Max Flow Vitality of Edges and Vertices in Undirected Planar Graphs

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Abstract. We study the problem of computing the vitality of edges and vertices with respect to st-max flow in undirected planar graphs, where the vitality of an edge/vertex in a graph with respect to max flow between two fixed vertices s, t is defined as the max flow decrease when the edge/vertex is removed from the graph. We show that a δ additive approximation of the vitality of all edges with capacity at most c can be computed in O(δn + n log log n) time, where n is the size of the graph. A similar result is given for the vitality of vertices. All our algorithms work in O(n) space.

Keywords: planar graphs · undirected graphs · max flow · vitality

1 Introduction

Max flow problems have been intensively studied in the last 60 years, we refer to [12] for a comprehensive bibliography. For general graphs, the currently best-known algorithms [16,20] compute the max flow between two vertices in O(mn) time, where m is the number of edges and n is the number of vertices.

Italiano et al. [15] solved the max flow problem for undirected planar graphs in O(n log log n) time. For directed st-planar graphs (i.e., graphs allowing a planar embedding with s and t on the same face) finding a max flow was reduced by Hassin [12] to the single source shortest path (SSSP) problem, that can be solved in O(n) time by the algorithm in [13]. For the planar directed case, Borradaile and Klein [8] presented an O(n log n) time algorithm. In the special case of directed planar unweighted graphs, a linear time algorithm has been proposed by Eisenstat and Klein [9].

The effect of arcs deletion on the max flow value has been studied since 1963, only a few years after the seminal paper by Ford and Fulkerson [10] in 1956. Wollner [24] presented a method for determining the most vital link (i.e., the arc whose deletion causes the largest decrease of the max flow value) in a railway network. A more general problem has been studied in [22], where an enumerative approach is proposed for finding the k arcs whose simultaneous removal causes the largest decrease in max flow. Wood [25] has shown that this problem is NP-hard in the strong sense, while its approximability has been studied in [4,21]. A survey on network interdiction problems can be found in [3].

In this paper, we deal with the computation of vitality of edges and vertices with respect to the max flow. The vitality of an edge e (resp., of a vertex v) measures the max flow decrease observed after the removal of edge e (resp., all edges incident to v) from the graph. A survey on vitality with respect to max flow problems can be found in [5]. In the same paper, it is shown that:

- the vitality of all edges in a general undirected graph can be computed by solving O(n) max flow instances, thus giving an overall O(n²m) time algorithm by applying the O(mn) max flow algorithms in [16,20];
– for \textit{st}-planar graphs (both directed or undirected) the vitality of all edges can be found in optimal \(O(n)\) time. The same result holds for determining the vitality of all vertices;
– the problem of determining the max flow vitality of a single edge is at least as hard as the max flow problem, both for general graphs and for the restricted class of \textit{st}-planar graphs.

Ausiello et al. \cite{6} proposed a recursive algorithm that computes the vitality of all edges in an undirected unweighted planar graph in \(O(n \log n)\) time.

The above cited paper of Italiano et al. \cite{15} also gave a dynamic algorithm that allows the operations described in the following theorem.

\textbf{Theorem 1 (\cite{15}).} There exists a data structure that after \(O(n \log r + \frac{n}{\sqrt{r}} \log n)\) preprocessing time supports: edge insertions and edge deletions in \(O((r + \frac{n}{\sqrt{r}}) \log^2 n)\) time; \(s\) to \(t\) distance queries in \(O((r + \frac{n}{\sqrt{r}}) \log^2 n)\) time; max \textit{st}-flow queries in \(O((r + \frac{n}{\sqrt{r}}) \log^2 n)\) time, where \(r \in [1, \ldots, n]\).

The dynamic structure given in Theorem 1 can be used to compute edge and vertex vitality. Thus by properly choosing \(r\), as implicitly shown in Łącki and Sankowski’s paper \cite{19}, the following corollary holds:

\textbf{Corollary 1 (\cite{15,19}).} It is possible to compute the vitality of \(h\) edges in \(O(\min\{hn \log \frac{\log n}{\log n}, hn^{2/3} \log^{8/3} n + n \log n\})\).

\textit{Our contribution} We propose fast algorithms for computing a guaranteed approximation of the vitality of all edges (resp., vertices) whose capacity is less than an arbitrary threshold \(c\).

Our main results are summarized in the following two theorems. For a graph \(G\), we denote by \(E(G)\) and \(V(G)\) its set of edges and vertices, respectively. Let \(c : E(G) \to \mathbb{R}^+\) be the \textit{edge capacity function}, we define the capacity \(c(v)\) of a vertex \(v\) as the sum of the capacities of all edges incident on \(v\). Moreover, for every \(x \in E(G) \cup V(G)\) we denote by \(\text{vit}(x)\) its vitality.

\textbf{Theorem 2.} Let \(G\) be a planar graph with positive edge capacities. Then for any \(c, \delta \geq 0\), we compute a value \(\text{vit}^\delta(e) \in (\text{vit}(e) - \delta, \text{vit}(e)]\) for all \(e \in E(G)\) satisfying \(c(e) \leq c\), in \(O(\frac{hn}{\log n} + n \log \log n)\) time.

\textbf{Theorem 3.} Let \(G\) be a planar graph with positive edge capacities. Then for any \(c, \delta \geq 0\), we compute a value \(\text{vit}^\delta(v) \in (\text{vit}(v) - \delta, \text{vit}(v)]\) for all \(v \in V(G)\) satisfying \(c(v) \leq c\), in \(O(\frac{hn}{\log n} + n \log n)\) time.

All our algorithms work in \(O(n)\) space. To explain the result stated in Theorem 2, we note that in the general case capacities are not bounded by any function of \(n\). Thus, in order to obtain useful approximations, \(c/\delta\) would seem to be very high. Despite this we explain that in many cases we can assume \(c/\delta\) constant, implying that the time complexity of Theorem 2 is equal to the best current time bound for computing the \textit{st}-max flow. The following remark is crucial, where \(c_{\text{max}} = \max_{e \in E(G)} c(e)\).

\textit{Remark 1.} [Bounding capacities] We can bound all edge capacities higher than \(MF\) to \(MF\), obtaining a new bounded edge capacity function. This change has no impact on the \textit{st}-max flow value or the vitality of any edge/vertex. Thus w.l.o.g., we can assume that \(c_{\text{max}} \leq MF\).

By using Remark 1 we can explain why \(c/\delta\) can be assumed constant in the case of uniform and power-law distribution of capacities.
• **Uniform case** (after bounding capacities as in Remark 1). If we set $c = c_{\text{max}}$ and $\delta = c/k$, for some constant $k$, then we obtain the capacities with an additive error less than $MF/k$, because of Remark 1. In many applications this error is acceptable even for small values of $k$.

• **Power-law case** (after bounding capacities as in Remark 1). Let $c = c_{\text{max}}^{\ell}$ for some constant $\ell$ and let $H_c = \{e \in E(G) | c(e) > c\}$. By power-law distribution, $|H_c|$ is small even for high value of $\ell$, and thus we compute the exact vitality of edges in $H_c$ by using Corollary 1. For edges with capacity less than $c$, we set $\delta = c/k$, for some constant $k$. By Remark 1 we compute the vitality of these edges with an additive error less than $MF/k$.

The above reasoning about Theorem 2 leads us to the following corollary.

**Corollary 2.** Let $G$ be a planar graph with positive edge capacities satisfying $c_{\text{max}} \leq MF$. Then for any $k, \ell \geq 0$, we compute a value $\overline{\text{vit}}(e) \in (\text{vit}(e) - \frac{MF}{k\ell}, \text{vit}(e))$ for all $e \in E(G)$ satisfying $c(e) \leq \frac{c_{\text{max}}}{\ell}$, in $O(kn + n \log \log n)$ time.

To apply the same argument to vertex vitality we have to make some observations. If $G$’s vertices have maximum degree $d$, then, after bounding capacities as in Remark 1 it holds $\max_{v \in V(G)} c(v) \leq dMF$. Otherwise, we note that a real-world planar graph is expected to have few vertices with high degree. The exact vitality of these vertices can be computed by Corollary 1 by removing the edges adjacent to a vertex one by one, or by using the following theorem.

**Theorem 4.** Let $G$ be a planar graph with positive edge capacities. Then for any $S \subseteq V(G)$, we compute $\text{vit}(v)$ for all $v \in S$ in $O(|S|n + n \log \log n)$ time.

If we denote by $E_S = \sum_{v \in S} \deg(v)$, where $\deg(v)$ is the degree of vertex $v$, then the result in Theorem 4 is more efficient than the result given in Corollary 1 if either $|S| < \log n$ and $E_S > |S| \log n$ or $|S| \geq \log n$ and $E_S > \frac{|S|n^{1/3}}{\log^{2/3}}$.

