Paper

Backward error bounds for 2 × 2 linear systems arising in the diagonal pivoting method

Kenta Kobayashi¹,³a) and Takeshi Ogita²,³b)

¹ Graduate School of Commerce and Management, Hitotsubashi University
2-1 Naka, Kunitachi, Tokyo 186-8601, Japan
² Division of Mathematical Sciences, Tokyo Woman’s Christian University
2-6-1 Zempukuji, Suginami-ku, Tokyo 167-8585, Japan
³ CREST, JST

a) kenta.k@r.hit-u.ac.jp
b) ogita@lab.twcu.ac.jp

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Abstract: Matrix factorizations such as LU, Cholesky and others are widely used for solving linear systems. In particular, the diagonal pivoting method can be applied to symmetric and indefinite matrices. Floating-point arithmetic is extensively used for this purpose. Since finite precision numbers are treated, rounding errors are involved in computed results. In this paper rigorous backward error bounds for 2 × 2 linear systems which arise in the factorization process of the diagonal pivoting method are given. These bounds are much better than previously known ones.

Key Words: rounding error analysis, backward error bound, floating-point arithmetic, diagonal pivoting method

1. Introduction and main result
In this paper we are concerned with the rounding error analysis on the diagonal pivoting method¹ when using floating-point arithmetic. Since finite precision numbers are used, rounding errors are involved in computed results.

For the backward error analysis on the diagonal pivoting method, it is necessary to derive a backward error bound for 2 × 2 linear systems during the factorization process. For the diagonal pivoting method there are several pivoting strategies such as Bunch–Parlett [1], Bunch–Kaufman [2] and so forth. Rounding error analyses for the diagonal pivoting method are presented in [3–5]. For example, see [5] for details.

Let \( \mathbb{F} \) be a set of floating-point numbers. We assume the use of the following standard model of floating-point arithmetic barring overflow and underflow: For \( a, b \in \mathbb{F} \) and \( \circ \in \{+, -, \ast, /\} \) it holds

¹It is also known as the block LDL\(^T\) factorization.
that
\[ fl(a \circ b) = (1 + \delta)(a \circ b), \quad |\delta| \leq u. \tag{1} \]

Throughout the paper we use the notation \( \delta_i \) with \( |\delta_i| \leq u, i = 1, 2, \ldots \), to treat the rounding errors in floating-point arithmetic. For rounding error analysis as in [6] we introduce the constant
\[ \gamma_n = \frac{n u}{1 - n u} \]
with the implicit assumption \( nu < 1 \). If \( nu \ll 1 \), then \( \gamma_n \sim nu \).

We refer to backward error bounds for a solution of a 2 \( \times \) 2 linear system which is a pivot in the factorization process of the diagonal pivoting method. We denote the pivot as the real symmetric 2 \( \times \) 2 matrix \( A \):
\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{F}^{2 \times 2}, \quad a_{12} = a_{21} =: \beta.
\]

We assume that the following conditions hold:
\[
|a_{11}| < \alpha |\beta|, \tag{2}
\]
\[
|a_{11}a_{22}| < \alpha^2 |\beta|^2, \tag{3}
\]
where \( \alpha \) is a constant for controlling the pivoting, e.g. \( \alpha = (1 + \sqrt{17})/8 \in [0.64, 0.641] \) in Bunch–Kaufman pivoting. In fact, the conditions (2) and (3) hold in most of pivoting strategies such as Bunch–Parlett, Bunch–Kaufman and others.

