On peculiar properties of generating functions of some orthogonal polynomials

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Abstract
We prove that for $|x| \leq |t| < 1$, $-1 < q \leq 1$ and $n \geq 0$:

$$\sum_{i \geq 0} t_i (q) h_{n+i}(x) = h_n(x|t) \sum_{i \geq 0} t_i (q) p_i(x),$$

where $h_n(x|t)$ and $h_n(x|q)$ are respectively the so-called $q$-Hermite and the big $q$-Hermite polynomials, and $(q)_n$ denotes the so-called $q$-Pochhammer symbol. We prove similar equalities involving big $q$-Hermite and Al-Salam–Chihara polynomials, and Al-Salam–Chihara and the so-called continuous dual $q$-Hahn polynomials. Moreover, we are able to relate in this way some other ‘ordinary’ orthogonal polynomials such as, e.g., Hermite, Chebyshev or Laguerre. These equalities give a new interpretation of the polynomials involved and moreover can give rise to a simple method of generating more and more general (i.e. involving more and more parameters) families of orthogonal polynomials. We pose some conjectures concerning Askey–Wilson polynomials and their possible generalizations. We prove that these conjectures are true for the cases $q = 1$ (classical case) and $q = 0$ (free case), thus paving the way to generalization of Askey–Wilson polynomials at least in these two cases.

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1. Introduction and auxiliary results

1.1. Introduction

In the paper we play around with the scheme:

$$\sum_{i \geq 0} t_i (q) p_{n+i}(x) = q_n(x|t) \sum_{i \geq 0} t_i (q) p_i(x),$$

where the series on both sides converge for $x, t$ from a certain Cartesian product of bounded intervals and $\{p_i\}_{i \geq 0}$ and $\{q_i\}_{i \geq 0}$ are certain families of orthogonal polynomials. So far we were able to relate in this scheme the $q$-Hermite (qH) and the big $q$-Hermite (bqH) polynomials,
the bqH and the Al-Salam–Chihara (ASC) polynomials, and also the ASC and the so-called continuous dual $q$-Hahn (c2h) polynomials; $(q)_n$ denotes the so-called $q$-Pochhammer symbol.

In fact, the idea of considering a ‘shifted generating function’, like the left-hand side of (1.1), and relating it to the same generating function without the shift (i.e. when $n = 0$), is not new and appeared in a version confined to two families of orthogonal polynomials (Hermite and Laguerre) in the book by Rainville [7] and for the Rogers–Szegő polynomials in [4].

In this paper we treat it as the general idea, show that it is useful and give more examples. These examples particularly concern polynomials satisfying the so-called Askey–Wilson (AW) scheme. Imagining the AW scheme as a sort of vertical structure with AW polynomials at the top, we treat this idea as the tool to ‘move up through’ the AW scheme and in particular as a tool towards possible generalization of the AW polynomials.

As we already pointed out above, this scheme can also be applied to some ‘classical’ orthogonal polynomials as Rainville did in his book [7] after some necessary modifications.

Can we continue this scheme and interpret in this way, e.g., AW polynomials? Can we ‘go beyond’ AW polynomials? From what we have been able to prove so far, it seems that we have a simple scheme to produce families of orthogonal polynomials with more and more parameters. AW polynomials provide the largest family (in the sense of the number of parameters) of orthogonal polynomials of one variable that has been relatively well described. Thus, there is now a chance to ‘go beyond’ these polynomials. Of course it requires further research and cannot be settled in one small article.

We interpret the obtained relationships between the above mentioned polynomials with the help of an ordinary $q$-difference operator, simply acting in the spirit of a recent attempt to ‘move up through’ the AW scheme as demonstrated by Atakishiyeva and Atakishiyev in [1], but with much simpler operators and more importantly indicating the way to ‘climb up through’ this scheme’ beyond AW polynomials.

The paper is organized as follows. In the next two subsections we provide a simple introduction to $q$-series theory, presenting the typical notation used and a few typical families of the so-called basic orthogonal polynomials. The word basic comes from the base, which is a parameter denoted in most cases by $q$ and such that $-1 < q \leqslant 1$. Then in section 2 we present our main results, open questions and remarks are in section 3, while less interesting laborious proofs are in section 4.

1.2. Notation

We use notation traditionally found in the so-called $q$-series theory. Since not all readers are familiar with it, we will now recall this notation.

