HODGE THEORETIC INVARIANTS DETECTING
TAUTOLOGICAL CYCLES IN MODULI SPACE OF CURVES

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Abstract. We apply the method of higher Abel-Jacobi invariants and cycle classes of Hodge theory to special tautological classes in the Chow ring of moduli space of curves with marked points. Specifically, based on the result of Green and Griffiths of non-triviality of Faber-Pandharipande cycle in self product of a curve of genus $g$, we show certain classes of tautological cycles that we call higher FP-cycles have non-trivial AJ-invariants.

1. Introduction

This paper is devoted to discuss a concept of interface between Hodge theory, moduli space of curves and quantum field theory. It discusses about non-triviality of certain tautological classes in the Chow groups of the moduli space of curves with $n$-ordered marked points using Hodge theoretic invariants detecting algebraic cycles. A historical background on this concept goes back to the work of M. Green, P. Griffiths \cite{8} on a special cycle presented by Faber and Pandharipande on the self product of a curve, calculating the infinitesimal invariants associated to the normal function of the spread of algebraic cycle in a generic family. They study the cycle

\[(1) \quad Z_D = D \times D - n.(\iota_\Delta C), D \in CH^2(C \times C).\]

where $C$ is a smooth curve of genus $g$ and $D$ is zero cycle on $C$ of degree $n$, and $\iota_\Delta$ stands for diagonal. Such a cycle clearly satisfy $\text{deg}(Z) = 0 = \text{Alb}(Z)$. They investigate the question whether $Z_D^{\text{rat}} = 0$. A specific result they prove is that $Z_K \neq 0$ (Theorem 2 in the paper) when $C$ is generic of genus $g \geq 4$, where $K = K_C$ is the canonical class. They also show that for generic divisors $D$ one has $Z_D^{\text{rat}} \neq 0$ when $g \geq 2$, (Theorem 1 in \cite{8}). Their method calculates the second infinitesimal invariant associated to a spread of the cycle (1), denoted $\mathfrak{Z}_D$, in a family $C \times C \to S$ whose generic fiber is $C$. In fact they show that, the second infinitesimal invariant of $\mathfrak{Z}_D$ not vanishes.

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Working on a projective family $\mathcal{X} \to S$ of relative dimension $n$, one considers the Deligne cycle class map

$$c_D : CH^n(\mathcal{X}) \to H^{2n}_D(\mathcal{X}, \mathbb{Q}(n))$$

J. Lewis [20] puts a decreasing Leray filtration $L^\bullet$ on $H^\bullet_D$, where on graded pieces he defines

$$\Psi_i : L^i CH^n(\mathcal{X}) \to \text{Gr}^i H^{2n}_D(\mathcal{X}, \mathbb{Q}(n))$$

One may think of $\Psi_i(3)$ as higher normal functions. The philosophy is to spread out the cycle $Z \in CH^n(X)$ in a generic family over a suitable (quasi) projective variety $S$. In other words $\mathfrak{3} = \text{spread}(Z)$, cf [15], [16], [17]. The simplest case will be $X = \mathcal{X} \times S$. Then

$$Z \in L^i \iff \Psi_0(\mathfrak{3}) = \Psi_1(\mathfrak{3}) = \ldots = \Psi_{i-1}(\mathfrak{3}) = 0$$

The invariants $\Psi_i$ break into parts via the short exact sequence

$$0 \to J^n(\mathcal{X}) \to H^{2n}_D(\mathcal{X}, \mathbb{Q}(n)) \to Hg^n(\mathcal{X}) \to 0$$

The image of $\Psi_i(\mathfrak{3})$ on $Hg^n(\mathcal{X})$ are denoted by $[\mathfrak{3}]_i$ and are called higher cycle classes of $\mathfrak{3}$, or the higher infinitesimal invariants of the normal function of $\mathfrak{3}$. The invariants successively characterize the derivatives of the lift of the normal function in the de Rham complex via the Leray (Hodge) filtration. The part of $\Psi_i(\mathfrak{3})$ contributing to $J^n(\mathfrak{3})$ is denoted by $[AJ(\mathfrak{3})]_{i-1}$ and are called higher Abel-Jacobi invariants of the cycle $\mathfrak{3}$.

The algebraic cycles under consideration in this text come from tautological relations in the Chow group of $M_g$ the moduli space of curve, and also the tautological ring of $C^n_g = C_g \times_{M_g} \ldots \times_{M_g} C_g$ the moduli space of curves with $n$-ordered marked points, where $C_g \to M_g$ is the universal curve of genus $g$. They are build up from the diagonals $\Delta_{ij}$, The $\kappa_i$ and $\psi_i$-classes. The aforementioned cycle $Z_K$ can be formulated as the pull back of the cycle

$$FP_1 := \pi_1^{\times 2}\Delta_{12}.\psi_1 \in CH^2(C^2_g)$$

under an embedding $S \hookrightarrow C^2_g$ for suitable $S$. The maps $\pi_i$, $i = 0, 1, 2$ are projectors of the Chow-Kunneth decomposition of the diagonal $C_g \hookrightarrow C_g \times_{M_g} C_g$. In fact as explained in [24], the restriction of $FP_1$ over $S$ is

$$\mathfrak{3} = \Delta_{12}.\psi_1 - \frac{1}{2g-2}\psi_1\psi_2 \in CH^2(C^2_g)$$

We generalize the definition (6) by defining
We first show that the restriction of this cycle over a generic family \( C \) of curves is non-trivial when \( g \geq 7 \). Our method employs some theorems on the behaviour of higher Abel-Jacobi invariants under product. The unexpected news is, this behaviour is completely non-trivial and complicated, [16]. In fact,

\[
[AJ(3)], \neq 0 \quad \text{and} \quad [AJ(\mathfrak{Y})]_j \neq 0 \quad \Rightarrow \quad [AJ(3 \times \mathfrak{Y})]_{i+j} \neq 0
\]

However under certain considerations, one obtains a valid implication. The paper also considers a study of the Hodge theoretic invariants in the Schur decomposition for self powers of an abelian type motive. Basically by a work of G. Ancona [1], followed by other people (see also [21]) one can obtain a representation theoretic decomposition

\[
h(C^n_g/M_g) \cong \bigoplus_{|\lambda| \equiv n \mod 2} CH^{p-n\lambda}(M_g, V_\lambda)
\]

where \( \lambda \) correspond to highest weights of the Lie group \( Sp(2g) \) and \( h \) stands for the functor of motives. Then the major idea is, this decomposition is compatible with the Leray (Lewis) filtration explained above. In fact we show that

\[
\bigoplus_{\lambda} [AJ(3)]^\lambda \leftrightarrow \Psi_i(3) = \bigoplus_{\lambda} \Psi_i(3)^\lambda \leftrightarrow \bigoplus_{\lambda} [3]^\lambda_i
\]

according to the decomposition in (10). With a bit of chance one may be able to discuss about the non-triviality of \( \Psi_i(3) \) through its weighted part in the right hand side of (11). Along this, we explain an idea due to [3], (see also [19]), using the theorem of Kostant or Borel-Bott-Weil in Lie algebra cohomology to find estimates on the Leray degree in cohomology of local systems appearing in Schur-Weyl construction (10). One may consider a sequence of generic embeddings \( S \hookrightarrow M_g \hookrightarrow A_g \) and look at

\[
H^i(S, V_\lambda) \leftrightarrow H^i(M_g, V_\lambda) \leftrightarrow H^i(A_g, V_\lambda)
\]

In order to discuss the vanishing of higher invariants \( \Psi_i^\lambda \) one may use Kostant to decide where the series of cohomologies above will show to be non-trivial. Of course this can be done when over \( A_g \), but by the result of Fakhruddin it is possible to determine the first non vanishing w.r.t \( i \) for all of them. In this way the existence of sections for the corresponding local systems could be partially discussed using the root systems of \( sp(2g) \).
Another tool explained in [24] is that the action of the correspondences on the Chow group of a tensor power of an abelian motive is given through the action of Brauer algebra, and simply may be studied by composition of Brauer diagrams. In fact in the reference there explained such a correspondence \((C_g^3)^{\times 4} \rightarrow C_g^2\)

\[(13) \quad \mathcal{GS} := \pi_1^{x_3} \Delta_{123}^{x_4} \rightarrow \pi_1^{x_2} \Delta_{12} \psi_1^2 = FP_2\]

relating the Gross-Schoen cycle to Faber-Pandharipande cycles by a correspondence defined with Brauer diagram compositions. It follows that one may deduce the non-triviality of FP-cycles from that of Gross-Schoen. The study of the relations in the tautological ring of \(M_{g,n}\) is one of the important problems in theory of moduli spaces and quantum field theory, [4], [5]. Usually the question of determining that a specific cycle in the Chow ring of an algebraic variety is non-trivial, is an extremely difficult question.

