Anomalous transport in velocity space: from Fokker–Planck to the general equation

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Abstract
The problem of anomalous diffusion in momentum (velocity) space is considered based on the master equation and the appropriate probability transition function (PTF). The approach recently developed for coordinate space in Trigger et al (2005 J. Phys., Conf. Ser. 11 37) is applied with necessary modifications to velocity space. A new general equation for the time evolution of the momentum distribution function in momentum space is derived. This allows the solution of various problems of anomalous transport when the PTF has a long tail in momentum space. For the opposite cases of the PTF rapidly decreasing as a function of transfer momenta (when large transfer momenta are strongly suppressed), the developed approach allows us to consider strongly non-equilibrium cases of the system evolution. The stationary and non-stationary solutions are studied. As an example, the particular case of the Boltzmann-type PTF for collisions of heavy and light particles with the determined (prescribed) distribution function, which can be strongly non-equilibrium, is considered within the proposed general approach. The appropriate diffusion and friction coefficients are found. The Einstein relation between the friction and diffusion coefficients is shown to be violated in these cases.

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1. Introduction

Interest in the anomalous diffusion is prompted by a large variety of applications, i.e. semiconductors, polymers, some granular systems, plasmas under specific conditions, various objects in biological systems, physical–chemical systems and others.

Deviation from the linear in time dependence $\langle r^2(t) \rangle \sim t$ of the mean-squared displacement has been experimentally observed, in particular, under essentially non-equilibrium conditions and in some disordered systems. The mean-squared separation of
a pair of particles passively moving in a turbulent flow anomalously increases (according to Richardson’s law) with the third power of time \[2\]. For diffusion, typical of glasses and related complex systems \[3\], the observed anomalous time dependence is slower than linear. These two types of anomalous diffusion are referred to as superdiffusion, \(\langle r^2(t) \rangle \sim t^\alpha (\alpha > 1)\), and subdiffusion \((\alpha < 1)\) \[4\]. To describe these two diffusion regimes, a number of efficient models and methods have been proposed. The continuous time random walk (CTRW) model of Scher and Montroll \[5\], leading to a strongly subdiffusive behavior, provides a basis for understanding photoconductivity in strongly disordered and glassy semiconductors. The Levy-flight model \[6\], leading to superdiffusion, describes various phenomena such as self-diffusion in micelle systems \[7\], reaction and transport in polymer systems \[8\] and has been applied even to the stochastic description of financial market indices \[9\]. For both cases, the so-called fractional differential equations in coordinate and time spaces were successfully applied \[10\].

However, recently a more general approach has been proposed in \[1, 11\], which reproduces the results of the standard fractional differentiation method (when it is applicable) and allows us to describe more complicated cases of anomalous diffusion in coordinate space. In \[12\], this approach has also been applied to diffusion in a time-dependent external field. The problem of anomalous diffusion in coordinate space in an external field has also been considered semi-phenomenologically on the basis of the Langevin equation in \[13, 14\].

In this paper, the problem of anomalous diffusion in momentum (velocity) space is considered by using the general approaches developed in \[1, 11\] for diffusion in coordinate space. In spite of the formal similarity, diffusion in momentum space is very different physically from coordinate space diffusion. Indeed, in the velocity space (V-space) we get for normal diffusion the Fokker–Planck equation with the friction term, whereas the corresponding equation in coordinate space is the simple diffusion equation without any friction (in the case of no external fields).

Some aspects of anomalous diffusion in V-space have been considered in several papers \[15–18\]. In general, in comparison with anomalous diffusion in coordinate space, anomalous diffusion in V-space is poorly studied. To our knowledge, there is still no corresponding way to describe anomalous diffusion in V-space self-consistently.

In this paper, a new kinetic equation for anomalous diffusion in V-space is derived (see also \[19, 20\]) based on the appropriate expansion of the probability transition function (PTF) (in spirit of the approach proposed in \[1\] for the diffusion in coordinate space) and some particular problems are studied on this basis.

The paper is organized as follows. The diffusion in V-space for the cases of the normal and anomalous behavior of the PTF is presented in section 2. Starting from argumentation based on the Boltzmann type of the PTF, we derive a new kinetic equation which then can be applied not only to the Boltzmann-type processes, but also to a wide class of processes occurring in nature with other types of PTF. The particular cases of anomalous diffusion for hard-sphere collisions with the specific power-type prescribed distribution function of light particles are analysed in section 3. More general examples of anomalous diffusion are considered in section 4. The non-stationary solution for the distribution function identical at \(t = 0\) to the initial distribution is found in section 5. In section 6 we show that in the particular cases of the power dependence of the coefficients in a general equation for anomalous diffusion the representation in fractional derivatives is possible. We establish the connection of general results with previously obtained distributions. In section 7, the Boltzmann-type equation is used to consider the effect of light particle drift on the PTF and on the kinetic equation structure. The method and results of this paper can easily be extended to describe plasma-like systems and anomalous processes in energy space.
Let us now consider the main problem formulated in the introduction, namely the description of diffusion in V-space. Such a description is based on a corresponding master equation for the distribution function \( f_g(p,t) \), which describes the balance of particles (we also use the term ‘grain’ in the general case of a heavy or/and the finite size particles), coming at and from the point \( p \) at the instant \( t \). The structure of this equation

\[
\frac{df_g(p,t)}{dt} = \int dq \{ W(q, p + q)f_g(p + q, t) - W(q, p)f_g(p, t) \}
\]

is formally similar to the master equation in coordinate space (see, e.g., [11]). Surely, for coordinate space, there is no evident manifestation of the momentum conservation law similar to that in momentum space. The PTF \( W(q, p') \) (the transition rate) describes the probability that a particle with momentum \( p' \) (point \( p' \)) passes from this point \( p' \) to the point \( p \) per unit time by transferring the momentum \( q = p' - p \) to the surrounding medium. Assuming initially that the characteristic transfer momentum \( q \) is smaller than \( p \), one may expand equation (1) with respect to \( q \) to the second order. Then, we arrive at the usual form of the Fokker–Planck equation for the density distribution function \( f_g(p,t) \) (see, e.g., [21]):

\[
\frac{df_g(p,t)}{dt} = \frac{\partial}{\partial p^\alpha} \left[ A_\alpha(p) f_g(p,t) + \frac{\partial}{\partial p^\beta} (B_{\alpha\beta}(p) f_g(p,t)) \right],
\]

\[
A_\alpha(p) = \int d^r q \sigma q^\alpha W(q, p), \quad B_{\alpha\beta}(p) = \frac{1}{2} \int d^r q \sigma q^\alpha q^\beta W(q, p).
\]

