Thermodynamics of Cyclic Quantum Amplifiers

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We develop a generic model for a cyclic quantum heat engine that makes it possible to coherently amplify a periodically modulated input signal without the need to couple the working medium to multiple reservoirs at the same time. Instead, we suggest an operation principle that is based on the spontaneous creation of population inversion in incomplete relaxation processes induced by periodic temperature variations. Focusing on Lindblad dynamics and systems with equally spaced energy levels, e.g. qubits or quantum harmonic oscillators, we derive a general working criterion for such cyclic quantum amplifiers. This criterion defines a class of candidates for suitable working media and applies to arbitrary control protocols. For the minimal case of a cyclic three-level amplifier, we show that our criterion is tight and explore the conditions for optimal performance.

Introduction. — Quantum amplifiers generate coherent electromagnetic energy using the stimulated emission of photons in a population-inverted medium [1–3]. Early on, Scovil and Schulz-DuBois realized that, when driven by a thermal gradient, such devices can be understood as quantum-mechanical heat engines, whose efficiency is subject to the Carnot bound [4]. In their approach, the working medium is a collection of three-level atoms, whose transitions are coupled either to a hot or a cold reservoir acting as a source of energy and a sink of entropy, respectively. A resonant driving field plays the role of a moving piston enabling the extraction of usable work in form of coherent radiation, see Fig. 1a–b.

Owing to its universal and transparent structure, this model has contributed significantly to our basic knowledge of energy conversion in the quantum regime [1–10]. At the same time, it has become a prototype for practical devices like photocells [11–14] and small-scale refrigerators [15–18]. Moreover, the three-level amplifier has served as a template for new types of thermal machines that utilize complex quantum effects such as lasing without inversion [19, 20], noise-induced coherence [21–23] or electromagnetically induced transparency [24]; recent proposals include even two-level variants that operate without population inversion using thermal evaporation [25], squeezed driving fields [26] or two-photon transitions [27]. These developments have led to profound theoretical insights over the last years. They might soon also be tested in practice as coherence-based heat engines can now be realized experimentally [28, 29].

The ideas of Scovil and Schulz-DuBois have shaped our perception of thermal quantum amplifiers as a distinct sort of heat engines, which operate in a steady state and use reversible energy filters to maintain a population-inverted working medium [30, 31]. In this article, we investigate an alternative strategy for coherent power generation: we develop and analyze a generic model for a cyclic quantum amplifier. Resembling a reciprocating heat engine, our device operates in a thermodynamic cycle [31–34], where heat is transferred periodically from a hot to a cold reservoir to create population inversion, see Fig. 1c–d. In contrast to earlier proposals, this working principle does not rely on energy filters. Instead, it requires at least one metastable energy level, which can be temporarily overpopulated while the system returns to equilibrium. To capture this condition quantitatively, we derive a general working criterion for cyclic quantum amplifiers, which makes it possible to characterize the applicable working systems without reference to a specific control protocol.
Cyclic Quantum Amplifiers.— Our setup consists of two basic components: a working medium with tunable Hamiltonian

$$H_t \equiv \sum_{n=1}^{N} E_n |n_t\rangle \langle n_t|$$

and a heat source to control the temperature $T_t$ of the environment. The time-dependence of the energy eigenstates $|n_t\rangle$ is determined by the input signal, while the energy levels $E_n$ are fixed. This condition ensures that the device exchanges only coherent power with the driving field [35]. At the same time, it reduces the accessible energy content of the system to the maximum amount of work that can be extracted through unitary operations. This quantity is given by the *ergotropy* [36–39]

$$\mathcal{E}_t \equiv \text{tr} [\rho_t H_t] - \min_U \text{tr} [\rho_t U H_t U^\dagger] \equiv E_t - E_t^{\text{res}} \geq 0, \quad (2)$$

determining the state of the system $\rho_t$ and setting $E_t$ to its total internal energy. The residual energy $E_t^{\text{res}}$ is found by evaluating the minimum over all unitary operators $U$.

