NUMERICAL METHODS FOR THE 2-HESSIAN ELLIPTIC
PARTIAL DIFFERENTIAL EQUATION

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ABSTRACT. The elliptic 2-Hessian equation is a fully nonlinear partial differential equation that appears in geometric surface providing an intrinsic curvature for three dimensional manifolds. In this article we explain why the naive finite difference method fails in general and provide explicit, semi-implicit and Newton solvers which perform better by enforcing a convexity type constraint needed for the ellipticity of the equation itself. We build a monotone wide stencil finite difference discretization, which is less accurate but provable convergent as a result of the Barles-Souganidis theory. Solutions with both discretizations are found using Newton’s method. Computational results are presented on a number of exact solutions which range in regularity from smooth to nondifferentiable and in shape from convex to non convex.

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1. Introduction

In this article we study numerical approximations of a fully nonlinear elliptic partial differential equation (PDE), the $k$-Hessian equation. The $k$-Hessian equations are a family of PDEs in $n$-dimensional space, which include the Laplace equation, when $k = 1$, and the Monge-Ampère equation, when $k = n$. We have already studied the Dirichlet problem for the Monge-Ampère equation [FO11a, FO11b, FO13]. Here we study the first instance of this equation which is neither the Laplacian, or the Monge-Ampère equation, which is 2-Hessian equation in three dimensions,

$$S_2[u] = u_{xx}u_{yy} + u_{xx}u_{zz} + u_{yy}u_{zz} - u_{xy}^2 - u_{xz}^2 - u_{yz}^2.$$

While the 2-Hessian equation is unfamiliar outside of Riemannian geometry and elliptic regularity theory, it is closely related to the scalar curvature operator, which provides an intrinsic curvature for a three dimensional manifold. Geometric PDEs have been used widely in image analysis [Sap06]. In particular, the Monge-Ampère equation in the context of Optimal Transportation has been used in three dimensional volume based image registration [HZTA04]. Scalar curvature equations have not yet been used in these contexts, perhaps because no effective solvers for PDEs involving this operator have yet been developed. The 2-Hessian operator also appears in conformal mapping problems. Conformal surface mapping has been used for two dimensional image registration [AHTK99, GWC04], but does not generalize directly to three dimensions. Quasi-conformal maps have been used in three dimensions [WWJ07, ZG11], however these methods are still being developed.

In this article we introduce a monotone discretization of the 2-Hessian equation in the three-dimensional case. A proof of convergence to the viscosity solution is provided. We build as well a second order accurate finite difference solver which demonstrates evidence of convergence even for singular solutions. Numerical results are presented.

We focus on the Dirichlet problem

$$\begin{cases} S_2[u] = f, & \text{in } \Omega, \\
 u = g, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega$ is a rectangular (three dimensional box) domain, which is natural for prescribed curvature problems. For other geometries, different boundary conditions need to be used. For the torus, periodic boundary conditions could be used. For the sphere, it is more complicated, but it is possible to patch together several cubic domains to obtain this topology.

1.1. Related work on curvature equations. The 2-Hessian equation is a curvature PDE in three dimensions, but it’s in two dimensions where one can find several works on the evolution of curves using curvature, going back to the seminal paper of Osher and Sethian [OS88]. In [Obe04], a finite difference monotone scheme is
given for the motion of level sets by mean curvature. The advantage of monotone discretizations is that they have a convergence proof, and convergent schemes are more stable and allow for faster solvers [Set95]. The surface evolver [Bra92] is a tool to evolve two dimensional surfaces by curvature based on the minimization of its energy. In [Sap06] one can find a relation between geometric PDEs and image analysis. For a review of the numerical methods for curvature flows see [DDE05].

In three dimensions there’s substantial less work done on curvature flows. It is well know that triangulated surfaces don’t provide curvature information and therefore finite element and related methods require higher order elements, which are quite difficult to implement in three dimensions [WMKG07, HPW06]. This makes finite differences natural, since they are simple to implement and don’t require surface discretization. Finite differences for the Monge-Ampère equation have been exhaustively studied [FO11a, FO11b, FO13]. Recent work by Mirebeau [BCM14] provides a two dimensional discretization of the Monge-Ampère, generalizable to three dimensions, using a mixture of finite difference and ideas from discrete geometry.

The Monge-Ampère problem is related to the problem of prescribed Gauss curvature. A numerical method for the problem of prescribed Gauss curvature can be found in [MO +14]. The Gauss curvature flow is also used in image process for surface fairing [EE07].

1.2. Related work on the 2-Hessian equation. Despite the number of applications of the 2-Hessian equation, there are very few publications devoted to solving it. In the early work of [SG10] a quadratically constrained eigenvalue minimization problem is solved to obtain the solution of the 2-Hessian equation. In [Awa14], an iterative method with quadratic convergence rate is proposed. Gauss-Seidel and semi-implicit solvers, that relate to the ones we present here, are also discussed.

1.3. Scalar curvature and the 2-Hessian equation. The Gaussian curvature of a two-dimensional surface is the product of the principal curvatures, $\kappa_1, \kappa_2$ of the surface. It is an intrinsic quantity: it does not depend on the embedding of the surface in space. Locally, the surface can be defined as the graph of the shape operator $u(x)$, chosen so that the gradient of the shape operator vanishes at $x$. Then the Gaussian curvature at $x$ is given by the determinant of the Hessian of the shape operator

$$\det(D^2u) = \kappa_1 \kappa_2,$$

which is the two dimensional Monge-Ampère operator applied to $u$ (if the gradient of $u$ does not vanish at $x$, additional first order terms appear).

The sign of the Gaussian curvature characterizes the surface, and relates how the area of a geodesic ball in a curved Riemannian surface deviates from that of the standard ball in Euclidean space (larger or smaller depending on the sign). The uniformization theorem of complex analysis establishes the fact that every surface has a conformal metric of constant Gaussian curvature: the sphere, the Euclidean plane, or hyperbolic space. The uniformization theorem can be proved by several different methods. A natural method is one that solves a semi-linear Laplace equation for the conformal map; see [MT02, Section 8].

**Curvature in three and higher dimensions** In dimensions greater than three, curvature is a tensor rather than a scalar quantity. The curvature tensor is defined by the sectional curvature, $K(p, x)$, which is given by the Gaussian curvature of
the geodesic surface defined by the tangent plane, \( p \), at \( x \). The scalar curvature (or the Ricci scalar), which is the trace of the curvature tensor, is the simplest curvature invariant of a Riemannian manifold. It can be characterized as a multiple of the average of the sectional curvatures. If we choose coordinates so that a three dimensional surface is given by the graph of a function \( u(x) \) whose gradient vanishes at \( x \), then the scalar curvature is given by a constant multiple of the 2-Hessian operator:

\[
\frac{1}{2} \left( \text{trace}(D^2 u)^2 - \text{trace}((D^2 u)^2) \right) = \kappa_1 \kappa_2 + \kappa_1 \kappa_3 + \kappa_2 \kappa_3
\]

where \( \kappa_1, \kappa_2, \kappa_3 \) are the three principal curvatures. Again, if the gradient of \( u \) does not vanish at \( x \), additional first order terms appear. However the equation above holds in general if we replace the principal curvatures with the eigenvalues of the Hessian. This leads to the 2-Hessian equation; see section 2 below.

