Novel approach to binary dynamics: application to the fifth post-Newtonian level

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(Dated: January 21, 2022)

We introduce a new methodology for deriving the conservative dynamics of gravitationally interacting binary systems. Our approach combines, in a novel way, several theoretical formalisms: post-Newtonian, post-Minkowskian, multipolar-post-Minkowskian, gravitational self-force, and effective one-body. We apply our method to the derivation of the fifth post-Newtonian dynamics. By restricting our results to the third post-Minkowskian level, we give the first independent confirmation of the recent result of Bern et al. [Phys. Rev. Lett. 122, 201603 (2019)]. We also offer checks for future fourth post-Minkowskian calculations. Our technique can, in principle, be extended to higher orders of perturbation theory.

Introduction.—The detection of the coalescence of compact binaries by the LIGO-Virgo collaboration \cite{1} demands an ever more accurate knowledge of the gravitational dynamics and radiation of binary systems. We propose here a new methodology for improving the theoretical description of the conservative dynamics of two-body systems in General Relativity. This methodology unifies in a novel way an array of previously developed theoretical tools, and combines it with some new insights. This allows one to reach in an expedient manner new high-order results of direct physical significance. Here, we exemplify the efficiency of our method by applying it to the first (essentially complete) computation of the conservative two-body dynamics (of two non-spinning masses $m_1, m_2$) at the fifth post-Newtonian (5PN) accuracy, \textit{i.e.} one order in $(v/c)^2$ beyond the last post-Newtonian (PN) order at which this dynamics has been heretofore fully derived \cite{2, 8}. [Our 5PN-level result cannot be compared with the recent 5PN-level works \cite{4, 10}, because the latter have computed only the small, and non gauge-invariant, subset of “static” contributions to the 5PN Hamiltonian.] As a by product of our calculation, we also compute the 5PN-level contribution to the (gauge-invariant) scattering angle of two bodies considered at the third post-Minkowskian (3PM) approximation. We find a result which is in agreement with the corresponding result recently derived from a two-loop scattering amplitude computation \cite{11, 12}, thereby providing the first confirmation of the latter result going beyond the 4PN-level checks derived from the gauge-invariant 4PN scattering \cite{13}. The result presented here is only a first application of a general methodology which can be extended to higher PN orders.

Let us motivate our new approach by considering the state of the art of the general relativistic two-body problem. The PN formalism has been the method of choice, during many years, for analytically tackling the dynamics of binary systems. However, it has recently reached a level of complexity which renders further progress acutely difficult. Most of the technical efficiency of the PN formalism comes from the fact that it systematically re-places the four-dimensional relativistic propagator

$$P_3(t, x, t', x') \equiv \Delta^{-1} = (\Delta - c^{-2} \partial^2_t)^{-1}, \quad (1)$$

entering the post-Minkowskian (PM) formalism, by its formal expansion in inverse powers of the velocity of light:

$$P_4^\text{PN}(t, x, t', x') = \left( \Delta^{-1} + \frac{1}{c^2} \partial^2_t \Delta^{-2} + \cdots \right) \delta(t - t'). \quad (2)$$

Here, we consider the time-symmetric propagator (as appropriate to the derivation of the conservative dynamics). Each term on the right-hand side (RHS) of the PN expansion (2) of $P_3(t, x, t', x')$ is local-in-time, in the sense that it involves a derivative of $\delta(t - t')$. Inserting the PN expansion (2) in the reduced action describing (after having integrated out the gravitational field) the PM-expanded dynamics of two worldlines \cite{4} allows one to express the relativistic gravitational interaction of two particles in terms of iterated integrals (given by Feynman-like diagrams) involving only the concatenation of the instantaneous three-dimensional propagator

$$P_3(t, x, t', x') = \delta(t - t') \Delta^{-1} = -\frac{1}{4\pi} \delta(t - t')/|x - x'|, \quad (3)$$

or of its descendants $\frac{1}{4\pi} \partial^2_t \Delta^{-2} \delta(t - t') + \cdots$. The use of such a PN-expanded propagator, together with the corresponding PN simplification of the nonlinear vertices generated by the Einstein-Hilbert action, leads to drastic simplifications (compared to a corresponding PM-expanded action involving the original 4-dimensional propagator $P_4^\text{PM}(t, x, t', x')$) in the computation of the reduced action, especially when using the Arnowitt-Deser-Misner Hamiltonian approach \cite{13}. Indeed, the concatenated massless propagators $\Delta^{-n}$ lead to 3-dimensional integrals containing only one length scale, namely (in $x$-space) the distance $r_{12} \equiv |x_1 - x_2|$ between the two bodies.

