A FRAMEWORK FOR TORSION THEORY COMPUTATIONS ON ELLIPTIC THREEFOLDS

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Abstract

We give a list of statements on the geometry of elliptic threefolds phrased only in the language of topology and homological algebra. Using only notions from topology and homological algebra, we recover existing results and prove new results on torsion pairs in the category of coherent sheaves on an elliptic threefold.

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1. Introduction

Given a smooth projective variety $X$, in order to study a moduli space of geometric objects on $X$, one often needs to fix a stability condition. In particular, when the geometric objects are coherent sheaves or chain complexes of coherent sheaves, a stability condition (for example, slope stability, tilt stability, or Bridgeland stability) is defined only after one fixes a t-structure on the derived category of coherent sheaves $D^b(\text{Coh}(X))$ on $X$. In other words, a t-structure is a precursor to a stability condition.

Since every torsion class in the abelian category $\text{Coh}(X)$ defines a t-structure on $D^b(\text{Coh}(X))$, the study of torsion classes in $\text{Coh}(X)$ and their relations can yield useful information on stability conditions on $X$ and ultimately moduli spaces of objects on $X$. For example, in the study of limits of Bridgeland stability conditions in [6] and their analogues on threefolds [8, 9], one needs to establish the Harder–Narasimhan properties of ‘limit stabilities’; the proof of the Harder–Narasimhan properties, in turn, is built on an understanding of t-structures arising from torsion classes in the category of coherent sheaves.

The t-structure computations in [6, 8, 9], however, are lengthy. Two observations quickly stand out while performing these computations:

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(A) there are many instances of recursions within the structures of proofs;
(B) many proofs appear to be independent of the algebro-geometric context they are in.

Observation (A) is partly due to the fibration structure of the algebraic varieties involved. Observation (B), on the other hand, suggests that some of the arguments may be useful for problems in other fields such as representation theory. Nonetheless, these observations beg the questions: can we extract these recursions from their context, simplify them, and make the proofs more efficient?

Specifically, by extracting the recursive arguments from their context, we mean ‘isolating’ the arguments that are independent of the algebro-geometric context. We partially achieve this in the present article by giving a list of ‘properties’ that can be treated as axioms. Using only concepts from triangulated categories, point-set topology, and homological algebra, the properties can be used to deduce and recover results on t-structures that appeared in [8, 9].

We envision that the axiomatic approach in this article can be expanded to other varieties with fibration structures, and of arbitrary dimensions (for example, elliptic fourfolds or surface-fibered threefolds). The axiomatic approach can help lower the computational complexity when similar computations need to be performed on different varieties.

1.1. The categories $C_{ij}$. For a smooth projective threefold $X$ with a Weierstraß fibration, we define subcategories $C_{ij}$ of $\text{Coh}(X)$ (see Section 4), which can be arranged in the following configuration:

$$
\begin{array}{c}
C_{40} & \rightarrow & C_{50} \\
\downarrow & & \downarrow \\
C_{20} & \rightarrow & C_{30} & \rightarrow & C_{51} \\
\downarrow & & \downarrow & & \downarrow \\
C_{00} & \rightarrow & C_{10} & \rightarrow & C_{31} & \rightarrow & C_{52} \\
\downarrow & & & & \downarrow & & \downarrow \\
C_{11} & \rightarrow & C_{32} & \rightarrow & C_{12} \\
\end{array}
$$

(1-1)

The configuration roughly reflects the order in which the $C_{ij}$ are enlisted in constructing the categories $T_{ij}$ below. We set a category $C_{ij}$ to be empty if it does not appear in (1-1). Of course, there are torsion classes other than those constructed from $C_{ij}$ that were used in [8, 9], and one should eventually consider all of these torsion classes in the study of stability conditions. We only consider $C_{ij}$ in this article since
they are the simplest categories of coherent sheaves that can be defined in terms of
dimension and the equivalence functor $\Phi$ (see Section 3.4).

The categories $C_{ij}$ offer a nontrivial level of complexity for testing our axiomatic
approach and reflect the underlying geometric structures. For instance, the category
$C_{00}$ is the category of coherent sheaves supported in dimension zero, that is, coherent
sheaves supported at a finite number of closed points on $X$; that $C_{00}$ is contained
in $W_{0,\Phi}$ (Property C0) encodes the fact that the equivalence of categories $\Phi$
is constructed using a universal family from a moduli problem (for example, see \cite{1, Section 6.2.3} or \cite{2}). As another example, the category $C_{31}$ contains coherent sheaves
supported in dimension two that vanish on the generic fiber of the Weierstraß fibration,
such that their transforms under $\Phi$ are sheaves supported in dimension one – such
sheaves are sometimes called ‘vertical sheaves’ and are important for understanding
the connection between one-dimensional and two-dimensional Donaldson–Thomas
invariants \cite{3}.

1.2. Results. We now describe the results of this article. Some of these results
were essential to the proof of the Harder–Narasimhan property of limit tilt stability on
elliptic threefolds in \cite{8, 9}; other results in this article, such as part (c) of Theorem 1.1
and Lemmas 4.24 and 4.25, are new.

Given any subcategories $S_1, \ldots, S_n$ of $\text{Coh}(X)$, we will write $\langle S_1, \ldots, S_n \rangle$ to denote
their extension closure in $\text{Coh}(X)$. For $n = 0, 2, 4$, we define
$$T_{n0} = \langle C_{ij} : i \leq n, 0 \leq j \leq 2 \rangle,$$
while for $n = 1, 3, 5$ we define
$$T_{n0} = \langle T_{(n-1)0}, C_{n0} \rangle,$$
$$T_{n1} = \langle T_{n0}, C_{n1} \rangle,$$
$$T_{n2} = \langle T_{n1}, C_{n2} \rangle,$$
and for $2 \leq i \leq 5$ we set $F_i = \langle C_{00}, C_{10}, \ldots, C_{i0} \rangle$. The results of this article can be
summarised in the following theorem.

**Theorem 1.1.** Let $X = C \times B$ be the product of an elliptic curve $C$ and a $K3$ surface $B$
of Picard rank one. The following 17 categories are all torsion classes in the category
of coherent sheaves on $X$:

(a) $T_{00}, T_{20}, T_{40}$;
(b) $T_{ij}$ for $i = 1, 3, 5$ and $0 \leq j \leq 2$;
(c) $F_i$ for $2 \leq i \leq 5$;
(d) $\langle C_{00}, C_{20} \rangle$.

The structure of this paper is as follows: in Section 2, we collect the key notions
from homological algebra we need. In Section 3, we give a list of ‘properties’ that
have algebraic geometry encoded in them; these properties are treated as axioms in
this paper. In Section 4, we use only the properties in Section 3 and concepts from
topology and homological algebra to prove a series of lemmas, which together give
Theorem 1.1.
2. Preliminaries on homological algebra

We assume that the reader is familiar with the basic properties of abelian categories, triangulated categories, and exact functors of triangulated categories.

2.1. Torsion pairs. Suppose that $\mathcal{A}$ is an abelian category. A pair of full subcategories $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$ is called a torsion pair (or a torsion theory) if [4]:

- every object $E \in \mathcal{A}$ fits in a short exact sequence in $\mathcal{A}$
  
  \[ 0 \to E' \to E \to E'' \to 0 \]
  
  for some $E' \in \mathcal{T}$ and $E'' \in \mathcal{F}$;

- for any $E' \in \mathcal{T}$ and any $E'' \in \mathcal{F}$, we have $\text{Hom}_{\mathcal{A}}(E', E'') = 0$.

Given a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, we will refer to $\mathcal{T}$ (respectively $\mathcal{F}$) as the torsion class (respectively torsion-free class) in the torsion pair. We will say that a subcategory $\mathcal{C}$ of $\mathcal{A}$ is a torsion class (respectively torsion-free class) in $\mathcal{A}$ if it is the torsion class (respectively torsion-free class) in a torsion pair.

If $(\mathcal{T}, \mathcal{F})$ is a torsion pair in $\mathcal{A}$, then $\mathcal{T}$, $\mathcal{F}$ are both extension-closed, and $\mathcal{T}$ is closed under quotients in $\mathcal{A}$, while $\mathcal{F}$ is closed under subobjects in $\mathcal{A}$.

In a noetherian abelian category, such as the category of coherent sheaves on an algebraic variety, a torsion class can be recognised via the following lemma.

**Lemma 2.1** [12, Lemma 1.1.3]. Let $\mathcal{C}$ be a noetherian abelian category. Then any full subcategory $\mathcal{T} \subseteq \mathcal{C}$ closed under quotients and extensions is a torsion class.

