STABILITY OF THE FOCAL AND GEOMETRIC INDEX IN SEMI-RIEMANNIAN GEOMETRY VIA THE MASLOV INDEX

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ABSTRACT. We investigate the problem of the stability of the number of conjugate or focal points (counted with multiplicity) along a semi-Riemannian geodesic $\gamma$. For a Riemannian or a non spacelike Lorentzian geodesic, such number is equal to the intersection number (Maslov index) of a continuous curve with a subvariety of codimension one of the Lagrangian Grassmannian of a symplectic space. Such intersection number is proven to be stable in a large variety of circumstances. In the general semi-Riemannian case, under suitable hypotheses this number is equal to an algebraic count of the multiplicities of the conjugate points, and it is related to the spectral properties of a non self-adjoint differential operator. This last relation gives a weak extension of the classical Morse Index Theorem in Riemannian and Lorentzian geometry. In this paper we reprove some results that were incorrectly stated by Helfer in [12]; in particular, a counterexample to one of Helfer’s results, which is essential for the theory, is given. In the last part of the paper we discuss a general technique for the construction of examples and counterexamples in the index theory for semi-Riemannian metrics, in which some new phenomena appear.

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1. Introduction

The original motivation for writing this paper was given by the following problem. Given a sequence $\gamma_j$ of geodesics in a semi-Riemannian manifold $(M, g)$ converging to a geodesic $\gamma_\infty$, what can be said about the convergence of the geometric index of $\gamma_j$ to that of $\gamma_\infty$? The question arose in the context of Lorentzian geometry, where the problem originated in an attempt to develop a Morse theory for lightlike geodesics as limit of the theory for timelike geodesics.

Recall that the geometric index of a geodesic $\gamma : [a, b] \mapsto M$ is the number of points that are conjugate to $\gamma(a)$ along $\gamma$, counted with multiplicity. Conjugate points along a geodesic correspond to the zeroes of (non trivial) Jacobi fields along $\gamma$, which are vector fields that annihilate the index form $I_\gamma$. The index form $I_\gamma$ is a symmetric bilinear form defined in the space of vector fields along $\gamma$ that is written in terms of the Levi-Civita connection and the curvature tensor of $g$; the celebrated Morse Index Theorem (see [2, 3, 7, 9, 15, 16, 18] for different versions of this theorem) states that the geometric index of a Riemannian or causal Lorentzian geodesic $\gamma$ is equal to the number (with multiplicity) of negative eigenvalues of $I_\gamma$, provided that the final point $\gamma(b)$ is not considered in the count of conjugate points. The number (with multiplicity) of negative eigenvalues of a symmetric bilinear form on a vector space is called the index of the form; a simple argument shows that if a symmetric bilinear form is continuous with respect to some norm in the vector space $V$, then its index does not change when one extends $I$ to the Banach space completion of $V$.

It is not too hard to prove that the convergence of a sequence of geodesics to a geodesic implies a strong convergence of the corresponding index forms. Hence, by considering suitable Hilbert space completions of the set of vector fields along $\gamma$ and representing $I_\gamma$ as a self-adjoint operator on this Hilbert space, the problem of convergence of the geometric index can be reformulated in terms of convergence of the index of a sequence of self-adjoint operators converging in the operator norm. However, it is very easy to give examples of sequences of (real) symmetric matrices with constant index converging to a symmetric matrix having a different index. In finite dimension, this phenomenon arises only when the limit matrix is non invertible, and in the case that, when passing to the limit, some negative eigenspace of the matrices of sequence falls into the kernel of the limit, causing a drop in the index. In the infinite dimensional case the situation is even worse, and one can have a sequence of self-adjoint operators converging to an invertible self-adjoint operator whose index is strictly less that the infimum of the indices of the approximating family.

Some questions concerning the continuity of the conjugate points in Riemannian geometry are studied in reference [17].

If $(M, g)$ is Riemannian, i.e., $g$ is positive definite, then, considering a suitable $H^1$-Sobolev completion of the space of vector fields along the geodesic, the self-adjoint operator associated to the index form is a compact perturbation of the identity. In this case, if the limit is an invertible operator, i.e., if the point $\gamma_\infty(b)$ is not conjugate to $\gamma_\infty(a)$ along $\gamma_\infty$, the geometric index of $\gamma_j$ is eventually constant, and equal to the geometric index of $\gamma_\infty$. The same conclusion holds for timelike Lorentzian geodesics,
provided that the limit be also timelike. In this case, we consider the index of the index form defined only in the space of vector fields which are everywhere orthogonal to the geodesic, and the Lorentzian metric is positive on such fields.

When one considers the case of lightlike geodesics, though, the situation is complicated by the fact that the index form always has a non trivial kernel, even when the final endpoint is non conjugate to the initial one. Namely, any vector field which is a multiple of the tangent field to the geodesic is in the kernel of the index form. The presence of the kernel in the lightlike case could be avoided by considering suitable quotients of the normal bundle, but then one loses the relation with the non lightlike geodesics, for which an index form cannot be defined on such quotient.

Thus, using abstract spectral arguments one only proves the semi-continuity of the index for Lorentzian causal geodesics.

A different technique to study the stability of the index is suggested by an analogy with the classical Sturm problem in the theory of ordinary differential equations. The Sturm oscillation theorem deals with second order differential systems of the form $-(px')' + rx = \lambda x$ where $p$ and $r$ are functions with $p > 0$, and $\lambda$ is a real parameter. The theorem states that, denoting by $C^1_o[\alpha, \beta]$ the space of $C^1$-functions on $[\alpha, \beta]$ vanishing at $\alpha$ and $\beta$, the index of the symmetric bilinear form $B(x, y) = \int_{a}^{b} [px'y' + rxy] \, dt$ in $C^1_o[a, b]$ is equal to the sum over $t \in [a, b]$ of the dimension of the kernel of the bilinear form $\int_{a}^{t} [px'y' + rxy] \, dt$ in $C^1_o[a, t]$.

The classical proof of the Sturm oscillation theorem ([6, Chapter 8]) is obtained by showing that the two quantities involved in the thesis can be obtained as the winding number of two homotopic closed curves in the real projective line. As a side effect of this theory, one obtains immediately that, since the winding number is stable by homotopies, and in particular by small $C^0$-perturbations, the index of $B$ is stable by small perturbations.

So, the stability of the index for a Sturm system is proven by relating the index form $B$ to some homotopical invariants of the problem. In this paper we exploit this method to obtain the stability of the geometrical index, or of suitable substitutes of it, for semi-Riemannian geodesics. More precisely, we investigate the notion of Maslov index for a Jacobi type system of ordinary differential equations, which is obtained as the intersection number between a curve and a subvariety of codimension one of a smooth manifold. The Maslov index of a system arising from the Jacobi equation of a Riemannian or a causal Lorentzian geodesic is then proven to be equal to the geometric index of the geodesic.

The idea and some of the results presented are not new.

In [5] and [8] the authors develop an approach to the index problem through topological methods. For instance, in [8], it is employed a similar technique to prove a generalization of the Sturm’s Theorems in the case of an arbitrary self-adjoint system of even order and subject to arbitrary self-adjoint boundary conditions. The main technical tool used in the proof is the notion of $U$-manifold, which is obtained from an even dimensional vector space $E$, endowed with a non degenerate Hermitian form $\psi$, as the set of all maximal subspaces of $E$ on which $\psi$ vanishes. Non trivial solutions of
the eigenvalue boundary value problem determine continuous curves in a $U$-manifold, and the proof of the result is based in studying the number of intersections of such curves with a subvariety of codimension one. The intersection theory developed by the author for $U$-manifolds is based on the relative homotopy theory.

In [12] (see also [13]), the author carries out a similar analysis for Morse–Sturm–Liouville systems, which are symmetric with respect to an indefinite inner product. In this context, the environment for the intersection theory is given by the set of all Lagrangians of a symplectic space which is naturally associated to the differential system. This set, the Lagrangian Grassmannian of the symplectic space, has a natural manifold structure, which is in general non orientable. A non trivial solution of the differential system gives a continuous curve in the Lagrangian Grassmannian, and the zeroes of the solution correspond to intersections of this curve with the subvariety of all Lagrangians $L$ which are not complementary to a given one $L_0$. The intersection theory employed in this situation is based on the notion of Maslov index of a curve, obtained using duality in homology theory (Arnol’d–Maslov cycle).

This approach does not seem to deal properly with the lack of orientability of the Lagrangian Grassmannian. Furthermore, several results of Helfer’s paper are incorrectly stated, due mainly to the lack of an essential assumption of nondegeneracy for the restrictions of certain symmetric bilinear forms. More precisely, in [12, Proposition 5.1 (b)] it is claimed the equality between the Maslov index and the sum of the signatures of the conjugate points; we give a counterexample to such equality in subsection 7.4. In [12, Proposition 6.2], the proof is incomplete, because the case of non simple eigenvalues is not treated properly (see Remark 6.2.2). Consequently, also the proof of [12, Proposition 7.1] is affected by these problems; a more restrictive statement of the Index Theorem is proven in Section 6.

It is important to observe that the possibility of such degeneracies, which do not occur in Riemannian or causal Lorentzian geodesics, is responsible for many of the new phenomena which arise in the general semi-Riemannian case, like for instance, the possibility of accumulation of the conjugate points along a geodesic. Curiously enough, also in the book by O’Neill [16, Exercise 8, page 299], the non degeneracy assumption is missing, and the author claims incorrectly that the set of focal points along a geodesic is discrete.

For this reasons, we have opted to provide an alternative, self-contained, presentation of some of Helfer’s results needed for our proof of the stability of the geometrical index. For the sake of completeness, the geometrical results of [12] that are being considered in this paper have been reproven in the slightly more general context of geodesics starting orthogonally to a given non degenerate submanifold $\mathcal{P}$ of $\mathcal{M}$. With such a generalization we are able to prove our stability results also for the focal index of a geodesic relative to a submanifold.

In order to keep our notation as uniform as possible and to make the results accessible to mathematicians and physicists from different areas, in the paper we make an effort to give a formal proof of almost everything we claim, even though this attitude has the disadvantage of not distinguishing between new and old results. For instance,
the statement and the proof of Proposition 2.3.1 in this paper, apart from the result of Lemma 2.3.2, is almost identical to [12, Section 3]; the proof of Lemma 6.2.1 is essentially contained in the proof of [12, Proposition 6.2], and also the proof of Theorem 6.2.4 is identical to the proof of a somewhat similar result proven in [12, Theorem 7.1]. It should also be remarked that many of the results concerning the geometry of the Lagrangian Grassmannian of a symplectic space presented in Section 3 may appear in similar forms on other references, like for instance [1, 11, 19].

We outline briefly the contents of this paper. In Section 2 we introduce the notations and we give a few preliminary results on the geometrical and the differential framework. In Section 3 we present a detailed description of the analytical structure of the Lagrangian Grassmannian of a symplectic space. All the results are given in an intrinsic, i.e., coordinate independent, form.

Section 4 is devoted to the intersection theory used for the definition and the properties of the Maslov index. The treatment presented is inspired by the intersection theory for $U$-manifolds of [8]; this approach has the advantage of avoiding the homology/cohomology duality issues in non orientable spaces. We remark that the Lagrangian Grassmannian is in general a non orientable manifold (see [10]). Some of the results presented in Section 4 are already proven in [1], although only in the case of closed curves. For the computation of the Maslov index of a loop, Arnol’d uses the isomorphism of the fundamental groups of the Lagrangian Grassmannian and of the circle $S^1$ induced by the square of the determinant function. For our purposes, the techniques developed in Section 4 using suitable coordinate charts, are more appropriate for the computation of the Maslov index (see subsection 4.3). Namely, we describe the Maslov index in terms of change of signatures of symmetric bilinear forms, obtaining a natural relation with the geometric index (see Theorem 5.1.2 and the following corollary).

In Section 5 we apply the results of the previous sections to the problem of the stability of the geometrical and focal indexes in semi-Riemannian geometry. A special attention is given to the case of the approximation of a lightlike geodesic by timelike geodesics in a Lorentzian manifold. If $\gamma_\infty$ is a lightlike geodesic in a Lorentzian manifold whose endpoints are not conjugate and $\gamma_n$ is a sequence of timelike geodesics converging to $\gamma_\infty$, then the geometric index of $\gamma_n$ is eventually constant and equal to the geometric index of $\gamma_\infty$.

In Section 6 we present a spectral theorem (Theorem 6.2.4), which is a weak generalization of the classical Morse Index Theorem for Riemannian or causal Lorentzian geodesics to the case of geodesics in an arbitrary semi-Riemannian manifold. The proof of the spectral theorem is obtained by a simple homotopy argument in the Lagrangian Grassmannian. It is interesting to observe that the proof of Theorem 6.2.4 gives an alternative and unifying proof of all the previous versions of the Morse Index Theorem in Riemannian and Lorentzian geometry (see [15]).

In Section 7, we study the problem of determining which curves of Lagrangians are associated to solutions of Jacobi systems. We give some necessary and sufficient
conditions for this occurrence, and we use these conditions to find examples of situations described generically in the rest of the paper. Remarkably, we give an example in which the equality between the Maslov index and the focal index of a Lorentzian spacelike geodesic $\gamma$ fails, due to the degeneracy of the metric on the space $\mathcal{J}[t]$ consisting of the evaluations at $t$ of the Jacobi fields along $\gamma$ that vanish at the initial instant.

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2. Preliminaries

Let $(M, g)$ be a smooth semi-Riemannian manifold, i.e., $M$ is a finite dimensional real $C^\infty$ manifold whose topology satisfies the second countability axiom and the Hausdorff separation axiom, and $g$ is a a smooth $(2, 0)$-tensor field on $M$ which defines a non degenerate symmetric bilinear form on each tangent space of $M$. We set $m = \dim(M)$; a non zero vector $v \in T_pM$ will be called spacelike, timelike or lightlike according to $g(v, v)$ being positive, negative or null.

We give some general definitions concerning symmetric bilinear forms for later use.

**Definition 2.0.1.** Let $V$ be any real vector space and $B : V \times V \mapsto \mathbb{R}$ a symmetric bilinear form. The **negative type number** (or index) $n_-(B)$ of $B$ is the possibly infinite number defined by

\[
\text{(2.0.1)}\quad n_-(B) = \sup \left\{ \dim(W) : W \text{ subspace of } V \text{ on which } B \text{ is negative definite} \right\}.
\]

The **positive type number** $n_+(B)$ is given by $n_+(B) = n_-(B)$; if at least one of these two numbers is finite, the **signature** $\text{sgn}(B)$ is defined by:

\[
\text{sgn}(B) = n_+(B) - n_-(B).
\]

The **kernel** of $B$, $\text{Ker}(B)$, is the set $V^\perp$ of vectors $v \in V$ such that $B(v, w) = 0$ for all $w \in V$; the **degeneracy** $\text{dgn}(B)$ of $B$ is the (possibly infinite) dimension of $\text{Ker}(B)$.

If $V$ is finite dimensional, then the numbers $n_+(B)$, $n_-(B)$ and $\text{dgn}(B)$ are respectively the number of $1$'s, $-1$'s and $0$'s in the canonical form of $B$ as given by the Sylvester’s Inertia Theorem. In this case, $n_+(B) + n_-(B)$ is equal to the codimension of $\text{Ker}(B)$, and it is also called the **rank** of $B$, $\text{rk}(B)$.

The semi-Riemannian manifold $(M, g)$ is said to be **Lorentzian** if the index of $g$ at every point of $M$ is equal to $1$. A four dimensional Lorentzian manifold is the mathematical model for a general relativistic spacetime; in this case, the timelike and lightlike geodesics in $M$, i.e., geodesics $\gamma$ with $\gamma'(t)$ timelike or lightlike for all $t$ respectively, represent the trajectories of massive and massless objects freely falling under the action of the gravitational field.

Let $\nabla$ denote the covariant derivative of the Levi–Civita connection of $g$ and let $\mathcal{R}(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ be the curvature tensor of $g$. If $V : [a, b] \mapsto T_pM$ is a vector field along a given curve $\gamma : [a, b] \mapsto M$, we will write $V'$ for the covariant derivative of $V$ along $\gamma$.

2.1. The geometrical problem

Let $P$ be a smooth submanifold of $M$, $p \in P$ and $n \in T_pP^\perp$, i.e., $n \in T_pM$ and $g(n, v) = 0$ for all $v \in T_pP$. The **second fundamental form** of $P$ at $p$ in the direction $n$ is the symmetric bilinear form $S_n : T_pP \times T_pP \mapsto \mathbb{R}$ given by:

\[
S_n(v_1, v_2) = g(\nabla_{v_1}V_2, n),
\]

\[
\text{SGN}(B) = n_+(B) - n_-(B).
\]
where $V_2$ is any extension of $v_2$ to a vector field on $\mathcal{P}$. If $T_p\mathcal{P}$ is non degenerate, i.e., if the restriction of $g$ to $T_p\mathcal{P}$ is non degenerate, then there exists a linear operator, still denoted by $S_n$, on $T_p\mathcal{P}$, such that $S_n(v_1, v_2) = g(S_n(v_1), v_2)$ for all $v_1, v_2 \in T_p\mathcal{P}$. Also, if $T_p\mathcal{P}$ is non degenerate, then the second fundamental form can be viewed as a $T_p\mathcal{P}^\perp$-valued symmetric bilinear form $S$ on $T_p\mathcal{P}$ defined by:

$$S(v_1, v_2) = \text{orthogonal projection of } \nabla v_1 V_2 \text{ onto } T_p\mathcal{P}^\perp,$$

so that

$$S_n(v_1, v_2) = g(S(v_1, v_2), n), \quad \forall v_1, v_2 \in T_p\mathcal{P}, \; n \in T_p\mathcal{P}^\perp.$$

Let $\gamma : [a, b] \mapsto \mathcal{M}$ be a non constant geodesic orthogonal to $\mathcal{P}$ at the initial point, i.e., $\gamma(a) \in \mathcal{P}$ and $\gamma'(a) \in T_{\gamma(a)}\mathcal{P}^\perp$. Let’s assume that $T_{\gamma(a)}\mathcal{P}$ is non degenerate; we will say that the family of objects $(\mathcal{M}, g, \gamma, \mathcal{P})$ is an admissible quadruple for the geometrical problem.

A Jacobi field along $\gamma$ is a smooth vector field $\mathcal{J}$ along $\gamma$ that satisfies the Jacobi equation:

$$(2.1.1) \quad \mathcal{J}'' = R(\gamma', \mathcal{J}) \gamma';$$

we say that $\mathcal{J}$ is a $\mathcal{P}$-Jacobi field if, in addition, $\mathcal{J}(a)$ satisfies:

$$(2.1.2) \quad \mathcal{J}(a) \in T_{\gamma(a)}\mathcal{P} \quad \text{and} \quad \mathcal{J}'(a) + S_{\gamma'(a)}[\mathcal{J}(a)] \in T_{\gamma'(a)}\mathcal{P}^\perp.$$

Observe that, if $\mathcal{P}$ is a single point of $\mathcal{M}$, then (2.1.2) reduces to $\mathcal{J}(a) = 0$. Geometrically, equation (2.1.1) means that $\mathcal{J}$ is the variational vector field corresponding to a variation of $\gamma$ by geodesics; condition (2.1.2) says that these geodesics are orthogonal to $\mathcal{P}$ at their initial points.

We define the following vector spaces:

$$(2.1.3) \quad \mathfrak{J} = \left\{ \mathcal{J} : \mathcal{J} \text{ is a } \mathcal{P}\text{-Jacobi field along } \gamma \right\};$$

$$(2.1.4) \quad \mathfrak{J}^\perp = \left\{ \mathcal{J} \in \mathfrak{J} : \mathcal{J}(t) \in \gamma'(t)^\perp, \; \forall t \in [a, b] \right\}.$$ 

Observe that if $\mathcal{J} \in \mathfrak{J}$ is such that $\mathcal{J}(t) \in \gamma'(t)^\perp$ for some $t \in [a, b]$, then $\mathcal{J} \in \mathfrak{J}^\perp$. Namely, for any Jacobi field $\mathcal{J}$, the function $g(\mathcal{J}(t), \gamma'(t))$ is affine on $[a, b]$, and if $\mathcal{J}$ is $\mathcal{P}$-Jacobi, then $g(\mathcal{J}(a), \gamma'(a)) = 0$; similarly, for a $\mathcal{P}$-Jacobi field $\mathcal{J}$, the condition $\mathcal{J}'(a) \in \gamma'(a)^\perp$ is equivalent to $\mathcal{J} \in \mathfrak{J}^\perp$. We conclude that $\mathfrak{J}^\perp$ can be described alternatively as the space of Jacobi fields along $\gamma$ satisfying the initial conditions:

$$(2.1.5) \quad \mathcal{J}(a) \in T_{\gamma(a)}\mathcal{P} \quad \text{and} \quad \mathcal{J}'(a) + S_{\gamma'(a)}[\mathcal{J}(a)] \in [T_{\gamma(a)}\mathcal{P} \oplus \gamma'(a)]^\perp.$$

It is easy to see that $\dim(\mathfrak{J}) = m$; namely, the dimension of the subspace of $T_{\gamma(a)}\mathcal{M} \oplus T_{\gamma(a)}\mathcal{M}$ defined by the initial conditions (2.1.2) is equal to $m$. Similarly, $\dim(\mathfrak{J}^\perp) = m - 1$, since the dimension of the subspace of $T_{\gamma(a)}\mathcal{M} \oplus T_{\gamma(a)}\mathcal{M}$ defined by the initial conditions (2.1.5) is equal to $m - 1$.

For all $t \in [a, b]$, we define the subspaces $\mathfrak{J}[t]$ and $\mathfrak{J}^\perp[t]$ of $T_{\gamma(t)}\mathcal{M}$ by

$$\mathfrak{J}[t] = \left\{ \mathcal{J}(t) : \mathcal{J} \in \mathfrak{J} \right\} \quad \text{and} \quad \mathfrak{J}^\perp[t] = \left\{ \mathcal{J}(t) : \mathcal{J} \in \mathfrak{J}^\perp \right\};$$
observe that $\mathfrak{J}^{-1}[t]$ should not be confused with $\mathfrak{J}[t]^{-1}$, which is the orthogonal complement of $\mathfrak{J}[t]$ in $T_{\gamma(t)}\mathcal{M}$. Precisely, the following relation holds:

\[(2.1.6) \quad \mathfrak{J}^{-1}[t] = \mathfrak{J}[t] \cap \gamma'(t)^\perp, \quad \forall t \in ]a, b];\]

this follows immediately from the observation that if $t \in ]a, b]$ and $\mathcal{J} \in \mathfrak{J}$ is such that $\mathcal{J}(t) \in \gamma'(t)^\perp$, then $\mathcal{J} \in \mathfrak{J}^{-1}$. The vector field $(t - a)\gamma'(t)$ is always a $\mathcal{P}$-Jacobi field, thus $\gamma'(t) \in \mathfrak{J}[t]$ for $t \in ]a, b]$; observe that $\gamma'(t) \in \mathfrak{J}^{-1}[t]$ if and only if $\gamma$ is lightlike.

We also remark that, for $t \in ]a, b]$, the following formula holds:

\[(2.1.7) \quad \dim(\mathfrak{J}[t]) = \dim(\mathfrak{J}^{-1}[t]) + 1, \quad t \in ]a, b].\]

Indeed, the codimension of $\mathfrak{J}^{-1}$ in $\mathfrak{J}$ is 1; moreover, for $t \in ]a, b]$, the linear operator $\mathcal{J} \mapsto \mathcal{J}(t)$ of evaluation at $t$ has the same kernel in $\mathfrak{J}$ and in $\mathfrak{J}^{-1}$.

The point $\gamma(t_0), t_0 \in ]a, b]$ is said to be a $\mathcal{P}$-focal point along $\gamma$ if there exists a non zero $\mathcal{P}$-Jacobi field $\mathcal{J}$ along $\gamma$ with $\mathcal{J}(t_0) = 0$. The multiplicity $\text{mul}(t_0)$ of $\gamma(t_0)$ is the dimension of the space of $\mathcal{P}$-Jacobi fields that vanish at $t_0$; we set $\text{mul}(t_0) = 0$ if $\gamma(t_0)$ is not $\mathcal{P}$-focal:

\[(2.1.8) \quad \text{mul}(t_0) = \dim \{ \mathcal{J} \in \mathfrak{J} : \mathcal{J}(t_0) = 0 \}.\]

Observe that if $\mathcal{J} \in \mathfrak{J}$ vanishes at $t_0$, then $\mathcal{J} \in \mathfrak{J}^{-1}$; since $\dim(\mathfrak{J}) = \dim(T_{\gamma(t_0)}\mathcal{M})$ and $\dim(\mathfrak{J}^{-1}) = \dim(\gamma'(t_0)^\perp)$, we conclude that $\text{mul}(t_0)$ is equal to the codimension of $\mathfrak{J}[t_0]$ in $T_{\gamma(t_0)}\mathcal{M}$, and also equal to the codimension of $\mathfrak{J}^{-1}[t_0]$ in $\gamma'(t_0)^\perp$.

We remark that (2.1.7) implies that $\mathfrak{J}^{-1}[t]$ is a proper subspace of $\mathfrak{J}[t]$, for $t \in ]a, b]$; this fact is trivial if $\gamma$ is not lightlike, but it has interesting consequences otherwise. For instance, if $\dim(\mathcal{M}) = 2$, this implies that there can be no focal points along any lightlike geodesics. Namely, if $t_0 \in ]a, b]$ is $\mathcal{P}$-focal, then $\mathfrak{J}[t_0]$ is at the most one dimensional, and therefore $\mathfrak{J}^{-1}[t_0] = \{0\}$, contradicting the fact that $\gamma'(t_0) \in \mathfrak{J}^{-1}[t_0]$.

If the number of $\mathcal{P}$-focal points along $\gamma$ is finite, one defines the geometric index, $i_{\text{geom}}(\gamma)$, of $\gamma$ relative to the initial submanifold $\mathcal{P}$ to be the sum of the multiplicities of the $\mathcal{P}$-focal points:

\[i_{\text{geom}}(\gamma) = \sum_{t \in ]a, b]} \text{mul}(t).\]

In order to extend to the semi-Riemannian case the classical Morse Theory, we need to introduce the concept of signature for a $\mathcal{P}$-focal point.

**Definition 2.1.1.** If $\gamma(t_0)$ is a $\mathcal{P}$-focal point, its signature $\text{sgn}(t_0)$ is defined to be the signature of the restriction of the metric $g$ to the space $\mathfrak{J}[t_0]^{-1}$. If $\gamma(t_0)$ is not a $\mathcal{P}$-focal point, we set $\text{sgn}(t_0) = 0$. The focal index $i_{\text{foc}}(\gamma)$ of the geodesic $\gamma$ relative to the initial submanifold $\mathcal{P}$ is defined by the sum:

\[i_{\text{foc}}(\gamma) = \sum_{t \in ]a, b]} \text{sgn}(t),\]

provided that the number of $\mathcal{P}$-focal points along $\gamma$ is finite.
Sufficient conditions for the finiteness of the number of $\mathcal{P}$-focal points will be discussed at the end of this Section (see Proposition 2.5.1, Corollary 2.5.2 and Remark 2.5.3).

2.2. The differential problem

Using a trivialization of the normal bundle along $\gamma$ by means of a parallel moving frame, we now reformulate the Jacobi problem given by (2.1.1) and (2.1.2) in terms of a second order linear differential equation in $\mathbb{R}^n$ with suitable initial conditions.

To this aim, we consider the following objects. Let $g$ be a non degenerate symmetric bilinear form in $\mathbb{R}^n$, and let $R(t)$, $t \in [a, b]$, be a continuous curve in the space of linear operators in $\mathbb{R}^n$ such that $R(t)$ is $g$-symmetric for all $t \in [a, b]$, i.e., $g(R(t)[v], w) = g(v, R(t)[w])$ for all $v, w \in \mathbb{R}^n$.