However, as had been anticipated years ago \cite{14}, the PN approach undergoes a fundamental conceptual (and technical) breakdown at the fourth post-Newtonian (4PN) level. At this level the naive PN expansion...
wave energy flux emitted by the system, namely (using version of the fractionally 1PN-accurate gravitational-
Here, $M$ and $F$ binary system, while related 5PN logarithmic terms. It reads, from Eq. (9.12) [23] (based on Ref. [24]). See also Refs. [25, 26] for the Ref. [2] (see also the related works [20–22]). The 5PN-an efficient tool for computing the non-local piece given PN accuracy, because the MPM formalism yields expansion [14]. The decomposition (4) makes sense, at a here, $S_{\text{tot}}$ [and its corresponding $H_{\text{loc}}$] differs from the one in Refs. [13, 23], where $H_{\text{loc}}$ included logarithmic contributions in its definition.

First step: computing the nonlocal-in-time piece of the action—The first step is to use results derived within the (PN-matched [14–18]) multipolar-post-Minkowskian (MPM) formalism [19] to decompose, at some given PN accuracy, the complete, (reduced) two-body conservative expansion [14]. The decomposition (4) makes sense, at a given PN accuracy, because the MPM formalism yields an efficient tool for computing the non-local piece $S_{\text{nonloc}}$. From Ref. [16] one knows that $S_{\text{nonloc}}$ starts at the 4PN level. The 4PN-accurate value of $S_{\text{nonloc}}$ was obtained in Ref. [2] (see also the related works [20–22]). The 5PN-accurate value of $S_{\text{nonloc}}$ was obtained in section IXA of [23] (based on Ref. [24]). See also Refs. [23, 26] for the related 5PN logarithmic terms. It reads, from Eq. (9.12) of [23] (see also the related recent work [27]),

\[
S_{\text{nonloc}}^{4+5\text{PN}}[x_1(s_1), x_2(s_2)] = \frac{G^2 M}{c^3} \int dt' \frac{\mathcal{F}_{\text{split}}^{1\text{PN}}(t,t')}{|t-t'|} \times \int dt' \frac{\mathcal{J}^{\text{split}}_{\text{1PN}}(t,t')}{5!}.
\]

Here, $M$ denotes the total conserved mass-energy of the binary system, while $\mathcal{F}_{\text{split}}^{1\text{PN}}(t,t')$ denotes the time-split version of the fractionally 1PN-accurate gravitational-wave energy flux emitted by the system, namely (using a superscript in parenthesis to denote a repeated time-derivative)

\[
\mathcal{F}_{\text{1PN}}^{\text{split}}(t,t') = \frac{G}{c^3} \left( \frac{1}{5} I_{ab}^{(3)}(t) I_{ab}^{(3)}(t') \right) + \frac{1}{189 c^2} I_{abc}^{(4)}(t) I_{abc}^{(4)}(t') + \frac{16}{45 c^2} I_{ab}^{(3)}(t) J_{ab}^{(3)}(t').
\]

The mass and spin multipole moments $I_{ab}$, $I_{abc}$, $J_{ab}$, entering the latter expression are the Blanchet-Damour (1PN-accurate) source multipole moments defined by explicit integrals over the stress-energy tensor of the source [14]. Their explicit expressions for a binary system can be found in Ref. [23]. Eq. (5) defines an explicit functional of the two worldlines, and subtracting it from the (in principle PN-computable) total action $S_{\text{tot}}$ defines the local-in-time contribution $S_{\text{loc}}^{4+5\text{PN}}$ to the twobody dynamics. The time-scale entering the partial finite operation (PF) used in (5) to define the logarithmically divergent integral over $t'$ has been fixed to be $2r_{12}^3/c$, where $r_{12}$ denotes the (harmonic-coordinate) radial distance between the two bodies. Note that the meaning here of $S_{\text{loc}}$ (and its corresponding $H_{\text{loc}}$) differs from the one in Refs. [13, 23].

