DISCRETE FOURIER ANALYSIS ON FUNDAMENTAL DOMAIN OF $A_d$
LATTICE AND ON SIMPLEX IN $d$-VARIABLES

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ABSTRACT. A discrete Fourier analysis on the fundamental domain $\Omega_d$ of the $d$-dimensional lattice of type $A_d$ is studied, where $\Omega_2$ is the regular hexagon and $\Omega_3$ is the rhombic dodecahedron, and analogous results on $d$-dimensional simplex are derived by considering invariant and anti-invariant elements. Our main results include Fourier analysis in trigonometric functions, interpolation and cubature formulas on these domains. In particular, a trigonometric Lagrange interpolation on the simplex is shown to satisfy an explicit compact formula and the Lebesgue constant of the interpolation is shown to be in the order of $(\log n)^d$. The basic trigonometric functions on the simplex can be identified with Chebyshev polynomials in several variables already appeared in literature. We study common zeros of these polynomials and show that they are nodes for a family of Gaussian cubature formulas, which provides only the second known example of such formulas.

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1. Introduction

The classical discrete Fourier analysis works with the periodic exponential functions \( e^{2\pi ikx} \) for \( k \) in the set \( \mathbb{Z}_n := \{0, 1, \ldots, n - 1\} \). The central piece is the discrete Fourier transform defined via an inner product over the set of equally spaced points \( \{k/n : k \in \mathbb{Z}_n\} \) in \([0,1]\), which agrees with the continuous Fourier transform for trigonometric polynomials of degree at most \( n \). This is equivalent to Gaussian quadrature formula for trigonometric polynomials, and it can be used to define trigonometric interpolation based on the equally spaced points. All three quantities, discrete Fourier transform, quadrature, and interpolation, are important tools in numerous applications and form an integrated part of the discrete Fourier analysis.

In several variables, periodicity of functions can be defined via a lattice \( L \), which is a discrete group described by \( AZ^d \), where \( A \) is a nonsingular matrix. A function \( f \) is periodic with respect to \( L = AZ^d \) if \( f(x) = f(x + Ak) \) for any \( k \in \mathbb{Z}^d \). The usual multiple Fourier analysis of \( \mathbb{R}^d \) uses periodicity defined by the lattice \( \mathbb{Z}^d \). One can develop a discrete Fourier analysis in several variables with a lattice periodicity, which has been defined in connection with signal processing and sampling theory (see, for example, [10, 14, 22]).

The domain on which the analysis takes place is the fundamental domain of the lattice, which is a bounded set \( \Omega \) that tiles \( \mathbb{R}^d \) in the sense that \( \Omega + AZ^d = \mathbb{R}^d \). It is known that the family of exponentials \( \{e^{2\pi i\alpha \cdot x} : \alpha \in L^\perp\} \), in which \( L^\perp \) is the dual lattice, forms an orthonormal basis for \( L^2(\Omega) \) and these exponentials are periodic with respect to \( L \). These two facts make it possible to establish a complete analogue of the classical Fourier analysis. Recent in [18], a discrete Fourier transform is defined and used to study cubature and trigonometric interpolation on the domain \( \Omega \), which provides a general framework for the discrete Fourier analysis with lattice tiling. Detail studies are carried out in [18] for the hexagon lattice, for which the fundamental domain is a regular hexagon, when \( d = 2 \), and in [19] for the face-centered cubic lattice, for which the fundamental domain is a rhombic dodecahedron, when \( d = 3 \). The regular hexagon and the rhombic dodecahedron are invariant under the reflection group \( A_2 \) and \( A_3 \), respectively.

In the present paper we shall consider the lattice of type \( A_d \) for all \( d \geq 2 \), for which the fundamental domain is the union of the images of a regular simplex in \( \mathbb{R}^d \) under the group action.

Our goal is then to establish a discrete Fourier analysis on the fundamental domain of the \( A_d \) lattice in \( \mathbb{R}^d \) that is comparable with the classical theory in one variable as outlined in the first paragraph of this introduction. However, this is only the first step. The framework developed in [18] applies to the fundamental domain \( \Omega \) that tiles \( \mathbb{R}^d \) without overlapping, which means that \( \Omega \) contains only part of its own boundary. For example, if the lattice is \( \mathbb{Z}^d \), then the domain is the half open and half closed \([-\frac{1}{2}, \frac{1}{2})^d \). The points on which the discrete Fourier transform is defined consist of lattice points in \( n\Omega \), which is not symmetric under the reflection group \( A_d \), as part of the boundary points are not included. Thus, as a second step, we will develop a discrete Fourier analysis that uses a symmetric set of points by working with the congruence of the boundary points under translation by the lattice and under the action of the group \( A_d \), which requires delicate analysis. Furthermore, by restricting to functions that are invariant under the group, the results on the set of symmetric points in the step 2 can be transformed to results on a simplex that makes up the fundamental domain. This is our third step, which establishes a Fourier analysis on trigonometric functions on a regular simplex in \( \mathbb{R}^d \). For the classical Fourier analysis, this step amounts to a discrete analysis on cosine and sine functions. We will define analogues of cosine and sine functions on a simplex in the step 3. The classical Chebyshev polynomials arise from cosine and sine functions. As our fourth step, we define generalized Chebyshev polynomials of the first and the second kind from those generalized cosine and sine functions, respectively, and study the discrete analysis of these algebraic polynomials.

In each of the steps outlined above, we will develop a concrete discrete Fourier analysis associated with the \( A_d \) lattice in \( \mathbb{R}^d \), providing explicit formulas and detailed analysis. The cases \( d = 2 \) and \( d = 3 \) studied in [18] and [19] provide a road map for our study. However,
the extension from the cases of two and three variables to the general \(d\)-variables is far from a trivial exercise. In order to carry out our program on the discrete analysis, it is essential that we have a complete understanding of the boundary of the fundamental domain and its congruent relations under the group. For the step 3, we also need to understand the boundary of the simplex and its relation with the fundamental domain. In \(\mathbb{R}^2\) and \(\mathbb{R}^3\), we can grasp the geometry of the hexagon and the rhombic dodecahedron by looking at their graphs and understand the relative positions between faces, edges and vertices, which are all parts of the boundary of the domain. This is not going to be the case in \(\mathbb{R}^d\) for \(d > 3\). For \(\mathbb{R}^d\) we will have to rely on the group theory and describe the boundary elements by working with subgroups.

The study in the cases of \(d = 2\) and \(d = 3\) has revealed much of what can be done and they will serve as examples throughout the paper.

The generalized cosine and sine functions on the simplex, as well as the generalized Chebyshev polynomials, have been studied before in the literature. In fact, in the case of \(d = 2\), they are first studied by Koornwinder in [15], who started with the eigenfunctions of the Laplace operator and showed that the generalized Chebyshev polynomials are orthogonal on the region bounded by the Steiner’s hypocycloid. Later these polynomials are generated and studied by several authors (see [1, 2, 3, 7, 8, 9, 12, 25] and the reference therein), often as eigenfunctions of certain differential operators. In [3], they appear as eigenfunctions of the radial part of the Laplacian-Beltrami operator on certain symmetric spaces. We highly recommend the article [3], which also includes a detailed account on the history of these polynomials. Furthermore, the generalized Chebyshev polynomials belong to large families of special functions (see, for example, [4]), just as the classical Chebyshev polynomials are the special cases of Jacobi polynomials. This, however, appears to have little bearing with our study in this paper. It should be mentioned, however, that our aim is to develop a discrete Fourier analysis on the generalized Chebyshev polynomials, which come down to study common zeros of these polynomials, a topic that does not seem to have been studied systematically before.

There are other types of discrete Fourier analysis in several variables. For the multiple Fourier analysis, with lattice \(\mathbb{Z}^d\), there is a rich collection of literature as it is essentially the tensor product of one dimensional results. Among others, we mention the recent work on antisymmetric exponential and trigonometric functions by Klimyk and Patera, see [16, 17] and the references therein.

The paper is organized as follows. In Section 2 we shall recall the basic facts on lattice and tiling, and describe the framework for the discrete Fourier analysis established in [18]. In Section 3 we study the case of \(A_d\) lattice and carry out both the step 2 and the step 3 of our program. The best way to describe the \(A_d\) lattice and its fundamental domain appears to be using homogeneous coordinates, which amounts to regard \(\mathbb{R}^d\) as the hyperplane \(t_1 + \ldots + t_{d+1} = 0\) of \(\mathbb{R}^{d+1}\). In fact, the group \(A_d\) becomes permutation group \(S_{d+1}\) in homogeneous coordinates. In Section 4 we establish a discrete Fourier analysis on the simplex, which studies trigonometric polynomials on the standard simplex in \(\mathbb{R}^d\). One of the concrete result is a Lagrange interpolation by trigonometric polynomial on equal spaced points on the simplex, which can be computed by a compact formula and has an operator norm, called the Lebesgue constant, in the order of \((\log n)^d\). Finally, in Section 5, we derive the basic properties of the generalized Chebyshev polynomials of the first and the second kind, and study their common zeros, which are related intrinsically to the cubature formula for algebraic polynomials. In particular, it is known that a Gaussian cubature exists if and only if the set of its nodes is exactly the variety of the ideal generated by the orthogonal polynomials of degree \(n\), which however happens rarely and only one single family of examples is known in the literature. We will show that this is the case for the generalized Chebyshev polynomials of the second kind, so that a Gaussian cubature formula exists for the weight function with respect to which the Chebyshev polynomials of the second kind are orthogonal, which provides the second family of examples for such a cubature.
2. Discrete Fourier analysis with lattice

In this section we recall results on discrete Fourier analysis associated with lattice. Background and the content of the first subsection can be found in [6, 10, 14, 22]. Main results, stated in the second subsection, are developed in [18] but will be reformulated somewhat, for which explanation will be given. We will be brief and refer the proof and further discussions to [18].

2.1. Lattice and Fourier series. We will consider lattices on \( \mathbb{R}^d \) but will use homogeneous coordinates, which amounts to identify \( \mathbb{R}^d \) with the hyperplane \( x_1 + \cdots + x_{d+1} = 0 \) in \( \mathbb{R}^{d+1} \). For convenience, we will state the results on the lattices on a \( d \)-dimensional subspace \( \mathcal{V}^d \) of the Euclidean space \( \mathbb{R}^m \) with \( m \geq d \). Since \( \mathcal{V}^d \) can be identified with \( \mathbb{R}^d \), all results proved for \( \mathbb{R}^d \) can be restated for \( \mathcal{V}^d \).

Let \( \mathcal{V}^d \) be a \( d \)-dimensional subspace of \( \mathbb{R}^m \) with \( m \geq d \). A lattice \( L \) of \( \mathcal{V}^d \) is a discrete subgroup that contains \( d \) linearly independent vectors,

\[
L := \{ k_1 a_1 + k_2 a_2 + \cdots + k_d a_d : \ k_i \in \mathbb{Z}, \ i = 1,2,\cdots, d \},
\]

where \( a_1, \cdots, a_d \) are linearly independent column vectors in \( \mathcal{V}^d \). Let \( A \) be the matrix of size \( m \times d \), which has column vectors \( a_1, \cdots, a_d \). Then \( A \) is called a generator matrix of the lattice \( L \). We can write \( L \) as \( AZ^d \); that is

\[
L = AZ^d := \{ Ak : \ k \in \mathbb{Z}^d \}.
\]

The dual lattice \( L^\perp \) of \( L \) is given by

\[
L^\perp := \{ x \in \mathcal{V}^d : \ x \cdot y \in \mathbb{Z} \text{ for all } y \in L \},
\]

where \( x \cdot y \) denotes the usual Euclidean inner product of \( x \) and \( y \). The generator matrix of \( L^\perp \), denoted by \( A^\perp \), is given by \( A^\perp = A(A^\top A)^{-1} \), which is the transpose of the Moore-Penrose inverse of \( A \). In particular, if \( m = d \), then the generator matrix of \( L^\perp \) is simply \( A^{-\text{tr}} \) and \( L^\perp = A^{-\text{tr}} \mathbb{Z}^d = \{ A^{-\text{tr}} k : \ k \in \mathbb{Z}^d \} \).

A bounded set \( \Omega \subset \mathcal{V}^d \) is said to tile \( \mathcal{V}^d \) with the lattice \( L \) if

\[
\sum_{\alpha \in L} \chi_{\Omega}(x + \alpha) = 1 \quad \text{for almost all } x \in \mathcal{V}^d,
\]

where \( \chi_{\Omega} \) denotes the characteristic function of \( \Omega \), which we write as \( \Omega + L = \mathcal{V}^d \). Tiling and Fourier analysis are closely related as demonstrated by the Fuglede theorem. Let \( \langle \cdot, \cdot \rangle_{\Omega} \) denote the inner product in \( L^2(\Omega) \),

\[
\langle f, g \rangle_{\Omega} := \frac{1}{|\Omega|} \int_{\Omega} f(x) \overline{g(x)} dx,
\]

where \( |\Omega| \) denotes the measure of \( \Omega \) and the bar denotes the complex conjugation. The following fundamental result was proved by Fuglede in [13].

**Theorem 2.1.** Let \( \Omega \subset \mathcal{V}^d \) be a bounded domain and \( L \) be a lattice of \( \mathcal{V}^d \). Then \( \Omega + L = \mathcal{V}^d \) if and only if \( \{ e^{2\pi i \kappa \cdot x} : \kappa \in L^\perp \} \) is an orthonormal basis with respect to the inner product (2.1).

Written explicitly, the orthonormal property states that

\[
\frac{1}{|\Omega|} \int_{\Omega} e^{2\pi i \kappa \cdot x} dx = \delta_{\kappa,0}, \quad \kappa \in L^\perp.
\]

Moreover, if \( L = AZ^d \), then the measure \( |\Omega| = \sqrt{\det(A^\top A)} \).

The set \( \Omega \) is called a spectral set for the lattice \( L \). If \( L = AZ^d \) we also write \( \Omega_A \) instead of \( \Omega \).
Because of Theorem 2.1 a function \( f \in L^1(\Omega) \) can be expanded into a Fourier series
\[
f(x) \sim \sum_{\kappa \in L^\perp} c_\kappa e^{2\pi i \kappa \cdot x}, \quad c_\kappa := \frac{1}{|\Omega|} \int_{\Omega} f(x) e^{-2\pi i \kappa \cdot x} dx.
\]
The Fourier transform \( \hat{f} \) of a function defined on \( L^1(V^d) \) and its inversion are defined by
\[
\hat{f}(\xi) := \int_{V^d} f(x) e^{-2\pi i \xi \cdot x} dx, \quad f(x) := \int_{V^d} \hat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi.
\]
Our first result is the following sampling theorem (see, for example, [14, 22]).

**Proposition 2.2.** Let \( \Omega \) be the spectral set of the lattice \( L \). Assume that \( \hat{f} \) is supported on \( \Omega \) and \( \hat{f} \in L^2(\Omega) \). Then
\[
f(x) = \sum_{\kappa \in L^\perp} \Phi_\Omega(x - \kappa) f(\kappa)
\]
in \( L^2(\Omega) \), where
\[
\Phi_\Omega(x) = \frac{1}{|\Omega|} \int_{\Omega} e^{2\pi i \xi \cdot x} d\xi.
\]
This theorem is a consequence of the Poisson summation formula. We notice that
\[
\Phi_\Omega(\kappa) = \delta_{0,\kappa} \quad \text{for all } \kappa \in L^\perp
\]
by Theorem 2.1, so that \( \Phi_\Omega \) can be considered as a cardinal interpolation function.

### 2.2. Discrete Fourier analysis and interpolation.

A function \( f \) defined on \( V^d \) is called **periodic** with respect to the lattice \( L \) if
\[
f(x + \alpha) = f(x) \quad \text{for all } \alpha \in L.
\]
The spectral set \( \Omega \) of the lattice \( L \) is not unique. In order to carry out the discrete Fourier analysis with respect to the lattice, we shall fix an \( \Omega \) such that \( \Omega \) contains 0 in its interior and we further require that \( \Omega \) tiles \( V^d \) with \( L \) without overlapping and without gap. In other words, we require that
\[
\sum_{\alpha \in L} \chi_\Omega(x + \alpha) = 1 \quad \text{for all } x \in V^d. \tag{2.3}
\]
For example, for the standard cubic lattice \( \mathbb{Z}^d \) we can take \( \Omega = [-\frac{1}{2}, \frac{1}{2})^d \). Correspondingly a spectral set satisfying (2.3) is called a **fundamental domain** of Lattice \( L \).

**Definition 2.3.** Let \( A, B \) and \( N \) be as in Definition 2.3. Then for any \( \kappa \in L^\perp_A \),
\[
\frac{1}{|\det(N)|} \sum_{\alpha \in \Lambda_N} e^{2\pi i \kappa \cdot \alpha} = \begin{cases} 1, & \text{if } \kappa \equiv 0 \pmod{B} \\ 0, & \text{otherwise} \end{cases}, \tag{2.4}
\]
and
\[
\frac{1}{|\det(N)|} \sum_{\kappa \in \Lambda_N^\perp} e^{-2\pi i \alpha \cdot \kappa} = \begin{cases} 1, & \text{if } \alpha \equiv 0 \pmod{A} \\ 0, & \text{otherwise} \end{cases}. \tag{2.5}
\]
Two points \( x, y \in V^d \) are said to be congruent with respect to the lattice \( AZ^d \), if \( x - y \in AZ^d \), and we write \( x \equiv y \pmod{A} \) or \( x \equiv y \pmod{AZ^d} \). The following two theorems are the central results for the discrete Fourier transform.

**Theorem 2.4.** Let \( A, B \) and \( N \) be as in Definition 2.3. Then for any \( \kappa \in L^\perp_A \),
\[
\frac{1}{|\det(N)|} \sum_{\alpha \in \Lambda_N} e^{2\pi i \kappa \cdot \alpha} = \begin{cases} 1, & \text{if } \kappa \equiv 0 \pmod{B} \\ 0, & \text{otherwise} \end{cases}, \tag{2.4}
\]
and
\[
\frac{1}{|\det(N)|} \sum_{\kappa \in \Lambda_N^\perp} e^{-2\pi i \alpha \cdot \kappa} = \begin{cases} 1, & \text{if } \alpha \equiv 0 \pmod{A} \\ 0, & \text{otherwise} \end{cases}. \tag{2.5}
\]
Theorem 2.5. Let $A$, $B$ and $N$ be as in Definition 2.3. Define the discrete inner product
\[
\langle f, g \rangle_N = \frac{1}{|\det(N)|} \sum_{\alpha \in \Lambda_N} f(\alpha)g(\alpha)
\]
for $f, g \in C(\Omega_A)$, the space of continuous functions on $\Omega_A$. Then
\[
\langle f, g \rangle_N = \langle f, g \rangle_{(2.6)}
\]
for all $f, g$ in the finite dimensional subspace $H_N := \text{span}\{\phi_\kappa : \phi_\kappa(x) = e^{2\pi i \kappa \cdot x}, \kappa \in \Lambda_N^\perp\}$.

Let $|E|$ denote the cardinality of the set $E$. Setting $\kappa = 0$ or $\alpha = 0$ in (2.4) or (2.5), respectively, we see that
\[
|\Lambda_N| = |\Lambda_N^\perp| = |\det(N)|.
\]
In particular, the dimension of $H_N$ is $|\Lambda_N^\perp| = |\det(N)|$.

