ANTI-WINDUP APPROXIMATIONS OF OBLIQUE PROJECTED DYNAMICS FOR FEEDBACK-BASED OPTIMIZATION*

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Abstract. In this paper we study how high-gain anti-windup schemes can be used to implement projected dynamical systems in control loops that are subject to saturation on a (possibly unknown) set of admissible inputs. This insight is especially useful for the design of autonomous optimization schemes that realize a closed-loop behavior which approximates a particular optimization algorithm (e.g., projected gradient or Newton descent) while requiring only limited model information. In our analysis we show that a saturated integral controller, augmented with an anti-windup scheme, gives rise to a perturbed projected dynamical system. This insight allows us to show uniform convergence and robust practical stability as the anti-windup gain goes to infinity. Moreover, for a special case encountered in autonomous optimization we show robust convergence, i.e., convergence to an optimal steady-state for finite gains. Apart from being particularly suited for online optimization of large-scale systems, such as power grids, these results are potentially useful for other control and optimization applications as they shed a new light on both anti-windup control and projected gradient systems.

Key words. Differential Inclusions, Autonomous Optimization, Anti-Windup Control

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1. Introduction. In recent years, the design of feedback controllers based on optimization algorithms has garnered significant interest as a new approach to real-time optimization of large-scale systems such as power grids [10, 20, 27, 33] and communication networks [22, 25]. The goal of autonomous (or feedback-based) optimization is to implement feedback systems that robustly solve nonlinear optimization problems in closed loop with a physical system, often without requiring explicit knowledge of the problem parameters, because the physical plant itself enforces certain constraints.

In this paper, we investigate a new approach to enforce constraints by exploiting physical saturation. More precisely, we study how anti-windup control, which is ubiquitous in feedback control to mitigate integrator windup, can be used to implement projected dynamical systems (PDS) which are at the basis of continuous-time algorithms for constrained optimization. In particular, PDS form a class of discontinuous dynamical systems that encompasses projected gradient flow [17], projected Newton flow [16], subgradient flow [9] and projected saddle-flows [5,14]. More generally, PDS arise in many contexts that include unilateral constraints, such as variational inequalities [12,28], evolutionary games [23], and complementarity systems [3,4].

The main contribution of this paper is to establish a rigorous connection between PDS and anti-windup controllers and to generalize [19]. From an abstract point of view, we consider a class of parametrized dynamical systems, termed anti-windup approximations (AWA), and we show uniform convergence of trajectories to the solution of a PDS as the anti-windup gain tends to infinity. Moreover, we establish semiglobal practical robustness of PDS with respect to anti-windup approximations. For the special case of strongly monotone vector fields we further show robust asymptotic stability for finite gains.

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Compared to [19] we make the following generalizations:

i) We do not require the feasible domain to be convex. Instead, we work with (non-convex) prox-regular sets and show, by means of a counter-example, that prox-regularity cannot, in general, be relaxed further.

ii) We consider oblique PDS that provide an additional degree in the form of a (Riemannian) metric that allows us to capture wider variety of dynamics (such as projected Newton flows) and that is required for coordinate-free formulations.

iii) We require solutions to be neither unique nor complete. In particular, our results allow for solutions with finite escape time. Although these results may not be of practical relevance, they illustrate the necessity of our assumptions.

iv) We establish requirements for the convergence of anti-windup approximations of monotone dynamics on convex domains, thus providing a (partial) solution to a previously open problem formulated in [19].

Finally, in a largely self-contained section, we illustrate the possibilities of the proposed anti-windup approximations of PDS and the applicability of our theoretical results in the context of autonomous optimization [6, 7, 17, 24, 32].

The rest of the paper is organized as follows: In section 2 we fix the notation and recall relevant notions from variational analysis and dynamical systems. In section 3 we define our problem and establish some technical lemmas. In sections 4 and 5 we present our first two main results (Theorems 4.5 and 5.2) on uniform convergence and semiglobal practical robust stability. In section 6 we provide a stronger stability guarantee (Theorem 6.3) for the special case of monotone vector fields. In section 7, we illustrate the consequences of our results in the context of feedback-based optimization. For this, we consider four different optimization dynamics (three gradient-based and one saddle-point flow) and discuss their convergence behavior observed in simulations. In section 8 we summarize our results and discuss open problems.

2. Preliminaries.

2.1. Notation. We consider $\mathbb{R}^n$ with the usual inner product $\langle \cdot, \cdot \rangle$ and 2-norm $\| \cdot \|$. We use $\mathbb{R}_{\geq 0}^n$ for the non-negative orthant. The closed (open) unit ball of appropriate dimension is denoted by $\mathbb{B}$ (int $\mathbb{B}$). For a sequence $\{K_n\}$, $K_n \to 0^+$ implies that $K_n > 0$ for all $n$ and $\lim_{n \to \infty} K_n = 0$. For a map $F : \mathbb{R}^n \to \mathbb{R}^m$, differentiable at $x \in \mathbb{R}^n$, $\nabla F(x) \in \mathbb{R}^{m \times n}$ denotes the Jacobian of $F$ at $x$. Given a set $C \subset \mathbb{R}^n$, its closure, boundary, and (closed) convex hull are denoted by $\text{cl} C$, $\partial C$, and $\text{co} C = (\text{cl} \, C \setminus \{\emptyset\})$, respectively. If $C$ is non-empty, we write $\|C\| := \sup_{v \in C} \|v\|$. The distance to $C$ is defined as $d_C(x) := \inf_{\tilde{x} \in C} \|x - \tilde{x}\|$, and the projection $P_C : \mathbb{R}^n \to C$ is given by $P_C(x) := \{\tilde{x} \in C | \|x - \tilde{x}\| = d_C(x)\}$. The domain of a set-valued map $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is defined as $\text{dom} \, H := \{x | H(x) \neq \emptyset\}$. We use the standard definitions of outer semicontinuity (osc), local boundedness, graphical convergence, etc. from set-valued analysis. In particular, unless noted otherwise, we follow the definitions and notation of [15, Chap. 5]. The identity matrix (of appropriate size) is denoted by $I$. Given a square symmetric matrix $G \in \mathbb{R}^{n \times n}$, $\lambda_G^{\max}$ and $\lambda_G^{\min}$ denote its maximum and minimum eigenvalue, respectively. The set of symmetric, positive definite matrices of size $n$ is denoted by $\mathbb{S}^n_+$. A metric on $C \subset \mathbb{R}^n$ is a map $G : C \to \mathbb{S}^n_+$. A metric $G$ induces an inner product $\langle u, v \rangle_{G(x)} := u^T G(x) v$ and an associated 2-norm $\|u\|_{G(x)} := (\langle u, u \rangle_{G(x)})^{1/2}$ for all $x \in C$ and all $u, v \in \mathbb{R}^n$. A metric is (Lipschitz) continuous if it is (Lipschitz) continuous as a map $G : C \to \mathbb{S}^n_+$ with respect to the $\lambda^{\max}$-norm on $\mathbb{S}^n_+$. 


2.2. Variational Geometry. We use the following, slightly simplified, notions of variational geometry. For a comprehensive treatment the reader is referred to [30].

Given a closed set \( C \subset \mathbb{R}^n \) and \( x \in C \), a vector \( v \) is a tangent vector to \( C \) at \( x \) if there exist sequences \( x_k \to x \) with \( x_k \in C \) for all \( k \) and \( \delta_k \to 0^+ \) such that \( \frac{x_k - x}{\delta_k} \to v \). The set of all tangent vectors at \( x \) is called the tangent cone at \( x \) and denoted by \( T_x C \). If the set-valued map \( x \mapsto T_x C \) is inner semicontinuous then \( C \) is Clarke regular (or tangentially regular) [30, Cor. 6.29]. If \( C \) is Clarke regular then \( T_x C \) is closed convex and the (Euclidean) normal cone at \( x \) is defined as the polar cone of \( T_x C \), i.e., \( N_x C := \{ \eta \mid \langle \eta, v \rangle \leq 0 \} \) [30, Cor. 6.30]. Further, the map \( x \mapsto N_x C \) is osc [30, Prop. 6.6]. We follow the convention that \( T_x C = N_x C = \emptyset \) for all \( x \notin C \).

We will mostly work with the special class of prox-regular sets. Given a Clarke regular set \( C \) and \( \alpha > 0 \), a normal vector \( \eta \in N_x C \) is \( \alpha \)-proximal if \( \langle \eta, y - x \rangle \leq \alpha \| y - x \|^2 \) for all \( y \in C \). The set \( C \) is \( \alpha \)-prox-regular at \( x \) if all normal vectors at \( x \) are \( \alpha \)-proximal. In other words, the normal cone coincides with the cone of \( \alpha \)-proximal normals. A set is \( \alpha \)-prox-regular if it is \( \alpha \)-prox-regular at all \( x \in C \) and \( \alpha \)-prox-regular if it is \( \alpha \)-prox-regular for some \( \alpha > 0 \). A key property of prox-regular sets is that the projection on \( C \) is locally well-defined [1, Thm. 2.2 & Prop. 2.3]:

**Proposition 2.1.** If \( C \subset \mathbb{R}^n \) is \( \alpha \)-prox-regular, then, for every \( x \in C + \frac{1}{\alpha^2} \text{int} \mathbb{B} \), the set \( P_C(x) \) is a singleton and \( d_C^2(x) \) is differentiable at \( x \) with \( \nabla(d_C^2(x)) = 2(x - P_C(x)) \). Further, \( P_C(x + v) = x \) holds for every \( x \in C \) and all \( v \in N_x C \cap \frac{1}{\alpha} \text{int} \mathbb{B} \).

