In this paper, we prove that the lower triangular matrix category $\Lambda = \begin{bmatrix} T & 0 \\ M & U \end{bmatrix}$, where $T$ and $U$ are quasi-hereditary Hom-finite Krull-Schmidt $K$-categories and $M$ is a $U \otimes_K T^{op}$-module that satisfies suitable conditions, is quasi-hereditary in the sense of \cite{26} and \cite{32}. Moreover, we solve the problem of finding quotients of path categories isomorphic to the lower triangular matrix category $\Lambda$, where $T = KR/\mathcal{J}$ and $U = KQ/I$ are path categories of infinity quivers modulo admissible ideals. Finally, we study the case where $\Lambda$ is a path category of a quiver $Q$ with relations and $U$ is the full additive subcategory of $\Lambda$ obtained by deleting a source vertex $*$ in $Q$ and $T = \text{add}\{*\}$. We then show that there exists an adjoint pair of functors $(R, E)$ between the functor categories $\text{mod } \Lambda$ and $\text{mod } U$ that preserve orthogonality and exceptionality; see \cite{2}. We then give some examples of how to extend classical tilting subcategories of $U$-modules to classical tilting subcategories of $\Lambda$-modules.

Introduction

Rings of the form $\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}$ where $T$ and $U$ are rings and $M$ is a $T-U$-bimodule have appeared often in the study of the representation theory of artin rings and algebras \cite{6} \cite{16} \cite{21} \cite{20}. Such rings are called triangular matrix rings and appear in the study of homomorphic images of hereditary artin algebras and of the decomposition of algebras and the direct sum of two rings. Triangular matrix rings and their homological properties have been widely studied. The so-called one-point extension is a special case of the triangular matrix algebra, and these types of algebras have been studied in several contexts; see for example \cite{12} \cite{18} \cite{22}. Recently in \cite{43}, B. Zhu studied the triangular matrix algebra $\Lambda$ where $T$ and $U$ are quasi-hereditary algebras, and he proved that under suitable conditions on $M$, $\Lambda$ is a quasi-hereditary algebra.

The notion of quasi-hereditary algebra and highest weight category were introduced and studied by E. Cline, B. Parshall and L. Scott \cite{38} \cite{11} \cite{41}. Highest
weight categories arise in the representation theory of Lie algebras and algebraic groups. For the setting of finite dimensional algebras, quasi-hereditary algebras were amply studied by V. Dlab and M. Ringel in [13, 16, 17, 39]. In addition, they introduced the set of standard modules \( \Lambda \Delta \) associated with an algebra \( \Lambda \). More recently, M. Ringel studied the homological properties of the category \( \mathcal{F}(\Lambda \Delta) \) of \( \Lambda \Delta \)-filtered \( \Lambda \)-modules and constructed the characteristic tilting module \( \Lambda T \) associated with \( \mathcal{F}(\Lambda \Delta) \).

On the other hand, the idea that additive categories are rings with several objects was developed by B. Mitchell, see [41], who showed that a substantial amount of non-commutative ring theory is still true in this generality and that familiar theorems for rings originate from the natural development of category theory. Following this approach, M. Auslander and I. Reiten worked exhaustively with the functor categories and found numerous applications to the development of the representation theory of artin algebras; see for instance [3, 4, 5, 6, 7], to name a few.

More recently, R. Martínez-Villa and Ø. Solberg, motivated by the work on functor categories by M. Auslander in [5, 3], studied the Auslander-Reiten components of finite dimensional algebras. They did so in order to establish when the category of graded functors is Noetherian [35, 36, 37]. Recently, R. Martínez-Villa and M. Ortíz studied tilting theory in arbitrary functor categories, in [34, 33]. They proved that most of the properties that are satisfied by a tilting module over an artin algebra also hold for functor categories. It is worth noting that the study of Auslander-Reiten components of finite dimensional algebras is related to the study of some infinite quivers that play an important role in the representation theory of coalgebras; see [9].

Following the line of the above mentioned works, M. Ortíz introduced in [32] the concept of quasi-hereditary category to study the Auslander-Reiten components of a finite dimensional algebra \( \Lambda \). Similarly, as the standard modules appear in the theory of quasi-hereditary algebras, the concept of standard functors appears in this context, which resulted in a generalization of the notion of standard modules. As a consequence, a connection is obtained between highest weight categories and quasi-hereditary categories as stated by H. Krause in [25].

Finally, following Mitchell’s philosophy, the concept of the triangular matrix category is introduced in [26, 27], as the analogue of the triangular matrix algebra to the context of rings with several objects, by V. Santiago, A. Leon and M. Ortíz, and they obtain some applications to path categories given by infinite quivers, the construction of recollements and the study of functorially finite subcategories in functor categories.

The aim of this paper is to show that infinite quiver path categories and quasi-hereditary categories can be constructed as triangular matrix categories, generalizing some of the results obtained by B. Zhu in [43]. Furthermore if, this type of matrices are of the one-point extension type, some homological
properties can be described. It is worth mentioning that recently in [31] similar results have been obtained in the context of standardly stratified lower triangular \( K \)-algebras with enough idempotents.

We outline the content of the paper section-by-section as follows.

In section 1, we recall basic results about path categories, functor categories, quasi-hereditary categories and triangular matrix categories.

In section 2, following the same arguments given by Leszczyński in [28, Lemma 1.3.], we prove that the tensor product of two path categories is again a path category as defined by Ringel [40]. Thus we will use this result to solve the problem of finding quotients of path categories isomorphic to the lower triangular matrix category \( \frac{K R}{J M K} / I Q \), where \( K R / J \) and \( K Q / I \) are path categories modulo admissible ideals and \( M \) is a functor from \( K Q / I \otimes (K R / J)^{op} \) to the category mod \( K \); see [42].

In section 3, we generalize a result given in [43, Theorem 3.1]. Specifically, we prove that if \( U \) and \( T \) are Hom-finite Krull-Schmidt quasi-hereditary categories with respect to filtrations \( \{U_j\}_{0 \leq j \leq n} \) and \( \{T_j\}_{j \geq 0} \) of \( U \) and \( T \), respectively, consisting of additively closed subcategories and \( M \in \mathcal{F}(U \Delta) \) for all \( T \in \mathcal{T} \), then \( \Lambda = (\frac{T}{M}, 0) \) is quasi-hereditary with respect to a certain filtration \( \{\Lambda_j\}_{j \geq 0} \). Moreover, we obtain a characterization of the category of the \( \Lambda \Delta \)-filtered \( \Lambda \)-modules.

In section 4, we consider a strongly locally finite quiver \( Q = (Q_0, Q_1) \) with relations \( I \). Then, given a source \( * \) in \( Q \) and by setting the full additive subcategories of \( C := K Q / I \), \( U = \text{add}(Q_0 - \{*\}) \) and \( T := \text{add}\{*\} \), we see that \( C \) is isomorphic to a triangular matrix category \( \Lambda \), and we prove that there exist an adjoint pair of additive functors \( R : \text{mod} \Lambda \rightarrow \text{mod} U \) and \( E : \text{mod} U \rightarrow \text{mod} \Lambda \) which preserve orthogonality and exceptionality, generalizing some results given in [2]. Further on we apply this result to extend classical tilting categories to functor categories.

1. Preliminaries

1.1. \( K \)-categories, Path Categories, Representations and Functor Categories. In this part, we recall some basic definitions to approach this work. The reader can consult [11] and [8] for more details.

\( K \)-Categories. Let \( K \) be a field. A category \( \mathcal{C} \) is a \( K \)-category if for each pair of objects \( X \) and \( Y \) in \( \mathcal{C} \), the set of morphisms \( \mathcal{C}(X, Y) \) is equipped with a \( K \)-vector space structure such that the composition \( \circ \) of morphisms in \( \mathcal{C} \) is a \( K \)-bilinear map. A \( K \)-category \( \mathcal{C} \) is called Hom-finite if \( \dim_K \mathcal{C}(X, Y) < \infty \).

Krull-Schmidt Categories. A Krull-Schmidt category is an additive category such that each object decomposes into a finite direct sum of indecomposable objects with local endomorphism rings.
IDEALS. Let \( C \) be an additive \( K \)-category. A class \( I \) of morphisms of \( C \) is a **two-sided ideal** in \( C \) if: (a) the zero morphism \( 0_X \in C(X, X) \) belongs to \( I \); (b) if \( f, g : X \rightarrow Y \) are morphisms in \( I \) and \( \lambda, \mu \in K \), then \( \lambda f + \mu g \in I \); (c) if \( f \in I \) and \( g \) is a morphism in \( C \) that is left-composable with \( f \), then \( g \circ f \in I \) and (d) if \( f \in I \) and \( h \) is a morphism in \( C \) that is right-composable with \( f \), then \( f \circ h \in I \). Equivalently, a two-sided ideal \( I \) of \( C \) can be considered as a subfunctor \( I(-, -) \subseteq C(-, -) : C^{op} \times C \rightarrow \text{Mod} K \), defined by assigning to each pair \((X, Y)\) of objects \( X, Y \) of \( C \) a \( K \)-subspace \( I(X, Y) \) of \( C(X, Y) \) such that: (i) if \( f \in I(X, Y) \) and \( g \in C(Y, Z) \), then \( gf \in I(X, Z) \); and (ii) if \( f \in I(X, Y) \) and \( h \in C(U, X) \), then \( fh \in I(U, Z) \).

Given a two-sided ideal \( I \) in an additive \( K \)-category \( C \), the **quotient category** \( C/I \) that will be the category with objects that are objects of \( C \) and the space of morphisms from \( X \) to \( Y \) in \( C/I \) is the quotient space \( (C/I)(X, Y) = C(X, Y)/I(X, Y) \) of \( C(X, Y) \). It is easy to see that the quotient category \( C/I \) is an additive \( K \)-category, and the projection functor \( \pi : C \rightarrow C/I \) assigning to each \( f : X \rightarrow Y \) in \( C \) the coset \( f + I \in C/I(X, Y) \) is a \( K \)-linear functor. Moreover, \( \pi \) is full and dense, and \( \ker(\pi) = I \).

The **(Jacobson) radical** of an additive \( K \)-category \( C \) is the two-sided ideal \( \text{rad}_C \) in \( C \) defined by the formula \( \text{rad}_C(X, Y) = \{ h \in C(X, Y) : 1_X - gh \text{ is invertible for any } g \in C(Y, X) \} \) for all objects \( X \) and \( Y \) of \( C \).

**Remark 1.1.** Let \( I \) be an ideal in a \( K \)-category \( C \). We clearly see then that \( f = (f_{ij}) : \oplus_{i=1}^n X_i \rightarrow \oplus_{j=1}^m Y_j \) lies in \( I(\oplus_{i=1}^n X_i, \oplus_{j=1}^m Y_j) \) if and only if \( f_{ij} \) lies in \( I(X_i, Y_j) \) for all \( i \) and \( j \), such that \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \).

**Tensor Product of \( K \)-Categories.** Let \( C \) and \( C' \) be \( K \)-categories. The tensor product \( C \otimes_K C' \) is the \( K \)-category whose class of objects is \( \text{Obj } C \times \text{Obj } C' \), where the set of morphisms from \((p_1, q_1)\) to \((p_2, q_2)\) is the ordinary tensor product \( C(p_1, p_2) \otimes C'(q_1, q_2) \). The composition

\[
C(p_1, p_2) \otimes C'(q_1, q_2) \times C(p_2, p_3) \otimes C'(q_2, q_3) \rightarrow C(p_1, p_3) \otimes C'(q_1, q_3)
\]

is given by the rule \(((f_1 \otimes g_1), (f_2 \otimes g_2)) \mapsto (f_2 f_1 \otimes g_2 g_1)\). This composition is bilinear; see [30).

**Quivers, Path Algebras and Path Categories.** A **quiver** is an oriented graph, formally denoted by a quadruple \( Q = (Q_0, Q_1, s, t) \), with a set of vertices \( Q_0 \) and a set of arrows \( Q_1 \), and two maps \( s, t : Q_1 \rightarrow Q_0 \), called source and target, defined by \( s(a \rightarrow b) = a \) and \( t(a \rightarrow b) = b \), respectively, if \( \alpha : a \rightarrow b \) is an arrow in \( Q_1 \).

A path of length \( l \geq 1 \) from \( a \) to \( b \) in a quiver \( Q \) is of the form \((a|\alpha_1, \ldots, \alpha_l|b)\) with arrows \( \alpha_i \) satisfying \( t(\alpha_i) = s(\alpha_{i+1}) \) for all \( 1 \leq i \leq l \) and \( a = s(\alpha_1) \) as well as \( b = t(\alpha_l) \). In addition, for any vertex \( a \) in \( Q_0 \), a path of length 0 from \( a \) to itself is denoted by \( e_a \).

Given a quiver \( Q \), it is the **path category** \( KQ \) is an additive category, with objects being direct sums of indecomposable objects. The indecomposable
be an ideal, we then say the representation \( f \) maps to a vector space and \( V \). All \( \rho \) \( \in \mathcal{I} \) \( \subseteq \mathcal{I} \) \( \in \mathcal{K}_Q \). Let \( I \) be an ideal in \( \mathcal{K}_Q \). Then given a pair of finite sets of vertices \( \{X_i\}_{i=1}^n \) \( \{Y_j\}_{j=1}^m \) we set \( \mathcal{I}(\oplus_{i=1}^n X_i, \oplus_{j=1}^m Y_j) = \{(f_{ij}) \in \mathcal{K}_Q(\oplus_{i=1}^n X_i, \oplus_{j=1}^m Y_j) | f_{ij} \in I\} \). This allows us to define an ideal \( \mathcal{I} \) in \( \mathcal{K}_Q \), and we refer to it as the ideal generated by \( I \). If \( I \subseteq \mathcal{K}_Q \) is generated by the set of paths \( \{\rho_i|i\} \), we say that \( \mathcal{I} \) is generated by the set of \( \{\rho_i|i\} \).

The ideal generated by all arrows is denoted by \( \mathcal{K}_Q^+ \). Note that \( (\mathcal{K}_Q^+)^n \) is the ideal generated by all paths of length \( \geq n \). Given vertices \( a,b \in Q_0 \), a finite linear combination \( \sum_w c_w w \) with \( c_w \in K \) where \( w \) is a path of length \( \geq 2 \) from \( a \) to \( b \) is called a relation on \( Q \). Any ideal \( I \subseteq (\mathcal{K}_Q^+)^2 \) can be generated, as an ideal, by relations. An ideal \( I \subseteq \mathcal{K}_Q \) is called admissible if it is generated by a set of relations. We then say that an ideal \( \mathcal{I} \) in \( \mathcal{K}_Q \) is admissible if it is generated by an admissible ideal in \( \mathcal{K}_Q \).

**Representations of Quivers.** A representation of a quiver \( Q \) is a pair \( V = (V_i)_{i \in Q_0}, (V_{ij})_{i,j \in Q_0} \), where each element of the family \( \{V_i\}_{i \in Q_0} \) is a vector space and \( V_{ij} : V_{i(a)} \to V_{j(a)} \) is a \( K \)-linear map. Let \( V \) and \( W \) be two representations of \( Q \). A morphism from \( V \) to \( W \) is a family of linear maps \( f = (f_i : V_i \to W_i)_{i \in Q_0} \) such that for each arrow \( \alpha : i \to j \) we have \( f_j V_{\alpha} = W_{\alpha} f_i \). We denote by \( \text{rep} \ Q \) the abelian category that has as objects the representations of \( Q \) and as morphisms just the morphisms of representations. Let \( \rho = (a|\alpha_1,\ldots,\alpha_l|b) \) a path in \( Q \), we set \( V_{\rho} = V_{a_1} \circ \cdots \circ V_{a_l} \). Let \( I \subseteq \mathcal{K}_Q \) be an ideal, then we say the representation \( V \) is bounded by \( I \) if \( V_{\rho} = 0 \) for all \( \rho \in I \). The full subcategory of \( \text{rep} \ Q \) consisting of representations bounded by \( I \) is denoted by \( \text{rep} \ (Q,I) \).

