WHEN SCHREIER TRANSVERSALS GROW WILD

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1 Introduction

Schreier formula for the rank of a subgroup of finite index of a finitely generated free group $F$ is generalized to an arbitrary (even infinitely generated) subgroup $H$ through the Schreier transversals of $H$ in $F$. The rank formula may also be expressed in terms of the cogrowth of $H$.

We introduce the rank-growth function $rk_H(i)$ of a subgroup $H$ of a finitely generated free group $F$. $rk_H(i)$ is defined to be the rank of the subgroup of $H$ generated by elements of length less than or equal to $i$ (with respect to the generators of $F$), and it equals the rank of the fundamental group of the subgraph of the cosets graph of $H$, which consists of the paths starting at 1 that are of length $\leq i$. When $H$ is supnormal, i.e. contains a non-trivial normal subgroup of $F$, we show that its rank-growth is equivalent to the cogrowth of $H$. A special case of this is the known result that a supnormal subgroup of $F$ is of finite index if and only if it is finitely generated. In particular, when $H$ is normal then the growth of the group $G = F/H$ is equivalent to the rank-growth of $H$.

A Schreier transversal forms a spanning tree of the cosets graph of $H$, and thus its topological structure is of a contractible spanning subcomplex of a simplicial complex. The $d$-dimensional simplicial complexes that contain contractible spanning subcomplexes have the homotopy type of a bouquet of $r$ $d$-spheres. When these complexes are also $n$-regular then $r$ can be computed by generalizing the rank formula (which applies to Schreier transversals) to higher dimensions.
Let us remark that part of the results here apply in a similar form also to Schreier transversals and Schreier bases of right ideals in free group algebras (see [9], [11], [10]).

2 Generalized Schreier Formula

Let $H$ be a non-trivial subgroup of a free group $F$. By the Nielsen-Schreier Theorem $H$ is free too (see, for example, [8]), and explicit free generators for it can be given. Suppose that $F$ is freely generated on a set $X$ (not necessarily finite). The Cayley graph of $F$ (with respect to $X$) has the form of a tree and is the universal covering of a space $Q$ which is a bouquet of $|X|$ loops ($|.|$ denotes cardinality throughout the paper). The covering space of $Q$ with regard to $H$ is the cosets graph $G$ of $H$, and it is obtained as the quotient of the Cayley graph of $F$ under the left action of $H$. Thus $H$ is the fundamental group of $G$. The set of vertices of $G$ is the set of right cosets of $H$ in $F$. A (double) edge which is labeled with $x \in X$ goes in the direction from the coset $Ht_1$ to the coset $Ht_2$ if and only if $Ht_2 = Ht_1 x$, and it is labeled with $x^{-1}$ in the direction from $Ht_2$ to $Ht_1$. This gives a connected graph with $|X \cup X^{-1}|$ edges at each vertex. It is more convenient to label the vertices of $G$ with specific coset representatives in the following way. Let $T$ be a spanning tree of $G$. The identity element 1 is chosen to represent the coset $H$ and defined to be the root of $T$, and each other vertex is labeled with the group element one gets by reading off the edge labels in a path in $T$ that starts at the root and ends at the given vertex. We also denote by $T$ the set of (the labels of) the vertices $V(T)$ of the tree $T$, that is the coset representatives of $H$. This set is a Schreier transversal for $H$ in $F$, which is characterized by the property that every initial segment of an element of $T$ is also in $T$. For each $1 \neq w \in H$ there exist $u, v \in T$ of maximal lengths such that $u$ is a prefix of $w$ and $v$ is a prefix of $w^{-1}$. Since $t_1 t_2^{-1} \notin H$ for every $t_1 \neq t_2$ in $T$, then $l(u) + l(v) < l(w)$, where $l$ denotes the length of the (reduced) element in $F$. The Schreier generators for $H$ with respect to $T$ are those $w \in H$ for which

$$l(u) + l(v) = l(w) - 1.$$  \hspace{1cm} (1)

Moreover, if $\phi$ is the coset map associated with $T$ then $H$ is freely generated by the non-trivial elements

$$tx(\phi(tx))^{-1},$$  \hspace{1cm} (2)

where $t$ ranges over $T$ and $x$ ranges over $X$ (see [9]). This set is called a Schreier basis for $H$. Since $tx = \phi(tx)$ only when $tx \in T$ then by [9] the rank of $H$ equals...
the cyclomatic number of $\mathcal{G}$, the cardinality of the “missing” edges in the directions of $X$ in $\mathcal{T}$, that is
\[
\text{rank}(H) = |\{e \in E(\mathcal{G}) - E(\mathcal{T})\}|	ag{3}
\]
where $E(\mathcal{G})$, $E(\mathcal{T})$ denote the set of edges of $\mathcal{G}$, $\mathcal{T}$ respectively. This is because each edge is labeled with some $x \in X$ in exactly one direction, and thus counted exactly once.

