Numerical solution of a parabolic optimal control problem with time-fractional derivative

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Abstract. The article deals with a linear-quadratic optimal control problem governed by a parabolic equation with time-fractional derivative taken in Caputo definition. We impose pointwise constraints on state and control functions. The derivative in time of state function is taken in the Caputo derivative form of fractional order $\alpha \in (0, 1)$. The mesh approximation is based on the using of finite-difference implicit scheme for the state equation. The Lagrange function’s saddle point satisfies the saddle point problem. Uzawa-type iterative method for numerical solution is used. The results of numerical tests are presented.

1. Introduction

Many physical and biological processes are described by differential differential equations of fractional order. There are some difficulties to find the exact solution of these equations in spite of results on the subject [4], [10], [14]. The research of boundary value problems for the equations of diffusion of fractional order in the differential and difference settings have been presented in [1], [2]. However, there are few articles devoted to parabolic optimal control problems with derivatives of fractional order (cf. [3], [5], [9], [11], [15], [16]). We solve one of these problems with Caputo derivative, using mesh approximation and preconditioned Uzawa-type method. An important task is to construct effective iterative methods for solving the discrete problem obtained. New numerical technique is presented in this work for a parabolic optimal control problem with Caputo fractional partial derivative in time of order $0 < \alpha < 1$ for state function. A saddle point problem is constructed for the mesh approximation of the optimal control problem using Lagrange function. A preconditioned Uzawa-type method with special form of a preconditioner is used for finding its solution. The results of numerical experiments are presented.

2. Problem and its approximation

Let $\Omega = (0, 1) \times (0, 1)$ with the boundary $\partial \Omega$, $Q_T = \Omega \times (0, T]$ and $\Sigma_T = \partial \Omega \times (0, T]$. A parabolic initial-boundary value problem in the form

$$\frac{C_D}{\alpha} D_t^\alpha y = u \text{ in } Q_T, \quad y = 0 \text{ on } \Sigma_T, \quad y = y_0(x) \text{ for } t = 0, x \in \Omega \quad (1)$$
Above $V$ and $y$ with a given observation function $C$ is a state problem. Here we denote by $C_D^α$ Caputo derivative operator defined as

$$C_D^α y(x,t) = \frac{1}{\Gamma(1-α)} \int_0^t (t-s)^{-α} \frac{∂y}{∂s}(x,s) \, ds, \quad 0 < α < 1.$$  

Note that if $y \in C^1[Q_T]$, then $\lim_{α→1-0} C_D^α y(x,t) = y(x,t)$ ∀$(x,t) ∈ Q_T$.

Traditionally, the approximation for the time-fractional derivative can be constructed by two ways: the L1 formula and the Gr¨ unwald-Letnikov formula. The first way is based on the piecewise linear interpolation with respect to $t$ for function $y(x,t)$ inside the integral in the Caputo fractional derivative sense, while the latter is often used to handle the Riemann-Liouville time-fractional derivative. For the $α$-th $(0 < α < 1)$ fractional derivative, the numerical accuracy of L1 formula is proved to be $2 - α$ (cf. [12], [8]), which is less than two, and that of the Gr¨ unwald-Letnikov formula depends on the choice of generating function of coefficients.

We define the sets of constraints for control $u$ and state $y$:

$$U_{ad} = \{u ∈ L_2(Q_T) : |u(x,t)| ≤ u_{max} \, a.e. \, (x,t) ∈ Q_T\},$$

$$Y_{ad} = \{y : y_{min} ≤ y(x,t) ≤ y_{max} \, a.e. \, (x,t) ∈ Q_T\}.$$  

Above $u_{max} > 0$ and $−∞ ≤ y_{min} < 0 ≤ y_{max} ≤ ∞$. Objective functional is defined as

$$J(y,u,q) = \frac{1}{2} \int_{Q_T} (y(x,t) − y_d(x,t))^2 dxdt + \frac{1}{2} \int_{Q_T} u^2 dxdt$$  

with a given observation function $y_d ∈ L_2(Q_T)$. Optimal control problem reads as follows:

$$\min_{(y,u)∈K} J(y,u),$$

$$K = \{(y,u) ∈ Y_{ad} × U_{ad} : \text{equation (1) holds}\}.$$  

It is a minimization problem of quadratical functional $J(y,u)$ on a closed, bounded and convex set $K ≠ ∅$, so, it has a unique solution (cf. [13]).

We will suppose for the simplicity that the function $y_d$ is continuous.

We construct a finite difference approximation of problem (3) on the uniform grid $ω_x × ω_T$ in $Q_T$, where $ω_x$ is a grid with a step $h$ while $ω_T = \{t_j = jτ, j = 0,1,\ldots; Mτ = T\}$. Let $V_h$ be the space of mesh functions defined on the grid $ω_x$ vanishing in the boundary nodes $\partialω_x$ and $y^j = y(x,t_j) ∈ V_h$ be a mesh function on a time level $t_j = jτ ∈ ω_T$. Later we use the same notations both for mesh functions and the vectors of their nodal values. By $N$ we denote the dimension of $V_h$ and by $∥⋅∥_x$ the euclidian norm of the vectors of nodal values in this space.

