Multi-triangulations as complexes of star polygons

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Brussels, March 2008
Definitions
Multi-triangulations

Let $k$ and $n$ be two integers with $n \geq 2k + 1$.
Let $V_n$ be the set of vertices of a convex $n$-gon.
Let $E_n$ be the set of the edges of the complete graph on $V_n$.

Two edges $[a, b]$ and $[c, d]$ cross if the corresponding open segments $]a, b[ \text{ and } ]c, d[$ intersect.
An $\ell$-crossing is a subset of $E_n$ of $\ell$ mutually intersecting edges.

A $k$-triangulation of the $n$-gon is a maximal subset of $E_n$ without $(k + 1)$-crossing.
The **length** of an edge $[a, b]$ is

$$\ell([a, b]) = \min(|a, b|, |b, a|).$$

The only edges that may appear in a $(k + 1)$-crossing are those of length $> k$. 
The length of an edge $[a, b]$ is

$$\ell([a, b]) = \min(||a, b||, ||b, a||).$$

The only edges that may appear in a $(k + 1)$-crossing are those of length $> k$.

We say that $[a, b]$ is a

(i) $k$-relevant edge if $\ell(\{a, b\}) > k$;
(ii) $k$-boundary edge if $\ell(\{a, b\}) = k$;
(iii) $k$-irrelevant edge if $\ell(\{a, b\}) < k$. 

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Any $k$-triangulation of the $n$-gon contains all the $k$-irrelevant and the $k$-boundary edges of $E_n$. 
Remarks & examples

A general construction

The complete graph $K_{2k+1}$ is the unique $k$-triangulation of the $(2k+1)$-gon.

All $k$-triangulations of the $(2k+2)$-gon are obtained by suppression of a long diagonal of the complete graph $K_{2k+2}$.
There are 14 2-triangulations of the heptagon:

There are 30 3-triangulations of the nonagon:
Remarks & examples

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ALREADY KNOWN RESULTS

Théorème.

1. A \( k \)-triangulation of the \( n \)-gon contains \( k(2n - 2k - 1) \) edges. \[\text{[Nak], [DKM]}\]
2. Any relevant edge can be flipped and the graph of flips is connected. \[\text{[Nak], [Jon]}\]
3. There exists a deletion/insertion operation that transforms a \( k \)-triangulation of the \((n + 1)\)-gon into a \( k \)-triangulation of the \( n \)-gon and reciprocaly. \[\text{[Nak], [Jon]}\]
4. The \( k \)-triangulations of the \( n \)-gon are counted by a Catalan determinant: \[\det(C_{n-i-j})_{i,j\leq k} \]. \[\text{[Jon]}\]
5. If \( n \geq 2k + 3 \), any \( k \)-triangulation of the \( n \)-gon has at least \( 2k \) ears. \[\text{[Nak]}\]

V. Capoyleas & J. Pach, A Turán-type theorem on chords of a convex polygon, 1992

T. Nakamigawa, A generalization of diagonal flips in a convex polygon, 2000

A. Dress, J. Koolen & V. Moulton, On line arrangements in the hyperbolic plane, 2002

J. Jonsson, Generalized triangulations and diagonal-free subsets of stack polyominoes, 2005
**Already known results**

**Théorème.**

1. A $k$-triangulation of the $n$-gon contains $k(2n - 2k - 1)$ edges. \[\text{[NAK], [DKM]}\]
2. Any relevant edge can be flipped and the graph of flips is connected. \[\text{[NAK], [JON]}\]
3. There exists a deletion/insertion operation that transforms a $k$-triangulation of the $(n+1)$-gon into a $k$-triangulation of the $n$-gon and reciprocally. \[\text{[NAK], [JON]}\]
4. The $k$-triangulations of the $n$-gon are counted by a Catalan determinant: $\det(C_{n-i-j})_{i,j \leq k}$. \[\text{[JON]}\]
5. If $n \geq 2k + 3$, any $k$-triangulation of the $n$-gon has at least $2k$ ears. \[\text{[NAK]}\]

**Two remarks.**

– undirect proofs :

\[
\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3} \\
\text{4} \\
\text{5}
\end{array}
\]

– generalisation of triangles?
Let $s_0, \ldots, s_{2k}$ be $2k + 1$ points of the unit circle in counterclockwise order.

