\{ \frac{J_0}{J_{\Delta t}} \right\} = \left( -\frac{a}{\Delta t} J_0 + \frac{a}{\Delta t} J_{\Delta t} \right) \delta J_0 \left( \frac{a}{\Delta t} J_0 - \frac{a}{\Delta t} J_{\Delta t} \right) \delta J_{\Delta t} \]

(29)

and

\[ - \int_0^{\Delta t} \hat{f} \delta u \, dt = - \int_0^{\Delta t} \left[ \delta u_0 \delta u_{\Delta t} \right] \left\{ \frac{L_0}{L_{\Delta t}} \right\} \left\{ \frac{\hat{f}_0}{\hat{f}_{\Delta t}} \right\} \, dt \]

\[ = - \int_0^{\Delta t} \delta u_0 \delta u_{\Delta t} \left[ \frac{L_0}{L_{\Delta t}} \frac{L_0}{L_{\Delta t}} \right] \left\{ \frac{\hat{f}_0}{\hat{f}_{\Delta t}} \right\} = - \int_0^{\Delta t} \delta u_0 \delta u_{\Delta t} \left[ \frac{\Delta t}{3} \frac{\Delta t}{3} \frac{\Delta t}{3} \right] \left\{ \frac{\hat{f}_0}{\hat{f}_{\Delta t}} \right\} \]

\[ = \left( -\frac{\Delta t}{3} \frac{\hat{f}_0}{\hat{f}_{\Delta t}} - \frac{\Delta t}{6} \frac{\hat{f}_{\Delta t}}{\hat{f}_{\Delta t}} \right) \delta u_0 + \left( -\frac{\Delta t}{6} \frac{\hat{f}_0}{\hat{f}_{\Delta t}} - \frac{\Delta t}{3} \frac{\hat{f}_{\Delta t}}{\hat{f}_{\Delta t}} \right) \delta u_{\Delta t} \]

(30)
After evaluating each integral and collecting terms with respect to corresponding variations, equation (24) is
discretized as follows

\[
\delta I_{[0,\Delta t]} = \left[ \frac{a}{\Delta t} (J_{\Delta t} - J_0) - \frac{1}{2} (u_{\Delta t} + u_0) \right] \delta J_0 + \left[ -\frac{a}{\Delta t} (J_{\Delta t} - J_0) + \frac{1}{2} (u_{\Delta t} + u_0) \right] \delta J_{\Delta t} + \frac{m}{\Delta t} (u_{\Delta t} - u_0) + \frac{c_1}{2} (u_{\Delta t} - u_0) + \frac{1}{2} (J_{\Delta t} - J_0) - \frac{\Delta t}{3} \hat{f}_0 - \frac{\Delta t}{6} \hat{f}_{\Delta t} - \hat{p}_0 + \Gamma \right] \delta u_0 \\
+ \left[ -\frac{m}{\Delta t} (u_{\Delta t} - u_0) + \frac{c_1}{2} (u_{\Delta t} - u_0) + \frac{1}{2} (J_{\Delta t} - J_0) - \frac{\Delta t}{6} \hat{f}_0 - \frac{\Delta t}{3} \hat{f}_{\Delta t} - \hat{p}_{\Delta t} + \Gamma \right] \delta u_{\Delta t} = 0
\]  

(31)

In equation (31), \( \Gamma \) represents a cubic function of \( u_0 \) and \( u_{\Delta t} \) given by

\[ \Gamma(u_0, u_{\Delta t}) = \frac{c_3}{2 \Delta t^2} (u_{\Delta t} - u_0)^3 + \frac{c_2}{2 \Delta t} (u_{\Delta t} - u_0)^2 \text{sgn}(u_{\Delta t} - u_0) \]  

(32)

Notice that while deriving equation (31), the enforced condition in equation (11) is utilized.

The discretized weak form of equation (31) also contains the sequentially assigning process of initial values meaning that all the variations can arbitrarily vary. In consequence, the following independent equations are obtained.

\[ \frac{a}{\Delta t} (J_{\Delta t} - J_0) - \frac{1}{2} (u_{\Delta t} + u_0) = 0 \]  

(33)

\[ \frac{m}{\Delta t} (u_{\Delta t} - u_0) + \frac{c}{2} (u_{\Delta t} - u_0) + \frac{1}{2} (J_{\Delta t} - J_0) - \frac{\Delta t}{3} \hat{f}_0 - \frac{\Delta t}{6} \hat{f}_{\Delta t} - \hat{p}_0 + \Gamma = 0 \]  