Suppose now that $F$ is finitely generated with $\text{rank}(F) = n$, and $H$ is of finite index $m$ in $F$. Then $|E(\mathcal{G})| = nm$ and $|E(\mathcal{T})| = m - 1$ (since $\mathcal{T}$ is a tree). By (3) we get that
\[
\text{rank}(H) = 1 + (n - 1)m.	ag{4}
\]
This is Schreier Formula (see [8]). When $H$ is not necessarily of finite index in $F$ and also not necessarily finitely generated, we give in the proposition below a formula that generalizes the above one. The rank is computed on a Schreier transversal, and the simpler form of the formula is given in Corollary 2.2 which expresses the rank in terms of the cogrowth (see below) of the subgroup. The common way of computing the rank of the subgroup as a limit of the ranks of the fundamental groups (the cyclomatic numbers) of finite subgraphs deals with counting edges. Whereas, what we are doing here is counting only vertices.

We use the following terminology and notation on graphs. A path in a graph $\mathcal{G}$ is a sequence $v_0, e_1, v_1, e_2, \ldots, v_i \in V(\mathcal{G})$, $e_i \in E(\mathcal{G})$, such that $e_i$ starts at the vertex $v_{i-1}$ and terminates at $v_i$. The length of a path $v_0, e_1, v_1, e_2, \ldots , v_n$ is $n$. A simple path is a path in which the vertices along it are distinct, except possibly for the first and last one, in which case it is a simple closed path or a simple circuit. We assume that each path is reduced, i.e. it is not homotopic to a shorter one when the initial and terminal vertices are kept fixed.

If $\mathcal{H} \subseteq \mathcal{G}$, i.e. $\mathcal{H}$ is a collection of vertices and edges of the graph $\mathcal{G}$, then we denote by $< \mathcal{H} >$ the subgraph generated by $\mathcal{H}$. It is the smallest subgraph of $\mathcal{G}$ which contains $\mathcal{H}$. That is, we add to $\mathcal{H}$ the endpoint vertices of all the edges in $\mathcal{H}$. On the other hand, the subgraph of $\mathcal{G}$ induced by $\mathcal{H}$ is the one whose vertices are those of $\mathcal{H}$ and whose edges are all the edges which join these vertices in $\mathcal{G}$. An induced subgraph is a subgraph which is induced by some $\mathcal{H} \subseteq \mathcal{G}$. If $\mathcal{H}_1, \mathcal{H}_2 \subseteq \mathcal{G}$ then $\mathcal{H}_1 - \mathcal{H}_2$ is the collection of vertices $V(\mathcal{H}_1) - V(\mathcal{H}_2)$ and edges $E(\mathcal{H}_1) - E(\mathcal{H}_2)$, and it does not necessarily form a subgraph of $\mathcal{G}$, even when $\mathcal{H}_1$ and $\mathcal{H}_2$ are subgraphs of $\mathcal{G}$. The boundary of the subgraph $\mathcal{H}$ of $\mathcal{G}$ is $\partial \mathcal{H} = \mathcal{H} \cap < \mathcal{G} - \mathcal{H} >$, and its interior is $\mathcal{H} = \mathcal{G} - \partial \mathcal{H}$. The outer boundary of $\mathcal{H}$ (in $\mathcal{G}$) is the set of vertices of $\mathcal{G} - \mathcal{H}$ which are adjacent to $\mathcal{H}$ in $\mathcal{G}$. Assume now that each edge of $\mathcal{G}$ is labeled with some $x \in X$ in one direction and with
for each $j$ this set equals the rank of $E$ possibly renaming the components of each $T$ for each $i$ and its edges are $E(H) = \bigcup_{i \geq 1} \bigcap_{j \geq 1} E(H_j)$, and its edges are $E(H) = \bigcup_{i \geq 1} \bigcap_{j \geq 1} E(H_j)$.

Finally, let $\alpha(H) = |\pi_0(H)|$ be the cardinality of the (connected) components of $H$.

**Proposition 2.1** Let $F$ be a free group of rank $n$ and let $H < F$. Let $T$ be a Schreier transversal for $H$ in $F$ and let $T_i$ be finite subgraphs of $T$ such that $T = \liminf T_i$. Then

$$\text{rank}(H) = \lim_{i \to \infty} \left( \alpha(T_i) + (n - 1)|V(T_i)| - \frac{1}{2} \sum_{j=1}^{\alpha(T_i)} |V(\partial_{out} T_{i,j})| \right), \quad (5)$$

where, for a fixed $i$, $\partial_{out} T_{i,j}$ is the outer boundary (in $T$) of the component $T_{i,j}$ of $T_i$, for $j = 1, \ldots, \alpha(T_i)$.

