Flux homomorphism on symplectic groupoids

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Abstract

For any Poisson manifold $P$, the Poisson bracket on $C^\infty(P)$ extends to a Lie bracket on the space $\Omega^1(P)$ of all differential one-forms, under which the space $Z^1(P)$ of closed one-forms and the space $B^1(P)$ of exact one-forms are Lie subalgebras. These Lie algebras are related by the exact sequence:

$$0 \to \mathbb{R} \to C^\infty(P) \xrightarrow{d} Z^1(P) \xrightarrow{f} H^1(P;\mathbb{R}) \to 0,$$

where $H^1(P;\mathbb{R})$ is considered as a trivial Lie algebra, and $f$ is the map sending each closed one-form to its cohomology class. The goal of the present paper is to lift this exact sequence to the group level for compact Poisson manifolds under certain integrability condition. In particular, we will give a geometric description of a Lie group integrating the underlying Poisson algebra $C^\infty(P)$. The group homomorphism obtained by lifting $f$ is called the flux homomorphism for symplectic groupoids, which can be considered as a generalization, in the context of Poisson manifolds, of the usual flux homomorphism of the symplectomorphism groups of symplectic manifolds introduced by Calabi.

1 Introduction

Poisson manifolds are a natural generalization of symplectic manifolds. In fact, one can think of a Poisson manifold as a union of symplectic manifolds of varied dimensions fitting together in a certain smooth way. A Poisson manifold, by definition, is a manifold $P$ on which the space of functions $C^\infty(P)$ carries a Poisson bracket: a Lie bracket satisfying the Leibniz rule $\{f, gh\} = \{f, g\}h + \{f, h\}g$ (see [13] for more details on this subject).

Associated to a given Poisson manifold $P$, there are various infinite dimensional Lie algebras, especially the Poisson algebra $C^\infty(P)$. It is very natural to ask if there exist Lie groups integrating these Lie algebras, and if yes, how these Lie groups can be described geometrically and how they are related. If $P$ is symplectic, closely related to the Poisson algebra $C^\infty(P)$ are the Lie algebra...
of symplectic vector fields $X_s(P)$ and that of all hamiltonian vector fields $X_h(P)$. Clearly, $C^\infty(P)$ is a one-dimensional central extension of $X_h(P)$. Globally, the group $Diff(P,\omega)$ of all symplectomorphisms has $X_s(P)$ as its Lie algebra, and its subgroup $Ham(P,\omega)$ consisting of hamiltonian symplectomorphisms can be considered as a Lie group integrating $X_h(P)$ \cite{2} \cite{3} \cite{11}. Then a Lie group integrating $C^\infty(P)$ will be a one-dimensional central extension of $Ham(P,\omega)$, which in fact can be interpreted as the group of all contact diffeomorphisms of the prequantum bundle over $P$ when $P$ is prequantizable \cite{2} \cite{17}. The flux homomorphism $F : \tilde{Diff}_0(P,\omega) \to H^1(P,\mathbb{R})$ was first introduced by Calabi \cite{3} as an invariant characterizing the hamiltonian symplectomorphisms, and was intensively studied by Banyaga \cite{1} and McDuff \cite{10} \cite{11}. Here $Diff_0(P,\omega)$ denotes the identity component of $Diff(P,\omega)$ and $\tilde{Diff}_0(P,\omega)$ refers to its universal covering. Since we will use the flux homomorphism repeatedly in the paper, we recall its definition below.

Any isotopy $\varphi_t$ of symplectomorphisms in $Diff_0(P,\omega)$ corresponds to a time-dependent vector field $Z_t$ defined by:

$$ \frac{d}{dt} \varphi_t = Z_t \circ \varphi_t. \tag{1} $$

The flux homomorphism is the map $F : \tilde{Diff}_0(P,\omega) \to H^1(P,\mathbb{R})$ defined by

$$ \{ \varphi_t \} \to \int_0^1 [Z_t \mathcal{L} \omega] dt, \tag{2} $$

where $\omega$ is the symplectic structure on $P$. Intrinsically, the flux homomorphism can be understood as the lifting, to the group level, of the Lie algebra homomorphism $f : X_s(P) \to H^1(P,\mathbb{R})$ given by:

$$ f : Z \to [Z \mathcal{L} \omega]. $$

While the kernel of $f$ is the Lie subalgebra $X_h(P)$, the kernel of $F$ should characterize the group of hamiltonian symplectomorphisms \cite{1} \cite{11}.

When $P$ is a general Poisson manifold, the Lie algebra $C^\infty(P)$ could be very far from either the Lie algebra of Poisson vector fields or the Lie algebra of hamiltonian vector fields. For example, when $P$ is a zero Poisson manifold, which is an extreme case, the Lie algebra of Poisson vector fields consists of all vector fields but the Lie algebra of hamiltonian vector fields contains only the zero element. Both of these Lie algebras are quite different from the Poisson algebra $C^\infty(P)$. It turns out that, for Poisson manifolds, it is more interesting to consider various Lie algebras of differential one-forms, instead of those of vector fields. In the case of symplectic manifolds, the space of one-forms is isomorphic to that of vector fields as Lie algebras. Therefore, everything will reduce to the previous case.

Recall that the Poisson tensor $\pi$ on a Poisson manifold $P$ naturally induces a Lie algebra structure on the space of all one-forms $\Omega^1(P)$, which is given by:

$$ [\omega, \theta] = L_{X_\omega} \theta - L_{X_\theta} \omega - d[\pi(\omega, \theta)], \tag{3} $$

where $X_\omega = \pi^# \omega$ and $X_\theta = \pi^# \theta$. Here $\pi^#$ is the bundle map from $T^*P$ to $TP$ defined by $< \pi^#(\omega), \theta > = \pi(\omega, \theta)$, for any $\omega, \theta \in T^*P$. In fact, $T^*P$ together with this bracket and the anchor map $\rho = \pi^# : T^*P \to TP$ becomes a Lie algebroid \cite{4} \cite{16}. It is simple to see that both the space $\mathcal{Z}^1(P)$ of closed one-forms and the space $B^1(P)$ of exact one-forms are closed under this
bracket, and therefore they are Lie subalgebras. On the Lie algebra level, one has the following exact sequence of Lie algebra homomorphisms:

$$0 \rightarrow \mathbb{R} \rightarrow C^\infty(P) \xrightarrow{d} Z^1(P) \xrightarrow{L} H^1(P, \mathbb{R}) \rightarrow 0,$$

(4)

where $H^1(P, \mathbb{R})$ is considered as a trivial Lie algebra, and $f$ is the map sending each closed one-form to its cohomology class. Our goal of the paper, roughly speaking, is to lift this exact sequence to an exact sequence on the level of groups, and to analyze the relation between their groups.

The essential object for the construction is the so called symplectic groupoid. Since the notion of symplectic groupoids appears almost everywhere in the paper, let us briefly recall it below. A symplectic groupoid is a Lie groupoid $\Gamma \rightarrow P$ (always assumed to be Hausdorff in our case) with a compatible symplectic structure in the sense that the graph of multiplication: $\Lambda = \{(x, y, z) | z = xy, \beta(x) = \alpha(y)\}$ is a lagrangian submanifold of $\Gamma \times \Gamma \times \Gamma$ (the bar means taking the opposite symplectic structure). It is well known that the base manifold $P$ of a symplectic groupoid naturally carries a Poisson structure. If a Poisson manifold $P$ can be realized as the base Poisson manifold of some symplectic groupoid, it is called integrable. Not every Poisson manifold is integrable. However, a fairly large class of Poisson manifolds, for example, Lie-Poisson structures, are known to be integrable. On the other hand, it was a theorem of Karasev and Weinstein that every Poisson manifold admits a local symplectic groupoid (we refer the reader to [4] [13] [16] for more details on the subject).

The relevance of the symplectic groupoid for this kind of construction was also noted some time ago by several people including Dazord, Karasev, Weinstein. The reason for such a relevance is quite simple. The Lie algebra of one-forms is very closely related to the Lie algebroid $T^*P$, and the global object of the latter is exactly the symplectic groupoid $\Gamma$ over $P$. So it is not surprising that Lie groups of $Z^1(P)$ and $B^1(P)$ can be described in terms of the groupoid $\Gamma$. In fact, it is known that the group $U(\Gamma)$ of lagrangian bisections of $\Gamma$ (see Section 2 for the definition of lagrangian bisections) is a group integrating $Z^1(P)$.

On the group level, the morphism $f$ should correspond to a group homomorphism $\mathcal{F} : \widetilde{U_0(\Gamma)} \rightarrow H^1(P, \mathbb{R})$, which will be called the flux homomorphism. To define $\mathcal{F}$ precisely, we will first embed $U(\Gamma)$ into the symplectomorphism group $Diff(\Gamma, \omega)$ by considering each lagrangian bisection as a symplectic diffeomorphism on $\Gamma$ via left translations, and then take the usual flux homomorphism for symplectomorphisms. An advantage of this approach is that one can directly apply various results about the usual flux homomorphism and avoid many tedious verifications. Discussion on the flux homomorphism and its properties occupies Section 2 and Section 3. In the case that $P$ is symplectic, the symplectic structure establishes an isomorphism between the cotangent bundle $T^*P$ and the tangent bundle $TP$. Hence the space of one-forms can be identified with the space of vector fields. Then $Z^1(P)$ and $B^1(P)$ are isomorphic, as Lie algebras, to the space of symplectic vector fields and that of hamiltonian vector fields, respectively. Moreover, the pair groupoid $\Gamma = P \times T^*\Gamma$ can be taken as a symplectic groupoid over $P$. The group $U(\Gamma)$ is thus isomorphic to the symplectomorphism group $Diff(P, \omega)$, and $\mathcal{F}$ reduces to the usual flux homomorphism.

