Approximation of BSDE with Hidden Forward Equation and Unknown Volatility

O.V. Chernoyarov\textsuperscript{1} and Yu.A. Kutoyants\textsuperscript{2}

\textsuperscript{1}National Research University “MPEI”, Moscow, Russia
\textsuperscript{2}Le Mans University, Le Mans, France
\textsuperscript{1}Maikop State Technological University, Maikop, Russia
\textsuperscript{1,2}Tomsk State University, Tomsk, Russia

Abstract

In the present paper the problem of approximating the solution of BSDE is considered in the case where the solution of forward equation is observed in the presence of small Gaussian noise. We suppose that the volatility of the forward equation depends on an unknown parameter. This approximation is made in several steps. First we obtain a preliminary estimator of the unknown parameter, then using Kalman-Bucy filtration equations and Fisher-score device we construct an one-step MLE-process of this parameter. The solution of BSDE is approximated by means of the solution of PDE and the One-step MLE-process. The error of approximation is described in different metrics.

Key words: BSDE, solution approximation, perturbed dynamical systems, volatility estimation.

1 Introduction

Backward stochastic differential equations (BSDE) were first introduced in the linear case by Bismuth [1]. The general (nonlinear) case was initiated by Pardoux and Peng [15]. Since then the BSDE attract high attention and are intensively developed due to their importance in financial mathematics and insurance (see, e.g. El Karoui \textit{et al.} [3], Ma and Yong [14], Shen and Wei [17], Sun et al. [18] and the references therein).
Let us recall what is the BSDE in the Markovian case following [3]. For the sake of simplicity only one-dimensional processes are considered. Let us consider a filtered probability space \((\Omega, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) with the filtration \((\mathcal{F}_t)_{t \in [0,T]}\) satisfying the usual conditions. Define the stochastic differential equation (called forward) by
\[
dx_t = S(t, X_t) \, dt + \sigma(t, X_t) \, dW_t, \quad 0 \leq t \leq T, \tag{1}
\]
where \(W_t, \mathcal{F}_t, 0 \leq t \leq T\) is the standard Wiener process and \(X_0\) is \(\mathcal{F}_0\) measurable initial value, respectively. The trend coefficient \(S(t,x)\) and the diffusion coefficient \(\sigma(t,x)\) satisfy the Lipschitz and linear growth conditions
\[
|S(t,x) - S(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq L |x - y|, \tag{2}
\]
\[
|S(t,x)| + |\sigma(t,x)| \leq C (1 + |x|), \tag{3}
\]
where \(L > 0\) and \(C > 0\) are constants. Under these conditions the stochastic differential equation has a unique strong solution (see Liptser and Shiryaev [13]).

The main problem is the following: Given two functions \(F(t,x,z,\sigma)\) and \(\Phi(x)\) we have to construct two processes \((Z_t, \Sigma_t, \mathcal{F}_t, 0 \leq t \leq T)\) such that the solution of the stochastic differential equation
\[
dZ_t = -F(t, X_t, Z_t, \Sigma_t) \, dt + \Sigma_t \, dW_t, \quad 0 \leq t \leq T, \tag{4}
\]
(called backward) has the terminal value \(Z_T = \Phi(X_T)\).

This equation is often written in integral form as follows
\[
Z_t = \Phi(X_T) + \int_t^T F(s, X_s, Z_s, \Sigma_s) \, ds - \int_t^T \Sigma_s \, dW_s, \quad 0 \leq t \leq T.
\]
We suppose that the functions \(F(t,x,y,z)\) and \(\Phi(x)\) satisfy the conditions
\[
|F(t,x,z_1,\sigma_1) - F(t,x,z_2,\sigma_2)| \leq L (|z_1 - z_2| + |\sigma_1 - \sigma_2|), \tag{5}
\]
\[
|F(t,x,z,\sigma)| + |\Phi(x)| \leq C (1 + |x|^p), \tag{6}
\]
where \(p \geq 1/2\).

This is the so-called Markovian case. For the existence and uniqueness of the solution see Pardoux and Peng [16].

The solution \((Y_t, Z_t, \mathcal{F}_t, 0 \leq t \leq T)\) could be constructed as follows. Suppose that \(u(t,x)\) is a solution of the partial differential equation
\[
\frac{\partial u}{\partial t} + S(t,x) \frac{\partial u}{\partial x} + \frac{\sigma(t,x)^2}{2} \frac{\partial^2 u}{\partial x^2} = -F(t,x,u,\sigma(t,x) \frac{\partial u}{\partial x}),
\]
2
with the terminal condition \( u(T,x) = \Phi(x) \).

Let us set \( Z_t = u(t,X_t) \), then by Itô’s formula we obtain
\[
dZ_t = \left[ \frac{\partial u(t,X_t)}{\partial t} + S(t,X_t) \frac{\partial u(t,X_t)}{\partial x} + \frac{\sigma(t,X_t)^2}{2} \frac{\partial^2 u(t,X_t)}{\partial x^2} \right] dt \\
+ \sigma(t,X_t) \frac{\partial u(t,X_t)}{\partial x} dW_t, \quad Y_0 = u(0,X_0).
\]

We use the notation
\[
\frac{\partial u(t,X_t)}{\partial x} = u_x(t,X_t) = \left. \frac{\partial u(t,x)}{\partial x} \right|_{x=X_t}.
\]

Hence if we denote \( \Sigma_t = \sigma(t,X_t) u_x(t,X_t) \), then this equation becomes
\[
dZ_t = -F(t,X_t,Z_t,\Sigma_t) dt + \Sigma_t dW_t, \quad Z_0 = u(0,X_0),
\]
and \( Z_T = u(T,X_T) = \Phi(X_T) \). Therefore the problem is solved and the equation (4) is obtained with given terminal value.

We are interested in the problem of the approximation of the solution \((Z_t, \Sigma_t,F_t, 0 \leq t \leq T)\) of BSDE in the case where the forward equation (1) contains an unknown finite-dimensional parameter \( \vartheta \):
\[
dX_t = S(\vartheta,t,X_t) dt + \sigma(\vartheta,t,X_t) dW_t, \quad X_0, \ 0 \leq t \leq T.
\]

Then the solution \( u \) of the corresponding partial differential equation depends on \( \vartheta \), i.e., \( u = u(t,x,\vartheta) \). The “natural” approximations \((\hat{Z}_t, \hat{\Sigma}_t, \hat{F}_t, 0 \leq t \leq T)\) could be constructed as follows. Suppose that \( u(t,x,\vartheta) \) is a solution of the partial differential equation
\[
\frac{\partial u}{\partial t} + S(\vartheta,t,x) \frac{\partial u}{\partial x} + \frac{\sigma(\vartheta,t,x)^2}{2} \frac{\partial^2 u}{\partial x^2} = -F\left(t,x,u(\vartheta,t) \frac{\partial u}{\partial x}\right), \quad (7)
\]
with the terminal condition \( u(T,x,\vartheta) = \Phi(x) \).

Of course, we can not set \( Z_t = u(t,X_t,\vartheta) \) since \( \vartheta \) is unknown. One way to obtain an approximation \((\hat{Z}_t, \hat{\Sigma}_t)\) of \((Z_t, \Sigma_t)\) is to find first an estimator-process \( \hat{\vartheta}_t \), \( 0 < t \leq T \) and then to set
\[
\hat{Z}_t = u(t,X_t,\hat{\vartheta}_t), \quad \hat{\Sigma}_t = u'_x(t,X_t,\hat{\vartheta}_t) \sigma(\hat{\vartheta}_t,t,X_t).
\]

If the estimator has good properties, say, \( \hat{\vartheta}_t - \vartheta \) is small in some sense, then the error
\[
\hat{Z}_t - Z_t = u(t,X_t,\hat{\vartheta}_t) - u(t,X_t,\vartheta) \approx \frac{\partial u(t,X_t,\vartheta)}{\partial \vartheta} (\hat{\vartheta}_t - \vartheta)
\]
is small as well.

Here \( \hat{\vartheta}_t, 0 \leq t \leq T \) is a good estimator-process of \( \vartheta \) in the sense that:
• The estimator $\hat{\vartheta}_t^*$ depends on $X^t = (X_s, 0 \leq s \leq t)$.

• It is easy to calculate for each $t \in (0, T]$.

• Provides the asymptotically efficient estimation of $Z_t$, i.e.,

$$E_{\vartheta} \left( \hat{Z}_t - Z_t \right)^2 \rightarrow \min.$$ 

Therefore the main problem is how to find a good estimator-process. Such problems were studied in the works [2], [6], [7], [12].

Different problems were solved following the same general procedure, which is illustrated as follows. Consider the forward equation with small volatility: $\sigma (t, X_t, \vartheta) = \varepsilon \sigma (t, X_t)$, where $\sigma (t, x)$ satisfies the conditions (2), (3), $\varepsilon \in (0, 1]$ is a small parameter, i.e. we consider asymptotics $\varepsilon \to 0$.

