Measuring $\sigma(e^+e^- \rightarrow \text{hadrons})$ with Tagged Photons
at Electron Positron Colliders

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A Monte Carlo generator has been constructed to simulate the reaction $e^+e^- \rightarrow \gamma + 2\pi$ and $\gamma + 4\pi$ where the photon is assumed to be observed in the detector. Predictions are presented for cms energies of 1GeV, 3GeV and 10GeV, corresponding to the energies of DAΦNE, BEBC and of $B$-meson factories. The event rates are sufficiently high to allow for a precise measurement of $R(Q^2)$ in the region of $Q$ between approximately 1GeV and 2.5GeV. Estimates for the kinematic breaking of isospin relations between different channels as a consequence of the $\pi^\pm - \pi^0$ mass difference are given.

1. Introduction

The precise determination of the cross section for electron positron annihilation into hadrons over a large energy range is one of the important tasks of current particle physics. The results are relevant for the analysis of electroweak precision measurements which are affected by the running of the electromagnetic coupling from the Thompson limit up to $M_Z$. Also the prediction for the anomalous magnetic moment of the muon depends critically on these data. Last not least the measurement of the energy dependence of $R(s)$ is one of the gold plated tests of QCD and allows for a precise determination of the strong coupling constant.

Depending on the energy region different techniques for the measurement of $R(s)$ have been applied up to date. At low energies, say from the two pion threshold up to roughly two GeV, exclusive channels are collected separately. For higher energies inclusive measurements start to become dominant. For energies below $m_\tau$ isospin invariance and CVC have traditionally been used to predict $\tau$ decays from electron positron annihilation [1-4]. Clearly this strategy can be inverted [5], pending irreducible uncertainties from isospin violation and radiative corrections [6].

To cover a large range of energies, results from many different experiments and colliders have to be combined, and energy scans have to be performed to obtain the full energy dependence. An attractive alternative is provided by the upcoming $\Phi-$ and $B-$ meson factories which operate at large luminosities, albeit at fixed energies. Events with radiated tagged photons give access to a measurement of $R$ over the full range of energies, from threshold up to the CMS energy of the collider. For events with tagged photons the invariant mass of the recoiling hadronic system is fixed by the photon energy which provides an important kinematic constraint.

To arrive at reliable predictions including angular and energy cutoffs as employed by realistic experiments, a Monte Carlo generator is indispensable. For hadronic states with invariant masses below two or even three GeV it is desirable to simulate the individual exclusive channels with two, three up to six mesons, i.e. pions, kaons, etas etc. which requires a fairly detailed parametrization of the various form factors.
In principle initial and final state radiation would be required for the complete simulation. Such a program has been constructed for the two pion case\[\Box]. There it is demonstrated that suitably chosen configurations, namely those with hard photons at small angles relative to the beam and well separated from the pions, are dominated by initial state radiation. In fact, this separation is possible \[\Box\] even when operating the \(\phi\) factory DA\(\Phi\)NE on top of the \(\phi\) resonance where direct radiative \(\phi\) decays cannot be ignored.

In a recent paper \[\Box\] this project has been continued with the construction of a generator for the radiative production of the four pion final state, including the \(\omega\)\(\rightarrow\pi^\pm\pi^\mp\pi^\pm\pi^\mp\) channel. This mode contributes a large fraction of the rate with invariant masses between one and two GeV. The energy region between 1.5 GeV and 2.5 GeV is difficult to access directly with current electron positron colliders. At the same time the experimental uncertainties are relatively large. This motivates the special effort devoted to this range.

The Monte Carlo program is constructed in a modular form such that the parametrization of the hadronic matrix element can easily be replaced by a more elaborate version. Different final states with three, four or five pions or kaons can be included. The present parametrization of the hadronic matrix element follows closely the form suggested in \[\Box\], correcting only some minor deficiencies. The four pion amplitude is assumed to be dominated by \(\rho^+\rightarrow\pi^+\pi^-\pi^+\pi^-\) plus a direct coupling \(\rho^+\rightarrow\phi\pi\pi\) and exhibits the proper behavior in the chiral limit.

