Isospin Fluctuations from a Thermally Equilibrated Hadron Gas

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Partition functions, multiplicity distributions, and isospin fluctuations are calculated for canonical ensembles in which additive quantum numbers as well as total isospin are strictly conserved. When properly accounting for Bose-Einstein symmetrization, the multiplicity distributions of neutral pions in a pion gas are significantly broader as compared to the non-degenerate case. Inclusion of resonances compensates for this broadening effect. Recursion relations are derived which allow calculation of exact results with modest computer time.

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I. INTRODUCTION

Motivated by the observation of large fluctuations of the ratio of neutral to charged particles in cosmic ray events [1, 2], numerous studies of isospin fluctuations have been undertaken during the last decade. It has been proposed that the melting and subsequent recondensation of the chiral condensate could provide a dynamical means for coherent pion emission where dozens of pions are emitted with the same isospin. If \( N \) pions are confined to a single quantum state in addition to being in an isosinglet [3], the probability of finding \( n_0 \) neutral pions takes a simple form in the limit of large \( N = n_+ + n_- + n_0 \),

\[
\frac{dN}{df} = \frac{1}{2\sqrt{f}} \ f \equiv n_0/N. \tag{1}
\]

One can obtain the same result by considering a coherent state

\[
\langle \vec{n} \rangle = \exp (\vec{n} \cdot \vec{\pi}) |0\rangle, \tag{2}
\]

where the pion field operators are \( \pi_0 = \pi_z, \pi_{\pm} = (\pi_x \pm i\pi_y)/\sqrt{2} \), and the direction of \( \vec{n} \) is averaged over all directions. The source of the field \( \vec{n} \) has been proposed to be the chiral condensate which might disorient itself in a quenching scenario [4]. This is often referred to as disoriented chiral condensate (DCC).

The dramatically broad isospin distribution of Eq. (1) relies on the assumption that the emission proceeds via a single quantum state. The inclusion of Bose-Einstein effects in the thermal emission of pions from a non-degenerate array of states was shown to broaden the isospin distribution [3], but not nearly as much as in Eq. (1). Neglecting isospin conservation in [3] accounted for the reduced broadening of the peak. A crude accounting for isospin conservation was suggested by considering the emission of neutral pion pairs (2/3 \( \pi^+\pi^- \) and 1/3 \( 2\pi_0 \)) [6], but came far short of considering the complete ensemble of isoscalar states in a multi-level system. The emission of pairs through a classical isoscalar field into non-degenerate single-particle levels, which may be considered as an oriented chiral condensate, has been studied as well [5].

The effects of exact charge conservation in canonical ensembles were also studied in other contexts with a projection method. Strangeness and baryon number conservation were found to enhance strangeness productions in \( pN \) collisions, particularly for small systems [6]. The confinement of the quark-gluon plasma to color-singlets was shown to lead to a reduction in the number of internal degrees of freedom, which could lead to measurable finite size effects in relativistic heavy ion collisions [7].

In this paper, we extend the sophistication of statistical treatments by considering the entire ensemble of isoscalar states available in a system with many single-particle levels. Methods for calculating isospin distribution are presented, which include Bose-Einstein symmetrization, the effects of resonances and the conservation of both total isospin \( I \) and its projection \( M \). We present sample calculations to illustrate the effects mentioned above and find that symmetrization effects are important for high quantum degeneracies, that isospin conservation has little impact when the size of the domain exceeds a dozen pions, and that resonances can strongly narrow the distribution.

The next section provides a description of the recursive methods developed to determine canonical partition functions, which include Bose-Einstein symmetrization, isospin conservation, and resonances. Techniques for exact calculation of isospin distributions, which are consistent with such ensembles, are presented in Sec. II. Results in Sec. III, and conclusions in IV.
small. In such a non-degenerate system, quantum statistics can be neglected. The partition function for a canonical ensemble of A particles conserving an additive quantum number or a vector of such quantities \( Q \) can be written as a product of single-particle partition functions

\[
Z_{A,Q} = \sum_{\{\nu_k\} \atop \sum \nu_k a_k = A} \prod_{k=1}^{\Omega} \frac{\omega_k^{\nu_k}}{\nu_k!},
\]

(3)

where \( N \) is the number of particle types, \( \nu_k \) is the occupation number of particle type \( k \), \( q_k \) is the charge of one particle, and the particle number \( a_k \) indicates how many times a particle contributes to the main conserved quantity \( A \). For example, if \( A \) is the number of pions then \( a_\rho = 2 \), since the \( \rho \) meson decays predominantly into two pions. The single-particle partition function \( \omega_k = g_k \sum_i \exp(-\epsilon_i^{(k)}/T) \) sums Boltzmann factors weighted by the the spin degeneracy \( g_k \) over all available single-particle levels \( i \).

Summation over the immense number of partitions in Eq. (3) can be avoided by rewriting the partition function as a recursion relation [11][12][13].

\[
Z_{A,Q} = \sum_{k=1}^{N} \frac{a_k \omega_k}{A} Z_{A-a_k,Q-q_k}.
\]

(4)

If intermediate values of the partition function are stored, the computations required for Eq. (4) scale linearly in both \( A \) and \( N \), thus making it possible to quickly calculate the canonical partition function numerically.