**Strongly Locally Finite Quivers.** Let \( Q \) be a quiver. For \( x \in Q_0 \), we denote by \( x^+ \) and \( x^- \) the set of arrows starting in \( x \) and the set of arrows ending in \( x \), respectively. Recall that \( x \) is a sink vertex or a source vertex if \( x^+ = \emptyset \) or \( x^- = \emptyset \). One says that \( Q \) is locally finite if \( x^+ \) and \( x^- \) are finite sets and interval finite if the set of paths from \( x \) to \( y \) is finite for any \( x,y \in Q_0 \). For short, we say that \( Q \) is strongly locally finite if it is locally finite and interval finite. In particular, \( Q \) contains no oriented cycle in case it is interval finite. Note that under these conditions, if \( Q \) is a strongly
locally finite quiver, the path category \( K \mathcal{Q} \) is a Hom-finite Krull-Schmidt \( K \)-category; see [3].

**Functor categories.** Recall that a category \( \mathcal{C} \) is said to be **skeletally small** if it has a small dense subcategory \( \mathcal{C}' \), see [3]. Let \( \mathcal{C} \) be a Hom-finite Krull-Schmidt and skeletally small \( K \)-category. The abelian category \((\mathcal{C}, \text{Ab})\) is the category of all additive covariant functors from \( \mathcal{C} \) to the category of abelian groups, which we will call \( \mathcal{C} \)-modules. Given two \( \mathcal{C} \)-modules \( F \) and \( G \), the set of morphisms \( \text{Hom}_{\mathcal{C}, \text{Ab}}(F, G) \) is denoted simply by \( \text{Hom}_C(F, G) \). Following [3, 4], \((\mathcal{C}, \text{Ab})\) is denoted by \( \text{Mod} \mathcal{C} \). We recall that a \( \mathcal{C} \)-module \( M \) is **finitely presented** if an exact sequence \( P_1 \to P_0 \to M \to 0 \) of \( \mathcal{C} \)-modules exist where \( P_0 \) and \( P_1 \) are projective finitely generated \( \mathcal{C} \)-modules.

We denote by \( \text{mod} \mathcal{C} \) the full subcategory of \( \text{Mod} \mathcal{C} \) consisting of finitely presented \( \mathcal{C} \)-modules. Let \( \mathcal{M} \) be a \( \mathcal{C} \)-module, so each \( \mathcal{C} \) in \( \mathcal{C} \) the abelian group \( M(\mathcal{C}) \) has a structure as a \( \text{End}_\mathcal{C}(\mathcal{C}) \)-module and hence as a \( K \)-module since \( \text{End}_\mathcal{C}(\mathcal{C}) \) is a \( K \)-algebra. We denote by \((\mathcal{C}, \text{mod} K)\) the full category of \( \text{Mod} \mathcal{C} \) of all \( \mathcal{C} \)-modules such that \( M(\mathcal{C}) \) is a finitely generated \( K \)-module. The category \((\mathcal{C}, \text{mod} K)\) is an abelian category with the property that the inclusion \((\mathcal{C}, \text{mod} K) \to \text{Mod} \mathcal{C} \) is exact and contains mod \( \mathcal{C} \) as a full subcategory.

Let \( Q \) be a quiver and \( I \) be an ideal \( I \subset K \mathcal{Q} \). Set \( \mathcal{C} = K \mathcal{Q}/I \). Then each representation \( V = ((V_i)_{i \in \mathcal{Q}}_0, (V_a)_{a \in \mathcal{Q}}_1) \) in \( \text{rep}(Q, I) \) defines a \( \mathcal{C} \)-module \( \tilde{V} \) in \( \mathcal{C} \), mod \( K \) by setting \( \tilde{V}(i) = V_i \) and \( \tilde{V}(a) = V_a \).

In general, the functor \( D : (\mathcal{C}, \text{mod} K) \to (\mathcal{C}^{\text{op}}, \text{mod} K) \) given by \( D(M)(X) = \text{Hom}_K(M(X), K) \) for all \( X \in \mathcal{C} \) defines a duality between \((\mathcal{C}, \text{mod} K)\) and \((\mathcal{C}^{\text{op}}, \text{mod} K)\), and we refer to it as the **standard duality**.

1.2. **Quasi-hereditary categories and triangular matrix categories.** Assume \( \mathcal{C} \) is a Hom-finite Krull-Schmidt \( K \)-category. In order to generalize the notion of quasi-hereditary category, the notion of heredity ideal and heredity chain is introduced in [32].

**Heredity ideals.** A two-sided ideal \( I \) in \( \mathcal{C} \) is called (left) **heredity** if the following conditions hold: (i) \( I^2 = I \), i.e., \( I \) is an idempotent ideal; (ii) \( I \text{rad} \mathcal{C}(-, ?) I = 0 \), and (iii) \( I(X, -) \) is a projective finitely generated \( \mathcal{C} \)-module for all \( X \in \mathcal{C} \). \( \mathcal{C} \) is called quasi-hereditary if there exist a chain \( \{I_j\}_{j \in J} \), where \( J \) is at most countable of two-sided ideals \( 0 = I_0 \subset I_1 \subset \cdots \subset \mathcal{C}(-, ?) \), which is exhaustive (that is, \( \cup_{j \in J} I_j = \mathcal{C}(-, ?) \)), and \( I_j/I_{j-1} \) is heredity in the quotient category \( \mathcal{C}/I_{j-1} \). Such a chain is called a **heredity chain**.

Let \( B \) be a full additive subcategory of \( \mathcal{C} \). Given \( \mathcal{C}, \mathcal{C}' \in \mathcal{C} \), we denote by \( I_B(\mathcal{C}, \mathcal{C}') \) the subset of \( \mathcal{C}(\mathcal{C}, \mathcal{C}') \) consisting of morphisms which factor through some object in \( B \). This allows us to define the two-sided ideal \( I_B(-, ?) \) which is an idempotent ideal in \( \mathcal{C} \). Moreover, if we denote by \( \text{Tr}_{\{\mathcal{C}(E, -)\}_{E \in B}} \mathcal{C}(X, -), X \in \mathcal{C} \), the trace of \( \{\mathcal{C}(E, -)\}_{E \in B} \) in \( \mathcal{C}(X, -) \), we have \( \text{Tr}_{\{\mathcal{C}(E, -)\}_{E \in B}} \mathcal{C}(X, -) = I_B(X, -) \).
QUASI-HEREDITARY CATEGORIES. Assume we have an exhaustive filtration \( \{B_j\}_{j \geq 0} \) of \( \mathcal{C} \) into additive full subcategories (that is, \( \cup_{j \geq 0} B_j = \mathcal{C} \)). We then have an exhaustive chain of two-sided idempotent ideals:

\[
\{0\} \supseteq I_{B_0} \supseteq I_{B_1} \supseteq \cdots \supseteq I_{B_j} \supseteq \cdots \supseteq C(-,?)
\]

Note that \( I_{B_{j-1}} / I_{B_j} \) is an idempotent ideal in the quotient category \( \mathcal{C} / I_{B_{j-1}} \) since \( I_{B_j} \) and \( I_{B_{j-1}} \) are idempotents in \( \mathcal{C} \) and

\[
\left( \frac{I_{B_j}}{I_{B_{j-1}}} \right)^2 = \frac{I_{B_j}}{I_{B_{j-1}}} \frac{I_{B_{j-1}}}{I_{B_{j-1}}} = \frac{I_{B_j}^2 + I_{B_{j-1}}}{I_{B_{j-1}}} = \frac{I_{B_j}}{I_{B_{j-1}}}.
\]

The above motivates us to introduce the principal definition in this section.

**Definition 1.2.** Let \( \mathcal{C} \) be a Hom-finite Krull-Schmidt \( K \)-category. Assume that \( \{B_j\}_{j \geq 0} \) is an exhaustive filtration of \( \mathcal{C} \) into full additive subcategories. We say that \( \mathcal{C} \) is quasi-hereditary with respect to \( \{B_j\}_{j \geq 0} \) if

\[
\{0\} \supseteq I_{B_0} \supseteq I_{B_1} \supseteq \cdots \supseteq I_{B_j} \supseteq \cdots \supseteq C(-,?)
\]

is a heredity chain.

Recall that a full additive subcategory \( \mathcal{B} \) of \( \mathcal{C} \) is called additively closed if it is closed under direct summands and isomorphisms. The following result given in [32] will be useful in the remainder of this work.

**Theorem 1.3.** Let \( \{B_j\}_{j \geq 0} \) be an exhaustive filtration of \( \mathcal{C} \) into additively closed full subcategories. Then \( \mathcal{C} \) is quasi-hereditary with respect to \( \{B_j\}_{j \geq 0} \) if and only if the following conditions hold:

(i) \( \text{rad}_\mathcal{C}(E, E') = I_{B_{j-1}}(E, E'), \) for all pairs of objects \( E, E' \in \text{Ind } B_j - \text{Ind } B_{j-1} \);

(ii) and for all \( X \in \mathcal{C} \) and \( j \geq 1, \) there exists an exact sequence

\[
\mathcal{C}(E_{j-1}, -) \longrightarrow \mathcal{C}(E_j, -) \longrightarrow I_{B_j}(X, -) \longrightarrow 0,
\]

with \( E_j \in B_j \) and \( E_{j-1} \in B_{j-1} \).

**Standard \( \mathcal{C} \)-modules.** Let \( \mathcal{C} \) be a quasi-hereditary category with respect to a family of additively closed subcategories \( \{B_j\} \). Each module \( c\Delta E(j) := \mathcal{C}(E, -)/I_{B_{j-1}}(E, -), \) \( E \in \text{Ind } B_j - \text{Ind } B_{j-1} \) is called standard, and \( c\Delta(j) \) denotes the category consisting of the standard \( \mathcal{C} \)-modules \( c\Delta E(j) \). In addition, \( c\Delta \) denotes the full subcategory consisting of the standard \( c\Delta \)-modules.

**Filtered \( \mathcal{C} \)-modules.** Let \( \mathcal{A} \) be an abelian category, and \( \mathcal{X} \subseteq \mathcal{A} \). We denote by \( \mathcal{X}^\text{fil} \) the class of objects of \( \mathcal{A} \), which are a finite direct sum of objects in \( \mathcal{X} \). We say that \( M \in \mathcal{A} \) is \( \mathcal{X} \)-filtered if there exists a chain \( \{M_j\}_{j \geq 0} \) of subobjects of \( M \) such that \( M_{j+1}/M_j \in \mathcal{X}^\text{fil} \) for \( j \geq 0 \). In case \( M = M_n \) for some \( n \in \mathbb{N} \), we say that \( M \) has a finite \( \mathcal{X} \)-filtration of length \( n \). We denote by \( \mathcal{F}(\mathcal{X}) \) the class of objects that are \( \mathcal{X} \)-filtered and by \( \mathcal{F}_f(\mathcal{X}) \) the class of
objects that have a finite filtration. For $M \in \mathcal{F}_f(\mathcal{X})$, the $\mathcal{X}$-length of $M$ can be defined as follows $l_\mathcal{X}(M) := \min\{n \in \mathbb{N} : M \text{ has an } \mathcal{X}\text{-filtration of length } n\}$.

By using the notion of $\mathcal{X}$-length and induction, the following useful remark can be proven.

**Remark 1.4.** Let $\mathcal{X}$ be a class of objects in an abelian category $\mathcal{A}$. Then, the class $\mathcal{F}_f(\mathcal{X})$ is closed under extensions.

Given a $\mathcal{C}$–module $F$, its trace filtration with respect to $\{B_j\}_{j \geq 0}$ is given by

$$\{0\} = F^{[0]} \subset F^{[1]} \subset F^{[2]} \subset \cdots \subset F^{[j]} \subset \cdots,$$

where $F^{[j]} := Tr_{\mathcal{C}(E_j,-)} F$ and $F = \bigcup_{j \geq 0} F^{[j]}$.

It is of interest to study the $\mathcal{C}$-modules $F$ that possess a trace filtration that satisfies $F^{[j]} \in c\Delta^{(j)}$ for all $j \geq 1$. It then follows that these $\mathcal{C}$-modules are $\Delta$-filtered. We denote the full subcategory of the $\Delta$-filtered modules by $\mathcal{F}(c\Delta)$.

The following result will be very useful in what follows.

**Lemma 1.5.** Let $F \in \mathcal{F}(\Delta)$.

(a) For all $j \geq 0$, $F^{[j]}$ has a presentation

$$(2) \quad \mathcal{C}(E_{j-1},-) \to \mathcal{C}(E_j,-) \to F^{[j]} \to 0, \quad E_{j-1} \in B_{j-1}, E_j \in B_j.$$

(b) $F^{[j]} \cong \mathcal{C} \otimes_{B_j}(F_{|B_j})$; see [13 Proposition A.3.2].

**Triangular Matrix Categories.** Following Mitchell’s philosophy [30], in [26, 27], the notion of triangular matrix categories is introduced in order to define the analogous of the triangular matrix algebras to the context of rings with several objects.

**Definition 1.6.** Given $\mathcal{T}$ and $\mathcal{U}$ additive $K$-categories and a functor $M : \mathcal{U} \otimes_K \mathcal{T}^{op} \to \text{Mod } K$, the triangular matrix category $\Lambda = (\mathcal{T} \mathcal{M} \mathcal{U})$ is the additive $K$-category whose collection of objects are the matrices $(\mathcal{T} \mathcal{M} \mathcal{U})$, where $\mathcal{U}$ and $\mathcal{T}$ are objects in $\mathcal{U}$ and $\mathcal{T}$, respectively.

Let $X = (\mathcal{T} \mathcal{M} \mathcal{U})$, $X' = (\mathcal{T} \mathcal{M} \mathcal{U})$ and $X'' = (\mathcal{T} \mathcal{M} \mathcal{U})$ be objects in $\Lambda$. The set of morphisms from $X$ to $X'$ is $\Lambda(X,X') = \left(\mathcal{T}(\mathcal{T}',\mathcal{M}(\mathcal{U},\mathcal{U}'))\right)$, where $\mathcal{T}(\mathcal{T}',\mathcal{M}(\mathcal{U},\mathcal{U}')) := \left\{\left(\frac{f}{g}\right) : f \in \mathcal{T}(\mathcal{T}',\mathcal{M}(\mathcal{U},\mathcal{U}')), g \in \mathcal{U}(\mathcal{U},\mathcal{U}'), m \in \mathcal{M}(\mathcal{U},\mathcal{U}')\right\}$, and the composition $\Lambda(X',X'') \times \Lambda(X,X') \to \Lambda(X,X'')$ is defined by

$$\left(\left(\frac{f_2}{g_2}, \frac{f_1}{g_1}\right), \frac{f}{g}\right) \mapsto \left(\frac{f_2 \circ f_1}{g_2 \circ g_1}\right)$$

with $m_2 \circ f_1 := M(1_{\mathcal{U}}, f_1) \circ (m_2)$ and $g_2 \circ m_1 := M(g_1 \circ 1_{\mathcal{T}})(m_1)$, where $M(1_{\mathcal{U}}, f_1) : M(U''', T') \to M(U'', T')$ and $M(g_2 \otimes 1_{\mathcal{T}}) : M(U'' \otimes T) \to M(U' \otimes T)$ are morphisms in $\text{Mod } K$. 


2. Tensor product of path categories and triangular categories over path categories

In [28, Lemma 1.3], Z. Leszczyński proves that the tensor product of two path algebras is again a path algebra. Following the same arguments given by Leszczyński, we see that the result can be extended to the context of path categories as defined by Ringel [40]. We give the proof of this result for the benefit of the reader. We will then use these results in the rest of this section to solve the problem of finding quotients of path categories isomorphic to the lower triangular matrix category

\[
\Lambda = \left(\frac{K\mathcal{R}/\mathcal{J}}{M}, \frac{K\mathcal{Q}/\mathcal{I}}{J_0}\right)
\]

where \(K\mathcal{R}/\mathcal{J}\) and \(K\mathcal{Q}/\mathcal{I}\) are path categories modulo admissible ideals and \(M\) is a functor from \(K\mathcal{Q}/\mathcal{I}\otimes(K\mathcal{R}/\mathcal{J})^{op}\) to the category \(mod\ K\); see [42].