**Proof.** If $H$ is of finite index $m$ in $F$ then there exists $i_0$ such that $T_i = T$ for every $i \geq i_0$, and then $\alpha(T_i) = 1$, $|V(T_i)| = m$ and $|V(\partial_{out} T_i)| = 0$. Thus (5) reduces to Schreier Formula.

Assume that $H$ is finitely generated but of infinite index. Denote as before by $G$ the cosets graph of $H$, which contains the Schreier transversal tree $T$. Let $C(G)$ be the core of $G$ (see [14]), that is the minimal deformation retract of $G$. It is the minimal connected subgraph of $G$ which contains all its simple circuits. Since $H$ is finitely generated $C(G)$ is finite, and there exists $i_0$ such that, after possibly renaming the components of each $T_i$, $V(C(G))$ is contained in $V(T_{i,1})$ for each $i \geq i_0$. Let us denote by $G_{i,j}$ the subgraph of $G$ induced by $T_{i,j}$. Then $E(G) = E(T) = E(G_{i,1}) = E(T_{i,1})$, for each $i \geq i_0$, and by (5) the cardinality of this set equals the rank of $H$. Hence it suffices to show that for each $i \geq i_0$ and for each $j$

$$|E(G_{i,j}) - E(T_{i,j})| = |E(G_{i,j}) - |E(T_{i,j})|$$

$$= 1 + (n - 1)|V(T_{i,j})| - \frac{1}{2}|V(\partial_{out} T_{i,j})|. \quad (7)$$

So assume $i \geq i_0$. Then $|E_{out}(G_{i,j})| = |E_{out}(T_{i,j})| = |V(\partial_{out} T_{i,j})|/2$ for each $j$, since all simple circuits of $G$ are in $G_{i,1}$. Each vertex in $G$ is the initial vertex of exactly $n$ edges in the directions $X$. Therefore

$$|E(G_{i,j})| = n|V(G_{i,j})| - |E_{out}(G_{i,j})|. \quad (8)$$
As for $T_{i,j}$, since it is a tree then
\[|E(T_{i,j})| = |V(T_{i,j})| - 1. \] (9)
Substituting in (8) gives (9).

Assume now that $H$ is not finitely generated. Then, because in general
\[|E_{\text{out}}(G_{i,j})| \geq |V(\partial_{\text{out}}T_{i,j})|/2, \] (10)
we get that for each $i, j$
\[|E(G_{i,j}) - E(T_{i,j})| \leq 1 + (n - 1)|V(T_{i,j})| - 1/2|V(\partial_{\text{out}}T_{i,j})|. \] (11)
Since $\text{rank}(H) = \lim_{i \to \infty} \left( \sum_{j=1}^{\alpha(T_i)} |E(G_{i,j}) - E(T_{i,j})| \right) = \infty$, equation (9) follows. \(\square\)

We remark that instead of taking finite subgraphs $T_i$ such that $T = \lim \inf T_i$, the rank formula can be clearly given as the supremum, over all finite subgraphs of $T$, of the expression appearing in (8).

A special case of Proposition 2.1 is when each component $T_{i,j}$ is a ball, that is its vertices are all the vertices of $T$ which lie at distance not greater than some fixed $k$ from some fixed vertex. If $\mathcal{H}$ is a subgraph of $\mathcal{G}$ and $|V(\mathcal{G})| > 1$ then we define $\delta(\mathcal{H})$ to be the number of components of $\mathcal{H}$ which consist of a single vertex, i.e. balls of radius 0. When $|V(\mathcal{G})| = 1$ then $\delta(\mathcal{G})$ is defined to be 0.

When the $T_i$ are concentric balls centered at the identity 1 then the values $|V(T_i)|, i = 0, 1, 2, \ldots$ relate to the growth function $\Gamma_\tau$ of $T$, as is defined below. By $l(g)$ we denote the length of $g \in F$, and we always assume that the group elements are written in reduced form with respect to the generating set $X$ of $F$. Then define

\[\gamma_\tau(i) = |\{v \in T \mid l(v) = i\}|, \] (12)
\[\Gamma_\tau(i) = |\{v \in T \mid l(v) \leq i\}|. \] (13)

