An affine Birkhoff–Kellogg-type result in cones with applications to functional differential equations

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In this short note, we prove, by means of classical fixed point index, an affine version of a Birkhoff–Kellogg-type theorem in cones. We apply our result to discuss the solvability of a class of boundary value problems for functional differential equations subject to functional boundary conditions. We illustrate our theoretical results in an example.

KEYWORDS
Birkhoff–Kellogg-type result, cone, fixed point index, functional boundary condition, retarded functional differential equation

MSC CLASSIFICATION
47H10, 34K10, 34B10, 34B18

1 | INTRODUCTION

The celebrated Birkhoff–Kellogg invariant direction theorem1 is a widely studied and applied tool of nonlinear functional analysis, also in view of its applicability to eigenvalue problems for ODEs and PDEs (see, e.g., the book2 and the recent papers3,4). Among the various extensions of the invariant direction theorem, one of them is set in the framework of cones and is due to Krasnosel’skii and Ladyženskiı.5 Before we state this latter result, let us recall that a cone

$$K$$

of a real Banach space

$$X$$

is a closed set with

$$K + K \subset K,
\lambda K \subset K$$

for all

$$\lambda \geq 0$$

and

$$K \cap (-K) = \{0\}.$$6

The Birkhoff–Kellogg-type theorem of Krasnosel’skii and Ladyženskiı reads as follows.

**Theorem 1.** Let

$$X, ||||$$

be a real Banach space,

$$U \subset X$$

be an open bounded set with

$$0 \in U,
K \subset X$$

be a cone,

$$T : K \cap \bar{U} \to K$$

be compact and suppose that

$$\inf_{x \in K \cap \partial U} ||Tx|| > 0.$$7

Then there exist

$$\lambda_0 \in (0, +\infty)$$

and

$$x_0 \in K \cap \partial U$$

such that

$$x_0 = \lambda_0 Tx_0.$$8

Here, by means of classical fixed point index, we prove a different version of the Birkhoff–Kellogg result, set within the context of affine cones. Our result is motivated by the study of retarded functional differential equations. In fact, when dealing with the solvability of a boundary value problem with delays and initial data, it is somewhat natural to rewrite it in the form of a perturbed integral equation and to seek the solutions of this equation in an affine cone. In particular,
Calamai and Infante\textsuperscript{7} proved, by means of fixed point index in an affine cone of continuous functions, the existence of multiple nontrivial solutions of the perturbed Hammerstein integral equations of the type
\[
    u(t) = \psi(t) + \int_{0}^{1} k(t, s)g(s)F(s, u_s)\,ds + \gamma(t)\alpha[u],
\]
where \(\alpha[\cdot]\) is a linear functional in the space \(C[0, 1]\) given by Stieltjes integral, namely,
\[
    \alpha[u] = \int_{0}^{1} u(s)\,dA(s).
\]

Here we discuss the solvability of the perturbed integral equations
\[
    u(t) = \psi(t) + \lambda \left( \int_{0}^{1} k(t, s)g(s)F(s, u_s)\,ds + \gamma(t)B[u] \right),
\]
where \(\lambda\) is a nonnegative parameter and \(B[\cdot]\) is a (not necessarily linear) functional in \(C^r([-r, 1], \mathbb{R})\). The functional \(B[\cdot]\) allows to cover the interesting case of nonlinear and nonlocal boundary conditions (BCs) that can occur in the differential problems; there exists a wide literature on these kinds of BCs; we refer the reader to the reviews\textsuperscript{8-14} and the manuscripts.\textsuperscript{15-17} We mention, in particular, the contributions of Mawhin and co-authors in this area of research; see, for example, an earlier study.\textsuperscript{18} Note that, in the applications, the functional \(B[\cdot]\) can also take into account of the past state of the system.

As a toy model, we discuss the solvability of the following class of third-order parameter-dependent functional differential equations with functional BCs.
\[
    u'''(t) + \lambda f(t, u_t, u'(t), u(t - r_1), u'(t - r_2)) = 0, \quad t \in [0, 1],
\]
with initial conditions
\[
    u(t) = \psi(t), \quad t \in [-r, 0],
\]
and one of the following BCs
\[
    u(0) = u'(0) = 0, \quad u(1) = \lambda B[u],
\]
\[
    u(0) = u'(0) = 0, \quad u'(1) = \lambda B[u],
\]
\[
    u(0) = u'(0) = 0, \quad u''(1) = \lambda B[u].
\]