**Outline of our approach** We adopt Itai and Shiloach’s approach [14], that first computes a modified version $\mathcal{D}$ of a dual graph of $G$, then reduces the computation of the max flow to the computation of shortest non-crossing paths between pairs of vertices of the infinite face of $\mathcal{D}$. We first study the effect on $\mathcal{D}$ of an edge or a vertex removal in $G$, showing that computing the vitality of an edge or a vertex can be reduced to computing some distances in $\mathcal{D}$ (see Proposition 2 and Proposition 3).

Then we compute required distance by solving SSSP instances. To decrease the cost we use a divide and conquer strategy: we slice $\mathcal{D}$ in regions delimited by some of the non-crossing shortest paths computed above. We choose shortest non-crossing paths with similar lengths, so that we compute a guaranteed approximation of each distance by looking into a single region instead of examining the whole graph $\mathcal{D}$ (see Lemma 3).

Finally we have all the machinery to compute an approximation of required distances of Proposition 2 and obtain edge vitalities. To deal with vertex vitalities through Proposition 3 we have to use Klein’s result [17] and introduce a partial order on the faces of $\mathcal{D}$.

**Structure of the paper** In Section 2 we report main results about how to compute max flow in planar graphs; we focus on the approach in [14] on which our algorithms are based. In Section 3 we show our theoretical results that allow us to compute edge and vertex vitality. In Section 4 we explain our divide and conquer strategy. In Section 5 and Section 6 we state our main results about edge and vertex vitality, respectively. In Section 7 conclusions and open problems are given.
2 Max flow in planar graphs

In this section we report some well-known results concerning max flow, focusing on planar graphs.

Given a connected undirected graph $G = (V(G), E(G))$ with $n$ vertices, we denote an edge $e = \{i, j\} \in E(G)$ by the shorthand notation $ij$, and we define $\text{dist}_G(u, v)$ as the length of a shortest path in $G$ joining vertices $u$ and $v$. Moreover, for two sets of vertices $S, T \subseteq V(G)$, we define $\text{dist}_G(S, T) = \min_{u \in S, v \in T} \text{dist}_G(u, v)$. We write for short $v \in G$ and $e \in G$ in place of $v \in V(G)$ and $e \in E(G)$, respectively.

Let $s, t \in G$, $s \neq t$, be two fixed vertices. A feasible flow in $G$ assigns to each edge $e = ij \in G$ two real values $x_{ij} \in [0, c(e)]$ and $x_{ji} \in [0, c(e)]$ such that: $\sum_{j : ij \in E(G)} x_{ij} = \sum_{j : ij \in E(G)} x_{ji}$, for each $i \in V(G) \setminus \{s, t\}$. The flow from $s$ to $t$ under a feasible flow assignment $x$ is defined as $F(x) = \sum_{j : sj \in E(G)} x_{sj} - \sum_{j : js \in E(G)} x_{js}$. The maximum flow from $s$ to $t$, denoted by $MF$, is the maximum value of $F(x)$ over all feasible flow assignments $x$.

An st-cut is a partition of $V(G)$ into two subsets $S$ and $T$ such that $s \in S$ and $t \in T$. The capacity of an st-cut is the sum of the capacities of the edges $ij \in E(G)$ such that $\{S \cap \{i, j\}\} = 1$ and $|T \cap \{i, j\}| = 1$. The well known Min-Cut Max-Flow theorem \cite{10} states that the maximum flow from $s$ to $t$ is equal to the capacity of a minimum st-cut for any weighted graph $G$.

We denote by $G - e$ the graph $G$ after the removal of edge $e$. Similarly, we denote by $G - v$ the graph $G$ after the removal of vertex $v$ and all edges adjacent to $v$.

**Definition 1.** The vitality $\text{vit}(e)$ (resp., $\text{vit}(v)$) of an edge $e$ (resp., vertex $v$) with respect to the maximum flow from $s$ to $t$, according to the general concept of vitality in \cite{18}, is defined as the difference between the maximum flow in $G$ and the maximum flow in $G - e$ (resp., $G - v$).

We deal with planar undirected graphs. A **plane graph** is a planar graph with a fixed embedding. The dual of a plane undirected graph $G$ is an undirected planar multigraph $G^*$ whose vertices correspond to faces of $G$ and such that for each edge $e \in G$ there is an edge $e^* = \{u^*, v^*\} \in G^*$, where $u^*$ and $v^*$ are the vertices in $G^*$ that correspond to faces $f$ and $g$ adjacent to $e$ in $G$. Length $w(e^*)$ of $e^*$ equals the capacity of $e$, moreover, for a subgraph $H$ of $G^*$ we define $w(H) = \sum_{e \in H} w(e)$.

We fix a planar embedding of the graph, and we work on the dual graph $G^*$ defined by this embedding. A vertex $v$ in $G$ generates a face in $G^*$ denoted by $f^*_v$. We choose in $G^*$ a vertex $v^*_s$ in $f^*_s$ and a vertex $v^*_t$ in $f^*_t$. A cycle in the dual graph $G^*$ that separates vertex $v^*_s$ from vertex $v^*_t$ is called an **st-separating cycle**. Moreover, we choose a shortest path $\pi$ in $G^*$ from $v^*_s$ to $v^*_t$.

**Proposition 1** (see \cite{14,23}). A (minimum) st-cut in $G$ corresponds to a (shortest) cycle in $G^*$ that separates vertex $v^*_s$ from vertex $v^*_t$.

2.1 Itai and Shiloach’s approach/decomposition

According to the approach in \cite{14} used to find a min-cut by searching for minimum st-separating cycles, graph $G^*$ is "cut" along a shortest path $\pi$ from $v^*_s$ to $v^*_t$, obtaining graph $D_G$, in which each vertex $v^*_i$ in $\pi$ is split into two vertices $x_i$ and $y_i$; when no confusion arises we omit the subscript $G$. Let us assume that $\pi = \{v^*_1, v^*_2, \ldots , v^*_k\}$, with $v^*_1 = v^*_s$ and $v^*_k = v^*_t$. For convenience, let $\pi_x$ be the duplicate of $\pi$ in $D$ whose vertices are $\{x_1, \ldots , x_k\}$ and let $\pi_y$ be the duplicate of $\pi$ in $D$ whose vertices are $\{y_1, \ldots , y_k\}$. For any $i \in [k]$, where $[k] = \{1, \ldots , k\}$, edges in $G^*$ incident on each $v^*_i$ from above $\pi$ are moved to $y_i$ and edges incident on $v^*_i$ from below $\pi$ are moved to $x_i$. Edges incident on $v^*_s$ and $v^*_t$ are considered above or below $\pi$ on the basis of two dummy edges: the first joining $v^*_s$ to a dummy vertex $\alpha$ inside face $f^*_s$ and the second joining $v^*_t$ to a dummy vertex $\beta$ inside face $f^*_t$.
For each $e^* \in \pi$, we denote by $e^*_x$ the copy of $e^*$ in $\pi_x$ and $e^*_y$ the copy of $e^*$ in $\pi_y$. In Figure 1, on the left there is graph $G$ in black and continuous line, $G^*$ in red and dashed lines, a shortest path $\pi$ from $v^*_s$ to $v^*_k$, on the right graph $D$. Note that, every $v \in V(G) \setminus \{s, t\}$ generates a face $f_v^D$ in $D$. There are not faces $f^*_s$ and $f^*_t$ because the dummy vertices $\alpha$ and $\beta$ are inside faces $f^*_s$ and $f^*_t$, respectively. Both faces $f^*_s$ and $f^*_t$ “correspond” in $D$ to the leftmost $x_1y_1$ path and to the rightmost $x_ky_k$ path. Since we are not interested in removing vertices $s$ and $t$, then faces $f^*_s$ and $f^*_t$ are not needed in $D$.

If $e^* \notin \pi$, then we denote the corresponding edge in $D$ by $e^D$. Similarly, if $v^*_i \notin \pi$ (that is, $i > k$), then we denote the corresponding vertex in $D$ by $v^*_i$.  

![Image](image.png)

Fig. 1: on the left graph $G$ in black and continuous line, $G^*$ in red and dashed lines, shortest path $\pi$ from $v^*_s$ ($v^*_t$) to $v^*_i$ ($v^*_k$) in green, $\alpha$ and $\beta$ are dummy vertices. On the right graph $D_G$.

### 3 Our theoretical results

In this section we show our main theoretical results (Proposition 2 and Proposition 3) that allow us to compute edge and vertex vitality. In Subsection 3.1 we show the effects in $G^*$ and $D$ of removing an edge or a vertex from $G$. In Subsection 3.2 we deal with crossings between $\pi$ and $st$-separating cycles and in Subsection 3.3 we state the two main propositions about edge and vertex vitality.