We start with the definition of a product of two quivers.

**Remark 2.1.** So all the path categories mentioned in this work satisfy the conditions of being Hom-finite Krull-Schmidt \(K\)-categories, all the quivers to which we refer will be considered strongly locally finite quivers.

**Definition 2.2.** Given two quivers \(Q = (Q_0, Q_1)\) and \(Q' = (Q'_0, Q'_1)\), the product quiver \((Q \times Q', (Q \times Q')_0, (Q \times Q')_1)\) is defined by

\[
(Q \times Q')_0 = Q_0 \times Q'_0 \text{ and } (Q \times Q')_1 = (Q_0 \times Q'_1) \cup (Q_1 \times Q'_0).
\]

Let \(Q\) and \(Q'\) be quivers, and consider their path algebras \(KQ\) and \(KQ'\). Assume that \(I\) and \(I'\) are admissible ideals in \(KQ\) and \(KQ'\), respectively. Let \(I\boxtimes I'\) be the ideal in \(K(Q \times Q')\) generated by \((Q_0 \times I') \cup (I \times Q'_0)\) and the set of relations

\[
(p', \beta)(\alpha, q) - (\alpha, q')(p, \beta), \quad \alpha \in Q_1, \beta \in Q'_1
\]

Similarly, denote by \(K(Q \times Q')\) the path category of \(Q \times Q'\) and by \(T \boxtimes T'\) is the ideal in \(K(Q \times Q')\) generated by the ideal \(T \boxtimes T'\).

**Proposition 2.3.** There exists an isomorphism of \(K\)-categories

\[
\frac{K(Q \times Q')}{T \boxtimes T'} \cong \frac{KQ}{I} \otimes_K \frac{KQ'}{I'}
\]

**Proof.** First we construct the isomorphism in the case \(I = 0, I' = 0\). Let \(I_G\) be the ideal in \(K(Q \times Q')\) generated by the elements \(\mathcal{B}\). Set \(\mathcal{B} = K(Q \times Q')\)
and $\mathcal{C} = KQ$, $\mathcal{C}' = KQ'$. Let $(p, r), (q, s) \in (Q \times Q')_0$ consider the map

$$F' : B((p, r), (q, s)) \to C(p, q) \otimes_K C'(r, s),$$

$$(p, r) | \gamma_1, \ldots, \gamma_n((q, s)) \mapsto F'_1(\gamma_1) \cdots F'_n(\gamma_n).$$

by setting $F'(\alpha, q) = \alpha \otimes \epsilon_q$ if $\gamma = (\alpha, q) \in Q_1 \times Q'_0$; $F'(p, \beta) = \epsilon_p \otimes \beta$, if $\gamma = (p, \beta) \in Q_0 \times Q'_1$ and $F'(\gamma(p, q)) = \epsilon_p \otimes \epsilon_q$ if $\gamma = (p, q) \in Q_0 \times Q'_0$. Extending by linearity the map $F'$ we get a full and dense functor $F : B \to C \otimes_K C'$. Since $I_C((p, r), (q, s)) \subseteq \text{Ker}(F)$, there exists an epimorphism $F : B((p, r), (q, s)) \to C(p, q) \otimes_K C(r, s)$ given by $F(w + I_C) = F(w)$ for all path $w$ from $(p, r)$ to $(q, s)$. The above allows us to obtain a full and dense functor $F : \frac{B}{I_C} \to C \otimes_K C'$.

Consider the canonical functor $\mathcal{C} \times \mathcal{C}' \to C \otimes_K C'$. We define a functor $H : \mathcal{C} \times \mathcal{C}' \to \frac{B}{I_C}$ by $H((p, r)) = (p, r)$ and

$$(4) \quad H : C(p, q) \times C'(r, s) \to \frac{B}{I_C}, ((p, r), (q, s));$$

$$(f, g) \mapsto ((q, s) \circ (f, r)) + I_C((p, r), (q, s)).$$

It follows from the relations $B$ that $H$ preserves compositions; moreover, $H$ is full. Thus, by the universal property of tensor product, $H$ induces a full functor $H : C \otimes_K C' \to \frac{B}{I_C}$ defined by $H((p, r)) = (p, r)$ and $H(f \otimes g) = H(f, g)$. Observe that $H$ is quasi-inverse to $F$.

Let $(f, g) \in C(p, q) \otimes C'(r, s)$. First, assume that $(f, g) \in I(p, q) \otimes C'(r, s)$. Then, by the canonical functor $H((p, r)) = (p, r)$ and $I(p, q) \subseteq I_C$, we get $I((p, q) \otimes C'(r, s)) \subseteq I((p, q) \otimes C'(r, s))$.

It is clear that $F((I \otimes C' + C \otimes I')((p, q), (r, s))) \subseteq I((I \otimes C' + C \otimes I')((p, q), (r, s))$ since $F((I \otimes I')((p, q), (r, s))) \subseteq I((I \otimes C' + C \otimes I')((p, q), (r, s)))$.

On the other hand, the quotient functors $C \to C/I$, $C' \to C'/I'$ induce a full functor $\pi : C \otimes_K C' \to C/I \otimes_K C'/I'$, with $\text{Ker}(\pi) = I \otimes C' + C \otimes I'$. Moreover, for each pair $(p, r), (q, s) \in Q_0 \times Q'_0$, the quotient functor $\pi_B : B/I_C \to \frac{B}{I_C}$ induces an isomorphism $\tilde{F} : C/I((p, q)) \otimes_K C'(r, s) \to \frac{B}{I_C}((p, r), (q, s))$ for which the following diagram commutes:
Thus, we have a faithful functor $\tilde{F} : KQ/\mathcal{I} \otimes_K KQ'/\mathcal{I}' \rightarrow K(Q \times Q')_{\mathcal{I} \cup \mathcal{J}}$, which induces an isomorphism of categories.

**Definition 2.4.** Let $Q, R$ be disjoint quivers. Consider their path $K$-categories $KQ$ and $KR$ with two-sided ideals $\mathcal{I} \subset KQ$ and $\mathcal{J} \subset KR$ generated by the sets of relations $\{\rho_v\}_{v \in \mathcal{U}}$ and $\{\sigma_v\}_{v \in \mathcal{V}}$. Let $M : KQ/\mathcal{I} \otimes_K (KR/\mathcal{J})^{\text{op}} \rightarrow \text{mod} \ K$ be a functor as described below. 

1. Define $\eta(-,-) : Q_0 \times R_0 \rightarrow \mathbb{N} \cup \{0\}$ by $\eta(i,j) := \text{dim}_K M(i,j)$.
2. For each pair $(i,j) \in Q_0 \times R_0$, let $B(i,j) = \{b^{(i,j)}_t\}_{0 \leq t \leq \eta(i,j)}$ be a $K$-basis for $M(i,j)$.
3. For each pair $(i,j) \in Q_0 \times R_0$, let $\overrightarrow{B}(j,i) = \{\overrightarrow{b}^{(j,i)}_t\}_{0 \leq t \leq \eta(i,j)}$ be a set of arrows $\overrightarrow{b}^{(j,i)}_t : j \rightarrow i$.
4. Define $B := \bigcup_{(i,j) \in Q_0 \times R_0} B(i,j)$ and $\overrightarrow{B} := \bigcup_{(j,i) \in R_0 \times Q_0} \overrightarrow{B}(j,i)$.
5. The augmented quiver of $R$ and $Q$ by $B$, which is denoted by $(R,B,Q)$, is the quiver with the following set of vertices and arrows:

   $$(R,B,Q)_0 := R_0 \cup Q_0,$$
   $$(R,B,Q)_1 := R_1 \cup Q_1 \cup \overrightarrow{B}.$$

**Remark 2.5.**

(i) Since $Q$ is a full subquiver of $(R,B,Q)$, any relation on $Q$ can be seen as a relation on $(R,B,Q)$, and $KQ$ is a full subcategory of $K(R,B,Q)$. The same for the subquiver $R \subset (R,B,Q)$.

(ii) We denote by $\mathcal{I} \cup \mathcal{J}$ the ideal of $K(R,B,Q)$ generated by the set of relations $\{\rho_v\}_{v \in \mathcal{U}} \cup \{\sigma_v\}_{v \in \mathcal{V}}$ seen as relations on $(R,B,Q)$.

(iii) Since $Q$ and $R$ are disjoint quivers, we naturally have additive fully faithful functors $G : KQ \rightarrow K(R,B,Q)_{\mathcal{I} \cup \mathcal{J}}$ and $H : KR \rightarrow K(R,B,Q)_{\mathcal{I} \cup \mathcal{J}}$. The functor $G$ is defined in objects by the inclusion map, $G(i) = i$, for all $i \in Q_0$ and in morphisms by $G : \text{Hom}_{KQ} (i,i') \rightarrow \text{Hom}_{K(R,B,Q)} (i,i')$ by $[f] \mapsto [[f]]$, for all pair $i,i' \in Q_0$.

If there is no risk of confusion, we simply write $[f]$ instead of $[[f]]$ to refer to the class of $f$ in $\text{Hom}_{K(R,B,Q)} (i,i')$. Analogously, the functor $H$ is defined.

In the rest of this section, we use the notation given in Remark 2.5. Let $(i,j) \in Q_0 \times R_0$. Define a $K$-linear map, which is an isomorphism:

$$\overline{(-)} : M(i,j) \xrightarrow{\cong} \text{Hom}_{K(R,B,Q)} (j,i)$$

induced by the map $B(i,j) \rightarrow \text{Hom}_{K(R,B,Q)} (j,i)$, $b^{(i,j)}_t \mapsto \overrightarrow{b}^{(j,i)}_t$, $0 \leq t \leq \eta(i,j)$. 
Remark 2.7. Let $X_0 = (\begin{smallmatrix} j_0 & 0 \\ M & i_0 \end{smallmatrix})$, $X_1 = (\begin{smallmatrix} j_1 & 0 \\ M & i_1 \end{smallmatrix}) \in \Lambda$, with $(i_0, j_0), (i_1, j_1) \in Q_0 \times R_0$. The map $\text{Hom}_\Lambda(X_0, X_1) \to \text{Hom}_{K[R,B,Q]}(j_0 \oplus i_0, j_1 \oplus i_1)$ given by

\[
\begin{pmatrix}
\text{Hom}_{K[R,B,Q]}(j_0, j_1) \\
\text{Hom}_{K[R,B,Q]}(i_1, j_0)
\end{pmatrix}
\begin{pmatrix}
0 \\
\text{Hom}_{K[R,B,Q]}(i_0, i_1)
\end{pmatrix}
\equiv
\begin{pmatrix}[r] 0 \\ m \end{pmatrix}
\to
\begin{pmatrix} H([r]) & 0 \\ \tilde{m} & G([q]) \end{pmatrix}
\]

is a morphism of abelian groups.

Proof. Straightforward. \qed

Consider the following set of relations in $\frac{K[R,B,Q]}{I \cup J}$:

\[
u(R,B,Q) := \{ [q] \bullet b - q \bullet \overrightarrow{b} \mid b \in B(i,j), q \in Q_1, i = s(q), \} \cup \{ [r] \bullet b - b \bullet \overrightarrow{r} \mid b \in B(i,j), r \in R_1, j = t(r) \} \}
\]

where $t(r)$ and $s(q)$ denote the target of $r$ and the source of $q$, respectively.

Remark 2.7. (i) Note that $\overrightarrow{b}$ and $\overrightarrow{r}$ are compositions of paths in $(R,B,Q)$. On the other hand, $q \bullet \overrightarrow{b}$ and $b \bullet \overrightarrow{r}$ are linear combinations of arrows taken from $\overrightarrow{B}(j,i')$ and $\overrightarrow{B}(j',i)$, respectively. For example, if $b \in B(i,j), [r] \in \frac{K[R,B,Q]}{I \cup J}(j',j), b \bullet \overrightarrow{r}$ is obtained through the following composition:

\[
\begin{pmatrix}
M(i,j) \\
M(i,j')
\end{pmatrix}
\begin{pmatrix}
\overrightarrow{b} \\
\overrightarrow{j}
\end{pmatrix}
\to
\begin{pmatrix}
\text{Hom}_{K[R,B,Q]}(j',i) \\
\overrightarrow{r}
\end{pmatrix}
\]

that is $M(1_i \otimes r^{op})(\overrightarrow{b}) = \overrightarrow{b} \bullet \overrightarrow{r}$.

(ii) Set $\mu = \langle \nu(R,B,Q) \rangle$. We have a projection

\[
\pi : \frac{K[R,B,Q]}{I \cup J} \to \frac{K[R,B,Q]}{I \cup J \cup \mu}.
\]

Moreover, $\pi \circ G$ and $\pi \circ H$ are fully faithful functors.

Theorem 2.8. With the above notation, there exists an isomorphism of categories

\[
F : \Lambda = \left( \begin{smallmatrix} K[R] & 0 \\ \frac{I}{M} & \frac{K[Q]}{J} \end{smallmatrix} \right) \to \frac{K[R,B,Q]}{I \cup J \cup \mu}.
\]

Proof. Let $X_0 = (\begin{smallmatrix} j_0 & 0 \\ M & i_0 \end{smallmatrix}), X_1 = (\begin{smallmatrix} j_1 & 0 \\ M & i_1 \end{smallmatrix}), X_2 = (\begin{smallmatrix} j_2 & 0 \\ M & i_2 \end{smallmatrix}) \in \Lambda$ with $(i_0, j_0)$, $(i_1, j_1)$, $(i_2, j_2) \in Q_0 \times R_0$. The map $F : \text{Hom}_\Lambda(X_0, X_1) \to \text{Hom}_{K[R,B,Q]}(j_0 \oplus i_0, j_1 \oplus i_1)$ is defined by composition of the following morphisms of abelian
groups:

\[ \text{Hom}_\Lambda(X_0, X_1) \rightarrow \text{Hom}_\Lambda(\mathcal{R}, \mathcal{S}, \mathcal{Q})(j_0 \oplus i_0, j_1 \oplus i_1) \]

\[ F \]

\[ \text{Hom}_\Lambda(\mathcal{R}, \mathcal{S}, \mathcal{Q})(j_0 \oplus i_0, j_1 \oplus i_1), \]

where the horizontal morphism is given in Lemma 2.0.

Let \( \begin{bmatrix} r \\ m \\ q \end{bmatrix} \in \text{Hom}_\Lambda(X_0, X_1) \), then \( F \left( \begin{bmatrix} r \\ m \\ q \end{bmatrix} \right) = \begin{bmatrix} \pi(r) \\ \pi(m) \\ \pi(q) \end{bmatrix} \). It is clear that \( F \) is additive because it is a composition of additive morphisms.

Now we see that \( F \) preserves composition. Indeed, we have that

\[ \text{Hom}_\Lambda(X_0, X_1) = \begin{pmatrix} \text{Hom}_\Lambda(\mathcal{R}, \mathcal{S}, \mathcal{Q})(j_0, j_1) & 0 \\ M(i_1, j_0) & \text{Hom}_\Lambda(\mathcal{R}, \mathcal{S}, \mathcal{Q})(i_0, i_1) \end{pmatrix}, \]

\[ \text{Hom}_\Lambda(X_1, X_2) = \begin{pmatrix} \text{Hom}_\Lambda(\mathcal{R}, \mathcal{S}, \mathcal{Q})(j_1, j_2) & 0 \\ M(i_2, j_1) & \text{Hom}_\Lambda(\mathcal{R}, \mathcal{S}, \mathcal{Q})(i_1, i_2) \end{pmatrix}. \]

Let \( \begin{bmatrix} r_1 \\ m_1 \\ q_1 \end{bmatrix} \in \text{Hom}_\Lambda(X_0, X_1) \) and \( \begin{bmatrix} r_2 \\ m_2 \\ q_2 \end{bmatrix} \in \text{Hom}_\Lambda(X_1, X_2) \). Then

\[ F \left( \begin{bmatrix} r_2 \\ m_2 \\ q_2 \end{bmatrix} \right) \circ F \left( \begin{bmatrix} r_1 \\ m_1 \\ q_1 \end{bmatrix} \right) = \begin{bmatrix} \pi \circ H([r_2 r_1]) \\ \pi((m_2 \bullet r_1 + q_2 \bullet m_1) \pi \circ G([q_2 q_1])) \end{bmatrix}. \]

On the other hand, we have

\[ F \left( \begin{bmatrix} r_2 \\ m_2 \\ q_2 \end{bmatrix} \right) \circ F \left( \begin{bmatrix} r_1 \\ m_1 \\ q_1 \end{bmatrix} \right) = \begin{bmatrix} \pi \circ H([r_2 r_1]) \\ \pi((m_2 \bullet r_1 + q_2 \bullet m_1) \pi \circ G([q_2 q_1])) \end{bmatrix}. \]

After writing \( m_1 \) and \( m_2 \) in terms of the elements of \( B(i_1, j_0) \) and \( B(i_2, j_1) \), respectively, we see that \( m_2 [r_1] - m_2 \bullet r_1 \) and \( [q_2] m_1 - q_2 \bullet m_1 \) lie in \( \mu \); therefore, \( \pi(m_2 [r_1]) = \pi(m_2 \bullet r_1) \) and \( \pi([q_2] m_1) = \pi(q_2 \bullet m_1) \).

