NON-UNIQUENESS OF THE HOMOTOPY CLASS OF BOUNDED CURVATURE PATHS

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Abstract. A bounded curvature path is a continuously differentiable piecewise $C^2$ path with bounded absolute curvature that connects two points in the tangent bundle of a surface. In this work we study the homotopy classes of bounded curvature paths for generic points in the tangent bundle of the Euclidean plane. In particular, we characterize the behavior of homotopies of such paths in terms of boundedness or unboundedness of their path length. Moreover, for a type of configuration of elements in the tangent bundle we prove the existence of a compact planar region in which no bounded curvature path lying on it can be made homotopic to a path outside of the region. In particular, we establish that for such type of configuration, the space of bounded curvature paths is divided into two disjoint components.

1. Motivation

Let $(x, X), (y, Y) \in T\mathbb{R}^2$, where $\mathbb{R}^2$ corresponds to the 2-dimensional Euclidean plane and $T\mathbb{R}^2$ is the tangent bundle. The problem of finding a path of minimal length starting at $x$, finishing at $y$, having tangents at these points $X$ and $Y$ respectively and satisfying a prescribed bounded curvature is well understood. We refer to this as the Dubins problem. The existence together with a geometrical characterization of such critical paths was introduced by L. Dubins in [7]. These paths, called Dubins paths, correspond to concatenations of arcs of circle with fixed radius, depending on the curvature, and at most one line segment. The arcs of circles have length less than $2\pi r$ and are denoted by $c$. The line segment is denoted by $s$. Dubins paths can be presented in a short form as $csc$ and $ccc$. Taking into account the path direction ($L$ denotes left directed and $R$ denotes right directed) the last expressions can be expanded to the six possible words given by $lsl$, $rsr$, $lsr$, $rsl$, $lrl$ and $rlr$. It is important to note that Dubins paths fit into a more general class of paths called bounded curvature paths. The latter have prescribed initial and final position and direction being continuously differentiable piecewise $C^2$ paths.

Given an arbitrary Dubins problem we analyze the extent to which its solution is extendible under continuous admissible deformations, paying special attention to the path topology and the distance between the given points in the plane. We will see that for the topology $csc$ together with a distance condition given in Section 4 that it is not possible to continuously deform embedded paths outside of a compact planar region while keeping the paths admissible throughout the deformation. Moreover, for a type of configuration of elements in $T\mathbb{R}^2$ we prove the existence of
a compact region such that no embedded Dubins path within it can be extended via an arbitrary homotopy between admissible paths.

Extensions are important in the case where reaching a prescribed lower bound for the length of an admissible path is needed. In the decline design of an underground mine, the mine is considered as a 3-dimensional network where the directed nodes correspond to the surface portal, access points, and draw points, and the links correspond to the centerlines of ramps and drives, compare [6]. In addition, the navigability conditions of turning radius for vehicles and gradient for ramps are required. The paper [10] gives an answer to the problem of finding minimal paths of bounded curvature in the 3-dimensional space with given initial and final directions, however the gradient constraint is not taken into consideration. The approach of minimizing the cost of the links corresponds to considering the projected problem in the horizontal plane. A planar path can be lifted into the 3-dimensional space while keeping a uniform gradient. The lifted path will satisfy the gradient constraint if and only if the length of the planar path reaches a lower bound dependent on the vertical displacement between the end points of the link. If the length of the minimum Dubins path is less than the given lower bound, we can attempt to extend the path to reach the required length. However suppose before achieving the desired length the homotopy reaches a locally maximum path. This paper will give explicit conditions for determining when such situation can occur.

2. Preliminaries

Let us denote by $T\mathbb{R}^2$ the tangent bundle of $\mathbb{R}^2$. The elements in $T\mathbb{R}^2$ correspond to pairs $(x,X)$ sometimes denoted just by $x$. As usual, the first coordinate corresponds to a point in $\mathbb{R}^2$ and the second to a tangent vector to $\mathbb{R}^2$ at $x$.

**Definition 2.1.** Given $(x,X), (y,Y) \in T\mathbb{R}^2$, we say that a path $\gamma : [0,s] \to \mathbb{R}^2$ connecting these points is a bounded curvature path if:

- $\gamma$ is $C^1$ and piecewise $C^2$.
- $\gamma$ is parametrized by arc length (i.e. $||\gamma'(t)|| = 1$ for all $t \in [0,s]$).
- $\gamma(0) = x, \gamma'(0) = X; \gamma(s) = y, \gamma'(s) = Y$.
- $||\gamma''(t)|| \leq \kappa$, for all $t \in [0,s]$ when defined, $\kappa > 0$ a constant.

Of course, $s$ is the arc-length of $\gamma$.

The first condition means that a bounded curvature path has continuous first derivative and piecewise continuous second derivative. For the third condition, without loss of generality, we can extend the domain of $\gamma$ to $(-\epsilon,s + \epsilon)$ for $\epsilon$ arbitrarily small. Sometimes we describe the third item as the endpoint condition. The last condition means that bounded curvature paths have absolute curvature bounded above by a positive constant. We denote the interval $[0,s]$ by $I$. Also, when more than one path is under consideration, we write $\gamma : [0,s_{\gamma}] \to \mathbb{R}^2$ to specify arc-length.

**Definition 2.2.** Given $x,y \in T\mathbb{R}^2$ and a maximum curvature $\kappa > 0$. The space of bounded curvature paths satisfying the given endpoint condition is denoted by $\Gamma(x,y)$.

It is important to note that the topological and geometrical properties of the space of bounded curvature paths $\Gamma(x,y)$ depends on the chosen elements in $T\mathbb{R}^2$. Properties such as compactness, connectedness, as well as the type of minimal length elements in $\Gamma(x,y)$, are intimately related with the endpoint condition.
Remark 2.3. In Definition 2.1 the curvature is bounded above by a positive constant. This constant can be normalized by choosing suitable scaling via a dilation (contraction) of the plane. Throughout this work we will always consider $\kappa = 1$ in Definition 2.1. Moreover, without loss of generality, consider the origin of our orthogonal global coordinate system as the base point $x$ and let $X$ lie in the space generated by the vector $\left(\frac{\partial}{\partial x}\right)$ in the tangent space $T_x \mathbb{R}^2$ denoted, from now, $\mathbb{R}^2$. This space is equipped with a natural projection $\pi : UT \mathbb{R}^2 \to \mathbb{R}^2$. For this map, the fiber $\pi^{-1}(x)$ corresponds to $S^1$ for all $x \in \mathbb{R}^2$. Therefore the space of endpoint conditions corresponds to a sphere bundle on $\mathbb{R}^2$ with fiber $S^1$.

Definition 2.4. Let $C_l(x)$ be the unit circle tangent to $x$ and to the left of $X$. An analogous interpretation applies for $C_r(x), C_l(y)$ and $C_r(y)$ (see Figure 2). These circles are called adjacent circles and their arcs are called adjacent arcs. We denote their centers with lower-case letters, so the center of $C_l(x)$ is denoted by $c_l(x)$.

![Figure 1. Examples of bounded curvature paths and adjacent circles.](image)

Remark 2.5. We employ the following convention: When a path is continuously deformed under parameter $p$, we reparametrize each of the deformed paths by its arc-length. Thus $\gamma : [0, s_p] \to \mathbb{R}^2$ describes a deformed path at parameter $p$, with $s_p$ corresponding to its arc-length.

Definition 2.6. Given $\gamma, \eta \in \Gamma(x,y)$. A bounded curvature homotopy between $\gamma : [0, s_0] \to \mathbb{R}^2$ and $\eta : [0, s_1] \to \mathbb{R}^2$ corresponds to a continuous one-parameter family of immersed paths $\mathcal{H}_t : [0, 1] \to \Gamma(x,y)$ such that:

- $\mathcal{H}_t(p) : [0, s_p] \to \mathbb{R}^2$ for $t \in [0, s_p]$ is an element of $\Gamma(x,y)$ for all $p \in [0, 1]$.
- $\mathcal{H}_t(0) = \gamma(t)$ for $t \in [0, s_0]$ and $\mathcal{H}_t(1) = \eta(t)$ for $t \in [0, s_1]$.