For any subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$, we will often write $\mathcal{C}^\perp$ to denote the subcategory of $\mathcal{A}$,

\[ \mathcal{C}^\perp = \{ E \in \mathcal{A} : \text{Hom}_{\mathcal{A}}(E', E) = 0 \text{ for all } E' \in \mathcal{C} \} . \]

For example, if $\mathcal{T}$ is a torsion class in an abelian category $\mathcal{A}$, then $\mathcal{T}^\perp$ coincides with the corresponding torsion-free class.

Given two subcategories $\mathcal{A}, \mathcal{B}$ of a category $\mathcal{C}$, we write $\text{Hom}_\mathcal{C}(\mathcal{A}, \mathcal{B}) = 0$ to mean that $\text{Hom}_\mathcal{C}(A, B) = 0$ for all $A \in \mathcal{A}, B \in \mathcal{B}$.

Given an abelian category $\mathcal{A}$, an extension-closed subcategory $\mathcal{C}$ of $\mathcal{A}$ is called a Serre subcategory if, given any object $E$ of $\mathcal{C}$ and any subobject or quotient $E'$ of $E$, the object $E'$ also lies in $\mathcal{C}$. By Lemma 2.1, any Serre subcategory of a noetherian abelian category is a torsion class.

2.2. Derived category. Given an abelian category $\mathcal{A}$, the bounded derived category $D^b(\mathcal{A})$ of $\mathcal{A}$ is a category in which the objects $E$ are chain complexes of objects in $\mathcal{A}$. That is, if $E$ is an object of $D^b(\mathcal{A})$, then $E$ is represented by a diagram in $\mathcal{A}$,

\[ \cdots \to E^{i-1} \xrightarrow{d^{i-1}} E^i \xrightarrow{d^i} E^{i+1} \xrightarrow{d^{i+1}} \cdots, \]

where each composition $d^{i+1}d^i$ is the zero map and the $E^i$ are zero for all but finitely many $i$. The morphisms $d^i$ are called the differentials and we sometimes write $d_E^i$.
for \( d^i \) to emphasise that the differentials are of the complex \( E \). For any integer \( i \), we will refer to the degree-\( i \) cohomology of \( E \) with respect to the standard t-structure on \( D^b(\mathcal{A}) \), defined as

\[
H^i(E) = \frac{\ker(d^i)}{\text{im}(d^{i-1})},
\]

simply as the cohomology of \( E \). We will also refer to \( D^b(\mathcal{A}) \) simply as the derived category of \( \mathcal{A} \).

The derived category \( D^b(\mathcal{A}) \) is a triangulated category. The reader may refer to references such as [5, 11, 13] for basic properties of a triangulated category. The main properties of a triangulated category that will be used in the computations in this article include the following properties.

- There is a shift functor \( [1] : D^b(\mathcal{A}) \to D^b(\mathcal{A}) \) that takes an object \( E \) as in (2-1) to the object \( E[1] \), where \( (E[1])^i = E^{i+1} \) and \( d^i_E[1] = d^{i+1}E \) for each \( i \).
- There is an embedding functor from \( \mathcal{A} \) into \( D^b(\mathcal{A}) \) that takes every object \( M \) of \( \mathcal{A} \) to the complex ‘concentrated in degree zero’, that is, the complex (2-1), where \( E^0 = M \), all the other \( E^i \) are zero, and all the morphisms \( d^i \) are zero maps.
- Via the embedding \( \mathcal{A} \to D^b(\mathcal{A}) \) above, for any \( A, B \in \mathcal{A} \),

\[
\text{Hom}_\mathcal{A}(A, B) \cong \text{Hom}_{D^b(\mathcal{A})}(A, B).
\]

- For every short exact sequence \( 0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \) in \( \mathcal{A} \), there is a corresponding exact triangle in \( D^b(\mathcal{A}) \),

\[
\cdots \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to A[1] \xrightarrow{\alpha[1]} B[1] \to \cdots.
\]

- Given any exact triangle

\[
\cdots \to E \to F \to G \to E[1] \to \cdots
\]

in \( D^b(\mathcal{A}) \), there is a corresponding long exact sequence of cohomology in \( \mathcal{A} \),

\[
\cdots \to H^{i-1}(G) \to H^i(E) \to H^i(F) \to H^i(G) \to H^{i+1}(E) \to \cdots.
\]

3. Properties encoding algebraic geometry

In this section, we give a list of statements on the Zariski topology of algebraic varieties, Fourier–Mukai functors between dual elliptic fibrations, and Chern classes. These statements are listed as ‘properties’, and will be treated as ‘axioms’ to perform the computations in Section 4.

3.1. Properties of dimension. For any smooth projective variety \( Y \), the coherent sheaves on \( Y \) form an abelian category, which we denote by \( \text{Coh}(Y) \) or \( \mathcal{A}_Y \). For any \( E \in \mathcal{A}_Y \), there is an associated topological subspace \( \text{supp}(E) \) of \( Y \) with respect to the Zariski topology on \( Y \). Given any morphism of projective varieties \( p : Y \to Z \), it induces a continuous map \( Y \to Z \) with respect to the Zariski topology, which we
also denote by $p$. For an object $E \in D^b(\mathcal{A}_Y)$, we define the support of $E$ to be $\text{supp}(E) = \bigcup_{i \in \mathbb{Z}} \text{supp}(H^i(E))$.

Given an object $E \in D^b(\mathcal{A}_Y)$, we also refer to the dimension of $\text{supp}(E)$ simply as the dimension of $E$. Properties of dimension include the following properties.

- **Property D0.** For every projective variety $Y$ of dimension $n$ and any topological subspace $W$ of $Y$ with respect to the Zariski topology, we can define the *dimension* of $W$, a nonnegative integer denoted by $\dim W$. The *codimension* of $W$ is defined to be $n - \dim W$. For any $E \in \mathcal{A}_Y$, we define the dimension (respectively codimension) of $E$, denoted $\dim E$ (respectively $\text{codim} E$), to be that of the support of $E$.

- **Property D1.** For any morphism of smooth projective varieties $p : Y \to Z$ and any short exact sequence in $\mathcal{A}_Y$,

$$0 \to M \to E \to N \to 0,$$

we have $p(\text{supp} E) = p(\text{supp} M) \cup p(\text{supp} N)$ and hence

$$\dim (p(\text{supp} E)) = \max \{ \dim (p(\text{supp} M)), \dim (p(\text{supp} N)) \}.$$

- **Property D2.** For any morphism of smooth projective varieties $p : Y \to Z$ and any $E \in \mathcal{A}_Y$, we have $\dim (p(\text{supp} E)) \leq \dim E$. When $p$ is flat of relative dimension $n$,

$$(\dim E - \dim (p(\text{supp} E))) \leq n.$$

Note that, if we take $p$ to be the identity morphism on $Y$ in Property D1, then it follows that $\text{supp}(E) = \text{supp}(M) \cup \text{supp}(N)$ and $\dim E = \max \{ \dim M, \dim N \}$.

We also have the following property on the support of a coherent sheaf.

- **Property Z1.** Suppose that $Y$ is a projective variety and $E \in \mathcal{A}_Y$. Let $W = \text{supp} E$ and let $W_1, \ldots, W_m$ denote the irreducible components of $W$ with respect to the Zariski topology. Then, for any $1 \leq i \leq m$, there exists a short exact sequence in $\mathcal{A}_Y$,

$$0 \to K \to E \to E|_{W_i} \to 0,$$

where $\text{supp}(E|_{W_i}) = W_i$ and $\text{supp}(K) = \bigcup_{j \neq i} W_j$. We refer to $E|_{W_i}$ as the *restriction* of $E$ to $W_i$.

### 3.2. Categories of coherent sheaves.

For any smooth projective variety $Y$ and any nonnegative integer $d$, we define

$$\mathcal{A}_Y^{\leq d} = \{ E \in \mathcal{A}_Y : \dim (E) \leq d \}.$$

For any morphism of smooth projective varieties $p : Y \to Z$ and integers $0 \leq e \leq d$, we define

$$\mathcal{A}^d(p)_e = \{ E \in \mathcal{A}_Y : \dim E = d, \dim (p(\text{supp} E)) = e \},$$

$$\mathcal{A}(p)_{\leq e} = \{ E \in \mathcal{A}_Y : \dim (p(\text{supp} E)) \leq e \}.$$

We also write $\mathcal{A}(p)_0 = \mathcal{A}(p)_{\leq 0}$. 
3.3. Our elliptic threefold. Unless otherwise stated, throughout this article, we will write \( X \) to denote a smooth projective threefold that admits a Weierstraß elliptic fibration \( \pi : X \to B \) in the sense of [1, Section 6.2]. That is, \( B \) is a smooth projective surface, the morphism \( \pi \) is flat of relative dimension one, all the fibers of \( \pi \) are integral and Gorenstein of arithmetic genus one (with the generic fiber being a smooth elliptic curve), and there exists a section \( \sigma : B \to X \) such that \( \Theta := \sigma(B) \) does not meet any singular point of any fiber of \( \pi \). A coherent sheaf on \( X \) that is set-theoretically supported on a finite union of fibers of \( \pi \) is called a fiber sheaf and so \( \mathcal{A}(\pi)_0 \) is the category of fiber sheaves on \( X \).

