The order of principal congruences of a bounded lattice.

AMS Fall Southeastern Sectional Meeting
University of Louisville, Louisville, KY
October 5-6, 2013

G. Grätzer
We characterize the order of principal congruences of a bounded lattice as a bounded ordered set. We also state a number of open problems in this new field.

arxiv: 1309.6712
Let $A$ be a lattice (resp., join-semilattice with zero). We call $A$ *representable* if there exist a lattice $L$ such that $A$ is isomorphic to the congruence lattice of $L$, in formula, $A \cong \text{Con} \ L$ (resp., $A$ is isomorphic to the join-semilattice with zero of compact congruences of $L$, in formula, $A \cong \text{Con}_c \ L$).
For over 60 years, one of lattice theory’s most central conjectures was the following:

*Characterize representable lattices as distributive algebraic lattices.*
For over 60 years, one of lattice theory’s most central conjectures was the following:

*Characterize representable lattices as distributive algebraic lattices.*

Or equivalently: Characterize representable join-semilattices as distributive join-semilattice with zero.
For over 60 years, one of lattice theory’s most central conjectures was the following:

*Characterize representable lattices as distributive algebraic lattices.*

Or equivalently: Characterize representable join-semilattices as distributive join-semilattice with zero.
This conjecture was refuted in F. Wehrung in 2007.
In this lecture, we deal with Princ $L$, the order of principal congruences of a lattice $L$. Observe that

(a) Princ $L$ is a directed order with zero.
In this lecture, we deal with Princ \( L \), the order of principal congruences of a lattice \( L \). Observe that

(a) Princ \( L \) is a directed order with zero.

(b) Con\(_c\) \( L \) is the set of compact elements of Con \( L \), a lattice theoretic characterization of this subset.
In this lecture, we deal with Princ $L$, the order of principal congruences of a lattice $L$. Observe that

(a) Princ $L$ is a directed order with zero.

(b) Con$_c$ $L$ is the set of compact elements of Con $L$, a lattice theoretic characterization of this subset.

(c) Princ $L$ is a directed subset of Con$_c$ $L$ containing the zero and join-generating Con$_c$ $L$; there is no lattice theoretic characterization of this subset.
This is the lattice $N_7$ and its congruence lattice $B_2 + 1$. Note that $\text{Princ } N_7 = \text{Con } N_7 - \{\gamma\}$, while in the standard representation $K$ of $B_2 + 1$ as a congruence lattice (G. Grätzer and E. T. Schmidt, 1962), we have $\text{Princ } K = \text{Con } K$. 
This is the lattice $N_7$ and its congruence lattice $B_2 + 1$. Note that $\text{Princ } N_7 = \text{Con } N_7 - \{\gamma\}$, while in the standard representation $K$ of $B_2 + 1$ as a congruence lattice (G. Grätzer and E. T. Schmidt, 1962), we have $\text{Princ } K = \text{Con } K$. This example shows that $\text{Princ } L$ has no lattice theoretic description in $\text{Con } L$. 
For a bounded lattice $L$, the order $\text{Princ } K$ is bounded. We now state the converse.
Theorem 1

For a bounded lattice $L$, the order $\text{Princ } K$ is bounded. We now state the converse.

**Theorem**

*Let $P$ be an order with zero and unit. Then there is a bounded lattice $K$ such that $P \cong \text{Princ } K$. If $P$ is finite, we can construct $K$ as a finite lattice.*
Problem

*Can we characterize the order $\text{Princ} \ L$ for a lattice $L$ as a directed order with zero?*
Problem

*Can we characterize the order Princ L for a lattice L as a directed order with zero?*

(Why directed?)
Can we characterize the order $\text{Princ } L$ for a lattice $L$ as a directed order with zero?

(Why directed?)
G. Czédli solved this problem for countable lattices
arXiv:1305.0965
Even more interesting would be to characterize the pair $P = \text{Princ } L$ in $S = \text{Con}_c L$ by the properties that $P$ is a directed order with zero that join-generates $S$. We have to rephrase this so it does not require a solution of the congruence lattice characterization problem.
Even more interesting would be to characterize the pair \( P = \text{Princ } L \)
in \( S = \text{Con}_c L \) by the properties that \( P \) is a directed order with zero that join-generates \( S \). We have to rephrase this so it does not require a solution of the congruence lattice characterization problem.

