ON A NONLINEAR PARABOLIC PROBLEM: STABILITY PROPERTIES OF GROUND STATES

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Abstract. We consider the Cauchy-problem for the following parabolic equation:
\[ u_t = \Delta u + f(u, |x|), \]
where \( x \in \mathbb{R}^n, n > 2, \) and \( f = f(u, |x|) \) is either critical or supercritical with respect to the Joseph-Lundgren exponent. Using a new unifying approach we extend to a larger class of nonlinear potentials \( f \), some known results concerning stability and weak asymptotic stability of positive Ground States.

1. Introduction

In this paper we discuss the stability properties of positive radial solutions for the following equation
\[ \Delta u + f(u, |x|) = 0, \]
which are positive steady states of the following Cauchy problem
\[
\begin{align*}
  u_t &= \Delta u + f(u, |x|), \\
  u(x, 0) &= \phi(x),
\end{align*}
\]
where \( x \in \mathbb{R}^n, n > 2, \) and \( f = f(u, |x|) \) is a potential which is null for \( u = 0 \), superlinear in \( u \), and supercritical in a sense that will be specified just below.

Let \( u(x, t; \phi) \) be the solution of (1.2)–(1.3). The analysis of the long time behavior of \( u(x, t; \phi) \) is strongly based on the separation properties of the radial solutions of (1.1). If \( u(x) \) is a radial solutions of (1.1), setting \( U(r) = u(x) \) when \( r = |x| \), we find that \( U = U(r) \) solves
\[
U'' + \frac{n-1}{r} U' + f(U, r) = 0,
\]
where ""'" denotes the derivative with respect to \( r \). In the whole paper we denote by \( U(r, \alpha) \) the unique solution of (1.4) with the initial condition \( U(0, \alpha) = \alpha > 0 \).

In the last decades the Cauchy problem (1.2)–(1.3) has raised a great interest, starting from the model case \( f(u, |x|) = u^{q-1} \), and it has been analyzed by several authors (see, e.g., [3, 10, 19, 20, 21, 29, 30, 31, 33, 34]). Since in the whole paper we are interested in positive solutions, there is no ambiguity in using the notation \( u^{q-1} \). It is well known that the behavior of solutions of (1.4), and consequently of (1.2), changes drastically as \( q \) passes through some critical values. In this paper we focus on the case where \( q > 2^* := \frac{2n}{n-2} \), so that for any \( \alpha > 0 \) the solution \( U(r, \alpha) \) of (1.4) is positive and bounded for any \( r > 0 \) (i.e. it is a Ground State), and

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especially on the case \( q \geq \sigma^* \), where
\[
\sigma^* := \begin{cases} 
\frac{(n-2)^2-4n+8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n > 10, \\
+\infty & \text{if } n \leq 10,
\end{cases}
\]
so that Ground States (GSs) gain some stability properties (see \[34\]). We recall that \( 2^* \) is the Sobolev critical exponent, which is related to the compactness of the embedding of \( L^q \) in \( H^1 \), while \( \sigma^* \) is the Joseph-Lundgren exponent, \[24\].

When \( 2^* < q < \sigma^* \) all the GSs intersect each other indefinitely, and this fact is used to construct suitable sub- and supersolution for (1.1). Then, it is possible to show that, in this range of parameters, GSs determine the threshold between solutions of (1.2) that blow up in finite time, and solutions that exist for any \( t \) and fade away.

**Theorem A.** \[34, 19\] Assume \( f(u, r) = u^{q-1}, 2^* < q < \sigma^* \). Then

1. If there is \( \alpha > 0 \) such that \( \phi(x) \geq U(|x|, \alpha) \), then there is \( T(\phi) \) such that
   \[
   \lim_{t \to T(\phi)} \|u(t, x; \phi)\|_\infty = +\infty.
   \]
2. If there is \( \alpha > 0 \) such that \( \phi(x) \leq U(|x|, \alpha) \), then
   \[
   \lim_{t \to +\infty} \|u(t, x; \phi)\|_\infty = 0.
   \]

On the other hand, when \( q \geq \sigma^* \), GSs are well ordered, and gain some stability properties as we will see just below.

In fact, already in \[34\], the whole argument was generalized to embrace the so called Henon-equation, i.e. \( f(u, r) = r^3u^{q-1} \), where \( \delta > -2 \). In this case there is a shift in the critical exponents, so we find convenient to introduce the following parameters (see Section 2 below, see also \[3\] for more details) which will be widely used through the whole paper:

\[
l_\sigma := \frac{2q + \delta}{2 + \delta} \quad \text{and} \quad m(l_\sigma) := \frac{2}{l_\sigma - 2} = \frac{2 + \delta}{q - 2}.
\]

In this context, the previous discussion is still valid, but we have stability whenever \( l_\sigma \geq \sigma^* \), and we lose it for \( 2^* < l_\sigma < \sigma^* \) (see \[34\]). Notice that \( l_\sigma \) reduces to \( q \) for \( \delta = 0 \). In both cases the GSs, \( U \), decay as \( U(r) \sim U(r, +\infty) = P_1 r^{-m(l_\sigma)} \) for \( r \to +\infty \), and \( U(r, +\infty) \) is the unique singular solution of (1.2).

To clarify the notion of stability we use, we need to introduce the definitions of suitable weighted norms (see, e.g., \[19\]). We set

\[
\|\psi\|_\lambda := \sup_{x \in \mathbb{R}^n} |(1 + |x|^\lambda)\psi(x)|,
\]

\[
\|\psi\|_\lambda := \sup_{x \in \mathbb{R}^n} \left| \frac{(1 + |x|^\lambda)}{\ln(2 + |x|)} \psi(x) \right|,
\]

where \( \psi \) is continuous and \( \lambda \in \mathbb{R}, k \in \mathbb{N} \).

**Definition 1.1.** We say that a GS, \( U(|x|) = U(|x|, \alpha) \), is stable with respect to some norm \( \| \cdot \|_\lambda \) if for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that, for \( \|\varphi - U\|_\lambda < \delta \), we have \( \|u(t, t, \varphi) - U(\cdot)\|_\lambda < \epsilon \) for all \( t > 0 \).

Further, we say that \( U(|x|) \) is weakly asymptotically stable with respect to \( \| \cdot \|_\lambda \) when \( U(|x|) \) is stable with respect to \( \| \cdot \|_\lambda \), and there exists \( \delta > 0 \) such that \( \|u(t, t, \varphi) - U(\cdot)\|_\lambda \to 0 \) as \( t \to +\infty \), if \( \|\varphi - U\|_\lambda < \delta \) (respectively, there exists \( \delta > 0 \) such that \( \|u(t, t, \varphi) - U(\cdot)\|_\lambda \to 0 \) as \( t \to +\infty \) for all \( \lambda' < \lambda \), if \( \|\varphi - U\|_\lambda < \delta \)).

Let us consider the quadratic equation in \( \lambda \)
\[
\lambda^2 + \left( n - 2 - \frac{4}{q - 2} \right) \lambda + 2 \left( n - 2 - \frac{2}{q - 2} \right) = 0.
\]

Equation (1.8) admits two real and negative solutions, say \( \lambda_2 = \lambda_1 < 0 \) if and only if \( q \geq \sigma^* \), which coincide if and only if \( q = \sigma^* \). Gui et al in \[19\] proved the following theorem.
Theorem B. Assume \( f(u, r) = u^{q-1} \), \( q \geq \sigma^* \). Let \( \lambda_2 \leq \lambda_1 \) be the roots of equation (1.8).

1. If \( q > \sigma^* \) any GS \( U(r, \alpha) \) is stable with respect to the norm \( \| \cdot \|_{m(q)+|\lambda_1|} \) and weakly asymptotically stable with respect to the norm \( \| \cdot \|_{m(q)+|\lambda_2|} \).

2. If \( q = \sigma^* \) any GS \( U(r, \alpha) \) is stable with respect to \( \| \cdot \|_{m(q)+|\lambda_1|} \) and weakly asymptotically stable with respect to the norm \( \| \cdot \|_{m(q)+|\lambda_2|} \).

Actually, there is a number of results meant to extend the previous analysis to more general potentials \( f \) (see, e.g., [1, 2, 3, 4, 5, 6, 7]). In particular the instability result given by Theorem A and the stability result Theorems B have been extended also to the following equation

\[
(1.9) \quad u_t = \Delta u + k(r)r^\delta u^{q-1}, \quad \text{where} \quad \delta > -2, \quad \text{and} \quad r = |x|
\]

assuming \( k(r) \) decreasing, uniformly positive and bounded, in the cases \( l_s > \sigma^* \) (see [8]), and \( l_s = \sigma^* \) (see [9]). In particular, these hypotheses implies that the singular radial solution \( U(r, +\infty) \) of (1.4) behaves like \( r^{-m(l_s)} \) both as \( r \to 0 \) and as \( r \to +\infty \).

In such a case \( q \) is replaced by \( l_s \) and also the values of \( \lambda_1, \lambda_2 \) change accordingly, i.e. they solve

\[
(1.10) \quad \lambda^2 + \left( n - 2 - \frac{2 + \delta}{q - 2} \right) \lambda + \frac{2 + \delta}{q - 2} \left( n - 2 - \frac{2 + \delta}{q - 2} \right) = 0.
\]

In [9] we proposed a unifying approach which allows to extend Theorem A to a more general class of nonlinearities \( f \), including (1.9), but also more involved dependence on \( u \).

The purpose of this paper is to continue the analysis of [3], extending the stability results found in Theorem B to a larger class of \( f \). This purpose is achieved with an approach obtained through the combination of the main ideas in [3, 4, 5, 6], techniques borrowed from the theory of non-autonomous dynamical systems (see [23, 3]), along with the use of some new arguments.

As far as (1.9) is concerned we are able to drop the assumption of boundedness on \( k \) replacing it by the following:

\[
(1.11) \quad k(r) \sim r^{-\eta}, \quad \text{as} \quad r \to 0, \quad \text{with} \quad 0 \leq \eta < 2 + \delta.
\]

Then, we can allow two different behaviors for singular and slow decay solutions (see [3]), namely: \( U(r) \sim r^{-m(l_s)} \) as \( r \to +\infty \) and \( U(r) \sim r^{-m(l_s)} \) as \( r \to 0 \), where

\[
(1.12) \quad l_s = \frac{2 + \delta + \eta}{2 + \delta - \eta} \quad \text{and} \quad m(l_s) = \frac{2 + \delta - \eta}{q - 2}.
\]

So we prove the following.

**Theorem 1.2.** Let \( f(u, r) \) be as in (1.9), where \( k(r) \in C^1 \) satisfies (1.11), is decreasing, and \( \lim_{r \to +\infty} k(r) > 0 \). Then

1. If \( l_s > \sigma^* \) any GS \( U(r, \alpha) \) is stable with respect to the norm \( \| \cdot \|_{m(l_s)+|\lambda_1|} \) and weakly asymptotically stable with respect to the norm \( \| \cdot \|_{m(l_s)+|\lambda_2|} \).

2. If \( l_s = \sigma^* \) any GS \( U(r, \alpha) \) is stable with respect to \( \| \cdot \|_{m(l_s)+|\lambda_1|} \) and weakly asymptotically stable with respect to the norm \( \| \cdot \|_{m(l_s)+|\lambda_2|} \).

In fact, our approach is flexible enough to consider also a finite sum of power in \( u \), i.e.

\[
(1.13) \quad f(u, |x|) = k_1(|x|)|u|^{q_1} + k_2(|x|)|u|^{q_2},
\]

where \( q_1 < q_2, k_i = k_i(|x|), i = 1, 2 \), are supposed to be \( C^1 \) (see Theorem 3.1 and Theorem 3.2 below).
Equation (1.13) has been already considered by Yang and Zhang in [36], but just in the particular situation of \( k_1(r) = k_2(r) \equiv 1 \). We emphasize that, even if it is not stated so clearly, in [36] it is required that

\[
(1.14) \quad \frac{2 + \delta_2}{q_2 - 2} (q_2 - q_1) + \delta_1 < 0,
\]

which excludes the important case \( \delta_1 = \delta_2 = 0 \). With these assumptions, Yang and Zhang were able to prove Theorem 13 (1), replacing \( q \) by \( l_s = 2^{q_2 + \delta_2} \), and changing the values of \( m(l_s) \) and of \( \lambda_i \) accordingly.

As a consequence of our main results we are able to generalize the results in [36] and to prove Theorem 13 allowing \( k_i \) to depend on \( r \), and even to be unbounded, i.e.

\[
(1.15) \quad k_1(r) \sim r^{-\eta_1} \quad \text{and} \quad k_2(r) \sim r^{-\eta_2}, \quad \text{as} \quad r \to 0,
\]

with \( 0 < \eta_i < 2 + \delta_i, \ i = 1, 2 \). However we still need to require (1.14).

**Theorem 1.3.** Let \( f(u, r) \) be as in (1.13), and assume (1.14) and (1.15). Suppose that both \( k_1(r) \sim r^{-\eta_1} \) and \( k_2(r) \sim r^{-\eta_2} \) are decreasing, \( k_1(r) \) is positive and \( k_2(r) \) is uniformly positive. Then, setting \( l_s = 2^{q_2 + \delta_2} \), we get the same conclusions as in Theorem 1.3.

Notice that we can deal with non-monotone functions \( k_1(r) \). Under, these assumptions we are able to prove Theorem 13 (2) which is new even in the case \( k_1(r) = k_2(r) \equiv 1 \) considered in [36].

The main ingredients to obtain our results on (1.2) are the separation and the asymptotic properties of GSs. The separation properties are a result of independent interest, and generalize the ones obtained in [7, Theorems 1,2], [35, Theorem 2]. As a consequence we also get Proposition 2.13 which gives an insight on the behavior of the singular solution of (1.4), which seems to play a key role in determining the threshold between blowing up and fading solutions (see the introduction in [34]).

To prove weak asymptotic stability, we need a suitable asymptotic expansion for GSs, which refines and generalizes the ones of [8, 36] (see Proposition 2.13 below). In fact in [8, 36] the highly nontrivial proof relies on an iterative scheme developed by [31] in a simpler (but still nontrivial) context. Here, we followed a different idea: in fact we have proved an asymptotic results for nonlinear systems of ODEs, which seems to be new to the best of our knowledge, and that, in our opinion, is of intrinsic mathematical interest (even for systems of ODEs). In this more general framework the statement assumes a more comprehensible aspect, and the proof is simplified, even if it is still quite cumbersome; We rely on the appendix for a detailed proof of this lemma.

Now, we review briefly some results which have been proved just in the setting of Theorems 113. First, using some sub- and super-solutions constructed on the self-similar solutions. [20, 26] proved that \( U(|x|, \alpha) \) is weakly asymptotically stable in the norm \( \| \cdot \| \) for any \( m(q) + \lambda_1 < l < m(q) + \lambda_2 + 2 \). Further Naito in [26] showed that this result is optimal, i.e. in this range asymptotic stability does not hold. Moreover Gui et al. in [20] proved that GSs are not even stable if we use too coarse, but surprisingly also too fine norms, namely for \( l < m(q) + \lambda_1 \) and for \( l \geq n \). Notice that we have stability for \( l = m(q) + \lambda_1 \), but still there is a small gap for \( m(q) + \lambda_2 + 2 < n \). Similarly the null solution is weakly asymptotically stable if \( m(q) \leq l < n \) and unstable otherwise, [20].

Moreover in a series of papers [9, 25, 26] the authors showed that the speed at which the solution \( u(t, x; \phi) \) converges depends linearly on the weight used to measure the distance with respect to the GS. Namely if \( \| \phi(x) - U(|x|, \alpha) \| \) is small enough then \( t^\nu \| u(t, x; \phi) - U(|x|, \alpha) \|_{r} \) is bounded for any \( t > 0 \), where
\[ \nu = \frac{1}{2} \max\{l - l', l - m(q) - \lambda_1\}, \] whenever \( m(q) + \lambda_1 < l < m(q) + \lambda_2 + 2 \) and \( 0 < l' < l \). The extension of these results to more general non-linearities will possibly be the object of future investigations.

To complete the picture we recall that, if either the assumptions of Theorem A or of Theorem B are satisfied, following [3] we can construct a family of subsolutions \( \phi \) for (1.4) with arbitrarily small \( L^\infty \) norm and decaying like \( r^{2-n} \) for large \( r \), and such that the solution \( u(t, x, \phi) \) blows up in finite time. This type of behavior contradicts the idea that the decay of the singular solution, i.e \( r^{-m(q)} \), is the critical one to determine the threshold between fading and blowing up solutions: The situation is more intricate. This results is in fact extended to more general nonlinearities \( f \), see [3].

To conclude, we briefly recall that, when the non-linearity \( f(u, r) \) becomes unbounded as \( r \to 0 \), it is not possible to find classical solutions of (1.2)–(1.3). However it is still possible to obtain mild solutions assuming that \( f(u, r)r^l \) is bounded for \( l > 2 \), and in fact the solutions \( u \) are classical for \( x \neq 0 \) and \( t > 0 \), and they are \( C^{\alpha, \alpha/2} \) also for \( x = 0 \) and \( t = 0 \) for any \( \alpha \in (0, l+2) \). For an exhaustive exposition about such a topic we refer to [3] (see also [3]).

**Plan of the paper.** The paper is divided as follows: In Section 2 we collect all the preliminary results concerning the solutions of (1.4). We prove ordering properties and asymptotic estimates for positive solutions of such a problem. Section 3 is devoted to the proof of the main results of the paper (from which Theorems 1.2 and 1.3 follow directly).

2. Ordering results and asymptotic estimates for the stationary problem.

The results of this section, which are crucial for our analysis, are obtained by applying the Fowler transformation to (1.4). For this purpose we need to introduce some quantities that will appear frequently in the whole paper, i.e.

\[ m(l) = \frac{2}{l^2 - 2}, \quad A(l) = n - 2 - 2m(l), \quad B(l) = m(l)[n - 2 - m(l)], \]

where \( l > 2 \) is a parameter (which is related to \( l_2 \) and \( l_4 \), in (1.6) and in (1.12), respectively) whose role will be explained few lines below. Set

\[ r = e^s, \quad y_1(s, l) = U(e^s)e^{m(l)s}, \quad y_2(s, l) = \dot{y}_1(s, l) \]

\[ g(y_1, s, l) = f(y_1 e^{-m(l)s}, e^s) e^{(m(l)+2)s} \]

Throughout the paper “\( \cdot \)” will denote the differentiation with respect to \( s \) (recall that “\( \cdot \)” indicates differentiation with respect to \( r \)). Using these transformations we pass from (1.4) to the following system:

\[ \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ B(l) & -A(l) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} 0 \\ g(y_1, s, l) \end{pmatrix}. \]

Here and in the sequel, we write \( y(s, \tau; Q; \bar{l}) \) = \( (y_1(s, \tau; Q; \bar{l}), y_2(s, \tau; Q; \bar{l})) \) to denote a trajectory of (2.3), where \( l = \bar{l} \), evaluated at \( s \) and departing from \( Q \in \mathbb{R}^2 \) at \( s = \tau \).