2. Higher cycle and Abel-Jacobi maps on Chow groups

1. Let \(X\) be a projective variety of dimension \(n\) defined over \(\mathbb{C}\), and \(Z \in CH^p(X)\) an algebraic cycle of codimension \(p\). We will consider the spread of \(Z\) on a family \(\pi : \mathfrak{X} \rightarrow S\), where \(S\) is quasi-projective. Let \(sp : CH^p(X) \cong CH^p(\mathfrak{X})\) be the spread map, [15], and set \(\mathfrak{Z} = sp(Z)\). Define

\[(14) \quad \psi : CH^p(X) \cong CH^p(\mathfrak{X}) \xrightarrow{\cong} H_{2p}^D(\mathfrak{X}, \mathbb{Q}(p))\]

where \(c_D\) is the Deligne cycle class map. J. Lewis [20], constructs a (decreasing) Leray filtration \(L^\bullet\) on Deligne cohomology of \(\mathfrak{X}\) in which its graded pieces fit into the short exact sequence

\[(15) \quad 0 \rightarrow Gr_{i-1} J^p(\mathfrak{X}) \xrightarrow{\alpha_{i-1}} Gr_{i} J^p H_{2p}^D(\mathfrak{X}, \mathbb{Q}(p)) \xrightarrow{\beta_i} Gr_{i} J^p g^p(\mathfrak{X}) \rightarrow 0\]

and defines the \(i\)th higher cycle class map

\[(16) \quad cl_X^i(Z) = \beta_i(Gr_{L}^i \psi(Z)) = \beta_i(Gr_{i} c_D(\mathfrak{Z})) = [c_D(\mathfrak{Z})]_i = [\mathfrak{Z}]_i = \psi_i(\mathfrak{Z})\]

When \([\mathfrak{Z}]_i = 0\) we set \(AJ_{L}^{p-1}(Z) = Gr_{L}^i \psi(Z) = Gr_{i} c_D(\mathfrak{Z}) = [AJ(\mathfrak{Z})]_{i-1}\). If we set

\[(17) \quad \psi_i : L^1 CH^p(\mathfrak{X}) \rightarrow Gr_{L}^i H_{2p}^D(\mathfrak{X}, \mathbb{Q}(p)), \quad Z \mapsto Gr_{L}^i \psi(Z)\]

then one defines \(L^\bullet := ker(\psi_{i-1})\) as a decreasing filtration by kernels on \(CH^p(X)\). It can be equivalently defined by \(L^1 CH^p(X) := \psi^{-1}(L^1 H_{2p}^D(\mathfrak{X}, \mathbb{Q}(p)))\), [15], [16], [17].

**Lemma 2.1.** [16] For a cycle \(Z \in CH^p(X)\) with \(\psi_i(Z) \neq 0\) then \(Z \not\sim 0\). Exactly one of the two following holds
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where $Z$

The lemma is a reformulation of the definition. If $Z \in L^i CH^p(X)$ then $\psi_i(Z) = [3]_i$ are called their invariants (higher cycle classes).

The filtration $L$ is characterized by the following properties

One may take $X = X \times S$, where $S$ is very general, (cf. [5]). The fundamental class $[3]_i \in H^i(S, \mathbb{C}) \otimes H^{2p-i}(X, \mathbb{C})$ is a Hodge class in $Gr^i_{L} H^p(S \times X) = H^i(S) \otimes H^{2p-i}(X)$. Therefore one can write

$$[3]_i = \sum_{l \leq i} \sum_{l \leq i} H^{l, l-i}(S) \otimes H^{p-l, p+l-i}(X)$$

The projected image of $[AJ(3)]_{i-1}$ under

$$Gr^i_{L} J^p(X \times S) \to J^d(H^{i-1}(S, \mathbb{C}) \otimes H^{2p-i}(X, \mathbb{C}))$$

is called the transcendental part of the higher Abel-Jacobi classes denoted by $[AJ(3)]_{i-1}^{tr}$. It can be calculated via the map $\{\int_{\partial^{-1} \mathbb{C}}\}$ modulo periods. One may further project

$$J^p(H^{i-1}(S, \mathbb{C}) \otimes H^{2p-i}(X, \mathbb{C})) \to J^p \left( \frac{H^{i-1}(S, \mathbb{C}) \otimes H^{2p-i}(X, \mathbb{C})}{SF^{1,p-i+1}} \right)$$

where

$$SF^{i,j}(H^{r}(S) \otimes H^{s}(X)) := F^i H^{r}(S) \otimes F^j H^{s}(X) + F^{i}H^{r}(S) \otimes \overline{F^{j}H^{s}(X)}$$
We denote the image of Abel-Jacobi map under this projection by $[AJ(3)]_{i-1}$, see [15], [16], [17] for more details.

2. The degeneration of the Leray spectral sequence for $\pi : X \to S$ implies the decomposition

$$H^k(X, \mathbb{Q}) = \bigoplus_{p+q=k} H^p(S, R^q\pi_*\mathbb{Q})$$

(22)

The summands form the graded Leray pieces $Gr^i_L H^k(X, \mathbb{C}) = H^i(S, R^k\pi_*\mathbb{Q})$ of the Leray filtration. By choosing a polarization $\mathcal{L}$ on $X$ one obtains an element $L \in H^0(S, R^2\pi_*\mathbb{Q})$ which is the image of $c_1(\mathcal{L})$. Then we have the Lefschetz decomposition $R^i\pi_*\mathbb{Q} = \bigoplus L^1.P_{q-i}$ where $P_l$ is the local system corresponding to the $l$-th primitive cohomology of fibers. The we also have

$$Gr^i_L H^k(X, \mathbb{C}) = H^i(S, R^{k-i}\pi_*\mathbb{Q}) = \bigoplus_l H^i(S, L^1.P_{k-i-2l})$$

(23)

This suggest that one may determine the graded Leray pieces $H^p(S, R^i\pi_*\mathbb{Q})$ just by having information of primitive parts. The primitive local systems $P_l$ appear as highest weight representations of $Sp(2g)$ and usually denoted as $V_\lambda$. [3].

3. HIGHER INVARIANTS FOR ZERO CYCLES ON PRODUCT OF CURVES

We try to investigate the behavior of the Lewis filtration, the higher cycle and Abel-Jacobi classes in the product families, and specially in self product of curves of genus $g$.

1. Let $C$ be a curve with base point $o$ and set

$$\pi_{C,2} = C \times o, \quad \pi_{C,0} = o \times C, \quad \pi_{C,1} = \Delta - \pi_{C,0} - \pi_{C,2}$$

Assume $X = C_1 \times ... \times C_n$ is a product of $n$ curves. For $\sigma \in S_n$, write $\sigma : (C_1 \times C_1) \times ... \times (C_n \times C_n) \to X \times X$ for the obvious action. Set

$$\pi_{X,\sigma} = \bigoplus_{\alpha_1 + ... + \alpha_n = n} \pi_{C_1,\alpha_1} \times ... \times \pi_{C_n,\alpha_n}$$

(25)

Then according to [15] the filtration $L^\bullet$ is given by

$$L^i CH^n(X) = \bigcap_{\sigma} \ker ((\pi_{X,\sigma})_*)$$

(26)

The projections onto the Chow-Kunneth components can be written as
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\[ P_{\lambda} = \sum_{\substack{j < \lambda \\sigma \in S_j}} (-1)^{\lambda - j} (i_{\sigma} \circ \pi_{\sigma}) \]

where \( \pi_{\sigma} : X \to X_{\sigma} = C_{\sigma(1)} \times \ldots \times C_{\sigma(i)} \) and \( i_{\sigma} : X_{\sigma} \to X \) is obvious maps, \([15]\).

