Exact $S$-matrices for AdS$_3$/CFT$_2$

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Abstract

We propose exact $S$-matrices for the AdS$_3$/CFT$_2$ duality between Type IIB strings on AdS$_3 \times S^3 \times M_4$ with $M_4 = S^3 \times S^1$ or $T^4$ and the corresponding two-dimensional conformal field theories. We fix the complete two-particle $S$-matrices for both those cases of AdS$_3$/CFT$_2$, on the basis of the symmetries $su(1|1)$ and $su(1|1) \times su(1|1)$, respectively preserved by their vacua. A crucial justification comes from the derivation of the all-loop Bethe ansatz matching exactly the recent conjecture proposed by [1] and [2].

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1 Introduction

The discovery of integrable structures on both sides of the AdS$_5$/CFT$_4$ correspondence [3, 4], was crucial in understanding and determine exactly some important physical quantities (see [5] and references therein) in $\mathcal{N} = 4$ Super-Yang-Mills and the IIB superstring theory on AdS$_5 \times S^5$ in the planar limit.

From the integrability point of view, one of the most recently investigated example of such gauge/string duality is the AdS$_3$/CFT$_2$ correspondence between IIB superstring theory on AdS$_3 \times S^3 \times S^3 \times S^1$ or AdS$_3 \times S^3 \times T^4$ backgrounds with RR fluxes and yet quite unknown two-dimensional CFTs. Indeed, while the NS AdS$_3$/CFT$_2$ was solved completely by implementing techniques typical of two-dimensional CFTs [6], the RR counterpart remains quite obscure: there the usual 2D CFT methods fail, then one of the most promising way to attack this problem is given by integrability techniques.

This investigation started in [1, 2], where a set of all-loop Bethe equations, describing in principle at any coupling the asymptotic spectrum of the string energies and the dimensions of the yet unknown gauge operators, were proposed on the basis of classical integrability of the corresponding supercoset sigma models. Unfortunately this approach cannot take into account the contribution of some (massless) modes of the full string theory. Some progress in the direction of incorporating them has been done very recently in [9], where a set of Bethe equations, completely decoupled from the others, has been proposed in order to describe the massless modes. Now, this would mean that the $S$-matrix between the massless and massive modes become trivial and lead to independent set of Bethe equations.

The aim of this Letter is to propose an $S$-matrix for the massive modes, in order to derive, on a firmer ground, the Bethe equations proposed in [1, 2]. We shall do this by using an analytic Bethe ansatz involving transfer matrix eigenvalues derived from the diagonalization of $su(1|1)$ and $su(1|1) \times su(1|1)$ invariant $S$-matrices, respectively for the AdS$_3 \times S^3 \times S^3 \times S^1$ and AdS$_3 \times S^3 \times T^4$ cases. Basically, this means that the $S^3 \times S^1$ ($T^4$) case has total symmetry $d(2,1;\alpha) \times d(2,1;\alpha)$ ($psu(1,1|2) \times psu(1,1|2)$), but the on-shell particle symmetry which preserves the vacuum is $su(1|1)$ ($su(1|1) \times su(1|1)$).

In the different context of open AdS$_5$/CFT$_4$ spin chains, the analytic Bethe ansatz built on an $su(1|1)$-invariant $S$-matrix, previously found in [10, 11], was already performed, without considering possible scalar factors, by [12] in order to determine the corresponding transfer matrix eigenvalues and Bethe equations. On the other hand, an $su(1|1) \times su(1|1)$-invariant $S$-matrix was proposed in [13] to describe the scattering of magnons in AdS$_3 \times S^3 \times T^4$; how-

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1In the case of AdS$_3 \times S^3 \times T^4$, the CFT dual is a $\mathcal{N} = (4,4)$ theory on a symmetric product of $T^4$.

2Some first tests of these Bethe equations against string energy calculations have been performed in [7].
ever, only magnons\(^3\) in the \(su(2)\) sector were analyzed there, in order to derive the dressing phase up to one-loop, and Bethe equations were not derived.

