Periodic solutions of a semilinear variable coefficient wave equation under asymptotic nonresonance conditions

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Abstract We consider the periodic solutions of a semilinear variable coefficient wave equation arising from the forced vibrations of a nonhomogeneous string and the propagation of seismic waves in nonisotropic media. The variable coefficient characterizes the inhomogeneity of media and its presence usually leads to the destruction of the compactness of the inverse of the linear wave operator with periodic-Dirichlet boundary conditions on its range. In the pioneering work of Barbu and Pavel (1997), they gave the existence and regularity of the periodic solution for Lipschitz, nonresonant and monotone nonlinearity under the assumption \( \eta u > 0 \) (see Section 2 for its definition) on the coefficient \( u(x) \) and left the case \( \eta u = 0 \) as an open problem. In this paper, by developing the invariant subspace method and using the complete reduction technique and Leray-Schauder theory, we obtain the existence of periodic solutions for such a problem when the nonlinear term satisfies the asymptotic nonresonance conditions. Our result needs neither requirements on the coefficient except the natural positivity assumption (i.e., \( u(x) > 0 \)) nor the monotonicity assumption on the nonlinearity. In particular, when the nonlinear term is an odd function and satisfies the global nonresonance conditions, there is only one (trivial) solution to this problem in the invariant subspace.

Keywords periodic solutions, wave equation, asymptotic nonresonance conditions

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1 Introduction

In the present paper, our concern is the existence of the periodic solutions of the semilinear variable coefficient wave equation

\[
\begin{align*}
  u(x)y_{tt} - (u(x)y_x)_x &= f(t, x, y), & (t, x) &\in \Omega := (0, T) \times (0, \pi)
\end{align*}
\] (1.1)

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with the periodic conditions
\[ y(0, x) = y(T, x), \quad y_t(0, x) = y_t(T, x), \quad x \in (0, \pi) \]  
(1.2)
and the Dirichlet boundary conditions
\[ y(t, 0) = y(t, \pi) = 0, \quad t \in (0, T), \]  
(1.3)
where the nonlinear term \( f(t, x, y) \in C(\Omega \times \mathbb{R}) \) is periodic in time \( t \) with a prescribed period \( T \) satisfying
\[ T = 2\pi \frac{p}{q} \]  
(1.4)
with \( p, q \in \mathbb{N}^+ = \{1, 2, 3, \ldots\} \) and their greatest common divisor being 1.

Equation (1.1) is a mathematical model to describe the forced vibrations of a nonhomogeneous string and the propagation of seismic waves in nonisotropic media (see, e.g., [1,19,34,35]). The variable coefficient \( \eta = (\mu)_{1/2} \), which is called the acoustic impedance function, arises naturally when the vibration equation \( \rho(l) y_{tt} - (\mu(l)) y_t \) is normalized into \( u(x) y_{tt} - (u(x)) y_t \) via the change of variables
\[ x = \int_0^l \left( \frac{\rho(s)}{\mu(s)} \right)^{1/2} ds. \]

It is obvious that the case of \( u(x) \equiv 1 \) corresponds to the classical (one-dimensional) wave equation. The problem of finding periodic solutions of such equations has been widely considered since the pioneering work of Rabinowitz [26]. For example, see [2,5,7,8,12,25,27] for the one-dimensional case and [3,4,6,11,33] for the higher-dimensional case. Most of these results are based on the spectral properties of the wave operator. In particular, for the Dirichlet boundary value problems, when the time period \( T \) is a rational multiple of string length \( \pi \), then 0 is the eigenvalue with infinite multiplicity of the wave operator \( \partial_{tt} - \partial_{xx} \), and the remaining eigenvalues are well separated and accumulate to infinity. This good separation property implies that the inverse of \( \partial_{tt} - \partial_{xx} \) is compact on its range. So the compactness method can be applied to estimating the component in the range space of a periodic solution. In general, to estimate the component in the kernel space needs to require that the nonlinearity be monotone. Thus, the nonlinear problems can be solved by using compactness and monotonicity methods (see, e.g., [9,13,14,30,31]). On the other hand, when \( T \) is an irrational multiple of \( \pi \), 0 is an accumulation point of the spectrum, and thus it involves the small divisor problem and the variational method does not work. So a quite different approach, the infinite-dimensional KAM (Kolmogorov-Arnold-Moser) theory, is proposed to deal with such problems (see [22,32]).

For the variable coefficient wave equations, Barbu and Pavel [1] firstly established the existence and regularity of the periodic solution for Lipschitz, nonresonant and monotone nonlinearity under the assumption
\[ \eta_u(x) = \frac{1}{2} \frac{u''}{u} - \frac{1}{4} \left( \frac{u'}{u} \right)^2 > 0 \]
on the coefficient \( u(x) \) and left the case \( \eta_u = 0 \) as an open problem. Subsequently, the research on periodic solutions of variable coefficient wave equations has gained more attention (see, e.g., [10,16,18,20,23,28,29]). It is worth mentioning that Ji and Li [21] obtained the existence of the periodic solution of the nonlinear wave equation with sublinear nonlinearity and constant or variable coefficients which may not satisfy the assumption \( \eta_u(x) > 0 \). Recently, Ji [17] found that for some types of boundary value conditions but not including the Dirichlet boundary value condition, the inverse of the variable coefficient wave operator is still compact on its range. Thus, without imposing the assumption \( \eta_u(x) > 0 \), Ji [17] got the existence of periodic solutions for the monotone and bounded nonlinearity.