For example, every closed convex set is Clarke regular as well as \( \alpha \)-prox-regular for all \( \alpha > 0 \). Further, every set of the form \( C = \{ x \mid h(x) \leq 0 \} \), where \( h : \mathbb{R}^n \to \mathbb{R}^m \) is differentiable, is Clarke regular if constraint qualifications hold [30, Thm. 6.14]. If, in addition, \( h \) has a globally Lipschitz derivative, then \( C \) is prox-regular [16, Ex. 7.7].

2.3. Dynamical Systems & Stability. Given a closed set \( C \subset \mathbb{R}^n \) and a set-valued map \( H : \mathbb{R}^n \nrightarrow \mathbb{R}^n \), we say that \( x : [0, T] \to C \) for some \( T > 0 \) is a (Carathéodory) solution of the (constrained) differential inclusion

\[
\dot{x} \in H(x), \quad x \in C
\]

if \( x \) is absolutely continuous, and \( x(t) \in C \) and \( \dot{x}(t) \in H(x(t)) \) hold for almost all \( t \in [0, T] \). A map \( x : [0, \infty) \to C \) is a complete solution, if its restriction to any compact subinterval \( [0, T] \) is a solution of (2.1).

**Definition 2.2.** An inclusion (2.1) is well-posed if \( C \) is closed, \( H \) is osc and locally bounded relative to \( C \), and \( H(x) \) is non-empty and convex for all \( x \in C \).

Standard results (e.g., [15, Lem. 5.26]) guarantee that (2.1) admits a solution for every initial condition \( x(0) \in C \) if it is well-posed and \( H(x) \cap T_x C \neq \emptyset \) for all \( x \in C \).

For convenience, we introduce the following notion of truncated solution:

**Definition 2.3.** Consider (2.1) with \( C = \mathbb{R}^n \). Given \( T, \epsilon > 0 \) and \( x_0 \in \mathbb{R}^n \), a solution \( x : [0, T'] \to \mathbb{R}^n \) of (2.1) with initial condition \( x(0) = x_0 \) is \((T, \epsilon)\)-truncated if \( x(t) \in x_0 + \epsilon B \) for all \( t \in [0, T'] \) and either \( T' = T \) or \( \| x(T') - x_0 \| = \epsilon \) holds.

Recall that on a compact domain, solutions of an (unconstrained) inclusion can always be extended up to the boundary of the domain [13, §7, Thm. 2];

**Theorem 2.4.** Let (2.1) be well-posed with \( C = \mathbb{R}^n \) and let \( A \subset \mathbb{R}^n \) be compact. Then, every solution \( x : [0, T] \to \mathbb{R}^n \) with \( x(0) \in A \) can be extended up to the boundary of \( A \), i.e., there is a solution for every \( T > 0 \) or there exists \( T \) such that \( x(T) \in \partial A \).

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3To allow for a more concise presentation, we limit ourselves to closed, Clarke regular subsets of \( \mathbb{R}^n \) which allow for an unambiguous definition of tangent and normal cones.
Therefore, by considering an augmented inclusion with \( \dot{H}(x) := (H(x), 1) \), initial condition \( \dot{x}(0) := (x(0), 0) \), \( \dot{C} = \mathbb{R}^n \times \mathbb{R} \), and \( \dot{A} = A \times [0, T] \), Theorem 2.4 guarantees the existence of truncated solutions for every \( T \) and every \( \epsilon \).

**Corollary 2.5.** Let (2.1) be well-posed with \( C = \mathbb{R}^n \). Then, for every \( T, \epsilon > 0 \) and every \( x(0) \in \mathbb{R}^n \) there exists a \( (T, \epsilon) \)-truncated solution to (2.1).

Hence, truncated solutions are convenient if finite escape times cannot be precluded, since their graph is always a compact subset of \([0, T] \times (x(0) + \epsilon \mathbb{B})\).

We also require the notion of \( \sigma \)-perturbation of an inclusion [15, Def. 6.27]:

**Definition 2.6.** Given \( \sigma > 0 \), the \( \sigma \)-perturbation of (2.1) is given by

\[
\dot{x} \in H_\sigma(x) \quad x \in C_\sigma
\]

where \( C_\sigma := C + \sigma \mathbb{B} \) and \( H_\sigma(x) := \overline{wH((x + \sigma \mathbb{B}) \cap C)} + \sigma \mathbb{B} \) for all \( x \in C_\sigma \).

Note in particular that, for \( \sigma' \geq \sigma \), we have \( C_\sigma \subset C_{\sigma'} \), \( H_\sigma(x) \subset H_{\sigma'}(x) \) for all \( x \in C_\sigma \), and every solution of the \( \sigma \)-perturbation is a solution of the \( \sigma' \)-perturbation.

Next, recall that \( \omega : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) is a \( \mathcal{K}_\infty \)-function (denoted by \( \omega \in \mathcal{K}_\infty \)) if \( \omega \) is continuous, strictly increasing, unbounded, and it holds that \( \omega(0) = 0 \). We require the following lemma about \( \mathcal{K}_\infty \)-functions:

**Lemma 2.7.** [31, Cor. 10] For every \( \omega \in \mathcal{K}_\infty \), there exist \( \sigma_1, \sigma_2 \in \mathcal{K}_\infty \) such that

\[
\omega(rs) \leq \sigma_1(r)\sigma_2(s) \quad \text{for all } r, s \geq 0.
\]

A function \( \beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) is a \( \mathcal{K} \mathcal{L} \)-function (denoted by \( \beta \in \mathcal{K} \mathcal{L} \)) if it is non-decreasing in its first argument, non-increasing in its second argument,

\[
\lim_{r \rightarrow 0^+} \beta(r, s) = 0 \quad \text{for each } s \in \mathbb{R}_{\geq 0}, \quad \text{and} \quad \lim_{s \rightarrow \infty} \beta(r, s) = 0 \quad \text{for each } r \in \mathbb{R}_{\geq 0}.
\]

A closed set \( A \subset \mathbb{R}^n \) is uniformly globally (pre-)asymptotically stable for (2.1) if there exists \( \beta \in \mathcal{K} \mathcal{L} \) such that for every solution \( x : [0, T] \rightarrow C \) of (2.1) it holds that

\[
d_A(x(t)) \leq \beta(d_A(x(0)), t) \quad \forall t \in [0, T].
\]

**Remark 2.8.** The term “pre-asymptotic” refers to the fact that solutions of (2.1) need not be complete for the above definition of stability to apply [15, Def. 3.6 & Thm 3.40]. However, if (2.1) is well-posed and \( A \) is compact it follows that, for any initial condition \( x(0) \in C \), the (compact) set \( \{x \mid \beta(d_A(x), 0) \leq \beta(d_A(x(0)), 0)\} \) is invariant, thus implicitly guaranteeing the existence of a complete solution.

### 2.4. Oblique Projected Dynamical Systems

PDS are continuous-time dynamical systems that are constrained to a set by projection of the vector field at the boundary of the domain. Compared to traditional definitions [3, 4, 8, 28], we incorporate the possibility of oblique projection directions by means of a variable metric [16]. Namely, we consider PDS as defined by the differential equation of the form

\[
\dot{x} = \Pi^C_x[f(x)](x) \quad x \in C,
\]

where \( C \subset \mathbb{R}^n \) is a Clarke regular set, \( G : C \rightarrow S^n_C \) is a metric on \( C \), and \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a vector field. Given \( x \in C \) and \( w \in \mathbb{R}^n \), the operator \( \Pi^C_x[w] \) projects \( w \) onto the tangent cone of \( C \) at \( x \) with respect to the metric \( G \), i.e.,

\[
\Pi^C_x[w](x) := \arg \min_{v \in T_xC} \|v - w\|_{G(x)}.
\]

Note that if \( x \in C \), then \( \Pi^C_x[w](x) \) is single-valued since \( C \) is assumed to be Clarke regular which implies that \( T_xC \) is closed convex. If \( x \notin C \), we have \( \Pi^C_x[w](x) = \emptyset \) and
therefore $\text{dom} \, \Pi^G_C[w] = C$ for all $w \in \mathbb{R}^n$. If $f$ is a vector field, we abuse notation and write $\Pi^G_C[f](x) := \Pi^G_C[f(x)](x)$ for brevity.

Given a metric $G$, we define the normal cone of $C$ at $x$ with respect to $G$ as $N^G_xC := \{ v \mid \forall v \in T_xC : \langle v, v \rangle_G \leq 0 \}$. Note in particular that we have

$$\tag{2.3} \eta \in N^G_xC \iff G^{-1}(x)\eta \in N^G_xC.$$ 

As a consequence of Moreau’s Theorem [21, Thm. 3.2.5] the operator $\Pi^G_C$ has the following crucial properties (see also [3, 4, 8]):

**Lemma 2.9.** [16, Lem. 4.5] If $C$ is Clarke regular then, for every $x \in C$, there exists a unique $\eta \in N^G_xC$ such that $\Pi^G_C[f](x) = f(x) - \eta$. Furthermore, $\Pi^G_C[f](x) = f(x) - \eta$ holds if and only if $\eta \in N^G_xC$ and $\langle f(x) - \eta, \eta \rangle_G = 0$. Using Cauchy-Schwarz, it also holds that $\|\eta\|_G(x) \leq \|f(x)\|_G(x)$.

Existence and uniqueness results for (2.2) without a variable metric can be found in [3, 8, 28] and others. For the case with a variable metric with bounded condition number, the following statement is a condensation of results in [16]:

**Theorem 2.10.** Consider (2.2) and let $C$ be Clarke regular, and $f$ and $G$ be continuous. Then, (2.2) admits a solution for every initial condition $x(0) \in C$.

