Opetopic bicategories: comparison with the classical theory  

Eugenia Cheng  

Department of Pure Mathematics, University of Cambridge  
E-mail: e.cheng@dpmms.cam.ac.uk  

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Abstract  

We continue our previous modifications of the Baez-Dolan theory of opetopes to modify the Baez-Dolan definition of universality, and thereby the category of opetopic $n$-categories and lax functors. For the case $n = 2$ we exhibit an equivalence between this category and the category of bicategories and lax functors. We examine notions of strictness in the opetopic theory.

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Introduction

The aim of this paper is to shed some light on the opetopic definition of weak $n$-category by examining the case $n=2$. Our previous work has been on the relationship between different approaches to the theory of $n$-categories. In this work we make a further gesture towards comparison, demonstrating an equivalence between the opetopic and classical approaches to bicategories.

The opetopic definition we use here is a modified version of the one given by Baez and Dolan in [3]. Their definition proceeds in two stages. First, a language for describing $k$-cells is set up. Then a concept of universality is introduced, to deal with composition and coherence.

Our previous work has focused on the construction of $k$-cell shapes, that is, the theory of opetopes. In [8] and [7] we show that the approach of Baez and Dolan is equivalent to those of Hermida/Makkai/Power ([9]) and Leinster ([14]), but only with a crucial modification to their definition. The key difference is the use of symmetric multicategories with a category (rather than a set) of objects. We refer the reader to ([8]) for the full details.

The definition of $n$-category that we use in this work includes the above modification, and we refer to the notion thus defined as ‘opetopic $n$-category’.

Any proposed definition of $n$-category should at least be in some way equivalent to the classical definitions as far as the latter are understood. We exhibit such equivalence for the cases $n \leq 2$.

In [6] we followed through the effects of our previous modifications to include the recursive definition of opetopic set. In the present paper we begin, in Section 1, by completing this process, to modify the definition of universality and hence of $n$-category. The structure of the definitions is exactly as given in [3]; we do not seek to propose a new approach. Thus, universality is defined in terms of factorisation properties as in [3]. An opetopic $n$-category is then defined to be an $I$-opetopic set in which every niche has a universal occupant, and composites of universals are universal. By this point, the words of the definition are exactly the same, as all the differences have been absorbed in the earlier constructions.

Note that in this setting, composites of cells are not necessarily uniquely defined. This is one of the key ways in which the theory differs from the classical theory.

In [3] an $n$-functor is defined to be a morphism of the underlying $I$-opetopic sets, preserving universality. However, we consider the more general notion of ‘lax $n$-functor’, in which universality is not required to be preserved; questions of strictness are discussed later. So we define the category

\[ \text{Opic-}n\text{-Cat} \]
of opetopic $n$-categories and lax functors. Baez and Dolan do not construct an $(n+1)$-category of $n$-categories, and we do not seek to construct one in this work. Although this leaves the theory of opetopic $n$-categories still incomplete, we will see that for $n=2$ a comparison with the classical theory is possible even without such a construction.

Finally in this section, we restate some useful results from [3], with the above modifications.

In Section 2 we begin to unravel the definitions for simple cases in which the interlocking recursion terminates quickly. First we examine the cases $n=0$ and $n=1$. We exhibit equivalences

$$\text{Opic-0-cat} \simeq \text{Set}$$

and

$$\text{Opic-1-cat} \simeq \text{Cat}.$$ 

To construct a category $C$ from an opetopic 1-category $X$, the general idea is:

- the objects of $C$ are the 0-cells of $X$
- the arrows of $C$ are the 1-cells of $X$
- composition is defined by 2-ary universal 2-cells in $X$
- identities are given by 0-ary universal 2-cells in $X$
- axioms are seen to hold by considering universal 3-cells in $X$.

This begins to give us a general flavour of how the comparison proceeds for higher dimensions.

We then discuss some of the properties of $n$-cells in an $n$-category, which will be useful later.

In Section 3 we examine the case $n=2$ and prove the main theorem of this work, giving an equivalence

$$\text{Opic-2-cat} \simeq \text{Bicat}$$

where $\text{Bicat}$ is the category of bicategories and lax functors.

In comparing the opetopic and classical approaches to bicategories there are two main issues.

1) An opetopic 2-category has $m$-ary 2-cells for all $m \geq 0$, that is, a 2-cell may have a string of $m$ composable 1-cells as its domain; however a 2-cell in a bicategory has only one 1-cell as its domain.

2) An opetopic 2-category does not have chosen universal 2-cells, that is, 1-cell composition is not uniquely defined; however, in a bicategory $m$-fold composition is uniquely defined for $m = 0, 2$ (identities are considered as 0-fold composites).
The first matter is dealt with in a relatively straightforward (albeit technically tedious) way, by generating the necessary sets of $m$-cells from 1-ary 2-cells and 1-cell composites.

The second point involves an issue of choice. To construct a bicategory $\mathcal{B}$ from an opetopic 2-category $\mathcal{X}$ we must make some choices to give the 0- and 2-fold composites. The general idea is:

- the 0-cells of $\mathcal{B}$ are the 0-cells of $\mathcal{X}$
- the 1-cells of $\mathcal{B}$ are the 1-cells of $\mathcal{X}$
- the 2-cells of $\mathcal{B}$ are the 1-ary 2-cells of $\mathcal{X}$

We then make certain choices of 0-ary and 2-ary universal 2-cells. Then

- 1-cell composition in $\mathcal{B}$ is given by the chosen 2-ary universal 2-cells
- 1-cell identities in $\mathcal{B}$ are given by the chosen nullary universal 2-cells
- constraints are induced from composites of the chosen universals
- axioms are seen to hold by examining 4-cells.

In fact, we define a category $\text{Opic-2-Cat}_b$ of ‘biased opetopic 2-categories’ whose objects are opetopic 2-categories equipped with the above choices, but whose morphisms are not required to preserve those choices. So the morphisms are simply morphisms of the underlying 2-categories, and we clearly have an equivalence $\text{Opic-2-Cat}_b \simeq \text{Opic-2-Cat}$.

We then exhibit an equivalence

$$\text{Opic-2-Cat}_b \simeq \text{Bicat},$$

deferring the more involved calculations to the Appendix.

Finally, in Section 3.2 we study notions of strictness in the opetopic theory. We demonstrate that, while the definition of ‘lax $n$-functor’ strictifies easily to ‘weak $n$-functor’ and ‘strict $n$-functor’, the definition of ‘weak $n$-category’ neither laxifies nor strictifies easily.

**Terminology**

Although we have modified concepts from [3] we have generally not changed the terminology or notation.

Thus, a symmetric multicategory $Q$ has a category of objects $o(Q)$; an arrow $f$ has source and target $s(f)$ and $t(f)$ respectively. The slice $Q^+$ of $Q$ is as defined in [3] rather than [3]. Baez and Dolan show how to construct a pullback multicategory given a set over $o(Q)$. Since with our modification $o(Q)$ is now a category, we may construct a pullback given a category $\mathcal{A}$ and functor $\mathcal{A} \rightarrow o(Q)$. We refer the reader to [3] for the full definitions.
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1 Definitions

In [3], weak $n$-categories are defined as opetopic sets satisfying certain universality conditions. In our previous work ([8], [7], [6]) we have examined the theory of opetopes and opetopic sets. It remains to discuss the notion of universality.

We briefly recall here that an opetopic set is a presheaf over the category of opetopes. We may think of this as a set of $k$-cells for each $k \geq 0$, where a $k$-cell is a $k$-opetope with all constituent $j$-opetopes labelled by $j$-cells in a way that respects sources and targets. For the full definition of opetopic sets we refer the reader to [6]; for some examples of $k$-cells for $k \leq 3$ see Section 2.2.

1.1 Universality

In the definition of opetopic $n$-category, it is universality that deals with composition, constraints, axioms and coherence. We now modify the Baez-Dolan definition of universality in the context of the results of our earlier work. Furthermore, with clarity in mind we state the definition in a terser form than in [3].

In Section 1.2 we will have the following definition: An opetopic $n$-category is an opetopic set in which

i) Every niche has an $n$-universal occupant.

ii) Every composite of $n$-universals is $n$-universal.

We use the word ‘composite’ in the following sense. Let $a_1, \ldots, a_r$ and $c$ be $k$-cells in an opetopic set $X$, with $k \geq 1$. Given a universal $(k + 1)$-cell

$$u : (a_1, \ldots, a_r) \rightarrow c$$

we say that $c$ is a composite of $a_1, \ldots, a_r$. In particular given a universal cell

$$u : (a, b) \rightarrow c$$

we say that $c$ is a composite of $a$ and $b$ and also that $u$ and $b$ give a factorisation of $c$ through $a$ (Similarly $u$ and $a$ give a factorisation of $c$ through $b$).

If $a$ and $b$ are pasted at the target of $b$, say, we may represent this as
Alternatively, regarding $a$, $b$ and $c$ as objects of a symmetric multicategory at the next dimension up, we may represent this as

$$
\begin{array}{c}
\begin{tikzpicture}
  \node (a) at (0,1) {$a$};
  \node (b) at (1,2) {$b$};
  \node (c) at (2,1) {$c$};
  \draw[->] (a) -- (b);
  \draw[->] (b) -- (c);
\end{tikzpicture}
\end{array}
$$

We now define $n$-universality for $k$-cells and for $k$-cell factorisations. The definition is by descending induction on $k$. Recall that a niche may be regarded as a potential domain for a cell; so a niche for a $k$-cell is a $(k-1)$-pasting diagram. We say a cell is ‘in’ a particular niche if it does indeed have this pasting diagram as its domain.

**Definition 1.1** A $k$-cell $\alpha$ is $n$-universal if either $k > n$ and $\alpha$ is unique in its niche, or $k \leq n$ and (1) and (2) below are satisfied:

1. Given any $k$-cell $\gamma$ in the same niche as $\alpha$, there is a factorisation $u : (\beta, \alpha) \rightarrow \gamma$.

2. Any such factorisation is $n$-universal.

**Definition 1.2** A factorisation $u : (b, a) \rightarrow c$ of $k$-cells is $n$-universal if $k > n$, or $k \leq n$ and (1) and (2) below are satisfied:
(1) Given any $k$-cell $b'$ in the same frame as $b$, and any $(k+1)$-cell

$$v : (b', a) \rightarrow c$$

with $b'$ and $a$ pasted in the same configuration as $b$ and $a$ in the source
of $u$, there is a factorisation of $(k+1)$-cells $(u, y) \rightarrow v$

(2) Any such factorisation is itself $n$-universal.