The coefficients \( A_\alpha \) and \( B_{\alpha\beta} \) describe the friction force and diffusion, respectively. The indices \( \alpha \) and \( \beta \) correspond to the coordinate axes. Let us consider the case when velocity of the particles, undergoing diffusion in V-space, is low in comparison with the characteristic velocities in the surrounding medium (see, e.g., [21]). For the particular example of the Boltzmann-type collisions (see below) this means that we consider two types of particles: heavy particles (undergoing diffusion with the characteristic velocity \( v \sim p/M \)) and light particles (‘medium’, with the characteristic velocity \( u \), in the particular case, e.g., \( u \sim \sqrt{T/m} \gg v \)). Then \( p \)-dependence of the PTF can be neglected for calculating the diffusion term, which in this case is a constant \( B_{\alpha\beta} = \delta_{\alpha\beta} B \). The constant \( B \) is equal to the integral

\[
B = \frac{1}{2^r} \int d^r q q^2 W(q),
\]

where \( r \) is the dimension of V-space. Neglecting the \( p \)-dependence of the PTF, we arrive at the coefficient \( A_\alpha = 0 \) (while the diffusion coefficient is constant).

As is well known this negligence is wrong and the coefficient \( A_\alpha \) for the Fokker–Planck equation can be determined using the assumption that the stationary distribution function is Maxwellian. Then, we arrive at the standard relation between the coefficients \( MTA_\alpha(p) = p_\alpha B \) (here \( M \) is the particle mass and \( T \) is the temperature of the particles in the equilibrium state). This relation is an analog of the Einstein relation in coordinate space. However, this argumentation is not applicable to systems far from equilibrium.

To find the coefficients in the kinetic equation, which are appropriate for more general non-equilibrium situations, e.g., for slowly decreasing PTF or (and) for strongly non-equilibrium kernels \( W \), let us use a more general way based on a certain small parameter (in the simplest case, e.g., on the difference of velocities of light and heavy particles for the Boltzmann-type collisions). To calculate the function \( A_\alpha \) we have taken into account that the function \( W(q, p) \)
is a scalar and can depend only on the variables $q, \mathbf{q} \cdot \mathbf{p}, p$. Expanding $W(q, \mathbf{p})$ in the product $\mathbf{q} \cdot \mathbf{p}$, one arrives at the approximate representation of the functions $W(q, \mathbf{p})$ and $W(q, \mathbf{p} + \mathbf{q})$:

\begin{align}
W(q, \mathbf{p}) &\simeq W(q) + \tilde{W}'(q) (\mathbf{q} \cdot \mathbf{p}) + \frac{1}{2} \tilde{W}''(q) (\mathbf{q} \cdot \mathbf{p})^2, \\
W(q, \mathbf{p} + \mathbf{q}) &\simeq W(q) + \tilde{W}'(q) (\mathbf{q} \cdot \mathbf{p}) + \frac{1}{2} \tilde{W}''(q) (\mathbf{q} \cdot \mathbf{p})^2 + q^2 \tilde{W}'(q).
\end{align}

Here we introduced the derivatives $\tilde{W}'(q) \equiv \partial W(q, \mathbf{q} \cdot \mathbf{p}, p)/\partial (\mathbf{q} \cdot \mathbf{p}) |_{\mathbf{q} \cdot \mathbf{p}=0, p=0}$ and $\tilde{W}''(q) \equiv \partial^2 W(q, \mathbf{q} \cdot \mathbf{p}, p)/\partial (\mathbf{q} \cdot \mathbf{p})^2 |_{\mathbf{q} \cdot \mathbf{p}=0, p=0}$.

Then, with necessary accuracy (which corresponds to the derivation of the well-known Fokker–Planck equation) for the coefficient $A_\alpha$ we find

$$A_\alpha(p) = \int d' q a q a p \tilde{W}'(q) = p a \int d' q a q a \tilde{W}'(q) = \frac{p a}{r} \int d' q a q a \tilde{W}'(q).$$

If the equality $\tilde{W}'(q) = W(q)/2MT$ is fulfilled for the function $W(q, \mathbf{p})$, we arrive at the usual Einstein relation for the coefficients $A_\alpha$ and $B_\alpha$:

$$MT A_\alpha(p) = p a B.$$

Let us check this relation for Boltzmann collisions which are described by the PTF $W(q, \mathbf{p}) = w_B(q, \mathbf{p})$ (the case $r = 3$) [11] (see also [22]):

$$w_B(q, \mathbf{p}) = \frac{2 \pi}{\mu^2 q} \int_0^\infty du \left[ \text{arccos} \left( 1 - \frac{q^2}{2 \mu^2 u^2} \right) - u \right] f_b(u^2 + v^2 - q \cdot v / \mu),$$

where $p = M v$, the quantities $d \sigma /do$, $\mu$ and $f_b$ are, respectively, the scattering differential cross-section, the mass and distribution function for light particles. The arguments $\chi$ and $u$ in the cross-section $d \sigma /do$ are the angle of the particle scattering in the center-of-mass coordinate system and the velocity of light particles before collision. In equation (9), we took into account the approximate equalities for light and heavy particle scattering, $q^2 \equiv (\Delta p)^2 = p'^2 (1 - \cos \theta)$ and $\theta \simeq \chi$ (where $p' = \mu u$ is the light particle momentum before collision). For the equilibrium Maxwellian distribution $f_b^0$, the equality $\tilde{W}'(q) = W(q)/2MT$ is evident and we arrive at the ordinary Fokker–Planck equation in V-space with the constant diffusion $D = B/M^2$ and friction $\beta = B/MT = DM/T$ coefficients which satisfy the Einstein relation.

The problem of determination of the $W$-function and the respective Fokker–Planck equation for the situations close to equilibrium is discussed in detail in the review [22]. However, even for quasi-equilibrium regimes, where long tails of the PTF are absent, the consideration in [22] is restricted to the hard-sphere interaction. The non-equilibrium forms of the $W$-function and the drastic changes in the Fokker–Planck equation structure for these situations were not considered, as well as the specific situations, when the long tail of the $W$-function exists. Therefore, anomalous diffusion in [22] is absent. As is easy to see, in the case of the hard-sphere cross-section and the equilibrium Maxwellian distribution $f_b^0$, equation (9) leads to the same result for the $W = w_B$ function as has been discussed in [22].