2 Spectrum and \(S\)-matrix

2.1 \(AdS_3 \times S^3 \times T^4\)

For the case of \(AdS_3 \times S^3 \times T^4\), the spectrum consists of eight massive modes whose energy-momentum dispersion relation is given by

\[
E = \sqrt{\frac{1}{4} + 4h^2(\lambda) \sin^2 \frac{p}{2}},
\]

where \(h\) is an almost unknown function of the 't Hooft coupling \(\lambda\): its strong coupling behavior has been predicted to be \(h(\lambda) \sim \sqrt{\lambda/2\pi}\) by [1], while the one-loop correction has been calculated recently by [8]. These are grouped into bifundamentals of \(su(1|1) \times su(1|1)\), which we refer to “A” and “B”. The \(S\)-matrices among these bifundamentals are given by tensor products of two \(su(1|1)\)-invariant \(S\)-matrices as follows:

\[
\begin{align*}
S^{(AA)}(p_1, p_2) &= S^{(BB)}(p_1, p_2) = S_0(p_1, p_2) \left[ \hat{S}(p_1, p_2) \otimes \hat{S}(p_1, p_2) \right], \\
S^{(AB)}(p_1, p_2) &= S^{(BA)}(p_1, p_2) = \tilde{S}_0(p_1, p_2) \left[ \hat{S}(p_1, p_2) \otimes \hat{S}(p_1, p_2) \right],
\end{align*}
\]

where [11, 12, 13]

\[
\hat{S}(p_1, p_2) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{x_1^+ - x_2^+}{x_1^+ - x_2^-} & \frac{x_1^+ - x_2^-}{x_1^- - x_2^-} & 0 \\
0 & \frac{x_1^+ - x_2^-}{x_1^- - x_2^-} & \frac{x_1^- - x_2^-}{x_1^- - x_2^-} & 0 \\
0 & 0 & 0 & \frac{x_1^- - x_2^+}{x_1^- - x_2^-}
\end{pmatrix},
\]

and we set \(\omega_{1,2} = \omega(p_{1,2}) = 1\). The \(x^\pm\) variables are the usual Zhukowski variables defined by

\[
\frac{x^+}{x^-} = e^{i\varphi}x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{h(\lambda)}.
\]

The \(S\)-matrix (2.4) satisfies the unitarity condition, but it does not have crossing symmetry [12]. An attempt to derive the crossing symmetry relations for the \(su(1|1)\) algebra has been put forward in [13] by using the antipode operation, but this implies, in this case,

\(^3\)Which were previously argued in [14] being BPS states in a centrally extended \(su(1|1) \times su(1|1)\) algebra.
a transformation on the kinematic variables \((x^\pm \rightarrow x^{\mp})\) that does not correspond to the particle-antiparticle transformation \((x^\pm \rightarrow 1/x^\pm)\). Then we guess a possible expression for the scalar factors on the basis of the unitarity and the final matching with the Bethe equations proposed by \([1,2]\):

\[
S_0(p_1,p_2) = \frac{x_1^+ - x_2^-}{x_1^- - x_2^+} \frac{1 - \frac{1}{x_1 x_2}}{1 - \frac{1}{x_1 x_2}} \sigma^2(p_1,p_2) \frac{x_1^- x_2^+}{x_1^+ x_2^-}, \quad \tilde{S}_0(p_1,p_2) = \sigma^{-2}(p_1,p_2) \frac{x_1^+ x_2^-}{x_1^- x_2^+},
\]

(2.6)

where \(\sigma(p_1,p_2)\) is the BES dressing phase \([13]\) and \(\bar{p}\) denotes the momentum of an antiparticle, such that \(x^\pm(\bar{p}) = 1/x^\pm(p)\). The scalar factors \([2,6]\) satisfy the relation \(S_0(p_1,p_2) = S_0(\bar{p}_1,\bar{p}_2), \tilde{S}_0(p_1,p_2) = \tilde{S}_0(\bar{p}_1,\bar{p}_2)\), that will be important later for the construction of the Bethe equations, and unitarity:

\[
S_0(p_1,p_2)S_0(p_2,p_1) = S_{su(2)}(p_1,p_2)\sigma^2(p_1,p_2) \frac{x_1^- x_2^+}{x_1^+ x_2^-} S_{su(2)}^{-1} S_{su(2)}^{-2}(p_1,p_2) \frac{x_2^- x_1^+}{x_2^+ x_1^-} = 1, \quad (2.7)
\]

\[
\tilde{S}_0(p_1,p_2)\tilde{S}_0(p_2,p_1) = \sigma^{-2}(p_1,\bar{p}_2) \frac{x_1^+ x_2^-}{x_1^- x_2^+} \sigma^{-2}(\bar{p}_2,p_1) \frac{x_1^- x_2^+}{x_1^+ x_2^-} = 1, \quad (2.8)
\]

where \(S_{su(2)}(p_1,p_2) = \frac{x_1^- - x_2^-}{x_1^- - x_2^+} \frac{1 - \frac{1}{x_1 x_2}}{1 - \frac{1}{x_1 x_2}}\).

