On entanglement of states and quantum correlations *

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Abstract
In this paper we present the novel qualities of entanglement of formation for general (so also infinite dimensional) quantum systems and we introduce the notion of coefficient of quantum correlations. Our presentation stems from rigorous description of entanglement of formation.

1 Introduction
The problem of quantum entanglement of mixed states has attracted much attention recently and that concept has been widely considered in different physical contexts (cf. [1], [2] and references therein, see also [3], [4], [5], [6]). Moreover, it is frequently argued that the nature of entangled states is strongly related to quantum correlations.

In this paper we are concerned with the generalization of the entanglement of formation, introduced in [7] as well as with the rigorous definition of a measure of quantum correlations. To this end, firstly we look more closely at the original definition of EoF. Namely, there is a difficulty in implementing the definition given by Bennett et al in the sense that it is not clear why the operation of taking \( \min \) over the set of all decomposition of the given state into finite convex combination of pure states is well defined (for details see [8]). To overcome this problem and to get a measure with nice properties we shall use the theory of decomposition which is based on the theory of compact convex sets and boundary integrals. Then, having rigorously described measure of entanglement we will discuss the concept of coefficient of quantum correlations. The paper is organized as follows. In Section II we set up notation and terminology, and we review some of the standard facts on the theory of decomposition. Section III contains our definition of entanglement of formation, EoF, with theorem 1 saying that EoF is equal to zero if and only if the state is a separable one.

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section IV we review properties of EoF. In the final section V, we present the concept of coefficient of quantum correlations with a discussion of its relations to entanglement.

2 Preliminaries

Let us consider a composite system “1 + 2” and its Hilbert space of pure states \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) where \( \mathcal{H}_i \) is the Hilbert space associated to subsystem \( i \) (\( i = 1, 2 \)). Let \( \mathcal{B}(\mathcal{H}) \) denote the set of all bounded linear operators on \( \mathcal{H} \). Unless otherwise stated, \( \mathcal{M} \) stands for a (unital) \( C^* \)-subalgebra of \( \mathcal{B}(\mathcal{H}_2) \). We will assume that \( \mathcal{H}_1 \) is a finite dimensional space (for a discussion how to dispense with that assumption see \([8]\)). \( \mathcal{H}_2 \) will be an arbitrary (infinite dimensional, separable) Hilbert space. In other words, the composite system consists of small subsystem and a big heat-bath, rather a typical situation for concrete physical problems.

 Turning to states we recall that any density matrix (positive operator of trace equal to 1) on \( \mathcal{H} \) determines uniquely a linear positive, normalized, functional \( \omega_\rho(\cdot) = \omega(\cdot) = Tr\{\rho \cdot\} \) on \( \mathcal{B}(\mathcal{H}) \) which is also called a normal state. We will assume Ruelle’s separability condition for \( \mathcal{M} \) (cf. \([9]\), \([10]\), \([11]\)): a subset \( \mathcal{F} \) of the set of all states \( \mathcal{S} \) of \( \mathcal{M} \) satisfies Ruelle’s separability [Note: this refers to topological properties, and is not related to the algebraic notion, which is the subject of this paper] condition if there exists a sequence \( \{\mathcal{M}_n\} \) of sub-\( C^* \)-algebras of \( \mathcal{M} \) such that \( \cup_{n \geq 1} \mathcal{M}_n \) is dense in \( \mathcal{M} \), and each \( \mathcal{M}_n \) contains a closed, two-sided, separable ideal \( \mathcal{I}_n \) such that

\[
\mathcal{F} = \{\omega; \omega \in \mathcal{S}, ||\omega||_{\mathcal{I}_n} = 1, n \geq 1\}
\]

We recall that this condition leads to a situation in which the subsets of states have good measurability properties (cf \([13]\)). Furthermore, one can easily verify that this separability condition is satisfied in our case provided that we restrict to the set of normal states on \( \mathcal{M} \) or \( \mathcal{M} \) is a separable \( C^* \)-algebra.

