Application range of crosstalk-affected spatial demultiplexing for resolving separations between unbalanced sources

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Abstract
Super resolution is one of the key issues at the crossroads of contemporary quantum optics and metrology. Recently, it was shown that for an idealized case of two balanced sources, spatial mode demultiplexing (SPADE) achieves resolution better than direct imaging even in the presence of measurement crosstalk (Gessner et al 2020 Phys. Rev. Lett. 125 100501). In this work, we consider arbitrarily unbalanced sources and provide a systematic analysis of the impact of crosstalk on the resolution obtained from SPADE. As we dissect, in this generalized scenario, SPADE’s effectiveness depends non-trivially on the strength of crosstalk, relative brightness and the separation between the sources. In particular, for any source imbalance, SPADE performs worse than ideal direct imaging in the asymptotic limit of vanishing source separations. Nonetheless, for realistic values of crosstalk strength, SPADE is still the superior method for several orders of magnitude of source separations.

1. Introduction

In recent years, considerable attention was devoted to the problem of resolving asymptotically small separations between two point-like light sources. In the case of separations below the so-called Rayleigh regime, which is determined by the optical apparatus’ point spread function width [1, 2], the efficiency of traditional measurement schemes relying on direct imaging drops significantly [3].

Over the course of the last two decades, a plethora of methods was developed to overcome the limitations of direct imaging and achieve super resolution. This includes photoactivated localization microscopy [4, 5], optical reconstruction microscopy [6], the use of superoscillations [7, 8] and inversion of coherence along an edge [9, 10], among others [11, 12]. In the particular case of estimating the distance between two incoherent light sources, such as a planet orbiting around a distant star [13], the optimal measurement is given by spatial demultiplexing (SPADE) in Hermite–Gauss modes [14, 15].

Unfortunately, due to the presence of noise and technical imperfections, such as apparatus misalignment, no measurement scheme is ideally implemented in experimental setups [16–25]. In the case of SPADE, there is always a small fraction of the measured mode that is not transmitted into the correct output, but to another mode instead: a phenomenon known as crosstalk. Recent efforts showed that, while crosstalk lowers SPADE’s efficiency significantly, the method is still typically superior even to ideal (i.e. noiseless and continuous) direct imaging [26]. These findings, however, were based on the assumption that the two light sources are exactly equally bright, which is often not the case, as in the aforementioned example of a planet–star system.

In this work, we assess the applicability of SPADE to resolving separations between unbalanced sources, i.e. sources of arbitrary relative brightness [27], in practical scenarios. As we find, the case of perfectly
Figure 1. Schematic depiction of the measurement setting. Two incoherent light sources of arbitrary relative brightness separated by distance $d$ lie in the source plane. The origin of the coordinate system in the image plane is centered between them.

balanced sources is the only one, for which SPADE is always more efficient than direct imaging. Otherwise, the effectiveness of SPADE has a non-trivial dependence on the relation between measured separations (normalized with respect to the point spread function width) and crosstalk strength, which we investigate in detail. Using analytical tools, we show that for asymptotically vanishing distances, SPADE is outperformed by direct imaging, while for distances much larger than crosstalk strength (which are still below the Rayleigh regime), SPADE remains approximately unaffected by crosstalk, constituting the ideal measurement scheme. The approximate range of separations, for which SPADE outperforms ideal direct imaging, is determined numerically.

This article is organized as follows. In section 2, we introduce the measurement setting and the necessary tools from estimation theory, as well as the notion of crosstalk. In section 3, we analyze the impact of crosstalk on the SPADE Fisher information for unbalanced sources. In section 4, we compare the crosstalk-influenced SPADE with direct imaging in experimentally relevant settings. We conclude in section 5.

2. Preliminaries

We begin by introducing the measurement setting and the necessary tools from estimation theory, as well as the notion of crosstalk.

2.1. Measurement setting

We consider two point-like, incoherent light sources of arbitrary relative brightness following the Poisson distribution, which is the case, e.g. for weak thermal light. We assume that the origin of the coordinate system in the image plane lies between the sources, so that their positions are given by

$$\vec{r}_\pm = \pm d/2(\cos \theta, \sin \theta)$$

for some $d > 0$, $\theta \in [0, 2\pi)$, as in figure 1.

To quantify the potential imbalance in the brightnesses of the two sources, we define the relative brightness

$$0 \leq \nu \leq 1.$$  \hspace{1cm} (1)

Here, $\nu = 1/2$ corresponds to equal brightnesses, while $\nu = 1$ ($\nu = 0$) corresponds to the source at $\vec{r}_+$ ($\vec{r}_-$) being the only visible one. For simplicity, further on we assume that the label $\vec{r}_+$ is given to the brighter source, so that $\nu \geq 1/2$ (given the opposite scenario, this can be achieved by a simple rotation of the coordinate system in the image plane by $\pi$)\(^6\). We remark that in our analysis we treat $\nu$ as a known parameter, i.e. we assume that the relative brightness of the two sources is known prior to the measurement. If this is not the case, our results still provide a meaningful upper bound to SPADE’s effectiveness in full multi-parameter estimation, for which the obtained resolution is necessarily smaller \cite{27}.

