Scoring a Goal Optimally in a Soccer Game Under Liouville-Like Quantum Gravity Action

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Abstract

In this paper, we present a new stochastic differential game-theoretic model of optimizing strategic behavior associated with a soccer player under the presence of stochastic goal dynamics by using a Feynman-type path integral approach, where the action of a player is on a \(\sqrt{8/3}\)-Liouville quantum gravity surface. Strategies to attack the oppositions have been used as control variables with extremes like excessive defensive and offensive strategies. Before determining the optimal strategy, we first establish an infinitary logic to deal with infinite variables on the strategy space, and then, a quantum formula of this logic is developed. As in a competitive tournament, all possible standard strategies to score goals are known to the opposition team, a player’s action is stochastic, and they would have some comparative advantages to score goals. Furthermore, conditions like uncertainties due to rain, dribbling and passing skills of a player, time of the game, home crowd advantages, and asymmetric information of action profiles are considered to determine the optimal strategy.

Keywords  Liouville-like quantum gravity · Soccer game · Infinitary logic and quantum formula · Stochastic goal dynamics · Lefschetz-Hopf fixed point theorem · Feynman-type path integral

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1 Introduction

Soccer is one of the most popular sports in the world because of its simplistic rules. Three major tournaments of this game are the World Cup, the Euro Cup, and the Copa America. Out of them, the World Cup is the most popular tournament. Following Santos [1], we know that after the early 1990s, Fédération Internationale de Football Association (FIFA) is worried about all the teams being defensive in terms of scoring goals. As a result, the total number of goals has been falling over the years which eventually leads to a fall in interest in this game. If we consider 2006 men’s soccer World champion Italy and 2010 champion Spain, they only allowed two goals in their entire seven matches in the tournament [1]. In March 17, 1994, in USA Today, FIFA specified its objective to encourage attacking and high-scoring matches. Since then, the objectives of some teams have changed, for example, in the 2014 men’s soccer World Cup semifinal between Brazil and Germany where Germany scored five goals in the first half. Furthermore, in the 2018 men’s soccer World cup, we see France chose some offensive strategies. Furthermore, the French national soccer team of 2018 men’s World cup had no big names (like Lionel Messi, Cristiano Ronaldo, Romelu Lukaku), and their average age was lower compared to other big teams. Therefore, all the players from the French national team played without any pressure and were able to show their natural skills which led them to win the World Cup. Conversely, some teams with big soccer stars did not perform well because all possible reaction functions of those players were known to others due to their presence in the game for a long time. Thus, mental stress influences the outcome of a game which leads to the fate of large sums of money for the winners and losers.

Soccer is dynamic and strategic and therefore provides plenty of room for analyzing an optimal strategic behavior [2]. In every game, two teams are pitched into direct opposition for a period of play of 90-min notional duration. At the beginning of a game, each team has one goalkeeper and ten outfield players. Coaches are free to deploy their outfield players in any formation of their choice and to adjust their teams’ formations and style of play at any stage of the game [2]. Each team’s objective is to attempt to score goals and prevent its opponent from scoring. There exists a high level of interdependence: both teams’ strategies have implications for both teams’ probabilities of scoring and conceding goals [2].

A significant contribution to the dynamic game-theoretic model of optimizing strategic behavior has been developed in [3] where two opposing soccer teams choose continuously between a defensive and an attacking formation [2]. Palomino et al. [3] determines that a team leading the game has a higher propensity to play defensive than it plays offensive when it is either level or trailing, especially during the second half of the game. Later, [4] under the similar dynamic game-theoretic environment as [3] shows the impact on team strategies in the National Hockey League of a recent change in the league points scoring system affecting matches that were tied at the end of regular time [2]. The main issues of this paper are the assumptions that are either counterfactual or controversial which
hinders them to do a more generalized dynamic strategic analysis. For example, [3] relies upon an unverified assumption that if one team adopts an attacking strategy and the other adopts a defensive strategy, the goal scoring rate of the attacking team increases by more than that of the defending team [2]. To verify if this assumption is true, we consider the 2014 men’s soccer semi-final between Germany and Brazil where Germany scores five goals in 29 min, with four goals scored within the first 6 min and subsequently bringing the score up to 7 - 0 in the second half. Therefore, although the German team took a very attacking strategy at the beginning of the first half, it scored only two more goals in the later 1 h of the game. [4] relaxes the assumption given in [3], by introducing a parameter allowing for comparative advantage in either attack or defense [2]. Both articles restrict the teams’ strategies into two options: attack and defense. Neither considers a second strategic dimension such as the choice between a violent and non-violent style of play [2]. A team that plays violently commits foul play. Furthermore, too much violence in the game might lead a player to get a red card. Other acts of violence are an attempt to disrupt or sabotage the opposing team [2]. Dobson and Goddard [2] assumes that the available strategic choices are discrete: teams choose between defensive and attacking formations and between non-violent and violent styles of play. This choice of strategy influences the probabilities of scoring and conceding goals at the current stage of the match and the probability that players get red cards and out of the game [2].

Doing numerical simulations [2] shows that the optimal strategic choices at each stage of the match are dependent on the current difference in scores and on the amount of time that has elapsed since the beginning of the game. Throughout this paper, we consider continuous strategy as the control variable where the control variable takes its highest value when a team’s strategy is attacking.

In this paper, we present a new game-theoretical model to find out an optimal weight associated with the control variable of a soccer player under the presence of stochastic goal dynamics by using the Feynman path integral method, where the actions of every player of both the teams are on $\sqrt{8/3}$-Liouville quantum gravity (LQG) surface [5–10]. LQG is a random fractal Riemannian surface in which areas, lengths, and other measures are given by exponentials of a Gaussian free field, and this surface can be viewed as generalized Brownian map [11–18]. One way to visualize this surface is to think of an airplane moving from one place to another. When an airplane flies from Chicago to Hong Kong, the pilot flies in a straight line connecting these two cities. Due to the spherical shape of the earth, this line connecting these cities turns out to be curved as a consequence of mapping a straight line on a sphere. If the earth were not round, but was instead a more complicated shape, possibly curved in wild random ways, then that airplane’s trajectory would appear more ergodic [19]. Since a player’s strategy for scoring a goal is dynamic and stochastic such that the strategy space is changing its size with every moment of the game, we choose this surface. Furthermore, the Feynman path integral approach with LQG surface can be used to obtain a solution for the stability of an economy after a pandemic crisis [20, 21], determine optimal bank profitability [22]. In a very competitive tournament like a soccer World Cup, all possible standard strategies to score goals are known to the opposition teams as reaction
functions. In this environment, if a player’s action is stochastic, then that player would have some comparative advantage which is also known to the opposition team, but the team does not know what type of stochastic action is going to take place where the complete stochastic action profile of the player is unknown. Apart from that, the conditions like uncertainties due to rain, dribbling and passing skill of a player, type of match (i.e., day match or a day-night match), and advantages of having a home crowd have been considered a stochastic component of the goal dynamics to determine the optimal weight.

The important contribution of this paper is to assume that a player’s strategy space follows a $\sqrt{8/3}$ LQG surface. To justify this assumption, let us discuss about human behaviors as automatons. Humans are well-equipped to survive in their natural environments by receiving information historically. By birth, they acquire skills such as breathing, digesting foods, and basic processing of sensory information and motor actions (hence, they are automaton in nature). Furthermore, over the time, they are able to acquire complex skills through learning strategies and historical experiences of their habitats. In a soccer game, a player acquires some basic skills to score a goal. On top of that, they develop more sophisticated skills based on past match experiences, suggestions from coaches or mentors (at different stages in soccer career), historical weather records of a certain venue, conditions of the ground, and the support from the crowd. This learning process is implemented at the neural level by adapting the synaptic connections between neurons and other processes. It is not completely understood how the billions of synapses are adapted without central control [23]. There are two main parts of a human brain which help to make decisions: the frontal and occipital lobes. While the frontal lobe manages thinking, emotions, personality, judgment, self-control, muscle control and movements, and memory storage, the occipital lobe sits at the back of the brain and is responsible for visual perception, including color, form, and motion. It is well understood that the occipital lobe sends signals to the frontal lobe through the synaptic system for making decisions, but how the signals are processed through the subatomic level by electrons is still a mystery. Since, billions of electrons are engaged to transfer signals from occipital to frontal lobes, even we consider eleven soccer players in a team, a total number of electrons involved in this transition process are trillions. Therefore, we can derive a game theoretic model from a continuum limit (as the number of electrons are very high) which is somewhat similar to classical mean field approaches in Statistical mechanics and physics (i.e., the derivation of Boltzmann or Vlasov equations in the kinetic theory of gases) or in Quantum Mechanics and Quantum Chemistry (i.e., density functional models like Hartree or Hartree-Fock models) [24]. Since this general signal processing is quantum in nature, the importance of the quantum field theory takes place through so-called diffusion tensor imaging, and indeed, the strategy space becomes a random surface [18, 25]. Given $2n$ unit equilateral triangles (where $n \in \mathbb{N}$), there are finitely many ways to glue each edge to a partner. A random spherical homeomorphic surface is obtained by taking uniform samples from the gluings which produce a topological surface [18]. If we allow $n \to \infty$, these random appropriately scaled surfaces converge in probability distributions (or simply probability law). The limit is a canonical
sphere-homeomorphic random surface, much the way Brownian motion is a canonical random path [18]. Depending on how the surface space and convergence topology are specified, the limit is the Brownian sphere (by generalized Donsker Theorem), the peanosphere, the pure $\sqrt{8/3}$ LQG, or a certain conformal field theory. Although all of the above objects have precise definitions, they are equivalent in some sense, but showing equivalence is mathematically difficult [18]. Out of these above approximations of random surfaces, we choose $\sqrt{8/3}$ LQG surface since it is comparatively easier to handle. One such property is that this surface homeomorphic and diffeomorphic, while Brownian motion is only homeomorphic. Since the space of synaptic signal processing is fractal in nature, the type of shot towards the goal or to a teammate is an outcome of the decision from this space. Furthermore, to determine a solution of a system, we construct a stochastic Lagrangian with $\sqrt{8/3}$ LQG metric and construct a Feynman action function for equal lengthed time sub-intervals. This leads to an optimal path on a random surface.

Recent literature talks about whether a soccer team should choose offensive or defensive strategies [1]. Some studies say that the relatively new “Three-point” rule and “Golden goal” do not necessarily create to break a tie and score goals [1, 26], and if asymmetry between two teams is big, then these two rules induce the weaker team to play more defensively [27]. There are some other studies combined that give mixed results on these two rules [1, 28–30]. Therefore, we do not include these two rules in our analysis. On the other hand, as scoring a goal on a given condition of a match is purely stochastic, a discounted reward on a player’s dynamic objective function can give them more incentive to score which is a common dynamic reward phenomenon in the animal kingdom [23].

The remainder of the paper is structured as follows: Section 2 motivates the development of the game-theoretical model with stochastic goal dynamics. Section 3 develops the background of this model with the construction of $\sqrt{8/3}$-LQG surface, $\sqrt{8/3}$-LQG metric, and environmental ergodicity of the field. Section 4 determines the optimal value of the control coefficient which is the main contribution of this paper. Section 5 presents all the proofs of the lemmas and propositions discussed in this paper. Finally, Section 6 concludes this paper with some future possibilities for research.

2 Construction of the Problem

In this section, we construct forward-looking stochastic goal dynamics under a Liouville-like quantum gravity action space with a conditional expected dynamic objective function. The objective function gives an expected number of goals for a match based on the total number of goals scored by a team at the beginning of each time interval. For example, at the beginning of a match, both of the teams start with 0 goals. Therefore, the initial condition is $Z_0 = 0$, where $Z_0$ represents the total number of goals scored by a team at time 0 of the interval $[0, t]$. The objective of player $i \in I$ at the beginning of $(M + 1)^{th}$ game is
\[
\text{OB}^i_a : \tilde{Z}_a(W, s) = h^i_0 + \max_{W_i \in W} \mathbb{E}_0 \left\{ \int_0^T \sum_{i=1}^I \sum_{m=1}^M \exp(-\rho^i s) W_i(s) h^i_0(s, w(s), z(s)) ds \right\},
\]

with \( \mathbb{E}_0[.] = \mathbb{E}[.| Z_0] \), where \( W_i \) is the strategy of player \( i \) (control variable), \( \alpha^i \in \mathbb{R} \) is constant weight, \( I \) is the total number of players in a team including those are at the reserve bench, \( \rho^i_s \in (0, 1) \) is a stochastic discount rate for player \( i \) with \( w \in \mathbb{R}^I \) and \( z \in \mathbb{R}^I \) are time dependent for all possible controls and goals available to player \( i \), \( h^i_0 \geq 0 \) is the initial condition of the function \( h^i_0 \), and \( \mathcal{F}_0^z \) is the filtration of goal dynamics starting at the beginning of the game. Therefore, for player \( i \), the difference between \( W_i \) and \( w \) is that strategy \( W_i \) is the subset of all possible strategies \( w \) available to them at time \( s \) before \( (M + 1)^{th} \) game starts. Furthermore, we assume \( h^i_0 \) is a known objective function to player \( i \), which is partly unknown to the opposition team because of incomplete and imperfect information in this system.

In Eq. (1), the function \( h^i_0 \) has an important contribution to \( \tilde{Z}_a(W, s) \). The function \( h^i_0 \) is the utility function of player \( i \), and it is increasing with respect to the control \( w(s) \) (i.e., \( \partial h^i_0 / \partial w > 0 \)) and goal \( z(s) \) (i.e., \( \partial h^i_0 / \partial z > 0 \)). In this paper, control variable \( w \) represents whether player \( i \) is using defensive or offensive strategies. If a player uses more offensive strategy, then that player might able to score more goals and gets more fans in the future. After feeling that, player \( i \) experiences an increase in the value of \( h^i_0 \). It is important to know that all the players in the game are assumed to be rational. Therefore, self satisfaction partially is not completely correlated for that player’s team to win a match. For example, if most of the player of a team try to play aggressive, then the defense would be weak and opposition might take advantages from that. This effect is observed frequently in Brazilian and Argentine soccer team in the most recent Men’s World Cup. On the other hand, positive correlation between \( h^i_0 \) and \( z \) is intuitive; if a player scores more goals, then they might be more favorite in the fan base, and in the future, they might get higher club contract. Hence, \( \partial h^i_0 / \partial z > 0 \). \( \mathcal{F}_0^z \) is the filtration of player \( i \) generated by the Brownian motion of the goal dynamics starting at time \( 0 \). Since, information is incomplete and imperfect, player \( i \) only knows their filtration and hence has no knowledge about the filtration processes of the other players including their team mates. In Eq. (1), \( W_i(s) \) is the strategy executed by player \( i \) at the \( M + 1^{th} \) match. If player \( i \) takes more aggressive strategy to score a goal, then for that player, \( W_i \) increases and vice versa. The weight \( \alpha^i \) has an important role to determine the objective function. If player \( i \) is too aggressive to score a goal while their team should play defensively, \( \alpha^i \) is adjusted so that the value of \( W_i(s) \) does not go too high. During a match, a coach determines this value and accordingly sends instruction to that player. Even after sending instruction, if player \( i \) does not change their strategy, either that player might be substituted or might get yellow or red cards. In this paper, we are going to determine this \( \alpha^i \). Therefore, the objective function represented by Eq. (1) is the expected, discounted strategy-weighted utility function of player \( i \) observed at time \( 0 \) of match \( M + 1 \).

Supposedly, the stochastic differential equation corresponding to goal dynamics is
\[ dZ(s, W) = \mu(s, W, Z(s, W))ds + \sigma(s, \hat{s}, W(s), Z(s, W))dB(s), \] (2)

where \( W_{I \times L'}(s) \subseteq W \) takes values from \( \mathbb{R}^{I \times L'} \) is a control matrix and \( Z_{I \times L'}(s) \subseteq Z \) takes values from \( \mathbb{R}^{I \times L'} \) is a matrix of scoring goals under soccer rules such that \( z \in Z, B_{I \times L'}(s) \) is a \( I \times L' \)-dimensional Brownian motion, \( \mu_{I \times L'} > 0 \) is the drift coefficient, and the positive semidefinite matrix \( \sigma_{I \times L'} > 0 \) is the diffusion coefficient such that

\[
\lim_{s \to \infty} \mathbb{E}[\mu(s, W, Z(s, W)) = Z^* > 0.
\]

The above argument states that if enough time is allowed in a game, then \( Z^* \) number of goals would be achieved which turns out to be a stable solution for this system. Finally,

\[
\sigma[s, \hat{s}, W(s), Z(s, W)] = \gamma \hat{s} + \sigma^*[s, W(s), Z(s, W)],
\] (3)

where \( \hat{s} > 0 \) comes from the strategies of the opposition team with the coefficient \( \gamma > 0 \), and \( \sigma^* > 0 \) comes from the weather conditions, venues, popularity of a club, or a team before starting of \((M + 1)^{th}\) game. The forward stochastic differential Eq. (3) is the core of our analysis. We use all possible important conditions during a game.