The density matrix \( \rho \) (state) on the Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) is called separable if it can be written or approximated (in the norm) by the density matrices (states) of the form:

\[
\rho = \sum p_i \rho^1_i \otimes \rho^2_i \quad \bigg( \omega(\cdot) = \sum p_i (\omega^1_i \otimes \omega^2_i)(\cdot) \bigg)
\]

where \( p_i \geq 0, \sum p_i = 1, \rho^\alpha_i \) are density matrices on \( \mathcal{H}_\alpha, \alpha = 1, 2, \) and \((\omega^1_i \otimes \omega^2_i)(A \otimes B) \equiv \omega^1_i(A) \cdot \omega^2_i(B) \equiv (Tr\rho^1_i A) \cdot (Tr\rho^2_i B) \equiv Tr\{\rho^1_i \otimes \rho^2_i \cdot A \otimes B\} \). In other words, separable states are the norm-closed convex hull of all product states on \( \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{M} \) (or more generally, on the tensor product of two \( C^* \)-algebras). It is well known (see e.g. \([12]\)) the state space of the tensor product \( \mathcal{A}_1 \otimes \mathcal{A}_2 (\mathcal{N}_1 \otimes \mathcal{N}_2) \) of two \( C^* \)-algebras \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) (two \( W^* \)-algebras \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \))
respectively) is not the norm-closed (weak*-closed) convex hull of all product states on $A_1 \otimes A_2$ ($N_1 \otimes N_2$). Thus, one can define

**Definition 1** Non-separable states are called entangled states. The set of entangled states is defined by

$$S_{\text{entangled}} \equiv S_{\text{en}} = S \setminus \{\text{separable states}\}$$

where $S$ stands for the state space.

Now, for the convenience of the reader, we introduce some terminology and give a short résumé of results from convexity and Choquet theory that we shall need in the sequel (for details see [13], [14], [15], and [11]). Let $A$ stand for a $C^*$-algebra. From now on we make the same assumption of Ruelle separability for $A$ which was posed for $M$. In next sections, by a slight abuse of notation we will write $A$ for $B(H_1) \otimes M$. By $S$ we will denote the state space of $A$, i.e. the set of linear, positive, normalized, linear functionals on $A$. We recall that $S$ is a compact convex set in the $*$-weak topology. Further, we denote by $M_1(S)$ the set of all probability Radon measures on $S$. It is well known that $M_1(S)$ is a compact subset of the vector space of real, regular Borel measures on $S$.

Further, let us recall the concept of barycenter $b(\mu)$ of a measure $\mu \in M_1(S)$:

$$b(\mu) = \int d\mu(\varphi)\varphi$$

where the integral is understood in the weak sense. The set $M_\omega(S)$ is defined as a subset of $M_1(S)$ with barycenter $\omega$, i.e.

$$M_\omega(S) = \{\mu \in M_1(S), b(\mu) = \omega\}$$

$M_\omega(S)$ is a convex closed subset of $M_1(S)$, hence compact in the weak $*$-topology. Thus, it follows by the Krein-Milman theorem that there are "many" extreme points in $M_\omega(S)$. We say the measure $\mu$ is simplicial if $\mu$ is an extreme point in $M_\omega(S)$. The set of all simplicial measures in $M_\omega(S)$ will be denoted by $\mathcal{E}(S)$.

### 3 Entanglement of Formation

Let us define, for a state $\omega$ on $B(H_1) \otimes M$ the following map:

$$(r_\omega)(A) \equiv \omega(A \otimes 1)$$

where $A \in B(H_1)$.

Clearly, $r_\omega$ is a state on $B(H_1)$. One has

Let $r_\omega$ be a pure state on $B(H_1)$ (so a state determined by a vector from $H_1$). Then $\omega$ can be written as a product state on $B(H_1) \otimes M$. 


The proof of that statement can be extracted from [16]. (For more details we refer the reader to [13], [17], [8]).

Conversely, there is another result in operator algebras saying that if \( \omega \) is a state on \( \mathcal{B}(\mathcal{H}_1) \) then there exists a state \( \omega' \) over \( \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{M} \) which extends \( \omega \). If \( \omega \) is a pure state of \( \mathcal{B}(\mathcal{H}_1) \) then \( \omega' \) may be chosen to be a pure state of \( \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{M} \) (cf. [11]). This observation is the crucial one for our definition of entanglement of formation which is phrased in terms of decomposition theory.

**Definition 2** Let \( \omega \) be a state on \( \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{M} \). The entanglement of formation, \( \text{EoF} \), is defined as

\[
E(\omega) = \inf_{\mu \in M(\mathcal{S})} \int_{\mathcal{S}} d\mu(\varphi) S(r\varphi) \tag{6}
\]

where \( S(\cdot) \) stands for the von Neumann entropy, i.e. \( S(\varphi) = -\text{Tr}\rho_\varphi \log \rho_\varphi \) where \( \rho_\varphi \) is the density matrix determining the state \( \varphi \).

