THE $C^*$-ALGEBRAS $qA \otimes K$ AND $S^2A \otimes K$ ARE ASYMPTOTICALLY EQUIVALENT

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Abstract. Let $A$ be a separable $C^*$-algebra. We prove that its stabilized second suspension $S^2A \otimes K$ and the $C^*$-algebra $qA \otimes K$ constructed by Cuntz in the framework of his picture of $KK$-theory are asymptotically equivalent. This means that there exists an asymptotic morphism from $S^2A \otimes K$ to $qA \otimes K$ and an asymptotic morphism from $qA \otimes K$ to $S^2A \otimes K$ whose compositions are homotopic to the identity maps. This result yields an easy description of the natural transformation from $KK$-theory to $E$-theory. Also by Loring’s result any asymptotic morphism from $qC$ to any $C^*$-algebra $B$ is homotopic to a $*$-homomorphism. We prove that the same is true when $C$ is replaced by any nuclear $C^*$-algebra $A$ and when $B$ is stable.

Introduction

Let $A$ be a separable $C^*$-algebra. Its first suspension is the $C^*$-algebra $SA = C_0(\mathbb{R}) \otimes A$. There are two other $C^*$-algebras associated to $A$ that are of importance in $KK$-theory of Kasparov: the second suspension $C^*$-algebra $S^2A = C_0(\mathbb{R}^2) \otimes A$ and the $C^*$-algebra $qA$ constructed by Cuntz [1] in the framework of his picture of $KK$-theory. Both $C^*$-algebras can replace $A$ in the definition of the $KK$-groups: for the second suspension this is Bott periodicity and for $qA$ this is Cuntz’s picture for $KK$-theory. These $C^*$-algebras are $E$-equivalent, i.e. their stabilized suspensions $S^3A \otimes K$ and $SqA \otimes K$ are equivalent in the category of separable $C^*$-algebras with morphisms being homotopy classes of asymptotic morphisms, where $K$ denotes the $C^*$-algebra of compact operators. In the present paper we show that they are equivalent in this category without taking the suspension of the stabilizations. More precisely we construct an asymptotic morphism from $S^2A \otimes K$ to $qA \otimes K$ and a $*$-homomorphism from $qA \otimes K$ to $S^2A \otimes K$ such that their compositions are homotopic to the identity maps. In general one says that two $C^*$-algebras are asymptotically equivalent if there exist asymptotic morphisms from each to the other whose compositions are homotopic to the identity maps. So the main result of this paper (Theorem 12) says that $C^*$-algebras $qA \otimes K$ and $S^2A \otimes K$ are asymptotically equivalent.

As a corollary (Corollary 13) we obtain a description of $E$-theory that is similar in form to Cuntz’s description of $KK$-theory. Cuntz ([1]) proved that $KK(A, B) = [qA, B \otimes K]$ (where [] means homotopy classes of $*$-homomorphisms). We assert that $E(A, B) = [[qA, B \otimes K]]$ (where [[]] means homotopy classes of asymptotic morphisms) and that the well known natural transformation $KK(A, B) \rightarrow E(A, B)$ is then nothing but the map that sends any $*$-homomorphism $qA \rightarrow B \otimes K$ to itself.

One more corollary (Corollary 14) concerns the question of when asymptotic morphisms are homotopic to $*$-homomorphisms. In [3] it was proved that any asymptotic morphism from $qC$ to any $C^*$-algebra $B$ is homotopic to a $*$-homomorphism. We prove that the same is true not only for $C$ but for any nuclear (even K-nuclear) $C^*$-algebra $A$ if $B$ is assumed to be stable. Recall that a $C^*$-algebra $B$ is called stable if $B \otimes K \cong B$.

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The plan of the paper is as follows. The first section contains all necessary information about $C^*$-algebra $qA$. In the second one we construct an asymptotic morphism $f^A : S^2A \otimes \mathcal{K} \to qA \otimes \mathcal{K}$ and a $*$-homomorphism $g^A : qA \otimes \mathcal{K} \to S^2A \otimes \mathcal{K}$ and show that $f^A$ induces a natural transformation from the $KK$-functor the to $E$-functor. In the third section we prove that $f^A$ and $g^A$ provide an asymptotic equivalence of the $C^*$-algebras $S^2A \otimes \mathcal{K}$ and $qA \otimes \mathcal{K}$ and obtain the corollaries described above.

1. Necessary Information about $qA$

Let $A$ and $B$ be two $C^*$-algebras. A $C^*$-algebra $C$ is called the free product of $A$ and $B$ if there are $*$-homomorphisms $i^A : A \to C$ and $i^B : B \to C$ with the following (universal) property: given $*$-homomorphisms $\phi_A : A \to D$ and $\phi_B : B \to D$ mapping $A$ and $B$ into the same $C^*$-algebra $D$, there is a unique $*$-homomorphism $\phi : C \to D$ such that $\phi \circ i^A = \phi_A$ and $\phi \circ i^B = \phi_B$. The $*$-homomorphisms $i^A$ and $i^B$ are referred to as the canonical inclusions. The free product of $A$ and $B$ will be denoted by $A * B$.

Consider $A * A$. Let $i^A_1 : A \to A * A$ and $i^A_2 : A \to A * A$ denote the two canonical inclusions of $A$ as a $C^*$-subalgebra of $A * A$. The $C^*$-algebra $qA$ constructed by Cuntz ([1]) is the closed ideal in $A * A$ generated by the set \( \{ i_1(x) - i_2(x) : x \in A \} \). One can prove that elements of the form

\[
(i^A_1(x_1) - i^A_2(x_1)) \ldots (i^A_1(x_N) - i^A_2(x_N))
\]

and

\[
\frac{i^A_1(x_1) - i^A_2(x_1)}{i^A_1(x_2) - i^A_2(x_2)} \ldots \frac{i^A_1(x_N) - i^A_2(x_N)}{i^A_1(x_{N-1}) - i^A_2(x_{N-1})},
\]

where $x_0, x_1, \ldots, x_N \in A$, $N \in \mathbb{N}$, span a dense $*$-subalgebra in $qA$.

Let $\phi, \psi : A \to B$ be two $*$-homomorphisms. By the universal property of $A * A$ there is a unique $*$-homomorphism $Q(\phi, \psi) : A * A \to B$ such that

\[
Q(\phi, \psi) \circ i^A_1 = \phi, \quad Q(\phi, \psi) \circ i^A_2 = \psi.
\]

Let $q(\phi, \psi)$ denote the restriction of $Q(\phi, \psi)$ to $qA$. Note that if $J$ is an ideal in $B$, then $Q(\phi, \psi)$ maps $qA$ into $J$ if and only if $\phi(x) - \psi(x) \in J$ for all $x \in A$. So in this case, $q(\phi, \psi) \in Hom(qA, J)$.