2. The radiative return

Hard photons observed at small angles relative to the electron or positron beam and at the same time well separated from charged particles in the final state can be used to reduce the effective center of mass energy at electron positron colliders. As shown in Table \[\Box\], the event rates are fairly high, of \(\mathcal{O}(10^6)\) both at DAPHNE and at B-meson factories. Performing a detailed analysis of the angular and energy distributions for the \(\pi^+\pi^-\) final state it has been shown that initial and final state radiation can be reasonably well separated \[\Box\]. For the four pion case we therefore restrict the discussion to initial state radiation only. The amplitude for the production of an arbitrary hadronic final state

\[
\mathcal{M} = i e^3 \bar{\psi}(p_1) \left[ \frac{1}{p - \frac{1}{m}} \gamma^{\nu} k^{\lambda} + \frac{1}{k - \bar{p} - m} \gamma^{\nu} k^{\lambda} \right] \frac{1}{Q^2} J^{em}_\nu
\]

involves the matrix element of the hadronic current

\[
J^{em}_\nu \equiv J^{em}_\nu(q_1, ..., q_n) \equiv \langle h(q_1), ..., h(q_n) | J^{em}_\nu(0) | 0 \rangle
\]

which has to be parameterized by form factors to be discussed below. For the two pion case the amplitude is determined by only one function, the pion form factor \(F_{2\pi}\).

\[
J^{em,2\pi}_\nu = (q_\nu^+ - q_\nu^-) F_{2\pi}(Q^2)
\]

The matrix element for the four pion case will be discussed below. After integrating the hadronic tensor \(H_{\mu\nu}\) over the hadronic phase space one gets

\[
\int J^{em}_\mu(J^{em}_\nu)^* \ d\Phi_n(Q; q_1, \ldots, q_n) = \frac{1}{\sqrt{6\pi}} (Q_\mu Q_\nu - g_{\mu\nu} Q^2) R(Q^2)
\]

where \(R(Q^2)\) is \(\sigma(e^+e^- \rightarrow \text{hadrons})/\sigma_{\text{point}}\). The additional integration of the differential cross section over the photon angles (the azimuthal angle is integrated over the full range and the polar angle within \(\theta_{\text{min}} < \theta < \pi - \theta_{\text{min}}\) ) leads to the differential distribution

\[
\frac{Q^2}{dQ^2} \frac{d\sigma}{3s} = \frac{4e^3}{3s} R(Q^2) \left\{ \frac{(s^2 + Q^4)}{s(s - Q^2)} \log \frac{1 + \cos \theta_{\text{min}}}{1 - \cos \theta_{\text{min}}} \right. \\
- \frac{(s - Q^2)}{s} \cos \theta_{\text{min}} \bigg\},
\]

which can be used to calculate the event rate observed for realistic photon energy and angular cuts (see Tab.1).

3. Isospin relations

The emphasis of \[\Box\] was towards final states consisting of four pions and a photon. Before entering a discussion of a model dependent parametrization of the form factors the
Table 1
Estimated number of radiative events $e^+e^- \rightarrow \text{hadrons} + \gamma$ for different center of mass energies. In the first two rows hadrons stands for $\pi^+\pi^-$ and the minimal photon energy is 0.1 GeV. The third row is obtained for a continuum contribution in the region $2 \text{ GeV} < \sqrt{Q^2} < 3.7 \text{ GeV}$ assuming a constant $R = 2.4$.

| Collider       | $\sqrt{s}$ | Integrated luminosity, fb$^{-1}$ | Event rates |
|----------------|------------|----------------------------------|-------------|
| DAΦNE         | 1.02       | 1                                | $\theta_{\text{min}} = 5^\circ$ |
| $B - \text{factory}$ | 10.6      | 100                              | $\theta_{\text{min}} = 7^\circ$ |
| $B - \text{factory}$ | 10.6      | 100                              | $\theta_{\text{min}} = 10^\circ$ |

constraints from isospin invariance must be recalled. They relate the amplitudes of the $e^+e^- \rightarrow 2\pi^+2\pi^-$ and $e^+e^- \rightarrow \pi^+\pi^- 2\pi^0$ processes and those for $\tau$ decays into $\pi^-3\pi^0$ and $\pi^+2\pi^-\pi^0$. The amplitude for $\tau$ decay

$$\mathcal{M}_\tau = \frac{G_F}{\sqrt{2}} \cos \theta_c \bar{v} (p_\nu) \gamma^\alpha (1 - \gamma_5) u (p_\tau) j^-_\alpha (6)$$

leads to the differential distribution

$$\frac{d\Gamma}{dQ^2} = 2 \Gamma_c \frac{\cos^2 \theta_c}{m_\tau^2} \left( 1 - \frac{Q^2}{m_\tau^2} \right) \left( 1 + 2 \frac{Q^2}{m_\tau^2} \right) R^\tau (Q^2)$$

with

$$\int J_{\mu} J_{\nu}^\ast \ d\Phi_n (Q; q_1, \ldots, q_n) = \frac{1}{3\pi} (Q_\mu Q_\nu - g_{\mu\nu} Q^2) R^\tau (Q^2)$$