To comment the above definition we recall that the map \( r \) and the function \( S \) are (\('\)-weakly) continuous. At this point we want to strongly emphasize that we use the entropy function \( S \) only to respect tradition. Namely, to have a well defined concept of EoF we need a concave non-negative continuous function which vanishes on pure states (and only on pure states). In our case, with the first subsystem being finite, the von Neumann entropy meets these conditions. Clearly, there are others functions satisfying these conditions. Our next remark is that we define EoF as infimum of integrals evaluated on continuous function and the infimum is taken over a compact set. Therefore, the infimum is attainable, i.e. there exists a measure \( \mu_0 \in M(\mathcal{S}) \) such that

\[
E(\omega) = \int_{\mathcal{S}} d\mu_0(\varphi) S(r\varphi) \tag{7}
\]

and

\[
\omega = \int_{\mathcal{S}} d\mu_0(\varphi) \varphi \tag{8}
\]

To argue that \( E(\omega) \) is a well defined measure of entanglement one should show that \( \mathcal{F} \ni \omega \mapsto E(\omega) \) is equal to 0 only for separable states (we recall that \( \mathcal{F} \) stands for the subset of states satisfying Ruelle’s condition, cf. Section II). This is the case. Namely, one can prove (see [8])

**Theorem 1** A state \( \omega \in \mathcal{F} \) is separable if and only if \( \text{EoF} E(\omega) \) is equal to 0.
4 Properties of EoF

In this section we list briefly properties of EoF. We start with

4.1 Convexity of EoF

Firstly, let us observe that the set $M_{\lambda_1\omega_1+\lambda_2\omega_2}(S)$ contains the sum of the sets $\lambda_1M_{\omega_1}(S)$ and $\lambda_2M_{\omega_2}(S)$ where $\lambda_1$ and $\lambda_2$ are non-negative numbers such that $\lambda_1 + \lambda_2 = 1$. To see this we recall (see e.g. [11] or [18]) that $\mu \in M_{\omega}(S)$ if and only if $\mu(f) \geq f(\omega)$ for any continuous, real-valued, convex function $f$. Thus

$$(\lambda_1\mu_1 + \lambda_2\mu_2)(f) \geq \lambda_1f(\omega_1) + \lambda_2f(\omega_2) \geq f(\lambda_1\omega_1 + \lambda_2\omega_2) \quad (9)$$

implies the above stated relation between sets. Hence

$$E(\lambda_1\omega_1 + \lambda_2\omega_2) = \inf_{\mu \in M_{\lambda_1\omega_1+\lambda_2\omega_2}(S)} \int d\mu(\varphi)S(\varphi) \leq \lambda_1 \inf_{\mu \in M_{\omega_1}(S)} \int d\mu(\varphi)S(\varphi)$$

$$+ \lambda_2 \inf_{\mu \in M_{\omega_2}(S)} \int d\mu(\varphi)S(\varphi) = \lambda_1 E(\omega_1) + \lambda_2 E(\omega_2) \quad (10)$$

Consequently, the function $S \ni \omega \mapsto E(\omega)$ is convex.

4.2 Subadditivity of EoF

To discuss this property, which seems to be important in quantum information (cf. [2]), we consider the tensor product of von Neumann algebras $B(H_1) \otimes M \otimes B(H_1) \otimes M$ and a state $\omega \otimes \omega$ over it where $\omega$ is a state on $B(H_1) \otimes M$. We observe

$$E(\omega \otimes \omega) = \inf_{\mu \in M_{\omega \otimes \omega}(S_T)} \int d\mu(\nu)S_{1+2}(r\nu) \leq$$

$$\inf_{\mu_1 \times \mu_2 \in M_{\omega}(S) \times M_{\omega}(S)} \int d\mu_1(\nu) \int d\mu_2(\nu')S_{1+2}(r \circ \nu \otimes \nu') \leq$$

$$\inf_{\mu_1 \times \mu_2 \in M_{\omega}(S) \times M_{\omega}(S)} \int d\mu_1(\nu) \int d\mu_2(\nu')(S_1(\nu) + S_1(\nu')) = 2E(\omega) \quad (11)$$

where $S_T$ denotes the set of all states on $B(H_1) \otimes M \otimes B(H_2) \otimes M$, $S_{1+2}$ ($S_1$) the von Neumann entropy on $B(H_1) \otimes B(H_1)$ ($B(H_1)$ respectively). The last inequality follows from subadditivity of the von Neumann entropy. Consequently, EoF has also a form of subadditivity. Applying the above argument to $E(\omega \otimes \ldots \otimes \omega)$ one can consider the “density” of EoF and treat $E(\omega)$ as an extensive (thermodynamical) quantity.
4.3 Continuity of EoF

As entanglement of formation, EoF, is a convex, real-valued function on the topological space $S$

$$S \ni \omega \mapsto E(\omega) \in \mathbb{R}$$  \hspace{1cm} (12)

it is natural to pose a question about its continuity. Going in that direction we proved (see [8])

**Proposition 1** EoF, $S \ni \omega \mapsto E(\omega)$, is a continuous function.

This result has the following important corollary. Namely, as $S \ni \omega \mapsto E(\omega)$ is a continuous convex function, an application of the Bauer maximum principle leads to:

**Corollary 1** $E(\omega)$ attains its maximum at an extremal point of $S$, so the family of maximally entangled states is a subset of pure states.

