Irrotational - fluid cosmologies in fourth-order gravity

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Abstract.

In this paper, we explore classes of irrotational cosmological models in the context of $f(R)$ gravity. In particular, we investigate the consistency of linearised dust models for shear-free cases as well as in the limiting cases when either the gravito-magnetic or gravito-electric components of the Weyl tensor vanish. We also discuss the existence and consistency of classes of non-expanding irrotational spacetimes.

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1. Introduction

The recent discovery of the accelerated rate of cosmic expansion has inspired a wave of new research into the nature of gravitational physics. New alternatives and/or generalisations to Einstein’s General Relativity (GR) theory abound already. One such generalisation comes in the form of $f(R)$-gravity theories. These are gravitational models based on the inclusion of higher-order curvature invariants in the Einstein-Hilbert action that result in fourth-order field equations [1, 2, 3, 4, 5, 6].

Recent attempts to explain away early-universe cosmic inflation [7] and late-time cosmic acceleration [8, 9, 10, 5, 11, 12, 13, 14, 15, 16, 17] using $f(R)$-gravity have brought the latter under severe scrutiny. Although the nonlinearity of the field equations has complicated the analysis of the detailed physics of these theories, recent cosmological applications include studies on the dynamics of homogenous cosmological models [18, 19, 20, 21, 22, 23] and the linear growth of large scale structures [24, 25, 26, 27, 28, 29, 30, 31, 32].

In order to understand the dynamics of nonlinear fluid flows in $f(R)$ theories, it is important to understand the relationship between their Newtonian and relativistic limits. This is relevant both in the physics of gravitational collapse and the late (nonlinear) stages of cosmic structure formation [33, 34, 35, 36, 37, 38]. The differential properties of time-like geodesics describe the fluid flows in cosmology [26, 39, 40]. The expansion $\Theta$, shear (distortion) $\sigma_{ab}$, rotation (vorticity) $\omega^a$, and acceleration $A_a$ of the four-velocity field $u^a$ tangent to the fluid flow lines describe kinematics of such fluid flows [as will be seen in Eqn (16) shortly]. The equations governing the fluid flows are of course obtained by contracting the Ricci identities along and orthogonal to $u^a$.

In this paper, we explore general properties of classes of irrotational spacetimes characterised by the vanishing of vorticity $\omega_a$, but generally non-vanishing energy density $\mu$, shear and a locally free gravitational field which is covariantly described by the gravito-electric (GE) and gravito-magnetic (GM) tensors, $E_{ab}$ and $H_{ab}$, respectively [35].

This paper is organised as follows: in Sec. 2 we give a covariant description and the general linearised (with respect to the Friedmann-Lemaître-Robertson-Walker (FLRW) background) field equations of $f(R)$-gravity models. In Sec. 3 we specialise to irrotational fluid spacetimes and analyse the properties of dust (in Sec. 3.1) and non-expanding (in Sec. 3.2) subclasses of spacetimes. Finally in Sec. 4 we discuss the results and give conclusions.

Natural units ($\hbar = c = k_B = 8\pi G = 1$) will be used throughout this paper, and Latin indices run from 0 to 3. The symbols $\nabla$, $\tilde{\nabla}$ and the overdot represent the usual covariant derivative, the spatial covariant derivative, and differentiation with respect to cosmic time. We use the $(-, +, +, +)$ signature and the Riemann tensor is defined by

$$R^a_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^c_{bd} \Gamma^a_{ce} - \Gamma^f_{be} \Gamma^a_{df},$$

where the $\Gamma^a_{bd}$ are the Christoffel symbols (i.e., symmetric in the lower indices), defined
Irrotational fluid cosmologies in fourth-order gravity

by

\[ \Gamma_{bd}^{a} = \frac{1}{2} g^{ae} (g_{be,d} + g_{ed,b} - g_{bd,e}) \, . \]  

(2)

The Ricci tensor is obtained by contracting the first and the third indices of the Riemann tensor:

\[ R_{ab} = g^{cd} R_{cadb} \, . \]

(3)

Unless otherwise stated, primes etc are shorthands for derivatives with respect to the Ricci scalar

\[ R = R^a_a \]

(4)

and \( f \) is used as a shorthand for \( f(R) \).