In this paper, we study the periodic solutions of the variable coefficient wave equation (1.1) with the periodic conditions (1.2) and the Dirichlet boundary value conditions (1.3). This is one type of the most basic boundary conditions, but was left unsolved in [17] due to the loss of compactness of the inverse of
the variable coefficient wave operator. Here, by further research on the spectrum of the variable coefficient wave operator, we construct an invariant subspace in which the spectrum has a good separation property. Then, under some symmetry condition on the nonlinear term, the problem (1.1)–(1.3) can be reduced onto this invariant subspace. Finally, by employing the Leray-Schauder type method, we obtain the existence of the periodic solution to the problem (1.1)–(1.3) when the nonlinear term satisfies the asymptotic nonresonance conditions in a nonuniform manner with respect to $t$ and $x$. The result we obtained does not require any restrictive conditions on the coefficient except the natural positivity assumption $u(x) > 0$, and it does not need the monotonicity assumption on the nonlinearity. In particular, when the nonlinear term is odd with respect to $y$ and satisfies the global nonresonance conditions (see (4.8)), the problem (1.1)–(1.3) has only one (trivial) solution in the invariant subspace.

Define

\[ \hat{f}(t,x,y) = \frac{f(t,x,y)}{u(x)}. \]

Throughout this paper we make the following assumptions:

(H1) $u(x) \in C^2[0,\pi]$ and $u(x) > 0$ for all $x \in [0,\pi]$.

(H2) For every constant $R > 0$, there exists $h_R(t,x) \in L^2(\Omega)$ such that

\[ |\hat{f}(t,x,y)| \leq h_R(t,x) \text{ for } |y| \leq R. \tag{1.5} \]

The rest of this paper is organized as follows. In Section 2, we introduce the definition of the weak periodic solution and characterize some properties of the spectrum of the variable coefficient wave operator. Then, two invariant subspaces of $L^2(\Omega)$ are constructed and some fundamental lemmas are given in Section 3. Finally, we establish and prove the main results (see Theorems 4.2 and 4.3) in Section 4.

2 Preliminaries

Define

\[ \Psi = \{ \psi \in C^\infty(\Omega) \mid \psi(t,0) = \psi(t,\pi) = 0, \psi(0,x) = \psi(T,x), \psi_t(0,x) = \psi_t(T,x) \} \]

and

\[ L^r(\Omega) = \left\{ y \left\| y \right\|_{L^r(\Omega)} = \left( \int_\Omega u(x) |y(t,x)|^r dt dx \right)^{\frac{1}{r}} < \infty \right\}. \]

In particular, for $r = 2$, $L^2(\Omega)$ is a Hilbert space with the inner product

\[ \langle y, z \rangle = \int_\Omega u(x) y(t,x) \overline{z(t,x)} dt dx, \quad \forall y, z \in L^2(\Omega). \]

For the sake of convenience, in what follows we use $\| \cdot \|$ to denote $\| \cdot \|_{L^2(\Omega)}$.

**Definition 2.1.** A function $y \in L^2(\Omega)$ is called a weak solution of the problem (1.1)–(1.3) if it satisfies

\[ \int_\Omega y(u(x) \psi_{tt} - (u(x) \psi_x)_x) dt dx - \int_\Omega f(t,x,y) \psi dt dx = 0, \quad \forall \psi \in \Psi. \]

For the study of the periodic solution of the problem (1.1)–(1.3), we need to analyze the spectrum of the variable coefficient wave operator, which is closely related to the following Sturm-Liouville problem:

\[ (u(x) \varphi_n'(x))' = -\lambda_n^2 u(x) \varphi_n(x), \quad n \in \mathbb{N}^+, \]

\[ \varphi_n(0) = \varphi_n(\pi) = 0. \]