A partition function conserving total isospin as well as additive quantum numbers is derived by adding a sum over all possible isospin configurations for a given partition, \( \{\nu_k\} \), and isospin weights \( \xi(I,M|\{\nu_k\}) \) to Eq. (3),

\[
\Omega_{A,I,M} = \sum_{\{\nu_k\} \atop \sum \nu_k a_k = A} \sum_{\nu_j} \xi(I,M|\{\nu_j\}) \prod_{j=1}^{\Omega} \frac{\omega_j^{\nu_j}}{\nu_j!}.
\]

(5)

To convert this partition function into a recursion relation insert \( \frac{1}{A} \sum_{k=1}^{N} a_k \nu_k = 1 \) into Eq. (3),

\[
\Omega_{A,I,M} = \sum_{k=1}^{N} \frac{a_k \omega_k}{A} \sum_{\nu_j} \xi(I,M|\{\nu_j\}) \prod_{j \neq k} \frac{\omega_j^{\nu_j}}{\nu_j!} \cdot \sum_{\nu_k} \nu_k.
\]

(6)

The entire system can be broken into two subsystems, a single particle with isospin \( I_k \) and projection \( m_k \) and a remainder system with isospin \( I' \) and projection \( M-m_k \), which are coupled with the appropriate Clebsch-Gordan coefficients. All possible values for the total isospin of the remainder system have to be summed over,

\[
\sum_{\nu_j} \xi(I,M|\{\nu_j\}) = \sum_{I'=[M-m_k]}^{I_k+1} \sum_{\nu_j} \xi(I',M|\{\nu_j\}) \cdot \langle I_k m_k; I', M - m_k | IM \rangle^2.
\]

(7)

With this modification, the summation indices in Eq. (6) can be switched,

\[
\nu_j' = \begin{cases} \nu_j & , j \neq k \\ \nu_j - 1 & , j = k \end{cases}
\]

(8)

and the partition function written as recursion relation

\[
\Omega_{A,I,M} = \sum_{k=1}^{N} \frac{a_k \omega_k}{A} \sum_{I'=[I-I_k]}^{I_k+I} \Omega_{A-a_k,I',M-m_k} \cdot (I_k m_k; I', M - m_k | IM)^2.
\]

(9)

Since the partition function is the trace of an isoscalar, i.e. \( e^{-H/T} \), it will not depend on the isospin projection \( M \). Hence, Eq. (9) can be further simplified by summing the RHS over all isospin projections \( m_k \) of an isospin multiplet,

\[
\Omega_{A,I} = \sum_{k'} \frac{a_k \omega_k}{A} \sum_{I'=[I-I_k]}^{I_k+I} \Omega_{A-a_k,I'},
\]

(10)

where the sum over \( k' \) includes iso-multiplets, not individual particles species.

\section*{B. Degenerate Systems}

In a degenerate system several particles might occupy the same quantum state, therefore, symmetrization of the wave function has to be accounted for. For the purpose of this paper we will restrict ourselves to studying Bose-Einstein particles. States with multiple particles have to be added to Eq. (3), which only contains states that are occupied by zero or one particle,

\[
Z_{A,M} = \sum_{n=1}^{\infty} \sum_{k=1}^{N} \frac{a_k \omega_k}{A} C_n(k) Z_{A-na_k,M-nm_k}.
\]

(11)

where the cycle diagram

\[
C_n(k) = \langle \tilde{\alpha} | e^{-H/T} | \alpha \rangle = \sum_{l} g_l \exp(-n\epsilon_l^{(k)}/T).
\]

(12)

Here, the state \( | \alpha \rangle \) refers to an \( n \)-particle state of distinguishable particles and \( | \tilde{\alpha} \rangle \) is the cyclic permutation of that state. The single-particle energy levels are \( \epsilon_l^{(k)} \) for particle type \( k \). A more rigorous derivation of Eq. (11) is given in [14].

Since the partition function for conserved total isospin \( \Omega_{A,I,M} \) is independent of the isospin projection \( M \), as mentioned above, a simple relation can be derived, \( Z_{A,M} = \sum_{I \geq M} \Omega_{A,I} \), which in turn leads to

\[
\Omega_{A,I} = Z_{A,M=I} - Z_{A,M=I+1}.
\]

