Geometry of cotangent bundle of Heisenberg group

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Abstract

In this paper the classification of left-invariant Riemannian metrics, up to the action of the automorphism group, on cotangent bundle of (2n+1)-dimensional Heisenberg group is presented. Also, it is proved that the complex structure on that group is unique and the corresponding pseudo-Kähler metrics are described and shown to be Ricci flat. It is well known that this algebra admits an ad-invariant metric of neutral signature. Here, the uniqueness of such metric is proved.

Introduction

Let us study the simply connected Lie group $G$. By the moduli space we consider the orbit space of the action of $\mathbb{R}^\times \text{Aut}(\mathfrak{g})$ on the space $\mathfrak{M}(G)$ of left invariant metrics on $G$, represented by its inner products on the corresponding Lie algebra $\mathfrak{g}$. Here, $\text{Aut}(\mathfrak{g})$ denotes the automorphism group of the corresponding Lie algebra and its orbits induce the isometry classes, while $\mathbb{R}^\times$ is the scalar group giving rise to the scaling.

There are two, in some way dual, approaches to the classification problem. The first one is to start from the (pseudo-)orthonormal basis of Lie algebra and act by the isometry group (and hence preserving the canonical form of the metric) on the commutators to make them as simple as possible. The second one is to start from basis of Lie algebra with simple commutators and act by the automorphism group of the Lie algebra (hence preserving the commutators) to make the metric take the most simple form. For the more detailed outline of each approach, we refer to [11, 13].

The moduli space approach to the problem of classification was first introduced in Milnor’s classical paper [15] where left invariant Riemannian metrics on three-dimensional unimodular Lie groups were considered. Much later, the classification was completed in Lorentz case [8]. In higher dimensions, there are numerous results in both Riemannian and pseudo-Riemannian setting. However, even in small dimensions, the moduli space of pseudo-Riemannian metrics can be quite large, hence is rarely practical or possible to obtain the complete classification of metrics.

Naturally, the case of nilpotent Lie groups is very thoroughly investigated (for example, see [3, 9, 12, 14, 18, 19]). Following our previous interests, in this paper the cotangent bundle of the Heisenberg group $H_{2n+1}$ is considered. Since the Heisenberg group is two-step nilpotent, its cotangent bundle has the same property. Although the moduli space of metrics on the cotangent bundle can be constructed using both nondegenerate and degenerate metrics on the original Lie group, in practice the process heavily depends on the subtle geometrical and algebraic analysis. For example, in case of Lie group $T^* H_3$ we obtained 19 non-equivalent families of left invariant metrics of arbitrary signature (see [17] for more details). Therefore, in higher dimensions it is feasible to consider Riemannian signature and only some special cases of pseudo-Riemannian metrics.

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The paper is organized as follows.

First, we briefly recall the construction of the cotangent bundle $T^*\mathfrak{h}_{2n+1}$ of the Lie algebra $\mathfrak{h}_{2n+1}$ corresponding to the Heisenberg group $H_{2n+1}$. The explicit form of the automorphism group is described in Lemma 1.1.

In Section 2 we classify all non-isometric left invariant Riemannian metrics on $T^*\mathfrak{h}_{2n+1}$ (see Theorem 2.1). Throughout the paper we identify the notion of metric on Lie group and the inner product on its Lie algebra. We obtained only one $n(2n+1)$-parameter family of Riemannian metrics. This family is correlated with the family of Riemannian metrics on Heisenberg group from [19].

Motivated by their applications in mathematics and physics, Section 3 is devoted to ad-invariant metrics. Cotangent Lie algebras are endowed with the canonical ad-invariant metric. This metric is of neutral signature and since the setting is two-step nilpotent, it is flat. We prove the uniqueness of this metric in Lemma 3.1.

In Section 4 we classify pseudo-Kähler metrics. Theorem 4.1 states that, up to the action of automorphisms and sign, there exists only one complex structure $J_0$ on $T^*\mathfrak{h}_{2n+1}$. This generalizes the result obtained by Salamon [16] in his classification of complex structures on nilpotent Lie algebras, where he considered the Lie algebra $T^*\mathfrak{h}_3$. This complex structure is three-step nilpotent (see [7]), but since the center of the algebra $T^*\mathfrak{h}_{2n+1}$ is odd dimensional, $J_0$ cannot be abelian complex structure, i.e. complex structure satisfying $[x,y] = [Jx,Jy]$. In Theorem 4.2 we find that the space of symplectic forms compatible with $J_0$ has a dimension $3n^2+n+2$, $n > 1$. Interestingly, for $n = 1$, the dimension of $J$-invariant closed 2-forms is five (see Remark 4.1). This case was considered in the paper [4]. Finally, we classify pseudo-Kähler metrics and show that they all belong to the same family of Ricci-flat metrics (Theorem 4.3).