It is of interest to establish under what conditions a bounded curvature path can be deformed, while keeping all the intermediate paths in $\Gamma(x,y)$, to a path of arbitrarily large length.

Definition 2.7. A bounded curvature path is said to be free if there exists a bounded curvature path arbitrarily long and a bounded curvature homotopy deforming one path into another.

The next remark summarizes well known facts about homotopy classes of paths on metric spaces. These facts are naturally adapted for elements in $\Gamma(x,y)$ for all
endpoint conditions. Our convention is to consider \( \Gamma(x,y) \) together with the \( C^1 \) metric.

**Remark 2.8.** For \( x,y \in T\mathbb{R}^2 \) then,

- Two bounded curvature paths are *bounded-homotopic* if there exists a bounded curvature homotopy from one path to another. Such a relation is an equivalence relation.
- A homotopy class on \( \Gamma(x,y) \) corresponds to an equivalence class on \( \Gamma(x,y) \) under the relation described above.
- The maximal path connected sets of \( \Gamma(x,y) \) are called the *homotopy classes* (or *path connected components*) of \( \Gamma(x,y) \).
- The homotopy classes of \( \Gamma(x,y) \) are nonempty, pairwise disjoint and their union is \( \Gamma(x,y) \).

The next definition can be easily adapted for arc length parametrized paths, we leave the details to the reader.

**Definition 2.9.** Suppose that \( \gamma, \delta : [0,1] \to \mathbb{R}^2 \) with \( \gamma(1) = \delta(0) \). The concatenation of \( \gamma \) and \( \delta \) is denoted by \( \delta \# \gamma \) and is defined to be,

\[
(\delta \# \gamma)(t) = \begin{cases} 
\gamma(2t) & 0 \leq t \leq \frac{1}{2} \\
\delta(2t-1) & \frac{1}{2} < t \leq 1 
\end{cases}
\]

If \( \gamma \) and \( \delta \) are arcs of circles, then \( \delta \# \gamma \) is such that \( \gamma \) and \( \delta \) are oppositely oriented arcs.

3. The Proximity Conditions

For given \( x,y \in T\mathbb{R}^2 \), the normalized bounded curvature constraint \( |\kappa| \leq 1 \) makes the four associated adjacent circles as barriers for local deformations of paths in \( \Gamma(x,y) \) around \( x \) and \( y \).

We extract four simple pairs of inequalities. These summarize the possible relations of distance between the adjacent circles. As we will see later, from one of these four relations we will obtain a necessary condition for the existence of bounded curvature paths having the same winding number not being in the same homotopy class in \( \Gamma(x,y) \). We denote by \( d \) the Euclidean metric in \( \mathbb{R}^2 \). It is easy to see that \( d(c_l(x), c_l(y)) = 2 \) and that \( d(c_r(y), c_r(x)) = 2 \).

Let an endpoint condition \( x,y \in T\mathbb{R}^2 \) be given. Taking into account the centers of the associated adjacent circles, four conditions of distance between them can be stated:

(i) \( d(c_l(x), c_l(y)) \geq 4 \) and \( d(c_r(x), c_r(y)) \geq 4 \)

(ii) \( d(c_l(x), c_l(y)) < 4 \) and \( d(c_r(x), c_r(y)) \geq 4 \)

(iii) \( d(c_l(x), c_l(y)) \geq 4 \) and \( d(c_r(x), c_r(y)) < 4 \)

(iv) \( d(c_l(x), c_l(y)) < 4 \) and \( d(c_r(x), c_r(y)) < 4 \)

These possible configurations for the centers of the adjacent circles \( C_l(x), C_r(x), C_l(y), \) and \( C_r(y) \) are illustrated in Figures 2 and 3. Observe that (ii) and (iii) give
the same planar configurations via a reflection (see Figure 2 bottom). In addition, (iv) implies that \(d(x, y) < 4\).

**Remark 3.1.** If the given endpoint condition \(x, y \in \mathbb{TR}^2\) satisfies proximity condition (iv) we have three possible scenarios (see Figure 3):

- \(x, y \in \mathbb{TR}^2\) is the endpoint condition of path consisting of a single arc of a unit circle of length less than \(\pi\) or \(x, y \in \mathbb{TR}^2\) is the endpoint condition of a path consisting of a concatenation of two arcs of unit circles each of length less than \(\pi\).
- The four adjacent circles associated with the endpoint condition enclose a region \(\Omega\).
- \(x, y \in \mathbb{TR}^2\) is the endpoint condition of free path.

**Definition 3.2.** The boundary of the region in Remark 3.1 corresponds to the concatenation of six unit circles. The orientation of these circles alternate (see Figure 3).

**Remark 3.3.** The boundary of the region in Remark 3.1 is a simple closed plane curve. By virtue of the Jordan curve theorem, the complement of such a curve consists of exactly two connected components in the plane. One of these components is bounded while the other component is unbounded, and the curve is the boundary of each of these components.

**Definition 3.4.** Define \(\Omega\) be the closure of the bounded component in Remark 3.3. We say that the endpoint condition carries a region \(\Omega\). Denote by \(\partial \Omega\) to the boundary of \(\Omega\).

In order to make the figures clearer, sometimes we omit the initial and final tangent vectors.
Figure 3. The two possible scenarios under proximity condition (iv). The dashed trace represents a path in $\Gamma(x,y)$.

**Definition 3.5.** The space $\Gamma(x,y)$ satisfies proximity condition A if its endpoint condition satisfies (i).

**Definition 3.6.** The space $\Gamma(x,y)$ satisfies proximity condition B if its endpoint condition satisfies (ii) or (iii).

**Definition 3.7.** The space $\Gamma(x,y)$ satisfies proximity condition C if its endpoint condition satisfies (iv) and $\Gamma(x,y)$ contains a path that has as a subpath:

- an arc of circle of length greater than or equal to $\pi$ or
- a line segment of length greater than or equal to 4.

In order to capture features that allow the existence of non-trivial homotopy classes of bounded curvature paths we introduce the next definition.

**Definition 3.8.** The space $\Gamma(x,y)$ satisfies proximity condition D if its endpoint condition satisfies (iv) and:

- The endpoint condition carries a region $\Omega$ or
- $\Gamma(x,y)$ contains a path which is an arc of a unit circle of length less than $\pi$ or
- $\Gamma(x,y)$ contains a path which is a concatenation of two oppositely oriented arcs of unit circles of length less than $\pi$ each.

The key point is that spaces of bounded curvature paths with endpoint condition satisfying proximity condition D have a non-trivial connected component. We will show that there exists a connected component with paths lying in $\Omega$.

The three items in Definition 3.8 are mutually exclusive. For example, it’s easy to see that no endpoint condition simultaneously allow the existence of a path which is an arc of a unit circle of length less than $\pi$ and a path which is a concatenation of two oppositely oriented arcs of unit circles of length less than $\pi$ each.

**Definition 3.9.**

- A path $\gamma : I \to \mathbb{R}^2$ is in $\Omega$ if $\gamma(t) \in \Omega$ for all $t \in I$.
- A path $\gamma : I \to \mathbb{R}^2$ is not in $\Omega$ if there exists $t \in I$ such that $\gamma(t) \notin \Omega$.