(13)
A second method for calculating the pion partition function constraining total isospin is obtained by evaluating the cycle diagram $C^{(k)}_{i,eta}$ in Eq. (12) for its isospin content. These new cycle diagrams are defined as

$$\zeta_{n,i} = \sum_\beta \langle \beta, n, i | e^{-H/T} | \beta, n, i \rangle,$$  \hspace{1cm} (14)

where the sum over $\beta$ represents a sum over all states with fixed particle number $n$ and isospin $i$. The particles are assumed to be distinguishable and $\tilde{\beta}$ represents a cyclic permutation of particles. The partition function for the pions in terms of this new cycle diagram is then

$$\Omega_{A,I} = \frac{1}{A} \sum_{n=1}^{A} \sum_{i=0}^{n} \sum_{I'=|I-i|}^{I+i} \zeta_{n,i} \Omega_{A-n,I'},$$  \hspace{1cm} (15)

where the new cycle diagrams $\zeta$ are yet to be determined. After obtaining a recursion relation for these functions from Eq. (13) itself,

$$\zeta_{A,I} = A \Omega_{A,I} - \sum_{n=1}^{A-1} \sum_{i=0}^{n} \sum_{I'=|I-i|}^{I+i} \zeta_{n,i} \Omega_{A-n,I'},$$  \hspace{1cm} (16)

we find that these cycle diagrams follow a simple pattern by considering a one-level system where $\Omega_{A,I}$ is easily calculated.

$$\zeta_{n,i} = \begin{cases} 
C_n, & i = n \\
-C_n, & i = n - 1 \\
C_n, & i = 0 \\
0, & \text{otherwise}
\end{cases}$$  \hspace{1cm} (17)

Since resonances are more massive and have lower phase space occupations, the probability of creating several resonances in the same state can be neglected except in the limit of extremely high densities. Therefore, resonances might be treated as independent, non-degenerate subsystems, for which a partition function can be obtained through Eq. (8). The partition functions of two subsystems, 1 and 2, can then be convoluted to obtain that of the entire system,

$$\Omega_{A,I,M} = \sum_{A'=0}^{A} \sum_{I'=0}^{I'} \sum_{I''=|I-I'|}^{I+I'} \Omega^{(1)}_{A',I'} \Omega^{(2)}_{A-I',I''}.$$  \hspace{1cm} (18)

If there are more than two subsystems Eq. (18) can be applied successively, i.e., partition functions of any two subsystems are convoluted first to obtain a new partition function, which is then convoluted with the partition function of another subsystem, and so on.

It should be pointed out that treating resonances as if they are in a different system is not consistent with the indistinguishability of pions from resonances and direct pions. For narrow resonances, one could uniquely identify the pions by constructing the invariant masses of the constituents. However, for broad resonances, one can not confidently identify the resonances. The criteria for resonances being narrow are identical to the criteria that they have sufficiently long lifetime to decay outside the collision region. For all the calculations considered in this paper, it is assumed that the resonances are separable. This assumption is excellent for pions from $\eta$ mesons, good for pions from $\omega$ resonances and questionable for pions from $\rho$ decays. Including the effects of symmetrizing the resonant and non-resonant pions remains an open question.

III. MULTICITY DISTRIBUTIONS AND ISOSPIN FLUCTUATIONS

A. Non-degenerate Systems

The multiplicity distribution can be calculated from a ratio of partition functions, where the numerator includes an extra constraint,

$$P_{A,M}(n_j) = \frac{Z_{A,M,n_j}}{Z_{A,M}}.$$  \hspace{1cm} (19)

Here the numerator represents a canonical ensemble with the appropriate conservation laws containing $n_j$ particles of type $j$. This additional constraint can be regarded as a “charge” and, therefore, is added to the indices of the partition function. The partition function in the numerator can be rewritten with the aid of Eq. (6),

$$Z_{A,M,n_j} = \sum_{k=1}^{N} \frac{a_k \omega_k}{A} Z_{A-a_k,M-m_k,n_j-d_{k,j}},$$  \hspace{1cm} (20)

where the feed-down factor $d_{k,j}$ indicates that a particle of type $k$ decays into $d_{k,j}$ particles of type $j$. This approach, however, will not work for a resonance that decays via more than one decay channel. In such a case, a pseudo-particle is included for each decay branch with its degeneracy $g_i$ scaled by the corresponding branching ratio.

It will prove more convenient to write equations in terms of the product of partition function and multiplicity distribution

$$W_{A,M}(n_j) = Z_{A,M} P_{A,M}(n_j)$$  \hspace{1cm} (21)

instead of the multiplicity distribution itself. For a non-degenerate system conserving only additive charges, the multiplicity distribution can be obtained from

$$W_{A,M}(n_j) = \sum_{k=1}^{N} \frac{a_k \omega_k}{A} W_{A-a_k,M-m_k}(n_j-d_{k,j}).$$  \hspace{1cm} (22)

The occupation number for particle type $j$ including feed-downs from resonance decays is determined by multiplying the occupation number of all particles, given in
by the feed-down-factor and summing over all resonances,

$$\langle n_j \rangle = \frac{1}{Z_{A,M}} \sum_{k=1}^{N} d_{k,j} \omega_k Z_{A-a_k,M-m_k}. \quad (23)$$

The second moment of the distribution will be needed for calculating isospin fluctuations as in Sec. [III C]

$$\langle n_j n_j' \rangle = \sum_{k,k'=1}^{N} \frac{d_{k,j} d_{k',j'}}{Z_{A,M}} \left\{ \delta_{k,k'} \omega_k Z_{A-a_k,M-m_k} + \omega_k \omega_{k'} Z_{A-a_{k'-a_k'},M-m_{k'-m_k}} \right\}. \quad (24)$$