For a proper morphism of varieties of relative dimension one (for example, an elliptic fibration), we have the following description of one-dimensional closed subvarieties of the domain, which follows from [7, Lemma 3.15].

- **Property Z2.** Suppose that \( p : Y \to Z \) is a proper morphism of varieties of relative dimension one, and \( W \) is an irreducible one-dimensional closed subset of \( Y \) with respect to the Zariski topology. Then \( W \) is either of the following two types:
  
  (a) \( W \) is contained in \( p^{-1}(a) \) for some \( a \in Z \);
  
  (b) for any \( b \in Z \), the intersection \( W \cap p^{-1}(b) \) is a finite number of points.

3.3.1. \( \mathcal{A}_h^{\leq 1} \), the category of sheaves with horizontal supports. When \( \pi : X \to B \) is a Weierstraß elliptic threefold, we define \( \mathcal{A}_h^{\leq 1} \) to be the full subcategory of \( \mathcal{A}_h^{\leq 1} \) consisting of one-dimensional sheaves \( E \) such that the one-dimensional irreducible components of \( \text{supp} \,(E) \) are all of type (b) in Property Z2. By construction, every nonzero \( E \in \mathcal{A}_h^{\leq 1} \) lies in \( \mathcal{A}^1(\pi)_1 \). Note that sheaves in \( \mathcal{A}_h^{\leq 1} \) are not necessarily pure one-dimensional sheaves.

3.4. Autoequivalence of \( D^b(X) \). For a Weierstraß elliptic threefold \( \pi : X \to B \), we have a pair of autoequivalences of the derived category \( D^b(\mathcal{A}_X) \),

\[
\Phi, \tilde{\Phi} : D^b(\mathcal{A}_X) \to D^b(\mathcal{A}_X).
\]

These two functors \( \Phi, \tilde{\Phi} \) are relative Fourier–Mukai transforms with kernels given by relative Poincaré sheaves (see [1, Section 6.2.3] or [2] for the precise definitions). For any object \( E \in D^b(\mathcal{A}_X) \), if \( \Phi E \in \mathcal{A}_X[-i] \) for some \( i \), then we say that \( E \) is \( \Phi \)-WIT\(_i\), and write \( \bar{E} \) to denote \( H^i(\Phi E) \). In this case, the object \( \bar{E} \) is unique up to isomorphism in \( D^b(\mathcal{A}_X) \). The notion of \( \Phi \)-WIT\(_i\) can similarly be defined.

The functor \( \Phi \) satisfies the following properties, with analogous properties satisfied by \( \tilde{\Phi} \).

- **Property A1.** For any \( E \in \mathcal{A}_X \), we have \( H^i(\Phi E) = 0 \) for all \( i \neq 0, 1 \).
- **Property A2.** \( \tilde{\Phi} \Phi \equiv \text{id}_{D^b(\mathcal{A}_X)[-1]} \equiv \Phi \tilde{\Phi} \).
- **Property A3.** For any \( E \in \mathcal{A}_X \),

\[
\dim(\pi(\text{supp } E)) = \dim(\pi(\text{supp } (\Phi E))).
\]

- **Property A4.** For any \( E \in D^b(\mathcal{A}_X) \), we have \( H^i(E) = 0 \) for all \( i \neq j \) if and only if \( E \) is isomorphic in \( D^b(\mathcal{A}_X) \) to some object in \( \mathcal{A}_X[-j] \).
Property A1 holds because \( \pi \) has relative dimension one and the kernel of \( \Phi \) is a sheaf \([1, \text{Section 6.1.1}]\). Property A2 is \([1, \text{Theorem 6.18}]\). Property A3 holds because \( \Phi, \hat{\Phi} \) are relative integral functors that satisfy the base-change property \([1, \text{Proposition 6.1}]\), while Property A4 holds for the derived category of any abelian category.

For \( i = 0, 1 \), we define \( W_{i,\Phi} \) to be the subcategory of \( \mathcal{A}_X \) consisting of all the \( \Phi \)-WIT \( i \) objects in \( \mathcal{A}_X \).

**Lemma 3.1.** For any \( E \in W_{i,\Phi} \), where \( 0 \leq i \leq 1 \), we have \( \hat{E} \in W_{1-i,\hat{\Phi}} \).

**Proof.** Suppose that \( E \in W_{i,\Phi} \), where \( 0 \leq i \leq 1 \). Then \( \Phi E \cong \hat{E}[-i] \) for some \( \hat{E} \in \mathcal{A}_X \). By Property A2, we have \( E[-1] \cong (\hat{\Phi} \hat{E})[-i] \), that is, \( \hat{\Phi} \hat{E} \cong E[-(1-i)] \) and the lemma follows. \( \square \)

### 3.5. Torsion classes in \( \mathcal{A}_X \)

We collect here basic examples of torsion classes in \( \mathcal{A}_X \).

- **Property TC1.** For any variety \( Y \) and any nonnegative integer \( d \), the category \( \mathcal{A}_Y^{\leq d} \) is a Serre subcategory of \( \mathcal{A}_Y \).

- **Property TC2.** For any morphism of varieties \( p : Y \to Z \) and any nonnegative integer \( e \), the category \( \mathcal{A}(p)_{\leq e} \) is a Serre subcategory of \( \mathcal{A}_Y \).

- **Property TC3.** \( (W_{0,\Phi}, W_{1,\Phi}) \) is a torsion pair in \( \mathcal{A}_X \). (In particular, \( W_{0,\Phi} \) is a torsion class in \( \mathcal{A}_X \).)

Property TC3 follows from \([2, \text{Lemma 9.2}]\) and properties of the heart of a t-structure. Since the category of coherent sheaves on any algebraic variety \( Y \) is a noetherian abelian category, any Serre subcategory of \( \mathcal{A}_Y = \text{Coh}(Y) \) is a torsion class in \( \mathcal{A}_Y \) by \([12, \text{Lemma 1.1.3}]\).  

### 3.6. Chern characters and the product threefold

In the special case where the elliptic threefold \( X \) is the product \( \mathbb{C} \times B \) of a smooth elliptic curve and a K3 surface \( B \) of Picard rank one, with the second projection \( \pi : X \to B \) as the fibration map, we have a matrix notation for the Chern characters of objects in \( D^b(\mathcal{A}_X) \). In this case, the Chern character \( \text{ch}(E) \) of any object \( E \in D^b(\mathcal{A}_X) \) can be represented by a \( 2 \times 3 \) matrix of integers by the second author’s joint work with Zhang \([10, \text{Section 4.2}]\) (see also \([9]\)) and we have the following properties on Chern characters. The labels of these properties end with a lower case ‘p’ to indicate that, as stated, they are specific to the product threefold case. We expect there to be slight modifications of these properties (taking into account twists by \( B \)-fields) that hold for a general Weierstraß elliptic threefold (for example, see \([8]\)).

- **Property CH0p.** For every \( E \in D^b(\mathcal{A}_X) \), there is an associated \( 2 \times 3 \) matrix

\[
\text{ch}(E) = (\alpha_{ij}) = \begin{pmatrix}
\alpha_{00} & \alpha_{01} & \alpha_{02} \\
\alpha_{10} & \alpha_{11} & \alpha_{12}
\end{pmatrix},
\]

where \( \alpha_{ij} \in \mathbb{Z} \) for \( 0 \leq i \leq 1, 0 \leq j \leq 2 \).
The six entries in the matrix correspond to the six generators of the algebraic cohomology ring of $X$ over $\mathbb{Z}$. For instance, in (3-1), the entry $\alpha_{00}$ represents the rank of $E$, while $\alpha_{10}$ represents the fiber degree of $E$, that is, $f\text{ch}_1(E)$, where $f$ denotes the fiber class of the fibration $\pi$.

- **Property CH1p.** For any $E \in \mathcal{A}_X$, 
  \[ \text{codim } E = \min \{ i + j : \alpha_{ij} \neq 0, 0 \leq i \leq 1, 0 \leq j \leq 2 \}. \]

- **Property CH2p.** For any $E \in \mathcal{A}_X$, 
  \[ \dim (\pi(\text{supp } E)) = \max \{ 2 - j : \alpha_{ij} \neq 0 \text{ for some } i \}. \]

- **Property CH3p.** There is a total ordering 
  \[ (0, 0) > (1, 0) > (0, 1) > (1, 1) > (0, 2) > (1, 2) \]
  such that, if $(i, j)$ is the largest element under this ordering satisfying $\alpha_{ij} \neq 0$, then $\alpha_{ij} > 0$ and, for any $(i', j') > (i, j)$, we have $\alpha_{i'j'} = 0$.