Now let \( f : C \to S \) be a family of smooth curves and set \( R = R^i f_* \mathbb{Q} \). Let \( C^n = C \times \ldots \times C \). A choice of isomorphism

\[ R^i f_* \mathbb{Q} \cong \bigoplus R^i[-i] \]

induces an isomorphism

\[ H^i(C^n) \cong \bigoplus_{\alpha_1 + \ldots + \alpha_n = k} H^{i-k}(S, R^{\alpha_1} \otimes \ldots \otimes R^{\alpha_n}), \quad \alpha_i = 0, 1, 2 \]

Let \( z \in CH^1(C) \) be a relative cycle in codimension 1 and define

\[ \pi_2 = z \times C, \quad \pi_0 = C \times (z - \pi^* \pi_* z^2), \quad \pi_1 = \Delta - \pi_0 - \pi_2 \]

as projectors \( Rf_* \mathbb{Q} \to Rf_* \mathbb{Q} \). They together define an isomorphism (16). Under this choice

\[ H^i(S, R^{\alpha_1} \otimes \ldots \otimes R^{\alpha_n}) = \text{Image}(\pi_{\alpha} = \pi_{\alpha_1} \times \ldots \times \pi_{\alpha_n} : C^n \to C^n) \]

There exist cohomology classes \( \tilde{\pi}_\alpha \in H^{2n}(C^n \times_S C^n, \mathbb{Q}) \) whose images in \( H^0(S, R^{2n}(\pi \times \pi)_* \mathbb{Q}) \) give the \( \pi_{\alpha} \). They have Dolbeault counterparts

\[ \tilde{\pi}^{n,n}_\alpha \in H^n(C^n \times_S C^n, \Omega_{C^n \times_S C^n}) \]

which act as correspondences on Dolbeault cohomologies of \( C^n \). They also act on Dolbeault cohomologies of the fibers in a way compatible with the Leray filtration

\[ pr_{2*} \circ \pi_{p+q-1} \circ pr_{1*} : L^iH^q(C^n, \Omega^p_{C^n}) \to L^iH^q(C^n, \Omega^p_{C^n}) \]

where \( pr_1, pr_2 \) are projections of \( C^n \times_S C^n \), \([26], [24]\).

2. An interesting question concerning the cohomology classes detecting zero cycles is their behavior under the product of cycles. We begin from the following lemma.

Lemma 3.1. \([16]\) If \( Z \in L^iCH_0(X) \) and \( W \in L^jCH_0(Y) \) Then \( Z \times W \in L^{i+j}CH_0(X \times Y) \).

Proof. It is for \( Z \times W = pr_1^*Z \cdot pr_2^*W \), and the sub-multiplicative property. \( \square \)
Proposition 3.2. [15], [16] Let $X$ and $Y$ be smooth projective varieties of dimensions $m$ and $n$ respectively. Let

- $Z \in L^i CH_0(X)$ with $cl_X^i(Z) \neq 0$ and
- $W \in L^j CH_0(Y)$ with $cl_Y^j(W) \neq 0$.

Then $Z \times W \in L^{i+j} CH_0(X \times Y)$ satisfies $cl_{X \times Y}^{i+j}(Z \times W) \neq 0$.

In fact one needs that the base of two spreads $\mathfrak{Z} \subset S_1 \times X$ and $\mathfrak{W} \subset S_2 \times Y$ are relatively generic. Otherwise there could be problems. The situation would be more complicated if we want to express similar problem by higher Abel-Jacobi classes. In fact if $[\mathfrak{Z}]_i = 0$, $[AJ(\mathfrak{Z})]_{i-1} \neq 0$ and $[\mathfrak{W}]_j = 0$, $[AJ(\mathfrak{W})]_{j-1} \neq 0$, then it will be never the case $[AJ(\mathfrak{Z} \times \mathfrak{W})]_{i+j-1} \neq 0$.

Theorem 3.3. [16] Let $X$ and $Y$ be smooth projective varieties of dimension $m$ and $n$ respectively and $Z \in L^i CH_0(X)$ with $cl_X^i(Z) \neq 0$ and $W \in L^{i+j} CH_0(Y)$ with $AJ_X^j(W)^{\text{rat}} \neq 0$, and $cl_Y^l(W) = 0$, $(l > j)$. Then $Z \times W \in L^{i+j} CH_0(X \times Y)$ has $AJ_{X \times Y}^{i+j}(Z \times W)^{\text{rat}} \neq 0$.

An ingredient of the proof of the above proposition is that, the spread $[\mathfrak{Z}] \in CH_0(\mathfrak{X} = S \times X)$ induces a non-trivial holomorphic map of $i$-forms $\Omega^i(X) \to \Omega^i(S)$ via the following identification

\begin{equation}
Gr^X_{\mathfrak{Z}} \mathcal{H}^\mathfrak{Z}(S \times X) = \text{Hom}_{\text{MHS}}(\mathbb{Q}(-n), H^i(X) \otimes H^{2n-i}(X)) \\
= \text{Hom}_{\text{MHS}}(H^i(X), H^i(S)) \to \text{Hom}_C(\Omega^i(X), \Omega^i(S))
\end{equation}

In fact $H^i(S) \otimes H^{2n-i}(X)$ has the factor $H^{i,0}(S) \otimes H^{n-i,n}(X)$ which produces the above projection. When $cl(\mathfrak{Z}) = [\mathfrak{Z}]_i \neq 0$ the above map on is non-trivial, [15], [16].

Corollary 3.4. [15] If $[\mathfrak{Z}] \in CH_0(\mathfrak{X})$ induces a non-zero map $\Omega^i(X) \to \Omega^i(S)$, and $\mathfrak{W} \in CH_0^{\text{rat}}(\mathfrak{Y})$ satisfies $AJ(\mathfrak{W}) := [AJ(\mathfrak{W})]_0 \neq 0$, Then $0 \notin \mathfrak{Z} \times \mathfrak{W} \in CH_0(\mathfrak{X} \times \mathfrak{Y})$ satisfies $[AJ(\mathfrak{Z} \times \mathfrak{W})]_i^{\text{rat}} \neq 0$.

We will use both of the theorems 3.3 and its corollary in the last section.

3. We end this section with a characterization of Lewis filtration in symmetric product of a curve $C$. Assume $S^n C$ is the $n$th symmetric power of a curve $C$. Define

\begin{equation}
o^{i \times i} : S^{n-i} C \to S^n C, \quad z \mapsto i.o + z
\end{equation}

Set $S_{k-1,k} = \{(z, z') \in S^{k-1} C \times S^k C; \ z \leq z'\}$ as a correspondence $S^{k-1} C \vdash S_k C$. 

Proposition 3.5. The decreasing filtration $N_i$ on $CH_0(S^nC)$ defined by

\begin{equation}
N_i CH_0 S^n C := \text{Image} (o^{x_i} : CH_0(S^{n-i}C) \to CH_0(S^nC))
\end{equation}

induces a splitting

\begin{equation}
CH_0(S^nC) = \bigoplus_{0 \leq i \leq n} (o^{x_i})_* CH_0(S^{n-i}C)^0
\end{equation}

where $CH_0(S^{n-i}C)^0 = \text{ker} \langle S_{n-i-1,n-i} \rangle_\ast$. The splitting induces a decomposition

\begin{equation}
N_j CH_0 (S^nC) = \bigoplus_{l \leq n-j} (o^{x_l})_\ast (CH_0(S^lC)^0)
\end{equation}

which is opposite to the Lewis filtration $L^i$ on $CH_0(S^nC)$.

