1. Introduction

Groups of maps $Map(M,G)$ where $M$ is a compact manifold and $G$ is a compact group form a natural generalization of loop groups $LG = Map(S^1, G)$. The latter have a well understood representation theory. The success of this theory rests mainly on the fact that the group $LG$ polarizes naturally to subgroups corresponding to positive or negative Fourier modes on the circle. This allows the concept of highest weight representation for a central extension of $LG$.

In higher dimension it is not immediately clear what would replace the Fourier decomposition. Of course, if for example $M = S^1 \times X$ for some manifold $X$, one could define the subgroups $N_\pm$ of $Map(M,G)$ consisting of positive (resp. negative) Fourier modes on the circle $S^1$. The group has a central extension defined by averaging the central extension in the $S^1$ direction over the manifold $X$, with respect to some measure on $X$. However, it seems both for mathematical and physical reasons that this is not the right concept. First the physical reason. In two dimensional quantum field theory the central extension of $LG$ arises because of the normal ordering regularization of bilinear expressions involving quantum fields at the same point. Similarly, in higher dimensions one has to introduce certain regularizations for the bilinears in order to get finite expressions for the physical currents. The net effect of the various regularizations is that the naive commutation relations get modified by the so-called Schwinger terms. The essential difference as compared to
the two dimensional situation is that extension defined by Schwinger terms is not a central one but an extension by an abelian ideal, [M1, M2].

A mathematical motivation for studying the abelian extensions is a kind of universality. The groups $Map(M, G)$ can be embedded into certain infinite-dimensional linear groups $GL_p$, modelled by Schatten ideals of degree $2p > \dim M$. The embedding comes from physics: It is the gauge action of $Map(M, G)$ on chiral fermions. When $\dim M = 1$ the index $2p = 2$, meaning that the group is modelled by Hilbert-Schmidt operators. All the groups $GL_p$ have topologically nontrivial abelian extensions generalizing the central extension $p = 1$, [MR]. The embedding $Map(M, G) \subset GL_p$ gives an abelian extension of the former group.

The Hilbert space $H_F$ of square integrable chiral fermion fields on $M$ splits into subspaces $H_{\pm}$ corresponding to positive (resp. negative) eigenvalues of the Dirac operator. This corresponds to the Fourier splitting on the circle or real line. An element $g \in GL_p$ can be written as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a : H_+ \rightarrow H_+$, $b : H_- \rightarrow H_+$ etc. In a highest weight representation there is a cyclic vector which is annihilated by the subgroup $g_+$ consisting of matrices $g$ with $c = 0$. Let $G_C$ be the complexification of $G$. When $M = S^1$ the intersection of $g_+$ with the subgroup $LG_C$ consists of loops with positive frequency and likewise the intersection of $g_-$ with $LG_C$ corresponds to negative frequencies. Thus the natural polarization on $LG$ is induced by the splitting $H = H_+ \oplus H_-$ of the "one-particle space". The polarization defines a complex structure on the subgroup of based loops. The complex structure makes available the holomorphic Borel-Bott-Weil induction method for constructing highest weight representations, [PS].

The situation is completely different in higher dimensions. The problem is that the intersection of $g_+$ or $g_-$ with the subgroup $Map(M, G_C) \subset GL_p$ is normally very small; it contains only the constant maps. Thus the splitting of $H$ does not induce a polarization on $Map(M, G)$. This is the case when the physical space $M$ has a positive definite Riemannian metric.

The purpose of the present paper is to show that if one chooses as the initial data surface not a space-like surface of the type $t = const.$ but the light cone $C = \{ x = (x_0, \mathbf{x}) | x_0^2 - x_1^2 - x_2^2 - x_3^2 = 0 \}$ the polarization in the initial data space, given by a Dirac or wave operator, indeed induces a polarization on $G = Map(C, G)$. We shall specialize to the physically most interesting four dimensional setting, but the discussion has an obvious generalization to higher dimensions. The subgroups $G_\pm$ corresponding to positive or negative frequencies of the Dirac (or wave) operator furthermore have the important property that the Schwinger terms defining the abelian extension vanish along directions of $G_\pm$. This is in accordance with the central term $c$ of a Kac-Moody algebra, $c(x, y) = 0$ if both $x$ and $y$ have positive (resp. negative) frequency. The results of this paper make it possible to extend (at least parts of) the theory of highest weight representations of loop groups into higher dimensions in a physically motivated manner.