Third-order functional differential equations with nonlocal boundary terms have been studied in the past; we mention here, for example, the work of Tsamatos\textsuperscript{19} and the subsequent papers.\textsuperscript{20-22}

As far as we are aware of, our Birkhoff–Kellogg-type result (Theorem 2 below) is new and complements the interesting topological results in affine cones proved by Djebali and Mebarki.\textsuperscript{23} On the other hand, we also complement the existence results of Calamai and Infante\textsuperscript{7}; this is illustrated in the case of a delay differential equation (DDE). In fact, here we can deal with equations of the type
\[
    u'''(t) = \lambda f(t, u(t), u'(t), u(t - r_1), u'(t - r_2)) = 0, \quad t \in [0, 1]
\]
in which we allow the dependence also in the derivative of the solution and we consider the presence of possibly different time lags.

\section*{2 | FIXED POINTS ON TRANSLATES OF A CONE}

In the proof of our results, we use the notion of fixed point index for compact maps, for classical references; see, for example, previous studies.\textsuperscript{6,24,25}
Let \((X, \| \cdot \|)\) be a real Banach space and \(K\) be a cone in \(X\). Given a bounded and open (in the relative topology) subset \(\Omega\) of \(K\), we denote by \(\bar{\Omega}\) and \(\partial \Omega\) the closure and the boundary of \(\Omega\) relative to \(K\). For \(y \in X\), the translate of the cone \(K\) is defined as
\[
K_y := y + K = \{ y + x : x \in K \}.
\]
Given an open bounded subset \(D\) of \(X\), we denote \(D_K = D \cap K\), an open subset of \(K\). Given a compact map \(\mathcal{G} : \bar{D}_K \to K\) such that \(x \neq \mathcal{G} x\) for \(x \in \partial D_K\), then the fixed point index \(i_K(\mathcal{G}, D_K)\) is well-defined. The index is an integer number which, roughly speaking, is obtained as an algebraic count of the fixed points of the map \(\mathcal{G}\) in \(D_K\). It is known that the properties of the fixed point index are analogous to those of the Leray–Schauder degree (among the fundamental ones there are Normalization, Additivity, Homotopy Invariance, and Solution properties).

Our Birkhoff–Kellogg-type result is a consequence of the Solution and Homotopy invariance properties of the index. The result reads as follows.

**Theorem 2.** Let \((X, \| \cdot \|)\) be a real Banach space, \(K \subset X\) be a cone, and \(D \subset X\) be an open bounded set with \(y \in D_K\) and \(\bar{D}_K \neq K\). Assume that \(\mathcal{F} : \bar{D}_K \to K\) is a compact map and consider the operator
\[
\mathcal{F}_{(y, \lambda)} := y + \lambda \mathcal{F},
\]
where \(\lambda \in \mathbb{R}\). Assume that there exists \(\tilde{\lambda} \in (0, +\infty)\) such that \(i_K(\mathcal{F}_{(y, \lambda)}, D_K) = 0\). Then there exist \(x^* \in \partial D_K\) and \(\lambda^* \in (0, \tilde{\lambda})\) such that \(x^* = y + \lambda^* \mathcal{F}(x^*)\).

**Proof.** First of all, note that we have \(i_K(y, D_K) = 1\) by the Solution property of the index. Consider the map \(H : [0, 1] \times \bar{D}_K \to E\) defined by \(H(t, x) = y + t\lambda \mathcal{F}(x)\). Note that \(H\) is a compact map with values in \(K\). If there exist \(t^* \in (0, 1)\) and \(x \in \partial D_K\) such that \(x = y + t^* \lambda \mathcal{F}(x)\), we are done. If it does not happen, the fixed point index is defined for \(y + t^* \lambda \mathcal{F}\) for every \(t \in [0, 1]\), and by the Homotopy invariance property, we obtain
\[
1 = i_K(y, D_K) = i_K(\mathcal{F}_{(y, \lambda)}, D_K) = 0.
\]
and the result follows. \(\square\)

**Remark 1.** We point out that a sufficient condition yielding that the index is equal to zero, as in the assumption of Theorem 2, is the following:

- Assume that \(\mathcal{G} : \bar{D}_K \to K\) is a compact map such that \(x \neq \mathcal{G} x\) for \(x \in \partial D_K\). If there exists \(e \in K \setminus \{0\}\) such that \(x \neq \mathcal{G} x + \sigma e\) for all \(x \in \partial D_K\) and all \(\sigma > 0\), then \(i_K(\mathcal{G}, D_K) = 0\).