To lift the homomorphism $C^\infty(P) \xrightarrow{d} Z^1(P)$, first of all, we need a Lie group integrating the Lie algebra $C^\infty(P)$. It turns out that such a Lie group can be described in terms of a prequantum bundle $E$ over the symplectic groupoid $\Gamma$ (properties of such prequantum bundles were studied in [20]). We will devote Section 4 to the discussion on this aspect.
Despite its conceptual clearness, the flux homomorphism is in general very difficult to compute. In Section 5, we will study an alternate map closely related to the flux homomorphism when there exists an invariant measure $\nu$ on the Poisson manifold $P$. This is the composition $\mu \circ F : U_0(\Gamma) \to H^1(P,\mathbb{R})^*$ of the flux homomorphism $F$ with a linear map $\mu : H^1(P,\mathbb{R}) \to H^1(P,\mathbb{R})^*$ induced from the invariant measure $\nu$. It turns out that this map $\mu \circ F$ can be explicitly expressed by using integration when $\Gamma$ is $\alpha$-simply connected. This becomes especially useful when $\mu$ is an isomorphism. In this case, one can indeed compute the flux homomorphism by computing $\mu \circ F$. As an example, this method will be applied to symplectic manifolds in Section 6.

For simplicity, we will restrict our attention to compact Poisson manifolds. However, many results should be easily modified for noncompact Poisson manifolds when various notions in the context are replaced by their compactly supported analogues.

Recently, it came to the author’s attention that the flux homomorphism (in a more general setting) was also been constructed by Dazord [6]. However, our approach is quite different. While Dazord deals with general noncompact Poisson manifolds with possibly non-Hausdorff symplectic groupoids, he proves his assertions directly. We will instead work within the framework of compact Poisson manifolds admitting Hausdorff symplectic groupoids. Within this more conventional framework, it allows us to apply many known results about the usual flux homomorphism of symplectomorphism groups [1] [11]. Since the original preprint, another related paper [3] has also appeared.

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2 The Flux homomorphism

Let $(\Gamma \to P, \alpha, \beta)$ be a symplectic groupoid over a compact Poisson manifold, with symplectic structure $\omega$. A lagrangian submanifold of $\Gamma$ for which the restrictions of $\alpha$ and $\beta$ are diffeomorphisms onto $P$ is called a lagrangian bisection. Under the multiplication of subsets induced from the product on $\Gamma$, the lagrangian bisections form a Lie group $U(\Gamma)$ in terms of Whitney $C^\infty$-topology. $U(\Gamma)$ is easily seen to be locally path connected by the local structure theorem of lagrangian submanifolds [14]. The unit of $U(\Gamma)$ is the identity space $P$ of the groupoid, and is denoted by 1. We denote by $U_0(\Gamma)$ the identity component of $U(\Gamma)$. For any $L \in U(\Gamma)$ and $u \in P$, $L(u)$ denotes the unique point in $L$ characterized by the condition that $\beta L(u) = u$. A lagrangian bisection $L$ induces a Poisson diffeomorphism on the base Poisson manifold $P$ by $u \mapsto \alpha(L(u))$, which will be denoted by $Ad_L(u)$.

Let $L_t$ be an isotopy in $U_0(\Gamma)$ starting at the identity section, i.e., $L_0 = 1$. For any fixed $t$, the variation of $L_t$ around the neighborhood of that fixed time defines a vector field $X_t$ along the bisection $L_t$ on $\Gamma$. That is,
\[ X_t(L_t(u)) = \frac{d}{ds}igr|_{s=t} L_s(u). \]  

(5)

It follows from the property \( \beta L_t(u) = u \) that

\[ \beta_* X_t = 0. \]

That is, \( X_t \) is tangent to \( \beta \)-fibers. This implies that there exists a time-dependent one-form \( \theta_t \in \Omega^1(P) \) on the base such that

\[ X_t \omega = \alpha^* \theta_t \]

(6)

along the bisection \( L_t \).

For any \( L \in U(\Gamma) \), we denote by \( \tau(L) \) the left translation on \( \Gamma \) by \( L \), which is a diffeomorphism of \( \Gamma \). In fact, \( \tau(L) \) is a symplectomorphism of \( \Gamma \). We thus obtain a map \( \tau : U(\Gamma) \rightarrow \text{Diff}(\Gamma, \omega) \), which is a group homomorphism. Clearly, \( \tau \) maps the identity component \( U_0(\Gamma) \) into \( \text{Diff}_0(\Gamma, \omega) \).

By \( \tilde{\tau} : \tilde{U}(\Gamma)_0 \rightarrow \tilde{\text{Diff}}_0(\Gamma, \omega) \), we denote its natural lifting to their universal coverings.

Let \( \varphi_t = \tau(L_t) \) be the family of symplectomorphisms of \( \Gamma \) corresponding to the lagrangian bisection isotopy \( L_t \). We denote by \( Y_t \) the time-dependent vector field on \( \Gamma \) which generates \( \varphi_t \), i.e.,

\[ \frac{d}{dt} \varphi_t = Y_t \circ \varphi_t. \]  

(7)

The usual flux homomorphism of \( \{ \varphi_t \} \), considered as a homotopy class of a symplectic isotopy of \( \Gamma \), is defined in terms of such a time-dependent vector field \( Y_t \). To find out the precise relation between \( Y_t \) and \( \theta_t \in \Omega^1(P) \), we first need to compute the one-form \( Y_t \omega \).

**Proposition 2.1** The time-dependent one-form \( Y_t \omega \) on \( \Gamma \) is equal to \( \alpha^* \theta_t \).

**Proof.** Fix any \( t = t_0 \),

\[ Y_{t_0} (\varphi_{t_0}(x)) = \left. \frac{d}{dt} \right|_{t=t_0} \varphi_t(x) \]

\[ = \frac{d}{dt} \bigl|_{t=t_0} (L_t(\alpha(x)) \cdot x) \]

\[ = (r_x)_* X_{t_0} |_{L_{t_0} \alpha(x)}, \]

where the right hand side makes sense since \( X_{t_0} \) is tangent to \( \beta \)-fibers.

Let \( \mathcal{X} \) be any lagrangian bisection through the point \( x \). Then,

\[ Y_{t_0} (\varphi_{t_0}(x)) \omega = [(r \cdot \mathcal{X})_* X_{t_0} |_{L_{t_0} \alpha(x)}] \omega = (r \cdot \mathcal{X})_* [X_{t_0} |_{L_{t_0} \alpha(x)}] \omega \]

\[ = (r \cdot \mathcal{X})_* \alpha^* [\theta_{t_0} |_{Ad_{L_{t_0} \alpha(x)}}] \omega \]

\[ = \alpha^* [\theta_{t_0} |_{Ad_{L_{t_0} \alpha(x)}}]. \]
By letting $y = \varphi_{t_0}(x)$, one gets immediately that

$$Y_{t_0}(y) \cdot \omega = \alpha^\ast[\theta_{t_0}(\alpha(y))].$$

This concludes our proof.

\[\square\]

An immediate consequence is the following

**Corollary 2.2** $\theta_t$ is closed for all $t$.

**Proof.** It follows from $L_{Y_t}\omega = 0$ that $Y_t \cdot \omega$ is closed for all $t$. This implies immediately that $\theta_t$ is closed.

\[\square\]

As a by-product, below we will see how the Lie algebra of $U(\Gamma)$ can be identified with the Lie algebra $Z^1(P)$ of closed one-forms for compact Poisson manifolds \cite{16}. Since we need to use such an identification frequently in the paper, we briefly recall it below. Given a smooth path of lagrangian bisections $\{L_t\}$ starting from 1, its derivative at $t = 0$ is given by the section $u \mapsto \frac{d}{dt}|_{t=0}L_t(u)$ of the vector bundle $T^2_P\Gamma$, which is isomorphic to the normal bundle $T_P\Gamma/TP$. Being contracted with the symplectic form $\omega$, this section can be naturally identified with a one-form on $P$. This is exactly the closed one-form $\theta_0$ as introduced earlier in this section. In this way, we obtain an identification of the tangent space of $U(\Gamma)$ at 1 with the space $Z^1(P)$ of closed one-forms.

We now turn to the discussion on the flux homomorphism. It follows from Proposition 2.1 that the flux homomorphism of $\{\varphi_t\}$ is given by

$$F(\{\varphi_t\}) = \alpha^\ast \int_0^1 [\theta_t]dt.$$ \hspace{1cm} (8)

This suggests the following

**Definition 2.3** We define the flux homomorphism on a symplectic groupoid $\Gamma$ to be the map $F : \widehat{U_0(\Gamma)} \longrightarrow H^1(P, \mathbb{R})$ given by:

$$F(\{L_t\}) = \int_0^1 [\theta_t]dt, \quad \text{for any} \quad \{L_t\} \in \widehat{U_0(\Gamma)}.$$ \hspace{1cm} (9)

**Proposition 2.4** Suppose that $P$ is a compact Poisson manifold. Then, $F$ is well-defined and surjective. Moreover, it is a group homomorphism when $H^1(P, \mathbb{R})$ being considered as an Abelian group. The derivative of $F$ is the map $f : Z^1(P) \longrightarrow H^1(P, \mathbb{R})$ which sends any closed 1-form to its cohomology class.
Proof. It follows from Equation (8) that the following diagram:

\[
\begin{array}{ccc}
\tilde{U}_0(\Gamma) & \xrightarrow{\tilde{\tau}} & \tilde{Diff}_0(\Gamma,\omega) \\
\downarrow & & \downarrow \\
F & \xrightarrow{\alpha^*} & H^1(\Gamma,\mathbb{R})
\end{array}
\]

commutes.