2. Light-cone initial data for the wave equation and the polarization
Since the cone $C$ is not a manifold (it has a singularity at the vertex $x = 0$) we have to specify what we mean by a smooth function on $C$. We call a function $f$ smooth if it extends to a smooth function in an open neighborhood of $C$ in $\mathbb{R}^4$. Let $G$ be a compact Lie group. The space $G$ of smooth maps $C \to G$ is a group under pointwise multiplication. Its Lie algebra is the space $Cg$ of smooth maps $C \to g$, where $g$ is the Lie algebra of $G$. The commutators are defined pointwise.

Let $g_C$ be the complexification of the Lie algebra $g$. We shall identify the space $Cg_C$ as the space of smooth $g_C$ valued solutions of the wave equation

\[(\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2)\phi = 0\]

A solution of (2.1) is uniquely determined by its initial data $\phi(0, x)\partial_0 \phi(0, x)$ on the surface $x_0 = 0$. We shall define a norm in the initial data space by

\[|| (\phi, \partial_0 \phi) ||^2 = \int_{x_0 = 0} < \bar{\phi}, (-\partial_1^2 - \partial_2^2 - \partial_3^2)^{1/2} \phi > d^3x + \int_{x_0 = 0} < \bar{\partial_0 \phi}, \partial_0 \phi > d^3x\]

where the positive square root is chosen for the Laplacian and $< \cdot, \cdot >$ is an invariant bilinear form on the Lie algebra $g$. We denote by $H$ the completion of the space of smooth initial data with compact support. The norm extends to an obvious inner product in $H$.

A pair of smooth solutions $\phi, \phi'$ of (2.1) (with appropriate vanishing conditions at infinity) defines a conserved current $j_\mu = < \bar{\phi}, \partial_\mu \phi' > - < \bar{\partial_\mu \phi}, \phi' >$. The bar means complex conjugation. The dual of the one-form $v \mapsto \nu^\mu j_\mu$ is a three-form and its integral over a space-like surface is denoted by $Q(\phi, \phi')$. The 'charge' $Q$ does not depend on the choice of the space-like surface. The light-cone $C$ can be thought of as a limiting case of a space-like surface and therefore the charge is

\[Q(\phi, \phi') = \int_C (< \bar{\phi}, D\phi' > - < D\bar{\phi}, \phi' >) \frac{d^3x}{x_0}\]

where $D = \nu^\mu \partial_\mu$ is the derivative along light rays. Note that the integration must be carried out over both the future light-cone $C_+$ and the past light-cone $C_-$. The charge $Q$ is continuous with respect to the norm in $H$ and therefore it extends to the whole of $H$. Furthermore, it is easy to see that $Q$ is nondegenerate. (The solutions of the wave equations with zero frequency are not contained in the Hilbert space $H$.)

Any vector in $H$ can be Fourier expanded as

\[\phi(x) = \frac{i}{2(2\pi)^{3/2}} \int_C e^{ip\cdot x} \hat{\phi}(p) \frac{d^3P}{p_0}\]

where the integration is over the light-cone in the momentum space. The norm is then given by

\[||\phi||^2 = \frac{1}{2} \int_C |\hat{\phi}(p)|^2 (1 + (p_1^2 + p_2^2 + p_3^2)^{1/2}) \frac{d^3P}{p_0}\]

and the charge is

\[Q(\phi, \phi') = \frac{1}{2} \int_C < \hat{\phi}(p), \hat{\phi}'(p) > \frac{d^3P}{p_0}\].
We shall use the following distribution in $\mathbb{R}^4$:

\begin{equation}
D(x) = \frac{i}{2(2\pi)^3} \int_{\mathbb{C}} e^{ip \cdot x} \frac{d^3p}{p_0}.
\end{equation}