A detailed proof can be found, for example, in Calamai and Infante.\(^7\)

As a Corollary of Theorem 2, we exhibit a norm-type Birkhoff–Kellogg result which can be useful in applications. In order to prove it, we make use of the following proposition.

**Proposition 1** (Proposition 2.1 of\(^2\)). Let \((X, \| \cdot \|)\) be a real Banach space, \(K \subset X\) be a cone, and \(D \subset X\) be an open bounded set with \(y \in D_K\) and \(\bar{D}_K \neq K\). Assume that \(\mathcal{F} : \bar{D}_K \to K\) is a compact map and assume that
\[
\begin{align*}
(a) & \quad \inf_{x \in \partial D_K} \| \mathcal{F}(x) \| > 0 \\
(b) & \quad \mathcal{F}(x) \neq \mu(x - y) \text{ for every } x \in \partial D_K \text{ and } \mu \in (0, 1].
\end{align*}
\]
Then, \(i_K(\mathcal{F}, D_K) = 0\).

We can now state our norm-type result, which can be seen as an affine version of Theorem 1.

**Corollary 1.** Let \((X, \| \cdot \|)\) be a real Banach space, \(K \subset X\) be a cone, and \(D \subset X\) be an open bounded set with \(y \in D_K\) and \(\bar{D}_K \neq K\). Assume that \(\mathcal{F} : \bar{D}_K \to K\) is a compact map and assume that
\[
\inf_{x \in \partial D_K} \| \mathcal{F}(x) \| > 0.
\]
Then there exist \(x^* \in \partial D_K\) and \(\lambda^* \in (0, +\infty)\) such that \(x^* = y + \lambda^* \mathcal{F}(x^*)\).
Proof. We make use of Proposition 1 with the map \( \tilde{\lambda} F \) in place of \( F \).

We proceed by contradiction and assume that there exist \( x_1 \in \partial D_K \) and \( \mu_1 \in (0, 1) \) such that \( \tilde{\lambda} F(x_1) = \mu_1(x_1 - y) \). Take \( R = \sup_{x \in D_K} ||x|| \), then we have

\[
\tilde{\lambda} \cdot \inf_{x \in D_K} ||F(x)|| \leq ||\tilde{\lambda} F(x_1)|| = ||\mu_1(x_1 - y)|| \leq ||x_1 - y|| \leq ||x_1|| + ||y|| \leq R + ||y||,
\]

a contradiction if

\[
\tilde{\lambda} > \frac{R + ||y||}{\inf_{x \in D_K} ||F(x)||}.
\]

Then, the result follows from Theorem 2. \( \square \)

3 | POSITIVE SOLUTIONS FOR A CLASS OF PERTURBED INTEGRAL EQUATIONS

Let \( I \subset \mathbb{R} \) be a compact real interval. By \( C^1(I, \mathbb{R}) \), we denote the Banach space of the continuously differentiable functions defined on \( I \) with the norm

\[
||u||_{I,1} := \max \{ ||u||_{I,\infty}, ||u'||_{I,\infty} \},
\]

where \( ||u||_{I,\infty} := \sup_{t \in I} |u(t)| \).

We adopt a standard notation, used in retarded functional differential equations (cf. Hale and Verduyn Lunel\( ^{26} \)), as follows. Given a positive real number \( r > 0 \), a continuous function \( u : J \to \mathbb{R} \), defined on a real interval \( J \), and given any \( t \in \mathbb{R} \) such that \( [t - r, t] \subseteq J \), by \( u_t : [-r, 0] \to \mathbb{R} \), we mean the function defined by \( u_t(\theta) = u(t + \theta) \).

We consider the following integral equation in the space \( C^1([-r, 1], \mathbb{R}) \):

\[
u(t) = \psi(t) + \lambda \int_0^1 k(t, s)g(s)F(s, u_s)ds + \gamma(t)B[u] =: \psi(t) + \lambda F u(t), \quad t \in [-r, 1]
\]  

where \( B \) is a suitable (possibly nonlinear) functional in the space \( C^1([-r, 1], \mathbb{R}) \).

We require the following assumptions on \( r \) as well as on the maps \( F, k, \psi, \gamma \), and \( g \) that occur in (1).

(C\(_1\)) The function \( \psi : [-r, 1] \to [0, +\infty) \) is continuously differentiable and such that \( \psi(t) = \psi'(t) = 0 \) for all \( t \in [0, 1] \).