The functor \( F \) is fully faithful. Let \( \begin{bmatrix} r \\ m \\ q \end{bmatrix} \in \text{Hom}_\Lambda(X_0, X_1) \) such that

\[ F \left( \begin{bmatrix} r \\ m \\ q \end{bmatrix} \right) = \begin{bmatrix} \pi(r) \\ \pi(m) \\ \pi(q) \end{bmatrix} = 0. \]

It follows that \( \pi \circ H([r]) = 0 \) and \( \pi \circ G([q]) = 0 \) imply \([r] = 0 \) and \([q] = 0 \) since \( \pi \circ H \) and \( \pi \circ G \) are fully faithful. After writing \( m \) in terms of the basis vectors in \( B(i_1, j_0) \), we have \( m = \sum_{t=1}^{n(i_1, j_0)} \lambda_t b_t^{i_1, j_0} \), and \( \overline{m} = \sum_{t=1}^{n(i_1, j_0)} \lambda_t b_t^{i_1, j_0} \). Thus \( \pi(\overline{m}) = 0 \) implies that \( \overline{m} \in \mu \), and it can be written as

\[ \overline{m} = \sum_u \alpha_u \left( [q_u] b_u - q_u \bullet b_u \right) + \sum_v \beta_v \left( b_v [r_v] - b_v \bullet r_v \right), \]
with $\alpha_u, \beta_v \in K$. Since the left side of the above equation is a linear combination of arrows and $r := \sum_u \alpha_u \left[ q_u | b_u \right] + \sum_v \beta_v \left( b_v | r_v \right)$ is a linear combination of path of length at least two, which are linearly independent in $\text{Hom}_{K(Q, R, \Sigma)}(j_0 \oplus i_0, j_1 \oplus i_1)$ because they lie not in $I \cup J$, we must have $r = 0$ and $\alpha_u = \beta_v = 0$, which implies $\overline{m} = 0$, and finally $\lambda_t = 0$, for $1 \leq t \leq \eta(i_1, j_0)$, since $(-) : M(i_1, j_0) \to \text{Hom}_{K(Q, R, \Sigma)}(j_0, i_1)$ is an isomorphism.

On the other hand, since $\pi \circ F$ and $\pi \circ G$ are full functors, it is sufficient to show that for every path $\gamma : j_0 \to i_1$ in $\text{Hom}_{K(Q, R, \Sigma)}(j_0, i_1)$ there exists an element $m \in M(i_1, j_0)$ such that $F((0 m 0)) = \gamma$. But $\gamma$ can be written as

$$\gamma = \overline{b}((j_1, i_1)|r) \quad \text{or} \quad \gamma = [q] \overline{b}(j_0, i_0),$$

as is shown in the picture

If we set $m = b(i_1, j_1) \circ r$ or $m = q \circ b(j_0, i_0)$, we get what we desire.

The last assertion follows from the fact that $F$ is an isomorphism on objects. \hfill \square

**Example 2.9.** Consider the quivers

$$Q : 1' \xleftarrow{\gamma_1} 2' \xrightarrow{\gamma_2} 3' \xleftarrow{\gamma_3} 4' \xrightarrow{\gamma_4} 5' \ldots$$

and

$$R : 1 \xrightarrow{\alpha_1} 2 \xleftarrow{\beta_1} 3 \xrightarrow{\alpha_2} 4 \xleftarrow{\beta_2} 5 \xrightarrow{\alpha_3} \ldots$$

with the set of relations

$$J = \{ \beta_1 \alpha_1 \text{ and } \alpha_{t+1} \alpha_t, \beta_t \beta_{t+1}, \alpha_t \beta_t - \beta_{t+1} \alpha_{t+1}, t \geq 1 \}.$$ 

Let $KQ$ and $KR/J$ be their respective path categories. Thus, by any functor $M : KQ \otimes KR^{op} \to \text{mod } K$ can be identified with a functor $M : K(Q \times R^{op})/0 \cap J \to \text{mod } K$, where $0 \cap J$ is generated by the sets of relations $Q_0 \times J$ and $\{(\gamma, t(\alpha))(s(\gamma), \alpha) - (t(\gamma), \alpha)(\gamma, s(\alpha)), (\gamma, t(\beta))(s(\gamma), \beta) - (t(\gamma), \beta)(\gamma, s(\beta)) : \gamma \in Q_1, \alpha, \beta \in R_1 \}$. In this example, we consider a representation on the right below that can be seen as a functor $M : KQ \otimes KR^{op} \to \text{mod } K$. 


and let $M(1, 1) = K^2 = \langle \varphi, \psi \rangle$, $M(1, 2) = K = \langle \theta \rangle$, where $\varphi = (1 \ 0)$, $\psi = (0 \ 1)$ and $\theta = 1$. Thus, $\overline{B} = \{ \overline{\varphi}, \overline{\psi} : 1 \to 1', \overline{\theta} : 2 \to 1' \}$ and

$$(R, B, Q) : \begin{array}{c}
1' \xleftarrow{\gamma_1} 2' \xrightarrow{\gamma_2} 3' \xrightarrow{\gamma_3} 4' \xrightarrow{\gamma_4} 5' \ldots \\
1 \xrightarrow{} 2 \xrightarrow{} 3 \xrightarrow{} 4 \xrightarrow{} 5 \ldots
\end{array}$$

On the other hand, $M(1 \otimes \alpha_{1}^{\mathrm{op}}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $M(1 \otimes \beta_{1}^{\mathrm{op}}) = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $M(1 \otimes \alpha_{1}^{\mathrm{op}})(\theta) = \psi$, $M(1 \otimes \beta_{1}^{\mathrm{op}})(\psi) = 0$, $M(1 \otimes \beta_{1}^{\mathrm{op}})(\varphi) = \theta$. Thus, $\mu = \{ \overline{\psi} \beta, \overline{\varphi} \beta - \overline{\theta}, \overline{\theta} \alpha - \overline{\psi} \}$. In this way

$$\Lambda = \begin{pmatrix} KQ & 0 \\ M & K\overline{B} \end{pmatrix} \cong \frac{K(R, B, Q)}{J \cup \mu}. $$

Although $J \cup \mu$ is not an admissible ideal in $K(R, B, Q)$, we can eliminate some arrows in $(R, B, Q)$ and obtain a quiver $Q'$ so that $\Lambda$ is isomorphic to a path category $KQ'$ modulo an admissible ideal; see [32].

Consider the quiver

$$Q' : \begin{array}{c}
\cdots \xrightarrow{} 3' \xleftarrow{\gamma_3} 2' \xrightarrow{\gamma_2} 1' \xleftrightarrow{\theta} 1 \xrightarrow{\alpha_1} 2 \xleftarrow{\beta_1} 3 \xrightarrow{\alpha_2} 4 \xleftarrow{\beta_2} 5 \ldots
\end{array}$$

and let $\mu' = \{ \theta \alpha_1, \theta - \varphi \beta_1 \}$ be a set of paths in $Q'$.

Therefore, the correspondence $[\psi] = [\theta \alpha_1] \leftrightarrow [\theta \alpha_1]$, $[\beta] \leftrightarrow [\beta]$, $[\alpha] \leftrightarrow [\alpha]$, $[\gamma] \leftrightarrow [\gamma]$ induces an equivalence $\frac{K(R, B, Q)}{J \cup \mu} \cong \frac{KQ'}{\mu' \cup J}$. Again if we consider the quiver,

$$Q'' : \begin{array}{c}
\cdots \xrightarrow{} 3' \xleftarrow{\gamma_3} 2' \xrightarrow{\gamma_2} 1' \xleftrightarrow{\psi} 1 \xrightarrow{\alpha_1} 2 \xleftarrow{\beta_1} 3 \xrightarrow{\alpha_2} 4 \leftarrow \beta_2 \ldots
\end{array}$$
then the correspondence $[θ] = [φ β_1] ⇔ [φ β_1], [β] ⇔ [β], [α] ⇔ [α], [γ] ⇔ [γ]$ induces an equivalence $KQ'/μ' ∪ J \cong KQ''/J$. In this way $Λ \cong KQ''/J$.

3. Triangular Matrix Categories over quasi-hereditary categories

Let $\mathcal{U}$ and $\mathcal{T}$ be Hom-finite Krull-Schmidt quasi-hereditary $K$-categories with respect to filtrations $\{U_j\}_{0 \leq j \leq n}$ and $\{T_j\}_{j \geq 0}$, respectively, consisting of full additively closed subcategories. Assume that the $K$-module $M$ induces an equivalence over $M$; see [32, Lemma 3.18]. Thus, $Λ = (T_M 0)$ is a Hom-finite Krull-Schmidt $K$-category; see [26, Proposition 6.9].

Consider the filtration of $Λ$ into subcategories $\{Λ_j\}_{j \geq 0}$ given by

$$\begin{align*}
Λ_0 &= (0_M 0) ; \\
Λ_j &= (0_M U_j) := \{ (0_M 0) : U \in \mathcal{U}_j \}, \text{ if } 1 \leq j \leq n; \\
Λ_{n+j} &= (T_M 0) = \{ (T_M 0) : T \in \mathcal{T}_j, U \in \mathcal{U} \}, \text{ if } j \geq 1.
\end{align*}$$

(5)

It is clear that $Λ_j \subseteq Λ$ is an additive full subcategory for all $j \geq 0$. Moreover, if we define

$$\begin{align*}
(0_M \text{Ind } U_j) &= \{ (0_M 0) : U \in \text{Ind } \mathcal{U}_j \}, \text{ if } 1 \leq j \leq n, \text{ and} \\
(\text{Ind } T_j 0) &= \{ (T_M 0) : T \in \mathcal{T}_j \}, \text{ if } j \geq 1.
\end{align*}$$

It follows that

$$\begin{align*}
\text{Ind } Λ_j &= (0_M \text{Ind } U_j), \text{ if } 1 \leq j \leq n, \text{ and} \\
\text{Ind } Λ_{n+j} &= (0_M \text{Ind } U = \text{Ind } U_0) \cup (\text{Ind } T_j 0), \text{ if } j \geq 1.
\end{align*}$$

In this way, we have that $Λ_j$, for $j \geq 0$, is additively closed. Moreover,

$$\text{Ind } Λ_j - \text{Ind } Λ_{j-1} = \begin{cases} 
\{ (0_M 0) : U \in \text{Ind } \mathcal{U}_j - \text{Ind } \mathcal{U}_{j-1} \}, & \text{if } 1 \leq j \leq n, \\
\{ (T_M 0) : T \in \text{Ind } \mathcal{T}_{j-n} - \text{Ind } \mathcal{T}_{j-n-1} \}, & \text{if } j > n.
\end{cases}$$

One of the main results of this section is the following; see [43, Theorem 3.1].

Theorem 3.1. Let $\mathcal{U}$ and $\mathcal{T}$ be Hom-finite Krull-Schmidt quasi-hereditary categories with respect to filtrations $\{U_j\}_{0 \leq j \leq n}$, $\{T_j\}_{j \geq 0}$ of $\mathcal{U}$ and $\mathcal{T}$, respectively, consisting of additively closed subcategories. Assume that $M_T \in \mathcal{F}(\mathcal{U}Δ)$ for all $T \in \mathcal{T}$. Then $Λ = (T_M 0)$ is quasi-hereditary with respect to the filtration $\{Λ_j\}_{j \geq 0}$ given in (5).
The proof of Theorem 3.1 will be a consequence of a series of results that are presented below.

Lemma 3.2. Let \( \mathcal{U} \) be a quasi-hereditary category with respect to a filtration \( \{ \mathcal{U}_j \}_{j \geq 0} \). Let \( M \) be a \( \mathcal{U} \)-module, and set \( M^{[\beta]} := \text{Tr}_{\mathcal{U}(U,-)} \mathcal{U}_j M \). In addition, assume that \( M \in \mathcal{F}(\mathcal{U}) \). Then for all \( U' \in \mathcal{U} \), \( M^{[\beta]}(U') = \{ m : m = M(s)(m') \text{ for some } s \in \mathcal{U}(U'', U'), \text{with } U'' \in \mathcal{U}_j, \text{ and } m' \in M(U'') \} \).

Proof. \( \subseteq \). By Yoneda’s isomorphism \( Y_{U'} : \text{Nat}((U', -), M^{[\beta]}) \cong M^{[\beta]}(U'), \eta \mapsto \eta_{U'}(1_{U'}) \). Let \( m \in M^{[\beta]}(U') \), then there exists \( \eta^m : (U', -) \to M^{[\beta]} \) such that \( \eta^m_{U'}(1_{U'}) = m \). On the other hand, by Lemma 1.5, there exists \( p : (U'', -) \to M^{[\beta]} \to 0 \), with \( U'' \in \mathcal{U}_j \) and therefore there exists a morphism \( s : U'' \to U' \) for which the following diagram is commutative:

\[
\begin{array}{ccc}
(U', -) & \xrightarrow{\eta^m} & (U'', -) \\
\downarrow{\exists (s, -)} & & \downarrow{p} \\
(U', -) & \xrightarrow{\eta^m} & M^{[\beta]} \\
\end{array}
\]

Again by Yoneda’s lemma we have the following commutative diagram:

\[
\begin{array}{ccc}
((U'', -), M^{[\beta]}) & \xrightarrow{(s, -)^*} & ((U', -), M^{[\beta]}) \\
\downarrow{Y_{U''}} & & \downarrow{Y_{U'}} \\
M^{[\beta]}(U'') & \xrightarrow{M^{[\beta]}(s)} & M^{[\beta]}(U') \\
\end{array}
\]

Let \( m' := Y_{U''}(p) \). Since \( M^{[\beta]} \) is a subfunctor of \( M_T \), we have \( m = M^{[\beta]}(s)(m') = M(s)(m') \), and we get what we desire.

\( \supseteq \). Let \( m \in M(U') \) and assume there exists \( s : U'' \to U' \), with \( U'' \in \mathcal{U}_j \), such that \( m = M(s)(m') \) for some \( m' \in M(U'') \). There then exist morphisms \( \eta^m : (U', -) \to M \) and \( p_{U''}^m : (U'', -) \to M \) such that \( \eta^m_{U'}(1_{U'}) = m \) and \( p_{U''}^m(1_{U''}) = m' \). Thus, by using the diagram (2), we have \( p_{U''}^m \circ (g, -) = \eta^m \). Note that \( \text{Im} \ p_{U''}^m \) is a subfunctor of \( M \) and is generated by \( (U'', -) \). Since \( U'' \in \mathcal{U}_j \), \( \text{Im} \ p_{U''}^m \) is contained in the largest submodule of \( M \) generated by \( \{ (U, -) : U \in \mathcal{U}_j \} \), namely \( M^{[\beta]} \); thus \( \text{Im} \ p_{U''}^m \subseteq M^{[\beta]} \). It follows that

\[
m = \eta^m_{U'}(1_{U'}) = p_{U''}^m \circ (s, -)_{U'}(1_{U'}) = p_{U''}^m(s) \in \text{Im} \ p_{U''}^m(U') \subseteq M^{[\beta]}(U') : \]

that is, \( m \in M^{[\beta]}(U') \).