We say that the polygon

- whose vertices are $s_0, \ldots, s_{2k}$,
- and whose edges are $[s_0, s_k], [s_1, s_{1+k}], \ldots, [s_k, s_{2k}], [s_{k+1}, s_0], \ldots, [s_{2k}, s_{k-1}]$ 

is a \textit{k-star}.

\begin{center}
\includegraphics[width=\textwidth]{diagram.png}
\end{center}
An angle of a subset $F$ of $E_n$ is a couple

$$\angle(u, v, w) = ([u, v], [v, w])$$

of edges of $F$ such that
- $u \prec v \prec w$ (for the counterclockwise order),
- for all $t \in [w, u]$, the edge $\{v, t\}$ is not in $F$.

$v$ is the vertex of the angle $\angle(u, v, w) = (\{u, v\}, \{v, w\})$.

For all $t \in [w, u]$, the edge $\{v, t\}$ is a bisector of $\angle(u, v, w)$.

An angle $\angle(u, v, w)$ is $k$-relevant if its edges are both either $k$-relevant, or $k$-boundary.
Results
Let $T$ be a $k$-triangulation.

Any angle of a $k$-star of $T$ is a $k$-relevant angle of $T$.

Reciprocally, any $k$-relevant angle of $T$ is contained in a $k$-star of $T$. 

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Corollary.
Let $e$ be an edge of a $k$-triangulation $T$. Then
1. if $e$ is a $k$-relevant edge, it belongs to exactly two $k$-stars of $T$,
2. if $e$ is a $k$-boundary edge, it belongs to exactly one $k$-star of $T$,
3. if $e$ is a $k$-irrelevant edge, it does not belong to any $k$-star of $T$.
**Theorem.**
Every pair of $k$-stars of a $k$-triangulation have a unique common bisector.

**Proposition.**
Let $T$ be a $k$-triangulation. Any edge which is not in $T$ is the common bisector of a unique pair of $k$-stars of $T$.

**Corollary.**
Any $k$-triangulation of the $n$-gon contains exactly $n - 2k$ $k$-stars and thus $k(2n - 2k - 1)$ edges.
**Flips**

**Theorem.**
Let $T$ be a $k$-triangulation of the $n$-gon. Let $e$ be an edge of $T$. Let $R$ and $S$ be the two $k$-stars of $T$ containing $e$. Let $f$ be the common bisector of $R$ and $S$.

Then $T$ and $T \triangle \{e, f\}$ are the only two $k$-triangulations of the $n$-gon containing $T \setminus \{e\}$.

The $k$-triangulation $T \triangle \{e, f\}$ is obtained by **flipping** the edge $e$ in the $k$-triangulation $f$. 
Let $G_{n,k}$ be the graph of flips of the set of $k$-triangulations of the $n$-gon.

**Theorem.**

The graph $G_{n,k}$ is connected, regular of degree $k(n - 2k - 1)$, and its diameter is at most $2k(n - 2k - 1)$.

**Remark.**

(i) if $n > 8k^3 + 4k^2$, the bound on the diameter can be improved to be $2nk - (8k^2 + 2k)$. [Nak]

(ii) for $k = 1$, this bound is optimal.

**D.D. Sleator, R.E. Tarjan & W.P. Thurston,**

Rotation distance, triangulations and hyperbolic geometry, 1988

For $k > 1$ and $n > 4k$, we only know that the diameter is at least $k(n - 2k - 1)$. 
Let assume here that $n > 2k + 3$.
A $k$-ear is an edge of length $k + 1$.
We say that a $k$-star is internal if it does not contain any $k$-boundary edge.