Taking the time derivative of Eq. (2) yields

$$\dot{\mathcal{E}}_t = \mathcal{J}_t - P_t, \quad (3)$$

This balance equation plays the role of the first law of thermodynamics for quantum amplifiers. The quantities

$$P_t \equiv -\text{tr} [\rho_t \dot{H}_t] = \sum_{n=1}^{N} \langle \dot{n}_t | [H_t, \rho_t] | n_t \rangle \quad \text{and} \quad (4)$$

$$\mathcal{J}_t \equiv \text{tr} [\dot{\rho}_t H_t] - \dot{E}_t^{\text{res}} = \sum_{n=1}^{N} \left( \langle n_t | \dot{\rho}_t | n_t \rangle - \langle r_n^m | \dot{\rho}_t | r_n^m \rangle \right) E_n$$

correspond to the instantaneous power output and the rate of reservoir-induced ergotropy production [35, 40]. Here, we have used the ordered spectral decomposition

$$\rho_t = \sum_{n=1}^{N} r_n^m | r_n^m \rangle \langle r_n^m |$$

of the state $\rho_t$ to evaluate the time derivative of the residual energy. The energy levels $E_n$ are thereby arranged in ascending order, i.e., $r_n^m \geq r_m^m$ and $E_n \leq E_m$ for $m > n$. The second expression for $P_t$ in Eq. (4) follows from Eq. (1) and vanishes if the system is in a quasi-classical state, i.e., if $\rho_t$ commutes with $H_t$. This observation shows that coherent power generation is a genuine quantum phenomenon, which requires the creation of superpositions between the energy levels of the medium [35].

Once the system has settled to a cyclic state, the ergotropy $\mathcal{E}_t$ becomes a periodic function of time. Thus, upon averaging Eq. (3) over one period $\tau$, the mean extracted work becomes

$$W = \int_0^{\tau} P_t \, dt = \int_0^{\tau} \mathcal{J}_t \, dt. \quad (6)$$

This relation shows that a cyclic quantum amplifier can only deliver finite output if the thermal ergotropy production $\mathcal{J}_t$ becomes positive during its operation cycle. Hence, it must be possible to drive the system into a population-inverted state by changing the temperature of its environment. In the following, we will further examine the necessary conditions for this effect.

Quantum Ladders.— We now specify the working medium as a quantum ladder with equally spaced energy levels, i.e., we set $E_n = n \hbar \omega$, where $\hbar \omega$ denotes the overall energy scale [41]. Such systems include, for example, qubits and quantum harmonic oscillators. In order to describe the interaction of the medium with its environment, we assume weak-coupling conditions and a proper separation of time-scales between the external driving and the relaxation dynamics of the reservoir. Under these conditions, the time evolution of the state $\rho_t$ is governed by a Markovian quantum master equation [40, 42],

$$\dot{\rho}_t = -\frac{i}{\hbar} [H_t, \rho_t] + \gamma \nu_t \left( |L_t \rho_t |L_t^\dagger \rangle \langle L_t \rho_t |L_t^\dagger | + |L_t^\dagger \rho_t |L_t \rangle \langle L_t^\dagger \rho_t |L_t \rangle \right)/2$$

$$+ \gamma (\nu_t + 1) \left( |L_t \rho_t |L_t \rangle \langle L_t \rho_t |L_t \rangle + |L_t^\dagger \rho_t |L_t^\dagger \rangle \langle L_t^\dagger \rho_t |L_t^\dagger \rangle \right)/2. \quad (7)$$

Here, the jump operators

$$L_t = \sum_{n=1}^{N} \hbar \omega (n+1) |(n+1)_t\rangle \langle n_t| \quad \text{and} \quad L_t^\dagger = \sum_{n=1}^{N} \hbar \omega |(n+1)_t\rangle \langle n_t|$$

describe the exchange of photons between system and reservoir, assuming for the sake of simplicity that the weighting factors $\nu_t$ are real. The rate $\gamma > 0$ determines the average frequency of emission and absorption events and the Bose-Einstein factors $\nu_t \equiv 1/(e^{\hbar \omega/k_B T_t} - 1)$ ensure thermodynamic consistency [43].

Working Criterion.— The operation principle of our engine relies on the possibility to thermally create population inversion in the system. In the following, we establish a kinetic criterion for this phenomenon. To this end, we derive a minimal condition on the weighting factors $\nu_t$ for $\mathcal{J}_t$ to become positive and thus identify a class of suitable candidates for the working media of cyclic quantum amplifiers. Mathematically, we proceed through the following steps. We first observe that, upon inserting Eq. (7) into Eq. (4), the thermal ergotropy production decomposes into two components, $\mathcal{J}_t = \hbar \omega \gamma (I_t^1 + I_t^2)$, where

$$I_t^1 \equiv \sum_{n,m=1}^{N} \nu_t (r_n^m - r_m^n)(1 + m - n) \langle r_n^m | L_t | r_m^n \rangle^2$$

$$I_t^2 \equiv \sum_{n,m=1}^{N} \nu_t (n - m - 1) \langle r_n^m | L_t | r_m^n \rangle^2. \quad (9)$$

