Mixing and observation for Markov operator cocycles

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Abstract
We consider generalized definitions of mixing and exactness for random dynamical systems in terms of Markov operator cocycles. We first give six fundamental definitions of mixing for Markov operator cocycles in view of observations of the randomness in environments, and reduce them into two different groups. Secondly, we give the definition of exactness for Markov operator cocycles and show that Lin’s criterion for exactness can be naturally extended to the case of Markov operator cocycles. Finally, in the class of asymptotically periodic Markov operator cocycles, we show the Lasota–Mackey type equivalence between mixing, exactness and asymptotic stability.

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1. Introduction

This paper concerns a cocycle generated by Markov operators, called a Markov operator cocycle. Let $(X, \mathcal{A}, m)$ be a probability space, and $L^1(X, m)$ (the quotient by equality
m-almost everywhere of) the space of all m-integrable functions on \( X \), endowed with the usual \( L^1 \)-norm \( \| \cdot \|_{L^1(X)} \). An operator \( P : L^1(X, m) \to L^1(X, m) \) is called Markov operator if \( P \) is linear, positive (i.e. \( Pf \geq 0 \) m-almost everywhere if \( f \geq 0 \) m-almost everywhere) and

\[
\int_X Pf \, dm = \int_X f \, dm \quad \text{for all} \quad f \in L^1(X, m).
\]

Markov operators naturally appear in the study of dynamical systems (as Perron–Frobenius operators; see (3)), Markov processes (as integral operators with the stochastic kernels of the processes), and random dynamical systems in the annealed regime (as integrations of Perron–Frobenius operators over environmental parameters). For these deterministic/stochastic dynamics, \( \{P^n f\}_{n \geq 0} \) is the evolution of density functions of random variables driven by the system. We refer to \([9, 13]\).

A Markov operator cocycle is given by compositions of different Markov operators which are provided with according to the environment \( \{\sigma^n(\omega)\}_{n \geq 0} \) driven by a measure-preserving transformation \( \sigma : \Omega \to \Omega \) on a probability space \((\Omega, \mathcal{F}, P)\),

\[
\mathbb{N} \times \Omega \times L^1(X, m) \to L^1(X, m) : (n, \omega, f) \mapsto P_{\sigma^n(\omega)}, \quad \omega \in \Omega
\]

(see definition 1.1 for more precise description). So, in nature it possess two kinds of randomness:

(a) The evolution of densities at each time are dominated by Markov operators \( P_\omega \),

(b) The selection of each Markov operators is driven by the base dynamics \( \sigma \).

The aim of this paper is to investigate how the observation of the randomness of the state space and the environment influences statistical properties of the system, and to give a step to understanding more complicated phenomenon in multi-stochastic systems.

Our focus lies on the mixing property. Recall that a Markov operator \( P : L^1(X, m) \to L^1(X, m) \) is said to be mixing if

\[
\int_X P^n f g \, dm \to \int_X f \, dm \int_X g \, dm \quad \text{as} \quad n \to \infty
\]

for any \( f \in L^1(X, m) \) and \( g \in L^\infty(X, m) \) (when \( P_{1_X} = 1_X \), see remark 1 for more general form). Due to (1), this means that two random variables \( P^n f \) and \( g \) are asymptotically independent so that the system is considered to ‘mix’ the state space well. In other words, the randomness of \( P \) in the sense of mixing can be seen through the observables \( f \) and \( g \). Hence, for Markov operator cocycles, the strength of the dependence of the observables on \( \omega \) expresses how one observes the randomness of the state space and the environment. Furthermore, more directly, we can consider different kinds of mixing properties according to whether the environment \( \omega \) is observed as a prior event to the observation of \( f, g \). According to these viewpoints, we will introduce six definitions of mixing for Markov operator cocycles (definition 1.2). In section 2, we show that five of them are equivalent when \( \Omega \) is a compact topological space, while at least one of them are different. In the case when the Markov operator cocycle is generated by a random dynamical system over a mixing driving system, we also show that all of them imply the (conventional) mixing property of the skew-product transformation induced by the random dynamical system.

We further investigate exactness for Markov operator cocycles. Since the observable \( g \) in (2) does not appear in the definition of exactness for a Markov operator \( P \) (recall that, when \( P_{1_X} = 1_X \), \( P \) is said to be exact if \( \lim_{n \to \infty} \|P^n f - \int_X f \, dm\|_{L^1(X)} = 0 \) for all \( f \in L^1(X, m) \); see also the remark following definition 1.3), in contrast to the mixing property, we only have
Figure 1. The relations between definitions in this paper. Here $B$, $C$ and $L^\infty$ are abbreviations of $B(\Omega, L^\infty(X, m))$, $C(\Omega, L^\infty(X, m))$ and $L^\infty(\Omega, L^\infty(X, m))$, respectively. The implications by a grey arrow represent trivial relations by definitions. The implication by a black arrow from prior mixing for homogeneous observables to prior mixing for inhomogeneous observables in $C$ holds when $\Omega$ is a compact topological space. Therefore, the above seven definitions are all equivalent when $\Omega$ is compact. Moreover, the below four definitions are equivalent for an asymptotically periodic Markov operator cocycle, while it is not true for general Markov operator cocycles due to corollary 2.5.

one definition of exactness for Markov operator cocycles (definition 1.3). We will show that Lin’s criterion for exactness can be naturally extended to the case of Markov operator cocycles (section 3), and finally, in the class of asymptotically periodic Markov operator cocycles, we prove Lasota–Mackey type equivalence between mixing, exactness and asymptotic stability, as well as their relationship with the existence of an invariant density map (section 4). See figure 1 for the summary.

1.1. Definitions of mixing and exactness

Let $D(X, m)$ and $L^1_0(X, m)$ be subsets of $L^1(X, m)$ given by

$D(X, m) = \left\{ f \in L^1(X, m) : f \geq 0 \text{ m-almost every } \right\}$,

$L^1_0(X, m) = \left\{ f \in L^1(X, m) : \int_X f \, dm = 0 \right\}$.

Note that $P : L^1(X, m) \rightarrow L^1(X, m)$ is a Markov operator if and only if $P(D(X, m)) \subset D(X, m)$.

One of the most important examples of Markov operators is the Perron–Frobenius operator induced by a non-singular transformation $T : X \rightarrow X$ (that is, $T, m$ is absolutely continuous with respect to $m$, where $T, m$ is the pushforward of $m$ given by $T, m(A) = m(T^{-1}A)$ for $A \in \mathcal{A}$). The Perron–Frobenius operator $L_T : L^1(X, m) \rightarrow L^1(X, m)$ of $T$ is defined by

$L_T f = \frac{dT, (f \, dm)}{dm}$ for $f \in L^1(X, m)$, (3)
where \( fm \) is a finite signed measure given by \((fm)(A) = \int_A f \, dm\) for \( A \in \mathcal{A} \) and \( dm/d\mu \) is the Radon–Nikodym derivative of an absolutely continuous finite signed measure \( \mu \). Note that for each \( X \)-valued random variable \( \chi \) whose distribution is \( fm \) with some \( f \in D(X, m) \), \( T(\chi) \) has the distribution \((LT f)m\) (and thus, \( LT \) is also called the transfer operator associated with \( T \)). It is straightforward to see that

\[
\int_X L_T f g \, dm = \int_X f g \circ T \, dm \quad \text{for} \quad f \in L^1(X, m) \quad \text{and} \quad g \in L^\infty(X, m),
\]

and that \( L_T \) is a Markov operator.

Recall that \((\Omega, \mathcal{F}, \mathbb{P})\) is a probability space, and \( \sigma : \Omega \to \Omega \) is a \( \mathbb{P} \)-preserving transformation. For a measurable space \( \Sigma \), we say that a measurable map \( \Phi : \mathbb{N}_0 \times \Omega \times \Sigma \to \Sigma \) is a random dynamical system on \( \Sigma \) over the driving system \( \sigma \) if

\[
\varphi^{(0)}_\omega = \text{id}_\Sigma \quad \text{and} \quad \varphi^{(n+m)}_\omega = \varphi^{(n)}_{\sigma^m \omega} \circ \varphi^{(m)}_\omega
\]

for each \( n, m \in \mathbb{N}_0 \) and \( \omega \in \Omega \), with the notation \( \varphi^{(n)}_\omega = \Phi(n, \omega, \cdot) \) and \( \sigma^m \omega = \sigma(\omega) \), where \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). A standard reference for random dynamical systems is the monographs by Arnold [2]. It is easy to check that

\[
\varphi^{(n)}_\omega = \varphi^{(n)}_{\sigma^{-1} \omega} \circ \varphi^{(n)}_{\sigma^{-2} \omega} \circ \cdots \circ \varphi^{(n)}_{\omega}
\]

with the notation \( \varphi^{(n)}_\omega = \Phi(1, \omega, \cdot) \). Conversely, for each measurable map \( \varphi : \Omega \times \Sigma \to \Sigma : (\omega, x) \mapsto \varphi_\omega(x) \), the measurable map \( (n, \omega, x) \mapsto \varphi^{(n)}_\omega(x) \) given by (5) is a random dynamical system. We call it a random dynamical system induced by \( \varphi \) over \( \sigma \), and simply denote it by \((\varphi, \sigma)\). When \( \Sigma \) is a Banach space and \( \varphi_\omega : \Sigma \to \Sigma \) is \( \mathbb{P} \)-almost surely linear, \((\varphi, \sigma)\) is called a linear operator cocycle. We give a formulation of Markov operators in random environments in terms of linear operator cocycles.

