On a generalisation of finite $T$-groups

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Abstract
Let $\sigma = \{\sigma_i | i \in I\}$ is some partition of all primes $\mathbb{P}$ and $G$ a finite group. A subgroup $H$ of $G$ is said to be $\sigma$-subnormal in $G$ if there exists a subgroup chain $H = H_0 \leq H_1 \leq \cdots \leq H_n = G$ such that either $H_{i-1}$ is normal in $H_i$ or $H_i/(H_{i-1})_{H_i}$ is a finite $\sigma_j$-group for some $j \in I$ for $i = 1, \ldots, n$. We call a finite group $G$ a $T_{\sigma}$-group if every $\sigma$-subnormal subgroup is normal in $G$.

In this paper, we analyse the structure of the $T_{\sigma}$-groups and give some characterisations of the $T_{\sigma}$-groups.

1 Introduction

Throughout this paper, all groups are finite and $G$ always denotes a finite group. $\mathbb{P}$ denotes the set of all primes and $\pi$ denotes a set of primes. If $n$ is an integer, then the symbol $\pi(n)$ denotes the set of all primes dividing $n$; as usual, $\pi(G) = \pi(|G|)$, the set of all primes dividing the order of $G$.

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1.1 \textit{T}-groups

A group $G$ is said to be a \textit{T}-group if every subnormal subgroup of $G$ is normal in $G$. The $T$-groups are clearly the groups in which normality is a transitive relation. The classical works by Gaschütz [5] and Robinson [17] reveal a very detailed picture of such groups.

Recall that $G$ is said to be a \textit{Dedekind group} if every subgroup of $G$ is normal in $G$; it is clear that a nilpotent group $G$ is a $T$-group if and only if every subgroup of $G$ is normal in $G$; that is, $G$ is a Dedekind group. More generally, Gaschütz proved the following result:

\textbf{Theorem 1.1.} (See Gaschütz [5])

Let $G$ be a group with $G^{\mathrm{nr}}$ the nilpotent residual of $G$, that is the intersection of all normal subgroups $N$ of $G$ with nilpotent quotient $G/N$. Then $G$ is a soluble $T$-group if and only if the following conditions hold:

(i) $G^{\mathrm{nr}}$ is a normal abelian Hall subgroup of $G$ with odd order;
(ii) $G/G^{\mathrm{nr}}$ is a Dedekind group;
(iii) Every subgroup of $G^{\mathrm{nr}}$ is normal in $G$.

Recall that a group $G$ satisfies the condition $\mathcal{R}_p$ [17] (where $p$ is a prime) if every subgroup of a Sylow $p$-subgroup $P$ of $G$ is normal in the normalizer of $P$. Robinson studied the structure of finite $T$-groups using the condition $\mathcal{R}_p$ and get the following theorem.

\textbf{Theorem 1.2.} (See Robinson [17])

A finite group $G$ which satisfies $\mathcal{R}_p$ for all $p$ if and only if $G$ is a soluble $T$-group.

Some other characterisations of the soluble $T$-groups have been researched. (See [2, 14, 15]).

1.2 The theory of $\sigma$-groups

In recent years, a new theory of $\sigma$-groups has been established by A. N. Skiba and W. Guo (See [7, 10, 19, 20]).

In fact, following L. A. Shemetkov [18], $\sigma = \{\sigma_i | i \in I\}$ is some partition of all primes $\mathbb{P}$, that is, $\mathbb{P} = \bigcup_{i \in I} \sigma_i$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$. $\Pi$ is always supposed to be a non-empty subset of $\sigma$ and $\Pi' = \sigma \setminus \Pi$. We write $\sigma(G) = \{\sigma_i | \sigma_i \cap \pi(G) \neq \emptyset\}$.

