Abstract

It has been known for some time that the SL(2,R) WZWN model reduces to Liouville theory. Here we give a direct and physical derivation of this result based on the classical string equations of motion and the proper string size. This allows us to extract precisely the physical effects of the metric and antisymmetric tensor, respectively, on the exact string dynamics in the SL(2,R) background. The general solution to the proper string size is also found. We show that the antisymmetric tensor (corresponding to conformal invariance) generally gives rise to repulsion, and it precisely cancels the dominant attractive term arising from the metric.

Both the sinh-Gordon and the cosh-Gordon sectors of the string dynamics in non-conformally invariant AdS spacetime reduce here to the Liouville equation (with different signs of the potential), while the original Liouville sector reduces to the free wave equation. Only the very large classical string size is affected by the torsion. Medium and small size string behaviours are unchanged.

We also find illustrative classes of string solutions in the SL(2,R) background: dynamical closed as well as stationary open spiralling
strings, for which the effect of torsion is somewhat like the effect of rotation in the metric. Similarly, the string solutions in the 2+1 BH-AdS background with torsion and angular momentum are fully analyzed.
1 Introduction

In this paper, we consider the exact classical string dynamics in the conformally invariant background corresponding to the SL(2,R) WZWN model. This background is locally 2 + 1-dimensional Anti de Sitter spacetime with non-vanishing parallelizing torsion.

Many mathematical aspects of the SL(2,R) WZWN model have been discussed in the literature (see for instance Refs. [1, 2, 3, 4]). In particular, it has been known for some time that the SL(2,R) WZWN model reduces to Liouville theory (for a review of the different methods, see [5] and references given therein). However, the physical aspects have still not really been extracted so far. The purpose of this paper is to investigate directly the physical effects of the conformal invariance on the generic exact classical string dynamics.

The conformal invariance of the SL(2,R) WZWN model is expressed as the torsion becoming parallelizing. Thus we consider the string equations of motion in a background consisting of the Anti de Sitter (AdS) metric plus an antisymmetric tensor representing the parallelizing torsion. Using the reduction method of [6, 7], we obtain directly from the classical equations of motion a simple differential equation, the Liouville equation, for the fundamental quadratic form $\alpha(\tau, \sigma)$, which determines the proper string size. By comparing with analogues results obtained in AdS but without torsion [3, 7, 8, 9], we can then precisely extract the physical effects of the conformal invariance on the exact dynamics of classical strings. We also compare with the results of [10] where, among other things, the effect of the conformal invariance was analysed, but for particular string configurations and for perturbative string solutions.

One essential point in this paper is the parametrization of the string equations of motion and constraints in terms of the proper string size. Then, associated potentials $V(\alpha)$ can be defined, and generic properties of the exact string dynamics can be extracted directly from the reduced equations of motion and potentials (without need of any solution). Previously [7], we have shown that the exact string dynamics in the non-conformally invariant AdS spacetime (without torsion), reduces to three different equations: sinh-Gordon, cosh-Gordon and Liouville equation, and all three must be considered in order to cover the generic string evolution.

In this paper we show that this reduction procedure beautifully generalizes and simplifies in the presence of torsion, corresponding to conformal invariance. In the conformally invariant AdS background, the presence of
torsion leads to a precise cancellation of the term $+\exp(\alpha)$ in the potentials $\cosh(\alpha)$, $\sinh(\alpha)$ and $\exp(\alpha)$ of the reduced equations. Thus, when including the torsion, the original sinh-Gordon and cosh-Gordon sectors reduce to the Liouville equation (with different signs of the potential), while the original Liouville sector reduces to the free wave equation (see Figs. 1, 2). Torsion generally produces a repulsive term $-\exp(\alpha)$, which precisely cancels the dominant attractive term arising from gravity. As a consequence, only the very large classical string size behaviour is affected by the torsion. Most of the string behaviour (medium and small string size behaviour) is unchanged.

We also find in this paper illustrative classes of string solutions in the conformally invariant AdS background. The ansatz we have introduced in Ref. [1] is applied here to this case. Dynamical closed strings as well as stationary open infinitely long strings are described. Here in the presence of torsion, the mathematics simplifies considerably; the solutions are expressed in closed form in terms of trigonometric and hyperbolic functions (in the non-conformally invariant case, the solutions generally involved elliptic functions [7]). Similarly, we find the string solutions in the 2+1 dimensional black hole anti de Sitter spacetime (BH-AdS) with torsion, and compare with the case of vanishing torsion.