**Definition 1.1.** We say that a linear operator cocycle \((P, \sigma)\) induced by a measurable map \( P : \Omega \times L^1(X, m) \to L^1(X, m) \) over \( \sigma \) is a Markov operator cocycle (or a Markov operator in random environments) if \( P_\omega = P(\omega, \cdot) : L^1(X, m) \to L^1(X, m) \) is a Markov operator for \( \mathbb{P} \)-almost every \( \omega \in \Omega \).

Let \((n, \omega, f) \mapsto P^{(n)}_\omega f \) be a Markov operator cocycle induced by \( P : \Omega \times L^1(X, m) \to L^1(X, m) \) such that \( P_\omega = P(\omega, \cdot) \) is the Perron–Frobenius operator \( L_T \) associated with a non-singular map \( T_\omega : X \to X \) for \( \mathbb{P} \)-almost every \( \omega \). Then, it follows from (4) that \( \mathbb{P} \)-almost surely

\[
\int_X P^{(n)}_\omega f g \, dm = \int_X f g \circ T^{(n)}_\omega \, dm, \quad \text{for} \quad f \in L^1(X, m) \quad \text{and} \quad g \in L^\infty(X, m),
\]

where \( T^{(n)}_\omega = T_{\sigma^{n-1} \omega} \circ T_{\sigma^{n-2} \omega} \circ \cdots \circ T_\omega \).

We are now in place to give definitions of mixing for Markov operator cocycles. Let \( K \) be a space consisting of measurable maps from \( \Omega \) to \( L^\infty(X, m) \).

**Definition 1.2.** A Markov operator cocycle \((P, \sigma)\) is called

(a) **Prior mixing for homogeneous observables** if for \( \mathbb{P} \)-almost every \( \omega \in \Omega \), any \( f \in L^1_0(X, m) \) and \( g \in L^\infty(X, m) \), it holds that

\[
\lim_{n \to \infty} \int_X P^{(n)}_\omega f g \, dm = 0;
\]

(b) **Posterior mixing for homogeneous observables** if for \( \mathbb{P} \)-almost every \( \omega \in \Omega \), any \( f \in L^1_0(X, m) \) and \( g \in L^\infty(X, m) \), it holds that

\[
\lim_{n \to \infty} \int_X P^{(n)}_\omega f g \, dm = 0.
\]
(b) Posterior mixing for homogeneous observables if for any \( f \in L^1_\omega(X, m) \), \( g \in L^\infty(X, m) \) and \( \mathbb{P}\)-almost every \( \omega \in \Omega \), (7) holds;

(c) Prior mixing for inhomogeneous observables in \( K \) if for \( \mathbb{P}\)-almost every \( \omega \in \Omega \), any \( f \in L^1_\omega(X, m) \) and \( g \in K \), it holds that

\[
\lim_{n \to \infty} \int_X P^{(n)}_\omega f g d\mu_\omega \, dm = 0;
\]  

(8)

(d) Posterior mixing for inhomogeneous observables in \( K \) if for any \( f \in L^1_\omega(X, m) \), \( g \in K \) and \( \mathbb{P}\)-almost every \( \omega \in \Omega \), (8) holds.

In the prior case (the posterior case), the observation of the environment \( \omega \) is a prior event (a posterior event, respectively) to the observation of \( f \) and \( g \). As the class of inhomogeneous observables \( K \) in definition 1.2, we will consider the following two fundamental classes.

(a) \( B(\Omega, L^\infty(X, m)) \): the set of all bounded and measurable maps from \( \Omega \) to \( L^\infty(X, m) \).

(b) \( C(\Omega, L^\infty(X, m)) \): the set of all bounded and continuous maps from \( \Omega \) to \( L^\infty(X, m) \) (when \( \Omega \) is a topological space and \( \mathcal{F} \) is its Borel \( \sigma \)-field).

Remark 1. The above definitions need not require an invariant density map for the Markov operator cocycle \((P, \sigma)\). We say that a measurable map \( h : \Omega \to D(X, m) \) is an invariant density map for \((P, \sigma)\) if \( P_\omega h = h_\omega \) holds for \( \mathbb{P}\)-almost every \( \omega \in \Omega \) where \( h_\omega = h(\omega) \). Now we assume that there exist an invariant density map \( h : \Omega \to D(X, m) \) for \((P, \sigma)\) such that for \( \mathbb{P}\)-almost every \( \omega \in \Omega \),

\[
\lim_{n \to \infty} m(\sup_{P^{(n)}_\omega} P^{(n)}_\omega 1_X \setminus \sup_{P^{(n)}_\omega} P^{(n)}_\omega h_\omega) = 0.
\]  

(9)

Then by (9) and the fact that \( P^{(n)}_\omega f - h_\omega = P^{(n)}_\omega (f - h_\omega) \in L^1_\omega(X, m) \) for \( f \in D(X, m) \) and \( \mathbb{P}\)-almost every \( \omega \in \Omega \), any \( f \in D(X, m) \) and \( g \in L^\infty(X, m) \), it holds that

\[
\lim_{n \to \infty} \int_X (P^{(n)}_\omega f - h_\omega g) \, dm = 0.
\]

Furthermore, when \( P_\omega \) is the Perron–Frobenius operator \( L_{T_\omega} \) associated with a non-singular map \( T_\omega : X \to X \), by (6), it is also equivalent to that for \( \mathbb{P}\)-almost every \( \omega \in \Omega \), any \( f \in L^1(X, \mu_\omega) \) and \( g \in L^\infty(X, m) \),

\[
\int_X f g \circ T^{(n)}_\omega \, d\mu_\omega - \int_X f \, d\mu_\omega \int_X g \, d\mu_\omega \to 0 \quad \text{as} \quad n \to \infty,
\]  

(10)

where \( \mu_\omega = h_\omega m \). Moreover, we can replace ‘for any \( f \in L^1(X, \mu_\omega) \)’ in the previous sentence with ‘for any measurable function \( f : \Omega \times X \to \mathbb{R} \) such that \( f_\omega = f(\omega, \cdot) \in L^1(X, \mu_\omega) \) \( \mathbb{P}\)-almost surely’, and ‘\( f \)’ in (10) with ‘\( f_\omega \)’. Similar equivalent conditions can be found for other types of mixing in definition 1.2.

All kinds of mixing in definition 1.2 were adopted in literature, especially in the form of (10) to discuss mixing for random dynamical systems. For instance, we refer to Baladi et al [4, 5] and Buzzi [6] for the definition 1, Dragičević et al [7] for the definition 2, Bahsoun et al [3] for the definition 3, and Gundlach [12] for the definition 4. Moreover, in the deterministic case (i.e. \( \Omega \) is a singleton), all the definitions are equivalent to the usual definition of mixing for a single Markov operator [13].
Remark 2. Another natural candidate for the class of inhomogeneous observable is the Bochner–Lebesgue space $L^\infty(\Omega, L^\infty(X, m))$, that is, the Kolmogorov quotient (by equality $\mathbb{P}$-almost surely) of the space of all $\mathbb{P}$-essentially bounded and Bochner measurable maps from $\Omega$ to $L^\infty(X, m)$ (and $\mathbb{8}$ is interpreted as it holds under the usual identification between an equivalent class and a representative of the class). However, in the case $K = L^\infty(\Omega, L^\infty(X, m))$, the prior version 3 does not make sense because one can find an equivalent class $[g] \in L^\infty(\Omega, L^\infty(X, m))$ and maps $g_1, g_2 \in [g]$ such that (8) holds for $g = g_1$ while $\mathbb{8}$ does not hold for $g = g_2$, see subsection 2.2. On the other hand, the posterior version 4 makes sense for $K = L^\infty(\Omega, L^\infty(X, m))$, and indeed, its relationship with posterior mixing for homogeneous observables will be discussed in theorem 2.2.

