Strong Solutions to Reflecting Stochastic Differential Equations with Singular Drift

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Abstract: In this paper, we prove that there exists a unique strong solution to reflecting stochastic differential equations with merely measurable drift giving an affirmative answer to the longstanding problem. This is done through Zvonkin transformation and a careful analysis of the transformed reflecting stochastic differential equations on non-smooth time-dependent domains.

Key Words: strong solution; reflecting stochastic differential equations; singular drift; Zvonkin transformation; time-dependent domain.

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1 Introduction

Let $D$ be a bounded domain in $\mathbb{R}^d$ with a $C^3$ boundary $\partial D$ and $b(t, x)$ a measurable $\mathbb{R}^d$-valued function bounded on $[0, T] \times D$ for every $T > 0$. In this paper, we are concerned with the strong solutions to reflecting stochastic differential equations (SDEs) on the domain $D$ with the singular drift $b$. The purpose is to give an affirmative answer to the longstanding problem of the existence and uniqueness of strong solutions.

More rigorously, given a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ satisfying the usual assumptions and a $d$-dimensional standard Brownian motion $W_t, t \geq 0$ on the probability space. Denote by $n(x)$ the unit inward normal to the boundary $\partial D$. We aim to show that for any $x \in \overline{D}$, there exists a unique pair of continuous adapted processes $(X_t, L_t)$ solving the reflecting stochastic differential equation below, namely, $X_t \in D$ for all $t \geq 0$, $P$-a.e., $L_t$ is a continuous process of bounded variation with values in $\mathbb{R}^d$, and the following equation holds:

$$
\begin{align*}
X_t &= x + W_t + \int_0^t b(s, X_s)ds + L_t, \\
|L|_t &= \int_0^t I_{\{X_s \in \partial D\}}d|L|_s, \\
L_t &= \int_0^t n(X_s)d|L|_s,
\end{align*}
$$

(1.1)

where $|L|_t$ is the total variation of $L_t$.

Reflecting SDEs have been investigated by many authors when the coefficients are smooth /lipschitz. H. Tanaka in [14] obtained the strong solutions of the reflecting SDEs in a convex domain based on solving the corresponding Skorokhod problem. P.L. Lions and A.S. Sznitman in [8] studied the reflecting SDEs by a penalized method in a $C^3$-domain. P. Dupuis and H. Ishii in [2] obtained the existence and uniqueness of the strong solutions to reflecting SDEs in a general domain, which only requires the directions of reflection to be $C^2$. In one-dimensional case, T.S. Zhang in [19] obtained the strong solution to reflecting SDEs with locally bounded drifts using crucially the comparison theorem. In multidimensional case, P. Marín-Rubio and J. Real in [10] obtained the strong solutions to reflecting SDEs when the drifts satisfy a certain monotonicity condition.

On the other hand, strong solutions have been studied by many people for stochastic differential equations with singular drift. In the celebrated work [20], Zvonkin introduced a quasi-isometric transformation of the phase space that can convert a stochastic differential equation with a non-zero singular drift into a SDE without drift. This method is now called Zvonkin transformation. There are many papers (particularly in recent years) devoted to extending the Zvonkin transformation in various ways to obtain the strong solutions of stochastic differential equations with singular coefficients. We mention [4], [12], [15], [16] and [17].

The purpose of this paper is to establish the existence and uniqueness of strong solutions of reflecting SDEs with drifts which are merely measurable. The existence of weak solutions of the reflecting SDE (1.1) is clear by using the Girsanov transform. To get the pathwise strong solution, the key is to prove the pathwise uniqueness of the equation (1.1). Because of the singularity of the drifts we could not rely on solving the deterministic Skorohod
problem, see for instance [2] and [8]. We will use the Zvonkin transformation. However difficulties immediately arise. Zvonkin transformation maps the domain $D$ into a family of time-dependent domains which are not as regular as the original one. Thus, after the transformation we are bound to establish the pathwise uniqueness of reflecting SDEs in time dependent, non-smooth domains. Moreover, the reflecting directions of the transformed process are not as smooth as the original inward normal and also the coefficients of the transformed reflecting SDEs are not Lipschitz. The existing results on reflecting SDEs in time dependent domains can not be applied. A large part of our work is to carry out a careful analysis of the transformed, time dependent domains and the time dependent reflecting directions to establish the necessary regularities required. To get the pathwise uniqueness, eventually we also need to construct a family of auxiliary test functions. This is done in a similar way as that in [2] and [9].

Throughout this paper, we assume $b(t,x)$ is bounded on $[0,T] \times D$ for every $T > 0$.

Now we describe the content and organization of the paper in more details. In Section 2, we consider the following parabolic partial differential equation (PDE) associated with the singular drift on the domain $(0,T) \times D$, equipped with the Neumann boundary condition:

\[
\begin{cases}
\partial_t u^T(t,x) + \frac{1}{2} \Delta_x u^T(t,x) + b(t,x) \cdot \nabla_x u^T(t,x) = 0, & \forall (t,x) \in (0,T) \times D, \\
\frac{\partial u^T}{\partial n}(t,x) = n(x), & \forall (t,x) \in (0,T) \times \partial D, \\
u^T(T,x) = x, & \forall x \in D.
\end{cases}
\]

We provide regularities of the solution $u^T(t,x)$, which will be used in subsequent sections. Especially, we show that there exists an open set $G \supset \bar{D}$ such that the extension of $u^T(t,\cdot)$ on $G$ is a homeomorphism for $t \in [0,T]$ and $\tilde{u}^T(t,x) := (t,u^T(t,x))$ is an open mapping on $(0,T) \times G$.

In Section 3, we study the time dependent domains $u^T(t,D)$, $t \in [0,T]$, the images of domain $D$ under the solution mappings $u^T(t,\cdot)$, $t \in [0,T]$. Among other things, we showed that the domains $u^T(t,D)$ satisfy the exterior and interior cone conditions when $T$ is sufficiently small. Regularities of the time dependent vector fields, $\gamma(t,x) := n((u^T)^{-1}(t,x))$, of the reflecting directions are also established. Here $(u^T)^{-1}(t,x)$ denotes the inverse function of $u^T(t,x)$.

In Section 4, we consider the flows associated with the time dependent vector fields of reflecting directions:

\[
\begin{cases}
y(t,x,0) = x, \\
\partial_r y(t,x,r) = \gamma(t,y(t,x,r)), & r \in \mathbb{R}.
\end{cases}
\]

We will provide a number of regularity results of the hitting times $\Gamma(t,x)$ of the flows on certain hyperplane. These hitting times will be used to construct test functions for proving the pathwise uniqueness of the transformed reflecting SDEs. Roughly speaking, since $\gamma$ only belongs to some Sobolev space on $\tilde{D} := \tilde{u}^T((0,T) \times D)$, to ensure the regularity of the hitting times $\Gamma(t,x)$, we need to prove that if $(t,x) \in \tilde{D}$, then $(t,y(t,x,r))$ lies in $\tilde{D}$ before $y(t,x,r)$ hits the hyperplane ($i.e.$ for $r \in (0,\Gamma(t,x)]$). At the end of this section, we will establish
some smooth approximations of $\Gamma(t,x)$, which will be used later to show that $y(\cdot,\cdot,\Gamma(\cdot,\cdot))$ belongs to some Sobolev space on $D$.

In Section 5, a family of auxiliary functions is constructed. We first construct the functions locally in some neighborhoods of the points on the boundary of the domain $D$ and then piece them together through a finite cover of the boundary. These test functions will be used to prove the pathwise uniqueness of the solutions of reflecting stochastic differential equations. We also introduce the stochastic Gronwall’s inequality and Krylov’s estimate.

In Section 6, we will establish the existence and uniqueness of strong solutions to the reflecting SDEs \( [1.1] \). The existence of a weak solution follows from the Girsanov theorem. The strong solution is obtained by proving the pathwise uniqueness of the solutions. To this end, we first establish a generalized Itô’s formula for the solution $X_t$ of the reflecting SDEs using the Krylov’s estimate. Then we will use the auxiliary functions to eliminate the local times of the transformed processes $u^T(t,X_t)$. Finally, with the help of the stochastic Gronwall’s inequality, the pathwise uniqueness is proved for the transformed processes and hence the pathwise uniqueness of the solutions $X_t$ follows.

The last part of the paper is the appendix which provides the proofs for some of the results in Section 4.

We close this introduction by mentioning some conventions used throughout this paper: \( |\cdot| \) or \( d(\cdot,\cdot) \) denotes the Euclidean norm in $\mathbb{R}^d$. \( \cdot \) denotes the inner product in $\mathbb{R}^d$. Use $B(x,r)$ to denote the ball in $\mathbb{R}^d$ centered at $x$ with radius $r$, and use $n^i(x)$ to denote the $i$-th component of the unit inward normal of $n(x)$ for $1 \leq i \leq d$. For $d \times d$ matrix $A$, we use $|A|$ to denote the determinant of $A$ and define $\|A\|_2 := \sup_{x \in B(0,1)} |Ax|$, $\|A\| := \sup_{1 \leq i,j \leq d} |a_{ij}|$. Let $D_xf(x)$ stand for the vector $(\partial_{x_1}f(x),\ldots,\partial_{x_d}f(x))$ if $f : \mathbb{R}^d \to \mathbb{R}$, and $D_xf(x)$ stand for the Jacobian matrix of $f$ if $f : \mathbb{R}^d \to \mathbb{R}^d$. For $x \in \mathbb{R}^d$, a unite vector $\gamma$ in $\mathbb{R}^d$ and $\theta, r > 0$, define $C(x,\gamma,\theta,r) := \{ y \in \mathbb{R}^d : 0 < |y - x| < r \text{ and } (y - x) \cdot \gamma > \cos \theta |y - x| \}$. For $-\infty < b < a < \infty$, we stile use the symbol $(a,b)$ to stand for $\{ t \in \mathbb{R}^d : b \leq t < a \}$ when there is no danger of causing ambiguity. For an open set $O \subset \mathbb{R}^d$ and measurable function $f : O \to \mathbb{R}^d$, denote $\|f\|_{L^{2d+2}(O)} := \|f\|_{L^{2d+2}(O)}$. For a bounded domain $O \subset (0,T) \times \mathbb{R}^d$, $W^{1,2}_{2d+2}(O)$ is a Sobolev space of functions $f(t,x)$ (with value in $\mathbb{R}^d$) such that

$$\|f\|_{W^{1,2}_{2d+2}(O)} := \|f\|_{L^{2d+2}(O)} + \|\partial_t f(t,x)\|_{L^{2d+2}(O)} + \sum_{1 \leq i,j \leq d} \|\partial_{x_i} \partial_{x_j} f(t,x)\|_{L^{2d+2}(O)} < \infty.$$  

Where $\partial_t f(t,x)$ stands for the first order weak derivative with respect to (w.r.t.) $t$ and $\partial_{x_i} \partial_{x_j} f(t,x)$ stands for the second order weak derivative w.r.t. $x$. In the sequel, we will also write $W^{1,2}_{2d+2}(O)$ for $W^{1,2}_{2d+2}(O;\mathbb{R}^d)$ with no danger of ambiguity. $c$ will denote a generic positive constant which may be different from line to line, and $a \lesssim b$ means $a \leq cb$ for some unimportant $c > 0$.

## 2 Parabolic PDEs associated with the singular drift

In this section, we consider parabolic PDEs associated with the singular drift on the domain $D$, equipped with the Neumann boundary condition. We will provide some results on the regularity of the solutions, which will be used in subsequent sections.
Moreover, by Theorem 7.20 in [7] we have

\[ \forall u \in \mathbb{R}^d : d(x, \partial D) < c \] for \( c > 0 \). Following the argument of Lemma 14.16 in [3], we see that for any \( x \in \Gamma_{b_0} \), there exists a unique \( \varphi(x) \in \partial D \) such that \( |x - \varphi(x)| = d(x, \partial D) \), moreover \( \varphi \in C^2(\Gamma_{b_0}) \). Thus we can extend \( n(x) \) to the whole space \( \mathbb{R}^d \) such that \( n \in C^2_0(\mathbb{R}^d) \) with \( |n(x)| \leq 1 \) on \( \mathbb{R}^d \) and \( |n(x)| = 1 \) on \( \Gamma_{b_0} \) (for example, take \( n(x) := n(\varphi(x))\phi(x) \) for some real function \( \phi \in C^2_0(\Gamma_{b_0}) \) with \( \phi(x) = 1 \) on \( \Gamma_{b_0} \)).

From Chapter 4 (Section 9) in [6], it is known that for any \( T > 0 \), there exists a unique weak solution \( u^T \in W^{1,2}_{2d+2}((0, T) \times D) \) to the following boundary value problem:

\[
\begin{align*}
\partial_t u^T(t, x) &+ \frac{1}{2} \Delta_x u^T(t, x) + b(t, x) \cdot \nabla_x u^T(t, x) = 0, \quad \forall (t, x) \in (0, T) \times D, \\
\frac{\partial u^T}{\partial n}(t, x) &\leq n(x), \quad \forall (t, x) \in (0, T) \times \partial D, \\
u^T(T, x) &= x, \quad \forall x \in D.
\end{align*}
\] (2.1)

We now consider smooth approximations of the drift vector field \( b(t, x) \).

Fix a nonnegative smooth function \( \psi \) on \( \mathbb{R}^{d+1} \) with compact support such that

\[
\int_{\mathbb{R}^{d+1}} \psi(t, x) dx dt = 1.
\]

For any positive integer \( n \), let \( \psi_n(t, x) := 2^n (t, 2^n x) \) and

\[
b_n(t, x) := \int_{\mathbb{R}^{d+1}} b(s, y) \psi(n, t - s, x - y) ds dy.
\]

Since \( b_n(t, x) \) is smooth, according to Theorem 5.18 in [7] there exists a unique \( u^T_n \in C_b^{1,2}(0, T) \times \partial D) \), that is the solution to the following boundary value problem:

\[
\begin{align*}
\partial_t u^T_n(t, x) &+ \frac{1}{2} \Delta_x u^T_n(t, x) + b_n(t, x) \cdot \nabla_x u^T_n(t, x) = 0, \quad \forall (t, x) \in (0, T) \times D, \\
\frac{\partial u^T_n}{\partial n}(t, x) &\leq n(x), \quad \forall (t, x) \in (0, T) \times \partial D, \\
u^T_n(T, x) &= x, \quad \forall x \in D.
\end{align*}
\]

Moreover, by Theorem 7.20 in [7] we have

\[
\lim_{n \to \infty} \|u^T_n - u^T\|_{W^{1,2}_{2d+2}((0, T) \times D)} = 0.
\] (2.2)

Set \( G := \{x \in \mathbb{R}^d, d(x, D) < \delta_0 \} \) and \( G' := \{x \in \mathbb{R}^d, d(x, D) < \delta_0 \} \). We have the following result.

**Lemma 2.1** There exist constants \( M_0 > 0 \) and \( 0 < \alpha_0 < 1 \), such that for any \( n \geq 1 \), \( 0 < T \leq 1 \), we can extend \( u^T \) and \( u^T_n \) to \([0, T] \times \mathbb{R}^d \), such that \( u^T \in C^{0,1}([0, T] \times G') \), \( u^T_n \in C^{1,1}(0, T) \times \mathbb{R}^d \), \( u^T(T, x) = u^T_n(T, x) = x \) on \( G' \) and

\[
\lim_{n \to \infty} \sup_{(t, x) \in [0, T] \times G'} \left( |u^T(t, x) - u^T_n(t, x)| + \|\nabla_x u^T_n(t, x) - \nabla_x u^T(t, x)\| \right) = 0.
\] (2.3)
Moreover, if $0 \leq s \leq t \leq T$ and $x \in G'$, then
\begin{align*}
|u^T(t, x) - u^T(s, x)| + \|\nabla_x u^T(t, x) - \nabla_x u^T(s, x)\| &\leq M_0|t - s|^\alpha_0, \quad (2.4) \\
\|\nabla_x u^T_n(t, x) - \nabla_x u^T_n(s, x)\| &\leq M_0|t - s|^\alpha_0. \quad (2.5)
\end{align*}

**Proof:** By (2.2) and Sobolev inequality (see Lemma II.3.3 in [6]), we know that

\[ u^T \in C^{0,1}([0, T] \times \bar{D}), \]

and (2.3)-(2.5) hold if $G'$ is replaced by $\bar{D}$.

Now we define the extensions of $u^T$ and $u^T_n$ on $[0, T] \times G' \setminus \bar{D}$ by
\begin{align*}
u^T(t, x) &:= 2u^T(t, \varphi(x)) - u^T(t, 2\varphi(x) - x), \\
u^T_n(t, x) &:= 2u^T_n(t, \varphi(x)) - u^T_n(t, 2\varphi(x) - x).
\end{align*}

Since $\varphi \in C^2(\Gamma_{\tilde{b}_0})$, it is easy to see that $u^T(T, x) = u^T_n(T, x) = x$ on $G'$ and

\[ u^T \in C^{0,1}([0, T] \times G' \setminus \bar{D}), \quad u^T_n \in C^{1,1}([0, T] \times G' \setminus \bar{D}). \]

Now we show that for any $x_0 \in \partial D$, $u^T(t, \cdot)$ and $u^T_n(t, \cdot)$ are differentiable in a neighborhood of $x_0$.

Since $\partial D \subset C^3$, for $x_0 \in \partial D$ there exist a neighborhood $U$ of $x_0$ and a $C^3$-diffeomorphism $\Psi$ that maps $U$ onto $B(\Psi(x_0), r)$ for some $r > 0$, such that $\Psi^{-1}(\partial^- (\Psi(x_0), r) = U \setminus D$, where $\Psi^{-1}$ is the inverses of $\Psi$ and $\partial^- (\Psi(x_0), r) := \{x = (x_1, x_2, \cdots, x_d) \in B(\Psi(x_0), r) : x_d \leq 0\}$. Set $v(t, x) := u^T(t, \Psi^{-1}(x))$ and $v_n(t, x) := u^T_n(t, \Psi^{-1}(x))$. Then for $x \in \partial^- (\Psi(x_0), r)$,
\begin{align*}
v(t, x) &= 2v(t, \Psi(\varphi^{-1}(x)))) - v(t, \Psi(2\varphi(\Psi^{-1}(x)) - \Psi^{-1}(x))), \\
v_n(t, x) &= 2v_n(t, \Psi(\varphi^{-1}(x)))) - v_n(t, \Psi(2\varphi(\Psi^{-1}(x)) - \Psi^{-1}(x))).
\end{align*}

One can verify that $v \in C^{0,1}([0, T] \times B(\Psi(x_0), r))$ and $v_n \in C^{1,1}([0, T] \times B(\Psi(x_0), r))$. Hence, we have
\[ u^T(t, x) = v(t, \Psi(x)) \in C^{0,1}([0, T] \times U), \]
and
\[ u^T_n(t, x) = v_n(t, \Psi(x)) \in C^{1,1}([0, T] \times U). \]

Note that $2\varphi(x) - x \in D$ for $x \in G' \setminus \bar{D}$, from the definition of $u^T$ and $u^T_n$ in (2.6), clearly (2.3)-(2.5) hold.

Set $\bar{u}^T(t, x) := (t, u^T(t, x))$ and $\bar{u}^T_n(t, x) := (t, u^T_n(t, x))$. Then we have the following proposition:

**Proposition 2.1** There exists a constant $T_0 \in (0, 1]$ such that for any $T \in (0, T_0]$, $\bar{u}^T$, $\bar{u}^T_n$ are open mappings on $(0, T) \times G$. Moreover, there exist positive constants $M_1, M_2, M_3$, such that for any $n \geq 1$, $0 \leq t \leq T \leq T_0$ and $x, y \in G$,
\begin{align*}
\frac{1}{2} \leq |D_x u^T(t, x)| &\leq 2, \quad \frac{1}{2} \leq |D_x u^T_n(t, x)| \leq 2, \quad (2.7) \\
M_1 |x - y| &\leq |u^T(t, x) - u^T(t, y)| \leq M_2 |x - y|, \quad (2.8) \\
M_1 |x - y| &\leq |u^T_n(t, x) - u^T_n(t, y)| \leq M_2 |x - y|. \quad (2.9)
\end{align*}

Furthermore, if $|x - y| < \frac{\alpha_0}{2}$, then
\[ |u^T(t, x) - u^T(t, y)| \wedge |u^T_n(t, x) - u^T_n(t, y)| \geq (1 - M_3 T_0^\alpha)|x - y|, \quad (2.10) \]
where $\alpha_0$ is the constant defined in Lemma 2.1.
Proof: We only give the proof of the properties of $u^T$ because the corresponding proof for $u_n^T$ is similar.

Since $D_xu^T(T, x)$ is the identity matrix, by Lemma 2.1 one can see that (2.7) holds if $T_0$ is sufficiently small. Without loss of generality, we assume $T_0 < (\frac{1}{dM_0})^{\frac{1}{\alpha_0}} \wedge (\frac{\delta_0}{8M_0})^{\frac{1}{\alpha_0}}$.

To show that $\tilde{u}^T$ is an open mapping on $(0, T) \times G$, it is sufficient to show that for any open set $A_1 \subset G$ and $(t_0, x_0) \in (0, T) \times A_1$, there exist constants $\eta, \delta > 0$ such that if $t \in (t_0 - \eta, t_0 + \eta)$, then

$$B(\tilde{u}^T(t_0, x_0), \delta) \subset \tilde{u}^T(t, A_1).$$

By (2.7) and the implicit function theorem, $u^T(t, \cdot)$ is an open mapping on $G$. Hence there exist a constant $\delta > 0$ and an open set $A_2 \subset A_1$ such that $u^T(t_0, A_2) = B(u^T(t_0, x_0), \delta)$ and $B(u^T(t_0, x_0), 2\delta) \subset u^T(t_0, A_1)$. By (2.4), there exists a constant $\eta > 0$ such that for any $t \in (t_0 - \eta, t_0 + \eta)$,

$$|u^T(t, x) - u^T(t_0, x)| < \frac{\delta}{4}, \forall x \in G'.$$

Suppose $B(u^T(t_0, x_0), \delta) \not\subset u^T(t, A_1)$ for some $t \in (t_0 - \eta, t_0 + \eta)$. Then there exists a point $x \in A_2$ such that $u^T(t_0, x) \in u^T(t, A_1)^c$. Since $u^T(t, x) \in u^T(t, A_1)$ and since $u^T(t, A_1)$ is an open set, we have

$$d(u^T(t_0, x), u^T(t, \partial A_1)) \leq |u^T(t_0, x) - u^T(t, x)| < \frac{\delta}{4}.$$

On the other hand, by (2.12),

$$d(u^T(t_0, x), u^T(t, \partial A_1)) \geq d(u^T(t_0, x), u^T(t_0, \partial A_1)) - d(u^T(t_0, \partial A_1), u^T(t, \partial A_1))$$

$$\geq d(B(u^T(t_0, x_0), \delta), u^T(t_0, \partial A_1)) - d(u^T(t_0, \partial A_1), u^T(t, \partial A_1))$$

$$\geq \delta - \delta = \frac{3\delta}{4},$$

which contradicts (2.13). Hence we have (2.11).

