RESOLVABILITY IN HYPERGRAPHS

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Abstract. This article emphasizes an extension of the study of metric and partition dimension to hypergraphs. We give a sharp lower bounds for the metric and partition dimension of hypergraphs in general and give exact values under specified conditions.

1. Introduction

A hypergraph $H$ is a pair $(V(H), E(H))$, where $V(H)$ is a finite non-empty set of vertices and $E(H)$ is a finite family of distinct non-empty subsets of $V(H)$, called hyperedges, with $\bigcup_{E \in E(H)} E = V(H)$. The “order” and the “size” of $H$ is denoted by $m$ and $k$, respectively. A subhypergraph $K$ of a hypergraph $H$ is a hypergraph with vertex set $V(K) \subseteq V(H)$ and edge set $E(K) \subseteq E(H)$. A hypergraph $H$ is linear if for distinct hyperedges $E_i, E_j \in E(H)$, $|E_i \cap E_j| \leq 1$, so for a linear hypergraph there are no repeated hyperedges of cardinality greater than one. A hypergraph $H$ such that no hyperedge is a subset of any other is called Sperner.

A vertex $v \in V(H)$ is incident with a hyperedge $E$ of $H$ if $v \in E$. If $v$ is incident with exactly $n$ hyperedges, then we say that the degree of $v$ is $n$; if all the vertices $v \in V(H)$ have degree $n$, then $H$ is $n$-regular. Similarly, if there are exactly $n$ vertices incident with a hyperedge $E$, then we say that the size of $E$ is $n$; if all the hyperedges $E \in E(H)$ have size $n$, then $H$ is $n$-uniform. A graph is simply a 2-uniform hypergraph. A hyperedge $E$ of $H$ is called a pendant hyperedge if for $E_i, E_j \in E(H)$, $E \cap E_i \neq \emptyset$ and $E \cap E_j \neq \emptyset$ implies $(E \cap E_i) \cap (E \cap E_j) \neq \emptyset$. A path of length $l$ from a vertex $v$ to another vertex $u$ in a hypergraph is a finite sequence of the form $v, E_1, w_1, E_2, w_2, \ldots, E_{l-1}, w_{l-1}, E_l, u$ such that $v \in E_1$, $w_i \in E_i \cap E_{i+1}$ for $i = 1, 2, \ldots, l - 1$ and $u \in E_l$. A hypergraph $H$ is called connected if there is a path between any two vertices of $H$. All hypergraphs considered in this paper are connected Sperner hypergraphs.

A hypergraph $H$ is said to be a hyperstar if there exists a subset $C$ of vertices such that $E_i \cap E_j = C \neq \emptyset$, for any $E_i, E_j \in E(H)$. Then $C$ is called the center of the hyperstar.

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hyperstar. If there exists a sequence of hyperedges $E_1, E_2, \ldots, E_k$ in a hypergraph $H$, then $H$ is said to be (1) a hyperpath if $E_i \cap E_j \neq \emptyset$ if and only if $|i - j| = 1$; (2) a hypercycle if, $E_i \cap E_j \neq \emptyset$ if and only if $i - j \in \{1, -1\}$ (mod $k$). A connected hypergraph $H$ with no hypercycle is called a hypertree. A subhypertree of a hypertree $H$ with edge set, say $\{E_{p_1}, E_{p_2}, \ldots, E_{p_l}\} \subset E(H)$, is called a branch of $H$ if $E_{p_1}$ (say) is the only hyperedge such that, for $E_i, E_j \in E(H) \setminus \{E_{p_1}, E_{p_2}, \ldots, E_{p_l}\}$, $E_i \cap E_j \neq \emptyset$ and $E_{p_1} \cap E_j \neq \emptyset$ implies $(E_{p_1} \cap E_i) \cap (E_{p_1} \cap E_j) \neq \emptyset$. The hyperedge $E_{p_1}$ is called the joint of the branch.

