UNIFORM EXPONENTIAL STABILISATION OF SERIALLY CONNECTED INHOMOGENEOUS EULER-BERNOULLI BEAMS

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Abstract. We consider a chain of Euler-Bernoulli beams with spatial dependent mass density, modulus of elasticity and area moment which are interconnected in dissipative or conservative ways and prove uniform exponential energy decay of the coupled system for suitable dissipative boundary conditions at one end and suitable conservative boundary conditions at the other end. We thereby generalise some results of G. Chen, M.C. Delfour, A.M. Krall and G. Payre from the 1980’s to the case of spatial dependence of the parameters.

Keywords: Euler-Bernoulli beam, inhomogeneous distributed parameter systems, serially connected beams, exponential stabilisation, frequency domain method.

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1. Introduction

Beam equations became the focus of attention for mathematical modelling, analysis and numerics of complex, multi-component systems, in particular mechanical systems for the modelling of airplanes, bridges, nowadays more and more electromechanical systems and robotics, at least since the 1980’s. Several types of partial differential equations serve as and compete as models for such vibrating beams or strings, from the wave equation, probably one of the most commonly and most detailed discussed model in mathematics, to the Rayleigh beam and the Euler-Bernoulli beam, to the Timoshenko beam and even more sophisticated models. In many cases these equations are non-linear in principle, but for the analysis and numerics of complex systems is is often useful, to consider the linear or linearised versions of these equations. In this article, we treat the linear Euler-Bernoulli beam model

$$\rho(\zeta)\omega_{tt}(t,\zeta) + (EI(\zeta)\omega_{\zeta\zeta})(\zeta) = 0, \quad t \geq 0, \quad \zeta \in (0,l)$$

where $\rho(\zeta)$ denotes the mass density times cross section area of a beam of length $l > 0$, and $E(\zeta)$ and $I(\zeta)$ its modulus of elasticity and area moment of the cross section, respectively. G. Chen and several coauthors [8], [9], [10] considered three particular important situations for the Euler-Bernoulli beam:

1. A single beam is being stabilised by dissipative boundary feedback at one end of the beam and conservative boundary conditions at the other end [9].
2. A pair of identical beams is damped via dissipative point feedback at the joint [10].
3. An arbitrary long, but finite chain of serially connected beams is damped at one end of the chain [8].

In all these cases the authors assume that the beam parameters $\rho$, $E$ and $I$ are constant along each of the beams. Since then for all three cases the corresponding articles inspired further mathematical research for more general models. E.g. in [10] and [24], situations have been considered where for a single Euler-Bernoulli beam the collocated feedback at the dissipative end is perturbed, i.e. the feedback input cannot be expressed solely by (the traces of) the energy variables $\omega_t$ and $EI\omega_{\zeta\zeta}$ and their spatial derivatives.

Other works, e.g. [3], [15], [2], [17], [1] further dealt with the problem of dissipative point feedback at the joint between two Euler-Bernoulli beams. These works highlighted that such a feedback law is not a good choice for exponential stabilisation (or, it is not a good model for such systems), because usually they gave the result that the property of asymptotic and uniform exponential stability depends on whether for the actuation point $\xi \in (0,l)$, at which the damper acts, the fraction $\frac{\xi}{l} \in (0,1)$ lies in some subset $\tilde{Q}$ of $\mathbb{Q} \cap [0,1]$ which is still dense in $[0,1]$. A very unsatisfactory results from engineering perspective.

At the same time more general networks defining the interconnection structure of Euler-Bernoulli beams and their stability properties have been considered, e.g. in [11], [22] and [23].

The methods used mostly for proving stability essentially break down into three more or less heavily used methods:

1. Construction of a suitable Lyapunov function: This method has been applied in [8], [3].
2. Analysis of the asymptotic behaviour of the (discrete) eigenvalues $\lambda_n$ for $n \to \infty$, see e.g. [9], [11], [15], [17], [22], [23].
3. Resolvent estimates on the imaginary axis based on the Gearhart-Greiner-Prüss-Huang Theorem, i.e. $\sup_{\beta \in \mathbb{R}} \| (i\beta - A)^{-1} \| < \infty$, e.g. [24], [2], [10].

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Each of this methods has its own advantages and disadvantages. E.g. the first method is suitable to allow for non-linear perturbations in the dissipative boundary feedback, but the method seems to be restricted to Euler-Bernoulli beams with almost homogeneous parameters $\rho$ and $EI$, cf. [5], and it is not clear at all whether all cases for which uniform exponential stabilisation is already known can be covered by this method as well.

Even more restrictive is the second method, which mainly can be used for homogeneous beam models, whereas the third method in generally is suitable for non-homogeneous beams as well (and will be applied in this article). At the same time, both the second and the third method are restricted to the case of linear boundary feedback, and leave stability questions concerning nonlinear feedback wide open.

Note that the papers listed above almost exclusively cover homogeneous beam equations, i.e. $\rho$ and $EI$ are constant, at least on each beam. This brings up the question: Is homogeneity of the beams only a technical restriction for the proofs? Can the general inhomogeneous case be reduced to the special homogeneous case? Does a (sufficiently regular) inhomogeneity influence well-posedness or stability at all? As it turns out, for the last question, which actually consists of two separate questions (well-posedness and stability), one of which has an easy answer, the other not so. In fact, for dissipative systems well-posedness (in the sense of semigroup generation, i.e. existence, uniqueness and continuous dependance on the initial datum for the corresponding abstract Cauchy problem) is invariant under perturbation by a coercive and continuous operator, see e.g. Lemma 7.2.3 in [18] or the much more general results in [7]. (For a background on strongly continuous semigroups $C_0$-semigroups), we refer to the monograph [13].) Does the same result hold if the term strongly continuous contraction semigroup is replaced by uniformly exponentially stable, strongly continuous contraction semigroup? Unfortunately not! Actually, there are already examples on finite dimensional Hilbert spaces which serve as counter examples, and in the class of infinite-dimensional port-Hamiltonian systems [27, 20, 18], i.e. a hyperbolic vector-valued PDE on an interval, in which form the Euler-Bernoulli beam can be rewritten, a striking counter example is known [12]. Though the example there is not a counter example for the class of Euler-Bernoulli beams, yet it motivates the standpoint we take in this paper: Stability of non-uniform beams should be addressed additionally to the question of stability for their homogeneous counterparts. Therefore, we generalise the results of [8] in this direction, which – to our knowledge – has not yet been achived up to now.

G. Chen et al. [3] investigated a system of Euler-Bernoulli beams which are serially interconnected (in a conservative or dissipative way), and which is damped at one of the two ends of the chain, e.g.

$$
\rho^j \omega_{tt}(t, \zeta) + (E^j P^{j+1} \omega_{\zeta \zeta})_\zeta(t, \zeta) = 0, \quad t \geq 0, \zeta \in (l^{j-1}, l^j), \quad j = 1, \ldots, m
$$

$$
\omega(t, 0) = 0, \quad t \geq 0
$$

$$
\omega_t(t, 0) = 0, \quad t \geq 0
$$

$$
\omega(t, l^j) = \omega(t, l^{j-1}), \quad t \geq 0, \quad j = 1, \ldots, m - 1
$$

$$
- E^j P^{j} \omega_{\zeta \zeta}(t, l^j) = - E^{j+1} P^{j+1} \omega_{\zeta \zeta}(t, l^{j+1}), \quad t \geq 0, \quad j = 1, \ldots, m - 1
$$

$$
- E^m P^m \omega_{\zeta \zeta}(t, L) = 0, \quad t \geq 0
$$

$$
- E^m P^m \omega_{\zeta \zeta}(t, L) = \kappa \omega_t(t, L), \quad t \geq 0
$$

$$
\omega(0, \zeta) = \omega_0(\zeta), \quad \zeta \in (l^{j-1}, l^j), \quad j = 1, \ldots, m
$$

$$
\omega_t(0, \zeta) = \omega_t(\zeta), \quad \zeta \in (l^{j-1}, l^j), \quad j = 1, \ldots, m
$$

where $0 = l^0 < l^1 < \ldots < l^m = L$ is a division of the interval $(0, L)$ for some $L > 0$ and $m \in \mathbb{N}$, and $\kappa > 0$ is some damping parameter. E.g. for the special case $m = 2$, $L = 1$ and $l^1 = l \in (0, 1)$ this system reads as

$$
\rho^1 \omega_{tt}(t, \zeta) + (E^1 P^2 \omega_{\zeta \zeta})_\zeta(t, \zeta) = 0, \quad t \geq 0, \zeta \in (0, l)
$$

$$
\rho^2 \omega_{tt}(t, \zeta) + (E^2 P^2 \omega_{\zeta \zeta})_\zeta(t, \zeta) = 0, \quad t \geq 0, \zeta \in (l, 1)
$$

$$
\omega(t, 0) = 0, \quad t \geq 0
$$

$$
\omega_t(t, 0) = 0, \quad t \geq 0
$$

$$
- (E^1 P^1 \omega_{\zeta \zeta})(t, l) = - (E^2 P^2 \omega_{\zeta \zeta})(t, l), \quad t \geq 0
$$

$$
(E^1 P^1 \omega_{\zeta \zeta})(t, l) = (E^2 P^2 \omega_{\zeta \zeta})(t, l), \quad t \geq 0
$$

$$
(E^2 P^2 \omega_{\zeta \zeta})(t, 1) = 0, \quad t \geq 0
$$

$$
(E^2 P^2 \omega_{\zeta \zeta})(t, 1) = \kappa \omega_t(t, 1), \quad t \geq 0.
$$

For first reading, the reader may always have this special case in mind since it already includes most of the relevant features of a chain of Euler-Bernoulli beams. The demonstration of the results in [3] is based on an energy multiplier method which gives a Lyapunov function for the Euler-Bernoulli beam system. For example,
in this case uniform exponential energy decay
\[ H(t) := \frac{1}{2} \sum_{j=1}^{m} \int_{\gamma_{j-1}}^{\gamma_j} \rho^j |\omega(t, \zeta)|^2 + E^j I^j |\omega_{\zeta\zeta}(t, \zeta)|^2 \, d\zeta \leq M e^{\eta t} H(0), \quad t \geq 0 \]
for some \( M \geq 1 \) and \( \eta < 0 \) independent of the initial data, has been shown for \( \kappa > 0 \) under the following additional structural constraints:

1. On each interval \( (\gamma^{j-1}, \gamma^j) \), the mass density times cross sectional area \( \rho(\zeta) = \rho^j \), the modulus of elasticity \( E(\zeta) = E^j \) and the area moment of the cross section \( I(\zeta) = I^j \) are constant.
2. The parameters \( \rho^1 > 0, E^j > 0 \) and \( I^j > 0 \) satisfy the monotonicity constraints
\[ \rho^j \leq \rho^{j+1}, \quad E^j I^j \geq E^{j+1} I^{j+1}, \quad j = 1, \ldots, m. \]

In this paper we are going to remove the first of these constraints, i.e. we show the same uniform exponential stability result for arbitrary piecewise Lipschitz-continuous and strictly positive \( \rho \in \text{Lip}((\gamma^{j-1}, \gamma^j); \mathbb{R}) \) and \( E I \in \text{Lip}((\gamma^{j-1}, \gamma^j); \mathbb{R}) \) (note that this implies that for each \( \gamma^j \), the one-sided limits \( \rho(\gamma^{-} - ) \) and \( \rho(\gamma^{+} + ) \) etc. exist) replacing the constant parameters \( \rho^j \) and \( E^j I^j \), but still satisfying the jump conditions
\[ \rho(\gamma^{-}) \leq \rho(\gamma^{+}), \quad (EI)(\gamma^{-}) \geq (EI)(\gamma^{+}). \]

This, we do in the framework of \( C_0 \)-semigroups, applying a simple case of the Arendt-Batty-Lyubich-Vu Theorem for asymptotic stability and the Gearhart-Prüss-Huang Theorem for uniform exponential stability. Also, we consider a slight generalisation by allowing dynamic boundary feedback via impedance passive finite-dimensional control systems which all are internally stable. For well-posedness (here: dissipativity of the interconnected system implies well-posedness with non-increasing energy for the system), we use abstract well-posedness results for so-called \emph{infinite-dimensional port-Hamiltonian systems} \cite{Lun95,FOS01,FOS02,FP15} (which rely on the Lumer-Phillips Theorem), and also employ the techniques used in \cite{FP15} for uniform exponential stabilisation of a (single) Euler-Bernoulli beam within the port-Hamiltonian framework to show the uniform exponential energy decay.

