SCHRÖDINGER EQUATIONS ON NORMAL REAL FORM SYMMETRIC SPACES

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Abstract. We prove dispersive and Strichartz estimates for Schrödinger equations on normal real form symmetric spaces. These estimates apply to the well-posedness and scattering for the non-linear Schrödinger equations.

1. Introduction and statement of the results

Let $G$ be a semisimple Lie group, and let $K$ be a maximal compact subgroup of $G$. Denote by $X$ the symmetric space $G/K$. It is well known [9] that $X$ is a Riemannian manifold. Denote by $\Delta$ the associated Laplace-Beltrami operator. In this short note we shall prove dispersive estimates for the Schrödinger operator $e^{it\Delta}$, $t \in \mathbb{R}$, for the class of normal real form symmetric spaces.

More precisely, let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$. Then, we can consider $\mathfrak{g}$ as a vector space $\mathfrak{g}\mathbb{R}$ over $\mathbb{R}$. Further, the multiplication by $i$ on $\mathfrak{g}$ defines a complex structure $J$ on $\mathfrak{g}\mathbb{R}$ and $\mathfrak{g}\mathbb{R}$ becomes a Lie algebra over $\mathbb{R}$, [9, pp.178-179]. A real form of $\mathfrak{g}$ is a subalgebra $\mathfrak{g}_0$ of the real algebra $\mathfrak{g}\mathbb{R}$ such that $\mathfrak{g}\mathbb{R} = \mathfrak{g}_0 \oplus J\mathfrak{g}_0$. Denote by $G_0$ the Lie group with Lie algebra $\mathfrak{g}_0$. Let also $K_0$ be a maximal compact subgroup of $G_0$ and denote by $\mathfrak{k}_0$ its Lie algebra and $\mathfrak{p}_0$ the complement of $\mathfrak{k}_0$ in $\mathfrak{g}_0$. A real form $\mathfrak{g}_0$ of $\mathfrak{g}$ is called normal if in each Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, the space $\mathfrak{p}_0$ contains a maximal abelian subalgebra of $\mathfrak{g}_0$ [9, p.426]. Then the Riemannian noncompact symmetric space $X_0 = G_0/K_0$ is called a normal real form symmetric space. Examples of normal real forms are $SL(n, \mathbb{R})/SO(n)$ and $Sp(n, \mathbb{R})/U(n)$. Further, in the classification [9, pp.451-455], the normal real forms include AI (including the hyperbolic plane), BDI with $|p-q| = 0$ or 1 (since then $\mathfrak{g}_0$ is a normal real form if and only if $p = q$, for $p+q$ even and $p = q+1$, for $p+q$ odd) and CI.

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In the present work we assume that the complex Lie group $G$ with Lie algebra $\mathfrak{g}$ is semisimple, connected, noncompact, with finite center. Let $S_t^0 = e^{it\Delta_0}$, $t \in \mathbb{R}$, be the Schrödinger operator on $X_0$, i.e. the heat kernel of imaginary time. The main result of the present paper is the following dispersive estimate.

**Theorem 1.** Assume that $X_0$ is a normal real form symmetric space of dimension $n$ and let $S_t^0$ be the corresponding Schrödinger operator. Then the estimates

\[
\|S_t^0\|_{L^q(X) \to L^{q'}(X)} \leq c|t|^{-n(1/2 - 1/q)}, \quad |t| < 1,
\]

and

\[
\|S_t^0\|_{L^q(X) \to L^{q'}(X)} \leq c|t|^{-n/2}, \quad |t| \geq 1,
\]

hold true for all $q \in (2, \infty]$.

As an application of Theorem 1, we shall obtain Strichartz estimates for the solution of the following linear Schrödinger equation:

\[
\begin{align*}
\left\{ \begin{array}{l}
  i\partial_t u(t, x) + \Delta_0 u(t, x) = F(t, x), \\
  u(0, x) = f(x)
\end{array} \right. \\
\end{align*}
\]

Consider the triangle

\[
T_n = \left\{ \left( \frac{1}{p}, \frac{1}{q} \right) \in (0, 1/2] \times (0, 1/2) : \frac{2}{p} + \frac{n}{q} \geq \frac{n}{2} \right\} \cup \left\{ (0, 1/2) \right\}.
\]

We say that the pair $(p, q)$ is admissible if $\left( \frac{1}{p}, \frac{1}{q} \right) \in T_n$.

