Rate of approximation of $zf'(z)$ by special sums associated with the zeros of the Bessel polynomials

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Abstract. Let $\alpha_{n1}, \ldots, \alpha_{nn}$ be the zeros of the $n$th Bessel polynomial $y_n(z)$ and let $a_{nk} = 1 - \alpha_{nk}/2$, $b_{nk} = 1 + \alpha_{nk}/2$ ($k = 1, \ldots, n$). We propose the new formula

$$zf'(z) \approx \sum_{k=1}^{n} (f(a_{nk}z) - f(b_{nk}z))$$

for numerical differentiation of analytic functions $f(z) = \sum_{m=0}^{\infty} f_m z^m$. This formula is exact for all polynomials of degree at most $2n$. We find the sharp order of nonlocal estimate of the corresponding remainder for the case when all $|f_m| \leq 1$. The estimate shows a high rate of convergence of the differentiating sums to $zf'(z)$ on compact subsets of the open unit disk, namely, $O(0.85^n n^{1-n})$ as $n \to \infty$.

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1 Introduction

Throughout the paper we assume that $f$ is a function, analytic in a neighborhood of the origin, and put $f(z) = \sum_{m=0}^{\infty} f_m z^m$. Consider the elementary 2-node differentiation formula

$$zf'(z) \approx f(3z/2) - f(z/2).$$

(1)

We can easily seen that

$$zf'(z) - \{f(3z/2) - f(z/2)\} = O(z^3) \quad \text{as} \quad z \to 0,$$

whereas for any complex numbers $a, b$ such that $a \neq 3/2$ and/or $b \neq 1/2$ the difference $zf'(z) - \{f(az) - f(bz)\}$ is only $O(z^\gamma)$, $\gamma \leq 2$. Hence the formula (1) is exact for all polynomials of degree at most 2 and is optimal (by the order of local approximation) in the class of all formulas of the form $zf'(z) \approx f(az) - f(bz)$. Moreover, the nodes $3z/2$ and $z/2$ in (1) are independent of $f$.

In this note we construct the $2n$-node generalization of (1) which preserves the main features of (1) — the exactness for polynomials of degree at most $2n$ ($2n$ is the number of nodes), the property that the nodes are independent of $f$ and that the values of $f$ in them are summed with the multipliers $+1$ or $-1$ (some known similar differentiation formulas are discussed in Sec. 4).

We also find the sharp in order of the quantity $n$ nonlocal estimate of the corresponding remainder for the case of bounded Taylor coefficients, $f_m$. The estimate shows a very high rate of convergence of the differentiating sums to $zf'(z)$ on compact subsets of the disk $|z| < 1$, namely, $O(0.85^n n^{1-n})$ as $n \to \infty$. 

1
The nodes in our differentiation formula linearly depend on the zeros \( \alpha_{nk} \) \((k = 1, \ldots, n)\) of the \( n \)th Bessel polynomial \( y_n(z) \), first introduced in [9]:

\[
y_n(z) = \sum_{k=0}^{n} \frac{(n + k)!}{(n - k)!k!} \left( \frac{z}{2} \right)^k.
\]

Recall that all \( \alpha_{nk} \) are simple [6, p. 75] and lie in the semi-annulus

\[
\frac{1}{n + 2/3} \leq |\alpha_{nk}| \leq \frac{2}{n + 1}, \quad \text{Re} \alpha_{nk} < 0 \tag{2}
\]

[5, Sec. 5], and there are algorithms for the accurate computation of these zeros [10]. We also need the following well-known results on the power sums of \( \alpha_{nk} \): if \( j = 1, 2, \ldots \) and \( \sigma^n_j = \alpha^n_{n1} + \cdots + \alpha^n_{nj} \), then

\[
\sigma^n_1 = -1, \quad \sigma^n_{2j+1} = 0 \quad (j = 1, \ldots, n - 1); \tag{3}
\]

\[
\sigma^n_{2n+1} = (-1)^n A^n n!^2 / (2n)!^2. \tag{4}
\]

The formulas (3) and (4) were proved in [2] and in [7], respectively.

2 The differentiation formula and its local properties

For \( n = 1, 2, \ldots \), define the quantities

\[
a_{nk} = 1 - \alpha_{nk}/2, \quad b_{nk} = 1 + \alpha_{nk}/2 \quad (k = 1, \ldots, n)
\]

and introduce the main differentiation formula:

\[
zf'(z) \approx \sum_{k=1}^{n} (f(a_{nk} z) - f(b_{nk} z)). \tag{5}
\]

Since \( y_1(z) = 1 + z \), then \( \alpha_{1,1} = -1 \), such that the formula (1) is actually the particular case of (5) with \( n = 1 \).