$q$ is a parameter such that $-1 < q \leqslant 1$ unless otherwise stated. Let us define $[0]_q = 0; [n]_q = 1 + q + \cdots + q^{n-1}, [n]_q! = \prod_{j=1}^{n}[j]_q$, with $[0]_q! = 1$ and

$$\begin{align*}
\begin{bmatrix} n \\ k \end{bmatrix}_q &= \begin{cases} [n]_q! \\ [n-k]_q! [k]_q! \\ 0 \end{cases} , & n \geqslant k \geqslant 0 \\
&= \begin{cases} [n]_q! \\ 0 \end{cases} , & \text{otherwise}
\end{align*}
$$

It will be useful to use the so-called $q$-Pochhammer symbol for $n \geqslant 1$ :

$$\begin{align*}
(a; q)_n &= \prod_{j=0}^{n-1}(1 - aq^j), \\
(a_1, a_2, \ldots, a_k; q)_n &= \prod_{j=1}^{k}(a_j; q)_n.
\end{align*}$$
with \((a; q)_0 = 1\). Often \((a; q)_n\), as well as \((a_1, a_2, \ldots, a_k; q)_n\), will be abbreviated to \((a)_n\) and \((a_1, a_2, \ldots, a_k)_n\), if it will not cause any misunderstanding.

It is easy to notice that \((q)_n = (1 - q)^n[n]_q\) and that
\[
\binom{n}{k}_q = \frac{(q)_n}{(q)_n-k(q)_k}, \quad n \geq k \geq 0
\]
otherwise

Also notice that \([n]_1 = n, [n]_1! = n!\), \[\binom{n}{0}_1 = \binom{n}{0}\], \(a; 1)_n = (1 - a)^n\) and 
\[n]_0 = \begin{cases} 1 & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}, \quad [n]_0! = 1, \quad \binom{n}{0}_0 = \begin{cases} 1 & \text{if } n = 0 \\ 1 - a & \text{if } n \geq 1 \end{cases}.

1.3. Auxiliary notions and results

1.3.1. Orthogonal polynomials. Following [5] or [6] we will define the following families of polynomials.

The \(q\)H polynomials denoted by \(\{h_n(x|q)\}_{n \geq 0}\) constitute the one parameter family of orthogonal polynomials satisfying the following three-term recurrence:
\[
h_{n+1}(x|q) = 2xh_n(x|q) - (1 - q^n)h_{n-1}(x|q),
\]
with \(h_{-1}(x|q) = 0\) and \(h_0(x|q) = 1\).

The \(bq\)H polynomials denoted by \(\{h_n(x|a, q)\}_{n \geq 0}\) constitute the two-parameter family of orthogonal polynomials that satisfy the following three-term recurrence:
\[
h_{n+1}(x|a, q) = (2x - aq^n)h_n(x|a, q) - (1 - q^n)h_{n-1}(x|a, q),
\]
with \(h_{-1}(x|a, q) = 0\), \(h_0(x|a, q) = 1\).

The ASC polynomials denoted by \(\{Q_n(x|a, b, q)\}_{n \geq 0}\) constitute the three-parameter family of orthogonal polynomials that satisfy the following three-term recurrence:
\[
Q_{n+1}(x|a, b, q) = (2x - (a + b)q^n)Q_n(x|a, b, q) - (1 - abq^{n-1})(1 - q^n)Q_{n-1}(x|\rho, \psi, q),
\]
with \(Q_{-1}(x|a, b, q) = 0, Q_0(x|a, b, q) = 1\).

For \(n = 0\) we set \((1 - abq^{n-1})(1 - q^n)Q_{n-1}(x|\rho, \psi, q) = 0\).

The \(c2h\) polynomials denoted by \(\{\psi_n(x|a, b, c, q)\}_{n \geq 0}\) constitute the four-parameter family of orthogonal polynomials that satisfy the following three-term recurrence:
\[
\psi_{n+1}(x|a, b, c, q) = (2x - d_n)\psi_n(x|a, b, c, q) - f_{n-1}(1 - q^n)\psi_{n-1}(x|a, b, c, q),
\]
with \(\psi_{-1}(x|a, b, c, q) = 0, \psi_0(x|a, b, c, q) = 1\) and coefficients \(d_n\) and \(f_n\) given by for \(n \geq 0\):
\[
d_n = (a + b + c)q^n + abcq^{n-1}(1 - q^n - q^{n+1}),
\]
\[
f_n = (1 - abq^n)(1 - acq^n)(1 - bcq^n).
\]
Again for \(n = 0\) we set \(f_{n-1}(1 - q^n)\psi_{n-1}(x|a, b, c, q) = 0\).

Orthogonality of all these polynomials takes place on \([-1, 1]\). From Favard’s theorem it follows that if for all \(n > 0\) : \((1 - abq^{n-1}) \geq 0\) and \((1 - abq^n)(1 - acq^n)(1 - bcq^n) \geq 0\) then respectively ASC and \(c2h\) polynomials are orthogonal with respect to a positive measure.

