EXTREMAL MILD SOLUTIONS FOR HILFER FRACTIONAL EVOLUTION EQUATION WITH MIXED MONOTONE IMPULSIVE CONDITIONS

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Abstract. The well established mixed monotone iterative technique that is used to study the existence and uniqueness of fractional order system is studied explicitly for impulsive system with Hilfer fractional order in this paper. The procedure of finding mild $L$-quasi solution of such impulsive evolution equation with noncompact semigroups involves measure of non-compactness and Sadovskii’s fixed point theorem as well. An example is provided to illustrate the main results.

1. Literature Motivation

Over the years, the basic study of differential equation mainly leads to the finding of its extremal solution. Conditions on when a differential system will have a unique solution is always a challenge for the researchers. Various methods are used to investigate the uniqueness of solution for the desired differential system. Successive approximation or the iterative technique are the typical methods used to determine such unique solution. Du and Lakshmikanthan [9] constructed an iterative procedure for an initial value problem,

$$x' = g(t, x); \quad x(0) = x_0,$$

where the unique solution was guaranteed by upper and lower solution in a closed set. Thereafter, many articles emerged in this direction based on their paper. The case when the order is preserved with respect to the image of the function, then it is monotonicity property. Whereas, the case leading to the decomposition of the function into monotonically decreasing function and monotonically increasing function guided Guo and Lakshimikantham [12] to introduce the concept of coupled fixed point. Their work focussed on the existence criteria for both continuous and discontinuous operators defined as $A : D \times D \to E$ where $D$ is the subset of the Banach space $E$ which is partially ordered by a cone $N$ (details regarding cone can be referred to [13] by Guo and Lakshimikantham) and $A(x, y)$ is non-decreasing in $x$ and non-increasing in $y$ which were termed as mixed monotone. Mixed monotonicity property finds itself useful mainly in the convergence analysis, global stability analysis, qualitative analysis etc. In continuation, Guo [14] investigated the existence and uniqueness of mixed monotone operator for a general case where the operator need not be continuous. Chang and Ma [4] studied the existence of coupled fixed points of set valued operators defined as $A : D \times D \to 2^E$. As an application, the authors considered the functional equation

$$g(x) = \sup_{y \in D}[f(x, y) + F(x, y, g(T(x, y)))], \quad x \in S,$$

with $S$ the state space, $D$ the decision space, $R = (-\infty, \infty)$, $T : S \times D \to S$, $f : S \times D \to R$ and $F : S \times D \times R \to R$ emerged in dynamic programming. Similarly Sun and Liu [20] improved the existing results on conditions on operator, where their conditions don’t require the operator to be continuous as well as the cone $N$ to be normal. The authors implemented their conditions

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to nonlinear Hammerstein integral equation given by
\[ g(x) = \int_G k(x,y) f(y, g(y)) dy, \]
with \( G \) the bounded closed subset of \( \mathbb{R}^n \), \( k(x,y) : G \times G \rightarrow \mathbb{R}^1 \) an non negative operator and \( f(y, g(y)) = f_1(y, g(y)) + f_2(y, g(y)) \) with \( f_1(y, g(y)) \) is non-decreasing in \( x \) and \( f_2(y, g(y)) \) is non-increasing in \( y \). Mean while Chang and Guo [5] studied the existence and uniqueness for fractional order system.

The author gave the existence and uniqueness of fixed points of such convex-concave mixed monotone operators. Application to differential equations in Banach spaces and to nonlinear partial differential equation and generalised many previous results in this direction.

Chen and Li [8] considered a fractional nonlocal evolution system of the form
\[ \begin{cases} \frac{CD^q_t x(t)}{\alpha x(t)} + Ax(t) = g(t, x(t), x(t)), & t \in J = [0, \omega], \\ x(0) = f(x, x), \end{cases} \]
where \( CD^q_t \) is the Caputo derivative of order \( q \in (0,1) \); \( A : D(A) \subset E \rightarrow E \) a closed linear operator such that \(-A\) generates \( Q(t) \) a uniformly bounded \( C_0\)-semigroup; \( g \in C(J \times E \times E) \) and \( f(x, x) \) is a nonlocal function. Using operator semigroup theory, theory of mixed monotone operator and some perturbation method, the authors obtained the coupled minimal and maximal mild \( L\)-quasi solutions of (1.2).

It can be noted that, when \( L \equiv 0 \), then the coupled minimal and maximal mild \( L\)-quasi solutions are equivalent to minimal and maximal mild quasi solutions of
For semilinear evolution equation with impulsive conditions, Li and Gou [20] studied the existence and mild $L$-quasi solutions of the periodic boundary value problem given by

$$
\begin{cases}
C D_0^\gamma x(t) = g(t, x(t), x(t)), & t \in [0, \omega], \quad t \neq t_k, \\
\Delta x|_{t=t_k} = J_k(x(t_k), x(t_k)), & k = 1, 2, \ldots, l, \\
x(0) = x(\omega),
\end{cases}
$$

where the operator, derivative and the function $g$ are defined as in (1.2); $J_k$ an impulsive function; $0 = t_0 < t_1 < \ldots < t_l < \omega$; $\Delta x|_{t=t_k} = x(t_k^+)-x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ corresponds to right and left limit of $x(t)$ at $t = t_k$. Another notable work is the work of Zhao and Wang [30] on the existence of solution of impulsive fractional system with Hilfer order.

The existence of solution of Hilfer fractional derivative which acts as a interim between the two classical fractional derivative Caputo and Riemann-Liouville derivative which was outlined by Hilfer [18] was examined by Furati et al. [10]. Consequently, Guo and Li [16] utilized fixed point theorem together with monotone iterative technique to study the existence of extremal solution of Hilfer fractional nonlocal evolution equation. Recently, Guo et al. investigated the existence of mild $L$-quasi-solutions using fixed point theorem along with mixed monotone iterative technique of nonlocal Hilfer fractional evolution equation defined as

$$
\begin{cases}
D_{0+}^{\rho,q} x(t) + A x(t) = g(t, x(t), G(x(t))), & t \in [0, \omega], \\
I_{0+}^{(1-p)(1-q)} x(0) = x_0 + \sum_{k=1}^{l} \lambda_k \tau_k, & \tau_k \in [0, \omega],
\end{cases}
$$

where $D_{0+}^{\rho,q} x(t)$ denotes the Hilfer fractional derivative of order $q$ and type $p$ with $\frac{1}{p} < q < 1$ and $0 \leq p \leq 1$; the operator $G$ is defined as $G x(t) = \int_0^t K(t, s) x(s) ds$; $\tau_k$ are prefixed points with $k = 1, 2, \ldots, l$ satisfying $0 \leq \tau_1 \leq \tau_2 \leq \ldots \leq \tau_l < \omega$; $\lambda_k$ are real numbers. But many physical problems are impulsive in nature, it is ideal to study an impulsive system. It is challenging to work on the existence of solution of impulsive fractional system with Hilfer fractional order. One can refer for the articles on approximate controllability of impulsive system with Hilfer fractional derivative by Debbouche and Antonov [2], by Ahmed et al. [1] and by Du et al. [3]. Hence it is worth to study the existence of coupled mild $L$-quasi solution of Hilfer fractional impulsive system as below

$$
\begin{cases}
D_{0+}^{\mu,\nu} x(t) + A x(t) = g(t, x(t), x(t)), & t \in [0, T], \quad t \neq t_k, \\
\Delta I_{0+}^{(1-\lambda)} x(t_k) = \phi_k(x(t_k), x(t_k)), & k = 1, 2, \ldots, l, \\
I_{0+}^{(1-\lambda)} x(0) = x_0,
\end{cases}
$$

where $D_{0+}^{\mu,\nu}$ denotes the Hilfer fractional derivative of order $0 < \mu < 1$ of type $0 \leq \nu \leq 1$ and $\lambda = \mu + \nu - \mu \nu$; $A : D(A) \subseteq E \to E$ is a closed linear operator and $-A$ generates a $C_0$-semigroup $Q(t) (t \geq 0)$ on a Banach space $E$. Let the impulse effect takes place at $t = t_k$, for $(k = 1, 2, \ldots, l)$: $\phi_k \in C(E \times E, E)$ determines the size of the jump at time $t_k$. In other words, the impulsive moments meet the relation $\Delta I_{t_k}^{1-\lambda} x(t_k) = I_{t_k^+}^{1-\lambda} x(t_k^+) - I_{t_k^-}^{1-\lambda} x(t_k^-)$, where $I_{t_k^+}^{1-\lambda} x(t_k^+)$ and $I_{t_k^-}^{1-\lambda} x(t_k^-)$ denotes the right and the left limit of $I_{t_k}^{1-\lambda} x(t)$ at $t = t_k$ with $0 = t_0 < t_1 < \ldots < t_l < t_{l+1} = T$; $g \in C(J \times E \times E, E)$; $x_0 \in E$.

