On Self-similar Solutions to Degenerate Compressible Navier–Stokes Equations

Pierre Germain\(^1\), Tsukasa Iwabuchi\(^2\), Tristan Léger\(^1\)

\(^1\) Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012-1185, USA. E-mail: pgermain@cims.nyu.edu; tleger@cims.nyu.edu

\(^2\) Mathematical Institute, Tohoku University, Sendai City 980-8578, Japan. E-mail: t-iwabuchi@tohoku.ac.jp

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Abstract: We study cavitating solutions to compressible Navier–Stokes equations with degenerate density-dependent viscosity. We consider two types of small radial solutions: forward self-similar (expanders), and backward self-similar (shrinkers). In the first case, we construct such solutions by a fixed-point argument. In the second case, we prove non-existence of such solutions using weighted energy estimates.

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1. Introduction

1.1. The equations. In this article we consider the compressible Navier–Stokes–Fourier system, which describes the motion of a heat conducting gas:

\[
\begin{align*}
\frac{\partial t}{\partial t} \rho + \text{div} (\rho u) &= 0, \\
\frac{\partial t}{\partial t} (\rho u) + \text{div} (\rho u \otimes u) + \nabla \pi &= \text{div}(\tau), \\
\frac{\partial t}{\partial t} \left[ \rho \left( \frac{|u|^2}{2} + e \right) \right] + \text{div} \left[ u \left( \rho \left( \frac{|u|^2}{2} + e + \pi \right) \right) \right] &= \text{div} (\tau \cdot u),
\end{align*}
\]

(1.1)

where \( \rho : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) denotes the density of the fluid, \( u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \) its velocity field, \( \pi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) its pressure, \( \tau : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{d \times d} \) its stress tensor, \( e : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) its internal energy and \( q : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \) its internal energy flux.

We take \( q \) to be proportional to the gradient of the temperature of the gas, in accordance with Fourier’s law: \( q = -\kappa \nabla \theta \), where \( \kappa > 0 \) denotes the thermal conductivity.

We will also restrict our attention to newtonian gases, for which \( \tau \) is given by

\[
\tau = \lambda \text{div} u \text{Id} + 2\mu D(u), \quad D(u) := \left( \frac{\partial_i u_j + \partial_j u_i}{2} \right)_{1 \leq i \leq d, 1 \leq j \leq d},
\]

where the Lamé coefficients \( \lambda, \mu \) are such that

\[
\mu > 0, \quad 2\mu + d\lambda \geq 0.
\]

Next we impose pressure and internal energy laws for the quantities \( \pi := \pi(\rho, \theta), e := e(\rho, \theta) \). The second law of thermodynamics requires (see for example the introduction of [2])

\[
\pi(\rho, \theta) = \rho^2 \frac{\partial e}{\partial \rho} \bigg|_\rho + \theta \frac{\partial \pi}{\partial \theta} \bigg|_\rho.
\]

This implies the existence of the entropy \( s := s(\rho, \theta) \) of the system, which is defined up to an additive constant as

\[
\frac{\partial s}{\partial e} \bigg|_\rho = \frac{1}{\theta}, \quad \frac{\partial s}{\partial \rho} \bigg|_\theta = -\frac{\pi}{\rho^2 \theta}.
\]

Finally we define the coefficient \( C_V \) as

\[
C_V := \frac{\partial e}{\partial \theta} \bigg|_\rho = -\frac{1}{\theta^2} \frac{\partial^2 s}{\partial e^2} \bigg|_\rho^{-1}.
\]

To satisfy these conditions, we will consider ideal gases (\( \pi = \rho R \theta \)) and Joule’s law for the internal energy \( e = C_V \theta \), where \( C_V \) is constant.

In this paper we will be interested in this system near vacuum. Then the dependence of the Lamé coefficients on the density becomes particularly relevant since it makes the equations degenerate. More precisely, we postulate the following laws:

\[
\lambda(\rho) = \lambda_0 \rho^\alpha, \quad \mu(\rho) = \mu_0 \rho^\alpha,
\]

(1.2)
where \( 0 < \alpha \leq 1 \). The resulting system can be viewed as a “zoomed in near vacuum” version of the model considered in [1,2] by Bresch and Desjardins, for ideal gases. Mathematically the laws for Lamé coefficients (1.2) satisfy their assumptions near vacuum. More precisely taking \( A = +\infty \), (1.2) satisfy equation (16) of [2] if we choose \((d-1)/d < \alpha < 1\). The authors also consider \( \kappa(\rho, \theta) = \kappa_0(1+\rho)(1+\theta^a) \), where \( a \geq 2 \). Since we are interested in a perturbative regime around the state \((\rho, \theta, u) = (0, 0, 0)\), we elected to take \( \kappa \) constant instead. Finally these authors considered ideal polytropic gases, which means that the pressure and internal energy laws differ from ours by a density dependent function (they take \( \pi = \rho R\theta + \pi_\infty(\rho) \) and \( e = C_V\theta + e_\infty(\rho) \) for some constants \( C_V, R \)).

Finally note that the special case \( \alpha = 1 \) corresponds to a shallow water type model (see for example [4,10]).

In conclusion, the equations read

\[
\begin{align*}
\partial_t \rho + \text{div} (\rho u) &= 0, \\
\partial_t (\rho u) + \text{div} (\rho u \otimes u) + \nabla (\rho R\theta) &= \text{div} (\lambda_0 \rho^a \text{div} u \text{ Id} + \mu_0 \rho^a (\nabla u + \nabla u^T)), \\
\partial_t \left[ \rho \left( \frac{|u|^2}{2} + C_V \theta \right) \right] + \text{div} \left[ u \left( \rho \left( \frac{|u|^2}{2} + C_V \theta \right) + \rho R\theta \right) \right] - \kappa \Delta \theta &= \text{div} (\lambda_0 \rho^a (\text{div} u) u + \mu_0 \rho^a (\nabla u + \nabla u^T) \cdot u).
\end{align*}
\]

(cNS)

Note that the system is scaling invariant: if \((\rho, u, \theta)\) denotes a solution, then \((\rho_\lambda, u_\lambda, \theta_\lambda)\), where

\[
\rho_\lambda(t, x) := \rho(\lambda^2 t, \lambda x), \quad u_\lambda(t, x) := \lambda u(\lambda^2 t, \lambda x), \quad \theta_\lambda(t, x) := \lambda^2 \theta(\lambda^2 t, \lambda x)
\]
is also a solution.

This leads us to considering self-similar solutions to (cNS). In this paper, we will therefore restrict our attention to self-similar, radially symmetric solutions. Moreover since a well-known difficulty in the case of degenerate viscosity is the presence of vacuum, we will also assume that our solutions exhibit cavitation at the origin. More precisely, we study both expanders of the form

\[
\begin{align*}
\rho(t, x) &= P \left( \frac{|x|}{\sqrt{t}} \right), & u(t, x) &= \frac{1}{\sqrt{t}} U \left( \frac{|x|}{\sqrt{t}} \right) \frac{x}{|x|}, \\
\theta(t, x) &= \frac{1}{t} \Theta \left( \frac{|x|}{\sqrt{t}} \right),
\end{align*}
\]

and shrinkers:

\[
\begin{align*}
\rho(t, x) &= P \left( \frac{|x|}{\sqrt{T-t}} \right), & u(t, x) &= \frac{1}{\sqrt{T-t}} U \left( \frac{|x|}{\sqrt{T-t}} \right) \frac{x}{|x|}, \\
\theta(t, x) &= \frac{1}{T-t} \Theta \left( \frac{|x|}{\sqrt{T-t}} \right),
\end{align*}
\]

where \( T > 0 \). We will require that \( P(0) = 0 \) to signify that there is cavitation at the origin.
1.2. Background.

1.2.1. Weak solutions  The system (1.1) and related models have been extensively studied in the literature. In the constant coefficient case, P.-L. Lions constructed weak solutions for barotropic compressible Navier–Stokes with large initial data and possibly vacuum states in [19]. His result was subsequently refined and extended to the full system by Feireisl et al. [9] and Feireisl [8].

In the case of degenerate density-dependent viscosity (for example of the type (1.2)), the situation becomes more involved. A work of major importance in this direction is the paper by Bresch et al. [3], where this problem is treated for various compressible models, including the shallow water model away from vacuum. Recently A. Vasseur and C. Yu were able to construct weak solutions to the barotropic compressible Navier–Stokes with vacuum in [24].

1.2.2. Strong solutions  In the constant coefficient case, A. Matsumura and T. Nishida proved global existence of small strong solutions in [20]. Note that in this work the density is bounded away from 0. R. Danchin improved this result in [7] by considering rough initial data. A similar result was proved in the density-dependent coefficient case in [6]. In the presence of vacuum, the compressible Euler equation has been studied by J. Jang and N. Masmoudi in [15]. Closer to the context of the present article, X. Huang and J. Li constructed (in [14]) strong solutions to (1.1) with constant coefficients and initial data with vacuum.

1.2.3. Self-similar solutions  Self-similar solutions have been the subject of investigation since the seminal work of Leray [18] on the incompressible Navier–Stokes equation. Indeed he noticed that the existence of a backward solution of this type would imply singularity formation. Forward self-similar solutions are also of interest since they are expected to describe the continuation of the backward solution after the singular time. Such small expander type solutions were constructed by Cannone and Planchon [5]. Recently V. Šverák and H. Jia were able to construct solutions to the incompressible Navier–Stokes equation with large self-similar initial data in [16]. Regarding backward self-similar solutions, J. Nečas, M. Růžička, V. Šverák proved non existence in the natural energy class of the Navier–Stokes equation (see [21]). This result was improved by Tsai [23]. He showed that it still holds if only local energy inequalities are assumed.

For compressible Navier Stokes equations, Z. Guo and S. Jiang showed in [13] that in the 1D isothermal barotropic case, there exist neither forward nor backward self-similar solutions. Qin-Su-Deng [22] proved the non-existence of forward and backward self-similar solutions associated with hyperbolic scaling in one dimension. The situation of parabolic scaling and space dimension higher than three has been investigated by the authors of the present paper: the first two constructed expanders both with and without cavitation in [11]. A complementary result is proved in the companion paper [12], where non-existence of small shrinkers is established. Focusing on vacuum states, it is physically more relevant to consider density-dependent coefficients. Prior to the present article, the density-dependent coefficient case had been considered in one space dimension by Li et al. [17].

1.3. Results.  We prove two types of results in this paper: existence of small expanders in Sect. 2, and non-existence of small shrinkers in Sect. 3. The construction of forward
solutions follows the same pattern as in the paper of the first two authors [11], although the proof is involved due to the degeneracy of the coefficients at vacuum. It requires a more precise control of the velocity profile near the origin. For technical reasons, we distinguish between the cases $0 < \alpha < 1$ and $\alpha = 1$. In the first situation ($0 < \alpha < 1$), here is simplified statement of our result:

**Theorem 1.1.** Let $d \geq 3$. Assume $2\mu_0 + d\lambda_0 = 0$. Then if

$$A + P_\delta + \delta + \frac{P_1^{1-\alpha}\Theta_0}{A} + \frac{A\delta}{\Theta_0} + \frac{A^2}{\Theta_0} + A\log\frac{1}{\delta^2 P_\delta^{1-\alpha}}$$

is small enough, there exists a solution to (cNS) of the form (1.3) such that

$$P(\delta) = P_\delta, \ U(0) = 0, \ U'(0) = A, \ \Theta(0) = \Theta_0 > 0, \ \Theta'(0) = 0.$$

In the second case ($\alpha = 1$), we prove

**Theorem 1.2.** Let $d \geq 3$. Assume $2\mu_0 + d\lambda_0 > 0$. Then if $A\log\delta^{-1} + P_\delta + \delta$ is small enough, there exists a solution to (cNS) of the form (1.3) such that

$$P(\delta) = P_\delta, \ U(0) = 0, \ U'(0) = A, \ \Theta(0) = \frac{2\mu_0 + \lambda_0}{R} A > 0, \ \Theta'(0) = 0.$$

Note that for $0 < \alpha < 1$, we construct a three parameter family of solutions, and for $\alpha = 1$ we obtain a two dimensional family (the choice of $\Theta_0$ is not free). In both cases we also obtain a precise description of the shape of the profiles. Details can be found in the full statements of these theorems in Sect. 2.

We emphasize that the conditions $2\mu_0 + d\lambda_0 = 0$ when $0 < \alpha < 1$ and $\Theta(0) = (2\mu_0 + \lambda_0)R^{-1}A > 0$ when $\alpha = 1$ are key to ensuring that the solutions we construct are continuous at the origin, and smooth away from it.

The constructions also apply in space dimensions 1 and 2. However we have not included these cases to the main statement since the initial condition $\Theta_\infty/r^2$ is not locally integrable, and therefore the equations are not well defined in a distributional sense.

