Criticality and Chaos in Systems of Communities

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Abstract. We consider a simple model of communities interacting via bilinear terms. After analyzing the thermal equilibrium case, which can be described by an Hamiltonian, we introduce the dynamics that, for Ising-like variables, reduces to a Glauber-like dynamics. We analyze and compare four different versions of the dynamics: flow (differential equations), map (discrete-time dynamics), local-time update flow, and local-time update map. The presence of only bilinear interactions prevent the flow cases to develop any dynamical instability, the system converging always to the thermal equilibrium. The situation is different for the map when unfriendly couplings are involved, where period-two oscillations arise. In the case of the map with local-time updates, oscillations of any period and chaos can arise as a consequence of the reciprocal “tension” accumulated among the communities during their sleeping time interval. The resulting chaos can be of two kinds: true chaos characterized by positive Lyapunov exponent and bifurcation cascades, or marginal chaos characterized by zero Lyapunov exponent and critical continuous regions.

1. Introduction
The fact that the speed of physical interactions is finite, as for light or sound, has well known crucial consequences in all physics, and it is the ultimate reason for the existence of waves. For example, if we artificially assume the speed of light as infinite in the Maxwell equations, the magnetic field would be zero and we would not have electromagnetic waves.

The situation is quite different when the equations contain non linear terms, which can result in erratic or even chaotic trajectories. Notice that, in the absence of mechanical kinetic terms, which is our framework, the presence of non linear terms is exclusively due to the presence of non bilinear interactions among the components of the system. For example, in the Kuramoto model [6]-[10], sinusoidal interactions lead to chaos [2]-[5] if the coupling constant $K$ is larger than some threshold $K_c$.

It should be emphasized that the presence of non linear terms is not always sufficient for the emergence of chaos. However, it has been observed that the introduction of time-delays among the components of a system augments the chance that the system develops chaos [9]-[13]. The idea at the base of such a mechanism is that time-delays among the system components makes it difficult for the system to set in a steady state. In particular, for linear flows, a fixed delay among the system components corresponds to a finite speed of interactions, which in turn leads to oscillations or waves. For non linear flows a delay can have more dramatic consequences, with chaos scenarios more probable.

In [17], we have introduced a simple discrete-time two components dynamical model, characterized by local-time updates. Local time updates means that each component updates...
its status only at certain times \( t \). In the specific model analyzed in [17], a component updates (sleeps) its status only at odd times \( t = 1, 3, \ldots \), (even times \( t = 0, 2, \ldots \)), while the other component updates (sleeps) its status only at even times \( t = 0, 2, \ldots \) (odd times \( t = 1, 3, \ldots \)). Remarkably, as we have shown in [17], for certain coupling values, this simple system exhibits chaos even if it is characterized only by bilinear interactions. In fact, when the couplings are in the steady state regime, the system can be described via a Hamiltonian which is quadratic in the system components. We are not aware of a similar case in the literature. Our understanding here is that there are three factors generating chaos: \( i \) the presence of unfriendly couplings, \( ii \) the discrete-time nature of the dynamics, and \( iii \) the existence of at least two interacting communities which rearrange their configurations at alternate times, \( i.e. \) via local-time updates. Notice that \( (iii) \) is not just a delay; it also implies that each component has a sleeping period. It is during such a sleeping period that the component accumulates a tension with respect to the other component when the reciprocal coupling is unfriendly. We are in fact dealing with a sort of dynamical frustration.

In this paper we review some aspects of the model which were only partially treated in [17]. In particular, we compare the analogous flow model with the present map and show that the above conditions \( (i)- (iii) \) are in fact necessary, and calculate the maximal Lyapunov exponent.

The reader not interested in the derivation of the dynamic equations may skip the next Section.