2. Covariant linearised Field Equations

The standard f(R) gravitational action with a matter field contribution to the Lagrangian, \( \mathcal{L}_m \), is given by

\[ \mathcal{A} = \frac{1}{2} \int d^4 x \sqrt{-g} [f(R) + 2 \mathcal{L}_m] \, . \]

(5)

Using the variational principle of least action with respect to the metric \( g_{ab} \), the generalised Einstein Field Equations (EFEs) can be given in a compact form as

\[ G_{ab} = \tilde{T}_{ab}^m + T_{ab}^R \equiv T_{ab} \, , \]

(6)

with

\[ \tilde{T}_{ab}^m \equiv \frac{T_{ab}^m}{f'} \, , \quad T_{ab}^R \equiv \frac{1}{f'} \left[ \frac{1}{2} (f - R f') g_{ab} + \nabla_b \nabla_a f' - g_{ab} \nabla_c \nabla^c f' \right] \]

(7)

defined as the effective matter and curvature (considered as a fluid) energy-momentum tensors (EMTs), respectively. Standard matter has an EMT given by

\[ T_{ab}^m = \mu^m u_a u_b + p^m (h_{ab} + q^m_{a} u_b + q^m_{b} u_a + \pi^m_{ab}) \, , \]

(8)

where \( \mu^m, p^m, q^m_{a} \), and \( \pi^m_{ab} \) are energy density, pressure, heat flux and anisotropic pressure respectively and \( u^a \equiv \frac{dx^a}{dt} \) is the 4-velocity of fundamental observers and is used to define the covariant time derivative for any tensor \( S^{a..b}_{c..d} \) along an observer’s worldlines:

\[ \dot{S}^{a..b}_{c..d} = u^c \nabla_c S^{a..b}_{c..d} \, , \]

(9)

The projection tensor into the tangent 3-spaces orthogonal to \( u^a \) is given by \( h_{ab} \equiv g_{ab} + u_a u_b \).
and is is used to define the fully orthogonally projected covariant derivative for any tensor $S^a_{c..d}$:

$$\tilde{\nabla}_e S^a_{c..d} = h^a_f h^p_i h^q_j h^r_s \nabla_r S^f_{p..q},$$

(10)

with total projection on all the free indices. The orthogonally projected symmetric trace-free (PSTF) part of tensors is defined as

$$V^{(a)} = h^a_b V^b, \quad S^{(ab)} = \left[ h^a_b h^b_a - \frac{1}{3} h^{ab} h_{cd} \right] S^{cd}.$$

(11)

and the element for the rest spaces orthogonal to $u^a$ is given by

$$\varepsilon_{abc} = u^d \eta_{abcd} = -\sqrt{\left| g \right|} \left[ \delta^0_a \delta^1_b \delta^2_c \delta^3_d - \frac{1}{3} h_{ab} h_{cd} \right] u^d \Rightarrow \varepsilon_{abc} = \varepsilon_{[abc]}, \quad \varepsilon_{abc} u^c = 0,$$

(12)

where $\eta_{abcd}$ is the 4-dimensional volume element with the properties:

$$\eta_{abcd} = \eta_{[abcd]} = 2 \varepsilon_{ab[cd]} u^d - 2 u^a \varepsilon_{bc][ab].$$

(13)

We define the covariant spatial divergence and curl of vectors and tensors as

$$\text{div} V = \tilde{\nabla}^a V_a, \quad (\text{div} S)_a = \tilde{\nabla}^b S_{ab},$$

(14)

$$\text{curl} V_a = \varepsilon_{abc} \tilde{\nabla}^b V^c, \quad \text{curl} S_{ab} = \varepsilon_{cd(a} \tilde{\nabla}^c S_{b)} d.$$

(15)

The first covariant derivative of $u^a$ can be split into its irreducible parts as

$$\nabla_a u_b = -A_a u_b + \frac{1}{3} h_{ab} \Theta + \sigma_{ab} + \varepsilon_{abc} \omega^c,$$

(16)

where $A_a \equiv \dot{u}_a$, $\Theta \equiv \tilde{\nabla}_a u^a$, $\sigma_{ab} \equiv \tilde{\nabla}_{(a} u_{b)}$, and $\omega^a \equiv \varepsilon^{abc} \tilde{\nabla}_b u_c$. The Weyl conformal curvature tensor $C_{abcd}$ is defined [41, 43]

$$C^{ab}_{cd} = R^{ab}_{cd} - 2 g^{[a}_{[c} R^{b]}_{d]} + \frac{R}{3} g^{[a}_{[c} g^{b]}_{d]},$$

(17)

and can be split into its “electric” and “magnetic” parts, respectively, as

$$E_{ab} \equiv C_{agbh} u^g u^h, \quad H_{ab} = \frac{1}{2} \eta_{ae} g^{gh} C_{ghbd} u^e u^d.$$

(18)

$E_{ab}$ and $H_{ab}$ represent the free gravitational field, enabling gravitational action at a distance (tidal forces and gravitational waves), and influence the motion of matter and radiation through the geodesic deviation for timelike and null vectors respectively [41].