1 Preliminaries

Let us briefly recall the construction of cotangent Lie algebra.

The cotangent algebra $T^*\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ is a semidirect product of $\mathfrak{g}$ and its cotangent space $\mathfrak{g}^*$ by means of the coadjoint representation

$$ T^*\mathfrak{g} := \mathfrak{g} \ltimes_{\text{ad}^*} \mathfrak{g}^*, $$

i.e. the commutators are defined by

$$ [(x, \phi), (y, \psi)] := ([x, y], \text{ad}^*(x)\psi - \text{ad}^*(y)\phi), \quad x, y \in \mathfrak{g}, \quad \phi, \psi \in \mathfrak{g}^*. \tag{1} $$

The coadjoint representation $\text{ad}^*: \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$ is given by

$$ (\text{ad}^*(x)(\phi))(y) := -\phi(\text{ad}(x)(y)) = -\phi([x, y]). $$

Now, we specialise this construction to the Heisenberg algebra $\mathfrak{h}_{2n+1}$ which is the Lie algebra of $(2n+1)$-dimensional Heisenberg group $H_{2n+1}$.

Let us denote the matrix of standard complex structure on $\mathbb{R}^{2n}$ by

$$ J = J_{2n} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \tag{2} $$

where $E$ is $n \times n$ identity matrix. The standard symplectic form in vector space $\mathbb{R}^{2n}$ can be written in the form

$$ \omega(x, y) = x^T J y, \quad x, y \in \mathbb{R}^{2n}. $$

The Heisenberg group is two-step nilpotent Lie group $H_{2n+1}$ defined on the base manifold $\mathbb{R}^{2n} \oplus \mathbb{R}$ by multiplication

$$ (x, \mu) \cdot (y, \lambda) := (x + y, \mu + \lambda + \omega(x, y)). $$
The corresponding Lie algebra
\[ \mathfrak{h}_{2n+1} = \mathbb{R}^{2n} \oplus \mathbb{R} = \mathbb{R}^{2n} \oplus \mathcal{Z} = \{(x, \lambda) \mid x \in \mathbb{R}^{2n}, \lambda \in \mathbb{R}\} \]
is given by the following commutator equation
\[ [(x, \mu), (y, \lambda)] = (0, \omega(x, y)). \quad (3) \]

Note that \( \mathcal{Z} = \mathbb{R}(z) \) is one-dimensional center and one-dimensional commutator subalgebra of \( \mathfrak{h}_{2n+1} \). To simplify notation we write (3) in the equivalent form
\[ [x + \mu z, y + \lambda z] = \omega(x, y)z. \]

If \( e_1, \ldots, e_n, f_1, \ldots, f_n \) represents the standard basis of \( \mathbb{R}^{2n} \), then nonzero commutators of \( \mathfrak{h}_{2n+1} \) are
\[ [e_i, f_i] = z, \quad i = 1, \ldots, n. \]

Let \( \mathbb{R}^{2n} = \mathbb{R}(e_1^*, \ldots, e_n^*, f_1^*, \ldots, f_n^*) \) and \( \mathcal{Z}^* = \mathbb{R}(z^*) \) be dual vector spaces of one-forms spanned by dual basis vectors. Denote the dual space of \( \mathfrak{h}_{2n+1} \) by
\[ \mathfrak{h}_{2n+1}^* = \mathbb{R}^{2n} \oplus \mathcal{Z}^* = \{(x^*, \mu^*) \mid x^* \in R^{2n}, \mu^* \in \mathbb{R}\} = \{x^* + \mu^* z^*\}. \]

Now, the cotangent space \( T^*\mathfrak{h}_{2n+1} \) of \( \mathfrak{h}_{2n+1} \), as a vector space, can be written as a direct sum
\[ \mathfrak{g} = T^*\mathfrak{h}_{2n+1} = \mathfrak{h}_{2n+1} \oplus \mathfrak{h}_{2n+1}^* = \mathbb{R}^{2n} \oplus \mathcal{Z}^* \oplus \mathbb{R}^{2n} \oplus \mathcal{Z} \cong \mathbb{R}^{4n+2}. \quad (4) \]

Note, that we changed the order of summands to better fit the structure of Lie algebra. Namely, one can check that according to (1) the commutator on \( T^*\mathfrak{h}_{2n+1} \) is given by
\[ [(x, \mu^* x^*, \mu), (y, \lambda^* y^*, \lambda)] = (0, 0, \lambda^* J^*(x) - \mu^* J^*(y), \omega(x, y)), \quad (5) \]

where
\[ J^* : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \quad J^*(X) := (JX)^* \]
is represented by matrix \( J \) given in (2), in basis \( e_1^*, \ldots, e_n^*, f_1^*, \ldots, f_n^* \).