All distributions will be defined in the space of smooth rapidly decreasing functions at infinity. An explicit formula for $D(x)$ is

\begin{equation}
D(x) = -\frac{1}{2\pi} \epsilon(x_0)\delta(x^2).
\end{equation}

We shall need also the distribution $\xi(p,q) = Q(e^{ip \cdot x}, e^{iq \cdot x})$. From the definition we get

\begin{equation}
\xi(p,q) = \int_{\mathbb{C}} e^{i(q-p) \cdot x} \frac{d^3x}{x_0} = \frac{d}{d\lambda}|_{\lambda=1} \int_{\mathbb{C}} e^{i(q-p)\cdot x} \frac{d^3x}{x_0} - \frac{d}{d\lambda}|_{\lambda=1} \int_{\mathbb{C}} e^{i(q+p)\cdot x} \frac{d^3x}{x_0}
\end{equation}

\begin{align}
&= 2i(2\pi)^2 \frac{d}{d\lambda}|_{\lambda=1} \left[ \epsilon((q_0-p_0)\delta((q-p)^2) - 2i(2\pi)^2 \epsilon(q_0 - \lambda p_0)\delta((q - \lambda p)^2)\right] \\
&= 2i(2\pi)^2 (q_0 + p_0) \delta((q-p)^2) + 2i(2\pi)^2 \epsilon(q_0 - p_0)2(q^2-p^2)\delta((p-q)^2).
\end{align}

Note in particular that $\xi(p,q)$ has support only on the set of light like separated points $(p,q)$. Another important property of $\xi$ is that its restriction to $C \times C$ is proportional to the Dirac $\delta$ distribution on $C$; this latter property can be read of also from the formula (2.6).

**Lemma 1.** Let $p,q$ be both on the future (resp. past) light-cone. Then the distribution $\xi(k, p+q)$, when restricted to $k \in C$, has support only on the future (resp. past) light-cone.

**Proof.** Suppose e.g. that $p,q \in C_+$. Now $p+q$ is a future pointing time or light like vector and thus $k \cdot (p+q)$ is nonpositive for $k \in C_-$. It follows that $(k - (p+q))^2 \geq 0$ and the equality sign can occur only when $p+q$ is light like, that is, when $p$ and $q$ are linearly dependent. When $(k - (p+q))^2 > 0$ we know that $\xi(k, p+q) = 0$ and in the latter case $(p+q)$ on the light-cone) $\xi(k, p+q)$ is proportional to $\delta(k-(p+q))$ which is zero when $0 \neq k \in C_+$. 

**Proposition 2.** The subspaces $b_{\pm}$ consisting of restrictions of solutions of the wave equation (2.1) to the light cone $C$, corresponding to either frequencies $p_0 \geq 0$ or $p_0 \leq 0$, are subalgebras of $\mathfrak{g}_{CC}$ under pointwise commutators.

**Proof.** The $k$:th Fourier component of a function $\phi : C \to \mathfrak{g}_{CC}$ (with $k \in C$) is given by the integral

\begin{equation}
\hat{\phi}(k) = \frac{1}{(2\pi)^{3/2}} Q(e^{ik \cdot x}, \phi)
\end{equation}

If a pair of solutions $\phi, \phi'$ has support only on the positive light-cone in momentum space then their pointwise commutator is an integral over Fourier modes $e^{ip \cdot x}$ where
$p$ is a time like vector and $p_0 \geq 0$. From the Lemma follows then that $[\phi, \phi']$, when restricted to the light cone, contains only momenta $k \in C_+$. In the case of a loop group the structure constants of the Lie algebra $Lg_C$ are simple. A basis is given by the Fourier modes $T_{a,n} = T_a e^{i n \phi}$, $n \in \mathbb{Z}$ and the $T_a$'s form a basis of $g$. One has then

\begin{equation}
[T_{a,n}, T_{b,m}] = c_{ab}^{\lambda k} \frac{\lambda^k_{nm} T_{f,k}}{2(2\pi)^3} \int_{C} \xi(k, p + q) e^{ik \cdot x} \frac{d^3k}{k_0}
\end{equation}

where both sides should be understood as distributions in $(p, q)$.