**Theorem 2.** Assume that $X_0$ is a normal real form symmetric space. Then, the solutions $u(t, x)$ of the Cauchy problem satisfy the Strichartz estimate

\[
\|u\|_{L^p_t L^q_x} \leq c \left\{ \|f\|_{L^p_x} + \|F\|_{L^{p'}_t L^{q'}_x} \right\},
\]

for all admissible pairs $(p, q)$ and $(\tilde{p}, \tilde{q})$ except when $2 < q \neq \tilde{q} \leq \frac{2n}{n-2}$ (resp. $q(\Gamma) < q < \tilde{q} < \infty$ if $n = 2$).

Let us say a few words for the proof of our results. In the case of $\mathbb{R}^n$, the first Strichartz estimate was obtained by Strichartz himself [10]. The Schrödinger equation has also been studied in various geometric settings, see for example [2, 3, 6] and the references therein. In particular, Theorems 1 and 2 are proved on the context of rank one symmetric spaces in [2, 3] and on rank one locally symmetric spaces in [6].

In the context of symmetric spaces, to prove estimates of the kernel of the Schrödinger operator, we use the spherical Fourier transform. We can define the spherical Fourier transform in the general class of
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symmetric spaces $X = G/K$, where $G$ is a semisimple Lie group, connected, noncompact, with finite center. In fact, we need a manipulable expression of the inverse spherical Fourier transform. This is the case when $X$ has rank one or when $G$ is complex. In particular, in the case when $G$ is complex, the spherical Fourier transform boils down to the Euclidean Fourier transform [11 p.1312], and thus its expression allows us to obtain precise estimates of the kernel $s_0^0$. To treat the case of normal real forms, we shall follow the idea of Anker and Lohoué in [1], where they first prove their results in the complex case and then they transfer them on normal real form symmetric spaces, by using the Flensted-Jensen transform (for details see Section 2).

This short note is organized as follows. In Section 2 we present some preliminaries we need for our proof, and in Section 3 we prove our results and present some of their applications in the non-linear Schrödinger equations.

2. Preliminaries

In this section we fix some notation and we recall some basic facts about symmetric spaces we need for our proofs. For more details see [1, 5, 9].

2.1. Symmetric spaces. Let $G$ be a semisimple Lie group, connected, noncompact, with finite center and let $K$ be a maximal compact subgroup of $G$. Denote by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of $G$ and $K$ and let $\mathfrak{p}$ be the subspace of $\mathfrak{g}$, orthogonal to $\mathfrak{k}$ with respect to the Killing form. Fix $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$ and denote by $\mathfrak{a}^*$ the real dual of $\mathfrak{a}$. Choose also a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$ and denote by $\Sigma^+$ a set of positive roots. Denote by $\rho$ the half sum of positive roots counted with their multiplicities.

Let $A$ be the analytic subgroup of $G$ with Lie algebra $\mathfrak{a}$. Put $A^+ = \exp \mathfrak{a}_+$. We have the Cartan decomposition $G = K(A^+)K = K(\exp \mathfrak{a}_+)K$. Let $k_1$, $k_2$ and $\exp H$ be the components of $g \in G$ in $K$ and $\exp \mathfrak{a}$ respectively, according to the Cartan decomposition. Then the Haar measure on $G$ is given by

$$dx = c\delta(H)dk_1dHdk_2,$$

and the modular function $\delta(H)$ satisfies the estimate

$$\delta(H) \leq ce^{2\rho(H)}, \quad H \in \mathfrak{a}_+. \quad (6)$$
Denote by $S(K\backslash G/K)$ the Schwartz space of $K$-bi-invariant functions on $G$. The spherical Fourier transform $\mathcal{H}$ on $G$ is defined by

$$\mathcal{H}f(\lambda) = \int_G f(x) \Phi_\lambda(x) \, dx, \quad \lambda \in \mathfrak{a}^*, \quad f \in S(K\backslash G/K),$$

where $\Phi_\lambda(x)$ is the elementary spherical function of index $\lambda$ on $G$.