The center and commutator subalgebra of \( T^*\mathfrak{h}_{2n+1} \) coincide with \( \mathbb{R}^{2n} \oplus \mathcal{Z} \). By that reason we have changed the order of summands in (4) to group central (and commutator) vectors together. Note that the algebra \( T^*\mathfrak{h}_{2n+1} \) is also two-step nilpotent.

To simplify the notation we write commutator (5) of \( T^*\mathfrak{h}_{2n+1} \) in equivalent form
\[ [x + \mu^* z^* + x^* + \mu z, y + \lambda^* z^* + y^* + \lambda z] = \lambda^* J^*(x) - \mu^* J^*(y) + \omega(x, y)z. \quad (6) \]

**Lemma 1.1.** The group of automorphism of algebra \( \mathfrak{g} = T^*\mathfrak{h}_{2n+1} \) of the form (4) in standard basis is
\[ \text{Aut}(T^*\mathfrak{h}_{2n+1}) = \left\{ F = \begin{pmatrix} F_1 & 0 \\ F_3 & F_4 \end{pmatrix} \mid F_1, F_4 \in \text{Gl}_{2n+1}(\mathbb{R}), \ F_3 \in \text{M}_{2n+1}(\mathbb{R}) \right\}, \quad (7) \]

\[ F_1 = \begin{pmatrix} \bar{F}_1 & v_1 \\ u^T & f_1 \end{pmatrix}, \quad F_3 = \begin{pmatrix} f_1 f_4 \bar{F}_1^{-T} - (J u_1)(J u_1)^T - f_4 \bar{F}_1^{-T} u_1 \\ -f_4 u_1^T \bar{F}_1^{-T} & f_4 \end{pmatrix}, \quad \bar{F}_1 J \bar{F}_1 = f_4 J, \quad (8) \]

\[ v_1, u_1 \in \mathbb{R}^{2n}, \quad f_1, f_4 \in \mathbb{R}\{0\}. \] Its dimension is \( \dim \text{Aut}(T^*\mathfrak{h}_{2n+1}) = 6n^2 + 9n + 3 \).

**Proof.** For element \( g \in \mathfrak{g} \) denote by \( A_g \) the matrix of operator \( ad(g) \) in standard basis. Each \( g \in \mathfrak{g} \) decomposes as \( g = \nu + \zeta, \nu \in \mathbb{R}^{2n} \oplus \mathcal{Z}^*, \zeta \in \mathbb{R}^{2n} \oplus \mathcal{Z} \). Note that \( ad(\zeta) = 0 \) for the central part. If \( \nu = x + \mu^* z^* \) the matrix of \( ad(g) \) is
\[ A_g = \begin{pmatrix} 0 & 0 \\ A_\nu & 0 \end{pmatrix}, \quad A_\nu = \begin{pmatrix} -\mu^* J & x^T J x \\ x^T J & 0 \end{pmatrix}. \quad (9) \]
The condition that $F \in \text{Aut}(\mathfrak{g})$ can we written in the equivalent form
\[ F[g, h] = [Fg, Fh], \quad g, h \in \mathfrak{g} \iff A_{Fg}F = FA_g, \quad g \in \mathfrak{g}. \quad (10) \]

We are looking for the matrix of automorphism in a block matrix form
\[ F = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}. \]

Automorphisms preserve the center $\mathbb{R}^{2n} \oplus \mathbb{Z}$ of $\mathfrak{g}$ and therefore $F_2 = 0$. Relations (10) impose no condition on $F_3$ and hence $F_3 \in M_{2n+1}(\mathbb{R})$ is arbitrary. The remaining condition is
\[ A_{F_1}F_1 = F_4A_\nu \text{ for each } \nu = x + \mu^*z^*. \quad (11) \]

We further decompose matrices $F_1$ and $F_4$ as
\[ F_k = \begin{pmatrix} F_k & v_k \\ u_k & f_k \end{pmatrix}, \quad (12) \]

$F_k \in M_{2n}(\mathbb{R}), v_k, u_k \in \mathbb{R}^{2n}, f_k \in \mathbb{R}, k = 1, 4$. Next, we use condition (11) for matrices (12) and operator (9). The bottom left component is
\[ (\bar{F}_1X + \mu^*v_1)^T J\bar{F}_1 = (-\mu^*u_4^T + f_4X^T)J \]

and has to be satisfied for $x \in \mathbb{R}^{2n}$ and $\mu^* \in \mathbb{R}$. It is equivalent to the following two conditions
\[ \bar{F}_1^T J\bar{F}_1 = f_4J, \quad u_4 = -f_4\bar{F}_1^{-1}v_1. \quad (13) \]

The bottom right component is
\[ (\bar{F}_1X)^T Jv_1 = u_4^T JX \]

and because of relations (13) it is satisfied for any $v_1, X \in \mathbb{R}^{2n}$. Therefore, $v_1 \in \mathbb{R}^{2n}$ is arbitrary.

