Propagation of correlations in Local Random Quantum Circuits.

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We derive a dynamical bound on the propagation of correlations in local random quantum circuits - lattice spin systems where piecewise quantum operations - in space and time - occur with classical probabilities. Correlations are quantified by the Frobenius norm of the commutator of two positive operators acting on space-like separated local Hilbert spaces. For times \( t = O(L) \) correlations spread to distances of the order of \( L \) growing, but at best, diffusively for any distance within that radius with exponentially suppressed distance dependent corrections whereas for \( t = o(L^2) \) all parts of the system get almost equally correlated with exponentially suppressed distance dependent corrections and approach the maximum amount of correlations that may be established asymptotically.

**Introduction.** In ensembles of quantum systems evolving in time under random local unitary interactions how do correlations of observable measurements spread on an average? For individual quantum systems on discrete lattices under local Hamiltonian dynamics the answer is provided in the form of the Lieb Robinson bound [1], and its generalizations [2, 3], that establish a dynamical bound on the operator norm of space-like separated observables under dynamics governed by a local Hamiltonian. With the concept of typicality [4] - properties that hold statistically on an average - playing an increasingly important role in explaining foundational as well as operational questions in quantum statistical mechanics such as quantum equilibration [5], thermodynamics [6], area laws for entanglement entropy [7] etc. it is important to have an estimate, in relevant families of quantum systems, for an average dynamical bound on the evolution of correlations in typical states produced via such random processes.

Here we obtain a dynamical bound on the average of Frobenius norm of the commutator of operators in a natural model of a Local Random Quantum Circuit (LRQC). These are stochastic circuits comprised of unitaries with finite support that respect the local-interaction structure of the defining underlying graph representing the constituents of a quantum many body system [7–10]. LRQCs are useful as tools to formulate statistical statements about ensembles of physical systems governed by random local interactions and have been studied to model efficiency and typicality of entanglement generation in random two-party processes [9], entanglement dynamics of time dependent local Hamiltonians [7], equilibration in quantum systems, as approximate polynomial designs [11, 12], to define quantum error-correcting codes and quite generally in the decoupling approach to quantum information [13].

In our model of a LRQC on a spin lattice, Fig. (1), at each discrete-time step, two random processes occur in succession. First, a support for interactions is chosen according to some distribution over the possible local interactions, followed by a choice of a unitary operator according to another probability distribution over unitary operators with the same support. We show that on an average over the circuit realizations, correlations spread at the rate of the size of the support of local unitaries per time step resulting in a strictly linear light cone. Within this light cone we determine a bound on the dynamical correlation in terms of the parameters of the two probability distributions defining the model - namely the probability distribution over the edges and the distribution over local unitaries, Fig. (2). We establish two distinct time regimes, one for times of the order of system size when average (over circuits) root mean (over basis states) square correlations grow diffusively for distances within the light cone and the other for times greater than the square of the system size where all parts of the system are almost equally correlated and close to the maximum achievable, which value we establish.

**Local Random Quantum Circuits.** A local random quantum circuit \( C^t[\mathcal{L}, \Xi, \eta^{(t)}] \), of depth \( t \) on a graph \( \mathcal{G} = (V, E) \), with an associated Hilbert space \( \mathcal{H}_V = \bigotimes_{v \in V} \mathcal{H}_v \), is specified by three quantities: local regions where interactions may take place, a distribution over unitaries acting on those local regions and a rule assigning probability weights to sequences of local regions chosen in \( t \) time steps. Mathematically one specifies subsets \( \mathcal{L} \) of the power set \( 2^V \) with elements \( S \in \mathcal{L} \) where unitary operations may take place, probability density