Let \( i : P \to \Gamma \) be the inclusion map. It follows from \( \alpha \circ i = id \) that \( i^* \alpha^* = id \), which implies that \( \alpha^* : H^1(P,\mathbb{R}) \to H^1(\Gamma,\mathbb{R}) \) is injective. Since \( F \) is well-defined on homotopy classes of symplectomorphisms, it follows that \( F \) is also well-defined. The same argument shows that \( F \) is indeed a homomorphism.

To prove the second assertion, we consider \( L_t = \exp t\theta \), \( 0 \leq t \leq 1 \), for any closed one-form \( \theta \). Here by definition, \( \exp t\theta \) is a family of lagrangian bisections obtained by moving the identity space \( P \) along the flows of \( X_{\alpha^*\theta} \). Clearly, \( L_t \) is defined for all \( t \) since \( X_{\alpha^*\theta} \) is a complete vector field. It is simple to see that \( F(L_t) = [\theta] \). The same argument also shows that the derivative of \( F \) is indeed given by the map: \( \theta \in Z^1(P) \to [\theta] \in H^1(P,\mathbb{R}) \). This concludes the proof of the proposition.

We end this section with some simple examples.

**Example 2.5** If \( P \) is a compact symplectic manifold, the pair groupoid \( \Gamma = P \times \overline{P} \) can be taken as its symplectic groupoid. Any lagrangian bisection of \( \Gamma \) can be naturally identified with the graph of a symplectic diffeomorphism. Therefore, \( U(\Gamma) \) can be identified with the group of symplectomorphisms. The flux homomorphism, in this case, reduces to the usual flux homomorphism of symplectomorphisms.

**Example 2.6** If \( P \) is a compact manifold with zero Poisson structure, the symplectic groupoid \( \Gamma \) is the cotangent space \( T^*P \). Then \( U(\Gamma) \) is the space of all closed one-forms, which is connected and simply connected. The group multiplication is simply the addition of one-forms. The flux homomorphism \( F : U_0(\Gamma) \to H^1(P,\mathbb{R}) \) is then the map sending any closed one-form to its cohomology class.

**Example 2.7** Let \( P \) be a regular Poisson manifold which is the product of a zero Poisson manifold \( Q \) with a symplectic manifold \( S \). Its symplectic groupoid is then the product groupoid \( T^*Q \times S \times \overline{S} \). A bisection of \( \Gamma \) consists of a pair \( \varphi \in C^\infty(S,\Omega^1(Q)) \) and \( \psi \in C^\infty(Q,Diff(S)) \). If it is a lagrangian bisection, its restrictions to both the inverse images \( \alpha^{-1}(\{q\} \times S) \) and \( \alpha^{-1}(Q \times \{x\}) \) are isotropic.
submanifolds for any $q \in Q$ and $x \in S$. This implies that the image of $\psi$ should lie in the symplectic
diffeomorphism group $Diff(S,\omega)$ and $\forall x \in S$,
$$d\varphi(x) + \psi_x^*\omega = 0,$$
where $\psi_x : Q \rightarrow S$ is the evaluation map at $x$ defined by $\psi_x(q) = \psi(q) \cdot x$. Therefore, $U(\Gamma)$ is a sub-
group of $C^\infty(S,\Omega^1(M)) \times C^\infty(M,Diff(S,\omega))$. The explicit expression of the flux homomorphism
is rather involved and we shall omit it here.

3 Exact lagrangian bisections

The Lie algebra of all closed one-forms contains the subspace $B^1(P)$ of all exact one-forms as a
Lie subalgebra. It is therefore expected that $U_0(\Gamma)$ contains a subgroup having $B^1(P)$ as its Lie
algebra. The purpose of the present section is to characterize the elements in this subgroup by
using the flux homomorphism introduced in the previous section.

Definition 3.1 A lagrangian bisection $L$ in $U_0(\Gamma)$ is said to be exact iff there is an isotopy $L_t \in U(\Gamma)$ connecting 1 and $L$ (i.e., $L_0 = 1$ and $L_1 = L$) such that $\tau(L_t)$ is a hamiltonian isotopy of $\Gamma$.

When $P$ is symplectic and $\Gamma$ is the pair groupoid $P \times \overline{P}$, a lagrangian bisection of $\Gamma$ corresponds
to a symplectomorphism on $P$. It is simple to see that exact lagrangian bisections will correspond
to hamiltonian symplectomorphisms.

We denote by $U_{ex}(\Gamma)$ the space of all exact lagrangian bisections. Since the product of two
hamiltonian isotopies is still a hamiltonian isotopy, $U_{ex}(\Gamma)$ is a subgroup of $U_0(\Gamma)$.

Proposition 3.2 $U_{ex}(\Gamma)$ is a path-connected normal subgroup of $U_0(\Gamma)$.

Proof. This follows from the fact that the group of hamiltonian symplectomorphisms $Ham(\Gamma,\omega)$
is a normal subgroup of $Diff_0(\Gamma,\omega)$.

The following proposition is an immediate consequence of Proposition 2.1.

Proposition 3.3 A lagrangian bisection $L \in U_0(\Gamma)$ is exact iff there is an isotopy $L_t \in U(\Gamma)$
connecting 1 and $L$ such that the corresponding time-dependent one-form $\theta_t$ defined by Equation
(6) is exact for all $t$.

The same argument also shows the following:

Proposition 3.4 For a compact Poisson manifold $P$, the Lie algebra of $U_{ex}(\Gamma)$ is the space $B^1(P)$
of all exact one-forms on $P$. 

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Proof. Let $L_t$ be any path in $U_{ex}(\Gamma)$ starting from 1, with $\theta_t$ being the corresponding time-dependent one-form on $P$. By definition, $\tau(L_t)$ is a hamiltonian isotopy in $\Gamma$, so $\alpha^*\theta_t$ is exact for all $t$ according to Proposition 2.1. Therefore, $\theta_t$ is all exact, so in particularly is $\theta_0$. Conversely, given any exact one-form $\theta$ on $P$, $\exp t\theta$ will be a path in $U_{ex}(\Gamma)$. This shows that the tangent space of $U_{ex}(\Gamma)$ at 1 is isomorphic to $B_1(P)$.

\[\square\]

Proposition 3.5 Assume that the base Poisson manifold $P$ is compact. Let $L_t$ be a path in $U(\Gamma)$ such that $L_0 = 1$ and $L_1 = L$. Then, $L_t$ lies in $U_{ex}(\Gamma)$ for all $t \in [0, 1]$ if and only if $\mathcal{F}(\{L_t\}) = 0$, for all $0 \leq T \leq 1$.

Proof. By definition, that $L_t$ is in $U_{ex}(\Gamma)$, for all $t$, is equivalent to saying that $\tau(L_t)$ lies in $\text{Ham}(\Gamma, \omega)$, for all $t$. The latter is equivalent to that $\tau(L_t)$ itself is a hamiltonian isotopy, which is equivalent to $F(\{\tau L_t\}) = 0, \forall 0 \leq T \leq 1$, according to Proposition II 3.3 in [1] or Theorem 5.2.4 in [11]. Now $F(\{\tau L_t\}) = \alpha^*\mathcal{F}(\{L_t\})$. The conclusion thus follows from the fact that $\alpha^*$ is injective.

\[\square\]

Remark Either Proposition II 3.3 in [1] or Theorem 5.2.4 in [11] requires that the symplectic manifold be compact. In our case, the symplectic groupoid $\Gamma$ is generally not compact even though $P$ is assumed to be compact. However, we note that the requirement for the compactness of the symplectic manifolds arises from the necessity that certain relevant vector fields be complete. In our case, the completeness of these vector fields is guaranteed by the compactness of $P$ according to a result of Kumpera and Spencer (see the lemma in Section 33 of Appendix of [7]).

By using the flux homomorphism, we now can characterize exact lagrangian bisections.

Theorem 3.6 Assume that the base Poisson manifold $P$ is compact. Then a lagrangian bisection $L$ is exact in $U_{ex}(\Gamma)$ if and only if there is an isotopy $L_t$ in $U(\Gamma)$ connecting 1 and $L$ such that $\mathcal{F}(\{L_t\}) = 0$.

Proof. Our proof here is essentially borrowed from that of Theorem 5.2.4 in [11]. One direction is quite straightforward by definition. Namely, if $L$ is exact, then there exists an isotopy $L_t$ in $U_{ex}(\Gamma)$ connecting 1 and $L$. Thus, it follows from Proposition 3.5 that $\mathcal{F}(\{L_t\}) = 0$.

Conversely, assume that $L_t$ is an isotopy in $U(\Gamma)$ connecting 1 and $L$ such that $\mathcal{F}(\{L_t\}) = 0$. Let $\theta_t$ be the corresponding time-dependent closed one-form on $P$ as defined by Equation (6). Then, $\int_0^1 \theta_t dt$ is exact.