The Bethe-Yang equations are derived from a periodic boundary condition (PBC). On a circle with circumference \(L\), we put \(N_A\) number of “A” particles with momenta \(\{p_A^1, p_A^2, \ldots, p_A^{N_A}\}\) and \(N_B\) number of “B” particles with momenta \(\{p_B^1, p_B^2, \ldots, p_B^{N_B}\}\). Now we choose an “A” particle with a momentum \(p_A^1\) and move it around the circle by scattering with all the other particles and similarly for a “B” particle with a momentum \(p_B^1\). Since this virtual process does not change any configuration, we arrive at PBC conditions

\[
e^{-ip_A^1 L} = \prod_{k=1,\neq j}^{N_A} S_0(p_A^j,p_k^A) \prod_{k=1}^{N_B} \tilde{S}_0(p_A^1,p_B^1) \left[ \hat{T}_{su(1|1)}(p_A^1|\{p_A^1, p_A^B\}) \otimes \hat{T}_{su(1|1)}(p_A^B|\{p_A^A, p_A^1\}) \right],
\]

\[
e^{-ip_B^1 L} = \prod_{k=1,\neq j}^{N_B} S_0(p_B^j,p_k^B) \prod_{k=1}^{N_A} \tilde{S}_0(p_B^1,p_A^1) \left[ \hat{T}_{su(1|1)}(p_B^1|\{p_B^1, p_B^A\}) \otimes \hat{T}_{su(1|1)}(p_B^A|\{p_B^B, p_B^1\}) \right],
\]

where \(\hat{T}_{su(1|1)}\) is a transfer matrix made of the \(su(1|1)\)-invariant \(S\)-matrix,

\[
\hat{T}_{su(1|1)}(p|\{p_A^1\},\{p_B^1\}) = \text{str}_a \left[ \hat{S}_{A_1}(p,p_A^1) \cdots \hat{S}_{A_{N_A}}(p,p_A^{N_A}) \hat{S}_{B_1}(p,p_B^1) \cdots \hat{S}_{B_{N_B}}(p,p_B^{N_B}) \right] \quad (2.9)
\]

and \(a, A_i\) and \(B_i\) stand for a two-dimensional vector space which the \(S\)-matrices act on.
2.2 AdS$_3 \times S^3 \times S^3 \times S^1$

The spectrum of AdS$_3 \times S^3 \times S^3 \times S^1$ is a bit more complicated. Denoting $l, R_1, R_2$ the radii of AdS$_3$ and the two $S^3$'s respectively, one has the following relation

$$\frac{1}{R_1^2} + \frac{1}{R_2^2} = \frac{1}{l^2}. \quad (2.10)$$

By defining $\alpha = l^2/R_1^2$, one can find two massive multiplets, each of which consists of two bosons and two fermions, with two different masses:

$$E_l = \sqrt{m_1^2 + 4\alpha^2 (\sin^2 \frac{p_l}{2})}, \quad l = 1, 3, \quad (2.11)$$

where

$$m_1 = \alpha, \quad m_3 = 1 - \alpha. \quad (2.12)$$

We propose that the four particles with mass $m_1$ are grouped into two fundamentals of $su(1|1)$, which we refer to “1” and “1̅”; and similarly the other four particles with mass $m_3$ into two additional fundamentals of $su(1|1)$, which we refer to “3” and “3̅”. In this case the Zhukowsky variables are defined as [2]:

$$x^+_{1,1} + \frac{1}{x^+_{1,1}} - x^-_{1,1} = \frac{i\alpha}{h(\lambda)}; \quad x^+_{3,3} + \frac{1}{x^+_{3,3}} - x^-_{3,3} = \frac{i(1 - \alpha)}{h(\lambda)}. \quad (2.13)$$

The $S$-matrices among these four doublets are given by single $su(1|1)$-invariant $S$-matrices as follows:

$$S^{(11)}(p_1, p_2) = S^{(33)}(p_1, p_2) = S^{(11)}(p_1, p_2) = S^{(33)}(p_1, p_2) = S_0(p_1, p_2) \hat{S}(p_1, p_2) \quad (2.14)$$

$$S^{(11)}(p_1, p_2) = S^{(33)}(p_1, p_2) = S^{(11)}(p_1, p_2) = S^{(33)}(p_1, p_2) = \tilde{S}_0(p_1, p_2) \tilde{S}(p_1, p_2) \quad (2.15)$$

$$S^{(13)}(p_1, p_2) = S^{(31)}(p_1, p_2) = S^{(13)}(p_1, p_2) = S^{(31)}(p_1, p_2) = \hat{S}(p_1, p_2) \quad (2.16)$$

$$S^{(13)}(p_1, p_2) = S^{(31)}(p_1, p_2) = S^{(13)}(p_1, p_2) = S^{(31)}(p_1, p_2) = \tilde{S}(p_1, p_2), \quad (2.17)$$

where $\hat{S}(p_1, p_2)$ is given in Eq. (2.14) and the scalar factors $S_0$ and $\tilde{S}_0$ are defined in Eq. (2.6).

