Ergodicity breaking and Localization of the Nicolai
supersymmetric fermion lattice model

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Abstract

We investigate dynamics of the spinless fermion lattice model proposed
by Hermann Nicolai [J. Phys. A. Math. Gen. 9 (1976)] which we shall call
the Nicolai model. This non-relativistic quantum many-body model is gener-
ated by spinless fermions without bosons. However, it has the same algebraic
structure as $N = 2$ supersymmetry. We explicitly provide its local fermionic
constants of motion that exist infinitely many irrespective of the dimension
of lattice. The existence of local constants implies breaking ergodicity of its
time evolution for all KMS (thermal equilibrium) states. At zero tem-
perature, there are infinitely many degenerated classical ground states on the
Fock space. Each of them breaks ergodicity as well. Adopting a viewpoint of
perturbation theory, we explain why delocalization is suppressed at zero tem-
perature despite its disorder-free translation-invariant quantum interaction.
Nevertheless the Nicolai model does not satisfy a criterion of the fully many-
body localization. We also discuss the meaning of quantum integrability of
the Nicolai model.

Key Words Supersymmetric fermion lattice model. Breaking ergodicity. Local
fermionic constants of motion. Quantum integrability. Many-body localization.
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1 Introduction

Ergodicity is a fundamental concept of dynamical systems rooted in statistical mechanics [18] [19] [30]. Despite its long history and several far-reaching mathematical results, it seems that ergodicity has not yet become either a well-founded hypothesis or a practical tool in statistical physics. Recently, quantum ergodicity has gained renewal interest stimulated by experimental findings of quantum dynamics, see e.g. [8] [20] [29] [49] [53]. Notably, quantum ergodicity and its breakdown have been discussed in connection to “thermalization and many-body localization (MBL)”. We refer to some review articles [1] [33] [41] on this growing subject.

As alluded to above there are only few concrete quantum many-body models that are known to be ergodic or non-ergodic. To get insight into general properties of quantum many-body dynamics it would be important to find new examples for which ergodicity can be rigorously proved or disproved.

In this paper, we investigate dynamics of a spinless fermion lattice model proposed by Nicolai [37]. This fermion lattice model has $\mathcal{N} = 2$ supersymmetry by definition, although it lacks basic ingredients of supersymmetry, a pair of fermions and bosons of the same quantum numbers except the spin [59]. In this paper we call this non-relativistic quantum many-body model the Nicolai model. Using a general $C^*$-algebraic formulation of supersymmetric fermion lattice models given in [32] we formulate the Nicolai model as a supersymmetric $C^*$-dynamical system. We aim to study its dynamical properties making use of several merits of the $C^*$-dynamical theory that enables rigorous treatment of quantum dynamics in infinitely extended systems [7] [55].

The Nicolai model exhibits certain phenomena characteristic for many-body localization (MBL). We point out two such phenomena. First, there exist infinitely
many local constants of motion that are frozen under the time evolution defined on the infinitely extended system. Second, there exist infinitely many classical eigen-
states (actually ground states) on the Fock space. The local constants of motion of the Nicolai model are notable by the following reasons. They are all fermionic. They are extensive, namely the number of them increases exponentially with respect to subsystems.

We readily show breaking ergodicity for the time evolution of the Nicolai model from the existence local constants of motion. Precisely, the ergodicity in the sense of Mazur [31] is broken for all KMS (thermal equilibrium) states [21] and all classical ground states.

We also discuss the eigenstate thermalization hypothesis (ETH) [16] [54] for the Nicolai model. ETH has been used frequently in theoretical physics (but not so much in mathematical physics) as a handy characterization of “ergodicity” [12] [41]. However, ETH is quite different from the basic concept of ergodicity: Ergodicity is defined for dynamical systems (pair of dynamics and invariant states), whereas ETH is a hypothesis for the discrete spectrum of finite Hamiltonians as it stands for. We take some set of local Hamiltonians that tends to the Hamiltonian on the whole system, and then verify that ETH is violated by the Nicolai model (which has been shown to be non-ergodic in the sense of Mazur).

It is widely believed that disorder or randomness is a central cause of localization for interacting quantum models [4] [24] as in the original Anderson localization model [2]. It is noted in [16] that random quantum many-body models showing many-body localization are qualitatively different from quantum integrable models and spin-glass models. The Nicolai model is exactly periodic (not quasi-periodic as in [26]) with no disorder term. Nevertheless it has several MBL-like properties as mentioned. Here we refer to a well-known fundamental question of many-body localization: Is MBL possible without disorder? This question has been discussed in several works under various physics setups [13] [15] [40] [47] [48] [52] [56]. We can see that the Nicolai model does not fit any of these works. In this paper we aim to compare this disorder-free non-ergodic model with known MBL models made by disorder.

We shall discuss the status of the infinitely many classical ground states of the Nicolai model whose degeneracy is extensive with respect to subsystems. In the physics literature many-body localization usually requires localization for all eigen-
states of the Hamiltonian, see [33]. (Recently, a less demanding notion of MBL is proposed in [23].) On the other hand, the above localization is merely for ground states at zero temperature. (Nevertheless, the Nicolai model has localized eigen-
states of any particle density.) Thus we consider that the Nicolai model can not be regarded as a MBL model in the usual sense.

The Nicolai model has natural binary codes by which its all infinitely many classical ground states can be expressed. This is certainly a MBL-like property. On the other hand, we can show that the criterion for the fully many-body localization (FMBL) proposed in [22] [33] is not satisfied by the Nicolai model. Precisely there is no complete set of “l-bits” by which the total Hamiltonian can be written as a sum of classical short-range interactions. It would be interesting to compare our no-go result with the work [39] that supports FMBL for certain supersymmetric fermion lattice models.
In [14] a general scenario of delocalization for disorder-free translation-invariant quantum Hamiltonians is proposed. This scenario makes use of perturbation argument imposing some general assumptions upon the translation-invariant Hamiltonian. We will show that this scenario can not apply to the Nicolai model in the following sense. We divide the Nicolai Hamiltonian into the classical interaction and the quantum hopping interaction. We take the classical interaction for our initial classical Hamiltonian. We then perturb the classical Hamiltonian by the hopping interaction. We take infinitely many classical ground states of the classical Hamiltonian as initial states. We see that those classical states are frozen under the hopping perturbation and resonant does not occur. We note that our argument is based on a specific perturbation, whereas the argument in [14] considers a generic perturbation. So there is no inconsistency between our result and the statements shown in [14].

Finally we address “quantum integrability” that the Nicolai model possesses. In [9] Caux-Mossel proposed a new definition of quantum integrability putting emphasis on quantum dynamics. The Nicolai model is categorized to the class named “constant quantum integrable”. A typical example in this class is non-interacting (free) fermion models. This class is more integrable than the class of “linear quantum integrable” to which the Heisenberg spin chain belongs.

Throughout this paper we work with the Nicolai model on one-dimensional integer lattice. However, all statements shown in this paper are valid for any dimensional integer lattice with some obvious modification.

2 Nicolai supersymmetric fermion lattice model

2.1 Basis for supersymmetry

We shall provide a brief summary of supersymmetry (SUSY) just necessary for the present work.

Suppose that $\mathcal{H}$ is a graded Hilbert space with a grading operator (or fermion parity operator) $(-1)^F$, where $F$ denotes a positive operator that counts a number of fermions. The Hilbert space is written as a product sum $\mathcal{H}_+ \otimes \mathcal{H}_-$, such that $(-1)^F$ has eigenvalue +1(-1) for any vector in $\mathcal{H}_+(\mathcal{H}_-)$, respectively. Consider a conjugate pair of linear operators $Q$ and $Q^*$ on $\mathcal{H}$, where * denotes the adjoint of linear operators. Assume that they are fermionic,

$$\{(−1)^F, Q\} = \{(−1)^F, Q^*\} = 0.$$  \hspace{1cm} (2.1)

Assume further that they are nilpotent,

$$Q^2 = 0 = Q^{*2}.$$  \hspace{1cm} (2.2)

By using the above fermion operators we define the Hamiltonian as

$$H := \{Q, Q^*\} \equiv QQ^* + Q^*Q.$$  \hspace{1cm} (2.3)

From (2.3) and (2.2) we see that

$$[H, Q] = [H, Q^*] = 0.$$  \hspace{1cm} (2.4)

The above algebraic structure satisfied by $\{Q, Q^*, H, (-1)^N, \mathcal{H}\}$ is called the $\mathcal{N} = 2$ supersymmetry, see general references of supersymmetry e.g. [59]. We will consider a genuine fermion model with no boson, however, has supersymmetry.
2.2 Fermion lattice system

We will give a general mathematical formulation of fermion lattice systems based on the CAR algebra. For definiteness we take integer lattice $\mathbb{Z}^\nu$ with any $\nu \in \mathbb{N}$. For any two sites $x = (x_i)$, $y = (y_i) \in \mathbb{Z}^\nu$ the distance between them is $|x - y| := \max_{1 \leq i \leq \nu} |x_i - y_i|$. For any subset $I$ of $\mathbb{Z}^\nu$ we denote the number of sites in $I$ by $|I|$. The notation $I \subset \mathbb{Z}^\nu$ means that a subregion $I \subset \mathbb{Z}^\nu$ has finite number of sites in it.

We consider interacting spinless fermions over $\mathbb{Z}^\nu$. Let $a_i$ and $a_i^*$ denote the annihilation operator and the creation operator of a spinless fermion at $i \in \mathbb{Z}^\nu$, respectively. Those satisfy the canonical anticommutation relations (CARs):

$$\{a_i^*, a_j\} = \delta_{i,j} 1, \quad \{a_i^*, a_j^*\} = \{a_i, a_j\} = 0. \quad (2.5)$$

For each site $i \in \mathbb{Z}^\nu$ we take the number operator:

$$n_i := a_i^* a_i. \quad (2.6)$$

For each $I \subset \mathbb{Z}^\nu$, let $\mathcal{A}(I)$ denote the finite-dimensional algebra generated by $\{a_i, a_i^* : i \in I\}$. It is isomorphic to $M_{2^{|I|}}(\mathbb{C})$, i.e. the algebra of all $2^{|I|} \times 2^{|I|}$ complex matrices. For any $I \subset J \subset \mathbb{Z}^\nu$, $\mathcal{A}(I)$ is naturally imbedded into $\mathcal{A}(J)$ as a subalgebra. We define

$$\mathcal{A}_o := \bigcup_{I \subset \mathbb{Z}^\nu} \mathcal{A}(I). \quad (2.7)$$

Taking the norm completion of the normed $*$-algebra $\mathcal{A}_o$ we obtain a $C^*$-algebra $\mathcal{A}$. It is well-known and called the CAR algebra. The dense $*$-subalgebra $\mathcal{A}_o$ will be called the local algebra.