It must be noticed that, in the absence of torsion, stationary strings in AdS spacetime are of “hanging string” type (that is, their shapes are simple generalizations of the shape of a rope hanging in a constant Newtonian potential). In the presence of torsion, these configurations become of ”spiralling string” type, and asymptotically, they are standard logarithmic spirals (see Figs. 3, 4). The effect of torsion on stationary strings in the AdS background thus appears quite similar to the effect of rotation in the Kerr-Newman spacetime [11].

This paper is organized as follows: In Section 2 we perform the reduction of the WZWN model to the Liouville equation, in terms of the string equations of motion and constraints and the proper string size, and we analyse the generic features of the exact string dynamics in the AdS background with torsion. In Section 3 we deal with particularly illustrative examples of string configurations and precise effects of the torsion and conformal invariance in this background, using a parametrization corresponding to global AdS spacetime. In Section 4, we discuss the analogous results obtained using a parametrization corresponding to the 2+1 dimensional BH-AdS spacetime with torsion. Conclusions and remarks are given in Section 5.
2 Reduction of the WZWN Model to the Liouville Equation

Our starting point is the sigma-model action including the WZWN term at level \( k \): \( S_{\sigma} = -\frac{k}{4\pi} \int_M d\tau d\sigma \, \eta^{\alpha\beta} \text{Tr}[g^{-1} \partial_\alpha g \, g^{-1} \partial_\beta g] - \frac{k}{6\pi} \int_B \text{Tr}[g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg]. \) (2.1)

Here \( M \) is the boundary of the manifold \( B \), and \( g \) is a group-element of \( SL(2, \mathbb{R}) \): \( g = \begin{pmatrix} a & c \\ -d & b \end{pmatrix}; \quad ab + cd = 1. \) (2.2)

Then, the action (2.1) takes the form \( S_{\sigma} = -\frac{k}{2\pi} \int_M d\tau d\sigma \, [\dot{a}b - a\dot{b}' + \dot{c}d - c\dot{d}'] - \frac{k}{\pi} \int_M d\tau d\sigma \, \log(c)[\dot{a}b' - a\dot{b}], \) (2.3)

where dot and prime denote derivative with respect to \( \tau \) and \( \sigma \), respectively. Let us introduce new coordinates \( (X, Y, U, T) \):

\[
\begin{align*}
  a &= H(U + X), \quad b = H(U - X), \quad c = H(T - Y), \quad d = H(T + Y),
  \end{align*}
\] (2.4)

where \( H \) is a constant (the Hubble constant). Then we get from (2.2):

\[
X^2 + Y^2 - U^2 - T^2 = -\frac{1}{H^2},
\] (2.5)

which is the standard embedding equation for the 2+1 AdS spacetime. Using a Lagrange multiplier \( \lambda \) to incorporate the condition (2.5), the action becomes:

\[
S_{\sigma} = -\frac{kH^2}{2\pi} \int_M d\tau d\sigma \, [\dot{U}^2 - U'^2 + \dot{T}^2 - T'^2 - \dot{X}^2 + X'^2 - \dot{Y}^2 + Y'^2 + 4\lambda(-T^2 - U^2 + X^2 + Y^2 + H^{-2}) + 4(\dot{X}U' - X'\dot{U}) \log(H(T - Y))].
\] (2.6)

It is also convenient to introduce the dimensionless 4-vector \( q^\mu \) and metric \( \eta_{\mu\nu} \) in the 4-dimensional embedding spacetime:

\[
q^\mu = H(T, U, X, Y), \quad \eta_{\mu\nu} = \text{diag}(-1, -1, 1, 1),
\] (2.7)
as well as world-sheet light-cone coordinates:

\[ \sigma^\pm = \tau \pm \sigma. \tag{2.8} \]

It is then straightforward to show that the classical equations of motion corresponding to the action (2.6) reduce to:

\[ q^\mu_+ + e^\alpha q^\mu_+ + \epsilon^\mu_{\nu \rho \sigma} q^\nu_+ q^\rho_- q^\sigma_- = 0, \quad (\mu = 0, 1, 2, 3), \tag{2.9} \]

where we introduced the fundamental quadratic form \( \alpha(\tau, \sigma) \):

\[ e^\alpha = -\eta_{\mu \nu} q^\mu_+ q^\nu_-, \tag{2.10} \]

and the antisymmetric tensor \( \epsilon^\mu_{\nu \rho \sigma} \) corresponding to the metric (2.7):

\[ \epsilon^{0123} = 1, \quad (\text{antisymmetric}). \tag{2.11} \]