### 2. The model in the flow, map and local-time versions

Here we consider a two components system. Generalization to an arbitrary number of components will be reported elsewhere. We consider two communities \([14]\) of agents \( \mathcal{N}^{(1)} \) and \( \mathcal{N}^{(2)} \), with cardinalities \( \mathcal{N}^{(1)} \) and \( \mathcal{N}^{(2)} \). The agent with index \( i \) can be in two possible states, \( \sigma_j = \pm 1 \), with \( i \in \mathcal{N}^{(1)} \cup \mathcal{N}^{(2)} \). According to the sign of the couplings, friendly or unfriendly, each agent \( i \) tends to follow or anti-follow its neighbors, by minimizing or maximizing the term \( \sigma_i \sum_j' \sigma_j \), where \( \sum_j' \) runs over the set of neighbors of \( i \). It is necessary to distinguish between intra- and inter-couplings, therefore we introduce the \( 2 \times 2 \) matrix \( J^{(l,m)} \). \( J^{(1,1)} \) and \( J^{(2,2)} \) are the intra-couplings, and \( J^{(1,2)} = J^{(2,1)} \) the inter-coupling. In the most general formulation we should introduce also the \( 2 \times 2 \) matrix \( \Gamma^{(l,m)} \) defined as the set of coupled spins \((i, j)\) within the same community (intra), or between the two communities (inter). Finally, we introduce a global factor \( \beta \) that rescales all the couplings in \( \beta J^{(l,m)} \).

Let us indicate the time by \( t \), which is supposed to range in the set of natural numbers \( \mathbb{N} \) for the map, while it can take any non negative real number for the flow. Inspired by the Glauber dynamics \([18]\), for flow and map we introduce the following transition rate probabilities for the spin with state \( \sigma_i \) to jump to the state \( \sigma'_i \)

\[
\nu^{(\text{FLOW}/\text{MAP})}(\sigma_i \rightarrow \sigma'_i; t) = \begin{cases} 
1 + \sigma'_i \tanh(\beta J^{(1,1)} \sum_{(i,j) \in \Gamma^{(1,1)}} \sigma_j + \beta J^{(1,2)} \sum_{(i,j) \in \Gamma^{(1,2)}} \sigma_j), & \text{for } i \in \mathcal{N}^{(1)}, \\
1 + \sigma'_i \tanh(\beta J^{(2,2)} \sum_{(i,j) \in \Gamma^{(2,2)}} \sigma_j + \beta J^{(2,1)} \sum_{(i,j) \in \Gamma^{(2,1)}} \sigma_j), & \text{for } i \in \mathcal{N}^{(2)},
\end{cases}
\]  

(1)

Let \((E, O)\) be any periodic partition of \( \mathbb{N} \), \( i.e., E \cup O = \mathbb{N}, E \cap O = \emptyset \), and \( E \) and \( O \) contain both infinite elements of \( \mathbb{N} \). For instance, \( E \) and \( O \) can be the set of even and odd numbers, respectively, and we will assume this choice in all the next examples. We now introduce the following local-time transition rate probabilities for the spin with state \( \sigma_i \) to jump to the state \( \sigma'_i \)
where we have introduced the global transition rates
\[ W(\sigma \rightarrow \sigma') = \prod_{i \in \mathcal{N}(1) \cup \mathcal{N}(2)} w(\sigma_i \rightarrow \sigma'_i), \]

are justified as they make each spin to follow the majority of its intra- and inter-neighbors and, thanks to the presence of the functions \(\tanh(\cdot)\), the rates are non negative and normalized at any time \(\sum_{\sigma} w(\sigma \rightarrow \sigma'; t) = 1/(\text{Time Unit})\). From a deeper viewpoint, Eqs. (1), (2), or (3), are based on the fact that, as we will see soon, in the case of positive couplings they lead to Boltzmann equilibrium governed by only quadratic interactions. More precisely, in the case of positive couplings these forms guarantee that at equilibrium the system satisfies the principle of detailed balance and the principle of maximal entropy for any quadratic interactions.

We formalize the discrete-time probabilistic dynamics induced by Eqs. (1), (2), or (3), as follows. Let \(N = \mathcal{N}(1) + \mathcal{N}(2)\). Let us introduce the spin vector \(\sigma = (\sigma_1, \ldots, \sigma_N)\), and the associated probability vector \(p(\sigma; t)\), i.e., the probability that the system is in the configuration \(\sigma\) at time \(t \in \mathbb{N}\). The master equation for flow and map (local-time or not) reads

\[
\frac{\partial p^{\text{FLOW}}(\sigma; t)}{\partial (\alpha t)} = - \sum_{\sigma'} p^{\text{FLOW}}(\sigma; t) W(\sigma \rightarrow \sigma') + \sum_{\sigma'} p^{\text{FLOW}}(\sigma'; t) W(\sigma' \rightarrow \sigma),
\]