We shall work out more explicitly the case of spherically symmetric functions. A spherically symmetric solution of the wave equation is an integral over the frequency $\lambda$ of the elementary complex valued solutions

\begin{equation}
\phi_{\lambda}(x) = e^{ix_0 \lambda} \frac{\sin(\lambda r)}{r}.
\end{equation}

Here $r^2 = x_1^2 + x_2^2 + x_3^2$. The restriction of $\phi_{\lambda}$ to the light cone depends only on the time coordinate $x_0$, $\phi_{\lambda}(x_0) = \phi_{\lambda}(x_0, r = |x_0|) = (e^{2i\lambda x_0} - 1)/2ix_0$. We can write

\begin{equation}
\phi_{\lambda} \phi_{\lambda'} = \int \phi_{\mu} \alpha(\mu, \lambda, \lambda') d\mu
\end{equation}

where, by simple Fourier analysis,

\begin{equation}
\alpha(\mu, \lambda, \lambda') = \epsilon(\lambda + \lambda' - \mu) - \epsilon(\lambda - \mu) - \epsilon(\lambda' - \mu) + \epsilon(-\mu),
\end{equation}

where $\epsilon(x) = +1$ for $x \geq 0$ and $\epsilon(x) = -1$ for $x < 0$. Note that if $\lambda, \lambda'$ are both positive (resp. negative) then $\alpha$ is nonzero only for positive (resp. negative) values of $\mu$.

In the general case a solution of the wave equation can be expanded in terms of the functions

\begin{equation}
\phi_{\lambda,j,m} = e^{i \lambda x_0} X_j(\lambda r) Y_{jm}(\theta, \psi)
\end{equation}

where the $Y_{jm}$'s are the normalized spherical harmonics ($j = 0, 1, 2, \ldots$ and $m = -j, -j + 1, \ldots, j$) and the $X_j$'s are related to Bessel functions, [CH],

\begin{equation}
X_j(z) = J_{j + \frac{1}{2}}(z)/\sqrt{z}.
\end{equation}

These solutions are smooth everywhere in $\mathbb{R}^4$. Their restrictions to $C$ are smooth functions, evaluated by setting $r = |x_0|$. In order to obtain explicit commutation relations for the Lie algebra $Cg$ one has to expand

\begin{equation}
\phi_{\lambda,j,m} \phi_{\lambda',j',m'} = \sum_{\ell,n} \int \phi_{\mu,\ell,n} \alpha(\mu, \ell, n; \lambda, j, m; \lambda', j', m') d\mu.
\end{equation}
In Fourier-Bessel analysis one can always expand a continuous function on the real half line \([0, \infty[,\) subject to the condition (2.21) below, in terms of Bessel functions of a given order \(\ell,\)

\[
(2.19) \quad f(x) = \int \lambda J_\ell(\lambda x) \hat{f}(\lambda) d\lambda,
\]

where the coefficients are given by

\[
(2.20) \quad \hat{f}(\lambda) = \int f(x) J_\ell(\lambda x) dx,
\]

where we consider only functions satisfying

\[
(2.21) \int_0^\infty x|f(x)|^2 dx < \infty.
\]

Since a smooth function on \(S^2\) can also be expanded in terms of spherical harmonics, we can easily show that a product of the functions \(\phi_{\lambda,j,m}\) on the light cone can indeed be expanded as a sum and integral over the solutions themselves.