This paper is organised as follows: In Section 2 we formally consider possible interconnection and boundary conditions leading to a dissipative system of joint Euler-Bernoulli beams. More precisely, we give classes of boundary control and observation maps leading to an open loop impedance passive system, thus leading to a dissipative system for dissipative (linear) closure relations. Using the abstract theory of infinite-dimensional port-Hamiltonian systems, this directly leads to well-posedness results in the sense that for any sufficiently regular initial data there is a unique solution with non-increasing energy and depending continuously on the initial data, in other words: The operator \( A \) governing the dynamics of the beam-observer-feedback-actuator system generates a strongly continuous contraction semigroup on a suitable energy state space \( X \). Then, Section 3 is devoted to the discussion of stability properties. We give sufficient conditions on the interconnection structure by means of dissipative static feedback or feedback via interconnection with an internally stable impedance passive finite-dimensional linear controller. Here, the main results of that section and this manuscript are Theorem 3.3 on asymptotic, i.e. strong, stability and Theorem 3.4 on uniform exponential stability, i.e. uniform exponential energy decay. In Example 3.7 we reformulate the previous well-posedness and stability results in the language of Euler-Bernoulli beam equations and show that our results cover the non-uniform beam versions of the uniform exponential stability results already presented in \cite{FP15}, especially including a discussion of several relevant conservative boundary conditions on the non-dissipative end of the series of Euler-Bernoulli beams, which already had been mentioned in \cite{FP15}, but under the restrictive condition of piecewise constant parameters \( \rho \) and \( E I \). We conclude the paper with some final remarks in Section 4.

## 2. WELL-POSEDNESS

We start by discussing the well-posedness of serially connected Euler-Bernoulli beams. Slightly generalising the setup in \cite{FP15}, we consider a decomposition \( 0 = \gamma^0 < \gamma^1 < \ldots < \gamma^m = L \) of an interval \( (0, L) \) \((L > 0)\) and on each subinterval \((\gamma^{j-1}, \gamma^j)\) the Euler-Bernoulli beam equation
\[ \rho(\zeta) \omega_{tt}(t, \zeta) + (EI(\zeta) \omega_{\zeta\zeta}(t, \zeta))_{\zeta\zeta} = 0, \quad \zeta \in (\gamma^{j-1}, \gamma^j) \]
where in contrast to the situation in \cite{FP15} we allow general spatial dependence of \( \rho \) and \( EI \) on \( \zeta \in (\gamma^{j-1}, \gamma^j) \). For the moment, it is be enough to let \( \rho, EI \in \text{Lip}_\infty((\gamma^{j-1}, \gamma^j)) \) be uniformly positive, i.e.
\[ \rho(\zeta), EI(\zeta) \geq \varepsilon > 0, \quad \text{a.e.} \ \zeta \in (\gamma^{j-1}, \gamma^j), \quad j = 1, \ldots, m. \]
The energy of the system is then defined as

\[ H(t) := \frac{1}{2} \int_0^L \rho(\zeta) |\omega_1(t, \zeta)|^2 + EI(\zeta) |\omega_{\zeta\zeta}(t, \zeta)|^2 \, d\zeta \]

\[ = \sum_{j=1}^m \frac{1}{2} \int_{\nu_j}^{\nu_{j+1}} \rho(\zeta) |\omega_1(t, \zeta)|^2 + EI(\zeta) |\omega_{\zeta\zeta}(t, \zeta)|^2 \, d\zeta =: \sum_{j=1}^m H_j(t). \]

Formally, for sufficiently regular solutions of the Euler-Bernoulli beam equations on each subinterval \((\nu_j-1, \nu_j)\), we obtain

\[
\frac{d}{dt} H^j(t) = \text{Re} \int_{\nu_j-1}^{\nu_j} \rho(\zeta) \omega(t, \zeta) \overline{\omega(t, \zeta)} + EI(\zeta) \omega_{\zeta\zeta}(t, \zeta) \overline{\omega_{\zeta\zeta}(t, \zeta)} \, d\zeta \\
= \text{Re} \int_{\nu_j-1}^{\nu_j} -(EI\omega_{\zeta\zeta})(t, \zeta) \omega(t, \zeta) + EI(\zeta)\omega_{\zeta\zeta}(t, \zeta) \overline{\omega_{\zeta\zeta}(t, \zeta)} \, d\zeta \\
= \text{Re} \left[ -(EI\omega_{\zeta\zeta})(t, \nu_j-1) \omega(t, \nu_j-1) + (EI\omega_{\zeta\zeta})(t, \nu_j-1) \overline{\omega_{\zeta\zeta}(t, \nu_j-1)} \right] \\
- \text{Re} \left[ - (EI\omega_{\zeta\zeta})(t, \nu_j) \omega(t, \nu_j) + (EI\omega_{\zeta\zeta})(t, \nu_j) \overline{\omega_{\zeta\zeta}(t, \nu_j)} \right] \\
- \text{Re} \left[ -(EI\omega_{\zeta\zeta})(t, \nu_j-1) \omega(t, \nu_j-1) + (EI\omega_{\zeta\zeta})(t, \nu_j-1) \overline{\omega_{\zeta\zeta}(t, \nu_j-1)} \right]
\]

where we denote by \(f(\zeta\pm)\) the left-sided and right-sided limits of a function \(f\) at \(\zeta\), respectively. Putting these equations together, we obtain

\[
\frac{d}{dt} H(t) = \sum_{j=1}^m \frac{d}{dt} H^j(t) = \text{Re} \left[ -(EI\omega_{\zeta\zeta})(t, \nu_j-1) \omega(t, \nu_j-1) + (EI\omega_{\zeta\zeta})(t, \nu_j-1) \overline{\omega_{\zeta\zeta}(t, \nu_j-1)} \right] \\
- \text{Re} \left[ - (EI\omega_{\zeta\zeta})(t, \nu_j) \omega(t, \nu_j) + (EI\omega_{\zeta\zeta})(t, \nu_j) \overline{\omega_{\zeta\zeta}(t, \nu_j)} \right] \\
+ \sum_{j=1}^{m-1} \text{Re} \left[ -(EI\omega_{\zeta\zeta})(t, \nu_j) \omega(t, \nu_j) + (EI\omega_{\zeta\zeta})(t, \nu_j) \overline{\omega_{\zeta\zeta}(t, \nu_j)} \right] \\
- \text{Re} \left[ - (EI\omega_{\zeta\zeta})(t, \nu_j-1) \omega(t, \nu_j-1) + (EI\omega_{\zeta\zeta})(t, \nu_j-1) \overline{\omega_{\zeta\zeta}(t, \nu_j-1)} \right].
\]

We see that for interconnection of the beams in a dissipative way, the most natural way to do so is by imposing dissipative (here, including conservative) boundary conditions at the left \((\zeta = \nu_j = 0)\) and right \((\zeta = \nu_j = L)\) end and a dissipative interconnection at the joint points \(\zeta = \nu_j \) \((j = 1, \ldots, m - 1)\). To make this possible, we demand that at every joint point \(\nu_j \) \((j = 1, \ldots, m - 1)\) we have

\[ \omega_1(t, \nu_j-1) = \omega_1(t, \nu_j) + (EI\omega_{\zeta\zeta})(t, \nu_j) \]

and

\[ \omega_{\zeta}(t, \nu_j-1) = \omega_{\zeta}(t, \nu_j) + (EI\omega_{\zeta\zeta})(t, \nu_j). \]

(In [25], [8] it has been discussed that for dissipativity of the system, at least one of two state variables which are dual (or complementary) to each other has to be continuous.) Writing \(f(\nu_j) := f(\nu_j) = f(\nu_j)\) whenever \(f(\nu_j) = f(\nu_j)\), this gives us four different cases, for which a static interconnection of the type

1. For \(\omega_1(t, \nu_j-1) = \omega_1(t, \nu_j) + (EI\omega_{\zeta\zeta})(t, \nu_j)\):

\[
\begin{pmatrix}
- (EI\omega_{\zeta\zeta})(t, \nu_j) \omega_1(t, \nu_j) + (EI\omega_{\zeta\zeta})(t, \nu_j) \\
(EI\omega_{\zeta\zeta})(t, \nu_j) - (EI\omega_{\zeta\zeta})(t, \nu_j)
\end{pmatrix} = -K^3 \begin{pmatrix}
\omega_1(t, \nu_j) \\
\omega_{\zeta}(t, \nu_j)
\end{pmatrix}
\]

2. For \(\omega_1(t, \nu_j-1) = \omega_1(t, \nu_j) + (EI\omega_{\zeta\zeta})(t, \nu_j)\):

\[
\begin{pmatrix}
- (EI\omega_{\zeta\zeta})(t, \nu_j-1) \omega_1(t, \nu_j-1) + (EI\omega_{\zeta\zeta})(t, \nu_j-1) \\
(EI\omega_{\zeta\zeta})(t, \nu_j-1) - (EI\omega_{\zeta\zeta})(t, \nu_j-1)
\end{pmatrix} = -K^3 \begin{pmatrix}
\omega_1(t, \nu_j) \\
\omega_{\zeta}(t, \nu_j)
\end{pmatrix}
\]

3. For \(- (EI\omega_{\zeta\zeta})(t, \nu_j-1) = -(EI\omega_{\zeta\zeta})(t, \nu_j) + \omega_{\zeta}(t, \nu_j-1) = \omega_{\zeta}(t, \nu_j) + (EI\omega_{\zeta\zeta})(t, \nu_j)\):

\[
\begin{pmatrix}
\omega_1(t, \nu_j-1) - \omega_1(t, \nu_j) \\
(EI\omega_{\zeta\zeta})(t, \nu_j-1) - (EI\omega_{\zeta\zeta})(t, \nu_j)
\end{pmatrix} = -K^3 \begin{pmatrix}
- (EI\omega_{\zeta\zeta})(t, \nu_j) \\
\omega_{\zeta}(t, \nu_j)
\end{pmatrix}
\]