The rest of the paper is organised in the following way. Section 2 includes essential definitions and lemma for the main results, while Section 3 encloses the main result under appropriate assumptions. Finally Section 4 possess an example to ascertain the main results.

### 2. Essential notions

This section includes some basic results, definitions and lemmas that are relevant to this paper.

**Definition 2.1.** [13] Let $E$ be a real Banach space. A nonempty convex closed set $N \subseteq E$ is called a cone if it satisfies the following two conditions:

1. $x \in N, \eta \geq \theta \Rightarrow \eta x \in N$. 

\[ \theta \leq x \leq y \Rightarrow \|x\| \leq \tilde{N}\|y\|. \]

It is to be noted that a normal cone is always convex and a cone is said to be positive if for \( x,y \in N, y-x \in N \forall x < y \).

Definition 2.3. \([14]\) An operator \( A : D \times D \to E \) with \( D \subset E \) is said to be mixed monotone if \( A(x,y) \) is non-decreasing in \( x \) and non-increasing in \( y \), that is for \( x_1,x_2,y \in D, x_1 \leq x_2 \) then, \( A(x_1,y) \leq A(x_2,y) \). Similarly, for \( y_1,y_2,x \in D, y_1 \leq y_2 \) then, \( A(x,y_1) \geq A(x,y_2) \). Also a point \((\hat{x},\hat{y}) \in D \times D\) is called a coupled fixed point of \( A \) if \( A(\hat{x},\hat{y}) = \hat{x} \) and \( A(\hat{y},\hat{x}) = \hat{y} \).

Let \( C(J,E) \) denote the space of all \( E \)-valued continuous function from \( J \) to \( E \) which is an ordered Banach space generated by the positive cone \( N \) and \( N' \) both are normal with the same normal constant \( \tilde{N} \). Let \( PC(J,E) \) be an ordered Banach space defined as \( PC(J,E) = \{ x : J \to E, x(t) \) is continuous at \( t \neq t_k \) and \( x(t_k^\pm) \) exists, \( k = 1,2,\ldots,l \}, \) with the norm \( \|x\|_{PC} = \sup\{\|x(t)\| : t \in J \}. \) As an impulsive system is considered, a piecewise continuous Banach space should be defined. Let \( PC_{1-\lambda}(J,E) = \{ x : (t-t_k)^{1-\lambda}x(t) \in C((t_k,t_{k+1}],E) \) and \( \lim_{t \to t_k} (t-t_k)^{1-\lambda}x(t), k = 1,2,\ldots,l \) exists with the norm \( \|x(t)\|_{PC_{1-\lambda}} = \max\{ \sup_{t \in (t_k,t_{k+1}]} (t-t_k)^{1-\lambda}\|x(t)\| : k = 0,1,\ldots,l \}. \)

The fractional integral of order \( \mu \) for an integrable function \( g \) is given as \([23]\),

\[ I_0^\mu g(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1}g(s)ds, \quad 0 < \mu < 1. \]

Here \( \Gamma(\cdot) \) is the gamma function. Also, the fractional derivative of Caputo and Riemann-Liouville of order \( \mu \), respectively are given by \([23]\),

\[ ^C D_0^\mu g(t) = \frac{1}{\Gamma(1-\mu)} \int_0^t \frac{g'(s)}{(t-s)^\mu}ds, \quad t > 0, \quad 0 < \mu < 1, \]

and

\[ ^L D_0^\mu g(t) = \frac{1}{\Gamma(1-\mu)} \left( \frac{d}{dt} \right) \int_0^t \frac{g(s)}{(t-s)^\mu}ds, \quad t > 0, \quad 0 < \mu < 1. \]

The Hilfer fractional derivative of order \( 0 < \mu < 1 \) and type \( 0 \leq \nu \leq 1 \) of function \( g(t) \) is given as

\[ D_0^{\mu,\nu} g(t) = I_0^{(1-\mu)} D_0^{(1-\nu)(1-\mu)} g(t), \]

where \( D := \frac{d}{dt} \). Gu and Trujillo \([11]\) can be referred for more details on Hilfer fractional derivative. Moreover, Riemann-Liouville and Caputo can be regarded as a special case of Hilfer fractional derivative, respectively as

\[ D_0^{\mu,\nu} = \begin{cases} D_0^{1-\mu} &= L \quad \nu = 0, \\ \frac{d}{dt} &= C \quad \nu = 1. \end{cases} \]

The parameter \( \lambda \) satisfies \( \lambda = \mu + \nu - \mu\nu, \quad 0 < \lambda \leq 1. \)

Definition 2.4. \([30]\) An operator family \( Q(t) : E \to E \) for \( t \geq 0 \) is supposedly positive if, for any \( u \geq N \), the inequality \( Q(t)u \geq \theta \) holds.
It can be referred [17] that the Kuratowski measure of non-compactness measure denoted by \( \alpha(\cdot) \) is defined on a bounded set. For any \( t \in J \) and \( B \subset C(J, E) \), define \( B(t) = \{ x(t) : x \in B \} \). If \( B \) is bounded in \( C(J, E) \), then \( B \) is bounded in \( E \). Also, \( \alpha(B(t)) \leq \alpha(B) \).

The following Lemmas are necessary for the proof of the main theorem in the next section.

**Lemma 2.1.** [15] Let \( E \) be a Banach space and let \( D \subset E \) be bounded. Then there exists a countable set \( D_0 \subset D \) such that \( \alpha(D) \leq 2\alpha(D_0) \).

**Lemma 2.2.** [17] Let \( B_p = \{ x_p \} \subset C(J, E) \), \( (p = 1, 2, \ldots) \) be a bounded and countable set. Then \( \alpha(B_p(t)) \) is Lebesgue integral on \( J \), and

\[
\alpha\left( \sum_{p=1}^{\infty} \int_{x_p(t)} dt \right) \leq 2 \int_{x_p(t)} dt.
\]

**Lemma 2.3.** [15] Let \( E \) be a Banach space and let \( D \subset C([b_1, b_2], E) \) be bounded and equicontinuous. Then \( \alpha(D(t)) \) is continuous on \([b_1, b_2]\) and

\[
\alpha(D(t)) = \max_{t \in [b_1, b_2]} \alpha(D(t)).
\]

**Lemma 2.4.** [24] (Sadovskii fixed point theorem) Let \( E \) be a Banach space and \( \Omega \) be a nonempty bounded convex closed set in \( E \). If \( Q : \Omega \rightarrow \Omega \) is a condensing mapping, then \( Q \) has a fixed point in \( \Omega \).

The subsequent Lemma is with reference to the generalized Gronwall inequality for fractional differential equation.

**Lemma 2.5.** [28] Suppose \( b \geq 0 \), \( \beta > 0 \) and \( a(t) \) is a nonnegative function locally integrable on \( 0 \leq t < T \) (some \( T \leq +\infty \)), and suppose \( x(t) \) is nonnegative and locally integrable on \( 0 \leq t < T \) with

\[
x(t) \leq a(t) + b \int_{0}^{t} (t-s)^{\beta-1}x(s)ds
\]
on this interval; then

\[
x(t) \leq a(t) + \int_{0}^{t} \left[ \sum_{n=1}^{\infty} \frac{b^n \Gamma(n)}{\Gamma(n+1)} (t-s)^n a(s) \right] ds, \quad 0 \leq t < T.
\]

**Definition 2.5.** [2] A function \( x \in PC_{1-\lambda}(J, E) \) is called the mild solution of system (1.3), if for \( t \in J \) it satisfies the following integral equation

\[
x(t) = S_{\mu, \nu}(t)x_0 + \sum_{i=1}^{k} S_{\mu, \nu}(t-t_i) \phi_i(x(t_i), x(t_i)) + \int_{0}^{t} (t-s)^{\mu-1}P_{\mu}(t-s)g(s, x(s), x(s))ds
\]

where,

\[
S_{\mu, \nu}(t) = I_0^\nu(1-\mu) K_{\mu}(t), \quad K_{\mu}(t) = t^{\mu-1}P_{\mu}(t), \quad P_{\mu}(t) = \int_{0}^{\infty} \mu \xi_\mu(\theta) Q(\nu \theta) d\theta,
\]

\[
\varpi_\mu(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-n\mu-1} \Gamma(n\mu+1)/n! \sin(n\pi\theta), \quad \theta \in (0, \infty)
\]

and \( \xi_\mu(\theta) = \frac{1}{\theta^{\mu-1}} \varpi_\mu(\theta^{-1}) \) is a probability density function defined on \((0, \infty)\), that is

\[
\xi_\mu(\theta) \geq 0 \quad \text{and} \quad \int_{0}^{\infty} \theta^{\mu-1} \xi_\mu(\theta) d\theta = 1.
\]

**Remark 2.1.**
Remark 3.1.

(1) From [10], when \( \nu = 0 \), the solution reduces to the solution of classical Riemann-Liouville fractional derivative, that is, \( S_{\mu,0}(t) = K_{\mu}(t) \).