We do not have an explicit formula for the distributional coefficients \(\alpha,\) but they can in principle be determined from the integral

\[
(2.22) \quad \alpha(\mu, \ell, n; \lambda, j, m; \lambda', j', m') = Q(\phi_{\mu, \ell, n}, \phi_{\lambda,j,m}\phi_{\lambda',j',m'}).\]

This follows from the orthogonality and completeness relations for the spherical harmonics and Bessel functions. The latter are, [W],

\[
(2.23) \quad \int_0^\infty xJ_\ell(\lambda x)J_\ell(\lambda' x) dx = \delta(\lambda - \lambda').
\]

By conservation of charge we have

\[
Q(\phi_{\lambda,j,m}, \phi_{\lambda',j',m'}) = \int_{x_0=0}^\infty (\lambda + \lambda')|\tilde{\phi}_{\lambda,j,m}\phi_{\lambda',j',m'}| r^2 dr d\Omega
= \delta(j - j')\delta(m - m') \int_0^\infty r J_j(\lambda r) J_j(\lambda' r) dr
= \delta(j - j')\delta(m - m')\delta(\lambda - \lambda').
\]

**3. Action on chiral fermions**

We shall consider the space of solutions of the Dirac-Weyl equation

\[
(3.1) \quad \sum_0^3 \sigma^\mu \partial_\mu \psi = 0,
\]
where the $\sigma$’s are complex hermitian Pauli matrices, $\sigma_0 = 1$, $\sigma_1^2 = 1$, and $\sigma_1\sigma_2 = -\sigma_2\sigma_1 = \sigma_3$, and cyclic permutations. A smooth solution in Minkowski space is completely determined by its Cauchy data on a space-like hyperplane. Alternatively, one can give initial data on the light-cone $x^2 = 0$. In the latter case there is a constraint between the two components of the spinor field. This is understood when looking at the Lorentz invariant inner product between solutions of (3.1) which are square integrable on a space-like hypersurface $S$. The inner product is

\begin{equation}
\langle \psi, \psi' \rangle = \int_S \psi^* \sigma^\mu \psi' n_\mu d^4x,
\end{equation}

where $n$ is the future pointing unit normal to the surface $S$. In the limiting case when the surface $S$ approaches the light-cone one gets

\begin{equation}
\langle \psi, \psi' \rangle = \int_C \psi^* x^\mu \sigma^\mu \psi' \frac{d^4x}{x_0}.
\end{equation}

On the cone $C$ the matrix $P(x) = \frac{1}{x_0} x_\mu \sigma^\mu$ is degenerate, its eigenvalues are 0 and 1. Thus only the one-component field $P(x)\psi$ contributes to the norm of the solution $\psi$ and therefore $\psi$ is completely determined by the values of $P\psi$ on $C$, the second component being given by the field equation.

The kernel of $P(x) - 1$ in $\mathbb{C}$ is a complex line depending only on the direction of the vector $(x_1, x_2, x_3)$. In this way we obtain a complex line bundle $E$ over $S^3$.

The independent initial data are sections of the line bundle, with coefficients in the space of complex valued functions on the real line $\mathbb{R}$, corresponding to the light ray parametrized by $\vec{x}/x_0 \in S^2$. The line bundle $E$ is the basic complex line bundle associated to the fibering $SU(2) \to SU(2)/S^1 = S^2$.

Suppose from now on that $\psi$ takes values in $\mathbb{C}^2 \times \mathbb{C}^N$ and a compact gauge group $G$ acts on the latter components in the tensor product. Because of the constraints on $C$, we cannot freely gauge transform an initial data through the pointwise multiplication $\psi'(x) = g(x)\psi(x)$ where $g$ takes values in (a representation of) $G$. However, we can multiply sections of the bundle $E \otimes \text{Map}(\mathbb{R}, \mathbb{C}^N)$ pointwise by a gauge transformation.

Let $H_F$ be the Hilbert space of square-integrable sections in $E \otimes \text{Map}(\mathbb{R}, \mathbb{C}^N)$. This space splits to subspaces $H_{\pm}$, defined by the nonnegative (resp., negative) eigenvalues of the Hamiltonian $h = i \sum \sigma^k \partial_k$ on the space of solutions of (3.1) (identified as the space of initial data $H_F$).

**Proposition 3.** A gauge transformation in $\mathcal{G}_+$ maps $H_+$ onto itself and similarly an element of $\mathcal{G}_-$ maps $H_-$ onto itself.

