Hamiltonian Stationary Tori in Kähler Manifolds

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Abstract

A Hamiltonian stationary Lagrangian submanifold of a Kähler manifold is a Lagrangian submanifold whose volume is stationary under Hamiltonian variations. We find a sufficient condition on the curvature of a Kähler manifold of real dimension four to guarantee the existence of a family of small Hamiltonian stationary Lagrangian tori.

1 Introduction and Statement of Results

Let $M^{2n}$ be a Kähler manifold with complex structure $J$, Riemannian metric $g$, and symplectic form $\omega$. The Lagrangian submanifolds of $M$, i.e. those $n$-dimensional submanifolds of $M$ upon which the pull-back of $\omega$ vanishes, are very natural and meaningful objects to consider when $M$ is studied from the symplectic point of view. To gain additional insight by studying $M$ from the metric point of view, it has been fruitful to consider those Lagrangian submanifolds of $M$ which are in some way well-adapted to the metric geometry of $M$. Indeed, it has been found that the Lagrangian submanifolds of $M$ (when $M$ is either Kähler-Einstein or Calabi-Yau) that are minimal with respect to the metric $g$ possess a rich mathematical structure and their study is an active area of research (see e.g. [6, 13]).

The minimal and Lagrangian submanifolds of $M$ are critical points of the $n$-dimensional volume functional with respect to compactly supported variations. It is possible to pose two other natural variational problems amongst Lagrangian submanifolds of $M$ whose critical points are also mathematically quite interesting. These variational problems are obtained by restricting the class of allowed variations. First, one can demand that the volume of $\Sigma$ is a critical point with respect to only those variations of $\Sigma$ which preserve the Lagrangian condition; in this case, $\Sigma$ is said to be Lagrangian stationary. Since it turns out that a smooth Lagrangian stationary submanifold is necessarily minimal (because the mean curvature vector field of $\Sigma$ is itself the infinitesimal generator of a Lagrangian variation, as indicated in [11]), points where a Lagrangian stationary submanifold fails to be minimal must be singular points, and what is of interest is the precise nature of the set of singularities. A second variational problem that one can pose is the following. There is a
natural sub-class of variations preserving the Lagrangian condition, namely the set of Hamiltonian transformations, which are generated by functions on $M$; hence one can also demand that the volume of $\Sigma$ is a critical point with respect to only Hamiltonian variations. In this case, $\Sigma$ is said to be Hamiltonian stationary, and there are indeed examples of non-trivial, smooth, Hamiltonian stationary submanifolds that are not minimal.

Hamiltonian stationary submanifolds of a Kähler-Einstein manifold $M$ have been studied by several authors, notably Oh [9, 10], Helein and Romon [3, 4, 5], Schoen and Wolfson [11, 12]. Oh initially posed the Lagrangian and Hamiltonian stationary variational problems and derived first and second variation formulæ. Hélein and Romon showed that $M$ is a Hermitian symmetric space of real dimension four, this stationarity condition can be reformulated as an infinite-dimensional integrable system whose solutions possess a Weierstraß-type representation. Moreover, they found all Hamiltonian stationary, doubly periodic immersions of $\mathbb{R}^2$ into $\mathbb{C}P^2$ using this representation. Finally, Schoen and Wolfson initiated the study of Lagrangian variational problems from the geometric analysis point of view, for the purpose of constructing minimal Lagrangian submanifolds as limits of volume-minimizing sequences of Lagrangian submanifolds.

The approach that is taken in this paper is to state a very general sufficient condition for the existence of a certain type of Hamiltonian stationary submanifold in a Kähler manifold $M$. Namely, we specify a condition at a point $p$ in $M$ which allows us to construct Hamiltonian stationary tori of sufficiently small radii optimally situated in a neighbourhood of the point $p$. Of course, a simple motivating example is $\mathbb{C}^n$ where one has the standard tori of any radii built with respect to any chosen unitary frame at any chosen point. These tori will be explicitly used in our construction and will be defined carefully below. But for a more significant example, we note that all Kähler toric manifolds contain Hamiltonian stationary Lagrangian tori of the type envisaged here. A Kähler toric manifold is a closed, connected $2n$-dimensional Kähler manifold $(M, g, \omega, J)$ equipped with an effective Hamiltonian holomorphic action $\tau : \mathbb{T}^n \to \text{Diff}(M)$ of the standard (real) $n$-torus $\mathbb{T}^n$. The orbits of the group action turn out to be Hamiltonian stationary Lagrangian submanifolds of $M$, essentially because the metric $g$ turns out to be equivariant under the action of $\tau$. Furthermore, we know that the image of the moment map of $\tau$ is a convex polytope in $\mathbb{R}^n$. If $\mu_\tau : M \to \mathbb{R}^n$ denotes the moment map and $M_0 := \mu_\tau^{-1}(\text{int}(P))$ then we know that $M_0$ is an open, dense subset of $M$ that is symplectomorphic to $\text{int}(P) \times \mathbb{T}^n$ upon which the action is free. The orbit tori located near the corners of the polytope turn out to have small volume tending to zero at the corners themselves. A discussion of the geometry of Kähler toric manifolds can be found in [1] and the specific example of $\mathbb{C}P^2$ will be presented below for the sake of building intuition.

On the other hand, in a general Kähler manifold $M$, one might expect that smooth, small Hamiltonian stationary tori are rather rare, with a condition depending in some way on the ambient geometry of $M$ governing their existence. The archetype for this kind of a result is an analogous construction of constant mean curvature hypersurfaces in a Riemannian manifold $M$. Indeed, Ye has shown that it is possible to perturb a sufficiently small geodesic sphere centered at the point
\( p \in M \) to a hypersurface of exactly constant mean curvature, provided that \( p \) is a non-degenerate critical point of the scalar curvature of \( M \) \[14\].

We now explain and state the Main Theorem to be proved in this paper. Let \( U_2(M) \) denote the unitary frame bundle of \( M \) and choose a point \( p \in M \) and a unitary frame \( U_p \in U_2(M) \) at \( p \). Let \((z^1, z^2)\) be geodesic normal complex coordinates for a neighbourhood of \( p \) whose coordinate vectors at the origin coincide with \( U_p \). Fix \( r := (r_1, r_2) \in \mathbb{R}_+^2 \), the open positive quadrant of \( \mathbb{R}^2 \), with small \( \| r \| \) and define the submanifold

\[
\Sigma_r(U_p) := \{ (r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) : (\theta_1, \theta_2) \in \mathbb{T}^2 \}.
\]

If \( M \) were \( C^2 \) then \( \Sigma_r(U_p) \) would be Hamiltonian stationary Lagrangian for all \( r \) and \( U_p \). In general, \( \Sigma_r(U_p) \) is almost Hamiltonian stationary Lagrangian when \( \| r \| \) is very small, as the ambient metric is nearly Euclidean in geodesic normal complex coordinates. Next, for any section \( X \in \Gamma(J(T \Sigma_r(U_p))) \)

define the deformed submanifold

\[
\mu_X(\Sigma_r(U_p)) := \{ (r_1(1 + X^1(\theta))e^{i\theta_1}, r_2(1 + X^2(\theta))e^{i\theta_2}) : (\theta_1, \theta_2) \in \mathbb{T}^2 \}.
\]

We now want to define a function on unitary frames which will be used to state the existence condition of the Main Theorem below. First observe that the unitary group acts on \( U_2(M) \) by matrix multiplication in the fiber direction. The subgroup of diagonal matrices \( \text{Diag} \subseteq U(2) \) thus acts on \( U_2(M) \) as well, and we define the function \( \mathcal{F}_r : U_2(M)/\text{Diag} \to \mathbb{R} \) by

\[
\mathcal{F}_r(U_p) := r_1^2 R_{11}^C(p) + r_2^2 R_{22}^C(p)
\]

where \( R_{11}^C \) and \( R_{22}^C \) are the components of the complex Ricci curvature computed with respect to the chosen frame at the point \( p \). Note that this makes sense since \( \text{Ric}^C(\theta, (\theta, \theta)) = \text{Ric}^C((\theta, \theta), (\theta, \theta)) \) for all \( \theta \in S^1 \). Furthermore, it is the case that \( \Sigma_r(U_p) = \Sigma_r(D \cdot U_p) \) for all diagonal matrices \( D \in \text{Diag} \) so that \( \mathcal{F}_r \) depends on only the information contained in \( U_p \) that relates to \( \Sigma_r(U_p) \).

In the statement of the Main Theorem below, the norm \( \| \cdot \|_{C^k,\alpha} \) is a weighted \( C^{k,\alpha} \) norm with respect to \( g \), defined by

\[
\| u \|_{C^k,\alpha} := \sup_{\Sigma_r} \| u \| + \| r \| \sup_{\Sigma_r} \| \nabla u \| + \cdots + \| r \|^k \sup_{\Sigma_r} \| \nabla^k u \| + \| r \|^{k+\alpha} [\nabla^k u]_{\Sigma_r}
\]

where \([ \cdot ]_{\Sigma_r} \) is the usual Hölder coefficient on \( \Sigma_r \). In addition, we take the metric on the frame bundle to be the natural metric inherited from \( g \).

**Main Theorem.** Let \((M, g, \omega, J)\) be a Kähler manifold, with \( \dim_{\mathbb{R}} M = 4 \). Suppose \( U_p \in U_2(M) \) is such that the equivalence class \([U_p] \in U_2(M) / \text{Diag}\) is a non-degenerate critical point of \( \mathcal{F}_r \). If \( \| r \| \) is sufficiently small, then there exists \( U_{p'} \in U_2(M) \) and a section \( X \in \Gamma(J(T \Sigma_r(U_p))) \) so that the submanifold \( \mu_X(\Sigma_r(U_p)) \) is smooth and Hamiltonian stationary Lagrangian. Moreover, for any \( k \in \mathbb{N} \) and \( \alpha \in (0, 1) \), we have \( \| X \|_{C^k,\alpha} \equiv O(\| r \|^2) \), and the distance between \( U_p \) and \( U_{p'} \) as points in \( U_2(M) \) is \( O(\| r \|^2) \).
We note as a direct corollary that it is possible to extend the Main Theorem slightly in order to answer a more general question. That is, the Main Theorem finds a Hamiltonian stationary submanifold that is a small perturbation of $\Sigma_r$ for $\|r\|$ sufficiently small. Now one can ask if it is possible to find neighbouring Hamiltonian stationary Lagrangian submanifolds which are perturbations of $\Sigma_{r'}$ with $r'$ sufficiently close to $r$. The answer to this question is that one can indeed find such submanifolds because the non-degenerate critical points of the family of functionals $F_{r'}$ with $r'$ varying in a neighbourhood of $r$ are stable. That is, if $r'$ is sufficiently close to $r$ then $F_{r'}$ has a non-degenerate critical point $[U_{p(r')}]$ near $[U_p]$. By the Implicit Function Theorem, moreover, the association $r' \mapsto [U_{p(r')}]$ is smooth and this can be lifted to a smooth association $r' \mapsto U_{p(r')}$.

**Corollary.** Let $r := (r_1, r_2) \in \mathbb{R}_+^2$ with $\|r\|$ sufficiently small and suppose $U_p \in U_2(M)$ is such that the equivalence class $[U_p] \in U_2(M)/\text{Diag}$ is a non-degenerate critical point of $F_r$. Then one can find a small neighbourhood $V \subset \mathbb{R}_+^2$ containing $r$ so that $\Sigma_{r'}(U_p)$ can be perturbed into a Hamiltonian stationary Lagrangian submanifold of $M$ for all $r' \in V$. Moreover, the mapping taking $r'$ to the associated Hamiltonian stationary Lagrangian submanifold is smooth.

The Main Theorem will be proved following broadly similar lines as the proof of Ye’s result. That is, for each $U_p$ and sufficiently small $\|r\|$, a section $X$ will be found so that $\mu_X(\Sigma_r(U_p))$ is almost Lagrangian and Hamiltonian stationary; in fact the small error will be arranged to lie in a certain finite-dimensional space. The discrepancy comes from the fact that the Hamiltonian stationary differential operator possesses an approximate co-kernel (coming from translation and $U(2)$-rotation) that constitutes an obstruction to solvability. Only when $\Sigma_r(U_p)$ is very special (such that the image of the Hamiltonian stationary differential operator acting on $\Sigma_r(U_p)$ is orthogonal to the associated co-kernel to lowest order in $\|r\|$) can a solution be found. The existence condition, as indicated in the Main Theorem, is that $[U_p]$ is a non-degenerate critical point of $F_r$. This condition is qualitatively similar to Ye’s condition in that it involves the ambient curvature tensor of $M$. But of course the condition here takes into account the freedom to choose the complex frame with respect to which $\Sigma_r(U_p)$ is built as well as the point $p$ where $\Sigma_r(U_p)$ is located.

As with Ye’s condition, it is not always the case that $F_r$ possesses non-degenerate critical points. For example, this occurs in the case of $\mathbb{C}P^2$ and of $\mathbb{C}^2$, despite the fact that both spaces contain small Hamiltonian stationary Lagrangian tori. These examples can be seen as analogues of the situation in $\mathbb{R}^n$, a space which fails to satisfy the non-degeneracy criterion of Ye and where constant mean curvature spheres come in great abundance. It should be noted that Pacard and Xu have recently strengthened Ye’s result by replacing the non-degeneracy condition appearing there with a different condition, from which they can deduce that every compact Riemannian manifold must have at least one point $p$ for which sufficiently small geodesic spheres centered at $p$ can be perturbed to hypersurfaces of constant mean curvature [10]. A similar strengthening should be possible in the Hamiltonian stationary Lagrangian case as well.

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2 Geometric Preliminaries

2.1 Kähler Manifolds

A complex manifold $M$ of real dimension $2n$ and integrable complex structure $J$ is said to be Kähler if it possesses a Riemannian metric $g$ for which $J$ is an isometry, as well as a symplectic form $\omega$ satisfying the compatibility condition $\omega(X,Y) = g(JX,Y)$ for all tangent vectors $X,Y$. Standard references for Kähler manifolds are [2] and [7]. What follows is a brief description, for the purpose of fixing terminology and notation, of those aspects of Kähler geometry that will be relevant for what follows.

The question of interest is the nature of the local geometry of a Kähler manifold. Consider first the simplest example of a Kähler manifold: this is $\mathbb{C}^n$ equipped with the standard Euclidean metric $\tilde{g} := \text{Re}(\sum_k dz^k \otimes d\bar{z}^k)$ and the standard symplectic form $\tilde{\omega} := -\text{Im}(\sum_k dz^k \otimes d\bar{z}^k)$ (both given in complex coordinates), as well as the standard complex structure (which coincides with multiplication by $\sqrt{-1}$ in complex coordinates). In a general Kähler manifold, it is a fact that it is always possible to find local complex coordinates for a neighbourhood $V$ of any point $p \in M$ in which the complex structure is standard everywhere in $V$, and the metric and symplectic form are standard at $p$ with vanishing derivatives. In fact, more is true: the metric and symplectic form possess special structure in such a coordinate chart.