4. For \(- (EI\omega_{\zeta\zeta})(t, \nu_j-1) = -(EI\omega_{\zeta\zeta})(t, \nu_j)\):

\[
\begin{pmatrix}
\omega_1(t, \nu_j-1) - \omega_1(t, \nu_j) \\
\omega_{\zeta}(t, \nu_j-1) - \omega_{\zeta}(t, \nu_j)
\end{pmatrix} = -K^3 \begin{pmatrix}
- (EI\omega_{\zeta\zeta})(t, \nu_j) \\
(EI\omega_{\zeta\zeta})(t, \nu_j)
\end{pmatrix}
\]
where $K^j \in \mathbb{K}^{2 \times 2}$ is a matrix with positive semidefinite symmetric part $\text{Sym} K = \frac{K+K^*}{2}$, makes the interconnection dissipative. (At first reading the reader might consider the case $m = 2$ and the particular interconnection condition

$$\omega_l(t^{1+}) = \omega_l(t^{1-})$$
$$\omega_l(t^{1+}) = \omega_l(t^{1-})$$

$$\left( (EI\omega_{\zeta\zeta})_l(t^{1+}) + (EI\omega_{\zeta\zeta})_l(t^{1+}) \right) = -K^1 \left( \omega_l(t^1) \right)$$

(to make the results easier digestible.) We reformulate this problem in a more abstract way to make the theory of infinite-dimensional port-Hamiltonian systems, cf. e.g. [20], [6], applicable. First, we write

$$H(\zeta) := \left[ \begin{array}{c} H_1(\zeta) \\ \vdots \\ H_m(\zeta) \end{array} \right] \in \mathbb{K}^{2m \times 2m},$$

$$x(\zeta) := \left( \begin{array}{c} x^1(\zeta) \\ \vdots \\ x^m(\zeta) \end{array} \right) \in \mathbb{K}^{2m},$$

$$P_2 = \left[ \begin{array}{ccc} 0 & -1 \\ 1 & 0 \end{array} \right],$$

$$P_1 = P_0 = 0 \in \mathbb{K}^{2m \times 2m},$$

the Euler-Bernoulli equations take the port-Hamiltonian form

$$\frac{\partial}{\partial t} x(t, \zeta) = \left( P_2 \frac{\partial^2}{\partial t^2} H(\zeta)x(t, \zeta) \right) := (Ax(t))(\zeta), \quad t \geq 0, \; \zeta \in (0, 1).$$

Therefore, by Theorem 4.1 in [20], the operator $A = \mathfrak{A}|_{D(A)}$ on a domain $D(A) \subseteq D(\mathfrak{A}) = \{ x \in L_2(0, 1; \mathbb{K}^{2m}) : Hx \in H^2(0, 1; \mathbb{K}^{2m}) \}$ which is dense in $L_2(0, 1; \mathbb{K}^{2m})$ generates a contractive $C_0$-semigroup on the energy state space $X = L_2(0, 1; \mathbb{K}^{2m})$ with inner product

$$(x \mid y)_X := \int_0^1 (x(\zeta) \mid H(\zeta)y(\zeta))_{\mathbb{K}^{2m}} \; d\zeta, \quad x, y \in X$$

if and only it is dissipative, i.e. the boundary (and here: interconnection) conditions restricting $D(\mathfrak{A})$ to a linear subspace $D(A)$ ensure that

$$\text{Re} \left( Ax \mid x \right)_X \leq 0, \quad x \in D(A).$$

Moreover, the same can also be said whenever $\mathfrak{A}$ is interconnected by a finite dimensional control system $\Sigma_c = (A_c, B_c, C_c, D_c)$, i.e.

$$\frac{d}{dt} x_c(t) = A_c x_c(t) + B_c u_c(t), \quad t \geq 0$$
$$y_c(t) = C_c x_c(t) + D_c u_c(t), \quad t \geq 0$$

where $x_c(t)$ lies in the controller state space $X_c$, a finite dimensional Hilbert space, and the Hilbert spaces $U_c$ and $Y_c$ are the finite dimensional control and observation spaces for the control system. To formulate this well-posedness result rigorously, we assume the following
Definition and Assumption 2.1 (Pointwise Control and Observation Operators). For \( x \in D(\mathfrak{A}) \) and matrices \( W_B^x, W_C^x \in \mathbb{K}^{2 \times 4} \) such that \( \left[ \begin{array}{c|c} W^x_B & W^x_C \end{array} \right] \in \mathbb{K}^{4 \times 4} \) is invertible, we define the linear operators \( \mathfrak{B}^j, \mathfrak{C}^j : D(\mathfrak{A}) \rightarrow \mathbb{K}^2 \) by

\[
\left( \begin{array}{c}
\mathfrak{B}_x^0 \\
\mathfrak{C}_x^0
\end{array} \right) := \left[ \begin{array}{c|c}
W_B^0 & W_C^0
\end{array} \right] \left( \begin{array}{c}
(\mathcal{H}^1 x^1)'(0) \\
\mathcal{P}_2(\mathcal{H}^1 x^1)'(0)
\end{array} \right) \equiv \left[ \begin{array}{c|c}
W_B^0 & W_C^0
\end{array} \right] \left( \begin{array}{c}
\omega_l(t, 0) \\
(\mathcal{E}I\omega_l)^{(1)}(t, 0)
\end{array} \right)
\]

\[
\left( \begin{array}{c}
\mathfrak{B}_x^m \\
\mathfrak{C}_x^m
\end{array} \right) := \left[ \begin{array}{c|c}
W_B^m & W_C^m
\end{array} \right] \left( \begin{array}{c}
(\mathcal{H}^m x^m)'(1) \\
\mathcal{P}_2(\mathcal{H}^m x^m)'(1)
\end{array} \right) \equiv \left[ \begin{array}{c|c}
W_B^m & W_C^m
\end{array} \right] \left( \begin{array}{c}
\omega_l(t, L) \\
(\mathcal{E}I\omega_l)^{(1)}(t, L)
\end{array} \right)
\]

and for each \( j \in \{1, \ldots, m-1\} \), depending on the case we are in, we define linear maps \( \mathfrak{B}_j^x, \mathfrak{B}_j^x \) and \( \mathfrak{C}_j^x \) as follows:

1. For \( \omega_l(t, V^-) = \omega_l(t, V^+) \) and \( \omega_l(t, V^-) = \omega_l(t, V^+) \):

\[
\mathfrak{B}_j^x := \left( \begin{array}{c}
(\mathcal{H}^1 x^1)'(0) + (\mathcal{H}^1 x^1)'(1) \\
(\mathcal{H}^1 x^1)'(0) - (\mathcal{H}^1 x^1)'(1)
\end{array} \right) \equiv \left( \begin{array}{c}
\omega_l(t, V^+) - \omega_l(t, V^-) \\
\omega_l(t, V^+) - \omega_l(t, V^-)
\end{array} \right)
\]

\[
\mathfrak{C}_j^x := \left( \begin{array}{c}
(\mathcal{H}^m x^m)'(1) \\
(\mathcal{H}^m x^m)'(1)
\end{array} \right) \equiv \left( \begin{array}{c}
\omega_l(t, V^+) - \omega_l(t, V^-) \\
\omega_l(t, V^+) - \omega_l(t, V^-)
\end{array} \right)
\]

where for any spatial dependent quantity \( f \) we write \( f(\zeta) := f(\zeta^+) = f(\zeta^-) \) whenever the two one-sided limits exist and coincide.

2. For \( \omega_l(t, V^-) = \omega_l(t, V^+) \) and \( (\mathcal{E}I\omega_l)^{(1)}(t, V^-) = (\mathcal{E}I\omega_l)^{(1)}(t, V^+) \):

\[
\mathfrak{B}_j^x := \left( \begin{array}{c}
(\mathcal{H}^1 x^1)'(0) - (\mathcal{H}^1 x^1)'(1) \\
(\mathcal{H}^1 x^1)'(0) + (\mathcal{H}^1 x^1)'(1)
\end{array} \right) \equiv \left( \begin{array}{c}
\omega_l(t, V^+) - \omega_l(t, V^-) \\
(\mathcal{E}I\omega_l)^{(1)}(t, V^+) - (\mathcal{E}I\omega_l)^{(1)}(t, V^-)
\end{array} \right)
\]

\[
\mathfrak{C}_j^x := \left( \begin{array}{c}
(\mathcal{H}^m x^m)'(1) \\
(\mathcal{H}^m x^m)'(1)
\end{array} \right) \equiv \left( \begin{array}{c}
(\mathcal{E}I\omega_l)^{(1)}(t, V^+) - (\mathcal{E}I\omega_l)^{(1)}(t, V^-) \\
\omega_l(t, V^+) - \omega_l(t, V^-)
\end{array} \right)
\]

3. For \( -(\mathcal{E}I\omega_l)^{(1)}(t, V^-) = -(\mathcal{E}I\omega_l)^{(1)}(t, V^+) \) and \( \omega_l(t, V^-) = \omega_l(t, V^+) \):

\[
\mathfrak{B}_j^x := \left( \begin{array}{c}
(\mathcal{H}^1 x^1)'(0) + (\mathcal{H}^1 x^1)'(1) \\
(\mathcal{H}^1 x^1)'(0) - (\mathcal{H}^1 x^1)'(1)
\end{array} \right) \equiv \left( \begin{array}{c}
-(\mathcal{E}I\omega_l)^{(1)}(t, V^-) + (\mathcal{E}I\omega_l)^{(1)}(t, V^+) \\
\omega_l(t, V^-) - \omega_l(t, V^+)
\end{array} \right)
\]

\[
\mathfrak{C}_j^x := \left( \begin{array}{c}
(\mathcal{H}^m x^m)'(1) \\
(\mathcal{H}^m x^m)'(1)
\end{array} \right) \equiv \left( \begin{array}{c}
-(\mathcal{E}I\omega_l)^{(1)}(t, V^-) + (\mathcal{E}I\omega_l)^{(1)}(t, V^+) \\
\omega_l(t, V^-) - \omega_l(t, V^+)
\end{array} \right)
\]

4. For \( -(\mathcal{E}I\omega_l)^{(1)}(t, V^-) = -(\mathcal{E}I\omega_l)^{(1)}(t, V^+) \) and \( (\mathcal{E}I\omega_l)^{(1)}(t, V^-) = (\mathcal{E}I\omega_l)^{(1)}(t, V^+) \):

\[
\mathfrak{B}_j^x := \left( \begin{array}{c}
(\mathcal{H}^1 x^1)'(0) - (\mathcal{H}^1 x^1)'(1) \\
(\mathcal{H}^1 x^1)'(0) - (\mathcal{H}^1 x^1)'(1)
\end{array} \right) \equiv \left( \begin{array}{c}
-(\mathcal{E}I\omega_l)^{(1)}(t, V^-) + (\mathcal{E}I\omega_l)^{(1)}(t, V^+) \\
(\mathcal{E}I\omega_l)^{(1)}(t, V^-) - (\mathcal{E}I\omega_l)^{(1)}(t, V^+)
\end{array} \right)
\]

\[
\mathfrak{C}_j^x := \left( \begin{array}{c}
(\mathcal{H}^m x^m)'(1) \\
(\mathcal{H}^m x^m)'(1)
\end{array} \right) \equiv \left( \begin{array}{c}
-(\mathcal{E}I\omega_l)^{(1)}(t, V^-) + (\mathcal{E}I\omega_l)^{(1)}(t, V^+) \\
(\mathcal{E}I\omega_l)^{(1)}(t, V^-) - (\mathcal{E}I\omega_l)^{(1)}(t, V^+)
\end{array} \right)
\]