#### 3.1 Effects on $G^*$ and $D$ of deleting an edge or a vertex of $G$

We observe that removing an edge $e$ from $G$ corresponds to contracting endpoints of $e^*$ into one vertex in $G^*$. With respect to $D$, if $e^* \notin \pi$, then the removal of $e$ corresponds to the contraction into one vertex of endpoints of $e^D$. If $e^* \in \pi$, then both copies of $e^*$ have to be contracted. In Figure 2, we show the effects of removing edge $eg$ from graph $G$ in Figure 1.

Let $v$ be a vertex of $V(G)$. Removing $v$ corresponds to contracting vertices of face $f^*_v$ in $G^*$ into a single vertex. In $D$, if $f^*_v$ and $\pi$ have no common vertices, then all vertices of $f^*_v$ are contracted into one. Otherwise $f^*_v$ intersects $\pi$ on vertices $\bigcup_{i \in I} \{v^*_i\}$ for some non empty set $I \subseteq [k]$. Then all vertices of $f^*_v$ are contracted into one vertex, all vertices of $\bigcup_{i \in I} \{x_i\}$ not belonging to $f^*_v$ are contracted into another vertex and all vertices of $\bigcup_{i \in I} \{y_i\}$ not belonging to $f^*_v$ are contracted into a third vertex. For convenience, we define $q^{x^D}_v = (\bigcup_{i \in I} \{x_i\}) \setminus V(f^*_v)$ and $q^{y^D}_v = (\bigcup_{i \in I} \{y_i\}) \setminus V(f^*_v)$. To better understand these definitions, see Figure 3. In Figure 4, it is shown what happens when we remove vertex $g$ of graph $G$ in Figure 1.
Fig. 2: starting from graph $G$ in Figure 1 we show on the left graph $G - eg$ and $(G - eg)^*$, and graph $D_{G - eg}$ on the right.

Fig. 3: a face $f_v^D$, for some $v \in V(G)$, and sets $q_x^D$ and $q_y^D$. Removing $v$ from $G$ corresponds in $D$ to contracting vertices of $f_v^D$, $q_x^D$, and $q_y^D$ in three distinct vertices.

Fig. 4: starting from graph $G$ in Figure 1 we show on the left graph $G - g$ and $(G - g)^*$, and graph $D_{G - g}$ on the right.

3.2 Single-crossing st-separating cycles

Itai and Shiloach [14] consider only shortest st-separating cycles that cross $\pi$ exactly once, that correspond in $D$ to paths from $x_i$ to $y_i$, for some $i \in [k]$. Formally, given two paths $p_1, p_2$ in a plane graph, a crossing between $p_1$ and $p_2$ is a minimal subpath of $p_1$ defined by vertices $v_1, v_2, \ldots, v_k$, with $k \geq 3$, such that vertices $v_2, \ldots, v_{k-1}$ are contained in $p_2$, and, fixing an orientation of $p_2$, edge $v_1v_2$ lies to the left of $p_2$ and edge $v_{k-1}v_k$ lies to the right of $p_2$, or vice-versa. We say $p_1$ crosses $p_2$ $t$ times if there are $t$ different crossings between $p_1$ and $p_2$.

In our approach, we contract vertices of an edge or a face of $G^*$. Despite this we can still consider only st-separating cycles that cross $\pi$ exactly once. The proof of this is the goal of this subsection.

**Lemma 1.** Let $\gamma$ be a simple st-separating cycle and let $S$ be either an edge or a face of $G^*$. Let $r = |V(\gamma) \cap V(S)|$. After contracting vertices of $S$ into one vertex, then $\gamma$ becomes the union of $r$ simple cycles and exactly one of them is an st-separating cycle.

**Proof.** Since an edge can be seen as a degenerate face, we prove the statement only in the case in which $S$ is a face $f$. Let $v^* \in V(\gamma)$ and let $u_1^*, u_2^*, \ldots, u_r^*$ be the vertices of $V(\gamma) \cap V(f)$ ordered in
clockwise order starting from $v^*$. For convenience, let $u^*_i = u^*_i$. For $i \in [r]$, let $q_i$ be the clockwise $u^*_i u^*_{i+1}$ path on $γ$. After contracting the vertices of $f$ into one, $q_i$ becomes a cycle. Every $q_i$'s joined with the counterclockwise $u^*_i u^*_{i+1}$ path on the border cycle of $f$ defines a region $R_i$ of $G^*$. We remark that if $q_i$ is composed by a single edge $e^*$, then $q_i$ becomes a self-loop and region $R_i$ is a composed only by $e^*$.

Cycle $γ$ splits graph $G^*$ into two regions: a region internal to $γ$ called $R_{in}$ and an external region called $R_{out}$. W.l.o.g., we assume that $s \in R_{in}$ and $t \in R_{out}$. Now we split the proof into two cases: $f \subseteq R_{in}$ and $f \subseteq R_{out}$.

- Case $f \subseteq R_{in}$. By above, it holds that $R_1, \ldots, R_r \subseteq R_{in}$ (see Figure 5 on the left). Being $γ$ an $st$-separating cycle, then there exists a unique $j \in [r]$ such that $s \in R_j$. Thus, after contracting vertices of $f$ into one, $p_j$ becomes the unique $st$-separating cycle, while all others $R_i$'s become cycles that split $G^*$ into two regions, and each region contains neither $s$ nor $t$ (see Figure 6 on the right).

- Case $f \subseteq R_{out}$. By above there exists a unique $j \in [r]$ such that $R_i \subseteq R_{out}$ for all $i \neq j$ and $R_{in} \subseteq R_j$ (see Figure 6 on the left). W.l.o.g., we assume that $j = r$. After contracting the vertices of $f$ into one, all regions $R_1, \ldots, R_{r-1}$ become regions inside $R_r$ because of the embedding (see Figure 6 on the right). We recall that $s \in R_{in}$, thus there are two cases: if $t \in R_i$ for some $i \in [r-1]$, then $p_i$ becomes the unique $st$-separating cycle; otherwise, $t \in R_{out}$, and thus $p_r$ becomes the unique $st$-separating cycle.

Let $Γ$ be the set of all $st$-separating cycles in $G^*$, and let $Γ_1$ be the set of all $st$-separating cycles in $G^*$ that cross $π$ exactly once. Given $γ \in Γ$ and either an edge or a face $S$ of $G^*$, thanks to Lemma 1 we can define $Δ_S(γ)$ as “the length of the unique $st$-separating cycle contained in $γ$ after contracting vertices of $S$ into one”. Being $MF$ equal to the length of a minimum $st$-separating cycle, the following relations hold:

\[
\text{for any } e \in E(G), \ vit(e) = MF - \min_{γ \in Γ} Δ_{e^*}(γ),
\]

\[
\text{for any } v \in V(G) \setminus \{s, t\}, \ vit(v) = MF - \min_{γ \in Γ} Δ_{f^v}(γ).
\]

Now we show that in the above equations we can replace set $Γ$ with set $Γ_1$. 7
Lemma 2. Let $e \in E(G)$ and $v \in V(G) \setminus \{s,t\}$. It holds that $\text{vit}(e) = MF - \min_{\gamma \in I_1} \Delta_{e^*}(\gamma)$ and $\text{vit}(v) = MF - \min_{\gamma \in I_1} \Delta_{f_v}(\gamma)$.

Proof. We recall that removing an edge $e$ from $G$ corresponds to contracting endpoints of $e^*$ into one vertex, while removing a vertex $v$ from $G$ corresponds to contracting all the vertices in face $f_v^*$ into one vertex. So we prove the thesis only in the more general case of vertex removal. For convenience, we denote $f_v^*$ by $f$. Let $\gamma \in \Gamma$ be such that $\text{vit}(f) = MF - \Delta_f(\gamma)$ and assume that $\gamma \not\in I_1$. If $V(\gamma) \cap V(f) = \emptyset$, then $\text{vit}(f) = 0$, hence it suffices to remove crossings between $\gamma$ and $\pi$, see [14]. Thus let us assume that $V(\gamma) \cap V(f) \neq \emptyset$.