By the definitions, we immediately see that the prior mixing implies the posterior mixing (that is, (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) in definition 1.2). It is also obvious that the prior (posterior) mixing for inhomogeneous observables in $B(\Omega, L^\infty(X, m))$ or $C(\Omega, L^\infty(X, m))$ implies the prior (posterior, respectively) mixing for homogeneous observables. Refer to figure 1.

We next define exactness for Markov operator cocycles.

**Definition 1.3.** A Markov operator cocycle $(P, \sigma)$ is called exact if for $\mathbb{P}$-almost every $\omega \in \Omega$ and any $f \in L^1_0(X, m)$, it holds that

$$\lim_{n \to \infty} \left\| P^{(n)} f \right\|_{L^1(X)} = 0. \quad (11)$$

As in remark 1, we can easily see that the exactness of a Markov operator cocycle $(P, \sigma)$ is equivalent to that for $\mathbb{P}$-almost every $\omega \in \Omega$ and any $f \in D(X, m)$,

$$\lim_{n \to \infty} \left\| P^{(n)} f - h_{\sigma^m} f \right\|_{L^1(X)} = 0.$$

In section 3, we will see another equivalent condition of the exactness in the case when $(P, \sigma)$ is associated with a random dynamical system on $X$. The relationship between mixing, exactness and asymptotic stability will be also discussed in section 4, see again figure 1 for a summary.

2. Mixing

2.1. Equivalence

We show the equivalence between prior/posterior mixing for homogeneous observables and prior/posterior mixing for inhomogeneous observables in $C(\Omega, L^\infty(X, m))$ when $\Omega$ is a compact topological space.

**Theorem 2.1.** Assume that $\Omega$ is a compact topological space. Then, the followings are equivalent:

(a) $(P, \sigma)$ is prior mixing for homogeneous observables.

(b) $(P, \sigma)$ is posterior mixing for homogeneous observables.

(c) $(P, \sigma)$ is prior mixing for inhomogeneous observables in $C(\Omega, L^\infty(X, m))$.

(d) $(P, \sigma)$ is posterior mixing for inhomogeneous observables in $C(\Omega, L^\infty(X, m))$.

**Proof.** As mentioned, the implications (c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (b) immediately follow from the definitions. We show (b) $\Rightarrow$ (a). Assume that $(P, \sigma)$ is posterior mixing for homogeneous observables, that is, for any $f \in L^1_0(X, m)$ and $g \in L^\infty(X, m)$, there is a measurable set $\Omega_0(f, g)$ such that $\mathbb{P}[(\omega, f, g)] = 1$ and (7) holds for each $\omega \in \Omega_0(f, g)$. By the simple function approximation with rational coefficients, we can find countable dense subsets $\{f_k\}_{k \in \mathbb{N}}$ of $L^1_0(X, m)$
and \( \{g_l\}_{l \in \mathbb{N}} \) of \( L^\infty(X, m) \). Define a measurable set \( \Omega_0 \) by
\[
\Omega_0 = \bigcap_{k \in \mathbb{N}} \bigcap_{l \in \mathbb{N}} \Omega_0(f_k, g_l).
\]
then it is straightforward to see that \( \mathbb{P}(\Omega_0) = 1 \) and (7) holds for any \( \omega \in \Omega_0 \), \( f \in L^1_1(X, m) \) and \( g \in L^\infty(X, m) \), i.e. \((P, \sigma)\) is prior mixing for homogeneous observables.

We next show (a) \( \Rightarrow \) (c). Assume that \((P, \sigma)\) is prior mixing for homogeneous observables, that is, there is a measurable set \( \Omega_0 \) with \( \mathbb{P}(\Omega_0) = 1 \) such that (7) holds for any \( \omega \in \Omega_0 \), \( f \in L^1_1(X, m) \) and \( g \in L^\infty(X, m) \). Fix such an \( \Omega_0 \). Fix also \( f \in L^1_1(X, m), g \in C(\Omega, L^\infty(X, m)) \) and \( \epsilon > 0 \). Then, since \( \Omega \) is compact, we get finitely many functions \( \{g_l\}_{l=1}^\infty \subset L^\infty(X, m) \) such that, for any \( \omega \in \Omega \) there is \( 1 \leq i(\omega) \leq I \) satisfying
\[
\|g_\omega - g_{i(\omega)}\|_{L^\infty(X)} < \epsilon.
\]
(Note that \( \{\omega \in \Omega : \|g_\omega - \tilde{g}\|_{L^\infty(X)} < \epsilon\} : \tilde{g} \in L^\infty(X)\) is an open covering of \( \Omega \) by virtue of the continuity of \( g \).) For convenience, let \( g_0 = 1_X \).

We further fix \( \omega \in \Omega_0 \). By applying (7) to \( g = g_i \) with \( 0 \leq i \leq I \), one can find \( N_i \equiv N_i(\omega, f) \in \mathbb{N} \) such that
\[
\left| \int_X P^{\sigma_0}_\omega f g_i \, dm \right| < \epsilon \quad \text{for all } n \geq N_i.
\]
Hence, for any \( n \geq \max_{0 \leq i \leq I} N_i \),
\[
\left| \int_X P^{\sigma_0}_\omega f g_{n(\omega)} \, dm \right| \leq \left| \int_X P^{\sigma_0}_\omega f g_{i(\omega)} \, dm \right| + \|g_{i(\omega)} - g_{n(\omega)}\|_{L^\infty(X)} \|P^{\sigma_0}_\omega f\|_{L^1(X)} < (1 + \|f\|_{L^1(X)}) \epsilon.
\]
Since \( \epsilon > 0 \) is arbitrary, we conclude
\[
\lim_{n \to \infty} \int_X P^{\sigma_0}_\omega f g_{n(\omega)} \, dm = 0 \quad \text{for all } \omega \in \Omega_0,
\]
which implies that \((P, \sigma)\) is prior mixing for inhomogeneous observables in \( C(\Omega, L^\infty(X, m)) \). This completes the proof.

Remark 3. As in the proof, the compactness of \( \Omega \) in theorem 2.1 is only needed to show the implication of prior mixing for inhomogeneous observables in \( C(\Omega, L^\infty(X, m)) \) from prior mixing for homogeneous observables.

We also can show the following equivalence for observables in \( L^\infty(\Omega, L^\infty(X, m)) \).

Theorem 2.2. If \((P, \sigma)\) is posterior mixing for homogeneous observables, then \((P, \sigma)\) is posterior mixing for inhomogeneous observables in \( L^\infty(\Omega, L^\infty(X, m)) \).

Proof. Assume that \((P, \sigma)\) is posterior mixing for homogeneous observables, i.e. for any \( f \in L^1_1(X, m) \) and \( g \in L^\infty(X, m) \), there is a measurable set \( \Omega_0(f, g) \) such that \( \mathbb{P}(\Omega_0(f, g)) = 1 \) and (7) holds for any \( \omega \in \Omega_0(f, g) \). Fix \( f \in L^1_1(X, m) \) and \( g \in L^\infty(\Omega, L^\infty(X, m)) \). We only consider the case when \( g_+ \) is positive for \( \mathbb{P} \times m\)-almost every \( (\omega, x) \in \Omega \times X \). (If not, we consider the usual decomposition \( g = g^+ - g^- \) with \( g^+_\omega(x) = \max\{g_\omega(x), 0\} \) and \( g^-_\omega(x) = \max\{-g_\omega(x), 0\} \).)
Thus, for any \( \text{Ni} \) homogeneous observables in \( B \), we give an example exhibiting prior mixing for homogeneous observables but not for inhomogeneous observables. Since \( g \in L^\infty(\Omega, L^\infty(X, m)) \) (in particular, \( g \) is Bochner measurable), there is a sequence of simple functions \( \{ g^k \}_{k \in \mathbb{N}} \subset L^\infty(\Omega, L^\infty(X, m)) \) of the form

\[
g^k(x) = \sum_{i=1}^{I(k)} h^k_i(x) 1_{\mathscr{F}}(\omega) \quad (g^k_i \in L^\infty(X, m), \ F^k_i \in \mathcal{F})
\]

and a \( \mathbb{P} \)-full measure set \( \Omega_1 \) such that \( \sup_{\omega \in \Omega_1} \| g^k - g^k_i \|_{L^\infty(X)} \to 0 \) as \( k \to \infty \). Define a \( \mathbb{P} \)-full measure set \( \Omega_0 \) by

\[
\Omega_0 = \bigcap_{k \in \mathbb{N}} \bigcap_{1 \leq i \leq I(k)} \Omega_0(f, g^k_i).
\]

Let \( \Omega_2 = \Omega_0 \cap (\bigcap_{n \geq 0} \sigma^{-n} \Omega_1) \), then \( \mathbb{P}(\Omega_2) = 1 \) by the invariance of \( \mathbb{P} \) for \( \sigma \).