If $n$ is clear from the context then we simply say ‘universal’.

Note that in the terminology of [3], the definition of ‘universal factorisation’ given above corresponds to a special case of ‘balanced punctured niche’. Furthermore, in each of the above definitions, each clause (1) and (2) corresponds to the assertion that a certain punctured niche is balanced.

Although we have still only defined ‘opetopic $n$-category’ in passing, the
following examples concerning particular cases in opetopic $n$-categories may
help to clarify the above definitions.

Examples 1.3

1) In an opetopic $n$-category the (unique) universal $1$-ary $(n+1)$-cells
have the form $x \rightarrow x$, since we have such universals given by the
targets of universal nullary $(n+2)$-cells

$$(\cdot) \rightarrow (x \rightarrow x).$$

2) In an opetopic $n$-category, a factorisation of $n$-cells is universal if and
only if it is unique. To see this, consider such a universal factorisation
$u : (b, a) \rightarrow c$. Now any $(n+1)$-cell is unique in its niche and hence
universal, so any $(n+1)$-cell $v : (b', a) \rightarrow c$ is a factorisation. But
then, by universality of the first factorisation, we have a (necessarily
universal) $(n+1)$-cell $y : b' \rightarrow b$ giving $b = b'$ and $u = v$, i.e. the
factorisation is unique.
3) In a 1-category, a 1-cell $x \xrightarrow{f} y$ is universal if and only if for any 1-cell $x \xrightarrow{g} z$ there is a \textit{unique} factorisation

$$
\begin{array}{c}
  y \\
  f \\
  \downarrow u \\
  x \\
  \quad \bar{g} \\
  \\
  \downarrow g \\
  \\
  z \\
\end{array}
$$

4) In a 2-category, a 1-cell $x \xrightarrow{f} y$ is universal if and only if for any 1-cell $x \xrightarrow{g} z$ there is a factorisation as above; however, we do not demand that such a factorisation be unique, but only universal. That is, given a 2-cell

$$
\begin{array}{c}
  y \\
  f \\
  \downarrow \theta \\
  x \\
  \quad h \\
  \\
  \downarrow g \\
  \\
  z \\
\end{array}
$$

there is a \textit{unique} factorisation

$$
\begin{array}{c}
  f \\
  \quad u \\
  \quad \bar{g} \\
  \quad \theta \\
  \quad h \\
  \\
  \downarrow g \\
  \\
  v \\
\end{array}
\quad \Downarrow \quad
\begin{array}{c}
  \downarrow \theta \\
\end{array}
$$

5) In a 3-category, $f$ as above is 3-universal if and only if any such factorisation $v$ as above is universal (rather than unique). That is, given any 3-cell

$$
\begin{array}{c}
  \Downarrow u \phi \\
\end{array}
\quad \Downarrow \theta \\
\end{array}
\quad \Downarrow \theta \\
\end{array}
\quad \Downarrow \theta \\
\end{array}
$$

there is a \textit{unique} factorisation

$$
\begin{array}{c}
  \quad \Downarrow u \phi \\
\end{array}
\quad \Downarrow \theta \\
\end{array}
\quad \Downarrow \theta \\
\end{array}
$$

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Definitions 1.4

- An $n$-coherent $Q$-algebra is a $Q$-opetopic set in which
  
i) Every niche has a universal cell in it (or universal ‘occupant’).
  
ii) Composites of universals are universal.

- A morphism of $n$-coherent $Q$-algebras is simply a morphism of their underlying $Q$-opetopic sets.

Observe that an $n$-coherent $Q$-algebra is specified uniquely up to isomorphism by the sets $X(k)$ and functions $f_k$ for $k \leq n+1$, since for $k \geq n+2$ the sets $X(k)$ and functions $f_k$ are induced. A morphism of such is then uniquely determined by the functions $F_k$ for $k \leq n$.

In [3] a morphism of $n$-coherent $Q$-algebras is required to preserve universality, yielding a stronger notion. We will later see that for $n = 2$ this gives weak rather than lax functors of bicategories. For the time being we consider the lax case only; we discuss strictness in Section 3.2.

1.2 Opetopic $n$-categories

We are now ready to state the definition of $n$-category. The statement here is exactly as in [3]; the differences have all been absorbed into the preliminary definitions. However, we note that the exact relationship between our complete modified definition and the exact Baez-Dolan original remains unclear.

Definitions 1.5

- An opetopic $n$-category is an $n$-coherent $I$-algebra.

- A lax $n$-functor is a morphism of $n$-coherent $I$-algebras.

We write $\text{Opic-}n\text{-Cat}$ for the category of opetopic $n$-categories and lax $n$-functors.

So an opetopic $n$-category is an opetopic set in which

i) Every niche has an $n$-universal occupant.

ii) Every composite of $n$-universals is $n$-universal.

We now restate, in this modified context, a useful proposition from [3]. This is a generalisation of the fact that in a category $\mathcal{C}$, for any objects $a,b$ we have a ‘homset’ $\mathcal{C}(a,b)$ of morphisms $a \to b$. Similarly, in a bicategory $\mathcal{B}$, we have ‘hom-categories’ $\mathcal{B}(a,b)$ whose objects are 1-cells and morphisms 2-cells; so we also have, for any 2-cells $\alpha,\beta$, homsets $\mathcal{B}(\alpha,\beta)$. 

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Thus in an $n$-category we expect to have ‘hom-$(n - m)$-categories’ of $m$-cells. However, since here the domain of an $m$-cell is not necessarily just a single $(m - 1)$-cell, instead of having just a pair of $(m - 1)$-cells as above, we need an $m$-frame to give the domain and codomain specifying the hom-category. (Recall that an $m$-frame consists of an $(m - 1)$-pasting diagram together with an $(m - 1)$-cell that might be the domain and codomain of an $m$-cell.)

**Proposition 1.6** Let $X$ be an $n$-coherent $Q$-algebra. Then for $m \leq n$ any $m$-frame determines an opetopic $(n - m)$-category.

The idea is first to restrict $X$ to cells of dimension $m$ and above; this is clearly still $(n - m)$-coherent. We can then restrict to only those cells in the given frame $\alpha$ by ‘pulling back’ along the morphism

$$1 \xrightarrow{\alpha} S(m).$$

So we follow Baez-Dolan and use the following construction of ‘pullback opetopic set’. Let $Q$ and $Q'$ be tidy symmetric multicategories with object-categories $C$ and $C'$ respectively, with $C \simeq S$ and $C' \simeq S'$ discrete. Let $X$ be a $Q$-opetopic set. Suppose we have a morphism $S' \to S$. Then we may construct a pullback opetopic set $X'$ by induction as follows. Let $X'(0)$ be given by the pullback

$$
nonumber X'(0) \longrightarrow X(0) \hspace{1cm} S'(0) \longrightarrow S(0). $$

Now we have equivalences

$$o(Q_{X(0)}^+) \sim S(1),$$

$$o(Q'_{X'(0)}^+) \sim S'(1)$$

where $S(1)$ and $S'(1)$ are discrete. So the morphism

$$X'(0) \to X(0)$$

induces a morphism

$$S'(1) \to S(1)$$

and we may form a pullback opetopic set of $X_1$ along this morphism; we set this to be $X'_1$, the underlying $Q'_{X'(0)}^+$-opetopic set of $X'$.
Proposition 1.7 (See [3], Proposition 45) If $X$ is $n$-coherent then $X'$ is $n$-coherent.

Proof. It is easy to check that a cell in $X'$ is universal if and only if the corresponding cell in $X$ is universal, and that a factorisation in $X'$ is universal if and only if the corresponding factorisation in $X$ is universal. □

Proof of Proposition 1.6. Let $\alpha$ be an $m$-frame in $X$ with $m \leq n$, so $\alpha \in S(m)$. Now $X$ determines an $(n-m)$-coherent $Q(m)$-algebra, and we have a morphism 

$$o(I) = 1 \xrightarrow{\alpha} S(m)$$

so we may form a pullback $I$-opetopic set along this morphism.

By Proposition 1.7, this is $(n-m)$-coherent, i.e. it is an opetopic $(n-m)$-category. □

Examples 1.8

1) In an $n$-category $X$, every 1-frame determines an $(n-1)$-category.

A 1-frame in $X$ is given by

\[
\begin{array}{ccc}
\alpha & \equiv & \theta \\
\alpha_1 & \equiv & \theta \\
\vdots & \equiv & \theta
\end{array}
\]

We denote the induced $(n-1)$-category by $\text{Hom}(a,b)$ or $X(a,b)$; its cells are of the form shown below.

0-cells

\[
\begin{array}{ccc}
a & \xrightarrow{f} & b
\end{array}
\]

1-cells

\[
\begin{array}{ccc}
a & \xrightarrow{	riangledown \alpha} & b
\end{array}
\]

2-cells ($k$-ary)

\[
\begin{array}{ccc}
\alpha_1 & \equiv & \theta \\
\alpha_2 & \equiv & \theta \\
\vdots & \equiv & \theta \\
\alpha_k & \equiv & \theta
\end{array}
\]

\[
\begin{array}{ccc}
\alpha & \equiv & \theta
\end{array}
\]

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2) Given a 2-frame

\[
\begin{array}{c}
\begin{array}{c}
  a \\
  \downarrow \\
  c \\
\end{array}
\end{array}
\]

say, we have an \((n - 2)\)-category whose cells are of the form shown below.

0-cells

\[
\begin{array}{c}
\begin{array}{c}
  a \\
  \downarrow \alpha \\
  c \\
\end{array}
\end{array}
\]

1-cells

\[
\begin{array}{c}
\begin{array}{c}
  a \\
  \downarrow \alpha \\
  c \\
\end{array}
\end{array} \equiv_{\theta} \begin{array}{c}
\begin{array}{c}
  a \\
  \downarrow \beta \\
  c \\
\end{array}
\end{array}
\]

2-cells (\(k\)-ary)

\[
\begin{array}{c}
\begin{array}{c}
  \alpha_0 \equiv_{\theta_1} \alpha_1 \equiv_{\theta_2} \alpha_2 \equiv_{\theta_3} \cdots \equiv_{\theta_k} \alpha_k \equiv_{\phi} \alpha_0 \equiv_{\theta} \alpha_k
\end{array}
\end{array}
\]

3) Given an \((n - 1)\)-frame we have a 1-category whose objects are \((n - 1)\)-cells and arrows are 1-ary \(n\)-cells.