For interconnection with a control system, or dissipative boundary feedback, it is convenient to have an impedance passive system. By our choice of the operators \( \mathfrak{B}^j \) and \( \mathfrak{C}^j \) and the energy balance \([1] \) we have for the operators \( \mathfrak{A}_0, \mathfrak{B} \) and \( \mathfrak{C} \) defined by

\[
\mathfrak{A}_0 = \mathfrak{A}|_{\ker \mathfrak{A}_0} = \mathfrak{A}|_{\oplus^{m-1} \ker \mathfrak{A}_0} : D(\mathfrak{A}_0) \subseteq X \rightarrow X
\]

\[
\mathfrak{B} = (\mathfrak{B}^j)^{m-1}_{j=0} : D(\mathfrak{A}_0) \subseteq X \rightarrow U := \mathbb{K}^{2(m+1)}
\]

\[
\mathfrak{C} = (\mathfrak{C}^j)^{m-1}_{j=0} : D(\mathfrak{A}_0) \subseteq X \rightarrow Y := \mathbb{K}^{2(m+1)}
\]
the energy balance
\[
\Re (\mathfrak{A}_0 x | x)_X \leq \sum_{j=1}^{m-1} \Re (\mathfrak{B}^j x | \mathfrak{C}^j x) + \Re \left( -\bar{P}_2 (\mathcal{H}^1 x^1)'(0) | (\mathcal{H}^1 x^1)(0) \right) \\
+ \Re \left( \bar{P}_2 (\mathcal{H}^m x^m)'(1) | (\mathcal{H}^m x^m)(1) \right), \quad x \in D(\mathfrak{A}_0).
\]
From here, a condition on the matrices \[ W_B^0 \begin{bmatrix} W_B^0 & W_B^m \end{bmatrix} \text{ and } \begin{bmatrix} W_B^0 & W_B^m \end{bmatrix} \text{ may lead to impedance passivity of the system.}

**Lemma 2.2.** The triple \((\mathfrak{A}_0, \mathfrak{B}, \mathfrak{C})\) is impedance passive, i.e.
\[
\Re (\mathfrak{A}_0 x | x)_X \leq \sum_{j=0}^{m} \Re (\mathfrak{B}^j x | \mathfrak{C}^j x)_{\mathbb{R}^2}, \quad x \in D(\mathfrak{A}_0),
\]
if and only if
\[
\begin{bmatrix} W_B^0 & W_B^m \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} W_B^0 & W_B^m \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} W_B^0 & W_B^m \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \geq 0
\]
are both positive semidefinite.

**Proof.** This can easily be proved using the energy balance [1], cf. Theorem 4.2 in [2] which straight-forward extends to the case \(\mathbb{K} = \mathbb{C}\). \(\square\)

**Assumption 2.3 (Impedance-passivity).** The triplet \(\mathfrak{S}_0 = (\mathfrak{A}_0, \mathfrak{B}, \mathfrak{C})\) is impedance passive.

The class control system of control systems we allow for linear closing of the open loop system \((\mathfrak{A}_0, \mathfrak{B}, \mathfrak{C})\) is always assumed to be a linear finite dimensional controller.

**Assumption 2.4 (Finite-dimensional, linear control system).** Let \(X_c\) be a finite dimensional Banach space (including the possible choice \(X_c = \{0\}\) leading to static boundary feedback) and (w.l.o.g.) let \(U_c = Y_c = \mathbb{K}^{2(m+1)}\) and \(A_c \in \mathcal{B}(X_c), B_c \in \mathcal{B}(U_c, X_c), C_c \in \mathcal{B}(X_c, Y_c)\) and \(D_c \in \mathcal{B}(U_c, Y_c)\) be bounded linear operators (i.e. in principle, matrices), defining the finite dimensional linear control system \(\Sigma_c = (A_c, B_c, C_c, D_c)\) with dynamics
\[
\begin{align*}
\frac{d}{dt} x_c(t) &= A_c x_c(t) + B_c u_c(t) \\
y_c(t) &= C_c x_c(t) + D_c u_c(t), \quad t \geq 0.
\end{align*}
\]

Now, let us consider the standard feedback interconnection between the Euler-Bernoulli beams and the controller given by
\[
\mathfrak{B} x(t) = -y_c(t), \quad u_c(t) = \mathfrak{C} x(t), \quad t \geq 0.
\]
The dynamics of the interconnected system is then described by the abstract Cauchy problem
\[
\begin{align*}
\frac{d}{dt}(x, x_c)(t) &= A(x, x_c)(t), \quad t \geq 0 \\
(x, x_c)(0) &= (x_0, x_{c,0})
\end{align*}
\]
for some given initial data \((x_0, x_{c,0}) \in \mathcal{X} := X \times X_c\), where the product Hilbert space \(\mathcal{X}\) is equipped with the inner product
\[
((x, x_c), (z, z_c))_{\mathcal{X}} = (x, z)_X + (x_c, z_c)_{X_c}, \quad (x, x_c), (z, z_c) \in \mathcal{X} = X \times X_c
\]
and the linear operator \(A : D(A) \subseteq \mathcal{X} \rightarrow \mathcal{X}\) is defined by
\[
A \begin{pmatrix} x \\ x_c \end{pmatrix} = \begin{bmatrix} \mathfrak{A} & 0 \\ B_c & A_c \end{bmatrix} \begin{pmatrix} x \\ x_c \end{pmatrix}
\]
\[
D(A) = \{ (x, x_c) \in D(\mathfrak{A}) \times X_c : B_0 x = 0, B x = -(C_c x_c + D_c x) \}
\]
where the linear operator \(\mathfrak{B}_0 : D(\mathfrak{B}_0) = D(\mathfrak{B}) \subseteq X \rightarrow \mathbb{K}^{2(m-1)}\) is defined by
\[
\mathfrak{B}_0 x = (\mathfrak{B}_0^j x^j)_{j=1}^{m-1}.
\]

**Proposition 2.5 (Well-posedness of the Abstract Cauchy Problem).** The operator \(A\) generates a contractive \(C_0\)-semigroup \((T(t))_{t \geq 0}\) on \(\mathcal{X}\) if and only if \(A\) is dissipative, i.e.
\[
\Re (A(x, x_c) | (x, x_c))_{\mathcal{X}} \leq 0, \quad (x, x_c) \in D(\mathfrak{A}) \times X_c.
\]
Moreover, in that case the operator \(A\) has compact resolvent. In particular, this is the case if \(\mathfrak{S}_0 = (\mathfrak{A}_0, \mathfrak{B}, \mathfrak{C})\) and \(\Sigma_c = (A_c, B_c, C_c, D_c)\) are impedance passive, i.e.
\[
\Re (\mathfrak{A}_0 x | x)_X \leq \Re (\mathfrak{B} x | \mathfrak{C} x)_{\mathbb{R}^2}, \quad x \in D(\mathfrak{A}_0)
\]
\[
\Re (A_c x_c + B_c u_c | x_c)_{X_c} \leq (C_c x_c + D_c u_c | u_c)_{U_c}, \quad x_c \in x_c, u_c \in U_c.
\]
Theorem 3.2 following asymptotic stability result for boundary damping at one of the two ends of the chain of Euler-Bernoulli beams.

Assumption 3.1. All control systems \( \Sigma_j \) on finite-dimensional controller state spaces \( X_j \) (then: \( X_j = \prod_{i=0}^m X_j \) and input and output spaces \( U_j = Y_j = \mathbb{R}^2 \)) for \( j = 1, \ldots, m \), for linear operators \( A_j \in \mathcal{B}(X_j), B_j \in \mathcal{B}(U_j, X_j), C_j \in \mathcal{B}(X_j; Y_j) \) and \( D_j \in \mathcal{B}(U_j; Y_j) \). The feedback interconnection then reads as

\[
\Sigma_j y_j = -y_j, \quad \mathcal{C}_j x = u_j^i
\]

and for the operator \( \mathcal{C}_j \) and input and output spaces \( U \). The feedback interconnection then reads as

\[
\Sigma_j x = \begin{bmatrix} \Sigma^0_j & & \\ & \ddots & \\ & & \Sigma^m_j \end{bmatrix}
\]

Next, we investigate stability properties under a slightly more restrictive condition on the finite dimensional control system. Namely, we assume that

\[
D_j = \begin{bmatrix} D^0_j & & \\ & \ddots & \\ & & D^m_j \end{bmatrix}
\]

is positive definite, for each \( j = 1, \ldots, m \).

Under this structural assumption and some slight regularity conditions on \( \rho \) and \( EI \) we can formulate the following asymptotic stability result for boundary damping at one of the two ends of the chain of Euler-Bernoulli beams.

Theorem 3.2 (Asymptotic Stability). Assume that \( \rho \) and \( EI \) are Lipschitz continuous on every interval \( (t', t) \) for \( j = 1, \ldots, m \), the control systems \( \Sigma_j \) are impedance passive with \( \sigma(A_j) \subseteq \mathbb{C}_0^+ \) and \( B_j = (C_j)^* \) (i.e. collocated input and output) for all \( j = 0, 1, \ldots, m \) and for the operator \( A \) defined above one has

\[
\text{Re} \left( A(x, x) \right) \leq -\kappa |x|^2, \quad x \in D(A)
\]

where \( \kappa > 0 \) and \( \Re : D(\mathfrak{A}) \to \mathbb{R}^4 \) is one of the functions

\[
\begin{bmatrix}
(H^0_{1x}(0) & (H^0_{1x})(0) \\
(H^1_{1x})(0) & (H^1_{1x})(0) \\
(H^2_{1x})(0) & (H^2_{1x})(0) \\
(H^m_{1x})(0) & (H^m_{1x})(0)
\end{bmatrix}
\]

\[
\begin{bmatrix}
(H^0_{2x'}(1) & (H^0_{2x'})(1) \\
(H^1_{2x'})(0) & (H^1_{2x'})(0) \\
(H^2_{2x'})(0) & (H^2_{2x'})(0) \\
(H^m_{2x'})(0) & (H^m_{2x'})(0)
\end{bmatrix}
\]

\[
\begin{bmatrix}
(H^0_{1x'}(0) & (H^0_{1x'})(0) \\
(H^1_{1x'})(0) & (H^1_{1x'})(0) \\
(H^2_{1x'})(0) & (H^2_{1x'})(0) \\
(H^m_{1x'})(0) & (H^m_{1x'})(0)
\end{bmatrix}
\]

\[
\begin{bmatrix}
(H^0_{2x''}(1) & (H^0_{2x''})(1) \\
(H^1_{2x''})(0) & (H^1_{2x''})(0) \\
(H^2_{2x''})(0) & (H^2_{2x''})(0) \\
(H^m_{2x''})(0) & (H^m_{2x''})(0)
\end{bmatrix}
\]

\[
\begin{bmatrix}
(H^0_{1x''}(1) & (H^0_{1x''})(1) \\
(H^1_{1x''})(0) & (H^1_{1x''})(0) \\
(H^2_{1x''})(0) & (H^2_{1x''})(0) \\
(H^m_{1x''})(0) & (H^m_{1x''})(0)
\end{bmatrix}
\]

\[
\begin{bmatrix}
(H^0_{2x'''}(1) & (H^0_{2x'''})(1) \\
(H^1_{2x'''})(0) & (H^1_{2x'''})(0) \\
(H^2_{2x'''})(0) & (H^2_{2x'''})(0) \\
(H^m_{2x'''})(0) & (H^m_{2x'''})(0)
\end{bmatrix}
\]

\[
\begin{bmatrix}
(H^0_{1x'''})(1) & (H^0_{1x'''})(1) \\
(H^1_{1x'''})(0) & (H^1_{1x'''})(0) \\
(H^2_{1x'''})(0) & (H^2_{1x'''})(0) \\
(H^m_{1x'''})(0) & (H^m_{1x'''})(0)
\end{bmatrix}
\]

\[
\begin{bmatrix}
(H^0_{2x''''}(1) & (H^0_{2x''''})(1) \\
(H^1_{2x''''})(0) & (H^1_{2x''''})(0) \\
(H^2_{2x''''})(0) & (H^2_{2x''''})(0) \\
(H^m_{2x''''})(0) & (H^m_{2x''''})(0)
\end{bmatrix}
\]

\[
\begin{bmatrix}
(H^0_{1x''''})(1) & (H^0_{1x''''})(1) \\
(H^1_{1x''''})(0) & (H^1_{1x''''})(0) \\
(H^2_{1x''''})(0) & (H^2_{1x''''})(0) \\
(H^m_{1x''''})(0) & (H^m_{1x''''})(0)
\end{bmatrix}
\]
Then the $C_0$-semigroup $(T(t))_{t \geq 0}$ generated by $A$ is asymptotically stable on $X$, i.e. for every initial value $(x_0, x_1^0) \in X$ one has
\[
T(t)(x_0, x_1^0) \to 0.
\]
In particular, $\sigma(A) = \sigma_p(A) \subseteq C_0^\infty := \{ \lambda \in \mathbb{C} : \text{Re } \lambda < 0 \}$.