**Definition 3.10.** We denote by $\Delta(\Omega) \subset \Gamma(x,y)$ the space of embedded bounded curvature paths in $\Omega$. And, by $\Delta'(\Omega) \subset \Gamma(x,y)$ the space of bounded curvature paths not in $\Omega$. 
Later on in Section 10 we will prove that the spaces of embedded bounded curvature paths in \( \Omega \) correspond to homotopy classes with nonempty interior. In particular, paths in \( \Delta(\Omega) \) cannot be deformed to paths having self intersections.

4. The Region \( \Omega \)

Via the proximity conditions we divide our study into cases. Our aim is to determine cases for which the endpoint condition leads to the existence of paths in \( \Gamma(x,y) \) that cannot be made bounded-homotopic one to another.

Consider \( x,y \in T\mathbb{R}^2 \) satisfying proximity condition D. As we already mentioned, spaces of bounded curvature paths may contain paths that can be made bounded-homotopic to paths of arbitrary large length. We are interested in capturing the features that lead to paths that cannot be made bounded-homotopic to paths of arbitrary length.

Remark 4.1. Throughout this work we will assume that the endpoint \( y \) has positive abscissa and is located in the complement of the union of the disks with boundary \( C_l(x) \) and \( C_r(x) \) of the coordinate system considered in Remark 2.3. We denote such a planar region as the forward region of \( x \in T\mathbb{R}^2 \).

As we will see later in Corollary 7.6, any bounded curvature path having its final position in the interior of the disks with boundary the circles \( C_l(x) \) or \( C_r(x) \) is a free path, i.e., is bounded-homotopic to a path which is arbitrarily long, where the intermediate paths have bounded curvature.

Remark 4.2. Suppose \( x,y \in T\mathbb{R}^2 \) satisfies proximity condition D. Consider a coordinate system for \( x \) as in Remark 2.3. Let \( D_l \) and \( D_r \) be two disks of radius 4 with centers \( c_l(x) \) and \( c_r(x) \) respectively. Note that the intersection of the forward region of \( x \) (see Remark 4.1) with the interior of \( D_l \) gives the range for possible positions for the center of \( C_l(y) \). Similarly, the intersection of the forward region of \( x \) with the interior of \( D_r \) gives the range for possible positions for the center of \( C_r(y) \). From the latter analysis we see that qualitatively we have three possible configurations that control the formation of \( \Omega \) (see Figure 4).

In general, the shape of \( \Omega \) depends on the endpoint condition.

For the description of \( \Omega \) we introduce more terminology. When two circles (exterior to each other) are tangent at only one point, a smooth curve given by two arcs, each in one of these circles, gives rise to an inflection point at the intersection point.
The four inflection points shown in Figure 5 are denoted by \( i \) and are indexed as elements in \( \mathbb{R}^2 \) with \(|i_k| < |i_{k+1}|\), for \( k = 1, 2, 3 \), where \(|| \) is the usual Euclidean length of a vector. Set \( \Theta_k \) as the smallest circular arc (whose length we denote as \( \theta_k \)) starting from \( x \) or \( y \) and finishing at \( i_k \) with \( 1 \leq k \leq 4 \), compare Figure 5.

![Figure 5. A region \( \Omega \) and its associated notation.](image)

Denote by \( L_1 \) and \( L_2 \) the lines joining the first two and the last two indexed inflection points respectively. Now we subdivide \( \Omega \) into three subregions as follows:

- Let \( R_1 \) be the closed portion of \( \Omega \) which is to the left of \( L_1 \).
- Let \( R_2 \) be the closed portion of \( \Omega \) which is between \( L_1 \) and \( L_2 \).
- Let \( R_3 \) be the closed portion of \( \Omega \) which is to the right of \( L_2 \).

If \( \theta_2 > \frac{\pi}{2} \), \( L_1 \) crosses twice the adjacent arc \( \theta_2 \). In this case we replace \( L_1 \) by the line joining \( i_1 \) and a point \( z \) in the adjacent arc \( \theta_2 \), such that \( L_1 \) is tangent to \( \theta_2 \) at \( z \). The same idea is applied if \( \theta_4 > \frac{\pi}{2} \).

**Remark 4.3.** Remember that when condition D applies, we have that:

\[
d(c_l(x), c_l(y)) < 4 \quad \text{and} \quad d(c_r(x), c_r(y)) < 4
\]

In particular we have that the center of the upper arc in \( \partial R_2 \) is located below the line segment with endpoints \( c_l(x) \) and \( c_l(y) \), and the center of the lower arc in \( \partial R_2 \) is located above the line segment with endpoints \( c_r(x) \) and \( c_r(y) \). Observe otherwise the middle arcs in \( \partial R_2 \) would have length greater or equal than \( \pi \) which contradicts the formation of \( \Omega \). We refer to this fact as the *arc condition*.

**Lemma 4.4.** The length of the circular arcs in \( \partial \Omega \) satisfies:

- if \( \theta_2 \geq \frac{\pi}{2} \), then \( \theta_1 < \frac{\pi}{4} \).
- if \( \theta_4 \geq \frac{\pi}{2} \), then \( \theta_3 < \frac{\pi}{4} \).

**Proof.** Suppose that \( \theta_1 \geq \frac{\pi}{2} \), with \( \theta_2 \geq \frac{\pi}{2} \). Consider the following subsets of \( \mathbb{R}^2 \):

\[
S^+ = \{(u, v) \in \mathbb{R}^2 \mid u \geq 0, v > 1\} \quad \text{and} \quad S^- = \{(u, v) \in \mathbb{R}^2 \mid u \geq 0, v < -1\}.
\]

Without loss of generality suppose that \( \theta_1 \) lies in \( C_l(x) \), then consider the line \( l_1 \) starting at \( c_l(x) \) and passing through the inflection point \( i_1 \). Note that the center...
of the boundary arc $\partial R_2$ adjacent to $C_1(x)$ must lie on $l_1$. Since the length of the boundary arc $\partial R_2$ adjacent to $C_1(x)$ is less than or equal to $\pi$ (this implies that $c_1(y)$ lies in the interior of the upper half space of $l_1$. Similarly, if $l_2$ is the line passing through $c_2(x)$ and $i_2$, then $c_r(y)$ lies in the interior of the lower half space of $l_2$.

Observe that $c_1(y)$ lies in the interior of $S^+$ and that $c_r(y)$ lies in the interior of $S^-$ implying that $d(c_1(y), c_r(y)) > 2$ which leads to a contradiction. An analogous approach proves the second statement.

An immediate consequence of Lemma 4.4 is the following result.

**Corollary 4.5.** The boundary of $\Omega$ contains at most two arcs $\Theta_k$ such that, $\theta_k > \frac{\pi}{2}$.

**Proof.** Suppose that $\theta_k > \frac{\pi}{2}$, for three indices. So, we have 4 possible arrangements satisfying the hypothesis, these are \{\theta_1, \theta_2, \theta_3\}, \{\theta_1, \theta_2, \theta_4\}, \{\theta_1, \theta_3, \theta_4\}, and \{\theta_2, \theta_3, \theta_4\}. Observe that by Lemma 4.4 if $\theta_1, \theta_2 > \frac{\pi}{2}$ immediately implies that $d(c_1(y), c_r(y)) > 2$. Similarly if $\theta_3, \theta_4 > \frac{\pi}{2}$ by the same argument implies that $d(c_1(x), c_r(x)) > 2$. Since $\theta_1, \theta_2$ or $\theta_3, \theta_4$ are contained in all the possible arrangements of three angles the result follows.

**Remark 4.6.** Observe that the combinations $\theta_1, \theta_3 > \frac{\pi}{2}; \theta_1, \theta_4 > \frac{\pi}{2}; \theta_2, \theta_3 > \frac{\pi}{2}; \theta_2, \theta_4 > \frac{\pi}{2}$ are the only possible configurations for the lengths of the adjacent arcs when $\Omega$ contains two arcs $\Theta_k$ with lengths greater than $\frac{\pi}{2}$.