Find the local representation of the remainder

\[
r_n(f; z) := zf'(z) - \sum_{k=1}^{n} (f(a_{nk} z) - f(b_{nk} z))
\]

of (5) near the origin. Obviously, we have

\[
r_n(f; z) = \sum_{m=1}^{\infty} m f_m z^m - \sum_{k=1}^{n} \sum_{m=1}^{\infty} (a^n_{nk} - b^n_{nk}) f_m z^m
\]

\[
= \sum_{m=1}^{\infty} (m - A^n_m) f_m z^m, \quad \text{where} \quad A^n_m := \sum_{k=1}^{n} (a^n_{nk} - b^n_{nk}). \tag{6}
\]

Proposition 1 For \( n = 1, 2, \ldots \),

\[
m - A^n_m = 0 \quad (m = 1, \ldots, 2n); \quad (2n + 1) - A^n_{2n+1} = (-1)^n \frac{n!^2}{(2n)!^2}.
\]
Proof. For $m = 1, 2, \ldots$, by applying the binomial formula, we have

$$(1 - t)^m - (1 + t)^m = -2 \sum_{j=0}^{M} \binom{m}{2j+1} t^{2j+1},$$

where $M = [(m - 1)/2]$ and $[x]$ denotes the greatest integer $\leq x$, so that

$$A_m^n = \sum_{k=1}^{n} \{(1 - \alpha_{nk}/2)^m - (1 + \alpha_{nk}/2)^m\} = -\sum_{j=0}^{M} \binom{m}{2j+1} \frac{\sigma_{2j+1}^n}{4^j}.$$ 

If $m \leq 2n$, then $M \leq n-1$ and $A_m^n = -\left(\frac{m}{1}\right)\sigma_1^n - 0 = m$ by (3). Analogously, if $m = 2n+1$, then $M = n$, so that

$$A_{2n+1}^n = -\left(\frac{2n+1}{1}\right)\sigma_1^n - 0 - \frac{2n+1}{2n+1} \frac{\sigma_{2n+1}^n}{4^n} = (2n+1) - (-1)^n \frac{n!}{(2n)!}$$

by (3) and (4). This completes the proof of Proposition 1.

Corollary 1 For $n = 1, 2, \ldots$, the remainder of (5) has the local form

$$r_n(f; z) = (-1)^n \frac{n!^2}{(2n)!^2} \cdot f_{2n+1} z^{2n+1} + O(z^{2n+2}) \quad \text{as} \quad z \to 0.$$ 

In particular, the sum in the right side of (5) interpolates the function $z f'(z)$ at zero with the multiplicity $2n + 1$ and $r_n(p; z) \equiv 0$ for all polynomials $p$ of degree at most $2n$.

We conclude this section with some properties of the quantities $a_{nk}, b_{nk}$.

Because of the properties of $\alpha_{nk}$ (see Sec. 1), all $a_{nk}$ and $b_{nk}$ ($k = 1, \ldots, n$) are pairwise distinct and tend to 1 as $n \to \infty$. We now prove

Proposition 2 For $n = 1, 2, \ldots$, $|a_{nk}| > 1$ and $|b_{nk}| < 1$ ($k = 1, \ldots, n$).

Proof. Since $\text{Re} \alpha_{nk} < 0$, then $|a_{nk}| \geq \text{Re} a_{nk} = 1 - \text{Re} \alpha_{nk}/2 > 1$.

To prove $|b_{nk}| < 1$ we need the more strong estimate [5, Eq. (6.2)]:

$$\text{Re} \alpha_{nk} < -\left[n^3(n + 1)\right]^{-1/2}.$$ 

Thus we actually have

$$|b_{nk}|^2 = (1 + \text{Re} \alpha_{nk}/2)^2 + (\text{Im} \alpha_{nk}/2)^2 = 1 + \text{Re} \alpha_{nk} + |\alpha_{nk}|^2/4$$

$$< 1 - \frac{1}{\sqrt{n^3(n + 1)}} + \frac{1}{(n + 1)^2} = 1 + \frac{1}{(n + 1)^2} \left[1 - \left(\frac{n + 1}{n}\right)^{3/2}\right] < 1,$$

and the assertion follows.
3 The nonlocal estimates of the interpolation error in (5)

Put \( n_0(x) = \max \{14; x^2(1 - x)^{-2}\} - 1. \)

**Theorem 1** Let \( x \in (0, 1) \) and all \( |f_m| \leq 1 \). If \( n \geq n_0(x) \), then
\[
|r_n(f; z)| \leq \frac{|z|^{2n+1}}{x - |z|} \frac{2x}{(1-x)^{2n+2}} \frac{n!^2}{(2n)!^2}, \quad |z| < x, \tag{7}
\]
and this estimate is sharp in order of the quantity \( n \) for \( |z| \approx x \).