The multiplicity distribution incorporating conserved total isospin can be derived in a similar manner as was employed for Eq. [9].

$$W_{A,I,M}(n_j) = \sum_{k=1}^{N} \frac{a_k \omega_k}{A} \sum_{I'} (\delta_{k,I} - \delta_{k,I'}) W_{A-a_k,I',M-m_k} \langle n_j - d_{k,j} \rangle. \quad (25)$$

**B. Degenerate Systems**

When considering only quantum numbers corresponding to additive charges, the multiplicity distribution is obtained through Eqs. [12] and [13] and we find

$$W_{A,M}(n_j) = \frac{1}{A} \sum_{k=1}^{N} \frac{a_k \omega_k}{A} \sum_{l=1}^{\infty} C_l^{(k)} W_{A-l a_k,M-l m_k} \langle n_j - l d_{k,j} \rangle, \quad (26)$$

where $C_l$ is the cycle diagram defined in Eq. [12]. Calculating the two-point function and the four-point functions allows one to obtain the first two moments of the multiplicity distribution. The 2-point function is

$$\langle a^+_i a_j \rangle = \frac{\delta_{ij}}{Z_{A,M}} \sum_{n} \exp(-n \epsilon_i / T) Z_{A-n_a,M-n_m}. \quad (27)$$

Summing over all particle types and states and multiplying by feed-down factors results in an expression for the occupation numbers,

$$\langle n_j \rangle = \frac{1}{Z_{A,M}} \sum_{k=1}^{N} d_{k,j} \sum_{l=1}^{\infty} C_l^{(k)} Z_{A-la_k,M-l m_k}. \quad (28)$$

Similarly, the 4-point function

$$\langle a^+_i a^+_j a_k a_l \rangle = \frac{\delta_{ij} \delta_{jk} + \delta_{ik} \delta_{jl}}{Z_{A,M}} \sum_{n_i,n_j} \exp(-n_i \epsilon_i / T) \exp(-n_j \epsilon_j / T) \cdot Z_{A-n_i a_i - n_j a_j,M-n_i m_i - n_j m_j}. \quad (29)$$

serves to obtain second moments of the multiplicity distribution

$$\langle n_j n_j' \rangle = \frac{\delta_{jj'}}{Z_{A,M}} \sum_{l,l'} C_{l+l'} Z_{A-\ell a_j,M-\ell m_j} + \frac{1}{Z_{A,M}} \sum_{l,l'} \left\{ \sum_{i} \delta_{j,i} C_i Z_{A-\ell a_k,M-l m_k} + \sum C_{l'} C_{l} Z_{A-\ell a_k-l' a_{k'},M-l m_k-l' m_{k'}} \right\}. \quad (30)$$

Neglecting the terms with $C_l$ where $\ell > 1$, one returns to the non-degenerate result, Eq. [24].

Calculating the multiplicity distribution for degenerate systems with the constraint of total isospin conservation becomes difficult because the analog of the cycle diagram, $C_l^{(k)}$ in Eq. [26], needs to be analyzed for isospin $i$ and charge $n_j$. Such a cycle diagram with $a$ particles and isospin projection $m$ can be written as

$$\chi_{a,i,m}(n_j) = \sum_{a,\beta} \langle \hat{a}, a, i, m|e^{-H/T} |\beta, a, n_j, m \rangle \cdot \langle \beta, a, n_j, m|a, a, i, m \rangle, \quad (31)$$

where the sums over $a$ and $\beta$ correspond to sums over all states with fixed $(a, i, m)$ and $(a, n_j, m)$, respectively, and $\hat{a}$ represents a cyclic permutation of particles, which are assumed to be distinguishable.

The multiplicity distribution can be calculated in terms of these cycle diagrams

$$W_{A,I,M}(n_j) = \frac{1}{A} \sum_{a,i,m,n_j',I'} \chi_{a,i,m}(n_j') \cdot W_{A-a,I',M-m}(n_j - n_j') \langle I', M - m; i, m|I, M \rangle^2. \quad (32)$$

To derive the cycle diagrams $\chi$, consider a single-level system, for which the probability distribution $W^{(1)}$ is calculated in Appendix A. It follows from Eq. (32) that

$$\chi^{(1)}_{A,I,M}(n_j) = A W^{(1)}_{A,I,M}(n_j) - \sum_{a < A,i,m,n_j,I'} \chi^{(1)}_{a,i,m}(n_j) \cdot W^{(1)}_{A-a,I',M-m}(n_j - n_j') \langle I', M - m; i, m|I, M \rangle^2 \quad (33)$$

The function $\chi$ which accounts for all levels can be easily generated from $\chi^{(1)}$,

$$\chi_{a,i,m,n_k} = \chi^{(1)}_{a,i,m,n_k} \sum_{\ell} g_{\ell} e^{-\omega_{\ell}}. \quad (34)$$

where $\ell$ indicates the single-particle energy levels with energy $E_{\ell}$.