Proof. (sketch) We proceed to prove $CH_0(S^nC) = \text{Image}(o_\ast) \oplus \text{ker}(S^\ast_{n-1,n})$. In fact using an induction argument one shows that any $z \in \text{Image} \cap \text{ker}$ lies in the Image$(o^{x_i})_\ast$. The induction step is based on the formula

\begin{equation}
S^\ast_{n-1,n} \circ o_\ast = \text{Id} + o_\ast \circ S^\ast_{n-2,n-1}
\end{equation}

Any point in $S^nC$ is the image of a zero cycle in $C^n$ and any point $x_1 \times \ldots \times x_n \in C^n$ as a zero cycle may be written as

\begin{equation}
(x_1, \ldots, x_n) = pr^\ast_1(x - o) \ldots pr^\ast_n(x - o) + z'
\end{equation}

where $z' = \sum n_i z'_i$ and each $z'_i$ has one factor equal to $o$, one can show that any zero cycle in $S^nC$ can be written as a the desired sum. The remainder of the proof is straightforward. □

Corollary 3.6. The Lewis filtration for $S^nC$ is given by

\begin{equation}
L^{n-i} CH_0(S^nC) = \bigoplus_{l \leq i} (o^{x_l})_\ast CH_0(S^{n-l}C)^0)
\end{equation}

The above theorem has been stated in [27] with further results for K3 surfaces. In fact the proof in the reference works for any abelian motive. We expressed it more special for curves. The interested reader may consult the reference. We may summarize in the following diagram similar to a discussion in [15].
where the top composition is inclusion.

4. Speculation on Schur functors

1. Let $\text{CHM}(S)_\mathbb{Q}$ be the $\mathbb{Q}$-linear, pseudo abelian, rigid, symmetric, tensor category of relative Chow motives, with the contravariant motive functor $M : \text{Var}/S \to \text{CHM}_\mathbb{Q}$, with $\text{Hom}(M(X), M(Y)) := \text{CH}^d(X \times_S Y)_\mathbb{Q}$, when $X, Y \in \text{Var}/S$ with $\text{reldim}_S X = d$. We denote $M(S)$ by $\mathbb{L}$ and $M(\mathbb{P}^1_S)$ by $\mathbb{L} = 1(-1)$. Assume

- $H^* = \oplus_n H^n : \text{Var}/S \to \text{Vec}_\mathbb{Q}^\perp$ is a contravariant functor into the twisted $\mathbb{Q}$-vector spaces.
- $R : \text{CHM}(S)_\mathbb{Q} \to \text{Vec}_\mathbb{Q}^\perp$ is the realization functor, i.e. $R \circ M = H^*$.

Let $\text{Lef}(A)$ be the Lefschetz group of an abelian motive $A$ and $B_{n,\mathbb{Q}}$ the algebra of correspondences, cf [1].

**Theorem 4.1.** [1] The realization functor induces an isomorphism of algebras

\[ R : B_{n,\mathbb{Q}} \xrightarrow{\cong} \text{End}_{\text{Lef}(A)}(H^1(A)^{\otimes n}) \]

Any decomposition of $H^1(A)^{\otimes n}$ as $\text{Lef}(A)$-representation arises canonically from a decomposition of $h^1(A)^{\otimes n}$ in $\text{CHM}(S)_F$. Moreover, two isomorphic sub-representations of $H^1(A)^{\otimes n}$ give rise to two isomorphic motives.

Every representations of $H^i(A^n, \mathbb{Q})$ correspond to a list of orthogonal projectors $\{p_\alpha\}$ of the algebra $\text{End}_{\text{Lef}(A)}(H^1(A)^{\otimes n})$ with pre-images $\pi_\alpha$ in $B_{n,\mathbb{Q}}$ which are unique. The theorem has the following consequences (cf. ref.).

- Any decomposition of $H^i(A^n, \mathbb{Q})$ into $\mathbb{Q}$-subHodge structures lifts canonically to a decomposition over the category $\text{CHM}(\mathbb{C})$ of Chow motives over $\mathbb{C}$, with $\mathbb{Q}$-coefficients.
- Assume $\pi : A^n \to S$ is a fibration. Then any decomposition of $R^i\pi_* \mathbb{Q}$ induced from $\rho : \text{Rep}(G) \to \text{VHS}(S(\mathbb{C}))$ has a canonical lift over the category $\text{CHM}(S)_\mathbb{Q}$, [1].

**Corollary 4.2.** [1], [21], [24] The $\mathbb{C} \to S$ be a relative curve. Assume $V = h^1(C/S)$ be the relative (abelian) motive of $C^n$. Then, one has the decomposition
(44) \[ h(C^n/S) \cong \bigoplus_{|\lambda| \leq n} V^{(\lambda)} \otimes L^{n_\lambda} \otimes \beta^*_\lambda \]

where \( h \) expresses the motive functor.

By Theorem (4.1) we deduce the following.

**Corollary 4.3.** [24] The decomposition in the corollary 3.2 lifts to \( CHM(M_g)_Q \), i.e. one has

\[
CH^p(C^n/S) = \bigoplus_{|\lambda| \leq n} CH^{p-n_\lambda}(S, V_\lambda)
\]

(45)

\[
H^{2p}(C^n/S) = \bigoplus_{|\lambda| \leq n} H^{2(p-n_\lambda)}(S, V_\lambda)
\]

One may write a more general formula as \( h(C^n/S) = \bigoplus h^{(p-n_\lambda)}(S, V_\lambda) \) in motives. The local system \( V_\lambda \) is the part of \((R^1)^{\otimes n}\) which transforms by the representation \( \sigma^*_\lambda \) at it is in the kernel of all the maps

\[
R^1 \otimes \ldots \otimes R^1 \rightarrow R^1 \otimes \ldots \otimes R^2 \otimes \ldots \otimes R^1
\]

When \( |\lambda| = n \) one can construct action of the Brauer algebra by correspondences of \( H^i(X = C^n) \). More specifically, let

(46) \[
\prod_{i,j} \Pi_{ij} \in CH^1(C^{2n}), \quad i, j \in \{1, \ldots, 2n\}
\]

where \( p_{ij} \) is the projection onto the \( i \)-th and \( j \)-th factor. Then any basis element of the Brauer algebra is given by a partition of \( \{1, \ldots, 2n\} \) into 2-element sets. We then correspond this to

(47) \[
\prod_{i,j} \Pi_{ij} \in CH^n(C^{2n})
\]

The explicit calculation of the projectors onto the factors \( H^i(S, V_\lambda) \) has been calculated in [22], see also [24].

We now return to our discussion on the higher cycle classes and Abel-Jacobi invariants.

**Proposition 4.4.** Assume \( X/S \) has a motive of abelian type. The functor \( Gr_L^\bullet \) of the filtration of Lewis (cf. section 1) commutes with Schur functor, i.e we have a commutative diagram
where the vertical sequences are exact in the middle and the last row is the usual Leray filtration. Moreover, under the assumption of HLC+BBC one has

\[ L^iCH^n(X) = \ker \left\{ \bigoplus_{|\lambda| \leq i} (\pi_\lambda)^* \right\} \]

where \( \pi_\lambda \) are the projections of the Schur functor and correspond to all \( \pi_{\alpha_1} \times \cdots \times \pi_{\alpha_n} \) where \( \alpha_1 + \alpha_2 + \cdots + \alpha_n = j \leq n \) and \( j \equiv n \pmod{2} \).

**Proof.** First note that the diagram is commutative without considering \( L^i \) or its graded pieces. This is a consequence of 3.1, 3.2 and 3.3, as well as the three horizontal isomorphisms. The exactness of the vertical sequence in the middle (with or without \( L^i \)) is by definitions. The existence of the projectors \( (\pi_\lambda)^* \) is guaranteed by Theorem 3.1. The relation (50) follows from the discussion in section 2. \( \Box \)

Based on the results in [24] we may also write similar diagram for tautological rings. The diagram (49) may be compared with the similar diagram [17], cf. part (4.13).