An ordered set $W$ of vertices of a connected graph $G$ is called a resolving set for $G$ if for every two distinct vertices $u, v \in V(G)$, there is a vertex $w \in W$ such that $d(u, w) \neq d(v, w)$. A resolving set of minimum cardinality is called a basis for $G$ and the number of vertices in a basis is called the metric dimension of $G$, denoted by $\dim(G)$. An ordered $t$-partition $\Pi = \{S_1, S_2, \ldots, S_l\}$ of $V(G)$ is called a resolving partition if for every two distinct vertices $u, v \in V(G)$, there is a set $S_i$ in $\Pi$ such that $d(u, S_i) \neq d(v, S_i)$, where $d(v, s) = \min_{s \in S} d(u, s)$. The minimum $t$ for which there is a resolving $t$-partition of $V(G)$ is called the partition dimension of $G$, denoted by $pd(G)$. In this article, we consider hypergraphs in the context of metric dimension and partition dimension, which are defined in Sections 2 and 3, respectively. We give sharp lower bounds for the metric and partition dimension of graphs. The metric dimension of some well-known families of hypergraphs such as hyperpaths, hypertrees and $n$-uniform linear hypercycles is investigated. Further, we find the metric and partition dimension of $3$-uniform linear hypercycles. We also characterize all the $n$-uniform (for all $n \geq 2$ and $n \neq 3$ when $k$ is even) linear hypergraphs with partition dimension $n$. Moreover, all the hypergraphs with metric dimension $1$ and partition dimension $2$ are characterized.

2. Metric Dimension of Hypergraphs

The metric dimension of a graph was first studied by Slater [14] and independently by Harary and Melter [8]. It is a parameter that has appeared in various applications, as diverse as combinatorial optimization, pharmaceutical chemistry, robot navigation and sonar. In recent years, a considerable literature has been developed (see [1, 5, 6, 9, 10, 11, 12, 13]). The problem of determining whether $\dim(H) < M$ ($M > 0$), where $H$ is a simple graph, is an NP-complete problem [7, 12]. The metric dimension of a hypergraph $H$ is defined as follows:

The distance between any two vertices $v$ and $u$ of a hypergraph $H$, $d(v, u)$, is the length of a shortest path between them and $d(v, u) = 0$ if and only if $v = u$. The diameter of $H$ is the maximum distance between the vertices of $H$, and is denoted by $\text{diam}(H)$. Two vertices $u$ and $v$ of $H$ are said to be “diametral” vertices if $d(u, v) = \text{diam}(H)$. The representation, $r(v|W)$, of a vertex $v$ of $H$ with respect to an ordered set $W = \{w_1, w_2, \ldots, w_q\} \subseteq V(H)$ is the $q$-tuple $r(v|W) = \ldots$
(d(v, w_1), d(v, w_2), ..., d(v, w_q)). The set W is called a resolving set for a hypergraph H if r(v|W) ≠ r(u|W) for any two different vertices v, u ∈ V(H). A resolving set with minimum cardinality is called a basis for H and that minimum cardinality is called the metric dimension of H, denoted by dim(H).

To determine whether a given set W ⊆ V(H) is a resolving set for a hypergraph H, W needs only to be verified for the vertices in V(H) \ W since every vertex w ∈ W is the only vertex of H whose distance from w is 0.

If we denote all the vertices of degree d in E_{i_1} ∩ E_{i_2} ∩ ... ∩ E_{i_d} by the class C(i_1, i_2, ..., i_d), then the collection of all such classes gives a partition of V(H). Thus, we have the following straightforward proposition:

**Proposition 2.1.** For any two distinct vertices u, v ∈ C(i_1, i_2, ..., i_d), we have d(u, w) = d(v, w) for any w ∈ V(H) \ {u, v}.

Thus, we extract the following Lemma related to the resolving set for H:

**Lemma 2.2.** If u, v ∈ C(i_1, i_2, ..., i_d) and W ⊆ V(H) resolves H, then at least one of the vertices u and v is in W. Moreover, if u ∈ W and v ∉ W, then (W \ {u}) ∪ {v} also resolves H.

Let us denote n(i_1, i_2, ..., i_d) = |C(i_1, i_2, ..., i_d)| − 1 when C(i_1, i_2, ..., i_d) ≠ ∅, otherwise we take n(i_1, i_2, ..., i_d) = 0. This notation helps us to write a lower bound for the metric dimension of hypergraphs in the following Proposition.

**Proposition 2.3.** For any hypergraph H with k hyperedges,

\[ dim(H) \geq \sum_{j=1}^{k} \sum_{i_1 < ... < i_j} n(i_1, i_2, ..., i_j). \]

**Proof.** It follows from the fact that if there are |C(i_1, i_2, ..., i_d)| number of vertices of degree d in E_{i_1} ∩ E_{i_2} ∩ ... ∩ E_{i_d}, then by Lemma 2.2 at least n(i_1, i_2, ..., i_d) vertices should belong to any basis W. □

**Remark 2.4.** By Proposition 2.3 it is clear that, in order to obtain a basis of any hypergraph H, it suffices to consider only one vertex, say v_{i_1,i_2,...,i_d}, from each class C(i_1, i_2, ..., i_d) if C(i_1, i_2, ..., i_d) ≠ ∅. We call this vertex, a representative vertex of the class C(i_1, i_2, ..., i_d). We denote the set of all representative vertices in a hypergraph H by R(H), and hence we always have, V(H) \ R(H) ⊆ W for any basis W of H.

