Adversaries in Online Learning Revisited:
with applications in Robust Optimization
and Adversarial training

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Abstract

We revisit the concept of “adversary” in online learning, motivated by solving robust optimization and adversarial training using online learning methods. While one of the classical setups in online learning deals with the “adversarial” setup, it appears that this concept is used less rigorously, causing confusion in applying results and insights from online learning. Specifically, there are two fundamentally different types of adversaries, depending on whether the “adversary” is able to anticipate the exogenous randomness of the online learning algorithms. This is particularly relevant to robust optimization and adversarial training because the adversarial sequences are often anticipative, and many online learning algorithms do not achieve diminishing regret in such a case.

We then apply this to solving robust optimization problems or (equivalently) adversarial training problems via online learning and establish a general approach for a large variety of problem classes using imaginary play. Here two players play against each other, the primal player playing the decisions and the dual player playing realizations of uncertain data. When the game terminates, the primal player has obtained an approximately robust solution. This meta-game allows for solving a large variety of robust optimization and multi-objective optimization problems and generalizes the approach of [Ben-Tal et al. 2013].

1 Introduction

This paper, motivated by solving robust optimization and (equivalently) adversarial training, revisits the concept of “adversary” in online learning. A significant amount of literature in online learning focuses on the so-called “adversarial” setup, where a learner picks a sequence of solutions against a sequence of inputs “chosen adversarially”, and achieves accumulated utility almost as good as the best fixed solution in hindsight. While those results are widely known and applied, we observe that the concept of “adversary” is often understood and applied in an incorrect way, partly because of a lack of a rigorous definition, which we address in this paper. Our observation is largely motivated by recent works applying online learning to solve robust
optimization and adversarial training, where diminishing regret, contrary to the claim, is not guaranteed to be achieved.

Robust optimization [see e.g., Ben-Tal and Nemirovski 1998, 2000, 2002, Bertsimas and Sim 2004, Ben-Tal et al. 2009, Bertsimas et al. 2011] is a powerful approach to account for data uncertainty when distributional information is not available. Taking a worst-case perspective in Robust Optimization (RO) we are interested in finding a good solution against the worst-case data realization, which leads to problems of the form

\[
\text{Minimize}_{\mathbf{x} \in \mathcal{X}} \quad \text{Maximize}_{\mathbf{u} \in \mathcal{U}} f(\mathbf{x}, \mathbf{u}),
\]

where \(\mathcal{X}\) is the decision set and \(\mathcal{U}\) is the uncertainty set and it is well known that robust optimization can be expressed in this standard form via appropriate choice of \(f\). We thus look for a robust solution \(\mathbf{x}\) that minimizes the cost \(f\) under the worst-case realization \(\mathbf{u}\). In particular, when the function \(f\) is convex in the decisions \(\mathbf{x}\) and concave in the uncertainty \(\mathbf{u}\) the resulting problem is convex and there is a plethora of approaches to solve these problems. In fact, robust optimization is tractable in many cases and can be solved often via methods from convex optimization as well as integer programming in the context of discrete decision; we refer the interested reader to Ben-Tal et al. [2009] for an introduction and to, e.g., Bertsimas and Sim [2003] for robust optimization with discrete decisions.

Robust optimization is also closely related to adversarial training, a subject that has drawn significant attention in machine learning research, and particularly in deep learning. It has been observed that a well-trained deep neural network can often be “fooled” by small perturbations. That is, if a data point (an image for example) is perturbed by a carefully designed adversarial noise, potentially imperceptible to the human being (i.e., with a small \(\ell_{\infty}\) norm in the case of images), the classification result can be completely changed. To mitigate this issue, adversarial training aims to train a neural network to be robust to such perturbations, which can be formulated as follows:

\[
\min_{\mathbf{w}} \max_{(\delta_1, \delta_2, \ldots, \delta_n)} \frac{1}{n} \sum_{i=1}^{n} l(y_i, \phi(\mathbf{w}, x_i + \delta_i)),
\]

where \(\mathbf{w}\) is the weight vector of the neural network, \((x_i, y_i)\) are the \(i\)-th data point, \(\phi(\mathbf{w}, x)\) is the prediction of the neural network on an input \(x\) and \(l(\cdot, \cdot)\) is the loss function. Thus, adversarial training essentially is an attempt to solve an (often non-convex and hard) robust optimization problem. Due to the lack of convexity, exactly solving the above formulation is intractable, and numerous heuristics has been proposed to address the computational issue.

Recently, several works explored a general framework to solve robust optimization and adversarial training via online learning. The main idea is the following: instead of solving the robust problem one-shot by exploiting convex duality, a sequence of scenarios \(\mathbf{u}_i \in \mathcal{U}\) is generated using online learning, and optimal decisions (or approximate solutions for very complicated functions, in the adversarial training case) for each scenarios are then averaged as the final output. Using theorems from online learning, it is shown that the final output is close to optimal (or achieves same approximation ratio, for adversarial training) for all scenarios in \(\mathcal{U}\).

The framework outlined above can be very appealing computationally. However, a close examination of the argument shows that because of the ambiguity on the concept “adversary” in online learning, some of the claimed results are invalid (see Section 2.2 for a concrete counter-example). Thus, we feel that it is necessary to characterize the concept “adversary” in a more rigorous way to avoid future confusion. This also enables us to develop new methods for solving robust optimization using online learning.

**Contribution.** We now summarize our contribution:
Clarification of concepts. The main contribution in this paper is to distinguish two types of adversaries in an online learning setup, which we termed “anticipatory” and “non-anticipatory”. In a nutshell, anticipatory adversaries are those that have access to inherent randomness of the online learning algorithm at the current step. The example that motivates this concept is when the adversary is chosen by solving an optimization problem whose parameters are the output of the online learning algorithm. Non-anticipatory adversaries do not have access to the inherent randomness of the current step (however can still be adaptive). Based on that, we further distinguish two types of online learning algorithms, which are both known in the literature to achieve diminishing regret with respect to adversarial input. Depending on whether such adversarial input can be anticipatory or not, we call the two classes strong learners and weak learners.

One-sided minimax problems via imaginary play. We then apply our model to the special case of solving robust optimization problems. We show how to solve problems of the form (1) by means of online learning with two weak learners. Slightly simplified, two learners play against each other solving Problem (1). However, in contrast to general saddle point problems, only one of the players can extract a feasible solution, as we considerably weaken convexity requirements both for the domains as well as the functions $f$ (or even drop them altogether). For this we present a general primal-dual setup that is then later instantiatted for specific applications by means of plugging-in the desired learners.

Biased play with asymmetric learners. We then show how to further gain flexibility by allowing asymmetry between the learners. Here one learner is weakened (in terms of requirements) to an optimization oracle, and consequently the other player is strengthened to allow anticipatory inputs.

Applications. Finally, we demonstrate how our approach can be used to solved a large variety of robust optimization problems. For example, we show how to solve robust optimization problems with complicated feasible sets involves integer programming. Another example is robust MDPs with non-rectangular uncertainty sets, where only the reward parameters are subject to uncertainty. Due to space constraints, we defer the applications to the appendix.

2 Preliminaries and motivation

In the following let $\Delta(n)$ denote the unit simplex in $\mathbb{R}^n$. We will use the shorthand $[m]$ to denote the set \{1, \ldots, m\}. For the sake of exhibition we will differentiate between Maximize and max, where the former indicates that we maximize a function via an algorithm, whereas the latter is simply indicating the maximum function without any algorithmic reference. Moreover, we will denote both the decision vector $x$ as well as the uncertainty vector $u$ in bold letters. All other notations are standard if not defined otherwise.

2.1 Conventional wisdom

In this work we consider games between two player and we will use robust optimization or adversarial training of the form (1) as our running example. For the sake of continuity, we adapt the notation of Ben-Tal et al. [2015], however we stress that we later will selectively relax some of the assumptions. Consider:

$$\text{Minimize}_x \ \{ f_0(x) \mid f_i(x, u_i) \leq 0, \ i \in [m]; \ x \in X \},$$

where $X \subseteq \mathbb{R}^n$ is the domain of feasible decisions and the $f_i$ with $i \in [m]$ are convex functions in $x$ that are parametrized via vectors $u_i \in \mathbb{R}^d$ for some $d \in \mathbb{N}$ for $i \in [m]$. The problem above is parametrized by a fixed choice of vectors $u_i$ with $i \in [m]$ and we will refer to a problem in this form as the nominal problem (with parameters $\{u_i\}_i$), which corresponds to the outer minimization problem given a realization of the adversary’s choice $\{u_i\}_i$. 

3
In robust optimization we *robustify* the nominal problem against the worst-case choice of \( \{u_i\}_i \) via the formulation:

\[
\text{Minimize } \{ f_0(x) \mid f_i(x, u_i) \leq 0, \forall u_i \in U_i, i \in [m]; x \in \mathcal{X} \},
\]

where the *uncertainty set* \( U_i \) is the set of possible choices of parameter \( u_i \). Thus, denote \( U = \prod_{i \in [m]} U_i \) and we have \( (u_1, \ldots, u_m) \in U \).

Recently there has been a line of work proposing methods to solve adversarial training and robust optimization via online learning methods or in an equivalent fashion (see e.g., Ben-Tal et al. [2015], Madry et al. [2017], Chen et al. [2017], Sinha et al. [2017]). In all cases the underlying meta-algorithm works as follows: the \( u \)-player takes \( \{x\} \) as input and generate a sequence of \( \{u\} \) according to an online learning algorithm which achieves diminishing regret against adversarial input. The \( x \)-player on the other hand, computes \( x_t \) by minimizing the loss function with \( u_t \) as input; the interpretation of the roles of the players depends on the considered problem.

In particular, in Ben-Tal et al. [2015], the authors proposed two methods along this line, using *online convex optimization* and Follow the Perturbed Leader (FPL) as the online learning algorithm, respectively. In Chen et al. [2017], the authors consider the case where \( U \) is a finite set, and proposed to use exponential weighting as the online learning algorithm (in the infinite case, they use online gradient descent), and then solve \( x_t \) by minimizing the loss function for the distributional problem. While superficially similar, these two approaches are markedly different as we will see.

### 2.2 A motivating counter example

Unfortunately, the outlined approach above can be easily flawed, for reasons that will be made clear later. We start with the following counterexample, and apply the second method (i.e., FPL based one) proposed in Ben-Tal et al. [2015].

Consider the following robust feasibility problem: Let \( U = \text{conv}\{((1, 1), (-1, -1), (2, 1))\} \) and \( \mathcal{X} = \text{conv}\{(0, 1), (-2, -1), (0, 0)\} \) does there exist \( x \in \mathcal{X} \) such that

\[
\min_{u \in U} u^\top x \geq 1?
\]

The answer is clearly negative, as for any \( x \), at least one of \((1, 1)^\top x\) and \((-1, -1)^\top x\) is less than or equal to 0. However, Theorem 2 of Ben-Tal et al. [2015] states that if we update \( u_t \) according to Follow the Perturbed Leader, compute \( x_t \) via

\[
x_t := \arg \max_{x \in \mathcal{X}} u_t^\top x,
\]

and if \( u_t^\top x_t \geq 1 \) for \( t = 1, 2, \ldots, T \) (which is true here as the objectives are linear and thus \( u_t \) is a vertex of \( U \)), then \( \bar{x} = \sum_{t=1}^T x_t \in \mathcal{X} \) is \( \epsilon \)-robust feasible, i.e.,

\[
\min_{u \in U} u^\top \bar{x} \geq 1 - \epsilon,
\]

where \( \epsilon = O(1/\sqrt{T}) \); this is clearly not true and we obtain the desired contradiction.

Furthermore, we also show that \( \bar{x} \) does not converge to the minimax solution: Since \( u_t \) is obtained by FPL and the objective is a linear function as before we have that \( u_t \) is a vertex of \( U \). Further notice that for \( u_t \in \{(1, 1), (-1, -1), (2, 1)\} \) we have \( x_t = \arg \max_{x \in \mathcal{X}} u_t^\top x \in \{(0, 1), (-2, -1)\} \). Therefore \( \bar{x} \) is on
the line segment between (0, 1) and (-2, -1). One can easily check that for any \( x \) on this line segment, we have
\[
\min_{u \in U} u^\top x \leq -1/5.
\]
On the other hand, clearly for \( x^* = (0, 0) \), we have
\[
\min_{u \in U} u^\top x^* = 0.
\]
Thus, \( \bar{x} \) does not converge to the minimax solution.

Interestingly, the first method proposed in Ben-Tal et al. [2015] turns out to be a valid method for this example. Also, the approach in Chen et al. [2017] does not suffer from this weakness as the Bayesian optimization oracle is applied to the output distribution, rather than a sampled solution (which would be problematic). Indeed, this is no coincidence. To clearly explain the different behaviors for various methods proposed in literature is the main motivation of this work.