\[
\frac{p^{\text{MAP}}(\sigma; t + 1) - p^{\text{MAP}}(\sigma; t)}{\alpha} = - \sum_{\sigma'} p^{\text{MAP}}(\sigma; t) W(\sigma \rightarrow \sigma') + \sum_{\sigma'} p^{\text{MAP}}(\sigma'; t) W(\sigma' \rightarrow \sigma),
\]

where we have introduced the global transition rates
\[ W(\sigma \rightarrow \sigma') = \prod_{i \in \mathcal{N}(1) \cup \mathcal{N}(2)} w(\sigma_i \rightarrow \sigma'_i), \]
where \( w(\sigma_i \to \sigma'_i) \) can be of the type Eqs. (1), (2), or (3), and where \( \alpha/2 > 0 \) may be interpreted as the rate at which, due to the interaction with an environment, a free spin \( J^{(l,m)} = 0 \) makes transitions from either state to the other. As we have proved in [15], it is necessary to impose the bound \( \alpha \leq 1 \) for \( p(\sigma; t) \) to be a probability at any time \( t \). By using Eqs. (2)-(6) it is easy to check that the stationary solutions \( p(\sigma) \) of Eq. (5) are given by the Boltzmann distribution \( p(\sigma) \propto \exp\left[-\beta H(\sigma)\right] \), where

\[
H = -J^{(1,1)} \sum_{(i,j) \in \Gamma^{(1,1)}} \sigma_i \sigma_j - J^{(2,2)} \sum_{(i,j) \in \Gamma^{(2,2)}} \sigma_i \sigma_j - J^{(1,2)} \sum_{(i,j) \in \Gamma^{(1,2)}} \sigma_i \sigma_j. \tag{7}
\]

Of course, the existence of a stationary solution \( p(\sigma) \) does not represent a sufficient condition for equilibrium. In fact, asymptotically the system can reach non-point-like attractors, and even aperiodic or chaotic regimes. However, as we have discussed in the Introduction, the fact that for certain values of couplings the system can be described via the Hamiltonian (7), means that we are dealing with only bilinear interactions. Here, the non-linearities that lead to chaos are caused by the local time updates.

From now on, we will omit the suffixes (FLOW), (MAP) and (LOCAL-TIME). It will be clear from the context what we are referring to. Eqs. (2)-(6) define the microscopic dynamics from which one can derive the macroscopic dynamics, i.e., the dynamics for the order parameters

\[
x^{(1)}(t) = \sum_{\sigma} p(\sigma; t) \frac{1}{N} \sum_{i \in \mathcal{N}^{(1)}} \sigma_i, \tag{8}
\]

\[
x^{(2)}(t) = \sum_{\sigma} p(\sigma; t) \frac{1}{N} \sum_{i \in \mathcal{N}^{(2)}} \sigma_i. \tag{9}
\]

3. Mean Field Limit

The mean-field limit is defined by the settings \( |\Gamma^{(1,1)}| = \binom{N^{(1)}}{2} \), \( |\Gamma^{(2,2)}| = \binom{N^{(2)}}{2} \), \( |\Gamma^{(1,2)}| = N^{(1)}N^{(2)} \), and the replacements \( J^{(1,1)} \rightarrow J^{(1,1)}/N^{(1)} \), \( J^{(2,2)} \rightarrow J^{(2,2)}/N^{(2)} \), \( J^{(1,2)} \rightarrow J^{(1,2)}(N^{(1)} + N^{(2)})/(2N^{(1)}N^{(2)}) \). We parametrize the size of the two communities as

\[
N^{(1)} = N \rho^{(1)}, \quad N^{(2)} = N \rho^{(2)}, \quad \rho^{(1)} + \rho^{(2)} = 1. \tag{10}
\]

Let us introduce the matrix \( \tilde{J} \)

\[
\tilde{J} = \begin{pmatrix} J^{(1,1)} & \frac{J^{(1,2)}}{2\rho^{(2)}} \\ \frac{J^{(2,1)}}{2\rho^{(1)}} & J^{(2,2)} \end{pmatrix}. \tag{11}
\]

As shown in [17], for \( N \to \infty \) we obtain the following deterministic evolution Eqs. for the order parameters (8)-(9)