If different particles, like resonances, are to be included in the ensemble, one can either insert a sum over species
into Eq. (12); or calculate \( W \) separately for each species and convolute them to find \( W \) for the entire system,

\[
W_{A,I,M}(n_j) = \sum_{A',I',M',n'_j,I''} W^{(1)}_{A',I',M',n'_j,I''} \cdot W^{(2)}_{A-A',I''-M-M'}(n_j - n'_j)(I'M'; I'', M - M'|IM)^2.
\]

In our calculations, \( W \) was calculated separately for resonances neglecting symmetrization and then convoluted with \( W \) calculated for pions with proper symmetrization.

C. Isospin Fluctuations

Given some system with quantum states \( \alpha \), pion isospin fluctuations can be defined as

\[
G^2 = \sum_{\alpha} \langle \alpha | (N_+ + N_- - 2N_0)^2 | \alpha \rangle,
\]

where \( N_+ \), \( N_- \) and \( N_0 \) are the number operators of the respective pions. These isospin fluctuations could be computed through multiplicity distributions or with the expressions for densities and higher moments that were given above. However, for total isospin should be conserved as well, multiplicity distribution calculations are slow and expressions for densities and higher moments are difficult to derive. Instead, we will derive the isospin fluctuation for a system in an isosinglet in terms of isospin projection states.

The operator in Eq. (39) is a product of two rank-2 spherical tensors components

\[
(N_+ + N_- - 2N_0)^2 = 6T_{20}T_{20},
\]

where

\[
T_{20} = \sum_{1} \frac{1}{\sqrt{6}} \left( \pi_{+,-,i} \pi_{+,i} + \pi_{-,i} \pi_{+,-,i} - 2\pi_{0,0,i} \pi_{0,0,i} \right),
\]

\[
T_{2\pm 1} = \sum_{1} \frac{1}{\sqrt{2}} \left( \pi_{0,0,i} \pi_{+,i} - \pi_{0,0,i} \pi_{+,i} \right),
\]

\[
T_{2\pm 2} = \sum_{1} \sqrt{\pi_{0,0,i}^2 \pi_{+,i}^2}.
\]

A product of spherical tensors can be decomposed into other spherical tensor components

\[
T_{20}T_{20} = \sum_{j,M} \langle 20; 20 | JM | A_{j,M} \rangle A_{j,M}.
\]

By the Wigner-Eckart theorem only \( A_{00} \) contributes when contracted into isosinglet states, and because \( A_{00} \) is an isoscalar we can write

\[
\langle G^2 \rangle = \frac{6}{\sqrt{5}} \sum_{I=0}^{I=0} \langle A_{00} \rangle = \frac{6}{\sqrt{5}} \left( \sum_{M=0}^{M=0} - \sum_{M=1}^{M=1} \right) \langle M | A_{00} | M \rangle.
\]

After some algebra we obtain

\[
A_{00} = \sum_{M} \langle 2M; 2, -M | 00 \rangle T_{2M}T_{2, -M} = A'_{00} + A''_{00},
\]

where

\[
A'_{00} = \frac{1}{\sqrt{5}} \left[ \frac{3}{2} N_+ + \frac{3}{2} N_- - N_0 + \frac{1}{6} (N_+ + N_- - 2N_0)^2 \right]
\]

and

\[
A''_{00} = \frac{1}{\sqrt{5}} \sum_{i,j} \left( 2\pi_{+,-,i}^\dagger \pi_{+,j} - \pi_{+,i}^\dagger \pi_{+,j} \pi_{+,i} + \pi_{+,-,i}^\dagger \pi_{+,j} \pi_{+,i} \right).
\]

The expectation of \( A'_{00} \) is non-zero when particles of different charges are in the same quantum state, or when two differently charged pions are produced into two different states with a quantum correlation arising from a resonance decay. The contribution to \( A''_{00} \) from the degenerate nature of the pion states can be determined via Eq. (39),

\[
\langle \pi_{k,i}^\dagger \pi_{k',j}^\dagger \pi_{k,j} \pi_{k',i} \rangle = \frac{1}{Z_{A,M}} \sum_{i,j} C_i(\pi) Z_{A-a_k-M-lm_k-M-lm_k}. \]

This contribution can be ignored in the non-degenerate limit, where occupation numbers are small.

Contributions to \( A''_{00} \) from the coherent correlation between pions from resonant decays can be found by expressing the resonances in terms of pion creation operators. For example, the \( \rho^+ \) meson, which is a member of an isotriplet, can be considered as one pion in an s wave and a second pion in a p wave. Referring to these two states as \( i \) and \( j \),

\[
| \rho^+ \rangle = \frac{1}{\sqrt{2}} \left( \pi_{+,-,i}^\dagger \pi_{0,0,i} - \pi_{+,j}^\dagger \pi_{0,0,i} \right) | 0 \rangle.
\]

How the states \( i \) and \( j \) are chosen is irrelevant since they are summed over in Eq. (13), but the coherent mixture of the two permutations, which is necessary for the \( \rho^+ \) to be a member of an isotriplet, results in a non-zero contribution to \( A''_{00} \),

\[
\langle A''_{00} \rangle = -\frac{1}{\sqrt{5}} \left( 2N_{\rho^+} + N_{\rho^+} - N_{\rho^+} \right).
\]