Now we show (2.11). For $0 \leq t \leq T \leq T_0$ and $x, y \in G$ with $\frac{1}{2} \leq ||x - y|| < \frac{\delta_0}{2}$, we have $\lambda x + (1 - \lambda)y \in G'$ for any $\lambda \in (0, 1)$. Hence by (2.4) and the fact that $\|\cdot\|_2 \leq \|\| \cdot \|$, we have

$$|u^T(t, x) - u^T(t, y)|$$

$$\geq |u^T(T, x) - u^T(T, y)| - \left| \int_0^1 (D_xu^T(T, \lambda x + (1 - \lambda)y) - D_xu^T(t, \lambda x + (1 - \lambda)y)) \cdot (x - y)d\lambda \right|$$

$$\geq |u^T(T, x) - u^T(T, y)| - \left| \int_0^1 d\|D_xu^T(T, \lambda x + (1 - \lambda)y) - D_xu^T(t, \lambda x + (1 - \lambda)y)\| \cdot |x - y|d\lambda \right|$$

$$\geq |x - y| - dM_0|T - t|^\alpha_0 \cdot |x - y| \geq (1 - dM_0T_0)|x - y|.$$

Finally we show (2.8). Let $0 \leq t \leq T \leq T_0$ and $x, y \in G$. When $|x - y| \geq \frac{\delta_0}{2}$, noting that $T_0 < (\frac{\delta_0}{8M_0})^{\frac{1}{\alpha_0}}$, by (2.4) we have

$$|u^T(t, x) - u^T(t, y)| \leq 2 \sup_{(t, x) \in [0, T] \times G} |u^T(t, x)| \leq \frac{4}{\delta_0} \sup_{(t, x) \in [0, T] \times G} |u^T(T, x)||x - y|,$$

$$|u^T(t, x) - u^T(t, y)| \leq 2 \sup_{(t, x) \in [0, T] \times G} |u^T(t, x)| \leq \frac{4}{\delta_0} \sup_{(t, x) \in [0, T] \times G} |u^T(T, x)||x - y|, \tag{2.14}$$
and
\[ |u^T(t, x) - u^T(t, y)| = |u^T(t, x) - u^T(T, x) - u^T(T, y) + x - y| \]
\[ \geq |x - y| - |u^T(t, x) - u^T(T, x)| - |u^T(t, y) - u^T(T, y)| \]
\[ \geq |x - y| - 2M_0T_0^\alpha \]
\[ \geq \frac{|x - y|}{2} - \left( \frac{\delta_0}{4} - 2M_0T_0^\alpha \right) \]
\[ > \frac{|x - y|}{2}. \]  
(2.15)

When \( |x - y| \leq \frac{\delta_0}{4} \), then \( \lambda x + (1 - \lambda)y \in G' \) for any \( \lambda \in (0, 1) \). Using the Lagrange mean value theorem and the boundness of \( \|D_xu^T(t, x)\| \), we have
\[ |u^T(t, x) - u^T(t, y)| \leq M_2|x - y|. \]  
(2.16)

Hence combining (2.10) with (2.14)-(2.16), we get (2.8).

\[ \blacksquare \]

3 Domain transformation and regularity of the reflecting directions

In this section we study the time dependent domains which are the images of domain \( D \) under the solution mappings \( u^T(t, \cdot) \), \( t \in [0, T] \). Regularities of the time dependent vector field of the reflecting directions will be established.

We start by showing the exterior and interior cone conditions for \( u^T(t, D) \) for sufficiently small \( T \). Set
\[ \delta_1 := M_1d(\partial D, \partial G) = \frac{M_1\delta_0}{2}, \]
\[ \tilde{G}_1^T := \{(t, x) : 0 \leq t \leq T, d(x, u^T(t, D)) < \frac{\delta_1}{2}\}, \]
and
\[ \tilde{G}_2^T := \{(t, x) : 0 \leq t \leq T, d(x, u^T(t, D)) < \frac{3\delta_1}{4}\}. \]
Recall the constant \( T_0 \) defined in the statement of Proposition 2.1. We first have the following Lemma.

Lemma 3.1 There exists an integer \( N_0 > 0 \) such that for \( n \geq N_0 \), \( 0 < T \leq T_0 \),
\[ \tilde{u}_n^T([0, T] \times \tilde{D}) \cup \tilde{u}_n^T([0, T] \times \tilde{D}) \subset \tilde{G}_1^T \subset \tilde{G}_2^T \subset \tilde{u}_n^T([0, T] \times G) \cap \tilde{u}_n^T([0, T] \times G). \]

Proof: By (2.3), there exists an integer \( N_0 > 0 \) such that for \( n \geq N_0 \), we have
\[ \sup_{(t, x) \in [0, T] \times G} |u_n^T(t, x) - u^T(t, x)| < \frac{\delta_1}{8}, \]
which implies \( \tilde{u}_n^T([0, T] \times \tilde{D}) \cup \tilde{u}_n^T([0, T] \times \tilde{D}) \subset \tilde{G}_1^T \subset \tilde{G}_2^T \). Noting that \( u^T(t, G) \) is an open set by Proposition 2.1 and the fact that \( \inf_{0 \leq t \leq T} d(u^T(t, D), u^T(t, \partial G)) \geq M_1d(\partial D, \partial G) = \delta_1 \), we have \( G_2^T \subset \tilde{u}^T([0, T] \times G) \).
Now we show that $\tilde{G}_n^T \subset \tilde{u}_n^T([0,T] \times G)$ for $n \geq N_0$. Suppose there exists a point $(t, x)$ belonging to $\tilde{G}_n^T \setminus \tilde{u}_n^T([0,T] \times G)$. Then $d(x, u_n^T(t, D)) \leq d(x, u^T(t, D)) + \frac{\delta_1}{8} < \frac{7\delta_1}{8}$. Therefore there exists a $y_1 \in u_n^T(t, D)$ such that $|x - y_1| < \frac{7\delta_1}{8}$. On the other hand, since $x \in u_n^T(t, G)$, $y_1 \in u_n^T(t, D)$ and since $u_n^T(t, G)$ is an open set, there exists a $\lambda \in (0, 1)$ such that $y_2 := \lambda x + (1 - \lambda)y_1 \in u_n^T(t, \partial G)$. Hence $\|y_2 - y_1\| < \frac{7\delta_1}{8}$, which contradicts the fact that $d(u_n^T(t, D), u_n^T(t, \partial G)) \geq M_1 d(D, \partial G) = \delta_1$ by (2.9).} 

The next result shows that $u^T(t, D)$ fulfils the exterior and interior cone conditions for sufficiently small $T$.

**Proposition 3.1** There exist constants $T_1 \in (0, T_0)$, $\theta_0 \in (0, \frac{\pi}{2})$ and $\delta_2 \in (0, \frac{\delta_1}{2})$ such that for any $t \in [0, T_1]$ and $x \in \partial D$,

$$ C(u^{T_1}(t, x), -n(x), \theta_0, \delta_2) \subset u^{T_1}(t, D)^c, $$

$$ C(u^{T_1}(t, x), n(x), \theta_0, \delta_2) \subset u^{T_1}(t, D). $$

**Proof:** We only prove (3.1), (3.2) can be proved similarly.

Since $\partial D$ is smooth, there exist constants $\theta \in (0, \frac{\pi}{2})$ and $r > 0$ such that for $x \in \partial D$, $C(x, -n(x), \theta, r) \subset \tilde{D}^c$. Choose $T_1 \in (0, T_0)$ to be sufficiently small so that $\frac{\cos \theta + dM_2 T_1^{\theta_0}}{1 - M_2 T_1^{\theta_0}} \in (0, 1)$. Set $\theta_0 := \arccos \frac{\cos \theta + dM_2 T_1^{\theta_0}}{1 - M_2 T_1^{\theta_0}}$ and $\delta_2 := \frac{\delta_1}{2} \wedge (M_1 r)$. Now we show that for $t \in [0, T_1]$ and $x \in \partial D$, (3.1) holds.

Take $y \in C(u^{T_1}(t, x), -n(x), \theta_0, \delta_2)$. Since $(t, y) \in \tilde{G}_1^T$, by (2.8) and Lemma 3.1 there exists a $y' \in G$ such that $y = u^{T_1}(t, y')$ and

$$ |x - y'| \leq \frac{1}{M_1} |u^{T_1}(t, x) - u^{T_1}(t, y')| < \frac{\delta_0}{4} \wedge r, $$

which implies $d(\lambda x + (1 - \lambda)y', x) < \frac{\delta_0}{4}$ for $\lambda \in (0, 1)$. From the definition of $G$, we see that $\lambda x + (1 - \lambda)y' \in G$. Together with (2.11) and (2.10) we have

$$ (x - y') \cdot n(x) $$

$$ = (u^{T_1}(t, x) - u^{T_1}(t, y')) \cdot n(x) + (x - u^{T_1}(t, x) - (y' - u^{T_1}(t, y'))) \cdot n(x) $$

$$ > \cos \theta_0 |u^{T_1}(t, x) - u^{T_1}(t, y')| - |u^{T_1}(T_1, x) - u^{T_1}(t, y) - u^{T_1}(T_1, y') - u^{T_1}(t, y')| $$

$$ \geq \cos \theta_0 |u^{T_1}(t, x) - u^{T_1}(t, y')| $$

$$ - \int_0^{\lambda_1} \| \nabla_x u^{T_1}(1, \lambda x + (1 - \lambda)y') - \nabla_x u^{T_1}(t, \lambda x + (1 - \lambda)y') \|_2 |x - y'| d\lambda $$

$$ \geq (1 - M_3 T_1^{\theta_0}) \cos \theta_0 |x - y'| - dM_2 T_1^{\theta_0} |x - y'| = \cos \theta |x - y'|, $$

which implies $y' \in C(x, -n(x), \theta, r) \subset \tilde{D}^c$. Hence $y = u^{T_1}(t, y') \in u^{T_1}(t, D)^c$, which implies (3.1).} 

Here and below, we fix $T_1 > 0$ as defined in Proposition 3.1. Denote $u(t, x) := u^{T_1}(t, x)$, $u_n(t, x) := u^{T_1}_n(t, x)$, $\tilde{u}(t, x) := \tilde{u}^{T_1}(t, x)$, $\tilde{u}_n(t, x) := \tilde{u}^{T_1}_n(t, x)$, $\tilde{G}_1 := \tilde{G}_1^{T_1}$, $\tilde{G}_2 := \tilde{G}_2^{T_1}$ and $\tilde{D} := \tilde{u}((0, T_1) \times D)$.

By Proposition 2.1 and Lemma 3.1, the inverses of $u$ and $u_n$ exist, denoted by $u^{-1}$ and $u_n^{-1}$. Moreover, it is easy to see that $u^{-1}, u_n^{-1}$ are continuous in $\tilde{G}_2$ w.r.t. $(t, x)$ for $n \geq N_0$.}
Take a smooth function $\phi(t, x) \in C^\infty_b([0, T_1] \times \mathbb{R}^d)$ such that $\phi(t, x) = 1$ on $\tilde{G}_1$ and $\phi(t, x) = 0$ on $[0, T_1] \times \mathbb{R}^d \setminus \tilde{G}_2$. For $n \geq N_0$, set

$$
\gamma_n(t, x) := (\gamma_n^1(t, x), \gamma_n^2(t, x), \ldots, \gamma_n^d(t, x)) := n(u_n^{-1}(t, x))\phi(t, x),
$$

$$
\gamma(t, x) := (\gamma^1(t, x), \gamma^2(t, x), \ldots, \gamma^d(t, x)) := n(u^{-1}(t, x))\phi(t, x).
$$

Then $\gamma_n$ and $\gamma$ are well defined in $[0, T_1] \times \mathbb{R}^d$. $\gamma$ will be the directions of reflection of the transformed reflecting SDEs. To obtain the regularity of $\gamma$, we need to study the convergence of $u_n^{-1}$, which is the content of the next lemma.

Set

$$
D(t, c) := \{x : d(x, u(t, D)^c) > c\},
$$

$$
\tilde{D}_c := \{(t, x) : t \in (0, T_1), \ x \in D(t, c)\} \text{ and } \tilde{D}'_c := \{(t, x) : t \in (0, T_1), \ d(x, D^c) > c\} \text{ for } c > 0.
$$

Then we have the following result.

**Lemma 3.2** For $n \geq N_0$, we have $u_n^{-1} \in C_b^{1,1}(\tilde{G}_2)$, $u^{-1} \in C_b^{0,1}(\tilde{G}_2)$ and

$$
\lim_{n \to \infty} \|u_n^{-1} - u^{-1}\|_{C_b^{0,1}(\tilde{G}_2)} = 0.
$$

Moreover, for any constants $\varepsilon, p > 0$ and functions $f \in L^p((0, T_1) \times D)$ and $g_k, g \in L^p(\tilde{D})$ with $\lim_{k \to \infty} \|g_k - g\|_{L^p(\tilde{D})} = 0$, $\tilde{D}$ is an open set in $\mathbb{R}^{d+1}$ and there exists an integer $N_0(\varepsilon) \geq N_0$ such that for any $n \geq N_0(\varepsilon)$, we have $\tilde{u}_n^{-1}(\tilde{D}_c) \subset \tilde{D}'_{2/n^2}$, $u_n^{-1} \in C_b^{1,2}(\tilde{D}_c)$, $|D_x u_n^{-1}(t, x)| \geq \frac{1}{2}$ for $(t, x) \in \tilde{D}_c$ and

$$
\lim_{n \to \infty} \|f(t, u_n^{-1}(t, x)) - f(t, u^{-1}(t, x))\|_{L^p(\tilde{D}_c)} = 0,
$$

$$
\lim_{n \to \infty} \|g(t, u_n(t, x)) - g(t, u(t, x))\|_{L^p(\tilde{D}_c)} = 0,
$$

$$
\lim_{k \to \infty} \|g_k(t, u_n(t, x)) - g(t, u_n(t, x))\|_{L^p(\tilde{D}_c)} = 0, \ \forall n \geq N_0(\varepsilon),
$$

where $\tilde{u}_n^{-1}$ is the inverse of $\tilde{u}_n$ and $M_2$ was the constant defined in Proposition 2.7.

**Proof:** First we show (3.4). For any $n \geq N_0$ and $(t, x) \in \tilde{G}_2$, we have $u_n^{-1}(t, x), u^{-1}(t, x) \in G$ by Lemma 3.1. Hence by (2.3) and (2.8), for $n \geq N_0$ we have

$$
\sup_{(t,x) \in \tilde{G}_2} |u_n^{-1}(t, x) - u^{-1}(t, x)| \leq \sup_{(t,x) \in \tilde{G}_2} \frac{1}{M_1} |u(t, u_n^{-1}(t, x)) - u(t, u^{-1}(t, x))|
$$

$$
= \frac{1}{M_1} \sup_{(t,x) \in \tilde{G}_2} |u(t, u_n^{-1}(t, x)) - x|
$$

$$
= \frac{1}{M_1} \sup_{(t,x) \in \tilde{G}_2} |u(t, u_n^{-1}(t, x)) - u_n(t, u_n^{-1}(t, x))|
$$

$$
\leq \frac{1}{M_1} \sup_{(t,y) \in [0,T_1] \times G} |u(t, y) - u_n(t, y)| \to 0,
$$

as $n \to \infty$. 

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On the other hand, by (2.7) and the implicit function theorem, it is easy to see that $u_i^{-1} \in C^0_b((0, T_1) \times D)$ and for any $(t, x) \in \hat{G}_2$, 
\begin{align*}
D_x u_i^{-1}(t, x) &= \frac{(D_x u(t, u^{-1}(t, x)))^{-1}}{|(D_x u(t, u^{-1}(t, x)))^{-1}|}, \\
\partial_t u_i^{-1}(t, x) &= -(D_x u^i(t, x)) \cdot (\partial_t u_n(t, u^{-1}(t, x))). 
\end{align*}
(3.9)
Combining this with (2.3), (2.7) and (3.8), we obtain
\[ \lim_{n \to \infty} \sup_{(t, x) \in \hat{G}_2} \|D_x u_i^{-1}(t, x) - D_x u_i^{-1}(t, x)\| = 0. \]
Hence we proved (3.10).

Given $\varepsilon > 0$, now we show that $\hat{D}_\varepsilon$ is an open set.

By Proposition 2.1, we know that $\hat{D}$ is open in $\mathbb{R}^{d+1}$. Take $(t_0, x_0) \in \hat{D}_\varepsilon \subset \hat{D}$, then there exist constants $\eta, \delta > 0$ such that $(t_0 - \eta, t_0 + \delta) \times B(x_0, 2\delta) \subset \hat{D}$ and $d(B(x_0, 2\delta), u(t_0, \partial D)) > \varepsilon$. By (2.7), there exists a positive constant $\eta' < \eta$ such that for any $(t, y) \in (t_0 - \eta', t_0 + \eta') \times G'$, $|u(t, y) - u(t, y)| < \frac{\delta}{2}$. Hence for $t \in (t_0 - \eta', t_0 + \eta')$, 
\[ d(B(x_0, \delta), u(t, \partial D)) \geq d(B(x_0, \delta), u(t_0, \partial D)) - d(u(t_0, \partial D), u(t, \partial D)) > \delta - \frac{\delta}{2} > \varepsilon. \]
Hence we have $(t_0 - \eta', t_0 + \eta') \times B(x_0, \delta) \subset \hat{D}_\varepsilon$, which proves that $\hat{D}_\varepsilon$ is open.

Noting that $\hat{u}(\hat{D}_\varepsilon) \subset \hat{D}_{\hat{M}_\varepsilon}$ and $\hat{u}^{-1}(\hat{D}_\varepsilon) \subset \hat{D}_{\hat{M}_\varepsilon}$ by (2.8), together with (2.3) and (3.8), there exists an integer $N_0(\varepsilon) \geq N_0$ such that for any $n \geq N_0(\varepsilon)$, we have
\[ \hat{u}_n(\hat{D}_\varepsilon) \subset \hat{D}_{\hat{M}_\varepsilon} \subset \hat{D}. \]
and
\[ \hat{u}_n^{-1}(\hat{D}_\varepsilon) \subset \hat{D}'_{\hat{M}_\varepsilon} \subset (0, T_1) \times D. \]
Hence by (2.7), (3.9) and (3.11) we have $u_i^{-1} \in C^{1,2}_b(\hat{D}_\varepsilon)$ and for any $(t, x) \in \hat{D}_\varepsilon$,
\begin{align*}
|D_x u_i^{-1}(t, x)| &= |(D_x u(t, u^{-1}(t, x)))^{-1}| \geq \frac{1}{2}, \\
|D_x u^{-1}(t, x)| &= |(D_x u(t, u^{-1}(t, x)))^{-1}| \geq \frac{1}{2}.
\end{align*}
(3.12)

By (2.7) and (3.10), it is easy to see that (3.11) holds.

Now we show (3.5). For any $\varepsilon_1 > 0$, there exists a function $\hat{f} \in C_b(\mathbb{R}^d)$ such that $\|f - \hat{f}\|_{L^p(\hat{D}'_{\hat{M}_\varepsilon})} < \varepsilon_1$. Combining this with (3.8), (3.11), (3.12) and a change of variable, we see that
\begin{align*}
&\lim_{n \to \infty} \|f(t, u^{-1}(t, x)) - f(t, u^{-1}(t, x))\|_{L^p(\hat{D}_\varepsilon)} \\
&\leq \lim_{n \to \infty} \|f(t, u^{-1}(t, x)) - \hat{f}(t, u^{-1}(t, x))\|_{L^p(\hat{D}_\varepsilon)} + \lim_{n \to \infty} \|\hat{f}(t, u^{-1}(t, x)) - \hat{f}(t, u^{-1}(t, x))\|_{L^p(\hat{D}_\varepsilon)} \\
&\leq \lim_{n \to \infty} \|f(t, u^{-1}(t, x)) - \hat{f}(t, u^{-1}(t, x))\|_{L^p(\hat{D}_\varepsilon)} \\
&\leq \lim_{n \to \infty} 2\|f(t, u^{-1}(t, x)) - \hat{f}(t, u^{-1}(t, x))\|_{L^p(\hat{D}_\varepsilon)} \\
&\leq \lim_{n \to \infty} 4\|f(t, x) - \hat{f}(t, x)\|_{L^p(\hat{D}'_{\hat{M}_\varepsilon})} < 4\varepsilon_1.
\end{align*}
Proposition 3.2 For $n \geq N_0$, we have
\[
\gamma_n \in C^{1,1}_b([0, T_1] \times \mathbb{R}^d), \quad \gamma \in W^{1,2}_{2d+2}(\tilde{D}) \cap C^{d,1}_b([0, T_1] \times \mathbb{R}^d),
\]
and
\[
\lim_{n \to \infty} \|\gamma_n - \gamma\|_{C^{0,1}_b([0, T_1] \times \mathbb{R}^d)} = 0. \tag{3.13}
\]
Moreover, for any $\varepsilon > 0$ and $n \geq N_0(\varepsilon)$, we have $\gamma_n \in C^{1,2}_b(\tilde{D}_\varepsilon)$ and
\[
\sup_{\varepsilon > 0} \sup_{n \geq N_0(\varepsilon)} \|\gamma_n\|_{W^{1,2}_{2d+2}(\tilde{D}_\varepsilon)} < \infty, \tag{3.15}
\]
where $N_0(\varepsilon)$ was defined in Lemma 3.2.

**Proof:** It is evident that $\gamma \in W^{1,2}_{2d+2}(\tilde{D})$ if (3.13), (3.15) hold. Hence by Lemma 3.2 we only need to prove (3.14) and (3.15).

First we show (3.14). By (2.7), Lemma 3.2 and a change of variable, we see that for any $1 \leq i, j \leq d,$
\[
\lim_{n \to \infty} \|\partial_{x_i} \partial_{x_j} u_n(t, u_n^{-1}(t, x)) - \partial_{x_i} \partial_{x_j} u(t, u^{-1}(t, x))\|_{L^{2d+2}(\tilde{D}_\varepsilon)} \\
\leq \lim_{n \to \infty} \|\partial_{x_i} \partial_{x_j} u_n(t, u_n^{-1}(t, x)) - \partial_{x_i} \partial_{x_j} u(t, u_n^{-1}(t, x))\|_{L^{2d+2}(\tilde{D}_\varepsilon)} \\
+ \lim_{n \to \infty} \|\partial_{x_i} \partial_{x_j} u(t, u_n^{-1}(t, x)) - \partial_{x_i} \partial_{x_j} u(t, u^{-1}(t, x))\|_{L^{2d+2}(\tilde{D}_\varepsilon)} \\
= \lim_{n \to \infty} \|\partial_{x_i} \partial_{x_j} u_n(t, u_n^{-1}(t, x)) - \partial_{x_i} \partial_{x_j} u(t, u_n^{-1}(t, x))\|_{L^{2d+2}(\tilde{D}_\varepsilon)} \\
\leq 2 \lim_{n \to \infty} \|\partial_{x_i} \partial_{x_j} u_n(t, u_n^{-1}(t, x)) - \partial_{x_i} \partial_{x_j} u(t, u_n^{-1}(t, x))\|_{L^{2d+2}(\tilde{D}_\varepsilon)} \\
\leq 2 \lim_{n \to \infty} \|\partial_{x_i} \partial_{x_j} u(t, x) - \partial_{x_i} \partial_{x_j} u(t, x)\|_{L^{2d+2}(\tilde{D}_\varepsilon)} = 0.
\]

Combining this with (2.7), (3.4) and (3.9), we obtain
\[
\lim_{n,m \to \infty} \|\partial_{x_i} \partial_{x_j} \gamma_n - \partial_{x_i} \partial_{x_j} \gamma_m\|_{L^{2d+2}(\tilde{D}_\varepsilon)} = 0. \tag{3.16}
\]
By (3.9), similar to the proof of (3.16), we also have
\[
\lim_{n,m \to \infty} \|\partial \gamma_n - \partial \gamma_m\|_{L^{2d+2}(\tilde{D}_\varepsilon)} = 0.
\]
Hence (3.14) follows.

Now we show (3.15). Note that by Lemma 3.2 and a change of variable,
\[
\sup_{\varepsilon > 0, n \geq N_0(\varepsilon)} \|\partial_{x_i} \partial_{x_j} u_n(t, u_n^{-1}(t, x))\|_{L^{2d+2}(\tilde{D}_\varepsilon)} + \|\partial_t u_n(t, u_n^{-1}(t, x))\|_{L^{2d+2}(\tilde{D}_\varepsilon)} \\
\leq 2 \sup_{\varepsilon > 0, n \geq N_0(\varepsilon)} \|\partial_{x_i} \partial_{x_j} u_n(t, x)\|_{L^{2d+2}(\tilde{D}_\varepsilon)} + \|\partial_t u_n(t, x)\|_{L^{2d+2}(\tilde{D}_\varepsilon)} \\
\leq 2 \sup_{n \geq 1} \|u_n(t, x)\|_{W^{1,2}_{2d+2}(0, T_1) \times \tilde{D}} < \infty,
\]
combining this with (2.7), (3.9) and the boundness of $\|\nabla x u_n^{-1}(t, x)\|$, we obtain (3.15).
Remark 3.1 As $u_n$ is not in $C^{1,2}((0, T_1) \times \mathbb{R})$, $\gamma_n$ does not belong to $C^{1,2}(\bar{D})$. 

By (3.6), (3.7) and Theorem 7.9 in [3], the following lemma is immediate.

Lemma 3.3 For any $F \in W^{1,2}_{2d+2}(\bar{D}) \cap C^{0,1}_{b}(\bar{D})$, we have

$$F(t, u(t, x)) \in W^{1,2}_{2d+2}((0, T_1) \times D),$$

and moreover the chain rule of weak differentiation holds for $F(t, u(t, x))$.

We close this section by showing the following Lemma. The estimates listed will be used in later sections.