Now we discuss some classes of hypergraphs for which the equality holds in the Proposition 2.3.

**Theorem 2.5.** For any hypergraph H with k hyperedges, if n(i) ≠ 0 for all E_i ∈ E(H), then \( dim(H) = \sum_{j=1}^{k} \sum_{i_1 < ... < i_j} n(i_1, i_2, ..., i_j) \). Moreover, there are \( \prod_{j=1}^{k} \prod_{i_1 < ... < i_j} (n(i_1, i_2, ..., i_j) + 1) \) basis for H.
Proof. Consider $W = V(H) \setminus R(H)$, we have to show that $W$ is a basis for $H$. Take any two different vertices $v,v' \in R(H)$. Since both the vertices $v$ and $v'$ are representative vertices of different classes, there exists a hyperedge $E_j$ such that $v' \in E_j$ and $v \not\in E_j$. It follows from $n(j) \neq 0$ that there exists a vertex of degree one $w_j \in V(H)$ such that $w_j \in E_j \cap W$. Clearly, $d(v',w_j) = 1$ and $d(v,w_j) \neq 1$, hence $W$ is a basis for $H$. Further, by Lemma 2.9 there are $\prod_{j=1}^{k} \prod_{i_1 < \ldots < i_j} (n(i_1, i_2, \ldots, i_j) + 1)$ such $W$.

For all $n \geq 4$, if $H$ is an $n$-uniform linear hypergraph with $k$ hyperedges, then $n(i, i + 1) = 0$ for every $i \in \{1, 2, \ldots, k\}$. Thus, we have the following corollary:

**Corollary 2.6.** For $n \geq 4$, let $H$ be an $n$-uniform linear hypergraph with $k$ hyperedges. If $n(i) \neq 0$ for all $E_i \in E(H)$, then $\dim(H) = \sum_{i=1}^{k} n(i)$.

We give two examples which show that the condition in Theorem 2.5 cannot be relaxed generally.

**Example 2.7.** Let $H$ be a hypergraph with vertex set $V(H) = \{v_1, v_2, v_3, v_4\}$ and edge set $E(H) = \{E_1, E_2\}$, where $E_1 = \{v_1, v_2, v_3\}$ and $E_2 = \{v_3, v_4\}$. Clearly, $n(2) = 0$ so $H$ does not satisfy the condition of Theorem 2.5. Without loss of generality, we can take the set of representative vertices $R(H) = \{v_1, v_3, v_4\}$, and hence $W = V(H) \setminus R(H) = \{v_2\}$. But, $W$ is not a resolving set for $H$ since $r(v_1|W) = r(v_3|W)$. In fact, $\dim(H) = 2 > 1$.

**Example 2.8.** Let $H$ be a hypergraph with vertex set $V(H) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and edge set $E(H) = \{E_1, E_2, E_3\}$, where $E_1 = \{v_1, v_2, v_3, v_4\}$, $E_2 = \{v_3, v_4, v_5, v_6\}$ and $E_3 = \{v_1, v_2, v_5, v_6\}$. Clearly, $n(i) = 0$ for all $i = 1, 2, 3$ and $n(1, 2) = n(2, 3) = n(3, 1) \neq 0$. Without loss of generality, we can take the set of representative vertices $R(H) = \{v_1, v_3, v_5\}$, and hence $W = V(H) \setminus R(H) = \{v_2, v_4, v_6\}$. But, $W$ is not a resolving set for $H$ since $r(v_1|W) = r(v_3|W) = r(v_5|W)$. In fact, $\dim(H) = 5 > 3$.

However, the condition in Theorem 2.5 can be reduced in some special cases as shown in the following results.

**Theorem 2.9.** Let $H$ be a hyperpath with $k$ hyperedges $E_1, E_2, \ldots, E_k$ in a canonical way. Then $\dim(H) = \sum_{i=1}^{k} n(i) + \sum_{i=1}^{k-1} n(i, i + 1)$ if both $n(1)$ and $n(k)$ are non-zero.