For illustrative purpose assume first \( f(u, r) = r^\delta u^q - 1 \), so we can set \( l = \frac{q - \delta}{2 + \delta} \) and (2.3) reduces to the following autonomous system

\[ \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ B(l) & -A(l) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} 0 \\ (y_1)^{q-1} \end{pmatrix}. \]

In this case we passed from a singular non-autonomous ODE to an autonomous system from which the singularity has been removed. Also note that when \( \delta = 0 \),
we can simply take \( l = q \). The sign of the constants \( A(l) \), \( B(l) \) defined in (2.1) determine respectively if the system is sub- or supercritical, if there are slow decay solutions \((B(l) > 0)\) or if they do not exist \((B(l) < 0)\).

**Remark 2.1.** We recall that, with the assumptions used in this paper, positive solutions \( U(r) \) of (1.4) have two possible behaviour as \( r \to 0 \):

- **Regular**, i.e. \( \lim_{r \to 0} U(r) = \alpha > 0 \), or **Singular**, i.e. \( \lim_{r \to 0} U(r) = +\infty \).

Similarly as \( r \to +\infty \) either \( \lim_{r \to +\infty} U(r)^{\frac{n-2}{2}} = \beta > 0 \) and we say that \( U(r) \) has fast decay, or \( \lim_{r \to +\infty} U(r)^{\frac{n-2}{2}} = +\infty \) and we say that \( U(r) \) has slow decay.

In fact the behavior of singular and slow decay solutions can be specified better, see Proposition 2.9 below), and Proposition 2.16.

In this article we restrict the whole discussion to the case \( l > 2^* \); therefore \( A(l) > 0 \) and \( B(l) > 0 \). System (2.4) admits three critical points for \( l > 2^* \): The origin \( O = (0,0) \), \( P = (P_1,0) \) and \( -P \), where \( P_1 = [B(l)]^{1/(q-2)} > 0 \). The origin is a saddle point and admits a one-dimensional \( C^1 \) stable manifold \( M^- \) and a one-dimensional \( C^1 \) unstable manifold \( M^+ \), see Figure 1. The origin splits \( M^+ \) in two relatively open components: We denote by \( M^s \) the component which leaves the origin and enters the semi-plane \( y_2 \geq 0 \). Since we are just interested in positive solutions, with a slight abuse of notation, we will refer to \( M^s \) as the unstable manifold.

**Remark 2.2.** The critical point \( P \) of (2.4) is a stable focus if \( 2^* < l < \sigma^* \) and a stable node if \( l \geq \sigma^* \).

As a consequence of some asymptotic estimates we deduce the following useful fact (see, e.g. [13]).

**Remark 2.3.** Let \( u(r) \) be a solution of (1.4) and let \( Y(s;l) \) be the corresponding trajectory of system (2.4), with \( l > 2^* \). Then \( u(r) \) is regular (respectively has fast decay) if and only if \( Y(s;l) \) converges to the origin as \( s \to -\infty \) (resp. as \( s \to +\infty \)), \( u(r) \) is singular (respectively has slow decay) if and only if \( Y(s;l) \) converges to \( P \) as \( s \to -\infty \) (resp. as \( s \to +\infty \)).

Using the Pohozaev identity introduced in 28, and adapted to this context in [12], we can draw a picture of the phase portrait of (2.4) (see Figure 1) below) and deduce information on positive solutions of (1.4). Then it is not hard to classify positive solutions: In the supercritical case \((l > 2^*)\) all the regular solutions are GSs with slow decay, there is a unique SGS with slow decay.

We stress that all the previous arguments concerning the autonomous Equation (2.3) still hold true for any autonomous super-linear system (2.3). More precisely, whenever \( g(y_1,s;l) \equiv g(y_1;l) \) and \( g(y_1;l) \) has the following property, denoted by \( G0 \) (see [13] for a proof in the general \( p \)-Laplace context, see also [3]).

![Figure 1. Sketches of the phase portrait of (2.3), for \( q > 2 \) fixed.](image-url)

\[ l > \sigma^* \]
**G0:** There is \( l > 2^* \) such that \( g(0; l) = 0 = \partial_{y_i} g(0, l) \) and \( \partial_{y_i} g(y_1, l) \) is a positive strictly increasing function for \( y_1 > 0 \) and \( \lim_{y_i \to +\infty} \partial_{y_i} g(y_1, l) = +\infty \).

When **G0** holds true, we denote by \( P_l \) the unique positive solution in \( y_1 \) of \( g(y_1; l) = B(l)y_1 \). Hence \( (P_l, 0) \) is again a critical point for (2.4). Further, we let \( \sigma_* < \sigma^* \) be the real solutions of the equation in \( l \) given by

\[
A(l)^2 - 4[\partial_{y_i} g(P_l, l) - B(l)] = 0,
\]

which reduces to \( A(l)^2 - 4(q - 2)B(l) = 0 \) for \( g(y_1) = (y_1)^{q-1} \). We emphasize that when \( f(u, r) = u^{q-1} \) the value of \( \sigma^* \) coincides with the one given in (1.3). Notice that Remarks 2.2, 2.3 continue to hold true in this slightly more general context (see (1.3)–(1.5)).

**2.1. Main assumptions and preliminaries.** We collect here below the assumptions used in our main results:

**G1:** There exists \( l_u \geq \sigma^* \) such that for any \( y_1 > 0 \) the function \( g(y_1; s; l_u) \) converges to a \( s \)-independent \( C^1 \) function \( g(y_1; -\infty; l_u) \neq 0 \) as \( s \to -\infty \), uniformly on compact intervals. The function \( g(y_1; s; l_u) \) satisfies **G0** for any \( s \in \mathbb{R} \). Further, there is \( \varepsilon > 0 \) such that \( \lim_{s \to -\infty} e^{-\varepsilon s} \partial_s g(y_1; s; l_u) = 0 \).

**G2:** There exists \( l_s \geq \sigma^* \) such that for any \( y_1 > 0 \) the function \( g(y_1; s; l_s) \) converges to a \( s \)-independent \( C^1 \) function \( g(y_1; +\infty, l_s) \neq 0 \) as \( s \to +\infty \), uniformly on compact intervals. The function \( g(y_1; s; l_s) \) satisfies **G0** for any \( s \in \mathbb{R} \). Further, there is \( \varepsilon > 0 \) such that \( \lim_{s \to +\infty} e^{+\varepsilon s} \partial_s g(y_1; s; l_s) = 0 \).

**G3:** Condition **G2** holds and \( g(y_1; s; l_s) \) and \( \partial_{y_i} g(y_1; s; l_s) \) are decreasing in \( s \) for any \( y_1 > 0 \).

**G4:** Condition **G2** is verified with \( \omega = \gamma \) satisfying

\[
g(P_l, s; l_s) = g(P_l, +\infty; l_s) + ce^{-\gamma s} + o(e^{-\gamma s})
\]

for a certain \( c \neq 0 \).

**K:** Either \( f \) is as in (1.9) or \( f \) is as in (1.13) and satisfies (1.14).

Hypotheses **G1**, **G2** are used to ensure that the phase portrait of (2.3) converges to an autonomous system of the form (2.4) (with \( l \geq \sigma^* \)), respectively as \( s \to \pm \infty \). **G3** is needed to prove ordering properties of positive solutions and generalizes the condition required in [3]. **G4** is needed to derive asymptotic estimates on slow decay solutions of (1.4), and it gives back the standard requirement when \( f(u, r) = k(r)u^{q-1} \), i.e. \( k(r) = k(\infty) + cr^{-1} + o(r^{-1}) \) (see [3]). Actually, condition **G4** is assumed for definiteness and may be weakened, at the price of some additional cumbersome technicalities. Finally, condition **K** is a technical requirement we are not able to avoid, which in fact is implicitly assumed also in [30]. It implies that there is \( c > 0 \) such that

\[
B(l_s) = \frac{g(P_l^{\ast}, +\infty; l_s)}{P_l^{\ast}} = c\frac{\partial_{y_i} g(P_l^{\ast}, +\infty; l_s)}{\eta - 1}
\]

with \( \eta = q \) in the case of (1.9), and \( \eta = q_2 \) for the potential (1.13).

**Remark 2.4.** Observe that **G1** and **G2** are satisfied, e.g., in the following cases:

- For equation (1.9) with \( k \) satisfying (1.11): \( l_s \) and \( l_u \) are as in (1.6) and (1.12), respectively.
- When \( f \) is as in (1.13) and (1.15) holds true: \( l_s \) is as in Theorem (1.3) i.e. \( l_s = \min \left\{ \frac{2q + \delta_i}{2 + \delta_i} \mid i = 1, 2 \right\} \) while \( l_u = \max \left\{ \frac{2q + \delta_i - \eta_i}{2 + \delta_i - \eta_i} \mid i = 1, 2 \right\} \). We also emphasize that, if we consider (1.13), then (1.14) amounts to ask for \( \frac{q_1 + \delta_i}{2 + \delta_i} \leq \frac{q_2 + \delta_i}{2 + \delta_i} \), so **K** is not satisfied if \( \delta_i = \eta_i = 0 \), since \( l_s = q_1 < q_2 = l_u \).

**Lemma 2.5.** Assume **G2** and **G3**, then we have the following
\( A^- \): The function \( G(y_1,s;2^*) := \int_0^{y_1} g(a,s;2^*) da \) is decreasing in \( s \) for any \( y_1 > 0 \) strictly for some \( s \).

Proof. Set \( G(z,s,l_s) = \int_0^z \frac{g(a,s,l_s)}{a} da, H(z,s) = G(z,s,l_s)/z. \) Then

\[
G(z,s,l_s) = \int_0^z \frac{g(a,s,l_s)}{a} da \leq \frac{g(z,s,l_s)}{z} \int_0^z da = \frac{z g(z,s,l_s)}{2}
\]

Therefore \( z g - G \geq z g - 2G \geq 0. \) Since \( \partial_z H = (z g - G)/z^2 \), then \( H(z,s) \) is increasing in \( z \) and decreasing in \( s \) for \( G3. \) Hence

\[
G(y_1,s,2^*) = G(y_1 e^{-\delta s},s,l_s) e^{\delta s} = H(y_1 e^{-\delta s},s)y_1,
\]

so we conclude that \( G(y_1,s,2^*) \) is decreasing in \( s \).

Observe that \( A^- \) means that the system is supercritical with respect to \( 2^* \), and this ensures the existence of GSs for (1.4) (see e.g. [3, Proposition 2.12]). In the sequel, in some cases, it will be convenient to use the slightly weaker condition \( A^- \), along with \( G2, \) in place of the combination of \( G2 \) and \( G3. \)

2.2. The stationary problem: the spatial dependent case. Now we turn to consider (2.3) in the \( s \)-dependent case. The first step is to extend invariant manifold theory to the non-autonomous setting.

Assume \( G1. \) We introduce the following 3-dimensional autonomous system, obtained from (2.3) by adding the extra variable \( z = e^{-\alpha t}, \) i.e.

\[
(2.7) \quad \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ B(l_u) & -A(l_u) & 0 \\ 0 & 0 & \infty \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ g(y_1, l_u, \ln(z); l_u) \\ 0 \end{pmatrix}.
\]

Similarly if \( G2 \) is satisfied we set \( l = l_u \) and \( \zeta(t) = e^{-\alpha t} \) and we consider

\[
(2.8) \quad \begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ B(l_u) & -A(l_u) & 0 \\ 0 & 0 & \infty \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \zeta \end{pmatrix} - \begin{pmatrix} 0 \\ g(y_1, l_u, \ln(\zeta); l_u) \\ 0 \end{pmatrix}.
\]

The technical assumptions at the end of \( G1, G2 \) are needed in order to ensure that the systems are smooth respectively for \( z = 0 \) and \( \zeta = 0. \)

We recall that if a trajectory of (2.3) does not cross the coordinate axes indefinitely then it is continent for any \( s \in \mathbb{R} \) (see e.g. [14, Lemma 3.9], [3]). Consider (2.7) (respectively (2.8)) each trajectory corresponding to a definitively positive solution \( u(r) \) of (1.4) is such that its \( \alpha \)-limit set is contained in the \( z = 0 \) plane (respectively its \( \omega \)-limit set is contained in the \( \zeta = 0 \) plane). Moreover such a plane is invariant and the dynamics reduced to \( z = 0 \) (respectively, \( \zeta = 0 \)) coincides with the one of the autonomous system (2.3) where \( g(y_1,s;l_u) \equiv g(y_1,-\infty;l_u) \) (respectively, \( g(y_1,s;l_u) \equiv g(y_1,+,\infty;l_u) \)).

Observe that the origin of (2.7) admits a 2-dimensional unstable manifold \( W^u(l_u) \) which is transversal to \( z = 0 \) (and a 1-dimensional stable manifold \( M^s \) contained in \( z = 0 \)).

Following (17) (see also [23]), for any \( \tau \in \mathbb{R} \) we have that

\[
W^u(\tau;l_u) = W^u(l_u) \cap \{ z = e^{\alpha \tau} \} \quad \text{and} \quad W^u(-\infty;l_u) = W^u(l_u) \cap \{ z = 0 \}
\]

are 1-dimensional immersed manifolds, i.e. the graph of \( C^1 \) regular curves. Moreover, they inherit the same smoothness as (2.7) and (2.8), that is: Let \( K \) be a segment which intersects \( W^u(\tau_0;l_u) \) transversally in a point \( Q(\tau_0) \) for \( \tau_0 \in [-\infty, +\infty) \), then there is a neighborhood \( I \) of \( \tau_0 \) such that \( W^u(\tau;l_u) \) intersects \( K \) in a point \( Q(\tau) \) for any \( \tau \in I \), and \( Q(\tau) \) is as smooth as (2.7).

Since we need to compare \( W^u(\tau;l_u) \) and \( W^u(\tau;l_u) \), we introduce the manifolds:

\[
(2.9) \quad W^u(\tau;l_u) := \{ R = Q \exp((m(l_u) - m(l_u)) \tau) \in \mathbb{R}^2 \mid Q \in W^u(\tau;l_u) \}.
\]
Note that \( W^u(\tau; l_u) \) and \( W^u(\tau; l_s) \) are homothetic, since they are obtained from each other simply multiplying by an exponential scalar. However, if \( l_u > l_s \), \( W^u(\tau; l_s) \) becomes unbounded as \( \tau \to -\infty \). In order to deal with bounded sets, we also define the following manifold which will be useful in Section 3 i.e.

\[
W^u(\tau; l_s) := \begin{cases} 
W^u(\tau; l_u) & \text{if } \tau \leq 0 \\
W^u(\tau; l_s) & \text{if } \tau \geq 0 
\end{cases}, \quad \xi(\tau) := \begin{cases} 
z(\tau) & \text{if } \tau \leq 0 \\
2 - \z(\tau) & \text{if } \tau \geq 0
\end{cases}
\]

(2.10)

and

\[
W^u(l_s) := \{ (Q, \xi(\tau)) \mid Q \in W^u(\tau; l_s) \}.
\]

The sets \( W^u(\tau; l_u) \) may be constructed also using the argument of [3, §13], simply requiring that \( \tau \) is \( C^1 \) in \( y \) uniformly with respect to \( t \) for \( t \leq \tau \) in a fixed neighborhood of the origin. In this case we see that the tangent to \( W^u(\tau; l_u) \) is the unstable space of the system obtained from \( \dot{y} = \tau y + Q \) linearizing in the origin. So we get the following.

**Remark 2.6.** Assume \( G1 \). Then, in the origin \( W^u(\tau; l_u) \) is tangent to the line \( y_2 = m(l_u)y_1 \), for any \( \tau \in \mathbb{R} \). Since \( W^u(\tau; l_u) \), \( W^u(\tau; l_s) \) and \( W^u(\tau; l_s) \) are homothetic, they are all tangent to \( y_2 = m(l_u)y_1 \) in the origin.

As in the \( s \)-independent case, we see that the regular solutions correspond to the trajectories in \( W^u \) (see [14, 13]). More precisely, from Lemma 3.5 in [14], we get the following.

**Lemma 2.7.** Assume \( G1, G2 \). Consider the trajectory \( y(s, \tau, Q; l_u) \) of \( \dot{y} = \tau y + Q \) with \( l = l_u \), the corresponding trajectory \( y(t, \tau, R; l_s) \) of \( \dot{y} = \tau y + R \) with \( l = l_s \) and let \( u(\tau) \) be the corresponding solution of (1.3). Then \( R = \exp[(m(l_s) - m(l_u))\tau] \).

Further \( u(\tau) \) is a regular solution if and only if \( Q \in W^u(\tau; l_u) \) or equivalently \( R \in W^u(\tau; l_u) \).

Now, we turn to consider singular and slow decay solutions of (1.3). Let \( P^-_1, P^+_1 \) be the unique positive solutions in \( y_1 \) respectively of \( B(l_u)y_1 = g(y_1, -\infty; l_u) \) and of \( B(l_u)y_1 = g(y_1, +\infty; l_u) \), and set \( P^\pm_1 = (P^\pm_1, 0) \). Then, it follows that \( (P^+, 0) \) and \( (P^-, 0) \) are respectively critical points of (2.7) and (2.8).

If \( l_u \geq 2^* \), then \( (P^+, 0) \) admits a 1-dimensional exponentially unstable manifold, transversal to \( z = 0 \) (the graph of a trajectory which will be denoted by \( y^*(s, s; l_u) \)) for system (2.7), while if \( l_u > 2^* \) then \( (P^+, 0) \) is stable for (2.8), so it admits a 3-dimensional stable manifold (an open set).

From [3, Proposition 2.12] we find the following

**Proposition 2.8.** Assume \( G1, G2 \), and \( A^- \). Then, all the regular solutions \( U(r, \alpha) \) of (1.4) are GSs with slow decay, there is a unique singular solution, say \( U(r, \infty) \), and it is a SGS with slow decay.

