Deformation Spaces for Affine Crystallographic Groups

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Abstract

We develop the foundations of the deformation theory of compact complete affine space forms and affine crystallographic groups. Using methods from the theory of linear algebraic groups we show that these deformation spaces inherit an algebraic structure from the space of crystallographic homomorphisms. We also study the properties of the action of the homotopy mapping class groups on deformation spaces. In our context these groups are arithmetic groups, and we construct examples of flat affine manifolds where every finite group of mapping classes admits a fixed point on the deformation space. We also show that the existence of fixed points on the deformation space is equivalent to the realisation of finite groups of homotopy equivalences by finite groups of affine diffeomorphisms. Extending ideas of Auslander we relate the deformation spaces of affine space forms with solvable fundamental group to deformation spaces of manifolds with nilpotent fundamental group. We give applications concerning the classification problem for affine space forms.

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Introduction

This work is devoted to some aspects of the study of the fundamental groups of affine space forms. An affine space form is a compact, complete, affinely flat manifold. In modern terminology, an affinely flat manifold is a locally homogeneous manifold which is modelled on the action of the affine group Aff(V) on some real vector space V. One defines more restricted kinds of affinely flat manifolds by considering any transitive subgroup A of Aff(V). For example, if A = E(V) is the group of Euclidean isometries of V, then one obtains the concept of a Riemannian flat manifold, and similarly one obtains geometrically interesting cases like Lorentz-flat manifolds or flat symplectic manifolds.

It is well known that a compact complete affine manifold arises as a quotient Γ\V, where Γ ≤ Aff(V) is a properly discontinuously and cocompactly acting subgroup of Aff(V). These groups are called affine crystallographic groups. An affine crystallographic subgroup Γ ≤ E(V) is called an Euclidean crystallographic group. The study of affine crystallographic groups has a long history which goes back to Bieberbach’s famous three theorems and Hilbert’s 18th problem on Euclidean crystallographic groups.

Much of the recent work on affine crystallographic groups is an effort to generalize the satisfactory theory of Euclidean crystallographic groups to the general case. This approach has encountered serious obstacles. A long standing conjecture (“Auslander’s conjecture”) states that any affine crystallographic group Γ is a virtually polycyclic group. Under the assumption that Auslander’s conjecture holds certain generalizations of Bieberbach’s theory are possible. Still, for example, the question which virtually polycyclic groups are isomorphic to an affine crystallographic group remains mysterious.

In this article, we study the deformation theory of complete affinely flat structures on a fixed compact manifold M. Let Γ be a group. An injective homomorphism ρ : Γ → Aff(V), such that ρ gives rise to a proper action of Γ on V with compact quotient Γ\V is called a crystallographic homomorphism. By the deformation theory of locally homogeneous spaces, developed by Thurston, the description of the topology of the deformation space of complete affinely flat structures on M roughly amounts to an analysis of the space of crystallographic homomorphisms of π₁(M) up to conjugacy by Aff(V). This approach, of course, has a long and fruitful history in geometry. One particular important example is the Teichmüller theory of surfaces.

The phenomena one could expect for the topology (and possibly geometry) of the deformation spaces of complete affine manifolds are, however, quite different from the situation of constant curvature geometric surfaces. The main theme here is that the deformation spaces of complete affine manifolds with virtually polycyclic fundamental group are of an algebraic and arithmetic nature.

1Benoist [17] gave a striking example of a finitely generated nilpotent group which is not isomorphic to an affine crystallographic group.
We provide a conceptual framework for the study of the set of affine crystallographic actions of a given virtually polycyclic group $\Gamma$. This builds on the strong interaction of affine crystallographic with the theory of representations of Lie- and algebraic groups. A basic result we obtain is that for any virtually polycyclic group $\Gamma$ the space $\text{Hom}_c(\Gamma, \text{Aff}(V))$ of crystallographic embeddings into $\text{Aff}(V)$ has a natural structure of a real algebraic variety defined over the rational numbers. We prove that $\text{Hom}_c(\Gamma, \text{Aff}(V))$ is Zariski-open in a certain closed subspace of the space of all homomorphisms of $\Gamma$ into the affine group. This result may be seen as an analogue of a classic theorem of Weil on discrete subgroups of Lie groups.

The deformation space $\mathcal{D}_c(\Gamma, \text{Aff}(V))$ arises as a quotient of $\text{Hom}_c(\Gamma, \text{Aff}(V))$ by the action of $\text{Aff}(V)$. The algebraic structure of $\text{Hom}_c(\Gamma, \text{Aff}(V))$ carries over to the deformation space which in some cases turns out to be homeomorphic to a semi-algebraic set, and in particular it is then a Hausdorff-space. This turns out to be true, for example, for any manifold which is finitely covered by a torus and also for certain solvmanifolds. In general, the structure of the quotient spaces $\mathcal{D}_c(\Gamma, \text{Aff}(V))$ seems particular hard to understand because the group actions which define the moduli problem are not reductive, and the usual techniques of geometric invariant theory do not apply.

Of particular interest in this context is furthermore the action of the mapping class group $\text{Out}(\pi_1(M))$ on the deformation space of complete flat affine structures, and its quotient, the moduli space of affine structures on $M$. Compared with the Teichmüller theory of conformal structures for surfaces, the general situation in our subject resembles more the case of a genus 1 surface than the situation for higher genus. In fact, if $\Gamma$ is a polycyclic group, then $\text{Out}(\Gamma)$ is an arithmetic group, similar to the model situation $\text{Out}(\pi_1(M)) = \text{GL}_2(\mathbb{Z})$. Moreover, we show that the action of $\text{Out}(\pi_1(M))$ extends to an action of an algebraic group on the deformation space.

In contrast to the situation for surfaces, however, the action of $\text{Out}(\pi_1(M))$ on the deformation space is not proper, and the moduli space may be highly singular.

We also consider the following type of question: Let $\Gamma$ be an affine crystallographic group and let $\Delta$ be a finite extension. Is $\Delta$ also isomorphic to an affine crystallographic group? Questions like this are known as realization problems. From classic work of Dehn, Nielsen and Poincare in surface theory it is known that there is a close interplay between such a question and Teichmüller theory.

We show in detail that, for virtually nilpotent groups, the solution of the above realization problem depends only on the action of the group of mapping classes for $\Gamma$ on the deformation space. This leads us in turn to a description of the deformation space $\mathcal{D}_c(\Delta, \text{Aff}(V))$ of $\Delta$ as a fixed point set in $\mathcal{D}_c(\Gamma, \text{Aff}(V))$ for the finite group of mapping classes which belongs to the extension $\Delta$. We mention here that our contribution to the realization problem may be understood as a suitable generalization of the classic Burckhardt-Zassenhaus theorem on Euclidean crystallographic groups. This is because there exists a fixed point for every finite subgroup of $\text{GL}_n(\mathbb{Z})$ on the deformation space of Euclidean tori. In this realm, we exhibit large classes of infranilmanifolds where the finite group of mapping classes have fixed points in $\mathcal{D}_c(\Gamma, \text{Aff}(V))$. Therefore the answer to the affine crystallographic realization problem for these manifolds is positive.

Another type of realization problem enters the stage, coming from Auslander’s nilshadow construction for solvmanifolds. Let $\Gamma$ be the fundamental group of a solvmanifold, and let $\Theta$ be its nilshadow. If $\Theta$ is an affine crystallographic group, is it true that $\Gamma$ is isomorphic to an affine crystallographic group? This question admits a treatment which is similar to the solution of the problem for finite extensions. The solution for this realization problem paves the way for a description of the de-

\footnote{This result is however highly non-trivial, see [13].}
formation spaces for affine solvmanifolds, and enables us to construct new examples of such manifolds.

**Broader context of the article** Let us mention briefly related work in the context of our article. The deformation theory of locally homogeneous spaces was formulated and developed by Thurston, see [75, 30, 26] for exposition. In similar spirit, but with different methods, is the local and global deformation theory of complex structures on compact manifolds, as pioneered by Kodaira and Spencer [47]. For compact surfaces both approaches lead to the same mathematical object, namely Teichmüller space, which is the analogue of our deformation space.

Traditionally locally homogeneous spaces are modeled on homogeneous spaces $G/H$ with $H$ compact. In this setup, discrete subgroups $\Gamma \leq G$ give rise to proper and, if $\Gamma$ is a uniform lattice, cocompact actions on $G/H$ in abundance. The case $H$ non-compact has been more or less ignored for quite some time, although its obvious relevance for relativity and Pseudo-Riemannian geometry. Motivated by phenomena discussed in the seminal paper [48], there has been lots of recent activity to understand locally homogeneous spaces modeled on $G/H$, where $G$ is reductive and $H$ is non-compact. (See, [40, 18, 63, 68] for important contributions.)

Kobayashi [44] more generally formulated the program to classify the deformation and moduli spaces of proper actions of a discrete group $\Gamma$ on a homogeneous space $G/H$ with non-compact $H$. The subject of this article with $G = \text{Aff}(V)$ and $H = \text{GL}(V)$, and $\Gamma$ crystallographic is an important special case.

Other recent activity in this program concerns the situation, where $G$ is nilpotent and $\Gamma$ is acting properly but not necessarily cocompact on $G/H$. (See, for example [6, 7, 46, 80].)

In complex geometry, there is related recent work of Catanese et al. [21, 22] to understand and classify the deformation of complex structures on compact nilmanifolds. The latter subject bears certain similarities and interactions with the deformation of affine structures on these manifolds.

**Structure of the article** There are three main chapters. In chapter 1 we introduce and develop the basic techniques in the study of crystallographic actions of solvable groups on affine space. The main result is the description of the space of crystallographic homomorphism as an algebraic variety defined over the rational numbers.

In chapter 2 we analyse the basic properties of the associated deformation spaces. We provide examples of such spaces which have interesting properties with respect to the action of the group of mapping classes. In particular, we study convexity and fixed point properties of deformation spaces.

Chapter 3 is technically most demanding and concerns the realisation questions for affine crystallographic groups, and the extension of the action of the group of mapping classes to an algebraic group action on the deformation space. These constructions are carried out in the context of nilmanifolds.

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3Margulis constructed proper non-uniform affine actions of finitely generated free groups on a three-dimensional vector space, see [54, 23].
Notational conventions

Groups
Z(G), the center of G
C_{g,h}(H), the centralizer of H in G
Fitt(Γ), Fitting subgroup, the maximal nilpotent normal subgroup of Γ
H ≤g G, H is a subgroup of finite index in G
[A,B] ≤ G, the commutator subgroup of A, B ⊆ G
G is said to be virtually P, if G has a finite index subgroup which satisfies P
G is called a wfn-group if G is without a non-trivial finite normal subgroup
f.t.n.-group, a finitely generated torsionfree nilpotent group

Algebraic sets
We will frequently consider affine algebraic varieties defined over the rationals or reals. Our special interest lies on the set of real points of these varieties. We call these sets real algebraic varieties.

M, the real Zariski closure of M ⊂ X, X a real algebraic variety

Linear algebraic groups
A Q-defined linear algebraic group G is a subgroup G ≤ GL_n(C) defined by polynomials with rational coefficients. For a subring R of C we put G_R = G ∩ GL_n(R).
g = g_s g_u, the Jordan decomposition for g ∈ G.
u(Γ), the maximal normal unipotent subgroup of Γ ≤ G
M_s = {m_s | m ∈ M}
M_u = {m_u | m ∈ M}, where M ⊆ G is a subset
M, the Zariski-closure of M in G ρ_u, the map γ ↦→ ρ(γ)u, where ρ : Γ → G is a homomorphism

Lie groups, real algebraic groups
A Lie group G is called real algebraic, if G is (a connected component of) the set of real points G_R of a linear algebraic group G
N(G), nilpotent radical of G, the maximal nilpotent normal connected subgroup
G^0, identity component in the (real) Zariski-topology, if G is real algebraic
G_0, identity component in the Hausdorff-topology
M, the real Zariski-closure of a linear group G ≤ GL_n,R
u(G), the unipotent radical, G ⊆ GL_n a linear group

Affine crystallographic groups
V, a finite dimensional real vector space
Aff(V) = V · GL(V), the group of affine transformations of V
A ≤ Aff(V), a Zariski closed subgroup of Aff(V), transitive on V
GL_A = A ∩ GL(V)
A_x = {a ∈ A | ax = x}, for x ∈ V
ρ : Γ → A a homomorphism, ρ^0(γ) = aρ(γ)a^{-1}

Definition Let Γ ≤ Aff(V) be a subgroup, Γ is called properly discontinuous if, for all compact subsets K ⊆ V, the set {γ ∈ Γ | γK ∩ K ≠ ∅} is finite
Γ ≤ A is called an affine crystallographic group of type A, if Γ is properly discontinuous, and if the quotient space Γ \ V is compact.
ACG, affine crystallographic group
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Chapter 1

Crystallographic Homomorphisms

Let $V$ denote a finite dimensional real vector space, and $A \leq \text{Aff}(V)$ a Zariski closed subgroup, which acts transitively on $V$. A subgroup $\Gamma \leq \text{Aff}(V)$ is called affine crystallographic if $\Gamma$ acts properly discontinuously and with a compact quotient on $V$. If $\Gamma \leq A$ then $\Gamma$ is called an affine crystallographic group of type $A$. We define the space of crystallographic homomorphisms as

$$\text{Hom}_c(\Gamma, A) = \{ \rho : \Gamma \to A \mid \rho \text{ is crystallographic} \}.$$ 

A homomorphism $\rho : \Gamma \to \text{Aff}(V)$ is called a crystallographic homomorphism if $\rho$ is injective and the image $\rho(\Gamma) \leq \text{Aff}(V)$ is a crystallographic subgroup. In this chapter we study the space of crystallographic homomorphisms

$$\text{Hom}_c(\Gamma, A) \subset \text{Hom}(\Gamma, A)$$

as a subset of the real algebraic variety $\text{Hom}(\Gamma, A)$ of all homomorphisms of $\Gamma$ into $A$. We establish that the space $\text{Hom}_c(\Gamma, A)$ may be described by algebraic equalities and inequalities, and carries itself a natural structure as a real algebraic variety.

1.1 Polycyclic groups, crystallographic groups

In this subsection we develop the foundations of affine crystallographic virtually polycyclic groups. We will need these results to build our later arguments on. Our basic references are [27] and [35] which are, however, written from different points of view. We put the notion of algebraic hull for a torsionfree polycyclic group (as developed by Mostow [57], see also, for example, [65, Chapter IV]) at the center of our considerations. We hope that our presentation clarifies some aspects of the theory of polycyclic affine crystallographic groups.

Solvable crystallographic groups First some further definitions and introductory remarks. A long standing conjecture of Louis Auslander states that affine crystallographic groups are virtually solvable groups. See Milnor’s paper [23] for an introduction to this problem, and related results. The conjecture is verified in certain special cases, in particular in low dimensions. No counter example is known.

\footnote{A generalisation in the context of affine actions on solvable Lie groups is obtained in [13]. The importance of the algebraic hull functor for arbitrary (that is not necessarily affine crystallographic virtually) polycyclic groups is now particularly evident from [15].}
See [1] for details. By a result of Mostow [56], a discrete solvable subgroup of a Lie group is a polycyclic group. Hence, a solvable affine crystallographic group is, in particular, a polycyclic group. Virtually polycyclic affine crystallographic groups are therefore the objects of our study.

We add that there are a few obvious mild restrictions for an abstract finitely generated group \( \Delta \) to be realised as a crystallographic subgroup of \( \text{Aff}(V) \): Let \( \Delta \leq \text{Aff}(V) \) be crystallographic. Then the quotient space \( \Delta \backslash V \) is a manifold if and only if \( \Delta \) is torsionfree. Moreover, by Selberg's lemma every affine crystallographic group contains a torsionfree normal subgroup of finite index. Since \( V \) is contractible, this implies that, the \textit{virtual cohomological dimension} of \( \Delta \) (see [20] for definition) satisfies \( \text{vcd} \Delta = \dim V \).

**Lemma 1.1** Let \( \Delta \leq \text{Aff}(V) \) be an ACG. Then every finite normal subgroup of \( \Delta \) is trivial.

**Proof.** Let \( N \) be a finite normal subgroup of \( \Delta \). Since \( N \) is finite, there exist a fixed point \( x \in V \) for \( N \). The set of fixed points for \( N \) is an affine subspace \( H \) of \( V \), and it contains the orbit \( \Delta x \). Since \( \Delta \) acts as an ACG on \( H \), we have \( \dim H = \text{vcd} \Delta = \dim V \). Therefore \( H = V \), and \( N = \{1\} \).

A group which satisfies the conclusion of the lemma will be called a \textit{wfn-group}.

**Hulls, splittings and shadows**

We start by providing some foundational material on algebraic and analytic hulls for polycyclic groups. Let us first recall the definition of polycyclic groups. Namely, a group \( \Gamma \) is called \textit{polycyclic} if \( \Gamma \) admits a finite normal series

\[
\Gamma = \Gamma_0 \geq \Gamma_1 \geq \cdots \geq \Gamma_k = \{1\}
\]

such that each quotient \( \Gamma_i/\Gamma_{i+1} \) is cyclic. We let \( \text{rank} \Gamma \) denote its rank which is the number of infinite cyclic quotients \( \Gamma_i/\Gamma_{i+1} \). (By some authors the rank of \( \Gamma \) is called the \textit{Hirsch-length}.) Then it is true that \( \text{rank} \Gamma = \text{vcd} \Gamma \). The rank of a virtually polycyclic group is defined as the rank of any finite index polycyclic subgroup.

**The algebraic hull for a virtually polycyclic group** Let \( H \) be a linear algebraic group. We put \( u(H) \) for its maximal unipotent normal subgroup. The group \( u(H) \) is Zariski-closed in \( H \) and is called the \textit{unipotent radical} of \( H \). It is customary to say that \( H \) has a \textit{strong unipotent radical} if \( C_H(u(H)) = Z(u(H)) \).

**Definition 1.2** A \( \mathbb{Q} \)-defined linear algebraic group \( H \) with a strong unipotent radical is called an \textit{algebraic hull for} \( \Gamma \) if \( \Gamma \) is a Zariski-dense subgroup of \( H \), \( \Gamma \leq H_{\mathbb{Q}} \), and \( \dim u(H) = \text{rank} \Gamma \).

If \( \Gamma \) is a virtually polycyclic wfn-group then algebraic hulls for \( \Gamma \) exist and are unique up to \( \mathbb{Q} \)-isomorphism. Also \( \Gamma \cap H_{\mathbb{Z}} \) has finite index in \( \Gamma \). This fact was proved by Mostow in the torsionfree polycyclic case, see also [65] Proposition 4.40. In section 3.2 of this article the case of finite extensions of f.t.n.-groups will be developed in detail. The more general construction for virtually polycyclic wfn-groups is given in [13]. Further applications of the theory of algebraic hulls are developed in [15].

Let us put \( H_{\Gamma} \) for the algebraic hull of \( \Gamma \). We will also need algebraic hulls over the real numbers.

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\(^2\)There is a rich and well developed theory of polycyclic groups, as is documented in the book [74]. For recent developments see also [65].
Definition 1.3 The group $H_\Gamma = H_{\Gamma, \mathbb{R}}$ is called a real algebraic hull for $\Gamma$.

Note that $\Gamma$ is a discrete subgroup in its real hull $H_\Gamma$ since $\Gamma \cap H_\mathbb{R}$ has finite index in $\Gamma$. Let us henceforth write f.t.n.-group for a finitely generated torsionfree nilpotent group. We understand the algebraic hull for a torsionfree polycyclic group $\Gamma$ as a substitute for the Malcev-completion\(^3\) of a f.t.n.-group. In fact, if $\Gamma$ is a f.t.n.-group then $H_\Gamma$ is the Malcev completion of $\Gamma$. Also the algebraic hull $H_\Gamma$ of a virtually polycyclic wfn-group $\Gamma$ satisfies the following rigidity property:

**Proposition 1.4** Let $H_\Gamma$ be an algebraic hull for $\Gamma$, $G$ a $\mathbb{Q}$-defined linear algebraic group with a strong unipotent radical. Let $\rho : \Gamma \to G$ be a homomorphism so that $\rho(\Gamma)$ is Zariski-dense in $G$. Then $\rho$ extends uniquely to a homomorphism $\rho_{H_\Gamma} : H_\Gamma \to G$. If $\rho(\Gamma) \leq G_k$, where $k$ is a subfield of $\mathbb{C}$, then $\rho_{H_\Gamma}$ is defined over $k$.

**Proof.** We will use the diagonal argument. Therefore we consider the subgroup

$$D = \{ (\gamma, \rho(\gamma)) \mid \gamma \in \Gamma \} \leq H \times G.$$ 

Let $\pi_1, \pi_2$ denote the projection morphisms on the factors of the product $H \times G$. Let $D$ be the Zariski-closure of $D$, and $U = u(D)$ the unipotent radical of $D$. Now $D$ is a solvable algebraic group, hence $U = D_0$. Let $\alpha = \pi_1 | D$. Since $\alpha$ is onto it follows that $\alpha$ maps $U$ onto $u(H)$. By [65, Lemma 4.36] we have $\dim U \leq \text{rank } \Gamma = \dim u(H)$, and hence $\dim U = \dim u(H)$. In particular it follows that the restriction $\alpha : U \to u(H)$ is an isomorphism. Thus the kernel of $\alpha$ consists only of semi-simple elements. Therefore every $x \in \ker \alpha$ centralizes $U$, and since $\pi_2(U) = u(G)$, $\pi_2(x)$ centralizes $u(G)$. Since $G$ has a strong unipotent radical, $x$ is in the kernel of $\pi_2$. Hence $x = 1$, and it follows that $\alpha$ is an isomorphism. We put $\rho_{H_\Gamma} = \pi_2 \circ \alpha^{-1}$ to get the required unique extension. If $\rho(\Gamma) \leq G_k$ then $D$ is $k$-defined, and hence also the morphism $\rho_{H_\Gamma}$. \hfill \Box

**Remark** The proposition shows that the condition that the Zariski-closure $\rho(\Gamma)$ has a strong unipotent radical forces the homomorphism $\rho$ to be well behaved. For example, $\rho$ must be unipotent on the Fitting subgroup of $\Gamma$. See Proposition 1.6 below.

**Corollary 1.5** The algebraic hull $H_\Gamma$ of $\Gamma$ is unique up to $\mathbb{Q}$-isomorphism. In particular, every automorphism $\phi$ of $\Gamma$ extends uniquely to a $\mathbb{Q}$-defined automorphism $\Phi$ of $H_\Gamma$.

Let us put Fitt($\Gamma$) for the Fitting subgroup of $\Gamma$, that is, the unique maximal nilpotent normal subgroup of $\Gamma$. We note another property of the algebraic hull:

**Proposition 1.6** Let $\Gamma \leq H_\Gamma$. Then $\text{Fitt}(\Gamma) \leq u(H_\Gamma)$. In particular, $u(\Gamma) = \text{Fitt}(\Gamma)$.

**Proof.** Let $F$ be the maximal nilpotent normal subgroup of $H_\Gamma$. Clearly, $F = F'$ is a Zariski-closed subgroup. Therefore $u(F') = u(H_\Gamma)$. Now since $F$ is nilpotent, $F_\gamma$ is a subgroup, $F_\gamma = u(F)$ and $F = F_\gamma \cdot u(F)$ is a direct product of groups. Since $H_\Gamma$ has a strong unipotent radical, $F_\gamma$ must be trivial, and it follows that $F = u(F')$. The Zariski-closure of Fitt($\Gamma$) is a nilpotent normal subgroup of $H_\Gamma$, and therefore Fitt($\Gamma$) is contained in $F$, hence Fitt($\Gamma$) $\leq u(H_\Gamma)$.

Recall that a Cartan subgroup of $H_\Gamma$ is the centralizer of a maximal torus in $H_\Gamma^0$. Let $C$ be a Cartan subgroup and $N = \text{Fitt}(\Gamma)$. We just proved that $N \leq u(H_\Gamma)$.

\(^3\)see [53, 63]


Lemma 1.7 Any Cartan subgroup $C$ is nilpotent, $C = C_s \cdot u(C)$. Moreover, $$u(H_G) = u(C)N.$$ 

Proof. For the first statement, see [19, 11.7]. Let $U = u(H_G)$. Since $H_G$ is solvable, $(H_G)_u = U$. Therefore $u(C) = C \cap U$. Since $[\Gamma, \Gamma] \leq \text{Fitt}(\Gamma)$, $[H_G, H_G] \leq N$. It is known ([19, 10.6]) that all maximal tori are conjugate by elements of $[H_G, H_G]$. Hence they are conjugate by elements of $N$. Therefore, if $T$ is a maximal torus, $u(H_G) = NC_G(T) = Nu(C)$. 

The algebraic hull and the semisimple splitting of a solvable Lie group

The notion of algebraic hull applies also to connected, simply connected solvable Lie groups $G$. We summarize, cf. [65, Proposition 4.40]:

Proposition 1.8 There exists a linear algebraic group $H_G$ with a strong unipotent radical, so that $G \leq H_G$ is a Zariski-dense subgroup, and $\dim u(H) = \dim G$.

As for discrete groups, the algebraic hull $H_G$ is unique up to isomorphism defined over $\mathbb{R}$, and it has analogous rigidity properties. Let $N$ be the nilpotent radical of $G$, i.e., the maximal, connected nilpotent normal subgroup of $G$. Then it follows (with the same proof as for Proposition 1.6) that $N \leq u(H_G)$. Note that $N$ is the connected component of $u(G) = G \cap u(H_G)$.

Definition 1.9 The group $H_G = H_{G,\mathbb{R}}$ is called the real algebraic hull of $G$, and $U_G = u(H_G)$ is called the unipotent shadow of $G$. If $N = u(G)$ then $G$ is called $u$-connected.

We remark further that $G$ is a normal subgroup of $H_G$. In fact, $N \leq U_G$ is Zariski-closed in $H_G$, and $[G, G] \leq N$ implies therefore that $[H_G, H_G] \leq N$. The existence of the algebraic hull for $G$ allows for the following:

Definition 1.10 A subset $X \leq G$ is called Zariski-dense in $G$, if $X$ is Zariski-dense in the algebraic hull of $G$.

In particular, a subgroup $\Gamma \leq G$ is Zariski-dense in $G$ if it is so in $H_G$. We consider next the (real) semisimple splitting construction. (Compare [24, Chapter 2, §3.6].) This idea goes back to Malcev’s and Auslander’s work on solvmanifolds.

Definition 1.11 Let $M = S \cdot U$ be a splittable, simply connected solvable Lie group, so that $U$ is the nilradical of $M$, and $S$ acts by semisimple automorphisms. $M$ is called a semisimple splitting for $G$, if $G$ is a normal subgroup of $M$, so that $M = S \cdot G$ (semidirect product), and $M = GN$ (product of subgroups).

Note that it is said that a group of automorphisms of $U$ is semisimple if the corresponding induced automorphism group of the Lie algebra acts by semisimple linear maps. Auslander (see [3]) proved that semisimple splittings for $G$ exist and are unique. We briefly show the existence of $M$ by realizing it inside the algebraic hull $H_G$.

Proposition 1.12 There exists a compatible embedding of the semisimple splitting $M$ into the real algebraic hull $H_G$, so that $U$ coincides with the unipotent shadow $U_G$.

Proof. Since $H_G$ is (real) algebraic and Zariski-connected, there exists a semidirect product decomposition $H_G = T \cdot U_G$, where $T$ is a maximal torus. We denote $\psi : H \to U_G$ the projection map on $U_G$ which is defined by the splitting. Let $C$ be a Cartan subgroup of $G$, so that $G = CN$, where $N \leq U_G$ is the nilradical of
$G$. We put $S = C_g = \{g_s \mid g \in C\}$, so that $C \leq SC_g$. $S$ is a Zariski-connected abelian subgroup of $H_G$, centralized by $C$. By the conjugacy of maximal tori, we may assume that $S \leq T$. Since $H_G = \overline{T} \leq TC_uN$, we conclude that $U_G = C_uN$. Therefore $U_G \leq SG$, and also $G \leq SU_G$. It follows that the crossed homomorphism $\psi : G \to U_G$ is surjective, in fact since $\dim U = \dim G$ it is a covering. Since $U$ is simply connected $\psi$ is a diffeomorphism. Therefore $S \cap G = \{1\}$ and we put $M = S \cdot U_G = S \cdot G = SC_uF = GU_G$. Note that $S$ acts faithfully on the strong unipotent radical $U_G$. Therefore $M \leq H_G$ is a semisimple splitting for $G$. \hfill $\Box$

**Syndetic hulls and standard groups** The notion of *syndetic hull* of a solvable subgroup of a linear group is due to Fried and Goldman, cf. [27, §1.6], and is an important tool in the study of discrete solvable groups, compare also [78, §5]. Fried and Goldman introduced this notion in the context of affine crystallographic groups. We use the slightly modified definition for the syndetic hull which is given in [35].

We then carry out some known constructions for ACGs in the algebraic hull of a general virtually polycyclic WFN-group.

**Definition 1.13** Let $\Gamma$ be a polycyclic subgroup of $GL(V)$, and $G$ a closed, connected subgroup of $GL(V)$. $G$ is called a *syndetic hull* of $\Gamma$ if $\Gamma$ is a Zariski-dense uniform lattice in $G$, and $\dim G = \rank \Gamma$.

We remark that a syndetic hull for $\Gamma$ is necessarily a connected, simply connected solvable Lie group. (One has to show that $G$ has no compact subgroups.) The construction of a syndetic hull for $\Gamma$ may take place in any (real) linear algebraic group $H$ which contains $\Gamma$, provided $H$ satisfies certain conditions.

**Definition 1.14** A polycyclic group $\Gamma \leq H$ is called standard in $H$ if $\Gamma$ is Zariski-dense, $\Gamma \leq H_0$, $\Gamma' \leq u(H)$, and $\Gamma/u(\Gamma)$ is torsionfree. $\Gamma$ is called a standard polycyclic group if $\Gamma$ is standard in its real algebraic hull $H_\Gamma$.

Note that for $\Gamma \leq H_\Gamma$ to be standard it is enough to assume that $\Gamma \leq (H_\Gamma)_0$, and that $\Gamma/u(\Gamma)$ is torsionfree. We recall from Proposition [13] that for the embedding $\Gamma \leq H_\Gamma$ we have $\Fitt(\Gamma) = u(\Gamma)$. So for a standard polycyclic group $\Gamma/Fitt(\Gamma)$ is torsionfree.

**Proposition 1.15** If $\Gamma$ is standard then $\Gamma$ has a $u$-connected syndetic hull $G$ in its real algebraic hull $H_\Gamma$.

**Proof.** The proposition is in fact true if $\Gamma$ is a discrete standard subgroup in a real algebraic group $H$. Since $\Gamma \leq H_{\Gamma,Z}$ has finite index in $\Gamma$, $\Gamma$ is a discrete subgroup of $H_\Gamma = H_{\Gamma,R}$. Now $\Gamma \leq H_\Gamma$ satisfies all the assumptions which are needed to carry out the construction given in [35, Proposition 4.1, Lemma 4.2]. \hfill $\Box$

We remark that $\Gamma$ is a standard polycyclic group if and only if $\Gamma$ may be realized as a Zariski-dense lattice in a $u$-connected simply connected solvable Lie group $G$. One half of this statement is implied by the previous proposition. The other is the following proposition.