Proof. Let $W = V(H) \setminus R(H)$. Then it follows from the facts $n(1) \neq 0$ and $n(k) \neq 0$ that there exists a vertex of degree one $w_1 \in E_1 \cap W$ and there exists a vertex of degree one $w_k \in E_k \cap W$. In order to prove the theorem, we only have to show that the representative vertices are resolved by the set $W$, and it yields from the fact that for any $1 \leq j \leq k$, we have $(d(v_j, w_1), d(v_j, w_k)) = (j, k - j + 1)$, and for any $1 \leq j < k - 1$, we have $(d(v_{j,j+1}, w_1), d(v_{j,j+1}, w_k)) = (j, k - j)$.
Theorem 2.10. Let $H$ be a hypertree with $k$ hyperedges and let $E_{p1}, E_{p2}, \ldots, E_{pt}$ be its pendant hyperedges. Then $\dim(H) = \sum_{j=1}^{k} \sum_{i_1 < i_2 < \cdots < i_j} n(i_1, i_2, \ldots, i_j)$ if $n(ps) \neq 0$ for all $s = 1, 2, \ldots, t$.

Proof. Consider $W = V(H) \setminus R(H)$, similarly as in the proof of Theorem 2.5, again we have to show that $W$ is a basis for $H$. Take any two different vertices $v, v' \in R(H)$, then both vertices are representative of two different classes, and hence there exists a hyperedge $E_j$ such that $v' \in E_j$ but $v \notin E_j$. Now, consider a hyperpath contained in the hypertree $H$ which starts and ends at the pendant hyperedges and contains both $v$ and $E_j$. By using the proof of Theorem 2.9 it can be seen that the vertices $v$ and $v'$ has different representations with respect to $W$, which proves the theorem. □

An $n$-uniform linear hyperstar $(n \geq 3)$ is a special case of hypertree in which $n(i) \neq 0$ for all $E_i \in E(H)$, so we have the following corollary:

Corollary 2.11. For $n \geq 3$, let $H$ be an $n$-uniform linear hyperstar with $k$ ($\geq 3$) hyperedges. Then $\dim(H) = k(n - 2)$.

Consider an $n$-uniform linear hypercycle $C_{k,n}$ with $k$ hyperedges. When $n \geq 4$, then $n(i) \neq 0$ for all $E_i \in E(C_{k,n})$ so, by Corollary 2.6 $\dim(C_{k,n}) = k(n - 3)$.

For the case $n = 3$, we have $n(i) = 0$ for all $E_i \in E(H)$, hence the lower bound given in Proposition 2.3 is zero and every vertex in $C_{k,3}$ is the representative vertex. We discuss this case in the following result:

Theorem 2.12. Let $C_{k,3}$ be a 3-uniform linear hypercycle with $k$ hyperedges. Then $\dim(C_{3,3}) = 2$ and for all $k \geq 4$,

$$\dim(C_{k,3}) = \begin{cases} 
2, & \text{if } k \text{ is even}, \\
3, & \text{if } k \text{ is odd}. 
\end{cases}$$

Proof. In $C_{k,3}$, each $v_j \in E_j$ represents a vertex of degree one and $v_{j,j+1} \in E_j \cap E_{j+1}$ with $v_{k,k+1} = v_{k,1}$. Clearly, $\dim(C_{k,3}) > 1$ for any $k$.

If $k$ is even, then we take $W = \{v_1, v_{k/2}\}$.

For $1 < j < \frac{k}{2}$, we have $r(v_j|W) = (j, \frac{k}{2} - j + 1)$ and for $1 \leq j < \frac{k}{2}$, $r(v_{j,j+1}|W) = (j, \frac{k}{2} - j)$. Now, if $\frac{k}{2} + 1 \leq j < k$, then $r(v_j|W) = (k + 2 - j, j - \frac{k}{2} + 1)$ and $r(v_{j,j+1}|W) = (k + 1 - j, j - \frac{k}{2} + 1)$ with $r(v_k|W) = (2, \frac{k}{2} + 1)$, $r(v_{k,k+1}|W) = (\frac{k}{2}, 1)$ and $r(v_{k,1}|W) = (1, \frac{k}{2})$. It is easy to see that the representations of all the vertices with respect to $W$ are distinct, hence $W$ forms a basis for $C_{k,3}$ and $\dim(C_{k,3}) = 2$.

For the special case when $k = 3$, the set $W = \{v_1, v_2\}$ forms a basis for $C_{3,3}$. Hence $\dim(C_{3,3}) = 2$.

If $k > 3$ is odd, then we first show that $\dim(C_{k,3}) > 2$. Suppose on contrary that $\dim(C_{k,3}) = 2$ and let $W$ is a basis of $C_{k,3}$. Let us call the vertices $v_{i,i+1}$, $i \in \{1, 2, \ldots, k\}$, of $C_{k,3}$, the common vertices. We have the following three possibilities:
(1) $W$ contains both common vertices. Without loss of generality, we may assume that one vertex is $v_{1,2}$ and the second vertex is $v_{j,j+1}$ ($2 \leq j \leq k$). Then $r(v_{j+1}|W) = r(v_{j+1,j+2}|W)$, for $2 \leq j < \frac{k+1}{2}$; $r(v_2|W) = r(v_{k,1}|W)$, for $j = \frac{k+1}{2}$; $r(v_1|W) = r(v_{2,3}|W)$, for $j = \frac{k+1}{2} + 1$ and $r(v_j|W) = r(v_{j-1,j}|W)$, for $\frac{k+1}{2} + 1 < j \leq k$, a contradiction.

