FIRST QUANTUM CORRECTION FOR THE MODULI SPACE OF
STABLE BUNDLES OVER A RIEMANN SURFACE

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Abstract. We compute some Gromov-Witten invariants of the moduli space $M_Σ$ of odd degree rank two stable vector bundles over a Riemann surface $Σ$ of genus $g \geq 2$. We thus find the first correction term for the quantum product of $M_Σ$ and hence get the two leading terms of the relations satisfied by the natural generators of the quantum cohomology of $M_Σ$. Finally, we use this information to get a full description of the quantum cohomology of $M_Σ$ when the genus of $Σ$ is $g = 3$.

1. Introduction

Let $Σ$ be a Riemann surface of genus $g \geq 2$ and let $M_Σ$ denote the moduli space of flat $SO(3)$-connections with nontrivial second Stiefel-Whitney class $w_2$.

This is a smooth symplectic manifold of dimension $6g - 6$. Alternatively, we can consider $Σ$ as a smooth complex curve of genus $g$. Fix a line bundle $Λ$ on $Σ$ of degree 1, then $M_Σ$ is the moduli space of rank two stable vector bundles on $Σ$ with determinant $Λ$, which is a smooth complex variety of complex dimension $3g - 3$. The symplectic deformation class of $M_Σ$ only depends on $g$ and not on the particular complex structure on $Σ$.

The manifold $X = M_Σ$ is a positive symplectic manifold with $π_2(X) = \mathbb{Z}$. For such a manifold $X$, its quantum cohomology, $QH^*(X)$, is well-defined (see [14] [15] [8] [12]). As vector spaces, $QH^*(X) = H^*(X)$ (rational coefficients are understood), but the multiplicative structure is different. Let $A$ denote the positive generator of $π_2(X)$, i.e. the generator such that the symplectic form evaluated on $A$ is positive. Let $N = c_1(X)[A] \in \mathbb{Z}_{>0}$. Then there is a natural $\mathbb{Z}/2\mathbb{Z}$-grading for $QH^*(X)$, which comes from reducing the $\mathbb{Z}$-grading of $H^*(X)$. (For the case $X = M_Σ$, $N = 2$, so $QH^*(M_Σ)$ is $\mathbb{Z}/4\mathbb{Z}$-graded). The ring structure of $QH^*(X)$, called quantum multiplication, is a deformation of the usual cup product for $H^*(X)$. For $α \in H^p(X)$, $β \in H^q(X)$, we

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define the quantum product of $\alpha$ and $\beta$ as
\[ \alpha \cdot \beta = \sum_{d \geq 0} \Phi_{dA}(\alpha, \beta), \]
where $\Phi_{dA}(\alpha, \beta) \in H^{p+r-2Nd}(X)$ is given by $< \Phi_{dA}(\alpha, \beta), \gamma > = \Psi^X_{dA}(\alpha, \beta, \gamma)$, the Gromov-Witten invariant, for all $\gamma \in H^{\dim X-p-q+2Nd}(X)$. One has $\Phi_0(\alpha, \beta) = \alpha \cup \beta$. The other terms are the correction terms and all live in lower degree parts of the cohomology groups. It is a fact [15] that the quantum product gives an associative and graded commutative ring structure.