Let $I_N f$ denote the Fourier expansion of $f \in C(\Omega_A)$ in $H_N$ with respect to the inner product $\langle \cdot, \cdot \rangle_N$. Then, analogous to the sampling theorem in Proposition 2.2, $I_N f$ satisfies the following formula
\[
I_N f(x) = \sum_{\alpha \in \Lambda_N} f(\alpha)\Phi_{\Omega_B}^A(x - \alpha), \quad f \in C(\Omega_A),
\]
where
\[
\Phi_{\Omega_B}^A(x) = \frac{1}{|\det(N)|} \sum_{\kappa \in \Lambda_N^\perp} e^{2\pi i \kappa \cdot x}.
\]
The following theorem shows that $I_N f$ is an interpolation function.

Theorem 2.6. Let $A$, $B$ and $N$ be as in Definition 2.3. Then $I_N f$ is the unique interpolation operator on $N$ in $H_N$; that is
\[
I_N f(\alpha) = f(\alpha), \quad \forall \alpha \in \Lambda_N.
\]
In particular, $|\Lambda_N| = |\Lambda_N^\perp|$. Furthermore, the fundamental interpolation function $\Phi_{\Omega_B}^A$ satisfies
\[
(2.8) \quad \Phi_{\Omega_B}^A(x) = \sum_{\kappa \in \Lambda_A^\perp} \Phi_{\Omega_B}(x + \kappa).
\]
Proof. Equation (2.8) was proved in [18]. Using (2.5) gives immediately $\Psi_{\Omega_B}^A = \delta_{\alpha,0}$ for $\alpha \in \Lambda_N$, so that the interpolation holds. This also shows that $\{\Psi_{\Omega_B}^A(x - \alpha) : \alpha \in \Lambda_N\}$ is linearly independent. By (2.7), $|\Lambda_N| = |\det(N)| = |\Lambda_N^\perp|$. Furthermore, (2.4) and (2.5) show that the interpolation matrix $M = (\phi_\kappa(\alpha))_{\kappa \in \Lambda_A^\perp, \alpha \in \Lambda_N}$ is invertible. Consequently the interpolation on points in $\Lambda_N$ is unique. 

This theorem is stated in [18] under the additional requirement that $\Lambda_N^\perp = \Lambda_{Nv}$, which holds in particular in the case that $A$ is a constant multiple of $B$. We prove the more general version here for future references.

The results state in this subsection provide a framework for the discrete Fourier analysis on the spectral set of a lattice. We will apply it to the $d$-dimensional lattice associated with the group $A_d$ in the following section. As mentioned in the introduction, the case $d = 2$ and $d = 3$ have been considered in detail in [18] and [19], respectively. These lower dimensional cases provide a roadmap for the $d$-dimensional results.
3. **Discrete Fourier analysis on the fundamental domain**

3.1. **Lattice $A_d$ and Fourier analysis.** For $d \geq 1$, we identify $\mathbb{R}^d$ with the hyperplane

$$\mathbb{R}^{d+1}_H := \{(t_1, t_2, \ldots, t_{d+1}) \in \mathbb{R}^{d+1} : t_1 + t_2 + \cdots + t_{d+1} = 0\}$$

of $\mathbb{R}^{d+1}$. The lattice $A_d$ that we will consider in this paper is simply constructed by

$$\mathbb{Z}^{d+1}_H := \mathbb{Z}^{d+1} \cap \mathbb{R}^{d+1}_H = \{(k_1, k_2, \ldots, k_{d+1}) \in \mathbb{Z}^{d+1} : k_1 + k_2 + \cdots + k_{d+1} = 0\}.$$ 

In other words, we will use homogeneous coordinates $t \in \mathbb{R}^{d+1}_H$ to describe our results in $d$-variables and $A_d = \mathbb{Z}^{d+1}_H$ in our homogeneous coordinates.

Throughout this paper, we adopt the convention of using bold letters, such as $t$ and $k$, to denote homogeneous coordinates. The advantage of homogeneous coordinates lies in preservation of symmetry. In fact, many of our formulas are symmetric and more transparent under homogeneous coordinates [18, 19, 27, 28].

The lattice $A_d$ is the root lattice of the reflection group $A_d$ [6, Chapter 4]. Under homogeneous coordinates, the group $A_d$ is generated by the reflections $\{\sigma_{ij} : 1 \leq i < j \leq d+1\}$, where $\sigma_{ij}$ is defined by

$$t_{\sigma_{ij}} := t - 2\frac{(t, e_{i,j})}{(e_{i,j}, e_{i,j})}e_{i,j} = t - (t_i - t_j)e_{i,j}, \quad \text{where } e_{i,j} := e_i - e_j.$$ 

The last equation shows that $\sigma_{i,j}$ is a transposition. Thus, the group $A_d$ is exactly the permutation group $S_{d+1}$ of $d+1$ elements. Let $G := S_{d+1}$. For $t \in \mathbb{R}^{d+1}_H$ and $\sigma \in G$, the action of $\sigma$ on $t$ is denoted by $t\sigma$, which means permutation of the element in $t$ by $\sigma$.

The fundamental domain that tiles $\mathbb{R}^{d+1}_H$ with lattice $A_d$ is the spectral set of $A_d$,

$$\Omega_H := \{t \in \mathbb{R}^{d+1}_H : -1 < t_i - t_j \leq 1, 1 \leq i < j \leq d+1\}. \quad (3.1)$$

The strict inequality in the definition of $\Omega_H$ reflects our requirement in (2.3). In the case of $d = 2$ and $d = 3$, the fundamental domain are hexagon and rhombic dodecahedron, respectively, and they are depicted in Figures 3.1-3.2, in which vertices are labeled in homogeneous coordinates.

![Figure 3.1. Fundamental domain — the regular hexagon for the lattice $A_2$.](image)

As described in [6, Chapter 21], the spectral set $\overline{\Omega}_H$ is the union of its fundamental simplex $\triangle_H$, defined by

$$\triangle_H := \{t \in \mathbb{R}^{d+1}_H : 0 \leq t_i - t_j \leq 1, 1 \leq i < j \leq d+1\}. \quad (3.2)$$
Let \( t, t, \ldots, t \) be denoted by \( \{ t \}_k \). The \( d \) non-zero vertices of \( \triangle_H \) are given by
\[
\mathbf{v}_k := \left( \frac{d+1-k}{d+1}, \frac{-k}{d+1} \right), \quad 1 \leq k \leq d,
\]
where \( \mathbf{v}^k := \left( \{ d + 1 - k \}_k, \{ -k \}_d+1-d \right) \).

Let \( t\mathcal{G} \) denote the orbit of \( t \) under \( \mathcal{G} \), that is, \( t\mathcal{G} := \{ t\sigma : \sigma \in \mathcal{G} \} \). Then \( \Omega_H \) is the union of the images of \( \triangle_H \) under \( \mathcal{G} \), that is
\[
\Omega_H = \triangle_H \mathcal{G} := \bigcup_{t \in \triangle_H} t\mathcal{G} = \bigcup_{\sigma \in \mathcal{G}} \{ t\sigma : t \in \triangle_H \}.
\]

Furthermore, the partition is non-overlapping, i.e., for any \( t, s \in \triangle_H \) and \( t \neq s \), \( t\mathcal{G} \cap s\mathcal{G} = \emptyset \).

For any \( t \in \mathbb{R}^{d+1} \), the stabilizer of \( t \) is denoted by \( \mathcal{G}_t := \{ \sigma \in \mathcal{G} : t\sigma = t \} \). Since \( \mathcal{G} \) is finite, \( |\mathcal{G}_t| \times |t\mathcal{G}| = |\mathcal{G}| \). For the vertices of \( \triangle_H \), \( |\mathbf{v}_k\mathcal{G}| = \frac{(d+1)!}{k!(d+1-k)!} \) and \( |\mathcal{G}_t\mathbf{v}_k| = k!(d+1-k)! \). It is worthwhile to note that \( \Omega_H \) is the convex hull of the \( \sum_{k=1}^d |\mathbf{v}_k\mathcal{G}| = 2^{d+1} - 2 \) vertices in
\[
\left\{ \frac{\mathbf{v}_k\sigma}{d+1} : \sigma \in \mathcal{G}, 1 \leq k \leq d \right\}.
\]

The generator matrix of the lattice \( A_d \) is given by the \( (d+1) \times d \) matrix
\[
A := \begin{pmatrix}
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 \\
0 & 0 & \ldots & 0 & 1 \\
-1 & -1 & \ldots & -1 & -1
\end{pmatrix}
\]
and the generator matrix of the dual lattice $L_d^\perp$ is the $(d+1) \times d$ matrix

$$A^\perp = A(A^\top A)^{-1} = \frac{1}{d+1} \begin{pmatrix} d & -1 & \ldots & -1 & -1 \\ -1 & d & \ldots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \ldots & d & -1 \\ -1 & -1 & \ldots & -1 & d \end{pmatrix}. $$

To describe $L_d^\perp$ explicitly, we introduce the notation

$$\mathbb{H} := \{ k \in \mathbb{Z}^{d+1} : k_1 \equiv k_2 \equiv \cdots \equiv k_{d+1} \pmod{d+1} \}. $$

For $j \in \mathbb{Z}^d$ set $k = (d+1)A^\perp j$. It follows readily that $k \in \mathbb{Z}^{d+1}_H$ and a quick computation shows $k \in \mathbb{H}$. On the other kind, the definition of $A^\perp$ leads to $j = A^\top k/(d+1)$, which shows $j \in \mathbb{Z}^3$ if $k \in \mathbb{H}$. Consequently, the dual lattice is given by

$$L_d^\perp = \left\{ \frac{k}{d+1} : k \in \mathbb{H} \right\}. $$

For $e^{2\pi i k \cdot t}$ with $k \in L_d^\perp$, we introduce the new notation

$$(3.4) \quad \phi_j(t) := e^{\frac{2\pi i}{d+1} k \cdot t} \quad \text{for} \quad j \in \mathbb{H}. $$

By the Fuglede theorem, $\{ \phi_j : j \in \mathbb{H} \}$ forms an orthonormal basis in $L^2(\Omega_H)$; that is,

$$(3.5) \quad \langle \phi_k, \phi_j \rangle = \delta_{k,j}, \quad k,j \in \mathbb{H},$$

where the inner product is defined by

$$(3.6) \quad \langle f, g \rangle := \frac{1}{|\Omega_H|} \int_{\Omega_H} f(t)g(t)dt = \frac{1}{\sqrt{d+1}} \int_{\Omega_H} f(t)g(t)dt.$$

**Definition 3.1.** A function $f$ is $H$-periodic if it is periodic with respect to the lattice $A_d$; that is, $f(t) = f(t + k)$ for all $k \in \mathbb{Z}^{d+1}_H$.

Evidently, the functions $\phi_j(t)$ in (3.4) are $H$-periodic. Furthermore, (3.5) shows that an $H$-periodic function $f$ can be expanded into a Fourier series

$$(3.7) \quad f \sim \sum_{k \in \mathbb{H}} \hat{f}_k \phi_k(t), \quad \text{where} \quad \hat{f}_k := \frac{1}{\sqrt{d+1}} \int_{\Omega_H} f(t)\phi_{-k}(t)dt.$$

Recall that $s \equiv t \pmod{H}$ means $s - t \in \mathbb{Z}^{d+1}_H$. The following lemma will be useful later.

**Lemma 3.2.** If $t, s \in \Omega_H$ and $s \equiv t \pmod{H}$, then $t = s$.

**Proof.** If $t, s \in \Omega_H$ and $t \equiv s \pmod{H}$, then $s - t \in \mathbb{Z}^{d+1}_H$ and, set $k := s - t$, $\ 1 \leq k_i - k_j \leq 1$ for all $1 \leq i, j \leq d + 1$. The last condition means that either $k_i \in \{0, 1\}$ for all $1 \leq i \leq d + 1$ or $k_i \in \{0, -1\}$ for all $1 \leq i \leq d + 1$. The homogenous condition $\sum_{i=1}^{d+1} k_i = 0$ then shows that $k = 0$ or $s = t$. \[\square\]

### 3.2. Structure of the fundamental domain
In this subsection, we concentrate on the structure of $\Omega_H$. For this purpose, we set $\mathbb{N}_{d+1} := \{1, 2, \ldots, d+1\}$ and start with the observation that $\Omega_H$ can be partitioned into $d + 1$ congruent parts.

**Lemma 3.3.** For $1 \leq j \leq d + 1$ define

$$\Omega_H^{(j)} := \{ t \in \mathbb{R}^{d+1}_H : 0 < t_i - t_j \leq 1 \ \text{and} \ 0 \leq t_i - t_j < 1 \ \text{for} \ 1 \leq i < j < l \leq d + 1 \}. $$

Then

$$\Omega_H = \bigcup_{1 \leq j \leq d+1} \Omega_H^{(j)} \quad \text{and} \quad \Omega_H^{(i)} \cap \Omega_H^{(j)} = \emptyset \quad \text{for} \ 1 \leq i \neq j \leq d+1. $$
Proof. For \( t \in \Omega_H \), let \( J = \{ k \in \mathbb{N}_{d+1} : t_k \leq t_i \text{ for } i \in \mathbb{N}_{d+1} \} \) and \( j = \min_{k \in J} k \). The definition shows that \( t_i - t_j \geq 0 \) for all \( i \in \mathbb{N}_{d+1} \); furthermore, if \( i < j < k \) then, as \( t \in \Omega_H \), \( 0 < t_i - t_j \leq 1 \) and \(-1 < t_j - t_k \leq 0 \), so that \( t \in \Omega_H^{(j)} \), which implies that \( \Omega_H \subseteq \bigcup_{j \in \mathbb{N}_{d+1}} \Omega_H^{(j)} \). Since \( \Omega_H^{(j)} \subset \Omega_H \) for each \( j \in \mathbb{N}_{d+1} \), it follows that \( \Omega_H = \bigcup_{j \in \mathbb{N}_{d+1}} \Omega_H^{(j)} \).

If \( t \in \Omega_H^{(i)} \cap \Omega_H^{(j)} \) with \( 1 \leq i < j \leq d+1 \), then \( t \in \Omega_H^{(i)} \) so that \( 0 \leq t_j - t_i \) and \( t \in \Omega_H^{(j)} \) so that \( 0 \leq t_i - t_j \). As a consequence, \( t_j < t_i \leq t_j \), which implies \( \Omega_H^{(i)} \cap \Omega_H^{(j)} = \emptyset \). \( \square \)

For \( d = 2 \), the partition of the hexagon is evident from Figure 3.1 the sets \( \Omega^{(1)}_H, \Omega^{(2)}_H, \Omega^{(3)}_H \) are the three parallelograms inside the hexagon, starting from the one in the upper left rotating clockwise. In the case \( d = 3 \), the rhombic dodecahedron is more complicated (see Figure 3.2), its decomposition is depicted in the Figure 3.3.

![Figure 3.3. Rhombohedral partition of the rhombic dodecahedron. Upper-Left: \( \Omega^{(1)}_H \); Upper-Right: \( \Omega^{(2)}_H \); Lower-Left: \( \Omega^{(3)}_H \); Lower-Right: \( \Omega^{(4)}_H \).](image)

The next lemma clarifies the intersection of the closure of \( \Omega^{(j)}_H \), for which we need to define \( \overline{\Omega}^{(j)}_H \) for a set \( J \subseteq \mathbb{N}_{d+1} \).

**Lemma 3.4.** For \( \emptyset \subset J \subseteq \mathbb{N}_{d+1} \), define

\[
\overline{\Omega}^{(j)}_H := \{ t \in \mathbb{R}^{d+1} : t_i = t_j, \forall i, j \in J; \text{ and } 0 \leq t_i - t_j \leq 1, \forall j \in J, \forall i \in \mathbb{N}_{d+1} \setminus J \}.
\]

Then

\[
\overline{\Omega}_H = \bigcup_{j \in \mathbb{N}_{d+1}} \overline{\Omega}^{(j)}_H \quad \text{and} \quad \overline{\Omega}'_H = \bigcap_{j \in J} \overline{\Omega}^{(j)}_H.
\]

**Proof.** The first equation follows easily from the previous lemma by taking closure.

If \( t \in \overline{\Omega}^{(j)}_H \cap \overline{\Omega}^{(j)}_H \), then \( 0 \leq t_i - t_j \leq 0 \); that is, \( t_i = t_j \). Hence, if \( t \in \bigcap_{j \in J} \overline{\Omega}^{(j)}_H \) then \( t_i = t_j \), \( \forall i, j \in J \), which implies \( \bigcap_{j \in J} \overline{\Omega}^{(j)}_H \subseteq \overline{\Omega}'_H \). Since \( \overline{\Omega}'_H \subset \overline{\Omega}^{(j)}_H \) whenever \( j \in J \) by definition, we conclude that \( \overline{\Omega}'_H = \bigcap_{j \in J} \overline{\Omega}^{(j)}_H \). \( \square \)
3.3. Boundary of the fundamental domain. In order to carry out the discrete Fourier analysis on the fundamental domain $\Omega_H$, we need to have a detailed knowledge of the boundary of the region. Much of this subsection is parallel to the study in [19], where the case $d = 3$ is discussed in detail. Some of the proofs, in fact, are essentially the same as in the case of $d = 3$ and often only minor adjustment is needed. In such cases, we shall point out the necessary adjustment and refer the proof to [19].

We use the standard set theoretic notations $\partial \Omega$, $\Omega^o$ and $\bar{\Omega}$ to denote the boundary, the interior and the closure of $\Omega$, respectively. Clearly $\bar{\Omega} = \Omega^o \cup \partial \Omega$. For $i, j \in \mathbb{N}_{d+1}$ and $i \neq j$, define

$$F_{i,j} = \{ t \in \bar{\Omega}_H : t_i = t_j = 1 \}.$$  

There are a total $d(d+1)$ distinct $F_{i,j}$, each stands for one facet of $d-1$ dimension, together with its boundary, of the fundamental domain.

The boundary of our $d$-dimensional polytope consists of lower dimensional sets, which can be obtained from the intersections of $d-1$ dimensional facets. For example, for $d = 3$, the boundary consists of faces, edges, and vertices; edges are intersections of faces and vertices are intersection of edges. In order to describe the intersections of facets, we define, for nonempty subsets $I, J$ of $\mathbb{N}_{d+1}$,

$$\Omega_{I,J} := \bigcap_{i \in I, j \in J} F_{i,j} = \{ t \in \bar{\Omega}_H : t_j = t_i - 1 \ for \ all \ i \in I, j \in J \}.$$

The main properties of these sets are summarized in the following lemma, the proof of which is given in [19].

**Lemma 3.5.** Let $I, J, I_1, J_1$ be nonempty subsets of $\mathbb{N}_{d+1}$. Then

(i) $\Omega_{I,J} = \emptyset$ if and only if $I \cap J \neq \emptyset$.

(ii) $\Omega_{I_1,J_1} \cap \Omega_{I_2,J_2} = \Omega_{I,J}$ if $I_1 \cup I_2 = I$ and $J_1 \cup J_2 = J$.

