STRONG LAW OF LARGE NUMBERS FOR RANDOM VARIABLES WITH MULTIDIMENSIONAL INDICES

BY

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Abstract. Let \( \{X_{\mathbf{n}} ; \mathbf{n} \in V \subset \mathbb{N}^2\} \) be a two-dimensional random field of independent identically distributed random variables indexed by some subset \( V \) of lattice \( \mathbb{N}^2 \). For some sets \( V \) the strong law of large numbers

\[
\lim_{n \to \infty} \frac{\sum_{k \in V; \mathbf{n} \leq \mathbf{k} < \mathbf{n}} X_k}{|n|} = \mu \text{ a.s.}
\]

is equivalent to

\[
EX_1 = \mu \quad \text{and} \quad \sum_{n \in V} P[|X_1| > |n|] < \infty.
\]

In this paper we characterize such sets \( V \).

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1. INTRODUCTION

Let \( \{X_{\mathbf{n}} ; \mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d\} \) be a family of independent identically distributed random variables indexed by \( \mathbb{N}^d \)-vectors, and let us put

\[
S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_k, \quad \mathbf{n} \in \mathbb{N}^d,
\]

where \( \mathbf{k} \leq \mathbf{n} \) iff \( k_j \leq n_j, j = 1, 2, \ldots, d \). In this paper we investigate the almost sure behavior of the sums \( S_{\mathbf{n}} \) when \( |n| \overset{\text{def}}{=} \prod_{j=1}^{d} n_j \to \infty \), i.e., the strong law of large numbers (SLLN).

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In the case of \( d = 1 \) the classical Kolmogorov’s SLLN result asserts that

\[
S_n \left\langle \frac{n}{n} \right\rangle \to \mu \quad \text{a.s.}
\]

(1.1)

is equivalent to

\[
EX = \mu, \quad E|X| < \infty,
\]

(1.2)

where here and in what follows \( X = X_1 \). The proof of Kolmogorov’s SLLN is based on the fact that for \( d = 1 \) the relation (1.1) is equivalent to

\[
\forall \varepsilon > 0 \quad P\left( \left| \frac{S_n}{n} - \mu \right| \geq \varepsilon, \text{infinitely often} \right) = 0.
\]

(1.3)

This is not the case if \( d > 1 \), since (1.1) is weaker than (1.3) even for i.i.d. random fields. Fortunately, Smythe [8] (Proposition 3.1, p. 913) observed that for i.i.d. random fields satisfying \( E|X| < \infty \) (this is obviously necessary for (1.1) to hold) relations (1.1) and (1.3) are equivalent. Moreover, Smythe [7] proved that (1.3) is equivalent to

\[
EX = \mu, \quad E|X| (\log_+ |X|)^{d-1} < \infty.
\]

(1.4)

Let us notice that the sufficiency of (1.4) was obtained in a more general setting of non-commutative ergodic transformations much earlier by Dunford [1] (see also Zygmund [10]).

It was Gabriel [2] who first observed that if we replace the whole lattice \( \mathbb{N}^d \) with a sector \( V_0^d = \{ \mathbf{n} : \theta n_i \leq n_j \leq \theta^{-1} n_i, i \neq j, i, j = 1, 2, \ldots, d \} \), then the situation is completely analogous to the one-dimensional case, namely \( E|X| < +\infty \) if and only if

\[
\lim_{V} \frac{S_n}{n} \text{ exists a.s.,}
\]

and then the limit is, of course, equal to \( EX \). Here \( \lim_{V} c_n = c_0 \) means that for every \( \epsilon > 0 \) we have \( |c_n - c_0| < \epsilon \) for all but a finite number of \( n \in V \). (We refer also to [3] for the sectorial Marcinkiewicz–Zygmund laws of large numbers.)

Later, Klesov and Rychlik [6] and Indlekofer and Klesov [4] proved that for a large class of subsets \( V \subset \mathbb{N}^d \) the SLLN along \( V \), i.e.

\[
\lim_{V} \frac{S_n}{n} = EX \quad \text{a.s.,}
\]

(1.5)

is equivalent to

\[
\sum_{n \in V} P(|X| \geq |n|) < +\infty.
\]

(1.6)
The relation (1.6) can be written in terms of the Dirichlet divisors. For \( V \subset \mathbb{N}^d \) let us define

\[
\tau_V(n) = \text{card}\{k \in V : |k| = n\}, \quad T_V(x) = \sum_{k \leq x} \tau_V(k).
\]

By the very definition we have

\[
\sum_{n \in V} P[|X| \geq |n|] = ET_V(|X|),
\]

hence (1.6) can be verified if we are able to determine the asymptotics of \( T_V \). For example, using methods of number theory, one can show that

\[
T_{\mathbb{N}^d}(x) \sim nw_{d-1}(\log x),
\]

where \( w_{k-1} \) is a polynomial of degree \( k - 1 \). This in turn leads to (1.4) as a necessary and sufficient condition for (1.1) and rediscovering a result of Smythe [8].