3 Definitions and Assumptions

**Assumption 1** For \( t > 0 \), let \( \mu(s, W, Z) : [0, t] \times \mathbb{R}^{I \times L'} \times \mathbb{R}^{I \times L'} \to \mathbb{R}^{I \times L'} \) and \( \sigma(s, \hat{s}, W, Z) : [0, t] \times S^{(I \times L')} \times \mathbb{R}^{I \times L'} \times \mathbb{R}^{I \times L'} \to \mathbb{R}^{I \times L'} \) be some measurable function with \((I \times L') \times t\)-dimensional two-sphere \( S^{(I \times L')} \times t \) and, for some positive constant \( K_1 \), \( W \in \mathbb{R}^{I \times L'} \) and, \( Z \in \mathbb{R}^{I \times L'} \) we have linear growth as

\[
|\mu(s, W, Z)| + |\sigma(s, \hat{s}, W, Z)| \leq K_1 (1 + |Z|),
\]

such that, there exists another positive, finite, constant \( K_2 \) and for a different score vector \( \tilde{Z}_{(I \times L') \times 1} \) such that the Lipschitz condition,

\[
|\mu(s, W, Z) - \mu(s, W, \tilde{Z})| + |\sigma(s, \hat{s}, W, Z) - \sigma(s, \hat{s}, W, \tilde{Z})| \leq K_2 |Z - \tilde{Z}|
\]

\( \tilde{Z} \in \mathbb{R}^{I \times L'} \) is satisfied and

\[
|\mu(s, W, Z)|^2 + |\sigma(s, \hat{s}, W, Z)|^2 \leq K_2^2 (1 + |\tilde{Z}|^2),
\]

where \( |\sigma(s, \hat{s}, W, Z)|^2 = \sum_{i=1}^{I} \sum_{j=1}^{L'} |\sigma^{ij}(s, \hat{s}, W, Z)|^2 \).

**Remark 1** The local Lipschitz condition and the linear growth of run dynamics in Assumption 1 guarantee that the SDE expressed in the Eq. (2) has a unique solution. Assumption 1 gives a unique solution as long as the drift part of the goal dynamics expressed in the Eq. (2) has a linear drift. Therefore, our main concentration on this paper is the goal dynamics following a linear SDE.
Assumption 2 There exists a probability space \((\Omega, \mathcal{F}, \mathcal{F}_s^\mathbb{Z}, \mathcal{P})\) with sample space \(\Omega\), Borel \(\sigma\)-algebra \(\mathcal{F}\), filtration at time \(s\) of goal \(\mathbb{Z}\) as \(\{\mathcal{F}_s^\mathbb{Z}\} \subset \mathcal{F}_s\), a probability measure \(\mathcal{P}\) and a \(l' \times l'\)-dimensional \(\{\mathcal{F}_s\}\)-Brownian motion \(B\) where the measure of aggressiveness by players \(W\) is an \(\{\mathcal{F}_s^\mathbb{Z}\}\)-Brownian motion adapted process such that Assumption 1 holds, for the feedback control measure of players there exists a measurable function \(h\) such that \(h : [0, t] \times C([0, t]) : \mathbb{R}^{l' \times l'} \rightarrow W\) for which \(W(s) = h(\mathbb{Z}(s, w))\) such that Eq. (2) has a unique fixed point [31].

Remark 2 Assumption 2 tells about the assumptions on the background probability space, and furthermore, it guarantees the existence of a unique fixed point. Assumptions 1 and 2 tell about the unique solution of the SDE expressed in the Eq. (2) through contraction mapping.

Assumption 3 (i). \(\mathbb{Z} \subset \mathbb{R}^{l' \times l'}\) such that a soccer player \(i\) cannot go beyond set \(\mathcal{Z}_i \subset \mathbb{Z}\) because of their limitations of skills. This immediately implies set \(\mathcal{Z}_i\) is different for different players.

(ii). The function \(h_0^i : [0, t] \times \mathbb{R}^{2l'} \rightarrow \mathbb{R}^{l'}\). Therefore, all players in a team at the beginning of \((M + 1)^{th}\) match have the objective function \(h_0^i : [0, t] \times \mathbb{R}^{l' \times l'} \rightarrow \mathbb{R}^{l' \times l'}\) such that \(h_0^i \subset h_0\) in functional spaces and both of them are concave which is equivalent to Slater condition [32].

(iii). There exists an \(\epsilon > 0\) such that for all \((W, \mathbb{Z})\) and \(i = 1, 2, ..., I\) such that

\[
\mathbb{E}_0 \left\{ \int_0^t \sum_{i=1}^I \sum_{m=1}^M \exp(-\rho^i_m)W_i(s)h_0^i(s, w(s), z(s)) ds \bigg| \mathcal{F}_0^\mathbb{Z} \right\} \geq \epsilon.
\]

Remark 3 Assumption 3 tells us that conditional expectation on inter-temporal weighted utility function has to be positive for player \(i\) to continue their career. The player retires when the value of this utility function is zero.

Definition 1 Suppose \(\mathbb{Z}(s, W)\) is a non-homogeneous Fellerian semigroup on time in \(\mathbb{R}^{l' \times l'}\). The infinitesimal generator \(A\) of \(\mathbb{Z}(s, W)\) is defined by

\[
Ah(z) = \lim_{s \to 0} \frac{\mathbb{E}_z[h(\mathbb{Z}(s, W))] - h(\mathbb{Z}(W))}{s},
\]

for \(Z \in \mathbb{R}^{l' \times l'}\) where \(h : \mathbb{R}^{l' \times l'} \rightarrow \mathbb{R}\) is a \(C^{1,2}\) function, \(Z\) has a compact support, and at \(\mathbb{Z}(W) > 0\) the limit exists where \(\mathbb{E}_z\) represents the soccer team’s conditional expectation of scoring goals \(Z\) at time \(s\). Furthermore, if the above Fellerian semigroup is homogeneous on times, then \(Ah\) is the Laplace operator.

Remark 4 The infinitesimal generator defined in Definition 1 describes the movement of the process in an infinitesimal time interval \([s, \tau]\) \(\subset [0, t]\) for all \(\tau = s + \epsilon\) and \(\epsilon = t/n\). Later, in this part of the paper, this definition is needed to construct a Wick-rotated Schrödinger-type equation.
Since, $h$ is a measurable function depending on $s$, there is a possibility that this function might have very large values and may be unstable. To stabilize $W$, we need to take the natural logarithmic transformation and define a characteristic-like quantum operator as in Definition 2.

**Definition 2** For a Fellerian semigroup $Z(s, W)$ for all $\varepsilon > 0$, the time interval $[s, \tau]$ with $\varepsilon \downarrow 0$, define a characteristic-like quantum operator starting at time $s$ is defined as

$$
\mathcal{A} h(Z) = \lim_{\varepsilon \downarrow 0} \frac{\log \mathbb{E}_s[\varepsilon^2 h(Z(s, W))] - \log \mathbb{E}_s(\varepsilon^2)}{\log \mathbb{E}_s(\varepsilon^2)},
$$

for $Z \in \mathbb{R}^{I \times J}$, where $h : \mathbb{R}^{I \times J} \to \mathbb{R}$ is a $C^{1,2}$ function, $\mathbb{E}_s[\cdot] = \mathbb{E}[\cdot|Z(s, \cdot)]$, for $\varepsilon > 0$ and a fixed $h$, we have the sets of all open balls of the form $B_\varepsilon(h)$ contained in $B$ (set of all open balls) and as $\varepsilon \downarrow 0$ then $\log \mathbb{E}_s(\varepsilon^2) \to \infty$.

**Remark 5** In the literature of stochastic processes, the Dynkin’s formula gives the expected value of any suitably smooth statistic of an Itô process at a stopping time. The above Definition 2 along with Dynkin’s Lemma give us the expression for a characteristic-like quantum generator for a $C^{1,2}$ function $h$ corresponding to the SDE expressed in the Eq. (2). The standard Feynman-Kac approach cannot give any solution if $(\sigma \sigma^T)_{ij} \frac{\partial^2 h(Z)}{\partial z_i \partial z_j}$ does not belong to a *Hilbert* space. Our method does not have this limitation.

**Definition 3** Following [33], a dynamic conditional expected objective function explained in Eq. (1) on the goal dynamics $Z \in \{Z_0, Z_1, ..., Z_t\}$ is a tuple $(s, \alpha^i, \{\text{OB}_{a}^i(W_i), \tau_{W_i}\}_{W_i \in W})$ where

(i) $w \subset W$ is a finite strategy space where player $i$ can choose strategy $W_i$ and $\alpha^i$ is all probabilities available to them from where they can choose $\alpha^i$.

(ii) For each strategy $W_i \in W$, $\text{OB}_{a}^i$ is a constrained objective function of soccer player $i$ such that Definition 2 holds.

(iii) For each strategy $W_i \in w$, define the rain or other environmental random factors which leads to a stoppage or termination of game $M + 1$ at time $s$ as $\tau_{W_i}$, which is a finitely-additive probability measure on the Borel $\sigma$-algebra on $\mathbb{R}^{I \times J}$ and is proper.

Following [34], two main types of logic are used in game theory: first-order logic and infinitary logic. First-order mathematical logic is built on finite base language based on the connective symbols such as conjunction, negation, conversion, inversion, and contrapositive: a countable collection of variables, quantifier symbols, constant symbols, predicate symbols, and function symbols [34]. Quantum formulae are of the form of $h(0, ..., i)$ such that the game operates in a quantum field with the characteristic-like quantum generator defined in Definition 2.
Comparing this statement with the interpretation of atomic formulae in [34], we can say \( h \) is a functional predicate symbol on terms 0, ..., \( t \). Our quantum formulae are a more generalized version of atomic formulae in the sense that quantum formulae consider improper differentiability on the 2-sphere continuous strategy available for all the players in both teams.

The problem with constructing a first-order logic is that it can only handle expressions of finite variables. Dealing with infinite variables in strategy space is the primary objective of our paper. When player \( i \) tries to score a goal, they change their actions based on the strategies of the opposition team and their skills. If player \( i \) is a senior player, then the opposition team has more information about that player’s strength or weakness which is also known to player \( i \). Therefore, at time \( s \) of match \( M + 1 \), player \( i \)’s action is mixed. Furthermore, we assume each player’s strategy set is a convex polygon with each side having the length of unity. The reason is that the probability of choosing a strategy is between 0 and 1. Therefore, if a player has three strategies, their strategy set is an equilateral triangle with each side of length unity.

To get a more generalized result, we extend the standard first-order logic to infinitary logic [34]. This logic considers mixed actions with infinite possible strategies such that each player can play a combination of infinitely many strategies at an infinite number of states. For example, a striker \( i \) gets the ball at time \( s \) either from their teammate or because of a missed pass from an opponent. As striker \( i \)’s objective is to kick the ball through the goal, their strategy depends on the total number of opponents between them and the goal. Therefore, striker \( i \) plays a mixed action, or striker \( i \) places weight \( \alpha \) on scoring strategy \( a \) at the total number of goals scored \( Z_s \) at time \( s \). For a countable collection of objective functions \( \{OB_i\}_{i=1}^\infty \) such that \( \bigwedge_{i=1}^\infty OB^i \) and \( \bigvee_{i=1}^\infty OB^i \) exist, converges to \( \bar{Z}_a \) in real numbers via the formula

\[
OB_a\left( (\bar{Z}_a^i)_{i=1}^\infty, \bar{Z}_a \right) = \forall \varepsilon \left[ \varepsilon > 0 \rightarrow \bigvee_{I \in \mathbb{N}} \bigwedge_{i > I} \left( \bar{Z}_a^i - \bar{Z}_a \right)^2 < \varepsilon^2 \right]
\]

or, without the quantifier the above statement becomes

\[
OB_a\left( (\bar{Z}_a^i)_{i=1}^\infty, \bar{Z}_a \right) = \bigwedge_{K \in \mathbb{N}} \bigvee_{I \in \mathbb{N}} \bigwedge_{i > I} K^2 \left( \bar{Z}_a^i - \bar{Z}_a \right)^2 < 1.
\]

### 3.1 $\sqrt{8/3}$ Liouville Quantum Gravity Surface

Following [35], we know a Liouville quantum gravity (LQG) surface is a random Riemann surface parameterized by a domain \( \mathcal{D} \subset \mathbb{S}^{(l \times d')\times l} \) with Riemann metric tensor \( e^{yk(l)}dZ \otimes d\tilde{Z} \), where \( \gamma \in (0, 2) \), \( k \) is some variant of the Gaussian free field (GFF) on \( \mathbb{D} \), \( l \) is some number coming from 2-sphere \( \mathbb{S}^{(l \times d')\times l} \), and \( dZ \otimes d\tilde{Z} \) is Euclidean metric tensor. In this paper, we consider the case where \( \gamma = \sqrt{8/3} \) because it corresponds to a uniformly random planer map. Because of incomplete and imperfect information, the strategy space is quantum in nature and each player’s decision is a point on a dynamic convex strategy polygon of that quantum strategy space. Furthermore, \( k : \mathbb{S}^{(l \times d')\times l} \rightarrow \mathbb{R}^{l \times d'} \) is a distribution such that each
player’s action can be represented by different shots to the goal including dribbling and passing. Although the strategy set is deterministic, the action on this space is stochastic.

**Definition 4** An equivalence relation $E$ on $S$ is smooth if there is another 2-sphere $S'$ and a distribution function with a conformal map $k' : S \rightarrow S'$ such that for all $l_1, l_2 \in S$, we have $l_1 E l_2 \iff k(l_1) = k(l_2)$.

Now, if $E$ is the equivalence relation of each player’s action, the smooth distribution function $k'$ is an auxiliary tool which helps determining whether $l_1$ and $l_2$ are in the same action component which occurs iff $k(l_1) = k(l_2)$.

**Example 1** Suppose $S' = C'$, where $C'$ represents a complex space. The relation given $l_1 \sim E l_2$ if and only if $l_1 - l_2 \in S'$ is smooth as the distribution $k : C' \rightarrow [0, 1]^l$ is defined by $k(l_1, ..., l_l) = (l_1 - |l_1|, ..., l_l - |l_l|)$, where $|l_m| = \max\{|l^* \in S'| l^* > l_m\}$ is the integer part of $l_m$, then $k(l_1) = k(l_2)$ if and only if $l_1 E l_2$.

Furthermore, $\sqrt{8/3}$-LQG surface is an equivalence class of action on 2-sphere $(D, k)$ such that $D \subset \mathbb{S}^{(\mathrm{id} \times \kappa)}_{\xi}$ is open and $k$ is a distribution function which is some variant of a GFF [35]. Action pairs $(D, k)$ and $(\hat{D}, \hat{k})$ are equivalent if there exists a conformal map $\xi : \hat{D} \rightarrow D$ such that $\hat{k} = k \circ \xi + Q \log |\zeta'|$, where $Q = 2/\gamma + \gamma/2 = \sqrt{3}/2 + \sqrt{2}/3$ [35].

Suppose $I$ is a non-empty finite set of players, $\mathcal{F}_s$ be the filtration of goal $Z, \Omega$ be a sample space. There for each player $i \in I$ an equivalence relationship $E_i \in \mathcal{E}$ on 2-sphere, called player $i$’s quantum knowledge. Therefore, $\sqrt{8/3}$-LQG player knowledge space at time $s$ is $(\Omega, \mathcal{F}_s, S, I, \mathcal{E})$. Given a $\sqrt{8/3}$-LQG player knowledge space $(\Omega, \mathcal{F}_s, S, I, \mathcal{E})$, the equivalence relationship $\mathcal{E}$ is the transitive closure of $\bigcup_{i \in I} E_i$.

**Definition 5** A knowledge space $(\Omega, \mathcal{F}, \mathcal{F}_s, S, I, \mathcal{E})$ such that $i \in I$, each equivalent class of $\mathcal{E}_i$ with Riemann metric tensor $v e^{8/3k(l)} dZ \otimes d\hat{Z}$ is finite, countably infinite or uncountable is defined as purely $\sqrt{8/3}$-LQG knowledge space which is purely quantum in nature (for detailed discussion about purely atomic knowledge, see [34]).

**Definition 6** For a fixed quantum knowledge space $(\Omega, \mathcal{F}_s, S, I, \mathcal{E})$, for player $i$, a dribbling and passing function $p_i$ is a mapping $p_i : \Omega \times S \rightarrow \mathcal{D}(\Omega \times S)$ which is $\sigma$-measurable and the equivalence relationship has some measure in 2-sphere.

Therefore, dribbling and passing space is a tuple $(\Omega, \mathcal{F}, \mathcal{F}_s, S, I, p)$ which is a type of $\sqrt{8/3}$-LQG knowledge space. There are other skills needed to score a goal such as power, speed, agility, shielding, tackling, trapping, and shooting, but we assume only dribbling and passing function is directly related to quantum knowledge. The rest of the uncertainties are coming from the stochastic part of the goal dynamics. The dribbling and passing space of player $i$ implicitly defines quantum
knowledge relationship $E_i$ of dribbling and passing functions. Hence, $(z, l)E_i(z', l')$ iff $p'_{z,l} = p_{z',l'}$, where $z$ is the goal situation according to player $i$’s perspective and $l$ is player $i$’s action on 2-sphere at time $s$.

**Definition 7** For player $i \in I$ and for all $z \in \Omega$, $l \in \mathbb{S}$ dribbling and passing space tuple $(\Omega, F, F_1^z, S, I, p)$ has a function $p_{z,l}^i$ which is purely quantum. Therefore, this space is purely quantum dribbling and passing space.