The Bethe-Yang equations can be written in a similar way as before. On a circle with circumference $L$, we put $N_1$ number of “1” particles with momenta $\{p^1_1, p^1_2, \ldots, p^1_{N_1}\}$, $N_1$ number of “1̅” particles with momenta $\{p^1_{1}, p^1_{2}, \ldots, p^1_{N_1}\}$, $N_3$ number of “3” particles with momenta $\{p^3_1, p^3_2, \ldots, p^3_{N_3}\}$, and $N_3$ number of “3̅” particles with momenta $\{p^3_1, p^3_2, \ldots, p^3_{N_3}\}$.
From these configurations, the PBC equations become

\[ e^{ip^j L} = \prod_{k=1,\neq j}^{N_1} S_0(p^1_j, p^1_k) \prod_{k=1}^{N_1} \tilde{S}_0(p^1_j, p^1_k) \cdot \tilde{T}_{su(1|1)}(p^1_j) \{ p^1_j, p^1_l, p^3_l, p^3_l \} \]  

(2.18)

\[ e^{ip^j L} = \prod_{k=1,\neq j}^{N_1} S_0(p^3_j, p^3_k) \prod_{k=1}^{N_1} \tilde{S}_0(p^3_j, p^3_k) \cdot \tilde{T}_{su(1|1)}(p^3_j) \{ p^1_j, p^1_l, p^3_l, p^3_l \} \]  

(2.19)

\[ e^{ip^3 L} = \prod_{k=1,\neq j}^{N_3} S_0(p^3_j, p^3_k) \prod_{k=1}^{N_3} \tilde{S}_0(p^3_j, p^3_k) \cdot \tilde{T}_{su(1|1)}(p^3_j) \{ p^1_j, p^1_l, p^3_l, p^3_l \} \]  

(2.20)

\[ e^{ip^3 L} = \prod_{k=1,\neq j}^{N_3} S_0(p^3_j, p^3_k) \prod_{k=1}^{N_3} \tilde{S}_0(p^3_j, p^3_k) \cdot \tilde{T}_{su(1|1)}(p^3_j) \{ p^1_j, p^1_l, p^3_l, p^3_l \} , \]  

(2.21)

where \( \tilde{T}_{su(1|1)} \) is given in Eq.(2.3).

### 3 Derivation of asymptotic Bethe ansatz equations

#### 3.1 Diagonalization of the transfer matrix

The \( su(1|1) \) transfer matrix has been diagonalized by the analytic Bethe ansatz method in [12]. The eigenvalues can be expressed by

\[ A(p\{p_\ell\}, \{\lambda_j\}) = \Lambda_0(p\{p_\ell\}) A(p\{\lambda_j\}) , \]  

(3.1)

\[ \Lambda_0(p\{p_\ell\}) = 1 - \prod_{\ell=1}^{N} \left( \frac{x^+(p) - x^+(p_\ell)}{x^+(p) - x^-(p_\ell)} \right) , \]  

(3.2)

\[ A(p\{\lambda_j\}) = \prod_{j=1}^{M} \left( \frac{x^-(p) - x^+(\lambda_j)}{x^+(p) - x^+(\lambda_j)} \right) , \]  

(3.3)

and the magnonic variables \( \lambda_j \) satisfy

\[ 1 = \prod_{\ell=1}^{N} \left( \frac{x^+(\lambda_j) - x^-(p_\ell)}{x^+(\lambda_j) - x^+(p_\ell)} \right) . \]  

(3.4)

Here, we have used a short notation that \( N = N_A + N_B \) and \( \{p_\ell\} = \{p^A_\ell, p^B_\ell\} \) for AdS_3 \( \times \) S^3 \( \times \) T^4; \( N = N_1 + N_2 + N_3 + N_3, \{p_\ell\} = \{p^1_\ell, p^7_\ell, p^3_\ell, p^3_\ell \} \) for AdS_3 \( \times \) S^3 \( \times \) S^3 \( \times \) S^1, respectively.

Inserting these into Eqs.(2.9-2.9), we get the asymptotic Bethe ansatz equations for
Figure 1: AdS$_3 \times S^3 \times T^4$: two momentum-carrying nodes (black dots) are connected to two magnonic nodes (circle).