In fact, the results of [4] and [6] were proved for the case \( d = 2 \) only. We shall describe them briefly. Let us introduce the following classes of nonnegative functions on \( \mathbb{N}^2 \):

\[
F_1 \overset{\text{def}}{=} \{ f : f \nearrow, x \leq f(x), f(x)/x \nearrow \}, \\
G_1 \overset{\text{def}}{=} \{ g : g \nearrow, g(x) \leq x, g(x)/x \searrow \}, \\
F_2 \overset{\text{def}}{=} \{ f : f \text{ is nondecreasing, } x \leq f(x) \}, \\
G_2 \overset{\text{def}}{=} \{ g : g \text{ is nondecreasing, } g(x) \leq x \}.
\]

By \( C(F_i, G_i), i = 1, 2 \), we will denote the class of subsets \( V \subset \mathbb{N}^2 \) of the form

\[
V = V(f, g) = \{ \underline{n} = (n_1, n_2) : g(n_1) \leq n_2 \leq f(n_1) \},
\]

where \( f \in F_i, g \in G_i \). Then the main result of [3] states that the class \( C(F_1, G_1) \) consists of good sets, i.e. such that (1.5) is equivalent to (1.6), while the paper [5] proves that a larger class \( C(F_2, G_2) \) has this property as well.

The purpose of the present paper is to indicate some other classes of subsets of \( \mathbb{N}^2 \), which are determined by classes of functions \( F_j \) and \( G_j \), exhibiting less regularity in comparison with \( C(F_2, G_2) \), but still containing \( C(F_2, G_2) \). In the next section we provide three theorems, each exploiting a different direction, as Example 2.1 shows:

(i) We smooth out the boundaries from up and down and evaluate the difference of series (1.6) for these boundaries.

(ii) We introduce the usual order for the boundaries with a finite number of oscillations.
(iii) We smooth on the boundaries from the bottom and evaluate the measure of area between the smoothed and original boundaries.

Throughout the paper, $c$ denotes the generic constants different in different places, perhaps. All functions in the families $F$ and $G$ considered in this paper always satisfy additionally $f(x) \geq x$, $x \in \mathbb{R}_+$, and $0 < g(x) \leq x$, $x \in \mathbb{R}_+$, respectively. We will use the inverse function for not necessarily strictly monotone and continuous functions putting $f^{-1}(y) = \inf\{x \in \mathbb{R}_+ : f(x-0) \leq y \leq f(x+0)\}$ and $f^{-\uparrow}(y) = \sup\{x \in \mathbb{R}_+ : f(x-0) \leq y \leq f(x+0)\}$. Furthermore, for an arbitrary graph $\Gamma = \{(x, f(x)) : x \in X\}$, where $X \subseteq \mathbb{R}$, we define the $\mathbb{N}^2$ boundary of $\Gamma$ by

$$\partial \Delta_f = \{(i, j) \in \mathbb{N}^2 : \exists (i_1, j_1), (i_2, j_2) \in \{(i, j), (i+1, j), (i, j+1), (i+1, j+1)\} \text{ s.t. } f(i_1) < j_1, f(i_2) > j_2\}$$

(obviously, this definition obeys the case when $f$ is a function). In the whole paper we note $x \lor y = \max\{x, y\}$, $x \land y = \min\{x, y\}$, $\log_+ x = \max\{\log x, 0\}$, and $\log x$ denotes the natural logarithm.

2. MAIN RESULTS

For an arbitrary function $f \in \mathbb{R}_+^{\mathbb{N}^2}$, we put

$$\underline{f}(x) = \inf_{u \geq x} f(u), \quad \overline{f}(x) = \sup_{0 \leq u \leq x} f(u).$$

It is easy to check that

(i) $\underline{f}(x)$ is nondecreasing, $\overline{f}(x)$ is nondecreasing,

(ii) $\underline{f}(x) \leq f(x) \leq \overline{f}(x)$, $x \in \mathbb{R}_+$,

(iii) for $f(x)$ nondecreasing or $f(x)$ nonincreasing, $\underline{f}(x) = f(x) = \overline{f}(x)$.

Furthermore, for two functions $f, g$ we put

$$\nabla = \nabla(f, g) = \nabla(\overline{f}, \overline{g}),$$

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(for fixed $f, g$ we will often omit arguments), and for arbitrary families of the functions $F$ and $G$ let us define

$$\overline{C}(F, G) = \{\nabla(f, g) : f \in F, g \in G\},$$

$$\overline{C}(F, G) = \{\nabla(f, g) : f \in F, g \in G\}.$$

Moreover, let us define the families of the functions $\{F_3, G_3\}$ as follows:

$$F_3 = \left\{ f : \int_0^\infty \frac{\log \left( \overline{f}(x) \lor e \right)}{x \lor 1} \, dx < \infty \right\}, \quad G_3 = \left\{ g : \int_0^\infty \frac{\log \left( \underline{g}(x) \lor e \right)}{x \lor 1} \, dx < \infty \right\}.$$
The class $C(F_3, G_3)$ consists of good sets.