In virtue of [15, Section 4], it is known that

\[ \lambda_n^2 = \left( n + \frac{\kappa}{2n\pi} + O\left( \frac{1}{n^3} \right) \right)^2. \]
with
\[ \kappa = \int_0^\pi \eta_u(x) dx, \quad \eta_u(x) = \frac{1}{2} u'' - \frac{1}{4} \left( \frac{u'}{u} \right)^2. \]
A direct calculation shows that
\[ \lambda_n^2 = n^2 + \frac{\kappa}{\pi} + O\left( \frac{1}{n^2} \right), \quad n \in \mathbb{N}^+. \]
Assume that \( \varphi_n(x) \) is the corresponding normalized eigenfunction which satisfies
\[ \int_0^\pi u(x)|\varphi_n(x)|^2 dx = 1. \]
Then the function system
\[ \left\{ T_m \varphi_n \cos \frac{2\pi}{T} mt, T_m \varphi_n \sin \frac{2\pi}{T} mt \right\}, \quad m \in \mathbb{N} = \mathbb{N}^+ \cup \{0\}, \quad n \in \mathbb{N}^+ \]
with \( T_0 = 1/\sqrt{T} \) and \( T_m = \sqrt{2/T} \) for \( m \in \mathbb{N}^+ \), is completely orthonormal in \( L^2(\Omega) \). Thus, in view of \( T = 2\pi \frac{p}{q} \) in (1.4), a function \( y \in L^2(\Omega) \) can be written as the Fourier series
\[ y = \sum_{n=1}^\infty \sum_{m=0}^\infty \left( a_{mn} \cos \frac{q}{p} mt + b_{mn} \sin \frac{q}{p} mt \right) \quad (2.1) \]
with
\[ a_{mn} = T_m \int_\Omega u(x) \varphi_n(x) y(t,x) \cos \frac{q}{p} mt \, dx, \]
\[ b_{mn} = T_m \int_\Omega u(x) \varphi_n(x) y(t,x) \sin \frac{q}{p} mt \, dx \]
for \( m \in \mathbb{N} \) and \( n \in \mathbb{N}^+ \).
Set
\[ \mathcal{D}(L) = \left\{ y \in L^2(\Omega) \left| \sum_{n=1}^\infty \sum_{m=0}^\infty \left( \lambda_n^2 - \left( \frac{q}{p} m \right)^2 \right)^2 (a_{mn}^2 + b_{mn}^2) < \infty \right. \right\}. \]
Then for any \( y \in \mathcal{D}(L) \) with the Fourier expansion (2.1), we define the operator \( L : \mathcal{D}(L) \to L^2(\Omega) \) as
\[ y \mapsto Ly = \sum_{n=1}^\infty \sum_{m=0}^\infty \left( \lambda_n^2 - \left( \frac{q}{p} m \right)^2 \right) T_m \varphi_n(x) \left( a_{mn} \cos \frac{q}{p} mt + b_{mn} \sin \frac{q}{p} mt \right). \]
It is obvious that
\[ \mu_{mn} = \lambda_n^2 - \left( \frac{q}{p} m \right)^2 \]
are the eigenvalues of \( L \), and their set is denoted by
\[ \Lambda(L) = \{ \mu_{mn} \mid m \in \mathbb{N}, n \in \mathbb{N}^+ \}. \]
Proposition 2.2. Assume that \( p \) is an even number. Then for odd \( m \), the eigenvalues \( \mu_{mn} \) are isolated and have finite multiplicity.
Proof. If \( p \) is even, then \( q \) is odd. Thus, for odd \( m \), we have \( np \neq mq \). Therefore, it is easy to know
\[ \mu_{mn} = \lambda_n^2 - \left( \frac{q}{p} m \right)^2 = \frac{(np - mq)(np + mq)}{p^2} + \frac{\kappa}{\pi} + O\left( \frac{1}{n^2} \right) \to \infty, \quad (2.2) \]
as \( m, n \to \infty \). The proof is completed.
3 Invariant subspaces of $L^2(\Omega)$ and some fundamental lemmas

In this section, we assume that $p$ is an even number. By Proposition 2.2, we may construct an invariant subspace of $L^2(\Omega)$ in which the spectrum of the restricted operator has a good separation property. Then we restrict the problem (1.1)–(1.3) in this invariant subspace and give several fundamental results.

**Definition 3.1.** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be the closed subspaces of $L^2(\Omega)$ such that $L^2(\Omega) = \mathcal{M}_1 \oplus \mathcal{M}_2$ and $\mathcal{M}_1 \perp \mathcal{M}_2$. The operator $L$ is called to be completely reduced by $\mathcal{M}_1$ and $\mathcal{M}_2$ if it satisfies

$$P_i(\mathcal{D}(L)) \subset \mathcal{D}(L), \quad LP_i(y) = P_iL(y), \quad i = 1, 2$$

for any $y \in \mathcal{D}(L)$, where $P_i$ is the projection onto $\mathcal{M}_i$ for $i = 1, 2$.

The restriction of $L$ on $\mathcal{M}_i$ is denoted by $L_i = L|_{\mathcal{M}_i \cap \mathcal{D}(L)}$ for $i = 1, 2$. Then $L_i$ inherits the properties of $L$ and $\Lambda(L_i) \subset \Lambda(L)$ for $i = 1, 2$.

Define

$$\mathcal{M}_o = \{ y \in L^2(\Omega) \mid y(t, x) = -y(t + T/2, x) \}$$

and

$$\mathcal{M}_e = \{ y \in L^2(\Omega) \mid y(t, x) = y(t + T/2, x) \}.$$  

It is not difficult to verify that

$$\mathcal{M}_o = \text{Span}\left\{ \varphi_n(x) \cos \frac{\alpha}{p} t, \varphi_n(x) \sin \frac{\alpha}{p} t \mid n \in \mathbb{N}^+, m \text{ is odd} \right\},$$

$$\mathcal{M}_e = \text{Span}\left\{ \varphi_n(x) \cos \frac{\alpha}{p} t, \varphi_n(x) \sin \frac{\alpha}{p} t \mid n \in \mathbb{N}^+, m \text{ is even} \right\}$$

and $L$ is completely reduced by $\mathcal{M}_o$ and $\mathcal{M}_e$.

For simplicity, define

$$L_o = L|_{\mathcal{M}_o \cap \mathcal{D}(L)} \quad \text{and} \quad L_e = L|_{\mathcal{M}_e \cap \mathcal{D}(L)}.$$  

Then by Proposition 2.2, it is known that $\dim(\ker L_o) < \infty$ and $\Lambda(L_o)$ is an unbounded discrete set. Moreover, in a standard way (see, e.g., [1,17]), it is easy to verify that $\mathcal{M}_o = \ker L_o \oplus \mathcal{R}(L_o)$ and $L_o^{-1}$ is compact on $\mathcal{R}(L_o)$, where $\mathcal{R}(L_o)$ denotes the range of $L_o$.