Let us also mention the so-called AW polynomials \(\{AW_n\}_{n \geq 0}\). They can be defined with the help of the basic hypergeometric function as was done in the original paper [3] by Askey and Wilson or by the three-term recurrence as in [6] or [5]. The polynomials that we will call AW and denote \(AW_n\) are in fact equal to \(2^n p_n(x)\) where polynomials \(p_n\) are defined by the
Lemma 1. Suppose \( a_i \)

then

\[
\sum_{i=0}^{n} \left[ \begin{array}{c}
\frac{n}{i}
\end{array} \right] (-a)^{n-i} q^{(i)} \left( \frac{bcq^i + bdq^i + cdq^i}{abcbd^q + cdq^i} \right) \psi_i(x|b, c, d, q),
\]

as proved in [10] formula (2.5). Again we understand that \( AW_{-1}(x|a, b, c, d, q) = 0 \) and \( AW_0(x|a, b, c, d, q) = 1 \).

Finally let us mention the Chebyshev polynomials \( \{U_n\}_{n \geq 0} \) of the second kind that satisfy the following three-term recurrence:

\[
2xU_n(x) = U_{n+1}(x) + U_{n-1}(x),
\]

with \( U_{-1}(x) = 0 \) and \( U_0(x) = 1 \). These polynomials will play an auxiliary role.

1.3.2. Properties of some orthogonal polynomials. We have the following elementary lemma.

Lemma 1. Suppose \( a_i(x) = \sum_{k=0}^{s} \beta_k U_{n-k}(x) \) for \( i = 1, 2, \ldots, n \) for some constants \( \beta_j \), \( j = 0, \ldots, s \) with \( n \geq s \). Suppose also that for \( m \geq n \) we have

\[
am_{m+1}(x) = 2x am(x) - a_{m-1}(x),
\]

then \( \forall m \geq n : \)

\[
am_m(x) = \sum_{k=0}^{s} \beta_k U_{m-k}(x).
\]

Proof. The proof is by induction. For \( m = n \) it is true by assumption. Let us assume that it is true for \( m = j \). Then for \( m = j + 1 \) we have

\[
a_{j+1}(x) = 2x \sum_{k=0}^{s} \beta_k U_{j-k}(x) - \sum_{k=0}^{s} \beta_k U_{j-1-k}(x) = \sum_{k=0}^{s} \beta_k (U_{j+k}(x) + U_{j-k}(x)) = \sum_{k=0}^{s} \beta_k U_{j+1-k}(x).
\]

As an immediate corollary we have the following proposition with some assertions already known. We present them here together in order to reveal the regularities and pave the way to possible generalizations. This proposition will help to justify the conjecture concerning generalization of our main result that will be presented below in section 3.

Proposition 1.

(i) \( \forall n \geq 0 : \)

\[
h_n(x|0) = U_n(x), h_n(x|a, 0) = U_n(x) - aU_{n-1}(x),
\]

(ii) \( \forall n \geq 1 : \)

\[
Q_n(x|a, b, 0) = U_n(x) - (a + b)U_{n-1}(x) + abU_{n-2}(x),
\]

(iii) \( \forall n \geq 1 : \)

\[
\psi_n(x|a, b, c, 0) = U_n(x) - (a + b + c)U_{n-1}(x) + (ab + cb + ac)U_{n-2}(x) - abcU_{n-3}(x),
\]

(iv) \( \forall n \geq 2 : \)

\[
AW_n(x|a, b, c, d, 0) = U_n(x) - (a + b + c + d)U_{n-1}(x) + (ab + ac + ad + bc + bd + cd)U_{n-2}(x) - (abc + abd + bcd + acd)U_{n-3} + abcdU_{n-4}(x),
\]

where \( AW_n \) denotes the Askey–Wilson polynomial as defined by (1.6).
Proof. Shifted to section 4.

Remark 1. Notice that \( \forall n \geq -1 \)

\[
h_n(x|0) = U_n(x), \quad h_n(x|0, q) = h_n(x|q), \quad Q_n(x|a, 0, q) = h_n(x|a, q),
\]

\[
\psi_{n+1}(x|a, b, 0, q) = Q_n(x|a, b, q).
\]

Remark 2. To support our intuition, let us remark, following e.g. [8], that

\[
\lim_{q \to 1^-} h_n \left( x \sqrt{\frac{1-q}{2}} | a \sqrt{1-q}, q \right) / (1-q)^{n/2} = He_n(x-a),
\]

where \( He_n \) denotes the \( n \)th so-called ‘probabilistic’ Hermite polynomial, i.e. the polynomial orthogonal with respect to measure with the density \( \exp(-x^2/2)/\sqrt{2\pi} \).

For completeness, let us mention that polynomials \( qH, bqH, ASC \) are related mutually by the following relationships (see e.g. [9] and [2, (4.9)]):

\[
h_n(x|a, q) = \sum_{k=0}^{n} \binom{n}{k} (-a)^k q^{(k)} h_{n-k}(x|q), \quad (1.8)
\]

\[
Q_n(x|a, b, q) = \sum_{k=0}^{n} \binom{n}{k} (-a)^k q^{(k)} h_{n-k}(x|b, q). \quad (1.9)
\]

Recently the \( c2h \) polynomials were related to the ASC polynomials by the following relationship (after slight modification of [2, (2.7)]):

\[
\psi_n(x|a, b, c, q) = \sum_{i=0}^{n} \binom{n}{i} (-a)^i q^{(i)} (bcq^{n-i}), Q_{n-i}(x|b, c, q). \quad (1.10)
\]

1.4. General result

We finish this section by presenting an auxiliary simple result that will be used several times in the following. We have the following proposition.

