Bifurcation Analysis of a Four-Dimensional System of Two Symmetric Coupled Nonlinear Oscillators with Clearance

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Abstract. The research gradually highlights vibration and dynamical analysis of symmetric coupled nonlinear oscillators model with clearance. The aim of this paper is the bifurcation analysis of the symmetric coupled nonlinear oscillators modeled by a four-dimensional nonsmooth system. The approximate solution of this system is obtained with aid of averaging method and Krylov-Bogoliubov (KB) transformation presented by new notations of matrices. The bifurcation function is derived to investigate its dynamic behaviour by singularity theory. The results obtained provide guidance for the nonlinear vibration of symmetric coupled nonlinear oscillators model with clearance.

1. Introduction
The application of symmetric structures, especially in bladed disks in turbomachines and large space deployable antenna [1], has galvanized efforts in aerospace engineering, nonlinear dynamics and other multiple fields. The research on vibration of symmetric structures is one of the key challenging in aerospace field, such as the vibration of mistuned bladed disk. Therefore, the dynamic analysis of symmetric structures to explore its properties of vibration is essential for improving its security and durability. Concerning symmetric structures, Papangelo et al [2] studied multistability and localization of cyclic symmetric structures. Fontanela et al. [3] investigated the nonlinear vibration localisation of two coupled beams modelled by a two degree-of-freedom piecewise smooth system using theoretical and experimental methods.

As a qualitative analysis method, singularity theory has an essential role in determining the stability of dynamic systems, which have been developed by several researchers [4-6] and references therein. Chen et al. [7-8] studied the bifurcation characteristics of practical engineering systems with aid of singularity theory. Guo and Zhang [9] studied bifurcation characters of composite laminated piezoelectric rectangular plate structure based on singularity theory. Furter [10] studied bifurcation problems using singularity theory introducing preserving path formulation. Han et al. [11] studied nonlinear mechanism of microbeam, especially static analysis via singularity theory. The bifurcation of symmetric system modelled by a class of two degree-of-freedom piecewise smooth system remains to be analysed.

Motivated by this, we study the bifurcation mechanism of the symmetric coupled beam with clearance modelled by piecewise smooth system. Our work is organized as follows. In section 2, the symmetric coupled nonlinear oscillators system is presented. In section 3, its approximate solution is obtained using averaging method. In section 4, bifurcation of two symmetric coupled nonlinear oscillators system is analysed.
2. Two Symmetric Coupled Nonlinear Oscillators System with Clearance

In this section, we begin with the two symmetric coupled nonlinear oscillators system with clearance, which is composed of two masses \( m \). The displacements of the two masses are \( x_1 \) and \( x_2 \) respectively. The masses are coupled by spring \( k \) and \( c \). The viscous damping between the mass and the ground is \( c \). The excitation of this system is \( \gamma \omega \cos \omega t \). The system is shown as follows [3].

\[
\begin{align*}
\ddot{x}_i + c \dot{x}_i + (k_i + k_c)x_i - k_c x_{i-1} + f_{nl}(x_i) = cY_0 \omega \sin \omega t + k_c Y_0 \cos \omega t \\
(1)
\end{align*}
\]

where \( i = 1, 2 \), the dimension of clearance is \( s \) and the nonlinear force \( f_{nl} \) is expressed as follows,

\[
f_{nl}(x_i) = \begin{cases} 
0 & |x_i| < s \\
k_c (x_i - s) & x_i > s \\
k_c (x_i + s) & x_i < -s 
\end{cases}
\]

Definition[12]: \( \tilde{c}_{ij}^{m,n}(M) \) is an \( m \times n \) block matrix, where the \( (i,j) \)-th block matrix is \( M \) and the others are zero matrices.