Since the second order terms pose the primary challenge in the solution of nonlinear elliptic equations, we focus on the 2-Hessian equation in this work. In a similar way, the Monge-Ampère equation can be related to the equation for Gauss curvature through the inclusion of appropriate first order terms. In [BFO14] we studied an extension of the Monge-Ampère equation with first order nonlinear terms; in that case the primary challenge was the boundary conditions.

1.4. Differential geometry and \( k \)-Hessian equations. Conformal changes of metric (multiplication of the metric by a positive function) have played an important role in surface theory [LP+87].

One of the foundational problems of Riemannian differential geometry is to generalize the uniformization theorem for surfaces to higher dimensions. The generalization of the uniformization theorem for surfaces to higher dimensional manifolds involves replacing constant Gauss curvature (which is a scalar in two dimensions) with constant scalar curvature (rather than constant tensor curvature). The resulting problem is called

**The Yamabe Problem** Given a compact Riemannian manifold \((M, g)\) of dimension \( n \geq 3 \), find a metric conformal to \( g \) with constant scalar curvature.

The solution of the Yamabe problem can be obtained by solving a nonlinear elliptic eigenvalue problem [Tru68]. Generalizations of the Yamabe problem to other curvatures result in \( k \)-Hessian type equations [Via00, Via99].

Also of interest is

**The Calabi-Yau problem** [GHJ03] Find a conformal mapping, given by \( u(x) \), which transforms a given metric \( g_{ij} \) to a new one \( \tilde{g}_{ij} \) given by

\[
\tilde{g}_{ij} = \exp(u) g_{ij}
\]

The function \( u(x) \) satisfies a real Monge-Ampère type PDE [Yau78]. In certain settings (for example, the quaternionic setting), the Calabi-Yau problem for a manifold which is even (\( d = 2n \)) dimensional, results in a \( k \)-Hessian type equation with \( k = d/2 \) [AV10].

Another interesting problem where the \( k \)-Hessian equation appears is the following:

**Christoffel-Minkowski Problem** Find a convex hypersurface with the \( k \)-th symmetric function of the principal radii prescribed on its outer normals.

Turns out the solution of the Christoffel-Minkowski problem corresponds to finding convex solutions of a \( k \)-Hessian equation on the \( n \)-sphere [GM03].
The 2-Hessian equation corresponds to scalar curvature, as we discuss above, and solving the 2-Hessian PDE (or a related one) allows for the construction of hyper-surfaces of prescribed curvatures, for example scalar curvature \[GG02\].

Also related are the problem of local isometric embedding of Riemannian surfaces in \(\mathbb{R}^3\) and the related Weyl problem \[TW08\].

2. Background on the equation

In this section, we present the background analysis for the \(k\)-Hessian equation, with particular focus on the 2-Hessian equation in the three dimensional case. We follow the review by Wang \[Wan09\].

The \(k\)-Hessian equation can be written as

\[(kH) S_k[u] = f\]

where \(1 \leq k \leq n\), \(S_k[u] = \sigma_k(\lambda(D^2u))\), \(\lambda(D^2u) = (\lambda_1, \ldots, \lambda_n)\) are the eigenvalues of the Hessian matrix \(D^2u\) and

\[\sigma_k(\lambda) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k}\]

is the \(k\)-th elementary symmetric polynomial.

In this paper, we focus on the the three-dimensional case with \(k = 2\)

\[(2H) S_2[u] = f\]

where

\[S_2[u] = \sigma_2(\lambda) = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3.\]

Admissible functions and ellipticity. When \(k\) is even, the \(k\)-Hessian equation lacks uniqueness: if \(u\) solves the \(k\)-Hessian equation, so does \(-u\). Thus an additional condition is needed to ensure solution uniqueness. Additionally, it is necessary to restrict the solutions to an appropriate class of functions in order to ensure that the equation is elliptic.

Set

\[\Gamma_k = \{ \lambda \in \mathbb{R}^n \mid \sigma_k(\lambda) > 0, j = 1, \ldots, k \}.\]

\(\Gamma_k\) is a symmetric cone, meaning that any permutation of \(\lambda\) is in \(\Gamma_k\). When \(k = 1\), \(\Gamma_1\) is the half space \(\{ \lambda \in \mathbb{R}^n \mid \lambda_1 + \ldots + \lambda_n > 0 \}\). When \(k = n\), \(\Gamma_n\) is the positive cone

\[\Gamma_n = \{ \lambda \in \mathbb{R}^n \mid \lambda_j > 0, j = 1, \ldots, n \}.\]

The result is a restriction to subharmonic functions for \(k = 1\) and convex functions for \(k = n\), as is usually done when studying the Laplace and Monge-Ampère equations, respectively.

**Definition 2.1.** A function \(u \in C^2\) is \(k\)-admissible if \(\lambda(D^2u) \in \overline{\Gamma}_k\).

We now take a closer look at the case we are interested: \(k = 2\) and \(n = 3\). We have

\[\Gamma_2 = \{ \lambda \in \mathbb{R}^3 \mid \lambda_1 + \lambda_2 + \lambda_3 > 0, \; \sigma_2(\lambda) > 0 \}\]

The following Proposition provides an alternative characterization of \(\Gamma_2\).

**Proposition 2.2.** Let

\[\Gamma = \{ \lambda \in \mathbb{R}^3 \mid \lambda_1 + \lambda_2 > 0, \; \lambda_1 + \lambda_3 > 0, \; \lambda_2 + \lambda_3 > 0 \}\]

Then

\[\Gamma_2 = \Gamma \cap \{ \lambda \in \mathbb{R}^3 \mid \sigma_2(\lambda) > 0 \}.\]
Proof. Proving the $\supseteq$ part is straightforward. We then prove the inclusion $\subseteq$. Suppose that $(\lambda_1, \lambda_2, \lambda_3) \in \Gamma_2$. Without loss of generality we can assume that $\lambda_1 \leq \lambda_2 \leq \lambda_3$. Thus, it’s enough to show that $\lambda_1 + \lambda_2 > 0$. Suppose that $\lambda_1 + \lambda_2 \leq 0$.

We consider two cases, each leading to a contradiction.

1. $\lambda_1 + \lambda_2 = 0$

   We have $\lambda_1 \lambda_2 \leq 0$. Hence
   
   $$\sigma_2(\lambda) = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3$$
   
   $$= \lambda_1 \lambda_2 + (\lambda_1 + \lambda_2) \lambda_3$$
   
   $$= \lambda_1 \lambda_2$$
   
   $$\leq 0,$$
   
   contradicting the assumption $\sigma_2(\lambda) > 0$.