To carry out a discrete Fourier analysis, we need to distinguish points on the boundary elements of different dimensions. It is also necessary to distinguish a closed boundary element and an open one. For example, for $d = 3$, we will distinguish a face with its boundary edges and a face without its edges. To make these more precise, we introduce the notation

$$\mathcal{K} = \{ (I, J) : I, J \subset \mathbb{N}_{d+1} ; I \cap J = \emptyset \}, \quad \mathcal{K}_0 = \{ (I, J) \in \mathcal{K} : i < j, \ for \ all \ (i,j) \in I \times J \}.$$

**Definition 3.6.** For $(I,J) \in \mathcal{K}$, the boundary element $B_{I,J}$ of the fundamental domain,

$$B_{I,J} := \{ t \in \Omega_{I,J} : t \notin \Omega_{I_1,J_1} \ for \ all \ (I_1, J_1) \in \mathcal{K} \ with \ |I| + |J| < |I_1| + |J_1| \},$$

is called a face of dimension $k$, or simply a $k$-face if $k = d + 1 - |I| - |J|$. In particular, it is called a facet if $k = d - 1$ (or $|I| + |J| = 2$), a ridge if $k = d - 2$ ($|I| + |J| = 3$), a face if $k = 2$ ($|I| + |J| = d - 1$), an edge if $k = 1$ ($|I| + |J| = d$) and a vertex if $k = 0$ ($|I| + |J| = d + 1$).

Figure 6.3 shows a 3-face, or a facet, of the fundamental domain when $d = 4$.

In the following, when we say a $k$-face we mean the open set, that is, without any of its boundaries. For $k$-faces with $k \geq 1$, the boundary elements $B_{I,J}$ represent the interiors. In fact, it is easy to see that $B_{(t_i), (t_j)} = F_{i,j}^o$ for distinct integers $i, j \in \mathbb{N}_{d+1}$ and $B_{I,J} = \Omega_{I,J}^o$ for $(I,J) \in \mathcal{K}$. By the definition of $\Omega_H$ and $\Omega_{I,J}$, we can write $B_{I,J}$ more explicitly as

$$B_{I,J} = \{ t \in \mathbb{R}_{d+1}^+: t_i > t_j > t_j = t_i - 1 \ for \ all \ i \in I, j \in J \ and \ l \in \mathbb{N}_{d+1} \setminus (I \cup J) \}.$$
Proposition 3.7. Let \((I, J)\) ∈ \(K\) and \((I_1, J_1)\) ∈ \(K\).

(i) \(B_{I,J} \cap B_{I_1,J_1} = \emptyset\), if \(I \neq I_1\) and \(J \neq J_1\).

(ii) \(\Omega_H \setminus \Omega_0^H = \bigcup_{(I,J)\in K} B_{I,J} = \bigcup_{0<i,j<i+j\leq d+1} B^{i,j}\).

(iii) \(\Omega_H \setminus \Omega_0^H = \bigcup_{(I,J)\in K_0} B_{I,J} = \bigcup_{0<i,j<i+j\leq d+1} B^{i,j}_0\).

The proof of this proposition is essentially the same as the one given in [19] for \(d = 3\).

The boundary elements can also be described by using symmetry. In fact, boundary elements of the same type can be transformed by the group \(G = S_{d+1}\). To make it more precise, we define, for \((I, J)\) ∈ \(K\) and \(\sigma \in G\),

\[B_{I,J}\sigma := \{t\sigma : t \in B_{I,J}\}.\]

Then it follows readily that

\[B^{(|I|,|J|)} = \bigcup_{t\in G} B_{I,J}\sigma := \{t\sigma : t \in B_{I,J}, \sigma \in G\}.\]

Recall that \(\sigma_{ij} = t - (t_i - t_j)e_{i,j}\) denotes the transposition in \(G\) that interchanges \(i\) and \(j\). We clearly have \(\sigma_{ij} = \sigma_{ji}\) and \(\sigma_{jj}\) is the identity element of the group. For a nonempty set \(I \subset \mathbb{N}_{d+1}\), define \(G_I := \{\sigma_{ij} : i, j \in I\}\). It is easy to verify that \(G_I\) forms a subgroup of \(G = S_{d+1}\) of order \(|I|\).

Lemma 3.8. Let \(#B^{i,j}\) and \(#B^{i,j}_0\) denote the number of distinct \((d+1-i-j)\)-faces contained in \(B^{i,j}\) and \(B^{i,j}_0\), respectively. Then

\[\#B^{i,j} = \frac{(d+1)!}{i!j!(d+1-i-j)!} \quad \text{and} \quad \#B^{i,j}_0 = \frac{(d+1)!}{(i+j)!(d+1-i-j)!}.\]
Proof. By definition, \( B^{i,j} \) and \( B_{ij}^{i,j} \) are unions of \((d+1-i-j)\)-faces of the fundamental domain. The first formula follows from the fact that \( B_{I,J} \sigma = B_{I,J} \) if \( \sigma \in \mathcal{G}_I \cup \mathcal{G}_J \cup \mathcal{G}_{I+1} \setminus \{(I,J)\} \). The second one follows from the fact that, for any \( K \subseteq \mathbb{N}_{d+1} \) and \( i,j \geq 1 \) such that \(|K| = i + j\), there is a unique \((I,J) \in \mathcal{K}^{i,j}_0 \) such that \( I \cup J = K \).

Using these formulas, we can derive by setting \( k = d+1-i-j \) that the fundamental domain \( \Omega_H \) has a total

\[
\sum_{j=1}^{d-k} \frac{(d+1)!}{j!(d+1-j-k)!} = (2^{d+1-k} - 2) \binom{d+1}{k}, \quad 0 \leq k \leq d-1,
\]
distinct \( k \)-faces, while \( \Omega_H \) has a total \((d-k)\binom{d+1}{k}\) distinct \( k \)-faces.

For the periodic functions, we will need to consider points on the boundary elements that are congruent modulus \( \mathbb{Z}_H^{d+1} \). For \((I,J) \subset \mathcal{K}\) we further define

\[
[B_{I,J}] := \{ B_{I,J} + k : k \in \mathbb{Z}_H^{d+1} \} \cap \Omega_H = \{ t + k \in \Omega_H : t \in B_{I,J}, k \in \mathbb{Z}_H^{d+1} \}.
\]

By lemma \ref{lemma:boundary}, the set \([B_{I,J}]\) consists of exactly those boundary elements that can be obtained from \( B_{I,J} \) by congruent modulus \( \mathbb{Z}_H^{d+1} \). As an example, in the case of \( d = 3 \), we have

\[
B_{\{1\},\{2,3\}} = \{ (t, t-1, t-1, 2t) : \frac{1}{2} < t < \frac{3}{4} \},
B_{\{1,2\},\{3\}} = \{ (1-t, 1-t, -t, 3t-2) : \frac{1}{2} < t < \frac{3}{4} \},
\]

and from this explicit description we can deduce, for example,

\[
[B_{\{1\},\{2,3\}}] = B_{\{1\},\{2,3\}} \cup (B_{\{1\},\{2,3\}} + (-1, 1, 0, 0)) \cup (B_{\{1\},\{2,3\}} + (-1, 0, 1, 0))
= B_{\{1\},\{2,3\}} \cup B_{\{2\},\{1,3\}} \cup B_{\{3\},\{1,2\}}.
\]

The last equation indicates that \([B_{I,J}]\) is a union of \( B_{I',J'} \), which is stated and proved in \cite{19} for \( d = 3 \) and the proof can be adopted with obvious replacement of \( d = 3 \) by general \( d \).

**Lemma 3.9.** Let \((I,J) \in \mathcal{K}\). Then

\[
[B_{I,J}] = \bigcup_{\sigma \in \mathcal{G}_{i,j}} B_{I,J} \sigma.
\]

Since \( \mathcal{K}^{i,j} \) can be obtained from \( \mathcal{K}^{i,j}_0 \) from the action of \( \mathcal{G} \), it follows that

\[
B^{i,j} = \bigcup_{(I,J) \in \mathcal{K}^{i,j}_0} [B_{I,J}] = \bigcup_{B \in \mathcal{K}^{i,j}_0} [B], \quad 0 < i, j < i + j \leq d + 1.
\]

We also note that \([B_{I,J}] \cap [B_{I',J'}] = \emptyset \) if \((I,J) \neq (I',J')\) for \((I,J) \in \mathcal{K}_0\) and \((I',J') \in \mathcal{K}_0\), which shows that \([B_{i,j}] \) is a non-overlapping partition.

To illustrate the above partitions, let us consider the case \( d = 2 \), for which

\[
B^{1,1} = [B_{\{1\},\{2\}}] \cup [B_{\{1\},\{3\}}] \cup [B_{\{2\},\{3\}}],
B^{1,2} = [B_{\{1\},\{2,3\}}],
B^{2,1} = [B_{\{1,2\},\{3\}}],
\]

where

\[
B_{\{1\},\{2\}} = \{ (1-t, 2t-1, -t) : \frac{1}{4} < t < \frac{2}{3} \},
B_{\{1\},\{3\}} = B_{\{1\},\{2\}} \sigma_{2,3},
B_{\{2\},\{3\}} = B_{\{1\},\{2\}} \sigma_{1,2},
B_{\{1,2\},\{3\}} = \{ \left( \frac{t}{2}, -\frac{3-2t}{2}, \frac{1}{2}, -\frac{1}{2} \right) \},
B_{\{1,2\},\{3\}} = \{ \left( \frac{1}{2}, -\frac{3-2t}{2}, \frac{1}{2}, -\frac{1}{2} \right) \}.
\]

In the case \( d = 3 \), the boundary elements are given explicitly in \cite{19}.
3.4. Point sets for discrete Fourier analysis for the lattice $A_d$. We apply the general result on discrete Fourier analysis in Section 2 to the lattice $A_d$ by choosing $A = A$ and $B = nA$ with $n$ being a positive integer. Let $\mathbb{1} = \{\{1\}\}$. Then the matrix

$$N := B^t A = nI + n\mathbb{1}^t \mathbb{1} = \begin{pmatrix} 2n & n & \ldots & n & n \\ n & 2n & \ldots & n & n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ n & n & \ldots & 2n & n \\ n & n & \ldots & n & 2n \end{pmatrix}$$

has integer entries. Since $N$ is now a symmetric matrix, $\Lambda_N = \Lambda_N^t$; moreover, it is easy to see that $\Lambda_N^\dagger = \Lambda_N$. We denote $\Lambda_N$ by $H_n$, which is given by

$$H_n := \{k \in \mathbb{H} : k \in \mathbb{H} : -n < \frac{k_i - k_j}{d+1} \leq n, \; 1 \leq i < j \leq d+1\}.$$

The finite dimensional space $\mathcal{H}_n$ of exponentials in Theorem 2.5 becomes

$$\mathcal{H}_n := \text{span} \{\phi_k : k \in \mathbb{H}_n\} \quad \text{with} \quad \dim \mathcal{H}_n = \det(N) = (d+1)n^d.$$

The points in $\mathbb{H}_n$ are not symmetric under $G$, since points on half of the boundary are not included. We further define the symmetric extension of $\mathbb{H}_n$ by

$$\mathbb{H}_n^* := \{k \in \mathbb{H} : \frac{k}{(d+1)n} \in \Omega_H \} = \{k \in \mathbb{H} : -n \leq \frac{k_i - k_j}{d+1} \leq n, \; 1 \leq i < j \leq d+1\}.$$

For $d = 2$ and $3$, the scope of $\mathbb{H}_n^*$ is depicted in Figure 3.5 and Figure 3.6, in which the vertices are labeled in homogeneous coordinates.

![Figure 3.5](image-url)  

**Figure 3.5.** $\mathbb{H}_n^*$ for $d = 2$.

For discrete Fourier analysis, it is essential to understand the sets $\mathbb{H}_n$ and $\mathbb{H}_n^*$, for which the main task lies in studying the structure of points on the boundary; in other words, we need to understand exactly when $\frac{k}{(d+1)n}$ belongs to a specific boundary elements of $\Omega_H$. Much of the work in the previous subsection is a preparation for this task. In the rest of this subsection, we develop results in this regard.

We start with the partition of $\mathbb{H}_n$ and $\mathbb{H}_n^*$ into $d+1$ congruent parts, each within a parallelepiped, corresponding to the partition of $\Omega_H$ and $\Omega_H^*$, as shown in Figures 3.3 for $d = 3$.

**Lemma 3.10.** For $1 \leq j \leq d+1$, define

$$H_n^{(j)} = \{k \in \mathbb{H} : 1 \leq \frac{k_i - k_j}{d+1} \leq n \text{ and } 0 \leq \frac{k_i - k_j}{d+1} \leq n-1 \text{ for } 1 \leq i < j < l \leq d+1\}.$$
Then

\[ H_n = \bigcup_{j \in \mathbb{N}_{d+1}} H_n^{(j)} \quad \text{and} \quad H_n^{(i)} \cap H_n^{(j)} = \emptyset \quad \text{for} \ 1 \leq i \neq j \leq d + 1. \]

Proof. By definition, \( H_n^{(j)} = \{ k \in H : \frac{k}{(d+1)n} \in \Omega_H^{(j)} \} \). Hence, this lemma is an immediate consequence of Lemma 3.3. \( \square \)

Similarly, as a consequence of Lemma 3.4, we have the following lemma.

Lemma 3.11. For \( \emptyset \subset J \subseteq \mathbb{N}_{d+1} \), define

\[ H_n^J := \{ k \in H : k_i = k_j, \ \forall i, j \in J; \ and \ 0 \leq k_i - k_j \leq (d+1)n, \ \forall j \in J, \ \forall i \in \mathbb{N}_{d+1} \setminus J \}. \]

Then

\[ H_n^* = \bigcup_{j \in \mathbb{N}_{d+1}} H_n^{(j)} \quad \text{and} \quad H_n^{*} = \bigcap_{j \in J} H_n^{(j)}. \]

Next we consider further partitions of \( H_n^* \). The set of interior points is defined by

\[ H_n^o := \{ j \in H : \frac{j}{(d+1)n} \in \Omega_H^o \}. \]

The set of points on the boundary of \( H_n^* \) is then \( H_n^* \setminus H_n^o \). Recall (3.9), we define, for \( 0 < i, j < i + j \leq d + 1 \),

\[ H_n^{i,j} := \{ k \in H : \frac{k}{(d+1)n} \in B^{i,j} \}, \quad H_n^{i,j,0} := \{ k \in H : \frac{k}{(d+1)n} \in B_0^{i,j} \}. \]

The index set \( H_n^{i,j} \) consists of those points \( j \) in \( H_n \) such that \( \frac{j}{(d+1)n} \) are in the boundary element \( B^{i,j} \) of \( \partial \Omega_H \). Using Proposition 3.7, it is easy to see that \( H_n^{i,j} \cap H_n^{i,j,1} = \emptyset \) if \( i \neq i_1 \) or \( j \neq j_1 \).

Lemma 3.12. For \( n \geq 1, \ 0 < i, j < i + j \leq d + 1 \),

\[ |H_n^o| = n^{d+1} - (n-1)^{d+1}, \quad |H_n^{i,j}| = \frac{(d+1)!}{i!j!(d+1-i-j)!} (n-1)^{d+1-i-j}. \]
Theorem 3.13. For \( n \geq 0 \),

\[
D_n^H(t) = \Theta_{n+1}(t) - \Theta_n(t), \quad \text{where} \quad \Theta_n(t) := \prod_{j=1}^{d+1} \frac{\sin \frac{\pi nt_j}{2^j\pi}}{\sin \frac{\pi t_j}{2^j\pi}}.
\]

This theorem is stated and proved in \[19\] for \( d = 3 \) and the proof carries over to general \( d \) with the obvious replacement of \( d + 1 = 4 \) by general \( d + 1 \). Recently a different derivation of this formula is given in \[30\].

As one consequence of the compact expression in (3.18), we obtain

\[
|H_n^*| = D_n^H(0) = (n + 1)^{d+1} - n^{d+1}.
\]

Another result that follows from the compact formula of the Dirichlet kernel is an upper bound for the Lebesgue constant, which is the norm of the partial sum \( S_n f \) in (3.16). Let \( \|f\|_\infty \) denote the uniform norm of \( f \in C(\overline{\Omega_H}) \) and let \( \|S_n\|_\infty \) denote the operator norm of \( S_n : C(\overline{\Omega_H}) \to C(\overline{\Omega_H}) \).

Theorem 3.14. There are positive constants \( c \) and \( C \) independent of \( f \) and \( n \) such that

\[
c(\log n)^d \leq \|S_n\|_\infty \leq C(\log n)^d.
\]

The upper bound \( \|S_n\|_\infty \leq C(\log n)^d \) was proved in \[19\] for \( d = 3 \), the proof extends to the general \( d \). It turns out, however, the upper bound can be derived from a general result in \[24\], which establishes the inequality

\[
\int_{[-\pi, \pi]^n} \left| \sum_{k, l \in E} e^{ikx} \right| dx \leq C(\log n)^d.
\]
for $E$ being a polyhedron in $\mathbb{R}^d$. Indeed, choosing a constant $\alpha$, if necessary, the set $\alpha \Omega$ can be imbedded inside $[-\pi, \pi]^d$ and $\alpha$ can be absorbed into the index set, so that (3.20) follows from the general result. The lower bound that matches (3.21) was established in [31] for $E$ being a convex polyhedron in $\mathbb{R}^d$. Upon choosing an appropriate dilation constant $\alpha$ so that $\alpha \Omega$ contains $[-\pi, \pi]^d$, the lower bound in (3.20) follows.

3.6. **Discrete Fourier analysis on the fundamental domain.** For the lattice $A_d$, the general result on the discrete inner product, Theorem 2.5, gives the following proposition in homogeneous coordinates:

**Proposition 3.15.** For $n \geq 0$, define

$$\langle f, g \rangle_n := \frac{1}{(d + 1)n^d} \sum_{j \in \mathbb{H}_n} f\left(\frac{j}{(d + 1)n}\right)g\left(\frac{j}{(d + 1)n}\right), \quad f, g \in C(\Omega_H).$$

Then

$$\langle f, g \rangle = \langle f, g \rangle_n, \quad f, g \in \mathcal{H}_n.$$

The inner product $\langle \cdot, \cdot \rangle_n$ is defined as a sum over the index set $\mathbb{H}_n$, which is not symmetric over $\Omega_H$ as only part of the points in the boundary of $\Omega_H$ are accounted for. More interesting to us is to consider an inner product based on the symmetric index set $\mathbb{H}_n^*$, which turns out to be equivalent to $\langle \cdot, \cdot \rangle_n$ for $H$-periodic functions.