To define the Gromov-Witten invariant, let $J$ be a generic almost complex structure compatible with the symplectic form. Then for every 2-homology class $dA$, $d \in \mathbb{Z}$, there is a moduli space $M_{dA}$ of pseudoholomorphic rational curves (with respect to $J$) $f : \mathbb{P}^1 \to X$ with $f_*[\mathbb{P}^1] = dA$. Note that $M_0 = X$ and that $M_{dA}$ is empty for $d < 0$. For $d \geq 0$, the dimension of $M_{dA}$ is $\dim X + 2Nd$. This moduli space $M_{dA}$ admits a natural compactification, $\overline{M}_{dA}$, called the Gromov-Uhlenbeck compactification [14] [15, section 3]. Consider now $r \geq 3$ different points $P_1, \ldots, P_r \in \mathbb{P}^1$. Then we have defined an evaluation map $ev : M_{dA} \to X^r$ by $f \mapsto (f(P_1), \ldots, f(P_r))$. This map extends to $\overline{M}_{dA}$ and its image, $ev(\overline{M}_{dA})$, is a pseudo-cycle [15]. So for $\alpha_i \in H^{p_i}(M_\Sigma)$, $1 \leq i \leq r$, with $p_1 + \cdots + p_r = \dim X + 2Nd$, we choose generic cycles $A_i$, $1 \leq i \leq r$, representatives of their Poincaré duals, and set
\[ (1) \Psi^{X}_{dA}(\alpha_1, \ldots, \alpha_r) = < A_1 \times \cdots \times A_r, [ev(\overline{M}_{dA})] > = \# ev_{P_1}(A_1) \cap \cdots \cap ev_{P_r}^r(A_r), \]
where $\#$ denotes count of points (with signs) and $ev_{P_i} : M_{dA} \to X$, $f \mapsto f(P_i)$. This is a well-defined number and independent of the particular cycles. Also, as the manifold $X$ is positive, $\coker L_f = H^1(\mathbb{P}^1, f^*c_1(X)) = 0$, for all $f \in M_{dA}$. By [14] the complex structure of $X$ is generic and we can use it to compute the Gromov-Witten invariants.

Also for $r \geq 2$, let $\alpha_i \in H^{p_i}(M_\Sigma)$, $1 \leq i \leq r$, then
\[ \alpha_1 \cdots \alpha_r = \sum_{d \geq 0} \Phi_{dA}(\alpha_1, \ldots, \alpha_r), \]
where the correction terms $\Phi_{dA}(\alpha_1, \ldots, \alpha_r) \in H^{p_1+\cdots+p_r-2Nd}(X)$ are determined by $< \Phi_{dA}(\alpha_1, \ldots, \alpha_r), \gamma > = \Psi^X_{dA}(\alpha_1, \ldots, \alpha_r, \gamma)$, for any $\gamma \in H^{\dim X+2Nd-(p_1+\cdots+p_r)}(X)$.

Returning to our manifold $X = M_\Sigma$, there is a classical conjecture relating the quantum cohomology $QH^\ast(M_\Sigma)$ and the instanton Floer cohomology of $\Sigma \times \mathbb{S}^1$, $HF^\ast(\Sigma \times \mathbb{S}^1)$ (see [10]). In [1] a presentation of $QH^\ast(M_\Sigma)$ was given using physical methods, and in [10] it was proved that such a presentation was a presentation of $HF^\ast(\Sigma \times \mathbb{S}^1)$ indeed. Siebert and Tian have a program [16] to find the presentation of $QH^\ast(M_\Sigma)$, which goes through proving a recursion formula for the Gromov-Witten invariants of $M_\Sigma$ in terms of the genus $g$. This will complete the proof of the conjectural isomorphism $QH^\ast(M_\Sigma) \cong HF^\ast(\Sigma \times \mathbb{S}^1)$.
The purpose of this paper is two-fold. On the one hand we aim to compute the Gromov-Witten invariants $\Psi^A_{M_\Sigma}$, relating them to the Donaldson invariants for the algebraic surface $S = \Sigma \times \mathbb{P}^1$. This gives the first correction term of the quantum product of $M_\Sigma$ and hence the first two leading terms of the relations satisfied by the generators of $QH^*(M_\Sigma)$. In particular we obtain the coefficient $c_g$ of [16, section 3.3]. On the other hand we infer a full presentation of $QH^*(M_\Sigma)$ when the genus of $\Sigma$ is $g = 3$ (recall that the case of genus $g = 2$ was worked out by Donaldson [3]). This is the starting point of the induction in [16].