AdS$_3 \times S^3 \times T^4$:

\[
e^{ip^A_j L} = \prod_{k=1, \neq j}^{N_A} S_0(p^A_j, p^A_k) \prod_{k=1}^{N_B} \frac{1}{\tilde{S}_0(p^A_j, p^B_k)} \\
\times \prod_{j=1}^{M} \left( \frac{x^{-}(p^A_j) - x^{+}(\lambda_j)}{x^{+}(p^A_j) - x^{+}(\lambda_j)} \right) \prod_{j=1}^{M} \left( \frac{x^{-}(p^A_j) - x^{+}(\bar{\lambda}_j)}{x^{+}(p^A_j) - x^{+}(\bar{\lambda}_j)} \right), \tag{3.5}
\]

\[
e^{ip^B_j L} = \prod_{k=1, \neq j}^{N_B} S_0(p^B_j, p^B_k) \prod_{k=1}^{N_A} \frac{1}{\tilde{S}_0(p^B_j, p^A_k)} \\
\times \prod_{j=1}^{M} \left( \frac{x^{-}(p^B_j) - x^{+}(\lambda_j)}{x^{+}(p^B_j) - x^{+}(\lambda_j)} \right) \prod_{j=1}^{M} \left( \frac{x^{-}(p^B_j) - x^{+}(\bar{\lambda}_j)}{x^{+}(p^B_j) - x^{+}(\bar{\lambda}_j)} \right), \tag{3.6}
\]

\[
1 = \prod_{l=1}^{N_A} \left( \frac{x^{+}(\lambda_j) - x^{-}(p^A_l)}{x^{+}(\lambda_j) - x^{+}(p^A_l)} \right) \prod_{l=1}^{N_B} \left( \frac{x^{+}(\lambda_j) - x^{-}(p^B_l)}{x^{+}(\lambda_j) - x^{+}(p^B_l)} \right), \tag{3.7}
\]

\[
1 = \prod_{l=1}^{N_A} \left( \frac{x^{+}(\bar{\lambda}_j) - x^{-}(p^A_l)}{x^{+}(\bar{\lambda}_j) - x^{+}(p^A_l)} \right) \prod_{l=1}^{N_B} \left( \frac{x^{+}(\bar{\lambda}_j) - x^{-}(p^B_l)}{x^{+}(\bar{\lambda}_j) - x^{+}(p^B_l)} \right). \tag{3.8}
\]

This can be represented pictorially by Fig.1.

Similarly, from Eqs.\,(2.18\textsuperscript{-}2.21), we obtain the asymptotic Bethe ansatz equations for
Figure 2: $\text{AdS}_3 \times S^3 \times S^3 \times S^1$: four momentum-carrying nodes (black dots) are connected to a single magnonic node (circle).

$\text{AdS}_3 \times S^3 \times S^3 \times S^1$:

$$e^{ip_k^1_L} = \prod_{k=1 \neq j}^{N_1} S_0(p_j^1, p_k^1) \prod_{k=1}^{N_1} \tilde{S}_0(p_j^1, p_k^1) \prod_{j=1}^{M} \frac{x^-(p_j^1) - x^+(\lambda_j)}{x^+(p_j^1) - x^+(\lambda_j)},$$  \hfill (3.9)

$$e^{ip_k^j_L} = \prod_{k=1 \neq j}^{N_1} S_0(p_j^1, p_k^1) \prod_{k=1}^{N_1} \tilde{S}_0(p_j^1, p_k^1) \prod_{j=1}^{M} \frac{x^-(p_j^1) - x^+(\lambda_j)}{x^+(p_j^1) - x^+(\lambda_j)},$$  \hfill (3.10)

$$e^{ip_k^3_L} = \prod_{k=1 \neq j}^{N_1} S_0(p_j^3, p_k^3) \prod_{k=1}^{N_1} \tilde{S}_0(p_j^3, p_k^3) \prod_{j=1}^{M} \frac{x^-(p_j^3) - x^+(\lambda_j)}{x^+(p_j^3) - x^+(\lambda_j)},$$  \hfill (3.11)

$$e^{ip_k^3_L} = \prod_{k=1 \neq j}^{N_1} S_0(p_j^3, p_k^3) \prod_{k=1}^{N_1} \tilde{S}_0(p_j^3, p_k^3) \prod_{j=1}^{M} \frac{x^-(p_j^3) - x^+(\lambda_j)}{x^+(p_j^3) - x^+(\lambda_j)},$$  \hfill (3.12)

$$1 = \prod_{\ell=1}^{N} \left( \frac{x^+(\lambda_j) - x^-(p_\ell)}{x^+(\lambda_j) - x^+(p_\ell)} \right).$$  \hfill (3.13)

These sets of Bethe ansatz equations can be represented pictorially by Fig.2.