5. The Diameter of $\Omega$

In order to establish under what requirements, bounded curvature homotopies of paths satisfying condition $D$ are confined or not in $\Omega$, it is important to have an estimate of the diameter of such a planar region. In Section 8, we will develop a criteria called the S-Theorem that will tell us whether or not a path remains in $\Omega$. Such a result will be intimately related with the understanding of the diameter of $\Omega$. The main difficulty is that such estimates must hold for all the $X,Y$ satisfying condition $D$.

**Definition 5.1.** The diameter of $\Omega$ is defined by:

$$\text{diam}(\Omega) = \max\{d(z, w) \mid z, w \in \Omega\}$$

**Definition 5.2.** The diameter of a path $\gamma$ is defined by:

$$\text{diam}(\gamma) = \max\{d(\gamma(t_1), \gamma(t_2)) \mid t_1, t_2 \in I\}$$

Due to the triviality of the following statement we omit its proof.

**Lemma 5.3.** For any points $z, w \in \Omega$ such that $d(z, w) = \text{diam}(\Omega)$ we have that $z, w \in \partial \Omega$.

In order to find an upper bound for the diameter for all the possible regions $\Omega$ we present the following results.

**Lemma 5.4.** For $x, y \in T\mathbb{R}^2$ satisfying condition $D$ such that $\langle X, Y \rangle > 0$ we have that $d(x, y)$ is a local maximum of length.

**Proof.** Perturb $y \in \mathbb{R}^2$ along $\partial \Omega$ on $\Theta_3$ (or $\Theta_4$) and denote such a perturbed point by $\tilde{y}$, with $d(y, \tilde{y}) \leq \epsilon$. Since $\langle X, Y \rangle > 0$, we have that $d(x, \tilde{y}) < d(x, y)$, since $\langle y - \tilde{y}, \tilde{y} - x \rangle > 0$ is easy to see. The same idea applies when fixing $y$ and perturbing $x$. \qed
It's not hard to see that if \( x, y \) satisfy condition D with \( \langle X, Y \rangle \leq 0 \) then \( d(x, y) \) is never a local maximum of length.

**Proposition 5.5.** Consider the two unit disks defined by the two middle arcs of \( \partial R_2 \) for a region \( \Omega \) satisfying condition D. These disks must have intersecting interiors.

**Proof.** Recall that \( \Omega \) is obtained when simultaneously,

\[
\text{dist}(\text{cl}(x), \text{cl}(y)) < 4 \quad \text{and} \quad \text{dist}(\text{cr}(x), \text{cr}(y)) < 4
\]

Suppose the two unit radius disks \( D_1, D_2 \subset \Omega \) defined by the middle arcs have disjoint interiors. The distance of their centers \( c_1, c_2 \) satisfies \( \text{dist}(c_1, c_2) \geq 2 \). Let \( Q \) be the quadrilateral with vertices \( \text{cl}(x), \text{cr}(x), \text{cl}(y), \text{cr}(y) \). Observe that, \( 2 = \text{dist}(c_1(x), c_1(y)) = \text{dist}(c_1(y), c_1(x)) = \text{dist}(c_r(x), c_2(y)) = \text{dist}(c_r(x), c_2(y)) \)

It is easy to see geometrically that \( c_1 \) and \( c_2 \) in \( Q \) and that \( \text{dist}(c_1, c_1) \geq 2 \) is impossible.

**Corollary 5.6.** If \( w \) and \( z \) belong to opposite components of \( \partial R_2 \) then,

\[
d(w, z) < 4
\]

**Proof.** By extending the middle arcs in \( \partial R_2 \) to circles, Proposition 5.5 ensures that these circles overlap. Therefore the distance of any two points in opposite arcs of \( \partial R_2 \) is bounded above by 4.

**Remark 5.7.** The two circles formed by extending the arcs of \( \partial R_2 \) in Corollary 5.6 may extend outside \( \Omega \). In such a case the result still holds.

**Remark 5.8.** Given two disjoint circles \( C_1 \) and \( C_2 \) in the plane. The line connecting their centers intersects the circles at four points \( \{x, y, z, w\} \); we denote by \( XY \) the line segment joining the points \( x \in C_1 \) and \( y \in C_2 \), which has interior disjoint from the circles, and by \( WZ \) the long segment joining the points \( w \in C_1 \) and \( z \in C_2 \). The following claims are valid since under the presented conditions there are only four critical points for the distance function between points on the circles, one maximum, one minimum and two saddle points:

- The maximum of the distances between points in \( C_1 \) and \( C_2 \) is given by the endpoints of the segment \( WZ \). That is,

\[
\max_{a \in C_1, b \in C_2} d(a, b) = d(z, w).
\]

- The minimum of the distances of points in \( C_1 \) and \( C_2 \) is given by the endpoints of the segment \( XY \). That is,

\[
\min_{a \in C_1, b \in C_2} d(a, b) = d(x, y).
\]

We leave the details to the reader.

**Theorem 5.9.** The diameter of \( \Omega \) is strictly bounded by 4.
Proof. Since the region $\Omega$ is compact and the Euclidean distance is continuous we have that $d : \Omega \times \Omega \to \mathbb{R}_{\geq 0}$ attains a maximum. By Lemma 5.3 the maximum of $d$ in $\Omega$ must be achieved in $\partial \Omega$. Lemma 5.4 establishes that $d(x, y)$ is a local maximum. Corollary 5.6 establishes an upper bound for the distances between the opposite arcs in $\partial R_2$ and, by the first item in Remark 5.8, such a value is a local maximum only if such arcs intersect the line joining the respective centers of the overlapping circles inside $\Omega$. The second item in Remark 5.8 establishes that the distances between opposite arcs in $\partial R_1$ and $\partial R_3$ are at most candidates to be saddles or local mimina. By considering $d(x, y)$ as in Lemma 5.4 and $d(z, w)$ as in Remark 5.8 we have that:

$$diam(\Omega) = \max\{d(x, y), d(z, w)\}$$

concluding that $diam(\Omega) < 4$ as desired. 

6. BOUNDED CURVATURE HOMOTOPIES VIA PARALLEL TANGENTS

The existence of parallel tangents allows us to construct bounded curvature homotopies for general bounded curvature paths.

**Definition 6.1.** We say that a bounded curvature path $\gamma$ has *parallel tangents* if there exist $t_1, t_2 \in I$, with $t_1 < t_2$, such that $\gamma'(t_1)$ and $\gamma'(t_2)$ are parallel and pointing in opposite directions (see Figure 6 left).

**Remark 6.2.** Let $\gamma$ be a $C^1$ path. The affine line generated by $\langle \gamma'(t) \rangle$ is called the tangent line at $\gamma(t), t \in I$. The ray containing $\gamma'(t)$ is called the positive ray.

**Proposition 6.3.** Bounded curvature paths having parallel tangents are free paths.

**Sketch of Proof.** Suppose $\gamma$ is a bounded curvature path having parallel tangents at parameters $t_1$ and $t_2$. Consider the subpath $\tilde{\gamma} : [t_1, t_2] \to \mathbb{R}^2$. Subdivide $\gamma$ in such a way that $\gamma = \gamma_2 \# \tilde{\gamma} \# \gamma_1$. Consider the parametrized lines $\psi_1, \psi_2 : [0, 1] \to \mathbb{R}^2$ defined by

$$\psi_1(r) = \tilde{\gamma}(t_1)(1 - r) + Pr$$

$$\psi_2(r) = \tilde{\gamma}(t_2)(1 - r) + Qr$$
where $P$ belongs to the positive ray of $\langle \gamma'(t_1) \rangle$ and $Q$ belongs to the negative ray of $\langle \gamma'(t_2) \rangle$ with $d(P, \gamma(t_1)) = d(Q, \gamma(t_2))$. Call $\delta_r$ the translation of $\gamma$ obtained by adding the vector from $\gamma(t_1)$ to $\psi_1(r)$. Define $\phi_r = \psi_3|_{[0, r]} \circ \delta_r \circ \psi_1|_{[0, r]}$ for each $r \in (0, 1]$ (see Definition 2.9), where $\psi_3(r) = \psi_2(1 - r)$. Define $H(r) = \gamma_2 \# \phi_r \# \gamma_1$. In this fashion we have that $H(0) = \gamma_2 \# \phi_0 \# \gamma_1 = \gamma$, and $H(r) = \gamma_2 \# \phi_r \# \gamma_1$ is a bounded curvature path for $r \in [0, 1)$. Since $P$ and $Q$ can be chosen to be arbitrarily far from $\gamma(t_1)$ and $\gamma(t_2)$ we conclude that $\gamma$ is a free path.