Let $\gamma$ denote the automorphism on the $C^*$-algebra $\mathcal{A}$ determined by

$$\gamma(a_i) = -a_i, \quad \gamma(a_i^*) = -a_i^*, \quad \forall i \in \mathbb{Z}^\nu. \quad (2.8)$$

Obviously, $\gamma \circ \gamma = \text{id}$. The total system $\mathcal{A}$ is decomposed into the even part and the odd part:

$$\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-,$$

$$\mathcal{A}_+ = \{ A \in \mathcal{A} | \gamma(A) = A \}, \quad \mathcal{A}_- = \{ A \in \mathcal{A} | \gamma(A) = -A \}. \quad (2.9)$$

The graded commutator on the graded algebra $\mathcal{A}$ is defined as

$$[F_+, G]_\gamma = [F_+, G] \text{ for } F_+ \in \mathcal{A}_+, \ G \in \mathcal{A},$$

$$[F_-, G_-]_\gamma = \{ F_-, G_- \} \text{ for } F_- \in \mathcal{A}_-, \ G_- \in \mathcal{A}_-. \quad (2.10)$$

As in (2.9) for each $I \subset \mathbb{Z}^\nu$ we may consider the grading structure:

$$\mathcal{A}(I) = \mathcal{A}(I)_+ \oplus \mathcal{A}(I)_-, \quad \text{where} \quad \mathcal{A}(I)_+ := \mathcal{A}(I) \cap \mathcal{A}_+, \quad \mathcal{A}(I)_- := \mathcal{A}(I) \cap \mathcal{A}_-. \quad (2.11)$$

similarly the local algebra has a natural graded structure:

$$\mathcal{A}_o = \mathcal{A}_o + \oplus \mathcal{A}_o -,$$

$$\mathcal{A}_o + := \mathcal{A}_o \cap \mathcal{A}_+, \quad \mathcal{A}_o - := \mathcal{A}_o \cap \mathcal{A}_-. \quad (2.12)$$
By the canonical anticommutation relations (2.5) we see the $\gamma$-locality:

$$[A, B]_\gamma = 0 \text{ for every } A \in \mathcal{A}(I) \text{ and } B \in \mathcal{A}(J) \text{ if } I \cap J = \emptyset, \ I, J \in \mathbb{Z}^\nu. \quad (2.13)$$

We introduce some basic transformations on the fermion lattice system. Let $\sigma$ denote the shift-translation automorphism group on $\mathcal{A}$. For each $k \in \mathbb{Z}^\nu$

$$\sigma_k(a_i) = a_{i+k}, \quad \sigma_k(a_i^*) = a_{i+k}^*, \quad \forall i \in \mathbb{Z}^\nu. \quad (2.14)$$

By $\gamma_\theta$ ($\theta \in [0, 2\pi]$) we denote the global $U(1)$-symmetry defined as

$$\gamma_\theta(a_i) = e^{-i\theta}a_i, \quad \gamma_\theta(a_i^*) = e^{i\theta}a_i^*, \quad \forall i \in \mathbb{Z}^\nu. \quad (2.15)$$

By definition $\gamma_\pi$ is equal to the grading $\gamma$ of (2.8). We may consider the particle-hole transformation $\rho$:

$$\rho(a_i) = a_i^*, \quad \rho(a_i^*) = a_i, \quad \forall i \in \mathbb{Z}. \quad (2.16)$$

### 2.3 $C^*$-dynamics of supersymmetric fermion lattice models

We shortly recall our general $C^*$-algebraic framework of supersymmetric fermion lattice models given in [32]. The basic building block is nilpotent superderivations defined on the local algebra of the fermion system.

First we consider bounded superderivations. For a non-zero odd element $q \in \mathcal{A}_-$ define the linear map from $\mathcal{A}$ into $\mathcal{A}$ by using the graded commutator

$$\delta_q(A) := [q, A]_\gamma \text{ for every } A \in \mathcal{A}. \quad (2.17)$$

This is a bounded superderivation defined on the whole system $\mathcal{A}$. We take its conjugate superderivation as

$$\delta_q^* := \delta_{q^*}. \quad (2.18)$$

Now we assume that $q$ is nilpotent (inevitably it is not self-adjoint).

$$q^2 = 0. \quad (2.19)$$

This implies that both $\delta_q$ and $\delta_q^*$ are nilpotent:

$$\delta_q \cdot \delta_q = 0 = \delta_q^* \cdot \delta_q^*. \quad (2.20)$$

Let

$$h := \{q^*, q\} \in \mathcal{A}_+. \quad (2.21)$$

By definition $h = h^*$ and $h \geq 0$. Define a bounded derivation

$$d_h(A) := [h, A] \text{ for every } A \in \mathcal{A}. \quad (2.22)$$

It is a $*$-derivation, namely a linear map defined on $\mathcal{A}$ satisfying the Leibniz rule such that

$$(d_h(A))^* = d_h(A^*) \text{ for every } A \in \mathcal{A}. \quad (2.23)$$
The above \( \{q, q^*, h\} \) satisfy the \( \mathcal{N} = 2 \) supersymmetry relation as defined in §2.1. We see the following identity

\[
d_h = \delta_q^* \cdot \delta_q + \delta_q \cdot \delta_q^*. \tag{2.24}
\]

This together with the nilpotent condition (2.20) represents a \( \mathcal{N} = 2 \) supersymmetry algebra in terms of (super)derivations \( \{\delta_q, \delta_q^*, d_h\} \).

Next we will consider the unbounded case of \( \mathcal{N} = 2 \) supersymmetry expressed by unbounded superderivations on the CAR algebra. Let \( \Psi \) be a map from the set of finite subsets \( \{I; I \subset \mathbb{Z}^\nu\} \) to the local algebra \( \mathcal{A}_o \) such that

\[
\Psi : I \mapsto \Psi(I) \in \mathcal{A}(I) - \text{ for every } I \subset \mathbb{Z}^\nu. \tag{2.25}
\]

Assume that it is uniformly bounded:

\[
\|\Psi\| := \sup_{I \subset \mathbb{Z}^\nu} \|\Psi(I)\| < \infty. \tag{2.26}
\]

The map \( \Psi \) is called an assignment of local fermion charges (over \( \mathbb{Z}^\nu \)), and each \( \Psi(I) \in \mathcal{A}(I) - \) is called a local fermion charge on \( I \). The conjugate of \( \Psi \) is defined as

\[
\Psi^* : I \mapsto \Psi(I)^* \in \mathcal{A}(I) - \text{ for every } I \subset \mathbb{Z}^\nu. \tag{2.27}
\]

We now impose several assumptions upon \( \Psi \). First it has a finite range. Namely assume that there exists \( r \in \mathbb{N} \cup \{0\} \) such that

\[
\Psi(I) = 0 \quad \text{whenever } \text{diam}(I) = \max_{x,y \in I} |x - y| > r. \tag{2.28}
\]

The minimum non-negative integer \( r \) satisfying the above condition (2.28) is called the range of \( \Psi \). Of course, \( \Psi^* \) has the same range \( r \) as that of \( \Psi \). Let \( \Psi \) be an assignment of local fermion charges satisfying the finite-range condition (2.28).

Then for every \( I \subset \mathbb{Z}^\nu \), by using the notation (2.17) let us take a linear map from \( \mathcal{A}(I) \) to \( \mathcal{A}_o \) as

\[
\delta_{\Psi}(A) := \sum_{X \cap I \neq \emptyset, X \subset \mathbb{Z}^\nu} \delta_{\Psi(X)}(A) = \sum_{X \cap I \neq \emptyset, X \subset \mathbb{Z}^\nu} [\Psi(X), A]_\gamma \text{ for every } A \in \mathcal{A}(I), \tag{2.29}
\]

where the summation is taken over all finite subsets \( \{X; X \subset \mathbb{Z}^\nu\} \) that have a non-trivial intersection with \( I \). Then the set of formulas (2.29) over all \( \{I, I \subset \mathbb{Z}^\nu\} \) consistently determines a unique linear map \( \delta_{\Psi} \) from \( \mathcal{A}_o \) into \( \mathcal{A}_o \):

\[
\delta_{\Psi}(A) := \sum_{X \subset \mathbb{Z}^\nu} [\Psi(X), A]_\gamma \text{ for every } A \in \mathcal{A}_o. \tag{2.30}
\]

For each \( A \in \mathcal{A}_o \) only a finite number of finite subsets \( \{X; X \subset \mathbb{Z}^\nu\} \) contribute to the above formula. We verify that \( \delta_{\Psi} \) is an odd linear map

\[
\delta_{\Psi} \cdot \gamma = -\gamma \cdot \delta_{\Psi} \text{ on } \mathcal{A}_o, \tag{2.31}
\]

and that it satisfies the graded Leibniz rule:

\[
\delta_{\Psi}(AB) = \delta_{\Psi}(A)B + \gamma(A)\delta_{\Psi}(B) \quad \text{for every } A, B \in \mathcal{A}_o. \tag{2.32}
\]
Thus $\delta_\Psi$ is a superderivation. The conjugate superderivation $\delta_\Psi^*$ for $\delta_\Psi$ is given as

$$\delta_\Psi^*(A) \equiv \delta_\Psi^*(A) = \sum_{X \in \mathbb{Z}^\nu} [\Psi^*(X), A]_\gamma \text{ for every } A \in \mathcal{A}_o. \quad (2.33)$$

By definition

$$\delta_\Psi(\mathcal{A}_{o\pm}) \subset \mathcal{A}_{o\mp}, \delta_\Psi^*(\mathcal{A}_{o\pm}) \subset \mathcal{A}_{o\mp}. \quad (2.34)$$

It is crucial to assume that the superderivation $\delta_\Psi$ associated to $\Psi$ satisfies the nilpotent condition:

$$\delta_\Psi \cdot \delta_\Psi = 0 \text{ on } \mathcal{A}_o. \quad (2.35)$$

If $\delta_\Psi$ is nilpotent, then the conjugate superderivation is automatically nilpotent:

$$\delta_\Psi^* \cdot \delta_\Psi^* = 0 \text{ on } \mathcal{A}_o. \quad (2.36)$$

If a state $\varphi$ on $\mathcal{A}$ is invariant under $\delta_\Psi$:

$$\varphi(\delta_\Psi(A)) = 0 \text{ for every } A \in \mathcal{A}_o, \quad (2.37)$$

then it is called a supersymmetric state for the supersymmetry model determined by $\Psi$. If there exists a supersymmetric state, then the model is said to be unbroken supersymmetry. If there exists no such state, then the model is said to be broken supersymmetry.

The assignment of local fermion charges $\Psi$ will be called ‘nilpotent’, if its associated superderivation $\delta_\Psi$ satisfies the nilpotent condition (2.35). A complete characterization of nilpotent assignments of local fermion charges is given in [32].

Let us define

$$d_\Psi := \delta_\Psi^* \cdot \delta_\Psi + \delta_\Psi \cdot \delta_\Psi^* \text{ on } \mathcal{A}_o. \quad (2.38)$$

It is easy to see that

$$(d_\Psi(A^*))^* = d_\Psi(A) \text{ for every } A, B \in \mathcal{A}_o, \quad (2.39)$$

$$d_\Psi \cdot \gamma = \gamma \cdot d_\Psi \text{ on } \mathcal{A}_o, \quad (2.40)$$

and

$$d_\Psi(AB) = d_\Psi(A)B + Ad_\Psi(B) \text{ for every } A, B \in \mathcal{A}_o. \quad (2.41)$$

By (2.38) (2.35) (2.36) the following commutativity relations hold:

$$\delta_\Psi \cdot d_\Psi = d_\Psi \cdot \delta_\Psi, \delta_\Psi^* \cdot d_\Psi = d_\Psi \cdot \delta_\Psi^* \text{ on } \mathcal{A}_o. \quad (2.42)$$

The $*$-derivation $d_\Psi$ on $\mathcal{A}_o$ uniquely determines a finite-range interaction over the fermion lattice system. In fact the range of $d_\Psi$ is at most $2r - 1$. Thus there exists a unique strongly continuous one parameter group of $*$-automorphisms $\alpha_t^\Psi$ ($t \in \mathbb{R}$) on $\mathcal{A}$ whose generator is equal to $d_\Psi$ on $\mathcal{A}_o$. By $\alpha_t^\Psi$ ($t \in \mathbb{R}$) we denote the time evolution of the Nicolai model.

We use the following two theorems shown in [32].

**Theorem 2.1.** Let $\Psi$ be a uniformly bounded nilpotent finite-range assignment of local fermion charges on the fermion lattice system $\mathcal{A}$. Then there exists a strongly continuous one parameter group of $*$-automorphisms $\alpha_t^\Psi$ ($t \in \mathbb{R}$) on $\mathcal{A}$ whose pre-generator is given by the derivation $d_\Psi \equiv \delta_\Psi^* \cdot \delta_\Psi + \delta_\Psi \cdot \delta_\Psi^*$ defined on the local algebra $\mathcal{A}_o$. 
Theorem 2.2. Let $Ψ$ be a uniformly bounded nilpotent finite-range assignment of local fermion charges on the fermion lattice system $A$. Suppose that a state $ϕ$ on $A$ is a supersymmetric state for the supersymmetry model determined by $Ψ$. Then $ϕ$ is also a ground state (in the sense of Definition 5.3.18 of [7]) for the one-parameter group of $*$-automorphisms $α_{Ψ}^{t}$ $(t ∈ R)$. In particular, $ϕ$ is invariant under $α_{Ψ}^{t}$ $(t ∈ R)$.