**Definition 8** For $(\Omega, F, F_1^z, S, I, p)$, if $p_{z,l}^i(z, l) > 0$ for all $i \in I$, $z \in \Omega$ and $l \in \mathbb{S}$, then it is positive. Furthermore, the dribbling and passing space is purely quantum and positive, and then the knowledge space is $\sqrt{8/3}$-LQG.

**Definition 9** A purely quantum space $(\Omega, F, F_1^z, S, I, p)$ is smooth if $\Omega$, the 2-sphere $S$ and the common quantum knowledge equivalence relation $E$ is smooth. As $k$ is a version of GFF, the quantum space is not smooth at the vicinity of the singularity, and for the time being, we exclude those point to make dribbling and the passing space smooth.

**Example 2** Suppose a male soccer team has three strikers A, B, and C such that player A is a left-wing, B is a center-forward and C is right-wing. Furthermore, at time $s$, player B faces an opposition center defensive midfielder (CDM) with at least one of the center backs (left or right) and decides to pass the ball to either of players A and C as they have relatively unmarked positions. Player A knows that if B passes him, based on goal condition $z_1$ based on his judgment, he will attempt to score a goal either by taking a long-distance shot with conditional probability $u(u_1 | z_1)$ or by running with the ball closer to the goal post with probability $u(u_2 | z_1)$, where for $k = 1, 2$, $u_k$ takes the value 1 when player A can score by using any of the two approaches. As the dribbling and passing space is purely quantum, player A’s expected payoff to score is $u(u_1 | z_1)e^{\sqrt{8/3}k_1(l_1)} + u(u_2 | z_1)e^{\sqrt{8/3}k_1(l_2)}$, where $l_k$ is the $k^{th}$ point observed on player A’s 2-sphere $S$. Similarly, player C has the payoff of scoring a goal is $u(u_3 | z_2)e^{\sqrt{8/3}k_2(l_3)} + u(u_4 | z_2)e^{\sqrt{8/3}k_2(l_4)}$, where $u_3$ and $u_4$ are the probabilities of taking a direct shot to goal and running closer to goal post and score. However, player B’s decision tends to be somewhat scattered, as he has to choose to pass either A or C or mixes up the calculation and decides to goal by himself. Consider player B mixes up strategy and decides to goal by himself with probability $\nu$ or he passes either of A and C with probability $(1 - \nu)$, which is known to players A and C.

Let us model the quantum knowledge space of these three players as $\Omega = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ and $S = \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$. Once, the payoff for left-wing A and right-wing C has been revealed, player B calculates the ratio of two expected payoffs. Therefore, player A knows that player B is is giving him a pass if

$$\frac{u(u_1 | z_1)e^{\sqrt{8/3}k_1(l_1)} + u(u_2 | z_1)e^{\sqrt{8/3}k_1(l_2)}}{u(u_3 | z_2)e^{\sqrt{8/3}k_2(l_3)} + u(u_4 | z_2)e^{\sqrt{8/3}k_2(l_4)}} > 1$$
with probability \((1 - \nu)\) and with probability \(\nu\) player B decides to score himself if the ratio is less than equal to 1. As both of the wing, players calculate the ratio of their expected payoffs, and it is enough to define \(\Omega = \mathbb{R}_+ \times \mathbb{R}_+\) and \(S = \mathbb{R}_+ \times \mathbb{R}_+\) for two wing players without losing any information. The quantum equivalent classes \(E^A\) which is player A’s knowledge can be represented as

\[
\begin{bmatrix}
0, u(u_1|z_1)e^{\sqrt{8/3}k_1(l_1)} + u(u_2|z_1)e^{\sqrt{8/3}k_1(l_2)} \\
u(u_3|z_2)e^{\sqrt{8/3}k_2(l_1)} + u(u_4|z_2)e^{\sqrt{8/3}k_2(l_2)} \\
\end{bmatrix} > 1, \quad (z_2 \in \mathbb{R}_+, l_1 \in \mathbb{R}_+, \text{ for all } k = 1, 2)
\]

and player B’s knowledge corresponding to the equivalent class \(E^B\) is

\[
\begin{bmatrix}
u(u_1|z_1)e^{\sqrt{8/3}k_1(l_1)} + u(u_2|z_1)e^{\sqrt{8/3}k_1(l_2)} \\
u(u_3|z_2)e^{\sqrt{8/3}k_2(l_1)} + u(u_4|z_2)e^{\sqrt{8/3}k_2(l_2)} \\
\end{bmatrix} > 1, 0, \quad (z_4 \in \mathbb{R}_+, l_2 \in \mathbb{R}_+, \text{ for all } k = 1, 2)
\]

Therefore, the belief of player A is

\[
p^A_{z,l} = \begin{cases} 1 - \nu & \text{if } \frac{u(u_1|z_1)e^{\sqrt{8/3}k_1(l_1)} + u(u_2|z_1)e^{\sqrt{8/3}k_1(l_2)}}{u(u_3|z_2)e^{\sqrt{8/3}k_2(l_1)} + u(u_4|z_2)e^{\sqrt{8/3}k_2(l_2)}} > 1, \\ \nu & \text{if } \frac{u(u_1|z_1)e^{\sqrt{8/3}k_1(l_1)} + u(u_2|z_1)e^{\sqrt{8/3}k_1(l_2)}}{u(u_3|z_2)e^{\sqrt{8/3}k_2(l_1)} + u(u_4|z_2)e^{\sqrt{8/3}k_2(l_2)}} \leq 1, \\ 0 & \text{otherwise,}
\end{cases}
\]

and similarly for the right wing B,

\[
p^B_{z,l} = \begin{cases} 1 - \nu & \text{if } \frac{u(u_3|z_2)e^{\sqrt{8/3}k_2(l_1)} + u(u_4|z_2)e^{\sqrt{8/3}k_2(l_2)}}{u(u_1|z_1)e^{\sqrt{8/3}k_1(l_1)} + u(u_2|z_1)e^{\sqrt{8/3}k_1(l_2)}} > 1, \\ \nu & \text{if } \frac{u(u_3|z_2)e^{\sqrt{8/3}k_2(l_1)} + u(u_4|z_2)e^{\sqrt{8/3}k_2(l_2)}}{u(u_1|z_1)e^{\sqrt{8/3}k_1(l_1)} + u(u_2|z_1)e^{\sqrt{8/3}k_1(l_2)}} \leq 1, \\ 0 & \text{otherwise.}
\end{cases}
\]

### 3.2 \(\sqrt{8/3}\) Liouville Quantum Gravity Metric

For domain \(D \subset S^{(l \times d)^m}\) and for a variant of GFF the distribution \(k\) suppose, \((D, k)\) is a \(\sqrt{8/3}\)-LQG action surface. As \(k\) is a variant of GFF, \(k\) induces a metric \(\omega_k\) on \(D\), and furthermore, if \((D, k)\) is a quantum sphere, the metric space \((D, \omega_k)\) is isometric.
to the Brownian map with two additional properties: a Brownian disk is obtained if \((\mathbb{D}, k)\) is a quantum disk, and a Brownian plane is obtained if \((\mathbb{D}, k)\) is a \(\sqrt{8/3}\) -quantum cone \([35–37]\). Following \([35]\), we know that \(\sqrt{8/3}\)-LQG surface can be represented as a Brownian surface with conformal structure. We assume the strategy space where the action is taken as the property like \(\sqrt{8/3}\)-LQG surface because each soccer player has radius \(r\) around themselves such that if an opposing player comes in this radius, he would be able to tackle. Furthermore, if \(r \to 0\), the player has complete control over the opponent with no mistakes and the same skill level. Therefore, the strategy space closer to the player (i.e., \(r = 0\)) bends towards himself in such a way that, the surface can be approximated to a surface on a 2 sphere and as the movement on this space is stochastic, it behaves like a Brownian surface with its convex strategy polygon changes its shape at every time point based on the condition of the game. At \(r = 0\), the surface hits essential singularity, and the player has infinite power to control the ball.

The \(\sqrt{8/3}\)-LQG metric will be constructed on a 2-sphere \((\mathbb{S}, k)\) which is also quantum in nature. Based on the distribution function \(k\), suppose \(C_i\) be the collection of i.i.d. locations of player \(i\) on the strategy space sampled uniformly from the area of their convex dynamic polygon \(\beta_i\). Furthermore, consider inside a polygon with area measure \(\beta_i\), player \(i\)’s action at time \(s\) is \(a_i^s\) and at time \(\tau\) is \(a_i^\tau\) such that, for \(\varepsilon > 0\), we define \(\tau = s + \varepsilon\). Therefore, \(a_i^s, a_i^\tau \in C_i\) is a quantum Loewner evolution growth process \(\{G^d_{s, \tau}\}_{s \geq \tau}^\infty\) starting from \(a_i^s\) and ending at \(a_i^\tau\) for all \(s \in [s, \tau]\).

Let there be two growth processes \(\{G^d_{s, \tau}\}_{s \geq \tau}^\infty\) and \(\{G'^d_{s, \tau}\}_{s \geq \tau}^\infty\) where both of them start at \(a_i^s\) and ends at \(a_i^\tau\). We will show on this surface that they are homotopic. Suppose \([p^1_i, p^2_i, ... , p^M_i]\) are \(M\)-partitions on player \(i\)’s finite convex strategy polygon \(\beta_i\) on \(\mathbb{S}((x, t))\). Hence, each of \(\beta_k \in \beta_i\) can be uniquely represented by \(\beta_k = \sum_{n=0}^M \theta_n(\beta_k)p^1_i\), where \(\theta_n(\beta_k) \in [0, 1]\) for all \(n = 0, ..., M\) and \(\sum_{n=0}^M \theta_n(\beta_k) = 1\). Thus,

\[
\theta_n(\beta_k) = \begin{cases} \text{barycentric coordinate of } \beta_k \text{ relative to } p^1_i \text{ if } p^1_i \text{ is a vertex of } \beta_k, \\ 0 \text{ otherwise.} \end{cases}
\]

Therefore, \(\theta_n(\beta_k) \neq 0\) are barycentric coordinates of \(\beta_k\). Furthermore, as \(\theta_n\) is identically equal to zero or the barycentric function of \(\beta_k\) relative to the vertex \(p^1_i\), \(\theta_n : \beta_k \to I\) is continuous, where \(I\) is the range of \(p^1_i\) which takes the value between 0 and 1. Now, assume the growth process has the map \(\{G^d_{s, \tau}\}_{s \geq \tau}^\infty : Y_i \to \beta_k\), where \(Y_i\) is an arbitrary topological space and it has a unique expression \(\{G^d_{s, \tau}\}_{s \geq \tau}^\infty(Y_i) = \sum_{n=0}^M \theta_n(p^n_iG^d_{s, \tau}(Y_i))\). As this growth process is a quantum Loewner evolution, it is continuous. Therefore, \(\theta_n \circ \{G^d_{s, \tau}\}_{s \geq \tau}^\infty : Y_i \to I\) is continuous.

**Lemma 1** Let \(\beta_k\) be a finite convex strategy polygon of player \(i\) in \(\mathbb{S}((x, t))\) such that the vertices are \([p^1_i, ..., p^M_i]\). Suppose \(Y_i\) be an arbitrary topological space and let \(\{G^d_{n, \tau}\}_{s \geq \tau}^\infty : Y_i \to \beta_k\) be a family of growth processes, one for each vertex \(p^1_i\), with \(\sum_{n=0}^M p^n_i\{G^d_{n, \tau}(Y_i)\}_{s \geq \tau}^\infty \in \beta_k\) for each \(y_i \in Y_i\). Then, \(y_i \mapsto \sum_{n=0}^M p^n_i\{G^d_{n, \tau}(Y_i)\}_{s \geq \tau}^\infty \in \beta_k\) is a continuous map \(\{G^d_{s, \tau}\}_{s \geq \tau}^\infty : Y_i \to \beta_k\) as \(\{G^d_{s, \tau}\}_{s \geq \tau}^\infty\) is a quantum Loewner evolution.
Proof As \( \{G_3^{d_i,a_i}\}_{i \geq 0} \) is a quantum Loewner evolution, it is continuous. Furthermore, as \( \{G_3^{d_i,a_i}\}_{i \geq 0} : Y_i \subset \beta_k \subset S^{(I \times I') \times t} \), the continuity of \( \{G_3^{d_i,a_i}\}_{i \geq 0} \) as a map of \( Y_i \) into \( \beta_k \) follows.

Lemma 2 Let \( \beta_k \) be a finite convex strategy polygon of player \( i \) in \( S^{(I \times I') \times t} \). Suppose \( Y_i \) be an arbitrary space such that \( \{G_3^{d_i,a_i}\}_{i \geq 0} \), \( \{G_3^{d_i,a_i}\}_{i \geq 0} : Y_i \to \beta_k \) is a quantum Loewner evolution. For each \( y_i \in Y_i \), suppose there is a smaller finite convex strategy polygon \( \rho \in \beta_k \) containing both \( \{G_3^{d_i,a_i}(y_i)\}_{i \geq 0} \) and \( \{G_3^{d_i,a_i}(y_i)\}_{i \geq 0} \). Then, these two growth processes \( \{G_3^{d_i,a_i}\}_{i \geq 0} \) and \( \{G_3^{d_i,a_i}\}_{i \geq 0} \) are homotopic.

Proof For each \( y_i \in Y_i \), the line joining two growth processes \( \{G_3^{d_i,a_i}(y_i)\}_{i \geq 0} \) and \( \{G_3^{d_i,a_i}(y_i)\}_{i \geq 0} \) is in \( \rho \). Therefore, this line segment is definitely inside \( \beta_k \). The representation of this segment is

\[
S(y_i, U) = \sum_{n=0}^{M} \left[ U \theta_n \left( \{G_3^{d_i,a_i}(y_i)\}_{i \geq 0} \right) + (1 - U) \theta_n \left( \{G_3^{d_i,a_i}(y_i)\}_{i \geq 0} \right) \right].
\]

Furthermore, as the mapping \( U \theta_n \circ \{G_3^{d_i,a_i}\}_{i \geq 0} + (1 - U) \theta_n \circ \{G_3^{d_i,a_i}\}_{i \geq 0} : Y_i \times I \to I \) is continuous, the function \( S : Y_i \times I \to I \) is continuous. This is enough to show homotopy.

Corollary 3 Let \( \beta_k \) be a finite convex strategy polygon of player \( i \) in \( S^{(I \times I') \times t} \). Suppose \( \{G_3^{d_i,a_i}\}_{i \geq 0}, \{G_3^{d_i,a_i}\}_{i \geq 0} : \beta_k \to \beta_k \) is a quantum Loewner evolution. For each \( \beta_k \in \beta_k \), suppose there is a smaller finite convex strategy polygon \( \rho \in \beta_k \) containing both \( \{G_3^{d_i,a_i}(\beta_k)\}_{i \geq 0} \) and \( \{G_3^{d_i,a_i}(\beta_k)\}_{i \geq 0} \). Then, these two growth processes \( \{G_3^{d_i,a_i}\}_{i \geq 0} \) and \( \{G_3^{d_i,a_i}\}_{i \geq 0} \) are homotopic.

From the above corollary, it is clear that any growth process in the interval \( \{d_i,a_i\} \) inside the finite strategy polygon \( \beta_k \) is homotopic. Therefore, without any other further restrictions, a soccer player can follow any path without hampering their payoff.

Definition 10 A collection \( k = \{k(I \times I') \} \) which are assumed to be homomorphisms \( k(I \times I') : C^{(I \times I') \times t}(\beta_k) \to C^{(I \times I') \times t}(\beta_k) \), one for each dimension \( (I \times I') \times t \geq 0 \), and such that \( \partial_{(I+1)(I')(I'+1)(t+1)} \circ k_{(I+1)(I')(I'+1)(t+1)} \circ \partial_{(I+1)(I')(I'+1)(t+1)} = k_{(I \times I') \times (I' \times I' + 1)} \circ \partial_{(I+1)(I')(I'+1)(t+1)} \) is called a chain transformation or chain map on \( \mathbb{S} \) where, \( \beta_k \) is a convex strategy polygon of player \( i \) other than \( \beta_k \).

For a finite convex polygon \( \beta_k \), we take the chains over 2-sphere \( \mathbb{S} \) and the chain transformation \( k \) of \( C_{(\beta_k;\mathbb{S})} \) into itself. Then, the following relationship of the trace is going to hold.
Lemma 4 (Hopf trace theorem, [38]) Suppose the dimension of player i’s strategy polygon $\beta_i$ has the dimension of $(I \times I') \times t$, and the collection of distribution function which follows

$$k_* : C_*(\beta_k, \mathbb{S}) \to C_*(\beta_k, \mathbb{S}),$$

by any chain transformation. Then,

$$\sum_{m=0}^{(I \times I') \times t} (-1)^m \text{tr}(k_m) = \sum_{m=0}^{(I \times I') \times t} (-1)^m \text{tr}(k_{m*}),$$

where $\text{tr}(.)$ represents trace of the argument.