**Proof.** Since the operator $A$ has compact resolvent and generates a strongly continuous contraction semigroup on $X$, by the Arendt-Batty-Lyubich-Vu Theorem, see e.g. Theorem V.2.21 in [13], the semigroup is asymptotically stable if and only if
\[
\sigma_p(A) \cap i\mathbb{R} = \emptyset.
\]
Therefore, let $(x, \beta) \in D(A) \times \mathbb{R}$ be such that $i\beta x = Ax$. Then, in particular
\[
0 = \text{Re} (i\beta x | x) = \text{Re} (Ax | x) \leq -\kappa |R^2 x|_k^2 \leq 0
\]
i.e. $\Re x = 0$. Then at least three of the four components of $((H^1x^1)(0), (H^1x^j)(0))$ are zero.

First, assume that $\beta \neq 0$. After possible multiplication by some $\alpha \in \mathbb{C} \setminus \{0\}$, we may and will assume that $(H^1x^1)(0) \geq 0$, $(H^1x^j)(0) \geq 0$, $\beta(H^2x^1)(0) \geq 0$ and $\beta(H^2x^j)(0) \geq 0$. By Lemma 4.2.9 in [4], then either $(H^1x^1) = 0$ on $[0, 1]$, or $(H^1x^1)(\zeta) > 0$, $(H^1x^j)(\zeta) > 0$, $\beta(H^2x^1)(\zeta) > 0$ and $\beta(H^2x^j)(\zeta) > 0$ for all $\zeta \in (0, 1)$. We show the following for $j = 1, \ldots, m - 1$:
\[
(H^jx^1)(1) = (H^jx^j)(1) = 0 \Rightarrow (H^{j+1}x^1)(0) = (H^{j+1}x^j)(0) = 0 \Rightarrow (H^{j+1}x^1)^j(\zeta) = 0, \ \zeta \in [0, 1].
\]
and
\[
(H^1x^1)(1) > 0, (H^1x^j)(1) > 0, 0, i\beta(H^2x^1)(1) > 0 \Rightarrow \beta(H^2x^j)(1) > 0 \Rightarrow \beta(H^2x^1)(1) > 0, \beta(H^2x^j)(1) > 0 \Rightarrow \beta(H^{j+1}x^1)^j(\zeta) > 0, (H^{j+1}x^j)^j(\zeta) > 0, \beta(H^{j+1}x^j)^j(\zeta) > 0, \ \zeta \in [0, 1].
\]
In the first case, one has $C^j x = 0$ and then
\[
x^1 = (i\beta - A_1)^{-1}B_1C^j x = 0 \Rightarrow B_1^j x = -C_1^j x^1 - D_1^j C^1 x = 0
\]
and thus $(H^{j+1}x^j)(0) = (H^1x^j)(1)$ and $(H^{j+1}x^j)^j(0) = (H^jx^j)(1)$, and the assertion follows from Lemma 4.2.9 in [4]. In the latter case, we find that
\[
0 = \text{Re} (i\beta x^j | x^j) = \text{Re} (A_1 x^j + B_1C^j x | x^j) \leq \text{Re} (C_1^j x^j + D_1^j C^j x | C^j x)_{k^2} = \text{Re} ((C_1^j (i\beta - A_1)^{-1}B_1^j + D_1^j)C^j x | C^j x)_{k^2}
\]
and since $\Sigma^j_k$ is impedance passive with $B_1^j = (C_1^j)^*$ it follows that
\[
\text{Sym} (C_1^j (i\beta - A_1)^{-1}B_1^j) \geq 0, \text{ Sym} (D_1^j) \geq 0,
\]
thus in particular $\text{Sym} (D_1^j)C^j x = 0$. In both cases $\text{Sym} (D_1^j)C^j x = 0$ (then: $D_1^j C^j x = \text{Sym}(D_1^j)C^j x = 0$) and $\text{Sym}(D_1^j)$ positive definite (then: $D_1^j C^j x = D_1^j \text{Sym}(D_1^j)C^j x = 0$, we have $D_1^j C^j x = 0$ and then $B_1^j C^j x = 0$, which implies that $B_1^j x = 0$, so that again $(H^{j+1}x^j)^j(0) = (H^jx^j)(1)$ and $(H^{j+1}x^j)^j(0) = (H^jx^j)(1)$, and the assertion follows by the same considerations as in Lemma 4.2.9 of [4]. As the condition
\[
((H^1x^1)^j)(1) > 0, (H^1x^j)^j(1) > 0, \beta(H^2x^1)^j(1) > 0, \beta(H^2x^j)^j(1) > 0
\]
is incompatible with the assumption $\Re x = 0$, this means that only the case $(H^1x^j)^j(0) = (H^1x^j)^j(0) = 0$, and thus $x = 0$ is possible, but then also $x_0 = (i\beta - A_1)^{-1}C^j x = 0$, so that $\beta \in \mathbb{R} \setminus \{0\}$ cannot be an eigenvalue of $A$.

For the case $\beta = 0$, one has $(H^1x^j)^j(0) = 0$ on $(0, 1)$ for each $j = 1, \ldots, m$, but as above one sees that $(H^{j+1}x^j)^j(0) = (H^jx^j)^j(1)$ and $(H^{j+1}x^j)^j(0) = (H^jx^j)^j(1)$, so that
\[
((H^1x^1)(\zeta), (H^1x^j)(\zeta)) = ((H^1x^1)(0), (H^1x^j)(0)) = ((H^1x^j)^j(1), (H^1x^j)^j(1)), \ \zeta \in [0, 1], j = 1, \ldots, m,
\]
but by the condition $\Re x = 0$ this can only be the case if $x = 0$, and then as before also $x_0 = 0$, hence $0$ is no eigenvalue of $A$ and we have shown that $\sigma(A) \cap i\mathbb{R} = \emptyset$, indeed. Asymptotic stability follows by the Arendt-Batty-Lyubich-Vu-Theorem.

In the language of [4], the proof of Theorem 3.2 shows the following:

**Lemma 3.3.** Let $\mathcal{G}_0$, $\Sigma_c$ and $\Re$ be as in Theorem 7.2. Then, the pair $(\mathcal{G}_0, \Re)$ has property ASP, i.e.
\[
\forall \beta \in \Re : \ x \in \ker(\mathcal{G}_0 - i\beta) \cap \ker \Re \Rightarrow x = 0.
\]
Moreover, let $\Re' = (\Re_1, \Re_2, \Re_3) : D(\Re') = D(\Re) = D(\mathcal{G}_0)$, then
\[
\forall \beta \in \Re \setminus \{0\} : \ x \in \ker(\mathcal{G}_0 - i\beta) \cap \ker \Re' \Rightarrow x = 0.
\]
In particular, in the situation of Theorem 3.2 we have that $\sigma_p(A) \cap i\mathbb{R} \subseteq \{0\}$, if the condition 2 is weakened to
\[
\text{Re} (A(x, x_0) | (x, x_0)) \leq -\kappa |R^2 x|_k^2, \ x \in D(A).
\]
Of course, asymptotic stability is just the first step towards exponential stability. To obtain uniform exponential stability as well, we have to

1. impose further restrictions on the boundary conditions at the left ($\zeta = \zeta_0 = 0$) and right end ($\zeta = \zeta_m = L$) of the chain of beams, and
2. impose monotonicity conditions on the parameter functions $\rho$ and $EI$ at their discontinuity points $\nu_j \in \mathbb{N}$, $j = 1, \ldots, m - 1$.

**Theorem 3.4 (Uniform Exponential Stability).** Assume that $\rho$ and $EI$ are Lipschitz continuous on every interval $(\nu_j^{-1}, \nu_j)$ for $j = 1, \ldots, m$, the control systems $\Sigma_k^\nu$ are impedance passive for all $j = 0, 1, \ldots, m$ and for the operator $A$ defined above one has

$$\Re (A(x, x_c) | (x, x_c))_X \leq -\kappa |x|_{\mathbb{K}^5}^2$$

where $\kappa > 0$ and $\Re : D(\mathfrak{A}) \to \mathbb{K}^5$ is of the form

$$\Re x = \begin{pmatrix}
(\mathcal{H}^1 x^1)'(0) \\
(\mathcal{H}^1 x^1)'(0) \\
(\mathcal{H}^1 x^1)'(0) \\
(\mathcal{H}^1 x^1)'(0)
\end{pmatrix}
\quad \text{or} \quad
\Re x = \begin{pmatrix}
(\mathcal{H}_1^m x^m)'(1) \\
(\mathcal{H}_1^m x^m)'(1) \\
(\mathcal{H}_1^m x^m)'(1) \\
(\mathcal{H}_1^m x^m)'(1)
\end{pmatrix}.$$}

Further, in the cases on the left side assume that $\mathcal{H}(1) \leq \mathcal{H}^{-1}(0)$ (i.e. $\mathcal{H}^{-1}(0) - \mathcal{H}(1)$ is positive semidefinite), or in the cases on the right side assume that $\mathcal{H}(1) \geq \mathcal{H}^{-1}(0)$, for all $j = 1, \ldots, m - 1$.

Then the operator $A$ generates a uniformly exponentially stable $C_0$-semigroup $(\mathcal{T}(t))_{t \geq 0}$ if and only if $(\mathcal{T}(t))_{t \geq 0}$ is asymptotically stable.

