Uniqueness of diffusion operators
and capacity estimates

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Abstract

Let $\Omega$ be a connected open subset of $\mathbb{R}^d$. We analyze $L_1$-uniqueness of real second-order partial differential operators $H = -\sum_{k,l=1}^d \partial_k c_{kl} \partial_l$ and $K = H + \sum_{k=1}^d c_k \partial_k + c_0$ on $\Omega$ where $c_{kl} = c_{lk} \in W^{1,\infty}_{loc}(\Omega), c_k \in L_{\infty,loc}(\Omega), c_0 \in L_{2,loc}(\Omega)$ and $C(x) = (c_{kl}(x)) > 0$ for all $x \in \Omega$. Boundedness properties of the coefficients are expressed indirectly in terms of the balls $B(r)$ associated with the Riemannian metric $C^{-1}$ and their Lebesgue measure $|B(r)|$.

First we establish that if the balls $B(r)$ are bounded, the Täcklind condition $\int_R^\infty dr r (\log |B(r)|)^{-1} = \infty$ is satisfied for all large $R$ and $H$ is Markov unique then $H$ is $L_1$-unique. If, in addition, $C(x) \geq \kappa (c^T \otimes c)(x)$ for some $\kappa > 0$ and almost all $x \in \Omega$, $\text{div } c \in L_{\infty,loc}(\Omega)$ is upper semi-bounded and $c_0$ is lower semi-bounded then $K$ is also $L_1$-unique.

Secondly, if the $c_{kl}$ extend continuously to functions which are locally bounded on $\partial \Omega$ and if the balls $B(r)$ are bounded we characterize Markov uniqueness of $H$ in terms of local capacity estimates and boundary capacity estimates. For example, $H$ is Markov unique if and only if for each bounded subset $A$ of $\overline{\Omega}$ there exist $\eta_n \in C_c^\infty(\Omega)$ satisfying $\lim_{n \to \infty} \|1_A \Gamma(\eta_n)\|_1 = 0$, where $\Gamma(\eta_n) = \sum_{k,l=1}^d c_{kl} (\partial_k \eta_n)(\partial_l \eta_n)$, and $\lim_{n \to \infty} \|1_A (\mathbb{1}_\Omega - \eta_n) \varphi\|_2 = 0$ for each $\varphi \in L_2(\Omega)$ or if and only if $\text{cap}(\partial \Omega) = 0$.

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1 Introduction

Let Ω be a connected open subset of $\mathbb{R}^d$ and define the second-order divergence-form operator $H$ on the domain $D(H) = C^\infty_c(\Omega)$ by

$$H = -\sum_{k,l=1}^d \partial_k c_{kl} \partial_l$$

where the $c_{kl} = c_{lk}$ are real-valued functions in $W^{1,\infty}_{\text{loc}}(\Omega)$, and the matrix $C = (c_{kl})$ is strictly elliptic, i.e. $C(x) > 0$ for all $x \in \Omega$. It is possible that the coefficients can have degeneracies as $x \to \partial \Omega$, the boundary of $\Omega$, or as $x \to \infty$.

The operator $H$ is defined to be $L_1$-unique if it has a unique $L_1$-closed extension which generates a strongly continuous semigroup on $L_1(\Omega)$. Alternatively, it is defined to be Markov unique if it has a unique $L_2$-closed extension which generates a submarkovian semigroup on the spaces $L_p(\Omega)$. Markov uniqueness is a direct consequence of $L_1$-uniqueness since distinct submarkovian extensions give distinct $L_1$-extensions. But the converse implication is not valid in general. The converse was established in [RS11a] for bounded coefficients $c_{kl}$ and the proof was extended in [RS11b] to allow a growth of the coefficients at infinity. The principal aim of the current paper is to establish the equivalence of Markov uniqueness and $L_1$-uniqueness of $H$ from properties of the Riemannian geometry defined by the metric $C^{-1}$ which give, implicitly, optimal growth bounds on the coefficients.

Our arguments extend to non-symmetric operators

$$K = H + \sum_{k=1}^d c_k \partial_k + c_0$$

with the real-valued lower-order coefficients satisfying the following three conditions:

1. $c_0 \in L_{2,\text{loc}}(\Omega)$ is lower semi-bounded,
2. $c_k \in L_{\infty,\text{loc}}(\Omega)$ for each $k = 1, \ldots, d$, div $c \in L_{\infty,\text{loc}}(\Omega)$ and div $c$ is upper semi-bounded,
3. there is a $\kappa > 0$ such that $C(x) \geq \kappa (c^T \otimes c)(x)$ for almost all $x \in \Omega$.

In the second condition $c = (c_1, \ldots, c_d)$ and $\text{div } c = \sum_{k=1}^d \partial_k c_k$ with the partial derivatives understood in the distributional sense. The third condition in (3) is understood in the sense of matrix ordering, i.e. $(c_{kl}(x)) \geq \kappa (c_k(x)c_l(x))$ for almost all $x \in \Omega$. These conditions together with the general theory of accretive sectorial forms are sufficient to ensure that $K$ has an extension which generates a strongly continuous semigroup on $L_1(\Omega)$ (see Section 2). As in the symmetric case $K$ is defined to be $L_1$-unique if it has a unique such extension.

The Riemannian distance $d(\cdot ; \cdot)$ corresponding to the metric $C^{-1}$ can be defined in various equivalent ways but in particular by

$$d(x ; y) = \sup \{ \psi(x) - \psi(y) : \psi \in W^{1,\infty}_{\text{loc}}(\Omega), \Gamma(\psi) \leq 1 \}$$
for all \( x, y \in \Omega \) where \( \Gamma \), the carré du champ of \( H \), denotes the positive map

\[
\varphi \in W^{1,2}_{\text{loc}}(\Omega) \mapsto \Gamma(\varphi) = \sum_{k,l=1}^{d} c_{kl}(\partial_k \varphi)(\partial_l \varphi) \in L^1_{\text{loc}}(\Omega).
\]

(5)

Since \( \Omega \) is connected and \( C > 0 \) it follows that \( d(x; y) \) is finite for all \( x, y \in \Omega \) but one can have \( d(x; y) \to \infty \) as \( x \) or \( y \) tends to the boundary \( \partial \Omega \). Throughout the sequel we choose coordinates such that \( 0 \in \Omega \) and denote the Riemannian distance to the origin by \( \rho \). Thus \( \rho(x) = d(x; 0) \) for all \( x \in \Omega \). The Riemannian ball of radius \( r > 0 \) centred at 0 is then defined by \( B(r) = \{ x \in \Omega : \rho(x) < r \} \) and its volume (Lebesgue measure) is denoted by \( |B(r)| \).

There are two properties of the balls \( B(r) \) which are important in our analysis. First, the balls \( B(r) \) must be bounded for all \( r > 0 \). It follows straightforwardly that this is equivalent to the condition that \( \rho(x) \to \infty \) as \( x \to \infty \), i.e. as \( x \) leaves any compact subset of \( \Omega \). Secondly, it is essential to have control of the growth of the volume \( |B(r)| \) (Lebesgue measure) of the balls. Our results are based on the Täcklind condition \( \text{Täc36} \),

\[
\int_{R}^{\infty} dr r (\log |B(r)|)^{-1} = \infty
\]

(6)

for all large \( R \). In particular this condition is satisfied if there are \( a, b > 0 \) such that \( |B(r)| \leq a e^{br^2 \log(1+r)} \) for all \( r > 0 \).

Täcklind established the Cauchy equation on \( \mathbb{R}^d \) has a unique solution within the class of functions satisfying a growth condition of the type \( \text{(6)} \). Moreover, uniqueness can fail if the growth bound is not satisfied. Subsequently Grigor’yan (see \( \text{Gri86} \), Theorem 1, or \( \text{Gri99} \), Theorem 9.1) used condition \( \text{(6)} \) to prove that the heat semigroup generated by the Laplace-Beltrami operator on a geodesically complete manifold is stochastically complete, i.e. it conserves probability. But stochastic completeness of the heat semigroup is equivalent to \( L^1 \)-uniqueness of the Laplace-Beltrami operator (see, for example, \( \text{Dav85} \) Section 2). Thus \( \text{(6)} \) suffices for \( L^1 \)-uniqueness of the Laplace-Beltrami operator. Our aim is to prove that the Täcklind condition and a variation of Grigor’yan’s arguments are sufficient to establish \( L^1 \)-uniqueness of \( H \) and \( K \). In our analysis Markov uniqueness of \( H \) plays the same role as geodesic completeness of the manifold.

**Theorem 1.1** Adopt the foregoing assumptions. Assume the Riemannian balls \( B(r) \) are bounded for all \( r > 0 \) and the Täcklind condition \( \text{(6)} \) is satisfied. Further assume that \( H \) is Markov unique. Then \( H \) and \( K \) are \( L^1 \)-unique.

The theorem extends results obtained in collaboration with El Maati Ouhabaz \( \text{OR11} \) based on conservation arguments which place more restrictive restrictions on the lower-order coefficients.

Theorem \( \text{1.1} \) will be proved in Section 3 after the discussion of some preparatory material in Section 2. Finally, in Section 4 we discuss the characterization of Markov uniqueness of \( H \) in terms of capacity estimates. These latter estimates give a practical method of establishing the Markov uniqueness property. They also establish that if the coefficients \( c_{kl} \) extend by continuity to locally bounded functions on \( \overline{\Omega} \) then Markov uniqueness is equivalent to the capacity of the boundary of \( \Omega \) being zero.

For background information and related results on uniqueness properties of diffusion operators we refer to Section 3.3 of \( \text{FOT94} \) together with the lecture notes of Eberle \( \text{Ebe99} \) and references therein.
2 Preliminaries

In this section we first recall some basic results on Markov uniqueness of the symmetric operator $H$ defined by (1). These results do not require any restrictions on the growth of the coefficients of $H$ or on the Riemannian geometry. Secondly, we discuss the accretivity properties, etc. of the non-symmetric operator $K$ and its Friedrichs extension together with continuity and quasi-accretivity properties of the associated positive semigroup. Although these results are formulated for the operators $H$ and $K$ they are to a large extent general properties of Dirichlet forms, symmetric [BH91] [FOT94] or non-symmetric [MR92]. Thirdly, we establish some basic regularity properties for solutions of the Cauchy equations associated with $H$ and $K$.