In the second part of the paper, we show that (cNS) does not have small solutions of shrinker type (1.4). We prove the following result (in simplified form here):

**Theorem 1.3.** Consider a cavitating solution to (cNS) of the form (1.4). Let $\varepsilon > 0$. Assume that

$$\sup_{r > 0} \left( (r)^2 \Theta + P^{1-\alpha} + \frac{U}{r\Theta} \right) + \sup_{r > \varepsilon} \left| \frac{U'}{r\Theta'} \right|$$

is sufficiently small.

Then $U \equiv \Theta \equiv 0$, $P = \text{Constant}.$

We show a similar result in the companion paper [12], when the Lamé coefficients are constant. Note however that the two approaches are different. Indeed some key cancellations are no longer available in the density-dependent viscosity case. Therefore we develop a different method tailored to our setting of solutions exhibiting cavitation in the present work.

The methods of this article could also be adapted to deal with a thermal conductivity of the form $\kappa = \kappa_0(1+\rho)$. However adding the temperature dependence as in [2] (namely taking $\kappa = \kappa_0(1+\rho)(1+\theta^a)$, $a \geq 2$) seems more challenging. Indeed this choice breaks the scale invariance of the system.
2. Existence of Expanders

In this section we construct expander solutions to (cNS), that is solutions of the type \((cNS)\). Plugging this ansatz into \((cNS)\), we obtain the following system of ODEs for the profiles \(P, U\) and \(\Theta\):

\[
\begin{align*}
&-\frac{1}{2} r P' + P' U + P \left( U' + \frac{d - 1}{r} U \right) = 0, \\
&-\frac{1}{2} P U - \frac{1}{2} (P U)' + (P U^2)' + \frac{d - 1}{r} P U^2 + (P R \Theta)' \\
&\quad = (2 \mu_0 + \lambda_0) P^\alpha \left( U'' + \frac{d - 1}{r} U' - \frac{d - 1}{r^2} U \right) \\
&\quad + (2 \mu_0 + \lambda_0) (P^\alpha)' U' + \lambda_0 (P^\alpha)' \frac{d - 1}{r} U, \\
&\quad - P \left( \frac{U^2}{2} + C_V \Theta \right) - \frac{1}{2} r \left( P \left( \frac{U^2}{2} + C_V \Theta \right) \right)' + \left( U P \left( \frac{U^2}{2} + C_V \Theta \right) + U P R \Theta \right)' \\
&\quad + \frac{d - 1}{r} \left( U P \left( \frac{U^2}{2} + C_V \Theta \right) + U P R \Theta \right) - \kappa \left( \Theta'' + \frac{d - 1}{r} \Theta' \right) \\
&\quad = 2 \mu_0 P^\alpha \left( (U')^2 + \frac{d - 1}{r^2} U^2 \right) + \lambda_0 P^\alpha \left( U' + \frac{d - 1}{r} U \right)^2 \\
&\quad + (2 \mu_0 + \lambda_0) P^\alpha (U'' + \frac{d - 1}{r} U' - \frac{d - 1}{r^2} U) U \\
&\quad + (2 \mu_0 + \lambda_0) \left( P^\alpha \right)' U' U + \lambda_0 \left( P^\alpha \right)' \frac{d - 1}{r} U^2. \\
\end{align*}
\] (2.1)

As announced in the introduction, we distinguish two cases for the existence of expanders: when \(0 < \alpha < 1\) and when \(\alpha = 1\). There are substantial differences between the two: we do not use the same integro-differential formulation for both, and require different conditions on the parameters. Most strikingly, \(\Theta_0\) is free for \(0 < \alpha < 1\) but not for \(\alpha = 1\).

2.1. Existence when \(0 < \alpha < 1\). In this section, we show the following:

**Theorem 2.1.** Let \(d \geq 3\) and \(0 < \alpha < 1\). Fix \((C_V, \kappa, R, \mu_0, \lambda_0) \in (0, \infty)^5\) such that \(2 \mu_0 + d \lambda_0 = 0\).

Then there exists a constant \(C(C_V, \kappa, R, \mu_0, \lambda_0) := C\) such that if

\[
P_\delta + \delta + \frac{P_\delta^{1-\alpha} \Theta_0}{A} + \frac{A \delta}{\Theta_0} + \frac{A}{P_\delta^\alpha} + \frac{A^2}{P_\delta \Theta_0} + A \log \frac{1}{\delta^2 P_\delta^{1-\alpha}} < C,
\]

there exists a solution \((P, U, \Theta) \in C^{\frac{d A}{1+\delta-\alpha}} \times C^1([0, \infty)) \times C^1([0, \infty))\) to (2.1) such that

\[
P(0) = 0, P(\delta) = P_\delta, U(0) = 0, U'(0) = A, \Theta(0) = \Theta_0 > 0, \Theta'(0) = 0.
\]

Near \(0\), the behavior of the solution is given by (here \(0 < \varepsilon < 1 - \alpha\)):

\[
P(r) = P_\delta \left( \frac{r}{\delta} \right)^{\frac{d A}{1+\delta-\alpha}} + O \left( \left( \frac{r}{\delta} \right)^{1+\frac{d A}{1+\delta-\alpha} + (1 - \alpha - \varepsilon)d A} \right).
\]
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\[ U(r) = Ar + O(r^{1+(1-\alpha-\varepsilon)dA}), \]
\[ \Theta(r) = \Theta_0 + O(r^2). \]

Moreover the profiles satisfy the following global bounds:
\[ P(r) \simeq P_\delta \min \left[ 1, \left( \frac{r}{\delta} \right)^{2dA} \right], \]
\[ |U(r)| \lesssim \frac{Ar}{(1 + P_\delta^{1/2-\alpha/2}r)^2}, \]
\[ |U'(r)| \lesssim \frac{A}{(1 + P_\delta^{1/2-\alpha/2}r)^2}, \]
\[ 0 \leq \Theta(r) \lesssim \frac{1}{(1 + \sqrt{P_\delta}r)^2}, \]
\[ |\Theta'(r)| \lesssim \frac{\sqrt{P_\delta}r}{(1 + \sqrt{P_\delta}r)^2}. \]

Finally as \( r \to +\infty \), there exist \( P_\infty > 0, U_\infty > 0, \Theta_\infty > 0 \) such that
\[ P(r) = P_\infty + O\left( \frac{1}{r^2} \right), \quad U(r) = \frac{U_\infty}{r} + O\left( \frac{1}{r^3} \right), \quad \Theta(r) = \frac{\Theta_\infty}{r^2} + O\left( \frac{1}{r^4} \right). \]

The proof is modelled after that of a similar result in [11] (Section 4): first we find an integro-differential formulation of the problem. Then we construct the solution locally near 0 using a fixed-point argument. After that, we prove global existence, and finally we study the asymptotic behavior of the solutions.

Note that the main difference compared to [11] is in the derivation of the integro-differential equation and the local existence part of the proof. These steps are more involved in the present paper due to the degeneracy of the Lamé coefficients at 0, and the presence of vacuum.

2.1.1. Integro-differential formulation  We are aiming at constructing solutions such that for \( 0 < A < 1/2 \) and \( \Theta_0 > 0 \),
\[ P(r) = O(r^{2dA}), \quad U(r) = Ar + O(r^{1+(1-\alpha-\varepsilon)\frac{2dA}{1-2\alpha}}), \quad \Theta(r) = \Theta_0 + O(r^2), \quad r \to 0. \]

First starting with the equation on \( P \), we directly obtain after integration that
\[ P(r) = P_\delta e^{V(r)-V(\delta)}, \]
where
\[ V(r) - V(\delta) = \int_\delta^r \frac{U'}{U} + \frac{d-1}{r} U - U \, dr. \] (2.2)

Next we move to the equation satisfied by \( U \):
\[ -\frac{1}{2} PU - \frac{1}{2} r(PU)' + (PU^2)' + \frac{d-1}{r} PU^2 + (PR\Theta)' = (2\mu_0 + \kappa_0) \left[ P^\alpha \left( U' + \frac{d-1}{r} U \right) \right]' - 2\mu_0 (P^\alpha)' \cdot \frac{d-1}{r} U. \]
Using the assumption \(2\mu_0 + d\lambda_0 = 0\), we can write the right-hand side
\[
(2\mu_0 + \lambda_0) P^\alpha \left\{ \left( U' + \frac{d-1}{r} U \right)' + \alpha \frac{U' + \frac{d-1}{r} U}{\frac{1}{2}r - U} \left( U' - \frac{1}{r} U \right) \right\}.
\]

After dividing by \(P^\alpha\), we have
\[
- \frac{1}{2} \left( r P^{1-\alpha} U \right)' - \frac{\alpha}{2} r P^{1-\alpha} \frac{U' + \frac{d-1}{r} U}{\frac{1}{2}r - U} - \frac{1}{P^\alpha} (PU^2)' + \frac{d-1}{r} P^{1-\alpha} U^2 + \frac{(PR\Theta)'}{P^\alpha}.
\]

The Eq. (2.3) can be written
\[
- \frac{1}{2} \left( r P^{1-\alpha} U \right)' - \frac{\alpha}{2} r P^{1-\alpha} \frac{U' + \frac{d-1}{r} U}{\frac{1}{2}r - U} - \frac{1}{P^\alpha} (PU^2)' + \frac{d-1}{r} P^{1-\alpha} U^2 + \frac{(PR\Theta)'}{P^\alpha}.
\]

Let
\[
\widetilde{V}(r) := \int_0^r \frac{U' - \frac{1}{r} U}{\frac{1}{2}r - U} d\tilde{r}.
\]

Multiplying the above by \(e^{\alpha \widetilde{V}(r)}\) we get
\[
- \frac{1}{2} \left( e^{\alpha \widetilde{V}(r)} r P^{1-\alpha} U \right)' + e^{\alpha \widetilde{V}(r)}
\]
\[
- \frac{\alpha}{2} r P^{1-\alpha} \frac{d}{r} U - \frac{1}{P^\alpha} (PU^2)' + \frac{d-1}{r} P^{1-\alpha} U^2 + \frac{(PR\Theta)'}{P^\alpha}.
\]

Now we integrate over \([0, r]\) and divide by \(e^{\alpha \widetilde{V}(r)}\):
\[
- \frac{1}{2} r P^{1-\alpha} U + e^{-\alpha \widetilde{V}(r)} \int_0^r e^{\alpha \widetilde{V}(r_2)}
\]
\[
\left\{ - \frac{\alpha}{2} r_2 P^{1-\alpha} \frac{d}{r_2} U - \frac{1}{P^\alpha} (PU^2)' + \frac{d-1}{r_2} P^{1-\alpha} U^2 + \frac{(PR\Theta)'}{P^\alpha} \right\} dr_2
\]
\[
= (2\mu_0 + \lambda_0) \left( U' + \frac{d-1}{r} U \right) - e^{-\alpha \widetilde{V}(r)} (2\mu_0 + \lambda_0) dA.
\]

Let
\[
W(r) := \frac{1}{2\mu_0 + \lambda_0} \int_0^r \frac{P(\tilde{r})^{1-\alpha}}{2} d\tilde{r}.
\]
Multiplication by $r^{d-1}e^{W(r)}$ then yields

$$(2\mu_0 + \lambda_0) \left( r^{d-1} e^{W(r)} U \right)' = r^{d-1} e^{W(r)} F_U(r),$$

where

$$F_U(r) := e^{-a\tilde{V}(r)}(2\mu_0 + \lambda_0) A + e^{-a\tilde{V}(r)} \int_0^r e^{a\tilde{V}(r_2)} \left\{ -\frac{a}{2} r_2 P^{1-a} \frac{d}{r_2^2} U + \frac{1}{p^a} (PU'^2)' + \frac{d-1}{r_2} P^{1-a} U^2 + \frac{(PR\Theta)'}{p^a} \right\} dr_2. \quad (2.6)$$

We obtain the desired integro-differential equation after integrating the above.

For the last equation on $\Theta$, we proceed as in [11]. Therefore we omit the details.

We obtain the following integro-differential formulation:

$$\begin{cases}
P(r) = P_\delta e^{V(r)-\tilde{V}(\delta)} & \text{for given } \delta > 0, \\
U(r) = \frac{r^{d+1}}{2\mu_0 + \lambda_0} \int_0^r \frac{r_1^{d-1}}{r_1^{d-1} e^{-W(r)+W(r_1)} F_U(r_1)} dr_1, \\
\Theta(r) = (d-2)r^{d+2} \int_0^r \frac{r_1^{d-3}}{r_1^{d-3} e^{-Z(r)+Z(r_1)} d_1 \Theta_0} dr_1, \\
-\frac{U^2}{2C_V} + \frac{r^{d+2}}{\kappa} \int_0^r \frac{d}{r_1} e^{-Z(r)+Z(r_1)} F_{\Theta}(r_1) dr_1,
\end{cases} \quad (2.7)$$

where $V, \tilde{V}, W$ and $F_U$ are defined as in (2.2), (2.4), (2.5), (2.6), and

$$F_{\Theta}(r_1) := UP \left( \frac{U^2}{2} + C_V \Theta \right) + UPR\Theta + \frac{d-2}{r_1} \int_0^{r_1} \left( UP \left( \frac{U^2}{2} + C_V \Theta \right) + UPR\Theta \right) dr_2$$

$$- (2\mu_0 + \lambda_0) \left( P^a UU' + \frac{d-2}{r_1} \int_0^{r_1} P^a UU' \right) dr_2$$

$$- \lambda_0 (d-1) \left( \frac{P^a U^2}{r_1} + \frac{d-2}{r_1} \int_0^{r_1} P^a U^2 \right)$$

$$+ \frac{\kappa}{C_V} \left( UU' + \frac{d-2}{2r_1} U^2 \right).$$

and

$$Z(r) := \frac{C_V}{\kappa} \int_0^r \frac{P(\tilde{r})}{2} d\tilde{r}.$$

2.1.2. Local existence In this section we construct a local solution to the above integro-differential formulation by a fixed point argument.