\(^6\) It is worth adding that the vast majority of our results concerns typical behavior of the measurement in the presence of crosstalk, which is obtained using either analytical or numerical statistical methods with large sample sizes. Since there is no universally advantageous side of the image plane, in which the brighter source should be placed to achieve better results, the emergent averaged findings are symmetric with respect to the point $\nu = 1/2$ anyway.
The electromagnetic field in the image plane is then conveniently described by two bases centered at the sources: \( u_{\pm nm} := u_{nm}(\vec{r} - \vec{r}_{\pm}) \), with \( u_{nm} \) being the Hermite–Gauss modes [28]:

\[
u_{nm}(\vec{r}) := \frac{H_m(\sqrt{2}r_{1}/w)H_m(\sqrt{2}r_{2}/w)}{w\sqrt{2\pi w}2^{n+m-1}\pi n!m!}e^{-(\vec{r}_{1}^2+\vec{r}_{2}^2)/w^2}.
\] (2)

Here, \( \vec{r} = (r_1, r_2) \), \( H_m(z) := (-1)^m e^{z^2} \partial_z^m e^{-z^2} \) are the Hermite polynomials. \( w \) is the width of the point spread function of the imaging system, which we assume to be Gaussian.

Each basis can be used to represent the electric field operator \( \hat{E}(\vec{r}) \) is a Hermitian conjugate of \( \hat{E}^{(+)}(\vec{r}) \)

\[
\hat{E}(\vec{r}) = \hat{E}^{(+)}(\vec{r}) + \hat{E}^{(-)}(\vec{r})
\] (3)

by expanding its positive-frequency part as

\[
\hat{E}^{(+)}(\vec{r}) = \sum_{nm} u_{+nm}(\vec{r}) \hat{b}^{+nm} = \sum_{nm} u_{-nm}(\vec{r}) \hat{b}^{−nm},
\] (4)

where \( \hat{b}^{\pm nm} \) are the annihilation operators associated with the modes \( u_{\pm nm} \).

The electromagnetic field in the image plane is described by \( M \) copies of the quantum state

\[
\hat{\rho}(d) = \nu \hat{\rho}_{+} + (1 - \nu) \hat{\rho}_{−},
\] (5)

with \( \hat{\rho}_{\pm} \) being the quantum states of the modes \( u_{\pm 00} \), such that [14]

\[
\text{Tr}(\hat{\rho}_{\pm} \hat{b}^{\dagger}_{\pm 00} \hat{b}^{\pm 00}) = \epsilon \ll 1.
\] (6)

Consequently, the total number of photons in the electromagnetic field is equal to \( N = Me \).

2.2. Fisher information and SPADE

According to estimation theory, the uncertainty \( \Delta d \) of estimation of the distance between the sources, based on \( N \) measured photons, is determined by the Cramér–Rao bound [16, 29, 30]

\[
\Delta d \geq 1/\sqrt{NF(d)}.
\] (7)

Here, \( F \) is the Fisher information per photon [31], which is a positive quantity dependent on the assumed measurement scheme. The larger the Fisher information, the fewer photons are needed to resolve a given distance.

One of the most widely-used methods of assessing the value of the distance is direct imaging, in which the estimation is based on spatially resolved measurements of intensity of light

\[
I(\vec{r}) := M\text{Tr} \left( \hat{\rho}(d) \hat{E}^{(-)}(\vec{r}) \hat{E}^{(+)}(\vec{r}) \right)
\] (8)

in the image plane. While it is a simple and relatively easy-to-implement method, the corresponding Fisher information per photon

\[
F_{D1}(d) = \int_{\mathbb{R}^2} d^2 \vec{r} \left( \frac{1}{p(\vec{r}|d)} \frac{\partial}{\partial d} p(\vec{r}|d) \right)^2,
\] (9)

where \( p(\vec{r}|d) = I(\vec{r})/N \), is significantly limited due to the presence of diffraction, especially in the sub-Rayleigh regime \( d/2w < 1 \).