2. Constructing the Asymptotic Equivalence between $S^2A \otimes \mathcal{K}$ and $qA \otimes \mathcal{K}$

Below all $C^*$-algebras are assumed to be separable.

For any two $C^*$-algebras $A$ and $B$ Connes and Higson define $E(A, B)$ to be the abelian group $[[S^2A \otimes \mathcal{K}, SB \otimes \mathcal{K}]]$ of homotopy classes of asymptotic morphisms from $SA \otimes \mathcal{K}$ to $SB \otimes \mathcal{K}$ ([2]). Recall that an asymptotic morphism from $A$ to $B$ is a family of maps $(\phi_t)_{t \in [0, \infty)} : A \to B$ satisfying the following conditions:

i) for any $a \in A$ the function $t \mapsto \phi_t(a)$ is continuous;

ii) for any $a, b \in A$, $\lambda \in \mathbb{C}$

- $\lim_{t \to \infty} \| \phi_t(a^*) - \phi_t(a)^* \| = 0$
- $\lim_{t \to \infty} \| \phi_t(a + \lambda b) - \phi_t(a) - \lambda \phi_t(b) \| = 0$;
- $\lim_{t \to \infty} \| \phi_t(ab) - \phi_t(a) \phi_t(b) \| = 0$.

In [2] it was also shown that $[[S^2A \otimes \mathcal{K}, SB \otimes \mathcal{K}]] \cong [[S^2A \otimes \mathcal{K}, B \otimes \mathcal{K}]]$ and we shall always mean by the $E$-group the group $[[S^2A \otimes \mathcal{K}, B \otimes \mathcal{K}]]$ of homotopy classes of asymptotic morphisms from $S^2A \otimes \mathcal{K}$ to $B \otimes \mathcal{K}$.

Let $\beta^C : C_0(\mathbb{R}^2) \otimes \mathcal{K} \to \mathcal{K}$ be the Bott asymptotic morphism. In fact it is the tensor product of the identity map $id_\mathcal{K} : \mathcal{K} \to \mathcal{K}$ with the restriction to $C_0(\mathbb{R}^2) \subset C(\mathbb{T}^2)$ of the family of maps from $C(\mathbb{T}^2)$ to $\mathcal{K}^+$ constructed in the Voiculescu’s example of almost
commuting unitaries ([7]), but here we shall not use an explicit form of $\beta^C$ but only the fact that it induces the identity map in the K-groups. Let

$$\beta^A = \beta^C \otimes id_A : S^2A \otimes K \to A \otimes K.$$ 

Obviously $\beta^A \in E(A,A)$. Note that since we always consider asymptotic morphisms up to homotopy we denote in the same way a class of homotopy equivalent asymptotic morphisms and any its representative.

For the KK-groups we will use Cuntz's approach ([1]) in which, as already was written, one regards $KK(A,B)$ as the group $[qA \otimes K, B \otimes K]$ of homotopy classes of $*$-homomorphisms from $qA \otimes K$ to $B \otimes K$. Let

$$\gamma^A = q(id_A,0) \otimes id_K : qA \otimes K \to A \otimes K.$$ 

Then $\gamma^A \in KK(A,A)$ and it is a unit element for the associative product $KK(A,B) \times KK(B,C) \to KK(A,C)$. Namely there exists a bilinear pairing $KK(A,B) \times KK(B,C) \to KK(A,C)$ such that $x \times \gamma^B = x = \gamma^A \times x$ for any $x \in KK(A,B)$ ([3]).

Let $A$ be a $C^*$-algebra. By [2] there exists a natural transformation from the functor $KK(A,-)$ into the functor $E(A,-)$ which is unique up to its value on $\gamma^A \in KK(A,A)$. Let

$$I_{A,B} : KK(A,B) \to E(A,B)$$

be such a natural transformation that $I_{A,A}(\gamma^A) = \beta^A$. Define an asymptotic morphism $f^A : S^2A \otimes K \to qA \otimes K$ by

$$f^A = I_{A,qA}(id_{qA \otimes K}).$$

The following easy theorem asserts that the asymptotic morphism $f^A$ induces the natural transformation $I_{A,B}$.

**Theorem 1.** $I_{A,B}(\phi) = \phi \circ f^A$ for any $\phi \in KK(A,B)$.

**Proof.** Since $\phi \in KK(A,B)$ is a $*$-homomorphism from $qA \otimes K$ to $B \otimes K$ it induces the maps $\phi_{KK} : KK(A,qA) \to KK(A,B)$ and $\phi_E : E(A,qA) \to E(A,B)$ in the KK-groups and the E-groups respectively. By the definition of a natural transformation of covariant functors the following diagram commutes

$$\begin{array}{ccc}
KK(A,B) & \xrightarrow{I_{A,B}} & E(A,B) \\
\phi_{KK} \uparrow & & \uparrow \phi_E \\
KK(A,qA) & \xrightarrow{I_{A,qA}} & E(A,qA)
\end{array}$$

Hence for the element $id_{qA \otimes K} \in KK(A,qA)$ we get

$$\phi_E(I_{A,qA}(id_{qA \otimes K})) = I_{A,B}(\phi_{KK}(id_{qA \otimes K})).$$

But $\phi_E(I_{A,qA}(id_{qA \otimes K})) = \phi \circ I_{A,qA}(id_{qA \otimes K}) = \phi \circ f^A$ and $I_{A,B}(\phi_{KK}(id_{qA \otimes K})) = I_{A,B}(\phi \circ id_{qA \otimes K}) = I_{A,B}(\phi).$\hfill $\square$

**Corollary 2.** $\gamma^A \circ f^A = \beta^A$.

**Proof.** By Theorem 1 $\gamma^A \circ f^A = I_{A,A}(\gamma^A)$. Since we have chosen a natural transformation to be equal $\beta^A$ on the element $\gamma^A$ we get $\gamma^A \circ f^A = \beta^A$.\hfill $\square$

Now we define a $*$-homomorphism $g^A : qA \otimes K \to S^2A \otimes K$ in the following way. Let $\pi_1, \pi_2 : \mathbb{C} \to C_0(\mathbb{R}^2)^+ \otimes M_2$ be two $*$-homomorphisms given by

$$\pi_1(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi_2(1) = p_{\text{Bott}} = \frac{1}{1 + zz} \begin{pmatrix} z & z \\ \bar{z} & 1 \end{pmatrix}.$$
(we identify \( \mathbb{R}^2 \) with \( \mathbb{C} \)). Fix once and for all some inclusion \( j : M_2 \rightarrow K \) and some isomorphism \( i : K \otimes K \rightarrow K \). Define \( \tilde{\pi}_1, \tilde{\pi}_2 : A \rightarrow A \otimes C_0(\mathbb{R}^2)^+ \otimes K \) by

\[
\tilde{\pi}_1 = (j \otimes id_{A \otimes C_0(\mathbb{R}^2)^+}) \circ (id_A \otimes \pi_1), \quad \tilde{\pi}_2 = (j \otimes id_{A \otimes C_0(\mathbb{R}^2)^+}) \circ (id_A \otimes \pi_2)
\]

respectively. Since

\[
\tilde{\pi}_1(a) - \tilde{\pi}_2(a) \in C_0(\mathbb{R}^2) \otimes K \otimes A = S^2 A \otimes K
\]

for any \( a \in A \), the \(*\)-homomorphism \( q(\tilde{\pi}_1, \tilde{\pi}_2) : qA \rightarrow S^2 A \otimes K \) is defined.