First, we assume that $\int_0^1 \theta_t dt = 0$.

Let $\gamma_t$ be a family of closed one-forms on $P$ defined by $\gamma_t = -\int_0^t \theta_s ds$, $0 \leq t \leq 1$ and $K_t = L_t \exp \gamma_t$, $0 \leq t \leq 1$. Here, the map $\exp$ is well defined since $P$ is compact. Then $K_t$ is an isotopy of lagrangian bisections connecting 1 and $L$. Moreover, for any $T \in [0, 1]$,
\[ F(\{K_t\}_{0 \leq t \leq T}) = F(\{L_t\}_{0 \leq t \leq T}) + F(\{\exp \gamma t\}_{0 \leq t \leq T}) = [\int_0^T \theta_t dt] + F(\{\exp \gamma T\}_{0 \leq t \leq 1}) = [\int_0^T \theta_t dt] + [\gamma T] = 0, \]

where the second equality follows from the homotopy equivalence between the two isotopies \( \{\exp \gamma t\}_{0 \leq t \leq T} \) and \( \{\exp t \gamma T\}_{0 \leq t \leq 1} \). This shows that \( K_t \) is a path in \( U_{ex}(\Gamma) \). In particular, \( K_1 \) is exact, which implies that \( L \) must be exact.

Next, let us assume that \( \int_0^1 \theta_t dt = dH \) for some function \( H \in C^\infty(P) \). Consider \( L'_t = \exp(-tdH) \ast L_t \), where the \( \ast \)-product means the following:

\[ \exp(-tdH) \ast L_t = \begin{cases} L_{2t}, & 0 \leq t \leq \frac{1}{2} \\ \exp(-(2t-1)dH) \cdot L_1, & \frac{1}{2} \leq t \leq 1. \end{cases} \]

Let \( \theta'_t \) denote its corresponding time-dependent one-form on \( P \) as defined by Equation (6). Then, it is clear that

\[ \theta'_t = \begin{cases} 2\theta_{2t}, & 0 \leq t \leq \frac{1}{2} \\ -2dH, & \frac{1}{2} \leq t \leq 1. \end{cases} \]

Therefore, \( \int_0^1 \theta'_t dt = 0 \). Hence, \( \exp(-dH) \cdot L \in U_{ex}(\Gamma) \), and then \( L \in U_{ex}(\Gamma) \).

As an immediate consequence, we have

**Corollary 3.7** Suppose that \( P \) is compact. The kernel of the flux homomorphism \( F : \tilde{U}_0(\Gamma) \rightarrow H^1(P, \mathbb{R}) \) is the universal covering of \( U_{ex}(\Gamma) \). I.e., \( \ker F = \tilde{U}_{ex}(\Gamma) \).

**Proof.** It is clear that \( \tilde{U}_{ex}(\Gamma) \subseteq \ker F \). To prove the other half, assume that \( L_t \) is an isotopy of lagrangian bisections satisfying \( F(\{L_t\}) = 0 \). Let \( \theta_t \in \Omega^1(P) \) be its corresponding time-dependent one-form on \( P \). Then, we have \( \int_0^1 \theta_t dt = dH \) for some function \( H \) on \( P \). According to the proof of the last proposition, \( \exp(-tdH) \ast L_t \) is an isotopy in \( U_{ex}(\Gamma) \). On the other hand, it is easy to see that \( L_t \) is homotopy equivalent to \( \exp(tdH) \ast [\exp(-tdH) \ast L_t] \). The latter is clearly a path entirely in \( U_{ex}(\Gamma) \), so it follows that \( \{L_t\} \in \tilde{U}_{ex}(\Gamma) \).

**Remark** For symplectic manifolds, the flux homomorphism is useful because it can be used to characterize hamiltonian symplectomorphisms. Hamiltonian symplectomorphisms, rather than
symplectomorphisms are what Arnold’s conjecture concerns about. It would be interesting to ask if there is something similar to Arnold’s conjecture for Poisson manifolds, which should concern about the intersection number of an exact lagrangian bisection with the identity space. If the Poisson manifold is a compact zero Poisson manifold, this would fall into the content of Morse theory. However, for a general Poisson manifold, it is not even clear whether an exact lagrangian bisection always intersects with the identity space, although there are some evidences that the answer is likely positive.

4 A Central extension of $U_{ex}(\Gamma)$

In this section, we will assume that the symplectic groupoid $(\Gamma, \omega)$ is prequantizable. That is, $\omega$ is of integer class. In this case, the group $U_{ex}(\Gamma)$ can be easily characterized with the help of the prequantum bundle over $\Gamma$. More importantly, such a prequantum bundle provides us with a natural one-dimensional central extension of $U_{ex}(\Gamma)$, whose corresponding Lie algebra is the Poisson algebra $C^\infty(P)$. Thus, we will obtain a geometric description for a Lie group integrating the Poisson algebra $C^\infty(P)$.

Suppose that $\Gamma$ is prequantizable so that we can construct a unique circle $(\mathbb{R}/\mathbb{Z})$-bundle $p : E \rightarrow \Gamma$ with a connection form $\theta$ satisfying the condition that the identity space $P$ has no holonomy. It is proved in [20] that $E$ has a groupoid structure once a horizontal lifting of $P$ is chosen in $E$. In the sequel, we will fix such a horizontal lift. Thus, $E$ carries a groupoid structure. As in [20], we will denote the source and target maps of $E \rightarrow P$ by $\alpha$ and $\beta$, while those of $\Gamma$ are denoted by $\alpha_0$ and $\beta_0$. Then, $\alpha = \alpha_0 \cdot p$ and $\beta = \beta_0 \cdot p$.

For a lagrangian bisection $L \subseteq U(\Gamma)$, the restriction of $E$ on $L$ is always flat. For any loop $\gamma(t)$ in $P$, $L \cdot \gamma(t)$ is a loop in $L$. We define $\Phi_{L}(\gamma)$ as its corresponding holonomy. Thus, we obtain a map $\Phi : L \rightarrow \Phi_{L}$ from $U(\Gamma)$ to $Hom(\pi_1(P), T^1)$. Here $\pi_1(P)$ denotes the fundamental group of $P$. It is clear that $Hom(\pi_1(P), T^1)$ is a left $U(\Gamma)$-module with the action: $(L \cdot \varphi)(\gamma) = \varphi(Ad_L \gamma)$, where $L \subseteq U(\Gamma), \varphi \in Hom(\pi_1(P), T^1)$ and $[\gamma] \in \pi_1(P)$.

**Proposition 4.1** The map $\Phi : U(\Gamma) \rightarrow Hom(\pi_1(P), T^1)$ satisfies the following 1-cocycle-like property:

$$\Phi_{KL} = \Phi_L + L \cdot \Phi_K.$$ 

**Proof.** Suppose that $\gamma(t)$ is any loop in $P$. Let $\gamma'(t) = Ad_{L} \gamma(t), \gamma_1(t) = L \cdot \gamma(t)$ and $\gamma_2(t) = K \cdot \gamma'(t)$. Then it is clear that $\beta \gamma_2(t) = \alpha \gamma_1(t) = \gamma'(t)$ and $\gamma_2(t) \gamma_1(t) = KL \cdot \gamma(t)$. Therefore, $(\gamma_2(t), \gamma_1(t), \gamma_2(t) \gamma_1(t))$ is a loop lying in the graph of groupoid multiplication $\Lambda \subseteq \Gamma \times \Gamma \times \Gamma$. According to Theorem 3.1 in [20], $\Lambda$ has no holonomy in the corresponding prequantum bundle $(E \times E \times \bar{E})/T^2 \rightarrow \Gamma \times \Gamma \times \Gamma$. It thus follows that $Hol(\gamma_2 \gamma_1) = Hol(\gamma_2) + Hol(\gamma_1)$. I.e., $\Phi_{KL}(\gamma) = \Phi_K(Ad_L \gamma) + \Phi_L(\gamma)$.

$\square$

An immediate consequence is the following:
Corollary 4.2 $\Phi : U_0(\Gamma) \rightarrow Hom(\pi_1(P), T^1)$ is a group homomorphism.

We will see below that $\Phi$ is closely related to the flux homomorphism. To describe their precise relation, let us introduce a group homomorphism $\rho : H^1(P, \mathbb{R}) \rightarrow Hom(\pi_1(P), T^1)$ by

$$\rho([\theta])([\gamma]) = \int_{\gamma} \theta \mod \mathbb{Z}, \quad \forall [\theta] \in H^1(P, \mathbb{R}), \text{ and } [\gamma] \in \pi_1(P).$$

Proposition 4.3 Suppose that $P$ is compact. For any isotopy $L_t \in U_0(\Gamma)$ connecting $1$ and $L$,

$$(\rho \circ F)(\{L_t\}) = \Phi(L).$$

Proof. Suppose that $\gamma(s), 0 \leq s \leq 1$ is any 1-cycle in $P$. Let $\lambda(t, s) = L_t^{-1}\gamma(s)$, and $\theta_t$ be the time-dependent closed 1-form on $P$ corresponding to $L_t$. As before, $Y_t$ denotes the time-dependent vector field on $\Gamma$ corresponding to the one family of diffeomorphisms $\varphi_t = \tau(L_t)$ defined by Equation (7).