4.4 Comparison with the Bennett, DiVincenzo, Smolin and Wooters definition of EoF

As our definition of EoF is a generalization of that given by Bennett et al (cf [7]), it is natural to compare these two definitions. Let us denote Bennett’s et al entanglement of formation by EoF$_B$. It is an easy observation that $EoF \leq EoF_B$.

To examine the converse inequality we start with another simple observation that

$$\inf_{\mu \in M(\omega(S))} \int d\mu(\nu)S(r\nu)$$ \hspace{1cm} (13)

$$= \inf \left\{ \sum_{i=1}^{n} \lambda_i S(r\nu_i) : \omega = \sum_{i=1}^{n} \lambda_i \nu_i \hspace{0.5cm} (\text{convex sum}) \right\}$$ \hspace{1cm} (14)

where the first infimum is attained for some $\mu \in M(\omega(S))$. The above observation follows from the fact that each measure $\mu$ can be (weakly) approximated by measures with finite support. On the other hand, measures concentrated on $S_p$, where $S_p$ is the set of all pure states, are known to be maximal with respect to the order $\mu \prec \nu$ (\mu \prec \nu if and only if $\mu(f) \leq \nu(f)$ for any convex, real-valued convex function $f$, cf. [B] or [R]), so minimal on the set of all concave functions.

It particular, such the measure is minimal on $S \circ r$. Thus to get the converse inequality, $EoF \geq EoF_B$ it would be enough to prove existence of very special type of decompositions, so called optimal decompositions. A decomposition $\omega = \sum_{j=1}^{n} \lambda_j \theta_j$, where $\{\theta_j\}$ are pure states, such that the infimum in the definition of EoF is attained will be called an optimal decomposition. In other words, the
infimum is attained by a measure \( \mu_0 \) with finite support contained in the set of all pure states. Thus, we want to have
\[
E(\omega) = \inf_{\mu \in \mathcal{M}_\omega(S)} \int_S S(r \varrho) d\mu(\varrho) = \int_S d\mu_0(\varrho) S(r \varrho)
\]
with \( \text{supp}\mu_0 = \{ \varrho_1, ..., \varrho_n \}, \ n < \infty \) and \( \varrho_i \in \mathcal{S}_p \). Here, \( \mu_0 = \sum_1^n \lambda_i \delta_{\varrho_i} \) where \( \delta_{\varrho} \) stands for the Dirac measure, \( \{ \varrho_i \} \) are pure states and \( \omega = \sum \lambda_i \varrho_i \). In (8) we proved:

**Proposition 2** The maximum of the set \( \{ \mu(-S \circ r); \mu \in \mathcal{M}_\omega(S) \} \) for a continuous convex function \(-S\) is attained by a simplicial boundary measure.

Then a straightforward application of the classical Carathéodory theorem (cf [18]) leads to

**Corollary 2** Assume that both Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are finite dimensional. Then, there exist optimal decompositions. Therefore, our definition of EoF and that given by Bennett et al are equal to each other. However, this is not true if the assumption on dimensionality of Hilbert spaces be dropped.

### 5 Quantum correlations

In this Section we introduce the notion of coefficient of quantum correlations and we will look more closely at relations between quantum correlations and entanglement. We wish to start with a generalization of the framework of the previous Sections. Let \( \mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \) be a (e.g. spatial) tensor product of \( \mathcal{C}^*\)-algebras \( \mathcal{A}_i \). We assume that each \( \mathcal{A}_i \) contains the identity \( \hat{1} \). Let \( \phi \) be a state on \( \mathcal{A} \). Again, the set of all states on \( \mathcal{A} \) will be denoted by \( \mathcal{S}(\mathcal{A}) \). The pair \( (\mathcal{A}, \phi) \) will be considered as a (quantum) probability system. Further, let \( (a_1, ..., a_m) \) be a system of elements of \( \mathcal{A} \) such that for every \( \nu = 1, 2, ..., m \) there is \( i_\nu \in \{ 1, ..., n \} \) such that \( a_\nu \in \mathcal{A}_{i_\nu} \). To measure any correlations of the system we have to analyze the evaluation of a state \( \phi \) on \( m \)-points \( a_1, ..., a_m \), i.e., \( \phi(a_1, ..., a_m) \).