For a finite convex strategy polygon $\beta_k$ in $\sqrt{8/3}$-LQG consider the map $\hat{k} : \beta_k \to \beta_k$. Using rational points on the surface $\mathbb{Q}$ as coefficients, then each induced homomorphism $\hat{k}_*^{(I \times I') \times t} : H^*(I \times I'; \mathbb{Q}) \to H^*(I \times I'; \mathbb{Q})$ is an endomorphism, where $\hat{k}_*$ is the linear transformation of the vector space. Following [38], each $\hat{k}_*^{(I \times I') \times t}$ has a trace $\text{tr} \left[ \hat{k}_*^{(I \times I') \times t} \right]$.

Definition 11 For the strategy polygon of player $i$ with $\dim(\beta_k) \leq (I \times I') \times t$ and the endomorphic map $\hat{k} : \beta_k \to \beta_k$, the Lefschetz number $\phi(\hat{k})$ is

$$\phi(\hat{k}) = \sum_{m=0}^{(I \times I') \times t} (-1)^m \text{tr}[\hat{k}_{m*}, H^m(\beta_k; \mathbb{Q})].$$

Lemma 5 (Granas and Dugundji [38]) For a strategy polygon $\beta_k$, if $\hat{k} : \beta_k \to \beta_k$ is continuous, then $\phi(\hat{k})$ depends on the homotopy class of $\hat{k}$. Furthermore, the Lefschetz number $\phi(\hat{k})$ is an integer irrespective of any $\sqrt{8/3}$-LQG field characteristics.

Proposition 1 (Lefschetz-Hopf fixed point theorem on $\sqrt{8/3}$-LQG surface) If player $i$ has a finite strategy polygon $\beta_k$ on $\mathbb{S}$ with the map $\hat{k} : \beta_k \to \beta_k$, then $\hat{k}$ has a fixed point for all Lefschetz number $\phi(\hat{k}) \neq 0$.

Proof We will prove this theorem by contradiction. Let $\hat{k}$ has no fixed points. As player $i$’s strategy polygon $\beta_k$, compact, for all $\beta_k \in \beta_k$, there $\exists \epsilon > 0$ such that the measure $d(\hat{k}(\beta_k), \beta_k) \geq \epsilon$. In this proof, a repeated barycentric subdivision of $\beta_k$ will be used with a fixed quadrilateral of mesh $< \epsilon/n_t$, where $n_t \in \mathbb{N}$.

Let $\beta_k^{(m)}$ be $m^\text{th}$ barycentric subdivision of $\beta_k$ such that the mapping $\psi : \beta_k^{(m)} \to \beta_k$ be a simplicial approximation of $\hat{k}$. Define $m^\text{th}$ barycentric subdivision’s map when chain characteristic is present in $\text{sub}^m : C_*(\beta_k) \to C_*(\beta_k^{(m)})$. It is enough to determine the trace of $\psi \text{sub}^m : C_q(\beta_k; \mathbb{Q}) \to C_q(\beta_k^{(m)}; \mathbb{Q})$ for each oriented $q$-sided smaller finite strategy polygons inside $\beta_k$, where $C_q(\beta_k; \mathbb{Q})$ is the chain characteristic on $q$-sided smaller strategy polygon. To determine trace we would use the Hopf trace theorem explained above. Suppose, for each $C_q(\beta_k; \mathbb{Q})$, there exists a basis $\{ \rho_i^q \}$ of all
oriented $q$-sided smaller strategy polygons in $\beta_k$. Expressing $\psi \text{sub}_m$ in terms of the basis function would be
\[
\text{sub}_m \rho_i^q = \sum \alpha_{ll'}^q k_p^q, \quad \alpha_{ll'} = 0, \pm 1, \ k_p^q \in \beta_k^{(m)}, \ k_p^q \subset \rho_i^q.
\]
Hence,
\[
\psi \text{sub}_m \rho_i^q = \sum \alpha_{ll'} \psi(k_m^q) = \sum \gamma_{ll'} \rho_i^q,
\]
where $\gamma_{ll'}$ is another basis. Now, suppose $v$ is the vertex of any $k_p^q \subset \rho_i^q$. Define $\nu_* := \beta_k \setminus \{ \rho \in \beta_k | \nu \notin \rho \}$. Then, for vertex $v$, we have $\hat{k}(v) \in \hat{k}(v_*) \subset \psi_*(v)$ such that $d[\hat{k}(v), \psi(v)] < \varepsilon/n_i$, where $\psi_* := \beta_k \setminus \{ \rho \in \beta_k | \nu \notin \rho \}$. Clearly,
\[
d[v, \psi(v)] \geq d[v, \hat{k}(v)] - d[\hat{k}(v), \psi(v)] \geq 2\varepsilon/n_i,
\]
such that if $\psi(v)$ belongs to $\rho_i^q$, then $\delta(\rho_i^q) \geq 2\varepsilon/n_i$, which is the contradiction of our argument that quadragulation is $< \varepsilon/n_i$. Therefore, $\text{tr}[\psi \text{sub}_m, C_\eta(\beta_k; \mathbb{Q})] = 0$ for each $q$-sided finite smaller strategy space. Finally, by Hopf trace theorem, we can say that Lefschetz number $\phi(\hat{k}) = 0$. This completes the proof.

According to [35], it is a continuum analog of the first passage percolation on a random planner map. Suppose for $\tau > 0$ and let $\eta_s^i$ be a whole plane of a Schramm-Loewner Evolution with the central charge $6$ (SLE$_6$) from $a_i^s$ to $a_i^\tau$ which is sampled independent with respect to $k$. Now, the plane $\eta_s^i$ has been run by terminal time interval $\tau$ such that $\forall \varepsilon > 0$ and determine it by $k$. For $\epsilon > 0$ and $s \in [s, \tau]$, suppose $G_s^d, a_i^s, \tau := \eta_s^i([s, \tau \land h^s])$, where $h^s$ is the first time $\eta_s^i$ hits $a_i^s$ when the player $i$ starts the process at $a_i^s$ and going towards $a_i^\tau$ [35]. Following [12], we know that, for $\epsilon \downarrow 0$, a growing family of sets $\{G_s^d, a_i^s\}_{s \geq 0}$ in the action interval $[a_i^s, a_i^\tau]$ can be found by taking almost sure limits of an appropriate chosen subsequence, which [35] calls as quantum Loewner evolution defined on $\sqrt{8/3}$-quantum sphere (QLE$(8/3, 0)$). For $\varepsilon \geq 0$, suppose $\mathcal{M}_s^d, a_i^s, a_i^\tau(\mathbf{Z}, \mathbf{W})$ be some length of the boundary of the connected set $\mathbb{S} \setminus G_s^d, a_i^s, \tau$, such that it contains the terminal action $a_i^\tau$, where $\mathbf{Z}$ be all possible the goals, $\mathbf{W}$ be all possible control strategies available to player $i$ at time $s$. Furthermore, assume the stopping time due to reach action $a_i^\tau$ is $\sigma^d, a_i^\tau > 0$ define a measure $\mathcal{M} \geq 0$ such that
\[
\mathcal{M}(\mathbf{Z}, \mathbf{W}) = \int_s^{\sigma^d, a_i^\tau} \frac{1}{\mathcal{M}_s^d, a_i^s, a_i^\tau(\mathbf{Z}, \mathbf{W})} d\tilde{s}.
\]
Define $\tilde{G}_\mathcal{M}^{d, a_i^s} := G_s^d, a_i^s, a_i^\tau$. Then, following [35], $\sqrt{8/3}$-LQG distance of $[a_i^s, a_i^\tau]$ is defined as
\[
\omega_k(a_i^s, a_i^\tau) := \inf \left\{ \mathcal{M}(\mathbf{Z}, \mathbf{W}) \geq 0; a_i^\tau \in \tilde{G}_{\mathcal{M}}^{d, a_i^s} \right\}.
\]
Finally, by [35] and [13], for all $\hat{r}_i \in \mathbb{R}$, the construction of this metric for $k + \hat{r}_i$ yields a scaling property such as

$$\omega_{k+\hat{r}_i}(d_s^i, d_e^i) = e^{\left(\sqrt{\frac{8}{3}}\right)^{\hat{r}_i/4}} \omega_k(d_s^i, d_e^i).$$

### 3.3 Interpretation of the Diffusion Part of Goal Dynamics

Equation (3) talks about two components of the diffusion part, $\hat{\sigma}$ which comes from the strategies from the opposition team, and $\sigma^*$ consists of the venue, the percentage of attendance of the home crowd, the type of match (i.e., day or day–night match), the amount of dew on the field, and the speed of wind (which gives an advantage to those free kickers who like make swings due to wind).

Firstly, consider the situation where a team is playing abroad. In this case, players feel relatively more stressed to score a goal than in their home environment. For example, if an Argentine or a Brazilian player plays in a European league, it is relatively tough for them to score a goal in a European environment as their playing style is very different than their native lands in Latin America. This is also true for the men’s soccer World Cup where Brazil is the only team from Latin America to be able to win the cup in Europe (1958 FIFA World Cup in Sweden). As playing abroad would create extra mental stress on players, we assume that pressure is a non-negative $C^{1,2}$ function $p(s, Z) : [0, t] \times \mathbb{R} \to \mathbb{R}$ at match $M + 1$ such that if $Z_{s-1} < E_{s-1}(Z)$ then $p$ takes a very high positive value. In other words, if the actual number of goals at time $s-1$ (i.e., $Z_{s-1}$) is less than the expected number of goals at that time, then the pressure to score a goal at $s$ is very high.

Secondly, the percentage of a home crowd in the total crowd of the stadium matters in the sense that if it is a home match for a team, then players get extra support from their fans and are motivated to score more. If a team has a mega-star, they will access more crowd. Generally, mega stars have fans all over the world, and therefore, they might get more crowds from the opposition team than their teammates. Define a positive finite $C^{1,2}$ function $A(u, W) : [0, t] \times \mathbb{R} \to \mathbb{R}$ with $\partial A / \partial W > 0$ and $\partial A / \partial s \geq 0$ depends on if at time $s$, player $i$ with valuation $W_i \in W$ is still playing, or is out of the field due to injury or other reasons.

Thirdly, if the match is a day match, then both teams have a comparative advantage in better visibility due to the sun. On the other hand, if the match is a day–night match, then a team’s objective is to choose the side of the field in such a way that they get better visibility and get an advantage by scoring more goals. Therefore, in this game, a team who loses the toss has a disadvantage in terms of the position of the field. Furthermore, because of the dew, the ball becomes wet and heavier at night and, and it would be harder to grip from a goalkeeper’s point of view as well as move the ball and score from a striker’s point of view. Therefore, if the toss-winning team scores some goals in the first half, that team surely has some comparative advantage to win the game. Hence, winning the toss is important. Furthermore, if a team loses the toss, then its decision to score in either of two halves depends on its opposition. Hence, define a function $\mathcal{B}(Z) \in \mathbb{R}$ such that
\[ \mathcal{B}(Z) = \frac{1}{2} \left[ \frac{1}{2} \mathbb{E}_0(Z_{p1}^2) + \frac{1}{2} \mathbb{E}_0(Z_{D}^1) + \frac{1}{2} \mathbb{E}_0(Z_{D}^2) \right], \]  

where for \( i = 1, 2, \) \( \mathbb{E}_0(Z_{pi}^i) \) is the conditional expectation of the goal of a team before the starting of the day match \( M + 1 \) with the total number of goals at \( i^{th} \) half \( Z_{D}^i \) and \( \mathbb{E}_0(Z_{DN}^i) \) is the conditional expectation of the goal before starting a day–night match \( M + 1 \). Furthermore, if a team wins the toss, then it will go for the payoff \[ \frac{1}{2} \left[ \frac{1}{2} \mathbb{E}_0(Z_{D}^2) + \frac{1}{2} \mathbb{E}_0(Z_{DN}^1) \right], \]  

and the later part of the Eq. (4) otherwise.

Finally, we consider the dew point measure and the speed of wind at time \( s \) as an important factor in scoring a goal. As these two are natural phenomena and represent ergodic behavior, we assume this can be represented by a Weierstrass function \( Z_e : [0, t] \rightarrow \mathbb{R} \) [39] defined as

\[ Z_e(s) = \sum_{\alpha = 1}^{\infty} (\lambda_1 + \lambda_2)^{(s-2)\alpha} \sin \left( (\lambda_1 + \lambda_2)^{\alpha} u \right), \]  

where \( s \in (1, 2) \) is a penalization constant of weather at over \( u, \lambda_1 \) is the dew point measure defined by the vapor pressure \( 0.6108 * \exp \left( \frac{17.27T_d}{T_d + 237.3} \right) \), where dew point temperature \( T_d \) in defined in Celsius, and \( \lambda_2 \) is the speed of wind such that \((\lambda_1 + \lambda_2) > 1\).

**Assumption 4**: \( \sigma^*(s, W, Z) \) is a positive, finite part of the diffusion component in Eq. (2) which satisfies Assumptions 1 and 2 and is defined as

\[ \sigma^*(s, W, Z) = p(s, Z) + A(s, W) + \mathcal{B}(Z) + Z_e(s) + \rho_1 p^T(s, Z)A(s, W) + \rho_2 A^T(s, W)\mathcal{B}(Z) + \rho_3 \mathcal{B}^T(Z)p(s, Z), \]  

where \( \rho_j \in (-1, 1) \) is the \( j^{th} \) correlation coefficient for \( j = 1, 2, 3 \), and \( A^T, B^T \) and \( p^T \) are the transposition of \( A, B, \) and \( p \) which satisfy all conditions with Eqs. (4) and (5). As the ergodic function \( Z_e \) comes from nature, a team does not have any control on it and its correlation coefficient with other terms in Eq. (6) are assumed to be zero.

The randomness \( \hat{\sigma} \) of Eq. (3) comes from the type of skill of a soccer star of the opposition team. There are mainly two main types of players: dribblers and tacklers and free kickers. Free kickers have two components, the speed of the ball after their kick \( s \in \mathbb{R}^{(lx)^x} \) in miles per hour, and the curvature of the ball path measured by the dispersion from the straight line connecting the goalkeeper and the striker measured by \( x \in \mathbb{R}^{(lx)^x} \) inches. Define a payoff function \( A_1(s, x, G) : \mathbb{R}^{(lx)^x} \times \mathbb{R}^{(lx)^x} \times [0, 1] \rightarrow \mathbb{R}^{2(lx)^x} \) such that, at time \( s \), the expected payoff after guessing a ball right is \( \mathbb{E}_A(s, x, G) \), where \( G \) is a guess function such that if a player \( i \) guesses the curvature and speed of the ball after an opposition player kicks, then \( G = 1 \), and if player \( i \) does not, then \( G = 0 \), and if player \( i \) partially guesses, then \( G \in (0, 1) \).

On the other hand, there is a payoff function \( A_2 \) for a dribbler such that \( A_2(s, x, \theta_1, G) : \mathbb{R}^{(lx)^x} \times \mathbb{R}^{(lx)^x} \times [\frac{1}{2} \pi, \frac{3}{2} \pi] \times [0, 1] \rightarrow \mathbb{R}^{2(lx)^x} \), where \( \theta_1 \) is the angle between the beginning and end points when an opposition player start dribbling.
and end it after player \( i \) gets the ball and \( k \in \mathbb{N} \). The expected payoff to score a goal at time \( s \) when the opposition player is a dribbler is \( \mathbb{E}_x A_2(s, x, \theta_1, G) \). Finally, an

tacker’s payoff function is \( A_3(s, x, \theta_1, \theta_2, G) : \mathbb{R}^{(l \times d') \times t} \times \mathbb{R}^{(l \times d') \times t} \times [-k\pi, k\pi] \times \mathbb{R} \to \mathbb{R}^{2(l \times d') \times t} \), where \( \theta_2 \) is the allowable tackle movement in terms of angle when they do either of block, poke, or slide tackles at time \( s \) as \( \mathbb{E}_x A_3(s, x, \theta_1, \theta_2, G) \). If \( \theta_2 \) is more than \( k\pi \), the opposition player gets a foul or a yellow card. As player \( i \) does not know who is what type of opposition they are going to face at a certain time during a game, their total expected payoff function at time \( s \) is \( A(s, x, \theta_1, \theta_2, G) = \mathcal{G}_1 \mathbb{E}_x A_1(s, x, G) + \mathcal{G}_2 \mathbb{E}_x A_2(s, x, \theta_1, G) + \mathcal{G}_3 \mathbb{E}_x A_3(s, x, \theta_1, \theta_2, G) \), where for \( j \in \{1, 2, 3\} \), \( \mathcal{G}_j \) is the probability of each of a dribbler, a tackler, and a free kicker with \( \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 = 1 \). Therefore, \( A(s, x, \theta_1, \theta_2, G) : \mathbb{R}^{(l \times d') \times t} \times \mathbb{R}^{(l \times d') \times t} \times [-k\pi, k\pi] \times [-k\pi, k\pi] \times [0, 1] \to \mathbb{R}^{2(l \times d') \times t} \).