By Lemma 1 there exist unique $a^*, b^* \in V(f) \cap V(\gamma)$ such that the clockwise $a^*b^*$ path $p$ on $\gamma$ becomes an $st$-separating cycle after the contraction of vertices of $f$ into one. Then we remove crossing between $p$ and $\pi$ in order to obtain a path $p'$ not longer than $p$ as above. Finally, let $\gamma' = p \circ q$, where $q$ is the clockwise $a^*b^*$ path on $f$. It holds that $\gamma' \in I_1$ and $\Delta_f(\gamma') \leq \Delta_f(\gamma)$, the thesis follows. \hfill \Box

3.3 Vitality vs. distances in $D$

The main results of this subsection are Proposition 2 and Proposition 3. The first proposition shows which distances in $D$ are needed to obtain edge vitality and in the latter proposition we do the same for vertex vitality. In Subsection 3.3.1 we have proved that removing an edge or a vertex from $G$ corresponds to contracting in single vertices some sets of vertices of $D$. The main result of Proposition 2 and Proposition 3 is that we can consider these vertices individually.

Let $e$ be an edge of $G$. The removal of $e$ from $G$ corresponds to the contraction of endpoints of $e^*$ into one vertex in $G^*$. Thus if an $st$-separating cycle $\gamma$ of $G^*$ contains $e^*$, then the removal of $e$ from $G$ reduces the length of $\gamma$ by $w(e^*)$. Thus $e$ has strictly positive vitality if and only if there exists an $st$-separating cycle $\gamma$ in $G^*$ whose length is strictly less than $MF + w(e^*)$ and $e^* \in \gamma$. This is the main idea to compute the vitality of all edges. Now we have to translate it to $D$.

We observe that capacities of edges in $G$ become lengths (or weights) in $D$. For this reason, we define $w(e^D) = c(e)$, for all edges $e \in G$ satisfying $e^* \not\in \pi$ and $w(e^D_x) = w(e^D_y) = c(e)$ for all edges $e \in G$ satisfying $e^* \in \pi$.

For $i \in [k]$, we define $d_i = \text{dist}_D(x_i, y_i)$. We observe that $MF = \min_{i \in [k]} d_i$. For a subset $S$ of $V(D)$ and any $i \in [k]$ we define $d_i(S) = \min\{d_i, \text{dist}_D(x_i, S) + \text{dist}_D(y_i, S)\}$. We observe that $d_i(S)$ represents the distance in $D$ from $x_i$ to $y_i$ if all vertices of $S$ are contracted into one.

For every $x \in V(G) \cup E(G)$ we define $MF_x$ as the max flow in graph $G - x$. By definition, $\text{vit}(x) = MF - MF_x$ and, trivially, $x$ has strictly positive vitality if and only if $MF_x < MF$.

Proposition 2. For every edge $e$ of $G$, if $e^* \not\in \pi$, then $MF_e = \min_{i \in [k]} \{d_i(e^D_x), d_i(e^D_y)\}$. If $e^* \in \pi$, then $MF_e = \min_{i \in [k]} \{ \min\{d_i(e^D_x), d_i(e^D_y)\} \}$.

Proof. Let $e$ be an edge of $G$. If $\text{vit}(e) = 0$, then $MF_e = MF$ and the thesis trivially holds. Hence let us assume $\text{vit}(e) > 0$, then by Lemma 2 there exists $\gamma \in I_1$ such that $w(\gamma) < MF + w(e^*)$ and $e^* \in \gamma$. If $e^* \not\in \pi$, then $e$ corresponds in $D$ to edge $e^D$, thus the thesis holds. If $e^* \in \pi$, then we note that every path in $D$ containing both $e^D_x$ and $e^D_y$ corresponds in $G^*$ to an $st$-separating cycle that passes through $e^*$ twice, thus its length is equal or greater than $MF + 2c(e)$. Thus we consider only paths that contain $e^D_x$ or $e^D_y$ but not both. The thesis follows. \hfill \Box

Now we deal with vertex vitality. Note that if $f^*_v$ and $\pi$ have some common vertices, then one among $q^x_{f^*_v}$ and $q^y_{f^*_v}$ could be empty. For convenience, we set $d_i(\emptyset) = +\infty$, for all $i \in [k]$. 


Proposition 3. For every vertex $v$ of $G$, if $f^*_v$ and $\pi$ have no common vertices, then $MF_v = \min_{i \in [k]} \{d_i(f)\}$, where $f = f^D_v$, otherwise

\[
MF_v = \min \left\{ \begin{array}{l}
\min_{i \in [k]} \{d_i(f)\}, \\
\min_{i \in [k]} \{d_i(q^v_f)\}, \\
\min_{i \in [k]} \{d_i(q^z_f)\}, \\
dist_D(f, q^v_f), \\
dist_D(f, q^z_f) \end{array} \right\}. \tag{1}
\]

Proof. If $f^*_v$ and $\pi$ have no common vertices, then the proof is analogous to the edge case. Thus let us assume that $f^*_v$ and $\pi$ have common vertices. Let $D'$ be the graph obtained from $D$ by adding a vertex $u, v, z$ connected with all vertices of $q^v_f$, of $q^z_f$, of $f$, respectively, with zero weight edges; for convenience we assume that $q^v_f$ and $q^z_f$ are both not empty. By Lemma 2 and discussion in Subsection 3.1, $MF_v = w(p)$, where $p$ is a shortest $x_i y_i$ path in $D'$, varying $i \in [k]$.

Note that after contracting vertices of $f$ into one vertex there exists an $x_i y_i$ path whose length is $dist_D(f, x_i)$, for all $x_i \in q^v_f$. In particular, there exists an $x_i y_i$ path whose length is $dist_D(f, q^v_f)$, for some $i$ satisfying $x_i \in q^v_f$. The same argument applies for $q^z_f$. This implies that if $vit(v) = 0$, then Equation (1) is correct. Hence we assume that $vit(v) > 0$, so at least one among $u, v$ and $z$ belongs to $p$.

If $u \in p$ and $v, z \notin p$ (resp., $v \in p$ and $u, z \notin p$), then $w(p) = \min_{i \in [k]} \{d_i(q^v_f)\}$ (resp., $w(p) = \min_{i \in [k]} \{d_i(q^z_f)\}$) . If $z \in p$ and $u, v \notin p$ then $w(p) = \min_{i \in [k]} \{d_i(f)\}$. We have analyzed all cases in which $p$ contains exactly one vertex among $u, v$ and $z$. To complete the proof, we prove that, for any $i \in [k]$, every $x_i y_i$ path that contains at least two vertices among $u, v$ and $z$ also contains a subpath whose length is at least $\min\{dist_D(f, q^v_f), dist_D(f, q^z_f)\}$.

Let $\ell$ be an $x_i y_i$ path, for some $i \in [k]$. If $u, z \in \ell$, then there exists a subpath $\ell'$ of $\ell$ from a vertex $x_j$ of $q^v_f$ to a vertex $r$ of $f$. If we add to $\ell'$ the two zero weighted edges $rz$ and $zy_j$ we obtain a $x_j y_j$ path whose length is at least $\min\{dist_D(f, q^v_f), dist_D(f, q^z_f)\}$. We can use a symmetric strategy if $v, z \in \ell$.

It remains only the case in which $u, v \in \ell$. If $q^v_f$ and $q^z_f$ are both non-empty, then $f$ splits $D$ and $D'$ into two or more parts and no part contains vertices of both $q^v_f$ and $q^z_f$ (see Figure 3). Thus if $u, v \in \ell$, then $\ell$ passes through at least one vertex of $f$, implying that $\ell$ has a subpath from a vertex of $f$ to a vertex of $q^v_f$, or $q^z_f$. As above, this path can be transformed in a $x_j y_j$ path shorter than $\ell$ whose length is at least $\min\{dist_D(f, q^v_f), dist_D(f, q^z_f)\}$, for some $j \in [k]$.

\[\square\]

4 Slicing graph $D$ preserving approximated distances

In this section we explain our divide and conquer strategy. We slice graph $D$ along shortest $x_i y_i$'s paths. If these paths have lengths that differ at most $\delta$, then we have a $\delta$ additive approximation of distances required in Proposition 2 and Proposition 3 by looking into a single slice instead of the whole graph $D$. This result is stated in Lemma 3. These slices can share boundary vertices and edges, implying that their dimension might be $O(n^2)$. In Lemma 4 we compute an implicit representation of these slices in linear time.

From now on, we mainly work on graph $D$, thus we omit the superscript $D$ unless we refer to $G$ or $G^*$. To work in $D$ we need a shortest $x_i y_i$ path and its length, for all $i \in [k]$. In the following theorem we show time complexities for obtaining elements in $D$. We say that two paths are single-touch if their intersection is still a path.

Given two graphs $A = (V(A), E(A))$ and $B = (V(B), E(B))$ we define $A \cup B = (V(A) \cup V(B), E(A) \cup E(B))$ and $A \cap B = (V(A) \cap V(B), E(A) \cap E(B))$. Note that if $A$ and $B$ are subgraphs of $G$, then $A \cup B$ and $A \cap B$ are subgraphs of $G$. 