From Theorem 2.1 the following crucial statement which we rely on follows.

Proposition 2.3. Let $δ_{Ψ}$ denote any nilpotent superderivation defined on $A$ generated by a uniformly bounded nilpotent finite-range assignment of local fermion charges $Ψ$ on the fermion lattice system. Suppose that an element $B ∈ A$ is annihilated by both superderivations $δ_{Ψ}$ and $δ_{Ψ}^{∗}$ associated to $Ψ$: $δ_{Ψ}(B) = 0, δ_{Ψ}^{∗}(B) = 0$. (2.43)

Then it is invariant under the time evolution $α_{Ψ}^{t}$ $(t ∈ R)$ associated to $Ψ$: $α_{Ψ}^{t}(B) = B$ for all $t ∈ R$. (2.44)

Proof. From the definition of $d_{Ψ}$ in (2.38) the assumption (2.43) yields $d_{Ψ}(B) = 0$. (2.45)

As $d_{Ψ}$ is a pre-generator for the strongly continuous one parameter group of $*$-automorphisms $α_{Ψ}^{t}$ $(t ∈ R)$ on $A$ by Theorem 2.1, the equation (2.44) follows from (2.45).

2.4 The Nicolai model

We shall define the supersymmetric fermion lattice model by Nicolai [37] as $C^*$-dynamical system based on the formulation given in the preceding subsection. We introduce the model on integer lattice $Z$ following the original work [37]. However, one can easily extend the Nicolai model to any dimensional integer lattice $Z^{ν}$, and the results which we are going to show are valid for any dimensional case with no essential change, see [7].

Take the assignment of local fermion charges over $Z$ as follows:

$$Ψ_{N}(\{2i - 1, 2i, 2i + 1\}) := a_{2i+1}a_{2i}^{∗}a_{2i-1} ∈ A(\{2i - 1, 2i, 2i + 1\}),$$

on $\{2i - 1, 2i, 2i + 1\}$ $(i ∈ Z)$, $Ψ_{N}(I) := 0$ on any other $I ∈ Z$. (2.46)

The above $Ψ_{N}$ has 2-periodicity by lattice translation and finite range of the length $r = 2$. The nilpotent condition (2.35) is satisfied by the superderivation $δ_{Ψ_{N}}$ associated to $Ψ_{N}$ because $Ψ_{N}(X_2)Ψ_{N}(X_1) = 0$ holds for any pair $X_1 ∈ Z$ and $X_2 ∈ Z$, see [32]. Thus $Ψ_{N}$ is a nilpotent finite-range assignment of local fermion charges over $Z$ and determines the superderivation $δ_{Ψ_{N}}$ on $A$. Let $d_{Ψ_{N}} := δ_{Ψ_{N}}^{∗}δ_{Ψ_{N}} + δ_{Ψ_{N}}δ_{Ψ_{N}}^{∗}$ on $A$. As in Theorem 2.1 we determine the supersymmetric fermion lattice model by $\{δ_{Ψ_{N}}, d_{Ψ_{N}}\}$. Let $α_{Ψ_{N}}^{t}$ $(t ∈ R)$ denote the time
evolution for the Nicolai model, i.e. the strongly continuous one parameter group of $^\ast$-automorphisms which is generated by the derivation $d_{\Psi_{\text{Nic}}} \equiv \delta_{\Psi_{\text{Nic}}}^\ast \cdot \delta_{\Psi_{\text{Nic}}} + \delta_{\Psi_{\text{Nic}}} \cdot \delta_{\Psi_{\text{Nic}}}^\ast$ defined on $\mathcal{A}_o$. We shall call this supersymmetric fermion lattice model the Nicolai model.

It is convenient to rewrite the Nicolai model in a format which is more common in physics literature. Let

$$Q_{\text{Nic}} := \sum_{X \in \mathbb{Z}} \Psi_{\text{Nic}}(X) = \sum_{i \in \mathbb{Z}} n_{2i} - n_{2i-1} n_{2i} - n_{2i} n_{2i+1} + n_{2i-1} n_{2i+1},$$

(2.47)

and

$$Q_{\text{Nic}}^\ast := \sum_{X \in \mathbb{Z}} \Psi_{\text{Nic}}^\ast(X) = \sum_{i \in \mathbb{Z}} - a_{2i}^\ast a_{2i+1}^\ast a_{2i+2}^\ast + a_{2i-1}^\ast a_{2i} a_{2i+1} a_{2i+2}^\ast$$

(2.48)

The Hamiltonian is defined as

$$H_{\text{Nic}} := \{Q_{\text{Nic}}, Q_{\text{Nic}}^\ast\}.$$

(2.49)

By direct computation we verify that

$$H_{\text{Nic}} = \sum_{i \in \mathbb{Z}} \left\{a_{2i}^\ast a_{2i-1} a_{2i} a_{2i+2} a_{2i+3}^\ast + a_{2i-1}^\ast a_{2i} a_{2i+3} a_{2i+2}^\ast + a_{2i}^\ast a_{2i+1} a_{2i+1}^\ast a_{2i+2} + a_{2i-1}^\ast a_{2i} a_{2i+1} a_{2i+1} a_{2i+2}^\ast \right\}.$$

(2.50)

For later sake we shall decompose the Hamiltonian into the classical term $H_{\text{classical}}$ and the hopping term $H_{\text{hop}}$ as

$$H_{\text{Nic}} = H_{\text{classical}} + H_{\text{hop}}$$

(2.51)

by setting

$$H_{\text{classical}} := \sum_{i \in \mathbb{Z}} n_{2i} - n_{2i-1} n_{2i} - n_{2i} n_{2i+1} + n_{2i-1} n_{2i+1},$$

(2.52)

and

$$H_{\text{hop}} := \sum_{i \in \mathbb{Z}} a_{2i}^\ast a_{2i-1} a_{2i+2} a_{2i+3}^\ast + a_{2i-1}^\ast a_{2i} a_{2i+3} a_{2i+2}^\ast.$$

(2.53)

Note that the notations $Q_{\text{Nic}}$, $Q_{\text{Nic}}^\ast$, and $H_{\text{Nic}}$ above are heuristic ones. Those will be realized as linear operators on some Hilbert space only after taking some appropriate representation. However, there is no delicate point here. By using those heuristic operators we can construct rigorously the conjugate pair of superderivations and the infinitesimal time generator:

$$\delta_{\Psi_{\text{Nic}}}(A) = \{Q_{\text{Nic}}, A\}_\gamma \quad \text{for every} \quad A \in \mathcal{A}_o,$$

(2.54)

and

$$\delta_{\Psi_{\text{Nic}}}^\ast(A) = \{Q_{\text{Nic}}^\ast, A\}_\gamma \quad \text{for every} \quad A \in \mathcal{A}_o,$$

(2.55)

and

$$d_{\Psi}(A) = [H_{\text{Nic}}, A] \quad \text{for every} \quad A \in \mathcal{A}_o.$$  

(2.56)
We will some obvious symmetries of the Nicolai model. First note that
\[ \gamma_{\theta}(\mathcal{H}_{\text{Nic}}) = \mathcal{H}_{\text{Nic}} \quad \forall \theta \in [0, 2\pi), \] (2.57)
and that
\[ \sigma_k(\mathcal{H}_{\text{Nic}}) = \mathcal{H}_{\text{Nic}} \quad \forall k \in 2\mathbb{Z}. \] (2.58)
Therefore the Nicola model has global $U(1)$-symmetry and $\mathbb{Z}_2$-translation symmetry in space (i.e. periodicity). From (5.9) and
\[ \rho(Q_{\text{Nic}}) = -Q_{\text{Nic}}^*, \quad \rho(Q_{\text{Nic}}^*) = -Q_{\text{Nic}}, \] (2.59)
we see the particle-hole invariance
\[ \rho(\mathcal{H}_{\text{Nic}}) = \mathcal{H}_{\text{Nic}}. \] (2.60)
Next we will see that the Nicolai model is an unbroken supersymmetry model. Let $|1\rangle_i$ and $|0\rangle_i$ denote the occupied state and the empty state of the spinless fermion at $i \in \mathbb{Z}$, respectively. The following relations hold for every $i \in \mathbb{Z}$:
\[ a_i|1\rangle_i = |0\rangle_i, \quad a_i|0\rangle_i = 0, \quad a_i^*|0\rangle_i = |1\rangle_i, \quad a_i|0\rangle_i = 0. \] (2.61)
We introduce the following translation invariant product vector over $\mathbb{Z}$:
\[ \eta_0 := \cdots \otimes |0\rangle \otimes |0\rangle_{-2} \otimes |0\rangle_{-1} \otimes |0\rangle_0 \otimes |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_0 \otimes |0\rangle_0 \otimes |0\rangle \cdots. \] (2.62)
This corresponds to the translation invariant Fock state $\psi_0$ determined by
\[ \psi_0(a_j^*a_j) = 0 \text{ for all } j \in \mathbb{Z}. \] (2.63)
Let
\[ \eta_1 := \cdots \otimes |1\rangle \otimes |1\rangle_{-2} \otimes |1\rangle_{-1} \otimes |1\rangle_0 \otimes |1\rangle_1 \otimes |1\rangle_2 \otimes |1\rangle_0 \otimes |1\rangle_0 \otimes |1\rangle \cdots, \] (2.64)
which corresponds to the fully occupied state $\psi_1$ determined by
\[ \psi_1(a_ja_j^*) = 0 \text{ for all } j \in \mathbb{Z}. \] (2.65)
For each $i \in \mathbb{Z}$
\[ 0 = a_{2i+1}a_{2i}^*a_{2i-1}\eta_0 = a_{2i+1}^*a_{2i}a_{2i-1}^*\eta_0, \]
\[ 0 = a_{2i+1}a_{2i}^*a_{2i-1}\eta_1 = a_{2i+1}^*a_{2i}a_{2i-1}^*\eta_1. \]
Hence we have
\[ 0 = Q_{\text{Nic}}\eta_0 = Q_{\text{Nic}}^*\eta_0, \]
and
\[ 0 = Q_{\text{Nic}}\eta_1 = Q_{\text{Nic}}^*\eta_1. \]
Thus both $\psi_0$ and $\psi_1$ are supersymmetric states. Later we will show more supersymmetric states of the Nicolai model.
3 Infinitely many local fermionic constants of motion

The purpose of this section is to systematically provide infinitely many local fermionic constants of motion for the Nicolai model.

3.1 Classical sequences that encode local fermionic constants of motion

In this subsection we introduce \{-1, +1\}-valued sequences that encode local fermionic constants of motion.

**Definition 3.1.** Let I be some interval of \( \mathbb{Z} \), i.e. \( I = [m, n] = \{ m, m+1, \cdots, n-1, n \} \) with \( m, n \in \mathbb{Z} \) \( (m < n) \). Let \( f \) be a \{-1, +1\}-valued function on I. If either

\[
f(2i - 1) = -1, \quad f(2i) = +1, \quad f(2i + 1) = -1, \tag{3.1}
\]

or

\[
f(2i - 1) = +1, \quad f(2i) = -1, \quad f(2i + 1) = +1, \tag{3.2}
\]

holds for some \( \{2i - 1, 2i, 2i + 1\} \subset I \ (i \in \mathbb{Z}) \), then \( f \) is said to be forbidden. Otherwise, if \( f \) does not include such subsequences \( (3.1), (3.2) \) anywhere in I, then it is said to be permitted. The set of all \{-1, +1\}-valued permitted sequences on I is denoted by \( \Xi_I \).