For some non-equilibrium (stationary or non-stationary) states, the PTF can have a long tail as a function of $q$. In this case, we derive a generalization of the Fokker–Planck equation in spirit of the consideration [1, 11] for the coordinate case. The necessity of this derivation arises since the diffusion and friction coefficients in the form (equations (4),(7)) diverge in the limit of large $q$, if the kernels of the functions $W$, $\tilde{W}'$ exhibit the asymptotic behavior, $W(q) \sim 1/q^{\alpha}$ with $\alpha \leq s + 2$ and (or) $\tilde{W}'(q) \sim 1/q^{\beta}$ with $\beta \leq s + 2$.

Let us substitute the expansions for $W$ (as an example, we choose $s = 3$; the arbitrary $s$ can be considered by the similar way) into equation (1). With an accuracy up to $(\mathbf{q} \cdot \mathbf{p})^2$, we
find
\[
\frac{df_g(p, t)}{dt} = \int dq \left\{ f_g(p + q, t) \left[ W(q) + \tilde{W}'(q) \left( q \cdot p \right) + \frac{1}{2} \tilde{W}''(q)(q \cdot p)^2 + q^2 \tilde{W}''(q) \right] \right.
\]
\[\left. - f_g(p, t) \left[ W(q) + \tilde{W}'(q) \left( q \cdot p \right) + \frac{1}{2} \tilde{W}''(q)(q \cdot p)^2 \right] \right\}.
\] (10)

After the Fourier transformation \( f_g(s, t) = \int \frac{dq}{2\pi} \exp(ips) f_g(p, t) \), equation (10) reads
\[
\frac{df_g(s, t)}{dt} = A(s) f_g(s, t) + B_\alpha(s) \frac{\partial f_g(s, t)}{\partial s_\alpha} + C_{\alpha\beta}(s) \frac{\partial^2 f_g(s, t)}{\partial s_\alpha \partial s_\beta}.
\] (11)

It should be noted that the variable \( s_\alpha \), arisen in equation (11), has the dimension \( p^{-1} \) and is a formal variable for the Fourier transformation.

The coefficients in equation (11) are given by (the last parts of the equalities are calculated for \( r = 3 \))

\[ A(s) = \int dq [\exp(-i(qs)) - 1] W(q) = 4\pi \int_0^\infty dq q^2 \left[ \left( \frac{\sin(qs)}{qs} - 1 \right) \right] W(q) \] (12)

\[ B_\alpha(s) \equiv s_\alpha B(s); \quad B(s) = -\frac{i}{s^2} \int dqqs [\exp(-i(qs)) - 1] \tilde{W}'(q) \]
\[ = \frac{4\pi}{s^2} \int_0^\infty dq q^2 \left[ \cos(qs) - \left( \frac{\sin(qs)}{qs} \right) \right] \tilde{W}'(q) \] (13)

\[ C_{\alpha\beta}(s) \equiv s_\alpha s_\beta C(s) = -\frac{1}{2} \int dq_{\alpha\beta} \left[ \exp(-i(qs)) - 1 \right] 2\tilde{W}''(q) \] (14)

\[ C(s) = -\frac{1}{2s^2} \int dq(qs)^2 [\exp(-i(qs)) - 1] 2\tilde{W}''(q) \]
\[ = \frac{2\pi}{s^2} \int_0^\infty dq q^4 \left[ \frac{2\sin(qs)}{q^2s^3} - 2\cos(qs) - \frac{\sin(qs)}{qs} + \frac{1}{3} \right] 2\tilde{W}''(q). \] (15)

For the isotropic function \( f(s) = f(s) \), one can rewrite equation (11) as
\[
\frac{df_g(s, t)}{dt} = A(s) f_g(s, t) + B(s) \frac{\partial f_g(s, t)}{\partial s} + C(s) \frac{\partial^2 f_g(s, t)}{\partial s^2}.
\] (16)

In the case where the PTF \( W(q) \) and the functions \( \tilde{W}'(q) \) and \( \tilde{W}''(q) \) strongly decrease for large values of \( q \), the exponents under the integrals in the functions \( A(s) \), \( B(s) \) and \( C(s) \) can be expanded as \( r = 3 \)

\[ A(s) \simeq -\frac{s^2}{6} \int dq q^2 W(q), \quad B(s) \simeq -\frac{1}{3} \int dq q^2 \tilde{W}'(q), \quad C(s) \simeq 0. \] (17)

Then, the simplified kinetic equation in V-space (based on the PTF which is non-equilibrium in the general case) is written as
\[
\frac{df_g(s, t)}{dt} = A_0 s^2 f_g(s) + B_0 s \frac{\partial f_g(s)}{\partial s},
\] (18)

where \( A_0 \equiv -1/6 \int dq q^2 W(q) \) and \( B_0 \equiv -1/3 \int dq q^2 \tilde{W}'(q) \) are defined by equation (17).

For \( C(s) = 0 \) the stationary solution of equation (16) is given by
\[
f_g(s) = C \exp \left[ -\int_0^s ds' A(s') \right] = C \exp \left[ -\frac{A_0 s^2}{2B_0} \right].
\] (19)
The respective normalized stationary momentum distribution is written as
\[ f_g(p) = \frac{N_g B_0^{3/2}}{(2\pi A_0)^{3/2}} \exp \left[ -\frac{B_0 p^2}{2A_0} \right]. \] (20)

Therefore, the constant in equation (19) is \( C = N_g \), where \( N_g \) is the density of heavy particles. Equation (18) and the obtained distribution are the generalization (for non-equilibrium situations) of the standard Fokker–Planck equation to the case of normal diffusion in V-space (see, e.g., [21]). The characteristic feature of these physical situations is the existence of the fixed (prescribed) kernel \( W(q, p) \) which is defined, e.g., by a certain non-Maxwellian distribution of small particles \( f_b \). To show this in other way, let us make the Fourier transformation of equation (11) with \( C = 0 \) and the respective coefficients
\[ \frac{df_g(s, t)}{dt} = -A(s) f_g(s) + B_\alpha(s) \frac{\partial f_g(s, t)}{\partial s^\alpha}, \] (21)

Therefore, we arrive at the Fokker–Planck-type equation with the friction coefficient \( \beta \equiv -B_0 \) and the diffusion coefficient \( D = -A_0/M^2 \). In general, these coefficients (equation (17)) do not satisfy the Einstein relation.

In the case of equilibrium (e.g., \( f_b = f_b^0 \), see above) for the \( W \)-function the equality \( W'(q) = W(q)/2MT_b \) is fulfilled. Then, with the necessary accuracy of the order \( \mu/M \), we find \( A(s)/B(s) \equiv A_0/B_0 = MT_b \). In this case, the Einstein relation between the diffusion and friction coefficients \( D = \beta T/M \) exists, and the standard Fokker–Planck equation is valid. For the case, described by equations (19) and (21) and under the condition that the stationary solution exists, instead the Einstein relation, one can only assert that the equilibrium distribution depends on one constant \( A_0/B_0 \neq MT_b \). This constant is determined by the properties of the model under consideration (on the structure of the PTF).