Let $B_f(y)$ denote the minimal family of connected subsets of the set $\{(x, y) : f(x) < y\}$ (minimal means that for every $B_1 \in B_f(y)$, $B_2 \in B_f(y)$, $B_1 \neq B_2$, $B_1 \cup B_2$ is disconnected). Let us note that all sets of the family $B_f(y)$ are subsets $[0, y] \times \{y\}$. Furthermore, let $K_f(y) := \text{card}\{B_f(y)\}$. Let us define

$$F_4 = \{f : \sup_{n \in \mathbb{N}} K_f(n) < \infty\}, \quad G_4 = \{g : \sup_{n \in \mathbb{N}} K_g(n) < \infty\}.$$

Theorem 2.2. The class $C(F_4, G_4)$ consists of good sets.

Now we consider the families:

$$F_5 = \left\{ f : \forall x, y \in \mathbb{R} \exists n \in \mathbb{N} : \left\lfloor \log (|y - f(x)|) \right\rfloor \log \left( \frac{|x - f(x)|}{|x - y|} \right) \leq cy \right\},$$

$$G_5 = \left\{ g : \forall x, y \in \mathbb{R} \exists n \in \mathbb{N} : \left\lfloor \log (|x - g(x)|) \right\rfloor \log \left( \frac{|y - g(x)|}{|x - y|} \right) \leq cy \right\},$$

$$F_6 = \left\{ f : \forall x \in \mathbb{N} : \left\lfloor \log (|f(x)|) \right\rfloor \log \left( \frac{|f(x)|}{|x|} \right) \leq cf(x) \right\},$$

$$G_6 = \left\{ g : \forall x \in \mathbb{N} : \left\lfloor \log (|g(x)|) \right\rfloor \log \left( \frac{|g(x)|}{|x|} \right) \leq cg(x) \right\},$$

$$F_7 = \left\{ f : \forall x, y \in \mathbb{R} \exists n \in \mathbb{N} : \left\lfloor \log (|f(x) - f(y)|) \right\rfloor \log \left( \frac{|f(x) - f(y)|}{|y|} \right) \leq cf^{-1}(y) \right\},$$

$$G_7 = \left\{ g : \forall x, y \in \mathbb{R} \exists n \in \mathbb{N} : \left\lfloor \log (|g(x) - g(y)|) \right\rfloor \log \left( \frac{|g(x) - g(y)|}{|y|} \right) \leq cg^{-1}(y) \right\}.$$

Theorem 2.3. The class $C(F_5, G_5)$ consists of good sets.

It is obvious that if $F \subset F', G \subset G'$, and the class $C(F', G')$ consists of good sets, then the class $C(F, G)$ consists also of good sets.

Remark 2.1. The following inclusions are true:

$$F_6 \cup F_7 \subset F_5, \quad G_6 \cup G_7 \subset G_5.$$

Because for $f$ nondecreasing and $g$ nondecreasing we have $f = f = \overline{f}$, $g = g = \overline{g}$ and $K_f(y) = 1$, $K_g(y) = 1$, we get

Corollary 2.1. The following inclusions are true:

$$F_1 \subset F_2 \subset F_i \quad \text{and} \quad G_1 \subset G_2 \subset G_i \quad \text{for } i = 3, 4, 5, 6, 7.$$

Therefore, all our Theorems 2.1, 2.2 generalize the main results of [4] and [6].
EXAMPLE 2.1. We will consider the class of functions
\begin{equation}
    f(x) = u(x) + g(x) \cos(h(x)\pi)
\end{equation}
for nondecreasing positive functions \(g\) and \(u\), with \(u(x) \geq x\), and an arbitrary function \(h\). Notice that we always have \(f(x) = u(x) + g(x)\) and \(f(x) = u(x)\).

(i) If \(u(x) = 2^x(\log_2 x)^2, g(x) = 2^x, h(x) = 2^x(\log_2 x)^2, x \in \mathbb{R}\), then the assumptions of Theorem 2.1 are satisfied, but those of Theorems 2.2 and 2.3 fail.

(ii) If \(u(x) = x, g(x) = x, h(x) = (x - 2^k)/2^{k-1}, x \in \mathbb{R}, k = \lceil \log_2 x \rceil\), then the assumptions of Theorem 2.2 hold, but those of Theorems 2.1 and 2.3 fail.

(iii) If \(u(x) = x, g(x) = x/\log x, h(x) = 2^x, x \in \mathbb{R}\), then the assumptions of Theorem 2.3 are satisfied, but those of Theorems 2.1 and 2.2 fail.