Set

\[
g^A = (id_{S^2 A} \otimes i) \circ (q(\tilde{\pi}_1, \tilde{\pi}_2) \otimes id_K).
\]

In the next section we show that \( f^A \) and \( g^A \) provide an asymptotic equivalence between \( S^2 A \otimes K \) and \( qA \otimes K \).

### 3. Proof of the main assertion

To prove that \( f^A \) and \( g^A \) provide an asymptotic equivalence between \( S^2 A \otimes K \) and \( qA \otimes K \) we are going to show that their compositions induce the identity maps in \( E\)-functor and in the functor \( G \) that will be introduced in subsection 3.2.

#### 3.1. The maps induced by \( f^A \) and \( g^A \) in \( E\)-functor.

**Lemma 3.** \( \beta^A \circ g^A \sim \gamma^A \).

**Proof.** Note first of all that \( g^A : qA \otimes K \rightarrow S^2 A \otimes K \) and \( \gamma^A : qA \otimes K \rightarrow A \otimes K \) factorize through the \( C^*\)-algebra \( q \mathbb{C} \otimes A \otimes K \). Namely let \( \eta_1, \eta_2 : A \rightarrow (q \mathbb{C} \otimes A) \otimes A \) be given by formulas

\[
\eta_1(a) = i_1^C(1) \otimes a, \quad \eta_2(a) = i_2^C(1) \otimes a
\]

for any \( a \in A \). Set

\[
s^A = q(\eta_1, \eta_2) : qA \rightarrow q \mathbb{C} \otimes A.
\]

It is easy to see that the diagrams

\[
\begin{array}{ccc}
qA \otimes K & \xrightarrow{\gamma^A} & A \otimes K \\
\downarrow{s^A \otimes id_K} & & \downarrow{\gamma^A \otimes id_A} \\
q \mathbb{C} \otimes A \otimes K & & \\
\end{array}
\]

and

\[
\begin{array}{ccc}
qA \otimes K & \xrightarrow{g^A} & S^2 A \otimes K \\
\downarrow{s^A \otimes id_K} & & \downarrow{g^A \otimes id_A} \\
q \mathbb{C} \otimes A \otimes K & & \\
\end{array}
\]

commute, that is

\[
\gamma^A = (\gamma^A \otimes id_A) \circ (s^A \otimes id_K), \quad g^A = (g^A \otimes id_A) \circ (s^A \otimes id_K).
\]

Since \( \beta^A = \beta^C \otimes id_A \) we have to establish the homotopy equivalence

\[
(\gamma^A \otimes id_A) \circ (s^A \otimes id_K) \sim (\beta^C \otimes id_A) \circ (g^C \otimes id_A) \circ (s^A \otimes id_K)
\]

or, equivalently,

\[
\gamma^C \sim \beta^C \circ g^C.
\]

For that we use K-theory. Let \( \gamma^C_* \) and \( (\beta^C_* \circ g^C)_* \) be the induced homomorphisms from \( K_0(q \mathbb{C}) \) to \( K_0(\mathbb{C}) \). For the generator \([i_1^C(1)] - [i_2^C(1)]\) of \( K_0(q \mathbb{C}) \) we have

\[
(\beta^C_* \circ g^C)_*[i_1^C(1)] - [i_2^C(1)] = \beta^C_*([[(1 0)]] - [p_{\text{Bott}}]) = [1],
\]
homotopic because, by Universal coefficients theorem, homomorphisms in K-theory. This implies that these asymptotic homomorphisms are induces the identity homomorphism in K-theory. So

\[ \gamma \]

We used here that \( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - [p_{Bott}] \) is a generator of \( K_0(S^2\mathbb{C}) \) and that the Bott map \( \beta^C \) induces the identity homomorphism in K-theory. So \( \gamma^C \) and \( \beta^C \circ \gamma^C \) induce the same homomorphisms in K-theory. This implies that these asymptotic homomorphisms are homotopic because, by Universal coefficients theorem,

\[ Hom(K_0(q\mathbb{C}), K_0(\mathbb{C})) \oplus Hom(K_1(q\mathbb{C}), K_1(\mathbb{C})) \cong KK(q\mathbb{C}, \mathbb{C}) \oplus KK(Sq\mathbb{C}, \mathbb{C}), \]

and since

\[ K_1(q\mathbb{C}) = K_1(\mathbb{C}) = 0, \quad KK(Sq\mathbb{C}, \mathbb{C}) = 0, \quad KK(q\mathbb{C}, \mathbb{C}) = [q\mathbb{C}, \mathbb{C}] \cong [[q\mathbb{C}, \mathbb{C}]] \]

we get

\[ Hom(K_0(q\mathbb{C}), K_0(\mathbb{C})) \cong [[q\mathbb{C} \otimes \mathbb{C}, \mathbb{C}]]. \]

\[ \square \]

Let \( B \) be any \( C^* \)-algebra. Let \( f_E^A : E(B, S^2A) \to E(B, qA) \) and \( g_E^A : E(B, qA) \to E(B, S^2A) \) be the maps induced by \( f^A \) and \( g^A \) respectively.

**Proposition 4.** \( f_E^A \circ g_E^A = id, \)

\[ g_E^A \circ f_E^A = id. \]

Here \( id \) means both the identity map from \( E(B, S^2A) \) into itself and the identity map from \( E(B, qA) \) into itself.