In the remainder of this section, we will assume that the categories \( \mathcal{U} \) and \( \mathcal{T} \) are Hom-finite Krull-Schmidt quasi-hereditary categories with respect to
filtrations of additively closed subcategories \( \{ \mathcal{U}_j \}_{0 \leq j \leq n} \) and \( \{ \mathcal{T}_j \}_{j \geq 0} \) of \( \mathcal{U} \) and \( \mathcal{T} \), respectively, and \( M_T \in \mathcal{F}(\mathcal{U} \Delta) \) for all \( T \in \mathcal{T} \).

**Proposition 3.3.** Let \( E = \left( \frac{T}{M} \; \frac{0}{U} \right) \) and \( E' = \left( \frac{T'}{M} \; \frac{0}{U'} \right) \) in \( \Lambda \). Then,

\[
I_{\Lambda_j}(E, E') = \begin{cases} 
0 & \text{if } 0 \leq j \leq n, \\
M^j_T(U') & \text{if } j > n,
\end{cases}
\]

\[
I_{\Lambda_j}(E, E') = \begin{cases} 
\text{Hom}_{\mathcal{U}}(U, U') & \text{if } j > n.
\end{cases}
\]

**Proof.** Let \( (\frac{f}{m} \; \frac{0}{h}) \in \text{Hom}_\Lambda(E, E') \). Therefore, \( f \in \text{Hom}_\mathcal{T}(T, T') \), \( m \in M(T, T') \) and \( h \in \text{Hom}_\mathcal{U}(U, U') \).

\((0 \leq j \leq n)\) The morphism \( (\frac{f}{m} \; \frac{0}{h}) \) lies in \( I_{\Lambda_j}(E, E') \) if and only there is a commutative diagram

\[
\begin{array}{ccc}
T & \xrightarrow{\frac{f}{m}} & T' \\
\downarrow{\frac{r}{h}} & & \downarrow{\frac{s}{h}} \\
M & \xrightarrow{\frac{0}{r}} & M
\end{array}
\]

with \( (\frac{0}{M} \; \frac{0}{U'}) \in \Lambda_j \), \( 1 \leq j \leq n \). Thus, \( U'' \in \mathcal{U}_j \) and \( (\frac{0}{M} \; \frac{0}{U'}) = (\frac{f}{m} \; \frac{0}{h}) \).

Therefore, \( f = 0 \), \( m = sm' \) and \( h = sr \). It is clear that \( h \in \mathcal{U}_j(U, U') \) because \( U'' \in \mathcal{U}_j \). In this way we conclude that \( (\frac{f}{m} \; \frac{0}{h}) \in I_{\Lambda_j}(E, E') \) if and only if \( h \in \mathcal{U}_j(U, U') \) and \( m = sm' = M_T(s)(m') \) where \( \mathcal{U} \xrightarrow{\sim} U'' \xrightarrow{\sim} U' \) and \( U'' \in \mathcal{U}_j \), in other words, \( h \in \mathcal{U}_j(U, U') \) and \( m \in M_T(U) \), by Lemma 3.2

Thus, \( I_{\Lambda_j}(E, E') = \left( \frac{0}{M} \; \frac{0}{U''} \right) \).

\((j > n)\) \( (\frac{f}{m} \; \frac{0}{h}) \in I_{\Lambda_j}(E, E') \) if and only if there is a commutative diagram

\[
\begin{array}{ccc}
T & \xrightarrow{\frac{r}{h}} & T' \\
\downarrow{\frac{s}{h}} & & \downarrow{\frac{0}{h}} \\
M & \xrightarrow{\frac{0}{r}} & M
\end{array}
\]

with \( (\frac{T''}{M} \; \frac{0}{U''}) \in \Lambda_j \), for \( j = n + (j - n) > n \), that is, \( T'' \in \mathcal{T}_{j-n} \) and \( U'' \in \mathcal{U} \).

Since \( (\frac{s}{m_2} \; \frac{0}{h_2}) (\frac{r}{m_1} \; \frac{h_1}{h_1}) = (\frac{f}{m} \; \frac{0}{h}) \), we get that \( f = sr \), \( m = m_2r + h_2m_1 \) and \( h = h_2h_1 \); moreover, \( m \in M(U', T) \), \( h \in \mathcal{U}(U', U') \), and \( f \in I_{\mathcal{T}_{j-n}}(T, T') \) since \( r \in \mathcal{T}(T, T') \) and \( T'' \in \mathcal{T}_{j-n} \). Therefore, \( (\frac{f}{m} \; \frac{0}{h}) \in I_{\Lambda_j}(E, E') \) if and only if \( m \in M(U', T) \), \( h \in \mathcal{U}(U', U') \), and \( f \in I_{\mathcal{T}_{j-n}}(T, T') \).

\( \square \)
Proposition 3.4. For each pair $E, E' \in \text{Ind } \Lambda_j - \text{Ind } \Lambda_{j-1}$, we have
\[ \text{rad}_\Lambda(E, E') = I_{\Lambda_{j-1}}(E, E'). \]

Proof. The proof is divided in two cases.

(1) $j \leq n$. Let $E = \begin{pmatrix} 0 & 0 \\ M & U \end{pmatrix}$ and $E' = \begin{pmatrix} 0 & 0 \\ M & U' \end{pmatrix}$ for which $U, U' \in \text{Ind } \mathcal{U}_j - \text{Ind } \mathcal{U}_{j-1}$. Therefore, by [26, Proposition 3.7]
\[ \text{rad}_\Lambda \left( \begin{pmatrix} 0 & 0 \\ M & U \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ M & U' \end{pmatrix} \right) = \left( \text{rad}_\mathcal{T}(0,0) \begin{pmatrix} 0 \\ M(U',0) \end{pmatrix}, \text{rad}_\mathcal{U}(U,U') \right) = \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right). \]

On the other hand, by Theorem [1.3] we have $\text{rad}_\mathcal{T}(U, U') = I_{\mathcal{U}_{j-1}}(U, U')$. Therefore, by Proposition 3.3 we conclude that
\[ \text{rad}_\Lambda \left( \begin{pmatrix} 0 & 0 \\ M & U \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ M & U' \end{pmatrix} \right) = I_{\Lambda_{j-1}} \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right). \]

(2) $j > n$. Let $E = \begin{pmatrix} T & 0 \\ M & 0 \end{pmatrix}$ and $E' = \begin{pmatrix} T' & 0 \\ M & 0 \end{pmatrix}$ such that $T, T' \in \text{Ind } \mathcal{U}_j - \text{Ind } \mathcal{U}_{j-n-1}$. By [26, Proposition 3.7] we have
\[ \text{rad}_\Lambda \left( \begin{pmatrix} T & 0 \\ M & 0 \end{pmatrix}, \begin{pmatrix} T' & 0 \\ M & 0 \end{pmatrix} \right) = \left( \text{rad}_\mathcal{T}(T,T') \begin{pmatrix} 0 \\ M(0,T) \end{pmatrix}, \text{rad}_\mathcal{U}(0,0) \right) = \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right). \]

Again by Theorem [1.3] we know that $\text{rad}_\mathcal{T}(T, T') = I_{\mathcal{T}_{j-n-1}}(T, T')$. It follows from Proposition 3.3 that
\[ \text{rad}_\Lambda \left( \begin{pmatrix} T & 0 \\ M & 0 \end{pmatrix}, \begin{pmatrix} T' & 0 \\ M & 0 \end{pmatrix} \right) = I_{\Lambda_{j-1}} \left( \begin{pmatrix} T & 0 \\ M & 0 \end{pmatrix}, \begin{pmatrix} T' & 0 \\ M & 0 \end{pmatrix} \right). \]

\[ \square \]

Remark 3.5. Recall, [26, Theorem 3.14], that there exists an equivalence of categories $\mathcal{f} : (\text{Mod } \mathcal{T}, \mathcal{G}(\text{Mod } \mathcal{U})) \to \text{Mod } \Lambda$, and, given a $\Lambda$-module $C$, there exists a pair of functors $C_1 : \mathcal{T} \to \text{Ab}$, $C_2 : \mathcal{U} \to \text{Ab}$ and $C_1 \xrightarrow{f} \mathcal{G}(C_2)$ such that $f((C_1, f, C_2)) \cong C$, where $C_1(T') := C \left( \begin{pmatrix} T' & 0 \\ M & 0 \end{pmatrix} \right)$ and $C_2(U') := C \left( \begin{pmatrix} 0 & 0 \\ M & U' \end{pmatrix} \right)$. Moreover, if there exists exact sequences
\[(T_0, -) \to (T_1, -) \to C_1 \to 0, \quad T_0, T_1 \in \mathcal{T} \quad \text{and} \quad (U_0, -) \to (U_1, -) \to C_2 \to 0, \quad U_0, U_1 \in \mathcal{U},\]
then there exists an exact sequence
\[(P_j, -) \to (T_0, 0) \to C_1 \to 0, \quad P_j := \begin{pmatrix} T_j & 0 \\ M & U_j \end{pmatrix}, \quad j \to (I_{\Lambda_j}, f, I_{\Lambda_j}^2) \to 0,
\]
where $P_j := \begin{pmatrix} T_j & 0 \\ M & U_j \end{pmatrix}$ is a projective $\Lambda$-module for $j = 0, 1$; see proof of [26, Proposition 6.3].

As a direct result of the above proposition, we have:
Lemma 3.6. Let $X = (\frac{T}{M} \, 0) \in \Lambda$. Let us identify $\text{I}_{\text{I}}(X, -)$ with 
\[ \left( I^{(1)}_{\text{I}}(X, -), f, I^{(2)}_{\text{I}}(X, -) \right) \text{ in } (\text{Mod } T, G(\text{Mod } U)), \]
then
(i) $I^{(1)}_{\text{I}}(X, -) = 0$ and $I^{(2)}_{\text{I}}(X, -) \cong M^{[j]} T(I_U U, -)$, if $0 \leq j \leq n$;
(ii) $I^{(1)}_{\text{I}}(X, -) \cong I_{\text{I}}(T, -)$ and $I^{(2)}_{\text{I}}(X, -) \cong M T U(U, -)$, if $j > n$.

Proof. Let $T' \in T$ and $U' \in U$. The proof is divided into two cases.

(1 $\leq j \leq n$). By Proposition 3.3 we have
\[ I^{(1)}_{\text{I}}(T') = I_{\text{I}}(X, (\frac{T'}{M} \, 0)) \approx (0 \, 0) \]
and
\[ I^{(2)}_{\text{I}}(U') = I_{\text{I}}(X, (\frac{0}{M} \, U')) \approx M^{[j]}(U') \prod I_U(U, U'). \]

$(j > n)$). By Proposition 3.3 we have
\[ I^{(1)}_{\text{I}}(T') = I_{\text{I}}(X, (\frac{T'}{M} \, 0)) \approx (I_{\text{I}}(T, T') \, 0) \approx I_{\text{I}}(T, T') \]
and
\[ I^{(2)}_{\text{I}}(U') = I_{\text{I}}(X, (\frac{0}{M} \, U')) \approx (I_{\text{I}}(U, U') \, 0) \approx M_{\text{I}} U(U, U'). \]

□

Proposition 3.7. Let $X = (\frac{T}{M} \, 0) \in \Lambda$, and assume $M_T \in F(\text{U} \Delta)$ for all $T \in T$. Then, for all $j \geq 1$ the following exact sequence exists:
\[ (E_{j-1}, -) \to (E_j, -) \to I_{\text{I}}(X, -) \to 0, \]
with $E_j \in \Lambda$ and $E_{j-1} \in \Lambda_{j-1}$.

Proof. The proof is divided into two cases.

(1 $\leq j \leq n$). By Lemma 1.5 and Theorem 1.3 there exist exact sequences
\[ (U'_{j-1}, -) \to (U'_j, -) \to I_U(U, -) \to 0, \]
\[ (U''_{j-1}, -) \to (U''_j, -) \to M^{[j]}(U, -) \to 0, \]
with $U'_j, U''_j \in U_U, U'_{j-1}, U''_{j-1} \in U_U$. Thus, we have an exact sequence
\[ (U_{j-1}, -) \to (U_j, -) \to M^{[j]}(U, -) \approx I^{(2)}_{\text{I}}(X, -) \to 0, \]
with $U_{j-1} = U'_{j-1} \prod U''_{j-1} \in U_U$ and $U_j = U'_j \prod U''_j \in U_U$. It follows that there exists an exact sequence of $\Lambda$-modules
\[ \left( \left( \frac{0}{M} \, U_{j-1}, \right), - \right) \to \left( \left( \frac{0}{M} \, U_j, \right), - \right) \to f \left( I^{(i)}_{\text{I}}(X, -), \right) \to 0, \]
with $f((I^{(i)}_{\text{I}}, f, I^{(2)}_{\text{I}})) \cong I_{\text{I}}(X, -), \left( \frac{0}{M} \, U_{j-1}, \right) \in \Lambda_{j-1}$ and $\left( \frac{0}{M} \, U_j, \right) \in \Lambda_j$. 
(j > n). Since $M_T \in \mathcal{F}(\Delta)$, $M_T$ is a finitely presented $\mathcal{U}$-module, then $M_T \bigoplus \mathcal{U}(U,-) \cong I^{(2)}_{\Lambda_j}(X,-)$ is finitely presented: an exact sequence of $\mathcal{U}$-modules $(U''',-)$ → $(U'',-)$ → $I^{(2)}_{\Lambda_j}(X,-)$ → 0 exists. On the other hand, an exact sequence of $\mathcal{T}$-modules exists:

$$ (T_{j-n} - ), (T_j - ), I_{T_{j-n}}(T, - ) \rightarrow 0, $$

with $T_{j-n} \in T_{j-n}$ and $T_{j-n-1} \in T_{j-n-1}$. Thus, we get an exact sequence

$$ \left( \left( \begin{array}{c} T_{j-n-1} \\ M \\ U'' \end{array} \right), - \right) \rightarrow \left( \left( \begin{array}{c} T_{j-n} \\ M \\ U' \end{array} \right), - \right) \rightarrow \mathcal{f} \left( \left( I^{(1)}_{\Lambda_j}, f, I^{(2)}_{\Lambda_j} \right) \right) \rightarrow 0, $$

$\mathcal{f}(I^{(1)}_{\Lambda_j}, f, I^{(2)}_{\Lambda_j}) \cong I_{\Lambda_j}(X, - )$, \left( \begin{array}{c} T_{j-n-1} \\ M \\ U'' \end{array} \right) \in \Lambda_{j-1}$ and \left( \begin{array}{c} T_{j-n} \\ M \\ U' \end{array} \right) \in \Lambda_j. \Box$

**Proof of Theorem 3.9.** It follows from Propositions 3.8 and 3.7. \Box

3.1. **The standard modules in** $\text{Mod } \Lambda$ **and** $\mathcal{F}(\Lambda \Delta)$. Assume that $\mathcal{T}$ and $\mathcal{U}$ are Hom-finite Krull-Schmidt and quasi-hereditary $K$-categories with respect to filtrations $\{ U_j \}_{0 \leq j \leq n}$ and $\{ T_j \}_{j \geq 0}$, respectively. If in addition $M : \mathcal{U} \otimes K \mathcal{T}^{op} \rightarrow \text{Mod } K$ is a functor such that $M_T = M(-, T) : \text{Mod } \mathcal{U} \rightarrow \text{mod } K$ is finitely presented $\mathcal{U}$-module, then by Theorem 3.1 the triangular matrix category $\Lambda = \left( \begin{array}{c} M \\ 0 \\ U \end{array} \right)$ is a quasi-hereditary $K$-category with respect to some filtration $\{ \Lambda_j \}_{j \geq 0}$. In this part, we study the relation between the full standard subcategories $\mathcal{U} \Delta$, $\mathcal{T} \Delta$, and $\Lambda \Delta$ of Mod $\mathcal{U}$, Mod $\mathcal{T}$ and Mod $\Lambda$, respectively. More concretely, we will show in Theorem 3.9 by using the notation of Remark 3.5 that

$$ \mathcal{F}_f(\Lambda \Delta) = \{ (F^{(1)}, f, F^{(2)}) : (F^{(1)} \in \mathcal{F}_f(\mathcal{T} \Delta) \text{ and } F^{(2)} \in \mathcal{F}_f(\mathcal{U} \Delta)) \}; $$

see [43, Theorem 3.1].