Following [19–21], $G$ is said to be: $\sigma$-primary if $|\sigma(G)| \leq 1$; $\sigma$-soluble if every chief factor of $G$ is $\sigma$-primary. $G$ is called $\sigma$-nilpotent if $G = G_1 \times \cdots \times G_n$ for some $\sigma$-primary groups $G_1, \cdots, G_n$. A subgroup $H$ of $G$ is said to be $\sigma$-subnormal in $G$ if there exists a subgroup chain $H = H_0 \leq H_1 \leq \cdots \leq H_n = G$ such that either $H_{i-1}$ is normal in $H_i$ or $H_i/(H_{i-1})_{H_i}$ is $\sigma$-primary for all $i = 1, \cdots, n$; An integer $n$ is said to be a $\Pi$-number if $\pi(n) \subseteq \bigcup_{\sigma_i \in \Pi} \sigma_i$; a subgroup $H$ of $G$ is called a $\Pi$-subgroup of $G$ if $|H|$ is a $\Pi$-number; a subgroup $H$ of $G$ is called a Hall $\Pi$-subgroup of $G$ if $H$ is a $\Pi$-subgroup of $G$ and $|G : H|$ is a $\Pi'$-number; a subgroup $H$ of $G$ is called a Hall
σ-subgroup of $G$ if $H$ is a Hall $\Pi$-subgroup of $G$ for some $\Pi \subseteq \sigma$. We use $\mathcal{N}_\sigma$ to denote the class of $\sigma$-nilpotent groups.

**Remark 1.3.** In the case when $\sigma = \{\{2\}, \{3\}, \cdots\}$ (we use here the notation in [20]),

1. $\sigma$-soluble groups and $\sigma$-nilpotent groups are soluble groups and nilpotent groups respectively.
2. A Hall $\sigma$-subnormal is subnormal.
3. A Hall $\sigma_i$-subgroup of $G$ is a Hall $\pi$-subgroup for some $\pi \subseteq \mathcal{P}$.
4. A Hall $\sigma_i$-subgroup of $G$ is a Sylow subgroup of $G$.

This new theory of $\sigma$-groups is actually the development and popularization of the famous Sylow theorem, the Hall theorem of the soluble groups and the Chunihin theorem of $\pi$-soluble groups. A series of studies have been caused (See, for example, [1, 3, 7–11, 13, 19–25]).

### 1.3 The $T_\sigma$-groups and the main results

Combined with the above two contents, we naturally reposed the following problem:

**Question 1.4.** Based on this new theory of $\sigma$-groups, could we establish the theory of generalised $T$-groups?

In this paper, we will solve this question. We first give the following definition:

**Definition 1.5.** We called a group $G$ a $T_\sigma$-group if every $\sigma$-subnormal of $G$ is normal in $G$.

Bearing in mind the results in [5, 17], it seems to be natural to ask:

**Question 1.6.** What is the structure of the $T_\sigma$-groups?

We will give a complete answer to this question in the case when $G$ is $\sigma$-soluble. It is clear that every subnormal subgroup is $\sigma$-subnormal in $G$ and so every $T_\sigma$-groups is a $T$-groups. However, the following example shows that the converse is not true.

**Example 1.7.** Let $A = C_3 \times C_2$ be a non-abelain subgroup of order 6 and let $G = A \times C_5$. Let $\sigma = \{\sigma_1, \sigma_2, \sigma_3\}$, where $\sigma_1 = \{2, 3\}$, $\sigma_2 = \{5\}$ and $\sigma_3 = \{2, 3, 5\}'$. Then $G$ is a $T$-group but is not a $T_\sigma$-group. In fact, obviously, $G$ is a $T$-group. However $G$ is not a $T_\sigma$-group since $C_2$ is a $\sigma$-subnormal subgroup of $G$ but is not normal in $G$.

In order to better describe the $T_\sigma$-groups, I give the following definition:

**Definition 1.8.** We called a group $G$ satisfies the condition $\mathcal{R}_{\sigma_i}$ if every subgroup $K$ of every Hall $\pi$-subgroup $H$ of $G$ (for $\pi \subseteq \sigma_i$) is normal in the normalizer $NG(H)$ of $H$.

**Remark 1.9.** In the case when $\sigma = \{\{2\}, \{3\}, \cdots\}$, the condition $\mathcal{R}_{\sigma_i}$ is just the condition $\mathcal{R}_p$. 

3
The following theorem gives an answer to Question 1.6.

**Theorem 1.10.** Let \( G \) be a group, \( D = G^{\sigma_0} \) and \( G \) is \( \sigma \)-soluble. Then the following statements are equivalent.

1. \( G \) is a \( T_\sigma \)-group;
2. \( G \) satisfies the conditions \( \mathcal{R}_{\sigma_i} \) for all \( i \).
3. \( G \) satisfies the following conditions:
   i. \( G = D \rtimes M \), where \( D \) is an abelian Hall subgroup of \( G \) of odd order, \( M \) is a Dedekind group;
   ii. every element of \( G \) induces a power automorphism on \( D \); and
   iii. \( O_{\sigma_i}(D) \) has a normal complement in a Hall \( \sigma_i \)-subgroup of \( G \) for all \( i \).