In the following we will consider those intervals whose edges are both even:

\[
I_{[2k, 2l]} \equiv [2k, 2k + 1, 2(k + 1), \cdots, 2(l - 1), 2l - 1, 2l], \quad k, l \in \mathbb{Z} \quad \text{such that} \quad k < l.
\tag{3.3}
\]

By definition \( I_{[2k, 2l]} \) \( (k, l \in \mathbb{Z}, k < l) \) of \( \mathbb{Z} \) has odd number of sites \( 2(l - k) + 1 \geq 3 \). We intend to find fermion operators on \( I_{[2k, 2l]} \) that are invariant under the time evolution. For this sake we introduce a subclass of permitted sequences on \( I_{[2k, 2l]} \) imposing some additional requirement upon the edges.

**Definition 3.2.** Let \( f \) be a \{-1, +1\}-valued permitted sequence on the interval \( I_{[2k, 2l]} \) with \( k, l \in \mathbb{Z} \) \( (k < l) \) of \( (3.3) \), namely \( f \in \Xi_{I_{[2k, 2l]}} \) as in Definition \( 3.1 \). Assume that \( f \) takes a constant on the left-end pair sites \( \{2k, 2k + 1\} \), and that \( f \) takes a constant on the right-end pair sites \( \{2l - 1, 2l\} \). Namely

\[
f(2k) = f(2k + 1) = +1 \quad \text{or} \quad f(2k) = f(2k + 1) = -1 \tag{3.4}
\]

and

\[
f(2l - 1) = f(2l) = +1 \quad \text{or} \quad f(2l - 1) = f(2l) = -1 \tag{3.5}
\]

are assumed. The set of all \{-1, +1\}-valued permitted sequences on \( I_{[2k, 2l]} \) satisfying the above marginal conditions on both edges is denoted by \( \hat{\Xi}_{k,l} \). The union of \( \hat{\Xi}_{k,l} \) over all \( k, l \in \mathbb{Z} \) \( (k < l) \) is denoted by \( \hat{\Xi} \). Each \( f \in \hat{\Xi} \) is called a local sequence of conservation for the Nicolai model.
Remark 3.3. The requirements (3.4) (3.5) on the edges of $I_{[2k,2l]}$ are essential to make conservations for the Nicolai model. Those can not be omitted or altered unlike the boundary conditions (like free, open, periodic, and so on) in statistical mechanics.

Remark 3.4. By crude estimate we see that the number of local sequences of conservation in $\hat{\Xi}_{k,l}$ is roughly $(2^{n-2})^{(l-k)} = 3^{l-k} = 3^{m/2}$, where $m = 2(l - k)$ denotes approximately the size of the system (i.e. the number of sites in $I_{[2k,2l]}$).

For some later purpose it is convenient to introduce the following subclasses of $\hat{\Xi}$.

**Definition 3.5.** For each $k, l \in \mathbb{Z}$ ($k < l$) let $r_{[2k,2l]}^+ \in \hat{\Xi}_{k,l}$ and $r_{[2k,2l]}^- \in \hat{\Xi}_{k,l}$ denote the constants over $I_{[2k,2l]}$ taking $+1$ and $-1$, respectively:

$$r_{[2k,2l]}^+(i) = +1 \forall i \in I_{[2k,2l]}, \quad r_{[2k,2l]}^-(i) = -1 \forall i \in I_{[2k,2l]}.$$  

(3.6)

These are called the $\pm$-characters supported on the segment $I_{[2k,2l]}$. The set $\{r_{[2k,2l]}^+\}$ over all $k, l \in \mathbb{Z}$ ($k < l$) will be denoted as $\hat{\Xi}_+^{\text{const.}}$, and the set $\{r_{[2k,2l]}^-\}$ over all $k, l \in \mathbb{Z}$ ($k < l$) will be denoted as $\hat{\Xi}_-^{\text{const.}}$. Set $\hat{\Xi}^{\text{const.}} := \hat{\Xi}_+^{\text{const.}} \cup \hat{\Xi}_-^{\text{const.}}$.

### 3.2 Assignment of local fermion operators

We shall give a rule to assign a local fermion operator for every local sequence of conservation in $\hat{\Xi}$ of Definition 3.2.

**Definition 3.6.** For each $i \in \mathbb{Z}$ let $\zeta_i$ denote the assignment from $\{-1, +1\}$ into the fermion annihilation-creation operators at $i$ as

$$\zeta_i(-1) := a_i, \quad \zeta_i(+1) := a_i^*.$$  

(3.7)

Take any pair of integers $k, l \in \mathbb{Z}$ such that $k < l$. For each $f \in \hat{\Xi}_{k,l}$, set

$$\mathcal{Q}(f) := \prod_{i=2k}^{2l} \zeta_i(f(i))$$

$$\equiv \zeta_{2k}(f(2k))\zeta_{2k+1}(f(2k + 1)) \cdots \zeta_{2l-1}(f(2l - 1))\zeta_{2l}(f(2l)) \in \mathcal{A}(I_{[2k,2l]}),$$  

(3.8)

where the multiplication is taken in the increasing order as above. The formulas (3.8) for all $k, l \in \mathbb{Z}$ ($k < l$) yield a unique assignment $\mathcal{Q}$ from $\hat{\Xi}$ into $\mathcal{A}_{\text{const.}}$.

By Definition 3.6 for $k, l \in \mathbb{Z}$ ($k < l$)

$$\mathcal{Q}(r_{[2k,2l]}^+) := a_{2k} a_{2k+1}^* \cdots a_{2l-1}^* a_{2l}^* \in \mathcal{A}(I_{[2k,2l]}),$$

$$\mathcal{Q}(r_{[2k,2l]}^-) := a_{2k} a_{2k+1} \cdots a_{2l-1} a_{2l} \in \mathcal{A}(I_{[2k,2l]}).$$  

(3.9)
Examples

We will give concrete examples for local sequences of conservation of Definition 3.2 and their associated local fermion operators of Definition 3.6. First we see that $\hat{\Xi}_{0,1}$ on $I_{[0,2]} \equiv [0, 1, 2]$ consists of two $\pm$-characters only.

| $\hat{\Xi}_{0,1}$ | 0 | 1 | 2 |
|-------------------|---|---|---|
| $r_{[0,2]}^{-}$   | -1| -1| -1|
| $r_{[0,2]}^{+}$   | +1| +1| +1|

By (3.8) of Definition 3.6 the corresponding local fermion operators are

$$
\mathcal{D}(r_{[0,2]}^{-}) = a_0 a_1 a_2 \in \mathcal{A}(I_{[0,2]})_{-}, \\
\mathcal{D}(r_{[0,2]}^{+}) = a_0^* a_1^* a_2^* \in \mathcal{A}(I_{[0,2]})_{-}.
$$

(3.10)

Next we consider the segment $I_{[0,4]} \equiv [0, 1, 2, 3, 4]$ by setting $k = 0$ and $l = 2$. The space $\hat{\Xi}_{0,2}$ on $I_{[0,4]}$ consists of the following five sequences of conservation:

| $\hat{\Xi}_{0,2}$ | 0 | 1 | 2 | 3 | 4 |
|-------------------|---|---|---|---|---|
| $r_{[0,4]}$       | -1| -1| -1| -1| -1|
| $u_{[0,4]}^{i}$   | -1| -1| -1| +1| +1|
| $u_{[0,4]}^{ii}$  | -1| -1| +1| +1| +1|
| $v_{[0,4]}^{i}$   | +1| +1| +1| -1| -1|
| $v_{[0,4]}^{ii}$  | +1| +1| -1| -1| -1|
| $r_{[0,4]}^{+}$   | +1| +1| +1| +1| +1|

Note that

$$
r_{[0,4]}^{-} = -r_{[0,4]}^{+}, \\
u_{[0,4]}^{i} = -v_{[0,4]}^{i}, \\
u_{[0,4]}^{ii} = -v_{[0,4]}^{ii}.
$$

(3.11)

By (3.8) of Definition 3.6 we have

$$
\mathcal{D}(r_{[0,4]}^{-}) = a_0 a_1 a_2 a_3 a_4 \in \mathcal{A}(I_{[0,4]})_{-}, \\
\mathcal{D}(u_{[0,4]}^{i}) = a_0 a_1 a_2 a_3^* a_4^* \in \mathcal{A}(I_{[0,4]})_{-}, \\
\mathcal{D}(u_{[0,4]}^{ii}) = a_0 a_1 a_2^* a_3^* a_4^* \in \mathcal{A}(I_{[0,4]})_{-}, \\
\mathcal{D}(v_{[0,4]}^{i}) = a_0^* a_1^* a_2 a_3 a_4 \in \mathcal{A}(I_{[0,4]})_{-}, \\
\mathcal{D}(v_{[0,4]}^{ii}) = a_0^* a_1^* a_2^* a_3 a_4 \in \mathcal{A}(I_{[0,4]})_{-}, \\
\mathcal{D}(r_{[0,4]}^{+}) = a_0^* a_1^* a_2 a_3^* a_4^* \in \mathcal{A}(I_{[0,4]})_{-}.
$$

(3.12)

We then consider the segment $I_{[0,6]} \equiv [0, 1, 2, 3, 4, 5, 6]$ taking $k = 0$ and $l = 3$. By definition it consists of $5 + 4 + 4 + 5 = 18$ sequences of conservation:
Note that $s_{[0,6]} \equiv r_{[0,6]}^-$ and $t_{[0,6]}^* \equiv r_{[0,6]}^+$ and that
\[
s_{[0,6]}^c = -t_{[0,6]}^*, \quad s_{[0,6]}^k = -t_{[0,6]}^k, \quad \forall k \in \{i, ii, iii, iv\}
\]
\[
u_{[0,6]}^k = -v_{[0,6]}^k, \quad \forall k \in \{i, ii, iii, iv\}. \quad (3.13)
\]
According to the rule we obtain the following list of 18 fermion operators associated
For every \( q \in \mathbb{Q} \), the local fermion operator \( \mathcal{D}f \in \mathcal{A}_{\omega_-} \) and its adjoint \( \mathcal{D}(f)^\ast \in \mathcal{A}_{\omega_-} \) are nilpotent:

\[
\mathcal{D}(f)^2 = 0 = \mathcal{D}(f)^\ast 2.
\]
For each \( f, g \in \hat{\Xi} \)
\[
\{ \mathcal{Q}(f), \mathcal{Q}(g) \} = 0 \quad (3.19)
\]
is satisfied unless the support of \( f \) and that of \( g \) have a non-empty intersection \( J \) and \( f = -g \) holds on \( J \).

**Proof.** Those can be verified by noting the form of \( f \in \hat{\Xi} \) as in Definition 3.2, the form of \( \mathcal{Q}(f) \) as in (3.8) of Definition 3.6 and basic identities of fermion operators: \( a_i a_i = a_i^* a_i^* = 0 \), \( a_i^* a_i = n_i \), and \( a_i a_i^* = 1 - n_i \) for any \( i \in \mathbb{Z} \).

From Proposition 3.8 we see that for any \( f \in \hat{\Xi} \) the pair of local fermion operators \( \{ \mathcal{Q}(f), \mathcal{Q}(f)^* \} \) satisfies the relation of \( \mathcal{N} = 2 \) supersymmetry as in §2.1. We will verify that those are hidden symmetries of the Nicolai model. Let us introduce algebras generated by theses local fermion operators.

**Definition 3.9.** Let \( C \) denote the \(*\)-subalgebra in \( \mathcal{A}_0 \) finitely generated by \( \{ \mathcal{Q}(f) \in \mathcal{A}_0 \mid f \in \hat{\Xi} \} \). For every \( k, l \in \mathbb{Z} \) such that \( k < l \) the \(*\)-subalgebra generated by \( \{ \mathcal{Q}(f) \in \mathcal{A}_0 \mid f \in \hat{\Xi}_{k', l}, k' < k \leq l \} \) is denoted by \( C(k, l) \).

By definition \( C(k, l) \subset \mathcal{A}([k, 2l]) \), and \( C(k, l) \supset C(p, q) \) if \( k \leq p < q \leq l \).