Introduce a learning interval $[0, \tau_\varepsilon]$, where $\tau_\varepsilon \to 0$ but slowly. Using observations $X^{\tau_\varepsilon} = (X_t, 0 \leq t \leq \tau_\varepsilon)$, a preliminary consistent estimator $\hat{\vartheta}_{t,\varepsilon}$ of $\vartheta$ is constructed. Then with the help of slightly modified Fisher-score device this estimator is improved up to the asymptotically ($\varepsilon \to 0$) efficient One-step MLE-process $\hat{\vartheta}_{t,\varepsilon}$, $\tau_\varepsilon, t \leq T$. Now the approximation of $Z_t, \Sigma_t$, is given by the relations

$$\hat{Z}_t = u \left( t, X_t, \hat{\vartheta}_{t,\varepsilon}^* \right), \quad \hat{\Sigma}_t = \varepsilon \sigma \left( t, X_t \right) u_x' \left( t, X_t, \hat{\vartheta}_{t,\varepsilon}^* \right)$$

It is shown that these approximations are asymptotically efficient. For the details see [6], [7], [12]. In [2] it is supposed that the volatility $\sigma (t, X_t, \vartheta)$ depends on $\vartheta$ and the forward equation (1) is observed in discrete times. Then a similar procedure of approximation was realized.

In all the previous problems the forward equation was assumed to be observed directly; however, in the present work we suppose that we have a partially observed linear system, where the forward equation is hidden and we observe its solution in the presence of white Gaussian noise.

# 2 Main result

## 2.1 Model of observations and BSDE

Suppose that the forward equation is

$$dY_t = -a (t) Y_t dt + b (\vartheta, t) dV_t, \quad Y_0 = y_0, \quad 0 \leq t \leq T.$$ (8)

The solution $Y^T = (Y_t, 0 \leq t \leq T)$ of this equation can not be observed directly and only the observations

$$dX_t = f (t) Y_t dt + \varepsilon \sigma (t) dW_t, \quad X_0 = 0, \quad 0 \leq t \leq T$$ (9)
are available. The parameter $\vartheta \in \Theta = (\alpha, \beta)$, where $|\alpha| + |\beta| < \infty$. Here $a(\cdot), b(\cdot), f(\cdot)$ and $\sigma(\cdot)$ are known functions and $\varepsilon \in (0, 1]$. These functions satisfy the following regularity conditions.

Conditions $A_1$. The functions $a(t), b(\vartheta, t), f(t)$ and $\sigma(t)$ have continuous derivatives w.r.t. $t \in [0, T]$.

$A_2$. The functions $b(\vartheta, t), f(t)$ and $\sigma(t)$ are separated from zero by a constant, which does not depend neither on $\vartheta$ nor on $t$.

We consider two functions $F(t, y, u, s)$, $\Phi(y)$ and observations $X^T = (X_t, 0 \leq t \leq T)$ and we aim at constructing the corresponding BSDE. Of course, we can not construct the BSDE

$$dZ_t = -F(t, Y_t, Z_t, s_t)dt + s_t dV_t, \quad Z_T = \Phi(Y_T), \quad 0 \leq t \leq T$$

for two reasons: first we have no access to the process $Y^T$ (no Wiener process $V_t$) and even if we have $Y^T$ the solution $U = U(t, y, \vartheta)$ of the corresponding PDE

$$\frac{\partial U}{\partial t} - a(t) y \frac{\partial U}{\partial y} + \frac{b(\vartheta, t)^2}{2} \frac{\partial^2 U}{\partial y^2} = -F(t, y, U, b(\vartheta, t) \frac{\partial U}{\partial y}),$$

$$U(T, y, \vartheta) = \Phi(y)$$

depends on the unknown parameter $\vartheta$. Therefore we can not set $Z_t = U(t, Y_t, \vartheta)$ since neither $Y_t$ nor $\vartheta$ are known.

As we have no solution $Y^T$ of the forward equation we reformulate the problem and propose BSDE based on the best in the mean squared estimator of this process. Introduce the conditional expectation $\hat{Y}^T = (\hat{Y}_t, 0 \leq t \leq T)$, where $\hat{Y}_t = E_\vartheta(Y_t | X_s, 0 \leq s \leq t)$. Now the corresponding BSDE becomes

$$dZ_t = -F(t, \hat{Y}_t, Z_t, s(t))dt + s(t) d\hat{W}_t, \quad Z_T = \Phi(\hat{Y}_T), \quad 0 \leq t \leq T, \quad (12)$$

where the Wiener process $\hat{W}_t, 0 \leq t \leq T$ is described below. To construct the equation (12) we need the equations of Kalman-Bucy filtration for $\hat{Y}_t$, which we remind here. It will be convenient to denote $\hat{Y}_t = m(\vartheta, t)$ in order to show the dependence on $\vartheta$. The equation for $m(\vartheta, t)$ is (see [13])

$$dm(\vartheta, t) = -\left[a(t) + \frac{\gamma(\vartheta, t)f(t)^2}{\varepsilon^2 \sigma(t)^2}\right] m(\vartheta, t) dt + \frac{\gamma(\vartheta, t)f(t)}{\varepsilon^2 \sigma(t)^2} dX_t. \quad (13)$$

Here $m(\vartheta, t) = 0$ and $\gamma(\vartheta, t) = E_\vartheta((Y_t - m(\vartheta, t))^2)$ is the solution of Riccati equation

$$\frac{\partial \gamma(\vartheta, t)}{\partial t} = -2a(t) \gamma(\vartheta, t) - \frac{\gamma(\vartheta, t)^2 f(t)^2}{\varepsilon^2 \sigma(t)^2} + b(\vartheta, t)^2, \quad \gamma(\vartheta, 0) = 0. \quad (14)$$
We further denote
\[
\gamma (\vartheta, t) = \frac{\gamma (\vartheta, t)}{\varepsilon}, \quad \gamma_0 (\vartheta, t) = \frac{b (\vartheta, t) \sigma (t)}{f (t)}, \quad A_\varepsilon (\vartheta, t) = \frac{\gamma (\vartheta, t) f (t)}{\sigma (t)^2},
\]
\[
A_0 (\vartheta, t) = \frac{b (\vartheta, t)}{\sigma (t)}, \quad q_\varepsilon (\vartheta, t) = a (t) + \frac{A_\varepsilon (\vartheta, t) f (t)}{\varepsilon}.
\]

The true value is denoted by \( \vartheta_0 \). The equation (13) for \( m (\vartheta_0, t) \) and Riccati equation (14) can be re-written as follows
\[
dm (\vartheta_0, t) = -a (t) m (\vartheta_0, t) dt + A_\varepsilon (\vartheta_0, t) \sigma (t) d\tilde{W}_t, \\
\frac{\partial \gamma_\varepsilon (\vartheta_0, t)}{\partial t} = -2a (t) \gamma_\varepsilon (\vartheta_0, t) - \frac{A_\varepsilon (\vartheta_0, t)^2 \sigma (t)^2}{\varepsilon} + \frac{b (\vartheta_0, t)^2}{\varepsilon},
\]
with initial values \( m (\vartheta_0, 0) = 0 \) and \( \gamma_\varepsilon (\vartheta_0, 0) = 0 \), respectively. Here \( \tilde{W}_t, \mathcal{F}_t, 0 \leq t \leq T \) is the innovation Wiener process defined by the relation
\[
dX_t = f (t) m (\vartheta_0, t) dt + \varepsilon \sigma (t) d\tilde{W}_t, \quad X_0 = 0
\]
(see [13], Theorem 7.12).

**Lemma 1.** Let the conditions \( \mathcal{A} \) be fulfilled. Then for any \( t_0 \in (0, T] \) we have the convergence
\[
\sup_{t_0 \leq t \leq T} |\gamma_\varepsilon (\vartheta, t) - \gamma (\vartheta, t)| \longrightarrow 0, \quad \sup_{t_0 \leq t \leq T} |A_\varepsilon (\vartheta, t) - A_0 (\vartheta, t)| \longrightarrow 0. \quad (15)
\]

**Proof.** See Lemma 2 in [8].

This lemma allows us to verify the following obvious result

**Lemma 2.** Let the conditions \( \mathcal{A} \) be fulfilled. Then for any \( t_0 \in (0, T] \) we have the convergence
\[
\sup_{t_0 \leq t \leq T} \mathbb{E}_{\vartheta_0} |m (\vartheta_0, t) - Y_t|^2 \leq C\varepsilon \rightarrow 0 \quad (16)
\]
as \( \varepsilon \rightarrow 0 \).