**Corollary 6.4.** Bounded curvature paths containing an arc of a circle of radius $r \geq 1$ with length greater than or equal to $\pi r$ are free paths.

**Proof.** The existence of an arc of a circle of radius $r \geq 1$ with length greater than or equal to $\pi r$ implies immediately the existence of parallel tangents. Then the result follows.

□

7. **Bounded Curvature Paths in $\Omega$**

We study the behavior of bounded curvature paths in $\Omega$ in terms of winding number, normalized curvature bound and endpoint condition $x, y \in T\mathbb{R}^2$. Our goal here is to establish a characterization of the spaces $\Gamma(x, y)$ satisfying condition D in terms of bounded curvature homotopies between elements in $\Delta(\Omega)$ and elements in $\Delta'(\Omega)$. The description of the elements in $\Delta(\Omega)$ requires an understanding of how these paths intersect $\partial\Omega$. From the latter, our analysis depends on the part of $\partial\Omega$ in which the intersection point lies.

**Definition 7.1.** Let $\gamma : \text{int}(I) \to \Omega$ with the property that its image does not coincide with $\partial\Omega$ in a subinterval of $I$. A point $\gamma(t) \in \partial\Omega$ is called **boundary tangent point** if $\gamma$ locally intersects $\partial\Omega$ in a single point or, given $\epsilon > 0$, there exists $m \in \mathbb{N}$ and a sequence $t_n \to t$ such that for $n > m$,

$$||\gamma(t_n) - \gamma(t)|| < \epsilon$$

with $\gamma(t_n) \in \partial\Omega$ (see Figure 7).

![Figure 7](image-url)

**Figure 7.** The points $\gamma(t)$ at the left and right illustrations correspond to boundary tangent points while $\gamma(p)$ at the center is not a boundary tangent point.

Bounded curvature paths can eventually leave $\Omega$ and then reenter it without violating the bounded curvature property. In addition, note that embedded bounded curvature paths in $\Omega$ containing points as $\gamma(p)$ in Figure 7 are easy to construct. Moreover, depending on the endpoint condition there may exist bounded curvature
paths in Ω allowing self intersections. In Theorem 9.4 we will prove that bounded curvature paths with self intersections are free paths.

For the sake of exposition we include without proof the following two results. Refer to [1] for their proofs and comments.

**Lemma 7.2.** A bounded curvature path $\gamma : I \to \mathcal{B}$ where,

$$\mathcal{B} = \{(x, y) \in \mathbb{R}^2 \mid -1 < x < 1, \, y \geq 0\}$$

cannot satisfy both:

- $\gamma(0), \gamma(s)$ are points on the $x$-axis.
- If $C$ is a unit circle with center on the negative $y$-axis, and $\gamma(0), \gamma(s) \in C$, then some point in $\text{Im}(\gamma)$ lies above $C$.

![Figure 8](image.png)

**Figure 8.** An illustration of Corollary 7.4, here $p$ and $q$ represent $\gamma(0)$ and $\gamma(s)$. Note that the parallel tangents are located in between $P$ and $P'$ and $Q$ and $Q'$. We obtain the left figure from the right one after rotating counterclockwise the band $\mathcal{B}$ with rotation axis the center of $C$.

Next we ensure the existence of parallel tangents for paths satisfying the hypothesis in Lemma 7.2 except that they are not confined to $\mathcal{B}$.

**Definition 7.3.** A line joining two points in a path, which are distant at least 2 apart is called a cross section.

**Corollary 7.4.** Suppose a bounded curvature path $\gamma : I \to \mathbb{R}^2$ satisfies:

- $\gamma(0), \gamma(s)$ are points on the $x$-axis.
- If $C$ is a unit circle with center on the negative $y$-axis, and $\gamma(0), \gamma(s) \in C$, then some point in $\text{Im}(\gamma)$ lies above $C$.

Then $\gamma$ admits parallel tangents and therefore a cross section.

The first goal is to establish if embedded bounded curvature paths in $\Omega$ admit boundary tangent points. As a first step we prove the following results. Their proofs are strongly dependent on the bounded curvature property together with the way these paths intersect $\partial \Omega$. 
Lemma 7.5. A bounded curvature path in a closed unit disk $\mathbb{D}$, which admits an intersection point with $\partial \mathbb{D}$, lies entirely in $\partial \mathbb{D}$.

Proof. Suppose there is a bounded curvature path $\gamma : I \to \mathbb{D}$ not entirely in $\partial \mathbb{D}$, such that $\gamma(p) \in \partial \mathbb{D}$, for some $p \in \text{int}(I)$. Since $\gamma$ is piecewise $C^2$, there are two cases. Firstly, if $\gamma''$ is defined (and continuous) in a neighborhood of $p$. Choose coordinates so that $\gamma(p)$ is the origin and $\gamma'(p)$ is the positive $x$-axis with $\mathbb{D}$ in the upper half plane. Then locally $\partial \mathbb{D}$ has the form $(t, 1 - \sqrt{1 - t^2})$. Write $\gamma$ in the form $(t, f(t))$. Then the curvature bound becomes $\frac{f''}{(1 + f'^2)^{3/2}} \leq 1$ using the well known formula for the curvature of a graph. Since $f(0) = 0$ and $f'(0) = 0$, integrating the differential inequality gives $f(t) \leq 1 - \sqrt{1 - t^2}$. But this implies $\gamma$ is disjoint from the interior of $\mathbb{D}$ near $p$. We conclude $\gamma$ coincides with $\partial \mathbb{D}$ near $p$. In the second case $\gamma''$ is not defined at $p$. If $\gamma(p)$ is an inflection point then $\gamma$ crosses $\partial \mathbb{D}$ at $\gamma(p)$, contrary to the assumption that $\gamma$ lies entirely in $\mathbb{D}$. If $\gamma(p)$ is not an inflection point we use a similar argument as the one when $\gamma''$ is defined (and continuous) in a neighborhood of $p$ to conclude the proof. \hfill $\Box$

Therefore the only bounded curvature paths in a unit disc $\mathbb{D}$ admitting intersection points with $\partial \mathbb{D}$ are arcs in $\partial \mathbb{D}$.

The next result justifies why we excluded the interior of the adjacent circles in Remark 4.1.

Corollary 7.6. A bounded curvature path having its final position in the interior of either of the disks with boundary $C_l(x)$ or $C_r(x)$, is a free path.

Proof. Since the distance between the initial and final points $x$ and $y$ satisfies $d(x, y) < 2$, by applying Corollary 7.4 the path admits parallel tangents and by virtue of Proposition 6.3 we conclude the path is free. \hfill $\Box$

Theorem 7.7.

- A bounded curvature path in $\Omega$ does not admit a boundary tangent point in $\mathcal{R}_1$ before first leaving to $\mathcal{R}_2$.
- A bounded curvature path in $\Omega$ does not admit a boundary tangent point in $\mathcal{R}_3$ after its last exit from $\mathcal{R}_2$.