**Definition 3.16.** For $n \geq 0$ define the symmetric discrete inner product

$$\langle f, g \rangle_n^* := \frac{1}{(d + 1)n^d} \sum_{j \in \mathbb{H}_n^*} c_j^{(n)} f\left(\frac{j}{(d + 1)n}\right)g\left(\frac{j}{(d + 1)n}\right), \quad f, g \in C(\Omega_H),$$

where $c_j^{(n)} = 1$ if $j \in \mathbb{H}_n^*$, and $c_j^{(n)} = \frac{1}{(\mathbb{H}_n^*)^j}$ if $j \in \mathbb{H}_n^*$. For instance, if $d = 2$ then $\Omega_H$ is a regular hexagon and we have

$$c_j^{(n)} = \begin{cases} 1, & j \in \mathbb{H}_n^0, \\
\frac{1}{2}, & j \in \mathbb{H}_n^1, \\
\frac{1}{4}, & j \in \mathbb{H}_n^{1,2} \cup \mathbb{H}_n^{2,1}, \\
\frac{1}{8}, & j \in \mathbb{H}_n^{1,3} \cup \mathbb{H}_n^{3,1}, \\
\frac{1}{16}, & j \in \mathbb{H}_n^{2,2}, \\
\end{cases}$$

if $d = 3$, then $\Omega_H$ is the rhombic dodecahedron and we have

$$c_j^{(n)} = \begin{cases} 1, & j \in \mathbb{H}_n^0, \\
\frac{1}{2}, & j \in \mathbb{H}_n^1, \\
\frac{1}{4}, & j \in \mathbb{H}_n^{2,1} \cup \mathbb{H}_n^{1,2}, \\
\frac{1}{8}, & j \in \mathbb{H}_n^{3,1} \cup \mathbb{H}_n^{1,3}, \\
\frac{1}{16}, & j \in \mathbb{H}_n^{2,2}, \\
\end{cases}$$

It is easy to verify that

$$\sum_{j \in \mathbb{H}_n^* \setminus \mathbb{H}_n^*} c_j^{(n)} = \sum_{0 < i, j < i + j \leq d + 1} \frac{|\mathbb{H}_n^{i, j}|}{(\mathbb{H}_n^*)^j} = \sum_{0 < i, j < i + j \leq d + 1} \frac{d + 1}{i + j} (n - 1)^{d + 1 - i - j} = \sum_{k = 2}^{d + 1} (k - 1) \binom{d + 1}{k} (n - 1)^{d + 1 - k}$$

$$= (d + 1) \sum_{k = 1}^{d} \binom{d}{k} (n - 1)^{d - k} \sum_{k = 2}^{d + 1} \binom{d + 1}{k} (n - 1)^{d + 1 - k}$$

$$= (d + 1) (n^d - (n - 1)^d - (n^{d + 1} - (n - 1)^{d + 1} - (d + 1)(n - 1)^d)$$

$$= (d + 1)n^d - n^{d + 1} + (n - 1)^{d + 1},$$
so that \((1,1)_n = \frac{1}{(d+1)n^d} (|\mathbb{H}_n^0| + (d + 1)n^{d+1} - n^{d+1} + (n-1)^{d+1}) = 1\) by (3.19).

**Theorem 3.17.** For \(n \geq 0\),

\[
\langle f, g \rangle = \langle f, g \rangle_n = \langle f, g \rangle_n^*, \quad f, g \in \mathcal{H}_n.
\]

**Proof.** Let \(f\) be an \(H\)-periodic function. Then

\[
\sum_{j \in \mathbb{H}_n^0} c_j^{(n)} f \left( \frac{j}{(d+1)n} \right) = \sum_{0 < i, k < i+k \leq d+1} \frac{1}{\binom{i+k}{i}} \sum_{j \in \mathbb{H}_n^0} f \left( \frac{j}{(d+1)n} \right)
\]

\[
= \sum_{0 < i, k < i+k \leq d+1} \frac{1}{\binom{i+k}{i}} \sum_{j \in \mathbb{H}_n^0} f \left( \frac{k}{(d+1)n} \right).
\]

Since \(|S_j| = \binom{i+k}{i}\) for \(j \in \mathbb{H}_n^0\), using the invariance of \(f\), we then conclude that

\[
\sum_{j \in \mathbb{H}_n^0} c_j^{(n)} f \left( \frac{j}{(d+1)n} \right) = \sum_{0 < i, k < i+k \leq d+1} \frac{1}{\binom{i+k}{i}} \sum_{j \in \mathbb{H}_n^0} \left( \binom{i+k}{i} \right) f \left( \frac{j}{(d+1)n} \right) = \sum_{j \in \mathbb{H}_n^0} f \left( \frac{j}{(d+1)n} \right).
\]

Since \(c_j^{(n)} = 1\) if \(j \in \mathbb{H}_n^0\), the proof is completed. \(\square\)

Since Theorem 3.17 shows that the integral of \(f\) over \(\mathcal{H}_n\) agrees with the discrete sum over \(\mathbb{H}_n\), it is not surprising that we have a cubature formula, which turns out to have a high degree of precision. To be more precise, we define by \(\mathcal{T}_n\) the space of generalized trigonometric polynomials,

\[
\mathcal{T}_n := \text{span} \{ \phi_k : k \in \mathbb{H}_n^* \}.
\]

**Theorem 3.18.** For \(n \geq 0\), the cubature formula

\[
\frac{1}{\sqrt{d+1}} \int_{\Omega} f(t) dt = \frac{1}{(d+1)n^d} \sum_{j \in \mathbb{H}_n^0} c_j^{(n)} f \left( \frac{j}{(d+1)n} \right)
\]

is exact for all \(f \in \mathcal{T}_{2n-1}\).

**Proof.** It suffices to prove that (3.22) is exact for \(f(t) = \phi_k(t)\) with any \(k \in \mathbb{H}^*_{2n-1}\). Since \(\Omega_H\) tiles \(\mathbb{R}^{d+1}_H\) with the lattice \(A_d\), there exist \(s \in \Omega_H\) and \(l \in \mathbb{Z}^{d+1}_H\) such that \(\frac{(k+l)}{(d+1)n} = s + l\).

Noting that \(k \in \mathbb{H}^*_{2n-1}\) and \(l \in \mathbb{Z}^{d+1}_H\) are vectors with integer entries, we further deduce that \(s = \frac{m}{(d+1)n}\) for certain \(m \in \mathbb{H}_n\), which states that \(k = m + (d+1)nl\) with certain \(m \in \mathbb{H}_n\) and \(l \in \mathbb{Z}^{d+1}_H\). Next we show that \(m = 0\) only if \(k = 0\). In fact \(k = (d+1)nI \in \mathbb{H}^*_{2n-1}\) clearly states that \(\frac{nl}{2n-1} \in \Omega_H\). Then the restriction \(\frac{1}{2} - 2 < l_i - l_j \leq 2 - \frac{1}{n}\) for \(1 \leq i < j \leq d + 1\) and the homogeneity of \(l\) imply that \(l = 0\), which gives \(k = 0\) in return. Now applying the periodicity of \(\phi_\delta\), Theorem 3.17 and (3.5) yields that

\[
\frac{1}{(d+1)n^d} \sum_{j \in \mathbb{H}_n^0} c_j^{(n)} \phi_k \left( \frac{j}{(d+1)n} \right) = \frac{1}{(d+1)n^d} \sum_{j \in \mathbb{H}_n^0} c_j^{(n)} \phi_m \left( \frac{j}{(d+1)n} \right)
\]

\[
= \langle \phi_m, \phi_0 \rangle = \delta_{m,0} = \delta_{k,0} = \frac{1}{\sqrt{d+1}} \int_{\Omega} \phi_k(t) dt.
\]

This completes the proof. \(\square\)

The cubature (3.22) is an analogue of the classical quadrature formula for trigonometric polynomials of one variable (see, for example, [34, Vol. II, p. 8]), which holds for trigonometric polynomials of one variable and has vast applications.
3.7. Interpolation on the fundamental domain. For the fundamental domain of \(A_d\), the general result on the interpolation, Theorem 2.6, becomes the following:

**Proposition 3.19.** For \(n > 0\), define

\[
I_n f(t) := \sum_{j \in \mathbb{H}_n^*} f\left(\frac{j}{(d+1)n}\right)\Phi_n(t - \frac{j}{(d+1)n}),
\]

where \(\Phi_n(t) = \frac{1}{(d+1)n^d} \sum_{k \in \mathbb{H}_n} \phi_k(t)\), for \(f \in C(\Omega_H)\). Then \(I_n f \in \mathcal{H}_n\) and

\[
I_n f\left(\frac{j}{(d+1)n}\right) = f\left(\frac{j}{(d+1)n}\right), \quad \forall j \in \mathbb{H}_n.
\]

The function \(I_n f\) interpolates on points in \(\mathbb{H}_n\), which is not symmetric as noted before. We are more interested in interpolation on all points in \(\mathbb{H}_n^*\). The operator \(I_n^*\) that we are able to define, however, does not interpolate at all points in \(\mathbb{H}_n^*\). On the other hand, \(I_n^* f\) can be used to derive an truly interpolation function for points on the fundamental simplex, which will be developed in the next section. Recall \(S_k\) defined in (3.15).

**Theorem 3.20.** For \(n \geq 0\) and \(f \in C(\Omega_H)\), define

\[
I_n^* f(t) := \sum_{j \in \mathbb{H}_n^*} f\left(\frac{j}{(d+1)n}\right)\ell_{j,n}(t),
\]

where

\[
\ell_{j,n}(t) = \Phi^*_n(t - \frac{j}{(d+1)n}) \quad \text{and} \quad \Phi^*_n(t) = \frac{1}{(d+1)n^d} \sum_{k \in \mathbb{H}_n^*} c_{k}^{(n)} \phi_k(t).
\]

Then \(I_n^* f \in \mathcal{I}_n\) and it satisfies

\[
I_n^* f\left(\frac{j}{(d+1)n}\right) = \begin{cases} f\left(\frac{j}{(d+1)n}\right), & j \in \mathbb{H}_n^0, \\ \sum_{k \in S_j} f\left(\frac{k}{(d+1)n}\right), & j \in \mathbb{H}_n^* \setminus \mathbb{H}_n^0, \end{cases}
\]

(3.23)

Furthermore, \(\Phi^*_n(t)\) is a real function and it satisfies

\[
\Phi^*_n(t) = \frac{1}{(d+1)n^d} \sum_{j=1}^{d+1} \left( \prod_{i \neq j} \sin \pi n t_i / \sin \pi t_i \right) \cos \pi n t_j \sum_{l \subseteq \mathbb{N}^{(d+1)}_{d+1}} |l|! |(d - |l|)!| \cos \pi (t_j + 2 \sum_{i \in l} t_i),
\]

(3.24)

where \(\mathbb{N}^{(d+1)}_{d+1} := \mathbb{N}_{d+1} \setminus \{j\}\).

**Proof.** By definition,

\[
\ell_{j,n}(\frac{k}{(d+1)n}) = \Phi^*_n(\frac{k-j}{(d+1)n}) = \frac{1}{(d+1)n^d} \sum_{i \in \mathbb{H}_n^*} c_{i}^{(n)} \phi_i(\frac{k-j}{(d+1)n}).
\]

Since \(\Omega_H\) tiles \(\mathbb{R}^{d+1}_H\), there exist \(m, l \in \mathbb{Z}^{d+1}_H\) such that \(\frac{m}{(d+1)n} \in \Omega_H\) and \(k - j = m + (d+1)n l\). Thus, by Theorem 3.17

\[
\ell_{j,n}(\frac{k}{(d+1)n}) = \frac{1}{(d+1)n^d} \sum_{i \in \mathbb{H}_n^*} c_{i}^{(n)} \phi_i(\frac{m}{(d+1)n}) = \frac{1}{(d+1)n^d} \sum_{i \in \mathbb{H}_n^*} c_{i}^{(n)} \phi_m(\frac{i}{(d+1)n})
\]

\[
= \langle \phi_m, \phi_0 \rangle_n^* = \langle \phi_m, \phi_0 \rangle = \delta_{m,0}.
\]

Equivalently we can write the above equation as

\[
\ell_{j,n}(\frac{k}{(d+1)n}) = \langle \phi_k, \phi_j \rangle_n^* = \begin{cases} 1, & k = j + (d+1)n l, l \in \mathbb{Z}^{d+1}_H, \\ 0, & \text{otherwise}, \end{cases}
\]

(3.25)

which proves (3.23).
To derive the compact formula for $\Phi_n^*(t)$ we start with the following symmetry argument,

$$\frac{1}{|G|} \sum_{\sigma \in \mathcal{G}} \sum_{k \in \mathbb{H}_n^*} \phi_k(t \sigma) = \frac{1}{|G|} \sum_{\sigma \in \mathcal{G}} \sum_{k \in \mathbb{H}_n^*} \phi_{k \sigma^{-1}}(t) = \frac{1}{|G|} \sum_{\sigma \in \mathcal{G}} \sum_{k \in \mathbb{H}_n^*} \phi_k(t)$$

Using the fact that $\bigcup_{\sigma \in \mathcal{G}} B^{i,j}_0 = B^{i,j}$ and (3.14), we deduce that

$$\sum_{\sigma \in \mathcal{G}} \sum_{k \in \mathbb{H}_n^* \sigma} \phi_k(t) = \sum_{0 < i, j < i + j \leq d + 1} \sum_{k \in \mathbb{H}_n^*} \phi_k(t) = \sum_{0 < i, j < i + j \leq d + 1} (d + 1)! \frac{(d + 1)!}{(i + j)!} \phi_k(t)$$

where the factor $\frac{(d + 1)!}{(i + j)!}$ in the third summand comes from the fact that, by Lemma 3.8

$$\sum_{\sigma \in \mathcal{G}} \sum_{k \in \mathbb{H}_n^* \sigma} \phi_k(t)$$

summarizes over $(d + 1)! \times \frac{(d + 1)!}{(i + j)!(d + 1 - i - j)!}$ terms whereas $\sum_{k \in \mathbb{H}_n^*}$ summarizes over $\frac{(d + 1)!}{(i + j)!}$ terms. As a result, we obtain that

$$\frac{1}{|G|} \sum_{\sigma \in \mathcal{G}} \sum_{k \in \mathbb{H}_n^*} \phi_k(t \sigma) = \sum_{k \in \mathbb{H}_n^*} c_k^{(n)} \phi_k(t) + \sum_{k \in \mathbb{H}_n^*} \phi_k(t) = \sum_{k \in \mathbb{H}_n^*} c_k^{(n)} \phi_k(t).$$

In other words, we have shown that

$$\Phi_n^*(t) = \frac{1}{|G|} \sum_{\sigma \in \mathcal{G}} \sum_{k \in \mathbb{H}_n^*} \phi_k(t \sigma)$$

We now evaluate the partial sum $\tilde{D}_n(t)$. Using Lemma 3.10 and the fact that $t_1 + \ldots + t_{d+1} = 0$, we obtain

$$\tilde{D}_n(t) = \sum_{j=1}^{d+1} \sum_{k \in \mathbb{H}_n^*} \phi_k(t) = \sum_{j=1}^{d+1} \sum_{k \in \mathbb{H}_n^*} e^{\frac{2\pi i}{n} \sum_{1 \leq i < j} (k_i - k_j) t_i + \sum_{j \leq 1 \leq d+1} (k_i - k_j) t_i}$$

$$= \sum_{j=1}^{d+1} \prod_{i=1}^{n-1} \sum_{t_i=1}^{d+1} e^{2\pi i t_i} = \sum_{j=1}^{d+1} \prod_{i=1}^{n-1} \frac{\sin \pi n t_i}{\sin \pi t_i} e^{\pi i t_j} e^{-\pi i t_j}.$$

Using symmetry and $t_1 + \ldots + t_{d+1} = 0$, we further derive that

$$\frac{1}{|G|} \sum_{\sigma \in \mathcal{G}} \tilde{D}_n(t \sigma) = \frac{1}{|G|} \sum_{i=1}^{d+1} \frac{\sin \pi n t_i}{\sin \pi t_i} \sum_{j=1}^{d+1} \frac{\sin \pi t_j}{\sin \pi n t_j} e^{-\pi i n t_j} \sum_{k \in \mathbb{H}_n^*} |G| \times |G_{\mathbb{H}_n^*}[t_j]| e^{\pi i t_j + 2 \sum_{i \in l} t_i}.$$

Since $\mathbb{H}_n^*$ is symmetric under the mapping $\tau_i : t \mapsto (t_1, \ldots, t_{i-1}, -t_i, t_{i+1}, \ldots, t_{d+1})$ and the definition of $c_k^{(n)}$ implies that $c_k^{(n)} = c_k^{(n)}$, we see that the function $\sum_{k \in \mathbb{H}_n^*} c_k^{(n)} \phi_k(t)$ is invariant under the action of $\tau_i$. As a consequence, this function must be real valued. Hence, taking the real part in the above expression, we conclude that

$$\sum_{k \in \mathbb{H}_n^*} c_k^{(n)} \phi_k(t) = \sum_{j=1}^{d+1} \left( \prod_{i < j \leq d+1} \frac{\sin \pi n t_i}{\sin \pi t_i} \cos \pi n t_j \sum_{l \subseteq \mathbb{H}_n^*[d+1]} \frac{|l|! (|l| - |l|)!}{(d+1)!} \cos \pi (t_j + 2 \sum_{i \in l} t_i),
$$

which completes the proof.
We note that the condition \( t_1 + \ldots + t_{d+1} = 0 \) implies various relations between trigonometric functions \( \sin \pi t_i \) and \( \cos \pi t_i \). For example, in the case of \( d = 2 \), we have that

\[
\sin 2t_1 + \sin 2t_2 + \sin 2t_3 = -4 \sin t_1 \sin t_2 \sin t_3 \\
\cos 2t_1 + \cos 2t_2 + \cos 2t_3 = 4 \cos t_1 \cos t_2 \cos t_3 - 1.
\]

Using these relations, it is possible to rewrite the compact formula in [3.24] in other forms. For instance, in the case of \( d = 2 \), the compact formula [3.24] becomes

\[
\Phi_n^*(t) = \frac{1}{3n^d} \sin \pi t_1 \sin \pi t_2 \cos \pi t_3 \left( \frac{1}{3} \cos \pi t_1 + \frac{1}{3} \cos \pi (t_1 - t_2) \right) + \frac{1}{3n^d} \sin \pi t_2 \sin \pi t_3 \cos \pi t_1 \left( \frac{2}{3} \cos \pi t_2 + \frac{1}{3} \cos \pi (t_2 - t_3) \right) + \frac{1}{3n^d} \sin \pi t_3 \sin \pi t_1 \cos \pi t_2 \left( \frac{2}{3} \cos \pi t_3 + \frac{1}{3} \cos \pi (t_3 - t_1) \right),
\]

whereas a different formula is derived in [18], given in terms of \( \Theta_n(t) \).

Using the compact formula of \( \Phi_n^* \), we can estimate the operator norm of \( I_n^* \).

**Theorem 3.21.** Let \( \|I_n^*\|_\infty \) denote the operator norm of \( I_n^* : C(\overline{\Omega}_H) \to C(\overline{\Omega}_H) \). Then there is a constant \( c \), independent of \( n \), such that

\[
\|I_n^*\|_\infty \leq c(\log n)^d.
\]

**Proof.** A standard procedure shows that

\[
\|I_n^*\|_\infty = \max_{t \in \overline{\Omega}_H} \sum_{k \in \mathbb{Z}^n} |\Phi_n^*(t - k/(d+1)n)|.
\]

Using the compact formula of \( \Phi_n^* \) in Theorem 3.20, it is easy to see that it suffices to prove that

\[
I_j := \frac{1}{(d+1)n^d} \max_{t \in \overline{\Omega}_H} \sum_{k \in \mathbb{Z}^n} \left| \prod_{1 \leq i \leq d+1} \frac{\sin \pi n(t_i - k_i/(d+1)n)}{\sin \pi(t_i - k_i/(d+1)n)} \right| \leq c(\log n)^d, \quad 1 \leq j \leq d+1, \ n \geq 0.
\]

For \( t \in \mathbb{R}^{d+1}_H \), let \( t^{(j)} := (t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_{d+1}) \in \mathbb{R}^d \). It is easy to see that \( \{t^{(j)} : t \in \overline{\Omega}_H \} \subset [-1, 1]^d \). Hence, enlarging the domain \( \mathbb{H}^n \) to \( \{k \in \mathbb{Z}^{d+1}_H : -(d+1)n \leq k_j \leq (d+1)n, k_i \equiv 0 \pmod{d+1}, 1 \leq i \leq d \} \), we see that

\[
I_j \leq \frac{1}{(d+1)n^d} \max_{t \in [-1,1]^d} \sum_{l=1}^{d} \sum_{k_i = -n}^{n} \left| \frac{\sin \pi n(t_l - k_l/(d+1)n)}{\sin \pi(t_l - k_l/(d+1)n)} \right| \leq \frac{d}{d+1} \left( \frac{1}{n} \sum_{k_i = -n}^{n} \frac{|\sin \pi(t_l - k_l/(d+1)n)|^d}{|\sin \pi(t_l - k_l/(d+1)n)|} \right) \leq c(\log n)^d,
\]

where the last step follows from the usual estimate of one variable (cf. [34] Vol. II, p. 19). \( \square \)

We expect that the estimate is sharp, that is, \( \|I_n^*\| \geq c(\log n)^d \), but do not have a proof at this point.