The upper left and upper right components are, respectively
\[ -(u_1^T X + \mu^*f_1)J\bar{F}_1 + J(\bar{F}_1X + \mu^*v_1)\mu_1^T = (-\mu^*\bar{F}_4 + v_4x^T)J, \quad \bar{F}_4JX = J(-u_1^T Xv_1 + f_1\bar{F}_1X), \]

for all $X \in \mathbb{R}^{2n}$. Combining them with previous relations yields
\[ \bar{F}_4 = f_1f_4\bar{F}_1^{-T} - (Jv_1)(Ju_1)^T, \quad v_1 = -f_4\bar{F}_1^{-T}u_1 \quad (14) \]

and $u_1 \in \mathbb{R}^{2n}$ is arbitrary. Numbers $f_1, f_4$ can’t be equal to zero since $F$ is nonsingular. The relations (13) and (14) imply that the automorphisms has form as in the statement of the Lemma.

Two calculate the dimension of the group note that matrix $\bar{F}_1$ is “almost” symplectic and therefore depends on $n(2n + 1)$ parameters. Since $F_3 \in M_{2n+1}(\mathbb{R}), u_1, v_1 \in \mathbb{R}^{2n}, f_1, f_4 \in \mathbb{R}\setminus\{0\}$ are arbitrary, and $F_4$ is completely dependent on $F_1$, the dimension is:
\[ n(2n + 1) + (2n + 1)^2 + 2n + 2n + 2 = 6n^2 + 9n + 3. \]

\[ \square \]

**Remark 1.1.** In [17] group of automorphism of Lie algebra $T^*\mathfrak{h}_3$ (special case for $n = 1$) are given. Group of automorphisms of Heisenberg algebra $\mathfrak{h}_{2n+1}$ is subgroup of $\text{Aut}(T^*\mathfrak{h}_{2n+1})$ and can be described as semidirect product of symplectic group $Sp(2n, \mathbb{R})$, subgroup of translations isomorphic to $\mathbb{R}^{2n}$ and 1-dimensional ideal. For more details, see [19].
2 Classification of left invariant Riemannian metrics

In this section we classify non-isometric left invariant Riemannian metrics on \( \mathfrak{g} = T^*\mathfrak{h}_{2n+1} \).

If \( \mathfrak{g} \) is a Lie algebra and \( \langle \cdot, \cdot \rangle \) inner product on \( \mathfrak{g} \) the pair \( (\mathfrak{g}, \langle \cdot, \cdot \rangle) \) is called a metric Lie algebra. The structure of metric Lie algebra uniquely defines left invariant Riemannian metric on the corresponding simple connected Lie group \( G \) and vice versa.

Two metric Lie algebras are isomorphic if there are isomorphic as Lie algebras and that isomorphism is isometry of their inner products.

Metric algebras are said to be isometric if there exists an isomorphism of Euclidean spaces preserving the curvature tensor and its covariant derivatives. This translates to the condition that metric algebras are isometric if and only if they are isometric as Riemannian spaces (see [1, Proposition 2.2]). Although two isometric metric algebras are also isometric, the converse is not true. In general, two metric algebras may be isometric even if the corresponding Lie algebras are non-isomorphic. The test to determine whether two given solvable metric algebras (i.e. solvmanifolds) are isometric was developed by Gordon and Wilson in [10]. However, by the results of Alekseevskii [1, Proposition 2.3], in the completely solvable case, isometric means isomorphic.

Since Lie algebra \( \mathfrak{g} \) is nilpotent and therefore completely solvable, non-isometric metrics on \( \mathfrak{g} \) are the non-isomorphic ones. They are obtained by action of group \( \text{Aut}(\mathfrak{g}) \) on space of metrics.

Fix basis of \( T^*\mathfrak{h}_{2n+1} \) described in Section 1. Riemannian inner product on \( \langle \cdot, \cdot \rangle \) is then represented by nonsingular symmetric matrix \( S \in M_{4n+2}(\mathbb{R}) \) of signature \((4n+2,0)\).

The action of \( F \in \text{Aut}(T^*\mathfrak{h}_{2n+1}) \) on inner products is given by

\[
S' = F^T SF. \tag{15}
\]

Matrix \( S \) has \((4n+3)(2n+1) = 8n^2 + 10n + 3\) parameters. According to \( \dim \text{Aut}(T^*\mathfrak{h}_{2n+1}) \) from Lemma 1.1 it is expected that moduli space of non-isometric inner products is of dimension \( n(2n+1) \). This is what we show in Theorem 2.1.