Proof. Consider the first statement. Let $\gamma$ be a bounded curvature path in $\Omega$. Suppose there exists $t \in \text{int}(I)$ such that:

$$\text{Im}(\gamma) \cap \partial \Omega = \{\gamma(t)\}$$

is an boundary tangent point in $\mathcal{R}_1$ before $\gamma$ reaches $\mathcal{R}_2$. Change coordinates so that the line through $x$ and $\gamma(t)$ is the new $x$-axis and a new origin $o$ is the midpoint between $x$ and $\gamma(t)$ with the positive $y$-axis passing through $\mathcal{R}_1$. Define $B$ as in Lemma 7.2. It is easy to check that $\mathcal{R}_1 \subset B$ and so the result follows from Lemma 7.2. An analogous method proves the second statement. \hfill $\Box$

The first statement in the last result establishes that no bounded curvature path in $\Omega$ admits a boundary tangent point with $\partial \Omega$ restricted to $\mathcal{R}_1$ before the path enters region $\mathcal{R}_2$. Symmetrically, no bounded curvature path in $\Omega$ admits a boundary tangent point with $\partial \Omega$ restricted to $\mathcal{R}_3$ after leaving $\mathcal{R}_2$ for the final time.
Theorem 7.8. Bounded curvature paths in \( \Omega \) do not admit boundary tangent points in \( \mathbb{R}^2 \).

Proof. Let \( \gamma \) be a bounded curvature path in \( \Omega \). Consider \( \partial R_2 \) as \( \partial \mathbb{D} \) in Lemma 7.5 then the proof follows. \( \square \)

8. The S-Theorem

In this section we prove a crucial result. In order to establish that embedded bounded curvature paths in \( \Omega \) do not escape, we prove that these paths do not have boundary tangent points. Once we exclude the existence of boundary tangent points on embedded bounded curvature paths in \( \Omega \) we can prove the existence of non-trivial homotopy classes of bounded curvature paths.

The S-Theorem gives a method for characterizing paths in \( \Omega \) via the turning map and its extremals. By contrast, the latter also gives an important property that only paths not in \( \Omega \) satisfy. Of special relevance will be the existence of maximal inflection points together with the results already developed about the diameter of \( \Omega \).

The definition of turning map played an important role in the classification of minimal length elements in spaces whose elements have a prescribed winding number (compare [2]). Consider the exponential map \( \exp : \mathbb{R} \to \mathbb{S}^1 \).

Definition 8.1. For a path \( \gamma : I \to \mathbb{R}^2 \). The turning map \( \tau \) is defined in the following diagram,

\[
\begin{array}{ccc}
I & \xrightarrow{\tau} & \mathbb{R} \\
& \searrow & \searrow \\
& & \mathbb{S}^1 \\
& \downarrow^{w} & \downarrow^{\exp} \\
\mathbb{R} & \rightarrow & \mathbb{S}^1
\end{array}
\]

The map \( w : I \to \mathbb{S}^1 \) is called the direction map, and gives the derivative \( \gamma'(t) \) of the path \( \gamma \) at \( t \in I \). The turning map \( \tau : I \to \mathbb{R} \) gives the turning angle the derivative vector makes at \( t \in I \) with respect to \( \exp(0) \) i.e., the turning angle \( \gamma'(t) \) makes with respect to the \( x \)-axis.

A path \( \gamma \in \Gamma(x,y) \) admits a negative direction if there exists \( t \in I \) such that \( \langle X, \gamma'(t) \rangle < 0 \). In order to ensure conditions for the existence of a negative direction we state the following intuitive result whose proof is left to the reader.

Lemma 8.2. A \( C^1 \) path \( \gamma(t) = (x(t), y(t)) \) with \( x : I \to \mathbb{R} \) not being a monotone function admits a negative direction.

Consequently, a path containing only non-negative directed points must have a non-decreasing coordinate function \( x : I \to \mathbb{R} \). Therefore such a path is confined to never travel backwards when projected to the \( x \)-axis.

Definition 8.3. A boundary tangent point \( \gamma(t) \in \partial R_1, t \in I \) is called a returning point if there exists \( r \in I \) such that \( r < t \) and \( \gamma(r) \in \mathbb{R}_2 \). A boundary tangent point \( \gamma(t) \in \partial R_3, t \in I \) is called a returning point if there exists \( r \in I \) such that \( t < r \) and \( \gamma(r) \in \mathbb{R}_2 \).

Bounded curvature paths in \( \Omega \) may or may not have a returning point. This depends on the shape of the region \( \Omega \) and therefore on the endpoint condition. In
Remark 8.7. The inflection normal. \( \gamma \)

line at particular, if \( \Omega \) contains an embedded unit radius disk \( \mathbb{D} \) such that \( \partial \mathcal{R}_1 \cap \mathbb{D} \) (or \( \partial \mathcal{R}_3 \cap \mathbb{D} \)) is non-empty then paths with self intersections having returning points can be constructed. However, these paths are not paths in \( \Delta(\Omega) \).

**Proposition 8.4.** A bounded curvature path in \( \Omega \) having a returning point in \( \partial \mathcal{R}_1 \) (or \( \partial \mathcal{R}_3 \)) admits a negative direction.

**Proof.** The returning point existence implies non monotonicity of the first component of \( \gamma \) and therefore by Lemma 8.2 we ensure the existence of a point \( r \in I \) such that \( \langle X, \gamma'(r) \rangle < 0 \). \( \square \)

**Definition 8.5.** A maximal inflection with respect to \( \partial \mathcal{R}_1 \) if \( \Omega \) contains an embedded unit radius disk \( \mathbb{D} \).

**Definition 8.8.** Suppose that \( \gamma \) crosses \( \mathcal{N} \) before and after the inflection point \( \gamma(t) \). Let \( F_1 \) be the last time \( \gamma \) crosses \( \mathcal{N} \) before reaching \( \gamma(t) \) and let \( F_2 \) be the first time \( \gamma \) crosses \( \mathcal{N} \) after reaching \( \gamma(t) \).

**Lemma 8.9.** Let \( \gamma \) be a bounded curvature path with maximal inflection point at \( \gamma(t) \) for some \( t \in I \).

- The path \( \gamma \) between \( F_1 \) and \( \gamma(t) \) does not cross the inflection tangent \( \mathcal{I} \) in a point other than \( \gamma(t) \).
- The path \( \gamma \) between \( \gamma(t) \) and \( F_2 \) does not cross the inflection tangent \( \mathcal{I} \) in a point other than \( \gamma(t) \).

**Proof.** Suppose that a bounded curvature path \( \gamma \) with maximal inflection point \( \gamma(t) \) crosses the inflection tangent \( \mathcal{I} \) in between \( F_1 \) and \( \gamma(t) \) (see Figure \([8]\)). In the generic case the path \( \gamma \) leaves and then reenters the quadrant III obtaining two adjacent intersections say at \( \gamma(t_1) \) and \( \gamma(t_2) \). Since \( \mathcal{I} \) corresponds to the \( x \)-axis, we have that the graph of \( \gamma \) in between \( t_1 \) and \( t_2 \) and in between \( t_2 \) and \( t \) admits minimum and maximum \( y \)-values respectively. In addition it is easy to see that the graph of \( \gamma \) at the minimum is concave and that at the maximum is convex. By virtue of the mountain pass theorem there exists an inflection point, say \( \gamma(t_i) \), other than \( \gamma(t) \) in between \( F_1 \) and \( \gamma(t) \), that is, \( \gamma(t_i) < \gamma(t) \) leading to a contradiction. If \( \gamma \) intersects \( \mathcal{I} \) in a single point the same method applies. The second assertion is proved using an analogous argument. \( \square \)

Figure \([8]\) illustrates the first statement in Lemma 8.9.

The next result is tackled via a proof by contradiction. The idea is to prove that paths admitting a retuning point must have a diameter bigger than the diameter of \( \Omega \).
Figure 9. The point $\gamma(t)$ at the left is a maximal inflection point but the point $\gamma(t)$ at right is not.