3. PROOFS

Proof of Theorem 2.1. From Theorem 1 in \[4\] we infer that for arbitrary families of the functions \(F, G\) the conditions for both the classes \(C(F, G)\) and \(\overline{C}(F, G)\) to consist of good sets are satisfied, i.e.

\[(\sum_{n \in V} P[|X| \geq |n|] < \infty \text{ and } EX = \mu) \iff \lim_{V} \frac{S_n}{|n|} = \mu, \]

and

\[(\sum_{n \in V} P[|X| \geq |n|] < \infty \text{ and } EX = \mu) \iff \lim_{V} \frac{S_n}{|n|} = \mu. \]

If additionally we show that, for every fixed \(f \in F_3, g \in G_3\),

\[(\sum_{n \in V} P[|X| \geq |n|] < \infty \text{ and } EX = \mu) \iff (\sum_{n \in V} P[|X| \geq |n|] < \infty \text{ and } EX = \mu) \]

then the assertion follows from the chain of implications

\[(\sum_{n \in V} P[|X| \geq |n|] < \infty \text{ and } EX = \mu) \implies (\sum_{n \in V} P[|X| \geq |n|] < \infty \text{ and } EX = \mu) \]

\[\implies (\lim_{V} \frac{S_n}{|n|} = \mu) \implies (\lim_{V} \frac{S_n}{|n|} = \mu) \implies (\lim_{V} \frac{S_n}{|n|} = \mu) \]

\[(\sum_{n \in V} P[|X| \geq |n|] < \infty \text{ and } EX = \mu) \implies (\sum_{n \in V} P[|X| \geq |n|] < \infty \text{ and } EX = \mu), \]

so that it is enough to prove (3.1). From the above considerations we may and do assume that \(EX = \mu\), i.e. \(E|X| < \infty\).
Because for each nonincreasing function $h$ and nondecreasing $t$ we have

$$\sum_{n=1}^{\infty} h(n) \leq \int_{0}^{\infty} h(x) \wedge h(1) dx, \quad \sum_{n \in \partial \triangle t} P[|X| \geq |n|] \leq E \sqrt{|X|}$$

(for the last inequality see the proof of Lemma 2 in [3]), and

$$\sum_{n \in \mathcal{V} \setminus \mathcal{V}} P[|X| \geq |n|] \leq \sum_{n \in \mathcal{V} \setminus \mathcal{V}} E|X|$$

we obtain

$$\sum_{n \in \mathcal{V} \setminus \mathcal{V}} P[|X| \geq |n|] \leq E|X|$$

$$\int_{0}^{\infty} \frac{1}{(x_1 \vee 1)(x_2 \vee 1)} dx_1 dx_2 \quad \sum_{n \in \partial \triangle t} P[|X| \geq |n|] + \sum_{n \in \partial \triangle t} P[|X| \geq |n|] + \sum_{n \in \partial \triangle t} P[|X| \geq |n|] + \sum_{n \in \partial \triangle t} P[|X| \geq |n|]$$

$$\leq E|X|I_1 + E|X|I_2 + 4E \sqrt{|X|}, \text{ say.}$$

Now we show how to evaluate $I_1$.

First we remark that because for $0 \leq a \leq b < \infty$ we have

$$\int_{0}^{b} \frac{1}{x \vee 1} dx = \begin{cases} \log(b/a) & \text{if } 1 \leq a \leq b, \\ \log(b) + (1 - a) & \text{if } a < 1 \leq b, \\ -a & \text{if } a \leq b \leq 1, \end{cases}$$

and for $a < 1$ we get $\log \frac{b \vee c}{a \vee 1} > 1$, the following inequality holds true:

$$\int_{0}^{b} \frac{1}{x \vee 1} dx \leq 2 \log \frac{b \vee c}{a \vee 1}.$$ 

Therefore,

$$I_1 \leq \int_{0}^{\infty} \int_{f(x)}^{x_1} \frac{1}{x_2 \vee 1} dx_2 \frac{1}{x_1 \vee 1} dx_1 \leq 2 \int_{0}^{\infty} \frac{\log \left( \frac{f(x) \vee c}{f(x) \vee 1} \right)}{x \vee 1} dx < \infty,$$

and similarly for $I_2 < \infty$. ■
For the proof of Theorem 3.1 let us notice that the functions \( f \) and \( g \) from the families \( F_1 \) and \( G_1 \), respectively, can be discontinuous. If, e.g., \( f(x_0 - 0) = y_0 < y_1 = f(x_0 + 0) \), then we “complete” the definition putting \( f(x_0) = [y_0, y_1] \) (the whole interval \([y_0, y_1]\)). Obviously, at this moment \( \Gamma = \{(x, f(x)), x \in \mathbb{R}\} \) is not a function, but a continuous graph, and \( f \) is a relation. However, we will write later “function \( f \)”, so that it does not cause misunderstanding. We say that the piecewise continuous graph \( \{(x, f(x)), x \in X\} \) for \( X \subset \mathbb{R} \) satisfies the condition \( G \) if

**Condition \( G \)**. If \( \{(x, f(x)), x \in (x_0, x_1)\} \) and \( \{(x, f(x)), x \in (x_2, x_3)\} \) are two pieces where the graph is continuous and \( x_1 < x_2 \), then \( f(x_0) \leq f(x_3) \).