(2) Similarly when \( \nu = 1 \), the solution reduces to the solution of classical Caputo fractional derivative, that is \( S_{\mu,1}(t) = S_{\mu}(t) \).

Lemma 2.6. [2] If the analytic semigroup \( Q(t)(t \geq 0) \) is bounded uniformly, then the operator \( P_{\mu}(t) \) and \( S_{\mu,\nu}(t) \) satisfies the following bounded and continuity conditions.

(1) \( S_{\mu,\nu}(t) \) and \( P_{\mu}(t) \) are linear bounded operators and for any \( x \in E \),

\[
\|S_{\mu,\nu}(t)x\|_E \leq \frac{M\lambda^{-1}}{\Gamma(\lambda)} \|x\|_E \quad \text{and} \quad \|P_{\mu}(t)x\|_E \leq \frac{M}{\Gamma(\mu)} \|x\|_E.
\]

(2) \( S_{\mu,\nu}(t) \) and \( P_{\mu}(t) \) are strongly continuous, which means that for any \( x \in E \) and \( 0 < t' < t'' \leq T \),

\[
\|P_{\mu}(t')x - P_{\mu}(t'')x\|_E \to 0 \quad \text{and} \quad \|S_{\mu,\nu}(t')x - S_{\mu,\nu}(t'')x\|_E \to 0 \quad \text{as} \ t'' \to t'.
\]

3. Main Results

To prove the main results of this paper, a perturbed equivalent system with constant \( C \geq 0 \) given below is taken for consideration.

\[
\begin{cases}
D_{0+}^{\nu,\mu}x(t) + (A+C)I(t) = g(t, x(t), x(t)) + Cx(t), \quad t \in J, \quad t \neq t_k, \\
\Delta x(t_{k}) = \phi_k(x(t_{k}), x(t_{k})), \quad k = 1, 2, \ldots, l, \\
I^{(1-\mu)}_{0+}x(0) = x_0.
\end{cases}
\]

(3.1)

Remark 3.1.

(1) With reference to [22], for any \( C \geq 0 \), \( -(A+C)I(t) \) generates an analytic semigroup \( R(t) = e^{-t}CQ(t) \) and for \( t \geq 0 \), \( R(t) \) is positive and \( \sup_{t \in [0, \infty)} \|R(t)\| \leq M^* \) for \( M^* \geq 1 \).

(2) Let \( S_{\mu,\nu}^*(t) \) and \( P_{\mu}^*(t) \) for \( t \geq 0 \) be two families of operators defined by

\[
S_{\mu,\nu}^*(t) = I_{0+}^{\nu(1-\mu)}K_{\mu}^*(t), \quad K_{\mu}^*(t) = t^{\mu-1}P_{\mu}^*(t), \\
P_{\mu}^*(t) = \int_{0}^{1} \mu^\theta \xi_{\mu}(\theta)R(t^\theta)\theta d\theta.
\]

(3) The above two operators are positive for \( t \geq 0 \) and for any \( x \in E \),

\[
\|S_{\mu,\nu}^*(t)x\|_E \leq \frac{M^*t^{1-\mu}}{\Gamma(\lambda)} \|P_{\mu}^*(t)x\|_E \leq \frac{M^*}{\Gamma(\mu)} \|K_{\mu}^*(t)x\|_E \leq \frac{M^*t^{1-\mu-1}}{\Gamma(\lambda)} \|P_{\mu}^*(t)x\|_E \leq \frac{M^*}{\Gamma(\mu)}.
\]

(4) \( S_{\mu,\nu}^*(t) \) and \( P_{\mu}^*(t) \) are strongly continuous, which means that for any \( x \in E \), \( 0 < t' < t'' \leq T \), and as \( t'' \to t' \),

\[
\|P_{\mu}^*(t')x - P_{\mu}^*(t'')x\|_E \to 0 \quad \text{and} \quad \|S_{\mu,\nu}^*(t')x - S_{\mu,\nu}^*(t'')x\|_E \to 0.
\]

Definition 3.1. A function \( x \in PC_{-\lambda}(J,E) \) is said to be a mild solution of the problem \( \star \star \star \) if \( x \) satisfies the following integral equation.

\[
x(t) = S_{\mu,\nu}^*(t)x_0 + \sum_{i=1}^{k} S_{\mu,\nu}^*(t - t_i)\phi_i(x(t_i), x(t_i)) \\
+ \int_{0}^{t} (t - s)^{\mu-1}P_{\mu}^*(t - s)\left[g(s, x(s), x(s)) + Cx(s)\right] ds.
\]

For \( y, z \in PC_{-\lambda}(J,E) \), \( [y, z] \) is used to denote the order interval \( \{x \in PC_{-\lambda}(J,E) : y \leq x \leq z\} \) and for \( t \in J \), \( [y(t), z(t)] \) denotes the order interval \( \{x(t) \in PC_{-\lambda}(J,E) : y(t) \leq x(t) \leq z(t)\} \).
Definition 3.2. If $y_0, z_0 \in PC_{1-\lambda}(J, E)$ satisfies all the inequalities of
\[
\begin{align*}
D^{0,\mu}_{0+} y_0(t) + A y_0(t) \leq g(t, y_0(t), z_0(t)) + L(y_0(t) - z_0(t)), & \quad t \in J, \ t \neq t_k \\
\Delta I^{(1-\lambda)}_{t_k} y_0(t_k) \leq \phi_k(y_0(t_k), z_0(t_k)), & \quad k = 1, 2, \ldots, l
\end{align*}
\]
for a constant $L \geq 0$, then $y_0$ and $z_0$ are called the coupled lower and upper $L$-quasi solution of the problem (1.3). If the inequalities are replaced by equality, then $y_0, z_0$ are called coupled $L$-quasi solution. And when $x_0 := y_0 = z_0$, then $x_0$ is called the solution of the problem (1.3).

The following theorem guarantees the existence of the extremal mild solution of the impulsive system (1.3).

Theorem 3.1. Let $E$ be an ordered Banach space with the positive normal cone $N$. Assume that $Q(t) \geq 0$ and the impulsive system (1.3) has both lower and upper solution given by $y_0$ and $z_0$ respectively, where $y_0, z_0 \in PC_{1-\lambda}$ and $y_0 \leq z_0$. By embracing the monotone iterative procedure and presuming the following assumptions, the impulsive system (1.3) has extremal solutions between $y_0$ and $z_0$.

A(1). There exist constants $C \geq 0$ and $L \leq 0$ such that
\[
g(t, y_2, z_2) - g(t, y_1, z_1) \geq -C(y_2 - y_1) - L(z_1 - z_2)
\]
and $y_0(t) \leq y_1(t) \leq y_2(t) \leq z_0(t), y_0(t) \leq z_2(t) \leq z_1(t) \leq z_0(t)$ for any $t \in J$.

A(2). The impulsive function for $t \in J$ satisfies
\[
\phi_k(y_1, z_1) \leq \phi_k(y_2, z_2), \quad k = 1, 2, \ldots, l.
\]

A(3). The sequence $\{y_p\} \subset [y_0(t), z_0(t)]$ and $\{z_p\} \subset [y_0(t), z_0(t)]$ for $t \in J$ is respectively increasing and decreasing monotonic sequences. In particular, there exists a constant $L_1 \geq 0$ such that
\[
\alpha(\{g(t, y_p, z_p)\}) \leq L_1(\alpha(\{y_p\}) + \alpha(\{z_p\})), \quad p = 1, 2, \ldots.
\]

A(4). Let $y_p = G(y_{p-1}, z_{p-1}), z_p = G(z_{p-1}, y_{p-1}), p = 1, 2, \ldots$, such that sequence $y_p(0)$ and $z_p(0)$ are convergent.