When $T$ is a minimal Schreier transversal tree, that is when it has also the property that every coset of $H$ is represented by an element of minimal length, then $\Gamma_\tau(i)$ is the cogrowth function of $H$, relative to the generating set of $F$, and is denoted by $\Gamma_{F/H}(i)$ (see [12]). We may look at $\Gamma_{F/H}(i)$ as representing the “volume” of the ball of radius $i$ with center 1 in the cosets graph of $H$ (with the metric induced by the word metric on $F$). If, in addition, $H$ is a normal subgroup of $F$ then the cogrowth function of $H$ equals the growth function of the group $F/H$, relative to the the generating set which is the canonical image of the generating set of $F$. (In this case the Schreier transversal for $H$ which is minimal with regard to a fixed ShortLex order on $F$ is also suffix-closed.)
Corollary 2.2 Let $F$ be a free group of rank $n$, let $H$ be a subgroup of $F$ and let $\mathcal{T}$ be a Schreier transversal for $H$ in $F$. Let $\mathcal{T}_i$ be induced finite subgraphs of $\mathcal{T}$, whose components $\mathcal{T}_{i,j}$ are balls, such that $\mathcal{T} = \liminf \mathcal{T}_i$. Then

$$\text{rank}(H) = \lim_{i \to \infty} \left( \alpha(\mathcal{T}_i) - \frac{1}{2} \delta(\mathcal{T}_i) + (n - 1)|V(\mathcal{T}_i)| - \frac{2n - 1}{2} |V(\partial \mathcal{T}_i)| \right)$$

(14)

$$= \lim_{i \to \infty} \left( \alpha(\mathcal{T}_i) - \frac{1}{2} \delta(\mathcal{T}_i) + (n - 1)|V(\mathcal{T}_i)| - \frac{1}{2} |V(\partial \mathcal{T}_i)| \right).$$

(15)

In particular,

$$\text{rank}(H) = 1 + \lim_{i \to \infty} \left( (n - 1)\Gamma_{\mathcal{T}}(i) - \frac{1}{2} \gamma_{\mathcal{T}}(i + 1) \right)$$

(16)

$$= 1 + \lim_{i \to \infty} \left( (n - 1)\Gamma_{F/H}(i) - \frac{1}{2} \gamma_{F/H}(i + 1) \right).$$

(17)

Proof. The corollary follows from the fact that when the core $C(\mathcal{G})$ is finite then for each $i$ large enough every vertex of $\partial \mathcal{T}_{i,j}$ is adjacent to $2n - 1$ vertices of $\mathcal{T} - \mathcal{T}_{i,j}$, unless $\mathcal{T}_{i,j}$ is a single vertex and then it is adjacent to $2n$ vertices of $\mathcal{T} - \mathcal{T}_{i,j}$. When $H$ is not finitely generated then we first notice that the expression we calculate for each ball is non-negative. Secondly, since $\mathcal{T} = \liminf \mathcal{T}_i$, then for every $r$ there exists $i_r$ such that, for every $i \geq i_r$, $\mathcal{T}_i$ has a component (ball) which contains the ball of radius $r$ around the identity. But the expression calculated on these balls tends to infinity whenever $H$ is of infinite rank, as shown below. This can also be concluded directly from Proposition 2.1. \hfill \square

3 Rank-growth

Given a Schreier transversal $\mathcal{T}$, let us define

$$r_\mathcal{T}(i) = 1 + (n - 1)\Gamma_{\mathcal{T}}(i) - \frac{1}{2} \gamma_{\mathcal{T}}(i + 1)$$

(18)

$$= 1 + \frac{2n - 1}{2} \Gamma_{\mathcal{T}}(i) - \frac{1}{2} \Gamma_{\mathcal{T}}(i + 1).$$

(19)

$r_\mathcal{T}(i)$ is an upper bound to the cyclomatic number of the subgraph of $\mathcal{G}$ which is induced by the vertices of $\mathcal{T}$ of distance at most $i$ from the root. In case $\mathcal{T}$ is a minimal Schreier transversal then $r_\mathcal{T}(i)$ is also denoted by $r_H(i)$:

$$r_H(i) = 1 + (n - 1)\Gamma_{F/H}(i) - \frac{1}{2} \gamma_{F/H}(i + 1).$$

(20)
The sequence \( r_T(i), i = 1, 2, \ldots \) is non-decreasing. This is because
\[
 r_T(i) - r_T(i - 1) = \frac{2n - 1}{2} \gamma_T(i) - \frac{1}{2} \gamma_T(i + 1),
\]
and each vertex of \( T \) of level \( i \) is adjacent to at most \( 2n - 1 \) vertices of level \( i + 1 \). Thus \( r_T(i) \) becomes eventually constant if and only if either \( T \) is finite, or for some \( i_0 \) each vertex of \( T \) of level \( i \geq i_0 \) has degree exactly \( 2n \), and this happens if and only if there are only finitely many edges in \( E(G) - E(T) \), or equivalently when \( H \) is finitely generated.