Let $P \subset \mathbb{R}^n$ be a subspace such that the restriction of $g$ to $P$ is non degenerate, and let $S$ be a symmetric bilinear form on $P$; then, there exists a $g$-symmetric linear operator on $P$, which we also denote by $S$, satisfying $S(v, w) = g(S[v], w)$ for all $v, w \in P$. We will say that the family $(g, R, P, S)$ is an admissible quadruple for the differential problem in $\mathbb{R}^n$.

We consider the following linear differential equation in $\mathbb{R}^n$:

\begin{equation}
J''(t) = R(t)[J(t)], \quad t \in [a, b];
\end{equation}

we will consider solutions $J$ of (2.2.1) that satisfy in addition the following initial conditions:

\begin{equation}
J(a) \in P, \quad J'(a) + S[J(a)] \in P^\perp,
\end{equation}

where $P^\perp$ is the $g$-orthogonal complement of $P$ in $\mathbb{R}^n$; such vector fields will be called $(P, S)$-solutions. Note that, if $P = \{0\}$ (and thus $S = 0$), a $(P, S)$-solution is simply a solution of (2.2.1) vanishing at $t = a$.

Let $\mathcal{J}$ be the space of all $(P, S)$-solutions:

\begin{equation}
\mathcal{J} = \left\{ J : J \text{ satisfies (2.2.1) and (2.2.2)} \right\};
\end{equation}

and, for $t \in [a, b]$, we set $\mathcal{J}[t] = \{ J(t) : J \in \mathcal{J} \}$.

Observe that $\dim(\mathcal{J}) = n$ since the subspace of $\mathbb{R}^n \oplus \mathbb{R}^m$ determined by (2.2.2) is $n$-dimensional.

**Definition 2.2.1.** An instant $t_0 \in [a, b]$ is $(P, S)$-focal if there exists a non null $(P, S)$-solution $J$ such that $J(t_0) = 0$. The multiplicity $\text{mul}(t_0)$ of $t_0$ is the dimension of the subspace of $\mathcal{J}$ consisting of such solutions; if $t_0$ is not $(P, S)$-focal we set $\text{mul}(t_0) = 0$:

\begin{equation}
\text{mul}(t_0) = \dim \left\{ J \in \mathcal{J} : J(t_0) = 0 \right\}.
\end{equation}

Since $\dim(\mathcal{J})$ is equal to $n$, then the multiplicity $\text{mul}(t_0)$ is the codimension of $\mathcal{J}[t_0]$ in $\mathbb{R}^n$:

\begin{equation}
\text{mul}(t_0) = \text{codim}(\mathcal{J}[t_0]) = \dim(\mathcal{J}[t_0])^\perp.
\end{equation}

In analogy with Definition 2.1.1, we now give the following:
Definition 2.2.2. The signature $\text{sgn}(t_0)$ of the $(P, S)$-focal instant $t_0$ is defined to be the signature of the restriction of $g$ to the space $\mathbb{J}[t_0]$. If $t_0$ is not $(P, S)$-focal, we set $\text{sgn}(t_0) = 0$; if the set of $(P, S)$-focal instants is finite, we define the focal index $i_{\text{foc}}$ of the quadruple $(g, R, P, S)$ to be the sum of the signatures of the $(P, S)$-focal instants:

$$i_{\text{foc}} = \sum_{t \in [a, b]} \text{sgn}(t).$$

2.3. Relations between the geometrical and the differential problem

Suppose that an admissible quadruple $(\mathcal{M}, g, \mathcal{P}, \gamma)$ for the geometrical problem is given. For all $t \in [a, b]$, the linear operator $\nu \mapsto \mathcal{R}(\gamma'(t), \nu)\gamma'(t)$ in $T_{\gamma(t)}\mathcal{M}$ is $g$-symmetric, and, by the usual symmetry properties of the curvature tensor, it takes values in $\gamma'(t)\perp$; we consider its restriction to $\gamma'(t)\perp$ and we denote it by $\mathcal{R}(t)$.

If we choose an arbitrary parallel moving frame that trivializes the normal bundle $(\gamma')\perp$ along $\gamma$, so that we have an isomorphism between $\gamma'(t)\perp$ and $\mathbb{R}^{m-1}$, we get a linear operator $\mathcal{R}(t)$ on $\mathbb{R}^{m-1}$ corresponding to $\mathcal{R}(t)$, and a symmetric bilinear form $g$ corresponding to (the restriction to $\gamma'(t)\perp$ of) $g$. Since $g$ is parallel, then $g$ is constant; obviously, $\mathcal{R}(t)$ is $g$-symmetric. Similarly, the subspace $T_{\gamma(a)}\mathcal{P} \subset \gamma'(a)\perp$ corresponds to a subspace $P$ of $\mathbb{R}^{m-1}$, and the second fundamental form $S'_{\gamma(a)}$ corresponds to a symmetric bilinear form $S$ on $P$; moreover, the restriction of $g$ to $P$ is non degenerate.

If $\gamma$ is not lightlike, then the restriction of $g$ to $\gamma'(t)\perp$ is non degenerate for all $t$, which implies that $g$ is non degenerate in $\mathbb{R}^{m-1}$ so that $(g, R, P, S)$ is an admissible quadruple for the differential problem in $\mathbb{R}^{m-1}$. We will say that $(g, R, P, S)$ is associated to the quadruple $(\mathcal{M}, g, \gamma, \mathcal{P})$ by the choice of a parallel trivialization of the normal bundle along $\gamma$.

If $(g, R, P, S)$ is associated to an admissible quadruple for the geometrical problem $(\mathcal{M}, g, \gamma, \mathcal{P})$, then $R$ is indeed a smooth map. Conversely, every quadruple $(g, R, P, S)$ with $R$ smooth arises in this way:

Proposition 2.3.1. If $(g, R, P, S)$ is an admissible quadruple for the differential problem in $\mathbb{R}^n$, with $R$ smooth, then there exists an admissible quadruple $(\mathcal{M}, g, \gamma, \mathcal{P})$ for the geometrical problem such that $(g, R, P, S)$ is associated to $(\mathcal{M}, g, \gamma, \mathcal{P})$ by some choice of a parallel trivialization of the normal bundle along $\gamma$. Moreover, the quadruple $(\mathcal{M}, g, \gamma, \mathcal{P})$ can be chosen with $\gamma$ timelike as well as spacelike, and $(\mathcal{M}, g)$ can be chosen to be conformally flat. If $g$ is positive definite, then $(\mathcal{M}, g)$ is Riemannian if $\gamma$ is spacelike and Lorentzian if $\gamma$ is timelike.

Proof. Consider $\mathcal{M} = \mathbb{R}^{n+1}$ with coordinates $(x_1, x_2, \ldots, x_{n+1})$ and canonical basis $\{e_1, e_2, \ldots, e_{n+1}\}$; let $\gamma : [a, b] \mapsto \mathcal{M}$ be the curve $\gamma(t) = t \cdot e_{n+1}$. We consider the non degenerate symmetric bilinear form $g_0$ on $\mathcal{M}$ given by $g_0(e_i, e_j) = g(e_i, e_j)$ for $i, j = 1, \ldots, n$, $g_0(e_{n+1}, e_{n+1}) = \pm 1$, and $g_0(e_i, e_j) = 0$ otherwise.

The choice of the sign of $g_0(e_{n+1}, e_{n+1})$ is done according to whether $\gamma$ should be timelike or spacelike, as desired.

Let $\mathcal{M}$ be endowed with the conformally flat metric $g = e^\Omega g_0$, where $\Omega$ is a smooth function in $\mathbb{R}^{n+1}$ that vanishes together with its partial derivatives on the $e_{n+1}$-axis.
The factor $\Omega$ will be chosen so that the corresponding metric $g$ will satisfy the required properties.

To this goal, we recall some formulas about the covariant derivative and the geodesic equation in general conformal metrics. Let $\nabla^{(0)}$ and $\nabla$ denote the covariant derivative or the gradient operators in the metrics $g_0$ and $g$ respectively; note that the covariant derivative $\nabla^{(0)}$ is the usual directional derivative in $\mathbb{R}^{n+1}$, although the gradient $\nabla^{(0)}$ is not the usual gradient in $\mathbb{R}^{n+1}$.

For smooth vector fields $X, Y$ in $\mathcal{M}$, we have:

\begin{equation}
\nabla_{X}Y = \frac{1}{2} \left[ g_0(\nabla^{(0)}\Omega, X) Y + g_0(\nabla^{(0)}\Omega, Y) X - g_0(X, Y) \nabla^{(0)}\Omega \right] + \nabla^{(0)}Y; \tag{2.3.1}
\end{equation}

moreover, the geodesic equation in $(\mathcal{M}, g)$ is:

\begin{equation}
\nabla^{(0)}_{c'}c' = \frac{1}{2} g_0(c', c') \nabla^{(0)}\Omega - g_0(\nabla^{(0)}\Omega, c') c'. \tag{2.3.2}
\end{equation}

Since $\nabla^{(0)}\Omega \equiv 0$ on $\gamma$, then $\gamma$ is a geodesic in $(\mathcal{M}, g)$; moreover, by (2.3.1), the parallel vector fields along $\gamma$ in $(\mathcal{M}, g)$ are just the constant vector fields. Hence, we trivialize the normal bundle along $\gamma$ in $(\mathcal{M}, g)$ by choosing the first $n$ vectors of the canonical basis as a parallel moving frame.

To compute the Jacobi equation along $\gamma$ in $(\mathcal{M}, g)$, we linearize the geodesic equation (2.3.2), obtaining:

\begin{equation}
J'' = \frac{1}{2} g_0(\gamma', \gamma') \text{Hess}^{(0)}_{\Omega}(J), \tag{2.3.3}
\end{equation}

where $J''$ is the ordinary second derivative in $\mathbb{R}^{n+1}$ and $\text{Hess}^{(0)}_{\Omega}$ is the $g_0$-symmetric linear operator given by $\text{Hess}^{(0)}_{\Omega}(v) = \nabla^{(0)}_{v} \nabla^{(0)}\Omega$. In the deduction of (2.3.3) we have used the fact that $\nabla^{(0)}\Omega$ and $\text{Hess}^{(0)}_{\Omega}(\gamma')$ vanish on $\gamma$.

Observing that the covariant derivative along $\gamma$ in $(\mathcal{M}, g)$ equals ordinary derivative in $\mathbb{R}^{n+1}$ and comparing equation (2.3.3) with the general Jacobi equation (2.1.1) we see that the curvature tensor $\mathcal{R}$ of $(\mathcal{M}, g)$ along $\gamma$ is given by:

$$
\mathcal{R}(\gamma', v) \gamma' = \frac{1}{2} g_0(\gamma', \gamma') \text{Hess}^{(0)}_{\Omega}(v).
$$

It is easily checked that:

$$
g_0(\text{Hess}^{(0)}_{\Omega}(e_i), e_j) = \frac{\partial^2 \Omega}{\partial x_i \partial x_j}, \quad \forall i, j = 1, \ldots, n, n+1;
$$

if we set:

$$
a_{ij}(t) = \frac{2}{g_0(\gamma', \gamma')} g_0(R(t) e_i, e_j), \quad i, j = 1, \ldots, n, \quad t \in [a, b]
$$

and consider an arbitrary smooth extension of $a_{ij}$ on $\mathbb{R}$, then the assignment

$$
\Omega(x_1, \ldots, x_{n+1}) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x_{n+1}) x_i x_j
$$
gives the required function.

To conclude the proof, we now need to exhibit a submanifold $\mathcal{P}$ of $\mathcal{M}$, passing through $\gamma(a)$ with tangent space $T_{\gamma(a)}\mathcal{P} = P \oplus \{0\}$, and whose second fundamental form in the normal direction $\gamma'(a)$ equals $S$. This will follow immediately from the next Lemma, in which we prove something slightly more general.

The last assertion in the statement of the proposition is totally obvious.

Lemma 2.3.2. Let $(\mathcal{M}, g)$ be a semi-Riemannian manifold, $p \in \mathcal{M}$, $P$ a non degenerate subspace of $T_p\mathcal{M}$ and $S : P \times P \mapsto P^\perp$ be a symmetric bilinear map. Then, there exists a smooth submanifold $\mathcal{P}$ of $\mathcal{M}$, with $p \in \mathcal{P}$, such that $T_p\mathcal{P} = P$ and such that the second fundamental form $S$ of $\mathcal{P}$ at $p$ equals $S$.

Proof. Let $U_0 \subset T_p\mathcal{M}$ be an open neighborhood of the origin such that the exponential map $\exp_p$ of $(\mathcal{M}, g)$ maps $U_0$ diffeomorphically onto an open neighborhood of $p$ in $\mathcal{M}$. Regarding $\exp_p$ as a coordinate map around $p$, it is well known that the Christoffel symbols of the Levi-Civita connection vanish at the point 0. Hence, the covariant derivative at this point coincide with the usual directional derivative in $T_p\mathcal{M}$. If $\mathcal{P}_0$ is a submanifold of $U_0$ passing through 0 and $\mathcal{P} = \exp_p(\mathcal{P}_0)$, then, since $d\exp_p(0)$ is the identity map, the tangent space $T_p\mathcal{P}$ is $T_p\mathcal{P}_0$; moreover, by the above observation about the covariant derivative, the second fundamental form of $\mathcal{P}$ at $p$ equals the second fundamental form of $\mathcal{P}_0$ at 0 in the flat space $T_p\mathcal{M}$.

We define $\mathcal{P}_0$ to be the smooth submanifold of $T_p\mathcal{M}$ given by the graph of the map $v \mapsto \frac{1}{2} S(v, v)$ in the decomposition $P \oplus P^\perp$, namely:

$$\mathcal{P}_0 = \left\{ v + \frac{1}{2} S(v, v) : v \in P \right\} \cap U_0.$$  

The conclusion follows from an elementary calculation of the second fundamental form of $\mathcal{P}_0$.

So far, we have associated an admissible quadruple for the differential problem only to quadruples $(\mathcal{M}, g, \gamma, \mathcal{P})$ with $\gamma$ spacelike or timelike. Indeed, if $\gamma$ is lightlike, then the symmetric bilinear form $g$ previously defined is degenerate on $\mathbb{R}^{n-1}$. One way to avoid this problem, following a customary procedure in Morse Theory (see [3] for the Lorentzian case), is to consider a suitable quotient of the normal bundle along the lightlike geodesic $\gamma$.

More precisely, given an admissible quadruple $(\mathcal{M}, g, \gamma, \mathcal{P})$ with $\gamma$ lightlike, for all $t \in [a, b]$ we consider the quotient space $\mathcal{N}(t) = \gamma'(t)^\perp / [R \gamma'(t)]$, where $R \gamma'(t)$ is the one dimensional subspace generated by $\gamma'(t)$. It is easy to see that $\mathcal{N} = \bigcup_{t \in [a, b]} \mathcal{N}(t)$ is a vector bundle along $\gamma$.

Since the kernel of the restriction of the metric $g$ to $\gamma'(t)^\perp$ is precisely $R \gamma'(t)$, then $g$ gives a well defined non degenerate symmetric bilinear form $\overline{g}$ on the quotient space $\mathcal{N}(t)$. Similarly, the linear operator $\mathcal{R}(t) = \mathcal{R} \gamma'(t)$ annihilates $R \gamma'(t)$, and therefore it defines a linear operator $\overline{\mathcal{R}}(t)$ on $\mathcal{N}(t)$. Obviously, $\overline{\mathcal{R}}$ is $\overline{g}$-symmetric.

The subspace $T_{\gamma(a)}\mathcal{P} \subset \gamma'(a)^\perp$ does not contain $\gamma'(a)$, because of our nondegeneracy assumption on $T_{\gamma(a)}\mathcal{P}$, hence it may be identified with a subspace of $\mathcal{N}(a)$, which will be denoted by the same symbol.
Let us now consider a trivialization of the normal bundle along \( \gamma \) by a parallel moving frame in such a way that the last vector field of the frame is the tangent vector \( \gamma' \) itself. The remaining \( m - 2 \) vector fields define a moving frame in the bundle \( \mathcal{N} \), and they induce a trivialization of \( \mathcal{N} \). We therefore get tensors \( R(t) \) and \( g \) on \( \mathbb{R}^{m-2} \) corresponding to the tensors \( \overline{R}(t) \) and \( \overline{g} \), as well as a subspace \( P \subset \mathbb{R}^{m-2} \) and a symmetric bilinear form \( S : P \times P \mapsto \mathbb{R} \) corresponding to the subspace \( T_{\gamma(0)} \mathcal{P} \) of \( \mathcal{N}(a) \) and the second fundamental form \( S_{\gamma'(a)} \) of \( \mathcal{P} \), respectively.

We have thus constructed an admissible quadruple \((g, R, P, S)\) for the differential problem in \( \mathbb{R}^{m-2} \) which we call the associated quadruple to \((\mathcal{M}, g, \gamma, \mathcal{P})\) in the case of a lightlike geodesic \( \gamma \).

**Remark 2.3.3.** Given an admissible quadruple \((\mathcal{M}, g, \gamma, \mathcal{P})\) for the geometric problem and an associated quadruple \((g, R, P, S)\) corresponding to some parallel trivialization of the normal bundle along \( \gamma \), we introduce a linear map \( \Phi \) that carries vector fields orthogonal to \( \gamma \) into vector fields in \( \mathbb{R}^n \), as follows. If \( \gamma \) is non lightlike, \( \Phi(v) \) is simply the set of \( m - 1 \) coordinates of \( v \) with respect to the chosen parallel moving frame. When \( \gamma \) is lightlike, \( \Phi(v) \) is the set of \( m - 2 \) coordinates of the projection of \( v \) in \( \mathcal{N} \) with respect to the chosen parallel basis of \( \mathcal{N} \).

If \( \gamma \) is non lightlike, such a map \( \Phi \) gives an isomorphism between \( \mathcal{J}^\perp \) and \( \mathcal{J} \); if \( \gamma \) is lightlike, \( \Phi \) maps \( \mathcal{J}^\perp \) onto \( \mathcal{J} \), and its kernel consists of affine multiples of \( \gamma' \). The surjectivity of \( \Phi \) in the lightlike case follows by observing that if \( \mathcal{J} \) is a solution of

\[\mathcal{J}'' = \mathcal{R}(\gamma', \mathcal{J}) \gamma' + f \gamma'\]

for some fixed smooth map \( f : [a, b] \mapsto \mathbb{R} \), then \( \mathcal{J} - F \gamma' \) is a Jacobi field along \( \gamma \), where \( F'' = f \).

The relation between the focal indexes of the geometric and differential problems is clarified by the following:

**Proposition 2.3.4.** Let \((\mathcal{M}, g, \gamma, \mathcal{P})\) be an admissible quadruple for the geometric problem and \((g, R, P, S)\) be an associated quadruple corresponding to some parallel trivialization of the normal bundle along \( \gamma \). Then, for all \( t_0 \in [a, b] \) there exists an isomorphism between \( \mathcal{J}[t_0]^\perp \) and \( \mathcal{J}[t_0]^\perp \) which carries the restriction of \( g \) to the restriction of \( \mathcal{P} \) to \( \mathcal{J}[t_0]^\perp \). In particular, for \( t_0 \in [a, b] \), \( \gamma(t_0) \) is a \( \mathcal{P} \)-focal point if and only if \( t_0 \) is a \( (P, S) \)-focal instant. In this case, its multiplicity and signature in the geometric and in the differential problem coincide, from which it follows that the focal indexes of the problems are equal.

**Proof.** If \( \gamma \) is not lightlike, let \( \phi : \gamma'(t_0)^\perp \mapsto \mathbb{R}^{m-1} \) be the isomorphism given by the chosen trivialization of the normal bundle to \( \gamma \). For a lightlike \( \gamma \), let’s denote by \( \phi : \mathcal{N}(t_0) \mapsto \mathbb{R}^{m-2} \) the isomorphism determined by the choice of the trivialization of the quotient bundle, as described above. By construction, \( \phi \) carries \( g \) (or \( \overline{g} \) for \( \gamma \) lightlike) to \( g \).

For \( \gamma \) non lightlike, it is easily checked that \( \phi \) carries \( \mathcal{J}^\perp[t_0] \) onto \( \mathcal{J}[t_0] \) by observing the correspondence between \( \mathcal{P} \)-Jacobi fields orthogonal to \( \gamma \) and \( (P, S) \)-solutions of
(2.2.1). Similarly, if $\gamma$ is lightlike, $\phi$ carries the quotient $\mathcal{J} [-][t_0]/[IR\gamma'(t_0)]$ onto $\mathbb{J}[t_0]$ (see Remark 2.3.3).

For $\gamma$ non lightlike, taking the orthogonal complements of $\mathcal{J} [-][t_0]$ in $\gamma'(t_0)$ and of $\mathbb{J}[t_0]$ in $IR^{m-1}$, using (2.1.6) we conclude that $\phi$ induces the desired isomorphism between $\mathcal{J}[t_0]$ and $\mathbb{J}[t_0]$.

If $\gamma$ is lightlike, we take the orthogonal complements of $\mathcal{J} [-][t_0]/[IR\gamma'(t_0)]$ in $\mathcal{N}(t_0)$ and of $\mathbb{J}[t_0]$ in $IR^{m-2}$. Again, using (2.1.6) we get that $\phi$ induces an isomorphism between the image of $\mathcal{J}[t_0]$ in the quotient space $\mathcal{N}(t_0)$ and $\mathbb{J}[t_0]$. To conclude the proof, we observe that (2.1.7) implies that $\gamma'(t_0)$ does not belong to $\mathcal{J}[t_0]$, which implies that it maps isomorphically into $\mathcal{N}(t_0)$.

$\square$

Remark 2.3.5. The main feature of the method we have described for associating quadruples $(g, R, P, S)$ to quadruples $(\mathcal{M}, g, \gamma, \mathcal{P})$ consists in the fact that, in the Lorentzian case, if $\gamma$ is non spacelike, then the bilinear form $g$ is positive definite. In particular, from Proposition 2.3.4 it follows that, if $(\mathcal{M}, g)$ is Lorentzian and $\gamma$ is non spacelike, then $g$ is positive definite in $\mathcal{J}[t_0]$. The positivity of $g$ in $IR^n$ will be used later (see Remark 6.2.5) to derive an alternative proof of the Morse index theorem for non spacelike Lorentzian geodesics.

A different way of associating a quadruple $(g, R, P, S)$ to a quadruple $(\mathcal{M}, g, \gamma, \mathcal{P})$ is to consider a trivialization of the entire tangent bundle along $\gamma$. Using this approach, one unifies the construction for the lightlike and the non lightlike case; we will need this construction in Section 5, where we will discuss a problem of approximation of lightlike geodesics by timelike geodesics. This construction will not introduce substantial modifications of the solution space for the differential problem. Namely, the statement of Proposition 2.3.4 is trivial if the association of quadruple is understood in this sense. It follows that, by using the two different associations of quadruples, we get the same $(P, S)$-focal instants, with the same multiplicities and signatures, and thus the same focal index.

We remark also that the statement and the proof of Proposition 2.3.1 can be adapted to the case that the association of quadruples is made by trivializing the whole tangent bundle.

2.4. The symplectic structure associated to a differential problem

Given the perfect analogy between the geometrical and the differential problem, as given by Proposition 2.3.1 and Proposition 2.3.4, we will henceforth concentrate our attention on an admissible quadruple for the differential problem $(g, R, P, S)$ in $IR^n$.

Given two solutions $J_1$ and $J_2$ of the differential equation (2.2.1), the quantity

\begin{equation}
\sigma(t) = g(J_1(t), J_2'(t)) - g(J_1'(t), J_2(t))
\end{equation}

is constant in $[a, b]$. Namely, a straightforward calculation using equation (2.2.1) shows that $\sigma'$ vanishes identically. This motivates the following definition:

Definition 2.4.1. The symplectic form $\omega$ on $IR^{2n}$ associated to $g$ is given by:

$$\omega[(x_1, x_2), (y_1, y_2)] = g(x_1, y_2) - g(x_2, y_1).$$
The nondegeneracy of \( \omega \) follows easily from the nondegeneracy of \( g \).

The initial conditions \( (J(a), J'(a)) \in B^{2n} \) determine uniquely a solution of (2.2.1), therefore the space of solutions of (2.2.1) can be identified with \( B^{2n} \). For all \( t \in [a, b] \), we have a linear automorphism \( \Psi(t) \) of \( B^{2n} \) satisfying

\[
\Psi(t)[(J(a), J'(a))] = (J(t), J'(t)),
\]

for every solution \( J \) of (2.2.1). This automorphisms are implemented by what is usually called the fundamental matrix of the first order linear differential system associated to (2.2.1). Observe that \( t \mapsto \Psi(t) \) is a curve of class \( C^1 \) in the general linear group \( GL(2n, B) \) which satisfies \( \Psi(0) = Id \).

Using the fact that the quantity (2.4.1) is constant, it is also easy to observe that \( \Psi(t) \) preserves the symplectic form \( \omega \) for all \( t \):

\[
\omega[\Psi(t)x, \Psi(t)y] = \omega[x, y], \quad \forall x, y \in B^{2n},
\]

hence, \( \Psi(t) \) is a curve in the symplectic group of \( B^{2n} \) corresponding to \( \omega \).

The important observation here is that \( \omega \) vanishes on the \( n \)-dimensional subspace of \( B^{2n} \) determined by the initial conditions (2.2.2). Namely, if \( J_1, J_2 \in J \), then \( J_i(a) \in P \) and \( J_i'(a) + S[J_i(a)] \in P\perp \) for \( i = 1, 2 \), and:

\[
\omega[(J_1(a), J_1'(a)), (J_2(a), J_2'(a))] = \omega(J_1(a), J_2'(a)) - \omega(J_1'(a), J_2(a)) = \omega(J_1(a), -S[J_2(a)]) - \omega(-S[J_1(a)], J_2(a)) = 0,
\]

where the last equality follows from the \( g \)-symmetry of \( S \).

Summarizing the facts that (2.4.1) is constant and that \( \omega \) vanishes on the space of initial conditions of \((P, S)\)-solutions, we have the following identity:

\[
g(J_1'(t), J_2'(t)) = g(J_1(t), J_2'(t)), \quad \forall J_1, J_2 \in J,
\]

for all \( t \in [a, b] \).

2.5. On the discreteness of the set of \((P, S)\)-focal instants

We give some conditions that guarantee the discreteness of the set of \((P, S)\)-focal instants.

**Proposition 2.5.1.** Let \((g, R, P, S)\) be an admissible quadruple for the differential problem in \( B^n \), and let \( t_0 \) be a \((P, S)\)-focal instant. If \( g \) is non degenerate on \( J[t_0] \), then there are no \((P, S)\)-focal instants other than \( t_0 \) in some neighborhood of \( t_0 \). Moreover, there are no \((P, S)\)-focal instants in some neighborhood of the initial instant \( a \).

**Proof.** Let \( mul(t_0) = n - k > 0 \) be the multiplicity of the focal instant \( t_0 \). Let \( J_1, J_2, \ldots, J_n \) be a basis of \( J \) such that \( J_1(t_0), \ldots, J_k(t_0) \) are a basis for \( J[t_0] \) and \( J_i(t_0) = 0 \) for \( i \geq k + 1 \).

The vectors \( J_{k+1}'(t_0), \ldots, J_n'(t_0) \) are a basis of \( J[t_0]^\perp \). To prove this, we first observe that they belong to \( J[t_0]^\perp \); namely, by (2.4.4), if \( i \in \{k + 1, \ldots, n\} \) and
Corollary 2.5.2. Let $(\mathcal{M}, g, \gamma, P)$ be an admissible quadruple for the geometric problem. Assume $(\mathcal{M}, g)$ is Riemannian or Lorentzian, and in the latter case, that $\gamma$ is non spacelike. Then, there are only a finite number of $P$-conjugate points along $\gamma$. Moreover, the focal index and the geometrical index of $\gamma$ coincide:

$$i_{\text{geom}}(\gamma) = i_{\text{foc}}(\gamma).$$

Proof. It is an easy consequence of Remark 2.3.5, Proposition 2.3.4 and Proposition 2.5.1.