It is possible to show that there is a function $F : \mathcal{V} \to \mathbb{R}$, called the Kähler potential, so that the metric and symplectic form are:

$$g = 2 \text{Re} \sum_{k,l} \left( \frac{\partial^2 F}{\partial z^k \partial \bar{z}^l} dz^k \otimes d\bar{z}^l \right) = \frac{1}{2} \sum_{k,l} \left( \frac{\partial^2 F}{\partial x^k \partial x^l} + \frac{\partial^2 F}{\partial y^k \partial y^l} \right) (dx^k \otimes dx^l + dy^k \otimes dy^l) + \frac{1}{2} \sum_{k,l} \left( \frac{\partial^2 F}{\partial y^k \partial x^l} - \frac{\partial^2 F}{\partial x^k \partial y^l} \right) (dy^k \otimes dx^l - dx^k \otimes dy^l)$$

$$\omega = -2 \text{Im} \sum_{k,l} \left( \frac{\partial^2 F}{\partial z^k \partial \bar{z}^l} dz^k \otimes d\bar{z}^l \right) = \frac{1}{2} \sum_{k,l} \left( \frac{\partial^2 F}{\partial x^k \partial x^l} + \frac{\partial^2 F}{\partial y^k \partial y^l} \right) (dx^k \otimes dy^l - dy^k \otimes dx^l) + \frac{1}{2} \sum_{k,l} \left( \frac{\partial^2 F}{\partial y^k \partial x^l} - \frac{\partial^2 F}{\partial x^k \partial y^l} \right) (dx^k \otimes dx^l + dy^k \otimes dy^l),$$

in the local complex coordinates $(z^1, \ldots, z^n)$ or local real coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ for $\mathcal{V}$, which are related by $z^k = x^k + iy^k$. Note that

$$\omega = \frac{1}{2} \sum_k d\left( \frac{\partial F}{\partial x^k} dy^k - \frac{\partial F}{\partial y^k} dx^k \right).$$
which is consistent with the fact that $d\omega = 0$, and locally, closed forms are exact. Write $\omega := d\alpha$, where $\alpha$ is called the Liouville form of $\omega$, and write $\hat{\omega} := \frac{1}{2} \sum_k (x^k dy^k - y^k dx^k)$ for the Liouville form of the standard symplectic form. Note also that the Kähler potential is unique up to the addition of a function $\varphi$ satisfying $\partial_z \partial_{\bar{z}} \varphi = 0$ for all $k,l$. One can additionally show that it is possible to choose $F$ near the origin having the form

$$F(z, \bar{z}) := \frac{1}{2} \|z\|^2 + \hat{F}(z, \bar{z})$$

where $\hat{F}$ vanishes at least to order four in $z$ and $\bar{z}$. Hence $\partial_z \partial_{\bar{z}} F = \delta_{kl} + O(\|z\|^2)$. Consequently, the Kähler structures near the origin are perturbations of the standard structures $\hat{g}$ and $\hat{\omega}$, whose Kähler potential is $\hat{F}(z, \bar{z}) := \frac{1}{2} \|z\|^2$.

The complexified curvature tensor of a Kähler manifold in local coordinates in $\mathcal{V}$ can be expressed in terms of the Kähler potential. Namely, the complexified curvature tensor satisfies

$$R^C_{klmn} = \frac{\partial^4 \hat{F}}{\partial z^k \partial \bar{z}^l \partial z^m \partial \bar{z}^n} - \sum_{u,v} g^{uv} \frac{\partial^3 \hat{F}}{\partial z^k \partial \bar{z}^u \partial z^m \partial \bar{z}^v} \frac{\partial^3 \hat{F}}{\partial z^l \partial \bar{z}^v \partial \bar{z}^u \partial \bar{z}^n}.$$ 

Since $\partial^3 F(0) = 0$, then we have

$$R^C_{klmn}(p) = \frac{\partial^4 \hat{F}(0)}{\partial z^k \partial \bar{z}^l \partial z^m \partial \bar{z}^n}. \quad (1)$$

### 2.2 Hamiltonian Stationary Lagrangian Submanifolds

Interesting submanifolds of a Kähler manifold can be characterized by the effect of the action of $J$ on tangent spaces. For instance, a complex submanifold of $M^{2n}$ is one whose tangent spaces are invariant under $J$. Two classes of submanifolds of importance in this paper are defined in terms of a complementary condition to that of a complex submanifold. An $n$-dimensional submanifold $\Sigma$ is called Lagrangian if $J(T_p \Sigma)$ is orthogonal to $T_p \Sigma$ for each $p \in \Sigma$. Hence a Lagrangian submanifold satisfies $\omega(X, Y) = 0$ for all $X, Y \in T_p \Sigma$ and $p \in \Sigma$. More generally, an $n$-dimensional submanifold $\Sigma$ for which $J(T_p \Sigma)$ is transverse to $T_p \Sigma$ for each $p \in \Sigma$ is called totally real.

We will be interested in diffeomorphisms of $M$ that preserve some or all aspects of its Kähler structure. The diffeomorphisms which preserve the full Kähler structure are the holomorphic isometries and are quite rare in general. In $\mathbb{C}^n$, though, there are holomorphic isometries: these are the $U(n)$-rotations. The diffeomorphisms which preserve the symplectic form but not necessarily the metric are called symplectomorphisms. Every Kähler manifold possesses symplectomorphisms; indeed, for each function $u : M \to \mathbb{R}$ the one-parameter family of diffeomorphisms obtained by integrating the vector field $X$ defined by $X \cdot \omega := du$ are symplectomorphisms. These diffeomorphisms are called Hamiltonian. The condition of being totally real or Lagrangian is preserved by symplectomorphisms.

Consider now a Lagrangian submanifold $\Sigma \subset M$. If $\Sigma$ is a critical point of the $n$-dimensional volume functional amongst all possible compactly supported variations, then $\Sigma$ is minimal, in
which case the mean curvature vector $\vec{H}_\Sigma$ of $\Sigma$ vanishes. Suppose, however, that $\Sigma$ is merely a critical point of the $n$-dimensional volume amongst only Hamiltonian variations, and thus is Hamiltonian stationary Lagrangian. By computing the Euler-Lagrange equations for $\Sigma$, it becomes clear that being Hamiltonian stationary is in general a strictly weaker condition than being minimal. Indeed, let $\phi_t$ be a one-parameter family of Hamiltonian diffeomorphisms of $M$ with infinitesimal deformation vector field $X$ satisfying $X \lrcorner \omega = du$ for $u : M \to \mathbb{R}$. Then

$$0 = \frac{d}{dt} \text{Vol}(\phi_t(\Sigma)) \bigg|_{t=0} = -\int_\Sigma g(\vec{H}_\Sigma, X) \, d\text{Vol}_\Sigma$$

$$= -\int_\Sigma \omega(X, J\vec{H}_\Sigma) \, d\text{Vol}_\Sigma$$

$$= -\int_\Sigma g(\nabla u, J\vec{H}_\Sigma) \, d\text{Vol}_\Sigma$$

$$= \int_\Sigma u \nabla \cdot (J\vec{H}_\Sigma) \, d\text{Vol}_\Sigma$$  \hspace{1cm} (2)

by Stokes’ Theorem. Here $\nabla$ is the connection associated with the ambient metric $g$ while $\nabla$ is the induced connection of $\Sigma$, and $\nabla \cdot$ is the divergence operator. Since (2) must hold for all functions $u$, it must be the case that the mean curvature of $\Sigma$ satisfies

$$\nabla \cdot (J\vec{H}_\Sigma) = 0.$$  \hspace{1cm} (3)

Equation (3) will be solved in this paper to find Hamiltonian stationary Lagrangian submanifolds.

Observe that since $\Sigma$ is Lagrangian and $\vec{H}_\Sigma$ is normal to $\Sigma$, then $J\vec{H}_\Sigma$ is tangent to $\Sigma$ and taking its divergence with respect to the induced connection makes sense. It is convenient to introduce some notation at this point so that the mean curvature (and second fundamental form) of a totally real submanifold can be treated in a similar manner. To this end, let $\Sigma$ be totally real and define the symplectic second fundamental form and the symplectic mean curvature of $\Sigma$ by the formulæ

$$B(X, Y, Z) := \omega((\nabla_X Y)^\perp, Z)$$

and

$$H(Z) := \text{Trace}(B(\cdot, \cdot, Z))$$

where $X^\perp$ denotes the orthogonal projection of a vector $X$ defined at a point $p \in \Sigma$ to the normal bundle of $\Sigma$ at $p$. The symplectic mean curvature is thus a one-form on $\Sigma$ and equation (3) becomes

$$\nabla \cdot H = 0,$$

where again $\nabla \cdot$ is the divergence operator.

**Remark:** The following observation about the symplectic second fundamental form is important. If $\Sigma$ is Lagrangian then $B(X, Y, Z) = \omega((\nabla_X Y)^\perp, Z)$ for all vector fields $X, Y, Z$ tangent to $\Sigma$ since $\omega((\nabla_X Y)^\parallel, Z) = 0$. Hence we have the usual symmetry $B(X, Y, Z) = B(Y, X, Z)$. In addition, we have $B(X, Z, Y) = g(J\nabla_X Z, Y) = g(\nabla_X JZ, Y) = -g(JZ, \nabla_X Y) = g(J\nabla_X Y, Z) = B(X, Y, Z)$. Consequently the symplectic fundamental form of a Lagrangian submanifold is fully symmetric in all of its slots.
2.3 Hamiltonian Stationary Lagrangian Submanifolds in $\mathbb{C}P^2$

We now discuss a simple example demonstrating that the Kähler manifold $\mathbb{C}P^2$, equipped with the Fubini-Study metric, contains a two-parameter family of Hamiltonian stationary Lagrangian tori that are not minimal; and that there are members of this family with arbitrary small radii. Therefore $\mathbb{C}P^2$ contains Hamiltonian stationary Lagrangian submanifolds of the type we intend to construct in this paper. As mentioned in the introduction, the existence of these tori is expected because $\mathbb{C}P^2$ is a toric Kähler manifold.

The family of tori in question will be obtained by projecting a family of three-dimensional tori in $S^5$ to $\mathbb{C}P^2$ using the Hopf projection. These are found by choosing three positive real numbers $r_1$, $r_2$, and $r_3$ satisfying $r_1^2 + r_2^2 + r_3^2 = 1$, and then setting

$$T_r := \{(r_1e^{i\theta}, r_2e^{i\theta}, r_3e^{i\theta}) : \theta \in \mathbb{S}^1\}.$$ 

Here we denote $r := (r_1, r_2, r_3)$. Notice that $T_r$ is foliated by the Hopf fibration: the fiber through the point $p := (r_1e^{i\theta_1}, r_2e^{i\theta_2}, r_3e^{i\theta_3})$ is $\{e^{i\theta}p : \theta \in \mathbb{S}^1\}$ which is clearly a subset of $T_r$. Moreover, this foliation is regular and thus $\Sigma_r := \pi_{\text{Hopf}}(T_r)$ is a two-dimensional submanifold of $\mathbb{C}P^2$, where $\pi_{\text{Hopf}} : S^5 \to \mathbb{C}P^2$ is the Hopf projection. Furthermore, it is clear that $\Sigma_r$ is a torus.

The torus $\Sigma_r$ is Hamiltonian stationary for the following reasons. First, recall the relationship between the symplectic form $\omega$ of $\mathbb{C}P^2$ and the Kähler structure of $\mathbb{C}^3$. That is, if $V_1$ and $V_2$ are two tangent vectors of $\mathbb{C}P^2$, then $\omega(V_1, V_2) := \text{Re}(\hat{g}(i\hat{V}_1, \hat{V}_2))$ where $\hat{g}$ is the Euclidean metric of $\mathbb{C}^3$ and $\hat{V}_i$ is the unique vector in $(\pi_{\text{Hopf}})^{-1}(V_i)$ that is orthogonal to the Hopf fiber. It follows that $\Sigma_r$ is Lagrangian because if $V_i$ is tangent to $\Sigma_r$ then

$$\hat{V}_i \in \text{span}_\mathbb{R}\left\{iz \frac{\partial}{\partial z^1}, iz \frac{\partial}{\partial z^2}, iz \frac{\partial}{\partial z^3}\right\}$$

and it is clear that $\text{Re}(\hat{g}(iX, Y)) = 0$ for all vectors $X, Y$ belonging to this space. Next, determining if $\Sigma_r$ is Hamiltonian stationary requires computing its second fundamental form. Now because $\Sigma_r$ is Lagrangian, it can be lifted to a Legendrian submanifold $\hat{\Sigma}_r \subseteq T_r$ of $S^5$ and this lifting is a local isometry. Furthermore, the second fundamental form of $\hat{\Sigma}_r$ coincides with the second fundamental form of $\Sigma_r$. Therefore it suffices to compute the second fundamental form of $\hat{\Sigma}_r$, which is a slightly simpler task and is done as follows. We can locally parametrize $\hat{v}$ by

$$\mathcal{A} : (\alpha^1, \alpha^2) \mapsto (r_1e^{iL^1(\alpha)}, r_2e^{iL^2(\alpha)}, r_3e^{iL^3(\alpha)})$$

where $L^k(\alpha) := \sum \alpha^sL_s^k\alpha^s$ is a linear function of $\alpha := (\alpha^1, \alpha^2)$ chosen so that the tangent vectors $V_s := \mathcal{A}_*(\frac{\partial}{\partial \alpha^s})$ are linearly independent and $\sum_{k=1}^3 r_s^2L_s^k = 0$ for $s = 1, 2$. This latter condition says that each $V_s$ is orthogonal to the Hopf vector field. Furthermore, one can check that any other choice of linear functions satisfying the aforementioned constraints amounts to a reparametrization of $\hat{\Sigma}_r$. The induced metric of the parametrization is

$$\hat{h} := \sum_{s,t} \text{Re}(\hat{g}(\hat{V}_s, \hat{V}_t))d\alpha^s \otimes d\alpha^t = \sum_{s,t} \left(\sum_{k=1}^3 r_s^2L_s^kL_t^k\right)d\alpha^s \otimes d\alpha^t$$
which is a flat metric. The second fundamental form of this parametrization can be deduced from
\[
\text{Re}(\hat{g}(\hat{\nabla}_V V_t, iV_u)) = \sum_k r_k^2 L_k^s L_t^k L_u^k
\]
where \(\hat{\nabla}\) is the Euclidean connection, which shows in particular that the second fundamental form is parallel with respect to the induced metric. Hence its divergence is zero. Consequently \(\Sigma_r\) is Hamiltonian stationary but not minimal.

Finally we would like to know the geometric dimensions of \(\Sigma_r\) in \(\mathbb{C}P^2\). Since we know the induced metric of \(\Sigma_r\), this amounts to finding the size of the smallest domain in \(\hat{\Sigma}_r\) that maps bijectively onto \(\Sigma_r\) under \(\pi_{\text{Hopf}}\). After some work, we find that this domain is the parallelogram in the \((\alpha^1, \alpha^2)\)-coordinates spanned by the vectors
\[
E_k := \sum_{s,t} \hat{h}^{st} \text{Re} \left( \hat{g} \left( iz_k \frac{\partial}{\partial z_k}, \hat{V}_t \right) \right) \frac{\partial}{\partial \alpha^s} \quad k = 1, 2.
\]
One can check that the volume of this parallelogram with respect to the induced metric \(\hat{h}\) is given by \(r_1 r_2 \sqrt{1 - r_1^2 - r_2^2}\). Hence one can consider \(\Sigma_r\) to be small when \(r_1\) or \(r_2\) tends to zero.

3 Constructing the Approximate Solution

Let us assume in this paper from now on that the real dimension of the ambient manifold is four and thus that the dimension of the Hamiltonian stationary Lagrangian submanifold is two, since this simplifies the presentation of the results and their proofs. We expect that most of the forthcoming calculations should generalize to higher dimensions and similar results will hold.