### 3.2 Comparison to the Bethe equations of [1, 2]

In order to translate Eqs.(3.5-3.8) into the notation of [1, 2], we have to replace $p_B$ by $\bar{p}_B$ and disentangle the two “magnonic” variables into four. In Eq.(3.5), for instance, the first step involves the second factor:

$$\prod_{k=1}^{N_B} \tilde{S}_0(p_j^A, \bar{p}_k^B) = \prod_{k=1}^{N_B} \sigma^{-2}(p_j^A, p_k^B) \frac{x_j^+ x_k^-}{x_j^- x_k^+}.$$  \hfill (3.14)
Now, since from the momentum constraint we have that \( \prod_{k=1}^{N_A} \frac{x^+_k}{x^-_k} \prod_{k=1}^{N_B} x^-_k = 1 \), and \( \frac{x^+_j}{x^-_j} = e^{i p_j} \), finally we get, ignoring for the moment the magnonic part (setting to zero both \( M \) and \( M' \)):

\[
e^{i p_j (L+\Lambda)} = \prod_{k=1, \ne j}^{N_A} \frac{x^+_j - x^-_k}{x^-_j - x^+_k} \frac{1 - \frac{1}{x^+_j x^-_k}}{1 - \frac{1}{x^-_j x^+_k}} \sigma^2(p^A_j, p^A_k) \prod_{k=1}^{N_B} \sigma^{-2}(p^B_j, p^B_k). \tag{3.15}
\]

In the case of Eq. (3.6), we get:

\[
e^{-i p_j (L+\Lambda)} = \prod_{k=1, \ne j}^{N_B} \frac{x^+_j - x^-_k}{x^-_j - x^+_k} \frac{1 - \frac{1}{x^+_j x^-_k}}{1 - \frac{1}{x^-_j x^+_k}} \sigma^2(p^B_j, p^B_k) \prod_{k=1}^{N_A} \sigma^{-2}(p^A_j, p^A_k). \tag{3.16}
\]

Now, in order to complete the comparison, we need also to redefine the magnonic variables (After this, Fig.1 changes to Fig.3):

\[
x^+(\lambda_j) = x_{1,j}; \quad j = 1, \ldots K_1; \quad x^+(\lambda_{K_1+j}) = 1/x_{1,j}; \quad j = 1, \ldots K_1; \quad M = K_1 + K_1 \tag{3.17}
\]

\[
x^+(\bar{\lambda}_j) = x_{3,j}; \quad j = 1, \ldots K_3; \quad x^+(\bar{\lambda}_{K_3+j}) = 1/x_{3,j}; \quad j = 1, \ldots K_3; \quad M = K_3 + K_3 \tag{3.18}
\]

Then the Eqs. (3.5-3.8) become:

\[
e^{i p_j^A(\Lambda+\Lambda)} = \prod_{k=1, \ne j}^{K_B} \frac{x^+_{1,j} - x^-_{1,k}}{x^-_{1,j} - x^+_{1,k}} \frac{1 - \frac{1}{x^+_{1,j} x^-_{1,k}}}{1 - \frac{1}{x^-_{1,j} x^+_{1,k}}} \sigma^2(p^A_{1,j}, p^A_k) \prod_{k=1}^{K_A} \sigma^{-2}(p^A_{1,j}, p^A_k) \tag{3.19}
\]

\[
e^{-i p_j^B(\Lambda+\Lambda)} = \prod_{k=1, \ne j}^{K_B} \frac{x^+_{1,j} - x^-_{1,k}}{x^-_{1,j} - x^+_{1,k}} \frac{1 - \frac{1}{x^+_{1,j} x^-_{1,k}}}{1 - \frac{1}{x^-_{1,j} x^+_{1,k}}} \sigma^2(p^B_{1,j}, p^B_k) \prod_{k=1}^{K_A} \sigma^{-2}(p^B_{1,j}, p^B_k) \tag{3.20}
\]

\[
1 = \prod_{l=1}^{N_A} \left( \frac{x^+_{1,j} - x^-_{1,k}}{x^+_{1,j} - x^+_{1,k}} \right) \prod_{l=1}^{N_B} \left( \frac{1 - \frac{1}{x^+_{1,j} x^-_{1,k}}}{1 - \frac{1}{x^-_{1,j} x^+_{1,k}}} \right), \tag{3.21}
\]

\[
1 = \prod_{l=1}^{N_A} \left( \frac{x^+_{3,j} - x^-_{1,k}}{x^+_{3,j} - x^+_{1,k}} \right) \prod_{l=1}^{N_B} \left( \frac{1 - \frac{1}{x^+_{3,j} x^-_{1,k}}}{1 - \frac{1}{x^-_{3,j} x^+_{1,k}}} \right), \tag{3.22}
\]