Taking the derivative of the identity $\varphi_t(L_t^{-1}x) = x$ at $t = t_0$, one gets:

$$Y_{t_0}(x) + (\varphi_{t_0})_* \frac{d}{dt}ig|_{t=t_0} L_t^{-1}x = 0.$$  

Since $x$ is an arbitrary point in $\Gamma$, by taking $x = \gamma(s)L_{t_0}$, we have

$$Y_{t_0}(\gamma(s)L_{t_0}) + (\varphi_{t_0})_* \frac{d}{dt}ig|_{t=t_0} L_t^{-1}\gamma(s)L_{t_0} = 0.$$  

I.e.,

$$Y_{t_0}(\gamma(s)L_{t_0}) = -L_{t_0} \frac{\partial \lambda(t, s)}{\partial t}ig|_{t=t_0} L_{t_0}.$$  

Therefore,

$$\dot{\gamma}(s) \big\langle \big\rangle \theta_t = \dot{\gamma}(s)L_t \big\langle \big\rangle \alpha^* \theta_t$$

$$= \omega(Y_t(\gamma(s)L_t), \dot{\gamma}(s)L_t)$$

$$= \omega(-L_t \frac{\partial \lambda}{\partial t} L_t, L_t \frac{\partial \lambda}{\partial s} L_t)$$

$$= -\omega(\frac{\partial \lambda}{\partial t}, \frac{\partial \lambda}{\partial s}),$$

where the last step follows from the fact that $L_t$ is lagrangian. Hence,

$$\langle \rho(F\{L_t\}), [\gamma] \rangle = \int_{\gamma} \int_0^1 \theta_t dt$$

$$= \int_0^1 \int_0^1 \dot{\gamma}(s) \big\langle \big\rangle \theta_t \ dt \, ds$$

$$= -\int_D \omega,$$

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where $D$ is the cylinder: $\lambda(t, s) = L_t^{-1} \gamma(s)$, $0 \leq t \leq 1$ and $s \in S^1$. Now the quantity: $\int_D \omega$ modulo $\mathbb{Z}$ is exactly the difference between the holonomies around the loops $\lambda(1, s)$ and $\lambda(0, s)$. The holonomy of the latter is zero because the loop lies in $P$, while the holonomy of $\lambda(1, s)$ is the minus of that of $L \cdot \gamma(s)$, which is $\Phi_L(\gamma(s))$. This completes the proof of the proposition.

\[\square\]

We denote by $\Omega$ the image of the fundamental group $\pi_1(U_0(\Gamma), 1)$ under the flux homomorphism $F$, i.e., $\Omega = F(\pi_1(U_0(\Gamma), 1))$. It follows from Proposition 4.3 that $\rho(\Omega) = 0$. Therefore, $\rho$ descends to a homomorphism: $H^1(P, \mathbb{R})/\Omega \to \text{Hom}(\pi_1(P), T^1)$, which will still be denoted by $\rho$. We note that the flux homomorphism induces a homomorphism: $U_0(\Gamma) \to H^1(P, \mathbb{R})/\Omega$, which will be denoted as $\tilde{F}$ below.

The following theorem is an alternate description of Proposition 4.3.

**Theorem 4.4** Assume that $P$ is a compact Poisson manifold and its symplectic groupoid $\Gamma$ is prequantizable. Then, the following diagram:

\[
\begin{array}{ccc}
U_0(\Gamma) & \xrightarrow{\tilde{F}} & H^1(P, \mathbb{R})/\Omega \\
\Phi & \downarrow & \downarrow \rho \\
\text{Hom}(\pi_1(P), T^1) & \xrightarrow{id} & \text{Hom}(\pi_1(P), T^1)
\end{array}
\] (12)

commutes.

Using this result, we can describe the group $U_{ex}(\Gamma)$ in terms of $\Phi$.

**Corollary 4.5** Under the same hypotheses of Theorem 4.4, the group $U_{ex}(\Gamma)$ coincides with the identity component of the kernel of $\Phi$. I.e.,

\[U_{ex}(\Gamma) = (\text{Ker} \Phi)_0.\]

**Proof.** For any $L$ in $U_{ex}(\Gamma)$, by definition, there exists an isotopy $L_t$ connecting $1$ and $L$ such that $\tau(L_t)$ is a hamiltonian isotopy. Hence, $F(\{L_s\}_{0 \leq s \leq t}) = 0$ for any $t \in [0, 1]$. In other words, $\tilde{F}(L_t) = 0$ for all $t \in [0, 1]$. It follows from Theorem 4.3 that $\Phi(L_t) = (\rho\tilde{F})(L_t) = 0$. Hence, $L_t$ is a path in $\text{Ker}\Phi$. This implies that $L \in (\text{Ker}\Phi)_0$.

Conversely, suppose that $L_t$, with $L_0 = P$ and $L_1 = L$, is an isotopy in $\text{Ker}\Phi$. Thus, $(\rho\tilde{F})(L_t) = \Phi(L_t) = 0$. This implies that for any loop $\gamma(s)$ in $P$, the double integral $\int_0^1 \theta_s ds$ is an integer for all $t$. However, since it is a continuous function with respect to $t$, it must be a constant and in our case it should be identically zero. Therefore, $F(\{L_s\}_{0 \leq s \leq t}) = 0$, $\forall t$. Hence, $L$ is in $U_{ex}(\Gamma)$.

\[\square\]
An immediate consequence is the following:

**Corollary 4.6** Under the same hypotheses, the Lie algebra of $\text{Ker} \Phi$ is $Z^1(P)$.

The prequantum bundle $E$ together with the connection form $\theta$ becomes a contact manifold. A bisection of the groupoid $E \to P$ is called a Legendre bisection if it is also a Legendre submanifold with respect to this contact structure (i.e. horizontal with respect to the connection $\theta$). It is clear that the space of all Legendre bisections of $E \to P$ is a Lie group. We denote by $U_0(E)$ its connected component of the unit. It is a subgroup of the group of all bisections of $E \to P$.

It is well known that left (or right) translations by Lagrangian bisections are symplectic diffeomorphisms of the symplectic groupoid. Similarly, left (or right) translations by Legendre bisections are contact diffeomorphisms.

**Proposition 4.7** A bisection $K$ of $E$ is Legendre iff the left translation $l_K$ (or the right translation $r_K$): $E \to E$ is a contact diffeomorphism.

**Proof.** Let $\psi$ denote the diffeomorphism $l_K : E \to E$. For any tangent vector $\delta_x$ in $E$, we take a path $x(t)$ starting from $x$ with $\dot{x}(0) = \delta_x$. Let $y(t) = K(\alpha(x(t)))$. Then $y(t)$ is a path in $K$ and $(y(t), x(t), \psi(x(t)))$ lies in the groupoid graph of $E$. Since the one-form $(\theta, \theta, -\theta)$ vanishes on the groupoid graph of $E$, and $K$ is a Legendre submanifold, it follows that $\delta_x \cdot \theta - \psi_\ast \delta_x \cdot \theta = 0$. Since $\delta_x$ is arbitrary, it follows that $\psi_\ast \theta = \theta$. That is, $\psi$ is a contact diffeomorphism. The converse is proved similarly. 

\[ \square \]

**Theorem 4.8** Suppose that the Poisson manifold $P$ is compact. Then, the Lie algebra of $U_0(E)$ is the Poisson algebra $C^\infty(P)$.

**Proof.** The Lie algebroid of $E$ is the central extension of the cotangent Lie algebroid $T^*P$ by the Poisson tensor $\pi$, considered as a Lie algebroid 2-cocycle. More precisely, the Lie algebroid of $E$ is isomorphic to $\tilde{A} = T^*P \oplus (P \times \mathbb{R})$, with the bracket being given by:

\[ [(\zeta, f), (\eta, g)] = ([\zeta, \eta], (\pi^\# \zeta)g - (\pi^\# \eta)f + \pi(\zeta, \eta)), \]

for any $\zeta, \eta \in \Omega^1(P)$, and $f, g \in C^\infty(P)$. Here a pair $(\zeta, f)$ for $\zeta \in \Omega^1(P)$ and $f \in C^\infty(P)$, is identified with a right invariant vector field in $E$ via the map $\lambda$:

\[ (\zeta, f) \to \tilde{X}_{\alpha_0^* \zeta} + (p^* \alpha^*_0 f) \xi, \]

where $\xi$ is the Euler vector field on $E$ generating the circle action. Let $G$ denote the group of all bisections of $E$. Then, the Lie algebra of $G$ can be identified with the space of sections of $\tilde{A}$ according to a general rule. More explicitly, the tangent space $T_1G$ at the unit is identified with $\Gamma(\tilde{A})$ as follows. Any isotomy $K_t$ in $G$ starting from 1 induces a family of diffeomorphisms on $E$ by the left translations $l_{K_t}$. The derivative of such a family of diffeomorphisms at $t = 0$ defines a left
invariant vector field $X$ on $E$, which can be canonically identified with a section $(\zeta, f)$ of $\tilde{A}$ by the map $\lambda$ above.

Now if $K_t$ is an isotopy in $U_0(E)$, the corresponding vector field $X$ will be contact according to Proposition 4.7. If $\lambda(\zeta, f) = X$, then $X = \tilde{X}_{\alpha_0^*\zeta} + (p^*\alpha_0^* f)\xi$. It is easy to see that $X$ is a contact vector field, if and only if $\zeta = -df$. Therefore, the Lie algebra of $U_0(E)$ can be identified with the Lie subalgebra of $\Gamma(\tilde{A})$ consisting of all elements of the form $(-df, f)$, which is clearly isomorphic to the Poisson algebra $C^\infty(P)$ via the identification $(-df, f) \mapsto -f$. This concludes our proof.

\[\square\]

The following result is then immediate.