Finally, for the general case of the PTF function, under the condition that the stationary non-equilibrium solution exists and has some known in advance universal character \( f_g(s) = f_g(0)(s) \), the functions \( A(s) \) and \( B(s) \) are connected according to equations (18) and (23) (see below) by the relation
\[ \frac{A(s)}{B(s)} = s_\alpha \frac{\partial \ln f_g(0)(s)}{\partial s^\alpha}. \] (22)

In fact, the approximation \( C(s) \simeq 0 \) is always practically applicable due to the small parameter (e.g. \( \mu/M \) for the Boltzmann-type PTF, see below, section 3). Therefore, the general kinetic equation (16) for the Fourier transform of the velocity distribution function takes the form
\[ \frac{df_g(s, t)}{dt} = A(s) f_g(s) + B_\alpha(s) \frac{\partial f_g(s, t)}{\partial s^\alpha}, \] (23)

where the coefficients \( A(s) \) and \( B_\alpha(s) \) are determined by (12) and (13), respectively.

### 3. The models of anomalous diffusion in V-space

Now we can calculate the coefficients for the particular cases of anomalous diffusion. For this we find (or suggest) some models of the PTF \( W(q, p) \). We will also estimate for the function \( W'' \) and the tensor \( C_{\alpha, \beta} \), although, at the end, we neglect these terms due to the small transfer momentum in the collision process.

At first, we calculate the simple model, i.e., the system of hard spheres with different masses \( m \) and \( M \gg m \), \( dr/d\sigma = a^2/4 \). Let us suppose that light particles in the model under consideration are described by the prescribed stationary distribution \( f_b = n_b \phi_b/\mu_0^3 \) (where \( n_b, \phi_b \) and \( \mu_0 \) are, respectively, the density, non-dimensional distribution and the
characteristic of light particles). The integration variable is defined by the equality $\xi \equiv (u^2 + v^2 - q \cdot v/\mu)/u_0^2$:

$$W_0(q, p) = \frac{n_b \alpha^2 \pi}{2 \mu^2 u_0 q} \int_{\alpha^2 u_0^2 q}^\infty \frac{d\xi \cdot \phi_0(\xi)}{u^2 + v^2 - q \cdot v/\mu}.$$  \hfill (24)

If the distribution $\phi_0(\xi) = 1/\xi^{\gamma}$ (where $\gamma > 1$) has a long tail, we get

$$W_0(q, p) = \frac{n_b \alpha^2 \pi}{2 \mu^2 u_0 q} \frac{\xi^{-1}}{(1 - \gamma)_{\xi_0}} = \frac{n_b \alpha^2 \pi}{2 \mu^2 u_0 q} \frac{\xi_0^{1 - \gamma}}{(1 - \gamma)}.$$  \hfill (25)

where $\xi_0 \equiv (q^2/4\mu^2 + v^2 - q \cdot v/\mu)/u_0^2$.

For the case $q = 0$, $\xi_0 \rightarrow 0$ we arrive at the expression for the anomalous PTF $W \equiv W_0$:

$$W_0(q, p = 0) = \frac{n_b \alpha^2 \pi}{23 - 2\gamma (1 - 1)\mu^4 - 2\gamma \mu q^2 - 1} = \frac{C_a}{2^{2\gamma - 1}}.$$  \hfill (26)

To determine the transport process and the kinetic equation in V-space, one should also find the functions $\tilde{W}(q)$ and $\tilde{W}''(q)$.

To find the functions $\tilde{W}'(q, p)$ and $\tilde{W}''(q, p)$ for $p \neq 0$, we use full-value $\tilde{\xi}_0 \equiv (q^2/4\mu^2 + p^2/M^2 - q \cdot p/M\mu)/u_0^2$ and the derivatives of these functions on $q \cdot p$ at $p = 0$, $\tilde{\xi}_0 = -1/M\mu u_0^2$ and $\tilde{\xi}_0'' = 0$. Then

$$\tilde{W}'(q, p) = \frac{dW}{dp} = \frac{n_b \alpha^2 \pi}{2 M^2 \mu^2 u_0^2 q^{2\gamma + 1}} \xi_0^{-\gamma}, \quad \tilde{W}''(q, p) = \frac{n_b \alpha^2 \pi \gamma}{2 M^2 \mu^2 u_0^2 q^{2\gamma + 1}} \xi_0^{-\gamma - 1}.$$  \hfill (27)

Therefore, for $p = 0 (\tilde{\xi}_0 \rightarrow \tilde{\xi}_0)$ we obtain the functions

$$\tilde{W}(q) = \frac{(4\mu^2 u_0^2)^\gamma n_b \alpha^2 \pi}{2 M^2 \mu^2 u_0^2 q^{2\gamma + 1}} \xi_0^{-\gamma}, \quad \tilde{W}''(q) = \frac{(4\mu^2 u_0^2)^\gamma n_b \alpha^2 \pi \gamma}{2 M^2 \mu^2 u_0^2 q^{2\gamma + 3}}.$$  \hfill (28)

The function $A(s)$, according to equation (12), is given by

$$A(s) = 4\pi \int_0^\infty dq q^2 \left[ \frac{\sin(q s)}{q s} - 1 \right] W(q) = 4\pi C_a \int_0^\infty dq \frac{1}{q^{2\gamma - 3}} \left[ \frac{\sin(q s)}{q s} - 1 \right].$$  \hfill (29)

Comparing the reduced equation (see below) in V-space with diffusion in coordinate space [11, 11], we can establish the correspondence $(2\gamma - 1 \leftrightarrow \alpha$ and $W(q) = C/q^{2\gamma - 1})$. This means that the convergence of the integral on the right-hand side of equation (29) (3D case) is provided if $3 < 2\gamma - 1 < 5$ or $2 < \gamma < 3$. The inequality $\gamma < 3$ provides the convergence for small values of the variable $q$ $(q \rightarrow 0)$, and the inequality $\gamma > 2$ provides the convergence for $q \rightarrow \infty$.

We now establish the conditions of the convergence of the integrals for $B(s)$ and $C(s)$ as

$$B(s) = \frac{4\pi}{s^2} \int_0^\infty dq q^2 \left[ \cos(q s) - \frac{\sin(q s)}{q s} \right] W(q).$$  \hfill (30)

The convergence of the function $B(s)$ exists for small values of $q$ if $\gamma < 2$ and for large values of $q$ $(q \rightarrow \infty)$ if $\gamma > 1/2$.