For such graphs we have

**Proposition 3.1**. Let \( \{(x, f(x)), x \in X\} \), where \( X \subset \mathbb{R} \), be a piecewise nonincreasing graph satisfying the condition \( G \). Then

\[
\sum_{(i,j) \in \partial \Delta_f} P[|X| > ij] \leq 4E|X|.
\]

**Proof of Proposition 3.1**. By \( Q(i, j) \) we denote the square \( \{(x, y) \in \mathbb{R}^2 : i < x \leq i + 1, j \leq y < j + 1\} \).

Let us consider one piece of the graph \( \Gamma = \{(x, f(x)), x \in (x_0, x_1)\} \) on which the graph is continuous (and it is not continuous or even does not exist at \( x_1 \)).

The boundary of this piece of the graph can be expressed as a subset \( P_1 \) (may be empty) of the path \( P = [(i, j), \ldots, (i + k, j - l)] \) for some positive integers \( i, j, k, l \), where if \((i_1, j_1)\) and \((i_2, j_2)\) are subsequent points, then \((i_2, j_2)\) is equal to \((i_1 + 1, j_1)\) or \((i_1, j_1 - 1)\), or \((i_1 + 1, j_1 - 1)\) according to the way the graph \( \Gamma \) “goes out” from \( Q(i_1, j_1) \) and “enters” \( Q(i_2, j_2) \). If the graph \( \Gamma \) does not “enter” the interior \( Q(i_2, j_2) \), then \((i_2, j_2) \notin P_1 \), but obviously \((i_2, j_2) \in P \).

For such paths \( P \) and \( P_1 \) we construct a function \( H \) defined on \( \Delta_f \) and taking values in \( \{(x, 1) : x \in \mathbb{N}\} \cup \{(1, y) : y \in \mathbb{N}\} \) as follows:

\[
H((i_1, j_1)) = (i_1, 1),
\]

\[
H((i_k, j_k)) = \begin{cases} (i_k, 1) & \text{if } i_k > i_{k-1}, \\ (1, j_k) & \text{if } i_k = i_{k-1}. \end{cases}
\]

On the piece \( (x_0, x_1) \) we have

\[
H(\Delta_{f|_{x \in (x_0, x_1)}}) \subset \{(i, 1), (i + 1, 1), \ldots, (i + k, 1), (1, j), (1, j - 1), \ldots, (1, j - l)\},
\]

and \( H \) is the injective function (in this area), where \( f|_{x \in (x_0, x_1)} \) denotes the restriction of the function \( f \) to the interval \( (x_0, x_1) \). Obviously, because for every point
\[(i, j) \in (\mathbb{N}\setminus\{0\})^2\] we have \(ij > \max\{i, j\}\), it follows that

\[\sum_{(i,j) \in \Delta_f \cap (x_0, x_1)} P[|X| > ij] < \sum_{(i,j) \in H(\Delta_f \cap (x_0, x_1))} P[|X| > ij].\]  

(3.3)

It may happen then that one continuous piece of the graph \(\Gamma\) has a path of boundaries \([(i, j), \ldots, (i + k, j - l)]\), whereas the next continuous piece of the graph contains a point \((i + k, j)\), and in this case the projection \(H\) may transform \((i + k, j)\) into the existing point \((i + k, 1)\) or \((1, j)\); consequently,

\[\sum_{i \leq 1} P[|X| > ij] < 2 \sum_{i \leq H(\partial_f)} P[|X| > ij] < 4 \sum_{i = 1}^{\infty} P[|X| > i] = |X|,

(3.4)

which completes the proof. □

**Proof of Theorem 2.2.** Without loss of generality we assume \(EX = 0\). We consider only the sector \(\{(m, n) \in \mathbb{R}^2 : m \leq n\}\) and the family of functions \(F_4\) since in the case \(G_4\) the proof runs similarly. For the function \(f : \mathbb{R} \rightarrow \mathbb{R}\), such that \(f(x) > x\) and every \(y \in \mathbb{R}\), we define the partition of the interval \([0, y]\) by \(B_f(y) + A_f(y)\) by \(B_f(y) = \{(x, y) : f(x) < y\}\), \(A_f(y) = \{(x, y) : f(x) \geq y\}\), and

\[B_f(y) = ([0, x_1) \times \{y\}) \cup ([x_2, x_3) \times \{y\}) \cup \ldots \cup ([x_{K_f(y) - 1}, x_{K_f(y)}) \times \{y\})\]

\[= \bigcup_{k = 1}^{K_f(y)} B_k(f, n), \]

\[A_f(y) = ([x_1, x_2) \times \{y\}) \cup ([x_3, x_4) \times \{y\}) \cup \ldots \cup ([x_{K_f(y)}, y) \times \{y\})\]