2.1 Markov uniqueness

The operator $H$ is positive(-definite) and symmetric on $L^2(\Omega)$. The corresponding positive, symmetric, quadratic form $h$ is given by

$$D(h) = C_\infty^\infty(\Omega)$$

and

$$h(\phi) = \sum_{k,l=1}^d (\partial_k \phi, c_{kl} \partial_l \phi)$$

where $(\cdot, \cdot)$ denotes the $L_2$-scalar product. The form is closable and its closure $h_D = \overline{h}$ determines a positive self-adjoint extension, the Friedricks’ extension, $H_D$ of $H$ (see, for example, [Kat80], Chapter VI). We use the notation $H_D$ since this extension corresponds to Dirichlet conditions on the boundary $\partial \Omega$. The closure $h_D$ is a Dirichlet form and consequently the $H_D$ generates a submarkovian semigroup $S$. (For details on Dirichlet forms and submarkovian semigroups see [BH91] [FOT94] [MR92].) In particular $S$ extends from $L^2(\Omega) \cap L^1(\Omega)$ to a positive contraction semigroup $S^{(1)}$ on $L^1(\Omega)$ and the generator $H_1$ of $S^{(1)}$ is an extension of $H$. Therefore $H$ has both a submarkovian extension and an $L^1$-generator extension.

Next we define a second Dirichlet form extension $h_N$ of $h$ as follows. First the domain $D(h_N)$ of $h_N$ is specified by

$$D(h_N) = \{\phi \in W^{1,2}_\text{loc}(\Omega) : \Gamma(\phi) + \phi^2 \in L^1(\Omega)\}$$

where $\Gamma$ denotes the positive map defined by (5). Then $h_N$ is given by

$$h_N(\phi) = \int_\Omega \Gamma(\phi) = \|\Gamma(\phi)\|_1$$

for all $\phi \in D(h_N)$. The form $h_N$ is closed as a direct consequence of the strict ellipticity assumption $C > 0$ (see [RS11b], Section 1, or [OR11], Proposition 2.1). The self-adjoint operator $H_N$ associated with $h_N$ is a submarkovian extension of $H$ which can be considered to correspond to Neumann boundary conditions. In general the two submarkovian extensions $H_D$ and $H_N$ of $H$ are distinct. The significance of the forms $h_D$ and $h_N$ is that they are the minimal and maximal Dirichlet form extensions of $h$.

**Proposition 2.1** Let $k$ be a Dirichlet form extension of $h$. Then $h_D \subseteq k \subseteq h_N$. Thus if $K$ is the submarkovian extension of $H$ corresponding to $k$ one has $H_N \leq K \leq H_D$.

In particular, $H$ is Markov unique if and only if $h_D = h_N$. 


Moreover, if \( k \in L^\infty(\Omega) \) then \( \chi \varphi \in D(\overline{H}) \) and

\[
k_\chi(\varphi) = (\varphi, \overline{H} \chi \varphi) - 2^{-1}(H \chi, \varphi^2).
\]

But if \( \chi_1 \in C_0^\infty(\Omega) \) with \( \chi_1 = 1 \) on \( \text{supp} \chi \) then \( \varphi_1 = \chi_1 \varphi \in D(\overline{H}) \subseteq W^{1,2}_{\text{loc}}(\Omega) \), where the last inclusion again uses elliptic regularity, and

\[ k_\chi(\varphi) = (\varphi_1, \overline{H} \chi \varphi_1) - 2^{-1}(H \chi, \varphi_1^2) = \int_\Omega \chi(\varphi_1) \]

by direct calculation. Combining these observations one has

\[
\int_\Omega \chi(\varphi_1) = k_\chi(\varphi) \leq k(\varphi)
\]

for all \( \varphi \in D(K) \cap L^\infty(\Omega) \). Then if \( V \) is a relatively compact subset of \( \Omega \) there is a \( \mu_V > 0 \) such that \( C(x) \geq \mu_V I \) for all \( x \in V \). Therefore choosing \( \chi \) such that \( \chi = 1 \) on \( V \) one deduces that \( \mu_V \int_V |\nabla \varphi|^2 \leq k(\varphi) \) for each choice of \( V \). Thus \( \varphi \in W^{1,2}_{\text{loc}}(\Omega) \). Moreover, \( \int_V \Gamma(\varphi) \leq k(\varphi) \) for each \( V \) so \( \varphi \in D(h_N) \). Consequently \( D(K) \cap L^\infty(\Omega) \subseteq D(h_N) \) and

\[
h_N(\varphi) = \sup_V \int_V \Gamma(\varphi) \leq k(\varphi)
\]

for all \( \varphi \in D(K) \cap L^\infty(\Omega) \). But since \( K \) is the generator of a submarkovian semigroup \( D(K) \cap L^\infty(\Omega) \) is a core of \( K \). In addition \( D(K) \) is a core of \( k \). Therefore the last inequality extends by continuity to all \( \varphi \in D(k) \). In particular \( D(k) \subseteq D(h_N) \). Hence \( k \subseteq h_N \) and \( H_N \leq K \).

The identity \( h_D = h_N \), in one guise or another, has been the basis of much of the analysis of Markov uniqueness (see, for example, [FOT94], Section 3.3, or [Ebe99], Chapter 3). Since \( h_N \) is an extension of \( h_D \) the identity is equivalent to the condition \( D(h_D) = D(h_N) \). But \( D(h_D) \) is the closure of \( C^\infty_c(\Omega) \) with respect to the graph norm \( \varphi \mapsto \| \varphi \|_{D(h_D)} = (h_D(\varphi) + \| \varphi \|^2)^{1/2} \). Therefore \( h_D = h_N \) if and only if \( C^\infty_c(\Omega) \) is a core of \( h_N \). Equivalently, \( h_D = h_N \) if and only if \( (D(h_D) \cap L^\infty(\Omega))_c \), the space of bounded functions in \( D(h_D) \) with compact support in \( \Omega \), is a core of \( h_N \).

It follows from the Dirichlet form structure that the subspace \( D(h_N) \cap L^\infty(\Omega) \) of bounded functions in \( D(h_N) \) is an algebra and a core of \( h_N \). Similarly \( D(h_D) \cap L^\infty(\Omega) \) is an algebra and a core of \( h_D \). The following observation on the algebraic structure is useful for various estimates.

**Proposition 2.2** The subalgebra \( D(h_D) \cap L^\infty(\Omega) \) of \( D(h_N) \cap L^\infty(\Omega) \) is an ideal, i.e.

\[
(D(h_D) \cap L^\infty(\Omega)) \cdot (D(h_N) \cap L^\infty(\Omega)) \subseteq D(h_D) \cap L^\infty(\Omega).
\]
Proof If $\eta \in D(h_D) \cap L_{\infty}(\Omega)$ then there is a sequence $\eta_n \in C_c^{\infty}(\Omega)$, with $\|\eta_n\|_2 \leq \|\eta\|_2$, which converges to $\eta$ in the $D(h_D)$-graph norm. But if $\varphi \in D(h_N) \cap L_{\infty}(\Omega)$ then $\eta_n \varphi \in W_0^{1,2}(\Omega)$. Further
\[ \lim_{n \to \infty} \|\eta_n \varphi - \eta \varphi\|_2 \leq \lim_{n \to \infty} \|\eta_n - \eta\|_2 \varphi\|_\infty = 0. \]
Moreover,
\[ h_D(\eta_n \varphi - \eta_m \varphi) \leq 2h_D(\eta_n - \eta_m) \|\varphi\|_\infty^2 + 2\int_\Omega \Gamma(\varphi)(\eta_n - \eta_m)^2. \]
Since $\Gamma(\varphi) \in L_1(\Omega)$ and $\eta_n$ is $L_2$-convergent it follows by equicontinuity that $\eta_n \varphi$ converges to $\eta \varphi$ in the $D(h_D)$-graph norm. Thus $\eta \varphi \in D(h_D)$. \qed

Although $D(h_N) \cap L_{\infty}(\Omega)$ is a core of $h_N$ it does not follow without further assumptions that $\langle D(h_N) \cap L_{\infty}(\Omega), \rangle$, the subspace of functions with compact support in $\overline{\Omega}$, is a core of $h_N$. Maz’ya gives an example with $\Omega = \mathbb{R}^d$ for which this property fails (see, [Maz85], Theorem 3 in Section 2.7). We will return to the discussion of this topic in Section 4.

2.2 Accretivity and continuity properties

Next we consider the non-symmetric operator $K$ defined by (2) with the lower order coefficients satisfying the three conditions of (3). In this subsection $K$ is viewed as an operator on the space of complex $L_2$-functions. Our aim is to establish accretivity and sectorial estimates which suffice to deduce that $K$ has a Friedrichs’ extension which generates a strongly continuous semigroup $T$ on $L_2(\Omega)$ and that the semigroup extends to the corresponding $L_\rho$-spaces. These estimates apply equally well to the formal adjoint $K^\dagger$ of $K$. The latter operator is defined as the restriction of the $L_2$-adjoint $K^*$ of $K$ to $C_c^{\infty}(\Omega)$. Therefore $K^\dagger$ is obtained from $K$ by the replacements $c \to -c$ and $c_0 \to c_0 - \text{div} \ c$.

After deriving the accretivity estimates we derive a local strong continuity property for the semigroup $T$ and the dual group $T^*$ generated by the Friedrichs’ extension of $K^\dagger$ both acting on $L_{\infty}(\Omega)$.