In this theorem, \( G^{\sigma_0} \) denotes the \( \sigma \)-nilpotent residual of \( G \), that is, the intersection of all normal subgroups \( N \) of \( G \) with \( \sigma \)-nilpotent quotient \( G/N \), and \( O_{\sigma_i}(D) \) denotes the maximal normal \( \sigma_i \)-subgroup.

**Remark 1.11.** In the case when \( \sigma = \{\{2\}, \{3\}, \cdots\} \), Theorems 1.1 and 1.2 are corollaries of our Theorem 1.10.

### 2 Preliminaries

**Lemma 2.1.** [19, Corollary 2.4 and Lemma 2.5] The class \( \mathfrak{N}_\sigma \) of all \( \sigma \)-nilpotent groups is closed under taking products of normal subgroups, homomorphic images and subgroups. Moreover, if \( E \) is a normal subgroup of \( G \) and \( E/E \cap \Phi(G) \) is \( \sigma \)-nilpotent, then \( E \) is \( \sigma \)-nilpotent.

**Lemma 2.2.** [19, Lemma 2.6(6)] every subgroup of a \( \sigma \)-nilpotent group \( G \) is \( \sigma \)-subnormal in \( G \).

The following lemma directly follows from Lemma 2.1 and [18, Lemma 1.2] (see also [6, Chap. 1, Lemma 1.1]).

**Lemma 2.3.** If \( N \) is a normal subgroup of \( G \), then \( (G/N)^{\sigma_0} = G^{\sigma_0} N/N \).

**Lemma 2.4.** [19, Lemma 2.6] Let \( A, K \) and \( N \) be subgroups of \( G \). Suppose that \( A \) is \( \sigma \)-subnormal in \( G \) and \( N \) is normal in \( G \). Then:

1. \( A \cap K \) is \( \sigma \)-subnormal in \( K \).
2. \( AN/N \) is \( \sigma \)-subnormal in \( G/N \).
3. If \( K \) is a \( \sigma \)-subnormal subgroup of \( A \), then \( K \) is \( \sigma \)-subnormal in \( G \).
4. If \( A \) is a \( \sigma \)-Hall subgroup of \( G \), then \( A \) is normal in \( G \).
5. If \( H \neq 1 \) is a Hall \( \sigma_i \)-subgroup of \( G \) and \( A \) is not a \( \sigma_i \)-group, then \( A \cap H \neq 1 \) and \( A \cap H \) is a Hall \( \sigma_i \)-subgroup of \( A \).
Lemma 2.5. (P. Hall [12]) Let $G$ be a soluble group and $\pi$ a set of primes. Then:

1. Hall $\pi$-subgroups of $G$ exist,
2. they form a conjugacy class of $G$, and
3. each $\pi$-subgroup of $G$ is contained in a Hall $\pi$-subgroup of $G$.

The following lemma is clear.

Lemma 2.6. (i) Every Dedekind group is nilpotent.

(ii) If $G = A \times B$, where $A$ is a Hall subgroup of $G$ and $A$ and $B$ are Dedekind groups, then $G$ is a Dedekind group.

(iii) Every subgroup and every quotient of a Dedekind group is a Dedekind group.

3 Proof of Theorems 1.10

(1) $\implies$ (2):

Suppose that $G$ is a $T_\sigma$-group and $K$ is a subgroup of a $\pi$-Hall subgroup $H$ of $G$, where $\pi \subseteq \sigma_i$ for some $i$. Since $H$ is a $\sigma_i$-group, $K$ is $\sigma$-subnormal in $H$ by Lemma 2.2. Note that $H$ is normal in $N_G(H)$. It implies that $K$ is $\sigma$-subnormal in $N_G(H)$ by Lemma 2.4(3). Hence $K$ is normal in $N_G(H)$ by the hypothesis. Consequently, $G$ satisfies the condition $\mathfrak{A}_{\sigma_i}$.

(2) $\implies$ (3):

Assume that this is false and let $G$ be a counterexample of minimal order. We proceed via the following steps.