**4. Discrete Fourier analysis on the simplex**

The fundamental domain of the lattice \( A_d \) is the union of the images of the fundamental simplex \( \triangle_H \) under the group \( \mathcal{G} \), as shown in (3.3). Hence, if we consider invariant functions under the group \( \mathcal{G} \), then the discrete Fourier analysis on the fundamental domain in the previous section can be carried over to the analysis on the simplex \( \triangle_H \), which is developed below.
4.1. Generalized sine and cosine functions. In the case of one-variable, the invariant and anti-invariant sums of the exponential functions are cosine and sine functions. We now consider their analogous in our setting. It turns out that these functions have already appeared in the literature, as mentioned in the introduction. It should be pointed out, however, that our study is on the discrete Fourier analysis, which has little overlap with the previous study in the literature. Most of the overlap will appear in the next section, when these trigonometric functions are transformed to Chebyshev polynomials.

Recall that the reflection group $A_d$ is the permutation group $G$. Denote the identity element in $G$ by $1$. It is easy to see that

\begin{equation}
\sigma_{ij}^2 = 1, \quad \sigma_{ij} \sigma_{jk} \sigma_{ij} = \sigma_{jk}, \quad i, j, k \in \mathbb{N}_{d+1}.
\end{equation}

For $\sigma \in G$, let $|\sigma|$ denote the number of inversions in $\sigma$. The group $G$ is naturally divided into two parts, $G^+ := \{ \sigma \in G : |\sigma| \equiv 0 \pmod{2} \}$ of elements with even inversions, and $G^- := \{ \sigma \in G : |\sigma| \equiv 1 \pmod{2} \}$ of elements with odd inversions. The action of $\sigma \in G$ on the function $f : \mathbb{R}^{d+1}_H \to \mathbb{R}$ is defined by $\sigma f(t) := f(\sigma t)$. A function $f$ in homogeneous coordinates is called invariant under $G$ if $\sigma f = f$ for all $\sigma \in G$, and it is called anti-invariant under $G$ if $\sigma f = \rho(\sigma) f$ with $\rho(\sigma) = 1$ if $\sigma \in G^+$ and $\rho(\sigma) = -1$ if $\sigma \in G^-$. The following proposition follows immediately from the definition.

**Proposition 4.1.** Define two operator $\mathcal{P}^+$ and $\mathcal{P}^-$ acting on $f(t)$ by

\begin{equation}
\mathcal{P}^\pm f(t) := \frac{1}{(d+1)!} \left[ \sum_{\sigma \in G^+} f(\sigma t) \pm \sum_{\sigma \in G^-} f(\sigma t) \right].
\end{equation}

Then the operators $\mathcal{P}^+$ and $\mathcal{P}^-$ are projections from the class of $H$-periodic functions onto the class of invariant, and respectively anti-invariant functions.

Recall that $\phi_k(t) = e^{2\pi i k \cdot t}$. Applying the operators $\mathcal{P}^\pm$ to these exponential functions gives the basic invariant and anti-invariant functions.

**Definition 4.2.** For $k \in \mathbb{H}$ define

\begin{align*}
\text{TC}_k(t) := \mathcal{P}^+ \phi_k(t) &= \frac{1}{(d+1)!} \left[ \sum_{\sigma \in G^+} \phi_k(\sigma t) + \sum_{\sigma \in G^-} \phi_k(\sigma t) \right], \\
\text{TS}_k(t) := \mathcal{P}^- \phi_k(t) &= \frac{1}{(d+1)!} \left[ \sum_{\sigma \in G^+} \phi_k(\sigma t) - \sum_{\sigma \in G^-} \phi_k(\sigma t) \right],
\end{align*}

and call them generalized cosine and generalized sine, respectively.

By definition, $\text{TC}_k$ is invariant and $\text{TS}_k$ is anti-invariant. Because of the symmetry, we only need to consider them on the fundamental simplex $\Delta_H$ defined in (3.2) or any other simplex $\Delta_H \sigma$, $\sigma \in G$, that makes up $\Omega_H$. We shall work with $\Delta_H$ below and recall that

\begin{equation}
\Delta_H = \{ t \in \mathbb{R}^{d+1}_H : 0 \leq t_i - t_j \leq 1, 1 \leq i \leq j \leq d + 1 \}.
\end{equation}

In the case of $d = 2$ and $d = 3$, it is an equilateral triangle and a regular tetrahedron, respectively; these regions are depicted in Figure 4.1, in which the corners are given in homogeneous coordinates.

By definition, $\phi_k$ and $\phi_{k\sigma}$ with $\sigma \in G$ lead to the same $\text{TC}_k$. In fact, we have

\begin{equation}
\text{TC}_{k\sigma}(t) = \text{TC}_k(t) = \text{TC}_k(t) \quad \text{for } t \in \Delta_H \text{ and } \sigma \in G.
\end{equation}

Thus, when working with $\text{TC}_k$, we can restrict $k$ to the index set

\begin{equation}
\Lambda := \{ k \in \mathbb{H} : k_1 \geq k_2 \geq \cdots \geq k_{d+1} \}.
\end{equation}
As for $TS_k$, it is easy to see that $TS_{k_\sigma}(t) = TS_k(t\sigma) = TS_k(t)$ for $\sigma \in G^+$ and $TS_{k_\sigma}(t) = TS_k(t\sigma) = -TS_k(t)$ for $\sigma \in G^-$. In particular, $TS_k(t) = 0$ whenever two or more components of $k$ are equal. Thus, when working with $TS_k$, we only need to consider $k \in \Lambda^o$, where

$$
\Lambda^o := \{ k \in \mathbb{H} : k_1 > k_2 > \cdots > k_{d+1} \},
$$

which is the set of the interior points of $\Lambda$. To describe the points on the boundary of $\Lambda$, we need to consider the compositions of the integer $d + 1$. A composition $p$ of $d + 1$ is a decomposition of $d + 1$, such that

$$
p = (p_1, \ldots, p_\ell) \in \mathbb{Z}^d, \quad p_i > 0, \quad 1 \leq i \leq \ell \leq d + 1, \quad |p| := p_1 + \cdots + p_\ell = d + 1,
$$

where $\ell = \ell(p)$ is the length of the composition. Notice that the order of $p_i$ matters, different orderings are deemed to be different compositions, which is the difference between a composition and a partition. We denote the collection of compositions of $d + 1$ by $C_{d+1}$; that is,

$$
C_{d+1} := \left\{ p \in \mathbb{Z}^{\ell(p)} : 1 \leq p_i \leq |p| = d + 1 \text{ for } 1 \leq i \leq \ell(p) \leq d + 1 \right\}.
$$

For $p \in C_{d+1}$ we further define

$$
\Lambda^p := \{ k = (\{k_1\}^{p_1}, \{k_2\}^{p_2}, \ldots, \{k_\ell\}^{p_\ell}) \in \mathbb{H} : k_i > k_j, 1 \leq i < j \leq \ell \}.
$$

Then evidently $\Lambda^o = \Lambda^{(1)^{d+1}}$ and $\Lambda = \bigcup_{p \in C_{d+1}} \Lambda^p$.

Recall that $kG = \{ k\sigma : \sigma \in G \}$ is the orbit of $k$ under $G$. The definition of $\Lambda$ implies that, for $k, j \in \Lambda$, $kG \cap jG = \emptyset$ whenever $k \neq j$. It follows that

$$
|kG| = \binom{d+1}{p} := \frac{(d+1)!}{p_1!p_2!\cdots p_\ell!} \quad \text{if } k \in \Lambda^p.
$$

Since $G_kG$ is isomorphic to $kG$, where $G_k$ is the stabilizer of $k$, we also have

$$
(4.4) \quad TC_k(t) = \frac{1}{|G|} \sum_{j \in kG} \sum_{\sigma \in G_j} \phi_{j\sigma}(t) = \frac{1}{|kG|} \sum_{j \in kG} \phi_j(t).
$$

The length $\ell = \ell(p)$ of $p \in C_{d+1}$ determines how many indices in $k \in \Lambda^p$ are repeated, which determines the dimension of the boundary elements of $\Lambda$. For instance, if $d = 2$ then $G_2 = \{(1, 1, 1), (1, 2), (2, 1), (3)\}$ and

$$
|kG| = \begin{cases} 
6, & k \in \Lambda^o := \Lambda^{1,1,1}, \\
3, & k \in \Lambda^c := \Lambda^{1,2} \cup \Lambda^{2,1}, \\
1, & k \in \Lambda^v := \Lambda^3,
\end{cases}
$$
where $\Lambda^e$ and $\Lambda^v$ consist of points in $\Lambda$ that are on the edges and vertices of $\Lambda$, respectively. If $d = 3$ then $G = \{(1, 1, 1, 1), (1, 1, 2), (1, 2, 1), (2, 1, 1), (1, 3), (3, 1), (2, 2), (4)\}$ and

$$|kG| = \begin{cases} 24, & k \in \Lambda^e := \Lambda^{1,1,1,1}, \\ 12, & k \in \Lambda^f := \Lambda^{1,1,2} \cup \Lambda^{1,2,1} \cup \Lambda^{2,1,1}, \\ 6, & k \in \Lambda^{e,1} := \Lambda^{2,2}, \\ 4, & k \in \Lambda^{e,2} := \Lambda^{1,3} \cup \Lambda^{3,1}, \\ 1, & k = 0 \in \Lambda^v := \Lambda^d, \end{cases}$$

where $\Lambda^f$, $\Lambda^e$ and $\Lambda^v$ consist of points in $\Lambda$ that are on the faces, edges and vertices of $\Lambda$, respectively.

We define an inner product on $\triangle_H$ by

$$(f, g)_{\triangle_H} := \frac{1}{|\triangle_H|} \int_{\triangle_H} f(t)g(t)dt = \frac{(d+1)!}{\sqrt{d+1}} \int_{\triangle_H} f(t)g(t)dt.$$ 

If $f \equiv g$ is invariant under $G$, then it follows immediately that $(f, g) = (f, g)_{\triangle_H}$, where $(\cdot, \cdot)$ is the inner product defined in (3.6) over $\Omega_H$. The generalized cosine and sine functions are orthogonal with respect to this inner product.

**Theorem 4.3.** For $k, j \in \Lambda$,

$$(TC_k, TC_j)_{\triangle_H} = \frac{\delta_{k,1}}{|kG|} = \frac{p_1!p_2! \cdots p_l!}{(d+1)!} \delta_{k,1}, \quad k \in \Lambda^p,$$

and for $k, j \in \Lambda^e$,

$$(TS_k, TS_j)_{\triangle_H} = \frac{1}{(d+1)!} \delta_{k,j}.$$ 

**Proof.** Both of these relations follow from the identity $(f, g) = (f, g)_{\triangle_H}$ for invariant functions. For (4.5), the invariance is evident and we only have to use the orthogonality of $\phi_k$ in (3.5) and (4.4). For (4.6), we use the fact that $TS_k(t)TS_j(t)$ is invariant under $G$ and the orthogonality of $\phi_k$ on $\Omega_H$. \qed

The definition and orthogonality of these trigonometric functions have appeared in the literature, we refer to [3] and its extensive references. However, the study in the literature is more on the side of algebraic polynomials, as will be discussed in Section 5 below.

**4.2. Discrete inner product on the simplex.** By Theorem 3.17 and (3.5), $\{\phi_k : k \in \mathbb{H}_d\}$ is an orthonormal set with respect to the symmetric inner product $\langle \cdot, \cdot \rangle_\ast$. Using the symmetry and the invariance of $TC_k$ and $TS_k$ under $G$, we can deduce a discrete orthogonality for the generalized cosine and sine functions. We define

$$(4.7) \quad \Lambda_n := \mathbb{H}_n^e \cap \Lambda = \{k \in \mathbb{H} : k_{d+1} \leq k_d \leq \ldots \leq k_1 \leq k_{d+1} + (d+1)n\}.$$ 

For $n = 4$, the point sets for $d = 2$ and $d = 3$ are depicted in Figure 4.1.

By the definition of $\mathbb{H}_n^e$, the set $\{\frac{k}{d+1} : k \in \Lambda_n\}$ contains points inside $\triangle_H$. We will also need to understand points on the boundary of $\triangle_H$, which can come from two types of points in $\mathbb{H}_n^e$. One part of the boundary points in $\triangle_H$ are also the boundary points of $\mathbb{H}_n^e$, whereas another part of the boundary points in $\triangle_H$ are points inside $\mathbb{H}_n^e$ but on the faces of $\triangle_H$. Accordingly, for $p \in C_{d+1}$, we define

$$\Lambda_n^{e,p} := \{(k_1)^{p_1}, (k_2)^{p_2}, \ldots, (k_\ell)^{p_\ell} \in \mathbb{H} : k_\ell + (d+1)n > k_1 > k_2 > \cdots > k_\ell\},$$

$$\Lambda_n^{0,p} := \{(k_1)^{p_1}, (k_2)^{p_2}, \ldots, (k_\ell)^{p_\ell} \in \mathbb{H} : k_\ell + (d+1)n = k_1 > k_2 > \cdots > k_\ell\}.$$
Evidently, \( \Lambda_n^o = \Lambda_{n,1}^o \) and
\[
\Lambda_n = \bigcup_{p \in \mathcal{C}_{d+1}} (\Lambda_n^{o,p} \cup \Lambda_n^{\partial,p}) .
\]

If \( p \neq \{1\}^{d+1} \), then \( \Lambda_n^{o,p} \) is a \((\ell - 1)\)-face of \( \Lambda_n \) which, however, is a subset of \( \mathbb{H}_n^o \), whereas \( \Lambda_n^{\partial,p} \) is a \((\ell - 2)\)-face of \( \Lambda_n \) which is also a face of \( \mathbb{H}_n^\partial \). More precisely, considering the orbits of the points in \( \Lambda_n \), we see that
\[
\mathbb{H}_n^o = \bigcup_{p \in \mathcal{C}_{d+1}} \Lambda_n^{o,p} \mathcal{G} \quad \text{and} \quad \mathbb{H}_n^\partial \setminus \mathbb{H}_n^o = \bigcup_{p \in \mathcal{C}_{d+1}} \Lambda_n^{\partial,p} \mathcal{G} .
\]

For \( d = 2 \), the sets of interior points, edge points and vertices of \( \triangle_H \) are given, in this order, explicitly by
\[
\begin{align*}
\Lambda_n^{1,1,1,1} & = \{ k \in \mathbb{H} : k_3 + 3n > k_1 > k_2 > k_3 \} , \\
\Lambda_n^{1,2} \cup \Lambda_n^{2,1} \cup \Lambda_n^{1,1,1,1} & = \{ (2k, -k, -k), (k, k, -2k), (3n - k, 2k - 3n, -k) : 0 < k < n \} , \\
\Lambda_n^{3} \cup \Lambda_n^{1,2} \cup \Lambda_n^{2,1} & = \{ (0, 0, 0), (2n, -n, -n), (n, n, -2n) \} .
\end{align*}
\]

In this case, the geometry in Figure 4.3 shows clearly that two of the three edges, \( \Lambda_n^{1,2} \cup \Lambda_n^{2,1} \), of the triangle \( \triangle_H \) are not edges of the hexagon, and the vertex \( \Lambda_n^{3} = \{(0, 0, 0)\} \) is not an vertex of the hexagon.
For $d = 3$, the boundary sets are given in detail in [19] and they are denoted by $\Lambda_n^0$, $\Lambda_n^1$, $\Lambda_n^{e,1}$, $\Lambda_n^{e,2}$ and $\Lambda_n^v$, which correspond to points of interior, faces, two type of edges, and vertices. In terms of the sets $\Lambda_n^{0,p}$ and $\Lambda_n^\partial$, they are expressed by

\begin{align*}
\Lambda_n^0 &= \Lambda_n^{0,1,1,1}, \\
\Lambda_n^1 &= \Lambda_n^{0,1,2,1} \cup \Lambda_n^{0,2,1,1} \cup \Lambda_n^\partial, \\
\Lambda_n^{e,1} &= \Lambda_n^{0,2,2} \cup \Lambda_n^{\partial,1,1,2}, \\
\Lambda_n^{e,2} &= \Lambda_n^{0,1,3} \cup \Lambda_n^{0,3,1} \cup \Lambda_n^{\partial,1,1,2} \cup \Lambda_n^{\partial,2,1,1}, \\
\Lambda_n^v &= \Lambda_n^{0,4} \cup \Lambda_n^{0,3,3} \cup \Lambda_n^{\partial,2,2} \cup \Lambda_n^{\partial,3,1}.
\end{align*}

We now define a discrete inner product $\langle \cdot, \cdot \rangle_{\triangle,n}$ by

$$\langle f, g \rangle_{\triangle,n} = \frac{1}{(d+1)n^d} \sum_{j \in \Lambda_n} \lambda_j^{(n)} f(\frac{j}{(d+1)n}) g(\frac{j}{(d+1)n}).$$

where, with $c_j^{(n)}$ defined in Definition (3.16),

$$\lambda_j^{(n)} := c_j^{(n)} \left( \frac{d+1}{p} \right) = \begin{cases} \frac{(d+1)!}{p_1!p_2! \cdots p_{d-1}!}, & j \in \Lambda_n^{0,p}, \\
\frac{(d+1)!}{(p_1+p_2)! \cdots (p_{d-1}+1)!}, & j \in \Lambda_n^\partial. \end{cases}$$

Let us verify the second equal sign in (4.9). The case $j \in \Lambda_n^{0,p}$ is easy, since then $j \in \mathbb{H}^n$ so that $c_j^{(n)} = 1$ and $\lambda_j^{(n)} = \binom{d+1}{p}$. In the case $j \in \Lambda_n^\partial$, we have $j_1 - j_l = (d+1)n$ which implies $j_i - j_l = (d+1)n$ for $i \in I_{p_1}$ and $l \in I_{p_k}$, where $I_{p_1} = \{1, \ldots, p_1\}$ and $I_{p_l} = \{d - p_l + 1, \ldots, d + 1\}$. Consequently, $\frac{1}{(d+1)n} \in B^{p_1, p_k}$ by the definition of $B^{i,j}$, which implies that $j \in \mathbb{H}^{p_1, p_k}$ so that $c_j^{(n)} = 1/(p_1 + p_k)$ and (4.9).