**Proof.** The initial data on $C$ for a solution corresponding to energy $\lambda$, angular momentum $j = \ell + 1/2$, and the third component of angular momentum equal to $j_3 = m + 1/2 (\ell = 0, 1, 2, \ldots$ and $m = -\ell, -\ell + 1, \ldots, \ell$) are functions of time $t = x_0$ and of spherical angles $\theta, \phi$ parametrizing the point $x/x_0$. An explicit formula is

\begin{equation}
\psi_{\lambda, j_3} = f_{\lambda, \ell+1/2}(t) \begin{pmatrix} Y_{\ell, m} \\ \gamma_+ Y_{\ell, m+1/2} \end{pmatrix} + f_{\lambda, \ell+1/2}(t) \begin{pmatrix} Y_{\ell+1, m} \\ \gamma_- Y_{\ell+1, m+1} \end{pmatrix},
\end{equation}

where $\gamma_+ = \sqrt{\ell-m}/\sqrt{\ell+m+1}$, $\gamma_- = -\sqrt{\ell+m+2}/\sqrt{\ell-m+1}$, $Y_{\ell, m}$’s are spherical harmonics, and

\begin{equation}
f_{\lambda, \ell}(t) = \frac{e^{i \lambda t}}{\sqrt{r}} f_{\ell}(r),
\end{equation}

where $r = x_0$. The initial data is given in the coordinates $x_0, \theta, \phi$ by $f_{\ell, m}(t) = \frac{1}{\sqrt{2\pi}}$. This provides an explicit formula for $f_{\lambda, \ell}(t)$.
where $r = |t|$ and the $j_{\lambda,p}$’s are, up to a normalization constant, again Bessel functions of order $p$.

The projection of $\psi$ onto the complex line $P(x)C$, as a function of $t$, is a linear combination of the functions $f_{\lambda,t+1/2}$ and $f_{\lambda,t+3/2}$. Both of these functions have only positive Fourier components when $\lambda$ is positive; this follows immediately from the integral representation

$$(3.6) \quad J_n(z) = \frac{z^n}{2\pi n!} \int_0^\pi \cos(z \cos \theta) \sin^{2n+1} \theta d\theta$$

for Bessel functions. Thus the product of $\psi$ with a positive energy wave function $\phi$ produces a function with only nonnegative Fourier components. On the other hand, a section corresponding to negative energy contains only negative Fourier components (as a function of time on $C$) and therefore $\phi \psi$ cannot contain negative energy components.

**Proposition 4.** Let $\phi : C \rightarrow G$ be a smooth bounded function such that $p_0^0 \hat{\phi}(p) \rightarrow 0$ for $p_0 \rightarrow \pm \infty$, where $\hat{\phi}$ is the Fourier transform of $\phi$. Write the operator corresponding to a pointwise multiplication by $\phi$ in $H^F$ as

$$T(\phi) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$ 

Then the off-diagonal blocks are in the Schatten ideal $L_4$, that is, $tr|c|^4 < \infty, tr|b|^4 < \infty$.

**Proof.** Here it is more convenient to work in the momentum basis. The solution of the Weyl equation with 4-momentum $p \in C$ is

$$(3.7) \quad \psi_p(x) = \left( \frac{p_1 + ip_2}{\sqrt{2p_0(p_0 - p_3)}} \right)^{1/2} e^{ip \cdot x}.$$ 

The norm of a square-integrable solution

$$(3.8) \quad \psi = \frac{1}{(2\pi)^{3/2}} \int_C \psi_p \alpha(p) \frac{d^3 p}{p_0}$$

is given by the Lorentz invariant integral

$$(3.9) \quad ||\psi||^2 = \int_C \alpha(p) \sigma^n \alpha(p) p_0 \frac{d^3 p}{p_0}.$$ 

We need an estimate for the integral

$$I(\phi) = \int_X |T(p,p')|^4 d^3 p d^3 p',$$

where $X = C_- \times C_+ \text{ or } X = C_+ \times C_-$ and $T(p,q)$ is the matrix element of a multiplication operator in the momentum basis. To start with, let $\phi(x) = e^{iq \cdot x}$ with $q \in C_+$. The integral $I$ is finite if and only if the integral

$$I'(\phi) = \int_{C_-} ||\pi_+ \phi_p||^4 d^3 p$$
is finite, where $\pi_{\pm}$ are projectors on the subspaces $H_{\pm}$.