![FIG. 1. (color online) Schematic action of a LRQC defined on circular chain of qudits (blue circles, top qudit interacts with the bottom one) with nearest neighbor connectivity. At discrete time intervals a single 2-body geometrically local random unitary acts on the chain. The circular topology makes the action of the LRQC translationally invariant over all the vertices of the graph.](image-url)
\[ \Xi = \{ d\mu_S \} \] over the group of unitaries \( U_S \) acting on those local regions \( S \in \mathcal{L} \), and a probability law \( q^{(t)} : \mathcal{L}^t \mapsto [0,1] \), assigning sequences of local regions of length \( t \) a probability weight. Given this data, a LRQC of depth \( t \) is essentially a unitary \( U_S \) which is a time ordered product of unitaries acting on the local regions i.e. \( U_S := U_t U_{t-1} \ldots U_1 \), where \( S = (S_i, S_{i-1}, \ldots, 2, 1) \in \mathcal{L}^t \) is the sequence of local regions and \( U_i \in \mathcal{U}(S_i) \forall i \in [1,t] \). Thus \( U_S \) is a unitary-valued random variable distributed according to the law \( q^{(t)}(S) dU_S = q^{(t)}(S) \prod_{i=1}^{t} d\mu_{S_i} \). The statistics \( (n\text{-th order moment}) \) of any observable \( \hat{O} \in \mathcal{B}(V) \) over the ensemble of circuits with fixed data \( \mathcal{L}, \Xi, q^{(t)} \), where \( B(V) \) is the algebra of bounded linear operators on \( \mathcal{H}_V \), may be obtained using a set of completely positive trace preserving maps \( R_n : \mathcal{B}(V)^{\otimes n} \mapsto \mathcal{B}(V)^{\otimes n} \) although for the purpose here we need only the first two such ensemble maps \( R_1 : \mathcal{B}(V) \mapsto \mathcal{B}(V) \) and \( R_2 : \mathcal{B}(V)^{\otimes 2} \mapsto \mathcal{B}(V)^{\otimes 2} \). These are the averaging superoperator and the second-moment superoperator respectively and have the following actions,

\[ R_1^{(t)}(\hat{O}_1) = \sum_{S \in \mathcal{L}^t} q^{(t)}(S) \int dU_S U_S^\dagger \hat{O}_1 U_S, \]

\[ R_2^{(t)}(\hat{O}_2^{\otimes 2}) = \sum_{S \in \mathcal{L}^t} q^{(t)}(S) \int dU_S (U_S^{\otimes 2})^\dagger \hat{O}_2^{\otimes 2} (U_S^{\otimes 2}), \]

i.e. they average, over sequences of local regions, the operator first and second moments w.r.t. a distribution of unitaries over those local regions. In our model of a LRQC, first a pair of nearest neighbor qudits \( a, b \in V \) is chosen uniformly with point supports \( (\mathbb{B}(a), \mathbb{B}(b)) \), \( p \subseteq \mathbb{S} \). This spread of support due to the LRQC dynamics leads to the building up of correlations between measurements of observables \( \hat{O}_p, \hat{O}_q \) at \( p \) and any other region \( q \in V \) with time \( t \) that depends on the lattice distance \( \mathcal{D} \) between \( p \) and \( q \), Fig. (2). For simplicity we choose positive operators \([14]\) with point supports \( p, q \in V \) and derive an upper bound on the average over the circuits (denoted by the overbar), \( \eta(D,t) = \| [C^t(\hat{O}_p), \hat{O}_q] \|_2 \), of the commutator Frobenius norm i.e.,

\[ \eta(D,t) = \sqrt{ \text{Tr} \{ [C^t(\hat{O}_p), \hat{O}_q] [C^t(\hat{O}_p), \hat{O}_q] \} } \]