Second step: computing the $O(\nu)$ piece of the time-averaged redshift ($z_1$) to sixth order in eccentricity—The second step of our approach consists in using gravitational Self-Force (SF) theory to compute to sufficient accuracy the first-order-self-force (1SF) contribution, say $\delta z_1 = O(\nu)$, to the time-averaged redshift ($z_1 = \langle ds_1/dt \rangle$ of the first body, considered as a function of the symmetric mass ratio $\nu$ and of the dimensionless radial and azimuthal frequencies $\Omega_1, \Omega_2$ of eccentric orbits [24, 30]. We denote $M \equiv m_1 + m_2$, $\mu = m_1 m_2/M, \nu = \mu/M = m_1 m_2/(m_1 + m_2)^2$. Ref. [31] has developed efficient tools for analytically computing $\delta z_1$ as a function of the inverse parameter of the elliptical orbit, $u_p = GM/(c^2 p)$, and of the eccentricity, $e$. Current results reached either high-orders in $e$ limited to 4PN accuracy [32], or high PN accuracy limited to fourth order in $e$ [33]. We crucially needed, for the present work, to extend the computation of $\delta z_1$ to the sixth order in $e$ and, to, at least, the 5PN accuracy, i.e., the sixth order in $u_p$. The result of our computation for the coefficient of $e^6$ in $\delta z_1$ reads, at 5PN accuracy,

\[
\delta z_1^{\text{cs}} = \nu \left[ \frac{1}{4} u_3^3 + \left( \frac{53}{12} \right) \frac{11^2}{128} u_4^2 + C_5 u_5^4 + C_6 u_6^5 + o(u_6^6) \right] + O(\nu^7),
\]

where

\[
C_5 = \frac{-38471}{360} + \frac{6455}{4096} \frac{\pi^2}{5} - \frac{178288}{5} \ln(2) + \frac{1994301}{160} - \frac{1953125}{288} \ln(5) + 16 \gamma + 8 \ln(u_p),
\]

\[
C_6 = \frac{-17344111}{5040} + \frac{782899}{4096} \frac{\pi^2}{135} + 66668054 \ln(2) + \frac{29268135}{448} \ln(3) - \frac{2027890625}{12096} \ln(5) - \frac{1694}{5} \gamma - \frac{847}{5} \ln(u_p).
\]
Third step: using the first law of binary dynamics to translate $\delta z_q^{\nu}$ into a corresponding $O(p_c^4)$-accurate, effective-one-body Hamiltonian—The first law of binary dynamics \cite{26, 35, 36} allows one to transform the gauge-invariant information contained in our new result \cite{7} (together with the previous $O(\epsilon^4)$ results \cite{32, 33}) into a corresponding knowledge of the (gauge-fixed) two-body Hamiltonian, as expressed in effective-one-body (EOB) theory \cite{37, 38}. To do this we had to extend the results \cite{36} to the sixth order in the ($\mu$-rescaled) radial momentum $p_r$. EOB theory expresses the two-body Hamiltonian $H = M c^2$ in terms of a rescaled “effective” Hamiltonian $\hat{H}_{\text{eff}}$ according to

$$ H = M c^2 \sqrt{1 + 2\nu (\hat{H}_{\text{eff}} - 1)}. \tag{9} $$

In turn, $\hat{H}_{\text{eff}}$ is expressed in terms of various bricks: two radial potentials $A(u; \nu)$, and $D(u; \nu) \equiv A(u; \nu) D(u; \nu) = (A(u; \nu) B(u; \nu))^{-1}$, and a momentum-dependent potential $Q(u, p_c, \nu)$, where $u \equiv GM/(c^2 \nu^2)$. Namely, henceforth setting $c = 1$,

$$ \hat{H}_{\text{eff}} = A(u; \nu)[1 + A(u; \nu) D(u; \nu) p_r^2 + p_r^2 u^2 + Q(u, p_c, \nu)], \tag{10} $$