Fix \( \omega \in \Omega_2 \) and \( \epsilon > 0 \). Fix also \( k \in \mathbb{N} \) such that

\[
\| g^k \sigma^n - g^k \sigma^{n+1} \|_{L^\infty(X)} < \epsilon \quad \text{for all } n \in \mathbb{N}.
\]

Calculate that

\[
\left| \int_X p^{01}_{\omega} f g_{\sigma^n \omega} \, dm \right| \leq \sum_{i=1}^{I(k)} \left| \int_X p^{01}_{\omega} f g^k_i \, dm \right| 1_{\mathscr{F}}(\sigma^n \omega)
\]

\[
\leq I(k) \max_{1 \leq i \leq I(k)} \left| \int_X p^{01}_{\omega} f g^k_i \, dm \right|.
\]

On the other hand, by the choice of \( \omega \), for any \( 1 \leq i \leq I(k) \) one can find a positive integer \( N_i = N_i(f, \omega, k) \) such that if \( n \geq N_i \), then

\[
\left| \int_X p^{01}_{\omega} f g^k_i \, dm \right| < \frac{\epsilon}{I(k)}.
\]

Thus, for any \( n \geq \max_{1 \leq i \leq I(k)} N_i \),

\[
\left| \int_X p^{01}_{\omega} f g_{\sigma^n \omega} \, dm \right| \leq \left| \int_X p^{01}_{\omega} f g^k_i \, dm \right| + \| g^k \sigma^n - g^k \sigma^{n+1} \|_{L^\infty(X)} \left\| p^{01}_{\omega} f \right\|_{L^1(X)}
\]

\[
< (1 + \left\| f \right\|_{L^1(X)}) \epsilon.
\]

Since \( \epsilon > 0 \) is arbitrary, we conclude that \( P, \sigma \) is prior mixing for inhomogeneous observables in \( L^\infty(\Omega, L^\infty(X, m)) \).

\[
\square
\]

2.2. Counterexamples

We give an example exhibiting prior mixing for homogeneous observables but not for inhomogeneous observables in \( B(\Omega, L^\infty(X, m)) \). Let \( T : X \to X \) be a measurably bijective map (up to zero \( m \)-measure sets) preserving \( m \) such that the Perron–Frobenius operator \( L_T \) associated with \( T \) is mixing (note that \( L_T 1_X = 1_X \) due to the invariance of \( m \) and recall (2)). Note that the baker map is well-known example as such map \( T \). Assume that there is a \( \mathbb{P} \)-positive measure set \( \Omega_0 \) such that the forward orbit of \( \omega \in \Omega_0 \) is measurable but not finite (e.g. \( \Omega = [0, 1] \), \( \mathbb{P} \) is the Lebesgue measure on \( \Omega \) and \( \sigma \) is the tent map), and that \( P_\omega = L_T \) for all
ω ∈ Ω₀. By construction, this Markov operator cocycle (P, σ) is prior mixing for homogeneous observables.

**Theorem 2.3.** The Markov operator cocycle (P, σ) given above is not prior mixing for inhomogeneous observables in B(Ω, L∞(X, m)).

**Proof.** We first note that the negation of prior mixing for inhomogeneous observables in B(Ω, L∞(X, m)) is that there is a measurable set Γ ⊂ Ω with |P(Γ)| > 0 such that for any ω ∈ Γ, there exist f = fω in L₀(X, m) and a bounded measurable map g = gω : Ω → L∞(X, m) : ω → gω satisfying

\[
\lim_{n→∞} \int_X P^{(n)} f g^{σ^n ω} dm = \lim_{n→∞} \int_X P^{(n)} f\tilde{g}^{σ^n ω} dm \neq 0.
\]

We emphasize that the observable g = gω may depend on ω ∈ Γ.

Let Γ = Ω₀ and fix ω ∈ Γ. Fix a measurable set A with m(A) = 1/2. Let f = 1_A − 1_{X\setminus A}.

Define \( g_ω = \begin{cases} L^n 1_A & \text{(when } \tilde{ω} = σ^n ω) \\ 0 & \text{(otherwise).} \end{cases} \)

Then, by construction, \( f ∈ L₀(X, m) \) and \( g : Ω → L∞(X, m) \) is a bounded and measurable map. Furthermore, since T is bijective, for every \( n ∈ \mathbb{N} \),

\[
L^n 1_A \cdot L^n 1_{X\setminus A} = 0 \quad m\text{-almost everywhere}
\]

(note that \( L^n 1_B = 1_{T^n(B)} \) for any measurable set B). Therefore, for every \( n ∈ \mathbb{N} \)

\[
\int_X P^{(n)} f g^{σ^n ω} dm = \int_X L^n (1_A − 1_{X\setminus A}) L^n 1_A dm
\]

\[= \int_X (L^n 1_A)^2 dm = m(T^n(A)) = \frac{1}{2} > 0. \]

In conclusion, (P, σ) is not prior mixing for inhomogeneous observables in B(Ω, L∞(X, m)). □

### 2.3. Skew-product transformations

In this subsection, we show that our definitions of mixing for Markov operator cocycles naturally lead to the conventional mixing property for skew-product transformations.

Recall that \((X, A, m)\) and \((Ω, F, P)\) are probability spaces, and \(σ : Ω → Ω\) is a \(P\)-preserving transformation. We further assume that \(σ\) is invertible and mixing. Let \((P, σ)\) be a Markov operator cocycle induced by the Perron–Frobenius operator corresponding to a non-singular transformation \(T_ω : X → X\) for \(P\)-almost every \(ω ∈ Ω\). Assume that there is an invariant density map \(h : Ω → D(X, m)\) of \((P, σ)\) and define a measurable family of measures \(\{μ_ω\}_{ω ∈ Ω}\) by \(μ_ω(A) = \int_A h_ω dm\) for \(A ∈ A\), so that we have \((T_ω)_*μ_ω = μ_ω\) due to (6).

Consider the skew-product transformation \(Θ : Ω × X → Ω × X\) defined by \(Θ(ω, x) = (σω, T_ω x)\) with the measure \(ν\) on \(Ω × X\),

\[
ν(A) = \int_Ω μ_ω(ν(A_ω)) dP(ω) \quad \text{for } A ∈ F ⊗ A,
\]

\[\text{where } ν(A) := \int_Ω μ_ω(A_ω) dP(ω) \quad \text{for } A ∈ F ⊗ A\]
where $A_\omega := \{ x \in X : (\omega, x) \in A \}$ denotes the $\omega$-section. Then, $(\Omega \times X, \mathcal{F} \otimes \mathcal{A}, \nu)$ becomes a probability space, and $\nu$ is an invariant measure for $\Theta$, namely the Perron–Frobenius operator $L_\Theta$ corresponding to $\Theta$ with respect to $\nu$ satisfies $L_\Theta 1_{\Omega \times X} = 1_{\Omega \times X} \mu$-almost everywhere.

**Theorem 2.4.** If $(P, \sigma)$ is prior mixing for inhomogeneous observables in $B(\Omega, L^\infty(X, m))$, then $\Theta$ is mixing, that is, for any $A, B \in \mathcal{F} \otimes \mathcal{A}$,

$$\lim_{n \to \infty} \nu(\Theta^{-n} A \cap B) = \nu(A) \nu(B).$$  \hspace{1cm} (12)

**Proof.** Let $1_{B_\omega}/\mu_\omega(B_\omega) \in D(X, \mu_\omega)$ so that $1_{B_\omega}/\mu_\omega(B_\omega) \in D(X, m)$. Assuming that $(P, \sigma)$ is prior mixing for inhomogeneous observables in $B(\Omega, L^\infty(X, m))$, we then know

$$\lim_{n \to \infty} \int_X \left( P^{(n)} \left( \frac{1_{B_\omega}/\mu_\omega(B_\omega)}{h_\omega} \right) - h_\omega \right) 1_{\lambda_{\sigma^m}} \, dm = 0.$$  