**Proof.** Consider the following diagram

\[
\begin{array}{c}
E(B, A) \\
\beta_E^A \downarrow \quad \quad \quad g_E^A \downarrow \quad \quad \quad \gamma_E^A \\
E(B, S^2A) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
An asymptotic morphism $\psi$ gives rise to a genuine $*$-homomorphism

$$F : D_1 \to C_0([0, \infty), D_2)/C_0([0, \infty), D_2)$$

given by

$$F(x) = \psi_t(x) + C_0([0, \infty), D_2)$$

for any $x \in D_1$. There are two $*$-homomorphisms $\overline{i_1, i_2} : C_0([0, \infty), D_2) \to C_b([0, \infty), D_2 \ast D_2)$ given by formulas

$$\overline{i_1}(f)(t) = i_1^{D_2}(f(t)), \quad \overline{i_2}(f)(t) = i_2^{D_2}(f(t)),$$

$f \in C_b([0, \infty), D_2)$. Since these $*$-homomorphisms send $C_0([0, \infty), D_2)$ to $C_0([0, \infty), D_2 \ast D_2)$ we have two $*$-homomorphisms

$$\hat{i_1, i_2} : C_b([0, \infty), D_2)/C_0([0, \infty), D_2) \to C_b([0, \infty), D_2 \ast D_2)/C_0([0, \infty), D_2 \ast D_2).$$

Set

$$\Phi = Q(\hat{i_1} \circ F, \hat{i_2} \circ F) : D_1 \ast D_1 \to C_b([0, \infty), D_2 \ast D_2)/C_0([0, \infty), D_2 \ast D_2).$$

Let $p : C_b([0, \infty), D_2 \ast D_2) \to C_b([0, \infty), D_2 \ast D_2)/C_0([0, \infty), D_2 \ast D_2)$ be the canonical surjection. Since

$$\Phi(i_1^{D_1}(a)) = p(i_1^{D_2}(\psi_t(a))), \quad \Phi(i_2^{D_1}(a)) = p(i_2^{D_2}(\psi_t(a)))$$

for any $a \in D_1$, and since $qD_1$ is the closed ideal generated by the set $\{i_1^{D_1}(a) - i_2^{D_1}(a) : a \in D_1\}$, we get

$$\Phi(qD_1) \subset p(C_b([0, \infty), qD_2)).$$

We shall denote the restriction of $\Phi$ to $qD_1$ also by $\Phi$. Define a $*$-homomorphism

$$\tau : p(C_b([0, \infty), qD_2)) \to C_b([0, \infty), qD_2)/C_0([0, \infty), qD_2)$$

by

$$\tau(p(f)) = f + C_0([0, \infty), qD_2),$$

$f \in C_b([0, \infty), qD_2)$. It is well-defined because for any $f \in C_b([0, \infty), qD_2)$ the condition $f \in C_b([0, \infty), D_2 \ast D_2)$ implies $f \in C_0([0, \infty), qD_2)$. So we have $\tau \circ \Phi : qD_1 \to C_b([0, \infty), qD_2)/C_0([0, \infty), qD_2)$. Choose a continuous section

$$s : C_b([0, \infty), qD_2)/C_0([0, \infty), qD_2) \to C_b([0, \infty), qD_2)$$

(it exists by Bartle-Graves theorem, [3] [4]) and define an asymptotic morphism $q\psi$ by

$$(q\psi)_t(x) = (s(\tau \circ \Phi(x)))(t).$$

Thus we get an asymptotic morphism $q\psi : qD_1 \to qD_2$ out of an asymptotic morphism $\psi : D_1 \to D_2$.

For any $C^*$-algebra $D$ let

$$\rho^D = q(i_1^{D} \otimes id_K, i_2^{D} \otimes id_K) : q(D \otimes K) \to qD \otimes K$$

and let $\theta_D : qD \otimes K \to q^2D \otimes K$ denote the isomorphism constructed in [1].

**Lemma 5.** The diagram

$$\begin{align*}
q(A \otimes K) \otimes K & \xrightarrow{\gamma_{A \otimes K} \otimes K} A \otimes K \otimes K \\
& \xrightarrow{id_A \otimes i} A \otimes K
\end{align*}$$

\[\rho^A \otimes id_K\]

$$\begin{align*}
qA \otimes K \otimes K & \xrightarrow{id_q \otimes i} qA \otimes K
\end{align*}$$

\[\gamma^A\]
is commutative, namely \( \gamma^A \circ (id_A \otimes i) \circ (\rho^A \otimes id_K) = (id_A \otimes i) \circ \gamma^{A \otimes K} \).

**Proof.** Since elements of the form

\[
(i_1^{A \otimes K}(a \otimes T) - i_2^{A \otimes K}(a \otimes T)) \otimes S
\]

and

\[
(i_1^{A \otimes K}(a_0 \otimes T_0) \, (i_1^{A \otimes K}(a \otimes T) - i_2^{A \otimes K}(a \otimes T))) \otimes S,
\]

where \( T, S, T_0 \in K, a, a_0 \in A \), span a dense subspace of \( q(A \otimes K) \otimes K \) (see [5], for example) it is enough to check that \( \gamma^A \circ (id_A \otimes i) \circ (\rho^A \otimes id_K) \) and \( (id_A \otimes i) \circ \gamma^{A \otimes K} \) coincide on elements of such form. For any \( T, S, T_0 \in K, a, a_0 \in A \) we have

\[
\gamma^A \circ (id_A \otimes i) \circ (\rho^A \otimes id_K) \left( (i_1^{A \otimes K}(a \otimes T) - i_2^{A \otimes K}(a \otimes T)) \otimes S \right) = \quad a \otimes i(T \otimes S) = \quad (id_A \otimes i) \circ \gamma^{A \otimes K} \left( (i_1^{A \otimes K}(a \otimes T) - i_2^{A \otimes K}(a \otimes T)) \otimes S \right),
\]

for another pair \( T_0 \in K, a_0 \in A \) we have

\[
\gamma^A \circ (id_A \otimes i) \circ (\rho^A \otimes id_K) \left( (i_1^{A \otimes K}(a_0 \otimes T_0) \, (i_1^{A \otimes K}(a \otimes T) - i_2^{A \otimes K}(a \otimes T))) \otimes S \right) = \quad a_0a \otimes i(T_0 T \otimes S) = \quad (id_A \otimes i) \circ \gamma^{A \otimes K} \left( (i_1^{A \otimes K}(a_0 \otimes T_0) \, (i_1^{A \otimes K}(a \otimes T) - i_2^{A \otimes K}(a \otimes T))) \otimes S \right)
\]

and we are done. \( \square \)

**Lemma 6.** Let \( \phi \in [[qB, A \otimes K]] \). Then the diagram

\[
\begin{array}{ccc}
q^2 B \otimes K & \xrightarrow{q^2 \otimes id_K} & q(A \otimes K) \otimes K & \xrightarrow{\gamma^{A \otimes K}} & A \otimes K \otimes K \\
\gamma^B & \downarrow & \phi \otimes id_K & & \\
qB \otimes K & & & & \\
\end{array}
\]

commutes, that is \( \gamma^{A \otimes K} \circ (q \phi \otimes id_K) = (\phi \otimes id_K) \circ \gamma^B \).