The paper is organised as follows. In section 2 we review the ordinary cohomology ring of $M_\Sigma$. In section 3 the moduli space of lines in $M_\Sigma$ is described, in order to compute the corresponding Gromov-Witten invariants in section 4. In section 5 we study the quantum cohomology of $M_\Sigma$ and finally determine it completely in the case of genus $g = 3$ in section 6.

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2. Classical cohomology ring of $M_\Sigma$

Let us recall the known description of the homology of $M_\Sigma$ [6] [17] [10]. Let $U \to \Sigma \times M_\Sigma$ be the universal bundle and consider the Künneth decomposition of

\[ c_2(\text{End}_0 U) = 2[\Sigma] \otimes \alpha + 4\psi - \beta, \]

with $\psi = \sum \gamma_i \otimes \psi_i$, where $\{\gamma_1, \ldots, \gamma_{2g}\}$ is a symplectic basis of $H^1(\Sigma; \mathbb{Z})$ with $\gamma_i \gamma_{i+g} = [\Sigma]$ for $1 \leq i \leq g$ (also $\{\gamma_i^\#\}$ will denote the dual basis for $H_1(\Sigma; \mathbb{Z})$). Here we can suppose without loss of generality that $c_1(U) = \Lambda + \alpha$ (see [17]). In terms of the map $\mu : H_*(\Sigma) \to H^{4-*}(M_\Sigma)$, given by $\mu(a) = -\frac{1}{4} p_1(\mathfrak{g}_U)/a$ (here $\mathfrak{g}_U \to \Sigma \times M_\Sigma$ is the associated universal $SO(3)$-bundle, and $p_1(\mathfrak{g}_U) \in H^4(\Sigma \times M_\Sigma)$ its first Pontrjagin class), we have

\[
\begin{align*}
\alpha &= 2 \mu(\Sigma) \in H^2 \\
\psi_i &= \mu(\gamma_i^\#) \in H^3, \quad 1 \leq i \leq 2g \\
\beta &= -4 \mu(x) \in H^4
\end{align*}
\]

where $x \in H_0(\Sigma)$ is the class of the point, and $H^i = H^i(M_\Sigma)$. These elements generate $H^*(M_\Sigma)$ as a ring [6] [19], and $\alpha$ is the positive generator of $H^2(M_\Sigma; \mathbb{Z})$. We can rephrase this as saying that there exists an epimorphism

\[ A(\Sigma) = \mathbb{Q}[\alpha, \beta] \otimes \Lambda(\psi_1, \ldots, \psi_{2g}) \to H^*(M_\Sigma) \]
(the notation $\mathbb{A}(\Sigma)$ follows that of Kronheimer and Mrowka [7]). The mapping class group $\text{Diff}(\Sigma)$ acts on $H^*(M_\Sigma)$, with the action factoring through the action of $\text{Sp}(2g, \mathbb{Z})$ on $\{\psi_i\}$. The invariant part, $H^*_I(M_\Sigma)$, is generated by $\alpha$, $\beta$ and $\gamma = -2\sum_{i=0}^g \psi_i \psi_{i+g}$. Then there is an epimorphism

$$H^*_I(M_\Sigma) = \mathbb{Q}[\alpha, \beta, \gamma] / I_g,$$

which allows us to write

$H^*_I(M_\Sigma) = \mathbb{Q}[\alpha, \beta, \gamma] / I_g$,

where $I_g$ is the ideal of relations satisfied by $\alpha$, $\beta$ and $\gamma$. Recall that $\deg(\alpha) = 2$, $\deg(\beta) = 4$ and $\deg(\gamma) = 6$. From [17], a basis for $H^*_I(M_\Sigma)$ is given by the monomials $\alpha^a \beta^b \gamma^c$, with $a + b + c < g$. For $0 \leq k \leq g$, the primitive component of $\Lambda^k H^3$ is

$$\Lambda^k H^3 = \ker(\gamma^{g-k+1} : \Lambda^k H^3 \to \Lambda^{2g-k+2} H^3).$$