Cosmological quantities that vanish in the background spacetime are considered to be first-order and gauge-invariant by virtue of the Stewart-Walker lemma [45].

In a multi-component fluid universe filled with standard matter fields (dust, radiation, etc) and curvature contributions, the total energy density, isotropic and anisotropic pressures and heat flux are given, respectively, by

$$\mu = \frac{\mu_m}{f^r} + \mu_R, \quad p = \frac{p_m}{f^r} + p_R, \quad \pi_{ab} = \frac{\pi_{ab}^m}{f^r} + \pi_{ab}^R, \quad q_a = \frac{q_a^m}{f^r} + q_a^R,$$

(19)
where the linearised thermodynamic quantities for the curvature fluid component are given by

\[ \mu_R = \frac{1}{f'} \left[ \frac{1}{2} (R f' - f) - \Theta f'' \dot{R} + f'' \nabla R \right], \]

\[ p_R = \frac{1}{f} \left[ \frac{1}{2} (f - R f') + f''' \dot{R} + f'' R + \frac{1}{3} \left( \Theta f'' \dot{R} - f'' \nabla R \right) \right], \]

\[ q_a^R = -\frac{1}{f} \left[ f''' \ddot{R} \nabla_a R + f'' \nabla \nabla_a R - \frac{1}{3} f'' \Theta \nabla_a R \right], \]

\[ \pi_{ab}^R = \frac{f''}{f} \left[ \nabla_{(a} \nabla_{b)} R - \sigma_{ab} \ddot{R} \right]. \]

Applying the 1 + 3 - covariant decomposition on the Bianchi and Ricci identities

\[ \nabla [_{a} R_{bc}] d^c = 0, \quad (\nabla_a \nabla_b - \nabla_b \nabla_a) u_c = R_{abc} d^d u_d \]

for the total fluid 4-velocity \( u^a \), the following linearised propagation (evolution) and constraint equations are obtained \([27, 35]\):

\[ \dot{\mu}_m = - (\mu_m + p_m) \Theta - \tilde{\nabla}^a q_a^m, \]

\[ \dot{\mu}_R = - (\mu_R + p_R) \Theta + \frac{\mu_m f''}{f^2} \dot{R} - \tilde{\nabla}^a q_a^R, \]

\[ \dot{\Theta} = - \frac{1}{3} \Theta^2 - \frac{1}{2} (\mu + 3p) + \tilde{\nabla} A^a, \]

\[ \dot{q}_a^m = - \frac{4}{3} \Theta q_a^m - \mu_m A_a, \]

\[ \dot{q}_a^R = - \frac{4}{3} \Theta q_a^R + \frac{\mu_m f''}{f^2} \tilde{\nabla} a R - \tilde{\nabla} P_R - \tilde{\nabla} b \pi_{ab}, \]

\[ \dot{\omega}_a = - \frac{2}{3} \Theta \omega_a - \frac{1}{2} \varepsilon_{abc} \tilde{\nabla} b A_c, \]

\[ \sigma_{ab} = - \frac{2}{3} \Theta \sigma_{ab} - E_{ab} + \frac{1}{2} \varepsilon_{abc} \tilde{\nabla} (a A_b), \]

\[ \dot{E}_{ab} + \frac{1}{2} \dot{\pi}_{ab} = \varepsilon_{cd(a} \tilde{\nabla}^c H_{b)} - \Theta E_{ab} - \frac{1}{2} (\mu + p) \sigma_{ab} - \frac{1}{2} \tilde{\nabla} (a q_b) - \frac{1}{6} \Theta \pi_{ab}, \]

\[ \dot{H}_{ab} = - \Theta H_{ab} - \varepsilon_{cd(a} \tilde{\nabla}^c E_{b)}^d + \frac{1}{2} \varepsilon_{cd(a} \tilde{\nabla}^c \pi_{b)}^d, \]

\[ (C^1) a := \tilde{\nabla} b \sigma_{ab} - \frac{2}{3} \tilde{\nabla} a \Theta - \varepsilon_{abc} \tilde{\nabla} b \omega^c + q_a = 0, \]

\[ (C^2)_{ab} := \varepsilon_{cd(a} \tilde{\nabla}^c \sigma_{b)}^d + \tilde{\nabla} (a \omega_b) - H_{ab} = 0, \]

\[ (C^3) a := \tilde{\nabla} b H_{ab} + (\mu + p) \omega_a + \frac{1}{2} \varepsilon_{abc} \tilde{\nabla} b q^c = 0, \]

\[ (C^4) a := \tilde{\nabla} b E_{ab} + \frac{1}{2} \tilde{\nabla} b \pi_{ab} - \frac{1}{3} \tilde{\nabla} a \mu + \frac{1}{3} \Theta q_a = 0, \]

\[ (C^5) a := \tilde{\nabla} a \omega_a = 0, \]

\[ (C^6) a := \tilde{\nabla} a p_m + (\mu_m + p_m) A_a = 0. \]