**Proof.** Let \( x \in qB, T \in K \). By the definition of \( q \phi \) we have

\[
(q \phi)_t \left( i_1^B(x) - i_2^B(x) \right) - \left( i_1^{A \otimes K}(\phi_t(x)) - i_2^{A \otimes K}(\phi_t(x)) \right) \to 0
\]

when \( t \to \infty \). Hence

\[
\lim_{t \to \infty} \left[ \gamma^{A \otimes K} \circ ((q \phi)_t \otimes id_K) \left( \left( i_1^B(x) - i_2^B(x) \right) \otimes T \right) - \left( \phi_t \otimes id_K \right) \circ \gamma^B \left( \left( i_1^B(x) - i_2^B(x) \right) \otimes T \right) \right] =
\]

\[
\lim_{t \to \infty} \left[ \gamma^{A \otimes K} \left( i_1^{A \otimes K}(\phi_t(x)) - i_2^{A \otimes K}(\phi_t(x)) \right) \otimes T \right] \phi_t(x) \otimes T \right] = 0.
\]

In a similar way we find that \( \gamma^{A \otimes K} \circ (q \phi \otimes id_K) \) and \( (\phi \otimes id_K) \circ \gamma^B \) asymptotically agree on elements \( \left( i_1^B(x_0) \, \left( i_1^B(x) - i_2^B(x) \right) \right) \otimes T \) when \( x_0, x \in qB, T \in K \). Since elements of the form \( \left( i_1^B(x) - i_2^B(x) \right) \otimes T \) and \( \left( i_1^B(x_0) \, \left( i_1^B(x) - i_2^B(x) \right) \right) \otimes T \) span a dense subspace of \( qB \otimes K \) we see that \( \gamma^{A \otimes K} \circ (q \phi \otimes id_K) = (\phi \otimes id_K) \circ \gamma^B \). \( \square \)
Lemma 7. Let $\phi \in [[qB, qA \otimes K]]$. Then the diagram

$$
q^2B \xrightarrow{\phi_B} q(qA \otimes K) \xrightarrow{\rho^A} q^2A \otimes K \xrightarrow{\gamma^A} qA \otimes K
$$

is commutative, that is $\gamma^A \circ \rho^A \circ q\phi = \phi \circ q(id_{qB}, 0)$.

Proof. Let $x \in qB$, $t \in [0, \infty)$. Writing $\phi_t(x)$ in the form

$$
\phi_t(x) = \lim_{k \to \infty} \sum_{i=1}^{N_k} z_i^{(k)}(t) \otimes T_i^{(k)}(t),
$$

where $z_i^{(k)}(t) \in qA$, $T_i^{(k)}(t) \in K$, we get

$$
\lim_{t \to \infty} \left[ \gamma^A \circ \rho^A \circ (q\phi)_t \left( i_1^{qB}(x) - i_2^{qB}(x) \right) \right] = \lim_{t \to \infty} \left[ \gamma^A \circ \rho^A \left( i_1^{qA\otimes K}(\phi_t(x)) - i_2^{qA\otimes K}(\phi_t(x)) \right) \right] = \lim_{t \to \infty} \sum_{i=1}^{N_k} \gamma^A \left( i_1^{qA}(z_i^{(k)}) \otimes T_i^{(k)} - i_2^{qA}(z_i^{(k)}) \otimes T_i^{(k)} \right) = 0.
$$

In a similar way we find that $\gamma^A \circ \rho^A \circ q\phi$ and $\phi \circ q(id_{qB}, 0)$ asymptotically agree on elements $i_1^{qB}(x_0) \left( i_1^{qB}(x) - i_2^{qB}(x) \right)$, where $x_0, x \in qB$. Since elements of the form $i_1^{qB}(x) - i_2^{qB}(x)$ and $i_1^{qB}(x_0) \left( i_1^{qB}(x) - i_2^{qB}(x) \right)$ span a dense subspace of $qB$ we conclude that the asymptotic morphisms $\gamma^A \circ \rho^A \circ q\phi$ and $\phi \circ q(id_{qB}, 0)$ coincide. \qed

Let $\psi \in G(B, A)$. There is an asymptotic morphism $\phi : qB \to A \otimes K$ such that $(id_A \otimes i) \circ (\phi \otimes id_K) \sim \psi$ ([2]). Define an asymptotic morphism $\Gamma(\psi) \in G(B, qA)$ by the following composition

$$
qB \otimes K \xrightarrow{\theta_B} q^2B \otimes K \xrightarrow{q\phi \otimes id_K} q(A \otimes K) \otimes K \xrightarrow{\rho^A \otimes id_K} qA \otimes K \otimes K \xrightarrow{id_{qA} \otimes i} qA \otimes K.
$$

Thus a map $\Gamma : G(B, A) \to G(B, qA)$ is defined by formula

$$
\Gamma(\psi) = (id_{qA} \otimes i) \circ (\rho^A \otimes id_K) \circ (q\phi \otimes id_K) \circ \theta_B
$$

for any $\psi \in G(B, A)$.

Let $\gamma^A_G : G(B, qA) \to G(B, A)$ be the map induced by $\gamma^A$.

Proposition 8. $\Gamma : G(B, A) \to G(B, qA)$ is a semigroup isomorphism with inverse $\gamma^A_G$.

Proof. Obviously $\Gamma$ and $\gamma^A_G$ are semigroup homomorphisms so we have to check only the following:

(i) $\Gamma(\gamma^A_G(\psi)) \sim \psi$ for any $\psi \in G(B, qA)$,

(ii) $\gamma^A_G(\Gamma(\psi)) \sim \psi$ for any $\psi \in G(B, A)$.

(i): Let $\psi \in G(B, qA)$ and $\phi : qB \to qA \otimes K$ be such an asymptotic morphism that $(id_{qA} \otimes i) \circ (\phi \otimes id_K) \sim \psi$. Then

$$
\Gamma(\gamma^A_G(\psi)) = (id_{qA} \otimes i) \circ (\rho^A \otimes id_K) \circ (q(\gamma^A \circ \phi) \otimes id_K) \circ \theta_B = (id_{qA} \otimes i) \circ (\rho^A \otimes id_K) \circ (q\gamma^A \otimes id_K) \circ (q\phi \otimes id_K) \circ \theta_B,
$$

where $\psi$ is the asymptotic morphism $\phi \circ q(id_{qB}, 0)$. Thus

$$
\Gamma(\gamma^A_G(\psi)) \sim \psi
$$

for any $\psi \in G(B, qA)$.\qed
because clearly \( q(\gamma^A \circ \phi) \otimes id_K = (q\gamma^A \otimes id_K) \circ (q\phi \otimes id_K) \).