### 4 Main Results

The components of stochastic differential games under \( \sqrt{8/3} \)-LQG with a continuum of states with dibbling and the passing function with finite actions are the following:

- \( I \) be a non-empty finite set of players and \( \mathcal{F}_s^Z \) be the filtration of goal \( Z \) with the sample space \( \Omega \).
- A finite set of actions at time \( s \) for player \( i \) such that \( a_i^j \in A_i^j \) for all \( i \in I \).
- A discount rate \( \rho_i^j \in (0, 1) \) for player \( i \in I \) with the constant weight \( \alpha_i \in \mathbb{R} \).
- The bounded objective function \( \mathcal{OB}_a^i \) expressed in Eq. (1) is Borel measurable. Furthermore, the system has a goal dynamics expressed in the Eq. (2).
- The game must be on a \( \sqrt{8/3} \)-LQG surface on 2-sphere \( \mathbb{S}^{(l \times d') \times t} \) with the Wiener metric tensor \( e^{\sqrt{8/3}k(l)} dZ \otimes d\hat{W} \), where \( k \) is some variant of GFF such that \( k : \mathbb{S}^{(l \times d') \times t} \to \mathbb{R}^{(l \times d') \times t} \).
- For \( \varepsilon > 0 \), there exists a transition function from time \( s \) to \( s + \varepsilon \) expressed as \( \Psi_{s,s+i}(Z) : \Omega \times \mathbb{S} \times \mathbb{R}^{(l \times d') \times t} \to \Delta(\Omega \times \mathbb{S} \times \mathbb{R}^{(l \times d') \times t}) \) which is Borel-measurable.
- For a fixed quantum knowledge space \( (\Omega, \mathcal{F}_s^Z, \mathbb{S}, I, \mathcal{E}) \), for player \( i \), a dribbling and passing function \( p_i^j \) is a mapping \( p_i^j : \Omega \times \mathbb{S} \to \Delta(\Omega \times \mathbb{S}) \) which is \( \sigma \)-measurable and the equivalence relationship has some measure in 2-sphere.

The game is played in continuous time. If \( Z_{l \times d'} \subset \mathbb{Z} \subset (\Omega \times \mathbb{S}) \) be the goal condition after the start of \( (M + 1)^{th} \) game and player \( i \) select an action profile at time \( s \) such that \( a_i^j \in \prod_i A_i^j \), then for \( \varepsilon > 0 \), \( \Psi_{s,s+i}(Z, a_i^j) \) is the conditional probability distribution of the next stage of the game. A stable strategy for a soccer player \( i \) is a behavioral strategy that depends on the goal condition at time \( s \). Therefore, we can say it is Borel measurable mapping associated with each goal \( Z \subset \Omega \) a probability distribution on the set \( A_i^j \).

**Definition 12** A stochastic differential game is purely quantum if it has countable orbits on 2-sphere. In other words,
For each goal condition $Z_{t \times l'} \subset Z \subset (\Omega \times S)$, every action profile of player $i$ starts at time $s_i, a_i^s \in A^i$ with the $\sqrt{8/3}$-LQG measure for a very small space on $S$ defined as $e^{v^{8/3k(l)}}$, $i^{th}$ player’s transition function $\Psi^{i}_{s,s+\varepsilon}(Z)$ is a purely quantum measure. Define

$$Q(Z) := \left\{ Z' \in (\Omega \times S) \mid \exists a_i^s \in A^i, e^{\sqrt{8/3k(l)}} \neq 0, \Psi^{i}_{s,s+\varepsilon}(Z'|Z, a_i^s) > 0 \right\}.$$

For each goal condition $Z_{t \times l'} \subset Z \subset (\Omega \times S)$, the set

$$Q^{-1}(Z) := \left\{ Z' \in (\Omega \times S) \mid \exists a_i^s \in A^i, e^{\sqrt{8/3k(l)}} \neq 0, \Psi^{i}_{s,s+\varepsilon}(Z'|Z, a_i^s) > 0 \right\}$$

is countable.

For purely atomic games, see [34].

**Proposition 2** A purely quantum game on $S$ with the objective function expressed in the Eq. (1) subject to the goal dynamics expressed in the Eq. (2) in which the orbit equivalence relation is smooth admits a measurable stable equilibrium.

We know a soccer game stops if rainfall is so heavy that it restricts the vision of the player, making it dangerous or the pitch becomes waterlogged due to the heavy rain. After a period of stoppage, the officials will determine the conditions with tests using the ball while the players wait off the field. Suppose a match stops after time $\tilde{t}$ because of the rain. After that, there are two possibilities: first, if the rain is heavy, the game will not resume; secondly, if the rain is not so heavy and stops after a certain point of time, then after getting water out of the field, the match might be resumed. Based on the severity of the rain and the equipment used to get the water out from the field, the match resumes for $[\tilde{t}, t - \varepsilon]$ where $\varepsilon \geq 0$. The importance of $\varepsilon$ is that if the rain is very heavy, $\varepsilon = t - \tilde{t}$ and on the other hand, for the case of very moderate rain, $\varepsilon = 0$. Therefore, $\varepsilon \in [0, t - \tilde{t}]$.

Now, we are going to construct a stochastic Itô-Henstock-Kurtzweil-McShane-Feynman-Liouville-type path integral in goal dynamics under rain. Since rain makes the environment more volatile, standard Riemann sum is not suitable. Instead, we construct a gauge integral on $\sqrt{8/3}$-LQG surface. First, we define a probability space $(\Omega, \mathcal{F}, \mathcal{F}^l, \mathbb{P})$ and create a $\delta$-gauge for time subinterval $[\tilde{t}, t - \varepsilon]$. Due to uncertainty in the environment caused by rain, the time partition becomes stochastic in nature. This leads to the incidence of Itô lemma in those intervals and leads to an Itô $\delta$-gauge. Based on this stochastic $\delta$ gauge, we construct a $\delta$-fine for that time subinterval. This gives the regular Henstock-Kurtzweil-McShane integral a stochastic structure. Furthermore, as our main interest is to study the Feynman action function with $\sqrt{8/3}$-LQG metric, the integral becomes a stochastic Itô-Henstock-Kurtzweil-McShane-Feynman-Liouville-type path integral. A detailed analysis of this integral is the following.
Definition 13 For a probability space \((\Omega, \mathcal{F}, \mathcal{F}_t^Z, \mathcal{P})\) with sample space \(\Omega\), filtration at time \(s\) of goal condition \(Z\) as \(\mathcal{F}_t^Z \subset \mathcal{F}_t\), a probability measure \(\mathcal{P}\) and a Brownian motion for rain \(B_t\), with the form \(B_t^{-1}(E)\) such that for \(s \in [\tilde{t}, t - \varepsilon]\), \(E \subseteq \mathbb{R}\) is a Borel set. If \(\tilde{t}\) is the game stopping time because of rain and \(b \in \mathbb{R}\) is a measure of rain in millimeters, then \(\tilde{t} := \inf\{s \geq 0 | B_s > b\}\).

Remark 6 After the rain, there are a lot of changes on the field. Rain gives extra moisture in the environment which gives strikes and free-kickers to swing their bows during a shot towards the opposition goalpost. Furthermore, because of the moisture, the ball will not come to a player according to that player’s prediction. This leads to difficulties to receive a pass from a team mate. As it is impossible to dry out the field completely, there are dew on the grass which reduces the speed of a ball after a shot. Therefore, a player has an extra advantage to swing a ball during a shot. Apart from that, as the field is slippery, it is very difficult to save the ball before crossing the boundary of the field and leads to a multiple throws. Therefore, we can conclude that the goal dynamics after the rain stoppage is different than before.

Definition 14 Let \(\delta_{\tilde{t}} : [\tilde{t}, t - \varepsilon] \rightarrow (0, \infty)\) be a \(C^2(s \in [\tilde{t}, t - \varepsilon])\) time-process of a match such that, it replaces stochastic process by Itô’s Lemma. Then, \(\delta_{\tilde{t}}\) is a stochastic gauge of that match if \(\tilde{t} := \hat{s} + \delta_{\tilde{t}}\) is a stopping time for each \(\hat{s} \in [\tilde{t}, t - \varepsilon]\) and \(B_{\hat{s}} > b\), where \(\hat{s}\) is the new time after resampling the stochastic interval \(\tilde{t}, t - \varepsilon\) on \(\sqrt{8/3}\)-LQG surface.

Remark 7 In order to understand the Definition 14 properly, let us assume \(\tilde{\mathcal{P}} = \{x_{0t}, x_1, ..., x_p\}\) is some partition of a stochastic time interval \([\tilde{t}, t - \varepsilon]\) with some sampling time-points \(\{\hat{s}_i\}_{i=1}^p\) such that, \(\hat{s}_i \in [x_{i-1}, x_i] \subset (\tilde{s}_i - \delta_{\tilde{t}}, \tilde{s}_i + \delta_{\tilde{t}}) \subset [\tilde{t}, t - \varepsilon]\). The importance of considering a stochastic gauge is that, after starting of the game at time \(\tilde{t}\), the environmental conditions are such that, under the worse case scenario, the rain might start again just after few minutes after \(\tilde{t}\) and the game needs to be stopped permanently. Therefore, we introduce the sample points \(\hat{s}\) instead of \(s\) because we only consider time \(s + 1\) if the measure of rain is less than \(b\) millimeter which makes \(s\) as a stochastic time. Here, the sample point time \(\hat{s}\) and the function \(\delta_{\hat{s}}\) replaces \(s\). In Itô’s sense, one might think this \(\hat{s}\) with its \(\delta_{\hat{s}}\) as a \(C^{1,2}\) function whose Laplacian operator with respect to \(s\) has all the information of \(s\). Since the game resumes at time \(\tilde{t}\) and ends at \(t - \varepsilon\), for all \(\varepsilon \geq 0\), we consider that these two times are stopping points such that \([\tilde{t}, t - \varepsilon] \subseteq [\tilde{t}, t]\).

Definition 15 Given a stochastic time interval of the soccer game \(\hat{I} = [\tilde{t}, t - \varepsilon] \subset \mathbb{R}\), a stochastic tagged partition of that match is a finite set of ordered pairs \(\mathcal{D} = \{(\hat{s}_i, \hat{I}_i) : i = 1, 2, ..., p\}\) such that \(\hat{I}_i = [x_{i-1}, x_i] \subset [\tilde{t}, t - \varepsilon]\), \(\hat{s}_i \in \hat{I}_i\), \(\bigcup_{i=1}^p \hat{I}_i = [\tilde{t}, t - \varepsilon]\) and for \(i \neq j\) we have \(\hat{I}_i \cap \hat{I}_j = \{\emptyset\}\). The point \(\hat{s}_i\) is the tag partition of the stochastic time-interval \(\hat{I}_i\) of the game.
Definition 16 If $\mathcal{D} = \{(\tilde{s}_i, \tilde{l}_i) : i = 1, 2, ..., p\}$ is a tagged partition of stochastic time-interval of the match $\tilde{l}$ and $\delta_\tilde{s}$ is a stochastic gauge on $\tilde{l}$, then $\mathcal{D}$ is a stochastic $\delta$-fine if $\tilde{l}_i \subset \delta(s_i)$ for all $i = 1, 2, ..., p$, where $\delta(s) = (\tilde{s} - \delta_\tilde{s}(\tilde{s}), \tilde{s} + \delta_\tilde{s}(\tilde{s}))$.

For a tagged partition $\mathcal{D}$ in a stochastic time-interval $\tilde{l}$, as defined in Definitions 15 and 16, and a function $\tilde{f} : [\tilde{l}, t - \varepsilon] \times \mathbb{R}^{2(t \times t')} \times \Omega \times \mathbb{S} \rightarrow \mathbb{R}^{(t \times t') \times \tilde{a}}$ the Riemann sum of $\mathcal{D}$ is defined as

$$ S(\tilde{f}, \mathcal{D}) = (D_\delta) \sum_{i=1}^{p} \tilde{f}(s_i, \tilde{l}_i, W, Z) = \sum_{i=1}^{p} \tilde{f}(s_i, \tilde{l}_i, W, Z), $$

where $D_\delta$ is a $\delta$-fine division of $\mathbb{R}^{(t \times t') \times \tilde{a}}$ with point-cell function $\tilde{f}(s_i, \tilde{l}_i, W, Z) = \tilde{f}(s_i, W, Z)\ell'(\tilde{l}_i)$, where $\ell'$ is the length of the over interval and $\tilde{l} = (t - \varepsilon) - \tilde{l}$ [40, 41].

Definition 17 An integrable function $\tilde{f}(s, \tilde{l}, W, Z)$ on $\mathbb{R}^{(t \times t') \times \tilde{a}}$, with integral

$$ a = \int_{\tilde{l}}^{t-\varepsilon} \tilde{f}(s, \tilde{l}, W, Z) $$

is stochastic Henstock-Kurzweil type integrable on $\tilde{l}$ if, for a given vector $\tilde{\varepsilon} > 0$, there exists a stochastic $\delta$-gauge in $[\tilde{l}, t - \varepsilon]$ such that for each stochastic $\delta$-fine partition $D_\delta$ in $\mathbb{R}^{(t \times t') \times \tilde{a}}$, we have

$$ \mathbb{E}_\tilde{s}\left\{\left|a - (D_\delta) \sum \tilde{f}(s, \tilde{l}, W, Z)\right|\right\} < \tilde{\varepsilon}, $$

where $\mathbb{E}_\tilde{s}$ is the conditional expectation on goal $Z$ at sample time $\tilde{s} \in [\tilde{l}, t - \varepsilon]$ of a non-negative function $\tilde{f}$ after the rain stops.

Proposition 3 Define

$$ h = \exp\left\{-\tilde{\varepsilon}\mathbb{E}_\tilde{s}\left[\int_{\tilde{s}}^{\tilde{s}+\tilde{\varepsilon}} \tilde{f}(s, \tilde{l}, W, Z)\right]\right\}\psi(Z)dZ. $$

If for a small sample time interval $[\tilde{s}, \tilde{s} + \tilde{\varepsilon}]$,

$$ \frac{1}{N_{\tilde{s}}} \int_{\mathbb{R}^{2(t \times t') \times \tilde{a}}} h $$

exists for a conditional gauge $\gamma = [\delta, \omega(\delta)]$, then the indefinite integral of $h$

$$ H(\mathbb{R}^{2(t \times t') \times \tilde{a}}) = \frac{1}{N_{\tilde{s}}} \int_{\mathbb{R}^{2(t \times t') \times \tilde{a}}} h $$
exists as Stieltjes function in $\mathbb{E}(\hat{s}, \hat{s} + \varepsilon] \times \mathbb{R}^{2(l \times l')} \times \Omega \times \mathbb{S} \times \mathbb{R}^I)$ for all $N_\varepsilon > 0$.

**Remark 8** Proposition 3 guarantees that a stochastic Itô-Henstock-Kurtzweil-McShane-Feynman-Liouville type path integral in goal dynamics exists through Stieltjes function. For more detailed discussion, see [40–43].

**Corollary 6** If $h$ is integrable on $\mathbb{R}^{2(l \times l') \times l}$ as in Proposition 3, then for a given small continuous sample time the interval $[\hat{s}, \hat{s}']$ with $\varepsilon = \hat{s}' - \hat{s} > 0$, there exists a $\gamma$-fine division $\mathcal{D}_\gamma$ in $\mathbb{R}^{2(l \times l') \times l}$ such that

$$\left| (\mathcal{D}_\gamma) h[\hat{s}, \hat{I}(Z), W, Z] - \mathcal{H}(\mathbb{R}^{2(l \times l') \times l}) \right| \leq \frac{1}{2} |\hat{s} - \hat{s}'| < \varepsilon,$$

where $\hat{I}(Z)$ is the interval of goal $Z$ in $\mathbb{R}^{2(l \times l') \times l}$. This integral is a stochastic Itô-Henstock-Kurtzweil-McShane-Feynman-Liouville type path integral in goal dynamics of a sample time after the beginning of a match after rain interruption.

As after the rain stops, the environment of the field changes, giving rise to a different probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}_\hat{s}$ is the new filtration at time $\hat{s}$ after rain. The objective function after rain becomes

$$\overline{\text{OB}}_a^i : \hat{Z}_a^i(W, \hat{s}) = h_i^a + \max_{w \in \mathcal{W}} \mathbb{E}_a^i \left\{ \int_{t-\varepsilon}^t \sum_{i=1}^I \sum_{m=1}^M \exp(-\rho_i^m) \alpha^i \mathbb{W}_i(\hat{s}) h_i^m[x, w(\hat{s}), Z(\hat{s})] d\hat{s} \right\}.$$  

(7)

Furthermore, if the match starts at time $\tilde{t}$ after the stoppage of the match due to rain, then Liouville-like action function on goal dynamics after the match starts after the rain is

$$\mathcal{L}_{\tilde{t}-\varepsilon}^i(Z) = \int_{\tilde{t}}^{\tilde{t}-\varepsilon} \mathbb{E}_{\hat{s}} \left\{ \sum_{i=1}^I \sum_{m=1}^M \exp(-\rho_i^m) \alpha^i \mathbb{W}_i(\hat{s}) h_i^m[\hat{s}, w(\hat{s}), Z(\hat{s})] ight\}.$$

(8)

$$+ \lambda_1 [\Delta Z(\hat{s}, W) d\hat{s} \mu[\hat{s}, W(\hat{s}), Z(\hat{s}, W)] d\hat{s}$$

$$- \sigma[\hat{s}, \hat{\sigma}, W(\hat{s}), Z(\hat{s}, W)] d\mathcal{B}(\hat{s})] + \lambda_2 e^{\sqrt{8/3} \xi(\hat{s})] d\hat{s}}.$$

The stochastic part of the Eq. (2) becomes $\sigma$ as $\lambda_1 > \lambda_1$. Equation (8) follows Definition 17 such that $a = \mathcal{L}_{\tilde{t}-\varepsilon}^i(Z)$, and it is integrable according to Corollary 6.