Regardless, we need the following result.

**Proposition 3.8.** The functor $\mathcal{f} : (\text{Mod } \mathcal{T}, \mathcal{G}(\text{Mod } \mathcal{U})) \rightarrow \text{Mod } \Lambda$ induce equivalences of full subcategories:

$$ (0, 0, \Delta(j)) \longleftrightarrow_{\Lambda} \Delta(j), \quad \text{if } 1 \leq j \leq n, \quad \text{and}, $$

$$ (\tau \Delta(j-n), 0, 0) \longleftrightarrow_{\Lambda} \Delta(j), \quad \text{if } j > n. $$

**Proof.** First, we identify $\Lambda \Delta_E = \frac{\Lambda(E_{\Lambda_j})}{\text{Ind } \Lambda_j - \text{Ind } \Lambda_j-1}$, with a triple $\left( \Lambda_E^{(1)}, g, \Lambda_E^{(2)} \right)$ with $\Lambda_E^{(1)} : \mathcal{T} \rightarrow \text{Ab}$ and $\Lambda_E^{(2)} : \mathcal{U} \rightarrow \text{Ab}$.

$(1 \leq j \leq n)$. Let $T' \in \mathcal{T}$ and $E = \left( \begin{array}{c} 0 \\ M \\ U \end{array} \right)$ with $U \in \text{Ind } \mathcal{U}_j - \text{Ind } \mathcal{U}_j-1$. Then

$$ \Lambda \Delta_E^{(1)}(T') = \Lambda \left( \left( \begin{array}{c} 0 \\ M \\ U \end{array} \right), \left( \begin{array}{c} T' \\ M \\ 0 \end{array} \right) \right) \cong 0. $$
On the other hand, if $U' \in \mathcal{U}$, we get
\[
\Lambda \Delta_E^{(2)}(U') = \frac{\Lambda(\binom{0}{M} U, \binom{0}{M} U)}{I_{\Lambda_j-1}(\binom{0}{M} U, \binom{0}{M} U)} \cong \frac{\mathcal{U}(U, U')}{I_{\Lambda_j-1}(U, U')}.
\]
In this way, $\Lambda \Delta_E^{(1)} \cong 0$ and $\Lambda \Delta_E^{(2)} \cong \frac{\mathcal{U}(U, -)}{I_{\Lambda_j-1}(U, -)} = \mathcal{U} \Delta_U$, with $U \in \text{Ind } \mathcal{U}_j - \text{Ind } \mathcal{U}_j - 1$.

$(j > n)$. Let $T' \in \mathcal{T}$ and $E = \binom{T}{0}$ with $T \in \text{Ind } \mathcal{T}_{j-n} - \text{Ind } \mathcal{T}_{j-n-1}$. Thus
\[
\Lambda \Delta_E^{(1)}(T') = \frac{\Lambda(\binom{T}{0}, \binom{T'}{0})}{I_{\Lambda_j-1}(\binom{T}{0}, \binom{T'}{0})} \cong \frac{T(T, T')}{I_{\Lambda_j-1}(T, T')}.
\]
If $U' \in \mathcal{U}$, we obtain
\[
\Lambda \Delta_E^{(2)}(U') = \frac{\Lambda(\binom{T}{0}, \binom{0}{M} U')}{I_{\Lambda_j-1}(\binom{T}{0}, \binom{0}{M} U')} \cong 0.
\]
Therefore $\Lambda \Delta_E^{(1)} \cong \frac{T(T, -)}{I_{\Lambda_j-1}(T, -)} \cong \mathcal{T} \Delta_T$, with $T \in \text{Ind } \mathcal{T}_{n-j} - \text{Ind } \mathcal{T}_{n-j-1}$ and $\Lambda \Delta_E^{(2)} \cong 0$.

\begin{flushright}
\rightline{\blacksquare}
\end{flushright}

**Theorem 3.9.** Let $F = (F^{(1)}, f, F^{(2)}) \in \text{Mod } \Lambda$, and consider its trace filtration $\mathcal{F}^{[j]} = \{F^{[j]}(1), f^{[j]}, (F^{[j]})^{(2)}\}_{j \geq 0}$ with respect to $\{\Lambda_j\}$. Then:

(i) $(F^{[j]})^{(1)} \cong 0$, if $0 \leq j \leq n$, and $(F^{[j]})^{(2)} \cong F^{(2)}$, if $j > 1$.

(ii) If $F = (F^{(1)}, f, F^{(2)}) \in \mathcal{F}(\Lambda \Delta)$ then $F^{(1)} \in \mathcal{F}(\tau \Delta)$ and $F^{(2)} \in \mathcal{F}(u \Delta)$.

(iii) $\mathcal{F}(\Lambda \Delta) = \{F^{(1)}, f, F^{(2)} : F^{(1)} \in \mathcal{F}(\tau \Delta) \text{ and } F^{(2)} \in \mathcal{F}(u \Delta)\}$.

**Proof:** (i) Assume we have an exact sequence of $\Lambda$-modules (*) $(X', -) \to (X, -) \to F \to 0$ with $X' = \binom{T}{0} U$ and $X = \binom{T}{0}$). Thus, we get an exact sequence (**) $I_{\Lambda_j}(X', -) \to I_{\Lambda_j}(X, -) \to F^{[j]} \to 0$. First, we identify (*) with the exact sequence
\[
(T(T', -), g', M_{T'} \text{ II } \mathcal{U}(U', -)) \xrightarrow{\text{(*)}} (T(T, -), g, M_{T} \text{ II } \mathcal{U}(U, -)) \xrightarrow{\text{(**)}} ((F^{(1)}, f, F^{(2)})) \to 0.
\]

In particular we have the exact sequence
\[
(M_{T'} \text{ II } \mathcal{U}(U', -)) \to M_{T} \text{ II } \mathcal{U}(U, -) \to F^{(2)} \to 0.
\]
Secondly, we identify the exact sequence (**) with the exact sequence

\[
(I_{\Lambda_0}^{(1)}(X', -), h, I_{\Lambda_0}^{(2)}(X', -)) \rightarrow (I_{\Lambda_0}^{(1)}(X, -), h, I_{\Lambda_0}^{(2)}(X, -)) \rightarrow (F^{(1)}M^{(1)}, f^{(1)}, (F^{(2)}M^{(2)}) 0.
\]

We then have exact sequences

\[
I_{\Lambda_0}^{(k)}(X', -) \rightarrow I_{\Lambda_0}^{(k)}(X, -) \rightarrow (F^{(j)})^{(k)} \rightarrow 0, \quad k = 1, 2.
\]

By Lemma 3.6 we have that \( I_{\Lambda_0}^{(1)}(X', -) \cong 0 \) and \( I_{\Lambda_0}^{(1)}(X, -) \cong 0 \) if \( 0 \leq j \leq n; \) therefore, \( (F^{(j)})^{(1)} \cong 0. \) On the other hand, if \( j > n \) we have \( I_{\Lambda_0}^{(2)}(X', -) \cong M_T \oplus \mathcal{U}(U', -) \) and \( I_{\Lambda_0}^{(2)}(X, -) \cong M_T \oplus \mathcal{U}(U, -). \) Thus, by (7), we get \( (F^{(j)})^{(2)} \cong F(2) \) if \( j > n. \)

In this way, if \( 1 \leq j \leq n \) we obtain that \( F^{(j)}M^{(j-1)} \) is a sum of copies of elements in \( \Lambda \Delta(1) \) and

\[
\frac{F^{(j)}}{F^{(j-1)}} \cong \begin{cases} (0, 0, (F^{(j)})^{(2)}/(F^{(j-1)})^{(2)}) , & if 1 \leq j \leq n; \\ ((F^{(j)})^{(1)}/(F^{(j-1)})^{(1)}, 0, 0) , & if j > 1. \end{cases}
\]

(ii) follows from by Proposition 3.8.

(iii) Let \( F = (F^{(1)}, f, F^{(2)}) \in \mathcal{F}_f(\Delta) \). By (iii), it only remains to prove that if \( F^{(1)} \in \mathcal{F}_f(\Delta) \) and \( F^{(2)} \in \mathcal{F}_f(\Delta) \), then \( F \in \mathcal{F}_f(\Delta) \). In fact, the \( \Lambda \)-modules \( (F^{(1)}, 0, 0) \) and \( (0, 0, F^{(2)}) \) are in \( \mathcal{F}_f(\Delta) \) by Proposition 3.8. It follows that we have a short exact sequence

\[
0 \rightarrow (F^{(1)}, 0, 0) \rightarrow (F^{(1)}, f, F^{(2)}) \rightarrow (0, 0, F^{(2)}) \rightarrow 0.
\]

Thus, \( (F^{(1)}, f, F^{(2)}) \) is in \( \mathcal{F}_f(\Delta) \) since \( \mathcal{F}_f(\Delta) \) is closed under extensions by Remark 1.4.

We end this section continuing with Example 2.9.

**Example 3.10.** Consider the quivers

\[
R: 1 \overset{\alpha_1}{\rightarrow} 2 \overset{\alpha_2}{\rightarrow} 3 \cdot \cdot \cdot \quad \text{and} \quad Q: 1' \overset{\beta_1}{\leftarrow} 2' \overset{\beta_2}{\leftarrow} 3' \overset{\beta_3}{\leftarrow} 4' \cdot \cdot \cdot \quad . \]

Let \( \mathcal{U} = KQ \) and \( \mathcal{T} = K\mathcal{R}/\mathcal{J} \) be the path categories of the above quivers, where \( \mathcal{J} \) is the ideal in \( K\mathcal{R} \) generated by the set of relations

\[
(8) \quad \{ \beta_1 \alpha_1 \text{ and } \alpha_{t+1} \alpha_t, \beta_t \beta_{t+1}, \alpha_t \beta_t - \beta_{t+1} \alpha_{t+1}, t \geq 1 \},
\]

First, we see that \( \mathcal{T} \) and \( \mathcal{U} \) are quasi-hereditary categories.

Set \( \mathcal{T}_0 = \{ 0 \} \), and let \( \mathcal{T}_j = \text{add}\{ t : 1 \leq t \leq j \}, \) for \( j \geq 1. \) Therefore, \( \{ 0 \} = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \) is a filtration of \( \mathcal{T} \) into additively closed subcategories. (i) It is clear that \( \text{rad}_\mathcal{T}(1, 1) = 0 \) because \( \beta_1 \alpha_1 = 0. \) Since \( \text{Ind}_\mathcal{T}_j - \text{Ind}_\mathcal{T}_{j-1} = \{ j \} \) for
all \( j \geq 1 \), we have that \( \text{rad}_T(j, j) = (\beta_j, \alpha_j) = (\alpha_{j-1} \beta_{j-1}) = I_{T_{j-1}}(j, j) \).

(ii) \( I_{T_{j}}(1, -) \cong (1, -), I_{T_{j}}(2, -) \cong (1, -) \) and \( I_{T_{j}}(j, -) = 0 \), if \( j \geq 3 \). For \( j \geq 2 \), we can readily check that there exists an exact sequence \( 0 \to I_{T_{j-1}}(j, -) \to T(j, -) \to I_{T_{j}}(j + 1, -) \to 0 \) and \( I_{T_{j}}(j + t, -) = 0 \) if \( t \geq 2 \).

Finally, the functor \( M \) given in Example 2.9 satisfies \( M_{T^*} : KQ \to \text{mod } K \) is projective for all \( T \) since \( M_1 \cong \mathcal{U}(1, -)^2 ; M_2 \cong \mathcal{U}(1, -) \), and \( M_t \cong \mathcal{U} \), for all \( t > 2 \), which are all in \( \mathcal{F}(\mathcal{U} \Delta) \) because \( \mathcal{U} \) is quasi-hereditary.

In this way, the matrix category \( (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}) \) is equivalent to the path category of the quiver \( Q'' \) modulo the ideal generated by the set relations \( \mathcal{R} \) and is quasi-hereditary with respect to the filtration \( \{0\} = \mathcal{U}_0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 = KQ \). Let \( \mathcal{U}_0 = \{0\} \) and \( \mathcal{U}_1 = \text{add}\{j' \in \mathbb{N} : j \text{ is odd} \} \) and \( \mathcal{U}_2 = \text{add}\{j' : j \in \mathbb{N}\} = KQ \). Thus \( KQ \) is quasi-hereditary with respect to the finite filtration \( \{0\} = \mathcal{U}_0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 = KQ \). The condition (i) clearly holds since \( \text{rad}_T(E, E') = I_{T_{j-1}}(E, E') = 0 \) for all pairs \( E, E' \in \text{Ind } \mathcal{U}_j - \text{Ind } \mathcal{U}_{j-1} \) and \( j = 1, 2 \). On the other hand, \( I_{T_{j}}(j, -) = (j, -) \) if \( j \) is odd and \( I_{T_{j}}(j, -) \cong (j - 1, -) \oplus (j + 1, -) \) if \( j \) is even.

4. One-Point Extensions by Projectives and Classical Tilting

Let us consider \( Q = (Q_0, Q_1) \) that is a strongly locally finite quiver and \( * \) is a source in \( Q \). Let \( Q' = (Q'_0, Q'_1) \) be the quiver obtained by removing the vertex \( * \) from \( Q \). Let \( I \subset KQ \), and set \( C = KQ/I \). We then have two full
subcategories of $C$, $\mathcal{U} = \text{add} Q_0^\ast$ and $\mathcal{T} = \text{add}\{\ast\}$. Let $M : \mathcal{U} \otimes \mathcal{T}^{op} \rightarrow \text{mod} K$ be the additive functor given by $M(u, \ast) = \mathcal{C}(\ast, u)$, where $(u, \ast) \in Q_0 \times \{\ast\}$, and given $(g \otimes f) \in \mathcal{U}(U, U') \otimes_K \mathcal{T}(\ast, \ast) = (\mathcal{U} \otimes_K \mathcal{T}^{op})((U, \ast), ((U', \ast))$, $M(g \otimes f) : M(U, \ast) \rightarrow M(U', \ast)$ is defined by $\mathcal{C}(f, g) : \mathcal{C}(\ast, U) \rightarrow \mathcal{C}(\ast, U')$ for all $U, U' \in \mathcal{U}$.

Consider the triangular matrix category $\Lambda = (\mathcal{T} \ 0 \ \mathcal{U})$. Let $X = (\ T \ 0 \ U \ )$, $X' = (\ T' \ 0 \ U' \ )$ and $X'' = (\ T'' \ 0 \ U'' \ )$ be objects in $\Lambda$. The set of morphisms $\Lambda(X, X') = (\ T(T', T') \ 0 \ U(U', U') \ )$ can be simply written as $(\mathcal{C}(T, T') \ 0 \ \mathcal{C}(U, U') \ )$. Thus, the composition $\Lambda(X', X'') \times \Lambda(X, X') \rightarrow \Lambda(X, X'')$ in $\Lambda$ is given by the map

$$\begin{align*}
& \left(\begin{array}{c}
\mathcal{C}(T', T') \\
\mathcal{C}(T, U')
\end{array}\right) \times \left(\begin{array}{c}
0 \\
\mathcal{C}(T, T') \mathcal{C}(U, U')
\end{array}\right) \rightarrow \left(\begin{array}{c}
\mathcal{C}(T, T'') \\
\mathcal{C}(T, U'') \mathcal{C}(U, U'')
\end{array}\right) \\
& \left(\begin{array}{c}
f' \ 0 \\
h' \ g'
\end{array}\right) \times \left(\begin{array}{c}
f \ 0 \\
h \ g
\end{array}\right) \rightarrow \left(\begin{array}{c}
f'f \ 0 \\
h'f + g'h' \ g'g
\end{array}\right).
\end{align*}$$

In this way, there is an isomorphism of categories $\mathcal{C} \cong \Lambda$.