First we define the functional space in which we solve the equation:

Let $(1-\alpha)/2 < \varepsilon < 1 - \alpha$. Define

$$\| (U, \Theta) \|^\delta := \sup_{0 < r < \delta} \left[ r^{-1} |U(r)| + |U'(r)| + \frac{A(r)}{P_\delta^{1-\alpha} \Theta_0} \left| \left( \frac{U(r)}{r} \right)' \right| + \frac{A}{\Theta_0^2} |\Theta(r)| + \frac{A}{\Theta_0} r^{-1} |\Theta'(r)| \right],$$

$$E^\delta = \{ (U, \Theta) \in C^1(0, \delta) \text{ such that } \Theta(0) = \Theta_0, \| (U, \Theta) \|^\delta < \infty \}.$$
Note that $E_{\delta}$ differs from the definition in [11]. Here we need better control of the profile $U$ near 0 to close the estimates.

We also define the map $\Phi$ on $B_{E_{\delta}}((Ar, \Theta_0), A/2)$ by the formula

$$\Phi : (U, \Theta) \mapsto \varepsilon^d e^{W(U, \Theta) + \frac{1}{2}U^2} - \int_0^r 1 - \int dV(\Theta) - \int dV(\Theta)$$

The remainder of the subsection is dedicated to the proof of the following lemma, which shows local existence of solutions to (2.7).

**Lemma 2.2.** Assume that

$$A \ll \frac{1}{2}, \quad P_\delta + \frac{P_{1-\alpha}^1}{A} + \frac{A\delta}{\Theta_0} + \frac{A^2}{\Theta_0} \ll 1.$$  

Then $\Phi$ is a contraction on $B_{E_{\delta}}((Ar, \Theta_0), A/2)$.

**Proof.** Stabilization: First we note that

$$|\tilde{V}(r)| = \int_0^r \frac{\tilde{r}(U')}{\tilde{r} - U} \leq P_\delta^{1-\alpha} \Theta_0 \int_0^r \tilde{r}^{-1} \left( \frac{1}{\delta} \right)^{(1-\alpha)\varepsilon} \tilde{r} d\tilde{r} \leq \frac{P_\delta^{1-\alpha} \Theta_0}{A(1 - \alpha + \varepsilon)} \left( \frac{r}{\delta} \right)^{(1-\alpha)\varepsilon} dA,$$

which implies

$$|e^{\delta \tilde{V}(r)} - 1| \lesssim \frac{\alpha}{1 - \alpha - \varepsilon} \frac{P_\delta^{1-\alpha} \Theta_0}{A} e^{\frac{r}{\delta} \Theta_0} \lesssim \frac{\alpha}{1 - \alpha - \varepsilon} \frac{P_\delta^{1-\alpha} \Theta_0}{A} \ll 1.$$

We have the estimates (the implicit constants do not depend on the parameters here)

$$P(r) \leq P_\delta \left( \frac{r}{\delta} \right)^{dA},$$

$$F_U(r) - d(2\mu_0 + \lambda_0)A \lesssim \left( \frac{\alpha(2\mu_0 + \lambda_0)}{1 - \alpha - \varepsilon} \frac{P_\delta^{1-\alpha} \Theta_0}{A} + P_\delta^{1-\alpha} Ar^2 + R \frac{P_\delta^{1-\alpha} \Theta_0}{A} \right) A,$$

$$F_\Theta(r) \lesssim P_\delta A^3 r^3 + (C_V + R) P_\delta A \Theta_0 r + (2\mu_0 + \lambda_0) P_\delta^2 A^2 r + \frac{\kappa}{C_V} A^2 r.$$

Define $(\tilde{U}, \tilde{\Theta}) := \Phi(Ar, \Theta_0)$. We deduce that (here the implicit constants depend on the parameters)

$$|\tilde{U} - Ar| \lesssim \left( P_\delta^{1-\alpha} \delta^2 + \frac{P_\delta^{1-\alpha} \Theta_0}{A} \right) Ar,$$

$$|\tilde{\Theta} - \Theta_0| \lesssim \left( P_\delta \Theta_0 + \frac{A^2}{\Theta_0} \right) r^2,$$

$$|\tilde{\Theta}'| \lesssim \left( P_\delta + \frac{A^2}{\Theta_0} \right) \Theta_0 r.$$

As for $(\tilde{U}(r)/r)'$, we write

$$-\frac{d}{r} U(r) + \frac{r^{-1}}{2\mu_0 + \lambda_0} F_U(r) - \frac{W'(r)}{r} U(r).$$

The third term satisfies (the implicit constants do not depend on parameters)

$$\left| \frac{W'(r)}{r} U(r) \right| \lesssim \frac{1}{2\mu_0 + \lambda_0} P_\delta^{1-\alpha} \left( \frac{r}{\delta} \right)^{(1-\alpha)dA} Ar \lesssim \frac{1}{2\mu_0 + \lambda_0} \left( \frac{Ar^2}{\Theta_0} \right) P_\delta^{1-\alpha} \Theta_0 r^{-1} \left( \frac{r}{\delta} \right)^{(1-\alpha)dA}.$$
By integration by parts, the sum of the first and the second terms is
\[
\left| -\frac{dU(r)}{r^2} + \frac{r^{-1}}{2\mu_0 + \lambda_0} F_U(r) \right|
\]
\[
= \left| \frac{r^{-d-1}}{2\mu_0 + \lambda_0} \int_0^r r_1^d e^{-W(r)+W(r_1)} \left( W'(r_1) F_U(r_1) + F_U'(r_1) \right) dr_1 \right|
\]
\[
\lesssim \frac{1}{2\mu_0 + \lambda_0} \left( \frac{A^2}{\Theta_0} \right) P^1_{\theta} \Theta_0 r^{-1} \left( \frac{r}{\delta} \right)^{(1-\alpha)dA} + \left| \frac{r^{-d-1}}{2\mu_0 + \lambda_0} \int_0^r r_1^d e^{-W(r)+W(r_1)} F_U'(r_1) dr_1 \right|
\]
To handle the last term, we write
\[
|F_U'(r)| \lesssim (2\mu_0 + \lambda_0) |\tilde{V}'| e^{-\alpha \tilde{V}} A + P^1_{\theta} A^2 r \left( \frac{r}{\delta} \right)^{(1-\alpha)dA}
\]
\[
+ R P^1_{\theta} A \Theta_0 r^{-1} \left( \frac{r}{\delta} \right)^{(1-\alpha)dA} + R P^1_{\theta} \Theta_0 r^{-1} \left( \frac{r}{\delta} \right)^{(1-\alpha)dA}
\]
\[
\lesssim \left( (2\mu_0 + \lambda_0) A + \frac{A^2}{\Theta_0} r^2 + R r^2 \right) P^1_{\theta} \Theta_0 r^{-1} \left( \frac{r}{\delta} \right)^{(1-\alpha-\epsilon)dA}
\]
and hence,
\[
\frac{r^{-d-1}}{2\mu_0 + \lambda_0} \int_0^r r_1^d e^{-W(r)+W(r_1)} F_U'(r_1) dr_1
\]
\[
\lesssim \left( (\alpha + \frac{R}{2\mu_0 + \lambda_0}) A + \frac{1}{2\mu_0 + \lambda_0} \frac{A^2}{\Theta_0} r^2 + \frac{R}{2\mu_0 + \lambda_0} r^2 \right) e^{-\frac{4d(\mu_0 + \lambda_0)}{\Theta_1}} P^1_{\theta} \Theta_0 r^{-1} \left( \frac{r}{\delta} \right)^{(1-\alpha-\epsilon)dA}
\]
We can then conclude that (the implicit constants depend on the parameters here)
\[
\left| \left( \frac{U(r)}{r} \right)' \right| \lesssim \left( \frac{A^2}{\Theta_0} + A + r^2 \right) P^1_{\theta} \Theta_0 r^{-1} \left( \frac{r}{\delta} \right)^{(1-\alpha-\epsilon)dA}
\]

Contraction: Let \((U_i, \Theta_i) \in B_{E^1}((Ar_i, \Theta_0), A/2), i = 1, 2,\)

Let \(D := \| (U_1, \Theta_1) - (U_2, \Theta_2) \|^2.\)

Denote \((U_i, \Theta_i) = \Phi((U_i, \Theta_i)), i = 1, 2.\)

We have
\[
|P^1_{\theta} - P^1_{\Theta_1}| \lesssim P^1_{\theta} D \ln \left( \frac{\delta}{r} \right) \left( \frac{r}{\delta} \right)^{(1-\alpha)dA}
\]
\[
\lesssim \frac{P^1_{\theta}}{\epsilon} \frac{D}{A} \left( \frac{r}{\delta} \right)^{(1-\alpha-\epsilon)dA},
\]
where we used that \(\sup_{r > 0} r^B \ln(r^{-1}) \lesssim B^{-1}.\)

We deduce the bounds (the implicit constants do not depend on the parameters)
\[
|W_1(r) - W_2(r)| \lesssim \frac{1}{2\mu_0 + \lambda_0} P^1_{\theta} \frac{D}{\epsilon} \left( \frac{r}{\delta} \right)^{(1-\alpha-\epsilon)dA},
\]
\[
|Z_1(r) - Z_2(r)| \lesssim \frac{CV}{\kappa} P^1_{\theta} D \left( \frac{r}{\delta} \right)^{(1-\epsilon)dA},
\]
\[
|W'_1(r) - W'_2(r)| \lesssim \frac{1}{2\mu_0 + \lambda_0} P^1_{\theta} \frac{D}{\epsilon} \left( \frac{r}{\delta} \right)^{(1-\alpha-\epsilon)dA}.
\]
Then we can estimate (the implicit constants depend on the parameters here)

\[|Z'_1(r) - Z'_2(r)| \lesssim \frac{C_V P_\delta D}{\kappa \varepsilon} A \left(\frac{r}{\delta}\right)^{(1-\varepsilon)dA},\]

\[|F_{U_1}(r) - F_{U_2}(r)| \lesssim \left(\frac{\alpha(2\mu_0 + \lambda_0) + R}{1 - \alpha - \varepsilon} + \frac{P_{\delta}^{1-\alpha} \Theta_0}{A} + P_{\delta}^{1-\alpha} A \delta^2\right) D,\]

\[|F'_{U_1}(r) - F'_{U_2}(r)| \lesssim \left(\frac{(2\mu_0 + \lambda_0)\alpha + R D + \frac{A}{\Theta_0} \delta^2 + R \frac{D}{A} \delta^2}{A} P_{\delta}^{1-\alpha} \Theta_0 r^{-1} \left(\frac{r}{\delta}\right)^{(1-\alpha-\varepsilon)dA},\]

\[|F_{\Theta_1}(r) - F_{\Theta_2}(r)| \lesssim \left((C_V + R) P_\delta + \frac{P_\delta A^2 \delta^2}{\Theta_0} + (2\mu_0 + \lambda_0) \frac{P_\delta^2 A}{\Theta_0} + \frac{\kappa A}{C_V \Theta_0}\right) D r \Theta_0.\]

In similar fashion, we have

\[|\widetilde{U}'_1(r) - \widetilde{U}'_2(r)| \lesssim \left(\delta^2 P_{\delta}^{1-\alpha} + \frac{P_{\delta}^{1-\alpha} \Theta_0}{A}\right) D r.\]  

(2.8)

Now we write

\[
\left|\left(\begin{array}{c} \widetilde{U}'_1(r) \\ \widetilde{U}'_2(r) 
\end{array}\right) - \left(\begin{array}{c} \widetilde{U}'_1(r) \\ \widetilde{U}'_2(r) 
\end{array}\right)\right|
\lesssim \frac{1}{r} |W'_1(r) U_1(r) - W'_2(r) U_2(r)|

+ r^{-d-1} \int_0^r \int_0^r |e^{-W_1(r)+W_1(r_1)} W'_1(r_1) F_{U_1}(r_1) - e^{-W_2(r)+W_2(r_1)} W'_2(r_1) F_{U_2}(r_1)| dr_1

+ r^{-d-1} \int_0^r \int_0^r |e^{-W_1(r)+W_1(r_1)} F'_{U_1}(r_1) - e^{-W_2(r)+W_2(r_1)} F'_{U_2}(r_1)| dr_1