- **Property A5p.** For any $E \in D^b(\mathcal{A}_X)$, if $\text{ch}(E) = (\alpha_{00} \alpha_{01} \alpha_{02} \alpha_{10} \alpha_{11} \alpha_{12})$, then 
  \[
  \text{ch}(\Phi E) = \begin{pmatrix}
  \alpha_{10} & \alpha_{11} & \alpha_{12} \\
  -\alpha_{00} & -\alpha_{01} & -\alpha_{02}
  \end{pmatrix},
  \]
  \[
  \text{ch}(E[1]) = \begin{pmatrix}
  -\alpha_{00} & -\alpha_{01} & -\alpha_{02} \\
  -\alpha_{10} & -\alpha_{11} & -\alpha_{12}
  \end{pmatrix}.
  \]

Properties CH1p and CH3p follow from [10, Proposition 5.13]. Property A5p is [10, Proposition 4.5].

**Lemma 3.2.** Let $\pi : X = C \times B \to B$ be the product elliptic threefold as in Section 3.6. Then 
\[ \mathcal{A}_X^{\leq 2} \cap W_{1, \Phi} \subseteq \mathcal{A}(\pi)_{\leq 1}. \]

**Proof.** Take any $E \in \mathcal{A}_X^{\leq 2} \cap W_{1, \Phi}$. Suppose that $\dim E \leq 1$; then Property D2 gives $E \in \mathcal{A}(\pi)_{\leq 1}$. So, let us suppose that $\dim E = 2$ from now on.

That $\dim E = 2$ implies that $\text{codim } E = 1$. By Property CH1p, we have $\alpha_{00} = 0$ and $\alpha_{10} \geq 0$. On the other hand, by Property A5p, we have $\text{ch}(\Phi E)[1] = (-\alpha_{10} -\alpha_{11} -\alpha_{12})$. That $E$ is $\Phi$-WIT$_1$ means that $\Phi E[1] \in \mathcal{A}_X$ and so $-\alpha_{10} \geq 0$ by Properties CH1p and CH3p. Overall, we have $\alpha_{10} = 0$. Property CH2p now gives $\dim (\pi(\text{supp } E)) \leq 1$, that is, $E \in \mathcal{A}(\pi)_{\leq 1}$. □

**4. Torsion classes in the category of coherent sheaves**

Some of the results in this section have already appeared in [8, 9]. All the proofs in this section, however, rely only on the properties listed in Section 3, Properties C0 and C1 below, and the preliminary notions in Section 2. Lemmas 4.3–4.22 in this section,
together with Remark 4.18, hold for a general Weierstraß threefold. Lemmas 4.23–4.28, on the other hand, hold only for the product elliptic threefold as defined in Section 3.6 and will be explicitly marked as such.

Let \( \pi : X \to B \) be a Weierstraß elliptic threefold. This assumption will be in place up to Lemma 4.22. We begin by introducing subcategories \( C_{ij} \) of \( \mathcal{A}_X \) that will form the building blocks of various torsion classes in \( \mathcal{A}_X \). The categories \( C_{ij} \) are defined for pairs \((i, j)\) taken from the collection

\[
(0, 0) \quad (1, 0) \quad (2, 0) \quad (3, 0) \quad (4, 0) \quad (5, 0) \quad (1, 1) \quad (3, 1) \quad (5, 1) \quad (4-1) \\
(1, 2) \quad (3, 2) \quad (5, 2)
\]

For each ordered pair \((i, j)\), we will also define a category \( \hat{C}_{ij} \) by replacing \( \Phi \) in the definition of \( C_{ij} \) with \( \hat{\Phi} \). Their definitions are

\[
C_{00} = \mathcal{A}^{\leq 0}_X, \\
C_{10} = \{ E \in \mathcal{A}(\pi)_0 \cap W_{0,\Phi} : \text{Hom}(C_{00}, E) = 0 \}, \\
C_{11} = \{ E \in \mathcal{A}(\pi)_0 \cap W_{1,\Phi} : \text{dim} \hat{E} = 0 \}, \\
C_{12} = \{ E \in \mathcal{A}(\pi)_0 \cap W_{1,\Phi} : \text{dim} \hat{E} = 1, \text{Hom}(C_{11}, E) = 0 \}, \\
C_{20} = \mathcal{A}^{\leq 1}_h.
\]

The category \( C_{00} \) is precisely the category of coherent sheaves on \( X \) supported at points. Note that \( C_{00} = \hat{C}_{00} \). By the classification theorem of semistable sheaves on integral genus-one curves [1, Proposition 6.38], the categories \( C_{10}, C_{11}, C_{12} \) are exactly the extension closures of: semistable fiber sheaves all of whose Harder–Narasimhan (HN) factors have strictly positive slopes; semistable fiber sheaves all of whose HN factors have slope zero; semistable fiber sheaves all of whose HN factors have strictly negative slopes. The category \( C_{20} \) is the category of pure one-dimensional sheaves \( F \) such that each irreducible component of \( \text{supp} F \) is ‘horizontal’. In the diagram notation of [8, 9], the categories \( C_{00}, C_{10}, C_{11}, C_{12}, C_{20} \) are

```
+  +  +  +  0  +  -  +  *
```

respectively. Continuing,

\[
C_{30} = \mathcal{A}^2(\pi)_1 \cap W_{0,\Phi}, \\
C_{31} = \{ E \in \hat{\Phi}(\hat{C}_{20}) : \text{dim} \hat{E} = 2 \}, \\
C_{32} = \{ E \in \mathcal{A}^2(\pi)_1 \cap W_{1,\Phi} : \text{dim} \hat{E} = 2 \}, \\
C_{40} = \{ E \in \mathcal{A}^2(\pi)_1 \cap W_{0,\Phi} : \text{dim} \hat{E} = 3 \}, \\
C_{50} = \mathcal{A}^3(\pi)_2 \cap W_{0,\Phi}, \\
C_{51} = \{ E \in \mathcal{A}^3(\pi)_2 \cap W_{1,\Phi} : \text{dim} \hat{E} = 2 \}, \\
C_{52} = \{ E \in \mathcal{A}^3(\pi)_2 \cap W_{1,\Phi} : \text{dim} \hat{E} = 3 \}.
\]
The categories $C_{30}, C_{31}, C_{32}, C_{40}, C_{50}, C_{51}, C_{52}$ are

$$
\begin{array}{cccccccc}
+ & + & + & + & + & + & + & + \\
+ & + & 0 & + & + & + & 0 & + \\
+ & + & - & + & + & + & - & + \\
\end{array}
$$

respectively, in the papers [8, 9].

Properties of the categories $C_{ij}$ (and similarly for $\widehat{C}_{ij}$, with $\Phi$ and $\widehat{\Phi}$ exchanged) include:

- **Property C0.** $C_{00} \subset W_{0,\Phi}$;
- **Property C1.** $C_{20} \subset W_{0,\Phi}$.

Property C0 follows from the fact that the kernel of the Fourier–Mukai functor $\Phi$ is a universal sheaf for a moduli problem on $X$ [2]. Property C1 is [7, Lemma 3.6]. Note that we have $C_{31} \subset W_{1,\Phi}$ by Property C1 and Lemma 3.1.

We omit the proofs of Lemmas 4.1 and 4.2 below since they are straightforward.

**Lemma 4.1.** Suppose that we have $A_X$-short exact sequences

$$
0 \rightarrow K \rightarrow A \rightarrow Q \rightarrow 0,
$$

$$
0 \rightarrow Q_0 \rightarrow Q \rightarrow Q_1 \rightarrow 0,
$$

where $Q_i \in W_{i,\Phi}$ for $i = 0, 1$. If $A$ is $\Phi$-WIT$_1$, then there exists an $A_X$-surjection $\widehat{A} \rightarrow \widehat{Q}_1$.

**Lemma 4.2.** Suppose that $T, C$ are subcategories of $A_X$, with $T$ being a torsion class in $A_X$. Suppose that for every $A \in C$ and every $A_X$-quotient $A \rightarrow A'$, we have $A' \in \langle T, C \rangle$. Then $\langle T, C \rangle$ is a torsion class in $A_X$.