First define $L$ and $M$ on $C_c^{\infty}(\Omega)$ by
\[ L\varphi = \sum_{k=1}^d c_k \partial_k \varphi \quad \text{and} \quad M\varphi = c_0 \varphi. \]

Then $K = H + L + M$. Let $k$ denote the corresponding sesquilinear form and quadratic form, i.e. $D(k) = C_c^{\infty}(\Omega)$, $k(\varphi, \psi) = (\varphi, K\psi)$ and $k(\varphi) = k(\varphi, \varphi)$ for $\varphi, \psi \in D(k)$. Further let $k^*$ denote the adjoint form, i.e. $D(k^*) = D(k)$ and $k^*(\varphi, \psi) = k(\psi, \varphi)$. The real part and imaginary parts of $k$ are defined by $\Re k = -1(k + k^*)$ and $\Im k = (2i)^{-1}(k - k^*)$, respectively. In particular
\[ (\Re k)(\varphi) = h(\varphi) + (\varphi, (c_0 - 2^{-1} \text{div} \ c) \varphi) \geq h(\varphi) + (\omega_0 - 2^{-1} \omega_1)\|\varphi\|_2^2 \] (7)
for all $\varphi \in C_c^{\infty}(\Omega)$ where $\omega_0 = \text{ess inf}_{x \in \Omega} c_0(x)$ and $\omega_1 = \text{ess sup}_{x \in \Omega} (\text{div} c)(x)$. Thus $\Re k$ is the form of a lower semi-bounded symmetric operator and consequently closable. Moreover, if $\omega = (\omega_0 - 2^{-1} \omega_1)$ then $k + \sigma$ is an accretive form for all $\sigma \geq -\omega$. Next
\[ (\Im k)(\varphi) = (2i)^{-1}((\varphi, L\varphi) - (L\varphi, \varphi)). \]
for all $\varphi \in C_c^\infty(\Omega)$. Hence
\begin{align}
|\langle 3k \rangle(\varphi)| & \leq \|\varphi\|_2 \|L\varphi\|_2 \leq \kappa^{-1/2}\|\varphi\|_2 h(\varphi)^{1/2} \\
& \leq (\varepsilon h(\varphi) + (4\varepsilon\kappa)^{-1}\|\varphi\|_2)
\end{align}
(8)
for all $\varphi \in C_c^\infty(\Omega)$ and $\varepsilon > 0$ where the second step uses the third condition of (3). It follows from (7) and (8) that $k + \sigma$ is a sectorial form for all $\sigma \geq (4\kappa)^{-1} - \omega$. Since $\Re k$ is closable it follows that $k + \sigma$ is closable with respect to the norm $\varphi \in C_c^\infty(\Omega) \mapsto \|\varphi\|_k = (\Re k)(\varphi) + \sigma\|\varphi\|_2^2)^{1/2}$ for any $\sigma > -\omega$. The closure of the form then determines a closed extension of $K + \sigma I$ (see [Kat80], Chapter VI or [Ouh05], Chapter 1). Therefore by subtracting $\sigma I$ one obtains a closed extension $K_D$ of $K$, the Friedrichs’ extension. The extension generates a strongly continuous semigroup $T$ on $L_2(\Omega)$ which satisfies the quasi-contractive bounds $\|T_t\|_{\infty} \leq e^{-\omega t}$, for all $t > 0$. The estimates (7) and (8) are also valid for the adjoint form $k^*$ which is associated with the formal adjoint $K^*$ of $K$. Therefore $K^*$ has a Friedrichs’ extension, $K_D^* = (K_D)^*$ and $K_D^*$ generates the adjoint semigroup $T^*$ on $L_2(\Omega)$.

It follows from the foregoing accretivity and sectorial properties that if $\sigma > (4\kappa)^{-1} - \omega$ then $k + \sigma$ satisfies the weak sector condition I(2.3) of Ma and Röckner [MR92] (see [Ouh05], Proposition 1.8). Therefore $k + \sigma$ is accretive, closable and satisfies the weak sector condition for all sufficiently large $\sigma$. Then it follows from [MR92], Section II.2d, that $k + \sigma$ is a (non-symmetric) Dirichlet form. Therefore $T$ is positive. Moreover, $T$ extends from $L_2(\Omega) \cap L_1(\Omega)$ to a strongly continuous semigroup on $L_1(\Omega)$, and from $L_2(\Omega) \cap L_\infty(\Omega)$ to a weakly$^*$ continuous semigroup on $L_\infty(\Omega)$. Similar conclusions are valid for the adjoint form $k^*$ and the adjoint semigroup $T^*$. Since one readily establishes that $K - (\omega_0 - \omega_1)$ and $K^* - \omega_0$ are both $L_1$-dissipative it then follows that $\|T_t\|_{1 \to 1} \leq e^{-(\omega_0 - \omega_1)t}$ and $\|T_t\|_{\infty \to \infty} = \|T^*_t\|_{1 \to 1} \leq e^{-\omega_0t}$ for all $t > 0$.

One can also define an extension $K_N$ of $K$ analogous to the extension $H_N$ of $H$ by form techniques. To this end one uses the lower semi-boundedness of $c_0$ and the third property of (3). The latter ensures that the first-order operator $L$ extends to $D(h_N)$ and that the corresponding form $l$ is relatively bounded by $h_N$ with relative bound zero. We omit the details.

The weak$^*$-continuity of the semigroup $T^{(\infty)}$ generated by $K_D$ on $L_\infty(\Omega)$ can be strengthened by general arguments which apply equally well to the semigroup generated by $K_N$.

**Proposition 2.3** The semigroup $T^{(\infty)}$ is $L_{p, loc}$-continuous for all $p \in [1, \infty)$.

**Proof** First we prove that $T^{(\infty)}$ is $L_{1, loc}$-continuous. It clearly suffices to prove that
\[\lim_{t \to 0} \|1_V(I - T_t^{(\infty)})\psi\|_1 = 0\]
for all relatively compact $V \subset \Omega$ and all positive $\psi \in L_\infty(\Omega)$.

Let $W$ be a second relatively compact subset of $\Omega$ with $\overline{V} \subset W$. Then
\[\|1_V(I - T_t^{(\infty)})\psi\|_1 \leq \|1_V(I - T_t^{(\infty)})1_W\psi\|_1 + \|1_V T_t^{(\infty)}(1_\Omega - 1_W)\psi\|_1\]
because $1_V(1_\Omega - 1_W) = 1_V 1_W = 0$. But $1_W\psi \in L_1(\Omega) \cap L_\infty(\Omega)$ and consequently
\[\limsup_{t \to 0} \|1_V(I - T_t^{(\infty)})1_W\psi\|_1 = \limsup_{t \to 0} \|1_V(I - T_t^{(1)})1_W\psi\|_1 \leq \lim_{t \to 0} \|I - T_t^{(1)}\|1_W\psi\|_1 = 0\]
by the strong continuity of $T^{(1)}$ on $L_1(\Omega)$. Next note that $1\in L_1(\Omega)$ and $1_{W\cap}\psi \in L_\infty(\Omega)$. But $1_{W^c}\psi = (1_\Omega - 1_W)\psi \geq 0$, since $\psi \geq 0$ by assumption. Moreover $T$ is positive. Therefore
\[
\limsup_{t \to 0} \|1_V T_t^{(\infty)} (1_\Omega - 1_W)\psi\|_1 = \limsup_{t \to 0} (1_V, T_t^{(\infty)} 1_{W^c}\psi) = 0
\]
by the weak* continuity of $T^{(\infty)}$. Combination of these conclusions completes the proof for $p = 1$.

Finally the continuity for $p \in (1, \infty)$ follows since
\[
\|1_V (I - T_t^{(\infty)})\psi\|_p \leq \|1_V (I - T_t^{(\infty)})\psi\|_1^{1/p} ((1 + e^{-\omega t}) \|\psi\|_\infty)^{1 - 1/p}
\]
by the Hölder inequality and the bounds $\|T_t^{(\infty)}\|_{\infty \to \infty} \leq e^{-\omega t}$. \hfill \Box

**Remark 2.4** The adjoint semigroup $T^*$ is also $L_{p,\text{loc}}$-continuous because it is the semigroup generated by the Friedrichs’ extension $K^\dagger_D$ of the formal adjoint $K^\dagger$ of $K$.

### 2.3 Parabolic regularity

Next we discuss some basic regularity properties of uniformly bounded solutions of the Cauchy equations corresponding to $H$ and $K$. The Cauchy equation is formally given by
\[
\partial_t \psi_t + H \psi_t = 0
\]
where $t > 0 \mapsto \psi_t$ is a function over $\Omega$ whose initial value $\psi_0$ is specified. A precise definition will be given in the following section. Analysis of the Cauchy equation requires consideration of functions over the $(d + 1)$-dimensional set $\Omega_+ = \mathbb{R}_+ \times \Omega$. We use the notation $u, v$, etc. for functions over $\Omega_+$ to avoid confusion with the functions $\varphi, \psi$, etc. over $\Omega$. We nevertheless use $(\cdot, \cdot)$ and $\|\cdot\|_2$ to denote the scalar product and norm on $L_2(\Omega_+)$ since this should not cause confusion. In particular
\[
\|u\|_2 = \left( \int_0^\infty dx_0 \int_\Omega dx |u(x_0, x)|^2 \right)^{1/2}.
\]
The tensor product structure ensures that the operators $H$ and $K$ and their various generator extensions act in a natural manner on $L_2(\Omega_+)$, e.g. $H_D$ on $L_2(\Omega)$ is replaced by $1_{\mathbb{R}_+} \otimes H_D$ on $L_2(\Omega_+)$. To avoid inessential complications we will use the same notation for the operators on the enlarged spaces, i.e. we identify $H_D$ with $1_{\mathbb{R}_+} \otimes H_D$ etc.