### 3 Anticipatory and non-anticipatory adversaries

In this section we provide definitions of the main concepts that we are introducing in this paper. There are two types of “adversaries”, namely anticipatory adversaries and non-anticipatory adversaries for online learning setups that need to be clearly distinguished. In a nutshell, slightly simplifying, the distinction is whether the adversary’s decision (who is potentially computationally unbounded) is independent of the private randomness \( \xi_t \) of the current round \( t \); if not the adversary might be able to anticipate the player’s decision \( x_t \).

**Definition 3.1.** An online learning setup is as follows: for \( t = 1, 2, \ldots \), the algorithm is given access to an external signal \( y_t \), and an exogenous random variable \( \xi_t \) which are independent with everything else, and furthermore \( \xi_i \) and \( \xi_j \) are independent for \( i \neq j \). An online learning algorithm is a mapping:
\[
x_t := L_x(x_1, y_1, x_2, y_2, \ldots, x_{t-1}, y_{t-1}, \xi_t).
\]
We say an online learning algorithm is deterministic if it is a mapping
\[
x_t := L_x(x_1, y_1, x_2, y_2, \ldots, x_{t-1}, y_{t-1}).
\]

In other words, an online learning algorithm picks an action at time \( t \) depending on past actions \( x_1, \ldots, x_{t-1} \), past signals \( y_1, \ldots, y_{t-1} \), and an exogenous randomness \( \xi_t \). And a deterministic online learning algorithm is independent of the exogenous randomness.

Existing analyses for online learning algorithms focus on two cases: either the external signal \( y_t \) is generated stochastically (and typically in an iid fashion), or it is generated “adversarially”. However, as we will show later, the term “adversarially” is loosely defined and causes significant confusion. Instead, we now define two types of adversary signals.

**Definition 3.2.** Recall an online learning algorithm is given access to exogenous random variables \( \xi_1, \xi_2, \ldots, \xi_t, \ldots \), and its output \( x_t \) may depend on \( \xi_1, \ldots, \xi_t \), but is independent to \( \xi_{t+1}, \xi_{t+2}, \ldots \). A sequence \( y_1, y_2, \ldots, y_t, \ldots \) is called non-oblivious non-anticipatory (NONA) with respect to \( \{x\} \) if \( y_t \) may depend on \( \xi_1, \ldots, \xi_{t-1} \), but is independent of \( \xi_t, \xi_{t+1}, \ldots \), for all \( t \). A sequence \( y_1, y_2, \ldots, y_t, \ldots \) is called anticipatory wrt \( \{x\} \) if \( y_t \) may depend on \( \xi_1, \ldots, \xi_t \), but is independent of \( \xi_{t+1}, \xi_{t+2}, \ldots \), for all \( t \).

We now provide some examples to illustrate the concept.
If \( \{ y \} \) is a sequence chosen arbitrarily, independent of \( \{ \xi \} \), then it is a NONA sequence.

(ii) If \( y_t \) is chosen according to \( y_t = \mathcal{F}_t(y_1, x_1, \ldots, y_{t-1}, x_{t-1}) \), for some function \( \mathcal{F}_t(\cdot) \), then \( \{ y \} \) is a NONA sequence.

(iii) If \( y_t \) is chosen according to \( y_t = \mathcal{F}_t(x_1, y_1, \ldots, x_{t-1}, y_{t-1}, x_t) \), for some function \( \mathcal{F}_t(\cdot) \), then \( \{ y \} \) is an anticipatory sequence. This is because \( x_t \) is (potentially) dependent to \( \xi_t \), and so is \( y_t \). As a special case, suppose \( y_t = \arg \max_{y \in Y} f(x_t, y) \), then \( \{ y \} \) is an anticipatory sequence.

(iv) If \( \{ x \} \) is the output of a deterministic online learning algorithm, and \( y_t \) is chosen according to \( y_t = \mathcal{F}_t(x_1, y_1, \ldots, x_{t-1}, y_{t-1}, x_t) \), for some function \( \mathcal{F}_t(\cdot) \), then \( \{ y \} \) is a NONA sequence. This is because \( x_t \) is independent of \( \xi_t \) since the online learning algorithm is deterministic. In this case, the following sequence is NONA as well: \( y_t = \arg \max_{y \in Y} f(x_t, y) \).

The standard target of online learning algorithms is to achieve diminishing regret vis a vis a sequence of external signals. Thus, depending on whether the external signal is anticipatory or not, we define two classes of learning algorithms.

**Definition 3.3.** Suppose \( \mathcal{X} \) is the feasible set of actions, and for \( x_t \) the action chosen at time \( t \), it is evaluated by \( f(x_t, y_t) \), with a smaller value being more desirable.

(i) We call an online learning algorithm \( \mathcal{L}_x \) for \( x \) a **weak learning algorithm** with regret \( R(\cdot, \cdot) \), if for any NONA sequence \( \{ y_t \}_{t=1}^{\infty} \), the following holds with a probability \( 1 - \delta \) (over the exogenous randomness of the algorithm), where \( \{ x_t \} \) are the output of \( \mathcal{L}_x \):

\[
\sum_{t=1}^{T} f(x_t, y_t) - \min_{x \in \mathcal{X}} \sum_{t=1}^{T} f(x, y_t) \leq R(T, \delta).
\]

(ii) We call an online learning algorithm \( \mathcal{L}_x \) for \( x \) a **strong learning algorithm** with regret \( R(\cdot, \cdot) \), if for any anticipatory sequence \( \{ y_t \}_{t=1}^{\infty} \), the following holds with a probability \( 1 - \delta \) (over the exogenous randomness of the algorithm), where \( \{ x_t \} \) are the output of \( \mathcal{L}_x \):

\[
\sum_{t=1}^{T} f(x_t, y_t) - \min_{x \in \mathcal{X}} \sum_{t=1}^{T} f(x, y_t) \leq R(T, \delta).
\]
To illustrate the subtle difference, let us consider the classical exponential weighting algorithms, where a set of experts are given, and the goal of the online learning algorithm is to predict unseen \( y_t \) according to the prediction of the experts, such that the algorithm does as good as the best among the experts. The algorithm maintains a weight vector over all experts depending on their performance in previous rounds, and outputs the weighted average of the prediction from the experts. Notice that this is a deterministic learning algorithm, i.e., the learning algorithm is independent of the exogenous randomness \( \xi_t \). Thus, whether \( y_t \) has access to \( \xi_t \) or not has no influence on the performance of the algorithm. As such, the exponential weighting algorithm (in this specific form) is a strong learning algorithm.

On the other hand, there is a variant of exponential weighting algorithm where instead of outputting the weighted average of the prediction, the algorithm outputs the prediction of one expert, based on a probability proportional to the weight. The common proof for this technique is that through randomization, the expected loss is upper bounded by the loss of the weighted average, and hence the regret of this variant is upper bounded by the regret of the vanilla version. Clearly, this argument implicitly uses an assumption that the realized \( y_t \) is independent of this randomness, and breaks down otherwise. Hence, this form of exponential weighting algorithm is a weak learning algorithm.

As a rule of thumb, it appears that for online learning algorithms that “work in the adversarial case”, all deterministic algorithms (e.g., Online Gradient Descent) are strong learning algorithms; whereas all algorithms which inherently require randomness (e.g., Follow the Perturbed Leader) are weak learning algorithms.

We remark that in the online learning literature, there is the concept of adaptive adversaries, which is a relevant concept that can better highlight the observation made in the paper. An adaptive adversary in online learning is allowed to adapt its choice at time \( t \) to the output of the online learning algorithm until time \( t - 1 \), but is independent of the exogenous randomness at time \( t \). Thus, it generates a non-anticipatory sequence. An online learning algorithm that achieves a diminishing regret against such an adversary is thus a weak learner, and not necessarily a strong learner.

## 4 Warmup: Minimax problem via Online Learning

We will first consider the case where we have one function \( f \). In principle this function \( f \) can be highly complex and could be, e.g., the maximum of a family of functions \( f_i \), however here the reader should be thinking of \( f \) as a relatively simple function. This will be made more precise below, where we specify the learnability requirements for \( f \), which ultimately limits the complexity of the considered functions. In Section 5, we will then consider the more general case of a family of (simple) functions \( \{f_i\}_i \), which arises naturally in robust optimization.

Thus, we are solving the following optimization problem

\[
\text{Minimize } \max_{x \in X} \max_{u \in U} f(x, u). \tag{2}
\]

**Assumption 4.1** (Problem structure). We will make the following assumptions regarding the domains and function \( f \) if not stated otherwise. Note that these assumptions only affect the \( x \)-player. (1) For any \( u \in U \), the function \( f(\cdot, u) \) is convex. (2) The set \( X \) is convex.

### 4.1 Parallel Weak Learners

Our first framework solves Problem (2) via weak online learning algorithms and imaginary play (i.e., both players can have full knowledge about the function \( f \)) to update \( x \) and \( u \) in parallel. In the following we
will always assume that the x-sequence is a sequence of elements in \( X \) and the u-sequence is a sequence of elements in \( U \).

**Assumption 4.2 (Weak Learnability).**

(i) There exists a weak online learning algorithm \( L_x \) for \( x \) with regret \( R_x(T, \delta) \). That is, for any NONA sequence \( \{u'_t\}_{t=1}^\infty \), the following holds with a probability \( 1 - \delta \), where \( \{x_t\} \) is the output of \( L_x \):

\[
\sum_{t=1}^T f(x_t, u'_t) - \min_{x \in X} \sum_{t=1}^T f(x, u'_t) \leq R_x(T, \delta).
\]

(ii) There exists a weak online learning algorithm \( L_u \) for \( u \) with regret \( R_u(T, \delta) \). That is, for any NONA sequence \( \{x'_t\}_{t=1}^\infty \), the following holds with a probability \( 1 - \delta \), where \( \{u_t\} \) is the output of \( L_u \):

\[
\max_{u \in U} \sum_{t=1}^T f(x'_t, u) - \sum_{t=1}^T f(x'_t, u_t) \leq R_u(T, \delta).
\]

As mentioned above, the learnability assumption constrains the complexity of the function \( f \). For example, if \( f(x, u) = \max_i \alpha f_i(x, u) \) for some family of functions \( \{f_i\} \) that are convex in \( x \) and concave in \( u \), then \( f \) might not be concave in \( u \) and the resulting Problem 2 might be intractable and the learnability assumption for \( u \) might be violated.

We are now ready to present the meta-algorithm, which is given in Algorithm 1. We would like to remark that we refer to these minimax problems as robust optimization as we only require to be able to produce an explicit (stationary) solution \( \bar{x} \) for the x-player.

**Algorithm 1:** Robust Optimization via Online Learning (ROOL)

**Input:** function \( f \), learners \( L_u, L_x \) satisfying Assumptions 4.1 and 4.2.

**Output:** point \( \bar{x} \)

1: for \( t = 1, \ldots, T \) do
2: \( x_t \leftarrow L_x(x_1, u_1, \ldots, x_{t-1}, u_{t-1}) \)
3: \( u_t \leftarrow L_u(x_1, u_1, \ldots, x_{t-1}, u_{t-1}) \)
4: end for
5: \( \bar{x} \leftarrow \frac{1}{T} \sum_{t=1}^T x_t \)

**Remark 4.3 (Dependence on \( f \)).** Note that in Algorithm 1 the function \( f \) does not explicitly occur. In fact, \( f \) is captured in Assumptions 4.1 and 4.2 and in particular, we make a priori no distinction what type of feedback (full information, semi-bandit, bandit, etc.) the learner observes. In principle, since we are assuming imaginary play both learners can have full knowledge about the function \( f \) while in actual applications the learners will only require limited information. For example, a learner might only require bandit feedback to ensure the learnability assumption with a given regret bound, while another might require full information depending on the setup. In the formulation above, Algorithm 1 is completely agnostic to this; also in all other algorithms, the situation will be analogous.

Observe that due to convexity of \( X \), we have \( \bar{x} \in X \). The theorem below shows that \( \bar{x} \) converges to \( x^* \), which achieves the best worst-case performance. Note that the guarantee is asymmetric as a saddle point may not exist as no assumptions on \( U \) or \( f(x, \cdot) \) are made. If indeed \( f \) is concave with respect to the
second argument and $U$ is convex, then the theorem reduces to the well known result of solving a zero-sum game via online learning in parallel [Freund and Schapire 1999]. The proof is similar and included in the supplementary material for completeness.