Lemma 3.4 Fix $\theta_1 \in (0, \frac{\pi}{2} \wedge \arctan \frac{1}{2})$ satisfying

$$\cos^2 \theta_1 + \left( \frac{\cos \theta_1 - (2 - 2 \cos \theta_1)^{\frac{1}{2}}}{1 + 12 \tan \theta_1} - (2 - 2 \cos \theta_1)^{\frac{1}{2}} \right)^2 \geq 1, \quad (3.17)$$

$$\frac{\cos \theta_1 - (2 - 2 \cos \theta_1)^{\frac{1}{2}}}{1 + 12 \tan \theta_1} - (2 - 2 \cos \theta_1)^{\frac{1}{2}} \geq \cos \frac{\theta_0}{2}, \quad (3.18)$$

$$\frac{(1 - 4 \tan^2 \theta_1)^{\frac{1}{2}} \cos \theta_1 - 2 \tan \theta_1 - \frac{1}{2}}{(1 + 4 \tan^2 \theta_1 + 2 \tan \theta_1 - (1 - 4 \tan^2 \theta_1)^{\frac{1}{2}} \cos \theta_1)^{\frac{1}{2}}} > \cos \theta_0, \quad (3.19)$$

Then there exist constants $0 < \delta_3 < \delta_2, \eta_0 > 0$ and an integer $N_1 \geq N_0$ such that for any $t_0 \in [0, T_1]$, $z_0 \in u(t_0, \partial D)$ and $n \geq N_1$, if $t, t' \in [(t_0 - \eta_0) \vee 0, (t_0 + \eta_0) \wedge T_1]$ and $x, x' \in B(z_0, \delta_3)$, then $|\gamma_n(t, x)| = |\gamma(t, x)| = 1$ and

$$\gamma(t, x) \cdot \gamma_n(t', x') \geq \cos \theta_1, \quad \gamma(t, x) \cdot \gamma(t', x') \geq \cos \theta_1, \quad \gamma_n(t, x) \cdot \gamma_n(t', x') \geq \cos \theta_1. \quad (3.21)$$

Proof: Recall that $\Gamma_c = \{ x \in \mathbb{R}^d : d(x, \partial D) < c \}$ for $c > 0$ and $|n(x)| = 1$ on $\Gamma_{\frac{4}{5}}$. By (2.8) and (3.4), there exists a constant $\delta \in (0, \delta_2)$ such that for sufficiently large $n$ and for $t \in [0, T_1]$, if $d(x, u(t, \partial D)) < \delta$, then $(t, x) \in G_1$ and $u_n^{-1}(t, x) \in \Gamma_{\frac{4}{5}}$. It implies that $|\gamma_n(t, x)| = |\gamma(t, x)| = 1$ by the definition of $\gamma_n$ and $\gamma$. (3.21) follows from (3.13).

4 Flows associated with the time dependent reflecting directions

In this section, we consider the flows associated with the time dependent vector fields of reflecting directions. We will provide a number of regularity results of the hitting times of the flows on certain hyperplane. These hitting times will be used to construct test functions in next section for proving the pathwise uniqueness of the transformed reflecting SDEs.

Let $N_0$ and $T_1$ be fixed as in Section 3. For $(t, x) \in [0, T_1] \times \mathbb{R}^d$ and $n \geq N_0$, let $y(t, x, \cdot)$ be the solution of the following ordinary differential equation:

$$\begin{cases}
y(t, x, 0) = x, \\
\partial_r y(t, x, r) = \gamma(t, y(t, x, r)), \quad r \in \mathbb{R}.
\end{cases}$$
and \( y_n(t, x, \cdot) \) the solution of the following ordinary differential equation:

\[
\begin{aligned}
&\begin{cases}
y_n(t, x, 0) = x, \\
\partial_r y_n(t, x, r) = \gamma_n(t, y_n(t, x, r)), \quad r \in \mathbb{R}.
\end{cases}
\end{aligned}
\]

Since \( \gamma \in C^{0,1}_b([0, T_1] \times \mathbb{R}^d) \) and \( \gamma_n \in C^{1,1}_b([0, T_1] \times \mathbb{R}^d) \) by Proposition 3.2, we see that \( y(\cdot, \cdot, \cdot) \) belongs to \( C^{0,1}_b([0, T_1] \times \mathbb{R}^d) \) and \( y_n(\cdot, \cdot, \cdot) \) belongs to \( C^{1,1}_b([0, T_1] \times \mathbb{R}^d \times \mathbb{R}) \). Moreover, 

\[ \psi^j_i(t, x, r) := \partial_x y^j_i(t, x, r) \]

is the solution to the following equation:

\[
\begin{aligned}
&\begin{cases}
\psi^j_i(t, x, 0) = \delta_i(j), \\
\partial_r \psi^j_i(t, x, r) = \sum_{1 \leq k \leq d} \partial_{y_k} \gamma^j_i(t, y(t, x, r)) \psi^k_i(t, x, r), \quad r \in \mathbb{R},
\end{cases}
\end{aligned}
\]  

(4.1)

\[ \psi^j_{n,i}(t, x, r) := \partial_x y^j_{n,i}(t, x, r) \]

is the solution to the following equation:

\[
\begin{aligned}
&\begin{cases}
\psi^j_{n,i}(t, x, 0) = \delta_i(j), \\
\partial_r \psi^j_{n,i}(t, x, r) = \sum_{1 \leq k \leq d} \partial_{y_k} \gamma^j_{n,i}(t, y_n(t, x, r)) \psi^k_{n,i}(t, x, r), \quad r \in \mathbb{R},
\end{cases}
\end{aligned}
\]  

(4.2)

and \( \Lambda_n(t, x, r) := \partial_t y_n(t, x, r) \) is the solution to the following equation:

\[
\begin{aligned}
&\begin{cases}
\Lambda_n(t, x, 0) = 0, \\
\partial_r \Lambda_n(t, x, r) = (\partial_t \gamma_n)(t, y_n(t, x, r)) + D_y \gamma_n(t, y_n(t, x, r)) \cdot \Lambda_n(t, x, r), \quad r \in \mathbb{R},
\end{cases}
\end{aligned}
\]  

(4.3)

where \( y^j_i(t, x, r) \) and \( y^j_{n,i}(t, x, r) \) are the \( j \)-th components of \( y(t, x, r) \) and \( y_n(t, x, r) \) respectively, \( \delta_i(j) := 1 \) if \( i = j \) and \( \delta_i(j) := 0 \) otherwise. By (3.13) and the Gronwall’s inequality, it is easy to see that for any \( e > 0 \) and \( 1 \leq i, j \leq d, \)

\[
\sup_{(t,x) \in [0,T_1] \times \mathbb{R}^d} \sup_{r \in (-c,c), n \geq N_0} |\psi^j_{n,i}(t, x, r)| < \infty,
\]  

(4.4)

\[
\lim_{n \to \infty} \sup_{(t,x) \in [0,T_1] \times \mathbb{R}^d} |y_n(t, x, r) - y(t, x, r)| = 0,
\]  

(4.5)

\[
\lim_{n \to \infty} \sup_{(t,x) \in [0,T_1] \times \mathbb{R}^d} |\psi^j_{n,i}(t, x, r) - \psi^j_i(t, x, r)| = 0.
\]  

(4.6)

First we have the following simple lemma.

**Lemma 4.1** There exists a constant \( \rho_0 \in (0, 1) \) such that for any \( n \geq N_0, r \in (-\rho_0, \rho_0) \), bounded measurable function \( f(x) \) and open set \( A \subset \mathbb{R}^d \), we have

\[
\int_A f(y_n(t, x, r)) dx \leq 2 \int_{y_n(t,A,r)} f(x) dx.
\]  

(4.7)

**Proof:** By (4.2), it is easy to see that for \( n \geq N_0 \) and \( r \in (-1, 1) \),

\[
\sup_{(t,x) \in [0,T_1] \times \mathbb{R}^d} |\psi^j_{n,i}(t, x, r) - \psi^j_{n,i}(t, x, 0)| \leq d |r| \|\gamma_n\|_{C^{0,1}_b([0,T_1] \times \mathbb{R}^d)} \sup_{(t,x) \in [0,T_1] \times \mathbb{R}^d} |\psi^k_{n,i}(t, x, \tau)| .
\]  

(4.8)

Since \( (\psi^j_{n,i}(t, x, 0))_{1 \leq i, j \leq d} \) is the identity matrix, it follows from (4.4) and (4.8) that there exists a constant \( \rho_0 \in (0, 1) \) such that for any \( n \geq N_0, (t,x) \in [0,T_1] \times \mathbb{R}^d \), and \( r \in (-\rho_0, \rho_0), \)
\[(\psi^j_n(t,x,r))_{1 \leq i,j \leq d} \geq \frac{1}{2}.\] Since \((\psi^j_n(t,x,r))_{1 \leq i,j \leq d}\) is the Jacobian matrix of \(y_n(t,\cdot,r)\), by a change of variable, we get \([4.7]\).

From now on, we fix \(t_0 \in [0, T_1]\) and \(z_0 \in u(t_0, \partial D)\). Set
\[
\rho_1 := \frac{\delta_3}{4} \wedge \rho_0,
\]
\[
H_{t_0,z} := \{ x \in \mathbb{R}^d : (x - z) \cdot \gamma(t_0, z) = 0 \},
\]
for some \(z \in \mathbb{R}^d\). Recall \(N_1\) is the constant defined in Lemma \ref{lem:4.2}. The next lemma states that in a neighborhood of \(z_0\), \(y(t, x, r)\) hits the hyperplane \(H_{t_0,z}\) at a unique point \(r = \Gamma^z(t, x)\), and so does \(y_n(t, x, r)\) for \(n \geq N_1\).

**Lemma 4.2** There exist constants \(\eta_1 \in (0, \eta_0)\) and \(\delta_4 \in (0, \frac{\delta_3}{2})\) such that for any \(n \geq N_1\), \((t, x) \in ((t_0 - \eta_1) \wedge 0, (t_0 + \eta_1) \wedge T_1) \times B(z_0, \delta_4)\) and \(z \in B(z_0, \delta_4)\) there exist unique \(\Gamma^z(t, x), \Gamma^z_n(t, x) \in (-\rho_1, \rho_1)\) such that \(y(t, x, \Gamma^z(t, x)), y_n(t, x, \Gamma^z_n(t, x)) \in H_{t_0,z}\). Moreover
\[
\begin{align*}
\Gamma^z_n(\cdot, \cdot) &\in C^{1,1}((t_0 - \eta_1) \wedge 0, (t_0 + \eta_1) \wedge T_1) \times B(z_0, \delta_4), \\
\Gamma^z(\cdot, \cdot) &\in C^{0,1}((t_0 - \eta_1) \wedge 0, (t_0 + \eta_1) \wedge T_1) \times B(z_0, \delta_4),
\end{align*}
\]
and
\[
\begin{align*}
\partial_x \Gamma^z_n(t, x) &= - (\gamma_n(t, y_n(t, x, \Gamma^z_n(t, x))) \cdot \gamma(t_0, z))^{-1} \sum_{1 \leq i \leq d} \psi^j_n(t, x, \Gamma^z_n(t, x)) \gamma^j(t_0, z), \\
\partial_t \Gamma^z_n(t, x) &= - (\gamma_n(t, y_n(t, x, \Gamma^z_n(t, x))) \cdot \gamma(t_0, z))^{-1} \Lambda_n(t, x, \Gamma^z_n(t, x)) \cdot \gamma(t_0, z), \\
\partial_x \Gamma^z(t, x) &= - (\gamma(t, y(t, x, \Gamma^z(t, x))) \cdot \gamma(t_0, z))^{-1} \sum_{1 \leq i \leq d} \psi^j(t, x, \Gamma^z(t, x)) \gamma^j(t_0, z), \\
\gamma_n(t, y_n(t, x, \Gamma^z_n(t, x))) \cdot \gamma(t_0, z) &\geq \cos \theta_1, \\
\lim_{n \to \infty} \sup_{(t,x) \in ((t_0-\eta_2)\wedge 0, (t_0+\eta_2)\wedge T_1) \times B(z_0, \delta_4)} |\Gamma^z_n(t, x) - \Gamma^z(t, x)| &= 0.
\end{align*}
\]

**Proof:** For \(n \geq N_1\), set \(H_{n,t}(x, z, r) := (y_n(t, x, r) - z) \cdot \gamma(t_0, z)\). Then for any \(x, z \in B(z_0, \frac{\delta_3}{2})\), \(t \in ((t_0 - \eta_1) \wedge 0, (t_0 + \eta_1) \wedge T_1)\) and \(r \in (-\frac{\delta_3}{2}, \frac{\delta_3}{2})\), we have \(|y_n(t, x, r) - z_0| < \delta_3\). Therefore by Lemma \ref{lem:3.4} we have
\[
\partial_r H_{n,t}(x, z, r) = \gamma_n(t, y_n(t, x, r)) \cdot \gamma(t_0, z) \geq \cos \theta_1 > 0.
\]
Note that \(H_{n,t}(z_0, z_0, 0) = 0\), hence applying the implicit function theorem to \(H_{n,t}(\cdot, \cdot, \cdot)\), there exist constants \(\eta_1 \in (0, \eta_0)\) and \(\delta_4 \in (0, \frac{\delta_3}{2})\) sufficiently small, such that for any \(n \geq N_1\) and \(t \in ((t_0 - \eta_1) \wedge 0, (t_0 + \eta_1) \wedge T_1)\), there exists a unique \(g_{n,t}(\cdot, \cdot, \cdot) \in C^{1,1}(B(z_0, \delta_4) \times B(z_0, \delta_4))\), such that \(H_{n,t}(x, z, g_{n,t}(x, z)) = 0\) and \(g_{n,t}(x, z) \in (-\rho_1, \rho_1)\) for \((x, z) \in B(z_0, \delta_4) \times B(z_0, \delta_4)\). Denote \(\Gamma^z_n(t, x) := g_{n,t}(x, z)\). Then it is easy to see that \([4.9], [4.11]\) and \([4.12]\) hold for any \(n \geq N_1\) and \(z \in B(z_0, \delta_4)\). Since \(|\Gamma^z_n(t, x)| < \rho_1 < \frac{\delta_3}{2}\), by \([4.16]\) we get \([4.14]\). Applying the implicit function theorem to \(H_{t}(x, z, r) := (y(t, x, r) - z) \cdot \gamma(t_0, z)\), we can choose common constants \(\eta_1, \delta_4 > 0\), such that for \((t, x) \in ((t_0 - \eta_1) \wedge 0, (t_0 + \eta_1) \wedge T_1) \times B(z_0, \delta_4)\) and \(r \in B(z_0, \delta_4)\), there exists a unique \(\Gamma^z(t, x) \in (-\rho_1, \rho_1)\) such that \((t, x, \Gamma^z(t, x)) \in H_{t_0,z}\). Moreover \([4.10]\) and \([4.13]\) hold.
Next we prove (4.15). By (4.5) we have
\[
\lim_{n \to \infty} \sup_{(t,x) \in ((t_0 - \eta_1) \vee 0, (t_0 + \eta_1) \wedge T_1) \times B(z_0, \delta_1)} |H_t(x, z, \Gamma^x_n(t, x)) - H_t(x, z, \Gamma^z(t, x))| = 0.
\] (4.17)

On the other hand, for \((t, x) \in ((t_0 - \eta_1) \vee 0, (t_0 + \eta_1) \wedge T_1) \times B(z_0, \delta_4)\) and \(r \in (-\rho_1, \rho_1)\), since \(|y(t, x, r) - z_0| \leq |r| + |x - z_0| < \delta_3\), together with Lemma 3.4 we have
\[
\sup_{(t,x) \in ((t_0 - \eta_1) \vee 0, (t_0 + \eta_1) \wedge T_1) \times B(z_0, \delta_4)} |\partial_r H_t(x, z, r)| \leq \frac{1}{\cos \theta_1} |\gamma(t, y(t, x, r) \cdot \gamma(t_0, z)| \geq \cos \theta_1.
\]
Hence, taking into account (4.17),
\[
|\Gamma^x_n(t, x) - \Gamma^x(t, x)| \leq \frac{1}{\cos \theta_1} |H_t(x, z, \Gamma^x_n(t, x)) - H_t(x, z, \Gamma^x(t, x))|,
\]
which yields (4.15). 

Define
\[
C(z, \delta) := \bigcup_{c \in \mathbb{R}} B(z - c\gamma(t_0, z), 2\delta \tan \theta_1) \cap B(z, \delta).
\] (4.18)
We have the following relationship.

**Lemma 4.3** For any \(\delta \in (0, \frac{\delta_1}{2}]\), \(t \in ((t_0 - \eta_1) \vee 0, (t_0 + \eta_1) \wedge T_1)\) and \(z := y(t_0, z_0, \frac{\delta}{2})\), we have
\[
|y(t, x, \Gamma^x(t, x)) - z| < 3\delta \tan \theta_1, \quad \forall x \in C(z, \delta),
\] (4.19)
and
\[
\{x \in B(z, \delta) : y(t, x, \Gamma^x(t, x)) \in B(z, \delta \tan \theta_1)\} \subset C(z, \delta).
\] (4.20)

**Proof:** Fix \(\delta \in (0, \frac{\delta_1}{2}]\), \(t \in ((t_0 - \eta_1) \vee 0, (t_0 + \eta_1) \wedge T_1)\), \(z := y(t_0, z_0, \frac{\delta}{2})\) and \(x \in B(z, \delta)\). Define \(P(\alpha) := (\alpha \cdot \gamma(t_0, z)\gamma(t_0, z)\) and \(Q(\alpha) := \alpha - P(\alpha)\) for \(\alpha \in \mathbb{R}^d\). It is easy to see that for any \(r \in (0, \Gamma^z(t, x))\), we have
\[
|y(t, x, r) - z_0| \leq |y(t, x, r) - x| + |x - z| + |z - z_0|
\leq |r| + \delta + \frac{\delta}{2} < \delta_3.
\]
Therefore by Lemma 3.4 and the fact that
\[
P(y(t, x, \Gamma^x(t, x)) - z) = 0,
\] (4.21)
we have
\[
|Q(\gamma(t, y(t, x, r)))|^2 = |\gamma(t, y(t, x, r)) - (\gamma(t, y(t, x, r)) \cdot \gamma(t_0, z))\gamma(t_0, z)|^2
\leq 1 - \cos^2 \theta_1
= \cos^2 \theta_1 \tan^2 \theta_1
\leq (\gamma(t, y(t, x, r)) \cdot \gamma(t_0, z))^2 \tan^2 \theta_1
= |P(\gamma(t, y(t, x, r)))|^2 \tan^2 \theta_1.
\]
and
\[ |P(y(t, x, \Gamma^z(t, x)) - x)| = |P(z - x) + P(y(t, x, \Gamma^z(t, x)) - z)| \]
\[ = |P(z - x)| \]
\[ \leq |z - x| < \delta. \]

Hence it follows that
\[ |Q(y(t, x, \Gamma^z(t, x)) - x)| \leq |\int_{0}^{\Gamma^z(t, x)} |Q(\gamma(y(t, x, r)))| dr| \]
\[ \leq |\int_{0}^{\Gamma^z(t, x)} |P(\gamma(y(t, x, r)))| \tan \theta_1 dr| \]
\[ = |\int_{0}^{\Gamma^z(t, x)} \gamma(t, y(t, x, r)) \cdot \gamma(t_0, z) dr| \tan \theta_1 \]
\[ = |(\int_{0}^{\Gamma^z(t, x)} \gamma(t, y(t, x, r)) dr \cdot \gamma(t_0, z))| \tan \theta_1 \]
\[ = |P(y(t, x, \Gamma^z(t, x)) - x)| \tan \theta_1 \]
\[ < \delta \tan \theta_1. \]

For \( x \in C(z, \delta) \), we easily see that \( |Q(z - x)| < 2\delta \tan \theta_1 \). Hence by (4.21) and (4.22) we get that
\[ |y(t, x, \Gamma^z(t, x)) - z| = |Q(y(t, x, \Gamma^z(t, x)) - z)| \]
\[ \leq |Q(y(t, x, \Gamma^z(t, x)) - x)| + |Q(z - x)| \]
\[ < \delta \tan \theta_1 + 2\delta \tan \theta_1 = 3\delta \tan \theta_1, \]
which is (4.19).

Next we prove (4.20). If \( x \in B(z, \delta) \setminus C(z, \delta) \), then we have
\[ |Q(z - x)| \geq 2\delta \tan \theta_1. \]

Hence by (4.21) and (4.22) we get that
\[ |y(t, x, \Gamma^z(t, x)) - z| = |Q(y(t, x, \Gamma^z(t, x)) - z)| \]
\[ \geq |Q(z - x)| - |Q(y(t, x, \Gamma^z(t, x)) - x)| \]
\[ > 2\delta \tan \theta_1 - \delta \tan \theta_1 = \delta \tan \theta_1, \]
which implies (4.20).

The following Lemma plays an important role in the proofs of Proposition 4.1 and Proposition 4.2, which establish the convergence of \( \Gamma^z_n(t, x) \) and \( y_n(t, x, \Gamma^z_n(t, x)) \) in some Sobolev spaces and provide further regularities of \( y(t, x, \Gamma^z(t, x)) \). Recall that
\[ D(t, c) = \{ x : d(x, u(t, D)^c) > c \}, \]
\( \theta_0 \) and \( N_0(\varepsilon) \) were defined in Proposition 5.1 and Lemma 8.2 respectively.
Lemma 4.4 There exist constants $\delta_5 \in (0, \delta_4)$ and $\eta_2 \in (0, \eta_1)$ such that for any $\varepsilon > 0$, there exists an integer $N_1(\varepsilon) > N_1 \vee N_0(\varepsilon)$ satisfying that for $z := y(t_0, z_0, \frac{\delta_5}{2})$, $n, m \geq N_1(\varepsilon)$, $t \in ((t_0 - \eta_2) \vee 0, (t_0 + \eta_2) \wedge T_1)$ and $x \in C(z, \delta_5) \cap D(t, \varepsilon)$, if $\Gamma^z(t, x) \neq 0$, then we have

$$y(t, x, r) \in D(t, (\varepsilon \sin \frac{\theta_0}{2}) \wedge (\frac{\delta_5}{16} \sin \frac{\theta_0}{2})) \text{ for } r \in (0, \Gamma^z(t, x)],$$

(4.23)

and

$$y_n(t, x, r) \in D(t, (\varepsilon \sin \frac{\theta_0}{2}) \wedge (\frac{\delta_5}{16} \sin \frac{\theta_0}{2})),$$

(4.24)

for $r \in (0, \Gamma^z_n(t, x)] \cup (\Gamma^z(t, x), \Gamma^z_m(t, x)] \cup (\Gamma^z_n(t, x), \Gamma^z(t, x])$.

Proof: Fix $\delta_5 := \frac{\delta_4 \sin \theta_0}{16}$, and $z := y(t_0, z_0, \frac{\delta_5}{2})$.

First we show that for $t \in ((t_0 - \eta_1) \vee 0, (t_0 + \eta_1) \wedge T_1)$ and $x \in C(z, \delta_5)$, if $\Gamma^z(t, x) \leq 0$ and $r \in [\Gamma^z(t, x), 0]$, then $d(y(t, x, r), u(t, D)^c) > \frac{\delta_5}{8} \sin \frac{\theta_0}{2}$.