The evolution equations uniquely determine the covariant quantities on some initial \( (t = t_0) \) hypersurface \( S_0 \) whereas the constraints restrict the initial data to be specified. For consistency, the constraint equations must remain satisfied on any hypersurface \( S_t \) for all comoving time \( t \).
3. Irrotational Spacetimes

Irrotational spacetimes are characterised by

\[ \omega_a = 0 . \]  

(40)

For a barotropic irrotational matter fluid, the evolution equations (25)-(33) can be rewritten as:

\[
\begin{align*}
\dot{\mu}_m &= - (1 + w) \mu_m \Theta - \nabla^a q_a^m , \\
\dot{\mu}_R &= - (\mu_R + p_R) \Theta + \frac{\mu m f''}{f^2} R - \nabla^a q_a^R , \\
\dot{\Theta} &= - \frac{1}{3} \Theta^2 - \frac{1}{2f} (1 + 3w) \mu_m - \frac{1}{2} (\mu_R + 3p_R) + \nabla_a A^a , \\
\dot{q}_a^m &= - \frac{4}{3} \Theta q_a^m - \mu_m A_a , \\
\dot{q}_a^R &= - \frac{4}{3} \Theta q_a^R + \frac{\mu m f''}{f^2} \nabla_a R - \nabla_a p_R - \nabla^b \pi_{ab} , \\
\dot{\sigma}_{ab} &= - \frac{2}{3} \Theta \sigma_{ab} - E_{ab} + \frac{1}{3} \pi_{ab} + \nabla_{(a} A_{b)} , \\
\dot{E}_{ab} + \frac{1}{2} \pi_{ab} &= \varepsilon_{cd(a} \nabla^c H_{b)d} - \Theta E_{ab} - \frac{1}{2} (\mu + p) \sigma_{ab} - \frac{1}{6} \nabla_{(a} q_{b)} - \frac{1}{3} \Theta \pi_{ab} , \\
\dot{H}_{ab} &= - \Theta H_{ab} - \varepsilon_{cd(a} \nabla^c E_{b)d} + \frac{1}{2} \varepsilon_{cd(a} \nabla^c \pi_{b)} , \\
\end{align*}
\]

and are constrained by the following equations:

\[
\begin{align*}
(C^{1s})_a &= \nabla^b \sigma_{ab} - \frac{2}{3} \nabla_a \Theta + q_a = 0 , \\
(C^{2s})_{ab} &= \varepsilon_{cd(a} \nabla^c \sigma_{b)d} - H_{ab} = 0 , \\
(C^{3s})_a &= \nabla^b H_{ab} + \frac{1}{2} \varepsilon_{abc} \nabla^b q^c = 0 , \\
(C^{4s})_a &= \nabla^b E_{ab} + \frac{1}{2} \nabla^b \pi_{ab} - \frac{1}{3} \nabla_a \mu + \frac{1}{3} \Theta q_a = 0 , \\
(C^{5s})_a &= w \nabla_a \mu_m + (1 + w) \mu_m A_a = 0 , \\
(C^{6s})_a &= \varepsilon_{abc} \nabla^b A^c = 0 \implies A_a = \nabla_a \psi \quad \text{for some scalar } \psi . \\
\end{align*}
\]

The new constraint (54) arises as a result of our irrotational restriction. To check for temporal consistency, we propagate this constraint to obtain

\[
\left( \varepsilon_{abc} \nabla^b A^c \right) = 0 ,
\]

(55)

which is an identity.

On the other hand, taking the curl of this constraint, one obtains

\[
\text{curl(curl}(A_a) = \nabla_a \left( \nabla^b A_b \right) - \nabla^2 A_a + \frac{2}{3} (\mu - \frac{1}{3} \Theta^2) A_a
\]

(56)

\[= \nabla_a \left( \nabla^b \nabla_b \psi \right) - \nabla^2 \nabla_a \psi + \frac{2}{3} (\mu - \frac{1}{3} \Theta^2) \nabla_a \psi \]

(57)

\[= \nabla_a \left( \nabla^2 \psi \right) - \nabla^2 \left( \nabla_a \psi \right) + \frac{2}{3} (\mu - \frac{1}{3} \Theta^2) \nabla_a \psi , \]