If, in addition, there exists $\kappa > 0$ such that $\sup_{x \in C} \lambda^\text{max}_{G(x)}/\lambda^\text{min}_{G(x)} \leq \kappa$ and $f$ is globally Lipschitz, then (3.4) admits a complete solution for every $x(0) \in C$.

If $C$ is prox-regular, and if $f$ and $G$ are (locally) Lipschitz, then (2.2) admits a unique solution for every initial condition $x(0) \in C$.

It is known that the solutions to (2.2) are equivalent to the solutions of $\dot{x} \in f(x) - N^G_xC$ (for $G = I$ see [2, 8]; for general $G$ see [16, Cor. 6.3]). In light of Lemma 2.9, we can show (next) that solutions of (2.2) are equivalent to solutions of

$$\tag{2.4} \dot{x} \in F(x) := f(x) - N^G_xC \cap \gamma \mathbb{B} \quad x \in C,$$

where $\gamma \geq \sup_{x \in C} \|f(x)\|_G(x)$ (assuming $\sup_{x \in C} \|f(x)\|_G(x) < \infty$). The advantage of this latter inclusion is that the mapping $F$ is bounded.

**Proposition 2.11.** If $C$ is Clarke regular and $x \mapsto \|f(x)\|_G(x)$ is bounded, then $x: [0, T] \to C$ with $T > 0$ is a solution of (2.2) if and only if it is a solution of (2.4).

**Proof.** Let $x : [0, T] \to C$ be a solution of (2.2). Then, $\Pi^G_C[f](x(t)) = f(x(t)) - \eta(t)$ for some $\eta(t) \in N^G_{x(t)}C$ satisfying $\|\eta(t)\|_G(x(t)) \leq \|f(x(t))\|_G(x(t)) \leq \gamma$ by Lemma 2.9 and therefore $\eta(t) \in N^G_{x(t)}C \cap \gamma \mathbb{B}$. Conversely, assume that $x$ solves (2.4). Whenever $\dot{x}(t)$ exists, it holds that $\dot{x}(t) \in T_{x(t)}C \cap -T_{x(t)}C$ [8, eq. 2.6] and $\dot{x}(t) = f(x(t)) - \eta(t)$ for some $\eta(t) \in N^G_{x(t)}C \cap \gamma \mathbb{B}$. Thus, we have

$$\langle f(x(t)) - \eta(t), \eta(t) \rangle_{G(x(t))} \leq 0 \quad \text{and} \quad \langle -f(x(t)) + \eta(t), \eta(t) \rangle_{G(x(t))} \leq 0,$$

and therefore $\langle f(x(t)) - \eta(t), \eta(t) \rangle_{G(x(t))} = 0$ which, in turn, implies that $f(x(t)) - \eta(t) = \Pi^G_C[f](x(t))$ by Lemma 2.9.\]

**Lemma 2.12.** If $f$ and $G$ are continuous and $C$ Clarke regular, (2.4) is well-posed.

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A solution $x : [0, T] \to C$ of (2.2) is unique if for every other solution $x' : [0, T'] \to C$ with the same initial condition it holds that $x(t) = x'(t)$ for all $t \in [0, \min\{T, T'\}]$. 

Proof. Non-emptiness and convexity of $F(x)$ are immediate because $N^G_C \cap \gamma B$ is non-empty (in particular, $0 \in N^G_C$ and convex for all $x \in C$ (and $f$ is single-valued). For outer semicontinuity recall that for a Clarke regular $C$ and continuous $G$ the mapping $x \mapsto N^G_C$ is osc [16, Lem. A.6]. It then follows that the truncation $N^G_C \cap \gamma B$ is osc and locally bounded [30, p.161]. Finally, since $f$ is continuous and single-valued, $x \mapsto f(x) - N^G_C \cap \gamma B$ is osc and locally bounded.

3. Problem Formulation & Technical Results. Throughout the paper, we consider the system given by the (unconstrained) inclusion

$$\dot{z} \in F_K(z) := f(z,P_Z(z)) - \frac{1}{n}G^{-1}(P_Z(z))(z - P_Z(z)),$$

where $Z \subset \mathbb{R}^n$ is a closed set, $f : \mathbb{R}^n \times Z \to \mathbb{R}^n$ is a continuous vector field, $G : Z \to S^n_+$ is a continuous metric, and $K > 0$ is a constant parameter. Because $P_Z$ is in general not single-valued (unless $Z$ is convex), (3.1) has to be treated as a differential inclusion.

Systems of the form (3.1) arise in the context of anti-windup control for feedback loops with integral controllers, as will be discussed in section 7. Hence, we will refer to (3.1) as an anti-windup approximation (AWA).

We study the behavior of solutions of (3.1) as $K \to 0^+$ and show that, under appropriate assumption on $Z,f,G$, and for an initial condition $z(0) \in Z$, these solutions converge uniformly to solutions of the projected dynamical system

$$\dot{z} = \Pi^Z_{\gamma}(\hat{f})(z), \quad z \in Z,$$

where we use $\hat{f}(z) := f(z,P_Z(z))$. Further, we show that a compact globally asymptotically set for (3.2) is semiglobally practically asymptotically stable for (3.1) in $K$. Namely, if $A$ is compact and asymptotically stable compact for the PDS (3.2), then for any compact set of initial conditions $B$ and any $\zeta > 0$, there exists $K > 0$ such that all trajectories of the AWA (3.1) starting in $B$ converge to a subset of $A + \zeta B$.

The key idea for studying (3.1) is to exploit $\alpha$-prox-regularity of $Z$ which, according to Proposition 2.1 guarantees, that $P_Z(z)$ is single-valued for all

$$z \in Z^\alpha := Z + \frac{1}{\alpha} \text{ int } B.$$

Hence, on $Z^\alpha$, (3.1) reduces to an ODE. Further, under appropriate conditions on the problem parameters and for small enough $K$, trajectories starting in $Z$ remain in $Z^\alpha$. This insight will be rigorously established in subsection 3.1. In subsection 3.2 we then show that the AWA (3.1) corresponds to a $\sigma$-perturbation of the PDS (3.2) as a function of $K$. We then apply standard results from [15] to establish uniform convergence and semiglobal practical asymptotic stability in sections 4 and 5, respectively.

3.1. Existence, Local Uniform Boundedness, and Equicontinuity. As a first step in studying (3.1), we prove the following lemma for future reference:

**Lemma 3.1.** Let $Z \subset \mathbb{R}^n$ be closed and $f : \mathbb{R}^n \times Z \to \mathbb{R}^n$ be continuous. Then, $z \mapsto \hat{f}(z) := f(z,P_Z(z))$ is locally bounded and osc. Furthermore, if $Z$ is $\alpha$-prox-regular for $\alpha > 0$, then $\hat{f}$ is single-valued and continuous for all $z \in Z^\alpha$.

**Proof.** The projection $P_Z : \mathbb{R}^n \to Z$ is osc and locally bounded [30, Ex. 5.23], and $P_Z(z)$ is non-empty and closed for all $z \in \mathbb{R}^n$ (since $Z$ is closed). By continuity of $f$ it follows that $\hat{f}$ is osc and locally bounded, since both properties are preserved under addition and composition [30, Prop. 5.51 & 5.52]. Using Proposition 2.1 it follows that $f$ is single-valued (hence continuous) for $z \in Z^\alpha$.

\[\square\]
Lemma 3.1 and Proposition 2.1 imply that, on $Z^o$, $F_K$ is single-valued and continuous. Consequently, standard results for ODEs guarantee that (3.1) admits a (local) solution for every initial condition $z(0) \in Z^o$. However, outside of $Z^o$, (3.1) is a differential inclusion for which the existence of solutions is not immediately guaranteed. Nevertheless, one can establish the existence of so-called Krasovskii solutions [15].

For the main result of this section we consider the following (local) setup:

**Assumption 3.2.** Consider (3.1) and $z_0 \in Z$. Let $M, \nu, \mu, \alpha, \epsilon > 0$ be such that

\begin{equation}
\|f(z, P_z(z))\| \leq M \quad \text{and} \quad \mu \|z\| \leq G^{-1}(P_z(z)) \leq \nu \|z\|
\end{equation}

hold for all $z \in (z_0 + \epsilon \mathbb{B}) \cap Z^o$, and $Z$ is $\alpha$-prox-regular at every $z \in (z_0 + \epsilon \mathbb{B}) \cap Z$.

Parameters $M, \nu, \mu, \epsilon$ that satisfy (3.3) can always be found for any $z_0 \in Z$ since $z \mapsto f(z, P_z(z))$ is locally bounded by Lemma 3.1, $G$ is continuous, and $P_z$ is single-valued on $(z_0 + \epsilon \mathbb{B}) \cap Z^o$.

Assumption 3.2 allows us to formulate the following proposition which combines the existence of truncated solutions, the invariance of a neighborhood of $Z$, and equicontinuity (i.e. uniform Lipschitz continuity):

**Proposition 3.3.** Let Assumption 3.2 be satisfied for $z_0 \in Z$. Given any $T > 0$ and $K < \frac{\nu}{2M}$, there exists a $(T, \epsilon)$-truncated solution $z$ for (3.1) with $z(0) = z_0$ (where $\epsilon$ stems from Assumption 3.2). Furthermore, $z$ satisfies, for almost all $t \in \text{dom } z$,

$$z(t) \in Z + \frac{KM}{\nu} \mathbb{B} \quad \text{and} \quad \|\dot{z}(t)\| \leq \left(1 + \frac{\nu}{M}\right) M.$$

**Proof.** First, we consider the existence of solutions: As mentioned, Lemma 3.1 and Proposition 2.1 imply that, on $(z_0 + \epsilon \mathbb{B}) \cap Z^o$, (3.1) reduces to a continuous ODE which is a well-posed inclusion (trivially). Hence, Theorem 2.4 guarantees the existence of a maximal solution $z : [0, T'] \to (z_0 + \epsilon \mathbb{B}) \cap Z^o$ starting at $z_0$ and with $x(T')$ on the boundary of $(z_0 + \epsilon \mathbb{B}) \cap Z^o$.

