Research Article

New Existence Results for Nonlinear Fractional Integrodifferential Equations

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Received 13 January 2021; Revised 23 February 2021; Accepted 3 March 2021; Published 31 March 2021

Academic Editor: Soheil Salahshour

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This paper discusses a boundary value problem of nonlinear fractional integrodifferential equations of order $1 < \alpha \leq 2$ and $1 < \beta \leq 2$ and boundary conditions of the form $x(0) = x(1) = cD^\beta x(1) = cD^\beta x(0) = 0$. Some new existence and uniqueness results are proposed by using the fixed point theory. In particular, we make use of the Banach contraction mapping principle and Krasnoselskii’s fixed point theorem under some weak conditions. Moreover, two illustrative examples are studied to support the results.

1. Introduction

Fractional differential equations are relevant in many fields of science, such as chemistry, fluid systems, and electromagnetic; for more details about the theory of fractional differential equations and their applications, we invite the readers to see [1–16] and the references therein.

Some physical applications of fractional differential equations include viscoelasticity, Schrodinger equation, fractional diffusion equation, and fractional relaxation equation; for more details, we refer the readers to [17].

In addition, fractional integrodifferential equations are used as an important tool to gain insight into some emerging problems from several science areas, for more details, we give the following references [18–23].

More recently, in [24], the existence and uniqueness of positive solutions for the fractional integrodifferential equation were proved.

In [25], the authors discussed the existence and uniqueness of solutions for nonlinear integrodifferential equations of fractional order with three-point nonlocal fractional boundary conditions. The existence of solutions for nonlinear fractional integrodifferential equations has been studied in [26].

Motivated by all these works and by the fact that there are no papers dealing with the new existence results for nonlinear fractional integrodifferential equations, in this work, we consider the existence and uniqueness of solutions for the following problem:

$$
\begin{align*}
D^\alpha (D^\beta) x(t) &= f(t, x(t), \Phi x(t), \psi x(t)), \quad t \in [0, 1], \\
x(0) &= x(1) = cD^\beta x(1) = cD^\beta x(0) = 0,
\end{align*}
$$

where $1 < \alpha \leq 2$, $1 < \beta \leq 2$, $D^\alpha$, $D^\beta$ are the Caputo fractional derivatives, $f : [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is a continuous function, and

$$
\begin{align*}
\Phi x(t) &= \int_0^t \lambda(t, s)x(s)ds, \\
\psi x(t) &= \int_0^t \delta(t, s)x(s)ds,
\end{align*}
$$

where $\lambda, \delta : [0, 1] \times [0, 1] \rightarrow [0, +\infty)$, with $\phi^* = \sup_{t \in [0, 1]} | \int_0^t \lambda(t, s)ds | < \infty$, $\psi^* = \sup_{t \in [0, 1]} | \int_0^t \delta(t, s)ds | < \infty.$
This paper is organized as follows. In Section 2, we present some preliminaries and notations that will be required for the later sections. After that, in Section 3, we establish the main results by using the fixed point theory. And, in the last section, we give two examples to illustrate the results.

2. Preliminaries and Notations

In this section, we give some notations, definitions, and lemmas which will be required for the rest of the paper.

Definition 1 [5]. The fractional integral of order $\alpha > 0$ with the lower limit zero for a function $f$ can be defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.$$  (3)

Definition 2 [5]. The Caputo derivative of order $\alpha > 0$ with the lower limit zero for a function $f$ can be defined as

$$^{c}D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds,$$  (4)

where $n \in \mathbb{N}$, $0 \leq n - 1 < \alpha < n$, $t > 0$.

Theorem 3 [27]. Let $M$ be a bounded, closed, convex, and nonempty subset of a Banach space $X$. Let $A$ and $B$ be two operators such that

1. $Ax + By \in M$ whenever $x, y \in M$
2. $A$ is compact and continuous
3. $B$ is a contraction mapping

Then, there exists $z \in M$ such that $z = Az + Bz$.