Proposition 2. Let \( \sigma_n(\rho, q) = \sum_{r \geq 0} q^r \xi_{r+n} \) for \( |\rho| < 1 \) and certain sequence \( \{\xi_m\}_{m \geq 0} \) such that \( \sigma_n \) exists for every \( n \). Then

\[
\sigma_n(\rho q^m, q) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} q^{(k)} \rho^k \sigma_{n+k}(\rho, q). \quad (1.11)
\]

Proof. An easy, not very interesting proof by induction is shifted to section 4.

2. Main results

Theorem 1. (i) For \( \forall n \geq 0; x^2 \leq 1; t^2 < 1 \):

\[
\sum_{i \geq 0} \frac{t^i}{(q)_i} h_{i+n}(x|q) = h_n(x|t, q) \sum_{i \geq 0} \frac{t^i}{(q)_i} h_i(x|q), \quad (2.1)
\]

where \( h_n(x|t, q) \) is the \( bqH \) polynomial defined by (1.3).

(ii) For \( \forall n \geq 0; x^2 \leq 1; t^2 < 1; |at| < 1 \):

\[
\sum_{i \geq 0} \frac{t^i}{(q)_i} h_{i+n}(x|a, t, q) = \frac{Q_n(x|a, t, q)}{(at)_n} \sum_{i \geq 0} \frac{t^i}{(q)_i} h_i(x|a, q), \quad (2.2)
\]

where \( Q_n(x|a, t, q) \) is the \( ASC \) polynomial defined by (1.4).
(iii) For \( \forall n \geq 0; |x| \leq 1; t^2 < 1; |at|, |bt| < 1:\)

\[
\sum_{i \geq 0} \frac{t^i}{(q)^i} Q_{i+n}(x|a, b, q) = \frac{\psi_n(x|a, b, t, q)}{(at, bt)_n} \sum_{i \geq 0} \frac{t^i}{(q)^i} Q_i(x|a, b, q), \tag{2.3}
\]

where polynomial \( \psi_n(x|a, b, t, q) \) is the \( c2h \) polynomial defined by (1.5).

\textbf{Proof.} Shifted to section 4. \( \square \)

It turns out that similar properties can be attributed to some other, classical orthogonal polynomials. In the case of Hermite and Laguerre polynomials, this was observed by Rainville in his book [7]. Let \( \{H_n(x)\}_{n \geq 1} \) and \( \{U_n(x)\}_{n \geq 1} \) denote classical and orthogonal polynomials, respectively Hermite and Chebyshev. Let us consider polynomials \( \{\lambda_n(x, \alpha)\}_{n \geq 1} \) that are monic versions of Laguerre polynomials. More precisely

\[
\lambda_n(x, \alpha) = (-1)^n n! L_n^{(\alpha)}(x),
\]

where \( L_n^{(\alpha)}(x) \) denotes traditional Laguerre polynomials e.g. defined by (1.11.1) of [6]. The three-term recurrences of polynomials \( \{H_n(x)\}_{n \geq 1}, \{\lambda_n(x, \alpha)\}_{n \geq 1} \) and \( \{U_n(x)\}_{n \geq 1} \) are given by formulae (1.13.4) and (1.11.4) of [6] and (1.7), respectively.

\textbf{Lemma 2.} For \( \forall n \geq 0, x, t \in \mathbb{R} \) with \( t \neq -1 \) in the case of (2.5) we have

\[
\sum_{j \geq 0} \frac{t^j}{j!} H_n(x+t) = H_n(x-t) \sum_{j \geq 0} \frac{t^j}{j!} H_j(x), \tag{2.4}
\]

\[
\sum_{j \geq 0} \frac{t^j}{j!} \lambda_n(x+t, \alpha) = \frac{\lambda_n(x, \alpha)}{(1+t)^n} \sum_{j \geq 0} \frac{t^j}{j!} \lambda_j(x, \alpha), \tag{2.5}
\]

\[
\sum_{j \geq 0} t^j U_n(x+t) = (U_n(x) - tU_{n-1}(x)) \sum_{j \geq 0} t^j U_j(x). \tag{2.6}
\]