**Lemma 4.3.** For any $0 \leq m \leq 2$, the category $\mathcal{A}_X^{\leq m}$ is a Serre subcategory of $A_X$, as is the category $A(\pi)_0$.\[m]

**Proof.** The assertions follow from Property D1. □

**Lemma 4.4.** $A(\pi)_0$ is a Serre subcategory of $A_X$.

**Proof.** This follows from Property D1. □

**Lemma 4.5.** $C_{00} \subset A(\pi)_0$.

**Proof.** For any $E \in C_{00}$, we have $\dim (\pi(\text{supp } E)) \leq 0$ by Property D2 and so $E \in A(\pi)_0$. □

**Lemma 4.6.** If $E \in C_{10}$, then any $A_X$-quotient $E'$ of $E$ lies in $\langle C_{00}, C_{10} \rangle$.

**Proof.** Let $E, E'$ be as described. Since $C_{00} = \mathcal{A}_X^{\leq 0}$ is a torsion class in $A_X$ and $C_{00}$ is contained in the abelian subcategory $A(\pi)_0$ of $A_X$, it follows that $C_{00}$ is a torsion class in $A(\pi)_0$. Hence, we have a short exact sequence in $A(\pi)_0$,

$$
0 \rightarrow E'_0 \rightarrow E' \rightarrow E'_1 \rightarrow 0,
$$

where $E'_0 \in C_{00}$, while $E'_1$ satisfies $\text{Hom}(C_{00}, E'_1) = 0$. Since $A(\pi)_0$ and $W_{0,\Phi}$ are both torsion classes in $A_X$, $E'_1$ must also lie in $A(\pi)_0 \cap W_{0,\Phi}$. Hence, $E'_1 \in C_{10}$. □
**Lemma 4.7.** $C_{11} = \Phi C_{00}$.

**Proof.** Since $C_{00} \subset W_{0,\Phi}$ by Property C0, we have $\Phi C_{00} \subset W_{1,\Phi}$ by Lemma 3.1. Moreover, for any $E \in C_{00}$, the transform $\tilde{E} := \Phi E$ is a coherent sheaf and $\tilde{E} = \Phi \Phi E[1] \cong E$ lies in $C_{00}$. Also, since $C_{00} \subset A(\pi)_0$ by Lemma 4.5, we have $\Phi C_{00} \subset \mathcal{A}(\pi)_0$ by Property A3. Hence, $\Phi C_{00} \subseteq C_{11}$.

To see the other inclusion, take any $F \in C_{11}$. Then $\tilde{F} = \Phi F[1] \in C_{00}$; that is, $\Phi C_{11}[1] \subseteq C_{00}$; that is, $C_{11} \subseteq \Phi C_{00}$. □

**Lemma 4.8.** $C_{12} = \Phi \hat{C}_{10}$.

**Proof.** By the same argument as in the proof of Lemma 4.7, we obtain $\Phi \hat{C}_{10} \subset \mathcal{A}(\pi)_0 \cap W_{1,\Phi}$. Also, if $E \in \Phi \hat{C}_{10}$ is a nonzero object, then $\tilde{E} \in \hat{C}_{10}$ and, by the definition of $\hat{C}_{10}$, the object $\tilde{E}$ cannot be supported in dimension zero, forcing dim $\tilde{E} = 1$. Now, for any $A \in \hat{C}_{10}$ and $B \in C_{11}$, we have $\text{Hom}(B, \Phi A) \cong \text{Hom}(\tilde{B}, A)$ since $\Phi$ is an equivalence and by Property A2. Since $\tilde{B} \in C_{00}$, we must have $\text{Hom}(\tilde{B}, A) = 0$ from the definition of $C_{10}$. Hence, $\text{Hom}(B, \Phi A) = 0$ and we have shown that $\Phi \hat{C}_{10} \subseteq C_{12}$.

To see the other inclusion, take any $E \in C_{12}$. The same argument as above shows that $\tilde{E} \in \mathcal{A}(\pi)_0 \cap W_{1,\Phi}$; we also have dim $\tilde{E} = 1$ from the definition of $C_{12}$. It remains to show that $\text{Hom}(C_{00}, \tilde{E}) = 0$. Suppose that we have a morphism $\alpha : A \to \tilde{E}$ for some $A \in C_{00}$. Since $A$ and $\tilde{E}$ are both $\Phi$-WIT0, the functor $\Phi$ takes $\alpha$ to the morphism $\Phi \alpha : A \to E$. Now we have $\tilde{A} \in C_{11}$ by Lemma 4.7 and so $\Phi \alpha$ and hence $\alpha$ must be zero from the definition of $C_{12}$. This completes the proof of the lemma. □

**Lemma 4.9.** $\langle C_{00}, C_{20} \rangle$ is a torsion class in $\mathcal{A}_X$.

**Proof.** We already know that $C_{00}$ is a torsion class from Lemma 4.3. Therefore, it suffices to take an arbitrary $E \in C_{20}$ and any $\mathcal{A}_X$-surjection $E \to E''$, and show that $E''$ lies in $\langle C_{00}, C_{20} \rangle$. If dim $E'' = 0$, then $E'' \in C_{00}$; so let us assume that dim $E'' = 1$.

Since $E \in C_{20} = \mathcal{A}_X^{(1)}$, all the one-dimensional irreducible components of $\text{supp}(E)$ are of type (b) in Property Z2. Since $\text{supp}(E'') \subseteq \text{supp}(E)$ by Property D1, the one-dimensional irreducible components of $\text{supp}(E'')$ are also of type (b) in Property Z2, that is, $E'' \in \mathcal{A}_X^{(1)} = C_{20}$. □

If a pair of integers $(i, j)$ is not part of the collection (4-1), we define $C_{ij}$ and $\hat{C}_{ij}$ to be empty. For $n = 0, 2, 4$, we define

$$T_{n0} = \langle C_{ij} : i \leq n, 0 \leq j \leq 2 \rangle.$$

For $n = 1, 3, 5$, we define

$$T_{n0} = \langle T_{(n-1), 0}, C_{n0} \rangle,$$
$$T_{n1} = \langle T_{n0}, C_{n1} \rangle,$$
$$T_{n2} = \langle T_{n1}, C_{n2} \rangle.$$
For instance,

\[ \mathcal{T}_{20} = \begin{pmatrix} C_{00} & C_{10} & C_{20} \\ C_{11} & \ast & C_{21} \\ C_{12} & \ast & \ast \end{pmatrix} \quad \text{and} \quad \mathcal{T}_{31} = \begin{pmatrix} C_{00} & C_{10} & C_{20} & C_{30} \\ C_{11} & \ast & C_{21} & \ast \\ C_{12} & \ast & \ast & \ast \end{pmatrix}. \]

For \( 2 \leq i \leq 5 \), we define \( \mathcal{F}_i = \langle C_{00}, C_{10}, \ldots, C_{i0} \rangle \).

We also define \( \hat{\mathcal{T}}_{ij} \) (respectively \( \hat{\mathcal{F}}_{ij} \)) using the same definition as \( \mathcal{T}_{ij} \) (respectively \( \mathcal{F}_i \)), with \( C_{ij} \) replaced with \( \hat{C}_{ij} \).

**Lemma 4.10.** Suppose that \( 1 \leq m \leq 3 \) is an integer and \( E \in \mathcal{A}^m(\pi)_{m-1} \cap W_{0,\Phi} \). For any \( \mathcal{A}_X \)-quotient \( E \to E' \) where \( \dim(E') = m \), we have \( E' \in \mathcal{A}^m(\pi)_{m-1} \cap W_{0,\Phi} \).

**Proof.** Let \( E, E' \) be as described. From Property TC3, we know that \( W_{0,\Phi} \) is a torsion class in \( \mathcal{A}_X \) and so \( E' \in W_{0,\Phi} \). Since \( \dim(E') = m \), we know that \( \dim(\pi(\text{supp } E')) \) is either \( m - 1 \) or \( m \) by Property D2. Property D1, however, gives that \( \dim(\pi(\text{supp } E')) \leq m - 1 \) and so \( \dim(\pi(\text{supp } E')) = m - 1 \) and we are done. \( \square \)

**Lemma 4.11.** \( \mathcal{A}(\pi)_0 \cap W_{0,\Phi} = \mathcal{T}_{10} \) and it is a torsion class in \( \mathcal{A}_X \).

**Proof.** We know that \( \mathcal{A}(\pi)_0 \) and \( W_{0,\Phi} \) are both torsion classes in \( \mathcal{A}_X \) from Lemma 4.4 and Property TC3. That the intersection of two torsion classes is again a torsion class follows from Lemma 2.1.

Recall that \( \mathcal{T}_{10} = \langle C_{00}, C_{10} \rangle \). We have \( C_{10} \subseteq \mathcal{A}(\pi)_0 \cap W_{0,\Phi} \) by definition, and \( C_{00} \subseteq \mathcal{A}(\pi)_0 \cap W_{0,\Phi} \) by Properties D2 and C0.