The \( \omega \) and \( \eta \) mesons are isosinglets and can be treated accordingly. For instance,

\[
| \eta \rangle = \frac{1}{\sqrt{6}} \sum_{i,j,k} \epsilon_{ijk} \pi_{+,-,i}^\dagger \pi_{0,0,i} \pi_{+,-,i}^\dagger | 0 \rangle,
\]

which adds another term to \( A''_{00} \),

\[
\langle A''_{00} \rangle = -\frac{4}{\sqrt{5}} (N_\omega + N_\eta).
\]
IV. RESULTS

In heavy ion reactions, and perhaps in pp reactions, pions reinteract with other pions in their neighborhood, or domain, and might be expected to sample a large portion of the available phase space. As isospin should be conserved in each domain, it seems reasonable to explore distributions for a few dozen pions rather than creating an ensemble of a few thousand pions, which could be treated as a grand canonical ensemble [13]. One of our study’s goals is to understand how many pions are required for conservation constraints to become irrelevant.

In the following, symmetrization and resonances are first ignored in order to focus on the effects of conserving isospin, subsequently, the effects of symmetrization and resonances are illustrated by considering a simple example.

A. Total Isospin Conservation

When quantum degeneracy and resonances are ignored, isospin distributions are independent of energy levels or temperature. Therefore, the results presented in the following are generic to any system where only pions are considered and the phase space occupation numbers are small. A random distribution ignoring isospin conservation, i.e. a mixed-event construction, will serve as a benchmark.

\[
P_{\text{random}}(n_0) = \left(\frac{1}{3}\right)^N \sum_{n_+ + n_- + n_0 = N} \frac{N!}{n_+! n_-! n_0!} \quad (49)
\]

Unlike distributions that conserve isospin, this distribution allows both even and odd numbers of neutral pions and is, therefore, scaled by a factor of two to compare the width with that of the other distributions. Secondly, when pion creation is constrained to isoscalar pairs, as in [6], the distribution can be considered as a binomial distribution of pairs where one third of the time the pair is comprised of two neutral pions and two thirds of the time the pair is comprised of a positive and negative pion.

\[
P_{\text{pairwise}}(n_0) = \left(\frac{1}{3}\right)^{n_0/2} \left(\frac{2}{3}\right)^{(N-n_0)/2} \frac{(N/2)!}{(n_0/2)!(N/2-n_0/2)!} \quad (50)
\]

This pairwise distribution is broader than the random distribution by a factor of \(\sqrt{2}\), as can be seen in Fig. 1.

Finally, the isospin distribution for non-degenerate particles is calculated with the methods of Sec. III A with all 12-pion isosinglet states being considered. The constraint of exact isospin conservation only modestly broadens the distribution relative to the random distribution as shown in Fig. 1.

These findings are underscored by comparing the isospin fluctuations as a function of total pion number, as shown in Fig. 2. When pion emission is constrained to isoscalar pairs, fluctuations are twice as large as compared to the random case for all system sizes. When considering all N-pion isosinglet states fluctuations are larger by a factor which falls from two to unity as N approaches infinity.
B. Including Symmetrization and Resonances

To illustrate the effects of Bose-Einstein symmetrization, total isospin conservation, and resonance decays, an assumption must be made about the available single-particle energy levels that are summed over in the single-particle partition function. Assuming the model system is confined to a cube of volume $V$, the energy states are obtained

$$\epsilon_{n,m,l} = \sqrt{\frac{\pi^2}{R^2}} (n^2 + m^2 + l^2) + M^2,$$

where $R = V^{1/3}$, $M$ is the mass of the particle, and $n, m, l$ are chosen to be half-integers. The choice of half integers, instead of the more usual integers, deemphasizes zero-point surface energy effects and seems more physical if the confinement to the volume does not arise from an infinite potential well. This becomes important when systems are confined to a small volume.

Isospin distributions were calculated for a system of 24 pions at two densities, 0.3 fm$^{-3}$, which is well above breakup densities for hadronic collisions and 0.1 fm$^{-3}$. The temperature was chosen to be 125 MeV. The Bose-Einstein nature of pions was taken into account when calculating multiplicities according to the formalism from Sec. III B. The multiplicity distributions were then convoluted with those of three resonant states, the isotriplet $\rho$ mesons and the isosinglet $\omega$ and $\eta$ mesons, which were treated as non-degenerate systems according to Sec. III A. In principle, strange mesons and baryonic resonances can be incorporated as well, but the calculation would be significantly lengthened by the inclusion of extra indices. Prospects for such calculations are discussed in the conclusions.

As expected, the isospin distributions for symmetrized pions in an isosinglet are broader than the random distribution when resonances are neglected, as shown in Fig. 3. This broadening is especially pronounced at high density. However, the inclusion of resonances more than compensates for the symmetrization effects and results in distributions that are narrower than the random distribution. Figure 4 displays fluctuations as a function of density for the 24-pion system. The dramatic broadening induced by symmetrization at high density is counteracted by a remarkable narrowing when resonant states are considered.