In the case when $G$ is complex, the inverse spherical Fourier transform is given by the following formula:

$$(7) \quad \mathcal{H}^{-1}f(\exp H) = c\Phi_0(\exp H)\int_{\mathfrak{p}^*} f(\lambda) e^{i\lambda(H)} d\lambda, \quad H \in \mathfrak{p},$$

(see [1, p.1312]).

2.2. Normal real forms. Let us now assume that $\mathfrak{g}$ is complex and let $\mathfrak{g}_0$ be a normal real form of $\mathfrak{g}$. Let $G_0$ be the Lie group with Lie algebra $\mathfrak{g}_0$, $K_0$ a maximal compact subgroup of $G_0$ and denote by $\mathfrak{k}_0$ its Lie algebra and by $\mathfrak{p}_0$ the complement of $\mathfrak{k}_0$ in $\mathfrak{g}_0$. Set $\mathfrak{k} = \mathfrak{k}_0 \oplus i\mathfrak{k}_0$ and $\mathfrak{u} = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$. Then $\mathfrak{g} = \mathfrak{u} \oplus i\mathfrak{u}$ and all the above decompositions are Cartan. We shall denote by $U \subset G$ the Lie group with Lie algebra $\mathfrak{u}$.

If $\mathfrak{a}_0$ is a maximal abelian subspace of $\mathfrak{p}_0$, then, in general, a maximal abelian subspace $\mathfrak{a}$ of $i\mathfrak{u}$, is a larger space than $\mathfrak{a}_0$. If $\mathfrak{g}_0$ is a normal real form, then we can assume that $\mathfrak{a} = \mathfrak{a}_0$. In this case, we have the same root system for $(\mathfrak{g}, \mathfrak{a})$ and $(\mathfrak{g}_0, \mathfrak{a}_0)$; only their multiplicity is different: 2 in $\mathfrak{g}$ and 1 in $\mathfrak{g}_0$. Hence

$$(8) \quad \rho = 2\rho_0.$$

The real (resp. complex), dual of $\mathfrak{a}$ will be denoted denoted by $\mathfrak{a}^*$ (resp. $\mathfrak{a}_0^*$), and similarly for $\mathfrak{a}_0$. Note that the Killing form $B_0$ on $\mathfrak{g}_0$, is just the restriction of the Killing form $B'$ of $\mathfrak{g}$. Then, the Killing form $B$ of $\mathfrak{g}$ as a real lie algebra is equal to $2 \Re B'$ [3, p.109]. We shall write $\|\cdot\|_0$ for the norm induced by $B_0$ on $\mathfrak{a}_0$, and $\|\cdot\|$ for the norm induced by $B$ on $\mathfrak{p}$. We have that,

$$(9) \quad \|H\| = \sqrt{2}\|H\|_0, \quad H \in \mathfrak{a}_0,$$

and

$$(10) \quad \|\lambda\|_0 = \sqrt{2}\|\lambda\|, \quad \lambda \in \mathfrak{a}_0^*.$$  

Note also that (10) is also valid for $\lambda \in (\mathfrak{a}_0)^*_\mathbb{C}$, [5, p.109]. Finally, the modular function $\delta_0$ on $G_0$ satisfies

$$(11) \quad \delta(H) = \delta_0(H)^2,$$

(see [4, pp.1309-1310], for more details).
2.3. **The Flensted-Jensen integral transform and the reduction to the complex case.** The relation between functions on the real semisimple Lie group $G_0$ and functions on the complex semisimple group $G$ was studied by Flensted-Jensen. In particular, in [5], the following integral transform

$$ (MF)(\exp 2Y) = \int_K F(k \exp Y)dk, \quad F \in L^1_{\text{loc}}(U \setminus G/U), $$

is introduced, which transforms a $U$-bi-invariant function $F$ on $G$ to a function $MF$ that is $K_0$-bi-invariant in $G_0$ [5, p.126].

Then, for the spherical Fourier transform $\mathcal{H}F$, it holds

$$ \mathcal{H}(MF)(\lambda) = \mathcal{H}F(2\lambda), $$

[1, p.1313], and

$$ M\Phi_0 = c\phi_0, $$

where $\Phi_0$ is the elementary spherical function of index 0 on $G$ and $\phi_0$ the elementary spherical function of index 0 on $G_0$, [1, p.1333].