**Lemma 3.2.** Let $\lambda, \tilde{\lambda} \in \Lambda(L_o)$ be two consecutive eigenvalues. Assume $\alpha, \beta \in \mathcal{M}_e \cap L^\infty(\Omega)$ satisfying

$$\lambda \leq \alpha(t, x) \leq \beta(t, x) \leq \tilde{\lambda} \quad \text{for a.e. } (t, x) \in \Omega. \quad (3.1)$$

Moreover, assume

$$\int_\Omega (\alpha - \lambda) w^2 \, dt \, dx > 0, \quad \forall w \in \ker(L_o - \lambda I) \backslash \{0\} \quad (3.2)$$

and

$$\int_\Omega (\tilde{\lambda} - \beta) w^2 \, dt \, dx > 0, \quad \forall w \in \ker(L_o - \tilde{\lambda} I) \backslash \{0\}. \quad (3.3)$$

Then there exist $\varepsilon > 0$ and $\delta > 0$ such that for any $\gamma \in \mathcal{M}_e \cap L^\infty(\Omega)$ satisfying

$$\alpha(t, x) - \varepsilon \leq \gamma(t, x) \leq \beta(t, x) + \varepsilon \quad \text{for a.e. } (t, x) \in \Omega, \quad (3.4)$$

we have $\|L_ow - \gamma y\| \geq \delta\|y\|$ for all $y \in \mathcal{D}(L_o) \cap \mathcal{M}_o$.

**Remark 3.3.** Here, the condition $\alpha, \beta, \gamma \in \mathcal{M}_e$ is imposed to make sure $\alpha y, \beta y, \gamma y \in \mathcal{M}_o$ for any $y \in \mathcal{M}_o$.

**Proof.** We prove the result by contradiction. In fact, if the result is false, we can find sequences $\{y_j\} \subset \mathcal{D}(L_o) \cap \mathcal{M}_o$ with $\|y_j\| = 1$ and $\{\gamma_j\} \subset \mathcal{M}_e \cap L^\infty(\Omega)$ such that

$$\alpha(t, x) - \frac{1}{j} \leq \gamma_j(t, x) \leq \beta(t, x) + \frac{1}{j}, \quad j \in \mathbb{N}^+ \quad (3.5)$$

but $\|L_ow - \gamma_j y_j\| < \delta\|y_j\|$. This contradicts the definition of $\lambda$ and $\tilde{\lambda}$.
for a.e. \((t,x) \in \Omega\),
\[
\gamma_j(t,x) \to \gamma_0(t,x), \quad \text{as } j \to \infty
\]
for some \(\gamma_0 \in M_e \cap L^\infty(\Omega)\) and a.e. \((t,x) \in \Omega\), and

\[
\|L_0 y_j - \gamma_j y_j\| \leq \frac{1}{j}. \tag{3.6}
\]

Let
\[
\xi_j = L_0 y_j - \gamma_j y_j. \tag{3.7}
\]
Then we have \(\|\xi_j\| \leq \frac{1}{j}\) for \(j \in \mathbb{N}^+\) and

\[
\alpha(t,x) \leq \gamma_0(t,x) \leq \beta(t,x) \tag{3.8}
\]
for a.e. \((t,x) \in \Omega\).

Define
\[
\mathcal{H}_1 = \{ y \in M_0 | \mu_{mn} \leq \lambda, m \in \mathbb{N}, n \in \mathbb{N}^+ \}
\]
and
\[
\mathcal{H}_2 = \{ y \in M_0 | \mu_{mn} \geq \bar{\lambda}, m \in \mathbb{N}, n \in \mathbb{N}^+ \}.
\]

Since \(\lambda\) and \(\bar{\lambda}\) are two consecutive eigenvalues, we have
\[
M_0 = \mathcal{H}_1 \oplus \mathcal{H}_2.
\]

Split \(y_j = y_{ij} + y_{2j}\) with \(y_{ij} \in D(L_0) \cap \mathcal{H}_1\) and \(y_{2j} \in D(L_0) \cap \mathcal{H}_2\). Let us multiply both sides of the equation (3.7) by \(y_{2j} - y_{ij}\), and then by taking the inner product in \(L^2(\Omega)\), we see

\[
\langle L_0 y_j - \gamma_j y_j, y_{2j} - y_{ij} \rangle = \langle \xi_j, y_{2j} - y_{ij} \rangle,
\]
i.e.,
\[
\langle L_0 y_{2j} - \gamma_j y_{2j}, y_{2j} \rangle - \langle L_0 y_{ij} - \gamma_j y_{ij}, y_{ij} \rangle = \langle \xi_j, y_{2j} - y_{ij} \rangle,
\]

Let \(\lambda^* < \lambda\) and \(\bar{\lambda} < \bar{\lambda}^*\) be two pairs of consecutive eigenvalues. Then \(\Delta^* < \Delta < \bar{\Delta} < \bar{\Delta}^*\). Decompose

\[
\mathcal{H}_i = \mathcal{H}_i^* \oplus \mathcal{H}_i^0, \quad i = 1, 2,
\]
where \(\mathcal{H}_i^*\) (resp. \(\mathcal{H}_i^0\)) and \(\mathcal{H}_i^0\) (resp. \(\mathcal{H}_i^0\)) are spanned by the eigenfunctions corresponding to the eigenvalues which satisfy \(\mu_{mn} \leq \lambda^*\) (resp. \(\mu_{mn} \geq \bar{\lambda}^*\)) and \(\mu_{mn} = \lambda\) (resp. \(\mu_{mn} = \bar{\lambda}\)), respectively. Consequently, we can rewrite

\[
y_{ij} = y_{ij}^* + y_{ij}^0 \quad \text{for } y_{ij}^* \in \mathcal{H}_i^*, \quad y_{ij}^0 \in \mathcal{H}_i^0, \quad i = 1, 2.
\]