(C\(_2\)) The kernel \( k : [-r, 1] \times [0, 1] \to [0, +\infty) \) is measurable, verifies \( k(t, s) = 0 \) for all \( t \in [-r, 0] \) and almost every (a. e.) \( s \in [0, 1] \), and for every \( \tilde{t} \in [0, 1] \), we have

\[
\lim_{t \to \tilde{t}} |k(t, s) - k(\tilde{t}, s)| = 0 \text{ for a.e. } s \in [0, 1].
\]

(C\(_3\)) For a.e. \( s \), the partial derivative \( \partial_s k(t, s) \) is continuous in \( t \) except at the point \( t = s \) where there can be a jump discontinuity; that is, right and left limits both exist, and there exists \( \Psi \in L^1(0, 1) \) such that \( |\partial_s k(t, s)| \leq \Psi(s) \) for \( t \in [0, 1] \) and a.e. \( s \in [0, 1] \).

(C\(_4\)) The function \( g : [0, 1] \to \mathbb{R} \) is measurable, \( g(t) \geq 0 \) a.e. \( t \in [0, 1] \), and satisfies that \( g\Phi \in L^1[0, 1] \) and \( \int_0^1 g(s)ds > 0 \).

(C\(_5\)) \( F : [0, 1] \times C^1([-r, 0], \mathbb{R}) \to [0, \infty) \) is an operator that satisfies some Carathéodory-type conditions (see also Hale and Verduyn Lunel\( ^{26} \)); namely, for each \( \phi, t \mapsto F(t, \phi) \) is measurable and for a.e. \( t, \phi \mapsto F(t, \phi) \) is continuous. Furthermore, for each \( R > 0 \), there exists \( \varphi_R \in L^\infty[0, 1] \) such that

\[
F(t, \phi) \leq \varphi_R(t) \text{ for all } \phi \in C^1([-r, 0], \mathbb{R}) \text{ with } \|\phi\|_{[-r,0],1} \leq R, \text{ and a.e. } t \in [0, 1].
\]

(C\(_6\)) The function \( \gamma : [-r, 1] \to [0, \infty) \) is continuous differentiable and such that \( \gamma(t) = \gamma'(t) = 0 \) for all \( t \in [-r, 0] \).
By \( K_0 \), we denote the following cone of nonnegative functions in the Banach space \( C^1([-r, 1], \mathbb{R}) \):
\[
K_0 = \{ u \in C^1([-r, 1], \mathbb{R}) : u(t) \geq 0 \text{ for every } t \in [-r, 1] \text{ and } u(t) = u'(t) = 0 \text{ for every } t \in [-r, 0] \}.
\]

Observe that the function
\[
w(t) = \begin{cases} 0, & t \in [-r, 0], \\ t^2, & t \in [0, 1], \end{cases}
\]
belongs to \( K_0 \), hence \( K_0 \neq \{0\} \).

Given \( \psi \in C^1([-r, 1], \mathbb{R}) \), let \( K_\psi \) be the following translate of the cone \( K_0 \),
\[
K_\psi = \psi + K_0 = \{ \psi + u : u \in K_0 \}.
\]

**Definition 1.** Given \( \psi \in C^1([-r, 1], \mathbb{R}) \) and \( \rho > 0 \), we define the following subsets of \( C^1([-r, 1], \mathbb{R}) \):
\[
K_{0, \rho} := \{ u \in K_0 : \| u \|_{[0,1], 1} < \rho \}, \quad K_{\psi, \rho} := \psi + K_{0, \rho}.
\]

The following theorem provides an existence result for Equation (1): Here we obtain a nontrivial solution within the cone \( K_\psi \) with fixed norm and a corresponding positive parameter.

**Theorem 3.** Let \( \rho \in (0, +\infty) \) and assume the following further conditions hold.

(a) There exist \( \delta_\rho \in C([0, 1], \mathbb{R}_+) \) such that
\[
F(t, \phi) \geq \delta_\rho(t), \text{ for every } (t, \phi) \in [0, 1] \times C^1([-r, 0], \mathbb{R}_+) \text{ with } \| \phi \|_{[-r, 0], 1} \leq \max \{ \rho, \| \psi \|_{[-r, 0], 1} \}.
\]

(b) \( B : K_{\psi, \rho} \to \mathbb{R}_+ \) is continuous and bounded. Let \( \eta \in (0, +\infty) \) be such that
\[
B[u] \geq \eta \rho, \text{ for every } u \in \partial K_{\psi, \rho}.
\]

(c) The inequality
\[
\sup_{t \in [0, 1]} \left\{ \gamma(t) \eta \delta_\rho - \int_0^1 k(t, s)g(s) \delta_\rho(s) ds \right\} > 0
\]
holds.