In principle, the state of light can be measured in any physically implementable basis \( \nu_{nm} \), corresponding to Fisher information per photon equal to

\[
F(d) = \sum_{n,m} \frac{1}{p(nm|d)} \left( \frac{\partial}{\partial d} p(nm|d) \right)^2,
\] (10)

where \( p(nm|d) \) is the conditional probability of detecting a photon in mode \( \nu_{nm} \) when the distance is equal to \( d \). An appropriate choice of \( \nu_{nm} \) can result in Fisher information that is larger than the one obtained from direct imaging. By optimizing over all possible measurements, we obtain the quantum Fisher information, which corresponds to the smallest uncertainty \( \Delta d \) allowed by quantum mechanics [32].
In the case at hand, the quantum Fisher information is achieved by spatial mode demultiplexing (SPADE) in the Hermite–Gauss basis centered at the origin, \( v_{nm} = u_{nm} \), followed by intensity measurement \[^{14}\]. Remarkably, the corresponding value of the Fisher information per photon is constant:

\[
w^2 F_{HG}(d) = 1, \tag{11}\]

both for balanced and unbalanced sources.

Note that in practice, only the first few modes, i.e. those given by \( n, m \in \{0, D - 1\} \), with \( D \geq 1 \) are measured. The smaller the source separation, the smaller \( D \) is required to obtain the ideal value \( \text{Eq. (11)} \) of the Fisher information. In particular, for distances far below the Rayleigh regime, \( d/2w \ll 1 \), this ideal value is obtained already by \( D = 2 \) \[^{26}\].

### 2.3. Crosstalk

In the case of SPADE, as already explained, there is a small but non-vanishing probability that a measured mode is transmitted into an incorrect output. This phenomenon is known as crosstalk. Because of crosstalk, the actual measurement basis deviates from the ideal one \[^{26}\]:

\[
v_{nm} = \sum_{k,l=0}^{D-1} c_{nm,kl} u_{kl}, \tag{12}\]

where we restricted ourselves to \( n, m \in \{0, D - 1\} \). It is assumed that the diagonal elements of the crosstalk matrix are close to identity and the remaining elements are of a much smaller order, so that, approximately, \( v_{nm} \approx u_{nm} \).

Assuming that the contribution from modes given by \( n \geq D \) or \( m \geq D \) is negligible, equation \( \text{Eq. (12)} \) formally defines a change of basis. This implies that the crosstalk matrix is unitary, and as such, can be written as

\[
c = e^{-i\mu \vec{\lambda} \cdot \vec{G}}, \tag{13}\]

where \( \mu \geq 0, \vec{\lambda} \in \mathbb{R}^{D^4 - 1} \) is normalized to one (i.e. \( \vec{\lambda} \cdot \vec{\lambda} = 1 \)) and \( \vec{G} \) is a vector of all \( D^4 - 1 \) generalized Gell–Mann matrices of size \( D^2 \times D^2 \) \[^{33, 34}\]. For \( \mu \ll 1 \) the crosstalk matrix is very close to the identity matrix and hence describes small imperfections in the measurement basis:

\[
c \approx 1 - i\mu \vec{\lambda} \cdot \vec{G}. \tag{14}\]

The crosstalk strength is defined by the mean off-diagonal matrix element:

\[
p_c := \frac{1}{D^2 (D^2 - 1)} \sum_{n,m,k,l=0}^{D-1} |c_{nm,kl}|^2. \tag{15}\]

For weak crosstalk matrices \( \text{Eq. (14)} \), the crosstalk strength is always proportional to \( \mu^2 \) and, as we show in appendix A, equals, on average,

\[
p_c(\mu) \approx \frac{2}{D^4 - 1} \mu^2. \tag{16}\]

In practice, we have access to crosstalk strength rather than the abstract parameter \( \mu \). For this reason, we base our considerations on \( p_c \) wherever possible.

### 3. Fisher information in the presence of crosstalk for unbalanced sources

It was shown recently \[^{26}\] that while the resolution obtained from SPADE suffers in the presence of crosstalk, it is still higher than that obtained from direct imaging. Here, we investigate how this result generalizes to the case of unbalanced sources.

We start by calculating the corresponding Fisher information. Each detector mode \( v_{nm} \) can be associated with its own field operator \( \hat{a}_{nm} \). These detector field operators can be expressed as functions of either of the source-centered field operators via

\[
\hat{a}_{nm} = \sum_{kl} f_{\pm nm,kl}(d) \hat{b}_{\pm kl}, \tag{17}\]
where

\[ f_{\pm nm,kl}(d) = \int_{\mathbb{R}^2} d^2 \mathbf{r} v_{\pm nm}(\mathbf{r}) \mathbf{u}_{\pm kl}(\mathbf{r}). \]  

(18)

The number of photons in detector mode \( v_{nm} \) is then equal to

\[ N_{nm} := \text{MT} \left[ \rho(d) \mathbf{a}_{nm} \mathbf{a}_{nm}^\dagger \right] = N \left[ \nu |f_{+ nm,00}(d)|^2 + (1 - \nu) |f_{- nm,00}(d)|^2 \right] \]

where the bottom line follows from equations (5), (6) and (17).