Remark 2.5.3. The $(P, S)$-focal instants coincide precisely with the zeroes of the function $r(t) = \det(J_1(t), \ldots, J_n(t))$, where $J_1, \ldots, J_n$ is a basis of $\mathbb{J}$. If $(g, R, P, S)$ is an admissible quadruple with $R(t)$ real analytic on $[a, b]$, then $r(t)$ is also analytic, and so its zeros are isolated. Observe indeed that $r(t)$ cannot vanish identically on $[a, b]$ because, by Proposition 2.5.1, $r(t)$ is non zero for $t$ sufficiently close to $a$, $t \neq a$. 

$j \in \{1, \ldots, k\}$, we have

$$g(J'_i(t_0), J_j(t_0)) = g(J_i(t_0), J'_j(t_0)) = g(0, J'_j(t_0)) = 0.$$

To prove the claim, we need to show that the vectors $J'_{k+1}(t_0), \ldots, J'_n(t_0)$ are linearly independent, because $\dim(\mathbb{J}[t_0]) = n - k$, by (2.2.5). To see this, observe that the fields $J_{k+1}, \ldots, J_n$ are linearly independent in $\mathbb{J}$, hence the pairs

$$(J_{k+1}(t_0), J'_{k+1}(t_0)), \ldots, (J_n(t_0), J'_n(t_0))$$

are linearly independent in $\mathbb{R}^{2n}$. The conclusion follows from the fact that $J_{k+1}(t_0) = \ldots = J_n(t_0) = 0$.

We now define a family of continuous vector fields $\tilde{J}_1, \ldots, \tilde{J}_n$ along $\gamma$, by setting:

$$\tilde{J}_j = J_j, \quad \text{for } j = 1, \ldots, k;$$

and

$$\tilde{J}_i(t) = \begin{cases} \frac{J_i(t)}{t - t_0}, & \text{if } t \neq t_0, \\ J'_i(t_0), & \text{if } t = t_0, \end{cases} \quad \text{for } i = k + 1, \ldots, n.$$ 

The vectors $\tilde{J}_1(t_0), \ldots, \tilde{J}_n(t_0)$ are now a basis for $\mathbb{R}^n$.

Namely, the first $k$ vectors $\tilde{J}_1(t_0), \ldots, \tilde{J}_k(t_0)$ are a basis for $\mathbb{J}[t_0]$, and the remaining $n - k$ vectors $\tilde{J}_{k+1}(t_0), \ldots, \tilde{J}_n(t_0)$ are a basis for $\mathbb{J}[t_0]^{-1}$; moreover, $g$ is non degenerate on $\mathbb{J}[t_0]$, which implies that $\mathbb{R}^n = \mathbb{J}[t_0] \oplus \mathbb{J}[t_0]^{-1}$.

By continuity, the vectors $\tilde{J}_1(t), \ldots, \tilde{J}_n(t)$ are a basis for $\mathbb{R}^n$ for $t$ sufficiently close to $t_0$. But that implies that, for $t$ sufficiently close to $t_0$ and $t \neq t_0$ the vectors $J_1(t), \ldots, J_n(t)$ are a basis for $\mathbb{R}^n$, which implies that there are no $(P, S)$-focal instants around $t_0$.

The case $t_0 = a$ is treated similarly, observing that $\mathbb{J}[a] = P$ and considering that $g$ is non degenerate on $P$.

We have the following immediate Corollary:
It follows easily that, if \((\mathcal{M}, \mathfrak{g}, \gamma, \mathcal{P})\) is an admissible quadruple for the geometric problem with \((\mathcal{M}, \mathfrak{g})\) analytic, then the set of \(\mathcal{P}\)-focal points along \(\gamma\) is finite.

3. GEOMETRY OF THE LAGRANGIAN GRASSMANNIAN

We have seen in Section 2 that the set \(\mathbb{J}\) can be identified with a Lagrangian subspace of the symplectic space \((\mathbb{R}^{2n}, \omega)\), i.e., a maximal subspace of \(\mathbb{R}^{2n}\) on which \(\omega\) vanishes. In view to future applications, in this Section we present the main properties and we discuss the geometrical structure of the collection of all Lagrangian subspaces of a symplectic space.

Throughout this section we will assume that \(V\) is a \(2n\)-dimensional real vector space, equipped with a symplectic form \(\omega\), i.e., a skew symmetric non degenerate bilinear form on \(V\).

3.1. Generalities on symplectic spaces

A symplectic basis of \((V, \omega)\) is a vector space basis \(e_1, \ldots, e_{2n}\) of \(V\) such that

\[
\omega[e_{n+j}, e_j] = -\omega[e_j, e_{n+j}] = 1
\]

for all \(j = 1, \ldots, n\), and \(\omega[e_i, e_j] = 0\) otherwise; the existence of a symplectic basis in \((V, \omega)\) is standard. We recall that a complex structure for \(V\) is a linear operator \(\mathcal{I} : V \mapsto V\) such that \(\mathcal{I}^2 = -\text{Id}\). A complex structure \(\mathcal{I}\) on \(V\) induces a complex vector space structure on \(V\), and \(\mathcal{I}\) becomes the scalar multiplication by the imaginary unit \(i\). A complex structure \(\mathcal{I}\) is compatible with the symplectic form \(\omega\) if the bilinear form \(\omega[\mathcal{I}\cdot, \cdot]\) is symmetric and positive definite on \(V\).

If \((V_i, \omega_i), i = 1, 2,\) are symplectic spaces of the same dimension \(2n\), a linear map \(T : V_1 \mapsto V_2\) is called a symplectomorphism if \(\omega_2(Tx, Ty) = \omega_1(x, y)\) for all \(x, y \in V_1\). Observe that a symplectomorphism \(T\) is always an isomorphism; namely, the \(n\)-th exterior powers \(\omega^n\) are volume forms in \(V_i, i = 1, 2\), which are preserved by \(T\).

We identify \(\mathbb{R}^{2n}\) with \(\mathbb{C}^n\) by considering the first \(n\) coordinates to be the real part, and the remaining coordinates to be the imaginary part. Therefore, we get a complex structure \(\mathcal{I}_0\) given by \(\mathcal{I}_0(e_j) = e_{n+j}, \mathcal{I}_0(e_{n+j}) = -e_j\), for \(j = 1, \ldots, n\), where \(\{e_i\}_{i=1}^{2n}\) is the canonical basis of \(\mathbb{R}^{2n}\). For \(x, y \in \mathbb{R}^{2n}\), we denote by \(x \cdot y\) the Euclidean inner product, and by \(\langle x, y \rangle\) the Hermitian product in \(\mathbb{C}^n \simeq \mathbb{R}^{2n}\) whose real part is \(x \cdot y\) and which is conjugate in the second variable. The canonical symplectic form \(\omega_0\) in \(\mathbb{R}^{2n}\) is the imaginary part of the Hermitian product. Observe that the canonical basis is a symplectic basis for \(\omega_0\) and \(\mathcal{I}_0\) is compatible with \(\omega_0\).

A subspace \(W\) of \(V\) will be called isotropic if \(\omega\) vanishes identically on \(W\) (by this we mean on \(W \times W\)); an \(n\)-dimensional isotropic subspace \(W\) will be called a Lagrangian subspace of \((V, \omega)\). It is easy to see that the Lagrangian subspaces coincide with the maximal isotropic subspaces of \((V, \omega)\).

Given a Lagrangian direct sum decomposition \(V = L_0 \oplus L_1\), i.e., both subspaces \(L_0\) and \(L_1\) are Lagrangian, we denote by \(\mathcal{O}_{L_0, L_1}\) the isomorphism from \(L_1\) to the dual
space $L_0^*$ given by:

$$\mathcal{D}_{L_0,L_1}(v) = \omega[v, \cdot] \bigg|_{L_0^*}, \quad \forall v \in L_1.$$  

The injectivity of $\mathcal{D}_{L_0,L_1}$ follows immediately from the non degeneracy of $\omega$. We observe that, by the anti-symmetry of $\omega$, the following identity holds:

$$\mathcal{D}_{L_1,L_0} = -(\mathcal{D}_{L_0,L_1})^*.$$  

Remark 3.1.1. The existence of a complex structure compatible with $(V, \omega)$ is proven easily. Namely, a complex structure compatible with $(V, \omega)$ is obtained as the pull-back of $\mathcal{I}_0$ by the symplectomorphism $V \mapsto \mathbb{H}^{2n}$ defined by a symplectic basis of $(V, \omega)$. Using a compatible complex structure $\mathcal{I}$, we can now prove that every Lagrangian subspace $L_0$ of $V$ admits a complementary Lagrangian subspace $L_1$. Namely, just define $L_1 = \mathcal{I}(L_0)$. Given any Lagrangian direct sum decomposition $V = L_0 \oplus L_1$, we construct a symplectic basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of $V$ by taking any linear basis $\{e_1, \ldots, e_n\}$ of $L_0$ and the linear basis $\{f_1, \ldots, f_n\}$ of $L_1$ whose image by $\mathcal{D}_{L_0,L_1}$ is the dual basis of $\{e_1, \ldots, e_n\}$. This implies that every linear isomorphism $\psi : L_0 \mapsto \mathbb{R}^n \oplus \{0\}$ extends to a symplectomorphism $\psi$ from $(V, \omega)$ to $(\mathbb{H}^{2n}, \omega_0)$ which carries $L_1$ to $\{0\} \oplus \mathbb{R}^n$.

The symplectic group $\text{Sp}(V, \omega)$ is the Lie subgroup of $\text{GL}(V)$ consisting of symplectomorphisms of $(V, \omega)$; its Lie algebra $\text{sp}(V, \omega)$ consists of all linear maps $H : V \mapsto V$ such that:

$$\omega(Hx, y) + \omega(x, Hy) = 0, \quad \forall x, y \in V.$$  

Equation (3.1.3) is equivalent to the symmetry of the bilinear form $\omega(H \cdot, \cdot)$ on $V$.

The group $\text{Sp}(\mathbb{H}^{2n}, \omega_0)$ is also denoted by $\text{Sp}(n, \mathbb{H})$; the subgroup of $\text{GL}(2n, \mathbb{H})$ consisting of unitary transformations with respect to the canonical Hermitian product is denoted by $\text{U}(n)$. Since $\omega_0$ is the imaginary part of the Hermitian product which is preserved by elements in $\text{U}(n)$, we see that $\text{U}(n)$ is a subgroup of $\text{Sp}(n, \mathbb{H})$.

By $\text{O}(n)$ we mean the orthogonal group in $\mathbb{R}^n$, and by $\text{SO}(n)$ the subgroup of $\text{O}(n)$ consisting of matrices with determinant equal to 1. Every linear map $\psi : \mathbb{R}^n \mapsto \mathbb{R}^n$ has a unique $\mathbb{C}$-linear extension to a map $\psi^\mathbb{C} : \mathbb{C}^n \mapsto \mathbb{C}^n$. If $\psi \in \text{O}(n)$, then $\psi^\mathbb{C} \in \text{U}(n)$, which identifies $\text{O}(n)$ with the subgroup of $\text{U}(n)$ consisting of those maps that preserve the subspace $\mathbb{R}^n \oplus \{0\} \in \mathbb{R}^{2n}$.

It is well known that $\text{U}(n)$, $\text{O}(n)$ and $\text{SO}(n)$ are compact Lie groups, and $\text{Sp}(n, \mathbb{H})$, or more in general $\text{Sp}(V, \omega)$, is a non compact Lie group.

3.2. The Lagrangian Grassmannian

For $k = 0, \ldots, 2n$, we denote by $G_k(V)$ the Grassmannian of all the $k$-dimensional subspaces of $V$. We will be interested in the subset $\Lambda = \Lambda(V, \omega) \subset G_n(V)$ consisting of all the Lagrangian subspaces of $(V, \omega)$:

$$\Lambda = \Lambda(V, \omega) = \left\{ L : L \text{ is a Lagrangian subspace of } (V, \omega) \right\}.$$
For simplicity, we will omit the argument \((V, \omega)\) whenever there is no risk of confusion, and we will write simply \(\Lambda\).

We recall that \(G_k(V)\) has the structure of a real analytic manifold of dimension \(k(2n - k)\); given a direct sum decomposition \(V = W_0 \oplus W_1\), where \(\dim(W_0) = k\), a local chart of \(G_k(V)\) is defined in an open neighborhood of \(W_0\) taking values in the vector space \(\mathcal{L}(W_0, W_1)\) of all linear maps \(T : W_0 \mapsto W_1\). Namely, to every \(W \in G_k(V)\) which is transversal to \(W_1\), i.e., \(W \cap W_1 = \{0\}\), one associates the unique 
\[T \in \mathcal{L}(W_0, W_1)\]
whose graph in \(W_0 \oplus W_1 = V\) is \(W\). We now give a description of the restrictions to \(\Lambda\) of the local charts defined on \(G_n(V)\) by this construction.

Given any real vector space \(Z\), we denote by \(B(Z, \mathbb{R})\) and \(B_{\text{sym}}(Z, \mathbb{R})\) respectively the space of bilinear forms and symmetric bilinear forms on \(Z\). There is an identification of \(B(Z, \mathbb{R})\) with \(\mathcal{L}(Z, Z^*)\) obtained by associating to each \(B \in B(Z, \mathbb{R})\) the map \(v \mapsto B(v, \cdot)\).

**Definition 3.2.1.** Given a Lagrangian direct sum decomposition \(V = L_0 \oplus L_1\), for all \(W \in G_n(V)\) transverse to \(L_1\), i.e., \(W \cap L_1 = \{0\}\), we define \(\phi_{L_0, L_1}(W) \in B(L_0, \mathbb{R}) \simeq \mathcal{L}(L_0, L_0^*)\) by

\[\phi_{L_0, L_1}(W) = \mathcal{D}_{L_0, L_1} \circ T,\]

where \(T\) is the unique linear operator \(T : L_0 \mapsto L_1\) whose graph in \(V = L_0 \oplus L_1\) is \(W\).

The map \(\phi_{L_0, L_1}\) is a diffeomorphism from the open set of \(G_n(V)\) consisting of subspaces transverse to \(L_1\) onto \(B(L_0, \mathbb{R})\).

Observe that \(\phi_{L_0, L_1}\) is simply one of the local charts on \(G_n(V)\) described above, up to the composition with the linear isomorphism \(\mathcal{D}_{L_0, L_1}\). We now show how the maps \(\phi_{L_0, L_1}\) induce a submanifold structure on \(\Lambda\).

**Proposition 3.2.2.** The set \(\Lambda\) is an analytic embedded submanifold of \(G_n(V)\) of dimension \(\frac{1}{2}n(n + 1)\); each map \(\phi_{L_0, L_1}\) restricts to a local chart on \(\Lambda\) which maps the open set of Lagrangian subspaces transverse to \(L_1\) onto \(B_{\text{sym}}(L_0, \mathbb{R})\).

For all \(L_0 \in \Lambda\) the tangent space \(T_{L_0}\Lambda\) is canonically isomorphic to \(B_{\text{sym}}(L_0, \mathbb{R})\); more precisely, this isomorphism is given by the differential at \(L_0\) of any coordinate map \(\phi_{L_0, L_1}\), and this isomorphism does not depend on the choice of the complementary Lagrangian \(L_1\).

Moreover, the isomorphisms \(T_{L_0}\Lambda \simeq B_{\text{sym}}(L_0, \mathbb{R})\) are natural in the sense that, given a symplectomorphism \(\psi\) of \((V, \omega)\), we have the following commutative diagram:

\[
\begin{array}{c}
\mathcal{T}_{L_0}\Lambda \xrightarrow{d\psi_{L_0}} \mathcal{T}_{\psi(L_0)}\Lambda \\
\downarrow \quad \downarrow \\
B_{\text{sym}}(L_0, \mathbb{R}) \xrightarrow{\psi_*} B_{\text{sym}}(\psi(L_0), \mathbb{R}),
\end{array}
\]  

where the vertical arrows are the canonical isomorphisms, \(\hat{\psi} : \Lambda \mapsto \Lambda\) is the diffeomorphism given by \(L \mapsto \psi(L)\), and \(\psi_*\) is the push-forward operator given by \(B \mapsto B(\psi^{-1}, \psi^{-1})\).
Proof. Let \( L \in G_n(V) \) be transverse to \( L_1 \), and let \( T : L_0 \mapsto L_1 \) be the linear operator whose graph in \( V = L_0 \oplus L_1 \) is \( L \). Then, \( L \) is Lagrangian if and only if \( \omega[v + T(v),w + T(w)] = 0 \) for all \( v,w \in L_0 \), i.e., if and only if
\[
\omega[v,T(w)] + \omega[T(v),w] = 0, \quad \forall v,w \in L_0.
\]
This is just the symmetry of the bilinear form \( \phi_{L_0,L_1}(L) = \mathfrak{D}_{L_0,L_1} \circ T \).

We now prove that the differential \( d\phi_{L_0,L_1}(L) \) does not depend on the choice of the complementary Lagrangian \( L_1 \); observe that, by Remark 3.1.1, we can always find complementary Lagrangians to \( L_0 \). To prove the claim, let \( L_1 \) and \( L_2 \) be two complementary Lagrangians to \( L_0 \); the two charts \( \phi_{L_0,L_1} \) and \( \phi_{L_0,L_2} \) map \( L_0 \) to the zero bilinear map. We have to prove that the differential of the transition map from \( \phi_{L_0,L_1} \) to \( \phi_{L_0,L_2} \) at 0 is the identity of \( \mathfrak{B}_{sym}(L_0,\mathbb{R}) \). The transition map is given by:
\[
\mathcal{L}(L_0, L_0^0) \ni B \longmapsto B \circ (\text{Id} + \rho \circ \mathfrak{D}_{L_0,L_1}^{-1} \circ B)^{-1} \in \mathcal{L}(L_0, L_0^0),
\]
where \( \rho \) is the restriction to \( L_1 \) of the projection \( L_0 \oplus L_2 \mapsto L_0 \) and \( \text{Id} \) is the identity on \( L_0 \). The differential of (3.2.2) at \( B = 0 \) is easily computed to be the identity.

It remains to prove the commutativity of (3.2.1). Let \( \Omega \) and \( \Omega' \) be the domains of the charts \( \phi_{L_0,L_1} \) and \( \phi_{\psi(L_0),\psi(L_1)} \) respectively. Then, it is easy to check the commutativity of the diagram:
\[
\begin{array}{ccc}
\Lambda \supset \Omega & \xrightarrow{\psi} & \Omega' \subset \Lambda \\
\phi_{L_0,L_1} \downarrow & & \downarrow \phi_{\psi(L_0),\psi(L_1)} \\
\mathfrak{B}_{sym}(L_0,\mathbb{R}) & \xrightarrow{\psi} & \mathfrak{B}_{sym}(\psi(L_0),\mathbb{R})
\end{array}
\]
The conclusion follows by differentiating (3.2.3). \( \square \)

The action of \( \text{Sp}(V,\omega) \) on \( \Lambda \) induces a map \( \text{sp}(V,\omega) \mapsto T_{L_0}\Lambda \) for every \( L_0 \in \Lambda \). This map is described in the following:

**Proposition 3.2.3.** Let \( L_0 \in \Lambda \); define the map \( \kappa_{L_0} : \text{Sp}(V,\omega) \mapsto \Lambda \) by \( \kappa_{L_0}(\psi) = \psi(L_0) \). The differential \( d\kappa_{L_0}(\text{Id}) \) of \( \kappa_{L_0} \) at the neutral element \( \text{Id} \in \text{Sp}(V,\omega) \) maps each \( H \in \text{sp}(V,\omega) \) to the symmetric bilinear form \( d\kappa_{L_0}(\text{Id})[H] \in \mathfrak{B}_{sym}(L_0,\mathbb{R}) \) given by the restriction of \( \omega(H\cdot,\cdot) \) to \( L_0 \).

**Proof.** Let \( L_1 \) be any complementary Lagrangian to \( L_0 \), \( V = L_0 \oplus L_1 \), and let \( \phi_{L_0,L_1} \) be the corresponding coordinate map around \( L_0 \). Recall that the differential \( d\phi_{L_0,L_1} \) at \( L_0 \) is the isomorphism used to identify \( T_{L_0}\Lambda \) with \( \mathfrak{B}_{sym}(L_0,\mathbb{R}) \) (see Proposition 3.2.2). Let \( \pi_0 : V \mapsto L_0 \) and \( \pi_1 : V \mapsto L_1 \) be the projections onto the summands.

In the chart \( \phi_{L_0,L_1} \), the map \( \kappa_{L_0} \) is given by:
\[
\psi \mapsto \phi_{L_0,L_1} \circ \kappa_{L_0}(\psi) = \mathfrak{D}_{L_0,L_1} \circ \psi_{10} \circ \psi_{00}^{-1},
\]
where \( \psi_{00} = \pi_0 \circ (\psi|_{L_0}) \) and \( \psi_{10} = \pi_1 \circ (\psi|_{L_0}) \). Formula (3.2.4) holds for \( \psi \) in a neighborhood of \( \text{Id} \in \text{Sp}(V,\omega) \), where \( \psi_{00} \) is invertible.

The differential of (3.2.4) is then easily computed as:
\[
\text{sp}(V,\omega) \ni H \longmapsto d\kappa_{L_0}(\text{Id})[H] = \mathfrak{D}_{L_0,L_1} \circ H_{10},
\]
where \( H_{10} = \pi_1 \circ (H|_{L_0}) \).

The conclusion follows at once from the definition of \( D_{L_0, L_1} \). \( \square \)

We recall that, if \( f : M \mapsto N \) is a smooth map between differentiable manifolds, two smooth vector fields \( X \) and \( Y \) on \( M \) and \( N \) respectively are said to be \( f \)-related if \( df(p)[X(p)] = Y(f(p)) \) for all \( p \in M \). If \( X \) and \( Y \) are \( f \)-related, then \( f \) maps integral curves of \( X \) into integral curves of \( Y \).

If a Lie group \( G \) acts on the left on the manifold \( M \), then to each \( X \) in the Lie algebra of \( G \) we associate a vector field \( X^* \) in \( M \) given by \( X^*(p) = \frac{d}{dt}\kappa_p(1)[X], \) where \( \kappa_p : G \mapsto M \) is the map \( g \mapsto g \cdot p \) and \( 1 \) is the neutral element of \( G \). For all \( p \in M \), the vector field \( X^* \) is \( \kappa_p \)-related to the right invariant vector field on \( G \) associated to \( X \).

Considering \( G = \text{Sp}(V, \omega) \) and \( M = \Lambda \), we are not motivated to give the following definition:

**Definition 3.2.4.** Let \( H \in \text{sp}(V, \omega) \), the vector field \( H^* \) in \( \Lambda \) associate to each \( L \in \Lambda \) the vector \( H^*(L) \in T_L\Lambda \simeq B_{\text{sym}}(L, \mathbb{R}) \) given by the restriction of \( \omega(H\cdot, \cdot) \) to \( L \).

The vector fields \( H^* \) will be used to project differential equations in \( \text{Sp}(V, \omega) \) to differential equations in \( \Lambda \).

Using group actions, we now give a description of the geometrical structure of \( \Lambda \) as a homogeneous space.

**Proposition 3.2.5.** \( \Lambda \) is diffeomorphic to \( U(n)/O(n) \); in particular, \( \Lambda \) is compact and connected.

**Proof.** By choosing a symplectic basis for \( (V, \omega) \), we reduce the problem to the case \( V = \mathbb{R}^{2n} \) and \( \omega = \omega_0 \). The group \( \text{Sp}(n, \mathbb{R}) \) acts smoothly on \( \Lambda \); we show that the restriction of this action to \( U(n) \) is transitive on \( \Lambda \). Let \( L_0, L_1 \in \Lambda \) be fixed; we consider bases \( B_0 \) and \( B_1 \) of \( L_0 \) and \( L_1 \) respectively, which are orthonormal relatively to the Euclidean inner product of \( \mathbb{R}^{2n} \). Since the imaginary part of the Hermitian product is \( \omega_0 \), and \( \omega_0 \) vanishes on both \( L_0 \) and \( L_1 \), then \( B_0 \) and \( B_1 \) are orthonormal basis of \( \mathbb{C}^n \simeq \mathbb{R}^{2n} \) with respect to the Hermitian product. Hence, there exists an element of \( U(n) \) that carries \( B_0 \) to \( B_1 \), and \( U(n) \) acts transitively on \( \Lambda \).

Obviously, the isotropy group of \( L_0 = \mathbb{R}^n \oplus \{0\} \) is \( O(n) \), which concludes the proof. \( \square \)

We now give the following definition:

**Definition 3.2.6.** Let \( L_0 \in \Lambda \) and \( k = 0, 1, \ldots, n \) be fixed. We denote by \( \Lambda_k(L_0) \) the subset of \( \Lambda \) consisting of Lagrangians \( L \) with \( \dim(L \cap L_0) = k \). We also define the sets \( \Lambda_{\leq k}(L_0) \) and \( \Lambda_{\geq k}(L_0) \) by:

\[
\Lambda_{\leq k}(L_0) = \bigcup_{i=0}^{k} \Lambda_i(L_0), \quad \Lambda_{\geq k}(L_0) = \bigcup_{i=k}^{n} \Lambda_i(L_0).
\]

**Remark 3.2.7.** Clearly, \( \Lambda_0(L_0) \) is precisely the set of all Lagrangians complementary to \( L_0 \). It is an open set of \( \Lambda \), since it is the domain of any coordinate map.
\( \phi_{L_1, L_0} \); moreover, it is diffeomorphic to a vector space by Proposition 3.2.2. For \( k = 0, \ldots, n \), we observe that \( \Lambda_{\leq k}(L_0) \) is open, and so \( \Lambda_{\geq k}(L_0) \) is closed in \( \Lambda \). Namely, let \( L \in \Lambda_{\leq k}(L_0) \); we prove that \( L \) admits a neighborhood in \( G_n(V) \) consisting only of subspaces \( W \) such that \( \dim(W \cap L_0) \leq k \). For, simply consider a subspace \( W_1 \) of \( V \) which is complementary to both \( L_0 \) and \( L \); then, given a linear operator \( T : L_0 \mapsto W_1 \), its graph in \( L_0 \oplus W_1 = V \) intersects \( L_0 \) in a subspace of dimension less than or equal to \( k \) if and only if \( \dim(\ker(T)) \leq k \). The conclusion follows easily by observing that the set of linear operators \( T \in \mathcal{L}(L_0, W_1) \) such that \( \dim(\ker(T)) \leq k \) is open.

Given \( L_0 \in \Lambda \), we denote by \( \text{Sp}(V, \omega, L_0) \) the closed subgroup of \( \text{Sp}(V, \omega) \) consisting of elements \( \psi \) such that \( \psi(L_0) = L_0 \); by \( \text{Sp}_+(V, \omega, L_0) \) we mean the subgroup of \( \text{Sp}(V, \omega, L_0) \) consisting of those \( \psi \) whose restriction to \( L_0 \) is orientation preserving.

The Lie algebra \( \text{sp}(V, \omega, L_0) \) of both \( \text{Sp}(V, \omega, L_0) \) and \( \text{Sp}_+(V, \omega, L_0) \) is the subalgebra of \( \text{sp}(V, \omega) \) consisting of those \( H \) such that \( H(L_0) \subset L_0 \).

Clearly, \( \text{Sp}(V, \omega, L_0) \) and \( \text{Sp}_+(V, \omega, L_0) \) act on all the spaces \( \Lambda_k(L_0) \) introduced in Definition 3.2.6. These actions are transitive on each \( \Lambda_k(L_0) \), as we prove in the following:

**Proposition 3.2.8.** For all \( k = 0, \ldots, n \), the group \( \text{Sp}_+(V, \omega, L_0) \) acts transitively on \( \Lambda_k(L_0) \).

