QUANTUM THEORY AND GALOIS FIELDS

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Abstract:

We discuss the motivation and main results of a quantum theory over a Galois field (GFQT). The goal of the paper is to describe main ideas of GFQT in a simplest possible way and to give clear and simple arguments that GFQT is a more natural quantum theory than the standard one. The paper has been prepared as a presentation to the ICSSUR’ 2005 conference (Besancon, France, May 2-6, 2005).

1 Modern physics and spacetime

The phenomenon of local quantum field theory (LQFT) has no analogs in the history of science. There is no branch of science where so impressive agreements between theory and experiment have been achieved. The theory has successfully predicted the existence of many new particles and even new interactions. It is hard to believe that all these achievements are only coincidences.

At the same time, the level of mathematical rigor in LQFT is very poor and, as a result, LQFT has several well known difficulties and inconsistencies. Probably the main inconsistency is that local quantum fields are meaningful only as operator-valued distributions, and products of such objects at the same spacetime points is ambiguous. The problem of how such products should be correctly defined is discussed in a wide literature but a universal solution has not been found yet.

The absolute majority of physicists believes that agreement with experiment is much more important than the lack of mathematical
rigor. The philosophy of most optimistic physicists is roughly as follows. The standard model describes almost everything and to construct the ultimate theory, this model should be only modified somehow at Planck distances. The content of the present paper will probably be of no interest for such physicists.

At the same time, some famous physicists are not so optimistic. For example, Weinberg believes [1] that the new theory may be 'centuries away’. Although he has contributed much to LQFT and believes that it can be treated the way it is, he also believes that it is a low energy approximation to a deeper theory that may not even be a field theory, but something different like a string theory [2].

Dirac was probably the least optimistic famous physicist. In his opinion [3]: The agreement with observation is presumably by coincidence, just like the original calculation of the hydrogen spectrum with Bohr orbits. Such coincidences are no reason for turning a blind eye to the faults of the theory. Quantum electrodynamics is rather like Klein-Gordon equation. It was built up from physical ideas that were not correctly incorporated into the theory and it has no sound mathematical foundation.

The main problem is the choice of strategy for constructing a new quantum theory. Since nobody knows for sure which strategy is the best one, different approaches should be investigated. In the present paper we are trying to follow Dirac's advice given in Ref. [3]: I learned to distrust all physical concepts as a basis for a theory. Instead one should put one’s trust in a mathematical scheme, even if the scheme does not appear at first sight to be connected with physics. One should concentrate on getting an interesting mathematics.

We believe that quantum theory over a Galois field (GFQT) is not only an interesting mathematical theory but it may be a basis for the ultimate quantum physics. The goal of the present paper is to describe main ideas of GFQT in a simplest possible way and to give clear and simple arguments which hopefully might convince some physicists that GFQT is a more natural quantum theory than the standard one. However, before discussing GFQT we would like to note the following.
The physicists claiming that only agreement with experiment is of any importance, typically do not pay attention not only to the lack of rigor in LQFT but also to the fact that the present approaches to elementary particle theories are not quite consistent. Let us try to elaborate the last statement.

If you ask modern physicists whether they believe in quantum theory, the absolute majority of them will answer 'yes' without any doubt. Then you could ask the next question: do you agree with one of the main statements of quantum theory that every physical quantity should be related to some (selfadjoint) operator? And again, the absolute majority will answer 'yes'. But in that case is time a well-defined physical quantity? It has been known for many years (see e.g. Ref. [4]) that there is no good operator, which can be related to time. In particular, we cannot construct a state which is the eigenvector of the time operator with the eigenvalue -5000 years BC or 2006 years AD.

It is also well known that, when quantum mechanics is combined with relativity, there is no operator satisfying all the properties of the spatial position operator (see e.g. Ref. [5]). In other words, the coordinate cannot be exactly measured by itself even in situations when exact measurement is allowed by uncertainty principle.

In the introductory section of the well-known textbook [6] simple arguments are given that for a particle with the mass $m$ the coordinate cannot be measured with the accuracy better than the Compton wave length $\hbar/mc$. Therefore exact measurement is possible only either in the nonrelativistic limit (when $c \to \infty$) or classical limit (when $\hbar \to 0$).

In particular, the quantity $x$ in the Lagrangian density $L(x)$ is only a parameter which becomes the coordinate in the nonrelativistic or classical limit. Note that even in the standard formulation of LQFT, the Lagrangian is only an auxiliary tool for constructing Hilbert spaces and operators. After this construction has been done, one can safely forget about Lagrangian and concentrate his or her efforts on calculating different observables. As Rosenfeld writes in his memoirs about Bohr [7]: *His (Bohr) first remark ... was that field components taken
at definite space-time points are used in the formalism as idealization without immediate physical meaning; the only meaningful statements of the theory concern averages of such fields components over finite space-time regions....Bohr certainly never showed any respect for the noble elegance of a Lagrangian principle.

The facts that the time and coordinate are not measurable were known already in 30th of the last century and became very popular in 60th (recall the famous Heisenberg S-matrix program). The authors of Ref. 6 claim that spacetime, Lagrangian and local quantum fields are rudimentary notions which will disappear in the ultimate quantum theory. Since that time, no arguments questioning those ideas have been given, but in view of the great success of gauge theories in 70th and 80th, these ideas became almost forgotten.