(2) $W$ contains one common vertex. Without loss of generality, we may assume that one vertex is $v_{1,2}$ and the second vertex is $v_j$ ($1 \leq j \leq k$). Then $r(v_{j+1}|W) = r(v_{j+1,j+2}|W)$, for $1 \leq j < \frac{k+1}{2}$; $r(v_1|W) = r(v_{k,1}|W)$, for $j = \frac{k+1}{2}$; $r(v_1|W) = r(v_2|W)$, for $j = \frac{k+1}{2} + 1$ and $r(v_2|W) = r(v_{2,3}|W)$, for $\frac{k+1}{2} + 1 < j \leq k$, a contradiction.

(3) $W$ contains no common vertex. Without loss of generality, we may assume that one vertex is $v_1$ and the second vertex is $v_j$ ($2 \leq j \leq k$). Then $r(v_{j+1}|W) = r(v_{j+1,j+2}|W)$, for $2 \leq j < \frac{k+1}{2}$; $r(v_{j-1}|W) = r(v_{j+1,j+2}|W)$, for $j = \frac{k+1}{2}$; $r(v_{j+1}|W) = r(v_{j-2,j-1}|W)$, for $j = \frac{k+1}{2} + 1$ and $r(v_{j-1}|W) = r(v_{j-2,j-1}|W)$, for $\frac{k+1}{2} + 1 < j \leq k$, a contradiction.

Now, we will show that $\text{dim}(C_{k,3}) \leq 3$. Take $W = \{v_1, v_2, v_{\frac{k+1}{2}}\}$. We note that, $r(v_{1,2}|W) = (1, 1, \frac{k-1}{2})$ and

$$r(v_j|W) = \begin{cases} (j, j-1, \frac{k+1}{2} - j + 1) & \text{for } 2 < j < \frac{k+1}{2}, \\ (\frac{k+1}{2}, \frac{k+1}{2}, 2) & \text{for } j = \frac{k+1}{2} + 1, \\ (k - j + 2, k - j + 3, j - \frac{k-1}{2}) & \text{for } \frac{k+1}{2} + 1 < j \leq k, \end{cases}$$

$$r(v_{j,j+1}|W) = \begin{cases} (j, j-1, \frac{k+1}{2} - i) & \text{for } 2 \leq j < \frac{k+1}{2}, \\ (\frac{k+1}{2}, \frac{k-1}{2}, 1) & \text{for } j = \frac{k+1}{2}, \\ (k - j + 1, k - j + 2, j - \frac{k-1}{2}) & \text{for } \frac{k+1}{2} < j \leq k. \end{cases}$$

One can see that all the vertices of $V(C_{k,3}) - W$ have distinct representations. This implies that $\text{dim}(C_{k,3}) = 3$ when $k > 3$ is odd. \qed

The primal graph, $\text{prim}(H)$, of a hypergraph $H$ is a graph with vertex set $V(H)$ and vertices $x$ and $y$ of $\text{prim}(H)$ are adjacent if and only if $x$ and $y$ are contained in a hyperedge. The middle graph, $M(H)$, of $H$ is a subgraph of $\text{prim}(H)$ obtained by deleting loops and parallel edges. Since the adjacencies between the vertices in $\text{prim}(H)$ are due to the adjacencies in the hypergraph $H$, so determining the length of a path between two vertices $u$ and $v$ in $\text{prim}(H)$ is equivalent to determine the length of a path between the vertices $u$ and $v$ in $H$. This fact yields the following result:

**Theorem 2.13.** Let $H$ be a hypergraph. Then

$$\text{dim}(H) = \text{dim}(\text{prim}(H)) = \text{dim}(M(H)).$$

The dual of $H = (\{v_1, v_2, \ldots, v_m\}, \{E_1, E_2, \ldots, E_k\})$, denoted by $H^*$, is the hypergraph whose vertices are $\{E_1, E_2, \ldots, E_k\}$ corresponding to the hyperedges of $H$ and with hyperedges $V_i = \{E_j : v_i \in E_j \text{ in } H\}$, where $i = 1, 2, \ldots, m$. In other
words, the dual $H^*$ swaps the vertices and hyperedges of $H$. The primal graph of the dual $H^*$ of a hypergraph $H$ is not a simple graph, in this case, the middle graph of $H^*$ is a simple graph. We discuss the metric dimension of dual hypergraphs separately in the following result, which also helps us to characterize all the hypergraphs with metric dimension one.