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Theorem 5 ([7],[11],[15]). If \( G \) is a positive edge-weighted planar graph,

1. we compute \( U = \bigcup_{i \in [k]} p_i \) and \( w(p_i) \) for all \( i \in [k] \), where \( p_i \) is a shortest \( x_iy_i \) path in \( D \) and \( \{p_i\}_{i \in [k]} \) is a set of pairwise non-crossing single-touch paths, in \( O(n \log \log n) \) time—see [12] for computing \( U \) and [7] for computing \( w(p_i) \)'s,
2. for every \( I \subseteq [k] \), we compute \( \bigcup_{i \in I} p_i \) in \( O(n) \) time—see [17] by noting that \( U \) is a forest and the paths can be found by using nearest common ancestor queries.

From now on, for each \( i \in [k] \) we fix a shortest \( x_iy_i \) path \( p_i \), and we assume that \( \{p_i\}_{i \in [k]} \) is a set of pairwise non-crossing shortest paths. Let \( U = \bigcup_{i \in [k]} p_i \), see Figure 7(b).

Given an \( ab \) path \( p \) and a \( bc \) path \( q \), we define \( p \circ q \) as the (possibly not simple) \( ac \) path obtained by the union of \( p \) and \( q \). Each \( p_i \)’s splits \( D \) into two parts as shown in the following definition and in Figure 7(b).

Definition 2. For every \( i \in [k] \), we define \( \text{Left}_i \) as the subgraph of \( D \) bounded by the cycle \( \pi_y[y_1, y_i] \circ p_i \circ \pi_z[x_i, x_1] \circ l \), where \( l \) is the leftmost \( x_1y_1 \) path in \( D \). Similarly, we define \( \text{Right}_i \) as the subgraph of \( D \) bounded by the cycle \( \pi_y[y_i, y_k] \circ r \circ \pi_z[x_k, x_i] \circ p_i \), where \( r \) is the rightmost \( x_ky_k \) path in \( D \).

Fig. 7: in (a) graph \( U \) in bold, in (b) subgraphs \( \text{Left}_i \) and \( \text{Right}_i \) are highlighted, in (c) subgraph \( \Omega_{i,j} \), for some \( i < j \).

Based on Definition 2 for every \( i, j \in [k] \), with \( i < j \), we define \( \Omega_{i,j} = \text{Right}_j \cap \text{Left}_j \), see Figure 7(c). We classify \((x_i, y_i)\)'s pairs according to the difference between \( d_y \) and \( MF \). Each class contains pairs for which this difference is about \( r \) times \( \delta \); where \( \delta > 0 \) is an arbitrarily fixed value.

For every \( r \in \mathbb{N} \), we define \( L_r = (\ell_r^1, \ldots, \ell_r^z) \) as the ordered list of indices in \([k]\) such that \( d_j \in [MF + \delta r, MF + \delta(r + 1)] \) for all \( j \in L_r \), and \( \ell_r^j < \ell_r^{j+1} \) for all \( j \in [z - 1] \). It is possible that \( L_r = \emptyset \) for some \( r > 0 \) (it holds that \( L_0 \neq \emptyset \)). If no confusion arises, we omit the superscript \( r \); thus we write \( \ell_i \) in place of \( \ell_r^i \).

The following lemma is the key of our slicing strategy. In particular, Lemma 3 can be applied for computing distances required in Proposition 2 and Proposition 3, since the vertex set of a face or an edge of \( D \) is always contained in a slice. An application is in the right part of Figure 8.

Lemma 3. Let \( r > 0 \) and let \( L_r = (\ell_1, \ell_2, \ldots, \ell_z) \). Let \( S \) be a set of vertices of \( D \) with \( S \subseteq \Omega_{\ell_i, \ell_{i+1}} \) for some \( i \in [z - 1] \). Then

\[
\min_{\ell \in L_r} d_{\ell}(S) > \min_{\ell \in L_r} \{d_{\ell_i}(S), d_{\ell_{i+1}}(S)\} - \delta.
\]

Moreover, if \( S \subseteq \text{Left}_{\ell_i} \) (resp., \( S \subseteq \text{Right}_{\ell_i} \)) then \( \min_{\ell \in L_r} d_{\ell}(S) > d_{\ell_i}(S) - \delta \) (resp., \( \min_{\ell \in L_r} d_{\ell}(S) > d_{\ell_i}(S) - \delta \)).

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Proof. We need the following crucial claim.

(i) Let $i < j \in \mathcal{L}_r$. Let $L$ be a set of vertices in $\mathcal{L}_i$ and let $R$ be set of vertices in $\mathcal{L}_j$.

Then $d_i(L) < d_j(L) + \delta$ and $d_i(R) < d_i(R) + \delta$.

Proof of (i): we prove that $d_i(L) < d_j(L) + \delta$. By symmetry, it also proves that $d_i(R) < d_i(R) + \delta$.

Let us assume by contradiction that $d_i(L) \geq d_j(L) + \delta$.

Let $\alpha$ (resp., $\epsilon$, $\mu$, $\nu$) be a path from $x_i$ (resp., $y_i$, $x_j$, $y_j$) to $z_\alpha$ (resp., $z_\epsilon$, $z_\mu$, $z_\nu$) whose length is $d(x_i, L)$ (resp. $d(y_i, L)$, $d(x_j, L)$, $d(y_j, L)$), see Figure 8 on the left. Being $x_j, y_j \in \mathcal{L}_i$ and $L \subseteq \mathcal{L}_i$, then $\mu, \nu$ cross $p_i$. Let $v$ be the vertex that appears first in $p_i \cap \mu$ starting from $x_j$ on $\mu$ and let $u$ be the vertex that appears first in $p_i \cap \nu$ starting from $y_j$ on $\nu$. An example of these paths is in Figure 8 on the left. Let $\zeta = p_i[y_i, u]$, $\theta = p_i[u, v]$, $\beta = p_i[x_i, v]$, $\kappa = \mu[x_j, v]$, $\iota = \nu[y_j, u]$, $\eta = \nu[u, z_\nu]$ and $\gamma = \mu[v, z_\mu]$, see Figure 8 in the middle.

Now $\ell(\beta) + \ell(\kappa) \geq \ell(\alpha)$, otherwise $\alpha$ would not be a shortest path from $x_i$ to $L$. Similarly $\ell(\zeta) + \ell(\eta) \geq \ell(\epsilon)$. Moreover, being $\ell(\zeta) + \ell(\theta) + \ell(\beta) = d_i$, then $\ell(\theta) \leq d_i - \ell(\alpha) + \ell(\gamma) - \ell(\epsilon) + \ell(\eta)$. Being $d_i(L) \geq d_j(L) + \delta$, then $\ell(\alpha) + \ell(\epsilon) \geq \ell(\mu) + \ell(\nu) + \delta$, this implies $\ell(\alpha) + \ell(\epsilon) \geq \ell(\kappa) + \ell(\gamma) + \ell(\iota) + \ell(\eta) + \delta$.

It holds that $\ell(\theta) + \ell(\kappa) + \ell(\iota) \leq d_j - \ell(\alpha) + \ell(\gamma) - \ell(\epsilon) + \ell(\eta) + \ell(\alpha) + \ell(\epsilon) + \ell(\eta) - \ell(\gamma) - \ell(\theta) - \ell(\kappa) = d_j - \delta < d_j$ because $i, j \in \mathcal{L}_r$ imply $|d_i - d_j| < \delta$. Thus $\kappa \circ \theta \circ \iota$ is a path from $x_j$ to $y_j$ strictly shorter than $d_j$.

End proof of (i).

Being $S \subseteq \mathcal{L}_i$ for all $j < i$ and $S \subseteq \mathcal{L}_j$ for all $j' > i + 1$, then the first part of the thesis follows from Claim (i). The second part follows also from Claim (i) by observing that if $S \subseteq \mathcal{L}_i$, then $S \subseteq \mathcal{L}_i$ for all $i \in \mathcal{L}_r$.

![Fig. 8](image)

To compute distances in $D$ we have to solve some SSSP instances in some $\Omega_{i,j}$’s subsets. These subsets can share boundary edges, thus the sum of their edges might be $O(n^2)$. We note that, by the single-touch property, if an edge $e$ belongs to $\Omega_{i,j}$ and $\Omega_{j,i}$ for some $i < j < \ell \in [k]$, then $e \in p_j$.