3.1 Rescaling the Ambient Manifold

Choose a point \(p \in M\) and find local complex coordinates so that a small neighbourhood \(\mathcal{V}\) of \(p\) maps to a small neighbourhood \(\mathcal{V}_0\) of the origin in \(\mathbb{C}^2\). Moreover, let these coordinates be such that the metric and symplectic form are of the type discussed in Section 2.1. Assume that the diameter of this neighbourhood is \(\rho_0 \in (0, 1)\); let \(r = (r_1, r_2)\), with \(\|r\| < \rho_0\), be the radii of the Hamiltonian stationary Lagrangian torus we intend to construct, and set \(\rho := \|r\|\). Now change coordinates in this neighbourhood and also re-scale the metric and symplectic form via
\[
z \mapsto \rho z \quad \text{and} \quad g \mapsto \rho^{-2} \varphi^* g \quad \text{and} \quad \omega \mapsto \rho^{-2} \varphi^* \omega. \tag{4}
\]
As a result, we obtain a new Kähler metric on a large neighbourhood \(\|r\|^{-1} \mathcal{V}_0\) of the origin in \(\mathbb{C}^2\), where the complex structure is standard and the Kähler potential is
\[
F_\rho(z, \bar{z}) := \frac{1}{2} \|z\|^2 + \rho^2 \hat{F}_\rho(z, \bar{z})
\]
with \(\hat{F}_\rho(z, \bar{z}) := \rho^{-4} \hat{F}(\rho z, \rho \bar{z})\). Furthermore, the Hamiltonian stationary Lagrangian condition is unchanged under this re-scaling and the torus \(\frac{1}{\rho} \Sigma_r\) has radii \((r_1, r_2)\) satisfying \(r_1^2 + r_2^2 = 1\). Therefore,
in order to construct a Hamiltonian stationary Lagrangian torus of small radii near \( p \), it is sufficient to construct a Hamiltonian stationary Lagrangian torus with unit radius vector near the origin in \( \mathbb{C}^2 \) with Kähler potential \( F_\rho \), but to take \( \rho \) sufficiently small. Finally, the weighted \( C^{k,\alpha} \) norm used in the statement of the Main Theorem is equivalent to the standard \( C^{k,\alpha} \) norm under the re-scaling.

**Remark:** The advantage of working with these scaled coordinates is that it is now possible to express the deviation of the background geometry from Euclidean space very efficiently using the parameter \( \rho \). In particular, \( \tilde{F}_\rho \) can be expanded in a power series in \( z \) and \( \bar{z} \) starting at order four that has coefficients depending on \( \rho \) but bounded uniformly by a constant of size \( O(\rho^2) \).

### 3.2 The Approximate Solution

Let \( U_2(M) \) denote the unitary frame bundle of \( M \) and choose a point \( p \in M \) and a unitary frame \( U_p \in U_2(M) \) at \( p \). Let \((z^1, z^2)\) be geodesic normal complex coordinates for a neighbourhood of \( p \) whose coordinate vectors at the origin coincide with \( U_p \). Now let \( r := (r_1, r_2) \) be some fixed vector belonging to \( \mathbb{R}^2_+ \), the open positive quadrant of \( \mathbb{R}^2 \), with \( \|r\| = 1 \). Define the 2-dimensional submanifold of \( \mathbb{C}^2 \) given by

\[
\Sigma_r(U_p) := \{ (r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) : (\theta_1, \theta_2) \in T^2 \} .
\]

Note that \( \Sigma_r(U_p) \) is the image of the \( T^2 \) under the embedding \( \mu_0 : (\theta_1, \theta_2) \mapsto (r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) \). We will denote \( \Sigma_r := \Sigma_r(U_p) \) when it is not necessary to speak explicitly of the frame \( U_p \) from which \( \Sigma_r(U_p) \) is built.

The following result motivates the use of \( \Sigma_r \) as an **approximate solution** of the problem of finding Hamiltonian stationary Lagrangian submanifolds in arbitrary Kähler manifolds.

**Lemma 1.** The submanifold \( \Sigma_r \) is Hamiltonian stationary Lagrangian with respect to the standard Kähler structure \((\hat{g}, \hat{\omega}, J)\) of \( \mathbb{C}^2 \). In fact, the symplectic second fundamental form \( \hat{B} \) and the symplectic mean curvature \( \hat{H} \) are parallel.

**Proof.** We include this standard calculation for the convenience of the reader. To begin, the tangent vectors of \( \Sigma_r \) can be found by differentiating in \( \theta \). In complex notation, these are \( E_k := i r_k e^{i\theta_k} \frac{\partial}{\partial z^k} \), for \( k = 1,2 \). From this we can immediately compute the components of the induced metric \( \hat{h} \) and those of \( \hat{\omega} \) restricted to \( \Sigma_r \). Indeed, since the Kähler potential is \( \hat{F}(z, \bar{z}) = \frac{1}{2} \|z\|^2 \), we can read off the induced metric and pullback of the symplectic form as the real and imaginary parts, respectively, of

\[
\sum_s dz^s \otimes dz^s(E_k, \hat{E}_l) = \sum_s r_k r_l i e^{i\theta_k} \delta_{sk} (-ie^{-i\theta_l} \delta_{sl}) = r_k^2 \delta_{kl}.
\]

Thus \( \hat{\omega} \) vanishes on \( \Sigma_r \), and so \( \Sigma_r \) is Lagrangian. The induced metric is given by \( \hat{h}_{kl} = r_k^2 \delta_{kl} \).

Let the ambient connection be \( \nabla \) (the bar does not denote complex conjugation here). The covariant derivatives of the tangent vector fields of the embedding with respect to \( \hat{g} \) in complex
notation, are
\[ \nabla_{E_k} E_l = \frac{\partial}{\partial \theta_k} (ir_1 e^{i\theta_l} \frac{\partial}{\partial z_l}) = -r_1 \delta_{kl} e^{i\theta_l} \frac{\partial}{\partial z_l} = \delta_{kl} J E_l. \]
Since \( \Sigma_r \) is Lagrangian, we therefore see that the parallel part \( (\nabla_{E_k} E_l) \parallel \) vanishes. We can now compute the symplectic second fundamental form. That is,
\[ \check{B}_{klj} = \check{\omega}(\nabla_{E_k} E_l, E_j) = \check{\omega}(\nabla_{E_k} E_l, E_j) = -r_1 \delta_{kl} e^{i\theta_l} \frac{\partial}{\partial z_l}, r_j e^{-i\theta_j} \frac{\partial}{\partial \bar{z}_j}) = r^2 \delta_{jm} \delta_{km} \delta_{lm}, \]
where \( m \) can be any of \( k, l \) or \( j \). This emphasizes the symmetry of \( \check{B} \) in its indices, as proved more generally above. From here we see \( \check{H}_j = \check{h}^{kl} \check{B}_{klj} = 1 \) for each \( j \).

**Remark:** Note that the previous line shows that these Hamiltonian stationary Lagrangian tori are not minimal.

Lemma 1 suggests that we should choose a point \( p \in M \), find local complex coordinates in a neighbourhood \( \mathcal{V} \) of \( p \) as in Section 2.1 scale these coordinates by a factor \( \rho \) as above. Then if we embed the submanifold \( \Sigma_r \) into the coordinate image of \( \mathcal{V} \), then it remains the case that \( \Sigma_r \) is Hamiltonian stationary Lagrangian with respect to the standard Kähler structure but it is no longer necessarily so with respect to the Kähler structure \((g, \omega, J)\) with Kähler potential \( F_\rho \). However, if \( \rho \) is sufficiently small, then \( \Sigma_r \) is totally real; moreover, it is close, in a sense that will be made more precise later on, to being Hamiltonian stationary Lagrangian.

### 3.3 The Equations to Solve

An exactly Hamiltonian stationary Lagrangian submanifold with respect to the Kähler structure \((g, \omega, J)\) near the submanifold \( \Sigma_r \) when \( \rho \) is sufficiently small will be found by perturbing \( \Sigma_r \) appropriately. This will be done by first defining a class of deformations of \( \Sigma_r \) and then selecting the appropriate deformation by solving a differential equation. Define these deformations as follows. For every function \( X : \mathbb{T}^2 \to \mathbb{R}^2 \) of suitably small norm, define an embedding \( \mu_X : \mathbb{T}^2 \hookrightarrow \mathbb{C}^2 \) by
\[ \mu_X : (\theta^1, \theta^2) \mapsto (r_1(1 + X^1(\theta))e^{i\theta^1}, r_2(1 + X^2(\theta))e^{i\theta^2}). \]
Note that the Euclidean-normal bundle of \( \Sigma_r \) coincides with the bundle \( J(T\Sigma_r) \) and is spanned by the Euclidean-orthonormal vector fields \( N_k := e^{i\theta_k} \frac{\partial}{\partial z_k} \) for \( k = 1, 2 \). Thus a geometric interpretation of this embedding is to view \( X \) as a section of the bundle \( J(T\Sigma_r) \) and \( \mu_X \) as the Euclidean-exponential map scaled by the radii \( r_1, r_2 \) in the different coordinate directions. We employ the slight abuse of notation \( \mu_X(\Sigma_r) := \mu_X(\mathbb{T}^2) \).
Finding \( X \in \Gamma(J(T\Sigma_r)) \) so that \( \mu_X(\Sigma_r) \) is Hamiltonian stationary Lagrangian with respect to the Kähler structure \((g, \omega, J)\) amounts to solving two equations:

\[
\begin{align*}
\mu_X^* \omega &= 0 \\
\nabla \cdot H(\mu_X(\Sigma_r)) &= 0
\end{align*}
\]

(5)

where \( H(\Sigma_r) \) is the symplectic mean curvature of \( \Sigma_r \). Thus one should consider the differential operator \( \Phi_\rho : \Gamma(J(T\Sigma_r)) \to \Lambda^2(\Sigma_r) \times \Lambda^0(\Sigma_r) \) given by

\[ \Phi_\rho(X) := (\mu_X^* \omega, \nabla \cdot H(\mu_X(\Sigma_r))) \]

and attempt to solve the equation \( \Phi_\rho(X) = (0, 0) \). Note that the first of these equations is first-order in the vector field \( X \) while the second equation is third-order in \( X \). Since \( \Sigma_r \) is generally not Hamiltonian stationary nor Lagrangian with respect to the Kähler structure \((g, \omega, J)\) when \( \rho > 0 \), then \( \Phi_\rho(0) \) is a non-vanishing tensor field on \( \Sigma_r \) depending continuously on \( \rho \) in some way that will be determined in the sequel. Certainly, however, one can assert that \( \Phi_0(0) = (0, 0) \).

It turns out that, as it stands, equation (5) does not represent a strictly elliptic problem. A few refinements are necessary in order to achieve this. First, an important observation to make is that the operator \( \Phi_\rho \) maps onto a much smaller space. In fact, it is true that the first component of \( \Phi_\rho(X) \) belongs to \( \text{d}\Lambda^1(\Sigma_r) \), the set of exact one-forms, which can be seen as follows. Observe that \( \mu_X^* \omega \) is closed and belongs to the same cohomology class as \( \mu_X^* \omega \) for all \( t \in [0, 1] \). But \( \mu_0^* \omega = \text{d}\alpha|_{\Sigma_r} \) where \( \alpha \) is the Liouville form, so that \( \mu_0^* \omega \) is exact. Therefore \( \mu_X^* \omega \) is exact as well. The second factor of \( \Phi_\rho(X) \) is a divergence; hence its integral against the volume form of \( \mu_X(\Sigma_r) \) must vanish.

Next, we make an Ansatz for the section \( X \) of the bundle \( J(T\Sigma_r) \). We write \( X := X^k J E_k \) where \( E_k := i r_k e^{i \theta_k} \frac{\partial}{\partial z^k} \) are the coordinate basis vectors of the tangent space \( T\Sigma_r \), and motivated by the Hodge decomposition, we split \( X \) into a gradient and a curl component with respect to the metric induced on \( \Sigma_r \) by the Euclidean ambient metric. More specifically, we choose \( X := \mathcal{X}(u, v) \) so that \( X \cdot \omega|_{\Sigma_r} = dv + i du \) for functions \( u, v : \Sigma_r \to \mathbb{R} \), where \( i \) is the Hodge star operator of \( \Sigma_r \) with respect to the Euclidean metric. By inspection, this outcome is achieved by the vector field

\[
\mathcal{X}(u, v) := \sum_k \frac{1}{r_k^2} \left( \frac{\partial v}{\partial \theta_k} + \sum_j \varepsilon_j^k \frac{\partial u}{\partial \theta_j} \right) r_k e^{i \theta_k} \frac{\partial}{\partial z^k}
\]

(6)

where \( \varepsilon_j^k \) satisfies \( \varepsilon_1^1 = \varepsilon_2^2 = 0 \) and \( \varepsilon_1^2 = -r_1/r_2 \) and \( \varepsilon_2^1 = r_2/r_1 \). Note that the mapping given by \((u, v) \mapsto \mathcal{X}(u, v)\) is linear in \((u, v)\) and independent of \( \rho \).

Using the Ansatz above, one can re-formulate (5) as a pair of equations for the functions \( u \) and \( v \) which will turn out to be elliptic. Since (5) is mixed a first- and third-order partial differential equation and \( \mathcal{X}(u, v) \) takes one additional derivative, the functions \( u \) and \( v \) will be assumed to lie in \( C^4,\alpha \). Moreover, since \( \mathcal{X}(u, v) \) clearly remains unchanged if a constant is added to either \( u \) or \( v \), we impose the normalization

\[
\int_{\Sigma_r} u \dVol_{\Sigma_r}^0 = \int_{\Sigma_r} v \dVol_{\Sigma_r}^0 = 0
\]
where $d\text{Vol}_{\Sigma_r}^r$ is the volume form of $\Sigma_r$ with respect to the metric induced on $\Sigma_r$ by the ambient Euclidean metric. Therefore define a new differential operator by

$$\Phi_\rho : C^4_0(\Sigma_r) \times C^4_0(\Sigma_r) \to C^2_0(\Sigma_r) \times C^0_0(\Sigma_r)$$

$$\Phi_\rho(u, v) := \Phi_\rho \circ \mathcal{X}(u, v).$$

where we use the zero subscript to denote a function space upon which our normalization has been imposed.

4 Analysis of the Hamiltonian Stationary Lagrangian Operator

In order to solve the equation $\Phi_\rho(u, v) = (0, 0)$ perturbatively, it is necessary to understand the mapping properties of the linearization of the operator $\Phi_\rho$ at $(0, 0)$. We will use the notation $L_\rho := D_{(0, 0)} \Phi_\rho$ as well as $L_\rho := D_0 \Phi_\rho$ in the remainder of the paper. Observe that $L_\rho = L_0 \circ \mathcal{X}$ by linearity. Furthermore, since $\Phi_\rho$ for $\rho > 0$ will often be compared with its Euclidean analogue at $\rho = 0$, we introduce the notation $\Phi := \Phi_0$ and $\Phi := \Phi_0$ in keeping with the convention of adorning objects associated with the Euclidean metric with “$\circ$”. Thus we shall denote the linearizations of these operators by $\tilde{L} := D_0 \Phi$ and $\tilde{L} := L \circ \mathcal{X}$, respectively. Again, note that $\tilde{L} = \tilde{L} \circ \mathcal{X}$.

This section contains the following material. First we compute linearized operator $\tilde{L}$ and determine its kernel. It will turn out that $\tilde{L}$ is not self-adjoint; hence we next compute the adjoint $\tilde{L}^*$ and compute its kernel. Finally, we compute $L_\rho$ with enough detail to be able to give estimates, in terms of $\rho$, for the difference $P_\rho := L_\rho - \tilde{L}$.

4.1 The Unperturbed Linearization

Let $\Phi$ be the Hamiltonian stationary Lagrangian differential operator with respect to the standard Kähler structure $(\bar{g}, \bar{\omega}, J)$. The task at hand is to compute its linearization at zero, denoted by $\tilde{L}$. Since $\tilde{\Phi} = \Phi \circ \mathcal{X}$ and $\mathcal{X}$ is linear, the main computation is to find the linearization at zero of $\tilde{\Phi}$ acting on sections of $J(T\Sigma_r)$, denoted by $\tilde{L}$. In the computations below, repeated indices are summed, a comma denotes ordinary differentiation and a semi-colon denotes covariant differentiation.