It is interesting to know also the rate in which the function \( r_T(i) \) grows. A preorder is defined on growth functions by
\[
 f_1(i) \preceq f_2(i) \iff \exists c > 0 \forall i \ [f_1(i) \leq cf_2(ci)].
\]
Then an equivalence relation is given by
\[
 f_1(i) \sim f_2(i) \iff f_1(i) \preceq f_2(i) \text{ and } f_2(i) \preceq f_1(i).
\]
(we refer to [6] for a survey on growth functions of groups and to Gromov’s [7] rich and beautiful geometric theory.) In Theorem 3.1 below we show that when the subgroup \( H \) of the free group \( F \) is supnormal, i.e. contains a non-trivial subgroup which is normal in \( F \), then for every Schreier transversal \( T \) of \( H \), its growth function \( \Gamma_T(i) \) is equivalent to the function \( r_T(i) \). This implies that the cogrowth of \( H \) is also equivalent to what we call the rank-growth of \( H \). We look at \( H \) as the direct limit of the subgroups
\[
 H_i = \langle \{ h \in H \mid l(h) \leq i \} \rangle,
\]
where \( l(h) \) is measured with respect to the generating set of \( F \). Then the rank-growth of \( H \) (with respect to the generators of \( F \)) is
\[
 rk_H(i) = \text{rank}(H_i).
\]
Clearly, if we choose another generating set for \( F \), we get an equivalent rank-growth function. Notice that \( H_i \) is the fundamental group of the subgraph of the cosets graph \( G \) of \( H \) which contains all paths starting at 1 of length \( \leq i \). Thus \( rk_H(i) \) is a non-decreasing function. If we define
\[
 \rho_H(i) = \text{rank}(\pi_1(B_i)),
\]
where \( B_i \) is (the induced subgraph which is) the ball of radius \( i \) centered at the vertex 1 of \( G \), then
\[
 \rho_H(i) = rk_H(2i + 1).
\]
Therefore $rk_H(i)$ and $\rho_H(i)$ are equivalent. Also $\rho_H(i) \sim r_H(i)$. In fact,
\begin{equation}
\rho_H(i) \leq r_H(i) \leq \rho_H(i + 1).
\end{equation}
More precisely,
\begin{equation}
r_H(i) = \rho_H(i) + \frac{1}{2}(|E_{\text{out}}^X(B_i)| - \gamma_{F/H}(i + 1)) \leq \rho_H(i + 1).
\end{equation}

**Theorem 3.1** Let $H$ be a supnormal subgroup of a finitely generated free group $F$, and let $T$ be a Schreier transversal for $H$ in $F$. Then
\begin{equation}
r_T(i) \sim \Gamma_T(i).
\end{equation}
In fact, if $H$ is not necessarily supnormal but has the property that $|F : N_F(A)| < \infty$ for some non-trivial $A < H$ then
\begin{equation}
rk_H(i) \sim r_H(i) \sim \Gamma_{F/H}(i).
\end{equation}

**Proof.** For every Schreier transversal of a subgroup of $F$ we have $r_T(i) \leq \Gamma_T(i)$. This follows immediately from the definition of $r_T(i)$ - see (19).

Suppose now that $H$ is supnormal. Let $h$ be a non-trivial element of a subgroup of $H$ which is normal in $F$, and let $m = l(h)$ (as usual, the length is with respect to the generators of $F$). Then at every vertex $v$ of the cosets graph $G$ of $H$, if we follow the path defined by $h$ we form a circuit. Therefore at every vertex of $T$ of level at most $i$, by following the path defined by $h$ we reach a vertex of $T$ of level at most $i + m$ where we must stop because the next edge is missing. The number of these missing edges is less then or equal to $r_T(i + m)$. Since at most $m$ vertices are the starting point of a tour defined by $h$ which reaches the same missing edge $h$ then
\begin{equation}
\Gamma_T(i) \leq mr_T(i + m).
\end{equation}
By the two inequalities we have
\begin{equation}
r_T(i) \sim \Gamma_T(i).
\end{equation}
Applying this result to a minimal Schreier transversal yields
\begin{equation}
rk_H(i) \sim r_H(i) \sim \Gamma_{F/H}(i).
\end{equation}
The condition of \( H \) being supnormal can be weakened. It suffices to demand that \( H \) contains a non-trivial subgroup \( A \) such that \( |F : N_F(A)| < \infty \), because then the cogrowth of \( H \) is equivalent to the growth (with respect to the generators of \( F \)) of the minimal coset representatives of \( H \cap N_F(A) \) in \( N_F(A) \). Even more, we need only the growth (again, with respect to the generators of \( F \)) of \( \{ g \in T \mid gAg^{-1} \subseteq H \} \), where \( T \) is a minimal Schreier transversal for \( H \), to be equivalent to the cogrowth of \( H \) in \( F \).