The success of gauge theories and new results in the string theory have revived the hope that Einstein’s dream about geometrization of physics could be implemented. Einstein said that the left-hand-side of his equation of General Relativity (GR), containing the Ricci tensor, is made from gold while the right-hand-side containing the energy-momentum tensor of the matter is made from wood. Since that time a lot of efforts have been made to derive physics from geometry of spacetime. The modern ideas in the (super)string theory are such that quantum gravity comes into play at Plank distances and all the existing interactions can be described if we find how the extra dimensions are compactified. These investigations involve very sophisticated methods of topology, algebraic geometry etc.

We believe that such investigations might be of mathematical interests and might give interesting results but cannot lead to ultimate quantum theory. It is rather obvious that geometrical and topological ideas originate from our macroscopic experience. For example, the water in the ocean seems to be continuous and is described with a good accuracy by equations of hydrodynamics. At the same time, we understand that this is only an approximation and in fact the water is discrete. As follows from the above discussion, the notion of space-time at Planck distances does not have any physical significance and
therefore methods involving geometry, topology, manifolds etc. at such distances cannot give a reasonable physics.

While the notion of spacetime can only be a good approximation at some conditions, the notion of empty spacetime fully contradicts the basic principles of quantum theory that only measurable quantities can have a physical meaning. Meanwhile the modern theories often begin with the background empty spacetime. Many years ago, when quantum theory was not known, Mach proposed his famous principle, according to which the properties of space at a given point depend on the distribution of masses in the whole Universe. This principle is fully in the spirit of quantum theory.

As described in a wide literature (see e.g. Refs. [8, 9, 1] and references therein), Mach’s principle was a guiding one for Einstein in developing GR, but when it has been constructed, it has been realized that GR does not contain Mach’s principle at all! As noted in Refs. [8, 9, 1], this problem is not closed.

Consider now how one should define the notion of elementary particles. Although particles are observable and fields are not, in the spirit of LQFT fields are more fundamental than particles, and a possible definition is as follows [10]: 'It is simply a particle whose field appears in the Lagrangian. It does not matter if it’s stable, unstable, heavy, light — if its field appears in the Lagrangian then it’s elementary, otherwise it’s composite'.

Another approach has been developed by Wigner in his investigations of unitary irreducible representations (IRs) of the Poincare group [11]. In view of this approach, one might postulate that a particle is elementary if the set of its wave functions is the space of a unitary IR of the symmetry group in the given theory.

Although in standard well-known theories (QED, electroweak theory and QCD) the above approaches are equivalent, the following problem arises. The symmetry group is usually chosen as a group of motions of some classical manifold. How does this agree with the above discussion that quantum theory in the operator formulation should not contain spacetime? A possible answer is as follows. One can notice that
for calculating observables (e.g. the spectrum of the Hamiltonian) we need in fact not a representation of the group but a representation of its Lie algebra by Hermitian operators. After such a representation has been constructed, we have only operators acting in the Hilbert space and this is all we need in the operator approach. The representation operators of the group are needed only if it is necessary to calculate some macroscopic transformation, e.g. time evolution. In the approximation when classical time is a good approximate parameter, one can calculate evolution, but nothing guarantees that this is always the case (for example, it is obviously unreasonable to describe states of a system every $10^{-1000}\text{sec}$ or consider translations by $10^{-1000}\text{cm}$). Let us also note that in the stationary formulation of scattering theory, the S-matrix can be defined without any mentioning of time (see e.g. Ref. [12]). For these reasons we can assume that on quantum level the symmetry algebra is more fundamental than the symmetry group.

In other words, instead of saying that some operators satisfy commutation relations of a Lie algebra $A$ because spacetime $X$ has a group of motions $G$ such that $A$ is the Lie algebra of $G$, we say that there exist operators satisfying commutation relations of the Lie algebra $A$ such that: for some operator functions $\{O\}$ of them the classical limit is a good approximation, a set $X$ of the eigenvalues of the operators $\{O\}$ represents a classical manifold with the group of motions $G$ and its Lie algebra is $A$. This is not of course in the spirit of famous Klein’s Erlangen program or LQFT.

Consider for illustration the well-known example of nonrelativistic quantum mechanics. Usually the existence of the Galilei spacetime is assumed from the beginning. Let $(r,t)$ be the spacetime coordinates of a particle in that spacetime. Then the particle momentum operator is $-i\partial/\partial r$ and the Hamiltonian describes evolution by the Schroedinger equation. In our approach one starts from an IR of the Galilei algebra. The momentum operator and the Hamiltonian represent four of ten generators of such a representation. If it is implemented in a space of functions $\psi(p)$ then the momentum operator is simply the operator of multiplication by $p$. Then the position operator can be $de$-
fined as \(i\partial/\partial p\) and time can be defined as an evolution parameter such that evolution is described by the Schroedinger equation with the given Hamiltonian. Mathematically the both approaches are equivalent since they are related to each other by the Fourier transform. However, the philosophies behind them are essentially different. In the second approach there is no empty spacetime (in the spirit of Mach’s principle) and the spacetime coordinates have a physical meaning only if there are particles for which the coordinates can be measured.