We also assume that the 2 \( \times \) 2 linear system \( Ax = b \) where \( b = (b_1, b_2)^T \in \mathbb{F}^2 \) is successfully solved by using floating-point arithmetic, and a computed solution \( \tilde{x} \) is obtained. According to the Oettli–Prager theorem [6, Theorem 7.3], the componentwise backward error defined as
\[
\omega(\tilde{x}) := \min_{\Delta A} \{ \epsilon : (A + \Delta A)\tilde{x} = b, \quad |\Delta A| \leq \epsilon |A| \}
\]
is characterized by
\[
\omega(\tilde{x}) = \max_i \frac{|(b - A\tilde{x})_i|}{(|A||\tilde{x}|)_i} \quad \text{(with } 0/0 := 1), \tag{4}
\]
where \( | \cdot | \) denotes the componentwise absolute value for vectors or matrices. We can see that \( \omega(\tilde{x}) \) depends on \( A, b \) and \( \tilde{x} \). Our goal is to provide an upper bound of \( \omega(\tilde{x}) \) for arbitrary \( A \) and \( b \) with the conditions (2) and (3) as the constant \( \epsilon_c \) satisfying
\[
(A + \Delta A)\tilde{x} = b, \quad |\Delta A| \leq \epsilon_c |A|. \tag{5}
\]
For any \( \epsilon_c \geq \omega(\tilde{x}) \), there exists the backward error \( \Delta A \) satisfying (5). Note that \( \Delta A \) is not uniquely determined in general.

The approximate solution \( \tilde{x} \) of \( Ax = b \) depends on how to solve it, and the constant \( \epsilon_c \) also does. There are mainly two ways; One is to apply Gaussian elimination with partial pivoting (GEPP) for solving \( Ax = b \). The other is to compute the explicit inverse \( A^{-1} \) of \( A \) such as
\[
A^{-1}b = \begin{bmatrix} a_{12} & -a_{21} \\ a_{22} & a_{11} \end{bmatrix} \begin{bmatrix} a_{11}a_{22} - a_{21}^2 \\ -a_{21} \end{bmatrix} b \tag{6}
\]
\[
= \begin{bmatrix} a_{11}a_{22} - a_{21}^2 \\ a_{22}a_{21} - a_{11} \end{bmatrix} \begin{bmatrix} a_{22} & -1 \\ a_{21} & \frac{1}{a_{21}} \end{bmatrix} b. \tag{7}
\]

In [3] estimations of \( \omega(\tilde{x}) \) for \( \alpha = (1 + \sqrt{17})/8 \) are given for the Bunch–Kaufman pivoting in cases of GEPP and the explicit inversion with scaling as in (7). The latter is adopted in the LAPACK auxiliary routine xLASYF.
\[ \epsilon_c = \begin{cases} 4\gamma_2 + O(u^2) & \text{(GEPP)} \\ \gamma_{180} + O(u^2) & \text{(explicit inversion with scaling as in (7))} \end{cases} \] (8)

\[ \sim \begin{cases} 8u & \text{(GEPP)} \\ 180u & \text{(explicit inversion with scaling as in (7))} \end{cases} \]

Note that these are derived with ignoring \( O(u^2) \) terms. Therefore we modify this point for the sake of completeness. Moreover, we also deal with the case of using the form (6) which does not apply scaling. We will show that using (6) gives smaller \( \epsilon_c \) than (7).

Although some rough estimates often suffice to show the backward stability of numerical algorithms, rigorous and computable ones are mandatory in verified numerical computations, in particular precise estimations are preferable. Therefore we refine backward error bounds for the \( 2 \times 2 \) linear system \( Ax = b \), i.e., we try to provide upper bounds of \( \omega(\tilde{x}) \) as small as possible.

By our rounding error analysis we will prove the following theorem.