**Theorem 2.14.** Let $H^*$ be the dual of a hypergraph $H$. Then

$$\dim(H^*) = \dim(M(H^*)).$$

**Proof.** By the definition of middle graph, for any two vertices $u$ and $v$ of $H^*$, a path $P$ is a shortest path between the vertices $u$ and $v$ in $H^*$ if and only if $P$ is a shortest path between $u$ and $v$ in $M(H^*)$. Thus a set $W \subseteq V(H^*)$ is a minimum resolving set for $H^*$ if and only if $W$ is a minimum resolving set for $M(H^*)$. □

The middle graph of $H^*$ is (1) a simple path $P_m$ if and only if $H$ is a hyperpath; (2) a simple cycle $C_m$ if and only if $H$ is a hypercycle. In [5], all the simple connected graphs having metric dimension one were characterized by proving the result “$\dim(G)$ is one if and only if $G$ is a simple path $P_m$ ($m \geq 1$)”.

Now, we characterize all the connected hypergraphs having the metric dimension 1. In fact, all these hypergraphs are the dual hypergraphs and have been characterized in the following consequence of Theorem 2.14.

**Corollary 2.15.** Let $H^*$ be the dual of a hypergraph $H$. Then $\dim(H^*) = 1$ if and only if $H$ is a hyperpath.

In [8], it was shown that the metric dimension of a simple cycle $C_m$ ($m \geq 3$) is two. Thus, we have the following corollary:

**Corollary 2.16.** Let $H^*$ be the dual of a hypercycle $H$. Then $\dim(H^*) = 2$.

### 3. Partition Dimension of Hypergraphs

Possibly to gain insight into the metric dimension, Chartrand et al. introduced the notion of a resolving partition and partition dimension [3][4]. To define the partition dimension, the distance $d(v,S)$ between a vertex $v$ in $H$ and $S \subseteq V(H)$ is defined as $\min_{s \in S} d(v,s)$. Let $\Pi = \{S_1, S_2, \ldots, S_t\}$ be an ordered $t$-partition of $V(H)$ and $v$ be any vertex of $H$. Then the representation, $r(v|\Pi)$, of $v$ with respect $\Pi$ is the $t$-tuple $r(v|\Pi) = (d(v,S_1), d(v,S_2), \ldots, d(v,S_t))$. The partition $\Pi$ is called a resolving partition for a hypergraph if $r(v|\Pi) \neq r(u|\Pi)$ for any two distinct vertices $v, u \in V(H)$. The partition dimension of a hypergraph $H$ is the cardinality of a minimum resolving partition, denoted by $pd(H)$.

From the definition of a resolving partition, it can be observed that the property of a given partition $\Pi$ of a hypergraph $H$ to be a resolving partition of $H$ can be verified by investigating the pairs of vertices in the same class. Indeed, $d(x,S_i) = 0$
for every vertex $x \in S_i$ but $d(x, S_j) \neq 0$ with $j \neq i$. It follows that $x \in S_i$ and $y \in S_j$
are resolved either by $S_i$ or $S_j$ for every $i \neq j$. From Proposition 2.1, we have the
following lemma:

**Lemma 3.1.** Let $\Pi$ be a resolving partition of $V(H)$. If $u, v \in C(i_1, i_2, \ldots, i_d)$ then
$u$ and $v$ belong to distinct classes of $\Pi$.

The following result gives the lower bound for the partition dimension of hyper-
graphs.

**Proposition 3.2.** Let $H$ be a hypergraph with $k$ hyperedges. Then $pd(H) \geq \lambda + 1$, where
$\lambda = \max |C(i_1, i_2, \ldots, i_d)|$ in $H$.

**Proof.** Since $\lambda = \max |C(i_1, i_2, \ldots, i_d)|$ in $H$, by Lemma 3.1, we have at least $\lambda$
disjoint classes $S_1, S_2, \ldots, S_\lambda$ of $V(H)$. Since $H$ is Sperner so there exists an edge
$E$ of $H$ such that $C(i_1, i_2, \ldots, i_d) \subseteq E$. Now, if $\Pi = \{S_1, \ldots, S_\lambda\}$ is a minimum
resolving partition of $V(H)$ then there exist two vertices $u$ and $v$ in $E$ such that
$u, v \in S_i$ (say) with $r(u|\Pi) = (1, \ldots, 0, \ldots, 1) = r(v|\Pi)$, where 0 is at the $i$th place,
a contradiction. Thus, $pd(H) \geq \lambda + 1$. $\square$

The lower bound given in Proposition 3.2 is sharp for an $n$-uniform linear hyper-
path.