**Theorem 4.4.** With probability $1 - 2\delta$, Algorithm 1 returns a point $\bar{x} = \frac{1}{T} \sum_{t=1}^{T} x_t$ satisfying

$$\max_{u' \in U} f(\bar{x}, u') - \min_{x^* \in \mathcal{X}} \max_{u \in U} f(x^*, u) \leq \frac{R_x(T, \delta) + R_u(T, \delta)}{T}.$$

We remark that the two weak learners framework superficially resembles the online learning based method to solve zero-sum games [Freund and Schapire 1999] where both players run an online learning algorithm. Yet, our setup and results depart from those in Freund and Schapire [1999], as we drop any requirement for the uncertainty $u$ and in particular $f$ is not necessarily concave with respect to $u$. In short, the minimax problem we solve is not a saddle-point problem, and as such only the $x$ player is able to extract a near-optimal solution. Notice that the lack of concavity with respect to $u$ arises naturally in robust optimization formulations and adversarial training (see Section 5 for details).

### 4.2 Biased Play with a Strong Learner

In our second framework, the structure of the problem is “biased” toward one player. Here one of the learners is particularly strong, allowing the other to break NONA-ness. We will consider the case where the $u$ learner is particularly strong. The case for $x$ is symmetric.

**Assumption 4.5 (Strong Learnability of $u$).**

(i) There exists a strong online learning algorithm $L_u$ for $u$ with regret $R_u(\cdot, \cdot)$. That is, for any anticipatory sequence $\{x'_t\}_{t=1}^{\infty}$, the following holds with a probability $1 - \delta$, where $\{u_t\}$ is the output of $L_u$:

$$\max_u \sum_{t=1}^{T} f(x'_t, u) - \sum_{t=1}^{T} f(x'_t, u_t) \leq R_u(T, \delta).$$

(ii) Given $u$, there is an optimization oracle that computes $x^*$.

**Algorithm 2** Robust Optimization via Strong Primal Learner

**Input:** function $f$, learners $L_x$ satisfying Assumptions 4.1 and 4.5.

**Output:** point $\bar{x}$

1. for $t = 1, \ldots, T$ do
2. $u_t \leftarrow L_u(x_t, u_1, \ldots, x_{t-1}, u_{t-1})$
3. $x_t \leftarrow \arg \min_{x \in \mathcal{X}} f(x, u_t)$
4. end for
5. $\bar{x} \leftarrow \frac{1}{T} \sum_{t=1}^{T} x_t$

We now a theorem similar to Theorem 4.4 for the case where $x$ is a strong learner; the proof is to be found in Supplementary Material.

**Theorem 4.6.** With probability $1 - \delta$, Algorithm 1 returns a point $\bar{x} = \frac{1}{T} \sum_{t=1}^{T} x_t$ satisfying

$$\max_{u' \in U} f(\bar{x}, u') - \min_{x^* \in \mathcal{X}} \max_{u \in U} f(x^*, u) \leq \frac{R_u(T, \delta)}{T}.$$
Some remarks are in order.

(i) Note that since $u_t$ is updated via solving an optimization problem determined by $x_t$, the sequence \{u_t\} is an anticipatory sequence. As such, it is crucial that a strong learning algorithm is required to update $x_t$. This explains the existence of the counter example in Section 2.2: both FPL and exponential weighting (with output randomly chosen) are weak learners, as opposed to online gradient descent which is a strong learner.

(ii) It is easy to extend the analysis to the case where an optimization oracle is replaced by a C-approximate optimization oracle, which given u computes $\tilde{x}^*$ such that

$$f(\tilde{x}^*, u) \leq C \cdot \min_{x \in X} f(x, u),$$

for some $C > 1$. In such a case, the statement in Theorem 3.6 is replaced by

$$\max_{u' \in \mathcal{U}} f(x, u') - C \cdot \min_{x^* \in X} \max_{u \in \mathcal{U}} f(x^*, u) \leq \frac{R_u(T, \delta)}{T}.$$

That is, the algorithm will return a C-approximate solution for the minimax problem.

Before concluding this section, we remark that the convexity requirement for the $x$ player can be relaxed if randomized actions are allowed; see Appendix B for details. Consequently, this allows us to solve robust optimization where both the feasible region $X$ and the uncertainty set $\mathcal{U}$ are represented as feasible regions of Integer Programming problems (see Appendix D for an example).

5 Multiple Objectives: Online Learning for Robust Optimization

Our general approach can be readily extended to the case where the primal player needs to satisfy multiple objectives simultaneously. Multi-objective decision-making naturally arises in many setups where the preference of decisions are multi-dimensional. In particular multi-objective decision-making can model robust optimization, where typically the decision maker aims to find a decision $x^*$ such that a set of robust constraints are satisfied, i.e.,

$$\max_{u' \in \mathcal{U}} f^i(x^*, u') \leq 0, \quad i = 1, 2, \ldots, n.$$

We consider solving the following general case:

$$\min_{x \in X} \left\{ \max_{\lambda \in \Delta} \max_{u_i \in \mathcal{U_i}, i = 1, \ldots, n} \sum_{i=1}^{n} \lambda_i f^i(x, u_i) \right\}, \quad (3)$$

where $\Delta \subseteq \Delta(n)$ is a closed convex set. For example, if $\Delta = \Delta(n)$, the n-dimensional unit simplex, then Problem (3) reduces to

$$\min_{x \in X} \max_{u \in \mathcal{U}} \left( \max_{u^1 \in \mathcal{U}} f^1(x, u^1), \ldots, \max_{u^n \in \mathcal{U}} f^n(x, u^n) \right),$$

where $\mathcal{U} = \mathcal{U}(n)$.
which corresponds to the aforementioned case of robust optimization. On the other hand, if \( \Lambda = \{ \lambda \} \) is a singleton, then (3) is equivalent to

\[
\text{Minimize}_{x \in X} \quad \text{Maximize}_{u^i \in \prod_i U_i} \left( \sum_{i=1}^{n} \lambda^i f^i(x, u^i) \right),
\]

and we solve our problem for a specific preference or weighing among the different objectives.

We first consider solving Problem (3) via parallel weak learners. We present the following two approaches both of which are based on imaginary play. Due to space constraints, the biased case with a strong learner is deferred to the supplementary material.

**Approach via Explicit Maximum**

In the first approach we model the maximum over the different functions \( f^i \) explicitly. To this end, let \( \tilde{u} \in \prod_i U_i \) denote the concatenation of \( u^i \), i.e., \( \tilde{u} = \{ u^1, u^2, \ldots, u^n \} \) and define the function

\[
F(x, \tilde{u}) \triangleq \max_{\lambda \in \Lambda} \lambda \cdot f^i(x, u^i).
\]

Thus, roughly speaking, the optimal \( x \) is approachable if weak learnability holds for both \( x \) and \( u^i \) with respect to \( F \). Due to space constraints we defer detailed results into the supplementary material.

**Approach via Distributional Maximum**

In this section we will present an alternative approach, where the maximum is only implicitly modeled via a distributional approach, which captures the maximum via a worst-case distribution.

In the following let

\[
g(x, \tilde{u}, \lambda) \triangleq \langle \lambda, f(x, \tilde{u}) \rangle.
\]

With this Problem (3) can be rewritten as

\[
\text{Minimize}_{x \in X} \quad \text{Maximize}_{u \in \prod_i U_i, \lambda \in \Lambda} g(x, \tilde{u}, \lambda).
\]

As before we specify the learnability requirement.

**Assumption 5.1 (Learnability).** We make the following assumptions for the learners:

(i) For every \( i \), there exists an online learning algorithm \( L^i \) for \( u^i \) for \( f^i(\cdot) \), i.e., for any NONA sequence of \( \{x^i_t\} \), the following holds with probability \( 1 - \delta \):

\[
\max_{u^i \in U^i} \sum_{t=1}^{T} f^i(x^i_t, u^i) - \sum_{t=1}^{T} f(x^i_t, u_t^i) \leq R^i_u(T, \delta),
\]

where \( \{ u_t^i \} \) is the output of \( L^i \).

(ii) There exists an online learning algorithm \( L^\lambda \) for \( \lambda \) for \( g(\cdot) \). That is, for any NONA sequences of \( \{x^i_t\} \) and \( \{ \tilde{u}^i_t\} \), the following holds with probability \( 1 - \delta \):

\[
\max_{\lambda \in \Lambda} \sum_{t=1}^{T} \langle \lambda, f(x^i_t, \tilde{u}^i_t) \rangle - \sum_{t=1}^{T} \langle \lambda_t, f(x^i_t, \tilde{u}^i_t) \rangle \leq R^\lambda(L, \delta),
\]

where \( \{ \lambda_t \} \) is the output of \( L^\lambda \).
There exists an online learning algorithm $\mathcal{L}_x$ for $x$ for $g(\cdot)$. That is, for any NONA sequences of $\{u_i^t\}$ and $\{\lambda_i^t\}$, the following holds with probability $1 - \delta$:

$$\sum_{t=1}^{T} \langle \lambda_i^t, f(x_t, u_i^t) \rangle - \min_{x \in \mathcal{X}} \sum_{t=1}^{T} \langle \lambda_i^t, f(x, u_i^t) \rangle \leq R_x(T, \delta),$$

where $\{x_t\}$ is the output of $\mathcal{L}_x$.

Note that the $\lambda$-learner and $u$-learner should be considered as the dual learners and $x$ as the primal learner. In fact, we show that the $\lambda$-learner and $u$-learner together give rise to a $(\bar{u}, \lambda)$-learner, which allows us then to reuse previous methodology. Further observe that, $\langle \lambda, f(x, \bar{u}) \rangle$ is a linear function of $\lambda$ and thus the second part of the assumption, for example, holds using the Follow the Perturbed Leader algorithm (see Kalai and Vempala [2005]).

**Proposition 5.2.** Suppose that Assumption 5.1 holds and that $\Lambda \subseteq \Delta(n)$. Then running $\mathcal{L}_u$ and $\mathcal{L}_x$ simultaneously is an online learning algorithm for $(\bar{u}, \lambda)$ of function $g(\cdot, \cdot)$. That is, for any NONA sequence $\{x_i^t\}$, let $\{u_i^t\}$ be the output of $\mathcal{L}_u$, and $\{\lambda_i^t\}$ be the output of $\mathcal{L}_x$, then with probability $1 - (n + 1)\delta$, we have

$$\max_{\lambda \in \Lambda, u \in \mathcal{U}_i^t} \sum_{t=1}^{T} \langle \lambda, f(x_t^t, \bar{u}) \rangle - \sum_{t=1}^{T} \langle \lambda_i^t, f(x_t^t, u_i^t) \rangle \leq \max_i R_u^i(T, \delta) + R_x(T, \delta).$$

By Proposition 5.2, there exist weak learners for both the primal and the dual player and solving Problem 2 reduces to solving Problem 1. Below we present the formal algorithm and the corresponding theorem with performance guarantees.

**Algorithm 3** Robust Optimization via Online Learning (ROOL) for maximum over functions (adaptive $\lambda$)

**Input:** function $f^1, \ldots, f^n$, learners $\mathcal{L}^1_u, \ldots, \mathcal{L}^n_u, \mathcal{L}_x$ satisfying Assumptions 4.1 and 5.1

**Output:** point $\bar{x}$

1: for $t = 1, \ldots, T$
2: \hspace{1em} $x_t \leftarrow \mathcal{L}_x(x_{t-1}, \bar{u}_1, \lambda_1, \ldots, x_{t-1}, \bar{u}_{t-1}, \lambda_{t-1})$
3: \hspace{1em} $u_i^t \leftarrow \mathcal{L}_u^i(x_t, \bar{u}_1^t, \lambda_1^t, \ldots, x_{t-1}, \bar{u}_{t-1}^t)$; \hspace{1em} $i = 1, \ldots, n$
4: \hspace{1em} $\lambda_i \leftarrow \mathcal{L}_\lambda(x_t, \bar{u}_1, \lambda_1, \ldots, x_{t-1}, \bar{u}_{t-1}, \lambda_{t-1})$
5: end for
6: $\bar{x} \leftarrow \frac{1}{T} \sum_{t=1}^{T} x_t$

**Theorem 5.3.** Suppose that Assumption 4.1 and 5.1 hold and that $\Lambda \subseteq \Delta(n)$. Then with probability $1 - (n + 2)\delta$, Algorithm 3 returns a point $\bar{x}$ satisfying

$$\max_{\bar{u}^t \in \prod \mathcal{U}_i^t, \lambda^t \in \Lambda} g(\bar{x}, \bar{u}^t, \lambda^t) - \min_{x \in \mathcal{X}} \max_{\bar{u}^t \in \mathcal{U}^t, \lambda^t \in \Lambda} g(x, \bar{u}^t, \lambda^t) \leq R_x(T, \delta) + \max_i R_u^i(T, \delta) + R_x(T, \delta).$$

To illustrate the result, let us consider the following example. Suppose all $f_i(x, u^t)$ are bilinear with respect to $x$ and $u^t$, as in the case of a robust linear programming, $\mathcal{X}$ and $\mathcal{U}_i^t$ are subsets of the Euclidean
space, and further suppose $\mathcal{X}$ is a convex set (notice that we make no such assumptions on the uncertainty sets $\mathcal{U}_i$). We say a set $\mathcal{Z} \subseteq \mathbb{R}^m$ is equipped with a linear optimization oracle, if given any $\theta \in \mathbb{R}^m$, we can compute $\text{Minimize} \quad \sum_{z \in \mathcal{Z}} \theta^\top z$. Thus, Assumption 5.1 holds if the followings are true:

(i) $\mathcal{U}_i$ is equipped with a linear optimization oracle for $i = 1, \ldots, n$.