3.1. FLOW

\[
\frac{\partial x^{(1)}(t)}{\partial (\alpha t)} = \tanh \left( \beta \tilde{J}^{(1,1)} x^{(1)}(t) + \beta \tilde{J}^{(1,2)} x^{(2)}(t) \right) - x^{(1)}(t), \tag{12}
\]

\[
\frac{\partial x^{(2)}(t)}{\partial (\alpha t)} = \tanh \left( \beta \tilde{J}^{(2,2)} x^{(2)}(t) + \beta \tilde{J}^{(2,1)} x^{(1)}(t) \right) - x^{(2)}(t), \tag{13}
\]
3.2. MAP

\[
\frac{x^{(1)}(t+1) - x^{(1)}(t)}{\alpha} = \tanh \left( \beta \tilde{J}^{(1,1)} x^{(1)}(t) + \beta \tilde{J}^{(1,2)} x^{(2)}(t) \right) - x^{(1)}(t),
\]

\[
\frac{x^{(2)}(t+1) - x^{(2)}(t)}{\alpha} = \tanh \left( \beta \tilde{J}^{(2,1)} x^{(1)}(t) + \beta \tilde{J}^{(2,2)} x^{(2)}(t) \right) - x^{(2)}(t),
\]

3.3. LOCAL-TIME FLOW

\[
\frac{\partial x^{(1)}(t)}{\partial (\alpha t)} = \begin{cases} 
0, & \text{if } \text{mod}[t, 2] \neq 0, \\
\tanh \left( \beta \tilde{J}^{(1,1)} x^{(1)}(t) + \beta \tilde{J}^{(1,2)} x^{(2)}(t) \right) - x^{(1)}(t), & \text{if } \text{mod}[t, 2] = 0,
\end{cases}
\]

\[
\frac{\partial x^{(2)}(t)}{\partial (\alpha t)} = \begin{cases} 
0, & \text{if } \text{mod}[t, 2] \neq 0, \\
\tanh \left( \beta \tilde{J}^{(2,1)} x^{(1)}(t) + \beta \tilde{J}^{(2,2)} x^{(2)}(t) \right) - x^{(2)}(t), & \text{if } \text{mod}[t, 2] = 0.
\end{cases}
\]

3.4. LOCAL-TIME MAP

\[
\frac{x^{(1)}(t+1) - x^{(1)}(t)}{\alpha} = \begin{cases} 
0, & t \in \mathcal{D}, \\
\tanh \left( \beta \tilde{J}^{(1,1)} x^{(1)}(t) + \beta \tilde{J}^{(1,2)} x^{(2)}(t) \right) - x^{(1)}(t), & t \in \mathcal{E},
\end{cases}
\]

\[
\frac{x^{(2)}(t+1) - x^{(2)}(t)}{\alpha} = \begin{cases} 
0, & t \in \mathcal{E}, \\
\tanh \left( \beta \tilde{J}^{(2,1)} x^{(1)}(t) + \beta \tilde{J}^{(2,2)} x^{(2)}(t) \right) - x^{(2)}(t), & t \in \mathcal{D}.
\end{cases}
\]

3.5. Stationary solutions

When the couplings are positive, there is little difference among the four dynamics, and they all asymptotically reach equilibrium according to the Boltzmann distribution \( p(\sigma) \propto \exp[-\beta H_{mf}(\sigma)] \), where \( H_{mf}(\sigma) \) is the mean-field analogous of Eq. (7), and \( x^{(1)}(t) \) and \( x^{(2)}(t) \) tend, for \( t \to \infty \), to the stationary solutions of Eqs. (14)-(17), i.e.,

\[
\begin{align*}
    x^{(1)} &= \tanh \left( \beta \tilde{J}^{(1,1)} x^{(1)} + \beta \tilde{J}^{(1,2)} x^{(2)} \right), \\
    x^{(2)} &= \tanh \left( \beta \tilde{J}^{(2,1)} x^{(1)} + \beta \tilde{J}^{(2,2)} x^{(2)} \right).
\end{align*}
\]

Eqs. (20) represent a particular case of the general result derived in [16] valid for \( n \) interacting communities at equilibrium. In particular, one can check that Eqs. (20) give rise to second order phase transitions whose critical surface is determined by the condition

\[
\det \left( I - \beta \tilde{J} \right) = 0.
\]