By (5, Lemma 5.1.11) \( \rho^A \circ q\gamma^A \sim \gamma^A \circ \rho^A \) and we have

\[
\Gamma(\gamma^A_G(\psi)) = (id_{qA} \otimes i) \circ (\gamma^A \otimes id_K) \circ (\rho^A \otimes id_K) \circ (q\phi \otimes id_K) \circ \theta_B \overset{\text{Lemma 7}}{=} (id_{qA} \otimes i) \circ (\phi \otimes id_K) \circ \gamma^B \circ \theta_B \sim \psi \circ \gamma^B \circ \theta_B. \tag{[1]}\]

(ii): Now let \( \psi \in G(B, A) \) and \( \phi : qB \to A \otimes K \) be such an asymptotic morphism that \( (id_A \otimes i) \circ (\phi \otimes id_K) \sim \psi \). Then

\[
\gamma^A_G(\Gamma(\psi)) = \gamma^A \circ (id_{qA} \otimes i) \circ (\rho^A \otimes id_K) \circ (q\phi \otimes id_K) \circ \theta_B \overset{\text{Lemma 5}}{=} (id_A \otimes i) \circ \gamma^{A \otimes K} \circ (q\phi \otimes id_K) \circ \theta_B \overset{\text{Lemma 6}}{=} (id_A \otimes i) \circ (\phi \otimes id_K) \circ \gamma^B \circ \theta_B \sim \psi \circ \gamma^B \circ \theta_B \sim \psi. \]

\[
\square
\]

**Lemma 9.** The diagram

\[
q(S^2A \otimes K) \otimes K \xrightarrow{q^A \otimes id_K} q(A \otimes K) \otimes K \xrightarrow{\rho^A \otimes id_K} qA \otimes K \otimes K
\]

\[
\gamma^{S^2A \otimes K}
\]

\[
S^2A \otimes K \otimes K \xrightarrow{id_{S^2A \otimes i}} qA \otimes K
\]

\[
\gamma^{S^2A \otimes K}
\]

\[
S^2A \otimes K \otimes K \xrightarrow{id_{S^2A \otimes i}} S^2A \otimes K
\]

commutes.

Namely \( q^A \circ (id_{qA} \otimes i) \circ (\rho^A \otimes id_K) \circ (q\beta^A \otimes id_K) \sim (id_{S^2A} \otimes i) \circ \gamma^{S^2A \otimes K} \).

**Proof.** We will prove the assertion by establishing the commutativity of the left and right triangles of the diagram

\[
q(S^2A \otimes K) \otimes K \xrightarrow{q^A \otimes id_K} q(A \otimes K) \otimes K \xrightarrow{\rho^A \otimes id_K} qA \otimes K \otimes K
\]

\[
\gamma^{S^2A \otimes K}
\]

\[
S^2A \otimes K \otimes K \xrightarrow{id_{S^2A \otimes i}} qA \otimes K
\]

\[
\gamma^{S^2A \otimes K}
\]

\[
S^2A \otimes K \otimes K \xrightarrow{id_{S^2A \otimes i}} S^2A \otimes K
\]

To prove the commutativity of the right triangle we have to prove

\[
g^A \circ (id_{qA} \otimes i) \sim (id_{S^2A} \otimes i) \circ (g^A \otimes id_K). \tag{3}
\]

Let \( h_1, h_2 : K \otimes K \otimes K \to K \) be the isomorphisms which send \( T_1 \otimes T_2 \otimes T_3 \) to \( i(T_1 \otimes i(T_2 \otimes T_3)) \) and \( i(i(A \otimes B) \otimes C) \) respectively for any operators \( T_1, T_2, T_3 \in K \). Then for any \( T, S \in K, a \in A \) we have

\[
(id_{S^2A} \otimes (h_2 \circ h_1^{-1})) \circ g^A \circ (id_{qA} \otimes i)\left((i^A_1(a) - i^A_2(a)) \otimes T \otimes S\right) = (id_{S^2A} \otimes (h_2 \circ h_1^{-1})) \left(a \otimes i\left(j\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \otimes i(T \otimes S)\right)\right) = a \otimes i\left(i\left(j\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \otimes T\right) \otimes S\right) = (id_{S^2A} \otimes i) \circ (g^A \otimes id_K)\left((i^A_1(a) - i^A_2(a)) \otimes T \otimes S\right).
\]
and for another $a_0 \in A$

\[
(id_{S^2 A} \otimes (h_2 \circ h_1^{-1})) \circ g^A \circ (id_{qA} \otimes i) \left( i^A_1(a_0) \left( i^A_1(a) - i^A_2(a) \right) \otimes T \otimes S \right) =
\]

\[
(id_{S^2 A} \otimes (h_2 \circ h_1^{-1})) \left( a_0 a \otimes i \left( j(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - p_{Bott}) \right) \otimes (T \otimes S) \right) =
\]

\[
a_0 a \otimes i \left( j(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - p_{Bott}) \otimes T \right) \otimes S
\]

\[
(id_{S^2 A} \otimes i) \circ (g^A \otimes id_K) \left( i^A_1(a_0) \left( i^A_1(a) - i^A_2(a) \right) \otimes T \otimes S \right).
\]

Since elements of the form

\[
(i^A_1(a) - i^A_2(a)) \otimes T \otimes S
\]

and

\[
i^A_1(a_0) \left( i^A_1(a) - i^A_2(a) \right) \otimes T \otimes S
\]

span a dense subspace of $qA \otimes K \otimes K$ we get

\[
(id_{S^2 A} \otimes (h_2 \circ h_1^{-1})) \circ g^A \circ (id_{qA} \otimes i) = (id_{S^2 A} \otimes i) \circ (g^A \otimes id_K).
\]

As well known any two isomorphisms from $K$ to itself are homotopic, hence $h_2 \circ h_1^{-1} \sim id_K$ and we obtain (3). Now to prove the commutativity of the left triangle of the diagram we have to prove that

\[
(g^A \otimes id_K) \circ (\rho^A \otimes id_K) \circ (q^A \otimes id_K) \sim \gamma^{S^2 A \otimes K}.
\]