**Proof.** For the difference \( \delta_t = m (\vartheta_0, t) - Y_t \) we have the equation
\[
d\delta_t = -a (t) \delta_t dt - b (\vartheta_0, t) dV_t + A_\varepsilon (\vartheta_0, t) \sigma (t) d\tilde{W}_t \\
= -q_\varepsilon (\vartheta_0, t) \delta_t dt - b (\vartheta_0, t) dV_t + A_\varepsilon (\vartheta_0, t) \sigma (t) dW_t,
\]
where
\[
q_\varepsilon (\vartheta_0, t) = a (t) + \varepsilon^{-1} A_\varepsilon (\vartheta_0, t) f (t).
\]
Hence
\[ \delta_t = -\int_0^t e^{-\int_s^t q(\vartheta_0,v)dv} b(\vartheta_0,s) dV_s + \int_0^t e^{-\int_s^t q(\vartheta_0,v)dv} A_\varepsilon(\vartheta_0,s) \sigma(s) dW_s \] (17)

and
\[ E_{\vartheta_0} | m(\vartheta_0,t) - Y_t|^2 = \int_0^t e^{-2\int_s^t q(\vartheta_0,v)dv} \left[ b(\vartheta_0,s)^2 + A_\varepsilon(\vartheta_0,s)^2 \sigma(s)^2 \right] ds \leq C \int_0^t e^{-\varepsilon(t-s)} ds \leq C \varepsilon \left[ 1 - e^{-\frac{\varepsilon}{2}} \right]. \]

Here we used the condition $A_2$ and the boundedness of all functions.

Therefore for small $\varepsilon$ the random process $m(\vartheta_0,t)$ is a good approximation of the solution $Y_t$ of the forward equation.

It is worth mentioning that if $\vartheta_0$ is known, then in order to construct (12) with innovation Wiener process we need the solution of the partial differential equation
\[
\frac{\partial u}{\partial t} - a(t) \frac{\partial u}{\partial y} + \frac{B_\varepsilon(\vartheta_0,t)^2}{2} \frac{\partial^2 u}{\partial y^2} = -F(t,y,u,B_\varepsilon(\vartheta_0,t) \frac{\partial u}{\partial y}),
\]
\[ u(T,y,\vartheta_0,\varepsilon) = \Phi(y), \] (18)

where $B_\varepsilon(\vartheta_0,t) = A_\varepsilon(\vartheta_0,t) \sigma(t)$. If the solution of this equation was available, then
\[ Z_t = u(t,m(\vartheta_0,t),\vartheta_0,\varepsilon), \quad s(t) = A_\varepsilon(\vartheta_0,t) \sigma(t) \frac{\partial u}{\partial y}(t,m(\vartheta_0,t),\vartheta_0,\varepsilon) \]
would form the equation (12).

We further denote
\[ \frac{\partial}{\partial t} u(t,y,\vartheta,\varepsilon) = u'_t(t,y,\vartheta,\varepsilon), \quad \frac{\partial}{\partial y} u(t,y,\vartheta,\varepsilon) = u'_y(t,y,\vartheta,\varepsilon), \]
\[ \frac{\partial}{\partial \vartheta} u(t,y,\vartheta,\varepsilon) = \dot{u}(t,y,\vartheta,\varepsilon), \quad \frac{\partial}{\partial \varepsilon} u(t,y,\vartheta,\varepsilon) = u'_\varepsilon(t,y,\vartheta,\varepsilon). \]

Suppose that we have some estimator-process $\hat{\vartheta}_{t,\varepsilon}$ which is consistent: for any $t \in (0,T]$ the estimator $\hat{\vartheta}_{t,\varepsilon} \rightarrow \vartheta_0$. Let us set
\[ \hat{Z}_t = u(t,\hat{\vartheta}_t,\hat{\vartheta}_{t,\varepsilon},\varepsilon), \quad \hat{s}(t) = A_\varepsilon(\hat{\vartheta}_{t,\varepsilon},t) \sigma(t) u'(t,\hat{\vartheta}_t,\hat{\vartheta}_{t,\varepsilon},\varepsilon), \]
where \( \hat{m}_t \) is an approximation of \( m(\vartheta, t) \). Then we have the following question: what is the relation between the solution \( Z_t \) of the equations (10), the solution \( Z_t \) of the equations (12) and the approximation \( \hat{Z}_t \) introduced above?

The convergences
\[
\gamma_*(\vartheta, t) \to \gamma_0(\vartheta, t), \quad A_\varepsilon(\vartheta, t) \to A_0(\vartheta, t), \quad m(\vartheta_0, t) \to Y_t,
\]
implies that the coefficient \( B_\varepsilon(\vartheta, t)^2 \) in the equation (18) converges to \( b(\vartheta, t)^2 \) in the equation (11). Hence under regularity conditions the solution \( u(\cdot, \cdot, \cdot) \) of (18) converges to the solution \( u(\cdot, \cdot, \cdot) \) of (11).

Note that we have no BSDE for the approximation process
\[
d\hat{Z}_t = -F(t, \hat{Z}_t, \hat{m}_t, \hat{s}(t))dt + \hat{s}(t)d\bar{W}_t, \quad \hat{Z}_T = \Phi(\hat{m}_T).
\]

The stochastic differential for the random process \( \hat{Z}_t = u(t, \hat{m}_t, \vartheta_\tau^*, \varepsilon, \varepsilon) \) could be written analytically (it is different of given above), but it is quite cumbersome and it is not used in the proofs. Our goal is to propose an approximation of the solution \( Z_t \) of the equation (10) and to study the error of approximation, say, \( E_{\vartheta_0}(Z_t - \hat{Z}_t)^2 \). Moreover, the optimality of such approximation is discussed.

### 2.2 Preliminary estimators

Our objective is to construct a good estimator-process \( (\hat{\vartheta}_{t, \varepsilon}, \tau \leq t \leq T) \) and for this construction we need a preliminary estimator \( \hat{\theta}_{\tau, \varepsilon} \) constructed by the first observations \( X^\tau = (X_t, 0 \leq t \leq \tau) \) on the (small) time interval \([0, \tau]\) where \( \tau \in (0, T] \). In this section we propose two such estimators. One is the MLE \( \hat{\vartheta}_{\tau, \varepsilon} \) and the other is the estimator of substitution which uses the estimator of the quadratic variation of the derivative of the limit of the observed process.

The likelihood ratio function is (see [13])
\[
L(\vartheta, X^\tau) = \exp \left\{ \int_0^\tau f(t) m(\vartheta, t) dt - \int_0^\tau \frac{f(t)^2 m(\vartheta, t)^2}{2\varepsilon^2 \sigma(t)^2} dt \right\}, \vartheta \in \Theta,
\]
and the corresponding maximum likelihood estimator (MLE) \( \hat{\vartheta}_{\tau, \varepsilon} \) is defined by
\[
L(\hat{\vartheta}_{\tau, \varepsilon}, X^\tau) = \sup_{\vartheta \in \Theta} L(\vartheta, X^\tau).
\]

In the sequel, let us introduce the notation
\[
\Gamma(\vartheta) = \int_0^\tau \frac{f(t) \hat{b}(\vartheta, t)^2}{2b(\vartheta, t) \sigma(t)} dt, \quad G_\tau(\vartheta, \vartheta_0) = \int_0^\tau \frac{f(t) [b(\vartheta, t) - b(\vartheta_0, t)]^2}{2b(\vartheta, t) \sigma(t)} dt.
\]
Conditions B.

B₁. The function \( b(\vartheta, t) \) has three continuous derivatives w.r.t. \( \vartheta \in \Theta \).

B₂. Identifiability condition: For any \( \tau \in (0, T] \) and \( \nu > 0 \)

\[
\inf_{\vartheta_0 \in \Theta} \inf_{|\vartheta - \vartheta_0| > \nu} G_{\tau}(\vartheta, \vartheta_0) > 0.
\]

B₃. Non degeneracy of Fisher information: For any \( \tau \in (0, T] \)

\[
\inf_{\vartheta \in \Theta} \Gamma(\vartheta) > 0.
\]

Note that if \( f(0) > 0 \) and \( \inf_{\vartheta \in \Theta} |\dot{b}(\vartheta, 0) b(\vartheta, 0)^{-1}| > 0 \), then the condition B₃ is fulfilled.

Proposition 1. The MLE \( \hat{\vartheta}_{\tau, \varepsilon} \) under regularity conditions A, B is consistent, asymptotically normal

\[
\sqrt{\frac{\Gamma(\vartheta_0)}{\varepsilon}} \left( \hat{\vartheta}_{\tau, \varepsilon} - \vartheta_0 \right) \Rightarrow \zeta \sim \mathcal{N}(0, 1),
\]

asymptotically efficient and the moments converge: for any \( p > 0 \)

\[
\left| \frac{\Gamma(\vartheta_0)}{\varepsilon} \right|^{p/2} E_{\vartheta_0} \left| \hat{\vartheta}_{\tau, \varepsilon} - \vartheta_0 \right|^p \longrightarrow E |\zeta|^p.
\]

For the proof see [8], Theorem 1.