**Proposition.**
The number of $k$-ears of a $k$-triangulation $T$ equals the number of internal $k$-stars plus $2k$.
In particular, $T$ contains at least $2k$ $k$-ears.
We say that a $k$-triangulation is $k$-colorable if there exists a coloration with $k$ color of its $k$-relevant edges such that there is no monochromatic crossing.

A $k$-accordion of $E_n$ is a set $Z = \{[a_i, b_i] \mid 1 \leq i \leq n - 2k - 1\}$ of $n - 2k - 1$ edges such that

- $b_1 = a_1 + k + 1$
- $[a_i, b_i] \in \{[a_{i-1}, b_{i-1} + 1], [a_{i-1} - 1, b_{i-1}]\}$, for all $i$.

**Proposition.**

Let $T$ be a $k$-triangulation, with $k > 1$. The following assertions are equivalent

(i) $T$ is $k$-colorable;

(ii) $T$ contains exactly $2k$ $k$-ears;

(iii) $T$ has no internal $k$-star;

(iv) the set of $k$-relevant edges of $T$ is the disjoint union of $k$ $k$-accordions.

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Flattening a $k$-star/inflattening a $k$-crossing

**Theorem.**

There is a bijection between

(i) the set of $k$-triangulations of the $(n + 1)$-gon with a marked boundary edge, and

(ii) the set of $k$-triangulations of the $n$-gone with a marked $k$-crossing with $k$ consecutives vertices.
Flattening a $k$-star/inflattting a $k$-crossing

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Further topics and open questions
**Multi-Dyck Paths**

**Theorem.**

The number of $k$-triangulations of the $n$-gon is

\[
\det(C_{n-i-j})_{1 \leq i,j \leq k} = \left| \begin{array}{cccc}
C_{n-2} & C_{n-3} & \cdots & C_{n-k} & C_{n-k-1} \\
C_{n-3} & C_{n-4} & \cdots & C_{n-k-1} & C_{n-k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
C_{n-k-1} & C_{n-k-2} & \cdots & C_{n-2k+1} & C_{n-2k} \\
\end{array} \right|,
\]

where

\[
C_m = \frac{1}{m + 1} \binom{2m}{m}.
\]

[Jon]

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Theorem.

The number of $k$-triangulations of the $n$-gon is

$$\det(C_{n-i-j})_{1 \leq i,j \leq k} = \begin{vmatrix} C_{n-2} & C_{n-3} & \ldots & C_{n-k} & C_{n-k-1} \\ C_{n-3} & C_{n-4} & \ldots & C_{n-k-1} & C_{n-k-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{n-k-1} & C_{n-k-2} & \ldots & C_{n-2k+1} & C_{n-2k} \end{vmatrix},$$

where $C_m = \frac{1}{m+1}{2m \choose m}$.

A Dyck path of semi-length $\ell$ is a lattice path using north steps $N = (0, 1)$ and east steps $E = (1, 0)$ starting from $(0, 0)$ and ending at $(\ell, \ell)$, and such that it never goes below the diagonal $y = x$.

The set of Dyck paths of semi-length $n - 2$ is in bijection with the set of triangulations of the $n$-gon.
The number of \( k \)-triangulations of the \( n \)-gon is

\[
\det(C_{n-i-j})_{1 \leq i,j \leq k} = \begin{vmatrix}
C_{n-2} & C_{n-3} & \ldots & C_{n-k} & C_{n-k-1} \\
C_{n-3} & C_{n-4} & \ldots & C_{n-k-1} & C_{n-k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
C_{n-k-1} & C_{n-k-2} & \ldots & C_{n-2k+1} & C_{n-2k}
\end{vmatrix},
\]

where \( C_m = \frac{1}{m+1} \binom{2m}{m} \).

A Dyck path of semi-length \( \ell \) is a lattice path using north steps \( N = (0,1) \) and east steps \( E = (1,0) \) starting from \((0,0)\) and ending at \((\ell,\ell)\), and such that it never goes below the diagonal \( y = x \).