\lesssim \frac{P_{\delta}^{1-\alpha} \Theta_0}{A} r^{-1} \left(\frac{r}{\delta}\right)^{(1-\alpha-\varepsilon)dA} D \left(\frac{A}{\Theta_0} \delta^2 + A + \delta^2\right).\]  

(2.10)

Similarly, we obtain the following estimates on \( \Theta \):

\[|\Theta_1(r) - \Theta_2(r)| \lesssim \left(\frac{P_\delta + \frac{P_\delta A^2 \delta^2}{\Theta_0} + \frac{P_\delta A}{\Theta_0} + \frac{A}{\Theta_0}}{\Theta_0}\right) D r^2 \Theta_0 + D A r^2,\]  

(2.11)

\[|\Theta'_1(r) - \Theta'_2(r)| \lesssim \left(\frac{P_\delta + \frac{P_\delta A^2 \delta^2}{\Theta_0} + \frac{P_\delta A}{\Theta_0} + \frac{A}{\Theta_0}}{\Theta_0}\right) D r \Theta_0 + D A.\]  

(2.12)

Putting all the estimates (2.8), (2.9), (2.10), (2.11) and (2.12) together, the desired result follows given our smallness assumptions. \( \square \)

To prove Hölder continuity of \( P \), we can repeat an argument from [11] (page 20). We omit the details here.
2.1.3. Global existence  We define:

\[ Z(r) = \sup_{0 < s < r} \left( \frac{(1 + P_\delta^{1/2-\alpha/2} s)^2}{M_1 s} |U(s)| + \frac{(1 + P_\delta^{1/2-\alpha/2} s)^2}{M_1'} |U'(s) + \frac{d - 1}{s} U(s)| + \frac{(1 + \sqrt{P_\delta s})^2}{M_2} \Theta(s) + \frac{(1 + \sqrt{P_\delta s})^2}{M_2 P_\delta s} |\Theta'(s)|, \right. \]

where the constants \( M_1, M_1', M_2 \) are chosen so that

\[ M_1 < M_1' \ll 1, \quad A \ll M_1, \quad M_1' \log \frac{1}{\delta^2 P_\delta^{1-\alpha}} \ll 1, \quad P_\delta^{1-\alpha} M_2 \ll M_1', \]

\[ \Theta_0 \ll M_2 \ll 1, \quad \frac{M_1^3}{P_\delta^{1-3\alpha}} \ll M_2, \quad \frac{M_1 M_1'}{P_\delta} \ll M_2, \quad M_1 M_2 + \frac{M_1^3}{P_\delta} + \frac{M_1 M_1'}{P_\delta} \ll \Theta_0. \]

For example one can take \( M_2 = \frac{A}{P_\delta^{1-\alpha}}, M_1 = \Lambda A, M_1' = \Lambda^2 A \) for some large \( \Lambda > 0 \).

We make use of \( Z \) and a bootstrap argument to prove global existence. This is contained in the following lemma:

**Lemma 2.3.** We have \( Z(\delta) \leq \frac{1}{2} \).

Moreover if \( Z(r) \leq 1 \) for some \( r > \delta \), then the stronger bound \( Z(r) \leq \frac{1}{2} \) holds.

**Proof.** Starting point:

To ensure \( Z(\delta) \ll 1 \), we require, as in [11]:

\[ A \ll M_1, \Theta_0 \ll M_2, A^2 \ll P_\delta M_2. \]

**Estimate on \( \tilde{V} \):** We note that for \( r > \delta \):

\[ |\tilde{V}(r)| \lesssim \int_0^\delta P_\delta^{1-\alpha} \Theta_0 r_1^{-1} (r_1)^{(1-\alpha-\varepsilon)} dA + \int_\delta^r \frac{M_1' + M_1}{r_1 (1 + P_\delta^{1/2-\alpha/2} r_1)^2} dr_1 \]

\[ \lesssim \frac{P_\delta^{1-\alpha} \Theta_0}{d (1 - \alpha - \varepsilon) A} + (M_1 + M_1') \left[ \log \frac{1}{P_\delta^{1-\alpha} \delta^2} + 1 \right] \lesssim 1, \]

provided \( (M_1' + M_1) \log \frac{1}{\delta^2 P_\delta^{1-\alpha}} = O(1) \).

By a similar reasoning, we can show that under that same assumption,

\[ P_\delta \lesssim P(r) \lesssim P_\delta, \]

for \( r > \delta \).

**Estimates on \( U \):** We begin with bounds on \( F_U \).

Using the fact that \( P(r) \leq P_\delta \left( \frac{r}{\delta} \right)^d A \) for \( r \in [0, \delta] \), we write

\[ \int_0^r e^{\alpha \tilde{V}(r_1)} \frac{(PR \Theta)'}{P^\alpha} dr_1 \lesssim \frac{P_\delta^{1-\alpha}}{(1 - \alpha) A} M_1 M_2 + P_\delta^{1-\alpha} M_2 \delta^2 + \left| \int_\delta^r e^{\alpha \tilde{V}(r_1)} \frac{(PR \Theta)'}{P^\alpha} dr_1 \right|. \]

After integrating by parts, we obtain

\[ \left| \int_\delta^r e^{\alpha \tilde{V}(r_1)} \frac{(PR \Theta)'}{P^\alpha} dr_1 \right| \lesssim P_\delta^{1-\alpha} RM_2 + RP_\delta^{1-\alpha} M_2 (M_1 + M_1') \log \frac{1}{P_\delta^{1-\alpha} \delta^2}. \]
We deduce from the above that (the implicit constants do not depend on the parameters)

\[ |F_U(r)| \lesssim (2\mu_0 + \lambda_0)A + P_\delta^{1/2-\alpha/2} M_1 M'_1 + R P_\delta^{1-\alpha} M_2 \]  
(2.13)

\[ |F'_U(r)| \lesssim \frac{(2\mu_0 + \lambda_0)M'_1 A + M_1 M'_1 + R P_\delta^{1-\alpha} M_2}{r}. \]  
(2.14)

Together with the fact that

\[ r^{1-d} \int_0^r r_1^{d-1} e^{-C P_\delta^{1-\alpha} (r^2-r_1^2)} dr_1 \lesssim \frac{r}{(1 + P_\delta^{1/2-\alpha/2} r)^2}, \]

(2.13) implies (the implicit constant depends on the parameters here)

\[ |U(r)| \lesssim \frac{r}{(1 + P_\delta^{1/2-\alpha/2} r)^2} \left( A + P_\delta^{1/2-\alpha/2} M_1 M'_1 + P_\delta^{1-\alpha} M_2 + P_\delta^{1-\alpha} M_2 (M_1 + M'_1) \log \frac{1}{P_\delta^{1-\alpha} \delta^2} \right). \]

Moving on to the derivative part, we first deal with \( r \geq \frac{1}{P_\delta^{1/2-\alpha/2}} \). We write that

\[ \left| U' + \frac{d - 1}{r} U \right| = \left| \frac{1}{2\mu_0 + \lambda_0} F_U(r) - W'(r) U(r) \right| \lesssim A + P_\delta^{1/2-\alpha/2} M_1 M'_1 + P_\delta^{1-\alpha} M_2 + M_1. \]

When \( r \geq \frac{1}{P_\delta^{1/2-\alpha/2}} \), we write, using an integration by parts:

\[ \left| U' + \frac{d - 1}{r} U \right| = \left| W'(r) \frac{r^{1-d}}{2\mu_0 + \lambda_0} \int_0^r e^{-W(r)+W(s)} \partial_s \left( \frac{s^{d-1} F_U(s)}{W'(s)} \right) ds \right|. \]

We have by (2.14)

\[ \left| \partial_s \left( \frac{s^{d-1} F_U(s)}{W'(s)} \right) \right| \lesssim s^{d-3} A + M_1 M'_1 + P_\delta^{1-\alpha} M_2 \left( \frac{1}{P_\delta^{1-\alpha}} \right)^2 \left( 1 + \left( \frac{s}{\delta} \right)^{-4dA(1-\alpha)} \right), \]

where in the above the implicit constant depends on the parameters.

Together with (2.15) and the fact that

\[ r^{1-d} \int_0^r r_1^{d-3} e^{-C P_\delta^{1-\alpha} (r^2-r_1^2)} dr_1 \lesssim \frac{1}{P_\delta^{1-\alpha} r^3} \text{ when } r \geq \frac{1}{P_\delta^{1/2-\alpha/2}}, \]

we conclude that

\[ \left| U' + \frac{d - 1}{r} U \right| \lesssim P_\delta^{1-\alpha} r \frac{1}{P_\delta^{1-\alpha} r^3} \frac{A + M_1 M'_1 + P_\delta^{1-\alpha} M_2}{P_\delta^{1-\alpha}} \lesssim \frac{1}{P_\delta^{1-\alpha} r^2} \left( A + M_1 M'_1 + P_\delta^{1-\alpha} M_2 \right). \]

Estimates on \( \Theta \) :
We start with the following estimates, where the implicit constants do not depend on the parameters

$$ |F_\Theta(r)| \lesssim \left( (C_V + R) P_\delta M_1 M_2 + P_\delta M_1^3 + ((2 \mu_0 + \lambda_0) P_\delta^{\alpha} + \frac{\kappa}{C_V}) M_1 M_1' \right) r, \quad \text{when } r \leq \frac{1}{\sqrt{P_\delta}}, $$

(2.16)

$$ |F_\Theta(r)| \lesssim \frac{1}{r} \left( \lambda_0 \frac{M_1^2}{P_\delta^{1-3\alpha}} + \left( \frac{\kappa}{C_V} P_\delta^{2\alpha-1} + (2 \mu_0 + \lambda_0) P_\delta^{3\alpha-1} \right) M_1 M_1' \right), \quad \text{when } r \geq \frac{1}{\sqrt{P_\delta}}. $$

(2.17)

We deduce from (2.16) that

$$ |\Theta(r)| \lesssim \Theta_0 + M_1 M_2 + M_1^3 + \frac{M_1 M_1'}{P_\delta}, $$

where the implicit constant depends on the parameters here. Now for the case of \( r \) large, using that

$$ r^{2-d} \int_0^r r_1^{d-3} e^{-C P_\delta (r^2-r_1^2)} dr_1 \lesssim \frac{1}{(1 + \sqrt{P_\delta r})^2}, $$

we deduce from (2.17) that

$$ |\Theta(r)| \lesssim \left( \frac{1}{\sqrt{P_\delta r}} \right)^2 \left( \Theta_0 + M_1 M_2 P_\delta^{\alpha} + M_1^3 + \frac{M_1 M_1'}{P_\delta^{1-2\alpha}} \right). $$

Similarly for the derivative, we use

$$ \left| \partial_r \left[ r^{2-d} \int_0^r r_1^{d-3} e^{-C P_\delta (r^2-r_1^2)} \right] dr_1 \right| \lesssim \frac{P_\delta r}{1 + P_\delta r^2}, $$

and deduce

$$ |\Theta'(r)| \lesssim P_\delta r \left( \Theta_0 + M_1 M_2 P_\delta^{\alpha} + \frac{M_1^3}{P_\delta^{1-3\alpha}} + \frac{M_1 M_1'}{P_\delta^{1-2\alpha}} \right), \quad \text{when } r \leq \frac{1}{\sqrt{P_\delta}}, $$

$$ |\Theta'(r)| \lesssim \frac{1}{r} \left( \Theta_0 + M_1 M_2 P_\delta^{\alpha} + \frac{M_1^3}{P_\delta^{1-3\alpha}} + \frac{M_1 M_1'}{P_\delta^{1-2\alpha}} \right), \quad \text{when } r \geq \frac{1}{\sqrt{P_\delta}}. $$

Global existence directly follows from the previous lemma by a standard continuation argument.

2.1.4. Asymptotic behavior

Regarding the asymptotic behavior of \( P, U \) and \( \Theta \), the arguments are almost identical to those of [11]. Therefore we only state the condition needed on \( M_1, M_1', M_2 \) to ensure positivity of \( \Theta \):

$$ M_1 M_2 + \frac{M_1^3}{P_\delta} + \frac{M_1 M_1'}{P_\delta} \ll \Theta_0. $$
2.2. Existence when $\alpha = 1$. In this section, we show an existence theorem in the case where $\alpha = 1$. Unlike in the previous section, we now assume $2\mu_0 + d\lambda_0 > 0$.

We show the following existence result:

**Theorem 2.4.** Let $d \geq 3$. Fix $(C_V, \kappa, R, \mu_0, \lambda_0) \in (0, \infty)^5$ such that $2\mu_0 + d\lambda_0 > 0$.

Then there exists a constant $C(C_V, \kappa, R, \mu_0, \lambda_0) := C > 0$ such that if

$$\frac{A}{P_\delta} + A \log \delta^{-1} + P_\delta + \delta < C,$$

there exists a solution $(P, U, \Theta) \in C^{\frac{2dA}{r + 1}} \times C^1([0, \infty)) \times C^1([0, \infty))$ to (2.1) such that $P(0) = 0$, $P(\delta) = P_\delta$, $U(0) = 0$, $U'(0) = A$, $\Theta(0) = \Theta_0 := \frac{1}{R}(2\mu_0 + d\lambda_0)A$, $\Theta'(0) = 0$.