## 2 Preliminary examples

Any proposed definition of \(n\)-category should at least be in some way equivalent to the classical definitions as far as the latter are understood. In \[3\] Baez and Dolan examine the case \(n = 1\) but do not explain how their definition is equivalent to the classical definition of bicategories in the case \(n = 2\). This is perhaps because, without the modifications described in this our earlier work, such an equivalence does not arise. The main results of this work gives an equivalence between the (modified) opetopic and the classical approaches to bicategories. We begin in this section with some examples to help clarify and motivate the later arguments; our general aim is to shed some light on the inescapable loops in the definition of universality, as well as to compare the resulting structures with the classical ones.
Note that for \( n \leq 1 \) the difference between our definition and the original Baez-Dolan definition is not yet apparent. The result for \( n = 1 \) is described in [3] (Example 42); we include it here (with more detail) for completeness.

### 2.1 Opetopic 0-categories

An opetopic 0-category \( X \) is determined, up to isomorphism, by the set \( X(0) \). For, given any 0-cell \( a \in A \), the following nullary 2-niche

\[
\begin{array}{c}
\downarrow \\
\bullet \\
\end{array} 
\]

must have a unique occupant, and so the unique occupant of the following 1-niche

\[
\begin{array}{c}
\bullet \\
\Rightarrow \\
\bullet \\
\end{array} 
\]

must have \( a \) as its target, and we can call the 1-cell \( 1_a \), giving

\[
X(1) \cong \{ a \rightarrow a : a \in A \}.
\]

**Proposition 2.1** There is an equivalence

\[
\text{Opic-0-Cat} \xrightarrow{\sim} \text{Set}
\]

surjective in the direction shown.

**Proof.** We construct such a functor, \( \zeta \). Let \( X \) be an opetopic 0-category. We put

\[ \zeta(X) = X(0). \]

A morphism \( f : X \rightarrow Y \) of opetopic 0-categories is uniquely specified by the function \( f_0 : X(0) \rightarrow Y(0) \) so we put

\[ \zeta(f) = f_0. \]

Conversely, given a set \( A \), we have an opetopic 0-category \( X \) such that \( \zeta(X) = A \); \( X \) is defined by

\[
\begin{align*}
X(0) &= A \\
X(1) &= \{ a \xrightarrow{1_a} a : a \in A \}.
\end{align*}
\]

So \( \zeta \) is surjective, and it is clearly full and faithful, giving an equivalence as required. \[\square\]
2.2 Opetopic 1-categories

We first clarify our notation. We draw

• 1-cells as arrows

• 2-cells as

• 3-cells as

These represent isomorphism classes of objects in the appropriate symmetric multicategory. We give below some typical examples of openings, niches, frames and cells.

1-opening

1-niche

1-frame

1-cell

3-ary

nullary

2-opening

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Where confusion is unlikely, we may omit some lower-dimensional labels once the higher-dimensional ones are in place, as in the following examples.
We begin by constructing a functor
\[ \zeta : \text{Opic-1-Cat} \rightarrow \text{Cat}; \]
we will eventually show that this functor is an equivalence.

- On objects

Given an opetopic 1-category \( X \) we define a category \( \mathcal{C} = \mathcal{C}_X \) as follows. First set \( \text{ob} \mathcal{C} = X(0) \). Then, given objects \( a, b \in X(0) \), let \( \mathcal{C}(a, b) \) be the preimage of \( a \xrightarrow{?} b \) under \( f_1 \). (Recall that we have a 0-category \( \text{Hom}(a, b) \), that is, a set.)

Composition and identities in \( \mathcal{C} \) are defined according to the 2-cells in \( X \) as follows. For composition consider 1-cells \( a \xrightarrow{f} b, b \xrightarrow{g} c \). We have the following 2-niche

\[
\begin{array}{c}
\xymatrix{ f \\
\downarrow^? \\
g }
\end{array}
\]

which has a unique occupant; we write it as

\[
\begin{array}{c}
\xymatrix{ f \\
\downarrow^u \\
gf }
\end{array}
\]

For identities we have already observed (Examples [3]) that in an opetopic \( n \)-category the universal 1-ary \((n+1)\)-cells are of the form \( a \rightarrow a \). Explicitly, for \( n = 1 \) we have for any \( a \in X(0) \) a nullary 2-niche

\[
\begin{array}{c}
\xymatrix{ a \\
\downarrow^? \\
a }
\end{array}
\]

which must have a unique occupant. So we write it as

\[
\begin{array}{c}
\xymatrix{ a \\
\downarrow^u \\
1_a \\
a }
\end{array}
\]

and check that this does indeed act as the identity with respect to the composition defined above. We seek the unique occupant of the niche.
Certainly we have the following 3-niche

\[ 1_a \downarrow u \rightarrow f, 1_a \uparrow u \rightarrow f.1_a \]

with a unique occupant. So by Example 1.3(1), we have \( f.1_a = f \) as required. Similarly \( 1_a.f = f \).

It remains to check that associativity holds. Given 1-cells

\[ a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d \]

we have the following universal 3-cells

\[ f \xrightarrow{(hg)f} h \iff f \xrightarrow{(hg)f} h \]

\[ f \xrightarrow{h(gf)} h \iff f \xrightarrow{h(gf)} h \]

But \( u_1 \) and \( u_2 \) are occupants of the same 2-niche; by uniqueness they must be the same, giving

\[ (hg)f = h(gf) \]

as required. So we have defined a category, and we set

\[ \zeta(X) = C_X. \]

Observe that we find composites and identities by considering universal 2-cells, and we check axioms by considering universal 3-cells.
On morphisms

Given a morphism of opetopic 1-categories $F : X \to Y$ we seek to define a functor $F : C_X \to C_Y$. We define the action of $F$ on objects and arrows by the functions

$$F_0 : X(0) \to Y(0)$$
$$F_1 : X(1) \to Y(1).$$

We check functoriality. By definition of morphisms of opetopic 1-categories, the following diagram commutes

\[
\begin{array}{ccc}
X(1) & \xrightarrow{o(I_X(0)^+) = o(Q(1))} & Y(1) \\
F_1 \downarrow & & \downarrow \\
Y(1) & \xrightarrow{o(I_Y(0)^+) = o(R(1))} & Y(1)
\end{array}
\]

giving

$$F(\text{dom } f) = \text{dom } (Ff)$$
$$\text{and } F(\text{cod } f) = \text{cod } (Ff).$$

Now the function

$$F_2 : X(2) \to Y(2)$$

makes the following diagram commute

\[
\begin{array}{ccc}
X(2) & \xrightarrow{o(Q(1)_{X(1)^+}} & Y(2) \\
F_2 \downarrow & & \downarrow \\
Y(2) & \xrightarrow{o(R(1)_{Y(1)^+}} & Y(2)
\end{array}
\]

so under the action of $F_2$ the following (universal) 2-cell in $X$

\[
\begin{array}{ccc}
f & \xrightarrow{\Delta u} & g \\
\downarrow & & \downarrow \\
gf & & gf
\end{array}
\]

gives the following 2-cell in $Y$
and so we have $F(g \circ f) = Fg \circ Ff$ by uniqueness of 2-niche occupants.

Similarly consider the following nullary 2-cell in $X$:

\[
\begin{array}{c}
\Downarrow \quad u \\
\quad a \\
\quad 1_a \\
\end{array}
\]

Under the action of $F_2$ we have the following 2-cell in $Y$:

\[
\begin{array}{c}
\Downarrow \quad F \ast \quad F_u \\
F a \\
F(1_a) \\
\end{array}
\]

and so we have $F(1_a) = 1_{Fa}$ by uniqueness of 2-niche occupants.

So $F$ is a functor as required. Observe that in the above construction we do not need to stipulate that universality be preserved.

Finally, before showing that $\zeta$ is an equivalence, we characterise universal 1-cells as invertibles.

**Proposition 2.2** A 1-cell $f$ in $X$ is universal if and only if it is invertible as an arrow of $\mathcal{C}_X$.

**Proof 1 (bare hands).** Let $a \xrightarrow{f} b$ be a universal 1-cell in $X$. We certainly have a 1-cell

\[
\begin{array}{c}
a \\
\quad 1_a \\
\quad a
\end{array}
\]

So by clause (1) of the definition of universal 1-cell we have a factorisation, that is a 1-cell

\[
\begin{array}{c}
b \\
\quad g \\
\quad a
\end{array}
\]

and a universal 2-cell
so we have $gf = 1_a$.

Now consider the 1-cell

$$a \xrightarrow{f} b.$$  

Similarly, we have a universal 2-cell

$$\begin{array}{cccc}
  & & b \\
  & f & \downarrow & u \\
 f & \rightarrow & 1_b & \\
 a & & f & \rightarrow b
\end{array}$$

Now by clause (1) of the definition of universal 2-cell, if we have a 2-cell

$$\begin{array}{cccc}
  & & f \\
  & \downarrow & h \\
 f & \rightarrow & \\
 a & & f
\end{array}$$

then we have a factorisation, so we certainly have a 2-cell

$$\begin{array}{cccc}
  & & h \\
  & \psi & \rightarrow & 1_b \\
 h & \rightarrow & \\
 a & & f
\end{array}$$

By uniqueness of 2-niche occupants, this gives

$$hf = f \Rightarrow h = 1_b.$$  

Now consider the following 3-cell

$$\begin{array}{cccc}
  & & g \\
  & f & \rightarrow & f \\
 f & \rightarrow & 1 & \rightarrow & f \\
 a & & f & \rightarrow b
\end{array} \Rightarrow \begin{array}{cccc}
  & & g \\
  & f & \rightarrow & f \\
 f & \rightarrow & f & \rightarrow b
\end{array}$$
giving $f(gf) = f$. But by associativity we have

$$f(gf) = (fg)f = f$$

so we have $fg = 1_b$. So if $f$ is universal in $X$ then $f$ is invertible in $C_X$.

Conversely, suppose $f$ is invertible in $C_X$, so we have in $X$ 2-cells

We now show that $f$ is universal:

i) Given any 0-cell $b' \in X(0)$ and 1-cell $a \xrightarrow{h} b'$ we have the following 3-cell

so by associativity the following universal 2-cell

giving a factorisation for $h$ as required.

ii) We show that any such factorisation is universal. Let

be such a factorisation. Then given any other 2-cell
we need to exhibit a factorisation

Now
\[ h = sf \Rightarrow hg = sfg = s \]
so we have \( s' = hg = s \) and 3-cell

as required. Any such factorisation is then trivially universal.