In the sequel, considering \( \phi(a_1, ..., a_m) \), we will always assume that \( a_i \in \mathcal{A}_i \) and indices are ordered. This is legitimate since each \( \mathcal{A}_i \) can be embeded in \( \mathcal{A} \) and then the tensor product structure implies that \( a_i \) commutes with \( a_j \) for \( i \neq j \), \( a_i \in \mathcal{A}_i, a_j \in \mathcal{A}_j \). Consequently, we will consider \( \phi(a_{\nu_1}, ..., a_{\nu_l}) \) where \( (\nu_1, ..., \nu_l) \subset \{ 1, ..., n \} \) is an ordered subset and \( a_{\nu_i} \in \mathcal{A}_{i_{\nu_i}} \).

Let us define, now in more general context, the restriction map \( r \) (cf. [11]). Let \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) be \( \mathcal{C}^*\)-subalgebras of the \( \mathcal{C}^*\)-algebra \( \mathcal{A} \). Assume that \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) and \( \mathcal{A} \) have a common identity, \( \mathcal{B}_1 \subseteq \mathcal{B}_2 \) and \( \mathcal{B}_1 \cup \mathcal{B}_2 \) generates \( \mathcal{A} \) as a \( \mathcal{C}^*\)-algebra. Define the map \( r : \mathcal{S}(\mathcal{A}) \to \mathcal{S}(\mathcal{B}_1) \) by
\[
(r \omega)(a) = \omega(a) \quad \text{for all} \quad a \in \mathcal{B}_1
\]
Specializing this definition for the tensor structure of \( A = \otimes_{i=1}^{N} A_i \), one has the following definition \( r_{\nu k} : S(A) \to S(A_{\nu k}) \),

\[
(r_{\nu k}\omega)(a) = \omega(\hat{1} \otimes \ldots \otimes \hat{1} \otimes a \otimes \hat{1} \otimes \ldots \otimes \hat{1})
\]  

(16)

Clearly, \( r_{\nu k} \) is an affine, \( w^* \)-continuous, “onto”, map. Again, as \( S(A) \) is \( w^* \)-weak compact set one can employ the Choquet theory. To this end, we denote by \( M_\omega(S(A)) \) the set of all positive, normalized Radon measures on \( S(A) \) with the barycenter \( \omega \). Furthermore, we denote by \( M^0_{r_{\nu k}\phi}(S(A_{\nu k})) \subset M_{r_{\nu k}\phi}(S(A_{\nu k})) \) the set of all finitely supported positive normalized Radon measures. Thus, if \( \mu_{\nu k} \) is in \( M^0_{r_{\nu k}\phi}(S(A_{\nu k})) \), then \( \mu_{\nu k} = \sum_{i=1}^{P} \lambda_i \delta_{\phi_i} \) with \( \sum_{i=1}^{P} \lambda_i \phi_i = r_{\nu k}\omega \). Again, \( \delta_{\phi_i} \) stands for the Dirac (or point) measure.

Turning to quantum correlations, we recall that the entanglement is often considered as a signature of quantum correlations. Although, the concept of quantum correlations is essential one for quantum statistical mechanics, there is still lack of its precise definition. To make an attempt to formulate a rigorous definition of quantum correlations, guided by the (classical) probability theory with its definition of coefficient of independence, we will define the coefficient of quantum correlations. Leaving aside for a moment the general framework, let us present the basic idea for the simplest composite system, i.e. a system consisting of two subsystems only. Thus, \( A = A_1 \otimes A_2 \). We note

**Remark 1** Let us consider a separable state \( \omega \) on \( A \equiv A_1 \otimes A_2 \), \( \omega(\cdot) \equiv Tr\{(\sum_i \lambda_i \hat{\rho}_i^1 \otimes \hat{\rho}_i^2)\} \) and observe that, in general, \( \omega(a \otimes \hat{1} \otimes \hat{1} \otimes b) \neq \omega(a \otimes \hat{1})\omega(\hat{1} \otimes b) \) for \( a \in A_1 \) and \( b \in A_2 \). Thus, the state \( \omega \) reflects some correlations. However, as the state \( \omega \) is separable, these correlations are considered to be of classical nature only. Namely, each (classical) probability measure can be \( (w^* \)-weakly) approximated by a net of probability measures with finite support. Hence, each (classical) probability measure on a composite system exhibits the basic properties of a separable state.