In particular, if \( n \geq 13 \) and \( x = x_n = \sqrt{n}/(1 + \sqrt{n}) \), then
\[
|r_n(f; z)| \leq \frac{|z|^{2n+1}}{x_n - |z|} \frac{0.92^{2n}}{n^{n-1}}, \quad |z| < x_n. \tag{8}
\]

**Remark 1.** Under the condition \( |f_m| \leq 1 \), the function \( r_n(f; \cdot) \) is analytic in the disk \( |z| < (n+1)/(n+2) \). Indeed, \( f \) is analytic in the open unit disk \( D \), whereas
\[
|b_{nk}| < 1, \quad |a_{nk}| \leq 1 + |\alpha_{nk}|/2 \leq (n+2)/(n+1)
\]
(use Proposition 2 and \( |\alpha_{nk}| \leq 2/(n+1) \)), hence all the nodes \( a_{nk}z \) and \( b_{nk}z \) belong to \( D \) if \( |z| < (n+1)/(n+2) \). In particular, \( r_n(f; \cdot) \) is analytic for \( |z| < x \), since by \( n \geq n_0(x) \) we have \( x \leq \sqrt{n+1}/(1 + \sqrt{n+1}) < (n+1)/(n+2) \). Note that, generally speaking, \( r_n(f; \cdot) \) is not analytic in the whole disk \( D \) by \( |\alpha_{nk}| > 1 \).

**Remark 2.** The choice \( x_n = \sqrt{n}/(1 + \sqrt{n}) \) is admissible, since \( n \geq n_0(x_n) \) for \( n \geq 13 \).

**Proof.** First consider the model function \( g(z) = z/(1-z) = z + z^2 + \ldots \). Obviously, \( r_n(g; z) \equiv zR_n(z) \), where
\[
R_n(z) = \frac{1}{(z-1)^2} - \sum_{k=1}^{n} \frac{a_{nk}}{1 - a_{nk}z} + \sum_{k=1}^{n} \frac{b_{nk}}{1 - b_{nk}z}. \tag{9}
\]
The following assertion will be established in Sec. 5 and this proves, in particular, the sharpness of (7) for \( |z| \approx x \).

**Proposition 3** Let \( x \in (0, 1) \). If \( n \geq n_0(x) \), then
\[
\frac{x^{2n}}{(1-x)^{2n+2}} \frac{n!^2}{(2n)!^2} \leq \max_{|z|=x} |R_n(z)| < \frac{2x^{2n}}{(1-x)^{2n+2}} \frac{n!^2}{(2n)!^2}. \tag{10}
\]

Now assume \( x \in (0, 1), |z| < x \) and \( n \geq n_0(x) \). Then we have \( |a_{nk}z| < 1, |b_{nk}z| < 1 \) (see Remark 1) and
\[
R_n(z) = \sum_{m=1}^{\infty} (m - A_m^n) z^{m-1} = \sum_{m=2n+1}^{\infty} (m - A_m^n) z^{m-1}
\]
(see Proposition 1), therefore, by using the Cauchy inequalities
\[
|m - A_m^n| \leq x^{1-m} \max_{|z|=x} |R_n(z)|
\]
and the upper estimate in (10), we get
Corollary 2 Let \( x \in (0, 1) \). If \( n \geq n_0(x) \), then

\[
|m - A^n_m| < \frac{2x^{2n+1-m} n!^2}{(1-x)^{2n+2} (2n)!^2} \quad (m \geq 2n + 1).
\] (11)

(The right side in (11) tends to infinity as \( m \to \infty \), and the example \( A^1_m = (3^m - 1)/2^m \) shows that the quantities \( |m - A^n_m| \) are indeed unbounded as \( m \to \infty \).)