Write \( S \) in block-matrix form

\[
S = \begin{pmatrix} S_1 & S_2 \\ S_2^T & S_4 \end{pmatrix}, \quad S_1^T = S_1, S_4^T = S_4.
\]

Let \( F \in \text{Aut}(T^*\mathfrak{h}_{2n+1}) \) be of the form (7). After the action (15) the top right matrix of \( S' \) is

\[
S_2' = F_1^T (S_2 F_4) + F_3^T (S_4 F_4).
\]

Matrix \( S_4 \) represents the restriction of the inner product on subspace \( \mathbb{R}^{2n} \oplus \mathbb{Z} \). Since inner product is Riemannian, the restriction is non-degenerate and thus \( S_4 \) is nonsingular. By choosing \( F_3 = -S_2 F_1 S_4^{-1} \) we achieve \( S_2' = 0 \) and therefore we can suppose that

\[
S = S' = \begin{pmatrix} S_1 & 0 \\ 0 & S_4 \end{pmatrix}, \quad S_1^T = S_1, \quad S_4^T = S_4. \tag{16}
\]

Now, the action of \( F \) on \( S \) with \( F_3 = 0 \) is given by

\[
S_k' = F_k^T S_k F_k, \quad k = 1, 4,
\]

where \( F_1 \) and \( F_4 \) are related by (8). We write \((2n+1) \times (2n+1)\) matrices \( F_k \) and \( S_k \) as block matrices

\[
F_k = \begin{pmatrix} \bar{F}_k & v_k \\ u_k & f_k \end{pmatrix}, \quad S_k = \begin{pmatrix} \bar{S}_k & t_k \\ t_k^\top & \omega_k \end{pmatrix}, \quad S_k^T = S_k, t_k \in \mathbb{R}^{2n}, \quad k = 1, 4.
\]

The top right vector in \( S_k' \) given is given by

\[
t_k' = u_k (t_k^\top v_k + \omega_k f_k) + F_k^T \bar{S}_k v_k + \bar{F}_k^T t_k f_k, \quad k = 1, 4.
\]
The number \( \omega_k, k = 1, 4 \) is different from zero (it is the norm of the vector \( z^* \) and \( z \), respectively). With appropriate choice of number \( f_k \) one can achieve that \( f_k^2 v_k + \omega_k f_k \neq 0, k = 1, 4 \).

Hence, there is a choice automorphism \( F \) with \( u_1 \) such that top right vector \( t'_1 \) in \( S'_1 \) is zero and with \( u_4 \), or equivalently \( v_1 \), such that top right vector \( t'_4 \) in \( S'_4 \) is zero.

Therefore, we can suppose that metric has form (16) with

\[
S_k = \begin{pmatrix} \tilde{S}_k & 0 \\ 0 & \omega_k \end{pmatrix}, \quad \tilde{S}_k^T = \tilde{S}_k, \quad k = 1, 4.
\]

To further simplify the metric consider automorphism \( F \) of the form (7) with \( u_1 = v_1 = 0 \) (and hence \( u_4 = v_4 = 0 \)). The action reduces to

\[
S'_k = \begin{pmatrix} \tilde{F}_k^T \tilde{S}_k \tilde{F}_k & 0 \\ 0 & f_k^2 \omega_k \end{pmatrix}, \quad k = 1, 4,
\]

with \( F_1^T J F_1 = f_4 J \).

As first, consider upper left block of \( S \), i.e. \( k = 1 \). Symplectic matrix \( \tilde{F}_1 \) can diagonalize positive symmetric matrix \( \tilde{S}_1 \) (see [19, 20] for details regarding independent action of \( \tilde{S}_1 \) and \( f_4 \)) to obtain

\[
\tilde{F}_k^T \tilde{S}_k \tilde{F}_k = D(\sigma) = \text{diag}(\sigma_1, \ldots, \sigma_n, \sigma_1, \ldots, \sigma_n), \quad \sigma_1 \geq \ldots \geq \sigma_n > 0.
\]

By choosing \( f_4 = \frac{1}{\sigma_n} \) one can achieve \( \sigma_n = 1 \) and for \( f_1 = \frac{1}{\sqrt{\omega_1}} \) we obtain

\[
S'_1 = \text{diag}(D(\sigma), 1) = \text{diag}(\sigma_1, \ldots, \sigma_n, 1, 1, \ldots \sigma_n, 1, 1).
\]