In the case of $d = 2$ and $d = 3$, the values of $\lambda_j^{(n)}$ are given by

$$\lambda_j^{(n)} = \begin{cases} 6, & j \in \Lambda_n^0, \\
3, & j \in \Lambda_n^e, & \text{if } d = 2, \\
1, & j \in \Lambda_n^v, \\
24, & j \in \Lambda_n^0, \\
12, & j \in \Lambda_n^e, \\
6, & j \in \Lambda_n^{e,1}, & \text{if } d = 3, \\
4, & j \in \Lambda_n^{e,2}, \\
1, & j \in \Lambda_n^v.
\end{cases}$$

We denote by $\mathcal{T}_C_n$ and $\mathcal{T}_S_n$ the spaces of the trigonometric polynomials

$$\mathcal{T}_C_n := \text{span} \{ \mathcal{T}_k : k \in \Lambda_n \}, \quad \mathcal{T}_S_n := \text{span} \{ \mathcal{T}_k : k \in \Lambda_n^0 \},$$

respectively. Since $\Lambda_n$ contains integer points in a regular simplex, it is easy to see that

$$\dim \mathcal{T}_C_n = |\Lambda_n| = \binom{n+d}{d} \quad \text{and} \quad \dim \mathcal{T}_S_n = |\Lambda_n^0| = \binom{n-1}{d}.$$

**Theorem 4.4.** For $f \bar{g} \in \mathcal{T}_C_{2n-1},$

$$\langle f, g \rangle_{\triangle} = \langle f, g \rangle_{\triangle,n}.$$  

Moreover, the following cubature formula is exact for all $f \in \mathcal{T}_C_{2n-1},$

$$\frac{1}{|\triangle_H|} \int_{\triangle_H} f(t) \, dt = \frac{1}{(d+1)n^d} \sum_{j \in \Lambda_n} \lambda_j^{(n)} f(\frac{j}{(d+1)n}).$$
In particular,

\[(4.13) \quad \langle TC_k, TC_j \rangle_{\Delta, n} = \frac{\delta_{k,j}}{\lambda_{k}^{n}}, \quad k, j \in \Lambda_n.\]

Proof. Let \( f \) be a function invariant under \( G \). According to (4.8),

\[
\sum_{j \in H_n} c_j^{(n)} f\left(\frac{j}{(d+1)n}\right) = \sum_{p \in C_{d+1}} \left(\begin{array}{c} d+1 \\ p \end{array}\right) \sum_{j \in \Lambda_n^p} c_j^{(n)} f\left(\frac{j}{(d+1)n}\right),
\]

\[
\sum_{j \in H_n^* \setminus H_n} c_j^{(n)} f\left(\frac{j}{(d+1)n}\right) = \sum_{p \in C_{d+1}} \left(\begin{array}{c} d+1 \\ p \end{array}\right) \sum_{j \in \Lambda_n^p} c_j^{(n)} f\left(\frac{j}{(d+1)n}\right).
\]

Adding these two expressions together, we conclude that

\[
(4.14) \quad \sum_{j \in H_n^*} c_j^{(n)} f\left(\frac{j}{(d+1)n}\right) = \sum_{p \in C_{d+1}} \left(\begin{array}{c} d+1 \\ p \end{array}\right) \left[ \sum_{j \in \Lambda_n^p} c_j^{(n)} f\left(\frac{j}{(d+1)n}\right) + \sum_{j \in \Lambda_n^{p, p}} c_j^{(n)} f\left(\frac{j}{(d+1)n}\right) \right],
\]

\[
= \sum_{j \in \Lambda_n} c_j^{(n)} \left(\begin{array}{c} d+1 \\ p \end{array}\right) f\left(\frac{j}{(d+1)n}\right) = \sum_{j \in \Lambda_n} \lambda_{j}^{(n)} f\left(\frac{j}{(d+1)n}\right).
\]

Replacing \( f \) by \( fg \), we have proved that \( \langle f, g \rangle_n = \langle f, g \rangle_{\Delta, n} \) whenever \( fg \) is invariant. Hence, (4.11) follows from Theorem 3.17. Furthermore, since \( \frac{1}{|G|} \int_{\Omega_n} f(t) dt = \frac{1}{|G|} \int_{\Delta, n} f(t) dt \) for all invariant \( f \), (4.12) follows from Theorem 3.18.

Furthermore, replacing \( f \) by \( TC_k TC_j \) in (4.14), we derive by (4.3) that

\[
\langle TC_k, TC_j \rangle_{\Delta, n} = \frac{1}{(d+1)n} \sum_{i \in H_n^*} c_i^{(n)} TC_k\left(\frac{i}{(d+1)n}\right) TC_j\left(\frac{i}{(d+1)n}\right)
\]

\[
= \frac{1}{(d+1)n} \sum_{i \in H_n^*} \left(\begin{array}{c} d+1 \\ p \end{array}\right) \sum_{\sigma \in \mathcal{G}} \phi_k\left(\frac{l_{\sigma}}{(d+1)n}\right) TC_j\left(\frac{l_{\sigma}}{(d+1)n}\right)
\]

\[
= \frac{1}{(d+1)n} \sum_{i \in H_n^*} \left(\begin{array}{c} d+1 \\ p \end{array}\right) \sum_{\sigma \in \mathcal{G}} \phi_k\left(\frac{1}{(d+1)n}\right) TC_j\left(\frac{l_{\sigma}}{(d+1)n}\right)
\]

\[
= \frac{1}{(d+1)n} \sum_{i \in H_n^*} \left(\begin{array}{c} d+1 \\ p \end{array}\right) \sum_{\sigma \in \mathcal{G}} \phi_k\left(\frac{1}{(d+1)n}\right) TC_j\left(\frac{l_{\sigma}}{(d+1)n}\right) = \langle \phi_k, TC_j \rangle_n^{*},
\]

Using (3.25) and abbreviating \( k \equiv j \mod (d+1)Z_{d+1} \) as \( k \equiv j \), we further deduce that

\[
\langle TC_k, TC_j \rangle_{\Delta, n} = \frac{1}{|G|} \sum_{\sigma \in \mathcal{G}} \langle \phi_k, \phi_{j\sigma} \rangle_n^{*} = \frac{1}{(d+1)!} \left| \left\{ \sigma \in \mathcal{G} : j\sigma \equiv k \right\} \right|
\]

\[
= \frac{\delta_{j,k}}{(d+1)!} \left| \left\{ \sigma \in \mathcal{G} : k\sigma \equiv k \right\} \right| = \frac{\delta_{j,k}}{\lambda_k^{(n)}},
\]

where the last equality follows readily from the definition of \( \Lambda_n \) and (4.9). \( \square \)

Since the above proof applies to invariant \( f \), it also applies to \( f, g \in TS_n \) since \( fg \) is invariant if both \( f \) and \( g \) are anti-invariant. Recall that \( TS_k\left(\frac{1}{(d+1)n}\right) = 0 \) when \( j \in \Lambda_n \setminus \Lambda_n^* \), we have also proven the following result.

**Theorem 4.5.** Let the discrete inner product \( \langle \cdot, \cdot \rangle_{\Delta, n} \) be defined by

\[
(4.11) \quad \langle f, g \rangle_{\Delta, n} = \frac{1}{d!} \sum_{j \in \Lambda_n} f\left(\frac{j}{(d+1)n}\right) g\left(\frac{j}{(d+1)n}\right).
\]
Then
\[
\langle f, g \rangle_{\Delta^*, n} = \langle f, g \rangle_{\Delta_H}, \quad f, g \in T \Sigma_n.
\]

4.3. **Interpolation on the simplex.** Using invariance and the fact that the fundamental simplex \(\Delta_H\) is the building block of the fundamental domain, we can also deduce results on interpolation on the fundamental simplex. For \(d = 3\) the results in this subsection have appeared in [19] and the proof there can be followed verbatim when \(3\) is replaced by \(d\). Thus, we shall omit the proof. Recall that the operator \(\mathcal{P}^\pm\) is defined in (4.1).

**Theorem 4.6.** For \(n > 0\), and \(f \in C(\Delta_H)\), define
\[
\mathcal{L}_n f(t) := \sum_{j \in \Lambda_n^+} f(\frac{j}{(d+1)n}) \ell_{j,n}(t), \quad \ell_{j,n}(t) := \frac{d!}{n^d} \sum_{k \in \Lambda_n^+} T S_k(t) T S_k(\frac{j}{(d+1)n}).
\]

Then \(\mathcal{L}_n f\) is the unique function in \(T S_n\) that satisfies
\[
\mathcal{L}_n f(\frac{j}{(d+1)n}) = f(\frac{j}{(d+1)n}), \quad j \in \Lambda_n^+.
\]
Furthermore, the fundamental interpolation function \(\ell_{j,n}^\pm\) is real and satisfies
\[
\ell_{j,n}^\pm(t) = \frac{d!}{n^d} \mathcal{P}_t^\pm \left[ \Theta_n(t - \frac{j}{(d+1)n}) - \Theta_{n-1}(t - \frac{j}{(d+1)n}) \right],
\]
where \(\mathcal{P}_t^\pm\) means that the operator \(\mathcal{P}_t^\pm\) is acting on the variable \(t\) and \(\Theta_n\) is defined in (3.18).

The function \(\mathcal{L}_n f\) interpolates at the interior points of \(\Lambda_n^+\). We can also consider interpolation on \(\Lambda_n\) by working with the operator \(I_n^* f\) in Theorem 3.20 which interpolates \(f\) on \(\mathbb{H}_n^*\) but interpolates a sum over congruent points on \(\mathbb{H}_n^* \subset \mathbb{H}_n^0\). Using invariance, however, \(I_n^* f\) leads to a genuine interpolation operator on \(\Lambda_n\).

**Theorem 4.7.** For \(n > 0\) and \(f \in C(\Delta_H)\) define
\[
\mathcal{L}_n f(t) := \sum_{j \in \Lambda_n} f(\frac{j}{(d+1)n}) \ell_{j,n}^\Delta(t), \quad \ell_{j,n}^\Delta(t) := \frac{\lambda_j^{(n)}}{(d+1)n^d} \sum_{k \in \Lambda_n} \lambda_k^{(n)} T C_k(t) T C_k(\frac{j}{(d+1)n}).
\]

Then \(\mathcal{L}_n f\) is the unique function in \(T C_n\) that satisfies
\[
\mathcal{L}_n f(\frac{j}{(d+1)n}) = f(\frac{j}{(d+1)n}), \quad j \in \Lambda_n.
\]
Furthermore, the fundamental interpolation function \(\ell_{j,n}^\Delta\) is given by
\[
\ell_{j,n}^\Delta(t) = \lambda_j^{(n)} \mathcal{P}_t^\pm \ell_{j,n}(t),
\]
where \(\ell_{j,n}\) is defined in Theorem 3.20 and has a compact formula.

Let \(\|\mathcal{L}_n\|\) and \(\|\mathcal{L}_n^*\|\) denote the operator norms of \(\mathcal{L}_n\) and \(\mathcal{L}_n^*\), respectively, both as operators from \(C(\Delta_H) \rightarrow C(\Delta_H)\). From Theorems 4.6 and 4.7, an immediate application of Theorem 3.21 yields the following theorem.

**Theorem 4.8.** There is a constant \(c\) independent of \(n\), such that
\[
\|\mathcal{L}_n\| \leq c(\log n)^d \quad \text{and} \quad \|\mathcal{L}_n^*\| \leq c(\log n)^d.
\]

It should be pointed out that the interpolation functions defined in these theorems are analogous of trigonometric polynomial interpolation on equally spaced points [24, Chapt. X]. These are trigonometric interpolation on equal spaced points in the simplex \(\Delta_H\), which can be easily transformed to interpolation on regular simplex, say \(\{y : 0 \leq y_d \leq \ldots \leq y_1 \leq 1\}\) in \(\mathbb{R}^d\). These interpolation functions are real, easily computable from their compact formulas, and have small Lebesgue constants. They should be the ideal tool for interpolation on the simplex in \(\mathbb{R}^d\).
5. Generalized Chebyshev Polynomials and their Zeros

The generalized sine and cosine functions can be used to define analogues of Chebyshev polynomials of the first and the second kind, respectively, which are algebraic orthogonal polynomials of $d$-variables, just as in the classical case of one variable. These polynomials have been defined in the literature, as noted in the Introduction. In the first subsection we define these polynomials and present their basic properties. Most of the results in this subsection are not new, however, we shall present a coherent and independent treatment, and some of the results on recurrence relations appear to be new. In the second subsection, we study the common zeros of these polynomials and use them to establish a family of Gaussian cubature formulas, which exist rarely.

5.1. Generalized Chebyshev polynomials. The generalized trigonometric functions can be transformed into polynomials under a change of variables $z : \mathbb{R}^{d+1} \to \mathbb{C}^d$, defined by

$$z_k = z_k(t) := TC_{\mathcal{V}}(t) = \frac{1}{|\mathcal{V}^{\mathcal{G}}|} \sum_{j \in \mathcal{V}^{\mathcal{G}}} e^{\frac{2\pi i}{|\mathcal{V}|} j \cdot t}, \quad k = 1, 2, \ldots, d,$$

where $v^k = (\{d+1-k\}^k, \{-k\}^{d+1-k})$, and $\frac{2\pi i}{|\mathcal{V}|} j$ are vertices of the fundamental triangle $\Delta_H$ defined in Section 2. It is easy to see that $z_k = z_{d+1-k}$. The homogeneity of $t$ shows that $v^k \cdot t = (d+1)(t_1 + \ldots + t_k)$ and, consequently,

$$z_k = \sum_{j \in \mathcal{V}^{\mathcal{G}}} e^{2\pi i \sum_{j \in \mathcal{V}} j \cdot t},$$

which shows that $z_1, \ldots, z_d$ are the first $d$ elementary symmetric polynomials of $e^{2\pi i t_1}, \ldots, e^{2\pi i t_{d+1}}$. The same change of variables are used in [3].

Since $TC_k(t)$, $k \in \Lambda$, is evidently a symmetric polynomial in $e^{2\pi i t_1}, \ldots, e^{2\pi i t_{d+1}}$, it is a polynomial in $z_1, \ldots, z_d$. Some of its properties can be derived from the recursive relations given below.

**Lemma 5.1.** The generalized sine and cosine functions satisfy the recurrence relations,

$$TC_j(t)TC_k(t) = \frac{1}{|\mathcal{G}|} \sum_{\sigma \in \mathcal{G}} TC_{k+j\sigma}(t), \quad j, k \in \Lambda,$$

$$TC_j(t)TS_k(t) = \frac{1}{|\mathcal{G}|} \sum_{\sigma \in \mathcal{G}} TS_{k+j\sigma}(t), \quad j, k \in \Lambda,$$

$$TS_j(t)TS_k(t) = \frac{1}{|\mathcal{G}|} \sum_{\sigma \in \mathcal{G}} (-1)^{|\sigma|} TC_{k+j\sigma}(t), \quad j, k \in \Lambda.$$

**Proof.** From the definition of $TC_k$, we obtain that

$$TC_k(t)TC_k(t) = \frac{1}{|\mathcal{G}|} \sum_{\sigma \in \mathcal{G}} \phi_{k\sigma}(t) \times \frac{1}{|\mathcal{G}|} \sum_{\sigma \in \mathcal{G}} \phi_{k\sigma}(t) = \frac{1}{|\mathcal{G}|^2} \sum_{\tau \in \mathcal{G}} \sum_{\sigma \in \mathcal{G}} \phi_{k\tau+j\sigma}(t)$$

$$= \frac{1}{|\mathcal{G}|^2} \sum_{\tau \in \mathcal{G}} \sum_{\sigma \in \mathcal{G}} \phi_{(k+j\sigma-1)\tau}(t) = \frac{1}{|\mathcal{G}|^2} \sum_{\tau \in \mathcal{G}} \sum_{\sigma \in \mathcal{G}} \phi_{(k+j\sigma)\tau}(t)$$

$$= \frac{1}{|\mathcal{G}|} \sum_{\sigma \in \mathcal{G}} TC_{k+j\sigma}(t),$$

which proves (5.3). The other two relations, (5.4) and (5.5), can be established similarly. \(\square\)

The polynomials defined by $TC_k(t)$ under (5.1) are analogue of Chebyshev polynomials of the first kind, to be defined formerly below. We will also define Chebyshev polynomial of the second kind, for which we need the following lemma.
Lemma 5.2. Let $\mathbf{v}^o := \left( \frac{(d+1)(d+1)}{2}, \frac{(d-2)(d+1)}{2}, \ldots, \frac{(2-d)(d+1)}{2}, \frac{-d(d+1)}{2} \right)$, then

\begin{equation}
\text{TS}_{\mathbf{v}^o}(t) = \frac{1}{(d+1)!} \prod_{1 \leq \mu < \nu \leq d+1} (e^{2\pi it_\mu} - e^{2\pi it_\nu}) = \frac{(2i)^{\frac{d(d+1)}{2}}}{(d+1)!} \prod_{1 \leq \mu < \nu \leq d+1} \sin \pi(t_\mu - t_\nu).
\end{equation}

Furthermore,

\begin{equation}
z_k \text{TS}_{\mathbf{v}^o}(t) = \frac{k!(d+1-k)!}{(d+1)!} \text{TS}_{\mathbf{v}^o + \mathbf{v}^k}(t), \quad 1 \leq k \leq d.
\end{equation}

Proof. Let $\beta := (d, d-1, \ldots, 1, 0) \in \mathbb{N}_0^{d+1}$. Then

\[ e^{\frac{2\pi i}{d\sigma} \mathbf{v}^o \cdot t} = e^{-\pi i(d_1 + \ldots + t_\beta)} e^{2\pi i \beta t} = e^{2\pi i \mathbf{t}}. \]

Using the well-known Vandermonde determinant (see, for example, [21, p. 40]),

\[ \sum_{\sigma \in S_{d+1}} \rho(\sigma) e^{\frac{2\pi i}{d\sigma} \mathbf{v}^o \cdot t} = \frac{1}{(d+1)!} \prod_{1 \leq \mu < \nu \leq d+1} (e^{2\pi it_\mu} - e^{2\pi it_\nu}). \]

and setting $x_j = e^{2\pi it_j}$, we obtain

\[ \text{TS}_{\mathbf{v}^o}(t) = \frac{1}{(d+1)!} \prod_{1 \leq \mu < \nu \leq d+1} (e^{2\pi it_\mu} - e^{2\pi it_\nu}). \]

Furthermore, since $t_1 + \ldots + t_{d+1} = 0$, the above equation immediately gives

\[ \text{TS}_{\mathbf{v}^o}(t) = \frac{1}{(d+1)!} \prod_{1 \leq \mu < \nu \leq d+1} (e^{2\pi it_\mu} - e^{2\pi it_\nu}). \]

from which the second equal sign of (5.6) follows.

Next we prove (5.7). Using (5.4), we have

\[ z_k \text{TS}_{\mathbf{v}^o}(t) = \text{TC}_{\mathbf{v}^k}(t) \text{TS}_{\mathbf{v}^o}(t) = \frac{1}{(d+1)!} \sum_{\sigma \in \mathcal{G}_k} \text{TS}_{\mathbf{v}^o + \mathbf{v}^k \sigma}(t). \]

Assume $\mathbf{I} := \mathbf{v}^k \sigma \neq \mathbf{v}^k$ for some $\sigma \in \mathcal{G}$. By the definition of $\mathbf{v}^k$, there exists an integer $1 \leq j \leq d$ such that $l_j = -k$ and $l_{j+1} = d+1-k$. Consequently, $v_j^o + l_j = (d+2-2j)(d+1) - k = (d-2j)(d+1) + (d+1-k) = v_{j+1}^o + l_{j+1}$, which implies that $\mathbf{v}^o + \mathbf{v}^k \sigma \in \partial \Lambda$ and $\text{TS}_{\mathbf{v}^o + \mathbf{v}^k \sigma}(t) = 0$.