1. Every Hall $\sigma_i$-subgroup of $G$ is a Dedekind group for all $i$. Hence $D \neq 1$ and $G$ is soluble.

Let $H$ be a Hall $\sigma_i$-subgroup and $K$ be a subgroup of $H$. Then $K$ is normal in $N_G(H)$ by the hypothesis and so $K$ is normal in $H$. Hence $H$ is a Dedekind group. This implies that $D \neq 1$. We now show that $G$ is soluble. In fact, since $G$ is $\sigma$-soluble, every chief factor $S/K$ of $G$ is $\sigma$-primary, that is, $S/K$ is a $\sigma_i$-group for some $i$. But as every every Hall $\sigma_i$-subgroup of $G$ is a Dedekind group, every Hall $\sigma_i$-subgroup is nilpotent. Hence $S/K$ is a elementary abelian group. It follows that $G$ is soluble.

2. Let $R$ be a non-identity minimal normal subgroup of $G$. Then the hypothesis holds for $G/R$. Hence $G/R$ satisfies statement (3) of the Theorem.

Let $H/R$ be a Hall $\pi$-Hall subgroup of $G/R$ where $\pi \subseteq \sigma_i$ and $K/R$ is a subgroup of $H/R$. Note that $R$ is a $p$-group since $G$ is soluble by Claim (1). Assume that $p$ belongs to $\pi$, then $H$ is a Hall $\pi$-subgroup of $G$. Hence $K$ is normal in $N_G(H)$ by hypothesis. Then $K/R$ is normal in $N_G(H)/R = N_{G/R}(H/R)$. If $p$ does not belong to $\pi$, then there are a Hall $\pi$-subgroup $V$ of $K$ and a Hall $\pi$-subgroup $W$ of $H$ such that $V \leq W$ by Lemma 2.5. It is clear that $W$ is also a Hall $\pi$-subgroup of $G$ since $H/R$ be a Hall $\pi$-Hall subgroup of $G/R$. Hence $V$ is normal in $N_G(W)$ by hypothesis and so $K/R = VR/R$ is normal in $N_G(W)/R = N_{G/R}(WR/R) = N_{G/R}(H/R)$.
(3) The hypothesis holds for every proper Hall subgroup $M$ of $G$ and $M^{\pi_i} \leq D$.

Let $M_i$ be a Hall $\sigma_i$-subgroup of $M$ and $K$ is a subgroup of $M_i$ for all $i$. Then $M_i$ is a Hall $\pi$-subgroup of $G$ where $\pi \subseteq \sigma_i$ since $M$ is a Hall subgroup. Hence $K$ is normal in $N_G(M_i)$, and so $K$ is normal in $N_M(M_i)$. Therefore $M$ satisfies the condition $\mathfrak{R}_{\sigma_i}$ for all $i$. This shows that the hypothesis for $M$. Moreover, since $G/D \in \mathfrak{R}_{\sigma}$ and $\mathfrak{R}_{\sigma}$ is subgroup closed by Lemma 2.1,

$$M/M \cap D \cong MD/D \in \mathfrak{R}_{\sigma}.$$ 

Hence $M^{\pi_i} \leq M \cap D \leq D$.

(4) $D$ is nilpotent.

Assume that this is false and let $R$ be a minimal normal subgroup of $G$. Then:

(a) $R = C_G(R) = O_p(G) = F(G) \leq D$ for some $p \in \sigma_i$. Hence $R$ is an unique minimal normal subgroup of $G$ and $R$ is not cyclic.

First note that $RD/R = (G/R)^{\pi_i}$ is abelian by Lemma 2.3 and Claim (2). Therefore $R \leq D$, and so $R$ is the unique minimal normal subgroup of $G$ and $R \not\in \Phi(G)$ by Lemma 2.1. Since $G$ is soluble, $R$ is an abelian subgroup. It follows that $R = C_G(R) = O_p(G) = F(G)$ for some $p \in \sigma_i$ by [4, Chap. A, 13.8(b)]. If $|R| = p$, then $G/R = C_G(R)$ is cyclic and so $G$ is supersoluble. But then $D = G^{\pi_i} \leq \sigma' \leq F(G)$ and so $D$ is nilpotent, a contradiction. Thus $R$ is not cyclic.

(b) Every Hall $\sigma_i$-subgroup is a Sylow $p$-subgroup, where $p \in \sigma_i$.

Let $H$ be a Hall $\sigma_i$-subgroup and $p \in \sigma_i$. Then $H$ is nilpotent and $R \leq H$ by Claim (1) and Lemma 2.6. Hence $H$ is a Sylow $p$-subgroup by Claim (a).

(c) $|\pi(G)| = 2$.

