Stability of the Ellis-Bronnikov-Morris-Thorne Wormhole

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ABSTRACT

The stability of one type of the static Ellis-Bronnikov-Morris-Thorne wormholes is considered. These wormholes filled with radial magnetic field and phantom dust with a negative energy density.

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I. INTRODUCTION

In general relativity, wormholes are topological tunnels joining separate regions of space in the Universe (see, for example, [1, 2, 3, 4, 5, 6]; or even regions in different universes, in multi-verse models [7]). Recently, wormholes have been actively studied in general-relativity theory using both analytical [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] and numerical [21, 22, 23, 24, 25, 26] methods. The accretion of phantom matter onto black holes is also considered in [27, 28, 29, 30]. These investigations have elucidated many important questions, but a whole series of problems remain unsolved.

One of the important problems is the problem of stability. In this paper we consider the problem of the stability of one specific type [31] of the Ellis-Bronnikov-Morris-Thorne wormhole [3, 5, 32]. At one time, static wormholes filled with exotic scalar fields and possessing zero mass were thought to be stable against small perturbations [33]. However, they were shown to be unstable in [23, 34]. This raises the question of the stability of other models for wormholes. Here, we analyze the stability of a static wormhole with zero mass filled with exotic matter comprised of a mixture of radial magnetic field and exotic dust.
with matter density $\varepsilon < 0$. The problem was analyzed also in [35].

II. MODEL

Because of the importance of the problem of stability we analyze this problem separately.

We analyze the stability of the static wormhole with exotic matter that is a mixture of radial magnetic field and exotic dust (with a negative matter density and zero pressure [31]). In this model there is a balance of the gravitational attraction and repulsion.

The net gravitational acceleration is everywhere exactly equal to zero, and the masses of both entrances are also equal to zero.

The general form for a spherical metric in a coordinate system co-moving with the dust, in this case is

$$ ds^2 = d\tau^2 - e^\lambda dR^2 - e^\eta (R^2 + Q^2) d\Omega^2, $$

Here $g_{\tau\tau} = 1$, $\lambda$ and $\eta$ are functions of $\tau$ and $R$, and $Q$ is a constant with the dimensions of length determined by the size of the throat of the wormhole. The Einstein equations for this model are

$$ 8\pi T_{\tau\tau}^\tau = -\frac{Q^2 e^{-\lambda}}{(Q^2 + R^2)^2} + \frac{e^{-\eta} - e^{-\lambda}}{Q^2 + R^2} + \frac{1}{4} (2\lambda_{,\tau} \eta_{,\tau} + \eta^2_{,\tau}) - e^{-\lambda} \left[ \eta_{,RR} + \frac{\eta_{,R}(3\eta - 2\lambda)_{,R}}{4} + \frac{R(3\eta - \lambda)_{,R}}{R^2 + Q^2} \right] $$

$$ 8\pi T_{\tau R}^R = \frac{e^{-\eta} - e^{-\lambda}}{Q^2 + R^2} + \eta_{,\tau\tau} + \frac{3}{4} \eta^2_{,\tau} + e^{-\lambda} \left[ \frac{Q^2}{(Q^2 + R^2)^2} - \frac{\eta^2_{,R}}{4} - \frac{R\eta_{,R}}{Q^2 + R^2} \right] $$

$$ 8\pi T^\theta_\theta = 8\pi T^\varphi_\varphi = \frac{\lambda_{,\tau\tau} + \eta_{,\tau\tau}}{2} + \frac{\lambda^2_{,\tau} + \eta^2_{,\tau} + \eta_{,\tau} \lambda_{,\tau}}{4} - e^{-\lambda} \left[ \frac{Q^2}{(Q^2 + R^2)^2} + \frac{2\eta_{,RR} + \eta_{,R}(\eta_{,R} - \lambda_{,R})}{4} + \frac{R(2\eta_{,R} - \lambda_{,R})}{2(Q^2 + R^2)} \right] $$

Here, $T^k_l$ are the components of the stress-energy tensor. The unperturbed, static solution whose stability we wish to investigate corresponds to a Morris-Thorne metric with $\eta = 0$. 

and \( \lambda = 0 \). In this model, the components of the stress-energy tensor have the form
\[
8\pi T^{MT}_\tau = -8\pi T^{MT}_R = 8\pi T^{MT}_\theta = 8\pi T^{MT}_\varphi = -\frac{Q^2}{(Q^2 + R^2)^2} \tag{6}
\]
In a general, non-static model with a radial magnetic field, the components of the energy-momentum tensor have the form
\[
8\pi T^{MAG}_\tau = 8\pi T^{MAG}_R = -8\pi T^{MAG}_\theta = -8\pi T^{MAG}_\varphi = \frac{Q^2}{e^{2\eta(Q^2 + R^2)^2}} \tag{7}
\]
where the constant \( Q \) characterizes the magnetic field strength. The components of the stress-energy tensor for exotic dust have the form
\[
8\pi T^{DUST}_\tau = \frac{-Q^2(1 + D_p)}{(Q^2 + R^2)^2}, \quad T^{DUST}_R = T^{DUST}_\theta = T^{DUST}_\varphi = 0. \tag{8}
\]
Here, the function \( D_p(R, \tau) \) describes the perturbation of the dust density of the energy (relative to the Morris-Thorne solution).