**Proof.** By choosing a symplectic basis of \( (V, \omega) \), we can reduce to the case \( V = \mathbb{R}^{2n} \), \( \omega = \omega_0 \) and \( L_0 = \mathbb{R}^n \oplus \{0\} \) (see Remark 3.1.1); let \( \{e_1, \ldots, e_{2n}\} \) be the canonical basis of \( \mathbb{R}^{2n} \). Let \( L \) be any Lagrangian such that \( \dim(L \cap L_0) = k \); we show that there is an element \( \psi \in \text{Sp}_+(V, \omega, L_0) \) such that \( \psi(L) \cap L_0 = \mathbb{R}^k \oplus \{0\} \). Let \( \psi \in \text{SO}(n) \) be a linear isometry of \( \mathbb{R}^n \) such that \( \psi(L \cap L_0) = \mathbb{R}^k \oplus \{0\} \); now consider the complex linear extension of \( \psi \) to \( \mathbb{C}^n \cong \mathbb{R}^{2n} \). Such a map has the required property.

Let \( L_1 \) be the subspace generated by \( \{e_1, \ldots, e_k, e_{n+k+1}, \ldots, e_{2n}\} \). Then, \( L_1 \) is Lagrangian, and \( L_1 \cap L_0 = \mathbb{R}^k \oplus \{0\} \). It remains to prove that, given a Lagrangian \( L \) with \( L \cap L_0 = \mathbb{R}^k \oplus \{0\} \), there exists an element \( \psi \in \text{Sp}_+(V, \omega, L_0) \) such that \( \psi(L) = L_1 \).

To prove this claim, we define the following spaces. Let \( V_1 \) be the space generated by \( \{e_1, \ldots, e_k, e_{n+1}, \ldots, e_{n+k}\} \); \( V_2 \) be generated by \( \{e_{k+1}, \ldots, e_n, e_{n+k+1}, \ldots, e_{2n}\} \) and \( S \) be generated by \( \{e_1, \ldots, e_n, e_{n+k+1}, \ldots, e_{2n}\} \). Observe that \( S \) is the orthogonal complement of \( \mathbb{R}^k \oplus \{0\} \) with respect to \( \omega_0 \); also, \( \mathbb{R}^{2n} = V_1 \oplus V_2 \), and \( \omega_0 \) restricts to the canonical symplectic forms of \( V_1 \cong \mathbb{R}^{2k} \) and of \( V_2 \cong \mathbb{R}^{n-k} \), that will be still denoted by \( \omega_0 \). Let \( \pi : S \mapsto V_2 \) be the restriction to \( S \) of the projection \( V_1 \oplus V_2 \mapsto V_2 \). It is easy to check that \( \omega_0(\pi(x), \pi(y)) = \omega_0(x, y) \) for all \( x, y \in S \). Since \( L \) is Lagrangian, we have \( L \subset S \); moreover, it is easily seen that \( \pi(L) \) is Lagrangian in \( V_2 \). Since \( L \cap L_0 = \mathbb{R}^k \oplus \{0\} \), we have that \( \pi(L) \) is complementary to \( \mathbb{R}^{n-k} \oplus \{0\} \) in \( V_2 \cong \mathbb{R}^{n-k} \oplus \mathbb{R}^{n-k} \). By Remark 3.1.1, there exists a symplectomorphism \( \varphi \) of \( (V_2, \omega_0) \) that is the identity on \( \mathbb{R}^{n-k} \oplus \{0\} \) and carries \( \pi(L) \) into \( \mathbb{R}^{n-k} \oplus \mathbb{R}^{n-k} = \pi(L_1) \).

Finally, the required element \( \psi \in \text{Sp}_+(V, \omega, L_0) \) is given by:

\[
\psi|_{V_1} = \text{Id}, \quad \psi|_{V_2} = \varphi.
\]
Indeed, \( \psi(L) = L_1 \), because \( \psi(L) \) and \( L_1 \) are both subspaces of \( S \) containing \( \text{Ker}(\pi) \) that have the same image under \( \pi \). This concludes the proof.

**Corollary 3.2.9.** Given any two Lagrangians \( L_0 \) and \( L \) in \( \Lambda \), there exists \( L_1 \in \Lambda \) which is complementary to both \( L_0 \) and \( L \). In particular, the domain of the coordinate map \( \phi_{L_0, L_1} \) contains both \( L_0 \) and \( L \).

**Proof.** By choosing a symplectic basis of \((V, \omega)\), we can reduce to the case \( V = \mathbb{R}^{2n} \), \( \omega = \omega_0 \) and \( L_0 = \mathbb{R}^n \oplus \{0\} \) (see Remark 3.1.1).

Let \( \{e_1, \ldots, e_{2n}\} \) be the canonical basis of \( \mathbb{R}^{2n} \) and \( L_2 \) be the subspace generated by \( \{e_1, \ldots, e_k, e_{n+k+1}, \ldots, e_{2n}\} \), where \( k = \dim(L_0 \cap L) \). Since \( L_2 \) and \( L \) are both in \( \Lambda_k(L_0) \), Proposition 3.2.8 gives a symplectomorphism \( \psi \) of \((\mathbb{R}^{2n}, \omega_0)\) such that \( \psi(L_0) = L_0 \) and \( \psi(L_2) = L \). Observe that the diagonal

\[
\Delta = \{(v, v) : v \in \mathbb{R}^n\}
\]

is a Lagrangian subspace of \( \mathbb{R}^{2n} \) which is complementary to both \( L_0 \) and \( L_2 \); the desired Lagrangian \( L_1 \) is, for instance, \( \psi(\Delta) \).

Although we will not need it, we observe that the existence of complementary Lagrangians can be proven in a much more general situation. Namely, using Baire’s Theorem, one proves that, given a sequence \( \{L_k\}_{k \in \mathbb{N}} \) of Lagrangians in \( \Lambda \), the set \( \bigcap_{k \in \mathbb{N}} \Lambda_0(L_k) \) of their common complementary Lagrangians is dense in \( \Lambda \). Each \( \Lambda_0(L_k) \) is open dense because its complement in \( \Lambda \) is a finite union of embedded submanifolds of lower dimension, as we will see next.

**Proposition 3.2.10.** For all \( k = 0, \ldots, n \) and all \( L_0 \in \Lambda \), \( \Lambda_k(L_0) \) is a connected embedded analytic submanifold of \( \Lambda \) having codimension equal to \( \frac{1}{2}k(k + 1) \). For \( L \in \Lambda_k(L_0) \), the tangent space \( T_L \Lambda_k(L_0) \subset B_{\text{sym}}(L, \mathbb{R}) \) is equal to the space of symmetric bilinear forms on \( L \) that vanish in \( L \cap L_0 \).

The submanifold \( \Lambda_1(L_0) \), that has codimension 1 in \( \Lambda \) has a transverse orientation in \( \Lambda \), namely, for \( L \in \Lambda_1(L_0) \), a vector \( B \in B_{\text{sym}}(L) \simeq T_L \Lambda \) is positive if \( B \) is positive definite on the one-dimensional space \( L \cap L_0 \). Moreover, the transverse orientation of \( \Lambda_1(L_0) \) in \( \Lambda \) is natural in the sense that, given \( \psi \in Sp(V, \omega, L_0) \), the diffeomorphism \( L \mapsto \psi(L) \) of \( \Lambda \) is orientation preserving.

**Proof.** To prove that \( \Lambda_k(L_0) \) is an embedded submanifold of \( \Lambda \), observe first that, by Proposition 3.2.8, \( \Lambda_k(L_0) \) is an orbit of the action of \( Sp_+(V, \omega, L_0) \). It follows that \( \Lambda_k(L_0) \) is an immersed submanifold, i.e., it does not necessarily have the relative topology. By [20, Theorem 2.9.7], an orbit is embedded if and only if it is locally closed, i.e., it is the intersection of an open and a closed set. Now, recall Remark 3.2.7 and simply observe that \( \Lambda_k(L_0) = \bigcap_{k \leq k} \bigcap_{k \leq k}(L_0) \).

We now compute the codimension of \( \Lambda_k(L_0) \) in \( \Lambda \).

Let \( L_1 \) be any Lagrangian complementary to \( L_0 \); the Lie group \( Sp(V, \omega, L_0) \) is diffeomorphic to \( GL(L_0) \times B_{\text{sym}}(L_1, \mathbb{R}) \). Namely, we have a diffeomorphism:

\[
F : GL(L_0) \times B_{\text{sym}}(L_1, \mathbb{R}) \mapsto Sp(V, \omega, L_0)
\]
that associates to each pair \((\alpha, \beta)\) the symplectomorphism \(\psi = F(\alpha, \beta)\) of \((V, \omega)\) whose restriction to \(L_0\) is \(\alpha\) and whose restriction to \(L_1\) is equal to:
\[
\psi_{|L_1} = \alpha \circ D_{L_1, L_0}^{-1} \circ \beta + D_{L_0, L_1}^{-1} \circ (\alpha^*)^{-1} \circ D_{L_0, L_1},
\]
where \(\alpha^* \in \text{GL}(L_0^*, IR)\) denotes the transpose map of \(\alpha\), and \(\beta\) is seen as a linear map \(\beta : L_1 \mapsto L_1^*\).

It follows that the dimension of \(\text{Sp}(V, \omega, L_0)\) is equal to \(n^2 + \frac{1}{2}n(n + 1)\). The group \(\text{Sp}_+(V, \omega, L_0)\) is the image under \(F\) of the product \(\text{GL}_+(L_0) \times B_{\text{sym}}(L_1, IR)\), where \(\text{GL}_+(L_0)\) is the group of orientation preserving isomorphisms of \(L_0\). It follows that \(\text{Sp}_+(V, \omega, L_0)\) and hence \(\Lambda_k(L_0)\) is connected.

Now, we choose an element \(L \in \Lambda_k(L_0)\) and we calculate the dimension of its isotropy group in \(\text{Sp}(V, \omega, L_0)\). To this aim, let \(S \subset L_0\) be any \(k\)-dimensional subspace and let \(S' \subset L_1\) be the image under \(D_{L_0, L_1}^{-1}\) of the annihilator of \(S\) in \(L_0^*\). Then, \(L = S \oplus S'\) is a Lagrangian in \(V\) and \(L \in \Lambda_k(L_0)\).

The isotropy group of \(L\) is the image under \(F\) of the set of pairs \((\alpha, \beta)\) such that \(\alpha(S) \subset S\) and \(\beta\) vanishes on \(S'\). It follows that the dimension of this isotropy group is \(n^2 + \frac{1}{2}k(k + 1)\). Hence, using Proposition 3.2.2, the codimension of \(\Lambda_k(L_0)\) in \(\Lambda\) is computed as \(\frac{1}{2}k(k + 1)\).

We now compute the tangent space \(T_L \Lambda_k(L_0)\) at any point \(L \in \Lambda_k(L_0)\). Such a space is given by the image of \(\text{sp}(V, \omega, L_0)\) under the differential \(d\kappa_L(\text{Id})\), defined in Proposition 3.2.3:
\[
T_L \Lambda_k(L_0) = \left\{ \omega(H \cdot , \cdot )_{|L} : H \in \text{sp}(V, \omega, L_0) \right\}.
\]

The elements of \(T_L \Lambda_k(L_0)\) vanish on \(L_0 \cap L\). A simple dimension counting shows that \(T_L \Lambda_k(L_0)\) consists precisely of those elements. This completes the proof of the first part of the statement.

We now consider the submanifold \(\Lambda_1(L_0)\); by the formula computed above, its codimension in \(\Lambda\) is equal to 1. The transverse orientation is well defined in the statement of the Proposition, and the naturality follows easily from the commutative diagram (3.2.1) in Proposition 3.2.2. \(\square\)

**Remark 3.2.11.** In Proposition 3.2.10 we have given a description of a tangent space \(T_L \Lambda_k(L_0)\) as a subspace of \(B_{\text{sym}}(L, IR)\), where \(B_{\text{sym}}(L, IR)\) is identified with \(T_L \Lambda\) by means of a coordinate map \(\phi_{L_0, L_1}\) (see Proposition 3.2.2).

In many situations we will have to deal with curves \(L(t)\) of Lagrangians, and to study the tangent space \(T_{L(t)} \Lambda\) it will be more convenient to work with a *fixed* coordinate map \(\phi_{L_0, L_1}\) rather than using variable charts \(\phi_{L(t), L_1}\).

For this reason, we now describe the transition map from a coordinate map \(\phi_{L_0, L_1}\) to \(\phi_{L, L_1}\), where \(L_1\) is a complementary Lagrangian to both \(L_0\) and \(L\) (see Corollary 3.2.9).

Let \(\eta : L_0 \mapsto L\) be the isomorphism obtained by the restriction to \(L_0\) of the projection \(L \oplus L_1 \mapsto L\). The transition map from \(\phi_{L_0, L_1}\) to \(\phi_{L, L_1}\) is now easily computed as:
\[
B_{\text{sym}}(L_0, IR) \ni B \mapsto \phi_{L, L_1}(L_0) + \eta_*(B),
\]
continuous curve such that

is defined as follows. If

where

homotopy class of the loop transition map is

orientation of

The isomorphism

point is given by

1 \in \text{group of}

wards this goal, we consider the transitive action of

The push-forward operator does not affect the positivity of a bilinear form, a given

in the statement of Proposition 3.2.10. Observe also that, for

\ell \in \Lambda_1(L_0), since the push-forward operator does not affect the positivity of a bilinear form, a given

B \in \text{B}_{\text{sym}}(L_0, \mathbb{R}) \text{ such that } d\phi_{L_0, L_1}(L)^{-1}[B] \text{ is a positive vector in the transverse orientation of } \Lambda_1(L_0) \text{ if and only if } B \text{ is positive definite on } L \cap L_0.

4. INTERSECTION THEORY: THE MASLOV INDEX

The purpose of this Section is to associate an integer number to each pair \((\ell, L_0)\), where \(\ell\) is a continuous curve in the Lagrangian Grassmannian \(\Lambda\) studied in the previous section, and \(L_0 \in \Lambda\). Such a number, that will be defined to be the Maslov Index of \(\ell\) with respect to \(L_0\), in the generic case will count (algebraically) the number of intersections of \(\ell\) with \(\Lambda_{\geq 1}(L_0)\).

We will assume throughout the Section that \((V, \omega)\) is a fixed symplectic space of dimension \(2n\), and that \(\Lambda\) is the associated Lagrangian Grassmannian.

4.1. The fundamental group of the Lagrangian Grassmannian

We begin with an easy result on the fundamental group of homogeneous spaces:

**Lemma 4.1.1.** Let \(G\) be a connected Lie group and \(K\) be a closed subgroup of \(G\); we denote by \(p : G \mapsto G/K\) the quotient map. Let \(q : \tilde{G} \mapsto G\) be the universal covering group of \(G\), \(\tilde{K} = q^{-1}(K)\) and \(\tilde{K}_0\) be the connected component of the neutral element \(1 \in \tilde{K}\). Then, the fundamental group \(\pi_1(G/K)\) is isomorphic to the quotient \(\tilde{K}/\tilde{K}_0\).

*The isomorphism*

\[ \zeta : \tilde{K}/\tilde{K}_0 \mapsto \pi_1(G/K, p(1)) \]

*is defined as follows. If \(g\tilde{K}_0\) is any element of \(\tilde{K}/\tilde{K}_0\), let \(c : [0, 1] \mapsto \tilde{G}\) be any continuous curve such that \(c(0) = 1 \in \tilde{G}\) and \(c(1) = g^{-1}\). Then, \(\zeta(g\tilde{K}_0)\) is the homotopy class of the loop \(p \circ q \circ c : [0, 1] \mapsto G/K\) based in \(p(1)\).*

*Proof.\* We start by determining the universal covering of the quotient \(G/K\). Towards this goal, we consider the transitive action of \(\tilde{G}\) on \(G/K\) given by \(g \cdot (xK) = (q(g)x)K\), for all \(g \in \tilde{G}\) and \(x \in G\). The isotropy group of \(p(1) = 1K\) is \(\tilde{K} = q^{-1}(K)\); we have therefore a diffeomorphism \(\tilde{G}/\tilde{K} \mapsto G/K\) given by \(g\tilde{K} \mapsto q(g)K\), for all \(g \in \tilde{G}\).

Since \(\tilde{K}/\tilde{K}_0\) is discrete, then the map \(\tilde{G}/\tilde{K}_0 \mapsto \tilde{G}/\tilde{K}\) given by \(g\tilde{K}_0 \mapsto g\tilde{K}\) is a covering map.
Considering the composition of the two maps above, we obtain a covering map \( \tilde{q} : \tilde{G}/\tilde{K}_0 \mapsto G/K \) given by \( \tilde{q}(g\tilde{K}_0) = q(g)K, \ g \in \tilde{G} \).

Since \( \tilde{G} \) is simply connected and \( \tilde{K}_0 \) is connected, the quotient \( \tilde{G}/\tilde{K}_0 \) is simply connected, and so \( \tilde{q} \) is the universal covering map of \( G/K \).

We now determine the group of covering automorphisms of \( \tilde{q} \), which is isomorphic to \( \pi_1(G/K) \). We recall that an automorphism \( \varphi \) of \( \tilde{q} \) is a homeomorphism of \( \tilde{G}/\tilde{K}_0 \) such that \( \tilde{q} \circ \varphi = \tilde{q} \). For all \( g \in \tilde{K} \), the map \( x\tilde{K}_0 \mapsto xg^{-1}\tilde{K}_0 \) is an automorphism of \( \tilde{q} \) which is trivial if \( g \in \tilde{K}_0 \). Thus, we have an action of \( \tilde{K}/\tilde{K}_0 \) in \( G/K \) by automorphisms of \( \tilde{q} \), which is transitive and simple (i.e., without fixed points) on the fibers of \( \tilde{q} \).

It follows that \( \tilde{K}/\tilde{K}_0 \) is the group of covering automorphisms of \( \tilde{q} \), concluding the proof of the first part of the statement.

To construct an explicit isomorphism between \( \tilde{K}/\tilde{K}_0 \) and \( \pi_1(G/K, p(1)) \), one uses the standard procedure of taking homotopy classes of loops obtained as the images under \( \tilde{q} \) of curves in \( \tilde{G}/\tilde{K}_0 \) that connect the point \( 1\tilde{K}_0 \) and a generic point in the fiber \( \tilde{q}^{-1}(1K) \).

We use Lemma 4.1.1 to compute the fundamental group of the Lagrangian Grassmanian \( \Lambda \):

**Corollary 4.1.2.** Let \( \{e_1, \ldots, e_{2n}\} \) be a symplectic basis of \( (V, \omega) \) and let \( L_0 \in \Lambda \) be the Lagrangian subspace generated by \( \{e_1, \ldots, e_n\} \). Then, the fundamental group of \( \Lambda \) with base point \( L_0, \pi_1(\Lambda, L_0) \), is isomorphic to \( \mathbb{Z} \). A generator of \( \pi_1(\Lambda, L_0) \) is given by the homotopy class of the loop \( \ell : [0, 1] \mapsto \Lambda \), where \( \ell(t) \) is the Lagrangian generated by the vectors \( e_1 \cdot \cos \pi t - e_{n+1} \cdot \sin \pi t, e_2, \ldots, e_n \).

**Proof.** We can clearly assume that \( V = \mathbb{R}^{2n} \), \( \omega = \omega_0 \), with \( \{e_1, \ldots, e_{2n}\} \) the canonical basis of \( \mathbb{R}^{2n} \), hence, \( L_0 = \mathbb{R}^n \oplus \{0\} \). We apply Lemma 4.1.1 to \( G = U(n) \), \( K = O(n) \); by Proposition 3.2.5, we can identify \( \Lambda \) with \( G/K \), and the quotient map \( p : U(n) \mapsto \Lambda \) is given by \( U(n) \ni \psi \mapsto \psi(L_0) \in \Lambda \).

Let \( SU(n) \) be the Lie group of unitary \( n \times n \) complex matrices having determinant equal to 1; the universal covering group of \( G \) is \( \tilde{G} = SU(n) \times \mathbb{R} \), with covering map \( q(A, t) = e^{it} \cdot A \).

The group \( \tilde{K} = q^{-1}(K) \) is easily computed as:

\[
(4.1.1) \quad \tilde{K} = \bigcup_{k \in \mathbb{Z}} \left[ e^{-i\frac{k\pi}{n}} O_k(n) \right] \times \left\{ \frac{k\pi}{n} \right\},
\]

where by \( O_k(n) \) we mean \( SO(n) \) if \( k \) is even and its complement \( O(n) \setminus SO(n) \) if \( k \) is odd.

The connected component \( \tilde{K}_0 \) equals \( SO(n) \times \{0\} \), and \( \tilde{K}/\tilde{K}_0 \) is isomorphic to \( \mathbb{Z} \). Such an isomorphism is given by mapping each term of the union in formula (4.1.1) into the integer \( k \).

As a generator for \( \tilde{K}/\tilde{K}_0 \), we choose the term in (4.1.1) corresponding to \( k = 1 \); such element is of the form \( g\tilde{K}_0 \), where \( g \) is chosen to be the pair \( g = (e^{-iA}, \frac{\pi}{n}) \), with
A the diagonal matrix:

\[
A = \begin{pmatrix}
\pi\left(\frac{1-n}{n}\right) & \frac{\pi}{n} & \cdots & \frac{\pi}{n}
\end{pmatrix}.
\]

Observe that $A$ is a traceless Hermitian matrix, so that $iA$ belongs to the Lie algebra $\text{su}(n)$ of $\text{SU}(n)$.

In order to determine a generator for $\pi_1(\Lambda, L_0)$, we choose the curve $c : [0, 1] \mapsto \tilde{G}$ given by $c(t) = (e^{itA}, -t \frac{\pi}{n})$, connecting the neutral element of $\tilde{G}$ with $g^{-1}$.

The curve $\ell = p \circ q \circ c$ in $\Lambda$ is now easily computed as given in the statement of the Corollary.

4.2. The intersection theory and the construction of the Maslov index

In order to develop our intersection theory, we are interested in the singular homology groups of $\Lambda$. The first homology group of a path connected topological space is isomorphic to the abelianization of its fundamental group, and therefore, it follows from Corollary 4.1.2 that $H_1(\Lambda)$ is isomorphic to $\mathbb{Z}$.

Each loop $\ell : [a, b] \mapsto \Lambda$ defines a homology class in $H_1(\Lambda)$. Given any $L_0 \in \Lambda$, we obtain a homomorphism $\pi_1(\Lambda, L_0) \mapsto H_1(\Lambda)$, which associates to the homotopy class of a loop based in $L_0$ its homology class. This is called the Hurewicz homomorphism and it is well known that it is surjective, and its kernel is the commutator subgroup of $\pi_1(\Lambda, L_0)$ (see [21, Proposition 4.21]).

The homology class of the curve $\ell : [0, 1] \mapsto \Lambda$ defined in Corollary 4.1.2 is therefore a generator of $H_1(\Lambda) \simeq \mathbb{Z}$, and, up to the choice of a sign, to each loop in $\Lambda$ we have a well defined integer associated to it. Such a number is to be interpreted as a sort of winding number of the loop around $\Lambda$.

Using the relative homology groups, we now show how to extend the above construction to curves in $\Lambda$ that are not necessarily closed. Let $L_0$ be a fixed Lagrangian in $\Lambda$; we will consider the relative homology group $H_1(\Lambda, \Lambda_0(L_0))$. We recall that $\Lambda_0(L_0)$ is the complement in $\Lambda$ of the set $\Lambda_{\geq 1}(L_0)$; every continuous curve $\ell : [a, b] \mapsto \Lambda$ with endpoints outside $\Lambda_{\geq 1}(L_0)$ defines a homology class in $H_1(\Lambda, \Lambda_0(L_0))$.

We observe that $\Lambda_0(L_0)$ is contractible, since any coordinate map $\phi_{L_0}$ maps $\Lambda_0(L_0)$ diffeomorphically onto a vector space (see Remark 3.2.7). Hence, by the long exact homology sequence of the pair $(\Lambda, \Lambda_0(L_0))$, we have an isomorphism $H_1(\Lambda) \mapsto H_1(\Lambda, \Lambda_0)$ induced by the inclusion $(\Lambda, \emptyset) \mapsto (\Lambda, \Lambda_0(L_0))$. This implies that $H_1(\Lambda, \Lambda_0(L_0)) \simeq \mathbb{Z}$, and a generator of this group is the homology class of the curve $\ell$ given in the statement of Corollary 4.1.2.

We make some remarks that follow from elementary properties of the homology theory. Let $\ell : [a, b] \mapsto \Lambda$ be a continuous curve with endpoints outside $\Lambda_1(L_0)$, and $\sigma : [c, d] \mapsto [a, b]$ a continuous map with $\sigma(c) = a$ and $\sigma(d) = b$. Then, the curves $\ell$ and $\ell \circ \sigma$ are homologous in $H_1(\Lambda, \Lambda_0(L_0))$. If $\sigma(d) = a$ and $\sigma(c) = b$, then $\ell \circ \sigma$ is homologous to the singular 1-chain $-\ell$ in $H_1(\Lambda, \Lambda_0(L_0))$. If $u \in [a, b]$ is
such that \( \ell(u) \not\in \Lambda_{\geq 1}(L_0) \), then \( \ell \) is homologous to the singular 1-chain \( \ell|_{[a,u]} + \ell|_{[u,b]} \).
Clearly, if the image of \( \ell \) does not intersect \( \Lambda_{\geq 1}(L_0) \), then \( \ell \) is homologous to zero in \( H_1(\Lambda, \Lambda_0(L_0)) \). Finally, if \( \ell_1, \ell_2 : [a, b] \mapsto \Lambda \) are continuous curves with endpoints outside \( \Lambda_{\geq 1}(L_0) \) and that are homotopic through curves with endpoints outside \( \Lambda_{\geq 1}(L_0) \), then, they are homologous in \( H_1(\Lambda, \Lambda_0(L_0)) \).

**Remark 4.2.1.** Each element in \( \text{Sp}(V, \omega) \) induces the identity homomorphism in the homology of \( \Lambda \), and each element of \( \text{Sp}(V, \omega, L_0) \) induces the identity in the relative homology of the pair \( (\Lambda, \Lambda_0(L_0)) \), in the following sense. Recall that a continuous map \( f \) between (pairs of) topological spaces induces homomorphisms between their (relative) homology groups, that will denoted by \( (f)_* \).

If \( \psi \in \text{Sp}(V, \omega) \) and \( \hat{\psi} : \Lambda \mapsto \Lambda \) is the diffeomorphism given by \( L \mapsto \psi(L) \), then \( (\hat{\psi})_* : H_1(\Lambda) \mapsto H_1(\Lambda) \) is the identity map. For, it is well known that \( \text{Sp}(V, \omega) \simeq \text{Sp}(n, \mathbb{R}) \) is connected, hence every \( \psi \) can be continuously connected to the neutral element of \( \text{Sp}(V, \omega) \), which gives a homotopy between \( \hat{\psi} \) and the identity of \( \Lambda \).

Moreover, if \( \psi \in \text{Sp}(V, \omega, L_0) \), then \( \hat{\psi} \) carries \( \Lambda_0(L_0) \) onto itself, and \( (\hat{\psi})_* \) is again the identity on \( H_1(\Lambda, \Lambda_0(L_0)) \). To see this, it suffices to observe that the following diagram commutes, by standard functoriality properties in homology:

\[
\begin{array}{ccc}
H_1(\Lambda, \Lambda_0(L_0)) & \xrightarrow{(\hat{\psi})_*} & H_1(\Lambda, \Lambda_0(L_0)) \\
\downarrow{(i)_*} & & \uparrow{(i)_*} \\
H_1(\Lambda) & \xrightarrow{\text{Id}=(\hat{\psi})_*} & H_1(\Lambda),
\end{array}
\]

where \( i : (\Lambda, \emptyset) \mapsto (\Lambda, \Lambda_0) \) is the inclusion.