To see the other inclusion, take any \( E \in \mathcal{A}(\pi)_0 \cap W_{0,\Phi} \). Since \( \mathcal{A}(\pi)_0 \) is a Serre subcategory of \( \mathcal{A}_X \), that \( C_{00} \) is a torsion class in \( \mathcal{A}_X \) (Lemma 4.3) implies that \( C_{00} \) is also a torsion class in the abelian category \( \mathcal{A}(\pi)_0 \); let \( F \) denote the corresponding torsion-free class in \( \mathcal{A}(\pi)_0 \). Then we have an \( \mathcal{A}(\pi)_0 \)-short exact sequence

\[ 0 \to E' \to E \to E'' \to 0, \]

where \( E' \in C_{00} \) and \( E'' \in F \). That \( \mathcal{A}(\pi)_0 \cap W_{0,\Phi} \) is a torsion class in \( \mathcal{A}_X \) (hence in \( \mathcal{A}(\pi)_0 \)) implies that \( E'' \in \mathcal{A}(\pi)_0 \cap W_{0,\Phi} \). Since \( \text{Hom}(C_{00}, E'') = 0 \) by construction, we have \( E'' \in C_{10} \), giving us \( E \in \mathcal{T}_{10} \). This completes the proof of the lemma. \( \square \)

**Lemma 4.12.** \( \mathcal{T}_{11} \) is a torsion class in \( \mathcal{A}^{c}_{X} \).

**Proof.** Recall that \( \mathcal{T}_{11} = \langle \mathcal{T}_{10}, C_{11} \rangle \). Take any \( E \in C_{11} \) and any \( \mathcal{A}_X \)-quotient \( E \to E' \). By Lemma 4.11, \( \mathcal{A}(\pi)_0 \cap W_{0,\Phi} \) is a torsion class in \( \mathcal{A}_X \) and hence in \( \mathcal{A}^{c}_{X} \); let \( F \) be the corresponding torsion-free class in \( \mathcal{A}^{c}_{X} \). Then we have an \( \mathcal{A}^{c}_{X} \)-short exact sequence

\[ 0 \to A_0 \to E' \to A_1 \to 0, \]

where \( A_0 \in \mathcal{A}(\pi)_0 \cap W_{0,\Phi} = \mathcal{T}_{10} \) and \( A_1 \in F \). By Lemmas 4.11 and 4.2, it suffices for us to show that \( A_1 \in \mathcal{T}_{11} \). That \( \mathcal{A}(\pi)_0 \) is a torsion class in \( \mathcal{A}_X \) (Lemma 4.4) implies that \( A_1 \in \mathcal{A}(\pi)_0 \). On the other hand, if \( A_{10} \) denotes the \( \Phi \)-WIT part of \( A_{10} \) in \( \mathcal{A}_X \), then \( A_{10} \in \mathcal{A}(\pi)_0 \) by Lemma 4.4 again; that is, \( A_{10} \in \mathcal{A}(\pi)_0 \cap W_{0,\Phi} \) and then \( A_{10} = 0 \).
by construction of $A_1$. That is, we have $A_1 \in \mathcal{A}(\pi)_0 \cap W_{1,\phi}$. By Lemma 4.1, we now have an $\mathcal{A}_X$-surjection $\widehat{E} \to \widehat{A}_1$. Since $\dim \widehat{E} = 0$ from the definition of $C_{11}$, we have $\dim \widehat{A}_1 = 0$ by Lemma 4.3. This shows that $A_1 \in C_{11}$ and we are done.

**Lemma 4.13.** $\mathcal{A}(\pi)_0 = \mathcal{T}_{12}$ is a Serre subcategory of $\mathcal{A}_X$ and hence a torsion class in $\mathcal{A}_X$.

**Proof.** We know that $\mathcal{A}(\pi)_0$ is a Serre subcategory of $\mathcal{A}_X$ from Lemma 4.4; since $\mathcal{A}_X$ is a noetherian abelian category, it follows that $\mathcal{A}(\pi)_0$ is a torsion class in $\mathcal{A}_X$. Hence, it suffices to show the equality $\mathcal{A}(\pi)_0 = \mathcal{T}_{12}$.

That $\mathcal{T}_{12} \subseteq \mathcal{A}(\pi)_0$ is clear from Lemma 4.5 and the construction of $\mathcal{T}_{12}$. To show the other inclusion, take any $E \in \mathcal{A}(\pi)_0$. Since $E \in \mathcal{A}^{X1}_X$ and $\mathcal{T}_{11}$ is a torsion class in $\mathcal{A}^{X1}_X$ by Lemma 4.12, there is an $\mathcal{A}^{X1}_X$-short exact sequence

$$0 \to E' \to E \to E'' \to 0,$$

where $E' \in \mathcal{T}_{11} \subset \mathcal{T}_{12}$, while $\text{Hom}(\mathcal{T}_{11}, E'') = 0$.

Since $\mathcal{A}(\pi)_0$ is a Serre subcategory of $\mathcal{A}_X$, the $\Phi$-WIT$_0$ component of $E''$ must lie in $\mathcal{A}(\pi)_0$; however, we have $\text{Hom}(\mathcal{E}_{10}, E'') = 0$ and so by Lemma 4.11 we must have $E'' \in W_{1,\phi}$. That is, $E'' \in \mathcal{A}(\pi)_0 \cap W_{1,\phi}$. Since $C_{11} \subseteq \mathcal{T}_{11}$, we also have $\text{Hom}(C_{11}, E'') = 0$. On the other hand, Property A3 gives $\widehat{E''} \in \mathcal{A}(\pi)_0$ and so $\dim \widehat{E''} \leq 1$. If $\dim \widehat{E''} = 1$, then $E'' \in \mathcal{C}_{12}$; if $\dim \widehat{E''} = 0$, then $E'' \in C_{11}$. Hence, $E'' \in \mathcal{T}_{12}$, implying that $E \in \mathcal{T}_{12}$ and we are done.

**Lemma 4.14.** $\mathcal{A}^{X1}_X \cap W_{0,\phi} = F_2$ and it is a torsion class in $\mathcal{A}_X$.

**Proof.** Since $\mathcal{A}^{X1}_X$ and $W_{0,\phi}$ are both torsion classes in $\mathcal{A}_X$ (by Lemma 4.3 and Property TC3), their intersection is also a torsion class in $\mathcal{A}_X$.

The inclusion $F_2 \subseteq \mathcal{A}^{X1}_X \cap W_{0,\phi}$ follows from Properties C0 and C1 and the definition of $C_{10}$. To see the inclusion $\mathcal{A}^{X1}_X \cap W_{0,\phi} \subseteq F_2$, take any $E \in \mathcal{A}^{X1}_X \cap W_{0,\phi}$. From Lemma 4.9, the extension closure $\langle C_{00}, C_{20} \rangle$ is a torsion class in $\mathcal{A}^{X1}_X$; let $F$ denote the corresponding torsion-free class in $\mathcal{A}^{X1}_X$. Then there exists an $\mathcal{A}^{X1}_X$-short exact sequence

$$0 \to E' \to E \to E'' \to 0,$$

where $E' \in \langle C_{00}, C_{20} \rangle$ and $E'' \in F$. Since $E' \in F_2$, it suffices to show that $E'' \in F_2$ when $E''$ is nonzero.

Since $E \in W_{0,\phi}$, we have $E'' \in W_{0,\phi}$ by Property TC3. By the definition of $F$, we have the vanishing $\text{Hom}(C_{00}, E'') = 0$ and so $\dim E''$ must be 1. By repeatedly applying Property Z1, and using Property Z2, we can construct an $\mathcal{A}_X$-short exact sequence

$$0 \to A_h \to E'' \to A_v \to 0,$$

where $A_h$ lies in either $C_{00}$ or $\mathcal{A}^{X1}_X = C_{20}$, and all the irreducible components of $\text{supp}(A_v)$ are contained in fibers of $\pi$ (that is, $A_v \in \mathcal{A}(\pi)_0$). Hence, $A_h \in F_2$, while $A_v \in \mathcal{A}(\pi)_0 \cap W_{0,\phi} = \mathcal{T}_{10} \subset F_2$ (by Lemma 4.11) and we are done. □
Lemma 4.15. \( \mathcal{T}_{20} \) is a torsion class in \( \mathcal{A}_X \).

Proof. We have \( \mathcal{T}_{20} = \langle \mathcal{T}_{12}, C_{20} \rangle \) by definition. For any \( E \in C_{20} \) and any \( \mathcal{A}_X \)-quotient \( E' \) of \( E \), we have \( E \in \mathcal{F}_2 \) and hence \( E' \in \mathcal{F}_2 \subset \mathcal{T}_{20} \) by Lemma 4.14. Then, by Lemmas 4.13 and 4.2, we conclude that \( \mathcal{T}_{20} \) is a torsion class in \( \mathcal{A}_X \).

Lemma 4.16. \( \mathcal{F}_3 \) is a torsion class in \( \mathcal{A}_X \).