(58)

which is another identity by virtue of Eqn (A.12).
3.1. Dust spacetimes

Pure dust spacetimes are characterised by

$$w = 0 = p_m, \dot{q}^m_a = 0 = A^m_a, \pi^m_{ab} = 0,$$

and the linearised evolution and constraint equations read:

$$\dot{\mu}_d = -\mu_d \Theta,$$
$$\dot{\mu}_R = -(\mu_R + p_R) \Theta + \frac{\mu_d f''}{f'^2} \tilde{R} - \tilde{\nabla}^a q^R_a,$$
$$\dot{\Theta} = -\frac{1}{3} \Theta^2 - \frac{1}{2 f} \mu_d - \frac{1}{2} (\mu_R + 3p_R),$$
$$\dot{q}^R_a = -\frac{4}{3} \Theta q^R_a + \frac{\mu_d f''}{f'^2} \nabla_a R - \tilde{\nabla}_a P_R - \tilde{\nabla}^b \pi^R_{ab},$$
$$\dot{\sigma}_{ab} = -\frac{2}{3} \Theta \sigma_{ab} - E_{ab} + \frac{1}{2} \pi^R_{ab},$$
$$\dot{E}_{ab} + \frac{1}{2} \pi^R_{ab} = \varepsilon_{cd(a} \nabla^c E^d_{b)},$$
$$\dot{H}_{ab} = -\Theta H_{ab} - \varepsilon_{cd(a} \nabla^c E^d_{b)} + \frac{1}{2} \varepsilon_{cd(a} \nabla^c \pi^R_{db}).$$

Notice that here no new constraints appear.

3.1.1. Shear-free spacetimes

If we turn off the shear, i.e., if we set

$$\sigma_{ab} = 0$$

in the above propagation equations, we get Eqn (64) turning into a new constraint

$$(C^{5d})_{ab} := \tilde{\nabla}^b \sigma_{ab} - \frac{2}{3} \tilde{\nabla}_a \Theta + q^R_a = 0,$$

the temporal and spatial consistencies of which have to be checked. Moreover, we see from Eqn (68) that $H_{ab}$ identically vanishes, thus resulting in another constraint from Eqn (66):

$$(C^{6d})_{ab} := \varepsilon_{cd(a} \nabla^c E^d_{b)} - \frac{1}{2} \varepsilon_{cd(a} \nabla^c \pi^R_{db}) = 0,$$

which is an identity by virtue of Eqn (72).

From (69), we see that $q^R_a$ is irrotational, and therefore can be written as the gradient of a scalar:

$$q^R_a = \tilde{\nabla}_a \phi.$$
However, we already know from (67) that \( q_a^R = \frac{2}{3} \tilde{\nabla}_a \Theta \). One can therefore conclude that in irrotational and shear-free dust spacetimes,

\[
\phi = \frac{2}{3} \Theta + C,
\]

for some spatially constant scalar \( C \).

Now to check for temporal consistency of Eqn (72), we take the time derivative of both sides of this equation to obtain the relation

\[
\dot{\pi}_a^R + \frac{2}{3} \Theta \pi_a^R - \frac{1}{2} \tilde{\nabla}_a (q_b^R) = 0 ,
\]

which, using \( q_a^R \) and \( \pi_a^R \) as defined by Eqns (22) and (23), can be rewritten as

\[
\left[ \frac{3}{2} \left( \frac{f'''}{f'} - \frac{f''^2}{f'^2} \right) \dot{R} - \frac{\Theta f''}{6f'} \right] \tilde{\nabla}_a \tilde{\nabla}_b R + \frac{3f'''}{2f'} \tilde{\nabla}_a \tilde{\nabla}_b \dot{R} = 0 .
\]

Thus irrotational shear-free dust spacetimes governed by \( f(R) \) gravitational physics evolve consistently if Eqn (77) is satisfied. In the case of GR, i.e., \( f(R) = R \), note that this equation becomes an identity since \( f'' = f''' = 0 \).

Now taking the curl of the above equation, we obtain

\[
\left[ \frac{3}{2} \left( \frac{f'''}{f'} - \frac{f''^2}{f'^2} \right) \dot{R} - \frac{\Theta f''}{6f'} \right] \varepsilon_{cda} \tilde{\nabla}^c \tilde{\nabla}_d \tilde{\nabla}_b R + \frac{3f'''}{2f'} \varepsilon_{cda} \tilde{\nabla}^c \tilde{\nabla}_d \tilde{\nabla}_b \dot{R} = 0 ,
\]

which is an identity by virtue of Eqn (A.3). Thus the temporal consistency condition (77) is satisfied on any initial hypersurface. Moreover, from Eqns (73) and (78), we can conclude that all irrotational shear-free dust spacetimes in \( f(R) \) gravity are spatially consistent.