3. **Proof of the results**

For the proof of our results we shall follow an idea of Anker and Lohoué [1, p.1308], where they first prove the results in the case when $G$ is complex and then they transfer them to the case of normal real form by using the integral transform of Flensted-Jensen [5].

3.1. **The Schrödinger kernel.** Denote by $s_t^0$ Schrödinger kernel on the symmetric space $X_0$. Then $s_t^0$ is a $K$-bi-invariant function and the Schrödinger operator $S_t^0 = e^{it\Delta_0}$ on $X_0$ is defined as a convolution operator:

$$ S_t^0 f(x) = \int_G s_t^0(y^{-1}x)f(y)dy = (s_t^0 \ast f)(x), \quad f \in C^\infty_0(X_0). $$

If $s_t$ is the Schrödinger kernel on $X$, then by (12) it follows that the Schrödinger kernel $s_t^0$ on the normal real form $X_0$ satisfies

$$ s_t^0(\exp H) = c \int_K s_{t/2}(k \exp(H/2))dk, $$

(see also [1, p.648]).

The Schrödinger kernel $s_t$ is given by

$$ s_t(\exp H) = (\mathcal{H}^{-1}w_t)(\exp H), \quad H \in \mathfrak{a}_+, $$

where

$$ w_t(\lambda) = e^{it(|\rho|^2+|\lambda|^2)}, \quad \lambda \in \mathfrak{p}^*, $$

and $\mathcal{H}^{-1}$ denotes the inverse spherical transform.
If $G$ is complex, then Gangolli proved in [7] that the heat kernel $h_t$ of $X$ is given by the formula

$$h_t(\exp H) = \phi_0(\exp H)(4\pi t)^{-n/2}e^{-t|\rho|^2}e^{-\|H\|^2/4t}.$$ 

But, $s_t$ is just the result of the substitution $t \to it$ in $h_t$. Thus, the Schrödinger kernel $s_t$ has the following explicit formula

$$s_t(\exp H) = \Phi_0(\exp H) (4\pi it)^{-n/2}e^{-it|\rho|^2}e^{-\|H\|^2/4it}, \quad H \in \mathfrak{a}_+, \ t \in \mathbb{R}.$$ 

Recalling that for $H \in \mathfrak{a}_+$,

$$e^{-\rho(H)} \leq \Phi_0(\exp H) \leq c(1 + \|H\|)^\alpha e^{-\rho(H)},$$

for some constants $c, a > 0$, [9, p.483], from (16), we obtain the following result.

**Proposition 3.** If $G$ is complex, then the Schrödinger kernel $s_t$ on $X = G/K$ satisfies the estimates

$$\|s_t(\exp H)\| \leq ct^{-n/2}\Phi_0(\exp H) \leq ct^{-n/2}(1 + \|H\|)^\alpha e^{-\rho(H)}, \quad H \in \mathfrak{a}_+, \ t \in \mathbb{R},$$

for some constants $c, a > 0$.

A consequence of (18) is the following Proposition.

**Proposition 4.** If $X_0$ is a symmetric space of normal real form, then the Schrödinger kernel $s^0_t$ on $X_0$ satisfies the estimate

$$|s^0_t(\exp H)| \leq ct^{-n/2}(1 + \|H\|_0)^\alpha e^{-\rho_0(H)}, \quad H \in \mathfrak{a}_+, \ t \in \mathbb{R},$$

for some constants $c, a > 0$.

**Proof.** Recall that the Schrödinger kernel is $K$-bi-invariant. Thus, from [15] and the fact that the total $dk$ measure of $K$ is normalized to 1, it follows that

$$s^0_t(\exp H) = cs_{t/2}(\exp(\sqrt{H}/2)), \quad H \in \mathfrak{a}_+, \ t \in \mathbb{R}.$$ 

Combining with Proposition 3 [8] and [9], we get that

$$|s^0_t(\exp H)| \leq c(t/2)^{-n/2}(1 + \|H/2\|)^\alpha e^{-\rho(H/2)} = ct^{-n/2}(1 + \sqrt{2}\|H/2\|_0)^\alpha e^{-2\rho_0(H/2)} \leq ct^{-n/2}(1 + \|H\|_0)^\alpha e^{-\rho_0(H)}.$$