In particular, this is the case if $\sigma(A) \subseteq \mathbb{C}_0^-$ for $j = 0, 1, \ldots, m$ and

$$\Re x = \begin{pmatrix}
(\mathcal{H}^1 x^1)'(0) \\
(\mathcal{H}^1 x^1)'(0) \\
(\mathcal{H}^1 x^1)'(0) \\
(\mathcal{H}^1 x^1)'(0)
\end{pmatrix}
\quad \text{or} \quad
\Re x = \begin{pmatrix}
(\mathcal{H}_1^m x^m)'(1) \\
(\mathcal{H}_1^m x^m)'(1) \\
(\mathcal{H}_1^m x^m)'(1) \\
(\mathcal{H}_1^m x^m)'(1)
\end{pmatrix}.$$}

**Proof.** First, we show that the pair $(\mathfrak{A}_0, (\Re, B))$ has property AIEP as introduced in Definition 2.8 of [6], i.e. for every sequence $(x_n, \beta_n)_{n \geq 1} \subseteq D(\mathfrak{A}_0) \times \mathbb{R}$ with $\sup_{n \in \mathbb{N}} \|x_n\|_X < \infty$ and $|\beta_n| \to 0$ such that

$$i_\beta x_n - \mathfrak{A}_0 x_n \to 0 \quad \text{in} \quad X, \quad \Re x_n \to 0 \quad \text{in} \quad \mathbb{K}^5, \quad B x_n \to 0 \quad \text{in} \quad \mathbb{K}^{2m}.$$}

it follows that $x_n \to 0$ in $X$. For this end, take $q^j \in C^2([0, 1]; \mathbb{R})$ for $j = 1, \ldots, m$, which we will specify in a moment. Then, by the proof of Proposition 4.3.19 in [4], for every sequence $(x_n, \beta_n)_{n \geq 1} \subseteq D(\mathfrak{A}_0) \times \mathbb{R}$ with $\sup_{n \in \mathbb{N}} \|x_n\|_X < \infty$, $|\beta_n| \to 0$ and $\mathfrak{A}_0 x_n - i_\beta x_n \to 0$ in $X$, it holds that

$$\Re (x_n^1 \mid (2q^j(\mathcal{H}^j - \mathcal{H}^{j+1})x_n^1)_{L_2} = o(1) - 2 \Re \left[ \left( - \mathcal{H}^{j+1} x_n^j(\zeta) \mid \frac{i q_j(\zeta)}{\beta_n} \mathcal{H}^j x_n^j(\zeta) \right) \right]_{K, 0}^1 + \left[ \left( x_n^j(\zeta) \mid q^j \mathcal{H}^j x_n^j(\zeta) \right) \right]_{K, 0}^1 \quad \text{and} \quad \Re (x_n^1 \mid (2q^{m}(\mathcal{H}^{m} - \mathcal{H}^{1})x_n^1)_{L_2} = o(1) - 2 \Re \left[ \left( - \mathcal{H}^{m} x_n^m(1) \mid \frac{i q^m(1)}{\beta_n} \mathcal{H}^1 x_n^m(1) \right) \right]_{K, 0}^1 + \left[ \left( x_n^m(1) \mid q^m \mathcal{H}^m(1)x_n^m(1) \right) \right]_{K, 0}^1$$

where $o(1)$ denotes further terms which tend to zero as $n \to \infty$. (Note that there is a typo in equation (4.27) in [4]. There actually should be a minus sign in front of the last line of the equation.) Summing up these equalities and writing $Q(\zeta) = \text{diag} (q^j(\zeta))_{j=1}^m$, we find that

$$\Re (x_n \mid (2Q^j(\mathcal{H} - \mathcal{H}^{j+1})x_n^j)_{L_2} = o(1) - 2 \Re \left[ \left( - \mathcal{H}^{j+1} x_n^j(1) \mid \frac{i q^j(1)}{\beta_n} \mathcal{H}^j x_n^j(1) \right) \right]_{K, 0}^1 + \left[ \left( x_n^j(1) \mid q^j \mathcal{H}^j(1)x_n^j(1) \right) \right]_{K, 0}^1$$

and

$$\Re x = \begin{pmatrix}
(\mathcal{H}^1 x^1)'(0) \\
(\mathcal{H}^1 x^1)'(0) \\
(\mathcal{H}^1 x^1)'(0) \\
(\mathcal{H}^1 x^1)'(0)
\end{pmatrix}
\quad \text{or} \quad
\Re x = \begin{pmatrix}
(\mathcal{H}_1^m x^m)'(1) \\
(\mathcal{H}_1^m x^m)'(1) \\
(\mathcal{H}_1^m x^m)'(1) \\
(\mathcal{H}_1^m x^m)'(1)
\end{pmatrix}.$$}
\[ + 2 \sum_{j=1}^{m-1} \left\{ \operatorname{Re} \left( -(H^j_{2,x_n,2})'(1) \mid iq^m(1) \beta_n \right) (H^j_{m,n,1})(1) \right\}_K - \left\{ \operatorname{Re} \left( -(H^{j+1}_{2,x_n,2})'(0) \mid i(q^{j+1})(0) \beta_n \right) (H^{j+1}_{1,x_n,1})(0) \right\}_K \]

\[ - 2 \sum_{j=1}^{m-1} \left\{ \operatorname{Re} \left( -(H^j_{2,x_n,2})'(1) \mid iq^m(1) \beta_n \right) (H^j_{m,n,1})(1) \right\}_K - \left\{ \operatorname{Re} \left( -(H^{j+1}_{2,x_n,2})'(0) \mid i(q^{j+1})(0) \beta_n \right) (H^{j+1}_{1,x_n,1})(0) \right\}_K \]

We show that for a suitable choice of the functions \( q^j \) and under the dissipation assumptions of the theorem that the terms on the right hand side all vanish as \( n \to \infty \). For this end, from Lemma 2.15 in [4], an interpolation argument and embedding theorems for Sobolev Slobodetskii spaces into \( H^k \)-spaces we conclude that

\[ \frac{H_{x_n}}{\beta_n} \to 0, \quad \text{in } C^1([0,1]; \mathbb{R}^{2m}). \]

Let us discuss the proper choice for the functions \( q^j \) in the situation where

\[ \Re x = \left( \begin{array}{c} (H^1_{x,1})(0) \\ (H^m_{1,x,1})(0) \end{array} \right). \]

For the other cases the result then follows by symmetry. So let \( \Re \) be given as above and additionally assume that \( \Re x_n \to 0 \).

(1) The terms

\[ -2 \operatorname{Re} \left\{ -(H^m_{2,x_n,2})'(1) \mid iq^m(1) \beta_n \right) (H^m_{1,x_n,1})(1) \rangle_{K} + [(q^m(1) \mid q^m H^m(1) x^m_{1}(1))]_{K=0} \]

not only tend to zero, but are zero for all \( n \in \mathbb{N} \), if we demand that

\[ q^m(1) = 0. \]

(2) For the third term, we have

\[ \left| \operatorname{Re} \left( -(H^m_{2,x_n,2})'(1) \mid iq^m(1) \beta_n \right) (H^m_{1,x_n,1})(1) \rangle_{K} \right| \leq |(q^m)'(1)| \beta_n \left| (H^m_{2,x_n,2})(1) \right| \left| (H^m_{1,x_n,1}) \right| \to 0 \]

as by \( \Re x_n \to 0 \) one of the terms \( |(H^m_{2,x_n,2})(1)| \) or \( |(H^m_{1,x_n,1})| \) tends to zero and in any case \( \frac{1}{\beta_n} \left| (H^m_{1,x_n,1}) \right| \) and \( \frac{1}{\beta_n} \left| (H^m_{2,x_n,2})(1) \right| \) tend to zero as well.

(3) Similarly,

\[ \left| \operatorname{Re} \left( -(H^m_{2,x_n,2})'(1) \mid iq^m(1) \beta_n \right) (H^m_{1,x_n,1})(1) \rangle_{K} \right| \leq |(q^m)'(1)| \beta_n \left| (H^m_{2,x_n,2})(1) \right| \left| (H^m_{1,x_n,1}) \right| \to 0 \]

as \( |(H^m_{2,x_n,2})(1)| \) or \( |(H^m_{1,x_n,1})| \) tends to zero due to \( \Re x_n \to 0 \).

(4) Since at least one of the terms \( (H^j_{2,x_n,2})'(0) \) or \( (H^j_{1,x_n,1})'(0) \) tends to zero,

\[ \left| \operatorname{Re} \left( -(H^j_{2,x_n,2})'(0) \mid iq^j(0) \beta_n \right) (H^j_{1,x_n,1})(0) \right| \to 0. \]

(5) For the terms

\[ - [(q^1(0) \mid q^j H^1(0) x^1_{n,1})]\] - \[ \operatorname{Re} \left( -(H^j_{2,x_n,2})'(0) \mid iq^j(0) \beta_n \right) (H^j_{1,x_n,1})(0) \] - \[ \operatorname{Re} \left( -(H^j_{2,x_n,2})'(0) \mid iq^j(0) \beta_n \right) (H^j_{1,x_n,1})(0) \]

we use that \( (H x_n)(0) \to 0 \).

This concludes the discussion of the boundary terms at the left and right end of the chain of beams. For the interconnection points, we use the continuity of two of the components, and obtain the following.

(6) For each \( j = 1, \ldots, m - 1 \) we have that

\[ + \operatorname{Re} \left( -(H^j_{2,x_n,2})'(0) \mid i(q^{j+1}) (0) \beta_n \right) (H^j_{1,x_n,1})(0) \] - \[ \operatorname{Re} \left( -(H^{j+1}_{2,x_n,2})'(0) \mid i(q^{j+1}) (0) \beta_n \right) (H^{j+1}_{1,x_n,1})(0) \]

\[ = \operatorname{Re} \left( \beta^j x_n \mid i(q^{j+1})(0) \beta_n \right) \]

if we choose \( q^j \) and \( q^{j+1} \) such that

\[ (q^j)'(1) = (q^{j+1})'(0), \quad j = 1, \ldots, m - 1 \]
and for all the allowed cases for the control and observation maps $\mathfrak{B}^j$ and $\mathfrak{C}^j$. However, since $\mathfrak{B}x_n \to 0$ by assumption and $\frac{x_n}{\beta_n} \to 0$, we find that these terms tend to zero as $n \to \infty$ as well:

$$\text{Re} \left( \frac{i(q^{j+1})'(0)}{\beta_n} \mathfrak{C}^j x \right) \to 0.$$  

(7) To handle the terms

$$\text{Re} \left( -(\mathcal{H}_2^j x_{n,2})'(1) \right) - \text{Re} \left( -(\mathcal{H}_2^j x_{n,2})'(0) \right) \to 0, \quad n \to \infty.$$

we note that both terms $$(\mathcal{H}_2^j x_{n,2})'(1) - (\mathcal{H}_2^j x_{n,2})'(0)$$ tend to zero (if the corresponding term is a component of $\mathfrak{B}^j$) or even equal zero (if the corresponding term constitutes a component of $\mathfrak{B}^j$). As $\frac{1}{\beta_n}(\mathcal{H}_1^j x_{n,2})'(0)$ and $\frac{1}{\beta_n}(\mathcal{H}_2^j x_{n,2})'(0)$ tend to zero as well since $\frac{x_n}{\beta_n} \to 0$ in $C^1([0,1]; \mathbb{K}^{2m})$, it then follows that also

$$\text{Re} \left( -\frac{i(q^{j+1})'(1)}{\beta_n} \mathfrak{C}^j x_{n,1} \right) \to 0, \quad n \to \infty.$$

(8) Lastly, the terms

$$(x_n^j(1) | (q^j \mathcal{H}^j(1)x_{n,1}(1)) - (x_n^{j+1}(0) | (q^{j+1} \mathcal{H}^{j+1})x_{n,1}^{j+1}(0)) \right)_\mathbb{K} = q^{j+1}(0) (x_n^j(1) | \mathcal{H}^j x_{n,1}(1)) - (x_n^{j+1}(0) | \mathcal{H}^{j+1}(0)x_{n,1}^{j+1}(0)) \right)_\mathbb{K}$$

have to be handled, where due to previous choices of $q^j$ and $q^{j+1}$ we do not have freedom left to choose the relation between $q^j(1) = q^{j+1}(0)$ in any other way. Therefore, for the moment we leave these terms as they are and demand monotonicity conditions on $\mathcal{H}^j(1) - \mathcal{H}^{j+1}(0)$ later on.