Then the $\text{Sp}(2g, \mathbb{Z})$-decomposition of $H^*(M_\Sigma)$ is [6]

$$H^*(M_\Sigma) = \bigoplus_{k=0}^g \Lambda^k H^3 \otimes \mathbb{Q}[\alpha, \beta, \gamma] / I_{g-k}.$$

**Proposition 1** ([17]). For $g = 1$, let $q_1^1 = \alpha$, $q_1^2 = \beta$, $q_1^3 = \gamma$. Define recursively, for $g \geq 1$,

$$\begin{cases}
q_{g+1}^1 = \alpha q_g^1 + g^2 q_g^2 \\
q_{g+1}^2 = \beta q_g^2 + \frac{2g}{g+1} q_g^3 \\
q_{g+1}^3 = \gamma q_g^3
\end{cases}$$

Then $I_g = (q_g^1, q_g^2, q_g^3)$, for all $g \geq 1$. Note that $\deg(q_g^1) = 2g$, $\deg(q_g^2) = 2g + 2$ and $\deg(q_g^3) = 2g + 4$.

3. **Holomorphic lines in $M_\Sigma$**

In order to compute the Gromov-Witten invariants $\Psi^M_A$, we need to describe the space of lines, i.e. rational curves in $M_\Sigma$ representing the generator $A \in H_2(M_\Sigma; \mathbb{Z})$,

$$\mathcal{M}_A = \{ f : \mathbb{P}^1 \to M_\Sigma / f \text{ holomorphic, } f_*[\mathbb{P}^1] = A \}.$$

Let us fix some notation. Let $J$ denote the Jacobian variety of $\Sigma$ parametrising line bundles of degree 0 and let $\mathcal{L} \to \Sigma \times J$ be the universal line bundle. If $\{\gamma_i\}$ is the basis of $H^1(\Sigma)$ introduced in section 2 then $c_1(\mathcal{L}) = \sum \gamma_i \otimes \phi_i \in H^1(\Sigma) \otimes H^1(J)$, where $\{\phi_i\}$ is a symplectic basis for $H^1(J)$. Thus $c_1(\mathcal{L})^2 = -2[\Sigma] \otimes \omega \in H^2(\Sigma) \otimes H^2(J)$, where $\omega = \sum_{i=1}^g \phi_i \wedge \phi_{i+g}$ is the natural symplectic form for $J$.

Consider now the algebraic surface $S = \Sigma \times \mathbb{P}^1$. It has irregularity $q = g \geq 2$, geometric genus $p_g = 0$ and canonical bundle $K \equiv -2\Sigma + (2g - 2)\mathbb{P}^1$. Recall that $\Lambda$ is a fixed line bundle of degree 1 on $\Sigma$. Fix the line bundle $L = \Lambda \otimes \mathcal{O}_{\mathbb{P}^1}(1)$ on $S$ (we omit all pull-backs) with $c_1 = c_1(L) \equiv \mathbb{P}^1 + \Sigma$, and put $c_2 = 1$. The ample cone of $S$
is \( \{ a \mathbb{P}^1 + b \Sigma / a, b > 0 \} \). Let \( H_0 \) be a polarisation close to \( \mathbb{P}^1 \) in the ample cone and \( H \) be a polarisation close to \( \Sigma \), i.e. \( H = \Sigma + t\mathbb{P}^1 \) with \( t \) small. We wish to study the moduli space \( \mathcal{M} = \mathcal{M}_H(c_1, c_2) \) of \( H \)-stable bundles over \( S \) with Chern classes \( c_1 \) and \( c_2 \).

**Proposition 2.** \( \mathcal{M} \) can be described as a bundle \( \mathbb{P}^{2g-1} \to \mathcal{M} = \mathbb{P}(E_\zeta) \to J \), where \( E_\zeta \) is a bundle on \( J \) with \( \text{ch} E_\zeta = 2g + 8\omega \). So \( \mathcal{M} \) is compact, smooth and of the expected dimension \( 6g - 2 \). The universal bundle \( \mathcal{V} \to S \times \mathcal{M} \) is given by

\[
0 \to \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{L} \otimes \lambda \to \mathcal{V} \to \Lambda \otimes \mathcal{L} \to 0,
\]

where \( \lambda \) is the tautological line bundle for \( \mathcal{M} \).