**Theorem 1.1** Suppose \( \alpha \in [0.64, 0.641] \). Under the conditions (2) and (3) we can choose

\[ \epsilon_c := \begin{cases} \frac{3(1 + 2\alpha^2)}{2} \gamma_2 & \text{for } u < 2^{-1} \quad \text{(GEPP)} \\ \frac{9 + \alpha^2}{13(1 - \alpha^2)} \gamma_{13} & \text{for } u \leq 2^{-7} \quad \text{(explicit inversion without scaling as in (6))} \\ \frac{10 + 3\alpha^2}{16(1 - \alpha^2)} \gamma_{16} & \text{for } u \leq 2^{-11} \quad \text{(explicit inversion with scaling as in (7))} \end{cases} \] (9)

and

\[ \epsilon_c \leq \begin{cases} 5.5u & \text{for } u \leq 2^{-9} \quad \text{(GEPP)} \\ 16u & \text{for } u \leq 2^{-13} \quad \text{(explicit inversion without scaling as in (6))} \\ 19.1u & \text{for } u \leq 2^{-13} \quad \text{(explicit inversion with scaling as in (7))} \end{cases} \] (10)

Note that \( \alpha = (1 + \sqrt{17})/8 \in [0.64, 0.641] \). It turns out that each of the bounds in the above theorem is considerably smaller than that in (8).

### 2. Rounding error analysis

For later use we first present the following lemma:

**Lemma 2.1** Let \( m,n \in \mathbb{N} \) and \( a,u \in \mathbb{R} \) with \( 0 < a < 1 \) and \( 0 < u < n^{-1} \) be given. For \( k \in \mathbb{N} \) satisfying

\[ k \geq \frac{m + an}{(1 - a)(1 - nu)}, \]

it holds that

\[ (1 - u)^m - a(1 + u)^n \geq (1 - ku)(1 - a). \]

**Proof.** It is straightforward that

\[ (1 - ku)(1 - a) \leq \left( 1 - \frac{(m + an)u}{(1 - a)(1 - nu)} \right)(1 - a) = 1 - a - \frac{(m + an)u}{1 - nu} \]

\[ = \left( 1 - \frac{mu}{1 - nu} \right) - a \left( 1 + \frac{nu}{1 - nu} \right) \leq (1 - mu) - a(1 - nu)^{-1} \]

\[ \leq (1 - u)^m - a(1 + u)^n, \]

since \( 1 - pu \leq (1 - u)^p \leq 1/(1 + u)^p \) for any \( p \in \mathbb{N} \). \( \square \)

In the following we will prove (9). The inequalities (10) immediately follow from (9). The proofs are given by carefully treating the rounding errors with considering their cancellations. Put \( \xi_k := 1 + \delta_k \).

Then \( 1 - u \leq |\xi_k| \leq 1 + u \). Moreover, we abbreviate \( \xi_{k_1} \xi_{k_2} \ldots \xi_{k_m} \) as \( \xi_{k_1,k_2,\ldots,k_m} \) for readability.
2.1 Gaussian elimination with partial pivoting

The process of solving the linear system $Ax = b$ by GEPP can be written as follows:

$$
\begin{align*}
    w & := \frac{a_{11}}{\beta}, & \quad z_1 & := b_2, & \quad z_2 & := b_1 - wz_1, \\
    x_2 & := \frac{z_2}{\beta - wz_2}, & \quad x_1 & := \frac{z_1 - a_{22} x_2}{\beta}. \\
\end{align*}
$$

Note that from the condition (2) the first and second rows are permuted. Taking the rounding errors into account we have

$$
\hat{w} := \frac{a_{11} \xi_1}{\beta}, \quad \hat{x}_2 := \frac{(b_1 - \hat{w} b_2 \xi_2) \xi_3}{(\beta - \hat{w} a_{22} \xi_4) \xi_5}, \quad \hat{x}_1 := \frac{(b_2 - a_{22} \hat{x}_2 \xi_7) \xi_8}{\beta} \xi_9.
$$