By (4.19), we get

$$|y(t, x, \Gamma^z(t, x)) - z_0| \leq |y(t, x, \Gamma^z(t, x)) - z| + |z - z_0| \leq 3\delta_5 \tan \theta_1 + |z - z_0|. \quad (4.25)$$

On the other hand, by Lemma 3.4 and the fact that $\cos \theta_1 > \frac{1}{4}$ we have

$$|z - z_0|^2 = \int_0^{\frac{\delta_5}{2}} \gamma(t_0, y(t_0, z_0, r)) \cdot (z - z_0) dr$$

$$= \int_0^{\frac{\delta_5}{2}} dr \int_0^{\frac{\delta_5}{2}} \gamma(t_0, y(t_0, z_0, r)) \cdot \gamma(t_0, y(t_0, z_0, \tau)) d\tau$$

$$> \int_0^{\frac{\delta_5}{2}} dr \int_0^{\frac{\delta_5}{2}} \frac{1}{4} d\tau = \frac{\delta_5^2}{16},$$

(4.26)

i.e., $\frac{\delta_5}{|z - z_0|} < 4$. By Lemma 3.4 and (4.25) we have

$$\gamma(t_0, y(t_0, \Gamma^z(t, x)), z_0) \cdot \gamma(t_0, z_0)$$

$$= (z - z_0) \cdot \gamma(t_0, z_0) + (y(t, x, \Gamma^z(t, x)) - z) \cdot \gamma(t_0, z_0)$$

$$= \int_0^{\frac{\delta_5}{2}} \gamma(t_0, y(t_0, z_0, \tau)) \cdot \gamma(t_0, z_0) d\tau + (y(t, x, \Gamma^z(t, x)) - z) \cdot (\gamma(t_0, z_0) - \gamma(t_0, z))$$

$$\geq \frac{\delta_5}{2} \cos \theta_1 - |y(t, x, \Gamma^z(t, x)) - z| |\gamma(t_0, z_0) - \gamma(t_0, z)|$$

$$\geq |z - z_0| \cos \theta_1 - |y(t, x, \Gamma^z(t, x)) - z| (2 - 2 \cos \theta_1)^{\frac{1}{2}}$$

$$\geq |z - z_0| (\cos \theta_1 - (2 - 2 \cos \theta_1)^{\frac{1}{2}}) - |y(t, x, \Gamma^z(t, x)) - z_0|(2 - 2 \cos \theta_1)^{\frac{1}{2}}$$

$$\geq (\frac{|z - z_0| (\cos \theta_1 - (2 - 2 \cos \theta_1)^{\frac{1}{2}})}{|z - z_0| + 3\delta_5 \tan \theta_1} - (2 - 2 \cos \theta_1)^{\frac{1}{2}}) |y(t, x, \Gamma^z(t, x)) - z_0|$$

$$> (\frac{\cos \theta_1 - (2 - 2 \cos \theta_1)^{\frac{1}{2}}}{1 + 12 \tan \theta_1} - (2 - 2 \cos \theta_1)^{\frac{1}{2}}) |y(t, x, \Gamma^z(t, x)) - z_0|,$$

where the fact $\frac{\delta_5}{|z - z_0|} < 4$ has been used in the last inequality. Note that for $\tau \in [\Gamma^z(t, x), 0]$,

$$|y(t, x, \tau) - z_0| \leq |y(t, x, \tau) - x| + |x - z| + |z - z_0| < \delta_3,$$

(4.28)
which implies that \( \gamma(t, y(t, \tau, \gamma)) \cdot \gamma(t_0, z_0) \geq \cos \theta_1 \) by Lemma 3.4 Together with (3.17) and (4.27) we get \( \gamma(t, y(t, \tau, \gamma)) \cdot (y(t, \Gamma^z(t, x)) - z_0) \geq 0 \). Hence
\[
|y(t, x, r) - z_0|^2 = |y(t, x, r) - y(t, x, \Gamma^z(t, x))|^2 + |y(t, x, \Gamma^z(t, x)) - z_0|^2
+ 2(y(t, x, r) - y(t, x, \Gamma^z(t, x))) \cdot (y(t, x, \Gamma^z(t, x)) - z_0)
\geq |y(t, x, \Gamma^z(t, x)) - z_0|^2 + 2 \int_{\Gamma^z(t, x)} \gamma(t, y(t, x, \tau)) \cdot (y(t, x, \Gamma^z(t, x)) - z_0) d\tau
\geq |y(t, x, \Gamma^z(t, x)) - z_0|^2.
\] (4.29)

Combining (4.19), (4.26), (4.29) and the fact that \( \theta < \arctan \frac{1}{24} \), we deduce that
\[
|y(t, x, r) - z_0| \geq |y(t, x, \Gamma^z(t, x)) - z_0|
\geq |z - z_0| - |y(t, x, \Gamma^z(t, x)) - z|
\geq \frac{\delta_5}{4} - 3\delta_5 \tan \theta_1 > \frac{\delta_5}{8}.
\] (4.30)

By Lemma 3.4 (4.27) and (4.28), we have
\[
(y(t, x, r) - z_0) \cdot \gamma(t_0, z_0)
= (y(t, x, r) - y(t, x, \Gamma^z(t, x))) \cdot \gamma(t_0, z_0) + (y(t, x, \Gamma^z(t, x)) - z_0) \cdot \gamma(t_0, z_0)
\geq \int_{\Gamma^z(t, x)} \gamma(t, y(t, x, \tau)) \cdot \gamma(t_0, z_0) d\tau + \left( \frac{\cos \theta_1 - (2 - 2 \cos \theta_1)^{1/2}}{1 + 12 \tan \theta_1} - (2 - 2 \cos \theta_1)^{1/2} \right)
\times |y(t, x, \Gamma^z(t, x)) - z_0|
\geq (r - \Gamma^z(t, x)) \cos \theta_1 + \left( \frac{\cos \theta_1 - (2 - 2 \cos \theta_1)^{1/2}}{1 + 12 \tan \theta_1} - (2 - 2 \cos \theta_1)^{1/2} \right)
\times |y(t, x, \Gamma^z(t, x)) - z_0|
\geq |y(t, x, r) - y(t, x, \Gamma^z(t, x))| \cos \theta_1 + \left( \frac{\cos \theta_1 - (2 - 2 \cos \theta_1)^{1/2}}{1 + 12 \tan \theta_1} - (2 - 2 \cos \theta_1)^{1/2} \right)
\times |y(t, x, \Gamma^z(t, x)) - z_0|
\geq |y(t, x, r) - z_0| \cos \frac{\theta_0}{2},
\] (4.31)

where \( \theta_1 < \frac{\theta_0}{2} \) and (3.18) have been used for the last inequality. Now, (4.28) and (4.31) implies that
\[
y(t, x, r) \in C(z_0, \gamma(t_0, z_0), \frac{\theta_0}{2}, \delta_2) \subset u(t_0, D).
\]

Combining this with (4.30) we obtain that
\[
d(y(t, x, r), u(t_0, D)) \geq d(y(t, x, r), \partial C(z_0, \gamma(t_0, z_0), \theta_0, \delta_2)) > \frac{\delta_5}{8} \sin \frac{\theta_0}{2}. \tag{4.32}
\]

Note that \( \hat{D}_{\frac{\delta_5}{10}} \sin \frac{\theta_0}{2} = \{(t, y) : t \in (0, T_1), y \in D(t, \frac{\delta_5}{10} \sin \frac{\theta_0}{2})\} \) is an open set in \( \mathbb{R}^{d+1} \) by Lemma 3.2 and \( \{t_0\} \times A \subset \hat{D}_{\frac{\delta_5}{10}} \sin \frac{\theta_0}{2} \) by (4.32), where
\[
A := \{y(t, x, r) : t \in ((t_0 - \eta_1) \vee 0, (t_0 + \eta_1) \wedge T_1), x \in C(z, \delta_3), \Gamma^z(t, x) \leq 0, r \in [\Gamma^z(t, x), 0] \}.
\]
Hence there exists a $\eta_2 \in (0, \eta_1 \wedge (\frac{\delta_1 \sin \theta_0}{8 \delta_0})^{1/\alpha_0})$ such that

$$(t_0 - \eta_2) \vee 0, (t_0 + \eta_2) \wedge T_1) \times A \subset \tilde{D}_{\frac{\delta_0 \sin \theta_0}{2}}.$$  

Obviously, for $t \in ((t_0 - \eta_2) \vee 0, (t_0 + \eta_2) \wedge T_1)$, $x \in C(z, \delta_5)$, if $\Gamma^z(t, x) \leq 0$ and $r \in [\Gamma^z(t, x), 0]$, then

$$d(y(t, x, r), u(t, D)^c) > \frac{\delta_5}{16} \sin \frac{\theta_0}{2}.$$  

(4.33)

Next we will prove (4.23). By (4.33) we just need to show that for $\varepsilon > 0$, $t \in ((t_0 - \eta_2) \vee 0, (t_0 + \eta_2) \wedge T_1)$ and $x \in C(z, \delta_5) \cap D(t, \varepsilon)$, if $\Gamma^z(t, x) > 0$ and $r \in (0, \Gamma^z(t, x)]$, then $d(y(t, x, r), u(t, D)^c) > \varepsilon \sin \frac{\theta_0}{2}$.

Set $z_0 := u(t, u^{-1}(t_0, z_0)) \in u(t, \partial D)$ and $F(a) := x + a \gamma(t, z_0)$ for $a \in \mathbb{R}$. Obviously, there exists a constant $a_1 < 0$ such that $F(a_1) \in \partial \cup_{r > 0} C(z_0, -\gamma(t, z_0), \theta_0, r)$). Noting $\eta_2 < (\frac{\delta_1 \sin \theta_0}{8 \delta_0})^{1/\alpha_0}$ and $\delta_5 = \frac{\delta_1 \sin \theta_0}{8}$, by (2.4) we have

$$|z_0 - z_0'| = |u(t_0, u^{-1}(t_0, z_0)) - u(t, u^{-1}(t_0, z_0))|$$

$$\leq M_0|t_0 - t|^\alpha_0 < \frac{\delta_1 \sin \theta_0}{8},$$  

and

$$|x - z_0'| \leq |x - z| + |z - z_0| + |z_0 - z_0'|$$

$$< 2\delta_5 + \frac{\delta_1 \sin \theta_0}{4} = \frac{\delta_1 \sin \theta_0}{4}. $$

Therefore

$$|F(a_1) - z_0'| = \frac{d(z_0', \{F(a) : a \in \mathbb{R}\})}{\sin \theta_0} \leq \frac{|x - z_0'|}{\sin \theta_0} < \frac{\delta_1}{4},$$

which implies that $F(a_1) \in \tilde{C}(z_0', -\gamma(t, z_0), \theta_0, \delta_2) \subset u(t, D)^c$. Since

$$F(a_1) \in u(t, D)^c \cap B(z_0', \frac{\delta_1}{4})$$

and

$$F(0) = x \in u(t, D) \cap B(z_0', \frac{\delta_1}{4}),$$

there exists a constant $a_2 \in (a_1, 0)$ such that $F(a_2) \in u(t, \partial D) \cap B(z_0', \frac{\delta_1}{4})$.

For any $\tau \in (0, \Gamma^z(t, x)]$, since $\tau < \rho_1 \leq \frac{\delta_1}{4}$ and $\delta_4 < \frac{\delta_1}{4}$, we have

$$|y(t, x, \tau) - F(a_2)| \leq |y(t, x, \tau) - x| + |x - F(a_2)|$$

$$\leq \tau + |x - F(a_2)|$$

$$\leq \tau + |x - F(a_1)|$$

$$\leq \tau + |x - z_0'| + |F(a_1) - z_0'|$$

$$< \tau + \frac{\delta_1 \sin \theta_0}{4} + \frac{\delta_1}{4} < \frac{\delta_3}{2},$$  

(4.35)
Combining this with Lemma 3.4, (4.34) and (4.36), we have
\[ |y(t, x, \tau) - z_0| \leq |y(t, x, \tau) - F(a_2)| + |F(a_2) - z'_0| + |z'_0 - z_0| \]
\[ < \frac{\delta_3}{2} + \frac{\delta_4}{4} + \frac{\delta_4 \sin \theta_0}{8} < \delta_3. \]  
(4.36)

Together with Lemma 3.4 (4.34) and the fact that \( x \in D(t, \varepsilon) \), we deduce that
\[ |y(t, x, r) - F(a_2)|^2 \]
\[ = |y(t, x, r) - x|^2 + |x - F(a_2)|^2 + 2(-a_2) \int_0^r \gamma(t, y(t, x, \tau)) \cdot \gamma(t, z'_0) d\tau \]
\[ \geq |x - F(a_2)|^2 \geq d(x, u(t, D)^c)^2 > \varepsilon^2. \]  
(4.37)

On the other hand, note that \( |F(a_2) - z_0| \leq |F(a_2) - z'_0| + |z'_0 - z_0| < \frac{\delta_4}{4} + \frac{\delta_4 \sin \theta_0}{8} < \delta_3. \)
Combining this with Lemma 3.4 (4.34) and (4.36), we have
\[ (y(t, x, r) - F(a_2)) \cdot \gamma(t, F(a_2)) \]
\[ = (y(t, x, r) - x) \cdot \gamma(t, F(a_2)) + (x - F(a_2)) \cdot \gamma(t, F(a_2)) \]
\[ = \int_0^r \gamma(t, y(t, x, \tau)) \cdot \gamma(t, F(a_2)) d\tau + (-a_2) \gamma(t, z'_0) \cdot \gamma(t, F(a_2)) \]
\[ \geq r \cos \theta_1 + (-a_2) \cos \theta_1 \]
\[ \geq \cos \theta_1 |y(t, x, r) - x| + \cos \theta_1 |x - F(a_2)| \]
\[ \geq \cos \theta_1 |y(t, x, r) - F(a_2)| \geq \cos \frac{\theta_0}{2} |y(t, x, r) - F(a_2)|. \]

Together with (4.35), we have \( y(t, x, r) \in C(F(a_2), \gamma(t, F(a_2)), \frac{\theta_0}{2}, \delta_2) \subset u(t, D) \). Hence combining this with (4.37) we obtain that
\[ d(y(t, x, r), u(t, D)^c) \geq d(y(t, x, r), \partial C(F(a_2), \gamma(t, F(a_2)), \theta_0, \delta_2)) \]
\[ \geq |y(t, x, r) - F(a_2)| \sin \frac{\theta_0}{2} > \varepsilon \sin \frac{\theta_0}{2}. \]

Finally we prove (4.24). By (4.5), there exists an integer \( \tilde{N}(\varepsilon) > N_1 \lor N_0(\varepsilon) \) such that for \( n > \tilde{N}(\varepsilon) \),
\[ |y_n(t, x, r) - y(t, x, r)| < \left( \frac{\varepsilon}{4} \sin \frac{\theta_0}{2} \right) \land \left( \frac{3\delta_5}{64} \sin \frac{\theta_0}{2} \right) \text{ for } r \in (0, \Gamma^z(t, x)]. \]

Using this and (4.23), we see that
\[ y_n(t, x, r) \in D(t, \left( \frac{3\varepsilon}{4} \sin \frac{\theta_0}{2} \right) \land \left( \frac{3\delta_5}{64} \sin \frac{\theta_0}{2} \right)) \text{ for } r \in (0, \Gamma^z(t, x)]. \]  
(4.38)

By (4.15), (4.38) and noting
\[ \sup_{(t, x) \in [0, \tau_1] \times \mathbb{R}^d, \ \tau \in (-1, x)_\infty, n \geq N_0} \ \left| \partial_r y_n(t, x, \tau) \right| < \infty, \]
for any \( \varepsilon > 0 \), there exists an integer \( N_1(\varepsilon) > \tilde{N}(\varepsilon) \) such that for \( n, m > N_1(\varepsilon) \) and \( r \in (\Gamma^z(t, x), \Gamma^z_n(t, x)] \cup (\Gamma^z_n(t, x), \Gamma^z_m(t, x)] \), we have
\[ |y_n(t, x, r) - y_n(t, x, \Gamma^z(t, x))| < \left( \frac{\varepsilon}{4} \sin \frac{\theta_0}{2} \right) \land \left( \frac{3\delta_5}{64} \sin \frac{\theta_0}{2} \right), \]
and $d(y_n(t, x, \Gamma^z(t, x)), u(t, D)^c) > \left(\frac{3\varepsilon}{4} \sin \frac{\theta_0}{2}\right) \setminus \left(\frac{3\varepsilon}{64} \sin \frac{\theta_0}{2}\right)$. Hence

$$d(y_n(t, x, r), u(t, D)^c) > d(y_n(t, x, \Gamma^z_n(t, x)), u(t, D)^c) - |y_n(t, x, r) - y_n(t, x, \Gamma^z_n(t, x))|$$

$$> \left(\frac{\varepsilon}{2} \sin \frac{\theta_0}{2}\right) \setminus \left(\frac{\delta_5}{32} \sin \frac{\theta_0}{2}\right).$$

Together with (4.38), we obtain (4.24). ■

From now on, we fix $z := y(t_0, z_0, \delta_5)$. Recall that $\psi^j_i$ and $\Lambda_n$ were defined in (4.1) and (4.2) respectively, $D(t, \varepsilon)$ and $C(z, \delta_5)$ were defined in (3.3) and (4.13) respectively. Set

$$O_\varepsilon := \{(t, x) : t \in ((t_0 - \eta_2) \wedge 0, (t_0 + \eta_2) \wedge T_1), x \in C(z, \delta_5) \cap D(t, \varepsilon)\}.$$

The regularity of $\Gamma^z(t, x)$ and the convergence of $\Gamma^z_n(t, x)$ are stated in the following two propositions. The proofs of theses results are quite lengthy. They are put in the appendix.

**Proposition 4.1** For any $1 \leq i, j \leq d$,

$$\psi^j_i(\cdot, \cdot, \Gamma^z(\cdot, \cdot)) \in W^{0,1}_{2d+2}(((t_0 - \eta_2) \wedge 0, (t_0 + \eta_2) \wedge T_1) \times C(z, \delta_5)) \cap \tilde{D},$$

and

$$\Gamma^z(\cdot, \cdot) \in W^{0,1}_{2d+2}(((t_0 - \eta_2) \wedge 0, (t_0 + \eta_2) \wedge T_1) \times C(z, \delta_5) \cap \tilde{D}).$$

**Proposition 4.2** Let $N_1(\varepsilon)$ be given in Lemma 4.4, then

$$\sup_{\varepsilon > 0} \sup_{n \geq N_1(\varepsilon)} \|\Lambda_n(\cdot, \cdot, \Gamma^z_n(\cdot, \cdot))\|_{L^{2d+2}(O_\varepsilon)} < \infty.$$  

Moreover, for any $\varepsilon > 0$, we have

$$\lim_{n, m \to \infty} \|\Lambda_n(\cdot, \cdot, \Gamma^z_n(\cdot, \cdot)) - \Lambda_m(\cdot, \cdot, \Gamma^z_m(\cdot, \cdot))\|_{L^{2d+2}(O_\varepsilon)} = 0,$$

and

$$\lim_{n, m \to \infty} \|\partial_t \Gamma^z_n(\cdot, \cdot) - \partial_t \Gamma^z_m(\cdot, \cdot)\|_{L^{2d+2}(O_\varepsilon)} = 0.$$  

**Remark 4.1** From (4.12), (4.14), (4.40), (4.41) and (4.43), we see that

$$\Gamma^z(\cdot, \cdot) \in W^{1,2}_{2d+2}(((t_0 - \eta_2) \wedge 0, (t_0 + \eta_2) \wedge T_1) \times C(z, \delta_5) \cap \tilde{D}).$$

5  

**Construction of test functions**

In this section, we will construct a family of auxiliary functions which will be used to prove the pathwise uniqueness of the solutions of reflecting stochastic differential equations. Recall that $\theta_1$ was defined in Lemma 3.4. Let $t_0$, $z_0$, $\delta_5$, $\eta_2$ and $z := y(t_0, z_0, \delta_5)$ be defined as in Section 4.

**Lemma 5.1** Let $u_0 \in C^0_0(B(z, \delta_5 \tan \theta_1) \cap H_{t_0, z})$ be nonnegative with $u_0(z) = 1$. Define $h(t, x) := u_0(y(t, x, \Gamma^z(t, x)))$. Then

(i). $h(t, t_0) = 1$,

(ii). $B(z, \delta_5) \cap \text{supp } h(t, \cdot) \subset C(z, \delta_5)$ for $t \in ((t_0 - \eta_2) \wedge 0, (t_0 + \eta_2) \wedge T_1)$,

(iii). $h$ belongs to the following space:

$$C^0_0(((t_0 - \eta_2) \wedge 0, (t_0 + \eta_2) \wedge T_1) \times B(z, \delta_5) \cap \tilde{D}) \cap \mathcal{W}^{1,2}_{2d+2}(((t_0 - \eta_2) \wedge 0, (t_0 + \eta_2) \wedge T_1) \times B(z, \delta_5) \cap \tilde{D}).$$
Proof: By Lemma 4.3, the choice of \( u_0 \) and the definition of \( \Gamma^z(t, x) \), we see that
\[
h(t_0, z_0) = h(t_0, z) = u_0(z) = 1,
\]
and
\[
B(z, \delta_5) \cap \text{supp } h(t, \cdot) \subset C(z, \delta_5), \quad \forall t \in ((t_0 - \eta_2) \lor 0, (t_0 + \eta_2) \land T_1), \quad (5.1)
\]
hence (i) and (ii) are proved.

By Lemma 5.1 and the fact that \( \gamma \in C_b^{0,1}([0, T_1] \times \mathbb{R}^d) \), we have for \( 1 \leq i \leq d \),
\[
h \in C_b^{0,1}(((t_0 - \eta_2) \lor 0, (t_0 + \eta_2) \land T_1) \times B(z, \delta_5)),
\]
and
\[
\partial_{x_i} h(t, x) = \sum_{1 \leq j \leq d} \partial_{y_j} u_0(g(t, x, \Gamma^z(t, x)))[\psi_j^1(t, x, \Gamma^z(t, x)) + \gamma(t, y(t, x, \Gamma^z(t, x))) \partial_{x_i} \Gamma^z(t, x)].
\]
From Proposition 4.1 and (5.1), it follows that
\[
h \in W^{0,2}_{2d+2}(((t_0 - \eta_2) \lor 0, (t_0 + \eta_2) \land T_1) \times B(z, \delta_5) \cap \hat{D}).
\]

Set \( h_n(t, x) := u_0(y_n(t, x, \Gamma_n^z(t, x))) \). Then \( h_n(t, x) \) converges to \( h(t, x) \) uniformly on \(((t_0 - \eta_2) \lor 0, (t_0 + \eta_2) \land T_1) \times B(z, \delta_5) \) and
\[
\partial_t h_n(t, x) = \nabla_y u_0(y_n(t, x, \Gamma_n^z(t, x))) \cdot \Lambda_n(t, x, \Gamma_n^z(t, x))
+ \nabla_y u_0(y_n(t, x, \Gamma_n^z(t, x))) \cdot \gamma_n(t, y_n(t, x, \Gamma_n^z(t, x))) \partial_t \Gamma_n^z(t, x). \quad (5.2)
\]
By (4.12), (4.14) and (4.11), we have
\[
\sup_{\varepsilon > 0} \sup_{n \geq N_1(\varepsilon)} \| \partial_t \Gamma_n^z \|_{L^{2d+2}(\mathcal{O}_\varepsilon)} < \infty. \quad (5.3)
\]
Therefore by (4.11), (5.2) and (5.3) we have \( \sup_{\varepsilon > 0} \sup_{n \geq N_1(\varepsilon)} \| \partial_t h_n \|_{L^{2d+2}(\mathcal{O}_\varepsilon)} < \infty \). On the other hand, by (4.12) and (4.13), we have for any \( \varepsilon > 0 \),
\[
\lim_{n,m \to \infty} \| \partial_t h_n - \partial_t h_m \|_{L^{2d+2}(\mathcal{O}_\varepsilon)} = 0.
\]
Hence \( h \in W^{0,1}_{2d+2}(((t_0 - \eta_2) \lor 0, (t_0 + \eta_2) \land T_1) \times B(z, \delta_5) \cap \hat{D}) \). The proof of (iii) is complete by combining the above statements about \( h \) together.

Now, we start to construct the first important class of test functions. The construction is inspired by [2] and [9].