Since the probabilities $r_n^m$ are arranged in descending order, the first term cannot be positive, i.e., $I_t^1 \leq 0$. Hence, we are left with analyzing the second contribution $I_t^2$. To
this end, we introduce the partial sums $R^n_i \equiv \sum_{m=1}^{n} r^m_i$. Using the inequality $r^m_i(n-m) \leq R^n_i - R^m_i$, we obtain

$$I^2 \leq \sum_{n=1}^{N} \left( R^n_i \langle r^n_i | [L_i, L_i'] | r^n_i \rangle - \sum_{n=1}^{N} r^n_i \langle r^n_i | [L_i, L_i'] | r^n_i \rangle \right) \equiv I^2_n,$$

(10)

In order to make this bound independent of the state $\rho$, we maximize the right-hand side of Eq. (10) with respect to orthonormal vectors $|r^n_i\rangle$ in two steps [36]. First, we note that each term of the form $\langle \psi | X | \psi \rangle$, with $X$ being a hermitian operator, is extremal as a function of the normalized vector $|\psi\rangle$ whenever this vector is an eigenvector of $X$. The orthonormal sets of vectors $|r^n_i\rangle$ that maximize the two sums in Eq. (10) must therefore form eigenbases of $[L_i, L_i']$ and $L_i L_i'$, respectively. For the second step, we apply the rearrangement inequality [44], which fixes the ordering of these bases. This procedure leads to

$$I^2 \leq \sum_{n=1}^{N} \left( R^n_i \pi_n \left( \{ \ell^2_{m} - \ell^2_{m-1} \} \right) - r^n_i \pi_n \left( \{ \ell^2_{m} - \ell^2_{m-1} \} \right) \right),$$

(11)

where the function $\pi_n(M)$ returns the $n$th-lowest element of the set $M$, the index $m$ assumes values $1 \leq m \leq N$, and $\ell_0 \equiv \ell_n \equiv 0$. Finally, upon expressing the partial sums $R^n_i$ in terms of the probabilities $r^n_i$ and rearranging the resulting expression, we find that

$$I^2 \leq \sum_{n=1}^{N} (r^n_i - r^{n+1}_i) \pi_n \text{ with }$$

$$\Phi_n \equiv \sum_{k=1}^{n} \left( (k-n) \pi_k \left( \{ \ell^2_{m-1} - \ell^2_{m} \} \right) - \pi_k \left( \{ \ell^2_{m-1} \} \right) \right),$$

and $r^{N+1}_i \equiv 0$. Since, by assumption, $r^n_i \geq r^{n+1}_i$, it follows that $I^2 \leq \Phi_n$, and thus $J_i$ and $W$, can become positive only if

$$\Phi_{\text{max}} \equiv \max_n \Phi_n > 0.$$

(13)

The criterion (13) is quite remarkable. It depends neither on the Hamiltonian $H_i$, the temperature profile $T_i$, nor on the specific state $\rho_i$. It can therefore be used to determine whether or not relaxation-induced population inversion can occur in a given system. Two natural choices of potential working media are qubits and the quantum harmonic oscillators. Applying our criterion to a qubit, i.e., $N = 2$, we find

$$\Phi_{\text{max}}^{QB} = \Phi_{\text{max}}^{QB} = 0.$$ 

(14)

For the harmonic oscillator, which features infinitely many energy levels and weighting factors $\ell_n = \sqrt{n}$, we similarly obtain

$$\Phi_{\text{max}}^{\text{HO}} = \sum_{k=1}^{n} (n - 2k + 1) = \Phi_{\text{max}}^{\text{HO}} = 0.$$ 

(15)

Hence, our criterion rules out both of these systems as working substances of cyclic quantum amplifiers. To find a suitable candidate, we now turn to a three-level system, where $\Phi_{\text{max}}^{\text{LS}} = \ell^2_1 - \ell^2_2$ if $\ell_2 \leq \ell_1$, $\Phi_{\text{max}}^{\text{LS}} = \ell^2_2 - 2\ell^2_1$ if $\ell_2 \geq \sqrt{2}\ell_1$, and $\Phi_{\text{max}}^{\text{LS}} = 0$ otherwise. Hence, three-level quantum ladders, whose jump weights satisfy

$$\ell_2 \leq \ell_1 \text{ or } \ell_2 \geq \sqrt{2}\ell_1$$

(16)

are suitable candidates for cyclic quantum amplification.