(ii) $\Lambda$ is equipped with a linear optimization oracle.

(iii) $\mathcal{X}$ is equipped with a linear optimization oracle.

Indeed, each of the three conditions ensures the learnability in Assumption 5.1 for $u^i$, $\lambda$, and $x$ respectively, via e.g., the Follow the Perturbed Leader algorithm. This is due to the fact that for each argument, its respective objective function is linear.

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A Proofs

A.1 Proofs from Section 4

Proof of Theorem 4.4. Since $x_t$ and $u_t$ are obtained by $\mathcal{L}_x$ and $\mathcal{L}_u$, we have that $\{x_t\}_t$ and $\{u_t\}_t$ are NONA. Thus, by Assumption 4.2,

$$\sum_{t=1}^T f(x_t, u_t) - \min_{x \in \mathcal{X}} \sum_{t=1}^T f(x, u_t) \leq R_x(T, \delta),$$

and

$$\max_{u \in U} \sum_{t=1}^T f(x_t, u) - \sum_{t=1}^T f(x_t, u_t) \leq R_u(T, \delta),$$

hold simultaneously with probability $1 - 2\delta$. Summing up the two inequalities leads to

$$\max_{u \in U} \sum_{t=1}^T f(x_t, u) - \sum_{t=1}^T f(x_t, u_t) \leq R_x(T, \delta) + R_u(T, \delta).$$

(6)

By convexity of $f(\cdot, u)$, we have

$$T f(x, u) \leq \sum_{t=1}^T f(x_t, u_t)$$

for all $u \in U$, so that

$$T \max_{u' \in U'} f(x', u') \leq \max_{u \in U} \sum_{t=1}^T f(x_t, u).$$

(7)

Since $u_t \in U$ for all $t$ we also have for any $x$ that $f(x, u_t) \leq \max_{u \in U} f(x, u)$, which implies $\sum_{t=1}^T f(x, u_t) \leq T \max_{u \in U} f(x, u)$, further leading to

$$\min_{x \in \mathcal{X}} \sum_{t=1}^T f(x, u_t) \leq T \min_{x' \in \mathcal{X}} \max_{u \in U} f(x', u).$$

(8)

Combining Equations (7) and (8) we obtain

$$T \max_{u' \in U'} f(x', u') - T \min_{x' \in \mathcal{X}} \max_{u \in U} f(x', u) \leq \max_{u \in U} \sum_{t=1}^T f(x_t, u) - \min_{x \in \mathcal{X}} \sum_{t=1}^T f(x_t, u_t),$$

which together with (6) establishes the theorem.

Proof of Theorem 4.6. By Assumption 4.5 with probability $1 - \delta$ we have

$$\max_{u \in U} \sum_{t=1}^T f(x_t, u) - \sum_{t=1}^T f(x_t, u_t) \leq R_u(T, \delta).$$

And by definition of $x_t$,

$$\sum_{t=1}^T f(x_t, u_t) = \sum_{t=1}^T \min_{x' \in \mathcal{X}} f(x, u_t) \leq \min_{x' \in \mathcal{X}} \sum_{t=1}^T f(x, u_t).$$
Combining the two inequalities leads to
\[
\max_{u \in U} T \sum_{t=1}^{T} f(x_t, u) - \min_{x \in X} T \sum_{t=1}^{T} f(x, u_t) \leq R_u(T, \delta).
\]

The argument follows now identically to the proof of Theorem 4.4.

We obtain an analogous statement for Randomized Robust Optimization via Strong Primal Learner whose proof is almost identical to the proof of Theorem 4.6 from above.

**Algorithm 4** Randomized Robust Optimization via Strong Primal Learner

**Input:** function \( f \), learners \( \mathcal{L}_x \) satisfying 4.5

**Output:** distribution \( \mu \)

1. for \( t = 1, \ldots, T \) do
2. \( x_t \leftarrow \mathcal{L}_x(x_{1:t-1}, u_{1:t-1}) \)
3. \( u_t \leftarrow \arg \max_{u \in U} f(x_t, u) \)
4. end for
5. \( \mu \leftarrow \) empirical distribution of \( (x_{1:T}) \)

**Theorem A.1.** With probability \( 1 - \delta \), Algorithm 4 returns a distribution \( \mu \) satisfying

\[
\max_{\nu^* \in \Delta(U)} \mathbb{E}_{x \sim \mu, u \sim \nu^*} f(x, u) - \min_{\mu^* \in \Delta(X)} \max_{\nu \in \Delta(U)} \mathbb{E}_{x \sim \mu^*, u \sim \nu} f(x, u) \leq \frac{R_x(T, \delta)}{T}.
\]

**A.2 Proofs from Section 5**

**Proof of Proposition 5.2.** By Assumption 5.1, with probability \( 1 - (n + 1)\delta \), we have for all \( i \)

\[
\max_{u^i \in U^i} \sum_{t=1}^{T} f_i'(x_{t}^i, u_{t}^i) - \sum_{t=1}^{T} f_i(x_{t}^i, u_{t}^i) \leq R_u^i(T, \delta), \tag{9}
\]

and

\[
\max_{\lambda \in \Lambda} \sum_{t=1}^{T} \langle \lambda, f(x_t^i, u_t^i) \rangle - \sum_{t=1}^{T} \langle \lambda_t, f(x_t^i, u_t^i) \rangle \leq R_\lambda(T, \delta).
\]
With the above the following holds:

\[
\max_{\lambda \in \Lambda} \mathbb{E}_{x \sim \mu} \sum_{t=1}^{T} \langle \lambda, f(x_t, \bar{u}) \rangle - \sum_{t=1}^{T} \langle \lambda_t, f(x'_t, \bar{u}_t) \rangle \\
\leq \max_{\lambda \in \Lambda} \left[ \mathbb{E}_{x \sim \mu} \sum_{t=1}^{T} \langle \lambda, f(x_t, \bar{u}) \rangle - \sum_{t=1}^{T} \langle \lambda_t, f(x'_t, \bar{u}_t) \rangle \right] + \left[ \mathbb{E}_{x \sim \mu} \sum_{t=1}^{T} \langle \lambda_t, f(x'_t, \bar{u}_t) \rangle - \sum_{t=1}^{T} \langle \lambda_t, f(x'_t, \bar{u}_t) \rangle \right] \\
= \max_{\lambda \in \Lambda} \left[ \sum_{t=1}^{T} \sum_{x \sim \mu} \mathbb{E}_{u \sim d_t} \langle \lambda, f(x_t, u) \rangle - \sum_{t=1}^{T} \langle \lambda_t, f(x'_t, u_t) \rangle \right] + \left[ \sum_{t=1}^{T} \sum_{x \sim \mu} \mathbb{E}_{u \sim d_t} \langle \lambda_t, f(x'_t, u_t) \rangle - \sum_{t=1}^{T} \langle \lambda_t, f(x'_t, u_t) \rangle \right] \\
\leq \max_{\lambda \in \Lambda} \left( \sum_{i=1}^{n} \lambda_i R_i^*(T, \delta) + R_\lambda(T, \delta) \right) \\
\leq \max \left( \sum_{i=1}^{n} \lambda_i R_i^*(T, \delta) + R_\lambda(T, \delta) \right),
\]

where (a) follows with (9) and (b) holds from \( \Lambda \subseteq \Delta(n) \), which completes the proof. \( \square \)

**Proof of Theorem 5.3.** Observe that \( \{ \bar{u}_t \}, \{ x_t \} \) and \( \{ \lambda_t \} \) are all NONA. By Proposition 5.2 we have

\[
\max_{\lambda \in \Lambda} \sum_{t=1}^{T} \sum_{x \sim \mu} g(x_t, \bar{u}, \lambda) - \sum_{t=1}^{T} \sum_{x \sim \mu} g(x_t, \bar{u}_t, \lambda_t) \leq \sum_{t=1}^{T} \sum_{x \sim \mu} g(x_t, \bar{u}, \lambda_t) \leq R_\lambda(T, \delta),
\]

and by Assumption 5.1(iii) we have

\[
\sum_{t=1}^{T} \sum_{x \sim \mu} g(x_t, \bar{u}, \lambda_t) \leq \min_{x \in X} \sum_{t=1}^{T} g(x, \bar{u}_t, \lambda_t) \leq R_\lambda(T, \delta),
\]

The rest of the proof follows the proof of Theorem 4.4 and is omitted to avoid redundancy. \( \square \)

**B Randomized Case**

Our main framework can be extended to the case where the \( x \) player is allowed to randomize her action. Indeed, under such a setup, we can further lift the convexity requirement for \( f(\cdot, u) \) and \( X \). Specifically, we consider solving the following optimization problem

\[
\begin{align*}
\text{Minimize} & \quad \max_{\mu \in \Delta(X)} \mathbb{E}_{x \sim \mu, u \sim \nu} f(x, u) \\
\text{Minimize} & \quad \max_{\nu \in \Delta(U)} f(x, u) \\
& = \min_{x \in X} \sum_{t=1}^{T} \sum_{x \sim \mu} g(x_t, \bar{u}_t, \lambda_t) \leq R_\lambda(T, \delta).
\end{align*}
\]

This can be useful in cases where the decision maker wants to make her decision against an unknown, but non-adaptive environment; or even against an adversarial agent, as long as the adversarial agent does not take
his action after observing the decision maker’s action, i.e., we allow for an oblivious (true) adversary not to be confused with the learnability assumption and the implied imaginary adversary.

In the setup described above, if the decision maker chooses a randomized policy following a distribution \( \mu \), then the expected loss she incurs is upper bounded by

\[
\max_{u \in U} \mathbb{E}_{x \sim \mu} f(x, u),
\]

which is at least as small as, and can be significantly smaller than the more pessimistic upper bound \( \mathbb{E}_{x \sim \mu} \max_{u \in U} f(x, u) \). We next show that our method can readily be applied to this setup while relaxing the convexity requirement for \( \mathcal{X} \) and \( f(\cdot, u) \).

**Algorithm 5** Randomized Robust Optimization via Online Learning (R2OOL)

**Input:** function \( f \), learners \( \mathcal{L}_u, \mathcal{L}_x \) satisfying Assumption 4.2

**Output:** distribution \( \mu \)

1. Run Algorithm 1 with changed output

\[
\mu \leftarrow \text{empirical distribution of } (x_1, \ldots, x_T)
\]

**Theorem B.1.** With probability \( 1 - 2\delta \), Algorithm 5 returns a distribution \( \mu \) satisfying

\[
\max_{v' \in \Delta(U)} \mathbb{E}_{x \sim \mu} f(x, u) - \min_{\mu^* \in \Delta(\mathcal{X})} \max_{v \in \Delta(U)} \mathbb{E}_{x \sim \mu^*} f(x, u) \leq R_x(T, \delta) + R_u(T, \delta).
\]

**Proof of Theorem B.1.** Since \( x_t \) and \( u_t \) are obtained by \( \mathcal{L}_x \) and \( \mathcal{L}_u \), we have that \( \{x_t\}_t \) and \( \{u_t\}_t \) are NONA. Thus, by Assumption 4.2

\[
\sum_{t=1}^{T} f(x_t, u_t) - \min_{x \in \mathcal{X}} \sum_{t=1}^{T} f(x, u_t) \leq R_x(T, \delta)
\]

\[
\max_{u \in U} \sum_{t=1}^{T} f(x_t, u) - \sum_{t=1}^{T} f(x_t, u_t) \leq R_u(T, \delta),
\]

holds simultaneously with probability \( 1 - 2\delta \). Summing up the two inequalities leads to

\[
\max_{u \in U} \sum_{t=1}^{T} f(x_t, u) - \min_{x \in \mathcal{X}} \sum_{t=1}^{T} f(x, u_t) \leq R_x(T, \delta) + R_u(T, \delta).
\]

Since \( \mu \) is the empirical distribution of \( (x_1, \ldots, x_T) \), we have

\[
\max_{u \in U} \sum_{t=1}^{T} f(x_t, u) = T \max_{u \in U} \mathbb{E}_{x \sim \mu} f(x, u) = T \max_{v' \in \Delta(U)} \mathbb{E}_{x \sim \mu, u \sim v'} f(x, u).
\]
Further let $\nu$ be the empirical distribution of $(u_1, \ldots, u_T)$. Then we have

$$\min_{x \in X} \sum_{t=1}^{T} f(x, u_t) = T \min_{x \in X} \mathbb{E}_{u \sim \nu} f(x, u) = T \min_{\mu^* \in \Delta(X)} \mathbb{E}_{x \sim \mu^*} \mathbb{E}_{u \sim \nu} f(x, u) \leq T \min_{\mu^* \in \Delta(X)} \max_{\nu \in \Delta(U)} \mathbb{E}_{x \sim \mu^*, u \sim \nu} f(x, u).$$

Substituting the two equations to Equation (13) establishes the theorem. □

As before note that the second summand of the statement in the theorem is Problem (10). We remark that Problem (10) cannot be reduced to Problem (2) with the decisions being $\mu$ and $\nu$, and the objective function being $g(\mu, \nu) = \mathbb{E}_{x \sim \mu} f(x, u)$; in particular Algorithm [1] does not apply. This is due to two key differences: first we do not assume that $\mu$ and $\nu$ are learnable. Furthermore, for given $\mu$ and $\nu$ we do not observe the value of $g(\mu, \nu)$, instead, we only observe a noisy realization whose expected value equals $g(\mu, \nu)$.