2. $\lambda_1 + \lambda_2 < 0$

   Since $\lambda_1 \leq \lambda_2$, we have $\lambda_1 < 0$. Moreover
   
   $$\sigma_2(\lambda) > 0 \iff \lambda_3 (\lambda_1 + \lambda_2) > -\lambda_1 \lambda_2 \iff \lambda_3 < -\frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}$$
   
   and
   
   $$\lambda_1 + \lambda_2 + \lambda_3 > 0 \iff \lambda_3 > -\lambda_1 - \lambda_2$$

   From the above two inequalities we get
   
   $$-\lambda_1 - \lambda_2 < -\frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}$$
   
   which we can rewrite as
   
   $$\lambda_1 (\lambda_1 + \lambda_2) + \lambda_2^2 < 0.$$

   Now, since $\lambda_1 < 0$ and $\lambda_1 + \lambda_2 < 0$, the left-end side of the inequality must be positive and we have thus derived a contradiction. □

It’s easy to show, using differentiation, that the function $\sigma_2$ is nondecreasing on the set $\Gamma$. This allows us to define the set of admissible functions as the set of functions where $S_2$ is elliptic. In general we have:

**Proposition 2.3.** If $u$ is $k$–admissible then the $k$–Hessian equation $(kH)$ is (degenerate) elliptic.

**Remark 2.1.** We allow the eigenvalues of $u$ to lie in the boundary of $\Gamma_k$ and in such case the $k$–Hessian equation may become degenerate elliptic.

The constraint $\sigma_2(\lambda) \geq 0$ will be enforced automatically in our schemes by taking a non-negative $f$ in the PDE $(2H)$. Therefore it’s enough to look at the set $\Gamma$ as defined in (3). We will refer to this restriction as plane-subharmonic since it corresponds to $u$ being subharmonic on every plane.

**Viscosity Solutions.** Well-posedness and regularity for the equation is studied in [CNS85]. Here we recall the definition of viscosity solutions.

**Definition 2.4.** A function $u \in C^0(\overline{\Omega})$ is called a viscosity subsolution (resp. super solution) of $(kH)$ if for any $\phi \in C^2(\Omega) \cap \overline{\Gamma_k}$

$$\sigma_k(\lambda(D^2 \phi(x))) \geq f \quad (\text{resp.}, \leq 0)$$
provided that \( u - \phi \) attains its maximum (resp., minimum) at \( x \in \Omega \). We call \( u \) a viscosity solution of \((kH)\) if it is both a viscosity subsolution and supersolution of \((kH)\).

**Alternative description of the 2-Hessian operator.** For a \(3 \times 3\) matrix \( M \), the characteristic polynomial is given by

\[
\det(M) - c(M)\lambda + \text{trace}(M)\lambda^2 - \lambda^3
\]

where \( c(M) \), the sum of the principal minors of \( M \), is given by

\[
c(M) = \frac{1}{2} \left( \text{trace}(M)^2 - \text{trace}(M^2) \right).
\]

If \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) are the eigenvalues of \( M \) then

\[
c(M) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3.
\]

Therefore, using (2), we conclude that (1) holds,

\[
S_2[u] = c(D^2u) = u_{xx}u_{yy} + u_{xx}u_{zz} + u_{yy}u_{zz} - u_{xy}^2 - u_{xz}^2 - u_{yz}^2.
\]

**Linearization.** The linearization of \( c \) in (5) is given by:

\[
\nabla c(M) \cdot N = \text{trace}(D^2u) \text{trace}(D^2v) - \text{trace}(D^2uD^2v).
\]

We can apply this to the linearization of the 2-Hessian operator. When \( u \in C^2 \) we can linearize this operator as:

\[
\nabla S_2[u] \cdot \nu = \text{trace}(D^2u) \text{trace}(D^2v) - \text{trace}(D^2uD^2v).
\]

**Lemma 2.5.** Let \( u \in C^2 \). The linearization of the 2-Hessian operator (5) is elliptic if \( u \) is 2-admissible.

**Remark 2.2.** When the function \( u \) fails to be “strictly” 2-admissible, the linearization can be degenerate elliptic, which affects the conditioning of the linear system (5). When \( u \) is not 2-admissible, the linear system can be unstable.

### 3. Discretization and solvers

In this section we explain why the naive finite difference method fails in general. We introduce explicit, semi-implicit, and Newton solvers for the naive finite difference method, which perform better by enforcing the plane-subharmonic constraint. This is similar to the solvers used in [BFO10] for the Monge-Ampère equation. Then we introduce a discretization which is monotone and thus provably convergent. While the monotone discretization is less accurate, it has the advantage that it gives a globally consistent, monotone discretization of the operator, meaning that we can apply the operator to non-admissible functions. This is useful because it circumvents the need for special initial data, and allows for the parabolic (time-dependent) operator to be define on an unconstrained class of functions.

In addition, we could combine the monotone discretization with the naive finite difference discretization to obtain provably convergent, accurate filtered finite difference schemes, using the ideas in [FO13]. This approach combines the advantages of both schemes, with little additional efforts. In this work, we were mainly interested in comparing the performance of the two schemes, so we did not implement the filtered scheme.
3.1. Naive finite difference scheme. We begin by discussing the naive finite difference discretization of the 2-Hessian. This is done by simply using standard finite differences to discretize the operator. Denote by \( D^{2,h}u \) the discretized Hessian using standard finite differences on a uniform grid with grid spacing \( h \), i.e.,

\[
D^{2,h}u_{ijk} = \begin{bmatrix}
D_{xx}u_{ij} & D_{xy}u_{ij} & D_{xz}u_{ij} \\
D_{yx}u_{ij} & D_{yy}u_{ij} & D_{yz}u_{ij} \\
D_{zx}u_{ij} & D_{zy}u_{ij} & D_{zz}u_{ij}
\end{bmatrix},
\]

where

\[
\begin{align*}
a_1 &= \frac{a_{i+1,j,k} + a_{i-1,j,k}}{2} & a_4 &= \frac{a_{i+1,j+1,k} + a_{i-1,j-1,k}}{2} & a_7 &= \frac{a_{i-1,j,k+1} + a_{i+1,j,k-1}}{2} \\
a_2 &= \frac{a_{i,j+1,k} + a_{i,j-1,k}}{2} & a_5 &= \frac{a_{i-1,j+1,k} + a_{i+1,j-1,k}}{2} & a_8 &= \frac{a_{i,j+1,k+1} + a_{i,j-1,k-1}}{2} \\
a_3 &= \frac{a_{i,j,k+1} + a_{i,j,k-1}}{2} & a_6 &= \frac{a_{i+1,j,k+1} + a_{i-1,j,k-1}}{2} & a_9 &= \frac{a_{i,j+1,k-1} + a_{i,j-1,k+1}}{2}
\end{align*}
\]

We then get the discrete version of the 2-Hessian operator \( S_2[u] \) as

\[
S_2^h[u] = c(D^{2,h}u)
\]

Since we are using centered finite differences, this discretization is consistent, and it is second order accurate if the solution is smooth (hence the superscript \( A \)). However, this scheme is not monotone due to the off-diagonal terms in the cross derivatives \( u_{xy}, u_{xz} \) and \( u_{yz} \). Therefore the Barles and Souganidis theory [BS91] does not apply and no convergence proof is available.