**Proposition 2.** Let $\Sigma \subset \mathbb{C}^2$ be Lagrangian for the standard symplectic structure. Let $X$ be a $C^3$ section of $N(\Sigma) = J(T\Sigma)$, and write $X := X^j J E_j$ where $E_1, E_2$ is a coordinate basis for the tangent space of $\Sigma$. Write $\tilde{L}(X) := (\tilde{L}^{(1)}(X), \tilde{L}^{(2)}(X))$. Then

$$\tilde{L}^{(1)}(X) := d(X \lrcorner \tilde{\omega})$$

$$\tilde{L}^{(2)}(X) = -(\tilde{\Delta}X^m)_{;m} - \tilde{h}^{lm} \tilde{h}^{sk} \tilde{H}_s (X^u B_{lk} u) ;_m + \tilde{h}^{km} \tilde{H}_k (X^u \tilde{H}_u) ;_m - \tilde{h}^{lm} \tilde{h}^{js} \tilde{h}^{kq} (X^u \tilde{H}_{sq} B_{jk} u) ;_m.$$

**Proof.** The formula for $\tilde{L}^{(1)}$ is straightforward. Recall that it is a standard computation involving the Lie derivative of a 2-form to show that $\frac{d}{dt} \mu^*_{X_t} \tilde{\omega} |_{t=0} = d(X \lrcorner \tilde{\omega}) + X \lrcorner d\tilde{\omega}$. Therefore since $d\tilde{\omega} = 0$ then $\tilde{L}^{(1)}(X) = d(X \lrcorner \tilde{\omega})$ as desired.
The remainder of the proof concentrates on the computation for \( \hat{L}^{(2)}(X) \). Let \( \Sigma \) be a Lagrangian submanifold of \( \mathbb{C}^2 \) carrying the Euclidean metric \( \hat{g} \), and let \( X \) be a section of the normal bundle of \( \Sigma \). Let \( \mu_t : \mathbb{C}^2 \to \mathbb{C}^2 \) be a one-parameter family of diffeomorphisms with \( \frac{d}{dt} \mu_t \bigg|_{t=0} = X \) and set \( \Sigma^t := \mu_t(\Sigma) \). Next, choose \( E_1, E_2 \) a local coordinate frame for \( \Sigma \) coming from geodesic normal coordinates at \( p_0 \in \Sigma \) in the induced metric \( \hat{h} \) at \( t = 0 \). Then \( JE_1, JE_2 \) is basis for the normal bundle of \( \Sigma \) at \( t = 0 \), because \( \Sigma \) is Lagrangian, but it does not necessarily hold for \( |t| \neq 0 \) since \( \mu_t \) is not assumed to be a family of symplectomorphisms. However, for \( p \) near \( p_0 \), and \( T_pM = T_p \Sigma \oplus J(T_p \Sigma) \). We write \( X \big|_{\Sigma} = X^j E_j \). Note that \( X \) and \( E_k \) commute along \( \mu_t \), and since \( X \) is transverse to \( \Sigma \), we can extend the fields \( E_k \) locally using the diffeomorphism \( \mu_t \) to a basis for \( T_{\mu_t(p_0)} \Sigma^t \), for \( |t| \) small. In these coordinates the matrix for \( \hat{h} \) on \( \Sigma^t \) is the same as that for \( \mu_t^* \hat{g} \) on \( T \Sigma \). The computations below are evaluated at \( p_0 \) at \( t = 0 \).

In terms of the local coordinates introduced above, we have

\[
\hat{\nabla} \cdot \hat{H}(\Sigma^t) = \hat{h}^{lm} \hat{h}^{jk} \hat{B}_{jkl;m}
\]

where \( \hat{h}_{kl} := \hat{g}(E_k, E_l) \) is the induced metric, \( \hat{h}^{jk} \) are the components of the inverse of the induced metric, \( \hat{\nabla} \) is the induced connection, and

\[
\hat{B}_{jkl} := \hat{\omega}(\hat{\nabla} E_j E_k E_l) = \hat{\omega}(\hat{\nabla} E_j E_k, E_l) - \hat{\Gamma}^{s}_{jk} \hat{\omega}(E_s, E_l)
\]

(7)

with \( \hat{\Gamma}^{s}_{jk} \) the Christoffel symbols of \( \hat{h}_{jk} \) and \( \hat{\nabla} \) the ambient Euclidean connection.

The terms in (7) all depend on \( t \). Since \( \hat{\nabla} \cdot \hat{H}(\Sigma^t) = \hat{h}^{lm} \hat{H}_{l;m} = \hat{h}^{lm} \hat{H}_{l,m} - \hat{h}^{lm} \hat{\Gamma}^{s}_{lm} \hat{H}_s \) where \( \hat{\Gamma}^{s}_{lm} := \hat{h}^{jk} \hat{B}_{jkl} \), differentiating (7) at \( t = 0 \) yields

\[
\frac{d}{dt} \hat{\nabla} \cdot \hat{H}(\Sigma^t) \bigg|_{t=0} = (\hat{h}^{lm})' \hat{H}_{l;m} - \hat{h}^{lm} (\hat{\Gamma}^{s}_{lm})' \hat{H}_s + \hat{h}^{lm} ((\hat{H}_l)')_{;m}
\]

where a prime denotes the value of the time derivative at zero.

Expressions for \( (\hat{h}^{lm})' \) and \( (\hat{\Gamma}^{s}_{lm})' \) and \( (\hat{H}_l)' \) are now required. To begin, it is straightforward to compute

\[
(\hat{h}^{lm})' = -2\hat{h}^{is} \hat{h}^{mq} X^u \hat{B}_{snu}
\]

\[
(\hat{\Gamma}^{s}_{lm})' = \hat{h}^{sq} \left[ (X^u \hat{B}_{lqu})_{;m} + (X^u \hat{B}_{mu})_{;l} - (X^u \hat{B}_{lmu})_{;q} \right].
\]

Next

\[
(\hat{H}_l)' = (\hat{h}^{jk})' \hat{B}_{jkl} + \hat{h}^{jk} (\hat{B}_{jkl})' = -2\hat{h}^{is} \hat{h}^{kq} X^u \hat{B}_{snu} \hat{B}_{jkl} + \hat{h}^{jk} (\hat{B}_{jkl})'
\]

and the fact that both \( \hat{\Gamma}^{s}_{jk}(p_0) \) and \( \hat{\omega} \bigg|_{\Sigma^t} \) vanish at \( t = 0 \) implies

\[
(\hat{B}_{jkl})' = \frac{d}{dt} \left( \hat{\omega}(\hat{\nabla} E_j E_k, E_l) - \hat{\Gamma}^{s}_{jk} \hat{\omega}(E_s, E_l) \right) \bigg|_{t=0}
\]

\[
= \frac{d}{dt} \left( \hat{\omega}(\hat{\nabla} E_j E_k, E_l) \right) \bigg|_{t=0}
\]
\[ \ddot{\omega} (\nabla_X \nabla_{E_k} E_k, E_t) + \omega (\nabla_{E_j} E_k, \nabla_X E_t) \]
\[ = \ddot{\omega} (\nabla_{E_j} \nabla_{E_k} X, E_t) + \omega (\nabla_{E_j} E_k, \nabla_E(X)) \]
\[ = E_j \ddot{\omega}(\nabla_{E_k} X, E_t) - \dot{\omega}(\nabla_{E_k} X, \nabla_{E_j} E_t) + \omega (\nabla_{E_j} E_k, \nabla_{E_i} X) \]
\[ = -E_j \ddot{g} (\nabla_{E_k} (\nabla^q E_q), E_t) + \ddot{g} (\nabla_{E_k} (\nabla^q E_q), \nabla_{E_j} E_t) + \ddot{g} (\nabla_{E_j} E_k, \nabla_{E_i} (\nabla^q E_q)) \]

where \( \nabla \) is the ambient connection; we have used that \( X \) commutes with \( E_k \) along \( \mu_t \), that the ambient curvature vanishes, and that \( \dot{\omega} \) is parallel. Now

\[ \ddot{\nabla}_E (\nabla^q E_q) = X^q E_q + X^q \nabla_E E_q = X^q E_t - X^q \dot{\gamma}^{uv} \bar{B}_{qu} J E_v . \]

Note that at \( t = 0, \nabla_{E_k} E_j \) is normal to \( \Sigma \) at \( p_0 \), and moreover \( \ddot{g} (\nabla_{E_j} E_k, J E_m) = -\dot{B}_{jk m} \) at \( p_0 \). Thus we have

\[ (\dot{B}_{jk l}) = -E_j \ddot{g} (X^q E_q - X^q \dot{\gamma}^{uv} \bar{B}_{k qu} J E_v, E_l) + \ddot{g} (X^q E_q - X^q \dot{\gamma}^{uv} \bar{B}_{k qu} J E_v, \nabla_{E_j} E_l) \]
\[ + \ddot{g} (\nabla_{E_j} E_k, X^q E_q - X^q \dot{\gamma}^{uv} \bar{B}_{k qu} J E_v, \nabla_{E_j} E_l) \]
\[ = -X^q \dot{\gamma}^{ijkl} \dot{\gamma} + X^q \bar{B}_{k qu} \bar{B}_{jlv} \dot{\gamma}^{uv} + X^q \bar{B}_{jku} \bar{B}_{lqu} \dot{\gamma}^{uv} . \]

Everything can now be put together:

\[ \ddot{L}^{(2)} (X) = -2 \dot{h}^{ls} \dot{h}^{mq} X^u \bar{B}_{sq u} \ddot{H}_{lm} \]
\[ - \dot{h}_{sq} \dot{H}_s (2 (X^u \bar{B}_{lu q} )_{,m} \dot{h}_{lm} - (X^u \ddot{H}_l )_{,q}) \]
\[ - 2 \dot{h}_{lm} \dot{h}_{sj} \dot{h}^{kj} (X^u \bar{B}_{su q} \ddot{B}_{jkl} )_{,m} \]
\[ + \dot{h}_{lm} \dot{h}_{jk} (-X^q \dot{\gamma}^{ijkl} \dot{\gamma} + X^q \bar{B}_{k qu} \bar{B}_{jlv} \dot{\gamma}^{uv} + X^q \bar{B}_{jku} \bar{B}_{lqu} \dot{\gamma}^{uv} )_{,m} \]
\[ = - (\Delta X^m )_{,m} - \dot{h}_{sq} \dot{H}_s ( (X^u \bar{B}_{lu q} )_{,m} \dot{h}_{lm} - (X^u \ddot{H}_l )_{,q}) \]
\[ - \dot{h}_{lm} \dot{h}_{sj} \dot{h}^{kj} (X^u \bar{B}_{su q} \ddot{B}_{jkl} )_{,m} \]

This is the desired formula. \( \square \)

To compute \( \dot{L}^{(2)} \) for the torus \( \Sigma_r \), note that both \( \dot{B} \) and \( \ddot{H} \) are parallel tensors in this case. Consequently the second fundamental form term in \( \dot{L}^{(2)} \) becomes simply \( X \mapsto -\dot{A}^l_k X^l_k \) where

\[ \dot{A}^{kl} := \dot{H}_s \dot{B}_{ls m} - \ddot{H}^l \dot{H}_m + \bar{B}_{sq m} \dot{B}_{sq m} \]

and furthermore, we can compute precisely: substituting and \( \dot{h}_{kl} = r^2 \delta_{kl} \) and \( \dot{B}_{jkl} = r^2 \delta_{sj} \delta_{sk} \delta_{st} \) for the induced metric and symplectic second fundamental form of \( \Sigma_r \) with respect to the Euclidean metric yields

\[ \dot{A}^{lm} = \frac{2 \delta^{lm}}{r^4 m} - \frac{1}{r^2 l^2 m^2} . \]

Now let \( X = X(t, v) \) as in \( [6] \) and substitute this into the formulæ of Proposition \( 2 \) to find the linearization \( \ddot{L} \).
Corollary 3. Let \((u, v) \in C^4_0(\Sigma_r) \times C^4_0(\Sigma_r)\). Write \(\tilde{L} = (\tilde{L}^{(1)}, \tilde{L}^{(2)})\). Then

\[
\tilde{L}^{(1)}(u, v) := d \ast du \\
\tilde{L}^{(2)}(u, v) := \dot{\Delta}(\dot{\Delta}v) + \dot{A}^{lm}v;_{;lm} + \dot{A}^{lm}\epsilon_l^{ik}u;_{;mk}.
\]

4.2 The Kernel of the Unperturbed Linearization

The determination of the kernel of the linearized operator \(\tilde{L}\) is best done in two stages. First one finds the kernel of \(\tilde{L}\) and then one takes into account the effect of \(X\). Thus the starting point is to express the formulæ of Proposition 2 explicitly in local coordinates. To this end, suppose that \(\Sigma_r\) is given in local coordinates by its standard embedding. Make the Ansatz \(X := \sum_k X_k(-r_k e^{i\theta_k} \frac{\partial}{\partial z_k})\) for the deformation vector field in the formulæ from Proposition 2 to obtain

\[
\tilde{L}(X) = -\left(\sum_{i,k} \frac{r^2}{k^2} X^k_i d\theta^i \wedge d\theta^k, \sum_{i,k} \frac{1}{r_k^2} (X^i_k - X^i_{-k}) + \sum_i \frac{2}{r_i^2} X^i_i\right).
\]

The operator \(\tilde{L}\) thus becomes a constant-coefficient differential operator on the torus. Solving the equation \(\tilde{L}(X) = (0, 0)\) for the kernel of \(\tilde{L}\) thus becomes a matter of Fourier analysis. (Note: this calculation appears in \([9]\) for the \(n\)-dimensional torus; it is included here for the sake of completeness.)

Proposition 4. Expressed in the local coordinates for the standard embedding of \(\Sigma_r\), the kernel of \(\tilde{L}\) consists of vector fields \(X := \sum_k X^k(-r_k e^{i\theta_k} \frac{\partial}{\partial z_k})\) where

\[
X^k = \lambda_k + \frac{1}{r_k^2} \frac{\partial f}{\partial \theta^k}
\]

with \(f(\theta) := a + \sum_j (b_{j1} \cos(\theta^j) + b_{j2} \sin(\theta^j)) + c_1 \cos(\theta^1 - \theta^2) + c_2 \sin(\theta^1 - \theta^2)\) and \(a, b_{js}, c_s, \lambda_k \in \mathbb{R}\).

Proof. The first equation in \(\tilde{L}(X) = (0, 0)\) implies either: that \(X^k\) is constant for every \(k\), and thus the one-form \(r_k^2 X^k d\theta^k\) is harmonic on \(\Sigma_r\); or else that there is a function \(f : \mathbb{T}^2 \rightarrow \mathbb{R}\) with

\[
X^k = \frac{1}{r_k^2} \frac{\partial f}{\partial \theta^k}.
\]

In the first case, the second equation in \(\tilde{L}(X) = (0, 0)\) is satisfied trivially. Note that a one-form of this type is not exact, implying that \(X\) is not induced by a Hamiltonian vector field. In the second case, insert \(X^k := r_k^2 \frac{\partial f}{\partial \theta^k}\) into the second equation to find

\[
\sum_{i,k} \frac{1}{r_i^2 r_k^2} (f;_{;ikk} - f;_{;ik}) + \sum_i \frac{2}{r_i^4} f;_{;ii} = 0.
\]
This is a constant-coefficient, fourth order elliptic equation on the torus which can be solved by taking the discrete Fourier transform. The Fourier coefficients \( \hat{f}(\vec{n}) := \langle f, e^{i\vec{n} \cdot \vec{\omega}} \rangle \) of the solutions must thus satisfy
\[
\left( \sum_{i,k} n_i^2 n_k^2 + n_i n_k \right) \frac{r_i^2 r_k^2}{2 n_i^2} - \left( \sum_i \frac{2n_i^2}{r_i^2} \right) \hat{f}(\vec{n}) = 0.
\]
The trivial solution of this equation is \( n_1 = n_2 = 0 \) and this corresponds to the constant functions. There are also non-trivial solutions of this equation: either \( n_i = \pm 1 \) for some fixed \( i \) and all other \( n_k = 0 \); or else \( n_i = \pm 1 \) and \( n_j = \mp 1 \) for \( i \neq j \). The fact that there are no other non-trivial solutions can be seen as follows. Summing over \( i, k \in \{1, 2\} \) explicitly and re-arranging terms yields the equation \( n_1^2 + n_1 + r_1^2 r_2^{-2} (n_2^2 + n_2) = 0 \). But since the quadratic \( x^2 \pm x + C^2 \) only has the integer roots \( x = 0, 1 \) when \( C = 0 \) and no integer roots when \( C \neq 0 \), it must be the case that \( (n_1, n_2) = (1, 0), (0, 1), (1, -1) \) or \( (-1, 1) \). Computing the inverse Fourier transform now yields the desired vector fields in the kernel of \( \hat{L} \).

