Modified Archimedes’ principle predicts rising and sinking of intruders in sheared granular flows

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We computationally determine the force on single spherical intruder particles in sheared granular flows as a function of particle size, particle density, shear rate, overburden pressure, and gravitational acceleration. The force scales similarly to, but deviates from, the buoyancy force predicted by Archimedes’ principle. The deviation depends only on the intruder to bed particle size ratio, but not the density ratio or flow conditions. We propose a simple force model that successfully predicts whether intruders rise or sink, knowing only the size and density ratios, for a variety of flow configurations in physical experiments.

Intruder particles in fluidized or flowing granular beds tend to segregate (rise or sink) due to their size or density difference with the bed particles [1-15]. Segregation in vibrofluidized systems, the Brazil nut effect [4], depends on various mechanisms [16,17] including buoyancy. With sufficient fluidization, the buoyancy force on an intruder follows Archimedes’ principle [5,9], thus explaining the phase transition between normal and reverse Brazil nut effects [6,7]. In contrast to this clear picture, the force driving segregation in sheared granular flows remains elusive. While extensive research has focused on segregation of flowing bidisperse mixtures from the continuum perspective [16,17], quantitative studies of the particle-scale segregation force are fewer and more recent. Guilard et al. [12] proposed a virtual spring based force model to simulate the force on a single intruder in a granular bed. A virtual spring measures the vertical force on the intruder. (b) Rheology: µ and φ vs. I. Error bars indicate ±1 standard deviation. Dashed lines are fits to the µ(I) rheology [19].
us to conveniently generate a wide range of shear flows with 600 Pa ≤ P0 ≤ 3000 Pa and 1 s⁻¹ ≤ γ ≤ 40 s⁻¹. As Fig. 1(b) shows, the inertial number I = γd/√ρ/σzz ranges from 0.005 to 0.25, where σzz is the vertical normal stress, and the effective friction μ and packing fraction φ follow the μ(φ) rheology 19.

An intruder of diameter d_i and density ρ_i is placed near the middle of the bed (initial height z_0), with size ratio R = d_i/d varying from 0.5 to 8 and density ratio R_ρ = ρ_i/ρ varying from 0.5 to 3. The same streamwise stabilizing force applies to the intruder. To measure the vertical force driving segregation, we follow Guillard et al. 12 and tether the intruder to a vertical spring (leaving free the other five degrees of freedom), which causes it to fluctuate about an equilibrium height z_eq [Fig. 1(a)].

In steady state, the net contact force exerted on the intruder by the neighboring bed particles, the bed force F, is balanced by the spring force and the gravitational force, i.e., F = k_{sp}(z_eq - z_0) + m_i g, where k_{sp} is the virtual stiffness and m_i is the intruder mass. The spring acts as a virtual force meter and the measurement of F is insensitive to k_{sp} 12,14. Uncertainties (error bars) of F are estimated considering temporal correlations 22 of the fluctuations of the intruder height about z_eq.

Results.—Figure 2(a) shows, for ρ_i = ρ, F (symbols) and m_i g (dashed curve) increase similarly with R. However, subtle differences between F and m_i g indicate imbalanced forces that drive segregation. To better visualize the differences, the ratio F/m_i g is plotted in Fig. 2(b). Focusing on ρ_i = ρ, F/m_i g is less than one for R < 1, i.e., a small intruder is pulled down by gravity. As R is increased above one, F/m_i g becomes greater than one, i.e., a large intruder is pushed up by the bed force. These scenarios are consistent with typical percolation and squeeze expulsion explanations for size segregation 13,21. Notably, F/m_i g falls slightly below one for R > 4, since m_i g increases more rapidly than F as R increases; thus, very large intruders sink. Such reverse segregation has been reported 15 but not yet quantitatively addressed 12,14.

Next, we vary R_ρ by changing ρ_i. The inset of Fig. 2(a) shows that F remains unchanged as ρ_i increases from 0.5ρ to 3ρ (different symbols), whereas m_i g obviously depends on ρ_i (dashed curves). Therefore, the intruder density (and the weight) does not directly affect the bed force but alters segregation behavior by changing the ratio F/m_i g. As shown in Fig. 2(b), a sufficiently heavy intruder (ρ_i = 3ρ) sinks regardless of its diameter, as the bed force can never support its weight; a light intruder (ρ_i = 0.5ρ) rises for R > 1, as its weight is less than the force pushing it upward.