Let $A$ be a matrix of mesh Laplace operator on the grid $ω_x$ with homogeneous Dirichlet boundary conditions. We takes the Caputo fractional derivative in the form

$$C_D^α y(x,t_j) = \frac{1}{\Gamma(1-α)} \sum_{k=1}^j \int_{t_{k-1}}^{t_k} (t_j − s)^{-α} \frac{∂y}{∂s}(x,s) \, ds,$$

and it’s L1 approximation has the following form

$$D^α y^j_k := \frac{1}{\Gamma(1-α)} \sum_{k=1}^j \frac{y^j_k − y^j(k-1)}{\tau} \, \int_{s=t_{k-1}}^{t_k} (t_j − s)^{-α} \, ds =$$

$$= \frac{τ^{-α}}{Γ(2-α)} \left( d_0 y^j_k − d_1 y^j_0 + \sum_{k=1}^{j-1} (d_{k+1} − d_k) y^{j-k}_k \right),$$

and it’s L1 approximation has the following form

$$\sum_{i=1}^N (d_{k+1} − d_k) y^{j-k}_k,$$
where \(d_k = k^{1-\alpha} - (k-1)^{1-\alpha}\) and \(d_k > d_{k+1} > 0, k = 1, \ldots, j\).

Let us approximate state problem (1) using implicit difference scheme:

\[
Ly_i^k = D^\alpha y_i^k + Ay_i^k = u_i^k, \quad y_0^i = y_0(x_i), \quad i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, M.
\]  

(4)

The approximation of the objective function (2) on the grid \(\omega_x \times \omega_r\) is the sum

\[
\frac{\tau h^2}{2} \left( \sum_{j=1}^{M} ||y_j^k - y_j^0||^2_x + \sum_{j=1}^{M} ||u_j^k||^2_e \right). \quad \text{We scaled it to derive the following mesh objective function:}
\]

\[
I(y, u) = \frac{1}{2} \left( \sum_{j=1}^{M} ||y_j^k - y_j^0||^2_x + \sum_{j=1}^{M} ||u_j^k||^2_e \right).
\]

(5)

The sets of the constraints for the mesh control and state functions we define as follows:

\[
U_{ad} = \{(u^1, \ldots, u^M) : u^j \in V_h, |u^j| \leq u_{\max}\},
\]

\[
Y_{ad} = \{(y^1, \ldots, y^M) : y^j \in V_h, y_{\min} \leq y^j \leq y_{\max}\}.
\]

Now, mesh optimal control problem reads as follows:

\[
\min_{(y, u) \in K_h} I(y, u),
\]

\[
K_h = \{(y, u) \in Y_{ad} \times U_{ad} : \text{equation (4) holds}\}.
\]

(6)

Problem (6) is a minimization problem of a quadratical function on a compact set, and it has a unique solution.

3. Iterative method for the mesh problem

Let \(N = N_x M\) be the dimension of the mesh functions of the variables \(x\) and \(t\), \(E \in \mathbb{R}^{N \times N}\) is unit matrix and \(L \in \mathbb{R}^{N \times N}\) is defined by the equality (4).

Let also \(\theta\) and \(\varphi\) be the indicator functions of the sets \(Y_{ad}\) and \(U_{ad}\), respectively, while \(\partial \theta\) and \(\partial \varphi\) are their subdifferentials. Mesh optimal control problem (6) is rewritten as

\[
\min_{L_y = u} \{I(y, u) + \psi(y) + \varphi(u)\}.
\]

We take Lagrange function for this problem in the form

\[
\mathcal{L}(y, u, \lambda) = I(y, u) + \psi(y) + \varphi(u) + (\lambda, L_y - u),
\]

where \((\cdot, \cdot)\) is euclidian inner product in \(\mathbb{R}^N\).

Saddle point of this Lagrange function satisfies [6] the following system:

\[
\begin{pmatrix}
E & 0 & L^T \\
0 & E & -E \\
L & -E & 0
\end{pmatrix}
\begin{pmatrix}
y \\
u \\
\lambda
\end{pmatrix}
+ \begin{pmatrix}
\partial \theta(y) \\
\partial \varphi(u) \\
0
\end{pmatrix}
\succeq \begin{pmatrix}
y_d \\
0 \\
0
\end{pmatrix}.
\]

(7)

With the notations \(z = (y, u)^T\), \(f = (y_d, 0, 0)^T\), \(\Psi(z) = (\theta(y), \varphi(u))^T\) and

\[
\begin{pmatrix}
A & B^T \\
B & 0
\end{pmatrix}
\begin{pmatrix}
z \\
\lambda
\end{pmatrix}
+ \begin{pmatrix}
\partial \Psi(z) \\
0
\end{pmatrix}
\succeq \begin{pmatrix}
f \\
0
\end{pmatrix}.
\]