**Problem**

*Let \( S \) be a representable join-semilattice. Let \( P \subseteq S \) be a directed order with zero and let \( P \) join-generate \( S \). Under what conditions is there a lattice \( K \) such that \( \text{Con}_c K \) is isomorphic to \( S \) and under this isomorphism \( \text{Princ } K \) corresponds to \( P \)?*
For a lattice $L$, let us define a valuation $v$ on $\text{Con}_c L$ as follows: for a compact congruence $\alpha$ of $L$, let $v(\alpha)$ be the smallest integer $n$ such that the congruence $\alpha$ is the join of $n$ principal congruences. A valuation $v$ has some obvious properties, for instance, $v(0) = 0$ and $v(\alpha \vee \beta) \leq v(\alpha) + v(\beta)$. Note the connection with Princ $L$:

$$\text{Princ } L = \{ \alpha \in \text{Con}_c L \mid v(\alpha) \leq 1 \}.$$
For a lattice $L$, let us define a valuation $v$ on $\text{Con}_c L$ as follows: for a compact congruence $\alpha$ of $L$, let $v(\alpha)$ be the smallest integer $n$ such that the congruence $\alpha$ is the join of $n$ principal congruences. A valuation $v$ has some obvious properties, for instance, $v(0) = 0$ and $v(\alpha \lor \beta) \leq v(\alpha) + v(\beta)$. Note the connection with Princ $L$:

\[
\text{Princ } L = \{ \alpha \in \text{Con}_c L \mid v(\alpha) \leq 1 \}.
\]

**Problem**

*Let $S$ be a representable join-semilattice. Let $v$ map $S$ to the natural numbers. Under what conditions is there an isomorphism $\varphi$ of $S$ with $\text{Con}_c K$ for some lattice $K$ so that under $\varphi$ the map $v$ corresponds to the valuation on $\text{Con}_c K$?*
Let $D$ be a finite distributive lattice. In G. Grätzer and E. T. Schmidt 1962, we represent $D$ as the congruence lattice of a finite lattice $K$ in which \textit{all congruences are principal} (that is, $\text{Con } K = \text{Princ } K$).
Let $D$ be a finite distributive lattice. In G. Grätzer and E. T. Schmidt 1962, we represent $D$ as the congruence lattice of a finite lattice $K$ in which all congruences are principal (that is, $\text{Con} K = \text{Princ} K$).

**Problem**

Let $D$ be a finite distributive lattice. Let $Q$ be a subset of $D$ satisfying $\{0, 1\} \cup \text{Ji } D \subseteq Q \subseteq D$. When is there a finite lattice $K$ such that $\text{Con} K$ is isomorphic to $D$ and under this isomorphism $\text{Princ} K$ corresponds to $Q$?
Example:
Let $D$ be the eight-element Boolean lattice. Let $Q$ be a subset of $D$ containing 0 and 1 and the three atoms (the join-irreducible elements).

**Lemma**

*If there is a finite lattice $K$ such that $\text{Con } K$ is isomorphic to $D$ and under this isomorphism $\text{Princ } K$ corresponds to $Q$, then $Q$ has seven or eight elements.*
In particular, let $Q = \text{Con} \, L$.

**Problem**

Let $\mathbf{K}$ be a class of lattices with the property that every finite distributive lattice $D$ can be represented as the congruence lattice of some finite lattice in $\mathbf{K}$. Under what conditions on $\mathbf{K}$ is it true that every finite distributive lattice $D$ can be represented as the congruence lattice of some finite lattice $L$ in $\mathbf{K}$ with the additional property: $\text{Con} \, L = \text{Princ} \, L$. 

G. Grätzer
G. Grätzer and E. T. Schmidt, *An extension theorem for planar semimodular lattices*. Periodica Mathematica Hungarica. arXiv: 1304.7489

**Theorem**

*Every finite distributive lattice* $D$ *can be represented as the congruence lattice of a finite, planar, semimodular lattice* $K$ *with the property that all congruences are principal.*

In fact, $K$ is constructed as a “rectangular lattice”. 

In the finite variant of the valuation problem, we need an additional property.

**Problem**

Let $S$ be a finite distributive lattice. Let $v$ be a map of $D$ to the natural numbers satisfying $v(0) = 0$, $v(1) = 1$, and $v(a \lor b) \leq v(a) + v(b)$ for $a, b \in D$. When is there an isomorphism $\varphi$ of $D$ with $\text{Con } K$ for some finite lattice $K$ such that under $\varphi$ the map $v$ corresponds to the valuation on $\text{Con } K$?
In Theorem 1, can we construct a semimodular lattice?
Problem 7

In Theorem 1, can we construct a semimodular lattice?

Remember Theorem 1:

Theorem

Let $P$ be an order with zero and unit. Then there is a bounded lattice $K$ such that

$$P \cong \text{Princ } K.$$

If $P$ is finite, we can construct $K$ as a finite lattice.
Problem

In Problems 2 and 3, in the finite case, can we construct a finite semimodular lattice $K$?
Problem 8

In Problems 2 and 3, in the finite case, can we construct a finite semimodular lattice $K$?

Remember Problems 2 and 3:

Problem

Let $S$ be a representable join-semilattice. Let $P \subseteq S$ be a directed order with zero and let $P$ join-generate $S$. Under what conditions is there a lattice $K$ such that $\text{Con}_c K$ is isomorphic to $S$ and under this isomorphism $\text{Princ} K$ corresponds to $P$?