The PN expansions of the potentials $A(u; \nu)$, and $D(u; \nu)$ are written as $A(u; \nu) = 1 - 2u + \sum_n a_n (\nu, \ln u) u^n$ and $D(u; \nu) = 1 + \sum_n d_n (\nu, \ln u) u^n$. In the gauge (hereafter called “$p_r$ gauge”) introduced in \cite{38}, the PN expansion of $Q(u, \nu)$ is given by a double expansion in $u$ and $p_r^2$, say $Q = q_0(u; \nu) p_r^2 + q_0(u; \nu) p_r^2 + q_0(u; \nu) p_r^2 + \ldots$, where $q_m(u; \nu) = \sum_n q_m (\nu, \ln u) n^n$. In addition, all the (logarithmically running) $\nu$-dependent coefficients $a_n (\nu, \ln u)$, $d_n (\nu, \ln u)$, $q_m (\nu, \ln u)$ are polynomials in $\nu$, starting at $\nu^1$, and of degree increasing with $n$. We derived the relation linking the ISF (of $O(\nu)$) piece in $q_0(u; \nu)$ to the ISF redshift $\delta z_1^{(u_p, \epsilon)} = \delta z_1^{(u_p)} + \delta z_1^{(u_p)} e^2 + \delta z_1^{(u_p)} e^4 + \delta z_1^{(u_p)} e^6$. This allowed us to extend the previous ISF knowledge of $q_0(u, p_c, \nu)$ \cite{32} to the $p_r^2$ level, namely

$$ q_0(u; \nu) = \nu q_0^{(1)} u^2 + \nu q_0^{(3)} u^3 + O(u^{7/2}) + O(u^2), \tag{11} $$

where $q_0^{(1)}$ is a known 4PN term \cite{23} and where

$$ q_0^{(3)} = \frac{2613083}{1050} + \frac{687574536}{4725} \ln(2) - \frac{23132628}{175} \ln(3) - \frac{101687500}{189} \ln(5), \tag{12} $$

is a new, 5PN level, result. See Ref. \cite{34} for the higher-order contributions in $u$ (up to $u^{19/2}$ included).

Fourth step: determining the ISF contribution to the local-in-time 5PN-accurate Hamiltonian by subtracting the nonlocal action—Inserting our new result \cite{11}, together with the previous high-PN 1SF knowledge of $A(u; \nu)$, $D(u; \nu)$ and $q_2(u; \nu)$, in Eqs. \cite{9} and \cite{10} determines the two-body Hamiltonian at the combined 1SF + 5PN accuracy. At the level of the unrescaled, total Hamiltonian $H$, Eq. \cite{9}, 1SF accuracy means knowing both the $\nu^1$ and the $\nu^2$ contributions. We can then subtract from the full Hamiltonian action $\int pdq - H(q, p)dt$ the nonlocal-in-time term \cite{3} to compute the local-in-time Hamiltonian action $\int pdq - H_{\text{loc}}(q, p)dt$. This is conveniently done by using the Delaunay averaging technique of the nonlocal action introduced in \cite{23}. This averaging technique leads to a gauge-invariant result which can then be expressed in the EOB-$p_r$ gauge. The so obtained 1SF + 5PN accurate local-in-time Hamiltonian $H_{\text{loc}}(q, p)$ can then be expressed (via the universal EOB energy map \cite{9}) in terms of corresponding 1SF + 5PN accurate EOB potentials $A_{\text{loc}}(u, \nu)$, $D_{\text{loc}}(u; \nu)$ and $Q_{\text{loc}}(u, p_c, \nu)$. All logarithmic dependence (including numerical logs, like $\ln 2$) has disappeared from these local potentials. For instance, the local contribution to $q_0(u; \nu)$ was found to be $q_0^{(\text{loc})} = -\frac{2}{3} \nu u^2 + \frac{121}{18} \nu u^3 + O(\nu^2)$. Here, the contribution $\frac{121}{18} \nu u^3$ is at the 5PN level.