(8)
The existence of a solution to (7) and convergence of a preconditioned Uzawa method were proved in [7], and iterative Uzawa-type method is written in form
\[
A z^{k+1} + \partial \Psi(z^{k+1}) \ni B^T \lambda^k + f,
\]
\[
\frac{1}{\rho} D(\lambda^{k+1} - \lambda^k) + Bz^{k+1} = 0, \quad \rho > 0
\]
with preconditioner \( D = D^T > 0 \) and iterative parameter \( \rho \) which satisfy the inequality
\[
D > \frac{\rho}{2} BA^{-1}B^T = \frac{\rho}{2}(LL^T + E).
\]
For method (9) we will use the preconditioner \( D = LL^T \) and iterative parameter \( \rho \in (0,1) \). The detailed form of the corresponding iterative method reads as follows:
\[
u^{k+1} + \partial \varphi(u^{k+1}) \ni \lambda^k,\]
\[
y^{k+1} + \partial \theta(y^{k+1}) \ni y_d - L^T \lambda^k,
\]
\[
\frac{LL^T}{\rho} \lambda^{k+1} - \lambda^k + u^{k+1} - Ly^{k+1} = 0.
\]

Implementation of every step in (11) consists of solving the inclusions to find \( u^{k+1} \) and \( y^{k+1} \), and solving a system of linear equations with the matrix \( LL^T \). Since the operators \( \partial \varphi \) and \( \partial \theta \) have diagonal forms, vectors \( u^{k+1} \) and \( y^{k+1} \) can be found component-wise by the explicit formulae. So, the most time consuming part of the algorithm is solution of the system with the matrix \( LL^T \). It should be noted that at each time level we must solve a tridiagonal linear system and have to use computed solutions at all previous time levels for inversion of \( L \) and \( L^T \).

4. Numerical results
For numerical tests we took following values: mesh \( N_x \times N_x = 50 \times 50 \), final time moment \( T = 0.5 \), steps \( h = \tau = 0.02 \), \( \rho = 0.9 \), observation function \( z_d(x,t) = e^t \sin(\pi x_1) \sin(\pi x_2) \) and initial condition for parabolic equation \( y_0(x) = \sin(\pi x_1) \sin(\pi x_2) \). Pointwise constraints on state function \( 0 \leq y(x,t) \leq 0.1 \) and control function \( |u(x,t)| \leq 0.05 \) for all \( (x,t) \in Q_T \). We inverse preconditioner matrix \( D = LL^T \) using inner iterative SOR-method with parameter \( \omega = 1.7 \), while Uzawa-type method is being used for computational error \( \|Ly^k - u^k\|_{D^{-1}} > \varepsilon = 5 \cdot 10^{-4} \).

The calculations results are presented in table 1 and table 2.

| \( \alpha \) | \( \tau \) | Iterations | \( J(y,u) \) | Elapsed time, s. |
|-----|-----|-----|-----|-----|
| \( \alpha = 0.25 \) | \( T/4 \) | 6 | 0.693 | 14.763 |
| | \( T/8 \) | 6 | 0.6932 | 105.913 |
| | \( T/12 \) | 7 | 0.6931 | 133.152 |
| | \( T/16 \) | 6 | 0.693 | 196.104 |
| \( \alpha = 0.5 \) | \( T/4 \) | 5 | 0.6931 | 14.8 |
| | \( T/8 \) | 5 | 0.6932 | 107.102 |
| | \( T/12 \) | 5 | 0.6931 | 130.87 |
| | \( T/16 \) | 6 | 0.6931 | 199.651 |
| \( \alpha = 0.75 \) | \( T/4 \) | 4 | 0.6932 | 14.512 |
| | \( T/8 \) | 5 | 0.6931 | 109.541 |
| | \( T/12 \) | 6 | 0.6931 | 131.036 |
| | \( T/16 \) | 5 | 0.6931 | 199.648 |
Table 2. Case with constraints on state and control.

| $\alpha$ | $\tau$ | Iterations | $J(y,u)$ | Elapsed time, s. |
|----------|--------|------------|----------|------------------|
| =0.25    | T/4    | 791        | 0.8553   | 34.54            |
|          | T/8    | 791        | 0.8554   | 247.016          |
|          | T/12   | 795        | 0.8555   | 339.814          |
|          | T/16   | 794        | 0.8554   | 925.1            |
| = 0.5    | T/4    | 788        | 0.8553   | 33.801           |
|          | T/8    | 788        | 0.8553   | 247.13           |
|          | T/12   | 789        | 0.8553   | 340.017          |
|          | T/16   | 788        | 0.8552   | 924.899          |
| = 0.75   | T/4    | 790        | 0.8554   | 34.491           |
|          | T/8    | 789        | 0.8554   | 249.42           |
|          | T/12   | 790        | 0.8553   | 342.571          |
|          | T/16   | 793        | 0.8553   | 925.034          |

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