Like in Lemma 3 we will reduce the general case to the case $A = \mathbb{C}$ using the map $s^A : qA \to q\mathbb{C} \otimes A$ that was introduced in the proof of Lemma 3. The right-hand side of (4) is

\[
\gamma^{S^2 A \otimes K} = (\gamma^C \otimes id_{S^2 A \otimes K}) \circ (s^{S^2 A \otimes K} \otimes id_K),
\]

that is the diagram

\[
\begin{array}{ccc}
q(S^2 A \otimes K) \otimes K & \xrightarrow{\gamma^{S^2 A \otimes K}} & S^2 A \otimes K \otimes K \\
\downarrow{s^{S^2 A \otimes K} \otimes id_K} & & \downarrow{\gamma^C \otimes id_{S^2 A \otimes K}} \\
q\mathbb{C} \otimes S^2 A \otimes K \otimes K & \xrightarrow{\gamma^C \otimes id_{S^2 A \otimes K}} & S^2 A \otimes K \otimes K
\end{array}
\]

commutes. It can be easily checked by comparing of the left-hand side and the right-hand side of (5) on elements of $q(S^2 A \otimes K) \otimes K$ of the form

\[
\left( i^A_1(\phi \otimes a \otimes S) - i^A_2(\phi \otimes a \otimes S) \right) \otimes T
\]

and

\[
i^A_1(\phi \otimes a \otimes S_0) \left( i^A_1(\phi \otimes a \otimes S) - i^A_2(\phi \otimes a \otimes S) \right) \otimes T,
\]

$\phi, \phi_0 \in S^2 \mathbb{C}$, $a, a_0 \in A$, $T, S, S_0 \in K$, that span a dense subspace in $q(S^2 A \otimes K) \otimes K$. Clearly the left-hand side of (4) is equal to $(g^A \circ \rho^A \circ q^A) \otimes id_K$. We assert that

\[
g^A \circ \rho^A \circ q^A \sim (g^C \otimes id_A) \circ (id_{q\mathbb{C}} \otimes \beta^C \otimes id_A) \circ s^{S^2 A \otimes K},
\]

(6)
that is that the diagram

\[
\begin{align*}
q(S^2A \otimes K) & \xrightarrow{qB} q(A \otimes K) \xrightarrow{\rho} qA \otimes K \\
\downarrow s_{S^2A \otimes K} & \quad & \downarrow g^A \\
qC \otimes S^2A \otimes K & \xrightarrow{id_q \otimes \beta \otimes id_A} qC \otimes A \otimes K \\
& \xrightarrow{qC \otimes id_A} qC \otimes K \\
& \xrightarrow{q \otimes id_A} qA \otimes K \\
S^2A \otimes K & \xrightarrow{id \otimes \beta \otimes id_A} S^2A \otimes K \\
& \xrightarrow{g^C \otimes id_A} qC \otimes A \otimes K
\end{align*}
\]

commutes. Indeed it is straightforward to show that the left-hand side and the right-hand side of (6) asymptotically agree on elements of \(q(S^2A \otimes K)\) of the form

\[i_1^{S^2A \otimes K}(\phi \otimes a \otimes T) - i_2^{S^2A \otimes K}(\phi \otimes a \otimes T)\]

and

\[i_1^{S^2A \otimes K}(\phi_0 \otimes a_0 \otimes T_0)(i_1^{S^2A \otimes K}(\phi \otimes a \otimes T) - i_2^{S^2A \otimes K}(\phi \otimes a \otimes T)),\]

\(\phi, \phi_0 \in S^2C, a, a_0 \in A, T, T_0 \in K,\) that span a dense subspace in \(q(S^2A \otimes K)\). Now, by (5), (6), to get (4) it remains to prove that

\[\gamma^C \otimes id_{S^2C} \sim (g^C \otimes id_A) \circ (id_q \otimes \beta^C \otimes id_A) \circ (s_{S^2A \otimes K} \otimes id_K)\]

or, equivalently,

\[\gamma^C \otimes id_{S^2C} \sim g^C \otimes (id_q \otimes \beta^C).\]

For that note that \(\gamma^C \otimes id_{S^2C}\) and \(g^C \otimes (id_q \otimes \beta^C)\) induce the same homomorphisms in K-theory. Indeed they both send the generator

\[\left(0 \quad 1 \right) \otimes (i_1^C(1) - i_2^C(1))\]

of \(K_0(S^2C \otimes qC)\) to the generator

\[\left(0 \quad 1 \right) - [p Bott]\]

of \(K_0(S^2C)\).

This implies that \(\gamma^C \otimes id_{S^2C}\) and \(g^C \otimes (id_q \otimes \beta^C)\) are homotopic because, as is well known,

\[[S^2C \otimes qC \otimes K, S^2C \otimes K] \cong Z \cong \text{Hom}(K_0(S^2C \otimes qC), K_0(S^2C)).\]

Let \(\psi \in G(B, A)\) and as before \(\phi : qB \to A \otimes K\) be an asymptotic morphism such that \((id_A \otimes i) \circ (\phi \otimes id_K) \sim \psi\). Define an asymptotic morphism \(b(\psi) \in G(B, S^2A)\) by the composition

\[qB \otimes K \xrightarrow{\theta_B} q^2B \otimes K \xrightarrow{q \otimes id_K} q(A \otimes K) \otimes K \xrightarrow{\rho \otimes id_K} qA \otimes K \otimes K \xrightarrow{id_A \otimes i} qA \otimes K \xrightarrow{g^A} S^2A \otimes K.\]

Thus the map \(b : G(B, A) \to G(B, S^2A)\) is defined by formula

\[b(\psi) = g^A \circ (id_A \otimes i) \circ (\rho \otimes id_K) \circ (q \otimes id_K) \circ \theta_B\]

for any \(\psi \in G(B, A)\).

Let \(\beta_G^A : G(B, S^2A) \to G(B, A)\) be the map induced by \(\beta^A\).

**Proposition 10.** \(b : G(B, A) \to G(B, S^2A)\) is a semigroup isomorphism with inverse \(\beta_G^A\).
Proof. Obviously $b$ and $\beta_G^A$ are semigroup homomorphisms so we have to check only the following:

(i) $(\beta_G^A \circ b)(\psi) \sim \psi$ for any $\psi \in G(B, A)$,

(ii) $(b \circ \beta_G^A)(\psi) \sim \psi$ for any $\psi \in G(B, S^2 A)$.

(i): Let $\psi \in G(B, A)$ and $\phi : qB \to A \otimes K$ be such an asymptotic morphism that $\psi \sim (id_A \otimes i) \circ (\phi \otimes id_K)$. Then

$$((\beta_G^A \circ b)(\psi)) = \beta^A \circ g^A \circ (id_{qA} \otimes i) \circ (\rho^A \otimes id_K) \circ (q\phi \otimes id_K) \circ \theta_B \sim \text{Lemma 3},$$

$$\gamma^A \circ (id_{qA} \otimes i) \circ (\rho^A \otimes id_K) \circ (q\phi \otimes id_K) \circ \theta_B \sim \text{Lemma 3},$$

$$(id_A \otimes i) \circ \gamma^{A \otimes K} \circ (q\phi \otimes id_K) \circ \theta_B \sim (id_A \otimes i) \circ (\phi \otimes id_K) \circ \gamma^{qB} \circ \theta_B \sim \psi \circ \gamma^{qB} \circ \theta_B \sim \text{Lemma 3},$$