Note the relative factor of 2 between the definitions in Eq. (4) and Eq. (3).

Final states with an even number of pions are produced through the isospin one part of the electromagnetic current only, whence

$$\sqrt{2} \ J^{\text{em}}_{\mu} (2\pi) = J^-_\mu (2\pi)$$

and $R(Q^2) = R^\tau (Q^2)$ for two pion final states.

A similar relation for the four pion final state is easily obtained [3,4]:

$$\langle \pi^+\pi^-\pi^0\pi^0 | J^3_\mu | 0 \rangle = J_\mu (p_1, p_2, p^+, p^-)$$

$$\langle \pi^+\pi^+\pi^-\pi^0 | J^3_\mu | 0 \rangle = J_\mu (p_1^+, p_2^+, p^1_1, p^2_1) + J_\mu (p_1^+, p_2^-, p^1_1, p^2_1)$$

$$\langle \pi^-\pi^0\pi^0\pi_0 | J^-_\mu | 0 \rangle = J_\mu (p_2, p_3, p^-, p_1) + J_\mu (p_2, p_3, p^-, p_3)$$

$$\langle \pi^-\pi^+\pi^0\pi^0 | J^-_\mu | 0 \rangle = J_\mu (p^- p_1, p^0_1, p_2, p_3) + J_\mu (p^+ p_1, p^- p_2, p_3)$$

which connects $\tau$ decay and electron positron annihilation ($J^{\text{em}} = \frac{1}{\sqrt{2}} J^3$). The (in $Q^2$) differential rates are then consistent with the familiar relations between $\tau$ decays and $e^+e^-$ annihilation into four pions:

$$R^\tau (-000) = \frac{1}{2} R (++ --)$$

$$R^\tau (-+ +0) = \frac{1}{2} R (++ --) + R (+ - 00)$$

4. The $\pi^\pm - \pi^0$ mass difference

All the relations obtained in the previous section are strictly applicable only in case all pions in the final states have the same mass, which is obviously not true. The relatively large (about 3.6%) $\pi^\pm - \pi^0$ mass difference will affect the $R(Q^2) \leftrightarrow R^\tau (Q^2)$ relations even if Eq. (3) and Eq. (4) still
< E [GeV] < 0.81 & only $\beta^3$ modified & $\beta^3$ and $\Gamma_\rho$ modified \\
\hline
0.28 < $E$ [GeV] < 0.81 & $a_\mu$ & 1.0163 & 1.0088 \\
& $\Delta\alpha(M_Z)$ & 1.0116 & 1.0027 \\
\hline
0.32 < $E$ [GeV] < 1.777 & $a_\mu$ & 1.0130 & 1.0058 \\
& $\Delta\alpha(M_Z)$ & 1.0096 & 1.0016 \\
\hline

Table 2
Kinematic correction factors for the predictions of $a_\mu$ and $\Delta\alpha$ from $\tau$ data.

hold. Let us retain these relations between the amplitudes and incorporate the $\pi^{\pm}\pi^0$ mass difference in the phase space considerations. This should give at least an indication of the size of these "kinematic" isospin violations.

To estimate these effects for the two-pion case assume that the cross section $\sigma(e^+e^- \rightarrow \pi^+\pi^-)$ is well measured and the squared form factor

$$|F(Q^2)|^2 = \frac{3Q^2}{\pi\alpha^2\beta^2_\tau} \sigma(e^+e^- \rightarrow \pi^+\pi^-)(Q^2), \quad (12)$$

$$\left(\beta_\tau = (1 - 4m_{\pi^-}^2/Q^2)^{1/2}\right)$$

then used to predict the $\tau^- \rightarrow \nu_\tau \pi^-\pi^0$ decay rate. The size of the corrections depends critically on the details of the assumptions.