**Corollary 4.5.** In the set up of proposition (4.4) if \( \mathfrak{Z} \in CH^n(X) \), then

\begin{itemize}
  \item \( \Psi_i(\mathfrak{Z}) = \bigoplus_{\lambda} \Psi_i(\mathfrak{Z})^\lambda \)
  \item \( [\mathfrak{Z}]_i = \bigoplus_{\lambda} [\mathfrak{Z}]_i^\lambda \)
  \item \( [AJ(\mathfrak{Z})]_{i-1} = \bigoplus_{\lambda} [AJ(\mathfrak{Z})]_{i-1}^\lambda \)
\end{itemize}

**Proof.** Follows from prop. 4.4. In fact from (15) we have

\[ \bigoplus [AJ(\mathfrak{Z})]_{i-1}^\lambda \hookrightarrow \Psi_i(\mathfrak{Z}) = \bigoplus_{\lambda} \Psi_i(\mathfrak{Z})^\lambda \twoheadrightarrow \bigoplus [\mathfrak{Z}]_i^\lambda \]

\( \Box \)

2. Let \( G/\mathbb{C} \) a reductive Lie group, \( P \) a parabolic subgroup and \( g \) and \( p \) be their Lie algebras. Set \( D = G/P = M/T \), the corresponding homogeneous manifold, where \( M \subset G \) is maximal compact, and \( T \) a torus. Assume \( \Phi_+ \) is a system of positive roots for \( g = n_- \oplus h \oplus n_+ \), \( m = \text{Lie}(M) \), and \( \rho \) is one half of the sum of the elements in \( \Phi_+ \).
Theorem 4.6. (Borel-Weil) \[10\] If \( \lambda + \rho \) is singular, i.e. \( (\lambda + \rho, \alpha) = 0 \) for some root in \( \Phi \), then \( H^i(D, V_\lambda) = 0 \) for all \( i \). If \( \lambda + \rho \) is non-singular, then there is unique element \( w \in W \) which takes it into the highest Weyl chamber. The cohomologies \( H^i(D, V_\lambda) \) vanish for \( i \neq l \), where \( l \) is the number of \( \alpha \in \Phi_+ \) s.t. \( w(\alpha) \) is negative. \( H^i(D, V_\lambda) \) is an irreducible \( M \)-module of highest weight \( w(\lambda + \rho) - \rho \).

N. Fakhruddin [3], compares the cohomologies \( H^i(S, V_\lambda) \) with that of \( n = g/p \) namely \( H^i(n, V_\lambda) \) for highest weights \( \lambda \) to establish a vanishing criterion. Set \( W' = \{ w | w(\Phi_-) \cap \Phi_+ \subset \Phi_+ \} \). It is the set of coset representatives for \( W_g/W_m \).

Theorem 4.7. (Kostant) Assume \( V_\lambda \) is an irreducible representation of \( g \) of highest weight \( \lambda \). Then \( H^i(n, V_\lambda) \) is a direct sum of representations of \( m \) with highest weight \( w(\lambda + \rho) - \rho \), the sum being over all \( w \in W' \) of lengths \( m \), with each representation occurring with multiplicity one.

In many places it is written as \( H^i(n, V_\lambda) = \bigoplus_{i(w) = i} C_{w(\lambda + \rho) - \rho} \). Kostant Theorem directly gives Borel-Weil

\[
H^i(D, V_{-\lambda}) = \bigoplus_{\mu} (V_\mu)^* \otimes H^i(n, V_\mu)_\lambda
\]

After pulling back the local system \( V_\lambda \), it becomes trivial and the period map is a local homeomorphism. So we may consider these local systems over the period domain \( D \) of a symplectic Hodge structure, at least for local computations. Specially this applies for the first \( i \) where \( H^i(S, P_l) \) does not vanish.

Proposition 4.8. [3] The cohomologies \( H^i(S, P_l) \) vanish while \( r + i/2 < l/2 \), and it is pure of weight \((i + l)/2\) when \( r + i/2 = l/2 \).

The number \( r = r(g, i, l) \) is related to the Kostant parameter of the representation of \( g \) analogous to Theorem 4.6. In [3] the number \( r \) is calculated explicitly one has

\[
r = \text{Max}\{q \in \mathbb{Z} | q(g - l) + q(g + 1)/2 \leq i \}
\]

The local systems \( P_l \) correspond to highest weight modules \( V_\lambda \). So I denote \( l_\lambda := l \). The Leray filtration on \( H^{2n}(X) \) is studied via the graded pieces \( H^j(S, R^{2n-i}) \). By what explained in the section (1) it can be decomposed further into terms of the form \( H^i(S, L^j \cdot P_{2n-i-2l}) \). Then using the proposition one finds that it vanishes when \( r(g, i, l) + i < n - l_\lambda \), cf. [3]. This provides a method to check out the vanishing of some of the cycle classes \( c^i(3) \) and also their corresponding \( AJ_{i-1}(3) \). Assume \( M_g \) \((\dim = 3g - 3)\) and \( A_g \) \((\dim = g(g + 1)/2)\) are the moduli space of curves of genus \( g \) resp. the moduli space of abelian varieties of dimension \( g \). We have the period map \( M_g \to A_g \) which is generically injective when \( g \geq 7 \).
Corollary 4.9. In a generic embedding as $S \rightarrow M_g \rightarrow A_g$ (when $g \geq 7$), the FIRST degree $i$ such that $L^i H^{2n} = H^i(-, R^{2n-i})$ is not vanishing in all terms of the sequence

\[
\begin{align*}
H^i(S, R^{2n-i}) & \leftarrow H^i(M_g, R^{2n-i}) \leftarrow H^i(A_g, R^{2n-i}) \\
H^i(S, V_\lambda) & \leftarrow H^i(M_g, V_\lambda) \leftarrow H^i(A_g, V_\lambda)
\end{align*}
\]

is the same and is defined by one of the following equivalent ways,

- $r(g, i, l) + i = n - l_\lambda$.
- $i = \sharp \{ \lambda \in \Phi_+ | w(\alpha) \text{ is negative} \}$, where $w \in W$ is so that $w(\rho_\lambda)$ belongs to the highest Weyl chamber.

Proof. The second map in the aforementioned embedding is the period map, and it is wellknown that it is generically injective when $g \geq 7$. The proof follows from the proposition 4.8 and [3] Lemma 4.1. The items follow from the discussion above and Borel-Weil Theorem. □

3. Theorem (4.1), together with the sub-multiplicative property of the filtration $L^\bullet$ give the following sub-multiplicative property

\[
L^i CH^\bullet(S, V_\lambda) \otimes L^j CH^\bullet(S, V_\mu) \subseteq L^{i+j} CH^\bullet(S, V_{\lambda+\mu})
\]

Similar relation exists for the Leray filtration in cohomologies. One can express an analogue of Theorem 3.2 in the decomposition into $\lambda$-pieces.

Proposition 4.10. Assume the spreads $3$ and $\mathfrak{W}$ satisfy

- $3 \in L^i CH_0(S, V_\lambda)$ and $[3]_l^3 \neq 0$
- $\mathfrak{W} \in L^j CH_0(S, V_\mu)$ such that
  - (a) $[\mathfrak{W}]_{ij}^l \neq 0$ OR
  - (b) $AJ_{\mathfrak{W}}(\mathfrak{W})^{tr} \neq 0$, and $[\mathfrak{W}]_l^m = 0$, $(l > j)$.

Then $0 \neq \text{rat} 3 \times \mathfrak{W} \in L^{i+j} CH_0(S, V_\lambda \otimes V_\mu = V_{\lambda+\mu})$.