Now consider bottom right block, i.e. \( k = 4 \). Since \( \tilde{F}_1, f_1, f_4 \) and hence \( \tilde{F}_4 \), are already chosen, there is no much freedom left. However, symplectic rotations \( R_\phi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} \)

\[
e_i \mapsto \cos \phi_i e_i - \sin \phi_i f_i, \quad f_i \mapsto \sin \phi_i e_i + \cos \phi_i f_i,
\]

by angles \( \phi_1, \ldots, \phi_n \in \mathbb{R} \) belong to \( \text{Aut}(\mathfrak{h}_{2n+1}) \subseteq \text{Aut}(T^* \mathfrak{h}_{2n+1}) \) and preserve inner product (17). These rotations induce rotations in \( \mathbb{R}^{2n} \). By choosing appropriate angles one can achieve that \( e_i^T \) and \( f_i^T \) are orthogonal, i.e. \( (\tilde{S}_4)_{i(n+i)} = 0, i = 1, \ldots, n \). Free parameters in the canonical form of the metric are: \( n - 1 \) parameter \( \sigma_i \), entries of symmetric \( 2n \times 2n \) matrix \( \tilde{S}_4 \) (\( n \) of them is equal to zero) and parameter \( \omega_4 \). Therefore the dimension of moduli space of non-equivalent metrics is

\[
(n - 1) + (n(2n + 1) - n) + 1 = n(2n + 1)
\]

as expected. Hence, we proved the following theorem.

**Theorem 2.1.** Dimension of the moduli space of Riemannian metrics on Lie algebra \( T^* \mathfrak{h}_{2n+1} \) is \( n(2n+1) \). Every such metric is represented by \( (4n + 2) \times (4n + 2) \) block matrix

\[
S = \begin{pmatrix} D(\sigma) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \tilde{S}_4 & 0 \\ 0 & 0 & 0 & \omega_4 \end{pmatrix},
\]

\( \omega_k > 0, D(\sigma) = \text{diag}(\sigma_1, \ldots, \sigma_n, 1, \sigma_1, \ldots, \sigma_n, 1), \sigma_1 \geq \ldots \geq \sigma_n \geq 1, \tilde{S}_4 \) is symmetric positive definite matrix of dimension \( 2n \times 2n \) satisfying \( (\tilde{S}_4)_{i(n+i)} = 0, i = 1, \ldots, n \).

**Remark 2.1.** Symplectic rotations (18) represent unique automorphisms preserving (17) if all \( \sigma_k \) are distinct. If some of them are equal, there exist wider class of automorphism preserving (17) that further simplifies the matrix \( S_4 \).
3 Ad-invariant metrics

The next step would be to consider pseudo-Riemannian case. However, as previously mentioned, the corresponding moduli space of metrics will grow quite large with the increase of the dimension. Nevertheless, it is known that every cotangent bundle of a Lie group \( G \) admits an ad-invariant metric defined by the duality pairing on the corresponding Lie algebra \( T^*g = g \times g^* \):

\[
\langle (x, x^*), (y, y^*) \rangle = x^*(y) + y^*(x), \quad x, y \in g, \quad x^*, y^* \in g^*.
\] (20)

The metric (20) is of neutral signature and it makes both algebras \( g \) and \( g^* \) totally isotropic, meaning \( g^\perp = g \) and \( g^{*, \perp} = g^* \).

**Lemma 3.1.** The metric (20) is unique ad-invariant metric on \( T^*h_{2n+1} \) and it is flat.

**Proof.** The condition of ad-invariance:

\[
\langle [x, y], z \rangle + \langle y, [x, z] \rangle = 0, \quad x, y, z \in T^*h_{2n+1},
\]

gives us that the corresponding metric is given by the symmetric matrix:

\[
S = \begin{pmatrix} \bar{S} & \alpha E \\ \alpha E & 0 \end{pmatrix}, \quad \bar{S} = S^T \in M_{2n+1}(\mathbb{R}), \quad \alpha \neq 0.
\]

If we apply the automorphism \( F \) of the form (7)-(8) with

\[
F_1 = E, \quad F_4 = \frac{1}{\alpha} E, \quad F_3 = -\frac{1}{2\alpha} \bar{S},
\]

we obtain that \( \bar{S} = 0 \) and \( \alpha = 1 \), i.e. every ad-invariant metric is equivalent to the metric (20).

The flatness of the metric follows trivially from the fact that the curvature tensor \( R \) for ad-invariant metric is given by

\[
R(u, v) = -\frac{1}{4} \text{ad}_{[u, v]}, \quad u, v \in T^*h_{2n+1}.
\]

Therefore, the ad-invariant metric is flat if and only if the corresponding Lie algebra is two-step nilpotent. \( \square \)

The previous results confirm the much extensive results on the uniqueness of the ad-invariant metrics recently obtained in [5].