Observe that there is a geometric interpretation of the kernel of \( \hat{L} \). The one-parameter families of complex structure-preserving isometries of \( C^2 \) are the unitary rotations and the translations. Each of these is a Hamiltonian deformation where the Hamiltonians are given by linear functions in the first case and quadratic polynomials of the form \( z \mapsto z^* \cdot A \cdot z \) in the second case, where \( A \) is a Hermitian matrix. Of these, only the non-diagonal Hermitian matrices generate non-trivial motions of \( \Sigma_r \). The restrictions of these Hamiltonian functions to \( \Sigma_r \) are the functions of the form
\[
f(\theta) = \sum_j \left( b_{j_1} \cos(\theta^1) + b_{j_2} \sin(\theta^1) \right) + c_1 \cos(\theta^1 - \theta^2) + c_2 \sin(\theta^1 - \theta^2) \quad a_k, b_{js}, c_s \in \mathbb{R} \tag{8}
\]
in the kernel of \( \hat{L} \). The remaining elements of the kernel of \( \hat{L} \) derive from another set of deformations of \( \Sigma_r \) which preserve both the Lagrangian condition and the Hamiltonian-stationarity. These arise from allowing the radii of \( \Sigma_r \) to vary — in other words the deformations \( \Sigma'_r := \Sigma_{r+at} \) for some \( a = (a_1, a_2) \).

The effect of the substitution \( X = \mathcal{X}(u, v) \) is to restrict to a space of deformations that are transverse to those deformations for which \( X \cup \hat{\omega} \) is closed but non-exact. In particular, this excludes the harmonic one-forms from the kernel of the operator \( \hat{L} \).

**Corollary 5.** The kernel of \( \hat{L} \) is
\[
\mathcal{K} := \{0\} \times \text{span}_{\mathbb{R}} \{ \cos(\theta^1), \cos(\theta^2), \sin(\theta^1), \sin(\theta^2), \cos(\theta^1 - \theta^2), \sin(\theta^1 - \theta^2) \}.
\]
Note: the constant functions are not in \( \mathcal{K} \) because the conditions \( \int_{\Sigma_r} u \, d\text{Vol}_{\Sigma_r}^0 = \int_{\Sigma_r} v \, d\text{Vol}_{\Sigma_r}^0 = 0 \) have been imposed on functions in the domain of \( \hat{L} \).

### 4.3 The Adjoint of the Unperturbed Linearization

The operator \( \hat{L} \) computed in Section 4.1 is not self-adjoint. Thus it is necessary to compute its adjoint and find the kernel of its adjoint in order to determine a space onto which \( \hat{L} \) is surjective.
Proposition 6. The formal $L^2$ adjoint of $\hat{L} : C^4_0(\Sigma_\rho) \times C^4_0(\Sigma_\rho) \to C^2_0(d\Lambda^1(\Sigma_\rho)) \times C^0_0(\Sigma_\rho)$ is the operator $\hat{L}^* := ((\hat{L}^*)^{(1)}, (\hat{L}^*)^{(2)}) : C^4_0(\Lambda^1(\Sigma_\rho)) \times C^4_0(\Sigma_\rho) \to C^2_0(\Sigma_\rho) \times C^0_0(\Sigma_\rho)$ where

$$
(\hat{L}^*)^{(1)}(u, v) := \Delta u + \hat{A}^{m\ell} \varepsilon_{i}^{k} v_{m\ell},
(\hat{L}^*)^{(2)}(u, v) := \hat{A}(\Delta v) + \hat{A}^{lm} v_{;lm}.
$$

and $\hat{A}^{lm} = 2r_{m}^{-4} \delta_{m}^{l} - r_{l}^{-2}r_{m}^{-2}$ as computed earlier.

Proof. Straightforward integration by parts based on the formulæ for $\hat{L}$ and $\mathcal{X}$. 

The kernel $\mathcal{K}^*$ of the adjoint $\hat{L}^*$ is now easy to find, given the formula (9). Consider the equation $\hat{L}^*(u, v) = (0, 0)$ for $(u, v) \in C^4_0(\Sigma_\rho) \times C^4_0(\Sigma_\rho)$. The second of these equations along with the calculations of Section 4.2 implies that $v$ is of the form (8) found before. Now $u$ can be determined from the first of these equations via $\hat{u} = -\hat{A}^{lm} \varepsilon_{i}^{k} v_{m\ell}$. Since the form of $\hat{A}^{lm}$ is known, one can in fact determine $u$ explicitly. Note that we will employ a slight abuse of notation below by identifying $C^0_0(\Sigma_\rho)$ with $C_0^0(d\Lambda^1(\Sigma_\rho))$ via the Hodge star operator.

Corollary 7. The kernel of $\hat{L}^*$ is

$$
\mathcal{K}^* := \text{span}_\mathbb{R}\{(0, 1)\} \oplus \text{span}_\mathbb{R}\{\cos(\theta^1) \cdot (1, r_1 r_2), \cos(\theta^2) \cdot (1, -r_1 r_2), \sin(\theta^1) \cdot (1, r_1 r_2), \sin(\theta^2) \cdot (1, -r_1 r_2), \cos(\theta^1 - \theta^2) \cdot (0, 1), \sin(\theta^1 - \theta^2) \cdot (0, 1)\}.
$$

Note that the projections of the $\cos(\theta^1 - \theta^2)$ and $\sin(\theta^1 - \theta^2)$ co-kernel elements to the first coordinate vanish; this fact will be used crucially later on.

4.4 The Perturbed Linearization

Let $\Phi_{\rho}$ be the Hamiltonian stationary Lagrangian differential operator with respect to the Kähler structure $(g, \omega, J)$ corresponding to the Kähler potential $F_{\rho}(z, \bar{z}) = \frac{1}{\rho} ||z||^2 + \rho^2 \hat{F}_{\rho}(z, \bar{z})$ with $\rho > 0$. The task at hand is to compute its linearization at zero, denoted by $L_{\rho}$ and express it as a perturbation of $\hat{L}$ in the form $L_{\rho} = \hat{L} + P_{\rho}$. Then the dependence of $P_{\rho}$ on $\rho$ must be analyzed. Since $\Phi_{\rho} = \Phi_{\rho} \circ \mathcal{X}$ and $\mathcal{X}$ is linear, once again it is best to start with the linearization of $\Phi_{\rho}$ acting on sections of $J(T\Sigma_\rho)$, denoted by $L_{\rho}$. In the computations below, repeated indices are summed, a comma denotes ordinary differentiation and a semi-colon denotes covariant differentiation with respect to the induced metric.

Proposition 8. Let $\Sigma$ be a totally real submanifold of $\mathbb{C}^2$ equipped with the Kähler metric $g$. Let $X$ be a $C^3$ section of $J(T\Sigma)$ and write $X := X^j E_j$ where $\{E_1, E_2\}$ is a coordinate basis for the tangent space of $\Sigma$. Write $L_{\rho}(X) := (L_{\rho}^{(1)}(X), L_{\rho}^{(2)}(X))$. Then

$$
L_{\rho}^{(1)}(X) := d(X \wedge \omega)
$$
\[ L^2_p(X) := \mathcal{E}_1(X) + \mathcal{E}_2(X) \]

where

\[
\begin{align*}
\mathcal{E}_1(X) := & -\langle \Delta X^m \rangle_{\Sigma} - h^{lm} X^s \tilde{R}_{sl} - h^{lm} h^{qu} H_{ql;m} X^s B_{usl} \\
& + h^{lm} h^{jk} h^{qu} (X^s (B_{ksq} B_{jul} - B_{ksq} B_{jul} - B_{qsk} B_{jul}));m \\
& - h^{lm} h^{qu} H_u (X^s B_{qsl});m + h^{lm} h^{qu} H_u (X^s B_{lsm};q)
\end{align*}
\]

\[
\begin{align*}
\mathcal{E}_2(X) := & -h^{lm} h^{qu} (h^{jk} B_{jkl};m) C(X)_{uq} - (h^{lm} h^{ju} h^{qk} C(X)_{uq} B_{jkl});m \\
& - \frac{1}{2} h^{lm} h^{jk} h^{sq} B_{jks} (C(X)_{ql;m} + C(X)_{rm;l} - C(X)_{lm;q}) \\
& + h^{lm} h^{jk} X^s \left( g(D((\nabla E_k E_s)^\perp), (\nabla E_j E_l)^\perp) + g((\nabla E_j E_k)^\perp, D((\nabla E_i E_s)^\perp)) \right);m \\
& - \frac{1}{2} h^{lm} h^{jk} h^{sq} \omega_{sl} (\beta(X)_{qij;k} + \beta(X)_{qik;j} - \beta(X)_{jkq}) + C(X)_{qij;k} + C(X)_{qik;j} - C(X)_{jkq};m
\end{align*}
\]

and also \( C(X)_{kl} := X^s \omega_{sl} + X^s \omega_{sk} \), \( \beta(X)_{kl} := X^s (B_{ksl} + B_{lks}) \), and \( D : TM \rightarrow TM \) is the operator giving the difference between the orthogonal projection of a vector \( W \in T_p M \) onto \( N_p \Sigma \) and its orthogonal projection onto \( J(T_p \Sigma) \).

Proof. The formula for \( L^1_p \) follows as before; thus consider \( L^2_p(X) \). In general, let \( \Sigma \) be a totally real submanifold of \( M \). Let \( X \) be a section of the bundle \( J(T \Sigma) \). Let \( \mu_t : M \rightarrow M \) be a one-parameter family of diffeomorphisms with \( \frac{d}{dt} \mu_t|_{t=0} = X \) and set \( \Sigma^t := \mu_t(\Sigma) \). Note that although \( X \) is always transverse to \( \Sigma \), it is not necessarily normal to \( \Sigma \) because \( \Sigma \) is not necessarily Lagrangian.

Next, choose \( E_1, E_2 \) a local coordinate frame for \( \Sigma \) coming from geodesic normal coordinates at \( p_0 \in \Sigma \) in the induced metric \( h \) at \( t = 0 \). Then \( J E_1, J E_2 \) is basis for \( J(T_p \Sigma) \) for \( p \) near \( p_0 \), and \( T_p M = T_p \Sigma \oplus J(T_p \Sigma) \) for such \( p \). We write \( X|_\Sigma = J(X^j E_j) = X^j J E_j \). Note that \( X \) and \( E_k \) commute along \( \mu_t \), and since \( X \) is transverse to \( \Sigma \), we can extend the fields \( E_k \) locally using the diffeomorphism \( \mu_t \) to a basis for \( T_{\mu_t(p)} \Sigma^t \), for \( |t| \) small. In these coordinates the matrix for \( h \) on \( \Sigma^t \) is the same as that for \( \mu^*_t g \) on \( T \Sigma \). The computations below are evaluated at \( p_0 \) at \( t = 0 \).

In terms of these coordinates, we have

\[
\nabla \cdot H(\Sigma^t) = h^{lm} h^{jk} B_{jkl;m}
\]

where \( h_{kl} := g(E_k, E_l) \) is the induced metric, \( h^{jk} \) are the components of the inverse of the induced metric, \( \nabla \) is the induced connection, and

\[
B_{jkl} := \omega((\nabla E_j E_k)^\perp, E_l) = \omega(\nabla E_j E_k, E_l) - \Gamma^s_{jk} \omega(E_s, E_l), \tag{10}
\]

where \( \Gamma^s_{jk} \) are the Christoffel symbols of \( h_{jk} \), and \( \nabla \) is the ambient connection of \( g \).

The terms in (10) all depend on \( t \). We will now compute the first derivative of (10) at \( t = 0 \). By writing

\[
\nabla \cdot H(\Sigma^t) = h^{lm} H_{l;m} = h^{lm} H_{l,m} - h^{lm} \Gamma^s_{lm} H_s
\]
we find
\[ \frac{d}{dt}(\nabla \cdot H(\Sigma')) \bigg|_0 = (h^{lm})' H_{lm} - \bar{h}^{lm} (\Gamma_{lm}')' H_s + \bar{h}^{lm} ((H'_t)')_m \]

where once again a prime denotes the value of the time derivative at zero.

We compute the first variation of the metric \( h \). The fact that \( \Sigma \) is not assumed to be Lagrangian for \( \omega \) influences the outcome of the computation. We have

\[
(h_{kl})' = g(\bar{\nabla}_X E_k, E_l) + g(E_k, \bar{\nabla}_X E_l) \\
= g(\bar{\nabla}_{E_k} X, E_l) + g(\bar{\nabla}_X E_k, E_l) \\
= X_{k}^{s} g(J E_s, E_l) + X_{l}^{s} g(J \bar{\nabla}_X E_s, E_l) + X_{k}^{s} g(\bar{\nabla}_E_t E_s, E_k) \\
= X_{k}^{\omega} \omega_{sl} + X_{l}^{\omega} \omega_{sk} + X^{s}(B_{ksl} + B_{ltsk}).
\]

Define \( \mathcal{C}(X)_{kl} := X_{k}^{s} \omega_{sl} + X_{l}^{s} \omega_{sk} \) and \( \beta(X)_{kl} := X^{s}(B_{ksl} + B_{ltsk}). \) Note that if \( \Sigma \) were Lagrangian with respect to \( \omega \) then \( \mathcal{C}(X) \) would vanish identically and \( \beta(X) \) would equal \( 2X^{s}B_{ksl} \). It is now straightforward to compute

\[
(h_{kl})' = -h^{km} h^{lq} h_{mq} = -h^{km} h^{lq} (\beta(X)_{mq} + \mathcal{C}(X)_{mq}) \\
(\Gamma_{lm}')' = \frac{1}{2} h^{kq} \left( \beta(X)_{ql,m} + \beta(X)_{qm,l} - \beta(X)_{lm,q} + \mathcal{C}(X)_{ql,m} + \mathcal{C}(X)_{qm,l} - \mathcal{C}(X)_{lm,q} \right). 
\]

Next we have

\[
(H_t)' = \frac{d}{dt} (h^{ik} B_{jkl}) \bigg|_{t=0} \\
= -h^{im} h^{kq} (\beta(X)_{mq} + \mathcal{C}(X)_{mq}) B_{jkl} + h^{ik} (B_{jkl})'.
\]