Fifth step: using EOB-PM theory to determine most of the nonlinear dependence on $\nu$ of the local Hamiltonian—At this stage, we have found the exact dependence of the nonlocal action \cite{31} on the two masses $m_1, m_2$, and therefore on $\nu$ for a given $M$, our use of SF technology has limited our determination of the local Hamiltonian $H_{\text{loc}}(q, p, \nu)$ to the ISF accuracy: $H_{\text{loc}} = M c^2 + \nu H_{\text{loc}}^{(1)} + \nu^2 H_{\text{loc}}^{(2)} + O(\nu^3)$. We can, however, determine most of the higher-order powers in $\nu$ by using results from the EOB formalism applied to PM-expanded scattering. More precisely, we can use two constraints.

On the one hand, the exact $\nu$ dependence of the EOB Hamiltonian has been determined both at the first post-Minkowskian (1PM) level \cite{39} and at the second post-Minkowskian (2PM) level \cite{40}. By transforming the latter results (obtained in a special “energy” gauge) into the (standard) EOB-$p_r$ gauge used above, we can determine the exact $\nu$ dependence of the 5PN-accurate (local and nonlocal) Hamiltonian for all the terms in the Hamiltonian which are either $\propto u^1$ or $\propto u^2$. For instance, we thereby found that the coefficient $q_{28}$ of $p_r^2 u^2$ in the $Q$ potential is $q_{28} = q_{28}^{(\text{loc})} = \frac{9}{14} \nu + \frac{15}{14} \nu^2 + \frac{7}{15} \nu^3 - 6 \nu^4$.

On the other hand, the general dictionary \cite{40} between the EOB Hamiltonian and the PM-expanded scattering function, $1/\chi(\bar{E}_{\text{eff}}, j) = \sum_n \chi_n (\bar{E}_{\text{eff}})^{j/n}$, where $j \equiv J/(Gm_1 m_2)$, has recently been used \cite{11} to show that the combination $(1 + 2\nu \bar{E}_{\text{eff}} - 1)^{u-1} \chi_n (\bar{E}_{\text{eff}})$ was a polynomial in $\nu$ of degree $d(n)$ equal to the integer part of $(n - 1)/2$. This yields a strong restriction on the $\nu$ dependence of the coefficients of the 5PN-level local Hamiltonian,

$$ H_{\text{loc}}^{(5)} = \sum_{m+n=6} h_{2m n}(\nu)(p_r^2)^m u^n. \tag{13} $$

[For notational simplicity, we use in Eq. \cite{13} $p_r^2$ to denote either $p_r^2$ or $p_r^4 \equiv p_r^2/v_1^2$. In order to apply this restriction, we computed (as a function of the coefficients $h_{2m n}(\nu)$) the scattering angle implied by the total, 5PN-accurate Hamiltonian (using the technique of Ref. \cite{13}).]
The contributions are keyed, on the horizon, to obtain the various contributions to the $5\text{PN}$-accurate local Hamiltonian. These contributions are contained (in our gauge) 36 unknown numerical coefficients, say $h_{2m n}^{\nu}$ parametrizing the powers of $\nu$ in the various contributions $h_{2m n}^{\nu}(\nu) = \sum_{k=1}^{k_{\text{max}}(m, n)} h_{2m n}^{\nu k}$ appearing in Eq. (13). [Here, we do not distinguish the coefficients of $p_r^2$ or $p_t^2$. If distinguished, there are 108 coefficients.] The degrees of these polynomials in $\nu$ are indeed found (when $m + n = 6$) to be all equal to $k_{\text{max}} = 6$ when $n = 6 - m = 1, \ldots , 6$.