Putting things together, and choosing the functions $q^j$ such that $q^j(\zeta) = q(j - 1 + \zeta)$ for some function $q \in C^2([0,1]; \mathbb{R})$ with $q(j) = 0$, we find that

$$\text{Re} \left( x_n | (2q^j q^j)(x_n) \right)_{L_2} = 2 \sum_{j=1}^{m-1} q(j) \left( (x_n^j(1) | (q^j \mathcal{H}^j(1)x_{n,1}(1))_\mathbb{K} - (x_n^{j+1}(0) | (q^{j+1} \mathcal{H}^{j+1})x_{n,1}^{j+1}(0))_\mathbb{K} \right) + o(1).$$

We want the term on the left hand side to define an equivalent inner product on $L_2(0,1; \mathbb{K}^{2m})$ and therefore choose $q \in C^2([0,1]; \mathbb{R})$ with $q(j) = 0$ and $q^j > 0$, $q \leq 0$ such that

$$2(q^j)' \mathcal{H}^j - q^j(\mathcal{H}^{j+1})' \geq 2m_0(q^j)' + q^j M_1 \geq \varepsilon_0$$

for

$$m_0 := \inf \{ \varepsilon > 0 : \mathcal{H}^j(\zeta) - \varepsilon I \text{ positive semidefinite for a.e. } \zeta \in (0,1), j = 1, \ldots, m \},$$

$$M_1 := \inf \{ \varepsilon > 0 : \mathcal{H}^j(\zeta)'(\zeta) \text{ positive semidefinite for a.e. } \zeta \in (0,1), j = 1, \ldots, m \}$$

and some $\varepsilon_0 > 0$. We then have

$$\varepsilon_0 \|x_n\|_{L_2}^2 \leq 2 \sum_{j=1}^{m-1} q(j) \left( (x_n^j(1) | (q^j(1)x_{n,1}(1))_\mathbb{K} - (x_n^{j+1}(0) | (q^{j+1}(0)x_{n,1}^{j+1}(0))_\mathbb{K} \right) + o(1).$$
Since for every \( j = 1, \ldots, m - 1 \), the term \( q(j) < 0 \) is negative the right hand side is less or equal \( o(1) \), if for all \( j = 1, \ldots, m - 1 \), we have
\[
\left( x_n^j(1) | \mathcal{H}^j(1)x_n^{j+1}(1) \right)_K = (x_n^{j+1}(0) | \mathcal{H}^{j+1}(0)x_n^{j+1}(0))_K \\
= ((\mathcal{H}^j x_n^j)(1) - (\mathcal{H}^{j+1} x_n^{j+1})(0)) (\mathcal{H}^j x_n^j)(1) - (\mathcal{H}^{j+1} x_n^{j+1})(0))_K \\
+ 2 ((\mathcal{H}^{j+1} x_n^{j+1})(0) | (\mathcal{H}^j(1)^{-1} - (\mathcal{H}^{j+1}(0)^{-1})((\mathcal{H}^j x_n^j)(1) - (\mathcal{H}^{j+1} x_n^{j+1})(0)))_K \\
= \frac{3}{2} ((\mathcal{H}^j x_n^j)(1) - (\mathcal{H}^{j+1} x_n^{j+1})(0)) | (\mathcal{H}^j(1)^{-1} - (\mathcal{H}^{j+1}(0)^{-1})((\mathcal{H}^j x_n^j)(1) - (\mathcal{H}^{j+1} x_n^{j+1})(0)))_K \\
+ \frac{1}{2} ((\mathcal{H}^{j+1} x_n^{j+1})(0) | (\mathcal{H}^j(1)^{-1} - (\mathcal{H}^{j+1}(0)^{-1})((\mathcal{H}^j x_n^j)(1) + (\mathcal{H}^{j+1} x_n^{j+1})(0)))_K.
\]

By equation assumption we have \( \mathfrak{B}_1 x_n \to 0 \) and \( \mathfrak{B}_2 x_n = 0 \). By this, the term \( (\mathcal{H}^j x_n^j)(1) - (\mathcal{H}^{j+1} x_n^{j+1})(0) \) and hence the first term converge to zero as \( n \to \infty \). For the second term, we can ensure that it is non-negative, if we use the additional structural assumption
\[
(\mathcal{H}^j(1)^{-1} - (\mathcal{H}^{j+1}(0)^{-1})
\]
(Recall that in terms of \( \rho \) and \( EI \) this means that \( \rho(l^j-1) \geq \rho(l^j+1) \) and \( (EI(l^j))^(-1) \leq (EI(l^j+1))^(-1) \). Then
\[
(x_n | (2Q^j \mathcal{H} - Q \mathcal{H}^j)x_n)_{L_2} \to 0
\]
and since this defines a norm (squared) equivalent to the standard \( L_2 \)-norm \( \|\cdot\|_{L_2} \) and the energy norm \( \|\cdot\|_X \) on \( X \), this implies that \( x_n \to 0 \) in \( X \) and we have successfully proved property AEIP. We record this preliminary result as the following

**Proposition 3.5.** Let the conditions of Theorem 3.4 be satisfied. Then the pair \((\mathfrak{A}_0, (\mathfrak{R}, \mathfrak{B})\) has property AEIP, i.e. for every sequence \((x_n, \beta_n)_{n \geq 1} \subseteq D(\mathfrak{A}_0) \times \mathbb{R}\) with
\[
\sup_{n \in \mathbb{N}} \|x_n\|_X < \infty,
\]
\[
|\beta_n| \to 0,
\]
\[
(\mathfrak{A}_0 - i \beta_n)x_n \to 0,
\]
\[
\forall x_n, \mathfrak{B}x_n \to 0
\]
it follows that
\[
\|x_n\|_{L_2} \to 0.
\]

**Proof of Theorem 3.4 (Continued).** We are now ready to show the assertion of Theorem 3.4. Assume that \((T(t))_{t \geq 0}\) is asymptotically stable, so that \( \sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \subseteq \mathbb{R}\) by the Arendt-Batty-Lyubich-Vu Theorem. Thanks to the Gearhart-Greiner-Prüss Huang Theorem, see e.g. Theorem V.1.10 in [13], it suffices to prove that
\[
\sup_{\beta \in \mathbb{R}} \left\| (i\beta - \mathcal{A})^{-1} \right\|_X < \infty
\]
and by e.g. Remark 2.7 in [6] this is equivalent to the statement:
\[
\left\{ \begin{array}{l}
\|\mathfrak{T}_n\|_X < \infty, \quad |\beta_n| \to \infty,
\end{array} \right\} \quad \Rightarrow \quad \|\mathfrak{T}_n\|_X \to 0.
\]
Let \((\mathfrak{T}_n, \beta_n)_{n \geq 1}\) be such a sequence. Then,
\[
0 \leftrightarrow \text{Re} \left( (A - i\beta_n)\mathfrak{T}_n | \mathfrak{T}_n \right) = \text{Re} \left( A\mathfrak{T}_n | \mathfrak{T}_n \right) \leq -\kappa |\Re x_n|_X,
\]
and hence \( \Re x_n \to 0 \). Having property AEIP for the pair \((\mathfrak{A}_0, (\mathfrak{R}, \mathfrak{B})\) at hand, we show that
(1) \( x_n \to 0 \) in \( X \), and
(2) \( \mathfrak{B}x_n \to 0 \) in \( \mathfrak{X}^{\mathbb{C}^{2(m+1)}} \)

The property \( x_n \to 0 \) in \( X \) and hence \( \mathfrak{T}_n \to 0 \) in \( X \) will then follow by property AEIP.

As we have seen previously the term \( \mathfrak{C}_n \) tends to zero as \( n \to \infty \) and hence we obtain convergence to zero for
\[
x_{c,n} = (i\beta_n - A_c)^{-1}(C x_n + o(1)) = \beta_n (i\beta_n - A_c)^{-1} \left( \frac{C x_n}{\beta_n} + \frac{1}{|\beta_n|} o(1) \right) \to 0,
\]
as the resolvent operators \( \beta(i\beta - A_c)^{-1} \) are uniformly bounded for \( \beta \in \mathbb{R} \). By the interconnection condition for \( \mathfrak{B}x \) this implies that
\[
\mathfrak{B}x_n = -(C_c(i\beta_n - A_c)^{-1}B_c + D_c)x_n + o(1).
\]
Next, we show that $\Re x_n \to 0$ using the impedance passivity of the systems $S = (\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ and $\Sigma_c = (A_c, B_c, C_c, D_c)$. The latter implies that $\text{Sym} \left( C_c(i\beta_n - A_c)^{-1}B_c + D_c \right) \geq 0$, hence

$$0 \leq \Re \left( \Re \left( \mathfrak{A} x_n - i\beta_n x_n \mid x_n \right) \right) \leq \Re \left( \Re \left( \mathfrak{A} x_n \mid x_n \right) \right)$$

First, this implies that $\text{Sym} \left( C_c(i\beta_n - A_c)^{-1}B_c + D_c \right) \mathfrak{C} x_n \to 0$. However, this also implies that $\text{Sym} \left( D_c \right) \mathfrak{C} x_n \to 0$ since

$$|\text{Sym} \left( C_c(i\beta_n - A_c)^{-1}B_c \right) \mathfrak{C} x_n| \leq \frac{|\mathfrak{C} x_n|}{|\beta_n|} \to 0.$$

To conclude that actually $D_c^j \mathfrak{C} x_n \to 0$ for $j = 0, 1, \ldots, m$ we use the additional structural constraints on the matrices $D_j \in \mathbb{K}^{2 \times 2}$:

$$D_c^j = \text{sym}, \text{positive semidefinite diagonal matrix or Sym } D_c^j \text{ is symmetric positive definite.}$$

If $D_c = \text{Sym } D_c$, then immediately $D_c \mathfrak{C} x_n = \text{Sym} \left( D_c \right) \mathfrak{C} x_n \to 0$, whereas in the case $\text{Sym } D_{c,j} > 0$ we get

$$D_c^j \mathfrak{C} x_n = D_c^j \left( \text{Sym } D_c^j \right)^{-1} \text{Sym } D_c^j \mathfrak{C} x_n \to 0$$

as well, i.e. in either case

$$\Re \mathfrak{C} x_n = - \left( C_c^j x_n^* + D_c^j \mathfrak{C} x_n \right) \to 0, \quad j = 0, 1, \ldots, m.$$

By property AIEP of the pair $(\mathfrak{A}, (\mathfrak{B}, \mathfrak{C})$ as shown above in Proposition 3.3, it follows that also $x_n \to 0$ in $X$, therefore $\dot{x}_n \to 0$ in $X$ and the assertion follows.