To overcome this problem we introduce subsets $\bar{\Omega}_{i,j}$ in the following way: for any $i < j \in [k]$, if $p_i \cap p_j$ is a non-empty path $q$, then we define $\bar{\Omega}_{i,j}$ as $\Omega_{i,j}$ in which we replace path $q$ by an edge with the same length; note that the single-touch property implies that all vertices in $q$ but its extremal have degree two. Otherwise, we define $\bar{\Omega}_{i,j} = \Omega_{i,j}$. Note that distances between vertices in $\bar{\Omega}_{i,j}$ are the same as in $\Omega_{i,j}$. Now we show how to compute all $\bar{\Omega}_{i,j}$’s in $O(n)$ time.

Lemma 4. Let $A = (a_1, a_2, \ldots, a_z)$ be any increasing sequence of indices of $[k]$. It holds that $\sum_{\ell \in [z - 1]} |E(\bar{\Omega}_{a_{\ell}, a_{\ell + 1}})| = O(n)$. Moreover, given $U$, we compute $\bar{\Omega}_{a_{\ell}, a_{\ell + 1}}$, for all $i \in [z - 1]$, in $O(n)$ total time.

Proof. For convenience, we denote by $\Omega_{\ell}$ the set $\Omega_{a_\ell, a_{\ell + 1}}$, for all $i \in [z - 1]$. We note that if $e \in \Omega_{\ell} \cap \Omega_{\ell + 1}$, then $e \in p_{\ell + 1}$. Thus, if $e$ belongs to more than two $\Omega_{\ell}$’s, then $e$ belongs to exactly
two $\tilde{\Omega}$’s because it is contracted in all other $\Omega_i$’s by definition of the $\tilde{\Omega}_i$’s. Thus $\sum_{i \in [z-1]} |E(\tilde{\Omega}_i)| = O(n) + O(z) = O(n)$ because $z \leq k \leq n$.

To obtain all $\tilde{\Omega}_i$’s, we compute $U_z = \bigcup_{a \in A} p_a$ in $O(n)$ time by Theorem 5. Then we preprocess all trees in $U_z$ in $O(n)$ time by using Gabow and Tarjan’s result in order to obtain the intersection path $p_a \cap p_{a+1}$ and its length in $O(1)$ time. Finally, we build $\tilde{\Omega}_i$ in $O(|E(\tilde{\Omega}_i)|)$, for all $i \in [z-1]$, with a BFS visit of $\Omega_i$ that excludes vertices of $p_a \cap p_{a+1}$.

\section{Computing edge vitality}

Now we can give our main result about edge vitality stated in Theorem 2. We need the following preliminary lemma that is an easy consequence of Lemma 3 and Lemma 4.

\begin{lemma}
Let $r \in \mathbb{N}$, given $U$, we compute a value $\alpha_r(e) \in \{\min_{i \in L_r} \{d_i(e)\}, \min_{i \in L_r} \{d_i(e)\} + \delta\}$ for all $e \in E(D)$ in $O(n)$ time.
\end{lemma}

\begin{proof}
We compute $U_r = \bigcup_{i \in L_r} p_i$ in $O(n)$ time by Theorem 5. Let $e \in E(D)$. If $e \notin U_r$, we set $\alpha_r(e) = MF + \delta (r+1) - w(e)$. If $e \in U_r$, then either $e \in \tilde{\Omega}_{t_i, t_{i+1}}$ for some $i \in [z-1]$, or $e \in \text{Left}_{t_i}$, or $e \in \text{Right}_{t_i}$.

For all $e \subseteq \text{Left}_{t_i}$, we set $\alpha_r(e) = d_{t_i}(e)$, similarly, for all $e \subseteq \text{Right}_{t_i}$, we set $\alpha_r(e) = d_{t_i}(e)$. Finally, if $e \in \tilde{\Omega}_{t_i, t_{i+1}}$, then we set $\alpha_r(e) = \min\{d_{t_i}(e), d_{t_{i+1}}(e)\}$. All these choices satisfy the required estimation by Lemma 5.

To compute required distances, it suffices to solve two SSSP instances with sources $x_i$ and $y_i$ to vertices of $\tilde{\Omega}_i \cup \Omega_{i+1}$, for every $i \in L_r$. In total we spend $O(n)$ time by Lemma 4 by using algorithm in [13] for SSSP instances.
\end{proof}

\begin{theorem}
Let $G$ be a planar graph with positive edge capacities. Then for any $c, \delta \geq 0$, we compute a value $\text{vit}^\delta(e) \in (\text{vit}(e) - \delta, \text{vit}(e)]$ for all $e \in E(G)$ satisfying $c(e) \leq c$, in $O(\frac{\omega}{n} + n \log \log n)$ time.
\end{theorem}

\begin{proof}
We compute $U$ in $O(n \log \log n)$ time by Theorem 5. If $d_i > MF + c(e)$, then $d_i(e^D) > MF$, so we are only interested in computing (approximate) values of $d_i(e^D)$ for all $i \in [k]$ satisfying $d_i < MF + c$. By Lemma 5 for each $r \in \{0, 1, \ldots, \lceil \frac{n}{\delta} \rceil\}$, we compute $\alpha_r(e^D) \in \{\min_{i \in L_r} \{d_i(e^D)\}, \min_{i \in L_r} \{d_i(e^D)\} + \delta\}$, for all $e^D \in E(D)$, in $O(n)$ time. Then, for each $e^D \in E(D)$, we compute $\alpha(e^D) = \min_{r \in \{0, 1, \ldots, \lceil \frac{n}{\delta} \rceil\}} \alpha_r(e^D)$; it holds that $\alpha(e^D) \in \{\min_{i \in [k]} \{d_i(e^D)\}, \min_{i \in [k]} \{d_i(e^D)\} + \delta\}$. Then, by Proposition 3, for each $e \in E(G)$ satisfying $c(e) \leq c$, we compute a value $\text{vit}^\delta(e) \in (\text{vit}(e) - \delta, \text{vit}(e)]$ in $O(1)$ time.
\end{proof}

\section{Computing vertex vitality}

In this section we show how to compute vertex vitality by computing an additive approximation of distances required in Proposition 3.

Let us denote by $F$ the set of faces of $D$. By Proposition 3, for every face $f \in F$ we need $\min_{i \in [k]} \{d_i(f)\}$, this is discussed in Lemma 6. For faces $f \in F^x = \{f \in F \mid f$ and $\pi_x$ have common vertices$\}$ we need also $\min_{i \in [k]} \{d_i(q^y_f)\}$ and $\text{dist}_D(f, q^y_f)$. Similarly, for faces $f \in F^y = \{f \in F \mid f$ and $\pi_y$ have common vertices$\}$ we need also $\min_{i \in [k]} \{d_i(q^y_f)\}$ and $\text{dist}_D(f, q^y_f)$.

We observe that there is symmetry between $q^y_f$ and $q^y_f$. Thus we restrict some definitions and results to the “y case” and then we use the same results for the “x case”. In this way, we have to show only how to compute $\text{dist}_D(f, q^y_f)$ (it is done in Subsection 6.1) and $\min_{i \in [k]} \{d_i(q^y_f)\}$ (see Subsection 6.2) for every face $f \in F$ that intersects $\pi_y$ on vertices.
By using the same procedure of Lemma 5 we can also computing $d_i(f)$ for $f \in F$. Thus we can state the following lemma.

**Lemma 6.** Let $r \in \mathbb{N}$, given $U$, we compute a value $\alpha_r(f) \in [\min_{i \in L_r} \{d_i(f)\}, \min_{i \in L_r} \{d_i(f)\} + \delta]$ for all $f \in F$ in $O(n)$ time.

### 6.1 Computing $\text{dist}_D(f, q_f^y)$

The unique our result of this subsection is stated in Lemma 7. To obtain it, we use the following result that easily derives from Klein’s algorithm about the multiple source shortest path problem [17].

**Theorem 6 ([17]).** Given an $n$ vertices undirected planar graph $G$ with nonnegative edge-lengths, given $r$ pairs $\{(a_i, b_i)\}_{i \in [r]}$ where the $b_i$’s are on the boundary of the infinite face and the $a_i$’s are anywhere, it is possible to compute $\text{dist}_{G}(a_i, b_i)$, for all $i \in [r]$, in $O(r \log n + n \log n)$ time and $O(n)$ space.

**Lemma 7.** We compute $\text{dist}_D(f, q_f^y)$, for all $f \in F^y$, in $O(n \log n)$ time.

**Proof.** For every $i \in [k]$ let $F_i \subseteq F^y$ be the set of faces such that $x_i \in f$, for all $f \in F_i$. We observe that if $|F_i| = m$, then $\deg_D(x_i) \geq m + 1$, where $\deg_D(x_i)$ is the degree of $x_i$ in $D$.