C Multiple Objectives: Online Learning for Robust Optimization — Additional Results —

In this appendix we provide omitted details for Section 5.

C.1 Approach via Explicit Maximum

In the first case we model the maximum over the different functions $f_i$ explicitly. To this end, let $\bar{u} \in \prod_i U_i$ denote the concatenation of $u_i$, i.e., $\bar{u} = \{u^1, u^2, \ldots, u^n\}$ and define the function

$$F(x, \bar{u}) \triangleq \max_{\lambda \in \Lambda} \lambda^i f(x, u^i).$$

As before we have to specify the learnability for the two players.

Assumption C.1 (Learnability),

(i) There exists an online learning algorithm $\mathcal{L}_x$ for $x$, such that for any NONA sequence $\{\bar{u}_t^i\}_{t=1}^{\infty}$, the following holds with a probability $1 - \delta$

$$\sum_{t=1}^{T} F(x_t, \bar{u}_t^i) - \min_{x \in X} \sum_{t=1}^{T} F(x, \bar{u}_t^i) \leq R_x(T, \delta),$$

where $\{x_t\}$ is the output of $\mathcal{L}_x$.

(ii) For each $i = 1, \ldots, n$, there exists an online learning algorithm $\mathcal{L}_u^i$ for $u^i$, such that for any NONA sequence $\{x_t^i\}_{t=1}^{\infty}$, the following holds with a probability $1 - \delta$

$$\max_{u^i \in U^i} \sum_{t=1}^{T} f^i(x_t^i, u^i) - \sum_{t=1}^{T} f(x_t^i, u_t^i) \leq R_u^i(T, \delta),$$

where $\{u_t^i\}$ is the output of $\mathcal{L}_u^i$. 

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It is important to observe that for the $u$ player, we do not require learnability for $F$ the maximum over the $f_i$ but only the learnability for each separate $f_i$. In fact, as we will see later, these two notions of learnability are not equivalent. In the remainder of this subsection we will work under Assumptions 4.1 and C.1.

We are ready to formulate our algorithm, which is similar in spirit to Algorithm 1.

**Algorithm 6** Robust Optimization via Online Learning (ROOL) for maximum over functions (optimal $\lambda$)

**Input:** function $f^1, \ldots, f^n$, learners $L_u, L_x$ satisfying Assumptions 4.1 and C.1

**Output:** point $\bar{x}$

1. for $t = 1, \ldots, T$
   2. $x_t \leftarrow L_x(x_1, \bar{u}_1, \ldots, x_{t-1}, \bar{u}_{t-1})$
   3. $u_{i_t} \leftarrow L_u(x_1, u_{i_1}, \ldots, x_{t-1}, u_{i_{t-1}}); \ i = 1, \ldots, n$
4. end for
5. $\bar{x} \leftarrow \frac{1}{T} \sum_{t=1}^T x_t$

We establish the following guarantee.

**Theorem C.2.** With probability $1 - (n + 1)\delta$, Algorithm 6 returns a point $\bar{x} = \frac{1}{T} \sum_{t=1}^T x_t$ satisfying

$$T \max_{\bar{u} \in \prod_i U_i} F(\bar{x}, \bar{u}) - T \min_{X \in X} \max_{\bar{u} \in \prod_i U_i} F(x, \bar{u}) \leq R_x(T, \delta) + \max_{i} R_{i u}(T, \delta).$$

**Proof.** For the sake of exposition we use $f(\bar{x}, \bar{u})$ to denote the vector $(f(\bar{x}, u_i))_i$. First observe that for any $\bar{u} \in \prod_i U_i$, we have

$$T F(\bar{x}, \bar{u}) = T \max_{\lambda \in \Lambda} \langle \lambda, F(\bar{x}, \bar{u}) \rangle \leq \max_{\lambda \in \Lambda} \langle \lambda, \frac{1}{T} \sum_{t=1}^T f(x_t, \bar{u}) \rangle = \max_{\lambda \in \Lambda} \sum_{t=1}^T \langle \lambda, f(x_t, \bar{u}) \rangle,$$

and taking maximization of $u$ over both sides, we obtain

$$T \max_{\bar{u} \in \prod_i U_i} F(\bar{x}, \bar{u}) \leq \max_{\bar{u} \in \prod_i U_i} \max_{\lambda \in \Lambda} \sum_{t=1}^T \langle \lambda, f(x_t, \bar{u}) \rangle. \quad (14)$$
Moreover, we have

\[
\max_{\lambda \in \Lambda} \max_{\vec{u} \in [1 \ldots L]^T} \sum_{t=1}^{T} \left( \lambda, f(x_t, \vec{u}) \right) - \sum_{t=1}^{T} F(x_t, \vec{u}_t) \\
= \max_{\lambda \in \Lambda} \max_{\vec{u} \in [1 \ldots L]^T} \sum_{t=1}^{T} \left( \lambda, f(x_t, \vec{u}) \right) - \sum_{t=1}^{T} \max_{\lambda_t \in \Lambda} \left( \lambda_t, f(x_t, \vec{u}_t) \right) \\
\leq \max_{\lambda \in \Lambda} \max_{\vec{u} \in [1 \ldots L]^T} \sum_{t=1}^{T} \left( \lambda, f(x_t, \vec{u}) \right) - \max_{\lambda_0 \in \Lambda} \sum_{t=1}^{T} \left( \lambda_0, f(x_t, \vec{u}_t) \right) \\
\leq \max_{\lambda \in \Lambda} \left\{ \max_{\vec{u} \in [1 \ldots L]^T} \sum_{t=1}^{T} \left( \lambda, f(x_t, \vec{u}) \right) - \sum_{t=1}^{T} \left( \lambda, f(x_t, \vec{u}_t) \right) \right\} \\
\leq \max_{\lambda \in \Lambda} \left\{ \sum_{i=1}^{n} \lambda_i \left[ \max_{\vec{u} \in [1 \ldots L]^T} \sum_{t=1}^{T} f_i(x_t, \vec{u}_t) - \sum_{t=1}^{T} f_i(x_t, \vec{u}_t) \right] \right\} \\
(a) \leq \max_{\lambda \in \Lambda} \left\{ \sum_{i=1}^{n} \lambda_i \left[ R^i_\lambda(T, \delta) \right] \right\} \\
(b) \leq \max_i R^i_\lambda(T, \delta),
\]

holds with probability \(1 - n\delta\). Here, (a) holds from the assumption that the \(u_i\) are learnable, and (b) holds because \(\Lambda \subseteq \Delta(n)\).

We also have

\[
\sum_{t=1}^{T} F(x_t, \vec{u}_t) - \min_{x \in X} \sum_{t=1}^{T} F(x, \vec{u}_t) \leq R_x(T, \delta),
\]

and

\[
\min_{x \in X} \sum_{t=1}^{T} F(x, \vec{u}_t) - T \min_{x \in X} \max_{\vec{u} \in [1 \ldots L]^T} F(x, \vec{u}) \\
= \min_{x \in X} \sum_{t=1}^{T} F(x, \vec{u}_t) - \min_{x \in X} \sum_{t=1}^{T} \max_{\vec{u} \in [1 \ldots L]^T} F(x, \vec{u}) \\
\leq 0.
\]

Summing up Equations (14), (15), (16), (17), we get

\[
T \max_{\vec{u} \in [1 \ldots L]^T} F(\vec{x}, \vec{u}) - T \min_{x \in X} \max_{\vec{u} \in [1 \ldots L]^T} F(x, \vec{u}) \leq \max_i R^i_\lambda(T, \delta) + R_x(T, \delta),
\]

with probability \(1 - (n + 1)\delta\). \(\Box\)

Remark C.3. We remark that Theorem C.2 exploits the asymmetric nature of the primal player and the dual player, and in particular it is not a direct consequence of Theorem 4.4. Indeed when \(u^i\) are outputs of learning
algorithms for the $f^i(\cdot, \cdot)$, then the concatenation of these outputs is not necessarily a learning algorithm for $F(\cdot, \cdot)$: for example, let $U^1 = U^2 = [-1, +1]$ and $X = \{-1, +1\}^2$. Furthermore we pick functions $f^1(x_i, u^1) = \frac{1}{2}|x_i^1 - u^1|$, and $f^2(x_i, u^2) = \frac{1}{2}|x_i^2 - u^2|$. Now suppose $\{x_i\}$ are an iid sequence uniformly sampled from $\{-1, +1\}^2$. It is easy to see that for $f^1$ and $f^2$ the choice $u^1_i = u^2_i = 0$ achieves zero regret. However, $\mathbb{E} \sum_{t=1}^T F(x_i, u^1_t) = \frac{3}{4}T$, whereas $\mathbb{E} \sum_{t=1}^T F(x_i, [1, 1]^T) = \frac{3}{2}T$ because with probability $3/4$, either $f^1(x_i, 1) = 1$ or $f^2(x_i, 1) = 1$. Thus, $\bar{u}_i$ is not a no-regret sequence for $F(x_i, \cdot)$.

**Remark C.4.** Notice that the learnability assumption in this approach is asymmetric: for the primal player we require $F(\cdot, \cdot)$ to be learnable, whereas for the dual players we only require $f^i(\cdot, \cdot)$ to be learnable. We discuss some implications of this asymmetry. Recall that taking the maximum over $\lambda \in \Lambda$ preserves convexity but not linearity w.r.t. the $x$ argument. Thus, if $f^i(\cdot, \cdot)$ are linear w.r.t. $x$ and learnable via *Follow the Perturbed Leader*, the same approach will not extend to $F(\cdot, \cdot)$. On the other hand, if $f^i(\cdot, \cdot)$ are convex w.r.t. $x$ and learnable via *online gradient descent*, then $F(\cdot, \cdot)$ is also learnable using *online gradient descent*. In particular, the gradient of $F(\cdot, \cdot)$ is obtained as $\langle \lambda^*, \nabla f^i(\cdot, \cdot) \rangle$ where $\lambda^*$ maximizes $\langle \lambda, f^i(\cdot, \cdot) \rangle$ over $\Lambda$.

### C.2 Biased Play with a Strong Learner

We briefly discuss the biased imaginary play case, where one player is equipped with a strong learner, and the other player is equipped with an optimization oracle. We start with the Explicit Maximum approach.

**Assumption C.5 (Strong learner).**

(i) For each $i = 1, \ldots, n$, there exists an online learning algorithm $L^i_u$ for $u^i$, such that for any anticipatory sequence $\{x^i_t\}_{t=1}^T$, the following holds with a probability $1 - \delta$

$$\max_{u^i \in \mathcal{U}^i} \sum_{t=1}^T f^i(x^i_t, u^i) - \sum_{t=1}^T f^i(x^i_t, u^i_0) \leq R^i_u(T, \delta),$$

where $\{u^i_t\}$ is the output of $L^i_u$.

(ii) Given $\bar{u}$, there exists an optimization oracle for $x$ w.r.t. $F(\cdot, \cdot)$, that computes

$$x^* = \arg \max_{x \in X} F(x, \bar{u}).$$

**Algorithm 7 Robust Optimization via Strong Dual Learner**

**Input:** function $f$, learners $L^i_u$ satisfying Assumptions 4.1 and C.5

**Output:** point $\bar{x}$

```
1: for $t = 1, \ldots, T$ do
2:    $u^i_t \leftarrow L^i_u(x_1, u^i_1, \ldots, x_{t-1}, u^i_{t-1})$, $i = 1, \ldots, n$
3:    $x_t \leftarrow \arg \max_{x \in X} F(x, \bar{u})$
4: end for
5: $\bar{x} \leftarrow \frac{1}{T} \sum_{t=1}^T x_t$
```

**Assumption C.6 (Strong $x$ learner).**

(i) There exists an online learning algorithm $L_x$ for $x$, such that for any anticipatory sequence $\{\bar{u}_t\}_{t=1}^\infty$, the following holds with a probability $1 - \delta$

$$\sum_{t=1}^T F(x_t, \bar{u}_t) - \min_{x \in X} \sum_{t=1}^T f^i(x_t, \bar{u}_t) \leq R_x(T, \delta),$$
where \( \{ x_t \} \) is the output of \( \mathcal{L}_x \).