**Proposition 4** If a team’s objective is to maximize Eq. (7) subject to the goal dynamics expressed in the Eq. (2) on the $\sqrt{8/3}$-LQG surface, such that Assumptions 1–4 hold with Propositions 1–3 and Corollary 6, then after a rain stoppage under a continuous sample time, the weight of player $i$ is found by solving
\[
\sum_{i=1}^{I} \sum_{m=1}^{M} \exp(-\rho g \lambda m) \alpha^i h_0^i[\mathbf{s}, w(\mathbf{s}), z(\mathbf{s})] \\
+ g_Z[\mathbf{s}, Z(\mathbf{s}, W)] \frac{\partial \{\mu[\mathbf{s}, W(\mathbf{s}), Z(\mathbf{s}, W)]\}}{\partial W} \\
+ \frac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{J} \frac{\partial \sigma^i_j[\mathbf{s}, \sigma, W(\mathbf{s}), Z(\mathbf{s}, W)]}{\partial W} \frac{\partial W}{\partial W_i} g_{Z_i Z_j}[\mathbf{s}, Z(\mathbf{s}, W)] = 0,
\]

with respect to \( \alpha_i \), where the initial condition before the first kick on the soccer ball after the rain stops is \( Z_{i,0} \). Furthermore, when \( \alpha_i = \alpha_j = \alpha^* \) for all \( i \neq j \), we get a closed form solution of the player weight as

\[
\alpha^* = - \left[ \sum_{i=1}^{I} \sum_{m=1}^{M} \exp(-\rho g \lambda m) \alpha^i h_0^i[\mathbf{s}, w(\mathbf{s}), z(\mathbf{s})] \right]^{-1} \\
\left[ \frac{\partial g[\mathbf{s}, Z(\mathbf{s}, W)]}{\partial Z} \frac{\partial \{\mu[\mathbf{s}, W(\mathbf{s}), Z(\mathbf{s}, W)]\}}{\partial W} \frac{\partial W}{\partial W_i} \\
+ \frac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{J} \frac{\partial \sigma^i_j[\mathbf{s}, \sigma, W(\mathbf{s}), Z(\mathbf{s}, W)]}{\partial W} \frac{\partial W}{\partial W_i} \frac{\partial g[\mathbf{s}, Z(\mathbf{s}, W)]}{\partial Z} \frac{\partial Z_i}{\partial Z_j} \right],
\]

where function \( g[\mathbf{s}, Z(\mathbf{s}, W)] \in C^2([t, t-\epsilon] \times \mathbb{R}^{2d \times P} \times \mathbb{R}^l) \) with \( Y(\mathbf{s}) = g[\mathbf{s}, Z(\mathbf{s}, W)] \) is a positive, non-decreasing penalization function vanishing at infinity which substitutes for the goal dynamics such that \( Y(\mathbf{s}) \) is an Itô process.

**Remark 9** The central idea of Proposition 4 is to choose an appropriate \( g \) function such that the system has a solution. Our main objective is to avoid the computation of very difficult value function. Since we are replacing the SDE (3) by the \( g \) function, it should be based on the integrating factor of the Eq. (3). Based on this approach in Example 3, we obtain optimal weight \( \alpha^* \).

**Example 3** Let the Eq. (2) be the form of

\[
dZ(\mathbf{s}, W) = \mu_1 Z(\mathbf{s}, W) d\mathbf{s} + \mu_2 [\mathbf{s}, W] d\mathbf{s} + \sigma[\mathbf{s}, \sigma_2, W] d\mathbf{B}(\mathbf{s}),
\]

where \( \mu_1 \) and \( \mu_2 \) are two \( I \times I' \)-dimensional matrices with constants and controls respectively so that the total drift of the system is

\[
\mu = \mu_1 Z(\mathbf{s}, W) + \mu_2 [\mathbf{s}, W].
\]

Multiplying \( \exp(-\mu_1 \mathbf{s}) \) on both sides of the Eq. (11) yields

\[
\exp(-\mu_1 \mathbf{s}) dZ(\mathbf{s}, W) - \exp(-\mu_1 \mathbf{s}) \mu_1 Z(\mathbf{s}, W) d\mathbf{s} \\
= \exp(-\mu_1 \mathbf{s}) [\mu_2 [\mathbf{s}, W] d\mathbf{s} + \sigma[\mathbf{s}, \sigma_2, W] d\mathbf{B}(\mathbf{s})].
\]

\( \square \) Springer
The left-hand side of the Eq. (13) can be written as 
\[ d\left[ \exp(-\mu_1 \hat{g})Z(\hat{g}, W) \right] = (\mu_1) \exp(-\mu_1 \hat{g})Z(\hat{g}, W)d\hat{g} + \exp(-\mu_1 \hat{g})d'Z(\hat{g}, W) \]
in Eq. (13) yields,
\[ \exp(-\mu_1 t - \epsilon)Z(t - \epsilon, W) - Z(t, W) = \int_t^{t-\epsilon} \exp(-\mu_1 \hat{g})\mu_2[\hat{g}, W]d\hat{g} + \int_t^{t-\epsilon} \exp(-\mu_1 \hat{g})\sigma[\hat{g}, \sigma_2^{*}, W]dB(\hat{g}). \]

Implementing Theorem 4.1.5 of [44] on Eq. (14) yields
\[ Z(t - \epsilon, W) = \exp(\mu_1(t - \epsilon)) \left[ Z(t, W) + \exp(-\mu_1(t - \epsilon))\sigma[t - \epsilon, \sigma_2^{*}, W]dB(t - \epsilon) \right. \]
\[ \left. + \int_t^{t-\epsilon} \exp(-\mu_1 \hat{g})\left[ \mu_2[\hat{g}, W]d\hat{g} + \sigma[\hat{g}, \sigma_2^{*}, W]B(\hat{g}) \right] d\hat{g} \right]. \]

Since \( \hat{g}(\hat{g}, Z) = \exp(-\mu_1 \hat{g})Z(\hat{g}, W) \) gives a solution to the SDE (11), for a given \( \lambda^* \) we choose \( g = \lambda^* \hat{g} \) function to find out the solution of the maximization problem. Since \( \frac{\partial g}{\partial \hat{g}} = \lambda^* (-\mu_1) \exp(-\mu_1 \hat{g})Z(\hat{g}, W), \frac{\partial g}{\partial Z} = \lambda^* \exp(-\mu_1 \hat{g}), \) and \( \frac{\partial^2 g}{\partial Z \partial Z} = 0, \) Eq. (9) implies
\[ \alpha^* = -\left[ \sum_{i=1}^{I} \sum_{m=1}^{M} \exp(-\rho^{i}_m m)\alpha^i h^i_0[\hat{g}, w(\hat{g}), z(\hat{g})] \right]^{-1} \left[ \lambda^* \exp(-\mu_1 \hat{g}) \frac{\partial \mu_2[\hat{g}, W]}{\partial W} \frac{\partial W}{\partial W_i} \right]. \]

After assuming \( \mu_2[\hat{g}, W] = WA^T_i + A_2, \) where \( A_1 \) is a \( I' \)-dimensional vector of constants and and \( A_2 \) is a \( I \times I' \)-dimensional constant matrix yields
\[ \alpha^* = -\sum_{i=1}^{I} \sum_{m=1}^{M} \frac{\lambda^* A_1 \exp(-\mu_1 \hat{g})}{\exp(-\rho^{i}_m m)\alpha^i h^i_0[\hat{g}, w(\hat{g}), z(\hat{g})]} \cdot \frac{\partial W_i}{\partial W_i} \cdot (p^i_{s+e})_{i\in I}. \]

5 Proofs

5.1 Proof of Proposition 2

Consider a stochastic differential game \[ \Omega \times S, k, \left( \alpha^i \right)_{i\in I}, \left( \rho^i_s \right)_{i\in I}, \left( \Psi_{s,s+e}^i \right)_{i\in I}, (p^i_{s+e})_{i\in I} \]. Let us define the goal space \( Z \) is a countable set such that all common knowledge
equivalence relations can be represented cardinaly in $Z$. First, we define a quantifier free formula whose free variables are beliefs, payoffs of a player, actions based on the goal condition at time $s$ of stochastic differential game on 2-sphere with goal space $Z$ such that at time $s$ for a given goal condition $Z_s$ and beliefs, the strategy polygon $\beta_k$ has a Lefschetz-Hopf fixed point on $\sqrt{8/3}$-LQG action space. Assume $z_1, z_2, z_3, z_4$ are indices in $Z$, $i, j$ are players in $I$, $a_j^i$ is player $i$’s action at time $s$ in $A^j_i$, and the action profile at time $s$ defined as $a_s := (a_1^i(p^i), a_2^i(p^j), ..., a_k^i(p^j))$ in $\prod_i A^i_j$, where $p^i$ is the dribbling and passing function.

For $j \in I$, $p^j : \Omega \times \$ → $\Delta(\Omega \times \$) and $z_3, z_4 \in Z$ the variable $\eta^j_{z_3,z_4,p^j}(s)$ is player $j$’s belief about the goal condition $z_3$ at time $s + \varepsilon$ while at the goal condition $z_4$ at time $s$. For $j \in I$, $z_2 \in Z$, $B(Z) \in \mathbb{R}^{(I \times D^t) \times \Omega}$ and $a_s \in \prod_j A^i_j$, the variable $\omega^j_{z_2,a_s,Z_e}(s)$ is defined as player $j$’s payoff at time $s$ with goal condition $z_2$ given their action profile and the expectation of goals based on whether their team is playing a day match or a day-night match defined by the function $B$ in the Eq. (4) in the previous section. Finally, For $j \in I$, $z_2 \in Z$, $Z_e : [0, t] → \mathbb{R}$ and $a_s \in A^i_j$, the variable $\alpha^j_{z_2,a_s,Z_e}(s)$ is the weight of player $j$ puts at time $s$ on their action $a_s$ when the goal condition is $z_2$ and at dew condition on the field $Z_e$ defined in the Eq. (5).

For $i \in I$ and for a mixed action, $\alpha_{a_i^j} \in \Delta A^i_j$ define a function

$$\tau^i(\alpha_{a_i^j}) = \left( \sum_{a_i^j \in A^i_j} \alpha_{a_i^j} = 1 \right) \bigwedge_{a_i^j \in A^i_j} (\alpha_{a_i^j} \geq 0),$$

and for $z_1, z_2 \in Z$ define

$$\gamma^j_{z_1,z_2,p^j} \left( \eta^j_{z_3,z_4,p^j}(s) \right)_{z_3,z_4,p^j} = \bigwedge_{z_3,p^j} \left( \eta^j_{z_3,z_4,p^j}(s) = \eta^j_{z_3,z_4,p^j}(s) \right),$$

where the above argument means for a given dribbling and passing function $p^j$ player $j$ has the same belief at both goal condition $z_1$ and $z_2$. Therefore, the function

$$\tau \left\{ \left[ a^j_{z_2,a_i^j,Z_e}(s) \right], \left[ \eta^j_{z_3,z_4,p^j}(s) \right] \right\}$$

$$= \left\{ \bigwedge_i \bigwedge_{z_2} \tau^i \left[ a^j_{z_2,a_i^j,Z_e}(s) \right] \right\} \bigwedge \left\{ \bigwedge_{z_1,z_2} \gamma^j_{z_1,z_2,p^j} \left( \eta^j_{z_3,z_4,p^j}(s) \right)_{z_3,z_4,p^j} \right\}$$

$$\rightarrow \bigwedge_{a_i^j} \left[ a^j_{z_1,a_i^j,Z_e}(s) = a^j_{z_2,a_i^j,Z_e}(s) \right],$$

exists iff mixed actions are utilized at every goal condition (i.e., $\bigwedge_i \bigwedge_{z_2} \tau^i \left[ a^j_{z_2,a_i^j,Z_e}(s) \right]$), and strategies are measurable with respect to player $i$’s knowledge of the game. Now,
for a transition function of player $i$ in the time interval $[s, s + \varepsilon]$ defined as $\Psi^i_{s,s+\varepsilon}(Z)$ with the payoff of them at the beginning of time $s$ is $\nu^i_s \geq 0$. For $k$ on $S$ define a state function

$$
\hat{Z}\left\{ \nu^i_s, \omega^i_{z_2, a_i, g}(s), \alpha^i_{z_2, a_i, z_v}(s), \Psi^i_{s,s+\varepsilon}(Z) \right\}
$$

$$
= \bigwedge_i \bigwedge_{z_1} \left[ \nu^i_s = \sum_{a_i} \left( \prod_{i \in I} \alpha^i_{z_1, a_i, z_v}(s) \right) \left( \omega^i_{z_2, a_i, g}(s) + (1 - \theta)\Psi^i_{s,s+\varepsilon}(Z)\nu^i_{s+\varepsilon} \right) \right],
$$

(17)
such that for the goal condition $z_1 \in Z$, the payoffs under mixed action profile of the game is $\nu^i_s$, where $\theta \in (0, 1)$ is a discount factor of this game. Furthermore, if we consider the objective function expressed in the Eq. (1) subject to the goal dynamics expressed in the Eq. (2) on the $\sqrt{8/3}$-LQG action space, the function defined in Eq. (17) in time interval $[s, s + \varepsilon]$ would be

$$
\hat{Z}_{s,s+\varepsilon}\left\{ \nu^i_s, \alpha^i_{z_2, a_i, z_v}(s), \Psi^i_{s,s+\varepsilon}(Z) \right\}
$$

$$
= \bigwedge_i \bigwedge_{z_1} \left\{ \nu^i_s = \sum_{a_i} \left( \prod_{i \in I} \alpha^i_{z_1, a_i, z_v}(s) \right) \right. \times \left. \left[ \mathbb{E}_s \int_s^{s+\varepsilon} \left( \sum_{i=1}^M \sum_{m=1}^M \exp(-\rho^i_m)W^i_h(s,w(s),z(s)) \right. \right. \right.

$$

$$
+ \lambda_1[\Delta Z(v, W) - \mu[v, W(v), Z(v, W)]d\nu - \sigma[v, \dot{v}, W(v), Z(v, W)]dB(v)]
$$

$$
+ \lambda_2\sqrt{8/3}(l) d\nu \right] \right\},
$$

where the expression inside the bracket $[.]$ is the quantum Lagrangian with two non-negative time independent Lagrangian multipliers $\lambda_1$ and $\lambda_2$ where the second stands for $\sqrt{8/3}$-LQG surface. Define

$$
\tilde{Z}\left\{ \nu^i_s, \omega^i_{z_2, a_i, g}(s), \alpha^i_{z_2, a_i, z_v}(s), \Psi^i_{s,s+\varepsilon}(Z) \right\}
$$

$$
= \bigwedge_i \bigwedge_{z_1} \bigwedge_{d \in A} \left[ \nu^i_s = \sum_{z_2} \sum_{a_i} \left( \prod_{j \neq i} \alpha^j_{z_1, a_j, z_v}(s) \right) \times \left( \omega^i_{z_2, a_i, g}(s) + (1 - \theta)\Psi^i_{s,s+\varepsilon}(Z)\nu^i_{s+\varepsilon} \right) \right],
$$

(18)
a goal condition $z_1 \in Z$ such that with no deviation in the game with continuation payoff $\nu^i_{s+\varepsilon}$ gets the payoff no more than the payoff at time $s$ or $\nu^i_s$. Therefore, for time interval $[s, s + \varepsilon]$, the goal condition should be
\[ \tilde{Z}_{s,t+\varepsilon}\{ v^i_s, \alpha^i_{z_2,d_2}, z_v(s), \Psi^i_{s,t+\varepsilon}(Z) \} \]
\[ = \bigwedge_i \bigwedge_{z_1} \bigwedge_{d \in \mathcal{A}} \left\{ v^i_s \geq \sum_{z_2} \sum_{a_j} \left( \prod_{j \neq i} \alpha^j_{z_1,d_1} z_v(s) \right) \right\} \times \left[ \mathbb{E}_s \int \sum_{i=1}^L \left( \sum_{m=1}^M \exp(-\rho_i^m) \alpha^m w_i(s, w(s), z(s)) \right) \right. \]
\[ + \lambda_i [\Delta Z(v, W) - \mu[v, W(v), Z(v, W)]d\nu - \sigma[v, \dot{\sigma}, W(v), Z(v, W)]d\mathbb{B}(v) ] \]
\[ + \lambda e \sqrt{8/3} \int d\nu \right\} \right\} \].