- $[(a)] \Rightarrow [(3 \times \mathfrak{W})]_l^{\lambda+\mu} \neq 0$.
- $[(b)] \Rightarrow [AJ(3 \times \mathfrak{W})]_{ij}^{tr} \neq 0$.

Proof. The proposition is analogous to proposition 3.2 and proposition 3.3. The proof is straight forward by proposition 4.4, corollary 4.5 with relation (55). □

It is natural to consider a situation that $V_\lambda \otimes V_\mu \supseteq V_\nu$. The multiplication map between cohomologies $H^\bullet(S, V_\lambda)$ and $H^\bullet(S, V_\lambda)$ depends to the choice of an interwiner $V_\lambda \otimes V_\mu \rightarrow V_\nu$, and the correct way to indicate a cup product is to write it as
Hodge theoretic invariants detecting tautological cycles in moduli space of curves

(56) \[ \text{Hom}_{Sp(2g)}(V_\lambda \otimes V_\mu, V_\nu) \otimes H^\bullet(S, V_\lambda) \otimes H^\bullet(S, V_\mu) \rightarrow H^\bullet(S, V_\nu) \]

Again by Theorem (4.1), it follows that any product as above is given by algebraic correspondences. One can consider the situation in which a tensor power \( V_\lambda^\otimes m \) splits in the Schur-Weyl construction.

(57) \[ V_\lambda^\otimes m = \bigoplus_{|\nu| \leq m} V_\nu \otimes \sigma_{\nu}^* \Rightarrow \pi_\nu^\lambda : CH^\bullet(S, V_\lambda^\otimes m) \to CH^\bullet(S, V_\nu) \]

In this case by the same reasoning any projections \( \pi_\nu^\lambda \) is induced by algebraic correspondences. Because the higher AJ-invariants and cycle classes are preserved functorially via pullback, pushforward and correspondences one can check the vanishing of these invariants via these maps. The same argument works in cohomologies. In particular

- \( (\pi_\nu^\lambda)_* L^i CH^\bullet(S, V_\lambda^\otimes m) \subset L^i CH^\bullet(S, V_\nu) \).
- \( [3]_i \neq 0 \Rightarrow [(\pi_\nu^\lambda)_* 3]_i \neq 0. \)
- \( [AJ(3)]_i \neq 0 \Rightarrow [AJ((\pi_\nu^\lambda)_*(3))]_i \neq 0. \)

are consequences of the properties listed in section (2).

5. THE SECOND FABER-PANDHARIPANDE CYCLE

1. Let \( C^n_g \) be the moduli of genus \( g \) curves with \( n \) ordered marked points. Alternatively it can be defined by \( C^n_g = C_g \times M_\mu \times \cdots \times M_\mu \) where \( C_g \) be the \( n \)-fold fibered power of the universal curve over \( M_g \). We consider the class

(58) \[ FP_1 := \pi_1^\times 2 \Delta_{12}.\psi_1 \in CH^2(C^2_g) \]

called the first Faber-Pandharipande cycle. It is a codimension 2 cycle in \( C^2_g \). The cycle has been studied in [24].

Lemma 5.1. ([24] section 6.3) The cycle \( FP_1 \) is given by the formula

(59) \[ FP_1 = \Delta_{12}.\psi_1 - \frac{1}{2g-2} \psi_1 \psi_2 - \frac{1}{2g-2} (\psi_1^2 + \psi_2^2) \]
\[ + \frac{1}{(2g-2)^2} \kappa_1 (\psi_1 + \psi_2) + \frac{1}{(2g-2)^2} \kappa_2 - \frac{1}{(2g-2)^3} \psi_1 \kappa_1^2 \]

Its restriction on \( CH^2(C^2) \) of a self product of a curve of genus \( g \) is given by \( \Delta_{12}.\psi_1 - \frac{1}{2g-2} \psi_1 \psi_2 \in CH^2(C^2). \)
Proof. The lemma is the content of [24] section 6.3, that is the result of applying the formula
\[ \pi_1 = \Delta_{12} - \frac{1}{2g-2}(\psi_1 + \psi_2) + \frac{1}{(2g-2)^2}\kappa_1 \]
in the definition (58). After the restriction the terms starting from the third term in (59) vanish, [24]. □

The restriction of the \( FP_1 \) is a special cycle worked by Green and Griffiths in [8]. They treat an infinitesimal invariant detecting the algebraic cycles, which are computed by applying the Gauss-Manin connection to a lift of their normal functions.

**Theorem 5.2.** [8] Assume \( C \) is a general curve of genus \( g \geq 4 \). Let \( i^\Delta : C \hookrightarrow C \times C \). The ordinary Faber-Pandharipande cycle

\[ Z_K := K_C \times K_C - (2g - 2)i^\Delta_! K_C \in CH^2(C \times C) \]
is \( \neq 0 \), where \( K_C \) is the canonical class of \( C \).

We sketch the strategy in [8] in brief. For \( g = 2, 3 \) it is known to be rationally equivalent to zero, (see the ref.). It is homologically and also AJ-equivalent to zero, that is \([Z_K] = Alb(Z_K) = 0\) for all \( g \). Some ingredients of the proof is as follows. The Leray filtration for the family \( \mathcal{C} \times_S \mathcal{C} \to S \) is induced in hypercohomology by

\[ \text{Image}\{\Omega^i_s \otimes \Omega^{*-i}_{\mathcal{C} \times_S \mathcal{C}} \to \Omega^*_{\mathcal{C} \times_S \mathcal{C}}\} \]

Let \( \mathfrak{Z} = \text{spread}(Z) \) over the family above. We have used the old terminology of Green and Griffiths below.

1. Because \( \deg(\mathfrak{Z}_s) = 0 \), it follows that the fundamental class \([\mathfrak{Z}]_0 \in H^0(S, R^4)\) vanishes.

2. \([\mathfrak{Z}]_1 \in H^1(S, R^3)\) defines a class in the cohomology

\[ R^3(\Omega^{\geq 2}) \xrightarrow{\nabla} \Omega^1_S \otimes R^3(\Omega^{\geq 1}) \xrightarrow{\nabla} \Omega^1_S \otimes R^3(\Omega^{\geq 0}) \]

Passing to the quotient \([\mathfrak{Z}]_1\) defines an invariant

\[ \delta^{(1)}(\nu_Z) \in \Omega^1_S \otimes \mathcal{H}^{1,2}/\nabla \mathcal{H}^{2,1} \]

Which is the (first) infinitesimal invariant of the normal function

\[ \nu : s \mapsto Alb_{C \times_S C}(\mathfrak{Z}_s) \]

But this function is zero as we said. Therefore \([\mathfrak{Z}]_1 = 0, [8]\).

3. Similarly, \([\mathfrak{Z}]_2 \in H^1(S, R^3)\) defines a class in the cohomology

\[ R^2(\Omega^{\geq 2}) \xrightarrow{\nabla} \Omega^1_S \otimes R^2(\Omega^{\geq 0}) \xrightarrow{\nabla} \Omega^1_S \otimes R^2(\Omega^{\geq \bullet}) \]
Therefore it defines a class

\[(66) \quad \delta^2(\nu_Z) \in \Omega^1_S \otimes \mathcal{H}^{0,2}/\nabla \mathcal{H}^{1,1}\]

Which is the second infinitesimal invariant of the normal function. Green-Griffiths calculate a non-vanishing identity for the principal part of this invariant as a functional on Schiffer variations. Let \(T = T_S, H = H^0(\Omega^1_C)\). By a duality argument one has

\[(67) \quad \delta^{(2)}(\nu_Z) \in \left\{ \text{ker} \left\{ \bigwedge^2 T \otimes \bigwedge^2 H \to T \otimes V \right\} \right\}^*\]

where \(V = (H \otimes H^*) \oplus (H^* \otimes H) \subset H^1(\Omega^1_C \times C), \ [8].\)

**Theorem 5.3.** \([8]\) Let \(p \neq q \in C\) and \(\phi, \omega \in H^0(\Omega^1_C(-p - q))\). Then

\[(68) \quad \delta^{(2)}(\nu_Z)(\theta_p \wedge \theta_q \otimes \omega \wedge \phi) = \omega'(p)\phi'(q) - \omega'(q)\phi'(p)\]

where \(\theta_p, \theta_q\) are Schiffer variations at \(p\) and \(q\).