Proposition 5.1 There exists a nonnegative function \( H \in C_b^{0,1}([0, T_1] \times \mathbb{R}^d) \cap W^{1,2}_{2d+2}(\hat{D}) \) such that for any \( t \in [0, T_1] \) and \( x \in u(t, \partial D) \)
\[
\nabla_x H(t, x) \cdot \gamma(t, x) \geq 1. \quad (5.4)
\]
Proof: By Lemma 5.1 we know that for any given \( t_0 \in [0, T_1] \) and \( z_0 \in u(t_0, \partial D) \), there exists a nonnegative function \( h(t, x) := u_0(y(t, x, \Gamma^z(t, x))) \) with \( h(t_0, z_0) = 1 \) and \( h \) belongs to the following space:
\[
C_b^{0,1}(((t_0 - \eta_2) \lor 0, (t_0 + \eta_2) \land T_1) \times B(z, \delta_5)) \cap W^{1,2}_{2d+2}(((t_0 - \eta_2) \lor 0, (t_0 + \eta_2) \land T_1) \times B(z, \delta_5) \cap \hat{D}),
\]

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where \( \delta_5, \eta_2 \) are dependent of \((t_0, z_0)\), and \( z := y(t_0, z_0, \frac{4\pi}{2}) \). Using the method of characteristics, we know that \( h(t, x) \) is the solution to the following Cauchy problem:

\[
\begin{align*}
\nabla_x h(t, x) \cdot \gamma(t, x) &= 0, \\
\gamma h(t, \hat{x})_{|H_{\hat{x}}}_{z} &= u_0.
\end{align*}
\]

(5.5)

By (3.19) and (3.20), there exists a constant \( \kappa \in (\frac{1}{2}, 1) \) such that

\[
\frac{(\kappa^2 - 4 \tan^2 \theta_1)^\frac{1}{2} \cos \theta_1 - 2 \tan \theta_1 - \frac{1}{2}}{(\frac{2}{4} + 4 \tan^2 \theta_1 + 2 \tan \theta_1 - (\kappa^2 - 4 \tan^2 \theta_1)^\frac{1}{2} \cos \theta_1)^\frac{1}{2}} > \cos \theta_0,
\]

(5.6)

and

\[
\frac{(\kappa^2 - 4 \tan^2 \theta_1)^\frac{1}{2} \cos \theta_1 - 2 \tan \theta_1 + \frac{1}{2} \cos \theta_1}{(\frac{2}{4} + 4 \tan^2 \theta_1 + 2 \tan \theta_1)^\frac{1}{2}} > \cos \theta_0.
\]

(5.7)

Now we show that

\[
u(t_0, \partial D) \bigcap (\bar{C}(z, \delta_5) \setminus B(z, \kappa \delta_5)) = \emptyset,
\]

(5.8)

where \( C(z, \delta_5) \) was defined in (4.18) and \( \bar{C}(z, \delta_5) \) is the closure of \( C(z, \delta_5) \).

Let \( x \in \bar{C}(z, \delta_5) \setminus B(z, \kappa \delta_5) \), \( \sigma := (x - z) \cdot \gamma(t_0, z) \) and \( \beta := x - z - \sigma \gamma(t_0, z) \). Then it is easy to see that

\[
|x - z_0| \leq |x - z| + |z - z_0| \leq \delta_5 + \frac{\delta_5}{2} < \delta_2,
\]

(5.9)

and

\[
|x - z_0| \geq |x - z| - |z - z_0| \geq \kappa \delta_5 - \frac{\delta_5}{2} > 0.
\]

(5.10)

Since \( x \in \bar{C}(z, \delta_5) \) we have

\[
|\beta| \leq 2 \delta_5 \tan \theta_1.
\]

(5.11)

Hence \( \delta_5^2 \geq |x - z|^2 \geq |\sigma|^2 = |x - z|^2 - |\beta|^2 \geq \kappa^2 \delta_5^2 - 4 \delta_5^2 \tan^2 \theta_1 \), which means that

\[
\frac{|\sigma|}{\delta_5} \in [(\kappa^2 - 4 \tan^2 \theta_1)^\frac{1}{2}, 1].
\]

(5.12)

If \( \sigma \leq 0 \), then by (5.11),

\[
|x - z_0|^2 \\
= |x - z|^2 + |z - z_0|^2 + 2 \int_{0}^{\frac{4\pi}{2}} (x - z) \cdot \gamma(t_0, y(t_0, z_0, \tau))d\tau \\
\leq \sigma^2 + |\beta|^2 + |z - z_0|^2 + 2 \int_{0}^{\frac{4\pi}{2}} |\beta||\gamma(t_0, y(t_0, z_0, \tau))|d\tau + 2\sigma \int_{0}^{\frac{4\pi}{2}} \gamma(t_0, z) \cdot \gamma(t_0, y(t_0, z_0, \tau))d\tau \\
\leq \sigma^2 + 4 \delta_5^2 \tan^2 \theta_1 + \frac{\delta_5^2}{4} + 2 \delta_5^2 \tan \theta_1 - |\sigma| \delta_5 \cos \theta_1,
\]

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which implies that
\[
\frac{|x - z_0|^2}{\delta_5^2} \leq \frac{\sigma^2}{\delta_5^2} + 4 \tan^2 \theta_1 + \frac{1}{4} + 2 \tan \theta_1 - \frac{|\sigma|}{\delta_5} \cos \theta_1.
\]
Combining this with (5.6), (5.11) and (5.12), we have
\[
(x - z_0) \cdot (\gamma(t_0, z_0) - (z - z_0) \cdot \gamma(t_0, z_0)) \\
= -\sigma \gamma(t_0, z) \cdot \gamma(t_0, z_0) - \beta \cdot \gamma(t_0, z_0) - |\beta| - |z - z_0| \\
\geq (|\sigma| \gamma(t_0, z) \cdot \gamma(t_0, z_0) - |\beta| - |z - z_0|) \\
\geq \left(\frac{|\sigma|}{\delta_5} \cos \theta_1 - 2 \tan \theta_1 - \frac{1}{2}\right) \delta_5 \\
\geq \frac{|\sigma|}{\delta_5} \cos \theta_1 - 2 \tan \theta_1 - \frac{1}{2} \\
\geq \left(\frac{\sigma^2}{\delta_5^2} + 4 \tan^2 \theta_1 + \frac{1}{4} + 2 \tan \theta_1 - \frac{|\sigma|}{\delta_5} \cos \theta_1\right) \frac{1}{2} |x - z_0| \\
\geq \frac{(\kappa^2 - 4 \tan^2 \theta_1) \frac{1}{2} \cos \theta_1 - 2 \tan \theta_1 + \frac{1}{2} \cos \theta_1}{\left(\frac{\sigma^2}{\delta_5^2} + 4 \tan^2 \theta_1 + 2 \tan \theta_1\right)^{\frac{1}{2}}} |x - z_0| \\
> |x - z_0| \cos \theta_0.
\]
This shows that \(x \in C(z_0, -\gamma(t_0, z_0), \delta_2, \theta_0) \subset u(t_0, D)^\circ\), which in particular implies \(x \not \in u(t_0, \partial D)\).

If \(\sigma > 0\), then by (5.11)
\[
|x - z_0|^2 \\
= |x - z|^2 + |z - z_0|^2 + 2 \int_0^{\frac{\delta_5}{\delta_5}} (x - z) \cdot \gamma(t_0, y(t_0, z_0, \tau)) d\tau \\
\leq \sigma^2 + |\beta|^2 + |z - z_0|^2 + 2 \int_0^{\frac{\delta_5}{\delta_5}} |\beta| |\gamma(t_0, y(t_0, z_0, \tau))| d\tau + 2\sigma \int_0^{\frac{\delta_5}{\delta_5}} \gamma(t_0, z) \cdot \gamma(t_0, y(t_0, z_0, \tau)) d\tau \\
\leq \sigma^2 + 4 \delta_5^2 \tan^2 \theta_1 + \frac{\delta_5^2}{4} + 2 \delta_5^2 \tan \theta_1 + \sigma \delta_5,
\]
which implies that
\[
\frac{|x - z_0|^2}{\delta_5^2} \leq \frac{\sigma^2}{\delta_5^2} + 4 \tan^2 \theta_1 + \frac{1}{4} + 2 \tan \theta_1 + \frac{\sigma}{\delta_5}.
\]
Combining this with (5.7), (5.11) and (5.12), we have
\[
(x - z_0) \cdot \gamma(t_0, z_0) \\
= \sigma \gamma(t_0, z) \cdot \gamma(t_0, z_0) + \beta \cdot \gamma(t_0, z_0) + (z - z_0) \cdot \gamma(t_0, z_0) \\
\geq \sigma \gamma(t_0, z) \cdot \gamma(t_0, z_0) - |\beta| + \int_0^{\frac{\delta_5}{\delta_5}} \gamma(t_0, y(t_0, z_0, \tau)) \cdot \gamma(t_0, z_0) d\tau \\
\geq \left(\frac{\sigma}{\delta_5} \cos \theta_1 - 2 \tan \theta_1 + \frac{1}{2} \cos \theta_1\right) \delta_5 \\
\geq \frac{\sigma}{\delta_5} \cos \theta_1 - 2 \tan \theta_1 + \frac{1}{2} \cos \theta_1 \\
\geq \left(\frac{\sigma^2}{\delta_5^2} + 4 \tan^2 \theta_1 + \frac{1}{4} + 2 \tan \theta_1 + \frac{\sigma}{\delta_5}\right) \frac{1}{2} |x - z_0| \\
\geq \frac{(\kappa^2 - 4 \tan^2 \theta_1) \frac{1}{2} \cos \theta_1 - 2 \tan \theta_1 + \frac{1}{2} \cos \theta_1}{\left(\frac{\sigma^2}{\delta_5^2} + 4 \tan^2 \theta_1 + 2 \tan \theta_1\right)^{\frac{1}{2}}} |x - z_0| \\
> |x - z_0| \cos \theta_0.
\]
Together with (5.3) and (5.10), we see that \( x \in C(z_0, \gamma(t_0, z_0), \delta_2, \theta_0) \subset u(t_0, D) \), which again implies \( x \not\in u(t_0, D) \). Hence we obtain (5.8).

By (2.4) and (5.8), there exists a \( \eta_2 \in (0, \eta_2) \) such that for \( t \in ((t_0 - \eta_2) \lor 0, (t_0 + \eta_2) \land T_1) \), we have \( u(t, \partial D) \cap (C(z, \delta_5) \setminus B(z, \kappa \delta_5)) = \emptyset \). Together with Lemma 5.1, we obtain that
\[
\begin{align*}
\{ t \in (t_0 - \eta_2) \lor 0, (t_0 + \eta_2) \land T_1) : x \in u(t, \partial D) \cap B(z, \delta_5) \cup \supp h(t, \cdot) \} = \emptyset.
\end{align*}
\]

Now take a nonnegative function \( \chi_1 \in C^{1,2}_0 \times B(z, \delta_5) \) such that \( \chi_1 = 1 \) on \( ((t_0 - \eta_3) \lor 0, (t_0 + \eta_3) \land T_1) \times B(z, \delta_5) \), and choose a constant \( M > 0 \) large enough, such that \( \chi_2(x) := (x - z_0) \cdot \gamma(t_0, z_0) + M \) is nonnegative on \( B(z, \delta_5) \). Set \( h_{t_0, z_0}(t, x) := h(t, x) \chi_1(t, x) \chi_2(x) \). We see that \( h_{t_0, z_0} \) belongs to the following space:
\[
C^{0,1}_0 \times B(z, \delta_5) \cap W^{1,2}_{2d+2}(t_0 - \eta_3) \lor 0, (t_0 + \eta_3) \land T_1) \times B(z, \delta_5) \cap \overline{D}).
\]

Note that by (5.11), \( \chi_1 = 1 \) on the neighborhood of
\[
\{ t \in ((t_0 - \eta_3) \lor 0, (t_0 + \eta_3) \land T_1) : x \in u(t, \partial D) \cap B(z, \delta_5) \cup \supp h(t, \cdot) \}.
\]

Using the above fact, Lemma 5.4 and (5.5) we have
\[
\begin{align*}
\nabla h_{t_0, z_0}(t, x) \cdot \gamma(t_0, z_0) = \chi_2(z_0) \nabla h_{t_0, z_0}(t, x) \cdot \gamma(t_0, z_0) + h(t, z_0) \nabla \chi_2(z_0) \cdot \gamma(t_0, z_0)
\end{align*}
\]
and
\[
\begin{align*}
\nabla h_{t_0, z_0}(t, x) \cdot \gamma(t, x) = \chi_2(x) \nabla h_{t_0, z_0}(t, x) \cdot \gamma(t, x) + h(t, x) \nabla \chi_2(x) \cdot \gamma(t, x)
\end{align*}
\]
\[
\begin{align*}
\geq 0,
\end{align*}
\]
for \( t \in ((t_0 - \eta_3) \lor 0, (t_0 + \eta_3) \land T_1) \) and \( x \in u(t, \partial D) \cap B(z, \delta_5) \). Now, by a standard compactness argument we can construct a nonnegative function \( H \in C^{0,1}_b \cap W^{1,2}_{2d+2}(0, T_1) \times \mathbb{R}^d \) such that (5.4) holds for any \( t \in [0, T_1] \) and \( x \in u(t, \partial D) \).

Following exactly the argument of Lemma 4.4 in [1], we have the next result.

**Lemma 5.2** There exist a function \( g \in C^1(\mathbb{R}^d) \cap C^2(\mathbb{R}^d \setminus \{0\} \times \mathbb{R}^d) \) and positive constants \( M_4, M_5 \), satisfying that for any \( \rho, \xi \in \mathbb{R}^d \) with \( |\xi| \leq 1 \) the following conditions hold:

\[
\begin{align*}
(i) \quad & g(0, \xi) = 0, \quad (5.15) \\
(ii) \quad & g(\rho, \xi) \geq M_4|\rho|^2, \quad (5.16) \\
(iii) \quad & |\nabla g(\rho, \xi)| \cdot \xi \geq 0, \quad \text{for } \rho \cdot \xi \geq -\cos \theta_0|\rho| \quad \text{and } |\xi| = 1, \quad (5.17) \\
(iv) \quad & |\nabla g(\rho, \xi)| \cdot \xi \leq 0, \quad \text{for } \rho \cdot \xi \leq -\cos \theta_0|\rho| \quad \text{and } |\xi| = 1, \quad (5.18) \\
v) \quad & |\nabla g(\rho, \xi)| \leq M_5|\rho|, \quad |\nabla g(\rho, \xi)| \leq M_5|\rho|^2, \quad \text{for } |\rho| \neq 0, \quad (5.19) \\
v) \quad & |\partial_\rho \partial_\xi g(\rho, \xi)| \leq M_5, \quad |\partial_\xi \partial_\xi g(\rho, \xi)| \leq M_5|\rho|, \quad (5.20)
\end{align*}
\]

where \( \theta_0 \) was defined in Proposition 3.4.
Take $\sigma \in C^2(\mathbb{R})$ such that $\sigma(t) = 1$ for $t \leq \frac{1}{2}$, $\sigma(t) = t$ for $t \geq 2$ and $\sigma'(t) \geq 0, \sigma(t) \geq t$ for $t \in \mathbb{R}$. It is easy to see that $\omega(\rho, \xi) := \sigma(g(\rho, \xi)) \in C^2(\mathbb{R}^{2d})$. Now we introduce the second important class of test functions. For $\epsilon > 0$, define

$$ f_\epsilon(t, x, y) := \epsilon \sigma(\frac{u(t, x) - u(t, y)}{\epsilon}, n(x)). \tag{5.21} $$

The following result holds.

**Proposition 5.2** There exist positive constants $M_6$ and $M_7$, which are independent of $\epsilon$, such that for $t \in [0, T_1]$ and $\rho, \xi \in \mathbb{R}^d$ with $|\xi| \leq 1$,

(i). $|\nabla_\rho \omega(\rho, \xi)| \leq M_6|\rho|, \ |\nabla_\xi \omega(\rho, \xi)| \leq M_6|\rho|^2,$ \tag{5.22}

(ii). $|\partial_{\rho_i} \partial_{\rho_j} \omega(\rho, \xi)| \leq M_6, \ |\partial_{\xi_i} \partial_{\rho_j} \omega(\rho, \xi)| \leq M_6|\rho|, \ |\partial_{\xi_i} \partial_{\xi_j} \omega(\rho, \xi)| \leq M_6|\rho|^2, \tag{5.23}

(iii). $M_7 \frac{|x - y|^2}{\epsilon} \leq f_\epsilon(t, x, y) \leq \epsilon + M_6 \frac{|x - y|^2}{\epsilon}, \text{ for } x, y \in D; \tag{5.24}

(iv). $\nabla_x f_\epsilon(t, x, y) \cdot n(x) \leq M_6 \frac{|x - y|^2}{\epsilon}, \text{ for } x \in \partial D \text{ and } y \in \bar{D}; \tag{5.25}

(v). $\nabla_y f_\epsilon(t, x, y) \cdot n(x) \leq M_6 \frac{|x - y|^2}{\epsilon}, \text{ for } x \in \bar{D} \text{ and } y \in \partial D. \tag{5.26}$

**Proof:** Let $\rho, \xi \in \mathbb{R}^d$ with $|\xi| \leq 1$, then by (5.19) and the boundedness of $\sigma'(t)$, it is easy to see that $|\nabla_\rho \omega(\rho, \xi)| \leq M_6|\rho|$ and $|\nabla_\xi \omega(\rho, \xi)| \leq M_6|\rho|^2$ for some positive constant $M_6$.

By (5.10), (5.19) and (5.20), we have

$$ |\partial_{\rho_i} \partial_{\rho_j} \omega(\rho, \xi)| = |\sigma''(g(\rho, \xi)) \partial_{\rho_i} g(\rho, \xi) \partial_{\rho_j} g(\rho, \xi) + \sigma'(g(\rho, \xi)) \partial_{\rho_i} \partial_{\rho_j} g(\rho, \xi)| $$

$$ \lesssim |\rho|^2 I_{g(\rho, \xi) \leq 2(\rho, \xi)} + 1 \leq |\rho|^2 I_{M_4 |\rho|^2 \leq 2(\rho, \xi)} + 1 \leq M_6, $$

Similarly, we also have $|\partial_{\xi_i} \partial_{\rho_j} \omega(\rho, \xi)| \leq M_6|\rho|$ and $|\partial_{\xi_i} \partial_{\xi_j} \omega(\rho, \xi)| \leq M_6|\rho|^2$.

Now we show (5.21). Let $x, y \in \bar{D}$. By (2.8), (5.10) and the fact that $\sigma(t) \geq t$, we have

$$ f_\epsilon(t, x, y) \geq \epsilon \sigma(\frac{u(t, x) - u(t, y)}{\epsilon}, n(x)) \geq \epsilon M_4 \frac{|u(t, x) - u(t, y)|}{\epsilon}^2 \geq M_7 \frac{|x - y|^2}{\epsilon}, \tag{5.27} $$

for some constant $M_7 > 0$. By (2.8), (5.15) and (5.19) we have

$$ f_\epsilon(t, x, y) $$

$$ = \epsilon \sigma((g(0, n(x)))) $$

$$ + \epsilon \int_0^1 \sigma'(g(\frac{\lambda(t, x) - u(t, y)}{\epsilon}, n(x))) \nabla_\rho g(\frac{\lambda(t, x) - u(t, y)}{\epsilon}, n(x)) \cdot \frac{u(t, x) - u(t, y)}{\epsilon} d\lambda $$

$$ \leq \epsilon + \epsilon \int_0^1 \lambda(\frac{u(t, x) - u(t, y)}{\epsilon}) \div \frac{u(t, x) - u(t, y)}{\epsilon} d\lambda \leq \epsilon + \frac{M_6 |x - y|^2}{\epsilon}. $$

Next we show (5.24). When $x \in \partial D$ and $y \in \bar{D}$ satisfying that $|x - y| < \frac{\delta_2}{M_2}$, by (2.8) we have $|u(t, x) - u(t, y)| < \delta_2$. Combining this with (3.1) we deduce that

$$ \frac{u(t, x) - u(t, y)}{\epsilon} \cdot n(x) \leq \cos \theta_0 \frac{|u(t, x) - u(t, y)|}{\epsilon}. $$

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In view of (2.8), (5.18), (5.19), taking into consideration of the facts $\nabla_x u^i(t, x) \cdot n(x) = n^i(x)$ and $\sigma'(t) \geq 0$, we have

$$\nabla_x f_x(t, x, y) \cdot n(x) = \sigma'(g(u(t, x) - u(t, y), n(x))) \sum_{i \leq i \leq d} [\partial_{\xi_i} g(u(t, x) - u(t, y), n(x)) \nabla_x u^i(t, x) \cdot n(x)$$

$$+ \varepsilon \partial_{\xi_i} g(u(t, x) - u(t, y), n(x)) \nabla_x n^i(x) \cdot n(x)]$$

$$\leq \sigma'(g(u(t, x) - u(t, y), n(x))) \nabla_x u^i(t, x) \cdot n(x)$$

$$+ \sup_{t \in \mathbb{R}} \sigma'(t) \varepsilon \sum_{i \leq i \leq d} |\partial_{\xi_i} g(u(t, x) - u(t, y), n(x)) \nabla_x n^i(x) \cdot n(x)|$$

$$\leq \sup_{t \in \mathbb{R}} \sigma'(t) \varepsilon M_0 \frac{|u(t, x) - u(t, y)|}{\varepsilon} \sum_{i \leq i \leq d} |\nabla_x n^i(x)| \leq M_0 \frac{|x - y|^2}{\varepsilon}.$$ 

When $x \in \partial D$ and $y \in \partial D$ satisfying that $|x - y| \geq \frac{\delta_2}{M_2}$, noting that $\nabla_x u^i(t, x) \cdot n(x) = n^i(x)$, by (2.8) and (5.19), we have

$$\nabla_x f_x(t, x, y) \cdot n(x) = \sigma'(g(u(t, x) - u(t, y), n(x))) \sum_{i \leq i \leq d} [\partial_{\xi_i} g(u(t, x) - u(t, y), n(x)) \nabla_x u^i(t, x) \cdot n(x)$$

$$+ \varepsilon \partial_{\xi_i} g(u(t, x) - u(t, y), n(x)) \nabla_x n^i(x) \cdot n(x)]$$

$$\leq \sup_{t \in \mathbb{R}} \sigma'(t) \frac{|\nabla_x g(u(t, x) - u(t, y), n(x))| + \varepsilon \sum_{i \leq i \leq d} |\partial_{\xi_i} g(u(t, x) - u(t, y), n(x))||\nabla_x n^i(x)|}{\varepsilon}$$

$$\lesssim \frac{|u(t, x) - u(t, y)|}{\varepsilon} + \frac{\varepsilon |u(t, x) - u(t, y)|^2}{\varepsilon}$$

$$\leq M_2 \frac{|x - y|}{\varepsilon} + M_2 \frac{|x - y|^2}{\varepsilon} \leq M_0 \frac{|x - y|^2}{\varepsilon}.$$ 

Finally we prove (5.26). When $x \in \partial D$ and $y \in \partial D$ satisfying that $|x - y| < \frac{\delta_2}{M_2}$, by (2.8) we have $|u(t, x) - u(t, y)| < \delta_2$. Combining this with (3.1) gives

$$\frac{u(t, x) - u(t, y)}{\varepsilon} \cdot n(y) \geq -\cos \theta_0 |\frac{u(t, x) - u(t, y)}{\varepsilon}|.$$
Assume Lemma 5.4 arguments as in the proof of Lemma 4.1 in [4], we have the following estimates:

\[
\nabla_y f_\varepsilon(t, x, y) \cdot n(y) = -\sigma'(g_\varepsilon(u(t, x) - u(t, y), n(x))) \sum_{1 \leq i \leq d} \partial_{y^i} g_\varepsilon(u(t, x) - u(t, y), n(x)) \nabla_y u^i(t, y) \cdot n(y)
\]

By (2.8), (5.17), (5.20), and the facts that \(\xi\) where

\[
\nabla_y f_\varepsilon(t, x, y) \cdot n(y) = -\sigma'(g_\varepsilon(u(t, x) - u(t, y), n(x))) \sum_{1 \leq i \leq d} \partial_{y^i} g_\varepsilon(u(t, x) - u(t, y), n(x)) \nabla_y u^i(t, y) \cdot n(y)
\]

and

\[
\nabla_y f_\varepsilon(t, x, y) \cdot n(y) = -\sigma'(g_\varepsilon(u(t, x) - u(t, y), n(x))) \sum_{1 \leq i \leq d} \partial_{y^i} g_\varepsilon(u(t, x) - u(t, y), n(x)) \nabla_y u^i(t, y) \cdot n(y)
\]

Using the Krylov’s estimate established in Lemma 5.1 in [11], and following the same arguments as in the proof of Lemma 4.1 in [4], we have the following estimates:

**Lemma 5.3** Let \(\xi_t\) and \(\eta_t\) be two nonnegative càdlàg adapted processes, \(A_t\) a continuous nondecreasing adapted process with \(A_0 = 0\), \(M_t\) a local martingale with \(M_0 = 0\). Suppose that

\[
\xi_t \leq \eta_t + \int_0^t \xi_s dA_s + M_t, \quad \forall t > 0.
\]

Then for any \(0 < q < p < 1\) and stopping time \(\tau > 0\), we have

\[
[E(\xi_\tau^*)^q]^{1/q} \leq \left(\frac{p}{p-q}\right)^{1/q} (Ee^{pA_\tau/(1-p)})^{(1-p)/p} E(\eta_\tau^*)^q,
\]

where \(\xi_\tau^* := \sup_{s \leq \tau} \xi_s\) and \(\eta_\tau^* := \sup_{s \leq \tau} \eta_s\).