Since \( \Gamma_T(i) \leq \Gamma_{F/H}(i) \) for every Schreier transversal \( T \) of a subgroup \( H \) of \( F \), then by Theorem 3.1 when \( H \) is supnormal in \( F \) then \( r_T(i) \leq r_{kH}(i) \). We also notice that a special case of Theorem 3.1 is the known result stating that a supnormal subgroup of a finitely generated group is of finite index if and only if it is finitely generated. And when \( H \) is normal in \( F \), then the growth \( \Gamma_G(i) \) of the group \( G = F/H \) is equivalent to the rank-growth of \( H \) and to the growth of

\[
r_H(i) = 1 + (n - 1)\gamma_G(i) - \frac{1}{2}\gamma_G(i + 1).
\]

The growth of the subgroup \( H \) is always exponential when it is of rank greater than 1, since it is free. But Grigorchuk showed (35) that when \( H \) is normal then its “growth exponent” \( \limsup_{i \to \infty} \Gamma_H^{(f)}(i)^{1/i} = 2n - 1 \), if and only if \( G = F/H \) is amenable, (in fact, Grigorchuk obtained more: a formula which connects the growth exponent of \( G \) with the spectral radius of a random walk on \( G \)), where \( n = \text{rank}(F) \) and \( \Gamma_H^{(f)}(i) \) represents the growth of \( H \) with respect to the generators of \( F \). (Recall that a group \( G \) is amenable if there exists an invariant mean on \( B(G) \), the space of all bounded complex-valued functions on \( G \) with the sup norm \( \| f \|_\infty \), see [4]). When \( G \) is non-amenable then the growth exponent of \( H \) is less than \( 2n - 1 \). But then the group \( G \) has exponential growth, and we have shown that in this case the rank-growth of \( H \) is also exponential, i.e. the maximal possible (up to equivalence). This seems at first sight contradictory. To illustrate this phenomenon we may think of a tree, called \( F \), that we prune its sides going from bottom upward. The number of branches we cut is called (half) the rank of \( H \), the tree that is left after the pruning is called \( G \), and (part of) what we cut is called \( H \). Then the further the cut is from the periphery and closer to the middle of the tree the larger \( H \) is, the smaller \( G \) is, and the rank of \( H \) also becomes smaller since we cut towards the main branches.

Although the rank of the subgroup of a free group can be expressed, as we have seen in Corollary 2.2, in terms of the growth function of any Schreier transversal of it, the growth function itself of one Schreier transversal of an infinitely generated subgroup may in general differ completely from that of another Schreier
transversal. This is shown in the next proposition.

**Proposition 3.2** There exists a subgroup of the free group of rank 2 with exponential cogrowth which has a Schreier transversal $T$ whose growth is $\Gamma_T(i) = i + 1$.

**Proof.** We will construct the cosets graph of such a subgroup inductively. Let $c$ be a positive integer which is large enough. First we make a simple circuit $\lambda_1$ of length $c$ that starts at the root 1. Then at the $n$-th step we construct a path $\lambda_n$ of length $2nc$, whose vertices, apart from the initial and terminal ones, are new. The initial vertex of $\lambda_n$ is the one before the last vertex in the path $\lambda_{n-1}$. The terminal vertex of $\lambda_n$ is chosen to be of minimal distance from the root among the vertices whose degree is less than 4.

If we delete the last edge of each path $\lambda_n$, then we get a linear Schreier transversal $T$, i.e. $\Gamma_T(n) = n + 1$. On the other hand, if we delete the middle edge of each $\lambda_n$, then we get a tree $T'$ with exponential growth, because each vertex of it has degree 4, except for a sequence of vertices $v_n$ of distances $\geq cn$ respectively from the root. Since the cogrowth function is greater than or equal to the growth function of any Schreier transversal of the subgroup, the result follows. \hfill \Box

It is shown in [12] that when $H = H_1 \cap H_2$ the cogrowth of $H$ satisfies

$$\Gamma_{F/H}(i) \leq \Gamma_{F/H_1}(i)\Gamma_{F/H_2}(i) \quad \text{for every } i.$$  \hfill (36)

The rank-growth of the intersection of two subgroups behaves similarly.

