On a Class of Fully Nonlinear Elliptic Equations on Closed Hermitian Manifolds

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Received: 27 April 2015 / Published online: 7 August 2015 © Mathematica Josephina, Inc. 2015

Abstract We study a class of fully nonlinear elliptic equations on closed Hermitian manifolds. We derive $C^\infty$ a priori estimates, and then prove the existence of admissible solutions. In the approach, a new Hermitian metric is constructed to launch the method of continuity.

Keywords Complex Monge–Ampère type · Cone condition · A priori estimates · Hermitian manifold · Continuity method

Mathematics Subject Classification 58J05 · 32W20 · 53C55

1 Introduction

Let $(M^n, \omega)$ be a compact Hermitian manifold of complex dimension $n \geq 2$ and $\chi$ a smooth Hermitian metric on $M^n$. In this paper we study the classical solvability of the fully nonlinear equation

$$\chi^n u = \psi \chi^{n-\alpha} \wedge \omega^\alpha \text{ on } M,$$

where $\psi \in C^\infty(M)$, $1 \leq \alpha \leq n$ and

$$\chi_u := \chi + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u.$$
We shall assume \( \psi > 0 \) and seek solutions with \( \chi_u > 0 \) so that Eq. (1) is elliptic; as in the literature we call such a function \textit{admissible} or \( \chi \)-\textit{plurisubharmonic}.

Equation (1) includes some of the most important partial differential equations in complex geometry and analysis. The case \( \alpha = n \) corresponds to the complex Monge–Ampère equations which play central roles in Kähler geometry as well as problems outside Kähler geometry, while for \( \alpha = 1 \) Eq. (1) appears in a problem proposed by Donaldson [5] in the setting of moment maps; see more discussions below.

There has been much interest recently in the complex Monge–Ampère equation on Hermitian manifolds, 20 years after the work by Cherrier [4]; see, e.g., [8,14,15]. In [15] Tosatti and Weinkove were able to extend Yau’s \( C^0 \) estimate [19] to the Hermitian case, solving the complex Monge–Ampère equation on closed Hermitian manifolds.

When \( \omega, \chi \) are both Kähler and \( \psi \) is constant, Eq. (1) was studied using a parabolic approach (\( J \)-flow) by Chen [2,3], Weinkove [17,18], Song and Weinkove [13] for \( \alpha = 1 \) in an attempt to solve Donaldson’s problem, and by Fang et al. [6] who extended the result to all \( 1 \leq \alpha < n \). It turns out, by integration and Kähler condition, that there is only one possibility of constant \( \psi \) when \( 1 \leq \alpha < n \):

\[
\psi = c := \frac{\int_M \chi^n}{\int_M \chi^{n-\alpha} \wedge \omega^\alpha}.
\]  

Our main purpose in this paper is to solve Eq. (1) for \( 1 \leq \alpha < n \) on closed Hermitian manifolds by the method of continuity. Following [6,10,13], we set

\[
[\chi] = \{ \chi_u : u \in C^2(M) \}, \quad [\chi]^+ = \{ \chi' \in [\chi] : \chi' > 0 \},
\]

and then define for \( \psi \in C^0(M), \psi > 0 \)

\[
\mathcal{C}_\alpha(\psi) := \left\{ [\chi] : \exists \chi' \in [\chi]^+, n \chi'^{m-1} > (n - \alpha) \psi \chi^{n-\alpha-1} \wedge \omega^\alpha \right\}.
\]

If \( [\chi] \in \mathcal{C}_\alpha(\psi) \), we say that \( \chi \) satisfies the cone condition with respect to \( \psi \). In other words, there is a real \( C^2 \) function \( u \) such that \( \chi_u \) is a Hermitian metric satisfying

\[
n \chi_u^{n-1} > (n - \alpha) \psi \chi_u^{n-\alpha-1} \wedge \omega^\alpha.
\]

For general Hermitian manifolds, we have the following result.

**Theorem 1** Let \((M^n, \omega)\) be a closed Hermitian manifold of complex dimension \( n \) and \( \chi \) a smooth Hermitian metric on \( M^n \). Suppose that \( \chi \) satisfies the cone condition with respect to \( \psi \). If

\[
\frac{\chi^n}{\chi^{n-\alpha} \wedge \omega^\alpha} \leq \psi,
\]

there exists a smooth function \( u \) on \( M \) and a unique constant \( b \) solving

\[
\chi_u^n = e^b \psi \chi_u^{n-\alpha} \wedge \omega^\alpha.
\]
Remark 1 For $\alpha = n$, Theorem 1 is proved by Tosatti and Weinkove [15]. Note that $\mathcal{C}_n(\psi) = \{[\chi] : [\chi]^+ \neq \emptyset\}$. In particular, $[\chi] \in \mathcal{C}_n(\psi)$ for any smooth positive function $\psi$, and thus we do not need condition (6).

Albeit the method is very standard in the study of elliptic equations, it is not easy to implement the method for Eq. (1) on closed complex manifolds.