**Theorem 3.10.** Let \( \alpha_t^{\Psi_{\text{Nic}}} (t \in \mathbb{R}) \) denote the time evolution for the Nicolai model given in §2.4. Then for every \( B \in C \)
\[
\alpha_t^{\Psi_{\text{Nic}}} (B) = B \quad \text{for all } t \in \mathbb{R}. \quad (3.20)
\]
In particular, for every \( f \in \hat{\Xi} \),
\[
\alpha_t^{\Psi_{\text{Nic}}} (\mathcal{Q}(f)) = \mathcal{Q}(f), \quad \alpha_t^{\Psi_{\text{Nic}}} (\mathcal{Q}(f)^*) = \mathcal{Q}(f)^* \quad \text{for all } t \in \mathbb{R}. \quad (3.21)
\]

**Proof.** By Proposition 2.3 it suffices to show that
\[
\delta_{\Psi_{\text{Nic}}} (B) = 0, \quad \delta_{\Psi_{\text{Nic}}}^* (B) = 0 \quad \text{for every } B \in C. \quad (3.22)
\]
Furthermore, by Definition 3.9 and the graded Leibniz rule of superderivations (2.32) it is enough to show that
\[
\delta_{\Psi_{\text{Nic}}} (\mathcal{Q}(f)) = 0, \quad \delta_{\Psi_{\text{Nic}}}^* (\mathcal{Q}(f)) = 0 \quad \text{for every } f \in \hat{\Xi}. \quad (3.23)
\]
From the formula of \( \Psi_{\text{Nic}} \) as in (2.46) and Definitions 3.1 3.2 3.6 by using the canonical anticommutation relations we see that for all \( X \in \mathbb{Z} \)
\[
\mathcal{Q}(f) \Psi_{\text{Nic}}(X) = 0 = \Psi_{\text{Nic}}(X) \mathcal{Q}(f), \quad \mathcal{Q}(f)^* \Psi_{\text{Nic}}(X) = 0 = \Psi_{\text{Nic}}^*(X) \mathcal{Q}(f). \quad (3.24)
\]
These yield (3.23).

**Theorem 3.10** can be rewritten in a more heuristic manner:
\[
\{ Q_{\text{Nic}}, \mathcal{Q}(f) \} = 0 = \{ Q_{\text{Nic}}^*, \mathcal{Q}(f) \} \quad \text{for every } f \in \hat{\Xi}, \quad (3.25)
\]
and
\[
[H_{\text{Nic}}, \mathcal{Q}(f)] = 0 \quad \text{for every } f \in \hat{\Xi}. \quad (3.26)
\]
We now provide some relevant terminologies.

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**Definition 3.11.** For each local sequence of conservation \( f \in \hat{\Xi} \), \( \mathcal{Q}(f) \) is called the local fermionic constant of motion associated to \( f \). The pair \( \{ \mathcal{Q}(f), \mathcal{Q}(f)^* \} \) is called the hidden fermion charge associated to \( f \). The \( * \)-algebra \( \mathcal{C} \) in Definition 3.9 is called the algebra of the constants of motion for the Nicolai model, and \( \mathcal{C}(k,l) \) with \( k, l \in \mathbb{Z} \) \((k < l)\) is called the algebra of the constants of motion within the segment \( \mathcal{I}_{[k,l]} \).

**Remark 3.12.** The subalgebras \( \mathcal{C} \) and \( \mathcal{C}(k,l) \) for any \( k, l \in \mathbb{Z} \) \((k < l)\) include many observables (i.e. self-adjoint operators). Some of them are \( U(1) \)-gauge invariant observables.

### 3.4 Quantum integrability

We now consider quantum integrability for the Nicolai model. In \([9]\) a new definition of “quantum integrability” intended for characterization of quantum many-body dynamics is proposed. Our argument below is based upon this definition. The level of rigor in this subsection follows \([9]\); it is not like the other parts of this work.

The definition of quantum integrability of \([9]\) consists of four requirements which is numbered from 1 to 4. The original definition is designed for bosonic (usual) symmetries or bosonic constants of motion. We also refer to \([60]\) where a set of “bosonic conservation laws” is discussed for some particular models. First we need to replace the commutator in the mentioned references by the anti-commutator as we deal with fermionic symmetries. In the following we check all the requirements of Caux-Mossel for the Nicolai model.

The set of nilpotent equations (3.18) and the anti-commutation relations (3.19) given in Proposition 3.8 will correspond to the first half of Requirement 1 of \([9]\). According to (3.26) in Theorem 3.10 any of \( \{ \mathcal{Q}(f) \in \mathcal{A}_o \mid f \in \hat{\Xi} \} \) commutes with the total Hamiltonian \( H_{\text{Nic}} \). This corresponds to the second half of Requirement 1. So we have verified Requirement 1 for our set of local constants of motion satisfied by the set of local fermion operators \( \{ \mathcal{Q}(f) \in \mathcal{A}_o \mid f \in \hat{\Xi} \} \).

From Theorem 3.10 and Remark 3.14 one sees that the number of the set of local constants of motion \( \{ \mathcal{Q}(f) \in \mathcal{A}_o \mid f \in \hat{\Xi} \} \) of the Nicolai model increases exponentially with respect to the volume of subsystems. However, counting independent operators needs some care. In fact the operators in \( \{ \mathcal{Q}(f) \in \mathcal{A}_o \mid f \in \hat{\Xi} \} \) are not algebraically independent. By using the CAR relations, one can verify that the operators in \( \{ \mathcal{Q}(f) \mid f \in \hat{\Xi}_{0,1} \} \) are algebraically independent (up to \( * \)-operation), and that the operators in \( \bigcup \{ \mathcal{Q}(f) \mid f \in \hat{\Xi}_{0,1} \cup \hat{\Xi}_{0,2} \} \) are algebraically independent as well. However, the operators in \( \bigcup \{ \mathcal{Q}(f) \mid f \in \hat{\Xi}_{0,1} \cup \hat{\Xi}_{0,2} \cup \hat{\Xi}_{0,3} \} \) are not algebraically independent any more. More detailed algebraic structure will be studied elsewhere. In any case, we can find new fermion operators outside of the specified region. Hence the Requirement 2 is satisfied in much the same reason as the free theories (described in Sec.5 of \([9]\)). We can see that the cardinality of \( \{ \mathcal{Q}(f) \in \mathcal{A}_o \mid f \in \hat{\Xi} \} \) is unbounded. Thus Requirement 3 is satisfied.

Requirement 4 is rather involved. So we refer the readers to the original paper. We will only state main points. Any operator in \( \{ \mathcal{Q}(f) \in \mathcal{A}_o \mid f \in \hat{\Xi} \} \) is a monomial of fermion creation and annihilation operators on the lattice. Hence in the language of Caux-Mossel it has the constant character of the preferred basis. Thus
Requirement 4 is satisfied. We shall state the conclusion.

**Proposition 3.13.** The Nicolai model belongs to the constant class of quantum integrability in the sense of Caux-Mossel.

**Remark 3.14.** Proposition 3.13 holds for the Nicolai model on any dimensional integer lattice as we will see in §7.

**Remark 3.15.** The definition of quantum integrability by Caux-Mossel does not require the complete set of constants of motion. Of course, however, the number of independent local constants of motion is essential, see [12, 43, 53]. Presently, Requirement 3 demands merely “unboundedness” of the number of independent local constants. If a homogeneous model has one strictly local constant of motion, then it automatically satisfies the criteria for the constant class of quantum integrability as we have seen essentially. The non-interacting free models as pointed out in [9], the Kitaev model [28], and the Nicolai model all belong to the same quantum-integrable class of [9]. We consider that more refined characterization would be desirable.

## 4 Highly degenerated classical supersymmetric ground states

In this section we give all classical supersymmetric ground states of the Nicolai model. Actually, there are many other supersymmetric ground states which are non-classical [27].

### 4.1 Classical configurations

As in §2.4 let $|0\rangle_i$ and $|1\rangle_i$ denote the empty-state vector and the occupied-state vector of the spinless fermion at $i \in \mathbb{Z}$, respectively. With $\{|0\rangle_i, |1\rangle_i; i \in \mathbb{Z}\}$ we can provide a concrete description of the Fock state representation as before.

**Definition 4.1.** Let $g(n)$ denote an arbitrary $\{0,1\}$-valued function over $\mathbb{Z}$. It is called a classical configuration over $\mathbb{Z}$. For any classical configuration $g(n)$ define

$$|g(n)_{n \in \mathbb{Z}}\rangle := \cdots \otimes |g(i-1)\rangle_{i-1} \otimes |g(i)\rangle_i \otimes |g(i+1)\rangle_{i+1} \otimes \cdots$$

(4.1)

This infinite product vector determines a state $\psi_{g(n)}$ on the fermion system $\mathcal{A}$ which will be called the classical state associated to the configuration $g(n)$ over $\mathbb{Z}$. Vice versa, any classical state on the fermion system $\mathcal{A}$ is given by the infinite product vector of (4.1) for some classical configuration $g(n)$ over $\mathbb{Z}$.

To each classical configuration over $\mathbb{Z}$ we assign an operator by the following rule.

**Definition 4.2.** For each $i \in \mathbb{Z}$ let $\hat{\kappa}_i$ denote the map from $\{0,1\}$ into $\mathcal{A}(\{i\})$ given as

$$\hat{\kappa}_i(0) := 1, \quad \hat{\kappa}_i(1) := a_i^*.$$  

(4.2)
For each classical configuration \( g(n) \) over \( \mathbb{Z} \) define the infinite-product of fermion field operators:

\[
\hat{O}(g) := \prod_{i \in \mathbb{Z}} \hat{\kappa}_i (g(i)) = \cdots \hat{\kappa}_{i-1} (g(i-1)) \hat{\kappa}_i (g(i)) \hat{\kappa}_{i+1} (g(i+1)) \cdots ,
\]

(4.3)

where the multiplication is taken in the increasing order. If \( g(n) \) has a compact support, then

\[
\hat{O}(g) \in \mathcal{A}_c. \quad (4.4)
\]

Otherwise \( \hat{O}(g) \) denotes a formal operator which is out of \( \mathcal{A} \).

One can naturally relate Definition 4.1 (product vectors) and Definition 4.2 (product operators) via the Fock representation.

**Proposition 4.3.** Let \( \eta_0 \) denote the Fock vector (no-particle wave function) given in (2.62). For any classical configuration \( g(n) \) over \( \mathbb{Z} \), the following identity holds:

\[
\hat{O}(g) \eta_0 = |g(n)_{n \in \mathbb{Z}}\rangle. \quad (4.5)
\]

**Proof.** This directly follows from Definition 4.1 and Definition 4.2 by noting (2.61). \( \square \)

**Remark 4.4.** Even when \( g(n) \) does not have a compact support, the identity (4.5) of Proposition 4.3 is valid. For illustration, take the constant \( \iota(n) := 1 \ \forall \ n \in \mathbb{Z} \) for the classical configuration over \( \mathbb{Z} \). By definition the support of \( \iota \) is non compact. Nevertheless, we have

\[
\hat{O}(\iota) \eta_0 = \cdots \a_2^* a_1^* a_0^* \cdots (\cdots \otimes |0\rangle_{-2} \otimes |0\rangle_{-1} \otimes |0\rangle_0 \otimes |0\rangle_1 \otimes |0\rangle_2 \cdots )
\]

\[
= \cdots a_2^* |0\rangle_{-2} \otimes a_1^* |0\rangle_{-1} \otimes a_0^* |0\rangle_0 \otimes a_1^* |0\rangle_1 \otimes a_2^* |0\rangle_2 \otimes \cdots 
\]

\[
= \cdots \otimes |1\rangle_{-2} \otimes |1\rangle_{-1} \otimes |1\rangle_0 \otimes |1\rangle_1 \otimes |1\rangle_2 \otimes \cdots \equiv \eta_1,
\]

where \( \eta_1 \) denotes the fully occupied wave function over \( \mathbb{Z} \) given in (2.64).