Using the Krylov’s estimate established in Lemma 5.1 in [11], and following the same arguments as in the proof of Lemma 4.1 in [4], we have the following estimates:

**Lemma 5.4** Assume \((X_t, L_t)\) and \((\bar{X}_t, \bar{L}_t)\) are solutions to the reflecting SDEs (1.1) with \(E[|L|_T] < \infty\) and \(E[|\bar{L}|_T] < \infty\) for \(T > 0\). Then there exists a positive constant \(M_8\) depending only on \(T, E[|L|_T], E[|\bar{L}|_T]\) and \(|b|_{L^{d+1}(0,T) \times D}\), such that for any \(f \in L^{d+1}(0,T)\times D)\),

\[
E|\int_0^T f(t, X_t) dt| \leq M_8 \|f\|_{L^{d+1}(0,T) \times D}).
\]

(5.27)
Moreover, there exists a positive constant $M_0$ depending only on $T$, $E[|L_T|]$, $E[|\tilde{L}_T|]$ and $\|b\|_{L^{d+1}((0,T) \times D)}$, such that for any $f \in L^{d+1}((0,T) \times \mathbb{R}^d)$ and $\alpha \in [0,1],$

$$E[\int_0^T |f(t, \alpha X_t + (1-\alpha)\tilde{X}_t)|dt] \leq M_0\|f\|_{L^{d+1}((0,T) \times \mathbb{R}^d)}.$$ (5.28)

6 Existence and uniqueness

In this section, we will establish the existence and uniqueness of strong solutions to the reflecting SDEs (1.1) with singular coefficients. The existence of a weak solution follows from the Girsanov theorem. The strong solution is obtained by proving the pathwise uniqueness of the solutions.

When the drift $b$ vanishes, the solution of equation (1.1) is the so called reflecting Brownian motion. The existence and uniqueness of reflecting Brownian motion $(X_t, L_t)$ is now well known (see e.g. [5]). Then using the Girsanov transformation, we easily obtain the following result.

**Proposition 6.1** For any $x \in \hat{D}$, there exists a unique weak solution $(X_t, L_t)$ to the reflecting SDEs (1.1) with $X_0 = x$, Moreover, $E_x[|L_T|] < \infty$.

To obtain the existence and uniqueness of strong solutions of the reflecting SDEs, according to the Yamada-Watanabe theorem it is sufficient to prove the pathwise uniqueness of the equation (1.1). The rest of this section is devoted to this goal.

Using the Krylov’s estimate in Lemma 5.4 we have the following generalized Itô’s formula:

**Lemma 6.1** Let $F \in W^{1,2}_q((0,T) \times D)$ for some $T > 0$ and $q > d + 2$. Let $X_t$ be a solution to the reflecting SDEs (1.1). Then we have for any $0 \leq t \leq T$,

$$F(t, X_t) = F(0, X_0) + \int_0^t \partial_t F(s, X_s)ds + \sum_{1 \leq i \leq d} \int_0^t \partial_{x_i} F(s, X_s) dW^i_s + \int_0^t \nabla_x F(s, X_s) \cdot b(s, X_s) ds + \int_0^t \nabla_x F(s, X_s) \cdot n(X_s) |dL_t| + \frac{1}{2} \int_0^t \Delta_x F(s, X_s) ds.$$ (6.1)

**Proof:** Since $F \in W^{1,2}_q((0,T) \times D)$ and the boundary of $D$ is smooth, there exists a sequence of functions $\{F_n\}_{n \geq 1} \subset C^{1,2}_b([0,T] \times \mathbb{R}^d)$ such that $F_n$ converges to $F$ in $W^{1,2}_q((0,T) \times D)$. By the Itô’s formula, we have

$$F_n(t, X_t) = F_n(0, X_0) + \int_0^t \partial_t F_n(s, X_s) ds + \sum_{1 \leq i \leq d} \int_0^t \partial_{x_i} F_n(s, X_s) dW^i_s + \int_0^t \nabla_x F_n(s, X_s) \cdot b(s, X_s) ds + \int_0^t \nabla_x F_n(s, X_s) \cdot n(X_s) |dL_t| + \frac{1}{2} \int_0^t \Delta_x F_n(s, X_s) ds.$$ (6.2)
Note that $F_n(t, x)$ and $\nabla_x F_n(t, x)$ converges to $F(t, x)$ and $\nabla_x F(t, x)$ uniformly on $[0, T] \times \bar{D}$ respectively by Sobolev inequality. Combining this with Lemma 5.4 letting $n \to \infty$ in (6.2), we get (6.1).

Recall that the constant $T_1$ was defined in Proposition 3.1 and the functions $H$ and $f_\varepsilon$ were defined in Proposition 5.1 and (5.21) respectively. For $0 \leq t \leq T_1$, set

$$F_\varepsilon(t, x, y) := \int_t^\infty f_\varepsilon(t, x, y) := e^{-\lambda[H(t, u(t,x)) + H(t, u(t,y))]} f_\varepsilon(t, x, y),$$

where $\varepsilon$ and $\lambda$ are some positive constants, and

$$M_t := -\lambda \int_0^t Z_s f_\varepsilon(s, X_s, \tilde{X}_s)[ \sum_{1 \leq i \leq d} (\partial_{x_i} H)(s, u(s, X_s)) \nabla_x u^i(s, X_s) \cdot dW_s$$

$$+ \sum_{1 \leq i \leq d} (\partial_{x_i} H)(s, u(s, \tilde{X}_s)) \nabla_x u^i(s, \tilde{X}_s) \cdot dW_s$$

$$+ \int_0^t Z_s \sum_{1 \leq i \leq d} \partial_{x_i} \omega(\frac{u(s, X_s) - u(s, \tilde{X}_s)}{\varepsilon}, n(X_s)) (\nabla_x u^i(s, X_s) - \nabla_x u^i(s, \tilde{X}_s)) \cdot dW_s$$

$$+ \varepsilon \int_0^t Z_s \sum_{1 \leq i \leq d} \partial_{x_i} \omega(\frac{u(s, X_s) - u(s, \tilde{X}_s)}{\varepsilon}, n(X_s)) \nabla_x n^i(X_s) \cdot dW_s,$$

$$A_t := -\lambda \int_0^t Z_s f_\varepsilon(s, X_s, \tilde{X}_s)[ (\nabla_x H)(s, u(s, X_s)) \cdot n(X_s) d|L|_s + (\nabla_x H)(s, u(s, \tilde{X}_s)) \cdot n(\tilde{X}_s) d|\tilde{L}|_s]$$

$$+ \int_0^t Z_s \nabla \omega(\frac{u(s, X_s) - u(s, \tilde{X}_s)}{\varepsilon}, n(X_s)) \cdot (n(X_s) d|L|_s - n(\tilde{X}_s) d|\tilde{L}|_s)$$

$$+ \varepsilon \int_0^t Z_s \sum_{1 \leq i \leq d} \partial_{x_i} \omega(\frac{u(s, X_s) - u(s, \tilde{X}_s)}{\varepsilon}, n(X_s)) \nabla_x n^i(X_s) \cdot n(X_s) d|L|_s,$$
\[ A_t^\varepsilon := -\lambda \int_0^t Z_s f_\varepsilon(s, X_s, \tilde{X}_s)[(\partial_s H)(s, u(s, X_s))ds + (\partial_s H)(s, u(s, \tilde{X}_s))ds] \]
- \[ \frac{\lambda}{2} \int_0^t Z_s f_\varepsilon(s, X_s, \tilde{X}_s) \sum_{1 \leq i,j \leq d} (\partial_{x_j} \partial_{x_i} H)(s, u(s, X_s))\nabla_x u^i(s, X_s) \cdot \nabla_x u^j(s, X_s)ds \]
- \[ \frac{\lambda}{2} \int_0^t Z_s f_\varepsilon(s, X_s, \tilde{X}_s) \sum_{1 \leq i,j \leq d} (\partial_{x_j} \partial_{x_i} H)(s, u(s, \tilde{X}_s))\nabla_x u^i(s, \tilde{X}_s) \cdot \nabla_x u^j(s, \tilde{X}_s)ds \]
+ \[ \varepsilon \int_0^t Z_s \sum_{1 \leq i \leq d} \partial \xi_i \omega \left( \frac{u(s, X_s) - u(s, \tilde{X}_s)}{\varepsilon}, n(X_s) \right) \nabla_x n^i(\tilde{X}_s) \cdot b(s, \tilde{X}_s) + \frac{1}{2} \Delta_x n^i(\tilde{X}_s)ds \]
+ \[ \frac{1}{2\varepsilon} \int_0^t Z_s \sum_{1 \leq i,j \leq d} \partial \xi_j \partial \rho_i \omega \left( \frac{u(s, X_s) - u(s, \tilde{X}_s)}{\varepsilon}, n(X_s) \right) \times (\nabla_x u^i(s, X_s) - \nabla_x u^i(s, \tilde{X}_s)) \cdot (\nabla_x u^j(s, X_s) - \nabla_x u^j(s, \tilde{X}_s))ds \]
+ \[ \frac{\varepsilon}{2} \int_0^t Z_s \sum_{1 \leq i,j \leq d} \partial \xi_j \partial \xi_i \omega \left( \frac{u(s, X_s) - u(s, \tilde{X}_s)}{\varepsilon}, n(X_s) \right) \nabla_x n^i(\tilde{X}_s) \cdot \nabla_x n^j(\tilde{X}_s)ds \]
+ \[ \frac{\lambda^2}{2} \int_0^t Z_s f_\varepsilon(s, X_s, \tilde{X}_s) \sum_{1 \leq i,j \leq d} (\partial_{x_j} H)(s, u(s, X_s)) (\partial_{x_i} H)(s, u(s, X_s)) \nabla_x u^i(s, X_s) \cdot \nabla_x u^j(s, X_s)ds \]
+ \[ \frac{\lambda^2}{2} \int_0^t Z_s f_\varepsilon(s, X_s, \tilde{X}_s) \sum_{1 \leq i,j \leq d} (\partial_{x_j} H)(s, u(s, \tilde{X}_s)) (\partial_{x_i} H)(s, u(s, \tilde{X}_s)) \nabla_x u^i(s, \tilde{X}_s) \cdot \nabla_x u^j(s, \tilde{X}_s)ds \]
+ \[ \frac{\lambda^2}{2} \int_0^t Z_s f_\varepsilon(s, X_s, \tilde{X}_s) \sum_{1 \leq i,j \leq d} (\partial_{x_j} H)(s, u(s, \tilde{X}_s)) (\partial_{x_i} H)(s, u(s, X_s)) \nabla_x u^i(s, \tilde{X}_s) \cdot \nabla_x u^j(s, X_s)ds \]
- \[ \lambda \int_0^t Z_s \sum_{1 \leq i,j \leq d} (\partial_{x_j} H)(s, u(s, X_s)) \partial \rho_i \omega \left( \frac{u(s, X_s) - u(s, \tilde{X}_s)}{\varepsilon}, n(X_s) \right) \times \nabla_x u^i(s, X_s) \cdot (\nabla_x u^j(s, X_s) - \nabla_x u^j(s, \tilde{X}_s))ds \]
- \[ \lambda \varepsilon \int_0^t Z_s \sum_{1 \leq i,j \leq d} (\partial_{x_j} H)(s, u(s, \tilde{X}_s)) \partial \xi_i \omega \left( \frac{u(s, X_s) - u(s, \tilde{X}_s)}{\varepsilon}, n(X_s) \right) \nabla_x u^i(s, X_s) \cdot \nabla_x n^j(\tilde{X}_s)ds \]
- \[ \lambda \int_0^t Z_s \sum_{1 \leq i,j \leq d} (\partial_{x_j} H)(s, u(s, \tilde{X}_s)) \partial \xi_i \omega \left( \frac{u(s, X_s) - u(s, \tilde{X}_s)}{\varepsilon}, n(X_s) \right) \times \nabla_x u^i(s, \tilde{X}_s) \cdot (\nabla_x u^j(s, X_s) - \nabla_x u^j(s, \tilde{X}_s))ds \]
- \[ \lambda \varepsilon \int_0^t Z_s \sum_{1 \leq i,j \leq d} (\partial_{x_j} H)(s, u(s, \tilde{X}_s)) \partial \xi_i \omega \left( \frac{u(s, X_s) - u(s, \tilde{X}_s)}{\varepsilon}, n(X_s) \right) \nabla_x u^i(s, \tilde{X}_s) \cdot \nabla_x n^j(\tilde{X}_s)ds. \]
Theorem 6.2 Assume \((X_t, L_t)\) and \((\tilde{X}_t, \tilde{L}_t)\) are two solutions to the reflecting SDEs (1.1). Then we have for any \(0 \leq t \leq T_1\),

\[
F_c(t, X_t, \tilde{X}_t) = F_c(0, X_0, \tilde{X}_0) + M_t + A_t^1 + A_t^2. \tag{6.3}
\]

**Proof:** Assume \((X_t, L_t)\) and \((\tilde{X}_t, \tilde{L}_t)\) are two solutions to reflecting SDEs (1.1). Applying Lemma [6.1] we obtain

\[
u(t, X_t) = u(0, X_0) + \int_0^t \nabla_x u(s, X_s) \cdot dW_s + \int_0^t n(X_s)d|L|_s,
\]

\[
u(t, \tilde{X}_t) = u(0, \tilde{X}_0) + \int_0^t \nabla_x u(s, \tilde{X}_s) \cdot dW_s + \int_0^t n(\tilde{X}_s)d|\tilde{L}|_s.
\]

Since \(\omega \in C^2(\mathbb{R}^{2d})\), using the Itô’s formula we have

\[
f_c(t, X_t, \tilde{X}_t) = f_c(0, X_0, \tilde{X}_0) + \int_0^t \sum_{1 \leq i \leq d} \partial_{x_i} \omega \left( \frac{u(s, X_s) - u(s, \tilde{X}_s)}{\varepsilon}, n(X_s) \right) \nabla_x u^i(s, X_s) \cdot dW_s
\]

\[+ \int_0^t \sum_{1 \leq i \leq d} \partial_{x_i} \omega \left( \frac{u(s, X_s) - u(s, \tilde{X}_s)}{\varepsilon}, n(X_s) \right) \nabla_x n^i(s, X_s) \cdot dW_s
\]

\[+ \sum_{1 \leq i \leq d} \partial_{x_i} \omega \left( \frac{u(s, X_s) - u(s, \tilde{X}_s)}{\varepsilon}, n(X_s) \right) \nabla_x n^i(s, X_s) \cdot n(X_s)d|L|_s + \frac{1}{2} \Delta_x n^i(s, X_s) \tag{6.4}
\]

\[+ \frac{1}{2\varepsilon} \int_0^t \sum_{1 \leq i, j \leq d} \partial_{x_i} \partial_{x_j} \omega \left( \frac{u(s, X_s) - u(s, \tilde{X}_s)}{\varepsilon}, n(X_s) \right)
\]

\[\times \left( \nabla_x u^i(s, X_s) - \nabla_x u^i(s, \tilde{X}_s) \right) \cdot \left( \nabla_x u^j(s, X_s) - \nabla_x u^j(s, \tilde{X}_s) \right) ds
\]

\[+ \sum_{1 \leq i, j \leq d} \partial_{x_i} \partial_{x_j} \omega \left( \frac{u(s, X_s) - u(s, \tilde{X}_s)}{\varepsilon}, n(X_s) \right) \nabla_x n^i(s, X_s) \cdot \nabla_x n^j(X_s) ds
\]

\[+ \frac{\varepsilon}{2} \sum_{1 \leq i, j \leq d} \partial_{x_i} \partial_{x_j} \omega \left( \frac{u(s, X_s) - u(s, \tilde{X}_s)}{\varepsilon}, n(X_s) \right) \nabla_x n^i(s, X_s) \cdot \nabla_x n^j(X_s) ds.
\]

On the other hand, since \(\tilde{H}(t, x) := H(t, u(t, x)) \in W^{1,2}_{2d+2}(0, T_1 \times D)\) by Lemma [3.3] we
can apply (2.1) and Lemma 6.1 to get

\[ H(t, u(t, X_t)) \]
\[ = H(0, u(0, X_0)) + \int_0^t \partial_x \tilde{H}(s, X_s)ds + \int_0^t \nabla_x \tilde{H}(s, X_s) \cdot dX_s + \frac{1}{2} \int_0^t \Delta_x \tilde{H}(s, X_s)ds \]
\[ = H(0, u(0, X_0)) + \int_0^t [\partial_x H(s, u(s, X_s)) + \sum_{1 \leq i \leq d} (\partial_{x_i} H)(s, u(s, X_s))\partial_x u^i(s, X_s)]ds \]
\[ + \int_0^t \sum_{1 \leq i \leq d} (\partial_{x_i} H)(s, u(s, X_s))\nabla_x u^i(s, X_s) \cdot dW_s + \nabla_x u^i(s, X_s) \cdot b(s, X_s)ds \]
\[ + \frac{1}{2} \int_0^t [ \sum_{1 \leq i \leq d} (\partial_{x_i} H)(s, u(s, X_s))\nabla_x u^i(s, X_s) \cdot \nabla_x u^j(s, X_s) \]
\[ + \sum_{1 \leq i \leq d} (\partial_{x_i} H)(s, u(s, X_s))\Delta_x u^i(s, X_s)]ds \]
\[ = H(0, u(0, X_0)) + \int_0^t (\partial_x H)(s, u(s, X_s))ds + \int_0^t \sum_{1 \leq i \leq d} (\partial_{x_i} H)(s, u(s, X_s))\nabla_x u^i(s, X_s) \cdot dW_s \]
\[ + n^i(X_s)d|L_s| + \frac{1}{2} \int_0^t \sum_{1 \leq i,j \leq d} (\partial_{x_j} \partial_{x_i} H)(s, u(s, X_s))\nabla_x u^i(s, X_s) \cdot \nabla_x u^j(s, X_s)ds, \]

and similarly,

\[ H(t, u(t, \tilde{X}_t)) \]
\[ = H(0, u(0, \tilde{X}_0)) + \int_0^t (\partial_x H)(s, u(s, \tilde{X}_s))ds + \int_0^t \sum_{1 \leq i \leq d} (\partial_{x_i} H)(s, u(s, \tilde{X}_s))\nabla_x u^i(s, \tilde{X}_s) \cdot dW_s \]
\[ + n^i(\tilde{X}_s)d|\tilde{L}_s| + \frac{1}{2} \int_0^t \sum_{1 \leq i,j \leq d} (\partial_{x_j} \partial_{x_i} H)(s, u(s, \tilde{X}_s))\nabla_x u^i(s, \tilde{X}_s) \cdot \nabla_x u^j(s, \tilde{X}_s)ds. \]

By (6.4), (6.5), (6.6) and the integration by parts, we easily deduce (6.3). \[ \Box \]

Now we are ready to prove the main result of the paper:

**Theorem 6.3** For any \( x \in \tilde{D} \), the reflecting SDEs (1.1) has a unique strong solution \((X_t, L_t)\) with \( X_0 = x \).

**Proof:** By Proposition 6.1, we know that the reflecting SDEs (1.1) has a unique weak solution. Hence by the Yamada-Watanabe theorem, it is sufficient to prove the pathwise uniqueness of the reflecting SDEs (1.1).

Assume \((X_t, L_t)\) and \((\tilde{X}_t, \tilde{L}_t)\) are two solutions to reflecting SDEs (1.1) with \( X_0 = \tilde{X}_0 = x \). Let \( A^1_t, A^2_t \) be defined as in (6.3). Then by Proposition 5.1 (5.24), (5.25) and (5.26), we
have for \( t \in [0, T_1] \),

\[
A_1^t = -\lambda \int_0^t Z_s f_\varepsilon(s, X_s, \tilde{X}_s) |\nabla_x H(s, u(s, X_s)) \cdot n(X_s)| d|L|_s + \int_0^t Z_s |\nabla_x f_\varepsilon(s, X_s, \tilde{X}_s)| d|L|_s
\]

\[
\leq \int_0^t Z_s (M_0 \frac{|X_s - \tilde{X}_s|^2}{\varepsilon} - \lambda M_1 \frac{|X_s - \tilde{X}_s|^2}{\varepsilon}) d|L|_s
\]

(6.7)

\[
+ \int_0^t Z_s (M_0 \frac{|X_s - \tilde{X}_s|^2}{\varepsilon} - \lambda M_1 \frac{|X_s - \tilde{X}_s|^2}{\varepsilon}) d|\tilde{L}|_s.
\]

Hence we can take \( \lambda := \frac{M_0}{M_1} \) so that \( A_1^t \leq 0 \) for any \( \varepsilon > 0 \).

By (2.8), (5.22), (5.23) and (5.24) and the Hölder inequality, we have for \( t \in [0, T_1] \),

\[
\lambda \int_0^t \left( \varepsilon + \frac{|X_s - \tilde{X}_s|^2}{\varepsilon} \right) |(\partial_x H)(s, u(s, X_s))| + |(\partial_s H)(s, u(s, \tilde{X}_s))| \right) ds
\]

\[
+ \lambda \sum_{1 \leq i,j \leq d} \int_0^t \left( \varepsilon + \frac{|X_s - \tilde{X}_s|^2}{\varepsilon} \right) |(\partial_x i, \partial_x j)(s, u(s, X_s))| + |(\partial_s i, \partial_x j)(s, u(s, \tilde{X}_s))| \right) ds
\]

\[
+ \varepsilon \int_0^t \frac{|u(s, X_s) - u(s, \tilde{X}_s)|}{\varepsilon} [\|b(s, X_s)\| + \frac{1}{2} \|\Delta u\|_{H^1}^2](X_s) \right) ds
\]

\[
+ \frac{1}{\varepsilon} \int_0^t \|\nabla_x u(s, X_s) - \nabla_x u(s, \tilde{X}_s)\|^2 ds
\]

\[
+ \lambda \int_0^t |u(s, X_s) - u(s, \tilde{X}_s)| \left( \|\nabla_x u(s, X_s) - \nabla_x u(s, \tilde{X}_s)\| \right) ds
\]

(6.8)

\[
\leq \varepsilon \int_0^t \left( \|\partial_x H\|_{L^1} + \|\partial_s H\|_{L^1} \right) ds
\]

\[
+ \varepsilon \sum_{1 \leq i,j \leq d} \int_0^t \left( \|\partial_x i, \partial_x j\|_{L^1} + \|\partial_s i, \partial_x j\|_{L^1} \right) ds
\]

\[
+ \frac{1}{\varepsilon} \int_0^t |X_s - \tilde{X}_s|^2 \left( \|\partial_x H\|_{L^1} + \|\partial_s H\|_{L^1} + \|b\|_{L^1} \right) ds
\]

\[
+ \frac{1}{\varepsilon} \sum_{1 \leq i,j \leq d} \int_0^t |X_s - \tilde{X}_s|^2 \left( \|\partial_x i, \partial_x j\|_{L^1} + \|\partial_s i, \partial_x j\|_{L^1} \right) ds
\]

\[
+ \frac{1}{\varepsilon} \int_0^t \|\nabla_x u(s, X_s) - \nabla_x u(s, \tilde{X}_s)\|^2 ds.
\]