Note that we can not use the MLE-process \( \hat{\vartheta}_{t, \varepsilon}, 0 < t \leq T \) as a good estimator-process since in order to solve equation (19) for all \( t \in (0, T] \) we need the solutions \( m(\vartheta, s), 0 \leq s \leq t \) of equations (13) for all \( \vartheta \in \Theta \) and all \( t \in (0, T] \). From a computational point of view, a good estimator-process would be one that could be easily computed. Therefore, \( \hat{\vartheta}_{\tau, \varepsilon} \) could be considered as a preliminary estimator. Its calculation is simpler because we need to solve (19) just once.

Note that even the calculation of the preliminary MLE \( \hat{\vartheta}_{\tau, \varepsilon} \) by (19) requires the solution of the filtration equations for many values of \( \vartheta \). Below we propose another estimator which requires much more simple calculations.

Let us consider another estimation procedure of the preliminary estimator based on the following property of the model. Remark that the observed process \( X^\tau \) converges with probability 1 to \( x^\tau \):

\[
\sup_{0 \leq t \leq \tau} |X_t - x_t| \longrightarrow 0,
\]
where \( x^\tau = (x_t, 0 \leq t \leq \tau) \) satisfies the limit \((\varepsilon = 0)\) relation
\[
x_t = \int_0^t f(s) Y_s \, ds, \quad 0 \leq t \leq \tau.
\]
Here \( Y_t \) is solution of the forward equation (8). Let us put \( N_t = f(t) Y_t \). Then by Itô formula
\[
\dot{N}_t^2 - 2 \int_0^\tau N_t \, dN_t = \int_0^\tau f(t)^2 b(\vartheta_0, t)^2 \, dt.
\]
We further define
\[
\Psi(\vartheta) = \int_0^\tau f(t)^2 b(\vartheta, t)^2 \, dt, \quad \tilde{N}_{\tau, \varepsilon} = \frac{1}{\varphi_{\varepsilon}} \int_0^\tau K_{\varepsilon} \left( \frac{s - t}{\varphi_{\varepsilon}} \right) \, dX_s,
\]
\[
N_{t, \varepsilon} = \frac{1}{\varphi_{\varepsilon}} \int_0^t K \left( \frac{s - t}{\varphi_{\varepsilon}} \right) \, dX_s, \quad 0 \leq t \leq \tau,
\]
\[
\dot{\Psi}_\varepsilon = \tilde{N}_{\tau, \varepsilon}^2 - 2 \int_0^\tau N_{t, \varepsilon} \, dN_{t, \varepsilon}, \quad \dot{\Psi}(\vartheta) = 2 \int_0^\tau f(t)^2 b(\vartheta, t) \dot{b}(\vartheta, t) \, dt.
\]
Here the one-sided kernels \( K_{\varepsilon}(\cdot) \) and \( K(\cdot) \) satisfy the usual conditions
\[
K_{\varepsilon}(u) \geq 0, \quad \int_{-1}^0 K_{\varepsilon}(u) \, du = 1, \quad K_{\varepsilon}(u) = 0, \text{ for } u \not\in [-1, 0],
\]
\[
K(u) \geq 0, \quad \int_0^1 K(u) \, du = 1, \quad K(u) = 0, \text{ for } u \not\in [0, 1].
\]
Further, suppose that the function \( \Psi(\vartheta), \vartheta \in \Theta \) is monotone increasing and denote
\[
\psi_m = \inf_{\vartheta \in \Theta} \Psi(\vartheta), \quad \psi_M = \sup_{\vartheta \in \Theta} \Psi(\vartheta), \quad \psi_m = \Psi(\alpha), \quad \psi_M = \Psi(\beta),
\]
\[
G(\psi) = \Psi^{-1}(\psi), \quad \psi_m < \psi < \psi_M, \quad \alpha < G(\psi) < \beta, \quad \eta_\varepsilon = G(\tilde{\Psi}_\varepsilon),
\]
\[
\mathbb{B}_m = \left\{ \omega : \tilde{\Psi}_\varepsilon \leq \psi_m \right\}, \quad \mathbb{B}_M = \left\{ \omega : \tilde{\Psi}_\varepsilon \geq \psi_M \right\},
\]
\[
\mathbb{B} = \left\{ \omega : \psi_m < \tilde{\Psi}_\varepsilon < \psi_M \right\}, \quad g(\nu) = \inf_{\vartheta_0 \in \Theta} \inf_{|\vartheta - \vartheta_0| > \nu} |\Psi(\vartheta) - \Psi(\vartheta_0)|.
\]
The substitution estimator (SE) is introduced as follows
\[
\tilde{\vartheta}_{\tau, \varepsilon} = \alpha \mathbb{1}_{\mathbb{B}_m} + \eta_\varepsilon \mathbb{1}_{\mathbb{B}} + \beta \mathbb{1}_{\mathbb{B}_M}.
\]
It has the following properties.
Proposition 2. Suppose that the conditions $A_1, B_1$ are fulfilled, for any (small) $\nu > 0$ we have $g(\nu) > 0$ and $\inf_{\vartheta \in \Theta} \Phi' (\vartheta) > 0$. Then the SE $\hat{\vartheta}_{\tau, \varepsilon}$ is uniformly consistent and for any $p > 0$ there exists a constant $C = C(p) > 0$ such that

$$\sup_{\hat{\vartheta}_0 \in \Theta} \varepsilon^{-p/2} E_{\hat{\vartheta}_0} \left| \hat{\vartheta}_{\tau, \varepsilon} - \vartheta_0 \right|^p \leq C. \tag{23}$$

For the proof and more general results see [10].

2.3 One-step MLE-process

Below we consider the MLE $\hat{\vartheta}_{\tau, \varepsilon}$ as a preliminary estimator. Following the same steps it could be shown that the SE $\tilde{\vartheta}_{\tau, \varepsilon}$ could also be used as the preliminary. Recall that this estimator is easier to calculate and the property (23) is sufficient for the proof of the Proposition 3.

Let us introduce the statistic

$$\vartheta^*_{\tau, \varepsilon} = \hat{\vartheta}_{\tau, \varepsilon} + \frac{1}{I'_T (\vartheta_{\tau, \varepsilon})} \int_T^t \frac{f(s) m_{\vartheta} (\hat{\vartheta}_{\tau, \varepsilon}, s)}{\varepsilon \sigma (s)^2} \left[ dX_s - f(s) m_{\vartheta} (\hat{\vartheta}_{\tau, \varepsilon}, s) ds \right],$$

where $I'_T (\vartheta)$ is the Fisher information

$$I'_T (\vartheta) = \int_T^t \frac{f(s) \dot{b} (\vartheta, s)^2}{2 b (\vartheta, s) \sigma (s)} ds.$$

We have now to precise how to calculate the values $\dot{m}_{\vartheta} (\hat{\vartheta}_{\tau, \varepsilon}, s)$ and $m_{\vartheta} (\hat{\vartheta}_{\tau, \varepsilon}, s)$ since according to (13) we have

$$m_{\vartheta} (\vartheta, t) = \int_0^t e^{-\int_0^t q_{\vartheta} (\vartheta, v) dv} \frac{\gamma (\vartheta, s) f (s)}{\varepsilon \sigma (s)^2} dX_s = \int_0^t e^{-\int_0^t q_{\vartheta} (\vartheta, v) dv} \frac{\gamma (\vartheta, s) f (s)}{\varepsilon \sigma (s)^2} dX_s$$

$$= h (\vartheta, t) \int_0^t H (\vartheta, s) dX_s.$$

As usual in such situations we replace the stochastic integral by an ordinary one as follows. We have

$$\int_0^t H (\vartheta, s) dX_s = H (\vartheta, t) X_t - \int_0^t X_s H'_{\vartheta} (\vartheta, s) ds.$$
Let us denote by \( N(\vartheta, t, X_t) \) the right hand side of this equality. Then we can set
\[
m(\hat{\vartheta}_{t, \varepsilon}, s) = h(\hat{\vartheta}_{t, \varepsilon}, s)N(\hat{\vartheta}_{t, \varepsilon}, s, X^*).
\]
The similar relation could also be written for \( \hat{m}(\hat{\vartheta}_{t, \varepsilon}, s) \).

Further, introduce the random processes
\[
\eta_{t, \varepsilon} = \frac{\hat{\vartheta}_{t, \varepsilon} - \vartheta_0}{\sqrt{\varepsilon}}, \quad \tau \leq t \leq T,
\]
\[
\eta_t = \frac{1}{\Gamma_t(\vartheta_0)} \int_\tau^t \hat{b}(\vartheta_0, s) \sqrt{f(s)} \, dw(s), \quad \tau \leq t \leq T,
\]
where \( w(s), 0 \leq s \leq T \) is some standard Wiener process.

In the sequel, we need an additional condition.