Proof. As $C > 0$, the problem (1.3) can be presented in the form of problem (3.1). So the proof of the existence of a unique mild solution for the problem (3.1) is sufficient. Define the operator $G : [y_0, z_0] \times [y_0, z_0] \to PC_{1-\lambda}(J, E)$ by
\[
G(y, z)(t) = S^{\ast}_{\mu, \nu}(t)x_0 + \int^1_0 (t - s)^{\mu-1}P^{\ast}_{\mu}(t-s) \left[ g(s, y(s), z(s)) + (C + L)y(s) - Lz(s) \right] ds, \quad t \in [0, t_1],
\]
\[
+ \int_{t_k}^t (t - s)^{\mu-1}P^{\ast}_{\mu}(t-s) \left[ g(s, y(s), z(s)) + (C + L)y(s) - Lz(s) \right] ds, \quad t \in (t_k, t_{k+1}], \quad k = 1, 2, \ldots, l.
\]

The following steps are required for the completion of the proof.

Step 1. To show $G(y_1, z_1) \leq G(y_2, z_2)$.
The condition $A(1)$ is used to reduce the below inequalities, which can be applied directly in the proof of the theorem. That is, $\forall t \in J'$,

$$y_0(t) \leq y_1(t) \leq y_2(t) \leq z_0(t), \quad y_0(t) \leq z_2(t) \leq z_1(t) \leq z_0(t).$$

$$\Rightarrow g(t, y_1(t), z_1(t)) + Cy_1(t) - Lz_1(t) \leq g(t, y_2(t), z_2(t)) + Cy_2(t) - Lz_2.$$

$$\Rightarrow g(t, y_1(t), z_1(t)) + (C + L)y_1(t) - Lz_1(t) \leq g(t, y_2(t), z_2(t)) + (C + L)y_2(t) - Lz_2.$$

(3.4)

Considering the case for $t \in J_0'$, for $J_0' = [0, t_1]$:

The operators $S^*_{\mu, \nu}(t)$ and $P^*_{\mu}(t)$ are positive operators, and hence when the mild solutions are compared, using [3.4], the following inequality is obtained.

$$\int_0^t (t - s)^{\mu - 1}P^*_{\mu}(t - s) \left[ g(s, y_1(s), z_1(s)) + (C + L)y_1(s) - Lz_1(s) \right] ds \leq$$

$$\int_0^t (t - s)^{\mu - 1}P^*_{\mu}(t - s) \left[ g(s, y_2(s), z_2(s)) + (C + L)y_2(s) - Lz_2(s) \right] ds.$$

In which case, for $\forall t \in J_k'$, with $J'_k = (t_k, t_{k+1})$, $k = 1, 2, \ldots, l$, applying the condition $A(2)$ yields

$$S^*_{\mu, \nu}(t)x_0 + \sum_{i=1}^k S^*_{\mu, \nu}(t - t_i)\phi_i(y_1(t_i), z_1(t_i))$$

$$+ \int_0^t (t - s)^{\mu - 1}P^*_{\mu}(t - s) \left[ g(s, y_1(s), z_1(s)) + (C + L)y_1(s) - Lz_1(s) \right] ds \leq$$

$$S^*_{\mu, \nu}(t)x_0 + \sum_{i=1}^k S^*_{\mu, \nu}(t - t_i)\phi_i(y_2(t_i), z_2(t_i))$$

$$+ \int_0^t (t - s)^{\mu - 1}P^*_{\mu}(t - s) \left[ g(s, y_2(s), z_2(s)) + (C + L)y_2(s) - Lz_2(s) \right] ds.$$

Eventually, $G(y_1, z_1)(t) \leq G(y_2, z_2)(t)$ for $t \in J$.

**Step 2.** To show $y_0 \leq G(y_0, z_0) : G(z_0, y_0) \leq z_0$:

For the case for which $t \in J_0'$:

Let $D_0^+z_0(t) + A_0z_0(t) + C_0z_0(t) = \xi(t), \quad \xi(t) \in PC_{1-\lambda}(J, E)$ By the Definition 3.2 of the coupled upper $L$-quasi solution, the mild solution of the system [133] can be written as

$$z_0(t) = S_{\mu, \nu}(t)z_0 + \int_0^t (t - s)^{\mu - 1}P^*_{\mu}(t - s)\xi(s) ds$$

$$\geq S_{\mu, \nu}(t)x_0 + \int_0^t (t - s)^{\mu - 1}P^*_{\mu}(t - s) \left[ g(s, z_0(s), y_0(s)) + (C + L)z_0(s) - Ly_0(s) \right] ds.$$

From (3.2), it can be observed that $z_0(t) \geq G(z_0, y_0)(t)$.

For $t \in J_1'$:

$$S^*_{\mu, \nu}(t)z_0 + S^*_{\mu, \nu}(t - t_1)\phi_1(z_0(t_1), y_0(t_1)) + \int_0^t (t - s)^{\mu - 1}P^*_{\mu}(t - s)\xi(s) ds$$

$$\geq S^*_{\mu, \nu}(t)x_0 + S^*_{\mu, \nu}(t - t_1)\phi_1(z_0(t_1), y_0(t_1)) + \int_0^t (t - s)^{\mu - 1}P^*_{\mu}(t - s)$$

$$\left[ g(s, z_0(s), y_0(s)) + (C + L)z_0(s) - Ly_0(s) \right] ds.$$

$$\Rightarrow z_0(t) \geq G(z_0, y_0)(t).$$
Progressing in the same manner, every $J_k^t$, yields $z_0(t) \geq G(z_0, y_0)(t)$. In the same manner, it can be proved that $y_0(t) \leq G(y_0, z_0)(t)$ by considering the lower $L$-quasi solutions. Altogether, it can be deduced that

$$y_0(t) \leq G(y_0, z_0)(t) \leq G(x, x)(t) \leq G(z_0, y_0)(t) \leq z_0(t).$$

Henceforth the conclusion may be drawn that

$$G : [y_0, z_0] \times [y_0, z_0] \to PC_{1-\lambda}(J, E)$$

is an increasing mixed monotonic operator. By means of the iterative pattern, two sequence $\{y_p\}$ and $\{z_p\}$ can be defined as,

$$y_p = G(y_{p-1}, z_{p-1}); \quad z_p = G(z_{p-1}, y_{p-1}); \quad p = 1, 2, \ldots, \quad (3.5)$$

Eventually, due to the monotonicity property of $G$, an increasing sequence is derived as,

$$y_0 \leq y_1 \leq y_2 \leq \ldots \leq y_p \leq \ldots \leq z_p \leq \ldots \leq z_2 \leq z_1 \leq z_0. \quad (3.6)$$

**Step 3.** Convergence of sequences $\{y_p\}$ and $\{z_p\}$ in $J'$.-

Let $B_p = \{y_p|p \in N\} + \{z_p|p \in N\}; \quad B_1 = \{y_{p-1}|p \in N\}; \quad B_2 = \{z_{p-1}|p \in N\}; \quad B_3 = \{(y_{p-1}, z_{p-1})|p \in N\}$ and $B_4 = \{(z_{p-1}, y_{p-1})|p \in N\}$. Equation (3.3) gives the relation $B_1 = G(B_3(t))$ and $B_2 = G(B_4(t))$. Let $\psi(t) := \alpha(B_p(t))$. Proving that $\psi(t) \equiv 0$ on every interval $J_k^t$ means that $\alpha(B_p(t)) \equiv 0$ for $k = 1, 2, \ldots, l$, and hence $\{y_p\} + \{z_p\}$ is precompact in $E$ for every $t \in J$. Ultimately, by the definition of precompact, $\{y_p\}$ and $\{z_p\}$ have converging subsequence in $E$. Thus it is necessary to prove that $\psi(t) \equiv 0$.

For $t \in J_0^t$ for $J_0^t = (0, t_1]$.-;

$$\psi(t) = \alpha(B_p(t)) = \alpha\left(B_1(t) + B_2(t)\right)$$

$$\psi(t) = \alpha\left(\left\{S_{\mu,\nu}^*(t)x_0 + \int_0^t (t-s)^{\mu-1} P_{\mu}^*(t-s)\left[g(s, y_{p-1}, z_{p-1}(s)) + (C + L)y_{p-1}(s) - Lz_{p-1}(s)\right] ds + S_{\mu,\nu}^*(t)x_0 + \int_0^t (t-s)^{\mu-1} P_{\mu}^*(t-s)\left[g(s, z_{p-1}(s), y_{p-1}(s)) + (C + L)z_{p-1}(s) - Ly_{p-1}(s)\right] ds\right\} : p \in N\right).$$

The below inequality is the consequence of Lemma 2.2

$$\psi(t) \leq 2\int_0^t \alpha\left\{\left[(t-s)^{\mu-1} P_{\mu}^*(t-s)\left[g(s, y_{p-1}(s), z_{p-1}(s)) + g(s, z_{p-1}(s), y_{p-1}(s)) + C(y_{p-1}(s) + z_{p-1}(s))\right] ds\right]\right\} : p = 1, 2, \ldots.$$ 

Applying the assumed conditions along with Lemma 2.6 results in

$$\psi(t) \leq \frac{2M^*}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} \left[(2L_1 + C)\left(\alpha B_1(s) + \alpha B_2(s)\right)\right] ds.$$ 

$$= \frac{2M^*}{\Gamma(\mu)} (2L_1 + C) \int_0^t (t-s)^{\mu-1} \psi(s) ds.$$ 

By Lemma 2.5, $\psi(t) \equiv 0$ on $J_0^t$. Thus $\{y_p(t)\} + \{z_p(t)\}$ is precompact and hence $\{y_p(t)\}$ and $\{z_p(t)\}$ are precompact for $t \in [0, t_1]$. In this regard, $\phi_1(B_3(t_1))$ and $\phi_1(B_4(t_1))$ are indeed precompact. Hence $\alpha(\phi_1(B_3(t_1))) = 0$ and $\alpha(\phi_1(B_4(t_1))) = 0$. 