**Proposition 1.16** Let $G$ be a connected simply connected solvable Lie group, and $\Gamma \leq G$ a Zariski-dense lattice. Then the real algebraic hull $H_G$ is a real algebraic hull for $\Gamma$. If $G$ is $u$-connected (in $H_G$) then $\Gamma$ is standard.

**Proof.** The Zariski-denseness of $\Gamma$ in $G$ implies by definition that $\Gamma$ is Zariski-dense in $H_G$. Moreover, $\rank \Gamma = \dim G$ since $\Gamma$ is cocompact. Therefore $H_G$ is a $\mathbb{R}$-defined algebraic hull for $\Gamma$. By an application of Proposition [1.3] $H_G$
is isomorphic over \(\mathbb{R}\) to \(H_\Gamma\). In particular, \(H_G\) is isomorphic to \(H_\Gamma\). Since \(\Gamma\) is contained in the connected group \(G \leq H_\Gamma\), \(\Gamma \leq (H_\Gamma)_0\).

Let \(N\) be the nilpotent radical of \(G\). It is known that \(\Gamma/(\Gamma \cap N) \leq A = G/N\) is torsionfree. Now if \(G\) is \(u\)-connected, \(N = G \cap u(H_\Gamma)\), and by Proposition 1.16, \(u(\Gamma) = \Gamma \cap u(H_\Gamma) = \Gamma \cap N\). We conclude that \(\Gamma/\Gamma(\Gamma)\) is torsionfree. In particular, \(\Gamma\) is standard. \(\square\)

The following fact is easy to see.

**Proposition 1.17** Let \(\Delta\) be a virtually polycyclic group. Then \(\Delta\) has a normal polycyclic subgroup \(\Gamma\) of finite index which is standard.

**Discrete shadows** The semisimple splitting construction for a discrete subgroup \(\Gamma\) of a connected, simply connected solvable Lie group \(G\) was introduced in the study of solvmanifolds. In particular, as Auslander showed (see [3]), the construction allows to associate to \(\Gamma\) certain lattices in the nilshadow of \(G\). A semisimple splitting for certain polycyclic groups \(\Gamma\) may be obtained with the help of nilpotent supplements. (See for example [71].) Maximal nilpotent supplements provide an analogue for the Cartan-subgroups in a solvable Lie-group \(G\). The use of nilpotent supplements allows to use the Jordan-decomposition in an algebraic group to construct the splitting. At this point, we only need the most elementary features of this construction.

Let \(\Gamma \leq G\) be a torsionfree polycyclic subgroup of some linear algebraic group \(G\). We further assume that \(\Gamma = CF\), where \(F\) is a normal subgroup of \(\Gamma\), and \(C\) is a nilpotent subgroup. \(C\) will be called a nilpotent supplement for \(F\). Let \(h : C \rightarrow G\) be a homomorphism, where \(C\) is nilpotent. For \(\gamma \in C\), we put \(h_u(\gamma) = h(\gamma)_u\). This defines a homomorphism \(h_u : C \rightarrow G\). Now we associate with \(\Gamma \leq G\) the group

\[
\Gamma_C^u = \langle C_u, F_u \rangle.
\]

We call \(\Gamma_C^u\) the unipotent shadow of \(\Gamma\) in \(G\) (with respect to \(C\)).

**Lemma 1.18** The group \(\Gamma_C^u \leq G_Q\) is a finitely generated, Zariski-dense subgroup of \(u(\Gamma) \leq G\).

**Proof.** By nilpotency, we get from the Jordan-decomposition \(C \leq C_sC_u\) and \(F \leq F_sF_u\). \(F_s\) is a central subgroup of \(F, F_u\) and consists of semisimple elements. Therefore \(\overline{F} = F_s\). Since \(C\) normalizes \(F\) it normalizes \(F\). Now \(\overline{\Gamma} = \overline{C}\overline{F} = (\overline{C})_u(\overline{F})_u(\overline{F})_u\). Since \(\Gamma\) is solvable, \(u(\overline{\Gamma}) = \overline{\Gamma}_u\). Therefore \(u(\overline{\Gamma}) = (\overline{C})_u(\overline{F})_u = C_uF_u\), and clearly \(\Gamma_C^u = \langle C_u, F_u \rangle\) is finitely generated and Zariski-dense. \(\square\)

Usually, we consider \(\Gamma \leq H_\Gamma\) as a subgroup of its algebraic hull, and \(\Gamma_C^u \leq H_\Gamma\) will be called the unipotent shadow of \(\Gamma\).

**Proposition 1.19** Let \(\Gamma = CF\) be a torsionfree polycyclic group, where \(F = \text{Fitt}(\Gamma)\) and \(C\) is a nilpotent supplement for \(F\). Then

\[
u(H_\Gamma) = \overline{C} + F.
\]

The unipotent shadow \(\Gamma_C^u \leq u(H_\Gamma)_Q\) is a Zariski-dense subgroup of \(u(H_\Gamma)\). In particular, \(\text{rank } \Gamma = \text{rank } \Gamma_C^u\).

**Proof.** Note first that by the properties of the Jordan decomposition \(\Gamma \leq H_\Gamma\) implies that \(\Gamma_C^u \leq u(H_\Gamma)_Q\). Since \(\overline{C}\) is nilpotent, \(C = S\overline{C}_u\), where \(S = \overline{C}_u\) centralizes the unipotent group \(u(\overline{C}) = (\overline{C})_u\). By Proposition 1.16, \(\text{Fitt}(\Gamma) \leq u(H_\Gamma)_Q\). Therefore \(\overline{F}\) is a unipotent normal subgroup of \(\overline{\Gamma} = H_\Gamma\). By the proof of the
Proposition 1.21 Let \( \rho : \Gamma \to G \) be a homomorphism. Then \( \rho \) descends to a unique homomorphism \( \rho^u : \Gamma^u \to G \). If \( \rho(\Gamma) \leq G_k \), where \( k \) is a subfield of \( \mathbb{C} \), then \( \rho(\Gamma^u) \leq G_k \).

Proof. We may assume that \( \rho(\Gamma) \) is Zariski-dense in \( G \). As in the proof of Proposition 1.21 we use the diagonal construction for the homomorphism \( \rho \). We proved there that, since \( \text{dim } u(\mathbb{H}_\Gamma) = \text{rank } \Gamma \), the projection map \( \alpha : D \to H_\Gamma \) induces a \( k \)-defined isomorphism \( \alpha : u(D) \to u(H_\Gamma) \) of the unipotent radicals. Since \( \Gamma^u \leq u(H_\Gamma) \) we can define \( \rho^u = \pi_2 \alpha^{-1} \). We have to show that \( \rho \) descends to \( \rho^u \) in the above sense. Since \( F \leq u(H_\Gamma) \), it is enough to verify that \( \pi_2 \alpha^{-1}(\gamma_u) = \rho(\gamma)_u \), for all \( \gamma \in \Gamma \). But, since \( d = (\gamma, \rho(\gamma))_u = (\gamma_u, \rho(\gamma)_u) \in u(D) \) it is clear that \( \alpha^{-1}(\gamma_u) = d \), and \( \pi_2(\alpha^{-1}(\gamma_u)) = \rho(\gamma)_u \). The rationality statement follows from well known properties of the Jordan-decomposition. \( \square \)

Proposition 1.21 implies that the correspondence \( \rho \mapsto \rho^u \) defines a map

\[
\gamma : \text{Hom}(\Gamma, G) \to \text{Hom}(\Gamma^u, G).
\]

Definition 1.22 We call \( \gamma : \text{Hom}(\Gamma, G) \to \text{Hom}(\Gamma^u, G) \) the shadow map for \( \Gamma \) (and \( G \)).

We will need the following result (compare [71, §7, Theorem 2]):

Proposition 1.23 If \( \Delta \) is a polycyclic group then \( \Delta \) has a characteristic subgroup \( \Gamma \) of finite index which admits a nilpotent supplement \( C \) for \( \text{Fitt}(\Gamma) \).

Let \( \Delta \leq H_\Delta \) be a virtually polycyclic wfn-group, and \( \Gamma \leq \Delta \) a characteristic subgroup of finite index which admits a nilpotent supplement \( C \) for \( \text{Fitt}(\Gamma) \). Since \( \text{Fitt}(\Gamma) \leq F = \text{Fitt}(\Delta) \), the group \( C \text{Fitt}(\Delta) \) is of finite index in \( \Delta \). \( C \) is then an almost nilpotent supplement for \( \Delta \). In any case, we can associate to \( \Delta \) the group

\[
\Delta^u_C = \langle C_u, F_u \rangle \leq H_\Delta,
\]

and we call it a unipotent shadow for \( \Delta \).

Crystallographic groups and simply transitive groups

In this section, we collect, and sometimes reformulate, basic results about virtually polycyclic ACGs. Our basic references are [27] and [35]. First, we explain the role of simply transitive groups.
Simply transitive hulls

The importance of syndetic hulls in our setting comes from the following central result, (compare [27], [35], §4):

Theorem 1.24 A virtually polycyclic subgroup \( \Delta \subset \text{Aff}(V) \) is an ACG if and only if there exists a simply transitive subgroup \( G \subset \text{Aff}(V) \) and \( \Gamma \leq_f \Delta \) such that \( G \) is a syndetic hull for \( \Gamma \).

The theorem describes the link between virtually polycyclic ACGs and certain simply transitive subgroups \( G \leq \text{Aff}(V) \). We will call \( G \) a simply transitive hull for the ACG \( \Gamma \). We remark that the drawback in considering the syndetic hull \( G \) is, in general, not uniquely determined by \( \Gamma \).

Extension property of simply transitive groups

The next proposition was noticed in [27]. Our aim in this section is to provide a converse.

Proposition 1.25 Let \( G \leq \text{Aff}(V) \) be a simply transitive group of affine motions. Then the embedding of \( G \) into its algebraic closure \( \overline{G} \leq \text{Aff}(V) \) is a real algebraic hull for \( G \).

Proof. Let \( U \) denote the unipotent radical of \( \overline{G} \). Auslander [4, §3] proved that \( U \) is simply transitive on \( V \). Hence \( \dim U = \dim V = \dim G \).

\( C_{\overline{G}}(U) \) is an algebraic subgroup of \( \overline{G} \). Since \( U \) is transitive, the elements of \( C_{\overline{G}}(U) \) act without fixed points on \( V \). Therefore \( C_{\overline{G}}(U) \) contains no semisimple elements. Hence \( C_{\overline{G}}(U) \) is a Zariski-closed normal unipotent subgroup of \( \overline{G} \). Hence \( C_{\overline{G}}(U) \leq u(\overline{G}) = U \). So \( \overline{G} \) has a strong unipotent radical. \( \square \)

Let us call a homomorphism \( i : H \to \text{Aff}(V) \) of real linear algebraic groups \( u\)-simply transitive if the unipotent radical \( u(i(H)) \) acts simply transitively on \( V \). We can characterise simply transitive groups now as follows.

Theorem 1.26 A connected, simply connected, solvable Lie subgroup \( G \leq \text{Aff}(V) \) is simply transitive on \( V \) if and only if the embedding of \( G \) into \( \text{Aff}(V) \) extends to a \( u\)-simply transitive embedding of algebraic groups \( H_G \to \text{Aff}(V) \).

Proof. The uniqueness of the real algebraic hull implies that the “only if” part of the theorem is just the previous proposition. So let us prove that \( G \) is simply transitive if \( \overline{G} \leq \text{Aff}(V) \) is a \( u\)-simply transitive algebraic hull for \( G \). By Proposition [1.12] there exists a torus \( S \leq \overline{G} \) such that \( U \leq GS \). The torus \( S \) has a fixed point \( x \in V \), so that \( Gx \supset Ux = V \). Therefore \( G \) acts transitively, and, since \( \dim G = \dim U = \dim V \), \( G \) acts a fortiori simply transitively on \( V \). \( \square \)

Extension property of crystallographic groups

We come now to the main result of this subsection.

Theorem 1.27 Let \( \Delta \leq \text{Aff}(V) \) be a virtually polycyclic \( \text{wfn}\)-group. Then \( \Delta \) is an ACG if and only if the embedding of \( \Delta \) into \( \text{Aff}(V) \) extends to a \( u\)-simply transitive embedding of algebraic groups \( H_\Delta \to \text{Aff}(V) \).

\(^4\) Auslander [4] observed that the nilpotent shadow of \( G \) acts simply transitively

\(^5\) the result was announced in [11, Theorem 3.2]
Proof. Let us assume that $\Delta$ is an ACG. By Theorem 1.24, $\Delta$ contains a simply transitive hull $G$ for a finite index subgroup $\Gamma$ of $\Delta$. Since $G$ is a syndetic hull for $\Gamma$, the algebraic closure of $\Gamma$ coincides with $G$, and rank $\Gamma = \dim G$. Therefore, by Theorem 1.26, $\dim u(\Delta) = \text{rank } \Gamma$. Since $G$ is $u$-simply transitive, $\Delta$ is $u$-simply transitive. Since $\Delta_0 = G$ has a strong unipotent radical, we can conclude that $\Delta$ is a real algebraic hull for $\Delta$.

For the converse, let us assume that $\Gamma \leq \text{Aff}(V)$ is a subgroup such that $\Gamma$ is a $u$-simply transitive real algebraic hull for $\Gamma$. Since $\Gamma$ has a standard subgroup of finite index, we may as well assume that $\Gamma$ is standard. Now if $\Gamma$ is standard, there exists, by Proposition 1.15, a syndetic hull $G \leq \Gamma$ for $\Gamma$. By Proposition 1.16, $G = \Gamma$ is an algebraic hull for $G$, and $u$-simply transitive by assumption. By Theorem 1.26, $G$ acts simply transitively on $V$. In particular, $\Gamma$ is an ACG. $\square$

Remark Theorem, together with Proposition 1.6, implies that any ACG $\Gamma \leq \text{Aff}(V)$ satisfies $\text{Fitt}(\Gamma) = u(\Gamma)$. This is the content of Lemma C in [35].

1.2 The variety of crystallographic homomorphisms

Let $\Gamma$ be a virtually polycyclic wfn-group, and $A \leq \text{Aff}(V)$ a Zariski closed subgroup. The purpose of this section is to study the space of crystallographic homomorphisms

$$\text{Hom}_c(\Gamma, A) \subset \text{Hom}(\Gamma, A)$$

as a subset of the real algebraic variety $\text{Hom}(\Gamma, A)$. The main result of this section, Theorem 1.41, establishes that the space $\text{Hom}_c(\Gamma, A)$ is described by algebraic equalities and inequalities, and carries itself a natural structure as a real algebraic variety which is defined over the rational numbers. We first prove the result for finitely generated torsionfree nilpotent groups. Our strategy is to use the unipotent shadow construction to extend from finitely generated torsionfree nilpotent groups to general virtually polycyclic groups.

Spaces of homomorphisms

Let us introduce here the spaces which we want to study. We start with some preliminary remarks.

Topological on the space of homomorphisms Let $H, G$ be locally compact groups, and $R(H, G)$ the space of all continuous homomorphisms from $H$ to $G$. We equip the space $R(H, G)$ with the compact open topology, meaning that a fundamental system of neighbourhoods in $R(H, G)$ is specified by all sets of the form

$$\mathcal{U}_{K, U} = \{ \rho \in R(H, G) \mid \rho(K) \subset U \},$$

where $K \subset H$ is compact, and $U \subset G$ is open.

Let us assume next that $\Gamma$ is a discrete group. We equip $\text{Hom}(\Gamma, G)$ with the subspace topology which is inherited from the product topology on $\text{Map}(\Gamma, G) = G^\Gamma$.

It is easy to see that this topology coincides with the compact open topology on $\text{Hom}(\Gamma, G) = R(\Gamma, G)$. If $\Gamma$ is finitely generated, and $S = \{ \gamma_1, \ldots, \gamma_n \}$ is a system of generators, then $\text{Hom}(\Gamma, G)$ is a subset of $G^n$ in a natural way. In fact, the inclusion

$$j : \text{Hom}(\Gamma, G) \longrightarrow G^n$$
which is given by $\rho \mapsto (\rho(\gamma_1), \ldots, \rho(\gamma_n))$ is a homeomorphism onto a closed subspace of $G^n$, see for example [60, 2.23]. Therefore, if $\Gamma$ is finitely generated, $\text{Hom}(\Gamma, G)$ carries also the subspace topology from $G^n$.

If $G = G_\mathbb{R}$ is a real (linear) algebraic group, $G$ carries the locally compact Hausdorff-topology and the Zariski-topology. So does $G^n$, and the embedding $\iota$ identifies $\text{Hom}(\Gamma, G)$ with a Zariski-closed subset. Aside from the Hausdorff-topology, the space $\text{Hom}(\Gamma, G)$ carries a natural structure of a real algebraic variety which is independent of the embedding $\iota$. This follows from:

**Proposition 1.28** Let $\Gamma$ be a finitely generated group, and $G$ a $\mathbb{Q}$-defined linear algebraic group. Then $\text{Hom}(\Gamma, G)$ has a natural structure of affine algebraic variety defined over $\mathbb{Q}$ and $\text{Hom}(\Gamma, G_k) = \text{Hom}(\Gamma, G)_k$ for any subfield $k \leq \mathbb{C}$.

See [52] for a proof. The space $\text{Hom}(\Gamma, G)$ is called a representation variety. Note that $G$ acts on homomorphisms by conjugation. This turns $\text{Hom}(\Gamma, G)$ into a $G$-variety.

**Crystallographic homomorphisms** Recall that a homomorphism $\rho : \Gamma \to \text{Aff}(V)$ is an affine crystallographic homomorphism, if $\rho$ is an isomorphism onto its image $\rho(\Gamma)$ and $\rho(\Gamma)$ is a crystallographic subgroup of $\text{Aff}(V)$. We put

$$\text{Hom}_c(\Gamma, \text{Aff}(V)) = \{ \rho : \Gamma \to \text{Aff}(V) \mid \rho \text{ is crystallographic} \}$$

for the space of crystallographic homomorphisms. We consider then $\text{Hom}_c(\Gamma, \text{Aff}(V))$ as a Hausdorff-topological space with the subspace topology inherited from the space $\text{Hom}(\Gamma, \text{Aff}(V))$. The space $\text{Hom}_c(\Gamma, \text{Aff}(V))$ has a natural action of $\text{Aff}(V)$, which is induced by conjugation on homomorphisms.

**Nilpotent crystallographic groups**

We prove here that for finitely generated torsionfree nilpotent groups the space of crystallographic homomorphisms is a Zariski-open subset of the space of unipotent homomorphisms. This result was developed in [10].

**Nilpotent crystallographic groups** Let $\Gamma$ be a f.t.n.-group. Our aim is to describe the inclusion

$$\text{Hom}_c(\Gamma, \text{Aff}(V)) \subset \text{Hom}(\Gamma, \text{Aff}(V)) .$$

**Remark** Using the fact that rank and cohomological dimension of $\Gamma$ coincide, it is evident that $\text{Hom}_c(\Gamma, \text{Aff}(V)) = \emptyset$, for rank$\Gamma \neq \dim V$.

Let us briefly specialise some facts of section [1.1]. Since $\Gamma$ is a f.t.n.-group, the real algebraic hull $U_\Gamma$ is just the real Malcev hull. The group $\Gamma$ is then a discrete, cocompact lattice in $U_\Gamma$. The following extension property is a special case of Proposition 1.24.

**Proposition 1.29** Let $\Gamma$ be a f.t.n.-group. Then every homomorphism $\rho : \Gamma \to U$ into a unipotent Zariski-closed subgroup $U \leq \text{Aff}(V)$ uniquely extends to a homomorphism of algebraic groups $\rho_U : U_\Gamma \to U$.

From Theorem 1.27 we deduce the following characterization of crystallographic f.t.n.-groups:

**Theorem 1.30** Let $\Gamma \subset \text{Aff}(V)$ be a f.t.n.-group. Then $\Gamma$ acts crystallographically on $V$ if and only if the algebraic closure $\overline{\Gamma} \subset \text{Aff}(V)$ is a unipotent simply transitive (real) Malcev hull for $\Gamma$. 

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The theorem will be useful together with

**Lemma 1.31** Let \( \Gamma \subset \text{Aff}(V) \) be a f.t.n.-group, and \( \text{rank} \Gamma \leq \dim V \). If the algebraic closure \( U = \overline{\Gamma} \subset \text{Aff}(V) \) is a unipotent simply transitive group then \( U \) is a real Malcev hull for \( \Gamma \).

**Proof.** Let \( U_\Gamma \) be the Malcev hull of \( \Gamma \). Let \( j : \Gamma \hookrightarrow \overline{\Gamma} = U \) be the inclusion homomorphism. Now \( \dim u(\overline{\Gamma}) \leq \text{rank} \Gamma = \dim U_\Gamma \). By Proposition 1.29, \( j \) extends to a surjective homomorphism \( j : U \Gamma \rightarrow U \) of unipotent algebraic groups. Since \( \dim U_\Gamma = \text{rank} \Gamma = \dim V = \dim U \), \( j : U_\Gamma \rightarrow U \) is an isomorphism. \( \square \)

Before stating the main result, we introduce some further notation. A function \( \delta \) on a \( G \)-variety \( V \), where \( G \) is a group, is called a relative \( G \)-invariant if there exists a character \( \chi \) of \( G \), so that, for all \( g \in G \) and \( v \in V \),

\[
\delta(g \cdot v) = \chi(g) \delta(v).
\]

For any continuous function \( \delta \), the set \( V_\delta \) defined as

\[
V_\delta = \{ v \in V \mid \delta(v) \neq 0 \}
\]

is called a special open subset of \( V \).

Let us put

\[
\text{Hom}_u(\Gamma, \text{Aff}(V)) = \{ \rho \mid \rho(\Gamma) \text{ is unipotent} \} \subset \text{Hom}(\Gamma, \text{Aff}(V))
\]

for the variety of unipotent representations.

**Lemma 1.32** The space \( \text{Hom}_u(\Gamma, \text{Aff}(V)) \) is a Zariski-closed in \( \text{Hom}(\Gamma, \text{Aff}(V)) \), and it is defined over \( \mathbb{Q} \).

**Proof.** If \( \Gamma \leq \text{GL}(V) \) is nilpotent then \( \Gamma_s \) and \( \Gamma_u \) are commuting subgroups, \( \Gamma \leq \Gamma_s \times \Gamma_u \) and the projection maps are homomorphisms. (See for example [71, Chapter 7, Proposition 3].) Therefore \( \rho(\Gamma) \) is unipotent if and only if \( \rho \) is unipotent on a set of generators, and we conclude that \( \text{Hom}_u(\Gamma, \text{Aff}(V)) \) is a Zariski-closed subset of \( \text{Hom}(\Gamma, \text{Aff}(V)) \). \( \square \)

Theorem 1.30 implies that

\[
\text{Hom}_c(\Gamma, \text{Aff}(V)) \subset \text{Hom}_u(\Gamma, \text{Aff}(V)).
\] (1.1)

Surprisingly, \( \text{Hom}_c(\Gamma, \text{Aff}(V)) \) is a special open subset of \( \text{Hom}_u(\Gamma, \text{Aff}(V)) \):

**Theorem 1.33** Let \( \Gamma \) be a f.t.n.-group which satisfies \( \text{rank} \Gamma = \dim V \). Then there exists a \( \mathbb{Q} \)-defined polynomial function \( \delta \) on the variety of unipotent representations \( \text{Hom}_u(\Gamma, \text{Aff}(V)) \), such that

\[
\text{Hom}_c(\Gamma, \text{Aff}(V)) = \text{Hom}_u(\Gamma, \text{Aff}(V))_\delta.
\]

Moreover, \( \delta \) is a relative invariant with respect to the natural \( \text{Aff}(V) \)-action on \( \text{Hom}_u(\Gamma, \text{Aff}(V)) \).

**Remark** The character which belongs to the relative invariant function \( \delta \) is the determinant on \( \text{Aff}(V) \), cf. [10].

We need some more preparations for the proof of the theorem.
The Malcev hull of a f.t.n.-group  Let $U_{\Gamma}$ be the Malcev hull of $\Gamma$. This means that $U_{\Gamma}$ is a $\mathbb{Q}$-defined unipotent linear algebraic group, and $\Gamma \leq (U_{\Gamma})_{\mathbb{Q}}$ is a Zariski-dense subgroup. Let $u$ be the Lie algebra of $U_{\Gamma}$. The exponential map
\[ \exp : u \longrightarrow U_{\Gamma} \]
is a polynomial map which is an isomorphism of algebraic varieties. The inverse map is $\log : U_{\Gamma} \rightarrow u$. It is known that there exists a $\mathbb{Q}$-structure on the complex Lie algebra $u$, so that $\log \Gamma \leq u_{\mathbb{Q}}$, and, in fact, $u_{\mathbb{Q}} = \mathbb{Q}\log \Gamma$ is the $\mathbb{Q}$-span of $\log \Gamma$. Moreover, $\exp$ and $\log$ are $\mathbb{Q}$-defined maps. (See [33] for more details.)

**Definition 1.34** Let $S = \{\gamma_1, \ldots, \gamma_n\}$ be a system of generators for $\Gamma$. $\Gamma$ is called a *Malcev-basis* of $\Gamma$, if the series of subgroups
\[ 1 \leq \langle \gamma_1 \rangle \leq \langle \gamma_1, \gamma_2 \rangle \leq \cdots \leq \langle \gamma_1, \ldots, \gamma_n \rangle = \Gamma \]
is a central series for $\Gamma$ with infinite cyclic factors.

By results of Malcev [53], every f.t.n.-group $\Gamma$ admits a Malcev-basis.

The following facts can be derived from the Baker-Campbell-Hausdorff formula: Put $u_i = \mathbb{Q}\log \Gamma_i$, where $\Gamma_i = \langle \gamma_1, \ldots, \gamma_i \rangle$. The complex span $u_i$ of $\log \Gamma_i$ is an ideal in the Lie algebra $u$ and the series
\[ 0 \subset u_1 \subset u_2 \cdots \subset u \]
is a central series for $u$ with one-dimensional factors. In particular,
\[ u_{\mathbb{Q}} = \langle \log \gamma_1, \ldots, \log \gamma_n \rangle_{\mathbb{Q}}. \]

**Simply transitive and étale unipotent actions** We have to consider simply transitive unipotent actions. Let $U \leq \text{Aff}(V)$ be a unipotent algebraic subgroup which satisfies $\dim U = \dim V$. $U$ is called *étale on $V$* if it has an open orbit on $V$. By a result of Rosenlicht [67] the orbits of the unipotent group $U$ on $V$ are all closed. Therefore $U$ is simply transitive if and only if $U$ is étale.

**Proof of Theorem 1.33** Let $a(V)$ denote the Lie algebra of $\text{Aff}(V)$, and let $a_{n}(V)$ denote the (algebraic) subset of nilpotent elements in $a(V)$. Recall that the exponential map
\[ \exp : a(V) \longrightarrow \text{Aff}(V) \]
defines a polynomial map which induces a *polynomial* equivalence from $a_{n}(V)$ onto the (algebraic) subset $\text{Aff}_{n}(V)$ of unipotent elements in $\text{Aff}(V)$. On $\text{Aff}_{n}(V)$, the map $\exp$ has a well defined polynomial inverse
\[ \log : \text{Aff}_{n}(V) \longrightarrow a_{n}(V). \]

Let $S = \{\gamma_1, \ldots, \gamma_n\}$ be a Malcev-basis for $\Gamma$, where $n = \dim V$. If $\rho \in \text{Hom}_{a}(\Gamma, \text{Aff}(V))$ then $\rho(\Gamma) \leq \text{Aff}(V)$ is a unipotent f.t.n.-group. The Lie algebra $u_{\rho}$ of $U_{\rho} = \rho(\Gamma)$ is therefore contained in $a_{n}(V)$. Since $u_{\rho}$ is a homomorphic image of $u$ it is spanned by the set $\{\log \rho(\gamma_i) \mid i = 1 \ldots n\}$. It follows from Proposition 1.30 together with Lemma 1.31 that, $\rho \in \text{Hom}_{a}(\Gamma, \text{Aff}(V))$ if and only if the algebraic closure $U_{\rho}$ is a simply transitive unipotent subgroup of $\text{Aff}(V)$.

Choose an arbitrary base point $x \in V$. The differential of the orbit map $A \mapsto A \cdot x$ of the action of $\text{Aff}(V)$ on $V$ defines a linear map $o_{x} : a(V) \rightarrow V$. Therefore $U_{\rho} \leq \text{Aff}(V)$ is étale in $x$ if and only if the restriction of $o_{x}$ to the Lie-algebra...
$u_\rho \subset \mathfrak{a}(V)$ is an isomorphism of vector spaces. It follows a fortiori that $U$ is simply transitive on $V$ if and only if the restriction $o_x : u_\rho \to V$ is an isomorphism.

Let us next consider the linear map $\tau_x(\rho) : \mathbb{R}^n \to \mathbb{R}^n$ which is defined by

$$\tau_x(\rho) : (a_1, \ldots, a_n) \mapsto a_x(\sum_{i=1}^n a_i \log \rho(\gamma_i)).$$

(After choosing an arbitrary basis in $U$ and is contained in the centralizer of $\delta$ and, clearly, $\delta$ is a polynomial on $\text{Hom}(\Gamma, \text{Aff}(V))$ with the property that $\delta(\rho) \neq 0$ if and only if $U_\rho = \rho(\Gamma) \leq \text{Aff}(V)$ is a simply transitive subgroup. In this case, the extension $\rho_{U_\Gamma} : U_\Gamma \to U_\rho$ is an isomorphism of algebraic groups, and $\rho(\Gamma)$ is a crystallographic group. Up to now we proved that

$$\text{Hom}_c(\Gamma, \text{Aff}(V)) = \text{Hom}_u(\Gamma, \text{Aff}(V))\delta.$$ 

Next we show that the function $\delta$ is independent of $x \in V$. To see this, we remark that our previous reasoning is also valid over the field of complex numbers. In particular if $\delta(\rho) \neq 0$ the corresponding unipotent algebraic group $U$ acts simply transitively on $\mathbb{C}^n$. If $\rho$ is fixed and $\delta(\rho) \neq 0$ then the function

$$x \mapsto \det \tau_x(\rho)$$

is a polynomial on $\mathbb{C}^n$ which does not vanish. Hence $\det \tau_x(\rho)$ is constant. To show that $\delta(\rho) = \det \tau_x(\rho)$ is a relative invariant on $\text{Hom}_c(\Gamma, \text{Aff}(V))$ we remark that by direct calculation, for all $g \in \text{Aff}(V)$, the formula

$$\det \tau_x(g^\rho) = (\det g) \det \tau_{g^{-1}x}(\rho)$$

holds. It follows therefore that $\det \tau_x(g^\rho) = (\det g) \det \tau_x(\rho)$.