We now give the following sufficient condition for two curves to be homologous in \( H_1(\Lambda, \Lambda_0(L_0)) \):

**Lemma 4.2.2.** Let \( \ell_1, \ell_2 : [a, b] \mapsto \Lambda \) be continuous curves with endpoints outside \( \Lambda_{\geq 1}(L_0) \). Suppose that there exists a Lagrangian \( L_1 \) complementary to \( L_0 \) such that the images of both \( \ell_1 \) and \( \ell_2 \) are entirely contained in the domain \( \Lambda_0(L_1) \) of \( \phi_{L_0, L_1} \).

Let \( \beta_i = \phi_{L_0, L_1} \circ \ell_i, i = 1, 2 \); then, if \( n_+(\beta_1(t)) = n_+(\beta_2(t)) \) for \( t = a, b \), it follows that \( \ell_1 \) and \( \ell_2 \) are homologous in \( H_1(\Lambda, \Lambda_0(L_0)) \).

**Proof.** We define the space:

\[
(4.2.1) \quad B_{\text{sym}}^{\geq 1}(L_0, \mathbb{R}) = \left\{ B \in B_{\text{sym}}(L_0, \mathbb{R}) : \text{dgn}(B) \geq 1 \right\},
\]

it is easy to see that \( \phi_{L_0, L_1}(\Lambda_{\geq 1}(L_0) \cap \Lambda_0(L_1)) = B_{\text{sym}}^{\geq 1}(L_0, \mathbb{R}) \). We also denote by \( B_{\text{sym}}^{0,i}(L_0, \mathbb{R}) \) the complement of \( B_{\text{sym}}^{\geq 1}(L_0, \mathbb{R}) \) in \( B_{\text{sym}}(L_0, \mathbb{R}) \); \( B_{\text{sym}}^{0,i}(L_0, \mathbb{R}) \) is given by the union of \( n+1 \) open connected components \( B_{\text{sym}}^{!}(L_0, \mathbb{R}) \), given by

\[
(4.2.2) \quad B_{\text{sym}}^{0,i}(L_0, \mathbb{R}) = \left\{ B \in B_{\text{sym}}(L_0, \mathbb{R}) : n_+(B) = i \right\}, \quad i = 0, \ldots, n.
\]
Observe that each $B^{0,i}_{\text{sym}}(L_0, IR)$ is indeed path connected, because, by Sylvester’s Inertia Theorem, it admits a transitive action of the connected group $GL_+(n, IR)$.

If we set $i = n_+((\beta_1(a)) = n_+((\beta_2(a))$ and $j = n_+((\beta_1(b)) = n_+((\beta_2(b))$, then we can find a continuous curve $\beta_3$ (and $\beta_4$) in $B^{0,i}_{\text{sym}}(L_0, IR)$ (in $B^{0,j}_{\text{sym}}(L_0, IR)$) from $\beta_2(a)$ to $\beta_1(a)$ (from $\beta_2(b)$ to $\beta_1(b)$).

Define $\ell_i = \phi^{-1}_{L_0,L_1} \circ \beta_i$, $i = 3, 4$. Then, since $\ell_3$ and $\ell_4$ do not intersect $\Lambda_{\geq 1}(L_0)$, the concatenation $\ell = \ell_3 \circ \ell_4$ is homologous to $\ell_1$ in $H_1(\Lambda, \Lambda_0(L_0))$.

Let $\beta = \phi_{L_0,L_1}^{-1} \circ \ell$; then, $\beta = \beta_3 \circ \beta_1 \circ \beta_4$. Since $\beta$ and $\beta_2$ have the same endpoints, they admit a fixed endpoint homotopy in the vector space $B_{\text{sym}}(L_0, IR)$. The composition of such homotopy with $\phi_{L_0,L_1}^{-1}$ gives a homotopy between $\ell$ and $\ell_2$ through curves with endpoints outside $\Lambda_{\geq 1}(L_0)$.

Hence, $\ell$ and $\ell_2$ are homologous in $H_1(\Lambda, \Lambda_0(L_0))$, and we are done.

In the next Lemma, we show how to compute the variation of the type numbers for a differentiable curve of symmetric bilinear forms:

**Lemma 4.2.3.** Let $Z$ be a finite dimensional real vector space and $A : [0, r] \to B_{\text{sym}}(Z, IR)$ be a map of class $C^1$. Suppose that the restriction $\tilde{A}$ of the derivative $A'(t)$ to the kernel $\ker(A(0))$ is non degenerate. Then, for $t > 0$ sufficiently small, $A(t)$ is non degenerate, and we have:

\[(4.2.3) \quad n_+(A(t)) = n_+(A(0)) + n_+((\tilde{A}), \quad n_-(A(t)) = n_-(A(0)) + n_-(\tilde{A}).\]

**Proof.** Let $N = \ker(A(0))$. We start with the case where $A(0)$ is positive semidefinite, i.e., $n_-(A(0)) = 0$, and $A$ is positive definite, i.e., $n_+(A) = \text{dgn}(\tilde{A}) = 0$. Let $S$ be a subspace of $Z$ which is complementary to $N$ and such that $A(0)$ is positive definite on $S$. We need to show that $A(t)$ is positive definite on $Z = N \oplus S$ for $t$ small enough. First, since $A(0)$ is positive definite on $S$, there is $\varepsilon > 0$ such that $A(t)$ is positive definite on $S$ for $t \in [0, \varepsilon]$ (the set of positive definite symmetric bilinear forms is open). Let $\| \cdot \|$ be an arbitrary norm on $Z$ and define:

\[(4.2.4) \quad c_0 = \inf_{x \in S, \|x\| = 1} A(t)[x, x] > 0, \quad c_1 = \inf_{y \in N, \|y\| = 1} A'(0)[y, y] > 0.\]

It is easy to see that, for all $t > 0$ small enough, we have

\[(4.2.5) \quad A(t)[y, y] \geq \frac{1}{2} c_1 t, \quad \forall y \in N, \|y\| = 1,\]

so that $A(t)$ is positive definite on both $N$ and $S$ for $t > 0$ small enough. We want to show that, if $t > 0$ is small enough, then for all $x \in S \setminus \{0\}$ and $y \in N \setminus \{0\}$, $A(t)$ is positive definite on the two dimensional subspace of $Z$ generated by $x$ and $y$. By the positivity on $S$ and $N$, it suffices to prove that, for $t > 0$ small enough, the following inequality holds:

\[(4.2.6) \quad A(t)[x, y]^2 \leq A(t)[x, x] \cdot A(t)[y, y],\]
for all \( x \in S, y \in N \). Obviously, we can assume \( \|x\| = \|y\| = 1 \). As \( A(0) \) vanishes on \( N \times S \), there exists \( c_2 > 0 \) such that, for all \( t > 0 \) small enough, we have:

\[
(4.2.7) \quad |A(t)[x, y]| \leq c_2 \cdot t,
\]

for all \( x \in S, y \in N \) with \( \|x\| = \|y\| = 1 \). By (4.2.4), (4.2.5) and (4.2.7), for all \( t > 0 \) small enough we get:

\[
A(t)[x, y]^2 \leq c^2_2 t^2 \leq \frac{1}{2} c_0 c_1 t \leq A(t)[x, x] \cdot A(t)[y, y],
\]

for all \( x \in S, y \in N \) with \( \|x\| = \|y\| = 1 \). This yields (4.2.6) and concludes the first part of the proof.

For the general case, we consider decompositions \( Z = S_+ \oplus S_- \oplus N \) and \( N = N_+ \oplus N_- \), where \( A(0) \) is positive definite on \( S_+ \), negative definite on \( S_- \), and \( A'(0) \) is positive definite in \( N_+ \) and negative definite in \( N_- \). We then apply the result proven in the first part of the proof to the restriction of \( A(t) \) to \( S_+ \oplus N_+ \) once, and again to the restriction of \( -A(t) \) to \( S_- \oplus N_- \). The conclusion follows by observing that \( A(t) \) is positive definite on \( S_+ \oplus N_+ \) and negative definite on \( S_- \oplus N_- \), which implies that \( n_+(A(t)) = \dim(S_+ \oplus N_+) \) and \( n_-(A(t)) = \dim(S_- \oplus N_-) \) for \( t > 0 \) small enough. Clearly, this also implies that \( A(t) \) is non degenerate.

We now go back to the study of the homology of the pair \((\Lambda, \Lambda_0(L_0))\) and of the intersection theory.

The choice of an isomorphism \( H_1(\Lambda, \Lambda_0(L_0)) \simeq \mathbb{Z} \) is equivalent to the choice of one of the two generators of \( H_1(\Lambda, \Lambda_0(L_0)) \). Using the canonical transverse orientation of \( \Lambda_1(L_0) \) (see Proposition 3.2.10), we now show how such a choice will be made.

In order to give a precise statement of our next Proposition, we give the following definitions. Let \( \ell : [a, b] \mapsto \Lambda \) be a smooth curve, with \( \ell(t_0) \in \Lambda_{\geq 1}(L_0) \) for some \( t_0 \in [a, b] \). We say that \( \ell \) intercepts \( \Lambda_{\geq 1}(L_0) \) transversally at the instant \( t_0 \) if \( \ell(t_0) \in \Lambda_1(L_0) \) and the tangent vector \( \ell'(t_0) \) does not belong to \( T_{\ell(t_0)}\Lambda_1(L_0) \). If \( \ell \) intersects \( \Lambda_{\geq 1}(L_0) \) transversally at \( t_0 \), we say that this intersection is positive or negative if \( \ell'(t_0) \) is respectively a positive or a negative vector in the sense of the transversal orientation of \( \Lambda_1(L_0) \) (see Proposition 3.2.10).

**Proposition 4.2.4.** Let \( \ell_1 \) and \( \ell_2 \) be smooth curves in \( \Lambda \) with both endpoints outside \( \Lambda_{\geq 1}(L_0) \). Suppose that both curves intercept \( \Lambda_{\geq 1}(L_0) \) only once, and that such intersections are both transverse and positive (or both negative). Then \( \ell_1 \) and \( \ell_2 \) define the same homology class in \( H_1(\Lambda, \Lambda_0(L_0)) \). Moreover, this homology class is a generator of \( H_1(\Lambda, \Lambda_0(L_0)) \) \( \simeq \mathbb{Z} \).

**Proof.** We consider the case of positive intersections; the other case is then easily obtained by passing to the backwards orientation. By reparameterizing, we can assume that both curves intercept \( \Lambda_1(L_0) \) at the same instant \( t_0 \in [a, b] \). By Proposition 3.2.8, there exists \( \psi \in \text{Sp}_+(V, \omega, L_0) \) such that \( \psi(\ell_2(t_0)) = \ell_1(t_0) \). Let \( \hat{\psi} \) denote the diffeomorphism of \( \Lambda \) given by \( L \mapsto \psi(L) \); we deduce from Proposition 3.2.10 that the curve \( \hat{\psi} \circ \ell_2 \) has a unique intersection with \( \Lambda_{\geq 1}(L_0) \), which is transverse and positive. Moreover, by Remark 4.2.1, \( \hat{\psi} \circ \ell_2 \) and \( \ell_2 \) are homologous in \( H_1(\Lambda, \Lambda_0(L_0)) \).
Without loss of generality, we can therefore assume that \( \ell_1(t_0) = \ell_2(t_0) \). Let \( L_1 \in \Lambda \) be a Lagrangian which is complementary to both \( L_0 \) and \( \ell_1(t_0) = \ell_2(t_0) \) (see Corollary 3.2.9). Since \( \ell_i, i = 1, 2 \), has a unique intersection with \( \Lambda_{\geq 1}(L_0) \) at \( t_0 \), then the restriction of \( \ell_i \) to any closed subinterval containing \( t_0 \) in its interior is homologous to \( \ell_i \) in \( H_1(\Lambda, \Lambda_0(L_0)) \). Thus, we can assume that the images of \( \ell_1 \) and \( \ell_2 \) are contained in the domain \( \Lambda_0(L_1) \) of the chart \( \phi_{L_0, L_1} \).

Let \( \beta_i = \phi_{L_0, L_1} \circ \ell_i, i = 1, 2 \); one checks easily that \( \operatorname{Ker}(\beta_i(t_0)) = \ell_i(t_0) \cap L_0 \) and, since the intersection of \( \ell_i \) with \( \Lambda_{\geq 1}(L_0) \) is unique, \( \beta_i(t) \) is non degenerate for all \( t \neq t_0 \). It follows that \( n_+(\beta_i(t)) \) is constant for \( t \in [a, t_0[ \) and for \( t \in ]t_0, b] \).

By the positivity of the intersection, the restriction of \( \ell'_i(t_0) \) to the one dimensional subspace \( \ell_i(t_0) \cap L_0 \) is positive definite (see Proposition 3.2.10). Moreover, by Remark 3.2.11, it follows that \( \beta'_i(t_0) \) is positive definite on \( \ell_i(t_0) \cap L_0 \).

The fact that \( \ell_1 \) is homologous to \( \ell_2 \) in \( H_1(\Lambda, \Lambda_0(L_0)) \) will follow from Lemma 4.2.2 once we prove that \( n_+(\beta_1(a)) = n_+(\beta_2(a)) \) and that \( n_+(\beta_1(b)) = n_+(\beta_2(b)) \).

Applying twice Lemma 4.2.3 around \( t_0 \), we obtain the following equalities for each \( i = 1, 2 \):

\[
 n_+(\beta_i(b)) = n_+(\beta_i(t_0)) + 1, \quad n_+(\beta_i(a)) = n_+(\beta_i(t_0));
\]

for the second equality we have applied Lemma 4.2.3 to the curve \( \beta_i \) reparameterized backwards. The conclusion follows from the fact that \( \beta_1(t_0) = \beta_2(t_0) \).

Using the first part of the Proposition, to conclude the proof we need to exhibit a smooth curve \( \ell \) whose homology class is a generator of \( H_1(\Lambda, \Lambda_0(L_0)) \), and that intersects \( \Lambda_{\geq 1}(L_0) \) exactly once, with such intersection transverse.

To this aim, let \( \{e_1, \ldots, e_{2n}\} \) be a symplectic basis of \((V, \omega)\) such that \( L_0 \) is the Lagrangian generated by \( \{e_{n+1}, \ldots, e_{2n}\} \) (see Remark 3.1.1). Consider the curve \( \ell : [0, 1] \to \Lambda \) introduced in the statement of Corollary 4.1.2. It intersects \( \Lambda_{\geq 1}(L_0) \) only at the instant \( t_0 = \frac{1}{2} \) and \( \ell(\frac{1}{2}) \in \Lambda_1(L_0) \), because \( \ell(\frac{1}{2}) \cap L_0 = \mathbb{R} \cdot e_{n+1} \).

To check the transversality, we make computations using the chart \( \phi_{L_0, L_1} \) (rather than \( \phi_{\ell(\frac{1}{2}), L_1} \), see Remark 3.2.11), where we choose \( L_1 \) to be the Lagrangian generated by the vectors \( e_i + e_{n+i}, i = 1, \ldots, n \), which is complementary to both \( L_0 \) and \( \ell(\frac{1}{2}) \). We set \( \beta = \phi_{L_0, L_1} \circ \ell \), and we obtain for each \( t \in [0, 1] \) a symmetric bilinear form in \( L_0 \) which, in the basis \( \{e_{n+1}, \ldots, e_{2n}\} \) is given by the diagonal matrix:

\[
\beta(t) \sim \begin{pmatrix} f(t) & 1 \\ \ldots & \ldots \\ 1 & 1 \end{pmatrix},
\]

with

\[
f(t) = \frac{\cos \pi t}{\cos \pi t + \sin \pi t}.
\]

Since \( \ell'(\frac{1}{2})(e_{n+1}, e_{n+1}) = f'(\frac{1}{2}) = -\pi \neq 0 \), then \( \ell \) intersects \( \Lambda_{\geq 1}(L_0) \) transversally (with negative intersection), and the proof is complete. \qed
Let $\mu_{L_0} : H_1(\Lambda, \Lambda_0(L_0)) \mapsto \mathbb{Z}$ be the unique isomorphism such that $\mu_{L_0}(\mathfrak{h}) = 1$ where $\mathfrak{h}$ is the homology class of any smooth curve $\ell$ in $\Lambda$, with endpoints outside $\Lambda_{\geq 1}(L_0)$ and intersecting only once $\Lambda_{\geq 1}(L_0)$, such intersection being transverse and positive. The fact that $\mu_{L_0}$ is well defined and that it is indeed an isomorphism follows directly from Proposition 4.2.4.

We can now define the Maslov index of a curve in $\Lambda$.

**Definition 4.2.5.** Let $\ell$ be any continuous curve in $\Lambda$ with endpoints outside $\Lambda_{\geq 1}(L_0)$. The Maslov index of $\ell$ (relatively to $L_0$) is the value of $\mu_{L_0}$ in the homology class of $\ell$. The Maslov index of $\ell$ will be denoted by $\mu_{L_0}(\ell)$.

If $\ell : [a, b] \mapsto \Lambda$ is any continuous curve such that $\{ t \in [a, b] : \ell(t) \in \Lambda_{\geq 1}(L_0) \}$ is contained in some closed interval $[c, d] \subset [a, b]$, then the Maslov index $\mu_{L_0}(\ell)$ of $\ell$ is defined to be the Maslov index of the restriction of $\ell$ to any such $[c, d]$.

Since the homology class of a concatenation of curves is equal to the sum of their homology classes, and since $\mu_{L_0}$ is a group homomorphism, it follows that the Maslov index of curves is additive by concatenation. Moreover, Proposition 4.2.4 gives us the following geometrical interpretation of the Maslov index of a curve. If $\ell$ is a smooth curve in $\Lambda$, with endpoints outside $\Lambda_{\geq 1}(L_0)$ and having only transverse intersections with $\Lambda_{\geq 1}(L_0)$, then the Maslov index of $\ell$ is the number of positive intersections minus the number of negative intersections of $\ell$ with $\Lambda_{\geq 1}(L_0)$.

If either one of the endpoints of $\ell$ do belong to $\Lambda_{\geq 1}(L_0)$, our definition of Maslov index simply says that these intersections are not counted.

4.3. **Computation of the Maslov index**

We now show how to compute the Maslov index of a curve having image entirely contained in the domain of a fixed chart.

**Proposition 4.3.1.** Let $\ell : [a, b] \mapsto \Lambda$ be any continuous curve with endpoints outside $\Lambda_{\geq 1}(L_0)$. If there exists a Lagrangian subspace $L_1$ complementary to $L_0$ and such that the image of $\ell$ is entirely contained in the domain $\Lambda_0(L_1)$ of the chart $\phi_{L_0, L_1}$, then:

\[
\mu_{L_0}(\ell) = n_+(\beta(b)) - n_+(\beta(a)),
\]

where $\beta = \phi_{L_0, L_1} \circ \ell$.

**Proof.** We start observing that, by Lemma 4.2.2, the Maslov index $\mu_{L_0}(\ell)$ depends only on the numbers $n_+(\beta(b))$ and $n_+(\beta(a))$.

To prove the statement, it suffices to exhibit for each $i, j = 0, \ldots, n$ a curve $\beta_{i,j} : [a, b] \mapsto B_{\text{sym}}(L_0, \mathbb{R})$, such that $n_+(\beta_{i,j}(a)) = i$, $n_+(\beta_{i,j}(b)) = j$, and such that the curve $\ell_{i,j} = \phi_{L_0, L_1}^{-1} \circ \beta_{i,j}$ has Maslov index equal to $j - i$. Clearly, since we can consider curves reparameterized backwards, it suffices to consider the case $i \leq j$. For $i = j$, a constant curve with positive type number equal to $i$ would do the job. It is indeed sufficient to exhibit curves $\beta_{i,i+1}$ as above for all $i = 0, \ldots, n - 1$. To prove this claim, observe in first place that, if such a curve $\beta_{i,i+1}$ is found and $\beta_{i,i+1}$ is any other curve having the same positive type numbers at the endpoints, then the corresponding
curves in $\Lambda$ have the same Maslov index. Now, if the curves $\tilde{\beta}_{i,i+1}$ are chosen in such a way that the endpoint of $\tilde{\beta}_{i,i+1}$ coincides with the initial point of $\beta_{i+1,i+2}$, then the concatenation $\beta_{i,j} = \tilde{\beta}_{i,i+1} \circ \tilde{\beta}_{i+1,i+2} \circ \cdots \circ \tilde{\beta}_{j-1,j}$ has the desired properties.

To complete the proof, we now show how to construct the curves $\beta_{i,i+1}$ as above. Choose any basis of $L_0$ and define a curve $\beta_{i,i+1} : [-1,1] \mapsto B_{\text{sym}}(L_0, IR)$ such that $\beta_{i,i+1}(t)$ is given in the chosen basis by the diagonal $n \times n$ matrix having diagonal vector $(1, \ldots , 1, t, -1, \ldots , -1)$. Let $\ell_{i,i+1} = \phi_{L_0,L_1}^{-1} \circ \beta_{i,i+1}$; we need to show that $i$ times

$$\mu_{L_0}(\ell_{i,i+1}) = 1.$$  

It is easy to see that every $\ell_{i,i+1}$ intersects $\Lambda_{\geq 1}(L_0)$ only once at $t_0 = 0$, and that $\ell_{i,i+1}(0) \cap L_0 = \text{Ker}(\beta_{i,i+1}(0))$. Since $\beta_{i,i+1}(0)$ is positive definite on the one dimensional space $\text{Ker}(\beta_{i,i+1})$, the intersection of $\ell_{i,i+1}$ with $\Lambda_{\geq 1}(L_0)$ is transverse and positive (see Proposition 3.2.10 and Remark 3.2.11). By definition, the Maslov index of $\ell_{i,i+1}$ is equal to 1, and we are done.

It is now easy to prove the following estimate for the Maslov index:

**Corollary 4.3.2.** Let $\ell : [a, b] \mapsto \Lambda$ be any continuous curve with endpoints outside $\Lambda_{\geq 1}(L_0)$. Then,

$$|\mu_{L_0}(\ell)| \leq \sum_{t \in [a,b]} \dim(\ell(t) \cap L_0).$$  

**Proof.** If there are infinitely many $t \in [a,b]$ such that $\ell(t) \in \Lambda_{\geq 1}(L_0)$, then the right hand side of (4.3.2) is infinite, and the statement of the Corollary is trivial. Otherwise, let $t_0 \in [a,b]$ be such that $\ell(t_0) \in \Lambda_{\geq 1}(L_0)$ and let $L_1 \in \Lambda$ be a Lagrangian complementary to both $L_0$ and $\ell(t_0)$ (see Corollary 3.2.9). Set $\beta = \phi_{L_0,L_1} \circ \ell$; then, $\beta$ is a curve in $B_{\text{sym}}(L_0, IR)$ defined in a neighborhood of $t_0$. It is easily seen that $\text{Ker}(\beta(t_0)) = \ell(t_0) \cap L_0$. By elementary arguments, we have that, for $t$ sufficiently close to $t_0$, the following inequality holds:

$$n_+(\beta(t)) \leq n_+(\beta(t)) \leq n_+(\beta(t_0)) + \text{dgn}(\beta(t_0)).$$

Hence, for $\varepsilon > 0$ small enough, we get:

$$|n_+(\beta(t_0 + \varepsilon)) - n_+(\beta(t_0 - \varepsilon))| \leq \text{dgn}(\beta(t_0)).$$

The conclusion follows easily from Proposition 4.3.1. \qed

Under a non degeneracy assumption, the Maslov index can be computed as a sum of signatures:

**Corollary 4.3.3.** Let $\ell : [a, b] \mapsto \Lambda$ be a curve of class $C^1$ having endpoints outside $\Lambda_{\geq 1}(L_0)$. If for all $t \in [a,b]$ such that $\ell(t) \in \Lambda_{\geq 1}(L_0)$ we have that $\ell'(t)$ is non degenerate on $\ell(t) \cap L_0$, then the number of intersections of $\ell$ with $\Lambda_{\geq 1}(L_0)$ is finite, and:

$$\mu_{L_0}(\ell) = \sum_{t \in [a,b]} \text{sgn}(\ell'(t)|_{\ell(t)\cap L_0}).$$
Proof. Let \( t_0 \in [a, b] \) be such that \( \ell(t_0) \in \Lambda_{\geq 1}(L_0) \) and let \( L_1 \in \Lambda \) be a Lagrangian complementary to both \( L_0 \) and \( \ell(t_0) \) (see Corollary 3.2.9). Set \( \beta = \phi_{L_0,L_1} \circ \ell; \) then, \( \beta \) is a curve in \( B_{\text{sym}}(L_0, IR) \) defined in a neighborhood of \( t_0 \). It is easily seen that \( \text{Ker}(\beta(t_0))) = \ell(t_0) \cap L_0 \), and it follows from Remark 3.2.11 that \( \beta'(t_0) \) and \( \ell'(t_0) \) coincide in \( \ell(t_0) \cap L_0 \).

Applying Lemma 4.2.3 around \( t_0 \), once to \( \beta \) and again to a backwards reparameterization of \( \beta \), we conclude that if \( \varepsilon > 0 \) is small enough, then \( \beta(t) \) is non degenerate for \( t \in [t_0 - \varepsilon, t_0 + \varepsilon] \setminus \{t_0\} \), and that:

\[
\begin{align*}
(4.3.4) & \quad n_+ (\beta(t_0 + \varepsilon)) = n_+ (\beta(t_0)) + n_+ (\beta'(t_0)\big|_{\text{Ker}(\beta(t_0)))}), \\
(4.3.5) & \quad n_+ (\beta(t_0 - \varepsilon)) = n_+ (\beta(t_0)) + n_- (\beta'(t_0)\big|_{\text{Ker}(\beta(t_0)))}).
\end{align*}
\]

Subtracting (4.3.5) from (4.3.4), we get

\[
(4.3.6) \quad n_+ (\beta(t_0 + \varepsilon)) - n_+ (\beta(t_0 - \varepsilon)) = \text{sgn}(\beta'(t_0)\big|_{\text{Ker}(\beta(t_0)))}).
\]

We have proven that the intersection of \( \ell \) with \( \Lambda_{\geq 1}(L_0) \) at \( t_0 \) is isolated, and, using Proposition 4.3.1, it follows from (4.3.6) that:

\[
\mu_{L_0}(\ell|_{[t_0 - \varepsilon, t_0 + \varepsilon]}) = \text{sgn}(\ell'(t_0)\big|_{\ell(t_0) \cap L_0}).
\]

The conclusion follows from the additivity of the Maslov index with respect to concatenation. \( \square \)

Obvious modifications can be made to the statements of Proposition 4.3.1 and Corollaries 4.3.2 and 4.3.3 to adapt them to the case of curves \( \ell \) with endpoints in \( \Lambda_{\geq 1}(L_0) \) for which the Maslov index \( \mu_{L_0}(\ell) \) is defined.

### 5. Applications of the Maslov Index: Stability of the Geometric Index

In this Section we apply the abstract theory developed in Sections 3 and 4 to the study of the indexes of the quadruples introduced in Section 2.

#### 5.1. The Maslov index of a differential problem

Let \( (g, R, P, S) \) be an admissible quadruple for the differential problem in \( IR^n \); we recall that, associated to \( (g, R, P, S) \), we have constructed a symplectic form \( \omega \) in \( IR^{2n} \) and, for each \( t \in [a, b] \) a symplectomorphism \( \Psi(t) \) of \( (IR^{2n}, \omega) \) (Definition 2.4.1 and equation (2.4.2)). The map \( t \mapsto \Psi(t) \) is a curve of class \( C^1 \) in \( \text{Sp}(IR^{2n}, \omega) \).