Since \( \partial D \) is smooth, there exist a function \( v \in W^{1,2}_{2d+2}((0, T_1) \times \mathbb{R}^d) \) and a sequence of functions \( \{v_n\}_{n \geq 1} \subset C^{1,2}_b((0, T_1) \times \mathbb{R}^d) \) such that \( v(t, x) = u(t, x) \) on \( (0, T_1) \times D \) and \( v_n \) converges to
\( v \) in \( W^{1,2}_{2d+2}((0,T_1) \times \mathbb{R}^d) \). Therefore, \( \nabla x v_n \) converges to \( \nabla x v \) uniformly on \((0,T_1) \times \mathbb{R}^d\) by Sobolev inequality. Hence by (5.28) we have

\[
\int_0^t \| \nabla_x u(s, X_s) - \nabla_x u(s, \tilde{X}_s) \|^2 ds \\
= \lim_{n \to \infty} \int_0^t \| \nabla_x v_n(s, X_s) - \nabla_x v_n(s, \tilde{X}_s) \|^2 ds \\
\leq \lim_{n \to \infty} \sum_{1 \leq i \leq d} \int_0^t \int_0^1 \nabla_x \partial_{x_i} v_n(s, \alpha X_s + (1 - \alpha) \tilde{X}_s) \cdot (X_s - \tilde{X}_s) da ds \bigg\|^2 ds \\
\leq \sum_{1 \leq i \leq d} \int_0^t \| X_s - \tilde{X}_s \|^2 \int_0^1 | \partial_{x_i} v(s, \alpha X_s + (1 - \alpha) \tilde{X}_s) |^2 da ds,
\]

and \( \int_0^t \int_0^1 \| \nabla_x \partial_{x_i} v(s, \alpha X_s + (1 - \alpha) \tilde{X}_s) |^2 da ds < \infty, P\text{-a.e.} \). Combing this with (5.34), (6.7), (6.8), (6.9) and Theorem 6.2 we have

\[
1 \leq \frac{1}{\varepsilon} | X_s - \tilde{X}_s |^2 \\
\lesssim \varepsilon + M_t + \varepsilon \int_0^t \bigg( |(\partial_s H)(s, u(s, X_s))| + |(\partial_s H)(s, u(s, \tilde{X}_s))| + 1 \bigg) ds \\
+ \varepsilon \sum_{1 \leq i \leq d} \int_0^t \bigg( |(\partial_{x_j} \partial_{x_i} H)(s, u(s, X_s))| + |(\partial_{x_j} \partial_{x_i} H)(s, u(s, \tilde{X}_s))| \bigg) ds \\
+ \frac{1}{\varepsilon} \int_0^t \| X_s - \tilde{X}_s \|^2 dC_t,
\]

where

\[
C_t := \int_0^t \bigg[ 1 + | b(s, X_s) | + |(\partial_s H)(s, u(s, X_s))| + |(\partial_s H)(s, u(s, \tilde{X}_s))| \\
+ \sum_{1 \leq i \leq d} \big( |(\partial_{x_j} \partial_{x_i} H)(s, u(s, X_s))| + |(\partial_{x_j} \partial_{x_i} H)(s, u(s, \tilde{X}_s))| \big) + \int_0^1 | \partial_{x_j} \partial_{x_i} v(s, \alpha X_s + (1 - \alpha) \tilde{X}_s) |^2 d\alpha \bigg] ds.
\]

Set \( \tau_R := \inf \{ t \geq 0 : C_t \geq R \} \cap T_1. \) Applying Lemma 5.3 to (6.10) with \( p = 2q = \frac{1}{2} \), we have

\[
\left[ E \left( \sup_{s \in [0,\tau_R]} | X_s - \tilde{X}_s |^{1/4 \pm} \right) \right]^4 \\
\lesssim 16 \varepsilon^2 e^{C \tau_R} \left( 1 + E \int_0^{T_1} \bigg( |(\partial_s H)(s, u(s, X_s))| + |(\partial_s H)(s, u(s, \tilde{X}_s))| \bigg) ds \\
+ \sum_{1 \leq i \leq d} E \int_0^{T_1} \bigg( |(\partial_{x_j} \partial_{x_i} H)(s, u(s, X_s))| + |(\partial_{x_j} \partial_{x_i} H)(s, u(s, \tilde{X}_s))| \bigg) ds \right) \\
\leq 16 \varepsilon^2 e^{R} \left( 1 + E \int_0^{T_1} \bigg( |(\partial_s H)(s, u(s, X_s))| + |(\partial_s H)(s, u(s, \tilde{X}_s))| \bigg) ds \\
+ \sum_{1 \leq i \leq d} E \int_0^{T_1} \bigg( |(\partial_{x_j} \partial_{x_i} H)(s, u(s, X_s))| + |(\partial_{x_j} \partial_{x_i} H)(s, u(s, \tilde{X}_s))| \bigg) ds \right).
\]
Note that \((\partial_s H)(s, u(s, x)), (\partial_x \partial_x H)(s, u(s, x)) \in L^{2d+2}([0, T_1] \times D)\). Hence by (5.27),

\[
E \int_0^{T_1} |(\partial_s H)(s, u(s, X_s)) + |(\partial_x H)(s, u(s, X_s))| ds \\
+ \sum_{1 \leq i, j \leq d} E \int_0^{T_1} |(\partial_{x_j} \partial_{x_i} H)(s, u(s, X_s)) + |(\partial_{x_i} \partial_{x_j} H)(s, u(s, X_s))| ds < \infty.
\]

Letting \(\varepsilon \to 0\) and \(R \to \infty\) in (6.11), we get \(E(\sup_{s \in [0, T_1]} |X_s - \tilde{X}_s|^{1/4}) = 0\), which implies \(X_t = \tilde{X}_t\) for all \(0 \leq t \leq T_1\), \(P\text{-a.e.}\). Since \(T_1\) is independent of the initial value \(x\), using a standard procedure, we can conclude \(X_t = \tilde{X}_t\) for all \(t > 0\), \(P\text{-a.e.}\). This completes the proof of the theorem.

7 Appendix

In this part, we provide the proofs of Proposition 4.1 and 4.2. Fix \(z := y(t_0, x_0, \frac{\delta}{2})\). Recall that \(D(t, \epsilon)\) and \(C(z, \delta)\) were respectively defined in (3.3) and (4.18), and

\[
O_\delta = \{(t, x) : t \in ((t_0 - \eta_2) \vee 0, (t_0 + \eta_2) \wedge T_1), x \in C(z, \delta) \cap D(t, \epsilon)\}.
\]

Set \(\varrho(\epsilon) := \left(\frac{\delta}{2} \sin \frac{\delta}{2}\right) \wedge \left(\frac{\delta}{2^2} \sin \frac{\delta}{2}\right)\) and

\[
O'_\delta := \{(t, x) : t \in ((t_0 - \eta_2) \vee 0, (t_0 + \eta_2) \wedge T_1), x \in D(t, \varrho(\epsilon))\}.
\]

Before giving the proofs of Proposition 4.1 and 4.2, we need a simple Lemma.

Lemma 7.1 For any constant \(p > 0\) and function \(f \in L^p(D)\), we have

\[
\lim_{n_1, n_2 \to \infty} \| \int_{-\rho_0}^{\rho_0} |f(t, y_{n_1}(t, x, r)) - f(t, y_{n_2}(t, x, r))| I_{u(t, D)}(y_{n_1}(t, x, r)) \\
\times I_{u(t, D)}(y_{n_2}(t, x, r)) dr \|_{L^p(D)} = 0.
\]

Proof: For any \(\varepsilon > 0\), there exists a function \(\tilde{f} \in C_b((0, T_1) \times \mathbb{R}^d)\) such that \(\|f - \tilde{f}\|_{L^p(D)} < \varepsilon\). Then by (4.5) and Lemma 4.1 we have for \(n_1, n_2 \geq N_0\),

\[
\lim_{n_1, n_2 \to \infty} \| \int_{-\rho_0}^{\rho_0} |f(t, y_{n_1}(t, x, r)) - f(t, y_{n_2}(t, x, r))| I_{u(t, D)}(y_{n_1}(t, x, r)) I_{u(t, D)}(y_{n_2}(t, x, r)) dr \|_{L^p(D)}
\leq \lim_{n_1, n_2 \to \infty} \int_D |f(t, y_{n_1}(t, x, r)) - \tilde{f}(t, y_{n_1}(t, x, r))| I_{u(t, D)}(y_{n_1}(t, x, r)) dx dt dr
\]

\[
+ \lim_{n_1, n_2 \to \infty} \int_D |f(t, y_{n_1}(t, x, r)) - \tilde{f}(t, y_{n_1}(t, x, r))| I_{u(t, D)}(y_{n_1}(t, x, r)) dx dt dr
\]

\[
+ \lim_{n_1, n_2 \to \infty} \int_D |f(t, y_{n_2}(t, x, r)) - \tilde{f}(t, y_{n_2}(t, x, r))| I_{u(t, D)}(y_{n_2}(t, x, r)) dx dt dr
\]

\[
\leq 4 \int_D |f(t, x) - \tilde{f}(t, x)|^p dx dt dr \leq 8\rho_0\varepsilon.
\]

Since \(\varepsilon\) is arbitrary, (7.1) follows.
Proof of Proposition 4.1

By Proposition 3.2 and (4.13), to show (4.40), we need only to show (4.39), i.e. for any $1 \leq i, j \leq d$,

$$\psi_{j}^{i}(\cdot, \cdot, \Gamma^{\varepsilon}(\cdot, \cdot)) \in W_{2d+2}^{0,1}((t_0 - \eta_2) \lor 0, (t_0 + \eta_2) \land T_1) \times C(\mathbb{R}, \delta_5)) \cap \tilde{D}).$$  \hfill (7.2)

For any given $\varepsilon > 0$, recall that $N_1(\varepsilon)$ is given in Lemma 4.4. Firstly, we show that for $n \geq N_1(\varepsilon)$, $\psi_{n,i}^{j}(t, x, \Gamma^{\varepsilon}(t, x)) \in C_{b}^{0,1}(\mathcal{O}_\varepsilon)$.

When $(t, x) \in \mathcal{O}_\varepsilon$ with $\Gamma^{\varepsilon}(t, x) \not= 0$, by (1.24), we have $y_n(t, x, r) \in D(t, \varepsilon)$ for $r \in (0, \Gamma^{\varepsilon}(t, x)]$. Hence together with Proposition 3.2 and Lemma 4.1 we can see that $\psi_{n,i}^{j}(t, x, \Gamma^{\varepsilon}(t, x)) \in C_{b}^{0,1}(\mathcal{O}_\varepsilon)$ and for $1 \leq m \leq d$,

$$d_{x_m} \frac{\partial \psi_{n,i}^{j}}{\partial x_m}(t, x, \Gamma^{\varepsilon}(t, x)) = \int_{0}^{\Gamma^{\varepsilon}(t, x)} \sum_{1 \leq k, l \leq d} \partial_{y_l} \partial_{y_k} \gamma^{j}_{n}(t, y_n(t, x, r)) \psi_{n,m}^{j}(t, x, r) dr
$$

$$+ \int_{0}^{\Gamma^{\varepsilon}(t, x)} \sum_{1 \leq k, l \leq d} \partial_{y_l} \gamma^{j}_{n}(t, y_n(t, x, r)) \partial_{x_m} \psi_{n,i}^{j}(t, x, r) dr
$$

$$+ \sum_{1 \leq k, l \leq d} \partial_{y_l} \gamma^{j}_{n}(t, y_n(t, x, \Gamma^{\varepsilon}(t, x))) \psi_{n,i}^{j}(t, x, \Gamma^{\varepsilon}(t, x)) \partial_{x_m} \Gamma^{\varepsilon}(t, x),$$

where $d \frac{\partial \psi_{n,i}^{j}}{\partial x_m}(t, x, \Gamma^{\varepsilon}(t, x))$ stands for the partial derivative of $\psi_{n,i}^{j}(t, x, \Gamma^{\varepsilon}(t, x))$ w.r.t. $x_m$.

Now we show that

$$\sup_{\varepsilon > 0} \sup_{n \geq N_1(\varepsilon)} \left\| \int_{0}^{\Gamma^{\varepsilon}(t, x)} |\nabla_{x} \psi_{n,i}^{j}(t, x, r)| dr \right\|_{L^{2d+2}(\mathcal{O}_\varepsilon)} < \infty. \hfill (7.4)$$

For $n \geq N_1(\varepsilon)$ and $(t, x) \in \mathcal{O}_\varepsilon$, by Lemma 1.1 and (1.24) we have for $1 \leq m, k \leq d$,

$$\left\| \int_{0}^{\Gamma^{\varepsilon}(t, x)} |\partial_{y_m} \partial_{y_k} \gamma^{j}_{n}(t, y_n(t, x, r))| dr \right\|_{L^{2d+2}(\mathcal{O}_\varepsilon)} \leq \int_{0}^{(t_0 + \eta_2) \lor (t_0 - \eta_2)} \int_{\mathbb{R}^d} |\partial_{y_m} \partial_{y_k} \gamma^{j}_{n}(t, y_n(t, x, r))|^{2d+2} d(x, t, r) \frac{1}{(2d+2)^{\frac{1}{2d+2}}}
$$

$$\leq \int_{0}^{(t_0 + \eta_2) \lor (t_0 - \eta_2)} \int_{\mathbb{R}^d} |\partial_{y_m} \partial_{y_k} \gamma^{j}_{n}(t, y_n(t, x, r))|^{2d+2} d(x, t, r) \frac{1}{(2d+2)^{\frac{1}{2d+2}}}
$$

$$\leq \int_{0}^{(t_0 + \eta_2) \lor (t_0 - \eta_2)} \int_{\mathbb{R}^d} |\partial_{y_m} \partial_{y_k} \gamma^{j}_{n}(t, y_n(t, x, r))|^{2d+2} d(x, t, r) \frac{1}{(2d+2)^{\frac{1}{2d+2}}}
$$

Together with (3.15), we deduce that

$$\sup_{\varepsilon > 0} \sup_{n \geq N_1(\varepsilon)} \left\| \int_{0}^{\Gamma^{\varepsilon}(t, x)} |\partial_{y_m} \partial_{y_k} \gamma^{j}_{n}(t, y_n(t, x, r))| dr \right\|_{L^{2d+2}(\mathcal{O}_\varepsilon)} < \infty. \hfill (7.5)$$

Applying the Gronwall’s inequality to equation (4.2), it is easy to see that

$$\sup_{\varepsilon > 0} \sup_{n \geq N_1(\varepsilon)} \sum_{1 \leq j, m, d \leq d} \left\| \int_{0}^{\Gamma^{\varepsilon}(t, x)} |\partial_{y_m} \partial_{y_k} \gamma^{j}_{n}(t, y_n(t, x, r))| dr \right\|_{L^{2d+2}(\mathcal{O}_\varepsilon)} < \infty.$$
Next we show that for any \( \varepsilon > 0 \),
\[
\sup_{n_1, n_2 \to \infty} \left\| \int_0^{t^*} \left( \nabla_x \psi_{n_1,j}^j(t, x, r) - \nabla_x \psi_{n_2,j}^j(t, x, r) \right) dr \right\|_{L^{2d+2} (\Omega_{\varepsilon})} = 0. \tag{7.6}
\]

In view of the boundness of \( |\psi_{n_1,j}^j(t, x, r)| \) and \( |\nabla_x \gamma_{n_1}^j(t, x)| \), Lemma 4.1 and (4.24), using the Gronwall’s inequality we have for \( n_1, n_2 \geq N_1(\varepsilon) \),
\[
\sum_{1 \leq j \leq d} \left\| \int_0^{t^*} \left( \nabla_x \psi_{n_1,j}^j(t, x, r) - \nabla_x \psi_{n_2,j}^j(t, x, r) \right) dr \right\|_{L^{2d+2} (\Omega_{\varepsilon})} \leq \sum_{1 \leq j, m, k \leq d} \left\| \int_{-\rho_1}^{\rho_1} \left| \partial_{y_m} \partial_{y_k} \gamma_{n_1}^j(t, y_{n_1}(t, x, r)) - \partial_{y_m} \partial_{y_k} \gamma_{n_2}^j(t, y_{n_1}(t, x, r)) \right| \right. \\
\times I_{D(t, \varepsilon)}(y_{n_1}(t, x, r)) dr \right\|_{L^{2d+2} (\Omega_{\varepsilon})} + \sum_{1 \leq j, m, k \leq d} \left\| \int_{-\rho_1}^{\rho_1} \left| \partial_{y_m} \partial_{y_k} \gamma_{n_2}^j(t, y_{n_2}(t, x, r)) - \partial_{y_m} \partial_{y_k} \gamma_{n_2}^j(t, y_{n_2}(t, x, r)) \right| \right. \\
\times I_{D(t, \varepsilon)}(y_{n_1}(t, x, r)) I_{D(t, \varepsilon)}(y_{n_2}(t, x, r)) dr \right\|_{L^{2d+2} (\Omega_{\varepsilon})} + \sum_{1 \leq j, m, k \leq d} \left\| \int_{-\rho_1}^{\rho_1} \left| \partial_{y_m} \partial_{y_k} \gamma_{n_2}^j(t, y_{n_2}(t, x, r)) - \partial_{y_m} \partial_{y_k} \gamma_{n_2}^j(t, y_{n_2}(t, x, r)) \right| \right. \\
\times \sup_{(t, x) \in \Omega_{\varepsilon}, |\varepsilon| < \rho_1, 1 \leq j \leq d} \left| \psi_{n_2,i}^j(t, x, r) - \psi_{n_2,i}^j(t, x, r) \right| \\
+ \sum_{1 \leq j, m, k \leq d} \left\| \int_{-\rho_1}^{\rho_1} \left| \partial_{y_m} \partial_{y_k} \gamma_{n_2}^j(t, y_{n_2}(t, x, r)) - \partial_{y_m} \partial_{y_k} \gamma_{n_2}^j(t, y_{n_2}(t, x, r)) \right| \right. \\
\times I_{D(t, \varepsilon)}(y_{n_2}(t, x, r)) dr \right\|_{L^{2d+2} (\Omega_{\varepsilon})} + \sum_{1 \leq k \leq d} \left\| \int_0^{t^*} \left( \nabla_x \psi_{n_1,i}^k(t, x, r) \right) dr \right\|_{L^{2d+2} (\Omega_{\varepsilon})} \\
\times \sup_{(t, x) \in \Omega_{\varepsilon}, |\varepsilon| < \rho_1, 1 \leq j \leq d} \left| \nabla_x \psi_{n_2,i}^j(t, x, r) - \psi_{n_2,i}^j(t, x, r) \right| \\
+ \sum_{1 \leq k \leq d} \left\| \int_0^{t^*} \left( \nabla_x \psi_{n_2,i}^k(t, x, r) \right) dr \right\|_{L^{2d+2} (\Omega_{\varepsilon})} \\
\times \sup_{(t, x) \in \Omega_{\varepsilon}, |\varepsilon| < \rho_1, 1 \leq j \leq d} \left| \nabla_x \psi_{n_2,i}^j(t, x, r) - \psi_{n_2,i}^j(t, x, r) \right| \\
\leq \sum_{1 \leq j \leq d} \left( \| \gamma_{n_1}^j - \gamma^j \|_{W^{0,2}_{2d+3} (\Omega_{\varepsilon})} + \| \gamma_{n_2}^j - \gamma^j \|_{W^{0,2}_{2d+3} (\Omega_{\varepsilon})} \right) \\
+ \sum_{1 \leq j, m, k \leq d} \left\| \int_{-\rho_1}^{\rho_1} \left| \partial_{y_m} \partial_{y_k} \gamma_{n_1}^j(t, y_{n_1}(t, x, r)) - \partial_{y_m} \partial_{y_k} \gamma_{n_2}^j(t, y_{n_2}(t, x, r)) \right| \right. \\
\times I_{D(t, \varepsilon)}(y_{n_1}(t, x, r)) I_{D(t, \varepsilon)}(y_{n_2}(t, x, r)) dr \right\|_{L^{2d+2} (\Omega_{\varepsilon})} + \sum_{1 \leq j \leq d} \left\| \gamma^j \|_{W^{0,2}_{2d+3} (\Omega_{\varepsilon})} \sup_{(t, x) \in \Omega_{\varepsilon}, |\varepsilon| < \rho_1, 1 \leq j \leq d} \left| \psi_{n_2,i}^j(t, x, r) - \psi_{n_2,i}^j(t, x, r) \right| \\
+ \sum_{1 \leq k \leq d} \left\| \int_0^{t^*} \left( \nabla_x \psi_{n_1,i}^k(t, x, r) \right) dr \right\|_{L^{2d+2} (\Omega_{\varepsilon})} \\
\times \sup_{(t, x) \in \Omega_{\varepsilon}, |\varepsilon| < \rho_1, 1 \leq j \leq d} \left| \nabla_x \psi_{n_2,i}^j(t, x, r) - \psi_{n_2,i}^j(t, x, r) \right| \\
+ \sum_{1 \leq k \leq d} \left\| \int_0^{t^*} \left( \nabla_x \psi_{n_2,i}^k(t, x, r) \right) dr \right\|_{L^{2d+2} (\Omega_{\varepsilon})} \\
\times \sup_{(t, x) \in \Omega_{\varepsilon}, |\varepsilon| < \rho_1, 1 \leq j \leq d} \left| \nabla_x \psi_{n_2,i}^j(t, x, r) - \psi_{n_2,i}^j(t, x, r) \right| \right\}
\]

Then by Proposition 3.2, (4.3), (4.6), (7.4) and Lemma 7.1 we get (7.6).
Next we show that
\[
\lim_{n_1, n_2 \to \infty} \left\| \psi_{n_1,1}^j(t, x, \Gamma^z(t, x)) - \psi_{n_2,1}^j(t, x, \Gamma^z(t, x)) \right\|_{W^{1,1}_{d+2}(\mathcal{O})} = 0. \tag{7.7}
\]
By (1.4), (4.24), (7.3), the boundness of \(|\nabla_x \Gamma^z(t, x)|\) and \(|\nabla_x \gamma_n(t, x)|\), for \(n_1, n_2 \geq N_1(\varepsilon)\) and \((t, x) \in \mathcal{O}_\varepsilon\), we have for \(1 \leq m \leq d\),
\[
|\frac{d}{dx_m} \psi_{n_1,1}^j(t, x, \Gamma^z(t, x)) - \frac{d}{dx_m} \psi_{n_2,1}^j(t, x, \Gamma^z(t, x))| \\
\leq \left| \int_{0}^{\Gamma^z(t, x)} \sum_{1 \leq k, l \leq d} |\partial_y \partial_y \gamma_{n_1}^j(t, y_n(t, x, r)) - \partial_y \partial_y \gamma_{n_2}^j(t, y_n(t, x, r))| dr \sup_{(t, x) \in \Omega_{\varepsilon}, \varepsilon < \rho_1, \gamma_n \in \mathcal{O}_{\varepsilon}(r) \leq n, \Gamma_n \leq \gamma_n} \left| \psi_{n,m}^l(t, x, r) \right|^2 \\
+ \left| \int_{0}^{\Gamma^z(t, x)} \sum_{1 \leq k, l \leq d} |\partial_y \partial_y \gamma_{n_2}^j(t, y_n(t, x, r)) - \partial_y \partial_y \gamma_{n_2}^j(t, y_n(t, x, r))| dr \sup_{(t, x) \in \Omega_{\varepsilon}, \varepsilon < \rho_1, \gamma_n \in \mathcal{O}_{\varepsilon}(r) \leq n, \Gamma_n \leq \gamma_n} \left| \psi_{n,m}^l(t, x, r) \right|^2 \\
+ \left| \int_{0}^{\Gamma^z(t, x)} \sum_{1 \leq k, l \leq d} |\partial_y \partial_y \gamma_{n_2}^j(t, y_n(t, x, r))| dr \sup_{(t, x) \in \Omega_{\varepsilon}, \varepsilon < \rho_1, \gamma_n \in \mathcal{O}_{\varepsilon}(r) \leq n, \Gamma_n \leq \gamma_n} \left| \psi_{n,m}^l(t, x, r) - \psi_{n_2,m}^l(t, x, r) \right| \\
\times \sup_{(t, x) \in \Omega_{\varepsilon}, \varepsilon < \rho_1, \gamma_n \in \mathcal{O}_{\varepsilon}(r) \leq n, \Gamma_n \leq \gamma_n} \left| \psi_{n,m}^l(t, x, r) \right| \\
+ \left| \int_{0}^{\Gamma^z(t, x)} \sum_{1 \leq k, l \leq d} |\nabla_x \psi_{n_1,1}^k(t, x, r) - \nabla_x \psi_{n_1,1}^k(t, x, r)| dr \sup_{(t, y) \in \bar{O}_\varepsilon} \left| \nabla_y \gamma_{n_1}^j(t, y) - \nabla_y \gamma_{n_2}^j(t, y) \right| \\
+ \left| \int_{0}^{\Gamma^z(t, x)} \sum_{1 \leq k, l \leq d} |\nabla_x \psi_{n_1,1}^k(t, x, r) - \nabla_x \psi_{n_1,1}^k(t, x, r)| dr \sup_{(t, y) \in \bar{O}_\varepsilon} \left| \nabla_y \gamma_{n_1}^j(t, y) \right| \\
+ \sum_{1 \leq k, l \leq d} |\partial_y \partial_y \gamma_{n_1}^j(t, y_n(t, x, r)) - \partial_y \partial_y \gamma_{n_2}^j(t, y_n(t, x, r))| \psi_{n_1,1}^k(t, x, r) \psi_{n_2,1}^k(t, x, r) | \partial_x \gamma_n(t, x)| \\
\times \int_{0}^{\rho_1} \sum_{1 \leq k, l \leq d} |\partial_y \partial_y \gamma_{n_1}^j(t, y_n(t, x, r)) - \partial_y \partial_y \gamma_{n_2}^j(t, y_n(t, x, r))| I_D(t, \varepsilon)(y_n(t, x, r)) dr \\
+ \left| \int_{0}^{\rho_1} \sum_{1 \leq k, l \leq d} |\partial_y \partial_y \gamma_{n_2}^j(t, y_n(t, x, r)) - \partial_y \partial_y \gamma_{n_2}^j(t, y_n(t, x, r))| \\
\times I_D(t, \varepsilon)(y_n(t, x, r)) I_D(t, \varepsilon)(y_n(t, x, r)) dr \\
+ \left| \int_{0}^{\rho_1} \sum_{1 \leq k, l \leq d} |\partial_y \partial_y \gamma_{n_2}^j(t, y_n(t, x, r))| I_D(t, \varepsilon)(y_n(t, x, r)) dr \\
\times \sup_{(t, x) \in \Omega_{\varepsilon}, \varepsilon < \rho_1, \gamma_n \in \mathcal{O}_{\varepsilon}(r) \leq n, \Gamma_n \leq \gamma_n} \left| \psi_{n_1,1}^l(t, x, r) - \psi_{n_2,1}^l(t, x, r) \right| \\
+ \sum_{1 \leq k, l \leq d} \left| \int_{0}^{\Gamma^z(t, x)} |\nabla_x \psi_{n_1,1}^k(t, x, r) - \nabla_x \psi_{n_2,1}^k(t, x, r)| dr \sup_{(t, y) \in \bar{O}_\varepsilon} \left| \nabla_y \gamma_{n_1}^j(t, y) - \nabla_y \gamma_{n_2}^j(t, y) \right| \\
+ \sum_{1 \leq k, l \leq d} \left| \int_{0}^{\Gamma^z(t, x)} |\nabla_x \psi_{n_1,1}^k(t, x, r) - \nabla_x \psi_{n_2,1}^k(t, x, r)| dr \\
+ \sum_{1 \leq k, l \leq d} \left| \partial_y \partial_y \gamma_{n_1}^j(t, y_n(t, x, r)) \psi_{n_1,1}^k(t, x, r) \psi_{n_2,1}^k(t, x, r) | \partial_x \gamma_n(t, x)| \\
- \partial_y \partial_y \gamma_{n_2}^j(t, y_n(t, x, r)) - \partial_y \partial_y \gamma_{n_2}^j(t, y_n(t, x, r)) | \psi_{n_2,1}^k(t, x, r) \psi_{n_2,1}^k(t, x, r) | \partial_x \gamma_n(t, x)| \\
\right| \right|.
Finally we show (7.2). From (4.4), (7.3), (7.4) and (7.5), we have

\[
\sup_{\epsilon > 0} \sup_{n \geq N_1(\epsilon)} \| \psi_n^j(t, x, \Gamma^z(t, x)) \|_{W^{\alpha+1}_{2d+2}(\Omega_\epsilon)} < \infty.
\]

Together with (4.6) and (7.4), we see that \( \psi_n^j(t, x, \Gamma^z(t, x)) \in W_{2d+2}(\Omega_\epsilon) \) and

\[
\sup_{\epsilon > 0} \| \psi_n^j(t, x, \Gamma^z(t, x)) \|_{W_{2d+2}^{\alpha+1}(\Omega_\epsilon)} < \infty.
\]

Since \( \cup_{\epsilon > 0} \Omega_\epsilon = (((t_0 - \eta_2) \cap (0, t_0 + \eta_2) \cap T_1) \times C(z, \delta_5)) \cap \hat{D} \), we have (7.2).