\textbf{Proof.} (2.4) is proved in [7, p 197, equation (1.1)], (2.5) is proved in [7, p 211, equation (1.9)] with an obvious modification such as the change of \( t \) to \( -t \). The proof of (2.6) is the following. Let \( v_n(x, t) = \sum_{j \geq 0} t^j U_{n+j}(x) \). We have

\[
2xv_n(x, t) = \sum_{j \geq 0} t^j (U_{n+j+1}(x) + U_{n+j-1}(x)) = v_{n+1}(x, t) + v_{n-1}(x, t),
\]

with

\[
v_1(x, t) = \sum_{j \geq 0} t^j U_{1+j}(x) = \sum_{j \geq 0} t^j (2xU_j(x) - U_{j-1}(x)) = 2xv_0(x, t) - t v_0(x, t) = (2x - t) v_0(x, t) = (U_1(x) - tU_0(x))v_0(x, t).
\]

Hence by proposition 1 we have: \( v_n(x, t)/v_0(x, t) = (U_n(x) - tU_{n-1}(x)). \) \( \square \)
3. Open problems and comments

Remark 3. Recalling the \(q\)-differentiation formula (see e.g. [5, (11.4.1), p 296]) \(D_{q,t}f(x) = f(qt) - f(x) \over x(q-1)\) we see that

\[
D_{q,t} \left( \frac{\rho^n}{(q)_n} \right) = \frac{\rho^{n-1}}{(1-q)(q)_{n-1}}.
\]

Let us define the \(n\)-fold composition of the operator \(D\).

\[
D^n_{q,t}f(x) = D_{q,t}(D_{q,t}(\cdots D_{q,t}f(x))).
\]

Hence we deduce that expressions of the form \(\phi_n(x|t, q) = \sum_{i \geq 0} \frac{t^i}{q^i} P_i(x|q)\) considered in the first three assertions of theorem 1 are in fact following (3.1) proportional to the appropriate \(q\)-derivatives of \(\phi_0(x|t, q)\) with respect to \(t\). Hence those assertions can be expressed in the following ‘\(q\)-Rodrigues’-like form:

\[
\begin{align*}
D^n_{q,t} \left( \sum_{i \geq 0} \frac{t^i}{(q)_i} h_i(x|q) \right) &= (1-q)^n h_n(x|t, q) \sum_{i \geq 0} \frac{t^i}{(q)_i} h_i(x|q), \\
D^n_{q,t} \left( \sum_{i \geq 0} \frac{t^i}{(q)_i} h_i(x|a, q) \right) &= (1-q)^n Q_n(x|a, t, q) \sum_{i \geq 0} \frac{t^i}{(q)_i} h_i(x|a, q), \\
D^n_{q,t} \left( \sum_{i \geq 0} \frac{t^i}{(q)_i} Q_i(x|a, b, q) \right) &= (1-q)^n \psi_n(x|a, b, t, q) \sum_{i \geq 0} \frac{t^i}{(q)_i} Q_i(x|a, b, q).
\end{align*}
\]

Thus it is natural to pose the following hypothesis.

Conjecture 1.

\[
D^n_{q,t} \left( \sum_{i \geq 0} \frac{t^i}{(q)_i} \psi_i(x|a, b, c, q) \right) = (1-q)^n AW_n(x|a, b, c, t, q) \sum_{i \geq 0} \frac{t^i}{(q)_i} \psi_i(x|a, b, c, q),
\]

where \(AW_n\) denotes the \(n\)th Askey–Wilson polynomial as defined in (1.6).

Remark 4. Let us notice that the above mentioned conjecture is almost trivially true for \(q = 1\) in view of (2.4) with an obvious modification that all polynomial families considered are modified in the following way. Instead of polynomials \(h_n\) we consider polynomials \(H_n(x|q) = h_n(x^{\sqrt{-q}}|q)/(1-q)^{n/2}\). Instead of polynomials \(h_n(x|a, q)\) we consider polynomials \(H_n(x|a, q) = h_n(x^{\sqrt{-q}}|a\sqrt{1-\bar{a}}, q)/(1-q)^{n/2}\). Instead of polynomials \(Q_n(x|a, b, q)\) we consider polynomials \(P_n(x|a, b, q) = Q_n(x^{\sqrt{-q}}|a\sqrt{1-\bar{a}}, b\sqrt{1-\bar{b}}, q)/(1-q)^{n/2}\). Instead of polynomials \(\psi_n(x|a, b, c, q)\) we consider polynomials \(\rho_n(x|a, b, c, q) = \psi_n(x^{\sqrt{-q}}|a\sqrt{1-\bar{a}}, b\sqrt{1-\bar{b}}, c\sqrt{1-\bar{c}}, q)/(1-q)^{n/2}\). Finally, instead of polynomials \(AW_n(x|a, b, c, d, q)\) we consider polynomials \(a_n(x|a, b, c, d) = AW_n(x^{\sqrt{-q}}|a\sqrt{1-\bar{a}}, b\sqrt{1-\bar{b}}, c\sqrt{1-\bar{c}}, d\sqrt{1-\bar{d}}, q)/(1-q)^{n/2}\) while all generating functions involved are defined as the sum \(\sum_{i \geq 0} \frac{t^i}{(q)^i} p_{n+i}(x)\) where instead of \(p_n\) we put \(H_n(x|q)\) or \(H_n(x|a, q)\) or \(P_n(x|a, b, q)\) or \(G_n(x|a, b, a, q)\). Then for \(q = 1\) we have \(H_n(x|1) = H_n(x)\), \(H_n(x|a, 1) = H_n(x-a)\), \(P_n(x|a, b, 1) = H_n(x-a-b)\), \(G_n(x|a, b, c, 1) = H_n(x-a-b-c)\), \(a_n(x|a, b, a, d, 1) = H_n(x-a-b-c-d)\), where \(H_n(x)\) denotes, as before, the probabilistic Hermite polynomial as described in remark 2.
Proposition 3. Conjecture 1 is true for \( q = 0 \).