A \( k \)-Dyck path of semi-length \( \ell \) is a \( k \)-tuple \((d_1, \ldots, d_k)\) of Dyck paths of semi-length \( \ell \) such that each \( d_i \) never goes above \( d_{i-1} \), for \( 2 \leq i \leq k \).
**Theorem.**

The number of $k$-triangulations of the $n$-gon is

\[
\det(C_{n-i-j})_{1 \leq i, j \leq k} = \left| \begin{array}{cccc}
C_{n-2} & C_{n-3} & \cdots & C_{n-k} & C_{n-k-1} \\
C_{n-3} & C_{n-4} & \cdots & C_{n-k-1} & C_{n-k-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
C_{n-k-1} & C_{n-k-2} & \cdots & C_{n-2k+1} & C_{n-2k}
\end{array} \right|,
\]

where $C_m = \frac{1}{m+1}\binom{2m}{m}$.

**Theorem.**

The number of $k$-Dyck paths of semi-length $n - 2k$ is $\det(C_{n-i-j})_{1 \leq i, j \leq k}$.  

**M. Desaïnt-Catherine & G. Viennot,**
Enumeration of certain Youg tableaux with bounded height, 1986

We have explicit bijections only when $k = 1$ and $k = 2$.

**S. Elizalde,** A bijection between 2-triangulations and pairs of non-crossing Dyck paths, 2006
**Rigidity**

A graph $G = (V, E)$, embedded in $\mathbb{R}^d$, is said to be rigid if any continuous movement of its vertices that preserves all edges lengths is an isometry of $\mathbb{R}^d$.

A triangulation is a **minimally rigid graph** of the plane.
RIGIDITY

A graph $G = (V, E)$, embedded in $\mathbb{R}^d$, is said to be rigid if any continuous movement of its vertices that preserves all edges lengths is an isometry of $\mathbb{R}^d$.

A triangulation is a minimally rigid graph of the plane.

**Conjecture.**
A $k$-triangulation is a minimally rigid graph in dimension $2k$.

**Two remarks.**
- $k$-triangulations have $2k$-Laman property.
- we have a proof for $k = 2$. 

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Let $\Delta_{n,k}$ be the complex of all subsets of $k$-relevant edges of $E_n$ that do not contain any $(k + 1)$-crossing.

When $k = 1$, this complex is known to be the boundary complex of the associahedron.

C. Lee, The associahedron and triangulations of an $n$-gon, 1989
Let $\Delta_{n,k}$ be the complex of all subsets of $k$-relevant edges of $E_n$ that do not contain any $(k + 1)$-crossing. When $k = 1$, this complex is known to be the boundary complex of the associahedron.

C. Lee, The associahedron and triangulations of an $n$-gon, 1989

When $k \geq 2$, we only know that $\Delta_{n,k}$ is topologically a sphere.

**Conjecture.**

There exists a simple polytope of dimension $k(n - 2k - 1)$ with boundary complex $\Delta_{n,k}$.

**Remark.** area of stars and rigidity can help.

L. Billera, P. Filliman & B. Sturmfels, Constructions and complexity of secondary polytopes, 1990

G. Rote, F. Santos & I. Streinu, Expansive motions and the polytope of pointed pseudo-triangulaitons, 2003

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Let $T$ be a $k$-triangulation of the $n$-gon.

The polygonal complex $\mathcal{C}(T)$ associated to $T$ is a polygonal decomposition of an orientable surface with boundary $S_{n,k}$.

The genus of $S_{n,k}$ is $g_{n,k} = \frac{1}{2}(2 - f + e - v - b) = \frac{1}{2}(2 - n + k + kn - 2k^2 - \gcd(n, k))$. 

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Surfaces

Flips define a morphism between

(i) the fundamental group $\pi_{n,k}$ of the graph of flips $G_{n,k}$ (i.e. the set of loops in $G_{n,k}$, up to homotopy), and

(ii) the mapping class group $\mathcal{M}_{n,k}$ of the surface $S_{n,k}$ (i.e. the set of diffeomorphisms of the surface $S_{n,k}$ into itself that preserve the orientation and that fix the boundary of $S_{n,k}$, up to isotopy).
Conclusion

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Conclusion

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arXiv: 0706.3121v2