**Proposition 2.9.** Assume \( G1, G2 \). Then if \( u(r) \) and \( v(r) \) are respectively a singular and a slow decay solution of (1.4) we have \( u(r)^{m(l_u)} \to P^-_1 \) as \( r \to 0 \) and \( u(r)^{m(l_u)} \to P^+_1 \) as \( r \to +\infty \).

### 2.3. Separation properties of stationary solutions

In this section we adapt the argument of [3] and of [36] to obtain separation properties of (1.4). We begin by the following Lemma which is rephrased from [35, Theorem 4.1], which is a slight adaption of [3, Lemma 2.11]. We emphasize that condition \( K \) is needed to prove estimate (2.15) below, and it is in fact implicitly required in [35, Theorem 4.1], even if it is not explicitly stated.
Lemma 2.10. Assume $G1, G2, G3, K$. Let $\bar{y}(s)$ be the trajectory of (2.10) corresponding to the GS $U(r, \alpha)$ of (131). Then, for any $s \in \mathbb{R}$ we have $\bar{y}_2(s) = \bar{y}_1(s) \geq 0$, $0 < \bar{y}_1(s) < P_1^+$ and

\begin{equation}
(2.11) \quad g(\bar{y}_1(s), s; l_s) < B(l_s)\bar{y}_1(s)
\end{equation}

Proof. Let us recall that all the regular solutions are GSs, this is a direct consequence of Proposition 2.6 and Lemma 2.7. Let $\bar{y}(s; l_u) = \bar{y}(s)e^{(\alpha_u - \alpha_1)s}$ be the corresponding trajectory of (2.6) where $l = l_u$, then, by standard fact in dynamical system theory, see [4], we see that there are $c_i > 0$ such that $\bar{y}_i(s; l_u)e^{-\alpha_1 s} \rightarrow c_i$ as $s \rightarrow -\infty$ for $i = 1, 2$. Hence $\bar{y}_i(s) \sim c_ie^{\alpha_1 s} \rightarrow 0$ as $s \rightarrow -\infty$ for $i = 1, 2$. So (2.11) is satisfied for $s < 0$.

Let us set

\begin{equation}
(2.12) \quad s_0 := \sup \left\{ S \in \mathbb{R} \mid g(\bar{y}_1(s), s; l_s) < B(l_s)\bar{y}_1(s) \text{ for any } s < S \right\},
\end{equation}

so that (2.11) holds for $s < s_0$.

It follows that $\dot{\bar{y}}_2(s) + A(l_s)\bar{y}_2(s) > 0$ for $s < s_0$, hence \( w(s) = \bar{y}_2(s)e^{A(l_s)s} \) is increasing for $s < s_0$. Since $w(s) \rightarrow 0$ as $s \rightarrow -\infty$ we find that $\bar{y}_2(s) > 0$, for $s \leq s_0$.

Further, assume by contradiction that there is $\bar{s} < s_0$ such that $\bar{y}_1(\bar{s}) = P_1^+$. Then, from $G3$, for $s < \bar{s}$ we have

\[ g(\bar{y}_1(\bar{s}), +\infty; l_s) \leq g(\bar{y}_1(\bar{s}), \bar{s}; l_s) < B(l_s)\bar{y}_1(\bar{s}) = g(P_1^+, +\infty; l_s). \]

Since $g(\cdot, +\infty; l_s)$ is increasing we get $\bar{y}_1(\bar{s}) < P_1^+$, and we have a contradiction. Thus, $0 < \bar{y}_1(s) < P_1^+$ for $s < s_0$.

Now, we show that $s_0 = +\infty$, so that (2.11) holds for any $s \in \mathbb{R}$ and the Lemma is proved. Assume by contradiction that $s_0 < +\infty$. Consider the curve $\bar{y}(s) = (\bar{y}_1(s), \bar{y}_2(s))$ defined for $s \leq s_0$. Since $\bar{y}_2(s) = \bar{y}_1(s) > 0$ for $s < s_0$, it follows that $\bar{y}(s)$ is a graph on the $y_1$-axis, and we can parametrize it by $\bar{y}_1$. Hence, we set \( Q(\bar{y}_1) := \bar{y}_1(\bar{y}_1) \) so that $\bar{y}(s)$ for $s \leq s_0$ and $\Gamma := \Gamma(\bar{y}_1) = (y_1, Q(\bar{y}_1))$ for $y_1 \in (0, \bar{y}(s_0))$ represent the same curve. As a consequence we have that

\begin{equation}
(2.13) \quad \frac{\partial Q}{\partial y_1} = \frac{\partial Q}{\partial s} \frac{\partial s}{\partial y_1} = \frac{\bar{y}_1}{\bar{y}_1} = -A(l_s) + \frac{B(l_s)\bar{y}_1(s; l_s)}{Q(\bar{y}_1)}. \tag{2.13}
\end{equation}

In the phase plane, consider the line $r(\mu)$ passing through $R = (\bar{y}_1(s_0), 0)$ with angular coefficient $-\mu$, i.e.

\[ r(\mu) := \{(y_1, y_2) \mid y_2 = \mu(\bar{y}_1(s_0) - y_1)\}. \]

Since $\bar{y}_2(s_0) = \bar{y}_1(s_0) > 0$, we see that $\Gamma(\bar{y}_1(s_0)) = (\bar{y}_1(s_0), \bar{y}_2(s_0))$ lies above $R$. By construction $r(\mu)$ intersects $\Gamma$ at least in a point, for any $\mu > 0$. We denote by $Y_1(\mu)$, $(\bar{y}(Y_1(s_0) - y_1(\mu)))$ the intersection with the smallest $Y_1$. Then, it follows that $Y_1 < \bar{y}_1(s_0)$ and $\partial^2 Q / \partial y_1^2(Y_1) \geq -\mu$. From these inequalities, along with (2.13), and using the fact that

\begin{equation}
(2.14) \quad B(l_s)\bar{y}_1(s_0) = g(\bar{y}_1(s_0), s_0; l_s)
\end{equation}
we get
\[-\mu \leq \frac{\partial Q}{\partial y_1}(Y_1) = -A(l_s) + \frac{B(l_s)[Y_1 - \bar{y}_1(s_0)] + [g(\bar{y}_1(s_0), s_0; l_s) - g(Y_1, s; l_s)]}{\mu[\bar{y}_1(s_0) - Y_1]} \leq -A(l_s) - \frac{B(l_s)}{\mu \bar{y}_1(s_0) - Y_1} \leq -A(l_s) + \frac{1}{\mu} \left[-B + \partial_y g(C, s_0; l_s)\right] \leq -A + \frac{1}{\mu} \left[-B + \frac{(\bar{q} - 1)g(C, s_0; l_s)}{\bar{y}_1(s_0)}\right] = -A + \frac{B(\bar{q} - 2)}{\mu}\]

(2.15)

where \(C \in (Y_1, \bar{y}_1(s_0))\) and we used the mean value theorem. Further \(\bar{q}\) stands for \(q\) if \(f\) is of type (1.9) and it stands for \(q_2\) if \(f\) is of type (1.13). Therefore, using (2.14) along with (2.14), we obtain
\[\mu^2 - A\mu + B(\bar{q} - 2) = \mu^2 - A\mu - B + \partial_y g(P_1^+, +\infty, l_s) > 0, \text{ for any } \mu > 0.\]

But this is verified if and only if
\[A^2 - 4B(\bar{q} - 2) = A^2 - 4[\partial_y g(P_1^+, +\infty, l_s) - B] < 0,\]

which is equivalent to \(l_s \in (\sigma^s, \sigma^+), \text{ cf } (2.5),\) so we have found a contradiction. Hence \(s_0 = +\infty.\) In particular, it follows that \(\bar{y}_1(s) < P_1^+, \bar{y}_1(s) > 0, \text{ for any } s \in \mathbb{R},\) and (2.11) holds true.

\[\text{Remark 2.11. Assume the hypotheses of Lemma 2.10. Assume further } K, \text{ then} (2.16) \]
\[\frac{\partial g}{\partial y_1}(\bar{y}_1(s), s; l_s) < \frac{\partial g}{\partial y_1}(P_1^+, +\infty; l_s)\]

Proof. From a straightforward computation we see that, when \(f\) is as in (1.9), then (2.14) implies (2.16). When \(f\) is as in (1.13) then
\[\partial_y g(y_1, s, l_s) = (q_1 - 1)k_1(e^s)y_1^q - 2 + (q_2 - 1)k_2(e^s)y_1^{q_2 - 2} \leq (q_2 - 1)g(y_1, s, l_s)/y_1.\]

So, let \(\bar{y}_1(s)\) be a trajectory corresponding to a GS of (1.4) as above; If \(K\) holds, from (2.11) we get
\[\frac{\partial g}{\partial y_1}(\bar{y}_1, s, l_s) \leq (q_2 - 1)\frac{g(\bar{y}_1(s), s; l_s)}{\bar{y}_1(s)} \leq (q_2 - 1)\frac{g(P_1^+, +\infty; l_s)}{P_1^+} \leq \frac{\partial g}{\partial y_1}(P_1^+, +\infty; l_s),\]

so (2.16) follows and the Lemma is proved.

\[\text{Proposition 2.12. Assume } G_1, G_2, G_3. \text{ Assume further } K. \text{ Then } U(r, \alpha_1) < U(r, \alpha_2) \text{ for any } r > 0, \text{ whenever } \alpha_1 < \alpha_2.\]

We emphasize that if \(g(y_1, s; l)\) is \(s\)-independent, as in [34], Lemma 2.10 implies Proposition 2.12. This fact follows directly by noticing that \(M^u\) is a graph on the \(y_1\)-axis, since \(y_1(s) = U(e^s, \alpha)\), as \(s\) is increasing in \(s, \text{ for any } \alpha > 0.\) In view of Lemma 2.10 we can parametrize the manifold \(M^u\) by \(\alpha\), then the ordering of the regular solutions \(U(r, \alpha)\) is preserved as \(s\) varies (i.e. as \(r\) varies), since they all move along a 1-dimensional object.

When we turn to consider an \(s\)-dependent function \(g(y_1, s; l)\), Proposition 2.12 needs a separate proof, which can be obtained by adapting the ideas developed in [8] [30]. In fact, in such a case \(M^u(r; l_s)\) is still one dimensional but may not be a graph on the \(y_1\)-axis, so a priori we may lose the ordering property.
Proof of Proposition 2.13. Let us set $Q(s) = e^{\lambda_1 s}$ and observe that

\begin{equation}
Q + AQ + [\partial_y g(P_1^+), +\infty; l_s) - B(Q = 0).
\end{equation}

Denote by $W(s) := [U(e^s, \alpha_2) - U(e^s, \alpha_1)]e^{m(l_s)s}$, and observe that

\begin{equation}
\dot{W} + A\dot{W} - BW + D(s) = 0,
\end{equation}

where $D(s) := g(U(e^s, \alpha_2))e^{m(l_s)s}, s; l_s) - g(U(e^s, \alpha_1))e^{m(l_s)s}, s; l_s)$.

Using continuous dependence on initial data we see that $U(r, \alpha_2) > U(r, \alpha_1)$ for $r$ small enough, so that $D(s) > 0$ for $s < 0$. Assume by contradiction that there is $\bar{r} = e^s > 0$ such that $U(r, \alpha_2) - U(r, \alpha_1) > 0$ for $0 \leq r < \bar{r}$, and $U(\bar{r}, \alpha_2) - U(\bar{r}, \alpha_1) = 0$. Then, $W(s)$, and $D(s)$ are positive for $s < \bar{s}$ and they are null for $s = \bar{s}$.

Setting $Z(s) := W(s)Q(s) - W(s)\dot{Q}(s)$, by direct calculation we can easily see that $\dot{Z}(s) = W(s)Q(s) - W(s)\dot{Q}(s)$. Then from (2.17), and (2.18) we get

\begin{equation}
\dot{Z} = -AZ(s) + Q(s)[\partial_y g(P_1^+, +\infty; l_s)W(s) - D(s)].
\end{equation}

Observe now that $W(s) \sim (\alpha_2 - \alpha_1)e^{m(l_s)s}$, as $s \to -\infty$, and also that

\begin{equation}
W(s) = m(l_s)W(s) + [U'(e^s, \alpha_2) - U'(e^s, \alpha_1)]e^{m(l_s)+m(l_s)s} \sim m(l_s)(\alpha_2 - \alpha_1)e^{m(l_s)s},
\end{equation}

as $s \to -\infty$. Hence, we get

\begin{equation}
Z(s) \sim (m(l_s) - \lambda_1(l_s))(\alpha_2 - \alpha_1)(e^{m(l_s)+\lambda_1(l_s)s}) \to 0 \text{ as } s \to -\infty.
\end{equation}

Moreover, $\lambda_1(l_s) + A(l_s) = -\lambda_2(l_s) > 0$ and $D(s) \to 0$ as $s \to -\infty$, hence $e^{\lambda_1 Q(s)} D(s) \in L^1(-\infty, \bar{s})$. Since $Z(s)$ is the unique solution of (2.19) satisfying (2.20) we find

\begin{equation}
Z(s) = \int_{-\infty}^{s} e^{-A(s-s')}Q(s')[\partial_y g(P_1^+, +\infty; l_s)W(s) - D(s)] ds.
\end{equation}

From the mean value theorem we find

\[\partial_y g(P_1^+, +\infty; l_s)W(s) - D(s) = \partial_y g(P_1^+, +\infty; l_s) - \partial_y g(U(s), s; l_s)]W(s),\]

where $U(s)$ lies between $U(r, \alpha_1)r^{m(l_s)}$ and $U(r, \alpha_2)r^{m(l_s)}$. Since $\partial_y g(y_1, s; l_s)$ is increasing in $y_1$, and using (2.10), for $s < \bar{s}$ we find

\[\partial_y g(P_1^+, +\infty; l_s)W(s) - D(s) \geq [\partial_y g(P_1^+, +\infty; l_s) - \partial_y g(U(e^s, \alpha_2)e^{m(l_s)s}, s; l_s)]W(s) > 0\]

Hence, from (2.21) and (2.22) we get

\[0 < Z(s) = \bar{W}(s)\dot{Q}(s) - W(s)\dot{Q}(s) = \bar{W}(s)\dot{Q}(s),\]

which gives $\bar{W}(s) > 0$. Thus, we find $W(s) < 0$ for $|s - \bar{s}|$ small enough, and this gives a contradiction. Therefore, $U(r, \alpha_2) - U(r, \alpha_1) > 0$ for any $r \geq 0$.

Now, we consider the singular solution $U(r, \infty)$.

Proposition 2.13. Assume the hypotheses of Proposition 2.13, then $U(r, \infty)r^{m(l_s)}$ is non-decreasing for any $r > 0$, and $U(r, \alpha) < U(r, \infty)$ for any $r > 0$, $\alpha > 0$.

Actually, this result is now even for $f$ of both types $f(u, r) = K(r)u^{q_1-1}$ and $f(u, r) = u^{q_1-1} + u^{q_2-1}$, which are considered in [3] and [36], respectively.

Proof. The result is well known when the system is autonomous: In fact in this case $U(r, \infty)r^{m(l_s)} = P_1^+$ and $W^u_{l_u} = W^u_{l_u}$ is a graph on the $y_1$-axis connecting the origin and $P^+$.

Now, we turn to consider the $s$-dependent setting. From the previous discussion we know that the manifold $M^u$ of the autonomous system (2.3), where $l = l_u$, $g = g(y_1, -\infty; l_u)$, is a graph on the $y_1$-axis connecting the origin and the critical point $P^+$. Moreover, observe that for any $r \in \mathbb{R}$ the manifold $W^u(r; l_u)$ is a graph
connecting the origin and the unique trajectory $y^*(s; l_u) U(r, \infty)$ (and such that $\lim_{s \to -\infty} y^*(s; l_u) = P^-$).

We claim that $W^u(\tau; l_u)$ is a graph on the $y_1$-axis, for any $\tau \in \mathbb{R}$. In fact let $Q, R \in W^u(\tau; l_u)$, with $Q = (Q_1, Q_2), R = (R_1, R_2)$, and let $U(r, \alpha_Q)$ and $U(r, \alpha_R)$ be the corresponding solution of (1.3). From Proposition 2.12 we know that if $\alpha_Q < \alpha_R$, then

$$Q_1 = U(e^\tau, \alpha_Q)e^{m(l_u)}r < U(e^\tau, \alpha_R)e^{m(l_u)}r = R_1,$$

so the claim follows.

Moreover, we also get $Q_1 < y_1^*(\tau; l_u)$. In fact by assumption by contradiction that $Q_1 > y_1^*(\tau; l_u)$. Then we can choose $R$ in the branch of $W^u(\tau; l_u)$ between $Q$ and $y^*(\tau; l_u)$, so that $\alpha_R > \alpha_Q$ and $Q_1 > R_1 > y_1^*(\tau; l_u)$; but this contradicts (2.24). Similarly if $Q_1 = y_1^*(\tau; s; l_u)$, then $R \in W^u(\tau; l_u)$ is such that $\alpha_R > \alpha_Q$, and $R_1 > Q_1 = y_1^*(\tau; s; l_u)$. But again we can choose $R$ in the branch of $W^u(\tau; l_u)$ between $R$ and $y^*(\tau; l_u)$, and reasoning as above we find again a contradiction. Therefore $U(r, \alpha) < U(r, \infty)$ for any $r > 0$, and any $\alpha > 0$.

Further, since $W^u(\tau; l_u)$ and $W^u(\tau; l_u)$ are homothetic, cf (2.24), then $W^u(\tau; l_u)$ is a graph on the $y_1$-axis, which connects the origin and the trajectory $y^*(s; l_u)$ corresponding to $U(\tau, \infty)$. Further $W^u(\tau; l_u) \subset \{(y_1, y_2) \mid 0 < y_1 < P_1^+, y_2 > 0\}$ (see Lemma 2.10). Therefore $y_2^*(s; l_u) \geq 0$ for any $s \in \mathbb{R}$. Hence $U(r, \infty)e^{m(l_u)}$ is non-decreasing for any $r > 0$, and the proof is concluded.

Proposition 2.13 is interpreted as follows in terms of system (2.23).

Remark 2.14. In the hypotheses of Proposition 2.12 hence of Proposition 2.13 we have that $W^u(\tau; l_u)$, $W^v(\tau; l_u)$, and $W^w(\tau; l_u)$ are graphs on the $y_1$-axis respectively for any $\tau \in \mathbb{R}$. Further they are contained in $y_2 > 0$ and connect the origin respectively with $y^*(\tau; l_u)$, $y^*(\tau; l_u)$, and $y^*(\tau; l_u)$.