Moreover the profiles satisfy the following global bounds:

$$P(r) \simeq P_\delta \min\left[1, \left(\frac{r}{\delta}\right)^{\frac{2dA}{r + 1}}\right],$$

$$|U(r)| \lesssim \frac{Ar}{(1 + r)^2}, \quad |U'(r)| \lesssim \frac{A}{(1 + r)^2},$$

$$0 \leq \Theta(r) \lesssim \frac{1}{(1 + \sqrt{P_\delta}r)^2}, \quad |\Theta'(r)| \lesssim \frac{\sqrt{P_\delta}r}{(1 + \sqrt{P_\delta}r)^2}.$$

Finally, there exist $P_\infty > 0$, $U_\infty > 0$, $\Theta_\infty > 0$ such that as $r \to +\infty$:

$$P(r) = P_\infty + O\left(\frac{1}{r^2}\right),$$

$$U(r) = \frac{U_\infty}{r} + O\left(\frac{1}{r^2}\right),$$

$$\Theta(r) = \frac{\Theta_\infty}{r^2} + O\left(\frac{1}{r^4}\right).$$

**Remark 2.5.** Note that compared to Theorem 2.4, the choice of $\Theta_0$ is not free.

**Proof.** We proof is similar to that of Theorem 2.4 above. Therefore we only outline the parts of the argument that differ, namely the derivation of the integro-differential equation, and the proof of local existence.

**Integro-differential formulation:**

We consider the following initial conditions:

$$P(0) = 0, \quad U(0) = 0, \quad U'(0) = A, \quad \Theta(0) = \Theta_0 := \frac{2\mu_0 + d\lambda_0}{R}A, \quad \Theta'(0) = 0.$$

First, we integrate the equation on $P$, and obtain for some $\delta > 0$:

$$P(r) = e^{V(r) - V(\delta)} P(\delta), \quad (2.18)$$

where

$$V(r) - V(\delta) = \int_\delta^r \frac{U' + \frac{d-1}{2r_1}U}{\frac{1}{2}r_1 - U}dr_1.$$
The equation on $U$ reads:

$$-\frac{1}{2}(rPU)'+(PU^2)' + \frac{d-1}{r}PU^2 - (2\mu_0 + \lambda_0)P \left( U' + \frac{d-1}{r}U \right)' + (PR\Theta)' = (2\mu_0 + \lambda_0)P'U' + \lambda_0P'd - \frac{1}{r}U.$$  

We integrate the previous expression, perform an integration by parts and divide by $P$ to obtain:

$$-\frac{1}{2}rU + U^2 + P^{-1}\int_0^r \frac{d-1}{r_1}PU^2dr_1 - (2\mu_0 + \lambda_0)(U' + \frac{d-1}{r}U) + R\Theta = -2\mu_0P^{-1}\int_0^r P'd - \frac{1}{r_1}Udr_1.$$  

The necessity of the relation between $A$ and $\Theta_0$ can be seen in this equation. Since we expect the behavior $U(r) \approx Ar$ near 0, equating the higher order terms on each side yields

$$-(2\mu_0 + \lambda_0)(U' + \frac{d-1}{r}U) + R\Theta + 2\mu_0 P^{-1}\int_0^r P'd - \frac{1}{r_1}Udr_1 \approx -(2\mu_0 + d\lambda_0)A + R\Theta_0.$$  

This directly gives $-(2\mu_0 + d\lambda_0)A + R\Theta_0 = 0$.

Now we multiply by $r^{d-1}e^{W(r)}$ (where $W(r) = \frac{r^2}{4(2\mu_0 + \lambda_0)}$) and obtain

$$-(2\mu_0 + \lambda_0)\left( r^{d-1}e^{W(r)}U \right)' = -r^{d-1}e^{W(r)} \left( R\Theta + U^2 + P^{-1}\int_0^r \frac{d-1}{r_1}PU^2dr_1 ight. 
+ 2\mu_0 P^{-1}\left. \int_0^r P'd - \frac{1}{r_1}Udr_1 \right).$$  

We conclude with the integro-differential satisfied by $U$:

$$U(r) = \frac{r^{1-d}}{2\mu_0 + \lambda_0} \int_0^r e^{W(r_1) - W(r)} r_1^{d-1}F_U(r_1)dr_1,$$

where

$$F_U(r) = R\Theta + U^2 + P^{-1}\int_0^r \frac{d-1}{r_1}PU^2dr_1 + 2\mu_0 P^{-1}\int_0^r P'd - \frac{1}{r_1}Udr_1.$$  

For $\Theta$, we do not detail the argument since it is similar to the above. We get

$$\Theta(r) = (d - 2)r^{2-d} \int_0^r r_1^{d-3}e^{-Z(r)+Z(r_1)}dr_1 \Theta_0 + \frac{r^{2-d}}{\kappa} \int_0^r r_1^{d-2}e^{Z(r_1)-Z(r)}F_\Theta(r_1)dr_1 - \frac{U^2}{2C_V},$$

where

$$Z(r) := \frac{C_V}{2\kappa} \int_0^r \tilde{r}P(\tilde{r})d\tilde{r},$$

$$F_\Theta(r) := UP \left( U^2 + C_V\Theta \right) + UPR\Theta + \frac{d-2}{r} \int_0^r \left( UP \left( \frac{U^2}{2} + C_V\Theta \right) + UPR\Theta \right)dr_2 
+ \left( \frac{\kappa}{C_V} - (2\mu_0 + \lambda_0) \right) \frac{(U^2)'}{2} + \frac{\kappa}{C_V} \frac{d-2}{2r}U^2.$$
We have the following estimates:
\[
\frac{2 \mu_0 + \lambda_0}{r} \int_0^r \left( (2 - d) r_1^{-d+1} P + P r_1^{-d+2} U' + \frac{d-1}{r} U \right) (r_1^{d-1} U U') r_1 dr_1
\]
\[
- \lambda \frac{d-1}{r} U^2 + \lambda_0 \frac{d-1}{r} \int_0^r (r_1^{-d+2} P)' (r_1^{d-2} U^2) dr_1
\]
\[
+ \frac{(2 \mu_0 + \lambda_0)}{r} \int_0^r r_1 P' U U' dr_1 + \frac{(d-1) \lambda_0}{r} \int_0^r P' U^2 dr_1.
\]

**Local existence:**
Fix \( \varepsilon > 0 \) such that \( A \ll \varepsilon \ll 1 \). Assume that \( \delta \) satisfies \( \delta^6 / \varepsilon \ll 1 \). We define the following norm (slightly different from the above):
\[
\| (U, \Theta) \|^\delta := \sup_{0 < r < \delta} \left[ r^{-1} |U(r)| + r^{1-\varepsilon} \left( \frac{U}{r} \right)' + \Theta(r) + r^{-1} |\Theta'(r)| \right].
\]

We apply the fixed point argument with the above norm in the space
\[
B_{\varepsilon^\delta}((Ar, \Theta_0), A/2) \cap \left\{ \lim_{r \to 0} \frac{U(r)}{r} = A, \lim_{r \to 0} \Theta(r) = \Theta_0 \right\}.
\]

**Stabilization:** We have the following estimates:
\[
|F_U(r) - d(2 \mu_0 + \lambda_0) A| \lesssim A \left( (R + A) \delta^2 + \mu_0 \frac{\delta^6}{\varepsilon} \right),
\]
\[
|F_\Theta(r)| \lesssim \left( P_\delta A^2 \delta^2 + (P_\delta (C_V + R + 2 \mu_0 + \lambda_0) + \frac{\kappa}{C_V} + 2 \mu_0 + \lambda_0) A \right) Ar,
\]

where we used that \( \Theta_0 = \frac{1}{R} (2 \mu_0 + d \lambda_0) A \). The implicit constants do not depend on the parameters.

Denote \((\hat{U}, \hat{\Theta}) = \Phi((Ar, \Theta_0))\).

We can deduce from the above that
\[
|\hat{U} - Ar| \lesssim Ar \left( \delta^2 + \frac{\delta^6}{\varepsilon} \right), \quad |\hat{\Theta} - \Theta_0| \lesssim (A^2 + P_\delta A)r^2, \quad |\hat{\Theta}'| \lesssim (A^2 + P_\delta A)r,
\]

where the implicit constants depend on the parameters.

For the second \( U \) -part of the norm we can write, using that \( P' > 0 \) on \((0, \delta)\):
\[
|F_U'(r)| \lesssim |\Theta'| + |UU'| + \left| \left( P^{-1} \int_0^r P U^2 r_1^dr_1 \right)' \right|
\]
\[
+ \frac{A}{r} P^{-1} \int_0^r P' \left| \frac{(d-1) U(r)}{r_1} \right| - \frac{(d-1) U(r)}{r} \right| dr_1
\]
\[
\lesssim ARr + A^2 r + \mu_0 \frac{A^2}{\varepsilon} r^{\varepsilon - 1}.
\]

We deduce the following estimate as we did in Sect. 2.1:
\[
\left| \left( \frac{\hat{U}}{r} \right)' \right| \lesssim Ar^{-1+\varepsilon} \left( \delta^2 - \varepsilon + \frac{A}{\varepsilon} \right).
\]

**Contraction:** Let \((U_i, \Theta_i) \in B_{\varepsilon^\delta}((Ar, \Theta_0), A/2)\). Denote \( D := \| (U_1, \Theta_1) - (U_2, \Theta_2) \|^\delta \).

Finally, let \((\bar{U}_i, \bar{\Theta}_i) := \Phi((U_i, \Theta_i))\).
We have, arguing as in the stabilization part of the proof:

\[
|\tilde{U}_1(r) - \tilde{U}_2(r)| \lesssim Dr\left(\delta^2 + \frac{\delta^e}{\varepsilon}\right),
\]

\[
\left|\left(\frac{\tilde{U}_1}{r}\right)' - \left(\frac{\tilde{U}_2}{r}\right)'ight| \lesssim Dr^{-1+\varepsilon}\left(\delta^{2-e} + \frac{A^2}{\varepsilon}\right),
\]

\[
|\Theta_1(r) - \Theta_2(r)| \lesssim ADr^2,
\]

\[
|\Theta'_1(r) - \Theta'_2(r)| \lesssim ADr.
\]

The proofs of global existence and asymptotic behavior of the solutions are done similarly as in Sect. 2.1, therefore details are omitted. □

3. Non-existence of Shrinkers

We now investigate shrinker-type solutions to the system \((\text{cNS})\), that is solutions of the type \((1.4)\).

Plugging the corresponding ansatz into the system, we obtain the following ODEs satisfied by the profiles \(P\), \(U\) and \(\Theta\):

\[
\left\{
\begin{align*}
\frac{1}{2} r P' + P'U + P \left( U' + \frac{d-1}{r} U \right) &= 0, \\
\frac{1}{2} PU + \frac{1}{2} r (PU)' + (PU^2)' + \frac{d-1}{r} PU^2 + (PR\Theta)' &= (2\mu_0 + \lambda_0) P^\alpha \left( U'' + \frac{d-1}{r} U' - \frac{d-1}{r^2} U \right) \\
&\quad - (2\mu_0 + \lambda_0) \alpha \frac{U' + \frac{d-1}{r} U}{\frac{1}{2} r + U} P^\alpha U' - \lambda_0 \alpha \frac{U' + \frac{d-1}{r} U}{\frac{1}{2} r + U} P^\alpha \frac{d-1}{r} U, \\
P \left( \frac{U^2}{2} + C_V \Theta \right) + \frac{1}{2} r \left( P \left( \frac{U^2}{2} + C_V \Theta \right) \right)' + \left( UP \left( \frac{U^2}{2} + C_V \Theta \right) + U PR\Theta \right)' &= (2\alpha \mu_0 P^\alpha \left( (U')^2 + \frac{d-1}{r^2} U^2 \right) + \lambda_0 P^\alpha \left( U' + \frac{d-1}{r} U \right)^2 \\
&\quad + (2\mu_0 + \lambda_0) P^\alpha \left( U' + \frac{d-1}{r} U' - \frac{d-1}{r^2} U \right) U \\
&\quad - (2\mu_0 + \lambda_0) \alpha \frac{U' + \frac{d-1}{r} U}{\frac{1}{2} r + U} P^\alpha U' U - \lambda_0 \alpha \frac{U' + \frac{d-1}{r} U}{\frac{1}{2} r + U} P^\alpha \frac{d-1}{r} U^2.
\end{align*}
\right.
\]

The main result of this section is that \((3.1)\) does not have non trivial small solutions.