Let $H_i$ be a Hall $\sigma_i$-subgroup of $G$, where for $p \in \sigma_i$. By Claim (b), $H_i$ is a Sylow $p$-subgroup denoted by $P$. If $|\pi(G)| = 1$, then $G$ is nilpotent, a contradiction. Assume that $|\pi(G)| \geq 3$. Then there exist two different primes belonging to $p'$, denoted by $q$ and $t$. Since $G$ is soluble by Claim (1), there are a Hall $t'$-subgroup $M_1$ of $G$ and a Hall $q'$-subgroup $M_2$ of $G$ by Lemma 2.5(1). Let $V_1 = M_1^{\pi_i}$ and $V_2 = M_2^{\pi_i}$. Suppose that $V_1 = 1$ or $V_2 = 1$. Assume without of generality that $V_1 = 1$. Then $M_1$ is $\sigma$-nilpotent. Let $Q$ be a Sylow $q$-subgroup of $M_1$. Then $Q \leq C_G(R) = R$ since $M_1$ is $\sigma$-nilpotent. This contradiction shows that $V_1 \neq 1$ and $V_2 \neq 1$. Since $M_1$ and $M_2$ are Hall subgroups of $G$, $M_1$ and $M_2$ satisfy the conditions $\mathfrak{R}_{\sigma_i}$ for all $i$ by Claim (3). The choice of $G$ implies that $V_1$ and $V_2$ are abelian Hall subgroups of $G$. Then $R \leq V_1 \cap V_2$. In fact, if $R \not\leq V_1$, then $R \cap V_1 = 1$. It follows that $V_1 \leq C_{M_i}(R) = R$, and so $R = V_1$, a contradiction. Hence $R \leq V_1$.

Similarly, we have $R \leq V_2$. Note that $R$ is not cyclic by Claim (a). Let $L < R$ and $|L| = p$. By Claim (3) and the choice of $G$, every element of $M_i$ $(i = 1, 2)$ induces a power automorphism on $V_i$. Hence $L$ are normal in $M_1$ and $M_2$. It follows that $L$ is normal in $\langle M_1, M_2 \rangle = G$, a contradiction. Hence we have Claim (c).

(d) The final contradiction for Claim (4).

By Claim (c), we may assume that $G = PQ$ where $P$ is a Sylow $p$-subgroup of $G$ and $Q$ is a
\( q \)-subgroup of \( G \). Since every Dedekind group of odd order is abelian by [16, Theorem 5.3.7], we have that either \( P \) is abelian or \( Q \) is abelian. If \( P \) is abelian, then \( RP = P \) is normal in \( G \) by Claim (a) and Theorem 3.2.28 in [2]. Hence \( D \leq P \) is nilpotent, a contradiction. If \( Q \) is abelian, then \( RQ \) is normal in \( G \) by Claim (a) and Theorem 3.2.28 in [2]. Hence by Frattini argument, \( G = RQN_G(Q) = RN_G(Q) \). Let \( N_p \) be a Sylow \( p \)-subgroup of \( N_G(Q) \). If \( N_p = 1 \), then \( R \) is a normal Sylow \( p \)-subgroup of \( G \). Therefore \( D \leq R \) is nilpotent, a contradiction. Assume that \( N_p \neq 1 \). Since \( RN_p \) is a Dedekind subgroup of \( G \) by Claim (1), \( R \leq RN_p \leq N_G(N_p) \). But since \( R \) is the unique minimal normal subgroup of \( G \) by Claim (a), we have

\[
R \leq N_p^G = N_p^{RN_G(N_p)} = N_p^{N_G(Q)} \leq N_G(Q).
\]

It follows that \( G = N_G(Q) \). Then \( Q \) is normal in \( G \). Therefore \( D \leq Q \) is nilpotent. This contradiction shows that Claim (4) holds.

(5) \( D \) is a Hall subgroup of \( G \).

Assume that this is false. Let \( P \) be a Sylow \( p \)-subgroup of \( D \) such that \( 1 < P < G_p \) for some prime \( p \) and some Sylow \( p \)-subgroup \( G_p \) of \( G \). Then \( p \mid [G : D] \). We can assume without loss of generality that \( G_p \leq H_1 \), where \( H_1 \) is a Hall \( \sigma_1 \)-subgroup of \( G \).

(a') \( D = P \) is a minimal normal subgroup of \( G \).