**Corollary 4.9**  
(i) The projection $p : E \to \Gamma$ induces a group homomorphism $U_0(E) \to U_0(\Gamma)$, denoted by the same symbol $p$, which lifts the Lie algebra homomorphism $d : C^\infty(P) \to Z^1(P)$.

(ii) $\text{Imp} = U_{ex}(\Gamma)$.

**Proof.** For any bisection $K$ of the groupoid $E$, $pK$ is clearly a bisection of $\Gamma$. Since $p : E \to \Gamma$ is a groupoid morphism [20], it follows that $p \circ K = l_{pK} \circ p$. If $K$ is a legendre bisection, we have $(l_K)^*\theta = \theta$ according to Proposition 4.7. By taking the exterior derivative, one gets that $(l_K)^* p^* \omega = p^* \omega$. It thus follows that $p^* l_{pK}^* \omega = p^* \omega$, which implies that $l_{pK}^* \omega = \omega$ since $p$ is a submersion. Therefore, $pK$ is a lagrangian bisection. This proves that we indeed get a map $p : U_0(E) \to U_0(\Gamma)$. This is a group homomorphism since $p : E \to \Gamma$ is a groupoid morphism. The last part of the proof of Theorem 4.8 has already indicated that the derivative of $p : U_0(E) \to U_0(\Gamma)$ is indeed the exterior differential.

To prove the second assertion, let $K$ be any bisection in $U_0(E)$. Then there is an isotopy $K_t$ in $U_0(E)$ such that $K_0 = 1$ and $K_1 = K$. Let $L_t = pK_t \in U_0(\Gamma)$. Then, $\Phi(L_t) = 0$ since $K_t$ is horizontal. That is, $L_t$ is in $\text{Ker} \Phi$. Therefore $L_1 \in (\text{Ker} \Phi)_0$. However, $(\text{Ker} \Phi)_0 = U_{ex}(\Gamma)$ according to Corollary 4.7. This shows that $\text{Imp} \subseteq U_{ex}(\Gamma)$.

Conversely, for any given $L \in U_{ex}(\Gamma) = (\text{Ker} \Phi)_0$, we assume that $L_t$ is an isotopy in $\text{Ker} \Phi$ connecting 1 and $L$. Then we can always find an isotopy of bisections $K_t$ in $E$ which is a parallel lifting of $L_t$ since $\Phi(L_t) = 0$. Then $K_t$ is an isotopy of legendre bisections. That is, $K_t \in U_0(E)$, and in particular $K_1 \in U_0(E)$. Hence, we obtain the other inclusion: $U_{ex}(\Gamma) \subseteq \text{Imp}$. This concludes the proof.

\[\square\]

Now we are ready to state the main theorem of the section.

**Theorem 4.10** When $P$ is a compact Poisson manifold and its symplectic groupoid $\Gamma$ is prequantizable, we have the following exact sequence of group homomorphisms:

\[0 \to T^1 \to U_0(E) \xrightarrow{p} U_0(\Gamma) \xrightarrow{\tilde{\pi}} H^1(P, \mathbb{R})/\Omega \to 0.\]
On the Lie algebra level, this corresponds to the exact sequence:

\[ 0 \rightarrow \mathbb{R} \rightarrow C^\infty(P) \xrightarrow{d} Z^1(P) \xrightarrow{\mathcal{F}} H^1(P, \mathbb{R}) \rightarrow 0. \]

**Proof.** It remains to prove that \( \ker \tilde{\mathcal{F}} = U_{\text{ex}}(\Gamma). \)

We already know that \( U_{\text{ex}}(\Gamma) \subseteq \ker \tilde{\mathcal{F}}. \) For the other direction, let us assume that \( L \in \ker \tilde{\mathcal{F}}. \) Let \( L_t \) be any isotopy connecting \( 1 \) and \( L. \) Then, by the definition of \( \tilde{\mathcal{F}}, \) there exists an isotopy \( S_t \) in \( U_0(\Gamma) \) with \( S_1 = S_0 = 1 \) such that \( \mathcal{F}(\{L_t\}) = \mathcal{F}(\{S_t\}). \) From this, it follows that \( \mathcal{F}(\{L_tS_t^{-1}\}) = 0. \) Therefore, \( L_1S_t^{-1} \) is an exact bisection, so is \( L = L_1. \)

\[ \square \]

**Remark.** It is worth pointing out that in general a prequantum bundle can be constructed whenever the periodic group of \( \Gamma \) is discrete. All the discussion in this section could be carried out similarly in this generalized context.

## 5 The flux homomorphism on \( \alpha \)-simply connected symplectic groupoids

In this section, again we will assume that \((P, \pi)\) is a compact Poisson manifold. Suppose that \( \nu \) is a finite measure on \( P, \) which is invariant under all hamiltonian flows. For instance, if \( P \) is a compact symplectic manifold with symplectic form \( \omega, \) one can take the volume form \( \omega^n \) as such a measure. Then \( \nu \) is invariant under all vector fields of the form \( X_\theta, \forall \theta \in Z^1(P). \) Therefore, it is also invariant under the adjoint action \( Ad_L : P \rightarrow P, \) for any \( L \in U(\Gamma). \) Define a pairing \( H^1(P, \mathbb{R}) \times H^1(P, \mathbb{R}) \rightarrow \mathbb{R} \) by

\[ <[\theta_1], [\theta_2]> = \int_\nu \pi(\theta_1, \theta_2). \]  

(13)

**Proposition 5.1** This pairing is well defined, skew-symmetric and bilinear.

**Proof.** It is clear that the rhs of Equation (13) is bilinear and skew-symmetric with respect to the arguments \( \theta_1 \) and \( \theta_2. \) To prove that its value only depends on their cohomology classes, it suffices to show that it vanishes when either \( \theta_1 \) or \( \theta_2 \) is exact. Assume that \( \theta_1 = df. \)

\[ <[\theta_1], [\theta_2]> = \int_\nu \pi(df, \theta_2) = - \int_\nu X_{\theta_2}(f). \]

Let \( \varphi_t \) be the flow on \( P \) generated by \( X_{\theta_2}. \) Since \( \nu \) is invariant under \( \varphi_t, \) we have \( \int_\nu \varphi_t^* f = \int_\nu f. \) By taking the derivative, one gets that \( \int_\nu X_{\theta_2}(f) = 0. \) This concludes the proof.

\[ \square \]
The pairing \(<\cdot, \cdot>\) induces a linear map \(\mu : H^1(P, \mathbb{R}) \rightarrow H^1(P, \mathbb{R})^*\). Consider its composition with the flux homomorphism: \(\varphi = \mu \circ F : \bar{U}_{\alpha}(\Gamma) \rightarrow H^1(P, \mathbb{R})^*\). In general, the flux homomorphism \(F\) is very difficult to compute. However, the map \(\varphi\) possesses a nice description when \(\Gamma\) is \(\alpha\)-simply connected (i.e. the \(\alpha\)-fibers of \(\Gamma\) are all simply-connected), as we will see below. This description will allow us to extract certain information about the flux homomorphism \(F\) even when an explicit expression is not available. This is particularly useful when \(\mu\) is an isomorphism. In this case, one can actually compute the flux homomorphism by first computing the map \(\varphi\).

To proceed, first we introduce a map \(\epsilon : U(\Gamma) \rightarrow H^1(\Gamma, \mathbb{R})^*\) as follows:

\[
<\epsilon(L), [J]> = \int_\nu J(L(u)), \quad J \in Z^1(\Gamma, \mathbb{R}),
\]

where \(H^1(\Gamma, \mathbb{R})\) is the first groupoid cohomology group with real coefficients (see [20]).

**Proposition 5.2** (i). The map \(\epsilon\) is a well-defined map.

(ii). \(\epsilon\) is a group homomorphism.

**Proof.** Suppose that \(J\) is a groupoid coboundary, i.e., \(J(x) = H(\alpha(x)) - H(\beta(x))\) for some \(H \in C^\infty(P)\). Then, \(J(L(u)) = H(Ad_Lu) - H(u)\) and \(\int_\nu J(L(u)) = \int_{Ad_L\nu} H = \int_\nu H = 0\), since the measure \(\nu\) is invariant. It is also simple to see that \(<\epsilon(L), J>\) is linear with respect to \(J\). Thus, \(\epsilon\) is indeed a well-defined map from \(U(\Gamma)\) to \(H^1(\Gamma, \mathbb{R})^*\).

To prove the second assertion, let \(L\) and \(K\) be any two lagrangian bisections. Then, \(\forall J \in Z^1(\Gamma, \mathbb{R})\),

\[
<\epsilon(LK), [J]> = \int_\nu J(L(Ad_Ku)K(u))
\]

\[
= \int_\nu J(L(Ad_Ku)) + \int_\nu J(K(u))
\]

\[
= \int_{Ad_K\nu} J(L(u)) + \int_{\nu} J(K(u))
\]

\[
= \int_\nu J(L(u)) + \int_{\nu} J(K(u))
\]

\[
= <\epsilon(L), [J]> + <\epsilon(K), [J]>. \]

This concludes the proof.

\[\square\]

According to Theorem 1.3 in [20], \(H^1(\Gamma, \mathbb{R})\) is isomorphic to the first Poisson cohomology \(H^1_\pi(P)\) when \(\Gamma\) is \(\alpha\)-simply connected. On the cochain level, this isomorphism \(\Psi : H^1_\pi(P) \rightarrow H^1(\Gamma, \mathbb{R})\) is established by:

\[
\Psi([X])(r) = \int_{\alpha(r)}^r \theta_X, \quad \forall r \in \Gamma,
\]

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where $X$ is any Poisson vector field on $P$, $\theta_X$ is its corresponding left-invariant one-form on $\alpha$-fibers defined by $\theta_X(X_{\beta \cdot f}) = X(f)$, $\forall f \in C^\infty(P)$, and the integration is over any path connecting $\alpha(r)$ and $r$ in the $\alpha$-fiber. On the other hand, the Poisson tensor $\pi$ induces a natural morphism $\pi^\# : H^1(P, \mathbb{R}) \rightarrow H^1_\pi(P)$. By taking their duals and composing with $\epsilon$, we obtain a map $\lambda : U(\Gamma) \rightarrow H^1_\pi(P, \mathbb{R})^*$ such that $\lambda = (\pi^\#)^* \circ \Psi^* \circ \epsilon$.

The following proposition gives an explicit expression for this map $\lambda$.

**Proposition 5.3** For any $L \in U(\Gamma)$ and $[\theta] \in H^1_\pi(P, \mathbb{R})$,

$$< \lambda(L), [\theta] > = \int_\nu (\int_u^{\alpha^{-1}(u) \cap L} \beta^* \theta),$$

where the interior integration is over any path in the $\alpha$-fiber connecting the point $u$ and its intersection with $L$: the point $\alpha^{-1}(u) \cap L$.

**Proof.** Given any closed one-form $\theta$, let $X = \pi^\# \theta$ and $J = (\Psi \circ \pi^\#) \theta$.

Since

$$(\beta^* \theta)(X_{\beta \cdot f}) = \theta(T \beta X_{\beta \cdot f}) = -\theta(X_f) = (\pi^\# \theta)f = Xf,$$

the restriction of $\beta^* \theta$ to $\alpha$-fibers will be the left-invariant one-form corresponding to the Poisson vector field $X$. Therefore, $\forall \gamma \in \Gamma$,

$$J(\gamma) = ((\Psi \circ \pi^\#) \theta)(\gamma) = \Psi([X])(\gamma) = \int_\alpha(\gamma) \beta^* \theta.$$

Hence,

$$< \lambda(L), [\theta] > = < \epsilon(L), (\Psi \circ \pi^\#)[\theta] >$$

$$= < \epsilon(L), [J] >$$

$$= \int_\nu J(L(u))$$

$$= \int_\nu (\int_{Ad_{L(u)}} \beta^* \theta)$$

$$= \int_\nu (\int_u^{\alpha^{-1}(u) \cap L} \beta^* \theta),$$

where the last step follows from the invariance of the measure $\nu$.

\[\square\]

We can now describe the main theorem of this section.

**Theorem 5.4** For any isotopy $L_t$ in $U_0(\Gamma)$ connecting $1$ and $L$, we have

$$\varphi(\{L_t\}) = \lambda(L).$$
Proof. Given any $[\xi] \in H^1(P, \mathbb{R})$, let $f(t) = <\lambda(L_t), [\xi]> = \int_v J(L_t(u))$, where $J = \Psi(\pi^\# \xi)$ is the corresponding groupoid 1-cocycle. By taking the derivative, one has

$$
\dot{f}(t) = \int_v X_t(J)(L_t(u))
$$

$$
= -\int_v <X_J, X_t \omega > (L_t(u))
$$

$$
= -\int_v <X_J, \alpha^* \theta_t > (L_t(u))
$$

$$
= -\int_v T\alpha X_J, \theta_t > (Ad_{L_t} u)
$$

$$
= -\int_v <\pi^\# \xi, \theta_t > (Ad_{L_t} u)
$$

$$
= \int_v \pi(\theta_t, \xi)(u),
$$

where the second from the last equality follows from the proof of Proposition 2.2 in [20].

Therefore,

$$
f(1) = f(1) - f(0) = \int_0^1 \dot{f}(t) = \int_0^1 \int_v \pi(\theta_t, \xi)(u)dt.
$$

On the other hand, it follows from the definition of $\varphi$ that

$$
<\varphi(\{L_t\}), [\xi]> = <\mu(\mathcal{F}(\{L_t\})), [\xi]>
$$

$$
= \int_v \pi(\mathcal{F}(\{L_t\}), \xi)
$$

$$
= \int_0^1 \int_v \pi(\theta_t, \xi)(u)dt.
$$

This concludes the proof of the theorem.

\[\square\]

An immediate consequence of Theorem 5.4 is the following:

**Corollary 5.5** When $\Gamma$ is $\alpha$-simply connected, the map $\varphi$ descends to a group homomorphism $\tilde{\varphi}: U_0(\Gamma) \to H^1(P, \mathbb{R})^*$. That is, $\varphi$ vanishes on $\pi_1(U_0(\Gamma), 1)$.

Another consequence is:

**Corollary 5.6** Suppose that $\Gamma$ is $\alpha$-simply connected. For any $\{L_t\} \in \tilde{U}_0(\Gamma)$ and $[\theta] \in H^1(P, \mathbb{R})$,

$$
<\mathcal{F}(\{L_t\}), \mu[\theta]> = -\int_v (\int_{\alpha^{-1}(u) \cap L_1} \beta^* \theta).
$$
When \( \mu \) is an isomorphism, the above corollary will enable us to carry out an explicit computation of the flux homomorphism.

Let us conclude this section by the following

**Corollary 5.7**  
(i). \( U_{ex}(\Gamma) \subseteq \text{Ker}\lambda \).

(ii). In particular, if \( \mu \) is an isomorphism, then  
\[
U_{ex}(\Gamma) = \text{Ker}(\lambda|_{U_0(\Gamma)}).
\]

### 6 The case of symplectic manifolds

As an example, we will consider compact symplectic manifolds in this section. In particular, we will investigate the relation between the flux homomorphism of their symplectic groupoids (being the fundamental groupoids in this case) and the usual flux homomorphism of symplectomorphisms.

Let \( M \) be a compact symplectic manifold. Then, \( M \) is equipped with a natural invariant volume form \( \omega^n \). We can take \( \nu \) to be its corresponding measure. The symplectic pairing, as often called in the literature [1] [3] [11], is the pairing in \( H^1(M, \mathbb{R}) \) given by:

\[
\sigma([\theta_1], [\theta_2]) = \int \theta_1 \wedge \theta_2 \wedge \omega^{n-1}, \quad \forall \theta_1, \theta_2 \in H^1(M, \mathbb{R}).
\]  

(14)

It is well known that the symplectic pairing is nondegenerate in the case of the symplectic structure subordinate to a compact Kahler manifold. The following lemma can be proved easily by using a local Darboux chart.

**Lemma 6.1** Up to a constant, \( \sigma \) coincides with the pairing \( \langle \cdot, \cdot \rangle \) defined by Equation (13).

For \( M \), its \( \alpha \)-simply connected symplectic groupoid \( \Gamma \) is the fundamental groupoid \( \Pi_1(M) \). For any \( [\theta] \in H^1(M, \mathbb{R}) \), let \( J \in C^\infty(\Gamma) \) be given by,

\[
J(\gamma) = \int_\gamma \theta, \quad \forall \gamma \in \Gamma = \Pi_1(M).
\]

Then \( (\Psi \circ \pi^\#)[\theta] = [J] \) according to [12] (in this case, \( (\Psi \circ \pi^\#) \) is an isomorphism). Therefore, \( \lambda : U_0(\Gamma) \to H^1(M, \mathbb{R})^\ast \) is given by

\[
<\lambda(L), [\theta]> = \int_M (\int_{L(u)} \theta) \omega^n = \int_{[0, 1] \times M} i^*(\theta_x \wedge \omega_y^n),
\]

where \( i \) is the embedding \( [0, 1] \times M \to M \times M \) defined as \( (t, u) \to (L(u)(t), u) \), and \( \theta_x \) is the one-form \( \theta \) considered as a one-form on the first component while \( \omega_y \) is \( \omega \) considered as a two-form on the second component.

Given any \( L \in U_0(\Gamma) \). For any fixed \( u \in M \), \( L(u) \) is an element in \( \Pi_1(M) \) with the end point \( u \). By an isotropic bisection, we mean a bisection of \( \Gamma \) whose all points lie in the isotropic groupoid.
of $Γ$. Given any isotropic bisection $L$ in $U_0(Γ)$, $L(u)$ is a homotopy class of a closed path with the end point $u$. It is simple to see that for any $u$ and $v$ in $M$, $L(u)$ and $L(v)$ are homotopy equivalent loops since $M$ is connected. Therefore, $\int_{L(u)} \theta = \int_{L(v)} \theta$, or $J(L(u)) = J(L(v))$. It thus follows that

$$<\lambda(L), [\theta]> = \int_M J(L(u))\omega^n = J(L(u)) \cdot Vol(M) = (\int_{L(u)} \theta) \cdot Vol(M).$$

An immediate consequence of this formula is the following:

**Proposition 6.2** An isotropic lagrangian bisection $L$ is in $\ker \lambda$ iff $L(u)$ is null-homologous for every $u \in M$. In particular, if an isotropic bisection $L \in U_0(Γ)$ is exact, then $L(u)$ is null-homologous.