Therefore, to define “pure” quantum correlations we should “subtract” classical correlations. Suppose that a measure \( \mu \) is in \( M^0_{\phi}(S(A)) \). So, \( \mu = \sum_{i=1}^{P} \lambda_i \delta_{\phi_i} \) and the corresponding decomposition of \( \phi \) is given by \( \phi = \sum_{i=1}^{P} \lambda_i \phi_i \). As \( r_1 (r_2) \) is an affine map of \( S(A) \) onto \( S(A_1) (S(A_2) \) respectively) one has

\[
r_1\phi = \sum_{i=1}^{P} \lambda_i \cdot r_1(\phi_i)
\]  

(17)

and

\[
r_2\phi = \sum_{i=1}^{P} \lambda_i \cdot r_2(\phi_i)
\]  

(18)
Consequently, the decomposition of \( \phi \) determined by \( \mu \) induces the corresponding decomposition of \( r_1 \phi \) and \( r_2 \phi \) (determined by \( \mu_1 = \sum \lambda_i \delta_{r_1 \phi_i} \) and \( \mu_2 = \sum \lambda_i \delta_{r_2 \phi_i} \), respectively). More generally, let us define \( \mu_I \) (\( \mu_{II} \)) on Borel subsets \( \mathcal{F}_I \subset \mathcal{S}(A_1) \) (\( \mathcal{F}_{II} \subset \mathcal{S}(A_2) \)) respectively by

\[
\mu_I(\mathcal{F}_I) = \mu(r_1^{-1}(\mathcal{F}_I))
\]

and

\[
\mu_{II}(\mathcal{F}_{II}) = \mu(r_2^{-1}(\mathcal{F}_{II}))
\]

where \( \mu \) is a measure in \( M_\phi(\mathcal{S}(A)) \). This can be done as for any Borel set \( F \) (for example, take as \( F \) the subset \( \mathcal{F}_I \subset \mathcal{S}(A_1) \), \( r^{-1}(F) \) is a Borel set in \( \mathcal{S}(A) \).

Suppose that \( \mathcal{F}_I^0 \) is a Borel subset in \( \mathcal{S}(A_1) \) such that \( \mathcal{F}_I^0 \subset \{ r_1 \rho_1, ..., r_1 \rho_P \} \) and consider \( \mu_I(\{ \mathcal{F}_I^0 \}) \equiv \mu(r_1^{-1}(\{ \mathcal{F}_I^0 \})) \), \( \mu \in M_\phi(\mathcal{S}) \). Clearly, \( \mathcal{F}_I^0 \equiv r_1^{-1}(\{ \mathcal{F}_I^0 \}) \supset \{ \rho_1, ..., \rho_P \} \). But, if \( \mu \) is supported by the subset \( \{ \rho_1, ..., \rho_P \} \) of \( \mathcal{F}_I^0 \) then \( \mu_I \) is supported on \( \{ r_1 \rho_1, ..., r_1 \rho_P \} \). Furthermore, assuming \( \mu \in M_\phi(\mathcal{S}(A)) \) and noting \( r_1 \phi = \int r_1 \xi d\mu(\xi) = \int \xi d\mu \circ r^{-1}(\xi) \) one has \( \int \xi d\mu(\xi) = r_1 \phi \). Here, we denoted \( r_1 \xi \) by \( \xi_r \). Clearly, the same argument can be applied for \( r_2 \) and \( \mu_{II} \).

In particular, one can easily note that

\[
\mu_I(\{ r_1 \rho_i \}) = \mu_{II}(\{ r_2 \rho_i \}), \quad i = 1, ..., P
\]

for any \( \mu \in M_\phi(\mathcal{S}) \). Having measures \( \mu_I \) on \( \mathcal{S}(A_1) \) and \( \mu_{II} \) on \( \mathcal{S}(A_2) \), both originating from the measure \( \mu \) on \( \mathcal{S}(A) \), we wish to define a new measure \( \boxtimes \mu \) on \( \mathcal{S}(A_1) \times \mathcal{S}(A_2) \) which encodes classical correlations between two subsystems described by \( A_1 \) and \( A_2 \) respectively. As the first step we define it for discrete measures. Let \( \mu^d \in M_\phi^d(\mathcal{S}) \), i.e. \( \mu^d = \sum_i \lambda_i^d \delta_{\rho_i^d} \) with \( \lambda_i^d \geq 0, \sum_i \lambda_i^d = 1, \rho_i^d \in \mathcal{S}(A) \). Then, the just given argument leads to \( \mu_I^d = \sum_i \lambda_i^d \delta_{r_1 \rho_i^d} \) and \( \mu_{II}^d = \sum_i \lambda_i^d \delta_{r_2 \rho_i^d} \). Define