The equations of motion (2.9) should, as usual, be supplemented by the string constraints:

\[ \eta_{\mu \nu} q^\mu_+ q^\nu_+ = 0, \tag{2.12} \]

and the embedding normalization condition (2.5):

\[ \eta_{\mu \nu} q^\mu q^\nu = -1. \tag{2.13} \]

Notice that \( \alpha \) determines the proper string size, as follows from the induced metric on the world-sheet:

\[ dS^2 = \frac{1}{H^2} \eta_{\mu \nu} dq^\mu dq^\nu = \frac{2}{H^2} e^\alpha (-d\tau^2 + d\sigma^2). \tag{2.14} \]

It is convenient to introduce the basis:

\[ \mathcal{U} = \{ q^\mu, q^\mu_+, q^\mu_-, l^\mu \}; \quad l^\mu \equiv e^{-\alpha} \epsilon^\mu_{\rho \sigma \delta} q^\rho_+ q^\sigma_- q^\delta, \tag{2.15} \]

\[ \eta_{\mu \nu} l^\mu l^\nu = 1. \tag{2.16} \]

The second derivatives of \( q^\mu \), expressed in the basis \( \mathcal{U} \), are given by:

\[ q^\mu_{++} = \alpha_+ q^\mu_+ + ul^\mu, \quad q^\mu_{--} = \alpha_- q^\mu_- + vl^\mu, \quad q^\mu_{+-} = -e^\alpha (q^\mu_+ + l^\mu), \tag{2.17} \]

where the functions \( u \) and \( v \) are implicitly defined by:

\[ u \equiv \eta_{\mu \nu} q^\mu_+ l^\nu, \quad v \equiv \eta_{\mu \nu} q^\mu_- l^\nu; \tag{2.18} \]
and satisfy:

\[ u_- = v_+ = 0 \quad \implies \quad u = u(\sigma^+), \; v = v(\sigma^-). \tag{2.19} \]

Then, by differentiating equation (2.10) twice, we get:

\[ \alpha_{+-} + u(\sigma^+)v(\sigma^-)e^{-\alpha} = 0. \tag{2.20} \]

If the product \( u(\sigma^+)v(\sigma^-) \) is positive definite, then the following conformal transformation on the world-sheet metric (2.14):

\[
\begin{align*}
\alpha(\sigma_+, \sigma_-) &= \hat{\alpha}(\hat{\sigma}_+, \hat{\sigma}_-) + \frac{1}{2}\log|u(\sigma^+)|v(\sigma^-)|, \\
\hat{\sigma}_+ &= \int \sqrt{|u(\sigma^+)|} \, d\sigma^+, \quad \hat{\sigma}_- = \int \sqrt{|v(\sigma^-)|} \, d\sigma^-, 
\end{align*}
\tag{2.21}
\]

reduces equation (2.20) to:

\[ \alpha_{+-} + e^{-\alpha} = 0, \tag{2.22} \]

which is just the Liouville equation (we skipped the hats).

It must be noticed, however, that for a generic string world-sheet, the product \( u(\sigma^+)v(\sigma^-) \) is neither positive nor negative definite. In the case that \( u(\sigma^+)v(\sigma^-) \) is negative, the conformal transformation (2.21) reduces equation (2.20) to:

\[ \alpha_{+-} - e^{-\alpha} = 0, \tag{2.23} \]

and including also the case when \( u(\sigma^+)v(\sigma^-) = 0 \), we conclude that the most general equation fulfilled by the fundamental quadratic form \( \alpha \) is:

\[ \alpha_{+-} + Ke^{-\alpha} = 0, \tag{2.24} \]

where:

\[
K = \begin{cases} 
+1, & u(\sigma^+)v(\sigma^-) > 0 \\
-1, & u(\sigma^+)v(\sigma^-) < 0 \\
0, & u(\sigma^+)v(\sigma^-) = 0 
\end{cases} \tag{2.25}
\]

Equation (2.24) is either the Liouville equation \((K = \pm 1)\), or the free wave equation \((K = 0)\).

Let us define a potential \( V(\alpha) \) by:

\[ \alpha_{+-} + \frac{dV(\alpha)}{d\alpha} = 0, \tag{2.26} \]
so that if $\alpha = \alpha(\tau)$, then $\frac{1}{2}(\dot{\alpha})^2 + V(\alpha) = \text{const}$. Then, it follows that:

$$V(\alpha) = \begin{cases} 
-e^{-\alpha}, & K = +1 \\
e^{-\alpha}, & K = -1 \\
0, & K = 0 
\end{cases}$$

(2.27)

The results (2.26)-(2.27) are represented in Fig.1., showing the different potentials.