**Remark 4.5.** In Proposition 4.3 the Fock state is used. Actually, however, it is possible to choose any other classical state as the background. For example, we may take the fully occupied state in place of the Fock (no-particle) state. Then the formula (4.2) of Definition 4.2 is to be replaced by

\[
\hat{\kappa}_i(0) := a_i, \quad \hat{\kappa}_i(1) := 1 \quad \text{for each } i \in \mathbb{Z},
\]

while the operator formula \( \hat{O}(g) \) in (4.3) is same as before using the above new \( \hat{\kappa}_i \). The identity (4.5) of Proposition 4.3 should be changed to

\[
\hat{O}(g) \eta_1 = |g(n)_{n \in \mathbb{Z}}\rangle.
\]
4.2 Classical supersymmetric ground states

We introduce the following special class of classical configurations.

**Definition 4.6.** Take any three-site subset \( \{2i - 1, 2i, 2i + 1\} \) centered at an even site \( 2i (i \in \mathbb{Z}) \). Among 8 configurations (\( \{0, 1\} \)-valued functions) on \( \{2i - 1, 2i, 2i + 1\} \), “0,1,0” and “1,0,1” are called forbidden triplets. If a classical configuration \( g(n) (n \in \mathbb{Z}) \) does not include such forbidden triplets over \( \mathbb{Z} \), then it is called a ground-state configuration for the Nicolai model over \( \mathbb{Z} \). The set of all ground-state configurations for the Nicolai model over \( \mathbb{Z} \) is denoted by \( \Upsilon \). The set of all ground-state configurations for the Nicolai model whose support is included in a finite region is denoted by \( \Upsilon_I \). The set of all ground-state configurations for the Nicolai model whose support is included in a finite region \( I \subseteq \mathbb{Z} \) is denoted by \( \Upsilon_I \).

We can classify all classical supersymmetric ground states by using Definition 4.6.

**Theorem 4.7.** A classical state on the fermion lattice system \( \mathcal{A} \) is a supersymmetric ground state of the Nicolai model if and only if its associated configuration \( g(n) \) over \( \mathbb{Z} \) is a ground-state configuration as defined in Definition 4.6 (i.e. \( g(n) \in \Upsilon \)). Every supersymmetric ground state on \( \mathcal{A} \) is invariant under the time evolution of the Nicolai model.

**Proof.** First we shall see the action of the local fermion charges of the Nicolai model upon classical states (via the GNS representation for the Fock state). As we need to consider non-trivial local fermion charges only, let us take \( \Psi_{\text{Nie}}(\{2i - 1, 2i, 2i + 1\}) \) \( (i \in \mathbb{Z}) \) as in (2.46). If the classical configuration \( g(n) \) over \( \mathbb{Z} \) satisfies \( g(2i - 1) = 1, g(2i) = 0, g(2i + 1) = 1 \), namely there includes the forbidden “1,0,1” on \( \{2i - 1, 2i, 2i + 1\} \), then

\[
\Psi_{\text{Nie}}(\{2i - 1, 2i, 2i + 1\})|g(n)_{n \in \mathbb{Z}}\rangle = \cdots \otimes |g(2i - 3)\rangle_{2i-3} \otimes |g(2i - 2)\rangle_{2i-2} \otimes |0\rangle_{2i-1} \otimes |1\rangle_{2i} \otimes |0\rangle_{2i+1} \otimes |g(2i + 2)\rangle_{2i+2} \otimes \cdots ,
\]

(4.6)

where any entry on the complement of \( \{2i - 1, 2i, 2i + 1\} \) in \( \mathbb{Z} \) is unchanged. For any other \( g(n) \) the corresponding classical vector \( |g(n)_{n \in \mathbb{Z}}\rangle \) is always deleted by \( \Psi_{\text{Nie}}(\{2i - 1, 2i, 2i + 1\}) \):

\[
\Psi_{\text{Nie}}(\{2i - 1, 2i, 2i + 1\})|g(n)_{n \in \mathbb{Z}}\rangle = 0.
\]

(4.7)

Similarly, consider the action of \( \Psi_{\text{Nie}}(\{2i - 1, 2i, 2i + 1\})^* \). If the classical configuration \( g(n) \) over \( \mathbb{Z} \) satisfies \( g(2i - 1) = 0, g(2i) = 1, g(2i + 1) = 0 \), namely there includes the forbidden “0,1,0” on \( \{2i - 1, 2i, 2i + 1\} \), then

\[
\Psi_{\text{Nie}}(\{2i - 1, 2i, 2i + 1\})^*|g(n)_{n \in \mathbb{Z}}\rangle = \cdots \otimes |g(2i - 3)\rangle_{2i-3} \otimes |g(2i - 2)\rangle_{2i-2} \otimes |1\rangle_{2i-1} \otimes |0\rangle_{2i} \otimes |1\rangle_{2i+1} \otimes |g(2i + 2)\rangle_{2i+2} \otimes \cdots ,
\]

(4.8)

where any entry on the complement of \( \{2i - 1, 2i, 2i + 1\} \) in \( \mathbb{Z} \) is unchanged. For any other \( g(n) \) the corresponding vector \( |g(n)_{n \in \mathbb{Z}}\rangle \) is always deleted:

\[
\Psi_{\text{Nie}}(\{2i - 1, 2i, 2i + 1\})^*|g(n)_{n \in \mathbb{Z}}\rangle = 0.
\]

(4.9)
The above relations (4.6) (4.7) (4.8) (4.9) tell that all local operators $\Psi_{\text{Nic}}$ and $\Psi_{\text{Nic}}^*$ given in (2.46) delete any classical vector $|g(n)_{n \in \mathbb{Z}}\rangle$ if $g(n)$ is a ground-state configuration. Now we have shown the if part of the statement.

We will show the only if part of the statement. Assume that a classical supersymmetric state $\psi_{g(n)}$ is given, where $g(n)$ is its associated configuration over $\mathbb{Z}$. From the assumption that $\psi_{g(n)}$ is supersymmetric, both $Q_{\text{Nic}}|g(n)_{n \in \mathbb{Z}}\rangle = 0$ and $Q_{\text{Nic}}^*|g(n)_{n \in \mathbb{Z}}\rangle = 0$ hold, where the supercharge operator $Q_{\text{Nic}}$ and its conjugate operator $Q_{\text{Nic}}^*$ are given in (2.47) (2.48). Note that the existence of such supercharge operators on the GNS Hilbert space (now the Fock space) for any (not necessarily classical) supersymmetric is guaranteed, see [32] for the detail. From (4.6) (4.7) the identity $Q_{\text{Nic}}|g(n)_{n \in \mathbb{Z}}\rangle = 0$ implies that $\Psi_{\text{Nic}}(I)|g(n)_{n \in \mathbb{Z}}\rangle = 0$ for all $I \in \mathbb{Z}$, since there is no cancellation among the actions of the local fermion charges $\Psi_{\text{Nic}}(I)$ for different $I \in \mathbb{Z}$ upon $|g(n)_{n \in \mathbb{Z}}\rangle$. Similarly, from (4.8) (4.9) the identity $Q_{\text{Nic}}^*|g(n)_{n \in \mathbb{Z}}\rangle = 0$ implies that $\Psi_{\text{Nic}}^*(I)|g(n)_{n \in \mathbb{Z}}\rangle = 0$ for all $I \in \mathbb{Z}$. Thus $|g(n)_{n \in \mathbb{Z}}\rangle$ should be annihilated by both $\Psi_{\text{Nic}}(I)$ and $\Psi_{\text{Nic}}^*(I)$ for any $I \in \mathbb{Z}$. This implies that $g(n)$ should be a ground-state configuration with no forbidden triplet included.

As any (both classical and non-classical) supersymmetric state is a ground state by Theorem 2.2, it is invariant under the time evolution. \hfill \qed

Remark 4.8. Let us mention an experimental work [49] on many-body localization for the 1D fermionic Aubry-André model [3]. Consider the following alternating configurations on $\mathbb{Z}$.

$$g_{\text{even}}(n) := \begin{cases} 1 & n \in \mathbb{Z} \text{ is even} \\ 0 & n \in \mathbb{Z} \text{ is odd} \end{cases}, \quad g_{\text{odd}}(n) := \begin{cases} 0 & n \in \mathbb{Z} \text{ is even} \\ 1 & n \in \mathbb{Z} \text{ is odd}. \end{cases}$$

Suppose that we let them evolve under some quantum time evolutions. For the Nicolai model both $g_{\text{even}}(n)$ and $g_{\text{odd}}(n)$ persist eternally under the time evolution since both $g_{\text{even}}(n)$ and $g_{\text{odd}}(n)$ are ground-state configurations of the Nicolai model.

On the other hand, for the 1D fermionic Aubry-André model these alternating states keep their form quite long time but not eternally under its time evolution. In passing the following configurations of the particle density $1/2$ are fixed under the time evolution of the Nicolai model:

$$g_{\text{left}}(n) := \begin{cases} 1 & n \in -\mathbb{N} \\ 0 & n \in \{0\} \cup \mathbb{N} \end{cases}, \quad g_{\text{right}}(n) := \begin{cases} 0 & n \in -\mathbb{N} \\ 1 & n \in \{0\} \cup \mathbb{N}. \end{cases}$$

4.3 Absence of fully many body localization

As shown in the preceding subsection, all classical ground states of the Nicolai model can be expressed by certain configurations (binary codes) satisfying a simple rule stated in Definition 4.6. However, the whole spectrum of the Nicolai is not available presently. It should be noted that the many-body localization which we now discuss is only at zero temperature.

Then is it possible to describe complete information of the Hamiltonian in terms of binary codes? The answer to this naïve question is no. We will show that there is no complete set of l-bits by which the total Hamiltonian can be written as a sum of classical short-range interactions. Here l-bits are generalization of binary codes
Hence the criterion for the fully many-body localization \[22\] \[33\] is not satisfied by the Nicolai model.

We need some preparation. We now provide another characterization of the classical supersymmetric ground states of the Nicolai model stated in Theorem \[4.7\].

For this aim we recall the formula \(H_{\text{Nic}} = H_{\text{classical}} + H_{\text{hop}}\) \[2.51\], where \(H_{\text{Nic}}\) is the total Hamiltonian given explicitly in \[2.50\], \(H_{\text{classical}}\) is the classical term given in \[2.52\], and \(H_{\text{hop}}\) is the hopping term given in \[2.53\]. We consider that a new classical spin lattice model (imbedded in the fermion lattice system) is determined by \(H_{\text{classical}}\).

**Proposition 4.9.** The set of all classical supersymmetric ground states for the Nicolai model over \(\mathbb{Z}\) is identical to the set of all ground states for the classical spin model over \(\mathbb{Z}\) corresponding to the classical part of the Nicolai model:

\[H_{\text{classical}} = \sum_{i \in \mathbb{Z}} n_{2i} - n_{2i-1}n_{2i} - n_{2i}n_{2i+1} + n_{2i-1}n_{2i+1}.\]

**Proof.** Take any three-site subset \(\{2k-1, 2k, 2k+1\}\) centered at an even site \(2k\) \((k \in \mathbb{Z})\). There are eight classical configurations on \(\{2k-1, 2k, 2k+1\}\). The local interaction within \(\{2k-1, 2k, 2k+1\}\) is \(m_{2k} := n_{2k} - n_{2k-1}n_{2k} - n_{2k}n_{2k+1} + n_{2k-1}n_{2k+1}\). The operator \(m_{2k}\) takes eigenvalue +1 upon the two forbidden triplets “0, 1, 0” and “1, 0, 1” on \(\{2k-1, 2k, 2k+1\}\), while it takes 0 on the other six classical configurations on \(\{2k-1, 2k, 2k+1\}\). As \(H_{\text{classical}}\) is the summation of these positive operators \(m_{2k}\), it is positive. \(H_{\text{classical}}\) is 0 on any classical configuration that does not include the forbidden triplets “0, 1, 0” and “1, 0, 1” anywhere over \(\mathbb{Z}\), while it is strictly positive on any other classical configuration. Thus \(H_{\text{classical}}\) takes its minimum value 0 only on the ground-state configurations defined in Definition \[1.6\]. By this combined with Theorem \[4.7\] we obtain the equivalence as stated.

As noted previously, \(H_{\text{Nic}}\) has non-classical (entangled) ground states in addition to the classical (product) ground states given in Theorem \[4.7\]. So Proposition \[4.9\] tells that \(H_{\text{classical}}\) has only partial information of the total Hamiltonian \(H_{\text{Nic}}\) at zero temperature. In fact we can verify that the Nicolai model does not satisfy the criterion of fully many-body localization proposed in \[22\] \[50\]. As the notion of l-bits essential for this criterion is defined by language of theoretical physics in those original works, we shall provide a rather heuristic proof from the standard of rigorous mathematics.

**Proposition 4.10.** The Nicolai model does not have a complete set of l-bits by which its Hamiltonian can be written as a summation formula.

**Proof.** All the classical ground states of the Nicolai model given in Theorem \[4.7\] are product states with respect to the set of complete l-bits \(\{n_i; i \in \mathbb{Z}\}\). (These l-bits coincide with the p-bits which are canonically determined.) If we transform them by some automorphism \(\alpha\), we get a new set of complete l-bits \(\{n'_i := \alpha(n_i); i \in \mathbb{Z}\}\). Then the ground states of Theorem \[4.7\] are not product type with respect to this new set of complete l-bits. Therefore \(\{n_i; i \in \mathbb{Z}\}\) is the unique choice of the set of complete l-bits. Note that \(H_{\text{classical}}\) is diagonalized with respect to \(\{n_i; i \in \mathbb{Z}\}\), however, the hopping term \(H_{\text{hop}}\) is not diagonalized with respect to \(\{n_i; i \in \mathbb{Z}\}\). Hence \(H_{\text{Nic}}\) is not diagonalized with respect to \(\{n_i; i \in \mathbb{Z}\}\). Thus there is no desired set of complete l-bits.

\[\Box\]
5 Ergodicity breaking

From the results of §3 and §4 it follows that the Nicolai model breaks ergodicity.

In the first subsection we consider the ergodicity due to Mazur [31]. It is given in terms of averaged temporal autocorrelation functions of invariant states. We prove that the Nicolai model breaks ergodicity in this sense. In the second subsection we consider the eigenstate thermalization hypothesis (ETH) [16] [54]. It has been used as a kind of ergodicity in physics literature. We show that the Nicolai model violates ETH.

Remark 5.1. There are some variant definitions of ergodicity although their basic ideas are similar. Even if we consider operator algebraic formalism only, there are some subtle differences among existing definitions cf. [5] [7] [35] [51] [55] [57]. (If a strong chaotic property of dynamics known as the asymptotic abelian condition is valid, then a clear formulation is available as noted in [55]. However, it is an unjustified hypothesis [36].) So when we speak about ergodicity, we have to specify the precise formulation of dynamical systems and the definition of ergodicity.

5.1 Ergodicity breaking in the sense of Mazur

We recall the definition of ergodicity due to Mazur [31] for a general $C^*$-dynamical system as stated in [51]. Consider a one-parameter group of automorphisms $\alpha_t$ ($t \in \mathbb{R}$) on a $C^*$-algebra $\mathcal{A}$. Assume that $\alpha_t$ ($t \in \mathbb{R}$) be strongly continuous:

$$\lim_{t \to 0} \|\alpha_t(A) - A\| \to 0 \quad \text{for every } A \in \mathcal{A}. \quad (5.1)$$

Suppose that a state $\omega$ on $\mathcal{A}$ is $\alpha_t$-invariant,

$$\omega(\alpha_t(A)) = \omega(A) \quad \text{for all } A \in \mathcal{A} \text{ and } t \in \mathbb{R}. \quad (5.2)$$

The triplet $(\mathcal{A}, \alpha_t, \omega)$ is called a quantum dynamical system ($C^*$-dynamical system).

By $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ we denote the GNS representation associated to the state $\omega$ of $\mathcal{A}$. Precisely, $\pi_\omega$ is a homomorphism from $\mathcal{A}$ into $\mathfrak{B}(\mathcal{H}_\omega)$ (the set of all bounded linear operators on the Hilbert space $\mathcal{H}_\omega$), and $\Omega_\omega \in \mathcal{H}_\omega$ is a cyclic vector such that $\omega(A) = (\Omega_\omega, \pi_\omega(A)\Omega_\omega)$ for all $A \in \mathcal{A}$.

By the continuity of $\alpha_t$ with respect to $t \in \mathbb{R}$, there exists a strongly continuous unitary group $\{U_\omega(t); t \in \mathbb{R}\}$ that implements $\alpha_t$ ($t \in \mathbb{R}$) on the GNS Hilbert space $\mathcal{H}_\omega$ as:

$$U_\omega(t) (\pi_\omega(A)) U_\omega(t)^{-1} = \pi_\omega(\alpha_t(A)) \quad \text{for all } A \in \mathcal{A} \text{ and } t \in \mathbb{R}. \quad (5.3)$$

By the Stone-von Neumann theorem [44], there exists a self-adjoint operator $H_\omega$ on $\mathcal{H}_\omega$ such that

$$U_\omega(t) = e^{itH_\omega} \quad \text{for all } t \in \mathbb{R}, \quad (5.4)$$

and

$$H_\omega \Omega_\omega = 0. \quad (5.5)$$

This self-adjoint operator $H_\omega$ is called a GNS Hamiltonian. (Sometimes it is called a modular Hamiltonian if $\omega$ is a KMS state with respect to $\alpha_t$ ($t \in \mathbb{R}$).)
Let $F_\omega$ denote the orthogonal projection on the $U_\omega(t)$-invariant vectors in $\mathcal{H}_\omega$, i.e. the projection in $\mathcal{H}_\omega$ with the range
\[
\{ \psi \in \mathcal{H}_\omega \mid U_\omega(t)\psi = \psi \; \text{for all} \; t \in \mathbb{R} \}.
\] (5.6)

With the above notations in hand, we shall introduce the notion of ergodicity. The following inequality holds for any $A \in \mathcal{A}$ as shown in [51].
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \omega(A^*\alpha_t(A)) \, dt \geq \omega(A^*)\omega(A).
\] (5.7)

The operator $A$ is called ergodic if this becomes an equality. Otherwise, $A$ is called a non-ergodic operator. If every operator of $\mathcal{A}$ is ergodic, then the quantum dynamical system $(\mathcal{A}, \alpha_t, \omega)$ is called ergodic. Otherwise $(\mathcal{A}, \alpha_t, \omega)$ is called non-ergodic.

**Theorem 5.2.** Let $\alpha_t^{\Psi_{\text{Nic}}}$ $(t \in \mathbb{R})$ denote the time evolution of the Nicolai model given in §2.4. For any KMS state with respect to $\alpha_t^{\Psi_{\text{Nic}}}$ $(t \in \mathbb{R})$ at any positive temperature $\beta \in \mathbb{R}$, the ergodicity in the sense of Mazur is broken. For any classical supersymmetric ground state in Theorem 4.7 the ergodicity is also broken.

**Proof.** In Theorem 2 of [51] the following criterion of ergodicity is given: The quantum dynamical system $(\mathcal{A}, \alpha_t, \omega)$ is ergodic if and only if $F_\omega$ is a one-dimensional projection.

First we consider the case of KMS states. For the precise definition and general properties of KMS states, we refer to [67]. Let $\omega$ denote a KMS with respect to $\alpha_t^{\Psi_{\text{Nic}}}$ $(t \in \mathbb{R})$. Take any $\alpha_t^{\Psi_{\text{Nic}}}$-invariant element $B \in \mathcal{C}$. Assume that it is not a scalar. Actually there are many non-scalar elements in $\mathcal{C}$ by Definition 3.9. As the fermion system $\mathcal{A}$ has no non-trivial ideal, any non-zero representation of $\mathcal{A}$ is injective, and so is any GNS representation of $\mathcal{A}$. Hence $\pi_\omega(B)\Omega_\omega \neq 0$. We normalize $B$ so that $\|\pi_\omega(B)\Omega_\omega\| = 1$. Let us denote this new normalized vector $\pi_\omega(B)\Omega_\omega$ by $\Omega_\omega^B$. It is known that the GNS vector $\Omega_\omega$ of any KMS state is a separating vector. Thus $\Omega_\omega^B$ and $\Omega_\omega$ are different rays that give rise to different states, namely $\Omega_\omega^B \neq \Omega_\omega$ up to $U(1)$-phase. As $\alpha_t^{\Psi_{\text{Nic}}}(B) = B$ and $U_\omega(t)^{-1}\Omega_\omega = \Omega_\omega$ for all $t \in \mathbb{R}$, we see
\[
U_\omega(t)\Omega_\omega^B = U_\omega(t)\pi_\omega(B)\Omega_\omega = U_\omega(t)\pi_\omega(B)U_\omega(t)^{-1}\Omega_\omega = \pi_\omega(\alpha_t^{\Psi_{\text{Nic}}}(B))\Omega_\omega = \pi_\omega(B)\Omega_\omega = \Omega_\omega^B.
\]

This tells that $\Omega_\omega^B$ is in the range of $F_\omega$. Hence the range of $F_\omega$ has more than one-dimension. We now conclude that $(\mathcal{A}, \alpha_t^{\Psi_{\text{Nic}}}, \omega)$ is non-ergodic.

Second we consider the case of classical supersymmetric ground states. Let $\omega$ denote any of such state given in Theorem 4.7. By Theorem 4.7 again, there are many other ground states which are identical to $\omega$ except on some finite region. Namely there are infinitely degenerated ground states in the same Hilbert space $\mathcal{H}_\omega$. Therefore the range of $F_\omega$ has more than one-dimension (in fact infinite dimension). We conclude that $(\mathcal{A}, \alpha_t^{\Psi_{\text{Nic}}}, \omega)$ is non-ergodic. \qed

### 5.2 Violation of Eigenstate thermalization hypothesis

We now start discussion on the eigenstate thermalization hypothesis (ETH) for the Nicolai model. As ETH is a set of requirements for Hamiltonians of discrete
spectrum, we need to prepare a sequence of local Hamiltonians of the Nicolai model. See original works \[16\] \[54\] as well as an extensive review \[12\] for general account of ETH.

Let \( J \) denote any finite subset of \( \mathbb{Z} \). Define

\[
Q_{\text{Nic}}(J) := \sum_{i \in \mathbb{Z}, \{2i-1,2i,2i+1\} \subset J} a_{2i+1}^* a_{2i} a_{2i-1} \in A(J) .
\]

(5.8)

Then the local supersymmetric Hamiltonian is defined as

\[
H_{\text{Nic}}(J) := \{ Q_{\text{Nic}}(J), Q_{\text{Nic}}(J)^* \} \in A(J) .
\]

(5.9)

We consider a set of increasing finite segments \( \{ J; J \subset \mathbb{Z} \} \) that eventually includes any finite subset of \( \mathbb{Z} \).

**Proposition 5.3.** Let \( \{ J; J \subset \mathbb{Z} \} \) be any infinite sequence of increasing finite segments that tends to the whole \( \mathbb{Z} \). The local Hamiltonians \( \{ H_{\text{Nic}}(J) \} \) of the Nicolai model violate ETH.

**Proof.** By Theorem \[4.7\] each \( g(n) \in \Upsilon_J \) gives a classical ground states on \( A(J) \) for \( H_{\text{Nic}}(J) \). So the dimension of the kernel of \( H_{\text{Nic}}(J) \) is extensive with respect to \( |J| \). As ETH requires non-degeneracy of eigenstates of local Hamiltonians, the Nicolai model violates ETH.