We now use the facts that \( \omega \) and \( J \) are parallel, that \( X \) and \( E_k \) commute along \( \mu_t \), and \( \Gamma_{jk}^{s}(p_0) \) vanishes at \( t = 0 \) to deduce

\[
(B_{jkl})' = \frac{d}{dt} \omega ((\bar{\nabla}_{E_j} E_k, E_l) \bigg|_{t=0} \\
= \omega (\bar{\nabla}_X \bar{\nabla}_{E_j} E_k, E_l) + \omega (\bar{\nabla}_{E_j} E_k, \bar{\nabla}_X E_l) - (\Gamma_{jk}^{s})' \omega_{sl} \\
= \omega (\bar{\nabla}_{E_j} \bar{\nabla}_{E_k} X, E_l) + \omega (\bar{\nabla}_{E_j} E_k, \bar{\nabla}_E_l X) + \omega (\bar{\nabla}_E_t E_k, E_l) - (\Gamma_{jk}^{s})' \omega_{sl} \\
= -E_j [g(\bar{\nabla}_{E_k} (X^{s} E_s), E_l)] + g(\bar{\nabla}_{E_k} (X^{s} E_s), E_l) + g(\bar{\nabla}_{E_j} E_k, \bar{\nabla}_E_l (X^{s} E_s)) \\
- X^{s} \bar{R}_{jkl} - (\Gamma_{jk}^{s})' \omega_{sl} \\
= -E_j [g(X^{s}_{k} E_s + (\bar{\nabla}_E_t (X^{s} E_s))^\perp, E_l)] + g(X^{s}_{k} E_s + (\bar{\nabla}_E_t (X^{s} E_s))^\perp, \bar{\nabla}_E_l E_l) \\
+ g(\bar{\nabla}_E_t E_k, X^{s}_{k} E_s + (\bar{\nabla} E_l (X^{s} E_s))^\perp) - X^{s} \bar{R}_{jkl} - (\Gamma_{jk}^{s})' \omega_{sl}
\]

where \( \bar{R}_{jkl} \) are the components of the ambient curvature tensor. Now using the fact that we’ve arranged to have \( \bar{\nabla}_{E_j} E_k \) orthogonal to \( \Sigma \) at \( p_0 \) at \( t = 0 \), we can deduce

\[
(B_{jkl})' = -E_j [g(X^{s}_{k} E_s, E_l)] + g((\bar{\nabla}_E_t (X^{s} E_s))^\perp, \bar{\nabla}_E_l E_l)
\]
\[ + g(\nabla E_j E_k, (\nabla E_i (X^s E_s))^\perp) - X^s \bar{R}_{j k l} - (\Gamma^s_{j k})' \omega_{s l} \]

\[= -X_{l k j} + X^s g((\nabla E_j E_s)_{i}^\perp, (\nabla E_j E_l)^\perp) + X^s g((\nabla E_j E_k)^\perp, (\nabla E_i E_s)^\perp) \]

\[-X^s \bar{R}_{j k l} - (\Gamma^s_{j k})' \omega_{s l}. \tag{11}\]

To deal with the $(\nabla E_j E_k)^\perp$ terms we introduce the operator $\mathcal{D}$ on $T_p M$ which is the difference between the orthogonal projection onto $N_p \Sigma$ and the orthogonal projection onto $J(T_p \Sigma)$. Now, for any $W \in N_p \Sigma$, we can write

\[W = h^{ij} g(W, JE_j) JE_i + \mathcal{D}(W) = -h^{ij} \omega(W, E_j) JE_i + \mathcal{D}(W).\]

where we’ve used the fact that $J$ is an isometry. Consequently (11) becomes

\[(B_{j k l})' = -X_{l k j} + X^q h^{u q} B_{k q s} B_{j l u} + X^s h^{u q} B_{l s q} B_{j k u} \]

\[+ X^s g(\mathcal{D}((\nabla E_i E_s)^\perp), (\nabla E_j E_l)^\perp) + X^s g((\nabla E_j E_k)^\perp, \mathcal{D}((\nabla E_i E_s)^\perp)) \]

\[-X^s \bar{R}_{j k l} - (\Gamma^s_{j k})' \omega_{s l}. \]

We have now computed all the separate constituents of $L_p^{(2)}(X)$. It remains only to put everything together. We find

\[L_p^{(2)}(X) = (h^{i m} H_{l ; m} - h^{i m} (\Gamma^i_{l m})' H_s + h^{i m} ((H_l)'')_{; m}\]

\[= -h^{i u} h^{m q} (\beta(X)_{u q} + \mathcal{C}(X)_{u q}) \]

\[-\frac{1}{2} h^{i m} h^{j k} h^{s q} B_{j k s} (\beta(X)_{q l ; m} + \beta(X)_{q m ; l} - \beta(X)_{l m ; q}) \]

\[-\frac{1}{2} h^{i m} h^{j k} h^{s u} B_{j k s} (\beta(X)_{q l ; m} + \beta(X)_{q m ; l} - \beta(X)_{l m ; q}) \]

\[-(h^{i m} h^{j u} h^{k q} B_{j k l} (\beta(X)_{u q} + \mathcal{C}(X)_{u q})_{; m} \]

\[+ h^{i m} h^{j k} \left( -X_{l k j} + X^q h^{u q} B_{k q s} B_{j l u} + X^s h^{u q} B_{l s q} B_{j k u} \right)_{; m} \]

\[+ h^{i m} h^{j k} X^s \left(g(\mathcal{D}((\nabla E_i E_s)^\perp), (\nabla E_j E_l)^\perp) + g((\nabla E_j E_k)^\perp, \mathcal{D}((\nabla E_i E_s)^\perp)) \right)_{; m} \]

\[-h^{i m} h^{j k} (X^s \bar{R}_{j k l} + (\Gamma^s_{j k})' \omega_{s l})_{; m} \]

\[= \mathcal{E}_1(X) + \mathcal{E}_2(X)\]

where $\mathcal{E}_1(X)$ and $\mathcal{E}_2(X)$ are as in the statement of the proposition. In attaining these expressions, we have expanded $\beta(X)_{i j} = X^s (B_{i a j} + B_{j a i})$ and we have denoted the components of the ambient Ricci tensor by $\bar{R}_{a i}$. The point of arranging the outcome of the calculation in this way is because the term $\mathcal{E}_1(X)$ has the same form as the linearization of the Hamiltonian stationary Lagrangian differential operator at a Lagrangian submanifold while the term $\mathcal{E}_2(X)$ vanishes at a Lagrangian submanifold.

The next step in the calculation is to determine the decomposition $L_p^{(s)}(X) = \mathcal{L}^{(s)}(X) + P_p^{(s)}(X)$ for $s = 1, 2$. Of course, $L_p^{(1)}(X) = d(X \mathcal{J} \omega)$ according to the usual Poincaré formula and so

\[P_p^{(1)}(X) = d(X \mathcal{J} \omega) - d(X \mathcal{J} \omega) = d(X \mathcal{J} (\omega + \hat{\omega})).\]
For $P^{(2)}_\rho$, observe that $\mathcal{E}_1(X)$ has the same form as $\hat{L}^{(2)}(X)$ and $\mathcal{E}_2(X)$ vanishes when $\rho = 0$. Thus formally we can decompose

$$P^{(2)}_\rho(X) = (\mathcal{E}_1(X) - \hat{L}^{(2)}(X)) + \mathcal{E}_2(X).$$

We will not determine the precise form of the operator $\mathcal{E}_1(X) - \hat{L}^{(2)}(X)$ since these details will not be needed in the sequel.

**Corollary 9.** The components of the operator $P_\rho$ are

$$P^{(1)}_\rho(X) := d(X \mathcal{J}(\omega - \hat{\omega}))$$

$$P^{(2)}_\rho(X) := (\mathcal{E}_1(X) - \hat{L}^{(2)}(X)) + \mathcal{E}_2(X)$$

with notation as in Proposition 8.

We now obtain a corresponding decomposition $L^{(s)}_\rho := \hat{L}^{(s)} + P^{(s)}_\rho$ where $P^{(s)}_\rho := P^{(s)}_\rho \circ X$.

### 4.5 Estimates for the Perturbed Linearization

The norms that will be used to estimate the various quantities involved in the proof of the Main Theorem will be the standard $C^{k,\alpha}$ norms; these will be taken with respect to the background metric $\hat{g}$ when the quantity being estimated is defined in $\mathbb{C}^2$ and with respect to the induced metric $\hat{h}$ when the quantity being estimated is defined on the submanifold $\Sigma_r$. Note that these norms are equivalent to those defined by the metrics $g$ and $h$ and coincide with the norms used in the statement of the Main Theorem when the re-scaling of Section 3.1 is reversed. Begin with the following lemma.

**Lemma 10.** Let $\Sigma$ be a totally real submanifold of $\mathbb{C}^2$ equipped with the Kähler metric $g$. Fix $\alpha \in (0,1)$ and $k \in \mathbb{N}$. There is a constant $C$ independent of $\rho$ so that for all $X \in \Gamma(J(T\Sigma))$ and $W \in \Gamma(N\Sigma)$ the following estimates hold:

$$\|g - \hat{g}\|_{C^{k,\alpha}(M)} \leq C\rho^2$$

$$\|\nabla \cdot X - \hat{\nabla} \cdot X\|_{C^{k,\alpha}(\Sigma_r)} \leq C\rho^2\|X\|_{C^{k,\alpha}(\Sigma_r)}$$

$$\|\omega - \hat{\omega}\|_{C^{k,\alpha}(M)} \leq C\rho^2$$

$$\|C(X)\|_{C^{k,\alpha}(\Sigma_r)} \leq C\rho^2\|X\|_{C^{k+1,\alpha}(\Sigma_r)}$$

$$\|B - \hat{B}\|_{C^{k,\alpha}(\Sigma_r)} \leq C\rho^2$$

$$\|D(W)\|_{C^{k,\alpha}(\Sigma_r)} \leq C\rho^2\|W\|_{C^{k,\alpha}(\Sigma_r)}$$

$$\|H - \hat{H}\|_{C^{k,\alpha}(\Sigma_r)} \leq C\rho^2$$

$$\|\mathcal{E}_2(X)\|_{C^{k,\alpha}(\Sigma_r)} \leq C\rho^2\|X\|_{C^{k+2,\alpha}(\Sigma_r)}.$$ 

Furthermore, the operator $\mathcal{D}$ vanishes if $\Sigma$ is Lagrangian.

**Proof.** The estimates mostly follow from the estimate of the Kähler potential $F_\rho(z, \bar{z}) := \frac{1}{2}\|z\|^2 + \rho^2 \hat{F}_\rho(z, \bar{z})$, where $\hat{F}_\rho(z, \bar{z}) := \rho^{-4} \hat{F}(\rho z, \rho \bar{z})$. Recall that for any multi-index $\alpha$ the derivative $\partial^\alpha \hat{F}(\zeta, \bar{\zeta})$ is $O(\|\zeta\|^{4-\alpha})$ for $|\alpha| \leq 4$, and $O(1)$ for $|\alpha| > 4$. This immediately gives the first two estimates. The estimate on the symplectic second fundamental form comes from the following (and then immediately implies the estimate on the mean curvature one-form):

$$B(X, Y, Z) - \hat{B}(X, Y, Z) = \omega((\nabla_X Y)^\perp, Z) - \hat{\omega}((\hat{\nabla}_X Y)^\perp, Z),$$

where $\omega$ is the standard symplectic form on $\mathbb{C}^2$.
Thus

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Further, since $\Sigma$ is Lagrangian for $\omega$, then $\omega$ is Lagrangian for $\omega$, which together with the equation above then yields the estimate of $B - \tilde{B}$, as well as the estimate on the divergence.

We now estimate $D$, which, together with the above estimates, will also yield the estimate of $E$, and thus complete the proof. Let $W \in N_{\rho} \Sigma$ be a unit vector. Recall from above that

$$\mathcal{D}(W) = W - h^{ij}g(W, JE_j)JE_i = W + h^{ij}\omega(W, E_j)JE_i.$$

If we use the orthogonal decomposition of $W$ with respect to the metric $\tilde{g}$, denoting it as $W = W^\perp + W^\parallel$, then since $g(W, E_j) = 0$, we have immediately $\tilde{g}(W, E_j) = \mathcal{O}(\rho^2)$. Thus $W^\parallel = \mathcal{O}(\rho^2)$.

Furthermore, since $\Sigma$ is Lagrangian for $\omega$, then $\tilde{W}^\perp = -\tilde{h}^{ij}\hat{\omega}(W^\perp, E_j)JE_i = -\tilde{h}^{ij}\hat{\omega}(W, E_j)JE_i.$

Thus $\mathcal{D}(W) - \tilde{W}^\parallel = \tilde{W}^\perp + h^{ij}\omega(W, E_j)JE_i = \mathcal{O}(\rho^2)$. \hfill \Box

Based on these elementary estimates, we have the following estimates of $P_{\rho}$ and $P_{\rho}$ on a totally real submanifold $\Sigma$.

**Proposition 11.** Let $\Sigma$ be a totally real submanifold of $\mathbb{C}^2$ equipped with the Kähler metric $g$. Fix $k \in \mathbb{N}$ and $\alpha \in (0,1)$. There is a constant $C$ independent of $\rho$ so that

$$\|P^{(1)}_{\rho}(X)\|_{C^{k,\alpha}} \leq C\rho^2\|X\|_{C^{k+1,\alpha}},$$

$$\|P^{(2)}_{\rho}(X)\|_{C^{k,\alpha}} \leq C\rho^2\|X\|_{C^{k+2,\alpha}},$$

$$\|P^{(1)}_{\rho}(u, v)\|_{C^{k,\alpha}} \leq C\rho^2\|(u, v)\|_{C^{k+1,\alpha} \times C^{k+1,\alpha}},$$

$$\|P^{(2)}_{\rho}(u, v)\|_{C^{k,\alpha}} \leq C\rho^2\|(u, v)\|_{C^{k+2,\alpha} \times C^{k+2,\alpha}}.$$

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5.1 Outline

In this final section of the paper, the equation $\Phi_{\rho}(u, v) = (0,0)$ will be solved for all $\rho$ sufficiently small using a perturbative technique. An initial difficulty that must be overcome is that it is not possible to find a suitable inverse for the linearized operator $L_{\rho} := D_{(0,0)}\Phi_{\rho}$ with $\rho$-independent norm because the operator $\tilde{L} := D_{(0,0)}\Phi$ has a non-trivial, six-dimensional kernel and fails to be surjective since its adjoint has a seven-dimensional kernel. This fact makes a three-step approach for solving $\Phi_{\rho}(u, v) = (0,0)$ necessary.

**Step 1.** The first step is to solve a projected problem wherein the difficulties engendered by the kernel and co-kernel of $\tilde{L}$ are avoided. Let $K$ be the kernel of $\tilde{L}$ and let $K^*$ be the kernel of $\tilde{L}^*$. Let $\pi : C^{2,\alpha}(d\Lambda^1(\Sigma_r)) \times C^{0,\alpha}(\Sigma_r) \to \left(C^{2,\alpha}(d\Lambda^1(\Sigma_r)) \times C^{0,\alpha}(\Sigma_r)\right) \cap [K^*]^\perp$ be the $L^2$-orthogonal projection onto $[K^*]^\perp$ with respect to the volume measure induced from the Euclidean ambient metric and consider the operator

$$\pi \circ \Phi_{\rho}|_{\mathcal{K}^\perp} : \left(C^{2,\alpha}_0(\Sigma_r) \times C^{0,\alpha}_0(\Sigma_r)\right) \cap \mathcal{K}^\perp \to \left(C^{2,\alpha}(d\Lambda^1(\Sigma_r)) \times C^{0,\alpha}(\Sigma_r)\right) \cap [K^*]^\perp.$$
The first step is thus to solve \( \pi \circ \Phi_\rho |_{K^\perp} (u, v) = (0, 0) \). The linearization of this new operator is \( \pi \circ L_\rho |_{K^\perp} \) which is by definition invertible at \( \rho = 0 \). This operator remains invertible for sufficiently small \( \rho > 0 \), and it will be shown below that a solution of the non-linear problem

\[
\pi \circ \Phi_\rho |_{K^\perp} (u, v) = (0, 0)
\]

can be found. We will denote the solution by \((u_\rho, v_\rho)\) and let \( \tilde{\Sigma}_r(U_\rho) := \mu_\rho U_\rho(\Sigma_r(U_\rho)) \) be the perturbed submanifold generated by this solution; we will abbreviate this by \( \tilde{\Sigma}_r \) when there is no cause for confusion.