Thus (7) immediately follows from (6), \( |f_m| \leq 1 \), Proposition 1 and (11):

\[
|r_n(f; z)| \leq \sum_{m=2n+1}^{\infty} |m - A^n_m||z|^m \leq \frac{2|x|^{2n+1} n!^2}{(1-x)^{2n+2} (2n)!^2} \sum_{m=0}^{\infty} |z|^m.
\]

To prove (8), put \( x = x_n = \sqrt{n}/(1 + \sqrt{n}) \) in (7) and apply the Stirling formula \( n! = \sqrt{2\pi n}(n/e)^n c_n \), \( 0 < c_n < 1/(12n) \), and also the inequalities \( x_n < 1, (1+1/\sqrt{n})\sqrt{n} < e, c_n + \sqrt{n} + n^{-1/2} < 0.3n \ (n \geq 13) \) and \( e^{1/3}/4 < 0.92 \):

\[
\frac{2x_n}{(1 - x_n)^{2n+2} (2n)!^2} < \frac{(1 + \sqrt{n})^{2n+2} e^{2n+2c_n}}{4^{2n} n^{n-1}} < \frac{e^{2(c_n + \sqrt{n} + n^{-1/2})}}{4^{2n} n^{n-1}} < \frac{0.92^{2n}}{n^{n-1}}.
\]

This completes the proof of Theorem 1. Note that \( 0.92^2 < 0.85 \).

Analogously, by using the relation \( \lim_{m \to \infty} |f_m|^{1/m} \leq 1 \) for an arbitrary analytic in \( D \) function \( f \), we get the some more general result (see also [4, Sec. 1]):

Theorem 2 Let \( x \in (0, 1) \) and \( f \) is analytic in \( D \). If \( n \geq n_0(x) \), then for any \( \varepsilon \in (0, 1) \)

\[
|r_n(f; z)| \leq C(f, \varepsilon) \frac{(1-\varepsilon/2)^{2n} x^{2n+1}}{(1-x)^{2n+2}} \frac{n!^2}{(2n)!^2}, \quad |z| \leq (1-\varepsilon)x.
\]

4 Similar differentiation formulas

Putting \( f(z) = f(0) + zh(z) \) in (5), we get

\[
(zh(z))' = \sum_{k=1}^{n} \left( a_{nk}h(a_{nk}z) - b_{nk}h(b_{nk}z) \right) + O(z^{2n}).
\] (12)

The similar formula was obtained in [4, Sec. 2.4]:

\[
(zh(z))' = H_n(h; z) + O(z^n), \quad H_n(h; z) := \sum_{k=1}^{n} \lambda_k h(\lambda_k z).
\]

Here \{\lambda_k\} is the (unique) solution of the system \( \sum_{k=1}^{n} \lambda_k^m = m \ (m = 1, \ldots, n) \); \( \sum_{k=1}^{n} \nu_k h(\nu_k z) \) are so-called \( h \)-sums, so that the sum in (12) is the difference of \( h \)-sums. For the case when \( h \) is analytic in \( D \), the rate of the uniform convergence \( H_n(h; z) \to (zh(z))' \) as \( n \to \infty \) on compact subsets of \( D \) is geometric [4].
More recently, the other similar formula was constructed in [3, Sec. 5.4]:

\[
zf'(z) = \sum_{k=1}^{n} \mu_k f(\lambda_k z) - pf_{n-1} z^{n-1} - qf_{2n-1} z^{2n-1} + r_n(z),
\]

\[
r_n(z) = \frac{6np}{(n-1)(n-2)} \cdot f_{2n} z^{2n} + O(z^{2n+1}) \quad (n \geq 3).
\]

Here \(\{\mu_k, \lambda_k\}\) is the solution of the discrete moment problem

\[
\sum_{k=1}^{n} \mu_k \lambda_k^m = c_m, \quad m = 0, 1, \ldots, 2n-1; \quad \mu_k \neq 0; \quad \lambda_k \neq \lambda_j \text{ for } k \neq j,
\]

where \(c_m = m (m \neq n-1, 2n-1)\), \(c_{n-1} = n-1 + p\), \(c_{2n-1} = 2n-1 + q\); \(p, q\) are constant parameters for the unique solvability of (14) (they are independent of \(f\)). The remainder in (13) is of quite high order, \(O(z^{2n})\), and this is achieved by knowing only \(n+2\) values of \(f\) and its fixed derivatives (while the order \(O(z^{2n+1})\) in (5) is achieved by knowing \(2n\) values of \(f\)). However, to use (13) we need to calculate these derivatives, \(f^{(n-1)}(0)\) and \(f^{(2n-1)}(0)\), very accurate. We are not aware of any effective estimates of the nonlocal error in (13).

The optimal “real” \(n\)-node formula for calculation of the first derivative of real functions at zero was obtained in [1] in the form

\[
f'(0)x \approx \sum_{k=1}^{n} w_k f(u_k x) \quad (x, w_k, u_k \in \mathbb{R}; \quad |u_k - u_j| \geq 1 \text{ for } k \neq j);
\]

where \(w_k, u_k\) satisfy \(d_0 = d_2 = \cdots = d_n = 0\), \(d_1 = 1\) and \(|d_{n+1}| = \text{minimum}\), where \(d_m := \sum_{k=1}^{n} w_k u_k^m\) (see also References in [1, 3]).