Then there exist \( \lambda_\rho \) and \( u_\rho \in \partial K_{\psi, \rho} \) such that the integral Equation (1) is satisfied.

**Proof.** Consider the operator \( F \) defined in (1). Due to the assumptions above, \( F \) maps \( K_{\psi, \rho} \) into \( K_0 \) and is compact. The compactness of the Hammerstein integral operator is a consequence of the regularity assumptions on the terms occurring in it combined with a careful use of the Arzelà-Ascoli theorem (see Webb\(^\text{27}\)), while the perturbation \( \gamma(t)B[\cdot] \) is a finite rank operator.

Take \( u \in \partial K_{\psi, \rho} \). Then we have
\[
\| Fu \|_{[-r, 1], \infty} = \sup_{t \in [0, 1]} \left| \int_0^1 k(t, s)g(s)F(s, u_\rho) ds + \gamma(t)B[u] \right| \geq \sup_{t \in [0, 1]} \left\{ \gamma(t) \eta \delta_\rho - \int_0^1 k(t, s)g(s) \delta_\rho(s) ds \right\}.
\]

Note that the RHS of (3) does not depend on the particular \( u \) chosen. Therefore, we have
\[
\inf_{u \in \partial K_{\psi, \rho}} \| Fu \|_{[-r, 1], 1} \geq \sup_{t \in [0, 1]} \left\{ \gamma(t) \eta \delta_\rho - \int_0^1 k(t, s)g(s) \delta_\rho(s) ds \right\} > 0,
\]
and the result follows by Corollary 1.
We now apply the previous results to the following class of third-order functional differential equations with functional BCs.

\[ u'''(t) + \lambda F(t, u_t) = 0, \ t \in [0, 1], \]

with initial conditions

\[ u(t) = \psi(t), \ t \in [-r, 0], \]

and one of the following BCs

\[ u(0) = u'(0) = 0, \ u(1) = \lambda B[u], \]

\[ u(0) = u'(0) = u'(1) = \lambda B[u], \]

\[ u(0) = u'(0) = u''(1) = \lambda B[u]. \]

We begin by considering some auxiliary problems.

First of all, note that the solution of the ODE \(-u''' = y\) under the BCs

\[ u(0) = u'(0) = u(1) = 0, \]

\[ u(0) = u'(0) = u'(1) = 0, \]

\[ u(0) = u'(0) = u''(1) = 0, \]

in the interval \([0, 1]\) is given by

\[ u(t) = \int_{0}^{1} \hat{k}_i(t, s)y(s)ds, \]

where Green’s function is

\[ \hat{k}_1(t, s) = \begin{cases} 
\frac{1}{2} \left( s(1 - t)(2t - ts - s), & s \leq t, \\
(1 - s)^2t^2, & s \geq t,
\end{cases} \]

in the case of the BCs (6),

\[ \hat{k}_2(t, s) = \begin{cases} 
\frac{1}{2} \left( (2t - t^2 - s)s, & s \leq t, \\
(1 - s)t^2, & s \geq t,
\end{cases} \]

for the BCs (7), and

\[ \hat{k}_3(t, s) = \begin{cases} 
\frac{1}{2} \left( s(2t - s), & s \leq t, \\
t^2, & s \geq t,
\end{cases} \]

for the BCs (8). Furthermore, note that the function

\[ \check{\gamma}_1(t) := t^2 \]

is the unique solution of the boundary value problem (BVP)

\[ \check{\gamma}'''(t) = 0, \ \check{\gamma}(0) = \check{\gamma}'(0) = 0, \ \check{\gamma}(1) = 1, \]

while the functions

\[ \check{\gamma}_2(t) \equiv \check{\gamma}_3(t) := \frac{1}{2}t^2 \]

solve the BVPs

\[ \check{\gamma}'''(t) = 0, \ \check{\gamma}(0) = \check{\gamma}'(0) = 0, \ \check{\gamma}'(1) = 1. \]

\[ \check{\gamma}'''(t) = 0, \ \check{\gamma}(0) = \check{\gamma}'(0) = 0, \ \check{\gamma}''(1) = 1. \]

By routine calculations (see also earlier studies\(^{28,29}\)), one obtains the following proposition.
Proposition 2. For every i = 1, 2, 3, we have the following:

1. \( \dot{k}_i \) is continuous and nonnegative in \([0, 1] \times [0, 1]\) and the partial derivative \( \partial_s k(t, s) \) is continuous in \( t \in [0, 1] \) for every \( s \in [0, 1] \).
2. \( \gamma_i \) is nonnegative and continuously differentiable in \([0, 1]\).