\( B_4. \) Non degeneracy of Fisher information: For any \( t_0 \in (\tau, T] \)
\[
\inf_{\vartheta \in \Theta} \Gamma_{t_0}(\vartheta) > 0.
\]

**Proposition 3.** Let the conditions \( A, B \) be fulfilled. Then the One-step MLE-process \( \hat{\vartheta}_{t, \varepsilon}, \tau < t \leq T \) is uniformly consistent: for any \( \nu > 0 \) and any \( t_0 \in (\tau, T) \)
\[
P_{\vartheta_0} \left( \sup_{t_0 \leq t \leq T} \left| \hat{\vartheta}_{t, \varepsilon} - \vartheta_0 \right| > \nu \right) \to 0,
\]
the stochastic process \( \eta_{t, \varepsilon}, t_0 \leq t \leq T \) converges in distribution in the measurable space \( (C[t_0, T], \mathcal{B}) \) to the random process \( \eta_t, t_0 \leq t \leq T \)
\[
\eta_{t, \varepsilon} \Rightarrow \eta_t, \quad \eta_t \sim N(0, \Gamma_t(\vartheta_0) - 1).
\]

**Proof.** Consider the normalized difference
\[
\frac{\hat{\vartheta}_{t, \varepsilon} - \vartheta_0}{\sqrt{\varepsilon}} = \hat{\vartheta}_{t, \varepsilon} - \vartheta_0 + \frac{1}{\Gamma_t(\hat{\vartheta}_{t, \varepsilon})} \int_\tau^t \frac{f(s)}{\sqrt{\varepsilon}} \hat{m}(\hat{\vartheta}_{t, \varepsilon}, s) \, dw_s + \frac{1}{\Gamma_t(\hat{\vartheta}_{t, \varepsilon})} \int_\tau^t \frac{f(s)^2}{\varepsilon \sigma(s)^2} \hat{m}(\hat{\vartheta}_{t, \varepsilon}, s) \, ds.
\]
We have the relations (Lemma 6 in [8])
\[
m(\hat{\vartheta}_{t, \varepsilon}, s) - m(\vartheta_0, s) = (\hat{\vartheta}_{t, \varepsilon} - \vartheta_0) \hat{m}(\hat{\vartheta}_{t, \varepsilon}, s),
\]
\[
m(\hat{\vartheta}_{t, \varepsilon}, s) - m(\vartheta_0, s) = (\hat{\vartheta}_{t, \varepsilon} - \vartheta_0) \hat{m}(\vartheta_0, s) + \frac{1}{2} (\hat{\vartheta}_{t, \varepsilon} - \vartheta_0)^2 \hat{m}(\hat{\vartheta}_{t, \varepsilon}, s),
\]
\[
\hat{m}(\vartheta_0, s) = \sqrt{\frac{\varepsilon \sigma(s)}{2b(\vartheta_0, s) f(s)}} \hat{b}(\vartheta_0, s) \xi_{s, \varepsilon} + \varepsilon R_{t, \varepsilon},
\]
\[
\Gamma_t(\hat{\vartheta}_{t, \varepsilon})^{-1} = \Gamma_t(\vartheta_0)^{-1} + (\hat{\vartheta}_{t, \varepsilon} - \vartheta_0) Q_{t, \varepsilon}.
\]
Here $\xi_{s,t}, s \in [\tau, T]$ are Gaussian, asymptotically independent random variables, i.e., $\xi_{s,t} \implies \xi_s \sim \mathcal{N}(0, 1)$, where $\xi_s, s \in [\tau, T]$ are mutually independent.

For any $\nu > 0$ we can write
\[
\mathbb{P}_{\vartheta_0} \left( \sup_{0 \leq t \leq T} |\nu^*_{t,\varepsilon} - \vartheta_0| > \nu \right) \leq \mathbb{P}_{\vartheta_0} \left( \left| \hat{\nu}_{t,\varepsilon} - \vartheta_0 \right| > \frac{\nu}{3} \right) + \mathbb{P}_{\vartheta_0} \left( \frac{1}{I^0_{t}(\hat{\nu}_{t,\varepsilon})} \int_\tau^t f(s) \frac{\hat{m}(\hat{\nu}_{t,\varepsilon}, s)}{\sigma(s)} d\tilde{W}_s \geq \frac{\nu}{3} \right) + \mathbb{P}_{\vartheta_0} \left( \frac{1}{I^0_{t}(\hat{\nu}_{t,\varepsilon})} \int_\tau^t f(s) \frac{\hat{m}(\hat{\nu}_{t,\varepsilon}, s)^2}{\sqrt{\varepsilon} \sigma(s)^2} d\tilde{W}_s \geq \frac{\nu}{3} \right).
\]

Now the convergence (24) follows from the consistency of $\hat{\nu}_{t,\varepsilon}$ and the following estimate of the moments of $\hat{m}(\cdot, \cdot)$: for any $p > 0$
\[
\sup_{\vartheta_0 \in \Theta} \sup_{\tau \leq t \leq T} \mathbb{E}_{\vartheta_0} |\hat{m}(\vartheta_0, t)|^p \leq C\varepsilon^{p/2}. \quad (26)
\]

The proof of this estimate follows from the proof of Lemma 6 in [8].

Moreover, we have the convergence results
\[
\int_\tau^t \frac{f(s)^2 \hat{m}(\hat{\nu}_{t,\varepsilon}, s)^2}{\varepsilon \sigma(s)^2} ds = \int_\tau^t \frac{f(s) b(\vartheta_0, s)^2 \xi_{s,t}^2}{2b(\vartheta_0, s) \sigma(s)} ds (1 + o(1)) \to I^c_t(\vartheta_0),
\]
\[
\int_\tau^t \frac{f(s) \hat{m}(\hat{\nu}_{t,\varepsilon}, s)}{\sqrt{\varepsilon} \sigma(s)} d\tilde{W}_s = \int_\tau^t \sqrt{\frac{f(s) b(\vartheta_0, s)}{2b(\vartheta_0, s)} \xi_{s,t} \sigma(s)} d\tilde{W}_s (1 + o(1)) \to \mathcal{N}(0, I^c_t(\vartheta_0)).
\]

The random processes $R_{t,\varepsilon}, Q_{t,\varepsilon}$ have bounded polynomial moments.

Hence we can write the representation
\[
\frac{\nu^*_{t,\varepsilon} - \vartheta_0}{\sqrt{\varepsilon}} = \frac{\hat{\nu}_{t,\varepsilon} - \vartheta_0}{\sqrt{\varepsilon}} + \frac{1}{I^c_t(\vartheta_0)} \int_\tau^t \frac{b(\vartheta_0, s) \sqrt{f(s)} \xi_{s,t} \sigma(s)}{2b(\vartheta_0, s) \sigma(s)} d\tilde{W}_s + o(1)
\]
\[
- \frac{\hat{\nu}_{t,\varepsilon} - \vartheta_0}{\sqrt{\varepsilon}} \frac{1}{I^c_t(\vartheta_0)} \int_\tau^t \frac{b(\vartheta_0, s)^2 f(s) \xi_{s,t}^2}{2b(\vartheta_0, s) \sigma(s)} ds + \frac{(\hat{\nu}_{t,\varepsilon} - \vartheta_0)^2}{\sqrt{\varepsilon}} P_{t,\varepsilon}
\]
\[
= \frac{1}{I^c_t(\vartheta_0)} \int_\tau^t \frac{b(\vartheta_0, s) \sqrt{f(s)} \xi_{s,t} \sigma(s)}{2b(\vartheta_0, s) \sigma(s)} d\tilde{W}_s + o(1) + \left( \frac{\hat{\nu}_{t,\varepsilon} - \vartheta_0}{\sqrt{\varepsilon}} \right)^2 P_{t,\varepsilon}. \quad (27)
\]

The random processes $P_{t,\varepsilon}$ has bounded polynomial moments. From this representation it follows that the One-step MLE-process is asymptotically
normal: for all $t \in (\tau, T]$

$$\frac{\hat{g}_{t, \varepsilon}^* - \hat{g}_0}{\sqrt{\varepsilon}} \Rightarrow \eta_t \sim \mathcal{N}(0, \Gamma^t_{\tau}(\hat{g}_0)^{-1}).$$

The representation (27) allows us to obtain the convergence of the finite dimensional distributions as well

$$(\eta_{t_1, \varepsilon}, \ldots, \eta_{t_k, \varepsilon}) \Rightarrow (\eta_{t_1}, \ldots, \eta_{t_k}), \quad (28)$$

for any $k \geq 2$ and any $t_0 \leq t_1 < \ldots < t_k \leq T$.