A further restriction one can make for such (shear-free) spacetimes is turning off \( E_{ab} \) in which case Eqn (65) changes to the (linearised) constraint

\[
\tilde{\nabla}_a q_b^R = 0 = \frac{1}{f'} \left[ \left( \dot{R} f''' - \frac{2}{3} \Theta f'' \right) \tilde{\nabla}_a \tilde{\nabla}_b R + f'' \tilde{\nabla}_a \tilde{\nabla}_b \dot{R} \right] .
\]

Eqn (64) implies

\[
\pi_{ab}^R = 0 = \frac{f'''}{f'} \tilde{\nabla}_a \tilde{\nabla}_b R ,
\]

which, for \( f'' \neq 0 \), leads to the conclusion that

\[
\tilde{\nabla}_a \tilde{\nabla}_b R = 0 .
\]

Using this and the relation (A.1), we see that Eqn (79) becomes an identity. Thus the linearised \( f(R) \) field equations in irrotational and shear-free dust spacetimes with vanishing Weyl tensor are consistent.
3.1.2. Dust solutions with div$H = 0$ The vanishing of the divergence of a non-zero $H_{ab}$ is a necessary condition for gravitational radiation [46, 47, 48]. Here we analyze the consistency of irrotational dust spacetimes with such restriction ($\nabla^b H_{ab} = 0$).

We see that there are no new constraints arising as a result of imposing a divergence-free $H_{ab}$ to the field equations, but as in the shear-free case discussed above, Eqn (69) implies that $q^a_a$ satisfies Eqns (74) and (75).

The so-called *Newtonian-like* spacetimes are described by the vanishing of the gravito-magnetic (GM) component of the Weyl tensor. Thus if the Weyl tensor is to have a purely ‘gravito-electric’ (GE) component, then we notice from Eqn (68) that a curl-free shear is required, i.e.,

$$H_{ab} = 0 \implies \varepsilon_{cd(a} \nabla^c \sigma_{b)}^d = 0,$$

and the constraint (73) is obtained from Eqn (66).

3.1.3. Purely gravito-magnetic spacetimes These are models with vanishing gravito-electric component of the Weyl tensor and are referred to as *anti-Newtonian*‡ models because they are considered to be the most extreme of non-Newtonian gravitational models [51]. The only anti-Newtonian solutions in GR are the FLRW spacetimes [51, 36], but a recent covariant consistency analysis [52] has shown that linearised anti-Newtonian universes are permitted by some models of $f(R)$ gravity.

As can be seen from the set of equations (60)-(70), no new constraint equations arise as a result of vanishing $E_{ab}$. This is because of the non-vanishing of $\pi^R_{ab}$ for generic $f(R)$ models; but in the GR limiting case Eqn (65) would have turned into a new constraint since $\pi^R_{ab} = 0$.

3.2. Non-expanding Spacetimes

In this section we explore cases where the background spacetime is not expanding, i.e., $\Theta = 0$. In this very special case, the linearised evolution equations (41)-(48) become

\begin{align*}
\dot{\mu}_m &= -\tilde{\nabla}^a q^m_a, \\
\dot{q}^m_a &= \frac{w}{1+w} \tilde{\nabla} a_\mu_m, \\
\dot{R} &= \frac{\mu_m f''}{f'^2} \hat{R} - \tilde{\nabla}^a q^R_a, \\
\dot{q}^R_a &= \frac{\mu_m f''}{f'^2} \tilde{\nabla} a R - \tilde{\nabla} a p_R - \tilde{\nabla} b \pi^R_{ab}, \\
\dot{\sigma}_{ab} &= -E_{ab} + \frac{1}{2} \pi_{ab} + \tilde{\nabla}_{(a} A_{b)} , \\
\dot{E}_{ab} + \frac{1}{2} \pi_{ab} &= \varepsilon_{cd(a} \tilde{\nabla}^c H^d_{b)} - \frac{1}{2} (\mu + p) \sigma_{ab} - \frac{1}{2} \tilde{\nabla}_{(a} q_{b)} , \\
\dot{H}_{ab} &= -\varepsilon_{cd(a} \tilde{\nabla}^c E^d_{b)} + \frac{1}{2} \varepsilon_{cd(a} \tilde{\nabla}^c \pi^d_{b)} .
\end{align*}