Summarizing our discussion, we assume that, by definition, on quantum level a Lie algebra is the symmetry algebra if there exist physical observables such that their operators satisfy the commutation relations characterizing the algebra. Then, a particle is called elementary if the set of its wave functions is a space of IR of this algebra by Hermitian operators. Such an approach is in the spirit of that considered by Dirac in Ref. [13]. By using the abbreviation ‘IR’ we will always mean an irreducible representation by Hermitian operators.

2 Galois fields vs. 'infinite' mathematics

The standard mathematics used in quantum physics is essentially based on the notion of infinity. Although any realistic calculation involves only a finite number of numbers, one usually believes that in principle any calculation can be performed with arbitrary large numbers and with any desired accuracy.

Suppose we wish to verify that \(100 + 200 = 200 + 100\). In the spirit of quantum theory it is insufficient to just say that \(100 + 200 = 300\) and \(200 + 100 = 300\). We should describe an experiment where these relations will be verified. In particular, we should specify whether we have enough resources to represent the numbers 100, 200 and 300.

We believe the following observation is very important: although standard mathematics is a part of our everyday life, people typically do not realize that standard mathematics is implicitly based on the assumption that one can have any desirable amount of resources. Also the standard mathematics is based on some assumptions the va-
lidity of which cannot be verified in principle (e.g. Zorn’s lemma or Zermelo’s axiom of choice). This obviously contradicts basic principles of quantum theory.

Suppose, however that our Universe is finite. Then the amount of resources cannot be infinite and it is natural to assume that there exists a fundamental number \( p \) such that all calculations can be performed only modulo \( p \).

One might think that division is a natural operation since we know from everyday experience that any macroscopic object can be divided by two, three and even a million parts. But is it possible to divide by two or three the electron or neutrino? It is obvious that if elementary particles exist, then division has only a limited sense. Indeed, let us consider, for example, the gram-molecule of water having the mass 18 grams. It contains the Avogadro number of molecules \( 6 \cdot 10^{23} \). We can divide this gram-molecule by ten, million, billion, but when we begin to divide by numbers greater than the Avogadro one, the division operation loses its sense. The obvious conclusion is that the notion of division has a sense only within some limits.

What conclusion should be drawn from the above observations? We essentially have the following dilemma. The first possibility is to accept that standard mathematics is nevertheless suitable for describing phenomena with any numbers but not all of the phenomena can occur in our Universe. Another possibility is to assume that there exists a number \( p \) such that no physical quantity can have a value greater than \( p \). In that case mathematics describing physics should obviously involve only numbers not greater than \( p \); in particular such a mathematics does not contain the actual infinity at all. It is clear that only the second possibility is in the spirit of quantum theory.

The above dilemma has a well known historical analogy. A hundred years ago nobody believed that there exists an absolute limit of speed. People did not see any reason which in principle does not allow any particle to have an arbitrary speed. The special theory of relativity does not show that the Newtonian mechanics is wrong: it is correct but only for phenomena where velocities are much less than the
Our next example is as follows. Suppose we wish to compute as many decimal digits of \( \pi \) as possible and we can build a computer as big as the Solar system which will compute \( \pi \) for thousands of years. It is clear that if \( N \) is the number of elementary particles in the Universe we will have no room for writing down \( N + 1 \) decimal digits of \( \pi \). Therefore all the digits cannot be computed even in principle.

It is well known that mathematics starts from natural numbers (recall the famous Kronecker expression: 'God made the natural numbers, all else is the work of man') which have a clear physical meaning. However only two operations are always possible in the set of natural numbers: addition and multiplication.

In order to make addition reversible, we introduce negative integers -1, -2 etc. Then, instead of the set of natural numbers we get the ring of integers where three operations are always possible: addition, subtraction and multiplication. However, negative numbers do not have a direct physical meaning (we cannot say, for example, 'I have minus two apples'). Their only role is to make addition reversible.

The next step is the transition to the field of rational numbers in which all four operations (except division by zero) are possible. However, as noted above, division has only a limited sense.

In mathematics the notion of linear space is widely used, and such important notions as the basis and dimension are meaningful only if the space is considered over a field or body. Therefore if we start from natural numbers and wish to have a field, we have to introduce negative and rational numbers. However, if, instead of all natural numbers, we consider only \( p \) numbers 0, 1, 2, \( p - 1 \) where \( p \) is prime (we treat zero as a natural number) then we can easily construct a field without adding any new elements. This construction, called Galois field, contains nothing that could prevent its understanding even by pupils of elementary schools.