Let \( R \) be a minimal subgroup of \( G \) contained in \( D \). Then by Claim (4), \( R \) is a \( q \)-group for some prime \( q \). Moreover, \( D/R = (G/R)^{\Phi_p} \) is a Hall subgroup of \( G/R \) by Claim (2) and Lemma 2.3. Suppose that \( PR/R \neq 1 \). Then \( PR/R \in Syl_p(G/R) \). If \( q \neq p \), then \( P \in Syl_p(G) \). This contradicts the fact that \( P < G_p \). Hence \( q = p \), so \( R \leq P \) and \( P/R \in Syl_p(G/R) \). We again get that \( P \in Syl_p(G) \). This contradiction shows that \( PR/R = 1 \), which implies that \( R = P \) is the unique minimal normal subgroup of \( G \) contained in \( D \). Since \( D \) is nilpotent by Claim (4), a \( p' \)-complement \( E \) of \( D \) is characteristic in \( D \) and so it is normal in \( G \). Hence \( E = 1 \), which implies that \( R = D = P \).

(b') \( D \not\leq \Phi(G) \). Hence for some maximal subgroup \( M \) of \( G \) we have \( G = D \times M \).

Note that \( G/D = G/G^{\Phi_p} \) is \( \sigma \)-nilpotent. If \( D \leq \Phi(G) \), then by lemma 2.1, \( G \in \mathcal{M}_\sigma \) and so \( D = 1 \), a contradiction.

(c') Let \( R \) be a minimal normal subgroup of \( G \). If \( D \neq R \), then \( G_p = D \times (G_p \cap R) \). Hence \( O_{p'}(G) = 1 \).

By Claims (2) and (a'), we have that \( DR/R \) is a Sylow \( p \)-subgroup of \( G/R \). It follows that \( DR/R = G_pR/R \). Hence \( G_p = D \times (G_p \cap R) \). Thus \( O_{p'}(G) = 1 \) since \( G \) is soluble and \( D < G_p \) by Claim (a').

(d') Let \( V = C_G(D) \cap M \). Then \( V \trianglelefteq G \) and \( C_G(D) = D \times V \leq H_1 \).

In view of Claims (a') and (b'), we have that \( C_G(D) = D \times V \) and \( V \) is a normal subgroup of \( G \). Moreover, \( V \cong VD/D \) is \( \sigma \)-nilpotent by Lemma 2.1. Let \( W \) be a \( \sigma_1 \)-complement of \( V \). Then \( W \) is characteristic in \( V \) and so it is normal in \( G \). Then \( W = 1 \) by Claim (c'). Hence we have Claim (d').
(e') $H_1 = G_p$ is a Sylow $p$-subgroup of $G$.

Since $G/D$ is $\sigma$-nilpotent and $D \leq H_1$ by Claim (a'), $H_1$ is normal in $G$. A $p'$-complement $E$ of $H_1$ is characteristic in $H_1$ since $H_1$ is nilpotent by Claim (1). Hence $E = 1$ by Claim (e'). It follows that $H_1 = G_p$ is a Sylow $p$-subgroup of $G$.

(f') $|\pi(G)| = 2$.

If $|\pi(G)| = 1$, then $G$ is nilpotent, a contradiction. Assume that $|\pi(G)| \geq 3$. Then there exist one more primes belonging to $p' = \sigma_i$ and let $q \in p'$. Since $G$ is soluble, $G$ has a Hall $\{p, q\}$-subgroup $H$ of $G$. Let $L = H^{p_0}$ and let $H = G_pQ$ where $Q$ is a $q$-subgroup of $G$. Note that $H < G$. If $L = 1$, then $H = P \times Q$ by Claim (e'). Consequently, $Q \leq C_G(D) \leq H_1 = G_p$ by Claim (d'), a contradiction. Hence $L \neq 1$. By Claim (3) and Claim (a'), $L \leq D = P$ and $L$ is a Hall subgroup of $H$ by the choice of $G$. Note that $L$ is a Hall subgroup of $G$ since $H$ is a Hall subgroup of $G$. Therefore $L = D$ is a Hall subgroup of $G$. The contradiction shows that Claim (f') holds.

(g') The final contradiction for Claim (5).

Let $\pi(G) = \{p, q\}$. Then $G$ satisfies the conditions $\mathfrak{R}_p$ and $\mathfrak{R}_q$ by Claim (e') and Claim (3). Hence $G$ is a $T$-group by Theorem 1.2. By Claims (e') and (f'), it is clear that $G^{\mathfrak{R}} = D$. Hence by Theorem 1.1, $D$ is a Hall subgroup of $G$. The contradiction completes the proof of Claim (5).