Finally, the convergence for the function $C(s)$ is defined by the equalities $\gamma < 2$ for small values of $q$ and $\gamma > 1$ for large values of $q$:

$$C(s) = \frac{2\pi}{s^2} \int_0^\infty dq q^4 \left[ \frac{2 \sin(q s)}{q^3 s^3} - \frac{2 \cos(q s)}{q^2 s^2} - \frac{\sin(q s)}{q s} + \frac{1}{3} \right] W(q).$$  \hfill (31)

Therefore, in the case under consideration, the convergence of the functions $A$, $B$ and $C$ for large values of $q$ is defined only by the convergence of the function $A$, which means $\gamma > 2$. 


To provide the convergence for small values of \( q \), it is sufficient to provide convergence for the functions \( B(s) \) and \( C(s) \), which means \( \gamma < 2 \). Therefore, for the purely power behavior of the function \( f_s(\xi) \), the simultaneous convergence of the coefficients \( A, B \) and \( C \) cannot be provided. However, in real physical models, the convergence of the coefficients in the integration region \( q \to 0 \) is always provided, e.g. by a finite value of \( v \) (see equation (25)) or due to the non-power behavior of the PTF \( W \) for small values of \( q \) (compare with the examples of anomalous diffusion in coordinate space [1]). Therefore, in the model under consideration, the ‘anomalous diffusion in V-space’ for the power behavior of \( W(q) \), \( W'(q) \) and \( W''(q) \) at large values of the variable \( q \) exists for the asymptotic behavior of the PTF \( W(q \to \infty) \sim 1/q^{2\gamma-1} \) if the inequality \( \gamma > 2 \) is fulfilled. In this case the integrals for the coefficients \( A(s), B(s) \) and \( C(s) \) converge. At the same time, the expansion of the exponential function in equations (12)–(15) under the integrals, which leads to the Fokker–Planck-type kinetic equation, is invalid for the power-type kernels \( W(q, p) \).

4. General conditions for anomalous diffusion in \( v \)-space

Now let us consider the more general model for which we will not connect the functions \( W(q) \), \( W'(q) \) and \( W''(q) \) with a concrete form of \( W(q, p) \) which is in general unknown. In this case one can suggest that the functions are independent from another power-type \( q \)-dependence.

As an example, this dependence can be taken as the power-type one for three functions \( W(q) \equiv a/q^\alpha, W'(q) \equiv b/q^\beta \) and \( W''(q) \equiv c/q^\eta \), where \( \alpha, \beta \) and \( \eta \) are independent and positive. Then, as follows from the above consideration, the convergence of the integral with the function \( W \) exists if \( 5 > \alpha > 3 \) (for asymptotically small and large \( q \), respectively). For the function \( W'(q) \), the convergence condition for asymptotically small and large values of \( q \) is provided if \( 5 > \beta > 2 \), respectively. Finally, for the function \( W''(q) \), the convergence condition is \( 7 > \eta > 5 \) (for asymptotically small and large values of \( q \), respectively).

For this example, the kinetic equation (equation (11)) reads

\[
\frac{df_g(s, t)}{dt} = P_0 s^{\alpha-1} f(s, t) + s^{\beta-5} P_1 s_j \frac{\partial}{\partial s_j} f(s, t) + s^{\eta-7} P_2 s_i s_j \frac{\partial^2}{\partial s_i \partial s_j} f(s, t),
\]

where

\[
P_0 = 4\pi a \int_0^\infty d\xi \xi^{2-\alpha} \left[ \frac{\sin \xi}{\xi} - 1 \right]
\]

\[
P_1 = 4\pi b \int_0^\infty d\xi \xi^{2-\beta} \left[ \cos \xi - \frac{\sin \xi}{\xi} \right]
\]

\[
P_2 = 4\pi c \int_0^\infty d\xi \xi^{4-\eta} \left[ \frac{\sin \xi}{\xi^3} - \frac{\cos \xi}{\xi^2} - \frac{\sin \xi}{2\xi} + \frac{1}{6} \right].
\]

As is easy to show, \( P_0 = -|P_0| \text{sgn} \ a < 0 \) and \( P_1 = -|P_1| \text{sgn} \ b \). Taking into account the isotropy in \( s \)-space we can rewrite equation (32) in the form

\[
\frac{df_g(s, t)}{dt} = P_0 s^{\alpha-3} f(s, t) + s^{\beta-4} P_1 s_j \frac{\partial}{\partial s_j} f(s, t) + s^{\eta-5} P_2 \frac{\partial^2}{\partial s^2} f(s, t).
\]

Naturally, equations (32) and (36) can be formally rewritten in momentum (or in velocity) space in terms of fractional derivatives of various orders. Therefore, as is easy to see, for the purely power behavior of the functions \( W(q) \), \( W'(q) \) and \( W''(q) \) the solution with the convergent coefficients exists for the values of the powers in the intervals mentioned above. In the case under consideration the universal type of the anomalous diffusion in V-space exists
if $5 > \alpha > 3$, $5 > \beta > 2$ and $7 > \eta > 5$. This takes place even in the cases, when the functions $W(q)$, $\tilde{W}'(q)$ and $\tilde{W}''(q)$ do not have cut-off for small values of $q$. Surely, the general description is also valid for more complicated functions $W$, $W'$ and $W''$, characterized by the non-power $q$-dependence for small values of $q$.

Now let us take into account an important circumstance; in general, there is a small parameter $\mu/M$ in the problem under consideration, which can simplify the description of velocity diffusion. As is easy to see, e.g. based on the particular cases (see equation (28)) for the convergent kernels of anomalous transport, the term with the second derivative $\tilde{W}''$ in general equations (11) and (16) for the distribution $f_q(p)$ is small (in comparison with the term with the first derivative). The similar smallness is observed for the case of normal diffusion in V-space. This smallness is of the order of the small ratio $\mu/M$ of particle masses.

Therefore, for the most physically important kernels describing anomalous velocity diffusion, the term with the second space derivative can be omitted and the non-stationary general diffusion equation is given by

$$\frac{df_g(s,t)}{dt} = A(s) f_g(s,t) + B(s) \frac{\partial f_g(s,t)}{\partial s}$$

(37)

or, for the isotropic case,

$$\frac{df_g(s,t)}{dt} = A(s) f_g(s,t) + B(s) s \frac{\partial f_g(s,t)}{\partial s}.$$  

(38)

In the case of the purely power behavior, $W(q) = a/q^\alpha$ and $W'(q) = b/q^\beta$, we have, as above, $A(s) = P_0 s^{\alpha-3}$ and $sB(s) = P_1 s^{\beta-4}$ (with the inequalities $5 > \alpha > 3$ and $5 > \beta > 2$).