Let us denote the set of numbers 0, 1, 2, \( p - 1 \) as \( GF(p) \). Define addition and multiplication as usual but take the final result modulo \( p \). For simplicity, let us consider the case \( p = 5 \). Then \( F_5 \) is
the set 0, 1, 2, 3, 4. Then 1 + 2 = 3 and 1 + 3 = 4 as usual, but 2 + 3 = 0, 3 + 4 = 2 etc. Analogously, 1 ⋅ 2 = 2, 2 ⋅ 2 = 4, but 2 ⋅ 3 = 1, 3 ⋅ 4 = 2 etc. By definition, the element \( y \in GF(p) \) is called opposite to \( x \in GF(p) \) and is denoted as \(-x\) if \( x + y = 0 \) in \( GF_p \). For example, in \( GF(5) \) we have -2=3, -4=1 etc. Analogously \( y \in GF(p) \) is called inverse to \( x \in GF(p) \) and is denoted as \( 1/x \) if \( xy = 1 \) in \( GF(p) \). For example, in \( GF(5) \) we have 1/2=3, 1/4=4 etc. It is easy to see that addition is reversible for any natural \( p > 0 \) but for making multiplication reversible we should choose \( p \) to be a prime. Otherwise the product of two nonzero elements may be zero modulo \( p \). If \( p \) is chosen to be a prime then indeed \( GF(p) \) becomes a field without introducing any new objects (like negative numbers or fractions). For example, in this field each element can obviously be treated as positive and negative simultaneously!

One might say: well, this is beautiful but impractical since in physics and everyday life 2+3 is always 5 but not 0. Let us suppose, however that fundamental physics is described not by ’usual mathematics’ but by ’mathematics modulo \( p \)’ where \( p \) is a very large number. Then, operating with numbers much smaller than \( p \) we will not notice this \( p \), at least if we only add and multiply. We will notice a difference between ’usual mathematics’ and ’mathematics modulo \( p \)’ only while operating with numbers comparable to \( p \).

We can easily extend the correspondence between \( GF(p) \) and the ring of integers \( Z \) in such a way that subtraction will also be included. Since the field \( GF(p) \) is cyclic (adding 1 successively, we will obtain 0 eventually), it is convenient to visually depict its elements by points of a circle of the radius \( p/2\pi \) on the plane \((x,y)\). In Fig. 1 only a part of the circle near the origin is depicted. Then the distance between neighboring elements of the field is equal to unity, and the elements 0, 1, 2,... are situated on the circle counterclockwise. At the same time we depict the elements of \( Z \) as usual such that each element \( z \in Z \) is depicted by a point with the coordinates \((z,0)\). We can denote the elements of \( GF(p) \) not only as 0, 1,... \( p-1 \) but also as 0, ±1, ±2,...±(\( p - 1 \))/2, and such a set is called the set of minimal residues.
Let $f$ be a map from $GF(p)$ to $\mathbb{Z}$, such that the element $f(a) \in \mathbb{Z}$ corresponding to the minimal residue $a$ has the same notation as $a$ but is considered as the element of $\mathbb{Z}$. Denote $C(p) = p^{1/(\ln p)^{1/2}}$ and let $U_0$ be the set of elements $a \in GF(p)$ such that $|f(a)| < C(p)$. Then if $a_1, a_2, \ldots, a_n \in U_0$ and $n_1, n_2$ are such natural numbers that

$$n_1 < (p - 1)/2C(p), \quad n_2 < \ln((p - 1)/2)/(\ln p)^{1/2}$$

then

$$f(a_1 \pm a_2 \pm \ldots a_n) = f(a_1) \pm f(a_2) \pm \ldots f(a_n)$$

if $n \leq n_1$ and

$$f(a_1a_2\ldots a_n) = f(a_1)f(a_2)\ldots f(a_n)$$

if $n \leq n_2$.

The meaning of the above relations is simple: although $f$ is not a homomorphism of rings $GF(p)$ and $\mathbb{Z}$, but if $p$ is sufficiently large, then for a sufficiently large number of elements of $U_0$ the addition, subtraction and multiplication are performed according to the same rules as for elements $z \in \mathbb{Z}$ such that $|z| < C(p)$. Therefore $f$ can be treated as a local isomorphism of rings $GF(p)$ and $\mathbb{Z}$.

The above discussion has a well known historical analogy. For many years people believed that our Earth was flat and infinite, and only after a long period of time they realized that it was finite and had a curvature. It is difficult to notice the curvature when we deal only with distances much less than the radius of the curvature $R$. Analogously one might think that the set of numbers describing physics has a curvature defined by a very large number $p$ but we do not notice it when we deal only with numbers much less than $p$. 

\[\text{Figure 1: Relation between } GF(p) \text{ and the ring of integers}\]
Let us note that even for elements from $U_0$ the result of division in the field $GF(p)$ differs generally speaking, from the corresponding result in the field of rational number $Q$. For example the element $1/2$ in $GF(p)$ is a very large number $(p + 1)/2$. For this reason one might think that physics based on Galois fields has nothing to with the reality. We will see in the subsequent section that this is not so since the spaces describing quantum systems are projective.