Let $\hat{P}_\omega : L^1(X, \mu_\omega) \to L^1(X, \mu_\omega)$ be the normalized Markov operator defined by

$$\hat{P}_\omega f(x) = \begin{cases} \frac{P_n(f h_\omega)(x)}{h_\omega(x)} & (x \in X^\omega) \\ 0 & \text{(otherwise)} \end{cases}$$

where $X^\omega := \text{supp} \ h_\omega$. Note that the relation $\hat{P}_\omega 1_{X^\omega} = 1_{X^\omega}$ holds for almost every $\omega \in \Omega$. Then we have,

$$\lim_{n \to \infty} \left( \int_X \hat{P}_\omega^{(n)} 1_{B_\omega} \cdot 1_{\lambda_{\sigma^m}} \, d\mu_\omega - \int_X 1_{\lambda_{\sigma^m}} \, d\mu_\omega \int_X 1_{B_\omega} \, d\mu_\omega \right) = 0.$$  \hspace{1cm} (13)

The first term can be calculated as

$$\int_X \hat{P}_\omega^{(n)} 1_{B_\omega} \cdot 1_{\lambda_{\sigma^m}} \, d\mu_\omega = \int_X \left( \hat{P}_\omega^{(n)} 1_{B_\omega} \right) \cdot h_\omega \cdot 1_{\lambda_{\sigma^m}} \, dm$$

$$= \int_X \left( \hat{P}_\omega^{(n)} 1_{B_\omega} \cdot h_\omega \right) \cdot 1_{\lambda_{\sigma^m}} \, dm$$

$$= \int_X 1_{B_\omega} \cdot h_\omega \cdot P_\sigma^{\star} \circ \cdots \circ P_{\sigma^{n-1}}^{\star} 1_{\lambda_{\sigma^m}} \, dm$$

$$= \int_X 1_{B_\omega} \cdot P_\sigma^{\star} \circ \cdots \circ P_{\sigma^{n-1}}^{\star} 1_{\lambda_{\sigma^m}} \, d\mu_\omega$$

Thus,

$$\lim_{n \to \infty} \left( \int_X 1_{B_\omega} \cdot P_\sigma^{\star} \circ \cdots \circ P_{\sigma^{n-1}}^{\star} 1_{\lambda_{\sigma^m}} \, d\mu_\omega - \int_X 1_{\lambda_{\sigma^m}} \, d\mu_\omega \int_X 1_{B_\omega} \, d\mu_\omega \right) = 0,$$

where $P_\sigma^{\star} : L^\infty(X, m) \to L^\infty(X, m)$ is the Koopman operator with respect to $T_\sigma$. Note that this implies the following natural mixing property for the random dynamical system $\{T_\omega\}_{\omega \in \Omega}$,

$$\lim_{n \to \infty} \left( \mu_\omega \left( T_\omega^{(-n)}A_{\sigma^m} \cap B_\omega \right) - \mu_{\sigma \omega}(A_{\sigma^m}) \mu_\omega(B_\omega) \right) = 0,$$  \hspace{1cm} (14)

where $T_\omega^{(-n)} = T_\sigma^{(-1)} \circ \cdots \circ T_{\sigma^{n-1}}^{(-1)}$. Since $\mathbb{P}$ is a probability measure, by the Lebesgue dominated convergence theorem,

$$\lim_{n \to \infty} \left( \int_\Omega \mu_\omega \left( T_\omega^{(-n)}A_{\sigma^m} \cap B_\omega \right) \, d\mathbb{P}(\omega) - \int_\Omega \mu_{\sigma \omega}(A_{\sigma^m}) \mu_\omega(B_\omega) \, d\mathbb{P}(\omega) \right) = 0.$$  \hspace{1cm} (15)
By using $\Theta^{-n}(A) = \bigcup_{\omega \in \Omega} (\sigma^{-n}_\omega A) = \bigcup_{\omega \in \Omega} (\sigma^{-n_1}_\omega \cdots \sigma^{-1}_\omega A_\omega)$ and $B = \bigcup_{\omega \in \Omega} (\omega, B_\omega) = \bigcup_{\omega \in \Omega} (\omega, B_{\sigma^{-n}_\omega})$, we have

$$
\nu(\Theta^{-n} A \cap B) = \int_{\Omega} \mu_{\sigma^{-n}_\omega}(T_{\sigma^{-1}_\omega} \cdots T_{\sigma^{-1}_\omega} A_\omega \cap B_{\sigma^{-n}_\omega})d\mathbb{P}(\omega)
= \int_{\Omega} \mu_{\omega}(T^{(-n)}_{\omega} A_{\sigma^{-n}_\omega} \cap B_\omega) \ d\mathbb{P}(\omega).
$$

On the other hand, since $\sigma$ is mixing, invertible and $\mathbb{P}$-preserving,

$$
\lim_{n \to \infty} \int_{\Omega} \mu_{\omega}(A_\omega)d\mathbb{P} \int_{\Omega} \mu_{\omega}(B_\omega)d\mathbb{P} = \lim_{n \to \infty} \int_{\Omega} \mu_{\omega}(A_\omega)L^n(\mu_{\omega}(B_\omega))d\mathbb{P}(\omega)
= \int_{\Omega} \mu_{\omega}(A_\omega)d\mathbb{P} \int_{\Omega} \mu_{\omega}(B_\omega)d\mathbb{P}(\omega)
= \nu(A)\nu(B),
$$

where $L_\sigma : L^1(\Omega, \mathbb{P}) \to L^1(\Omega, \mathbb{P})$ is the Perron–Frobenius operator of $\sigma$. Therefore we obtain $\nu(\Theta^{-n}A \cap B) \to \nu(A)\nu(B)$ as $n \to \infty$ for $A, B \in \mathcal{F} \otimes A$.

**Remark 4.** The converse of theorem 2.4 is not true in general due to the example given in subsection 2.2. Indeed, let $\Theta$ be the direct product of two baker maps with $\Omega = (0, 1]^2$, Leb$(0, 1]^2$). Then, the Markov operator cocycle $(P, \sigma)$ induced by $\Theta$ is not prior mixing for inhomogeneous observables in $B(\Omega, L^\infty(X, m))$ due to theorem 2.3. On the other hand, $\Theta$ is mixing because the backer map is mixing and the direct product of two same mixing systems is also mixing (cf. [15]).

**Remark 5.** In the case of prior mixing for homogeneous observables, as in the proof of theorem 2.4, we can derive the convergence

$$
\nu(\Theta^{-n}(F_1 \times A_1) \cap (F_2 \times A_2)) \to \nu(F_1 \times A_1)\nu(F_2 \times A_2) \quad (n \to \infty) \quad (15)
$$

for any $F_1, F_2 \in \mathcal{F}$ and $A_1, A_2 \in \mathcal{A}$.

On the other hand, (15) implies the conventional mixing of $\Theta$ by a standard approximation of measurable sets in $\mathcal{F} \otimes \mathcal{A}$ by finite union of direct product sets\footnote{Fix $A, B \in \mathcal{F} \otimes \mathcal{A}$ and $\epsilon > 0$. Then, one can find $\{F_{i,1}^{(j)}\}_{j=1}^J, \{F_{i,2}^{(j)}\}_{j=1}^J \subset \mathcal{F}$ and $\{A_{i,1}^{(j)}\}_{j=1}^J, \{A_{i,2}^{(j)}\}_{j=1}^J \subset \mathcal{A}$ with $J \in \mathbb{N}$ such that both $\{A_{i,1}^{(j)} \times F_{i,2}^{(j)}\}_{j=1}^J$ and $\{A_{i,2}^{(j)} \times F_{i,1}^{(j)}\}_{j=1}^J$ are pairwise disjoint and both $\nu(A_{i,1}^{(j)} \times F_{i,2}^{(j)})$ and $\nu(A_{i,2}^{(j)} \times F_{i,1}^{(j)})$ are bounded by $\epsilon$. By taking finer partition if necessary, one can assume that $\{F_{i,1}^{(j)}\}_{j=1}^J$ is also pairwise disjoint. Since $\theta$ is invertible, $\{\Theta^n(F_{i,1}^{(j)} \times A_{i,2}^{(j)})\}_{j=1}^J$ is again pairwise disjoint for each $n \geq 0$. Hence, $\nu(\Theta^{-n} A \cap B) \to \nu(A)\nu(B)$ is $\epsilon$-close to

$$
\sum_{j=1}^J (\nu(\Theta^{-n}(F_{i,1}^{(j)} \times A_{i,2}^{(j)}) \cap (F_{i,2}^{(j)} \times A_{i,1}^{(j)})) - \nu(F_{i,1}^{(j)} \times A_{i,2}^{(j)})\nu(F_{i,2}^{(j)} \times A_{i,1}^{(j)}))
$$

whose absolute value is smaller than $\epsilon$ for any sufficiently large $n$ by (15). Since $\epsilon$ is arbitrary, this implies (12), that is, the mixing of $\Theta$.}.
Therefore, if $\Theta$ is mixing, then for any $f \in L^1_0(X, m)$, $g \in L^\infty(X, m)$ and $\rho \in L^\infty(\Omega, \mathbb{P})$, by applying (16) to

$$\tilde{f}(\omega, x) = \begin{cases} f(x) & (x \in \text{supp}(h_\omega)) \\ 0 & (x \notin \text{supp}(h_\omega)) \end{cases} \text{ and } \tilde{g}(\omega, x) = \rho(\omega) g(x),$$

it follows from (4) and the invariance of $\mathbb{P}$ for $\theta$ that

$$\int_{\Omega} \rho(\omega) \left( \int_X P_{\sigma^n \omega}^\theta f \, g \, dm \right) d\mathbb{P}(\omega) \to 0 \quad (n \to \infty).$$

Since $\rho$ is arbitrary, this immediately implies the prior mixing of $(P, \sigma)$.

