Rigidity, Functional Equations and the Calogero-Moser Model

H. W. Braden

Department of Mathematics and Statistics,
The University of Edinburgh,
Edinburgh, UK

Abstract

Suppose we have a natural Hamiltonian $H$ of $n$ particles on the line, centre of mass momentum $P$ and a further independent quantity $Q$, cubic in the momenta. If these are each $S_n$ invariant and mutually Poisson commute we have the Calogero-Moser system with potential $V = \frac{1}{6} \sum_{i \neq j} \wp(q_i - q_j) + \text{const.}$

1 Introduction

The following note deals with many particle Hamiltonian systems on the line and their integrability. Although such systems arise in many physical settings and have been extensively studied there still is no simple way to determine their integrability or otherwise. General arguments [17] tell us that many particle Hamiltonian systems for sufficiently repulsive potentials are integrable, yet there appear few direct methods of actually solving for such systems. The integrable systems we can actually solve seem to form a very privileged class. The result presented here sheds some light on this state of affairs. We will follow a less well known route to the study of integrable systems, that employing functional equations.

Here we address the following question: What $S_n$ invariant, natural Hamiltonian systems of $n$-particles on the line and conserved centre of mass momentum admit a third independent, $S_n$ invariant, mutually Poisson commuting quantity, cubic in the momenta? (The precise statement and explanation of these terms will be given below.) Our answer is somewhat surprising. This data characterises the $a_n$ Calogero-Moser systems. This is the “rigidity” of our title. Although no restrictions were placed on further Poisson commuting invariants we arrive at a system for which sufficient exist to yield complete integrability. The mixture of symmetry and polynomial momentum is powerful. Such natural requirements and our result go some way in explaining the ubiquity of this class of models. The situation is somewhat reminiscent of the original work of Ruijsenaars and Schneider [28] who, when demanding certain commutation properties, discovered a class of Hamiltonian systems that proved to be integrable. It is also analogous to what one encounters with W and related algebras, where a few commutation relations specify the whole structure. Indeed, the connections between Conformal Field Theory and these models may mean this is more than analogy [23, 18].

There are obvious generalisations to this work which will be taken up in the discussion. Before turning to the statement and proof of the result (given in the following two sections) it is perhaps worth making some general remarks on connections between integrable systems and functional equations. Functional equations have of course a long and interesting history in

*E-mail:hwb@ed.ac.uk
connection with mathematical physics and touch upon many branches of mathematics \cite{1,2}. They have arisen in the context of completely integrable systems in several different ways. We have already mentioned the work of Ruijsenaars and Schneider. Hietarinta similarly derived a functional equation when seeking a second quartic integral for two particle systems on the line \cite{19}. A further way in which they arise is by assuming an ansatz for a Lax pair, the consistency of the Lax pair yielding functional and algebraic constraints. In this manner Calogero discovered the elliptic Calogero-Moser model \cite{13} and Bruschi and Calogero constructed Lax pairs for the Ruijsenaars models \cite{8,9}. The functional equations found by this route appear \cite{4} as particular examples of

$$\begin{vmatrix} \phi_1(x + y) &=& \phi_2(x) & \phi_2(y) \\ \phi_3(x) & \phi_3(y) \\ \phi_4(x) & \phi_4(y) \\ \phi_5(x) & \phi_5(y) \end{vmatrix} = 0.$$ 

The general analytic solution of this has been given by Braden and Buchstaber \cite{5}. Interestingly, Novikov’s school have shown that the Hirzebruch genera associated with the index theorems of known elliptic operators arise as solutions of functional equations which are particular examples of this. The string inspired Witten index was shown by Ochanine to be described by Hirzebruch’s construction where now the elliptic function solutions were important \cite{20}. A similar approach based upon an ansatz and consequent functional equations was used by Inozemtsev \cite{21} to construct generalisations of the Calogero-Moser models, while in \cite{6} this route was used to construct new solutions to the WDVV equations. Various functional equations have also arisen when studying the properties of wave-functions for associated quantum integrable problems. Gutkin found several functional relations by requiring a nondiffractive potential \cite{16} while Calogero \cite{14} and Sutherland \cite{29,30} obtained functional relations by seeking factorizable ground-state wave-functions. A recurring equation in this latter approach is

$$f(x) + g(y) + h(z) = 0, \quad x + y + z = 0.$$ 

This finds general solution in \cite{10,3}. In our present work we will make use of the particular case of this equation \cite{10,3}: 

**Theorem 1** Let \(f\) be a three-times differentiable function satisfying the functional equation

$$\begin{vmatrix} 1 & 1 & 1 \\ f(x) & g(y) & h(z) \\ f'(x) & g'(y) & h'(z) \end{vmatrix} = 0, \quad x + y + z = 0. \tag{1}$$

Up to the manifest invariance

$$f(x) \to \alpha f(\delta x) + \beta,$$

the solutions of (4) are one of \(f(x) = \wp(x + d)\), \(f(x) = e^x\) or \(f(x) = x\). Here \(\wp\) is the Weierstrass \(\wp\)-function and \(3d\) is a lattice point of the \(\wp\)-function.