We now consider the operator $\mathcal{H} = -\partial_0 + H$ acting on $C_0^\infty(\Omega_+)$. The formal adjoint is then given by $\mathcal{H}^\dagger = \partial_0 + H$. Next we introduce the Sobolev space
\[
V^{1,2}(\Omega_+) = \{ \psi \in L_2(\Omega_+) : \partial_k \psi \in L_2(\Omega_+) \text{ for all } k = 1, \ldots, d \} = L_2(\mathbb{R}_+) \otimes W^{1,2}(\Omega)
\]
and the weighted, or anisotropic, space
\[
V^{2,2}(\Omega_+) = \{ \psi \in V^{1,2}(\Omega_+) : \partial_0 \psi, \partial_k \partial_0 \psi \in L_2(\Omega_+) \text{ for all } k, l = 1, \ldots, d \}
\]
with the usual norms. Then the spaces $V^{-1,2}(\Omega_+)$ and $V^{-2,2}(\Omega_+)$ of distributions are defined by duality (see, for example, [Gri09] Section 6.4).

The principal regularity property used in the subsequent discussion of $L_1$-uniqueness of $H$ is the following.
Proposition 2.5 If $\mathcal{H}^*$ denotes the $L_2$-adjoint of $\mathcal{H}$ then $D(\mathcal{H}^*) \subseteq V^2_{loc}(\Omega+)$. 

Proof The proposition is a corollary of Lemma 6.19 in [Gri09]. The discussion of parabolic regularity properties in the latter reference is for a strongly elliptic symmetric operator $P$ with smooth coefficients interpreted as acting on distributions from $\mathcal{D}'(\Omega+)$. But since the estimates are local only local strong ellipticity is necessary and this follows from the strict ellipticity of the matrix $C$ of coefficients of $H$. Moreover, the proof of Lemma 6.19 only uses the assumption that the coefficients of $P$ are locally Lipschitz. Therefore the proof of Lemma 6.19 is applicable with $P$ replaced by $\mathcal{H}^*$. □

In the discussion of $L_1$-uniqueness of $K$ it is convenient to introduce the operator $K_0 = H + L$ on $C_c^\infty(\Omega)$ and the corresponding operator $K_0^* = -\partial_0 + K_0$ on $C_c^\infty(\Omega+)$. Note that the formal adjoint of $K_0$ is given by $K_0^† = H - L + M_0$ where $M_0$ is the operator of multiplication by the locally bounded function $-\text{div } c$.

Proposition 2.6 If $K_0^*$ denotes the $L_2$-adjoint of $K_0$ then $D(K_0^*) \subseteq V^2_{loc}(\Omega+)$. 

Proof The proof of the proposition is a repetition of the argument used to prove Lemma 6.19 in [Gri09]. The operator $\mathcal{P}$ in the latter reference is now replaced by $K_0^*$. Therefore one has the terms corresponding to $\mathcal{H}^*$ together with additional first-order and zero-order terms. The additional first-order terms $-\sum_{k=1}^d c_k \partial_k$ cause no problem since they combine with the terms $-\sum_{k,l=1}^d (\partial c_{lk}) \partial_k$. The zero-order term, i.e. multiplication by $-\text{div } c$, also causes no problem since $\text{div } c \in L_{\infty,loc}(\Omega)$ by assumption. □

3 $L_1$-uniqueness

In this section we prove Theorem [11]. We adopt the Cauchy equation approach of Grigor’yan in his analysis of operators on manifolds. Grigor’yan’s argument relies essentially on the geodesic completeness of the manifold but in the following proof this is replaced by Markov uniqueness of $H$. The latter property is equivalent, by Proposition [2.1] to $C_c^\infty(\Omega)$ being a core of $h_N$ and this suffices for the application of Grigor’yan’s techniques.

First for $\tau > 0$ set $\Omega_\tau = (0, \tau) \times \Omega$. Denote a general point in $\Omega_\tau$ by $(t, x)$. So $\partial_0$ denotes the partial derivative with respect to the first variable $t$. A function $u \in L_{\infty}(\Omega_\tau)$ is defined to be a bounded weak solution of the Cauchy equation corresponding to $K$ on $\Omega_\tau$ with initial value $\psi \in L_{\infty}(\Omega)$ if

$$(u, (-\partial_0 + K)v) = 0 \quad \text{for all } v \in C_c^\infty(\Omega_\tau)$$

and

$${\lim_{t \to 0}} \int_{V} dx |u(t, x) - \psi(x)|^2 = 0 \quad \text{for all relatively compact subsets } V \subseteq \Omega.$$ 

The ‘time-dependent’ criterion for $L_1$-uniqueness of $K$ is formulated in terms of weak solutions of the Cauchy equation with zero initial value.

Proposition 3.1 If for some $\tau > 0$ the only bounded solution of Cauchy equation (9) on $\Omega_\tau$ with initial value 0 in the $L_{2,loc}$-sense (10) is the zero solution then $K$ is $L_1$-unique.
\textbf{Proof} It follows from an extension of the Lumer–Phillips theorem (see [Ebe99], Theorem 1.2 in Appendix A of Chapter 1) that $K$ is $L_1$-unique if and only if the $L_1$-closure of $K$ is the generator of a strongly continuous semigroup on $L_1(\Omega)$. But this is the case if and only if the range of $\lambda I + K$ is $L_1$-dense for all large $\lambda > 0$.

Assume that $K$ is not $L_1$-unique. Thus for each large $\lambda$ there is a non-zero $\psi \in L_\infty(\Omega)$ such that $(\psi, (\lambda I + K) \varphi) = 0$ for all $\varphi \in C_c^\infty(\Omega)$. Then define $u_1$ on $\Omega_\tau$ by $u_1(t, x) = e^{\lambda t} \psi(x)$ for all $t \in (0, \tau)$ and all $x \in \Omega$. It follows that $u_1$ is a solution of the Cauchy equation (9) on $\Omega_\tau$ with $\|u_1\|_\infty \leq e^{\lambda \tau} \|\psi\|_\infty$. Moreover, $u_1$ has initial value $\psi$ in the $L_{2,\text{loc}}$-sense (10).

Next define $u_2$ on $\Omega_\tau$ by $u_2(t, x) = (T^*_t \psi)(x)$ for all $t \in (0, \tau)$ and $x \in \Omega$ where $T^*$ is the adjoint of the semigroup $T$ generated by the Friedrichs’ extension $K_D$ of $K$. The adjoint semigroup $T^*$ acts on $L_\infty(\Omega)$ and $\|T^*_s\|_{\infty\rightarrow\infty} = \|T_s\|_{1\rightarrow 1} \leq e^{-(\omega_0 - \omega_1)s}$ for all $s > 0$ by the discussion of Subsection 2.2. Therefore $u_2$ is also a solution of the Cauchy equation (9) on $\Omega_\tau$ with $\|u_2\|_\infty \leq e^{\omega \tau} \|\psi\|_\infty$ where $\omega = (-\omega_0 + \omega_1) \lor 0$. But the adjoint semigroup $T^*$ on $L_\infty(\Omega)$ is $L_{2,\text{loc}}$-continuous by Proposition 2.3 and Remark 2.4. Thus $u_2$ has initial value $\psi$ in the $L_{2,\text{loc}}$-sense (10).

Finally

$$\sup_{x \in \Omega} |u_1(t, x)| \geq e^{(\lambda - \omega)t} \sup_{x \in \Omega} |u_2(t, x)|$$

for all $t \in (0, \tau)$. Thus if $\lambda > \omega$ one must have $u_1 \neq u_2$ and so $u_1 - u_2$ is a non-zero bounded weak solution of the Cauchy equation (9) with initial value zero in the $L_{2,\text{loc}}$-sense (10).

Therefore the proposition follows by negation. \qed

The key result in the proof of $L_1$-uniqueness, the analogue of Theorem 2 in [Gri86], Theorem 9.2 in [Gri99] or Theorem 11.9 in [Gri09], can now be formulated as follows.

\textbf{Proposition 3.2} Assume $H$ is Markov unique and that the balls $B(r)$ are bounded for all $r > 0$. Let $u \in L_\infty(\Omega_\tau)$ be a bounded weak solution of the Cauchy equation (9) with zero initial value in the $L_{2,\text{loc}}$-sense (10). Further assume

$$\int_0^\tau dt \int_{B(r)} dx |u(t, x)|^2 \leq e^{\sigma(r)}$$

for all large $r$ where $v$ is a positive increasing function on $(0, \infty)$ such that

$$\int_R^\infty dr \sigma(r)^{-1} = \infty$$

for all large $R > 0$. Then $u \equiv 0$.

This proposition in combination with Proposition 3.1 immediately gives conditions for $L_1$-uniqueness of $K$ or $H$.

\textbf{Corollary 3.3} Assume the balls $B(r)$ are bounded for all $r > 0$ and that the Täcklind condition (6) is satisfied. It follows that if $H$ is Markov unique then both $H$ and $K$ are $L_1$-unique.

\textbf{Proof} Assume $H$ is Markov unique. If $u$ is a bounded weak solution of (9) and (10) then

$$\int_0^\tau dt \int_{B(r)} dx |u(t, x)|^2 \leq \tau \|u\|_\infty^2 |B(r)| \ .$$

(11)
It follows that the hypothesis of the proposition are fulfilled with \( \sigma(r) = \log(\tau \| u \|_\infty |B(r)|) \). Therefore \( u = 0 \) by Proposition 3.2 and \( K \) is \( L_1 \)-unique by Lemma 3.1. But setting the lower-order coefficients equal to zero one simultaneously deduces that \( H \) is \( L_1 \)-unique. \( \square \)

The proof of Theorem 1.1 is now reduced to proving Proposition 3.2. Once this is established the theorem follows from Corollary 3.3.

**Proof of Proposition 3.2** It suffices to prove that if \( r \) is large and \( \delta \in (0, \tau] \) satisfies \( \delta \leq r^2/(16\sigma(r)) \) then there is a \( b > 0 \) such that

\[
\int_{B(r)} dx |u(\tau, x)|^2 \leq \int_{B(2r)} dx |u(\tau - \delta, x)|^2 + b r^{-2}. \tag{12}
\]

The rest of the proof then follows by direct repetition of Grigoryan’s argument [Gri99] pages 186 and 187 or [Gri09] pages 306 and 307. In this part of the proof, which we omit, the \( L_{2,\text{loc}} \)-initial condition is crucial. Any weaker form of the initial condition is insufficient. Now we concentrate on establishing (12).