\[= \bigcup_{k = 1}^{K_f(y)} A_k(f, y), \quad 0 < x_1 < x_2 < x_3 < \ldots < x_{K_f(y)} < y,\]

for some finite (the definition of the family \(F_4\)) integers \(K_f(y) \in \mathbb{N}\). We put \(K = \sup\{K_f(y) : y \in \mathbb{R}\}\). For each \(y\) we complete the families \(B(f, y) = \{B_k(f, y), 1 \leq k \leq K_f(y)\}\) putting \(B_k(f, y) = \emptyset\) for \(k = K_f(y) + 1, K_f(y) + 2, \ldots, K\). Immediately, from the definition of this family we have the property

\[\forall y_1 < y_2 \forall 1 \leq i \leq K \exists 1 \leq j \leq B_k(f, y_1) \subset B_i(f, y_2).\]

Thus, on the base of the family \(B(f, y)\) we define the family

\[\Gamma_k(y) = \bigcup_{i = 1}^{k} \bigcup_{1 \leq t \leq y \cup B_i(f, t) \subset B(f, y), 1 \leq j \leq K} B_j(f, t), \quad 1 \leq k \leq K.\]
Furthermore, for every $1 \leq k \leq K$ we put

$$A(k) = \bigcup_{y \in \mathbb{R}} A_k(f, y), \quad k = 1, 2, 3, \ldots, K.$$  

We explain the introduced families in Figure 1.

![Figure 1: The partition of the graph on the areas $A(i), 1 \leq i \leq K$](image)

It is easy to check that Lemma 1 and the proof of Theorem 1 in [4] hold for the sequences $\{\tau_k, k \in \mathbb{N}\} \subset A(k)$ and the increasing sequences of sums of random variables

$$Y_n(k) = \sum_{m \in \Gamma_k(n_2) \cap \mathbb{N}^2} X_m = \sum_{m \in [1, n_1] \times [1, n_2] \cap \mathbb{B}} X_m, \quad n \in A(k),$$

iff only $A(k)$ is not bounded for $k = 1, 2, 3, \ldots, K$. Some comments are required about the fulfilling of Lemma 2 in [4] for the boundaries of our sets $A(k)$. The boundary of such sets can be divided by at most $K$ graphs $\Xi_i, 1 \leq i \leq K$, piece-wise continuous and increasing (in Figure 1 we mark three such graphs: $a$, $b$, and $c$, respectively) and at most $K$ graphs $\Upsilon_i, 1 \leq i \leq K$, piecewise continuous and decreasing (in Figure 1 we mark two such graphs: $d$ and $e$, respectively). For each graph from the family $\Xi_i, 1 \leq i \leq K$, we intermediately use Lemma 2 of [4], whereas for the graphs from the family $\Upsilon_i, 1 \leq i \leq K$, we use our Proposition 3.1.

Thus, using the notation of [4],

$$\lim_{n \in A(k)} \frac{Y_n(k)}{[1, n_1] \times [1, n_2] \cap \mathbb{B}} = 0, \quad k = 1, 2, 3, \ldots, K,$$
and because each subsequence \( \mathcal{N} = \{ n_i \in A, i \in \mathbb{N} \} \) can be divided into \( K \) sub-
sequences \( \mathcal{N} \cap A(k) \), the assertion holds. ■

Note that in the above proof we use only the definitions of \( \{ A_i(f, y), B_i(f, y), \Gamma_i(y) \} \) for integer \( y \)'s. Therefore, we restrict ourselves in the definitions of \( F_i \) and \( G_i \), and \( K_f(y) \) and \( K_y(y) \) for integer \( y \)'s, only.

**Proof of Theorem 2.3.** We show that if

\[
\lim_{V \to \infty} \frac{S_k}{n} = EX,
\]

then

\[
\lim_{V \to \infty} \frac{S_k}{n} = EX.
\]

Obviously, (3.5) follows from Theorem 1 in [4]. Then we have \( E|X| < \infty \). Furthermore, we define four functions:

\[
\begin{align*}
M_1 & : V \to V, \quad M_1((k_1, k_2)) = (k_1, [f(k_1)]), \\
M_2 & : V \to V, \quad M_2((k_1, k_2)) = ([f^{-1}(k_2)], k_2), \\
M_3 & : V \to V, \quad M_3((k_1, k_2)) = (k_1, [g(k_1)]), \\
M_4 & : V \to V, \quad M_4((k_1, k_2)) = ([g^{-1}(k_2)], k_2).
\end{align*}
\]

Obviously, as \( M_i(k_1, k_2) \in V, i = 1, 2, 3, 4 \), from (3.5) we have

\[
\lim_{|n| \to \infty} \frac{S_{M_i(n)}}{|M_i(n)|} = EX, \quad i = 1, 2, 3, 4.
\]