Combining all the previous tools and results, we were able to determine $34$ of the $a \text{ priori}$ unknown numerical coefficients $h_{2m n}^{\nu}$, Fig. 1 indicates the source of information having allowed us to determine each one of these 34 coefficients: the test-particle limit determines the $\nu^1$ row; the 1SF computations determine the $\nu^2$ row; the first two columns are respectively determined by the 1PM and 2PM exact EOB Hamiltonians; the $\nu^{\geq 3}$ dependence of the next third and fourth columns (respectively corresponding to 3PM and 4PM) are completely determined by the EOB-PM scattering constraint mentioned above. The latter constraint determines the coefficients in the last two columns (5PM and 6PM) except for the two coefficients $h_{25}^{p_r^2}$ and $h_{06}^{p_t^2}$. [Distinguishing $p_r^2$ and $p_t^2$, we determine 106 coefficients among 108.] When using the (more compact) EOB parametrization of the local Hamiltonian the full description of the local-in-time $5\text{PN}$-accurate Hamiltonian $H_{\text{loc}}^{5\text{PN}}$ is obtained by inserting in the EOB map the effective Hamiltonian $\hat{H}_{\text{eff}}$ defined by the following (logarithm-free) values of the local pieces of the EOB building blocks $A(u; \nu)$, $D(u; \nu)$ and $Q(u, p_r; \nu)$:

$$
A_{\text{loc}} = 1 - 2u + 2u^3 + \nu \left( \frac{94}{3} - \frac{41}{32} \pi^2 \right) u^4 + a_{10}^{\text{loc}} u^5 + a_{12}^{\text{loc}} u^6 ,
$$

$$
D_{\text{loc}} = 1 + 6u^2 + (52\nu - 6\nu^2) u^3 + d_{11}^{\text{loc}} u^4 + d_{12}^{\text{loc}} u^5 ,
$$

$$
Q_{\text{loc}} = p_r^2 \left[ 2(5\pi - 1) u^2 + q_{12}^{\text{loc}} u^3 + q_{14}^{\text{loc}} u^4 \right] + p_r^2 (q_{62}^{\text{loc}} u^2 + q_{63}^{\text{loc}} u^3) + q_{82}^{\text{loc}} p_r^2 u^2 ,
$$

with

$$
a_{10}^{\text{loc}} = \left( \frac{4237}{60} + \frac{2275}{512} \pi^2 \right) \nu + \left( \frac{41}{32} \pi^2 - \frac{221}{6} \right) \nu^2 ,
$$

$$
a_{12}^{\text{loc}} = \left( -\frac{1026301}{1575} + \frac{246367}{3072} \pi^2 \right) \nu + a_{12}^{\text{loc}} \nu^2
$$

+ $4\nu^3$, 

$$
a_{11}^{\text{loc}} = \left( \frac{1679}{9} - \frac{23761}{1536} \pi^2 \right) \nu + \left( -260 + \frac{123}{16} \pi^2 \right) \nu^2 ,
$$

$$
a_{12}^{\text{loc}} = \left( \frac{331054}{175} - \frac{63707}{512} \pi^2 \right) \nu + d_{12}^{\text{loc}} \nu^2
$$

+ \left( \frac{1069}{3} - \frac{205}{16} \pi^2 \right) \nu^3 ,

and

$$
a_{43}^{\text{loc}} = 20\nu - 83\nu^2 + 10\nu^3 ,
$$

$$
a_{44}^{\text{loc}} = \left( \frac{1580641}{3150} - \frac{93031}{1536} \pi^2 \right) \nu
$$

+ \left( \frac{2075}{3} + \frac{31633}{512} \pi^2 \right) \nu^2 + \left( 640 - \frac{615}{32} \pi^2 \right) \nu^3 ,
$$

$$
a_{62}^{\text{loc}} = \frac{9}{5} \nu - \frac{27}{5} \nu^2 + 6\nu^3 ,
$$

$$
a_{63}^{\text{loc}} = \frac{123}{10} \nu - \frac{69}{5} \nu^2 + 116\nu^3 - 14\nu^4 ,
$$

$$
a_{62}^{\text{loc}} = \frac{6}{7} \nu + \frac{18}{7} \nu^2 + \frac{24}{7} \nu^3 - 6\nu^4 .
$$

Modulo the two undetermined coefficients $a_{33}^{\text{loc}}$ and $d_{13}^{\text{loc}}$, the full $5\text{PN}$-accurate dynamics is given by adding to the local action defined by $H_{\text{loc}}^{5\text{PN}}$ the 4+5PN nonlocal one $\hat{H}_{\text{loc}}^{5\text{PN}}$.