According to \(\|y_j\|^2 = \|y_{ij}\|^2 + \|y_{2j}\|^2 = 1\), we have

\[
\|y_{1j} - y_{2j}\| \leq \|y_{1j}\| + \|y_{2j}\| \leq 2 \left( \frac{\|y_{1j}\|^2 + \|y_{2j}\|^2}{2} \right)^{\frac{1}{2}} \leq \sqrt{2}.
\]

Thus, we have

\[
\frac{\sqrt{2}}{j} \geq \langle \xi_j, y_{2j} - y_{1j} \rangle
\]
\[
= \langle L_0 y_{2j} - \gamma_j y_{2j}, y_{2j} \rangle - \langle L_0 y_{1j} - \gamma_j y_{1j}, y_{1j} \rangle
\]
\[
= \langle L_0 (y_{2j}^* + y_{2j}^0) - \gamma_j (y_{2j}^* + y_{2j}^0), (y_{2j}^* + y_{2j}^0) \rangle - \langle L_0 (y_{1j}^* + y_{1j}^0) - \gamma_j (y_{1j}^* + y_{1j}^0), (y_{1j}^* + y_{1j}^0) \rangle
\]
\[
=: I_1 - I_2 \tag{3.9}
\]
with

\[
I_1 = \langle L_0 y_{2j}^* - \gamma_j y_{2j}^*, y_{2j}^* \rangle + \langle (\bar{\lambda} - \gamma_j) y_{2j}^0, y_{2j}^0 \rangle, \tag{3.10}
\]
\[
I_2 = \langle L_0 y_{1j}^* - \gamma_j y_{1j}^*, y_{1j}^* \rangle + \langle (\lambda - \gamma_j) y_{1j}^0, y_{1j}^0 \rangle.
\]
Here, we make use of the orthogonality of $\mathcal{H}_i^*$ and $\mathcal{H}_i^0$.

By (3.1) and (3.5), we have

$$I_1 \geq \left\langle L_0 y_{2j}^0 - \left(\bar{\lambda} + \frac{1}{j}\right) y_{2j}^*, y_{2j}^* \right\rangle - \frac{1}{j} \langle y_{2j}^0, y_{2j}^0 \rangle,$$
(3.12)

and

$$I_2 \leq \left\langle L_0 y_{1j}^0 - \left(\Lambda - \frac{1}{j}\right) y_{1j}^*, y_{1j}^* \right\rangle + \frac{1}{j} \langle y_{1j}^0, y_{1j}^0 \rangle.$$
(3.13)

Therefore, from (3.9), (3.12) and (3.13), we have

$$\frac{\sqrt{2}}{j} \geq \frac{\langle L_0 y_{2j}^0 - \lambda y_{2j}^*, y_{2j}^* \rangle}{\langle y_{2j}^0, y_{2j}^0 \rangle} - \frac{1}{j} \langle y_{2j}^0, y_{2j}^0 \rangle \geq (\lambda^* - \bar{\lambda})\|y_{2j}^*\|^2 + (\Lambda - \lambda^*)\|y_{1j}^*\|^2 - \frac{1}{j}.$$

Thus, it follows that

$$y_{1j}^0 \to 0, \quad y_{2j}^0 \to 0, \quad \text{as } j \to \infty.$$
(3.14)

Since $\{y_j\}$ is a bounded sequence and $\dim \mathcal{H}_i^0 < \infty$ ($i = 1, 2$) provided by Proposition 2.2, there exist $y_{1j}^0 \in \mathcal{H}_i^0$ and $y_{2j}^0 \in \mathcal{H}_i^2$ such that

$$y_{1j}^0 \to y_1^0, \quad y_{2j}^0 \to y_2^0, \quad \text{as } j \to \infty.$$

Thus, in view of (3.14) and $y_{i,j} = y_{i,j}^* + y_{i,j}^0$ ($i = 1, 2$), we have

$$y_{ij} \to y_i^0, \quad y_{2j} \to y_2^0, \quad \text{as } j \to \infty$$

and

$$y_j \to y_\infty := y_1^0 + y_2^0, \quad \text{as } j \to \infty$$

with $\|y_\infty\|^2 = \|y_1^0\|^2 + \|y_2^0\|^2 = 1$ provided by $\|y_j\| = 1$.

According to (3.6), we have $\|L_0 y_{ij}^0\|$ ($i = 1, 2$) is bounded. Thus, by (3.14) and $\gamma_j(t, x) \to \gamma_0(t, x)$, we have

$$\langle L_0 y_{2j}^0 - \gamma_j y_{2j}^*, y_{2j}^* \rangle \to 0 \quad \text{and} \quad \langle L_0 y_{1j}^0 - \gamma_j y_{1j}^*, y_{1j}^* \rangle \to 0, \quad \text{as } j \to \infty.$$

Consequently, from (3.9)–(3.11), we have

$$\int_\Omega (\gamma_0 - \bar{\lambda})|y_{1j}^0|^2 \, dt \, dx + \int_\Omega (\bar{\lambda} - \gamma_0)|y_{2j}^0|^2 \, dt \, dx = 0.$$