Since the standard quantum theory is based on complex numbers, the question arises whether it is possible to construct a finite analog of such numbers. By analogy with the field of complex numbers, we can consider a set $GF(p^2)$ of $p^2$ elements $a + bi$ where $a, b \in GF(p)$ and $i$ is a formal element such that $i^2 = 1$. The question arises whether $GF(p^2)$ is a field, i.e. we can define all the four operations excepting division by zero. The definition of addition, subtraction and multiplication in $GF(p^2)$ is obvious and, by analogy with the field of complex numbers, one could define division as $1/(a + bi) = a/(a^2 + b^2) - ib/(a^2 + b^2)$ if $a$ and $b$ are not equal to zero simultaneously. This definition can be meaningful only if $a^2 + b^2 \neq 0$ in $GF(p)$ for any $a, b \in GF(p)$ i.e. $a^2 + b^2$ is not divisible by $p$. Therefore the definition is meaningful only if $p$ cannot be represented as a sum of two squares and is meaningless otherwise. We will not consider the case $p = 2$ and therefore $p$ is necessarily odd. Then we have two possibilities: the value of $p \pmod 4$ is either 1 or 3. The well known result of number theory (see e.g. the textbooks [14]) is that a prime number $p$ can be represented as a sum of two squares only in the former case and cannot in the latter one. Therefore the above construction of the field $GF(p^2)$ is correct if $p = 3 \pmod 4$. In that case the above local homomorphism of the rings $Z$ and $GF(p)$ can be extended to the homomorphism between the rings $Z + iZ$ and $GF(p^2)$ if we consider a set $U$ such that $a + bi \in U$ if $a \in U_0$ and $b \in U_0$.

The first impression is that if Galois fields can somehow replace the conventional field of complex numbers then this can be done only for $p$ satisfying $p = 3 \pmod 4$ and therefore the case $p = 1 \pmod 4$ is of no interest for this purpose. It can be shown however, [15] that
correspondence between complex numbers and Galois fields containing $p^2$ elements can also be established if $p = 1 \pmod{4}$. Nevertheless, arguments given in Refs. [16, 17] indicate that if quantum theory is based on a Galois field then $p$ is probably such that $p = 3 \pmod{4}$ rather than $p = 1 \pmod{4}$. In general, it is well known (see e.g. Ref. [14]) that any Galois field consists of $p^n$ elements where $p$ is prime and $n > 0$ is natural. The numbers $p$ and $n$ define the field $F_{p^n}$ uniquely up to isomorphism and $p$ is called the characteristic of the Galois field.

As discussed in Sect. 1, one of the main problems in modern theory is the existence of infinities. A desire to have a theory without divergencies is probably the main motivation for developing modern theories extending LQFT, e.g. loop quantum gravity, noncommutative quantum theory, string theory etc. For example, the main idea of the string theory is that a string is a less singular object than a point and this gives hope that such a theory will have no divergencies. At the same time, in GFQT divergencies cannot exist in principle since any Galois field is finite.

The idea to replace the field of complex numbers in quantum theory by something else is well known. There exists a wide literature where quantum theory is based on quaternions, p-adic numbers or other constructions. However, as noted above, if we accept that the future quantum physics should not contain the actual infinity at all then the only possible choice is a Galois field or even a Galois ring.

3 Correspondence between standard theory and GFQT

The usual requirement for any new theory is that at some conditions the theory should reproduce well known results of the standard theory. In other words, there should exist a correspondence principle between the new and standard theories. The existence of such a principle does not mean of course that the theories should be absolutely identical; for some phenomena the predictions of the theories may be essentially different. The above discussion gives ground to believe that the standard
quantum theory could be treated as a limit of GFQT when \( p \to \infty \) in the same sense as classical nonrelativistic mechanics is a limit of classical relativistic mechanics when \( c \to \infty \) and a limit of nonrelativistic quantum mechanics when \( \hbar \to 0 \). The correspondence between GFQT and the standard quantum theory has been discussed in detail in Refs. \[15, 17\] and below we describe the main ideas of this correspondence.

A well known historical fact is that originally quantum theory has been proposed in two formalisms which seemed to be essentially different: the Schroedinger wave formalism and the Heisenberg operator (matrix) formalism. It has been shown later by Born, von Neumann and others that the both formalisms are equivalent and, in addition, the path integral formalism has been developed. A direct correspondence between GFQT and the standard theory is rather straightforward if the standard theory is considered in the operator formalism.

We define GFQT as a theory where quantum states are represented by elements of a linear projective space over a field \( GF(p^2) \) and physical quantities are represented by linear operators in that space. Then a Lie algebra \( \mathcal{A} \) over \( GF(p) \) is called the symmetry algebra if the operators in \( GF(p^2) \) representing the observables belong to a representation of \( \mathcal{A} \) in \( GF(p^2) \). If this representation is irreducible then the system is called elementary particle.

At the same time, in the standard theory quantum systems are described by representations of real Lie algebra in projective Hilbert spaces. We first have to understand how the correspondence between projective Hilbert spaces and projective linear spaces over \( GF(p^2) \) can be established.