**Theorem 8.10** (The S-theorem). Embedded bounded curvature paths in $\Omega$ do not admit returning points.

**Proof.** Suppose there exists an embedded bounded curvature path $\gamma : I \to \mathbb{R}^2$ in $\Omega$ with a returning point at $\partial R_1$. Since $\theta_1$ and $\theta_2$ have length less than $\pi$, so by considering $x = p$ and the returning point to be $q$ in Corollary 7.4 (see Figure 9) we can homotope $\gamma$ until it touches $\partial R_3$, so the homotoped path has two returning points after applying Proposition 6.3. Let $\gamma(t)$ be a maximal inflection point with coordinate system as in Remark 8.7. By virtue of Lemma 8.9 we have that the path $\gamma$ between $F_1$ and $\gamma(t)$ does not cross the inflection tangent $T$ in a point other than $\gamma(t)$ and symmetrically $\gamma$ also does not cross $T$ in between $\gamma(t)$ and $F_2$. In other words since a maximal inflection point is an extremal of the turning map $\tau$ we have that the two returning points must lie in opposite quadrants.

Our strategy is to establish that none of the trajectories from $x$ to $y$ are possible for embedded paths in $\Omega$ under the hypothesis of a returning point. The first scenario occurs when $\gamma$ crosses the inflection normal $N$. In other words we want to analyze the behavior of $\gamma$ as its trajectory travels from the quadrant IV to III and from the quadrant I to II, see Figure 10. We separate this into cases:

1. $d(F_1, \gamma(t)) \geq 2$ and $d(F_2, \gamma(t)) \geq 2$.
2. $d(F_1, \gamma(t)) < 2$ and $d(F_2, \gamma(t)) < 2$.
3. $d(F_1, \gamma(t)) < 2$ and $d(F_2, \gamma(t)) \geq 2$.
4. $d(F_1, \gamma(t)) \geq 2$ and $d(F_2, \gamma(t)) < 2$.

The proof of the first case is trivial since it immediately implies that $\text{diam}(\gamma) > 4$ and therefore by Theorem 5.9 $\gamma$ is not in $\Omega$.

For the second case we apply Corollary 7.4 to $\gamma$ where the circle $C$ is the right adjacent circle of $\gamma(t)$ and the parallel lines are $L_1 = T$ and $L_2$ satisfies the equation $y = 2$. We conclude that $\gamma$ must contain a point of the line $L_2$, otherwise it crosses $T$ contradicting the hypothesis that $\gamma(t)$ is a maximal inflection point, using Lemma 8.9.

By an analogous argument applying Corollary 7.4 to $\gamma$ where the circle $C$ is the left adjacent circle of $\gamma(t)$ and parallel lines $L_1 = T$ and $L_2$ satisfies equation $y = -2$, we conclude that $\gamma$ must contain a point of the line $L_2$, otherwise it crosses $T$ contradicting the hypothesis that $\gamma(t)$ is a maximal inflection point, using Lemma 8.9.
So, we obtain that $\text{diam}(\gamma) > 4$ therefore $\gamma$ is a path not in $\Omega$. In addition note that the third and fourth cases are identical and also lead to $\text{diam}(\gamma) > 4$ by a combination of the first and second case.

**Figure 10.** A bounded curvature path with diameter bigger than 4.

In conclusion we have excluded the following trajectories for embedded bounded curvature paths in $\Omega$. Here $\to$ indicates direction of traveling for $\gamma$ containing the first and the last element in the sequence $x$ and $y$ respectively:

- $I \to IV \to III \to I \to II$.
- $I \to IV \to III \to I \to II \to III$.
- $IV \to III \to I \to II$.
- $IV \to III \to I \to II \to III$.

Next we will establish that no embedded bounded curvature path in $\Omega$, under the hypothesis of a returning point, has initial point $x$ belonging to $III$. Therefore we will exclude of the following trajectories:

- $III \to I$.
- $III \to I \to II$.
- $III \to I \to II \to III$.

Let $x, y \in \mathbb{T}^2$ such that $x$ lies in $III$ and $y$ lies in $I$. Since $\gamma$ admits a returning point, say at $\gamma(t)$ for some $t \in I$, its maximal inflection point is such that $\langle X, \gamma'(t) \rangle < 0$. Now, let $L_1$ be the line satisfying equation $y = 2$ and $L_2$ be the line satisfying equation $y = -2$ in the coordinate system considered in Remark 8.7. Consider the following cases:

- The point $x$ lies in the band between $\mathcal{I}$ and $L_1$ and $y$ lies in the band between $\mathcal{I}$ and $L_2$.
- The point $x$ lies in the upper half plane with boundary $L_1$ and $y$ lies in the lower half plane with boundary $L_2$.
- The point $x$ lies in the band between $\mathcal{I}$ and $L_1$ and $y$ lies in the lower half plane with boundary $L_2$.
- The point $x$ lies in the upper half plane with boundary $L_1$ and $y$ lies in the band between $\mathcal{I}$ and $L_2$. 
If \( x \) belongs to the band in between \( I \) and \( L_1 \) by virtue of Corollary 7.4, \( \gamma \) contains a point in the upper half plane bounded by the line \( L_1 \). The same argument applies when if \( y \) belongs to the band in between \( I \) and \( L_2 \). Therefore we have that \( \text{diam}(\gamma) > 4 \) implying that \( \gamma \) is a path not in \( \Omega \).

The case when \( x \) lies in the upper half plane bounded by \( L_1 \) and \( y \) lies in the lower half plane bounded by \( L_2 \) trivially implies that \( \text{diam}(\gamma) > 4 \), therefore \( \gamma \) is a path not in \( \Omega \). The proof of third and fourth statements involve just a combination of the first and second statements.

The validity of the \( S \)-Theorem for the three remaining configurations for the position of the returning point at \( \partial \mathcal{R}_1 \) or \( \partial \mathcal{R}_3 \) is proven using an identical argument as above. Note that the different signs of the curvature at \( \gamma(t) \) induce different arrangements for the quadrants. □

**Remark 8.11.** When \( \theta_2 > \frac{\pi}{2} \), (\( \theta_4 > \frac{\pi}{2} \)) the returning point may lie on \( \partial \mathcal{R}_2 \) in an extension of an arc of \( \partial \mathcal{R}_1 \), (\( \partial \mathcal{R}_3 \)). In any such cases, Theorem 8.10 also applies.

**Theorem 8.12.** Paths in \( \Delta(\Omega) \) have bounded length.

**Proof.** Let \( \gamma \in \Delta(\Omega) \) be a path of arbitrarily large length. Suppose \( \gamma \) does not admits a negative direction. Since \( \Omega \subset \mathbb{R}^2 \) is bounded and \( \gamma \) is a path of arbitrary large length that only travels forward, we have that \( \gamma \) must leave \( \Omega \) leading to a contradiction. Suppose \( \gamma \) admits a negative direction, since the turning map \( \tau : I \to \mathbb{R} \) is a continuous function defined on a compact domain, \( \tau \) admits a maximal inflection point. As a consequence of Lemma 8.9, the trajectory of \( \gamma \) after the maximal inflection point (see Figure 11):

- Lies in between the lines \( I \) and \( L_2 \) traveling forward.
- Turns back and lies in between the lines \( L_1 \) and \( L_2 \).
- Crosses \( L_2 \).
- Turns back and crosses \( L_1 \) twice.