**Step 2.** The previous step shows that a solution \((u, v) := (u_\rho, v_\rho)\) of the projected problem on \( \Sigma_r \) can always be found so long as \( \rho \) is sufficiently small. One should realize that the solution \((u_\rho, v_\rho)\) that has been found depends implicitly on the point \( p \in M \) and the choice of unitary frame \( U_\rho \) at \( p \) out of which \( \Sigma_r \) has been constructed. Moreover, this dependence is smooth as a standard consequence of the fixed-point argument used to find \((u_\rho, v_\rho)\). The solution is such that \( \Phi_\rho(u_\rho, v_\rho) \) is an *a priori* non-trivial but small quantity that belongs to \( K^* \).

In the second step of the proof of the Main Theorem, it will be shown that when an existence condition is satisfied at the point \( p \in M \), there exists \( U_\rho \) so that \( \Phi_\rho(u_\rho, v_\rho) \) vanishes except for a component in the space \( \text{span}_\mathbb{R}\{(0, 1)\} \). We set this up as follows. First, write \( K^* = \text{span}_\mathbb{R}\{(0, 1)\} \oplus K_0^* \) where \( K_0^* := \text{span}_\mathbb{R}\{f^{(1)}v^{(1)}, \ldots, f^{(6)}v^{(6)}\} \) and the \( v^{(i)} \) are constant vectors determined in Corollary 7 normalized so that the second component \( v^{(2)}_2 = 1 \). Therefore

\[
\Phi(u_\rho, v_\rho) = a(0, 1) + \sum_{j=1}^{6} b_j f^{(j)}v^{(j)} \quad \text{for some } a, b_1, \ldots, b_6 \in \mathbb{R}
\]

Now define a smooth mapping on the unitary 2-frame bundle \( U_2(M) \) over \( M \), given by

\[
G_\rho : U_2(M) \rightarrow \mathbb{R}^6
\]

\[
G_\rho(U_\rho) := (I_\rho^{(1)}, \ldots, I_\rho^{(6)})
\]

where

\[
I_\rho^{(i)}(U_\rho) := \int_{\Sigma_r} (f^{(i)} - c^{(i)})v^{(i)} \cdot \Phi(u_\rho, v_\rho)d\text{Vol}_{\Sigma_r}
\]

and \( c^{(i)} \) has been chosen to ensure that \( \int_{\Sigma_r} (f^{(i)} - c^{(i)})d\text{Vol}_{\Sigma_r} = 0 \). We now have

\[
I_\rho^{(i)}(U_\rho) = \sum_{i=1}^{6} b_i \int_{\Sigma_r} f^{(j)}f^{(i)}d\text{Vol}_{\Sigma_r}
\]

and would now like to find \( U_\rho \) so that \( G_\rho(U_\rho) \equiv 0 \). This will turn imply that \( b_i = 0 \) for all \( i \) because the matrix whose coefficients are the integrals \( \int_{\Sigma_r} f^{(j)}f^{(i)}d\text{Vol}_{\Sigma_r} \) is an invertible matrix.

The idea for locating a zero of \( G_\rho \) is first to find \( U_\rho \) so that \( G_\rho(U_\rho) \) vanishes to lowest order in a Taylor expansion in powers of \( \rho \), but in such a way that \( G_\rho \) remains locally surjective at this \( U_\rho \). The implicit function theorem for finite-dimensional manifolds can then be invoked to find a nearby \( U_{\rho'} \) for which \( G_\rho(U_{\rho'}) \equiv 0 \) exactly.
Step 3. The previous step shows that the only non-vanishing component of $\nabla \cdot H(\tilde{\Sigma}_r)$ is perhaps the projection of $\nabla \cdot H(\tilde{\Sigma}_r)$ to span\(_{\mathbb{R}}\{(0,1)\}$). But the divergence theorem can now be invoked to show that this component must vanish as well, thereby completing the proof of the Main Theorem.

5.2 Estimates for the Approximate Solution

To begin, we must compute the size of $\|\Phi_\rho(0,0)\|_{C^2_0,\alpha} \times C^0_0,\alpha}$ which must be sufficiently small for the perturbation method of Step 1 to succeed.

**Proposition 12.** There is a constant $C > 0$ independent of $\rho$ so that

$$\|\Phi_\rho(0,0)\|_{C^2_0,\alpha} \times C^0_0,\alpha \leq C \rho^2.$$  

**Proof.** By Lemma 1, we have $\Phi(0,0) = (0,0)$. By Lemma 10, we have $\|\omega - \hat{\omega}\|_{C^2_0,\alpha} \leq C \rho^2$. Furthermore, by writing

$$\nabla \cdot H = \nabla \cdot \Hat{H} + (\nabla - \Hat{\nabla}) \cdot \Hat{H} + \nabla \cdot (H - \Hat{H}),$$

we have

$$\|\nabla \cdot H\|_{C^0_0,\alpha} \leq C \rho^2 \|\Hat{H}\|_{C^0_0,\alpha} + \|H - \Hat{H}\|_{C^1_0,\alpha} \leq C \rho^2,$$

again using the estimates of Lemma 10. \qed

5.3 Solving the Projected Problem

This section proves that Step 1 from the outline above can be carried out.

**Theorem 13.** For every $\rho$ sufficiently small, there is a solution $(u_\rho, v_\rho) \in \left( C^4_0,\alpha(\Sigma_r) \times C^4_0,\alpha(\Sigma_r) \right) \cap \mathcal{K}^\perp$ that satisfies

$$\pi \circ \Phi_\rho(u_\rho, v_\rho) = (0,0).$$

Moreover, the estimate $\|(u_\rho, v_\rho)\|_{C^4_0,\alpha} \times C^4_0,\alpha} \leq C \rho^2$ holds.

**Proof.** The solvability of the equation $\pi \circ \Phi_\rho(u,v) = (0,0)$ is governed by the behaviour of the linearized operator $\pi \circ L_\rho$ between the Banach spaces given in the statement of the theorem, as well as on the size of $\|\Phi_\rho(0,0)\|_{C^2_0,\alpha} \times C^0_0,\alpha}$, which we know to be $O(\rho^2)$ by Proposition 12.

First, by standard elliptic theory, the operator $\pi \circ L$ is invertible between $\mathcal{K}^\perp$ and $[\mathcal{K}^\ast]^\perp$ with the estimate

$$\|\pi \circ L(u,v)\|_{C^2_0,\alpha} \times C^0_0,\alpha} \geq C \|(u,v)\|_{C^4_0,\alpha} \times C^4_0,\alpha}$$

where $C$ is a constant independent of $\rho$. Consequently, if $\rho$ is sufficiently small, then the operator $\pi \circ L_\rho$ is uniformly injective with the estimate

$$\|\pi \circ L_\rho(u,v)\|_{C^2_0,\alpha} \times C^0_0,\alpha} \geq \frac{C}{2} \|(u,v)\|_{C^4_0,\alpha} \times C^4_0,\alpha}.$$
Hence by perturbation, the operator \( \pi \circ L_\rho \) is also surjective onto \( [\mathcal{K}^*]^{-1} \) and the inverse is bounded above independently of \( \rho \).

The remainder of the proof uses the contraction mapping theorem. First, write

\[
\pi \circ \Phi_\rho(u, v) := \pi \circ \Phi_\rho(0, 0) + \pi \circ L_\rho(u, v) + \pi \circ Q_\rho(u, v)
\]

where \( Q_\rho \) is the quadratic remainder (in \( u \) and \( v \)) \( \Phi_\rho \). It is fairly straightforward to show that \( Q_\rho \) satisfies the estimate

\[
\|Q_\rho(u_1, v_1) - Q_\rho(u_2, v_2)\|_{C^2,0,0,0,0} \leq C\|u_1 + u_2, v_1 + v_2\|_{C^4,0,0,0,0} \| (u_1 - u_2, v_1 - v_2)\|_{C^4,0,0,0,0}
\]

for some constant \( C \) independent of \( \rho \), provided \( \rho \) is sufficiently small. This is because such an estimate is certainly true for the quadratic remainder of \( \Phi \). Now let \( L_\rho^{-1} : [\mathcal{K}^*]^{-1} \rightarrow \mathcal{K}^\perp \) denote the inverse of \( L_\rho \) onto \( \mathcal{K}^\perp \). By proposing the Ansatz \( (u, v) := L_\rho^{-1} \left(- (w, \xi) - \pi \circ \Phi_\rho(0, 0)\right) \), for \( (w, \xi) \in [\mathcal{K}^*]^{-1} \), the equation \( \pi \circ \Phi_\rho(u, v) = (0, 0) \) becomes equivalent to the fixed-point problem for the map

\[
\mathcal{N}_\rho : (w, \xi) \mapsto \pi \circ Q_\rho(L_\rho^{-1} \left(- (w, \xi) - \pi \circ \Phi_\rho(0, 0)\right))
\]

on \( [\mathcal{K}^*]^{-1} \). For small enough \( \rho \), the non-linear mapping \( (w, \xi) \mapsto \pi \circ Q_\rho(L_\rho^{-1} \left(- (w, \xi) - \pi \circ \Phi_\rho(0, 0)\right)) \) verifies the estimates required to find a fixed point in a closed ball \( B \subset \mathcal{K}^\perp \) of radius equal to \( \|\Phi_\rho(0, 0)\|_{C^2,0,0,0,0} = O(\rho^2) \) by virtue of the \( \rho \)-independent estimates that have been found for \( L_\rho^{-1} \) and \( Q_\rho \). For example, for \( (w, \xi) \in B \),

\[
\|\mathcal{N}_\rho(w, \xi)\|_{C^2,0,0,0,0} \leq C\|\Phi_\rho(0, 0)\|_{C^2,0,0,0,0}^2 \leq \|\Phi_\rho(0, 0)\|_{C^2,0,0,0,0}^2 \leq C\rho^2
\]

for \( \rho \) small enough; hence the set \( B \) is mapped to itself under \( \mathcal{N}_\rho \). Furthermore, \( \mathcal{N}_\rho \) is a contraction on \( B \) as a result of the bilinear estimate on \( Q_\rho \) given above. Consequently, \( \mathcal{N}_\rho \) must have a fixed point \( (w, \xi) \in B \) which thus satisfies \( \|(w, \xi)\|_{C^2,0,0,0,0} \leq C\rho^2 \) for some constant \( C \) independent of \( \rho \). The desired estimate follows.

**Remark:** The solution \((u_\rho, v_\rho)\) is in fact smooth by elliptic regularity theory and the estimate \( \|(u_\rho, v_\rho)\|_{C^k,0,0,0,0} \leq C\rho^2 \) holds for all \( k \in \mathbb{N} \), where \( C \) is independent of \( \rho \).

### 5.4 Derivation of the Existence Condition

The remainder of the proof begins with a more careful investigation of the integrals \( \langle 12 \rangle \) for all choices of \( f \) spanning \( \mathcal{K}_0^* \). Recall that such \( f \) come from translation and \( U(2) \)-rotation in the local coordinates at the point \( \rho \); one can thus construct a basis for \( \mathcal{K}_0^* \) as follows. Let \( (U, \tau) \cdot \) denote the motion of \( \mathbb{C}^2 \) given by \( z \mapsto U(z) + \tau \) where \( U \in U(2) \) and \( \tau \in \mathbb{C}^2 \). Then we consider the six-dimensional parameter family of motions of \( M \) given by

\[
\mathcal{R} := \{ (\exp(i\tau_5 K_1 + i\tau_6 K_2), \tau) \cdot : \tau_5, \tau_6 \in \mathbb{R} \text{ and } \tau := (\tau_1, \ldots, \tau_4) \in \mathbb{R}^4 \}\]
where

\[ K_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad K_2 := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \]

are elements in the Lie algebra of \( U(2) \) that generate all non-trivial \( U(2) \)-rotations of \( \Sigma_r \). Note that the orbit of \( \mathcal{U}_p \) under a small neighbourhood of the identity in \( \mathcal{R} \) projects diffeomorphically onto a neighbourhood of \([\mathcal{U}_p] \in U_2(M)/\text{Diag}\). Denote by \( \mu^{(i)}_t \) for \( i = 1, \ldots 6 \) those motions which correspond to \( \tau_i = t \) and \( \tau_{i'} = 0 \) for \( i' \neq i \). Note that each \( \mu^{(i)} \) is Hamiltonian with respect to the Euclidean Kähler structure, with \( J \nabla f^{(i)} := \frac{d}{dt} \mu^{(i)}(t) \vert_{t=0} \). Moreover the restriction of \( f^{(i)} \) to \( \Sigma_r \) belongs to \( \mathcal{K}^0_\Sigma \). Indeed, the translations \( \mu^{(1)}, \ldots, \mu^{(4)}_t \) yield the functions \( \cos(\theta^s) \) and \( \sin(\theta^s) \) for \( s = 1, 2 \) while the \( U(2) \)-rotations \( \mu^{(5)}_t \) and \( \mu^{(6)}_t \) yield the functions \( \sin(\theta^1 - \theta^2) \) and \( \cos(\theta^1 - \theta^2) \).

We can relate the integrals \( I_r^{(i)}(\mathcal{U}_p) \) to the ambient geometry of \( M \) to lowest order in \( \rho \) using the first variation formula along with Stokes’ theorem. Let \( v^{(i)} := (v_1^{(i)}, 1) \) and note that \( v_1^{(5)} = v_1^{(6)} = 0 \).