Finally, define
\[ \tilde{Z}^*_{s,t+\varepsilon}\{ v^i_s, \alpha^i_{z_2,d_2}, z_v(s), \Psi^i_{s,t+\varepsilon}(Z) \} \]
\[ = \bigwedge_{j \in I} \bigwedge_{z_1 \in Z} r^i \left[ \alpha^i_{z_2,d_2} z_v(s) \right] \wedge \tilde{Z}_{s,t+\varepsilon}\{ v^i_s, \alpha^i_{z_2,d_2}, z_v(s), \Psi^i_{s,t+\varepsilon}(Z) \} \]
\[ \wedge \tilde{Z}_{s,t+\varepsilon}\{ v^i_s, \alpha^i_{z_2,d_2}, z_v(s), \Psi^i_{s,t+\varepsilon}(Z) \}, \]

such that each player is playing a probability distribution in each goal condition subject to a goal dynamics, dew condition of the field, whether the match is a day or day-night match, and the strategy profiles in goal condition \[ z_1 \in Z \] give the right payoff using equilibrium strategies on \( \sqrt{8/3} \)-LQG action space. By [45], we know that a stochastic game with countable states has a stationary equilibrium, and by Theorem 4.7 of [34], we conclude that the system admits a stable equilibrium under 2-sphere.

\[ 5.2 \text{ Proof of Proposition 3} \]

Define a gauge \( \gamma = [\delta, \omega(\delta)] \) for all possible combinations of a \( \delta \)-gauge in \([\tilde{t}, t] \times \mathbb{R}^{2(|\mathcal{I}'|)\times \Omega} \) and \( \omega(\delta) \)-gauge in \( \mathbb{R}^{2(|\mathcal{I}'|)\times \Omega} \) such that it is a cell in \([\tilde{t}, t] \times \mathbb{R}^{2(|\mathcal{I}'|)\times \Omega} \) and \( \omega(\delta) \) is at least a \( C^1 \) function. The reason behind considering \( \omega(\delta) \) as a function of \( \delta \) is because, after rain stops, if the match proceeds on time \( s \), then we can get a corresponding sample time \( \bar{s} \) and a player has the opportunity to score a goal. Let \( D_{\gamma} \) be a stochastic \( \gamma \)-fine in cell \( E \) in \([\tilde{t}, t - \varepsilon] \times \mathbb{R}^{2(|\mathcal{I}'|)\times \Omega} \times \mathbb{R}^{2(|\mathcal{I}'|)\times \Omega} \times \mathbb{R}\). For any \( \varepsilon > 0 \) and for a \( \delta \)-gauge in \([\tilde{t}, t - \varepsilon] \times \mathbb{R}^{2(|\mathcal{I}'|)\times \Omega} \times \mathbb{R}^{2(|\mathcal{I}'|)\times \Omega} \times \mathbb{R}\) choose a \( \gamma \) so that
\[ \frac{1}{N_{\delta}} (D_{\gamma}) \sum_{\mathcal{H}} \bar{h} - H(\mathbb{R}^{2(|\mathcal{I}'|)\times \Omega}) < \frac{1}{2} |\bar{s} - \bar{s}'|, \]

where \( \bar{s}' = \bar{s} + \varepsilon \). Assume two disjoint sets \( E_a \) and \( E_b = [\bar{s}, \bar{s} + \varepsilon] \times \mathbb{R}^{2(|\mathcal{I}'|)\times \Omega} \times \mathbb{R} \times \{ R \} \). As the domain of \( \bar{f} \) is a 2-sphere, Theorem 3 in [41] implies there is a gauge \( \gamma_a \) on set \( E_a \) and a gauge \( \gamma_b \) for set \( E_b \) with \( \gamma_a < \gamma \) and

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\( \gamma_b < \gamma \), so that both the gauges conform in their respective sets. For every \( \delta \)-fine in \([\hat{s}, \hat{s}'] \times \mathbb{R}^{2(I \times I')} \times \Omega \times \Xi \) and a positive \( \bar{\epsilon} = |\hat{s} - \hat{s}'| \), if a \( \gamma_b \)-fine division \( D_{\gamma_b} \) is of the set \( E^a \) and \( \gamma_b \)-fine division \( D_{\gamma_b} \) is of the set \( E^b \), then by the restriction axiom we know that \( D_{\gamma_a} \cup D_{\gamma_b} \) is a \( \gamma \)-fine division of \( E \). Furthermore, as \( E^a \cap E^b = \emptyset \)

\[
\frac{1}{N_{\hat{s}}} (D_{\gamma_a} \cup D_{\gamma_b}) \sum \hat{s} = \frac{1}{N_{\hat{s}}} \left( (D_{\gamma_a}) \sum \hat{s} + (D_{\gamma_b}) \sum \hat{s} \right) = \alpha + \beta.
\]

Let us assume that for every \( \delta \)-fine, we can subdivide the set \( E^b \) into two disjoint subsets \( E^b_1 \) and \( E^b_2 \) with their \( \gamma_b \)-fine divisions given by \( D_{\gamma_b}^1 \) and \( D_{\gamma_b}^2 \), respectively. Therefore, their Riemann sum can be written as \( \beta_1 = \frac{1}{N_{\hat{s}}} (D_{\gamma_b}^1) \sum \hat{s} \) and \( \beta_2 = \frac{1}{N_{\hat{s}}} (D_{\gamma_b}^2) \sum \hat{s} \), respectively. Hence, for a small sample time interval \([\hat{s}, \hat{s}']\),

\[
|\alpha + \beta_1 - H(\mathbb{R}^{2(I \times I')} \times \mathbb{I} | \leq \frac{1}{2} |\hat{s} - \hat{s}'|
\]

and

\[
|\alpha + \beta_2 - H(\mathbb{R}^{2(I \times I')} \times \mathbb{I})| \leq \frac{1}{2} |\hat{s} - \hat{s}'|.
\]

Therefore,

\[
|\beta_1 - \beta_2| = \left| \left[ \alpha + \beta_1 - H(\mathbb{R}^{2(I \times I')} \times \mathbb{I}) \right] - \left[ \alpha + \beta_2 - H(\mathbb{R}^{2(I \times I')} \times \mathbb{I}) \right] \right| \\
\leq |\alpha + \beta_1 - H(\mathbb{R}^{2(I \times I')} \times \mathbb{I})| + |\alpha + \beta_2 - H(\mathbb{R}^{2(I \times I')} \times \mathbb{I})| \\
\leq |\hat{s} - \hat{s}'|.
\]

Equation (19) implies that the Cauchy integrability of \( \hat{s} \) is satisfied, and

\[
H(\mathbb{R}^{2(I \times I')} \times \mathbb{I}) = \frac{1}{N_{\hat{s}}} \int_{\mathbb{R}^{2(I \times I')} \times \mathbb{I}} \hat{s}.\]

Now, consider two disjoint set \( M^1 \) and \( M^2 \in \mathbb{R}^{2(I \times I')} \times \mathbb{I} \) such that \( M = M^1 \cup M^2 \) with their corresponding integrals \( H(M^1), H(M^2), \text{ and } H(M) \). Suppose \( \gamma \)-fine divisions of \( M^1 \) and \( M^2 \) are given by \( D_{\gamma_a}^1 \) and \( D_{\gamma_a}^2 \), respectively, with their Riemann sums for \( \hat{s} \) are \( m_1 \) and \( m_2 \). Equation (19) implies \( |m_1 - H(M^1)| \leq |\hat{s} - \hat{s}'| \) and \( |m_2 - H(M^2)| \leq |\hat{s} - \hat{s}'| \). Hence, \( D_{\gamma_a}^1 \cup D_{\gamma_a}^2 \) is a \( \gamma \)-fine division of \( M \). Let \( m = m_1 + m_2 \), then Eq. (19) implies \( |m - H(M)| \leq |\hat{s} - \hat{s}'| \) and

\[
\left| [H(M^1) + H(M^2)] - H(M) \right| \leq |m - H(M)| + |m_1 - H(M^1)| \\
+ |m_2 - H(M^2)| \\
\leq 3|\hat{s} - \hat{s}'|.
\]

Therefore, \( H(M) = H(M^1) + H(M^2) \), and it is Stieltjes. \( \square \)
5.3 Proof of Proposition 4

For a positive Lagrangian multipliers \( \lambda_1 \) and \( \lambda_2 \), with initial goal condition \( Z_t \), the goal dynamics are expressed in Eq. (2) such that such that Definition 17, Propositions 1–3 and Corollary 6 hold. Subdivide \([\bar{t}, t - \varepsilon]\) into \( n \) equally distanced small time-intervals \([\bar{s}, \bar{s}'\]) such that \( \varepsilon \downarrow 0 \), where \( \bar{s}' = \bar{s} + \varepsilon \). For any positive \( \varepsilon \), and normalizing constant \( N_{\bar{s}} > 0 \), define a goal transition function as

\[
\Psi_{\bar{s}, \bar{s}+\varepsilon}(Z) = \frac{1}{N_{\bar{s}}} \int_{\mathbb{R}^{2|x'| \times n}} \exp \left\{ -\varepsilon \mathcal{L}_{\bar{s}, \bar{s}+\varepsilon}(Z) \right\} \Psi_{\bar{s}}(Z) dZ,
\]

where \( \Psi_{\bar{s}}(Z) \) is the goal transition function at the beginning of \( t \), \( N_{\bar{s}}^{-1}dZ \) is a finite Wiener measure which satisfies Proposition 3, and for \( k^{th} \) sample time interval, a goal transition function of \([\bar{t}, t - \varepsilon]\) is

\[
\Psi_{\bar{t}, \bar{t}-\varepsilon}(Z) = \frac{1}{(N_{\bar{s}})^n} \int_{\mathbb{R}^{2|x'| \times n}} \exp \left\{ -\varepsilon \sum_{k=1}^{n} \mathcal{L}_{\bar{s}, \bar{s}+\varepsilon}^{k}(Z) \right\} \Psi_{0}(Z) \prod_{k=1}^{n} dZ^k,
\]

with finite Wiener measure \( (N_{\bar{s}})^{-n} \prod_{k=1}^{n} dZ^k \) satisfying Corollary 6 with its initial goal transition function after the rain stops as \( \Psi_{\bar{t}}(Z) > 0 \) for all \( n \in \mathbb{N} \). Define \( \Delta Z(v) = Z(v + d\nu) - Z(v) \), then Fubini’s Theorem for the small interval of time \([\bar{s}, \bar{s}'\]) with \( \varepsilon \downarrow 0 \) yields,

\[
\mathcal{L}_{\bar{s}, \bar{s}'}(Z) = \int_{\bar{s}}^{\bar{s}'} \mathbb{E}_z \left\{ \sum_{i=1}^{I} \sum_{m=1}^{M} \exp(-\rho_i^m_m a_i^m W_i(v) h_0^i[v, w(v), z(v)]) \right.
+ \lambda_1 [\Delta Z(v, W) d\nu - \mu[v, W(v), Z(v, W)] d\nu - \sigma[v, \bar{s}, W(v), Z(v, W)] dB(v)]
+ \lambda_2 e^{\sqrt{8/3} k(v)} d\nu \bigg\}.
\]

As we assume the goal dynamics have drift and diffusion parts, \( Z(v, W) \) is an Itô process and \( W \) is a Markov control measure of players. Therefore, there exists a smooth function \( g[v, Z(v, W)] \in C^2([\bar{t}, t - \varepsilon] \times \mathbb{R}^{2|x'| \times n} \times \mathbb{R}^{l}) \) such that \( \mathcal{Y}(v) = g[v, Z(v, W)] \) with \( \mathcal{Y}(v) \) being an Itô’s process. Assume

\[
g[v + \Delta v, Z(v, W) + \Delta Z(v, W)] = \lambda_1 [\Delta Z(v, W) - \mu[v, W(v), Z(v, W)] d\nu - \sigma[v, \bar{s}, W(v), Z(v, W)] dB(v)]
+ \lambda_2 e^{\sqrt{8/3} k(v)} d\nu.
\]

For a very small sample over-interval around \( \bar{s} \) with \( \varepsilon \downarrow 0 \) generalized Itô’s Lemma gives
\[ \hat{L}_{\hat{\alpha}, \hat{\beta}}(Z) = E_{\hat{\alpha}} \left\{ \sum_{i=1}^{I} \sum_{m=1}^{M} \hat{\epsilon} \exp(-\rho_\alpha^m m) \alpha^m W_i(\hat{\alpha}) h_i^0(\hat{\alpha}, w(\hat{\alpha}), z(\hat{\alpha})) \\
+ \hat{\epsilon} g[\hat{\alpha}, Z(\hat{\alpha}, W)] + \hat{\epsilon} g_{Z}[\hat{\alpha}, Z(\hat{\alpha}, W)] + \hat{\epsilon} g_{Z}[\hat{\alpha}, Z(\hat{\alpha}, W)] \{ \mu[\hat{\alpha}, W(\hat{\alpha}), Z(\hat{\alpha}, W)] \} \\
+ \hat{\epsilon} g_{Z}[\hat{\alpha}, Z(\hat{\alpha}, W)] \sigma[\hat{\alpha}, \hat{\sigma}, W(\hat{\alpha}), Z(\hat{\alpha}, W)] \Delta \mathbf{B}(\hat{\alpha}) \\
+ \frac{1}{2} \hat{\epsilon} \sum_{i=1}^{I} \sum_{j=1}^{J} \sigma_{ij}[\hat{\alpha}, \hat{\sigma}, W(\hat{\alpha}), Z(\hat{\alpha}, W)] g_{Z_{ij}}(\hat{\alpha}, Z(\hat{\alpha}, W)) + o(\hat{\epsilon}) \right\} , \]

where \( \sigma_{ij}[\hat{\alpha}, \hat{\sigma}, W(\hat{\alpha}), Z(\hat{\alpha}, W)] \) represents \((i,j)^{th}\) component of the variance-covariance matrix, \( g_{\hat{\alpha}} = \partial g / \partial \hat{\alpha} , g_Z = \partial g / \partial Z , g_{Z_{ij}} = \partial^2 g / \partial Z_i \partial Z_j \), \( \Delta \mathbf{B}_i \Delta \mathbf{B}_j = \delta_{ij} \hat{\epsilon} \), \( \Delta Z_\hat{\alpha} \hat{\epsilon} = \hat{\epsilon} \Delta Z_\hat{\alpha} = 0 \), and \( \Delta Z_\hat{\alpha} \hat{\epsilon} \Delta Z_\hat{\alpha} \hat{\epsilon} = \hat{\epsilon} \), where \( \delta_{ij} \) is the Kronecker delta function. As \( E_{\hat{\alpha}}[\Delta \mathbf{B}(\hat{\alpha})] = 0 \), \( E_{\hat{\alpha}}[o(\hat{\epsilon})] / \hat{\epsilon} \rightarrow 0 \) and for \( \hat{\epsilon} \downarrow 0 \),

\[ \mathcal{L}_{\hat{\alpha}, \hat{\beta}}(Z) = \sum_{i=1}^{I} \sum_{m=1}^{M} \exp(-\rho_\alpha^m m) \alpha^m W_i(\hat{\alpha}) h_i^0(\hat{\alpha}, w(\hat{\alpha}), z(\hat{\alpha})) \\
+ g[\hat{\alpha}, Z(\hat{\alpha}, W)] + g_{\hat{\alpha}}[\hat{\alpha}, Z(\hat{\alpha}, W)] + g_Z[\hat{\alpha}, Z(\hat{\alpha}, W)] \{ \mu[\hat{\alpha}, W(\hat{\alpha}), Z(\hat{\alpha}, W)] \} \\
+ g_{Z}[\hat{\alpha}, Z(\hat{\alpha}, W)] \sigma[\hat{\alpha}, \hat{\sigma}, W(\hat{\alpha}), Z(\hat{\alpha}, W)] \Delta \mathbf{B}(\hat{\alpha}) \\
+ \frac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{J} \sigma_{ij}[\hat{\alpha}, \hat{\sigma}, W(\hat{\alpha}), Z(\hat{\alpha}, W)] g_{Z_{ij}}(\hat{\alpha}, Z(\hat{\alpha}, W)) + o(1) . \]

There exists a vector \( \xi_{(2(\times I')(\times I))} \) so that

\[ Z(\hat{\alpha}, W)_{(2(\times I')(\times I))} = Z(\hat{\alpha}', W)_{(2(\times I')(\times I))} + \xi_{(2(\times I')(\times I))} . \]

Assume \( |\xi| \leq \eta \hat{\epsilon} |Z^T(\hat{\alpha}, W)|^{-1} \), then

\[ \Psi_{\hat{\alpha}}(Z) + \hat{\epsilon} \frac{\partial \Psi_{\hat{\alpha}}(Z)}{\partial \hat{\alpha}} + o(\hat{\epsilon}) \]

\[ = \frac{1}{N_{\hat{\alpha}}} \int_{\mathbb{R}^{2(\times I')\times I}} \left[ \Psi_{\hat{\alpha}}(Z) + \xi \frac{\partial \Psi_{\hat{\alpha}}(Z)}{\partial Z} + o(\hat{\epsilon}) \right] \\
\times \exp \left\{ -\hat{\epsilon} \left[ \sum_{i=1}^{I} \sum_{m=1}^{M} \exp(-\rho_\alpha^m m) \alpha^m W_i(\hat{\alpha}) h_i^0(\hat{\alpha}, w(\hat{\alpha}), z(\hat{\alpha}) + \xi) \\
+ g[\hat{\alpha}, Z(\hat{\alpha}, W) + \xi] + g_{\hat{\alpha}}[\hat{\alpha}, Z(\hat{\alpha}, W) + \xi] \\
+ g_Z[\hat{\alpha}, Z(\hat{\alpha}', W) + \xi] \{ \mu[\hat{\alpha}, W(\hat{\alpha}), Z(\hat{\alpha}', W) + \xi] \} \\
+ \frac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{J} \sigma_{ij}[\hat{\alpha}, \hat{\sigma}, W(\hat{\alpha}), Z(\hat{\alpha}', W) + \xi] g_{Z_{ij}}(\hat{\alpha}, Z(\hat{\alpha}', W) + \xi) \right\} d\xi + o(\hat{\epsilon}^{1/2}) . \]