Let $D'$ be the graph obtained by adding a new vertex $u_f$ for each face $f \in F^y$ and connecting $u_f$ to all vertices of $f$ by an edge of length $L$, where $L = \sum_{e \in E(D)} w(e)$. Thus $\text{dist}_D(y_i, f) = \text{dist}_{D'}(y_i, u_f) - L$.

We compute $d_{D'}(y_i, u_f)$, for $i \in [k]$ and $f \in F_i$, by using Klein’s result [17]. Being $|V(D')| = O(n)$, we spend $O(\log n \sum_{i \in [k]} |F_i| + n \log n) \leq O(\log n \sum_{i \in [k]} (\deg_D(x_i) - 1) + n \log n) = O(n \log n + n \log n) = O(n \log n)$ time. Finally, for all $f \in F^y$,

$$
\text{dist}_D(f, q_f^y) = \min_{\{i \in [k] \mid x_i \in f\}} \text{dist}_D(y_i, f) = \min_{\{i \in [k] \mid x_i \in f\}} \text{dist}_{D'}(y_i, u_f) - L.
$$

Thus we obtain what we need adding no more time than $\sum_{\{i \in [k] \mid x_i \in f\}} O(1) \leq O(\sum_{f \in F^y} |V(f)|) = O(n)$. 

### 6.2 Computing $d_i(q_f^y)$

We note that for computing the $d_i(q_f^y)$’s we can not directly use Lemma 3 as we have done for the $d_i(c)$’s and the $d_i(f)$’s. Indeed, it is possible that vertices in $q_f^y$ are not contained in any slice $\Omega_{i,j}$, with $i, j$ consecutive indices in $L_r$. To overcome this, we have to introduce a partial order on faces of $D$.

For all $f \in F^y$, we define $f^-$ and $f^+$ as the minimum and maximum indices in $[k]$, respectively, such that $x_{f^-, f^+} \in V(f)$. Now we introduce the concept of maximal face. Let $f \in F^y$ and let $p_f$ and $q_f$ be the two subpaths of the border cycle of $f$ from $x_{f^-}$ to $x_{f^+}$. We say that $g < f$ if $g$ is contained in the region $R$ bounded by $\pi_x[x_{f^-}, x_{f^+}] \circ p_f$, this implies that $g$ is also contained in the region $R'$ bounded by $\pi_x[x_{f^-}, x_{f^+}] \circ q_f$, thus the definition does not depend on the choice of $p_f$ and $q_f$. Finally, we say that $f$ is maximal if it does not exist any face $g \in F^y$ satisfying $f < g$, and we define $F_{max} = \{f \in F^y \mid f$ is maximal\}, see the left part of Figure 9. We find $F_{max}$ in $O(n)$ time.

Given $r \in \mathbb{N}$ and $f \in F^y$, we define $f^+_r$ as the smallest index in $L_r$ such that $f^+ > \ell^*_r$ (if $f^+ > \ell^*_r$, then we define $f^+_r = \ell^*_r$). Similarly, we define $f^-_r$ as the largest index in $L_r$ such that $f^- > \ell^*_r$ (if $f^- < \ell^*_r$, then we define $f^-_r = \ell^*_r$), see the right part of Figure 9.

Now we deal with computing $d_i(q_f^y)$, for all $f \in F^y$. By following Equation (1), we can restrict only to the easier case in which $f$ satisfies $d_i(q_f^y) \leq \text{dist}_D(f, q_f^y)$; indeed, if $f$ does not satisfy it, then we are not interested in the value of $d_i(q_f^y)$.
Lemma 8. Let \( r \in \mathbb{N} \). Given \( \text{dist}_D(f, q_f^y) \) and given \( U \), for all \( f \in F^y \) satisfying \( \min_{i \in L_r} d_i(q_f^y) < \text{dist}_D(f, q_f^y) \) we compute a value \( \beta_r(f) \in \{ \min_{i \in L_r} d_i(q_f^y), \min_{i \in L_r} d_i(q_f^y) + \delta \} \) in \( O(n) \) total time.

Proof. Let \( f \in F^y \). We observe that if \( i \in [f^-, f^+] \), then every path from \( x_i \) to \( q_f^y \) passes through either \( x_{f^-} \) or \( x_{f^+} \). Thus, for every \( i \in [f^-, f^+] \), it holds that \( d_i(q_f^y) \geq \text{dist}_D(f, q_f^y) \). Hence for any \( f \in F^y \) satisfying \( \min_{i \in L_r} d_i(q_f^y) < \text{dist}_D(f, q_f^y) \) it holds that

\[
\min_{i \in L_r} d_i(q_f^y) = \min_{i \in L_r, i \in [f^-, f^+]} \{ d_i(q_f^y) \},
\]

being \( q_f^y \subseteq \text{Right}_{f^-} \) and \( q_f^y \subseteq \text{Left}_{f^+} \), then Lemma [3] and Equation [2] imply

\[
\min_{i \in L_r} d_i(q_f^y) = \min_{i \in L_r, i \in [f^-, f^+]} \{ d_i(q_f^y) \} \geq \min \{ d_{f^-}(q_f^y), d_{f^+}(q_f^y) \} - \delta.
\]

To complete the proof, we need to show how to compute \( d_{f^-}(q_f^y) \) and \( d_{f^+}(q_f^y) \), for all \( f \in F^y \) satisfying \( \min_{i \in L_r} d_i(q_f^y) < \text{dist}_D(f, q_f^y) \) in \( O(n) \) time. In the following claim we prove it by removing the request that every face \( f \in F^y \) has to satisfy \( \min_{i \in L_r} d_i(q_f^y) < \text{dist}_D(f, q_f^y) \).

(ii) We compute \( d_{f^-}(q_f^y) \) and \( d_{f^+}(q_f^y) \), for all \( f \in F^y \), in \( O(n) \) time.

Proof of (ii): we recall that \( d_i(q_f^y) = \text{dist}_D(x_i, q_f^y) + \text{dist}_D(y_i, q_f^y) \), for all \( i \in [k] \) and \( f \in F^y \). Being \( q_f^y \subseteq V(\pi_y) \) we compute \( \text{dist}_D(y_i, q_f^y) \) in \( O(|V(q_f^y)|) \) time. Thus we have to compute only \( \text{dist}_D(x_i, q_f^y) \), for required \( i \in L_r \) and \( f \in F^y \).

For every \( f \in F^y \), let \( R_f = \Omega_{f^-}, f^+, \) and let \( \mathcal{R} = \bigcup_{f \in F_{max}} R_f \). We observe that, given two maximal faces \( f \) and \( g \), it is possible that \( R_f = R_g \). This happens if and only if \( f^- = g^- \) and \( f^+ = g^+ \) (see face \( i \) and face \( j \) in Figure [10]). We overcome this abundance by introducing \( \tilde{F} \) as a minimal set of faces such that \( \mathcal{R} = \bigcup_{f \in \tilde{F}} R_f \) and \( R_f \neq R_g \), for all distinct \( f, g \in \tilde{F} \) (see Figure [10] for an example of \( \tilde{F} \)).

For every \( f \in \tilde{F} \), it holds that \( \pi_y(f^-, f^+) \subseteq R_f \). Thus, by the above argument, if \( g \in F^y \) and \( R_g \subseteq R_f \), then \( q_y^g \subseteq R_f \). We solve 4 SSSP instances in \( R_f \) with sources \( x_j \), for all \( j \in \{ f^-, f^+, f^-, f^+ \} \) (possibly, \( f^- = f^+ \) and/or \( f^+ = f^- \)). Now we have to prove that this suffices to compute \( d_{f^-}(q_y^g) \) and \( d_{f^+}(q_y^g) \), for all \( f \in F^y \). In particular we show that, after solving the SSSP instances, we compute \( d_{g^-}(q_y^g) \) and \( d_{g^+}(q_y^g) \) in \( O(|V(g)|) \) time, for each \( g \in F^y \).

Let \( g \in F^y \), and let \( f \in \tilde{F} \) be such that \( g \subseteq R_f \). There are two cases: either \( g^- = f^- \) and \( g^+ = f^+ \), or \( g^- \neq f^- \) and/or \( g^+ \neq f^+ \).
We compute all these distances by the solutions of previous SSSP instances in $O(|V(g)|)$ time, because $q_y^g \subseteq R_f$ and $|V(q_y^g)| < |V(g)|$. Otherwise, w.l.o.g., we assume that $g_r^- \neq f_r^-$ (if $g_r^+ \neq f_r^+$, then the proof is similar). By definitions of $\bar{F}$, $\Omega_g$, and $\Omega_f$, it holds that $g < f$. Thus $g_r^- \in [f^-, f^+]$, therefore every path from $x_{g_r^-}$ to $q_y^g$ passes through either $f^-$ or $f^+$ (see $g_3$ and $f_5$ in Figure 10). By this discussion, it follows that

$$
\text{dist}_D(x_{g_r^-}, q_y^g) = \min \left\{ \text{dist}_D(x_{g_r^-}, x_{f^-}) + \text{dist}_D(x_{f^-}, q_y^g), \text{dist}_D(x_{g_r^-}, x_{f^+}) + \text{dist}_D(x_{f^+}, q_y^g) \right\} 
$$

We compute all these distances by the solutions of previous SSSP instances in $O(|V(q_y^g)|)$ time, and thus we compute $\text{dist}_D(x_{g_r^-}, q_y^g)$ in $O(|V(q_y^g)|)$ time. By symmetry, the same cost is required to compute $\text{dist}_D(x_{g_r^+}, q_y^g)$.