Let us verify the condition

$$\mathbb{E}_{\theta_0} |\eta_{t_1, \varepsilon} - \eta_{t_2, \varepsilon}|^4 \leq C |t_1 - t_2|^2 \quad (29)$$

which along with the convergence (28) provide the weak convergence (25) of the random process $\eta_{t, \varepsilon}, t_0 \leq t \leq T$. We introduce

$$J_1(t) = \int_{\tau}^{t} \frac{f(s) \hat{m}(\hat{g}_{\tau, \varepsilon}, s)}{\sqrt{\varepsilon} \sigma(s)} d\tilde{W}_s, \quad K(t) = \Gamma^t_{\tau}(\hat{g}_{\tau, \varepsilon})^{-1},$$

$$J_2(t) = \int_{\tau}^{t} f(s)^2 \hat{m}(\hat{g}_{\tau, \varepsilon}, s) [m(\hat{g}_{0, \varepsilon}, s) - m(\hat{g}_{\tau, \varepsilon}, s)] \frac{d\sigma(s)}{\varepsilon^2}.$$

Then we can write

$$\mathbb{E}_{\theta_0} |\eta_{t_1, \varepsilon} - \eta_{t_2, \varepsilon}|^4 \leq C \mathbb{E}_{\theta_0} |K(t_1) J_1(t_1) - K(t_2) J_1(t_2)|^4$$

$$+ C \mathbb{E}_{\theta_0} |K(t_1) J_2(t_1) - K(t_2) J_2(t_2)|^4$$

$$\leq C \mathbb{E}_{\theta_0} |(K(t_1) - K(t_2)) J_1(t_1)|^4$$

$$+ C \mathbb{E}_{\theta_0} |(J_1(t_1) - J_1(t_2)) K(t_2)|^4$$

$$+ C \mathbb{E}_{\theta_0} |(K(t_1) - K(t_2)) J_2(t_1)|^4$$

$$+ C \mathbb{E}_{\theta_0} |(J_2(t_1) - J_2(t_2)) K(t_2)|^4.$$

Using once again the estimates (26) we obtain

$$\mathbb{E}_{\theta_0} |K(t_1) - K(t_2)|^8 \leq C |t_2 - t_1|^8, \quad \mathbb{E}_{\theta_0} |J_1(t_2)|^8 \leq C,$$

$$\mathbb{E}_{\theta_0} |(J_1(t_1) - J_1(t_2))|^8 \leq C |t_2 - t_1|^4, \quad \mathbb{E}_{\theta_0} |K(t_2)|^8 \leq C,$$

$$\mathbb{E}_{\theta_0} |(J_2(t_1) - J_2(t_2))|^8 \leq C |t_2 - t_1|^8.$$

These estimates and the Cauchy-Schwartz inequality allow us to verify (29) and therefore to obtain (25).
2.4 Approximation

Consider the family of solutions \( u(t, y, \vartheta, \varepsilon), \vartheta \in \Theta, \varepsilon \in (0, 1] \) of the equations

\[
\begin{align*}
    u_t' - a(t) y u_y' + \frac{1}{2} B_\varepsilon(\vartheta, t)^2 u_y'' &= -F(t, y, u, B_\varepsilon(\vartheta, t) u_y'), \\
    u(T, y, \vartheta, \varepsilon) &= \Phi(y).
\end{align*}
\] (30)

and the equation

\[
\begin{align*}
    U_t' - a(t) y U_y' + \frac{1}{2} b(\vartheta, t)^2 U_y'' &= -F(t, y, U, b(\vartheta, t) U_y'), \\
    U(T, y, \vartheta) &= \Phi(y).
\end{align*}
\] (31)

We suppose that by continuity \( U(t, y, \vartheta) = u(t, y, \vartheta, 0) \). Recall that \( B_\varepsilon(\vartheta, t) \to b(\vartheta, t) \) as \( \varepsilon \to 0 \).

Conditions C

\( C_1 \). The functions \( F(t, y, u, s) \) and \( \Phi(y) \) satisfy the conditions (5), (6).

\( C_2 \). The function \( u(t, y, \vartheta, \varepsilon), t \in (0, T], y \in \mathcal{R}, \vartheta \in \Theta, \varepsilon \in [0, 1] \) has continuous derivatives \( u'_y(\cdot), \dot{u}(\cdot), u''_y(\cdot) \).

It is worth noting that \( Z_t = u(t, m(\vartheta_0, t), \vartheta_0, \varepsilon) \) is a solution of BSDE

\[
dZ_t = -F(t, m(\vartheta_0, t), Z_t, s(t)) \, dt + s(t) \, d\hat{W}_t, \quad Z_T = \Phi(m(\vartheta_0, T)),
\]

where \( s(t) = B_\varepsilon(\vartheta_0, t) u'_y(t, m(\vartheta_0, t), \vartheta_0, \varepsilon) \). As \( u(\cdot) \to U(\cdot) \) the corresponding limit BSDE is

\[
d\hat{Z}_t = -F(t, Y_t, Z_t, s_t) \, dt + s_t \, dV_t, \quad \hat{Z}_T = \Phi(Y_T),
\]

where \( Z_t = U(t, Y_t, \vartheta_0), s_t = b(\vartheta_0, t) U'_y(t, Y_t, \vartheta_0) \).

We do not set \( m(\vartheta^*_t, t) \) and \( \hat{Z}_t = u(t, m(\vartheta^*_t, t), \vartheta^*_t, \varepsilon) \) since in this case we need to solve the equations (13) for many values of \( \vartheta \) and the relevant computational cost is very high. Introduce the recurrent equation

\[
d\hat{m}_t = -q_\varepsilon(\vartheta^*_t, t) \hat{m}_t \, dt + \varepsilon^{-1} A_\varepsilon(\vartheta^*_t, t) \, dX_t, \quad \tau < t \leq T
\] (32)

where the initial value is \( \hat{m}_\tau = m(\vartheta^*_\tau, \tau) \).

Let us set

\[
\hat{Z}_t = u(t, \hat{m}_t, \vartheta^*_t, \varepsilon), \quad \hat{s}_t = B_\varepsilon(\vartheta^*_t, t) u'_y(t, \hat{m}_t, \vartheta^*_t, \varepsilon)
\]

The main result of this work is the following theorem.
Theorem 1. Let the conditions $A, B, C$ be fulfilled. Then

$$
\frac{\dot{Z}_t - Z_t}{\sqrt{\varepsilon}} \implies U_y(t, Y_t, \vartheta_0) \sqrt{\frac{b(\vartheta_0, t) \sigma(t)}{2f(t)}} \left[ \dot{\zeta}_t - \dot{\xi}_t \right] + \frac{\tilde{U}(t, Y_t, \vartheta_0)}{U_y(t, \vartheta_0)} \int^t_\tau \sqrt{\frac{b(\vartheta_0, s)^2 f(s)}{2b(\vartheta_0, s)\sigma(s)}} \, dw(s). \tag{33}
$$

Here $\dot{\zeta}_t \sim N(0, 1)$, $\dot{\xi}_t \sim N(0, 1)$ are mutually independent random variables and $w(s), \tau \leq s \leq T$ is a Wiener process.

Proof. First we notice

$$
\dot{Z}_t - Z_t = u(t, \tilde{m}_t, \vartheta^*_{t,\varepsilon}, \varepsilon) - U(t, Y_t, \vartheta_0)
= u(t, \tilde{m}_t, \vartheta^*_{t,\varepsilon}, \varepsilon) - u(t, m(\vartheta_0, t), \vartheta^*_{t,\varepsilon}, \varepsilon) + u(t, m(\vartheta_0, t), \vartheta^*_{t,\varepsilon}, \varepsilon) - u(t, m(\vartheta_0, t), \vartheta_0, \varepsilon) + u(t, m(\vartheta_0, t), \vartheta_0, \varepsilon) - u(t, Y_t, \vartheta_0, \varepsilon) + u(t, Y_t, \vartheta_0, \varepsilon) - u(t, Y_t, \vartheta_0, 0)
= u_y(t, \tilde{m}_t, \vartheta^*_{t,\varepsilon}, \varepsilon) (\tilde{m}_t - m(\vartheta_0, t)) + u_y(t, \tilde{m}_t, \vartheta^*_{t,\varepsilon}, \varepsilon) (m(\vartheta_0, t) - Y_t) + \dot{u}(t, m(\vartheta_0, t), \vartheta, \varepsilon) (\vartheta^*_{t,\varepsilon} - \vartheta_0) + u'(t, Y_t, \vartheta, \varepsilon). \tag{34}
$$