Next, by Proposition 2.1, we have $\nabla d_Z^2(z) = 2(z - P_z(z))$ for all $z \in Z^o$. Hence, the Lie derivative of $d_Z^2$ along $F_K$ for all $z \in (z_0 + \epsilon \mathbb{B}) \cap Z^o$ is well-defined and satisfies

$$\mathcal{L}_{F_K}(\frac{1}{2} d_Z^2(z)) = (z - P_z(z))^T (f(z, P_z(z)) - \frac{1}{K} G^{-1}(P_z(z))(z - P_z(z)))$$

$$\leq d_z(z) \|f(z, P_z(z))\| - \frac{1}{K} \|z - P_z(z)\|^T G^{-1}(P_z(z))(z - P_z(z))$$

$$\leq d_z(z) \|f(z, P_z(z))\| - \frac{K}{\nu} d_Z^2(z) = d_z(z)(M - \frac{K}{\nu} d_Z^2(z)).$$

It follows that $\mathcal{L}_{F_K}(\frac{1}{2} d_Z^2(z)) < 0$ whenever $d_z(z) > \frac{KM}{\nu}$. Since $K < \frac{\nu}{2M}$ and using an invariance argument, it follows that $z(t) \in Z + \frac{KM}{\nu} \mathbb{B} \subset Z^o$ for all $t \in [0, T']$.

In other words, for small enough $K$, any solution of (3.1) starting at $z_0$ remains within a neighborhood of $Z$ on which the projection $P_z$ is single-valued.

Since $z(T')$ lies on the boundary of $(z_0 + \epsilon \mathbb{B}) \cap Z^o$, but at the same time $z(T') \in Z + \frac{KM}{\nu} \mathbb{B}$, it follows that $\|z(T') - z_0\| = \epsilon$. In other words, $z(T')$ lies on the boundary of $z_0 + \epsilon \mathbb{B}$ (rather than the boundary of $Z^o$). Hence, (after restricting $z$ to $[0, T]$ if $T' > T$) it can be concluded that $z$ is a $(T, \epsilon)$-truncated solution of (3.1).

Finally, we have that for all $z \in (Z + \frac{KM}{\nu} \mathbb{B}) \cap (z_0 + \epsilon \mathbb{B})$ it holds that

$$\left\| \frac{1}{K} G(z)^{-1}(z - P_z(z)) \right\| \leq \frac{1}{K} \nu \frac{KM}{\nu} \leq M \frac{\nu}{M}.$$

It then follows from the definition of $M$ and the triangle inequality that $\|F_K(z)\| \leq M + M \frac{\nu}{M}$, thus establishing the bound on $\|\dot{z}(t)\|$. \qed
The proof of Proposition 3.3 suggests that the prox-regularity assumption on $Z$ is primarily required for $d_Z(z)$ to have a single-valued derivative in a neighborhood of $Z$. The following example shows, however, that prox-regularity is a more fundamental requirement which, in general, cannot be avoided.

**Example 3.4.** Consider the set $Z := \{(z_1, z_2) \in \mathbb{R}^2 \mid \|z_2\| \geq \max\{0, z_1\}\}^\kappa$ for any $\frac{1}{2} < \kappa < 1$. Further assume that $G(z) = I$ and $f(z) = (1, 0)$ for all $z \in \mathbb{R}^n$. Hence, we can choose $M = \nu = \mu = 1$ and any $\epsilon > 0$ to satisfy Assumption 3.2. Note, however, that $Z$ is not prox-regular at $(0,0)$. Namely, every point on the positive $z_1$-axis has a non-unique projection onto $Z$ as illustrated in Figure 1a.

We claim that for every $K > 0$ there exists a Krasovskii solution (i.e., a solution of the inclusion $\dot{z} \in \hat{F}(z) := f(z, P_Z(z)) - N^G_z Z \cap \gamma B$) starting on the $z_1$-axis that leaves the set $Z + \frac{K \mu}{\nu} B$ established in Proposition 3.3. This can be deduced graphically from Figure 1b. Namely, let $z_0 = (z_{01}, 0)$ be such that $d_Z(z_0) = K$. Then, there exists $v = (v_1, 0)$ with $v_1 > 0$ in the Krasovskii-regularization of $F_K(z_0)$, i.e., $v \in \overline{co} F_K(z_0)$. In other words, on the boundary of $Z + K \overline{B}$, the vector $v$ points out of the (supposedly) invariant set. This, in turn, can be used to rigorously establish that the set $Z + K \overline{B}$ is not invariant, illustrating that the conclusion of Proposition 3.3 does not hold without prox-regularity of $Z$, even when considering more general Krasovskii solutions.

**3.2. Anti-Windup Trajectories as Perturbed PDS.** As a key technical result, we establish that solutions of the AWA (3.1) are also solutions of a $\sigma$-perturbation of the PDS in its alternate form (2.4). To prove this claim, consider $z_0 \in Z$, and let $M, \mu, \nu, \alpha, \epsilon > 0$ be such that Assumption 3.2 is satisfied. It follows from Proposition 2.11 that, for some $T > 0$, every $(T, \epsilon)$-truncated solution $z : [0, T') \to (z_0 + \epsilon \overline{B})$ of the PDS (3.2) with $z(0) = z_0$ is also a $(T, \epsilon)$-truncated solution of the inclusion

$$\dot{z} \in \hat{F}(z) := f(z, P_Z(z)) - N^G_z Z \cap \gamma \overline{B} \quad \text{where} \quad \gamma := \max\left\{\frac{M}{\sqrt{\mu}}, \frac{\mu}{\nu} M\right\}$$ (3.4)

and vice versa. This choice of $\gamma$ will be convenient in the proof of Proposition 3.5 below. For now, note that using Cauchy-Schwarz, it holds that

$$\sup_{z \in z_0 + \epsilon \overline{B}} \|f(z, P_Z(z))\|_{G(z)} \leq \sup_{z \in z_0 + \epsilon \overline{B}} \sqrt{\|G(z)\|} \sup_{\leq 1/\sqrt{\mu}} \leq M$$

We cannot rely on the existence of Carathéodory solutions because $Z$ is not prox-regular and Proposition 3.3 does not apply, but every Carathéodory solution (if it exits) is a Krasovskii solution.
thus satisfying the condition on $\gamma$ in (2.4) and Proposition 2.11.

Furthermore, given $z_0 \in \mathcal{Z}$, let Assumption 3.2 hold with some $\epsilon > 0$. By Lemma 3.1 we have that $z \mapsto f(z, P_Z(z))$ is continuous on $z \in \mathcal{Z}_0$ and hence uniformly continuous on the bounded set $\mathcal{Z}_0 \cap (z_0 + \epsilon \mathcal{B})$. As a consequence of uniform continuity there exists $\omega \in \mathcal{K}_\infty$ such that, for all $z, z' \in \mathcal{Z}_0 \cap (z_0 + \epsilon \mathcal{B})$, we have

$$
\|f(z, P_Z(z)) - f(z', P_Z(z'))\| \leq \omega(\|z - z'\|).
$$

**Proposition 3.5.** Consider $z_0 \in \mathcal{Z}$ and let Assumption 3.2 hold with $M, \nu, \mu, \alpha$ and $\epsilon$. Further, let $K < \frac{\mu}{KM}$. Then, for some $T > 0$, every $(T, \epsilon)$-truncated solution $z : [0, T'] \to (z_0 + \epsilon \mathcal{B})$ of (3.1) is a solution of the $\sigma$-perturbation of (3.4) with $\sigma := \max \left\{ \frac{KM}{\mu}, \omega \left( \frac{KM}{\mu} \right) \right\}$, where $\omega \in \mathcal{K}_\infty$ satisfies (3.5).

**Proof.** We need to show that the $(T, \epsilon)$-truncated solution $z$ satisfies

$$
\dot{z}(t) \in \hat{F}_\sigma(z(t)), \quad z(t) \in \mathcal{Z}_\sigma
$$

for almost all $t \in [0, T']$, where $\mathcal{Z}_\sigma := \mathcal{Z} + \sigma \mathcal{B}$ and $\hat{F}_\sigma(z) := \text{co} \hat{F}((z + \sigma \mathcal{B}) \cap \mathcal{Z}) + \sigma \mathcal{B}$ for all $z \in \mathcal{Z}_\sigma$ and with $\hat{F}$ defined in (3.4). Note that for $z \in \mathcal{Z}_\sigma$ we have that

$$
P_Z(z) \subset (z + \sigma \mathcal{B}) \cap \mathcal{Z}.
$$

Proposition 3.3 guarantees that $z(t) \in \mathcal{Z} + \frac{KM}{\mu} \mathcal{B}$, and since $\sigma \geq \frac{KM}{\mu}$ it follows that $z(t) \in \mathcal{Z}_\sigma$ for all $t \in [0, T']$. For the remainder of the proof we omit the argument of $z(t)$ to simplify notation. All statements hold for almost all $t \in [0, T']$.