Perhaps one reason for the underlying connection between integrability and functional equations is that fact that Baker-Akhiezer functions obey such relations. Such connections between integrable functional equations and algebraic geometry have been studied by Buchstaber and Krichever \cite{11} and Dubrovin, Fokas and Santini \cite{15}. Whatever, these connections between functional equations and complete integrability warrant further investigation.
2 The Result

The result discovered is the following:

**Theorem 2** Let $H$ and $P$ be the (natural) Hamiltonian and centre of mass momentum

$$H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + V, \quad P = \sum_{i=1}^{n} p_i. \quad (2)$$

Denote by $Q$ an independent third order quantity

$$Q = \frac{n}{6} \sum_{i=1}^{n} p_i^4 + \frac{1}{6} \sum_{i \neq j, k} d_{ijk} p_ip_jp_k + \sum_{i \neq j} d_{ij} p_i^2 p_j + \frac{1}{2} \sum_{ij} a_{ij} p_ip_j + \sum_i b_ip_i + c. \quad (3)$$

If these are $S_n$ invariant and Poisson commute,

$$\{P, H\} = \{P, Q\} = \{Q, H\} = 0,$$

then $V = \frac{1}{6} \sum_{i \neq j} \varphi(q_i - q_j) + \text{const}$ and we have the Calogero-Moser system.

Some explanatory remarks are in order. Here $S_n$ invariance means that for any coefficient $\alpha_{ij}(q_1, q_2, \ldots, q_n)$ in the expansions above we have $\alpha_{\sigma(i)\sigma(j)}(q_{\sigma(1)}, q_{\sigma(2)}, \ldots, q_{\sigma(n)})$ for all $\sigma \in S_n$. In particular $V(q_1, q_2, \ldots, q_n) = V(q_{\sigma(1)}, q_{\sigma(2)}, \ldots, q_{\sigma(n)})$ for all $\sigma \in S_n$. We remark that had we begun with particles of possibly different particle masses, $H = \frac{1}{2} \sum_{i=1}^{n} m_i p_i^2 + V$, the effect of $S_n$ invariance is such as to require these masses to be the same. Thus we are assuming the $S_n$ invariant Hamiltonian of the introduction. Finally, by “an independent third order quantity” $Q$ we mean one functionally independent of $H$ and $P$ and for which one cannot obtain an invariant of lower degree by subtracting multiples of $P^3$ and $PH$. We are not dealing with quadratic conserved quantities here.

3 The Proof

Our proof has five steps. We begin by noting that the Poisson commutativity $\{Q, H\} = 0$ yields (with $\{q_i, p_j\} = \delta_{ij}$)

$$\begin{align*}
0 &= \frac{1}{6} \sum_{l} \sum_{i \neq j \neq k} (\partial_l d_{ijk}) p_ip_jp_k + \sum_{l} \sum_{i \neq j} (\partial_l d_{ij}) p_i^2 p_j p_l + \sum_{i,j,l} (\partial_l a_{ij}) p_ip_j p_l \\
&+ \sum_{i,j} (\partial_l b_i) p_ip_j - 3 \sum_{i} p_i^2 (\partial_l V) - \frac{1}{2} \sum_{i \neq j \neq k} d_{ijk} (\partial_l V) p_ip_j \\
&- \sum_{i \neq j} d_{ij} (2 (\partial_l V) p_ip_j + (\partial_l V) p_i^2) - \sum_{i,j} a_{ij} (\partial_l V) p_j + \sum_{i} (\partial_l c) p_i - \sum_{i} b_i \partial_l V.
\end{align*} \quad (4)$$

The steps then are:

1. First we show that the $d_{ijk}$ and $d_{ij}$ terms in (4) may be taken to be zero.

Having made this simplification we then focus on the terms remaining in (4) independent and quadratic in the momenta:

$$\begin{align*}
\partial_i b_i + \partial_j b_j &= 0, \quad i \neq j, \quad (5) \\
\partial_i b_i - 3 \partial_i V &= 0, \quad (6) \\
\sum_{i} b_i \partial_i V &= 0. \quad (7)
\end{align*}$$
2. Second, using (3) we show that $b_j$ may be written in the form

$$b_j = \sum_{i \neq j} W(q_i - q_j) + U(q_j),$$

where $W$ is an even function.