**Proof.** For the polarisation \( H_0 \), the moduli space of \( H_0 \)-stable bundles with Chern classes \( c_1, c_2 \) is empty by [13]. Now for \( p_1 = -4c_2 + c_1^2 = -2 \) there is only one wall, determined by \( \zeta \equiv -\mathbb{P}^1 + \Sigma \) (here we fix \( \zeta = 2\Sigma - L = \Sigma - c_1(\Lambda) \) as a divisor), so the moduli space of \( H \)-stable bundles with Chern classes \( c_1, c_2 \) is obtained by crossing the wall as described in [9]. First, note that the results in [9] use the hypothesis of \(-K\) being effective, but the arguments work equally well with the weaker assumption of \( \zeta \) being a good wall [9, remark 1] (see also [5] for the case of \( q = 0 \)). In our case, \( \zeta \equiv -\mathbb{P}^1 + \Sigma \) is a good wall (i.e. \( \pm \zeta + K \) are both not effective) with \( l_\zeta = 0 \). Now with the notations of [9], \( F \) is a divisor such that \( 2F - L \equiv \zeta \), i.e. \( F = \Sigma \). Also \( \mathcal{F} \to S \times J \) is the universal bundle parametrising divisors homologically equivalent to \( F \), i.e. \( \mathcal{F} = \mathcal{L} \otimes \mathcal{O}_{\mathbb{P}^1}(1) \). Let \( \pi : S \times J \to J \) be the projection. Then \( \mathcal{M} = E_\zeta = \mathbb{P}(E_\zeta^\vee) \), where

\[
E_\zeta = \text{Ext}^1_\pi(\mathcal{O}(L - \mathcal{F}), \mathcal{O}(\mathcal{F})) = R^1\pi_* (\mathcal{O}(\zeta) \otimes \mathcal{L}^2). \]

Actually \( \mathcal{M} \) is exactly the set of bundles \( E \) that can be written as extensions

\[
0 \to \mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{L} \to E \to \Lambda \otimes \mathcal{L} \to 0
\]

for a line bundle \( \mathcal{L} \) of degree 0. The Chern character is [9, section 3] \( \text{ch} E_\zeta = 2g + e_{K-2\zeta} \), where \( e_\alpha = -2(\mathbb{P}^1, \alpha)\omega \) (the class \( \Sigma \) defined in [9, lemma 11] is \( \mathbb{P}^1 \) in our case). Finally, the description of the universal bundle follows from [9, theorem 10]. \( \square \)