Thus, by the definition of the stabilizer $\mathcal{G}_k$,

\[ z_k \text{TS}_{\mathbf{v}^o}(t) = \frac{1}{(d+1)!} \sum_{\sigma \in \mathcal{G}_k} \text{TS}_{\mathbf{v}^o + \mathbf{v}^k \sigma}(t) = \frac{k!(d+1-k)!}{(d+1)!} \text{TS}_{\mathbf{v}^o + \mathbf{v}^k}(t), \]

as the stabilizer of $\mathbf{v}^k$ has cardinality $k!(d+1-k)!$.\]

The same argument that proves (5.6) also shows that, for each $\mathbf{k} \in \Lambda$,

\[ \text{TS}_{\mathbf{k} + \mathbf{v}^o}(t) = \det \left( x_i^{\lambda_i + \beta} \right)_{1 \leq i, j \leq d+1}, \quad \lambda := (k_1 - k_{d+1}, k_2 - k_{d+1}, \ldots, k_d - k_{d+1}, 0), \]

where $x_i = e^{2\pi it_i}$. The definition of $\Lambda$ shows that $\lambda$ is a partition. Hence, according to [21, p. 40], $\text{TS}_{\mathbf{k} + \mathbf{v}^o}(t)$ is divisible by $\text{TS}_{\mathbf{v}^o}$ in the ring $\mathbb{Z}[x_1, \ldots, x_d]$, and the quotient

\[ s_\lambda(x_1, \ldots, x_d) = \text{TS}_{\mathbf{k} + \mathbf{v}^o}(t) / \text{TS}_{\mathbf{v}^o}(t) \]

is a symmetric polynomial in $x_1, \ldots, x_d$, which is the Schur function in the variables $x_1, \ldots, x_d$ corresponding to the partition $\lambda$. Since this ratio is a symmetric polynomial, it is then a polynomial in the elementary symmetric polynomials $z_1, \ldots, z_d$; more precisely, it is a polynomial in $z$ of degree $(k_1 - k_{d+1})/(d+1)$ as shown by the formula [21 (3.5)]. This is our analogue of Chebyshev polynomials of the second kind.

To simplify the notation, we find it convenient to change the index and define the Chebyshev polynomials of the first and the second kind formally as follows:
Define the index mapping \( \alpha : \Lambda \to \mathbb{N}_0^d \),
\[
\alpha_i = \alpha_i(k) := \frac{k_i - k_{i+1}}{d+1}, \quad 1 \leq i \leq d.
\]
Under the change of variables \((5.1)\) and \((5.8)\), define
\[
T_\alpha(z) := \text{T}_k(t) \quad \text{and} \quad U_\alpha(z) := \frac{\text{T}_{k+v}(t)}{\text{T}_{v}(t)}, \quad \alpha \in \mathbb{N}_0^d.
\]
We call \( T_\alpha(z) \) and \( U_\alpha(z) \) Chebyshev polynomials of the first and the second kind, respectively.

It is easy to see that the mapping \((5.8)\) is an isomorphism. Indeed, since \( k_1 + \ldots + k_{d+1} = 0 \) for \( k \in \Lambda \), it is easy to see that the inverse of \( \alpha \) is given by
\[
k_i = \sum_{\mu=1}^d \sum_{\nu=1}^\mu \alpha_\nu - (d + 1) \sum_{\nu=1}^{i-1} \alpha_\nu, \quad 1 \leq i \leq d + 1.
\]
We also note that \( \alpha(v^k) = c_k := (\{0\}^{k-1}, 1, \{0\}^{d-k}) \), the \( k \)-th element of the standard basis for the Euclidean space \( \mathbb{R}^d \).

In the recurrence relation that we shall state in a moment, we will need the definition of \( T_\alpha(z) \) in which \( \alpha \) can have negative components. If \( \alpha \) has a negative component, say \( \alpha_i < 0 \), then \( k_i < k_{i+1} \), so that \( k \) in \((5.8)\) does not belong to \( \Lambda \). If \( k \in \mathbb{N} \), we define by \( k^+ \) the rearrangement of \( k \) such that \( k^+ \in \Lambda \). By the definition of \( T_k \), we have \( T_{k^+}(t) = T_{k^+}(t) \) for all \( k \in \mathbb{N} \). Thus, if \( \alpha \) has a negative component, then we define
\[
T_\alpha(z) = T_{\alpha^+}(z) \quad \text{where} \quad \alpha^+ \text{ corresponds to } k^+ \text{ by } \text{(5.8)}.
\]

Both \( T_\alpha(z) \) and \( U_\alpha(z) \) are polynomials of degree \( |\alpha| = \alpha_1 + \ldots + \alpha_d \) in \( z \). Moreover, both of them satisfy a simple recursive relation, which we summarize in the following theorem.

Theorem 5.4. Let \( P_\alpha \) denote either \( T_\alpha \) or \( U_\alpha \). Then
\[
(5.9) \quad P_\alpha(z) = P_{\alpha_1,\alpha_2,\ldots,\alpha_1}(z), \quad \alpha \in \mathbb{N}_0^d,
\]
and they satisfy the recursion relation
\[
(5.10) \quad \binom{d+1}{i} z_i P_\alpha(z) = \sum_{j \in \mathbb{N}^d} P_{\alpha_1 \alpha_2 \ldots \alpha_1}(j), \quad \alpha \in \mathbb{N}_0^d,
\]
in which the components of \( \alpha(j) \), \( j \in \mathbb{N}^d \), have values in \( \{-1,0,1\} \), \( U_\alpha(z) = 0 \) whenever \( \alpha \) has a component \( \alpha_i = -1 \), and
\[
T_\alpha(z) = 1, \quad T_{k_1}(z) = z_k, \quad 1 \leq k \leq d
\]
and
\[
U_\alpha(z) = 1, \quad U_{k_1}(z) = \binom{d+1}{k} z_k, \quad 1 \leq k \leq d.
\]

Proof. The relation \((5.9)\) follows readily from the fact that \(-k_i = k_j \) and \((5.8)\). The relation \((5.10)\) follows immediately from \((5.3)\) and \((5.4)\). The values of \( P_0 \) and \( P_k \) follow from definitions and \((5.7)\). If \( \alpha \) has a component \( \alpha_i = -1 \) then, by \((5.8)\), \( k_i(\alpha) = k_{i+1}(\alpha) - (d + 1) \). A quick computation shows then \( k_i(\alpha) + v_1^i = k_{i+1}(\alpha) + v_i^1 \), which implies that \( k + v^0 \in \partial \Lambda \), so that \( \text{T}_{k+v}(t) = 0 \) and \( U_\alpha(z) = 0 \).

It is easy to see that \( |\alpha(j)|, j \in \mathbb{N}^d \), also takes value in \( \{-1,0,1\} \). As a result, in terms of the total degree \( |\alpha| \) of \( P_\alpha \), the right hand side of \((5.10)\) contains only polynomials of degree \( n - 1, n \) and \( n + 1 \), so that it is a three-term relation in that sense. For \( d = 2 \), the relation \((5.10)\) can be rewritten as a recursive relation,
\[
P_{\alpha+e_1}(z) = 3z_1 P_\alpha(z) - P_{\alpha+(1,1)}(z) - P_{\alpha-e_2}(z),
\]
\[
P_{\alpha+e_2}(z) = 3z_2 P_\alpha(z) - P_{\alpha+(1,1)}(z) - P_{\alpha-e_1}(z)\]
which can then be used to generate a polynomial $P_\alpha$ from lower degree polynomials recursively. The same, however, cannot be said for $d \geq 3$. For example, if $d = 3$ then (5.10) for $k = 2$ is

\[6z_2 P_\alpha(z) = P_{\alpha + 2}(z) + P_{\alpha + (1, -1, 1)}(z) + P_{\alpha + (0, 0, -1)}(z) + P_{\alpha + (0, 0, 1)}(z),\]

which has two polynomials of $|\alpha| + 1$ in the right hand side, $P_{\alpha + 2}(z)$ and $P_{\alpha + (1, -1, 1)}(z)$. It is possible to combine the relations in (5.10) to write $P_{\alpha + 2}(z) = \sum_{i=1}^d (d+1)_i z_i P_\alpha(z) + Q(z)$, where $Q(z)$ contains only linear combinations of \{\(P_\beta\) for $|\beta| = |\alpha|$ and $|\beta| = |\alpha| - 1$. In lower dimension, the exact forms of $Q$ can be easily determined (for $d = 3$ see [29]), but the general formula for a generic $d$ appears to be complicated and we shall not pursue it here.

Next we show that $T_\alpha$ and $U_\alpha$ are orthogonal polynomials. The integral of the orthogonality is taken over the region that is the image of $\Delta_H$ with respect to a measure, or weight function, that comes from the Jacobian of the change of variables. Furthermore, since $z_k \mapsto z_{d+1-k}$, we can consider real coordinates, denoted by $x$,

\[x_k = z_k + \frac{z_{d+1-k}}{2}, \quad x_{d+1-k} = \frac{z_k - z_{d+1-k}}{2i} \quad \text{for} \quad 1 \leq k \leq \lfloor \frac{d}{2} \rfloor,\]

(5.11)

and

\[x_{d+1-k} = \frac{1}{\sqrt{2}} \frac{z_{d+1}}{2} \quad \text{if} \quad d \text{is odd.}\]

Combining (5.1) and (5.11), our change variable becomes $\mathbb{R}_H^{d+1} \mapsto \mathbb{R}^d; t \mapsto x$. Let us define

\[w(x) = w(x(t)) := \prod_{1 \leq \mu \leq \nu \leq d+1} \sin^2 \pi (t_\mu - t_\nu).\]

**Lemma 5.5.** The Jacobian of the changing variable $t \mapsto x$ is given by

\[\det \left[ \frac{\partial(x_1, x_2, \ldots, x_d)}{\partial(t_1, t_2, \ldots, t_d)} \right] = 2^{d(d+2)} \prod_{k=1}^{d} \frac{\pi}{(d+1)} |w(x)|^{\frac{k}{2}}.\]

**Proof.** Let $\partial_i$ denote the partial derivative with respect to $t_i$ for $1 \leq i \leq d$. We first prove that

\[\det \left[ \frac{\partial(z_1, z_2, \ldots, z_d)}{\partial(t_1, t_2, \ldots, t_d)} \right] = \prod_{k=1}^{d} 2 \pi i \left( \frac{d+1}{k} \right) \prod_{1 \leq \mu \leq \nu \leq d+1} \sin \pi (t_\mu - t_\nu).\]

(5.13)

Using the Jacobian of the elementary symmetric polynomials with respect to its variables and $t_1 + \ldots + t_{d+1} = 0$, this can be shown as in [3] (5.9). We give an inductive proof below.

Regarding $t_1, t_2, \ldots, t_{d+1}$ as independent variables, one sees that

\[\frac{\partial z_k}{\partial t_j} = \frac{2 \pi i}{(d+1)} \sum_{j \in I \subseteq [n_{d+1}^d], |I| = k} e^{2 \pi i \sum_{\nu \in I} t_\nu}.\]

For each fixed $j$, split $I \subseteq [n_{d+1}^d]$ as two parts, one contains \{\(j, d+1\)\} and one does not, so that after canceling the common factor, we obtain

\[\frac{\partial z_k}{\partial t_j} - \frac{\partial z_k}{\partial t_{d+1}} = \frac{2 \pi i}{(d+1)} \left( \sum_{j \in I \subseteq [n_{d+1}^d], |I| = k} e^{2 \pi i \sum_{\nu \in I} t_\nu} - \sum_{d+1 \in I \subseteq [n_{d+1}^d], |I| = k} e^{2 \pi i \sum_{\nu \in I} t_\nu} \right) \]

\[= \frac{2 \pi i}{(d+1)} \left(e^{2 \pi i t_j} - e^{2 \pi i t_{d+1}}\right) \sum_{I \subseteq [n_{d+1}^d], |I| = k-1} e^{2 \pi i \sum_{\nu \in I} t_\nu}.\]

Hence, setting $f^{d,k}_{j,k} := \sum_{I \subseteq [n_{d+1}^d], |I| = k-1} e^{2 \pi i \sum_{\nu \in I} t_\nu}$ and defining the matrix $F_d := (f^{d,k}_{j,k})_{1 \leq j,k \leq d}$, we have shown that

\[\det \left( \frac{\partial(z_1, z_2, \ldots, z_d)}{\partial(t_1, t_2, \ldots, t_d)} \right) = \prod_{k=1}^{d} \frac{2 \pi i}{(d+1)} \times \prod_{j=1}^{d} \left(e^{2 \pi i t_j} - e^{2 \pi i t_{d+1}}\right) \times \det(F_d).\]
By definition, \( f_{d,1}^d = 1 \); splitting \( \mathbb{N}^{(d)} \) according to if the part contains \( d \) or not, it follows that
\[
f_{d,k}^d - f_{d,k}^{d-1} = (e^{2\pi i t_d} - e^{2\pi i t_d}) \sum_{I \subseteq \mathbb{N}^{(d)},|I|=k-2} e^{2\pi i \sum_{i \in I} t_i} = (e^{2\pi i t_d} - e^{2\pi i t_d}) f_{d,k-1}^{d-1},
\]
which allows us to use induction and show that
\[
\det(F_d) = \prod_{1 \leq \mu < d-1} (e^{2\pi i t_\mu} - e^{2\pi i t_\mu}) \times \det(F_{d-1}) = \ldots = \prod_{1 \leq \mu < \nu \leq d} (e^{2\pi i t_\mu} - e^{2\pi i t_\nu}).
\]
This proves (5.13) upon using (5.6). Now, a quick computation using (5.11) shows that
\[
\left| \det \left[ \frac{\partial}{\partial(x_1, x_2, \ldots, x_d)} \right] \right| = \left| \det \left[ \frac{\partial}{\partial(z_1, z_2, \ldots, z_d)} \right] \right| \times \left| \det \left[ \frac{\partial}{\partial(z_1, x_2, \ldots, x_d)} \right] \right|
\]
\[
= 2^{d(d+2)} \prod_{k=1}^{d} \frac{\pi}{k} \prod_{1 \leq \mu < \nu \leq d+1} \left| \sin \pi (t_\mu - t_\nu) \right|
\]
which is exactly (5.12).

Under this change of variables, the domain \( \Delta_H \) is mapped to
\[\Delta^* := \left\{ x = x(t) \in \mathbb{R}^d : t \in \mathbb{R}^{d+1}, \prod_{1 \leq i < j \leq d+1} \sin \pi (t_i - t_j) \geq 0 \right\},\]
which is the image of \( \Delta_H \) under the mapping \( t \mapsto x \). In the case of \( d = 2 \) and \( d = 3 \), \( \Delta^* \) is depicted in the Figure 5.1.

\[\text{Figure 5.1. The region } \Delta^* \text{ for } d = 2 \text{ and } d = 3\]

We will need the cases of \( \mu = -1/2 \) and \( \mu = 1/2 \) of the weighted inner product
\[\langle f, g \rangle_\omega := c_\alpha \int_{\Delta^*} f(z) g(z) w^\alpha(z) dx,\]
where \( c_\alpha \) is a normalization constant, \( c_\alpha := 1/\int_{\Delta^*} w^\alpha(z) dx \). The change of variables \( t \mapsto x \) shows immediately that
\[\frac{1}{|\Delta_H|} \int_{\Delta_H} f(t) dt = c_{-1/2} \int_{\Delta^*} f(z) w^{-1/2}(z) dx,\]
where
\[c_{-1/2} = \frac{1}{|\Delta_H|} \times \frac{\omega^{1/2}(z)}{\left| \det \left( \frac{\partial(x_1, x_2, \ldots, x_d)}{\partial(t_1, t_2, \ldots, t_d)} \right) \right|} = 2^{-d(d+2)} \sqrt{d+1} d! \prod_{k=1}^{d} \frac{(d+1)_k}{\pi}.\]
Furthermore, by (5.6) and (5.5),
\[
w(z) = \left(\frac{-1}{2}\right)^{\frac{d(d+1)}{4}} [(d+1)!]^2 \left[TS_\nu(t)\right]^2 = \left(\frac{-1}{2}\right)^{\frac{d(d+1)}{2}} (d+1)! \sum_{\sigma \in \mathcal{U}} (-1)^{\sigma_1} TC_{\nu^+ \nu^+ \sigma}(t),
\]
which shows that \( w \) is a polynomial in \( z \) with a total degree of \( 2d \), and furthermore, by (4.6),
\[
c_z = \frac{1}{\int_{\Delta^d} w(x(t))dx} = \frac{1}{\int_{\Delta^d} w(x(t))dt} = \sqrt{\frac{2d^2}{d+1} \prod_{k=1}^{d} \frac{(d+1)_k}{\pi}}.
\]

The orthogonality of \( T_\kappa \) and \( U_\alpha \), respectively, then follows from the orthogonality of \( TC_k \) and \( TS_k \) with the change of variables. More precisely, Theorem 4.3 leads to the following theorem:

**Theorem 5.6.** The polynomials \( T_\alpha \) and \( U_\alpha \) are orthogonal polynomials with respect to \( w^{-1/2} \) and \( w^{1/2} \), respectively, and

\[
\langle T_\alpha, T_\beta \rangle_{w^{-1/2}} = d_\alpha \delta_{\alpha, \beta}, \quad d_\alpha = \frac{1}{(d+1)^{\frac{1}{2}}} \quad \text{if} \ \alpha \in \Lambda^p.
\]

\[
\langle U_\alpha, U_\beta \rangle_{w^{1/2}} = \delta_{\alpha, \beta} \left(\frac{d}{d+1}\right)!, \quad \alpha, \beta \in \mathbb{N}_0^d,
\]

where \( \Lambda^p \) is the image of \( \Lambda^p \) under the index mapping (5.8),

\[
\Lambda^p = \left\{ \alpha \in \mathbb{N}_0^d : \alpha = \alpha(k), \quad k \in \Lambda^p \right\}.
\]

The orthogonality of these polynomials has been established in the literature; we refer to [3] and the references therein. By (4.10), the set \( \{T_\alpha : \alpha \in \mathbb{N}_0^d\} \) and, respectively, \( \{U_\alpha : \alpha \in \mathbb{N}_0^d\} \) forms a mutually orthogonal basis for the space \( \Pi^d \) of polynomials in \( d \)-variables with respect to \( w^{-1/2} \) and, respectively \( w^{1/2} \).

### 5.2. Zeros of Chebyshev polynomials of the second kind and Gaussian cubature.

It is well known that zeros of orthogonal polynomials of one variable are nodes of the Gaussian quadrature. We consider its extension in several variables in this subsection. First we review the background in several variables.

Let \( w \) be a nonnegative weight function defined on a compact set \( \Omega \) in \( \mathbb{R}^d \). A cubature formula of degree \( 2n-1 \) for the integral with respect to \( w \) is a sum of point evaluations that satisfies

\[
\int_{\Omega} f(x) w(x) dx = \sum_{j=1}^{N} \lambda_j f(x^j), \quad \lambda_j \in \mathbb{R}
\]

for every \( f \in \Pi^{\frac{d}{2n-1}} \). The points \( x^j = (x^j_1, x^j_2, \ldots, x^j_d) \) are called nodes of and the numbers \( \lambda_j \) are called weights of the cubature. It is well-known that a cubature formula of degree \( 2n-1 \) exists only if \( N \geq \dim \Pi^{\frac{d}{2n-1}} \). A cubature that attains such a lower bound is called a Gaussian cubature.