2.4. Asymptotic estimates for slow decay solutions. In this subsection we state the asymptotic estimates for slow decay solutions of (1.3), which are crucial to prove our main results: We always assume $G1, G2$, and $G4$.

In fact, we generalize the results obtained for $f(u, r) = k(\tau)u^{q-1}$ in [8, §3] for $q > \sigma^*$, and in [6] for $q = \sigma^*$. The main argument in [8] has been re-used in [30], and it is an adaptation to the non-autonomous context of the scheme introduced by Li in [22] (and developed in [19]). Here, we follow a different approach, so we give an interpretation in terms of general facts in ODE theory of the argument behind the whole [3, §3], which thereafter becomes clearer in our opinion.

Due to assumption $G4$ we can now set $\zeta = e^{-\gamma s}$ in (2.18), and obtain a smooth system which has $\mathcal{P} := (P_1^+, 0, 0)$ as critical point. In this subsection we consider this system and its linearization around $\mathcal{P}$ so we leave the explicit dependence on $l_u$ unsaid.

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ B - \partial_{y_1} g^{1\infty}(P_1^+) & -A & 0 \\ 0 & 0 & -\gamma \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \zeta \end{pmatrix}$$

Let us denote by $A$ the matrix in (2.24): It has 3 negative eigenvalues $\lambda_2 \leq \lambda_1 < 0$ and $-\gamma < 0$ (G4 is needed in order to guarantee smoothness of the system (2.23) for $\zeta = 0$). Therefore the critical point $\mathcal{P}$ of (2.3) is a stable node.

Assume first that the 3 eigenvalues are simple, then we have 3 eigenvectors, respectively $v_1 = (1, -m + \lambda_1, 0), v_2 = (1, -m + \lambda_2, 0)$, and $v_3 := v_3 = (0, 0, 1)$. Any solution $\ell(t)$ of (2.24) can be written as

$$\ell(s) = \bar{a}v_1 e^{\lambda_1 s} + \bar{b}v_2 e^{\lambda_2 s} + z v_3 e^{-\gamma s}$$

for some $\bar{a}, \bar{b}, z \in \mathbb{R}$. 
By standard facts in invariant manifold theory (see [4 §13]), any trajectory 
\((y(s), \zeta(s))\) of (2.23) converging to \(P\) can be seen as a non-linear perturbation of a solution \(\ell(s)\) of (2.24). More precisely set \(n(s) = (n_1(s), n_2(s)) = (y_1(s) - P^+_1, y_2(s))\), then \(N(s) := (n_1(s), n_2(s), \zeta(s)) = \ell(s) + O(\ell(s)^2)\). Therefore
\[
n_1(s) = a_1e^{\lambda_1s} + b_1e^{\lambda_2s} + ze^{-\gamma s} + O(e^{2\lambda_1s} + e^{2\lambda_2s} + e^{-2\gamma s})
\]
In the appendix we prove that the expansion can be continued to an arbitrarily large order: This is the contained of Proposition 2.15 and of its general form containing resonances, i.e. Proposition 2.16. Let us rewrite (2.23) as
\[
(2.26) \quad \vec{x} = A\vec{z} + N(\vec{x})
\]
where \(\vec{x} = (y_1, y_2, \zeta)\), and \(A\) is as the matrix in (2.24).

**Proposition 2.15.** Assume for simplicity \(\vec{N} \in C^\infty\) and that the eigenvalues of \(A\) are real, negative and simple and are rationally independent, i.e there is no \(\chi = (\chi_1, \chi_2, \chi_3) \in \mathbb{Z}^3 \setminus \{0, 0, 0\}\) such that \(\chi_1|\lambda_1| + \chi_2|\lambda_2| + \chi_3|\gamma| = 0\), so that no resonances are possible. Further assume for definiteness that \(|\lambda_1| < \gamma\).

Then for any \(k \in \mathbb{N}\) we can find a polynomial \(P\) of degree \(k\) in 3 variables such that
\[
y_1(t) = P(e^{\lambda_1t}, e^{\lambda_2t}, e^{-\gamma t}) + o(e^{(k+1)|\lambda_1+c|t})
\]
as \(t \to +\infty\), for \(\varepsilon > 0\) small enough.

We remand the interested reader to the Appendix for details.

Now we state the result in a form which is more suitable for our purpose; Set
\[
(2.27) \quad I_\theta = \left\{ \chi = (\chi_1, \chi_2, \chi_3) \in \mathbb{N}^3 : \chi_1|\lambda_1| + \chi_2|\lambda_2| + \chi_3|\gamma| \leq \theta \right\}
\]
Then, we can expand \(n_1(s)\) as follows
\[
(2.28) \quad n_1(s) = a_1e^{\lambda_1s} + b_1e^{\lambda_2s} + ze^{-\gamma s} + P_\theta(s) + o(e^{-\theta s}),
\]
where the function \(P_\theta(s)\) is completely determined by the values of the coefficients \(a, b, z\).

As a first case, assume that \(\gamma, |\lambda_1|, |\lambda_2|\) are rationally independent. Then, there are constants \(c^\chi \in \mathbb{R}\) such that
\[
(2.29) \quad P_\theta(s) = \sum_{\chi \in I_\theta, \chi \geq 2} c^\chi e^{(\chi_1\lambda_1 + \chi_2\lambda_2 - \chi_3\gamma)s} \quad \text{with} \quad \chi = (\chi_1, \chi_2, \chi_3)
\]
and \(|\chi| = \chi_1 + \chi_2 + \chi_3\).

Let us now consider the resonant cases, i.e. when there are \(M^0, M^1, \ldots, M^j\), (a \(j\)-ple resonance) \(M^i = (\chi_1^i, \chi_2^i, \chi_3^i) \in I_\theta, |M^i| > 0\) for \(i = 1, \ldots, j\), such that
\[
\chi_1^i|\lambda_1| + \chi_2^i|\lambda_2| + \chi_3^i|\gamma| = \theta \leq \theta
\]
Then, we have to replace \(\sum_{i=0}^j c_{M^i}s^i e^{(\chi_1^i\lambda_1 + \chi_2^i\lambda_2 - \chi_3^i\gamma)s}\) in the function \(P_\theta\),
\[
(2.30) \quad \sum_{i=0}^j c_{M^i}s^i e^{(\chi_1^i\lambda_1 + \chi_2^i\lambda_2 - \chi_3^i\gamma)s}
\]
(notice that we have included the possible case of resonances with the linear terms, e.g., \(\chi_2\) multiple of \(\chi_1\) etc...). The same happens when we have resonances within the linear terms, e.g. \(|\lambda_1| = |\lambda_2|\) (i.e. \(I_\theta = \sigma^\gamma\)), or \(|\lambda_1| = \gamma\): We replace the terms as done in (2.30).

Before collecting all these facts in Proposition 2.16 below, we need some further notation. Let us introduce the following sets, i.e.
\[
(2.31) \quad J_{|\lambda_1|} = \{ \chi = (0, 0, \chi_3) \in \mathbb{N}^3 \mid 0 < \chi_3\gamma < |\lambda_1| \},
\]
\[
(2.32) \quad J_{|\lambda_2|} = \{ \chi = (\chi_1, \chi_2, 0) \in \mathbb{N}^3 \mid |\lambda_1| < \chi_1|\lambda_1| + \chi_3\gamma < |\lambda_2| \}.
\]
Observe that \( J_{\lambda_1} \) is empty if \(|\lambda_1| \leq \gamma \), and \( J_{\lambda_2} \) is empty, e.g., if \(|\lambda_2| < 2|\lambda_1| \), and \(|\lambda_2| \leq \gamma \). We denote by

\[
\Psi(s) = \sum_{\chi = (0, 0, \lambda) \in J_{\lambda_1}} e^{\lambda \varphi} e^{-\chi \cdot \gamma s} + \chi_r(s) e^{\lambda_1 s}
\]

where \( \chi_r(s) = 0 \) if \(|\lambda_1|/\gamma \notin \mathbb{N} \), and \( \chi_r(s) = \chi_r s \) if \(|\lambda_1|/\gamma \in \mathbb{N} \) and \( l_s > \sigma_r^* \), while \( \chi_r(s) = \chi_r s^2 \) if \(|\lambda_1|/\gamma \in \mathbb{N} \) and \( l_s = \sigma_r^* \), for a certain \( \chi_r \in \mathbb{R} \).

Proposition 2.16. Assume \( G1, G2, G4 \). Let \( I_\theta = I_\theta \cap \{(1, 0, 0), (0, 1, 0)\} \cup J_{\lambda_1} \cup J_{\lambda_2} \). Any trajectory \((y_1(s), y_2(s), y_3(s))\) converging to \( \mathcal{P} \) is such that \( y_1(s) \) has the following expansion if \( l_s > \sigma_r^* \):

\[
y_1(s) = P_1^+ + \Psi(s) + a e^{\lambda_1 s} + Q_{1,\theta}^+(s) + b e^{\lambda_2 s} + Q_{2,\theta}^+(s) + o(e^{-\theta s}), \quad \text{where}
\]

\[
Q_{1,\theta}(s) = 2 \sum_{\chi = (0, 0, \lambda) \in J_{\lambda_1}} e^{\chi \cdot \lambda_1 + \chi \cdot \lambda_2 - \chi \cdot \gamma s}, \quad \text{with} \quad \chi = (\chi_1, \chi_2, \chi_3), \quad \text{and}
\]

\[
Q_{2,\theta}(s) = \sum_{\chi \in I_\theta} e^{\chi \cdot \lambda_1 + \chi \cdot \lambda_2 - \chi \cdot \gamma s}
\]

as \( s \to +\infty \), if we do not have resonances, otherwise we need to replace the resonant terms in \( Q_{1,\theta}(s) \) according to \( (2.30) \).

If \( l_s = \sigma_r^* \) so that \( \lambda_1 = \lambda_2 \) we have

\[
y_1(s) = P_1^+ + \Psi(s) + a e^{\lambda_1 s} + b e^{\lambda_1 s} + Q_{2,\theta}^+(s) + o(e^{-\theta s})
\]

as \( s \to +\infty \), again if we do not have resonances, otherwise we need to replace the resonant terms in \( Q_{2,\theta}(s) \) according to \( (2.30) \).

Remark 2.17. We emphasize that \( Q_{1,\theta}(s) \) contains terms which are negligible with respect to \( a e^{\lambda_1 s} \) while \( Q_{2,\theta}(s) \) contains terms which are negligible with respect to \( b e^{\lambda_2 s} \). Further if \(|\lambda_1| < \gamma \) then \( \Psi(s) \) is identically null by definition.

The proof is developed in the Appendix in a general framework, by showing a result on asymptotic expansions for ODEs, which seems to be new to the best of our knowledge. In fact we borrow some of the ideas from \([3, 56]\).

Remark 2.18. Fix \( Q \) and \( \tau \in \mathbb{R} \); then \( y_1(t, \tau, Q; l_s) \) admits an expansion either of the form \( (2.31) \) or of the form \( (2.35) \). All the coefficients in the expansions are determined by the choice of \( a, b \), which are in fact smooth functions of \( Q \), i.e., \( a = a(Q), \ b = b(Q) \).

In fact, all the coefficients in \( \Psi(s) \) are determined when the non-linearity \( g \) and \( \tau \) are fixed; the coefficients in \( Q_{1,\theta} \) are assigned (and can be determined) once \( a \) is fixed, while \( Q_{2,\theta} \) is assigned once \( a \) and \( b \) are assigned.

Remark 2.19. Fix \( Q \) and \( \tau \), the coefficients \( a = a(Q), \ b = b(Q) \) may be evaluated through the method explained in \( [3] \). However from the previous discussion we have the following. Let \( a_1, b_1, z_1 \) be such that \( (Q - P^+, e^{-\gamma \tau}) = a_1 v_1 + b_1 v_1 + z_1 v_2 \). Then

\[
a = a_1 + O(|Q - P^+|^2) \quad \text{and} \quad b = b_1 + O(|Q - P^+|^2).
\]

The proof of these two remarks is provided in the Appendix. For further details about these points see \([?, \text{Remarks 4.12, 4.16}]\).

Now, we translate Proposition \( (2.16) \) for the original equation \( (2.14) \).

Lemma 2.20. Assume \( G1, G2 \) with \( l_s \geq l_s \geq \sigma_r^*, \ G3, G4 \). Consider a GS \( U(r, \alpha) \) for \( \alpha > 0 \), or the SGS \( U(r, \infty) \) then there are continuous functions \( \mathcal{A} : \)
(0, +∞) → ℝ, B : (0, +∞) → ℝ, such that A is monotone decreasing, and if l_s > σ^*

\[ U(r, \alpha) = \frac{P^+}{r^m} + \frac{\Psi(r)}{r^m} + A(\alpha)r^{\lambda_1-m} + \frac{Q_1\theta(\ln(r))}{r^m} + B(\alpha)r^{\lambda_2-m} + \frac{Q_2\theta(\ln(r))}{r^m} + o(r^{-\theta-m}) \]

(2.36)

as \( r \to +\infty \). If \( l_s = \sigma^* \) we have

\[ U(r, \alpha) = \frac{P^+}{r^m} + \frac{\Psi(r)}{r^m} + A(\alpha)ln(r)r^{\lambda_1-m} + B(\alpha)r^{\lambda_2-m} + \frac{Q_2\theta(\ln(r))}{r^m} + o(r^{-\theta-m}). \]

(2.37)

\[ \frac{\lambda_1}{2} < \alpha < \lambda_2, \]

\[ \lambda_2 - \lambda_1 > \frac{\lambda_1}{2}, \]

\[ |\lambda_1| < |\lambda_2|, \]

\[ \lambda_1 < \lambda_2 < 0. \]

Remark 2.21. If we replace \( G3 \) with the weaker assumption \( A^- \) in Lemma 2.20 then we still get the expansions in (2.36), (2.37), but we cannot ensure that \( A \) is monotone decreasing.

Proof of Lemma 2.20. Fix \( \tau \in \mathbb{R} \); let \( y(s, \tau, Q(\alpha); l_s) \) be the trajectory of (2.23) corresponding to \( U(r, \alpha) \), so that \( Q(\alpha) \in W^2_\nu(r) \). Then we can apply Proposition 2.19 to \( y(s, \tau, Q(\alpha); l_s) \) and we find the expansions (2.36), (2.37), where, according toRemark 2.18 the coefficients \( a, b \) are \( a = a(Q(\alpha)) \) and \( b = b(Q(\alpha)) \). We set

\[ A(\alpha) = a(Q(\alpha)), \quad B(\alpha) = b(Q(\alpha)). \]

It follows that \( A : (0, +\infty) \to \mathbb{R} \) and \( B : (0, +\infty) \to \mathbb{R} \) are continuous functions. Finally if \( G3 \) holds \( U(r, \alpha_1) < U(r, \alpha_2) \) if \( \alpha_1 < \alpha_2 \) for any \( r > 0 \), and in particular for \( r \) large, so \( A(\alpha) \) is monotone increasing.

\[ \frac{\lambda_1}{2} < \alpha < \lambda_2, \]  

\[ \lambda_2 - \lambda_1 > \frac{\lambda_1}{2}, \]  

\[ |\lambda_1| < |\lambda_2|, \]  

\[ \lambda_1 < \lambda_2 < 0. \]

3. Main results: Stability and asymptotic stability

Let us state Theorems 3.1 and 3.2 from which Theorems 1.1, 1.2, 1.3 follow directly. Let \( r > 0 \), we denote by \( [r] := \{ k \in \mathbb{N} \mid k - 1 < r \leq k \} \). We have the following results

Theorem 3.1. Suppose \( f \) is \( C^k \), where \( k = \left[ \frac{|\lambda_1|}{\gamma} \right] \). Assume \( K, G1, G2, G3, G4 \). Then any radial GS \( U(r, \alpha) \) of (1.2) is stable with respect to the norm \( \| \cdot \|_{m(l_s) + \lambda_1} \) if \( l_s > \sigma^* \), and with respect to the norm \( \| \cdot \|_{m(l_s) + |\lambda_1|} \) if \( l_s = \sigma^* \).

Theorem 3.2. Assume the hypotheses of Theorem 3.1. Then any radial GS \( U(r, \alpha) \) of (1.2) is weakly asymptotically stable with respect to the norm \( \| \cdot \|_{m(l_s) + |\lambda_2|} \) if \( l_s > \sigma^* \), and with respect to the norm \( \| \cdot \|_{m(l_s) + \lambda_1} \) if \( l_s = \sigma^* \).

Let us recall that the stability of positive GS \( U(|x|, \alpha) \) of (1.2) has been analyzed in a number of papers, (see [8, 15, 19, 20, 24]). In [19], when \( f(u, |x|) = u^{q-1} \) and \( q > \sigma^* \), the authors proved that the positive GS of (1.2) are stable in the norm \( \| \cdot \|_{m + \lambda_1} \) and weakly asymptotically stable with respect to \( \| \cdot \|_{m + |\lambda_2|} \). These results have been subsequently extended in [8] to functions \( f(u, |x|) \) of the form \( k(|x|)u^{q-1} \) where \( K \) is a monotone decreasing uniformly positive and bounded function. Here, we are able to prove asymptotic stability in place of weak asymptotic stability. Further, we drop the assumption that \( k \) is bounded. This will allow us to consider potential giving rise to singular solutions \( U(r, \infty) \) having two different behaviors as \( r \to 0 \) (i.e. \( U(r, \infty) \sim P^{-r^{-m(l_s)}} \)) and as \( r \to \infty \) (i.e. \( U(r, \infty) \sim P^{r^{-m(l_s)}} \)).
3.1. Proof of Theorem 3.1. We first introduce some standard definition.

**Definition 3.3.** We say that \( \bar{\phi} \) is a super-solution of (1.1) if \( \Delta \bar{\phi} + f(\bar{\phi}, |x|) \leq 0 \); analogously \( \bar{\phi} \) is a sub-solution if \( \Delta \bar{\phi} + f(\bar{\phi}, |x|) \geq 0 \).