So if \( f \) is invertible then \( f \) is universal, and the proposition is proved. \( \square \)

Although the above calculations may help in understanding the definitions, the proposition may be proved more quickly using the Yoneda Lemma as follows.

**Proof 2 (Yoneda).** \( f \) is universal in \( X \) if and only if

1) Given any arrow \( b \xrightarrow{g} c \) there is an arrow \( b \xrightarrow{g} c \) such that \( \bar{g}f = g \)
and

2) \( h_1f = h_2f \Rightarrow h_1 = h_2 \)
i.e. for all \( c \in \text{ob} \ C \) the function

\[ f^* : \ C(b,c) \rightarrow C(a,c) \]

\[ h \mapsto h \circ f \]
is an isomorphism. But this is true if and only if \( f \) is isomorphism since the Yoneda embedding is full and faithful. \( \square \)

In a later work ([5]) we propose a characterisation of universality that generalises the above Yoneda result.
Proposition 2.3 \textit{The functor }$\zeta$\textit{ exhibits an equivalence of categories}

\[ \text{Opic-1-Cat} \xrightarrow{\sim} \text{Cat} \]

\textit{surjective in the direction shown.}

\textbf{Proof.} We have defined a functor

\[ \zeta : \text{Opic-1-Cat} \rightarrow \text{Cat} \]

above, and it is clearly full and faithful; we show that it is surjective.

Given any (small) category $\mathcal{C}$, we may construct an opetopic 1-category $X$ with $X(0) = \text{ob } \mathcal{C}$ and $X(1) = \text{arr } \mathcal{C}$. We see immediately that every 1-niche has a universal occupant $a \xrightarrow{1_a} a$. The set $X(2)$ is defined as follows.

Every nullary 2-niche

\[ \begin{array}{c}
\downarrow \\
 a \rightarrow a
\end{array} \]

has a unique occupant

\[ \begin{array}{c}
\downarrow \\
 a \xrightarrow{1_a} a
\end{array} \]

and every $m$-ary 2-niche

\[ f_1 \downarrow \cdots \downarrow f_m \]

has a unique occupant

\[ f_1 \downarrow \cdots \downarrow f_m \]

Furthermore, since a 1-cell is universal if and only if it is invertible as an arrow of $\mathcal{C}$, composites of universals are universal.

So $X$ is 1-coherent, and clearly $\zeta(X) = \mathcal{C}$. \hfill \Box
The definition of universality works from the top down: universal cells are understood via cells in the dimension above, and the starting point is that all cells in dimensions higher than \((n + 1)\) are trivial. So in effect, \(n\)-cells result from the ‘first’ step of the induction; we now make some general observations about \(n\)-cells, which will be useful later.

Recall (Example 1.8(3)) that every \((n-1)\)-frame determines an opetopic 1-category. So we have an opetopic 1-category of \((n-1)\)-cells and 1-ary \(n\)-cells, or, by Proposition 2.3, a category.

Let \(X\) be an opetopic \(n\)-category. First recall that composites of \(n\)-cells in \(X\) are uniquely determined, since occupants of \((n+1)\)-niches are unique. Also, composition of \(n\)-cells is strictly associative and a morphism of opetopic \(n\)-categories must be strictly functorial on \(n\)-cell composites. (In fact, we have a symmetric multicategory of \((n-1)\)-cells and \(n\)-cells.)

Now consider an \(n\)-niche \(\alpha\) in \(X\). Then, given any universal occupant \(u\), every occupant \(f\) of \(\alpha\) factors uniquely as 

\[
f = g \circ u
\]

where \(g\) is a 1-ary \(n\)-cell. So, for any such universal, we may express the set of occupants of \(\alpha\) as 

\[
g \circ u \text{ such that } g \in X(n)_1 \text{ and } s(g) = t(u)
\]

where \(X(n)_1\) is the set of 1-ary \(n\)-cells. Given any other universal occupant \(u'\), we then have 

\[
u' = x \circ u
\]

for some (unique) universal \(x\). So we have 

\[
\{g' \circ u'\} = \{g \circ u\}
\]

since 

\[
g' \circ u' = g' \circ (x \circ u) = (g' \circ x) \circ u.
\]

More generally, given any non-empty set \(U\) of universal occupants of \(\alpha\), the set of occupants of \(\alpha\) may be expressed as 

\[
\{g \circ u : u \in U, g \in X(n)_1, s(g) = t(u)\} \sim
\]

Here \(\sim\) is the equivalence relation generated by 

1) \(g \circ u \sim g' \circ u' \iff g = g' \circ x_{uu'}\)

2) \(1 \circ u \sim u\)

where for any \(u, u' \in U\), \(x_{uu'}\) is the unique universal such that 

\[
u' = x_{uu'} \circ u.
\]
3 Bicategories

We are now ready to turn our attention to the case $n = 2$. We show how to construct a classical bicategory from an opetopic 2-category, leading to the main theorem, which shows how the opetopic and classical theories of bicategories are equivalent.

An important difference between this construction and that for the case $n = 1$ is that an element of choice now arises. The universality condition stipulates that every niche should have a universal occupant, but does not specify such universals. This approach differs from the approach of Leinster ([15]), for example, in which contractibility is defined as a property but specific contractions are then given.

This approach also differs from the classical approach to bicategories, in which binary and nullary composites of 1-cells are specified, even though $m$-fold composites are not, for $m > 2$. (Note that 1-cell identities are considered as ‘nullary composites’.) Leinster refers to this theory as being ‘biased’ towards binary composites; in [14], he introduces the notion of unbiased bicategory. The theory of bicategories is made ‘unbiased’ by specifying $m$-fold composites for all $m$. This theory turns out to be equivalent to the classical one ([15]). Leinster also comments that, provided at least one choice has been made for each of $k = 0$ and some $k \geq 2$, an equivalent theory of bicategories may be formed.

Another way of eliminating bias from a bicategory might be to choose no specified composites. We will later see that this is how the opetopic approach may be interpreted. Once we have shown that this theory is equivalent to the classical one, it is easy to see which choices give rise to a theory of bicategories, and it follows immediately that all such theories are equivalent. This issue turns out to be related to the question of strictness, and we discuss these notions in Section 3.2.

3.1 The main theorem: equivalence with the classical theory

We show that the opetopic and classical theories of bicategories are equivalent, in the following sense.

**Theorem 3.1** Write $\text{Bicat}$ for the category of bicategories and morphisms (lax functors). Then

$$\text{Opic-2-Cat} \simeq \text{Bicat}.$$ 

Given an opetopic 2-category $X$, we seek to construct a bicategory $B$ (using the definition given in [13]). To do this we need to make some choices of universal 2-cells. The general idea is

- the 0-cells of $B$ are the 0-cells of $X$
• the 1-cells of \( B \) are the 1-cells of \( X \)
• the 2-cells of \( B \) are the 1-ary 2-cells of \( X \).

We then choose a universal occupant for each 0-ary and 2-ary 2-niche in \( X \). Then

• 1-cell composition in \( B \) is given by the chosen 2-ary universal 2-cells in \( X \)
• 1-cell identities in \( B \) are given by the chosen nullary universal 2-cells in \( X \)
• constraints are induced from composites of the chosen universals
• axioms are seen to hold by examining 4-cells.

In fact, we define a category of ‘biased opetopic 2-categories’ in which these choices have already been made.

Definitions 3.2

• A biased opetopic 2-category is an opetopic 2-category together with a chosen universal occupant for every nullary and 2-ary 2-niche.
• A morphism of biased opetopic 2-categories is simply a morphism of the underlying 2-categories.

We write \( \text{Opic-2-Cat}_b \) for the category of biased opetopic 2-categories and morphisms.

Note that the choice of universal 2-cells is free, that is, the chosen cells are not required to satisfy any axioms. Furthermore, no preservation condition is imposed on the morphisms in this category.

Proposition 3.3  There is an equivalence

\[ \text{Opic-2-Cat}_b \cong \rightarrow \text{Opic-2-Cat} \]

surjective in the direction shown.

Proof. Clear from the definitions. \( \square \)

So in fact, we prove the following proposition:

Proposition 3.4  There is an equivalence

\[ \text{Opic-2-Cat}_b \cong \rightarrow \text{Bicat} \]

surjective in the direction shown.
Finally we will make some comments about the choices made in forming a biased opetopic 2-category.

For the longer calculations in this subsection, and for an explanation of the ‘shorthand’ used in manipulating 2-cells, we refer the reader to Appendix C.

**Proof of Proposition 3.4.** We construct a functor

\[ \zeta : \text{Opic-2-Cat}_b \rightarrow \text{Bicat} \]

and show that it is surjective, full and faithful.

- We define the action of \( \zeta \) on objects.

Let \( X \) be a biased opetopic 2-category. So in addition to the usual data, we have

i) for each object \( A \in X(0) \) a chosen universal 2-cell

\[ \begin{array}{c}
\downarrow \iota_A \\
A \quad \xrightarrow{\sim} \quad A
\end{array} \]

ii) for each pair \( f, g \) of composable 1-cells, a chosen universal 2-cell

\[ \begin{array}{c}
f \quad c_{fg} \quad g
\end{array} \]

We may indicate these chosen 2-cells by \( \sim \) as in

\[ \begin{array}{c}
\downarrow \sim \\
A \quad \xrightarrow{\sim} \quad A
\end{array} \]

We now define a bicategory \( B = B_X \) as follows. First set

\[ \text{ob}(B) = X(0). \]

Recall (Proposition 4.0) that given objects \( A, B \in X(0) \), we have an opetopic 1-category \( \text{Hom}(A, B) \). Let \( B(A, B) \) be the category corresponding
to Hom(A, B) according to Proposition 2.3. So we have 1-cells given by
1-cells of X
\[ a \xrightarrow{f} b \]
and 2-cells given by 1-ary 2-cells of X
\[ \begin{array}{c}
  f \\
  \Downarrow \alpha \\
  g
\end{array} \]

2-cell composites are given by the (unique) 3-cell occupants, for example
\[ \begin{array}{c}
  \alpha \\
  \Downarrow \beta
\end{array} \quad \equiv \quad \begin{array}{c}
  \beta \\
  \Downarrow \alpha
\end{array} \]

and 2-cell identities by nullary 3-cells
\[ f \quad \equiv \quad f \Downarrow 1 \]

Now for any objects \( A, B, C \in \text{ob} \, \mathcal{B} \) we need a functor
\[ c_{ABC} : \mathcal{B}(B, C) \times \mathcal{B}(A, B) \rightarrow \mathcal{B}(A, C) \]
\[ (g, f) \rightarrow g \circ f = gf \]
\[ (\beta, \alpha) \rightarrow \beta \ast \alpha. \]

We define \( g \circ f \) to be the target 1-cell of the chosen universal \( c_{gf} \), so we have
\[ \begin{array}{c}
  f \\
  \Downarrow c_{gf} \\
  g \circ f
\end{array} \]

Note that for each composable pair \( f, g \), we have specified a 2-cell \( c_{gf} \); this
is crucially stronger than merely specifying a 1-cell \( g \circ f \).