---

### 6 Failure of delocalization

We shall give account for the failure of delocalization of the Nicolai model at zero temperature. In \[14\] a general scenario of delocalization for generic disorder-free translation-invariant quantum Hamiltonians is proposed. Concretely we will apply this scenario by De Roeck-Huveneers to the Nicolai model, and show that this does not work: the resonant shown there does not happen. To this end we recall that the Nicolai model has a natural decomposition \( H_{\text{Nic}} = H_{\text{classical}} + H_{\text{hop}} \) (2.51), where \( H_{\text{classical}} \) is the classical term given in (2.52), and \( H_{\text{hop}} \) is the hopping term given in (2.53).

**Proposition 6.1.** Let \( H_{\text{classical}} \) denote the initial classical Hamiltonian. Consider its perturbation by the quantum interaction \( H_{\text{hop}} \). Then all ground states of \( H_{\text{classical}} \) (that exist infinitely many) are invariant under any order of the perturbation by \( \lambda H_{\text{hop}} \) (\( \lambda \in \mathbb{R} \)). No resonant in the sense of \[14\] happens.

**Proof.** By Proposition \[4.3\] Theorem \[4.7\] and Proposition \[4.9\] any ground state of \( H_{\text{classical}} \) is represented by a vector \( |g(n)_{n \in \mathbb{Z}}\rangle \) with some \( g(n) \in \Upsilon \) defined in Definition \[4.6\]. As it is a ground state for both \( H_{\text{Nic}} \) and \( H_{\text{classical}} \), the following identities hold:

\[
H_{\text{Nic}}|g(n)_{n \in \mathbb{Z}}\rangle = 0 = H_{\text{classical}}|g(n)_{n \in \mathbb{Z}}\rangle.
\]

(6.1)

From (6.1) and \( H_{\text{hop}} = H_{\text{Nic}} - H_{\text{classical}} \) we have

\[
H_{\text{hop}}|g(n)_{n \in \mathbb{Z}}\rangle = 0.
\]

(6.2)
This implies that for any $k \in \mathbb{N}$ and any $\lambda \in \mathbb{R}$

$$
(\lambda H_{\text{hop}})^k |g(n)_{n \in \mathbb{Z}}\rangle = 0.
$$

(6.3)

So we obtain “no-resonant”

$$
\langle h(n)_{n \in \mathbb{Z}} | (\lambda H_{\text{hop}})^k |g(n)_{n \in \mathbb{Z}}\rangle = 0,
$$

(6.4)

where $h(n)$ is any classical configuration over $\mathbb{Z}$. Of course, this holds for any $h(n) \in \mathbb{Y}$.

**Remark 6.2.** We note that the high degeneracy of ground states of the Nicolai model is not harmful for delocalization; this would even make resonance happen easier. The model given in [14] is a (generic) interacting boson lattice model, whereas our model is a fermion lattice model. As the spinless fermion lattice model has much fewer degrees of freedom at each site (only two, up and down) that the boson lattice model, more resonant spots will appear for the former than the latter. So this difference is not harmful.

**Remark 6.3.** For $\lambda = 1$, the model is the Nicolai model itself. Recall that it has high quantum integrability as shown in Proposition 3.13. For general $\lambda \in \mathbb{R}$, we can not assure that $H_{\text{classical}} + \lambda H_{\text{hop}}$ still has such quantum integrability.

**Remark 6.4.** The formula (6.4) telling no-resonant has been shown for one particular perturbation. Resonant may happen for a generic quantum hopping perturbation according to [14].

### 7 Generalization to multi-dimensional lattice

We have studied the Nicolai model on one-dimensional lattice $\mathbb{Z}$. In this section, we indicate that our results given so far can be easily extended to the Nicolai model on $\mathbb{Z}^\nu$ for arbitrary $\nu \in \mathbb{N}$. In the following we discuss $\mathbb{Z}^2$ since this essentially represents all $\mathbb{Z}^\nu$.

We shall define an assignment of local fermion charges $\Psi_{\text{Nic}}$ over $\mathbb{Z}^2$. Of course, this new $\Psi_{\text{Nic}}$ should be a natural generalization of (2.46) given in § 2.4 for the one-dimensional lattice $\mathbb{Z}$. Let

$$
J_{(2i,2j)} := \{(2i - 1, 2j), (2i, 2j - 1), (2i, 2j), (2i + 1, 2j), (2i, 2j + 1)\} \quad \text{for } i, j \in \mathbb{Z}.
$$

(7.1)

Namely $J_{(2i,2j)} \in \mathbb{Z}^2$ consists of its center $(2i, 2j) \in (2\mathbb{Z})^2$ and four sites next to the center. Define

$$
\Psi_{\text{Nic}}(J_{(2i,2j)}) := a_{(2i-1,2j)} a_{(2i,2j-1)} a^*_{(2i,2j)} a_{(2i+1,2j)} a_{(2i,2j+1)} \in \mathcal{A}(J_{(2i,2j)})
$$

$$
\Psi_{\text{Nic}}(J) := 0 \quad \text{on any other } J \in \mathbb{Z}^2.
$$

(7.2)

The superderivation $\delta_{\Psi_{\text{Nic}}}$ associated to the above $\Psi_{\text{Nic}}$ is nilpotent because of the identity $\Psi_{\text{Nic}}(X_2)\Psi_{\text{Nic}}(X_1) = 0$ for any $X_1 \in \mathbb{Z}^2$ and $X_2 \in \mathbb{Z}^2$. Thus by Theorem 2.1 we obtain a nilpotent superderivation $\delta_{\Psi_{\text{Nic}}}$ and a strongly continuous one parameter group of $*$-automorphisms $\alpha_t^{\Psi_{\text{Nic}}} \ (t \in \mathbb{R})$ on the fermion lattice system $\mathcal{A}$ over $\mathbb{Z}^2$. 

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As done in \[3\] we can systematically construct infinitely many local fermionic constants of this new time evolution $\alpha_t^{\Psi_{\text{Nic}}}$ ($t \in \mathbb{R}$). To this end we replace Definition 3.1 for $\mathbb{Z}$ by the following one.

**Definition 7.1.** Let I be some rectangle of $\mathbb{Z}^2$. Let $f$ be a $\{-1, +1\}$-valued function on I. If on some $J_{(2i,2j)} \subset I$, either
\begin{equation}
\begin{aligned}
f((2i, 2j)) &= +1, \\
f((2i - 1, 2j)) &= -1, \\
f((2i, 2j - 1)) &= -1, \\
f((2i + 1, 2j)) &= -1, \\
f((2i, 2j + 1)) &= -1,
\end{aligned}
\end{equation}

or
\begin{equation}
\begin{aligned}
f((2i, 2j)) &= -1, \\
f((2i - 1, 2j)) &= +1, \\
f((2i, 2j - 1)) &= +1, \\
f((2i + 1, 2j)) &= +1, \\
f((2i, 2j + 1)) &= +1,
\end{aligned}
\end{equation}

is satisfied, then $f$ is called forbidden. Otherwise, $f$ is called permitted.

With this new definition, we can immediately generalize Definition 3.2 to $\mathbb{Z}^2$ and get local configurations of conservation for the Nicolai model. We set up an analogous rule given in \[3.2\] for $\mathbb{Z}^2$ and provide local fermionic operators as done in \[3.3\]. For explanation of this procedure we shall give an example. Let
\begin{equation}
I_{[0,2l]\times[0,2m]}^2 \equiv \{(x,y) \in \mathbb{Z}^2; 0 \leq x \leq 2l, 0 \leq y \leq 2m\} \quad (l,m \in \mathbb{N}).
\end{equation}

As in Definition 3.5 take the simplest local configurations:
\begin{equation}
r_{[0,2l]\times[0,2m]}^+(i) = +1 \quad \forall i \in I_{[0,2l]\times[0,2m]}^2, \quad r_{[0,2l]\times[0,2m]}^-(i) = -1 \quad \forall i \in I_{[0,2l]\times[0,2m]}^2.
\end{equation}

The assignment of local fermion operators from local configurations of conservation will be denoted by the same symbol $\mathcal{D}$ as in Definition 3.6. Then we have
\begin{equation}
\mathcal{D}(r_{[0,2l]\times[0,2m]}^+) = \prod_{i \in I_{[0,2l]\times[0,2m]}^2} a_i^\dagger \in \mathcal{A}(I_{[0,2l]\times[0,2m]}^2)^-, \quad \mathcal{D}(r_{[0,2l]\times[0,2m]}^-) = \prod_{i \in I_{[0,2l]\times[0,2m]}^2} a_i \in \mathcal{A}(I_{[0,2l]\times[0,2m]}^2)^-,
\end{equation}

where we specify certain order of products. Repeating almost the same argument as in \[3\] we can show that both of them are invariant under the time evolution $\alpha_t^{\Psi_{\text{Nic}}}$ ($t \in \mathbb{R}$).

From the above generalization of the results in \[3\] to the multi-dimensional integer lattice we can easily derive analogous results shown in \[4\] \[5\] \[6\] for the Nicolai model on $\mathbb{Z}^2$.

8 Discussion

We have studied dynamics of the Nicolai supersymmetric fermion lattice model \[37\] for any dimensional integer lattice. We have shown that the Nicolai model has infinitely many local fermionic constants of motion. Precisely, those are hidden
local supersymmetries that the Nicolai model possesses in addition to its defining
dynamical supersymmetry. The number of those local constants is extensive. As a
consequence of these local constants of motion, we have shown breaking ergodicity
in the sense of Mazur for all KMS states and all classical supersymmetric ground
states. We have derived violation of ETH for the Nicolai model, which is another
indication of ergodicity breaking.

We have determined all classical supersymmetric ground states for the Nicolai
model establishing a simple rule of construction from classical configurations (bi-
ary codes). Those infinitely many classical ground states on the Fock space look
like many-body localization. However, since many-body localization (MBL) usually
refers to complete localization for positive “temperature” \[38\] \[33\], the localization
at only zero temperature of the Nicolai model can not be regarded as genuine MBL.

To understand quantum dynamical in detail, we need to know the whole spec-
trum of the Nicolai Hamiltonian. We have only studied ground states not excited
states at all. However, we have given a relevant result beyond ground states: The
Nicolai model does not satisfy the criterion of the fully many-body localization, as
it lacks a complete set of “l-bits” \[22\] \[33\]. We shall shortly recall our derivation
of this no-go statement given in \[4.3\]. A complete set of classical l-bits is uniquely
determined by the highly degenerated classical ground states. On the other hand,
there are many fermionic l-bits determined by the time evolution. Those two sets
of l-bits do not coincide. This disagreement implies non-existence of a complete set
of l-bits for the Nicolai model.

It has been discussed in \[10\] \[22\] \[25\] \[34\] \[46\] \[50\] that a complete set of local
integrals of motion (l-bits) characterizes the fully many-body localized phase. This
may suggest that the extensive number of local fermionic constants of the Nicolai
model should not be complete. A less demanding notion of many-body localization
that does not require completeness of l-bits is proposed in \[23\]. It would be inter-
esting to estimate incompleteness of the set of local fermionic constants of motion
of the Nicolai model given in this paper.

As noted in \[31\] there is a notable question of MBL: Does MBL always necessitate
disorder? We have seen that the Nicolai model, which is a disorder-free interacting
fermion model, has certain properties similar to MBL, however, it is not a genuine
MBL model. We have no idea whether our conclusion has any relevance to this
fundamental question.

Finally we shall pose some future problems which are not touched in this paper.

1. It would be interesting to determine which operators are ergodic or non-
ergodic for the time evolution of the Nicolai model. Of course, those are
state-dependent. Similarly, we may discuss which operators are thermalized
or frozen under the time evolution.

2. How does quench dynamics look like for the Nicolai model? It would be inter-
esting to see how such high integrability of the model effect quench dynamics.
We may refer to the work \[11\].

3. How is a generalized Gibbs ensemble \[45\] for the Nicolai model? A notable
point is that its many constants of motion are fermionic. How should we
incorporate these non-observables into steady states?
4. We speculate a random version of the Nicolai model by changing its constant coefficients by random variables. How does the randomness change (enhance) MBL-character of the Nicolai model? It would be interesting to compare this randomized Nicolai model with the supersymmetric Sachdev-Ye-Kitaev model [17].

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