(ii): Let $\psi \in G(B, S^2 A)$ and $\phi : qB \to S^2 A \otimes K$ be such an asymptotic morphism that $\psi \sim (id_{S^2 A} \otimes i) \circ (\phi \otimes id_K)$. Then

$$(b \circ \beta_G^A)(\psi) = g^A \circ (id_{qA} \otimes i) \circ (\rho^A \otimes id_K) \circ (q(\beta^A \circ \phi) \otimes id_K) \circ \theta_B =$$

$$g^A \circ (id_{qA} \otimes i) \circ (\rho^A \otimes id_K) \circ (q(\beta^A \circ \phi) \otimes id_K) \circ \theta_B \sim \text{Lemma 4},$$

$$(id_{S^2 A} \otimes i) \circ (\phi \otimes id_K) \circ \gamma^{S^2 A \otimes K} \circ \theta_B \sim (id_{S^2 A} \otimes i) \circ (\phi \otimes id_K) \circ \gamma^{qB} \circ \theta_B \sim$$

$$\psi \circ \gamma^{qB} \circ \theta_B \sim \text{Lemma 4}.$$

□

Proposition 11. $f_G^A \circ g_G^A = id,$

$g_G^A \circ f_G^A = id.$

Here $id$ means both the identity map from $G(B, S^2 A)$ into itself and the identity map from $G(B, qA)$ into itself.

Proof. Consider the following diagram

\[ \begin{array}{ccc}
G(B, A) & \xrightarrow{\beta_G^A} & G(B, A) \\
\downarrow{g_G^A} & & \downarrow{g_G^A} \\
G(B, S^2 A) & \xrightarrow{f_G^A} & G(B, qA)
\end{array} \]

We shall prove that it commutes and this will imply the statement of the proposition.

By Propositions [10] and [8] $\beta_G^A$ and $\gamma_G^A$ are isomorphisms. By Lemma [3] $\beta_G^A \circ g_G^A = \gamma_G^A$ whence

$$g_G^A = (\beta_G^A)^{-1} \circ \gamma_G^A \quad (7)$$

By Corollary [2] $\gamma_G^A \circ f_G^A = \beta_G^A$ whence

$$f_G^A = (\gamma_G^A)^{-1} \circ \beta_G^A \quad (8)$$

From (7) and (8) we obtain the assertions of the proposition. □
3.3. Main result.

Theorem 12. 
(i) \( g^A \circ f^A \sim id_{S^2A \otimes K} \).
(ii) \( f^A \circ g^A \sim id_{qA \otimes K} \).

Proof. (i) By Proposition 4 \( g_E^A \circ f_E^A = id \) whence \( g^A \circ f^A \circ \phi \sim \phi \) for any \( \phi \in E(B, S^2A \otimes K) \). Set \( B = A \otimes K, \phi = id_{S^2A \otimes K} \). Then

\[
\text{id}_{S^2A \otimes K} \sim g^A \circ f^A \circ \text{id}_{S^2A \otimes K} = g^A \circ f^A.
\]

(ii) By Proposition 11 \( f_G^A \circ g_G^A = id \) whence \( f^A \circ g^A \circ \phi \sim \phi \) for any \( \phi \in [[qB \otimes K, qA \otimes K]] \).

Setting \( B = A, \phi = id_{qA \otimes K} \) we get

\[
\text{id}_{qA \otimes K} \sim f^A \circ g^A.
\]

\[\Box\]

So \( C^*\)-algebras \( S^2A \otimes K \) and \( qA \otimes K \) are asymptotically equivalent and we obtain immediately

Corollary 13. \( E(A, B) = [[qA, B \otimes K]] \) for every \( C^*\)-algebras \( A \) and \( B \).

Corollary 14. Let \( A \) be a nuclear \( C^*\)-algebra and \( B \) be any \( C^*\)-algebra. Then every asymptotic morphism from \( qA \) to \( B \otimes K \) is homotopic to a \( * \)-homomorphism from \( qA \) to \( B \otimes K \).

Proof. Let \( \phi_t \in [[qA, B \otimes K]] \). Since \( A \) is nuclear \( I_{A,B} \) is an isomorphism (2). Define a \( * \)-homomorphism \( \psi_0 : qA \otimes K \rightarrow B \otimes K \) by

\[
\psi_0 = I_{A,B}^{-1} ((i_B \otimes i) \circ (\phi_t \otimes id_K) \circ f^A). \tag{9}
\]

By 5 there exists a \( * \)-homomorphism \( \psi : qA \rightarrow B \otimes K \) such that

\[
(i_B \otimes i) \circ (\psi \otimes id_K) \sim \psi_0. \tag{10}
\]

We will prove that \( \phi_t \sim \psi \). By Theorem 11

\[
I_{A,B}(\psi_0) = \psi_0 \circ f^A. \tag{11}
\]

By 9 the left-hand side of (11) is \( I_{A,B}(\psi_0) = (i_B \otimes i) \circ (\phi_t \otimes id_K) \circ f^A \). By (10) the right-hand side of (11) is \( \psi_0 \circ f^A \sim (i_B \otimes i) \circ (\psi \otimes id_K) \circ f^A \).

So

\[
(i_B \otimes i) \circ (\phi_t \otimes id_K) \circ f^A \sim (i_B \otimes i) \circ (\psi \otimes id_K) \circ f^A,
\]

and by Theorem 12 we obtain

\[
(i_B \otimes i) \circ (\phi_t \otimes id_K) \sim (i_B \otimes i) \circ (\psi \otimes id_K)
\]

whence \( \phi_t \sim \psi \). \[\Box\]

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References

[1] J. Cuntz, A new look at $KK$-theory, *K-theory* 1 (1987), 31-51.

[2] A. Connes, N. Higson, Deformations, morphismes asymptotiques et K-theorie bivariante, *C. R. Acad. Sci. Paris*, Ser.I Math. 311 (1990), 101-106.

[3] T. Loring, Perturbation questions in the Cuntz Picture of K-theory, *K-theory* 11(1997), 161-193.

[4] T. Loring, Almost multiplicative maps between $C^*$-algebras, *Operator Algebras and Quantum Field Theory*, Internat.Press, 1997, 111-122.

[5] K. Thomsen, K. K. Jensen, *Elements of KK-theory*, Birkhauser, Boston, 1991.

[6] N. Higson, A characterization of KK-theory, *Pacific J. of Math.*, Vol.126, No.2 (1987).

[7] D. Voiculescu, Asymptotically commuting finite rank unitary operators without commuting approximants, *Acta Sci. Math. (Szeged)* 45 (1983), 429-431.

[8] R. G. Bartle, L. M. Graves, Mappings between function spaces, *Trans. Amer. Math. Soc.* 72 (1952), 400-413.

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