We now show how horizontal 2-cell composition is induced. Consider 2-cells
\[ \begin{array}{c}
  f_1 \\
  \Downarrow \alpha \\
  f_2
\end{array} \quad , \quad \begin{array}{c}
  g_1 \\
  \Downarrow \beta \\
  g_2
\end{array} \]
we seek a 2-cell

\[
\begin{tikzpicture}

\node (A) at (0,0) {$g_1 f_1$};
\node (B) at (1,1) {$\beta \ast \alpha$};
\node (C) at (2,0) {$g_2 f_2$};
\draw (A) to [bend left] (B);
\draw (B) to (C);
\end{tikzpicture}
\]

We have a 3-cell

\[
\begin{tikzpicture}

\node (A) at (0,0) {$f_1 \sim g_1$};
\node (B) at (1,1) {$g_1$};
\node (C) at (2,0) {$g_1 f_1$};
\draw (A) to (B);
\draw (B) to (C);
\end{tikzpicture}
\]

unique in its niche, and a universal 2-cell

\[
\begin{tikzpicture}

\node (A) at (0,0) {$f_1 \sim g_1$};
\node (B) at (1,1) {$g_1 f_1$};
\end{tikzpicture}
\]

inducing, by definition of universality, a 2-cell

\[
\begin{tikzpicture}

\node (A) at (0,0) {$g_1 f_1$};
\node (B) at (1,1) {$\downarrow \theta$};
\node (C) at (2,0) {$g_2 f_2$};
\draw (A) to [bend left] (B);
\draw (B) to (C);
\end{tikzpicture}
\]

unique such that there is a 3-cell

\[
\begin{tikzpicture}

\node (A) at (0,0) {$\sim$};
\node (B) at (1,1) {$\theta$};
\end{tikzpicture}
\]

\[
\begin{tikzpicture}

\node (A) at (0,0) {$\phi$};
\end{tikzpicture}
\]

Put $\beta \ast \alpha = \theta$. We check functoriality, that is

i) $1_g \ast 1_f = 1_{gf}$

ii) $(\beta_2 \circ \beta_1) \ast (\alpha_2 \circ \alpha_1) = (\beta_2 \ast \alpha_2) \circ (\beta_1 \ast \alpha_1)$ (middle 4 interchange)

(see Appendix, Lemma A.1).

Next we need, for each object $A$, a 1-cell $A \xrightarrow{\iota_A} A$. We define this to be the target of the chosen universal $\iota_A$, so we have

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Note that, as before, we have specified a universal 2-cell, not just the 1-cell $I_A$.

We now seek natural isomorphisms $a, r, l$. Each of these is induced uniquely from the chosen universals $\iota$ and $c$. For $a$, consider 1-cells

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D.$$  

We seek a natural isomorphism

$$a_{hgf} : (hg)f \cong h(gf).$$

We have

$$f \hline
\begin{array}{c}
g \hline
(hg)f \equiv f \hline
\begin{array}{c}
g \hline
(\iota_A) \end{array}
\end{array}
$$

and

$$f \hline
\begin{array}{c}
g \hline
h(gf) \equiv f \hline
\begin{array}{c}
g \hline
\phi \end{array}
\end{array}
$$

$\theta$ and $\phi$ are composites of universals, so universal. Universality of $\theta$ induces a unique 2-cell $\alpha$ such that

$$\theta \alpha = \phi$$

so

$$f \hline
\begin{array}{c}
g \hline
h(gf) \equiv f \hline
\begin{array}{c}
g \hline
\alpha \end{array}
\end{array}
$$
Put \( a_{hgf} = \alpha \). We see from universality of \( \phi \) that \( a_{hgf} \) is an isomorphism; we check that it satisfies naturality (see Appendix, Lemma A.2).

Next we seek a natural transformation \( r \), so we need for any 1-cell \( A \xrightarrow{f} B \) a 2-cell

\[
\begin{array}{c}
\text{f.I}_A \\
\downarrow r \\
\text{f}
\end{array}
\]

Now we have a 3-cell

\[
\begin{array}{c}
\text{I}_A \\
\sim \\
\text{f}
\end{array}
\Rightarrow
\begin{array}{c}
\text{f.I}_A \\
\downarrow \alpha \\
\text{f.I}_A
\end{array}
\]

and the target 2-cell \( \alpha \) is universal since it is the composite of universals. (Note that this is not the same \( \alpha \) as above.) So \( \alpha \) induces

\[
\begin{array}{c}
\alpha \\
\uparrow r \\
\text{I}_f
\end{array}
\Rightarrow
\begin{array}{c}
\text{I}_f
\end{array}
\]

so

\[
\begin{array}{c}
\text{I}_A \\
\sim \\
\text{f}
\end{array}
\Rightarrow
\begin{array}{c}
\text{f.I}_A \\
\downarrow \alpha \\
\text{f.I}_A
\end{array}
\]

Since \( \alpha \) is universal it is an isomorphism with \( r_f \) as its inverse; so \( r_f \) is also an isomorphism. We also check naturality (see Appendix, Lemma A.3). The construction of and result for \( l \) follow similarly.

Finally we check the axioms for a bicategory (see Appendix, Lemma A.4). So we have defined a bicategory \( B_X \) and we put \( \zeta(X) = B_X \).

- We define the action of \( \zeta \) on morphisms.

Let \( F : X \to X' \) be a morphism of opetopic 2-categories, so for each \( k \) we have

\[
\begin{array}{c}
X(k) \\
\downarrow F_k \\
X'(k)
\end{array}
\xrightarrow{f_k}
\begin{array}{c}
S(k) \\
\downarrow f'_k \\
S'(k)
\end{array}
\]

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We construct from $F$ a lax functor

$$(F, \phi) : \mathcal{B}_X \to \mathcal{B}_{X'}.$$  

The action of $F$ on objects is given by the function

$$F_0 : X(0) \to X'(0);$$

we also need, for any objects $A, B \in \text{ob } \mathcal{B}_X$ a functor

$$F_{AB} : \mathcal{B}_X(A, B) \to \mathcal{B}_{X'}(FA, FB).$$

Now for any $A, B \in \text{ob } \mathcal{B}_X$ we have an opetopic 1-category Hom$(A, B)$, and restricting $F$ to this gives a morphism of opetopic 1-categories

$$\text{Hom}(A, B) \to \text{Hom}(FA, FB)$$

so by Proposition 2.3 we have a functor $F_{AB}$ as required.

Next we seek a natural transformation $\phi_{ABC}$, so for any 1-cells

$$A \xrightarrow{f} B \xrightarrow{g} C$$

we need a 2-cell

$$\phi_{gf} : Fg \circ Ff \to F(g \circ f).$$

We have in $X$ a chosen universal 2-cell

$$\begin{array}{ccc}
    f & c & g \\
    gf
  \end{array}$$

so under the action of $F$ we have in $X'$ a 2-cell

$$\begin{array}{ccc}
    Ff & Fc & Fg \\
    F(gf)
  \end{array}$$

But in $X'$ we have a chosen universal 2-cell

$$\begin{array}{ccc}
    Ff & \sim & Fg \\
    Fg.Ff
  \end{array}$$
which, by definition of universality, induces a 2-cell

![Diagram](image)

unique such that

\[ Ff \triangle Fg = Ff \triangle Fc \]

we check that this satisfies naturality (see Appendix, Lemma A.5).

We now seek a natural transformation \( \phi_A \) for each object \( A \), so we seek a 2-cell

![Diagram](image)

We have in \( X \) a chosen universal 2-cell

\[ A \xrightarrow{I_A} A \]

so applying \( F \) gives a 2-cell in \( X' \)

\[ FA \xrightarrow{F I_A} FA \]

Now the chosen universal in \( X' \)

\[ FA \xrightarrow{I_{FA}} FA \]

induces, by universality, a 2-cell
unique such that

\[ \iff A \iff \phi A \iff F I A \]

and there is no non-trivial naturality to check.

Finally we check that the axioms for a lax functor hold (see Appendix, Lemma A.6). So \((F, \phi)\) is indeed a lax functor, and we set \(\zeta(F) = (F, \phi)\).

It is clear that the above construction of \(\zeta\) is functorial, so we have defined a functor

\[ \zeta : \text{Opic-2-Cat}_b \longrightarrow \text{Bicat}; \]

it remains to show that \(\zeta\) is surjective, full and faithful.

- We show that \(\zeta\) is surjective.

Given a bicategory \(\mathcal{B}\), we construct an opetopic 2-category \(X\) such that \(\zeta(X) = \mathcal{B}\). The idea is

i) The 0-cells of \(X\) are the 0-cells of \(\mathcal{B}\).

ii) The 1-cells of \(X\) are the 1-cells of \(\mathcal{B}\).

iii) The 1-ary 2-cells of \(X\) are the 2-cells of \(\mathcal{B}\).

iv) For \(m \neq 1\), certain \(m\)-ary universals are fixed according to \(m\)-fold composites in \(\mathcal{B}\); the remaining cells are then generated to ensure that these do indeed satisfy universality.

v) The 3-cells of \(X\) are determined from 2-cell composition in \(\mathcal{B}\).

Put \(X(0) = \text{ob} (\mathcal{B})\) and set \(X(1)\) to be the set of 1-cells of \(\mathcal{B}\); the function \(f_1 : X(1) \longrightarrow S(1)\) is defined so that the preimage of the frame \(A \rightarrow B\) is the set of objects of the category \(\mathcal{B}(A, B)\).