\[
1 = \prod_{l=1}^{N_B} \left( \frac{x^+_{1,j} - x^-_{1,k}}{x^+_{1,j} - x^+_{1,k}} \right) \prod_{l=1}^{N_A} \left( \frac{1 - \frac{1}{x^+_{1,j} x^-_{1,k}}}{1 - \frac{1}{x^-_{1,j} x^+_{1,k}}} \right), \tag{3.23}
\]

\[
1 = \prod_{l=1}^{N_B} \left( \frac{x^+_{3,j} - x^-_{1,k}}{x^+_{3,j} - x^+_{1,k}} \right) \prod_{l=1}^{N_A} \left( \frac{1 - \frac{1}{x^+_{3,j} x^-_{1,k}}}{1 - \frac{1}{x^-_{3,j} x^+_{1,k}}} \right), \tag{3.24}
\]
which match exactly the equations conjectured by [1, 2], if we define \( L_{[1,2]} = L + K_A - K_B + K_1 + K_3 \).

In analogy with the case of \( \text{AdS}_3 \times T^4 \), in order to get the Bethe equations written in the notation of [1, 2], we need to change \( p^{1,3} \rightarrow \bar{p}^{1,3} \) in Eqs. (3.9-3.13) and to redefine the magnonic nodes corresponding to the variable 2 into two sets of 2 and \( \bar{2} \) variables,

\[
x^+(\lambda_j) = x_{2,j}; \quad j = 1, \ldots, K_2; \quad x^+(\lambda_j + K_2) = 1/x_{2,j}; \quad j = 1, \ldots, K_2,
\]

as illustrated in Fig.4:

\[
e^{i p^1_j (L + N_1 - N_1 + K_2)} = e^{i(p_1 - p_1)} \prod_{k=1,\neq j}^{K_1} \left( \frac{x^+_{1,j} - x^-_{1,k}}{x^+_{1,j} - x^-_{1,k}} \frac{1 - \frac{1}{x^+_{1,j}x^-_{1,k}}}{1 - \frac{1}{x^+_{1,j}x^-_{1,k}}} \right)^{N_1} \sigma^2(x_{1,j}, p_{1,k}) \prod_{k=1}^{K_1} \sigma^{-2}(p_{1,j}, p_{1,k})
\]

\[
\times \prod_{j=1}^{K_2} \left( \frac{x^-_{1,j} - x^-_{2,j}}{x^+_{1,j} - x^-_{2,j}} \right)^{K_3} \prod_{j=1}^{K_2} \left( \frac{x^-_{1,j} - x^-_{2,j}}{x^+_{1,j} - x^-_{2,j}} \right)^{K_3} \left( \frac{1 - \frac{1}{x^-_{1,j}x^-_{2,j}}}{1 - \frac{1}{x^-_{1,j}x^-_{2,j}}} \right)^{N_1} \sigma^2(x_{2,j}, p_{2,k}) \prod_{k=1}^{K_2} \sigma^{-2}(p_{2,j}, p_{2,k})
\]

\[
e^{-i p^1_j (L + N_1 - N_1 + K_2)} = e^{i(p_1 - p_1)} \prod_{k=1,\neq j}^{K_1} \left( \frac{x^+_{1,j} - x^-_{1,k}}{x^+_{1,j} - x^-_{1,k}} \frac{1 - \frac{1}{x^+_{1,j}x^-_{1,k}}}{1 - \frac{1}{x^+_{1,j}x^-_{1,k}}} \right)^{N_1} \sigma^2(x_{1,j}, p_{1,k}) \prod_{k=1}^{K_1} \sigma^{-2}(p_{1,j}, p_{1,k})
\]

\[
\times \prod_{j=1}^{K_2} \left( \frac{x^-_{1,j} - x^-_{2,j}}{x^+_{1,j} - x^-_{2,j}} \right)^{K_3} \prod_{j=1}^{K_2} \left( \frac{x^-_{1,j} - x^-_{2,j}}{x^+_{1,j} - x^-_{2,j}} \right)^{K_3} \left( \frac{1 - \frac{1}{x^-_{1,j}x^-_{2,j}}}{1 - \frac{1}{x^-_{1,j}x^-_{2,j}}} \right)^{N_1} \sigma^2(x_{2,j}, p_{2,k}) \prod_{k=1}^{K_2} \sigma^{-2}(p_{2,j}, p_{2,k})
\]

Figure 3: \( \text{AdS}_3 \times S^3 \times T^4 \): two momentum-carrying nodes (black dots) are connected to four magnonic nodes (circle) after redefinition.
We proposed $su(1|1) \times su(1|1)$- and $su(1|1)$-invariant $S$-matrices for the massive modes of IIB string theory on $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$, respectively. From these...