4 Classsification of pseudo-Kähler metrics on \( T^*h_{2n+1} \)

Almost complex structure \( J \) is linear map \( J : T^*h_{2n+1} \to T^*h_{2n+1} \) that satisfies \( J^2 = -Id \). Recall that complex structure \( J \) is integrable complex structure, i.e. it satisfies \( N_J(X, Y) = 0 \) for all \( X, Y \in T^*h_{2n+1} \) where \( N_J \) is the Nijenhuis tensor defined by

\[
N_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY].
\] (21)

If \( \Omega \) is 2-form and \( J \) is an almost complex structure on \( g \) we say that \( \Omega \) is \( J \)-invariant if

\[
\Omega(X, Y) = \Omega(JX, JY),
\] (22)

for all \( X, Y \in g \).

The metric \( \langle \cdot, \cdot \rangle \) defined by complex structure \( J \) and \( J \)-invariant closed 2-form \( \Omega \) by

\[
\langle X, Y \rangle := \Omega(JX, JY)
\] (23)

is called pseudo-Kähler.
**Theorem 4.1.** Up to automorphisms the complex structure on $T^*h_{2n+1}$ is up to a sign unique, and represented by matrix

$$
J_0 = \begin{pmatrix}
J & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & J & 0 \\
0 & -1 & 0 & 0
\end{pmatrix},
$$

where $J$ is the $2n \times 2n$ matrix of standard complex structure given by (2).

**Proof.** Vanishing of Nienhuis tensor (21) can be equivalently expressed in terms of ad operator

$$
\text{ad}(X) - J \text{ad}(JX) - J \text{ad}(X)J - \text{ad}(JX)J = 0,
$$

for all $X \in T^*h_{2n+1}$.

Each $X \in T^*h_{2n+1}$ decomposes as $X = \nu + \zeta, \nu \in \mathbb{R}^{2n} \oplus \mathbb{Z}^*, \zeta \in \mathbb{R}^{2n} \oplus \mathbb{Z}$ and the matrix $A_X \circ \text{ad}(X)$ depends only on $\nu$, as in (9). With respect to the same decomposition write the matrix of $\mathcal{J}$ in the form

$$
\mathcal{J} = \begin{pmatrix}
J_1 & J_2 \\
J_3 & J_4
\end{pmatrix}.
$$

The condition (25) is now equivalent to the following matrix equations:

$$
J_2A_\nu J_2 = 0, \quad J_2A_{J_1+J_2} + J_2A_\nu J_1 = 0,
$$

$$
A_{J_1+J_2}J_2 - J_4A_\nu J_2 = 0, \quad A_\nu + J_4A_{J_1+J_2} + J_4A_\nu J_1 = A_{J_1+J_2}J_1 = 0,
$$

for all $\nu \in \mathbb{R}^{2n} \oplus \mathbb{Z}^*, \zeta \in \mathbb{R}^{2n} \oplus \mathbb{Z}$. Careful analysis of these equations yields the following solution

$$
J_1 = \begin{pmatrix}
\varepsilon J & 0 \\
0 & n_1
\end{pmatrix}, \quad J_2 = \begin{pmatrix}
0 & 0 \\
0 & n_2
\end{pmatrix}, \quad J_3 = \begin{pmatrix}
\bar{J}_3 & k_3 \\
K_3 & n_3
\end{pmatrix}, \quad J_4 = \begin{pmatrix}
\varepsilon J & 0 \\
0 & 0
\end{pmatrix},
$$

with $\varepsilon = \pm 1, n_1, n_2, n_3 \in \mathbb{R}, k_3, m_3 \in \mathbb{R}^{2n}, \bar{J}_3 \in M_2n(\mathbb{R})$. If we apply condition $J^2 = -Id$, we obtain

$$
m_3 = k_3 = 0, \quad n_1 = 0, \quad n_3 = -\frac{1}{n_2}, \quad \bar{J}_3J + J\bar{J}_3 = 0.
$$

Now, we apply the automorphism $F$ of the form (7)-(8) with

$$
\bar{F}_1 = J, \quad u_1 = v_1 = 0, \quad f_1 = n_2, f_4 = \varepsilon, \quad F_3 = \begin{pmatrix}
\frac{J_3}{2} & 0 \\
0 & 0
\end{pmatrix},
$$

to the structure $\mathcal{J}$ of the form (26) and (27) satisfying (28), and obtain

$$
F^{-1}JF = \varepsilon J_0, \quad \varepsilon = \pm 1,
$$

as in the statement of the theorem. \hfill \square

Complex structure $\mathcal{J}$ is said to be *Hermitian* if it preserves the metric:

$$
\langle \mathcal{J}u, \mathcal{J}v \rangle = \langle u, v \rangle, \quad u, v \in g.
$$

In the fixed basis where $J_0$ has the form (24), the direct computation shows that the following lemma holds.

**Lemma 4.1.** For the Hermitian complex structure $J_0$ the corresponding Riemannian metric is given by (19) with $\omega_4 = 1$ and $\bar{S}_4$ being positive definite matrix of dimension $2n \times 2n$ satisfying $(\bar{S}_4)i_{(n+i)} = 0, i = 1, \ldots, n$, and $J\bar{S}_4 = \bar{S}_4J$. 