Therefore, the conditional probability of detecting a photon in the mode \( v_{nm} \), under the condition that distance value is \( d \), equals

\[ p_\nu(nm|d) := \frac{N_{nm}}{N} = \nu |f_{+ nm,00}(d)|^2 + (1 - \nu) |f_{- nm,00}(d)|^2, \]

(20)

where, in the presence of crosstalk [26],

\[ f_{\pm nm,00}(d) = \sum_{k,l=0}^{D-1} c_{nm,kl} \beta_{\pm kl}. \]

(21)

Here,

\[ \beta_{\pm kl} = \frac{1}{\sqrt{\kappa}}(|\pm x|)^{k+1} \cos \theta \sin \theta e^{-x^2/2}, \]

(22)

where we introduced the short-hand notation \( x := d/(2w) \) (note that \( x \) is dimensionless).

Thus, the final expression for the SPADE Fisher information per photon for unbalanced sources reads (cf equation (10))

\[ F_\nu(d) = \sum_{n,m=0}^{D-1} \frac{1}{p_\nu(nm|d)} \left( \frac{\partial}{\partial dp_\nu(nm|d)} \right)^2. \]

(23)

Using this formula with equation (14) at the input of equation (21), we can compute the Fisher information in the presence of generic, i.e. arbitrary unitary, weak crosstalk.

To investigate the behavior of such Fisher information at small distances, we first observe that in the limit \( x \to 0 \), the Fisher information is dominated by terms given by \( n,m \in \{0,1\} \). In other words, it is enough to consider \( D = 2 \). The corresponding vectors \( \lambda, \mathbf{G} \) defining the crosstalk matrix via equation (13) are 15-dimensional, allowing for explicit analytical treatment. The resulting Fisher information can be then in principle expanded in \( x \) to obtain an approximation valid for small separations.

However, such straightforward expansion will not give us the full picture. To see why, let us consider a specific model of crosstalk given by uniform crosstalk:

\[ (c_{nm})_{nm,kl} = \delta_{nk}\delta_{ml} + (1 - \delta_{nk}\delta_{ml}) \sqrt{\beta_c}. \]

(24)

For such crosstalk matrix, the conditional probability \( p_\nu(10|x) \) obtained for small separations \( x \ll 1 \) and weak crosstalk strength \( \beta_c \ll 1 \) reads

\[ p_\nu(10|x) \approx \cos^2 \theta x^2 + 2 \cos \theta (2\nu - 1) x \sqrt{\beta_c} + p_c. \]

(25)

Crucially, the dominating terms on the r.h.s. depend on the relation between the separation and crosstalk strength. For \( x \gg \sqrt{\beta_c} \), the last term is comparatively insignificant, while for \( x \ll \sqrt{\beta_c} \), this is true for the first term. Consequently, the inverse probability entering the Fisher information (23) has two completely different approximate forms:

\[ \frac{1}{p_\nu(10|x)} \approx \begin{cases} \frac{1}{\cos \theta x} & x \gg \sqrt{\beta_c}, \\ \frac{1}{\cos \theta x} \left[ 1 - \frac{2(2\nu-1)\sqrt{\beta_c}}{\cos \theta x} \right] & x \ll \sqrt{\beta_c}. \end{cases} \]

(26)

To obtain this equation, we simply expanded the l.h.s. in the smallest relevant parameter: \( \sqrt{\beta_c} \) in the top line, and \( x \) in the bottom line. For \( x \approx \sqrt{\beta_c} \), neither term in equation (25) can be discarded, making it difficult to obtain a simple approximation.
As seen, if we expand in \( x \), we obtain a result that is valid only in the regime \( x \ll \sqrt{\nu} \). One can check that this dependence on the ratio between separation and crosstalk strength extends to the other conditional probabilities, and thus the whole Fisher information. Crucially, while above we used the simplified uniform crosstalk model for illustrative purposes, the same qualitative behavior is observed for any crosstalk.

Performing the full calculation for the SPADE Fisher information subject to arbitrary crosstalk, we find that

\[
\begin{aligned}
w^2 F_\nu(x) &\approx \begin{cases} 
1 & x \gg \sqrt{\nu} \\
q_0 + q_1 x + q_2 x^2 & x \ll \sqrt{\nu},
\end{cases}
\end{aligned}
\]

where \( q_i \) are functions of \( \nu, \theta \) and crosstalk. Let us discuss the impact of this result separately in the three ranges \( x \gg \sqrt{\nu}, x \ll \sqrt{\nu} \), and \( x \approx \sqrt{\nu} \).