Let the sequence \( \{ n_k = (n_{1,k}, n_{2,k}), k \in \mathbb{N} \} \subset V \cap V \) be such that \( |n_k| \to \infty \), and let

\[
\{ n_{i,k} \in \mathbb{N} \} = \bigcup_{i=1}^{4} \{ n_{i,k}^{(i)} = (n_{1,k}^{(i)}, n_{2,k}^{(i)}) \},
\]

be four subsequences such that

\[
\begin{align*}
[f(n_{1,k}^{(1)}) - f(n_{1,k}^{(1)})] \log_+ (n_{1,k}^{(1)} [f(n_{1,k}^{(1)}) - f(n_{1,k}^{(1)})]) & \leq c f(n_{1,k}^{(1)}), \\
[f^{-1}(n_{2,k}^{(2)}) - f^{-1}(n_{2,k}^{(2)})] \log_+ (n_{2,k}^{(2)} [f^{-1}(n_{2,k}^{(2)}) - f^{-1}(n_{2,k}^{(2)})]) & \leq c f^{-1}(n_{2,k}^{(2)}), \\
[g(n_{1,k}^{(3)}) - g(n_{1,k}^{(3)})] \log_+ (n_{1,k}^{(3)} [g(n_{1,k}^{(3)}) - g(n_{1,k}^{(3)})]) & \leq c g(n_{1,k}^{(3)}), \\
[g^{-1}(n_{2,k}^{(4)}) - g^{-1}(n_{2,k}^{(4)})] \log_+ (n_{2,k}^{(4)} [g^{-1}(n_{2,k}^{(4)}) - g^{-1}(n_{2,k}^{(4)})]) & \leq c g^{-1}(n_{2,k}^{(4)}).
\end{align*}
\]
At least one of the above-defined subsequences is infinite (we denote the set of such subsequences by $I$).

Let us remark that for $x > y > 0$ we have $\lfloor x \rfloor - \lfloor y \rfloor \leq [x - y]$. Indeed, if $x - y$ is an integer, then $\lfloor x \rfloor - \lfloor y \rfloor = x - y = [x - y]$. On the other hand, since for arbitrary $z \in (0, 2)$ we have $\lfloor z \rfloor /|z| = 1$, it follows that

$$\lfloor x \rfloor - \lfloor y \rfloor = \lfloor x - y \rfloor = \lfloor x - y + \{x\} \rfloor = [x - y + \{x\}] = [x - y + \{x\} + \{y\}] = \lfloor x - y + \{x\} + \{y\} \rfloor \leq [x - y] + 1 = [x - y].$$

Therefore, the subsequences defined as above satisfy

(3.8) $$\lim\sup_{k \to \infty} \left( \frac{|n_{ik}| - |M_i(n_{ik})|}{|n_{ik}|} \left( \log_+ \left( \frac{|n_{ik}| - |M_i(n_{ik})|}{|n_{ik}|} \right) \lor 1 \right) \right) \leq c < \infty, \quad i \in I,$$

and, in consequence, because $\lim_{V \to \infty} \log_+ \left( \frac{|n_{ik}| - |M_i(n_{ik})|}{|n_{ik}|} \right) = +\infty$ or $|n_{ik}| = |M_i(n_{ik})|$, $k \in \mathbb{N}$, we obtain

(3.9) $$\lim\sup_{k \to \infty} \frac{|M_i(n_{ik})|}{|n_{ik}|} = 1, \quad i \in I.$$

On the other hand, let us remark that

$$S_{n_k} - S_{M_i(n_k)} \sim S_{n_k} - M_i(n_k),$$

and from Theorem 1 in [5] we have

$$\lim_{k \to \infty} \frac{S_{n_k} - ES_{n_k} - S_{M_i(n_k)} + ES_{M_i(n_k)}}{\left( |n_{ik}| - |M_i(n_{ik})| \right) \left( \log_+ \left( \frac{|n_{ik}| - |M_i(n_{ik})|}{|n_{ik}|} \right) \lor 1 \right)} = 0, \quad i \in I.$$

Because for $i \in I$

(3.10) $$\lim_{k \to \infty} \frac{-ES_{n_{ik}} + ES_{M_i(n_{ik})}}{\left( |n_{ik}| - |M_i(n_{ik})| \right) \left( \log_+ \left( \frac{|n_{ik}| - |M_i(n_{ik})|}{|n_{ik}|} \right) \lor 1 \right)} = \lim_{k \to \infty} \frac{-EX}{\log_+ \left( \frac{|n_{ik}| - |M_i(n_{ik})|}{|n_{ik}|} \right) \lor 1} = 0,$$
and
\[ \lim_{k \to \infty} S_{n_k} = \lim_{k \to \infty} \left\{ S_{M_i(n_k)} - S_{M_i(n_k)} \right\}\frac{|n_k| - |M_i(n_k)|}{|n_k|} \left( \log^+ (|n_k| - |M_i(n_k)|) \lor 1 \right) \]
\[ \times \left( \frac{|n_k| - |M_i(n_k)|}{|n_k|} \left( \log^+ (|n_k| - |M_i(n_k)|) \lor 1 \right) \right) \]
\[ = EX \cdot 1 + 0 \cdot c = EX, \quad i \in I, \]