In conclusion, the prior mixing of $(P, \sigma)$ for homogeneous observables, (15) and the mixing of $\Theta$ are equivalent, and thus, due to the relationship summarized in figure 1, the prior/posterior mixing for (in)homogeneous observables in every class considered in this paper implies the conventional mixing of $\Theta$.

By theorems 2.1 and 2.2 together with remark 3, prior mixing for homogeneous observables is equivalent to posterior mixing for inhomogeneous observables in $L^\infty(\Omega, L^\infty(X, m))$, which is equivalent to posterior mixing for inhomogeneous observables in $B(\Omega, L^\infty(X, m))$ by definition (refer to remark 2). Therefore, by remarks 4 and 5 we obtain the following corollary.

**Corollary 2.5.** Let $(P, \sigma)$ be the Markov operator cocycle given in remark 4. Then $(P, \sigma)$ is posterior mixing for inhomogeneous observables in $B(\Omega, L^\infty(X, m))$ but not prior mixing for inhomogeneous observables in $B(\Omega, L^\infty(X, m))$.

**Remark 6.** As we will see in section 4, when $(P, \sigma)$ is an asymptotically periodic Markov operator cocycle, then the posterior mixing for inhomogeneous observables in $B(\Omega, L^\infty(X, m))$ is equivalent to the prior mixing for inhomogeneous observables in $B(\Omega, L^\infty(X, m))$. This is contrastive to corollary 2.5. On the other hand, the backer map, being the fibre dynamics of the counterexample in corollary 2.5, is a well-known example whose Perron–Frobenius operator is not asymptotically periodic.

### 2.4. Problems

We finally propose a related problem. Our definitions of mixing were given in terms of the decay of the correlation between $P_{\sigma^nf}$ and $g$ (or $g_{\sigma^nf}$), and it is of great importance to evaluate the rate of decay, as seen in the previous works [3, 4, 7, 12]. Thus, we pose the following problem:

**Problem 1.** Investigate the relationship between decay rates of correlations for each type of mixing in definition 1.2.

All results introduced in remark 1 established not only mixing property but also exponential mixing (for expanding or hyperbolic maps). As mentioned there, these results include both prior and posterior mixing for both homogeneous and inhomogeneous observables. We also remark that Froyland et al recently developed a multiplicative ergodic theorem for semi-invertible operator cocycles in [10, 11], which enabled one to consider the ‘quasi-compactness’ of transfer operator cocycles in terms of Lyapunov exponents and played the key role in the establishment of exponential mixing (and its consequences such as several limit theorems) for random expanding or hyperbolic dynamical systems in [7, 8].
3. Exactness

As a characterization of exactness which is well-known for one non-singular transformation (see [1]), we have the generalization of Lin’s theorem [14] as follows. For each $\omega \in \Omega$, $P_\omega$ denotes the adjoint operator of $P_\omega$, defined by

$$\int_X P_\omega f g \, dm = \int_X f P^*_\omega g \, dm$$

for $f \in L^1(X, m)$ and $g \in L^\infty(X, m)$, and we will use the notation

$$P^{(n)}_\omega = P^*_\omega \circ P^*_\omega \circ \cdots \circ P^*_\omega$$

for $\omega \in \Omega$ and $n \geq 1$.

**Theorem 3.1.** Let $(P, \sigma)$ be a Markov operator cocycle and $S = \{g \in L^\infty(X, m) : \|g\|_{L^\infty} \leq 1\}$ the unit ball in $L^\infty(X, m)$. Then the following are equivalent for each $\omega \in \Omega$.

(a) $f \in L^1(X, m)$ satisfies $\|P^{(n)}_\omega f\|_{L^1(X)} \to 0$ as $n \to \infty$;

(b) $f \in L^1(X, m)$ satisfies $\int_X f g \, dm = 0$ for any $g \in \bigcap_{n \geq 1} P^{(n)}_\omega S$.

Consequently, $(P, \sigma)$ is exact if and only if $\bigcap_{n \geq 1} P^{(n)}_\omega S = \{c1_X : c \in \mathbb{R}\}$ for $P$-almost every $\omega \in \Omega$.

**Proof.** First of all, notice that for any $\omega \in \Omega$, $P^*_\omega S \subset S$ and $P^{(n)}_\omega = P^*_\omega \circ P^*_\omega \circ \cdots \circ P^*_\omega$ enable us to have the decreasing sequence in $L^\infty(X, m)$:

$$S \supset P^*_\omega S \supset P^{(2)}_\omega S \supset \cdots \supset \bigcap_{n \geq 1} P^{(n)}_\omega S.$$

Now we assume (a) is true. Then for each $g \in \bigcap_{n \geq 1} P^{(n)}_\omega S$, there is a sequence $\{g_n\}_n \subset S$ so that $P^{(n)}_\omega g_n = g$ and for $f$ in the condition (a),

$$\int_X f g \, dm = \int_X f P^{(n)}_\omega g_n \, dm = \int_X P^{(n)}_\omega f g_n \, dm \leq \|P^{(n)}_\omega f\|_{L^1(X)} \to 0$$

as $n \to \infty$. Thus (b) is valid.

Next, suppose that (b) holds. By the Banach–Alaoglu theorem and continuity of $P^*_\omega$ on the weak-* topology in $L^\infty(X, m)$, $S$ is compact in weak-* and so is $P^{(n)}_\omega S$. For $f$ in the condition (b), taking $g_n = \text{sgn} \left( P^{(n)}_\omega f \right) \in S$ where $\text{sgn}(\phi) = 1$ on $\{\phi \geq 0\}$ and $\text{sgn}(\phi) = -1$ otherwise, we have

$$\|P^{(n)}_\omega f\|_{L^1(X)} = \int_X P^{(n)}_\omega f g_n \, dm = \int_X f P^{(n)}_\omega g_n \, dm.$$

Let $g$ be an accumulation point of $\{P^{(n)}_\omega g_n\}_n$ which belongs to $\bigcap_{n \geq 1} P^{(n)}_\omega S$. Then we have $\int_X f g \, dm = 0$ by assumption (b) and for some subsequence $\{n_i\}_i \subset \mathbb{N}$, we have

$$\lim_{i \to \infty} \|P^{(n_i)}_\omega f\|_{L^1(X)} = \lim_{i \to \infty} \int_X f P^{(n_i)}_\omega g_{n_i} \, dm = \int_X f g \, dm = 0.$$

Since $P_\omega$ is Markov, $\|P^{(n)}_\omega f\|_{L^1(X)} \leq \|P^{(n)}_\omega f\|_{L^1(X)}$ for $n \geq n_i$. Therefore we have the condition (a).
Finally, considering the case when \( f \in L^1(X, m) \), we have the equivalent condition for exactness of \((P, \sigma)\) and the proof is completed. □

As an immediate corollary of theorem 3.1, we have:

**Corollary 3.2.** If a Markov operator cocycle \((P, \sigma)\) is derived from non-singular transformations \( T_\omega \), that is, each \( P_\omega \) is the Perron–Frobenius operator associated to \( T_\omega \). Then \((P, \omega)\) is exact if and only if for \( P\)-almost every \( \omega \in \Omega \),

\[
\bigcap_{n \geq 1} (T_\omega^{(n)})^{-1} \mathcal{A} = \{0, X\} \quad (\text{mod } m).
\]

**Proof.** Since \( P_\omega \) is the Koopman operator of \( T_\omega \), characteristic functions are mapped to characteristic functions. Thus we can consider \( P_\omega \) on \( \{ g \in L^\infty(X, m) : \|g\|_{L^\infty(X)} = 1 \} \) and we prove the corollary. □

4. Asymptotic periodicity

In the arguments of conventional Markov operators, it is known that mixing and exactness are equivalent properties in the asymptotically periodic class [13]. In this section, we consider a similar result to the conventional one for our definitions of mixing and exactness for Markov operator cocycles under the following definition of asymptotic periodicity. Moreover, in the sequel of the section, we introduce the relation between the asymptotic periodicity and exactness from the viewpoint of the existence of an invariant density.