As mentioned above, given the dependence laws that we have adopted for the Lamé coefficients, it is natural to restrict to solutions that exhibit cavitation. More precisely, we will make the following mild assumptions on the density profile \(P\) throughout this section:

- \(\exists \varepsilon, P_\varepsilon > 0, \forall r \geq \varepsilon, P(r) \geq P_\varepsilon\)
• \( \exists \Lambda > 0, \forall r \in (0, \varepsilon), \frac{r \int_0^r P^{1-\alpha}(r_1)dr_1}{\int_0^r P^{1-\alpha}(r_1)r_1dr_1}, \frac{\int_0^r P^\alpha}{\int_0^r P^{1-\alpha}} \leq \Lambda. \) (3.2)

Note that the functions \( P \) constructed in Sect. 2 satisfy this condition. Since the forward and backward systems are essentially equivalent near the origin, a shrinker exhibiting cavitation would be expected to behave similarly near 0.

The main theorem of this section is:

**Theorem 3.1.** Let \( d \geq 3 \) and \( 0 < \alpha < 1 \). Fix \( (C_V, \kappa, R, \mu_0, \lambda_0) \in (0, \infty)^5 \) such that \( C_V \leq \frac{\kappa}{2\mu_0+\lambda_0} \).

Assume that the density function \( P \) satisfies the above condition (3.2).

Then there exists a constant \( C(C_V, \kappa, R, \mu_0, \lambda_0, \varepsilon, \Lambda) := C > 0 \) such that if

\[
\sup_{r > 0} \left( \langle r \rangle^2 \Theta + P^{1-\alpha} + \left| \frac{U}{r\Theta} \right| \right) + \sup_{r > \varepsilon} \left| \frac{U'}{\Theta r'} \right| < C,
\]

then \( U \equiv \Theta \equiv 0, P = \text{Constant} \).

**Remark 3.2.** The assumption \( d \geq 3 \) comes from the fact that we use Hardy’s inequality in the proof.

**Remark 3.3.** Note that by scaling we expect the behavior \( U(r) \sim \frac{U_\infty}{r} \) and \( \Theta \sim \frac{\Theta_\infty}{r^2} \) at infinity. This makes the assumptions in the theorem critical.

**Remark 3.4.** The proof can be adapted with minor changes when \( \alpha = 1 \), or when the Lamé coefficients are not density dependent. For example this method could be applied in the case where \( C_V \geq \frac{\kappa}{2\mu_0+\lambda_0} \). In this case the smallness condition would be written in terms of \( \sup_{r > \varepsilon} \left| \frac{\Theta}{rU} \right| \) and \( \sup_{r > \varepsilon} \left| \frac{\Theta'}{rU'} \right| \).

**Remark 3.5.** Note that our smallness assumptions imply that there exists \( 0 < b \ll 1 \) such that

\[
\sup_{r > 0} \left| \frac{U}{r} \right| \leq b.
\]

**Remark 3.6.** We notice that for any \( r > 0 \), \( C_V \leq \frac{1}{2} \frac{\kappa}{2\mu_0+\lambda_0} \frac{1}{P\alpha} \), given the smallness assumption on \( P \).

The remainder of this section is dedicated to the proof of Theorem 3.1.

**3.1. Set-up.** Let us recall \( \mu = \mu_0 P^\alpha, \lambda = \lambda_0 P^\alpha \). We start by writing the equations on \( \Theta \) and \( U \) from (3.1) in \( 2 \times 2 \) matrix form:

\[
\begin{pmatrix}
C_V P(\frac{\xi}{2} + U) \frac{d}{dr} & PR \Theta \frac{d}{dr} \\
PR \frac{d}{dr} & P(\frac{\xi}{2} + U) \frac{d}{dr}
\end{pmatrix}
\begin{pmatrix}
\Theta \\
U
\end{pmatrix}
-
\begin{pmatrix}
\kappa(\frac{d-1}{r} \frac{d}{dr} - \frac{d^2}{dr^2}) & 0 \\
0 & (2\mu + \lambda_0)(\frac{d-1}{r} \frac{d}{dr} - \frac{d^2}{dr^2})
\end{pmatrix}
\begin{pmatrix}
\Theta \\
U
\end{pmatrix}
\]
\[
\begin{align*}
&= \left( -P(r) \Theta(r) \left( C_V + R \frac{d-1}{r} U \right) + 2\mu \left( (U')^2 + \left( \frac{d-1}{r} U \right)^2 \right) + \lambda \left( U' + \frac{d-1}{r} U \right)^2 \right) \\
&\quad + \left( -\frac{1}{2} P(r) U(r) - (2\mu + \lambda) \frac{d-1}{r^2} U \right) \\
&\quad + \left( -(2\mu + \lambda) \alpha \frac{U' + \frac{d-1}{r} U}{\frac{1}{2} r + U} U' U - \lambda \alpha \frac{U' + \frac{d-1}{r} U}{\frac{1}{2} r + U} \frac{d-1}{r} U^2 \right) \\
&\quad - \left( 2\mu + \lambda \alpha \frac{U' + \frac{d-1}{r} U}{\frac{1}{2} r + U} U' - \lambda \alpha \frac{U' + \frac{d-1}{r} U}{\frac{1}{2} r + U} \frac{d-1}{r} U \right) \\
\end{align*}
\]

We denote

\[
B = \begin{pmatrix} \kappa & 0 \\ 0 & 2\mu + \lambda \end{pmatrix},
\]

\[
\tilde{A}(r) = \begin{pmatrix} C_V P \left( \frac{d-1}{2} U \right) PR \Theta \\ PR \end{pmatrix}.
\]

The previous equation can then be written

\[
\tilde{A}(r) \frac{d}{dr} \begin{pmatrix} \Theta \\ U \end{pmatrix} - B \left( \frac{d-1}{r} - \frac{d^2}{dr^2} \right) \begin{pmatrix} \Theta \\ U \end{pmatrix} = \left( -P(r) \Theta(r) \left( C_V + R \frac{d-1}{r} U \right) + 2\mu \left( (U')^2 + \left( \frac{d-1}{r} U \right)^2 \right) + \lambda \left( U' + \frac{d-1}{r} U \right)^2 \right) \\
\quad + \left( -\frac{1}{2} P(r) U(r) - (2\mu + \lambda) \frac{d-1}{r^2} U \right) \\
\quad + \left( -(2\mu + \lambda) \alpha \frac{U' + \frac{d-1}{r} U}{\frac{1}{2} r + U} U' U - \lambda \alpha \frac{U' + \frac{d-1}{r} U}{\frac{1}{2} r + U} \frac{d-1}{r} U^2 \right) \\
\quad - \left( 2\mu + \lambda \alpha \frac{U' + \frac{d-1}{r} U}{\frac{1}{2} r + U} U' - \lambda \alpha \frac{U' + \frac{d-1}{r} U}{\frac{1}{2} r + U} \frac{d-1}{r} U \right).
\]

Hence after multiplication by \( B^{-1} \) and \( \exp \left( - \int_0^r B^{-1} \tilde{A}(r') dr \right) \), we obtain:

\[
-\nabla \cdot \left( \exp \left( - \int_0^r B^{-1} \tilde{A}(r') dr \right) \nabla \begin{pmatrix} \Theta \\ U \end{pmatrix} \right) = \exp \left( - \int_0^r B^{-1} \tilde{A}(r') dr \right) \\
\times \left( -P(r) \Theta(r) \left( C_V + R \frac{d-1}{r} U \right) + 2\mu \left( (U')^2 + \left( \frac{d-1}{r} U \right)^2 \right) + \lambda \left( U' + \frac{d-1}{r} U \right)^2 \right) \\
\quad + \exp \left( - \int_0^r B^{-1} \tilde{A}(r') dr \right) \\
\times \left( -(2\mu + \lambda) \alpha \frac{U' + \frac{d-1}{r} U}{\frac{1}{2} r + U} U' U - \lambda \alpha \frac{U' + \frac{d-1}{r} U}{\frac{1}{2} r + U} \frac{d-1}{r} U^2 \right) \\
\quad \times \left( -\alpha \frac{U' + \frac{d-1}{r} U}{\frac{1}{2} r + U} U' - \lambda \alpha \left( U' + \frac{d-1}{r} U \right) \frac{d-1}{r} U \right). (3.3)
\]

Denote from now on
exp(A(r)) := \exp \left( - \int_0^r B^{-1} \tilde{A}(r') dr' \right) 
:= \exp \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right).

3.2. Basic properties of the matrix A. In this section we collect some elementary facts about the matrix A.

First note that we have explicit expressions for its coefficients:
\[ \alpha = -\frac{C}{\kappa} \int_0^r P(r') \left( \frac{r'^2}{2} + U \right) dr', \]
\[ \beta = -\frac{R}{\kappa} \int_0^r P \Theta_1 dr', \]
\[ \gamma = -R \int_0^r \frac{P}{2\mu + \lambda} dr', \]
\[ \delta = -\int_0^r P(r') \left( \frac{r'^2}{2} + U \right) dr'. \]

**Remark 3.7.** Given the assumption \( CV \leq \frac{\kappa}{2\mu + \lambda} \), we have \( \delta < \alpha \).

The next lemma shows that the matrix \( A \) can be diagonalized.

**Lemma 3.8.** We have the following decomposition for \( A \):
There exists a real-valued \( 2 \times 2 \) matrix \( Q \), with diagonal elements equal to one, such that:
\[ A(r) = Q^{-1}(r) \begin{pmatrix} -\lambda_{\text{min}} & 0 \\ 0 & -\lambda_{\text{max}} \end{pmatrix} Q(r), \]
where \( -\lambda_{\text{max}} < -\lambda_{\text{min}} \) denote the two real valued eigenvalues of \( A \).

**Proof.** The characteristic polynomial of that matrix \( A \) is \( X^2 - (\delta + \alpha) X + \alpha \delta - \beta \gamma \).
Its discriminant is
\[ \Delta = (\delta + \alpha)^2 - 4(\alpha \delta - \beta \gamma) = (\alpha - \delta)^2 + 4\beta \gamma. \]

Given that both \( \beta \) and \( \gamma \) are negative, \( \Delta > 0 \).

From this we can deduce the two eigenvalues:
\[ -\lambda_{\text{max}} := \frac{1}{2} (\alpha + \delta - \sqrt{\Delta}), \quad -\lambda_{\text{min}} := \frac{1}{2} (\alpha + \delta + \sqrt{\Delta}). \]

We find the corresponding eigenvectors, and deduce that the matrix \( A(r) \) can then be diagonalized after introducing the matrix \( Q(r) \) defined as
\[ Q(r) = \begin{pmatrix} 1 & -\frac{\lambda_{\text{max}} - \delta}{\gamma} \\ \frac{\gamma}{\lambda_{\text{min}} - \delta} & 1 \end{pmatrix}, \]
\[ Q^{-1}(r) = \frac{1}{D} \begin{pmatrix} 1 & -\frac{\lambda_{\text{max}} - \delta}{\gamma} \\ \frac{\gamma}{\lambda_{\text{min}} - \delta} & 1 \end{pmatrix}, \]
where \( D := 1 - \frac{\lambda_{\text{max}} - \delta}{\lambda_{\text{min}} - \delta} \). \( \square \)
We write
\[ Q(r) = \begin{pmatrix} 1 & q_{12} \\ q_{21} & 1 \end{pmatrix}, \quad Q^{-1}(r) = \frac{1}{D} \begin{pmatrix} 1 & -q_{12} \\ -q_{21} & 1 \end{pmatrix}, \quad D = 1 - q_{12}q_{21}. \]

The following lemma justifies the fact that the matrix \( Q \) is a perturbation of the identity for large values of \( r \):

**Lemma 3.9.** We have the following estimates, valid for any \( r > 0 \):
\[
|q_{12}| \leq \frac{2(2\mu_0 + \lambda_0) R \Lambda}{\kappa (1/2 - b)} P_{1/2},
\]
\[
|q_{21}| \leq \frac{2R\Lambda}{(1/2 - b)r},
\]
\[
|D| \geq 1.
\]

**Proof.** We have
\[
|q_{12}| = \left| \frac{-\lambda_{\text{max}} - \delta}{\gamma} \right| = \frac{2\beta}{\alpha - \delta + \sqrt{\Delta}} \leq \frac{\beta}{\alpha - \delta}.
\]

Using Remark 3.6 and the condition (3.2), we obtain the desired bound.

Similarly, we obtain the bound on \( q_{21} \).