New results at $3\text{PM}$ and $4\text{PM}$— As one can see on Fig. 1, our results give a complete description of the $5\text{PN}$ dynamics at the $3\text{PM}$ and $4\text{PM}$ levels (fourth and fifth columns in Fig. 1). This means in particular that our findings allow us to compute, with $5\text{PN}$ accuracy, the $3\text{PM}$ ($O(G^3)$) and $4\text{PM}$ ($O(G^4)$) terms, $\chi_3$ and $\chi_4$, in the scattering angle. The computation at $5\text{PN}$ accuracy of $\chi_3$ from our results for the full loc + nonloc dynamics (with $\chi_3^{\text{nonloc}} = 0$) yields (denoting $p_\infty \equiv \sqrt{2\tilde{c}_\text{eff}^2 - 1}$)

$$
\chi_3 = -\frac{1}{3} p_\infty^3 + \frac{4}{p_\infty} + \left( -8\nu + 24 \right) p_\infty
$$

as shown in (12).

![FIG. 1: Schematic representation of the theoretical tools used to obtain the various contributions to the $5\text{PN}$-accurate local Hamiltonian. These contributions are contained (in our gauge) 36 unknown numerical coefficients, say $h_{2m n}^{\nu}$ parametrizing the powers of $\nu$ in the various contributions $h_{2m n}^{\nu}(\nu) = \sum_{k=1}^{k_{\text{max}}(m, n)} h_{2m n}^{\nu k}$ appearing in Eq. (13). Here, we do not distinguish the coefficients of $p_r^2$ or $p_t^2$. If distinguished, there are 108 coefficients.]
\[
\begin{align*}
\chi_{4,}\text{loc} & = \chi_{4,}\text{nonloc} + \chi_{\text{value for the 4PM-level scattering angle}}
\end{align*}
\]

In this expression the last term \( \propto p_\infty^7 \) is the 5PN contribution to \( \chi_3 \). Importantly, we checked that this newly derived result is in agreement with the corresponding 5PN-level term in the PN expansion of the (partly conjectural) 3PM-level recent result of [11, 12]. This is the first independent, partial confirmation of the latter result.

In addition, our results yield an explicit 5PN-accurate value for the 4PM-level scattering angle \( \chi_{4,}\text{loc} = \chi_{4,}\text{nonloc} \). Let us only cite here the 5PN-level term in the local contribution \( \chi_{4,}\text{loc} (p_\infty) \):

\[
\begin{align*}
\chi_{4,}\text{loc} (p_\infty) &= \pi \left( -\frac{94899}{32768} \pi^2 \nu^2 + \frac{93031}{32768} \pi^2 \nu^4 
- \frac{1945583}{33600} \nu^3 + \frac{1937}{16} \nu^4 - \frac{2895}{32} \nu^5 + \frac{525}{64} \nu^6 
+ \frac{1845}{2048} \nu^7 \right) p_\infty^7.
\end{align*}
\]

The complementary nonlocal contribution is derivable by the methods of [13].

Conclusions.— We have introduced a new methodology (based on combining several different theoretical tools) for analytically computing the conservative dynamics of two bodies in General Relativity. We have applied our approach to deriving a nearly complete expression for the 5PN-level action. It is given by the sum of a 4PN+5PN nonlocal action, Eq. (5), and of a local one \( \int \rho \mathrm{d}q - H_{5,\text{nonloc}}^{\leq 5\text{PN}} \). We determined the full functional structure of \( H_{5,\text{nonloc}}^{\leq 5\text{PN}} \), except for two \( (\nu^3) \)-level unknown coefficients. Our results give access to the 5PN-accurate \( O(G^3) \) and \( O(G^4) \) scattering angles. This provided the first independent confirmation of the recent 3PM result of Refs. [11, 12].

Our work opens promising avenues for further progress on the dynamics of binary systems. Indeed, the technique we defined here can be extended, in principle, to higher PN orders. Our work also offers new motivations for doing targeted, partial computations able to determine the two currently missing numerical coefficients. We can think of several ways in which they could be determined: second-order self-force computation; partial computation of 5PN dynamics by traditional techniques aiming only at terms having selected mass dependence; or, eventually, high-accuracy numerical computation.

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