(6) $G = D \times M$ where $M$ is a Dedekind group.

Since $D$ is a normal subgroup of $G$, by Schur-Zassenhaus Theorem, $G = D \times M$ and $M$ is a Hall subgroup of $G$. But since $D = G^{\mathfrak{R}}$, $M$ is $\sigma$-nilpotent. Then by Claim (1) and Lemma 2.6(ii), we have that $M$ is a Dedekind group.

(7) Let $H_i$ be a Hall $\sigma_i$-subgroup of $G$ for each $\sigma_i \in \sigma(D)$. Then $H_i = O_{\sigma_i}(D) \times S$ for some subgroup $S$ of $H_i$.

By Claim (1), $H_i$ is nilpotent. By Claims (4) and (5), $D$ is a nilpotent Hall subgroup of $G$. Hence we have Claim (7).

(8) Every subgroup $H$ of $D$ is normal in $G$. Hence every element of $G$ induces a power automorphism on $D$.

Since $D$ is nilpotent by Claim (4), it is enough to consider the case when $H \leq O_{\sigma_i}(D) = H_i \cap D$ for some $\sigma_i \in \sigma(D)$. By condition (2), $H$ is normal in $N_G(O_{\sigma_i}(D))$. But clearly $N_G(O_{\sigma_i}(D)) = G$. Therefore $H$ is normal in $G$.

(9) $|D|$ is odd.

Suppose that 2 divides $|D|$. Then by Claims (4) and (7), $G$ has a chief factor $D/K$ with $|D/K| = 2$. This implies that $D/K \leq Z(G/K)$. Since $D$ is a normal Hall subgroup of $G$ by Claim (5), it has a complement $M$ in $G$. Hence $G/K = D/K \times MK/K$, where $MK/K \cong M \cong G/D$ is $\sigma$-nilpotent. Therefore $G/K$ is $\sigma$-nilpotent by Lemma 2.1 and Claim (4). But then $D \leq K < D$, a contradiction. Hence we have (9).

(10) $D$ is abelian.
By Claim (8), $D$ is a Dedekind group. But $D$ is odd order by Claim (9). Hence $D$ is abelian by [16, Theorem 5.3.7].

(11) Final contradiction.

Claims (5), (6), (7), (8), (9) and (10) show that the conclusion (3) holds for $G$. This final contradiction completes the proof of $(2) \implies (3)$.

$(3) \implies (1)$:

Suppose that $G$ satisfies the conditions (i), (ii) and (iii) of (3). Then $G$ is soluble. Now we need to prove that every $\sigma$-subnormal subgroup $H$ of $G$ is normal in $G$. Suppose that this is false, that is, some $\sigma$-subnormal subgroup $H$ of $G$ is not normal in $G$. Let $G$ be a counterexample with $|G| + |H|$ minimal. Then by the condition (i) and Lemma 2.6(i), we see that $D \neq 1$. We now proceed the proof via the following steps.

(I) The hypothesis holds for every quotient $G/N$ of $G$, where $N$ is a proper normal subgroup of $G$.

By the condition (i), we have that $G/N = (DN/N) \times (MN/N)$, where $DN/N \cong D/D \cap N$ is an abelian Hall subgroup of $G/N$ of odd order and $MN/N \cong M/M \cap N$ is a Dedekind-group by Lemma 2.6(iii). Hence condition (i) holds for $G/N$. Suppose that $V/N$ is any subgroup of $DN/N$, then $V = N(D \cap V)$. Since $D \cap V$ is normal in $G$ by condition (ii), $V/N$ is normal in $G/N$. Hence the condition (ii) holds for $G/N$. Since $D$ is nilpotent, clearly $O_{\sigma_i}(D)N/N = O_{\sigma_i}(DN/N)$. Condition (iii) implies that $O_{\sigma_i}(D)$ has a normal complement $S$ in a Hall $\sigma_i$-subgroup $E$ of $G$ for every $i$. Then $EN/N$ is a Hall $\sigma_i$-subgroup of $G/N$ and $SN/N$ is normal in $EN/N$. Hence

$$(SN/N)(O_{\sigma_i}(DN/N)) = (SN/N)(O_{\sigma_i}(D)N/N) = EN/N$$

and

$$(SN/N) \cap O_{\sigma_i}(DN/N) = (SN/N) \cap (O_{\sigma_i}(D)N/N) = N(S \cap O_{\sigma_i}(D)N) / N$$

$$= N(S \cap O_{\sigma_i}(D))(S \cap N) / N = N/N.$$ 

Hence condition (iii) also holds on $G/N$.