Extremal mild solutions for Hilfer fractional evolution equation with mixed monotone Impulsive conditions 9
Now for \( t \in J_1' \), for \( J_1' = (t_1, t_2) \):

\[
\psi(t) = \alpha(B_p(t)) = \alpha \left( G(B_3(t)) + G(B_4(t)) \right) \\
= \alpha \left( G(y_{p-1}, z_{p-1})(t) + G(z_{p-1}, y_{p-1})(t) \right)
\]

\[
\psi(t) = \alpha \left( S^*_{\mu, \nu}(t) x_0 + S^*_{\mu, \nu}(t) \phi_1 \left( y_{p-1}(t_1), z_{p-1}(t_1) \right) + \int_0^t (t-s)^{\mu-1} \\
P^*_\mu(t-s) \left[ g(s, y_{p-1}(s), z_{p-1}(s)) + (C+L)y_{p-1}(s) - Lz_{p-1}(s) \right] ds \\
+ S^*_{\mu, \nu}(t) x_0 + S^*_{\mu, \nu}(t) \phi_1 \left( z_{p-1}(t_1), y_{p-1}(t_1) \right) + \int_0^t (t-s)^{\mu-1} \\
P^*_\mu(t-s) \left[ g(s, z_{p-1}(s), y_{p-1}(s)) + (C+L)z_{p-1}(s) - Ly_{p-1}(s) \right] ds \right) \\
\]

\[
\psi(t) \leq \frac{2M^*b^{1-\lambda}}{\Gamma(\lambda)} \left[ \alpha \left( \phi_1(B_3(t_1)) + \phi_1(B_4(t_1)) \right) \right] \\
+ \frac{2M^*}{\Gamma(\mu)} (2L_1 + C) \int_0^t \int_0^t (t-s)^{\mu-1} \psi(s) ds.
\]

By Lemma 2.5, \( \psi(t) \equiv 0 \) in \( J_1' \). By Proceeding the same way interval by interval, it can be proved that \( \psi(t) \equiv 0 \) on every interval \( J_k, k = 1, 2, \ldots l \). Thus \( \{y_p\} \) and \( \{z_p\} \) are precompact and eventually for \( p = 1, 2, \ldots, \{y_p\} \) and \( \{z_p\} \) has a converging subsequence and from (3.6), it can be observed that \( \{y_p\} \) and \( \{z_p\} \) are converging sequences and hence there exists \( \varphi(t) \), \( \varphi(t) \in E \) such that

\[
\lim_{p \to \infty} y_p(t) \to \varphi(t), \quad \lim_{p \to \infty} z_p(t) \to \varphi(t), \quad t \in J.
\]

From (3.6) and using the fact that \( y_p(t) = G(y_{p-1}, z_{p-1})(t) \), (3.2) can be represented as below

\[
y_p(t) = \begin{cases} 
S^*_{\mu, \nu}(t) x_0 + \int_0^t (t-s)^{\mu-1} P^*_\mu(t-s) \left[ g(s, y_{p-1}(s), z_{p-1}(s)) \\
+ (C+L)y_{p-1}(s) - Lz_{p-1}(s) \right] ds, & t \in [0, t_1] \\
+ \sum_{i=1}^k S^*_{\mu, \nu}(t) \phi_i \left( y_{p-1}(t_i), z_{p-1}(t_i) \right) \\
+ \int_0^t (t-s)^{\mu-1} P^*_\mu(t-s) \left[ g(s, y_{p-1}(s), z_{p-1}(s)) \\
+ (C+L)y_{p-1}(s) - Lz_{p-1}(s) \right] ds, & t \in (t_k, t_{k+1}], \ k = 1, 2, \ldots l.
\end{cases}
\]

Using Lebesgue dominated convergence theorem, as \( p \to \infty \)

\[
\varphi(t) = \begin{cases} 
S^*_{\mu, \nu}(t) x_0 + \int_0^t (t-s)^{\mu-1} P^*_\mu(t-s) \left[ g(s, \varphi(s), \varphi(s)) \\
+ (C+L)\varphi(s) - L\varphi(s) \right] ds, & t \in [0, t_1], \\
+ \sum_{i=1}^k S^*_{\mu, \nu}(t) \phi_i (\varphi(t_i), \varphi(t_i)) \\
+ \int_0^t (t-s)^{\mu-1} P^*_\mu(t-s) \left[ g(s, \varphi(s), \varphi(s)) \\
+ (C+L)\varphi(s) - L\varphi(s) \right] ds, & t \in (t_k, t_{k+1}], \ k = 1, 2, \ldots l.
\end{cases}
\]
It can be observed that \( g(t) \in PC_{1-\lambda}(J, E) \) and \( g(t) = G(x, \pi)(t) \). In a similar manner, it can be proved that \( \exists \pi(t) \in PC_{1-\lambda}(J, E) \) such that \( \pi(t) = G(x, \pi)(t) \). With the monotonicity property of \( G \), it can be concluded that \( y_0 \leq x \leq \pi \leq z_0 \). This proves that there exists minimal and maximal solutions \( \bar{x} \) and \( \pi \) respectively in \([y_0, z_0]\) for the given impulsive system (1.3).

The existence of mild solution of (1.3) can be discussed by replacing the conditions \( A(2) \) and \( A(3) \) by the below given conditions.

\( A(2^*) \). The impulsive function \( \phi_k(\cdot, \cdot) \) satisfies

\[
\phi_k(y_1, z_1) \leq \phi_k(y_2, z_2), \quad k = 1, 2, \ldots, l.
\]

for \( t \in J \) and \( y_0(t) \leq y_1 \leq y_2 \leq z_0(t), \ y_0(t) \leq z_2 \leq z_1 \leq z_0(t) \). Also there exists \( M_k > 0 \) satisfying the condition given by

\[
\sum_{k=1}^{l} M_k \leq \frac{\Gamma(\mu)\Gamma(\mu + 1) - 4M^*(2L_1 + C)T^\mu}{4M^*T^\lambda - 1\Gamma(\mu - 1)} \tag{3.7}
\]

such that \( \alpha(\phi_k(D_1, D_2)) \leq M_k[\alpha(D_1) + \alpha(D_2)] \). In the above condition (3.7), the denominator does not jump to infinity as the term \( M_k \) takes its value from \( J_k, \ k > 0 \). That is \( T \neq 0 \).

\( A(3^*) \). There exists a constant \( L_1 < 0 \) such that

\[
\alpha(g(t, D_1, D_2)) \leq L_1(\alpha(D_1) + \alpha(D_2)), \quad t \in J
\]

with the countable sets \( D_1 = \{y_p\} \) and \( D_2 = \{z_p\} \) in \([y_0(t), z_0(t)]\). The following theorem utilizes both the above inequalities where the existence of at least one mild solution between the coupled \( L \)-quasi upper and lower solutions is investigated. Also the semigroup generated by the operator \(-A\) is assumed to be equicontinuous.

**Theorem 3.2.** Let \( E \) be an ordered Banach space with the positive normal cone \( N \). Assume that \( Q(t) \geq 0 \) which is equicontinuous on \( E; \ g \in C(J \times E \times E, E); \ x_0 \in E \). Let the impulsive system (1.3) has coupled \( L \)-quasi lower and upper solution, given by \( y_0 \) and \( z_0 \) respectively, where \( y_0, z_0 \in PC_{1-\lambda} \) and \( y_0 \leq z_0 \). If the assumptions \( A(1), A(2^*) \) and \( A(3^*) \) are satisfied, then the impulsive system (1.3) has coupled minimal and maximal \( L \)-quasi mild solution between \([y_0, z_0]\) and at least one mild solution in \([y_0, z_0]\) between \( \bar{x} \) and \( \pi \) such that for \( p \to \infty, \ y_p(t) \to \bar{x}; \ z_p(t) \to \pi, \ t \in J \).