‡ An earlier use of the word ‘anti-Newtonian’ exists [49, 50], where the word is used to refer to an earlier stage of the Universe when the dimension of irregularity exceeds the cosmological (Hubble) horizon.
whereas the revised constraint equations are given by

\begin{align}
\left(C^{1s}\right)_a &:= \hat{\nabla}^b \sigma_{ab} + q_a = 0 , \\
\left(C^{2s}\right)_{ab} &:= \varepsilon_{cd(a} \hat{\nabla}^c \sigma_{b)d} - H_{ab} = 0 , \\
\left(C^{3s}\right)_a &:= \hat{\nabla}^b H_{ab} + \frac{1}{2} \varepsilon_{abc} \hat{\nabla}^b q^c = 0 , \\
\left(C^{4s}\right)_a &:= \hat{\nabla}^b E_{ab} + \frac{1}{2} \hat{\nabla}^b \pi_{ab} - \frac{1}{3} \hat{\nabla}_a \mu = 0 , \\
\left(C^{5s}\right)_a &:= w \hat{\nabla}_a \mu_m + (1 + w) \mu_m A_a = 0 , \\
\left(C^{6s}\right) &:= \hat{\nabla}_a A^a - \frac{1}{27} (1 + 3w) \mu_m - \frac{1}{3} (\mu_R + 3 p_R) = 0 .
\end{align}

Eqn (84) has been obtained by using Eqn (53) into (44).

### 3.2.1. Dust Solutions

In the case of dust ($A_a = 0 = q^m_a$), the active gravitational mass $\mu + 3p = 0$ because of Eqn (95). Since (83) implies $\mu_a(t) = \text{const}$, we notice that

$$\mu_R + 3 p_R = \text{const}$$

as well. From the definitions (20) and (21) for $\mu_R$ and $p_R$ and the trace equation

$$3 f'' \hat{R} + 3 \hat{R}^2 f''' + 3 \Theta \hat{R} f'' - 3 f'' \hat{\nabla}^2 R - R f' + 2 f - \mu + 3 p = 0 ,$$

we conclude that (96) implies\(^8\)

$$f - 2 f'' \hat{\nabla}^2 R = \text{const} .$$

### 3.2.2. Shear-free Solutions

If we make the shear-free assumption, the propagation equation (87) turns into the constraint

$$\left(C^{7s}\right)_{ab} := E_{ab} - \frac{1}{2} \pi_{ab} - \hat{\nabla}_{(a} A_{b)} ,$$

whereas the constraint equations (90) and (91) imply $q_a = 0$ and $H_{ab} = 0$. This means that Eqn (88) reduces to

$$\dot{E}_{ab} + \frac{1}{2} \pi_{ab} = 0 .$$

If we differentiate the new constraint (99) with respect to cosmic time, and solve simultaneously with Eqn (100) we obtain

$$\dot{E}_{ab} - (\hat{\nabla}_{(a} A_{b)})^c = 0 .$$

On the other hand, if we take the gradient of (99) and solve simultaneously with (93), we obtain

$$\hat{\nabla}^b E_{ab} - \frac{1}{6} \hat{\nabla}_a \mu - \frac{1}{2} \hat{\nabla}^b \hat{\nabla}_{(a} A_{b)} = 0 .$$

\(^8\) In GR, where $f(R) = R$, this translates into stating the obvious result that a constant $\mu_a$ implies a constant $R$ since $f'' = 0$. 

*Irrotational - fluid cosmologies in fourth-order gravity*
Moreover, the curl condition of (99) is identically satisfied by virtue of Eqns (89), (54) and (A.3).

If we consider the special case of dust \( A_a = 0 = q_a^m \), \( \pi_{ab}^m = 0 \) in this shear-free setting, then (101) implies \( E_{ab} = \text{const} \) in time. However, since \( E_{ab} \) is related to \( \pi_{ab}^R \) via Eqn (99), then \( \pi_{ab}^R = \text{const} \) as well. This dictates that, because of (23), the term \( \mathcal{L}_T \nabla_a (\nabla_b R) \) be constant in cosmic time. It is also no coincidence that (98) is recovered for this subclass as a result of (95).

Another interesting point to note about non-expanding, shear-free dust spacetimes is that since \( q_a = 0 \implies q_a^R = 0 \), we are dictated by Eqn (22) to conclude:

\[
\left( f'' \nabla_a R \right) = 0 .
\] (103)

4. Discussions and Conclusion

A completely general covariant analysis of irrotational fluids in \( f(R) \) cosmology requires taking nonlinear effects into account. This paper is meant to be a first step in that direction, and we have looked at the limits of irrotational fluid spacetimes for some specialised solutions linearised around FLRW background.