Since $z - P_Z(z) \subset N_{P_Z(z)} \mathcal{Z}$ for all $z \in \mathbb{R}^n$ [30, Ex. 6.16] and using (2.3) we have

$$
\frac{1}{K} G(P_Z(z))^{-1} (z - P_Z(z)) \in N_{P_Z(z)} \mathcal{Z}.
$$

Furthermore, since $z \in \mathcal{Z} + \frac{KM}{\mu} \mathcal{B}$ and using $\gamma$ as defined in (3.4) we have that

$$
\frac{1}{K} G(P_Z(z))^{-1} (z - P_Z(z)) \leq \frac{1}{K} \nu \left( \frac{KM}{\mu} \right) = \frac{\nu M}{\mu} \leq \gamma.
$$

Combining (3.8) and (3.9) we have

$$
\dot{z} \in f(z, P_Z(z)) - N_{P_Z(z)} \mathcal{Z} \cap \gamma \mathcal{B}.
$$

Note that, in contrast to (3.4), the normal cone is evaluated at $P_Z(z)$.

Next, using the fact that $\omega$, as defined in (3.5), is strictly increasing, and exploiting the definition of $\sigma$, we have

$$
\|f(z, P_Z(z)) - f(P_Z(z), P_Z(z))\| \leq \omega(\|z - P_Z(z)\|) \leq \omega \left( \frac{KM}{\mu} \right) \leq \sigma.
$$

Therefore, in summary, using (3.11) on (3.10), as well as (3.7), we have that

$$
\dot{z} \in f(z, P_Z(z)) - N_{P_Z(z)} \mathcal{Z} \cap \gamma \mathcal{B} \subset f(P_Z(z), P_Z(z)) + \sigma \mathcal{B} - N_{P_Z(z)} \mathcal{Z} \cap \gamma \mathcal{B} = \hat{F}(P_Z(z)) + \sigma \mathcal{B} \subset \hat{F}((z + \sigma \mathcal{B}) \cap \mathcal{Z}) + \sigma \mathcal{B} \subset \hat{F}_\sigma(z).
$$

Hence, $z(\cdot)$ satisfies (3.6) which completes the proof. \(\square\)
4. Uniform Convergence. We establish the graphical/uniform convergence of solutions of the anti-windup approximation (3.1) to solutions of the projected dynamics (3.2). This proof requires two arguments: On the one hand, we need to show that a graphically convergent sequence of solutions of (3.1) converges to a solution of (3.2). On the other hand, we need that such a graphically convergent sequence exists.

Starting with the latter requirement, we first recall that from a bounded sequence of sets, we can always extract a graphically convergent subsequence [15, Thm. 5.7]. This applies in particular to a sequence of (uniformly) truncated solutions:

**Lemma 4.1.** Consider a sequence $K_n \to 0^+$ and $z_0 \in Z$. Given $T, \epsilon > 0$, any sequence $\{z_n\}$ of $(T, \epsilon)$-truncated solution of (3.1) with $K = K_n$ and $z_n(0) = z_0$ has a graphically convergent subsequence.

**Lemma 4.1** is purely set-theoretic and does not imply that the limit $\text{gph lim}_{n \to \infty} z_n$ is a single-valued map. Hence, we need the following simplification of [15, Thm. 5.29]:

**Lemma 4.2.** Let the inclusion (2.1) be well-posed and $z_0 \in Z$. Further, given any $T, \epsilon, \rho > 0$ and $\delta_i \to 0^+$, let $z_i : [0, T_i] \to X_i$ denote a $(T, \epsilon)$-truncated solution of the $\delta_i\sigma$-perturbation of (2.1). If the sequence $\{z_i\}$ converges graphically, then convergence is to a solution $z : [0, T] \to X$ of (2.1), where $T = \lim_{i \to \infty} T_i$.

**Remark 4.3.** In the context of Lemma 4.2, graphical convergence implies uniform convergence to $z$ on every subinterval of $[0, T]$ [15, Lem. 5.28]. Furthermore, if $T_i \geq T$ for all $i$, then convergence is uniform on $[0, T]$.

Since, by Proposition 3.5, solutions of (3.1) are solutions of a $\sigma$-perturbation of an alternate form PDS (3.4) we can use Lemma 4.2 to establish the following result:

**Proposition 4.4.** Given $z_0 \in Z$, let Assumption 3.2 be satisfied. Consider $T > 0$ and a sequence $K_i \to 0^+$, and assume that a sequence of $(T, \epsilon)$-truncated solutions $z_i$ of the AWA (3.1) with $K = K_i$ and $z_i(0) = z_0$ for all $i$ converges graphically. Then, the limit is a $(T, \epsilon)$-truncated solution of the PDS (3.2).

**Proof.** Let $M, \mu, \nu > 0$ and $\omega \in \mathcal{K}_\infty$ be defined as in Assumption 3.2 and (3.5), respectively. Using Lemma 2.7, there exist $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$ such that $\omega(rs) \leq \sigma_1(r)\sigma_2(s)$ for all $r, s \geq 0$. Hence, we define $\delta_i := \max\{K_i, \sigma_1(K_i)\}$ and $\rho := \max\left\{\frac{M}{\mu}, \sigma_2\left(\frac{M}{\mu}\right)\right\}$.

Proposition 3.5 states that for every $K_i$, the solution $z_i$ of (3.1) is also a solution of the $\sigma$-perturbation of (3.4) with $\sigma := \max\left\{\frac{M}{\mu}, \omega\left(\frac{K_i M}{\mu}\right)\right\}$. It follows that $z_i$ is also a solution of every $\sigma'$-perturbation of (3.4) with $\sigma' \geq \sigma$. In particular, we can set

$$
\sigma' := \delta_i \rho = \max\{K_i, \sigma_1(K_i)\} \max\left\{\frac{M}{\mu}, \sigma_2\left(\frac{M}{\mu}\right)\right\} \geq \sigma ,
$$

and thus we have that $z_i$ is a solution of the $\delta_i \rho$-perturbation of (3.4).

Since, by assumption, $\{z_i\}$ converges graphically to $z$ it follows from Lemma 4.2 that $z$ is a solution of (3.4), and, by Proposition 2.11, $z$ is a solution of (3.2).

Finally, we need to show that $z : [0, T] \to (z_0 + \epsilon B)$ is a $(T, \epsilon)$-truncated solution. Namely, we need to show that either $T = T'$ or $\|z(T) - z_0\| = \epsilon$. This requirement is equivalent to $(T', z(T'))$ lying on the boundary of the cylinder $\mathcal{X} := [0, T] \times (z_0 + \epsilon B)$. Since, by definition, for every $i$, $z_i$ is a $(T, \epsilon)$-truncated solution of (3.1) we have that $(T_i, z_i(T_i)) \in \partial \mathcal{X}$ for all $i$. Since $\partial \mathcal{X}$ is closed, it follows that the limit also lies in $\partial \mathcal{X}$.

---

\(^4\)We require only the first of the two statements of the original theorem. Further, we consider the case where $\rho$ is constant. Finally, we work with truncated solutions which have, by definition, a compact domain (and thus are trivially locally eventually bounded [15, Def. 5.24]).
Now, we can immediately combine Lemma 4.1 and Proposition 4.4 to arrive at our first main result about the graphical convergence of truncated solutions (i.e., local) solutions of anti-windup approximations to a projected dynamical system:

**Theorem 4.5.** Let Assumption 3.2 be satisfied for some $z_0 \in \mathcal{Z}$. Given any $T > 0$ (and $\epsilon > 0$ from Assumption 3.2), consider a sequence $K_n \rightarrow 0^+$ and let $\{z_n\}$ denote a sequence of $(T, \epsilon)$-truncated solutions of the AWA (3.1) with $K = K_n$ and $z_n(0) = z_0$. Then, there exists a subsequence of $\{z_n\}$ that converges graphically to a $(T, \epsilon)$-truncated solution of the PDS (3.2).

Under certain circumstances, it can be useful to know that, rather than a subsequence of gains $\{K_n\}$, any sequence $K_n \rightarrow 0^+$ will lead to a converging sequence of solutions. This is guaranteed if it is known that the PDS (3.2) has a unique solution:

**Corollary 4.6.** Let Assumption 3.2 be satisfied for some $z_0 \in \mathcal{Z}$. Given any $T > 0$ (and $\epsilon > 0$ from Assumption 3.2), assume that the PDS (3.2) admits a unique $(T, \epsilon)$-truncated solution $z$ with $z(0) = z_0$. Then, any sequence $\{z_n\}$ of $(T, \epsilon)$-truncated solutions of the AWA (3.1) with $z_n(0) = z_0$ and $K = K_n$ with $K_n \rightarrow 0^+$ converges graphically to the (unique) $(T, \epsilon)$-truncated solution of the PDS (3.2).

**Proof.** Assume, for the sake of contradiction, that $\{z_n\}$ does not converge to the unique solution $z$ of (3.2). This implies that there exists $\nu > 0$ and a subsequence $\{z_m\}$ of $\{z_n\}$ such that $d_\infty(gph \ z_m, gph \ z) \geq \nu$ for all $m$ where $d_\infty$ denotes the Hausdorff distance between two sets. (In particular, since $z$ is a truncated solution $gph \ z$ is compact and thus graphical convergence is equivalent to convergence with respect to $d_\infty$.) However, by Lemma 4.1, the sequence $\{z_m\}$ has a convergent subsequence that converges to some limit $\tilde{z}$. By Proposition 4.4, $\tilde{z}$ is a solution of (3.2), but we also have $\|\tilde{z} - z\|_\infty \geq \epsilon$ which contradicts the uniqueness of $z$.