□
3.2. **Proof of Theorem 1.** For the proof of the large time estimate \((2)\), we shall use the Kunze-Stein phenomenon \([8]\) which asserts that if \(\kappa\) is \(K\)-bi-invariant, then

\[
\| * \kappa \|_{L^2(G) \to L^2(G)} = C \int_G |\kappa(g)| \Phi_0(g) dg
\]

\((20)\)

\[
= C \int_{a^+} |\kappa(\exp H)| \Phi_0(\exp H) \delta_0(H) dH.
\]

Using \((20)\), it is proved in \([3, \text{Theorem 4.2}]\) that

\[
\| S_t^0 \|_{L^q(X_0) \to L^q(X_0)} \leq c \int_{a^+} |s_t^0(\exp H)|^{q/2} \varphi_0(\exp H) \delta_0(H) dH.
\]

But, from \((11)\) and \((8)\) we have

\[
\delta_0(H) \leq e^{2\rho_0(H)}.
\]

Also, by \((13)\), \((9)\), and \((17)\) it follows that

\[
\varphi_0(\exp H) \leq c(1 + \|H\|_0)^\alpha e^{-\rho_0(H)}.
\]

Combining \((21)\), \((22)\), \((23)\) and Proposition \(4\), we obtain for \(q > 2\), that

\[
\| S_t^0 \|_{L^q(X_0) \to L^q(X_0)} \leq c t^{-nq/4} \int_{a^+} (1 + \|H\|_0)^\alpha (1+ (q/2)) e^{-\rho_0(H)(q/2)-1} dH
\]

\[
\leq c t^{-nq/4} \int_{a^+} (1 + \|H\|_0)^\alpha (1+ (q/2)) e^{-\rho_0(H)(q/2)-1} dH
\]

\[
\leq c t^{-nq/4}
\]

and the large time \((2)\) estimate follows.

For the proof of the small time estimate \((1)\), we shall use the elementary estimate

\[
\| e^{it\Delta_0} \|_{L^1(X_0) \to L^\infty(X_0)} = \| s_t \|_{L^\infty} \leq c |t|^{-n/2},
\]

and the \(L^2\)-conservation

\[
\| e^{it\Delta_0} \|_{L^2(X_0) \to L^2(X_0)} = 1.
\]

By interpolation from \((24)\) and \((25)\), it follows that for \(\theta \in (0, 1)\),

\[
\| e^{it\Delta_0} \|_{L^{p_\theta}(X_0) \to L^{q_\theta}(X_0)} \leq \| e^{it\Delta_0} \|_{L^2(X_0) \to L^2(X_0)} \| e^{it\Delta_0} \|_{L^1(X_0) \to L^\infty(X_0)} \]

\[
\leq c |t|^{-n\theta/2},
\]

where

\[
\frac{1}{p_\theta} = \frac{1 + \theta}{2}, \quad \frac{1}{q_\theta} = \frac{1 - \theta}{2}.
\]
Choosing $\theta = 1 - (2/q)$ we get that
$$\|e^{it\Delta_0}\|_{L^{q'}(X_0)\to L^q(X_0)} \leq c|t|^{-n(\frac{1}{2} - \frac{1}{q})}.$$

As an application of the Strichartz estimates and Theorem 1, we shall consider the non-linear Schrödinger equation (NLS)
\begin{equation}
\left\{ \begin{array}{l}
    i\partial_t u(t, x) + \Delta_0 u(t, x) = F(u(t, x)), \\
    u(0, x) = f(x).
\end{array} \right.
\end{equation}

Assume that $F$ has a power-like nonlinearity of order $\gamma$, i.e.
$$|F(u)| \leq c|u|^{\gamma}, \quad |F(u) - F(v)| \leq c \left(|u|^{\gamma-1} + |v|^{\gamma-1}\right) |u - v|.$$

Then, using the previous Strichartz estimates, we can prove well-posedness of \(26\) and scattering for the NLS for small $L^2$ data when $\gamma \in (1, 1 + \frac{4}{n})$ (see [2, Theorem 4.2, Theorem 5.1]). The proofs are standard and thus omitted (see for example [3, pp.12-14]).

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