Let \( \rho_r(x) = \inf_{y \in B(r)} d(x, y) \) denote the Riemannian distance from \( x \) to the ball \( B(r) \). Set \( \xi_t = \nu \rho_r^2(t-s)^{-1} \) where \( \nu, s > 0 \) are fixed with \( t \neq s \). The values of \( s \) and \( \nu \) will be chosen later. In particular the choice of \( \nu \) depends on the lower-order coefficients. It follows that the partial derivative \( \xi_t^l \) with respect to \( t \) is given by \( \xi_t^l = -\nu \rho_r^2(t-s)^{-2} \) and \( \Gamma(\rho_r^2) = 4 \rho_r^2 \Gamma(\rho_r) \leq 4 \rho_r^2 \). Therefore \( \Gamma(\xi_t) \leq 4 \nu^2 \rho_r^2 (t-s)^{-2} \) and

\[
\xi_t^l + (4 \nu)^{-1} \Gamma(\xi_t) \leq 0. \tag{13}
\]

(An auxiliary function of this type was introduced by Aronson, [Aro67] Section 3, in his derivation of Gaussian bounds on the heat kernel.)

First we consider the case that \( u \in L_\infty(\Omega_\tau) \) is a weak solution of the Cauchy equation (9) corresponding to \( H \) and aim to deduce \( L_1 \)-uniqueness of \( H \). The argument for \( K \) is very similar but the lower-order terms introduce additional computational complications.

In the notation of Subsection 2.3 the Cauchy equation for \( H \) states that \( \langle u, H v \rangle = 0 \) for all \( v \in C_c^\infty(\Omega_\tau) \). Therefore \( u \in D(\mathcal{H}^*) \). But \( D(\mathcal{H}^*) \subseteq V_{\text{loc}}^{2,2}(\Omega_\tau) \) by Proposition 2.5. Thus the Cauchy equation can be explicitly written as

\[
(\partial_0 u, v) - \sum_{k,l=1}^d (\partial_k c_{kl} \partial_l u, v) = 0 \tag{14}
\]

for all \( v \in C_c^\infty(\Omega_\tau) \). But (14) extends to all \( v \in L_2(\Omega_\tau) \) with compact support because \( u \in V_{\text{loc}}^{2,2}(\Omega_\tau) \). Now define \( \psi_t \) by \( \psi_t(x) = u(t, x) \) and let \( \psi_t^l \) denote its partial derivative with respect to \( t \). Then set \( v \) equal to the restriction of \( \eta^2 e^{\xi_t} \psi_t \) to \( \langle \tau - \delta, \tau \rangle \times \Omega \) with \( \eta \in C_c^\infty(\Omega) \). Thus \( \text{supp} \, v \subseteq [\tau - \delta, \tau] \times \text{supp} \, \eta \) is compact. It follows, after an integration by parts in the \( x \)-variables, that

\[
\int_{\tau - \delta}^\tau dt \left( \psi_t^l, \eta^2 e^{2\xi_t} \psi_t \right) = -\sum_{k,l=1}^d \int_{\tau - \delta}^\tau dt \left( \partial_l \psi_t, c_{kl} \partial_k (\eta^2 e^{2\xi_t} \psi_t) \right) \\
= \int_{\tau - \delta}^\tau dt \left( \psi_t, \Gamma(\eta e^{\xi_t}) \psi_t \right) - \sum_{k,l=1}^d \int_{\tau - \delta}^\tau dt \left( \partial_l (\eta e^{\xi_t} \psi_t), c_{kl} \partial_k (\eta e^{\xi_t} \psi_t) \right) \\
= \int_{\tau - \delta}^\tau dt \left( \psi_t, \Gamma(\eta e^{\xi_t}) \psi_t \right) - \int_{\tau - \delta}^\tau dt h_D(\eta e^{\xi_t} \psi_t). \tag{15}
\]
Since $\eta \in C_c^\infty(\Omega)$ there are no boundary terms. But one also has
\[
(\psi_t, \Gamma(\eta e^{\xi t})\psi_t) \leq 2 (e^{\xi t}\psi_t, \Gamma(\eta)e^{\xi t}\psi_t) + 2 (\eta\psi_t, \Gamma(e^{\xi t})\eta\psi_t)
= 2 (e^{\xi t}\psi_t, \Gamma(\eta)e^{\xi t}\psi_t) + 2 (\eta e^{\xi t}\psi_t, \Gamma(\xi_t)\eta e^{\xi t}\psi_t)
\]
(16)
for all $\eta \in C_c^\infty(\Omega)$. Combination of (15) and (16) immediately leads to the inequality
\[
2^{-1} \int_{\tau-\delta}^\tau dt \int_\Omega \eta^2 e^{2\xi t}(\psi_t^2)' \leq 2 \int_{\tau-\delta}^\tau dt (e^{\xi t}\psi_t, \Gamma(\eta)e^{\xi t}\psi_t) + 2 \int_{\tau-\delta}^\tau dt (\eta e^{\xi t}\psi_t, \Gamma(\xi_t)\eta e^{\xi t}\psi_t)
\]
for all $\eta \in C_c^\infty(\Omega)$. Then integrating by parts in the $t$-variable and rearranging gives
\[
\left[\|\eta e^{\xi_t}\psi_t\|_2^2\right]_{\tau-\delta}^\tau \leq 2 \int_{\tau-\delta}^\tau dt (e^{\xi t}\psi_t, \Gamma(\eta)e^{\xi t}\psi_t) + \int_{\tau-\delta}^\tau dt (\eta e^{\xi t}\psi_t, (\xi_t' + 2\Gamma(\xi_t))\eta e^{\xi t}\psi_t)
\]
\[
\leq 2 \int_{\tau-\delta}^\tau dt (e^{\xi t}\psi_t, \Gamma(\eta)e^{\xi t}\psi_t)
\]
(17)
for all $\eta \in C_c^\infty(\Omega)$ where the last step uses (13) with $\nu$ chosen equal to $8^{-1}$. Next we use the Markov uniqueness of $H$ to extend (17) to a larger class of $\eta$.

First choose $s = \tau + \delta$ in the definition of $\xi_t$ so with the previous choice of $\nu = 8^{-1}$ one has $\xi_t = -8^{-1}\rho^2/(\tau + \delta - t)^{-1} \leq 0$ for all $t \in (0, \tau]$. Therefore
\[
\|\eta e^{\xi_t}\psi_t\|_2 \leq \|\eta\psi_t\|_2 \leq \|\psi_t\|_{\infty} \|\eta\|_2 \leq \|u\|_{\infty} \|\eta\|_2
\]
and
\[
0 \leq (e^{\xi_t}\psi_t, \Gamma(\eta)e^{\xi_t}\psi_t) \leq (\psi_t, \Gamma(\eta)\psi_t) \leq \|u\|_{\infty} h_D(\eta)
\]
for all $\eta \in C_c^\infty(\Omega)$ and all $t \in (0, \tau]$. Since $H$ is Markov unique $h_D = h_N$ and $C_c^\infty(\Omega)$ is a core of $h_N$ by Proposition 277. It then follows by continuity that (17) extends to all $\eta \in D(h_N)$. Thus one concludes that
\[
\|\eta e^{\xi_t}\psi_t\|_2 \leq \|\eta e^{\xi_t-\delta}\psi_{\tau-\delta}\|_2 + 2 \int_{\tau-\delta}^\tau dt (e^{\xi t}\psi_t, \Gamma(\eta)e^{\xi t}\psi_t)
\]
(18)
for all $\eta \in D(h_N)$.

Next let $\theta \in C_c^\infty(\mathbb{R})$ satisfy $0 \leq \theta \leq 1$, $\theta(s) = 1$ if $s \in [0, 3/2]$, $\theta(s) = 0$ if $s \geq 2$ and $|\theta'| \leq 3$. Then set $\theta_s = \theta \circ (r^{-1}\rho)$. It follows that $\theta_s \in D(h_N) \cap L_\infty(\Omega)$. Moreover, $\theta_s = 1$ if $\rho \leq 3r/2$ and $\theta_s = 0$ if $\rho \geq 2r$. Thus supp $\theta_s \subseteq B(2r)$ which is a bounded subset of $\Omega$ by assumption. But $\Gamma(\rho) \leq 1$. So one also has $\|\Gamma(\theta_s)\|_{\infty} \leq 9r^{-2}$. Hence replacing $\eta$ in (18) by $\theta_s$ one has
\[
\int_{B(r)} |e^{\xi t}\psi_t|^2 \leq \int_{B(2r)} |e^{\xi t-\delta}\psi_{\tau-\delta}|^2 + 18(a/r)^2 \int_{\tau-\delta}^\tau dt \int_{B(2r)\setminus B(3r/2)} dx |(e^{\xi t}\psi_t)(x)|^2
\]
(19)
But if $x \in B(r)$ then $\xi_t = 0$. Moreover, $\xi_{\tau-\delta} \leq 0$. Further if $x \in B(2r)\setminus B(3r/2)$ then $\rho_t(x) \geq r/2$ and so $\xi_t(x) \leq -r^2/(16\delta)$ for $t \in (\tau-\delta, \tau)$. Then it follows from (19) and the hypothesis of the proposition that
\[
\int_{B(r)} |\psi_t|^2 \leq \int_{B(2r)} |\psi_{\tau-\delta}|^2 + 18(a/r)^2 \int_{\tau-\delta}^\tau dt \int_{B(2r)} dx |\psi_t|^2 e^{-r^2/(16\delta)}
\]
\[
\leq \int_{B(2r)} |\psi_{\tau-\delta}|^2 + 18(a/r)^2 e^{-(r^2/(16\delta)) + \sigma(2r)}.
\]
(20)
Finally choosing \( \delta \leq r^2/(16\sigma(2r)) \) one has
\[
\int_{B(r)} |\psi_{\tau}|^2 \leq \int_{B(2r)} |\psi_{\tau-\delta}|^2 + 18(a/r)^2.
\]
Thus we have established (12) and the proposition follows for a solution of the Cauchy equation corresponding to \( H \). Thus \( H \) is \( L_1 \)-unique.