(34)

\[ -\frac{m}{\Delta t} (u_{\Delta t} - u_0) + \frac{c_1}{2} (u_{\Delta t} - u_0) + \frac{1}{2} (J_{\Delta t} - J_0) - \frac{\Delta t}{6} \hat{f}_0 - \frac{\Delta t}{3} \hat{f}_{\Delta t} + \hat{p}_{\Delta t} + \Gamma = 0 \]  

(35)

MAPLE software\(^{31}\) can analytically resolve nonlinear simultaneous equations (33) to (35), and solutions are
given by

\[ u_{\Delta t} = \text{RootOf}(\Gamma_1 \text{ or } \Gamma_2) \]  

(36)

\[ J_{\Delta t} = J_0 + \frac{\Delta t}{2 a} (u_0 + u_{\Delta t}) \]  

(37)

\[ \hat{p}_{\Delta t} = \frac{1}{6 \Delta t} \left[ 12 m (u_{\Delta t} - u_0) + \Delta t^2 \left( \hat{f}_{\Delta t} - \hat{f}_0 \right) - 6 \Delta t \hat{p}_0 \right] \]  

(38)

where

\[ \Gamma_1 : A u_{\Delta t}^3 + B_1 u_{\Delta t}^2 + C_1 u_{\Delta t} + D_1 = 0 \quad \text{for } u_{\Delta t} > u_0 \]  

(39)

\[ \Gamma_2 : A u_{\Delta t}^3 + B_2 u_{\Delta t}^2 + C_2 u_{\Delta t} + D_2 = 0 \quad \text{for } u_{\Delta t} < u_0 \]  

(40)

with

\[ A = 6 c_3 a \]  

(41)

\[ B_1 = -18 c_3 u_0 a + 6 c_2 a \Delta t \]  

(42)
\[
C_1 = 12 m a \Delta t + 3 \Delta t^3 + 18 c_3 u_0^2 a + 6 c_1 a \Delta t^2 - 12 c_2 u_0 a \Delta t
\]  
(43)

\[
D_1 = 3 u_0 \Delta t^3 - 6 c_3 a u_0^3 - 12 m a u_0 \Delta t - 4 f_0 a \Delta t^3 - 6 c_1 a u_0 \Delta t^2 + 6 c_2 a u_0^2 \Delta t - 12 a \hat{p}_0 \Delta t^3 - 2 a \hat{f}_M \Delta t^3
\]  
(44)

\[
B_2 = -18 c_3 u_0 a - 6 c_2 a \Delta t
\]  
(45)

\[
C_2 = 12 m a \Delta t + 3 \Delta t^3 + 18 c_3 u_0^2 a + 6 c_1 a \Delta t^2 + 12 c_2 u_0 a \Delta t
\]  
(46)

\[
D_2 = 3 u_0 \Delta t^3 - 6 c_3 a u_0^3 - 12 m a u_0 \Delta t - 4 f_0 a \Delta t^3 - 6 c_1 a u_0 \Delta t^2 - 6 c_2 a u_0^2 \Delta t - 12 a \hat{p}_0 \Delta t^3 - 2 a \hat{f}_M \Delta t^3
\]  
(47)

Without any nonlinear source in damping (thus, \(c_2 = 0\) and \(c_3 = 0\)), both \(\Gamma_1\) and \(\Gamma_2\) are reduced to the linear equation \(L\)

\[
L : C_L u_M + D_L = 0
\]  
(48)

with

\[
C_L = 12 m a \Delta t + 3 \Delta t^3 + 6 c_1 a \Delta t^2
\]  
(49)
Figure 3. Responses of the cubic damping system under $\dot{f}_{\text{har}}$. (a) C1 ($c_3 = 0.1$) and (b) C5 ($c_3 = 0.5$).

Figure 4. Responses of the quadratic-cubic damping system under $\dot{f}_{\text{har}}$. (a) Q1C1 ($c_2 = 0.1, c_3 = 0.1$), (b) Q1C5 ($c_2 = 0.1, c_3 = 0.5$), (c) Q5C1 ($c_2 = 0.5, c_3 = 0.1$), and (b) Q5C5 ($c_2 = 0.5, c_3 = 0.5$).

Figure 5. Responses of the quadratic damping system under $\dot{f}_{\text{rec}}$. (a) Q1 ($c_2 = 0.1$) and (b) Q5 ($c_2 = 0.5$).