Here $\tilde{m}_t, \tilde{m}_t, \tilde{\vartheta}$ are some intermediate points in the corresponding expansions. We further study the quantities $\delta_t = \tilde{m}_t - m(\vartheta_0, t)$ and $m(\vartheta_0, t) - Y_t$. Recall the equations for $m(\vartheta_0, t)$ and $\tilde{m}_t$

$$
dm(t, \vartheta_0, t) = -a(t) m(\vartheta_0, t) dt + \frac{\gamma_*(\vartheta_0, t) f(t)}{\sigma(t)} d\tilde{W}_t, \quad m(\vartheta_0, 0) = 0,
$$

$$
d\tilde{m}_t = -a(t) \tilde{m}_t dt - \frac{\gamma_*(\vartheta^*_{t,\varepsilon}, t) f(t)^2}{\varepsilon \sigma(t)^2} \delta_t dt + \frac{\gamma_*(\vartheta^*_{t,\varepsilon}, t) f(t)}{\sigma(t)} d\tilde{W}_t,
$$

where $\tilde{m}_t = m(\vartheta^*_{t,\varepsilon}, \tau)$. Therefore for $\delta_t$ we obtain the equation

$$
d\delta_t = -q_\varepsilon(\vartheta^*_{t,\varepsilon}, t) \delta_t dt + \frac{[\gamma_*(\vartheta^*_{t,\varepsilon}, t) - \gamma_*(\vartheta_0, t)] f(t)^2}{\sigma(t)^2} d\tilde{W}_t, \quad \tau < t \leq T,
$$

where $\delta_t = m(\vartheta^*_{t,\varepsilon}, \tau) - m(\vartheta_0, \tau)$ and

$$
q_\varepsilon(\vartheta^*_{t,\varepsilon}, t) = a(t) + \frac{\gamma_*(\vartheta^*_{t,\varepsilon}, t) f(t)^2}{\varepsilon \sigma(t)^2}.
$$
The solution of this equation on the time interval $[\tau, T]$ is
\[
\delta_t = \delta_{\tau} e^{-\int_{\tau}^t q_{\varepsilon} (\vartheta_{*,\varepsilon,v}) dv} + e^{-\int_{\tau}^t q_{\varepsilon} (\vartheta_{*,\varepsilon,v}) dv} \int_{\tau}^t e^{\int_{\tau}^s q_{\varepsilon} (\vartheta_{*,\varepsilon,v}) dv} \frac{\gamma_{*} (\vartheta_{*,\varepsilon,s} - \gamma_{*} (\vartheta_{0, s})) f (s)^2}{\sigma (s)^2} dW_s.
\]
Note that at the vicinity of the point $t$ we have the expansion
\[
q_{\varepsilon} (\vartheta_{*,\varepsilon,s}) = \frac{1}{\varepsilon} \frac{\gamma_{*} (\vartheta_{0,t}) f (t)^2}{\sigma (t)^2} \left( 1 + O (\varepsilon) + O (s-t) + O (\sqrt{\varepsilon}) \right),
\]
where the relation $\vartheta_{*,\varepsilon} - \vartheta_{0} = O (\sqrt{\varepsilon})$ is used. Let us denote $K (\vartheta, t) = \gamma_{*} (\vartheta_{0,t}) f (t)^2 \sigma (t)^{-2}$ and notice that
\[
q_{\varepsilon} (\vartheta_{*,\varepsilon,v}) = q_{\varepsilon} (\vartheta_{0,v}) (1 + O (\sqrt{\varepsilon})).
\]
The same arguments as in the proof of Lemma 2 in [8] for the stochastic integral lead us to
\[
\int_{\tau}^t e^{\int_{\tau}^s q_{\varepsilon} (\vartheta_{*,\varepsilon,v}) dv} \frac{\gamma_{*} (\vartheta_{*,\varepsilon,s} - \gamma_{*} (\vartheta_{0, s})) f (s)^2}{\sigma (s)^2} dW_s = \int_{\tau}^t e^{\int_{\tau}^s \frac{\dot{\gamma}_{*} (\vartheta_{0,v}) (\vartheta_{*,\varepsilon,s} - \vartheta_{0,v}) f (s)^2}{\sigma (s)^2} dW_s} (1 + o (1))
\]
\[
= \int_{\tau}^t e^{\int_{\tau}^t \frac{\dot{\gamma}_{*} (\vartheta_{0,v}) \eta_{*,\varepsilon} f (t)^2}{\sigma (t)^2} (1 + o (1))}
\]
\[
= \frac{\dot{\gamma}_{*} (\vartheta_{0,t}) \eta_{*,\varepsilon} f (t)^2}{\sigma (t)^2} \int_{\tau}^t e^{\int_{\tau}^t \frac{\dot{\gamma}_{*} (\vartheta_{0,v}) \eta_{*,\varepsilon} f (t)^2}{\sigma (t)^2} dW_s} (1 + o (1))
\]
\[
= \frac{\dot{\gamma}_{*} (\vartheta_{0,t}) \eta_{*,\varepsilon} f (t)^2}{\sigma (t)^2} (1 + o (1))
\]
Here $\zeta_{t,\varepsilon} \sim N (0, 1), t \in (\tau, T]$ are independent random variables. We remind that by Lemma 1 we have
\[
\gamma_{*} (\vartheta_{0,t}) \rightarrow b (\vartheta_{0,t}) \frac{\sigma (t)}{f (t)}, \quad \dot{\gamma}_{*} (\vartheta_{0,t}) \rightarrow \frac{b (\vartheta_{0,t}) \sigma (t)}{f (t)}.
\]
The second limit here could be obtained similarly to the first one in Lemma 2 [8].
For the initial value we have
\[ \delta_{\tau} = m(\hat{\vartheta}, \tau) (\hat{\vartheta}_{\tau, \varepsilon} - \vartheta_0) e^{-\int_{0}^{t} u_{e}^{(\vartheta_{e}, \varepsilon)} dv} \]
\[ = m(\vartheta_0, \tau) (\hat{\vartheta}_{\tau, \varepsilon} - \vartheta_0) e^{-\int_{0}^{t} u_{e}(\vartheta_{0}, v) dv} (1 + o(1)) \]
\[ = m(\vartheta_0, \tau) \hat{\vartheta}_{\tau, \varepsilon} e^{-\frac{1}{2} \int_{0}^{t} K(\vartheta_{0}, v) dv} \sqrt{\varepsilon} (1 + o(1)) = O \left( e^{-\frac{1}{2}(t-\tau)} \right), \]
where \( c_{s} = \inf_{\tau < \nu \leq \tau} K(\vartheta_0, v). \)

Finally, we obtain the representation
\[ \hat{m}_t - m(\vartheta_0, t) = \sqrt{\frac{b(\vartheta_0, t)^2 f(t)}{2b(\vartheta_0, t) \sigma(t)}} \hat{\vartheta}_{\tau, \varepsilon} \sqrt{\varepsilon} (1 + o(1)). \tag{35} \]

For the difference \( m(\vartheta_0, t) - Y_t \) we have the representation (17)
\[ m(\vartheta_0, t) - Y_t = \int_{0}^{t} e^{-\frac{1}{2} K(\vartheta_0, \tau) (t-s)} \frac{\gamma_{*}(\vartheta_0, \tau) f(s)}{\sigma(s)} dW_s (1 + o(1)) \]
\[ - \int_{0}^{t} e^{-\frac{1}{2} K(\vartheta_0, \tau) (t-s)} b(\vartheta_0, s) dV_s (1 + o(1)) \]
\[ = \frac{\gamma_{*}(\vartheta_0, t) f(t)}{\sigma(t)} \int_{0}^{t} e^{-\frac{1}{2} K(\vartheta_0, \tau) (t-s)} dW_s (1 + o(1)) \]
\[ - b(\vartheta_0, t) \int_{0}^{t} e^{-\frac{1}{2} K(\vartheta_0, \tau) (t-s)} dV_s (1 + o(1)) \]
\[ = \frac{\gamma_{*}(\vartheta_0, t) f(t)}{\sigma(t)} \hat{\xi}_{t, \varepsilon} \sqrt{\varepsilon} (1 + o(1)) \]
\[ - b(\vartheta_0, t) \sqrt{2K(\vartheta_0, t)} \hat{\xi}_{t, \varepsilon} \sqrt{\varepsilon} (1 + o(1)) \]
\[ = \sqrt{\frac{b(\vartheta_0, t) \sigma(t)}{2f(t)}} \left[ \hat{\xi}_{t, \varepsilon} - \hat{\xi}_{t, \varepsilon} \right] \sqrt{\varepsilon} (1 + o(1)). \]

From the convergences \( m(\vartheta_0, t) \to Y_t, \ \vartheta_{t, \varepsilon} \to \vartheta_0 \) as \( \varepsilon \to 0 \) and the continuity of derivatives we obtain the representation
\[ \frac{\hat{Z}_t - Z_t}{\sqrt{\varepsilon}} = u'(t, Y_t, \vartheta_0, 0) \sqrt{\frac{b(\vartheta_0, t) \sigma(t)}{2f(t)}} \left[ \hat{\xi}_{t, \varepsilon} - \hat{\xi}_{t, \varepsilon} \right] (1 + o(1)) \]
\[ + \frac{\hat{u}(t, Y_t, \vartheta_0, 0)}{1_{t}^{(\vartheta_0)}} \int_{\tau}^{t} \sqrt{\frac{b(\vartheta_0, s)^2 f(s)}{2b(\vartheta_0, s) \sigma(s)}} \xi_{s, \varepsilon} d\tilde{W}_s (1 + o(1)). \tag{36} \]
Therefore

\[
\frac{\hat{Z}_t - Z_t}{\sqrt{\varepsilon}} \Rightarrow u'_y(t, Y_t, \vartheta_0, 0) \sqrt{b(\vartheta_0, t) \sigma(t) \varepsilon} \left[ \hat{\zeta}_t - \hat{\xi}_t \right]
\]

\[
+ \frac{\dot{u}(t, Y_t, \vartheta_0, 0) - \dot{u}(\vartheta_0, t)}{\Pi^*_T(\vartheta_0)} \int_{t}^{T} \sqrt{\frac{b(\vartheta_0, s)^2 f(s)}{2b(\vartheta_0, s) \sigma(s)}} \, dw(s).
\]

Here \( \hat{\zeta}_t \sim N(0, 1), \hat{\xi}_t \sim N(0, 1) \) are mutually independent random variables and \( w(s), 0 \leq s \leq T \) is the Wiener process.