Remark We may also define $\delta(\rho) = \delta(\rho_u)$. If $\rho \in \text{Hom}(\Gamma, \text{Aff}(V))$, such that $\delta(\rho) \neq 0$, then, by the Theorem, $\rho_u$ is in $\text{Hom}_c(\Gamma, \text{Aff}(V))$. In particular $U = \rho_u(\Gamma)$ is a simply transitive subgroup. But since $\rho(\Gamma)_u$ consists of semisimple elements, and is contained in the centralizer of $U$, it must be trivial. Therefore $\rho = \rho_u$, and a fortiori $\rho \in \text{Hom}_c(\Gamma, \text{Aff}(V))$. Hence,

$$\text{Hom}_c(\Gamma, \text{Aff}(V)) = \text{Hom}(\Gamma, \text{Aff}(V))\delta.$$ 

Note however that the function $\delta$ is not continuous on $\text{Hom}(\Gamma, \text{Aff}(V))$.

The next example illustrates the theorem in the simplest possible situation:

**Example 1.2.1** Let $V = \mathbb{R}$. Then

$$\text{Hom}(\mathbb{Z}, \text{Aff}(\mathbb{R})) = \text{Aff}(V) = \left\{ \begin{pmatrix} \epsilon & v \\ 0 & 1 \end{pmatrix} \mid \epsilon \neq 0 \right\} \leq \text{GL}(2, \mathbb{R})$$

$$\text{Hom}_u(\mathbb{Z}, \text{Aff}(\mathbb{R})) = V = \left\{ \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \right\}$$

and

$$\text{Hom}_c(\mathbb{Z}, \text{Aff}(\mathbb{R})) = \left\{ \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \mid v \neq 0 \right\}.$$
Virtually polycyclic crystallographic groups

After some more preparations we come to the statement and proof of our main result on the space of crystallographic homomorphisms, Theorem 1.41.

The crystallographic shadow map Let $\Gamma$ be a virtually polycyclic fn-group. The main step in our description of the space $\text{Hom}_c(\Gamma, A)$ is to characterise the elements $\rho \in \text{Hom}_c(\Gamma, A)$ by properties which may be verified on the unipotent shadow of $\Gamma$. Let us assume here that $\Gamma$ is polycyclic, and $\Gamma = CF \leq \text{Aff}(V)$ admits a nilpotent supplement for $F = \text{Fitt}(\Gamma)$. Recall then from Proposition 1.19 that there is a certain unipotent group $\Gamma^u \leq \text{Aff}(V)$, the unipotent shadow of $\Gamma$ in $\text{Aff}(V)$, which is associated with $\Gamma$.

**Proposition 1.35** Let $\Gamma \leq \text{Aff}(V)$ be a torsionfree polycyclic group which satisfies $\text{rank} \Gamma = \dim V$. Moreover, we assume that $\Gamma = CF$ admits a nilpotent supplement $C$ for $F = \text{Fitt}(\Gamma)$. Then $\Gamma$ is an ACG if and only if the unipotent shadow $\Gamma^u$ of $\Gamma$ in $\text{Aff}(V)$ is an ACG.

Proof. Let us first assume that $\Gamma$ is an ACG. Then, by Theorem 1.27, the algebraic closure $\overline{\Gamma}$ is a $u$-simply transitive real algebraic hull for $\Gamma$. By Proposition 1.19, $\Gamma^u = C_u F$ is a lattice in $U = u(\overline{\Gamma})$. Since $U$ is simply transitive, $\Gamma^u$ is an ACG.

Now let us assume that $\Gamma^u = C_u F_u$ is an unipotent ACG. It follows that $U = \overline{\Gamma^u} = C_u F_u$ is a unipotent simply transitive subgroup of $\overline{\Gamma}$. Since $\overline{\Gamma}$ is solvable, $U \leq u(\overline{\Gamma})$. By a lemma of Mostow (cf. [19, 4.36]) it is known that $\dim u(\overline{\Gamma}) \leq \text{rank} \Gamma = \dim V$, and hence it follows that $u(\Gamma) = U$. So $\Gamma$ is in fact $u$-simply transitive, and $\dim u(\overline{\Gamma}) = \text{rank} \Gamma$. The centralizer $C_{\overline{\Gamma}}(U)$ of the simply transitive normal subgroup $U$ is a unipotent subgroup of $\overline{\Gamma}$. Therefore $C_{\overline{\Gamma}}(U) \leq u(\overline{\Gamma}) = U$. Hence $\overline{\Gamma}$ has a strong unipotent radical, and is a $u$-simply transitive algebraic hull for $\Gamma$. By Theorem 1.27, $\Gamma$ is an ACG.\[\square\]

Recall from Definition 1.20 that, for $\Gamma = CF$ as above, a homomorphism $\rho : \Gamma \rightarrow G$ is said to descend to the unipotent shadow $\Gamma^u \leq H$ if there is a homomorphism $\rho^u : \Gamma^u \rightarrow G$ so that $\rho^u|_{F} = \rho_u$, and $\rho^u|_{C_u}(\gamma_u) = \rho_u(\gamma)$, for all $\gamma \in C$.

**Proposition 1.36** Let $\Gamma$ be a torsionfree polycyclic group which satisfies $\text{rank} \Gamma = \dim V$. We assume also that $\Gamma = CF$ admits a nilpotent supplement $C$ for $F = \text{Fitt}(\Gamma)$. Let $\rho : \Gamma \rightarrow \text{Aff}(V)$ be a homomorphism. Then $\rho$ is a crystallographic homomorphism if and only if $\rho$ descends to a crystallographic homomorphism $\rho^u$ on the unipotent shadow $\Gamma^u$ of $\Gamma$.

Proof. Let us first assume that $\rho$ is crystallographic. By Theorem 1.27, $\rho$ extends to a $u$-simply transitive morphism of (real) algebraic groups $\rho_{H_u} : H_u \rightarrow \text{Aff}(V)$. As in the proof of Proposition 1.21, $\rho$ descends to the restriction of $\rho_{H_u}$ to $\Gamma^u \leq u(H_u)$. Since $u(\rho(H_u))$ is simply transitive, this homomorphism is crystallographic.

Conversely, let us assume that $\rho$ descends to a crystallographic homomorphism $\rho^u$ on $\Gamma^u \leq u(H)$. Then, since $\rho^u(\Gamma^u)$ is a unipotent shadow for $\rho(\Gamma)$ in $\text{Aff}(V)$, Proposition 1.35 implies that $\rho(\Gamma)$ is crystallographic on $V$. In particular, $\text{rank} \rho(\Gamma) = \dim V = \text{rank} \Gamma$. It follows that the kernel of $\rho$ is finite. Since $\Gamma$ is torsionfree, $\rho$ is injective, hence $\rho$ is a crystallographic homomorphism.\[\square\]

Proposition 1.36 implies that the correspondence $\rho \mapsto \rho^u$ defines a map
\[s_u : \text{Hom}_c(\Gamma, A) \rightarrow \text{Hom}_c(\Gamma^u, A)\]

**Definition 1.37** We call $s_u : \text{Hom}_c(\Gamma, A) \rightarrow \text{Hom}_c(\Gamma^u, A)$ the crystallographic shadow map for $\Gamma$.\[19\]
In fact, the crystallographic shadow map for $\Gamma$ is just the restriction of the shadow map $s_u : \text{Hom}(\Gamma, A) \to \text{Hom}(\Gamma^u, A)$, see Proposition 1.21.

**Adapted systems of generators** Let $\Gamma = CF \leq H_\Gamma$ be a torsionfree polycyclic group which admits a nilpotent supplement $C$ for $F = \text{Fitt}(\Gamma)$.

**Definition 1.38** Let $S = \{\gamma_i \in \Gamma, i = 1, \ldots, l\}$, be a system of generators for $\Gamma$. $S$ is called adapted to the supplement $C$, if $C = \langle \gamma_i, i = 1, \ldots, s \rangle$, and $F = \langle \gamma_i, i = s + 1, \ldots, l \rangle$.

Let $S$ be an adapted system of generators for $\Gamma = C \text{Fitt}(\Gamma)$. Then it follows that

$$S_u = \{\theta_i = (\gamma_i)_u \in u(H_\Gamma)\}$$

is a set of generators for $\Gamma^u$. Assume now also that $\text{rank} \Gamma = \text{dim} V$.

**Proposition 1.39** Let $\rho : \Gamma \to \text{Aff}(V)$ be a homomorphism, $S$ an adapted system of generators. Then $\rho$ is a crystallographic homomorphism if and only if the assignment

$$\theta_i \mapsto - \rho(\gamma_i)_u, \theta_i \in S_u$$

defines a crystallographic homomorphism $\rho^u \in \text{Hom}_u(\Gamma^u, \text{Aff}(V))$.

**Proof.** By Proposition 1.21 there exists a unique homomorphism

$$\rho^u \in \text{Hom}(\Gamma^u, \text{Aff}(V)),$$

so that $\rho$ descends to $\rho^u$, and moreover $\rho^u(\theta_i) = \rho(\gamma_i)_u$. The proposition then follows from Proposition 1.36. \hfill $\square$

**Remark** By the remark following Theorem 1.27 every crystallographic group has a unipotent Fitting subgroup. Astonishingly enough, we do not have to assume this necessary condition on $\rho$ to ensure that $\rho$ is crystallographic. In fact, the proof of Proposition 1.35 shows that if $\rho^u$ is crystallographic then also $\rho(\gamma) = \rho(\gamma)_u$, for all $\gamma \in \text{Fitt}(\Gamma)$.

**The crystallographic restriction** A nilpotent crystallographic group $\Gamma$ is necessarily unipotent. This lead us to consider the space of unipotent homomorphisms $\text{Hom}_u(\Gamma, \text{Aff}(V))$. For a general virtually polycyclic wfn-group $\Gamma$, a corresponding restriction arises from the internal structure of the algebraic hull.

For $g \in H_\Gamma$, we let $c_U(g) \in \text{Aut}(U_\Gamma)$ denote conjugation with $g$ on $U_\Gamma = u(H_\Gamma)$. Taking the differential of $c_U(g)$ in the identity defines a representation

$$\alpha_U : H_\Gamma \longrightarrow \text{Aut}(u_\Gamma)$$

on the Lie algebra $u_\Gamma$ of $U_\Gamma$. Since $H_\Gamma$ has a strong unipotent radical, the kernel of $\alpha_U$ is contained in $U$. For $\gamma \in H_\Gamma$, we define the characteristic polynomial

$$\chi_U(\gamma, T) = \text{det} (T\text{id} - \alpha_U(\gamma)),$$

and, correspondingly for $g \in \text{Aff}(V)$, we let $\chi(g, T)$ denote the characteristic polynomial of $g$ as an element of $\text{GL}(V \oplus \mathbb{R})$. The crystallographic restrictions for $\Gamma$ are described by the following:

**Lemma 1.40** Let $\Gamma$ be virtually polycyclic wfn-group. If $\rho : \Gamma \to \text{Aff}(V)$ is a crystallographic homomorphism then, for all $\gamma \in \Gamma$,

$$\chi(\rho(\gamma), T) = \chi_U(\gamma, T) (T - 1).$$

(1.2)
Proof. Since \( \rho \) is crystallographic there exists a \( u \)-simply transitive homomorphism \( \rho_T : H \to \text{Aff}(V) \) which extends \( \rho \), so that \( U = \rho_T(U) \) is a simply transitive subgroup of \( \text{Aff}(V) \). \( \rho(U) \) acts by conjugation on \( U \), as well as on the Lie algebra \( u \subset \text{End}(V \oplus \mathbb{R}) \) of \( U \). Let \( \phi_r : R \to u \) be the differential of the homomorphism \( \rho \). The isomorphism \( \phi_r \) is equivariant with respect to conjugation, i.e., \( \phi_r \) satisfies, for all \( X \in u, g \in H \),

\[
\phi_r(\alpha(g)X) = \rho(g)\phi_r(X)\rho(g)^{-1}.
\]

Since \( \chi_U(\gamma, T) = \chi_U(\gamma_s, T) \) we are allowed to assume that \( \gamma = \gamma_s \) is a semisimple element. We may also assume that \( \rho(\gamma_s) = \rho(\gamma)_s \in \text{GL}(V) \). Let \( o_0 : u \to V \) be the differential of the orbit map of \( U \) in \( 0 \) (the evaluation map in \( 0 \)). An immediate calculation shows that the isomorphism \( o_0 \phi_r : u \to V \) is equivariant with respect to the action of \( \gamma_s \), i.e., for all \( X \in u \),

\[
o_0 \phi_r(\alpha(\gamma_s)X) = \rho(\gamma_s) o_0 \phi_r(X)
\]

Hence, \( \chi_U(\gamma_s, T) = \chi_{\text{GL}(V)}(\rho(\gamma)_s, T) \), and the lemma follows. \( \square \)

We define now a certain Zariski-closed subspace of \( \text{Hom}(\Gamma, A) \). Namely, we put

\[
\text{Hom}_\chi(\Gamma, A) = \{ \rho \mid \chi(\rho(\gamma), T) = \chi_U(\gamma, T)(T - 1) \} \text{, for all } \gamma \in \Gamma
\]

for the subvariety of homomorphisms of type \( A \) which satisfy the crystallographic restrictions. If \( \Gamma \) is nilpotent, then \( \text{Hom}_\chi(\Gamma, A) = \text{Hom}_u(\Gamma, A) \). From the lemma we infer that

\[
\text{Hom}_c(\Gamma, A) \subset \text{Hom}_\chi(\Gamma, A).
\]

We will show next that \( \text{Hom}_c(\Gamma, A) \) is Zariski-open in \( \text{Hom}_\chi(\Gamma, A) \).

The variety of crystallographic homomorphisms We are ready now to generalize Theorem 1.33 from nilpotent groups to virtually polycyclic groups. Also we consider the situation now for any Zariski-closed subgroup \( A \leq \text{Aff}(V) \). If \( \Gamma \) is a polycyclic finite index subgroup of \( \Delta \), we let \( \text{Hom}_\chi(\Delta, A) \subset \text{Hom}(\Delta, A) \) be the Zariski-closed subset of homomorphisms which satisfy the crystallographic restrictions (1.2) on \( \Gamma \).

**Theorem 1.41** Let \( \Delta \) be a virtually polycyclic wfn-group which satisfies \( \text{rank } \Delta = \dim V \). Then there exists a unique \( \mathbb{Q} \)-defined polynomial function \( \delta \) on the real algebraic variety \( \text{Hom}_\chi(\Delta, A) \), such that

\[
\text{Hom}_c(\Delta, A) = \text{Hom}_\chi(\Delta, A)\delta.
\]

Moreover, \( \delta \) is a relative invariant for the conjugation action of \( A \) on \( \text{Hom}_\chi(\Delta, A) \).

**Proof.** Since \( \text{Hom}(\Delta, A) \) is Zariski-closed in \( \text{Hom}(\Delta, \text{Aff}(V)) \), it is clearly enough to prove the result in the case \( A = \text{Aff}(V) \). Observe that \( \rho(\Delta) \) is a crystallographic group if and only if a finite index subgroup of \( \rho(\Delta) \) is a crystallographic group. Since \( \Delta \) is a wfn-group, \( \rho \) is crystallographic (in particular injective) if and only if it is so on a finite index subgroup \( \Gamma \). Since the restriction map \( \text{Hom}(\Delta, \text{Aff}(V)) \to \text{Hom}(\Gamma, \text{Aff}(V)) \) is algebraic, it is therefore enough to show the theorem for \( \Gamma \). By Proposition 1.23 there exists a finite index subgroup \( \Gamma \) of \( \Delta \) which is torsionfree and admits a nilpotent supplement \( C \) for \( \text{Fitt}(\Gamma) \). We now show the result for \( \Gamma \).

By Theorem 1.33 there exists a polynomial function \( \delta_u \) on \( \text{Hom}_u(\Gamma_C, \text{Aff}(V)) \) which is a relative invariant and tests crystallography for \( \text{Hom}(\Gamma_C, \text{Aff}(V)) \). The shadow map for \( \Gamma \) is a map

\[
s_u : \text{Hom}(\Gamma, \text{Aff}(V)) \to \text{Hom}_u(\Gamma_C, \text{Aff}(V)).
\]
Therefore (compare Proposition 1.39) the function \( \delta = \delta_u \circ s_u \) tests crystallography on \( \text{Hom}(\Gamma, \text{Aff}(V)) \), and in particular on \( \text{Hom}_\chi(\Gamma, \text{Aff}(V)) \).

We now show that \( s_u \) is continuous on \( \text{Hom}_\chi(\Gamma, \text{Aff}(V)) \). Let \( S = \{ \gamma_i \} \) be an adapted system of generators for \( \Gamma \). By Proposition 1.39 the shadow map \( s_u \) is obtained by computing \( \rho(\gamma_i)_u \) for each generator \( \theta_i = (\gamma_i)_u \) of the unipotent shadow \( \Gamma_u^\theta \). Now, for \( g \in \text{GL}(W) \), we may compute the unipotent part of the multiplicative Jordan decomposition of \( g \) by the formula

\[
g_u = I + g_n^{-1} g_n,
\]

where \( g = g_u + g_n \) is the additive Jordan-decomposition in \( \text{End}(W) \). Moreover \( g_n = P(g), \ g_n = Q(g) \), where \( P(T), Q(T) \in \mathbb{Q}[T] \) are certain polynomials which depend only on the characteristic polynomial of \( g \in \text{GL}(W) \). If \( \rho \in \text{Hom}_\chi(\Gamma, \text{Aff}(V)) \) satisfies the crystallographic restriction then \( \rho(\gamma_i) \) has characteristic polynomial \( \chi_v(\gamma, T)(T - 1) \) which does not depend on \( \rho \). Therefore, there exist polynomials \( P_i(T), Q_i(T) \) such that, for every \( \rho \in \text{Hom}_\chi(\Gamma, \text{Aff}(V)) \),

\[
\rho(\gamma_i)_u = I + P_i(\rho(\gamma_i))^{-1} Q_i(\rho(\gamma_i)).
\]

This means that \( s_u \) is a \( \mathbb{Q} \)-defined polynomial map on \( \text{Hom}_\chi(\Gamma, \text{Aff}(V)) \), and so is \( \delta = \delta_u \circ s_u \). It is easy to see that \( s_u \) is \( \text{Aff}(V) \)-equivariant. Hence, \( \delta \) is a \( \mathbb{Q} \)-defined polynomial function which is a relative invariant, and the theorem is proved. \hfill \Box

**Remark** Let \( \Delta \) be as in Theorem 1.41. The theorem implies that the existence problem for crystallographic subgroups of \( \text{Aff}(V) \) which are isomorphic to \( \Delta \) may be reduced to the existence problem for real solutions of a certain system of algebraic equations with rational coefficients. Therefore, by the Tarski-Seidenberg Theorem this decision problem admits an effective solution.

In particular, Theorem 1.41 shows that \( \text{Hom}_c(\Delta, A) \) is a Zariski-open subset of \( \text{Hom}_\chi(\Delta, A) \). The theorem therefore also establishes a certain rigidity property for affine crystallographic homomorphisms: If \( \rho \in \text{Hom}_c(\Gamma, \text{Aff}(V)) \) then any nearby \( \rho' \in \text{Hom}_\chi(\Gamma, \text{Aff}(V)) \) is in \( \text{Hom}_c(\Gamma, \text{Aff}(V)) \). In his celebrated paper [77], A. Weil proved the following:

Let \( \Gamma \leq G \) be a discrete cocompact subgroup of the Lie group \( G \). Then the space

\[
R(\Gamma, G) = \{ \rho : \Gamma \to G \mid \rho \text{ is injective and } \rho(\Gamma) \text{ is discrete and cocompact} \}
\]

is open in the space of all homomorphisms from \( \Gamma \) to \( G \). (The space \( R(\Gamma, G) \) is now called the Weil space.)

Our Theorem 1.41 thus may be interpreted as an analogue of the result of Weil in the specific situation of crystallographic affine actions for virtually polycyclic group.

We state some first consequences now. The main applications, however, concern the structure and topology of the deformation spaces of affine crystallographic actions. This will be described in the following chapter of this article.

**Corollary 1.42** Let \( \Gamma \) be a virtually polycyclic group. Then \( \text{Hom}_c(\Gamma, A) \) is a locally closed subset with respect to the Zariski-topology on \( \text{Hom}(\Gamma, A) \).

As a special open subset of an affine real algebraic variety the set of crystallographic homomorphisms has a natural algebraic structure itself:

\[\text{Abels (see [1]) gave generalizations of Weil’s theorem in the general context of proper actions on homogeneous spaces of Lie groups.}\]
**Corollary 1.43** \( \text{Hom}_c(\Gamma, \text{Aff}(V)) \) has the structure of a \( \mathbb{Q} \)-defined real affine algebraic variety. In particular, \( \text{Hom}_c(\Gamma, \text{Aff}(V)) \) has only finitely many connected components also in the Hausdorff-topology.

Another important application of the reasoning in the proof of Theorem 1.41 is:

**Corollary 1.44** Let \( \Theta \) be a unipotent shadow for \( \Gamma \). Then the shadowmap

\[
    s_u : \text{Hom}_c(\Gamma, A) \rightarrow \text{Hom}_c(\Theta, A)
\]

is a morphism of real algebraic varieties defined over \( \mathbb{Q} \). In particular, \( s_u \) is a continuous map.

**Proof.** In fact, in the course of the proof of Theorem 1.41 we proved that the shadow map

\[
    s_u : \text{Hom}_\chi(\Gamma, A) \rightarrow \text{Hom}_u(\Theta, A)
\]

is given by certain rational polynomials with respect to an adapted system of generators of \( \Gamma \). Therefore \( s_u \) is algebraic also on \( \text{Hom}_c(\Gamma, A) \subset \text{Hom}_\chi(\Gamma, A) \). \( \square \)
Chapter 2

Deformation Spaces

In this chapter we study properties of the deformation spaces of affine crystallographic actions of virtually polycyclic groups. These spaces are defined in a purely algebraic way as quotients of spaces of homomorphisms with an appropriate topology. If $M$ is a fixed compact smooth manifold then the deformation space $D_c(M,A)$ of complete affine $A$-structures on $M$ is the space of such structures up to diffeomorphism, equipped with the $C^\infty$-topology from the space of developing maps. If $\Gamma$ is the fundamental group of $M$ the deformation theorem of Thurston links the spaces $D_c(M,A)$ and $D_c(\Gamma,A)$ via a continuous and open map, the holonomy map

$$hol : D_c(M,A) \rightarrow D_c(\Gamma,A).$$

Note that the theory of Thurston covers the general situation of locally homogeneous manifolds $M$, modelled on a homogeneous space $G/H$, see [74], and, in particular, [26, 30] for further explanation and proofs. In [13] implications are discussed for the deformations of compact complete affine manifolds.

In this realm, Kobayashi [44], more generally formulates the program to determine the deformation spaces for proper, not necessarily cocompact, actions of a group $\Gamma$ on a homogeneous space $G/H$. The situation $G = \text{Aff}(V)$, and $G/H = V$ is an important special case in this program.

If $\Gamma$ is a torsionfree virtually polycyclic ACG of type $A$ then the quotient space $M = \Gamma \backslash V$ is a compact complete affine manifold which admits an affine atlas with coordinate changes in the group $A$. It is also a smooth aspherical compact manifold with fundamental group $\Gamma$. Results of the previous chapter (cf. Proposition 1.15) imply that the smooth manifold $M$ falls into the class of infrasolvmanifolds. In this class of smooth manifolds the diffeomorphism type of $M$ is determined by the fundamental group $\Gamma$ alone. This is proved in [13]. This smooth rigidity for compact complete affine manifolds implies that the holonomy map is a homeomorphism, compare [10]. Therefore, the algebraic viewpoint on the deformation space $D_c(M,A)$ captures the whole picture. In this sense, we allow ourselves at some points to speak about the deformation spaces of affine structures on certain manifolds, although we only give results on the affine crystallographic actions of their fundamental groups.

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1 See [6, 7, 46, 80] for recent contributions on this problem.

2 A result in [27] shows that any two compact complete affine manifolds with virtually solvable fundamental group are (polynomially) diffeomorphic.
2.1 The geometry of the deformation space

Let $\Gamma$ be a group, and let $\text{Hom}_c(\Gamma, A)$ be the space of crystallographic homomorphisms of type $A$. The group $A$ acts by conjugation on the space of homomorphisms. Our principal object of interest is the deformation space of crystallographic homomorphisms of type $A$ which is defined as

$$D_c(\Gamma, A) = \text{Hom}_c(\Gamma, A) / A.$$  

In this situation, the group $\text{Aut}(\Gamma)$ of automorphisms of $\Gamma$ acts freely on $\text{Hom}_c(\Gamma, A)$ commuting with the action of $A$. The quotient topological space

$$C_c(\Gamma, A) = \text{Aut}(\Gamma) \backslash \text{Hom}_c(\Gamma, A)$$

is the space of crystallographic subgroups of $A$ which are isomorphic to $\Gamma$. This space is sometimes called the *Chabauty space*. The group $A$ acts by conjugation on $C_c(\Gamma, A)$, and the action of the group $\text{Aut}(\Gamma)$ on $\text{Hom}_c(\Gamma, A)$ factorizes to an action of the outer automorphism group $\text{Out}(\Gamma) = \text{Aut}(\Gamma) / \text{Inn}(\Gamma)$ on $D_c(\Gamma, A)$. The quotient space

$$M_c(\Gamma, A) = \text{Out}(\Gamma) \backslash D_c(\Gamma, A)$$

is called the *moduli space*. The situation is described in the following commutative diagram of maps and spaces:

Basic problems in the deformation theory for $\Gamma$ are to understand the topology and geometry of the spaces involved in this diagram.

We restrict ourselves here to the case that $\Gamma$ is a virtually polycyclic group. Recall from chapter [1] that then the affine crystallographic Weil space $\text{Hom}_c(\Gamma, A)$ carries a Hausdorff-topology, and also a Zariski-topology which is induced from $\text{Hom}(\Gamma, A)$. This is the essence of our Theorem [1.41]. The next step is to learn more about the remaining spaces. Then the following interesting picture emerges:

**Algebraic and arithmetic nature of deformation spaces** The deformation space arises as a quotient of the real points of the $\mathbb{Q}$-defined algebraic variety $\text{Hom}_c(\Gamma, A)$ by a compatible $\mathbb{Q}$-defined algebraic action of $A$. The moduli space $M_c(\Gamma, A)$ arises as a quotient of the deformation space by a compatible $\mathbb{Q}$-defined algebraic action of the *arithmetic group* [1] $\text{Out}(\Gamma)$. In fact, as is proved recently in [13], the outer automorphism group of a virtually polycyclic group is an arithmetic group.

\(^3\text{see } [72] \text{ for a discussion of the notion of arithmetic group}\)
Moreover, the action of $\text{Aut}(\Gamma)$ on $\text{Hom}_c(\Gamma, A)$ extends to an algebraic $\mathbb{Q}$-defined action of the algebraic automorphism group $\text{Aut}_a(H_\Gamma)$, and this action factors over the algebraic inner group $\text{Out}_a(H_\Gamma) = \text{Aut}_a(H_\Gamma)/\text{Inn}(H_\Gamma)$ to an action on $\mathcal{D}_c(\Gamma, A)$, which extends the $\text{Out}(\Gamma)$-action on $\mathcal{D}_c(\Gamma, A)$. These facts are announced in [11]. In chapter 3 of this article we will give a proof for (virtually) f.t.n.-groups $\Gamma$. (The proof for the general fact needs more of the machinery which relates the groups $\text{Aut}(\Gamma)$ and $\text{Aut}_a(H_\Gamma)$, as is developed in [13, 15].)

**Topology of deformation spaces**  The deformation space, as well as the moduli space have nice topological properties in some well known geometric situations. For example they are Hausdorff-manifolds in some cases. This is in particular true for the deformation spaces of constant curvature Riemannian metrics, and the related Teichmüller theory of complex analytic structures on surfaces, see for example [66]. Such properties were suspected to be no longer true for the deformation spaces of affine crystallographic groups, see [30].

To the contrary it was shown in [9] that the deformation space $\mathcal{D}_c(\mathbb{Z}^2, \text{Aff}(\mathbb{R}^2))$ is a Hausdorff-space, in fact, homeomorphic to $\mathbb{R}^2$. Whereas the moduli space $\mathcal{M}_c(\mathbb{Z}^2, \mathbb{R}^2) = \text{GL}_2(\mathbb{Z}) \setminus \mathbb{R}^2$ is highly non Hausdorff.

Note that via the holonomy map the space $\mathcal{D}_c(\mathbb{Z}^2, \text{Aff}(\mathbb{R}^2))$ is homeomorphic to the deformation space of complete affine structures on the two torus $T^2$. A geometric interpretation of the coordinates of $\mathcal{D}_c(\mathbb{Z}^2, \text{Aff}(\mathbb{R}^2))$ in terms of periods of developing maps for affine structures on $T^2$ is given in [11].

This example also nicely illustrates the arithmetic and algebraic nature of the deformation space. For an interpretation of rational points in $\mathbb{R}^2$ in terms of geometric properties of the corresponding affine structures on $T^2$, see [9]. It is shown there that rationality of an affine structure is equivalent to the existence of closed geodesics.

In this chapter we concentrate our effort to a further study of the deformation spaces $\mathcal{D}_c(\Gamma, A)$. We will show that in some cases the deformation spaces for affine crystallographic groups are in fact Hausdorff and homeomorphic to a semi-algebraic set. Let us mention first the following basic result on the separation properties of deformation spaces:

**Theorem 2.1** Let $\Gamma$ be a virtually polycyclic group, and $A \leq \text{Aff}(V)$ a Zariski-closed subgroup. Then the deformation space $\mathcal{D}_c(\Gamma, A)$ is a $T_1$-topological space$^4$ with finitely many connected components.

**Proof.** By Corollary [1.43] the space $\text{Hom}_c(\Gamma, A)$ has only finitely many components. Therefore, the same holds for the quotient space $\mathcal{D}_c(\Gamma, A)$.

To see that points in $\mathcal{D}_c(\Gamma, A)$ are closed, we shall need some of the machinery, which will be developed in the sequel. We can argue as follows: By the realisation theorem, Theorem [3.49] it is enough to prove separation properties for deformation spaces of f.t.n.-groups. If $\Gamma$ is a f.t.n.-group then, by Proposition [3.63] the deformation space $\mathcal{D}_c(\Gamma, A)$ may be represented as the quotient space of a real algebraic variety by a unipotent action of the real Malcev hull $U_\Gamma$. By the Rosenlicht theorem [67] every orbit of $U_\Gamma$ is closed. Therefore, the points in $\mathcal{D}_c(\Gamma, A)$ are closed.  

$^4$meaning that every point is closed
Geometric notions for deformation spaces

Geometric notions are introduced by the natural group actions on our spaces. Let \( \rho \in \text{Hom}_c(\Gamma, A) \) be a crystallographic homomorphism. An important concept concerns the dimension of the deformation space.