Finally, we can state the following ready-to-use result about uniform convergence in the case when the existence of unique complete solutions is guaranteed:

**Corollary 4.7.** Consider the AWA (3.1), let $Z$ be prox-regular, $f$ globally Lipschitz, and there exist $\mu, \nu > 0$ such that $\mu I \leq G^{-1}(z) \leq \nu I$ for all $z \in \mathbb{R}^n$. Given $z_0 \in Z$ and a sequence $K_n \rightarrow 0^+$, every sequence of complete solutions $z_n$ of the AWA (3.1) with initial condition $z_0$ and $K = K_n$ converges uniformly to the unique complete solution of the PDS (3.2) on every compact interval $[0, T]$.

**Proof.** Note that the assumptions on $Z, G, \text{ and } f$ guarantee that for every initial condition (3.2) admits a unique complete (Carathéodory) solution (Theorem 2.10).

Hence, given any $T > 0$, let $z : [0, T] \rightarrow Z$ denote the unique solution of the PDS (3.2) and define $\epsilon > \sup_{t \in [0, T]} \|x(t) - x_0\|$. Since $f$ is continuous and hence bounded over a compact set, Assumption 3.2 is satisfied with $\nu, \mu, \alpha$ and by choosing $M := \max_{z \in z_0 + B} \|f(z, P_{\mathcal{Z}}(z))\|$. Theorem 4.5 guarantees convergence of a subsequence to the $(T, \epsilon)$-truncated solution $z : [0, T'] \rightarrow Z$ of (3.2). Moreover, for the same reason as in Corollary 4.6 the sequence itself converges.

Finally, by definition of $\epsilon$, we have that $z$ is defined on $[0, T']$ with $T' = T$ and $\|z(T) - z_0\| < \epsilon$ and, in this case, graphical convergence of $(T, \epsilon)$-truncated solutions implies their uniform convergence on $[0, T]$ (see Remark 4.3).

**Remark 4.8.** Theorem 4.5 and its corollary can be slightly generalized, albeit at the expense of additional technicalities. For instance, instead of considering a single initial condition $z_0 \in Z$, it is in general possible to consider a sequence of initial conditions (under some additional restrictions) that converges to $z_0$. ■
5. Semiglobal Practical Robust Stability. Since anti-windup approximations can be seen as perturbations of projected dynamical systems, we can establish semiglobal practical asymptotic stability in $K$ with the following simplified lemma:

**Lemma 5.1.** [15, Lem. 7.20] Let the inclusion (2.1) be well-posed and let $A \subset X$ be a compact and asymptotically stable set for (2.1), i.e., $d_A(x(t)) \leq \beta(d_A(x(0)), t)$ for all $t \geq 0$ holds for some $\beta \in KL$ and any (complete) solution $x$ of (2.1). Then, for every $\rho > 0$, every compact $B \subset \mathbb{R}^n$, and every $\zeta > 0$ there exists $\delta \in (0, 1)$ such that every solution $x_\delta$ of the $\delta$-$\rho$-perturbation of (2.1) starting in $B \cap C_{\delta\rho}$ satisfies $d_A(x_\delta(t)) \leq \beta(d_A(x_\delta(0)), t) + \zeta$ for all $t \geq 0$.

Hence, using Proposition 3.5, we arrive at the following second main result:

**Theorem 5.2.** Consider a PDS (3.2) where $C$ is Clarke regular, $f$ and $G$ are continuous, and for which the compact set $A \subset Z$ is globally asymptotically stable, i.e., there is $\beta \in KL$ such that for every solution $z$ it holds that $d_A(z(t)) \leq \beta(d_A(z(0)), t) \quad \forall t \geq 0$.

Then, for every $\zeta > 0$ and every compact $B \subset Z$ there exists $K^* > 0$ such that for all $K \in (0, K^*)$ every solution $z_K$ of the AWA (3.1) with $z_K(0) \in B$ satisfies $d_A(z_K(t)) \leq \beta(d_A(z_K(0)), t) + \zeta \quad \forall t \geq 0$.

**Proof.** First, we establish that Assumption 3.2 holds for every $z_0 \in B$. Since $B$ is compact, let $\mathcal{B} := \max_{z \in B} \beta(d_A(z), 0)$. Since $\beta$ is strictly increasing and unbounded, and, since $A$ is compact, the set $\mathcal{V} := \{z | \beta(d_A(z), 0) \leq \mathcal{B}\}$ is compact. Hence, we can choose $\epsilon > 0$ such that $\mathcal{V} \subset \mathcal{B} + \epsilon B$. It follows that any solution of (3.2) starting in $\mathcal{B}$ remains in $\mathcal{B} + \epsilon B$. By continuity over the compact set $\mathcal{B} + \epsilon B$, we can further choose $M, \mu, \nu > 0$ such that $\|f(z, P_\omega(z))\| \leq M$ and $\mu \leq G(z) \leq \nu$ holds for all $z \in \mathcal{B} + \epsilon B$. Thus, Assumption 3.2 is satisfied for all $z_0 \in B$. Further, every (complete) solution of the PDS (3.2) starting in $\mathcal{B}$ remains in $\mathcal{B} + \epsilon B$ and hence can be written in its alternate form (3.4). Next, fix any $\rho > 0$. Lemma 5.1 implies that for every $\zeta > 0$ and every compact $B \subset Z$ there exists $\delta \in (0, 1)$ such that the $\delta$-$\rho$-perturbation is $\zeta$-practically pre-asymptotically stable. Given such a $\delta$, we conclude that there exists $K^* > 0$ that, for all $K' < K^*$, $\max\{K' P, \omega(K' P)\} \leq \delta \rho$ since $\omega$ is strictly increasing and $\omega(0) = 0$. Thus, Proposition 3.5 states that the solution of (3.1) with $K = K'$ is a solution of the $\sigma$-perturbation of (3.4) with $\sigma = \max\{K' P, \omega(K' P)\}$. Moreover, it is also solution to any $\sigma'$-perturbation with $\sigma' \geq \sigma$ and, in particular, for $\sigma' = \delta \rho$.

Since the asymptotic stability of $\mathcal{A}$ can often be established with a smooth Lyapunov function (see [15, Thm. 3.18]), we can also state the following corollary:

**Corollary 5.3.** Consider the PDS (3.2) where $C$ is Clarke regular, $f$ and $G$ are continuous. Further, consider a compact set $A \subset Z$ for which there exists a Lyapunov function. Then, for every $\zeta > 0$ and every compact set $B \subset \mathbb{R}^n$, there exists $K^*$ such that for all $K \in (0, K^*)$ every solution of (3.1) converges to a subset of $A + \zeta B$.

---

5We consider only global asymptotic stability, which allows us to use the distance function instead of more general indicator functions. Further, we limit ourselves to $\rho$ being a positive constant instead of a function. As noted in Remark 2.8, compactness and stability of $A$ guarantee the existence of complete solutions since finite-time escape is not possible.

6Namely, $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a Lyapunov function for $A$ if it is differentiable everywhere on $Z$, there exist $\alpha, \sigma, \varpi \in K_\omega$ such that $\alpha(d_A(z)) \leq V(z) \leq \varpi(d_A(z))$ for all $z \in Z$, and $\langle \nabla V(z), \Pi_{d_A(z)}^G[f](z) \rangle \leq -\alpha(z)$ for all $z \in Z$ where $\alpha : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is continuous and positive definite with respect to $A$, i.e., $\alpha(z) > 0$ for all $z \notin A$ and $\alpha(z) = 0$ for all $z \in A$.  

6. Preservation of Equilibria & Robust Convergence. Finally, we consider
the special case of (3.1) when \( f \) depends only on \( P_Z(z) \), i.e., we study the system
\[
\dot{z} \in F_K(z) := f(P_Z(z)) - \frac{1}{K} G^{-1}(P_Z(z))(z - P_Z(z))
\]
where, as before, \( Z \) is an \( \alpha \)-prox-regular set, \( G : Z \to \mathbb{S}_+^n \) is a continuous metric,
\( K > 0 \) is a scalar, and \( f : Z \to \mathbb{R}^n \) is a continuous vector field. All of the previous
results for (3.1) also apply to (6.1). In particular, as \( K \to 0^+ \), trajectories of (6.1)
converge uniformly to solutions of the PDS (3.2). Also, the practical stability results
of section 5 apply, but we show next that a stronger result can be derived for (6.1).

In the following, \( z^* \) is a weak equilibrium of (6.1) if the constant trajectory \( z \equiv z^* \)
is a solution of (6.1). Since we consider only Carathéodory solutions, \( z^* \) is a weak
equilibrium of (6.1) if and only if \( 0 \in F_K(z^*) \).

An important advantage of (6.1) over the more general system (3.1) is that equi-
libria of (3.2) are preserved in the following sense (which generalizes [19, Prop. 4]):

**Proposition 6.1.** If \( \tau^* \in Z \) is a weak equilibrium point of the PDS (3.2), then
there exists \( K^* > 0 \) such that for all \( K \in (0, K^*) \) there exists a weak equilibrium point
\( z_K^* \in \tau^* + N_{\tau^*}Z \cap \frac{1}{K^*} \text{int} B \) for the AWA (6.1). Conversely, if \( z_K^* \in Z_O^\tau \) is a weak
equilibrium of (6.1) for some \( K \), then \( P_Z(z_K^*) \) is a weak equilibrium of (3.2).

**Proof.** Given a weak equilibrium \( \tau^* \in Z \) of (3.2), let \( z_K^* := \tau^* - K G(\tau^*) f(\tau^*) \).
For \( K \in (0, K^*) := 1/(2\alpha\|G(\tau^*) f(\tau^*)\|) \), we have \( z_K^* \in Z_O^\tau \).