Let us introduce the random process

\[
z(t, \vartheta_0, Y_t) = \dot{u}(t, Y_t, \vartheta_0, 0) - \Pi^*_T(\vartheta_0) \int_{t}^{T} \sqrt{\frac{b(\vartheta_0, s)^2 f(s)}{2b(\vartheta_0, s) \sigma(s)}} \, dw(s)
\]

and notice that the Gaussian process \( Y_t, 0 \leq t \leq T \) and Wiener process \( w(t), 0 \leq t \leq T \) are independent.

We will further introduce a new condition.

\( \mathcal{D} \). The derivatives \( u'_y(\cdot), \dot{u}(\cdot), u'_\varepsilon(\cdot) \) have polynomial majorants in \( y \).

**Corollary 1.** Let the conditions \( \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \) be fulfilled. Then for any continuous functions \( h(\cdot) \) we have the relation

\[
\varepsilon^{-1/2} \int_{\tau}^{T} h(t) \left( \frac{\hat{Z}_t - Z_t}{\sqrt{\varepsilon}} \right) \, dt \Rightarrow \int_{\tau}^{T} h(t) z(s, \vartheta_0, Y_s) \, ds.
\]

**Proof.** The proof follows from the limits

\[
\int_{\tau}^{T} r(Y_s, s) \hat{\zeta}_{s, \varepsilon} \, ds \rightarrow 0, \quad \int_{\tau}^{T} g(Y_s, s) \hat{\xi}_{s, \varepsilon} \, ds \rightarrow 0
\]

for any continuous functions \( r(\cdot), g(\cdot) \) with finite moments (see [8]).

Indeed for the process

\[
\hat{\zeta}_{t, \varepsilon} = \sqrt{\frac{2K(\vartheta_0, t)}{\varepsilon}} \int_{0}^{t} e^{-\frac{1}{2}K(\vartheta_0, t)(t-s)} \, dW_s
\]

we have

\[
E_{\vartheta_0} \hat{\zeta}_{t_1, \varepsilon} \hat{\zeta}_{t_2, \varepsilon} = \sqrt{\frac{K_1 K_2}{K_1 + K_2}} \left[ e^{-\frac{1}{2}K(\vartheta_0, t_1-t_2)} - e^{-\frac{1}{2}(K_1 t_1 + K_2 t_2)} \right] \rightarrow 0,
\]

where \( K_i = K(\vartheta_0, t_i), i = 1, 2 \) and set \( K = K_1 \mathbb{I}_{\{t_1 > t_2\}} + K_2 \mathbb{I}_{\{t_1 \leq t_2\}}. \)

\[\square\]
Corollary 2. Let the conditions $A, B, C, D$ be fulfilled. Then

$$
\varepsilon^{-1} \mathbb{E}_{\vartheta_0} \left( \hat{Z}_t - Z_t \right)^2 \rightarrow \frac{b(\vartheta_0, t) \sigma(t)}{f(t)} \mathbb{E}_{\vartheta_0} U_y'(t, Y_t, \vartheta_0)^2 + \frac{\mathbb{E}_{\vartheta_0} \hat{U} (t, Y_t, \vartheta_0)^2}{I_t'(\vartheta_0)}.
$$

Proof. The proof follows from the same arguments as in Corollary 1.

\[ \square \]

3 Discussion

It is shown that from four components of the error of approximation (33) only two of them have main contribution (Theorem 1). Moreover, if we consider the integrated error, then we have just one term (Corollary 1).

The contribution of the approximation of the conditional expectation $m(\vartheta_0, t)$ by the values $\hat{m}_t$ of solution of recurrent equation (adaptive filtration) is negligible. This means that for the observational model (8), (9) with unknown parameter of volatility function, the equation (32) proposing the approximation $\hat{m}_t$ of $m(\vartheta_0, t)$ has error of order $\varepsilon$. This result could be applied in filtration theory.

Several possible generalizations could be made quite easily. For example, if we suppose as in [8] that the function $f(t) = f(\vartheta, t)$, then any construction of approximation of $Z_t$ would be close to this one given here.

Another statement of the problem could be obtained if $b(\vartheta, t)$ is replaced by $\psi_\varepsilon b(\vartheta, t)$, where $\psi_\varepsilon = \varepsilon^\delta, \delta \in (0, 1/3]$. The problem of parameter estimation for such models was studied in [11] and the problem of approximation of the solution of BSDE for such models could be considered as well.

Acknowledgment. This research (sections 2.1-2.3) was financially supported by the Ministry of Education and Science of the Russian Federation (project no. FSWF-2020-0022) and the research (section 2.4) was carried out with support by RSF project no. 20-61-47043.

References

[1] Bismut, J.M. (1973) Conjugate convex functions in optimal stochastic control. J. Math. Anal. Appl., 44, 384-404.

[2] Gasparyan, S. and Kutoyants, Yu.A. (2015) On approximation of the BSDE with unknown volatility in forward equation. Armenian J. of Mathematics, 7, 1, 59-79.
[3] El Karoui N., Peng S. and Quenez M. (1997) Backward stochastic differential equations in finance. Math. Fin., 7, 1-71.

[4] Kutoyants, Yu.A. (1994) Identification of Dynamical Systems with Small Noise, Kluwer Academic Publisher, Dordrecht.

[5] Kutoyants, Yu.A. (2004) Statistical Inference for Ergodic Diffusion Processes. Springer, London.

[6] Kutoyants, Yu.A. (2014) Approximation of the solution of the backward stochastic differential equation. Small noise, large samples and high frequency cases. Proceedings of the Steklov Institute of Mathematics, 287, 133-154.

[7] Kutoyants Yu. A. (2016) On approximation of BSDE and Multi-step MLE-processes. Probability, Uncertainty, and Quantitative Risk, 1, 1, 1-22.

[8] Kutoyants, Yu. A. (2019) On parameter estimation of hidden Ornstein-Uhlenbeck process. Journal of Multivariate Analysis, 169, 1, 248-263.

[9] Kutoyants, Yu. A. (2020) Parameter estimation for continuous time hidden Markov processes. Automation and Remote Control, 81, 3, 446-469.

[10] Kutoyants, Yu. A. (2020) Quadratic variation estimation of hidden Markov process and related problems. Submitted.

[11] Kutoyants, Yu. A. (2020) Hidden Markov model where higher noise makes smaller errors. Submitted.

[12] Kutoyants, Yu.A. and Zhou, L. (2014) On approximation of the backward stochastic differential equation. J. Stat. Plann. Infer. 150, 111-123. (arXiv:1305.3728)

[13] Liptser, R. and Shiryaev, A.N. (2005) Statistics of Random Processes. v. 2, 2-nd ed. Springer, N.Y.

[14] Ma, J. and Yong, J. (1999) Forward-Backward Stochastic Differential Equations and their Applications. Lecture Notes in Mathematics. Springer, Berlin.

[15] Pardoux, E. and Peng, S. (1990) Adapted solution of a backward stochastic differential equation. System Control Letter, 14, 55-61.
[16] Pardoux, E. and Peng, S. (1992) Backward stochastic differential equation and quasilinear parabolic differential equations. In *Stochastic Partial Differential Equations and Their Applications*, Lecture Notes in Control and Information Sciences, 176, 200-217.

[17] Shen, Y. and Wei, J. (2016) Optimal investment-consumption-insurance with random parameters. *Scandinavian Actuaral Journal*, 2016, 1, 37-62.

[18] Sun, Z., Zhang, X. and Yuen, K.C. (2020) Mean-variance asset-liability management with affine diffusion factor process and a reinsurance option. (2020) *Scandinavian Actuarial Journal*, 2020, 3, 218-244.