Definition 2.2 The action \( \rho \) is called (locally) \( A \)-rigid if the orbit \( A\rho \) is an open set in \( \text{Hom}_c(\Gamma, A) \).

Local rigidity of \( \rho \) is equivalent to the fact that the point \( [\rho] \) constitutes a connected component of the deformation space \( D_c(\Gamma, A) \). The action \( \rho \) is rigid if the deformation space is a point. It is well known that rigidity occurs for certain geometric structures, for example hyperbolic structures in dimension \( n > 2 \). We expect that local rigidity fails for most affine crystallographic actions.

Another important notion is

Definition 2.3 The deformation space \( D_c(\Gamma, A) \) is called convex if every finite subgroup of \( \text{Out}(\Gamma) \) has a fixed point. A special case occurs if the group \( \text{Out}(\Gamma) \) has fixed points. Then we will call \( D_c(\Gamma, A) \) fixed-pointed.

The convexity of the deformation space of \( \Gamma \) is in particular important because of the role it plays in the realization problem for finite extensions, see chapter 3. In Theorem 2.13 below we show that natural classes of f.t.n.-groups, and as a consequence many f.t.n.-by finite and torsionfree polycyclic groups have fixed-pointed or convex deformation spaces.

2.2 Models for deformation spaces

The purpose of this section is to provide some examples of deformation spaces for affine crystallographic groups which are Hausdorff topological spaces. We first give some details on the structure of deformation spaces for tori. The results of chapter 3 allow then to construct more examples of deformation spaces which are Hausdorff. In particular we show that the deformation spaces of complete affine structures of certain three-manifolds are Hausdorff spaces.

Deformation spaces for affine tori

In [10] it was proved that the deformation spaces \( D_c(\mathbb{Z}^n, \text{Aff}(V)) \) of affine tori are homeomorphic to a semi-algebraic set in some Euclidean space. Here we want to study more closely these deformation spaces and the structure of the algebraic varieties associated to them.

Theorem 1.41 asserts that the space \( \text{Hom}_c(\mathbb{Z}^n, A) \) has a natural structure as a real algebraic variety. Here we want to describe the algebraic variety \( \text{Hom}_c(\mathbb{Z}^n, A) \), that is, the space of crystallographic homomorphisms of type \( A \), and the associated deformation space \( D_c(\mathbb{Z}^n, A) \) in more detail. A basic result on these deformation spaces is:

Theorem 2.4 The deformation space \( D_c(\mathbb{Z}^n, A) \) is homeomorphic to a semi-algebraic set, and in particular \( D_c(\mathbb{Z}^n, A) \) is a Hausdorff space.

Proof. The proof of [10] Corollary 2.9 for \( A = \text{Aff}(V) \) carries over almost verbatim to a more general \( A \) in the case that \( \text{GL}_A = A \cap \text{GL}(V) \) is a real reductive group. Let us briefly recall the argument. Since \( \mathbb{Z}^n \) is abelian the centralizer of every crystallographic action is a simply transitive subgroup of \( A \), and \( D_c(\mathbb{Z}^n, A) \) is
homeomorphic to the orbit space $\text{Hom}_c(\mathbb{Z}^n, A)/\text{GL}_A$. It follows from Lemma 3.66 that $\text{GL}_A$ acts freely on $\text{Hom}_c(\mathbb{Z}^n, A)$, and in particular all orbits are closed. It is known that the space of closed orbits of the reductive group $\text{GL}_A$ is homeomorphic to a semi-algebraic set. For a more general $A$ the theorem will follow from Proposition 2.6 below.

Using a fixed choice of basis we embed $\mathbb{Z}^n$ as a lattice in $V$. As a real algebraic group, $V$ identifies then with the reductive automorphism group of $\mathbb{Z}^n$. As shown in section 3.6 the automorphism group of $V$, that is the group $\text{GL}(V)$, acts freely on the space $\text{Hom}_c(\mathbb{Z}^n, A)$. For $g \in \text{GL}(V)$ the action is given by

$$\rho \mapsto \tilde{\rho} g^{-1} j ,$$

where $\tilde{\rho} : V \to A$ is the unique simply transitive representation of $V$ which extends $\rho$, and $j : \mathbb{Z}^n \to V$ is the fixed embedding of $\mathbb{Z}^n$ in $V$. This action of $\text{GL}(V)$ commutes with the conjugation action of $A$. The quotient space

$$G_{st}(V, A) = \text{GL}(V) \setminus \text{Hom}_c(\mathbb{Z}^n, A)$$

is the space of simply transitive abelian subgroups of $A$. We describe the structure of the algebraic variety $\text{Hom}_c(\mathbb{Z}^n, A)$ as follows:

**Proposition 2.5** The space $G_{st}(V, A)$ is homeomorphic to a real algebraic variety and there is an isomorphism of algebraic varieties

$$\text{Hom}_c(\Gamma, A) = \text{GL}(V) \times G_{st}(V, A) ,$$

such that the action of $\text{GL}(V)$ on $\text{Hom}_c(\Gamma, A)$ is given by right multiplication on the first factor. The action of $\text{GL}_A$ by conjugation on $\text{Hom}_c(\Gamma, A)$ is given by its action on $G_{st}(V, A)$ and by left multiplication on $\text{GL}(V)$.

**Proof.** Each $\rho \in \text{Hom}_c(\mathbb{Z}^n, A)$ extends to a homomorphism $U_{\mathbb{Z}^n} = V \to A$. The translational components of its derivative define a linear isomorphism $\tilde{t}(\rho) : V \to V$, since $V$ acts simply transitively. Also it is easy to see that the $(\text{GL}(V)$- (change of basis) action on $\text{Hom}_c(\mathbb{Z}^n, A)$ corresponds to (transposed) right multiplication on $\text{GL}(V)$, i.e., $\tilde{t}(\rho g) = \tilde{t}(\rho)(g^{-1})^t$. In particular, each element of $G_{st}(V, A)$ has a unique representative $\rho_{id} \in \text{Hom}_c(\mathbb{Z}^n, A)$ which satisfies $\tilde{t}(\rho_{id}) = id_V$, and this representative is computed by the formula $\rho_{id} = \rho \tilde{t}(\rho)^{-1}$. Therefore $G_{st}(V, A)$ identifies with a Zariski-closed subspace $\text{Hom}_{id}(\mathbb{Z}^n, A)$ of $\text{Hom}_c(\mathbb{Z}^n, A)$, and the natural map

$$\text{GL}(V) \times \text{Hom}_{id}(\mathbb{Z}^n, A) \to \text{Hom}_c(\mathbb{Z}^n, A)$$

given by $(g, \rho) \mapsto \rho g$ is an algebraic isomorphism. The equivariance statements are easy to verify. Hence the proposition follows.

We let $\text{GL}_A$ act on $\text{GL}(V)$ by left multiplication and put

$$X_A = \text{GL}_A \setminus \text{GL}(V) .$$

The following result reveals the structure of the deformation spaces $D_c(\mathbb{Z}^n, A)$ and the nature of the $\text{GL}(V)$-action on $D_c(\mathbb{Z}^n, A)$ more precisely:

**Proposition 2.6** Let $F = G_{st}(V, A)$ be the algebraic variety of simply transitive abelian subgroups of $A$ with the natural action of $\text{GL}(A)$. Then

$$D_c(\mathbb{Z}^n, A) = \text{GL}(V) \times_{\text{GL}_A} F$$

is homeomorphic to a fiber product, and in particular a bundle over the homogeneous space $X_A$ with fiber $F$. The $\text{GL}(V)$-action on $D_c(\mathbb{Z}^n, A)$ corresponds to the natural $\text{GL}(V)$-action on the fiber product which is induced by right multiplication of $\text{GL}(V)$ on itself.

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Proof. We already remarked in the proof of Theorem 2.4 that $D_c(\mathbb{Z}^n, A)$ is homeomorphic to the quotient by $\text{GL}_A$. Therefore the proposition follows from Proposition 2.6. □

As a consequence we see that $D_c(\mathbb{Z}^n, \text{Aff} (V))$ is actually homeomorphic to a real algebraic variety.

**Corollary 2.7** The deformation space $D_c(\mathbb{Z}^n, \text{Aff} (V))$ is homeomorphic to the real algebraic variety $G_{st}(V, \text{Aff} (V))$.

**Proof.** The group $\text{GL}_{\text{Aff}(V)}(V) = \text{GL}(V)$ acts by conjugation on $\text{Hom}_c(\mathbb{Z}^n, \text{Aff} (V))$, so that $t(\rho g) = gt(\rho)$. Therefore each orbit intersects $\text{Hom}_{id}(\mathbb{Z}^n, \text{Aff} (V))$ precisely once. The corollary now follows from Proposition 2.6. □

**Example 2.2.1** The deformation space $D_c(\mathbb{Z}^2, \text{Aff}(\mathbb{R}^2))$ is homeomorphic to $\mathbb{R}^2$ and the actions of $\text{GL}(\mathbb{R}^2)$ and $\text{GL}_2(\mathbb{Z})$ on $D_c(\mathbb{Z}^2, \text{Aff}(\mathbb{R}^2))$ correspond to the canonical linear actions. The fixed point $o \in D_c(\mathbb{Z}^2, \text{Aff}(\mathbb{R}^2))$ corresponds to the natural action of $\mathbb{Z}$ by translations. (See [9])

We next cover a few more special cases:

**Example 2.2.2** Let $q$ be an inner product on $V$, $O(q)$ the group of linear $q$-isometries, and $A(q)$ the group of affine isometries for $q$. It is easy to see and well known that $G_{st}(V, A(q))$ consists of the groups of translations only. (A proof may be found in [12]) It follows that $D_c(\mathbb{Z}^n, A(q))$ is homeomorphic to $X_{A(q)}$.

The symplectic case is already more interesting. Let $V$ be a vector space with a nondegenerate alternating product $\omega$. We let $\text{Sp}(\omega)$ denote the group of linear $\omega$-isometries of $V$, and $A(\omega)$ the group of affine isometries for $\omega$. Let $U_k$ denote the tautological vector bundle over the Grassmanian of $k$-dimensional isotropic subspaces of the symplectic vector space $(V, \omega)$. It is a natural $\text{Sp}(\omega)$-variety. Let $2n = \dim V$. The following is proved in [12]:

**Proposition 2.8** The variety $G_{st}(V, A(\omega))$ admits a $\text{Sp}(\omega)$-invariant stratification

$$G_{st}(V, A(\omega)) = \bigcup_{k=0}^{n} G_{st}(V, A(\omega))_k.$$  

Each stratum is an open subbundle of the third symmetric power $S^3 U_k$ of $U_k$.

Together with Proposition 2.6 we obtain a model for the space $D_c(\mathbb{Z}^n, A(\omega))$.

**Remark** As is observed in [12] the study of simply transitive abelian symplectic affine groups plays a role in the construction of flat models for special Kähler geometry, a particular geometry which arises in supersymmetric quantum field theory.

**Semi-algebraic deformation spaces**

We proved up to now that the spaces $D_c(\mathbb{Z}^n, A)$ are homeomorphic to semi algebraic sets. The results of chapter [3] allow us to further generalize this result. A finite effective extension group of $\mathbb{Z}^n$ is traditionally called a Bieberbach group.

**Corollary 2.9** Let $\Delta$ be a Bieberbach group. Then the deformation space $D_c(\Delta, A)$ is homeomorphic to a semi algebraic set and is a Hausdorff space.
This implies corresponding results on the deformation spaces of affine space forms which are finitely covered by a torus. More details and the proofs will be given in section 3.3. A further result is:

**Corollary 2.10** Let $\Gamma$ be a virtually torsion-free polycyclic group such that the unipotent shadow of $\Gamma$ is abelian. Then the deformation space $D_c(\Gamma, A)$ is homeomorphic to a semi algebraic set and is a Hausdorff space.

The proof will be given in section 3.4.

**Deformation spaces of complete affine three manifolds** The following question was raised by Bill Goldman:

Let $M$ be a closed 3-manifold which is a torus bundle over $S^1$. Is the deformation space of complete affine structures on $M$ Hausdorff?

As is proved in [27] if $M$ is a closed complete affine 3-manifold then $M$ is finitely covered by a torus bundle over $S^1$. The previous results imply a partial answer to the above question:

**Theorem 2.11** Let $M$ be a closed complete affine three manifold, and $A \leq \text{Aff}(3)$ a Zariski-closed subgroup.

i) If $M$ is finitely covered by a torus, then the deformation space $D_c(M, A)$ is Hausdorff.

ii) If $M$ is a torus bundle over $S^1$ where the attaching map is hyperbolic with a positive trace then $D_c(M, A)$ is Hausdorff.

**Proof.** The first claim follows from Corollary 2.9 above, keeping in mind the nontrivial fact that the deformation space of complete affine structures on $M$ is homeomorphic to the deformation space of crystallographic actions of the fundamental group.

For the second claim, we remark that if the condition on the bundle is satisfied, the fundamental group $\pi_1(M)$ of $M$ is a Zariski-dense lattice in the 3-dimensional Lie group $\text{Sol}$. Therefore, the unipotent shadow of $\Gamma = \pi_1(M)$ is abelian, and Corollary 2.10 applies.

**Remark** It would be further interesting to understand the separation properties of f.t.n.-groups which are lattices in the 3-dimensional Heisenberg group.

### 2.3 Convexity properties of deformation spaces

The purpose of this section is to provide some examples of deformation spaces for affine crystallographic groups which are convex in the sense of Definition 2.3.

**Strong convexity** Let $\Gamma$ be a f.t.n.-group and let $U_\Gamma$ be a real Malcev hull for $\Gamma$. The automorphism group of $U_\Gamma$ naturally acts on the space $\text{Hom}_c(\Gamma, A)$ and on the deformation space $D_c(\Gamma, A)$. Moreover, there is an induced action of $\text{Out}(U_\Gamma) = \text{Aut}(U_\Gamma)/\text{Inn}(U_\Gamma)$ on $D_c(\Gamma, A)$ which extends the natural $\text{Out}(\Gamma)$-action. This is explained in section 3.6.

We may as well interpret the fixed point properties of the $\text{Aut}(U_\Gamma)$-action as a convexity property of the deformation space $D_c(\Gamma, A)$. For a f.t.n.-group $\Gamma$, we will henceforth understand Definition 2.3 in the following stronger sense:

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Definition 2.12 Let \( \Gamma \) be a f.t.n.-group. The space \( \mathcal{D}_c(\Gamma, A) \) is called strongly convex if every reductive subgroup of \( \text{Aut}(U_\Gamma) \) has a fixed point in \( \mathcal{D}_c(\Gamma, A) \). If the group \( \text{Aut}(U_\Gamma) \) has a fixed point, \( \mathcal{D}_c(\Gamma, A) \) will be called fixed pointed.

Note, if \( \mathcal{D}_c(\Gamma, A) \) is fixed pointed in the sense of Definition 2.12 then every finite subgroup of \( \text{Out}(\Gamma) \) has a fixed point on \( \mathcal{D}_c(\Gamma, A) \). In fact, every finite subgroup \( \mu \leq \text{Out}(\Gamma) \) may be lifted to a finite subgroup of \( \text{Aut}(U_\Gamma) \).

Example 2.3.1 Let \( \Gamma = \mathbb{Z}^n \leq V \) be a lattice, so that \( U_\Gamma = V \). The space \( \mathcal{D}_c(\mathbb{Z}^n, \text{Aff}(V)) \) has a natural fixed point for all automorphisms. The fixed point for \( \text{Out}(U_{\mathbb{Z}^n}) = \text{GL}(V) \) is given by the natural action of \( \mathbb{Z}^n \) on \( V \) via translations. (Compare also Proposition 2.6)

For the moment being, we restrict our interest to the principal case \( A = \text{Aff}(V) \). Our aim here is to construct examples of groups with fixed pointed deformation spaces \( \mathcal{D}_c(\Gamma, \text{Aff}(V)) \), and more generally with convex deformation spaces. The first step is to consider f.t.n.-groups.

Conditions for strong convexity Let \( \Gamma \) be an f.t.n.-group. Recall that the nilpotency class of \( \Gamma \) is the length of a shortest central series for \( \Gamma \). (For example, if \( \Gamma \) is abelian then \( \Gamma \) is of class 1.) By abuse of language we say that \( \Gamma \) admits an invariant grading if the Lie algebra of the Malcev hull \( U_\Gamma \) has an invariant grading. (See Definition 2.22) Let us consider now the following conditions for \( \Gamma \):

i) \( \Gamma \) is of nilpotency class \( \leq 2 \),

ii) \( \Gamma \) is of nilpotency class \( \leq 3 \),

iii) \( \Gamma \) admits a positive invariant grading.

It is known from constructions given by Scheuneman that if one of the conditions i)-iii) is satisfied \( \Gamma \) admits affine crystallographic actions. This will be explained further below.

We will prove below:

Theorem 2.13 Let \( \Gamma \) be a f.t.n.-group. If \( \Gamma \) satisfies condition i) then the deformation space \( \mathcal{D}_c(\Gamma, \text{Aff}(V)) \) is fixed pointed. If one of the conditions ii) or iii) is satisfied by \( \Gamma \) then \( \mathcal{D}_c(\Gamma, \text{Aff}(V)) \) is strongly convex.

The proof of the theorem will show that it is possible to construct affine crystallographic actions which are fixed by reductive groups if the algebraic structure of the f.t.n.-group \( \Gamma \) is not too complicated. In particular if the rank of \( \Gamma \) is small, for example, rank \( \Gamma \leq 5 \), conditions ii) or iii) are satisfied. This follows from known classification results for nilpotent Lie algebras. On the other hand, it is known that for a “sufficiently generic” nilpotent group connected reductive groups of automorphisms do not exist at all, compare [31]. This leaves us with the open problem:

Does there exist an affine crystallographic f.t.n.-group \( \Gamma \) with a non-convex deformation space?

The answer to this question is in particular important in the light of the solution to the realization problems in chapter 3. For example, by Theorem 3.30, the convexity of \( \mathcal{D}_c(\Gamma, A) \) implies that \( \Gamma \) satisfies the realization property for finite extensions. In particular, every finite effective extension \( \Delta \) of \( \Gamma \) admits an affine crystallographic actions.

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6Using Theorem 3.30
action. Therefore the question above is more or less equivalent to the question:

Does there exist an affine crystallographic f.t.n.-group $\Gamma$ with a finite effective extension $\Delta$ which is not isomorphic to an affine crystallographic group?

By Theorem 3.39 the convexity of the deformation space is inherited to finite extensions of $\Gamma$. Therefore Theorem 2.13 also yields:

**Corollary 2.14** Let $\Delta$ be a finite extension of an f.t.n.-group, $\Gamma = \text{Fitt}(\Delta)$. Then the deformation space $D_c(\Delta, \text{Aff}(V))$ is fixed pointed if condition i) is satisfied by $\Gamma$. The deformation space $D_c(\Delta, \text{Aff}(V))$ is convex if the group $\Gamma$ satisfies one of ii) or iii).

We remark that the corollary in particular establishes the existence of affine crystallographic actions for $\Delta$. (See also section 3.5) By Theorem 3.54 the convexity of the deformation space is also inherited from unipotent shadows. This yields:

**Corollary 2.15** Let $\Gamma$ be a torsionfree polycyclic group, $\Theta$ a unipotent shadow for $\Gamma$. Then the deformation space $D_c(\Gamma, \text{Aff}(V))$ is fixed pointed if condition i) is satisfied by $\Theta$. $D_c(\Gamma, \text{Aff}(V))$ is convex if the shadow $\Theta$ satisfies one of ii) or iii).

The previous corollaries are an immediate consequence of Theorem 2.13 and the above mentioned inheritance results of chapter 3. Theorem 2.13 follows from the existence of certain well understood simply transitive unipotent actions constructed by Scheuneman, see below.

**Proof of Theorem 2.13** For a simply transitive subgroup $U \leq \text{Aff}(V)$ we let $\text{Aut}_{\text{Aff}(V)}(U) \leq \text{Aut}(U)$ be the image of the normalizer of $U$. (cf. section 3.6).

Let $\Gamma$ be of nilpotency class $\leq 2$, and $U_\Gamma$ its real Malcev hull. By Proposition 2.20 $U_\Gamma$ embeds into $\text{Aff}(V)$ so that the image $U \leq \text{Aff}(V)$ is a simply transitive subgroup with the property that $\text{Aut}_{\text{Aff}(V)}(U) = \text{Aut}(U)$. This representation corresponds to an element $\rho \in \text{Hom}_c(\Gamma, \text{Aff}(V))$.

Recall now that $\text{Aut}(U_\Gamma)$ acts on the deformation space $D_c(\Gamma, \text{Aff}(V))$, extending the action of $\text{Aut}(\Gamma)$. By Proposition 3.67 the subgroup $\text{Aut}_{\text{Aff}(V)}(U) \leq \text{Aut}(U)$ corresponds to a subgroup of the stabiliser $\text{Aut}(U_\Gamma)_{[\rho]}$ of $[\rho] \in D_c(\Gamma, \text{Aff}(V))$. Therefore, $\text{Aut}_{\text{Aff}(V)}(U) = \text{Aut}(U)$ implies that $\text{Aut}(U_\Gamma)_{[\rho]} = \text{Aut}(U_\Gamma)$. This means that $[\rho]$ is a fixed point for $\text{Aut}(U_\Gamma)$.

The second statement is proved similarly. If $\Gamma$ satisfies condition ii) or condition iii), Scheuneman’s examples (cf. Theorem 4.24 Theorem 2.21) provide us with crystallographic homomorphisms $\rho$ for $\Gamma$ so that $\text{Aut}_{\text{Aff}(V)}(U)$ contains a Levi subgroup of $\text{Aut}(U)$. This shows that $\text{Aut}(U_\Gamma)_{[\rho]}$ contains a Levi subgroup $L$ of $\text{Aut}(U_\Gamma)$. Therefore $[\rho]$ is a fixed point for $L$. Hence any reductive subgroup of $\text{Aut}(U_\Gamma)$ has a fixed point in the $\text{Aut}(U_\Gamma)$-orbit of $[\rho]$. In particular, this holds for finite subgroups of $\text{Aut}(U_\Gamma)$. 

**Scheuneman’s examples**

Scheuneman (cf. [69, 70]) constructed unipotent simply transitive actions on affine space to give new examples of compact complete affine manifolds. We analyze here the automorphism groups of these actions. Some of the examples we study were known already to Elie Cartan (cf. [37]). Scheuneman’s examples provide the only known general method to establish the existence of affine crystallographic actions on large classes of (reasonably well behaved) torsionfree nilpotent groups.

To present the constructions, we need to establish first the infinitesimal picture of simply transitive actions.
**Simply transitive affine actions of Lie algebras**

Let $g$ be a Lie algebra, $\phi : g \to \mathfrak{gl}(V)$ a representation of $g$ on a vector space $V$. A map $D : g \to V$ is called a derivation for $\phi$ if

$$D([X,Y]) = \phi_X DY - \phi_Y DX , \quad \text{for all } X, Y \in g.$$ 

$D$ will be called **nonsingular** if $D$ is an isomorphism of vector spaces. We call the pair $(\phi, D)$ an **affine representation** on $V$. Let $a(V) = \mathfrak{gl}(V) \oplus V$ be the Lie algebra of $\text{Aff}(V)$. Every Lie algebra homomorphism $\phi : g \to a(V)$ is, with respect to the splitting of $a(V)$, a sum $\phi = \phi + D$, such that $(\phi, D)$ is an affine representation.

Let $U$ denote a simply connected nilpotent Lie group with Lie algebra $u$. The next proposition is well known (compare [39]):

**Proposition 2.16** An affine representation $(\phi, D) : u \to a(V)$ is the differential of a unipotent simply transitive representation of $U$ if and only if $\phi$ is a representation of nilpotent linear operators and $D$ is a nonsingular derivation for $\phi$.

If $U$ is a unipotent simply transitive subgroup of $\text{Aff}(V)$, we call the Lie algebra $u \subset a(V)$ a **simply transitive subalgebra** of $a(V)$.

**Normalisers of simply transitive actions** Let $U$ be a unipotent (real) linear algebraic group with Lie algebra $u$. Recall that the automorphism group $\text{Aut}(U)$ of $U$ identifies with the group $\text{Aut}(u)$ of Lie algebra automorphisms of $u$ by taking the differential in the identity. (This identification also defines the natural algebraic group structure on $\text{Aut}(U)$.)

Let $U \leq A$ be a subgroup of the affine group. We define (see also section 3.6) the group

$$\text{Aut}_A(U) \leq \text{Aut}(U)$$

of $A$-automorphisms of $U$ as the natural image of $N_{\text{GL}_A}(U)$ in $\text{Aut}(U)$, and

$$\text{Aut}_A(u) \leq \text{Aut}(u)$$

as the corresponding group of Lie algebra automorphisms of $u$.

Note that $\text{Aut}_A(u)$ may also be computed as the image of $N_{\text{GL}_A}(u)$ in $\text{Aut}(u)$, since $u$ is a subalgebra of $a$, where $\text{GL}_A$ acts by conjugation on $a$. Now let $\varphi : u \to a$ be a simply transitive representation of $u$, $\varphi = (\phi, D)$. We put

$$\text{Aut}_A(u, \varphi) = \text{Aut}_A(\varphi(u))^{\varphi^{-1}} .$$

To compute $\text{Aut}_A(u, \varphi)$ it is convenient to identify $V$ with the underlying vector space of $u$, i.e., we consider simply transitive representations $\varphi$ of $u$ on itself. For elements $D, g \in \text{GL}(u)$ let us define $(D, g) := g^{-1} D g D^{-1}$. The group $\text{Aut}_A(u)$ may now be characterized in terms of $\varphi$ as follows:

**Proposition 2.17** Let $A \leq \text{Aff}(u)$ be a transitive subgroup. Let $\varphi : u \to a(u)$ be a a unipotent simply transitive representation of $u$ on itself. Let $g \in \text{Aut}(u)$. Then $g \in \text{Aut}_A(u, \varphi)$ if and only if

$$D g D^{-1} \in N_{\text{GL}_A}(\varphi(u))$$

and

$$(D, g) \phi_X (D, g)^{-1} = g^{-1} \phi g_X g , \quad \text{for all } X \in g .$$
Proof. Let \( h \in N_{GL_A}(\phi(u)) \). If \( \varphi = (\phi, D) \) then the conjugate representation is \( \varphi^h = (\phi^h, hD) \). The image \( \Phi_h \) of \( h \) in \( \text{Aut}_A(u, \varphi) \) is defined by the condition that
\[
\varphi \Phi_h = \varphi^h.
\]
This is equivalent to \( \phi^h = \phi \Phi_h \), and \( hD = D\Phi_h \). By the latter condition it follows that
\[
D\Phi_h D^{-1} = h \in N_{GL_A}(\phi(u)) .
\]
The proposition is immediate with the first condition. \( \square \)

This gives rise to the following

**Definition 2.18** Let \( \phi: u \to \mathfrak{gl}(u) \) be a representation of Lie algebras, and \( g \in \text{Aut}(u) \). Then \( g \) is called compatible with \( \phi \) if
\[
g^{-1} \phi gX g = \phi X, \quad \text{for all } X \in \mathfrak{g} .
\]
We remark that a particular special case is the adjoint representation of \( u \) on itself, \( \phi = \text{ad} \). By its very definition, \( \text{Aut}(u) \) is compatible with \( \text{ad} \).

**Basic examples of unipotent simply transitive actions**

We come now to a first family of examples which was known to Cartan already. These examples are fully invariant and may be seen as a generalization of the natural simply transitive representation of an abelian Lie algebra.

**Definition 2.19** We call a simply transitive representation \( \varphi \) of \( u \) invariant if \( \text{Aut}(u, \varphi) = \text{Aut}(u) \). We call \( \varphi \) invariant by the subgroup \( H \leq \text{Aut}(u) \) if
\[
H \leq \text{Aut}(u, \varphi) .
\]

**Proposition 2.20** Let \( u \) be a 2-step nilpotent Lie algebra. Then \( u \) has a natural invariant simply transitive representation.

**Proof.** We take \( \phi = \frac{1}{2} \text{ad} \), and \( D = \text{id}_u \), where \( \text{ad} \) denotes the adjoint representation. \( \square \)

This construction was generalized by Scheuneman (69) as follows:

**Proposition 2.21** Let \( u \) be a nilpotent Lie algebra, which admits a nonsingular derivation \( D \). Then \( u \) has a simply transitive affine representation \( \varphi = (\text{ad}, D) \). The representation \( \varphi \) is invariant by the centralizer of \( D \) in \( \text{Aut}(U) \).

**Proof.** Since \( \text{ad} \) is compatible with \( \text{Aut}(u) \), the centralizer
\[
C_{\text{Aut}(u)}(D) = \{ g \in \text{Aut}(u) \mid (g, D) = \text{id}_u \}
\]
is contained in \( \text{Aut}(u, \varphi) \), by Proposition 2.17. \( \square \)

**Construction of derivations** The proposition raises the particular problem to study the centralizers of certain elements in \( \text{Aut}(u) \). A semisimple element \( g \in \text{Aut}(u) \) is called expanding if all eigenvalues have absolute value \( > 1 \). We construct now examples of Lie algebras which contain expanding elements in the center of the Levi subgroups of \( \text{Aut}(u) \). (We call a maximal reductive subgroup of the linear algebraic group a Levi subgroup, cf. Theorem 3.31.)

For preparation we have to introduce a new concept.
Definition 2.22 A (positive) filtration on a Lie algebra \( \mathfrak{g} \) is a nested sequence of subspaces
\[
\mathfrak{g} = \mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \mathfrak{g}_3 \supseteq \cdots
\]
such that \([\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}\). The filtration is called invariant if it is preserved by \( \text{Aut}(\mathfrak{g}) \).

For each positive filtration on \( \mathfrak{g} \) there is an associated graded Lie algebra
\[
\mathfrak{g} = \bigoplus_{i=1}^{\infty} \mathfrak{g}_i / \mathfrak{g}_{i+1}
\]
where the Lie product on \( \mathfrak{g} \) is defined by setting, for \( x \in \mathfrak{g}_i, y \in \mathfrak{g}_j \),
\[
[x, y] = [x, y] \in \mathfrak{g}_{i+j} / \mathfrak{g}_{i+j+1}.
\]

A grading
\[
\mathfrak{g} = \bigoplus_{i=1}^{k} V_i
\]
is called invariant by a subgroup \( G \leq \text{Aut}(\mathfrak{g}) \) if every \( g \in G \) preserves the decomposition. If \( \mathfrak{g} \) has a positive grading \( \mathfrak{g} = \bigoplus_{i=1}^{k} V_i \) then the nested sequence of ideals \( \mathfrak{g}_i = \bigoplus_{i=1}^{k} V_i \) defines an associated positive filtration.

Note that every positive grading is preserved by a one parameter group \( l_\lambda \) of expanding automorphisms in \( \text{Aut}(\mathfrak{g}) \). In fact, for \( \lambda > 1 \), \( l_\lambda \) is given by
\[
l_\lambda(v) = \lambda^i v, \quad v \in V_i.
\]

Proposition 2.23 Let \( \mathfrak{u} \) be a nilpotent Lie algebra which has a positive grading with an invariant associated filtration. Let \( L \) be a Levi-subgroup of the linear algebraic group \( \text{Aut}(\mathfrak{u}) \). Then \( L \) contains a one parameter group of expanding automorphisms in \( \text{Aut}(\mathfrak{g}) \).