Since \( \tau^* \) is an equilibrium of (3.2) (by assumption) and using Lemma 2.9, we have
\( f(\tau^*) \in -N_{\tau^*}G Z \). It follows from (2.3) that \( -K G(\tau^*) f(\tau^*) \in N_{\tau^*}Z \) and consequently
\( z_K^* \in \tau^* + N_{\tau^*}Z \). By Proposition 2.1, it follows that \( P_Z(z_K^*) = \tau^* \) and therefore
\[
F_K(z_K^*) = f(\tau^*) - \frac{1}{K} G^{-1}(\tau^*)(\tau^* - K G(\tau^*) f(\tau^*) - \tau^*) = 0.
\]
Thus, \( z_K^* \) is a weak equilibrium of (6.1). The converse case follows the same ideas. \( \square \)

Although equilibria of the PDS (3.2) are preserved by the AWA (6.1) (after pro-
jection), it is not clear whether convergence properties are preserved, especially since
we are primarily interested in the convergence of \( t \mapsto P_Z(z(t)) \) rather than the con-
vergence of the solution \( z \) of (6.1). Theorem 5.2 suggests that, in general, convergence
is only within a neighborhood of asymptotically stable equilibria of the PDS (3.2).

However, as we shown below, under additional conditions on \( f, G \) and \( Z \), the
projected solutions \( t \mapsto P_Z(z(t)) \) do indeed converge to an equilibrium of (3.2).

6.1. Anti-Windup Approximations of Monotone Dynamics. Next, we
show that if \(-f\) is monotone and \( G \equiv I \), then \( F_K \) as defined in (6.1), is monotone for
small enough \( K \). This, in turn, allows us conclude asymptotic stability of (6.1).

Since we require only monotonicity of \( f \), the following results can be used not only
when \( f \) is chosen as the gradient of a convex cost function, but also for saddle-point
flows (see subsection 7.2), and pseudo-gradients for Nash-equilibrium seeking [11,28].

Given a set \( C \subset \mathbb{R}^n \), recall that a map \( F : C \Rightarrow \mathbb{R}^n \) is (strictly; \( \beta \)-strongly)
monotone if for all \( x, x' \in C \) and all \( v \in F(x) \) and \( v' \in F(x') \) it holds that
\[
\langle v - v', x - x' \rangle \geq 0 \quad (\text{strictly; } \beta \geq 2\alpha
\|
x - x'\|^2).
\]
Further, if \( C \) is \( \alpha \)-prox-regular, the map \( x \mapsto N_xC \) has a hypomono-
tone localization [30, Ex. 13.38], i.e., for all \( x, x' \in C \), all \( \eta \in N_xC \cap \mathbb{B} \), and all \( \eta' \in N_{x'}C \cap \mathbb{B} \) we have
\[
\langle \eta - \eta', x - x' \rangle \geq -2\alpha \|
x - x'\|^2.
\]
In particular, if \( C \) is convex, we have \( \langle \eta' - \eta, x' - x \rangle \geq 0 \) and \( x \mapsto N_xC \) is monotone.
Proposition 6.2. Consider $F_K$ as defined in (6.1) with $G \equiv 1$ and $C$ is assumed to be $\alpha$-prox-regular. Let $-f$ be $\beta$-strongly monotone and globally $L$-Lipschitz. Then $-F_K$ is strictly monotone on $Z_0^\alpha$ for all $0 < K < 4(\beta - 2\alpha)/L^2$.

Proof. Given any $z, z' \in Z_0^\alpha$, let $\varpi := P_Z(z)$ and $\varpi' := P_Z(z')$. Further, let $\eta := z - \varpi \in N_Z\varpi$ and $\eta' := z' - \varpi' \in N_Z\varpi'$. We can work directly with the monotonicity of $f$, the hypomonotonicity of $z \mapsto N_Zz$, and Cauchy-Schwarz to derive

$$
\langle z - z', F_K(z) - F_K(z') \rangle = \langle z - z', f(\varpi) - f(\varpi') \rangle - \frac{1}{K} \langle (z - \varpi) + \frac{1}{K} (z' - \varpi') \rangle \\
= \langle \varpi - \varpi', f(\varpi) - f(\varpi') \rangle - \frac{1}{K} \langle (\eta - \eta'), \varpi - \varpi' \rangle \\
+ \frac{1}{L} \langle \eta - \eta', f(\varpi) - f(\varpi') \rangle - \frac{1}{K} \langle \varpi - \varpi', \eta - \eta' \rangle \\
\leq -(\beta - 2\alpha)\|\varpi - \varpi'\|^2 + \frac{1}{L} \|\varpi - \varpi'\|\|\eta - \eta'\| - \frac{1}{K} \|\eta - \eta'\|^2.
$$

A sufficient condition for the righthand side to be negative for all $\varpi \neq \varpi'$ is that $\beta - 2\alpha > 0$ and that the determinant $\frac{1}{K}(\beta - 2\alpha) - \frac{1}{L}^2$ is positive, i.e., if $0 < K < 4(\beta - 2\alpha)/L^2$. \hfill \Box

This leads us to our third theoretical result which establishes convergence of anti-windup approximations for strongly monotone dynamics on convex sets:

Theorem 6.3. Consider the AWA (6.1) with $G \equiv 1$ and let $C$ be closed convex. Assume that $-f$ is $\beta$-strongly monotone and globally $L$-Lipschitz. Then, for all $K < 4\beta/L^2$, every trajectory of (6.1) converges to an equilibrium point $z^*$ (which is unique) such that $P_Z(z^*)$ is the unique equilibrium of the PDS (3.2).

Proof. Because of convexity of $Z$, $P_Z(z)$ is single-valued and continuous for all $z \in \mathbb{R}^n$ and globally 1-Lipschitz (i.e., non-expansive). As a consequence, $F_K$ is globally Lipschitz continuous and there exists a unique complete solution of (6.1) for every initial condition $z(0) \in \mathbb{R}^n$. Furthermore, since $K < 4\beta/L^2$ and $Z$ is convex (which lets us take $\alpha \to 0^+$), Proposition 6.2 guarantees that $F_K$ is strictly monotone on $\mathbb{R}^n$.

Next, recall that the strong monotonicity of $-f$ and convexity of $Z$ imply that (3.2) has a unique equilibrium $z^*$ [28, Thm. 2.3]. Consequently, Proposition 6.1 guarantees the existence of an equilibrium point $z^*$ of (6.1) such that $P_Z(z^*) = z^*$. Furthermore, $z^*$ is unique by [28, Thm. 2.2]. In particular, strict monotonicity of $F_K$ implies that $V(z) := \frac{1}{2}\|z - z^*\|^2$ is a Lyapunov function for (6.1) which can be used to establish global asymptotic stability of $z^*$.

Theorem 6.3 can, presumably, be generalized to prox-regular sets as well as general metrics $G$. However, in that case, additional restriction on $z(0)$ are required, the threshold value for $K$ is less easily quantifiable, and convergence is likely only local.

7. Application: Anti-Windup for Autonomous Optimization. Next, we show how the AWA (3.1) models physical systems and how anti-windup implementations can be used in the context of autonomous optimization to approximate closed-loop optimization dynamics that are formulated as projected dynamical systems.

First, consider the feedback control loop illustrated in Figure 2. Namely, we study a plant controlled by an integral feedback controller that is subject to input saturation modelled as an Euclidean projection. An anti-windup scheme is in place to avoid integrator windup. More precisely, we consider a dynamical system of the
\[ \dot{x} = f(x, u) \quad x \in \mathbb{R}^m \]
\[ \dot{u} \in k(x, u, P_U(u)) - \frac{1}{K} \tilde{G}^{-1}(P_U(u))(u - P_U(u)) \quad u \in \mathbb{R}^p \]

where \(U \subset \mathbb{R}^p\) is prox-regular, \(\tilde{f} : \mathbb{R}^m \times U \to \mathbb{R}^m\) and \(k : \mathbb{R}^m \times \mathbb{R}^p \times U \to \mathbb{R}^p\) are continuous vector fields, \(\tilde{G} : U \to \mathbb{S}^p_+\) is a continuous metric, and \(K > 0\).

The system (7.1) can be brought into the form of an AWA (3.1) with \(n = m + p\) by defining \(z := [x, u]\), \(Z := \mathbb{R}^m \times U\), and \(G(z) := \begin{bmatrix} I & 0 \\ 0 & G(u) \end{bmatrix}\). Thus, we further have
\[ P_Z(z) = \begin{bmatrix} x \\ P_U(u) \end{bmatrix} \quad \text{and} \quad f(z, P_Z(z)) := \begin{bmatrix} \tilde{f}(x, P_U(u)) \\ k(x, u, P_U(u)) \end{bmatrix}. \]

With these definitions, the PDS (3.2) takes the form
\[ \dot{x} = \tilde{f}(x, u) \quad x \in \mathbb{R}^m \]
\[ \dot{u} = \Pi_{P_U}^G[k(x, u, u)](u) \quad u \in U, \]

where we can ignore the projection onto \(U\) in the third argument of \(k\), because any solution of the PDS (3.2) is viable (i.e., remains in \(U\)) by definition.

Remark 7.1. Figure 2 shows one limitation of our problem setup: Compared to existing work on anti-windup control \([34, 35]\), we do not model any proportional controller subject to input saturation. This is motivated, on one hand, by theoretical necessity. On the other hand, for our application scenario of autonomous optimization discussed below, stability of the physical plant is usually a prerequisite.

7.1. Feedback-based Gradient Schemes for Quadratic Programs. To illustrate the design opportunities for autonomous optimization, we present three anti-windup schemes that approximate projected gradient flows for a quadratic program (QP). We consider the relatively simple problem of solving a QP as it allows for a concise presentation, easy implementation, and comparability. However, needless to say, our theoretical results in the previous sections cover much more general setups.