It is interesting to compare these results with the analogue results obtained in AdS but without torsion [3, 5, 4, 7]. In that case, instead of eq.(2.27), we have found [7]:

$$\tilde{V}(\alpha) = \begin{cases} 
2\sinh \alpha, & K = +1 \\
2\cosh \alpha, & K = -1 \\
e^\alpha, & K = 0 
\end{cases}$$

(2.28)

which is shown in Fig.2. As discussed in more detail in [4], it means that for large proper string sizes (large $\alpha$), the potential $\tilde{V}(\alpha)$ is always attractive. The positive increasing potential for positive $\alpha$ in AdS spacetime prevents the string from growing indefinitely. That is, gravity as represented by the metric will (not surprisingly in AdS) generally tend to contract a large string.

By comparing equations (2.27) and (2.28), we see that the effect of conformal invariance is to precisely cancel the term $e^\alpha$ in the potential. This holds for all three cases ($K = 0, \pm 1$). That is, when including the parallelizing torsion, the original sinh-Gordon and cosh-Gordon equations reduce to the Liouville equation (with different signs of the potential), while the original Liouville equation reduces to the free wave equation.

Thus the physical effect of conformal invariance (represented via the parallelizing torsion) is to precisely cancel the dominant attractive part of the potential arising from the metric. In other words, the parallelizing torsion generally gives rise to a repulsive term $-e^\alpha$ in the potential. The combined effect of gravity and torsion eventually gives rise to either attraction or repulsion, but for large proper string size $\alpha$, the potential $V(\alpha)$ vanishes exponentially in all cases, Fig.1.

On the other hand, for small proper string size $\alpha$, the potential is not affected by the parallelizing torsion.

These results complete and generalize results obtained in [10] for particular string configurations (circular strings), and in [7] for the non-conformally
invariant case.

Finally, it should be noticed that the general solution of the Liouville equation (2.24) is known in closed form (say, $K = 1$):

$$\alpha(\sigma^+, \sigma^-) = \log \left\{ \frac{(f(\sigma^+) + g(\sigma^-))^2}{2f'(\sigma^+)g'(\sigma^-)} \right\}$$  \hspace{1cm} (2.29)

where $f(\sigma^+)$ and $g(\sigma^-)$ are arbitrary functions of the indicated variables. The proper string size, $S(\tau)$, is then:

$$S(\tau) = \int d\sigma \ s(\tau, \sigma),$$  \hspace{1cm} (2.30)

where, using equations (2.14) and (2.29),

$$s(\tau, \sigma) = \frac{\sqrt{2}}{H} e^{\alpha/2} = \frac{f(\sigma^+) + g(\sigma^-)}{H \sqrt{f'(\sigma^+)g'(\sigma^-)}}.$$  \hspace{1cm} (2.31)

This is the general solution ($K = 1$) to the string size in the conformally invariant AdS background. The full string dynamics in this background is exactly integrable. However, it is still a highly non-trivial problem to obtain the explicit expression for the coordinates $q^\mu$, taking into account the constraints (2.12) and the normalization condition (2.13).

### 3 Examples

In this section we consider in detail some illustrative examples of string configurations. It is convenient to first introduce the standard parametrization (see for instance [14]) of 2 + 1 AdS in terms of static coordinates $(t, r, \phi)$:

$$X = r \cos \phi, \quad U = \frac{1}{H} \sqrt{1 + H^2 r^2} \cos(\phi t),$$

$$Y = r \sin \phi, \quad T = \frac{1}{H} \sqrt{1 + H^2 r^2} \sin(\phi t),$$  \hspace{1cm} (3.1)

which automatically fulfils the normalization condition (2.13). Next we make the following ansatz [7]:

$$r = r(\xi^1),$$

$$t = t(\xi^1) + c_1 \xi^2,$$

$$\phi = \phi(\xi^1) + c_2 \xi^2,$$  \hspace{1cm} (3.2)
where \((c_1, c_2)\) are arbitrary constants while \((\xi^1, \xi^2)\) are the two world-sheet coordinates, to be specified later.