For the last estimate on \( D \), we notice that
\[
1 - \frac{\alpha - \delta - \sqrt{\Delta}}{\alpha - \delta + \sqrt{\Delta}} = 1 - \frac{-4\beta\gamma}{(\alpha - \delta + \sqrt{\Delta})^2} \geq 1,
\]
since both \( \beta \) and \( \gamma \) are negative. \( \square \)

In what follows, we will systematically decompose \( Q \) as the sum of the identity matrix, and the matrix of its off-diagonal terms that will be treated like an error term. The following basic computation will be used repeatedly in the sequel:

**Lemma 3.10.** We have, for real numbers \( a, b, c, d \):
\[
\begin{aligned}
\left\{ Q^{-1} \begin{pmatrix} e^{-\lambda_{\text{min}}} & 0 \\ 0 & e^{-\lambda_{\text{max}}} \end{pmatrix} Q \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right\} \\
= \left\{ \left( 1 + \frac{q_{12}q_{21}}{D} \right) e^{-\lambda_{\text{min}}} - \frac{q_{12}q_{21}}{D} e^{-\lambda_{\text{max}}} \right\} ac \\
+ \left\{ \left( 1 + \frac{q_{12}q_{21}}{D} \right) e^{-\lambda_{\text{max}}} - \frac{q_{12}q_{21}}{D} e^{\lambda_{\text{min}}} \right\} bd \\
+ (e^{-\lambda_{\text{min}}} - e^{-\lambda_{\text{max}}} \frac{q_{12}q_{21}}{D} bc + (-e^{-\lambda_{\text{min}}} + e^{-\lambda_{\text{max}}} \frac{q_{21}}{D} ad).
\end{aligned}
\]

We also have
\[
\frac{1}{2} \leq 1 + \frac{q_{12}q_{21}}{D} \leq 1, \quad 0 < -q_{12}q_{21} \leq 1.
\]

We end this section with rough bounds that are relevant to treat the case where \( r \) is close to 0:
Lemma 3.11. We have the following estimate for any $r > 0$:

$$|q_{12}| \leq \sqrt{\frac{2\mu_0 + \lambda_0}{\kappa}} \sup_{r > 0} P^\alpha \Theta.$$  

Moreover, we have

$$\sup_{0 < r < \varepsilon} e^{\sqrt{\Delta}} \leq 2, \quad \sup_{0 < r < \varepsilon} e^{\lambda_{\min}} \leq 2.$$  

Proof. We write that

$$|q_{12}| = \left| \frac{-\lambda_{\max} - \delta}{\gamma} \right| = \frac{2|\beta|}{\alpha - \delta + \sqrt{(\alpha - \delta)^2 + 4\beta \gamma}} \leq \frac{\beta}{\gamma} \leq \sqrt{\frac{2\mu_0 + \lambda_0}{\kappa}} \sup_{r > 0} P^\alpha \Theta.$$  

For the last two estimates, we use the following crude bounds:

$$\sqrt{\Delta} \leq \sqrt{\frac{4R^2}{\kappa} \frac{\varepsilon^2}{2\mu_0 + \lambda_0} \sup_{r > 0} P \Theta \sup_{r > 0} P^{1-\alpha} + \varepsilon^4 (\sup_{r > 0} P^{1-\alpha})^2},$$  

$$\lambda_{\min} \leq \frac{\varepsilon^2}{2\mu_0 + \lambda_0} \sup_{r > 0} P^{1-\alpha}.$$  

The estimates then follow from the smallness assumptions.  

3.3. Weighted energy estimate. Now we take the inner product of Eq. (3.3) with $(\Theta; U)^T$, and integrate by parts in the left-hand side.

Then we split the expression between small and large values of $r$. Overall we get

$$\text{LHS} = \int_0^\varepsilon \langle \exp(A(r)) (\Theta', U') \rangle dx$$  

$$+ \int_\varepsilon^\infty \langle \exp(A(r)) (\Theta', U') \rangle dx$$  

$$:= \text{LHS}_1 + \text{LHS}_2.$$  

We prove the following estimate on the left-hand side:

Lemma 3.12. We have

$$\text{LHS} \geq \int_0^\varepsilon e^{-\lambda_{\max}} (\Theta'^2 + U'^2) dx + \frac{1}{4} \left( \int_\varepsilon^{+\infty} e^{-\lambda_{\min}} \Theta'^2 dx + \int_\varepsilon^{+\infty} e^{-\lambda_{\max}} U'^2 dx \right).$$  

Proof. Bound on $\text{LHS}_1$:

In this case the bound is straightforward:

$$\text{LHS}_1 \geq \int_0^\varepsilon e^{-\lambda_{\max}} (U'^2 + \Theta'^2) dx.$$
Bound on $\text{LHS}_2$:

As announced above, in this range, we use the diagonalization of $A$ and we write the matrices $Q$ and $Q^{-1}$ as the sum of the identity and an error term. This yields, using Lemma 3.10:

\[
\text{LHS}_2 = \int_{\varepsilon}^{\infty} \left\{ \left( 1 + \frac{q_{12}q_{21}}{D} \right) e^{-\lambda_{\min}} - \frac{q_{12}q_{21}}{D} e^{-\lambda_{\max}} \right\} \Theta^2 dx
+ \int_{\varepsilon}^{\infty} \left\{ \left( 1 + \frac{q_{12}q_{21}}{D} \right) e^{-\lambda_{\max}} - \frac{q_{12}q_{21}}{D} e^{-\lambda_{\min}} \right\} U'^2 dx
+ \int_{\varepsilon}^{\infty} \left( e^{-\lambda_{\min}} - e^{-\lambda_{\max}} \right) \frac{q_{12} - q_{21}}{D} U' \Theta' d\chi.
\]

We notice that

\[
\frac{1}{2} \leq 1 + \frac{q_{12}q_{21}}{D} \leq 1, \quad -q_{12}q_{21} > 0,
\]

which makes the first two terms the main contribution. The last term is treated as an error.

Using the bounds from Lemma 3.9 and the smallness assumptions, we write that

\[
\left| \int_{\varepsilon}^{\infty} \frac{q_{12}}{D} \left| e^{-\lambda_{\min}} - e^{-\lambda_{\max}} \right| U' \Theta' d\chi \right|
\leq \sup_{r \geq \varepsilon} \left| \frac{U'}{r \Theta'} \right| \frac{2(2\mu_0 + \lambda_0) R \Lambda \sup_{r \geq \varepsilon} P^\alpha (\Theta)}{\kappa (1/2 - b)} \int_{\varepsilon}^{+\infty} e^{-\lambda_{\min} \Theta'^2} d\chi
\leq \frac{1}{100} \int_{\varepsilon}^{+\infty} e^{-\lambda_{\min} \Theta'^2} d\chi,
\]

\[
\left| \int_{\varepsilon}^{\infty} \frac{q_{21}}{D} \left| e^{-\lambda_{\min}} - e^{-\lambda_{\max}} \right| \Theta' U' d\chi \right|
\leq \sup_{r \geq \varepsilon} \left| \frac{U'}{r \Theta'} \right| \frac{2 R \Lambda}{1/2 - b} \int_{\varepsilon}^{+\infty} e^{-\lambda_{\min} \Theta'^2} d\chi
\leq \frac{1}{100} \int_{\varepsilon}^{+\infty} e^{-\lambda_{\min} \Theta'^2} d\chi.
\]

The other terms are easier to bound, therefore we omit the details. \(\square\)

Now we move on to the right-hand side. We show that

**Lemma 3.13.** We have

\[
\text{RHS} \leq \frac{1}{20} \left( \int_{\mathbb{R}^d} e^{-\lambda_{\min} \Theta'^2} d\chi + \int_{\mathbb{R}^d} e^{-\lambda_{\max} U'} d\chi - \int_{\mathbb{R}^d} e^{-\lambda_{\max}} \frac{P}{2\mu + \lambda} U^2 d\chi \right)
- \frac{C_V}{4\kappa} \int_{\mathbb{R}^d} e^{-\lambda_{\min} P \Theta'^2} d\chi.
\]

**Proof.** We start by decomposing the right-hand side into three parts:

\[
\left\{ \text{RHS of (3.3)}, \left( \frac{\Theta}{U} \right) \right\}
\]
= - \left\{ \exp \left( A(r) \left( \begin{array}{c} 0 \\ \frac{d-1}{r^2} U \end{array} \right) \right), \left( \begin{array}{c} \Theta \\ U \end{array} \right) \right\} \\
- \left\{ \exp \left( A(r) \left( \frac{C_{\nu} P(r) \Theta(r)}{\kappa} \right) \right), \left( \begin{array}{c} \Theta \\ U \end{array} \right) \right\} \\
+ \left\{ \exp \left( A(r) \left( \frac{N_1(P, \Theta, U)}{N_2(P, \Theta, U)} \right) \right), \left( \begin{array}{c} \Theta \\ U \end{array} \right) \right\} \\
:= \text{RHS}_1 + \text{RHS}_2 + \text{RHS}_3,

where

\begin{align*}
N_1(P, \Theta, U) &= - \frac{R}{\kappa} P \Theta \frac{d-1}{r} U + 2 \frac{\mu}{\kappa} \left( (U')^2 + \left( \frac{d-1}{r} U \right)^2 \right) + \frac{\lambda}{\kappa} \left( U' + \frac{d-1}{r} U \right)^2 \\
&\quad - \frac{2 \mu + \lambda}{\kappa} \frac{U'}{2 r + U} U' U - \frac{\lambda}{\kappa} \frac{U'}{2 r + U} U d - \frac{1}{r} U^2, \\
N_2(P, \Theta, U) &= - \frac{\lambda}{2 \mu + \lambda} \frac{U'}{2 r + U} U' U - \frac{\lambda}{2 \mu + \lambda} \frac{U'}{2 r + U} U d - \frac{1}{r} U.
\end{align*}

Next we estimate \text{RHS}_1, \text{RHS}_2 and \text{RHS}_3. The result from Lemma 3.13 will then follow.

**Bound on \text{RHS}_1:**

As we did above, we use the elementary computation from Lemma 3.10 to obtain the expression:

\begin{align*}
\text{RHS}_1 &= - \int_{\mathbb{R}^d} \left\{ \left( 1 + \frac{q_{12} q_{21}}{D} \right) e^{-\lambda_{\max}} - \frac{q_{12} q_{21}}{D} e^{-\lambda_{\min}} \right\} \frac{d-1}{r^2} U^2 \, dx \\
&\quad - \int_{\mathbb{R}^d} \left( e^{-\lambda_{\min}} - e^{-\lambda_{\max}} \right) \frac{q_{12}}{D} \frac{d-1}{r^2} U \Theta \, dx.
\end{align*}

We know that the first term is negative. For the second term, we will distinguish between \( r \) large and \( r \) small. We introduce a cut-off function \( \chi \) such that:

- \( \chi = 1 \) on \([0, \varepsilon]\).
- \( \chi \) is supported on \([0, 2\varepsilon]\).
- \( |\chi'| \leq C 1_{\{\varepsilon \leq |x| \leq 2\varepsilon\}} \) for some numerical constant \( C \). Here \( 1_A \) denotes the characteristic function of the set \( A \).

Using this cut-off, we write:

\begin{align*}
|\text{(3.5)}| &\leq \int_{\mathbb{R}^d} \frac{|q_{12}|}{D} e^{-\lambda_{\min}} \frac{d-1}{r^2} - \frac{\chi^2 |U| \Theta}{2} \, dx \\
&\quad + \int_{\mathbb{R}^d} \frac{|q_{12}|}{D} e^{-\lambda_{\min}} \frac{d-1}{r^2} (1 - \chi^2) |U| \Theta \, dx.
\end{align*}

**Bound on (3.6):**

We write, using Lemma 3.11, Young’s inequality, as well as the mean value theorem, we obtain the bound:

\begin{align*}
|\text{(3.6)}| &\leq \frac{1}{2} \sqrt{\frac{2 \mu_0 + \lambda_0}{\kappa} \sup_{r > 0} P^\alpha \Theta \sup_{0 < r < \varepsilon} (e^{2\sqrt{\lambda}}) \int_0^{+\infty} e^{-\lambda_{\max}} \chi^2 \left[ \frac{(d-1)U^2}{r^2} + \frac{(d-1)^2}{r^2} \right] \, dx.
\end{align*}
The first part containing $U$ can be absorbed into (3.4) by requiring $(2\mu_0 + \lambda_0) \sup_{r > 0} P^a \Theta \ll \kappa$. For the part containing $\Theta$, we use that $-\lambda_{\text{max}} \leq 0$ as well as Hardy’s inequality, and the properties of the cut-off $\chi$ to write:

$$\int_{\mathbb{R}^d} e^{-\lambda_{\text{max}}} \chi^2 \frac{(d-1)\Theta^2}{r^2} \, dx \leq C' \int_{\mathbb{R}^d} (\chi(\Theta))^2 \, dx$$

$$\leq 2C' \int_{\mathbb{R}^d} \Theta^2 \, dx + 2C' \int_{\mathbb{R}^d} \chi^2 \Theta^2 \, dx$$

$$\leq 2C' \sup_{0 < r < \varepsilon} e^{\lambda_{\text{max}}} \int_{\mathbb{R}^d} e^{-\lambda_{\text{max}}} \Theta^2 \, dx$$

$$+ 2C' C^2 \sup_{0 < r < \varepsilon} e^{\lambda_{\text{min}}} \frac{1}{P \varepsilon} \int_{\mathbb{R}^d} e^{-\lambda_{\text{min}}} P \Theta^2 \, dx,$$

where $C'$ denotes the constant in Hardy’s inequality. Imposing the smallness condition

$$\sqrt{2\mu_0 + \lambda_0} \sup_{r > 0} P^a \Theta \ll \frac{P \varepsilon C_V}{\kappa}, \quad P \varepsilon$$

the above estimates give acceptable contributions in view of the desired estimate in Lemma 3.13.