Rewriting equation (2.2.1) as a first order linear system, we get the following Cauchy problem satisfied by \( \Psi \):

\[
(5.1.1) \quad \Psi'(t) = H(t) \Psi(t), \quad \Psi(0) = \text{Id},
\]

where \( H(t) : IR^{2n} \mapsto IR^{2n} \) is the linear map defined for all \( t \in [a, b] \) by:

\[
(5.1.2) \quad H(t)[(x, y)] = (y, R(t)[x]), \quad \forall x, y \in IR^n.
\]
Since $R(t)$ is $g$-symmetric, $t \mapsto H(t)$ defines a continuous curve in $\text{sp}(\mathbb{R}^{2n}, \omega)$; equation (5.1.1) says that $\Psi(t)$ is equal to the evaluation at $\Psi(t)$ of the right invariant vector field determined by $H(t)$ in $\text{Sp}(\mathbb{R}^{2n}, \omega)$.

We now define the following Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega)$:

$$\ell_0 = \{(x, y) \in \mathbb{R}^{2n} : x \in P, y + S[x] \in P^\perp\},$$

and

$$L_0 = \{0\} \oplus \mathbb{R}^n.$$

Observe that $\ell_0$ is the subspace of $\mathbb{R}^{2n}$ determined by the initial conditions (2.2.2); the fact that it is a Lagrangian subspace is proven in formula (2.4.3). We also define the $C^1$-curve $\ell : [a, b] \mapsto \Lambda$ by:

$$\ell(t) = \Psi(t)[\ell_0].$$

The crucial observation here is that the curve $\ell$ intercepts $\Lambda_{\geq 1}(L_0)$ at $t_0 \in ]a, b]$ if and only if $t_0$ is a $(P, S)$-focal instant. Observe also that $\ell(a) \in \Lambda_{\geq 1}(L_0)$, unless $P = \mathbb{R}^n$. More in general, if $t_0 \in [a, b]$, then we have:

$$\ell(t_0) \cap L_0 = \left\{(0, J'(t_0)) : J \in \mathbb{J}, J(t_0) = 0 \right\} = \{0\} \oplus \mathbb{J}[t_0]^-.$$

The last equality in (5.1.6) follows easily from (2.4.4), arguing as in the proof of Proposition 2.5.1. Recalling (2.2.5), we have that, for $t_0 \in ]a, b]$, $\ell(t_0) \in \Lambda_k(L_0)$ if and only if $t_0$ is a $(P, S)$-focal instant of multiplicity $k$.

It follows from Proposition 3.2.3 and formula (5.1.1) that $\ell$ satisfies the following Cauchy problem:

$$\ell'(t) = H(t)^* [\ell(t)], \quad \ell(a) = \ell_0;$$

where $H^*$ is the vector field introduced in Definition 3.2.4. In the notation of Proposition 3.2.3, the curve $\ell$ is equal to $\kappa_{t_0} \circ \Psi$.

By Proposition 2.5.1, $t_0 = a$ is an isolated intersection of $\ell$ with $\Lambda_{\geq 1}(L_0)$, and therefore we can give the following definition:

**Definition 5.1.1.** Let $(g, R, P, S)$ be an admissible quadruple for the differential problem such that the final instant $t_0 = b$ is not $(P, S)$-focal. Its Maslov index $\mu_{L_0}((g, R, P, S))$ is the Maslov index $\mu_{L_0}(\ell)$, where $L_0$ is the Lagrangian defined in (5.1.4) and $\ell$ is the curve defined in (5.1.5).

The condition that $t_0 = b$ is not $(P, S)$-focal means that the curve $\ell$ does not intersect $\Lambda_{\geq 1}(L_0)$ at its final endpoint; possibly, one could extend the definition of Maslov index for quadruples where $t_0 = b$ is an isolated $(P, S)$-focal instant.

We have the following relation between the Maslov index and the focal index of a quadruple $(g, R, P, S)$:

**Theorem 5.1.2.** Let $(g, R, P, S)$ be an admissible quadruple for the differential problem in $\mathbb{R}^n$ such that:

1. $t_0 = b$ is not a $(P, S)$-focal instant;

...
2. for every \((P,S)\)-focal instant \(t_0 \in ]a,b[\), the restriction of \(g\) to \(\mathbb{J}[t_0]\) is non degenerate.

Then, the focal index \(i_{\text{foc}}\) of \((g,R,P,S)\) is well defined, and it equals the Maslov index:
\[
i_{\text{foc}} = \mu(g,R,P,S).
\]

**Proof.** Using Proposition 2.5.1, hypothesis 2 implies that the number of \((P,S)\)-focal instants is finite, and so \(i_{\text{foc}}\) is well defined (see Definition 2.2.2). Now, using equations (5.1.2), (5.1.6), (5.1.7) and Definitions 2.4.1 and 3.2.4, we compute as follows:
\[
\ell'(t_0)[(0,x),(0,y)] = H(t_0)^* [\ell(t_0)][(0,x),(0,y)] = \\
= \omega[H(t_0)[(0,x)],(0,y)] = \\
= \omega[(x,R(t_0)[0]),(0,y)] = g(x,y),
\]
for all \(t_0 \in ]a,b[\) and for all \(x,y \in \mathbb{R}^n\) such that the pairs \((0,x)\) and \((0,y)\) belong to \(\ell(t_0) \cap L_0 = \{0\} \oplus \mathbb{J}[t_0]_\perp\), i.e., for all \(x,y \in \mathbb{J}[t_0]_\perp\).

By hypothesis 2 and equation (5.1.9), \(\ell'(t_0)\) is non degenerate on \(\ell(t_0) \cap L_0\); moreover, we have:
\[
\text{sgn}(\ell'(t_0)|_{\ell(t_0) \cap L_0}) = \text{sgn}(g|_{\mathbb{J}[t_0]_\perp}).
\]
The conclusion follows from Corollary 4.3.3. 

Observe that Theorem 5.1.2 gives also an alternative proof of Proposition 2.5.1, as Corollary 4.3.3 guarantees that the number of intersections of \(\ell\) with \(\Lambda_{\geq 1}(L_0)\) is finite.

We now apply the above result to Riemannian or causal Lorentzian geodesics, obtaining the following:

**Corollary 5.1.3.** Let \((\mathcal{M},g,P,\gamma)\) be an admissible quadruple for the geometric problem such that \(\gamma(b)\) is not a \(P\)-focal point. Assume \((\mathcal{M},g)\) is Riemannian or Lorentzian, and in the latter case, \(\gamma\) is not spacelike. Let \((g,R,P,S)\) be any associated quadruple to \((\mathcal{M},g,P,\gamma)\). Then, the geometric index of \(\gamma\) equals the Maslov index of \((g,R,P,S)\):
\[
i_{\text{geom}}(\gamma) = \mu(g,R,P,S).
\]

**Proof.** By Proposition 2.3.4, since \(\gamma(b)\) is not \(P\)-focal, then \(t_0 = b\) is not a \((P,S)\)-focal instant. The conclusion follows at once from Proposition 2.3.4, Remark 2.3.5, Corollary 2.5.2 and Theorem 5.1.2.

5.2. Stability of the indexes

We now want to study the stability of the Maslov and the focal index for the differential problem and for the geometrical problem. We begin by introducing a notion of convergence for quadruples \((g,R,P,S)\); in particular, we will describe the topological structure of the set of pairs \((P,S)\) as a suitable fiber bundle.

For \(k = 0,\ldots,n\), let \(\text{GB}_k(n,\mathbb{R})\) be the set of pairs \((P,S)\), where \(P \subset \mathbb{R}^n\) is a \(k\)-dimensional subspace and \(S \in \text{B}_{\text{sym}}(P,\mathbb{R})\). We define in \(\text{GB}_k(n,\mathbb{R})\) the structure of a vector bundle over the Grassmannian \(G_k(\mathbb{R}^n)\), whose fiber over \(P \in G_k(\mathbb{R}^n)\) is
the vector space $B_{\text{sym}}(P, \mathbb{R})$. To define local trivializations of $GB_k(n, \mathbb{R})$ we argue as follows. Let $\mathbb{R}^n = W_0 \oplus W_1$ be a direct sum decomposition, where $W_0$ is a $k$-dimensional subspace. As in Section 3, we define a chart in $G_k(\mathbb{R}^n)$ by associating to each $P \in G_k(\mathbb{R}^n)$ transverse to $W_1$ the only linear map $T : W_0 \mapsto W_1$ whose graph in $W_0 \oplus W_1 = \mathbb{R}^n$ is $P$. Then, a local trivialization of $GB_k(n, \mathbb{R})$ is defined by mapping each $S \in B_{\text{sym}}(P, \mathbb{R})$ to the bilinear map $\hat{T}_*(S) = S(T^{-1} \cdot, \hat{T}^{-1} \cdot) \in B_{\text{sym}}(W_0, \mathbb{R})$, where the isomorphism $\hat{T} : W_0 \mapsto P$ is given by $v \mapsto v + T(v)$.

As to the geometrical problem, we now define the following space. Let $M$ be any smooth manifold of dimension $m$ and let $k = 0, \ldots, m$ be fixed. We denote by $GB_k(M)$ the set of triples $(p, P, S)$, where $p$ is a point of $M$, $P$ is a $k$-dimensional subspace of $T_pM$ and $S$ is a symmetric bilinear form on $P$. The space $GB_k(M)$ has an obvious structure of a fiber bundle over $M$ with projection $(p, P, S) \mapsto p$; namely, any local trivialization of the tangent bundle $TM$ around $p_0 \in M$ induces a bijection from the fiber of $GB_k(M)$ over $p$ (in a neighborhood of $p_0$) and the manifold $GB_k(m, \mathbb{R})$. These bijections give a local trivialization of $GB_k(M)$ around $p_0$. Observe that, in the case $k = 0$, the typical fiber $GB_0(m, \mathbb{R})$ reduces to a point and the fiber bundle $GB_0(M)$ is diffeomorphic to $M$.

The notion of convergence in the bundles $GB_k(n, \mathbb{R})$ and $GB_k(M)$ can be described in elementary terms, using convergence of linear basis and matrices. Namely, a sequence $(P_j, S_j)$ in $GB_k(n, \mathbb{R})$ converges to $(P, S)$ if and only if for each $j \in \mathbb{N}$ there exists a basis \{e^j_1, \ldots, e^j_k\} of $P_j$ such that $e^j_i \to e_i$ as $j \to \infty$ for all $i$, with \{e_1, \ldots, e_k\} a basis of $P$, and such that $S_j(e^j_\alpha, e^j_\beta) \to S(e_\alpha, e_\beta)$ as $j \to \infty$ for all $\alpha, \beta = 1, \ldots, k$.

The convergence $(p_j, P_j, S_j) \to (p, P, S)$ in $BG_k(M)$ is equivalent to the convergence of $p_j$ to $p$ in $M$ and to the convergence of $(P_j, S_j)$ to $(P, S)$ in $GB_k(m, \mathbb{R})$, when one considers a local trivialization of the tangent bundle $TM$ around $p$.

Alternatively, the manifold structure of $GB_k(n, \mathbb{R})$ and $GB_k(M)$ can be described in terms of principle and associated bundles.\(^1\)

We can now prove the following results about the stability of the Maslov index in the differential problem and of the focal index in the geometrical problem:

**Theorem 5.2.1.** For each $j \in \mathbb{N} \cup \{\infty\}$, let $(g_j, R_j, P_j, S_j)$ be an admissible quadruple for the differential problem in $\mathbb{R}^n$.

Assume that $(g_j, R_j, P_j, S_j)$ tends to $(g_\infty, R_\infty, P_\infty, S_\infty)$ as $j \to \infty$, in the following sense:

1. $\dim(P_j) = k$ for all $j = 1, \ldots, \infty$;

\(^1\)The open subset $L_{\text{inj}}(\mathbb{R}^k, \mathbb{R}^n)$ in the vector space $L(\mathbb{R}^k, \mathbb{R}^n)$ consisting of injective linear maps is the total space of a $GL(k, \mathbb{R})$-principal bundle over $G_k(\mathbb{R}^n)$. Namely, the projection is given by $T \mapsto T(\mathbb{R}^k)$ and the action of $GL(k, \mathbb{R})$ on $L_{\text{inj}}(\mathbb{R}^k, \mathbb{R}^n)$ is given by composition on the right. Moreover, we have an action of $GL(k, \mathbb{R})$ on the left on $B_{\text{sym}}(\mathbb{R}^k, \mathbb{R})$ given by $\varphi \cdot B = B(\varphi^{-1} \cdot, \varphi^{-1} \cdot)$. It’s easily seen that the associated bundle obtained from this principal bundle and this action is (isomorphic to) the vector bundle $GB_k(n, \mathbb{R})$. The fiber bundle $GB_k(M)$ can also be seen as an associated bundle to the $GL(m, \mathbb{R})$-principal bundle of referentials in $M$ and to the action of $GL(m, \mathbb{R})$ on the manifold $GB_k(m, \mathbb{R})$ on the left defined in the obvious way.
2. \((P_j, S_j) \mapsto (P_\infty, S_\infty)\) in \(\text{GB}_k(n, \mathbb{R})\) as \(j \to \infty\);
3. \(g_j \mapsto g_\infty\) in \(\text{B}_{\text{sym}}(\mathbb{R}^n, \mathbb{R})\) as \(j \to \infty\);
4. \(R_j \mapsto R_\infty\) uniformly on \([a, b]\) as \(j \to \infty\).

If \(t_0 = b\) is not \((P_\infty, S_\infty)\)-focal for \((g_\infty, R_\infty, P_\infty, S_\infty)\), then, for \(j \in \mathbb{N}\) sufficiently large, \(t_0 = b\) is not \((P_j, S_j)\)-focal for \((g_j, R_j, P_j, S_j)\), and:

\[
\mu(g_j, R_j, P_j, S_j) = \mu(g_\infty, R_\infty, P_\infty, S_\infty).
\]

**Proof.** For each \(j = 1, \ldots, \infty\), define the objects \(\Psi_j, H_j, (\ell_0)_j\) and \(\ell_j\) relative to the quadruple \((g_j, R_j, P_j, S_j)\) as in formulas (2.4.2), (5.1.2), (5.1.3) and (5.1.5) respectively. A simple calculation using the charts described for the compact-open topology. Since \(G_{\text{sym}}(\mathbb{R}^{2n})\) shows that \((\ell_0)_j \to (\ell_0)_\infty\) in \(G_{\text{sym}}(\mathbb{R}^{2n})\), and therefore in \(\Lambda\).

Obviously, \(H_j\) tends to \(H_\infty\) uniformly on \([a, b]\); by standard results about the continuous dependence on the data for ordinary differential equations, from (5.1.1) we get that \(\Psi_j\) tends to \(\Psi_\infty\) uniformly (actually, in the \(C^1\)-topology) as \(j \to \infty\).

By the continuity of the action of \(\text{Sp}(\mathbb{R}^{2n}, \omega)\) in \(\Lambda\), it follows that \(\ell_j\) tends to \(\ell_\infty\) in the compact-open topology. Since \(\ell_\infty(b) \in \Lambda_0(L_0)\) and \(\Lambda_0(L_0)\) is open in \(\Lambda\), we have that \(\ell_j(b) \in \Lambda_0(L_0)\), i.e., \(t_0 = b\) is not \((P_j, S_j)\)-focal for \((g_j, R_j, P_j, S_j)\) for \(j \in \mathbb{N}\) sufficiently large.

It is not hard to prove (see Remark 5.2.3 below) that there exists an \(\varepsilon > 0\) such that there are no \((P_j, S_j)\)-focal instants on the interval \([a, a + \varepsilon]\) relatively to the quadruple \((g_j, R_j, P_j, S_j)\), for all \(j = 1, \ldots, \infty\). Hence, the curve \(\ell_j\) does not intersect \(\Lambda_{\geq 1}(L_0)\) in the interval \([a, a + \varepsilon]\), for all \(j = 1, \ldots, \infty\). The Maslov index \(\mu(g_j, R_j, P_j, S_j)\) is by definition equal to the Maslov index \(\mu_{L_0}\) of the restriction of \(\ell_j\) to \([a + \varepsilon, b]\).

Since \(\Lambda\) and \(\Lambda_0(L_0)\) are locally path connected, and since \(\Lambda\) is locally simply connected, the convergence of \(\ell_j\) to \(\ell_\infty\) (over the interval \([a + \varepsilon, b]\)) in the compact-open topology implies that, for \(j \in \mathbb{N}\) sufficiently large, \(\ell_j\) is homotopic to \(\ell_\infty\) through curves with endpoints outside \(\Lambda_{\geq 1}(L_0)\).

Therefore, \(\mu_{L_0}(\ell_j) = \mu_{L_0}(\ell_\infty)\) for \(j \in \mathbb{N}\) large enough, and we are done. \(\square\)

**Corollary 5.2.2.** Let \((\mathcal{M}, g)\) be a Riemannian or Lorentzian manifold; for each \(j \in \mathbb{N} \cup \{\infty\}\) let \(\mathcal{P}_j\) be a \(k\)-dimensional smooth submanifold of \(\mathcal{M}\) and let \(\gamma_j : [a, b] \mapsto \mathcal{M}\) be a non constant geodesic in \(\mathcal{M}\), with \(\gamma_j(a) \in \mathcal{P}_j\) and \(\gamma'_j(a) \in T_{\gamma_j(a)} \mathcal{P}_j^\perp\).

If \((\mathcal{M}, g)\) is Lorentzian, we also assume that \(\gamma_j\) is non spacelike and that \(\gamma'_j(a) \notin T_{\gamma_j(a)} \mathcal{P}_j\) for all \(j = 1, \ldots, \infty\).

Let \(S_j\) denote the second fundamental form of \(\mathcal{P}_j\) at \(\gamma_j(a)\) in the normal direction \(\gamma'_j(a)\). Suppose that

- \(\lim_{j \to \infty} \gamma'_j(a) = \gamma'_\infty(a)\) in \(T \mathcal{M}\),
- \(\lim_{j \to \infty} (\gamma_j(a), T_{\gamma_j(a)} \mathcal{P}_j, S_j) = (\gamma_\infty(a), T_{\gamma_\infty(a)} \mathcal{P}_\infty, S_\infty)\) in \(\text{GB}_k(\mathcal{M})\).

Then, if \(\gamma_\infty(b)\) is not \(\mathcal{P}_\infty\)-focal, it follows that, for \(j \in \mathbb{N}\) sufficiently large, \(\gamma_j(b)\) is not \(\mathcal{P}_j\)-focal, and the geometrical index of \(\gamma_j\) relative to \(\mathcal{P}_j\) is equal to the geometrical index of \(\gamma_\infty\) relative to \(\mathcal{P}_\infty\):

\[
\text{i}_{\text{geom}}(\gamma_j) = \text{i}_{\text{geom}}(\gamma_\infty), \quad \forall j \gg 0.
\]
Proof. We choose a local trivialization of the tangent bundle $T\mathcal{M}$ around $\gamma_\infty(a)$ by linearly independent smooth vector fields $X_1, \ldots, X_m$. Since $\gamma_j(a) \to \gamma_\infty(a)$ as $j \to \infty$, we can assume without loss of generality that $\gamma_j(a)$ is in the domain of the $X_i$'s, for all $j = 1, \ldots, \infty$.

Now, we trivialize the tangent bundle along each $\gamma_j$, $j = 1, \ldots, \infty$, by considering the parallel transport of the vectors $X_i(\gamma_j(a))$ along $\gamma_j$. Associated to these trivializations, we produce quadruples $(g_j, R_j, P_j, S_j)$ admissible for the differential problem in $\mathbb{R}^m$, $j = 1, \ldots, \infty$. We emphasize that we are considering trivializations of the entire tangent bundle along the geodesics $\gamma_j$; recall Remark 2.3.5 for a discussion about this issue. We also observe that the condition that $\gamma'_j(a) \notin T_{\gamma_j(a)}P_j$ implies in particular that $g$ is non degenerate on $T_{\gamma(a)}P_j$.

Clearly, under our hypothesis, $(g_j, R_j, P_j, S_j)$ tends to $(g_\infty, R_\infty, P_\infty, S_\infty)$ as $j \to \infty$ in the sense of Proposition 5.2.1.

The conclusion follows now easily from Corollary 5.1.3 and Proposition 5.2.1. \qed

Remark 5.2.3. Let $(g_\lambda, R_\lambda, P_\lambda, S_\lambda)$ be an admissible quadruple for the differential problem in $\mathbb{R}^n$, that depends continuously on a parameter $\lambda$ varying in a compact topological space. This means that $\dim(P_\lambda) = k \in \mathcal{N}$ for all $\lambda$ and that the maps $\lambda \mapsto g_\lambda \in \mathbb{B}_{\text{sym}}(\mathbb{R}^n, \mathbb{R})$, $(t, \lambda) \mapsto R_\lambda(t) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ and $\lambda \mapsto (P_\lambda, S_\lambda) \in \mathbb{GB}_k(\mathbb{R}^n)$ are continuous. A minor modification in the argument of the proof of Proposition 2.5.1 shows that we can find $\varepsilon > 0$ such that there are no $(P_\lambda, S_\lambda)$-focal instants on the interval $[a, a + \varepsilon]$ for all $\lambda$. Namely, the vector fields $J_i$ and $\hat{J}_i$ appearing in the proof of Proposition 2.5.1 may be chosen to depend continuously on $(t, \lambda)$, and the conclusion follows easily.

6. The Spectral Index.

Some Remarks on a Possible Extension of the Morse Index Theorem

In this section we will define the spectral index of an admissible quadruple for the differential problem $(g, R, P, S)$. Such number is related to the spectral properties of the unbounded operator associated to the differential equation (2.2.1) with boundary conditions (2.2.2) and $J(b) = 0$. Under suitable hypotheses, we will prove that this index equals the Maslov index of the quadruple. If $(g, R, P, S)$ arises from a Riemannian or non spacelike Lorentzian geodesic, the equality of the spectral index and the Maslov index of $(g, R, P, S)$ gives an equivalent form of the classical Morse index theorem.

6.1. Eigenvalues of the differential problem and the spectral index

Let’s fix an admissible quadruple $(g, R, P, S)$ for the differential problem in $\mathbb{R}^n$; we will consider the space $\mathcal{H} = L^2([a, b], \mathbb{R}^n)$ of $\mathbb{R}^n$-valued square-integrable vector fields on $[a, b]$; rather than choosing a specific inner product on $\mathcal{H}$, we will only regard it as a Hilbertable space, since all our statements on $\mathcal{H}$ will only depend on its topological structure.
On $\mathcal{H}$ we define the following bounded symmetric bilinear form $\hat{g}$:

$$\hat{g}(u, v) = \int_a^b g(u(t), v(t)) \, dt; \quad (6.1.1)$$

from the nondegeneracy of $g$ and the fundamental theorem of Calculus of Variations, it follows easily that $\hat{g}$ is non degenerate on $\mathcal{H}$.

Let $\hat{R} : \mathcal{H} \mapsto \mathcal{H}$ be the bounded linear operator given by:

$$\hat{R}[v](t) = R(t)[v(t)], \quad t \in [a, b], \quad (6.1.2)$$

and let $\mathcal{A}$ be the densely defined unbounded operator given by

$$\mathcal{A} = -\frac{d^2}{dt^2} + \hat{R}, \quad (6.1.3)$$

defined in the domain $D \subset \mathcal{H}$:

$$D = \left\{ u \in C^2([a, b], \mathbb{R}^n) : u(a) \in P, \ u'(a) + S[u(a)] \in P^\perp, \ u(b) = 0 \right\}. \quad (6.1.4)$$

It is easily seen that the operator $\mathcal{A}$ is $\hat{g}$-symmetric, in the sense that

$$\hat{g}(\mathcal{A}u, v) = \hat{g}(u, \mathcal{A}v), \quad (6.1.5)$$

for all $u, v \in D$. However, it is in general impossible to choose a Hilbert space product on $\mathcal{H}$ that makes $\mathcal{A}$ symmetric. Hence, the spectrum of $\mathcal{A}$ will not in general be real, and for this reason we need to introduce a complexification of $\mathcal{H}$. Indeed, we need to investigate the holomorphy properties of our differential problem in order to establish the discreteness of the set of eigenvalues of $\mathcal{A}$.

Let $\mathcal{H}^C$ be the complex Hilbert space $L^2([a, b], \mathbb{C}^n)$; we regard $\mathcal{H}$ as a subspace of $\mathcal{H}^C$. The space $\mathcal{H}^C$ is a complexification of $\mathcal{H}$, in the sense that $\mathcal{H}^C = \mathcal{H} \oplus i\mathcal{H}$. To each subspace $\mathcal{W} \subset \mathcal{H}$ we associate its complexification $\mathcal{W}^C = \mathcal{W} \oplus i\mathcal{W}$, which is the complex subspace of $\mathcal{H}^C$ generated by $\mathcal{W}$.

Moreover, every linear operator on $\mathcal{H}$ (bounded or unbounded) has a unique complex linear extension to $\mathcal{H}^C$. For simplicity, we will maintain the same notations for linear operators on $\mathcal{H}$ and their complex linear extensions to $\mathcal{H}^C$. In particular, we will consider the complex linear extension of $\hat{R}$ to $\mathcal{H}^C$ and of $\mathcal{A}$ to $D^C$.

Let $\lambda \in \mathbb{C}$; we consider the eigenvalue problem for $\mathcal{A}$ in $D^C$:

$$u'' = \hat{R}[u] - \lambda u. \quad (6.1.6)$$

Observe that (6.1.6) is the differential equation (2.2.1) corresponding to the quadruple $(g, R_\lambda, P^C, S)$, where

$$R_\lambda(t) = R(t, \lambda) = R(t) - \lambda \cdot \text{Id}. \quad (6.1.7)$$

Here, we are considering an obvious extension of the notion of admissible quadruples for the differential problem to complex spaces. For such an extension, one identifies the space $\mathcal{L}(\mathbb{R}^k, \mathbb{R}^k)$ with the subspace of $\mathcal{L}_c(\mathbb{C}^k, \mathbb{C}^k)$ consisting of all the complex linear operators on $\mathbb{C}^k$ that preserve the real subspace $\mathbb{R}^k$. For instance, $R(t)$ and $R_\lambda(t)$ are seen as complex linear operators on $\mathbb{C}^n$. 
In analogy with (2.4.2) and (5.1.2), we define complex linear operators $\Psi_\lambda(t)$ and $H_\lambda(t)$ on $\mathbb{C}^{2n}$, given by:

(6.1.8) \[ \Psi_\lambda(t)[(u(a), u'(a))] = (u(t), u'(t)), \quad H_\lambda(t)[(x, y)] = (y, R_\lambda(t)[x]), \]

where $u : [a, b] \mapsto \mathbb{C}^n$ is a solution of (6.1.6) and $x, y \in \mathbb{C}^n$. Formulas (6.1.7) and (6.1.8) define maps:

$$ R : [a, b] \times \mathbb{C} \mapsto \mathcal{L}_\mathbb{C}(\mathbb{C}^n, \mathbb{C}^n), \quad \Psi, H : [a, b] \times \mathbb{C} \mapsto \mathcal{L}_\mathbb{C}(\mathbb{C}^{2n}, \mathbb{C}^{2n}); $$

these maps are continuous, and they are holomorphic on the second variable. For the holomorphy of $\Psi$, we are using well known regularity results for the solutions of differential equations. Indeed, $\Psi$ and $\frac{\partial \Psi}{\partial \lambda}$ satisfy the following Cauchy problems:

(6.1.9) \[ \frac{d}{dt} \Psi(t, \lambda) = H(t, \lambda) \Psi(t, \lambda), \quad \Psi(a, \lambda) = \text{Id}; \]

(6.1.10) \[ \frac{d}{dt} \frac{\partial \Psi}{\partial \lambda}(t, \lambda) = \frac{\partial H}{\partial \lambda}(t, \lambda) \Psi(t, \lambda) + H(t, \lambda) \frac{\partial \Psi}{\partial \lambda}(t, \lambda), \quad \frac{\partial \Psi}{\partial \lambda}(a, \lambda) = 0. \]

From (6.1.9) and (6.1.10), we see that $\Psi(t, \lambda)$ is differentiable in $t$, the derivative $\frac{d}{dt} \Psi(t, \lambda)$ is jointly continuous in the two variables and holomorphic in $\lambda$.