Essentially, it is very hard to make the inverse function theorem work. To resolve the issue, we further develop the method carried out in [14]. In the approach, a new Hermitian metric is constructed, which is a crucial new technique in applying the method of continuity. The new technique reveals the importance and necessity of studying Hermitian metrics, since the metric is Hermitian even if both $\chi$ and $\omega$ are Kähler.

Moreover, we notice that the technique can be applied to more general equations than Eq. (1). In Sect. 4, the method of continuity is carried out without any specific property of the equation. Indeed, given a priori estimates, the continuity method can be carried out for a family of geometric equations

$$F(\chi_u) = f(\lambda(\chi_u)) = \psi > 0,$$

where $\lambda(\chi_u) = (\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of $\chi_u$ with respect to $\omega$, which include complex Hessian equations, the complex Monge–Ampère equation and complex quotient equations. The work in Sect. 4 can be conveniently used in the future study of elliptic equations in complex geometry.

The pointwise condition (6) is imposed to guarantee the validity of a priori estimates for a family of solutions in the method of continuity. For general Hermitian manifolds, we do not have a weaker condition now because of the lack of study on Hermitian geometry. Should we have more knowledge, there would be chances to obtain deeper results. When $\chi$ and $\omega$ are both Kähler, we are able to solve Donaldson’s problem formerly proven by the flow method in [6,18].

**Theorem 2** Let $(M^n, \omega)$ be a closed Kähler manifold of complex dimension $n$ and $\chi$ also a Kähler metric. Suppose that $\chi$ satisfies the cone condition with respect to $\psi$, and $\psi \geq c$ for all $x \in M$ where $c$ is defined in (2). Then there exists a smooth function $u$ on $M$ and a unique constant $b$ solving (7).

**Remark 2** The methods used in [6,18] crucially rely on the assumption that $\psi$ is a Kähler class constant defined by (2). So Theorem 2 is new for $1 \leq \alpha < n$ in the Kähler case (when both $\omega$ and $\chi$ are Kähler). The extension to general $\psi$ requires substantial new techniques.

Since the essential cone condition is dependent on $\psi$ and the cokernel is not fixed, we cannot obtain a priori estimates for the whole family of solutions in the method of continuity. To overcome the difficulty, we piecewise use the continuity method. This idea can be applied to all complex quotient equations under the cone condition. In Theorem 2, we technically set up an intermediate step and apply the method twice. In the first stage, Theorem 1 is applied.
In order to carry out the method of continuity and then prove the existence results, we need to obtain a priori estimates in advance, which are the fundamental of the study.

**Theorem 3** Under the assumptions of Theorem 1, there are uniform $C^\infty$ a priori estimates of smooth admissible solution $u$. In particular,

$$\Delta u + tr \chi \leq Ce^{A(u - \inf_M u)}$$

and consequently

$$||u||_{C^0} \leq C,$$

where $A, C$ are uniform constants. Here we denote $tr \chi = \sum_{i, j} g^{ij} \chi_{ij}$.

In [10], we proved the second order estimate

$$\Delta u + tr \chi \leq Ce^{A(\sup_M (u - u) - \inf_M (u - u))} - e^{A(\sup_M (u - u) - (u - u))}$$

for the Dirichlet problem. Our approach is quite different. In this paper, we significantly improve the $C^2$ estimate, which is the foundation of all estimates. Indeed, Tosatti and Weinkove [14] showed that, to obtain the $C^0$ estimate it suffices to prove an improved version of the $C^2$ estimate. To prove the sharp estimate on general Hermitian manifolds without any other condition, a key trick due to Phong and Sturm [12] is applied.

This paper is organized as follows. In Sect. 2, we state some preliminary and necessary knowledge related to Eq. (1). In Sect. 3, we establish the crucial sharp $C^2$ estimate. In Sect. 4, we apply the new technique to the method of continuity, and reduce the feasibility of the method to two simple conditions. In Sect. 5, we solve Eq. (1) on general Hermitian manifolds and Kähler manifolds, respectively.

**2 Preliminary**

We denote by $\nabla$ the Chern connection of $g$. As in [8, 9], in local coordinates $z = (z^1, \ldots, z^n)$ we have

$$g_{ij} = g \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j} \right), \quad \left[ g^{ij} \right] = \left[ g_{ij} \right]^{-1}.$$

Therefore, the Christoffel coefficients are

$$\Gamma^i_{lj} = \sum_m g^{im} \frac{\partial g_{jm}}{\partial z^l}, \quad \Gamma^i_{lj} = \Gamma^i_{lj} = \sum_m g^{mi} \frac{\partial g_{mj}}{\partial z^l}. $$

$\square$ Springer
The torsion and curvature tensors of $\nabla$ are defined by
\begin{align}
T(X, Y) &= \nabla_X Y - \nabla_Y X - [X, Y], \\
R(X, Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z,
\end{align}
respectively. In local coordinates, the coefficients are
\begin{align}
T^k_{ij} &= \Gamma^k_{ij} - \Gamma^k_{ji} = \sum_l g^{k\bar{l}} \left( \frac{\partial g_{j\bar{l}}}{\partial z^i} - \frac{\partial g_{i\bar{l}}}{\partial z^j} \right), \\
R_{i\bar{j}k\bar{l}} &= -\sum_m g_{m\bar{l}} \frac{\partial \Gamma^m_{ik}}{\partial \bar{z}^j} = -\frac{\partial^2 g_{k\bar{l}}}{\partial z^i \partial \bar{z}^j} + \sum_{p,q} g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z^i} \frac{\partial g_{p\bar{l}}}{\partial \bar{z}^j}.
\end{align}

Let us consider the covariant derivatives of $\chi_u$. For convenience, we express
\begin{align}
X := \chi_u,
\end{align}
and thus
\begin{align}
X_{i\bar{j}} = \nabla \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j} \right) = X_{i\bar{j}} = \frac{\partial}{\partial z^i} \frac{\partial}{\partial \bar{z}^j}.
\end{align}

Also, we denote the coefficients of $X^{-1}$ by $X^{i\bar{j}}$. We use the indices to represent covariant derivatives with respect to the connection $\nabla$: for example,
\begin{align*}
X_{i\bar{j}k} &= \nabla X \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial z^k} \right), \\
X_{j\bar{i}k} &= \nabla X \left( \frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial z^k} \right), \\
X_{i\bar{j}k\bar{l}} &= \nabla^2 X \left( \frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial \bar{z}^k}, \frac{\partial}{\partial z^\ell} \right), \ldots .
\end{align*}

It is easy to see that
\begin{align}
X_{i\bar{j}k} &= X_{j\bar{i}k}.
\end{align}

Assume that at the point $p$, $g_{i\bar{j}} = \delta_{ij}$ and $X_{i\bar{j}}$ is diagonal in a specific chart. For later reference, we call such local coordinates *normal* coordinate charts. Therefore,
\begin{align}
X_{i\bar{j}j} - X_{j\bar{i}i} &= R_{j\bar{i}i} X_{i\bar{j}} - R_{i\bar{i}j} X_{j\bar{i}} + 2 \Re \left\{ \sum_p T_{i\bar{j}}^p X_{i\bar{p}j} \right\} \\
&\quad - \sum_p T_{i\bar{j}}^p T_{i\bar{j}}^p X_{p\bar{p}} - G_{i\bar{i}j\bar{j}},
\end{align}
where
\[
G_{i\bar{j}j} = \chi_{i\bar{j}i\bar{j}} - \chi_{i\bar{i}j\bar{j}} + \sum_p R_{j\bar{i}p\bar{j}}^p + \sum_p R_{i\bar{j}p\bar{j}}^p
+ 2\Re \left\{ \sum_p T_{i\bar{j}}^p \chi_{i\bar{j}p\bar{j}} \right\} - \sum_{p,q} T_{i\bar{j}}^p T_{i\bar{j}}^{\bar{q}} \chi_{p\bar{q}}^P.
\]  

(19)

Let \( S_\alpha(\lambda) \) denote the \( \alpha \)-th elementary symmetric polynomial of \( \lambda \in \mathbb{R}^n \),
\[
S_\alpha(\lambda) = \sum_{1 \leq i_1 < \cdots < i_\alpha \leq n} \lambda_{i_1} \cdots \lambda_{i_\alpha}.
\]  

(20)

For a Hermitian matrix \( A \), define \( S_\alpha(A) = S_\alpha(\lambda(A)) \) where \( \lambda(A) \) denote the eigenvalues of \( A \). Further, write \( S_\alpha(X) = S_\alpha(\lambda_+(X)) \) and \( S_\alpha(X^{-1}) = S_\alpha(\lambda^+(X^{-1})) \) where \( \lambda_+(A) \) and \( \lambda^+(A) \) denote the eigenvalues of a Hermitian matrix \( A \) with respect to \( \{g_{ij}\} \) and to \( \{g^{ij}\} \), respectively. Unless otherwise indicated we shall use \( S_\alpha \) to denote \( S_\alpha(X^{-1}) \) when no possible confusion would occur. In local coordinates, we can write the complex Monge–Ampère type equations (1) in the form
\[
\frac{S_n(\chi_u)}{S_{n-\alpha}(\chi_u)} = \frac{\psi}{C_n^\alpha},
\]  

(21)

or equivalently,
\[
S_\alpha(\chi_u^{-1}) = \frac{C_n^\alpha}{\psi}.
\]  

(22)

As in [10], differentiating the equation twice and applying the strong concavity of \( S_\alpha \) [11], we have
\[
C_n^\alpha \partial_l (\psi^{-1}) = - \sum_i S_{\alpha-1;i}(X_i\bar{i})^2 X_{\bar{i}i}l
\]  

(23)

and
\[
C_n^\alpha \bar{\partial}_l (\psi^{-1}) \geq \sum_{i,j} S_{\alpha-1;i}(X_i\bar{i})^2 X_j\bar{j} X_{\bar{i}i}j - \sum_i S_{\alpha-1;i}(X_i\bar{i})^2 X_{i\bar{i}i},
\]  

(24)

where for \( \{i_1, \ldots, i_s\} \subseteq \{1, \ldots, n\} \),
\[
S_{k;i_1\cdots i_s}(\lambda) = S_k(\lambda|_{\lambda_{i_1} = \cdots = \lambda_{i_s} = 0}).
\]  

(25)

Choosing a local chart such that \( g_{i\bar{j}}(p) = \delta_{ij} \) at a fixed point \( p \), inequality (5) is equivalent to
\[
\frac{C_n^\alpha}{\psi} > S_{\alpha;k}(\chi_u^{-1})
\]  

(26)

for all \( k \). We may assume
\[
\epsilon \omega \leq \chi_u \leq \epsilon^{-1} \omega
\]  

(27)

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for some $\epsilon > 0$.

The following lemma is the key to constructing the barrier function in the second order estimate. An equivalent form of Lemma 1 and its proof are given in [6].

**Lemma 1** Let $u \in C^2(M)$ be a solution to Eq. (1) and $w = \Delta u + tr \chi$. There exist constants $N, \theta > 0$ such that when $w \geq N$ at a point $p$ where $g_{ij} = \delta_{ij}$ and $X_{ij}$ is diagonal,

$$
\sum_i S_{\alpha-1;i}(X^{i\bar{i}})^2(u_{i\bar{i}} - u_{\bar{i}i}) \geq \theta \sum_i S_{\alpha-1;i}(X^{i\bar{i}})^2 + \theta.
$$

(28)

3 The Second Order Estimate

In this section, we prove the sharp second order estimate.

**Proposition 1** Let $u \in C^4(M)$ be a solution to Eq. (1) and $w = \Delta u + tr \chi$. Then there are uniform positive constants $C$ and $A$ such that

$$
\sup_M w \leq Ce^{A(u - \inf_M u)},
$$

(29)

where $C, A$ depend on the given geometric quantities.

**Proof** Let us consider the function $e^{\phi}w$ where $\phi$ is to be specified later. Suppose that $e^{\phi}w$ attains its maximal value at some point $p \in M$. Choose a local chart near $p$ such that $g_{ij} = \delta_{ij}$ and $X_{ij}$ is diagonal at $p$. Therefore, at the point $p$, we have

$$
\frac{\partial_i w}{w} + \partial_i \phi = 0,
$$

(30)

$$
\bar{\partial}_i w + \bar{\partial}_i \phi = 0,
$$

(31)

and

$$
\frac{\bar{\partial}_i \partial_i w}{w} - \frac{\bar{\partial}_i w \partial_i w}{w^2} + \bar{\partial}_i \partial_i \phi \leq 0.
$$

(32)

It is easy to see that

$$
\partial_i w = \sum_i X_{\bar{i}i},
$$

(33)

and

$$
\bar{\partial}_i \partial_i w = \sum_i X_{\bar{i}i\bar{i}}.
$$

(34)
By (18) and (24), we have the following inequality at the point $p$,

$$
0 \geq -2 \sum_{i,j,l} S_{\alpha-1;i}(X^{ii})^2 \Re \{ \overline{T_{ij}^{i}} X_{ij}^{l} \} + \sum_{i,j,l} S_{\alpha-1;i}(X^{ii})^2 X^{ij} X_{ji}^{ij} X_{ij}^{ij}
$$

$$
+ \sum_{i,j,l} S_{\alpha-1;i}(X^{ii})^2 T_{ji}^{i} T_{ij}^{j} X_{jj}^{j} - \frac{1}{w} \sum_{i} S_{\alpha-1;i}(X^{ii})^2 |\partial_{i} w|^2
$$

$$
- C + \sum_{i,j} S_{\alpha-1;i}(X^{ii})^2 \left( - R_{j ii}^{j} X_{ii}^{i} + R_{j jj}^{i} X_{jj}^{j} + G_{i j j}^{i} \right)
$$

$$
+ w \sum_{i} S_{\alpha-1;i}(X^{ii})^2 \partial_{i} \partial_{i} \phi.
$$

(35)

As in [4,16], direct calculation shows that

$$
\left| X_{ij}^{i} - X_{ji}^{j} \overline{\partial_{i} w} - T_{ii}^{i} X_{jj}^{j} \right|^2
$$

$$
= X_{ij}^{i} X_{ji}^{j} + X_{ji}^{j} X_{ij}^{i} \frac{ |\partial_{i} w|^2 }{ w^2 } + T_{ij}^{i} T_{ij}^{j} X_{jj}^{j} - 2 \Re \{ X_{ij}^{i} X_{ij}^{i} \overline{\partial_{i} w} \}\right)
$$

$$
- 2 X_{jj}^{j} \Re \{ X_{ij}^{i} T_{ij}^{i} \} + 2 X_{jj}^{j} \Re \{ X_{ij}^{i} \overline{w} T_{ij}^{i} \}.
$$