In order to conclude that \( K \) is \( L_1 \)-unique it remains to prove Proposition 3.2 for a solution of the Cauchy equation (9) corresponding to \( K = H + L + M \). In particular we have to consider the estimation of the lower-order terms. But now with the notation of Subsection 2.3 the Cauchy equation states that
\[
(u, K_0 v) + (u, M v) = 0
\]
for all \( v \in C_c^\infty(\Omega_\tau) \). Let \( V_\tau = (0, \tau) \times V \) where \( V \) is a relatively compact subset of \( \Omega \). It follows that
\[
|(u, K_0 v)| \leq \|u\|_\infty \|Mv\|_1 \leq \tau^{1/2} \|u\|_\infty \|c_0\|_{L_2(V)} \|v\|_2
\]
for all \( v \in C_c^\infty(V_\tau) \) because of the assumption that \( c_0 \in L_{2,loc}(\Omega) \). Hence \( u \) is in the domain of the adjoint of \( K_0|_{C_c^\infty(V_\tau)} \). Then one deduces from Proposition 2.6 that \( u \in V_{loc}^{2,2}(\Omega_\tau) \). Therefore one can argue as before. First the Cauchy equation (14) is replaced by
\[
(\partial_0 u, v) - \sum_{k,l=1}^d (\partial_k c_{kl} \partial_t u, v) + (u, L v) + (u, M v) = 0
\]
for all \( v \in C_c^\infty(\Omega_\tau) \). Then (15) is replaced by
\[
\int_{\tau-\delta}^\tau dt \left( \psi_t', \eta^2 e^{2\xi_t} \psi_t \right) = \int_{\tau-\delta}^\tau dt \left( \psi_t, \Gamma(\eta^\xi_t) \psi_t \right) - \int_{\tau-\delta}^\tau dt \ h_D(\eta^\xi_t \psi_t) \]
\[
- \int_{\tau-\delta}^\tau dt \left( \psi_t, L \eta^2 e^{2\xi_t} \psi_t \right) - \int_{\tau-\delta}^\tau dt \left( e^{\xi_t} \eta^\xi_t, M e^{\xi_t} \eta^\xi_t \psi_t \right). \quad (22)
\]
The first term on the right hand side is again estimated by (16) and it remains to estimate the terms originating with the lower-order terms \( L \) and \( M \). But
\[
(\psi_t, L \eta^2 e^{2\xi_t} \psi_t) = (\eta^\xi_t \psi_t, L \eta^\xi_t \psi_t) + (\psi_t, [L, \eta^\xi_t] \eta^\xi_t \psi_t)
\]
Further
\[
(\psi_t, [L, \eta^\xi_t] \eta^\xi_t \psi_t) = (e^{\xi_t} \psi_t, \eta L(\eta) e^{\xi_t} \psi_t) + (e^{\xi_t} \eta^\xi_t \psi_t, L(\xi_t) \eta^\xi_t \psi_t)
\]
It follows, however, from the third condition in (3) that
\[
\|L(\eta)\|_2 \leq \kappa^{-1}(\varphi, \Gamma(\eta)\varphi)
\]
for all \( \varphi \in L_2(\Omega) \). Therefore
\[
\begin{align*}
|\langle \eta^\xi_t \psi_t, L \eta^\xi_t \psi_t \rangle| & \leq h_D(\eta^\xi_t \psi_t) + (4\kappa)^{-1}(\eta^\xi_t \psi_t, \eta^\xi_t \psi_t), \\
|\langle e^{\xi_t} \psi_t, \eta L(\eta) e^{\xi_t} \psi_t \rangle| & \leq 2^{-1}(\eta e^{\xi_t} \psi_t, \eta e^{\xi_t} \psi_t) + (2\kappa)^{-1}(e^{\xi_t} \psi_t, \Gamma(\eta) e^{\xi_t} \psi_t), \\
|\langle e^{\xi_t} \eta^\xi_t \psi_t, L(\xi_t) \eta^\xi_t \psi_t \rangle| & \leq 2^{-1}(\eta e^{\xi_t} \psi_t, \eta e^{\xi_t} \psi_t) + (2\kappa)^{-1}(\eta e^{\xi_t} \psi_t, \Gamma(\xi_t) \eta^\xi_t \psi_t).
\end{align*}
\]
(23)
Combining estimates (16), (22) and (23) one deduces that

\[(\psi^\prime_t, \eta^2 e^{\xi_t} \psi_t) \leq 2\gamma (e^{\xi_t} \psi_t, \Gamma(\eta)e^{\xi_t} \psi_t) + 2\gamma (\eta e^{\xi_t} \psi_t, \Gamma(\xi_t) e^{\xi_t} \psi_t)
- (\eta e^{\xi_t} \psi_t, (c_0 - 1 - \gamma)\eta e^{\xi_t} \psi_t)\]

with \(\gamma = (1 + (4\kappa)^{-1})\). Now \(L_1\)-uniqueness of \(K\) is equivalent to \(L_1\)-uniqueness of \(K + \omega I\) for any \(\omega \in \mathbb{R}\). Therefore, replacing \(c_0\) by \(c_0 + \omega\) one may assume \(\omega_0 \geq 1 + \gamma\). Hence \(c_0 - 1 - \gamma \geq 0\) and one concludes that

\[(\psi^\prime_t, \eta^2 e^{2\xi_t} \psi_t) \leq 2\gamma (e^{\xi_t} \psi_t, \Gamma(\eta)e^{\xi_t} \psi_t) + 2\gamma (\eta e^{\xi_t} \psi_t, \Gamma(\xi_t) e^{\xi_t} \psi_t).\]

Integrating by parts and rearranging gives

\[
\left[\|\eta e^{\xi_t} \psi_t\|_2^2\right]_{\tau-\delta}^\tau \leq 2\gamma \int_{\tau-\delta}^\tau dt (e^{\xi_t} \psi_t, \Gamma(\eta)e^{\xi_t} \psi_t) \\
+ \int_{\tau-\delta}^\tau dt (\eta e^{\xi_t} \psi_t, (\xi_t + 2\gamma \Gamma(\xi_t)) e^{\xi_t} \psi_t)
\]

(24)

for all \(\eta \in C_c^\infty(\Omega)\). Then setting \(\nu = (8\gamma)^{-1}\) in the definition of \(\xi_t\) one has \(\xi_t + 2\gamma \Gamma(\xi_t) \leq 0\) by (13). Therefore one concludes that

\[
\|\eta e^{\xi_t} \psi_t\|_2^2 \leq \|\eta e^{\xi_{\tau-\delta}} \psi_{\tau-\delta}\|_2^2 + 2\gamma \int_{\tau-\delta}^\tau dt (e^{\xi_t} \psi_t, \Gamma(\eta)e^{\xi_t} \psi_t)
\]

(25)

for all \(\eta \in C_c^\infty(\Omega)\) in direct analogy with (17). In particular this estimate is valid with \(s = \tau + \delta\) in the definition of \(\xi_t\). Since \(H\) is Markov unique (25) extends to all \(\eta \in D(h_N)\) by repetition of the previous reasoning. The rest of the proof is exactly the same as the earlier proof for \(H\). Using (25) in place of (18) one establishes Proposition 3.2 for \(K\) and thereby concludes that \(K\) is \(L_1\)-unique. \(\square\)

The foregoing ‘time-dependent’ argument to deduce \(L_1\)-uniqueness from Markov uniqueness appears to be quite different to the ‘time-independent’ arguments of [RST1a] and [RS11b] for the symmetric operator \(H\). The two methods are, however, related. The time-independent proof uses Davies–Gaffney off-diagonal Gaussian bounds [Gal59] [Dav92] and one derivation of the latter bounds is by a variation of the foregoing time-dependent argument. (See [Gri99] Chapter 12.) The time-dependent argument is based on the Täcklind condition (6) on \(|B(r)|\) but the time-independent method for \(H\) requires the stronger condition \(|B(r)| \leq a e^{br^2}\) for some \(a, b > 0\) and all \(r > 0\). The latter restriction is essential because the argument uses the Davies–Gaffney off-diagonal bounds.

One may extend Theorem 1.1 to operators \(K\) for which the coefficients \(c_k\) and \(c_0\) are complex-valued. But then the assumptions (3) have to be appropriately modified, e.g. it is necessary that \(\text{Re } c_0\) is lower semi-bounded and \(\text{Re } \text{div } c\) is upper semi-bounded. Moreover, the third condition in (3) has to be replaced by \(C(x) \geq \kappa ((\tau^T \otimes c + c^T \otimes \tau)(x)\) for almost all \(x \in \Omega\). The proof is essentially the same but the spaces involved are complex.

4 Markov uniqueness

The basic ingredients in the foregoing analysis of \(L_1\)-uniqueness were the growth restrictions on the Riemannian geometry and the Markov uniqueness of \(H\). In this section we consider
the characterization of the latter property by capacity conditions. The first result of
this nature is due to Maz'ya (see [Maz85], Section 2.7) for the case $\Omega = \mathbb{R}^d$. Maz'ya
demonstrated that the identity $h_D = h_N$ is equivalent to a family of conditions on sets of
finite capacity. More recently it was established in [RS11a] and [RS11b] that Markov
uniqueness is equivalent to the capacity of the boundary of $\Omega$ being zero. Our aim is to
establish that both these capacity criteria are valid for $H$ and for general open $\Omega$ whenever
the Riemannian balls $B(r)$ are bounded for all $r > 0$. But this requires in part a slightly
stronger assumption on the properties of the coefficients $c_{kl}$.

First we define a subset $A$ of $\overline{\Omega}$ to have finite capacity, relative to $H$, if there is an
$\eta \in D(h_N)$ such that $\eta = 1$ on $A$. Each relatively compact subset of $\Omega$ has finite capacity
by Urysohn's lemma. Moreover, each set of finite capacity $A$ must have finite volume, i.e.
$|A| < \infty$, but one can have unbounded sets with finite capacity (see, [Maz85], Section 2.7).