Proof. Since the filtration \( \mathfrak{u}_i \) of \( \mathfrak{u} \) associated with the grading is invariant, \( L \) acts on the factor spaces \( \mathfrak{u}_i / \mathfrak{u}_{i+1} \). Therefore \( L \) acts by automorphisms on the associated graded Lie algebra \( \mathfrak{g} \) such that the grading is preserved. Since \( L \) is reductive the action of \( L \) on \( \mathfrak{g} \) is also faithful.

Now the fact that \( \mathfrak{u} \) is graded with respect to the filtration \( \mathfrak{u}_i \) implies that there is an isomorphism of filtered Lie algebras \( \pi : \mathfrak{u} \rightarrow \mathfrak{g} \). Hence it follows that \( \text{Aut}(\mathfrak{u}) \) has a Levi subgroup which preserves a grading compatible which is compatible with the original filtration. Since all Levi subgroups are conjugate in \( \text{Aut}(\mathfrak{u}) \), every Levi subgroup \( L \) preserves a corresponding positive grading of \( \mathfrak{u} \). The one parameter family \( l_\lambda \) of expanding automorphisms which belongs to this positive grading commutes with \( L \). \( \Box \)

As a consequence we obtain

Theorem 2.24 Let \( \mathfrak{u} \) be a nilpotent Lie algebra which has an invariant positive grading, and let \( L \) be a Levi-subgroup of \( \text{Aut}(\mathfrak{U}) \). Then there exists an affine simply transitive representation which is invariant by \( L \).

Remark It is easily checked from certain lists of nilpotent Lie algebras that every nilpotent Lie algebra of dimension less or equal to five has an invariant positive grading.
Scheuneman representations for 3-step nilpotent Lie algebras  A little more refined construction which builds on the previous cases was given by Scheuneman in [70]. The construction is compelling because it works for every 3-step nilpotent Lie algebra.

**Theorem 2.25** Let $u$ be a 3-step nilpotent Lie algebra, and let $L$ be a Levi-subgroup of $\text{Aut}(U)$. Then there exists an affine simply transitive representation which is invariant by $L$.

**Proof.** Following [70], we consider the filtration

$$u = u_1 \supseteq u_2 \supseteq u_3 \supseteq \{0\}$$

which is given by the descending central series, i.e., $u_2 = [u, u]$, $u_3 = [u, u_2]$. We choose a compatible decomposition of $u = \bigoplus_{l=1}^{3} V_l$ as a direct sum of vector spaces, $V_3 = u_3$. We define a diagonal element $g = g_{\alpha, \beta, \gamma} \in \text{GL}(u)$ which preserves the grading by $gv = \alpha v$, for $v \in V_1$, $gv = \beta v$, for $v \in V_2$, and $gv = \gamma v$, for $v \in V_3$. Correspondingly, we define $D = D_{r,s,t}$. It is easy to see (compare [70]) that there exist $\alpha, \beta, \gamma > 0$, and $r, s, t > 0$ so that $D$ is a derivation for the representation $\phi = \text{ad} g$ of $u$ on itself. The pair $(\phi, D)$ defines then a simply transitive representation $\varphi$ of $u$. Since the filtration we chose is invariant, we may assume that the spaces $V_l$ are invariant by automorphisms of a Levi subgroup $L$ of $\text{Aut}(u)$. In this case $g$ commutes with $L$, and the representation $\phi$ is compatible with $L$ in the sense of Definition 2.18. Since also $D$ commutes with $L$, it follows that $L \leq \text{Aut}(u, \varphi)$, by Proposition 2.17.

**Remark** When applied to the 2-step nilpotent case the method recovers just the natural invariant simply transitive action.
Chapter 3

Geometric Realization Problems

3.1 Nielsen’s realization problem

Nielsen considered in [59] the following question which is now called the Nielsen realization problem. Namely, can any finite group $G$ of isotopy classes of self-homeomorphisms of a surface be realized as a finite group of self-homeomorphisms?

As it turned out, Nielsen’s question is equivalent to the following: Let $\Delta$ be a finite effective extension of the fundamental group $\pi_1(S)$ of a surface $S$. Is it true, that $\Delta$ acts discontinuously on the plane?

Geometry gives a positive answer to this question. For example, every effective finite extension of $\mathbb{Z}^2$ acts as an Euclidean crystallographic group. These are the well known so called wallpaper groups. Therefore the realization problem for Euclidean surfaces has a positive solution. For hyperbolic surfaces the problem was settled not long ago. Kerckhoff’s [38] celebrated fixed point theorem states: Any finite group of mapping classes of a closed orientable surface $M_g$ of genus $g \geq 2$ has a fixed point in the Teichmüller space of hyperbolic structures. A particular consequence of the fixed point theorem is: Let $\Delta$ be a finite effective extension of $\pi_1(M_g)$, then $\Delta$ is isomorphic to a discontinuous group of hyperbolic motions. These results give a beautiful example of the significance of geometric methods in topology.

Recall, that a homomorphism $\alpha : F \to \text{Out}(\Gamma)$ from a group $F$ to the outer automorphism group of a group $\Gamma$ is called an abstract kernel for $\Gamma$. Every group $F$ of homeomorphisms of a manifold naturally defines an abstract kernel $\alpha : F \to \pi_1(M)$. If this is the case, we say that $\alpha$ is realized by a group of homeomorphisms. The topological realization problem may be posed as follows: Let $M$ be a manifold and let $\alpha$ be a finite abstract kernel for $\pi_1(M)$. Is it possible to realize $\alpha$ by a group of homeomorphisms of $M$?

Under suitable topological conditions on the manifold $M$, for example if $M$ is aspherical, a necessary condition for $\alpha$ to be realizable as a group of homeomorphisms is the existence of a finite effective extension group $\Delta$ for $\Gamma = \pi_1(M)$ which induces $\alpha$. The group $\Delta$ is then an algebraic realization of the kernel $\alpha$. As illustrated above in the case of surfaces, geometry may help to realize $\Delta$ as properly discontinuous group of homeomorphisms of the universal cover $\tilde{M}$ of $M$. Still, geometric realization questions are interesting in their own right. An answer to such a question may show for example how well a particular geometry is adapted to the topology of a manifold.
A classic example which extends beyond the surface case is the following: Bieberbach proved in 1912 his famous theorems on the structure of the Euclidean crystallographic groups. Later Burckhardt and Zassenhaus showed that Euclidean crystallographic groups are characterized by their algebraic properties. In particular, their remark implies (see [82][§25]) a positive answer to the geometric realization problem for Euclidean crystallographic groups:

**Theorem 3.1** Let $\Gamma$ be isomorphic to an Euclidean crystallographic group. If $\Delta$ is a finite effective extension of $\Gamma$ then $\Delta$ is also isomorphic to an Euclidean crystallographic group.

Since, by Bieberbach’s second theorem, flat Euclidean manifolds are determined by their fundamental group up to affine diffeomorphism this implies:

**Theorem 3.2** Let $M$ be compact Euclidean space form, and $\alpha$ an injective abstract kernel for $\pi_1(M)$. Then the following are equivalent:

i) $\alpha$ has an algebraic realization $\Delta$,

ii) $\alpha$ can be realized by a group of affine diffeomorphisms of $M$.

### The realization problem for affine crystallographic groups

In our work we are concerned with the realization problem for affine crystallographic groups. We restrict ourselves here to the case that $\Gamma$ is a virtually nilpotent ACG.

Every affine space form $M$ with fundamental group isomorphic to $\Gamma$ is finitely covered by a nilmanifold. (A nilmanifold is diffeomorphic to the quotient of a nilpotent Lie group by a closed subgroup. $M$ itself has an infranilmanifold structure.) For certain aspherical manifolds the topological realization question was solved by Lee and Raymond (cf. [50]) with methods from the theory of Seifert fiber spaces. One particular result is:

**Theorem 3.3** Let $M$ be an infranilmanifold, $\alpha$ an abstract kernel for $\pi_1(M)$. If there exists an algebraic realization of $\alpha$ then $\alpha$ may be realized as a group of homeomorphisms of $M$.

However, the following geometric question remains: Which of the finite quotient manifolds of an affine space form $M$ are again an affine space form? Our basic result shows that if $M$ has a virtually nilpotent fundamental group then the solution to the geometric realization problem is a question about the fixed points of abstract kernels on the deformation space.

**Theorem 3.4** Let $\Lambda$ be virtually nilpotent and isomorphic to an affine crystallographic group of type $A$. Then a finite effective extension $\Delta$ of $\Lambda$ is isomorphic to an affine crystallographic group of type $A$ if and only if the kernel associated to the extension has a fixed point in the deformation space $D_c(\Lambda, A)$.

We will also consider the question for which crystallographic groups the analogue of Theorem 3.1 holds. These are precisely those affine crystallographic groups $\Gamma$ which satisfy the following realization property.

**Definition 3.5** We say that $\Gamma$ has the affine realization property if every finite effective extension $\Delta$ of $\Gamma$ is isomorphic to an affine crystallographic group.

---

1Most of our results here generalise to virtually polycyclic ACGs in the context of infrasolv-manifolds. See [12][95] for some of the general constructions.
We remark that if the fundamental group $\pi_1(M)$ of an affine space form $M$ satisfies the realization property then, by some standard reasoning\footnote{using that compact complete affine manifolds with isomorphic fundamental group are polynomially diffeomorphic, see \cite{27}}, the following generalization of Theorem 3.2 holds:

**Theorem 3.6** Let $M$ be a compact affine space form with a virtually nilpotent fundamental group $\pi_1(M)$, and $\alpha$ an injective abstract kernel for $\pi_1(M)$. If $\pi_1(M)$ has the realization property then the following are equivalent:

i) $\alpha$ has an algebraic realization $\Delta$.

ii) $\alpha$ can be realized by a group of polynomial diffeomorphisms of $M$.

Theorem \cite{34} implies that $\Gamma$ satisfies the realization property if the deformation space $D_c(\Gamma, \text{Aff}(V))$ is convex in the sense of Definition 2.3. In particular, compare Corollary \cite{35,37} the examples of chapter \cite{2} show that some natural classes of virtually nilpotent crystallographic groups satisfy the realization property.

**Corollary 3.7** Let $\Gamma$ be a virtually nilpotent affine crystallographic group. If the deformation space $D_c(\Gamma, \text{Aff}(V))$ is convex then $\Gamma$ has the realization property.

This fact should provide us with enough motivation to study the deformation spaces for affine space forms.

### 3.2 The Realization of finite extensions for f.t.n.-groups

Let $\Gamma$ be a group which is isomorphic to an ACG of type $A$, and let $\beta : F \rightarrow \text{Out}(\Gamma)$ be an abstract kernel for $\Gamma$. As a subgroup of $\text{Out}(\Gamma)$, $\beta(F)$ acts on the deformation space $D_c(\Gamma, A)$. Let us assume that there exists a finite normal extension $\Gamma \leq_f \Delta$ which realizes the abstract kernel $\beta$. This means that $\beta$ coincides with the natural homomorphism $F = \Delta/\Gamma \rightarrow \text{Out}(\Gamma)$ which is associated to the extension. The following is a fundamental observation:

**Proposition 3.8** Let $\beta : F \rightarrow \text{Out}(\Gamma)$ be the kernel associated to the normal extension $\Gamma \leq_f \Delta$. Assume that there exists a crystallographic homomorphism $\rho_{\Delta} \in \text{Hom}_c(\Delta, A)$. Let $\rho$ denote the restriction of $\rho_{\Delta}$ to $\Gamma$. Then $\rho$ is in $\text{Hom}_c(\Gamma, A)$, and $[\rho] \in D_c(\Gamma, A)$ is a fixed point for the action of $F$ on $D_c(\Gamma, A)$.

**Proof.** Clearly, the restriction of $\rho_{\Delta}$ to $\Gamma$ remains crystallographic. For $g \in F$, we have $\beta(g)[\rho] = [\rho^{\phi_g}]$, where $\phi_g \in \text{Aut}(\Gamma)$ represents $\beta(g) \in \text{Out}(\Gamma)$. But, since $\Delta$ realizes the kernel $\beta$, there exists $\delta \in \Delta$ such that $\beta(g)$ is represented by conjugation with $\delta$. Therefore, we may assume that $\phi_g(\gamma) = \delta \gamma \delta^{-1}$, for all $\gamma \in \Gamma$. It follows that $\rho^{\phi_g}(\gamma) = \rho_{\Delta}(\delta)\rho(\gamma)\rho_{\Delta}(\delta)^{-1}$. Since $\rho_{\Delta}(\delta) \in A$, we conclude $\beta(g)[\rho] = [\rho^{\phi_g}] = [\rho]$. \hfill $\square$

From now on let us assume that $\Gamma$ is a f.t.n.-group.

**Definition 3.9** A normal extension group $\Delta$ of $\Gamma$ is called\footnote{effectively} effective if the associated kernel $\beta : \Delta/\Gamma \rightarrow \text{Out}(\Gamma)$ is an injective homomorphism.

We are going to show that the converse of Proposition \cite{38} holds for all finite effective extensions $\Delta$ of $\Gamma$. Namely, the existence of fixed points for $\beta$ on the deformation space of $\Gamma$ is the only obstruction to realize $\Delta$ as an affine crystallographic group.
Theorem 3.10 Let $\Gamma$ be a f.t.n.-group which is isomorphic to an affine crystallographic group of type $A$, and let $\Gamma \leq_f \Delta$ be an effective finite extension with associated abstract kernel $\beta : F \longrightarrow \text{Out}(\Gamma)$. Then $\Delta$ may be realized as an affine crystallographic group of type $A$ if and only if the kernel $\beta$ has a fixed point in $D_c(\Gamma, A)$.

Remark In the course of the proof of Theorem 3.10 we will show a stronger statement: Every $\rho \in \text{Hom}_c(\Gamma, A)$ which is a fixed point for $\beta$ may be extended to a homomorphism $\rho_\Delta \in \text{Hom}_c(\Delta, A)$.

Theorem 3.4 follows from Theorem 3.10. The proof of Theorem 3.4 will be given below in section 3.3. The purpose of the remainder of this section is to prove Theorem 3.10. We will proceed as follows:

The first step is to consider the related problem of realizing a finite extension $\Delta$ of $\Gamma$ as a Zariski dense subgroup of a linear algebraic group. The solution to this problem (Proposition 3.15) leads us to the construction of an algebraic hull for $\Delta$ which satisfies functorial properties with respect to the representations of $\Delta$. As one application of the construction we deduce a splitting result for the extension $\Gamma \leq \Delta$ (Proposition 3.24). Another application is a certain characterization of the finite effective nilpotent extensions of $\Gamma$. (Corollary 3.18)

The second step is the proof of the affine crystallographic realization for the finite effective extensions $\Delta$ of $\Gamma$. This breaks into two distinct parts: The case that $\Delta$ itself is a f.t.n.-group, and the case that the extension $\Gamma \leq \Delta$ splits.

We will use frequently certain results from the theory of f.t.n.-groups. For a general reference on the relevant aspects of the theory, see [24, Chapter 2], [65, Chapter II], and in particular the book [71]. We start with some preparatory material.

The algebraic hull of an extension

Automorphisms and the Malcev hull Let $\Gamma$ be a f.t.n.-group. It is known that there exists a $\mathbb{Q}$-defined unipotent algebraic group $U$ (called the Malcev completion of $\Gamma$), and an embedding $j : \Gamma \rightarrow U_\mathbb{Q}$ such that $\Gamma$ is a Zariski-dense subgroup in $U$. Every automorphism of $\Gamma$ extends to a unique $\mathbb{Q}$-defined automorphism of $U$. In fact, the automorphism group $\text{Aut}(U)$ of $U$ is a $\mathbb{Q}$-defined linear algebraic group, in such a way that $\text{Aut}(U)_\mathbb{Q} = \text{Aut}(U)$, and the extension defines an injective homomorphism

$$
\epsilon_j : \text{Aut}(\Gamma) \longrightarrow \text{Aut}(U)_\mathbb{Q}
$$

which satisfies $\epsilon_j(\phi) j = j \phi$, for all $\phi \in \text{Aut}(\Gamma)$. By factorization there exists a homomorphism $o_j : \text{Aut}(\Gamma) \rightarrow \text{Out}(U)$ such that the diagram

$$
\begin{align*}
\text{Aut}(\Gamma) & \xrightarrow{\epsilon_j} \text{Aut}(U) \\
& \downarrow \\
\text{Out}(\Gamma) & \xrightarrow{o_j} \text{Out}(U)
\end{align*}
$$

is commutative. Note that $o_j$ is, in general, not injective. The preimage of its kernel is $\epsilon_j^{-1}(\text{Inn}(U))$. Let us therefore denote

$$
\text{Inn}_\mathbb{Q}(\Gamma) = \epsilon_j^{-1}(\text{Inn}(U)) \quad \text{and} \quad \text{Out}_\mathbb{Q}(\Gamma) = \text{Aut}(\Gamma)/\text{Inn}_\mathbb{Q}(\Gamma) .
$$

Definition 3.11 Let $\beta : F \rightarrow \text{Out}(\Gamma)$ be an abstract kernel. $\beta$ is called radicably effective if $\beta$ induces an embedding $F \hookrightarrow \text{Out}_\mathbb{Q}(\Gamma)$. $\beta$ is called radicably trivial if $\beta$ induces the trivial homomorphism to $\text{Out}_\mathbb{Q}(\Gamma)$.
Let \( \Gamma \leq \Delta \) be a finite extension with \( \Gamma \) normal in \( \Delta \), and let \( c : \Delta \to \text{Aut}(\Gamma) \) be the homomorphism which is induced by conjugation. We put \( \Gamma^r = c^{-1}(\text{Im} q(\Gamma)) \). So \( \Gamma \leq \Delta \) is radicably effective if and only if \( \Gamma^r = \Gamma \). We remark that if the extension is finite and effective \( \Gamma^r \) is a nilpotent normal subgroup in \( \Delta \). In fact, it follows (Corollary [3.17]) that \( \Gamma^r \) is the Fitting subgroup of \( \Delta \).

Later we will need the following lemma:

**Lemma 3.12** Let \( \mu \leq \text{Aut}(U_\mathbb{Q}) \) be a finite subgroup, and \( \Gamma \leq U_\mathbb{Q} \) a finitely generated subgroup. Then there exists \( \Gamma_f \leq U_\mathbb{Q} \) such that \( \Gamma \leq_f \Gamma_f \), and \( \Gamma_f \) is normalized by \( \mu \).

**Proof.** Let \( u \) be the Lie algebra of \( U \). \( u \) is defined over \( \mathbb{Q} \), and the exponential map \( \exp : u \to U \) identifies \( u_\mathbb{Q} \) with \( U_\mathbb{Q} \). The group \( \mu \) acts as a subgroup of automorphisms in \( \text{GL}(u_\mathbb{Q}) \) on \( u_\mathbb{Q} \), and to show that \( \mu \) normalizes \( \Gamma \leq U_\mathbb{Q} \) is equivalent to show that \( \mu \) normalizes the set \( \log \Gamma \subseteq u_\mathbb{Q} \). Replacing, if necessary, \( \Gamma \) with a finite extension group, we may assume ([71][6 B, Theorem 3]) that \( \Gamma \) is a lattice subgroup, such that \( \Lambda = \log \Gamma = \mathbb{Z} \log \Gamma \) is a full lattice in \( u_\mathbb{Q} \). Since \( \mu \leq \text{GL}(u_\mathbb{Q}) \) is finite there exists an integer \( m \) such that, for all \( g \in \mu \), \( g \frac{1}{m} \Lambda \subseteq \frac{1}{m} \Lambda \). Let \( \Gamma^\mathbb{Q} \) be the group which is generated by the set \( \{ \exp u \mid u \in \frac{1}{m} \Lambda \} \). Now \( \Gamma_f = \Gamma^\mathbb{Q} \) is a finite extension of \( \Gamma \), and is normalized by \( \mu \).

The Malcev extension functor As before let the \( \mathbb{Q} \)-defined algebraic group \( U \) denote the Malcev completion of \( \Gamma \). We need to consider the extension functor (Malcev-rigidity) from the unipotent representations of \( \Gamma \) to representations of \( U \). Remark, that this is just a special case of Proposition [1.4]

**Proposition 3.13** Let \( \rho : \Gamma \to H \) be a homomorphism of \( \Gamma \) to a unipotent \( \mathbb{Q} \)-defined linear algebraic group \( H \). Then there exists a unique morphism of algebraic groups

\[
\rho_U : U \to H
\]

which extends \( \rho \). If \( \rho(\Gamma) \leq H_k \), where \( k \) is a subfield of \( \mathbb{C} \), then \( \rho_U \) is defined over \( k \).

**Proof.** Consider the subgroup \( D = \{ (\gamma, \rho(\gamma)) \mid \gamma \in \Gamma \} \) of the product \( U \times H \). Let \( \pi_1, \pi_2 \) denote the projection morphisms on the factors of the product. Let \( D \) be the Zariski-closure of \( D \). By [63][Theorem 2.10] we have \( \dim D = \text{rank} \Gamma = \dim U \), and it follows that \( \pi_1 : D \to U \) is an isomorphism. Hence \( \rho_U = \pi_2 \circ \pi_1^{-1} \) is the unique extension. If \( \rho(\Gamma) \leq H_k \) then \( D \) is \( k \)-defined, and hence also the morphism \( \rho_U \).

At some point later we consider the spaces \( \text{Hom}(\Gamma, H_{\mathbb{R}}) \) as topological spaces equipped with the compact-open topology. (Compare section [1.2]) Now let \( \Gamma^* \) be a f.t.n.-group which is a finite extension of \( \Gamma \). There exists a unique embedding of \( \Gamma^* \) into \( U_\mathbb{Q} \) which is the identity on \( \Gamma \), see [71][86A]. We note:

**Corollary 3.14** Let \( H \) be a unipotent \( \mathbb{Q} \)-defined linear algebraic group. Then every \( \rho \in \text{Hom}(\Gamma, H_{\mathbb{R}}) \) has a unique extension \( \rho_{\Gamma^*} \in \text{Hom}(\Gamma^*, H_{\mathbb{R}}) \). The extension functor induces a homeomorphism

\[
e : \text{Hom}(\Gamma, H_{\mathbb{R}}) \to \text{Hom}(\Gamma^*, H_{\mathbb{R}}) .
\]

**Proof.** We use the embedding \( \Gamma^* \hookrightarrow U_\mathbb{Q} \). Malcev rigidity implies that every \( \rho \in \text{Hom}(\Gamma, H_{\mathbb{R}}) \) extends to a homomorphism \( \rho_U : U_{\mathbb{R}} \to H_{\mathbb{R}} \). The restriction of \( \rho_U \) to \( \Gamma^* \) is the unique extension \( e(\rho) = \rho_{\Gamma^*} \in \text{Hom}(\Gamma^*, H_{\mathbb{R}}) \). The map \( e \) is then clearly a bijection.
Let $G$ be a discrete group. For any $V \subset H^\mathbb{R}$ which is open in the Hausdorff-topology on $H^\mathbb{R}$, and any $\gamma \in G$ we put
\[ U_G(\gamma, V) = \{ \rho \in \text{Hom}(G, H^\mathbb{R}) \mid \rho(\gamma) \in V \} . \]
The compact open topology on $\text{Hom}(G, H^\mathbb{R})$ is then generated by all sets $U_G(\gamma, V)$. It is immediate that the restriction map $e^{-1}$ is continuous. To prove that $e$ is continuous we take the following approach: We can assume that $\Gamma$ is a subgroup of $\text{Hom}(G, H^\mathbb{R})$, is a homomorphism on $G$, we may assume that $\Gamma$ is contained in the subgroup $U_{\mathbb{Z}}$ of integer points of $U$, and any $\gamma \in \Gamma$. We put $V^\gamma = \{ \rho \mid \rho(\gamma) \in V \}$, $\gamma \in \Gamma$. Now
\[ e^{-1}(U_{\mathbb{Z}}(\gamma, V)) = \{ \rho \mid \rho(\gamma) \in V^\gamma \} = U_{\mathbb{Z}}(\gamma, V^\gamma) . \]
Hence, $e$ is continuous.

\textbf{The algebraic hull of an effective extension} Let $\Gamma$ be a f.t.n.-group. Then $\Gamma \leq U_{\mathbb{Q}}$, where $U$ is the Malcev completion of $\Gamma$. In fact, we may assume that $\Gamma \leq U_{\mathbb{Z}}$, i.e., $\Gamma$ is contained in the subgroup $U_{\mathbb{Z}}$ of integer points of $U$. Now let $\Delta$ be a finite normal extension group of $\Gamma$, i.e., $\Gamma$ is a normal subgroup $\Delta$. Since the extension $\Gamma \leq \Delta$ is effective, we may further refine Proposition 6.15. Let $\text{Fitt}(\Delta)$ be the Fitting subgroup of $\Delta$, i.e., its maximal nilpotent normal subgroup. We show that $\text{Fitt}(\Delta)$ is a f.t.n.-group and embeds into $U_{\mathbb{Q}}$.

\textbf{Proposition 3.15} There exists a $\mathbb{Q}$-defined linear algebraic group $I(U, \Delta)$ with $U$ its component of identity, and an embedding $\psi : \Delta \rightarrow I(U, \Delta)_{\mathbb{Q}}$ which is the identity on $\Gamma$, such that $I(U, \Delta) = \psi(\Delta)U$ and $\psi(\Delta) \cap U = \Gamma$.

\textbf{Proof.} We may assume that $\Gamma \leq U_{\mathbb{Z}}$, i.e., $\Gamma$ is contained in the subgroup $U_{\mathbb{Z}}$ of integer points of $U$. Let $\Delta = \Gamma r_1 \cup \cdots \cup \Gamma r_m$ be a decomposition of $\Delta$ into left cosets. By Malcev-rigidity, conjugation with $r_i$ on $\Gamma$ extends to $\mathbb{Q}$-defined rational homomorphisms $f_i$ of $U$. A straightforward application of the construction used to prove Proposition 2.2 in [34] implies the result.

In the case that the extension $\Gamma \leq \Delta$ is effective we may further refine Proposition 6.15. Let $\text{Fitt}(\Delta)$ be the Fitting subgroup of $\Delta$, i.e., its maximal nilpotent normal subgroup. We show that $\text{Fitt}(\Delta)$ is a f.t.n.-group and embeds into $U_{\mathbb{Q}}$.

\textbf{Proposition 3.16} If the finite extension $\Gamma \leq \Delta$ is effective then there exists a $\mathbb{Q}$-defined linear algebraic group $I^*(U, \Delta)$ with $U$ its component of identity, and an embedding $\psi : \Delta \rightarrow I^*(U, \Delta)_{\mathbb{Q}}$ which is the identity on $\Gamma$, such that $I^*(U, \Delta) = \psi(\Delta)U$ and $\psi(\Delta) \cap U = \psi(\text{Fitt}(\Delta))$. Moreover the centralizer of $U$ in $I^*(U, \Delta)$ is contained in $U$.

\textbf{Proof.} By Proposition 3.15 we may assume that $\Delta$ is a subgroup of $G_{\mathbb{Q}}$, where $G$ is a linear algebraic group with $u(G) = U$, and $\Gamma \leq U$. Let $\Gamma^* = \text{Fitt}(\Delta)$. Since $\Gamma$ is a subgroup of finite index in $\Gamma^*$, the group $\Gamma^*_s = \{ \gamma_s \mid \gamma \in \Gamma^* \}$ is finite. Since $\Gamma^*$ is normal in $\Delta$, $\Gamma^*_s$ is normalized by $\Gamma$. The centralizer of $\Gamma^*_s$ is a Zariski-closed subgroup of $G$ which contains a finite index subgroup of $\Gamma$. Therefore the centralizer $C_G(\Gamma^*_s)$ contains $U$, in particular $\Gamma^*_s$ centralizes $\Gamma$. Consider the homomorphism $\psi_u : \Gamma^* \rightarrow U_{\mathbb{Q}}$ given by $\gamma \mapsto \gamma_u$. Since the extension $\Gamma \leq \Gamma^*$ is effective, the homomorphism $\psi_u$ is injective. Therefore $\Gamma^*$ is a f.t.n.-group, and embeds as a subgroup of $U_{\mathbb{Q}}$ containing $\Gamma$. We obtain $I^*(U, \Delta)$ by applying Proposition 3.15 to the extension $\Gamma^* \leq \Delta$.

The centralizer of $U$ in $I^*(U, \Delta)$ splits as a direct product $C_{I^*(U, \Delta)}(U) = H \times Z(U)$ where $H$ is a finite group of semisimple elements. It follows that the set $X_\Delta = \{ \gamma \in \Delta \mid \gamma_s \in H \}$ is a normal subgroup of $\Delta$, and the map $\psi_u : \gamma \mapsto \gamma_u$ is a homomorphism on $X_\Delta$. Since the extension $\Gamma \leq \Delta$ is effective and $\Gamma \leq U$, $\psi_u$.
is injective on $X_{\Delta}$. Therefore $X_{\Delta}$ is nilpotent and hence contained in $\text{Fitt}(\Delta)$. By construction $\text{Fitt}(\Delta)$ is unipotent in $I^*(U, \Delta)$ and hence no $\gamma_s \neq 1$ centralizes $U$. Since $I^*(U, \Delta) = \Delta U$, $H$ is trivial. \hfill \Box

Let $c : \Delta \to \text{Aut}(\Gamma)$ be the homomorphism which is induced by conjugation, and $\Gamma^r = c^{-1}(\text{Inn}_Q(\Gamma))$. Then there are finite extensions

$$\Gamma \leq \Gamma^r \leq \Delta.$$ 

Recall that the extension $\Gamma \leq \Delta$ is called radicably effective (see Definition 3.11) if and only if $\Gamma^r = \Gamma$.

**Corollary 3.17** Let $\Gamma$ be a f.t.n.-group and $\Gamma \leq \Delta$ an effective finite extension. Then $\Gamma^r = \text{Fitt}(\Delta)$, and in particular, the extension $\text{Fitt}(\Delta) \leq \Delta$ is radicably effective.

**Proof.** We consider $\Gamma^r \leq I^*(U, \Delta)$. Since $c(\Gamma^r) \leq \text{Inn}_Q(\Gamma)$ consists of unipotent automorphisms the semisimple parts of the elements of $\Gamma^r$ centralize $U$. Hence, by Proposition 3.16 $\Gamma^r = U$ and therefore $\Gamma^r = \Delta \cap U = \text{Fitt}(\Delta)$.