We refer to \([34]\) or to \([3, \S3]\) for an extension of this definition to weak and mild solutions. Also, depending on a number of very relevant factors (for instance, the type of domain and of the boundary conditions, the regularity of the forcing term, etc... ) the notion of weak solution for parabolic equations can change considerably as described, e.g., in \([21, 17, 18, 33]\). In particular, we mention that, a dynamical approach to study a generalized parabolic equation on an unbounded strip-like domain is given in \([32]\). In this case a suitable definition of weak solutions, on weighted Sobolev (and Bochner) spaces, is considered and the author proved the existence of a global attractor. Then, this situation is further generalized in \([2]\).

Both Theorems 3.1, 3.2 depend strongly on the following well known fact, proved in \([31, \text{Theorem 2.4}], \) see also \([9, \text{Theorem 3.10}].\)

**Lemma 3.4.** Assume \( G_1, G_2 \), and let \( U_1(r) \) and \( U_2(r) \) be positive solutions of (1.3) respectively for \( r \leq R_1 \) and for \( r \geq R_2 \), where \( R_1 > R_2 \), and let \( R \in (R_2, R_1) \) be such that \( U_1(R) = U_2(R) \). Consider

\[
\phi(x) = \begin{cases} U_1(r) & \text{if } 0 < |x| \leq R, \\ U_2(r) & \text{if } |x| \geq R. \end{cases}
\]

We have that

- If \( U_1(R) \geq U_2'(R) \), then \( \phi(x) \) is a continuous weak super-solution of (1.1).
- If \( U_1(R) \leq U_2'(R) \), then \( \phi(x) \) is a continuous weak sub-solution of (1.1).

**Lemma 3.5.** Assume \( G_1, G_2 \);

(i) If the initial value \( \phi \) in (1.3) is a continuous weak super-(sub-) solution of (1.1), then the solution \( u(t, x; \phi) \) of (1.2)–(1.3) is non-increasing (non-decreasing) in \( t \) as long as it exists, for any \( x \); strictly if \( \phi \) is not a solution.

(ii) If \( \phi \) is radial, then \( u(t, x; \phi) \) is radial in the \( x \) variable for any \( t > 0 \).

To prove Theorem 3.1 we adapt the main ideas developed in \([19, 8, 36]\).

As a consequence of the proof of Proposition 2.12 we get the following result which will be useful to prove the stability of the solutions, and replaces a longer elliptic estimate performed in \([8, \text{Lemma 4.3}], \) and adapted in \([36, 6]\). We stress that in fact the proof in the critical case, considered in \([4]\), suffers from a flaw.

**Lemma 3.6.** Assume \( K, G_1, G_2, G_3, G_4 \). Assume \( \beta > \alpha \) then \( A(\beta) > A(\alpha) \).

**Proof.** Since \( U(r, \beta) > U(r, \alpha) \) for any \( r > 0 \) (see Lemma 2.12), we already know that \( A(\beta) \geq A(\alpha) \), so we just need to prove that the inequality is strict. Set \( h(s) = [U(e^s, \alpha_2) - U(e^s, \alpha_1)]e^{(r(t)) - \lambda_1}s \), and, following the notation of Proposition 2.12, \( Q(s) = e^{\lambda_1 s} \). Following the main line in the proof of Proposition 2.12 we see that \( h(s) = Z(s)/Q^2(s) \). In particular, from (2.21) and (2.22), \( h(s) > 0 \) for any \( s \in \mathbb{R} \). Since \( \lim_{s \to +\infty} h(s) = 0 \) we see that \( h(s) > 0 \) for any \( s \in \mathbb{R} \), and \( \lim_{s \to +\infty} h(s) > 0 \).

If \( l_0 > \sigma^* \), then \( \lim_{s \to +\infty} h(s) = A(\beta) - A(\alpha) > 0 \), and the proof is concluded.

Assume now \( l_0 = \sigma^* \), and also assume by contradiction that \( A(\beta) = A(\alpha) \). In this case we see that \( \lim_{s \to +\infty} h(s) = B(\beta) - B(\alpha) \in (0, +\infty) \). However, from (2.21), since \( A = -2\lambda_1 \), for any \( s \in \mathbb{R} \) we find

\[
\hat{h}(s) = \int_{-\infty}^{s} e^{\lambda_1 s} Q(s)[\partial_{t_1} g(P^+_1, +\infty; l_0) W(s) - D(s)] > 0
\]
Therefore \( \liminf_{s \to +\infty} \dot{h}(s) \geq \dot{h}(0) > 0 \), hence \( B(\beta) - B(\alpha) = \lim_{s \to +\infty} h(s) = +\infty \), but this is a contradiction. Hence \( A(\beta) > A(\alpha) \).

**Lemma 3.7.** Assume \( K, G_1, G_2, G_3, G_4 \), and \( l_\alpha \geq l_\beta \). If \( l_\alpha > \sigma^+ \), then \( \| U(r, \beta) - U(r, \alpha) \|_{m+|\lambda_1|} \to 0 \) as \( \beta \to \alpha \), while if \( l_\beta = \sigma^+ \), then \( \| U(r, \beta) - U(r, \alpha) \|_{m+|\lambda_1|} \to 0 \) as \( \beta \to \alpha \).

**Proof.** We develop the proof assuming \( l_\alpha > \sigma^+ \), the case \( l_\beta = \sigma^+ \) is completely analogous. It is well known that, for any fixed \( R > 0 \) and any \( \varepsilon > 0 \), there is \( \delta_1(\varepsilon, \alpha, R) > 0 \) such that

\[
\sup\{\|U(r, \beta) - U(r, \alpha)\| \mid 0 \leq r \leq R\} < \varepsilon
\]

whenever \( |\beta - \alpha| < \delta_1 \) (this is a continuous dependence on initial data argument for the singular equation (1.1)). Further from (2.36) we see that for \( r \) large enough we have

\[
\| U(r, \beta) - U(r, \alpha) \| \leq \|A(\beta) - A(\alpha)\| + o(r^{\lambda_2 - \lambda_1/2})
\]

Thus, for any \( \varepsilon > 0 \) there exists \( M(\varepsilon) \) such that \( |\alpha(r^{\lambda_2 - \lambda_1/2})| \leq C\varepsilon \), when \( r \geq M(\varepsilon) \). Further from Lemma (2.20) we see that for any \( \varepsilon > 0 \) we can find \( \delta_2(\varepsilon, \alpha) > 0 \) such that \( |\alpha(r) - \alpha(\alpha)| \leq \varepsilon \) if \( |\beta - \alpha| < \delta_2 \). Therefore

\[
\| U(r, \beta) - U(r, \alpha) \| \leq \varepsilon, \quad \text{for } r \geq M
\]

The proof then follows from (3.2), (3.3), choosing \( M = R \) and \( \delta(R, \alpha, \varepsilon) = \min\{\delta_1, \delta_2\} \).

**Proof of Theorem 3.1.** We give the proof just in the \( l_\alpha > \sigma^+ \) case, in the \( l_\beta = \sigma^+ \) case is completely analogous. Fix \( \alpha > 0 \) and \( \varepsilon > 0 \) (small); let \( \phi(x) \) be such that \( \|U(|x|, \alpha) - \phi(x)\|_{m+|\lambda_1|} = \delta \), where \( \delta > 0 \) will be chosen below.

Let \( \eta < \alpha \) and set

\[
z(r, \eta) = [U(r, \alpha + \eta) - U(r, \alpha)](1 + r^{m-\lambda_1})
\]

Observe that \( z(0, \eta) = \eta \) and \( \lim_{r \to +\infty} z(r, \eta) = A(\alpha + \eta) - A(\alpha) \). So we can set

\[
\underline{z}(\eta) = \min\{z(r, \eta) \mid r > 0\} \quad \text{and} \quad \overline{z}(\eta) = \max\{z(r, \eta) \mid r > 0\}.
\]

Moreover \( z(r, \eta) \) is uniformly positive (respectively negative) for any \( r > 0 \) if \( \eta > 0 \) (resp. \( \eta < 0 \)), so \( \underline{z}(\eta) > 0 \) if \( \eta \neq 0 \); This follows from Lemmas (2.12), (3.8).

Finally, from Lemma (3.7) we know that \( \lim_{\eta \to 0} \underline{z}(\eta) = \lim_{\eta \to 0} \overline{z}(\eta) = 0 \). Then, for any \( \varepsilon > 0 \) we can find \( d = d(\varepsilon) > 0 \) such that \( \overline{z}(-d) < \varepsilon \), and \( \underline{z}(d) < \varepsilon \). Set \( \alpha_1 = \alpha - d, \alpha_2 = \alpha + d \), and choose \( \delta = \min\{\underline{z}(-d), \overline{z}(d)\} \). Then

\[
U(|x|, \alpha_1) < \phi(x) < U(|x|, \alpha_2)
\]

\[
\|U(|x|, \alpha_i) - U(|x|, \alpha)\|_{m-\lambda_1} \leq \varepsilon \quad \text{for } i = 1, 2
\]

Therefore, from the comparison principle (see, e.g., [15] Appendix[i]), we have that

\[
U(|x|, \alpha_1) < u(t, x; \phi) < U(|x|, \alpha_2), \quad \text{for any } t \geq 0, x \in \mathbb{R}^n,
\]

and the proof is concluded.

3.2. **Weak asymptotic stability.** To prove weak asymptotic stability we follow the outline of the proof of [19] Theorem 4.1] and adapted in [8] [6].

**Proposition 3.8.** Assume we are under the hypotheses of Theorem (3.1) and consider the stationary problem (1.1). Then, for any radial GS \( U(\cdot, d) \) of (1.1), there is a sequence of radial strict super-solutions \( U^{(1)}(\cdot, c_1) > U^{(2)}(\cdot, c_2) > \ldots > U(\cdot, d) \) of (1.1) and a sequence of radial strict sub-solutions \( U^{(1)}(\cdot, c_1) < U^{(2)}(\cdot, c_2) < \ldots < U(\cdot, d) \) of (1.1).
... < U(·, d) such that U(·, d) is the only solution of (1.1) satisfying $\underline{U}^{(k)}(·, c^k) < U(·, d) < \overline{U}^{(k)}(·, c^k)$, for every $k$. Moreover

$$\lim_{k \to \infty} T^{(k)}(·, c^k) = U(·, d) = \lim_{k \to \infty} \overline{U}^{(k)}(·, c^k).$$

**Proof.** Let $h : [0, +\infty) \to [0, 1]$ be a monotone decreasing $C^\infty$ function such that $h(0) = 1$ and $h(r) \equiv 0$ for $r \geq 1$. Let $\mathcal{G}(y_1, s; l_s) = g(y_1, s; l_s) - g(y_1, +\infty; l_s)$ and observe that $\mathcal{G}(y_1, s; l_s) \geq 0$ and it is decreasing in $s$ for any $y_1, s$.

**Assume first** $\mathcal{G}(y_1, s) \equiv 0$, i.e. consider the generic case, and denote by

$$\mathcal{G}^{(k)}(y_1, s) = g(y_1, s; l_s) + \frac{h(c^k)}{h(c^k)} \mathcal{G}(y_1, s; l_s)$$

and let $\overline{f}^{(k)}, \underline{f}^{(k)}$ be the corresponding functions obtained via (2.2). Notice that by construction $\mathcal{G}^{(k)}(y_1, s)$, and $\mathcal{G}^{(k)}(y_1, s)$ are both decreasing in $s$ for any $k \geq 1$; hence $\underline{f}^{(k)} \geq f \geq \overline{f}^{(k)}$ satisfy $G_1, G_2, G_3, G_4, K$ so that Lemma 2.10 and Proposition 2.12 hold true. In particular all the regular solutions of the respective problem (1.1), say $\overline{U}^{(k)}(r, \alpha), U(r, \alpha), \underline{U}^{(k)}(r, \alpha)$, are GSs. Further the corresponding trajectories of (2.3), say $\mathcal{G}^{(k)}(s, \alpha), \mathcal{G}(s, \alpha), \mathcal{G}^{(k)}(s, \alpha)$ are monotone increasing in their first component and converge to $P^+$, and have the asymptotic expansion as described in Proposition 2.10. More precisely they both have either the expansion (2.30) or (2.37), where the function $\Psi(\ln(r))$ coincide for $r \geq 1$, while the coefficients $a = \mathcal{A}^{(k)}(\alpha), a = \mathcal{A}^{(k)}(\alpha)$ and $b = \mathcal{B}^{(k)}(\alpha), b = \mathcal{B}^{(k)}(\alpha)$ are different, see Lemma 2.20. Further by construction, $\overline{U}^{(k)}(r, \alpha), U^{(k)}(r, \alpha)$ are respectively super and sub-solutions for the original problem (1.1).

We divide our argument in several steps.

**Step 1.** If there is $R > 0$ such that $U(R, d) = \overline{U}^{(k)}(R, e)$ (respectively $U(R, d) = \underline{U}^{(k)}(R, e)$), then $U(r, d) \geq \overline{U}^{(k)}(r, c)$ (respectively $U(r, d) \leq \underline{U}^{(k)}(r, c)$) for any $r \geq R$.

Let $\tau(\xi) : (0, 2) \to \mathbb{R}$ be the inverse of the function $\xi(\tau)$ defined in (2.10). We consider

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ B(l_s) & -A(l_s) & 0 & 0 \\ 0 & 0 & \xi & 0 \\ g(y_1, \xi) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \tau(\xi) \\ l_s \end{pmatrix},$$

where $A(l_s), B(l_s)$ coincide with $A(l_n), B(l_n)$ for $s \leq 0$ and with $A(l_s), B(l_s)$ for $s \geq 0$, and similarly $g(y_1, \xi) \equiv g(y_1, \frac{\ln(\xi)}{\varepsilon}; l_s)$ for $\xi \leq 1$ (i.e. $s \leq 0$) and $g(y_1, \frac{\ln(1-\xi)}{\varepsilon}; l_s)$ for $\xi \geq 1$ (i.e. $s \geq 0$). Notice that (3.9) coincides with (2.7) when $\xi \leq 1$ (i.e. $s \leq 0$) and it is equivalent to (2.8) when $\xi \geq 1$ (i.e. $s \leq 0$ and $\xi \leq 1$, it differs from (2.8) just in the fact that $\xi = 2 - \xi$). Further we recall that the unstable manifold $W^u(l_s)$ defined in (2.10) has dimension 2 and connects the $\xi$-axis and the graph of $y^*(s, l_s)$; further it is a graph on the $y_2 = 0$ plane, see Remark 2.13. Moreover $W^u(l_s)$ splits the set

$$E = \{(y_1, y_2, \xi) \mid 0 < y_1 < y^*_1(\xi, l_s), 0 < \xi < 2\}$$

in 2 open components, say $E^+$ and $E^-$ (the one with larger and smaller $y_2$).

By construction the flow of the modified system (3.9) where $g$ is replaced respectively by $\mathcal{G}^{(k)}$ and by $\mathcal{G}^{(k)}$ on $W^u(l_s)$ points towards $E^-$ and $E^+$ respectively for $s \leq 0$, and it is tangent to $E^0$ for $s \geq 0$. So the corresponding manifolds $W^{u,(k)}_+(l_s)$ and $W^{u,(k)}_-(l_s)$ lie respectively in $E^+$ and $E^-$. 

Now assume $U(R, d) = U^{(k)}(R, c)$ and consider the corresponding trajectories $y(s; l_s)$ and $\overline{y}^{(k)}(s; l_s)$: Then $y_1(ln(R); l_s) = \overline{y}_1^{(k)}(ln(R); l_s)$ and $y_2(ln(R); l_s) \geq \overline{y}_2^{(k)}(ln(R); l_s)$. Hence $y_1(s; l_s) \geq \overline{y}_1^{(k)}(s; l_s)$ for $s$ in a right neighborhood of $ln(R)$.

Then the claim in Step 1 concerning $U^{(k)}(r, c)$ follows. The claim concerning $\overline{U}^{(k)}(r, c)$ is analogous.

We continue the discussion for later purposes. We know that $y_2(ln(R); l_s) \geq \overline{y}_2^{(k)}(ln(R); l_s)$, assume first $y_2(ln(R); l_s) > \overline{y}_2^{(k)}(ln(R); l_s)$. Then $y_1(s; l_s) > \overline{y}_1^{(k)}(s; l_s)$ for $s$ in a right neighborhood of $ln(R)$.

Assume now $y_2(ln(R); l_s) = \overline{y}_2^{(k)}(ln(R); l_s)$: Then $R \geq 1$. In fact assume for contradiction that $0 < R < 1$, then $\overline{y}_2(ln(R); l_s) = Q = \overline{y}^{(k)}(ln(R); l_s)$ is such that $(Q, ξ(ln(R))) ∈ E^0$, but from (2.3) we get $\overline{y}_2(ln(R); l_s) < \overline{y}^{(k)}(ln(R); l_s)$. Hence $\overline{y}^{(k)}(r; l_s)$ crosses transversally $E^0$ at $s = ln(R)$, going from $E^+$ to $E^-$, in particular it is in $E^+$ when $s$ is in a sufficiently small left neighborhood of $ln(R)$. But $(\overline{y}^{(k)}(s; l_s), ξ(s)) ∈ \overline{W}^{\text{ns}}(k)(l_s) ⊂ E^-$, and this is a contradiction, so $R > 1$.

Observe that if $R > 1$ then $\overline{y}^{(k)}(s; l_s)$ and $y(s; l_s)$ are solutions of the same equation (2.3) for $s \geq 0$ which coincide for $s = ln(R)$, so they coincide for $s \geq 0$.

In fact we have already proved the following, i.e.

**–Step 2.** For any $0 < r < 1$ we have that

\[
U^{(k)}(r, d) < U(r, d) < \overline{U}^{(k)}(r, d)
\]

and either (3.10) holds for any $r > 0$ or the functions coincide for any $r \geq 1$. Moreover $\overline{A}^{(k)}(d) ≤ A(d) ≤ A^{(k)}(d)$.

**–Step 3.** Fix $d$ and the corresponding coefficient $A(d)$. It is possible to choose $c^k < d < e^k$ so that $A^{(k)}(c^k) = \overline{A}^{(k)}(e^k) = A(d)$. Then, from Step 1 it follows that $U(r, d)$ is the unique solution of the original equation (1.4) such that

\[
U^{(k)}(r, c^k) ≤ U(r, d) ≤ \overline{U}^{(k)}(r, e^k), \text{ for any } r \geq 0
\]

Fix $τ > 0$ and $0 < c < d < e$: let $y(s, τ, P; l_s)$, $y(s, τ, Q; l_s)$, $y(s, τ, R; l_s)$ be the trajectories of (2.3) corresponding to the solutions $U(r, c)$, $U(r, d)$, $U(r, e)$ of (1.4). It follows that $P, Q, R$ are points in $W^u(τ, l_s)$ and $P, R$ are respectively the closest to and the farthest from the origin. Let us consider the lines $ℓ^p, ℓ^r$ parallel to the $y_2$-axis and passing through $P$ and $R$ respectively: We denote by $\overline{P}^{(k)}$ and $\overline{R}^{(k)}$, the intersections of $\overline{W}^{\text{ns}}(k)(τ, l_s)$ respectively with $ℓ^p$ and with $ℓ^r$.