A 2-uniform hypercycle $C_{k,2}$ is a simple connected cycle on $m$ vertices and it
was shown that the partition dimension of a simple connected cycle is 3 [4], so
$pd(C_{k,2}) = 3$. In the next result, we investigate the partition dimension of 3-uniform
hypercycle $C_{k,3}$, $k \geq 3$.

**Theorem 3.3.** Let $C_{k,3}$ be a 3-uniform linear hypercycle with $k \geq 3$ hyperedges. Then
$pd(C_{k,3}) = 3$.

**Proof.** For all $k \geq 3$, we denote the vertices of $C_{k,3}$ by $v_j^i$, where $j$ ($1 \leq j \leq k$)
represents the hyperedge number of $C_{k,3}$ and $i$ ($1 \leq i \leq 3$) represents the vertex
number of the $i$th hyperedge. Each $v_j^i \in E_j$; represents the vertex of degree one and
$v_3^j = v_3^{j+1} \in E_j \cap E_{j+1}$ represents a vertex of degree 2 with $v_3^k = v_3^1$.

If we put all the vertices of $C_{k,3}$ into two classes $S_1$ and $S_2$, then they do not form
a resolving partition $\Pi$ of $V(C_{k,3})$, because one can easily check that there exist two
vertices $u, v$ of $C_{k,3}$ in a class such that $r(u|\Pi) = (0, 1) = r(v|\Pi)$. Thus $pd(C_{k,3}) \geq 3$.
On the other hand, $pd(C_{k,3}) \leq 3$, because we have a resolving partition of cardinality
3 for $pd(C_{k,3})$ in each of the following case:

For $k \equiv 0$ (mod 6), we have a resolving partition for $pd(C_{k,3})$ as

$$\Pi = \{\{v_1^1, \ldots, v_3^1\}, \{v_2^1, \ldots, v_3^2\}, \{v_2^3, \ldots, v_3^4\}, \{v_2^5, \ldots, v_3^6\}\}.$$ 

For $k \equiv 1, 4$ (mod 6), we have a resolving partition for $pd(C_{k,3})$ as

$$\Pi = \{\{v_1^1, \ldots, v_2^\frac{k+2}{6}\}, \{v_3^1, \ldots, v_3^{\frac{k+2}{6}-1}\}, \{v_3^{\frac{k+2}{6}}, \ldots, v_3^k\}\}.$$
For $k \equiv 2 \pmod{6}$, we have a resolving partition for $pd(C_{k,3})$ as
\[
\Pi = \{\{v_1^1, \ldots, v_2^{\frac{k+1}{2}}\}, \{v_2^{\frac{k+1}{2}} + 1, \ldots, v_3^{\frac{k+1}{2} - 1}\}, \{v_3^{\frac{k+1}{2} - 1}, \ldots, v_2^{\frac{k+1}{2}}\}\}.
\]
For $k \equiv 3 \pmod{6}$, we have a resolving partition for $pd(C_{k,3})$ as
\[
\Pi = \{\{v_2^{\frac{k+1}{2}}, \ldots, v_2^{\frac{k+1}{2} + 1}\}, \{v_3^{\frac{k+1}{2}}, \ldots, v_3^{\frac{k+1}{2} + 1}\}, \{v_3^{\frac{k+1}{2} + 1}, \ldots, v_2^{\frac{k+1}{2}}\}\}.
\]
For $k \equiv 5 \pmod{6}$, we have a resolving partition for $pd(C_{k,3})$ as
\[
\Pi = \{\{v_2^{\frac{k+1}{2}}, \ldots, v_2^{\frac{k+1}{2} + 1}\}, \{v_2^{\frac{k+1}{2} + 1}, \ldots, v_2^{\frac{k+1}{2} + 1}\}, \{v_3^{\frac{k+1}{2} + 1}, \ldots, v_3^{\frac{k+1}{2}}\}\}.
\]

In [3], it was shown that a 2-uniform linear hyperpath (simple path) has partition dimension 2. Now, we generalize this result by proving that if $H$ is an $n$-uniform linear hyperpath ($n \geq 2$), then the partition dimension of $H$ is $n$.

**Theorem 3.4.** For $n \geq 2$, let $H$ be an $n$-uniform linear hypergraph with $k$ hyperedges. Then, for a 3-uniform linear hyperpath $H$ with even hyperedges, $pd(H) = 3$ and for all other values of $n$, $pd(H) = n$ if and only if $H$ is a hyperpath.