By introducing coordinate transformation \( y_1 = x_1, y_2 = \dot{x}_1, y_3 = x_2, y_4 = \dot{x}_2 \), the system (1) is transformed into the following form.

\[
y = By + \varepsilon F(y, t)
\]

where \( 0 < \varepsilon \ll 1 \), \( p_i = \frac{c_i}{m}, \quad p_2 = \frac{k_i + k_c}{m}, \quad p_3 = \frac{k_c}{m}, \quad p_4 = \frac{c \omega Y_0}{m}, \quad p_5 = k_c \omega Y_0 \).

\( y = (y_1, y_2, y_3, y_4)^T \), \( B = \tilde{c}_{11}^{4,4}(I) + \tilde{c}_{11}^{4,4}(p_1) + \tilde{c}_{11}^{4,4}(p_2) + \tilde{c}_{11}^{4,4}(p_3) + \tilde{c}_{11}^{4,4}(-p_2) \),

\( F(y, t) = \tilde{c}_{11}^{4,4}(F_1) + \tilde{c}_{11}^{4,4}(F_2), \ F_1 = -p_1 y_2 - p_2 y_1 - p_4 f_{nl}(y_4) - p_3 \sin \omega t + p_5 \cos \omega t \),

\( F_2 = -p_3 y_4 - p_4 f_{nl}(y_1) - p_5 \sin \omega t + p_6 \cos \omega t \).

3. The Approximate Solution Using Averaging Method

When \( \varepsilon = 0 \), the fundamental solution of Eq.(2) is as follows.

\[
\phi_{1k} = \cos \lambda_t, \quad \phi_{2k} = -\lambda_t \sin \lambda_t, \quad \phi_{3k} = -\frac{\lambda_t^2}{p_3} \cos \lambda_t, \quad \phi_{4k} = -\frac{\lambda_t^2}{p_3} \sin \lambda_t, \quad k = 1, 2 
\]

(3a)

\[
\phi_{1k}' = \sin \lambda_t, \quad \phi_{2k}' = \lambda_t \cos \lambda_t, \quad \phi_{3k}' = \frac{\lambda_t^2}{p_3} \sin \lambda_t, \quad \phi_{4k}' = \frac{\lambda_t^2}{p_3} \cos \lambda_t, \quad k = 1, 2
\]

(3b)

The fundamental solution of conjugate equation of Eq. (2) is presented as follows.

\[
\psi_{1k} = \cos \lambda_t, \quad \psi_{2k} = -\frac{1}{\lambda_t} \sin \lambda_t, \quad \psi_{3k} = -\frac{\lambda_t^2}{p_3} \cos \lambda_t, \quad \psi_{4k} = -\frac{\lambda_t^2}{p_3} \sin \lambda_t, \quad k = 1, 2
\]

(4a)

\[
\psi_{1k}' = \sin \lambda_t, \quad \psi_{2k}' = \frac{1}{\lambda_t} \cos \lambda_t, \quad \psi_{3k}' = \frac{\lambda_t^2}{p_3} \sin \lambda_t, \quad \psi_{4k}' = \frac{\lambda_t^2}{p_3} \cos \lambda_t, \quad k = 1, 2
\]

(4b)

Introduce the transformation \( y_s = (\tilde{c}_{11}^{s,2}(A_1) + \tilde{c}_{11}^{s,2}(A_2))(\tilde{c}_{11}^{s,1}(\varphi_1, \theta_1) + \tilde{c}_{11}^{s,1}(\varphi_2, \theta_2)), \ s = 1, 2, 3, 4 \), where the new variables \( A_1, A_2 \) are amplitude and \( \theta_1, \theta_2 \) are phase that depend on time \( t \). The standard equations are obtained.

\[
\tilde{c}_{11}^{s,1}(\frac{dA_1}{dt}) + \tilde{c}_{11}^{s,1}(\frac{d\theta_1}{dt}) = \tilde{c}_{11}^{s,1}(\lambda_1) + \varepsilon (\tilde{c}_{11}^{s,1}(\Phi_1(A, \theta)) - \tilde{c}_{11}^{s,1}(\Phi_1^*(A, \theta)))
\]

(5)
where $\Delta_i' = 1 + \frac{\lambda_i^2}{p_3}$, $\Phi_i(A, \theta) = \frac{(F_{2}\psi_{2i} + F_{3}\psi_{3i})}{\Delta_i'}$, $\Phi_i'(A, \theta) = \frac{(F_{2}\psi_{2i} + F_{3}\psi_{3i})}{\Delta_i' A_i}$.