3.2. Failure of the parabolic solver for the naive finite differences. In this section we give a simple example to illustrate that the use of the naive finite difference scheme (6) together with a parabolic solver fails to converge.

The parabolic solver is given by

\[
u^{n+1} = u^n - dt(S_2^h[u] - f).
\]

Consider the solution of (2H) in \([0,1]^3\), given by

\[
u(x) = \frac{x^2}{2}, \quad f(x) = 3.
\]

The iteration is initialized with the exact solution with noise from a uniform distribution \( U(-0.01,0.01) \). The result after performing two iterations with the parabolic solver (7) with time step \( dt = dx^4 \) and the initial guess are illustrated in Figure 1. Regardless of the time step chosen (\( dt = dx^4/10 \) and \( dt = dx^4/100 \) were also used), after a sufficient number of iterations the solution behaves like in the example of Figure 1, until it eventually blows up. This tells us that the instability of the parabolic solver is inherent from the discretization rather than being the result of a poorly chosen time step. This instability is due to the fact that there is no mechanism to pick the right solution. The discretization, being a quadratic equation as we will see in subsubsection 3.3.1, has two solutions: the 2-convex solution we are looking for and the negative of this.

3.3. Solvers for the naive finite difference scheme. In this section we present three different solvers for the naive finite difference scheme: a Jacobi type solver obtained by solving the discretization for the reference variable; a semi-implicit solver based on an identity that relates the Laplacian and the 2-Hessian operator; a Newton solver.
3.3.1. Jacobi solver. The accurate discretization of (2H) leads to a quadratic equation for the reference variable at each grid point. To see this we introduce the notation (8)

\[ a_1 = \frac{u_{i+1,j,k} + u_{i-1,j,k}}{2}, \quad a_2 = \frac{u_{i,j+1,k} + u_{i,j-1,k}}{2}, \quad a_3 = \frac{u_{i,j,k+1} + u_{i,j,k-1}}{2}. \]

Using (6), \( S_A^2[u] = f \) can be rewritten as

\[ \frac{4}{h^4} \left( \sum_{i_1 < i_2 \leq 3} (a_{i_1} - u_{i,j,k})(a_{i_2} - u_{i,j,k}) \right) = f_{ijk} + \frac{1}{4h^4} \sum_{p=2}^4 (a_{2p} - a_{2p+1})^2. \]

Solving for \( u_{ijk} \) and selecting the smaller root (in order to select the locally more plane-subharmonic solution), we obtain (J)

\[ u_{ijk} = \frac{a_1 + a_2 + a_3}{3} - \frac{1}{12} \sqrt{8 \sum_{i_1 < i_2 \leq 3} (a_{i_1} - a_{i_2})^2 + 3 \sum_{p=2}^4 (a_{2p} - a_{2p+1})^2 + 12 f_{ijk} h^4}. \]

We can now use a Jacobi iteration to find the fixed point of (J). Notice that the plane-convexity constraint is not enforced beyond the selection of the smaller root in (J).

**Remark 3.1.** Formula (J) can be rewritten as

\[ u_{ijk} = \frac{a_1 + a_2 + a_3}{3} - \frac{h^2}{6} \sqrt{\text{trace}(D^2 h u_{ijk})^2 + 3 \left( f_{ijk} - S_h^2 A[u] \right)}. \]
Remark 3.2. Formula (J) can also be used in a Gauss-Seidel iteration, which should converge faster than the Jacobi iteration. We choose not to implement it here since all computational results were obtained in MATLAB, which is known to be slow with loops.

In order to prove the convergence of the above solver, it is sufficient to prove that it is monotone, which in this case is the same as showing that the value $u_{ijk}$ is a non-decreasing function of the neighboring values [Obe06]. However, this is not the case for (J).

3.3.2. Semi-implicit solver. The next solver we discuss is a semi-implicit one, which involves solving a Laplace equation at each iteration.

We begin with the following identity for the Laplacian in three dimensions:

$$| \Delta u | = \sqrt{(\Delta u)^2} = \sqrt{u_{xx}^2 + u_{yy}^2 + u_{zz}^2 + 2u_{xx}u_{yy} + 2u_{xx}u_{zz} + 2u_{yy}u_{zz}}.$$ 

If $u$ solves the 2-Hessian equation, then

$$| \Delta u | = \sqrt{(\Delta u)^2} = \sqrt{u_{xx}^2 + u_{yy}^2 + u_{zz}^2 + 2u_{xy}^2 + 2u_{xz}^2 + 2u_{yz}^2 + 2f} = \sqrt{|D^2 u|^2 + 2f}.$$ 

This leads to a semi-implicit scheme for solving the 2-Hessian equation given by

$$\Delta u^{n+1} = \sqrt{|D^2 u^n|^2 + 2f}.$$ 

Note that if $u$ is a 2-admissible function, then $\Delta u \geq 0$, a condition the scheme enforces.

A good initial value for the iteration is given by the solution of

$$\Delta u^0 = \sqrt{2f}.$$ 

3.3.3. Newton solver. To solve the discretized equation

$$S_{2}^{A}[u] = f$$

we can use a damped Newton iteration

$$u^{n+1} = u^n - \alpha v^n$$

where $0 < \alpha \leq 1$. The damping parameter $\alpha$ is chosen at each step to ensure that the residual $\|S_{2}^{A}[u^n] - f\|$ is decreasing. (In practice we can often take $\alpha = 1$, but damping is sometimes needed.) The corrector $v^n$ solves the linear system

$$(\nabla u S_{2}^{A}[u^n]) v^n = S_{2}^{A}[u^n] - f.$$ 

To setup the above equation we need the Jacobian of the scheme.

The Jacobian of the 2-Hessian operator, discretized using standard finite differences, is given by

$$\nabla u S_{2}^{A}[u] = \sum_{\nu_1, \nu_2 \in \{x,y,z\}, \nu_1 \neq \nu_2} (D_{\nu_1,\nu_2}u)D_{\nu_2}v - (D_{\nu_1,\nu_2}u)D_{\nu_1}v$$

which is a discrete version of the linearization of the 2-Hessian equation (5).
3.4. **Monotone finite difference scheme.** In this section we construct a monotone finite difference scheme. As we saw before, the naive approach of simply using finite differences for the terms in the Hessian matrix will not work because the cross derivative terms $u_{xy}, u_{xz}$ and $u_{yz}$ are not monotone. Instead the idea is to use wide stencils and a rotated coordinate system in which the Hessian matrix is diagonal. However, this coordinate system must be found in a monotone way. This section is divided in three parts: first, we extend the function $\sigma^2$ to be non-decreasing in $\mathbb{R}^3$; second, we find an expression for the 2-Hessian operator $S_2[u]$ which can be discretized in a monotone manner; and third, we present the monotone finite difference scheme.