Here \( y_p \) and \( z_p \) are given as \( y_p = \bar{G}(y_{p-1}, z_{p-1}), \ z_p = \pi(G(z_{p-1}, y_{p-1}), \ t \in J \).

\[
y_0(t) \leq y_1(t) \leq \cdots \leq y_p(t) \leq \cdots \leq \bar{x} \leq \pi \leq \cdots \leq z_p(t) \leq z_1(t) \leq z_0(t).
\]

**Proof.** It can be verified that the assumption \( A(3^*) \Rightarrow A(3) \). Hence by the theorem (3.1), the impulsive system (1.3) has minimal and maximal \( L \)-quasi lower \( \bar{x} \) and upper \( \pi \) solutions in \([y_0, z_0]\). From the normality definition of the cone \( P \), there exists \( M > 0 \) such that

\[
\|g(t, y(t), z(t)) + (c + L)y(t) - Lz(t)\| \leq \tilde{M} \tag{3.8}
\]

The proof of the theorem terminates in finding at least one mild solution in \([y_0, z_0]\). First, let the operator \( F \) be defined as \( F : [y_0, z_0] \to [y_0, z_0] \) such that \( Fx = \bar{G}(x, x) \). It is evident that \( F \) is continuous and the fixed point of the operator \( F \) is equivalent to the mild solution of the system (1.3).

For the case for which \( t \in J_0 \);
Let \( s_1, s_2 \in [0, t_1] \) such that \( 0 < s_1 < s_2 \leq t_1 \). The following inequality determines the equicontinuous of the operator \( \mathcal{F} \).

\[
\left\| s_2^{-\lambda}(\mathcal{F}x)(s_2) - s_1^{-\lambda}(\mathcal{F}x)(s_2) \right\| \leq \left\| s_2^{-\lambda}\mathcal{G}(x, x)(s_2) - s_1^{-\lambda}\mathcal{G}(x, x)(s_2) \right\|
\]

\[
\leq \left\| s_2^{-\lambda}S_{\mu, \nu}^*(s_2)x_0 - s_1^{-\lambda}S_{\mu, \nu}^*(s_1)x_0 \right\| + \left\| \int_0^{s_2} s_2^{-\lambda}(s_2 - s)^{-\mu-1} P_{\mu}^*(s_2 - s) \left[ g(s, x(s), x(s)) + (C + L)x(s) - Lx(s) \right] ds \right\|
\]

For convenience let \( g(s, x(s), x(s)) + Cx(s) \) be denoted by \( \zeta(s) \).

\[
\left\| s_2^{-\lambda}(\mathcal{F}x)(s_2) - s_1^{-\lambda}(\mathcal{F}x)(s_2) \right\| \leq \left\| s_2^{-\lambda}S_{\mu, \nu}^*(s_2)x_0 - s_1^{-\lambda}S_{\mu, \nu}^*(s_1)x_0 \right\|
\]

\[
+ \left\| s_2^{-\lambda} \int_{s_1}^{s_2} (s_2 - s)^{-\mu-1} P_{\mu}^*(s_2 - s) [\zeta(s)] ds \right\|
\]

\[
+ \left\| \int_0^{s_1} (s_2^{-\lambda}(s_2 - s)^{-\mu-1} - s_1^{-\lambda}(s_1 - s)^{-\mu-1}) P_{\mu}^*(s_2 - s) [\zeta(s)] ds \right\|
\]

\[
+ \left\| s_1^{-\lambda} \int_0^{s_1} (s_2 - s)^{-\mu-1} (P_{\mu}^*(s_2 - s) - P_{\mu}^*(s_1 - s)) [\zeta(s)] ds \right\|
\]

\[
= \sum_{i=1}^{5} \left\| I_i \right\|
\]

Now for \( i = 1, 2, \ldots, 5 \), \( I_i \) can be calculated individually as below. For \( I_1 \), using Remark \((3.1)\), it can be observed that

\[
I_1 = \left\| s_2^{-\lambda}S_{\mu, \nu}^*(s_2)x_0 - s_1^{-\lambda}S_{\mu, \nu}^*(s_1)x_0 \right\|
\]

\[
\leq \left\| s_2^{-\lambda}(S_{\mu, \nu}^*(s_2) - S_{\mu, \nu}^*(s_1)) \right\| \| x_0 \|
\]

\[
\rightarrow 0, \text{ as } s_2 \rightarrow s_1.
\]

For \( I_2 \), using Remark \((3.1)\), the following observation can be made similar to \( I_1 \).

\[
I_2 = \left\| s_2^{-\lambda}S_{\mu, \nu}^*(s_1)x_0 - s_1^{-\lambda}S_{\mu, \nu}^*(s_1)x_0 \right\|
\]

\[
\leq \frac{M^* T^{\lambda-1}}{\Gamma(\lambda)} \left\| s_2^{-\lambda} - s_1^{-\lambda} \right\| \| x_0 \|
\]

\[
\rightarrow 0, \text{ as } s_2 \rightarrow s_1.
\]

\( I_3 \) can be evaluated using Remark \((3.1)\) as below.

\[
I_3 = \left\| s_2^{-\lambda} \int_{s_1}^{s_2} (s_2 - s)^{-\mu-1} P_{\mu}^*(s_2 - s) [\zeta(s)] ds \right\|
\]

\[
\leq \frac{M^* \tilde{M}}{\Gamma(\mu)} \left\| \int_{s_1}^{s_2} (s_2 - s)^{-\mu-1} ds \right\|
\]

\[
\rightarrow 0, \text{ as } s_2 \rightarrow s_1.
\]
\[ I_4 = \left\| \int_0^{s_1} \left( s_2^{1-\lambda} (s_2 - s)^{\mu-1} - s_1^{1-\lambda} (s_1 - s)^{\mu-1} \right) P_\mu^* (s_2 - s) \right\| \]

\[ \Rightarrow I_4 \leq \frac{M M^*}{\Gamma(\mu)} \left\| \int_0^{s_1} \left( s_2^{1-\lambda} (s_2 - s)^{\mu-1} - s_1^{1-\lambda} (s_1 - s)^{\mu-1} \right) ds \right\| \]

\[ \rightarrow 0, \text{ as } s_2 \rightarrow s_1. \]

Similarly for \( \epsilon \in (0, s_1) \), \( I_5 \) can be evaluated as below.

\[ I_5 = \left\| \int_0^{s_1 - \epsilon} s_1^{1-\lambda} (s_1 - s)^{\mu-1} \left( P_\mu^* (s_2 - s) - P_\mu^* (s_1 - s) \right) \right\| \]

\[ \Rightarrow I_5 \leq \frac{M M^*}{\Gamma(\mu)} \left\| \int_{s_1 - \epsilon}^{s_1} s_1^{1-\lambda} (s_1 - s)^{\mu-1} ds \right\| \]

\[ \leq \frac{2 M M^* t_1^{1-\lambda} \epsilon^\mu}{\Gamma(\mu + 1)} \]

\[ \rightarrow 0, \text{ as } \epsilon \rightarrow 0 \text{ and } s_2 \rightarrow s_1. \]

Thus the following conclusion can be drawn for \( J_0^* \).

\[ \Rightarrow \left\| s_2^{1-\lambda} (F(x))(s_2) - s_1^{1-\lambda} (F(x))(s_2) \right\| \rightarrow 0. \]

For \( J_k^* = (t_k, t_{k+1}] \), let \( s_1, s_2 \in (t_k, t_{k+1}] \) such that \( t_k < s_1 < s_2 \leq t_{k+1} \), for which the following equality is evaluated.

\[ \left\| (s_2 - t_k)^{1-\lambda} (F(x))(s_2) - (s_1 - t_k)^{1-\lambda} (F(x))(s_2) \right\| \]

\[ \Rightarrow \left\| (s_2 - t_k)^{1-\lambda} G(x, x)(s_2) - (s_1 - t_k)^{1-\lambda} G(x, x)(s_2) \right\| \rightarrow 0, \text{ as } s_2 \rightarrow s_1 \]

Calculations similar to \( J_0^* \) are performed to obtain the following observation.

\[ \Rightarrow \left\| G(x, x)(s_2) - G(x, x)(s_2) \right\| \rightarrow 0, \text{ as } s_2 \rightarrow s_1. \]

Consequently, \( \left\| (F(x))(s_2) - (F(x))(s_2) \right\| \rightarrow 0 \) independently of \( x \in [y_0, z_0] \) as \( s_2 \rightarrow s_1 \), which implies that \( (F(x)) : [y_0, z_0] \rightarrow [y_0, z_0] \) is equicontinuous. In this regard, for any \( D \subset [y_0, z_0] \), \( F(D) \subset [y_0, z_0] \) is bounded and equicontinuous. By Lemma 2.4 it is evident that there exists a countable set \( D_0 = \{ x_p \} \subset D \), such that

\[ \alpha(F(D)) \leq 2\alpha(F(D_0)). \]