We now construct \(X(2)\) bearing in mind the comments in Section 2.3. Write \(X(2)_m \subset X(2)\) for the set of \(m\)-ary 2-cells. First we define the set \(X(2)_1\) of 1-ary 2-cells to be the set of 2-cells of \(\mathcal{B}\).

For 0-ary 2-cells, we first define for each \(A \in X(0)\) a 2-cell
We then define the set of occupants of the same niche to be
\[ \{ \alpha \circ 
abla, \alpha \in X(2)_1, s(\alpha) = I_A \} \]
that is, cells of the form

\[ \begin{array}{c}
A \xrightarrow{\alpha \circ \nabla} A
\end{array} \]

where we put \(1 \circ \nabla = \nabla\).

Similarly for \(X(2)_2\) we first define for each composable pair of 1-cells \(f, g\) a 2-cell

\[ f \xrightarrow{g \circ f} A \]

where \(g \circ f\) is the composite in \(\mathcal{B}\). We then define the set of occupants of this niche to be
\[ \{ \alpha \circ g \circ f : \alpha \in X(2)_1, s(\alpha) = g \circ f \} \]
that is, cells of the form

\[ f \xrightarrow{g \circ f} \]

where we put \(1 \circ \nabla = \nabla\).

For \(X(2)_m, m > 2\), consider a 2-niche of the form
We have no preferred \(m\)-fold composite in \(\mathcal{B}\); instead, for each composite \(\gamma(f_1, \ldots, f_m)\) we define a 2-cell \(u_\gamma\) which is to be universal:

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\gamma(f_1, \ldots, f_m)
\end{array}
\]

Now, suppose we have composites \(\gamma(f_1, \ldots, f_m)\) and \(\gamma'(f_1, \ldots, f_m)\). Then we have a unique invertible \(a_{\gamma\gamma'}: \gamma(f_1, \ldots, f_m) \Rightarrow \gamma'(f_1, \ldots, f_m)\) given by composing components of the associativity constraint \(a\). (Uniqueness follows from coherence for a bicategory.)

We then generate occupants of this niche as

\[
\{\alpha \circ u_\gamma : \alpha \in X(2), s(\alpha) = \gamma(f_1, \ldots, f_m)\} / \sim
\]

where \(\sim\) is the equivalence relation generated by

i) \(\alpha \circ u_\gamma = \beta \circ u_{\gamma'} \iff \beta \circ a_{\gamma\gamma'} = \alpha \in \mathcal{B}\)

ii) \(1 \circ u_\gamma = u_\gamma\).

Note in particular that since \(1 \circ a_{\gamma\gamma'} = a_{\gamma\gamma'}\) we have

\[
a_{\gamma\gamma'} \circ u_\gamma = u_{\gamma'}.
\]

So, given any \(\gamma\), every occupant of the niche is uniquely expressible as \(\alpha \circ u_\gamma\), with \(\alpha \in X(2)_1\). This shows that \(u_\gamma\) is indeed universal, and completes the definition of \(X(2)\).

Note that the universality of the \(u_\gamma\) follows from coherence for classical bicategories, as it depends on the fact that any two composites of given 1-cells are uniquely isomorphic.

We now construct \(X(3)\). We must specify a unique 3-cell for any 3-niche, that is, a unique composite 2-cell for any formal composite of 2-cells.

1) First, composites of 1-ary 2-cells are determined by 2-cell composition in \(\mathcal{B}\).

2) Next we consider any composite of the form \(c \circ \iota\). We define the composites by
3) Now consider a composite of the form

\[
\begin{array}{c}
\alpha \\
\sim
\end{array}
\]

where \(\alpha\) is any 1-ary 2-cell. We put

\[
\sim \alpha \equiv \sim 1 \ast \alpha
\]

and similarly

\[
\sim \alpha \equiv \sim \alpha \ast 1
\]

4) Now consider a formal composite of chosen 2-ary 2-cells \(c_{gf}\). Such a diagram uniquely determines a composite \(\gamma\) in \(B\) of its boundary 1-cells. So we set the composite 2-cell in \(X\) to be \(u_\gamma\). Conversely, any 2-cell \(u_\gamma\) thus arises as the composite of some 2-cells \(c\).

5) Finally, since we require that 2-cell composition be strictly associative, we have determined all 3-cells in \(X\). For, using the above cases, any nullary, 2-ary or \(m\)-ary composite can be written in the form
respectively, where \( \alpha \) is a composite of 1-ary 2-cells which we can then compose in \( \mathcal{B} \).

This completes the definition of the opetopic set \( X \); it remains to check that \( X \) is 2-coherent. Certainly, every 3-niche has a unique occupant by construction. A 2-cell \( \alpha \circ \iota, \alpha \circ c \) or \( \alpha \circ u_\gamma \) is universal if and only if \( \alpha \) is universal, that is, if and only if \( \alpha \) is invertible in \( \mathcal{B} \). So every 2-niche has a universal occupant and composites of universal 2-cells are universal.

We can check that a 1-cell in \( X \) is universal if and only if it is an (internal) equivalence in \( \mathcal{B} \); this follows by an analogous argument to the ‘Yoneda’ proof of Proposition 2.2. So every 1-niche has a universal occupant \( I_A \), and composites of universal 1-cells are universal.

So \( X \) is a biased opetopic 2-category, with chosen universal 2-cells \( \iota \) and \( c \), and it is clear that \( \zeta(X) = \mathcal{B} \). So \( \zeta \) is surjective.

- We show that \( \zeta \) is full.

Let \( X \) and \( X' \) be biased opetopic 2-categories, and suppose we have a morphism of bicategories

\[
(G, \phi) : \mathcal{B}_X \rightarrow \mathcal{B}_{X'}.
\]

We define a morphism \( F : X \rightarrow X' \) as follows. For \( k = 0 \) and \( k = 1 \) the functions

\[
F_k : X(k) \rightarrow X'(k)
\]

are given by the action of \( G \) on objects and 1-cells respectively. We construct \( F_2 \) as follows. The action of \( F_2 \) on 1-ary 2-cells is the action of \( G \) on 2-cells of \( \mathcal{B}_X \). For 0-ary 2-cells, we observe that any such is expressible uniquely as

\[
\Downarrow \iota_A
\]

where \( \iota_A \) is the chosen universal for \( X \). Then we define
where $\iota_{F,A}$ is the appropriate chosen universal for $X'$; this assignation is well-defined by uniqueness.

For $m \geq 2$, any $m$-ary 2-cell is expressible in the form

$$f_2 \circ \cdots \circ f_1 \circ \theta \circ f_m$$

Here $\theta$ is the composite of some configuration of chosen universals $c$, determining a 1-cell composite $\gamma(f_1, \ldots, f_m)$ in $B$, and $\alpha : \gamma \Rightarrow g$. Then we define

$$F_m : \quad \begin{array}{l}
\begin{array}{c}
\downarrow \iota_A \\
\Downarrow f_1 \circ \cdots \circ f_m \circ \theta \circ \alpha
\end{array}
\end{array} \quad \rightarrow \quad \begin{array}{c}
\begin{array}{c}
\downarrow \iota_{FA} \\
\Downarrow f_1 \circ \cdots \circ f_m \circ \theta \circ \alpha
\end{array}
\end{array}$$

$$= \quad \begin{array}{c}
\begin{array}{c}
\downarrow \iota_{FA} \\
\Downarrow \phi_A
\end{array}
\end{array}$$

where $\Phi$ is the appropriate composite of components of the constraint $\phi$. This assignation is well-defined by uniqueness and the axioms for a morphism of bicategories.
It is clear from the construction that this is a morphism of biased opetopic 2-categories, and that
\[ \zeta(F) = (G, \phi). \]
So \( \zeta \) is full.

- We show that \( \zeta \) is faithful.

Consider morphisms \( F, F' \) of unbiased opetopic 2-categories, such that \( \zeta(F) = \zeta(F') \). Write \( \zeta(F) = (G, \phi) \) and \( \zeta(F') = (G', \phi') \).

Certainly since \( G = G' \) on objects and 1-cells we have \( F_0 = F'_0 \) and \( F_1 = F'_1 \). Similarly, \( G = G' \) on (bicategorical) 2-cells gives \( F_2 = F'_2 \) on (opetopic) 1-ary 2-cells. For \( m \)-ary 2-cells with \( m \neq 1 \) consider again the above presentation of 2-cells. Then \( \phi = \phi' \) gives \( F_2 = F'_2 \) on all opetopic 2-cells. So \( \zeta \) is faithful.

So finally we may conclude that \( \zeta \) exhibits an equivalence

\[ \text{Opic-2-Cat}_b \overset{\sim}{\longrightarrow} \text{Bicat} \]
as required. \( \square \)

**Proof of Theorem 3.1.** By Proposition 3.4 we have

\[ \text{Opic-2-Cat}_b \overset{\sim}{\longrightarrow} \text{Bicat} \]

and by Proposition 3.3 we have

\[ \text{Opic-2-Cat}_b \overset{\sim}{\longrightarrow} \text{Opic-2-Cat} \]

so we have an equivalence

\[ \text{Opic-2-Cat} \simeq \text{Bicat} \]
as required. \( \square \)

**Remarks 3.5**

1) Note that the final equivalence is not surjective in *either* direction. Left-to-right involves a choice of universal 2-cells; right-to-left involves generating sets of 3-cells and \( k \)-ary 2-cells (for \( k \neq 1 \)) which are only defined up to isomorphism. Observe that a different choice of universal 2-cells yields a bicategory non-trivially isomorphic but with the same cells.
2) The term ‘biased opetopic 2-category’ is used in the spirit of Leinster’s work on biased and unbiased bicategories ([15]). Rather than pick universal \(m\)-ary 2-cells for just \(m = 0, 2\), we might pick universals for all \(m \geq 0\). Again with no further stipulations on morphisms, this yields an equivalent category of ‘unbiased opetopic 2-categories’. By a straightforward modification of the above proof, we may see that this corresponds to the theory of unbiased bicategories; Leinster has shown directly that the biased and unbiased theories are equivalent.

3) In fact, we may choose any number of universal \(m\)-ary 2-cells for each \(m\) and define a category obviously equivalent to \(\text{Opic-2-Cat}\), by making no stipulation on morphisms. We might then ask: when does this yield a theory of bicategories? In order to modify the above construction as required, we need enough chosen universals to give a complete presentation of the 2-cells of \(X\). From the observations in Section 2.3 we see that this is possible provided we have chosen at least one 0-ary universal, and at least one \(m\)-ary universal for some \(m > 1\) (for each appropriate niche). This idea is discussed in [15] (Appendix A); in the opetopic setting it is immediate that each resulting category of ‘bicategories’ is equivalent.