Problem

Let $S$ be a representable join-semilattice. Let $\nu$ map $S$ to the natural numbers. Under what conditions is there an isomorphism $\varphi$ of $S$ with $\text{Con}_c K$ for some lattice $K$ so that under $\varphi$ the map $\nu$ corresponds to the valuation on $\text{Con}_c K$?
In E. T. Schmidt 1962 (see also G. Grätzer and E. T. Schmidt 2003), for a finite distributive lattice $D$, a countable modular lattice $M$ is constructed with $\text{Con} M \cong D$.

**Problem**

*In Theorem 1, for a finite $P$, can we construct a countable modular lattice $K$?*
Some of these problems seem to be of interest for algebras other than lattices as well.

**Problem**

*Can we characterize the order $\text{Princ} \mathcal{A}$ for an algebra $\mathcal{A}$ as an order with zero?*
Problem 9

Problem

For an algebra $\mathcal{A}$, how is the assumption that the unit congruence $1$ is compact reflected in the order $\text{Princ}\mathcal{A}$?
Problem 10

Problem

Let $\mathcal{A}$ be an algebra and let $\text{Princ}\, \mathcal{A} \subseteq Q \subseteq \text{Con}_c \, \mathcal{A}$. Does there exist an algebra $\mathcal{B}$ such that $\text{Con}\, \mathcal{A} \cong \text{Con}\, \mathcal{B}$ and under this isomorphism $Q$ corresponds to $\text{Princ}\, \mathcal{B}$?
Problem

Extend the concept of valuation to algebras in general. State and solve Problem 3 for algebras.
Problem 11

Problem

Extend the concept of valuation to algebras in general. State and solve Problem 3 for algebras.

Remember Problem 3:

Problem

Let $S$ be a representable join-semilattice. Let $v$ map $S$ to the natural numbers. Under what conditions is there an isomorphism $\varphi$ of $S$ with $\text{Con}_c K$ for some lattice $K$ so that under $\varphi$ the map $v$ corresponds to the valuation on $\text{Con}_c K$?
Problem 12

Problem

Can we sharpen the result of G. Grätzer and E. T. Schmidt 1960: every algebra $\mathcal{A}$ has a congruence-preserving extension $\mathcal{B}$ such that $\text{Con} \mathcal{A} \cong \text{Con} \mathcal{B}$ and $\text{Princ} \mathcal{B} = \text{Con}_c \mathcal{B}$.

I do not even know whether every algebra $\mathcal{A}$ has a proper congruence-preserving extension $\mathcal{B}$.
Problem 12

Problem

Can we sharpen the result of G. Grätzer and E. T. Schmidt 1960: every algebra $A$ has a congruence-preserving extension $B$ such that $\text{Con} A \cong \text{Con} B$ and $\text{Princ} B = \text{Con}_c B$.

I do not even know whether every algebra $A$ has a proper congruence-preserving extension $B$. 
For a bounded order $Q$, let $Q^-$ denote the order $Q$ with the bounds removed. Let $P$ be the order in Theorem 1. Let $0$ and $1$ denote the zero and unit of $P$, respectively. We denote by $P^d$ those elements of $P^-$ that are not comparable to any other element of $P^-$, that is,

$$P^d = \{ x \in P^- \mid x \parallel y \text{ for all } y \in P^-, \ y \neq x \}.$$
We first construct the lattice $F$ consisting of the elements $o$, $i$ and the elements $a_p$, $b_p$ for every $p \in P$, where $a_p \neq b_p$ for every $p \in P^-$ and $a_0 = b_0$, $a_1 = b_1$. The lattice $F$: 

Proof by Picture: The Lattice $F$
Proof by Picture: The Lattice $K$

We are going to construct the lattice $K$ (of Theorem 1) as an extension of $F$. For $p \prec q$, between the edges $[a_p, b_p]$ and $[a_q, b_q]$ we insert the lattice $S = S(p, q)$:

![Lattice Diagram]

The principal congruence of $K$ representing $p \in P^-$ will be $\text{con}(a_p, b_p)$.
Proof by Picture: The Orders $C$, $V$, and $H$

For $x \in S(p, q)$ and $y \in S(p', q')$, $p \prec q$, $p' \prec q'$ we have to find $x \lor y$ and $x \land y$.

If $\{p, q\} \cap \{p', q'\} = \emptyset$, then $x$ and $y$ are complimentary.

If $\{p, q\} \cap \{p', q'\} \neq \emptyset$, then $\{p, q\} \cup \{p', q'\}$ form a three element order $C$, $V$, or $H$:
We form $x \lor y$ and $x \land y$ in the appropriate lattices,

$S_C = S(p < q, q < q')$, $S_V = S(p < q, p < q')$ with $q \neq q'$, and

$S_H = S(p < q, p' < q)$ with $p \neq p'$. 

G. Grätzer

The order of principal congruences of a bounded lattice
The lattice $S_C = S(p < q, \; q < q')$:
The lattice $S_V = S(p < q, \ p < q')$ with $q \neq q'$:
The lattice $S_H = S(p < q, \ p' < q)$ with $p \neq p'$: