Abstract—In this paper, we first introduce the concept of elementary linear subspace, which has similar properties to those of a set of coordinates. Using this new concept, we derive properties of maximum rank distance (MRD) codes that parallel those of maximum distance separable (MDS) codes. Using these properties, we show that the decoder error probability of MRD codes with error correction capability $t$ decreases exponentially with $t^2$ based on the assumption that all errors with the same rank are equally likely. We argue that the channel based on this assumption is an approximation of a channel corrupted by crisscross errors.

I. INTRODUCTION

The Hamming metric has often been considered the most relevant metric for error-control codes so far. Recently, the rank metric [1] has attracted some attention due to its relevance to space-time coding [2] and storage applications [3]. In [4], space-time block codes with good rank properties have been proposed. Rank metric codes are used to correct crisscross errors that can be found in memory chip arrays and magnetic tapes [3]. Rank metric codes have been used in public-key cryptosystems as well [5].

In [1], a Singleton bound on the minimum rank distance of rank metric codes was established, and codes that attain this bound were called maximum rank distance (MRD) codes. An explicit construction of MRD codes (these codes are referred to as Gabidulin codes) was also given in [1], and this construction was extended in [6]. Also, a decoding algorithm that parallels the extended Euclidean algorithm (EEA) was proposed for MRD codes.

In this paper, we investigate the performance of MRD codes when used to protect data from additive errors based on two assumptions. First, we assume all errors with the same rank are equally likely. We argue that the channel based on this assumption is an approximation of a channel corrupted by crisscross errors (see Section IV for details). Second, we assume that a bounded rank distance decoder is used, with error correction capability $t$. If the error has rank no more than $t$, the decoder gives the correct codeword. When the error has rank greater than $t$, the output of the decoder is either a decoding failure or a wrong codeword, which corresponds to a decoder error. Note that the decoder error probability of maximum distance separable (MDS) codes was investigated in [7], where all errors with the same Hamming weight were assumed to be equiprobable. The main contributions of this paper are:

- We introduce the concept of elementary linear subspace (ELS). The properties of an ELS are similar to those of a set of coordinates.
- Using elementary linear subspaces, we derive useful properties of MRD codes. In particular, we prove the combinatorial property of MRD codes, derive a bound on the rank distribution of these codes, and show that the restriction of an MRD code on an ELS is also an MRD code. These properties parallel those of MDS codes.
- Using the properties of MRD codes, we derive a bound on the channel error probability of MRD codes that decreases exponentially with $t^2$. Our simulation results are consistent with our bound.

The rest of the paper is organized as follows. Section II gives a brief review of the rank metric, Singleton bound, and MRD codes. In Section III we first introduce the concept of elementary linear subspace and study its properties, and then obtain some important properties of MRD codes. Section IV derives the bound on the decoder error probability of MRD codes when all errors with the same rank are equiprobable. Finally, in Section V our bound on the decoder error probability is confirmed by simulation results.

II. PRELIMINARIES

A. Rank metric

Consider an $n$-dimensional vector $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ in $\text{GF}(q^m)^n$. Assume $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ is a basis set of $\text{GF}(q^m)$ over $\text{GF}(q)$, then for $j = 0, 1, \ldots, n - 1$, $x_j$ can be written as $x_j = \sum_{i=0}^{m-1} x_{i,j} \alpha_i$, where $x_{i,j} \in \text{GF}(q)$ for $i = 0, 1, \ldots, m - 1$. Hence, $x_j$ can be expanded to an $m$-dimensional column vector $(x_{0,j}, x_{1,j}, \ldots, x_{m-1,j})^T$ with respect to the basis set $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$. Let $\mathbf{X}$ be the $m \times n$ matrix obtained by expanding all the coordinates of $\mathbf{x}$. That is,

$$
\mathbf{X} = \begin{pmatrix}
x_{0,0} & x_{0,1} & \cdots & x_{0,n-1} \\
x_{1,0} & x_{1,1} & \cdots & x_{1,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
x_{m-1,0} & x_{m-1,1} & \cdots & x_{m-1,n-1}
\end{pmatrix},
$$

where $x_j = \sum_{i=0}^{m-1} x_{i,j} \alpha_i$. The rank norm of the vector $\mathbf{x}$ (over $\text{GF}(q)$), denoted as $\text{rk}(\mathbf{x}|\text{GF}(q))$, is defined to be the rank of the matrix $\mathbf{X}$ over $\text{GF}(q)$, i.e., $\text{rk}(\mathbf{x}|\text{GF}(q)) \overset{\text{def}}{=} \text{rank}(\mathbf{X})$ [1]. The rank norm of $\mathbf{x}$ can also be viewed as
the smallest number of rank 1 matrices $B_i$ such that $X = \sum_i B_i$ [8]. All the ranks are over the base field $GF(q)$ unless otherwise specified in this paper. To simplify notations, we denote the rank norm of $x$ as $rk(x)$ henceforth. Accordingly, $\forall x, y \in GF(q^m)^n, d(x, y) \overset{\text{def}}{=} rk(x - y)$ is shown to be a metric over $GF(q^m)^n$, referred to as the rank metric henceforth [1]. Hence, the minimum rank distance $d$ of a code of length $n$ is simply the minimum rank distance over all possible pairs of distinct codewords. A code with a minimum rank distance $d$ can correct errors with rank up to $t = \lfloor (d - 1)/2 \rfloor$.

B. The Singleton bound and MRD codes

The minimum rank distance of a code can be specifically bounded. First, the minimum rank distance $d$ of a code over $GF(q^m)$ is obviously bounded by $m$. Codes that satisfy $d = m$ are studied in [9]. Also, it can be shown that $d \leq d_v$ [1], where $d_v$ is the minimum Hamming distance of the same code. Due to the Singleton bound on the minimum Hamming distance of block codes, the minimum rank distance of an $(n, M)$ block code over $GF(q^m)$ thus satisfies

$$d \leq n - \log_q m + 1. \quad (1)$$

In this paper, we refer to the bound in (1) as the Singleton bound for rank metric codes and codes that attain the bound as MRD codes. Note that (1) implies that the cardinality of any MRD code is a power of $q^m$.

III. PROPERTIES OF MRD CODES

In order to derive the bound on decoder error probability, we need some important properties of MRD codes, which will be established next using the concept of elementary linear subspace.

A. Elementary linear subspaces

Many properties of MDS codes are established by studying sets of coordinates. These sets of coordinates may be viewed as linear subspaces which have a basis of vectors with Hamming weight 1. We first define elementary linear subspaces, which are the counterparts of sets of coordinates.

Definition 1 (Elementary linear subspace (ELS)): A linear subspace $V$ of $GF(q^m)^n$ is said to be elementary if it has a basis $B$ consisting of row vectors in $GF(q)^n$. $B$ is called an elementary basis of $V$.

We remark that $V$ is an ELS if and only if it has a basis consisting of vectors of rank 1. We also remark that not all linear subspaces are elementary. For example, the span of a vector of rank $u > 1$ has dimension 1, but requires $u$ vectors of rank 1 to span it. Next, we show that the properties of elementary linear subspaces are similar to those of sets of coordinates.

Definition 2 (Rank of a linear subspace): The rank of a linear subspace $L$ of $GF(q^m)^n$ is defined to be the maximum rank among the vectors in $L$

$$rk(L) \overset{\text{def}}{=} \max_{x \in L} \{rk(x)\}.$$
elements of \(B\) and those of \(\bar{B}\). This may be expressed as
\[
\sum_{b_i \in B \cup \bar{B}} y_i b_i = 0. \tag{2}
\]
Applying the trace function to each coordinate on both sides of (2), we obtain \(\sum_{b_i \in B \cup \bar{B}} \text{Tr}(y_i) b_i = 0\), which implies a linear dependence over \(\text{GF}(q)\) of the vectors in \(B\) and \(\bar{B}\). This contradicts the fact that \(B \cup \bar{B}\) is a basis. Therefore, \(W \oplus V = \text{GF}(q)^n\).

**Definition 3 (Restriction of a vector):** Let \(L\) be a linear subspace of \(\text{GF}(q)^n\) and let \(\bar{L}\) be complementary to \(L\), i.e., \(L \oplus \bar{L} = \text{GF}(q)^n\). Any vector \(x \in \text{GF}(q)^n\) can then be represented as \(x = x_L \oplus x_{\bar{L}}\), where \(x_L \in L\) and \(x_{\bar{L}} \in \bar{L}\). We will call \(x_L\) and \(x_{\bar{L}}\) the restrictions of \(x\) on \(L\) and \(\bar{L}\), respectively.

Note that for any given linear subspace \(L\), its complementary linear subspace \(\bar{L}\) is not unique. Furthermore, the restriction of \(x\) on \(L, x_L\) depends on not only \(L\) but also \(\bar{L}\). Thus, \(x_L\) is well defined only when both \(L\) and \(\bar{L}\) are given. All the restrictions in this paper are with respect to a fixed pair of linear subspaces complementary to each other.

**Definition 4:** Let \(x \in \text{GF}(q)^n\) and \(V\) be an ELS. If there exists an ELS \(\bar{V}\) complementary to \(V\) such that \(x = x_{\bar{V}} \oplus 0\), we say that \(x\) vanishes on \(V\).

**Lemma 3:** Given a vector \(x \in \text{GF}(q)^n\) of rank \(u\), there exists an ELS with dimension \(n - u\) on which \(x\) vanishes. Also, \(x\) does not vanish on any ELS with dimension greater than \(n - u\).

**Proof:** By Lemma 1 \(x \in A\), where \(A\) is an ELS with dimension \(u\). Let \(\bar{A}\) be an ELS with dimension \(n - u\) that is complementary to \(A\). Thus, \(x\) may be expressed as \(x = x_A \oplus x_{\bar{A}} = x_A \oplus 0\). That is, \(x\) vanishes on \(\bar{A}\). Also, suppose \(x\) vanishes on an ELS \(B\) with dimension greater than \(n - u\). Then there exists an ELS \(\bar{B}\) with dimension \(u\) such that \(x \in \bar{B}\), which contradicts Lemma 1.

**B. Properties of MRD codes**

We now derive some useful properties for MRD codes, which will be used in our derivation of the decoder error probability. In this subsection, let \(C\) be an MRD code over \(\text{GF}(q)^n\) with length \(n\) \((n \leq m)\) and cardinality \(q^{nk}\). Note that \(C\) may be linear or nonlinear. First, we derive the basic combinatorial property of MRD codes.

**Lemma 4 (Basic combinatorial property):** Let \(K\) be an ELS of \(\text{GF}(q)^n\) with dimension \(k\), and fix \(\bar{K}\), an ELS complementary to \(K\). Then, for any vector \(k \in K\), there exists a unique codeword \(c \in C\) such that its restriction on \(\bar{K}\) satisfies \(c_{\bar{K}} = k\).

**Proof:** Suppose there exist \(c \neq d \in C\) such that \(c_{\bar{K}} = d_{\bar{K}}\). Their difference \(c - d\) is in \(\bar{K}\), and hence has rank at most \(n - k\) by Proposition 1, which contradicts the fact that \(C\) is MRD. Then all the codewords lead to different restrictions on \(\bar{K}\). Also, \(|C| = |K| = q^{nk}\), thus for any \(k\), there exists a unique \(c\) such that \(c_{\bar{K}} = k\).

This property allows us to obtain a bound on the rank distribution of MRD codes.

**Lemma 5 (Bound on the rank distribution):** Let \(A_u\) be the number of codewords of \(C\) with rank \(u\). Then, for the redundancy \(r = n - k\) and \(u \geq d\),
\[
A_u \leq \binom{n}{u} (q^m - 1)^{n-u}. \tag{3}
\]

**Proof:** From Lemma 3 any codeword \(c\) with rank \(u \geq d\) vanishes on an ELS with dimension \(v = n - u\). Thus (3) can be established by first determining the number of codewords vanishing on a given ELS, and then multiplying by the number of such ELS’s. \[\binom{n}{u}\]. Let \(V\) be an arbitrary ELS with dimension \(v\). First, since \(v \leq k - 1\), \(V\) is properly contained in an ELS \(K\) with dimension \(k\). According to the combinatorial property, \(c\) is completely determined by \(c_K\). Hence, if we specify that a codeword vanishes on \(V\), we may specify \(k - v\) other nonzero components arbitrarily. There are at most \((q^m - 1)^{k-v}\) ways to do so, implying that there are at most \((q^m - 1)^{d-v}\) vectors that vanish on \(V\).

Note that the exact formula for the rank distribution of linear MRD codes was derived in [1]. However, the bound in (3) is more convenient for the present application.

It is well known that a punctured MDS code is an MDS code [11]. We will show that the restriction of an MDS code to an ELS is also MDS. Let \(V\) be an ELS with dimension \(v \geq k\), an elementary basis \(\{b_0, b_1, \ldots, b_{v-1}\}\), and a complementary ELS \(\bar{V}\). For any codeword \(c\), suppose \(c_V = \sum_{i=0}^{v-1} a_i b_i\), where \(a_i \in \text{GF}(q^m)\). Let us define a mapping \(r : \text{GF}(q^m)^v \rightarrow \text{GF}(q^m)^v\) given by \(c \mapsto r(c) = (a_0, \ldots, a_{v-1})\). Then \(C_v = \{r(c) | c \in C\}\) is called the restriction of \(C\) on \(V\).

**Lemma 6 (Restriction of an MRD code):** For an ELS \(V\) with dimension \(v \geq k\), \(C_v\) is an MRD code.

**Proof:** Clearly, \(C_v\) is a code over \(\text{GF}(q)^v\) with length \(v\) \((v \leq m)\) and cardinality \(q^{mk}\). Assume \(c \neq d \in C_v\) and consider \(x = c - d\). Then we have \(rk(r(c) - r(d)) = rk(x_V) \geq rk(x) - rk(x_{\bar{V}}) \geq n - k + 1 - (n - v) = v - k + 1\).

The Singleton bound on \(C_V\) completes the proof.

**IV. DECODER ERROR PROBABILITY OF MRD CODES IN CASE OF CRISCCROSS ERRORS**

Let \(C\) be a linear \((n, k)\) MRD code over \(\text{GF}(q)^m\) \((n \leq m)\) with minimum rank distance \(d = n - k + 1\) and error correction capability \(t = \lfloor (d - 1)/2 \rfloor\). We assume that \(C\) is used to protect data from additive errors with rank \(u\). That is, the received word corresponding to a codeword \(c\) of \(C\) is \(c + e\). We argue that the additive error with rank \(u\) is an approximation of crisscross errors. Let us assume \(q = 2\) and a codeword can be represented by an \(m \times n\) array of bits. Suppose some of the bits are recorded erroneously, and the error patterns are such that all corrupted bits are confined to a number of rows or columns or both. Such an error model, referred to as crisscross errors, occurs in memory chip arrays or magnetic tapes [3]. Suppose the errors are confined to a row (or column), then such an error pattern can be viewed as the addition of an error array which has non-zero coordinates on only one row (or column) and hence has rank 1. We may reasonably assume each row is corrupted equally likely and so is each column. Thus, all the errors that are restricted to
$u > 1$ rows (or columns) are equally likely. Finally, if we assume the probability of a corrupted row is the same as that of a corrupted column, then all crosscrossover errors with weight $u$ [3] are equally likely. The weight of the crosscrossover error is no less than the rank of the error [3]. However, in many cases the weight $u$ equals the rank. Hence, assuming all errors with the same rank are equiprobable is an approximation of crosscrossover errors.