We start with \( x \ll \sqrt{\nu} \). The qualitative effect of crosstalk on the Fisher information in this range (more precise quantitative analysis is provided in the next section) is once again accurately captured by the uniform crosstalk model, for which the coefficients in the bottom line of equation (27) have simple explicit forms:

\[
\begin{aligned}
q_0 &\approx (2\nu - 1)^2, \\
q_1 &\approx \nu \frac{1}{\sqrt{\nu}} (1 - \nu)(2\nu - 1) (\sin^3 \theta + \cos^3 \theta), \\
q_2 &\approx -\frac{\nu}{\sqrt{\nu}} (1 - \nu)(4\nu - 1) (4\nu - 3) (3 + 4\cos \theta),
\end{aligned}
\]

where we restricted ourselves to leading terms in crosstalk strength. In the most radical case of \( \nu = 1/2 \), we can see that \( q_0 = q_1 = 0 \) and the Fisher information becomes dominated by the quadratic term, making it vanish with decreasing separation as \( F_\nu \propto x^2 \), a phenomenon originally investigated in [26]. For other values of \( \nu \), the Fisher information \( F_\nu \), eventually approaches a positive constant value, which however is always smaller than the ideal one (11). This shows the influence of crosstalk on the measurement for \( x \ll \sqrt{\nu} \) is significant.

In the opposite range, \( x \gg \sqrt{\nu} \), we observe a completely different behavior. There, the Fisher information coincides, to a good approximation, with the optimal value (11). As such, in this range, crosstalk can be effectively ignored. To see what this means in practice, we performed extensive numerical analysis, finding that, typically, the relation \( x \gg \sqrt{\nu} \) corresponds to \( x > 3\sqrt{\nu} \). In other words, for \( x > 3\sqrt{\nu} \) the crosstalk-affected SPADE Fisher information is approximately optimal. We remark that for small crosstalk strengths the point \( x = 3\sqrt{\nu} \) lies deep within the sub-Rayleigh regime \( x < 1 \), meaning that the discussed range is practically relevant. See appendix B for details.

Finally, let us discuss the case of \( x \approx \sqrt{\nu} \). Just as in the case of the conditional probability (26), in equation (27) we did not provide an approximation for the range \( x \approx \sqrt{\nu} \). In this range, an accurate description requires the full expression. To see this, we observe that both lines of equation (27) are extracted from an infinite series in \( x \) and \( \sqrt{\nu} \), that contains, in particular, terms of the form \((x/\sqrt{\nu})^n \). Clearly, the closer the ratio \( x/\sqrt{\nu} \) to one, the fewer terms like these can be discarded, making approximations hard to obtain. Therefore, for \( x \approx \sqrt{\nu} \), we study the Fisher information numerically.

4. Comparison with direct imaging

To assess the practical impact of our findings, we compare the crosstalk-affected SPADE with ideal direct imaging in the sub-Rayleigh regime.

We begin our analysis with the asymptotic limit of vanishing distances, \( x \to 0 \), for which we obtain exact analytical results. From the definition of light intensity (8), making use of equations (4)–(6), we calculate that

\[
I(\vec{r}|d) = N \left( \nu |u_{+0}|^2 + (1 - \nu) |u_{-0}|^2 \right).
\]

Substituting this into the definition (9) of the Fisher information for direct imaging, expanding the integrand in \( x \) and integrating yields

\[
w^2 F_{\text{DI}} (x \to 0) \approx (2\nu - 1)^2.
\]

This value coincides with the constant term \( q_0 \) for the SPADE Fisher information (27) subject to uniform crosstalk (28). However, we stress that uniform crosstalk constitutes only a particular model of crosstalk, not necessarily representative of typical crosstalk (indeed, this is exactly what we find below). Estimating the
quantitative impact of crosstalk on a typical measurement requires more precise tools. To this end, we instead consider the mean value of the SPADE Fisher information subject to arbitrary crosstalk of a fixed strength. As was the case with uniform crosstalk, in the limit of \( x \to 0 \), the Fisher information approaches the constant term \( q_0 \) from the bottom line of equation (27). The goal is to calculate the mean value of \( q_0 \) averaged over all possible realizations of crosstalk.