Lemma 4 [5]. Let $\alpha, \beta \geq 0$; then, the following relation hold:

$$I^\alpha t^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)} t^{\alpha + \beta}.$$  (5)

Lemma 5 [5]. Let $n \in \mathbb{N}$ and $n - 1 < \alpha < n$. If $f$ is a continuous function, then we have

$$^{c}D^\alpha D^\beta f(t) = f(t) + a_0 + a_1 t + a_2 t^2 + \cdots + a_{n-1} t^{n-1}.$$  (6)

Lemma 6. Let $h \in C([0, 1], \mathbb{R})$. Then, the unique solution of the problem

$$\begin{cases} ^{c}D^\alpha ^{c}D^\beta x(t) = h(t), & t \in [0, 1], \\ x(0) = x(1) = ^{c}D^\beta x(1) = ^{c}D^\beta x(0) = 0, \end{cases}$$  (7)

is given by

$$x(t) = \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha + \beta - 1} h(s) ds$$

$$- \frac{t^{\beta+1}}{\Gamma(\alpha) \Gamma(\beta + 2)} \int_0^1 (1-s)^{\alpha - 1} \times h(s) ds$$

$$- \frac{t}{\Gamma(\alpha + \beta)} \int_0^1 (1-s)^{\alpha + \beta - 1} h(s) ds.$$  (8)

Proof. By applying Lemma 5, we have

$$^{c}D^\beta x(t) = I^\beta h(t) + a_0 + a_1 t,$$  (9)

$$x(t) = I^\beta h(t) + I^\beta a_0 + I^\beta a_1 t + a_2 t,$$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}$. So

$$x(t) = \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha + \beta - 1} h(s) ds + \frac{t^{\beta+1}}{\Gamma(\beta + 1)} a_0$$

$$+ \frac{t^{\beta+1}}{\Gamma(\beta + 2)} a_1 + a_2 + a_3 t.$$  (10)

And by using $^{c}D^\beta x(0) = x(0) = 0$, we obtain $a_0 = a_2 = 0$. As a result of $^{c}D^\beta x(1) = 0$, we have that

$$a_1 = - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha - 1} h(s) ds.$$  (11)

Now, we use $x(1) = 0$ to get

$$a_3 = - \frac{1}{\Gamma(\alpha + \beta)} \int_0^1 (1-s)^{\alpha + \beta - 1} h(s) ds + \frac{1}{\Gamma(\alpha) \Gamma(\beta + 2)} \int_0^1 (1-s)^{\alpha - 1} h(s) ds.$$  (12)

By substituting the value of $a_0, a_1, a_2, a_3$, we obtain the following

$$x(t) = \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha + \beta - 1} h(s) ds$$

$$- \frac{t^{\beta+1}}{\Gamma(\alpha) \Gamma(\beta + 2)} \int_0^1 (1-s)^{\alpha - 1} \times h(s) ds$$

$$- \frac{t}{\Gamma(\alpha + \beta)} \int_0^1 (1-s)^{\alpha + \beta - 1} h(s) ds.$$  (13)

Conversely, by direct computations, we obtain the desired result.

3. Main Results

Let $X$ be the Banach space of all continuous functions from $[0, 1] \rightarrow \mathbb{R}$ endowed with the norms $\|y\| = \sup \{ |y(t)| : t \in [0, 1] \}$ and $\|y\|_v = \sup \{ |y(t)| e^\phi t^\nu : \nu > 0 \phi^* (\Gamma(\alpha + \beta)) ||\sigma|| $, and $\sigma$ will be defined later.
We consider the ball \( B = \{ y \in X : \| y \| \leq r \} \) with \( r \geq \left( \frac{\| \theta \|}{u} \right) \left( \frac{(2e^\gamma - 1)}{(\Gamma(\alpha + \beta)) (\Gamma(\alpha + \beta + 2))} \right) \). We define the operators \( F = F_1 + F_2 \) on \( B_r \), where
\[
F_{1y}(t) = \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), \psi y(s)) ds,
\]
and
\[
F_{2y}(t) = -\frac{\Gamma^{\beta+1} - t}{\Gamma(\alpha)(\beta + 2)} \int_0^t (1-s)^{\alpha-1} f(s, y(s), \psi y(s)) ds - \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (1-s)^{\alpha-1}(s) ds.
\]
For \( x, y \in B_r \), we have
\[
\| F_{1y} \| \leq \sup_{t \in [0,1]} \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), \psi x(s), \psi x(s)) ds,
\]
and
\[
\| F_{2y} \| \leq \sup_{t \in [0,1]} \frac{1}{\Gamma(\alpha + \beta)} \int_0^t (1-s)^{\alpha-1} f(s, y(s), \psi y(s), \psi y(s)) ds.
\]
Therefore,
\[
\| F_1 x + F_2 y \| \leq \frac{\| \theta \|}{u} \left( \frac{2(e^\gamma - 1)}{\Gamma(\alpha)(\beta + 2)} + \frac{e^\gamma}{\Gamma(\alpha + \beta)} \right).
\]
Now, we prove that \( F_1 \) is a contraction. For \( x, y \in B_r \), we have
\[
\| F_1 y(t) - F_1 x(t) \| \leq \sup_{t \in [0,1]} \frac{1}{\Gamma(\alpha + \beta) e^{\epsilon t}} \int_0^t (t-s)^{\alpha-1} f(s, y(s), \psi y(s)) ds.
\]
By using the condition of the new norm, we conclude that \( F_1 \) is a contraction.