We have shown that the only constraint arising as a result of the irrotational fluid assumption in \( f(R) \)-gravity is that the 4-acceleration \( A_a \) be given as a gradient of some scalar \( \psi \) (see Eqn (54)). Upon specialising to dust-fluid cases, we have seen that no new constraints arise and hence the limiting field equations propagate consistently. But if one further specialises to shear-free dust, then we get a vanishing \( H_{ab} \) and a temporally and spatially consistent constraint equation (72). We have shown that, as a result of the shear-free condition, \( q_a^R \) is irrotational and can be given as a gradient of some scalar function \( \phi = \frac{4}{3} \Theta + C \) for some spatially constant \( C \). We have also shown that the linearised field equations of \( f(R) \)-gravity in irrotational shear-free dust spacetimes with vanishing \( E_{ab} \) are consistent.

Another subclass of irrotational spacetimes we looked at are the non-expanding cases, where the new constraint (95) appears as a result of the \( \Theta = 0 \) restriction. Dust spacetimes in this subclass are further constrained by Eqn (98) whereas general shear-free cases are constrained by Eqn (99). Moreover, the very special case of shear-free dust solutions results in \( E_{ab} \) and \( \pi_{ab}^R \) being constants over cosmic time as well as in the vanishing of \( q_a^R \).

In short, we have explored some [sub]classes of irrotational cosmological models and shown how these models put restrictions on the possible forms of the underlying \( f(R) \) gravitational theory.

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Appendix A. Useful linearised differential identities \cite{27, 35, 44}

For all scalars $f$, vectors $V_a$ and tensors that vanish in the background, $S_{ab} = S_{(ab)}$, the following linearised identities hold:

\[
\left( \tilde{\nabla}_{(a} \tilde{\nabla}_{b)} f \right)^{\cdot} = \tilde{\nabla}_{(a} \tilde{\nabla}_{b)} f - \frac{2}{3} \Theta \tilde{\nabla}_{(a} \tilde{\nabla}_{b)} f + \dot{f} \tilde{\nabla}_{(a} A_{b)} \tag{A.1}
\]

\[
\tilde{\varepsilon}^{abc} \tilde{\nabla}_b \tilde{\nabla}_c f = 0 \tag{A.2}
\]

\[
\tilde{\varepsilon}_{cd} \tilde{\nabla}^c \tilde{\nabla}_{(b} \tilde{\nabla}_{d)} f = \tilde{\varepsilon}_{cd} \tilde{\nabla}^c \tilde{\nabla}_b \tilde{\nabla}^d f = 0 \tag{A.3}
\]

\[
\tilde{\nabla}^2 \left( \tilde{\nabla}_a f \right)^{\cdot} = \tilde{\nabla}_a \left( \tilde{\nabla}^2 f \right) + \frac{1}{3} \tilde{R} \tilde{\nabla}_a f \tag{A.4}
\]

\[
\tilde{\nabla}_a f \cdot = \tilde{\nabla}_a \dot{f} - \frac{1}{3} \Theta \tilde{\nabla}_a f + \dot{f} A_a \tag{A.5}
\]

\[
\tilde{\nabla}_a S_{b...} \cdot = \tilde{\nabla}_a \dot{S}_{b...} - \frac{1}{3} \Theta \tilde{\nabla}_a S_{b...} \tag{A.6}
\]

\[
\left( \tilde{\nabla}^2 f \right)^{\cdot} = \tilde{\nabla}^2 \dot{f} - \frac{2}{3} \Theta \tilde{\nabla}^2 f + \dot{f} \tilde{\nabla}^a A_a \tag{A.7}
\]

\[
\tilde{\nabla}_{[a} \tilde{\nabla}_{b]} V_c = - \frac{1}{3} \tilde{R} V_{[a b]} c \tag{A.8}
\]

\[
\tilde{\nabla}_{[a} \tilde{\nabla}_{b]} S_{c d} = - \frac{1}{3} \tilde{R} S_{a [c} h_{b] d} \tag{A.9}
\]

\[
\tilde{\nabla}^a \left( \tilde{\varepsilon}_{abc} \tilde{\nabla}^b V^c \right) = 0 \tag{A.10}
\]

\[
\tilde{\nabla}_b \left( \tilde{\varepsilon}^{c d (a} \tilde{\nabla}_{c} S_{d)} \right) = \frac{1}{2} \tilde{\varepsilon}^{a b c} \tilde{\nabla}_b \left( \tilde{\nabla}^a S_{c} \right) \tag{A.11}
\]

\[
curl \curl V_a = \tilde{\nabla}_a \left( \tilde{\nabla}^2 V_a \right) - \tilde{\nabla}^2 V_a + \frac{2}{3} (\mu - \frac{1}{3} \Theta^2) V_a \tag{A.12}
\]

where $\tilde{R} \equiv 2 (\mu - \frac{1}{3} \Theta^2)$ is the 3-curvature scalar.

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