The first observation is that Hilbert spaces in quantum physics contain a big redundancy of elements. Indeed, with any desired accuracy any element of the Hilbert space can be approximated by a finite linear combination

\[
\psi = \tilde{c}_1 \tilde{e}_1 + \tilde{c}_2 \tilde{e}_2 + ... \tilde{c}_N \tilde{e}_N
\]  

(2)

where the \( \tilde{e}_1, \tilde{e}_2, ... \) are the basis elements and the coefficients \( \tilde{c}_1, \tilde{c}_2, ... \) are complex rational numbers (it is well known that in any separable
Hilbert space the elements (2) are dense in this space). However, even the set (2) contains too many elements which are not needed. Indeed, the Hilbert spaces in quantum theory are projective, i.e. $\psi$ and $c\psi$ represent the same state. This is a consequence of the fact that only ratios of probabilities are meaningful while the probability by itself have no significance. In particular, the usual normalization by one is only a matter of convenience for those who like such a normalization. Therefore we can multiply the both parts of Eq. (2) by the common denominator of the coefficients. In other words, it is sufficient to consider only such elements where the coefficients have the form $\tilde{c}_j = \tilde{a}_j + i\tilde{b}_j$, $\tilde{a}_j$ and $\tilde{b}_j$ are integers and $i$ is the imaginary unity.

Consider now the elements in $GF(p^2)$, which have the form

$$x = c_1e_1 + c_2e_2 + ... c_Ne_N \quad (3)$$

such that $f(c_j) = \tilde{c}_j$ where the map $f$ is defined in the preceding section. We also can supply the space over $GF(p^2)$ by a scalar product $(x, y) \in GF(p^2)$ such that

$$(x, y) = \overline{(y, x)}, \quad (cx, y) = \overline{c(x, y)}, \quad (x, cy) = c(x, y) \quad (4)$$

where the complex conjugation in $GF(p^2)$ is fully analogous to the standard complex conjugation if $p = 3 \pmod{4}$. If $f((e_j, e_k)) = (\tilde{e}_j, \tilde{e}_k)$ then there exists the correspondence between the elements given by Eqs. (2) and (3).

Analogously we can define the correspondence between the operators in projective Hilbert spaces and projective spaces over $GF(p^2)$ [15, 17]. The idea of the correspondence is rather transparent: we first transform the wave functions to make the coefficients in Eq. (2) complex integers and if the magnitude of the coefficients is much less than $p$ than such states are practically indistinguishable from elements from a linear space over $GF(p^2)$. If $p$ is very large then there exist many states corresponding to each other.

We believe that the above construction also sheds light on the fact that the notion of probability is a good illustration of the Kronecker expression. Indeed, suppose that we have conducted an experiment $N$
times, the first event occurred \( n_1 \) times, the second \(- n_2 \) times etc. such that \( n_1 + n_2 + \ldots = N \). Therefore the experiment is fully described by a set of natural numbers. However, people define rational numbers \( w_i(N) = n_i/N \) and then define the limit when \( N \to \infty \).

The idea to apply Galois fields to quantum physics has been considered by several authors (see e.g. Refs. [18]). We believe that our proposal is extremely natural and straightforward: to take the standard Heisenberg operator approach to quantum theory and replace complex numbers by a Galois field. To the best of our knowledge, such an approach has not been discussed in the literature.

4 Poincare invariance vs. de Sitter invariance

The next problem in constructing GFQT is the choice of the symmetry algebra. Consider first the choice of such an algebra in the standard theory.

As follows from our definition of symmetry on quantum level, the standard theory is Poincare invariant if the representation operators for the system under consideration satisfy the well-known commutation relations

\[
[P^\mu, P^\nu] = 0, \quad [M^{\mu\nu}, P^\rho] = -2i(g^{\mu\rho}P^\nu - g^{\nu\rho}P^\mu),
\]

\[
[M^{\mu\nu}, M^{\rho\sigma}] = -2i(g^{\mu\rho}M^{\nu\sigma} + g^{\nu\sigma}M^{\mu\rho} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma}) \quad (5)
\]

where \( \mu, \nu, \rho, \sigma = 0, 1, 2, 3 \), \( P^\mu \) are the four-momentum operators, \( M^{\mu\nu} \) are the representation operators of the Lorenz algebra and the metric tensor has the nonzero components \( g^{00} = -g^{11} = -g^{22} = -g^{33} = 1 \).

Eq. (5) is written in units \( \hbar/2 = c = 1 \). Then the spin of any particle is integer, the spin of fermions is odd and the spin of bosons is even. Such a choice is convenient for establishing correspondence with GFQT (in units where \( \hbar = 1 \) the spin of fermions is half-integer and, as noted in Sect. 3, the value of \( 1/2 \) in a Galois field is a big number \( (p + 1)/2 \).
The question arises whether Poincare invariant quantum theory can be a starting point for its generalization to GFQT. The answer is probably ‘no’ and the reason is the following. GFQT is discrete and finite because the only numbers it can contain are elements of a Galois field. Those elements cannot have a dimension and operators in GFQT cannot have the continuous spectrum.

One might argue that quantum physics should describe the results of measurements and, by definition, any measurement is performed by a classical observer. So any quantum theory should necessarily contain three quantities with the dimensions of mass, time and length.

We believe that a possible counterargument is as follows. Any quantum theory should contain two parts: 1) a parts describing universal relations (which do not depend on whether some phenomenon takes place on the Earth now or at the very early stage of the Universe when classical measurements were not possible at all); 2) a part describing how classical measurements are interpreted in terms of those relations. It is natural to believe that part 1) should not depend on any dimensional constants (e.g. $\hbar, c, G$ etc.) and in particular on whether those constants are really constant in time.