In general, \((x^{(1)}, x^{(2)}) = (0, 0)\) is stable when the eigenvalues of \( \beta \tilde{J} \) are inside the interval \((-1, 1)\) (disordered phase, or no consensus), otherwise the system reaches a spontaneous ordered phase \((x^{(1)}, x^{(2)}) \neq (0, 0)\) (frozen phase, or consensus). However, Eq. (21) is exact only for the flow and the map. In general, due to the sleeping time, the critical value of \( \beta \) can differ from the one given by Eq. (21). This was erroneously not stressed in [17]. The correct way to to derive the exact critical value of \( \beta \) must use the full expression of the Jacobian, which is the subject of Section 5.
4. Comparing flow, map and local-time versions

From now on we shall always assume $\alpha = 1$ and $N^{(1)} = N^{(2)}$.

In this Section we compare the four dynamics numerically.

In Figs (1a)-(2b) we compare the normal flow and map. In Figs. (1a) and (1b) we consider a situation with positive couplings at small and large values of $\beta$; in Figs. (2a) and (2b) we consider a situation with positive and negative couplings at small and large values of $\beta$, respectively. It is evident that, if the couplings are positive, the flow and the map remain roughly close at any time, and tend to coincide for $t \to \infty$ to the same stationary solution provided by Eq. (20). When instead some of the couplings are negative, at small $\beta$ the map tends to make damping oscillations around the flow (which for the present case is null), whereas, at large $\beta$, the map tends to make regular oscillations around the flow solution (which for the present case is null).
Figure 2: (Color online) Plot of $x^{(1)}(t)$ and $x^{(2)}(t)$, solution of the flow, Eqs. (12)-(13), and the map, Eqs. (14)-(15) (here with $\alpha = 1$), for a case with positive and negative couplings; below (Panel a) and above (Panel b) the threshold $\beta_c$. Initial conditions are the same in all cases. The lines of the map are guides for the eyes.

In Figs. (3a)-(3c) we compare the flow and map with local-time updates for a case with positive and negative couplings and three values of $\beta$, respectively. Fig. (3a) provides the behavior for $\beta = 1.5$ where the map follows damping oscillations and both the map and the flow tend to the same stationary value provided by Eq. (20). Fig. (3b) provides the behavior for a relatively larger value of $\beta$, $\beta = 2.5$, where the map follows regular oscillations of period 2, while the flow tends to a stationary value. Fig. (3c) provides the behavior for $\beta = 4.5$, where the map follows regular oscillations of period 4, while the flow tends to a stationary value. Increasing further $\beta$ leads to larger and larger periods (not shown).
Figure 3: (Color online) Plot of $x^{(1)}(t)$ and $x^{(2)}(t)$, solution of the local-time flow, Eqs. (16)-(17), and the local-time map, Eqs. (18)-(19) (here with $\alpha = 1$), $\beta = 1.5$ (a), $\beta = 2.5$ (b), and $\beta = 4.5$ (c). Initial conditions are the same in all cases. The lines of the map are guides for the eyes.
Figure 4: (Color online) Plot of $x_t^{(1)}(\beta)$ (red dots), solution of the local-time map, Eqs. (18)-(19) (here with $\alpha = 1$) as a function of $\beta$, with $T$ high enough to remove temporary transients. In this case $T = 10^4$. We plot also the maximum Lyapunov exponent $\lambda(\beta)$ (black highly irregular line) calculated from Eqs. (25)-(27). Each value of $\beta$ is in correspondence with a random initial condition for a total of $10^5$ samples. Panel (a) is a case with bifurcation cascade, Panel (b) is a case with marginal chaos.

5. Chaos

The case analyzed in Figs. (3) shows that, for large values of $\beta$, the local-time map can lead to erratic and, possibly, chaotic trajectories. In general, for each dynamics, the major quantity of interest is the Jacobian $J^{(l,m)}$, where $l, m \in \{1, 2\}$:

5.1. FLOW

$$J^{(l,m)} = \left[ 1 - \left( x_t^{(l)} \right)^2 \right] \beta J^{(l,m)},$$

(22)
5.2. MAP

\[ J_{t+1}^{(l,m)} = \left[ 1 - (x_t^{(l)})^2 \right] \beta J_{t}^{(l,m)}, \]  

(23)