A little computation yields

$$
\hat{x} = \frac{1}{(\beta^2 - a_{11} a_{22} \xi_{1,4}) \xi_5} \begin{bmatrix}
    -a_{22} \xi_{3,6,7,8,9} \\
    \beta \xi_{3,6} \\
    -a_{11} \xi_{1,2,3,6}
\end{bmatrix} b,
$$

which implies

$$
\hat{A} \hat{x} = b, \quad \hat{A} := \begin{bmatrix}
    \hat{a}_{11} & \hat{a}_{12} \\
    \hat{a}_{21} & \hat{a}_{22}
\end{bmatrix} = \begin{bmatrix}
    \frac{a_{11} \xi_{1,2}}{\xi_{8,9}} & \frac{\beta \xi_5 + a_{11} a_{22} \xi_1 (\xi_2,3,6,7 - \xi_{4,5})}{\beta \xi_{3,6}} \\
    \frac{\beta \xi_5}{\beta \xi_{3,6}} & \frac{a_{22} \xi_7}{\beta}
\end{bmatrix}.
$$

Here

$$
|\hat{a}_{11} - a_{11}| = \left| a_{11} \left( \frac{\xi_{1,2}}{\xi_{8,9}} - 1 \right) \right| \leq \left( \frac{(1 + \mathbf{u})^2}{(1 - \mathbf{u})^2} - 1 \right) |a_{11}| \leq 2 \gamma_2 |a_{11}|,
$$

$$
|\hat{a}_{12} - a_{12}| = \left| \beta \left( \frac{\xi_5}{\xi_{3,6}} - 1 \right) + \frac{a_{11} a_{22} \xi_1}{\beta} \left( \frac{\xi_{2,7} - \xi_{4,5}}{\xi_{3,6}} \right) \right| \leq \left( \frac{1 + \mathbf{u}}{(1 - \mathbf{u})^2} - 1 \right) + \alpha^2 (1 + \mathbf{u}) \left( \frac{(1 + \mathbf{u})^2}{(1 - \mathbf{u})^2} - 1 \right) |a_{12}| \leq \frac{1}{1 - 2 \mathbf{u}} \left( \left( 3 \mathbf{u} - \mathbf{u}^2 \right) + \alpha^2 (6 \mathbf{u} + 2 \mathbf{u}^2) \right) |a_{12}| \leq \frac{3(1 + 2 \alpha^2)}{2} \gamma_2 |a_{12}|, \tag{11}
$$

$$
|\hat{a}_{21} - a_{21}| = \left| \beta \left( \frac{1}{\xi_{8,9}} - 1 \right) \right| \leq \left( \frac{1}{(1 - \mathbf{u})^2} - 1 \right) \left| \beta \right| \leq \gamma_2 |a_{21}|,
$$

and

$$
|\hat{a}_{22} - a_{22}| = \left| a_{22} (\xi_7 - 1) \right| \leq \mathbf{u} |a_{22}|.
$$

Note that we utilize $1 - 2 \alpha^2 > 0$ in (13). Hence we have

$$
\epsilon_c := \frac{3(1 + 2 \alpha^2)}{2} \gamma_2,
$$

which implies $\epsilon_c \leq 5.5 \mathbf{u}$ for $\mathbf{u} \leq 2^{-9}$.
2.2 Explicit inversion without scaling

As is the case with the previous subsection, let \( \hat{x} \) denote a computed solution of \( Ax = b \) obtained by using (6). The process of solving \( Ax = b \) via (6) can be written as follows:

\[
\mu := \beta^2 - a_{11}a_{22}, \quad g_1 := -\frac{a_{22}}{\mu}, \quad g_12 = g_21 := \frac{\beta}{\mu}, \quad g_22 := -\frac{a_{11}}{\mu}.
\]

\[
x_1 := g_11b_1 + g_12b_2, \quad x_2 := g_21b_1 + g_22b_2.
\]