The eigenvalues of $\mathcal{A}$ in $D^\mathbb{C}$, i.e., the complex numbers $\lambda$ for which equation (6.1.6) admits non trivial solutions in $D^\mathbb{C}$, can be described as the zeroes of a suitable entire function. For instance, they are the zeroes of the function

$$ r(\lambda) = \det(\pi \circ \Psi(b, \lambda)[e_1], \ldots, \pi \circ \Psi(b, \lambda)[e_n]), $$

where $e_1, \ldots, e_n$ is a basis of the vector space $\ell_0 \subset \mathbb{R}^{2n}$ defined in (5.1.3) and $\pi : \mathbb{C}^{2n} \mapsto \mathbb{C}^n$ is the projection onto the first $n$ coordinates. Hence, the set of eigenvalues of $\mathcal{A}$ in $D^\mathbb{C}$ is either $\mathbb{C}$ or a discrete subset of $\mathbb{C}$. We will establish next that the real eigenvalues of $\mathcal{A}$ in $D^\mathbb{C}$ (or equivalently in $D$) are bounded from below, and so the set of eigenvalues of $\mathcal{A}$ in $D^\mathbb{C}$ is discrete in $\mathbb{C}$.

We need the following technical Lemma:

**Lemma 6.1.1.** Let $Z$ be a finite dimensional real (or complex) vector space equipped with a positive definite inner (or Hermitian) product $\langle \cdot, \cdot \rangle$, and let the corresponding norm be denoted by $\| \cdot \|$. Let $u : [a, b] \mapsto Z$ be a $C^1$-function such that $u(b) = 0$. Then, the following inequality holds:

(6.1.11) \[ \int_a^b \| u(t) \| \, dt \cdot \int_a^b \| u'(t) \| \, dt \geq \frac{1}{4} \| u(a) \|^4. \]

**Proof.** Using the Cauchy–Schwarz inequality, we compute easily:

$$ \| u(a) \|^2 = - \int_a^b \frac{d}{dt} \langle u(t), u(t) \rangle \, dt = -2 \int_a^b \langle u'(t), u(t) \rangle \, dt \leq 2 \left( \int_a^b \| u(t) \|^2 \, dt \right)^{\frac{1}{2}} \left( \int_a^b \| u'(t) \|^2 \, dt \right)^{\frac{1}{2}}. $$

The inequality (6.1.11) follows easily. \qed

We can now prove the following:
Proposition 6.1.2. The real part of the eigenvalues of $A$ in $D^C$ is bounded from below.

Proof. Since $g$ is non-degenerate on $P$, then $P$ and $P^\perp$ are complementary subspaces in $\mathbb{R}^n$; let $\langle \cdot, \cdot \rangle$ be any positive definite inner product on $\mathbb{R}^n$ which makes $P$ and $P^\perp$ orthogonal, and denote also by $\langle \cdot, \cdot \rangle$ its extension to a Hermitian product in $\mathbb{C}^n$. We denote by $\langle \cdot, \cdot \rangle_2$ the corresponding Hermitian product in $L^2([a, b], \mathbb{C}^n)$.

Using integration by parts, for all $u \in D^C$ we have the following:

\begin{equation}
-\frac{d^2}{dt^2} u(t) + \lambda u(t) = \int_a^b \langle u'(t), u'(t) \rangle \, dt + \langle u(a), u'(a) \rangle.
\end{equation}

Moreover, for all $u \in D^C$, we have:

\begin{equation}
\|u(t)\|_2^2 = \int_a^b \|u(t)\|^2 \, dt = 1; \text{ we apply to such a function Lemma 6.1.1, obtaining:}
\end{equation}

\begin{equation}
\int_a^b \|u'(t)\|^2 \, dt \geq \frac{\|u(a)\|^4}{4}, \quad \forall u \in D^C, \|u\|_2 = 1.
\end{equation}

Using (6.1.13) and (6.1.14), we obtain that the right side of (6.1.12) is bounded from below for $u \in D^C$ with $\|u\|_2 = 1$, i.e., there exists $k_0 \in \mathbb{R}$ such that:

\begin{equation}
\langle -\frac{d^2}{dt^2} u, u \rangle_2 \geq k_0, \quad \forall u \in D^C, \|u\|_2 = 1.
\end{equation}

Let now $\lambda$ be any eigenvalue of $A$ in $D$ and $u \in D^C$ be a corresponding eigenvector with $\|u\|_2 = 1$. From (6.1.15) we compute easily:

\[
\text{Re}(\lambda) = \text{Re}(\langle \lambda u, u \rangle_2) = \langle -\frac{d^2}{dt^2} u, u \rangle_2 + \text{Re}(\langle \hat{R}[u], u \rangle_2) \geq k_0 - \|\hat{R}\|_2 > -\infty,
\]

where $\|\hat{R}\|_2$ is the operator norm of $\hat{R}$ in $L^2([a, b], \mathbb{C}^n)$. This concludes the proof. \qed

As we have observed previously, the set of eigenvalues of $A$ in $D^C$ is discrete, and so Proposition 6.1.2 gives us the following corollary:

Corollary 6.1.3. The operator $A$ has only a finite number of real negative eigenvalues in $D$. \qed

Remark 6.1.4. In what follows, we will have to consider the operators $A_t$ and $\hat{R}_t$ on $L^2([a, t], \mathbb{R}^n)$, for a fixed $t \in [a, b]$, defined in analogy with (6.1.2) and (6.1.3) considering the restriction of $R$ to $[a, t]$. The domain of $A_t$ is meant to be the subspace $D_t$ defined as in (6.1.4) by replacing the endpoint $b$ with $t$.

Clearly, Proposition 6.1.2 and Corollary 6.1.3 remain valid for $A_t$; as a matter of fact, one can choose a lower bound for the real eigenvalues of $A_t$, which is independent of $t \in [a, b]$. This can be easily seen by considering that the constant $k_0$ in the inequality (6.1.15) does not depend on $t$ and that $\|\hat{R}_t\|$ is bounded from above by the supremum norm of $R : [a, b] \mapsto \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$. 


From now on, we will disregard the complexified spaces introduced, and we will only deal with the real eigenvalues of $A$ in $D$. So, we will look at the maps $H$ and $\Psi$ only in their real domains and counterdomains:

$$H : [a, b] \times \mathbb{R} \mapsto \text{sp}(\mathbb{R}^{2n}, \omega), \quad \Psi : [a, b] \times \mathbb{R} \mapsto \text{Sp}(\mathbb{R}^{2n}, \omega),$$

where $\omega$ is the symplectic form of Definition 2.4.1.

Keeping in mind formulas (5.1.3), (5.1.4) and (5.1.5), we define $\ell : [a, b] \times \mathbb{R} \mapsto \Lambda$ by:

$$\ell(t, \lambda) = \Psi(t, \lambda)[\ell_0].$$

If $\lambda$ is a real eigenvalue of $A$ in $D$, we denote by $H_\lambda = \text{Ker}(A - \lambda \cdot \text{Id}) \subset D$ the corresponding eigenspace. We observe that $H_\lambda$ is the set of $(P, S)$-solutions relative to the quadruple $(g, R_\lambda, P, S)$ vanishing at $t_0 = b$. It follows that $\lambda \in \mathbb{R}$ is an eigenvalue of $A$ if and only if $t_0 = b$ is a $(P, S)$-focal instant for such a quadruple. Moreover, the dimension of $H_\lambda$ coincides with the multiplicity of $t_0 = b$ as a $(P, S)$-focal instant for the quadruple $(g, R_\lambda, P, S)$, and therefore it is finite:

$$\dim(H_\lambda) < +\infty. \quad (6.1.16)$$

We now look at the Maslov index $\mu_{L_0}$ of the curve $\lambda \mapsto \ell(b, \lambda)$; we observe that $\ell(b, \lambda) \in \Lambda_{\geq 1}(L_0)$ if and only if $\lambda$ is an eigenvalue of $A$ in $D$. Moreover, in analogy with (5.1.6), we have:

$$\ell(b, \lambda) \cap L_0 = \left\{(0, u'(b)) : u \in \mathbb{J}_\lambda, \ u(b) = 0 \right\} = \{0\} \oplus \mathbb{J}_\lambda[b] ^\perp, \quad (6.1.17)$$

where $\mathbb{J}_\lambda$ is the space of all $(P, S)$-solutions relative to the quadruple $(g, R_\lambda, P, S)$.

We can now give the following:

**Definition 6.1.5.** The spectral index $i_{\text{spec}}$ of the quadruple $(g, R, P, S)$ is the sum of the signatures of the restrictions of $\hat{g}$ to the eigenspaces relative to the negative eigenvalues of $A$:

$$i_{\text{spec}} = \sum_{\lambda < 0} \text{sgn}(\hat{g} | n_\lambda).$$

Observe that, by Corollary 6.1.3 and formula (6.1.16), $i_{\text{spec}}$ is a finite integer number.

### 6.2. A generalized Morse Index Theorem

We want to prove that, under suitable hypotheses, $i_{\text{spec}}$ is equal to the Maslov index $\mu_{L_0}$ of the curve $\lambda \mapsto \ell(b, \lambda)$. We start with the following:

**Lemma 6.2.1.** Let $\lambda \in \mathbb{R}$ be an eigenvalue of $A$ in $D$. The map $H_\lambda \ni u \mapsto (0, u'(b)) \in \ell(b, \lambda) \cap L_0$ is a linear isomorphism which carries the restriction of $\hat{g}$ to the restriction of the symmetric bilinear form $\frac{\partial}{\partial x}(b, \lambda)$. 
Proof. The map \( u \mapsto (0, u'(b)) \) is clearly injective on \( \mathcal{H}_\lambda \), and it is onto by (6.1.17).

We compute the derivative \( \frac{\partial \ell}{\partial \lambda} \); recalling Proposition 3.2.3 and Definition 3.2.4, we have:

\[
\frac{\partial \ell}{\partial \lambda}(t, \lambda) = [\partial \Psi / \partial \lambda \Psi^{-1}]^* \ell(t, \lambda) = \omega(\partial \Psi / \partial \lambda \Psi^{-1} \cdot \cdot \cdot |_{\ell(t, \lambda)});
\]

the pull-back of \( \frac{\partial \ell}{\partial \lambda}(t, \lambda) \) by \( \Psi(t, \lambda) \) is a symmetric bilinear form on \( \ell_0 \) given by:

\[
(6.2.1) \quad \frac{\partial \ell}{\partial \lambda}(t, \lambda)[\Psi \cdot, \Psi \cdot] = \omega(\partial \Psi / \partial \lambda \cdot \cdot \cdot |_{\ell_0});
\]

We want to calculate the derivative of the pull-back (6.2.1) with respect to \( t \). First, we differentiate \( \Psi^{-1} \partial \Psi / \partial \lambda \):

\[
\frac{d}{dt} \left[ \Psi^{-1} \frac{\partial \Psi}{\partial \lambda} \right] = -\Psi^{-1} \frac{\partial \Psi}{\partial t} \Psi^{-1} \frac{\partial \Psi}{\partial \lambda} + \Psi^{-1} \frac{\partial H}{\partial \lambda} \Psi + \Psi^{-1} H \frac{\partial \Psi}{\partial \lambda} =
\]

\[
(6.2.2) \quad = -\Psi^{-1} H \frac{\partial \Psi}{\partial \lambda} + \Psi^{-1} \frac{\partial H}{\partial \lambda} \Psi + \Psi^{-1} H \frac{\partial \Psi}{\partial \lambda} = \Psi^{-1} \frac{\partial H}{\partial \lambda} \Psi,
\]

where in the first equality we have used (6.1.10) and in the second one we have used (6.1.9). Hence, the derivative of the pull-back (6.2.1) is given by:

\[
(6.2.3) \quad \frac{d}{dt} \omega(\partial \Psi / \partial \lambda \cdot \cdot \cdot |_{\ell_0}) = \frac{d}{dt} \omega(\Psi^{-1} \frac{\partial H}{\partial \lambda} \Psi \cdot \cdot \cdot |_{\ell_0}) = \omega(\partial \Psi / \partial \lambda \cdot \cdot \cdot |_{\ell_0}),
\]

where in the first and in the third equality we have used the fact that \( \Psi \) is a symplectomorphism and in the second one we have used (6.2.2).

Observe that, by (6.1.7) and (6.1.8), \( \frac{\partial H}{\partial \lambda} \) is simply given by:

\[
(6.2.4) \quad \frac{\partial H}{\partial \lambda}[(x, y)] = (0, -x), \quad \forall x, y \in \mathbb{R}^n.
\]

Integration of (6.2.3) on the interval \([a, b]\) using (6.2.1) gives:

\[
(6.2.5) \quad \int_a^b \omega(\partial \Psi / \partial \lambda \cdot \cdot \cdot |_{\ell_0}) dt = \frac{\partial \ell}{\partial \lambda}(b, \lambda)[\Psi(b, \lambda) \cdot, \Psi(b, \lambda) \cdot],
\]

because \( \frac{\partial \Psi}{\partial \lambda}(a, \lambda) = 0 \) (see equation (6.1.10)).

Finally, let \( u, v \) be elements of \( \mathcal{H}_\lambda \); evaluating (6.2.5) in the pairs \((u(a), u'(a))\) and \((v(a), v'(a))\) in \( \ell_0 \), we get:

\[
\int_a^b g(u(t), v(t)) dt = \frac{\partial \ell}{\partial \lambda}(b, \lambda)[(u(b), u'(b)), (v(b), v'(b))] =
\]

\[
= \frac{\partial \ell}{\partial \lambda}(b, \lambda)[(0, u'(b)), (0, v'(b))],
\]

which was obtained by using Definition 2.4.1 and formulas (6.1.8) and (6.2.4).

This concludes the proof. \( \square \)
Remark 6.2.2. The computation presented in the proof of Lemma 6.2.1 also appears in the proof of [12, Proposition 6.2]. However, in drawing the final conclusion of the Proposition, the author identifies the intersection \( \ell(t, \lambda) \cap L_0 \) with the \textit{generalized eigenspace} \( \bigcup_{k \geq 1} \ker(\mathcal{A} - \lambda)^k \). This is clearly \textit{not} the case if \( \lambda \) is not a simple eigenvalue of \( \mathcal{A} \); it is not clear whether the case of non simple eigenvalues can be treated by other arguments.

Corollary 6.2.3. Suppose that the restriction of \( \hat{g} \) to \( \mathcal{H}_\lambda \) is non degenerate for every negative eigenvalue \( \lambda \) of \( \mathcal{A} \) in \( D \). Then, if \( \lambda_0 < 0 \) is smaller than the minimum eigenvalue of \( \mathcal{A} \) in \( D \), the Maslov index \( \mu_{\lambda_0} \) of the curve \( [\lambda_0, 0] \ni \lambda \mapsto \ell(b, \lambda) \) is equal to the spectral index \( i_{\text{spec}} \).

\[ \mu_{\lambda_0} = \mu(g, R, P, S) \]

Proof. It follows immediately from Corollary 4.3.3 and Lemma 6.2.1.

In the following theorem we relate the spectral index with the Maslov index of a quadruple \((g, R, P, S)\). Recalling Theorem 5.1.2, we then obtain an equality also of the focal and the spectral indexes.

Theorem 6.2.4. Let \((g, R, P, S)\) be an admissible quadruple for the differential problem in \( \mathbb{R}^n \); assume that \( t_0 = b \) is not a \((P, S)\)-focal instant. Suppose that the restriction of \( \hat{g} \) to \( \mathcal{H}_\lambda \) is non degenerate for every negative eigenvalue \( \lambda \) of \( \mathcal{A} \) in \( D \). Then, the spectral and the Maslov indexes of \((g, R, P, S)\) coincide:

\[ i_{\text{spec}} = \mu(g, R, P, S). \]

Proof. Let \( \lambda_0 < 0 \) be chosen so that \( \ell(t, \lambda) \notin \Lambda_{\geq 1}(L_0) \) for all \( t \in [a, b] \) and all \( \lambda \leq \lambda_0 \). By Remark 6.1.4, to such purpose it suffices to take \( |\lambda_0| \) large enough. By Remark 5.2.3, we can find \( \varepsilon > 0 \) small enough, so that \( \ell(t, \lambda) \notin \Lambda_{\geq 1}(L_0) \) for all \( t \in ]a, a + \varepsilon[ \) and for all \( \lambda \in [\lambda_0, 0] \). We will consider the restriction of \( \ell \) to the rectangle \( [a + \varepsilon, b] \times [\lambda_0, 0] \). Now, the Maslov index \( \mu_{\lambda_0} \) of the curve \( t \mapsto \ell(t, 0), t \in [a + \varepsilon, b] \) is by definition the Maslov index of the quadruple \((g, R, P, S)\); the Maslov index \( \mu_{\lambda_0} \) of the curve \( \lambda \mapsto \ell(b, \lambda), \lambda \in [\lambda_0, 0] \) is equal to the spectral index, by Corollary 6.2.3. Finally, the image by \( \ell \) of the remaining two sides of the rectangle \([a + \varepsilon, b] \times [\lambda_0, 0] \) is disjoint from \( \Lambda_{\geq 1}(L_0) \) by our choice of \( \varepsilon \) and \( \lambda_0 \). The conclusion follows from the homotopy invariance of the Maslov index.

Remark 6.2.5. Theorem 6.2.4 can be seen as a generalization of the classical Morse Index Theorem in Riemannian or Lorentzian geometry, in the following sense. Let’s assume that \((g, R, P, S)\) is a given admissible quadruple for the differential problem in \( \mathbb{R}^n \), with the property that \( g \) is \textit{positive definite} in \( \mathbb{R}^n \). This is the case when \((g, R, P, S)\) arises from an admissible quadruple \((\mathcal{M}, \mathcal{g}, \gamma, \mathcal{P})\), with \((\mathcal{M}, \mathcal{g})\) Riemannian or Lorentzian, and in the latter case, with \( \gamma \) non spacelike (see Proposition 2.3.1 and Remark 2.3.5). In the Lorentzian case, the bilinear form \( g \) is positive definite when one considers trivializations of the normal bundle along \( \gamma \), if \( \gamma \) is timelike, and of the quotient bundle \( \mathcal{N} \) for a lightlike geodesic \( \gamma \) (see Section 2).

For a positive definite \( g \), the corresponding bilinear form \( \hat{g} \) is a Hilbert space inner product on \( \mathcal{H} \); by \((6.1.5), A \) is symmetric, and it admits a closed self-adjoint extension \( \mathcal{A}_0 \) to some suitable Sobolev space \( D_0 \). More precisely, \( D_0 \) is easily seen to be the
space of all \( C^1 \)-functions \( u : [a, b] \mapsto \mathbb{R}^n \) with absolutely continuous derivative and square integrable second derivative, satisfying the boundary conditions:

\[
(6.2.6) \quad u(a) \in P, \quad u'(a) + S[u(a)] \in P^\perp, \quad u(b) = 0.
\]

An explicit integral formula for the resolvent of \( A_o \), using the method of variations of constants, shows that the spectrum \( \sigma(A_o) \) coincides with the set of eigenvalues of \( A \) in \( D \).

By the spectral theorem for (unbounded) self-adjoint operators, we get a direct sum decomposition

\[
\mathcal{H} = \bigoplus_{\lambda \in \sigma(A_o)} \mathcal{H}_\lambda,
\]

where \( \mathcal{H}_\lambda \) is, as before, the eigenspace of \( A \) corresponding to \( \lambda \).

We introduce the index form \( I \) by:

\[
(6.2.7) \quad I(u, v) = \int_a^b \left[ g(u', v') + g(R[u], v) \right] \, dt - S[u(a), v(a)],
\]

which is a symmetric bilinear form in \( D_o \); we observe that a simple integration by parts shows that:

\[
I(u, v) = \hat{g}(A_o[u], v) = \int_a^b g(-u'' + R[u], v) \, dt, \quad \forall u, v \in D_o.
\]

From the spectral decomposition it follows easily that the index of the bilinear form \( I \) on \( D_o \) (in the sense of Definition 2.0.1) is the sum of the dimensions of \( \mathcal{H}_\lambda \) for negative \( \lambda \). As \( \hat{g} \) is positive definite, this number coincides with the spectral index of \((g, R, P, S)\).

The bilinear form \( I \) of formula (6.2.7) can be naturally extended to a continuous bilinear form on the Hilbert space \( \mathcal{H}^1 \) consisting of all absolutely continuous functions with square integrable first derivative and satisfying the boundary conditions (6.2.6). The Hilbert space inner product on \( \mathcal{H}^1 \) that makes \( I \) continuous is, for instance, \((u, v) \mapsto \hat{g}(u', v')\). The space \( D_o \) is a dense linear subspace of \( \mathcal{H}^1 \); since \( I \) is continuous in \( \mathcal{H}^1 \), a simple density argument shows that the index of \( I \) on \( \mathcal{H}^1 \) is the same as the index of \( I \) on any dense subspace of \( \mathcal{H}^1 \). For instance, in the classical proof of the Morse Index Theorem (see [7, 15]), one considers the space of piecewise smooth functions.

Now, if \( t_0 = b \) is not a \((P, S)\)-focal instant, Theorem 6.2.4 gives us an equality between the spectral index and the Maslov index of \((g, R, P, S)\); observe that the non degeneracy assumption of \( \hat{g} \) on each \( \mathcal{H}_\lambda \) is automatically satisfied.

If \((g, R, P, S)\) is associated to the quadruple \((\mathcal{M}, g, \gamma, \mathcal{P})\), where \((\mathcal{M}, g)\) is either Riemannian or Lorentzian, and in the latter case, \( \gamma \) is non spacelike, then, under the assumption that \( \gamma(b) \) is not a \( \mathcal{P} \)-focal point, Corollary 5.1.3 gives the equality between the geometrical index of \( \gamma \) and the Maslov index of \((g, R, P, S)\), i.e., the index of the bilinear form \( I \) on the space \( \mathcal{H}^1 \) (or any of its dense subspaces).

It is not difficult to prove that the index of \( I \) on \( \mathcal{H}^1 \) is equal to the index of the second variation of the energy functional on the set of curves connecting the submanifold \( \mathcal{P} \).
with the point \( \gamma(b) \). The equality of this index with the geometrical index of \( \gamma \) is precisely the statement of the classical Morse Index Theorem.

7. Curves of Lagrangians originating from differential problems

In this section we discuss some necessary conditions for a given curve of Lagrangians \( \ell : [a, b] \mapsto \Lambda \) to arise from an admissible quadruple for the differential problem. In some cases, in order to produce examples or counterexamples we will also give sufficient conditions, and in particular we will exhibit a counterexample to Theorem 5.1.2 when the hypothesis (2) is dropped. Namely, we give an example of an admissible quadruple \( (g, R, P, S) \) in \( \mathbb{R}^2 \), with \( R \) real analytic, \( n_+(g) = 1 \), \( P = \{0\} \), having a unique \((P, S)\)-focal instant, and whose Maslov index is equal to \(-1\), while the focal index is zero. The geometric realization of this example (see Proposition 2.3.1) is given by a spacelike geodesic in a three-dimensional real analytic Lorentzian manifold.

7.1. Differential problems determine curves of Lagrangians that are tangent to distributions of affine spaces

Let \( g \) be a fixed non degenerate symmetric bilinear form in \( \mathbb{R}^n \), and let \( \omega \) be the symplectic form in \( \mathbb{R}^{2n} \) given in Definition 2.4.1. Let \( \Psi : [a, b] \mapsto \text{Sp}(\mathbb{R}^{2n}, \omega) \) be a C\(^1\)-curve such that \( \Psi(a) = \text{Id} \); a necessary and sufficient condition for such a curve to arise from a quadruple \( (g, R, P, S) \) in the sense of (2.4.2) can be given as follows. Let \( \mathring{D}_0 \) be the right invariant distribution of vector spaces in \( \text{Sp}(\mathbb{R}^{2n}, \omega) \) whose value \( \mathring{D}_0(\text{Id}) \) at the identity element is the subspace of \( \text{sp}(\mathbb{R}^{2n}, \omega) \) consisting of the linear operators \( H : \mathbb{R}^{2n} \mapsto \mathbb{R}^{2n} \) of the form:

\[
H(x, y) = (0, R[x]),
\]

for some \( g \)-symmetric linear operator \( R : \mathbb{R}^n \mapsto \mathbb{R}^n \). Observe that \( \mathring{D}_0 \) is a distribution of rank \( \frac{1}{2}n(n + 1) \). We also define a right invariant distribution \( \mathring{D} \) of affine subspaces in \( \text{Sp}(\mathbb{R}^{2n}, \omega) \) whose value \( \mathring{D}(\text{Id}) \) is the affine translation of the vector space \( \mathring{D}_0(\text{Id}) \) by the vector \( \mathring{H} \in \text{sp}(\mathbb{R}^{2n}, \omega) \) given by

\[
\mathring{H}(x, y) = (y, 0).
\]

(7.1.1)

Keeping in mind formulas (5.1.1) and (5.1.2), it is easily seen that a necessary and sufficient condition for \( \Psi \) to arise from a quadruple \( (g, R, P, S) \) is that \( \Psi'(t) \in \mathring{D}(\Psi(t)) \) for all \( t \in [a, b] \); we will refer to this situation by saying that \( \Psi \) is horizontal with respect to the distribution \( \mathring{D} \).

We are now going to project the distributions \( \mathring{D}_0 \) and \( \mathring{D} \) to distributions \( D_0 \) and \( D \) in \( \Lambda \). Recalling Proposition 3.2.3 and Definition 3.2.4, we consider the map:

\[
\Lambda \times \text{sp}(\mathbb{R}^{2n}, \omega) \ni (L, H) \mapsto H^*(L) = d\kappa_L(\text{Id})[H] \in T\Lambda;
\]

(7.1.2)

it is obvious that this map is a vector bundle morphism, and, due to the transitivity of the action of \( \text{Sp}(\mathbb{R}^{2n}, \omega) \) on \( \Lambda \), it is surjective on each fiber. Moreover, for each \( L \in \Lambda \), the kernel of (7.1.2) (restricted to the fiber over \( L \)) is the Lie algebra \( \text{sp}(\mathbb{R}^{2n}, \omega, L) \).
Moreover, comparing (7.1.4) with Proposition 3.2.10, we have that, for $H \in \text{sp}(\mathbb{R}^{2n}, \omega)$ such that $H(L) \subset L$, we define $\mathcal{D}$ and $\mathcal{D}_0$ to be the images of $\Lambda \times \hat{\mathcal{D}}(\text{Id})$ and of $\Lambda \times \hat{\mathcal{D}}_0(\text{Id})$ respectively under the map (7.1.2). Using the right invariance property of $\hat{\mathcal{D}}_0$ and $\mathcal{D}$, it is easily seen that, if $\ell_0 \in \Lambda$ and $\psi \in \text{Sp}(\mathbb{R}^{2n}, \omega)$ are given, then setting $L = \psi(\ell_0)$, one has:

\[ \mathcal{D}(L) = d\kappa_{\ell_0}(\psi)[\hat{\mathcal{D}}(\psi)], \quad \mathcal{D}_0(L) = d\kappa_{\ell_0}(\psi)[\hat{\mathcal{D}}_0(\psi)]. \]

Observe that, for each $L \in \Lambda$, $\mathcal{D}(L)$ is an affine subspace of $T_L\Lambda$ whose parallel vector subspace is $\mathcal{D}_0(L)$; we emphasize that the dimension of $\mathcal{D}_0(L)$ is non constant for $L \in \Lambda$, and so we have distributions of non constant rank.