(ii) Given \( x \), for each \( i = 1, \ldots, n \), there exists an optimization oracle for \( u^i \) w.r.t. \( f^i(\cdot) \), that computes

\[
u^i = \arg \max_{u^i \in U^i} f^i(x, u^i).
\]

Algorithm 8 Robust Optimization via Strong Primal Learner

**Input:** function \( f \), learners \( \mathcal{L}_x \) satisfying Assumptions 4.1 and C.6

**Output:** point \( \overline{x} \)

1: for \( t = 1, \ldots, T \) do
2: \( x_t \leftarrow \mathcal{L}_x(x_1, u_1^1, \ldots, x_{t-1}, u_{t-1}^i) \)
3: \( u_t^i \leftarrow \arg \max_{u^i \in U^i} f^i(x_t, u^i), \ i = 1, \ldots, n \)
4: end for
5: \( \overline{x} \leftarrow \frac{1}{T} \sum_{t=1}^T x_t \)

**Theorem C.7.**
(i) Suppose Assumptions 4.1 and C.5 hold, then with probability \( 1 - n\delta \), Algorithm 7 returns a point \( \overline{x} = \frac{1}{T} \sum_{t=1}^T x_t \) satisfying

\[
T \max_{\overline{u} \in \prod_i U^i} F(\overline{x}, \overline{u}) - T \min_{x \in X} \max_{\overline{u} \in \prod_i U^i} F(x, \overline{u}) \leq \max_i R^i_u(T, \delta).
\]

(ii) Suppose Assumptions 4.1 and C.6 hold, then with probability \( 1 - \delta \), Algorithm 8 returns a point \( \overline{x} = \frac{1}{T} \sum_{t=1}^T x_t \) satisfying

\[
T \max_{\overline{u} \in \prod_i U^i} F(\overline{x}, \overline{u}) - T \min_{x \in X} \max_{\overline{u} \in \prod_i U^i} F(x, \overline{u}) \leq R_x(T, \delta).
\]

**Proof.** The proof of first claim follows by adapting the proof of Theorem C.2 by replacing Equation 16 by

\[
\sum_{i=1}^T F(x_t, \overline{u}_t) - \min_{x \in X} \sum_{t=1}^T F(x, \overline{u}_t) = 0,
\]

using the fact that \( x_t \) is obtained via the optimization oracle.

The proof of the second claim is by adapting the proof of Theorem C.2 specifically, replacing (a) of Equation 13 to

\[
\max_{\lambda \in \Lambda} \left\{ \sum_{i=1}^n \lambda_i \left[ \max_{u^i \in U^i} \sum_{t=1}^T f^i(x_t, u^i) - \sum_{t=1}^T f^i(x_t, \overline{u}_t^i) \right] \right\} = 0,
\]

using the fact that \( \overline{u} \) is a result of the optimization oracle. \( \blacksquare \)

Similarly, extending the distributional maximum approach to the biased imaginary play is straightforward as well. Indeed, using the fact that Problem 3 can be rewritten as

\[
\text{Minimize }_{x \in X} \text{ Maximize }_{\overline{u} \in \prod_i U^i, \lambda \in \Lambda} g(x, \overline{u}, \lambda),
\]

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we immediately conclude that if there exists a strong learner for $x$ w.r.t. $g(\cdot)$, and optimization oracle both for $u_i$ w.r.t $f^i(\cdot)$, and for $\lambda$ w.r.t $g(\cdot)$, which implies an optimization oracle for $(\bar{u}, \lambda)$ jointly w.r.t $g(\cdot)$, then the problem reduces to the biased imaginary play for a single objective function discussed in Section [4.2]. On the other hand, suppose strong online learners exist for both $u_i$ and for $\lambda$, then following a similar argument as that of Proposition [5.2], it can be established that playing these two learning algorithms simultaneously is a strong learner for $(\bar{u}, \lambda)$ w.r.t $g(\cdot)$. Therefore, Problem [5] reduces to the biased imaginary play for a single objective function if an optimization oracle for $x$ w.r.t $g(\cdot)$ exists. Notice that since $g(\cdot)$ is linear to $\lambda$, constructing a strong learner for $\lambda$ is relatively easy (e.g., using online gradient descent).

### C.3 Randomized robust optimization

In the single objective function case we have shown that it is possible to relax Assumption [4.1], i.e., allowing the loss function $f(\cdot)$ and $x'$ to be non-convex to obtain a randomized solution that is optimal. In general, this relaxation cannot be extended to the multi-function case. To see that, take the approach via the distributional maximum as an example. Directly adapting the results from Section [B] leads to an algorithm which outputs a distribution $\mu$ such that

$$\max_{\lambda, \bar{u}} E_{x \sim \mu} g(x, \bar{u}, \lambda) - \min_{\mu'} \max_{\lambda', \bar{u}'} E_{x' \sim \mu'} g(x', \bar{u}', \lambda') \to 0.$$  

However, the quantity of interest that we want to compare to is $\max_{\bar{u}} E_{x \sim \mu} \left[ \max_{\lambda} g(x, \bar{u}, \lambda) \right]$: if the algorithm outputs a solution $x$ randomly, then the corresponding worst-case $\lambda$ should adapt to the random choice of $x$.

Here, we consider the following special case, for which extension to multiple functions is possible. Specifically, we consider the case that $f_j(x, u^i) \geq 0$ for all $i, x, u^i$, and seek a solution $x^*$ such that $\max_i \max_{\lambda_i} f_i(x^*, u^i) = 0$. This is motivated by the robust feasibility problem where $f_i(x, u^i)$ is the violation of the $i$-th constraint under parameter realization $u^i$. In particular, we assume $\Lambda = \Delta(n)$.

We will show that in this case, both explicit maximum and distributional maximum approach can be extended. The algorithms are identical to Algorithms [6] and [7] except for the last stage, where we output a distribution $\mu$ which is the empirical distribution of $x^i$ instead of the average. We call the resulting algorithm randomized explicit maximum and randomized distributional maximum respectively.

**Theorem C.8.** Suppose Assumption [C.1] holds, with probability $1 - (n + 1)\delta$, the randomized explicit maximum algorithm returns a distribution $\mu$ satisfying

$$\max_{\bar{u} \in \mathbb{I}_1^{t \times d}} E_{x \sim \mu} F(x, \bar{u}) - (n + 1) \min_{\mu' \in \Delta(\mathcal{X})} \max_{\bar{u}' \in \mathbb{I}_1^{t \times d}} E_{x' \sim \mu'} F(x', \bar{u}') \leq \frac{(n + 1) R_x(T, \delta) + \sum_{i=1}^n R_{x_i}(T, \delta)}{T}.$$  

Moreover, if the problem is infeasible, i.e., there exists $\varepsilon > 0$, such that for any $\mu' \in \Delta(\mathcal{X})$,

$$\max_{\bar{u} \in \mathbb{I}_1^{t \times d}} E_{x' \sim \mu'} F(x', \bar{u}') > \varepsilon,$$

then with probability $1 - n\delta$,

$$\liminf_{T \to \infty} \frac{1}{T} \sum_{t=1}^T F(x_t, \bar{u}_t) \geq \frac{1}{n + 1}\varepsilon.$$  

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Proof. For any \( \bar{u} \),

\[
\sum_{t=1}^{T} F(x_t, \bar{u}) - \sum_{t=1}^{T} F(x_t, \bar{u}_t)
= \sum_{t=1}^{T} \max_{i_t \in \{1,2,\ldots,n\}} f_i^t(x_t, u^i_t) - \sum_{t=1}^{T} \max_{i_t \in \{1,2,\ldots,n\}} f_i^t(x_t, u_{t}^{i_t})
\leq \sum_{t=1}^{T} \max_{i_t \in \{1,2,\ldots,n\}} \left[ f_i^t(x_t, u^i_t) - f_i^t(x_t, u_{t}^{i_t}) \right].
\]

Denote by \( i_t^* \triangleq \arg\max_{i_t \in \{1,2,\ldots,n\}} \left[ f_i^t(x_t, u^i_t) - f_i^t(x_t, u_{t}^{i_t}) \right] \), then the right hand side equals

\[
\sum_{t=1}^{T} \left[ f_{i_t^*}^t(x_t, u_{i_t^*}^t) - f_i^t(x_t, u_{t}^{i_t}) \right]
= \sum_{t=1}^{T} \sum_{i=1}^{n} \left[ f_i^t(x_t, u^i_t) - f_i^t(x_t, u_{t}^{i_t}) \right] - \sum_{t=1}^{T} \sum_{i \not= i_t^*} \left[ f_i^t(x_t, u^i_t) - f_i^t(x_t, u_{t}^{i_t}) \right]
\leq \sum_{t=1}^{T} \sum_{i=1}^{n} \left[ f_i^t(x_t, u^i_t) - f_i^t(x_t, u_{t}^{i_t}) \right] - \min_{\bar{u}} \sum_{t=1}^{T} \sum_{i \not= i_t^*} \left[ f_i^t(x_t, u^i_t) - f_i^t(x_t, u_{t}^{i_t}) \right]
\leq \sum_{t=1}^{T} \sum_{i=1}^{n} \left[ f_i^t(x_t, u^i_t) - f_i^t(x_t, u_{t}^{i_t}) \right] + \sum_{t=1}^{T} \sum_{i=1}^{n} f_i^t(x_t, u_{t}^{i_t}),
\]

where we use non-negativity of \( f(\cdot) \) for the last inequality. Take maximum over \( \bar{u} \), we have

\[
\max_{\bar{u}} \sum_{t=1}^{T} F(x_t, \bar{u}) - \sum_{t=1}^{T} F(x_t, \bar{u}_t)
\leq \max_{\bar{u}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left[ f_i^t(x_t, u^i_t) - f_i^t(x_t, u_{t}^{i_t}) \right] + \sum_{i=1}^{n} \sum_{t=1}^{T} f_i^t(x_t, u_{t}^{i_t})
\leq \sum_{i=1}^{n} R_i^T(T, \delta) + n \sum_{t=1}^{T} F(x_t, \bar{u}_t)
\]

holds with probability \( 1 - n \delta \). Here, the last equality holds by Assumption C.1 requiring that the \( u_i \) are learnable.

We also have

\[
\sum_{t=1}^{T} F(x_t, \bar{u}_t) - \min_{x \in \mathcal{X}} \sum_{t=1}^{T} F(x, \bar{u}_t) \leq R_x(T, \delta).
\]

Summing up the two inequalities, we obtain

\[
\max_{\bar{u}} \sum_{t=1}^{T} F(x_t, \bar{u}) - \min_{x \in \mathcal{X}} \sum_{t=1}^{T} F(x, \bar{u}_t) \leq \sum_{i=1}^{n} R_i^T(T, \delta) + n \sum_{t=1}^{T} F(x_t, \bar{u}_t) + R_x(T, \delta)
\leq \sum_{i=1}^{n} R_i^T(T, \delta) + n \min_{x \in \mathcal{X}} \sum_{t=1}^{T} F(x, \bar{u}_t) + (n + 1) R_x(T, \delta),
\]
which leads to
\[
\max_{\vec{u}} \sum_{t=1}^{T} F(x_t, \vec{u}) - (n+1) \min_{x \in X} \sum_{t=1}^{T} F(x, \vec{u}_t) \leq \sum_{i=1}^{n} R^i_u(T, \delta) + (n+1) R_x(T, \delta),
\]

The rest of the proof for the first claim follows similarly as that of Theorem B.1.

To establish the second statement, notice that when the problem is infeasible, by definition there exists \( \epsilon > 0 \) such that
\[
\frac{1}{T} \sum_{t=1}^{T} F(x_t, \vec{u}) \geq \epsilon.
\]

Combining this with Equation (18) we have with probability \( 1 - n\delta \),
\[
\liminf_{T} \frac{1}{T} \sum_{t=1}^{T} F(x_t, \vec{u}_t) \geq \frac{\epsilon}{n+1}.
\]

\( \square \)

**Theorem C.9.** Suppose that Assumption 5.1 holds and that \( \Lambda = \Delta(n) \). Then with probability \( 1 - (n+2)\delta \), the randomized distributional maximum algorithm returns a distribution \( \mu \) satisfying
\[
\max_{\vec{u} \in \Pi, \lambda \in \Lambda} \mathbb{E}_{x \sim \mu} g(x, \vec{u}, \lambda) - n \min_{\lambda' \in \Delta(X)} \max_{\vec{u}' \in \Pi} \mathbb{E}_{x' \sim \mu'} g(x', \vec{u}', \lambda') \leq \frac{n \{ R_x(T, \delta) + \max_i R^i_u(T, \delta) + R_\lambda(T, \delta) \}}{T}.
\]

Moreover, \( \mu \) is the empirical distribution over the \( x_t \in X \) played by the x-player.