Due to the above setting, the functional boundary value problem (FBVP) (4)–(6) can be rewritten in the form (1), where \( \gamma(t) := H(t)\gamma_1(t) \) and \( k(t, s) := H(t)\dot{k}_1(t, s) \) with

\[
H(r) = \begin{cases} 
1, & r \geq 0, \\
0, & r < 0,
\end{cases}
\]

and provided that \( \psi, F, B \) possess a suitable behavior, Theorem 3 can be applied directly; this fact holds also in the case of the FBVPs (4), (5), and (7) and (4), (5), and (8).

Let us now show briefly how our theory can be applied to DDEs. Namely, let \( f : [0, 1] \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \to [0, \infty) \) be a given Carathéodory map. Consider the third-order DDE with two time lags

\[
\dddot{u}(t) = \lambda f(t, u(t), u'(t), u(t - r_1), u'(t - r_2)) = 0, \quad t \in [0, 1],
\]

(12)

where \( r_1 \) and \( r_2 \) are positive and fixed (possibly different). We can apply the techniques developed in this paper to Equation (12) with initial condition (5) along with one of the BCs (6)–(8). To see this, observe that (12) is a special case of the functional Equation (4), in which taking \( r := \max\{r_1, r_2\} \), the operator \( F : [0, 1] \times C([-r, 0], \mathbb{R}) \to [0, \infty) \) is defined by

\[
F(t, \phi) = f(t, \phi(0), \phi'(0), \phi(-r_1), \phi'(-r_2)).
\]

Such an operator satisfies the above condition \((C_3)\) provided that the following assumption on the map \( f \) is verified: \([\text{(}C_3^r)\text{]}\) For each \( R > 0 \), there exists \( \varphi^*_R \in L^\infty([0, 1]) \) such that

\[
f(t, u, v, p, q) \leq \varphi^*_R(t) \text{ for all } (u, v, p, q) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}
\]

with \( 0 \leq u, p \leq R, |v| \leq R, |q| \leq R, \) and a.e. \( t \in [0, 1] \).

To better illustrate the growth conditions, we now provide a specific example.

Example 4. We adapt the nonlinearities studied in Example 2.6 of Infante\(^4\) to the context of delay equations by consider the family of FBVPs

\[
\dddot{u}(t) + \lambda t e^{u(t)} \left( u(t) \right)^2 \left( 1 + \left( u(t) \right)^2 + u \left( t - \frac{1}{3} \right) \right)^2, \quad t \in (0, 1),
\]

(13)

with the initial condition

\[
u(t) = \psi(t), \quad t \in \left[ -\frac{1}{2}, 0 \right],
\]

(14)

with \( \psi(t) = H(-t)t^2 \) and one of the three BCs (9)–(11), where we fix

\[
B[u] = \frac{1}{1 + \left( u \left( \frac{1}{2} \right) \right)^2} + \int_{-\frac{1}{2}}^{1} t^2(u'(t))^2 dt.
\]

Now choose \( \rho \in (0, +\infty) \). Thus, we may take

\[
\eta(t) = \frac{1}{1 + \rho^2}, \quad \delta(t) = t.
\]
Therefore, for every \( i = 1, 2, 3 \), we have

\[
\sup_{\tau \in [0, 1]} \left\{ \gamma(t) \frac{1}{1 + \rho^2} + \int_{0}^{1} k_i(t, s)s \, ds \right\} \geq \frac{1}{2(1 + \rho^2)} > 0,
\]

which implies that (2) is satisfied for every \( \rho \in (0, +\infty) \).

Thus, we can apply Theorem 3, obtaining uncountably many pairs of solutions and parameters \((u_\rho, \lambda_\rho)\) for the FBVPs (13), (14), and (9); (13), (14), and (10); and (13), (14), and (11).

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

AUTHOR CONTRIBUTIONS

All the authors contributed equally and significantly in writing this article. All the authors read and approved the final manuscript.

FINANCIAL DISCLOSURE

None reported.

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