Introduce KB transformation
\[
\begin{aligned}
\tilde{\varphi}_{x,i}^1(z_i) + \tilde{\varphi}_{x,i}^1(v_i) = \tilde{\varphi}_{x,i}^1(\lambda_i) + \tilde{\varphi}_{x,i}^1(\lambda_i t + v_i) + \epsilon(\tilde{\varphi}_{x,i}^1(U_x(t, z, \nu)) + \tilde{\varphi}_{x,i}^1(V_x(t, z, \nu)))
\end{aligned}
\]
(6)

And the derivative of the new variable $z_i, v_i$ is required
\[
\begin{aligned}
\tilde{\varphi}_{x,i}^1(z_i) + \tilde{\varphi}_{x,i}^1(v_i) = \tilde{\varphi}_{x,i}^1(\lambda_i - \omega) + \epsilon(\tilde{\varphi}_{x,i}^1(Y_x(z, \nu)) + \tilde{\varphi}_{x,i}^1(Z_x(z, \nu)))
\end{aligned}
\]
(7)

Substitute Eq. (6) into Eq. (5), considering Eq. (7). That is
\[
\begin{aligned}
\tilde{\varphi}_{x,i}^1(z_i) + \frac{\partial U_x}{\partial t} + \tilde{\varphi}_{x,i}^1(z_i) + \frac{\partial V_x}{\partial t} = \tilde{\varphi}_{x,i}^1(\Phi_i) + \tilde{\varphi}_{x,i}^1(-\Phi_i)
\end{aligned}
\]
(8)

For convenience, $f_{ni}(x_i)$ will be denoted as $f_{ni}(A, \theta)$ and $f_{ni}(x_2)$ will be denoted as $f_{ni}^2(A, \theta)$.

It's computed without difficulty that the Fourier series expansion of $f_{ni}(x_i)$ and $f_{ni}(x_2)$ as the following form.
\[
\begin{aligned}
f_{ni}(A_2, \theta_2) &= \frac{2k_n}{\pi}(A_2 \cos \arccos \frac{1-Z}{\sqrt{1-Z^2}} + 2k_n A_2 \sum_{p=0}^{2}(\frac{Z^p}{2} \sin((2p+q)\theta_2)),
\end{aligned}
\]
\[
\begin{aligned}
f_{ni}^2(A_2, \theta_2) &= \frac{-2k_n}{p_3}(A_2 \cos \arccos \frac{1-N}{\sqrt{1-N^2}} + 2k_n A_2 \sum_{p=0}^{2}(\frac{N^p}{2} \sin((2p+q)\theta_2)),
\end{aligned}
\]
where $Z = \frac{A_2}{s}, N = \frac{A_2^2}{sp}$.

The approximate equation of the four-dimensional symmetric coupled nonlinear oscillators model with clearance under the condition of $k = 2$ is obtained as
\[
\begin{aligned}
\dot{\varphi}_{x,i}^1 \dot{\varphi}_{x,i}^2 + \ddot{\varphi}_{x,i}^1(\dot{\varphi}_{x,i}^2) = \ddot{\varphi}_{x,i}^1(\lambda_i - \omega) + \frac{\epsilon}{2(p_i^2 + \lambda_i^2)}(\varphi_{x,i}^1(A_2, \nu_2) + \varphi_{x,i}^1(H_2(A_2, \nu_2)))
\end{aligned}
\]
(9)

where $H_1(A_2, \nu_2) = p_1(A_2 - \lambda_i^2) - p_1(\nu_2 \cos \nu_2 + \nu_2 \sin \nu_2)\nu_2$, $H_2(A_2, \nu_2) = -p_1(\nu_2 \cos \nu_2 + \nu_2 \sin \nu_2)\nu_2 - p_1(p_3 - \lambda_i^2)\nu_2 \cos \nu_2 - p_1(p_3 - \lambda_i^2)\nu_2 \sin \nu_2 - p_1(\nu_2 \cos \nu_2 + \nu_2 \sin \nu_2)\nu_2$.