(36)

Note also that

$$
X_{j ji}^{i} - X_{i j j}^{i} = \sum_{k} \hat{T}_{ij}^{k} X_{k j}^{k} - \sum_{k} \hat{T}^{k}_{ij} X_{k j}^{k},
$$

(37)

where $\hat{T}$ denotes the torsion with respect to the Hermitian metric $\chi$. We compute the sum straightforwardly and obtain

$$
\sum_{i,j,l} S_{\alpha-1;i}(X^{ii})^2 X^{ij} \left| X_{ij}^{i} - X_{ji}^{j} \overline{\partial_{i} w} - T_{ii}^{i} X_{jj}^{j} \right|^2
$$

$$
= \sum_{i,j,l} S_{\alpha-1;i}(X^{ii})^2 X^{ij} X_{ji}^{ij} X_{ij}^{ij} - \frac{1}{w} \sum_{i} S_{\alpha-1;i}(X^{ii})^2 |\partial_{i} w|^2
$$

$$
+ \sum_{i,j,l} S_{\alpha-1;i}(X^{ii})^2 T_{ji}^{i} T_{ij}^{j} X_{jj}^{j} - 2 \sum_{i,j} S_{\alpha-1;i}(X^{ii})^2 \Re \{ \sum_{k} \hat{T}_{ij}^{k} X_{k j}^{k} \overline{\partial_{i} w} \}
$$

$$
- 2 \sum_{i,j,l} S_{\alpha-1;i}(X^{ii})^2 \Re \{ X_{ij}^{i} T_{ij}^{i} \}.
$$

(38)

Equation (38) implies that
\[
\frac{2}{w} \sum_{i,j} S_{\alpha-1;i}(X^{ij})^2 \Re \left\{ \sum_k \hat{T}^{k}_{ji} X_k j \partial_i w \right\} \\
\leq \sum_{i,j,l} S_{\alpha-1;i}(X^{ij})^2 X^{jl} X_{ji} - \frac{1}{w} \sum_{i,j,l} S_{\alpha-1;i}(X^{ij})^2 |\partial_i w|^2 \\
+ \sum_{i,j,l} S_{\alpha-1;i}(X^{ij})^2 T^{j}_{il} T^{l}_{ij} X_{ji} - 2 \sum_{i,j,l} S_{\alpha-1;i}(X^{ij})^2 \Re \left\{ \sum_k \hat{T}^{k}_{ji} X_k j \partial_i w \right\}. 
\]  

(39)

Combining (35) and (39), we have

\[
0 \geq -C + \sum_{i,j} S_{\alpha-1;i}(X^{ij})^2 \left( -R_{jji} X_{ji} + R_{ii} X_{jj} + G_{ii} \right) \\
+ w \sum_{i} S_{\alpha-1;i}(X^{ij})^2 \partial_i \hat{\partial}_i \phi + \frac{2}{w} \sum_{i,j} S_{\alpha-1;i}(X^{ij})^2 \Re \left\{ \sum_k \hat{T}^{k}_{ji} X_k j \partial_i w \right\}. 
\]  

(40)

By applying a trick due to Phong and Sturm [12], we use the function

\[
\phi := -A(u - u) + \frac{1}{u - u - \inf_M (u - u) + 1}. 
\]  

(41)

Without loss of generality, we assume \( C, A \gg 1 \) throughout this section.

It is easy to see that

\[
\partial_i \phi = -A(\partial_i u - \partial_i u) - \frac{\partial_i u - \partial_i u}{(u - u - \inf_M (u - u) + 1)^2} 
\]  

(42)

and

\[
\partial_i \hat{\partial}_i \phi = -A(\partial_i \hat{\partial}_i u - \partial_i \hat{\partial}_i u) - \frac{\partial_i \hat{\partial}_i u - \partial_i \hat{\partial}_i u}{(u - u - \inf_M (u - u) + 1)^2} \\
+ \frac{2\partial_i (u - u) \hat{\partial}_i (u - u)}{(u - u - \inf_M (u - u) + 1)^3}. 
\]  

(43)

Then, the third term in (40) turns into

\[
w \sum_{i} S_{\alpha-1;i}(X^{ij})^2 \partial_i \hat{\partial}_i \phi \\
= w \sum_{i} S_{\alpha-1;i}(X^{ij})^2 \frac{2\partial_i (u - u) \hat{\partial}_i (u - u)}{(u - u - \inf_M (u - u) + 1)^2} \\
- \left( Aw + \frac{w}{(u - u - \inf_M (u - u) + 1)^2} \right) \sum_{i} S_{\alpha-1;i}(X^{ij})^2 (u_{ii} - u_{i\bar{i}}); 
\]  