**Proof.** By Proposition 5.2 we have
\[
\max_{\lambda \in \Lambda, \vec{u} \in \Pi, \lambda' \in \Lambda} \sum_{t=1}^{T} g(x_t, \vec{u}, \lambda) - \sum_{t=1}^{T} g(x_t, \vec{u}_t, \lambda_t) \leq \max_i R^i_u(T, \delta) + R_\lambda(T, \delta),
\]
and by Assumption 5.1(iii) we obtain a distribution \( \mu \)
\[
\sum_{t=1}^{T} g(x_t, \vec{u}_t, \lambda_t) - \min_{x \in X} \sum_{t=1}^{T} g(x, \vec{u}_t, \lambda_t) \leq R_x(T, \delta).
\]

Following the proof of Theorem B.1 we have
\[
\max_{\vec{u} \in \Pi, \lambda \in \Lambda} \mathbb{E}_{x \sim \mu} g(x, \vec{u}, \lambda) - \min_{\lambda' \in \Delta(X)} \max_{\vec{u}' \in \Pi} \mathbb{E}_{x' \sim \mu'} g(x', \vec{u}', \lambda') \leq \frac{R_x(T, \delta) + \max_i R^i_u(T, \delta) + R_\lambda(T, \delta)}{T}.
\]

Now by the non-negativity assumption of the \( f_i(\cdot, \cdot) \) (and hence \( g(\cdot) \)), we have
\[
\max_{\lambda \in \Lambda} g(x, \vec{u}, \lambda) \leq \sum_{i=1}^{n} g(x, \vec{u}, e_i)
\]

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where $e_i$ is the $i$-th basis vector (i.e., the $i$-th entry equals 1, and the rest equals 0). This leads to

$$
\max_{\bar{u} \in \prod_{I} U} \max_{\lambda \in \Lambda} \mathbb{E}_{x \sim \mu} g(x, \bar{u}, \lambda)
\leq \max_{\bar{u} \in \prod_{I} U} \mathbb{E}_{x \sim \mu} \sum_{i=1}^{n} g(x, \bar{u}, e_i)
\leq (a) \max_{\bar{u} \in \prod_{I} U} \left[ n \cdot \max_{\lambda \in \Lambda} \mathbb{E}_{x \sim \mu} g(x, \bar{u}, \lambda) \right],
$$

where (a) holds because $\Lambda = \Delta(n)$. Using

$$
\min_{\mu' \in \Delta(X)} \max_{\bar{u}' \in \prod_{I} U} \max_{\lambda' \in \Lambda} \mathbb{E}_{x' \sim \mu'} g(x', \bar{u}', \lambda') \leq \min_{\mu' \in \Delta(X)} \max_{\bar{u}' \in \prod_{I} U} \max_{\lambda' \in \Lambda} \mathbb{E}_{x' \sim \mu'} g(x', \bar{u}', \lambda'),
$$

the theorem follows via

$$
\begin{align*}
&\max_{\bar{u} \in \prod_{I} U} \mathbb{E}_{x \sim \mu} g(x, \bar{u}, \lambda) - n \min_{\mu' \in \Delta(X)} \max_{\bar{u}' \in \prod_{I} U} \max_{\lambda' \in \Lambda} \mathbb{E}_{x' \sim \mu'} g(x', \bar{u}', \lambda') \\
&\leq \max_{\bar{u} \in \prod_{I} U} \mathbb{E}_{x \sim \mu} g(x, \bar{u}, \lambda) - n \min_{\mu' \in \Delta(X)} \max_{\bar{u}' \in \prod_{I} U} \max_{\lambda' \in \Lambda} \mathbb{E}_{x' \sim \mu'} g(x', \bar{u}', \lambda') \\
&\leq n \max_{\bar{u} \in \prod_{I} U} \mathbb{E}_{x \sim \mu} g(x, \bar{u}, \lambda) - n \min_{\mu' \in \Delta(X)} \max_{\bar{u}' \in \prod_{I} U} \max_{\lambda' \in \Lambda} \mathbb{E}_{x' \sim \mu'} g(x', \bar{u}', \lambda') \\
&\leq n \left( R_x(T, \delta) + \max_i R^i_x(T, \delta) + R_T(T, \delta) \right).
\end{align*}
$$

\[
\square
\]

The above result bounds the expected performance of a random $x$ according to $\mu$ versus $n$ times the performance of the optimal $\mu'$; this notion is very close to the concept of $\alpha$-regret. However, in our context this approximation does not matter: there exists a robust feasible $x^*$ if and only if

$$
\min_{\mu' \in \Delta(X)} \max_{\bar{u}' \in \prod_{I} U} \max_{\lambda' \in \Lambda} \mathbb{E}_{x' \sim \mu'} g(x', \bar{u}', \lambda') = 0.
$$

Thus, the multiplicative factor has no impact in the feasibility case, if the problem is feasible, then we are able to output a randomized solution whose constraint violation is sublinear in $T$. Note however, that this will likely reduce the convergence rate by some factor, which is slower but still sublinear in $T$ for reasonable learners.

## D Application: RO with IP-representable sets

In this section we provide an application, which is naturally captured by Problem \([10]\). In specific, we instantiate our framework for the case where both the feasible region $\mathcal{X}$ as well as the uncertainty set $\mathcal{U}$ can be represented via an (Mixed-) Integer Programming Problem. In many cases optimizing over such sets is NP-hard and in particular no good characterization in terms of valid inequalities can be provided (unless coNP = NP), however even very large instances can be solved in reasonable time using state-of-the-art integer programming solvers such as e.g., CPLEX or Gurobi. In these cases the assumption that we have a
linear programming oracle for these sets is justified and we assume in this section that we have access to both the feasible region as well as the uncertainty set by means of a linear optimization oracle.

We assume that \( f \) is linear in \( x \) and \( u \), i.e., we let \( f(x, u) = u^T A x \) for some matrix \( A \) of appropriate dimension. For this setup both learners can use the Follow the Perturbed Leader algorithm [see Kalai and Vempala (2005)] that works for online learning of linear functions over arbitrary (non-convex) sets as long as we have access to a linear programming oracle for the set. The FPL algorithm has the following high-probability regret guarantee (Corollary D.2), which follows almost immediately from [Neu and Bartók (2013)].

**Theorem D.1** (Neu and Bartók (2013)). Let \( S \subseteq \mathbb{R}^d_+ \) with \( \|s\|_1 \leq m \) for all \( s \in S \). Assume that the absolute values of all losses are bounded by \( 1 \). Then FPL over \( S \) achieves an expected regret of \( O(m^{3/2} \sqrt{T \log d}) \).

From the above, via standard arguments, we can easily obtain a high-probability version of the regret bound of FPL:

**Corollary D.2** (High-probability FPL). Let \( \delta > 0 \) and let \( S \subseteq \mathbb{R}^d_+ \) with \( \|s\|_1 \leq m \) for all \( s \in S \). Assume that the absolute values of all losses are bounded by \( 1 \). Then with probability at least \( 1 - \delta \), algorithm FPL over \( S \) achieves regret \( O(m^{3/2} \sqrt{T \log d} + \log \frac{1}{\delta}) \).

**Proof.** For any sequence of losses \( \{\ell_t\}_{t \in [T]} \) with \( |\ell_t| \leq 1 \) for all \( s \in S, t \in [T] \) by Theorem D.1 FPL produces a sequence of actions \( \{s_t\}_{t \in [T]} \subseteq S \) with

\[
\mathbb{E}_t \left[ \sum_{t \in [T]} \ell_t s_t \right] - \min_{s \in S} \sum_{t \in [T]} \ell_t s \leq O(m^{3/2} \sqrt{T \log d}),
\]

where \( \mathbb{E}_t [\cdot] \) is the conditional expectation conditioned on FPL’s internal randomization up to round \( t - 1 \). The random variables \( Z_t = \sum_{t \in [T]} \ell_t s_t - \mathbb{E}_t \left[ \sum_{t \in [T]} \ell_t s_t \right] \) form a martingale difference sequence with respect to FPL’s internal randomization for \( t \in [T] \) and we have \( Z_t \leq 2 \). Therefore with probability at least \( 1 - \delta \), we obtain:

\[
\sum_{t \in [T]} \ell_t s_t - \mathbb{E}_t \left[ \sum_{t \in [T]} \ell_t s_t \right] \leq 2 \sqrt{T \log \frac{1}{\delta}},
\]

via Azuma’s inequality and hence the claim follows by summing up both inequalities.

We would like to remark that FPL also admits an approximate variant; see [Ben-Tal et al. (2015)].

**Setup.**

(i) **Learner** \( \mathcal{L}_x \): FPL over \( \mathcal{X} \) and access to a linear optimization oracle for \( \mathcal{X} \).

(ii) **Learner** \( \mathcal{L}_y \): FPL over \( \mathcal{U} \) and access to a linear optimization oracle for \( \mathcal{U} \).

(iii) **Function** \( f \): Require that \( f(\cdot, u) \) is linear for all \( u \in \mathcal{U} \) and \( f(x, \cdot) \) is linear for all \( x \in \mathcal{X} \), i.e., \( f(x, u) = u^T A x \).

(iv) **Feedback**: Player \( x \) observes \( u^T A \) and player \( u \) observes \( Ax \).

Combining the above with Theorem 4.4 we obtain:
Corollary D.3. Let the setup be given as above. With probability $1 - 2\delta$, Algorithm 3 returns an empirical distribution $\tilde{\mu}$ satisfying

$$\max_{\nu' \in \Delta(\mathcal{U})} \mathbb{E}_{\nu \sim \tilde{\mu}} f(x, u) - \min_{\mu^* \in \Delta(\mathcal{X})} \max_{\nu' \in \Delta(\mathcal{U})} \mathbb{E}_{\nu \sim \mu^*, u \sim \nu'} f(x, u) \leq O \left( \frac{m^{3/2}_X \sqrt{\log d_X + \log \frac{1}{\delta}} + m^{3/2}_U \sqrt{\log d_U + \log \frac{1}{\delta}}}{\sqrt{T}} \right),$$

where $m_X, d_X, m_U, d_U$ are the dimensions and $\ell_1$-diameters of $\mathcal{X}$ and $\mathcal{U}$ respectively.

Remark D.4 (Infeasibility of dualization approach). Note that this setup is a good example where we cannot solve the robust problem via dualizing the uncertainty set. If the uncertainty set $\mathcal{U}$ is intractable (e.g., optimizing over $\mathcal{U}$ is NP-hard or $\text{conv}(\mathcal{U})$ has high extension complexity), then a tractable dual formulation cannot exist. However, having access to linear optimization oracles still allows us to solve the robust optimization problem.

Remark D.5 (Non-adaptivity of $u$). We would like to stress, that we do not solve $\min_{x \in \mathcal{X}} \max_{u \in \mathcal{U}} f(x, u)$ with solution pair $(x, u) \in \mathcal{X} \times \mathcal{U}$ but rather $\tilde{x} \in \text{conv}(\mathcal{X})$, i.e., it is a mixed strategy and $\min_{x \in \mathcal{X}} \max_{u \in \mathcal{U}} f(x, u)$ should be considered the value that we benchmark against. We showed that $\tilde{x}$ can achieve the same value (with vanishing regret) as compared to choosing the minimal $x^*$, i.e.,

$$\max_{u' \in \mathcal{U}} \min_{x^* \in \mathcal{X}} f(x^*, u') \leq \min_{x^* \in \mathcal{X}} \max_{u \in \mathcal{U}} f(x^*, u) + o(1),$$

as $u$ is not adaptive with respect to $x$. Here non-adaptivity refers to the $u$-player having to make her decision $u_t$ in round $t$ without knowing what the $x$-player will play as $x_t$ in round $t$ (and also vice versa with swapped roles). Put differently, the non-adaptivity ensures that we solve the problem

$$\min_{\mu \in \Delta(\mathcal{X})} \max_{u \in \mathcal{U}} \mathbb{E}_{x \sim \mu, u \sim \mathcal{X}} [f(x, u)],$$

which is equivalent to solving the problem over the respective convex hulls of $\mathcal{X}$ and $\mathcal{U}$.

We now present a sample application.

Example D.6 (Robust Routing in Unrealiable Networks). Let $G = (V, E)$ be an undirected graph with distinguished source $s$ and sink $t$ with $s, t \in V$. We consider the setup of robust minimal cost routing in $G$ with unrealiable edges: we want to find a route (without revisiting edge) of minimum cost from $s$ to $t$ in $G$ under various scenarios where edges might become unavailable.