Therefore, the first approximate solution of equation (2) is obtained.

4. **Bifurcation Analysis via Singularity Theory**

The approximation equation is expressed as
\[
\begin{aligned}
\dot{\varphi}_{x,i}^1 + \ddot{\varphi}_{x,i}^1(\dot{\varphi}_{x,i}^2) = \dot{\varphi}_{x,i}^1(\dot{\varphi}_{x,i}^1(A_2) + \ddot{\varphi}_{x,i}^1(\lambda_i - \omega) + \frac{B}{2(p_i^2 + \lambda_i^2)}(\varphi_{x,i}^1(A_2, \cos(\nu_2 - \beta)) + \varphi_{x,i}^1(\sin(\nu_2 - \beta)))
\end{aligned}
\]
(10)

where
\[ \delta_1(A_z) = -\frac{1}{2(p_z^2 + \lambda_z^2)}(-p_sp_z^2 + p_s^2 \lambda_z^2), \tan \beta = -\frac{p_sp_0 - p_s^2 \lambda_z^2}{p_s^2 \lambda_z^2 - p_0 p_z}, \]

\[ p_s(A_z) = \lambda_z + \frac{1}{2(p_z^2 + \lambda_z^2)A_z}[-\frac{2k_d p_3 p_s^4}{\pi}(Z \arccos \frac{1}{Z} - \sqrt{1 - \frac{1}{Z^2}}), \]

\[ + \frac{2k_d p_s^4}{\pi p_s} (N \arccos \frac{1}{N} + \sqrt{1 - \frac{1}{N^2}})]] \]

\[ B = p_s \sqrt{(p_z \lambda_z^2 - p_0 p_z)^2 + (p_0 p_0 - p_s \lambda_z^2)^2} \]

The bifurcation equation under the condition of primary resonance \( \lambda_z - \omega = \epsilon \sigma \) is obtained, where \( \sigma \) is the tuning parameter. The bifurcation equation is as follows.

\[ G = [2(p_z^2 + \lambda_z^2)\lambda_z \delta_1(A_z)A_z]^2 + [2(p_s(A_z) - \omega)(p_z^2 + \lambda_z^2)A_z \lambda_z]^2 - B^2 \]

(11)

The bifurcation equation is expanded by Taylor series, and we can obtain that

\[ A^4 + \alpha A^3 s + \alpha A^2 s^2 + \alpha A^1 + \alpha A^0 s^3 + \alpha A^0 s^2 - \alpha^0 = 0 \]

(12)

where \( A = A_z, \alpha_1 = \frac{d_2}{d_1}, \alpha_2 = \frac{d_3}{d_1}, \alpha_3 = \frac{d_4}{d_1}, \alpha_4 = \frac{d_5}{d_1}, \alpha_5 = \frac{d_6}{d_1}, \alpha_0 = \frac{B}{d_1}, \)

\[ d_i = 4k_d^2(h_1 g_1 + b_2 g_2)^2, d_2 = 8k_d^2(a_1 g_1 + a_2 g_2)(h_1 g_1 + b_2 g_2), \]

\[ d_3 = 4k_d^2(a_1 g_1 + a_2 g_2), d_4 = 8k_d^2 \sigma(h_1 g_1 + b_2 g_2), \]

\[ d_4 = 8k_d^2 \sigma(a_1 g_1 + a_2 g_2), d_6 = 4k_d^2 \sigma^2 + k_d^2, \]