Note that the square of the Frobenius norm of a commutator for two observables \( A, B \) with spectral resolutions \( A = \sum_{i} a_i |a_i \rangle \langle a_i|, B = \sum_{i} b_i |b_i \rangle \langle b_i| \) is given by,

\[ \text{Tr} \{ [A, B]^\dagger [A, B] \} = \sum_{i,j} (a_i - a_j)^2 (a_i |b_j|)^2, \]

which sums the modulus square of matrix elements of \( B \) linking two eigenstates of \( A \) weighted by the square of the difference of the corresponding eigenvalues - quantifying the sum of the square of correlations in measurements of \( A \) and \( B \) over a complete basis. Normalizing first the quantity in (4) by the dimension of the space and then taking the square root gives the root mean square (RMS) correlation along any basis state of the space. The quantity of interest, for us, is thus (3) normalized by \( d^{L/2} \) (root of the dimension of \( \mathcal{H}_V \)) i.e. \( \eta(D,t)/d^{L/2} \) which gives the average (over the circuits) RMS (over the Hilbert space) correlations due to the stochastic unitary dynamics of LRQCs. We note that the Frobenius norm also provides a (weak) upper bound on the operator norm [15].

**Outline of the derivation.-** First, \( \eta(t) \) is bound by a functional of iterated ensemble maps \( R_1^t \) and \( R_2^t \) acting on \( O_p^t \) and \( O_q^{\otimes 2} \) respectively. Next, we show that \( R_1^t(\hat{O}_p^{\otimes 2}) \) is independent of \( t \) and the entire spatio-temporal dependence of the commutator two-norm is contained in \( R_2^t(\hat{O}_q^{\otimes 2}) \). This dependence is analysed using a matrix representation \( M \) for \( R_2 \) in an operator basis consisting of swap operators \( T_S \) [16] on all subsets \( S \subseteq V \) - leading to an algebra over subsets of \( V \). Finally, we estimate the matrix entries \( M_{i,j}^t \) by considering the symmetry of a graph derived from the adjacency matrix of \( M \) with the weights on edges representing the one-step matrix action. This yields all ingredients for evaluating an upper bound on \( \eta(t) \).

**Bounding the commutator norm.-** We start by using the concavity of the square root function to take the average in Eq. (3)
under the square-rootifying,

$$
\eta(t) \leq \sqrt{\text{Tr}[[C'(\hat{O}_p), \hat{O}_q]][C'(\hat{O}_p), \hat{O}_q]]} = \sqrt{2\text{Tr}(\hat{R}_1(\hat{O}_p^2)\hat{O}_q^2) - 2\text{Tr}(\hat{R}_2(\hat{O}_p^2)\hat{O}_q^2\hat{T}_V)}
$$

(5)

which expresses the dynamical bound on the average commutator two-norm as the square root of a linear combination of traces w.r.t. $\hat{O}_p^2$ and $\hat{O}_q^2\hat{T}_V$ of the images of $\hat{O}_p^2$ and $\hat{O}_q^2$ under the ensemble maps $\hat{R}_1$ and $\hat{R}_2$ respectively (see supp. mat.). It turns out that $\text{Tr}(\hat{R}_1(\hat{O}_p^2)\hat{O}_q^2) = \text{Tr}(\hat{O}_p^2)\text{Tr}(\hat{O}_q^2)^t = \text{Tr}(\hat{O}_p^2)\text{Tr}(\hat{O}_q^2)^t = L^{-1/2}$ is independent of time (see supp. mat.) while $\text{Tr}(\hat{R}_2(\hat{O}_p^2)\hat{O}_q^2\hat{T}_V) = (\hat{R}_2(\hat{O}_p^2)\hat{O}_q^2\hat{T}_V)$ decreases monotonically with $t$ eventually reaching an asymptotic value for large times. To show this we need $\hat{R}_2(\hat{O}_p^2)$ which can be obtained by following the iterations of $\hat{R}_2(\hat{O}_p^2)$ for $t$ time steps (see supp. mat.) and requires evaluation of $\hat{R}_2(\hat{T}_{p+1})$ and $\hat{R}_2(\hat{T}_{p+1})$ for $n = 1, 2, \ldots, t-1$ i.e. iterated $\hat{R}_2$ maps on swap operators $\hat{T}_{p+1}$ supported on nearest neighbors of node $p$. The iterated version gives,

$$
\hat{R}_2^n(\hat{O}_p^2) = r^n\hat{O}_p^2 + \frac{2A}{L} 1 - r^n \frac{1}{1 - r^n} \hat{I}_V
$$