In the theorem \(\omega'(p) = f'(0)\) for \(\omega\) locally defined by \(\omega = f(z)dz, f(0) = 0\), (cf. ref).

(4). The hypothesis for the genus is related to the fact that the two tangents at two general points of the canonical curve defined by the canonical map \(\phi_K\) into the projective space do not intersect when \(g \geq 4, \ [8].\)

Putting every thing in our language of higher AJ-maps, it follows that the Abel-Jacobi invariant \(AJ^1([-3]) \neq 0\). We have listed these results below,

- \(cl^0_{C \times C}(Z_K) = [Z_K] = 0\)
- \(cl^1_{C \times C}(Z_K) = 0\)
- \(AJ^0_{C \times C}(Z_K) := AJ_{C \times C}(Z_K) = 0\)
- \(Z_K \in L^2CH^2(C \times C)\)
- \(cl^2_{C \times C}(Z_K) = 0\)
- \(AJ^1_{C \times C}(Z_K) \neq 0\)
- \(AJ^1_{C \times C}(Z_K)^{tr} \neq 0 \neq AJ^1_{C \times C}(Z_K)^{tr}\).

**Definition 5.4.** Define the 2th Faber-Pandharipande cycle

\[(69) \quad FP_2 = \pi_1^{\ast}(\Delta^2_{12}.\psi_1) \in CH^3(C^2_g)\]

We should already assume \(\dim(C^2_g) = 2g - 2 + 2 = 2g \geq 3 = \text{codim}_{C^2_g}FP_2\). The cycle \(FP_2\) has codimension 3 in \(C^2_g\). The second Faber-Pandharipande cycle is a nontrivial algebraic cycle in the tautological ring of \(C^2_g\) when \(g\) is big enough. We will give a simple proof of this based on the behavior of higher Abel-Jacobi invariants for zero
cycles in the product varieties, i.e proposition 3.2 and Theorem 3.3. We proceed to check out the AJ-invariants for the restriction of \(FP_2\) over a generic family \(C \times_S C\) where \(\dim(S) = 1\).

**Theorem 5.5.** Let \(C \to S\) be a generic family of smooth curves of genus \(g\), and \(\dim(S) = 2\). Then the restriction of the algebraic cycle \(FP_2\) over \(C \times_S C\) is \(\neq 0\) when \(g \geq 7\).

**Proof.**

**Step 1:** First we show that the restriction of the cycle \(FP_1 \times FP_1\) over \(C^2 \times_S C^2\) is \(\neq 0\). In fact this follows from the Theorem 3.2 and the invariants of the cycle \(Z_K\) listed above. It follows that

\[(70) \quad c_{C_S^2 \times_S C^2}^2(3K \times 3K) = 0, \quad AJ_{C_S^2 \times_S C^2}^3(3K \times 3K) \neq 0\]

This tells that \(3K \times 3K \neq 0\) and \(3K \times 3K \in L^{4}CH^{4}_{C_S^2}(C_S^2 \times_S C^2)\).

**Step 2:** Next we note that \(3K, 3K = (\iota_{\Delta_S})^*(3K \times 3K) \in CH^{4}(C_S^2)\) where \(\iota_{\Delta_S} : C_S^2 \to C_S^2 \times_S C_S^2\) is the relative diagonal. By properties of the filtration \(L^\bullet\) it follows that \(\iota_{\Delta_S}^* : C_S^2 \to C_S^2 \times_S C_S^2\) is \(\neq 0\).

**Step 3:** Let \(r : S \hookrightarrow M_g\) as in the Theorem. The cycle \(\mathfrak{Z} = r^*FP_1 = \pi_1^{x_2} \circ r^*(\Delta_{12}).r^*(\psi_1)\) is given by the similar formula on the product family \(C_S^2\). Set \(\mathfrak{W} = r^*(FP_2)\) be the restriction of the second FP-cycle on \(C_S^2\). It does as \(\mathfrak{Z} = 3K\) does for \(FP_1\). Assume \(\mathfrak{W}^{rat} \equiv 0\). We show it causes a contradiction. Let

\[(71) \quad \iota_{\Delta_S} : C_S^2 \hookrightarrow C_S^2 \times_S ... \times_S C_S^2\]

be the diagonal. Then we have

\[(72) \quad (\iota_{\Delta_S})^* \circ \{(\pi_1^{x_2})^t \times (\pi_1^{x_2})^t\}(2^{x_2} \times 2^{x_2}) = (\iota_{C_S^2})^* \circ \{(\pi_1^{x_2})^t \times (\pi_1^{x_2})^t \times (\pi_1^{x_2})^t\}(3^{x_3})\]

where they equal \(r^*(\Delta_{12}^{x_2}.\psi_1^{x_2}) = r^*(\Delta_{12}^{x_3}.\psi_1^{x_3})\). The operators used in equation (72) are obviously algebraic correspondences. Therefore the assumption \(\mathfrak{W}^{rat} \equiv 0\) implies \(3^{x_3} \equiv^{rat} 0\) and hence \(\mathfrak{Z}^{rat} \equiv 0\) by projecting onto one of the factors. This is a contradiction.

**Step 4:** The proof in steps 1, 2 and 3 is based on the non-triviality proof of \(\mathfrak{Z}\) and that of the method by Green-Griffiths in [8] we briefly mentioned above. There, the genus condition that is \(g \geq 4\) is coming from the embedding dimension of the canonical curve in \(\mathbb{P}^r\), where the genus condition translates into \(r \geq 3\). The method of the proof in step 3 shows that one need to consider the canonical embedding compatible through a diagonal map of type
\[ C^2_S \hookrightarrow C^2_S \times C^2_S \hookrightarrow \mathbb{P}^{2r} \]

which by the above consideration requires \( 2r \geq 6 \), which is equivalent to \( g \geq 7 \). \( \square \)

The codimension 3 cycle \( FP_2 \) lies lower down the Leray filtration. Specifically: \( FP_2 \) is a codimension 3 cycle in \( C^2_g \) over \( M_g \). The Leray spectral sequence degenerates at \( E_2 \) by the Deligne theorem. So the associated graded of the Leray filtration \( L^\bullet \) are

\[ H^6(C^2_g, \mathbb{C}) = \bigoplus_t H^{6-t}(M_g, R^t f_\ast \mathbb{Q}) \]

The class of the cycle \( FP_2 \) will determine a class in \( H^6(C^2_g, \mathbb{C}) \). If it lies in \( L^s H^6(C^2_g) \), it will determine a class in \( H^s(M_g, R^{6-s} f_\ast \mathbb{Q}) \). Then by the above proof \( FP_2 \) lies in \( L^3 \) or \( L^4 \), so that it is detected by a class in \( H^3(M_g, R^3) \) or \( H^4(M_g, R^2) \).

**Definition 5.6.** Define the \( n^{th} \) Faber-Pandharipande cycle

\[ FP_n = \pi^{x^2}(\Delta^n_{12}, \psi_1) \in CH^{n+1}(C^2_g) \]

Again we assume \( \dim(C^2_g) = 2g - 2 + 2 = 2g \geq n + 1 = \text{codim}_{C^2_g} FP_n \). The cycle \( FP_n \) has codimension \( n + 1 \) in \( C^2_g \).

**Corollary 5.7.** Assume \( r : S \hookrightarrow M_g \) and \( \dim(S) = n - 1 \), then the restriction of the higher Faber-Pandharipande cycles defined by \( FP_n = \pi^{x^2}(\Delta^n_{12}, \psi_1) \) over \( C \times S \mathbb{C} \)

\[ \text{rat} \neq 0 \] when \( g \geq 3n + 1 \). The same holds for the cycles \( FP_n \in CH^{n+1}(C^2_g) \).