The stationary solution of equation (38) (see, also (19)) for the case under consideration is written as

$$f_{g}^{st}(s) = C \exp \left[ - \int_{s_0}^{s} ds' \frac{A(s')}{s' B(s')} \right] = C \exp \left[ - \frac{P_0 s^{\alpha-\beta+2}}{P_1 (\alpha - \beta + 2)} \right].$$

(39)

where $5 > \alpha - \beta + 2 > 0$.

5. Solution for velocity distribution in the non-stationary case

To find the solution in the isotropic non-stationary case, equation (38) should be written as

$$\frac{dX(s,t)}{dt} - B(s) s \frac{\partial}{\partial s} X(s,t) = A(s),$$

(40)

where $X(s,t) \equiv \ln f_g(s,t)$. The general non-stationary solution of this equation can be written as the sum of the general solution of the homogeneous equation $Y(s,t)$ (equation (40)), where the function $A(s)$ is taken zero,

$$Y(s,t) = \Phi(\xi), \quad \xi \equiv t + \int_{s_0}^{s} ds' \frac{1}{s' B(s')}.$$  

(41)

and the particular solution $Z(s,t)$ of the inhomogeneous equation (equation (40)). Here $\Phi$ is the arbitrary function of the variable $\xi$. The particular solution $Z(s,t)$ of the inhomogeneous equation (40) reads

$$Z(s,t) \equiv f_{g}^{st}(s) = - \int_{s_0}^{s} ds' \frac{A(s')}{s' B(s')}.$$  

(42)

Therefore, we find

$$f_g(s,t) = \exp [X(s,t)] \equiv \exp [Y + Z] = L(\xi) f_{g}^{st}(s).$$  

(43)
where \( L(\xi) \) is the arbitrary function of \( \xi \), which should be found from the initial condition \( f_g(s, t = 0) = \phi_0(s) \).

The variable \( \xi \equiv \xi(s, t) \) equals

\[
\xi(s, t) = t + \int_{s_0}^s \frac{1}{s'B(s')} = t + \frac{s^{-\beta}}{P_1(5 - \beta)} + c, 
\]

(44)

where \( c \) is the arbitrary constant which can be omitted due to the presence of the arbitrary function \( L \).

The general non-stationary solution for the case under consideration reads

\[
f_g(s, t) = L\left( t + \frac{s^{-\beta}}{P_1(5 - \beta)} \right) \exp \left[ -\frac{P_0s^{\alpha - \beta + 2}}{P_1(\alpha - \beta + 2)} \right]. 
\]

(45)

The inequalities for the combinations of the above coefficients are \( 5 > \alpha - \beta + 2 > 0, \ 3 > 5 - \beta > 0 \). The unknown function \( L \) can be found from equation (45) and the initial condition \( f_g(s, 0) = \phi_g(s) \):

\[
L\left( \frac{s^{-\beta}}{P_1(5 - \beta)} \right) \exp \left[ -\frac{P_0s^{\alpha - \beta + 2}}{P_1(\alpha - \beta + 2)} \right] = \phi_g(s). 
\]

(46)

The function \( \phi_g(s) = \int d^3p \exp(ip\cdot s) f_g(p, t = 0) \) is the Fourier component of the initial distribution in momentum space. Using the notation \( \zeta \equiv s^{5 - \beta}/[P_1(5 - \beta)] \) (which means \( s(\zeta) \equiv [P_1(5 - \beta)\zeta]^{1/(5 - \beta)} \)), we find

\[
L(\zeta) = \phi_g[s(\zeta)] \exp \left\{ \frac{P_0s(\zeta)^{\alpha - \beta + 2}}{P_1(\alpha - \beta + 2)} \right\}. 
\]

(47)

Therefore, the time-dependent solution is

\[
f_g(s, t) = \phi_g[s(\zeta + t)] \exp \left\{ \frac{P_0s(\zeta + t)^{\alpha - \beta + 2}}{P_1(\alpha - \beta + 2)} \right\} \exp \left[ -\frac{P_0s^{\alpha - \beta + 2}}{P_1(\alpha - \beta + 2)} \right], 
\]

(48)

where we express the value \( s(\zeta + t) \equiv [P_1(5 - \beta)(\zeta + t)]^{1/(5 - \beta)} \) as the evident function of variables \( s, t \):

\[
s(\zeta + t) \equiv [P_1(5 - \beta)(\zeta + t)]^{1/(5 - \beta)} \equiv [s^{5 - \beta} + P_1(5 - \beta)t]^{1/(5 - \beta)} \equiv [s^{5 - \beta} + P_1(5 - \beta)t]^{1/(5 - \beta)}. 
\]

(49)

or, finally,

\[
f_g(s, t) = \phi_g([s^{5 - \beta} + P_1(5 - \beta)t]^{1/(5 - \beta)}) 
\times \exp \left\{ \frac{P_0[s^{5 - \beta} + P_1(5 - \beta)t]^{(\alpha - \beta + 2)/(5 - \beta)} - P_0s^{\alpha - \beta + 2}}{P_1(\alpha - \beta + 2)} \right\}. 
\]

(50)

It should be emphasized that the real solution for the fractional powers \( 1/(5 - \beta) \) and \( (\alpha - \beta + 2)/(5 - \beta) \) exists only if \( P_1 > 0 \). The limit \( t \to \infty \) for the solution can be identical to the stationary solution only for specific initial conditions. For these cases, the stationary solution can be non-equilibrium.

6. Particular cases: representation of the diffusion equation in fractional derivatives

For the power dependence of the functions \( W(q) \) and \( \tilde{W}'(q) \), equation (37) can be formally written in terms of fractional derivatives:

\[
\frac{d f_g(p, t)}{dt} = P_0 D^{\nu} f_g(p, t) - P_1 (3 + \gamma) D^{\nu} f_g(p, t) + P_1 p_\alpha D^{\nu + 1} f_g(p, t), 
\]

(51)

where \( \nu \equiv \alpha - 3, \gamma \equiv \beta - 5 \) and \( D^{\nu + 1} f_g(p, t) \equiv i \int d^3s \exp(-ips)s^{\nu} f_g(s, t) \).
Let us now consider formally the particular case of anomalous diffusion in V-space, when the specific structure of the PTF $W(q, p)$ provides a rapid (let us say, exponential) decrease in the functions $W'(q)$ and $W''(q)$. Therefore, the exponential function under the integrals in the coefficients $B(s)$ and $C(s)$ can be expanded, which means $B(s) = B_0$ and $C(s) \simeq 0$ respectively. At the same time, the function $W(q) = a/q^\alpha$ has a purely power dependence on $q$.