A bounded distance decoder, which looks for a codeword within rank distance $t$ of the received word, is used to correct the error. Clearly, if $e$ has rank no more than $t$, the decoder gives the correct codeword. When the error rank has greater than $t$, the output of the decoder is either a decoding failure or a decoder error. We denote the probabilities of error and failure — for error correction capability $t$ and a given error rank $u$ — as $P_{E}(t; u)$ and $P_{F}(t; u)$ respectively. If $u \leq t$, then $P_{E}(t; u) = P_{E}(t; u) = 0$. When $u > t$, $P_{E}(t; u) = 0$ and $P_{F}(t; u) = 1$. In particular, if $t < u < d - t$, then $P_{E}(t; u) = 0$ and $P_{F}(t; u) = 1$; Thus we only need to investigate the case where $u \geq d - t$.

Since $C$ is linear and hence geometrically uniform, we assume without loss of generality that the all-zero codeword is sent. Thus, the received word can be any vector with rank $u$ with equal probability. We call a vector decodable if it lies within rank distance $t$ of some codeword. If $D_{u}$ denotes the number of decodable vectors of rank $u$, then for $u \geq t + 1$ we have

$$P_{E}(t; u) = \frac{D_{u}}{N_{u}} = \frac{D_{u}}{\binom{n}{u}} A(m, u),$$

where $N_{u}$ denotes the number of vectors of rank $u$ and $A(m, u) \triangleq \binom{m - 1}{u}^{m - 1}$. Hence the main challenge is to derive upper bounds on $D_{u}$. We have to distinguish two cases: $u \geq d$ and $u < d$. The approach we use to bound $D_{u}$ is similar to that in [7].

**Proposition 3:** For $u \geq d$, then

$$D_{u} \leq \binom{n}{u} (q^{m} - 1)^{u-r} V_{t},$$

where $V_{t} = \sum_{i=0}^{t} N_{i}$ is the volume of a ball of rank radius $t$.

**Proof:** Each decodable vector can be written uniquely as $c + e$, where $c \in C$ and $rk(e) \leq t$. For a fixed $e$, $C + e$ is an MRD code, so it satisfies Equation [3]. Therefore, the number of decodable words of rank $u$ is at most $\binom{n}{u} (q^{m} - 1)^{u-r}$ multiplied by the number of possible error vectors, $V_{t}$.

**Lemma 7:** Given $y \in GF(q^{m})^{w}$ with rank $w$, there are at most $\binom{n}{s-w}(m, s-w) A_{q^{m}}(m, s-w)$ vectors $z \in GF(q^{m})^{n-w}$ such that $x = (y, z) \in GF(q^{m})^{n}$ has rank $s$.

**Proof:** The vector $x$ has $s$ linearly independent coordinates. Since $w$ of them are in $y$, then $s - w$ of them are in $z$. Thus $z$ has $s - w$ linearly independent coordinates which do not belong to $\mathcal{S}(y)$. Without loss of generality, assume those coordinates are on the first $s - w$ positions of $z$, and denote these coordinates as $z'$. For $s - w + 1 \leq i \leq n - v$, $z_{i}$ is a linear combination of the coordinates of $y$ and the first $s - w$ coordinates of $z$. Hence, we have $z_{i} = a_{i} + b_{i}$, where $a_{i} \in \mathcal{S}(y)$ and $b_{i} \in \mathcal{S}(z')$. There are $q^{w(u-s+w)}$ choices for the vector $a = (0, \ldots, 0, a_{s-w+1}, \ldots, a_{u})$. The vector $b$ has rank $s - w$, so there are at most $\binom{s-w}{u} A_{q^{m}}(m, s-w)$ choices for $b$.

**Proposition 4:** For $d - t \leq u < d$, we have

$$D_{u} \leq \binom{n}{u} \sum_{w=d-u}^{t} \binom{w}{u} (q^{m} - 1)^{w-r} \cdots \binom{u}{s-w} A_{q^{m}}(m, s-w).$$