Our goal is to design a feedback controller that steers a plant to a steady state that solves the optimization problem
\[
\begin{align*}
\text{minimize} & \quad \Phi(x) := \frac{1}{2} x^T Q x + c^T x + d \\
\text{subject to} & \quad x = h(u) := H u + w \\
& \quad u \in U := \{ v \mid A_u v \leq b_u \}
\end{align*}
\] (7.2)
where $x \in \mathbb{R}^m$ and $u \in \mathbb{R}^p$ denote the system state and control input, respectively, and $Q \in \mathbb{S}^m$, $A_u \in \mathbb{R}^{r \times p}$ and the remaining parameters are of appropriate size. The map $h$ denotes the steady-state input-to-state map of the plant subject to the disturbance $w$.\footnote{In contrast to~\eqref{eq:state_map}, we assume for~\eqref{eq:input_map} that the physical plant is described by a steady-state input-to-state map $x = h(u)$ that satisfies $f(h(u), u) = 0$ for all $u \in \mathcal{U}$. This approximation can be motivated by singular perturbation ideas~\cite{18, 26} which stipulate that the interconnection of fast decaying plant dynamics and slow optimization dynamics is asymptotically stable. The results in this section can be generalized to a dynamic plant accordingly.}

The set $\mathcal{U}$ defines constraints which are enforced by physical saturation.

For solving~\eqref{eq:input_map} we aim at approximating the projected gradient flow $\dot{u} = \Pi_C^U[-G^{-1}(u)\nabla \Phi(u)](u)$, where we have defined $\Phi(u) := \Phi(h(u))$ to eliminate the state variable $x$. In particular, we have $\nabla \Phi(u) = H^T \nabla \Phi(h(u))$. In the following, the metric $G$ will be either $G \equiv I$ or $G \equiv Q$ (the latter yielding a projected Newton flow).

To approximate $\dot{u} = \Pi_C^U[-G^{-1}(u)\nabla \Phi(u)](u)$, we consider three systems that fall into the class of anti-windup approximations defined by \eqref{eq:awa}, two of which can be implemented in a feedback loop as in Figure 2. Their convergence behavior for the same problem instance and varying $K$ is illustrated in Figure 3 and discussed below.

i) **Penalty Gradient Flow:** As a reference system we consider the gradient flow of the potential function $\Psi(u) := \Phi(u) + \frac{1}{2K} d_U^2(u)$ which is given by

\begin{equation}
\dot{u} = -\nabla \Psi(u) = -H^T \nabla \Phi(h(u)) - \frac{1}{K} (u - P(u)).
\end{equation}

In this case, we have $G \equiv I$ and $K > 0$ takes the role of a penalty parameter for the soft penalty term $d_U^2$ that approximately enforces the input constraint $u \in \mathcal{U}$.\footnote{The penalty $d_U^2$ is illustrative in the context of autonomous optimization, however, it is not generally practical for numerical optimization, because evaluating $\nabla d_U^2$ requires computing $P_U$. Instead, in numerical applications, it is more common to use a penalty $\|\max\{A_u u - b_u, 0\}\|^2$.}

The system \eqref{eq:penalty_flow} is a special case of the AWA \eqref{eq:awa} and, as a consequence, Theorems 4.5 and 5.2 (uniform convergence and robust practical stability) and their corollaries apply as $K \to 0^+$. However, \eqref{eq:penalty_flow} is not of the special form \eqref{eq:linearized_awa} and convergence of to the optimizer of the problem \eqref{eq:input_map} is not guaranteed for positive $K > 0$. Neither does \eqref{eq:penalty_flow} lend itself to a feedback implementation, because $\nabla \Phi$ is evaluated at $h(u)$ rather than at $h(P_U(u))$ (which is the actual system state for the saturated input).

ii) **Anti-Windup Gradient Scheme:** As a second type of dynamics we consider

\begin{equation}
\dot{\pi} = -H^T \nabla \Phi(\pi) - \frac{1}{K} (u - \pi) \quad \text{controller} \quad \pi := P_U(u) \quad \text{physical system} \quad \pi := h(\pi)
\end{equation}

which can be implemented in closed loop because the quantities $\pi$ and $\pi$ are “evaluated” by the physical system at no computational cost (and are assumed to be measurable), which is one of the key features of autonomous optimization. Furthermore, because $\mathcal{U}$ is convex and $\Phi$ is strongly convex (which implies strong monotonicity), Theorem 6.3 is applicable and guarantees that $\pi = (\pi, \pi)$ converges to the optimizer of \eqref{eq:input_map}. This is confirmed in Figure 3.

iii) **Anti-Windup Newton Scheme:** As the final gradient-based anti-windup scheme we consider an anti-windup approximation with $G \equiv Q$ and which is given by

\begin{equation}
\dot{\pi} = -Q^{-1} (H^T \nabla \Phi(\pi) - \frac{1}{K} (u - \pi)) \quad \text{controller} \quad \pi := P_U(u) \quad \text{physical system} \quad \pi := h(\pi).
\end{equation}
The anti-windup dynamics are simulated with MATLAB using a fixed-stepsize forward Euler scheme. The projection on $U$ is evaluated using quadprog. The nominal PDS is approximated using a projected forward Euler scheme as $u^{k+1} = P_U(u^k + \alpha f(u^k))$ which is guaranteed to converge uniformly as $\alpha \to 0^+$ [28].
A trajectory of (7.4) is not affected by the value of $K$ and is equivalent to the convergence rate of the nominal projected gradient flow. In contrast, the convergence rate of the anti-windup Newton scheme (7.5) does depend on $K$ and one can recover the rate of projected Newton flow only in the limit $K \to 0^+$. An analysis of this observation remains, however, outside the scope of this paper.

7.2. Feedback-based Saddle-Flows with Anti-Windup. In autonomous optimization, constraints on the system state (or output) cannot be enforced directly because they are not directly controllable and often subject to disturbances affecting the physical plant (e.g. an unknown value of $w$). For the purpose of enforcing state or output constraints, projected saddle-point flows have been proven effective [10,29,32]. In this section, we indicate how anti-windup approximations can be combined with this type of dynamical system, even though this leads us slightly outside the scope of our theoretical results. We consider quadratic program

$$\begin{align*}
\text{minimize} \quad & \Phi(x) \\
\text{subject to} \quad & x = h(u), \ u \in \mathcal{U} \\
& x \in \mathcal{X} := \{x \mid A_x x \leq b_x\},
\end{align*}$$

(7.6)

where $\Phi$, $h$, and $\mathcal{U}$ are defined as in (7.2) and $\mathcal{X}$ denotes a set of state constraints with $A_x \in \mathbb{R}^{s \times m}$ and $b_x \in \mathbb{R}^s$. To solve (7.6), we consider the projected saddle-point flow

$$\begin{align*}
\dot{u} &= \Pi_{\mathcal{U}} \left[ -H^T \nabla \Phi(h(u)) - H^T A_x^T \mu \right] \\
\dot{\mu} &= \Pi_{\mathbb{R}^s_{\geq 0}} \left[ A_x h(u) - b_x \right],
\end{align*}$$

(7.7)

where $\mu \in \mathbb{R}^s$ denotes the dual multipliers associated with the output constraints. The system (7.7) (and special cases in which either primal or dual variables are not projected) has been extensively studied and convergence is guaranteed, for instance, under strict convexity of $\Phi$. We refer the reader to [5,14] and references therein.

We approximate (7.7) with a (partial) anti-windup implementation as

$$\begin{align*}
\dot{u} &= -H^T \nabla \Phi(x) - H^T A_x^T \mu - \frac{1}{K} (u - \bar{u}) \\
\dot{\mu} &= \Pi_{\mathbb{R}^s_{\geq 0}} \left[ A_x \bar{x} - b_x \right],
\end{align*}$$

(7.8)

where $\bar{u} := P_U(u)$ and $\bar{x} := h(\bar{u})$.

We do not approximate the projected integration of the dual variables with an anti-windup term, since the dual variables are often internal variables of the controller and the projection on the non-negative orthant is easily implementable.

Figure 4 illustrates the behavior of (7.7) and (7.8). Similarly to the results for the gradient anti-windup approximations, we observe that $u$ does not, in general, converge to its optimal value. However, the saturated control input $P_U(u)$ (and thereby the actual system state) and the dual variable $\mu$ converge to the solution of (7.6).

Theorem 6.3 (robust convergence) does not apply to (7.8). First, while the projected saddle-flow (7.7) is monotone, strong monotonicity is usually not guaranteed [5,14]. Second, by applying only a partial anti-windup approximation, the vector field remains discontinuous because of the projection of $\mu$ on $\mathbb{R}^s_{\geq 0}$.

8. Conclusion. In this paper we have studied a general class of dynamical systems which are inspired by classical anti-windup control schemes. We have rigorously established that these systems approximate oblique projected dynamical systems in terms of uniform convergence and semiglobal practical robust stability. Furthermore, we have shown that for a special case, and under an additional monotonicity assumption, these anti-windup approximations exhibit robust convergence to the equilibria of
the limiting projected dynamical system. We have further illustrated several ways in which our results apply in the context of autonomous optimization. In particular, we have shown how physical saturation can be exploited to drive a plant to an optimal steady state without explicit knowledge of the physically-enforced input domain.

Several points remain open: First, it is unclear whether our analysis can be extended to consider control laws that incorporate a proportional control component. Second, the strong monotonicity requirement for robust convergence to equilibria of a projected dynamical systems can presumably be relaxed. Third, our simulations suggest that certain anti-windup gradient schemes retain the same convergence rate as the limiting projected gradient flow, independently of the anti-windup gain. Fully understanding this surprising phenomenon requires further work.

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