From Lemma 2.4 it can be observed that

\[ \alpha(F(D_0)) = \max_{t \in J} \alpha(F(D_0))(t). \]
For $t \in J_0$, by Lemma 2.2 Equation (3.2), and from the assumption $A(3^*)$, the following inequality is evaluated.

$$\alpha(\mathcal{F}(D_0)(t)) = \alpha \left( \left\{ S_{\mu,\nu}^*(t)x_0 + \int_0^t (t-s)^{\mu-1} P_\mu^*(t-s) \left[ g(s, x_p(s), x_p(s)) + C x_p(s) \right] ds \right\} \right)$$

$$\leq \alpha \left( \left\{ S_{\mu,\nu}^*(t)x_0 \right\} \right) + 2M^* \frac{M^* T^{\lambda-1}}{\Gamma(\mu)} \alpha \left( \left\{ \phi_1(D_0(t_i), D_0(t_i)) \right\} \right)$$

$$+ \frac{2M^* (2L_1 + C)}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} \alpha \left( D_0(s) \right) ds$$

$$\leq 2M^* \frac{(2L_1 + C) T^\mu}{\Gamma(\mu + 1)} \alpha(D).$$

For the case $J_1$, $t \in (t_1, t_2]$, the inequality is evaluated as below using the Lemma 2.2 Equation (3.2), and the assumptions $A(2^*)$ and $A(3^*)$

$$\alpha(\mathcal{F}(D_0)(t)) = \alpha \left( \left\{ S_{\mu,\nu}^*(t)x_0 + S_{\mu,\nu}^*(t-t_1) \phi_1(x_p(t_1), x_p(t_1)) \right\} \right)$$

$$+ \int_0^t (t-s)^{\mu-1} P_\mu^*(t-s) \left[ g(s, x_p(s), x_p(s)) + C x_p(s) \right] ds \right\} \right)$$

$$\leq \alpha \left( \left\{ S_{\mu,\nu}^*(t)x_0 \right\} \right) + \frac{M^* T^{\lambda-1}}{\Gamma(\mu)} \alpha \left( \left\{ \phi_1(D_0(t_i), D_0(t_i)) \right\} \right)$$

$$+ \frac{2M^* (2L_1 + C)}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} \alpha \left( D_0(s) \right) ds$$

$$\leq 2M^* \frac{(2L_1 + C) T^\mu}{\Gamma(\mu + 1)} \alpha(D).$$

Now for a general case, that is for $J_k$, $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots, l$ the inequality is calculated below using the Lemma 2.2 Equation (3.2), and the assumptions $A(2^*)$ and $A(3^*)$.

$$\alpha(\mathcal{F}(D_0)(t)) = \alpha \left( \left\{ S_{\mu,\nu}^*(t)x_0 + \sum_{i=1}^k S_{\mu,\nu}^*(t-t_i) \phi_1(x_p(t_i), x_p(t_i)) \right\} \right)$$

$$+ \int_0^t (t-s)^{\mu-1} P_\mu^*(t-s) \left[ g(s, x_p(s), x_p(s)) + C x_p(s) \right] ds \right\} \right)$$

$$\leq \alpha \left( \left\{ S_{\mu,\nu}^*(t)x_0 \right\} \right) + \frac{M^* T^{\lambda-1}}{\Gamma(\mu)} \alpha \left( \left\{ \sum_{i=1}^k \phi_1(D_0(t_i), D_0(t_i)) \right\} \right)$$

$$+ \frac{2M^* (2L_1 + C)}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} \alpha \left( D_0(s) \right) ds$$

$$\leq 2M^* \frac{\sum_{i=1}^k M_i T^{\lambda-1}}{\Gamma(\lambda)} + \frac{(2L_1 + C) T^\mu}{\Gamma(\mu + 1)} \alpha(D).$$

By Lemma 2.3, since $\mathcal{F}(D_0)$ is bounded and equicontinuous, the above inequality results in,

$$\alpha(\mathcal{F}(D_0)(t)) \leq 4M^* \frac{\sum_{i=1}^k M_i T^{\lambda-1}}{\Gamma(\lambda)} + \frac{(2L_1 + C) T^\mu}{\Gamma(\mu + 1)} \alpha(D) \leq \alpha(D).$$

Let $4M^* \left( \frac{\sum_{i=1}^k M_i T^{\lambda-1}}{\Gamma(\lambda)} + \frac{(2L_1 + C) T^\mu}{\Gamma(\mu + 1)} \right) = \eta$
Thus the condition \( A \) above equation reduces to, \( L \) and \( N \) from the definition of the normal cone with the normality constant \( \tilde{\eta} \). The solution with the normality constant \( \tilde{\eta} \) in an ordered Banach space \( E \) for \( t \in [0, T] \) into \( n \) parts such that \( \Delta_n = 0 = \tilde{\eta}_0 < \tilde{\eta}_1 < \ldots < \tilde{\eta}_n = T \). Here the points \( \tilde{\eta}_0, \tilde{\eta}_1, \ldots, \tilde{\eta}_n \) are not the impulse points such that the below condition holds.

\[
4M^* \left( \frac{\sum_{i=1}^{k} M_i \|\Delta_n\|^{\lambda-1}}{\Gamma(\lambda)} + \frac{(2L_1 + C)\|\Delta_n\|^\mu}{\Gamma(\mu + 1)} \right).
\]

In the interval \([0, \tilde{\eta}_0]\), according to the above statements (i) and (ii), there exists a mild solution \( x_1(t) \in [0, \tilde{\eta}_0] \). Now in the interval \([\tilde{\eta}_0, \tilde{\eta}_1], \tilde{\eta}_1, \tilde{\eta}_2 \) with initial condition \( x(\tilde{\eta}_1) = x_1(\tilde{\eta}_1) \), it has a mild solution \( x_2(t) \in [\tilde{\eta}_1, \tilde{\eta}_2] \). Thus, the mild solution of the equation is extended from \([0, \tilde{\eta}_1]\) to \([0, \tilde{\eta}_2]\). Subsequently, continuing this process, the mild solution of the equation is extended to \([0, T]\). Thus the impulsive system \( (1.3) \) has a mild solution \( x \in PC_{1-\lambda}(J, E) \) that satisfies \( x(t) = x_i(t) \) such that \( \tilde{\eta}_{i-1} \leq t \leq \tilde{\eta}_i \), for \( i = 1, 2, \ldots, n \).

Since \( x = Fx = G(x, x) \) for \( y_0 \leq x \leq z_0 \), with respect to the mixed monotone property, the conclusion can be drawn as \( y_1 = G(y_0, z_0) \leq G(x, x) \leq G(z_0, y_0) = z_1 \). In a similar way, it is true for \( y_2 \leq x \leq z_2 \) and in general, \( y_p \leq x \leq z_p \). It is clear that, letting \( p \to \infty \) reduces to \( \tilde{\nu} \leq x \leq \tilde{\pi} \). Hence it can be concluded that the impulsive system \( (1.3) \) has at least one mild solution between \( \tilde{\nu} \) and \( \tilde{\pi} \).

**Corollary 3.1.** In an ordered Banach space \( E \), let \( N \) be the positive cone with normal constant \( \tilde{\eta} \). With the assumption that the operator \( Q(t) \) is positive for \( t \in J \), if the assumptions \( A(1) \) and \( A(2) \) are satisfied combined with the condition given below, then the condition \( A(3) \) is automatically true.

**A(5).** There exists a constant \( C^* \) and \( L^* \) such that

\[
g(t, y_2, z_2) - g(t, y_1, z_1) \leq C^*(y_2 - y_1) + L^*(z_1 - z_2)
\]

and \( y_0(t) \leq y_1(t) \leq y_2(t) \leq z_0(t) \), \( y_0(t) \leq y_2(t) \leq z_0(t) \) for any \( t \in J \).