\[
\boxtimes \mu = \sum_i \lambda_i^d \delta_{r_1 \rho_i^d} \times \delta_{r_2 \rho_i^d}
\]

where we have used (21). Now, let us take an arbitrary measure \( \mu \) in \( M_\phi(\mathcal{S}) \). Then, there exists net \( \mu_k \) such that \( \mu_k \in M_\phi^d(\mathcal{S}) \) and \( \mu_k \rightarrow \mu \) (*-weakly). Defining \( \mu^k_I \) (\( \mu^k_{II} \)) analogously as \( \mu_I \) (\( \mu_{II} \)) respectively; cf (14) one has \( \mu^k_I \rightarrow \mu_I \) and \( \mu^k_{II} \rightarrow \mu_{II} \) where the convergence is taken in *-weak topology. Then define, for each \( k \), \( \boxtimes \mu^k \) as in (22). One can verify that \( \{ \boxtimes \mu^k \} \) is convergent to a measure on \( \mathcal{S}(A_1) \times \mathcal{S}(A_2) \), so taking the weak limit we arrive to the measure \( \boxtimes \mu \) on \( \mathcal{S}(A_1) \times \mathcal{S}(A_2) \). All that leads to

**Definition 3** 1. Let \( A \) be a C*-algebra with two W*-subalgebras \( B_1, B_2 \) satisfying conditions given prior to formula (14) supplemented by the condition \( B_2 \subset B_1' \). The coefficient of quantum correlations for the state \( \phi \) evaluated
on \( a_1 a_2, \phi(a_1 a_2), a_i \in \mathcal{B}_i, i = 1, 2 \), is defined as

\[
CQC(\phi; a_1, a_2) = \inf_{\mu \in M_\infty(S(\mathcal{A}))} |(\int \xi d(\mu)(\xi)) (a_1 a_2) - (\int \xi d(\otimes \mu)(\xi)) (a_1 a_2)|
\]

(23)

where \( a_i \in \mathcal{B}_i, i = 1, 2 \).

2. Assume that \( \mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \). Then, specializing the definition of CQC to the tensor structure of the C* algebra \( \mathcal{A} \) we have the degree of quantum correlations for the state \( \phi \) evaluated on \( a_1 \otimes a_2 \). It is defined as

\[
d(\phi; a_1, a_2) = \inf_{\mu \in M_\infty(S(\mathcal{A}_1))} |\phi(a_1 \otimes a_2) - (\int \xi d(\otimes \mu)(\xi)) (a_1 \otimes a_2)|
\]

(25)

where \( a_i \in \mathcal{A}_i, i = 1, 2 \).

To comment on the above definition, firstly we note that the definition of CQC makes no appeal to the tensor structure of \( \mathcal{A} \). Therefore, such a definition seems to be very well adapted to the general theory of quasi-local algebras, so to the general theory of quantum systems (cf. \cite{11}). Clearly, we followed the classical definition of coefficient of independence. The main difference between the classical and quantum approaches (apart from the existence of extra correlations) relies on the fact that in the quantum setting, the set of states does not form a simplex. Therefore, there is no uniqueness in decomposition of a (quantum) state. Hence, to carry out our construction we are forced to take the infimum operation over the set of all “good” decompositions.

Secondly, to have the framework well adapted to an analysis of separable (so also entangled) states it is necessary to take into account the tensor structure of the algebra. To distinguish these two cases, we give two different names to a measure of quantum correlations: coefficient (degree respectively) of quantum correlation.

Turning to separable states we have

**Proposition 3** A state \( \phi \) is separable one if and only if \( d(\phi; a_1, a_2) = 0 \) for any \( a_1, a_2 \).

**Proof:** Recall that \( \phi \) is separable iff \( EoF(\phi) = 0 \) (cf. Section III). Hence, there exists a measure \( \mu^0 \) such that \( \phi = \int \xi d\mu^0(\xi) \) with the property that \( suppmu \) is contained in the following set

\[
\{ \xi_1 \otimes \xi_2; \xi_1 \in S(\mathcal{A}_1), \xi_2 \in S(\mathcal{A}_2) \}
\]

An application of the restriction maps \( r_1 \) and \( r_2 \) to the measure \( \mu^0 \) lead to measures \( \mu^0_1 \) and \( \mu^0_2 \). Then, considering the \((\ast\text{-}weak)\) approximation one has

\[
\mu^0 = lim \mu^0_k \text{ with } \mu^0_k = \sum \lambda^k_1 \delta_{\phi_{a,k}^1} \times \delta_{\phi_{a,k}^2} \text{ where } \phi_{a,k} \in S(\mathcal{A}_a), a \in \{1, 2\}.
\]