Next, we will prove that \( F_2 \) is compact and continuous. Continuity of \( f \) implies that the operator \( F_2 \) is continuous. Also, \( F_2 \) is uniformly bounded on \( B_r \) as
\[
\| F_2 y \| \leq \frac{\| \theta \| (e^\gamma - 1)}{\Gamma(\alpha + \beta) v} \left( \frac{2}{\Gamma(\alpha)(\beta + 2)} + \frac{1}{\Gamma(\alpha + \beta)} \right).
\]
Suppose that \( 0 \leq t_1 < t_2 \leq 1 \). We have
\[
| F_2 y(t_2) - F_2 y(t_1) | \leq \frac{| t_2^{\beta+1} - t_1^{\beta+1} |}{\Gamma(\alpha)(\beta + 2)} + \frac{| t_2 - t_1 |}{\Gamma(\alpha + \beta)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s), \psi x(s), \psi x(s)) ds.
\]
Then, \( | F_2 y(t_2) - F_2 y(t_1) | \rightarrow 0 \), as \( t_1 \rightarrow t_2 \) independently from \( y \in B_r \).

This shows that the operator \( F_2 \) is relatively compact on \( B_r \). Thus, by the Arzela Ascoli theorem, we obtain that \( F_2 \) is compact on \( B_r \).

By the Krasnoselskii fixed point theorem, the problem (1) has at least one solution on \( B_r \).
Theorem 8. Suppose that \( f : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R} \) is a continuous function satisfying

\((H_1)\) for all \( t \in [0, 1] \) and \( x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R} \), we have \( f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3) \leq \sigma(t) \left| (x_1 - y_1) + (x_2 - y_2) + (x_3 - y_3) \right| \) with \( \sigma(t) \in L^1([0, 1] \times [0, \infty)) \). Then, there exists a unique solution for the problem (1) under the following condition: \( r_1 < 1 \), where \( r_1 = 2(1 + \phi + \psi + \sigma) \). Then, there exists a unique solution for the problem (1) under the following condition: \( r_1 < 1 \), where \( r_1 = 2(1 + \phi + \psi + \sigma) \).

Proof. Define \( F : X \rightarrow X \) by

\[
F(x) = \frac{1}{\Gamma(\alpha + \beta)} \int_0^1 (t-s)^{\alpha-1} f(s, x(s), \phi(s), \psi(s), \psi(s)) ds - \frac{t^\beta}{\Gamma(\alpha)\Gamma(\beta + 2)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s), \phi(s), \psi(s)) ds - \frac{t}{\Gamma(\alpha + \beta)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s), \phi(s), \psi(s)) ds.
\]

(22)

Setting \( \sup_{t \geq 0} |f(t, 0, 0, 0)| = P \).

We consider the set \( B_r = \{ x \in X : \| x \| \leq r \} \), where \( r \geq (r_2/(1 - r_1)) \), with

\[
r_2 = 2P \left( \frac{1}{\Gamma(\alpha + \beta)} + \frac{1}{\Gamma(\alpha)\Gamma(\beta + 2)} \right).
\]

(23)

For each \( t \in [0, 1] \) and \( x \in B_r \), we have

\[
|F(x(t))| \leq \frac{1}{\Gamma(\alpha + \beta)} \int_0^1 (t-s)^{\alpha-1} f(s, x(s), \phi(s), \psi(s)) ds + \frac{t^\beta}{\Gamma(\alpha)\Gamma(\beta + 2)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s), \phi(s), \psi(s)) ds + \frac{t}{\Gamma(\alpha + \beta)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s), \phi(s), \psi(s)) ds.
\]

(24)

Then, \( \| F(x) \| \leq r \).