**Lemma 14.** The following formula holds.

\[
I_r^{(i)}(\mathcal{U}_p) = \left. \frac{d}{dt} \text{Vol}(\mu^{(j)}_t(\Sigma_r)) \right|_{t=0} + v_1^{(j)} \int_{\Sigma_r} f^{(j)} \cdot (\omega - \bar{\omega}) + \mathcal{O}(\rho^4). \tag{13}
\]

**Proof.** After a careful computation, we find

\[
\int_{\Sigma_r} (f^{(j)} - c^{(j)}) \Phi(u_\rho, v_\rho) \cdot v^{(j)} d\text{Vol}_{\Sigma_r}
\]

\[
= \int_{\Sigma_r} \nabla \cdot H(\Sigma_r)(f^{(j)} - c^{(j)}) d\text{Vol}_{\Sigma_r} + v_1^{(j)} \int_{\Sigma_r} (f^{(j)} - c^{(j)})(\omega - \bar{\omega})
\]

\[
+ \int_{\Sigma_r} (f^{(j)} - c^{(j)}) L(u_\rho, v_\rho) \cdot v^{(j)} d\text{Vol}_{\Sigma_r}
\]

\[
+ \int_{\Sigma_r} (f^{(j)} - c^{(j)}) L(\rho, v_\rho) \cdot v^{(j)} (d\text{Vol}_{\Sigma_r} - d\text{Vol}_{\Sigma_r}^0)
\]

\[
+ \int_{\Sigma_r} (f^{(j)} - c^{(j)}) P_\rho(u_\rho, v_\rho) \cdot v^{(j)} d\text{Vol}_{\Sigma_r} + \int_{\Sigma_r} (f^{(j)} - c^{(j)}) Q_\rho(u_\rho, v_\rho) \cdot v^{(j)} d\text{Vol}_{\Sigma_r}
\]

\[
= \left. \frac{d}{dt} \text{Vol}(\mu^{(j)}_t(\Sigma_r)) \right|_{t=0} + v_1^{(j)} \int_{\Sigma_r} (f^{(j)} - c^{(j)})(\omega - \bar{\omega})
\]

\[
+ \int_{\Sigma_r} (f^{(j)} - c^{(j)}) L(u_\rho, v_\rho) \cdot v^{(j)} d\text{Vol}_{\Sigma_r} + \mathcal{O}(\rho^4).
\]

Here we have used the expansion \( \Phi_\rho(u_\rho, v_\rho) = \Phi_\rho(0, 0) + L_\rho(u_\rho, v_\rho) + \mathcal{Q}_\rho(u_\rho, v_\rho) \), where \( L_\rho = \tilde{L} + P_\rho \) and \( Q_\rho \) is the quadratic remainder of the operator \( \Phi_\rho \), along with the following facts:

- \( \|(u_\rho, v_\rho)\| \) and \( \|\tilde{L}^{(2)}(u_\rho, v_\rho)\|_{C^0} \) and \( \|\nabla \cdot H(\Sigma_r)\|_{C^0} \) are all \( \mathcal{O}(\rho^2) \)
- \( \|P_\rho(u_\rho, v_\rho)\|_{C^0} \leq C \rho^2 \|(u_\rho, v_\rho)\|_{C^{2,\alpha} \times C^{2,\alpha}} = \mathcal{O}(\rho^4) \)
- \( \|Q_\rho(u_\rho, v_\rho)\|_{C^0} \leq C \|(u_\rho, v_\rho)\|_{C^{4,\alpha} \times C^{4,\alpha}}^2 = \mathcal{O}(\rho^4) \)
- the difference between any of the volume forms appearing above is \( \mathcal{O}(\rho^2) \)
\[ \int_{\Sigma_r} f^{(i)} d\text{Vol}_{\Sigma_r}^2 = 0 \] which implies \( |c^{(i)}| = \mathcal{O}(\rho^2) \)

along with Stokes' Theorem. To complete the proof of the lemma, we note that the second term vanishes since \( (f^{(i)} - c^{(j)}) \nu^{(j)} \) belongs to the kernel of \( \hat{L}^* \).

Now, let \( \{\Xi^{(1)}, \ldots, \Xi^{(6)}\} \) be the vectors in \( T_{U_p}(U_2(M)/\text{Diag}) \) corresponding to motions \( \{\mu_t^{(1)}, \ldots, \mu_t^{(6)}\} \) above.

**Proposition 15.** Define the smooth mapping

\[ \mathcal{F}_r : U_2(M)/\text{Diag} \rightarrow \mathbb{R} \]

\[ \mathcal{F}_r([U_p]) := r_1^2 R_{11}^C(p) + r_2^2 R_{22}^C(p) \]

where the components of the complex Ricci curvature \( R_{11}^C \) and \( R_{22}^C \) are computed with respect to the chosen frame. Then the mapping \( G_\rho : U_2(M) \rightarrow \mathbb{R}^6 \) defined by \( G_\rho(U_p) := (f^{(1)}(U_p), \ldots, f^{(6)}(U_p)) \) satisfies

\[ G_\rho(U_p) = 4\pi^2 r_1 r_2 \rho^2 D \mathcal{F}_r([U_p]) \cdot (\Xi^{(1)}, \ldots, \Xi^{(6)}) + \mathcal{O}(\rho^3). \]

**Proof.** We expand the terms appearing in \( [13] \). We begin with the derivative of the volume since it is the more involved quantity. We have

\[ \text{Vol}(\mu_t^{(i)}(\Sigma_r)) = \text{Vol}((U_t, \tau_t) \cdot \Sigma_r) = \int_{(U_t, \tau_t) \cdot \Sigma_r} (\det(h_{F_\rho, t}))^{1/2} d\theta^1 \wedge d\theta^2 + \mathcal{O}(\rho^4) \]

where \( (U_t, \tau_t) \cdot \Sigma_r \) is the motion corresponding to \( \mu_t^{(i)} \) while \( h_{F_\rho, t} \) is the induced metric of \( (U_t, \tau_t) \cdot \Sigma_r \) with respect to the Kähler metric whose Kähler potential is \( F_\rho \). But

\[ \int_{(U_t, \tau_t) \cdot \Sigma_r} (\det(h_{F_\rho, t}))^{1/2} d\theta^1 \wedge d\theta^2 = \int_{\Sigma_r} (\det(h_{F_\rho}))^{1/2} d\theta^1 \wedge d\theta^2 \]

where \( h_{F_\rho} \) is the induced metric of \( \Sigma_r \) with respect to the Kähler metric whose Kähler potential is \( F_\rho := F_\rho \circ (U_t, \tau_t) \), as can be checked fairly easily. Therefore to complete the calculation, one must find the first few terms of the Taylor series of \( (\det(h_{F_\rho}))^{1/2} \) in \( \rho \) and allow the integration over the torus to pick out certain terms.

To this end, note that if \( f : \mathbb{C}^2 \rightarrow \mathbb{R} \) is a real-valued function then elementary Fourier analysis shows that its restriction to the torus satisfies

\[ \int_{\mathbb{T}^2} f(e^{i\theta_1}, e^{i\theta_2}) d\theta^1 \wedge d\theta^2 = 4\pi^2 \left( f(0) + r_1^2 f_{11}(0) + r_2^2 f_{22}(0) \right) + Q^{(4)}(r_1, r_2) \]

where \( Q^{(4)} \) consists only of terms coming from fourth and higher-order Fourier coefficients of \( f \big|_{\mathbb{T}^2} \).

This formula can be seen by writing \( f(z, \bar{z}) := f(0) + \frac{\partial f}{\partial z}(0) z^k + \frac{\partial f}{\partial \bar{z}}(0) \bar{z}^k + \cdots \) and substituting \( z^k = r^k e^{i\theta^k} \); the integration over the torus then causes all odd-order combinations of \( z^k \) and \( \bar{z}^k \) to vanish while giving exactly the terms in \( [13] \) at order two. To apply this to the calculation at hand, first compute

\[ F_\rho^t(z, \bar{z}) := \frac{1}{2} \|U_t(z) + \tau_t\|^2 + \rho^2 (F_\rho)(z, \bar{z}) + \rho^3 \mathcal{O}(\|z\|^{5}) \]
where \((\hat{F}_\rho)^{(4)}(z, \bar{z})\) is the \(O(\|z\|^4)\) term in the Taylor series expansion of \(\hat{F}_\rho \circ (U_t, \tau_t)\). Now let \(Q^{(3)}(z_1, z_2)\) denote a cubic polynomial in its arguments and observe

\[ h_{F_\rho} = \text{Re} \sum_{a,b} \left( r_a^2 \delta_{ab} + \rho^2 (\hat{F}_\rho)^{(4)}_{ab} r_a r_b e^{i(\theta^a - \theta^b)} + \rho^3 Q^{(3)}(r_1 e^{i\theta^1}, r_2 e^{i\theta^2}) + O(\rho^4) \right) d\theta^a \otimes d\theta^b. \]

The \(O(\rho^4)\) term is quartic and higher in \(r_k e^{i\theta_k}\). Integrating and taking advantage of the fact that the cubic terms in the expansion of \((\det(h_{F_\rho}))^{1/2}\) must vanish we can express

\[
\int_{\Sigma_r} (\det(h_{F_\rho}))^{1/2} d\theta^1 \wedge d\theta^2 = \int_{\Sigma_r} r_1 r_2 \left( 1 + \frac{\rho^2}{2} \sum_c (\hat{F}_\rho)^{(4)}_{,cc} \right) d\theta^1 \wedge d\theta^2 + O(\rho^4). \tag{16}
\]

Next, we write the first few terms of the Fourier expansion of the integrand (via the Taylor expansion) and integrate these to re-write the \(O(\rho^2)\) part of (16) as

\[
 r_1 r_2 \sum_{c,u,v} r_u r_v \text{Re} \left( (\hat{F}_\rho)_{,ccuv}(0) e^{i(\theta^u + \theta^v)} + (\hat{F}_\rho)_{,ccuv}(0) e^{i(\theta^u - \theta^v)} \right) d\theta^1 \wedge d\theta^2.
\]

Performing this integral yields

\[
\text{Vol}((U_t, \tau_t) \cdot \Sigma_r) = 4\pi^2 r_1 r_2 \left( 1 + \rho^2 \sum_{c,u} (\hat{F}_\rho)^{(4)}_{,ccuv}(0) \right) + O(\rho^4)
\]

\[
= 4\pi^2 r_1 r_2 \left( 1 + \rho^2 \left( r_1^2 (\hat{F}_\rho)^{(4)}_{,1111}(0) + (r_1^2 + r_2^2) (\hat{F}_\rho)^{(4)}_{,1212}(0) + r_2^2 (\hat{F}_\rho)^{(4)}_{,2222}(0) \right) \right) + O(\rho^4)
\]

after explicitly expanding the sums over \(c\) and \(u\). Therefore the lowest-order term in the expansion of \(\frac{d}{dt} \text{Vol}((U_t, \tau_t) \cdot \Sigma_r)|_{t=0}\) in \(\rho\) is

\[
\frac{d}{dt} \bigg|_{t=0} 4\pi^2 r_1 r_2 \left( r_1^2 (\hat{F}_\rho)^{(4)}_{,1111}(0) + (r_1^2 + r_2^2) (\hat{F}_\rho)^{(4)}_{,1212}(0) + r_2^2 (\hat{F}_\rho)^{(4)}_{,2222}(0) \right).
\tag{17}
\]

Using (11), the expression (17) can be re-phrased in terms of the complex Ricci curvature of \(M\) as

\[
\frac{d}{dt} \bigg|_{t=0} 4\pi^2 r_1 r_2 \left( r_1^2 R_{11}^\rho((U_t, \tau_t) \cdot p) + r_2^2 R_{22}^\rho((U_t, \tau_t) \cdot p) \right).
\]

We now turn to the difference of symplectic forms term. The expression \(\omega - \hat{\omega}\) has leading order \(\rho^2\) and the leading order part is an antisymmetric 2-tensor whose coefficients are homogeneous quadratic polynomials in \(z\) and \(\bar{z}\). Pulling this back to \(\Sigma_r\) yields an expression whose leading order part is a homogeneous fourth degree polynomial in \(\cos(\theta^s)\) and \(\sin(\theta^s)\) for \(s = 1, 2\). Multiplying this by \(f^{(i)}\) for \(i = 1, 2, 3\) or 4 produces a fifth degree polynomial in these quantities. This always integrates to zero over the torus. Note that it is not necessary to consider the integrals against \(f^{(5)}\) or \(f^{(6)}\) since \(v_{1}^{(5)} = v_{1}^{(6)} = 0\). Hence the magnitude of \(\nu^{(i)} \int_{\Sigma_r} f^{(i)} \cdot (\omega - \hat{\omega})\) is determined by the next-to-leading terms in the expansion of \(\omega - \hat{\omega}\). These are all \(O(\rho^3)\). Expression (14) follows. \(\Box\)
5.5 The Proof of the Main Theorem

In this section, we conclude the proof of the Main Theorem by showing that if the mapping $\mathcal{F}_r$ has a non-degenerate critical point $[U_p]$ in $U_2(M)/\text{Diag}$, then $\tilde{\Sigma}_r(U_p)$ can be further perturbed into an exactly Hamiltonian stationary Lagrangian submanifold. This will then complete the proof of the Main Theorem.

**Theorem 16.** Suppose $[U_p]$ is a non-degenerate critical point of the functional $\mathcal{F}_r$ defined in the previous section. If $\rho$ is sufficiently small, then there is $U_p'$ near $U_p$ so that the submanifold $\tilde{\Sigma}_r(U_p')$ that was obtained via Theorem 13 from the torus $\Sigma_r(U_p)$ is a Hamiltonian stationary Lagrangian submanifold. The distance between $U_p$ and $U_p'$ is $O(\rho^2)$.

**Proof.** We must to find $U_p'$ so that $G_\rho(U_p')$ vanishes identically. But the estimate of Proposition 15 says that $G_\rho(U_p') = 4\pi^2 r_1 r_2 \rho^2 D \mathcal{F}_r([U_p]) \cdot (\Xi^{(1)}, \ldots, \Xi^{(6)}) + O(\rho^4)$.

Suppose now that $D \mathcal{F}_r([U_p]) = 0$ and $D^2 \mathcal{F}_r([U_p])$ is non-degenerate. Since the norm of the inverse of $D^2 \mathcal{F}_r([U_p])$ must be bounded above by a constant independent of $\rho$, then the implicit function theorem for maps between finite-dimensional manifolds implies that it is possible to find a neighbouruing $U_p'$ so that $G_\rho(U_p') \equiv 0$ provided $\rho$ is sufficiently small. Furthermore the distance between $U_p$ and $U_p'$ as points in $U_2(M)$ is $O(\rho^2)$, which is a consequence of the fact that the error term in the equation $G_\rho(U_p) = 0$ is $O(\rho^4)$. As indicated above, this now implies that $\nabla \cdot H(\tilde{\Sigma}_r)$ is constant. Then the divergence theorem implies that it must vanish. \hfill \Box

**References**

[1] Miguel Abreu, *Kähler geometry of toric manifolds in symplectic coordinates*, Symplectic and contact topology: interactions and perspectives (Toronto, ON/Montreal, QC, 2001), Fields Inst. Commun., vol. 35, Amer. Math. Soc., Providence, RI, 2003, pp. 1–24.

[2] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, New York, 1978.

[3] Frédéric Hélein and Pascal Romon, *Hamiltonian stationary Lagrangian surfaces in Hermitian symmetric spaces*, Differential geometry and integrable systems (Tokyo, 2000), Contemp. Math., vol. 308, Amer. Math. Soc., Providence, RI, 2002, pp. 161–178.

[4] ———, *Hamiltonian stationary Lagrangian surfaces in $\mathbb{C}^2$*, Comm. Anal. Geom. 10 (2002), no. 1, 79–126.

[5] ———, *Hamiltonian stationary tori in the complex projective plane*, Proc. London Math. Soc. (3) 90 (2005), no. 2, 472–496.

[6] Dominic Joyce, *Lectures on special Lagrangian geometry*, Global theory of minimal surfaces, Clay Math. Proc., vol. 2, Amer. Math. Soc., Providence, RI, 2005, pp. 667–695. MR MR2167283 (2006j:53077)

[7] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Vol. II*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1996, Reprint of the 1969 original, A Wiley-Interscience Publication.
[8] Yong-Geun Oh, *Second variation and stabilities of minimal Lagrangian submanifolds in Kähler manifolds*, Invent. Math. **101** (1990), no. 2, 501–519.

[9] ———, *Volume minimization of Lagrangian submanifolds under Hamiltonian deformations*, Math. Z. **212** (1993), no. 2, 175–192.

[10] F. Pacard and X. Xu, *Constant mean curvature spheres in Riemannian manifolds*, To appear in Manuscripta Mathematica. Preprint: http://perso-math.univ-mlv.fr/users/pacard.frank.

[11] R. Schoen and J. Wolfson, *Minimizing volume among Lagrangian submanifolds*, Differential Equations: La Pietra 1996 (Shatah Giaquinta and Varadhan, eds.), Proc. of Symp. in Pure Math., vol. 65, 1999, pp. 181–199.

[12] ———, *Minimizing area among Lagrangian surfaces: the mapping problem*, J. Diff. Geom. **58** (2001), 1–86.

[13] Richard Schoen, *Special Lagrangian submanifolds*, Global theory of minimal surfaces, Clay Math. Proc., vol. 2, Amer. Math. Soc., Providence, RI, 2005, pp. 655–666. MR MR2167282 (2006j:53080)

[14] Rugang Ye, *Foliation by constant mean curvature spheres*, Pacific J. Math. **147** (1991), no. 2, 381–396.