3.4.1. **Non-decreasing extension of the operator.** In this section we find a non-decreasing extension of $\sigma^2$ from $\Gamma$ to $\mathbb{R}^3$. Our ultimate goal is to build a monotone finite difference approximation of the 2−Hessian equation,. Since we know that the eigenvalues of admissible solutions $u$ belong to the set $\Gamma$, we are free to redefine $\sigma^2$ outside of $\Gamma$ in order to ensure convergence. We then require an extension of $\sigma^2$ that is non-decreasing in $\mathbb{R}^3$, which is accomplished in the following Lemma.

**Lemma 3.1.** The function $\bar{\sigma} = f \circ \text{sort}$ where sort denotes the sorting function and $f$ is given by

$$f(x, y, z) = x \max(y, |x|) + x \max(z, |x|) + \max(y, |x|) \max(z, |x|)$$

extends $\sigma^2$ on $\Gamma$ and is non-decreasing in $\mathbb{R}^3$.

**Proof.** Without loss of generality, we assume that $x \leq y \leq z$ since sorting the values is monotone. Moreover, we can rewrite $f$ as

$$f(x, y, z) = \max(\max(x, |x|) + x, \max(z, |x|) + \max(x, |x|) - x^2$$

Suppose $(x, y, z) \in \Gamma$, then we recover $\sigma^2(x, y, z)$.

Next we show that $\bar{\sigma}$ is non-decreasing as a function of $(x, y, z)$. We have two cases to consider:

- $x + y \geq 0$
- $x + y < 0$

Since $x \leq y \leq z$, $(x, y, z) \in \Gamma$ and so we recover $\sigma^2$ which we know to be a non-decreasing function in $\Gamma$.

- $x + y < 0$

Since $x \leq y \leq z$, $x < 0$. We then get $\bar{\sigma}(x, y, z) = -x^2$, which is increasing since $x < 0$.

Hence $\bar{\sigma}$ is non-decreasing. \qed

3.4.2. **Elliptic expression for the operator.** In this section we build an expression that can be discretized in a monotone way.

The idea is to mimic what was done for the Monge-Ampère equation in [FO11b]: use a matrix identity to obtain a monotone expression for the operator.

First note that trace($M$) is invariant over conjugation $O^T M O$ by orthogonal matrices $O$. Second note that trace($M^2$) = $\sum_{ij} m_{ij}^2 \geq \sum_i m_{ii}^2$ with equality when $M$ is diagonal. Hence we have

$$\text{trace}(M)^2 - \text{trace}(M^2) \leq \text{trace}(O^T M O)^2 - \sum_i (O^T M O)_{ii}$$
and therefore
\[ c(M) = \min_{O^2 O = I, \, R = O^T MO} \left\{ \left( \sum_i r_{ii} \right)^2 - \sum_i r_{ii}^2 \right\}, \]
which can be rewritten as
\[
c(M) = \min_{O^2 O = I, \, R = O^T MO} \sigma_2(\text{diag}(R)),
\]
where \( \text{diag}(R) = (r_{11}, r_{22}, r_{33}) \) is the vector which is the diagonal of the matrix \( R \) and \( \sigma_2 \) is defined by (2). Thus, we have just proved the following Lemma.

**Lemma 3.2.** Let \( M \) be a \( 3 \times 3 \) symmetric matrix and \( V \) be the set of all orthonormal bases of \( \mathbb{R}^3 \):
\[
V = \{ (\nu_1, \nu_2, \nu_3) | \nu_i \in \mathbb{R}^3, \nu_i \perp \nu_j \text{ if } i \neq j, \| \nu_i \|_2 = 1 \}.
\]
Then
\[
c(M) = \min_{(\nu_1, \nu_2, \nu_3) \in V} \sigma_2 \left( \nu_1^T M \nu_1, \nu_2^T M \nu_2, \nu_3^T M \nu_3 \right)
\]
We can now use Lemma 3.2 to characterize the 2-Hessian operator of a \( C^2 \) function by expressing it in terms of second directional derivatives of \( u \) as follows:
\[
S_2[u] = \min_{(\nu_1, \nu_2, \nu_3) \in V} \sigma_2 \left( \frac{\partial^2 u}{\partial \nu_1^2}, \frac{\partial^2 u}{\partial \nu_2^2}, \frac{\partial^2 u}{\partial \nu_3^2} \right).
\]

**3.4.3. Monotone operator.** We now present the monotone discretization of the 2-Hessian operator.

We approximate the second derivatives using centered finite differences which leads to a spatial discretization with parameter \( h \). In addition, we consider a finite number of possible directions \( \nu \) that lie on the grid, thus introducing the directional discretization with parameter \( d \theta \). We denote this set of orthogonal vectors by \( \mathcal{G} \).

We then have
\[
(2H)^M \quad S_2^M[u] = \min_{(\nu_1, \nu_2, \nu_3) \in \mathcal{G}} \tilde{\sigma} (\mathcal{D}_{\nu_1 \nu_1} u, \mathcal{D}_{\nu_2 \nu_2} u, \mathcal{D}_{\nu_3 \nu_3} u)
\]
where \( \mathcal{D}_{\nu \nu} \) is the finite difference operator for the second directional derivative in the direction \( \nu \) which lies on the finite difference grid and are given by
\[
\mathcal{D}_{\nu \nu} u(x_i) = \frac{1}{|\nu|^2 h^2} (u(x_i + h \nu) + u(x_i - h \nu) - 2u(x_i)).
\]
Depending on the direction of the vector \( \nu \), this may involve a wide stencil.

We define \( d \theta \) as
\[
d \theta = \max_{(w_1, w_2, w_3) \in V} \min_{(\nu_1, \nu_2, \nu_3) \in \mathcal{G}} \max \{ \cos(w_1^T \nu_1), \cos(w_2^T \nu_2), \cos(w_3^T \nu_3) \}
\]

**Lemma 3.3.** The finite difference scheme given by \((2H)^M\) is degenerate elliptic.

**Proof.** From the definition, the discrete second directional derivatives \( \mathcal{D}_{\nu \nu} \) are nondecreasing functions of the differences between neighboring values and reference values, \( u_j - u_i \), where \( u_j \) is one of the neighboring values of \( u_i \) in the direction \( \nu \). The scheme \((2H)^M\) is a nondecreasing combination of the operators \( \min \) and \( \tilde{\sigma} \) (the latter proved in Lemma 3.1 to be nondecreasing) applied to the degenerate elliptic terms \( \mathcal{D}_{\nu \nu} \), and so it is also degenerate elliptic. \( \square \)
Lemma 3.4. Let \( x_0 \in \Omega \) be a reference point on the grid and \( \phi \) be a twice continuously differentiable function that is defined in a neighborhood of the grid. Then the scheme \( S^2_M[\phi] \) defined in (2H) approximates (2H) with accuracy

\[
S^2_M[\phi] = S_2[\phi] + O(h^2 + d\theta)
\]

Proof. From a simple Taylor series computation we have

\[
\text{where the vectors } v^j \text{ scheme } S \text{ are orthogonal unit vectors, which may not be in the set of grid vectors } G. \text{ However there is a set of vectors } w \in G \text{ such that } |w_j|v_j + dv_j = w_j \text{ with } |dv_j| = O(d\theta).
\]