Therefore, \( FB_r \subseteq B_r \).

Next, we show that \( F \) is a contraction mapping. For \( x, y \in B_r \), we have

\[
|F(x(t)) - F(y(t))| \leq \frac{1}{\Gamma(\alpha + \beta)} \int_0^1 (t-s)^{\alpha-1} f(s, x(s), \phi(s), \psi(s)) ds + \frac{t^\beta}{\Gamma(\alpha)\Gamma(\beta + 2)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s), \phi(s), \psi(s)) ds + \frac{t}{\Gamma(\alpha + \beta)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s), \phi(s), \psi(s)) ds.
\]

(25)
Since \( r_1 < 1 \), then \( F \) is a contraction. Therefore, the system (1) has a unique solution.

4. Examples

In this section, we give two examples to show the applicability of our results.

Example 1. Consider the following problem:

\[
\begin{align*}
\left\{ \begin{array}{l}
\mathcal{D}_t^\alpha \left( \mathcal{D}_t^\beta \right) x(t) = \frac{t^3}{400} \left( \frac{|x(t)e^{-t}|}{1 + |x(t)|} + \int_0^t \frac{(t+s)^3|s|(|\cos (s) + \sin (s))}{400(1 + |x(s)|)} ds \right), \\
x(0) = x(1) = \mathcal{D}_t^\alpha x(0) = \mathcal{D}_t^\beta x(1) = 0.
\end{array} \right.
\end{align*}
\]  

(26)

Here,

\[
\beta = \frac{16}{11},
\]

\[
\alpha = \frac{17}{11},
\]

\[
f(t, x, y, z) = \frac{t^3}{400} \left( \frac{|x|e^{-t}}{1 + |x|} + \frac{|y| \cos (t)}{1 + |y|} + \frac{|z| \sin (t)}{1 + |z|} \right),
\]

\[
\lambda(t, s) = \delta(t, s) = \frac{(t+s)^3}{400},
\]

\[
\sigma(t) = \frac{t^3}{400},
\]

\[
\theta(t) = \frac{3t^3}{400}.
\]

It follows that

\[
\phi^* = \psi^* = \frac{15}{1600},
\]

\[
\sigma^* = \frac{1}{1600}.
\]

By Theorem 8, we conclude that the problem (26) has a unique solution.

Example 2. Consider the following system:

\[
\begin{align*}
\left\{ \begin{array}{l}
\mathcal{D}_t^\alpha \left( \mathcal{D}_t^\beta \right) x(t) = \frac{t^2}{200} \left( \frac{1}{1 + |x(t)|} + \frac{1}{100} \int_0^t t^3s|s|ds \right), \\
x(0) = x(1) = \mathcal{D}_t^\alpha x(0) = \mathcal{D}_t^\beta x(1) = 0.
\end{array} \right.
\end{align*}
\]

(29)

Here,

\[
B = \frac{11}{7},
\]

\[
\alpha = \frac{10}{7},
\]

\[
f(t, x, y, z) = \frac{t^2}{200} \left( \frac{1}{1 + |x|} + y(t) + z(t) \right),
\]

\[
\lambda(t, s) = \delta(t, s) = \frac{t^4s^3}{200},
\]

\[
\sigma(t) = \frac{t^2}{200}.
\]

It is clear that

\[
\phi^* = \psi^* = \frac{1}{800},
\]

\[
\sigma^* = \frac{1}{600},
\]

\[
r_1 = 0, 0036.
\]

By Theorem 8, we conclude that the problem (29) has a unique solution.

5. Conclusion

In this paper, we proved the existence and uniqueness of solutions for nonlinear fractional integrodifferential equations of order \( 1 < \alpha \leq 2 \) and \( 1 < \beta \leq 2 \) using the Banach contraction mapping principle and Krasnoselskii’s fixed point theorem under some weak conditions. Furthermore, we provided two examples to illustrate the main results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.
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