A section corresponding to positive (negative) energy is characterized by the property that as a function of time $t = x_0$ it contains only positive (negative) Fourier components. Thus we can write

$$\pi_+ e^{iq \cdot x} \psi_p = \begin{cases} e^{iq \cdot x} \psi_p & \text{when } (p + q) \cdot x/t \geq 0 \\ 0 & \text{when } (p + q) \cdot x/t < 0 \end{cases}.$$ 

Denote by $A(p, q)$ the area on $S^2$ of the set of points $x/t$ with $(p + q) \cdot x/t \geq 0$. By separating the angular and time variables in the definition of the norm in $H_F$ we observe that for large $p_0$ the integral $I'$ behaves like

$$I''(\phi) = \int A(p, q) d^3p.$$ 

Denote $\hat{x} = x/x_0$. An estimate for the area $A$ can be reduced from the inequalities

$$(p + q) \cdot \hat{x} \geq 0 \text{ or } 1 - \hat{p} \cdot \hat{x} \leq q \cdot \hat{x}/p_0.$$ 

Denoting by $\theta$ the angle between $\hat{x}$ and $\hat{p}$ we get

$$\cos \theta \geq 1 - \frac{|q_0|}{p_0}.$$ 

For large $p_0$ the area $A(p, q)$ behaves like $\theta^2 \sim |q_0/p_0|$ and therefore the integrand in $I''$ can be substituted by $q_0^4/p_0^4$. The integral of this with the measure $d^3p = p_0^2 dp_0 d\Omega$ converges at $p_0 \to \infty$.

The general case is handled using a Fourier decomposition of $\phi$ in terms of the functions $e^{iq \cdot x}$ on the light-cone; by the assumption, the integral of $\hat{\phi}$ with $q_0^4$ is converging.

In quantum theory the gauge transformations on $C$ are naively effected by the component $x^\mu j_\mu := \psi(x) x^\mu \sigma_\mu \psi(x)$ of the current operator $j_\mu$. Here the dots refer to a normal ordering with respect to the free fermionic Hamiltonian. However, in $3 + 1$ dimensions even the normal ordered operators are ill-defined in the fermionic Fock space. In general, applying the smeared current $\int j_\mu(x)f(x)dx$ to a vector in the Fock space produces a vector of infinite norm. This is an indication of the fact that a gauge transformation tends to send a vector in a representation of canonical anticommutation relations (CAR) to a vector in a different, inequivalent, representation of CAR.

One can make sense of the gauge transformations as unitary maps between different CAR representations, [M2]. Alternatively, one can consider the gauge transformations as sesquilinear forms in a Fock space, [R], with a suitably defined twisted product, [L].

The Proposition 4 shows that the gauge transformations on the light cone can be embedded to the group $GL_2$ consisting of operators $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $H_F = H_+ \oplus H_-$ with $b, c \in L_4$. The group $GL_2$ acts through an abelian extension $\hat{GL}_2$ in a bundle $\hat{F}$ of fermionic Fock spaces $F_W$ parametrized by certain representations of CAR. The base space is an infinite-dimensional Grassmannian $Gr_2$ consisting of closed subspaces $W \subset H_F$ such that the orthogonal projection $W \to H_+$ is Fredholm and
the projection $W \rightarrow H_-$ is in $L_4$. To each $W \in Gr_2$ there corresponds an irreducible representation of CAR characterized by the existence of a vacuum vector $|W> \in H$ with the property

$$a^*(u)|W> = a(v)|W> \forall u \in H_-, v \in H_+$$

where the CAR is generated by creation operators $a^*(u)$ and annihilation operators $a(v)$ with the only nonvanishing anticommutation relations