3. Third, using $\{P, Q\} = 0$, we may set $U = 0$.

4. Fourth, that we may rewrite (3) in the form

$$0 = \sum_{i < j < k} \begin{vmatrix} 1 & 1 & 1 \\ W(q_i - q_j) & W(q_j - q_k) & W(q_k - q_i) \\ W'(q_i - q_j) & W'(q_j - q_k) & W'(q_k - q_i) \end{vmatrix}.$$ 

5. Finally we argue that each term in the sum (3) itself vanishes and so we arrive at an equation of the form (4). The result then follows simply.

**Step 1.** We begin by focusing on the terms in (3) quartic in the momenta. For $l$ different from $i, j, k$ we see that $\partial_i d_{ijk} = 0$, and so $d_{ijk} = d'_{ijk}(q_i, q_j, q_k)$. Further, from the coefficients of $p_i^3 p_j^2$, $p_i^2 p_j p_k$ (for $i, j, k$ distinct) respectively, we find

$$\partial_i d_{ij} = 0, \quad \partial_j d_{ij} = 0, \quad \partial_j d_{ik} + \partial_k d_{ij} + \partial_i d_{ijk} = 0.$$  

The first and third of these together show $\partial^2 d_{ijk} = 0$ and so $d_{ijk}$ is at most linear in $q_i$. By symmetry

$$d_{ijk} = \alpha q_i q_j q_k + \beta (q_i q_j q_k + q_j q_i q_k + q_k q_i q_j) + \gamma (q_i + q_j + q_k) + \delta.$$ 

Now using $\{P, Q\} = 0$ shows $\alpha = \beta = \gamma = 0$. Thus $d_{ijk}$ is a constant. This fact, together with the second and third equations of (11), shows $\partial^2 d_{ij} = 0$. Therefore $d_{ij}$ is at most linear in $q_k$ (for $k \neq i, j$). The first two equations of (11) show $d_{ij}$ is independent of $q_i$ and $q_j$. Now a similar argument employing $\{P, Q\} = 0$ yields $d_{ij}$ to be constant. By subtracting appropriate multiples of $P^3$ and $PH$ we may then remove the $d$ terms from $Q$. Our assumption of independence means that the leading term of $Q$ does not vanish when doing this. Thus (after such a subtraction and a possible rescaling) we may set the $d$ terms in $Q$ to be zero. Henceforth we will assume this simplification has been made.

**Step 2.** Suppose $i, j, k$ are distinct. Then from (3) we obtain ($\partial_{ij} = \partial_i \partial_j$ etc.)

$$\partial_j b_i + \partial_i b_j = 0, \quad \partial_k b_i + \partial_i b_k = 0.$$ 

Taking the difference of these we see $\partial_i (\partial_k b_j - \partial_j b_k) = 0$ and so

$$-\partial_k b_j + \partial_j b_k = 2F(q_j, q_k).$$ 

Combining this with $\partial_k b_j + \partial_j b_k = 0$ we obtain

$$\partial_j b_k = F(q_j, q_k) = -F(q_k, q_j) = -\partial_k b_j.$$ 

We wish to further restrict the form of $F$. If we apply $\partial_i$ to (3) and then use (3) we see

$$-\partial_i \partial_i b_j = \partial_i \partial_j b_i = 3\partial_i \partial_j V = \partial_i \partial_j b_j$$ 

and so

$$(\partial_i + \partial_j) \partial_j b_j = 0.$$
Therefore
\[ \partial_i b_j = F(q_i - q_j), \quad F(x) = -F(-x). \] (11)
Upon integrating we obtain (8) where \( W'(x) = F(x) \) and \( W(x) = W(-x) + \tilde{c} \). \textit{A priori} we cannot argue that the integration constant \( \tilde{c} \) vanishes if \( W(0) \) is not defined, as happens for singular potentials. However setting \( W(x) = \frac{1}{2} (\tilde{W}(x) + \tilde{W}(-x)) \) again yields (8) up to a constant, which at this stage may be incorporated into the arbitrary function \( U \).) We have employed the \( S_n \) symmetry throughout this step to identify each of the possibly different functions \( F, W \) and \( U \) arising from each pair as the same.