\[ k_1 = \lambda z p_s(p_s^2 - \lambda_z^2), k_2 = (p_z^2 + \lambda_z^2)\lambda z, g_1 = -\frac{k_d p_3 p_s^4}{(p_z^2 + \lambda_z^2)\lambda z \pi}, g_2 = \frac{k_d p_3}{(p_z^2 + \lambda_z^2)\lambda z p_s \pi}, \]

\[ a_1 = -\frac{2(Z_0^2 - 1)}{Z_0}, b_1 = \frac{1}{(\pi - 2 \arcsin(\frac{1}{Z_0}) + \sqrt{Z_0^2 - 1})}, \]

\[ a_2 = -\frac{2}{N_0 \sqrt{N_0^2 - 1}}, b_2 = \frac{1}{(\pi - 2 \arcsin(\frac{1}{N_0}) + \frac{2(N_0^2 + 1)}{N_0^2 \sqrt{N_0^2 - 1}})}. \]

**Lemma:** The transition set is \( \Sigma = B \cup H \cup D [7] \).

(1) Bifurcation point set:

\[ B = \{ \alpha \in \mathbb{R}^4 \mid \exists (A, s) \in \mathbb{R} \times \mathbb{R}, s.t. G(A, s, \alpha) = G_A(A, s, \alpha) = G_s(A, s, \alpha) = 0 \} \]

(2) Hysteresis set:

\[ H = \{ \alpha \in \mathbb{R}^4 \mid \exists (A, s) \in \mathbb{R} \times \mathbb{R}, s.t. G(A, s, \alpha) = G_A(A, s, \alpha) = G_s(A, s, \alpha) = 0 \} \]

(3) Double limit point set:

\[ D = \{ \alpha \in \mathbb{R}^4 \mid \exists (A, A_1, s) \in \mathbb{R} \times \mathbb{R}, A_1 \neq A_2, s.t. G(A_j, s, \alpha) = G_A(A_j, s, \alpha) = 0, j = 1, 2 \} \]

We obtain transition set when \( \alpha_1 = 0, \alpha_2 = -1, \alpha_3 = 0 \).

(1) Bifurcation point set: \((4\alpha_2^2 + 4\alpha_3)(64\alpha_4^2 + 256\alpha_5^2 + 1) + 2048\alpha_6^2 = 0 \)

(2) Hysteresis set: \( \alpha_6 = 0 \).
(3) Double limit point set: empty set.

![Figure 1](image1.png)

**Figure 1.** Transition set of system under the condition $\alpha_i = 0$, $\alpha_2 = -1$, $\alpha_3 = 0$

The transition set of Eq. (12) is shown as Fig.1. In order to present the regions more comprehensively, Fig.1 is the sketch of transition set from different perspectives the transition set separate three-parameter space of $\alpha_4$, $\alpha_5$ and $\alpha_6$, which divided into six regions. The bifurcation of two symmetric coupled nonlinear oscillators system with parameters alternated from regions I to VI are depicted in Fig.2. The bifurcation diagrams are as follows.

![Figure 2](image2.png)

**Figure 2.** Bifurcation diagram of system.
Fig. 2 indicates the bifurcation characters of regions I to VI. We selected the dimension of clearance $s$ as the bifurcation parameter. The results show that there are abundant dynamical behaviors in symmetric coupled nonlinear oscillators system with clearance. The bifurcation diagram and regions of transition set could provide useful tool to make option to parameters for design and analysis of mechanics and engineering.

5. Conclusions
In this paper, the approximate solution is obtained by averaging method and KB transformation. The bifurcation function is derived to investigate its dynamic behavior by singularity theory. The bifurcation mechanism of the symmetric coupled nonlinear oscillators model with clearance is investigated with aid of transition set and bifurcation diagram. The analytical result could be important for design of symmetric coupled nonlinear oscillators model with clearance.

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