+ \frac{B}{L} \sum_{S \in W} \{a_1(S, t) + a_2(S, t)\} \hat{T}_S
$$

(6)

where $r = (L - 2)/L$ is the probability with which $\hat{R}_2$ leaves $\hat{O}_p^2$ invariant. $A = \text{Tr}(\hat{O}_p^2)(x^2d^2 - 1)/d(d^2 - 1)$ and $B = \text{Tr}(\hat{O}_p^2)(d - x^2)/(d^4 - 1)$ are constants (w.r.t. size $L$) that depend only the norm properties of $\hat{O}_p$. through $x = ||\hat{O}_p||_2/||\hat{O}_p||_2$. In Eq. (6) we have denoted by $W \subset 2^V$, the set with elements that are sets of contiguous vertex labels in $V$ i.e. $W = \{S \subset V|S = \{\phi\}, \{1\}, \{1, 2\}, \ldots, \{2, 3\}, \{2, 3, 4\}, \{1, 2, 3, \ldots, L\}, \ldots\}$. As the set of swaps $\hat{T}_S$ supported on $W$ forms an invariant subspace under $\hat{R}_2$ we expand $\hat{R}_2(\hat{T}_{p+1}) = \sum_{S \in W} c_2(S)\hat{T}_S$.

We estimate $a_1(S, t) = \sum_{i=0}^n c_{i,1}(S) t^i \geq 0$ are obtained by evaluating $c_1(S) = (M^n|a_0(S)|^2)_S, n \in [0, t-1]$, that in general is difficult to obtain exactly [17]. However a lower estimate for $a_1(S, t)$ is enough to lower bound Eq. (6) and in turn upper bound Eq. (5).

We estimate the elements of $M^n$ using a graph $G(W, E')$, derived from the adjacency structure of $M$ whose vertices are the sets of supports of operators that form the basis for $M$ and whose edges are pairs of elements in $W$ with corresponding non-zero entry in $M$, Fig. (3). This derived graph shows the dependence of the coefficients $c_{1/2, t}$ at time $t$ on those at the previous time step as $c_{1/2, t} S \in W$ depends only on those $c_{1,2, t-1}$ for which $S, S'$ are connected by an edge. Starting from $c_1(S, t) = \delta(S, S_t = \{p, p-1\}) + \delta(S, S_t = \{p, p+1\})$ i.e. unity at fiducial nodes $S_t$, corresponding to the operators $\hat{T}_{p+1}$ and $\hat{T}_{p-1}$, applying $M$ transfers weights to other nodes $S' \in W$. All these weights eventually flow to the sink nodes $S = \{\phi\}, \{V\}$, that span the unit eigenspace for $M$ [8]. Removing the sink nodes (and the edges including these nodes) from $G$ amounts to omitting from $M$ the rows and columns corresponding to $S = \{\phi\}, \{V\}$ yielding a positive, symmetric matrix $M'$ with a corresponding graph $G'(W \setminus \{\{\phi\}, \{V\}, \{E\})$. This is useful since $M'$ has only two kinds of entries: $r = (L - 2)/L$ on the diagonal (self loops in $G'$) while $u = N_d/L = d/(d+1)L$ for the other non-zero off-diagonal entries (edges in $G'$). Then it can be shown that $c_{1/2n} = (M^n)|S, S_t|_S \geq \langle\hat{T}_S, \hat{O}_p^2\hat{T}_V\rangle|S, S_t\rangle$ by $\langle\hat{T}_S, \hat{O}_p^2\hat{T}_V\rangle|S, S_t\rangle$ Dist$(S_i, S_j)$ the graph theoretic distance between nodes labeled by $S_i, S_j$ on the derived graph $G'$ (different from the distance $D$ between nodes $p, q$ on the circular chain) resulting in $a_1,2(S, t) \geq u^{D}\langle\hat{T}_S, \hat{O}_p^2\hat{T}_V\rangle$ for the non-sink nodes while the same for the sink nodes is given by $a_1,2(S, t) = 2u \sum_{S \in W} |S, S_t\rangle|S, S_t\rangle \langle\hat{T}_S, \hat{O}_p^2\hat{T}_V\rangle$ for $u$ times the sum over nearest neighbors of the sinks of their coefficient values for all previous times.