Taking the rounding errors into account yields

\[
\hat{\mu} := (\beta^2 \xi_1 - a_{11}a_{22} \xi_2) \xi_3, \quad \hat{g}_1 := -\frac{a_{22}}{\hat{\mu}} \xi_4, \quad \hat{g}_12 = \hat{g}_21 := \frac{\beta}{\hat{\mu}} \xi_5, \quad \hat{g}_22 := -\frac{a_{11}}{\hat{\mu}} \xi_6
\]

\[
\hat{x}_1 := (\hat{g}_11b_1 \xi_7 + \hat{g}_12b_2 \xi_8) \xi_9, \quad \hat{x}_2 := (\hat{g}_21b_1 \xi_{10} + \hat{g}_22b_2 \xi_{11}) \xi_{12}.
\]

Hence, we have

\[
\hat{x} = \frac{1}{\hat{\mu}} \left[ -a_{22} \xi_{4,7,9}, \beta \xi_{5,8,9}, -a_{11} \xi_{6,11,12} \right] b.
\]

Then

\[
\hat{A}\hat{x} = b, \quad \hat{A} := \psi \left[ \begin{array}{c} \xi_{6,11} a_{11} \\ \xi_9 \\ \xi_{5,10} a_{11} \\ \xi_9 \\ \beta \xi_{4,7} a_{22} \\ \xi_{12} \end{array} \right]
\]

with

\[
\psi := \frac{(\beta^2 \xi_1 - a_{11}a_{22} \xi_2) \xi_3}{\beta^2 \xi_{5,5,8,10} - a_{11}a_{22} \xi_{4,6,7,11}}.
\]

It follows that

\[
|\hat{A} - A| \leq \varepsilon |A|, \quad \varepsilon := \max_{(a,b,c) = (6,11,9),(5,8,12),(5,10,9),(4,7,12)} \left| \psi \frac{\xi_{a,b}}{\xi_c} - 1 \right|.
\]

Substituting \( \xi_1 \) by \( (1 - u) \) or \( (1 + u) \) to maximize \( \varepsilon \) yields

\[
\varepsilon \leq \frac{\beta^2 (1 + u) - |a_{11}a_{22}| (1 - u)}{\beta^2 (1 - u) - |a_{11}a_{22}| (1 + u)} \cdot \frac{(1 + u)^3}{1 - u} - 1
\]

\[
= \frac{\beta^2 ((1 + u)^4 - (1 - u)^5) + |a_{11}a_{22}| (1 - u)(1 + u)^3 u}{\beta^2 ((1 - u)^4 - |a_{11}a_{22}| (1 + u)^4)(1 - u)}
\]

\[
\leq \left\{ \frac{(1 + u)^4 - (1 - u)^5}{1 - u} + \alpha^2 (1 + u)^3 u \right\} \left( (1 - u)^4 - \alpha^2 (1 + u)^4 \right)^{-1}.
\]

Here we have

\[
\frac{(1 + u)^4 - (1 - u)^5}{1 - u} + \alpha^2 (1 + u)^3 u \leq \frac{9 u}{1 - u} + \frac{u}{1 - 3u} \alpha^2 = \frac{u}{1 - 3u} (9 + \alpha^2).
\]

Moreover, by Lemma 2.1 it holds for \( u \leq 2^{-7} \) that

\[
(1 - u)^4 - \alpha^2 (1 + u)^4 \geq (1 - 10u) (1 - \alpha^2).
\]

Thus

\[
\varepsilon \leq \frac{u}{(1 - 3u)(1 - 10u)} \cdot \frac{9 + \alpha^2}{1 - \alpha^2} \leq \frac{9 + \alpha^2}{13(1 - \alpha^2)} \gamma_{13} =: \epsilon_c,
\]

which implies \( \epsilon_c \leq 16u \) for \( u \leq 2^{-13} \).
2.3 Explicit inversion with scaling