Consider, for example such a physical quantity as angular momentum. In units $\hbar/2 = 1$ any angular momentum can be only an integer. So we can describe the value of the angular momentum either by an integer or in units $kg \cdot m^2/s$. It is natural to believe that only the first description is universal while the second one reflects only our macroscopic experience.

The angular momentum operators and Lorenz boost operators are dimensionless in units $\hbar/2 = c = 1$ but then the momentum operators have the dimension of the inverse length. In addition, the momentum operators and the operators of the Lorenz boosts contain the continuous spectrum.

Let us recall however the well-known fact that conventional Poincare invariant theory is a special case of de Sitter invariant one. The symmetry algebra of the de Sitter invariant quantum theory can
be either so(2,3) or so(1,4). Those algebras are the Lie algebras of symmetry groups of the four-dimensional manifolds in the five-dimensional space, defined, respectively, as
\[ \pm x_5^2 + x_0^2 - x_1^2 - x_2^2 - x_3^2 = \pm R^2 \] (6)
where a constant \( R \) has the dimension of length. We use \( x_0 \) to denote the conventional time coordinate and \( x_5 \) to denote the fifth coordinate. The notation \( x_5 \) rather than \( x_4 \) is used since in the literature the latter is sometimes used to denote \( ix_0 \).

The quantity \( R^2 \) in the two cases of Eq. (6) is often written, respectively, as \( R^2 = \mp 3/\Lambda \) where \( \Lambda \) is the cosmological constant. The existing astronomical data show that it is very small. In the literature the latter case is often called the de Sitter (dS) space while the former is called the anti de Sitter (AdS) one.

The both de Sitter algebras are ten-parametric, as well as the Poincare algebra. However, in contrast to the Poincare algebra, all the representation operators of the de Sitter algebras are dimensionless (in units \( \hbar/2 = c = 1 \)). The commutation relations can now be written in the form of one tensor equation
\[ [M^{ab}, M^{cd}] = -2i(g^{ac}M^{bd} + g^{bd}M^{cd} - g^{ad}M^{bc} - g^{bc}M^{ad}) \] (7)
where \( a, b, c, d \) take the values 0,1,2,3,5 and the operators \( M^{ab} \) are antisymmetric. The diagonal metric tensor has the components \( g^{00} = -g^{11} = -g^{22} = -g^{33} = 1 \) as usual, while \( g^{55} = 1 \) for the algebra so(2,3) and \( g^{55} = -1 \) for the algebra so(1,4).

When \( R \) is very large, the transition from the de Sitter symmetry to Poincare one (this procedure is called contraction [19]) is performed as follows. We define the operators \( P^\mu = M^{\mu5}/2R \). Then, when \( M^{\mu5} \to \infty, R \to \infty \), but their ratio is finite, Eq. (7) splits into the set of expressions given by Eq. (5).

Note that our definition of the de Sitter symmetry on quantum level does not involve the cosmological constant at all. It appears only if one is interested in interpreting results in terms of the de Sitter spacetime or in the Poincare limit. Since all the operators \( M^{ab} \) are
dimensionless in units $\hbar/2 = c = 1$, the de Sitter invariant quantum theories can be formulated only in terms of dimensionless variables.

If one assumes that spacetime is fundamental then in the spirit of GR it is natural to think that the empty space is flat, i.e. that the cosmological constant is equal to zero. This was the subject of the well-known dispute between Einstein and de Sitter described in a wide literature (see e.g. Refs. [9, 20] and references therein). In the LQFT the cosmological constant is given by a contribution of vacuum diagrams, and the problem is to explain why it is so small. On the other hand, if we assume that symmetry on quantum level in our formulation is more fundamental, then the cosmological constant problem does not arise at all. Instead we have a problem of why nowadays Poincare symmetry is so good approximate symmetry.

Summarizing the above discussion, we see that elementary particles in GFQT can be investigated by considering IRs of the so(2,3) or so(1,4) algebras over a Galois field $GF(p^2)$. The case so(2,3) has been discussed in detail in Ref. [17] and the main results are described below.

5 Results and discussion

The original motivation for investigating GFQT was as follows. Let us take the standard QED in dS or AdS space, write the Hamiltonian and other operators in angular momentum representation and replace standard IRs for the electron, positron and photon by corresponding IRs over $GF(p^2)$. Then we will have a theory with a natural cutoff $p$ and all renormalizations will be well defined. In other words, instead of the standard approach, which, according to Polchinski’s joke [21], is essentially based on the formula $\infty - \infty = \text{physics}$, we will have a well defined scheme. One might treat this motivation as an attempt to substantiate standard momentum regularizations (e.g. the Pauli-Villars regularization) at momenta $p/R$ (where $R$ is the radius of the Universe). In other terms this might be treated as introducing fundamental length of order $R/p$. We now discuss reasons explaining why this naive attempt fails.
Consider first the construction of IR over $GF(p^2)$ for the electron. We start from the state with the minimum energy (where energy=mass) and gradually construct states with higher and higher energies. In such a way we are moving counterclockwise along the circle on Fig. 1 in Sect. 2. Then sooner or later we will arrive at the left half of the circle, where the energy is negative, and finally we will arrive at the point where energy=-mass. In other words, instead of the analog of IR describing only the electron, we obtain an IR describing the electron and positron simultaneously. In general the following conclusion can be drawn: IRs of the AdS algebra over a Galois field have the following properties:

i) The representation space of any IR necessarily contains states describing both, a particle and its antiparticle. In particular, there are no IRs such that their representation space describes only a particle without its antiparticle and vice versa;

ii) There are no IRs describing neutral particles i.e. particles which do not have distinct antiparticles.