In one variable, a Gaussian quadrature always exists and its nodes are zeros of orthogonal polynomials. For several variables, however, Gaussian cubatures exist only rarely. In fact, the first family of weight functions that admit Gaussian cubature were discovered only relatively recent in [9]. When they do exist, however, their nodes are common zeros of orthogonal polynomials. To be more precise, let \( \mathbb{P}_n := \{ P_\kappa : \kappa \in \mathbb{N}_0^d, \ |\kappa| = n \} \) denote a basis of orthonormal polynomials of degree \( n \) with respect to \( w(x)dx \). Then, it is known that a Gaussian cubature exists if and only if \( \mathbb{P}_n \) (every element in \( \mathbb{P}_n \)) has \( \dim \Pi^{\frac{d}{2n-1}} \) real common zeros; in other words, the polynomial ideal generated by \( \{ P_\kappa : \ |\kappa| = n \} \) has a zero dimensional variety of size \( \dim \Pi^{\frac{d}{2n-1}} \). See [11, 23, 24] for these results and further discussions.

Below we shall show that a Gaussian cubature exists for \( w^{1/2} \), which makes \( w^{1/2} \) the second family of examples that admits Gaussian cubature. First we study the zeros of the Chebyshev
polynomials of the second kind. Because of $\bar{z}_k = z_{d+1-k}$ and (5.9), the real and complex parts of the polynomials in $\{U_{\alpha}(\xi) : |\alpha| = n\}$ form a real basis for the space of orthogonal polynomials of degree $n$. Hence, we can work with the zeros of the complex polynomials $U_{\alpha}(\xi)$.

Let $Y_n$ and $Y_n^\circ$ be the image of $\left\{ \frac{j}{(d+1)n} : j \in \Lambda_n \right\}$ and $\left\{ \frac{j}{(d+1)n} : j \in \Lambda_n^\circ \right\}$ under the mapping $t \mapsto x$ respectively,

$$Y_n := \left\{ \frac{j}{(d+1)n} : j \in \Lambda_n \right\} \quad \text{and} \quad Y_n^\circ := \left\{ \frac{j}{(d+1)n} : j \in \Lambda_n^\circ \right\}.$$ 

**Theorem 5.7.** The set $Y_{n+d}^\circ$ is the variety of the polynomial ideal $\langle U_{\alpha}(x) : |\alpha| = n \rangle$.

**Proof.** The definition of $U_{\alpha}$ and (5.8) shows that if $U_{\alpha}$ has degree $n$, that is, $|\alpha| = n$, then $k \in \Lambda^\circ$ in $TS_{\psi_{-k}}/TS_{\psi}$ satisfies $k_1 - k_{d+1} = (d + 1)n$. Hence, in order to show that $U_{\alpha}$ vanishes on $Y_{n+d}^\circ$, it suffices to show, by the definition of $\psi$, that

$$TS_k \left( \frac{j}{(d+1)n} \right) = 0 \quad \text{for} \quad k \in \Lambda^\circ \text{ and } k_1 - k_{d+1} = (d + 1)n.$$ 

Recall that $\sigma_{i,j}$ denotes the transposition that interchanges $i$ and $j$. A simply computation shows that

$$\phi_k \left( \frac{j}{(d+1)n} \right) - \phi_k \left( \frac{j_1 + j_{d+1}}{(d+1)n} \right) = 2i \sin \frac{\pi(j_1 - j_{d+1})(k_1 - k_{d+1})}{(d+1)^2 n} e^{\frac{\pi i}{2d+1}((j_1+j_{d+1})(k_1+k_{d+1})+2\sum_{\nu=\nu_1}^d j_\nu k_\nu)},$$

which is 0 whenever $j \in \Lambda_n$ and $k \in \Lambda_n$ satisfies $k_1 - k_{d+1} = (d + 1)n$. Hence, it follows that

$$TS_k \left( \frac{j}{(d+1)n} \right) = \frac{1}{2|G|} \sum_{\sigma \in G} \varepsilon_\sigma \left[ \phi_k \left( \frac{j}{(d+1)n} \right) - \phi_k \left( \frac{j_1 + j_{d+1}}{(d+1)n} \right) \right] = 0,$$

where $\varepsilon_\sigma$ takes value of either $-1$ or $1$. \hfill $\square$

By (4.10), $|Y_{n+d}^\circ| = |\Lambda_{n+d}^\circ| = \binom{n+d+1}{d+1} = \dim \Pi_{n-1}^d$. According to the general theory, Theorem 5.7 implies that the Gaussian cubature exists for the weight function $w^\frac{1}{2}$ on $\Delta^*$. In fact, the precise form of this cubature is known.

**Theorem 5.8.** For $w^\frac{1}{2}$ on $\Delta^*$, a Gaussian cubature formula exists, which is given explicitly by

(5.16) $$c^\frac{1}{2} \int_{\Delta^*} f(x) w^\frac{1}{2}(x) dx = \frac{2(d+1)!}{(d+1)(n+d)!} \sum_{y \in Y_{n+d}^\circ} w(y) f(y), \quad \forall f \in \Pi_{2n-1}^d,$$

where the normalization constant $c^\frac{1}{2} = \sqrt{\pi}^d \prod_{k=1}^d \left( \frac{d+1}{2} \right)^k$.

**Proof.** The change of variables $t \mapsto x$ in (5.1) and (5.11) shows

$$c^\frac{1}{2} \int_{\Delta^*} f(z) w^\frac{1}{2}(z) dz = \frac{(d+1)!}{|\Delta_H|} \int_{\Delta_H} f(z(t)) |TS_{\psi}(t)|^2 dt$$

$$= \frac{(d+1)!}{(n+d)!} \sum_{j \in \Lambda_{n+d}^\circ} |TS_{\psi} \left( \frac{j}{(n+d)(d+1)} \right)|^2 f(z(\frac{j}{(n+d)(d+1)})],$$

where the last step follows from the fact that $TS_{\psi}(t)$ vanishes on the boundary of $\Delta$. This is exactly the cubature (4.12) applied to the function $f(z(t)) |TS_{\psi}(t)|^2$. \hfill $\square$

The set of nodes of a Gaussian cubature is poised for polynomial interpolation, that is, there is a unique polynomial $P$ in $\Pi_{n-1}^d$ such that $P(\xi) = f(\xi)$ for all $\xi \in Y_{n+d}^\circ$ and a generic function $f$. In fact, let $K_n(\cdot, \cdot)$ be the reproducing kernel of $\Pi_{n}^d$ in $L^2 w^\frac{1}{2}$, which can be written as

$$K_n(x, y) := \sum_{|\alpha|=n} (d + 1)U_{\alpha}(z)U_{\alpha}(w)$$
by (5.15), where \(x, y\) is related to \(z, w\) according to (5.11), respectively. Then it follows from the general theory (cf. \([11]\)) that the following result holds:

**Proposition 5.9.** The unique interpolation polynomial of degree \(n\) on \(Y_{n+d}^\circ\) in \(\Pi_{n-1}^d\) is given by

\[
L_n f(x(t)) = \sum_{j \in \Lambda_{n+d}^*} f(x\left(\frac{j}{(n+d)(d+1)}\right))w(x\left(\frac{j}{(n+d)(d+1)}\right))K_n^*(t, x, y),
\]

where \(K_n^*(t, s) = K_n(x, y)\) with \(x, y\) related to \(t, s\) by (5.1) and (5.11), respectively.

In fact, this interpolation operator is exactly the one in Theorem 5.6 under the change of variables \(t \mapsto x\). In particular, we can derive a compact formula for \(K_n(x, y)\) as indicated in that theorem.

5.3. Cubature formula and Chebyshev polynomials of the first kind. For the Chebyshev polynomials of the first kind, the polynomials in \(\{T_\alpha : |\alpha| = n\}\) do not have enough common zeros in general. In fact, as shown in [18], in the case of \(d = 2\), the three orthogonal polynomials of degree 2 has no common zero at all. As a consequence, there is no Gaussian cubature for the weight function \(w^{-\frac{1}{2}}\) on \(\Delta^*\) in general.

However, the change of variables \(t \mapsto x\) shows that (4.12) leads to a cubature of degree \(2n-1\) with respect to \(w^{-\frac{1}{2}}\) based on the nodes of \(Y_n\).

**Theorem 5.10.** For the weight function \(w^{-\frac{1}{2}}\) on \(\Delta^*\) the cubature formula

\[
c_{-\frac{1}{2}} \int_{\Delta^*} f(x)w^{-\frac{1}{2}}(x)dx = \frac{1}{(d+1)n^d} \sum_{j \in \Lambda_n} \lambda_j^{(n)} f\left(x\left(\frac{j}{n(d+1)}\right)\right), \quad \forall f \in \Pi_{2n-1},
\]

holds, where \(c_{-\frac{1}{2}} = 2^{-\frac{d(d+1)}{4}} \sqrt{d+1} d! \prod_{k=1}^d \frac{(d+1)}{\pi} \) and \(\lambda_j^{(n)}\) is given by (4.9).

It follows from (4.10) that the \(|Y_n| = \binom{n+d}{d} = \dim \Pi_n^d\) and \(Y_n\) includes points on the boundary of \(\Delta^*\), hence, the cubature in (5.18) is an analog of the Gauss-Lobatto type cubature for \(w^{-\frac{1}{2}}\) on \(\Delta^*\). The number of nodes of this cubature is more than the lower bound of \(\dim \Pi_{n-1}^d\). Such a formula can be characterized by the polynomial ideal that has the set of the nodes as its variety. Indeed, according to a theorem in [23], the set of nodes \(Y_n\) of the formula (5.18) must be the variety of a polynomial ideal generated by \(\dim \Pi_n^d\) linearly independent polynomials of degree \(n+1\), and these polynomials are necessarily quasi-orthogonal in the sense that they are orthogonal to all polynomials of degree \(n-2\). Given the complicated relation between ideals and varieties, it is of interesting to identify the polynomial ideal that generates the cubature (5.18).

This is given in the following theorem.

**Theorem 5.11.** Let \(\nu^* = \kappa^{-1}(\nu^*) = (d+1, \{0\}^{d-1}, -d-1)\). For \(\alpha = \alpha(k)\) as in (5.8) and let \(\alpha^* := \alpha((k - \nu^*)^+)\). Then \(Y_n\) is the variety of the polynomial ideal

\[
\langle T_\alpha(x) - T_{(\alpha^*)}(x) : |\alpha| = n + 1 \rangle.
\]

Furthermore, the polynomial \(T_\alpha(x) - T_{(\alpha^*)}(x)\) are of degree \(n + 1\) and orthogonal to all polynomials in \(\Pi_{n-2}\) with respect to \(w^{-\frac{1}{2}}\).

**Proof.** From the definition of \(TC_k\), we have

\[
TC_k(t) - TC_{k-v^*}(t) = \frac{1}{|G|} \sum_{\sigma \in G} \left( \phi_k + \phi_k\sigma_{1,d+1} \right) \left( \phi_{k-v^*} + \phi_k\sigma_{1,d+1+v^*} \right)(t\sigma).
\]
A direct computation shows that
\[
\phi_k(t) + \phi_k\sigma_{1,d+1}(t) - \phi_k \cdot \phi_{k\sigma_{1,d+1}+\nu}(t)
= \frac{\pi i}{\sin \pi t} \left( \exp \left( \sum_{\sigma=0}^{d-1} 2k\sigma t + (k_1 + k_{d+1})(t_1 + t_{d+1}) \right) e^{\pi i(t_1-t_{d+1})} \right)
\times \frac{d}{d+1}
\left( \frac{k_1-k_{d+1}+1}{d+1} \right)
= -4 \sin \pi (t_1 - t_{d+1}) \sin \pi \left( \frac{k_1-k_{d+1}+1}{d+1} \right) (t_1 - t_{d+1}) \frac{d}{d+1}
\left( \sum_{\nu=2}^{d} 2k\nu t + (k_1 + k_{d+1})(t_1 + t_{d+1}) \right).
\]
Hence, for any \( j \in \mathbb{N} \) and \( k \in \mathbb{H} \) with \( k_1 - k_{d+1} = (n+1)(d+1) \),
\[
(\phi_k + \phi_k \sigma_{1,d+1} - \phi_k \cdot \phi_{k\sigma_{1,d+1}+\nu})(\frac{1}{(d+1)n}) = 0,
\]
which yields that
\[
(TC_k - TC_{k\cdot\nu})(\frac{1}{(d+1)n}) = 0 \quad \text{for } k \in \mathbb{H} \text{ with } k_1 - k_{d+1} = (n+1)(d+1).
\]
As in the proof of Theorem 5.7 by 5.8, this shows that \( T_\alpha - T_{\alpha\cdot} \) vanishes on \( Y_n \).
Moreover, suppose \( k \in \mathbb{N} \) and set \( j := (k - \nu)^+ \); since \( k_1 \geq \ldots \geq k_{d+1} \) and \( k_i = k_j \mod d+1 \) for \( i \geq j \), it follows that \( j_1 = k_1 = (d+1) \) if \( k_1 > k_2 \) and \( j_1 = k_1 \) if \( k_1 = k_2 \), and \( j_{d+1} = k_{d+1} + (d+1) \) if \( k_{d+1} < k_d \) and \( j_{d+1} = k_{d+1} \) if \( j_{d+1} = k_d \). Consequently, \( (j_1 - j_{d+1}) - (k_1 - k_{d+1}) \in \{0, -d-1, -2d-2\} \), which shows that \( |\alpha^+| \in \{|\alpha|, |\alpha| - 1, |\alpha| - 2\} \), so that \( T_{\alpha\cdot} \) is a Chebyshev polynomial of degree at least \( n-1 \) and \( T_\alpha - T_{\alpha\cdot} \) is orthogonal to all polynomials in \( \Pi_n^d \).

We note that the general theory on cubature formulas in view of ideals and varieties also shows that there is a unique interpolation polynomial in \( \Pi_n^d \) based on the points in \( Y_n \). This interpolation polynomials, however, is exact the interpolation trigonometric polynomial in Theorem 4.7. We shall not stay the result formally.

**References**

[1] H. Bacry, Generalized Chebyshev polynomials and characters of GL(N, C) and SL(N, C), *Group Theoretical Methods in Physics, Lecture Notes in Phys.*, vol. 201, p. 483-485, Springer-Verlag, Berlin, 1984.

[2] H. Bacry, Zeros of polynomials and generalized Chebyshev polynomials, *Group theoretical methods in physics*, Vol. I (Yumrula, 1985), 481-494, VNU Sci. Press, Utrecht, 1986.

[3] R. J. Beerends, Chebyshev polynomials in several variables and the radial part of the Laplace-Beltrami operator, *Trans. Amer. Math. Soc.*, 328 (1991), 779-814.

[4] R. J. Beerends and E. M. Opdam, Certain hypergeometric series related to the root system BC, *Trans. Amer. Math. Soc.*, 339 (1993), 581-609.

[5] H. Berens, R. Schmiedl, and Yuan Xu, Multivariate Gaussian cubature formula, *Arch. Math. 64* (1995), 26-32.

[6] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd ed. Springer, New York, 1999.

[7] A. Debiard, Polynomes de Tchebychev et de Jacobi dans un espace euclidien de dimension p, *C. R. Acad. Sci. Paris 296* (1983), 529-532.

[8] K. B. Dunn and R. Lidl, Multi-dimensional generalizations of the Chebyshev polynomials, *Proc. Japan Acad. 56* (1980), 154-165.

[9] K. B. Dunn and R. Lidl, Generalizations of the classical Chebyshev polynomials to polynomials in two variables, *Czechoslovak Math. J.* 32 (1982), 516-528.

[10] D. E. Dudgeon and R. M. Mersereau, *Multidimensional Digital Signal Processing*, Prentice-Hall Inc, Englewood Cliffs, New Jersey, 1984.

[11] C. F. Dunkl and Yuan Xu, *Orthogonal polynomials of several variables*, Encyclopedia of Mathematics and its Applications, vol. 81, Cambridge Univ. Press, 2001.

[12] R. Eier and R. Lidl, A class of orthogonal polynomials in k variables, *Math. Ann.* 260 (1982), 93-99.

[13] B. Fuglede, Commuting self-adjoint partial differential operators and a group theoretic problem, *J. Functional Anal.* 16 (1974), 101-121.

[14] J. R. Higgins, *Sampling theory in Fourier and Signal Analysis, Foundations*, Oxford Science Publications, New York, 1996.
[15] T. Koornwinder, Orthogonal polynomials in two variables which are eigenfunctions of two algebraically independent partial differential operators, I - IV, *Nederl. Acad. Wetensch. Proc. Ser. A* 77 = *Indag. Math.* 36 (1974), 48-66 and 357-381.

[16] A. U. Klimyk and J. Patera, (Anti)symmetric multivariate exponential functions and corresponding Fourier transforms. *J. Phys. A* 40 (2007), 10473–10489.

[17] A. U. Klimyk and J. Patera, (Anti)symmetric multivariate trigonometric functions and corresponding Fourier transforms. *J. Math. Phys.* 48 (2007), no. 9, 093504, 24 pp.

[18] H. Li, J. Sun and Yuan Xu, Discrete Fourier analysis, cubature and interpolation on a hexagon and a triangle, *SIAM J. Numer. Anal.*, 46 (2008) 1653-1681.

[19] H. Li and Yuan Xu, Discrete Fourier analysis on a dodecahedron and a tetrahedron, *Mathematics of Computation*, to appear (preprint, 2008).

[20] R. Lidl, Tschebyscheff polynome in mehreren variablen, *J. Reine .Angew. Math.* 273 (1975), 178-198.

[21] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed, Oxford Univ. Press Inc., New York, 1995.

[22] R. J. Marks II, *Introduction to Shannon Sampling and Interpolation Theory*, Springer-Verlag, New York, 1991.

[23] I. P. Mysovskikh, *Interpolatory cubature formulas*, Nauka, Moscow, 1981.

[24] A. N. Podkorytov, Order of growth of Lebesgue constants of Fourier sums over polyhedra (Russian. English summary). *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.* 1982, 110–111, 127.

[25] P. E. Ricci, Chebyshev polynomials in several variables, (Italian), *Rend. Mat.* 11 (1978), 295-327.

[26] A. Stroud, *Approximate calculation of multiple integrals*, Prentice-Hall, Englewood Cliffs, NJ, 1971.

[27] J. Sun, Multivariate Fourier series over a class of non tensor-product partition domains, *J. Comput. Math.* 21 (2003), 53-62.

[28] J. Sun, Multivariate Fourier transform methods over simplex and super-simplex domains, *J. Comput. Math.* 24 (2006), 305-322.

[29] J. Sun, A new class of three-variable orthogonal polynomials and their recurrences relations, *Science in China, Series A*, 51 (2008), 1071-1092.

[30] X. Sun, Approximation on the Voronoi cells of the $A_d$ lattice, *manuscript*, 2008.

[31] V. A. Yudin, Lower bound of the Lebesgue constants. (Russian) *Math. Notes* 25 (1979), no. 1-2, 63 - 65.

[32] Yuan Xu, On orthogonal polynomials in several variables, *Special Functions, q-series and Related Topics*, The Fields Institute for Research in Mathematical Sciences, Communications Series, Volume 14, 1997, p. 247-270.

[33] Yuan Xu, Polynomial interpolation in several variables, cubature formulae, and ideals, *Advances in Comp. Math.*, 12 (2000), 363–376.

[34] A. Zygmund, *Trigonometric series*, Cambridge Univ. Press, Cambridge, 1959.

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