The mathematical motivation for this ansatz is that it reduces the string equations of motion (2.9) to ordinary differential equations, as we now show (see also Ref. [7]). In fact, the equations (2.9) reduce to:

\[
\frac{d^2 t}{(d \xi^1)^2} + \frac{2H^2 r}{1 + H^2 r^2} \left( \frac{dt}{d \xi^1} \right)^2 + \frac{2H r}{1 + H^2 r^2} \left( \frac{dr}{d \xi^1} \right) c_2 = 0, \tag{3.3}
\]

\[
\frac{d^2 \phi}{(d \xi^1)^2} + \frac{2}{r} \left( \frac{d \phi}{d \xi^1} \right) \left( \frac{dr}{d \xi^1} \right) + \frac{2H}{r} \left( \frac{dr}{d \xi^1} \right) c_1 = 0, \tag{3.4}
\]

\[
\frac{d^2 r}{(d \xi^1)^2} + H^2 r (1 + H^2 r^2) \left( \left( \frac{dt}{d \xi^1} \right)^2 - c_1^2 \right) - r (1 + H^2 r^2) \left( \left( \frac{d \phi}{d \xi^1} \right)^2 - c_2^2 \right)
- \frac{H^2 r}{1 + H^2 r^2} \left( \frac{dr}{d \xi^1} \right)^2 + 2H r (1 + H^2 r^2) \left( c_2 \left( \frac{dt}{d \xi^1} \right) - c_1 \left( \frac{d \phi}{d \xi^1} \right) \right) = 0, \tag{3.5}
\]

while the constraints (2.12) become:

\[
(1 + H^2 r^2) \left( \frac{dt}{d \xi^1} \right) c_1 = r^2 \left( \frac{d \phi}{d \xi^1} \right) c_2, \tag{3.6}
\]

\[
\frac{1}{1 + H^2 r^2} \left( \frac{dr}{d \xi^1} \right)^2 - (1 + H^2 r^2) \left( \left( \frac{dt}{d \xi^1} \right)^2 + c_1^2 \right) + r^2 \left( \left( \frac{d \phi}{d \xi^1} \right)^2 + c_2^2 \right) = 0. \tag{3.7}
\]

The above equations of motion and constraints are consistently integrated to:

\[
\frac{dt}{d \xi^1} = \frac{k_1 - H c_2 r^2}{1 + H^2 r^2}, \quad \frac{d \phi}{d \xi^1} = \frac{k_2 - H c_1 r^2}{r^2}, \tag{3.8}
\]

\[
r^2 = \frac{(H^2 k_2^2 - k_1^2)(c_2^2 + 2H c_1 k_2)}{r^2 k_2^2} \left( r^2 - \frac{k_2^2}{c_1^2 + 2H c_1 k_2} \right) \left( r^2 + \frac{k_2^2}{H^2 k_2^2 - k_1^2} \right), \tag{3.9}
\]

where the integration constants \((k_1, k_2)\) fulfil:

\[
c_1 k_1 = c_2 k_2. \tag{3.10}
\]

The equations (3.8)-(3.9) can be solved explicitly in closed form in terms of trigonometric or hyperbolic functions. This is a great simplification compared
to the case without torsion. In that case, the solution generally involved elliptic functions \[7\].

As for the fundamental quadratic form \(\alpha\), we get:

\[
e^{\alpha} = \pm \frac{H^2}{2} \left[ r^2 c_2^2 - (1 + H^2 r^2) c_1^2 \right], \tag{3.11}
\]

where the sign must be chosen in accordance with eq.(2.14). Then we get from equations (3.3)-(3.10):

\[
\frac{d^2 \alpha}{(d\xi^1)^2} \pm \left[ H^2 \left( c_2^2 - H^2 c_1^2 \right) \left( k_1^2 - (c_1 + H k_2)^2 \right) \right] e^{-\alpha} = 0. \tag{3.12}
\]

This corresponds to equation (2.24) after a constant redefinition of \(\xi^1\). The different values of \(K\) will appear depending on the sign of the square bracket in (3.12).

In the following subsections, we consider some more explicit examples to clarify the physics of the ansatz (3.2).