A pair of modules $X$ and $Y$ is called orthogonal if $\text{Ext}_\Lambda^j(X, Y) = \text{Ext}_\Lambda^j(Y, X) = 0$ for all $j \geq 1$ and exceptional if additionally $\text{pd}_\Lambda X < \infty$ for all $X \in \text{mod} \Lambda$.

We then obtain the main result in this section, which is similar to that given in [2].

**Theorem 4.1.** An adjoint pair of additive functors $\mathcal{R} : \text{mod} \Lambda \rightarrow \text{mod} \mathcal{U}$ and $\mathcal{E} : \text{mod} \mathcal{U} \rightarrow \text{mod} \Lambda$ exists, which preserve orthogonality and exceptionality.

Since $Q$ is locally finite, there is a finite number of neighbours, $u_1, \ldots, u_n$ of $\{\ast\}$. Set $U_0 := \oplus_{i=1}^n u_i$, and consider the morphism $h : \ast \rightarrow U_0$ induced by the arrows $\alpha_i : \ast \rightarrow u_i$, $1 \leq i \leq n$. Thus, for all $u \in Q_0$, we have an isomorphism $\mathcal{C}(U_0, u) \cong \mathcal{C}(\ast, u)$, $g \mapsto gh$, which induces an isomorphism of $\mathcal{U}$-modules:

$$\mathcal{U}(U_0, \ast) \cong \mathcal{C}(\ast, \ast)|\mathcal{U}. \tag{9}$$

Consider the projective $\Lambda$-modules $P = ((\ast \ 0 \ 0), \ )$ and $P_0 = ((\ 0 \ 0 \ U_0), \ )$. Thus, given $X = (\ T \ 0 \ U \ ) \in \Lambda$, we have an isomorphism of $K$-modules:

$$\varphi : P_0(X) = \left(\begin{array}{cc}
0 & 0 \\
0 & \mathcal{C}(U_0, U)
\end{array}\right) \rightarrow \text{rad} P(X) = \left(\begin{array}{cc}
0 & 0 \\
0 & \mathcal{C}(\ast, U)
\end{array}\right) \left(\begin{array}{cc}
0 & 0 \\
0 & g
\end{array}\right) \rightarrow \left(\begin{array}{cc}
0 & 0 \\
0 & \tilde{h}
\end{array}\right) \left(\begin{array}{cc}
0 & 0 \\
0 & g\tilde{h}
\end{array}\right),$$

which is natural in $X$. Thus, we get an isomorphism $\varphi : P_0 \rightarrow \text{rad} P$ of $\Lambda$-modules.
Set $S = \frac{P}{\text{rad}P}$. It is clear that $S$ is a simple $\Lambda$–module, and it is injective in $(\Lambda, \text{mod} K)$ since $S \cong D(\Lambda, (\Lambda_0^0))$, where $D$ is the standard duality. Therefore, we have an exact sequence
\begin{equation}
0 \rightarrow P_0 \rightarrow P \rightarrow S \rightarrow 0.
\end{equation}

In what follows, we will refer to objects of $(\mathcal{U}, \text{mod} K)$ and $(\Lambda, \text{mod} K)$ simply as $\mathcal{U}$-modules and $\Lambda$-modules, respectively.

We now define a pair of additive functors, $\mathcal{E} : (\mathcal{U}, \text{mod} K) \rightarrow (\Lambda, \text{mod} K)$ and $\mathcal{R} : (\Lambda, \text{mod} K) \rightarrow (\mathcal{U}, \text{mod} K)$ called extension and restriction, respectively, and they are defined as follows.

Let $G$ be a $\mathcal{U}$-module, and consider the pair of objects $(\Lambda_0^0)$ and $(\Lambda_0^0)$ in $\Lambda$. Thus,
\[
(\mathcal{E}G) \left( \begin{array}{c} * \\ M \\ 0 \end{array} \right) := \text{Hom}_\mathcal{U} \left( (\ast, \ast), (\ast, \ast) \right) \cong \text{Hom}_\mathcal{U} \left( \mathcal{U}(U_0, \ast), G \right) \cong G(U_0)
\]
\[
(\mathcal{E}G) \left( \begin{array}{c} 0 \\ M \\ U \end{array} \right) := \text{Hom}_\mathcal{U} \left( \mathcal{U}(U, \ast), G \right) \cong G(U).
\]

Let $F$ be a $\Lambda$-module and $U \in \mathcal{U}$. Therefore, $(\mathcal{R}F)(U) := F \left( \begin{array}{c} 0 \\ M \\ U \end{array} \right)$.

Clearly, $(\mathcal{R}, \mathcal{E})$ is an adjoint pair of functors, and both are exact.

**Proposition 4.2.** The functor $\mathcal{R}$ sends finitely presented $\Lambda$-modules into finitely presented $\mathcal{U}$-modules.

**Proof.** Set $Q_\ast := \Lambda((\Lambda_0^0), \ast)$ and $Q_U := \Lambda((\Lambda_0^0)^\ast)$, $U \in \mathcal{U}$ we have isomorphisms of $K$-vector spaces $\mathcal{U}(U_0, U) \cong U(\ast, \ast) \cong \mathcal{R}Q_\ast(U), \theta \mapsto (\begin{array}{c} 0 \\ \theta \\ 0 \end{array})$ and $\mathcal{U}(U, \ast) \cong \mathcal{R}Q_U(U), \theta \mapsto (\begin{array}{c} 0 \\ \theta \\ 0 \end{array})$, which are natural in $U$. Thus, we have isomorphisms of $\mathcal{U}$-modules:

\[\mathcal{R}Q_U \cong \mathcal{U}(U, \ast)\] and \[\mathcal{R}Q_\ast \cong \mathcal{U}(U_0, \ast).
\]

Therefore, the restriction $\mathcal{R}Q$ of a projective finitely generated $\Lambda$-module $Q$ is a projective finitely generated $\mathcal{U}$-module. The rest of the proof follows from the fact that $\mathcal{R}$ is an exact functor.

Later we will see that the extension functor $\mathcal{E}$ sends finitely presented $\mathcal{U}$-modules into finitely presented $\Lambda$-modules.

Given a $\mathcal{U}$-module $G$, we can see it as a $\Lambda$-module by setting $G \left( \begin{array}{c} \ast \\ M \end{array} \right) = G(U)$. In this way we consider $(\mathcal{U}, \text{mod} K)$ embedded in $(\Lambda, \text{mod} K)$ by identifying it with the full subcategory of $(\Lambda, \text{mod} K)$ consisting of the $\Lambda$–modules such that $G \left( \begin{array}{c} \ast \\ M \end{array} \right) = G \left( \begin{array}{c} \ast \\ M \end{array} \right)$.

Let $X$ be a $\Lambda$-module. We claim that $\mathcal{R}X$ is a submodule of $X$. In fact, we have that
\[
X \left( \begin{array}{c} \ast \\ M \\ U \end{array} \right) = X \left( \begin{array}{c} \ast \\ M \\ 0 \end{array} \right) \prod X \left( \begin{array}{c} 0 \\ M \\ U \end{array} \right) \cong X \left( \begin{array}{c} \ast \\ M \\ 0 \end{array} \right) \prod (\mathcal{R}X) \left( \begin{array}{c} \ast \\ M \\ U \end{array} \right).
\]
for all \((\begin{array}{c} 0 \\ U \\ 0 \end{array})\) ∈ \(\Lambda\). Thus, \(R\) is a subfunctor of the identity functor \(\text{Id}_{\text{mod} \, \Lambda}\).

**Lemma 4.3.** The following statements hold.

(a) The functor \(R\) is the torsion radical of the torsion pair \((\text{mod} \, U, \text{add} \, S)\) in \(\text{mod} \, \Lambda\).

(b) Let \(X\) be a \(\Lambda\)-module. The canonical sequence in this torsion pair

\[
0 \longrightarrow RX \longrightarrow X \longrightarrow S^{rx} \longrightarrow 0
\]

satisfies that \(r_X = \dim_K \text{Hom}_\Lambda(X, S)\).

**Proof.** (a) By that mentioned above, a \(\Lambda\)-module \(X\) can be seen as a \(U\)-module if and only if \(X \circ (\begin{array}{c} T \\ M \end{array} 0 U) \sim X \circ (\begin{array}{c} 0 \\ M \end{array} 0 U)\), for all \((\begin{array}{c} T \\ M \end{array} 0 U)\) ∈ \(\Lambda\). Since \(X \circ (\begin{array}{c} 0 \\ 0 \end{array} M U) = RX \circ (\begin{array}{c} T \\ M \end{array} 0 U)\), we conclude that a \(\Lambda\)-module \(X\) can be seen as a \(U\)-module if and only if \(X \sim RX\).

On the other hand, \(RS = 0\) since \((RS)(U) = S \circ (\begin{array}{c} 0 \\ M \end{array} 0 U) = 0\). Moreover, if \(RX = 0\), then \(X \circ (\begin{array}{c} 0 \\ M \end{array} 0 U) = 0\) for all \(U \in \mathcal{U}\), and therefore \(X \in \text{add} \, S\). Thus \(RX = 0\) if and only if \(X \in \text{add} \, S\), and it is clear that \(R^2 X = RX\). Applying the exact functor \(R\) to the short exact sequence of \(\Lambda\)-modules \(0 \rightarrow RX \rightarrow X \rightarrow X/RX \rightarrow 0\) yields \(R(X/RX) = 0\).

(b) The statement follows after applying the functor \(\text{Hom}_\Lambda(-, S)\) to the canonical sequence (11):

\[
0 \longrightarrow \text{Hom}_\Lambda(S^{rx}, S) \longrightarrow \text{Hom}_\Lambda(X, S) \longrightarrow \text{Hom}_\Lambda(RX, S) = 0
\]

\[
\longrightarrow \text{Ext}^1_\Lambda(S^{rx}, S) = 0.
\]

□

**Corollary 4.4.** For any \(\Lambda\)-module \(X\), the \(U\)-module \(RX\) is projective (in which case, \(\text{pd}_\Lambda X \leq 1\)) or else \(\text{pd}_\Lambda X = \text{pd}_U RX\).

**Proof.** For all \(U \in \mathcal{U}\), the \(U\)-projective module \(U(U, -)\) can be identified under the full embedding of \((U, \text{mod} K)\) in \((\Lambda, \text{mod} K)\), with the projective \(\Lambda\)-module \((\begin{array}{c} 0 \\ M \end{array} 0 U), -)\). If \(RX\) is a projective \(U\)-module, then \(\text{pd}_\Lambda S \leq 1\) implies \(\text{pd}_\Lambda X \leq 1\). In other case, if \(\text{pd}_U RX = d\), thus \(\text{pd}_\Lambda RX = d\), and the sequence (11) yields \(\text{pd}_\Lambda X = d\). □

**Lemma 4.5.** Let \(G\) be a \(U\)-module. There is an isomorphism of \(K\)-vector spaces

\[
\text{Ext}^1_\Lambda(S, G) \cong \text{Hom}_\Lambda(P_0, G).
\]

**Proof.** By applying \(\text{Hom}_\Lambda(-, G)\) to the exact sequence

\[
0 \longrightarrow P_0 \longrightarrow P \longrightarrow S \longrightarrow 0
\]

we get

\[
0 \longrightarrow \text{Hom}_\Lambda(S, G) \longrightarrow \text{Hom}_\Lambda(P, G) \longrightarrow \text{Hom}_\Lambda(P_0, G)
\]

\[
\longrightarrow \text{Ext}^1_\Lambda(S, G) \longrightarrow \text{Ext}^1_\Lambda(P, G) = 0.
\]
Now the desired result follows from the fact
\[ \text{Hom}_\Lambda(P, G) = \text{Hom}_\Lambda((\begin{pmatrix} \ast & 0 \\ 0 & 0 \end{pmatrix}), G) \cong G((\begin{pmatrix} \ast & 0 \\ 0 & 0 \end{pmatrix})) = G((\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix})) = 0. \]

Note that for all \( \mathcal{U} \)-module \( G \) we have
\[ (12) \quad \text{Hom}_\Lambda(S, G) = 0. \]

\[ \square \]

Since \( (\mathcal{R}, \mathcal{E}) \) is an adjoint pair of functors, there are, associated with it, a counit \( \epsilon : \mathcal{R}\mathcal{E} \rightarrow \text{Id}_{\text{Mod } \mathcal{U}} \) and a unit \( \delta : \text{Id}_{\text{Mod } \Lambda} \rightarrow \mathcal{E}\mathcal{R} \).

We have the following result that extends to the one given in \[2\].

**Proposition 4.6.** The adjoint pair \( (\mathcal{R}, \mathcal{E}) \) satisfies the following.

(a) The counit \( \epsilon \) is a functorial isomorphism.
(b) For all \( \Lambda \)-module \( X \), the kernel and cokernel \( \delta_X \) lie in \( \text{add } S \).
(c) Let \( X \in \text{Mod } \Lambda \). The following are equivalent:
   (i) \( \delta_X \) is a monomorphism,
   (ii) \( S \) is not a direct summand of \( X \),
   (iii) \( \text{Hom}_\Lambda(S, X) = 0 \).

**Proof.** We only prove (a). The rest is analogous to the proof given in \[2\] Proposition 2.5. It is clear that \( \epsilon \) is an isomorphism since, for all \( U \), we have
\[ (\mathcal{R}\mathcal{E}G)(U) = (\mathcal{E}G) \begin{pmatrix} 0 & 0 \\ M & U \end{pmatrix} \cong \text{Hom}_\mathcal{U}(U(U, -), G) \cong G(U). \]

\[ \square \]

It follows from the above result that \( \mathcal{E} \) is a fully faithful functor, \[29\] Theorem (IV. 3.3)]. The right perpendicular category of \( S \) is the full subcategory of \( (\Lambda, \text{mod } K) \) defined by
\[ S^{\text{perp}} = \{ X \in (\Lambda, \text{mod } K) \mid \text{Hom}_\Lambda(S, X) = 0, \text{Ext}_\Lambda^1(S, X) = 0 \}. \]

**4.1. Homological properties of functors \( (\mathcal{R}, \mathcal{E}) \).** It then follows that \( \delta_X \) is a functorial isomorphism for all \( X \in S^{\text{perp}} \); see \[2\] Lemma 3.1]. Let \( G \) be a \( \mathcal{U} \)-module. Thus, by Proposition 4.6 there exists an exact sequence called the extension sequence
\[ (13) \quad 0 \rightarrow G \xrightarrow{k_G} \mathcal{E}RG \xrightarrow{\epsilon_G} S^{\text{perp}}G \rightarrow 0, \]

which coincides with the restriction sequence for \( \mathcal{E}RG \cong \mathcal{E}G \). In particular, \( e_G = \epsilon_G \).

**Proposition 4.7.** Let \( G \) be a \( \mathcal{U} \)-module. The extension sequence satisfies the following properties

(a) \( e_G = \dim_K \text{Ext}_\Lambda^1(S, G) \).
(b) The connecting morphism \( \text{Hom}_\Lambda(S, S^{\text{perp}}G) \rightarrow \text{Ext}_\Lambda^1(S, G) \) is an isomorphism.
(c) $EG \in S^{\perp}$.

Proof. We only prove (a). The rest of the proof is similar to that given in [2]. Evaluating the exact sequence (13) in the object $(S_{\Lambda}^\triangledown) \in \Lambda$, we have

$$0 \rightarrow G(U) \rightarrow (C(\ast, -)|_{U}, G) \rightarrow S \left( \begin{array}{c} S_{\Lambda} \\ U \end{array} \right) \rightarrow 0$$

since $E G \cong EG$. By Lemma 4.5, we obtain $e_G = \dim_K (C(\ast, -)|_{U}, G) = \dim_K Hom_{\Lambda}(P_0, G) = \dim_K Ext^1_{\Lambda}(S, G)$. □

It follows that mod $U$ and $S^{\perp}$ are equivalent categories.

Corollary 4.8. The functors $E$ and $R$ induce an equivalence between mod $U$ and $S^{\perp}$, and $E$ sends finitely presented $U$-modules into finitely presented $\Lambda$-modules.

Proof. The first statement is a consequence of Propositions 4.6 (a), 4.7 (c) and [2, Lemma 3.1].