**Proof.** The proof is shifted to section 4. \( \square \)

**Remark 5.** A successful attempt to obtain different polynomials from the AW scheme was made in [1]. It was done through modification of the so-called AW divided \( q \)-difference operator, a complicated \( q \)-differentiation scheme, applied straightforwardly to polynomials themselves as well as to the densities of measures that make these polynomials orthogonal. Our differentiation scheme presented in remark 3 involves a much simpler \( q \)-difference operator not applied directly to the polynomials involved but to the characteristic functions of these polynomials.

**Problem 1** (Open problems). Continuing the line of generalizations presented in theorem 1 and conjecture 1 one can pose the following question. Is it true that:

\[
D^n_{q,t} \left( \sum_{i \geq 0} t^i \text{AW}_i(x|a, b, c, d, q) \right) = \text{GAW}_n(x|a, b, c, d, q, t) \sum_{i \geq 0} t^i \text{AW}_i(x|a, b, c, d, q),
\]

where \( \{\text{GAW}_n(x|a, b, c, d, q, t)\}_{n \geq -1} \), constitute a family of orthogonal (i.e. satisfying some three-term recurrence) polynomials in \( x \)?

If it was true then naturally polynomials GAW\(_n\) could be regarded as a generalization of the AW polynomials. Then of course one could pose many further questions concerning the properties of these polynomials. One such question could be to compare this attempt to generalize AW polynomials with another one presented in [11].

4. Proofs

**Proof of proposition 1.** Assertions (i), (ii), (iii) and assertion (iv) for complex parameters were already mentioned in [8]; however, in view of lemma 1, the proof can be reduced to the following two arguments. First of all, notice by examining equations (1.2)–(1.5) and proposition 2 of [10] that for \( n \geq 2 \) for the first three assertions and for \( n \geq 3 \) for the AW polynomials for \( q = 0 \), the three-term recurrences satisfied by all polynomials mentioned in the proposition are in fact identical with (1.7). Now it remains to check (by direct computation performed, for example, with the help of the package Mathematica) that indeed for \( n = 0, 1, 2, 3 \) all polynomials in question have the form mentioned in proposition 1. Another justification of all assertions follows formulae (1.9), (1.8), (1.10) and (1.6), from which it follows for \( q = 0 \) that for \( n \geq 1 \) : \( \text{AW}_n(x|a, b, c, d, 0) = \psi_n(x|b, c, d, 0) = n! \psi_{n-1}(x|b, c, d, 0) \) and similarly for the other polynomials considered. One has to be cautious with the case \( n = 1 \) and AW polynomials. One can easily check that for \( n = 1 \) assertion (iv) is not true. \( \square \)

**Proof of proposition 2.** First we will prove that

\[
\sigma_n(\rho q^m, q) = \sigma_n(\rho q^{m-1}, q) - \rho q^{m-1} \sigma_{n+1}(\rho q^{m-1}, q).
\]

We have:

\[
\sigma_n(\rho q^m, q) = \sum_{j \geq 0} q^{(m-1)j} \xi_{n+j} = \sum_{j \geq 0} q^{(m-1)j} \rho^j \xi_{n+j} - \sum_{j \geq 0} q^{(m-1)j} (1 - q) \rho^j \xi_{n+j} = \sigma_n(\rho q^{m-1}, q) - \rho q^{m-1} \sum_{j \geq 0} q^{j(n-1)} \xi_{n+j+1}.
\]
Then we prove (1.11) by induction with respect to $\rho$. We see that it is true for $m = 1$. Hence let us assume that it is true for $m \leq k$. Let us consider $m = k + 1$. We have:

\[
\sigma_n(\rho q^{k+1}, q) = \sigma_n(\rho q^k, q) - \rho q^k \sigma_{n+1}(\rho q^k, q)
\]

\[
= \sum_{j=0}^{k} (-1)^j \left[ \begin{array}{l} k \\ j \end{array} \right] q^{j} \rho^j \sigma_{n+j}(\rho, q) - \rho q^k \sum_{j=0}^{k} (-1)^j \left[ \begin{array}{l} k \\ j \end{array} \right] q^{j} \rho^j \sigma_{n+1+j}(\rho, q)
\]

\[
= \sigma_n(\rho, q) + (-1)^{k+1} \rho^{k+1} q^{(k+1)} \sigma_{n+k+1}(\rho, q)
\]

\[
+ \sum_{j=1}^{k} (-1)^j \left[ \begin{array}{l} k \\ j \end{array} \right] q^{j} \rho^j \sigma_{n+j}(\rho, q)
\]

\[
+ \sum_{j=0}^{k-1} (-1)^{j+1} \left[ \begin{array}{l} k \\ j \end{array} \right] q^{j} \rho^{k-j} \sigma_{n+k+1}(\rho, q).
\]