## 3.1 Circular Strings

Circular strings are obtained from the above general formalism by setting \((\xi^1, \xi^2) = (\tau, \sigma)\), as well as:

\[
c_1 = 0, \quad k_2 = 0, \quad c_2 = 1, \quad k_1 \equiv E. \tag{3.13}
\]

Then equations (3.8)-(3.9) become:

\[
\begin{align*}
\phi &= \sigma, \\
\dot{t} &= \frac{E - H r^2}{1 + H^2 r^2}, \\
\dot{r}^2 + (1 + 2EH) r^2 &= E^2.
\end{align*} \tag{3.14}
\]

Here we must take \(E \geq 0\) to ensure that \(\dot{t} \geq 0\) (the string is propagating forward in time). Then (3.14) describes a circular string oscillating between \(r = 0\) and \(r = r_{\text{max}}\):}

\[
r_{\text{max}} = \frac{E}{\sqrt{1 + 2EH}}. \tag{3.15}
\]
The explicit solution of (3.14), in closed form, is:

\[ \phi = \sigma, \]
\[ Ht = \arctan \left( \frac{1 + EH}{\sqrt{1 + 2EH}} \tan(\sqrt{1 + 2EH} \tau) \right) - \tau, \]
\[ r = \frac{E}{\sqrt{1 + 2EH}} \left| \sin(\sqrt{1 + 2EH} \tau) \right|, \]  \tag{3.16}

where we took initial conditions \((r(0) = 0, t(0) = 0)\). Here,

\[ e^\alpha = \frac{H^2 r^2}{2} \]  \tag{3.17}

and the string size, equations (2.30)-(2.31), is:

\[ S(\tau) = 2\pi r(\tau). \]  \tag{3.18}

These circular strings have been discussed in more detail in Ref.[10].

### 3.2 Stationary Strings

Stationary strings are obtained from the general formalism by setting \((\xi^1, \xi^2) = (\sigma, \tau)\), as well as:

\[ c_2 = 0, \quad k_1 = 0, \quad c_1 = 1, \quad k_2 \equiv L. \]  \tag{3.19}

Then equations (3.8)-(3.9) become:

\[ t = \tau, \]
\[ \phi' = \frac{L - Hr^2}{r^2}, \]  \tag{3.20}
\[ r'^2 = \frac{H^2(1 + 2HL)}{r^2} \left( r^2 + \frac{1}{H^2} \right) \left( r^2 - \frac{L^2}{1 + 2HL} \right). \]

It follows that we must have \(1 + 2HL > 0\) to ensure that a region exists where \(r'^2 \geq 0\). There is also a ”turning point” \((r' = 0)\) at \(r = r_{\text{min}}:\)

\[ r_{\text{min}} = \frac{|L|}{\sqrt{1 + 2HL}}, \]  \tag{3.21}
thus the stationary string stretches out from \( r = r_{\text{min}} \) to \( r = \infty \). The explicit solution of (3.18), in closed form, is:

\[
\begin{align*}
t &= \tau, \\
\phi &= \arctan\left(\frac{\sqrt{1+2HL}}{HL} \tanh(H\sqrt{1+2HL} \sigma)\right) - H\sigma, \\
r &= \sqrt{\frac{(1+HL)^2 \sinh^2(H\sqrt{1+2HL} \sigma) + H^2L^2}{H^2(1+2HL)}},
\end{align*}
\]

(3.22)

where we took initial conditions \((r(0) = r_{\text{min}}, \phi(0) = 0)\). Here,

\[
e^\alpha = \frac{1}{2}(1 + H^2r^2),
\]

(3.23)

where a factor \( H \) has been absorbed in \( \tau \) and \( \sigma \). Then the string size, equations (2.30)-(2.31), is:

\[
S(\tau) = S = \int_{-\infty}^{+\infty} \sqrt{1+H^2r^2} \, d\sigma = \infty.
\]

(3.24)

Notice that:

\[
\phi(-\infty) = \infty, \quad \phi(\infty) = -\infty,
\]

(3.25)

so that the stationary strings are open infinitely long clockwise spirals. An example is shown in Fig.3. Asymptotically \((|\sigma| \to \infty)\), the stationary strings are standard logarithmic spirals:

\[
\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} r \cos \phi \\ r \sin \phi \end{pmatrix} \sim \begin{pmatrix} \cos(H\sigma) \\ \sin(H\sigma) \end{pmatrix} e^{\pm H\sqrt{1+2HL} \sigma},
\]

(3.26)

up to a constant scaling and a rotation. The simplest explicit example is obtained for \( L = 0 \):

\[
\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos(H\sigma) \\ -\sin(H\sigma) \end{pmatrix} \sinh(H\sigma),
\]

(3.27)

which is shown in Fig.4.

Again it is interesting to compare with the case of stationary strings in AdS spacetime, but without torsion [15]. In that case, the stationary strings were of the "hanging string" type, that is, the shape of the stationary strings was a simple generalization of the shape of a rope hanging in a constant
Newtonian potential. Here, in the presence of torsion, the stationary strings are instead of the "spiralling string" type. It follows that the effect of torsion is somewhat similar to the effect of rotation in the metric: The effect of rotation in the metric on the shape of stationary strings was first investigated in Ref.\cite{11}. It was shown that stationary strings in the Schwarzschild background are of the "hanging string" type, while in the Kerr background, stationary strings could also be of the "spiralling string" type. Thus, we have seen that the effect of torsion in AdS spacetime is quite similar.