**Definition 4.1 (asymptotic periodicity).** A Markov operator cocycle \((P, \sigma)\) is said to be asymptotically periodic if there exist an integer \( r \), finite collections \( \{\lambda_i\}_{i=1}^r \subset B(\Omega, (L^1(X, m))) \) and \( \{\varphi_i\}_{i=1}^r \subset B(\Omega, D(X, m)) \) satisfying that \( \{\varphi_i\}_{i=1}^r \) have mutually disjoint supports for \( P\)-almost every \( \omega \in \Omega \), and there exists a permutation \( \rho_\omega \) of \( \{1, \ldots, r\} \) such that

\[
P_\omega \varphi_i \omega = \varphi_{\rho_\omega(i)} \omega \quad \text{and} \quad \lim_{n \to \infty} \left\| P_\omega^{(n)} \left( f - \sum_{i=1}^r \lambda_i(f) \varphi_i \omega \right) \right\|_{L^1(X)} = 0 \quad (17)
\]

for every \( f \in L^1(X, m) \), \( 1 \leq i \leq r \) and \( P\)-almost every \( \omega \in \Omega \), where \( \lambda_i \omega = \lambda_i(\omega) \), \( \varphi_i \omega = \varphi_i(\omega) \).

Furthermore, if in addition \( r = 1 \), then \((P, \sigma)\) is said to be asymptotically stable.

Note that when \((P, \sigma)\) is asymptotically periodic,

\[
h_\omega = \frac{1}{r} \sum_{i=1}^r \varphi_i \omega
\]

becomes an invariant density for \((P, \sigma)\).

For an asymptotically periodic single Markov operator, exactness and mixing coincide with \( r = 1 \) for the representation of asymptotic periodicity (see theorems 5.5.2 and 5.5.3 in [13]). The following theorem and proposition are Markov operator cocycles version of them.

**Theorem 4.2.** Let \((P, \sigma)\) be an asymptotically periodic Markov operator cocycle. Then the followings are equivalent.

(a) \((P, \sigma)\) is exact;

(b) \((P, \sigma)\) is prior mixing for inhomogeneous observables in \( B(\Omega, L^\infty(X)) \);
(c) $(P, \sigma)$ is posterior mixing for inhomogeneous observables in $B(\Omega, L^\infty(X))$;
(d) $(P, \sigma)$ is asymptotically stable.

**Proof.** (a) $\Rightarrow$ (b) $\Rightarrow$ (c): obvious.

(c) $\Rightarrow$ (d): suppose $(P, \sigma)$ is posterior mixing for inhomogeneous observables in $B(\Omega, L^\infty(X,m))$ and $r > 1$ (recall that $r$ is the period of the asymptotically periodic Markov operator cocycle $(P, \sigma)$ given in definition 4.1). By asymptotic periodicity of $(P, \sigma)$, we have an invariant density $h_\omega = \frac{1}{r} \sum_{i=1}^{r} \varphi_i^\omega$. Set $g_\omega = 1_{\text{supp } \varphi_i^\omega}$ and write

$$
\int_X P_\omega^n (h_\omega - \varphi_i^\omega) \cdot g_{\sigma^n \omega} \, dm = \int_{\text{supp } \varphi_i^\omega} (h_{\sigma^n \omega} - \varphi_{\sigma^n \varphi_i^{(1)}}^\omega) \, dm
$$

$$
= \frac{1}{r} - \int_{\text{supp } \varphi_i^\omega} \varphi_{\sigma^n \varphi_i^{(1)}}^\omega \, dm
$$

$$
= \begin{cases} 
\frac{1}{r} & (\rho_{\omega}^n(1) \neq 1) \\
\frac{1}{r} - 1 & (\rho_{\omega}^n(1) = 1)
\end{cases}
$$

for each $n \geq 0$. This contradicts posterior mixing for inhomogeneous observables in $B(\Omega, L^\infty(X))$ of $(P, \sigma)$.

(d) $\Rightarrow$ (a): we assume $(P, \sigma)$ is asymptotically periodic with $r = 1$. That is, for any $f \in L^1(\Omega, X,m)$, $\|P_\omega^n(f - \lambda^\omega(f) \varphi)^\omega)\|_{L^1(\Omega)} \rightarrow 0$ as $n \rightarrow \infty$. Since $P_\omega$ is Markov and $\varphi_\omega \in D(X, m)$ for each $\omega \in \Omega$, for any $f \in D(X, m)$ we have $\lambda^\omega(f) = 1$. Thus $(P, \sigma)$ is exact by remark 1. □

The following proposition reveals the relationship between two kinds of mixing and exactness. Namely, in the setting of asymptotically periodic systems mixing for homogeneous observables, mixing for inhomogeneous measurable observables and exactness are equivalent under certain topological assumption of $\Omega$.

**Proposition 4.3.** Let $(P, \sigma)$ be an asymptotically periodic Markov operator cocycle. Suppose $\sigma$ preserves a regular probability measure $\mathbb{P}$ on a metric space $(\Omega, F, \mathbb{P})$ with metric $d_\Omega$ and for $\mathbb{P}$-almost every $\omega \in \Omega$, each component of invariant densities $\varphi_i^\omega$ belongs to $C(\Omega, L^\infty(X, m))$. Then the condition that $(P, \sigma)$ is prior mixing for homogeneous observables is necessary and sufficient for each condition in theorem 4.2.

**Proof.** Necessity: obvious.

Sufficiency: we show $r = 1$ in the representation of asymptotic periodicity. Assume $r > 1$ contrarily. By our assumption, for $\mathbb{P}$-almost every $\omega \in \Omega$ and any $\epsilon > 0$ there exists $\delta > 0$ such that $d_\Omega(\omega, \omega') < \delta$ implies $\|\varphi_i^\omega - \varphi_i^{\omega'}\|_{L^1(\Omega)} < \epsilon$ for $i = 1, \ldots, r$. Also, we have that for $i \neq j$, if $d_\Omega(\omega, \omega') < \delta$ then

$$
\int_{\text{supp } \varphi_i^\omega} \varphi_j^{\omega'} \, dm \leq \|\varphi_i^\omega - \varphi_i^{\omega'}\|_{L^1(\Omega)} + \int_{\text{supp } \varphi_i^\omega} \varphi_j^\omega \, dm < \epsilon.
$$

Set a $\delta$-ball $B_\delta(\omega)$ centred at a given $\omega \in \Omega$ which is of positive measure since $\mathbb{P}$ is regular. Poincaré’s recurrence theorem tells us that $\sigma^n \omega$ visits $B_\delta(\omega)$ infinitely many times and let $\{n_k\}_k$ satisfy $\sigma^{n_k} \omega \in B_\delta(\omega)$. Then for an invariant density $h_\omega = \frac{1}{r} \sum_{i=1}^{r} \varphi_i^\omega$ and for $A = \text{supp } \varphi_i^\omega$, if $\omega'$
satisfies $d_{\Omega}(\omega, \omega') < \delta$,
\[
\left| \int_A h_{\omega'} \, dm \right| = \frac{1}{r} \left| \int_A \left( \varphi_{\omega'}^i - \varphi_{\omega'}^1 + \varphi_{\omega'}^i \right) \, dm + \sum_{i \neq 1} \int_A \varphi_{\omega'}^i \, dm \right|
\]
and we have
\[
\frac{1}{r} - \epsilon \leq \left| \int_A h_{\omega'} \, dm \right| \leq \frac{1}{r} + \epsilon.
\]
Therefore, taking $0 < \epsilon < \frac{1}{2}$ we have
\[
\left| \int_A P^{(\sigma)}_{\omega'} (\varphi_{\omega'}^i - h_{\omega}) \, dm \right| = \left| \int_A \left( \varphi_{\rho^i_{\omega'}(1)} - h_{\sigma^i_{\omega}} \right) \, dm \right|
\]
\[
\geq \begin{cases} 
\frac{1}{r} - 2\epsilon & (\rho^i_{\omega'}(1) \neq 1) \\
(1 - 2\epsilon) - \frac{1}{r} & (\rho^i_{\omega'}(1) = 1)
\end{cases}
\]
\[
> 0.
\]
This contradicts prior mixing of $(P, \sigma)$ for homogeneous observables. \hfill \Box

In order to relate asymptotic periodicity and exactness together with the existence of an invariant density, we give the definition of quasi-constrictiveness.