**Proof:** Recall that a decodable vector of rank $u$ can be expressed as $e + e$, where $e \in C$ and $rk(e) \leq t$. This vector vanishes on an ELS $V$ with dimension $v = n - u$ by Lemma 3. Fix $V$, an ELS complementary to $V$. We have $w \triangleq rk(r_{v}(e)) \leq t$. $C_{V}$ is an MRD code by Lemma 6 hence $d = u$. By Lemma 5 and denoting $r' = r - u$, the number of codewords of $C_{V}$ with rank $w$ is at most $\binom{u}{w} (q^{m} - 1)^{w-r'}$. Each codeword $c$ such that $rk(r_{v}(c)) = w$, we must count the number of error vectors $e$ such that $r_{v}(e) + r_{v}(e) = 0$. Suppose that $e$ has rank $s \geq w$, and denote $g = r_{v}(e)$ and $f = r_{v}(e)$. Note that $e$ is completely determined by $f$. The vector $(g, f)$ has rank $s$, hence by Lemma 4 there are at most $\binom{s}{w} A(m, s-w)$ choices for the vector $f$.

The total number $D_{V}$ of decodable vectors vanishing on $V$ is then at most

$$D_{V} \leq \binom{n}{u} \sum_{w=d-u}^{t} \binom{w}{u} (q^{m} - 1)^{w-r} \cdots \binom{u}{s-w} A_{q^{m}}(m, s-w).$$

The number of possible ELS’s of dimension $v$ is $\binom{n}{v}$. Multiplying the bound on $D_{V}$ by $\binom{n}{v}$, the number of possible ELS’s of dimension $v$, we get the result.

**Corollary 2:** For $d - t \leq u < d$, then $D_{u} \leq \frac{q^{u}}{q^{w}} \binom{n}{u} (q^{m} - 1)^{u-r} V_{v}$.

**Proof:** We shall use Equation [7]. We have

$$D_{V} \leq (q^{m} - 1)^{u-r} \sum_{w=d-u}^{t} \binom{w}{u} \binom{u}{s-w} \cdots q^{w(u-s+w)} A_{q^{m}}(m, s-w) (q^{m} - 1)^{w}$$

$$< (q^{m} - 1)^{u-r} \sum_{w=d-u}^{t} \binom{u}{s-w} q^{w(u-s+w)}.$$  (8)

Using the following combinatorial relation [10]:

$$\sum_{s=d-u}^{u} \binom{u}{s-w} q^{w(u-s+w)} = \binom{u+w}{u},$$

we obtain

$$D_{V} \leq (q^{m} - 1)^{u-r} \sum_{s=d-u}^{t} q^{w(s)}.$$  (9)
\[\frac{q^2}{q^2-1}(q^m-1)^{-r} \sum_{s=d-u}^{t} A(m,s)[s] < \frac{q^2}{q^2-1}(q^m-1)^{-r} V_i.\]

We can eventually derive a bound on the decoder error probability.

**Proposition 5:** For \(d - t \leq u < d\), the decoder error probability satisfies
\[P_E(t;u) < \frac{q^2}{q^2-1}(q^m-1)^{-r} V_i.\]  \hspace{1cm} (10)

For \(u \geq d\), the decoder error probability satisfies
\[P_E(t;u) < \frac{(q^m-1)^{-r}}{A(m,u)} V_i.\]  \hspace{1cm} (11)

**Proof:** Directly from Proposition 3 and Corollary 2. Before deriving an upper bound for \(P_E(t;u)\), we need to establish two lemmas.

**Lemma 8:** For \(0 \leq u \leq m\), \(A(m,u) \geq q^{nu-\sigma(q)}\), where \(\sigma(q) = \frac{1}{\ln(q)} \sum_{k=1}^{\infty} \frac{1}{(q^k-1)}\) is a decreasing function of \(q\) with \(\sigma(2) \approx 1.7919\).

**Proof:** We have \(A(m,u) = q^{mu-M_m}\), where \(M_m = -\sum_{j=m-u+1}^{\infty} \log_q(1-q^{-j})\). \(M_m\) is an increasing function of \(u\), with maximum equal to \(M_m = \frac{1}{\ln(q)} \sum_{k=1}^{\infty} \frac{1}{k(q^k-1)} \leq \sigma(q)\).