Now we consider the discretized problem

\[
S^2_M[\phi] = \min_{(v_1,v_2,v_3) \in G} \sigma_2(D_{v_1,v_1,\phi}, D_{v_2,v_2,\phi}, D_{v_3,v_3,\phi})
\]

\[
\leq \sigma_2(D_{w_1,w_1,\phi}, D_{w_2,w_2,\phi}, D_{w_3,w_3,\phi})
\]

\[
= \sigma_2\left(\frac{\partial^2\phi}{\partial w_1^2}, \frac{\partial^2\phi}{\partial w_2^2}, \frac{\partial^2\phi}{\partial w_3^2}\right) + O(h^2)
\]

\[
= \sigma_2\left(\frac{\partial^2\phi}{\partial v_1^2}, \frac{\partial^2\phi}{\partial v_2^2}, \frac{\partial^2\phi}{\partial v_3^2}\right) + O(h^2 + d\theta)
\]

\[
= \min_{(v_1,v_2,v_3) \in V} \sigma_2\left(\frac{\partial^2u}{\partial v_1^2}, \frac{\partial^2u}{\partial v_2^2}, \frac{\partial^2u}{\partial v_3^2}\right) + O(h^2 + d\theta),
\]

where we used the fact that

\[
\frac{\partial^2\phi}{\partial w_j^2} = \frac{\partial^2\phi}{\partial v_j^2} + O(d\theta).
\]

In addition, since the set of grid vectors \( G \) is a subset of the set of all orthogonal vectors \( V \), we find that

\[
\min_{(v_1,v_2,v_3) \in G} \sigma_2(D_{v_1,v_1,\phi}, D_{v_2,v_2,\phi}, D_{v_3,v_3,\phi}) \leq \min_{(v_1,v_2,v_3) \in V} \sigma_2(D_{v_1,v_1,\phi}, D_{v_2,v_2,\phi}, D_{v_3,v_3,\phi})
\]

\[
= \min_{(v_1,v_2,v_3) \in V} \sigma_2\left(\frac{\partial^2u}{\partial v_1^2}, \frac{\partial^2u}{\partial v_2^2}, \frac{\partial^2u}{\partial v_3^2}\right) + O(h^2).
\]

Combining the two inequalities deduced above, we conclude the proof. \( \square \)

Theorem 3.5. Let the PDE (2H) have a unique viscosity solution. The solutions of the scheme (2H) converge to the viscosity solution of (2H) as \( h, d\theta \to 0 \).

Proof. The convergence follows from verifying consistency and degenerate ellipticity, as in [FO11a], by the Barles and Souganidis theory [BS91]. This is accomplished in Lemmas 3.3 and 3.4. \( \square \)

3.5. Solvers for the monotone finite difference scheme. In this section we present two solvers for the monotone finite difference scheme.
3.5.1. Parabolic solver. Using the monotone discretization $S^M_2[u]$, the simplest solver for the 2-Hessian equation is to use the fixed point method

$$u^{n+1} = u^n - \alpha(S^M_2[u] - f)$$

which corresponds to the discrete version of the parabolic equation $u_t = -S_2[u] + f$ using a forward Euler step. The fixed point iteration will be a contraction in the maximum norm provided that we choose $\alpha$ small enough, as dictated by the nonlinear CFL condition [Obe06], which in this case means $\alpha = O(h^4)$. This will make the solver very slow. Moreover, since we extended $\sigma_2$ to be degenerate elliptic in $\mathbb{R}^3$, this is a global solver, meaning that it will converge regardless of the initial guess we choose.

3.5.2. Newton solver. As with the standard finite difference scheme, one can also use a (damped) Newton solver. In this case the Jacobian for the monotone discretization is obtained by using Danskin’s Theorem [Ber03] and the product rule:

$$\nabla_u S^M_2[u] = \begin{cases} -2(D_{\nu_1^*\nu_1^*}u)D_{\nu_1^*\nu_1^*} & \text{if } D_{\nu_1^*\nu_1^*}u + D_{\nu_2^*\nu_2^*}u < 0 \\ \sum_{\nu_1, \nu_2 \in \{\nu_1^*, \nu_2^*, \nu_3^*\}} (D_{\nu_1,\nu_1}u)D_{\nu_2,\nu_2} & \text{otherwise} \end{cases}$$

where $\nu_1^*$ are the directions active in the minimum in $(2H)^M$ with $D_{\nu_1^*\nu_1^*}u \leq D_{\nu_2^*\nu_2^*}u \leq D_{\nu_3^*\nu_3^*}u$. Unlike the previous solver, this is a local solver, meaning that we need a good initial guess in order to have convergence.

4. Computational results

In this section we summarize the results of a number of different examples using the solvers described in the previous section. These computations are performed on a $N \times N \times N$ grid on the cube $[0,1]^3$. Unless otherwise mentioned, all solvers were initialized with an initial guess provided by the explicit method (J), which we iterate until $|S^A_2[u^n] - f| < 10^{-1}$. The initial guess for the explicit method (J) was the exact solution with some noise from a uniform distribution. As stopping criteria for the Newton solver we used $|S^H_2[u^n] - f| < 10^{-10}$ where $H \in \{A, M\}$. Solutions were also computed using (J) and (9) and very similar results to the ones obtained with the Newton solver were obtained. For that reason, we choose not to display them here.

Remark 4.1. Notice that at points near the boundary of the domain some values required by the wide stencil will not be available. For these reason and to simplify things we set the exact solution at those points. However it is important to point out that we can use interpolation at the boundary to construct a (lower) accuracy stencil, thus avoiding the need to initialize with the exact solution.

Example 4.1 (Quadratic function). We consider the case where $u$ is a non-convex (but 2-admissible function) given by

$$u(x) = x_1^2 - \frac{1}{2}x_2^2 + 2x_3^2, \quad f(x) = 2.$$ 

with $x = (x_1, x_2, x_3)$. In Table 1, we compare the results obtained using standard finite differences and the monotone schemes with different stencil sizes. For this example, we used the Newton solver for all schemes.
All methods provide machine accuracy which is expected since the standard finite
differences are exact for quadratic functions and the monotone schemes computed
the desired directional derivative.

| N   | Standard          | Monotone (27-point) | Monotone (99-point) | Monotone (291-point) |
|-----|-------------------|---------------------|---------------------|---------------------|
| 15  | $4.441 \times 10^{-16}$ | $4.441 \times 10^{-16}$ | $4.441 \times 10^{-16}$ | $4.441 \times 10^{-16}$ |
| 20  | $4.441 \times 10^{-16}$ | -0.00 | $8.882 \times 10^{-16}$ | -2.27 | $8.882 \times 10^{-16}$ | -2.27 | $6.661 \times 10^{-16}$ | -1.33 |
| 25  | $4.441 \times 10^{-16}$ | -0.00 | $8.882 \times 10^{-16}$ | -0.00 | $8.882 \times 10^{-16}$ | -0.00 | $8.882 \times 10^{-16}$ | -1.23 |
| 30  | $4.441 \times 10^{-16}$ | -0.00 | $1.332 \times 10^{-15}$ | -2.14 | $8.882 \times 10^{-16}$ | -0.00 | $8.882 \times 10^{-16}$ | -0.00 |
| 35  | $4.441 \times 10^{-16}$ | -0.00 | $1.332 \times 10^{-15}$ | -0.00 | $8.882 \times 10^{-16}$ | -0.00 | $1.110 \times 10^{-15}$ | -1.40 |

**Table 1.** Accuracy in the $l\infty$ norm and order of convergence of
the schemes for the first example using the Newton solver.