Now we change the index of summation from $j = s - 1$ and obtain:

\[
\sigma_n(\rho q^s, q) - \rho q^s \sigma_{n+1}(\rho q^s, q)
\]

\[
= \sigma_n(\rho, q) + (-1)^{k+1} \rho^{k+1} q^{(k+1)} \sigma_{n+k+1}(\rho, q) + \sum_{j=1}^{k} (-1)^j \left[ \begin{array}{l} k \\ j \end{array} \right] q^{j} \rho^j \sigma_{n+j}(\rho, q).
\]

since \(\binom{k}{j} + j - 1 = \binom{k}{j+1}\). Now we use the fact that \(\binom{k}{j} + q^{j+1} \binom{k}{j+1} = \binom{k+1}{j+1}\). Hence we see that (1.11) is true.

\[\square\]

**Proof of theorem 1.** Notice that (i) and (ii) follow (iii) by remark 1. Thus, it is enough to prove (iii).

(iii) Let us denote $\chi_n(x|a, b, t, q) = \sum_{i \geq 0} t^i q^i Q_{n+i}(x|a, b, q)$. We use (1.11) on the way.

\[
2x \chi_n(x|a, b, t, q) = \sum_{i \geq 0} t^i (2x - (a + b)q^{t+i} + (a + b)q^{n+i}) Q_{n+i}(x|a, b, q)
\]

\[
= (a + b)q^n \chi_n(x|a, b, t, q) + \sum_{i \geq 0} \frac{t^i}{q^i} Q_{n+i+1}(x|a, b, q)
\]

\[
+ (1 - abq^{n+i-1}) (1 - q^{n+i}) Q_{n+i-1}(x|a, b, q)
\]

\[
= (a + b)q^n \chi_n(x|a, b, t, q) - (a + b) q^n t \chi_{n+1}(x|a, b, t, q) + \chi_{n+1}(x|a, b, t, q)
\]

\[
+ (1 - q^n) \sum_{i \geq 0} \frac{t^i}{q^i} (1 - abq^{n+i-1}) Q_{n+i-1}(x|a, b, q)
\]

\[
+ q^n \sum_{i \geq 0} \frac{t^i}{q^i} (1 - q^n) (1 - abq^{n+i-1}) Q_{n+i-1}(x|a, b, q).
\]

Further, we have

\[
2x \chi_n(x|a, b, t, q) = (a + b)q^n \chi_n(x|a, b, t, q) + (1 - (a + b)q^t) \chi_{n+1}(x|a, b, t, q)
\]

\[
+ (1 - q^n) (1 - abq^{n-1}) \chi_{n-1}(x|a, b, t, q)
\]

\[
+ (1 - q^n) abq^{n-1} \sum_{i \geq 0} \frac{t^i}{q^i} (1 - q^n) Q_{n+i-1}(x|a, b, q)
\]
\[ +q^n(1 - abq^n) \sum_{i \geq 0} \frac{t^i}{(q)_i} (1 - q^n)Q_{n+i-1}(x|a, b, q) \]
\[ +q^nabq^n \sum_{i \geq 0} \frac{t^i}{(q)_i} (1 - q^n)(1 - q^{n-1})Q_{n+i-1}(x|a, b, q) \]
\[ = \chi_{n+1}(x|a, b, t, q)(1 - (a + b)q^n) + q^nabq^{n-1}t + q^n(1 - abq^n)t \]
\[ + (1 - q^n)(1 - abq^{n-1})\chi_{n-1}(x|a, b, t, q). \]

Hence we have the following equation:
\[ (2x - (a + b + t)q^n - abtq^{n-1}(1 - q^n - q^{n+1}))\chi_n(x|a, b, t, q) \]
\[ = (1 - (a + b)q^n)q^t + q^{2n}abt^2) \chi_{n+1}(x|a, b, t, q) \]
\[ + (1 - q^n)(1 - abtq^{n-1})\chi_{n-1}(x|a, b, t, q). \]

Notice that \(1 - (a + b)q^n + q^{2n}abt^2) = (1 - atq^n)(1 - btq^n).\)