Therefore, by (3.1) and (3.8), it follows that

$$\int_\Omega (\gamma_0 - \bar{\lambda})|y_{1j}^0|^2 \, dt \, dx = 0 \quad \text{and} \quad \int_\Omega (\bar{\lambda} - \gamma_0)|y_{2j}^0|^2 \, dt \, dx = 0.$$

Now, set

$$\Omega_i = \{ (t, x) \in \Omega \mid y_i^0 \neq 0 \} \quad \text{for } i = 1, 2.$$

Thus, it follows that

$$\gamma_0 = \bar{\lambda} \quad \text{on } \Omega_1 \quad \text{and} \quad \gamma_0 = \tilde{\lambda} \quad \text{on } \Omega_2.$$

Furthermore, with the help of $\bar{\lambda} < \tilde{\lambda}$, we have $\Omega_1 \cap \Omega_2 = \emptyset$.

We now focus the attention on the discussion of the two cases where $\Omega_1 = \emptyset$ and $\Omega_1 \neq \emptyset$, and then we shall encounter a contradiction and come to the desired conclusion.

If $\Omega_1 = \emptyset$, we have $y_\infty = y_1^0$. Then

$$0 = \int_\Omega (\mu - \gamma_0)|y_{2j}^0|^2 \, dt \, dx \geq \int_\Omega (\mu - \beta)|y_{2j}^0|^2 \, dt \, dx.$$
By (3.3), we have \( y_\infty = y_1^0 = 0 \), which contradicts \( \|y_\infty\| = 1 \).
If \( \Omega_1 \neq \emptyset \), we have
\[
0 = \int_{\Omega} (\gamma_0 - \lambda)|y_1^0|^2 \, dt \, dx = \int_{\Omega_1} (\gamma_0 - \lambda)|y_1^0|^2 \, dt \, dx.
\]
\[
\geq \int_{\Omega_1} (\alpha - \lambda)|y_1^0|^2 \, dt \, dx = \int_{\Omega} (\alpha - \lambda)|y_1^0|^2 \, dt \, dx.
\]
By (3.2), we have \( y_1^0 = 0 \) on \( \Omega_1 \), which contradicts the definition of \( \Omega_1 \).
Thus, for either \( \Omega_1 = \emptyset \) or \( \Omega_1 \neq \emptyset \), we derive a contradiction, so the conclusion is true. \( \square \)

**Proposition 3.4 (See [24]).** Let \( X \) and \( Y \) be two real normed vector spaces, and let \( \Omega \subset X \) be an open bounded set. Assume that \( F = L + G \), \( L : \mathcal{D}(L) \subset X \to Y \) is a linear Fredholm operator with zero index and \( G : X \to Y \) is a linear and \( L \)-completely continuous mapping, where \( \mathcal{D}(L) \) is the domain of \( L \). If \( \ker F \) is trivial and \( 0 \in \Omega \), then
\[
D_L(F, \Omega) = \pm 1,
\]
where \( D_L(F, \Omega) \) denotes the degree of \( F \) in \( \Omega \) relative to \( L \).

**Lemma 3.5.** Let \( \gamma \in \mathcal{M}_c \cap L^\infty(\Omega) \) satisfy (3.4). Then for every open ball \( B_r \subset \mathcal{M}_o \) with center 0 and radius \( r > 0 \),
\[
D_{L_o}(L_o - \gamma I, B_r) = \pm 1,
\]
where \( I \) is the identity mapping.

**Proof.** Noting that \( \dim(\ker L_o) < \infty \), \( \mathcal{M}_o = \ker L_o \oplus \mathcal{R}(L_o) \) and \( L_o^{-1} \) is compact on \( \mathcal{R}(L_o) \), we have that \( L_o \) is a Fredholm operator with zero index and \( \gamma I \) is \( L_o \)-completely continuous on \( \mathcal{M}_o \).

By Lemma 3.2, the equation
\[
L_o y - \gamma y = 0
\]
has only one trivial solution in \( \mathcal{D}(L_o) \cap \mathcal{M}_o \), which implies that \( \ker(L_o - \gamma I) \) is trivial. Thus, for every open ball \( B_r \subset \mathcal{M}_o \) with center 0 and radius \( r \), by Proposition 3.4, we have
\[
D_{L_o}(L_o - \gamma I, B_r) = \pm 1.
\]
The proof is completed. \( \square \)

4 The main results

To prove the main results, we introduce a continuation theorem of the Leray-Schauder type.

**Lemma 4.1 (See [24]).** Let \( X \) and \( Y \) be two real normed vector spaces. Assume that \( L : \mathcal{D}(L) \subset X \to Y \) is a linear Fredholm operator with zero index and \( N : \bar{B}_r \to Y \) is an \( L \)-compact operator, where \( \mathcal{D}(L) \) is the domain of \( L \) and \( \bar{B}_r \subset X \) is an open ball with center 0 and radius \( r \). If there exist a constant \( r > 0 \) and an \( L \)-compact operator \( A : X \to Y \) such that
(i) for every \( (y, s) \in (\mathcal{D}(L) \cap \partial B_r) \times (0, 1) \),
\[
Ly - (1 - s)Ay - sNy \neq 0,
\]
(ii) \( 0 \notin (L - A)(\mathcal{D}(L) \cap \partial B_r) \), and
(iii) \( D_L(L - A, B_r) \neq 0 \),
then the equation
\[
Ly - Ny = 0
\]
possesses at least one solution in \( \mathcal{D}(L) \cap \bar{B}_r \).
Let $F: L^2(\Omega) \to L^2(\Omega)$ be the Nemytskii operator induced by $\hat{f}$, i.e., $F(y)(t,x) = \hat{f}(t,x,y)$.