As is the case with the previous subsection, let \( \hat{x} \) denote a computed solution of \( Ax = b \) obtained by using (7). The process of solving \( Ax = b \) via (7) can be written as follows:

\[
\begin{align*}
  w_1 & := \frac{a_{11}}{\beta}, \quad w_2 := \frac{a_{22}}{\beta} \\
  \mu & := \beta(w_1 w_2 - 1), \quad g_{11} := \frac{w_2}{\mu}, \quad g_{12} = g_{21} := -\frac{1}{\mu}, \quad g_{22} := \frac{w_1}{\mu} \\
  x_1 & := g_{11} b_1 + g_{12} b_2, \quad x_2 := g_{21} b_1 + g_{22} b_2.
\end{align*}
\]

Taking the rounding errors into account yields

\[
\begin{align*}
  \tilde{w}_1 & := \frac{a_{11}}{\beta} \xi_1, \quad \tilde{w}_2 := \frac{a_{22}}{\beta} \xi_2 \\
  \tilde{\mu} & := \beta(\tilde{w}_1 \tilde{w}_2 \xi_3 - 1) \xi_{4,5}, \quad \tilde{g}_{11} := \frac{\tilde{w}_2}{\tilde{\mu}} \xi_6, \quad \tilde{g}_{12} = \tilde{g}_{21} := -\frac{1}{\tilde{\mu}} \xi_7, \quad \tilde{g}_{22} := \frac{\tilde{w}_1}{\tilde{\mu}} \xi_8 \\
  \tilde{x}_1 & := (\tilde{g}_{11} b_1 \xi_9 + \tilde{g}_{12} b_2 \xi_{10}) \xi_{11}, \quad \tilde{x}_2 := (\tilde{g}_{21} b_1 \xi_{12} + \tilde{g}_{22} b_2 \xi_{13}) \xi_{14}.
\end{align*}
\]

Hence, we have

\[
\tilde{x} = \frac{1}{(a_{11} a_{22} \xi_{1,2,3} - \beta^2) \xi_{4,5}} \begin{bmatrix}
  a_{22} \xi_{2,6,9,11} & -\beta \xi_{7,10,11} \\
  -\beta \xi_{7,12,14} & a_{11} \xi_{1,8,13,14}
\end{bmatrix} b.
\]

Then

\[
\hat{A} \hat{x} = b, \quad \hat{A} := \psi \begin{bmatrix}
  \xi_{1,8,13,14} & \xi_{7,10,11} \\
  \xi_{7,12,14} & \xi_{6,9,13,14}
\end{bmatrix} \xi_{d} = \psi \beta \xi_{1,2,3} \xi_{4,5}.
\]

It follows that

\[
| \hat{A} - A | \leq \max(\varepsilon_1, \varepsilon_2) |A|,
\]

where

\[
\varepsilon_1 := \max_{(a,b,c,d) \in \{(1,8,13,14), (2,6,9,14)\}} \left| \psi, \frac{\xi_{a,b,c}}{\xi_{d}} - 1 \right| \quad \text{and} \quad \varepsilon_2 := \max_{(a,b,c) \in \{(7,10,14), (7,12,11)\}} \left| \psi, \frac{\xi_{a,b}}{\xi_{c}} - 1 \right|.
\]

In a similar way to the previous subsection, it holds that

\[
\max(\varepsilon_1, \varepsilon_2) \leq \left| \frac{\beta^2 - |a_{11} a_{22}| \{(1 + u)^5 - (1 - u)^5 \}}{\beta^2 |a_{11} a_{22}| (1 + u)^6} \right| \frac{(1 + u)^5}{1 - u} - 1.
\]

Therefore, we get

\[
\frac{(1 + u)^5}{1 - u} + 3a^2(1 + u)^5 u < \frac{10u}{1 - 5u} + \frac{3u}{1 - 5u} a^2 = \frac{u}{1 - 5u} (10 + 3a^2).
\]

Moreover, by Lemma 2.1 it holds for \( u \leq 2^{-11} \) that

\[
(1 - u)^4 - \alpha^2(1 + u)^6 \geq (1 - 11u)(1 - \alpha^2).
\]

Thus

\[
\max(\varepsilon_1, \varepsilon_2) \leq \frac{u}{(1 - 5u)(1 - 11u)} \frac{10 + 3a^2}{1 - \alpha^2} < \frac{10 + 3a^2}{16(1 - \alpha^2)} \gamma_{16} := \varepsilon_c,
\]

which implies \( \varepsilon_c \leq 19.1u \) for \( \alpha \in [0.64, 0.641] \) and \( u \leq 2^{-13} \).
Fig. 1. Distribution histogram of $\omega(\hat{x})$ for $10^6$ data sets of normally distributed pseudo-random numbers among 500 bins.