Proof. We have \( \mathcal{F}_3 = \langle \mathcal{F}_2, C_{30} \rangle \) by definition. Take any \( E \in C_{30} \) and any \( \mathcal{A}_X \)-surjection \( E \twoheadrightarrow E' \). By Lemmas 4.14 and 4.2, it suffices to show that \( E' \in \mathcal{F}_3 \).

By Property TC3, we have \( E' \in W_{0,\Phi} \), while by Property D1 \( \dim(E') \leq 2 \). If \( \dim(E') \leq 1 \), then \( E' \in \mathcal{F}_2 \) by Lemma 4.14; if \( \dim(E') = 2 \), then, by Lemma 4.10, we have \( E' \in C_{30} \). This completes the proof.

Lemma 4.17. \( \mathcal{T}_{30} \) is a torsion class in \( \mathcal{A}_X \).

Proof. We have \( \mathcal{T}_{30} = \langle \mathcal{T}_{20}, C_{30} \rangle \) by definition. Take any \( E \in C_{30} \) and any \( \mathcal{A}_X \)-surjection \( E \twoheadrightarrow E' \). By Lemmas 4.15 and 4.2, it suffices to show that \( E' \in \mathcal{T}_{30} \). Since \( C_{30} \subset \mathcal{F}_3 \), by Lemma 4.16, we have \( E' \in \mathcal{F}_3 \subset \mathcal{T}_{30} \) and we are done.

Remark 4.18. We have \( C_{31} \subset \mathcal{A}(\pi)_{\leq 1} \). To see this, note that \( C_{20} = \mathcal{A}_{b}^{\leq 1} \) is contained in \( \mathcal{A}(\pi)_{\leq 1} \) by Property D2. Then, by Properties C1 and A3, we have \( C_{31} \subseteq \hat{\Phi}(C_{20}) \subset 0 \).

Lemma 4.19. \( \mathcal{T}_{31} \) is a torsion class in \( \mathcal{A}_X \).

Proof. We have \( \mathcal{T}_{31} = \langle \mathcal{T}_{30}, C_{31} \rangle \) by definition. Take any \( E \in C_{31} \) and any \( \mathcal{A}_X \)-surjection \( E \twoheadrightarrow E' \). By Lemmas 4.17 and 4.2, it suffices for us to show that \( E' \in \mathcal{T}_{31} \).

By Property TC3, there is an \( \mathcal{A}_X \)-short exact sequence

\[ 0 \to A_0 \to E' \to A_1 \to 0, \]

where \( A_i \in W_{i,\Phi} \). Note that \( \dim A_0, \dim A_1 \leq 2 \) by Property D1.

If \( \dim A_0 \leq 1 \), then \( A_0 \in \mathcal{F}_2 \) by Lemma 4.14 and so \( A_0 \in \mathcal{T}_{31} \). Now suppose that \( \dim A_0 = 2 \); since \( E \in \mathcal{A}(\pi)_{\leq 1} \) by Remark 4.18, it follows that \( A_0 \in \mathcal{A}(\pi)_{\leq 1} \) by Lemma 4.3, that is, \( A_0 \in \mathcal{A}(\pi)_{\leq 1} \cap W_{0,\Phi} \). Note that \( \dim (\pi(\text{supp} A_0)) \) must be exactly 1 by Property D2 and so \( A_0 \in C_{30} \subseteq \mathcal{T}_{31} \).

As for \( A_1 \), we have \( A_1 \in \mathcal{T}_{31} \) if \( \dim A_1 \leq 1 \) as above, so let us assume that \( \dim A_1 = 2 \). Note that \( C_{31} \) is contained in \( W_{1,\Phi} \) by Property C1 and Lemma 3.1. Applying Lemma 4.1 to the composite \( \mathcal{A}_X \)-surjection \( E \twoheadrightarrow E' \to A_1 \) then gives \( \hat{A}_1 \in \hat{C}_{20} \), which shows that \( A_1 \in C_{31} \subseteq \mathcal{T}_{31} \).

Overall, we have \( E' \in \mathcal{T}_{31} \), completing the proof.

Lemma 4.20. \( \mathcal{T}_{20} = \mathcal{A}_{X}^{\leq 1} \).
PROOF. The inclusion $T_{20} \subseteq \mathcal{A}_X^{\leq 1}$ is clear. To see the other inclusion, take any $E \in \mathcal{A}_X^{\leq 1}$. By Properties Z1 and Z2, there is an $\mathcal{A}_X^{\leq 1}$-short exact sequence

$$0 \to E_h \to E \to E_v \to 0,$$

where $E_h$ lies in $C_{00}$ (if dim $E_h = 0$) or $C_{20}$ (if dim $E_h = 1$) and $E_v$ lies in $\mathcal{A}(\pi)_0$. Since $\mathcal{A}(\pi)_0 = T_{12}$ by Lemma 4.13, we see that $E$ lies in $T_{20}$. □

**Lemma 4.21.** Suppose that $Q \in W_{1,\Phi}$ satisfies $\widehat{Q} \in \mathcal{A}_X^{\leq 1}$. Then $Q \in T_{31}$.

**Proof.** As in the proof of Lemma 4.20, there is an $\mathcal{A}_X^{\leq 1}$-short exact sequence

$$0 \to A_h \to \widehat{Q} \to A_v \to 0,$$  \hspace{1cm} (4-2)

where $A_h \in C_{00} \cup C_{20}$ and $A_v \in \mathcal{A}(\pi)_0$. Note that $\widehat{Q} \in W_{0,\widehat{\Phi}}$ by Lemma 3.1 and so $A_v \in W_{0,\widehat{\Phi}}$ by Property TC3. By Property A3 and Lemma 3.1, we now have $\widehat{A}_v \in \mathcal{A}(\pi)_0$ and so $\widehat{A}_v \in \widehat{T}_{12} \subset \widehat{T}_{31}$ by Lemma 4.13.

On the other hand, note that $A_h \in W_{0,\Phi}$ by Properties C0 and C1. If $A_h \in C_{00}$, then $\widehat{A}_h \in \widehat{C}_{11} \subset \widehat{T}_{31}$ by Lemma 4.7. If $A_h \in C_{20}$, then there are two cases for $\widehat{A}_h$: (i) if dim $\widehat{A}_h = 2$, then $\widehat{A}_h \in \widehat{C}_{31}$ by the definition of $\widehat{C}_{31}$, in which case $\widehat{A}_h \in \widehat{T}_{31}$; (ii) if dim $\widehat{A}_h \leq 1$, then $\widehat{A}_h \in \widehat{T}_{20}$ by Lemma 4.20 and we still have $\widehat{A}_h \in \widehat{T}_{31}$.

Thus, both $\widehat{A}_h, \widehat{A}_v$ lie in $\widehat{T}_{31}$. Since $\Phi$ takes the short exact sequence (4-2) to the $\mathcal{A}_X$-short exact sequence

$$0 \to \widehat{A}_h \to \widehat{Q} \to \widehat{A}_v \to 0,$$

we see that $Q$ itself lies in $T_{31}$. □

**Lemma 4.22.** $T_{32}$ is a torsion class in $\mathcal{A}_X$.

**Proof.** We have $T_{32} = \langle T_{31}, C_{32} \rangle$ by definition. Take any $E \in C_{32}$ and any $\mathcal{A}_X$-surjection $E \twoheadrightarrow E'$. By Lemmas 4.19 and 4.2, it suffices for us to show that $E' \in T_{32}$. By Property TC3, there is an $\mathcal{A}_X$-short exact sequence

$$0 \to A_0 \to E' \to A_1 \to 0,$$

where $A_i \in W_{i,\Phi}$. Property D1 implies that dim($A_0$), dim($A_1$) $\leq 2$. Since $\mathcal{A}_X^{\leq 1} = T_{20} \subset T_{32}$ by Lemma 4.20, we can further assume that dim $A_0 = $ dim $A_1 = 2$.

That $E' \in C_{32}$ means that $E' \in \mathcal{A}(\pi)_{\leq 1}$ and so, by Properties D1 and D2, we have dim $(\pi(\text{supp} A_0)) = $ dim $(\pi(\text{supp} A_1)) = 1$. Hence, $A_0 \in C_{30} \subset T_{32}$.

On the other hand, we have $A_1 \in \mathcal{A}(\pi)_1 \cap W_{1,\Phi}$. By Properties A3 and D2, we know that dim($\widehat{A}_1$) is either 1 or 2. If dim($\widehat{A}_1$) = 1, then $A_1 \in T_{31} \subset T_{32}$ by Lemma 4.21; if dim($\widehat{A}_1$) = 2, then $A_1 \in C_{32} \subset T_{32}$. Thus, $E' \in T_{32}$ and we are done. □

From here on to the end of Section 4, we will assume that $X$ is the product threefold as in Section 3.6.