To deal with the problem (1.1)–(1.3) in the subspace $\mathcal{M}_\alpha$, it is necessary to require $F(\mathcal{M}_\alpha) \subset \mathcal{M}_\alpha$. By direct verification, we have $F(\mathcal{M}_\alpha) \subset \mathcal{M}_\alpha$ if and only if $\hat{f} = \hat{f}_1 + \hat{f}_2$, where $\hat{f}_1$ and $\hat{f}_2$ satisfy

$$
\left\{
\begin{aligned}
\hat{f}_1(t,x,y) &= \hat{f}_1(t + T/2, x, y), \\
\hat{f}_1(t,x,y) &= -\hat{f}_1(t,x,-y),
\end{aligned}
\right.

\quad \text{and} \quad
\left\{
\begin{aligned}
\hat{f}_2(t,x,y) &= -\hat{f}_2(t + T/2, x, y), \\
\hat{f}_2(t,x,y) &= \hat{f}_2(t,x,-y).
\end{aligned}
\right.

(4.1)

Since $\hat{f}_1$ is odd with respect to $y$, it is obvious that $y^{-1}\hat{f}_1(t,x,y)$ has the same asymptotic behavior for $y \to +\infty$ and $y \to -\infty$.

**Theorem 4.2.** Assume that (H1)–(H2) hold, $p$ is an even number, $\lambda, \bar{\lambda} \in \Lambda(L_\alpha)$ are two consecutive eigenvalues, and $\alpha, \beta \in \mathcal{M}_\alpha \cap L^\infty(\Omega)$ satisfy (3.1)–(3.3). If $\hat{f} = \hat{f}_1 + \hat{f}_2$, $\hat{f}_1$ and $\hat{f}_2$ satisfy the symmetry conditions (4.1),

$$
\lim_{|y| \to +\infty} \frac{\hat{f}_2(t,x,y)}{y} = 0
$$

(4.2)

uniformly for a.e. $(t,x) \in \Omega$, and

$$
\alpha(t,x) \leq \liminf_{|y| \to +\infty} \frac{\hat{f}_1(t,x,y)}{y} \leq \limsup_{|y| \to +\infty} \frac{\hat{f}_1(t,x,y)}{y} \leq \beta(t,x)
$$

(4.3)

uniformly for a.e. $(t,x) \in \Omega$, then the problem (1.1)–(1.3) has at least one periodic solution lying in $\mathcal{M}_\alpha$.

**Proof.** Since $\hat{f} = \hat{f}_1 + \hat{f}_2$ and $\hat{f}_1$ and $\hat{f}_2$ satisfy (4.1), $F(\mathcal{M}_\alpha) \subset \mathcal{M}_\alpha$. Thus, we consider the problem (1.1)–(1.3) in the subspace $\mathcal{M}_\alpha$.

By (4.2) and (4.3), there is an $R > 0$ such that

$$
\alpha(t,x) - \varepsilon \leq y^{-1}\hat{f}(t,x,y) \leq \beta(t,x) + \varepsilon \quad \text{for } |y| \geq R,
$$

(4.4)

where $\varepsilon$ is presented in Lemma 3.2. The estimate (4.4) combining (H2) gives

$$
|\hat{f}(t,x,y)| \leq C|y| + h_R(t,x)
$$

for all $y \in \mathbb{R}$ and a.e. $(t,x) \in \Omega$, where the constant $C > 0$ depends on $\varepsilon$ and $R$. Thus, the operator $F_\alpha := F|_{\mathcal{M}_\alpha}$ is continuous and maps a bounded set into a bounded set. Then $y \in \mathcal{D}(L_\alpha) \cap \mathcal{M}_\alpha$ is a weak solution of the problem (1.1)–(1.3) if and only if

$$
L_\alpha y - F_\alpha y = 0.
$$

(4.5)

By recalling that $L_\alpha$ is a Fredholm operator with zero index, $\mathcal{M}_\alpha = \ker L_\alpha \oplus \mathcal{R}(L_\alpha)$ and $L_\alpha^{-1}$ is compact on $\mathcal{R}(L_\alpha)$, we know that $F_\alpha$ is $L_\alpha$-compact. Thus, by Lemma 3.5, for every $r > 0$, it follows that

$$
D_{L_\alpha}(L_\alpha - \alpha I, B_r) = \pm 1.
$$

In addition, by Lemma 3.2, $0 \notin (L_\alpha - \alpha I)(\mathcal{D}(L_\alpha) \cap \partial B_r)$. Consequently, by Lemma 4.1, the equation (4.5) will possess at least one solution if the set of all the possible solutions of

$$
L_\alpha y - (1-s)\alpha y - sF_\alpha(y) = 0, \quad s \in (0,1)
$$

(4.6)