Given any symplectomorphism $f$, by the graph of $f$, we mean the submanifold $\{(f(x), x) | x \in M\}$ of $M \times M$. Then the graph of a symplectomorphism is clearly a lagrangian submanifold. Let $p$ denote the projection $p = \alpha \times \beta : \Pi_1(M) \rightarrow M \times M$. The following result reveals the relation between the usual flux homomorphism of symplectomorphisms and the flux homomorphism of symplectic groupoids.

**Proposition 6.3** Let $h_t$ be a symplectic isotopy in $Diff_0(M, ω)$ such that $h_0 = id$. Suppose that $h_t$ lifts to an isotopy of bisections $L_t$ in $Γ$ with $L_0 = 1$ (i.e., $L_t$ goes to the graph of $h_t$ under the projection $p$). Then,

(i) $L_t$ is a lagrangian bisection;

(ii) $\mathcal{F}(\{L_t\}) = \mathcal{F}(\{h_t\})$.

**Proof.** Let $\tilde{ω}$ be the symplectic form on $Γ$. Then, $\tilde{ω} = \alpha^*ω - \beta^*ω$. Let $i$ and $i'$ denote the inclusions: $i : L_t \rightarrow Γ$ and $i' : (the \ graph \ of \ h_t) \rightarrow M \times \overline{M}$, respectively. Then, $i^*\tilde{ω} = (p|_{L_t})^*(i')^*(ω \ominus ω) = 0$, since the graph of $h_t$ is lagrangian. This shows that $L_t$ is a lagrangian bisection.

Let $Y_t$ be the corresponding time-dependent vector field on $Γ$ defined by Equation (7), and $θ_t$ the time-dependent one-form on $P$ as defined by Equation (8). Thus,

$$Y_t \lhd \tilde{ω} = (Y_t \lhd \alpha^*ω) - (Y_t \lhd \beta^*ω) = \alpha^*(T\alpha Y_t \lhd ω).$$

Since $\alpha L_t(m) = h_t(m)$ for any $m \in M$, it follows that $T\alpha Y_t = \dot{h}_t$. Hence, $Y_t \lhd \tilde{ω} = \alpha^*(\dot{h}_t \lhd ω)$. In other words, $θ_t = \dot{h}_t \lhd ω$. Thus, $\mathcal{F}(\{L_t\}) = \int_0^1 θ_t dt = \int_0^1 (\dot{h}_t \lhd ω) dt = \mathcal{F}(\{h_t\})$.

$\square$

By combining with Theorem 5.4, this proposition leads to the following:
Theorem 6.4 For \( \{h_t\} \in \widetilde{Diff}_0(M, \omega) \), suppose that \( L_t \) is any isotopy in \( U_0(\Gamma) \) lifting \( \{h_t\} \). Then,
\[
(\mu \circ F)\{h_t\} = \lambda(L_1).
\]
(15)

In particular, if the symplectic pairing is nondegenerate, we have
\[
F(\{h_t\}) = \mu^{-1} \lambda(L_1).
\]

This theorem suggests a useful method for computing the flux homomorphism for symplectomorphisms. The rhs of Equation (15) can be essentially expressed by an integration over the symplectic manifold, which is much easier to handle. In the case when \( \mu \) is an isomorphism, this will enable us to carry out an explicit computation for the usual flux homomorphism \( F \).

We note that any symplectic isotopy \( h_t \) of \( M \) can always be lifted to an isotopy of lagrangian bisections in \( \Gamma \). For example, for any fixed \( t \), set \( L_t = \{[h_t - s(m)], 0 \leq s \leq t | m \in M \} \). Then it is simple to see that \( L_t \) is a lifting of \( h_t \).

Let \( A : \widetilde{Diff}_0(M, \omega) \rightarrow U_0(\Gamma) \) be the map which sends \( \{h_t\} \) to \( L = \{[h_{1-s}(m)], 0 \leq s \leq 1 | m \in M \} \).

The following corollary follows directly from Theorem 6.4.

Corollary 6.5 The following diagram:

\[
\begin{array}{ccc}
\widetilde{Diff}_0(M, \omega) & \xrightarrow{A} & U_0(\Gamma) \\
F \downarrow & & \downarrow \lambda \\
H^1(M, \mathbb{R}) & \xrightarrow{\mu} & H^1(M, \mathbb{R})^*
\end{array}
\]

(16)

commutes.

Combining Proposition 6.2 with the corollary above leads the following result of McDuff [10]:

Corollary 6.6 A loop of symplectomorphisms \( \{\varphi_t\}_{0 \leq t \leq 1} \in \pi_1(Diff_0(M, \omega), 1) \) is in the kernel of \( \mu \circ F \) if and only if the loop \( u \rightarrow \varphi_t(u) \in M \) is null-homologous for every \( u \in M \).

Remark (1). Clearly, \( A \) maps the fundamental group \( \pi_1(Diff_0(M, \omega), 1) \) into the group \( I \) of isotropy lagrangian bisections of \( \Gamma \). It would be interesting to know if \( A \) is surjective. It is also interesting to ask if \( I \) is discrete.

(2). Suppose that \( \varphi_t \) with \( \varphi_1 = \varphi_0 = id \) is a loop in \( Ham(M, \omega) \) for a symplectic manifold \( P \). It was asked in [11] whether the loop \( u \rightarrow \varphi_t(u) \) is contractible for every \( u \in P \). We may ask the following more general question: does the intersection of \( U_{ex}(\Gamma) \) with \( I \) consist only the trivial element 1. Note that this fails for general Poisson manifolds. For instance, for zero Poisson manifolds, it is simple to see that \( U_{ex}(\Gamma) \cap I = U_{ex}(\Gamma) \).
Proposition 6.7 Suppose that the image of $\pi_1(\text{Diff}_0(M,\omega),1)$ under $\eta$ coincides with the space $I$ of all isotropy lagrangian bisections and $\mu$ is an isomorphism. Then, the flux homomorphism $F$ descends to a homomorphism $F' : \text{Diff}_0(M,\omega) \to H^1(M,\mathbb{R})/\Omega$, where $\Omega = (\mu^{-1}\lambda)(I)$ and the map $F'$ is given by $F'(h) = [(\mu^{-1}\lambda)(L)] \in H^1(M,\mathbb{R})/\Omega$, $\forall h \in \text{Diff}_0(M,\omega)$. Here $L$ is any bisection in $U_0(\Gamma)$ such that $p(L) =$ the graph of $h$.

We end this section by applying the results above to the symplectic torus $T^{2n}$, which was treated in [1] by a different method.

Example 6.8 Consider the torus $T^{2n} \cong \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ with the canonical symplectic structure. Let $\varphi_t$ be a symplectic isotopy with a lift $\psi_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ such that $\psi_t(x + l) = \psi_t(x) + l$, $\forall l \in \mathbb{Z}^{2n}$ and $\psi_0 = \text{id}$ for any $x \in \mathbb{R}^{2n}$. The symplectic groupoid $\Gamma$ can be taken as the cotangent space $T^*T^{2n} \cong T^{2n} \times \mathbb{R}^{2n}$ with the standard symplectic structure and the source and target maps are given respectively by $\alpha(x,p) = x - \frac{1}{2}p$, $\beta(x,p) = x + \frac{1}{2}p$ (see [1]). Clearly, the first groupoid cohomology $H^1(\Gamma,\mathbb{R})$ is generated by $I_i(x,p) = p_i$, $1 \leq i \leq 2n$. Take $L_t = \{(\frac{1}{2}(x + \psi_t(x)),x - \psi_t(x)) \in T^{2n} \times \mathbb{R}^{2n} | x \in \mathbb{R}^{2n}\}$. It is simple to see that both $\alpha$ and $\beta$ are diffeomorphisms when restricted to $L_t$ and $p(L_t) =$ graph($\varphi_t$). In other words, $L_t$ is a family of lagrangian bisections of $\Gamma$, which lifts the graph of $\varphi_t$. Now,

$$<\epsilon(L_1), [J_i>] = \int_{T^n} J_i(L_1(x)) = \int_{T^n} (x^i - \psi_1^i(x))dx.$$ 

On the other hand, it is simple to see that $(\Psi \circ \pi^#)[dx_i] = [J_i]$, $i = 1, \cdots, 2n$. Therefore,

$$<\lambda(L_1), [dx_i] > = <\epsilon(L_1), (\Psi \circ \pi^#)[dx_i] > = <\epsilon(L_1), [J_i] > = \int_{T^n} (x^i - \psi_1^i(x))dx.$$ 

Hence, according to Theorem 6.4, the flux homomorphism $F$ maps $\{\varphi_t\}$ to $[\Sigma_{j=1}^{2n} a_j dx_j]$, where $a = (a_1,\cdots,a_{2n}) = (\int_{T^{2n}} (x^1 - \psi_1^1(x))dx,\cdots,\int_{T^{2n}} (x^{2n} - \psi_1^{2n}(x))dx) \cdot J$, and $J$ is the canonical symplectic matrix on $\mathbb{R}^{2n}$.

Remark The $\alpha$-simply connected symplectic groupoid of a compact Kahler manifold $P$ with negative holomorphic constant curvature is isomorphic to the cotangent bundle $T^*P$ [18]. It would be interesting to generalize the computation above to such a manifold. It is reasonable to expect that in this case the flux homomorphism should be related to certain “center of mass” of the manifold.

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