$$a^*(u)a(u') + a(u')a^*(u) = <u, u'> \text{ with } u, u' \in H.$$

Two representations parametrized by $W, W'$ are equivalent if and only if the projection of $W'$ to $W_\perp$ is Hilbert-Schmidt, [A]. The group $GL_2$ acts in the base in a natural way but in the case of Weyl fermions one has to “twist” the action in the total space $F$, [M2]. Infinitesimally, this means that the normal operator commutators are replaced by the commutators, [MR],

$$[X, Y]_c = [X, Y] + c_2(X, Y; W),$$

where the 2-cocycle $c_2$ is a function of the "background" $W$,

$$c_2(X, Y; W) = \frac{1}{8}\text{tr}(\epsilon - F)[[\epsilon, X], [\epsilon, Y]],$$

where $F : H \rightarrow H$ is the linear operator characterized by $F u = u$ for $u \in W$ and $F u = -u$ for $u \in W_\perp$, and $\epsilon$ is the sign of the free Hamiltonian $h$, $\epsilon = h/|h|$ (we use the convention that the zero eigenvalue of $h$ corresponds to sign +1).

The cocycle $c_2$ has the property that it vanishes if both $X$ and $Y$ belong to the subalgebra $g_+ \subset gl_2$ (resp., the subalgebra $g_-$) characterized by vanishing of the off-diagonal block $c$ (resp., $b$). Now $b_+ \subset g_+$ and $b_- \subset g_-$, therefore we have:

**Proposition 5.** The restriction of the cocycle $c_2$ to either of the subalgebras $b_\pm \subset Cg_C$ vanishes.

Note that the restriction of $c_2$ to $Cg_C$ differs from the simple local cocycle

$$c'_2(X, Y; f) = \int_C \text{tr}(df f^{-1})^2(XdY - YdX),$$

where the trace is computed in a finite-dimensional representation of $G$ and $f \in G$, [M1, Chapter 4].

Although we have an unitary action of $\hat{GL}_2$ between fibers, we do not have an unitary representation of the group in a (separable) Hilbert space since there is no invariant measure on the base, [P1].

In order to define the action of the other three components of the 4-current $j_\mu$ we have to find an embedding to the the Lie algebra of $GL_2$. Let $n \in \mathbb{R}^4$ be any time-like unit vector and let $S(n)$ be the plane in $\mathbb{R}^4$ with unit normal equal to $n$. For a smooth function $f$ on $S(n)$ define formally

$$j(n, f) = \int :\psi(x)n^\mu\sigma_\mu\psi(x): f(x)dS(n),$$
where $S(n)$ is the translation invariant measure induced by the measure $d^4x$ on $\mathbb{R}^4$.

We would like to interpret the $j(n,f)$'s as generators of gauge transformations for initial data on the space-like surface $S(n)$. In the non second quantized formalism there are no problems with this. Again, we split the first quantized Hilbert space $H_F$ into $H_+ \oplus H_-$ with respect to the free Hamiltonian $h$, but now we are going to identify the space of solutions of the Weyl equation as the space of square-integrable initial data on $S(n)$, with the inner product

$$<\psi, \psi'> = \int \psi(x)^* n^\mu \sigma_\mu \psi(x) d^3x.$$  

(3.15)

Each smooth $G$ valued function $\phi$ on $S(n)$, with bounded derivatives, defines an unitary operator in $H$ by pointwise multiplication. This operator $T(\phi)$ belongs to $GL_2$. The proof is even simpler than in the case of the cone. One has to show that the commutator $[\epsilon, T(\phi)] \in L_4$. This follows from the fact that 1) the operator $[h, T(\phi)]$ is bounded (as a multiplication operator), and 2) the operator $|h|^{-1}$ raised to power 4 is trace class (remember that the number of states in the volume $d^3p$ is momentum space).

The important point in the discussion above is that the current $n^\mu j_\mu$ should be smeared with a smooth function on its characteristic surface $S(n)$. For example, the space components $j_k$ integrated over the time slice $x_0 = \text{const.}$ are not in the Lie algebra of $GL_2$. By a direct computation one can show that they do not even have compact off-diagonal blocks (for a generic smearing function!)

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