We let the $x$ learner choose paths over $G(V, E)$, which can be represented as $\mathcal{X} \subseteq \{0, 1\}^E$. The corresponding optimization problem over $\mathcal{X}$ for nonnegative costs $\{c_e\}_{e \in E}$ can be solved e.g., using network flows or shortest path computations. The $u$ learner is playing edge removal subject to budget and connectivity constraints: The set $\mathcal{U}$ is given by

$$\mathcal{U} := \{C \subseteq E \mid |C| \leq K, G[E \setminus C] \text{ is connected}\},$$

i.e., the adversary can remove a small number of edges as long as the graph remains connected, e.g., removing valuable shortcuts between nodes but the adversary cannot adapt to the chosen route.
Another application of our framework is solving robust MDPs with parametric uncertainty. An MDP is defined by a 6-tuple: \( (T, \gamma, S, A, p, r) \), where \( T \) is the (possibly infinite) decision horizon, \( \gamma \in (0, 1] \) is the discount factor, \( S \) is the state space, \( A \) is the action space, and \( p \) and \( r \) are the transition probability and reward respectively. The decision criterion is to find a policy \( \pi \) that maximizes the expected cumulative discounted reward. See the classical textbook for more details [Puterman, 1994].

Robust MDPs [Nilim and El Ghaoui, 2005], [Iyengar, 2005] are concerned with solving MDPs under parametric uncertainty: here \( p \) and \( r \) are unknown, but instead a so-called “uncertainty set” \( \mathcal{U} \) is given such that \( (p, r) \in \mathcal{U} \), and the decision criterion is to find an optimal policy \( \pi \) for the worst-case parameter realization in \( \mathcal{U} \):

\[
\text{Maximize} \quad \min_{(p, r) \in \mathcal{U}} \{ \mathbb{E}_{\pi, p, r} \left[ \sum_{t=1}^{T} \gamma^{t-1} r(s_t, a_t) \right] \}. \tag{19}
\]

This problem is in general hard even when \( \mathcal{U} \) is a relatively simple set [Wiesemann et al., 2013]. The two special cases where the problem can be solved efficiently are (1) when \( \mathcal{U} \) is rectangular, i.e., it is a Cartesian product of uncertainty sets of parameters of each state; or (2) only the reward parameters \( r \) are subject to uncertainty.

Specifically, when \( p \) is known and only \( r \) is subject to uncertainty with \( r \in \mathcal{U} \), Problem (19) can be reformulated as:

\[
\max_{x} \min_{r \in \mathcal{U}} \sum_{s \in S} \sum_{a \in A} r(s, a) x(s, a) \tag{20}
\]

\[
\text{s.t.} \quad \sum_{a \in A} x(s', a) - \sum_{s \in S} \sum_{a \in A} \gamma p(s' | s, a) x(s, a) = 0, \forall s', \forall a,
\]

\[
x(s, a) \geq 0, \forall s, \forall a.
\]

Here \( a(\cdot) \) is the distribution of the initial state. Let \( x^* \) be the optimal solution to the above problem, then the optimal policy to (19) is obtained by \( q_\delta(a) = x^*(s, a) / (\sum_{a' \in A} x^*(s, a')) \) where \( q_\delta(a) \) stands for the probability of choosing action \( a \) at state \( s \).

\[^1\text{for simplicity we consider } T = \infty, \text{as the finite horizon case is easily converted into an infinite horizon case}\]
Notice that when \( \mathcal{S} \) and \( \mathcal{A} \) are even moderately large, solving (20) can be computationally expensive, even when \( \mathcal{U} \) is a nice and simple set. For example, suppose \( \mathcal{U} = \{ r \mid \| r - r_0 \|_2 \leq c \} \) for given \( r_0 \) and \( c \), i.e., the set of all reward vectors that are close (in the sense of Euclidean distance) to a “nominal parameter” \( r_0 \), then Problem (20) is a second order cone programming with \( |\mathcal{S}| \cdot |\mathcal{A}| \) linear constraints.

We now discuss how to solve Problem (20) using our proposed framework. Observe that Problem (20) can be rewritten as

\[
\begin{align*}
\text{Maximize} & \quad r^\top x \\
\text{Minimize} & \quad x \in \mathcal{X}
\end{align*}
\]

where \( \mathcal{X} \) is the feasible set of the constraints of (20). Thus, we can apply Algorithm 1 to solve (20). The key observation is that the objective function is linear with respect to \( x \), which means FPL is a weak learner for the primal player. Furthermore, each iteration of FPL solves Maximize \( \tilde{f}^\top x \) for some given \( \tilde{f} \). In the MDP context, we can solve such a problem by first finding the optimal deterministic policy \( \pi^* \) of the MDP where the reward vector is \( \tilde{r} \) (by value iteration or policy iteration), and then find \( x^* \) corresponding to \( \pi^* \). In particular, one can obtain \( x^* \) via first computing \( y^* \triangleq (1 - \gamma P_{\pi^*})^{-1} \alpha = \sum_{i=1}^{\infty} \gamma^{i-1} P_{\pi^*}^{i-1} \alpha \), where \( P_{\pi^*} \in \mathbb{R}^{(|\mathcal{S}| \times |\mathcal{S}|)} \) is the transition matrix of the Markov chain induced by \( \pi^* \), and then set

\[
x^*(s,a) = \begin{cases} y^*(s,a) & \text{if } a = \pi^*(s) \\ 0 & \text{otherwise.} \end{cases}
\]

Thus, the computation required for obtaining an \( \epsilon \)-optimal solution is \( O \left( (|\mathcal{S}|^2 + |\mathcal{S}| |\mathcal{A}|) \log 1/\epsilon \right) \) by combination with the computational requirement for value iteration and for finding \( y^* \). It is also worthwhile to mention that \( \|x\|_1 = \frac{1}{1 - \gamma} \), which is independent of \( |\mathcal{S}| \) and \( |\mathcal{A}| \).

For the dual player, online gradient descent is a learner (see e.g., Hazan [2016]):

**Theorem E.1.** Let \( f_1(\cdot), f_2(\cdot), \ldots, f_n(\cdot), \ldots \) be a sequence of arbitrary convex loss functions, possibly anticipatory, and let \( z_t \) be the output at \( t \)-stage of online gradient descent, then

\[
\sum_{t=1}^{T} f_t(z_t) - \min_{z \in \mathcal{Z}} \sum_{t=1}^{T} f_t(z) \leq \frac{3}{2} G D \sqrt{T},
\]

where \( G \) is an upper bound of the Lipschitz continuities of \( f_t(\cdot) \) and \( D \) is the diameter of \( \mathcal{Z} \).

Notice that for the dual player, \( G \leq \max \|x\|_2 \leq \|x\|_1 \leq \frac{1}{1 - \gamma} \). Also notice that computing the gradient for the dual player is trivial as it is just \( x_t \).

Invoking Theorem 4.4 together with Corollary D.2 and Theorem E.1, we obtain the following. We remark that the regret bound is almost independent of the dimensionality \( |\mathcal{S}| |\mathcal{A}| \).

**Corollary E.2.** Let the setup be given as above. With probability \( 1 - 2\delta \), Algorithm 7 returns a point \( x = \frac{1}{T} \sum_{t=1}^{T} x_t \) satisfying

\[
\max_{u \in \mathcal{U}} f(x, u') - \min_{u \in \mathcal{U}} \max_{x \in \mathcal{X}} f(x^*, u) \\
\leq O \left( \sqrt{\log(|\mathcal{S}| \cdot |\mathcal{A}|)} + \log \frac{1}{\delta} + \frac{D}{(1 - \gamma) \sqrt{T}} \right) + \frac{D}{(1 - \gamma) \sqrt{T}} M,
\]

where \( D \) is the diameter of \( \mathcal{U} \), and \( M = \max_{x,r} \|r^\top x\| \leq \max_r \|r\|_{\infty} / (1 - \gamma) \).
Remark E.3. In many cases $D$ is relatively small. For example, when $\mathcal{U} = \{r \mid \|r - r_0\|_2 \leq c\}$, then $D \leq c$. Another interesting case is when the perturbations are sparse, e.g., $\mathcal{U} = \{r \mid \|r - r_0\|_\infty \leq c, \|r - r_0\|_0 \leq d\}$. That is, only $d$ out of $|S||A|$ entries of $r$ are allowed to deviate from its nominal value, and each entry can deviate at most by $c$. Note that $\mathcal{U}$ is not a convex set. However, due to linearity of the objective function, we can optimize instead over the convex hull of $\mathcal{U}$, denoted by $\hat{\mathcal{U}}$, and the optimal solution to (20) over $\mathcal{U}$ and $\hat{\mathcal{U}}$ coincides. Moreover, the diameter of $\hat{\mathcal{U}}$ is bounded above by $2 \cdot \text{diameter}(\mathcal{U}) \leq 2\sqrt{dc}$.

F Computational Experiments

In this section we report computational experiments and while not a focus of this paper, we demonstrate the real-world practicality of our approach for Example D.6. Following the approach of Section D we run two FPL learners where the primal player is solving a min-cost flow problem over a graph, whose polyhedral formulation is integral and the dual player is playing the uncertainty.

Here we report results for three representative instances, which where completed within a few minutes of computational time. The smaller one has $n = 50$ nodes, the second one is a very dense graph instance with $n = 100$ nodes, and the larger one has $n = 1000$ nodes and is relatively sparse. We plot $\max_{u' \in \mathcal{U}} f(x, u') - \min_{x^* \in \mathcal{X}} \max_{u \in \mathcal{U}} f(x^*, u)$ (which we refer to as ‘regret’ slightly abusing notions) against iterations $t$ in Figures 1 and 2. In all cases our algorithm based on imaginary play converges very fast (note that the figures report in log scale) and in fact much faster as predicted by the conservative worst-case bound given in Example D.6.

![Figure 1: Regret (log scale) vs. Iterations. Left: Robust min cost flow with $n = 50$ nodes run for 1000 iterations. Right: Same instance run for 5000; approaching zero regret.](image-url)
F.1 Implementation details

We now provide implementation details for the computational experiments reported above for reproducibility of results. The implementation is in python 3.5 using the network simplex algorithm from the networkx library. All computational experiment were completed in a few minutes on a standard laptop with a 2.7 GHz Intel Core i5 processor; time per iteration is between 0.2s and 1s depending on the size of the graph (note that the network simplex in networkx is a pure python implementation).

The graph instances were generated using networkx’s fast_gnp_random_graph function with edge probabilities \( p \) varying between 1% and 30%. Demand was set to be 1 unit to be sent from node 0 to node 1, without loss of generality due to symmetry of the random graph generation. Both nominal edge capacities as well as costs where set to uniform random values between 0 and 1 inducing non-trivial min cost flows in the nominal problem for the uncertainty realizations.

Following the approach of Section 4 we run two FPL learners where the primal player is solving a min-cost flow problem over the graph, whose polyhedral formulation is integral and the dual player is playing uncertainty in terms of congestion, with high cost \( M \) corresponding to edge removal. The value of \( M \) is computed dynamically for each instance. The dual player plays congestion uncertainty either (a) from an ellipsoidal uncertainty set initialized with a random matrix (here the worst-case uncertainty can be computed directly via algebraic manipulations), or (b) from a budget constraint set of the form \( \{ x \geq 0 \mid \sum_i x_i \leq K \} \) (here the worst-case uncertainty can be obtained via sorting); the results and timing are very similar for both uncertainty types. Our FPL implementations are those of Kalai and Vempala (2005) as we are in the full-information case.

In all figures we plot \( \max_{u' \in U} f(x, u') - \min_{u^* \in X} \max_{u \in U} f(x^*, u) \) in log-scale (which we refer to as ‘regret’ slightly abusing notions) against iterations \( t \). The right hand side value \( \min_{u^* \in X} \max_{u \in U} f(x^*, u) \) has been computed a priori to enable plotting of the regret; in actual implementations this is of course not needed as the number of iterations imply a strong bound on the quality of the solution but here we wanted to
make convergences to zero regret explicit.

Generally it can be observed that in all cases the algorithm based on imaginary play converges very fast and in fact much faster as predicted by the conservative worst-case bound given in Example D.6.

We provide additional computations in Figures 3.
Figure 3: Regret (log scale) vs. Iterations. Left/Top: Robust min cost flow with $n = 400$ nodes and $p = 0.1$ run for 5000 iterations with $\ell_2$-uncertainty set with radius $5$. Right/Top: Robust min cost flow with $n = 200$ and $p = 0.3$ nodes run for 5000 iterations with $\ell_2$-uncertainty set with radius $20$. Left/Bottom: Robust min cost flow with $n = 200$ and $p = 0.3$ nodes run for 5000 iterations with $\ell_2$-uncertainty set with radius $100$. Right/Bottom: Robust min cost flow with $n = 200$ and $p = 0.3$ nodes run for 5000 iterations with budgeted uncertainty with $K = 50$. 