Clearly, \( \mu_{a,k}^0 \equiv \mu^0_k \circ r_a = \sum \lambda^k_1 \delta_{\phi_{a,k}^1} \), \( a \in \{I, II\} \). Therefore, \( \mu^0_k = \otimes \mu^0_k \).
Hence, \( d(\phi; a_1, a_2) \) is equal to 0. Conversely, suppose that \( d(\phi; a_1, a_2) = 0 \) for any \( a_1, a_2 \). As \( M_\phi(S(A)) \) is compact, then \( \inf \) in definition of \( d(\phi; a_1, a_2) \) is attainable. Therefore, there exist two measures \( \mu_I \) and \( \mu_{II} \) defining \( \boxplus \mu \) such that
\[
\phi(a_1 \otimes a_2) = \left( \int \xi d(\boxplus \mu)(\xi) \right) (a_1 \otimes a_2) \tag{26}
\]
However, this proves the separability.

The Proposition may be summarized by saying that any separable state contains classical correlations only. Therefore, an entangled state contains “non-classical” (or quantum) correlations.

**Remark 2** CQC yields information about quantum correlations and therefore it makes legitimate to apply CQC for an analysis of quantum stochastic dynamics. However, this topic exceeds the scope of this paper and it will be present in another paper (see [20], and [6])

Turning to the general case, \( A = \otimes_{i=1}^N A_i \), to each state \( \phi \) on \( A \) we will assign the family of product states
\[
\{ \phi_{\lambda_1}^\lambda \otimes ... \otimes \phi_{\lambda_N}^\lambda \}_{\lambda \in \Lambda} \tag{27}
\]
where for each \( \lambda \in \Lambda, \phi_{\lambda_i}^\lambda \in \text{supp}\mu_k \) for some \( \mu_k \in M_{r_{\nu_k}}(S(A_{\nu_k})) \). We recall that \( M_{r_{\nu_k}}(S(A_{\nu_k})) \) stands for all all normalized positive Radon measures with barycenter of the restricted state \( r_{\nu_k} \phi \) on \( A_{\nu_k} \). Let us define
\[
\text{conv}\{ \phi_{\lambda_1}^\lambda \otimes ... \otimes \phi_{\lambda_N}^\lambda \}_{\lambda \in \Lambda} \equiv S_{cc} \tag{28}
\]
where, by a slight abuse of notation we denote an extension of \( \phi_{\nu_1}^\lambda \otimes ... \otimes \phi_{\nu_l}^\lambda \) to a state over \( A \) by the same letter.

We have observed that one can interpret a state in \( S_{cc} \) as a state encoding classical correlations only. Therefore, a state in \( S_{cc} \) will be called a \( c \)-dependent state. As CQC measures the deviation of correlations of a state from classical correlations, going in that direction, we propose

**Definition 4** Let \( A = \otimes_{i=1}^N A_i \) and let a state \( \phi \) be in \( S(A) \). Then

1. \[
d(\phi, S_{cc}) = \inf_{\psi \in S_{cc}} \| \phi - \psi \| \tag{29}
\]
will be called the uniform degree of quantum correlation (UDQC).

2. \[
d_\phi(a_{\nu_1}, ..., a_{\nu_l}) = \inf_{\psi \in S_{cc}} | \phi(a_{\nu_1}, ..., a_{\nu_l}) - \psi(a_{\nu_1}, ..., a_{\nu_l}) | \tag{30}
\]
will be called the weak degree of quantum correlation (WDQC). Here, we recall that \((\nu_1, ..., \nu_l) \subset \{1, ..., N \}\) is an ordered subset.
We close this Section with some remarks on Definition 4. Firstly, it is an easy observation that $d(\phi, S_{cc}) = 0$ if and only if $\phi$ is a separable state. Secondly, the equality $d_\phi(a_1, \ldots, a_m) = 0$ can be treated as a definition of quantum independence of subsystems of a composite system. However, we would like to emphasize that the subsystems still can have a “classical” correlations. Finally, let us specialize Definition 4.2 to a quantum chain, i.e. $A = \bigotimes_{i \in \mathbb{Z}} M_d(\mathbb{C}) \equiv A_{\mathbb{Z}}$, $A_1 = A_{(-\infty,0]}$, $A_2 = A_{[1]}$, ..., $A_{N-1} = A_{[N-2]}$, $A_N = A_{(N-1,\infty]}. Here, the algebra $M_d(\mathbb{C})$ associated with each site $i$ is taken to be the full algebra of $d \times d$ matrices. Then, the subset of states with $WDQC > 0$ can be called finitely quantum correlated states (cf. [21]).

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