We have proved that, after solving the described SSSP instances, we compute $d_{f^-}(q_y^g)$ and $d_{f^+}(q_y^g)$, for all $f \in F_y$, in $O(|V(f)|)$ time for each $f \in F_y$. Being $\sum_{f \in F_y} |V(f)| = O(n)$, it remains to show that we can solve all the previous SSSP instances in $O(n)$ time. We want to use Lemma 4 (we recall that, for our purposes, distances in $\Omega_{f^-, f^+}$ are the same in $\Sigma_{f^-, f^+}$).

Let us fix $i \in [h]$ and let $a = f_i$, $b = f_{i+1}$, $c = f_{i+2}$ and $d = f_{i+3}$. We can not use directly Lemma 4 because it is possible that $a_r^- < b_r^+$ (see in Figure 10 $a = f_3$ and $b = f_4$, thus $b_r^- = \ell_4 < \ell_5 = a_r^+$) and thus we might have not an increasing set of indices. But, by definition of $\bar{F}$, it holds that $a_r^- \leq d_r^-$, indeed $a_r^+ \in [b_r^-, c_r^+]$ otherwise $R_b = R_c$; these relations do not depend on $i$. Similarly, $d_r^- \geq a_r^+$. Thus we solve first the SSSP instances in $R_{f_i}$, for all $i \in [h]$ such that $i \equiv 0$ mod 3; then for $i \equiv 1$ mod 3 and finally for $i \equiv 2$ mod 3. By Lemma 4 it costs $O(n)$ time. End proof of (i)

Finally, by Equation 3, we set $\beta_r(f) = \min\{d_{f^-}(q_y^g), d_{f^+}(q_y^g)\}$, for all $f \in F_y$ satisfying $\beta_r(f) < \text{dist}_D(f, q_y^g)$ and we ignore faces in $F_y$ that do not satisfy it.

![Fig 10: assume that $L_r = (\ell_1, \ldots, \ell_7)$. A possible $\bar{F}$ is $\bar{F} = \{f_1, \ldots, f_6\}$. Moreover, $g_1, g_3, g_4$ are not in $F_{\max}, g_2 \in F_{\max}$ and $R_{g_2} = R_{f_4}$ thus $g_2 \not\in \bar{F}$.](image)

6.3 Computational complexity of vertex vitality

Now we give our theorems about vertex vitality. To prove Theorem 3 we follow the same approach used in Theorem 2 by referring to Proposition 3 in place of Proposition 2.

We recall that the result stated in Theorem 3 is more efficient than the result in Corollary 1 if either $|S| < \log n$ and $E_S > |S| \log n$ or $|S| \geq \log n$ and $E_S > \frac{|S|^{1/3}}{\log^{1/3}}$, where $E_S = \sum_{v \in S} \deg(v)$.
Theorem 3. Let $G$ be a planar graph with positive edge capacities. Then for any $c, \delta \geq 0$, we compute a value $\text{vit}^\delta(v) \in (\text{vit}(v) - \delta, \text{vit}(v)]$ for all $v \in V(G)$ satisfying $c(v) \leq c$, in $O\left(\frac{n}{\delta} \log n + n \log \log n\right)$ time.

Proof. We compute $D$ and $U$ in $O(n \log \log n)$ time by Theorem 5. If $c(v) < c$, then $w(f^D_v) < c$. For convenience, we denote $f^D_v$ by $f$. By Lemma 7, we compute $\text{dist}_D(f, q^y_f)$ (resp., $\text{dist}_D(f, q^x_f)$) in $O(n \log n)$ time, for all $f \in F^y$ (resp., for all $f \in F^x$). Now we have to show how to obtain $\min_{i \in [k]} \{d_i(f)\}$, $\min_{i \in [k]} \{d_i(q^y_f)\}$ and $\min_{i \in [k]} \{d_i(q^x_f)\}$ that we may compute with an error depending on $\delta$.

We note that $\min_{i \in [k]} \{d_i(f)\} = \min_{i \in [k]} \{d_i < MF + w(f)\} d_i(f)$. Indeed, if $d_i(f) = MF - z$, for some $z > 0$, then $d_i$ is at most $MF - z + w(f)$. For the same reason, $\min_{i \in [k]} \{d_i(q^x_f)\}$ and, similarly, $\min_{i \in [k]} \{d_i(q^y_f)\}$.

By using Lemma 5 for each $r \in \{0, 1, \ldots, \left[\frac{r}{\delta}\right]\}$, we compute a value $\alpha_r(f) \in \left[\min_{i \in L_r} d_i(f), \min_{i \in L_r} d_i(f) + \delta\right]$, for all $f \in F$ in $O(n)$ time. Then, for each $f \in F$, we compute $\alpha(f) = \min_{r \in \{0, 1, \ldots, \left[\frac{r}{\delta}\right]\}} \alpha_r(f)$. By above, for any $f \in F$ satisfying $w(f) < c$, it holds that $\alpha(f)$ satisfies $\alpha(f) \in \left[\min_{i \in [k]} \{d_i(f)\}, \min_{i \in [k]} \{d_i(f)\} + \delta\right]$.

With a similar strategy, by replacing Lemma 5 with Lemma 8, for each $f \in F^y$ satisfying $w(f) < c$ and $\min_{i \in L_r} d_i(q^y_f) < \text{dist}_D(f, q^y_f)$, we compute a value $\beta(f) \in \left[\min_{i \in [k]} \{d_i(q^y_f)\}, \min_{i \in [k]} \{d_i(q^y_f)\} + \delta\right]$. The same results hold for the "x case": for each $f \in F^x$ satisfying $w(f) < c$ and $\min_{i \in L_r} d_i(q^x_f) < \text{dist}_D(f, q^x_f)$, we compute a value $\gamma(f) \in \left[\min_{i \in [k]} \{d_i(q^x_f)\}, \min_{i \in [k]} \{d_i(q^x_f)\} + \delta\right]$.

Then, by Proposition 3 for each $v \in V(G)$ satisfying $c(v) \leq c$ $(w(f) < c)$ we compute a value $\text{vit}^{\delta}(v)$ satisfying $\text{vit}^{\delta}(v) \in (\text{vit}(v) - \delta, \text{vit}(v)]$ in $O(1)$ time by using $\text{dist}_D(f, q^y_f)$, $\text{dist}_D(f, q^x_f)$, $\alpha(f)$, $\beta(f)$ and $\gamma(f)$.

\[ \Box \]

Theorem 4. Let $G$ be a planar graph with positive edge capacities. Then for any $S \subseteq V(G)$, we compute $\text{vit}(v)$ for all $v \in S$ in $O(|S|n + n \log \log n)$ time.

Proof. We compute $D$ and $U$ in $O(n \log \log n)$ time by Theorem 5. For convenience, we denote $f^D_v$ by $f$. To compute $\min_{i \in [k]} \{d_i(f)\}$, $\text{dist}_D(f, q^y_f)$ and $\text{dist}_D(f, q^x_f)$ we put a vertex $u_f$ in the face $f$ and we connect it to all vertices of $f$ with zero weighted edges. Then we solve an SSSP instance with source $u_f$ and we compute $\min_{i \in [k]} \{d_i(f)\}$, $\text{dist}_D(f, q^y_f)$ and $\text{dist}_D(f, q^x_f)$ in $O(n)$. With a similar strategy, we compute $\min_{i \in [k]} \{d_i(q^y_f)\}$ and $\min_{i \in [k]} \{d_i(q^x_f)\}$ in $O(n)$ time. Finally, by Proposition 3 for each $v \in S$, we compute $\text{vit}(v)$ in $O(1)$ time.

\[ \Box \]

7 Conclusions and open problems

We proposed algorithms for computing guaranteed approximation of the vitality of all edges or vertices with bounded capacity with respect to the max flow from $s$ to $t$ in undirected planar graphs.

It is still open the problem of computing the exact capacity of all edges of an undirected planar graph within the same time bound as computing the max flow value, as is already known for the $st$-planar case.

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