Using continuous dependence on initial data of ODE we see that $\overline{P}^{(k)} → P$ and $\overline{R}^{(k)} → R$ as $k → ∞$. Since $a(Q)$ is continuous, see Remark 2.18 and 2.35, we see that $a(\overline{P}^{(k)}) → a(P) = A(c) < A(d)$, while $a(\overline{R}^{(k)}) → a(R) = A(e) > A(d)$. Therefore we can choose $N$ large enough so that $a(\overline{P}^{(k)}) < A(d) < a(R^{(k)})$ for any $k ≥ N$. Hence we can find $Q^{(k)} ∈ \overline{W}^{\text{ns}}(k)(τ, l_s)$ between $\overline{P}^{(k)}$ and $\overline{R}^{(k)}$ such that $a(\overline{Q}^{(k)}) = A(d)$. Correspondingly we find $e^k$ such that $A(k)(e^k) = a(Q^{(k)}) = A(d)$. Note that in view of Step 2 we have $c^k ≥ d$. The proof for $\overline{A}^{(k)}(c^k)$ is analogous.

**–Step 4.** Formula (3.3) and the following Remark hold true.

**Remark 3.9.** $\overline{B}^{(k)}(c^k)$ and $\overline{B}^{(k)}(e^k)$ are respectively strictly decreasing and increasing in $k$ and they both converge to $B(d)$.

**Proof.** To prove (3.3) it is enough to observe that, by construction, the functions $\overline{U}^{(k)}(r, c^k)$ and $\overline{U}^{(k)}(r, e^k)$ are bounded and monotonically respectively increasing and decreasing in $k$. Then, from standard elliptic estimates and Step 3 we see that the limit of both is the solution $U(r, d)$ of the original problem (1.4).
Now, we turn to consider Remark 3.9. Let us recall that, by construction, the following relation holds true, i.e.: $A(\varepsilon) = B(\varepsilon) = A(\varepsilon^2)$ (see Step 3). Further we also infer that $B(\varepsilon) = A(\varepsilon^2)$ and $B(\varepsilon^2)$ are respectively decreasing and increasing and converge to $B(\varepsilon)$. As usual we just prove the last inequality, the others being analogous. Let $\bar{w}(x)$ be the radial function defined by $\bar{w}(x) = \bar{U}(\|x\|)$. Observe that $\Delta [\bar{w}(x)] - \bar{w}(x) \leq 0$, hence from standard arguments (see [27, Theorem 3.8]), we see that there is $C > 0$ such that $\bar{U}(r, \varepsilon) > c(r, \varepsilon - \varepsilon^2)$ for any $r$, from the construction in Lemma 2.20, it follows that $\bar{U}(r, \varepsilon^2) > c(\varepsilon)$ for any $\varepsilon > 0$, i.e., $\bar{U}(r, \varepsilon^2) = \bar{U}(r, \varepsilon^2)$ for $r \geq 0$, but this is a contradiction and the Lemma is proved.

From Remark 3.9 we see that the inequalities in (3.11) are strict for $r$ large. Then, from Step 1 we conclude.

Assume now $G(y_1, s) \equiv 0$, this is the case, e.g., when $f(u, r) = cu|u|^{q-2}$. Following [19] we denote by $f^{(1)}(u, r) := [1 - \mu h(r)/k] f(u, r)$ and $f^{(2)}(u, r) := [1 + \mu h(r)/k] f(u, r)$, for $k \in \mathbb{N}$ and where $\mu > 0$ is chosen small enough so that $f^{(1)}(u, r)$ satisfies $A^-$; then it is easy to check that $f^{(1)}(u, r)$ and $f^{(2)}(u, r)$ satisfy $A^-$ for any $k \in \mathbb{N}$. So Proposition 3.3 holds true, and in all the cases the regular solutions of (1.1), denoted respectively by $U^{(k)}(r, \alpha), U(r, \alpha)$, are GSs, but a priori they might not be ordered. However repeating the argument of Step 1 in [19, Theorem 4.1], it is easy to prove that

\begin{equation}
(3.12) \quad U^{(k)}(r, \alpha) \leq U(r, \alpha) \leq U^{(k)}(r, \alpha)
\end{equation}

for any $r > 0$ and any $\alpha > 0$. The proof might be concluded arguing as in [19, Theorem 4.1]. However notice that we can also repeat the argument at the end of Step 1 of this proof to get (3.14) for any $r > 0$, and then carry on through Step 2, 3, 4 of this proof and conclude also in this case, with no further changes.

From the previous discussion we easily find the following result.

3.3. Proof of the weak asymptotic stability. Now we consider $d > 0$ fixed, and we use the shorthand notation $U^{(1)}(r, \alpha) = U(r, \alpha)$, $U^{(1)}(r, \alpha) = U(r, \alpha)$.

Lemma 3.10. Assume that we are in the hypotheses of Theorem 3.3. Then $\bar{u}(t, x) \rightarrow U(|x|, d)$ and $u(t, x) \rightarrow U(|x|, d)$ as $t \rightarrow +\infty$, with the norm $\| \cdot \|_1$, for any $0 \leq l < m + |\lambda_2|$.

Notice that if $l_0 = \sigma^*$ and $\|m + |\lambda_1| = \|m + \lambda_2|$.

Proof. Let us set $B := \lim_{|x| \times \rightarrow +\infty} |U(|x|) - U(|x|)|/|x|^{m + |\lambda_2|}$ and notice that $B > 0$ is finite, see Proposition 3.5 and Remark 3.9. Fix $0 \leq l < m + |\lambda_2|$ and observe that for any $\varepsilon > 0$ we can find $\rho > 0$ such that

\begin{equation}
(3.13) \quad \|U(|x|) - U(|x|)\|^{l} < 2B|x|^{m + |\lambda_2|} < \varepsilon/2
\end{equation}

for $t > \rho$.

Since $\bar{U}(|x|)$ and $U(|x|)$ are respectively a radial super and sub-solution of (1.1), then $\bar{u}(t, x)$ and $u(t, x)$ are respectively radially symmetric super and sub-solution of (1.2). Further they are resp. monotone decreasing and increasing in $t$, so they converge to a radial solution of (1.1), see Lemma 3.5. From Lemma 3.8 we know
that $U(r, d)$ is the unique solution of (3.14) between $\tilde{U}(r)$ and $\bar{U}(r)$, so $\tilde{u}(x, t)$ and $\bar{u}(t, x)$ converge monotonically to $U(|x|, d)$ as $t \to +\infty$, for any fixed $x \in \mathbb{R}^n$. Then, from the equiboundedness of the functions involved and of their derivatives we see that the convergence is uniform in any ball of radius $R > 0$ fixed. Hence setting $R = \rho > 0$, for any $\varepsilon > 0$ we find $T(\varepsilon) > 0$ such that

$$
|\tilde{u}(x, t) - \bar{u}(x, t)| |x|^l \leq \varepsilon/2
$$

for any $|x| \leq \rho$. Further from (3.13) and the comparison principle we easily find

$$
|\tilde{u}(x, t) - \bar{u}(x, t)| |x|^l \leq |U(|x|) - \bar{U}(|x|)| |x|^l \leq \varepsilon/2
$$

for $|x| \geq \rho$. Hence the Lemma follows from (3.14) and (3.15). \hfill \Box

**Proof of Theorem 3.2** Assume for definiteness $l_+ > \sigma^*$, the case $l_+ \geq \sigma^*$ being analogous. Fix $d > 0$ and denote by

$$
W(r, d) = [\bar{U}(r) - U(r, d)](1 + r^{m+|\lambda_2|}) \quad \bar{\delta} = \inf_{r > 0} W(r, d)
$$

$$
W(r, d) = [U(r, d) - \bar{U}(r)](1 + r^{m+|\lambda_2|}) \quad \hat{\delta} = \inf_{r > 0} W(r, d)
$$

Observe that $\tilde{W}(r, d)$, $\bar{W}(r, d)$ are both positive for any $r > 0$, see Proposition 3.3. Further $\tilde{W}(0, d) = e^{\bar{\delta}} - d > 0$, $\bar{W}(0, d) = d - e^{\hat{\delta}} > 0$, $\lim_{r \to +\infty} \tilde{W}(r, d) = \frac{1}{B^{(1)}}(e^{\bar{\delta}} - B(d) > 0$, $\lim_{r \to +\infty} \bar{W}(r, d) = B(d) - \frac{1}{B^{(1)}(e^{\hat{\delta}} > 0$, see Remark 3.4. It follows that $\delta = \min(\bar{\delta}, \hat{\delta}) > 0$.

Now let us consider $\phi$ such that $||\phi - U(\cdot, d)||_{m+|\lambda_2|} < \delta$: by construction we have $\tilde{U}(|x|) \leq \phi(x) \leq \bar{U}(|x|)$, for any $x \in \mathbb{R}^n$. Therefore

$$
\tilde{u}(t, x) \leq u(t, x; \phi) \leq \bar{u}(t, x)
$$

for any $t > 0$ and any $x \in \mathbb{R}^n$. So from Lemma 3.10 we easily conclude. \hfill \Box

**Appendix A. Proof of Proposition 2.16**

In what follows we develop a constructive argument to prove the asymptotic expansion results of subsection 2.4. This result, to the best of our knowledge, seems to be new and it is of independent interest for the ODEs theory. Further, we borrow some of the ideas from [31, 36]. The purpose is to approximate a generic solution converging to a critical point of a non-linear system, by a recursive sequence of solutions of approximating linear non-homogeneous systems.

Let us consider an equation of the form

$$
\dot{x} = f(x) = Lx + N(x)
$$

where $x \in \mathbb{R}^n$, $N(x)$ is at least $C^2$ and such that $N(0) = N_2(0) = 0$. More regularity will be required in the second part of the proof.

We denote by $\lambda_i$ the eigenvalues of $L$ and by $m$ the number of eigenvalues with distinct real parts. We set $\Lambda := -\Re(\lambda_i)$, and we assume for definiteness $\Lambda_i < \Lambda_{i+1}$ for any $1 \leq i \leq m-1$. We also assume $\Lambda_1 > 0$, hence (A.1) is exponentially stable. Further let $l_i$ be the number of eigenvalues (counted with multiplicity) such that $-\Re(\lambda_j) = \Lambda_i$, so that $\sum_{i=1}^m l_i = n$.

We can assume without loss of generality that $L$ is block diagonalized, and that each block, $L_1, \ldots, L_m$ is in Jordan form. We denote by $P_j$ the matrix which is the identity in the $j$-th block and the null matrix in the other blocks, so that $L P_j = P_j L = L_j$. Then we set $P = \sum_{j=1}^m P_j$: So the matrices $P_i$ and $P$ are projections on eigenspaces.
Note that any solution $\ell(t)$ of the linear equation $\dot{y} = Ly$ takes the form

$$\ell(t) = (e^{L_{i+1}t}, e^{L_{i+2}t}, \ldots, e^{L_m t}) \ell(0) = \sum_{i=1}^m \ell_i(t)$$

(A.2)

where

$$\ell_i(t) = (0, \ldots, 0, e^{L_{i+1}t}, 0, \ldots, 0) \ell(0)$$

In the whole appendix the notation $f(t) = o(g(t))$ and $f(t) = O(h(t))$ means respectively that $\|f(t)\|/\|g(t)\| \to 0$ and $\|f(t)\|/\|h(t)\|$ remains bounded as $t \to +\infty$. Further $\varepsilon$ is a positive constant which is taken as small as needed, and it may change from line-to-line.

**Remark A.1.** Notice that, for any $\varepsilon > 0$ we have $e^{-\Lambda_{i}t} \leq \|e^{L_{i}t}\| \leq e^{(-\Lambda_{i}+\varepsilon)t}$ whence $t > 0$, and $i = 1, \ldots, m$. In fact there is $c > 0$ such that $e^{-\Lambda_{i}t} \leq \|e^{L_{i}t}\| \leq c[1 + t^{1-\varepsilon}]e^{-\Lambda_{i}t}$ for any $t > 0$, and any $i = 1, \ldots, m$.

We start from the following technical Lemma.

**Lemma A.2.** Let $y(t) = O(e^{-kt})$ where $k > \Lambda_{i}$, then the integral

$$I(t) = \int_{t}^{+\infty} e^{L(t-s)}P_{1}y(s)ds$$

is well defined and $I(t) = O(e^{-kt})$.

**Proof.** First notice that, for any $j = 1, \ldots, i$, and any sufficiently small $\varepsilon > 0$, we have

$$\left\| \int_{t}^{+\infty} e^{L(t-s)}P_{j}y(s)ds \right\| \leq \int_{t}^{+\infty} e^{(\Lambda_{j}+\varepsilon)(s-t)}Ce^{-ks}ds = \frac{C}{k - \Lambda_{j} - \varepsilon}e^{-kt}$$

Then observe that the first $i$ blocks of $I(t)$ satisfy the previous estimate, while the last $m - i$ ones are null. So the Lemma immediately follows. \(\Box\)

Now we recall the following standard result: We sketch the proof since it gives the **Step 0** of our approximating procedure.

**Lemma A.3.** [4 §13-Theorem 4.5] Let $f$ be $C^2$, and let $x(t)$ be a solution of (A.1) such that $x(t) \to 0$ as $t \to +\infty$. Then, there is $\bar{\ell}$ such that the solution $\ell(t) := e^{t\bar{\ell}}\ell$ of the linear equation $\dot{y} = Ly$, satisfies

$$\|x(t) - \ell(t)\|e^{2(\Lambda_{i}-\varepsilon)T} \to 0, \quad \text{as } t \to +\infty,$$

for any $\varepsilon > 0$. Further

$$\bar{\ell} = x(0) + \dot{\mathfrak{R}}_{1}, \quad \text{where } \mathfrak{R}_{1} = \int_{0}^{+\infty} e^{-L_{s}}P_{1}N(x(s))ds.$$  

**Sketch of the proof.** From [4 §13] we already know that any solution $x(t)$ of (A.1) satisfies $x(t)e^{(\Lambda_{i}-\varepsilon)t} \to 0$ as $t \to +\infty$, for any $\varepsilon > 0$. Therefore the integral in (A.4) defining $\mathfrak{R}_{1}$ is convergent and $\bar{\ell}$ is well defined, see Lemma A.2. Then observe that the solution $x(t)$ of (A.1) can be rewritten as follows

$$x(t) = e^{t\bar{\ell}}x(0) + \int_{0}^{t} e^{L(t-s)}N(x(s))ds.$$

Hence we get

$$x(t) - \ell(t) = \int_{0}^{t} e^{L(t-s)}(1 - P_{1})N(x(s))ds - \int_{t}^{+\infty} e^{L(t-s)}P_{1}N(x(s))ds.$$  

Since $N(x(t)) = o(e^{-2(\Lambda_{i}-\varepsilon)t})$, using also Lemma A.2 we get (A.3) and conclude the proof. \(\Box\)
Since $k_e$ we stress that by construction (A.12) Denote by $\mathcal{R}_a$ (A.11) By construction (A.13) $x(t) = A^k_1(t) + R^k_1(t)$, where $A^k_1(t) = \sum_{j=0}^{k_2} a^j_1(t)$ 

(A.8) $a^j_1(t) = o(e^{-[(1+j)k_1-\epsilon]t})$, and $R^k_1(t) = o(e^{-k_2\epsilon t})$

To help the reader with the notation we emphasize that the apex indicates the step of the iteration, while the subscript indicates the eigenvalue we are dealing with.

Proof. Let us start from $A^0_1(t) = a^0_1(t) := \ell_1(t), R^0_1(t) := x(t) - \ell_1(t)$. Repeating the computation of Lemma A.2 (with $\ell_1$ replacing $\ell$ we see that

(A.9) $a^0_1(t) = o(e^{-k_1\epsilon t}), R^0_1(t) = o(e^{-(2k_1-\epsilon)\epsilon t} + e^{-k_2\epsilon t})$.

If $k_2 = 0$ we have $R^0_1(t) = o(e^{-(k_2-\epsilon)\epsilon t})$ and Lemma A.5 is proved. Otherwise we go through the following steps:

- **Step 1.** Since $N(a^0_1(t)) = o(e^{-(k_2+\epsilon)\epsilon t})$ as $t \to +\infty$, using Lemma A.2 we can define

$$M^1_1(s) := N(a^0_1(s)), \quad a^1_1(t) := \int_0^t e^{L(t-s)}(1 - \mathcal{P}_1)M^1_1(s)ds - \int_t^{+\infty} e^{L(s-t)}\mathcal{P}_1M^1_1(s)ds$$

By construction

$$a^1_1(t) = o(e^{-(2k_1-\epsilon)t})$$

Denote by $A^1_1(t) = a^1_1(t) + a^0_1(t)$, and by $R^1_1(t) = x(t) - A^1_1(t)$; From (A.6), (A.11) we see that $R^1_1(t)$ can be written as follows:

$$R^1_1(t) := J_1(t) + K^1_1(t), \quad where \ J_1(t) = e^{Lt}(\mathbb{I} - \mathcal{P}_1)x(0), \ and$$

$$K^1_1(t) := \int_0^t e^{L(t-s)}(\mathbb{I} - \mathcal{P}_1)M^{1,R}_1(s)ds - \int_t^{+\infty} e^{L(s-t)}\mathcal{P}_1M^{1,R}_1(s)ds$$

By construction

$$a^1_1(t) = o(e^{-(k_2+\epsilon)t})$$

Denote by $A^1_1(t) = a^1_1(t) + a^0_1(t)$, and by $R^1_1(t) = x(t) - A^1_1(t)$; From (A.6), (A.11) we see that $R^1_1(t)$ can be written as follows:

$$R^1_1(t) := J_1(t) + K^1_1(t), \quad where \ J_1(t) = e^{Lt}(\mathbb{I} - \mathcal{P}_1)x(0), \ and$$

$$K^1_1(t) := \int_0^t e^{L(t-s)}(\mathbb{I} - \mathcal{P}_1)M^{1,R}_1(s)ds - \int_t^{+\infty} e^{L(s-t)}\mathcal{P}_1M^{1,R}_1(s)ds$$

We stress that by construction $J_1(t) = o(e^{-(k_2+\epsilon)t})$. Since $R^1_1(t) = R^0_1(t) - a^1_1(t) = o(e^{-(2k_1-\epsilon)t})$, cf. (A.9), (A.11), we find that $M_{2,R}(t) = O(A^0_1(t)R^1_1(t)) = o(e^{-(3k_1-\epsilon)t})$. Therefore, from (A.12) and Lemma A.2 we find $K^1_1(t) = o(e^{-(3k_1-\epsilon)t})$ so that $R^1_1(t) = o(e^{-(k_2+\epsilon)t} + e^{-(3k_1-\epsilon)t})$.