**Figure 4**: $AdS_3 \times S^3 \times S^3 \times S^1$: four momentum-carrying nodes (black dots) are connected to two magnonic nodes (circle) after redefinition.

$$
e^{-ip_3^3(L + N_3 - N_3 + K_2)} = e^{i(p_3 - P_3)} \prod_{k=1}^{N_3} \frac{x^+_{3,k} - x^-_{3,k}}{x^-_{3,j} - x^+_{3,k}} \frac{1}{1 - \frac{1}{x^+_{3,j}x^-_{3,k}}} \sigma^2(p_{3,j}, p_{3,k}) \prod_{k=1}^{N_3} \sigma^{-2}(p_{3,j}, p_{3,k})$$

$$\times \prod_{j=1}^{K_2} \left( \frac{x^-(p_j^3) - x_{2,j}}{x^+(p_j^3) - x_{2,j}} \right) \prod_{j=1}^{K_2} \left( \frac{1}{1 - \frac{1}{x^-(p_j^3)x_{2,j}}} \right), \quad (3.29)$$

$$1 = \prod_{\ell=1}^{K_3} \left( \frac{x^-(p_{3,\ell}^3) - x_{2,j}}{x_{2,j} - x^+(p_{3,\ell}^3)} \right) \prod_{\ell=1}^{K_3} \left( \frac{x^+(p_{3,\ell}^3) - x^-_{2,j}}{x_{2,j} - x^+(p_{3,\ell}^3)} \right) \prod_{\ell=1}^{K_3} \left( \frac{1}{1 - \frac{1}{x^+(p_{3,\ell}^3)x_{2,j}}} \right), \quad (3.30)$$

In this case, to have full agreement with [11 [2], we have to redefine the parameter $L$ in different ways in each equation for the momentum-carrying variables:

$$L_1 \equiv L + N_1 - N_1 + K_2; \quad L_3 \equiv L + N_3 - N_3 + K_2;$$

$$L_1 \equiv L + N_1 - N_1 + K_2; \quad L_3 \equiv L + N_3 - N_3 + K_2.$$

This could be useful to solve the apparent disagreement between string results and predictions from the Bethe equations for energies of solutions belonging to the $su(2) \times su(2)$ sector, pointed out by [7], where they would need independent definitions of the spin chain lengths in each $su(2)$ subsector.

**4 Discussion**

We proposed $su(1|1) \times su(1|1)$- and $su(1|1)$-invariant $S$-matrices for the massive modes of IIB string theory on $AdS_3 \times S^3 \times T^4$ and $AdS_3 \times S^3 \times S^3 \times S^1$, respectively. From these
we derived the Bethe equations proposed in [1, 2]. The derivation involved, among other steps, the particle-antiparticle transformation on some momenta of the “massive” variables and the doubling of the fermionic variables in a fashion similar to the AdS$_5$CFT$_4$ [16] and AdS$_4$/CFT$_3$ [18] cases.

Some scalar factors remain undetermined and we were able to guess them by requiring the unitarity of the $S$-matrix and the matching with the conjectured BAEs. Because of the apparently missing crossing relations [?] for the $su(1|1)$ algebra [12, 13], a more solid derivation of such scalar factors remains as an open problem.

Another open problem is to incorporate the massless modes into the $S$-matrix formulation. In a relativistic theory, the massless limit can be obtained by shifting the rapidity to $\pm\infty$, which often makes the $S$-matrices between massive and massless modes trivial. While this mechanism seems not applicable in our non-relativistic case, we believe a similar argument may provide a clue.

Albeit these unsolved problems, we believe that our findings can play a rôle similar the $su(2|2)$ $S$-matrix [17] in both AdS$_5$/CFT$_4$ and AdS$_4$/CFT$_3$ [18], and to lead some deeper understanding of the yet quite unexplored AdS$_3$/CFT$_2$.

Finally, it would be interesting to investigate possible exact $S$-matrices for the analogous case of AdS$_2$/CFT$_1$, for which a set of all-loop Bethe equations has been recently proposed in [19]. It will be also interesting to check our proposals in certain perturbative computations. One immediate way is to compute the worldsheets $S$-matrix based on a gauge-fixed string action for the strong coupling limit. On the other hand, it would be also important to check the reflectionless of our $S$-matrices, as predicted by [13], through some weak coupling perturbative calculations, for example, or along the lines of [21].

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4See also [20] for the derivation of the classical equations in this case and a review about finite-gap integration in various AdS$_d$ backgrounds.
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