**Example 4.2** (smooth convex radial function). We consider now the case where
$u$ is given by

$$(15) \quad u(x) = \exp \left( \frac{\|x - x_0\|^2}{2} \right), \quad f(x) = (3 + 2\|x - x_0\|^2) \exp(\|x - x_0\|^2).$$

The maximum errors are given in Table 2. As in the previous example we used
the Newton solver for all schemes.

The standard finite differences provided second order convergence, which was
expected since the solution is smooth. The monotone schemes provided only first
order convergence (or close to it).

| N   | Standard          | Monotone (27-point) | Monotone (99-point) | Monotone (291-point) |
|-----|-------------------|---------------------|---------------------|---------------------|
| 15  | $2.393 \times 10^{-4}$ | -3.472 \times 10^{-4} | -2.167 \times 10^{-4} | -1.302 \times 10^{-4} |
| 20  | $1.298 \times 10^{-4}$ | 2.00 | $2.225 \times 10^{-4}$ | 1.46 | $1.518 \times 10^{-4}$ | 1.17 | $1.034 \times 10^{-4}$ | 0.75 |
| 25  | $8.197 \times 10^{-5}$ | 1.97 | $1.650 \times 10^{-4}$ | 1.28 | $1.165 \times 10^{-4}$ | 1.13 | $8.552 \times 10^{-5}$ | 0.81 |
| 30  | $5.607 \times 10^{-5}$ | 2.01 | $1.346 \times 10^{-4}$ | 1.08 | $9.357 \times 10^{-5}$ | 1.16 | $7.216 \times 10^{-5}$ | 0.90 |
| 35  | $4.091 \times 10^{-5}$ | 1.98 | $1.259 \times 10^{-4}$ | 0.42 | $7.809 \times 10^{-5}$ | 1.14 | $6.247 \times 10^{-5}$ | 0.91 |

**Table 2.** Accuracy in the $l\infty$ norm and order of convergence of
the schemes for the second example using the Newton solver.

**Example 4.3** (smooth non-convex radial function). We consider now the case
where $u$ is given by

$$(16) \quad u(x) = \exp \left( 2x_1^2 - x_2^2 + 4x_3^2 \right), \quad f(x) = 8 \left( 1 + 12x_1^2 + 6x_2^2 + 16x_3^2 \right) \exp \left( 4x_1^2 - 2x_2^2 + 8x_3^2 \right)$$

The maximum errors are given in Table 3. Once again the solutions were com-
puted with a Newton solver for all schemes.

The standard finite differences demonstrate again second order convergence. For
the monotone schemes the error only decreases when considering wider stencils.
This tells us that the directional resolution error dominates the spatial resolution
error, explaining why the accuracy tapers off with the grid size.
convergence (before it tapers off in the case of the 99-point stencil). Moreover, the monotone schemes with wider stencils also exhibit second order convergence and only with wider stencils we see a decrease in error with the grid size. As in the previous example, standard finite differences provide second order accuracy in the $l^\infty$ norm and order of convergence of the schemes for the third example using the Newton solver.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
N   & Standard & Monotone (27-point) & Monotone (99-point) & Monotone (291-point) \\
\hline
15  & 3.028 $\times 10^{-4}$ & 3.287 $\times 10^{-2}$ & 1.110 $\times 10^{-2}$ & 5.044 $\times 10^{-3}$ \\
20  & 1.669 $\times 10^{-5}$ & 1.95  & 1.211 $\times 10^{-2}$ & 6.171 $\times 10^{-3}$ & -0.29 \\
25  & 1.052 $\times 10^{-6}$ & 1.98  & 1.260 $\times 10^{-2}$ & 6.920 $\times 10^{-3}$ & -0.22 \\
30  & 7.218 $\times 10^{-7}$ & 1.99  & 1.306 $\times 10^{-2}$ & 6.396 $\times 10^{-3}$ & -0.41 \\
35  & 5.262 $\times 10^{-8}$ & 1.99  & 1.339 $\times 10^{-2}$ & 6.703 $\times 10^{-3}$ & -0.29 \\
\hline
\end{tabular}
\caption{Accuracy in the $l^\infty$ norm and order of convergence of the schemes for the third example using the Newton solver.}
\end{table}

Example 4.4 (smooth non-convex radial function). We consider another example of smooth radial function which is non convex but 2-admissible:

\begin{equation}
(17) \quad u(x) = \log(2 + ||x||), \quad f(x) = -\frac{4(-6 + ||x||^2)}{(2 + ||x||^2)^3}
\end{equation}

This example was considered in [Awa14] as well.

The maximum errors are given in Table 4. Once again the solutions were computed with a Newton solver, regardless of the scheme.

As in the previous example, standard finite differences provide second order convergence and only with wider stencils we see a decrease in error with the grid size. Moreover, the monotone schemes with wider stencils also exhibit second order convergence (before it tapers off in the case of the 99-point stencil).

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
N   & Standard & Monotone (27-point) & Monotone (99-point) & Monotone (291-point) \\
\hline
15  & 4.723 $\times 10^{-5}$ & 1.664 $\times 10^{-3}$ & 3.882 $\times 10^{-4}$ & 4.909 $\times 10^{-4}$ \\
20  & 2.564 $\times 10^{-5}$ & 2.00  & 1.787 $\times 10^{-4}$ & 2.500 $\times 10^{-4}$ & 2.21 \\
25  & 1.615 $\times 10^{-5}$ & 1.98  & 1.007 $\times 10^{-4}$ & 1.462 $\times 10^{-4}$ & 2.30 \\
30  & 1.111 $\times 10^{-5}$ & 1.98  & 8.617 $\times 10^{-5}$ & 9.063 $\times 10^{-5}$ & 2.53 \\
35  & 8.052 $\times 10^{-6}$ & 2.02  & 6.170 $\times 10^{-5}$ & 6.506 $\times 10^{-5}$ & 2.08 \\
\hline
\end{tabular}
\caption{Accuracy in the $l^\infty$ norm and order of convergence of the schemes for the fourth example using the Newton solver.}
\end{table}

Example 4.5 (non smooth convex function). We consider now the case where $u$ is given by

\begin{equation}
(18) \quad u(x) = \frac{1}{2} \left( (||x - x_0|| - 0.2)^2 \right), \quad f(x) = \left( 3 \frac{1}{25||x - x_0||^2} - \frac{4}{5||x - x_0||} \right) 1_{||x - x_0|| > 0.2}(x).
\end{equation}

The maximum errors are given in Table 5. Due to its degenerate ellipticity, the monotone schemes required the used of the damped Newton solver.