Figure 6. Responses of the cubic damping system under $\dot{f}_{\text{rec}}$. (a) C1 ($c_3 = 0.1$) and (b) C5 ($c_3 = 0.5$).
\[ D_L = 3 u_0 \Delta t^3 - 12 m a u_0 \Delta t - 4 f_0 a \Delta t^3 - 6 c_1 a u_0 \Delta t^2 - 12 a \hat{p}_0 \Delta t^2 - 2 a \hat{f}_0 \Delta t^3 \] (50)

Notice that the difference in equation (39) and equation (40) results from the existence of a signum function in equation (32). If a system has only one damping source from the velocity-squared, one obtains

\[ \bar{Q}_1 : \bar{B}_1 (u_M - u_0)^2 + C_L u_M + D_L = 0 \quad \text{for} \quad u_M > u_0 \quad (51) \]

\[ \bar{Q}_2 : \bar{B}_1 (u_M - u_0)^2 + C_L u_M + D_L = 0 \quad \text{for} \quad u_M < u_0 \quad (52) \]

with

\[ \bar{B}_1 = 6 c_2 a \Delta t \] (53)

For positive values of \( c_2, a, \) and \( \Delta t, \) \( \bar{B}_1 \) in equation (53) is always greater than 0.

With considering quadratic function and linear function separately, equations (51) and (52) can be schematically visualized in a function space as depicted in Figure 1.
Coefficients such as $C_L$ and $D_L$ in equations (49) and (50) are dependent on the time-step $\Delta t$, thus, by varying $\Delta t$, one can adjust the linear function $F_2$ in Figure 1 to have real root(s) of $u_{\Delta t}$. One critical question arises. What would be a numerical solution of $u_{\Delta t}$ if the equation has two real-valued roots? In a time-marching numerical method, a smaller size of the time-step $\Delta t$ generally improves accuracy and one can expect small changes in $u_{\Delta t}$ compared to the previous step solution $u_0$. Accordingly, among two real-valued roots of quadratic equations, $u_{\Delta t}$ can be refined with the following criterion.

Take $u_{\Delta t}$ that makes $|u_{\Delta t} - u_0|$ smallest. \hspace{1cm} (54)
This criterion still holds when a cubic damping exists: cubic equations in equations (39) and (40) always have at least one real-valued root and $u_D$ is selected based on this statement.

Overall, the following algorithm is adopted in the present method.

Algorithm: Low-order temporal finite element method

1. Identify input parameters.
   • System parameters ($m; a; c_1; c_2; c_3$)
   • Initial conditions ($u_0; \dot{p}_0; J_0 = 0$)
   • Known applied force ($f_0; f_D$)
   • Time parameters (time-step $\Delta t$ and total analysis time $t$)

2. Solve $u_{\Delta t}, J_{\Delta t}, \dot{p}_{\Delta t}$ with equations (36)–(38) and the statement (54).
3. Store and update solutions ($u_{\Delta t}, J_{\Delta t}, \dot{p}_{\Delta t}$ these are set to initial values for next step).
4. Repeat [2] to [3] until the total analysis time $t$.

Several points must be clarified in this algorithm. First, the algorithm obtains numerical solution through a single time-marching method by generalizing equation (24) as

$$\delta I[0, t] = \sum_{n=1}^{N} \delta I[(n-1)\Delta t, n\Delta t] \text{ with } N\Delta t = t$$  \hspace{1cm} (55)
where $\delta I_{([n-1]t_n, t_n+n\Delta t)} = 0$ in each time interval. After each time-step solutions, numerical results are stored and updated until the end of analysis. Second point is that any value can be taken for the initial value of $J_0$. Physically meaningful variable is the spring force $J$. The variable $J$ only appears in the discretized process (see equations (26) and (27) and equations (33)–(35)) with providing the reference to

Figure 10. Convergence of the present method (quadratic-cubic damping system under $f_{har}$). (a) Q1C1 ($c_2 = 0.1$, $c_3 = 0.1$), (b) Q1C5 ($c_2 = 0.1$, $c_3 = 0.5$), (c) Q5C1 ($c_2 = 0.5$, $c_3 = 0.1$), and (b) Q5C5 ($c_2 = 0.5$, $c_3 = 0.5$).

Figure 11. Convergence of the present method (quadratic damping system under $f_{rec}$). (a) Q1 ($c_2 = 0.1$) and (b) Q5 ($c_2 = 0.5$).

Figure 12. Convergence of the present method (cubic damping system under $f_{rec}$). (a) C1 ($c_3 = 0.1$) and (b) C5 ($c_3 = 0.5$).