(44)

and the fourth term is
\[
\frac{2}{w} \sum_{i,j} S_{\alpha-1;i}(X^{i\bar{j}})^2 \Re \left\{ \sum_k \hat{T}^k_{ji} \chi_{k\bar{j}} \bar{\partial}_i w \right\} \\
= -2 \sum_{i,j} S_{\alpha-1;i}(X^{i\bar{j}})^2 \Re \left\{ \sum_k \hat{T}^k_{ji} \chi_{k\bar{j}} \bar{\partial}_i \phi \right\} \\
\geq -\frac{w}{(u - \bar{u} - \inf_M (u - \bar{u}) + 1)^3} \sum_i S_{\alpha-1;i}(X^{i\bar{i}})^2 |\partial_i (u - \bar{u})|^2 \\
- CA^2 \frac{(u - \bar{u} - \inf_M (u - \bar{u}) + 1)^3}{w} \sum_i S_{\alpha-1;i}(X^{i\bar{i}})^2.
\]

Therefore,

\[
0 \geq \left( Aw + \frac{w}{(u - \bar{u} - \inf_M (u - \bar{u}) + 1)^2} \right) \sum_i S_{\alpha-1;i}(X^{i\bar{i}})^2 (u_{i\bar{i}} - u_{\bar{i}\bar{j}}) \\
- CA^2 \frac{(u - \bar{u} - \inf_M (u - \bar{u}) + 1)^3}{w} \sum_i S_{\alpha-1;i}(X^{i\bar{i}})^2 \\
- Cw \sum_i S_{\alpha-1;i}(X^{i\bar{i}})^2 - C.
\]

For \( A \gg 1 \) which is to be determined later, there are two cases in consideration:

1. \( w > A(u - \bar{u} - \inf_M (u - \bar{u}) + 1)^{\frac{3}{2}} \geq A > N \), where \( N \) is the crucial constant in Lemma 1;
2. \( w \leq A(u - \bar{u} - \inf_M (u - \bar{u}) + 1)^{\frac{3}{2}} \).

In the first case, by Lemma 1,

\[
0 \geq Aw\theta \left( \sum_i S_{\alpha-1;i}(X^{i\bar{i}})^2 + 1 \right) - Cw \sum_i S_{\alpha-1;i}(X^{i\bar{i}})^2 - C.
\]

This gives a bound \( w \leq 1 \) at \( p \) if we choose \( A\theta > C \). It contradicts the assumption \( A \gg 1 \).

In the second case,

\[
we^\phi \leq we^\phi|_p \leq A(u - \bar{u} - \inf_M (u - \bar{u}) + 1)^{\frac{3}{2}} e^{-A(u - \bar{u}) + 1}|_p \\
\leq Ae^2 e^{-A \inf_M (u - \bar{u})}
\]

and hence

\[
w \leq Ae^2 e^{A(u - \bar{u}) - \frac{1}{u - \bar{u} - \inf_M (u - \bar{u}) + 1} - A \inf_M (u - \bar{u})} \\
\leq Ae^2 e^{A(u - \inf_M u) - A \inf_M (u - \bar{u})} \leq Ce^{A(u - \inf_M u)}.
\]

\( \square \)
4 Method of Continuity

For elliptic differential equations on closed manifolds, a crucial step is to make the method of continuity work, especially the inverse function theorem. In this section, we define a new metric and apply the approach in [14].

Define $\varphi > 0$ by

$$\chi^n = \varphi \chi^{n-\alpha} \land \omega^\alpha.$$  \hfill (50)

We use the continuity method and consider the family of equations

$$\chi_{\alpha t}^n = \psi\varphi^{1-t} e^{b_t} \chi_{\alpha t}^{n-\alpha} \land \omega^\alpha, \quad \text{for } t \in [0, 1],$$  \hfill (51)

with $\chi_{\alpha t} > 0$, where $b_t$ is a constant for each $t$. We consider the set

$$T := \left\{ t' \in [0, 1] \mid \exists u_t \in C^{2,\alpha}(M) \text{ and } b_t \text{ solving (51) for } t \in [0, t'] \right\}.$$  \hfill (52)

In this section, we assume: (1) $0 \in T$, i.e., $b_0$ is known; (2) we have uniform $C^\infty$ estimates for all $u_t$.

The first assumption $0 \in T$ tells us that $T$ is not empty. So it suffices to show that $T$ is both open and closed in $[0, 1]$.

First, we prove that $T$ is closed. At the point $x_1$ where $u_t$ achieves its minimum, $\chi_{\alpha t} - \chi \geq 0$ and thus

$$\frac{S_n(\chi_{\alpha t})}{S_{n-\alpha}(\chi_{\alpha t})} \geq \frac{S_n(\chi)}{S_{n-\alpha}(\chi)}.$$  \hfill (53)

This inequality implies that $b_t \geq t (\ln \varphi(x_1) - \ln \psi(x_1))$. Similarly, we have $b_t \leq t (\ln \varphi(x_2) - \ln \psi(x_2))$ if $u_t$ achieves its maximum at $x_2$. So

$$|b_t| \leq \sup_M |\ln \varphi - \ln \psi|.$$  \hfill (54)

The closedness of $T$ follows from the uniform bound for $b_t$ and uniform $C^\infty$ estimates of $u_t$.