**Proposition 3.3** Let $H_1, H_2$ be non-trivial subgroups of a finitely generated free group $F$ and let $H = H_1 \cap H_2$. Then

$$rk_H(i) \leq 1 + 2(rk_{H_1}(i) - 1)(rk_{H_2}(i) - 1) - \min(rk_{H_1}(i), rk_{H_2}(i))$$ \hfill (37)

for every $i$. Hence

$$rk_H(i) \preceq rk_{H_1}(i)rk_{H_2}(i).$$ \hfill (38)

**Proof.** This follows immediately from the best general bound for the intersection of finitely generated subgroups in free groups, which is due to Burns ([2]). \hfill \Box
4 The Generalized Word Problem

Given a subgroup \( H \) of a group \( G \) it is interesting to know the *distorsion* of \( H \) with respect to \( G \), that is a bound \( f(i) \) of the length, with respect to a finite set of generators of \( H \), of an element of \( H \) whose length is \( i \) with respect to a finite set of generators of \( G \) (see [7], [3]). When \( F \) is free then it is known that every element of a subgroup \( H \) of it, whose length is \( i \) in \( F \), has length at most \( i \) with respect to a Schreier basis of \( H \) (or a Nielsen-reduced basis, which is no other than a minimal Schreier basis), thus the distorsion is linear. A bit more precise description is obtained by using \( d(w, T) \), the distance of \( w \in F \) from a Schreier transversal \( T \). That is

\[ d(w, T) = \min \{ l(t^{-1}w) \mid t \in T \}, \] (39)

where \( l \) denotes the length in \( F \). Notice that \( d(w, T) \leq l(w) \) since \( 1 \in T \). Then if \( B_T \) is a corresponding Schreier basis for the subgroup \( H < F \) then every \( w \in F \) can be written in the form

\[ w = b_1^{\varepsilon_1} \cdots b_k^{\varepsilon_k} \bar{w}, \] (40)

with \( b_j \in B_T \), \( \varepsilon_j = \pm 1 \) and \( \bar{w} \in T \), such that \( k \leq d(w, T) \). To see it, let \( t \in T \) be the maximal prefix of \( w \) in \( T \), i.e. \( l(t^{-1}w) = d(w, T) \). If \( t = w \) we are done. Otherwise, there exists \( x \in X \cup X^{-1}, X \) the generating set of \( F \), such that \( b_1^{\varepsilon_1} = tx(\phi(tx))^{-1} \in B_T \cup B_T^{-1}, \phi \) the coset map, and such that \( w = txw' \) when written in reduced form. Thus

\[ w = b_1^{\varepsilon_1} \phi(tx)w'. \] (41)

But \( d(\phi(tx)w', T) \leq l(w') = d(w, T) - 1 \) and we proceed by induction.

The above shows that when we are given a Schreier transversal \( T \) and a corresponding Schreier basis \( B_T \) for \( H < F \) then it is possible to obtain algorithmically a “normal form” modulo \( H \) for every element of \( F \), i.e. its coset representative in \( T \), and this demonstrates the importance of Schreier generating sets (whose shape and role is similar to those of Gröbner bases for algebras, see [10]). Thus the generalized word problem for \( H \) in \( F \) is then solvable. In fact, whenever \( G = < X \mid R > \), and \( H \) is a subgroup of \( G \) generated by a set \( S \subseteq F = < X > \), then the generalized word problem for \( H \) in \( G \) is solvable when the underlying set of the subgroup \( < N, S > \) of \( F \), where \( N = < R >^F \) is the normal closure of \( R \) in \( F \), as well as a set \( T \) of coset representatives for \( H \) in \( G \), are recursively enumerable (r.e.) sets. For, by listing the elements of \( < N, S > \) and those of \( T \) we can list all elements of \( F \), and also find which coset is the coset \( < N, S > \) in \( F \). Hence, both the set \( < N, S > \) and its complement in \( F \) are r.e. and therefore \( < N, S > \) is recursive.
Proposition 4.1 Let $G = < X \mid R >$ be a finitely generated group and let $H$ be a subgroup of $G$ generated by a set $S$. Suppose also that the subgroup $A = < N, S >$ of the free group $F = < X >$, where $N = < R >^F$, is r.e. Then the generalized word problem for $H$ in $G$ is solvable whenever one of the functions $\Gamma_{F/A}(i), r_A(i)$ or $rk_A(i)$ is recursive.

Proof. First we remark that $A$ is r.e. for example when $R$ and $S$ are r.e. We construct inductively $B_i$, the concentric balls of radius $i$, of the cosets graph of $A$ in $F$. We start with the vertex 1. Then assuming that $B_i$ was constructed, we first extend it to level $i + 1$ without forming new circuits. If the number of vertices at level $i + 1$ agrees with $\Gamma_{F/A}(i + 1)$ or with $r_A(i + 1)$ or if $\pi_1(B_{i+1}) = rk_A(2i + 1)$ then we are done. Otherwise, by listing the elements of $A$, each defining a circuit in the cosets graph, we stop until we reach the desired values of our functions. □

5 Contractable Spanning Subcomplexes

If we look at Proposition 2.1 we see that it makes little use of the group structure. It is mainly a statement about $m$-regular graphs, i.e. graphs whose vertices have all the same degree $m$. We may then try to generalize this theorem from such graphs to special simplicial complexes. When $C$ is a simplicial complex then we denote by $|C|$ the topological space of $C$ (as in [13]), and whenever we relate some topological properties to $C$ they describe, in fact, those of $|C|$. We Call a subcomplex $D$ of a $d$-dimensional simplicial complex $C$ a spanning subcomplex if

(i) $D$ contains $C^{(d-1)}$, the $(d-1)$-skeleton of $C$;

(ii) every principal simplex of $D$ (i.e. a simplex which is not a face of another simplex of $D$ of higher dimension) is also principal in $C$.