4) Like Leinster, we might observe that the equivalence of categories 
\[
\text{Opic-2-Cat} \simeq \text{Bicat}
\]
is two levels ‘better’ than we might have asked; we have a comparison at the 1-dimensional level without having to invoke 3- or even 2-dimensional structures. So the theory might already be seen as fruitful despite the lack of an \((n + 1)\)-category of \(n\)-categories.

In summary, we have the following equivalences, surjective in the directions shown:
\[
\text{Opic-2-Cat} \xleftarrow{\sim} \text{Opic-2-Cat}_b \xrightarrow{\sim} \text{Bicat}.
\]

3.2 Strictness

In this section we discuss (informally) various possible notions of strictness in the opetopic setting, and compare these with the classical biased and unbiased settings.

In the classical theory of bicategories, ‘strictness’ (of bicategories or their morphisms) is determined by the ‘strictness’ of the constraints; in general ‘lax’ for plain morphisms, ‘weak’ for isomorphisms and ‘strict’ for identities.

In the opetopic theory we cannot make such definitions, since we do not have those constraints unless we have chosen universal 2-cells. Even then the constraints are not explicitly given. So we must define strictness by
some other means; we may define stricter and weaker notions in terms of universals.

We first turn our attention to morphisms. Recall that the original Baez-Dolan definition demanded that a morphism preserve universality; this is stronger than the general morphisms we use in our definition of \textbf{Opic-2-Cat}.

**Proposition 3.6** Recall (Proposition 3.4) that we have an equivalence

$$\zeta : \text{Opic-2-Cat}_0 \xrightarrow{\sim} \text{Bicat}.$$  

Let \( F \) be a morphism of opetopic 2-categories. Then \( F \) preserves universals \iff \( \zeta(F) \) is a weak functor (homomorphism) of bicategories.

**Proof.** Suppose \( F : X \to X' \) preserves universals. Then the chosen universal in \( X \)

\[
\begin{array}{c}
  f \\
  \downarrow \phi \\
  c \\
  \downarrow g \\
  g f \\
  \end{array}
\]

becomes, under the action of \( F \), a universal in \( X' \)

\[
\begin{array}{c}
  F f \\
  \downarrow \phi' \\
  F c \\
  \downarrow F g \\
  F(g f) \\
  \end{array}
\]

inducing

\[
\begin{array}{c}
  F f \\
  \downarrow \phi^\sim \\
  F c \\
  \downarrow F g \\
  \end{array}
\sim
\begin{array}{c}
  F f \\
  \downarrow \sim \\
  F g \\
  \downarrow F g.F f \\
  \end{array}
\]

so \( \phi_{ABC} \) is an isomorphism.

Conversely suppose \( \phi_{gf} \) and \( \phi_A \) are invertible for all \( f, g, A \). First note that 1-ary universal 2-cells are always preserved (clear from the case \( n = 1 \)). Now, any universal can be expressed as

\[
\begin{array}{c}
  \theta \\
  \downarrow \alpha \\
  \end{array}
\]
where $\theta$ is some composite of chosen universals and $\alpha$ is universal. Now applying $F$ we have

\[
\begin{array}{c}
F_{\alpha}, F_{\alpha}', \ldots
\end{array}
\]

which is universal since $F\alpha$ is universal.

The result for 1-cells follows (with some effort). □

**Definition 3.7** We write $\text{Opic-2-Cat}\text{(weak)}$, $\text{Opic-2-Cat}_b\text{(weak)}$ and $\text{Bicat}\text{(weak)}$ for the lluf subcategories with only weak morphisms.

**Proposition 3.8** The equivalences given in the proofs of Propositions 3.3 and 3.4 restrict to equivalences

\[
\text{Opic-2-Cat}\text{(weak)} \cong \text{Opic-2-Cat}_b\text{(weak)} \cong \text{Bicat}\text{(weak)}
\]
surjective in the directions shown.

**Proof.** The first equivalence is clear from the definitions and the second follows from Proposition 3.6. Since these are lluf subcategories the functors are clearly still surjective. □

Since we have still made no stipulation about the action of morphisms on chosen universals, it is clear that we will still have a result of the form ‘all theories are equivalent’ (cf [16]). That is, regardless of the number of universals chosen, the category-with-weak-morphisms will remain equivalent to the category $\text{Opic-2-Cat}\text{(weak)}$. This ceases to be so in the strict case.

There is no obvious way of further strengthening the conditions imposed on morphisms in $\text{Opic-2-Cat}\text{(weak)}$, but if we consider $\text{Opic-2-Cat}_b\text{(weak)}$, we can further demand that chosen universals be preserved.

**Proposition 3.9** Let $F$ be a weak morphism of biased opetopic 2-categories. Then $F$ preserves chosen universals iff $\zeta(F)$ is strict.

**Proof.** ‘$\Rightarrow$’ is clear from the definition of $\zeta$. Now for any morphism $(F, \phi)$ of opetopic 2-categories we have

\[
\cdots
\]

\[
F_{\alpha}
\]

\[
\begin{array}{c}
F_{\alpha}, F_{\alpha}', \ldots
\end{array}
\]
Definition 3.10 We call a weak morphism of biased opetopic 2-categories strict if it preserves chosen universal 2-cells.

Write $\text{Opic-2-Cat}_b(\text{str})$ and $\text{Bicat}(\text{str})$ for the lluf subcategories with only strict morphisms.

Proposition 3.11 The previously defined equivalence restricts to an equivalence

$$\text{Opic-2-Cat}_b(\text{str}) \sim \rightarrow \text{Bicat}(\text{str})$$

surjective in the direction shown.

Proof. Follows immediately from Proposition 3.9

We now consider the possibility of altering the structures of the 2-categories themselves. Considering the structures used so far as ‘weak’, we might try to find either lax or strict opetopic $n$-categories.

In the lax direction we might consider removing the condition that universals compose to universals. Observe that in the case $n = 1$ we do not use this condition to prove

$$\text{Opic-1-Cat} \simeq \text{Cat}$$

so a ‘lax opetopic 1-category’ would be just the same as a weak one, as we would hope.

However, for $n = 2$ it is not clear that this ‘laxification’ produces a useful structure for the general or biased theories. Consider instead the case in which $m$-ary universal 2-cells have been chosen for every $m \geq 0$. That is,
we define an ‘unbiased opetopic 2-category’ to be one in which every 2-niche has a chosen universal occupant.

If we now remove the condition that composites of universals be universals, we have certain 2-cell ‘constraints’ induced by the chosen universals. For example we have

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
g \\
\sim \\
f \\
\sim \\
h
\end{array}
\end{array}
\end{array}
\quad \Rightarrow \quad
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
g \\
\sim \\
f \\
\sim \\
h
\end{array}
\end{array}
\end{array}
\]

and thus an induced 2-cell

\[
\gamma : hgf \Rightarrow (hg)f.
\]

This produces a structure something like a ‘lax unbiased bicategory’ in the sense of Leinster (15) except that the constraints \( \gamma \) are acting in the opposite direction.

For strictness there is likewise no obvious way of imposing stronger conditions on an opetopic 2-category. Once we have chosen universals, we might demand that the chosen universals compose to chosen universals, but this will certainly not be possible unless we have chosen \( m \)-ary universals for all \( m \geq 0 \). So once again we find ourselves in the unbiased theory.

If we have one chosen universal for each 2-niche, the above condition forces strict associativity and left and right unit action. So we have a 2-category; this is to be expected since Leinster has already observed that unbiased 2-categories are in one-to-one correspondence with 2-categories. (There is a possibility of more interesting structure if a niche has more than one chosen universal.)

**Remarks**

From this informal discussion we see that the theory of opetopic 2-categories neither laxifies nor strictifies particularly naturally. In the lax direction, this is perhaps consistent with the fact that there is no very satisfactory lax version of classical bicategories. In the strict direction, this demonstrates why we have found it hard to state a coherence theorem of the form ‘every bicategory is biequivalent to a 2-category’; we simply do not know what a ‘strict opetopic 2-category’ is. (Note however that statements of the form ‘all diagrams commute’ are much less problematic.)

We have already observed that there are (at least) two possible ways of removing the bias in a bicategory: we may choose \( m \)-ary composites for no \( m \), or all \( m \). It appears that, although the former philosophy may be viewed as being more egalitarian towards all universal cells, the latter provides more footholds for exploring the theory.
3.3 Conclusions

We might regard the category of opetopic 2-categories (with no choices made) as being the most general of all the theories discussed in this work. However we will also observe that in describing the 2-cells, performing calculations or exploring the theory further, it is often more practical to use some presentation of 2-cells, that is, to make choices of universals either explicitly or implicitly.

In the opetopic setting the choice of universals is ‘free’ in the sense that no axioms are required; all axioms are subsumed into the conditions for $n$-coherence. So in each separate case the axioms do not have to be stated explicitly.

This was suggested in [3] as one of the motivations for the opetopic approach to $n$-categories, since as $n$ increases, the axioms for an $n$-category increase in complexity with fiendish rapidity. This work demonstrates a sense in which this idea is realised.

A Calculations for Section 3

In this appendix we perform the calculations deferred from Section 3. However, we first introduce some shorthand to deal with some of the more unwieldy parts of the algebra.