Now we show that $T$ is open. Assuming that $\hat{t} \in T$, we need to show that there exists small $\varepsilon > 0$ such that $t \in T$ for any $t \in [\hat{t}, \hat{t} + \varepsilon]$.

Set $F(u) := \frac{S_n(\chi_u)}{S_{n-\alpha}(\chi_u)}$. We have

$$\frac{F(ut)}{F(ut)} = \psi^t \varphi^{t-b_t} e^{b_t-b_t}.$$  \hfill (55)

Note that $F$ here is different from that in the previous section.

We define a new Hermitian metric corresponding to $u$,

$$\Omega_t := \frac{\sqrt{-1}}{2} F(u_t) \sum_{i,j} F_{ij}(u_t) dz^i \land d\bar{z}^j.$$  \hfill (56)
and
\[
\hat{\Omega} = \Omega_i = \frac{\sqrt{-1}}{2} F(u_i) \sum_{i,j} F_{i\bar{j}}(u_i) dz^i \wedge d\bar{z}^j, \tag{57}
\]
where \(F^{i\bar{j}} = \frac{\partial F}{\partial u_i} \) and \([F_{i\bar{j}}]^n_{n\times n}\) is the inverse of \([F^{i\bar{j}}]^n_{n\times n}\).

Applying Gauduchon’s theorem [7] to \(\hat{\Omega}\), there exists a function \(\hat{f}\) such that \(\hat{\Omega}_G = e^{\hat{f}} \hat{\Omega}\) is Gauduchon, i.e., \(\partial \bar{\partial} (\hat{\Omega}_G^{n-1}) = 0\). By adding a constant to \(\hat{f}\), we may assume
\[
\int_M e^{(n-1)\hat{f}} \hat{\Omega} = 1. \tag{58}
\]

We try to solve the equation
\[
\frac{F(u_t)}{F(u_i)} = \left( \int_M \frac{F(u_t)}{F(u_i)} e^{(n-1)\hat{f}} \hat{\Omega} \right) \psi^{t-i} \varphi^{t-i} e^{c_t}, \tag{59}
\]
where \(c_t\) is chosen so that
\[
\int_M \psi^{t-i} \varphi^{t-i} e^{(n-1)\hat{f}} \hat{\Omega} = 1. \tag{60}
\]

Obviously, \(c_t = 0\).

Define two Banach manifolds \(B_1\) and \(B_2\) by
\[
B_1 := \left\{ \eta \in C^{2,\alpha}(M) \mid \chi_{\eta+u_i} > 0, \int_M \eta e^{(n-1)\hat{f}} \hat{\Omega} = 0 \right\},
\]
\[
B_2 := \left\{ h \in C^{\alpha}(M) \mid \int_M h e^{(n-1)\hat{f}} \hat{\Omega} = 1 \right\}.
\]

It is easy to see that
\[
T_0B_1 = B_1 \text{ and } T_0B_2 = \left\{ \rho \in C^{\alpha}(M) \mid \int_M \rho e^{(n-1)\hat{f}} \hat{\Omega} = 0 \right\}.
\]

Also define a map \(\Psi\) mapping \(B_1\) into \(B_2\) by
\[
\Psi(\eta) := \log F(\eta + u_i) - \log F(u_i) - \log \left( \int_M \frac{F(\eta + u_i)}{F(u_i)} e^{(n-1)\hat{f}} \hat{\Omega} \right).
\]

Note that \(\Psi(0) = 0\). By the inverse function theorem, we only need to show that
\[
(D\Psi)_0 : T_0B_1 \to T_0B_2
\]
is invertible. Direct calculation shows that

\[
(D\Psi)_0(\eta) = \frac{1}{F(u_t)} \sum_{i,j} F^{i\bar{j}}(u_t) \eta_i \eta_j - \int_M \frac{1}{F(u_t)} \sum_{i,j} F^{i\bar{j}}(u_t) \eta_i \eta_j e^{(n-1)\hat{\Omega}}
\]

\[
= \Delta \hat{\Omega} \eta - n \int_M e^{(n-1)\hat{\Omega}} \hat{\Omega}^{n-1} \wedge \frac{\sqrt{-1}}{2} \partial \bar{\partial} \eta = \Delta \hat{\Omega} \eta.
\]

(61)

By a result in [1], the equation \(\Delta \hat{\Omega} G u = v\) is solvable if \(\int_M v \hat{\Omega}^n_G = 0\). Given \(\rho \in T_0 B_2\), we have

\[
\int_M \rho e^{(n-1)\hat{\Omega}} \hat{\Omega}^n = \int_M \rho e^{-\hat{\Omega}} \hat{\Omega}^n = 0.
\]

Therefore we can solve Eq. (59) for \(t \in [\hat{t}, \hat{t} + \epsilon)\), and hence \(T\) is open.