Figure 13. Convergence of the present method (quadratic-cubic damping system under $f_{rec}$). (a) Q1C1 ($c_2 = 0.1$, $c_3 = 0.1$), (b) Q1C5 ($c_2 = 0.1$, $c_3 = 0.5$), (c) Q5C1 ($c_2 = 0.5$, $c_3 = 0.1$), and (b) Q5C5 ($c_2 = 0.5$, $c_3 = 0.5$).
compute $J_{\Delta t}$, whereas independent initial conditions such as $u_0$ and $\dot{p}_0$ are explicitly used. For this reason, $J_0 = 0$ is taken, here. Finally, notice that the present method utilizes the non-iterative algorithm exploiting characteristics of a polynomial equation and the solution refined technique among multiple real-valued roots.
Numerical simulation

Three cases of nonlinearity are taken to verify the present TFEM: linear-quadratic, linear-cubic, and linear-quadratic-cubic damping force. For each case, non-dimensional model parameters are taken such that \( m = 1 \), \( a = 4 \), \( c_1 = 0.01 \), \( \dot{u}(0) = 0 \), and \( \ddot{u}(0) = 0 \). Then, let nonlinear damping parameters \( c_2 \) and \( c_3 \) have 0.1 and 0.5. Each simulation case is named after nonlinear source. For example, Q1, C5, and Q5C1 represent quadratic damping with \( c_2 = 0.1 \), cubic damping with \( c_3 = 0.5 \), and quadratic-cubic damping with \( c_2 = 0.5 \) and \( c_3 = 0.1 \), respectively. Also, in simulation, two external loadings such as \( \dot{f}_{\text{har}} \) and \( \dot{f}_{\text{rec}} \) are considered.

\[
\dot{f}_{\text{har}} = 5 \sin(0.5 t) \quad (56)
\]

\[
\dot{f}_{\text{rec}} = 10 \quad \text{for} \quad 0 \leq t \leq 5\pi \quad \text{and} \quad \dot{f}_{\text{rec}} = 0 \quad \text{for} \quad t > 5\pi \quad (57)
\]

with a total analysis time \( t = 200 \). For the reference, results from the RKF45 method in MATLAB (ode45 solver)\(^{32}\) are provided, where \( \Delta t = 0.01 \) is taken in the present method, while default error tolerances such as \( \text{AbsTol} = 1\text{e-6} \) (absolute error tolerance) and \( \text{RelTol} = 1\text{e-3} \) (relative error tolerance) are adopted in the RKF45 method.

Numerical solutions of the present method and the RKF45 are presented in a phase space as depicted in Figures 2 to 7.

As shown in figures, for all the simulation cases studied, both methods produce essentially the same numerical solutions. It must be noted that a different number of total time steps is adopted in each numerical method. The present TFEM obtains 20,000 time-step solutions regardless of simulation case (\( \Delta t = 0.01 \) for \( t = 200 \)), whereas a total number of time steps in the RKF45 differs from case to case as indicated in Table 1.

In fact, the RKF45 method in MATLAB (ode45 solver) utilizes an adaptive time-step scheme. More specifically, the solver estimates error at a step and checks error tolerance. If error is greater than the error tolerance, changes the step-size and repeat this until criteria are satisfied. Such adaptive time-step scheme may show the further direction to refine one-step time marching algorithm in the present TFEM with taking optimal time-step size. However, at this moment, for every simulation case, convergence property of the present TFEM is investigated with varying time step, where the analysis time is fixed as \( t = 40 \). Simulation results are presented in Figures 8 to 13, and these figures indicate the good convergence characteristics of the present method. Also, one can see that the present method with time-step size \( \Delta t = 0.2 \) produces numerical results with sufficient accuracy for every case.

Conclusions

In this paper, SDOF (Single-Degree-Of-Freedom) nonlinear systems involving polynomial powers of dissipative damping force are encapsulated in the context of EHP (Extended framework of Hamilton’s Principle). This formalism can properly consider initial conditions and all the governing differential equations. Based upon the present weak formulation, various TFEMs (Temporal Finite Element Methods) can be developed. For its viability, the TFEM using low-order shape functions is developed for quadratic, cubic, and quadratic-cubic damped systems. The present TFEM utilizes non-iterative algorithm, and some numerical examples are provided to validate this approach. For every example studied, the present TFEM using small time-step produces essentially the same results to those obtained from the adaptive time-step RKF45 method with default error tolerances in MATLAB. Also, additional simulation shows the good convergence characteristics of the present method.

There still remain some challenging issues to be addressed. In particular, future work will be directed toward the use of optimal time-step size, development of higher order time-stepping methods, and extension to other nonlinear sources\(^{8,33–35}\).

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