In this particular case, the kinetic equation (36) reads

$$\frac{df_g(s, t)}{dt} = P_0 s^{\alpha-3} f_g(s, t) + B_0 \frac{\partial}{\partial s_j} f_g(s, t),$$

or, formally, in the momentum space

$$\frac{df_g(p, t)}{dt} = P_0 D^\nu f_g(p, t) - B_0 \frac{\partial}{\partial p_i} [p_i f_g(p, t)].$$

where $\nu \equiv (\alpha - 3)(2 > \nu > 0)$, and we introduced the fractional differentiation operator $D^\nu f(p, t) \equiv \int ds \exp(-isp) f(s, t)$ in momentum space. This is a very particular case of the general equations (23) and (36). In this case the result is similar to the particular phenomenological consideration [17], where a model of the Langevin equation with a constant friction frequency $\nu_0 = -B_0$ has been considered. In [18] a similar model with the fractional derivative $\nu = 3/2$ has been applied to estimate the fusion rate in a hot rarified plasma. The more elaborated analysis of the anomalous velocity diffusion in the plasma system on the basis of the self-consistent approach [19] is done in [25].

The stationary solution of equation (52) is given by

$$f_g(s) = C \exp \left[ -\frac{P_0 s^\nu}{v B_0} \right],$$

$$f_g(p) = C \int d^3 s \exp(-isp) \exp \left[ -\frac{P_0 s^\nu}{v B_0} \right] \approx \frac{4\pi C}{p} \int_0^{\infty} ds \sin(ps) \exp \left[ -\frac{P_0 s^\nu}{v B_0} \right].$$

As an example, let us choose the case $\nu = 1$. Then, we find $f_g(p)$

$$f_g(p) = \frac{8\pi C P_0}{B_0 \left[ (p^2 + 4P_0^2/B_0^2)^{3/2} \right]}.$$  

In the case $\nu = 1$ the long tail of the distribution is proportional to $p^{-4}$ and the distribution $f_g(p)$ corresponds with the Cauchy–Lorentz distribution. Normalization of the distribution $f_g(p)$ leads to the value $C = n_g/(2\pi)^3$, where $n_g = N_g/V$ is the average density of particles undergoing diffusion in V-space. A similar approach can be taken for other types of anomalous diffusion in V-space.

7. Diffusion model based on Boltzmann collisions with drift

Let us consider the simplest case of the non-equilibrium but stationary distribution $f_b$, i.e. the shifted velocity distribution.

The evident generalization of the PTF $w^b_\theta(q, p)$ for this case (characterized by the drift velocity $u_d$) in the distribution function $f_b$ is given by

$$w^b_\theta(q, p, u_d) = \frac{2\pi}{\mu^2 q} \int_{q/2\mu}^{\infty} du u \cdot \frac{d\sigma}{d\omega} \left[ \arccos \left( 1 - \frac{q^2}{2\mu^2 u^2} \right), u \right] \times f_b(u^2 + (v - u_d)^2 - q \cdot (v - u_d)/\mu).$$

$$\text{(57)}$$
where $\mathbf{p} = M\mathbf{v}$. Again, as in section 2, to find the coefficients in the kinetic equation, let us use the difference between the velocities of light and heavy particles. At the same time, the drift velocity $u_d$, generally speaking, is not small in comparison with the current characteristic velocities $u$ and $\mathbf{q}/\mu$ of small particles.

To calculate the function $A_\alpha$, we have taken into account that the scalar function $w^d_{q}(\mathbf{q}, \mathbf{p}, u_d)$ can in general be written in the form $W(\mathbf{q}, \mathbf{p}, u_d) = W(q, p, u_d, l, \xi, \eta)$ (here $l \equiv \langle \mathbf{q} \cdot \mathbf{p} \rangle$) and expanded on $\xi \equiv \langle \mathbf{q} \cdot \mathbf{p} \rangle$ and $\eta \equiv \langle M\mathbf{u}_d \cdot \mathbf{p} \rangle = \langle \mathbf{p}_d \cdot \mathbf{p} \rangle$. In fact, it is the expansion in the velocity $\mathbf{v} = \mathbf{p}/M$, which is small in comparison with the other characteristic velocities $\mathbf{q}/\mu$, $u$ and, in the general case, $u_d$. As shown above (for the case $u_d = 0$), to arrive at the simple and solvable equation for the distribution function $f_\alpha$, taking into account the smallness of $\mathbf{v}$ in comparison with characteristic velocities and $v^2$ in comparison with $\mathbf{v}$, we approximate the function $W(q, p, u_d, l, \xi, \eta) \approx W(q, v = 0, u_d, l, \xi, \eta)$, because we are mainly interested in high values of $q$ for the anomalous transport. At the same time, after this approximation and expansion on $\xi$ and $\eta$ (for $u_d \neq 0$), we arrive (for the special case of kernels with the purely power-type $q$-dependence for small $q$) at the divergence of some coefficients of the diffusion equation. A similar situation takes place also for anomalous diffusion in coordinate space. This divergence is absent for the realistic PTF, which provides the cut-off for small values of $q$. This cut-off has the physical nature and is not related to the approximation $W(q, p, u_d) \equiv W(q, p, u_d, l, \xi, \eta) \approx W(q, v = 0, u_d, l, \xi, \eta)$.

Let us expand the PTF $W(q, p, u_d, l, \xi, \eta)$ in $\xi$ and $\eta$ similarly to the more simple case (5):

$$W(q, p, u_d) = W(q, p, u_d, l, \xi, \eta) \approx W_0(q, p, u_d, l) + U_1(q, p, u_d, l) \xi + U_2(q, p, u_d, l) \xi^2$$

Here $W_0(q, p, u_d, l) = W(q, p, u_d, l, \xi, \eta) |_{\xi, \eta = 0}$. Then, introducing the functions $V_1(q, p, u_d, l) = \partial W/\partial \xi |_{\xi, \eta = 0}$, $U_1(q, p, u_d, l) = \partial W/\partial \eta |_{\xi, \eta = 0}$, $V_2(q, p, u_d, l) = \partial^2 W/\partial \xi^2 |_{\xi, \eta = 0}$, $U_2(q, p, u_d, l) = \partial^2 W/\partial \xi \partial \eta |_{\xi, \eta = 0}$, $W_2(q, p, u_d, l) = \partial^2 W/\partial \xi \partial \eta |_{\xi, \eta = 0}$, we can rewrite equation (58) in the form

$$W(q, p, u_d, l, \xi, \eta) \approx W_0(q, p, u_d, l) + V_1(q, p) + U_1(p_d \cdot p) + V_2(q \cdot p)^2 + U_2(p_d \cdot p)^2 + W_2(q \cdot p)(p_d \cdot p)^2 \approx W_0(q, p = 0, u_d, l) + V_1(q, p = 0, u_d, l)(q \cdot p) + U_1(q, p = 0, u_d, l)(p_d \cdot p).$$

Finally, we set $p = 0$ in the coefficients of equation (59) and omit the terms with the second-order derivatives due to the existence of the small parameter (e.g. $\mu/M$). This means that only the terms of the order of $\sqrt{\mu/M}$ are essential in the expansion of $W(q, p, u_d)$. Let us calculate $W(q, p + q, u_d)$ taking into account the difference between the values of the characteristic momenta $q$, $p + q$, $u_d$ and the characteristic values of velocities.