We also vary R_ρ by changing ρ at constant ρ_i such that m_i g remains the same but F varies significantly. Combined with the data in Fig. 2(b), plots for different ρ_i and ρ collapse on curves distinguished only by R_ρ [Fig. 2(c)], i.e., whether an intruder rises or sinks is determined only by the relative diameter and density.

Finally, Fig. 2(d) shows that flow conditions P_0, γ, and g have no significant impact on F/m_i g over a wide range of variation. As illustrated in Fig. 2(d) inset, F/m_i g is essentially independent of I for 0.05 < I < 0.25, a range encompassing typical inertial flows 25. Reducing I toward the quasistatic limit (typically 10⁻³) may enhance the segregation force 12, a point we address below.

Scaling.—We now focus on the scaling of F and test an Archimedes buoyancy-like force scale, φρgV_i, where V_i is the intruder volume, viewing the flow as a “fluid” of bulk density φ with normal stress gradient φρg. Figure 3(a) shows F/φρgV_i vs. R for 204 distinct simulations. All data collapse on a master curve, confirming that F scales with the buoyancy force. However, the master curve deviates from F/φρgV_i = 1; it starts below one for R < 1, increases and reaches a maximum of about 2.5 at R ≈ 2, and approaches one as R increases to large values.

The deviation between F and φρgV_i appears to originate in geometric effects at the particle level. For relatively large intruders [e.g., R = 8 in Fig. 3(b) inset], a large number of contacting neighbor particles (blue) transmit contact stress in a nearly uniform manner, consistent with the fluid buoyancy analogy F ≈ φρgV_i 14. As R is decreased to intermediate values (e.g., R = 2), contact uniformity breaks down significantly; this is characterized by the time-averaged number ratio of contacting neighbor particles (N_c) to all “nearby” particles (N_u)
Figure 3. (a) $F/\phi \rho g V_i$ vs. $R$, with varying $\rho_i$, $\rho$, $P_0$, $\dot{\gamma}$, and $g$. Solid curve is a fit to Eq. (1). Dashed curves show the two exponential terms defining $f(R)$. Inset: extreme cases with $I = 0.005$ ($P_0 = 3000$ Pa, $\dot{\gamma} = 1$ s$^{-1}$) and $\rho_i = 0$ ($P_0 = 1800$ Pa, $\dot{\gamma} = 20$ s$^{-1}$), respectively. (b) $N_c/N_{\infty}$ vs. $R$ ($P_0 = 2200$ Pa, $\dot{\gamma} = 20$ s$^{-1}$). Inset: $x$-$z$-plane view of contact network at various $R$, showing intruders (red), contacting neighbor particles (blue), and noncontacting neighbor particles (gray).

defined within a distance $d + d_i/2$ from the intruder center, which decreases rapidly as $R$ decreases [Fig. 3(b)]. Consequently, noncontacting neighbor particles (gray) are more likely to lose connection in stress transmission, which in turn leads to more contact forces passing through the intruder and thus a higher net force compared to the uniform limiting case, i.e., $F > \phi \rho g V_i$. As $R$ is further decreased below one (e.g., $R = 0.5$), brief collisions dominate [20] and the intruder tends to percolate through voids without enduring contacts [13], resulting in a net contact force smaller than the uniform limiting case, i.e., $F < \phi \rho g V_i$.

The geometric effects are associated with the frictional nature of granular contacts. For frictionless intruders [$\mu_i = 0$ in Fig. 3(a) inset], $F/\phi \rho g V_i$ collapses toward one, explaining previous observations that large intruders do not rise with low friction [13][14]. In nearly quasistatic flows [$I = 0.005$ in Fig. 3(a) inset], $F/\phi \rho g V_i$ is higher likely due to enhanced frictional resistance to deformation near yielding [27]. This effect tends to plateau above yielding, explaining the insensitivity of $F/m_{\infty} g$ to $I$ in Fig. 2(d). A similar trend of enhanced segregation force only at very low $I$ was found in previous two-dimensional simulations [12].