**Step 3.** Now upon employing \( \{Q, P\} = 0 \) we see \( \sum_{i=1}^{n} \partial_i b_j = 0 \). Using (8) we deduce that \( \partial_i U(q_i) = 0 \) and so \( U(q_i) \) is a constant. Such a constant may be removed altogether by subtracting an appropriate multiple of \( P \) from \( Q \), or simply incorporated into a redefinition of \( W(x) \). Whatever, we may take \( U = 0 \). Then
\[ b_i^2 = \sum_{j \neq i} W^2(q_j - q_i) + 2 \sum_{j \neq k \neq i} W(q_j - q_i) W(q_k - q_i). \] (12)

**Step 4.** Now employing (11, 12) we see \( 0 = \sum_{i} \partial_i b_i^2 \). Using \( (12) \) we obtain
\[ \partial_i b_i^2 = -2 \sum_{j \neq i} W(q_j - q_i) F(q_j - q_i) + 2 \sum_{j \neq k \neq i} \partial_i (W(q_j - q_i) W(q_k - q_i)) \]

When we sum this expression over \( i \) the first term will vanish using oddness and evenness properties. Thus we arrive at
\[ 0 = \sum_{i \neq j \neq k} \partial_i (W(q_j - q_i) W(q_k - q_i)). \]

Define \( A_{ijk} \) by
\[ A_{ijk} = \partial_i (W_{ji} W_{ki}) + \partial_j (W_{ij} W_{kj}) + \partial_k (W_{ik} W_{jk}) = \begin{vmatrix} 1 & 1 & 1 \\ W_{ij} & W_{jk} & W_{ki} \\ F_{ij} & F_{jk} & F_{ki} \end{vmatrix}, \]
where we use the shorthand \( W_{ij} = W(q_i - q_j) \). Then from the functional form of \( W \) we know
\[ A_{ijk} = A_{ikj} = A_{jki} = \Psi(q_i - q_j, q_j - q_k, q_k - q_i) \] (13)
and is fully symmetric in \( i, j, k \). Thus
\[ 0 = \sum_{i < j < k} A_{ijk} = \sum_{i < j < k} \begin{vmatrix} 1 & 1 & 1 \\ W(q_i - q_j) & W(q_j - q_k) & W(q_k - q_i) \\ W'(q_i - q_j) & W'(q_j - q_k) & W'(q_k - q_i) \end{vmatrix}. \] (14)
which is equation (11).

**Step 5.** We now wish to argue that \( A_{ijk} = 0 \). If we apply \( \partial_{ijk} \) to (14) we find that \( \partial_{ijk} A_{ijk} = 0 \).
as only one term in the sum depends on \(i, j, k\). Thus \(\partial_{jk}A_{ijk}\) is independent of \(q_i\), and consequently due to the functional form \([13]\) it must be a function of \(q_j - q_k\) only. Therefore we must have

\[
\partial_{jk}A_{ijk} = B(q_j - q_k),
\]

and so, after integration and use of symmetry,

\[
A_{ijk} = E(q_i - q_j) + E(q_j - q_k) + E(q_k - q_i)
\]

(where \(E(x) = E(-x)\) and \(E''(x) = -B(x)\)). We may therefore rewrite \([14]\) as

\[
0 = \sum_{i<j} E(q_i - q_j).
\]

Taking the partial derivative \(\partial_{ij}\) of this expression then gives \(\partial_{ij}E(q_i - q_j) = 0\), as only this term depends on both \(i\) and \(j\). This, together with the evenness of \(E\) tells us that \(E = 0\). That is \(A_{ijk} = 0\). Therefore for each distinct triple \(i, j, k\)

\[
0 = \begin{vmatrix}
1 & 1 & 1 \\
W(q_i - q_j) & W(q_j - q_k) & W(q_k - q_i) \\
W'(q_i - q_j) & W'(q_j - q_k) & W'(q_k - q_i)
\end{vmatrix}.
\]

But this is none other than \([11]\). The even solution of this is \(W(x) = \wp(x)\), up to a constant. Finally, using \([8]\) and \([9]\) we obtain the stated conclusion.

4 Discussion

Our result may be interpreted as a rigidity theorem for the \(a_n\) Calogero-Moser system and in part explains this models’ ubiquity: demanding a cubic invariant together with \(S_n\) invariance necessitates the model. A detailed scrutiny of our proof shows several generalisations possible. A natural generalisation is to replace the \(S_n\) invariance with the invariance of a general Weyl group \(W\) and make connection with the Calogero-Moser models associated to other root systems \([24, 25]\). Quite a bit is known about the quantum generalisations in this regard. Given a putative integrable Hamiltonian, what might the invariants look like? The present work tells us that for \(S_n\) invariant systems not of Calogero-Moser type we should look for conserved quantities quartic and above in the momenta.
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