Define a \( C^2 \) function
\[
\begin{align*}
f[\mathbf{s}, \mathbf{W}(\mathbf{s}), \xi] &= \sum_{i=1}^{I} \sum_{m=1}^{M} \exp(-\rho_{s}^{i} m) \alpha^{i} W_{i}(\mathbf{s}) h_{ij}[\mathbf{s}, \mathbf{w}(\mathbf{s}), \mathbf{z}(\mathbf{s}) + \xi] \\
&\quad + g[\mathbf{s}, \mathbf{Z}(\mathbf{s}'), \mathbf{W}] + \xi] + g_{\mathbf{s}}[\mathbf{s}, \mathbf{Z}(\mathbf{s}', \mathbf{W}) + \xi] \\
&\quad + g_{Z}[\mathbf{s}, \mathbf{Z}(\mathbf{s}', \mathbf{W}) + \xi] \{ \mu[\mathbf{s}, \mathbf{W}(\mathbf{s}), \mathbf{Z}(\mathbf{s}', \mathbf{W}) + \xi] \\
&\quad + \frac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{I} \sigma^{ij}[\mathbf{s}, \mathbf{W}(\mathbf{s}), \mathbf{Z}(\mathbf{s}', \mathbf{W}) + \xi] \\
&\quad \times g_{Z_{ij}}[\mathbf{s}, \mathbf{Z}(\mathbf{s}', \mathbf{W}) + \xi].
\end{align*}
\]

Hence,
\[
\begin{align*}
\nabla_{\mathbf{s}} f(\mathbf{s}) + \hat{\varepsilon} \frac{\partial \Psi_{\mathbf{s}}(\mathbf{Z})}{\partial \mathbf{s}} &= \frac{\Psi_{\mathbf{s}}(\mathbf{Z})}{N_{\hat{\varepsilon}}} \int_{\mathbb{R}^{2(l \times d') \times d}} \exp\left\{-\hat{\varepsilon} f[\mathbf{s}, \mathbf{W}(\mathbf{s}), \mathbf{z}]\right\} d\xi \\
&\quad + \frac{1}{N_{\hat{\varepsilon}}} \frac{\partial \Psi_{\mathbf{s}}(\mathbf{Z})}{\partial \mathbf{Z}} \int_{\mathbb{R}^{2(l \times d') \times d}} \xi \exp\left\{-\hat{\varepsilon} f[\mathbf{s}, \mathbf{W}(\mathbf{s}), \mathbf{z}]\right\} d\xi + o(\varepsilon^{1/2}).
\end{align*}
\]

For \(\hat{\varepsilon} \downarrow 0\), \(\Delta \mathbf{Z} \downarrow 0\)
\[
\begin{align*}
f[\mathbf{s}, \mathbf{W}(\mathbf{s}), \xi] &= f[\mathbf{s}, \mathbf{W}(\mathbf{s}), \mathbf{Z}(\mathbf{s}', \mathbf{W})] \\
&\quad + \sum_{i=1}^{I} f_{Z_{ij}}[\mathbf{s}, \mathbf{W}(\mathbf{s}), \mathbf{Z}(\mathbf{s}', \mathbf{W})][\mathbf{\xi}_{j} - \mathbf{Z}_{i}(\mathbf{s}, \mathbf{W})] \\
&\quad + \frac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{I} f_{Z_{ij}}[\mathbf{s}, \mathbf{W}(\mathbf{s}), \mathbf{Z}(\mathbf{s}', \mathbf{W})][\mathbf{\xi}_{j} - \mathbf{Z}_{i}(\mathbf{s}', \mathbf{W})][\mathbf{\xi}_{j} - \mathbf{Z}_{i}(\mathbf{s}', \mathbf{W})] + o(\hat{\varepsilon}).
\end{align*}
\]

There exists a symmetric, positive definite and non-singular Hessian matrix \(\Theta(2(l \times d') \times d) \times (2(l \times d') \times d)\) and a vector \(\mathbf{R}(2(l \times d') \times d) \times \mathbf{1}\) such that
\[
\int_{\mathbb{R}^{2(l \times d') \times d}} \exp\left\{-\hat{\varepsilon} f[\mathbf{s}, \mathbf{W}(\mathbf{s}), \mathbf{z}]\right\} d\xi
\]
\[
= \sqrt{\frac{(2\pi)^{2(l \times d') \times d}}{\varepsilon |\Theta|}} \exp \left\{-\frac{1}{2} f[\mathbf{s}, \mathbf{W}(\mathbf{s}), \mathbf{z}] + \frac{1}{2} \mathbf{R}^{T} \Theta^{-1} \mathbf{R} \right\},
\]

where
\[
\int_{\mathbb{R}^{2(l \times d') \times d}} \xi \exp\left\{-\hat{\varepsilon} f[\mathbf{s}, \mathbf{W}(\mathbf{s}), \mathbf{z}]\right\} d\xi
\]
\[
= \sqrt{\frac{(2\pi)^{2(l \times d') \times d}}{\varepsilon |\Theta|}} \exp\left\{-\frac{1}{2} f[\mathbf{s}, \mathbf{W}(\mathbf{s}), \mathbf{z}] + \frac{1}{2} \mathbf{R}^{T} \Theta^{-1} \mathbf{R} \right\}
\]
\[
\times [\mathbf{Z}(\mathbf{s}', \mathbf{W}) + \frac{1}{2} (\Theta^{-1} \mathbf{R})].
\]

Therefore,
\[
\Psi_\delta(Z) + \hat{\epsilon} \frac{\partial \Psi_\delta(Z)}{\partial \delta} = \frac{1}{N_\delta} \sqrt{\frac{(2\pi)^{2d} \times \epsilon |\Theta|}{\epsilon |\Theta|}} \\
\times \exp\left\{ -\hat{\epsilon} f[\delta, W(\delta), Z(\delta', W)] + \frac{1}{2} \hat{\epsilon} R^T \Theta^{-1} R \right\} \\
\times \left\{ \Psi_\delta(Z) + [Z(\delta', W) + \frac{1}{2} (\Theta^{-1} R)] \frac{\partial \Psi_\delta(Z)}{\partial Z} \right\} + o(\hat{\epsilon}^{1/2}).
\]

Assuming \( N_\delta = \sqrt{(2\pi)^{2d} \times \epsilon |\Theta|} > 0 \), we get Wick rotated Schrödinger type equation as,

\[
\Psi_\delta(Z) + \hat{\epsilon} \frac{\partial \Psi_\delta(Z)}{\partial \delta} = \left\{ 1 - \hat{\epsilon} f[\delta, W(\delta), Z(\delta', W)] + \frac{1}{2} \hat{\epsilon} R^T \Theta^{-1} R \right\} \\
\times \left\{ \Psi_\delta(Z) + [Z(\delta', W) + \frac{1}{2} (\Theta^{-1} R)] \frac{\partial \Psi_\delta(Z)}{\partial Z} \right\} + o(\hat{\epsilon}^{1/2}).
\]

As \( Z(\delta, W) \leq \eta \hat{\epsilon} |\xi^{T}|^{-1} \), there exists \( |\Theta^{-1} R| \leq 2 \eta \hat{\epsilon} |1 - \xi^{T}|^{-1} \) such that for \( \hat{\epsilon} \downarrow 0 \), we have \( |Z(\delta', W) + \frac{1}{2} (\Theta^{-1} R)| \leq \eta \hat{\epsilon} \), and hence,

\[
\frac{\partial \Psi_\delta(Z)}{\partial \delta} = [-f[\delta, W(\delta), Z(\delta', W)] + \frac{1}{2} R^T \Theta^{-1} R] \Psi_\delta(Z).
\]

For \( |\Theta^{-1} R| \leq 2 \eta \hat{\epsilon} |1 - \xi^{T}|^{-1} \) and at \( \hat{\epsilon} \downarrow 0 \),

\[
\frac{\partial \Psi_\delta(Z)}{\partial \delta} = -f[\delta, W(\delta), Z(\delta', W)] \Psi_\delta(Z),
\]

and

\[
-\frac{\partial}{\partial W_i} f[\delta, W(\delta), Z(\delta', W)] \Psi_\delta(Z) = 0. \tag{20}
\]

In Eq. (20), \( \Psi_\delta(Z) \) is the transition wave function and cannot be zero; therefore,

\[
\frac{\partial}{\partial W_i} f[\delta, W(\delta), Z(\delta', W)] = 0.
\]

We know, \( Z(\delta', W) = Z(\delta, W) - \xi \) and for \( \xi \downarrow 0 \) as we are looking for some stable solution. Hence, \( Z(\delta', W) \) can be replaced by \( Z(\delta, W) \) and

\[
f[\delta, W(\delta), Z(\delta, W)] = \sum_{i=1}^{I} \sum_{m=1}^{M} \exp(-\rho_i^{j} m) a_r W_i(\delta) h_0[\delta, w(\delta), z(\delta)] \\
+ g[\delta, Z(\delta, W)] + g_\delta[\delta, Z(\delta, W)] \\
+ g_\delta Z[\delta, Z(\delta, W)] \{ \mu[\delta, W(\delta), Z(\delta, W)] \} \\
+ \frac{1}{2} \sum_{i=1}^{I} \sum_{j=1}^{l} a_r^{ij} [\delta, \delta, W(\delta), Z(\delta, W)] g_{Z_r} \{ \delta, Z(\delta, W) \}.
\]

\( \therefore \) Springer
so that

\[
\sum_{i=1}^{l} \sum_{m=1}^{M} \exp(-\rho^i_s m) a^i h^i_0[\bar{s}, w(\bar{s}), z(\bar{s})] \\
+ g_Z[\bar{s}, Z(\bar{s}), W] \frac{\partial \{\mu[\bar{s}, W(\bar{s}), Z(\bar{s}), W]\}}{\partial W} \frac{\partial W}{\partial W_i} \\
+ \frac{1}{2} \sum_{i=1}^{l} \sum_{j=1}^{l} \frac{\partial \sigma^{ij}[\bar{s}, \hat{\sigma}, W(\bar{s}), Z(\bar{s}), W]}{\partial W} \frac{\partial W}{\partial W_i} g_{Z_i Z_j}[\bar{s}, Z(\bar{s}), W] = 0.
\]

If in Eq. (21) we solve for \( \alpha_i \), we can get a solution of the weight attached to \( W_i(\bar{s}) \).
In order to get a closed form solution, we have to assume \( \alpha_i = \alpha_j = \alpha^* \) for all \( i \neq j \) which yields

\[
\alpha^* = - \left[ \sum_{i=1}^{l} \sum_{m=1}^{M} \exp(-\rho^i_s m) a^i h^i_0[\bar{s}, w(\bar{s}), z(\bar{s})] \right]^{-1} \\
\times \left[ \frac{\partial g[\bar{s}, Z(\bar{s}), W]}{\partial Z} \frac{\partial \{\mu[\bar{s}, W(\bar{s}), Z(\bar{s}), W]\}}{\partial W} \frac{\partial W}{\partial W_i} \\
+ \frac{1}{2} \sum_{i=1}^{l} \sum_{j=1}^{l} \frac{\partial \sigma^{ij}[\bar{s}, \hat{\sigma}, W(\bar{s}), Z(\bar{s}), W]}{\partial W} \frac{\partial W}{\partial W_i} \frac{\partial^2 g[\bar{s}, Z(\bar{s}), W]}{\partial Z_i \partial Z_j} \right].
\]

Expression in Eq. (22) is a unique closed form solution. The sign of \( \alpha^* \) varies along with the signs of all the partial derivatives.

\[
6 \text{ Discussion}
\]

In this paper, we obtain a weight \( \alpha^* \) for a soccer match with rain interruption. This coefficient tells us how to select a player to score goals at a certain position based on the condition of that match. It is a common practice to hide a new talented player until the 15 min of the game. As the player is new, opposition team has less information about them, and as they are playing for the first time in the game, they have more energy than a player who is playing for the last 75 min. Our model will determine the weight associated with these players under more generalized and realistic conditions of the game.

We use a Feynman path integral technique to calculate \( \alpha^* \). In the later part of the paper, we focus on the more volatile environment after the rain stops. We assume that after a rain stoppage, the occurrence of each kick towards the goal strictly depends on the amount of rain at that sample time. If it is more than \( b \in \mathbb{R} \) millimeters, then it is very hard to move with the ball on the field, and it is extremely difficult for a goalkeeper to grip the ball which results in not resuming the match again. Using Itô’s lemma, we define a \( \delta_\phi \)-gauge which generates a sample time \( \phi \) instead of an actual time \( s \) is assumed to follow a Wiener process. Furthermore, we
assume the action space of a soccer player has a $\sqrt{8/3}$-LQG surface, and we construct a stochastic Itô-Henstock-Kurzweil-McShane-Feynman-Liouville-type path integral to solve for the optimal weight associated with them. As before rain, the environment does not offer extra moisture, a technically sound player does not need to predict the behavior of an opponent which is not true for the case of a match after a rain stoppage.

Feynman path integral is concerned with the mechanical action of a physical system and is represented by the integral of the Lagrangian function. We use this approach to determine the Wick-rotated Schrödinger-type equation under $\sqrt{8/3}$-LQG surface. First, we subdivide the time interval $[0, t]$ into small, equal-lengthed sub-intervals $[s, \tau]$. Second, a stochastic Lagrangian with $\sqrt{8/3}$-LQG metric is constructed for each time-interval; third, the functional form of the SDE has been replaced by a $C^1,2$ function $g$ and Itô lemma is implemented; and fourth, a Feynman action function is constructed for each $[s, \tau]$. Since our problem is a Lagrangian control problem (i.e., objective function without a terminal condition), at the beginning of each time interval, no future information is available. Therefore, the conditional expectation at $[s, \tau]$ is the expectation at time $s$. Fifth, we perform a few first-order Taylor series expansions and then Gaussian integrals to obtain the Wick-rotated Schrödinger-type equation. Finally, the first-order condition of the Schrödinger-type equation gives us the optimal weight of the control $\alpha^*$. Our approach is different than the analytical path integral or so-called path integral control approach based on Feynman-Kac lemma. Consider the simple Cauchy problem

$$\begin{align*}
\frac{\partial v}{\partial s}(s, Z(s, W)) &+ \mu(s, W(s), Z(s, W)) \frac{\partial v}{\partial Z}(s, Z(s, W)) \\
+ \frac{1}{2} \sigma^2(s, W(s), Z(s, W)) \frac{\partial^2 v}{\partial Z^2}(s, Z(s, W)) &= 0,
\end{align*}$$

with the terminal condition $v(t, Z(t, W)) = \Psi(Z(t, W))$, where $\mu(\cdot), \sigma(\cdot),$ and $\Psi(\cdot)$ are given, and we wish to determine the value function $v(\cdot)$. Let $v(s, Z)$ solves the boundary problem expressed in the Eq. (23) and the process

$$\sigma(s, W(s), Z(s, W)) \frac{\partial v}{\partial Z}(s, Z(s, W)) \in L^2, \text{ for } s \leq t, Z \in \Omega,$$

where $L^2$ represents the Hilbert space, and $Z$ has the following evaluation:

$$dZ(s, W) = \mu(s, W(s), Z(s, W))ds + \sigma(s, W(s), Z(s, W))dB(s),$$

then $v$ has the stochastic Feynman-Kac representation

$$v(s, Z) = \mathbb{E}_Z[\Psi(Z(t, W))].$$

The main problem of this approach is that it is impossible to verify Eq. (24). Therefore, a previous knowledge of $v$ is required which is not the case for most of the real-life problems. If the Eq. (23) does not have sufficiently integrable solution, the solution does not have any sense. Finally, solution obtained by Feynman-Kac
representation theorem is not unique. Apart from that, this approach requires computation of the very difficult value function with the terminal condition.

Since we consider a Lagrangian control problem, our approach does not consider a terminal condition. Furthermore, our approach is based on $C^{1,2}$ Itô process $g$ derived by an integrating factor of the SDE. These are the advantages of the Feynman-type path integral method. On the hind side, the solution from our approach is unique only when goal dynamics expressed by the SDE (2) has a linear drift. For non-linear SDEs, there are more than one solutions, and our approach supplies a sufficiently integrable solution. The remaining solutions must be found using other techniques. This is one limitation of this approach under generalized SDE. Other limitations are the Feynman-type path integral approach that requires continuous finite time intervals. For discrete-time problems as well as infinite horizon cases, this approach fails to give any solution. Our approach can be used for any stochastic control problem where the SDEs are non-linear. In Example 3, we find an explicit expression for $\alpha^*$ which concludes that rain interruption cannot affect the control coefficient. Intuitively, as the prediction of rain accurately is almost impossible before the game starts, a team sticks their strategies based on the opposition players’ previous performances. In Eq. (16), in Example 3, we see $\alpha^*$ is negatively related with the utility function $h_i^0$, which concludes that a higher self satisfaction does not lead to an increase in the weight of the control which is true in the sense that, under a losing condition, a team’s extreme aggressive strategy leads to a complete defeat.

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