This result is extremely simple and beautiful since it shows that in GFQT the very existence of antiparticles immediately follows from the fact that any Galois field is finite.

In the standard theory a particle and its antiparticle are described by different IRs but they are combined together by a local covariant equation (e.g. the Dirac equation). We see that in GFQT the idea of the Dirac equation is implemented without assuming locality but already at the level of IRs.

Our construction immediately explains why a particle and its antiparticle have the same mass and spin but opposite charges. While in the standard theory this is a consequence of the CPT theorem, which in particular involves locality, in GFQT no locality is required.

One might immediately conclude that since, as a consequence of ii), the photon in GFQT cannot be elementary, this theory cannot be realistic and does not deserve attention. We believe however, that the nonexistence of neutral elementary particles in GFQT shows that the photon, the graviton and other neutral particles should be considered
on a deeper level. For example, several authors considered a model where the photon is a composite state of Dirac singletons [22].

In my discussions with physicists, some of them commented GFQT as follows. This is the approach where a cutoff (the characteristics $p$ of the Galois field) is introduced from the beginning and for this reason there is nothing strange in the fact that the theory does not have infinities. It has a large number $p$ instead and this number can be practically treated as infinite.

Consider, however the vacuum energy problem. In the standard theory the vacuum energy of the electron-positron field equals $-\infty$. To avoid such an undesirable behavior it is additionally required that all operators in question should be taken in normal ordering. However, the requirement of normal ordering does not follow from the theory; it is simply an extra requirement aiming to obtain the correct value of the vacuum energy. Therefore, if GFQT were simply a theory with a cutoff $p$, one would expect the vacuum energy to be of order $p$. However, since the rules of arithmetic in Galois fields are different, one can prove that [17]

\[ iii) \text{The vacuum is the eigenvector of all the representation operators with the eigenvalues zero without imposing an artificial requirement that the operators should be written in the normal form.} \]

This calculation can be treated as the first example when the quantity, which in the standard theory is infinite, is calculated beyond perturbation theory. The vacuum energy problem is discussed in practically every textbook on LQFT and it is well known that the result $E_{\text{vac}} = -\infty$ was a motivation for Dirac’s hole theory.

The result of GFQT related to the spin-statistics theorem can be formulated as follows [17]

\[ iv) \text{The normal spin-statistics connection simply follows from the requirement that quantum theory should be based on complex numbers. This is a consequence of the famous and elegant fact of number theory that if } p > 2 \text{ then } -1 \text{ can be a square modulo } p \text{ if and only if } p = 1 \text{ (mod } 4\text{). Therefore if } p = 3 \text{ (mod } 4\text{) then the relation } zz^* = -1 \text{ can be valid only if the residue field } \mathbb{Z}/p\mathbb{Z} \text{ is extended.} \]
Recall that in the standard theory the proof of the theorem involves locality. Moreover, there are reasons to believe that GFQT indicates a stronger requirement than the spin-statistics theorem:

v) If the numbers of the physical and nonphysical states should be the same (in the spirit of our understanding of antiparticles) then only fermions can be elementary.

Such a possibility has been discussed in a wide literature. In particular, Heisenberg discussed a possibility that there exists only one fundamental fermion field with the spin 1/2. In our recent Ref. [23] another arguments are given that only fermions could be elementary.

The results i) - v) are based on the consideration of IRs of the so(2,3) algebra over a Galois field \( GF(p^2) \). It is also very interesting to investigate IRs of the so(1,4) algebra over \( GF(p^2) \). In particular, a problem arises whether gravity might be a manifestation of the number \( p \). This problem will be discussed elsewhere.

We see that GFQT sheds a new light on the fundamental problems of physics. We believe, however, that not only this makes GFQT an extremely interesting theory. For centuries, scientists and philosophers have been trying to understand why mathematics is so successful in explaining physical phenomena (see e.g. Ref. [24]). However, such a branch of mathematics as number theory and, in particular, Galois fields, have practically no implications in particle physics. Historically, every new physical theory usually involved more complicated mathematics. The standard mathematical tools in modern quantum theory are differential and integral equations, distributions, analytical functions, representations of Lie algebras in Hilbert spaces etc. At the same time, very impressive results of number theory about properties of natural numbers (e.g. the Wilson theorem) and even the notion of primes are not used at all! The reader can easily notice that GFQT involves only arithmetic of Galois fields (which are even simpler than the set of natural numbers). The very possibility that the future quantum theory could be formulated in such a way, is of indubitable interest.

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