The conditions for pions to prefer forming a resonance are similar to the conditions for symmetrization to be important, i.e., a high phase space density. At low temperature, where the mass penalty for resonant formation would play a larger role, resonant effects would become relatively less important than symmetrization. However, for the temperature of 125 MeV considered here, the resonant effects overwhelm the effects of symmetrization at all densities.

Finally, it should be noted that the isospin fluctuation $G^2$ was calculated both from the distributions themselves and from the methods described in Sec. III C. Although the two sets of numerical calculations have little in common aside from the functions used to generate the single-particle levels, the two sets of moments agreed with each other within the numerical accuracy of the computer.

V. CONCLUSIONS

For the first time, expressions were obtained for multiplicity distributions and isospin fluctuations for a canonical ensemble, in which total isospin as well as additive quantum numbers are exactly conserved. The formalism has been extended to include both pions and resonances and can account for Bose-Einstein symmetrization of the pion wavefunction. Numerical calculations were then performed to study the effects of total isospin conservation, quantum symmetrization, and resonance decays on the width of the multiplicity distribution, which can be squared to obtain the isospin fluctuations. Direct expressions for the isospin fluctuations permit a much quicker
FIG. 4: Isospin fluctuations, or squared width of the isospin distributions, as a function of density scaled by the width of a random distribution. Calculations were performed for 24 pions restricted to isosinglet states at a temperature of 125 MeV. At high density, the distributions are broadened by including symmetrization and narrowed by including resonances.

calculation than those for the multiplicity distributions.

It was found that conservation of total isospin and its projection has little effect on the width of the multiplicity distributions, when the systems are larger than a dozen particles. It should be noted that conserving only the projection, not the total isospin, would result in distributions where the average number of neutral pions would not equal one third of the total. At high phase space densities, including Bose-Einstein symmetrization leads to a multiplicity distribution that is much broader than a random distribution. However, addition of resonances more than compensates for this broadening and narrows the multiplicity distribution below the width of the random distribution. Both effects are small when the phase space density is below 0.1 fm$^{-3}$.

The widths of the multiplicity distributions are largely dominated by the behavior of the tails, thus making it imperative to perform exact calculations. These calculations were made possible by using recursion relations that circumvent summing over the immense number of partitions in the partition functions.

The calculations in this paper were largely schematic and included only three meson resonance, $\rho$, $\eta$, and $\omega$. For a more realistic calculation more resonances like strange mesons and baryonic resonances need to be added along with strangeness and baryon number conservation. The formalism presented in this paper scales linearly with the number of particle species if the resonances were to be included without additional conserved charges. Theoretically, any amount of quantum numbers can be conserved as long as they commute with the isospin operator. Practically, every new index, which has to be added to the partition function and multiplicity distributions, to conserve another charge increments the number of loops by one. The increase in runtime has little consequence for calculations of partition functions and direct calculations of isospin fluctuations, which are virtually instantaneous, whereas any additional index would significantly slow down calculations of multiplicity distributions, which take on the order of ten minutes.

The aforementioned caveats are not expected to become major obstacles in the application of the presented formalism. Although the particle multiplicities are high, in the thousands, in possible physical applications like relativistic heavy ion collisions, the local nature of charge conservation would limit the number of particles considered at any given time. The system under consideration would have to be broken into domains, in which the charges are conserved locally, then calculations proceed one domain at a time. Each domain would have a relatively small number of particles with which to cope.

Before tackling the more numerically challenging problem of including strangeness and baryon number, one should consider the limitations of any comparison with experiment. Most importantly, it difficult to count neutral pions as each neutral pion decays into two photons. Furthermore, one should consider the ability to identify neutrons and kaons, especially those kaons which then decay into pions. Given the inherent complexity of any such measurement, we felt that it was proper to stop short of performing more complicated calculations without a commensurate consideration of the measurement. Nonetheless, several valuable lessons were gained from the schematic calculations presented here.

Another possible application of partition functions with exact quantum number and isospin conservation, one that has not been explored in this paper, are Monte Carlo algorithms for particle generation. Recently, there has been much interest in modeling relativistic heavy ion collisions with hybrid models, in which early, dense stages of the collision are described by a hydrodynamical model before switching to a hadronic cascade to simulate the freeze-out stage. The change of degrees of freedom at the interface between the two models from the energy-momentum tensor to hadrons is generally modeled by a grand-canonical ensemble, which conserves charges only in the average over many events. However, event-by-event charge conservation is essential to calculating observables like fluctuations and balance functions, which have been proposed as possible signal for the quark-gluon-plasma.