Model. The master scaling curve in Fig. 3(a) suggests a modified Archimedes’ principle of the form

$$F = f(R)\phi \rho g V_i,$$

where $f(R)$ is a dimensionless scale factor. Based on two geometric effects that dominate in different ranges of $R$, i.e., percolation-induced force weakening for $R < 1$ and nonuniformity-induced force strengthening for $R > 1$, we propose $f(R) = (1 - c_1 e^{-R/R_1})(1 + c_2 e^{-R/R_2})$, where $c_1$, $c_2$, $R_1$, and $R_2$ are fitting parameters. The first term [lower dashed curve in Fig. 3(a)] represents stronger percolation (thus smaller bed force) as $R$ decreases; its exponential form is chosen to reconcile the exponential dependency of percolation probability [21] and percolation velocity [28] on $R$. The second term [upper dashed curve in Fig. 3(a)] decreases toward one as $R$ increases, accounts for decreased uniformity of contacts around the intruder at small $R$ [Fig. 3(b)]. Fitting to the data in Fig. 3(a) gives $c_1 = 1.43$, $c_2 = 3.55$, $R_1 = 0.92$, and $R_2 = 2.94$, where $R_1$ and $R_2$ are characteristic size ratios for the two effects to dominate. The two terms together recover the continuum argument, $f(\infty) \rightarrow 1$, and the force balance in monodisperse flows, $f(1) = 1/\phi$ (i.e., $F = \rho g V_i$ at $R = 1$). Although $\phi$ is case specific, the fitting results in $\phi = 1/f(1) = 0.55$, a value agreeing with Fig. 1(b), which further supports the model.

Despite the empirical formulation of $f(R)$, the model adopts a minimum number of parameters to describe the data over the full range of $R$, clearly indicates two geometric effects, each associated with physically reasonable characteristic size ratios, and is appropriately constrained by limiting cases. Moreover, it provides a simple means to predict segregation based only on $R$ and $R_{\rho}$. An intruder in a sheared bed is “neutral” when the bed force $f(R)\rho g V_i$ offsets its weight $\rho_i g V_i$, i.e., $R_{\rho} = \phi f(R)$, which describes a curve dividing the $R$-$R_{\rho}$ space into “rise” (below the curve) and “sink” (above the curve) zones; see Fig. 4. To validate this phase diagram, we simulate single unperturbed intruders with varying $d_i$ and $\rho_i$, observing whether they rise, sink, or neither (i.e., mean displacement less than $3d$) over $500$ s of simulation. The predictions are in excellent agreement with the simulation results [Fig. 3(a)].

To further demonstrate the generality of the segregation transition predicted by Eq. (1), we compare it with experiments by Félix and Thomas [15], who studied segregation of tracer particles of different sizes and densities in various configurations, i.e., rotating drums, chute flows, and heap flows. Despite the different flow geometries, the predictions are in excellent agreement with the simulation results [Fig. 3(a)].
tries, the segregation direction in the experiments agrees remarkably well with the predictions of our phase diagram [Fig. 4(b)], showing the capability of our model to predict segregation for varying size and density ratios as well as different flow conditions. The few mismatches occurring near the neutral curve are mainly from chute and heap flows, where only loose criteria for the segregation direction were applied in the experiments [18].

Discussion.—Our segregation force model respects the continuum limit ($R \gg 1$) in that whether an intruder rises or sinks depends only on its density relative to the surrounding flow ($\rho_i/\rho \phi$), noting $f(\infty) \to 1$. For intruders somewhat larger than the bed particles ($R > 1$), discrete particle interactions result in a positive deviation from Archimedes’ principle, an extra lift effect underlying the rise of large particles in many size segregation studies [16, 17]. The maximum deviation at $R \approx 2$ explains the optimal segregation rate at $R \approx 2$ and the saturation of segregation velocity for $R > 2$ [29–31]. The modified Archimedes’ principle described here bridges segregation mechanisms in noncontinuum situations with the continuum buoyancy force, suggesting a unifying framework for understanding forces in granular media. Indeed, Archimedes’ principle with appropriate corrections applies to dense granular shear flows (this work), creeping granular fluids [32], vibrofluidized granular gases [8, 9], and plastic granular solids near yielding [27].

The model proposed here enables prediction of intruder segregation for various flow configurations based on size and density ratios. The finding that $F/\phi \rho g V_i$ is insensitive to external flow conditions is not to be confused with the known effects of shear rate and confining pressure on segregation velocity [22, 30–33]. While the direction of segregation is determined by competition between the bed force and the gravitational force, the segregation velocity depends further on resistive forces (often viewed as drag). Understanding the drag force has proved challenging due to the difficulty in isolating driving and drag terms from contact forces [10, 15, 55]. Now with the generalized driving force model we provide, it is possible to calculate the drag force on moving intruders. It is also relevant to consider varying the particle species concentration around the tethered intruders to account for general industrial and geophysical settings [36, 42] where the segregation force depends on the particle species concentration [11, 31].

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