So we can expand $x(t)$ as $x(t) = A^1_1(t) + R^1_1(t)$ and $R^1_1(t) = o(e^{-(k_2+\epsilon)t} + e^{-(3k_1-\epsilon)t})$. If $k_2 = 2$, then $R^1_1(t) = o(e^{-(k_2+\epsilon)t})$ and Lemma A.3 is proved, while if $k_2 > 1$, then $R^1_1(t) = o(e^{-(3k_1-\epsilon)t})$ and we go to the next step.

- **Step 2.** Since $N(A^1_1(t)) = N(A^0_1(t)) = o(e^{-(3k_1-\epsilon)t})$, we can define

$$M^2_1(s) := N(A^1_1(s)) - N(A^0_1(s)), \quad and$$

$$a^2_1(t) := \int_0^t e^{L(t-s)}(1 - F_1)M^2_1(s)ds - \int_t^{+\infty} e^{L(t-s)}\mathcal{P}_1M^2_1(s)ds.$$
Notice that
\[ a_1^2(t) = o(e^{-(3\lambda_1-\varepsilon)t}) \] as \( t \to +\infty \).
Denote by \( A_1^2(t) = A_1^2(t) + a_1^2(t) \), and by \( R_1^2(t) = x(t) - A_1^2(t) \). From (A.6), (A.10), (A.13) we see that \( R_1^2(t) \) solves the following fixed point problem:
\[ R_1^2(t) := J_1(t) + K_1^2(t) \]
where
\[ K_1^2(t) = \int_0^t e^{L(t-s)}(1 - \mathcal{P}_1)M_1^2(s)ds - \int_0^{+\infty} e^{L(t-s)}\mathcal{P}_1M_1^2R(s)ds \]
\[ M_1^2(s) = N(x(s)) - N(A_1^1(s)) = N(A_1^1(s) + R_1^2(s)) - N(A_1^1(s)) \]
Since \( R_1^2(t) = R_1^1(t) - A_1^1(t) = o(e^{-(3\lambda_1-\varepsilon)t}) \), cf. (A.14), we find that \( M_1^2(t) = O(A_1^0(t)R_1^2(t)) = o(e^{-(4\lambda_1-\varepsilon)t}) \). Therefore from (A.15) and Lemma (A.2) we find that \( K_1^2(t) = o(e^{-(4\lambda_1-\varepsilon)t}) \).
So we can expand \( x(t) \) as \( x(t) = A_1^2(t) + R_1^2(t) \) and \( R_1^2(t) = o(e^{-(\lambda_1-\varepsilon)t} + e^{-(4\lambda_1-\varepsilon)t}) \). If \( k_2 = 2 \) then \( R_1^2(t) = o(e^{-(2\lambda_2-\varepsilon)t}) \) and Lemma (A.2) is proved, while if \( k_2 > 2 \) then \( R_1^2(t) = o(e^{-(4\lambda_1-\varepsilon)t}) \) we iterate the argument of Step j below, till \( j = k_2 \).

**Step j.** Since \( N(A_1^{-1}(t)) - N(A_1^{-2}(t)) = o(e^{-(j+1)\lambda_1-\varepsilon}t) \) as \( t \to +\infty \), we can define
\[ M_1^j(t) := N(A_1^{-1}(s)) - N(A_1^{-j}(s)), \quad \text{and} \]
\[ a_1^j(t) := \int_0^t e^{L(t-s)}(1 - \mathcal{P}_1)M_1^j(s)ds - \int_0^{+\infty} e^{L(t-s)}\mathcal{P}_1M_1^jR(s)ds \]
Notice that
\[ a_1^j(t) = o(e^{-(j+1)\lambda_1-\varepsilon}t) \] as \( t \to +\infty \).
Denote by \( A_1^j(t) = A_1^{-1}(t) + a_1^j(t) \), and by \( R_1^j(t) = x(t) - A_1^j(t) \). From (A.6), (A.10), (A.13), (A.19) we see that \( R_1^j(t) \) can be written as follows:
\[ R_1^j(t) := J_1(t) + K_1^j(t) \]
where
\[ K_1^j(t) = \int_0^t e^{L(t-s)}(1 - \mathcal{P}_1)M_1^jR(s)ds - \int_0^{+\infty} e^{L(t-s)}\mathcal{P}_1M_1^jR(s)ds \]
\[ M_1^jR(s) = N(x(s)) - N(A_1^{-1}(s)) = N(A_1^j(s) + R_1^j(s)) - N(A_1^{-1}(s)) \]
Since \( R_1^j(t) = R_1^{j-1}(t) - a_1^j(t) = o(e^{-(j+1)\lambda_1-\varepsilon}t) \), cf (A.17), we find that the term \( M_1^jR(t) = O(A_1^0(t)R_1^j(t)) = o(e^{-(j+2)\lambda_1-\varepsilon}t) \). Thereafter, from (A.13) and Lemma (A.2) we find \( K_1^j(t) = o(e^{-(j+1)\lambda_1-\varepsilon}t), \) and \( R_1^j(t) = o(e^{-(j+1)\lambda_1-\varepsilon}t + e^{-(3\lambda_1-\varepsilon)t}) \).

In the next table we enumerate the terms \( a_1^j(t) \), and the related asymptotic behaviors, that we are going to use in the remaining part of the proof.

| \( a_1^j(t) \) | \( e^{-\lambda_1 t} \) | \( e^{-3\lambda_1 t} \) | \( e^{-5\lambda_1 t} \) | \( \ldots \) | \( e^{-(2j+1)\lambda_1 t} \) |
|---------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( a_2^j(t) \) | \( e^{-2\lambda_1 t} \) | \( e^{-4\lambda_1 t} \) | \( e^{-6\lambda_1 t} \) | \( \ldots \) | \( e^{-(2j+2)\lambda_1 t} \) |
| \( a_3^j(t) \) | \( e^{-3\lambda_1 t} \) | \( e^{-5\lambda_1 t} \) | \( e^{-7\lambda_1 t} \) | \( \ldots \) | \( e^{-(2j+3)\lambda_1 t} \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( a_m^j(t) \) | \( e^{-m\lambda_1 t} \) | \( e^{-(m+2)m\lambda_1 t} \) | \( e^{-(m+4)m\lambda_1 t} \) | \( \ldots \) | \( e^{-(2j+m)\lambda_1 t} \) |

**Table 1.** Terms \( a_1^j(t) \): We omit the \( \varepsilon \)-shift of the exponents so that \( e^{-\lambda t} \) stands for \( o(e^{-(\lambda-\varepsilon)t}) \).
In the next table we list the terms $R_l^j(t)$, and the related asymptotic behaviors.

| $R_l^j(t)$ | $R_l^j(t)$ | ... | $R_l^{k_j-1}(t)$ | $R_l^{k_j}(t)$ |
|------------|------------|-----|-----------------|---------------|
| $e^{-(k_2+k_1+1)\lambda_1 t}$ | $e^{-(k_2+k_1+2)\lambda_1 t}$ | ... | $e^{-(k_2+k_1)\lambda_1 t}$ | $e^{-(k_2)\lambda_1 t}$ |
| $R_l^0(t)$ | $R_l^1(t)$ | $R_l^{k_j-1}(t)$ | $R_l^{k_j}(t)$ |
| $e^{-(k_2+k_1+1)\lambda_1 t}$ | $e^{-(k_2+k_1+2)\lambda_1 t}$ | ... | $e^{-(k_2+k_1)\lambda_1 t}$ | $e^{-(k_2)\lambda_1 t}$ |
| $R_l^0(t)$ | ... | ... | ... | ... |

Table 2. Terms $R_l^j(t)$: Also in this case we omit the $e$-shift of the exponents so that $e^{-\lambda t}$ stands for $o(e^{-\lambda t})$.

It is worthwhile to observe that $a_1(t)$ and $a_2(t)$ are solutions respectively of the (explicitly solvable) linear non-homogeneous problems

\[(A.19) \quad \dot{y} = Ly + M_1(t), \quad \text{and} \quad \dot{y} = Ly + M_2(t),\]

where $M_1(t)$ and $M_1^j(t)$ are defined in (A.10) and in (A.16), respectively. The initial conditions can be written respectively as

\[a_1(0) = - \int_0^{+\infty} e^{-Ls}P_1M_1(s)ds, \quad \text{and} \quad a_j(0) = - \int_0^{+\infty} e^{-Ls}P_1M_j^j(s)ds.\]

However $a_1(0)$, $a_2(0)$ are the real unknown of the problem, since they depend on $\bar{\ell}_1$, which is evaluated by using a fixed point argument and not by a closed formula. In fact $\bar{\ell}_1$ can just be approximated as $\bar{\ell}_1 = P_1x(0) + O(x^2(0))$.

Also, the remainder terms $K_1^j(t)$ and $K_1^j(t)$ solve the non-linear differential equations

\[(A.20) \quad \dot{y} = Ly + M_1^{R,j}(t), \quad \text{and} \quad \dot{y} = Ly + M_1^{R,j}(t),\]

where $M_1^{R,j}(t)$ and $M_1^{R,j}(t)$ are defined in (A.12) and in (A.13), respectively (and are non linear function of $y$).

In order to proceed with the expansion beyond $e^{-\lambda_2 t}$ we need to take into account the contribution of the linear part once again. For this reason we set

\[(A.21) \quad \bar{\ell}_2 = P_2(x(0)) + \bar{\Theta}_2, \quad \text{where} \quad \bar{\Theta}_2 := \int_0^{+\infty} e^{-Ls}P_2[N(x(s)) - N(A_2^{k_2}(s))]ds.\]

Notice that $\bar{\Theta}_2$ is well defined since

\[(A.22) \quad M_2^{0,R}(t) := N(x(t)) - N(A_2^{k_2}(t)) = o(e^{-[(k_2+k_1+1)\lambda_1-\varepsilon]t}).\]

Then, we iterate the previous argument, and we prove the following.

**Lemma A.6.** Let us recall that $\lambda_1 \left( \sum_{i=1}^{3} k_i \right) \leq \lambda_3 < \lambda_1 \left( 1 + \sum_{i=1}^{3} k_i \right)$, cf. (A.7), and set $a_2(0) := \ell_2(t) = e^{Lt}\bar{\ell}_2$. Then we can expand $x(t)$ as follows

\[(A.23) \quad x(t) = A_2^{k_2}(t) + \sum_{j=1}^{k_3} a_2(t) + R_2^{k_3}(t), \quad \text{where} \quad a_2(t) = o(e^{-\lambda_2-1-\lambda_1-\varepsilon}t), \quad \text{and} \quad R_2^{k_3}(t) = o(e^{-\lambda_3-\varepsilon}t).\]

**Proof.** Let us set $A_2^{k_2}(t) = A_2^{k_2}(t)$, $A_2^{k_2}(t) = A_2^{k_2}(t) + a_2(t)$. From (A.5), (A.13), we get

\[R_2^{k_3}(t) = x(t) - A_2^{k_2}(t) = e^{Lt}(\mathbb{I} - P_1)x(0) - e^{Lt}P_2x(0) + \int_0^t e^{L(t-s)}(\mathbb{I} - P_1)(\mathbb{I} - P_2)R_2^{0,R}(s)ds - \int_0^{+\infty} e^{L(t-s)}(P_1 + P_2)R_2^{0,R}(s)ds.\]
and consequently
\[
R_3^2(t) = x(t) - A_2^0(t) = J_2(t) + K_2^0(t),
\]
where \( J_2(t) = e^{Lt}(1 - \mathcal{P}_2)x(0) \)

(A.24)
\[
K_2^0(t) = \int_0^t e^{L(t-s)}(1 - \mathcal{P}_2)M_2^0R(s)ds - \int_t^{+\infty} e^{L(t-s)}\mathcal{P}_2M_2^0R(s)ds.
\]
Notice that \( a_3^0(t) = o(e^{-((k_2+k_1+1)\lambda_1-\epsilon)t}) \), \( J_2(t) = o(e^{-((k_3-\epsilon)t)}) \). Further, using Lemma A.2 and (A.22), we find \( K_2^0(t) = o(e^{-(k_2+k_1+1)\lambda_1-\epsilon)t}) \) and

(A.25)
\[
R_2^2(t) = o(e^{-(k_2+k_1+1)\lambda_1-\epsilon)t} + e^{-(\lambda_3-\epsilon)t}).
\]
Whence, if \( k_3 = 0 \), then \( R_2^0(t) = o(e^{-(\lambda_3-\epsilon)t}) \), and the Lemma is proved.

If \( k_3 > 0 \), we proceed as above; We denote by

(A.26)
\[
M_j^1(s) := N(A_j^0(s) - N(A_j^{-1}(s)),
\]
and in general for any \( 1 \leq j \leq k_3 \) we set

(A.27)
\[
a_j^0(t) := \int_0^t e^{L(t-s)}(1 - \mathcal{P}_2)M_j^1(s)ds - \int_t^{+\infty} e^{L(t-s)}\mathcal{P}_2M_j^1(s)ds.
\]
Then, using again Lemma A.2, we find that

(A.28)
\[
M_j^1(t) = O(a_j^0(t) \cdot A_j^{-1}(t)) = O(a_j^0(t) \cdot a_j^0(t)) = o(e^{-(\lambda_2+\lambda_1-\epsilon)t})

a_j^0(t) = o(e^{-(\lambda_2+\lambda_1-\epsilon)t})
\]
Analogously, through an inductive argument, for any \( j \in \{1, \ldots, k_3\} \) we get
\[
M_j^1(t) = O(a_j^{-1}(t) \cdot A_j^{-2}(t)) = O(a_j^{-1}(t) \cdot a_j^0(t)) = o(e^{-(\lambda_2+j\lambda_1-\epsilon)t})
\]
\[
a_j^0(t) = o(e^{-(\lambda_2+j\lambda_1-\epsilon)t}).
\]
As above the remainder term \( R_3^1(t) := x(t) - A_2^2(t) \) can be written as follows

(A.29)
\[
R_3^1(t) := J_2(t) + K_2^1(t),
\]
where
\[
K_2^1(t) := \int_0^t e^{L(t-s)}(1 - \mathcal{P}_2)M_2^{1R}(s)ds - \int_t^{+\infty} e^{L(t-s)}\mathcal{P}_2M_2^{1R}(s)ds
\]
\[
M_2^{1R}(s) = N(x(s) - N(A_2^0(s)) = N(A_2^0(s) + R_2^0(s)) - N(A_2^0(s))
\]
Since \( R_2^0(t) = R_2^0(t) - a_j^0(t) = o(e^{-(k_2+k_1)\lambda_1-\epsilon)t}) \), see (A.25), (A.28), we find
\[
M_2^{1R}(t) = O(a_j^0(t)R_2^0(t)) = O(a_j^0(t)R_2^0(t)) = o(e^{-(k_2+k_1)\lambda_1-\epsilon)t}) \]
Hence, from Lemma A.2 we find \( K_2^1(t) = o(e^{-(k_2+k_1+2)\lambda_1-\epsilon)t}) \). So we get

(A.30)
\[
R_2^2(t) = o(e^{-((k_2+k_1+2)\lambda_1-\epsilon)t} + e^{-(\lambda_3-\epsilon)t})
\]
Whence, if \( k_3 = 1 \), then \( R_2^3(t) = o(e^{-(\lambda_3-\epsilon)t}) \), and the Lemma is proved.

Otherwise we proceed by induction assuming that \( R_2^j(t) = o(e^{-(k_2+k_1+j)\lambda_1-\epsilon)t} + e^{-(\lambda_3-\epsilon)t}) \) and proving that \( R_2^j(t) = o(e^{-(k_2+k_1+j+1)\lambda_1-\epsilon)t} + e^{-(\lambda_3-\epsilon)t}) \), for \( 1 \leq j \leq k_3 \). Note that

(A.31)
\[
K_2^j(t) := \int_0^t e^{L(t-s)}(1 - \mathcal{P}_2)M_2^{jR}(s)ds - \int_t^{+\infty} e^{L(t-s)}\mathcal{P}_2M_2^{jR}(s)ds
\]
\[
M_2^{jR}(s) = N(x(s) - N(A_2^{j-1}(s)) = N(A_2^j(s) + R_2^j(s)) - N(A_2^{j-1}(s))
\]
In fact $R_{2}^{j}(t) = R_{2}^{j-1}(t) - a_{2}^{j}(t) = o(e^{-[(k_{2} + k_{1} + j)\wedge 1]t })$, whence
\[ M_{2}^{R}(t) = O(R_{2}^{j}(t)A_{2}(t)) = O(R_{2}^{j}(t)a_{0}^{j}(t)) = o(e^{-[(k_{2} + k_{1} + j + 1)\wedge 1]t }). \]
Therefore, from Lemma A.2, we find $R_{2}^{j+1}(t) = o(e^{-[(k_{2} + k_{1} + j + 1)\wedge 1]t + e^{-(\lambda_{2} - \epsilon)t})$, so the claim is proved. In particular, iterating the argument till $j + 1 = k_{3}$ we find $R_{2}^{k_{3}}(t) = o(e^{-(\lambda_{2} - \epsilon)}t )$, and we conclude the proof of the Lemma.

**Remark A.7.** We stress that $R_{2}^{j}(t)$ is in general negligible with respect to $a_{2}^{j}(t)$ for any $j$. In fact $a_{2}^{j}(t) = o(e^{-(\lambda_{2} + j\wedge 1 - \epsilon)t})$, $R_{2}^{j}(t) = o(e^{-[(k_{2} + k_{1} + j)\wedge 1]t})$, and $\lambda_{2} + j\wedge 1 \leq (k_{2} + k_{1} + 1)\wedge 1 + j\wedge 1$.