Since we are interested only in the limit \( x \to 0 \), we can once again set \( D = 2 \). We assume that crosstalk matrices are distributed according to equation (13) with \( \mu \) related to crosstalk strength via equation (16) and \( \lambda \) being a 15-dimensional vector distributed according to a normal probability measure. This allows for uniform sampling from the 15-dimensional hypersphere after normalization of \( \tilde{\lambda} \) [35]. Substituting the weak crosstalk matrix (14) into equation (23), we find

\[
q_0 \approx \frac{(2\nu - 1)^2}{36\nu^2} \left[ \sin^2 \theta [3\nu^2 \alpha_1^2 + 12\mu \alpha_2 \alpha_3] \right] \left[ \frac{-6\alpha_1 + 2\sqrt{3}\alpha_1 + \sqrt{6}\alpha_1 + \mu \alpha_2 \alpha_3}{\alpha_1^2 + \alpha_2^2} \right] + \frac{\cos^2 \theta [3\nu^2 \alpha_1^2 + 12\sqrt{3}\mu \alpha_2 \alpha_3] \left[ -4\alpha_1 + 2\sqrt{2}\alpha_1 \right]}{\alpha_1^2 + \alpha_2^2}.
\]

where we introduced the parametrization \( \alpha_k = n_k \) with \( n = \sqrt{\alpha \cdot \alpha} \) to account for the normalization of \( \tilde{\lambda} \).

We stress that, being interested in weak crosstalk (14), we discard terms of order \( \mu^2 \) and higher.

We observe that terms linear in \( \mu \) are also odd in at least one random variable. Due to the fact that \( \alpha_k \) are normally distributed, these terms have vanishing mean values and can be discarded. As a result, we get

\[
q_0 \to (2\nu - 1)^2 \left( \frac{\alpha_1^2 \sin^2 \theta}{\alpha_1^2 + \alpha_2^2} + \frac{\alpha_2^2 \cos^2 \theta}{\alpha_1^2 + \alpha_2^2} \right).
\]

We remark that the normalization factor \( n \) disappears. The mean value can be now easily obtained by integrating over all random variables \( \alpha_k \) with Gaussian weight:

\[
\langle q_0 \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d\alpha_1 d\alpha_2 \ldots d\alpha_{15} e^{-\alpha_1^2/2} q_0 = \frac{1}{2} (2\nu - 1)^2.
\]

The corresponding standard deviation can be calculated in a similar fashion, yielding

\[
\sigma_{q_0} := \sqrt{\langle q_0^2 \rangle - \langle q_0 \rangle^2} = \frac{1}{4} (2\nu - 1)^2 \sqrt{\frac{3 + \cos 4\theta}{2}}.
\]

Combining the last two equations, we thus find that the mean Fisher information averaged over generic crosstalk of fixed strength approaches, in the asymptotic limit \( x \to 0 \),

\[
w^2 \langle F_{gen} (x \to 0) \rangle \approx \frac{1}{2} (2\nu - 1)^2 \left( 1 \pm \frac{1}{2} \sqrt{\frac{3 + \cos 4\theta}{2}} \right),
\]

where the term after \( \pm \) stands for one standard deviation.

Comparing this with equation (30), we find that the average value of crosstalk-influenced SPADE Fisher information is lower than the one obtained from ideal direct imaging even after adding one standard deviation to the former. This result stands in contrast to the case of perfectly balanced sources [26], where it was found that SPADE and direct imaging both approach zero for \( x \to 0 \), with SPADE scaling more favorably. Let us stress, however, that equation (30) holds for ideal direct imaging only. In reality one should expect direct imaging to perform worse due to finite pixel size and experimental noise, similarly to how SPADE suffers from crosstalk. Thus, in practice, we should expect SPADE to perform better in comparison with direct imaging.

This is indeed what we find when we consider the sub-Rayleigh regime outside of the very limit \( x \to 0 \). According to our numerical simulations, performed for crosstalk strength in the range \( p_c \in [10^{-3}, 10^{-1}] \) and described in detail in appendix C, the crosstalk-averaged SPADE Fisher information is larger than its ideal direct imaging counterpart for all separations larger than the threshold point \( x_{th} \), with \( x_{th} \) ranging from approximately \( x_{th} \approx 0.05 \sqrt{p_c} \) for \( \nu = 0.55 \) to \( x_{th} \approx 0.2 \sqrt{p_c} \) for \( \nu = 0.7 \). Notably, the threshold point decreases as \( \nu \) approaches 1/2, and in particular equals zero at this point, reproducing known results for balanced sources.

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7 We stress that here, e.g. \( \alpha_{13} \) means the thirteenth component of the vector \( \alpha \), i.e. 13 does not stand for a double index.
This definition follows from the Cramér–Rao bound (the smallest solution distinguishable by the device for a fixed number of measured photons). Mathematically, the MRD is given by the equation:

\[ d_{\text{min}} = \frac{1}{\sqrt{NF(\nu)}}. \] (36)

Unfortunately, calculating MRD analytically is typically not feasible. For a single realization of crosstalk, in the asymptotic region \( x \to 0 \), the Fisher information may be expanded up to the quadratic term, as in the bottom line of equation (27). It is straightforward to see that the resulting equation (36) is equivalent to a quartic polynomial equation in \( d_{\text{min}} \), which can be solved using the quartic root formula. However, neither the Fisher information nor the MRD obtained in such a way can be analytically averaged over crosstalk due to the corresponding integrals having non-elementary functional forms. For this reason, we calculate MRD numerically from definition, as before employing the full expression (23) for the SPADE Fisher information for \( D = 3 \).