(II) $H_G = 1$.

Assume $H_G \neq 1$. The hypothesis holds for $G/H_G$ by Claim (I). On the other hand, $H/H_G$ is $\sigma$-subnormal in $G/H_G$ by Lemma 2.4(2), so $H/H_G$ is normal in $G/H_G$ by the choice of $G$. But then $H$ is normal in $G$, a contradiction. Hence we have Claim (II).

(III) $H$ is a $\sigma_i$-group for some $i$ and $H \in \sigma^x$ for all $x \in G$.

Claim (II) and the condition (ii) imply that $H \cap D = 1$. Since $H \cong HD/D \leq G/D$, $H$ is $\sigma$-nilpotent by Lemma 2.1. Hence $H = A_1 \times \cdots \times A_n$ for some $\sigma$-primary groups $A_1, \cdots, A_n$. Then $H = A_i$ is a $\sigma_i$-group for some $i$ since otherwise $H$ is normal in $G$ by the choice of $(G, H)$. Note that $G = D \times M$ by the condition (i). Let $M_i$ be the Hall $\sigma_i$-subgroup of $M$ and $E$ be a Hall $\sigma_i$-subgroup of $G$ containing $M_i$. Lemma 2.4(5) implies that $H \leq E^x$ for all $x \in G$. If $E \cap D = 1$, then $M_i$
is a Hall $\sigma_i$-subgroup of $G$, and so $H \leq M^x$ for all $x \in G$. Now suppose that $E \cap D \neq 1$. Then $H \leq E^x = O_{\sigma_i}(D) \times M^x_i$ by condition (iii). But since $H \cap D = 1$, we have also that $H \leq M^x_i \leq M^x$ for all $x \in G$.

(IV) The Hall $\sigma_j$-subgroups of $G$ are Dedekind-groups for all $j$.

Let $A$ be a Hall $\sigma_j$-subgroup of $G$. If $A \cap D = 1$, then $A \cong AD/D \leq G/D$, where $G/D$ is a Dedekind group by the condition (i). Hence $A$ is a Dedekind group by Lemma 2.6(iii). Now assume that $A \cap D \neq 1$. Then $A = (A \cap D) \times S$ by condition (iii), where $A \cap D = O_{\sigma_j}(D)$ and $S$ is a normal complement of $A \cap D$ in $A$. Then $A$ is a Dedekind group by Lemma 2.6(ii) because $A \cap D$ and $S \cong DS/D \leq G/D$ are Dedekind groups.

(V) $D$ is also a $\sigma_i$-group.

Assume that this is false. Note that $D$ is an abelian group. Assume without of generality that $O_{\sigma_j}(D) \neq 1$, where $j \neq i$. Then by Claim (1), $HO_{\sigma_j}(D)/O_{\sigma_j}(D)$ is normal in $G/O_{\sigma_j}(D)$, and so $HO_{\sigma_j}(D)$ is normal in $G$. But by Lemma 2.4(1), $H$ is $\sigma$-subnormal in $HO_{\sigma_j}(D)$. Hence by Lemma 2.4(4), $H$ is normal in $HO_{\sigma_j}(G)$. Then $H$ is characteristic in $HO_{\sigma_j}(G)$. It follows that $H$ is normal in $G$, a contradiction. Hence we have Claim (V).

(VI) Final contradiction.

Since $D$ is a $\sigma_i$-group by Claim (V) and $G = D \rtimes M$ by the condition (i), we have that $G = H_iM$, where $H_i$ is a Hall $\sigma_i$-group of $G$. Since $G$ is soluble and $H$ is a $\sigma_i$-group by Claim (III), we can assume without of generality that $H \leq H_i$ by Lemma 2.5(3). Then $H_i \leq N_G(H)$ by Claim (IV). On the other hand, since $M$ is a Dedekind group by the hypothesis, we have $M \leq N_G(H)$ by Claim (III). Hence $G = H_iM \leq N_G(H)$. This shows that $H$ is normal in $G$. This contradiction completes the proof for $(3) \implies (1)$.

In summary, the Theorem 1.10 is proved.

\[\square\]

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