Since $\Gamma \leq \text{Fitt}(\Delta)$, the extension $\text{Fitt}(\Delta) \leq \Delta$ is effective. Since $\text{Fitt}(\Delta)$ is a f.t.n.-group, the first part of the corollary implies that $\text{Fitt}(\Delta)^r = \text{Fitt}(\Delta)$. \hfill \Box

**Corollary 3.18** Let $\Gamma$ be a f.t.n.-group and $\Gamma \leq \Delta$ an effective finite extension. Then $\Delta$ is a nilpotent group if and only if the extension is radicably trivial. If $\Delta$ is nilpotent then $\Delta$ is a f.t.n.-group.

The following lemma shows that the class of finite effective extensions of f.t.n.-groups is closed under the operation of taking effective extensions.

**Lemma 3.19** Let $\Lambda$ be a finite effective extension group of the f.t.n.-group $\Theta$, and $\Delta$ a finite effective extension of $\Lambda$. Let $\Gamma = \text{Fitt}(\Delta)$. Then $\Gamma$ is a f.t.n.-group and the extension $\Gamma \leq \Delta$ is effective.

**Proof.** We may assume that $\Theta = \text{Fitt}(\Lambda)$ and embed $\Delta$ as a subgroup of a linear algebraic group $I(U, \Delta)$ such that $U \cap \Delta = \Theta$. It follows that $Z(\Theta)_s = Z(\Delta(U))_s$ is a finite subgroup of $I(U, \Delta)$ which is normalized by the Zariski-closure $\overline{\Lambda}$ of $\Lambda$. Since $\Lambda$ is a normal subgroup of $U \cap \Delta = \Theta$, $\overline{\Lambda}$ is also normalized by $Z(\Delta(\Theta))_s$. Therefore $C = [Z(\Theta)_s, \Lambda] \leq \overline{\Lambda} \cap Z(\Delta(\Theta))_s$. Since $C \leq \overline{\Lambda}$ centralizes $U$, Proposition 3.16 implies that $C$ is unipotent. Hence $C$ is trivial, and $Z(\Delta(\Theta))_s$ centralizes $\Lambda$. Since $\Delta$ is an effective extension of $\Lambda$ the map $\gamma \mapsto \gamma_u$ is an injective homomorphism on $Z(\Delta(\Theta))$. Therefore $Z(\Delta(\Theta))$ is torsionfree, and also $\Gamma = Z(\Delta(\Theta)) \Theta$ is a f.t.n.-group. Since the extension $\Gamma \leq \Delta$ is effective, $\Delta$ is a finite effective extension of a f.t.n.-group. The lemma follows. \hfill \Box

**Definition 3.20** Let $\Delta$ be an effective finite extension of some f.t.n.-group. A $Q$-defined linear algebraic group $U_\Delta$ is called an algebraic hull for $\Delta$, if the following hold: $\Delta$ is a Zariski-dense subgroup of $U_\Delta$, so that $\Delta \leq U_\Delta Q$, $\text{Fitt}(\Delta)$ is contained in the unipotent radical $u(U_\Delta)$, and $\dim u(U_\Delta) = \text{rank } \text{Fitt}(\Delta)$. \hfill \Box

**Corollary 3.21** Every $\Delta$ as above admits an algebraic hull, and the algebraic hull is unique up to $Q$-isomorphism. \hfill \Box

---

3See Definition 1.2 for generalisation in the more general context of finite extensions of polycyclic groups.

4In fact, every virtually polycyclic wfn-group has an algebraic hull. See [13], and chapter 1 of this article.
The existence of the algebraic hull for $\Delta$ is proved by Proposition 3.16. In fact $U_\Delta = I^*(U, \Delta)$, where $U$ is the Malcev completion of $\text{Fitt}(\Gamma)$. The uniqueness of the algebraic hull follows from the following functorial property.

**Proposition 3.22** Let $U_\Delta$ be an algebraic hull for $\Delta$, $G$ a $\mathbb{Q}$-defined linear algebraic group. Then every homomorphism $\rho : \Delta \to G_\mathbb{Q}$, such that $\rho(\text{Fitt}(\Delta))$ is contained in a $\mathbb{Q}$-defined unipotent subgroup $H$ of $G$, extends uniquely to a $\mathbb{Q}$-defined homomorphism $\rho U_\Delta : U_\Delta \to G$.

**Proof.** Let $\Gamma = \text{Fitt}(\Delta)$, and let $U$ be the unipotent radical of $U_\Delta$. By the Malcev extension property the induced homomorphism $\rho : \Gamma \to H_\mathbb{Q}$ extends to a $\mathbb{Q}$-defined homomorphism $\rho U : U \to H$. Now every $g \in U_\Delta$ may be written as a product $g = u\delta$, where $u \in U$, $\delta \in \Delta$. Therefore any extension $\rho U_\Delta$ of $\rho$ must satisfy

$$\rho U_\Delta(g) = \rho U(u)\rho(\delta).$$

We claim that this expression also defines a homomorphism from $U_\Delta$ to $G$. We have to verify that the expression is well defined on $U_\Delta$: Assume therefore that $u'\delta = u\delta'$. Then $\delta(\delta')^{-1} = u'u^{-1} \in U \cap \Delta$. Since $U \cap \Delta = \Gamma$ it follows that $\rho U(u')\rho U(u^{-1}) = \rho(u'u^{-1}) = \rho(\delta)\rho(\delta')^{-1}$. Hence, $\rho U(u)\rho(\delta) = \rho U(u')\rho(\delta')$.

Similarly, we note that, by Zariski-denseness of $\Gamma$ in $U$, for all $\delta \in \Delta$, $u \in U$, the identity

$$\rho U(\delta u\delta^{-1}) = \rho(\delta)\rho U(u)\rho(\delta^{-1})$$

holds. This shows that the expression for $\rho U_\Delta$ defines a homomorphism. We see that $\rho U_\Delta$ is a $\mathbb{Q}$-defined morphism on $U_\Delta$, by computing $\rho U_\Delta$ on the product of varieties $U_\Delta = \mu \times U$, where $\mu \leq U_\Delta$ is a finite subgroup of $U_\Delta$. $\square$

In summary, the algebraic hull for $\Delta$ satisfies the same rigidity properties as the Malcev-completion for $\Gamma$ does. The proposition implies in particular:

**Corollary 3.23** Every automorphism of $\Delta$ extends to a unique $\mathbb{Q}$-defined automorphism of $U_\Delta$.

**Splitting extensions** We use the algebraic hull of $\Delta$ to show that the extension $\Gamma \leq \Delta$ splits in a finite extension. Let us first introduce some terminology.

Let $\Delta$ be a group, $\hat{\Gamma} \leq \Delta$ a normal subgroup, and assume there is a subgroup $\mu \leq \Delta$ such that $\Delta = \hat{\Gamma} \cdot \mu$ is a semidirect product. Let us put $F = \Delta / \hat{\Gamma}$. The normal extension $\Gamma \leq \Delta$ is said to split (in the extension $\hat{\Gamma} \leq \Delta$) if there exists an embedding $\psi : \Delta \hookrightarrow \hat{\Delta}$ and a commutative diagram of homomorphisms

$$
\begin{array}{cccc}
1 & \longrightarrow & \Gamma & \longrightarrow & \Delta & \longrightarrow & F & \longrightarrow & 1 \\
& & \psi & & \downarrow \text{id}_F & & & & \\
1 & \longrightarrow & \hat{\Gamma} & \longrightarrow & \Delta & \longrightarrow & F & \longrightarrow & 1
\end{array}
$$

(3.1)

Now consider $\Delta$ as a subgroup of its algebraic hull $U_\Delta$. We recall that $U_\Delta = \Delta U$, where $U$ is the unipotent radical of $U_\Delta$. It is known (c.f. Theorem 3.30) that there exists a finite subgroup $\mu \leq U_{\Delta, \mathbb{Q}}$ such that $U_{\Delta, \mathbb{Q}} = \mu \cdot U_\mathbb{Q}$ is a semi-direct product. This implies, in fact, that the extension $\Gamma \leq \Delta$ splits in the extension $U_\mathbb{Q} \leq U_{\Delta, \mathbb{Q}}$.

**Proposition 3.24** There exists a finite normal extension $\Gamma_f \leq \Delta_f$, where $\Gamma_f$ is a f.t.n.-group, and an embedding $\Delta \leq f \Delta_f$, such that the extension $\Gamma \leq \Delta$ splits in the extension $\Gamma_f \leq \Delta_f$. 
Proof. Let $\Delta = \Gamma_1 \cup \cdots \cup \Gamma_m$ be a decomposition of $\Delta$ in left cosets. It follows from the splitting of $U_{\Delta,Q}$ that $r_i = u_ig_i$ with $u_i \in U_Q$, $g_i \in \mu$, $i = 1, \ldots, m$. Let $\Gamma$ be the subgroup of $U_Q$ generated by the $u_i$ and $\Gamma$. Since $u_i \in U_Q$, $\Gamma$ is a finite extension of $\Gamma$. Moreover, since the finite group $\mu$ normalizes $U_Q$, Lemma 3.12 implies that there exists a subgroup $\Gamma_f \leq U_Q$ which is a finite extension of $\Gamma$ and which is normalized by $\mu$. A finitely generated subgroup of $U_Q$ is a f.t.n.-group, hence $\Gamma_f$ is a f.t.n.-group. Let $\Delta_f = \Gamma_f \mu$ then it follows from the construction that $\Delta$ is contained in $\Delta_f$. \hfill \Box

The splitting group $\Delta_f$ inherits the functorial properties of $U_{\Delta}$.

**Proposition 3.25** Let $\rho : \Delta \rightarrow G_Q$ be a homomorphism of $\Delta$ to a $Q$-defined linear algebraic group $G$ such that $\rho(\Gamma)$ is contained in $H_Q$, where $H$ is a $Q$-defined unipotent subgroup of $G$. Then $\rho$ extends to a homomorphism $\rho_f : \Delta_f \rightarrow G_Q$.

**Proof.** By Proposition 3.22, $\rho$ extends to a homomorphism $\rho_{U_{\Delta}} : U_{\Delta} \rightarrow G$. Now by construction of $\Delta_f$ we have inclusions $\Delta \leq \Delta_f \leq U_{\Delta}$, and the restriction of $\rho_{U_{\Delta}}$ to $\Delta_f$ induces the required extension $\rho_f$. \hfill \Box

**Realization as affine crystallographic group**

We are now dealing with the proof of Theorem 3.10. Let $\Delta$, $\Gamma$ satisfy the assumptions of Theorem 3.10, i.e., $\Gamma$ is an f.t.n.-group, $\Gamma \leq_f \Delta$ is an effective normal extension with associated kernel $\beta$. We assume that $\Gamma$ is isomorphic to an ACG of type $A$.

Let us first consider the special case, where the extension $\Gamma \leq \Delta$ is radicably trivial. Proposition 3.60 implies that, in this case, the kernel of the extension acts trivially on $D_c(\Gamma, A)$. Also, by Proposition 3.10 $\Delta$ is a f.t.n.-group. Therefore let us consider now a (not necessarily normal) finite extension $\Gamma \leq_f \Gamma^*$, where $\Gamma^*$ is a f.t.n.-group. The following result is an immediate consequence of Proposition 3.14.

**Proposition 3.26** Every $\rho \in \text{Hom}_c(\Gamma, A)$ has an extension $\rho_{\Gamma^*} \in \text{Hom}_c(\Gamma^*, A)$, and this extension is unique.

Now let us consider a tower of finite extensions $\Gamma \leq \Gamma^* \leq \Delta$, where the extensions $\Gamma \leq \Delta$, $\Gamma^* \leq \Delta$ are normal. For later purposes we note:

**Lemma 3.27** If $[\rho] \in D_c(\Gamma, A)$ is a fixed point for $\Delta/\Gamma$ then $[\rho_{\Gamma^*}] \in D_c(\Gamma^*, A)$ is a fixed point for $\Delta/\Gamma^*$.

**Proof.** Let $c_g : \Gamma^* \rightarrow \Gamma^*$ denote conjugation with $g \in \Delta$. Since $[\rho] \in D_c(\Gamma, A)$ is a fixed point for $\Delta/\Gamma$ there exists $a \in A$, such that, for all $\gamma \in \Gamma$,

$$\rho_{\Gamma^*}(c_g(\gamma)) = \rho(c_g(\gamma)) = \rho^a(\gamma) = \rho^a_{\Gamma^*}(\gamma).$$

By the uniqueness of extensions we must have $\rho_{\Gamma^*} \circ c_g = \rho^a_{\Gamma^*}$. Therefore $[\rho_{\Gamma^*}]$ is a fixed point for $\Delta/\Gamma^*$. \hfill \Box

We turn now to the realization of radicably effective extensions. We start by considering split extensions $\Delta = F \cdot \Gamma$, where $F \leq \Delta$ is a finite subgroup with $F \cap \Gamma = \{1\}$. Recall from section 3.6 Proposition 3.65 that there exists a natural homomorphism $c_\rho : N_A(\rho(\Gamma)) \rightarrow \text{Aut}(\Gamma)[\rho]$ which is onto.

**Proposition 3.28** Let $F \leq \text{Aut}(\Gamma)$ be a finite subgroup. If $[\rho] \in D_c(\Gamma, A)$ is a fixed point for $F$, then there exists an embedding

$$\iota : F \rightarrow N_A(\rho(\Gamma))$$
which satisfies $c_\rho(\iota(g)) = g$, for all $g \in F$. Any two embeddings $\iota, \iota' : F \to N_A(\rho(\Gamma))$ with the latter property are conjugate by an element of $C_A(\rho(\Gamma))$.

**Proof.** We will identify $\text{Aut}(\Gamma)$ with a subgroup of $\text{Aut}(U_\Gamma)$, where $U_\Gamma$ is a real hull of $\Gamma$, as in section 3.6. From the assumption that $[\rho]$ is a fixed point we deduce that $F \leq \text{Aut}(\Gamma)[\rho]$. We are going to apply some of the facts which are collected in section 3.6.

The group $U = \overline{\text{Aut}(\Gamma)}$ is a simply transitive subgroup of $A$. By Proposition 3.64 and Proposition 3.65 the map $c : N_A(U) \to \text{Aut}(U_\Gamma)[\rho]$ is onto with kernel $C_A(U)$, and restricts to a surjective map $c_\rho : N_A(\rho(\Gamma)) \to \text{Aut}(\Gamma)[\rho]$. Now let $H = c^{-1}(F)$. $H$ is a Zariski-closed subgroup of $N_A(U)$. Since $F \leq \text{Aut}(\Gamma)[\rho]$ it follows that $H \leq N_A(\rho(\Gamma))$. $H$ contains $C_A(U)$ as a normal subgroup of finite index. In fact, $H/C_A(U)$ is isomorphic to $F$. Since $C_A(U)$ is unipotent, $C_A(U)$ is the unipotent radical of $H$. Now by splitting of algebraic groups there exists a subgroup $\mu \leq H$ such that $H = \mu \cdot C_A(U)$. Since the restriction of $c$ to $\mu$ is an isomorphism onto $F$, we can set $\iota = c^{-1}$. We thus proved the existence of $\iota$.

We prove now the conjugacy statement. Let us remark first that every homomorphism $\iota : F \to N_A(\rho(\Gamma))$ which satisfies the assumption maps $F$ into the group $H$. Moreover $\iota$ is uniquely determined by its image $\iota(F) \leq H$. Since both $\iota(F)$ and $\iota'(F)$ are Levi-subgroups in $H$, $\iota'(F)$ is conjugated to $\iota(F)$ by an element of $C_A(U)$. Hence also the homomorphisms $\iota$ and $\iota'$ are conjugate.

**Corollary 3.29** Let $F \leq \text{Aut}(\Gamma)$ be a finite subgroup, and $\Delta = F \cdot \Gamma$ be the corresponding split extension of $\Gamma$. Then $\rho \in \text{Hom}_c(\Gamma, A)$ extends to a homomorphism $\rho_\Delta \in \text{Hom}_c(\Delta, A)$, if and only if $[\rho] \in D_c(\Gamma, A)$ is a fixed point for $F$. Any two extensions $\rho_\Delta, \rho'_\Delta \in \text{Hom}_c(\Delta, A)$ of $\rho$ are conjugate by an element of $C_A(\rho(\Gamma))$.

**Proof.** By Proposition 3.28 there exists an embedding $\iota : F \to N_A(\rho(\Gamma))$ such that $c_\rho(\iota(g)) = g$, for all $g \in F$. For $g \in F, \gamma \in \Gamma$ we can define $\rho_\Delta(g\gamma) = \iota(g)\rho(\gamma)$, to get the required extension $\rho_\Delta$. Now, if $\rho_\Delta$ is any extension of $\rho$ to $\Delta$, then $\iota(g) = \rho_\Delta(g)$ defines a homomorphism $\iota : F \to N_A(\rho(\Gamma))$ which satisfies $c_\rho(\iota(g)) = g$, for all $g \in F$. Therefore the conjugacy statement of Proposition 3.28 implies that any two extensions of $\rho$ to $\Delta$ are conjugate by an element $u \in C_A(\rho(\Gamma))$. 

We are now ready to finish the proof of Theorem 3.10.

**Proof of Theorem 3.10** Let $\rho \in \text{Hom}_c(\Gamma, A)$ be such that $[\rho] \in D_c(\Gamma, A)$ is a fixed point for $\beta$. Let $\Gamma^* = \text{Fitt}(\Delta)$. By Proposition 3.26 $\rho$ extends to a unique homomorphism $\hat{\rho} \in \text{Hom}_c(\Gamma^*, A)$. By Lemma 3.27 $[\hat{\rho}] \in D_c(\Gamma^*, A)$ is a fixed point for the kernel of the extension $\Gamma^* \leq \Delta$.

So we may assume now that $\hat{\Gamma} = \text{Fitt}(\Delta)$, or equivalently that the extension $\Gamma \leq \Delta$ is radicably effective. Let $F = \Delta/\Gamma$. We apply Proposition 3.24 to embed $\Delta$ in an extension group $\Delta_f = \mu \cdot \Gamma_f$ which splits. $\Gamma$ embeds as a subgroup of the f.t.n.-group $\Gamma_f$, and $\mu$ is isomorphic to $F$. By Proposition 3.26 $\rho$ uniquely extends to $\rho_f \in \text{Hom}_c(\Gamma_f, A)$. We remark that, by a similar argument as given in the proof of Lemma 3.27 $[\rho_f] \in D_c(\Gamma_f, A)$ is a fixed point for $\mu$. Moreover, since $\Gamma \leq \Delta$ is radicably effective, the extension $\Gamma_f \leq \Delta_f$ is a (radically) effective extension. So we may view $\mu$ as a subgroup of $\text{Aut}(\Gamma_f)$. By Corollary 3.29 the affine crystallographic representation of $\Gamma_f$ extends to $\Delta_f$. Since $\Delta$ is a subgroup of $\Delta_f$ we have, in particular, extended the original representation $\rho$ of $\Gamma$ to a representation $\rho_\Delta$ of $\Delta$. 

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Appendix: Splitting of algebraic groups  In the preceding proofs we made crucial use of the existence of Levi subgroups in linear algebraic groups and the fact that the Levi subgroups are all conjugated, a result which is originally due to Mostow. Let us recall the precise statement of the result, cf. [10] [5.1]:

**Theorem 3.30**  Let $G$ be a $\mathbb{Q}$-defined linear algebraic group. Then there exists a reductive $\mathbb{Q}$-defined subgroup $L$ of $G$ such that $G = Lu(G)$ and $L \cap u(G) = \{1\}$. Any two such subgroups $L$ and $L'$ are conjugate by an element of $u(G)_\mathbb{Q}$. Moreover, each $\mathbb{Q}$-defined reductive subgroup $H$ of $G$ is conjugate to a subgroup of $L$ by an element of $u(G)_\mathbb{Q}$.

From the theorem corresponding rational splitting and conjugacy statements are easily deduced for any subfield $k \leq \mathbb{C}$. In fact, by the theorem, the variety $G$ is a product of varieties $L$ and $u(G)$, where $L$ is isomorphic to $G/u(G)$. The same statement holds then for the rational points of these varieties. In particular we deduce that $(G/u(G))_\mathbb{Q} = G_\mathbb{Q}/u(G)_\mathbb{Q}$, and $G_\mathbb{Q} = L_\mathbb{Q}u(G)_\mathbb{Q}$, with $L_\mathbb{Q} \cap u(G)_\mathbb{Q} = \{1\}$.

The application of this result replaces in our setting the explicit use of cohomological reasoning which is pervasive in most treatments of the classical Bieberbach theory.

### 3.3 Deformation spaces of finite extensions

Let $\Gamma$ be a f.t.n.-group which is isomorphic to an ACG of type $A$. The extension results of the previous section are particular well suited to understand the deformation spaces of the finite extensions of $\Gamma$. We continue to use the notational conventions of the previous section.

Let $\beta : F \rightarrow \text{Out}(\Gamma)$ be an abstract kernel for $\Gamma$ which is realized by a finite extension $\Gamma \leq_f \Delta$. The restriction map $r_\Gamma : \text{Hom}_c(\Delta, A) \rightarrow \text{Hom}_c(\Gamma, A)$ factorizes to a continuous map

$$r_\Gamma : D_c(\Delta, A) \rightarrow D_c(\Gamma, A)$$

on the deformation spaces. We denote the set of fixed points for the kernel $\beta$ on $D_c(\Gamma, A)$ with $D_c(\Gamma, A)^F$, and we put $\text{Hom}_c(\Gamma, A)^F$ for its preimage in $\text{Hom}_c(\Gamma, A)$. We topologize $D_c(\Gamma, A)^F$ with the induced subspace topology from $D_c(\Gamma, A)$. It follows from Proposition 3.28 that the image of $r_\Gamma$ is contained in $D_c(\Gamma, A)^F$. The following is then a commutative diagram of (continuous) maps:

$$\text{Hom}_c(\Delta, A) \xrightarrow{r_\Gamma} \text{Hom}_c(\Gamma, A)^F \xrightarrow{r_\Gamma} D_c(\Delta, A) \xrightarrow{r_\Gamma} D_c(\Gamma, A)^F$$

We apply the results of the previous section to study the properties of the map $r_\Gamma$. Let $\Gamma$ be an f.t.n.-group and $U$ its Malcev completion. Recall from section 3.6 that $\text{Out}(U_\mathbb{R})$ acts naturally on $D_c(\Gamma, A)$. We consider first the case of commensurable f.t.n.-groups. Their deformation spaces are essentially identical.

**Proposition 3.31**  Let $\Gamma$ and $\Gamma'$ be commensurable f.t.n.-groups. Then there exists an $\text{Out}(U_\mathbb{R})$-equivariant homeomorphism $h : D_c(\Gamma, A) \xrightarrow{\sim} D_c(\Gamma', A)$.

**Proof.** $\Gamma$ and $\Gamma'$ contain a common subgroup of finite index which is a f.t.n.-group. It is therefore enough to prove the statement in the case of a finite extension $\Gamma \leq \Gamma'$.
Now by Proposition 3.26 and Proposition 3.14 the restriction map from $\Gamma'$ to $\Gamma$ is a homeomorphism from $\text{Hom}_c(\Gamma', A)$ to $\text{Hom}_c(\Gamma, A)$. This homeomorphism is easily seen to be compatible with the actions of $\text{Aut}(U_R)$ on these spaces. □

However, this phenomenon is not restricted to commensurability. More generally the following holds:

**Proposition 3.32** Let $\Gamma$ and $\Gamma'$ be lattices in the real Lie group $U_R$. Then there exists an $\text{Out}(U_R)$-equivariant homeomorphism $h : D_c(\Gamma, A) \rightleftharpoons D_c(\Gamma', A)$.

We leave the proof to the reader. In fact, Theorem 3.62 implies that the properties of the deformation space $D_c(\Delta, A)$ depend only on the algebraic hull $U_\Delta$ of $\Delta$.

**Corollary 3.33** Let $\Gamma$ be a f.t.n.-group and $\Gamma \leq_f \Delta$, where $\Gamma$ is normal in $\Delta$, be a finite, effective extension, $F = \Delta/\Gamma$. Then there is a continuous bijection

$$\bar{r}_\Gamma : D_c(\Delta, A) \longrightarrow D_c(\Gamma, A)^F.$$ 

**Remark** We do not expect in general that the map $\bar{r}_\Gamma$ will be a homeomorphism. Instead, we expect that there is a stratification of $D_c(\Delta, A)$ such that $\bar{r}_\Gamma$ will induce a homeomorphism on the strata.

Before we give the proof of the corollary, let us exemplify the previous remark by considering a particular important stratum in the deformation space. Recall from section 3.6 that $\rho \in \text{Hom}_c(\Gamma, A)$ is called symmetric if the centralizer of $\rho(\Gamma)$ in $V$. Let $\text{Hom}_c(\Gamma, A)_s$ denote the set of all symmetric crystallographic homomorphisms of $\Gamma$. $\text{Hom}_c(\Gamma, A)_s$ is a closed $A$-invariant subspace of $\text{Hom}_c(\Gamma, A)$, and the deformation space $D_c(\Gamma, A)$ embeds as a closed subspace into $D_c(\Gamma, A)$. Let us denote $D_c(\Delta, A)_s = \bar{r}_\Gamma^{-1}(D_c(\Gamma, A)_s)$.

**Theorem 3.34** The map $\bar{r}_\Gamma$ induces a homeomorphism

$$D_c(\Delta, A)_s \rightleftharpoons D_c(\Gamma, A)^F_s.$$ 

**Proof of Corollary 3.33** We want to show that the map $\bar{r}_\Gamma : D_c(\Delta, A) \longrightarrow D_c(\Gamma, A)^F$ is a bijection. Now $\Delta$ is a subgroup of $\Delta_f$, where $\Delta_f = \Gamma \cdot F$ is as in Proposition 3.24 and it is shown in the proof of Theorem 3.10 that there is a commutative diagram of restriction maps

$$
\begin{array}{cccc}
D_c(\Gamma, A)^F & \rightleftharpoons & D_c(\Delta, A) \\
\downarrow & & \downarrow \\
D_c(\Gamma_f, A)^F & \rightleftharpoons & D_c(\Delta_f, A)
\end{array}
$$

By Proposition 3.26 the left upgoing arrow is a bijection. The map $\bar{r}_{\Gamma_f}$ is a bijection by the conjugacy statement of Corollary 3.29 and the right upgoing arrow is onto by Proposition 3.25. It follows that $\bar{r}_\Gamma$ is a bijection. □

For the proof of the theorem we need a lemma.

**Lemma 3.35** Let $\tilde{F} \leq \text{Aut}(\Gamma_f)$ be a finite subgroup, and $\Delta_f = \tilde{F} \cdot \Gamma_f$ be the corresponding split extension. Then the restriction map $r_{\Gamma_f} : \text{Hom}_c(\Delta_f, A) \rightarrow \text{Hom}_c(\Gamma_f, A)$ admits a (semi-algebraic) continuous cross section $s : \text{Hom}_c(\Gamma_f, A)^F \rightarrow \text{Hom}_c(\Delta_f, A)$.
Proof. By Proposition \[3.71\] there exists a continuous map \( s : \text{Hom}_c(\Gamma, A)^F \rightarrow \text{Hom}(F, A) \) such that \( c(s_F(\rho, g)) = g \), for all \( g \in F \). Let \( \rho \in \text{Hom}_c(\Gamma, A)^F \). By Corollary \[3.29\] there exists a corresponding \( \rho \Delta_f = s(\rho) \in \text{Hom}_c(\Delta_f, A) \) which restricts to \( \rho \).

Proof of Theorem \[3.34\] We consider again the diagram in the proof of Corollary \[3.33\]. By Proposition \[3.14\] the restriction map from \( \text{Hom}_c(\Gamma f, A) \) to \( \text{Hom}_c(\Gamma, A) \) is a homeomorphism, and consequently the induced map from \( D_c(\Gamma f, A) \) to \( D_c(\Gamma, A) \) is a homeomorphism. The same is true for the bijection from \( D_c(\Delta_f, A) \) to \( D_c(\Delta, A) \).

Now it follows from the lemma that \( \bar{r}_{\Gamma f} : D_c(\Delta_f, A) \rightarrow D_c(\Gamma f, A) \) is a homeomorphism.

Some applications

Moduli spaces of finite extensions Let \( \Gamma \) be a f.t.n.-group. Recall that the moduli space \( M_c(\Gamma, A) \) is just the space \( D_c(\Gamma, A) / \text{Out}(\Gamma) \) with the quotient topology. Let \( \Delta \) be a finite effective extension group of \( \Gamma \). There is a natural induced restriction map \( \tilde{r} : M_c(\Delta, A) \rightarrow M_c(\Gamma, A) \) of moduli spaces. In general, this map is not injective and does not give a precise picture for the space \( M_c(\Delta, A) \).

We give now a description of \( M_c(\Delta, A) \) in the case that the extension is radicably effective, i.e., we assume that \( \Gamma = \text{Fitt}(\Delta) \). We may view \( F = \Delta / \Gamma \) as a subgroup of \( \text{Out}(\Gamma) \). Since \( \Gamma \) is an invariant normal subgroup of \( \Delta \), there is a natural homomorphism \( \bar{\nu} : \text{Out}(\Delta) \rightarrow \text{N}_{\text{Out}(\Gamma)}(F) / F \) which comes associated with the extension. Note that the group \( \text{N}_{\text{Out}(\Gamma)}(F) / F \) acts as a group of homeomorphisms on \( D_c(\Gamma, A)^F \). In fact, this action describes the action of \( \text{Out}(\Delta) \) on \( D_c(\Delta, A) \), and it is possible to recover \( M_c(\Delta, A) \) from \( D_c(\Gamma, A) \) and the action of \( F \leq \text{Out}(\Gamma) \).

Theorem \[3.36\] Let \( \Delta \) be an effective finite extension of the f.t.n.-group \( \Gamma \), and assume that \( \Gamma = \text{Fitt}(\Delta) \). Let \( \text{Out}(\Delta) \) act on \( D_c(\Delta, A) \) with respect to the homomorphism \( \nu \). Then the embedding \( \tilde{r}_\Gamma : D_c(\Delta, A) \rightarrow D_c(\Gamma, A)^F \) is \( \text{Out}(\Delta) \)-equivariant. In particular, there is a continuous bijection from the moduli space \( M_c(\Delta, A) \) to the quotient space \( D_c(\Gamma, A)^F / \bar{\nu}(\text{Out}(\Delta)) \). The bijection induces a homeomorphism on the moduli space \( M_c(\Delta, A)_s \) of symmetric crystallographic homomorphisms to \( D_c(\Gamma, A)^F / \bar{\nu}(\text{Out}(\Delta)) \).

Proof. That the map \( \tilde{r}_\Gamma \) is \( \text{Out}(\Delta) \)-equivariant is a routine calculation from the definitions. We omit it therefore. The remaining statements follow then from the previous results on deformation spaces.

Inheritance of the Hausdorff-property Though the topology on \( D_c(\Delta, A) \) is potentially finer than the topology induced from \( D_c(\text{Fitt}(\Delta), A)^F \) we still can deduce an immediate useful consequence from Corollary \[3.33\].