On the other hand, assume that $G \in \text{mod } U$. $G$ is then seen as a $\Lambda$-module. Since $E RG \cong EG$, it follows from the exact sequence (13) that $EG$ belongs to mod $\Lambda$ since $S$ lies in mod $\Lambda$. □

The following result extends the one given in [2], which has a similar proof.

Proposition 4.9. Let $X$ and $Y$ be in mod $\Lambda$. Then:

(a) There is an epimorphism $Ext^1_{\Lambda}(X, Y) \rightarrow Ext^1_{U}(RX, RY)$.

(b) There is an isomorphism $Ext^j_{\Lambda}(X, Y) \cong Ext^j_{U}(RX, RY)$, for each $j \geq 2$.

(c) If $Y \in S^{\perp}$, then the epimorphism of (a) is an isomorphism.

Proof of Theorem 4.1. The first part is a consequence of Proposition 4.2 and Corollary 4.8. Let $M$ and $N$ be orthogonal $U$-modules. By Proposition 4.7 (c), $E N, E M \in S^{\perp}$ so that, by Corollary 4.8.

$$Ext^j_{U}(EM, EN) \cong Ext^j_{U}(RE M, RE N) \cong Ext^j_{U}(M, N) = 0.$$ Therefore, $EM$ and $EN$ are orthogonal.

Let $X$ and $Y$ be orthogonal $\Lambda$-modules. Therefore, Proposition 4.9 (b) yields, $Ext^j_{\Lambda}(X, Y) \cong Ext^j_{U}(RX, RY) \cong 0$, for each $j \geq 2$, and by 4.9 (a) $Ext^1_{\Lambda}(X, Y) = 0$ implies $Ext^1_{U}(RX, RY) = 0$. Thus, $RX$ and $RY$ are orthogonal. The statement about exceptionality follows from Corollary 1.4 □

We finish this section showing with a couple of examples how we can extend classical tilting categories in functor categories of path categories by using the developed theory.
We now recall the definition of classical tilting subcategories in functor categories. Recall that an annuli variety is an additive category where the idempotents split \[3\]. The following definition is given in \[34\].

**Definition 4.10.** Let \(\mathcal{C}\) be an annuli variety. A subcategory \(\mathcal{T}\) of \(\text{Mod}\ \mathcal{C}\) is a classical tilting category if the following holds:

(i) \(\text{pd}\ \mathcal{T} \leq 1\).

(ii) \(\text{Ext}^1_{\mathcal{C}}(T_i, T_j) = 0\), for all pair of objects \(T_i, T_j \in \mathcal{T}\).

(iii) For all object \(C \in \mathcal{C}\), there exists an exact sequence

\[
0 \to C(C, -) \to T_1 \to T_2 \to 0,
\]

with \(T_1, T_2 \in \mathcal{T}\).

A subcategory of \(\text{Mod}\ \mathcal{C}\) that satisfies (i) and (ii) is called partial tilting.

**Example 4.11.** Let \(\mathcal{T}\) be a classical tilting category in \(\text{Mod}\ \mathcal{U}\). Thus the full category \(E(\mathcal{T})\) of \(\text{Mod}\ \Lambda\) consisting of the objects \(E_T, T \in \mathcal{T}\) can be extended to a classical tilting category in \(\text{Mod}\ \Lambda\).

(i) Since \(\text{pd}\ \mathcal{T} \leq 1\) and \(\text{pd}\ S^T \leq 1\), it follows from the exact sequence

\[
0 \to T \to E_T \to S^T \to 0
\]

that \(\text{pd}\ E_T \leq 1\).

(ii) By Theorem \[4.7\], \(\text{Ext}^1_{\Lambda}(E_T, E_T') = 0\), for every pair of objects \(T, T' \in \mathcal{T}\). Thus, \(E_T\) is a partial tilting category in \(\text{Mod}\ \Lambda\). Thus, by using the Bonnart argument \[10\], \[34, Theorem 7\], we can obtain a classical tilting category in \(\text{Mod}\ \Lambda\) from the partial tilting category \(E(\mathcal{T})\) in \(\text{Mod}\ \mathcal{U}\). A similar argument allows to obtain a classical tilting category \(R(\mathcal{T})\) in \(\text{Mod}\ \mathcal{U}\) from a classical tilting category \(\mathcal{T}\) in \(\text{Mod}\ \Lambda\) by using the restriction functor \(R\).

**Example 4.12.** Label the vertices of \(\mathbb{Z}\mathbb{A}_\infty\) in the following way:

\[
\begin{array}{cccccc}
(1, -1) & (1, 0) & (1, 1) & (1, 2) & (1, 3) \\
(2, -2) & (2, -1) & (2, 0) & (2, 1) & (2, 2) \\
(3, -3) & (3, -2) & (3, 0) & (3, 1) & (3, 2) \\
(4, -4) & (4, -3) & (4, 0) & (4, 1) & (4, 2) \\
(5, -5) & (5, -4) & (5, 0) & (5, 1) \\
\end{array}
\]

Let \((r, s) \in \mathbb{N} \times \mathbb{Z}\). Define the representation \(T(r, s) = (V_{ij}, f_\alpha)_{(i,j) \in \mathbb{N} \times \mathbb{Z}}\), given by

\[
V_{ij} = \begin{cases} 
K & \text{if } i \geq r \text{ and } -(i-r-s) \leq j \leq s; \\
0 & \text{in other case,}
\end{cases}
\]

and \(f_\alpha = 1_K : V_{rt} \to V_{uv}\) if \(V_{rt} = V_{uv} = K\) and \(f_\alpha = 0\) in other case:
Thus, $\Lambda M$ is given by $\text{Ext}(\text{morphisms } T K)$, since $K$ is a classical tilting category in $\text{Mod}(K(A_\infty, \sigma))$. Consider the path category $K(A_\infty, \sigma)$, modulo the ideal generated by the mesh relations.

Thus, $\Lambda \cong \begin{pmatrix} * & 0 \\ M & K(A_\infty, \sigma) \end{pmatrix}$, where $M = K(A_\infty, \sigma) \otimes \{ \ast \to \mod K \}$ is given by $M(U, \ast) = \Lambda(\ast, U)$ for all vertex $U$ in $A_\infty$.

We see that $\{ \mathcal{E} T(r, s) \prod S \}_{(r, s) \in \mathbb{N} \times \mathbb{Z}}$ is a classical tilting category in $\text{Mod } \Lambda$.

(i) Since $pd S \leq 1$, it follows that $pd(\mathcal{E} T(r, s) \prod S) \leq 1$.

(ii) Let $(r, s), (r', s') \in \mathbb{N} \times \mathbb{Z}$. First note that by Corollary 3.5 we have $\text{Ext}_\Lambda^1(S, \mathcal{E} T(r', s')) \cong \text{Ext}_\Lambda^1(RS, T(r', s')) = 0$ because $RS = 0$. Moreover, since $\mathcal{E}$ preserves orthogonality and $S$ is an injective $\Lambda$–module, we have $\text{Ext}_\Lambda^1(\mathcal{E} T(r, s) \prod S, \mathcal{E} T(r', s') \prod S) = 0$.

(iii) Set $E^j_i$ the vertex $(i, j) \in (A_\infty)_0$. Note that $\Lambda(\ast, -)|_{K(A_\infty, \sigma)} \cong K(A_\infty, \sigma)(E^1_1, -)$. Let $(r, s) \in \mathbb{N} \times \mathbb{Z}$. By Yoneda's lemma, we have isomorphisms $(\mathcal{E} T(1, 1)) \begin{pmatrix} * & 0 \\ M & E^1_1 \end{pmatrix} \cong T(1, 1)(E^1_1)$ and $(\mathcal{E} T(1, 1)) \begin{pmatrix} * & 0 \\ M & 0 \end{pmatrix} \cong T(1, 1)(E^1_1) \cong K$; therefore, $\mathcal{E} T(1, 1) \cong T(1, 1) \prod S$. On the other hand, if

\[
\begin{array}{cccccc}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]
(r, s) \neq (1, 1), one can see that ET(r, s) \cong T(r, s). Thus, T(1, 1) is the unique tilting object that suffers a change under the functor E.

For each (r, s) \in \mathbb{N} \times \mathbb{Z}, we can see T(r, s) as an object in Mod Λ, and we have a resolution in Mod Λ of the form

\[
0 \to (E^*_r, -) \to T(1, r + s - 1) \to T(r + 1, s - 1) \to 0.
\]

In addition, for projective (\ast, -) we have the resolution

\[
0 \to (\ast, -) \to T(1, 1) \amalg S \to T(2, 0) \to 0.
\]

References

[1] Assem I., Simson D., Skowroński A. Elements of the Representation Theory of Associative Algebras I: Techniques of Representation Theory, London Mathematical Society Student Texts 65, Cambridge University Press, Cambridge, 2006.

[2] Assem I., Happel D., Trepode S. Extending Tilting Modules to One-Point Extensions by Projectives, Communications in Algebra, 35:10, 2983-3006, DOI: 10.1080/00927870701404556 , 2007.

[3] Auslander M. Representation Theory of Artin Algebras I, Comm. in algebra, 3(1), 177-268, 1974.

[4] Auslander M., Reiten I. Stable equivalence of dualizing R-varieties*, Advances in Mathematics, Vol. 12, Issue 3, 1974.

[5] Auslander M. A functorial approach to representation theory, Representations of algebras, Lecture Notes in Mathematics, Vol. 944, Springer-Verlag, 105-179, 1982.

[6] Auslander M., Plitezek M. I., Reiten I. Coxeter functors without diagrams, Trans. Amer. Math. Soc. 250, 1-46, 1979.

[7] Auslander M., Reiten I., Smalø S. Representation theory of artin algebras, Studies in Advanced Mathematics 36, Cambridge University Press, 1995.

[8] Barot M. Introduction to the Representation Theory of Algebras, Springer International Publishing, X, 179, 2015.
[9] Bautista R., Liu S., Paquette Ch. *Representation theory of strongly locally finite quivers*, Proceedings of the London Mathematical Society, Volume 106, Issue 1, 2013.

[10] Bongartz K. *Tilted algebras*, Representations of Algebras, Lecture Notes in Mathematics, Vol. 903, Springer, Berlin-New York, Auslander M., Lluis E. (eds), https://doi.org/10.1007/BFb0092982, 26-38, 1981.

[11] Cline E., Parshall B. J., and Scott L. L. *Algebraic stratification in representation categories*, Journal of algebra, 117, Issue 2, 504-521, 1988.

[12] Chase S. U. *A generalization of the ring of triangular matrices*, Nagoya Math. J. 18, 13-25, 1961.

[13] Dlab V. *Quasi-hereditary algebras*, Appendix to Drozd Y. A., Kirichenko V. V. *Finite dimensional algebras*, Springer-Verlag, 1993.

[14] Dlab V. *Quasi-hereditary algebras revisited*, An. St. Univ. Ovidius. Constanța 4, 43-54, 1996.

[15] Dlab V., Ringel C. M. *Representations of graphs and algebras*, Memoirs of the A.M.S., No. 173, 1976.

[16] Dlab V., Ringel C. M. *Quasi-hereditary algebras*, Illinois Journal of Mathematics 33, No. 2, 280-291, 1989.

[17] Dlab V., Ringel C. M. *The Module Theoretical Approach to Quasi-hereditary Algebras*, Repr. Theory and Related Topics, London Math. Soc. LNS, 168, 200-224, 1992.

[18] Frey P. *Representations in abelian category*, Proceedings of the conference on categorical algebra, La Jolla, 95-120, 1996.

[19] Fossen R. M., Griffith P. A., Reiten I. *Trivial Extensions of Abelian Categories*, Lecture Notes in Mathematics, No. 456, Springer, Berlin, Heidelberg, New York, 1975.

[20] Gordon R., Green E. L. *Modules with cores and amalgamations of indecomposable modules*, Memoirs of the A.M.S., No. 187, 1978.

[21] Haghany A., Varadarajan K. *Study of formal triangular matrix rings*, Comm. in algebra 27 (11), 5507-5525, 1999.

[22] Heller A. *Homological algebra in abelian categories*, The Annals of Math, Second Series, Vol. 68, No. 3, 484-525, 1958.

[23] Krause H. *Krull-Schmidt categories and projective covers*, Expo. Math., 33, Num. 4, 535-549, 2015.

[24] Krause H. *Highest weight categories and recollements*, Annales de l’Institut Fourier, Vol. 67, No. 6, 2679-2701, 2017.

[25] Leon-Galeana A., Ortiz-Morales M., Santiago V. *Triangular Matrix Categories: Dualizing Varieties and Generalized one-point extension*, Preprint [arXiv:1903.05914]

[26] Leonard A., Ortiz-Morales M., Santiago V. *Triangular Matrix Categories: Recollements and functorially finite subcategories*, Preprint [arXiv:1903.03020]

[27] Leszczyński Z. *On the representation type of tensor product algebras*, Fundamenta Mathematicae, 144, 1994.

[28] MacLane S. *Categories for the working mathematician*, Springer-Verlag, New York, Graduate Texts in Mathematics, Vol. 5, 1971.

[29] Mitchell B. *Rings with several objects*, Advances in Math., 8, 1-161, 1972.

[30] Marcos E., Mendoza O., Sáenz C., Santiago V., *Standardly stratified lower triangular K-algebras with enough idempotents*, Preprint [arXiv:2101.10879]

[31] Ortiz M. *The Auslander-Reiten components seen as quasi-hereditary categories*. Applied categorical Structures 26, 239-285, 2018.

[32] Martínez-Villa R., Ortiz-Morales M. *Tilting theory and functor categories II. Generalized Tilting*, Applied categorical structures, Vol. 21, 311-348, 2013.

[33] Martínez-Villa R., Ortiz-Morales M. *Tilting theory and functor categories I. Classical Tilting*, Applied categorical structures, Vol. 22, 595-646, 2014.
[35] Martínez-Villa R., Solberg Ø. *Graded and Koszul categories*, Applied categorical structures, Vol. 18, 615-652, 2010.

[36] Martínez-Villa R., Solberg Ø. *Artin-Schelter regular algebras and categories*, Journal of pure and applied algebra, Vol. 215, 546-565, 2011.

[37] Martínez-Villa R., Solberg Ø. *Noetherianity and Gelfand-Kirillov dimension of components*, Journal of algebra, 323, No. 5, 1309-1407, 2010.

[38] Parshall B. J., and Scott L.L. *Derived categories, quasi-hereditary algebras and algebraic groups*, Algebra, Proc. Workshop, Ottawa/Can. 1987, Math. Lect. Note Ser., Expo. Math., CRAF, Carleton Univ. 3, 105, 1988.

[39] Ringel C. M. *The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences*, Math. Z. 208, 209-223, 1991.

[40] Ringel C. M. *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Mathematics, Springer-Verlag Berlin Heidelberg, XVI, 380, 1984.

[41] Scott L. L. *Simulating algebraic geometry with algebra I: The algebraic theory of derived categories*, in The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986) Proc. Sympos. Pure Math. AMS, 47, pp. 2, 71-281, 1987.

[42] Skartsætherhagen I. Ø. *Quivers and admissible relations of tensor products and trivial extensions*, Master of Science in Mathematics, Norwegian University of Science and Technology, Department of Mathematical Sciences, 2011. [http://hdl.handle.net/11250/258909](http://hdl.handle.net/11250/258909)

[43] Zhu B. *Triangular Matrix Algebras Over Quasi-Hereditary Algebras*, Tsukuba J. Math., Vol. 25, No. 1, 1-11, 2001.

Rafael Francisco Ochoa de la Cruz:
Instituto de Matemáticas, Universidad Nacional Autónoma de México
Circuito Exterior, Ciudad Universitaria, C.P. 04510, México, D.F. MEXICO.
rafaelfochoa88@gmail.com

Martin Ortíz Morales:
Facultad de Ciencias, Universidad Autónoma del Estado de México
Campus Universitario “El Cerrillo, Piedras Blancas”, Carretera Toluca-Ixtlahuaca Km. 15.5, Estado de México. CP 50200.
mortizmo@uaemex.mx