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APPENDIX A: SINGLE-LEVEL MULTIPLICITY DISTRIBUTION CONSERVING TOTAL ISOSPIN

By definition, the distribution \( W_{a,i,m}^{(1)}(n_j) \), needed in Eq. (33), can be written as the product of the partition function and the multiplicity distribution,

\[
W_{a,i,m,n_j}^{(1)} = \omega_{a,i,m}^{(1)} p_{a,i,m}^{(1)}(n_j),
\]

where the partition function for \( a \) particles in a single level with energy \( E \) is

\[
\omega_{a,i,m}^{(1)} = \begin{cases} \exp(-aE/T), & \text{if } a + i \text{ even} \\ 0, & \text{if } a + i \text{ odd} \end{cases} \quad (A2)
\]

\( p_{a,i,m}^{(1)}(n_j) \) is the probability of observing \( n_j \) pions in a single-state system that contains a total of \( a \) pions with total isospin \( i \) and projection \( m \). This probability distribution has to be calculated for only one type of pions because the pion occupation numbers are related through

\[
m = n_+ - n_-, \quad (A3)
\]

\[
a = n_0 + n_+ + n_. \quad (A4)
\]

In the following, probability distributions will be derived for positive pions.

The isospin wave function of the system can be written in terms of eigenstates of the number operators

\[
|a, i, m\rangle = \sum_{n_+ = m}^{(a+m)/2} \alpha_{a,i,m,n_+}|a + m - 2n_+\rangle|n_+\rangle|n_-\rangle, \quad (A5)
\]

where \( n_0 = a + m - 2n_+ \) and \( n_- = n_+ - m \). The coefficients in Eq. (A5) are related to the probability distribution by

\[
p_{a,i,m}^{(1)}(n_+) = (\alpha_{a,i,m,n_+})^2. \quad (A6)
\]

The isospin wave function \( |a, a, a\rangle \) can only be constructed if all pions in the state are positive, i.e., \( n_+ = a \), therefore

\[
\alpha_{a,a,a,n_+} = \begin{cases} 1, & \text{if } n_+ = a \\ 0, & \text{otherwise} \end{cases} \quad (A7)
\]

Leaving the pion number \( a \) and isospin \( i = a \) fixed, we can apply the isospin lowering operator \( I_- = \sqrt{2} (\pi^+_0 + \pi^-_0 \pi^+_0) \) to reach lower values of \( m \),

\[
I_- |a, i, m\rangle = \sqrt{i(i+1) - m(m-1)} |a, i, m - 1\rangle. \quad (A8)
\]

The LHS of Eq. (A8) expands to

\[
L_- \sum_{n_+} \alpha_{a,a,m,n_+} |a + m - 2n_+\rangle|n_+\rangle|n_-\rangle
\]

\[
= \sqrt{2} \sum_{n_+ = m}^{(a+m)/2} \sqrt{a + m - 2n_+} \sqrt{n_+ - m + 1} \alpha_{a,a,m,n_+} |a + (m - 1) - 2n_+\rangle|n_+\rangle|n_- - (m - 1)\rangle
\]

\[
+ \sqrt{2} \sum_{n_+ = m - 1}^{(a+m)/2 - 1} \sqrt{a + (m - 1) - 2n_+} \sqrt{n_+ + 1} \alpha_{a,a,m,n_+ + 1} |a + (m - 1) - 2n_+\rangle|n_+\rangle|n_- - (m - 1)\rangle. \quad (A9)
\]

When the coefficients on the LHS are match with those on the RHS of Eq. (A8) a recursion relation is obtained,

\[
\alpha_{a,a,m-1,n_+} = \sqrt{\frac{2}{i(i+1) - m(m-1)}} \left\{ \sqrt{(a + m - 2n_+)(n_+ - m + 1)} \alpha_{a,a,m,n_+}
\right.
\]

\[
+ \sqrt{(a + m - 1 - 2n_+)(n_+ + 1)} \alpha_{a,a,m,n_+ + 1} \right\} \quad (A10)
\]

So far, we have only found the coefficients \( \alpha_{a,i,m,n_+} \) for \( a = i \). With the help of the isoscalar operator

\[
U_2 = 2\pi_{+0}^+ \pi_0^- - \pi_{-0}^+ \pi^+_0 \quad (A11)
\]

that creates two pions without altering the isospin, higher values of \( a \) can be reached,

\[
U_2 |a, i, m\rangle = N_{a,i} |a + 2, i, m\rangle, \quad (A12)
\]

where the normalization constant is

\[
N_{a,i} = \sqrt{(a + 2)(a + 3) - i(i+1)}. \quad (A13)
\]

Matching the coefficients on both sides of Eq. (A12) to each other leads to
\[ \alpha_{a+2, i, m,n_+} = \frac{1}{N} \left\{ 2\sqrt{n_+ (n_+ - m)} \alpha_{a,i,m,n_+} - 1 - \sqrt{(a + m - 1 - 2n_+)(a + m - 2 - 2n_+)} \alpha_{a,i,m,n_+} \right\}. \]  \quad (A14)

In case, \( a + i \) is odd, no combination of pions yields the isospin wave function \( |a, i, m\rangle \), as is evident from Eq. (A2). Therefore

\[ \alpha_{a,i,m,n_+} = 0, \quad \text{if} \quad a + i \text{ odd.} \quad (A15) \]

Equations (A7), (A10), (A14), and (A15) completely determine the coefficients \( \alpha_{a,i,m,n_+} \), which in turn define the probability distributions \( p^{(1)}_{a,i,m}(n_+) \) and, therefore, the single-level partition functions \( W^{(1)}_{a,i,m,n_j} \).

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