Corollary \[3.37\] Let \( \Delta \) be a virtually nilpotent ACG, and let \( \Gamma = \text{Fitt}(\Delta) \) be the Fitting subgroup. If \( D_c(\Gamma, A) \) is a Hausdorff topological space, then \( D_c(\Delta, A) \) is a Hausdorff space too.
Inheritance of convexity to finite extensions

If $\Delta$ is a virtually nilpotent ACG then the Fitting subgroup $\Gamma = \text{Fitt}(\Delta)$ is a crystallographic f.t.n.-group, and is also a characteristic subgroup of $\Delta$. Therefore there is a natural restriction homomorphism

$$\nu : \text{Aut}(\Delta) \to \text{Aut}(\Gamma),$$

and, evidently, the map $r_\Gamma : \text{Hom}_c(\Delta, A) \to \text{Hom}_c(\Gamma, A)$ is equivariant with respect to $\nu$. It follows

Proposition 3.38 Let $\Delta$ be virtually nilpotent ACG, and let $\Gamma = \text{Fitt}(\Delta)$. Then the induced inclusion on deformation spaces

$$D_c(\Delta, A) \hookrightarrow D_c(\Gamma, A)$$

is equivariant with respect to the actions of $\text{Aut}(\Delta)$ and $\text{Aut}(\Gamma)$.

As a particular consequence we see that convexity properties of $D_c(\text{Fitt}(\Delta), A)$ will be inherited to $\Delta$.

Theorem 3.39 Let $\Delta$ be a virtually nilpotent ACG, and let $\Gamma = \text{Fitt}(\Delta)$. Then the following hold:

i) If $D_c(\Gamma, A)$ is fixed pointed then $D_c(\Delta, A)$ is fixed pointed too.

ii) If $D_c(\Gamma, A)$ is convex then $D_c(\Delta, A)$ is convex too.

Proof. Let $F = \Delta/\Gamma$. If $D_c(\Gamma, A)$ has a fixed point $[\rho_0]$ for $\text{Aut}(\Gamma)$ then, by Corollary 3.33, $\tilde{r}_\Gamma(D_c(\Delta, A))$ meets $[\rho_0]$. By equivariance and the injectivity of the map $\tilde{r}_\Gamma$, $[\rho_0]$ is a fixed point for $\text{Aut}(\Delta)$ on $D_c(\Delta, A)$.

We assume now that $D_c(\Gamma, A)$ is convex. The restriction homomorphism $\nu$ induces a map

$$\tilde{\nu} : \text{Out}(\Delta) \longrightarrow N_{\text{Out}(\Gamma)}(F)/F.$$ 

Let $\mu$ be a finite subgroup of $\text{Out}(\Delta)$, and let $\tilde{\mu} \leq N_{\text{Out}(\Gamma)}(F)$ be the preimage of $\tilde{\nu}(\mu)$. Since $F$ is finite so is $\tilde{\mu}$ which is a finite normal extension of $F$. If $[\rho_0] \in D_c(\Gamma, A)$ is a fixed point for $\tilde{\mu}$ then, since $F \leq \tilde{\mu}$, $\tilde{r}_\Gamma(D_c(\Delta, A))$ meets $[\rho_0]$. By the equivariance properties of the embedding of $D_c(\Delta, A)$ into $D_c(\Gamma, A)$, $[\rho_0]$ is a fixed point for $\mu$ too.

The realization theorem

We now come to the proof of the realization theorem for finite extensions of a virtually nilpotent affine crystallographic group:

Proof of Theorem 3.4 Since the virtually nilpotent group $\Lambda$ is isomorphic to an affine crystallographic group, $\Theta = \text{Fitt}(\Lambda)$ is a f.t.n.-group, and the centralizer of $\Theta$ is contained in $\Theta$. Hence the extension $\Theta \leq \Lambda$ is effective. It follows from Lemma 3.1 that $\Delta$ is an effective extension of $\Gamma = \text{Fitt}(\Delta)$, and also that $\Gamma = \text{Fitt}(\Delta)$ is a f.t.n.-group. Since we already proved the realization theorem for effective extensions of f.t.n.-groups, to show that $\Delta$ is isomorphic to a crystallographic group, it is enough to show that the kernel $F \leq \text{Out}(\Gamma)$ which is associated to the extension $\Gamma \leq \Delta$ has a fixed point in $D_c(\Gamma, A)$.

We remark that $\Theta = \text{Fitt}(\Lambda)$ has finite index in $\Gamma = \text{Fitt}(\Delta)$. Therefore (compare Proposition 3.31) $\text{Out}(\Theta)$ and the kernel $F_\Theta \leq \text{Out}(\Theta)$ which is associated to $\Theta \leq \Lambda$ act also on $D_c(\Gamma, A)$, and the set of fixed points of $F_\Theta$ corresponds to $D_c(\Lambda, A)$. The action of $\text{Out}(\Theta)$ on $D_c(\Gamma, A)$ factorizes over $\text{Out}(U_R)$. The action of $\text{Out}(\Lambda)$ on $D_c(\Lambda, A) \subseteq D_c(\Gamma, A)$ is then encoded in the homomorphism

$$\tilde{\nu} : \text{Out}(\Lambda) \longrightarrow N_{\text{Out}(U_R)}(F_\Theta)/F_\Theta.$$
Let \( \mu \leq \text{Out}(\Lambda) \) be the kernel which is associated to the extension \( \Lambda \leq \Delta \), and let \( \bar{\mu} \leq \text{Out}(\mathbb{U}_\mathbb{R}) \) be the preimage of \( \bar{\nu}(\mu) \) in \( \text{NOut}(\mathbb{U}_\mathbb{R})(F_\Theta) \). By our assumption, \( \mu \) has a fixed point in \( \mathcal{P}_c(\Lambda, A) \) and this implies that \( \bar{\mu} \) has a fixed point in \( \mathcal{P}_c(\Gamma, A) \). It is easy to see that the kernel \( F \leq \text{Out}(\mathbb{U}_\mathbb{R}) \) is contained in \( \bar{\mu} \). Therefore \( F \) has a fixed point in \( \mathcal{P}_c(\Gamma, A) \).

\[ \square \]

**Manifolds of Euclidean type**

Let \( M \) be a closed manifold which admits a Riemannian metric of constant curvature zero, i.e., a flat Riemannian metric. In the spirit of the theory of geometrization, \( M \) is called a *manifold of Euclidean type*. By Bieberbach theory \( M \) is finitely covered by a torus \( T^n \). The fundamental group \( \pi_1(M) \) is isomorphic to a finite extension of \( \mathbb{Z}^n \).

It was remarked previously in [10] that the deformation space \( \mathcal{D}_e(T^n, \text{Aff}(\mathbb{R}^n)) \) of affine structures on \( T^n \) is a Hausdorff-space and homeomorphic to a semi-algebraic set. We obtain the following generalization of this result:

**Corollary 3.40** Let \( M \) be a closed manifold of Euclidean type. Then the deformation space of complete affine structures of type \( A \) is a Hausdorff space and homeomorphic to a semi-algebraic set.

**Proof.** First let us remark that, by Theorem 2.4, \( \mathcal{D}_e(\mathbb{Z}^n, A) \) is a semi-algebraic set. Now \( \pi_1(M) \) is a normal extension of \( \mathbb{Z}^n \) with some finite group \( F \). We remark next that the crystallographic actions of \( \mathbb{Z}^n \) are all symmetric. By Corollary 3.34 it follows that \( \mathcal{D}_e(M, A) \) is homeomorphic to \( \mathcal{D}_e(\mathbb{Z}^n, A)^F \). Therefore \( \mathcal{D}_e(M, A) \) is homeomorphic to a closed subspace of \( \mathcal{D}_e(\mathbb{Z}^n, A) \), and also it is Hausdorff. Since the action of \( \text{GL}(n, \mathbb{Z}) \) on \( \mathcal{D}_e(\mathbb{Z}^n, A) \) is algebraic \( \mathcal{D}_e(\mathbb{Z}^n, A)^F \) is a semi-algebraic set.

\[ \square \]

**The Burckhardt-Zassenhaus theorem** We let \( \text{GL}_A \) act on \( \text{GL}(V) \) by left multiplication and put

\[ X_A = \text{GL}_A \backslash \text{GL}(V) . \]

Let \( F = G_{st}(V, A) \) be the algebraic variety of simply transitive abelian subgroups of \( A \). \( \text{GL}(A) \) acts by conjugation on \( F \). It was proved in Proposition 2.4 that

\[ \mathcal{D}_e(\mathbb{Z}^n, A) = \text{GL}(V) \times_{\text{GL}_A} F \]

is a fiber product, and in particular a bundle over the homogeneous space \( X_A \) with fiber \( F \). The \( \text{GL}(V) \)-action on \( \mathcal{D}_e(\mathbb{Z}^n, A) \) corresponds to the natural \( \text{GL}(V) \)-action on the fiber product which is induced by right multiplication of \( \text{GL}(V) \) on itself.

If \( A \) contains the subgroup \( V \) of translations we note that \( \text{GL}_A \) fixes \( V \) as an element of \( G_{st}(V, A) \). Therefore, provided \( A \) contains the subgroup \( V \) of translations, \( \mu \leq \text{GL}(V) \) fixes a point in \( \mathcal{D}_e(\mathbb{Z}^n, A) \) if and only if it fixes a point on \( X_A \).

It follows that, in this case, the existence problem for fixed points in the deformation spaces of tori is only a matter of the action of \( \text{GL}(V) \) on \( X_A \). In particular we recover from Theorem 3.10 the Burckhardt-Zassenhaus theorem (see 3.4) on Euclidean crystallographic groups:

**Corollary 3.41** Every finite effective extension of \( \mathbb{Z}^n \) acts as an Euclidean crystallographic group on affine space \( \mathbb{A}^n \).

**Proof.** Every finite subgroup \( \mu \) of \( \text{GL}_n \) has a fixed point in \( X = O_n \backslash \text{GL}_n \), as is well known. Since \( G_{st}(V, E(n)) \) contains the subgroup of translations, \( \mu \) has a fixed point on \( \mathcal{D}_e(\mathbb{Z}^n, E(n)) \).

\[ \square \]
In general, it seems to be a nontrivial problem to give a precise description of the variety $G_{st}(V, A)$, and therefore also of the deformation spaces for complete affine structures on manifolds of Euclidean type. Next we cover some accessible special cases.

**Complete affine surfaces** It is well known that a closed affine surface is diffeomorphic to the two-torus or the Klein bottle. In [9] the following result is proved:

**Proposition 3.42** The deformation space $D_c(T^2, \text{Aff}(\mathbb{R}^2))$ of the two-torus is homeomorphic to $\mathbb{R}^2$. In these coordinates the action of $\text{GL}_2(\mathbb{Z})$ corresponds to the natural linear action. The fixed point $0 \in \mathbb{R}^2$ represents the flat Euclidean structure on $T^2$.

It is easy to see from [9] that also the natural $\text{GL}_2(\mathbb{R})$-action is linear in the above model for $D_c(T^2, \text{Aff}(\mathbb{R}^2))$. In a sense, our model encodes all information on the affine crystallographic groups in dimension two. We apply our method to study the two dimensional affine crystallographic groups up to affine conjugacy: Recall that there exist up to isomorphism seventeen finite effective extensions of $\mathbb{Z}^2$. All these groups act as Euclidean crystallographic groups on the plane. These are the famous **wall paper groups**. As crystallographic subgroups of the Euclidean group, the wall paper groups are unique up to conjugacy with affine transformations. Let us call an affine crystallographic group **non-Euclidean** if it is not conjugated to an Euclidean crystallographic group by an affine transformation. So which of the finite effective extensions of $\mathbb{Z}^2$ admit non-Euclidean affine crystallographic actions?

**Proposition 3.43** Let $\Gamma$ be a finite effective extension of $\mathbb{Z}^2$ which acts as a non-Euclidean affine crystallographic group. If $\Gamma$ preserves orientation then $\Gamma$ is isomorphic to $\mathbb{Z}^2$. If $\Gamma$ does not preserve orientation then $\Gamma$ contains $\mathbb{Z}^2$ as a translation subgroup of index two. There are three isomorphism classes of the latter type and for these groups the deformation space $D_c(\Gamma, \text{Aff}(\mathbb{R}^2))$ is homeomorphic to the real line.

**Proof.** Let $\mu$ be a finite subgroup of $\text{GL}_2(\mathbb{Z})$ which fixes a line in $\mathbb{R}^2$. Every nonidentity element of $\mu$ has eigenvalues 1 and $-1$ and is of order two. It follows that $\mu$ is conjugate in $\text{GL}_2(\mathbb{Z})$ to

$$\mu_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{or} \quad \mu_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

From the classification (see for example [82][§22, §23]) it follows that there are two (effective) extensions of $\mathbb{Z}^2$ by $\mu_1$, and only one by $\mu_2$. The torsionfree extension of $\mathbb{Z}^2$ by $\mu_1$ is the fundamental group of the Klein bottle. $\square$

**Corollary 3.44** The deformation space $D_c(K, \text{Aff}(\mathbb{R}^2))$ of complete affine structures on the Klein bottle $K$ is homeomorphic to the real line. The moduli space $\mathcal{M}_c(K, \text{Aff}(\mathbb{R}^2))$ coincides with the deformation space.

**Proof.** Let $\Delta$ be the Klein-bottle group, $q \in \mu_1$ the non-identity element of $\mu_1$, $a, b$ the standard generators for the lattice $\mathbb{Z}^2$. Then $\Delta$ is generated by $q, a, b$ with the relations $qag^{-1} = a$, $qbg^{-1} = -b$ and $q^2 = a$. The group $\mu_1$ fixes a line in $\mathbb{R}^2$ which is the deformation space for $\Delta$. It is easy to see that the image of the map from $\text{Out}(\Delta)$ to the normalizer of $\mu_1$ in $\text{GL}_2(\mathbb{Z})$ is $\mu_1$ itself. By Theorem 3.36 the fixed line in $\mathbb{R}^2$ is therefore also the moduli space. $\square$
Metric and symplectic affine structures on tori  
We turn our attention to some particular interesting subgeometries, namely complete affine structures with a parallel inner product (flat pseudo-Riemannian manifolds) or a parallel nondegenerate alternating product (symplectic affine manifolds). In these cases we can give a precise description of the spaces $G_{st}(V, A)$, see section 2.2. This amounts to a good understanding of the deformation spaces for these geometries on manifolds of Euclidean type.

3.4 The realization problem for unipotent shadows

Every torsionfree polycyclic group $\Gamma$ has an unipotent shadow $\Theta$ in its real algebraic hull $H_{\Gamma}$. Recall from Proposition 1.36, that if $\Gamma$ is an affine crystallographic group then the unipotent shadow $\Theta$ is an affine crystallographic group as well. Therefore the following realization problem makes sense:

Let $\Gamma$ be a torsionfree polycyclic group with unipotent shadow $\Theta$. Assume that $\Theta$ is isomorphic to an affine crystallographic group. Is it true that $\Gamma$ is isomorphic to an affine crystallographic group?

We need the following

Definition 3.45 Let $H_{\Gamma}$ be a real algebraic hull for $\Gamma$. We choose a splitting $H_{\Gamma} = T \cdot U_{\Gamma}$, where $T$ is a Levi subgroup of $H_{\Gamma}$ and put $U_{\Gamma} = u(H_{\Gamma})$. Then $\Theta \leq U_{\Gamma}$, and $U_{\Gamma}$ is a real Malcev hull for $\Theta$. Since $H_{\Gamma}$ has a strong unipotent radical, conjugation embeds $T$ as a subgroup of $\text{Aut}(U_{\Gamma})$. We call the corresponding image $T_{\Gamma} \leq \text{Out}(U_{\Gamma})$ of $T$ the semisimple kernel associated to $\Gamma$.

Recall (see section 3.6) that, since $U_{\Gamma}$ is a Malcev-hull for $\Theta$, $\text{Out}(U_{\Gamma})$, and hence also $T_{\Gamma}$, act on the deformation space $D_{c}(\Theta, A)$ of $\Theta$. The main result of this section is the following theorem. It shows that the answer to the question above lies in the action of the semisimple kernel $T_{\Gamma}$ of $\Gamma$ on the deformation space of $\Theta$.

Theorem 3.46 Let $\Gamma$ be a torsionfree polycyclic group with unipotent shadow $\Theta$. Assume that $\Theta$ is an ACG and let $\tau \in \text{Hom}_{c}(\Theta, A)$. If $[\tau] \in D_{c}(\Theta, A)$ is a fixed point for the semisimple kernel $T_{\Gamma}$ of $\Gamma$ then there exist $\rho \in \text{Hom}_{c}(\Gamma, A)$ which descends to $\tau$.

An immediate consequence is the following realization result. It gives an answer to the question which polycyclic groups can act as affine crystallographic groups in terms of the set of crystallographic actions of the unipotent shadow.

Corollary 3.47 Let $\Gamma$ be a torsionfree polycyclic group, $\Theta \leq H_{\Gamma}$ a unipotent shadow for $\Gamma$. Then $\Gamma$ may be realized as an affine crystallographic group of type $A$ if and only if there exists a fixed point for the action of the semisimple kernel $T_{\Gamma}$ in the deformation space $D_{c}(\Theta, A)$.

Remark  Auslander (compare [1]) remarked that the unipotent shadow of a crystallographic group is crystallographic. The realization criterion of the Corollary may be seen as providing a converse to his result.

Applications to the existence problem for crystallographic actions on large classes of polycyclic groups follow. (See section 3.5)
The shadow map on deformation spaces

Let $\Gamma$ be a torsionfree polycyclic group, $\Gamma \leq H_1$ and $\Theta = \Gamma_c \leq \text{u}(H_1)$ a unipotent shadow for $\Gamma$ with respect to some (almost) nilpotent supplement $C$ for $\text{Fitt}(\Gamma)$. Recall from Proposition 1.36 that there exists a canonical crystallographic shadow map

$$s_u : \text{Hom}_c(\Gamma, A) \rightarrow \text{Hom}_c(\Theta, A).$$

We remark first (see Proposition 3.48) that the crystallographic shadow map induces a shadow map

$$\sigma_u : D_c(\Gamma, A) \rightarrow D_c(\Theta, A)$$

on the deformation spaces. We study here the properties of the shadow map $\sigma_u$ in order to obtain information about the deformation spaces $D_c(\Gamma, A)$. The results and methods are analogous to those in section 3.2 on the realization of finite extension for f.t.n.-groups. We can interpret them also as a solution to a suitable realization problem for polycyclic groups with a prescribed shadow.

The induced shadow map

Let $H_\Gamma$ be a real algebraic hull for $\Gamma$, and let $\Theta$ be a unipotent shadow. We choose a splitting $H_\Gamma = T \cdot U_\Gamma$ of $H_\Gamma$. We may view $T$ as a subgroup of $\text{Aut}(U_\Gamma)$, and $T$ acts on $D_c(\Theta, A)$ via the semisimple kernel $T_\Gamma \leq \text{Out}(U_\Gamma)$.

**Proposition 3.48** The crystallographic shadow map $s_u$ induces a continuous map

$$\sigma_u : D_c(\Gamma, A) \rightarrow D_c(\Theta, A)^T.$$

**Proof.** To see that there is an induced map $\sigma_u$ on deformation spaces which is given by $\sigma_u([\rho]) = [s_u(\rho)]$, it is enough to verify that, for all $g \in A$, $\rho \in \text{Hom}_c(\Gamma, A)$,

$$s_u(\rho^g) = s_u(\rho)^g.$$

To verify this, we have to recall the construction of the homomorphism $s_u(\rho) = \rho^u$. If $\Gamma = C\text{Fitt}(\Gamma)$ then, by Proposition 1.21, $\rho^u : \Gamma_c \rightarrow A$ is determined by the conditions $\rho^u(\gamma_u) = \rho(\gamma)_u$, for all $\gamma \in C$, $\gamma \in \text{Fitt}(\Gamma)$. Therefore the above formula is immediate since the Jordan-decomposition in $A$ is preserved by conjugation. The map $\sigma_u$ is continuous since (Corollary 1.44) $s_u$ is continuous.

Next we have to prove that $\sigma_u(\rho) \in D_c(\Theta, A)^T$. But this is an immediate consequence of Theorem 1.27. The homomorphism $\rho : \Gamma \rightarrow A$ extends to a $u$-simply transitive embedding $\rho_{H_\Gamma} : H_\Gamma \rightarrow A$. By Proposition 1.21, $\rho_{H_\Gamma}$ restricts to $s_u(\rho) = \rho^u$ on $\Theta = \Gamma_c \leq U_\Gamma$. Then the action of $T \leq \text{Aut}(U_\Gamma)$ on $\text{Hom}_c(\Theta, A)$ is induced by conjugation via $\rho_{H_\Gamma}(T)$. Hence, $T \leq \text{Aut}(U_\Gamma)\sigma_u([\rho])$.

We are going to prove here

**Theorem 3.49** Let $\Gamma$ be a torsionfree polycyclic group, $\Theta \leq U_\Gamma$ a unipotent shadow for $\Gamma$, and $T$ the semisimple kernel associated to $\Gamma$. Then the induced shadow map

$$\sigma_u : D_c(\Gamma, A) \rightarrow D_c(\Theta, A)^T$$

is a continuous bijection onto $D_c(\Theta, A)^T$.

It does not seem to be clear whether the map $\sigma_u$ is also a homeomorphism onto its image. However, this is the case on a certain stratum of $D_c(\Gamma, A)$. Let $D_c(\Theta, A)_s$ be the closed stratum of symmetric structures in the deformation space $D_c(\Theta, A)$. (Compare Theorem 3.34). We define a stratum $D_c(\Gamma, A)_s = \sigma_u^{-1}(D_c(\Theta, A)_s)$ in $D_c(\Gamma, A)$.
Theorem 3.50 The map $\sigma_u$ induces a homeomorphism

$$D_c(\Gamma, A) \xrightarrow{\approx} D_c(\Theta, A)^T_s.$$  

We postpone the proof of Theorem 3.46, Theorem 3.49 and Theorem 3.50 for a moment in order to derive some consequences.

Inheritance of the Hausdorff-property Though the topology on $D_c(\Gamma, A)$ is potentially finer than the topology induced from $D_c(\Theta, A)^T_s$ we still can deduce an immediate useful consequence.

Corollary 3.51 Let $\Gamma$ be a torsionfree polycyclic ACG, and let $\Theta$ be a unipotent shadow for $\Gamma$. If $D_c(\Theta, A)$ is a Hausdorff topological space, then $D_c(\Gamma, A)$ is Hausdorff too.

Some applications of this fact were already described in chapter 2. If $\Theta$ is abelian then $D_c(\Theta, A) = D_c(\Theta, A)_s$. Therefore it follows from Theorem 3.50.

Corollary 3.52 Let $\Gamma$ be a torsionfree polycyclic ACG, so that the unipotent shadow $\Theta$ is abelian. Then $D_c(\Gamma, A)$ is homeomorphic to a semi-algebraic set.

Inheritance of convexity Let us recall that every automorphism $\phi \in \text{Aut}(\Gamma)$ extends to a unique algebraic automorphism $\Phi$ of $H_\Gamma$. Since $\Phi$ is algebraic it restricts to an automorphism $\Phi^u$ of the unipotent radical $U_\Gamma$ of $H_\Gamma$. Therefore there is a natural induced homomorphism

$$\nu^u : \text{Aut}(\Gamma) \rightarrow \text{Aut}(U_\Gamma).$$

Since $U_\Gamma = U_\Theta$ is a Malcev hull for $\Theta$, the group $\text{Aut}(U_\Gamma)$ acts on the Deformation space $D_c(\Theta, A)$ and it is easy to see that the shadow map $s_u : \text{Hom}_c(\Gamma, A) \rightarrow \text{Hom}_c(\Theta, A)$ is equivariant with respect to $\nu^u$. We summarize:

Proposition 3.53 Let $\Gamma$ be a polycyclic ACG, $\Theta$ a unipotent shadow for $\Gamma$. Then the shadow map on deformation spaces

$$\sigma_u : D_c(\Gamma, A) \rightarrow D_c(\Theta, A)$$

is equivariant with respect to the actions of $\text{Aut}(\Gamma)$ and $\text{Aut}(U_\Gamma)$ and the homomorphism $\nu^u$.

As a particular consequence we see that the convexity properties of the unipotent shadow $\Theta$ will be inherited.

Theorem 3.54 Let $\Gamma$ be a polycyclic ACG and $\Theta$ a unipotent shadow for $\Gamma$. Then the following hold:

i) If $D_c(\Theta, A)$ is fixed pointed then $D_c(\Gamma, A)$ is fixed pointed too.

ii) If $D_c(\Theta, A)$ is convex then $D_c(\Gamma, A)$ is convex too.

Proof. If $D_c(\Theta, A)$ has a fixed point $[\rho]$ for $\text{Aut}(U_\Gamma)$, Theorem 3.49 implies that $\sigma_u(D_c(\Gamma, A))$ meets $[\rho]$. By equivariance and injectivity of the map $\sigma_u$, $[\rho]$ is a fixed point for $\text{Aut}(\Gamma)$ in $D_c(\Gamma, A)$.

We assume now that $D_c(\Theta, A)$ is convex. Let $T_\Gamma \leq \text{Out}(U_\Gamma)$ be the semisimple kernel for the shadow $\Theta$. It is easy to see that there is a well defined homomorphism

$$\nu^u : \text{Out}(\Delta) \rightarrow \text{N}_{\text{Out}(U_\Gamma)}(T_\Gamma)/T_\Gamma.$$  

Let $\mu \leq \text{Aut}(\Gamma)$ be a finite subgroup, and let $\overline{\mu} \leq \text{N}_{\text{Out}(\Gamma)}(T_\Gamma)$ be the preimage of $\nu^u(\mu)$. Since $T_\Gamma$ is reductive so is $\overline{\mu}$ which is a finite normal extension of $T_\Gamma$. If $[\rho]$ is a fixed point for $\overline{\mu}$ then, since $T_\Gamma \leq \overline{\mu}$, $\sigma_u(D_c(\Gamma, A))$ meets $[\rho]$. Moreover $[\rho]$ is a fixed point for $\mu$ too. Therefore $D_c(\Gamma, A)$ is convex. $\square$
Surjectivity of the shadow map  We start now with the proofs of the main theorems. Let $\Theta$ be a f.t.n.-group, $U_\Theta$ a real Malcev-hull for $\Theta$. For $\tau \in \text{Hom}_c(\Theta, A)$ we put $U = \tau(\Theta)$ to denote the unipotent simply transitive hull of $\tau(\Theta) \leq A$. Recall from Proposition \ref{prop:3.56} that in this situation conjugation in $A$ defines a natural map

$$c_\tau : N_A(U) \to \text{Aut}(U_\Theta)[\tau].$$

The map $c_\tau$ is then onto with kernel $C_A(U)$.

**Proposition 3.55** Let $T \leq \text{Aut}(U_\Theta)$ be a reductive algebraic subgroup. If $[\tau] \in D_c(\Theta, A)$ is a fixed point for $T$, then there exists an embedding of (real-) linear algebraic groups

$$j : T \to N_A(U)$$

which satisfies $c(j(t)) = t$, for all $t \in T$. Any two embeddings $j, j' : T \to N_A(U)$ with the latter property are conjugate by an element of $C_A(U)$.

**Proof.** By assumption $T \leq \text{Aut}(U_\Theta)[\tau]$. Let $H = c^{-1}(T)$. $H$ is a Zariski-closed subgroup of $N_A(U)$, and $H$ contains the unipotent group $C_A(U)$ as a normal subgroup. In fact, since $H / C_A(U)$ is isomorphic to $T$, $C_A(U)$ is the unipotent radical of $H$. Now by splitting of algebraic groups there exists a subgroup $T_H \leq H$ such that $H = T_H \cdot C_A(U)$. Since the restriction of $c$ to $T_H$ is an isomorphism of algebraic groups onto $T$, we can set $j = c^{-1} : T \to T_H \leq H$. We thus proved the existence of $j$.

We prove now the conjugacy statement. Let us remark first that every homomorphism $j : T \to N_A(U)$ which satisfies the assumptions maps $T$ into the group $H$. Moreover $j$ is uniquely determined by its image $j(T) \leq H$. Since $j(T)$ and $j'(T)$ are Levi-subgroups in $H$, $j'(T)$ is conjugated to $j(T)$ by an element of $C_A(U)$. Hence also the homomorphisms $j$ and $j'$ are conjugate.

Let $\Gamma$ be a torsionfree polycyclic group with real algebraic hull $H_\Gamma$, $\Theta \leq U_\Gamma$ a unipotent shadow for $\Gamma$. We say that $\rho_{H_\Gamma} : H_\Gamma \to A$ extends a homomorphism $\tau \in \text{Hom}_c(\Theta, A)$ if $\tau$ is the restriction of $\rho_{H_\Gamma}$ to $\Theta$.

**Corollary 3.56** A homomorphism $\tau \in \text{Hom}_c(\Theta, A)$ extends to a $u$-simply transitive embedding of linear algebraic groups $\rho_{H_\Gamma} : H_\Gamma \to A$ if and only if $[\tau] \in D_c(\Theta, A)$ is a fixed point for the semisimple kernel $T_\Gamma$. Any two such extensions $\rho_{H_\Gamma}, \rho_{H_\Gamma}'$ of $\tau$ are conjugate by an element of $C_A(\tau(\Theta))$.

**Proof.** Since $H_\Gamma = T_\Gamma U_\Gamma$ has a strong unipotent radical, we may view the reductive group $T_\Gamma$ as a subgroup of $\text{Aut}(U_\Gamma)$. Let us assume that $[\tau] \in D_c(\Theta, A)$ is a fixed point for $T_\Gamma$. By Proposition \ref{prop:3.55} there exists an embedding $j : T \to N_A(U)$, $U = \tau(\Theta) \leq A$, such that $c(j(t)) = t$, for all $t \in T$. We already used implicitly that, since $U_\Gamma$ is a real Malcev hull for $\Theta$, the homomorphism $\tau \in \text{Hom}_c(\Theta, A)$ extends to a $u$-simply transitive homomorphism $\tau_{U_\Gamma} : U_\Gamma \to A$. For $t \in T, u \in U_\Gamma$ we can then define

$$\rho_{H_\Gamma}(tu) = j(t) \tau_{U_\Gamma}(u).$$

By the properties of $j$, this defines a homomorphism $\rho_{H_\Gamma} : H_\Gamma \to A$ of algebraic groups which is clearly injective and $u$-simply transitive. Thus we proved the “if” part of the first statement. The “only if” we saw in Proposition \ref{prop:3.48}.