Further, using Eq. (6) to obtain $\langle\hat{R}_2^n(\hat{O}_p^2), \hat{O}_p^2\hat{T}_V\rangle$ requires evaluation of $Q_S = \langle\hat{T}_S, \hat{O}_p^2\hat{T}_V\rangle$ which value depends on whether the set $S$ includes the node labelled $q$ or not. Indeed we have (with $|S|$ the size of the subset) $Q_S = dL^{-|S|}2\text{Tr}(\hat{O}_q^2) = Q_{1/2S}$ for $q \cap S = q$ and $Q_S = \ldots$
\[ d^{L+1} - \text{Tr}(\hat{Q}_2^2) = Q_{2s} \text{ for } q \cap \{S\} = \phi, \text{ which let's us write} \]
\[ \sum_{S \in W} a(S,t)Q_S = \sum_{S \in W} a(S,t)Q_{2s} - \sum_{S \in W} a(S,t)\Delta_S \]  
(7)

with \( a(S,t), Q_{2s}, \Delta_S \) and the asymptotic value \( \eta \) of the maximum contribution to the 2-Rényi entropy growth \[8\]. The time-scale \( T_1 = (1 + 1/d^2)(L - 2) \sim O(L) \) of the weight in the expansion for \( \mathcal{R}_2(\mathcal{T}_{p,p+1}) \) is associated with the fiducial nodes as \( a(S,t) \sim (udt/D)^2 \) falls exponentially with distance \( D \) on the graphed graph \( \text{since } u < 1 \).

We thus keep only the fiducial node contributions for the first term in Eq. (7) and upper bound the second term using contributions from the nodes \( S, S \cap q = q \) nearest to \( S_L, R \) each giving the dynamical bound,

\[
\eta(D,t) \leq \Theta(t-D)[m_1(1-r^t) - m_2 r^{udt/2} + m_3 r^{(udt/2)^2}]^{1/2}
\]  
(8)

where \( \Theta(t-D) \) is the asymptotic value \( \eta(D,t \to \infty) \), and with \( m_1 = \{d(d-x^2)(d-y^2)/d^2\}\phi_{\mathcal{O}[2]}||\mathcal{O}||_2^2 \) is the asymptotic value \( \eta(D,t \to \infty) \), and with \( m_2 = \{d(d-x^2)(d-y^2)/d^2\}\phi_{\mathcal{O}[2]}||\mathcal{O}||_2^2 \) and \( m_3 = \{d(d-x^2)(d-y^2)/m(d^2-1)\}\phi_{\mathcal{O}[2]}||\mathcal{O}||_2^2 \) are quantities that depend only on the properties of \( \mathcal{O}_{\mathcal{O}[2], \Phi} \).

The bound (8) implies a linear increase in the spread of the correlations to distances \( D \) at the exponential rate \( \Theta(t-D) \) due to the step function \( \Theta(t-D) \). For \( t \ll L \) the rate of correlation growth for any \( D \leq t \) is essentially determined by \( (1-r^t) \) which is the probability of the map \( \mathcal{R}_2 \) acting non-trivially on \( \mathcal{O}[2] \) in \( t \)-time steps. Since the second and third terms under the root in bound (8) are extensively suppressed due to the factor of \( d(L^2 + 1) \) for \( l \ll D \ll t \ll L \) only the first term contributes significantly, and one has \( \sim O(L^2) \) implying a diffusive growth for the bound on average RMS correlations within the linear light cone. The square of this quantity gives the mean square correlation (not RMS) implying a ballistic growth of the same in this spatio-temporal regime - consistent with the 2-Rényi entropy growth \[8\]. The time-scale \( T_2 \) thus estimates the diffusive regime for the growth of average RMS correlations.