In figure 3, we plot the MRD obtained from crosstalk-influenced SPADE and direct imaging as a function of relative brightness. We used photon numbers \( N = \{10^2, 10^3, 10^4\} \) in the orders of magnitude used in contemporary experimental setups [3, 9, 36] and crosstalk strength \( \rho_c = 0.01 \), the largest value reported so far [37]. From the figure we observe that there is always a non-zero range starting at the point of equal source brightnesses \( \nu = 1/2 \), at which SPADE is superior to ideal direct imaging, despite the fact that the strength of crosstalk is relatively high. Especially at smaller photon numbers SPADE outperforms ideal direct imaging for most values of relative brightness, which suggests it to be an efficient tool for measurements with relatively low number of photons collected.

From the physical point of view, this behavior may be seen as a consequence of the fact that, as discussed previously, SPADE becomes superior to direct imaging with growing source separation, where the effects of crosstalk become increasingly negligible. However, large MRD correspond to low photon numbers (this can be seen from equation (36) and is also obvious, since fewer measured photons must give us less
Figure 3. Minimal resolvable distance for $\theta = 0$ and photon numbers $10^2$ (top), $10^4$ (bottom left), $10^6$ (bottom right). The two lines denote: direct imaging (red) and SPADE averaged over 2000 random crosstalk matrices of strength $p_c = 0.01$ (blue dashed). Both plots were obtained numerically from definition (36), where for SPADE we used the Fisher information (23) with $D = 3$. As seen, there is always a region (denoted by green color) beginning at the point $\nu = 1/2$, at which SPADE outperforms ideal direct imaging. The smaller the photon number, the larger the width of this region. See main text for additional analysis.

information]. Hence, for a fixed value of relative brightness, SPADE becomes superior to direct imaging with shrinking photon number, which is exactly what we see in figure 3.

5. Concluding remarks

We assessed the impact of crosstalk on resolving sub-Rayleigh separations between two unbalanced light sources by SPADE. Using statistical methods, we found that the effectiveness of SPADE depends on the relationship between crosstalk strength and the measured separation (normalized with respect to the width of the point spread function). For separations much larger than the square root of crosstalk strength, SPADE still provides the optimal measurement scheme, while for asymptotically vanishing separations its performance is reduced. Numerical simulations performed for realistic values of crosstalk strength and relative brightness show that, while crosstalk-affected SPADE no longer achieves the optimal value of Fisher information, it is still superior to direct imaging for a wide range of experimentally relevant sub-Rayleigh source separations. It is worth noting that the real applicability of SPADE is likely higher due to the fact that we were comparing it to direct imaging with no imperfections.

Data availability statement

The data cannot be made publicly available upon publication because they are not available in a format that is sufficiently accessible or reusable by other researchers. The data that support the findings of this study are available upon reasonable request from the authors.

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Appendix A. Crosstalk strength and random crosstalk matrices

To reliably generate random crosstalk matrices of a given strength, it is necessary to know the relation between crosstalk strength $p_c$ and the number $\mu$ used in the exponential representation (13). In this appendix, we derive the relation (16) between these parameters and discuss the generation of random crosstalk matrices.

It follows from equations (14) and (15) that in the lowest order $p_c \propto \mu^2$. In fact, from the construction of the generalized Gell–Mann matrices $[34]$ one can easily calculate that in the lowest order

$$p_c(\mu) \approx \frac{2\mu^2}{D^2(D^2-1)} \sum_{k \in \mathcal{S}_{\text{nd}}} \lambda_k^2,$$

where the summation is over $k$ corresponding to non-diagonal matrices $G_k$.

To see how $p_c$ typically depends on $\mu$ (when averaged over all possible crosstalk matrices), we assume that each $\lambda_k$ is generated with the same, normal weight. Thus, on average, every $\lambda_k$ must contribute equally to equation (A1):

$$\langle \lambda_k^2 \rangle = \frac{1}{D^4-1},$$

as $\vec{\lambda}$ has $D^4-1$ components. Consequently, since $D^2(D^2-1)$ out of these components correspond to non-diagonal matrices $G_k$, we have

$$\langle \sum_{k \in \mathcal{S}_{\text{nd}}} \lambda_k^2 \rangle = \frac{D^2(D^2-1)}{D^4-1}.$$  

Taking the average of equation (A1) and using the above equation immediately yields equation (16).

To show that the approximate equation (16) accurately captures the relation between crosstalk strength and $\mu$ for weak crosstalk, we compare the two in figure A1, finding our approximation and numerical data agree up to one standard deviation.