5 Proof of Proposition 3

Consider the normalized polynomial

\[
Q_n(z) = \frac{n!}{(2n)!} \sum_{k=0}^{n} \frac{(2n-k)!}{(n-k)!} \frac{z^k}{k!} \quad (Q_n(0) = 1),
\]

and let \(z_{n1}, \ldots, z_{nn}\) be the zeros of \(Q_n\). By a simple observation,

\[
Q_n(z) \equiv \frac{n!}{(2n)!} \cdot z^n y_n \left(\frac{2}{z}\right),
\]

therefore \(z_{nk} = 2/\alpha_{nk} (k = 1, \ldots, n)\). We need

Lemma 1 ([8]) Let \(\rho > 0\). If \(n \geq \max\{14; \rho^2\} - 1\), then

\[
\frac{1}{2} < |Q_n(z)Q_n(-z)| < \frac{3}{2}, \quad |z| \leq \rho,
\]

and if \(n > \max\{2; (\rho^2 + 4)/8\}\), then

\[
0 < 1 - \rho^2(8n-4)^{-1} \leq Q_n(x)Q_n(-x) \leq 1, \quad -\rho \leq x \leq \rho.
\]
For the proof of (15) and (16), see [8, Sec. 6] and [8, Sec. 7], respectively.

Define the polynomials

\[ M_n(z) = (1 - z)^n Q_n \left( \frac{z}{z - 1} \right), \quad G_n(z) = (1 - z)^n Q_n \left( \frac{z}{1 - z} \right) \]

and put \( u_{nk} = a_{nk}^{-1}, \ v_{nk} = b_{nk}^{-1} \) \((k = 1, \ldots, n)\); the last numbers are pairwise distinct, as well as differ of 1, since \( \alpha_{nk} \) are pairwise distinct and \( \text{Re} \alpha_{nk} < 0 \).

Obviously, \( \{u_{nk}\} \) and \( \{v_{nk}\} \) are all the roots of \( M_n \) and \( G_n \), respectively, for example, \( M_n(u_{nk}) = (1 - u_{nk})^n Q_n(z_{nk}) = 0 \). Hence (9) takes the form

\[
R_n(z) = \frac{1}{(z - 1)^2} + \sum_{k=1}^{n} \left( \frac{1}{z - u_{nk}} - \frac{1}{z - v_{nk}} \right) = \frac{1}{(z - 1)^2} + \frac{d}{dz} \left( \ln \frac{M_n(z)}{G_n(z)} \right)
\]

\[
= \frac{1}{(z - 1)^2} - \frac{1}{(z - 1)^2} \frac{d}{dw} \left( \ln \frac{Q_n(w)}{Q_n(-w)} \right),
\]

where \( w = z/(z - 1) \). Now we need the formula

\[
1 - \frac{d}{dz} \left( \ln \frac{Q_n(z)}{Q_n(-z)} \right) = (-1)^n \frac{z^{2n} \gamma_n}{Q_n(z)Q_n(-z)}, \quad \gamma_n := \frac{n!^2}{(2n)!^2}
\]

(see Sections 2, 5 in [8]; we can also obtain this formula from [2, Eq. (17)]). By using this formula, we get the identity

\[
R_n(z) = (-1)^n \frac{z^{2n} \gamma_n}{(z - 1)^{2n+2} Q_n(w)Q_n(-w)}, \quad w = z/(z - 1).
\]

Let \( x \in (0, 1) \). If \( |z| \leq x \), then \( |w| \leq \rho_0 \) with \( \rho_0 := x/(1 - x), \ \rho_0 > 0 \). Note that \( \max\{14; \rho_0^2\} - 1 \equiv n_0(x) \), hence we have

\[
|Q_n(w)Q_n(-w)| > 1/2
\]

for \( n \geq n_0(x) \) by (15). Thus the upper bound in (10) follows from (17).

Analogously, the lower bound in (10) follows from (17) and (16):

\[
\max_{|z| = x} |R_n(z)| \geq |R_n(x)| = \frac{x^{2n} \gamma_n}{(1 - x)^{2n+2} Q_n(-\rho_0)Q_n(\rho_0)} \geq \frac{x^{2n} \gamma_n}{(1 - x)^{2n+2}}
\]

(see that \( n_0(x) = \max\{14; \rho^2\} - 1 > \max\{2; (\rho_0^2 + 4)/8\} \)).

This completes the proof of Proposition 3.

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