Notice that we can continue this iterative scheme up to an arbitrary order. In fact, recalling the definitions of $a_{1}^{j}(t)$, $A_{1}^{j}(t)$, $A_{2}^{j}(t)$ of Lemmas A.5, A.6, and the definitions of the $k_{i}$ given in (A.7), we set
\[ A_{i}^{j}(t) = A_{i-1}^{k_{1}}(t) + \sum_{j=0}^{\mu} a_{i}^{j}(t), \quad \text{for any } 0 \leq \mu \leq k_{i+1}, \ i = 2, \ldots, m \]
(setting also $k_{m+1} = +\infty$), where the functions $a_{i}^{j}(t)$ are defined as follows
\[ a_{i}^{0}(t) := e^{-Lt}[P_{i}x(0) + \bar{\mathfrak{H}}_{i}], \quad \text{for } i = 2, \ldots, m \]
where
\[ \bar{\mathfrak{H}}_{i} := \int_{0}^{+\infty} e^{-Ls}P_{i}[N(x(s)) - N(A_{i-1}^{k_{1}-1}(s))]ds. \]
If $k_{i} > 0$ we also have
\[ M_{i}^{1}(s) := N(A_{i}^{0}(s)) - N(A_{i-1}^{k_{1}}(s)); \]
\[ a_{i}^{1}(t) := \int_{0}^{t} e^{L(t-s)}(\| - P_{i})M_{i}^{1}(s)ds - \int_{t}^{+\infty} e^{L(t-s)}P_{i}M_{i}^{1}(s)ds. \]
If $k_{i} \geq j > 1$ we also have
\[ M_{i}^{j}(s) := N(A_{i}^{j-1}(s)) - N(A_{i}^{j-2}(s)) \quad \text{for } 1 \leq j \leq k_{i+1} \]
\[ a_{i}^{j}(t) := \int_{0}^{t} e^{L(t-s)}(\| - P_{i})M_{i}^{j}(s)ds - \int_{t}^{+\infty} e^{L(t-s)}P_{i}M_{i}^{j}(s)ds. \]
We have the following result, see also Table I and Table II.

**Lemma A.8.** Assume $N \in C^{2}$ and let $x(t)$ be a solution of (A.1) such that $x(t) \to 0$ as $t \to +\infty$. Let $r \in \mathbb{N}$, and set $\mu = r + \sum_{i=1}^{m} k_{i}$, then we can expand $x(t)$ as follows
\[ x(t) = \sum_{j=0}^{k_{2}} a_{1}^{j}(t) + \ldots + \sum_{j=0}^{k_{m}} a_{m}^{j}(t) + \sum_{j=0}^{\mu} a_{m}^{j}(t) + R_{m}^{\mu}(t), \quad \text{where} \]
\[ a_{2}^{j}(t) = o(e^{-(\lambda_{2} + j\wedge 1 - \epsilon)t}), \quad \text{and} \ R_{m}^{\mu}(t) = o(e^{-(\mu + 1 - \epsilon)\wedge 1}t). \]
The proof is simply an iteration of the previous scheme.

Now, using the fact that all the functions appearing in the definition of $a_{2}^{j}(t)$ are exponentials, possibly multiplied by polynomials, we can improve the estimates of Lemma A.3. To proceed we need more regularity in order to replace the function $N$ of (A.1) by its Taylor polynomial of degree $\mu \geq 2$, say $N_{\mu}$. Let us fix $r \in \mathbb{N}$ and $\mu = r + \sum_{i=1}^{m} k_{i}$, and go back to Step 1. We rename $\tilde{a}_{1}^{j}(t) := a_{1}^{j}(t)$, see (A.3), then we denote by $\tilde{a}_{2}^{j}(t)$ the function $a_{2}^{j}(t)$ of (A.10) but where $N$ is replaced by $N_{\mu}$. Notice that $\tilde{a}_{1}^{j}(t) - a_{1}^{j}(t) = o(e^{-(\mu + 1 - \epsilon)\wedge 1}t)$ so that $\tilde{R}_{1}^{j}(t) = x(t) - \tilde{a}_{1}^{j}(t) \sim R_{1}^{j}(t)$.
Then we iterate the scheme above replacing everywhere $N$ by $N_\mu$ and obtaining new functions denoted by “$\tilde{\cdot}$”. I.e. we set $\tilde{a}_1^0(t) = a_1^0(t)$,

$$\tilde{A}_1^r(t) = \sum_{j=0}^r \tilde{a}_j^1(t) \quad \text{for} \ 2 \leq r \leq k_2, \quad \tilde{A}_2^r(t) = \tilde{A}_1^{k_2}(t) + \sum_{j=0}^r \tilde{a}_j^2(t) \quad \text{for} \ 0 \leq r \leq k_3,$$

$$\tilde{A}_r^i(t) = \tilde{A}^i_{r-1}(t) + \sum_{j=0}^r \tilde{a}_j^i(t) \quad \text{for} \ 0 \leq r \leq k_i$$

where $\tilde{N}_1$, $\tilde{M}_i^j(t)$, $\tilde{a}_i^j(t)$ are defined as $\tilde{N}_i$, $M_i^j(t)$, $a_i^j(t)$ in (A.32), (A.33), (A.34) but with $N(\cdot)$, $A_i^j(t)$ replaced by $N_\mu(\cdot)$ and $\tilde{A}_i^j(t)$. Then we have the following.

**Lemma A.9.** Let $r \in \mathbb{N}$, set $\mu = r + \sum_{i=1}^m k_i$, and assume $N \in C^{\mu+1}$; denote by $N_\mu(x)$ the Taylor polynomial of degree $\mu$ centered in the origin. Let $x(t)$ be a solution of (A.1) such that $x(t) \to 0$ as $t \to +\infty$, then set $\tilde{N}_1 := \tilde{N}_1$, see (A.1), and

$$\tilde{N}_i := \int_0^{+\infty} e^{-Ls} P_i[N(x(s)) - N_\mu(\tilde{A}_i^1(s))] ds \quad \text{for} \ i = 2, \ldots, m$$

$$\tilde{a}_i^0(t) := e^{\epsilon t} [P_i x(0) + \tilde{N}_i], \quad \text{for} \ i = 1, \ldots, m.$$  

Then we set $\tilde{A}_i^{1}(s) := \tilde{A}_i^{k_i}(s)$ for $i = 2, \ldots, m$, and for $1 \leq j \leq k_{i+1}$

$$\tilde{M}_j^i(s) := N_\mu(\tilde{A}_i^{j-1}(s)) - N_\mu(\tilde{A}_i^{j-2}(s))$$

$$\tilde{a}_i^j(t) := \int_0^{e^{\epsilon(t-s)}} (I - P_i) \tilde{M}_j^i(s) ds - \int_t^{+\infty} e^{L(t-s)} P_i \tilde{M}_j^i(s) ds$$

$(k_{m+1} = +\infty)$. Then we can expand $x(t)$ as follows:

$$x(t) = \tilde{A}_m^\mu(t) + \tilde{R}_m^\mu(t)$$

where again $\tilde{R}_m^\mu(t) = O(e^{-[(\mu+1)\lambda_1-\epsilon]t}) = O(e^{-[(t+r+\sum_{i=1}^m k_i)\lambda_1-\epsilon]t})$ but $\tilde{A}_m^\mu(t)$ is as above.

**Proof.** Formula (A.38) simply follows by observing that, at each step, $\tilde{A}_i^j(t) - A_i^j(t) = o(e^{-(\mu+1-\epsilon)\lambda_1 t})$ so that $\tilde{R}_m^\mu(t) = x(t) - \tilde{A}_m^\mu(t)$ and $R_m^\mu(t) = x(t) - A_m^\mu(t)$ have the same order.

Our purpose now is to observe that each coordinate of $\tilde{A}_m^\mu(t)$ is made up by sum of exponentials, possibly multiplied by polynomials, in the resonant cases. Therefore we introduce the following assumptions:

**R1:** All the eigenvalues $\lambda_i$ of $L$ are real and distinct, hence $k_i = 1$ for any $i$, and $m = n$.

**R2:** R1 holds and for any $i \in 1, \ldots, n$ and $(\chi_1, \ldots, \chi_{i-1}) \in \mathbb{N}^{i-1}$, then $\lambda_i \neq \sum_{j=1}^{i-1} \lambda_j \chi_j$.

We assume first **R2** so that our iterative scheme contains no resonances at all.

**Lemma A.10.** Let $r \in \mathbb{N}$, and $\mu = r + \sum_{i=1}^m k_i$; Assume $N \in C^{\mu+1}$, and **R2**; then the function $\tilde{A}_m^\mu(t)$ of Lemma A.9 is a vectorial polynomial of degree (at most) $\mu$, i.e. each coordinate $\tilde{A}_{m,s}^\mu(t)$ of $\tilde{A}_m^\mu(t)$ is a polynomial in the functions $e^{-\lambda_1 t}$. More precisely, there is a polynomial $p_\mu^s(f_1, \ldots, f_n)$ of degree $\mu$ such that

$$\tilde{A}_{m,s}^\mu(t) = p_\mu^s(e^{-\lambda_1 t}, \ldots, e^{-\lambda_n t}).$$

**Proof.** Requiring **R1** we see that there is $c_{i,s} \in \mathbb{R}$ such that the $s$th coordinate of the linear terms $\tilde{a}_i^0(t)$ is $c_{i,s} e^{-\lambda_1 t}$ while the others are null (we do not have linear resonances). **R2** is needed to avoid also nonlinear resonances, i.e. resonances in the integral (A.37) defining $\tilde{a}_i^j(t)$ for $j \geq 1$. In fact it is easy to check that the first
coordinate of \( \tilde{M}_1(s) \) defined in (A.37) (or in (A.10) but with \( N_\mu \) replacing \( N \)) is a polynomial in \( e^{-\lambda t} \). Then \( \textbf{R2} \) guarantees that each coordinate \( s \) of \( \tilde{a}_1(s) \) takes the form \( p(0) = (e^{-\lambda_1 t}, \ldots, e^{-\lambda_n t}) \) where \( p_k(0) \) is a polynomial of degree \( \mu \). Then we go through all the iterative scheme and we get the thesis of the Lemma. □

Let us introduce some notation in line with Section 2.4.

\[
I_\Theta = \{ \chi = (\chi_1, \ldots, \chi_m) \in \mathbb{N}^m : \chi_1 \land_1 + \ldots + \chi_m \land_m \leq \Theta \}.
\]

and observe that if \( \textbf{R1} \) holds then \( m = n \).

Now we reformulate the previous Lemma in a way that is more suitable to be used for the parabolic problem studied in this article.

**Proposition A.11.** Assume \( N \in C^\infty \), \( \textbf{R2} \) and let \( x(t) = (x_1(t), \ldots, x_n(t)) \to 0 \) as \( t \to +\infty \); then for any \( \Theta > 0 \) we can find coefficients \( c^\chi = (c_1^\chi, \ldots, c_n^\chi) \in \mathbb{R}^n \), \( \chi \in I_\Theta \) such that for any \( s = 1, \ldots, n \)

\[
(A.40) \quad x_s(t) = \sum_{\chi \in I_\Theta} c_s^\chi e^{-(\chi_1 \land_1 + \ldots + \chi_n \land_n)t} + o(e^{-(\Theta + \varepsilon)t}) \quad \text{with} \quad \chi = (\chi_1, \ldots, \chi_n)
\]

Moreover, let us denote by \( d_i = c^\chi \) where \( \chi \) is such that \( \chi_i = 1 \) and \( \chi_j = 0 \) for \( j \neq i \). Then the coefficients \( d_i \) just depend on \( x(0) \) and the coefficients \( c^\chi \) where \( |\chi| := \sum \chi_i > 1 \) are determined by the coefficients \( d_i \). In fact, for any \( \chi \) such that \( \chi_j = 0 \) for \( j \geq i + 1 \), then \( c^\chi \) is determined by \( d_1, \ldots, d_i \).

Lemma A.11 is a straightforward consequence of Lemma A.10. The property of the coefficients is a straightforward consequence of the construction.

From the discussion just after the proof of Lemma A.5 we get the following result, which proves Lemma 2.19 (in the non-resonant case).

**Remark A.12.** The coefficients \( d_i \) (from which all the other coefficients can be determined exactly) can be written as \( d_i = P_i(x(0)) - \tilde{R}_i \). However notice that they are the real unknown of this procedure, since \( \mathcal{R}_i \) is defined by a fixed point argument and we can just say that \( d_i = P_i(x(0)) + O(x(0)^2) \).

**Remark A.13.** In fact in the previous Lemma we can allow \( N \) to be just \( C^\mu \) where \( \mu = \max \{|\chi| = \sum \chi_i | \chi = (\chi_1, \ldots, \chi_n) \in I_\Theta\} \)

If we drop the assumptions \( \textbf{R1}, \textbf{R2} \) we have to deal with resonances, which at the end forces us to replace constants \( c^\chi \) by polynomials. Assume first \( \textbf{R1} \) but drop \( \textbf{R2} \). It might happen, e.g., that \( \lambda_3 = 2 \lambda_2 \). In this case in the functions \( \tilde{M}_1(t) \) defined in (A.36) we find terms of type \( e^{-k \lambda_2 t} = e^{-\lambda_3 t} \), which are in resonance with \( e^{kt} \). Thus in \( \tilde{a}_1(t) \) in (A.36) we find resonant terms of the form \( q(t)e^{-\lambda_3 t} \).

Further, if \( \textbf{R1} \) does not hold, the functions \( \tilde{a}_1(t) \) satisfies just the estimate in Remark A.1. We assume that the eigenvalues \( -\lambda_i \) are all real, this is the main case of interest in this article. It follows that each coordinate of \( \tilde{a}_1(t) \) equals \( q(t)e^{-\lambda_i t} \), where \( q(t) \) is a polynomial of degree at most \( k_i - 1 \) (and will depend on the coordinate). Then, plugging in the iterative scheme, we see that each coordinate \( \tilde{a}_1(t) \) of \( \tilde{a}_1(t) \) (for \( s = 1, \ldots, n \)) will be sum of exponentials, possibly multiplied by polynomials. With this observation in mind we get the following adaption of Lemma A.11.

**Proposition A.14.** Assume \( N \in C^\infty \), and that all the eigenvalues of \( L \) are real and negative. Let \( x(t) \to 0 \) as \( t \to +\infty \); then for any \( \Theta > 0 \) and any \( s = 1, \ldots, n \) we can find polynomials \( c^\chi(t) \), \( \chi \in I_\Theta \) such that

\[
(A.41) \quad x_s(t) = \sum_{\chi \in I_\Theta} c_s^\chi(t)e^{-(\chi_1 \land_1 + \ldots + \chi_n \land_m)t} + o(e^{-\Theta t}) \quad \text{with} \quad \chi = (\chi_1, \ldots, \chi_m)
\]
Again denote by $d_i(t) = (c_1^i(t), \ldots, c_m^i)$ where $\chi$ is such that $\chi_i = 1$ and $\chi_j = 0$ for $j \neq i$. Then the vector of polynomials $d_i(t)$ just depend on $x(0)$ and the vector of polynomials $e^{\xi}(t)$ where $|\chi| := \sum_{j=1}^{m} \chi_j > 1$ are determined by the $d_i(t)$. Further if $\chi_j = 0$ for $j \geq i + 1$ then $e^{\xi}(t)$ is determined by $d_1, \ldots, d_i$.

**Remark A.15.** If $R_2$ does not hold then there are $i$ and $\chi = (\chi_1, \ldots, \chi_m)$ such that $\chi_1 + \ldots + \chi_i - 1 < \chi_1$, i.e. for any $\chi \in I_{\gamma_1 - \varepsilon}$.

**Remark A.16.** We emphasize that, even if we drop $R_1$ and $R_2$, Remarks A.12 and A.13 continue to hold, by construction. However $e^{\xi t}$ may contain terms which are polynomials multiplied by exponentials.

**Remark A.17.** A result analogous to Lemma A.14 can be obtained also in the case where the eigenvalues of $L$ may be complex and conjugate, say $\lambda_i = -\chi_1 \pm i \omega$ (with negative real parts, $\chi_1 > 0$). In this case the corresponding functions $d_i(t)$ will be sum of terms of the form $e^{-\lambda t}[c \cos(\omega t) + d \sin(\omega t)]$. Then each coordinate of the function $e^{\xi}(t)$ might be a polynomial possibly multiplied by exponential and sinusoidal functions.

We emphasize that all this discussion in this article is in fact applied to the case of a 2-dimensional non-autonomous system (2.8) which can be regarded as a 3-dimensional autonomous system (2.8), which has the critical point $P^+$. The eigenvalues of the linearization of (2.8) in $P^+$ are $\lambda_1$, $\lambda_2$ and $-\gamma$, and are real and negative.

However the dimension corresponding to $\gamma$ is special, since the corresponding equation can be solved independently from the others (the solution is in fact $\zeta(s) = ce^{-\gamma s}$), and does not depend on the initial conditions. Let us go back to the notation of Section 2.4.

**Remark A.18.** Consider a solution $y(s, \tau; Q, l_s)$ of (2.3) converging to $P$, and assume $-\gamma \leq \lambda_1$; Then the function $\Psi(s)$ defined in (2.3) and Proposition 2.10 does not depend on $Q$ (in fact it depends on $\tau$).

**Remark A.19.** Consider a solution $y(s, \tau; Q, l_s)$ of (2.3) converging to $P$, and assume $-\gamma = \lambda_1$. If $\lambda_1 < \lambda_2$ the coefficient $a$ in (2.3) depend on $Q$ while $\Psi(s)$ contains a resonant term of the form $c_{(0,0,1)} s^2 e^{\lambda_1 s}$ which does not depend on $Q$. If $\lambda_1 = \lambda_2$ the coefficient $a$ in (2.3) depends on $Q$ while $\Psi(s)$ contains a doubly resonant term of the form $c_{(0,0,1)} s^3 e^{\lambda_1 s}$ which does not depend on $Q$.

**Proof.** Remark A.19 immediately follows from the construction in this Appendix, see in particular Remark A.18. To prove Remark A.19 recall that the determination of $c_{(0,0,1)}$ and $a$ in both the cases goes back to Lemma A.3 so it just involves the resolution of the linearization of (2.3) in $P$. This latter problem at the end can be regarded as a linear differential equation with constant coefficients and a forcing term which is a resonant exponential. Then it trivially follows that the most resonant part just depends on the forcing term, so we get the conclusion.

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