In the energy state space formulation we used for the proof of well-posedness, asymptotic and exponential stability these are only four cases because in the energy state space no destruction is made between the cases $\omega(t, 1) = 0$ and $\omega(t, 1) = 0$ (i.e. $\omega(t, 1) = c$), cf. the first and fifth case, and between the cases $\omega(t, 1) = 0$ and $\omega(t, 1) = 0$ (i.e. $\omega(t, 1) = c$), cf. the third and last case, respectively. In energy state space these conditions above therefore read as:

(1) $(\mathcal{H}^m x_n(1)) = 0$, 
(2) $(\mathcal{H}^m x_n(1)) = (\mathcal{H}^m x_n)'(1) = 0$, 
(3) $(\mathcal{H}^m x_n)'(1) = 0$, 
(4) $(\mathcal{H}^m x_n)'(1) = 0$. 

**Example 3.7** (Stability of Serially Connected Euler-Bernoulli Beams). Let $L > 0$ and $0 = l^0 < l^1 < \ldots < l^m = L$ for some $m \in \mathbb{N}$. Let $\rho, EI : (0, L) \setminus \bigcup_{j=1}^{m-1} \{l^j \} \to \mathbb{R}$ be uniformly positive and Lipschitz-continuous on every subinterval $(l^j, l^{j+1})$ for $j = 0, 1, \ldots, m-1$, and assume that for every $j = 1, \ldots, m-1$ one has the jump conditions:

$$\rho(l^{j-}) \geq \rho(l^{j+}), \quad (EI(l^{j-}) \leq (EI(l^{j+}), \quad j = 1, \ldots, m-1.$$

On each interval $(l^j, l^{j+})$ consider the Euler-Bernoulli beam equation

$$\rho(\zeta) \omega_{\zeta}(t, \zeta) + (EI(\omega_{\zeta})_{\zeta} \zeta(0, \zeta) = 0, \quad t \geq 0, \zeta \in (l^j, l^{j+}), \quad j = 0, 1, \ldots, m-1.$$

Further assume that at the left end $\zeta = l^0 = 0$ the following dissipative boundary conditions is imposed:

$$\begin{pmatrix}
(EI(\omega_{\zeta})(0)) & - (EI(\omega_{\zeta})(0)) \\
(\omega_{\zeta}(t, 0)) & (\omega_{\zeta}(t, 0))
\end{pmatrix} = - K_0 \begin{pmatrix}
(\omega_{\zeta}(t, 0)) \\
(\omega_{\zeta}(t, 0))
\end{pmatrix}$$

for some $K_0 \in \mathbb{K}^{2 \times 2}$ such that

$$K_0 = \begin{pmatrix}
k_{11}^0 & 0 \\
0 & 0
\end{pmatrix} \quad \text{or Sym } (K_0) \quad \text{is symmetric positive definite}.$$

and assume that at the right end conservative boundary conditions of the form

(1) $\omega(t, 1) = (EI\omega_{\zeta})(t, 1) = 0$ (simply supported or pinned right end) if Sym $(K_0)$ is positive definite,
(2) $\omega_{\zeta}(t, 1) = (EI(\omega_{\zeta})_{\zeta})(t, 1) = 0$ (free right end) if Sym $(K_0)$ is positive definite,
are imposed. Further assume that for every $j \in \{1, \ldots, m-1\}$ one of the following interconnection conditions hold true:

1. $\omega_t(t, V^-) = \omega_t(t, V^+)$, $\omega_{\xi_t}(t, V^-) = \omega_{\xi_t}(t, V^+)$ and
   \[
   \left( - (EI\omega_{\zeta\zeta})_\zeta (t, V^-) + (EI\omega_{\zeta\zeta})_\zeta (t, V^+) \right) / (EI\omega_{\zeta\zeta})_\zeta (t, V^-) - (EI\omega_{\zeta\zeta})_\zeta (t, V^+) = -K_j \left( \frac{\omega_t(t, V)}{\omega_{\xi_t}(t, V)} \right)
   \]
2. $\omega_t(t, V^-) = \omega_t(t, V^+)$, $(EI\omega_{\zeta\zeta})_\zeta (t, V^-) = (EI\omega_{\zeta\zeta})_\zeta (t, V^+)$ and
   \[
   \left( - (EI\omega_{\zeta\zeta})_\zeta (t, V^-) + (EI\omega_{\zeta\zeta})_\zeta (t, V^+) \right) / \omega_{\xi_t}(t, V^-) - \omega_{\xi_t}(t, V^+) = -K_j \left( \frac{\omega_t(t, V)}{\omega_{\xi_t}(t, V)} \right)
   \]
3. $-(EI\omega_{\zeta\zeta})_\zeta (t, V^-) = -(EI\omega_{\zeta\zeta})_\zeta (t, V^+)$, $\omega_{\xi_t}(t, V^-) = \omega_{\xi_t}(t, V^+)$ and
   \[
   \omega_t(t, V^-) - \omega_t(t, V^+) = (EI\omega_{\zeta\zeta})_\zeta (t, V^-) / \omega_{\xi_t}(t, V^-) - (EI\omega_{\zeta\zeta})_\zeta (t, V^+) / \omega_{\xi_t}(t, V^+) = -K_j \left( \frac{\omega_t(t, V)}{\omega_{\xi_t}(t, V)} \right)
   \]
4. $-(EI\omega_{\zeta\zeta})_\zeta (t, V^-) = -(EI\omega_{\zeta\zeta})_\zeta (t, V^+)$, $(EI\omega_{\zeta\zeta})_\zeta (t, V^-) = (EI\omega_{\zeta\zeta})_\zeta (t, V^+)$ and
   \[
   \omega_t(t, V^-) - \omega_t(t, V^+) = (EI\omega_{\zeta\zeta})_\zeta (t, V^-) / \omega_{\xi_t}(t, V^-) - (EI\omega_{\zeta\zeta})_\zeta (t, V^+) / \omega_{\xi_t}(t, V^+) = -K_j \left( \frac{\omega_t(t, V)}{\omega_{\xi_t}(t, V)} \right)
   \]
where $K_j \in \mathbb{R}^{2 \times 2}$ is a diagonal positive semidefinite symmetric matrix or has a positive definite symmetric part $\text{Sym}(K_j) > 0$. Then, for every initial datum
\[
(\omega(0, \cdot), \omega_t(0, \cdot)) = (\omega_0, \omega_1) \in H^2(0, L) \setminus \{0\}
\]
there is a unique strong solution
\[
\omega \in C([0, \infty); H^2(0, L) \setminus \{0\})
\]
of the Euler-Bernoulli-beam system, depending continuously on the initial data, and there are $M \geq 1$ and $\eta < 0$ independent of the initial data, such that the energy
\[
H(t) := \int_0^1 \rho(\zeta) |\omega_t(\zeta)|^2 + EI(\zeta) |\omega_{\xi_t}(\zeta)|^2 \, d\zeta
\]
decays uniformly exponentially, i.e.
\[
H(t) \leq M e^{-\eta t} H(0), \quad t \geq 0.
\]

**Proof.** In either case we have
\[
\text{Re} \left( A\hat{x} \mid \hat{x} \right) \leq -\kappa \left( |(H_j^2 x_j)^{(0)}|^2 + |(H_1^1 x_1)^{(0)}|^2 \right), \quad x \in D(A)
\]
for some $\kappa > 0$. For $\beta \in \sigma_p(A)$ to be excluded by demanding that $\text{Sym}(K_0) > 0$, so that
\[
\text{Re} \left( A\hat{x} \mid \hat{x} \right) \leq -\kappa \left( |(H_1^1 x_1)^{(0)}|^2 + |(H_j^2 x_j)^{(0)}|^2 \right), \quad x \in D(A).
\]
Then, in all cases uniform exponential stability follows by Theorem 3.4. \qed

**Remark 3.8.** For the second and third case and $K_0 = \text{diag}(k_{11}^1, 0)$, either $\sigma_p(A) \cap i\mathbb{R} = \emptyset$, i.e. if also
\[
\text{Re} \left( A\hat{x} \mid \hat{x} \right) \leq -\kappa \left( |(H_j^1 x_j)^{(0)}|^2 \right), \quad x \in D(A)
\]
for some $j = 1, \ldots, m$, i.e. by suitable damping in one of the joints at $V$, and then by Theorem 3.4 the system is uniformly exponentially stable, or $\sigma_p(A) \cap i\mathbb{R} = \emptyset$. (The proof of Theorem 3.2 shows that $\sigma_p(A) \cap i\mathbb{R} \subseteq \{0\}$ already for $\mathfrak{R} = ((H_j^2 x_j)^{(0)}, (H_1^1 x_1)^{(0)}))$. In fact, then the only candidate for an eigenfunction is given by
\[
(H_1^1 x_1)(\zeta) = 0, \quad (H_j^1 x_j)(\zeta) = \zeta (H_1^1 x_1)(\zeta), \quad i \in \mathbb{Z}
\]
i.e. from $(H_1^1 x_1)^{(1)} = (H_1^1 x_1^{j+1})^0(0)$ it follows that
\[
(H_j^1 x_j)(\zeta) = (j + 1 + \zeta)(H_1^1 x_1)^{(0)} = (j + 1 + \zeta) c, \quad \zeta \in [0, 1], \quad j = 1, \ldots, m
\]
for some $c \in \mathbb{K}$ and the eigenspace $\ker(A)$ is one-dimensional. This corresponds to the dynamical solution
\[
\omega_t(t, \zeta) = (j + 1 + \zeta) c + \omega_1(0, \zeta), \quad \text{clearly a solution which for moderately large } t > 0 \text{ does not satisfy the modelling conditions for a linear Euler-Bernoulli beam, in particular, the assumption that } |\omega_t(t, \zeta) - \omega^{\text{ref}}(\zeta)| \ll 1 \text{ for some reference configuration } \omega^{\text{ref}}.
\]
A physical interpretation of this eigenstate would be a beam which is rotating in
the transversal flat. Such phenomena are, as already stated, not covered by the linear beam model. However, after restricting the initial data to \(\ker(A^+)^\perp\), i.e.

\[
\sum_{j=1}^m \int_0^1 (j - 1 + \zeta)(H_j(x')\xi(0, \zeta) d\zeta = \sum_{j=1}^m \int_0^1 (j - 1 + \zeta)\tilde{H}_j((1 - \zeta)p + \zeta^{p+1})\tilde{\xi}_1((1 - \zeta)p + \zeta^{p+1})
\]

\[
= \sum_{j=1}^m \int_0^p \int_{\omega_j}^{\omega_{j+1}} (j - 1 + \zeta) \omega_i(0, \zeta) \, d\zeta = 0,
\]

by linearity, compactness of the resolvent \((\sigma_p(A) \cap B_1(0) \text{ is discrete!})\) and Theorem 3.4 also for the second and third case, the solution tends uniformly exponentially to zero, also in the case \(K_0 = \text{diag}(k_{11}, 0)\) for some \(k_{11} > 0\).

4. Conclusion

In this paper we presented a proof via the resolvent method for the uniform stabilisation of a chain of serially connected non-uniform Euler-Bernoulli beams with damping at one end. We considered several possible interconnection conditions and pairs of dissipative / conservative boundary conditions at the ends of the chain which enforce uniform exponential energy decay for the beam system. We thereby not only generalised the results in [8] to the case of non-uniform beams (which in this generality seems not to be possible by their method), but identified several other possible combinations of dissipative-conservative pairs boundary conditions at the left and right end of the chain leading to exponential energy decay as well. Moreover, we showed that instead of static boundary or feedback interconnections, dynamic feedback interconnections with finite dimensional control systems can be used as well to achieve well-posedness and stability results.

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