To generate random generic crosstalk matrices with desired crosstalk strength, we generate vectors $\vec{\lambda}$ according to Gaussian distribution and substitute into equation (13) with $\mu$ calculated from equation (16).

Appendix B. Optimal measurement region

As discussed in section 3, the relation $x \gg \sqrt{p_c}$ determines the range of separations at which the measurement apparatus performs with the highest effectiveness in the presence of crosstalk. To make this relation useful for practical predictions, in this appendix, we give it a precise quantitative meaning.

To this end, we investigate the values of the ratio

$$k := x/\sqrt{p_c},$$

at which the crosstalk-affected Fisher information is approximately equal to at least 90% of its maximal value. In figure B1(a) we present the SPADE Fisher information averaged with respect to generic crosstalk as
observe that for the case of approximately an order of magnitude smaller than \(k\) the median is in the range 1.3–3.1. Furthermore, even in the worst cases, the required ratio value does not exceed \(k\). These results indicate that for typical crosstalk strengths, the relation \(\nu \approx \frac{90}{k}\) for \(\nu \in \{0.5, 0.6, 0.7\}\) is enough to obtain 90% of the maximal Fisher information for various \(\nu\) with \(\nu\) randomized across samples. Black dots represents outliers. For each box 200 samples were used. (c) Average ratio of Fisher information at the point \(x = 3\sqrt{\nu}\) and maximal Fisher information as a function of \(\nu\).

Figure B1. (a) SPADE Fisher information averaged with respect to generic crosstalk as a function of \(k = x/\sqrt{\nu}\) for \(\nu \in \{10^{-2}, 10^{-3}, 10^{-4}\}\) (red dot–dashed, green, blue short–dashed, respectively). \(\theta = 0, D = 3\) and \(\nu = 0.6\). Horizontal dashed lines correspond to 90% of the respective maximal values of Fisher information. (b) Box and whisker plot of \(k\) required to obtain 90% (yellow brighter boxes) and 95% (blue darker boxes) of maximal Fisher information for various \(\nu\) with \(\nu\) randomized across samples. These results show clearly that, if \(\nu\) efficiency is considered sufficient, the optimal measurement region for most crosstalk matrices starts at separations as low as \(x = 3\sqrt{\nu}\) and is independent of \(\nu\). This is additionally supported by figure B1(c) on which we present the average ratio of Fisher information at the point \(x = 3\sqrt{\nu}\) and maximal Fisher information as a function of \(\nu\) for \(\nu \in \{0.5, 0.6, 0.7\}\).

Appendix C. Range of advantage of SPADE over ideal direct imaging

In this appendix, we determine the range of separations, at which SPADE is superior to ideal direct imaging. More precisely, we consider the threshold point, or threshold point, \(x_t\) for which the Fisher information for both methods is equal.

Once again, we consider the ratio \(k\) between separation and crosstalk strength as defined in equation (B1). Figure C1(a) shows the average value of \(k\) corresponding to threshold points for 200 crosstalk realizations as a function of the average crosstalk strength for relative brightnesses \(\nu \in \{0.55, 0.6, 0.7\}\). We can see that for the three values of \(\nu\), the threshold point equals approximately \(x_t = \{0.05\sqrt{\nu}, 0.10\sqrt{\nu}, 0.2\sqrt{\nu}\}\), in order of increasing \(\nu\). Furthermore, the threshold point has no significant dependence on crosstalk strength in the considered range. We stress that even for the relatively large relative brightness \(\nu = 0.7\) the transition point is approximately an order of magnitude smaller than \(\sqrt{\nu}\).

Figure C1(b) shows a box plot of \(k\) corresponding to threshold points for different \(\nu\) averaged over random crosstalk of varying strength. From this figure we can see that as \(\nu \rightarrow 1/2\), the median of \(k\)
Figure C1. (a) Mean ratio $k$ corresponding to threshold points averaged over 200 random crosstalk realizations as a function of crosstalk strength $p_c$ for $\nu \in \{0.55, 0.6, 0.7\}$ represented by red triangles, blue circles, green squares respectively. (b) Box and whisker plot of $k$ corresponding to threshold points for different $\nu$. For each $\nu$ 200 crosstalk matrices of random strength were generated. Randomization of $p_c$ was realized by generating a random real $r$ from the interval $[\ln 0.1, \ln 0.8]$ with uniform distribution and setting $\mu = \exp r$. This roughly corresponds to the range of crosstalk strengths from figure C1(a).

approaches 0, which corresponds to SPADE being always better than ideal direct imaging. This is of course expected given the findings for balanced sources, since in such a case there is no threshold point [26]. In addition, the median decreases faster near $\nu = 1/2$. This shows SPADE is especially effective for $\nu$ close to 1/2 (see also figure 3).

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