**Bound on (3.7):**

Using Lemma 3.11, we obtain:

$$| (3.7) | \leq \sup_{r > \varepsilon} \frac{U}{r \Theta} \left( \frac{2 R \Lambda (2\mu_0 + \lambda_0) \sup_{r > 0} P^a \Theta}{\kappa (1/2 - b)} \right) \int_{\mathbb{R}^d} (1 - \chi^2) e^{-\lambda_{\text{min}}} \frac{d - 1}{r^2} \Theta^2 \, dx$$

$$\leq \frac{2 R \Lambda (2\mu_0 + \lambda_0) \sup_{r > 0} P^a \Theta}{\varepsilon^2 \kappa P \varepsilon (1/2 - b)} \sup_{r > \varepsilon} \left( \frac{U}{r \Theta} \right) \int_{\mathbb{R}^d} e^{-\lambda_{\text{min}}} P \Theta^2 \, dx,$$

which will be controlled by a part of RHS$_2$. We can conclude with the smallness assumptions.

**Bound on RHS$_2$:**

We use a similar reasoning for this part. With Lemma 3.10, we obtain:

$$\text{RHS}_2 = -\frac{C_V}{\kappa} \int_{\mathbb{R}^d} \left\{ \left( \frac{q_{12} q_{21}}{D} \right) e^{-\lambda_{\text{min}}} - \frac{q_{12} q_{21}}{D} e^{-\lambda_{\text{max}}} \right\} P \Theta^2 \, dx$$

$$- \int_{\mathbb{R}^d} \frac{P}{2(2\mu + \lambda)} \left\{ \left( \frac{q_{12} q_{21}}{D} \right) e^{-\lambda_{\text{max}}} - \frac{q_{12} q_{21}}{D} e^{-\lambda_{\text{min}}} \right\} U^2 \, dx$$

$$+ \int_{\mathbb{R}^d} \left\{ (e^{-\lambda_{\text{min}}} - e^{-\lambda_{\text{max}}}) \frac{1}{D} \left( \frac{q_{12}}{2(2\mu + \lambda)} - \frac{q_{21} C_V}{\kappa} \right) \right\} P U \Theta \, dx.$$ (3.10)

Now we estimate (3.10).

We only estimate the terms with the slowest exponential, the other two terms can clearly be estimated in the same way.

As we did previously, we distinguish between $r$ small and $r$ large.

We start with the slowest exponential term in (3.10):

$$\int_{\mathbb{R}^d} e^{-\lambda_{\text{min}}} \frac{q_{12}}{D} \frac{1}{2\mu + \lambda} P U \Theta \, dx$$
\[ \int_0^\varepsilon e^{-\lambda_{\text{min}}} \frac{q_{12} P}{D} \frac{U}{2\mu + \lambda} \, dx \]  
(3.11)
\[ + \int_{\varepsilon}^{+\infty} e^{-\lambda_{\text{min}}} \frac{q_{12} P}{D} \frac{U}{2\mu + \lambda} \, dx, \]  
(3.12)
and we estimate both pieces separately.

In the case where \( r \) is small, we use Lemma 3.11 (more precisely the bound \(|q_{12}| \leq \sqrt{\frac{P}{\Theta}} \) in the proof), and get:

\[ |(3.11)| \leq \int_0^\varepsilon e^{-\lambda_{\text{min}}} \sqrt{\int_0^r \frac{P}{\Theta}} \frac{\sqrt{P}}{\sqrt{2\mu + \lambda}} |U| \sqrt{P} \, dx \]
\[ \leq \frac{1}{2} \int_0^\varepsilon e^{-\lambda_{\text{min}}} \sqrt{\int_0^r \frac{P}{\Theta}} \left( \frac{P}{2\mu + \lambda} U^2 + P \Theta^2 \right) \, dx \]
\[ \leq \frac{1}{2} \int_0^\varepsilon e^{-\lambda_{\text{min}}} \left( \Lambda \sup_{r \geq 0} \Theta \right) \frac{P}{\kappa} \Theta^2 \, dx \]
\[ + \frac{1}{2} \sup_{0 < r < \varepsilon} \lambda_{\text{max}} \int_0^\varepsilon e^{-\lambda_{\text{min}}} \sqrt{\int_0^r \frac{P}{\Theta}} \left( \frac{P}{2\mu + \lambda} U^2 \right) \, dx. \]

The first term gives an acceptable contribution to the desired inequality from Lemma 3.13 by requiring the condition
\[ \sqrt{\frac{\Lambda \sup_{r \geq 0} \Theta}{\kappa}} \ll \frac{CV}{\kappa}. \]

The second term can be absorbed into (3.9) by a similar condition, replacing the right-hand side by 1.

Now we move on to the second part, using Lemma 3.9:

\[ |(3.12)| \leq \frac{2R \Lambda \sup_{r \geq 0} \Theta}{\kappa (1/2 - b)} \sup_{r \geq \varepsilon} \left| \frac{U}{r \Theta} \right| \int_{\mathbb{R}^d} e^{-\lambda_{\text{min}}} \Theta P \, dx, \]

and we can conclude using the smallness assumptions.

For the remaining term, we write
\[ \frac{C V}{\kappa} \int_{\mathbb{R}^d} \frac{q_{21}}{D} e^{-\lambda_{\text{min}}} P \Theta U \, dx \leq \frac{2 R \Lambda C V}{\kappa (1/2 - b)} \sup_{r > 0} \left| \frac{U}{r \Theta} \right| \int_{\mathbb{R}^d} e^{-\lambda_{\text{min}}} \Theta P \, dx. \]

Bound on RHS

All these terms are treated as error terms, and the proofs are easier or similar to the above. Therefore we only show how to estimate the main terms.

**Main contribution from \( N_1 \):**

For the first term, we simply use Remark 3.5 and write
\[ \int_{\mathbb{R}^d} \frac{R}{\kappa} e^{-\lambda_{\text{min}}} \Theta^2 \frac{d}{r} - 1 \frac{U}{r} \, dr \leq \frac{R(d - 1)b}{\kappa} \int_{\mathbb{R}^d} e^{-\lambda_{\text{min}}} \Theta^2 \, dr. \]

Next, we write
\[ \int_{\mathbb{R}^d} \frac{\mu}{\kappa} e^{-\lambda_{\text{min}}} U \Theta \, dx \leq \frac{\mu_0}{\kappa} \sup_{r > 0} \Theta \sup_{0 < r < \varepsilon} e^{2\sqrt{\Lambda}} \int_0^\varepsilon e^{-\lambda_{\text{max}}} U^2 \, dx \]
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\[ + \frac{\mu_0}{\kappa} \sup_{r > \varepsilon} \left| \frac{U'}{r \Theta'} \right| \sup_{r > 0} P^\alpha r^2 \Theta \int_\varepsilon^{+\infty} e^{-\lambda_{\min}} \Theta'^2 \, dx. \]

This is acceptable if we require \( \mu_0 \sup_{r > 0} P^\alpha r^2 \ll \kappa \).

Similarly

\[ \int_{\mathbb{R}^d} \frac{\mu}{\kappa} e^{-\lambda_{\min}} \frac{(d - 1) U^2}{r^2} \Theta \, dx \leq \frac{\mu_0}{\kappa} \sup_{r > 0} P^\alpha \Theta \sup_{0 < r < \varepsilon} e^{2\sqrt{\lambda_{\min}}} \int_0^\varepsilon e^{-\lambda_{\max}} \frac{(d - 1)^2 U^2}{r^2} \, dx \]

\[ + \frac{\mu_0}{\kappa} \sup_{r > \varepsilon} P^{\alpha - 1} \Theta \frac{U^2}{r^2 \Theta^2} \int_\varepsilon^{+\infty} e^{-\lambda_{\min}} P \Theta'^2 \, dx. \]

The other main terms are bounded in the same way.

**Main contribution from \( N_2 \):**

We write that

\[ \int_{\mathbb{R}^d} e^{-\lambda_{\max}} \frac{U}{r/2 + U} U'^2 \, dx \leq \frac{a b}{\frac{1}{2} - b} \int_{\mathbb{R}^d} e^{-\lambda_{\max}} U'^2 \, dx. \]

The other main terms are bounded in the same way. \( \square \)

**Conclusion of the proof of Theorem 3.1.** The result follows directly by putting together Lemmas 3.12 and 3.13. \( \square \)

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**References**

1. Bresch, D., Desjardins, B.: Weak Solutions for the Compressible Navier–Stokes Equations with Density Dependent Viscosities, Handbook of Mathematical Analysis in Mechanics of Viscous Fluids, 1437–1599. Springer, Cham (2018)
2. Bresch, D., Desjardins, B.: On the existence of global weak solutions to the Navier–Stokes equations for viscous compressible and heat conducting fluids, J. Math. Pures Appl. (9) 87(1), 57–90 (2007)
3. Bresch, D., Desjardins, B., Lin, C.-K.: On compressible fluid models: Korteweg, lubrication and shallow water systems. Commun. Partial Differ. Equ. 28(3–4), 843–868 (2003)
4. Bresch, D., Noble, P.: Mathematical justification of a shallow water model. Methods Appl. Anal. 14(2), 87–117 (2007)
5. Cannone, M., Planchon, F.: Self-similar solutions for Navier–Stokes equations in \( \mathbb{R}^3 \). Commun. Partial Differ. Equ. 21(1–2), 179–193 (1996)
6. Chikami, N., Danchin, R.: On the well-posedness of the full compressible Navier–Stokes system in critical Besov spaces. J. Differ. Equ. 258(10), 3435–3467 (2015)
7. Danchin, R.: Global existence in critical spaces for flows of compressible viscous and heat-conductive gases. Arch. Ration. Mech. Anal. 160(1), 1–39 (2001)
8. Feireisl, E.: Dynamics of Viscous Compressible Fluids. Oxford Lecture Series in Mathematics and Its Applications, vol. 26. Oxford University Press, Oxford (2004)
9. Feireisl, E., Novotný, A., Petzeltová, H.: On the existence of globally defined weak solutions to the Navier–Stokes equations. J. Math. Fluid Mech. 3(4), 358–392 (2001)
10. Gerbeau, J.-F., Perthame, B.: Derivation of viscous Saint–Venant system for laminar shallow water; numerical validation. Discrete Contin. Dyn. Syst. Ser. B I(1), 89–102 (2001)
11. Germain, P., Iwabuchi, T.: Self-similar solutions for compressible Navier–Stokes equations. arXiv:1903.09958
12. Germain, P., Iwabuchi, T., Léger, T.: Backward self-similar solutions for compressible Navier–Stokes equations. arXiv:1911.00339

13. Guo, Z., Jiang, S.: Self-similar solutions to the isothermal compressible Navier–Stokes equations. IMA J. Appl. Math. 71, 658–669 (2006)

14. Huang, X., Li, J.: Global classical and weak solutions to the three-dimensional full compressible Navier–Stokes system with vacuum and large oscillations. Arch. Ration. Mech. Anal. 227(3), 995–1059 (2018)

15. Jang, J., Masmoudi, N.: Well-posedness of compressible Euler equations in a physical vacuum. Commun. Pure Appl. Math. 68(1), 61–111 (2015)

16. Jia, H., Šverák, V.: Local-in-space estimates near initial time for weak solutions of the Navier–Stokes equations and forward self-similar solutions. Invent. Math. 196(1), 233–265 (2014)

17. Li, T., Chen, P., Xie, J.: Self-similar solutions of the compressible flow in one-space dimension. J. Appl. Math. 5, Art. ID 194704 (2013)

18. Leray, J.: Sur le mouvement d’un liquide visqueux emplissant l’espace. Acta Math. 63, 193–248 (1934)

19. Lions, P.-L.: Mathematical Topics in Fluid Mechanics. Vol. 2. Compressible Models. Oxford Lecture Series in Mathematics and Its Applications, vol. 10. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York (1998)

20. Matsumura, A., Nishida, T.: The initial value problem for the equations of motion of viscous and heat-conductive gases. J. Math. Kyoto Univ. 20(1), 67–104 (1980)

21. Nečas, J., Ružička, M., Šverák, V.: On Leray’s self-similar solutions of the Navier–Stokes equations. Acta Math. 176, 283–294 (1996)

22. Qin, Y., Su, X., Deng, S.: Remarks on self-similar solutions to the compressible Navier–Stokes equations of a 1D viscous polytropic ideal gas. Appl. Math. Sci. (Ruse) 2, 1493–1506 (2008)

23. Tsai, T.-P.: On Leray’s self-similar solutions of the Navier–Stokes equations satisfying local energy estimates. Arch. Ration. Mech. Anal. 143, 29–51 (1998)

24. Vasseur, A., Yu, C.: Existence of global weak solutions for 3D degenerate compressible Navier–Stokes equations. Invent. Math. 206(3), 935–974 (2016)

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