**Proof.** Let \( \{y_p\}, \{q_p\} \) be two set of increasing sequences such that \( \{y_p\}, \{q_p\}, \{y_q\}_p, \{q_q\} \subset [y_0(t), z_0(t)] \),

for \( t \in J \) and \( p \leq q \). By the condition \( A(1) \) and \( A(5) \),

\[
\theta \leq g(t, y_q, z_q) - g(t, y_p, z_p) + C(y_q - y_p) + L(z_p - z_q) \\
\leq (C^* + C)(y_q - y_p) + (L^* + L)(z_p - z_q).
\]

From the definition of the normal cone with the normality constant \( \tilde{\eta} \) of the positive cone \( N \), the equation further reduces to,

\[
\|g(t, y_q, z_q) - g(t, y_p, z_p) + C(y_q - y_p) + L(z_p - z_q)\| \\
\leq \langle \tilde{\eta} \{(C^* + C)(y_q - y_p) + (L^* + L)(z_p - z_q)\} \rangle.
\]

\[
\Rightarrow \|g(t, y_q, z_q) - g(t, y_p, z_p)\| \\
\leq (\tilde{\eta}C^* + \tilde{\eta}C + C)\|y_q - y_p\| + (\tilde{\eta}L^* + \tilde{\eta}L + L)\|z_p - z_q\|.
\]

Let \( L_1 = \tilde{\eta}(C^* + C + L^* + L) + C + L \). By the definition of measure of non-compactness the above equation reduces to,

\[
\alpha \left( \{g(t, y_p, z_p)\} \right) \leq L_1 \left( \alpha \left( \{y_p\} \right) + \alpha \left( \{z_p\} \right) \right), \quad p = 1, 2, \ldots.
\]

Thus the condition \( A(3) \) is reduced.

**Theorem 3.3.** An impulsive fractional system \( (1.3) \) is said to have an unique mild solution that lie between \( [y_0, z_0] \), where \( y_0 \in PC_{1-\lambda} \) and \( z_0 \in PC_{1-\lambda} \) are the coupled \( L \)-quasi lower and upper solution with \( y_0 \leq z_0 \), if the conditions \( A(1), A(2), A(4) \) and \( A(5) \) holds.
Proof. If \( x \) and \( \varphi \) are the maximal and the minimal solution of the impulsive system (1.3), then to prove the uniqueness, it has to be proved that \( x = \varphi \). Let \( t \in J_0^+ \). Using (3.3) in both the solutions results in,

\[
\theta \leq \varphi(t) - x(t) = G(\varphi, x)(t) - G(x, \varphi)(t)
\]

\[
= \int_0^t (t-s)^{\mu-1} P^*_\mu(t-s) \left[ (g(s, \varphi(s), x(s)) - g(s, x(s), \varphi(s)) \right]
\]

\[
+ (C + 2L)(|\varphi(s) - x(s)|)\,ds
\]

\[
\leq \int_0^t (t-s)^{\mu-1} P^*_\mu(t-s) \left[ (\varphi(s) - x(s)) \right]
\]

\[
+ (C + 2L)(|\varphi(t) - x(t)|)\,ds.
\]

Using the normality of the positive cone \( N \), the above inequality reduces to,

\[
||\varphi(t) - x(t)|| \leq \frac{MM^*(C^* + L^* + C + 2L)}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} ||\varphi(t) - x(t)||\,ds.
\]

By Gronwall inequality, \( ||\varphi(t) - x(t)|| = 0 \). Which implies \( \varphi(t) = x(t) \).

For every interval \( J_k \), as \( \phi_k(\varphi(t_k), x(t_k)) = \phi_k(\varphi(t_k), x(t_k)) \), the calculation is similar and it results in \( \varphi(t) = x(t) \) for \( t \in J_k \), for \( k = 1, 2, \ldots, l \). The uniqueness is thus proved.

4. Observation

This paper is based on finding extremal solution of impulsive system with Hilfer fractional derivative using mixed monotone iterative technique. Theorem 3.1 guarantees the existence of minimal and maximal solution for the considered system and Theorem 3.2 discusses the condition such that there exists at least one mild solution between the minimal and maximal solution. Finally, Theorem 3.3 ensure the uniqueness of such mild solution. The results are proved considering that the semigroup generated by the operator is a non-compact semigroup and an equicontinuous semigroup.

The results can further be extended in studying the case when the semigroup generated is compact and for the case when the coupled upper and lower quasi solution does not exist. Also, this article can lead to the study of impulsive system with nonlocal conditions. It is to be noted that the Theorem 3.1 is true for the case when the cone \( N \) which is normal is replaced with positive cone which is regular. For detailed proof [21 Corollary 3.3] may be referred.

5. Example

An example is provided in this section which illustrates the main results.

Example 1. Let \( E = L^p(\Lambda) \) for \( 1 < p < \infty \) be generated by a positive cone \( N \) defined as \( N = \{ x \in L^p(\Lambda) : x(y) \geq \theta, \text{ a.e } y \in \Lambda \} \), where \( \theta \) is the zero element. Here \( \Lambda \subset R^N \), \( N \geq 1 \) is a bounded domain with a sufficiently smooth boundary \( \partial \Lambda \). An impulsive Hilfer fractional parabolic partial differential equation with above conditions is considered as below.

\[
\begin{cases}
D^\mu_x(t, w) - \nabla^2 x(t, w) = g(t, w, x(t, w), x(t, w)), & (t, y) \in J \times \Lambda \\
\Delta^\lambda\lambda I^{1-\lambda}_{\mu+} x(t_k) = \phi_k(x(w, t_k), x(w, t_k)), & k = 1, 2, \ldots, l, y \in \Lambda \\
I^{1-\lambda}_{\mu+}\lambda x(0, w) = x_0
\end{cases}
\]

(5.1)

where \( D^\mu_x \) is the Hilfer fractional derivative with order \( 0 < \mu < 1 \) and type \( 0 \leq \nu \leq 1 \), \( t \in [0, T] \), \( \nabla^2 \) is the Laplace operator such that \( -Ax = \nabla^2 x \), \( J = [0, T] \) with impulsive points at \( t_k \) for \( k = 0, 1, \ldots, l \) such that \( J' = J \setminus \{ 0, t_1, t_2, \ldots, t_l \} \). Let \( -A \) generates a equicontinuous analytic semigroup \( Q(t) \) for \( t \geq 0 \) and it is defined as \( A : D(A) \subset E \rightarrow E \). Here, \( D(A) = W^2 \cap W^1_0(\Lambda) \).

The continuous function \( g \) is defined as \( g : J \times \Lambda \times \Lambda \rightarrow E \) and the impulsive function is defined as \( \phi_k : E \times E \rightarrow E \).

Now, the Example 5.1 can be given as an abstract form similar to (1.3).
Theorem 5.1. Let the Hilfer fractional system given in Example 1 satisfy the following conditions with $x_0 \geq 0$.

$E(1)$. There exists a function $z = z(t, w) \in PC_{1-\lambda}(J, \Lambda)$ such that

$$
\begin{align*}
D_{0+}^\alpha z(t, w) - \nabla^2 z(t, w) &\geq g(t, w, z(t, w), z(t, w)), \quad t \in J, \\
\varphi(t_k, w) &\geq \phi_k(z(t_k, w), z(t_k, w)), \quad k = 1, 2, \ldots
\end{align*}
$$

$E(2)$. There exist constants $C \geq 0$ and $L \leq 0$ such that

$$
\begin{align*}
g(t, w, y_2(t, w), z_2(t, w)) - g(t, w, y_1(t, w), z_1(t, w)) \\
&\geq -C(y_2(t, w) - y_1(t, w)) - L(z_1(t, w) - z_2(t, w))
\end{align*}
$$

and $y_0(t, w) \leq y_1(t, w) \leq y_2(t, w) \leq z_0(t, w)$, $y_0(t, w) \leq z_2(t, w) \leq z_1(t, w) \leq z_0(t, w)$ for any $t \in J$.

$E(3)$. The impulsive function for $t \in J$ satisfies

$$
\phi_k(y_1(t_k, w), z_1(t_k, w)) \leq \phi_k(y_2(t_k, w), z_2(t_k, w)), \quad k = 1, 2, \ldots, l.
$$

$E(4)$. For $t \in J$, the sequence $\{y_p(t, w)\} \subset [y_0(t, w), z_0(t, w)]$ is an increasing monotonic sequence and $\{z_p(t, w)\} \subset [y_0(t, w), z_0(t, w)]$ is a decreasing monotonic sequences. In particular, there exists a constant $L_1 \geq 0$ such that for $p = 1, \ldots$,

$$
\alpha\left(\{g(t, y_p(t, w), z_p(t, w))\}\right) \leq L_1\left(\alpha\left(\{y_p(t, w)\}\right) + \alpha\left(\{z_p(t, w)\}\right)\right).
$$

Then using the monotone iterative procedure initiating from 0 to $z(t, w)$, the system has minimal and maximal solutions.

Proof. From the assumption $E(1)$, it can be concluded that the lower and upper solution lies between 0 and $z(t, w)$. Also the Example 1 satisfies all the assumptions of Theorem 3.3, it can be concluded that there exists a unique solution between 0 and $z(t, w)$.

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