**Definition 4.4.** Let $(P, \sigma)$ be a Markov operator cocycle. Then $(P, \sigma)$ is called quasi-constrictive if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $E \in \mathcal{A}$ with $m(E) < \delta$, it holds that
\[
\lim_{n \to \infty} \sup_{\omega \in \Omega} \int_{E} P^{(\sigma)} f \, dm < \epsilon \quad \text{for any } f \in D(X, m).
\]

For the sake of convenience, for an asymptotically periodic Markov operator cocycle $(P, \sigma)$ and each component of the invariant density $\varphi_{\omega}$, we denote a measurable map $\omega \mapsto P^{(\sigma)}_{\omega} |_{\text{supp}\varphi_{\omega}}$ by $P^{(\sigma)}_{\text{supp}\varphi_{\omega}}$ for $k \in \mathbb{N}$ and $i = 1, \ldots, r$. In the following two propositions, we can see that (i): asymptotic periodicity of $(P, \sigma)$ is equivalent to (ii): the existence of an invariant density and exactness of $(P^{(\sigma)}_{\text{supp}\varphi_{\omega}}, \sigma^k)$ for some $k \in \mathbb{N}$ and all $i = 1, \ldots, r$.

**Proposition 4.5.** Let $(P, \sigma)$ be a Markov operator cocycle such that $P$ is strongly continuous i.e., $\omega \mapsto P_{\omega} f$ is continuous for each $f \in L^1(X, m)$. Suppose $\Omega$ is compact, $(P, \sigma)$ has an invariant density $h_{\omega}$ and $\{\mu_{\omega}\}_{\omega \in \Omega}$ is uniformly absolutely continuous with respect to $m$ where $d\mu_{\omega} = h_{\omega} \, dm$. If $(P^{(\sigma)}, \sigma^k)$ is exact for some $k \in \mathbb{N}$, then $(P, \sigma)$ is quasi-constrictive.

**Proof.** By the assumption, $(P, \sigma)$ admits an invariant density $\{h_{\omega}\}_{\omega}$ and there exists $k \in \mathbb{N}$ such that, for any $\psi \in L^1(X, m)$ and $\mathbb{P}$-almost every $\omega$, we have $\left\| P^{(\sigma)}_{\omega} \psi \right\|_{L^1(X)} \to 0$ as $n \to \infty$. We show $(P, \sigma)$ is quasi-constrictive. For any $N$ sufficiently large, $f \in D(X, m)$ and $E \in \mathcal{A}$,
\[
\sup_{n > N} \sup_{\omega \in \Omega} \int_{E} P^{(\sigma)}_{\omega} f \, dm
\]
\[
= \sup_{n_0, k > N} \sup_{\omega \in \Omega} \left\{ \int_{E} P^{(\sigma)}_{\omega} f \, dm, \int_{E} P^{(\sigma)}_{\omega} f \, dm, \ldots, \int_{E} P^{(\sigma)}_{\omega} f \, dm \right\}.
\]
For $j \in \{0, \ldots, k - 1\}$, we have
\[
\int_E P_{\omega}^{(nk+j)} f \, dm = \int_E P_{\sigma_{\omega}}^{(nk)} p_{\sigma_{\omega}}^{(j)} f \, dm \leq \left\| P_{\sigma_{\omega}}^{(nk)} \left( P_{\omega}^{(j)} f - h_{\sigma_{\omega}} \right) \right\|_{L^1(X)} + \int_E P_{\sigma_{\omega}}^{(nk)} h_{\sigma_{\omega}} \, dm.
\]
Thus, since $P_{\omega}^{(j)} f - h_{\sigma_{\omega}} \in L^1(X, m)$ and $\Omega$ is compact, we have
\[
\limsup_{n \to \infty} \sup_{\omega \in \Omega} \left\| P_{\omega}^{(nk)} \left( P_{\omega}^{(j)} f - h_{\sigma_{\omega}} \right) \right\|_{L^1(X)} \leq \limsup_{n \to \infty} \max_{0 \leq j \leq k-1} \left\| P_{\sigma_{\omega}}^{(nk)} \left( P_{\omega}^{(j)} f - h_{\sigma_{\omega}} \right) \right\|_{L^1(X)} + \max_{0 \leq j \leq k-1} \mu_{\omega_{\sigma}}(E).
\]
Indeed, if $\limsup_{n \to \infty} \sup_{\omega \in \Omega} \max_{0 \leq j \leq k-1} \left\| P_{\sigma_{\omega}}^{(nk)} \left( P_{\omega}^{(j)} f - h_{\sigma_{\omega}} \right) \right\|_{L^1(X)} \neq 0$, there exist $\psi \in L^1(X, m)$, $\epsilon_0 > 0$, $\{\omega_i\} \subset \Omega$ and $\{n_i\} \subset \mathbb{N}$ such that $\left\| P_{\omega_i}^{(nk)} \psi \right\|_{L^1(X)} \geq \epsilon_0$. Compactness of $\Omega$ ensures that there exists further subsequences $\omega'_{i \lambda} = \omega_i$ and $n'_{i \lambda} = n_i$, such that $\omega'_{i \lambda} \to \omega$ for some $\omega \in \Omega$. To $s$ tends to $\infty$. Now strong continuity of $P$ and exactness of $P^\epsilon$ imply that
\[
\left\| P_{\omega'_{i \lambda}}^{(nk)} \psi \right\|_{L^1(X)} \leq \left\| P_{\omega'_{i \lambda}}^{(nk)} \psi - P_{\omega_i}^{(nk)} \psi \right\|_{L^1(X)} + \left\| P_{\omega_i}^{(nk)} \psi \right\|_{L^1(X)} \to 0
\]
as $s \to \infty$ and this leads contradiction.

Uniform absolute continuity of $\{\mu_{\omega}\}$ with respect to $m$ implies that for each $\epsilon > 0$ there exists $\delta > 0$ such that if $m(E) < \delta$ then (18) is less than $\epsilon$. Therefore the desired result is obtained.

**Proposition 4.6.** Let $(P, \sigma)$ be an asymptotically periodic Markov operator cocycle such that the permutation $\mu_{\omega} \equiv \rho$ in definition 4.1 is constant $\mathbb{P}$-almost everywhere. Then $(P, \sigma)$ admits an invariant density and there exists a natural number $k$ such that $(P^k|_{\text{supp} \, \omega}, \sigma^k)$ is exact for $i = 1, \ldots, r$.

**Proof.** Obviously $h_{\omega} = \frac{1}{\tau} \sum_{i=1}^{\tau} \varphi_{i \omega}^\tau$ is an invariant density for $P_{\omega}$.

Now let $k$ be the smallest number satisfying $\rho^k = \text{id}$. Then, setting $A^\sigma_{i \omega} = \text{supp} \, \varphi_{i \omega}^\sigma$, $P^k_{\omega_{A^\sigma_{i \omega}}} = A^\sigma_{i \omega}$ is a Markov operator from $L^1(A^\omega_{i \omega}, m)$ into $L^1(A^\omega_{i \omega}, m)$. By representation of asymptotic periodicity of $P_{\omega}$, for any $f \in D(A^\omega_{i \omega}, m)$ we have that
\[
\lambda_{ij}^\omega(f) = \begin{cases} 1 & (j = i) \\ 0 & (j \neq i). \end{cases}
\]
This implies that for any $f \in D(A^\omega_{i \omega}, m)$,
\[
\lim_{n \to \infty} \left\| P_{\omega_{A^\sigma_{i \omega}}}^{(nk)} (f - \varphi_{ij}^\sigma) \right\|_{L^1(X)} = 0.
\]
Therefore we conclude $(P^k|_{\text{supp} \, \omega}, \sigma^k)$ is exact by remark 1. \(\square\)
Remark 7. As is well known (see [13] and references therein), quasi-constrictive single Markov operators were shown to be asymptotically periodic. If we can generalize their result for Markov operator cocycles, as a consequence of propositions 4.5 and 4.6 we expect the following when $\Omega$ is compact.

(a) If $(P, \sigma)$ admits an invariant density bounded below and above and $(P^k, \sigma^k)$ is exact for some $k \geq 1$, then $(P, \sigma)$ is asymptotically periodic.

(b) Conversely, if $(P, \sigma)$ is asymptotically periodic, then $(P, \sigma)$ admits an invariant density and there exists $k \geq 1$ such that $(P^k|_{\text{supp} \, \phi}, \sigma^k)$ is exact for $i = 1, \ldots, r$.

In particular, $(P, \sigma)$ is asymptotically periodic with period 1 if and only if $(P, \sigma)$ admits a unique invariant density and is exact.

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