Let now $P$ be a $g$-nondegenerate subspace of $\mathbb{R}^{2n}$ and $S$ be a symmetric bilinear form in $P$; we denote by $\ell_0$ the Lagrangian in $\Lambda$ determined by $(P, S)$ in (5.1.3). Let $\ell : [a, b] \mapsto \Lambda$ be a $C^1$-curve with $\ell(a) = \ell_0$. Clearly, a necessary condition for $\ell$ to arise from a quadruple $(g, R, P, S)$ in the sense of (5.1.5) is that $\ell$ be horizontal with respect to $\mathcal{D}$, i.e., $\ell'(t) \in \mathcal{D}(\ell(t))$ for all $t \in [a, b]$. We will show that this condition is in general not sufficient.

We now compute explicitly $\mathcal{D}$ and $\mathcal{D}_0$. Let $L_0 \in \Lambda$ be fixed. Using Definition 3.2.4, we see that $\mathcal{D}_0(L)$ consist of the restrictions to $L$ of the bilinear forms $\omega(H, \cdot, \cdot)$, where $H$ runs through $\mathcal{D}_0(\text{Id})$. We compute easily:

\[ (7.1.3) \quad \{ \omega(H, \cdot, \cdot) : H \in \mathcal{D}_0(\text{Id}) \} = \{ B \in B_{\text{sym}}(\mathbb{R}^{2n}, \mathbb{R}) : L_0 \subset \ker(B) \}, \]

where $L_0 = \{0\} \oplus \mathbb{R}^n$. Moreover, the image of $(L, H)$ under (7.1.2) is the restriction to $L$ of the bilinear form $0 \oplus g$ in $\mathbb{R}^{2n}$. Hence, we have the following description of $\mathcal{D}_0$ and $\mathcal{D}$:

\[ (7.1.4) \quad \mathcal{D}_0(L) = \{ B \in B_{\text{sym}}(L, \mathbb{R}) : L \cap L_0 \subset \ker(B) \}, \quad \mathcal{D} = (0 \oplus g)|_L + \mathcal{D}_0(L). \]

Now it is easy to compute the dimensions of $\mathcal{D}_0(L)$, for varying $L \in \Lambda$. Namely, if $L \in \Lambda_0(L_0)$, then $L \cap L_0 = \{0\}$, and therefore $\mathcal{D}_0(L) = \mathcal{D}(L) = T_L\Lambda$. More in general, if $L \in \Lambda_k(L_0)$, then

\[ \dim(\mathcal{D}_0(L)) = \dim(B_{\text{sym}}(\mathbb{R}^{n-k}, \mathbb{R})) = \frac{1}{2}(n-k)(n-k+1), \quad L \in \Lambda_k(L_0); \]

moreover, comparing (7.1.4) with Proposition 3.2.10, we have that, for $L \in \Lambda_k(L_0)$, $\mathcal{D}_0(L) \subset T_L(\Lambda_k(L_0))$.

We consider the surjective linear map

\[ (7.1.5) \quad \hat{\mathcal{D}}_0(\text{Id}) \longrightarrow \mathcal{D}_0(L) \]

given by the restriction of (7.1.2).

For $L \in \Lambda_0(L_0)$, then $\dim(\hat{\mathcal{D}}_0(\text{Id})) = \dim(\mathcal{D}_0(L))$, hence (7.1.5) is an isomorphism. More in general, for $L \in \Lambda_k(L_0)$, the dimension of the kernel of (7.1.5) is equal to the codimension of $\mathcal{D}_0(L)$ in $T_L\Lambda$. We also consider the surjective affine map:

\[ (7.1.6) \quad \hat{\mathcal{D}}(\text{Id}) \longrightarrow \mathcal{D}(L), \]
defined similarly.

Suppose now that \( \ell : [a, b] \mapsto \Lambda \) is a \( C^1 \)-curve with \( \ell(a) = \ell_0 \) which is horizontal with respect to \( \mathcal{D} \). By the surjectivity of (7.1.6), for all \( t \in [a, b] \) there exists a (possibly non unique) \( H(t) \in \mathcal{D}(\text{Id}) \) mapped onto \( \ell'(t) \). Observe that every such element \( H(t) \) defines uniquely a \( g \)-symmetric linear map \( R(t) \) on \( \mathbb{R}^n \) via the formula (5.1.2). The only obstruction for \( \ell \) to arise from a quadruple \( (g, R, P, S) \) consists precisely in the fact that one may not be able to make a continuous choice of the maps \( H(t) \). Such obstruction may only occur at the jumps of the function \( \dim(\mathcal{D}(\ell(t))) \).

If, for \( t \) in a subinterval of \([a, b]\), \( \ell(t) \in \Lambda_0(L_0) \), then there is a unique choice of \( H(t) \) on such interval, which is clearly continuous (such \( H(t) \) has the same regularity as \( \ell' \)). As a matter of facts, one can prove easily that a continuous choice of \( H(t) \) can be made on every interval for which \( \dim(\mathcal{D}_0(\ell(t))) \) is constant, even though the choice of \( H(t) \) may not be unique.

### 7.2. A study of curves of Lagrangians in local coordinates

In order to determine sufficient conditions for a curve \( \ell \) to arise from a quadruple \( (g, R, P, S) \), we need to study derivatives of \( \ell \) of higher order at the points of intersection with \( \Lambda_{\geq 1}(L_0) \). To this aim, we now consider a local chart \( \phi_{L_0, L_1} \) (see Definition 3.2.1) where \( L_1 \) is any Lagrangian complementary to \( L_0 \); we will consider a restriction of \( \ell \) whose image lies in the domain \( \Lambda_0(L_1) \) of \( \phi_{L_0, L_1} \).

Let \( \beta \) be the composition \( \phi_{L_0, L_1} \circ \ell \); we write differential equation in (5.1.7) (recall formula (5.1.2)) in terms of \( \beta \).

By Remark 3.2.11, the isomorphism \( B_{\text{sym}}(\ell(t), \mathbb{R}) \simeq B_{\text{sym}}(L_0, \mathbb{R}) \) given by the differential \( d \phi_{L_0, L_1}(\ell(t)) \) is the pull-back \( \eta^* \) by the isomorphism \( \eta : L_0 \mapsto \ell(t) \) given by the restriction of the projection \( \ell(t) \oplus L_1 \mapsto \ell(t) \). To simplify the notations, whenever possible we will omit the variable \( t \) in the computations that follow.

Recalling Definition 3.2.4, the expression in coordinates of the right side of the differential equation in (5.1.7) is therefore the bilinear form on \( L_0 \) given by \( \omega(H\eta \cdot, \eta \cdot) \). Writing \( \ell(t) \) as the graph of a linear map \( T : L_0 \mapsto L_1 \), we have:

\[
(7.2.1) \quad \beta = \mathfrak{D}_{L_0, L_1} \circ T.
\]

It is now easily seen that \( \eta(v) = v + Tv \) for all \( v \in L_0 \). Let \( \pi_i : L_0 \oplus L_1 \mapsto L_i \), \( i = 0, 1 \), be the projections; we write \( H_{ij} = (\pi_i \circ H)|_{L_j} \), for \( i, j = 0, 1 \). We now compute \( \omega(H\eta v, \eta w) \) for all \( v, w \in L_0 \) as follows:

\[
\omega(H\eta v, \eta w) = [\mathfrak{D}_{L_0, L_1} \circ H_{10}(v)](w) + [\mathfrak{D}_{L_1, L_0} \circ H_{00}(v)](Tw) + [\mathfrak{D}_{L_0, L_1} \circ H_{11}(Tv)](w) + [\mathfrak{D}_{L_1, L_0} \circ H_{01}(Tv)](Tw).
\]

Using (3.1.2) and (7.2.1), by the above formula we get:

\[
\omega(H\eta \cdot, \eta \cdot) = \mathfrak{D}_{L_0, L_1} \circ H_{10} - \beta \circ H_{00} + \mathfrak{D}_{L_0, L_1} \circ H_{11} \circ \mathfrak{D}_{L_0, L_1}^{-1} \circ \beta + \mathfrak{D}_{L_0, L_1}^{-1} \circ \beta - \beta \circ H_{01} \circ \mathfrak{D}_{L_0, L_1}^{-1} \circ \beta.
\]
Since $H \in \text{sp}(\mathbb{R}^{2n}, \omega)$, it follows easily $\mathcal{D}_{L_0, L_1} \circ H_{11} \circ \mathcal{D}_{L_0, L_1}^{-1} = -H_{00}^*$; hence the differential equation for $\beta$ is given by:

$$\beta' = \mathcal{D}_{L_0, L_1} \circ H_{10} - \beta \circ H_{00} - H_{00}^* \circ \beta - \beta \circ \mathcal{D}_{L_0, L_1}^{-1} \circ \beta. \quad (7.2.2)$$

By Definition 2.4.1 and (5.1.2), it is easily checked that:

$$\mathcal{D}_{L_0, L_1} \circ H_{10} = 0 \oplus g. \quad (7.2.3)$$

Moreover, writing $L_1$ as the graph of a $g$-symmetric linear map $Z : \mathbb{R}^n \mapsto \mathbb{R}^n$:

$$L_1 = \{(x, Z(x)) : x \in \mathbb{R}^n\}, \quad (7.2.4)$$

we get:

$$H_{00} = -0 \oplus Z, \quad H_{01} \circ (\mathcal{D}_{L_0, L_1})^{-1} = 0 \oplus [(R - Z^2) \circ g^{-1}], \quad (7.2.5)$$

where in the above formulas $g$ is considered as the map $\mathbb{R}^n \mapsto (\mathbb{R}^n)^*$ given by $x \mapsto g(x, \cdot)$.

Using (7.2.3) and (7.2.5) where $L_0 = \{0\} \oplus \mathbb{R}^n$ is identified with $\mathbb{R}^n$, we rewrite (7.2.2) as:

$$\beta' = g + \beta \circ Z + Z^* \circ \beta - \beta \circ (R - Z^2) \circ g^{-1} \circ \beta. \quad (7.2.6)$$

Equation (7.2.6) is the translation in coordinates of the differential equation in (5.1.7); a $C^1$-curve $\beta$ is such that the corresponding curve $\ell$ is horizontal with respect to $\mathcal{D}$ if and only if for each $t$ there exists a $g$-symmetric $R(t) : \mathbb{R}^n \mapsto \mathbb{R}^n$ satisfying (7.2.6).

We now concentrate our attention to the problem of determining conditions on $\beta$ that guarantee the existence of a continuous choice of maps $R(t)$ as above satisfying (7.2.6). A first necessary condition to the existence of $R(t)$ is obtained by evaluating (7.2.6) at a pair $(v, w) \in \text{Ker}(\beta) \times \mathbb{R}^n$:

$$\beta'(v, w) = g(v, w) + \beta(Zv, w), \quad \forall v \in \text{Ker}(\beta), \; w \in \mathbb{R}^n. \quad (7.2.7)$$

Condition (7.2.7) is simply a coordinate version of the horizontality of $\ell$ (compare with (7.1.4)). Let’s assume now that $\beta$ is a curve of class $C^2$; we determine a necessary condition for the existence of a curve $R(t)$ of class $C^1$ satisfying (7.2.6). We differentiate (7.2.6) and we evaluate at a pair of vectors $v, w \in \text{Ker}(\beta)$, obtaining:

$$\beta''(v, w) = \beta'(Zv, w) + \beta'(v, Zw), \quad \forall v, w \in \text{Ker}(\beta).$$

Using (7.2.7), the above formula becomes:

$$\beta''(v, w) = 2g(Zv, w) + 2\beta(Zv, Zw), \quad \forall v, w \in \text{Ker}(\beta). \quad (7.2.8)$$

At the instants $t$ where $\beta$ is invertible, the unique $(g$-symmetric) map $R(t)$ satisfying (7.2.6) is computed as:

$$R = \beta^{-1} \circ (g - \beta' + \beta \circ Z + Z^* \circ \beta) \circ \beta^{-1} \circ g + Z^2. \quad (7.2.9)$$

Observe that $\text{Ker}(\beta(t)) = \ell(t) \cap L_0$; hence, the condition that $\beta$ be invertible means that $\ell(t) \in \Lambda_0(L_0)$. As we have observed earlier, in this case there is no obstruction to the existence of the map $R(t)$.

For simplicity, we will now restrict to the case that $\beta$ is smooth, that it is not invertible for only a finite number of instants $t$ and that $\det(\beta)$ has only zeroes of finite
order. For instance, this is the case if $\beta$ is real analytic and if $\det(\beta)$ is not identically zero.

Let $t_0$ be a fixed instant at which $\beta$ is not invertible. For $t \sim t_0$, $t \neq t_0$, we write (7.2.9) in matrix form (using a suitable basis of $\mathbb{R}^n$); the entries of $R(t)$ will then be given by quotients of smooth functions of $t$. A necessary and sufficient condition for the existence of a smooth extension of $R$ at the instant $t_0$ is that in these quotients the order of zero of the functions appearing at the numerator be greater than or equal to the order of zero of the functions at the denominator. In this situation, it is obvious that necessary and sufficient conditions for the existence of a smooth extension of $R$ can be given in terms of certain nonlinear systems of equations involving higher order derivatives of the coefficients of $\beta$ at $t = t_0$. It is interesting to observe that, if $\beta$ is real analytic, then so is $R$.

7.3. The case where $g$ is nondegenerate on $\text{Ker}(\beta(t_0))$

We temporarily make the extra assumption that $g$, or equivalently $\beta'(t_0)$, be nondegenerate on $\text{Ker}(\beta(t_0))$. We can prove then that conditions (7.2.7) and (7.2.8) are sufficient. Towards this goal, let $e_1, \ldots, e_n$ be a basis of $\mathbb{R}^n$ such that $e_1, \ldots, e_k$ is a basis of $\text{Ker}(\beta(t_0))$; the restriction of $\beta(t_0)$ to the space spanned by the $e_{k+1}, \ldots, e_n$ is clearly non degenerate. We will now think of all our bilinear forms as matrices relative to this basis.

For $t \neq t_0$, let $\tilde{\beta}(t)$ be the matrix obtained by dividing the first $k$ columns of $\beta(t)$ by $(t - t_0)$; we define $\tilde{\beta}(t_0)$ by replacing the first $k$ columns of $\beta(t_0)$ by the first $k$ columns of $\beta'(t_0)$. It is easy to see that $\tilde{\beta}$ is smooth.

If we define $D_k(a) = \begin{pmatrix} a \cdot I_k & 0 \\ 0 & I_{n-k} \end{pmatrix}$, where $I_j$ denotes the $j \times j$ identity matrix and $a \in \mathbb{R}$, then we can write:

$$\beta(t) = \tilde{\beta}(t)D_k(t - t_0), \quad \forall t \neq t_0,$$

hence:

$$\beta(t)^{-1} = D_k \left( \frac{1}{t - t_0} \right) \tilde{\beta}(t)^{-1}, \quad \forall t \neq t_0. \quad (7.3.1)$$

Since $g$ is non degenerate on $\text{Ker}(\beta(t_0))$, it follows from (7.2.7) that $\tilde{\beta}(t_0)$ is invertible, and so $\tilde{\beta}(t)^{-1}$ is smooth. By (7.3.1), the last $n - k$ lines of $\beta(t)^{-1}$ are smooth, and the first $k$ lines have a singularity of order at the most one at $t = t_0$, i.e., they are the quotient of smooth functions by $t - t_0$.

By the symmetry of $\beta(t)^{-1}$, it actually follows that the last $n - k$ columns of $\beta(t)^{-1}$ are smooth, from which it follows that the singularities of $\beta(t)^{-1}$ are concentrated in the upper left $k \times k$ block, and all the singularities are of order at the most one.

We denote by $Q$ the following symmetric bilinear form:

$$Q = g - \beta' + \beta \circ Z + Z^* \circ \beta;$$
formula (7.2.9) can be rewritten in terms of $Q$ as:

\begin{equation}
R = \beta^{-1} \circ Q \circ \beta^{-1} \circ g + Z^2.
\end{equation}

We observe that $Q$ is smooth, its coefficients of the first $k$ lines and of the first $k$ columns have zeroes of order at least one at $t = t_0$, and that the coefficients of the upper left $k \times k$ block have zeroes of order at least two at $t = t_0$.

From (7.3.2) it now follows easily that $R$ is smooth, which proves the claim.

For the above argument, the crucial hypothesis of nondegeneracy of the restriction of $\beta(t_0)$ cannot be avoided; if this condition is not satisfied, in order to get to the conclusion one needs to analyze the behavior of derivatives of higher order of $\beta$ at $t = t_0$.

Under the assumption that $g$ be positive definite, i.e., when $g$ is related to a Riemannian or a causal Lorentzian geodesic problem, by the above argument (7.2.7) and (7.2.8) characterize completely the curves $\beta$ arising from a quadruple $(g, R, P, S)$.

7.4. A counterexample to the equality $\mu(g, R, P, S) = i_{\text{loc}}$

As announced at the beginning of the section, we now pass to the construction of a counterexample to Theorem 5.1.2 when the hypothesis (2) is dropped.

We consider the following setup. Let $n = 2$, the objects $g$, $Z$ and $P$ that we consider are:

\begin{align*}
g &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
Z &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad P = \{0\};
\end{align*}

all the matrices involved are relative to the canonical basis of $\mathbb{R}^2$. Observe that $Z$ is $g$-symmetric, i.e., the matrix $gZ$ is symmetric.

For our purposes, we will construct a curve $\ell$ in $\Lambda$ whose image is entirely contained in the domain of the chart $\phi_{L_0, L_1}$, where $L_0 = 0 \oplus \mathbb{R}^2$ and $L_1$ is defined by (7.2.4). It now suffices to describe the curve $\beta$ in $B_{\text{sym}}(\mathbb{R}^2, \mathbb{R}) \simeq \mathbb{R}^3$; we write:

\begin{equation}
\beta(t) = \begin{pmatrix} x(t) \\ z(t) \\ y(t) \end{pmatrix},
\end{equation}

where $x$, $y$ and $z$ are real analytic scalar functions on an interval $[a, b]$, with $a < 0 < b$, such that the following properties are satisfied:

1. $\beta(a) = 0$, which means that the initial condition $\ell(a) = \ell_0 = L_0$ is satisfied;
2. $\beta'(a) = g$ and $\beta''(a) = 2gZ = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, i.e., conditions (7.2.7) and (7.2.8) are satisfied at $t = a$;
3. $\det(\beta(t))$ has zeroes precisely at $t = a$ and at $t = 0$;
4. $\text{Ker}(\beta(0))$ is generated by the first vector of the canonical basis of $\mathbb{R}^2$, i.e., $x(0) = z(0) = 0$ and $y(0) \neq 0$;
5. $x'(0) = 0$, $z'(0) = 1 + y(0)$ and $x''(0) = 2 + 2y(0)$, i.e., conditions (7.2.7) and (7.2.8) are satisfied at $t = 0$;
6. $\det(\beta(t))$ has a zero of order precisely 3 at $t = 0$, and its third derivative is positive at $t = 0$. This is equivalent to $y(0) = -1$ and $x''(0) < 0$;
the function $R$ given by (7.3.2), or equivalently the function $\beta^{-1} \circ Q \circ \beta^{-1}$ is non
singular at $t = 0$.

Conditions (5) e (6) imply also $\varepsilon'(0) = x''(0) = 0$; conditions (1) and (2) imply that
$\det(\beta(t))$ has a zero of order two at $t = a$.

We will now show that it is possible to determine polynomial functions $x$, $y$ and $z$
satisfying all the above conditions. We proceed by steps as follows.

Our interval $[a, b]$ will be of the form $[-1, b]$, with $b > 0$ sufficiently small. Once
made a choice of functions $x$, $y$ and $z$ so that (1)—(7) are satisfied, the endpoint $b$
will be chosen in such a way that $\det(\beta(t))$ is strictly positive in $[0, b]$ (observe that this is
possible by condition (6)).

We denote by $\hat{\beta}$ the following matrix:

$$
\hat{\beta}(t) = \begin{pmatrix}
    y(t) & -z(t) \\
    -z(t) & x(t)
\end{pmatrix},
$$

so that $\beta^{-1} = (\det(\beta))^{-1} \cdot \hat{\beta}$ whenever $\beta$ is invertible. By condition (6)
above, $\det(\beta)$
has a zero of order 3 at $t = 0$, so, in order to satisfy (7), a necessary and sufficient
condition is that the entries of the matrix $\hat{\beta} \circ Q \circ \hat{\beta}$ have zeroes of order at least 6 at
$t = 0$. If we write:

$$
\hat{\beta} \circ Q \circ \hat{\beta} \circ g = \begin{pmatrix}
    p_1(t) & p_2(t) \\
    p_3(t) & p_1(t)
\end{pmatrix},
$$

we have:

$$
\begin{align*}
p_1 &= x'yz - yz^2 + xy - xyz' + xy^2 + z^2 - z^2 z' + xy'z, \\
p_2 &= -x'y^2 - 2yz + 2yzz' - y'z^2, \\
p_3 &= -x'z^2 + 2z^3 - 2xz + 2xz z' - 2xyz - x^2 y''.
\end{align*}
$$

The conditions (4) through (7) above are satisfied, for instance, with the choice:

$$
(7.4.1) \quad x(t) = -2t^3 - \frac{54}{5}t^5, \quad y(t) = -1 - 6t + 18t^2 - 54t^3, \quad z(t) = -3t^2, \quad t \sim 0.
$$

It is easy now to see that it is possible to choose smooth functions $x$, $y$ and $z$
that coincide with the polynomials given in (7.4.1) around $t = 0$ and such that conditions
(1)—(3) are also satisfied. By what we have observed so far, such choice provides
a smooth counterexample to Theorem 5.1.2 when the nondegeneracy assumption is
dropped.

For the final step of our real analytic counterexample we now argue abstractly using
a density argument of polynomials, as follows.

Let $x$, $y$ and $z$ be given smooth functions so that conditions (1)—(7) are satisfied; we
start observing that if $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ are smooth functions having the first six derivatives
at $t = 0$ and the first two derivatives at $t = -1$ equal to the corresponding derivatives
of $x$, $y$ and $z$, then, replacing $x$, $y$ and $z$ by $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$, only condition (3) may fail to
hold. If such a replacement is done in such a way that $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ are sufficiently close
to $x$, $y$ and $z$ in the $C^3$-topology, then also condition (3) will remain true. To prove
this, we apply the next lemma to the function \( f = xy - z^2 \) on the interval \([-1, 0]\) with \( k = 3 \):

**Lemma 7.4.1.** Let \( k \in \mathbb{N} \) and let \( f : [a, b] \mapsto \mathbb{R} \) be a function of class \( C^{k+1} \). Assume that \( f \) has zeroes precisely at the endpoints \( a, b \) and that these zeroes are of order at the most \( k \). Then, there exists a neighborhood \( U \) of \( f \) in the \( C^{k+1} \)-topology such that, every \( g \in U \) having the same order of zeroes as \( f \) at \( a \) and \( b \) has no zeroes in \([a, b]\).

**Proof.** Let \( i \) and \( j \) be the order of zeroes of \( f \) at \( a \) and \( b \) respectively, \( i, j \in \{1, \ldots , k\} \). Define the following constant:

\[
M = \max\{\|f^{(i+1)}\|_\infty, \|f^{(j+1)}\|_\infty\},
\]

and let \( \delta_1, \delta_2 > 0 \) be such that \( a < a + \delta_1 < b - \delta_2 < b \) and

\[
\delta_1 < \frac{(i+1)!|f^{(i)}(a)|}{2(M+1)}, \quad \delta_2 < \frac{(j+1)!|f^{(j)}(b)|}{2(M+1)}.
\]

Finally, let \( \alpha > 0 \) be the infimum of \( |f| \) on the interval \([a + \delta_1, b - \delta_2]\). The desired neighborhood of \( f \) is defined by requiring that \( g \in U \) if and only if:

\[
|g^{(i)}(a)| > \frac{f^{(i)}(a)}{2}, \quad |g^{(j)}(b)| > \frac{f^{(j)}(b)}{2},
\]

\[
\|g^{(i+1)} - f^{(i+1)}\|_\infty < 1, \quad \|g^{(j+1)} - f^{(j+1)}\|_\infty < 1, \quad \|g - f\|_\infty < \alpha.
\]

To check that this choice of \( U \) works, let \( g \in U \) be chosen so that \( g \) has a zero of order \( i \) at \( a \) and a zero of order \( j \) at \( b \). Using the \( i \)-th order Taylor polynomial of \( g \) around \( a \), we get:

\[
g(t) = (t - a)^i \left( \frac{f^{(i)}(a)}{i!} + r(t) \right),
\]

where \( r(t) \) satisfies:

\[
|r(t)| \leq \sup_{[a,t]} \frac{|g^{(i+1)}|}{(i+1)!} \cdot (t - a) \leq \frac{M + 1}{(i+1)!} \cdot (t - a).
\]

By our choice of \( \delta_1 \), it follows that \( g \) has no zeroes in \([a, a + \delta_1]\); similarly, \( g \) has no zeroes in \([b - \delta_2, b]\).

From \( \|g - f\|_\infty < \alpha \), it follows that \( g \) has no zeroes in \([a + \delta_1, b - \delta_2]\), which concludes the proof. \( \Box \)

Finally, for the construction of our analytic counterexample, we use the observations above, and a simple density result which is contained in the following Lemma:

**Lemma 7.4.2.** Let \( k \in \mathbb{N} \) and \( a_i, b_i \in \mathbb{R}, \ i = 0, \ldots , k, \) be fixed. Consider the following subsets of \( C^k([a, b], \mathbb{R}) \):

\[
A = \left\{ f \in C^k([a, b], \mathbb{R}) : f^{(i)}(a) = a_i, \ f^{(i)}(b) = b_i, \ i = 0, \ldots , k \right\},
\]

\[
B = \left\{ f \in A : f \text{ is a polynomial} \right\}
\]
Then, $B$ is dense in $A$ in the $C^k$-topology.

We can therefore build a real analytic curve $\ell : [-1,b] \mapsto \Lambda$ which arises from an admissible quadruple $(g, R, P, S)$, with $R$ real analytic. By condition (3) the only $(P, S)$-focal instant occurs at $t = 0$; the restriction of $g$ to the $\ell(0) \cap L_0$ is zero by condition (4). It follows that the focal index of the quadruple is zero. On the other hand, by condition (6), the sign of $\det(\beta(t))$ changes from negative to positive as $t$ passes through 0; moreover, the trace of $\beta(t)$ is negative around $t = 0$. By Proposition 4.3.1, this implies that the Maslov index of the quadruple is $-1$.

7.5. Instability of the focal index

Let $(g, R, P, S)$ be the quadruple constructed in the previous subsection. A small perturbation of $(g, R, P, S)$ preserves the Maslov index, by Theorem 5.2.1. However, we observe that the focal index may change, by the following arguments.

If we identify $\beta$ with a curve in $\mathbb{R}^3$, then the $(P, S)$-focal instants occur precisely at the intersections of this curve with the double cone $xy - z^2 = 0$. Given one such intersection $\beta(t_0)$, the degeneracy of $g$ on $\ker(\beta(t_0))$ means that $\beta(t_0) \neq 0$ belongs to one of the straight lines $x = z = 0$ and $y = z = 0$.

Such condition is evidently unstable by small perturbations, and a quadruple obtained from $(g, R, P, S)$ by a small perturbation will generically satisfy the hypotheses of Theorem 5.1.2 and therefore, its focal index will be equal to $-1$.

7.6. Instability of focal points with signature zero

Let’s assume that $n = 2$ and that $g$ is symmetric bilinear form of signature 0, i.e., $\det(g) < 0$. An instant $t_0$ such that $\beta(t_0) = 0$ is a $(P, S)$-focal instant of signature 0. Again, it is fairly obvious that a small perturbation of the curve $\beta$ may not intersect the double cone $xy - z^2$ around $t_0$, which amounts to say that a $(P, S)$-focal instant with signature 0 may evaporate by small perturbations of the quadruple $(g, R, P, S)$.

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