**Conclusions and Discussion** - We showed how average RMS correlations in our model of a LRQC depend on the defining probability distributions. Here, because the sequence of local regions picked and the unitaries were independently and identically chosen at each time step from the uniform measure, this model of a LRQC may be considered to be a particular example of a system undergoing general quantum Markovian dynamics \[3\] but one where the relaxation time scales with the system size resulting in no clustering of correlations. Correlated choices of local regions and/or local operations can infact model Non-Markovian random piece-wise quantum processes \[8, 11\]. The present results may be relevant to understand, for eg., the effect of local noise in an algorithm distributed over many qubits due to improper rotations and/or faulty spatial addressing of gates \[20\] or quantum computation with bounded depth quantum circuits with intervening classical layers \[21\]. Generalizing these results to other random quantum circuit models to account for experimentally relevant situations would thus be important in the context of fault tolerant quantum information processing.

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AVERAGING OVER THE CIRCUITS AS ENSEMBLE MAPS

Ineq. (5) requires the average over the circuits of the square of the Frobenius norm of the commutator $[C^t(\hat{O}_p), \hat{O}_q]$ and may be expressed as,

$$\text{Tr}[(C^t(\hat{O}_p), \hat{O}_q)](C^t(\hat{O}_p), \hat{O}_q)]$$

$$= \text{Tr}[(C^t(\hat{O}_p)\hat{O}_q - \hat{O}_q C^t(\hat{O}_p))](C^t(\hat{O}_p)\hat{O}_q - \hat{O}_q C^t(\hat{O}_p))]$$

$$= \text{Tr}[(\hat{O}_q C^t(\hat{O}_p) - C^t(\hat{O}_p)\hat{O}_q)](C^t(\hat{O}_p)\hat{O}_q - \hat{O}_q C^t(\hat{O}_p))]$$

$$= 2\text{Tr}[\hat{O}_q C^t(\hat{O}_p) - C^t(\hat{O}_p)\hat{O}_q] C^t(\hat{O}_p)\hat{O}_q - \hat{O}_q C^t(\hat{O}_p))$$

where in the fifth line above we have used the cyclic-ity under trace and in the sixth line used the swap trick for trace for a product of operators $A,B \in B(\mathcal{H}_V)$ as a trace of their tensor product times the swap operator on two copies of the same space i.e. $\text{Tr}_{H_1}(AB) = \text{Tr}_{H_2}(A \otimes B \bar{\mathcal{T}}_V)$. Also, $\text{C}(\hat{O}_p)^2 = \text{U}_S^p \hat{O}_p U^p_S \hat{O}_p U^p_S = \text{U}_S^p \hat{O}_p U^p_S$ and $\text{C}(\hat{O}_p) \otimes \text{C}(\hat{O}_p) = \text{U}_S^p \hat{O}_p U^p_S \otimes \text{U}_S^p \hat{O}_p U^p_S = \text{U}_S^{p \otimes 2} \hat{O}_p^{\otimes 2} (U^p_S)^{\otimes 2}$.

THE ACTION OF THE MAPS $R_1$ AND $R_2$

The superoperator $R_1$ acting on an operator $\hat{O}_p$ acting at site $p$ is obtained by expanding Eq. (2):

$$R_1(\hat{O}_p^2) = \begin{array}{l}
\frac{L}{L} \sum_{i=1}^{L} \int dU_{i,i+1} U_{i,i+1} \hat{O}_p^2 U_{i,i+1}^\dagger \\
\end{array}$$

$$= r \hat{O}_p^2 + \begin{array}{l}
\frac{L}{L} \int dU_{p+1,p+1} \hat{O}_p^2 U_{p+1,p+1}^\dagger + \frac{1}{L} \int dU_{p,p+1} U_{p,p+1} \hat{O}_p^2 U_{p,p+1}^\dagger
\end{array}$$

$$= r \hat{O}_p^2 \otimes \mathbb{I}_{p+1} + \frac{2 \text{Tr} (\hat{O}_p^2)}{L} \mathbb{I}_p \otimes \mathbb{I}_{p+1}$$

$$= (r \hat{O}_p^2 + w \mathbb{I}_p) \otimes \mathbb{I}_{p+1}, \text{ where } r = \frac{L - 2}{L}, w = \frac{2 \text{Tr} (\hat{O}_p^2)}{L}\left(\frac{L}{d}\right)$$