The appropriate expansion for the function $W(q, p + q, u_d)$ with the necessary accuracy is written as

$$W(q, p + q, u_d) \approx W(q, p, u_d) + \left[q_a \partial/\partial p_a + \frac{1}{2} q_a q_m \frac{\partial^2}{\partial p_a \partial p_m} \right] W(q, p, u_d)$$

or

$$W(q, p + q, u_d) \approx W_0(q, p = 0, u_d, l) + V_1(q, p = 0, u_d, l)(q \cdot p) + U_1(q, p = 0, u_d, l)(q \cdot p_d) + U_1(q, p = 0, u_d, l)[q \cdot (p + q)]$$

$$+ U_1(q, p = 0, u_d, l)[p_d \cdot (p + q)].$$

(61)
Then the collision term of the kinetic equation can be given by the formula
\[
\frac{d}{dt}f_g(p, t) = \int dq \left\{ W_0(q, p = 0, u_d, l) + V_1(q, p = 0, u_d, l)(q \cdot p) \\
+ U_1(q, p = 0, u_d, l)(p_d \cdot p) + V_1(q, p = 0, u_d, l)q^2 \\
+ U_1(q, p = 0, u_d, l)(p_d \cdot (q \cdot p)) - \left[ W_0(q, p = 0, u_d, l) \\
+ V_1(q, p = 0, u_d, l)(p_d \cdot p) + U_1(q, p = 0, u_d, l)(p_d \cdot p) \right] f_g(p, t) \right\},
\]
(62)

After the Fourier transformation (10), equation for the distribution function
\[
f_g(s, t) = \int \frac{d}{dp} \left[ \exp(i ps) \right] f_g(p, t)
\]
can be written in the form
\[
\frac{d}{dt}f_g(s, t) = \int dq \left\{ \exp(-i qs) \left[ W_0(q, p = 0, u_d, l) - i V_1(q, p = 0, u_d, l) \left( q \cdot \frac{\partial}{\partial s} \right) \\
- i U_1(q, p = 0, u_d, l) \left( p_d \cdot \frac{\partial}{\partial s} \right) \right] f_g(s, t) - \left[ W_0(q, p = 0, u_d, l) \\
- i V_1(q, p = 0, u_d, l) \left( q \cdot \frac{\partial}{\partial s} \right) - i U_1(q, p = 0, u_d, l) \left( p_d \cdot \frac{\partial}{\partial s} \right) \right] f_g(s, t) \right\}.
\]
(63)

Therefore, we arrive at the equation similar to equation (37), but with the evidently non-isotropic structure:
\[
\frac{d}{dt}f_g(s, t) = A_d(s, p_d) f_g(s, t) + B_d(s, p_d) \frac{\partial}{\partial s} f_g(s, t),
\]
(64)

where the coefficients are expressed as the integrals
\[
A_d(s, p_d) = \int dq \left[ \exp(-i qs) - 1 \right] W_0(q, p = 0, u_d, l),
\]
(65)
\[
B_d(s, p_d) = -i \int dq \left[ \exp(-i qs) - 1 \right] \left[ V_1(q, p = 0, u_d, l) q + U_1(q, p = 0, u_d, l) p_d \right] \equiv s B'_d(s, p_d) + p_d B''_d(s, p_d).
\]
(66)

This equation will be analyzed in detail in a separate paper.

8. Conclusions

In this paper, the problem of anomalous diffusion in momentum (velocity) space has been consistently considered. The new kinetic equation for anomalous diffusion in V-space has been derived for the general case of the non-equilibrium PTF.

It should be emphasized that these types of anomalous diffusion are not related to the time dispersion (the non-local in time PTF). Anomalous diffusion in V-space created by the non-local in time master equation has been considered in a few papers (see, e.g., [15, 16, 23, 24]) on the basis of the well-known Fokker–Planck kernel in p-space.

The similar problem with the non-local in time PTFs, which at the same time have long q-tails, has been considered for diffusion in V-space separately, based on the results obtained in this paper.

The evolution of the distribution function can lead to the stationary non-equilibrium limit or be asymptotically time dependent. The model of anomalous diffusion in V-space is described based on the respective expansion of the kernel in the master equation. The conditions of the convergence of the coefficients of the new kinetic equation have been found
for particular cases. The wide variety of anomalous processes in V-space exists, since two (C is usually small) different coefficients enter the general diffusion equation even in the isotropic case.

The examples of anomalous diffusion for heavy particles, based on the Boltzmann kernel with the prescribed equilibrium and non-equilibrium distribution function for the light particles, have been studied. The PTF for the Boltzmann kernel depends on the appropriate cross-section for scattering of two types of colliding particles and on the prescribed velocity distribution for one sort (light in the case under consideration) of particles. The particular case of the hard-sphere interaction has been considered. The Einstein relation is applicable only to the equilibrium PTF (hence, the equilibrium prescribed distribution of light particles). In the general case, even for the stationary solution, the Einstein relation between the diffusion and friction coefficients is absent. For normal diffusion, the friction and diffusion coefficients have been explicitly found for the non-equilibrium case. The stationary and non-stationary (with an initial condition) solutions of the general kinetic equation have been found. In the equilibrium case, the known Fokker–Planck equation is reproduced as a particular case. We also have shortly reviewed anomalous diffusion in coordinate space.

The kinetic equation obtained in this paper is applicable to various systems in physics (including plasma physics [25]), chemistry and biology. Special attention has to be paid to the recent experiments which indicate superdiffusion in two-dimensional (r = 2) complex plasma [26–28] and simulations which could explain the dimensionality dependence [29] and prove the transient character of superdiffusion in thermodynamic equilibrium [27]. We will consider these problems in a separate publication devoted to the plasma and based on the theory developed in [19, 25] and in this paper.

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