5.3. LOCAL-TIME FLOW

\[
J_{t}^{(l,m)} = \begin{cases} 
\delta_{l,m}, & \text{mod}[t + l - 1, 2] \neq 0, \\
\left[ 1 - (x_t^{(l)})^2 \right] \beta J_{t}^{(l,m)}, & \text{mod}[t + l - 1, 2] = 0,
\end{cases}
\]

(24)

5.4. LOCAL-TIME MAP

\[
J_{t+1}^{(l,m)} = \begin{cases} 
\delta_{l,m}, & \text{mod}[t + l - 1, 2] \neq 0, \\
\left[ 1 - (x_t^{(l)})^2 \right] \beta J_{t}^{(l,m)}, & \text{mod}[t + l - 1, 2] = 0.
\end{cases}
\]

(25)

The knowledge of \( J_{t}^{(l,m)} \) allows to calculate the maximum Lyapunov exponent \( \lambda \). It turns out that in the map case (local-time or not), due to the fact that - via recurrence - we have the explicit solution of the Eqs., the evaluation of \( \lambda \) is easier if compared to the flow case. Since the flow does not show any signs of chaos or erratic trajectories, we can limit the evaluation of the Lyapunov exponent to the map, where we can use

\[
\lambda = \lim_{t \to \infty} \frac{1}{2t} \log(\Lambda_t),
\]

(26)

where \( \Lambda_t \) is the largest eigenvalue of the following matrix

\[
Q_t = L_t L_t^\dagger, \quad \text{where} \quad L_t = \prod_{t'=0}^{l} J_{t'}.
\]

(27)

Given a set of couplings, we let the system to evolve toward high enough values of \( t = T \) in order to remove temporary transients and repeat the numerical experiment for several values of \( \beta \), each \( \beta \) being associated to a random initial condition. Hence, we plot \((x^{(1)}(T), x^{(2)}(T))\) as functions of \( \beta \), and we indicate these functions as \((x^{(1)} T(\beta), x^{(2)} T(\beta))\). In our examples and for the range of \( \beta \) considered, \( T \geq 10^3 \) turns out to be enough high. In fact, the variable \( \beta \) plays the role of a time-scaling: the higher \( \beta \), the higher the necessary transient \( T \). In general, the functions \((x^{(1)}_T(\beta), x^{(2)}_T(\beta))\) look multi-valued functions due to the existence of bifurcation points. There exist two kinds of bifurcations: doubling period and phase transition. However, it is clear that in either case, doubling period or phase transition, the presence of bifurcations increases the chance to develop chaos. Therefore, for our aims here, it is more interesting, as well as highly more efficient, to show the plots that include all the bifurcation points. Figs. (4a)-(4b) show two different scenarios. The case of Panel (a) is a “classical” bifurcation cascade [1], where, after a threshold \( \beta_{\text{Chaos}} \), bifurcation takes place increasingly, up to windows of stability. In the case of Panel (b) we observe a different situation which we call “marginal chaos”. In this case, after a threshold \( \beta_{\text{Marginal}} \), the dynamics is characterized by totally erratic trajectories with no period, up to windows of stability. Strictly speaking, even out of the windows of stability, the system is not chaotic, since the maximum Lyapunov exponent is \( \lambda = 0 \). However, in such regions, the system turns out to be critical over continuous intervals; a situation which is not typical in models characterized by discrete symmetries like the Ising model.
6. Conclusions
We have analyzed and compared four kinds of dynamics of a simple two component Ising-like system: flow, map, local-time flow, and local-time map. Remarkably, even if only bilinear interactions are present, the local-time map gives rise to erratic trajectories and, depending on the set of couplings, two chaotic scenarios take place: bifurcation cascades or “marginal chaos”, i.e., criticality extended over continuous intervals of $\beta$, the time-scale parameter. The analogous local-time flow does not present any of such behaviors, not even oscillations, confirming all the scenarios we have already discussed in [17]. We stress that this local-time update is not just a time-delay: here we have also a sleeping time during which, in the case of some unfriendly couplings, there is an accumulation of frustration among the system components, giving rise to instability and chaos, even for a system characterized by only bilinear interactions. As we have discussed in [17], we believe that maps with local-time updates are a quite common feature in complex systems. Urgent issues will be to understand how this scenario generalizes to large systems with many, say $q$ components, how the probability to have chaos changes with $q$, and if the mean-field (or more precisely fully connected) picture remains robust.

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