and, in consequence,
\[ \lim_{k \to \infty} S_{n_k} = EX, \]

the proof is completed. ■

**Proof of Example 2.1.** In all the three cases we have
\[ \int_0^\infty \log \left( \frac{f(x)\lor e}{f(x)\lor 1} \right) dx = \int_0^\infty \log \left( \frac{(u(x) + g(x))\lor e}{u(x)\lor 1} \right) dx, \]
\[ \log^+ \left( x \left| f(x) - f(x) \right| \right) \log^+ \left( x \right) \left| f(x) - f(x) \right| \log^+ \left( x \right) \left| g(x) \cos \left( h(x) \pi \right) \right| \log^+ \left( x \right) \left| g(x) \cos \left( h(x) \pi \right) \right| \]

In the case (i), because \( \log(1 + x) \leq x \), we have
\[ \int_1^\infty \log \left( 1 + 1/(\log x)^2 \right) dx \leq \int_1^\infty \frac{1}{x(\log x)^2} dx < \infty. \]

Let us define the sequence \( \{x_n, n \geq 1\} \) divergent to infinity, so that, for \( i \geq 1 \), \( 2^{x_i}(\log x_i)^2 \in \mathbb{N} \) (it is possible as the function \( 2^{x_i}(\log x)^2 \) is continuously increasing to infinity for \( x > 1 \)). Then for every constant \( c \) there exists \( i_0 \) such that, for every \( i > i_0 \),
\[ \left| 2^{x_i} \cos \left( 2^{x_i}(\log x_i)^2 \pi \right) \right| \log^+ \left( x_i \left| 2^{x_i} \cos \left( 2^{x_i}(\log x_i)^2 \pi \right) \right| \right) = 2^{x_i} \log x_i + x_i 2^{x_i} \log 2 \geq c(2^{x_i}(\log x_i)^2 + 2^{x_i}); \]
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thus the assumptions of Theorem 2.1 are satisfied, whereas the assumptions of Theorem 2.3 fail. Let us remark that, for arbitrary \( x \in \mathbb{N} \) in the interval \((x, y)\), the function \( f \) has at least \( 2^y(\log y)^2 - 2^x(\log x)^2 - 2 \) oscillations, where \( 2^y(\log y)^2 = 2^x[(\log x)^2 + 1] \). Therefore, for \( y > e \),

\[
K_f(y) \geq 2^y(\log y)^2 - 2^x(\log x)^2 - 2 \geq 2^x - 2,
\]

and \( K_f(y) \to \infty \) as \( y \to \infty \), so that the assumptions of Theorem 2.2 fail.

In the case (ii) we have

\[
\int_{1}^{\infty} \frac{\log(2)}{x} dx = \infty.
\]

Furthermore, it is easy to check that \( |\cos(h(x)\pi)| \) is equal to one only for \( x = 2^k \) or \( x = 3 \cdot 2^{k-1} \) and it is equal to zero only for \( x = 5 \cdot 2^{k-2} \) and \( x = 7 \cdot 2^{k-2} \) for \( k \in \mathbb{N} \). Thus, in the interval \( x \in [2^k, 2^{k+1}] \) the function \( f \) has two local minima at \( x = 5 \cdot 2^{k-2} \) and \( x = 7 \cdot 2^{k-2} \) equal to \( 5 \cdot 2^{k-2} \) and \( 7 \cdot 2^{k-2} \), respectively, and two local maxima at \( x = 2^k \) and \( x = 3 \cdot 2^{k-1} \) equal to \( 2^k+1 \) and \( 3 \cdot 2^k \), respectively, so that for every \( x \in \mathbb{R} \) we have \( K_f(x) \leq 4 \), and the assumptions of Theorem 2.3 are fulfilled. Taking \( x = k \in \mathbb{N} \), we see that for every constant \( c \) there exists a sufficiently large \( k \in \mathbb{N} \) such that

\[
[k|\cos(k\pi)|] \log_+ \left( k[k|\cos(k\pi)|] \right) = 2k \log k \geq ck;
\]

thus the assumptions of Theorem 2.3 fail.

In the case (iii) we have

\[
\int_{1}^{\infty} \frac{\log(1 + 1/\log x)}{x} dx = \infty,
\]

so that the assumptions of Theorem 2.3 fail. Failure of the assumptions of Theorem 2.2 follows from analogous considerations to those for the point (i). From

\[
\frac{x}{\log x} |\cos(2^x\pi)| \log \left( \frac{x^2}{\log x} |\cos(2^x\pi)| \right) \leq \frac{x}{\log x} \log x^2 = 2x \leq 2 \left( x + \frac{x}{\log x} |\cos(2^x\pi)| \right)
\]

we see that the assumptions of Theorem 2.3 are satisfied with \( c = 2 \).

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