Let us now consider perturbations of the static solution. We introduce the dimensional coordinates:
\[
x \equiv R/Q, \quad y \equiv \tau/Q, \quad Y \equiv 1 + x^2
\]
Let us consider Eqs. (2)-(5) linearized in the variables \( \eta \ll 1 \) and \( \lambda \ll 1 \). First of all we have for the stress-energy tensor
\[
8\pi Q^2 \delta T^\tau_\tau = -\frac{2D_p}{Y^2} - \frac{2\eta}{Y^2}, \tag{9}
\]
\[
8\pi Q^2 \delta T^R_\tau = -8\pi Q^2 \delta T^\theta_\theta = -\frac{2\eta}{Y^2}, \tag{10}
\]
Using (5) we have for \( T^R_\tau \)
\[
8\pi Q^2 T^R_\tau = \eta_{,xy} + \frac{x(\eta - \lambda)_{,y}}{Y} = 0, \tag{11}
\]
\[
\eta - \lambda = \frac{Y}{x}(F(x) - \eta_x), \tag{12}
\]
where \( F(x) \) is an arbitrary function. Equations (2)-(4) now are written in the following form:
\[
\eta_{,xx} + \frac{3x\eta_{,x} + \eta - x\lambda_x - \lambda}{Y} - \frac{2\eta + \lambda}{Y^2} = \frac{2D_p}{Y^2}, \tag{13}
\]
\[ \eta_{yy} \frac{x \eta_x}{Y} + \frac{x^2 \lambda}{Y^2} + \frac{1 - x^2}{Y^2} \eta = 0, \quad (14) \]

\[ \frac{\lambda_{yy} + \eta_{yy} - \eta_{xx}}{2} + \frac{x(\lambda - 2 \eta)_x}{2Y} - \frac{2 \eta - \lambda}{Y^2} = 0. \quad (15) \]

Substituting (12) into (13)-(15) yields

\[ \frac{\eta_x}{xY} + \frac{3 \eta}{Y^2} - F_x - \frac{1 + 2x^2}{xY} F = - \frac{2D_p}{Y^2}, \quad (16) \]

\[ \eta_{yy} + \frac{\eta}{Y^2} - \frac{x}{Y} F = 0, \quad (17) \]

\[ \left(2 \eta + \frac{Y}{x} \eta_x\right)_{yy} + \frac{\eta_x}{xY} - \frac{2 \eta}{Y^2} = \frac{F_x}{x} - \frac{F}{x} = 0. \quad (18) \]

We now introduce a new function \( \alpha \). Using the following relation

\[ \eta \equiv \alpha + xY F \quad (19) \]

Substituting this function into (16)-(17) yields

\[ \frac{\alpha_x}{xY} + \frac{3 \alpha}{Y^2} + 4 \frac{x}{Y} F = - \frac{2D_p}{Y^2}, \quad (20) \]

\[ \alpha_{yy} + \frac{\alpha}{Y^2} = 0, \quad (21) \]

and

\[ \alpha = A \sin \psi, \quad \psi = y/Y + y_0(x), \quad (22) \]

where \( A(x) \) and \( y_0(x) \) are the arbitrary amplitude and oscillation phase, respectively. Hence, the function \( \eta(y, x) \) always remains finite. However, the dust density and the function \( \lambda \) grow linearly with time [see (12) and (20)]:

\[ 2D_p = -\alpha_x Y/x - 3 \alpha - 4xY F \xrightarrow{y \gg 1} 2Ay \cos \psi / Y \propto y, \quad (23) \]

\[ \lambda \xrightarrow{y \gg 1} \propto y. \]

The physically slow growth is proportional to time, proportional to the perturbation of the radial co-moving coordinate, and the corresponding perturbations of the dust density. It is the consequence of the inertial motions of the dust after the initial perturbations.
III. CONCLUSION

We have considered the problem of the stability of the static wormhole model proposed by us earlier in [31].

In this model the wormhole is filled with radial magnetic field and exotic dust with a negative mass density. Here are no gravitational accelerations in the unperturbed system and the effective mass is equal to zero. We have shown that this model is stable for all spherical perturbation modes except for the radial motion of the dust due to its inertia. The growth of this mode is very slow, and is proportional to the time. It is obvious that this mode can easily be suppressed. The model with these properties can be important for the astrophysical applications [1, 2, 3].

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