Then the analogue of a spanning tree in graph theory is a spanning subcomplex $D$ whose topological space is contractible. We call such a subcomplex a contractible spanning subcomplex. In this case condition (ii) becomes redundant. This is because if $\sigma_1^{d-1}$ is principal in $D$ but is a face of a $d$-simplex $\sigma_d^2$ of $C$ then by considering the boundary of $|\sigma_2^2|$ in $|D|$ we get that $\pi_{d-1}(|D|)$ is not trivial and thus $D$ is not contractible. Or we can look at the homology of $D$, and see that $H_{d-1}(D)$ does not vanish since $\sigma_1^{d-1}$ does not appear in $B_{d-1}(D)$ but is a summand of a cycle.
We come now to the analogue of Proposition 2.1 for simplicial complexes. The formula we give, however, is not so nice as the one in the one-dimensional case, where all terms involve only the zero-dimensional skeleton. We use the following additional notation and definitions. Let \( C \) be a simplicial complex and let \( D \) be a subcomplex of it. The collection of \( k \)-simplices of \( C \) is denoted by \( F^k(C) \), and its cardinality is denoted by \( \beta_k(C) \). When \( X \) is a collection of simplices of \( C \) we denote by \( < X > \) the subcomplex generated by \( X \). If \( D_1, D_2 \) are subcomplexes of \( C \) then \( D_1 - D_2 \) is the collection of simplices \( D_1 - D_2 = \{ \sigma | \sigma \in D_1, \sigma \notin D_2 \} \), and it does not necessarily form a subcomplex. The boundary of the subcomplex \( D \) of \( C \) is \( \partial D = D \cap < C - D > \), and its interior is \( \hat{D} = D - \partial D \). If \( \sigma^k \) is a \( k \)-simplex of \( C \) we define its degree to be \( \deg(\sigma^k) = |\{ \sigma^{k+1} \in F^{k+1}(C) | \sigma^k \subset \sigma^{k+1} \}| \). If \( X \) is a collection of simplices of \( C \) then \( \deg(X) \) is the total degree of the members of \( X \).

Theorem 5.1 Let \( C \) be a countable \( d \)-dimensional simplicial complex which contains a contractible spanning subcomplex \( D \). Then \( |C| \) is homotopic to a bouquet of \( r \) \((r \text{ can be } \infty) \) \( d \)-spheres.

If, in addition, \( C \) is \( n \)-regular and \( D_i \) are finite subcomplexes of \( D \) such that \( D = \lim \inf D_i \) then

\[
r = \lim_{i \to \infty} \left( \frac{1}{d+1} (n\beta_{d-1}(D_i) - \deg_{<D-D_i>} (F^{d-1}(\partial D_i))) - \beta_d(D_i) \right).
\]

Proof. We define a contractible space to be homotopic to a bouquet of zero \( d \)-spheres. So assume that \(|C|\) is not contractible. Since \(|D|\) is contractible, \(|C|/|D|\) is homotopic to \(|C|/|D|\). If there are \( r \) \((r \text{ can be } \infty) \) \( d \)-simplices \( \sigma^d \) which are not in \( D \), then since the boundary of \(|\sigma^d|\) is in \(|D|\), we get that \(|C|\) is homotopic to a bouquet of \( r \) \( d \)-spheres (see also \[\cit\] for shellable complexes, where every shellable complex contains a contractible spanning subcomplex but not necessarily the other way round).

The second part of the proof is similar to that of Proposition 2.1. In fact we are only dealing with the simplices of \( C \) of dimensions \( d - 1 \) and \( d \), and then we compute the number of \( d \)-simplices of \( C - D \). \(\square\)
We remark that in case each subcomplex $D_i$ in (42) has contractible connected components then \( \beta_d(D_i) \) may be expressed in terms of the \( \beta_j(D_i), j = 0, \ldots, d - 1 \) using the (topological) Euler characteristic

\[
\chi(D_i^{(d-1)}) = \sum_{i=0}^{d-1} (-1)^i \beta_i(D_i^{(d-1)}).
\]

(43)

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