Example.—To explore the physical picture behind the condition (16), we now apply the protocol of Fig. 1c to a three-level system. Hence, in the first stroke, the state $\rho_i$ follows the master equation (7) with $T_i \equiv T_h$ for the time $t_h$. The temperature is then abruptly reduced to the cold level $T_c < T_h$. The system relaxes at this temperature until its ergotropy becomes maximal, i.e., until the time $T$, at which the difference $p^2_t - p^1_t$ of popula-
tions $p_i^0 = \langle n_i | \rho_i | n_2 \rangle$ is maximal. At this time, the relaxation process is terminated and a $\pi$-pulse is applied, which swaps the populations of the lowest and the second level, thus generating the coherent work

$$W = E_T = \hbar\omega (p_2^0 - p_1^0)$$

(17)

and restoring the initial state of the system.

The results of our analysis are summarized in Fig. 2, which reveals two key effects. First, beyond a certain threshold value, which depends on $T_h$ and $t_h$, the average work $W$ grows monotonically as a function of the ratio $\tau_2/\tau_1$. This behavior arises from an increasing separation between the characteristic relaxation times $\tau_2 = 1/ (\gamma \ell_2)$ and $\tau_1 = (1/\gamma \ell_1)$ of the upper and the lower level. If $\tau_2 \ll \tau_1$, the population of level 3 can be essentially transferred to level 2 before level 1 is significantly affected. In this regime, a pronounced population inversion emerges, leading to a large output $W$. Second, we find that an increasing amount of work can be extracted if either the hot temperature $T_h$ is raised or if the duration $t_h$ of the input stroke is extended; at the same time, the threshold value of $\tau_2/\tau_1$ decreases. This phenomenon can be understood by observing that the level populations after stroke 1 become more homogeneous for larger values of $T_h$ and $t_h$. Hence, less population has to be redistributed during the second stroke to create a strong inversion. In the limiting case $T_h, t_h \to \infty$, all three levels are equally populated after the first stroke and the bound (16) becomes tight.

**Extensions.**—Having studied the three-level system in detail, we now consider a more general type of quantum ladder, for which $N \geq 3$ is arbitrary and the weights $\ell_n$ depend algebraically on the level index, i.e., $\ell_n = n^\alpha$. The corresponding coefficients $\Phi_{\text{max}}^\alpha$ are plotted in Fig. 3 for different numbers of energy levels. Notably, we find that, irrespective of $N$,

$$\Phi_{\text{max}}^\alpha = 0 \quad \text{for} \quad 0 \leq \alpha \leq 1/2.$$  

(18)

This result suggest that the squared weights $\ell_n^\alpha$ must either decrease with $n$ or increase at least linearly to enable the spontaneous creation of population inversion. Whether or not this observation can be corroborated for more complicated relations between $\ell_n$ and $n$ remains as an open question.

Turning to more general situations, we note that the versatile technique that we have developed to derive our working criterion (13) can be easily adapted for setups with multiple reservoirs or composite working systems that consist of a collection of non-interacting quantum ladders. Further extensions of our scheme might even make it possible to consider non-equilibrium reservoirs [20, 45–48] or strongly driven systems, for which the master equation (7) has to be replaced by a Floquet-Lindblad equation [49–51]. In principle, the qualitative behavior observed in our case study can be expected to persist for more general systems; that is, a strong separation of relaxation time scales and the preparation of the working system in a state with nearly flat level populations during the input stroke should generically improve the performance of cyclic quantum amplifiers.

**Outlook.**—These possibilities show that our work provides both a new approach to assess the viability of coherence-based heat engines and a valuable starting point for future investigations seeking to further explore the mechanisms of thermal energy conversion in the quantum regime. Our results thereby corroborate the emerging picture that coherent power generation is a technically demanding process, which requires a well-tailored setup. In fact, it was shown only recently that thermodynamic cycles cannot produce coherent work in the limits of linear [35] and adiabatic [52] response, i.e., if either the overall amplitude or the frequency of the driving field and temperature variations is small. Here, we have taken a first step towards a more complete characterization of the necessary working conditions for periodic thermal devices that deliver coherent energy output. In particular, our working criterion (13) shows that, even far from equilibrium, cyclic quantum amplifiers are subject to much stronger restrictions than conventional cyclic heat engines.

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