Table I. Statistical results of $\omega(\hat{x})$ for $10^6$ data sets of normally distributed pseudo-random numbers.

|                      | GEPP                          | Explicit inversion without scaling | Explicit inversion with scaling |
|----------------------|-------------------------------|-----------------------------------|-------------------------------|
| Median               | $0.65 \times 10^{-16}$        | $0.86 \times 10^{-16}$            | $0.89 \times 10^{-16}$        |
| Maximum value        | $3.97 \times 10^{-16} \sim 3.57u$ | $7.54 \times 10^{-16} \sim 6.79u$ | $8.03 \times 10^{-16} \sim 7.23u$ |
| Standard deviation   | $0.40 \times 10^{-16}$        | $0.62 \times 10^{-16}$            | $0.64 \times 10^{-16}$        |

3. Sharpness of the error bounds

Recall from Theorem 1.1 that

$$\epsilon_c \leq \begin{cases} 
5.5u & \text{for } u \leq 2^{-9} \\
16u & \text{for } u \leq 2^{-13} \\
19.1u & \text{for } u \leq 2^{-13}
\end{cases} \text{ (GEPP)}$$

holds for $\alpha = (1 + \sqrt{17})/8$. In this section we show how sharp the error bounds are. We assume the use of IEEE 754 binary64 floating-point arithmetic with rounding to nearest (ties to even), and then $u = 2^{-53}$.

Let us first consider the following example: For $v := 2^{-27}$, let

$$\begin{align*}
a_{11} &:= 23(1 + 9v) \\
a_{22} &:= -36(1 + 20v + 16u) \\
\beta &= a_{12} = a_{21} := 46(1 - 14v) \\
b_1 &= 16 - 15v \\
b_2 &= -32(1 - 16v + u).
\end{align*}$$

In case of using GEPP, we have $|\hat{a}_{12} - a_{12}| \leq \epsilon_c = 5.816 \cdots \times 10^{-16} \sim 5.24u$, which is slightly smaller than 5.5u but shows the sharpness of the error bound in a way. On the other hand, $\omega(\hat{x}) = 4.403 \cdots \times 10^{-16} \sim 3.97u$. As mentioned before, the difference between $\omega(\hat{x})$ and $\epsilon_c$ is due to the arbitrariness of $\Delta A$. Thus it turns out that the bound 5.5u for GEPP is very reasonable with respect to $\epsilon_c$, but not necessarily to $\omega(\hat{x})$. In a similar way, we also think we may find examples that nearly attain the other bounds of $\epsilon_c$ for explicit inversions.

Next, we evaluate $\epsilon_c$ in Theorem 1.1 as an upper bound of $\omega(\hat{x})$ from a statistical perspective. In Fig. 1, we display a distribution histogram of $\omega(\hat{x})$ for $10^6$ data sets $(a_{11}, a_{22}, \beta, b_1, b_2)$ of normally distributed pseudo-random numbers. Note that all the data sets satisfy the conditions (2) and (3). As expected, we can see from Fig. 1 that the results seem to be log-normally distributed. The medians, maximum values and standard deviations are also displayed in Table I. It can be seen from the results that $\epsilon_c$ in Theorem 1.1 is at most about three times larger than $\omega(\hat{x})$. We think this is reasonable as an a priori estimate for $\omega(\hat{x})$. 

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