(10) establishing that $R_1(\hat{O}_p^2) \in \text{Span}(\hat{O}_p^2 \otimes \mathbb{I}_{p+1}, \mathbb{I}_p \otimes \mathbb{I}_{p+1})$. This implies that the non-trivial support of the order does not change under the map $R_1$ with the number of iterations. Because $R_1$ is also trace-preserving, $\text{Tr} [R_1(\hat{O}_p^2)] = \text{Tr} (\hat{O}_p^2)$

$$= r \text{Tr} \hat{O}_p^2 + w, \text{ yielding } w = \frac{(1-r) \text{Tr} (\hat{O}_p^2)}{d}.$$

Iterations of $R_1$ give,

$$R_1^t (\hat{O}_p^2) = r^t \hat{O}_p^2 \otimes \mathbb{I}_{V \setminus p} + \frac{1 - r^t}{d} \text{Tr} (\hat{O}_p^2) \mathbb{I}_p \otimes \mathbb{I}_{V \setminus p}$$

(11)
resulting in $\langle R_1^i(\hat{O}^2_p), \hat{O}^2_q \rangle = \text{Tr}(\hat{O}^2_p)\text{Tr}(\hat{O}^2_q)$ $d^{t-2}$ which is independent of time. Next, expanding Eq. (2) for $R_2$ we get,

$$R_2(\hat{O}^2_p) = \frac{1}{L} \sum_{i=1}^{L} \int dU_{i+i+1} U_{i,i+1}^T \hat{O}^2_p (U^T)^{\hat{O}^2_q}_{i,i+1}$$

(12)

where using the result for integration over tensor product representations of unitary groups

$$\int dU_{i+i+1} U_{i,i+1}^T \hat{O}^2_p (U^T)^{\hat{O}^2_q}_{i,i+1} = \frac{2^\text{Tr}(\hat{O}^2_p)\Pi_{p,p+1}}{d^2(d^2+1)} \Pi_{p,p+1}^+ + \frac{2^\text{Tr}(\hat{O}^2_p)\Pi_{p,p+1}}{d^2(d^2-1)} \Pi_{p,p+1}^-$$

onto the totally symmetric and antisymmetric spaces of $\mathcal{H}_d^2$ respectively and simplifying we get for the one-step action,

$$R_2(\hat{O}^2_p) = r\hat{O}^2_p + \frac{2A}{L} I + \frac{B}{L} (\hat{T}_{p,p+1} + \hat{T}_{p,p-1}),$$

(13)

which implies that $R_2(\hat{O}^2_p) \in \text{Span}(\hat{O}^2_p, I, \hat{T}_{p,p+1}, \hat{T}_{p,p-1})$. In general for a fixed operator $\hat{O}^2_p$ the image $R_2(\hat{O}^2_p)$ lies in a $(2^V+1)$-dimensional space of operators $\hat{O}^2_p \cup \hat{T}_{S_V}$ where the $\hat{T}_S$ are swap operators acting on subsets $S \subset V$ that swap the corresponding copies in the doubled Hilbert space $\mathcal{H}_V \oplus \mathcal{H}_V$. For example, $\hat{T}_{S=\phi} = \mathbb{I}_V$, $\hat{T}_{S=S(1)} = \mathbb{I}_1 \otimes \mathbb{I}_{\{V\backslash 1\}}$ etc. However for the specific model of LQCs considered here there is an exponential reduction in the size of the invariant subspace to swap operators with support only on contiguous subsets $W \subset V$. We note that the set of operators $\hat{O}^2_p, \hat{T}_S, S \in W$ form an invariant subspace under the action of the map $R_2$, i.e., $R_2(\hat{O}^2_p) \in \text{Span}(\hat{O}^2_p, \mathbb{I}, \hat{T}_{p,p+1}, \hat{T}_{p,p-1})$. One can then evaluate $R_2(\hat{O}^2_p)$ by writing down a difference equation,

$$R_2^{t+1}(\hat{O}^2_p) - rR_2^t(\hat{O}^2_p) = \frac{2A}{L} + \frac{B}{L} (R_2^t(\hat{T}_{p,p+1}) + R_2^t(\hat{T}_{p,p-1})),$$