Let \( \tilde{\chi}_n(x|a, b, t, q) = \chi_n(x|a, b, t, q)(at, bt)_n. \) After multiplying both sides of (4.2) by \((at, bt)_n,\) we then have
\[ \tilde{\chi}_n(x|a, b, t, q)(2x - (a + b + t)q^n - abtq^{n-1}(1 - q^n - q^{n+1})) \]
\[ = \tilde{\chi}_{n+1}(x|a, b, t, q) + (1 - q^n)(1 - abtq^{n-1}(1 - atq^n) \times (1 - btq^{n-1})\tilde{\chi}_{n-1}(x|a, b, t, q). \]

with \( \tilde{\chi}_0(x|a, b, t, q) = \chi_0(x|a, b, t, q). \) Hence \( \tilde{\chi}_n \) satisfies the same three-term recurrence as \( \psi_n \) compare (1.5). Besides we have
\[ \chi_1(x|a, b, t, q) = \sum_{i \geq 0} \frac{t^i}{(q)_i} Q_{1+i}(x|a, b, q) \]
\[ = \sum_{i \geq 0} \frac{t^i}{[i]_q} q^{i}[(2x-(a+b)q^n)Q_{i}(x|a, b, q)-(1 - q^n)q(1-\chi_{i-1}(x|a, b, q) \]
\[ = (2x - t)\chi_0(x|a, b, t, q) - (a + b)\chi_0(x|a, b, t, q) + t\chi_0(x|a, b, t, q) \]
\[ = (2x - a - b - t + ab)\chi_0(x|a, b, t, q) \]
\[ + (a + b)\chi_1(x|a, b, t, q) - t^2ab\chi_1(x|a, b, t, q). \]

So:
\[ \chi_1(x|a, b, t, q) = (2x - a - b - t + ab) \]
\[ (1 - (a + b)t + t^2ab) \chi_0(x|a, b, t, q). \]

Consequently \( \tilde{\chi}_1(x|a, b, t, q) = (2x - a - b - t + ab)\tilde{\chi}_0(x|a, b, t, q) \) and we deduce that \( \tilde{\chi}_{-1}(x|a, b, t, q) = 0. \) Thus we deduce from examining equation (4.4) that:
\( \tilde{\chi}_n(x|a, b, t, q)/\tilde{\chi}_0(x|a, b, t, q) \) satisfies the three-term recurrence the same as the one satisfied by continuous dual \( q \)-Hahn polynomials, with the same initial conditions.

**Proof of proposition 3.** By proposition 1, (iii) we have
\[ \sum_{j \geq 0} t^j\psi_{j+n}(x|a, b, c, 0) = \]
\[ \sum_{j \geq 0} t^j(U_{n+j}(x) - (a + b + c)U_{n+j-1}(x) + (ab + ac + bc)U_{n+j-2}(x) - abcU_{n+j-3}(x)). \]
Let us denote \( \chi_n(x|a, b, c, t) = \sum_{j \geq 0} t^j \psi_{j+n}(x|a, b, c, 0) \). Hence:

\[
\chi_n(x|a, b, c, t) = \alpha_n(x, t) - (a + b + c)\alpha_{n-1}(x, t) + (ab + ac + bc)\alpha_{n-2}(x, t) - abc\alpha_{n-3}(x, t),
\]

where we denoted \( \alpha_n(x, t) = \sum_{j \geq 0} t^j U_{j+n}(x) \). \( \alpha_n(x, t) \) was already calculated in lemma 2, (2.6) and is equal to \( \alpha_n(x, t)(U_n(x) - tU_{n-1}(x)) \).

Hence

\[
\chi_n(x|a, b, c, t) = (U_n(x) - tU_{n-1}(x) - (a + b + c)U_{n-1} + (at + bt + ct)U_{n-2}
\]

\[
+ (ab + ac + bc)U_{n-2} - (abt + act + bct)U_{n-3}
\]

\[
- abcU_{n-3} + abctU_{n-4})\alpha_0(x, t)
\]

\[
= (U_n(x) - (a + b + c + t)U_{n-1}(x) + (ab + ac + bc + at + bt + ct)U_{n-2}(x)
\]

\[
- (abc + tab + tac + bct)U_{n-3}(x) + abctU_{n-4}(x))\alpha_0(x, t)
\]

\[= AW_n(x|a, b, c, t, 0)\alpha_0(x, t),\]

by proposition 1, (iv). On the other hand we have:

\[
\chi_0(x|a, b, c, t) = \sum_{j \geq 0} t^j(U_j(x) - (a + b + c)U_{j-1}(x)
\]

\[
+ (ab + ac + bc)U_{j-2}(x) - abcU_{j-3}(x))
\]

\[= \alpha_0(x, t)(1 - (a + b + c)t + (ab + ac + bc)t^2 - abct^3)
\]

\[= \alpha_0(x, t)(1 - at)(1 - bt)(1 - ct).
\]

So

\[
\chi_n(x|a, b, c, t) = AW_n(x|a, b, c, t, 0) \frac{\chi_0(x|a, b, c, t)}{(1 - at)(1 - bt)(1 - ct)}. \]

\[
\Box
\]

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