Here the inflection tangent \( I \) and \( L_1 \), \( L_2 \) are parallel lines tangent to unit radius circles. Since the length of \( \gamma \) is chosen to be arbitrarily large, is easy to see that under the possible trajectories of \( \gamma \) after the maximal inflection point a diameter bigger than 4 is always achieved. The first two cases run with an analogous argument as applied to paths that only travel forward (see Figure 11 top). The third case immediately implies \( \text{diam}(\gamma) > 4 \), the last case implies the existence of parallel tangents, by Theorem 6.3 \( \gamma \) is a free path and by Theorem 9.1 these are paths not in \( \Omega \) (see Figure 11 bottom). Since the possible unbounded length paths always have diameter bigger than 4, by Theorem 5.9 the result follows. □

9. **Non-Trivial Homotopy classes of Bounded Curvature Paths**

As we previously observed, the critical configurations for the endpoint condition are given when proximity condition \( D \) is satisfied. A compact planar region \( \Omega \) is formed and its shape depends on the endpoint condition. In addition, recall that the boundary of \( \Omega \) is formed by two bounded curvature paths. In the previous sections, we proved results on the behavior of bounded curvature paths in \( \Omega \) via arguments concerning the way a bounded curvature path may touch the boundary of \( \Omega \). In this section we put together the previous ideas to prove the main results of this work.
Figure 11. Possible trajectories for $\gamma$ after the maximal inflection point $\gamma(t)$. By Lemma 8.9, after the maximal inflection point, the path $\gamma$ does not cross the inflection tangent $I$ before intersecting $N$ (compare Figure 9).

**Theorem 9.1.** Embedded bounded curvature paths in $\Omega$ do not admit boundary tangent points.

**Proof.** By virtue of Theorem 7.7 bounded curvature paths do not admit boundary tangent points with $\partial R_1$ before entering $R_2$ and $\partial R_3$ after leaving $R_2$ for the final time. Due to the S-Theorem, embedded bounded curvature paths do not admit boundary tangent points at $\partial R_1$. By a symmetrical argument, embedded bounded curvature paths do not admit boundary tangent points in $\partial R_3$. By Theorem 7.8 bounded curvature paths do not admit boundary tangent points with $\partial R_2$. Since $\partial \Omega$ is conformed by $\partial R_1$, $\partial R_2$ and $\partial R_3$, the result follows.

**Corollary 9.2.** Suppose the endpoint condition $x,y \in T\mathbb{R}^2$ carries a region $\Omega$. Then the space $\Gamma(x,y)$ is partitioned into the spaces $\Delta(\Omega)$ and $\Delta'(\Omega)$. That is, $\Delta(\Omega)$ and $\Delta'(\Omega)$ belong to different homotopy classes. In particular the elements in $\Delta(\Omega)$ are not free paths.

**Proof.** Consider a path $\gamma \in \Delta(\Omega)$. Suppose there exists a bounded curvature path $\delta \in \Delta'(\Omega)$ together with a bounded curvature homotopy $H_t : [0,1] \rightarrow \Gamma(x,y)$ such that $H_t(0) = \gamma$ and $H_t(1) = \delta$. So, there exists $p \in I$ such that $H_t(p)$ has a boundary tangent point contradicting via Theorem 9.1 the continuity of $H_t$. By Theorem 8.12 the elements in $\Delta(\Omega)$ have bounded length and therefore are not free paths.
By the last result we have that bounded curvature paths in $\Delta(\Omega)$ cannot be made bounded-homotopic to bounded curvature paths in $\Delta'(\Omega)$. In particular, the spaces $\Gamma(x,y)$ satisfying proximity condition $D$ are not path connected implying immediately the existence of at least two different homotopy classes in $\Gamma(x,y)$ when $D$ is satisfied.

The following result motivates why in Definition 3.8 condition $(iv)$ is not sufficient for the formation of non-trivial homotopy classes in $\Gamma(x,y)$. In addition, the next result shows why non-trivial homotopy classes in $\Gamma(x,y)$ must not admit paths containing arcs of unit circles of length greater than or equal to $\pi$ as sub paths.

**Corollary 9.3.** Paths in $\Delta(\Omega)$ do not admit parallel tangents. In particular they do not admit an arc of a unit circle of length greater than or equal to $\pi$ as a sub path.

**Proof.** This is an immediate consequence of Proposition 6.3 and Theorem 9.1. □

**Theorem 9.4.** Bounded curvature paths with self intersections are free paths.

**Proof.** Consider a bounded curvature path having a self intersection at $\gamma(t_1) = \gamma(t_2)$ with $t_1 < t_2$. Consider $\gamma(t_1 + \delta)$ and $\gamma(t_2 - \delta)$ for sufficiently small $\delta > 0$. Consider a unit disc containing in its boundary $\gamma(t_1 + \delta)$ and $\gamma(t_2 - \delta)$. By Corollary 7.4 $\gamma$ must contains a pair of parallel tangents and by Proposition 6.3 $\gamma$ is a free path. □

It is easy to see that there exist bounded curvature paths with self intersection in $\Omega$. By virtue of Theorem 9.4 these are free paths.

**Corollary 9.5.** Paths in $\Delta(\Omega)$ cannot be made bounded-homotopic to a bounded curvature paths with self intersections.

**Proof.** Consider a bounded curvature path $\gamma \in \Delta(\Omega)$ and a homotopy of bounded curvature paths $H_t$ such that $H_t(0) = \gamma$, and $H_t(1)$ is a path in $\Omega$ having a self intersection. Due to the continuity of homotopies, there exists $r \in I$ such that $H_t(r) = \sigma$ is the first path admitting a self intersection in $H_t$. By Theorem 9.4 we have that $\sigma$ is a free path. Therefore, by the continuity of $H_t$, the path $\gamma$ is also a free path, contradicting Theorem 9.1. □

### 10. An Application to Motion Planning

Homotopies of bounded curvature paths are crucial in the case where reaching a prescribed lower bound for the length of a bounded curvature path with given endpoint condition is needed. In the decline design of an underground mine, the mine is considered as a 3-dimensional network (strategic locations in the mine are represented as directed nodes with links establishing connections between such locations). The directed nodes represent the surface portal, access points, and draw points, and the links correspond to the centerlines of ramps and drives. Additional restrictions on the topology of the 3-dimensional network are manifested via navigability conditions of bounded turning radius for vehicles and the maximum gradient that the ramps may allow.

The Melbourne University research group on underground mine optimization led by Professors Doreen Thomas and Hyam Rubinstein is mainly focused on the investigation of how to optimize the network of declines; that is, the systems of ramps and drives which satisfy operational gradient and curvature constraints so that the cost of construction and haulage over the life of the mine is minimized.
The work in [10] gives an answer to the problem of finding minimal paths of bounded curvature in a 3-dimensional space with given initial and final directions; however, the gradient constraint is not taken into consideration. The approach of minimizing the cost of the links corresponds to considering the projected problem in the horizontal plane. A planar path can be lifted into the 3-dimensional space while keeping a uniform gradient. The lifted path will satisfy the gradient constraint if and only if the length of the planar path reaches a lower bound dependent on the vertical displacement between the end points of the link. If the length of the minimum length path is less than the given lower bound, we can attempt to extend the path to reach the required length. If the projected directed nodes $x,y \in T\mathbb{R}^2$ are such that a region $\Omega$ is obtained, then by virtue of Theorem 9.1 we have that such paths are trapped in $\Omega$, so the desired lower bound for the length of the paths between the projected nodes may not be achieved.

**Figure 12.** An underground mine design.

As an immediate consequence of Corollary 9.2 and in the context of the problem described in the previous paragraphs we have the next result.

**Corollary 10.1.** Suppose $x,y \in T\mathbb{R}^2$ satisfies proximity condition $D$. Then the minimal length element in $\Gamma(x,y)$ is not bounded-homotopic to a path of arbitrary large length.

The algorithm implemented in DOT assumes the validity of Corollary 10.1. So that, in the case when the projected nodes satisfy condition $D$ a region $\Omega$ is obtained. The algorithm searches for the second shortest Dubins path between the given endpoint condition. In this fashion the algorithm in DOT becomes optimal.

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