Despite the lack of smoothness of the solution, the Newton solver with standard finite differences still converged. As for the monotone scheme, there was a significant increase in the number of iterations required: the wider the stencil, the more iterations required (around 10 times more iterations when compared to the Newton solver for the naive finite differences in the worst cases).
Notice that the surface plots of the level sets of the solution with the standard finite differences and monotone scheme with \(c \in \{ -0.01, -0.03, -0.07 \} \). Note that the zero level set \((c = 0)\) is the cube \([0, 1]^3\) where the zero Dirichlet boundary conditions are prescribed. The surface plots become spheres as \(c\) decreases, with \(c = -0.01\) being the only where there’s a tangible difference between the two schemes, most likely due to the expected higher accuracy from the standard finite differences. In Figure 3, we plot the curve \(u(t, t, t)\) with \(t \in [0, 1]\) and see that there’s a small difference between the solutions from the standard finite differences and the monotone scheme.

**Example 4.8.** We consider as well the example with \(f \equiv 1\) and \(g \equiv 0\) Dirichlet boundary conditions but with a different domain \(\Omega = \Omega_1 \cup \Omega_2\) where

\[
\Omega_1 = \{(x, y, z) \in \mathbb{R}^3 : (x - 0.35)^2 + (y - 0.35)^2 + (z - 0.5)^2 < 0.3^2 \},
\]

\[
\Omega_2 = \{(x, y, z) \in \mathbb{R}^3 : (x - 0.65)^2 + (y - 0.65)^2 + (z - 0.5)^2 < 0.3^2 \}.
\]

**Example 4.7.** We consider as well the example with \(f \equiv 1\) and \(g \equiv 0\) Dirichlet boundary conditions. No exact solution is known. In Figure 2, we illustrate some of the surface plots of the level sets \(u = c\) of the solution with the standard finite differences and monotone scheme with \(c \in \{ -0.01, -0.03, -0.07 \} \). Note that the zero level set \((c = 0)\) is the cube \([0, 1]^3\) where the zero Dirichlet boundary conditions are prescribed. The surface plots become spheres as \(c\) decreases, with \(c = -0.01\) being the only where there’s a tangible difference between the two schemes, most likely due to the expected higher accuracy from the standard finite differences. In Figure 3, we plot the curve \(u(t, t, t)\) with \(t \in [0, 1]\) and see that there’s a small difference between the solutions from the standard finite differences and the monotone scheme.

**Example 4.6 (example with blow-up).** We considered as well the case

\[
(19) \quad u(x) = -\sqrt{3 - \|x\|^2}, \quad f(x) = -\frac{9 + \|x\|^2}{(-3 + \|x\|^2)^2}
\]

Notice that \(f\) is unbounded at the boundary point \((1, 1, 1)\) and \(u\) will be singular at that point as well. Despite that the Newton solver still converged, but with a smaller rate of convergence (approximately 0.3). It is important to observe that in the case of the Monge-Ampère, the Newton solver failed to converge in the analogue example. This may be because the Monge-Ampère equation is more strongly nonlinear than the 2-Hessian equation. The better accuracy of the wider monotone schemes is explained by the fact that the exact solution is prescribed at more grid points near the boundary of the (computational) domain, in particular, where \(u\) is singular and \(f\) is unbounded.

**Example 4.8.** We consider as well the example with \(f \equiv 1\) and \(g \equiv 0\) Dirichlet boundary conditions but with a different domain \(\Omega = \Omega_1 \cup \Omega_2\) where

\[
\Omega_1 = \{(x, y, z) \in \mathbb{R}^3 : (x - 0.35)^2 + (y - 0.35)^2 + (z - 0.5)^2 < 0.3^2 \},
\]

\[
\Omega_2 = \{(x, y, z) \in \mathbb{R}^3 : (x - 0.65)^2 + (y - 0.65)^2 + (z - 0.5)^2 < 0.3^2 \}.
\]
Figure 2. Surface plots of the level sets of the solution to Example 4.7 on a $30 \times 30 \times 30$ grid with the naive finite differences (left) and the 27-point monotone scheme (right).

Figure 3. Plot of the curves $t \mapsto u(t,t,t)$ of the solution of Example 4.7 on a $30 \times 30 \times 30$ grid.

$$\Omega_2 = \{(x,y,z) \in \mathbb{R}^3 : (x-0.65)^2 + (y-0.65)^2 + (z-0.5)^2 < 0.3^2\}.$$ No exact solution is known. In Figure 4, we illustrate some of the surface plots of the level sets $u = c$ of the solution with the standard finite differences and
monotone scheme with $c \in \{0, -0.01, -0.02, -0.03, -0.035, -0.039\}$. In this case the zero level set is not convex, with the level sets $u = c$ becoming more convex with smaller values of $c$. In this case the difference between the standard finite differences and monotone scheme is even smaller than in Example 4.7, as we can see in Figure 5, where we plot the curve $u(t, t, t)$ with $t \in [0, 1]$.

Figure 4. Surface plots of the level sets of the solution to Example 4.8 on a $30 \times 30 \times 30$ grid with the naive finite differences (left) and the 27-point monotone scheme (right).

5. Conclusions

The 2-Hessian equation is a fully nonlinear Partial Differential Equation which is elliptic provided the solutions are restricted to a convex cone, which we called plane-subharmonic. It is natural to compare this equation with the Monge-Ampère PDE, which is elliptic on the cone of convex functions, and which has been studied numerically in previous work by two of the authors. The elliptic 2-Hessian equation is more challenging because the constraints for ellipticity are less restrictive.

We gave two different discretizations for the 2-Hessian equation in the three-dimensional case: a naive one obtained by simply using standard finite differences to discretize the Hessian and a monotone discretization that takes advantage of a characterization of the operator using a matrix inequality (12). The monotone discretization is provably convergent but less accurate, because the monotone discretization required the use of a wide stencil. Computational results were provided using exact solutions of varying regularity and shape, from smooth to non differentiable, and from convex to nonconvex.
The naive discretization failed, unless we introduced a mechanism for selecting the correct 2-admissible (plane-subharmonic) solution. Once this mechanism was introduced, experimental results on a variety of solutions demonstrated that the method appeared to converge. The standard finite difference discretization failed using a standard parabolic solver. Two alternative solvers were presented, which enforced the “plane-subharmonic” restriction and proved to work numerically for all the examples considered. Additionally, a Newton solver was also implemented, converging for all examples considered, even for degenerate ones or with singular right-hand sides, whenever initialized with a good initial guess. For smooth examples, we obtained second order convergence.

The monotone discretization, less accurate due the introduction of a directional resolution to make it monotone, is stable and provably convergent. Numerical examples show that the directional resolution easily dominates the spacial resolution, a natural consequence of the three dimensional setting.

Moreover, one could have implemented filtered schemes, previously introduced in [FO13], which would provide schemes that are provably convergent but with greater accuracy than the monotone schemes. However, we did not implement them here, since our main goal was to compare the two different discretizations presented and, moreover, the accurate scheme by itself proved to be convergent for all the examples considered, even degenerate ones.

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