If the form factor and the form of the current

$$J^{\text{cm}}(2\pi) = \sqrt{2} J^{\text{cm}}(2\pi) = (q_{1,\mu} - q_{2,\mu}) F(Q^2) \quad (13)$$

remain unchanged, the integral for the $\tau$ rate receives an $S$-wave contribution which can be eliminated by replacing Eq. (13) by

$$\sqrt{2} J^{\text{cm}}(2\pi) = \left(q_{1,\mu} - q_{2,\mu} - \frac{q_{1,\mu} + q_{2,\mu}}{Q^2} Q \cdot (q_1 - q_2)\right) F(Q^2) \quad (14)$$

Numerically the contribution to the integral of this latter term is tiny – nevertheless we shall adopt the second choice.

The introduction of the $\pi^{\pm}\pi^0$ mass difference raises the prediction for the $\tau$ decay rate by 0.86%. It seems, however, plausible, that the energy dependent width of the $\rho$-meson, which is present in the form factor, has to be modified accordingly, leading to an effective increase of $\Gamma_\rho$ by 0.74%. The two effects nearly compensate in the integral. Hence the relation between $\tau^- \rightarrow \nu_\tau \pi^-\pi^0$ partial decay width and the $e^+e^- \rightarrow \pi^+\pi^-$ cross section would only be corrected by 0.06%. However, a sizable $Q^2$ dependence of the ratio of the two spectral functions ($V(Q^2) \sim |F(Q^2)|^2\beta^3$) is expected, with a reduction approximately 0.74% close to the peak of the $\rho$ resonance and enhancements at the tails.

At present, however, $\tau$ data provide an important input for the prediction of the QED coupling at the scale of $M_Z$ and the hadronic contribution to $g-2$. Let us, for the moment, assume that the aforementioned kinematic effects are indeed present. If the $e^+e^-$ cross section is deduced from $\tau$ data through Eq. (13)

$$\frac{1}{\Gamma_e} \frac{d\Gamma(\tau^- \rightarrow \nu_\tau \pi^-\pi^0)}{dQ^2} = \frac{3\cos^2 \theta_e}{2\pi\alpha^2m_\tau^2}(15)$$

and the contributions to $g-2$ and $\alpha(M_Z)$ are evaluated without kinematic corrections of phase space and form factor the former are overestimated by 0.58% and 0.16% respectively (Tab.2).

The situation is more complicated for the four pion case. To estimate the size of the effect the quantities $R(\ldots)$ etc. were evaluated once assuming that all masses are equal to $m_-$ and once taking the real masses. The corrections to the integrated quantities modify the integrated
It has, however, the advantage of being universal and no change of the variance reduction method is required with each modification of the hadronic current. The process to be simulated by the program in its final stage is $e^+ e^- \rightarrow \gamma + \text{hadrons}$ with an exclusive description of final states, even if till now only $\pi^+ \pi^-$, $2\pi^0 \pi^+ \pi^-$ and $2\pi^+ 2\pi^-$ hadronic final states are implemented. The LL radiative QED corrections were taken into account using structure function method as developed in \cite{10} and limited to the initial emission only. In fact the program can run in one of two modes (chosen by a user) one with collinear radiation and one without it. Hard large angle photon emission is limited to initial state radiation, which is justified by \cite{10} where it was demonstrated for the $\pi^+ \pi^-$ hadronic state that the contribution from the final state emission as well as the initial-final state interference can be reduced to a negligible level by applying suitable cuts.

Let us now discuss the cuts, which reduce the contribution of final state radiation to a negligible level. We recall that in \cite{10} it was shown that the following set of angular cuts

cuts1 : 

\[ (7^\circ < \theta_\gamma < 20^\circ \quad \text{or} \quad 160^\circ < \theta_\gamma < 173^\circ) \]
\quad \text{and} \quad 30^\circ < \theta_\pi < 150^\circ ,

( $\theta_\gamma$ ( $\theta_\pi$ ) is the photon (pion) polar angle) fulfills this requirement for the $e^+ e^- \rightarrow \pi^+ \pi^- \gamma$ cross section at DAPHNE. It reduces, however, the observed cross section significantly. This starts to become dramatic, when one runs at energies well above 1 GeV. The following set of cuts

cuts2 : 

\[ (7^\circ < \theta_\gamma < 20^\circ \quad \text{and} \quad 30^\circ < \theta_\pi < 173^\circ) \]
\quad \text{or} \quad (160^\circ < \theta_\gamma < 173^\circ \quad \text{and} \quad 7^\circ < \theta_\pi < 150^\circ) \]

also reduces the contribution from final state radiation to a negligible level due to the fact that the pions and photon are well separated as in the previous case.