5 Solving the Complex Monge–Ampère Type Equations

In this section, we give proofs of the existence results stated in Sect. 1. Observe that in Sect. 4, we make two assumptions to carry out the method of continuity. It suffices to show that the two assumptions are fulfilled. The obstacle for the uniform estimates of \(u_t\) is that the cone condition generally does not work for all \(\psi_t \varphi^{1-t} e^{b_t}\).

Proof (Proof of Theorem 1) Under the given conditions, we begin the method of continuity from \(\chi\). It is easy to see that \(b_0 = 0\).

Since

\[
\frac{\chi^n}{\chi^{n-\alpha} \wedge \omega^\alpha} \leq \psi,
\]

(63)

we have \(\varphi \leq \psi\). At the maximum point of \(u_t\),

\[
\psi_t \varphi^{1-t} e^{b_t} \leq \varphi,
\]

(64)

so

\[
b_t \leq 0.
\]

(65)

This means on \(M\)

\[
\psi_t \varphi^{1-t} e^{b_t} \leq \psi.
\]

(66)

Then the cone condition \(C_\alpha(\psi)\) is uniform for all \(u_t\). As a result, we have uniform \(C^\infty\) estimates of \(u_t\).

When \(\chi\) and \(\omega\) are both Kähler, we have more information relative to the equations. This helps us to obtain deeper results. Since the equation flow is restricted by the initial cone condition, we need to carefully choose the path.

Proof (Proof of Theorem 2) In order to prove Theorem 2, we need to apply the method of continuity twice.
First, since $\chi \in C_\alpha(\psi)$, there must be a function $u$ satisfying

$$\chi u = \chi + \frac{\sqrt{-1}}{2} \partial \bar{\partial} u > 0$$

(67)

and

$$n \chi_{n-1}^u > (n - \alpha) \psi \chi_{n-1}^u \wedge \omega^\alpha.$$  

(68)

Define $\varphi$ by

$$\chi u = \varphi \chi_{n-1}^u \wedge \omega^\alpha.$$  

(69)

It is easy to see that

$$n \chi_{n-1}^u > (n - \alpha) \varphi \chi_{n-1}^u \wedge \omega^\alpha.$$  

(70)

And hence

$$n \chi_{n-1}^u > (n - \alpha) (\max\{\psi, \varphi\} + \delta) \chi_{n-1}^u \wedge \omega^\alpha$$

(71)

for sufficiently small $\delta > 0$. By approximation, we can find a smooth function $v$ such that

$$\frac{\chi_v^u}{\chi_v^{n-1} \wedge \omega^\alpha} \leq \frac{\chi_u^u}{\chi_u^{n-1} \wedge \omega^\alpha} + \frac{\delta}{2} = \varphi + \frac{\delta}{2}$$

(72)

and

$$n \chi_v^{n-1} > (n - \alpha) \varphi_0 \chi_v^{n-1} \wedge \omega^\alpha,$$  

(73)

where $\varphi_0$ is a smooth function satisfying $\varphi_0 \geq \max\{\psi, \varphi\} + \frac{\delta}{2}$. So we have $\chi_v \in C_\alpha(\varphi_0)$ and

$$\frac{\chi_v^n}{\chi_v^{n-1} \wedge \omega^\alpha} \leq \varphi_0.$$  

(74)

By Theorem 1, there exists an admissible solution $u_0$ of

$$\chi u = \varphi_0 e^{b_0} \chi_{n-1}^u \wedge \omega^\alpha,$$  

(75)

for some $b_0 \leq 0$.

Second, we start the method of continuity from $\chi u_0$ and consider the family of equations

$$\chi_{\tilde{u}_t} = \psi_t \varphi_0^{1-t} e^{b_t} \chi_{\tilde{u}_t}^{n-1} \wedge \omega^\alpha,$$  

for $t \in [0, 1].$  

(76)

Note that $b_0$ has been found out in the first stage.

Integrating Eq. (76),

$$\int_M \chi^n = \int_M \psi_t \varphi_0^{1-t} e^{b_t} \chi_{\tilde{u}_t}^{n-1} \wedge \omega^\alpha \geq c e^{b_t} \int_M \chi^{n-1} \wedge \omega^\alpha,$$  

(77)

which implies

$$b_t \leq 0.$$  

(78)

As a consequence,

$$\psi_t \varphi_0^{1-t} e^{b_t} \leq \psi_t \psi_0^{1-t} \leq \varphi_0.$$  

(79)

Therefore, we have uniform $C^\infty$ estimates of $\tilde{u}_t$. 

$\square$
Acknowledgments  The author would like to thank Bo Guan for constant support and advice. The main part of the work was done at The Ohio State University. The author also wishes to thank Ben Weinkove for his helpful suggestions and comments.

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