We begin by establishing that there are an abundance of sets of finite capacity.

Proposition 4.1  The subspace $(D(h_N) \cap L_\infty(\Omega))_{cap}$ of bounded functions in $D(h_N)$ whose
supports have finite capacity is a core of $h_N$.

Proof  It suffices to prove that each $\varphi \in D(h_N) \cap L_\infty(\Omega)$ can be approximated in the $D(h_N)$-graph norm by a sequence $\varphi_n \in (D(h_N) \cap L_\infty(\Omega))_{cap}$. Clearly one may assume that $\varphi \geq 0$. But if $\lambda > 0$ the set $A_\lambda = \{x \in \Omega : \varphi(x) > \lambda\}$ has finite capacity. This is a consequence of the Dirichlet form structure by the following argument of Maz'ya. Define
$\varphi_\lambda$ by $\varphi_\lambda(x) = \lambda^{-1}(\varphi(x) \wedge \lambda)$. Then $\varphi_\lambda \in D(h_N)$, $0 \leq \varphi_\lambda \leq 1$, $\varphi_\lambda = 1$ on $A_\lambda$ and $h_N(\varphi_\lambda) \leq \lambda^{-2}h_N(\varphi)$, where the latter bounds follow from the Dirichlet property of $h_N$. Therefore $A_\lambda$ has finite capacity. Now consider the sequence $\varphi_m = \varphi - \varphi \wedge m^{-1} \in D(h_N) \cap L_\infty(\Omega)$. Since $\text{supp} \varphi_m = A_m^{-1}$ it follows that $\varphi_m \in (D(h_N) \cap L_\infty(\Omega))_{cap}$. But the $\varphi_m$ converge in the $D(h_N)$-graph norm to $\varphi$ as $m \to \infty$ by [FOT94], Theorem 1.4.2(iv).

Secondly, to formulate suitable versions of Mazya's approximation criterion for Markov
uniqueness we introduce the condition $C_A$ for each subset $A$ of $\overline{\Omega}$ by

$$
C_A: \left\{ \begin{array}{l}
\text{there exist } \eta_1, \eta_2, \ldots \in D(h_D) \text{ such that} \\
\lim_{n \to \infty} \|1_A (1_\Omega - \eta_n) \varphi\|_2 = 0 \text{ for each } \varphi \in L_2(\Omega) \\
\text{and } \lim_{n \to \infty} \|1_A \Gamma(\eta_n)\|_1 = 0.
\end{array} \right.
$$

Although the approximating sequence in this condition is formed by functions $\eta_n \in D(h_D)$
one can, equivalently, choose $\eta_n \in C_c^\infty(\Omega)$. This follows because $C_c^\infty(\Omega)$ is a core of $h_D$.
Explicitly, for each $\eta_n \in D(h_D)$ there is a $\chi_n \in C_c^\infty(\Omega)$ such that $\|\eta_n - \chi_n\|_{D(h_D)} \leq n^{-1}$. Therefore

$$
\|1_A (1_\Omega - \chi_n) \varphi\|_2 \leq \|1_A (1_\Omega - \eta_n) \varphi\|_2 + n^{-1} \|\varphi\|_\infty
$$

for all $\varphi \in L_2(\Omega) \cap L_\infty(\Omega)$ and

$$
\|1_A \Gamma(\chi_n)\|_1 \leq \|1_A \Gamma(\eta_n)\|_1 + h_D(\chi_n - \eta_n) \leq \|1_A \Gamma(\eta_n)\|_1 + n^{-1}.
$$

Hence the $C_A$-convergence criteria for the $\chi_n$ are inherited from the $\eta_n$. Alternatively, one
may assume, without loss of generality, that the $\eta_n$ satisfy $0 \leq \eta_n \leq 1$. This follows because
$\zeta_n = (0 \vee \eta_n) \wedge 1 \in D(h_N)$,

$$
\|1_A (1_\Omega - \zeta_n) \varphi\|_2 \leq \|1_A (1_\Omega - \eta_n) \varphi\|_2
$$
for all \( \varphi \in L_2(\Omega) \) and \( \| \mathbb{1}_A \Gamma(\zeta_n) \|_1 \leq \| \mathbb{1}_A \Gamma(\eta_n) \|_1 \) (see [BH91], Proposition 4.1.4). Therefore the \( \zeta_n \) inherit the \( \mathcal{C}_A \)-convergence properties of the \( \eta_n \).

The next proposition is a local version of Maz’ya’s result [Maz85], Theorem 1 in Section 2.7 (see also [FOT94], Theorem 3.2.2). Note that it is independent of any constraints on the Riemannian geometry.

**Proposition 4.2** The following conditions are equivalent:

I. \( H \) is Markov unique,

II. \( \mathcal{C}_A \) is satisfied for each subset \( A \) of \( \overline{\Omega} \) with finite capacity.

**Proof** \( \text{I} \rightarrow \text{II} \) If \( A \subseteq \overline{\Omega} \) is a set of finite capacity there exists an \( \eta \in D(h_N) \) with \( \eta = 1 \) on \( A \). But \( h_N = h_D \), by Markov uniqueness. Therefore \( \eta \in D(h_D) \). Then the constant sequence \( \eta_n = \eta \) satisfies \( \mathcal{C}_A \).

\( \text{II} \rightarrow \text{I} \) It suffices to prove that each \( \varphi \in D(h_N) \) can be approximated in the \( D(h_N) \)-graph norm by a sequence \( \varphi_n \in D(h_D) \cap L_\infty(\Omega) \). But \( (D(h_N) \cap L_\infty(\Omega))_{\text{cap}} \) is a core of \( h_N \), by Proposition 4.1. Therefore one may assume that \( \varphi \in (D(h_N) \cap L_\infty(\Omega))_{\text{cap}} \). Set \( A = \text{supp} \varphi \) and let \( \eta_n \in D(h_D) \cap L_\infty(\Omega) \) be the corresponding \( \mathcal{C}_A \)-sequence. Then let \( \varphi_n = \eta_n \varphi \). It follows from Proposition 2.2 that \( \varphi_n \in D(h_D) \cap L_\infty(\Omega) \). But

\[
\lim_{n \to \infty} \| \varphi - \varphi_n \|_2 = \lim_{n \to \infty} \| \mathbb{1}_A(\mathbb{1}_\Omega - \eta_n) \varphi \|_2 = 0.
\]

In addition \( \nabla(\varphi_n - \varphi) = (\nabla \eta_n) \varphi + (1 - \eta_n)(\nabla \varphi) \). Therefore

\[
\Gamma(\varphi_n - \varphi) \leq 2 \Gamma(\eta_n) \varphi^2 + 2(1 - \eta_n)^2 \Gamma(\varphi).
\]

Then since \( \text{supp} \varphi_n \subseteq \text{supp} \varphi \) it follows that

\[
h_N(\varphi - \varphi_n) = \| \mathbb{1}_A \Gamma(\varphi_n - \varphi) \|_1 \leq 2 \| \mathbb{1}_A \Gamma(\eta_n) \|_1 \| \varphi \|_\infty^2 + 2 \| \mathbb{1}_A(\mathbb{1}_\Omega - \eta_n) \chi \|_2^2
\]

where \( \chi = \Gamma(\varphi)^{1/2} \in L_2(\Omega) \). Therefore \( h_N(\varphi - \varphi_n) \to 0 \) as \( n \to \infty \). This establishes that \( D(h_D) \cap L_\infty(\Omega) \) is a core of \( h_N \). Hence \( h_D = h_N \) and \( H \) is Markov unique. \( \square \)

Next we discuss improvements to the foregoing results with two additional assumptions. First we assume the Riemannian balls \( B(r) \) are bounded for all \( r > 0 \). This immediately gives an improved version of Proposition 4.4.

**Proposition 4.3** Assume \( B(r) \) is bounded for all \( r > 0 \). Then \( (D(h_N) \cap L_\infty(\Omega))_c \), the subspace of bounded functions in \( D(h_N) \) with compact support, is a core of \( h_N \).

**Proof** The proof is essentially identical to the proof of Lemma 2.3 in [ORT91]. First, the \( B(r) \) are bounded if and only if \( \rho(x) \to \infty \) as \( x \to \infty \) where \( \rho \) is again the Riemannian distance from the origin. Secondly, let \( \tau \in C_c^\infty(\mathbb{R}) \) satisfy \( 0 \leq \tau \leq 1 \), \( \tau(s) = 1 \) if \( s \in [0,1] \), \( \tau(x) = 0 \) if \( s \geq 2 \) and \( |\tau'| \leq 2 \). Then set \( \tau_n = \tau \circ (n^{-1} \rho) \). It follows that \( \tau_n \) has compact support. Moreover, \( \tau_n(x) \to 1 \) as \( n \to \infty \) for all \( x \in \Omega \). But \( \Gamma(\rho) \leq 1 \). So one also has \( \| \Gamma(\tau_n) \|_\infty \leq 4n^{-2} \). Thirdly, if \( \varphi \in D(h_N) \cap L_\infty(\Omega) \) and \( \varphi_n = \tau_n \varphi \) then \( \varphi_n \in (D(h_N) \cap L_\infty(\Omega))_c \) by Proposition 2.2. But

\[
\| \varphi_n - \varphi \|^2_{D(h_N)} \leq 2 \int_\Omega \Gamma(\tau_n) \varphi^2 + 2 \int_\Omega (\mathbb{1}_\Omega - \tau_n)^2 \Gamma(\varphi) + \int_\Omega (\mathbb{1}_\Omega - \tau_n)^2 \varphi^2
\]
and all three terms on the right converge to zero as $n \to \infty$ by the dominated convergence theorem. Therefore $\varphi \in D(h_N) \cap L_\infty(\Omega)$ is the limit of the $\varphi_n \in (D(h_N) \cap L_\infty(\Omega))_c$ with respect to the $D(h_N)$-graph norm. Since $D(h_N) \cap L_\infty(\Omega)$ is a core of $h_N$ it follows that $(D(h_N) \cap L_\infty(\Omega))_c$ is also a core. \hfill $\square$

Secondly we assume that the $c_{kl} \in W^{1,\infty}_{\text{loc}}(\overline{\Omega})$, the space of restrictions to $\Omega$ of functions in $W^{1,\infty}_{\text{loc}}(\mathbb{R}^d)$. This ensures that the coefficients extend by continuity to functions which are uniformly locally bounded on $\overline{\Omega}$. Therefore each bounded subset $A$ of $\overline{\Omega}$ has finite capacity. This is again a consequence of Urysohn’s lemma. Now one can establish an improved version of Proposition 4.1.

