Matrix Parameterized Pseudo-differential Calculi on Modulation Spaces

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Abstract. We consider a broad matrix parameterized family of pseudo-differential calculi, containing the usual Shubin’s family of pseudo-differential calculi, parameterized by real numbers. We show that continuity properties in the framework of modulation space theory, valid for the Shubin’s family extend to the broader matrix parameterized family of pseudo-differential calculi.

0. Introduction

A pseudo-differential calculus on $\mathbb{R}^d$ is a rule which takes any appropriate function or distribution, defined on the phase space $T^*\mathbb{R}^d \simeq \mathbb{R}^{2d}$ into a set of linear operators acting on suitable functions or distributions defined on $\mathbb{R}^d$. There are several other situations with similar approaches. For example, a main issue in quantum mechanics concerns quantization, where observables in classical mechanics (which are functions or distributions on the phase space) carry over to corresponding observables in quantum mechanics (which usually are linear operators on subspaces of $L^2(\mathbb{R}^d)$). A somewhat similar situations can be found in time-frequency analysis. Here the phase space corresponds to the time-frequency shift space, and the filter parameters for (non-stationary filters) are suitable functions or distributions on the time-frequency shift space, while the corresponding filters are linear operators acting on signals (which are functions or distributions, depending on the time).

A common family of pseudo-differential calculi concerns $a \mapsto \text{Op}_t(a)$, parameterized by $t \in \mathbb{R}$. If $a \in \mathcal{S}(\mathbb{R}^{2d})$, then the pseudo-differential operator $\text{Op}_t(a)$ is defined by

$$\text{Op}_t(a)f(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} a(x - t(x - y), \xi) f(y) e^{i(x - y, \xi)} dyd\xi,$$

when $f \in \mathcal{S}(\mathbb{R}^d)$ (cf., e.g., [28]).
In the paper we consider as in [3] a slightly larger family of pseudo-differential calculi, compared to the situations above, which are parameterized by matrices instead of the real number $t$. More precisely, if $a \in \mathcal{S}(\mathbb{R}^{2d})$ and $A$ is a real $d \times d$ matrix, then the pseudo-differential operator $\text{Op}_A(a)$ is defined by

$$\text{Op}_A(a)f(x) = (2\pi)^{-d} \int_{\mathbb{R}^{2d}} a(x - A(x - y), \xi) f(y) e^{i(x - y, \xi)} \, dy \, d\xi,$$

when $f \in \mathcal{S}(\mathbb{R}^d)$ (cf., e.g., [28]). We note that $\text{Op}_A(a) = \text{Op}_t(a)$ when $A = t \cdot I$, where $I$ is the $d \times d$ identity matrix. On the other hand, in [1], D. Bayer considered a more general situation, where each pseudo-differential calculus is parameterized by four matrices instead of one.

The definition of $\text{Op}_A(a)$ extends in several directions. In Section 2 we discuss such extensions within the theory of modulation spaces. That is, we deduce continuity for such operators between different modulation spaces, when $a$ belongs to (other) modulation spaces. Similar analysis and results can be found in, e.g., [17, 19, 20, 30, 32, 35, 38] in the more restricted case $A = t \cdot I$, and we emphasize that all results are obtained by using the framework of these earlier contributions. Furthermore, some results here are in some cases contained in certain results in Chapters 1 and 2 in [1].

In Section 3 we also give examples on how these operators might be used in quantization, by taking the average of $\text{Op}_A(a)$ with $A = \frac{1}{2} \cdot I + r \cdot U$, overall $r \in [0, 1]$ and unitary matrices $U$ with real entries.

1. Preliminaries

In this section we introduce some notations and discuss basic results. We start by recalling some facts concerning Gelfand–Shilov spaces. Thereafter we recall some properties about pseudo-differential operators. Especially we discuss the Weyl product and twisted convolution. Finally we recall some facts about modulation spaces. The proofs are in general omitted, since the results can be found in the literature.

We start by considering Gelfand–Shilov spaces. Let $0 < h, s \in \mathbb{R}$ be fixed. Then $\mathcal{S}_{s,h}(\mathbb{R}^d)$ consists of all $f \in C^\infty(\mathbb{R}^d)$ such that

$$\|f\|_{\mathcal{S}_{s,h}} \equiv \sup_{x, \beta} |x^\beta \partial^\alpha f(x)| h^{\alpha |\alpha| + |\beta|} \alpha! s^\beta ! s$$

is finite. Here the supremum should be taken over all $\alpha, \beta \in \mathbb{N}^d$ and $x \in \mathbb{R}^d$.

Obviously $\mathcal{S}_{s,h} \hookrightarrow \mathcal{S}$ is a Banach space which increases with $h$ and $s$. Here and in what follows we use the notation $A \hookrightarrow B$ when the topological spaces $A$ and $B$ satisfy $A \subseteq B$ with continuous embeddings. Furthermore, if $s > 1/2$ and $s_0 = 1/2$, then $\mathcal{S}_{s,h} \cup_{h \to 0} \mathcal{S}_{s_0,h}$ contain all finite linear combinations of Hermite functions. Since such linear combinations are dense in $\mathcal{S}$, it follows that the dual $(\mathcal{S}_{s,h})'(\mathbb{R}^d)$ of $\mathcal{S}_{s,h}(\mathbb{R}^d)$ is a Banach space which contains $\mathcal{S}'(\mathbb{R}^d)$. 