At the same time the cross section reduction is much smaller, especially for higher beam energies.
Figure 1. Diagrams contributing to the hadronic current.

| Mode                              | Ref | Present Model | Experiment       |
|-----------------------------------|-----|---------------|------------------|
| \( \tau^- \rightarrow \nu_{\tau} 2\pi^- \pi^+ \pi^0 \) | 3.11% | 4.33% | 4.20(8)%        |
| \( \tau^- \rightarrow \nu_{\tau} \pi^- \omega (\pi^- \pi^+ \pi^0) \) | 1.20% | 1.48% | 1.73(11)%       |
| \( \tau^- \rightarrow \nu_{\tau} \pi^- 3\pi^0 \) | 0.98% | 1.14% | 1.08(10)%       |

Table 3
Branching ratios of \( \tau \) decay modes. Results of Ref. and the present current are compared to experimental data.
and higher energies of the observed photons. The effect of the two sets of cuts on the cross sections with four charged pions in the final state is presented in Fig. 2 for $2E_{\text{beam}} = 3$ GeV. “No cuts” corresponds to $7^\circ \leq \theta_\gamma \leq 173^\circ$ and not cuts on the pions. For higher beam energies the effect of the cross section reduction is much stronger and at $2E_{\text{beam}} = 10$ GeV the cross sections for the cuts specified in Eq. (19) is reduced almost to zero. For the cuts specified in Eq. (20) the reduction remains tolerable (Table 3).

One may even restrict photon and pions detection angles to the central region, e.g. $25^\circ < \theta_\gamma < 155^\circ$ and $30^\circ < \theta_\pi < 150^\circ$ respectively if a minimal angle of $20^\circ$ between photon and charged and neutral pions is required in order to suppress final state radiation and to clearly separate neutral pions and the photon. With this cut one obtains $(\sqrt{s} = 10$ GeV and $\mathcal{L} = 100$ fb$^{-1}$ ) a rate of $1.17 \cdot 10^5$ events with $2\pi^+2\pi^-\gamma$ and $1.53 \cdot 10^5$ events with $2\pi^0\pi^+\pi^-\gamma$.

Additional collinear emission always present in the real experiment reduces slightly the cross sections. Its actual size depends on the cuts on the invariant mass of the $4\pi + \gamma$ system. The effect is similar for different energies and for both charge modes.

7. Summary

As an alternative to a direct measurement of the cross section for $e^+e^- \to$ hadrons at the relevant energy one may use initial state radiation to reduce the effective energy of electron positron colliders, exploiting the large luminosity of “factories” and accessing thus a continuum of hadronic final states.

With this motivation a Monte Carlo generator has been constructed to simulate the reaction $e^+e^- \to \gamma + 2\pi$ and $\gamma + 4\pi$, where the photon is assumed to be observed in the detector. Once more accurate data become available, the modular structure of the program will allow for modification or replacement of the hadronic current in a simple way. Additional collinear photon radiation has been incorporated with the technique of structure functions.

Predictions are presented for cms energies of 1GeV, 3GeV and 10GeV, corresponding to the energies of DAΦNE, BEBC and of $B$-meson factories. Even after applying realistic cuts the event rates are sufficiently high to allow for a precise measurement of $R(Q^2)$ in the region of $Q$ between approximately 1GeV and 2.5GeV.

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| √s (GeV) | Integrated luminosity, fb⁻¹ | Event rates |
|---------|---------------------------|-------------|
| 1       | 1                         | 1.04 · 10⁴ | 1.13 · 10⁴ |
| 3       | 1                         | 4.66 · 10⁴ | 5.72 · 10⁴ |
| 10      | 100                       | 1.86 · 10⁵ | 2.33 · 10⁵ |

Table 4
Estimated number of radiative events \(e^+e^- \rightarrow 4\pi + \gamma\) for different center of mass energies. The minimal photon energy is: 0.05 GeV (first row), 0.1 GeV (second row), 0.2 GeV (third row). The angular cuts of Eq. (20) were applied.

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