Now, if $\rho_{H_\Gamma}$ is any $u$-simply transitive algebraic group embedding which extends $\tau$ to $H_\Gamma$, then $j(t) = \rho_{H_\Gamma}(t)$ defines a homomorphism $j : T \to N_A(U)$, $U = \tau(\Theta)$, which satisfies $c(j(t)) = t$, for all $t \in T$. Therefore the conjugacy statement of Proposition \ref{prop:3.55} implies that any two $u$-simply transitive extensions of $\tau$ to $H_\Gamma$ are conjugate by an element $u \in C_A(U)$. \hfill \Box
Proof of Theorem 3.46  By the previous corollary, \( \tau \in \text{Hom}_c(\Theta, A) \) extends to a \( u \)-simply transitive algebraic group homomorphism \( \rho : H \to A \). Let \( \rho \) be the restriction of \( \rho_H \) to \( \Gamma \). Let \( \gamma \in \Gamma \) be so that \( \gamma_u \in \Theta \). Since \( \rho_H \) preserves the Jordan-decomposition, \( \tau(\gamma_u) = \rho_H(\gamma_u) = \rho(\gamma)_u \). Therefore \( \tau \) descends to \( \rho \) in the sense of Definition 1.29. By Theorem 1.27, \( \rho \) is a crystallographic homomorphism for \( \Gamma \). \( \square \)

Proof of Theorem 3.49  It follows from Theorem 3.46 that the map \( \sigma_u \) is onto \( D_c(\Theta, A) \). Let \( \tau \in \text{Hom}_c(\Theta, A) \). By Theorem 1.27 every extension \( \rho \in \text{Hom}_c(\Gamma, A) \) of \( \tau \) extends uniquely to a \( u \)-simply transitive homomorphism \( \rho_H : H \to A \) which then is also an extension of \( \tau \). Therefore the conjugacy statement of Corollary 3.56 implies that \( \sigma_u \) is injective. By Corollary 1.44, the shadow map \( s_u : \text{Hom}_c(\Gamma, A) \to \text{Hom}_c(\Theta, A) \) is algebraic, in particular \( s_u \) is continuous, hence also \( \sigma_u \). \( \square \)

Proof of Theorem 3.50  To prove that \( \sigma_u \) is an open mapping on \( D_c(\Gamma, A) \) we show that \( s_u : \text{Hom}_c(\Gamma, A) \to \text{Hom}_c(\Theta, A) \) admits a continuous section. This is done in a manner completely analogous to the proof of Lemma 3.35. We leave the details to the reader. \( \square \)

3.5 Applications to the existence of affine crystallographic actions

One of the main questions of the subject is to decide whether a given virtually polycyclic group \( \Delta \) may act as an affine crystallographic group. Not many general results on this question seem to be known. (But see [32] for the classification of torsionfree polycyclic groups which act crystallographically by affine Lorentz-transformations.) Our methods here apply when some information on the unipotent shadow of \( \Delta \) is available. In general, this suggests that the main difficulty of the problem lies in the existence and structure of crystallographic actions of torsionfree nilpotent groups. In section 2.3 we exhibited some classes of well understood f.t.n.-groups. Since these examples have strongly convex deformation spaces our realization results provide a positive answer for the existence of crystallographic actions for their finite extensions and also certain associated polycyclic groups. (See Corollary 2.14 Corollary 2.15)

Crystallographic actions of finite extensions  Let \( \Gamma \) be a f.t.n.-group. We say that \( \Gamma \) admits an invariant grading if the Lie algebra of the real Malcev hull \( U_\Gamma \) has a positive grading which is preserved by a Levi subgroup of \( \text{Aut}(U_\Gamma) \). (Compare Definition 2.22)

Corollary 3.57 Let \( \Gamma \) be a f.t.n.-group which satisfies one of the following conditions

i) \( \Gamma \) is of nilpotency class \( \leq 3 \),

ii) \( \Gamma \) has rank \( \leq 5 \),

iii) \( \Gamma \) admits an invariant grading.

Then \( \Gamma \) is isomorphic to an affine crystallographic group which satisfies the realization property. Moreover, if \( \Delta \) is a finite effective extension of \( \Gamma \) then \( \Delta \) is isomorphic to an affine crystallographic group, and \( \Delta \) has the realization property.

Proof.  By Theorem 2.13 the deformation space of \( \Gamma \) is convex, in particular, it is not empty. If \( \Delta \) is a finite effective extension of \( \Gamma \), then the associated kernel
$\beta : \Delta/\Gamma \to \text{Out}(\Gamma)$ has a fixed point in $\mathcal{D}_c(\Gamma, \text{Aff}(V))$. By Theorem 3.10 $\Delta$ admits an affine crystallographic action. Since the convexity of the deformation space is inherited to $\Delta$, also $\Delta$ has the realization property.

Remark Our corollary includes the result of Lee ([49]) who proved that a finitely generated torsionfree virtually nilpotent group $\Gamma$ with rank $\Gamma \leq 3$ acts as an affine crystallographic group.

Examples of affine crystallographic polycyclic groups Let $\Gamma$ be a torsionfree polycyclic group. We associated to $\Gamma$ its unipotent shadow $\Theta \leq U_\Theta$, and the semisimple kernel $T_\Gamma \leq \text{Out}(U_\Theta)$. Let $u_\Theta$ be the Lie algebra of $U_\Theta$. We say that $T_\Gamma$ centralizes a nonsingular derivation $D$ of $u_\Theta$ if there exists a subgroup $T \leq \text{Aff}(U_\Theta)$ which projects onto $T_\Gamma$ and centralizes $D$.

Corollary 3.58 Let $\Gamma$ be a torsionfree polycyclic group with unipotent shadow $\Theta$. We assume that $\Theta$ satisfies one of the following conditions

i) $\Theta$ is of nilpotency class $\leq 3$,

ii) $\Theta$ admits an invariant grading,

iii) The semisimple kernel $T_\Gamma$ of $\Gamma$ centralizes a nonsingular derivation $D$ of $u_\Theta$.

Then $\Gamma$ is isomorphic to an affine crystallographic group.

Proof. By Theorem 2.13 conditions i) and ii) imply that $\mathcal{D}_c(\Theta, \text{Aff}(V))$ is strongly convex. In particular, the semisimple kernel $T_\Gamma \leq \text{Out}(U_\Theta)$ has a fixed point in $\mathcal{D}_c(\Theta, \text{Aff}(V))$. If condition iii) is satisfied it follows from Proposition 2.21 that there exists an affine crystallographic action $\rho$ of $\Theta$ so that $T_\Gamma \leq \text{Out}(U_\Theta)[\rho]$, that is, $T_\Gamma$ fixes the point $[\rho] \in \mathcal{D}_c(\Theta, \text{Aff}(V))$. Therefore, in all three cases, it follows from Theorem 3.46 that $\Gamma$ admits an affine crystallographic action.

To illustrate the corollary, we consider the particular case that $\Gamma$ is a Zariski-dense lattice in a semi-direct product. Then the conditions on the unipotent shadow $\Theta$ of $\Gamma$ become in particular transparent.

Example 3.5.1 (semi-direct products) Let $G$ be a connected simply connected solvable Lie group, and $\Gamma \leq G$ a Zariski-dense lattice. We assume that $G$ is a semisimple semi-direct product, that is, the Lie algebra $\mathfrak{g}$ of $G$ splits as a direct sum

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n},$$

where $\mathfrak{n}$ is the nilpotent radical of $\mathfrak{g}$, and $\mathfrak{a}$ is an abelian subalgebra which acts on $\mathfrak{n}$ by semisimple transformations. The unipotent shadow $U$ of $G$ is obtained by killing the torus action. That is, the Lie algebra $\mathfrak{u}$ of $U$ is just the direct product

$$\mathfrak{u} = \mathfrak{b} \oplus \mathfrak{n},$$

of an abelian ideal $\mathfrak{b}$ and the ideal $\mathfrak{n}$. The Lie algebra $\mathfrak{a}$ acts on $\mathfrak{u}$ by centralizing $\mathfrak{b}$, and by the adjoint action on $\mathfrak{n}$. Moreover, the Zariski closure of the semisimple abelian subgroup $T_A \leq \text{Aut}(U)$, which belongs to $\mathfrak{a}$, is a torus $T \leq \text{Aut}(U)$ which projects onto the semisimple kernel $T_\Gamma \leq \text{Out}(U)$.

Therefore we get:

Corollary 3.59 Let $G$ be a simply connected solvable Lie group which is a semisimple semi-direct product $G = T_A N$, where $N$ is the nilradical of $G$. Assume that $G$ has a Zariski-dense lattice $\Gamma$, and assume further that one of the following conditions is satisfied:
Proof. Let $\Phi$.

We show now that the action of $\text{Aut}(\Delta)$ extends to an action of $\text{Aut}(\Delta)$. By Proposition 3.22, there exists, for every $\rho_\Delta$ algebraic hull $\rho_\Delta$, a unique homomorphism $\rho_\hat{} : U_\Delta \to A$ which satisfies $\rho_\hat{} \circ j = \rho$. For $\rho \in \text{Hom}_c(\Delta, A)$, $\Phi \in \text{Aut}(U_\Delta)$, we put then

$$\rho^\Phi = \rho_\hat{} \circ \Phi^{-1} \circ j.$$ 

Since $\rho^\Phi(\Gamma)$ is a lattice in the unipotent simply transitive group $\rho(\Gamma)$, it follows that $\rho^\Phi(\Gamma)$ is an ACG, hence $\rho^\Phi \in \text{Hom}_c(\Delta, A)$.

**Proposition 3.60** The correspondence $\rho \mapsto \rho^\Phi$ extends the $\text{Aut}(\Delta)$-action on $\text{Hom}_c(\Delta, A)$ to an action of $\text{Aut}(U_\Delta)$ on $\text{Hom}_c(\Delta, A)$. The $\text{Aut}(U_\Delta)$-action on $\text{Hom}_c(\Delta, A)$ is free, and induces an action of $\text{Out}(\Delta)$ on $\text{Hom}_c(\Delta, A)$.

In particular the action of $\text{Out}(\Delta)$ factorizes over $\text{Out}(U_\Delta)$.

**Proof.** Let $\Phi, \Psi \in \text{Aut}(U_\Delta)$. We calculate $(\rho^\Phi)^\Psi = \rho^\Phi \circ \Psi^{-1} \circ j$, where $\rho^\Phi : U_\Delta \to A$ is the extension of $\rho^\Phi : \Delta \to A$. Therefore $\rho^\Phi = \rho \circ \Phi^{-1}$, and we get $(\rho^\Phi)^\Psi = \rho \circ (\Phi^{-1} \circ \Psi^{-1} \circ j) = \rho^{\Phi \circ \Psi}$. So, in fact, $\text{Aut}(U_\Delta)$ acts on $\text{Hom}_c(\Delta, A)$. Obviously, the $\text{Aut}(U_\Delta)$-action extends the $\text{Aut}(\Delta)$-action, in the sense that, for $\phi \in \text{Aut}(\Delta)$, $\rho^\phi = \rho^{\phi \circ \Phi}$. Since $\Delta$ is Zariski-dense in $U_\Delta$, $\text{Aut}(U_\Delta)$ acts freely.

For $h \in U_\Delta$, $\gamma \in \Delta$, let $c_h : U_\Delta \to U_\Delta$ denote conjugation with $h$. We get $\rho^{\Phi \circ \gamma}(h) = \rho(h) \rho(\gamma)^{-1} \rho(h)^{-1}$. Therefore $\text{Out}(U_\Delta)$ acts on $\text{Hom}_c(\Delta, A)$.

**Remark.** The $\text{Aut}(U_\Delta)$-action depends on the embedding $j : \Delta \to U_\Delta$. Nevertheless, the image of $\text{Aut}(U_\Delta)$ as a group of morphisms of $\text{Hom}_c(\Delta, A)$ is independent from the choice of embedding. In particular, $\text{Out}(U_\Delta)$ maps to a well defined subgroup of homeomorphisms on the deformation space $\text{D}_c(\Delta, A)$.

3.6 Group actions on deformation spaces

Let $\Gamma$ be an f.t.n.-group, and $\Delta$ a finite effective extension group of $\Gamma$. We let $U$ denote the Malcev completion of $\Gamma$, and $U_\Delta$ the algebraic hull for $\Delta$. In this section we view $\Gamma$ as a lattice in the connected simply connected (real) Lie group $U_\Gamma = U_R$, and $\Delta$ as a subgroup of $U_\Delta = U_{\Delta R}$. The purpose of this final section is to exploit the functorial properties of the real algebraic hull $U_\Delta$ in the study of the deformation space $\text{D}_c(\Delta, A)$, and to provide some auxiliary results which we need at various places in this work.

**The hull functor on deformation spaces** Recall that the group $\text{Aut}(\Delta)$ acts in a natural way on $\text{Hom}_c(\Delta, A)$. In fact, the action of $\phi \in \text{Aut}(\Delta)$ is described for all $\rho \in \text{Hom}_c(\Delta, A)$ by

$$\rho \mapsto \rho^\phi = \rho \circ \phi^{-1}.$$

The $\text{Aut}(\Delta)$-action on $\text{Hom}_c(\Delta, A)$ induces then an $\text{Out}(\Delta)$-action on the deformation space $\text{D}_c(\Delta, A)$. We fix an embedding $j : \Delta \hookrightarrow U_\Delta$ of $\Delta$ as a lattice in its real algebraic hull $U_\Delta$. By Corollary 3.23 the embedding $j$ induces an embedding

$$\epsilon_j : \text{Aut}(\Delta) \longrightarrow \text{Aut}(U_\Delta).$$

We show now that the action of $\text{Aut}(\Delta)$ extends to an action of $\text{Aut}(U_\Delta)$. By Proposition 3.22 there exists, for every $\rho \in \text{Hom}_c(\Delta, A)$, a unique homomorphism $\rho_\hat{} : U_\Delta \to A$ which satisfies $\rho_\hat{} \circ j = \rho$. For $\rho \in \text{Hom}_c(\Delta, A)$, $\Phi \in \text{Aut}(U_\Delta)$, we put then

$$\rho^\Phi = \rho_\hat{} \circ \Phi^{-1} \circ j.$$ 

Since $\rho^\Phi(\Gamma)$ is a lattice in the unipotent simply transitive group $\rho(\Gamma)$ it follows that $\rho^\Phi(\Gamma)$ is an ACG, hence $\rho^\Phi \in \text{Hom}_c(\Delta, A)$.

**Proposition 3.60** The correspondence $\rho \mapsto \rho^\Phi$ extends the $\text{Aut}(\Delta)$-action on $\text{Hom}_c(\Delta, A)$ to an action of $\text{Aut}(U_\Delta)$ on $\text{Hom}_c(\Delta, A)$. The $\text{Aut}(U_\Delta)$-action on $\text{Hom}_c(\Delta, A)$ is free, and induces an action of $\text{Out}(\Delta)$ on $\text{D}_c(\Delta, A)$. In particular the action of $\text{Out}(\Delta)$ factorizes over $\text{Out}(U_\Delta)$.

**Proof.** Let $\Phi, \Psi \in \text{Aut}(U_\Delta)$. We calculate $(\rho^\Phi)^\Psi = \rho^\Phi \circ \Psi^{-1} \circ j$, where $\rho^\Phi : U_\Delta \to A$ is the extension of $\rho^\Phi : \Delta \to A$. Therefore $\rho^\Phi = \rho \circ \Phi^{-1}$, and we get $(\rho^\Phi)^\Psi = \rho \circ (\Phi^{-1} \circ \Psi^{-1} \circ j) = \rho^{\Phi \circ \Psi}$. So, in fact, $\text{Aut}(U_\Delta)$ acts on $\text{Hom}_c(\Delta, A)$. Obviously, the $\text{Aut}(U_\Delta)$-action extends the $\text{Aut}(\Delta)$-action, in the sense that, for $\phi \in \text{Aut}(\Delta)$, $\rho^\phi = \rho^{\phi \circ \Phi}$. Since $\Delta$ is Zariski-dense in $U_\Delta$, $\text{Aut}(U_\Delta)$ acts freely.

For $h \in U_\Delta$, $\gamma \in \Delta$, let $c_h : U_\Delta \to U_\Delta$ denote conjugation with $h$. We get $\rho^{\Phi \circ \gamma}(h) = \rho(h) \rho(\gamma)^{-1}$. Therefore $\text{Out}(U_\Delta)$ acts on $\text{D}_c(\Delta, A)$.

**Remark.** The $\text{Aut}(U_\Delta)$-action depends on the embedding $j : \Delta \to U_\Delta$. Nevertheless, the image of $\text{Aut}(U_\Delta)$ as a group of morphisms of $\text{Hom}_c(\Delta, A)$ is independent from the choice of embedding. In particular, $\text{Out}(U_\Delta)$ maps to a well defined subgroup of homeomorphisms on the deformation space $\text{D}_c(\Delta, A)$.
The presence of the Aut($U_\Delta$)-action on the deformation space $D_c(\Delta, A)$ comes from a correspondence of the affine crystallographic representations of $\Delta$ with certain representations of its real hull $U_\Delta$. We want to briefly explain this now:

Let $H$ be a real algebraic group which contains a normal unipotent subgroup $U$ of finite index, and assume that the centralizer of $U$ is contained in $U$. Note that the assumptions on $H$ are satisfied by the algebraic hull $U_\Delta$.

**Definition 3.61** A homomorphism $\bar{\rho} : H \to A$ is called crystallographic if $\bar{\rho}$ is injective and if $U$ acts simply transitively on $V$.

We then define the space $\text{Hom}_c(H, A)$ of crystallographic homomorphisms of $H$ and the corresponding deformation space $D_c(H, A) = \text{Hom}_c(H, A)/A$. The space $\text{Hom}(H, A)$ is a topological space with the compact open topology. Also by the results of chapter 1 there is a natural structure of real algebraic variety on $\text{Hom}_c(H, A)$.

We fix an embedding $j : \Delta \hookrightarrow U_\Delta$ of $\Delta$ in its real algebraic hull $U_\Delta$. Recall that every $\rho \in \text{Hom}_c(\Delta, A)$ is unipotent on the Fitting subgroup $\Gamma$ of $\Delta$. By Proposition 3.22, $\rho \in \text{Hom}_c(\Delta, A)$ extends to a representation $\rho_{\Delta}$ of $U_\Delta$. Without a proof we state:

**Theorem 3.62** The correspondence $\rho \mapsto \rho_{\Delta}$ defines a homeomorphism

$$h_j : \text{Hom}_c(\Delta, A) \xrightarrow{\approx} \text{Hom}_c(U_\Delta, A)$$

which commutes with the natural actions of automorphism groups on both spaces. In particular, $h_j$ induces a homeomorphism $\tilde{h}_j : D_c(\Delta, A) \xrightarrow{\approx} D_c(U_\Delta, A)$ which commutes with the natural outer automorphism actions.

The theorem implies that the topology and structure of the deformation space $D_c(\Delta, A)$ for $\Delta$, and also of the moduli space $M_c(\Delta, A)$, depend only on the properties of the algebraic hull for $\Delta$.

Here is another application. Look at the variety $C = \text{Hom}_c(U_\Delta, A) \times V$. Then $A$ acts on $C$, where $g(\rho, v) = (\rho^g, gv)$. Moreover, Aut($U_\Delta$) acts on the first factor of $C$, commuting with the action of $A$. Define $\mathcal{LC}(U) = C/A$.

**Proposition 3.63** The deformation space $D_c(U, A)$ is a quotient of the variety $\mathcal{LC}(U)$ by the induced unipotent action of Inn($U_\Delta$) on $\mathcal{LC}(U)$

**Proof.** Since $A$ acts transitively on $V$, we can deduce that

$$D_c(U, A) = \mathcal{LC}(U)/\text{Inn}(U_\Delta).$$

Furthermore, $\mathcal{LC}(U)$ is the set of $A$-conjugacy classes of étale representation with basepoint, and this corresponds to the algebraic variety of left-symmetric algebra products on the Lie algebra $u$. (See, for example, [12], for discussion.)

**Stabilizers in deformation space** For a moment we restrict our considerations to f.t.n.-groups. Let $\Gamma \leq U_\Gamma$ be a f.t.n.-group and assume that $\Gamma$ is an ACG. Let $\rho \in \text{Hom}_c(\Gamma, A)$ be an affine crystallographic representation, and $[\rho] \in D_c(\Gamma, A)$ the corresponding point in the deformation space. We let $U = \bar{\rho}(U_\Gamma) \leq A$ be the unipotent simply transitive hull for $\rho(\Gamma)$. The group Aut($U_\Gamma$) acts on $D_c(\Gamma, A)$. The stabilizer Aut($U_\Gamma$)$_{[\rho]} \leq \text{Aut}(U_\Gamma)$ of $[\rho]$ may be described in terms of the normalizer $N_A(U)$ of the unipotent simply transitive hull $U$. 


Proposition 3.64 There are canonical isomorphisms
\[
N_A(U) / C_A(U) \xrightarrow{\cong} \text{Aut}(U)[\rho],
\]
\[
N_A(U) / (C_A(U)U) \xrightarrow{\cong} \text{Out}(U)[\rho].
\]

Proof. $N_A(U) / C_A(U)$ acts by conjugation as a group of automorphisms of $U$. Since $\bar{\rho}$ is an isomorphism, there exists for each $g \in N_A(U)$ an element $\Phi_g \in \text{Aut}(U)$ such that $\rho^g = \rho^{\bar{\rho} g}$. This is, precisely, the condition that $[\rho^g] = [\rho]$. Therefore the map $g \mapsto \Phi_g$ defines a homomorphism of $N_A(U)$ onto $\text{Aut}(U)[\rho]$ with kernel $C_A(U)$. This factorizes to an isomorphism $N_A(U) / (C_A(U)U) \xrightarrow{\cong} \text{Out}(U)[\rho]$. \qed

We consider next the $\text{Aut}(\Gamma)$-action on $\mathcal{D}_c(\Gamma, A)$. Let us put
\[
\text{Out}_{A,\rho}(\Gamma) = N_A(\rho(\Gamma))/ (C_A(U)\rho(\Gamma)).
\]
We describe the stabilizer of $[\rho]$ in terms of the normalizer $N_A(\rho(\Gamma)) \leq A$.

Proposition 3.65 There are canonical isomorphisms
\[
N_A(\rho(\Gamma))/ C_A(U) \xrightarrow{\cong} \text{Aut}(\Gamma)[\rho],
\]
\[
\text{Out}_{A,\rho}(\Gamma) \xrightarrow{\cong} \text{Out}(\Gamma)[\rho].
\]

Proof. As in Proposition 3.60, we view $\text{Aut}(\Gamma)$ as a subgroup of $\text{Aut}(U)$, and with this identification $\text{Aut}(\Gamma)[\rho] = \text{Aut}(\Gamma) \cap \text{Aut}(U)[\rho]$. By Proposition 3.64, conjugation on $U$ induces a surjective homomorphism $c : N_A(U) \twoheadrightarrow \text{Aut}(U)[\rho]$. Therefore
\[
\text{Aut}(\Gamma)[\rho] = \text{Aut}(\Gamma) \cap c(N_A(U)) = c(N_A(\rho(\Gamma))).
\]
This shows that $N_A(\rho(\Gamma))$ is mapped onto $\text{Aut}(\Gamma)[\rho]$. The first isomorphism follows.

Now $\text{Out}(\Gamma)[\rho]$ is just the image of $\text{Aut}(\Gamma)[\rho]$ in $\text{Out}(\Gamma)$. Hence, $\text{Out}(\Gamma)[\rho]$ is a quotient of $N_A(\rho(\Gamma))$, and clearly the kernel is $C_A(U)\rho(\Gamma)$.

The normalizer of a unipotent simply transitive group Let $A \leq \text{Aff}(V)$ be a Zariski-closed subgroup, and $U \leq A$ a simply transitive unipotent group. We discuss some of the structure of the normalizer $N_A(U)$ of $U$ in $A$.

Lemma 3.66 The centralizer $C_A(U) \leq A$ is a unipotent normal subgroup of $N_A(U)$ and acts freely on $V$. The group $C_A(U)U$ is a unipotent normal subgroup in $N_A(U)$.

Proof. $C_A(U)$ acts freely since $U$ is simply transitive, and clearly $C_A(U)$ is a normal Zariski-closed subgroup of $A$. Since any reductive subgroup of $C_A(U)$ has fixed points on $V$, $C_A(U)$ is unipotent.

There are some subgroups of $\text{Aut}(U)$ which are associated to the normalizer $N_A(U)$, and which are important in our context. (These groups also carry some (differential-) geometric interpretations in terms of the simply transitive group action of $U$ and associated flat left invariant connection on the real Lie group $U$.) We do not go into these details here, but see a related discussion in [27, §3.11].) We consider the homomorphism
\[
c_U : N_A(U) \rightarrow \text{Aut}(U),
\]
where $c_U(g) : U \rightarrow U$ is given by conjugation with $g \in N_A(U)$. Let us define subgroups $\text{Aff}_A(U)$, $\text{Aut}_A(U)$, $\text{Inn}_A(U) \leq \text{Aut}(U)$ as follows
\[
\text{Aff}_A(U) = c(N_A(U)) \quad (\cong N_A(U)/C_A(U))
\]
\[
\text{Aut}_A(U) = c(N_{GL_A}(U)) \quad (\cong N_A(U)/U)
\]
\[
\text{Inn}_A(U) = c(N_{GL_A}(U)) \cap C_A(U)U \quad (\cong (C_A(U)U)/U)
\]

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Finally, we put \( \text{Out}_A(U) \) for the image of \( N_A(U) \) in \( \text{Out}(U) \). Note that the group \( \text{Aut}_A(U) \) projects onto \( \text{Out}_A(U) \). Proposition 3.64 implies now

**Proposition 3.67** Let \( \Gamma \) be a f.t.n.-group and \( \rho \in \text{Hom}_c(\Gamma, A) \). If \( U \leq A \) is the simply transitive hull for \( \rho(\Gamma) \) then there are natural isomorphisms

\[
\text{Aff}_A(U) \xrightarrow{\approx} \text{Aut}(U)[\rho], \\
\text{Out}_A(U) \xrightarrow{\approx} \text{Out}(U)[\rho].
\]

**Proof.** In fact, the representation \( \bar{\rho} \) defines an isomorphism \( \rho : U_{\bar{1}} \to U \), and the proof of Proposition 3.64 shows that \( \text{Aut}(U)[\rho] = \rho^{-1}\text{Aff}_A(U)\rho \). The statement for \( \text{Out}_A(U) \) follows as well.

The following are geometrically interesting special cases:

**Definition 3.68** Let \( U \leq A \) be a simply transitive unipotent subgroup. \( U \) is called \( A \)-symmetric if \( C_A(U) \) is transitive on \( V \). We call \( U \) fully \( A \)-symmetric if \( U \) is \( A \)-symmetric and \( c_U(\text{Out}_A(U)) = \text{Out}(U) \). Finally, we call \( U \) \( A \)-invariant if \( c_U(\text{Out}_A(U)) = \text{Out}(U) \), and \( U \) is called \( A \)-convex if \( c_U(\text{Out}_A(U)) \) contains a Levi subgroup of \( \text{Out}_A(U) \).

In our context these conditions are interpreted in terms of group actions on the deformation spaces.

**The normalizer of a unipotent ACG** Let us consider now the normalizer \( N_A(\rho(\Gamma)) \) of \( \rho(\Gamma) \) in \( A \). The Zariski-closure of \( U = \rho(\Gamma) \) is a simply transitive unipotent subgroup of type \( A \). By Zariski-denseness of \( \rho(\Gamma) \) in \( U \), we have \( N_A(\rho(\Gamma)) \leq N_A(U) \), as well as \( C_A(\rho(\Gamma)) = C_A(A) \). For \( x \in V \), we put \( N_{A,x}(\rho(\Gamma)) = N_A(\rho(\Gamma)) \cap A_x \). Our interest will be in the image of \( N_{A,x}(\rho(\Gamma)) \) in \( \text{Out}_A(\rho(\Gamma)) \). In a special case, the projection from \( N_{A,x} \) to \( \text{Out}_A(\rho(\Gamma)) \) is surjective, for all \( x \in V \).

**Lemma 3.69** If \( U \) is an \( A \)-symmetric simply transitive group then, for all \( x \in V \), the natural map

\[
N_{A,x}(\rho(\Gamma)) \to N_A(\rho(\Gamma))/C_A(U)
\]

is an isomorphism.

**Proof.** Put \( H = N_A(\rho(\Gamma))/C_A(U) \). \( H \) acts on the orbit space \( C_A(U)/V \). It is easy to see that \( N_{A,x}(\rho(\Gamma)) \) projects onto \( H_{[x]} \), where \( H_{[x]} \) is the stabilizer of \( [x] = C_A(U)x \). \( U \) is fully symmetric then \( Z_A(U) \) acts transitively on \( V \). Hence \( H = H_{[x]} \), \( N_{A,x}(\rho) \cong N_A(\rho(\Gamma))/C_A(U) \), \( \square \).

**Definition 3.70** \( \rho \in \text{Hom}_c(\Gamma, A) \) is called symmetric, if \( U = \rho(\Gamma) \) is \( A \)-symmetric.

Let \( c_\rho : N_A(\rho(\Gamma)) \to \text{Aut}(\Gamma) \) denote the conjugation homomorphism, and \( \text{Hom}_c(\Gamma, A) \), the set of symmetric crystallographic homomorphisms. The next fact was required in section 3.6.

**Proposition 3.71** Let \( F \leq \text{Out}(\Gamma) \) be a finite subgroup. There exists a continuous map \( s_x : \text{Hom}_c(\Gamma, A)^F_x \to \text{Hom}(F, A_x) \) such that \( c_\rho(s_x(\rho, g)) = g \), for all \( g \in F \).

**Proof.** For every \( \rho \in \text{Hom}_c(\Gamma, A) \), \( \bar{\rho}(U_{\bar{1}}) \) is a simply transitive subgroup of \( A \). Therefore there exists, for every \( g \in \text{Out}(\Gamma) \), a unique polynomial diffeomorphism \( \Phi_g \) of \( V \) which satisfies \( \Phi_g\bar{\rho}(u)x = \bar{\rho}(gu)x \). In particular, \( \Phi_g\bar{\rho}(u)\Phi_g^{-1} = \bar{\rho}(gu) \) holds, for all \( u \in U_{\bar{1}} \). Now, if \( \rho \) is fixed by \( F \), and \( \rho \in \text{Hom}_c(\Gamma, A) \), then by Proposition 3.65 and Lemma 3.69 there exists a \( \phi \in N_{A,x}(\rho(\Gamma)) \) such that \( \phi\bar{\rho}(u)\phi^{-1} = \bar{\rho}(gu) \).

We remark that, since \( C_A(U) \) is transitive on \( V \), it coincides with the centralizer of
$U$ in the group of diffeomorphisms of $V$. We conclude that $\Phi^{-1} \in C_A(U)$, and in particular $\Phi \in N_A(x)(U)$. So we set $s_x(\rho, g) = \Phi g$. The map $s_x$ is easily seen to be continuous, since it is the restriction of the continuous map on $\text{Hom}_c(\Gamma, A)$ which assigns to $\Phi g(\rho)$ its first derivative in $x$.

We remark that symmetric simply transitive actions may be constructed for many unipotent Lie groups $U$.

Example 3.6.1 Let $\mathcal{A}$ be a nilpotent associative algebra, finite dimensional over $\mathbb{R}$ and $u$ the commutation Lie algebra of $\mathcal{A}$. Then it is known, compare [4, 23], that the corresponding simply connected Lie group $U$ admits a simply transitive unipotent representation. It is possible to show that this representation is also symmetric, and that every symmetric simply transitive representation arises from this construction.
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