**Proof of Proposition 4.2**

We first show that for any \( \epsilon > 0 \),

\[
\lim_{n, m \to \infty} \| \int_{\Gamma_n^z(t, x)} (\partial_t \gamma_n)(t, y_n(t, x, \tau)) |d\tau| \|_{L^{2d+2}(\Omega_\epsilon)} = 0. \tag{7.8}
\]

By Lemma 4.1, (4.24) and the Hölder inequality, we have for any \( M > 0 \) and \( n \geq N_1(\epsilon) \),

\[
\| \int_{\Gamma_n^z(t, x)} (\partial_t \gamma_n)(t, y_n(t, x, \tau)) |d\tau| \|_{L^{2d+2}(\Omega_\epsilon)} \\
\leq \| \int_{\Gamma_n^z(t, x)} (\partial_t \gamma_n)(t, y_n(t, x, \tau)) - (\partial_t \gamma)(t, y_n(t, x, \tau)) |d\tau| \|_{L^{2d+2}(\Omega_\epsilon)} \\
+ \| \int_{\Gamma_n^z(t, x)} (\partial_t \gamma)(t, y_n(t, x, \tau)) |d\tau| \|_{L^{2d+2}(\Omega_\epsilon)} \\
\leq \| \int_{-\rho_1}^{\rho_1} (\partial_t \gamma_n)(t, y_n(t, x, \tau)) - (\partial_t \gamma)(t, y_n(t, x, \tau)) |I_D(t, \epsilon(t))| \|_{2d+2}(\Omega_\epsilon) \\
+ \| \int_{\Gamma_n^z(t, x)} (\partial_t \gamma)(t, y_n(t, x, \tau)) |d\tau| \|_{L^{2d+2}(\Omega_\epsilon)} \\
\leq \| \partial_t \gamma_n - \partial_t \gamma \|_{L^{2d+2}(\Omega_\epsilon')} \\
+ \left( \int_{(t_0 + \eta_2) \cap T_1} \int_{C(z, \delta_5) \cap D(t, x)} \| (\partial_t \gamma)(t, y_n(t, x, \tau)) \|_{2d+2} |d\tau| |dxdt \right)^{\frac{1}{2d+2}} \\
\leq \| \partial_t \gamma_n - \partial_t \gamma \|_{L^{2d+2}(\Omega_\epsilon')} + M(\int_{(t_0 - \eta_2) \cap T_1} \int_{C(z, \delta_5)} \| (\partial_t \gamma)(t, y_n(t, x, \tau)) \|_{2d+2} |d\tau| |dxdt \right)^{\frac{1}{2d+2}} \\
+ \left( \int_{(t_0 - \eta_2) \cap T_1} \int_{C(z, \delta_5) \cap D(t, x)} \int_{-\rho_1}^{\rho_1} |(\partial_t \gamma)(t, y_n(t, x, \tau))| \|I_D(t, \epsilon(t))| |d\tau| |dxdt \right)^{\frac{1}{2d+2}} \\
\leq \| \partial_t \gamma_n - \partial_t \gamma \|_{L^{2d+2}(\Omega_\epsilon')} + M(\int_{(t_0 - \eta_2) \cap T_1} \int_{C(z, \delta_5)} \| \Gamma_n^z(t, x) - \Gamma_m^z(t, x) |dxdt \|_{2d+2} \\
+ \left( \int_{-\rho_1}^{\rho_1} \int_{(t_0 - \eta_2) \cap T_1} \int_{D(t, \epsilon(t))} |(\partial_t \gamma)(t, x)| \|I_D(t, \epsilon(t))| |dxdt \right)^{\frac{1}{2d+2}}. 
\]

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Combining this with Proposition 3.2 and (4.15), we obtain that
\[
\lim_{n,m \to \infty} \left\| \int_{\Gamma_n^*(t,x)} (\partial_t \gamma_n)(t, y_n(t, x, \tau)) d\tau \right\|_{L^{2d+2}(\Omega_\varepsilon)} \\
\lesssim (\int_{(t_0-\eta_2)^2 T_1} \int_{D(t, \hat{\rho}(\varepsilon))} |(\partial_t \gamma)(t, x)| d\tau)^{1/2}.
\] (7.9)

Since \( \gamma \in W^{1,2}_{2d+2}(\tilde{D}) \), letting \( M \to \infty \) in (7.9), we get (7.8).

Next we show that for any \( \varepsilon > 0 \),
\[
\lim_{n,m \to \infty} \left\| \int_{0}^{\Gamma_m^*(t,x)} \left| (\partial_t \gamma_n)(t, y_n(t, x, \tau)) - (\partial_t \gamma_m)(t, y_m(t, x, \tau)) \right| d\tau \right\|_{L^{2d+2}(\Omega_\varepsilon)} = 0.
\] (7.10)

For \( (t, x) \in \Omega_\varepsilon \) and \( n, m \geq N_1(\varepsilon) \), by (4.24) we have
\[
\left| \int_{0}^{\Gamma_m^*(t,x)} \left| (\partial_t \gamma_n)(t, y_n(t, x, \tau)) - (\partial_t \gamma_m)(t, y_m(t, x, \tau)) \right| d\tau \right| \\
\leq \left| \int_{0}^{\Gamma_m^*(t,x)} \left| (\partial_t \gamma_n)(t, y_n(t, x, \tau)) - (\partial_t \gamma)(t, y_n(t, x, \tau)) \right| d\tau \right| + \left| \int_{0}^{\Gamma_m^*(t,x)} \left| (\partial_t \gamma)(t, y_n(t, x, \tau)) - (\partial_t \gamma_m)(t, y_m(t, x, \tau)) \right| d\tau \right| \\
\leq \int_{-\rho_1}^{\rho_1} \left| (\partial_t \gamma_n)(t, y_n(t, x, \tau)) - (\partial_t \gamma)(t, y_n(t, x, \tau)) \right| I_D(t, \hat{\rho}(\varepsilon))(y_n(t, x, \tau)) d\tau \\
+ \int_{-\rho_1}^{\rho_1} \left| (\partial_t \gamma_m)(t, y_m(t, x, \tau)) - (\partial_t \gamma)(t, y_m(t, x, \tau)) \right| I_D(t, \hat{\rho}(\varepsilon))(y_m(t, x, \tau)) d\tau \\
+ \int_{-\rho_1}^{\rho_1} \left| (\partial_t \gamma)(t, y_n(t, x, \tau)) - (\partial_t \gamma)(t, y_m(t, x, \tau)) \right| I_D(t, \hat{\rho}(\varepsilon))(y_n(t, x, \tau)) I_D(t, \hat{\rho}(\varepsilon))(y_m(t, x, \tau)) d\tau.
\]
which implies
\[
\left\| \int_0^\Gamma_n (t, x) \left| (\partial_t \gamma_n)(t, y_n(t, x, \tau)) - (\partial_t \gamma_m)(t, y_m(t, x, \tau)) \right| d\tau \right\|_{L^{2d+2}(\mathcal{O}_\varepsilon)} \lesssim \left( \int_{-\rho_1}^{\rho_1} \int_{(t_0-\eta_2)\vee 0}^{(t_0+\eta_2)\wedge T_1} \int_{\mathbb{R}^d} \left| (\partial_t \gamma_n)(t, y_n(t, x, \tau)) - (\partial_t \gamma)(t, y_n(t, x, \tau)) \right|^{2d+2} \right. \\
\left. \times I_{D(t, \varepsilon)}(y_n(t, x, \tau)) dxdtd\tau \right)^{\frac{1}{2d+2}} + \left( \int_{-\rho_1}^{\rho_1} \int_{(t_0-\eta_2)\vee 0}^{(t_0+\eta_2)\wedge T_1} \int_{\mathbb{R}^d} \left| (\partial_t \gamma)(t, y_n(t, x, \tau)) - (\partial_t \gamma)(t, y_m(t, x, \tau)) \right|^{2d+2} \right. \\
\left. \times I_{D(t, \varepsilon)}(y_n(t, x, \tau)) dxdtd\tau \right)^{\frac{1}{2d+2}} + \left( \int_{-\rho_1}^{\rho_1} \int_{(t_0-\eta_2)\vee 0}^{(t_0+\eta_2)\wedge T_1} \int_{\mathbb{R}^d} \left| (\partial_t \gamma)(t, y_m(t, x, \tau)) - (\partial_t \gamma)(t, y_m(t, x, \tau)) \right|^{2d+2} \right. \\
\left. \times I_{D(t, \varepsilon)}(y_n(t, x, \tau)) dxdtd\tau \right)^{\frac{1}{2d+2}} \lesssim \left| \partial_t \gamma_n - \partial_t \gamma \right|_{L^{2d+2}(\mathcal{O}_\varepsilon)} + \left| \partial_t \gamma_m - \partial_t \gamma \right|_{L^{2d+2}(\mathcal{O}_\varepsilon)} \\
+ \left( \int_{-\rho_1}^{\rho_1} \int_{(t_0-\eta_2)\vee 0}^{(t_0+\eta_2)\wedge T_1} \int_{\mathbb{R}^d} \left| (\partial_t \gamma)(t, y_n(t, x, \tau)) - (\partial_t \gamma)(t, y_m(t, x, \tau)) \right|^{2d+2} \right. \\
\left. \times I_{D(t, \varepsilon)}(y_n(t, x, \tau)) I_{D(t, \varepsilon)}(y_m(t, x, \tau)) dxdtd\tau \right)^{\frac{1}{2d+2}} \to 0,
\]
as \( n, m \to \infty \) by Proposition \ref{proposition 3.2}, Lemma \ref{lemma 4.1} and Lemma \ref{lemma 7.1}.

Now we show \ref{lemma 4.1}, i.e.
\[
\sup_{\varepsilon > 0} \sup_{n \geq N_1(\varepsilon)} \left\| \Lambda_n(t, x, \Gamma_n^\varepsilon(t, x)) \right\|_{L^{2d+2}(\mathcal{O}_\varepsilon)} < \infty.
\]
(7.11)

Applying the Gronwall’s inequality to \ref{lemma 4.1}, we get that for \( (t, x) \in \mathcal{O}_\varepsilon \) and \( n \geq N_1(\varepsilon) \),
\[
\left\| \sup_{\tau \in (0, \Gamma_n^\varepsilon(t, x) \cup (\Gamma_n^\varepsilon(t, x), \Gamma_m^\varepsilon(t, x))] \right\|_{L^{2d+2}(\mathcal{O}_\varepsilon)} \left| \Lambda_n(t, x, \tau) \right| \right\|_{L^{2d+2}(\mathcal{O}_\varepsilon)} \lesssim \left\| \sup_{\tau \in (0, \Gamma_n^\varepsilon(t, x) \cup (\Gamma_n^\varepsilon(t, x), \Gamma_m^\varepsilon(t, x))]} \int_0^\tau \left| (\partial_t \gamma_n)(t, y_n(t, x, \tau')) \right| d\tau' \right\|_{L^{2d+2}(\mathcal{O}_\varepsilon)} \\
\lesssim \left( \int_{\rho_1}^{-\rho_1} \int_{(t_0-\eta_2)\wedge 0}^{(t_0+\eta_2)\wedge T_1} \int_{\mathbb{R}^d} \left| (\partial_t \gamma_n)(t, y_n(t, x, \tau')) \right|^{2d+2} I_{D(t, \varepsilon)}(y_n(t, x, \tau')) dxdtd\tau' \right)^{\frac{1}{2d+2}} \\
\lesssim \left| \partial_t \gamma_n \right|_{L^{2d+2}(\mathcal{O}_\varepsilon)},
\]
where the last inequality follows from Lemma \ref{lemma 4.1} Together with \ref{equation 3.15}, we have (7.11).

Next we show that for any \( \varepsilon > 0 \),
\[
\lim_{n, m \to \infty} \left\| \Lambda_n(t, x, \Gamma_m^\varepsilon(t, x)) - \Lambda_m(t, x, \Gamma_m^\varepsilon(t, x)) \right\|_{L^{2d+2}(\mathcal{O}_\varepsilon)} = 0.
\]
(7.13)
Since for \( r \in \mathbb{R} \),
\[
|\Lambda_n(t, x, r) - \Lambda_m(t, x, r)| \\
\leq \int_0^r \left| (\partial_t \gamma_n)(t, y_n(t, x, \tau)) - (\partial_t \gamma_m)(t, y_m(t, x, \tau)) \right| d\tau \\
+ \int_0^r \left| (\nabla_y \gamma_n(t, y_n(t, x, \tau)) - \nabla_y \gamma(t, y_n(t, x, \tau))) \cdot \Lambda_n(t, x, \tau) \right| d\tau \\
+ \int_0^r \left| (\nabla_y \gamma_m(t, y_m(t, x, \tau)) - \nabla_y \gamma(t, y_m(t, x, \tau))) \cdot \Lambda_n(t, x, \tau) \right| d\tau \\
+ \int_0^r \left| (\nabla_y \gamma(t, y_m(t, x, \tau)) - \nabla_y \gamma(t, y_m(t, x, \tau))) \cdot \Lambda_n(t, x, \tau) \right| d\tau \\
+ \sup_{(t, y) \in [0, T_1] \times \mathbb{R}^d} \left| \nabla_y \gamma_m(t, y) \right| \int_0^r \left| (\Lambda_n(t, x, \tau) - \Lambda_m(t, x, \tau)) \right| d\tau,
\]
by the Gronwall’s inequality and Lemma 4.4, we see that for \( (t, x) \in \mathcal{O}_\varepsilon \) and \( n, m \geq N_1(\varepsilon) \),
\[
|\Lambda_n(t, x, \Gamma^z_{m}(t, x)) - \Lambda_m(t, x, \Gamma^z_{m}(t, x))| \\
\leq \int_0^{\Gamma^z_{m}(t, x)} \left| (\partial_t \gamma_n)(t, y_n(t, x, \tau)) - (\partial_t \gamma_m)(t, y_m(t, x, \tau)) \right| d\tau \\
+ \int_0^{\Gamma^z_{m}(t, x)} \left| (\nabla_y \gamma_n(t, y_n(t, x, \tau)) - \nabla_y \gamma(t, y_n(t, x, \tau))) \cdot \Lambda_n(t, x, \tau) \right| d\tau \\
+ \int_0^{\Gamma^z_{m}(t, x)} \left| (\nabla_y \gamma_m(t, y_m(t, x, \tau)) - \nabla_y \gamma(t, y_m(t, x, \tau))) \cdot \Lambda_n(t, x, \tau) \right| d\tau \\
+ \int_0^{\Gamma^z_{m}(t, x)} \left| (\nabla_y \gamma(t, y_m(t, x, \tau)) - \nabla_y \gamma(t, y_m(t, x, \tau))) \cdot \Lambda_n(t, x, \tau) \right| d\tau \\
(7.14)
\]
\[
\leq \int_0^{\Gamma^z_{m}(t, x)} \left| (\partial_t \gamma_n)(t, y_n(t, x, \tau)) - (\partial_t \gamma_m)(t, y_m(t, x, \tau)) \right| d\tau \\
+ \rho_1 \sup_{(t, y) \in [0, T_1] \times \mathbb{R}^d} \left| \nabla_y \gamma_n(t, y) \right| \sup_{\tau \in (0, \Gamma^z_{m}(t, x))} \left| \Lambda_n(t, x, \tau) \right| \\
+ \rho_1 \sup_{(t, y) \in [0, T_1] \times \mathbb{R}^d} \left| \nabla_y \gamma_m(t, y) \right| \sup_{\tau \in (0, \Gamma^z_{m}(t, x))} \left| \Lambda_n(t, x, \tau) \right| \\
+ \rho_1 \sup_{(t, x) \in [0, T_1] \times \mathbb{R}^d, \quad \tau \in (-\rho_1, \rho_1)} \left| \nabla_y \gamma(t, y_n(t, x, \tau)) - \nabla_y \gamma(t, y_m(t, x, \tau)) \right| \sup_{\tau \in (0, \Gamma^z_{m}(t, x))} \left| \Lambda_n(t, x, \tau) \right|.
\]
Hence by Proposition 3.2, (4.5), (7.10), (7.12) and (7.14),
\[
\| \Lambda_n(t, x, \Gamma_m^z(t, x)) - \Lambda_m(t, x, \Gamma_n^z(t, x)) \|_{L^{2d+2}(O_\epsilon)} \\
\leq \| \int_0^{\Gamma_n^z(t, x)} \| (\partial_t \gamma_n)(t, y_n(t, x, \tau)) - (\partial_t \gamma_m)(t, y_m(t, x, \tau)) \| d\tau \|_{L^{2d+2}(O_\epsilon)} \\
+ \sup_{(t, y) \in [0, T_1] \times \mathbb{R}^d} \| \nabla_y \gamma_n(t, y) - \nabla_y \gamma(t, y) \| \| \partial_t \gamma_n \|_{L^{2d+2}(O_\epsilon')} \\
+ \sup_{(t, y) \in [0, T_1] \times \mathbb{R}^d} \| \nabla_y \gamma_m(t, y) - \nabla_y \gamma(t, y) \| \| \partial_t \gamma_m \|_{L^{2d+2}(O_\epsilon')} \\
+ \sup_{(t, x, \tau) \in [0, T_1] \times \mathbb{R}^d, \tau \in (-\rho_1, \rho_1)} \| \nabla_y \gamma(t, y_n(t, x, \tau)) - \nabla_y \gamma(t, y_m(t, x, \tau)) \| \| \partial_t \gamma_n \|_{L^{2d+2}(O_\epsilon)} \\
\rightarrow 0,
\]
as \( n, m \to \infty \).

Now we show (4.42), i.e.
\[
\lim_{n, m \to \infty} \| \Lambda_n(t, x, \Gamma_n^z(t, x)) - \Lambda_m(t, x, \Gamma_m^z(t, x)) \|_{L^{2d+2}(O_\epsilon)} = 0. \tag{7.15}
\]
By (7.13), to prove (7.15), we need only to prove that
\[
\lim_{n, m \to \infty} \| \Lambda_n(t, x, \Gamma_n^z(t, x)) - \Lambda_n(t, x, \Gamma_m^z(t, x)) \|_{L^{2d+2}(O_\epsilon)} = 0.
\]
By the definition of \( \Lambda_n(t, x, r) \), (4.13), (7.8) and (7.12), we have
\[
\| \Lambda_n(t, x, \Gamma_m^z(t, x)) - \Lambda_n(t, x, \Gamma_n^z(t, x)) \|_{L^{2d+2}(O_\epsilon)} \\
= \| \int_{\Gamma_m(t, x)} \| (\partial_t \gamma_n)(t, y_n(t, x, r)) + \partial_y \gamma_n(t, y_n(t, x, r)) \|_{L^{2d+2}(O_\epsilon)} \| dr \|_{L^{2d+2}(O_\epsilon)} \\
\leq \| \int_{\Gamma_n(t, x)} \| (\partial_t \gamma_n)(t, y_n(t, x, r)) \| dr \|_{L^{2d+2}(O_\epsilon)} \\
+ \| \sup_{r \in \Gamma_n(t, x)} \| \Lambda_n(t, x, r) \|_{L^{2d+2}(O_\epsilon)} \| \sup_{(t, x) \in O_\epsilon} | \Gamma_n^z(t, x) - \Gamma_m^z(t, x) | \rightarrow 0,
\]
as \( n, m \to \infty \).

Finally we show (4.43), i.e.
\[
\lim_{n, m \to \infty} \| \partial_t \Gamma_n^z(t, x) - \partial_t \Gamma_m^z(t, x) \|_{L^{2d+2}(O_\epsilon)} = 0. \tag{7.16}
\]
By (4.12) and (4.14), we have for \((t,x) \in O_\varepsilon\) and \(n, m \geq N_1\),
\[
|\partial_t \Gamma^z_n(t,x) - \partial_t \Gamma^z_m(t,x)| \\
\leq \left| \frac{\Lambda_n(t,x,\Gamma^z_n(t,x)) - \Lambda_m(t,x,\Gamma^z_m(t,x))}{\gamma_n(t,y_n(t,x,\Gamma^z_n(t,x))) \cdot \gamma(t_0,z)} - \frac{1}{\gamma_m(t,y_m(t,x,\Gamma^z_m(t,x))) \cdot \gamma(t_0,z)} \right| \Lambda_m(t,x,\Gamma^z_m(t,x)) \\
\leq \cos^{-1} \theta |\Lambda_n(t,x,\Gamma^z_n(t,x)) - \Lambda_m(t,x,\Gamma^z_m(t,x))| \\
+ \cos^{-2} \theta |\gamma_n(t,y_n(t,x,\Gamma^z_n(t,x))) - \gamma_m(t,y_m(t,x,\Gamma^z_m(t,x)))| |\Lambda_n(t,x,\Gamma^z_n(t,x))| \\
\lesssim |\Lambda_n(t,x,\Gamma^z_n(t,x)) - \Lambda_m(t,x,\Gamma^z_m(t,x))| \\
+ |\Lambda_m(t,x,\Gamma^z_m(t,x))| \sup_{(t,x) \in O_\varepsilon} |\gamma_n(t,y_n(t,x,\Gamma^z_n(t,x))) - \gamma_m(t,y_m(t,x,\Gamma^z_m(t,x)))|.
\]

So together with (3.13), (4.5), (4.15), (7.11) and (7.15), we obtain (7.16).

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