**Lemma 9:** For \(m \geq 1\) and \(t \leq m/2, V_i \leq q^{(m-n-t)+\sigma(q)}\).

**Proof:** First, we need to prove the following claim.

Claim: For \(m \geq 1\) and \(i \leq t \leq m/2\), we have
\[\left[\begin{array}{c} m \\ i-1 \end{array}\right] q^{-(t-i)} \geq 1.\]

An exhaustive search proves the result for \(m < 4\). We shall assume that \(m \geq 4\) herein. The case \(i = t\) being trivial, we hence assume that \(i < t\). Using Lemma 6, we find that \(\left[\begin{array}{c} m \\ i \end{array}\right] \geq q^{(t-i)(m-t+i)-\sigma(q)}\), hence \(\left[\begin{array}{c} m \\ i-1 \end{array}\right] q^{-(t-i)} \geq q^{(t-i)(m-t+i)-\sigma(q)} \geq q^{m/2-\sigma(q)} \geq 1\).

The claim implies \(A(m,i) \leq q^{mi} \leq \left[\begin{array}{c} m \\ i-1 \end{array}\right] q^{(m-t+i)}\). Since \(V_i = \sum_{i=0}^{m} n A(m,i)\), the bound on \(A(m,i)\) allows us to derive the bound on \(V_i\).

The result in Proposition 5 may be weakened in order to find a bound on the decoder error probability which only depends on \(t\).

**Proposition 6:** For \(u \geq d-t\), the decoder error probability satisfies
\[P_E(t;u) < q^{-t^2+2\sigma(q)}.\]  \hspace{1cm} (12)

**Proof:** First suppose that \(u \geq d\). From Proposition 5, we have for \(u \geq d-t\): \(P_E(t;u) < q^{(m-n-t)+\sigma(q)} V_i\). The bounds in Lemmas 6 and 7 lead to \(P_E(t;u) < q^{-mr+(m+n-t)+2\sigma(q)}\). Since \(n \leq m\) and \(2t \leq r\), it follows that \(P_E(t;u) < q^{-t^2+2\sigma(q)}\). For \(d-t \leq u < d\), we find that \(P_E(t;u) < q^{2(q-1)-mr+(m+n-t)+2\sigma(q)} < q^{-mr+(m+n-t)+2\sigma(q)}\). Using the same reasoning as above, we find the same conclusion. \hspace{1cm} ■

**V. SIMULATION RESULTS**

In this section, we use numerical simulations to verify our bound given in Proposition 5. In our simulations, we used a special family of MRD codes called Gabidulin codes [1] with the following parameters: \(q = 2, m = n = 16\), and \(d = 2t + 1 = n - k + 1\). The simulations were based on the following process: first a random message word in \(GF(q^m)^k\) is encoded using the generator matrix of the Gabidulin code, then a random error vector with rank \(u > t\) is added to the codeword, and finally the EEA [1] is used to decode the received word. Since \(u > t\), the decoding results in either a failure or an error. Similarly to the decoding of Reed-Solomon codes, decoder failure is declared based on the output of the EEA. Different values for \(t\) and \(u\) were used in our simulations to verify our bound. Each value of the decoder error probability is computed after at least 15 occurrences of decoder errors to ensure reliability of simulation results.

Note that our bound given in Proposition 5 does not depend on \(u\), and decreases exponentially with \(t^2\). In Figure 1, the decoder error probability is viewed as a function of \(t\) as \(t\) varies from 1 to 4 and \(u\) is set to \(n = 16\). Note that when \(t = 1\), the bound in Proposition 5 is trivial. We observe that both the bound and the simulated decoder error probability decrease exponentially with \(t^2\). In Figure 2, the decoder error probability is viewed as a function of \(u\) as \(u\) is set to either 2 or 3 and \(u\) varies from \(t + 1\) to \(n = 16\). Clearly, the decoder error probability varies with \(u\) somewhat, but the bound in Proposition 5 is applicable to all values of \(u\).

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Fig. 2. Decoder error probability of an MRD code as a function of $u$, with $q = 2$, $m = n = 16$, and $t$ equal to 2 or 3.

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