Another effect of the torsion on stationary strings in AdS spacetime concerns the multi-string property: In AdS spacetime without torsion \cite{15}, it was shown that the solution corresponding to the ansatz (3.2), (3.17) actually describes a multi-string, that is, one single world-sheet, determined by one set of initial conditions, describes a finite or even an infinite number of different and independent stationary strings. Here, in the presence of torsion, the multi-string property is lost for stationary strings: for $\sigma \in [-\infty, \infty[$ the solution (3.20) describes only one stationary string.

4 Strings in the BH-AdS background with Torsion

In Section 3, we have been concerned with global 2+1 AdS spacetime. However, the general results obtained in Section 2 hold for any parametrization of the SL(2,R) WZWN model. That is to say, everything in Section 2 is valid also for strings in the 2+1 black hole anti de Sitter spacetime (BH-AdS) \cite{16}. The 2+1 BH-AdS spacetime is obtained by replacing the SL(2,R) parametrization (3.1) by \cite{17}:

\[
\begin{align*}
    a &= \sqrt{\frac{r^2 - r_-^2}{r^2_+ - r_-^2}} e^{H(r_+ \phi - H r_- t)} , \\
    b &= \sqrt{\frac{r^2 - r_-^2}{r^2_+ - r_-^2}} e^{-H(r_+ \phi - H r_- t)} , \\
    c &= \sqrt{\frac{r^2 - r_-^2}{r^2_+ - r_-^2}} e^{H(r_+ t - r_- \phi)} ,
\end{align*}
\]
\[ d = -\sqrt{\frac{r^2 - r_+^2}{r_+^2 - r_-^2}} e^{-H(r_+ \phi - r_- \phi)}, \] (4.1)

where we used the notation of eq.(2.2). In these expressions \( r_{\pm} \) are the outer and inner horizons:

\[ r_{\pm}^2 = \frac{M}{2H^2} \left( 1 \pm \sqrt{1 - H^2 J^2 / M^2} \right), \] (4.2)

where \((M, J)\) represent the mass and angular momentum of the black hole, respectively. Finally we used the notation \( H^{-1} = l \) (the length scale) for comparison with sections 2, 3. Notice also that eq.(4.1) is only valid for \( r > r_+ \), but analogous expressions hold in the other regions. For more details about the BH-AdS spacetime, we refer the readers to the original papers [16, 17].

It is now straightforward to perform the analysis of circular and stationary strings in the background of 2+1 BH-AdS with torsion, c.f. the analysis of Section 3, so here we just give the main results.

### 4.1 Circular Strings

The equations of motion and constraints for circular strings are solved by:

\[ t = \int^r \frac{E - Hr^2}{H^2 r^2 - M + J^2 / 4r^2} \, d\tau, \] (4.3)

\[ \phi = \sigma + \int^r \frac{J(E - Hr^2)}{2r^2(H^2 r^2 - M + J^2 / 4r^2)} \, d\tau, \] (4.4)

\[ \dot{r}^2 + V(r) = 0 \iff \tau = \pm \int^r \frac{dr}{\sqrt{-V(r)}}, \] (4.5)

where:

\[ V(r) = -(M - 2EH)r^2 - (E^2 - J^2 / 4), \] (4.6)

and \( E \) is an integration constant. Here:

\[ e^{\alpha} = \frac{H^2 r^2}{2}, \] (4.7)

and we have:

\[ \ddot{\alpha} + H^2 (E^2 - J^2 / 4) e^{-\alpha} = 0. \] (4.8)
Now taking into account that \( M > 0, \ H > 0, \ |J| \leq M/H \) as well as the physical requirement that the string propagates forward in time (at least outside the horizon), we must have:

\[
2EH - M > 0, \tag{4.9}
\]

which also implies that:

\[
E^2 > J^2/4. \tag{4.10}
\]

It follows that the potential is a monotonically increasing function, and that the circular string contracts from \( r = r_{\text{max}} \) to \( r = 0 \), where:

\[
r_{\text{max}} = \sqrt{\frac{E^2 - J^2/4}{2EH - M}}. \tag{4.11}
\]

Depending on \( E \), the maximal string radius can be larger or smaller than \( r_+ \). In the first case, the string will initially be outside the horizon, but will then contract, fall into it and collapse into \( r = 0 \). In the latter case, the string is always inside the horizon and collapses into \( r = 0 \).