is a priori bounded independently of $s$. Set

$$
g(t,x,y) = \begin{cases} 
 y^{-1}\hat{f}(t,x,y) & \text{for } |y| \geq R, \\
 R^{-1}\hat{f}(t,x,R)\frac{y}{R} + \left(1 - \frac{y}{R}\right)\alpha(t,x) & \text{for } 0 \leq y < R, \\
 R^{-1}\hat{f}(t,x,-R)\frac{y}{R} + \left(1 + \frac{y}{R}\right)\alpha(t,x) & \text{for } -R \leq y \leq 0.
\end{cases}
$$
Obviously, 
\[ \alpha(t,x) - \varepsilon \leq g(t,x,y) \leq \beta(t,x) + \varepsilon \]
for all \( y \in \mathbb{R} \) and a.e. \((t,x) \in \Omega\). Let \( \Theta(t,x,y) = f(t,x) - g(t,x,y)y \). Thus, we have
\[ |\Theta(t,x,y)| \leq 2h_R(t,x) + |\alpha(t,x)| \]
for all \( y \in \mathbb{R} \) and a.e. \((t,x) \in \Omega\). Define
\[ (G(y)z)(t,x) = g(t,x,y)z, \quad \forall y,z \in L^2(\Omega). \]
For every \( y \in \mathcal{M}_o \), let \( G_o(y) = G(y) |_{\mathcal{M}_o} \). Since \( g(t,x,y(t,x)) = g(t+T/2,x,y(t+T/2,x)) \) for \( y \in \mathcal{M}_o \), \( G_o(y) : \mathcal{M}_o \rightarrow \mathcal{M}_o \). Define \( \Theta_o(y)(t,x) = \Theta(t,x,y) \) for \( y \in L^2(\Omega) \) and let \( \Theta_o = \Theta |_{\mathcal{M}_o} \). Hence,
\[ F_o(y) = G_o(y)g + \Theta_o(y), \quad \forall y \in \mathcal{M}_o. \]
Thus the equation (4.6) is equivalent to
\[ L_o y - ((1-s)\alpha I + sG_o(y))(y) = s\Theta_o(y), \quad s \in (0,1). \]
Let \( \tilde{\gamma} = (1-s)\alpha + sg(t,x,y) \). We have
\[ \alpha(t,x) - \varepsilon \leq \tilde{\gamma} \leq \beta(t,x) + \varepsilon. \]
Thus, with the aid of Lemma 3.2, it follows that
\[ \delta\|y\| \leq \|L_o y - \tilde{\gamma}y\| = \|s\Theta(y)\| \leq 2\|h_R\| + \pi T\|\alpha\|_{L^\infty(\Omega)}, \]
i.e.,
\[ \|y\| \leq \frac{1}{\delta}(2\|h_R\| + \pi T\|\alpha\|_{L^\infty(\Omega)}). \]

The proof is completed. \[ \square \]

Now, we consider that \( \tilde{f} \) has the properties
\[ \tilde{f}(t,x,y) = \tilde{f}(t+T/2,x,y) \quad \text{and} \quad \tilde{f}(t,x,-y) = -\tilde{f}(t,x,y), \quad (4.7) \]
which imply \( \tilde{f}_2 \equiv 0 \) and \( \tilde{f} = \tilde{f}_1 \) for any \((t,x,y) \in \Omega \times \mathbb{R}\). Obviously, \( F(\mathcal{M}_o) \subset \mathcal{M}_o \) and 0 is a trivial solution. The following theorem shows that under the global nonresonance conditions, the problem (1.1)–(1.3) does not possess other solutions in the invariant subspace \( \mathcal{M}_o \).

**Theorem 4.3.** Assume that (H1)–(H2) hold, \( p \) is an even number, \( \Lambda, \hat{\lambda} \in \Lambda(L_o) \) are two consecutive eigenvalues, and \( \alpha, \beta \in \mathcal{M}_c \cap L^\infty(\Omega) \) satisfy (3.1)–(3.3). If \( \tilde{f} \) satisfies (4.7) and
\[ \alpha(t,x) \leq \frac{\tilde{f}(t,x,y) - \tilde{f}(t,x,z)}{y - z} \leq \beta(t,x) \quad (4.8) \]
for \( y \neq z \) and a.e. \((t,x) \in \Omega\), then the problem (1.1)–(1.3) has only one (trivial) solution in \( \mathcal{M}_o \).

**Proof.** Since \( \tilde{f} \) satisfies (4.7), \( \tilde{f}_2 \equiv 0 \) and (4.2) is verified. By (4.8), the estimate (4.3) holds. Thus, by Theorem 4.2, there exists at least one periodic solution to the problem (1.1)–(1.3) in \( \mathcal{M}_o \).

Notice that 0 is a trivial solution to the problem (1.1)–(1.3). Now we show that the problem (1.1)–(1.3) does not possess other solutions in \( \mathcal{M}_o \) by contradiction. Suppose that \( y_0 \neq 0 \) is a solution to the problem (1.1)–(1.3). Then \( y_0 \) satisfies
\[ L_o y_0 = \tilde{f}(t,x,y_0) = 0, \quad (4.9) \]
Set
\[ \xi(t,x,y_0) = y_0^{-1}\tilde{f}(t,x,y_0) \in \mathcal{M}_c. \]
Then by (4.8), we have
\[ \alpha(t, x) - \varepsilon \leq \xi(t, x, y_0) \leq \beta(t, x) + \varepsilon \]
for a.e. \((t, x) \in \Omega\), where \(\varepsilon\) is presented in Lemma 3.2. Therefore, by Lemma 3.2 and the equation (4.9), we have
\[ \delta \| y_0 \| \leq \| L_0 y_0 - \xi(t, x, y_0) y_0 \| = 0. \]
So we have \(y_0 = 0\), and it yields a contradiction. This is the result we desire.

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