**Proof.** Let $H$ be an $n$-uniform linear hypergraph. Then it is a routine exercise to verify that a partition $\Pi = \{S_1, S_2, \ldots, S_n\}$ of $V(H)$, where each $S_i, 1 \leq i \leq n - 1$, contains the $i$th vertex of every hyperedge of $H$ and $S_n$ contains the $n$th vertex of the $k$th hyperedge, is a minimum resolving partition.

Conversely, suppose that $\Pi = \{S_1, S_2, \ldots, S_n\}$ be a minimum resolving partition of $V(H)$ and $H$ is an $n$-uniform linear hypergraph. For $n = 2$, $H$ is a 2-uniform linear hyperpath since the partition dimension of a graph is 2 if and only if the graph is a simple path (2-uniform linear hyperpath) [3]. For $n = 3$, $k$, odd and for all $n \geq 4$, if $H$ is not a hyperpath then either $H$ contains a hypercycle or $H$ is a hypertree. Suppose that $H$ contains a hypercycle, then by using the similar arguments as given in the proof of Theorem [3,3], we can see that $pd(H) \geq n + 1$, a contradiction. Now, suppose that $H$ is a hypertree. Consider a path $P : v, E_1, v_1, E_2, w_2, \ldots, E_{l-1}, w_{l-1}, E_1, u$ between two diametral vertices $v$ and $u$ in $H$. Then $P$ contains either a pendant hyperedge, say $E_p$, or a branch with joint $E_{p_1}$ (say), or both a pendant hyperedge and a branch. In the first case, if $|E_p \cap (E_i \cap E_j)| = 1$ ($i \neq j$), then there exist two vertices $x, y$ in $H$, either $x \in E_p$ and $y \in E_i$ or $E_j$, or $x \in E_i$ and $y \in E_j$, such that $x, y \in S_i$ (say) and have $r(x|\Pi) = (1, \ldots, 0, \ldots, 1) = r(y|\Pi)$, where 0 is at the $t$th place. If $|E_p \cap E_i| = 1$ for all $i \neq 1, l$, then there are two vertices $x \in E_p$ and $y \in E_i$ such that $x, y \in S_j$ (say) and have $r(x|\Pi) = (1, \ldots, 0, \ldots, 1) = r(y|\Pi)$, where 0 is at the $j$th place, a contradiction to the fact that $\Pi$ is resolving partition. Similarly, in the second and third case, we can see that a partition of cardinality $n$ is not a resolving partition of $V(H)$. Thus $H$ is an $n$-uniform linear hyperpath. \qed
The rank of a hypergraph $H$ is the maximum number of vertices in a hyperedge. One might think that the partition dimension of $H$ is always greater than or equal to the rank of $H$. This is true for an $n$-uniform linear hyperpath and an $n$-uniform linear hypercycle $C_{k,3}$. But, in general, it is not true as shown in the following example:

**Example 3.5.** Let $H$ be a hypergraph with vertex set $V(H) = \{v_i : 1 \leq i \leq 11\}$ and edge set $E(H) = \{E_1, E_2\}$, where $E_1 = \{v_i: 1 \leq i \leq 7\}$ and $E_2 = \{v_i: 6 \leq i \leq 11\}$. Clearly, $\text{rank}(H) = 7$, $\lambda = 5$ and $\Pi = \{S_i = \{v_i, v_{i+5}\}: 1 \leq i \leq 5, S_6 = \{v_{11}\}\}$ is a minimum resolving partition of $V(H)$. This implies that $\text{pd}(H) = 6 \neq \text{rank}(H)$.

Likewise the results on the metric dimension of the primal and dual graph of a hypergraph, we have the following two results on the partition dimension of the primal and dual graph of a hypergraph, respectively:

**Theorem 3.6.** Let $H$ be a hypergraph. Then $\text{pd}(H) = \text{pd}(\text{prim}(H))$.

**Theorem 3.7.** Let $H^*$ be the dual of a hypergraph $H$ and $M(H^*)$ be the middle graph of $H^*$. Then $\text{pd}(H^*) = \text{pd}(M(H^*))$.

Since, it was shown that the simple paths $P_m$ are the only graphs with $\text{pd}(P_m) = 2$ [3] and the partition dimension of the simple cycles $C_m$ is $3$, so, by Theorem [3,7] we have the following corollaries:

**Corollary 3.8.** Let $H^*$ be the dual of a hypergraph $H$. Then $\text{pd}(H^*) = 2$ if and only if $H$ is a hyperpath.

**Corollary 3.9.** Let $H^*$ be the dual of a hypercycle $H$. Then $\text{pd}(H^*) = 3$.

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