A.1 Shorthand for calculations

The following shorthand is used for calculations in an opetopic 2-category.

i) Since 3-niche occupants are unique, we may omit the target of a 3-cell without ambiguity. We then write an equality to indicate that the 3-cells have the same target. For example we might write

\[
\begin{array}{c}
\alpha \\
\beta
\end{array}
= 
\begin{array}{c}
\gamma \\
\delta
\end{array}
\]

meaning

\[
\begin{array}{c}
\alpha \\
\beta
\end{array} 
\Rightarrow 
\begin{array}{c}
\theta
\end{array}
\]
and

\[
\begin{align*}
\begin{array}{c}
\gamma \\
\delta
\end{array}
\end{align*}
\Rightarrow
\begin{align*}
\begin{array}{c}
\theta
\end{array}
\end{align*}
\]

ii) Recall that, by uniqueness of 3-niche occupants, we have associativity of 2-cell composition. So we may substitute ‘equal’ (in the above sense) 2-cell composites in part of the domain of another 3-cell. For example, given

\[
\begin{align*}
\begin{array}{c}
\alpha \\
\beta
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\gamma \\
\delta
\end{array}
\end{align*}
\]

and a 3-cell

\[
\begin{align*}
\begin{array}{c}
\alpha \\
\beta
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\gamma \\
\delta
\end{array}
\end{align*}
\]

we have

\[
\begin{align*}
\begin{array}{c}
\alpha \\
\beta
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\gamma \\
\delta
\end{array}
\end{align*}
\]

This is shorthand for the following

\[
\begin{align*}
\begin{array}{c}
\alpha \\
\beta
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\phi
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\phi
\end{array}
\end{align*}
\begin{align*}
\begin{array}{c}
\theta
\end{array}
\end{align*}
\]

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iii) Recall that 2-cell identities act as identities on $k$-ary 2-cells for all $k$ (not only 1-ary 2-cells), for example

$$
\begin{array}{c}
\gamma \delta \Rightarrow \phi \\
\phi \Rightarrow \theta
\end{array}
$$

and

$$
\begin{array}{c}
\alpha \\
\Rightarrow \theta
\end{array}
$$

so we have $\alpha = \theta$, that is

$$
\begin{array}{c}
\alpha \\
\Rightarrow \theta
\end{array}
$$

iv) Note that if $u$ is any universal 2-cell, we have
by definition of universality. This also holds if $\theta$ and $\phi$ are 2-cell composites, for example

\[
\begin{array}{c}
\theta \\
\Rightarrow \\
\alpha \\
\beta \\
\phi
\end{array}
\]

and

\[
\begin{array}{c}
\alpha \\
\beta \\
\Rightarrow \\
\gamma \\
\delta \\
\phi
\end{array}
\]

Furthermore, this holds if $u$ is a composite of universals, since a composite of universals is universal, for example if $u_1$ and $u_2$ are universal then

\[
\begin{array}{c}
\begin{array}{c}
\alpha \\
\beta \\
\Rightarrow \\
\gamma \\
\delta
\end{array}
\end{array}
\]

and in particular

\[
\begin{array}{c}
\begin{array}{c}
\alpha \\
\Rightarrow \\
1
\end{array}
\end{array}
\]

### A.2 Calculations

Throughout this subsection, we use the notation and constructions exactly as given in Section 3.
Lemma A.1

i) $l_g \ast 1_f = l_{gf}$

ii) $(\beta_2 \circ \beta_1) \ast (\alpha_2 \circ \alpha_1) = (\beta_2 \ast \alpha_2) \circ (\beta_1 \ast \alpha_1)$ (middle 4 interchange)

Proof.

i) We have

\[ \sim 1 = 1 \sim 1 = \sim \]

by the action of 1 and definition of $\ast$, so

\[ 1 \ast 1 = 1 \]

as required.

ii) Given

\[ \begin{array}{c}
\text{f}_1 \\
\text{f}_2 \\
\text{f}_3 \\
\hline
\text{u}_1 \\
\text{u}_2 \\
\text{u}_3 \\
\hline
\text{g}_1 \\
\text{g}_2 \\
\text{g}_3 \\
\hline
\text{g}_1f_1 \\
\text{g}_2f_2 \\
\text{g}_3f_3 \\
\end{array} \]

we write

\[ \begin{array}{c}
f_1 \\
g_1 \\
\hline
\text{f}_1u_1 \\
g_1f_1 \\
\hline
f_2 \\
g_2 \\
\hline
f_3 \\
g_3 \\
\hline
\text{f}_3u_3 \\
g_3f_3 \\
\end{array} \]

for the chosen universal 2-cells as shown. Then we have

\[ \begin{array}{c}
u_1 \\
\hline
\alpha_2 \circ \beta_1 \\
\alpha_2 \circ \beta_1 \ast \beta_1 \\
\hline
\text{u}_2 \\
\text{u}_3 \\
\hline
\beta_1 \circ \beta_1 \\
\beta_1 \ast \beta_1 \circ \beta_1 \\
\end{array} \]

balanced
by definition, but also

\[
\begin{align*}
\alpha & = \beta \\
\alpha & = \beta \\
\alpha & = \beta \\
\alpha & = \beta
\end{align*}
\]

by definition, hence the result.

\[\square\]

**Lemma A.2** \(a\) is natural

**Proof.** Given 2-cells

\[
\begin{array}{ccc}
\alpha & \beta & \gamma \\
\alpha & \beta & \gamma \\
\alpha & \beta & \gamma \\
\alpha & \beta & \gamma \\
\end{array}
\]

we need to show that the following naturality square commutes

\[
\begin{array}{ccc}
(h_1 g_1) f_1 & \overset{a}{\longrightarrow} & h_1 (g_1 f_1) \\
\downarrow & & \downarrow \\
(h_2 g_2) f_1 & \overset{a}{\longrightarrow} & h_2 (g_2 f_2)
\end{array}
\]

We have

\[
\begin{align*}
\alpha & \beta \\
\alpha & \beta \\
\alpha & \beta \\
\alpha & \beta \\
\end{align*}
\]

so by uniqueness we have
as required. \hfill \Box

**Lemma A.3** $r$ is natural

**Proof.** Given a 2-cell

$$
\begin{array}{c}
A \\ \\
\alpha \\ \\
f_2 \\
\alpha \\
B
\end{array}
\xrightarrow{f_1}
\begin{array}{c}
A \\ \\
\alpha \\ \\
f_1 \\
\alpha \\
B
\end{array}
$$

we need to show that the following naturality square commutes

$$
\begin{array}{ccc}
f_1 \circ I_A & \xrightarrow{r} & f_1 \\
\downarrow \alpha \ast 1 & & \downarrow \alpha \\
f_2 \circ I_A & \xrightarrow{r} & f_2
\end{array}
$$

Writing chosen composites as

$$
\begin{array}{c}
\begin{array}{c}
I_A \\
\downarrow f_1 \circ I_A
\end{array}
\begin{array}{c}
\triangle \\
u_1
\end{array}
\end{array}
\quad , 
\begin{array}{c}
\begin{array}{c}
I_A \\
\downarrow f_2 \circ I_A
\end{array}
\begin{array}{c}
\triangle \\
u_2
\end{array}
\end{array}
$$

we have

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
I_A \\
\downarrow f_1 \circ I_A
\end{array} \\
\begin{array}{c}
\triangle \\
u_1
\end{array}
\end{array}
\end{array}
\equiv
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
I_A \\
\downarrow f_2 \circ I_A
\end{array} \\
\begin{array}{c}
\triangle \\
u_2
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
$$

so by uniqueness
\[ \alpha \ast 1 \quad = \quad \begin{array}{c} r \\ \alpha \end{array} \]
as required. \hfill \square

**Lemma A.4** \( a, l \) and \( r \) satisfy the axioms for a bicategory.

**Proof.**

i) associativity pentagon

\[ \begin{array}{c} ((kh)g)f \\ a \\ k(h(gf)) \end{array} = \begin{array}{c} 1 \ast a \\ a \ast 1 \end{array} \]
as required.

ii) unit triangle
Lemma A.5 $\phi$ is natural.

Proof. Given 2-cells $f_1, f_2, g_1, g_2$ with

![Diagram](image)

we need to show that the following diagram commutes

$$
\begin{array}{ccc}
F g_1 \circ F f_1 & \xrightarrow{\phi_{g_1 f_1}} & F (g_1 \circ f_1) \\
| \quad F \beta \ast F \alpha | & & | \quad F(\beta \ast \alpha) \\
F g_2 \circ F f_2 & \xrightarrow{\phi_{g_2 f_2}} & F (g_2 \circ f_2)
\end{array}
$$

as required.

$\square$
We write the chosen universal 2-cells as

\[
\begin{align*}
&f_1 \triangleright v_1 \triangleright g_1, \quad f_2 \triangleright v_2 \triangleright g_2, \\
&Ff_1 \triangleright u_1 \triangleright Fg_1, \quad Ff_2 \triangleright u_2 \triangleright Fg_2
\end{align*}
\]

so

\[
\begin{align*}
&\phi \quad \quad = \quad \quad Fv_1 \\
&\phi \quad \quad = \quad \quad Fv_2
\end{align*}
\]

We have

\[
\begin{align*}
&\frac{F\alpha}{F\beta} \quad u_2 \quad \frac{F\beta}{F\alpha} = \quad \frac{F\beta \ast F\alpha}{\phi_{g_2f_2}} \\
&\phi_{g_2f_2}
\end{align*}
\]

in \(X'\), and in \(X\) we have

\[
\begin{align*}
&\alpha \quad v_2 \quad \beta = \quad \frac{v_1}{\beta \ast \alpha}
\end{align*}
\]

so applying \(F\), we have, since \(F\) is strictly functorial on 2-cells,

\[
\begin{align*}
&\frac{F\alpha}{F\beta} \quad Fv_2 \quad \frac{F\beta}{F\alpha} = \quad \frac{Fv_1}{F(\beta \ast \alpha)} = \quad \frac{u_1}{\phi_{g_2f_2}}
\end{align*}
\]

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so by uniqueness we have

\[
\begin{array}{c}
F \beta \ast F \alpha \\
\phi_{g_2f_2}
\end{array}
\]

\[
\begin{array}{c}
\phi_{g_1f_1} \\
F(\beta \ast \alpha)
\end{array}
\]

as required.

\[\square\]

**Lemma A.6** \((F, \phi)\) satisfies the axioms for a morphism of bicategories.

**Proof.** We have in \(X\)

\[
\begin{array}{c}
g \\
f \sim \sim h \\
a
\end{array}
\]

so applying \(F\), we get in \(X'\)

\[
\begin{array}{c}
Fg \\
Ff \sim \phi \\
\phi_{g_2f_2} \\
F \alpha \\
a
\end{array}
\]

\[
\begin{array}{c}
\phi_{g_1f_1} \\
F(\beta \ast \alpha)
\end{array}
\]

as required. For \(r\) we have in \(X\)

\[
\begin{array}{c}
\sim \\
r
\end{array}
\]

\[
\begin{array}{c}
\sim \\
a \\
1 \ast \phi \\
\phi
\end{array}
\]

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so applying $F$, we get in $X'$

\[
\begin{array}{c}
\phi \sim Ff \\
\phi \parallel Ff \\
\phi \sim Ff \\
\end{array}
\]

as required. The axiom for $l$ holds similarly. $\Box$

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