**Theorem 4.4** Assume $c_{kl} \in W^{1,\infty}_{\text{loc}}(\overline{\Omega})$. Consider the following conditions:

I. $H$ is Markov unique,

II. $\mathcal{C}_A$ is satisfied for each bounded subset $A$ of $\overline{\Omega}$.

Then [I] $\Rightarrow$ [II]. Moreover, if $B(r)$ is bounded for all $r > 0$ then [II] $\Rightarrow$ [I] and the conditions are equivalent.

**Proof** [I] $\Rightarrow$ [II] If $A \subseteq \Omega$ is bounded then there is an $\eta \in C_c^\infty(\mathbb{R}^d)$ with $\eta = 1$ on $A$. Since the $c_{kl} \in W^{1,\infty}_{\text{loc}}(\overline{\Omega})$ it follows that $\eta$, or more precisely the restriction of $\eta$ to $\Omega$, is in $D(h_N)$. But $h_N = h_D$, by Markov uniqueness. Therefore $\eta \in D(h_D)$ and the constant sequence $\eta_n = \eta$ satisfies Condition $\mathcal{C}_A$.

This establishes the first statement in Theorem 4.4. Next we assume that the balls $B(r)$ are bounded and consider the converse reasoning.

[II] $\Rightarrow$ [I] It is necessary to prove that $D(h_N) = D(h_D)$. But since the Riemannian balls $B(r)$ are bounded $(D(h_N) \cap L_\infty(\Omega))_c$ is a core of $h_N$ by Proposition 4.3. Therefore it suffices to prove that each $\varphi \in (D(h_N) \cap L_\infty(\Omega))_c$ can be approximated in the $D(h_N)$-graph norm by a sequence $\varphi_n \in D(h_D) \cap L_\infty(\Omega)$.

Let $A = \text{supp } \varphi$. If $\eta_n \in D(h_D)$ is the $\mathcal{C}_A$-sequence corresponding to the bounded set $A$ define $\varphi_n$ by $\varphi_n = \eta_n \varphi$. Since we may assume $\eta_n \in D(h_D) \cap L_\infty(\Omega)$ it follows that $\varphi_n \in D(h_D) \cap L_\infty(\Omega)$ by Proposition 4.2. But then the argument used to prove [II] $\Rightarrow$ [I] in Proposition 4.2 establishes that $\varphi_n$ converges to $\varphi$ in the $D(h_N)$-graph norm. Therefore $D(h_D) \cap L_\infty(\Omega)$ is a core of $h_N$. Hence $h_D = h_N$ and $H$ is Markov unique. \hfill $\square$

The assumption that the balls $B(r)$ are bounded is essential for the implication [II] $\Rightarrow$ [I] in Theorem 4.4. Maz’ya has constructed an example for $\Omega = \mathbb{R}^d$ (see, [Maz85], Theorem 3 in Section 2.7) in which the coefficients grow rapidly in a set with an infinitely extended cusp. The growth is such that the Riemannian distance to infinity along the axis of the cusp is finite and consequently the balls $B(r)$ are not bounded for all sufficiently large $r$. In this example Condition [II] is satisfied but $h_D \neq h_N$, i.e. Condition [I] is false.

Condition [II] of Theorem 4.4 is related to the boundary capacity condition established in Theorem 1.2 in [RS11a] as a characterization of Markov uniques. We conclude this section with a brief discussion of the relationship. The capacity of a general subset $A$ of $\overline{\Omega}$ is defined by

$$\text{cap}(A) = \inf \left\{ \| \psi \|^2_{D(h_N)} : \psi \in D(h_N), \ 0 \leq \psi \leq 1 \text{ and there exists an open set } U \subseteq \mathbb{R}^d \text{ such that } U \supseteq A \text{ and } \psi = 1 \text{ on } U \cap \Omega \right\}$$
with the convention that $\text{cap}(A) = \infty$ if the infimum is over the empty set. (This definition is analogous to the canonical definition of the capacity associated with a Dirichlet form [BH91] [FOT94] and if $\Omega = \mathbb{R}^d$ the two definitions coincide.) A slight extension of the arguments of [RS11a] then gives the following characterization of Markov uniqueness.

**Proposition 4.5** Assume $c_{kl} \in W^{1,\infty}_{\text{loc}}(\overline{\Omega})$. Consider the following conditions:

I. $H$ is Markov unique,

II. $\text{cap}(\partial \Omega) = 0$.

Then $\text{I} \Rightarrow \text{II}$. Moreover, if the balls $B(r)$ are bounded for all $r > 0$ then $\text{II} \Rightarrow \text{I}$ and the conditions are equivalent.

**Proof** $\text{I} \Rightarrow \text{II}$ It follows from the general properties of the capacity that if $A = \bigcup_{k=1}^{\infty} A_k$ then $\text{cap}(A) \leq \sum_{k=1}^{\infty} \text{cap}(A_k)$. Therefore it suffices to prove that $\text{cap}(B) = 0$ for each bounded $B \subset \partial \Omega$. But $\text{cap}(B) < \infty$, because $c_{kl} \in W^{1,\infty}_{\text{loc}}(\overline{\Omega})$. Therefore there is an open subset $U$ of $\mathbb{R}^d$ containing $B$ and a $\psi \in D(h_N)$ with $\psi = 1$ on $U \cap \Omega$. Then by Markov uniqueness one can find $\varphi_n \in (D(h_D) \cap L_{\infty}(\Omega))$, such that $\|\psi - \varphi_n\|_{D(h_N)} \to 0$ as $n \to \infty$. Therefore there are open subsets $U_n$ of $\mathbb{R}^d$ containing $B$ with $\psi - \varphi_n = 1$ on $U_n \cap \Omega$. Then $\varphi_n = 0 \lor (\psi - \varphi_n) \land 1 \in D(h_N)$, because $h_N$ is a Dirichlet form, $0 \leq \varphi_n \leq 1$, $\varphi_n = 1$ on $U_n \cap \Omega$ and $\|\varphi_n\|_{D(h_N)} \leq \|\psi - \varphi_n\|_{D(h_N)} \to 0$ as $n \to \infty$ again by the Dirichlet property. Thus $\text{cap}(B) = 0$.

$\text{II} \Rightarrow \text{I}$ Assume the balls $B(r)$ are bounded. Then $(D(h_N) \cap L_{\infty}(\Omega))_c$ is a core of $h_N$ by Proposition 4.3. Therefore it suffices to prove that each $\varphi \in (D(h_N) \cap L_{\infty}(\Omega))_c$ can be approximated in the $D(h_N)$-norm by a sequence $\varphi_n \in (D(h_D) \cap L_{\infty}(\Omega))_c$. If $A = (\text{supp} \varphi) \cap \partial \Omega$ then $\text{cap}(A) = 0$ and one may choose $\eta_n \in (D(h_N) \cap L_{\infty}(\Omega))$ and open sets $U_n \subset \mathbb{R}^d$ such that $A \subset U_n$, $0 \leq \eta_n \leq 1$, $\eta_n = 1$ on $U_n \cap \Omega$ and $\|\eta_n\|_{D(h_N)} \to 0$ as $n \to \infty$. Then set $\varphi_n = (1 - \eta_n) \varphi$. It follows that $\varphi_n \in (D(h_D) \cap L_{\infty}(\Omega))_c$. Moreover, by estimates similar to those used to prove Proposition 2.2 one deduces that $\|\varphi_n\|_{D(h_N)} \to 0$ as $n \to \infty$. Hence $H$ is Markov unique.

**Corollary 4.6** Assume $c_{kl} \in W^{1,\infty}_{\text{loc}}(\overline{\Omega})$ and that the balls $B(r)$ are bounded for all $r > 0$. Then the following conditions are equivalent:

I. $C_A$ is satisfied for each bounded subset $A$ of $\overline{\Omega}$.

II. $\text{cap}(\partial \Omega) = 0$.

**Proof** It follows from Theorem 4.4 that Condition I is equivalent to Markov uniqueness of $H$ and it follows from Proposition 4.3 that Markov uniqueness of $H$ is equivalent to Condition II.

The proof of the corollary is indirect but if $\Omega$ is bounded then there is a simple direct proof which shows that the two conditions of the corollary are complementary. Condition II is valid for bounded $\Omega$ if it is valid for $A = \overline{\Omega}$, i.e. the condition is equivalent to the existence of $\eta_n \in D(h_D)$ such that $\lim_{n \to \infty} h_D(\eta_n) = 0$ and $\lim_{n \to \infty} \|\eta_n\|_{D(h_N)} = 0$. Then, however, $\psi_n = 1 - \eta_n \in D(h_N)$, $\psi_n = 1$ near $\partial \Omega$ and $\|\psi_n\|_{D(h_N)} \to 0$ as $n \to \infty$. Thus $\text{cap}(\partial \Omega) = 0$. Conversely if $\text{cap}(\partial \Omega) = 0$ then there exist $\psi_n \in D(h_N)$ with $\psi_n = 1$ near $\partial \Omega$ such that $\|\psi_n\|_{D(h_N)} \to 0$ as $n \to \infty$. Then setting $\eta_n = (1 - \psi_n)\psi_n$ one has $\eta_n \in D(h_D)$ and these functions satisfy Condition II of the corollary.
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