In the case without torsion \[13\], the potential \( V(r) \) was quartic in \( r \) and the solutions involved elliptic functions. As a consequence, the possibility \( E^2 < J^2/4 \) (Sinh-Gordon sector) also appeared, and a potential barrier between the inner horizon \( r_- \) and \( r = 0 \), preventing the string from collapsing into \( r = 0 \), was present.

### 4.2 Stationary Strings

The equations of motion and constraints for stationary strings are solved by:

\[
t = \tau - \int_{\sigma}^\sigma \frac{J(L - Hr^2)}{2r^2(H^2r^2 - M + J^2/4r^2)} \, d\sigma, \tag{4.12}
\]

\[
\phi = \int_{\sigma}^\sigma \frac{(L - Hr^2)(H^2r^2 - M)}{r^2(H^2r^2 - M + J^2/4r^2)} \, d\sigma, \tag{4.13}
\]

\[
r^2 + U(r) = 0 \quad \Leftrightarrow \quad \sigma = \pm \sqrt{-U(r)} \int_{r}^{r} \frac{dr}{\sqrt{-U(r)}}, \tag{4.14}
\]

where:

\[
U(r) = (H^2r^2 - M) \left[ \frac{L^2 - J^2/4}{r^2} + (M - 2LH) \right]. \tag{4.15}
\]
and $L$ is an integration constant. Here:

$$e^{\alpha} = \frac{1}{2}(H^2r^2 - M), \quad (4.16)$$

and we have:

$$\alpha'' + H^2 \left[ M(2HL - M) - H^2(L^2 - J^2/4) \right] e^{-\alpha} = 0. \quad (4.17)$$

Now taking into account that $M > 0$, $H > 0$, $|J| \leq M/H$ as well as the physical requirement that the stationary string (at least a part of) must be outside the static limit, we must have:

$$2LH - M > 0, \quad (4.18)$$

which also implies that:

$$L^2 > J^2/4. \quad (4.19)$$

It follows that the stationary string stretches out to infinity, but there is a "turning point" ($r' = 0$) at $r = r_{\min}$:

$$r_{\min} = \sqrt{L^2 - J^2/4 \over 2LH - M}. \quad (4.20)$$

Depending on $L$, the turning point can be outside or inside the static limit $r_{st} = \sqrt{M/H}$. In the first case, the string will be of "hanging string" type, with both ends at infinity, while in the latter case it will be of "spiralling string" type with one end at infinity, crossing the static limit and spiralling into the black hole. In the limiting case, when $r_{\min}$ is equal to $r_{st}$, corresponding to:

$$L = \frac{M}{H} \pm \frac{|J|}{2}, \quad (4.21)$$

the solution just fulfills the free wave equation, interpolating between the "hanging string" and the "spiralling string" types.

5 Conclusion

Using a physical approach, working directly with the classical string equations of motion and the proper string size, we reduced the SL(2,R) WZWN
model to Liouville theory. This allowed us to extract the precise physical effects of the parallelizing torsion on the generic string dynamics. We showed that the parallelizing torsion, corresponding to conformal invariance, generally led to repulsion. In fact, the parallelizing torsion gives rise to a repulsive term that *precisely* cancels the dominant attractive term arising from the metric. As a consequence, the sinh-Gordon and cosh-Gordon sectors of the non-conformally invariant AdS background reduce to the Liouville equation (with different signs of the potential), while the original Liouville sector reduces to the free wave equation. Thus, the dynamics of the classical large size strings is affected by the torsion, but most of the string size behaviour (intermediate and small sizes) is quite the same. We also gave the general solution to the proper string size.

We then analysed in detail the circular and stationary strings in the AdS spacetime and in the 2+1 BH-AdS spacetime, both with parallelizing torsion. These results confirmed our generic results (as they should), and we compared with the case of vanishing torsion. In particular, it was shown that the effect of torsion on the stationary strings is quite similar to the effect of rotation in the metric.
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Figure 1: The potential (2.27). The three curves represent, from above, $K=-1$, $K=0$ and $K=1$, respectively.
Figure 2: The potential (2.28). The three curves represent, from above, $K=-1$, $K=0$ and $K=1$, respectively.
Figure 3: The stationary string (3.20) corresponding to the case $H_L = -3/8$. 
Figure 4: The stationary string (3.27) corresponding to the case HL=0.