(14)

Iterating which for $n = t - 1, t - 2, t - 3, \ldots, 0$ and adding those up we get,

$$R_2^n(\hat{O}^2_p) = r^n(\hat{O}^2_p) + \frac{2A}{L} \sum_{i=0}^{t} r^i + \frac{B}{L} (R_2^{t-1}(\hat{T}_{p,p+1}) + rR_2^{t-2}(\hat{T}_{p,p+1}) + \ldots + r^{t-1}\hat{T}_{p,p+1}) + \frac{B}{L} (R_2^{t-1}(\hat{T}_{p,p-1}) + \ldots + r^{t-1}\hat{T}_{p,p-1})$$

(15)

by keeping the most significant contributions for both. For the distance independent term $\sum_{S \in W} a(S,t)Q_S$ for short times $a(S,t) \approx 1$ for the fiducial nodes whereas $a(S,t) = O(1/L)$ for all other ones whereas for long times $a(S,t)$ is exponentially small for all nodes except for the sink nodes $S = \{\phi\}$, Fig. (5). In both time-regimes we keep only the leading order contributions in time for the distance dependent term $\sum_{S \in W} a(S,t)\Delta S$.

**SPATIO-TEMPORAL BEHAVIOR OF a(S,t)**

The behavior of $a_{1,2}(S,t)$ with increasing distance $D = \text{Dist}(S,S_{L,R})$ from the respective fiducial nodes show two distinct regimes. For $t < T_1 = \frac{L}{d^2} = (1 + 1/d^2)(L-2) = O(L)$ there is monotonic decrease of $a_{1,2}(D,t)$ with distance while for $t > T_2 = c(L+1)(L-2)(d^2+1) = O(L^2)$ there is monotonic increase of $a_{1,2}(D,t)$ with $D$, Fig. (4). We use this fact in extracting the leading contributions to the dynamical bound.

$$\sum_{S \in W} a(S,t)Q_S = \sum_{S \in W} a(S,t)Q_1S + \sum_{S \in W} a(S,t)Q_2S$$

$$= \sum_{S \in W} a(S,t)Q_1S + \sum_{S \in W} a(S,t)Q_2S - \sum_{S \in W} a(S,t)Q_2S$$

$$= \sum_{S \in W} a(S,t)Q_2S - \sum_{S \in W} a(S,t)\Delta S, \Delta S = (Q_1S - Q_2S)$$

(16)
FIG. 4. (color online) Spatial dependence of $a^* (S, t)$ at various fixed instants of time for a circular chain with $L = 10$. The blue ($t = 10 < T_1$) curve shows monotonic decrease of $a^* (S, t)$ for discrete various of $D = 0, 1, 2, ..., (L - 1)$. The Red ($t = 20$), Green ($t = 200$) and Magenta ($t = 380$) curves are for times $T_1 < t < T_2$ which show the a maxima for intermediate values of $D$, $0 < D < D_{\text{max}} = (L - 1)$. The Cyan ($t = 500 > T_2$) curve shows monotonic increase with $D$. The actual values have been amplified by various factors (Blue:$\times 10^0$, Red:$\times 10^1$, Green:$\times 10^3$, Magenta:$\times 10^{37}$, Cyan:$\times 10^{49}$) to set comparable orders of amplitude in order for their features to be explained on the same plot.

FIG. 5. (color online) Asymptotic contributions from the sink nodes for a chain with $L = 10$. The contributions from the $S = \{V\}$ node is multiplied by a factor of 100 to bring it to a comparable scale to be shown on the same graph.