---

8
Now, we describe all \( \mathcal{J}_0 \)-invariant closed 2-forms on the Lie algebra \( g = T^*h_{2n+1} \).

In Section 1 we introduced the standard basis of Lie algebra \( T^*h_{2n+1} \):

\[
e_1, \ldots, e_n, f_1, \ldots, f_n, z^*, e_1^*, \ldots, e_n^*, f_1^*, \ldots, f_n^*, z.
\]

Denote the dual basis on \( g^* = (T^*h_{2n+1})^* \) by

\[
e^1, \ldots, e^n, f^1, \ldots, f^n, \zeta^*, e_1^*, \ldots, e_n^*, f_1^*, \ldots, f_n^*, \zeta.
\]

**Theorem 4.2.** All \( \mathcal{J}_0 \) invariant closed 2-forms on \( T^*h_{2n+1} \), \( n > 1 \) are of the form

\[
\Omega = A^1_{ij}(e^i \wedge e^j + f^i \wedge f^j) + A^2_{ij}(e^i \wedge f^j) + K_{ij}(e^i \wedge e^j + f^i \wedge f^j)
\]

\[
+ di e^i \wedge f^i - \frac{\mu}{2}(e^i \wedge e^i + f^i \wedge f^i) + \mu \zeta^* \wedge \zeta,
\]

where \( A^1_{ij} = -A^1_{ji}, A^2_{ij} = A^2_{ji}, K_{ij} = -K_{ji}, d_i \in \mathbb{R}, \mu \in \mathbb{R} \) and summation is assumed over repeated indices.

The dimension of the space of such forms is \( \frac{3n^2 + n + 2}{2}, n > 1 \).

**Proof.** The commutators (6) can be written in the form

\[
de = 0, \quad df^i = 0,
\]

\[
de^i = f^1 \wedge \zeta^*, \quad df^i = -e^i \wedge \zeta^*, \quad d\zeta = e^1 \wedge f^i.
\]

The proof is by straightforward computation and will be omitted. Start from the general 2-form \( \Omega \) over the basis (29). Impose the condition (22) for complex structure \( J = \mathcal{J}_0 \) given by (24). Finally, the condition \( d\Omega = 0 \) bring us to the form (30).

**Remark 4.1.** For \( n = 1 \), i.e. in case of 6-dimensional Lie algebra \( T^*h_3 \), the result is obtained in [17]. The dimension of \( J \)-invariant closed 2-forms is five. Such forms has general for (30) with additional terms

\[
a_1(e^i \wedge \zeta^* - f^1 \wedge \zeta) + a_2(f^1 \wedge \zeta^* + e^1 \wedge \zeta), \ a_1, a_2 \in \mathbb{R}.
\]

These terms come from a different decomposition of the space of 3-forms in basis (29).

As noted in the Introduction, the complex structure \( \mathcal{J}_0 \) is not abelian, hence the Hermitian metric from Lemma 4.1 cannot be pseudo-Kähler (see [2, Theorem A] and [6]). However, we show that there exists a family of Ricci-flat pseudo-Kähler metrics.

**Theorem 4.3.** The Lie algebra \( T^*h_{2n+1} \) admits Ricci-flat pseudo-Kähler metrics that are not flat. Every pseudo-Kähler metric on \( T^*h_{2n+1} \) is equivalent to \( S = -\mathcal{J}_0 \Omega \), where \( \mathcal{J}_0 \) is complex structure (24) and \( \Omega \) is symplectic form given by (30).

**Proof.** The form of the pseudo-Kähler metric follows directly from (23). Since, we have already fixed the basis in a way that the complex form \( J \) takes the form (24), the choice of automorphisms preserving \( \mathcal{J}_0 \) is quite restricted. The obtained simplification is insignificant comparing to the difficult notation required, hence we choose not to perform it.

Next, we need to examine the curvature properties of those metrics. The Ricci tensor \( \rho \) on the nilpotent Lie group can be expressed in terms of operators ad and \( \text{ad}^* \):

\[
\rho(u, v) = -\frac{1}{4} \text{tr}(j_u \circ j_v) - \frac{1}{2} \text{tr}(\text{ad}_u \circ \text{ad}_v^*),
\]

where \( j_u v = \text{ad}_v^* u \) for arbitrary left invariant vector fields \( u, v \). By direct calculation we get that both summands are equal to zero, hence the metric is Ricci-flat. Now, the only thing left is to show that at least one component of curvature tensor \( R \) is non-zero. For example, long, but straightforward computation gives us that \( R(e_1, f_1) \) contains non-trivial components. 

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