**Proof.** The proof is analogous inductive repetition of the proof of Theorem 5.5. The genus condition also follows similar to the step 4, by induction. The last part follows from Lemma 4.1 in \( \cite{3} \). \( \square \)

2. Consider the cycle \( \mathfrak{G} \mathfrak{S} = \pi^{x^3}_{\ast} \Delta_{123} \in CH^2(C^2_g) \) called the Gross-Schoen cycle, and originally studied in \( \cite{11} \). B. Gross and C. Schoen study the cycle

\[ Y = \Delta_{123} - \Delta_{12} - \Delta_{13} - \Delta_{23} + \Delta_1 + \Delta_2 + \Delta_3 \]

where \( \Delta_I = \{(x_1, ..., x_n) : x_i = x_j \text{ if } i, j \in I \text{ and } x_i = 0 \text{ if } i \notin I \} \). Let \( \mathfrak{Y} \) be the spread of \( Y \) over \( S \). As it is explained in \( \cite{24} \)

\[ (r^3_S)^{\ast} \mathfrak{G} \mathfrak{S} - \mathfrak{Y} = \text{sum of } FP_1 \text{-cycles} \]

where \( (r^3_S)^{\ast} \) is inclusion. It follows that

- \( [\mathfrak{Y}]_0 = 0 \).
- \( [AJ(\mathfrak{Y})]_0 \neq 0 \).
The result of \[1\] predicts that when \(C\) is a general curve of genus \(g \geq 3\), then \(\mathcal{W} \neq 0\).

There is an alternative way to see the non triviality of \(\mathcal{W}\) via correspondences. That is, According to [24] Lemma (12.4), one may construct a correspondence

\[
\Gamma^*: H^4(M_g, V^\otimes 12) \rightarrow H^2(M_g, V^\otimes 2), \quad (\pi_1^{x^3} \Delta_{123})^x \mapsto \pi_1^{x^2} \Delta_{12} \cdot \psi_1^2 = FP_2
\]

via Brauer diagrams. This shows \(FP_2\) can not be equivalent to zero. Otherwise its pre-image is also equivalent to zero. However its pre-image is the self product of Gross-Schoen cycle, and as we said \(\mathcal{W} \neq 0\). Working over generic families one constructs a correspondence \((C_g^3)^4 \dashv C_g^2\) to get a commutative diagram

\[
\begin{array}{ccc}
CH^8((C_g^3)^4) & \overset{\Gamma^*}{\longrightarrow} & CH^3(C_g^3) \\
\uparrow & & \uparrow \\
CH^8((C_g^3)^4) & \overset{\Gamma^*}{\longrightarrow} & CH^3(C_g^2)
\end{array}
\]

It follows that the same holds for the pullback over \(S\) of the above cycles. More general, one can construct a correspondence with same method which takes

\[
(\pi_1^{x^3} \Delta_{123})^{x^2m} \mapsto \pi_1^{x^2} \Delta_{12} \cdot \psi_1^m = FP_m
\]

Lets take a more general situation by defining a cycle namely

\[
\mathfrak{F}(n,m) := \pi_1^{x^n}(\Delta_{12...n} \cdot \psi_1^m) \in CH^{n+m-1}(C_g^n)
\]

It is the image of the cycle \(\pi_1^{x^3}(\Delta_{123})^{x(n+2m-2)}\) under the map constructed in ([24], Lemma 12.4, page 43). One obtains a map (given by algebraic correspondences) constructed via the projectors in the Brauer algebra such that

\[
H^{n+2m-2}(M_g, V^\otimes 3(n+2m-2)) \rightarrow H^{n+2m-2}(M_g, V^\otimes n)
\]

Applying Theorem (3.3)-Corollary (3.4) again with the observation above gives

\[
[AJ(\mathcal{W}^x)^{(n+m-1)}]_{n+m-1} \neq 0 \quad \Rightarrow \quad [AJ(\mathfrak{F}(n,m)^S)]_{n+m-1} \neq 0
\]

which proves similar non triviality of \(\mathfrak{F}(n,m)^S\) in appropriate generic case.

3. (a). Associated to the higher normal function of algebraic cycles is their Picard-Fuchs equation. To the algebraic cycle \(Z \in CH^n(X = C^n)\) with \(\mathfrak{Z} = \text{spread}(Z)\) one
associates the higher normal function \( \nu(t) := [AJ(Z_t)]_{p-1} \). Then the Picard-Fuchs differential equation defined by \( \nu \) should split into \( \lambda \)-pieces

\[
D_{PF}\tilde{\nu} = \bigoplus_{\lambda} D_{PF}\lambda \cdot \tilde{\nu}_{\lambda} = 0
\]

where \( \tilde{\nu} \) is a lift of the higher normal function \( \nu_{Z_t} \), see [2].

(b). Another is to consider the height pairings

\[
\langle \cdot, \cdot\rangle_\lambda : Gr^{\rho_{\lambda}} LCH^r(S, V_\lambda)_\mathbb{Q} \times Gr^{r-n-\rho_{\lambda}} LCH^r(S, V_\lambda)_\mathbb{Q} \to \mathbb{R}
\]

which are non-degenerate and satisfy a Hard Lefschetz property. If \( L \in CH^1(C^n) \) is the operator of hyperplane section and \( x \in CH^r_{\text{hom}}(C^n) \) is such that \( L^n - 2r + 2 \cdot x = 0 \), then

\[
(-1)^{r}\langle x_\lambda, L^n - 2r + 1 \rangle_\lambda > 0
\]

will give a method to check the non-triviality of \( x \), see [2].

(c). The specific formulas calculating higher Abel-Jacobi invariants are given by regulator maps on the Milnor K-groups

\[
R : K^n_n(C^{\times n}) = CH^n(C^{\times n}, n) \to H^n_\mathbb{D}(C^{\times n}, \mathbb{Z}(n))
\]

\[
R(f = [f_1 \otimes ... \otimes f_n]) = \sum_{j=1}^{n} (\pm 2\pi \sqrt{-1})^j \log f_j d\log f_{j-1} \wedge ... \wedge d\log f_n \delta_{T_{f_1} \cap ... \cap T_{f_{j-1}}}
\]

where \( T_f := f^{-1}\mathbb{R}^- \), \( \pm = (-1)^{n-1} \) and \( \delta \) is the delta function, being 1 on the set, zero elsewhere. The invariants \( \Psi_i \) are calculated via formulas of the type

\[
\Psi_i(Z) = \sum_{\lambda} R(f_\lambda) \otimes [(\pi_X^*)^\lambda](\wedge^\lambda d\log z)^\vee
\]

where \( f \) is determined by \( Z \), see [15], [18].

(d). It is interesting to look at the special case of hyperelliptic curves. For those, the decompositions in section (4) could well be "multiplicative", that is compatible with intersection product. This is due to the vanishing of the Gross-Schoen cycle for hyperelliptic curves (and choice of a Weierstrass point as base point). In addition, it could be that the higher FP-cycles vanish for suitable families of hyperelliptic curves.

**Remark 5.8.** The aforementioned invariants can also be applied to the tautological cycles in the moduli of quasi-polarized K3 surfaces \((X, H)\) of degree 2l, denoted \( F_l \). One has the universal surface \( \pi : \mathcal{X} \to F_l \) with universal quasi polarization \( H \to \mathcal{X} \).
The Hodge bundle is $R^2\pi_*\Omega^2_{X/F_l} \to F_l$. In this case the motive is still of abelian type and one has a decomposition

$$CH^k(X^n) = \bigoplus \lambda \ CH^{k-n}_l(F_l, V_\lambda)$$

by the Theorem of G. Ancona. Then it would be interesting to find sort of cycles in $CH^k(X^n)$ which their non-triviality could be checked in this case, cf. [6], [25], [28].

Data available on request from the author: The data that support the findings will be available on [researchgate.net] at the author name [Mohammad Reza Rahmati] from the date of publication to allow for public access research findings.

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