**Lemma 4.23.** Let $\pi : X \to C$ be the product elliptic threefold in Section 3.6. Then $\mathcal{A}_X^{\geq 2} = T_{40}$. 
A framework for torsion theory computations on elliptic threefolds

That $\mathcal{F}_{40} \subseteq \mathcal{A}_X^{\leq 2}$ is clear. To show the other inclusion, take any $E \in \mathcal{A}_X^{\leq 1}$. Since $\mathcal{A}_X^{\leq 2}$ is a Serre subcategory of $\mathcal{A}_X$ by Lemma 4.3, Property TC3 implies that there is an $\mathcal{A}_X^{\leq 2}$-short exact sequence

$$0 \to E_0 \to E \to E_1 \to 0,$$

where $E_i \in W_i$. Given Lemma 4.20, it suffices to assume that $\dim E_0 = \dim E_1 = 2$.

Lemma 3.2 and Property D2 together now give $\dim (\text{supp } E_1) = 1$ and so $E_1 \in \mathcal{A}^2(\pi_1) \cap W_{1,\Phi}$. By Properties A3 and D2 again, we know that $\dim (\hat{E}_1)$ is either 1 or 2. If $\dim (\hat{E}_1) = 1$, then Lemma 4.21 says that $E_1 \in \mathcal{T}_{31} \subset \mathcal{T}_{40}$; if $\dim (\hat{E}_1) = 2$, then $E_1 \in \mathcal{T}_{32} \subset \mathcal{T}_{40}$.

On the other hand, Property D2 implies that $\dim (\pi(\text{supp } E_0)) = 1$ and so $E_0 \in \mathcal{A}^2(\pi_1) \cap W_{0,\Phi} = C_{30} \subset \mathcal{T}_{40}$. Hence, $E \in \mathcal{T}_{40}$, as wanted. □

**Lemma 4.24.** Let $\pi : X \to C$ be the product elliptic threefold in Section 3.6. Then $\mathcal{A}_X^{\leq 2} \cap W_{0,\Phi} = \mathcal{F}_4$ and it is a torsion class in $\mathcal{A}_X$.

**Proof.** Since $\mathcal{A}_X^{\leq 2}$ and $W_{0,\Phi}$ are both torsion classes in $\mathcal{A}_X$ by Lemma 4.3 and Property TC3, their intersection is also a torsion class in $\mathcal{A}_X$. Hence, it remains to show the equality $\mathcal{A}_X^{\leq 2} \cap W_{0,\Phi} = \mathcal{F}_4$. The inclusion $\mathcal{F}_4 \subseteq \mathcal{A}_X^{\leq 2} \cap W_{0,\Phi}$ is clear from the definition of $\mathcal{F}_4$ and Properties C0 and C2.

To show the inclusion $\mathcal{A}_X^{\leq 2} \cap W_{0,\Phi} \subseteq \mathcal{F}_4$, take any $E \in \mathcal{A}_X^{\leq 2} \cap W_{0,\Phi}$. Note that $\mathcal{F}_3$ is a torsion class in $\mathcal{A}_X$ by Lemma 4.16; let $\mathcal{F}$ denote the corresponding torsion-free class in $\mathcal{A}_X^{\leq 2}$. Then we have an $\mathcal{A}_X^{\leq 2}$-short exact sequence

$$0 \to E' \to E \to E'' \to 0,$$

where $E' \in \mathcal{F}_3 \cap \mathcal{F}_4$ and $E'' \in \mathcal{F}$. Since $W_{0,\Phi}$ is a torsion class in $\mathcal{A}_X$ (Property TC3), we have $E'' \in \mathcal{A}_X^{\leq 2} \cap W_{0,\Phi}$. We will be done once we show that $E'' \in \mathcal{F}_4$.

If $\dim E'' \leq 1$, then $E'' \in \mathcal{F}_2 \subset \mathcal{F}_4$ by Lemma 4.14, so we can assume that $\dim E'' = 2$. Then Property D2 implies that $\dim (\pi(\text{supp } E'')) = 1$ or 2. If $\dim (\pi(\text{supp } E'')) = 1$, then $E'' \in C_{30} \subset \mathcal{T}_4$, so we can further assume that $\dim (\pi(\text{supp } E'')) = 2$.

Properties A3 and D2 together now imply that $\dim \hat{E}''$ is either 2 or 3. If $\dim \hat{E}'' = 2$, then Lemma 3.2 gives $\dim (\pi(\text{supp } E'')) \leq 1$, which is a contradiction. If $\dim \hat{E}'' = 3$, then $E'' \in C_{40} \subset \mathcal{T}_4$, completing the proof. □

**Lemma 4.25.** Let $\pi : X \to C$ be the product elliptic threefold in Section 3.6. Then $W_{0,\Phi} = \mathcal{F}_5$.

**Proof.** By Lemma 4.24, we have $\mathcal{F}_4 \subseteq W_{0,\Phi}$; since $C_{50} \subset W_{0,\Phi}$ by definition, we have $\mathcal{F}_5 \subseteq W_{0,\Phi}$. To show the other inclusion, take any $E \in W_{0,\Phi}$. Since $\mathcal{F}_4$ is a torsion class in $\mathcal{A}_X$ (Lemma 4.24), we have an $\mathcal{A}_X$-short exact sequence

$$0 \to E' \to E \to E'' \to 0,$$

where $E' \in \mathcal{F}_4 \subset \mathcal{F}_5$ and $\text{Hom}(\mathcal{F}_4, E'') = 0$. Since $W_{0,\Phi}$ is a torsion class in $\mathcal{A}_X$, we have $E'' \in W_{0,\Phi}$. If $\dim E'' \leq 2$, then $E'' \in \mathcal{F}_4$ by Lemma 4.24 again and so $E'' \in \mathcal{F}_5$. If $\dim E'' = 3$, then Property CH2p implies that $\dim (\pi(\text{supp } E'')) = 2$, that is, $E'' \in C_{50} \subset \mathcal{F}_5$. Overall, we have $E \in \mathcal{F}_5$. □
**Lemma 4.26.** Let $\pi : X \to C$ be the product elliptic threefold in Section 3.6. Then $T_{50}$ is a torsion class in $A_X$.

**Proof.** We have $T_{50} = \langle T_{40}, C_{50} \rangle$ by definition. Take any $E \in C_{50}$ and any $A_X$-surjection $E \to E'$. By Lemmas 4.23 and 4.2, it suffices to show that $E' \in T_{50}$. Since $E \in T_5$, Lemma 4.25 and Property TC3 together give $E' \in T_5 \subset T_{50}$, completing the proof.

**Lemma 4.27.** Let $\pi : X \to C$ be the product elliptic threefold in Section 3.6. Then $T_{51}$ is a torsion class in $A_X$.

**Proof.** We have $T_{51} = \langle T_{50}, C_{51} \rangle$ by definition. Take any $E \in C_{51}$ and any $A_X$-surjection $E \to E'$. By Lemmas 4.26 and 4.2, it suffices to show that $E' \in T_{51}$. Since $E \in F_5$, Lemma 4.25 and Property TC3 together give $E' \in F_5 \subset T_{51}$, completing the proof.

**Lemma 4.28.** Let $\pi : X \to C$ be the product elliptic threefold in Section 3.6. Then $A_X = T_{52}$.

**Proof.** Take any $E \in A_X$. Property TC3 asserts there is an $A_X$-short exact sequence

$$0 \to A_0 \to E' \to A_1 \to 0,$$

where $A_i \in W_{i,\Phi}$. Then $A_0 \in T_5 \subset T_{51}$ by Lemma 4.25, so it remains to show that $A_1 \in T_{51}$.

If $\dim A_1 \leq 2$, then $A_1 \in T_{40} \subset T_{51}$ by Lemma 4.23; so let us assume that $\dim A_1 = 3$ from now on. Since $E \in C_{51}$, we have $E \in W_{1,\Phi}$. Thus, by Lemma 4.1, there is an $A_X$-surjection $\widehat{E} \to \widehat{A}_1$. By definition of $C_{51}$, we have $\dim \widehat{E} = 2$ and so $\dim \widehat{A}_1 \leq 2$ by Property D1.

Suppose that $\dim \widehat{A}_1 \leq 1$; then $\dim (\pi(\text{supp} \widehat{A}_1)) \leq 1$ by Property D2 and so $\dim (\pi(\text{supp} A_1)) = \dim (\pi(\text{supp} \widehat{A}_1)) \leq 1$ by Property A3. By Property D2 again, however, we obtain $\dim A_1 \leq 2$, which is a contradiction. Thus, we must have $\dim \widehat{A}_1 = 2$, meaning that $A_1 \in C_{51} \subset T_{51}$ and we are done.

**Proof of Theorem 1.1.** This follows from Lemmas 4.3, 4.9, 4.11–4.17, 4.19, and 4.22–4.27.
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