**Proposition 3.** There is a well defined map \( \mathcal{M}_A \to \mathcal{M} \).

**Proof.** Every line \( f : \mathbb{P}^1 \to M_\Sigma \) gives a bundle \( E = (\text{id}_\Sigma \times f)^* \mathcal{U} \) over \( \Sigma \times \mathbb{P}^1 \) by pulling-back the universal bundle \( \mathcal{U} \to \Sigma \times M_\Sigma \). Then for any \( t \in \mathbb{P}^1 \), the bundle \( E|_{\Sigma \times t} \) is defined by \( f(t) \). Now, by equation (2), \( p_1(E) = p_1(\mathcal{U})|_{\Sigma \times A} = -2a[A] = -2 \). Since \( c_1(E) = (\text{id}_\Sigma \times f)^* c_1(\mathcal{U}) = \Lambda + \Sigma \), it must be \( c_2 = 1 \). To see that \( E \) is \( H \)-stable, consider any sub-line bundle \( L \hookrightarrow E \) with \( c_1(L) \equiv a\mathbb{P}^1 + b\Sigma \). Restricting to any \( \Sigma \times t \subset \Sigma \times \mathbb{P}^1 \) and using the stability of \( E|_{\Sigma \times t} \), one gets \( a \leq 0 \). Then \( c_1(L) \cdot \Sigma < \frac{c_1(E, \Sigma)}{2} \), which yields the \( H \)-stability of \( E \) (recall that \( H \) is close to \( \Sigma \)). So \( E \in \mathcal{M} \). \( \square \)
Now define \( N \) as the set of extensions on \( \Sigma \) of the form
\[
0 \to L \to E \to \Lambda \otimes L^{-1} \to 0,
\]
for \( L \) a line bundle of degree 0. Then the groups \( \text{Ext}^1(\Lambda \otimes L^{-1}, L) = H^1(L^2 \otimes \Lambda^{-1}) = H^0(L^{-2} \otimes \Lambda \otimes K) \) are of constant dimension \( g \). Moreover \( H^0(L^2 \otimes \Lambda^{-1}) = 0 \), so the moduli space \( N \) which parametrises extensions like (1) is given as \( N = \mathbb{P}(E^\vee) \), where \( E = \text{Ext}^1_p(\Lambda \otimes \mathcal{L}^{-1}, \mathcal{L}) = R^1p_*\mathcal{L}^2 \otimes \Lambda^{-1} \), \( p : \Sigma \times J \to J \) the projection. Then we have a fibration \( \mathbb{P}^{g-1} \to N = \mathbb{P}(E^\vee) \to J \). The Chern character of \( E \) is
\[
\text{ch} (\mathcal{E}) = \text{ch} (R^1p_*\mathcal{L}^2 \otimes \Lambda^{-1}) = -\text{ch} (p_*\mathcal{L}^2 \otimes \Lambda^{-1}) = -p_*((\text{ch} \mathcal{L})^2 (\text{ch} \Lambda)^{-1} \text{Todd} T\Sigma) = -p_*((1 + c_1(\mathcal{L}) + \frac{1}{2}c_1(\mathcal{L})^2)(1 - \Lambda)(1 - \frac{1}{2}K)) = -p_*((1 - \frac{1}{2}K + 2c_1(\mathcal{L}) - 4\omega \otimes [\Sigma] - \Lambda) = g + 4\omega.
\]
It is easy to check that all the bundles in \( N \) are stable, so there is a well-defined map
\[
i : N \to M_{\Sigma}.
\]

Now we wish to construct the space of lines in \( N \). Note that \( \pi_2(N) = \pi_2(\mathbb{P}^{g-1}) = \mathbb{Z} \), as there are no rational curves in \( J \). Let \( L \in \pi_2(N) \) be the positive generator. We want to describe
\[
\mathcal{N}_L = \{ f : \mathbb{P}^1 \to N/f \text{ holomorphic, } f_*[\mathbb{P}^1] = L \}.
\]
For the projective space \( \mathbb{P}^n \), the space of lines \( H_1 \) is the set of algebraic maps \( f : \mathbb{P}^1 \to \mathbb{P}^n \) of degree 1. Such an \( f \) has the form \( f[x_0, x_1] = [x_0u_0 + x_1u_1], [x_0, x_1] \in \mathbb{P}^1 \), where \( u_0, u_1 \) are linearly independent vectors in \( \mathbb{C}^{n+1} \). So
\[
H_1 = \mathbb{P}(\{(u_0, u_1)/u_0, u_1 \text{ are linearly independent}\}) \subset \mathbb{P}((\mathbb{C}^{n+1} \oplus \mathbb{C}^{n+1})^\vee) = \mathbb{P}^{2n+1}.
\]
The complement of \( H_1 \) is the image of \( \mathbb{P}^n \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^{2n+1}, ([u], [x_0, x_1]) \mapsto [x_0u, x_1u] \), which is a smooth \( n \)-codimensional algebraic subvariety. Now \( \mathcal{N}_L \) can be described as
\[
\mathbb{P}^{2g-1} \cap \mathbb{P}^{2n+1} \to \mathbb{P}((\mathcal{E} \oplus \mathcal{E})^\vee) \to J
\]
\[
\mathcal{N}_L \to J
\]
\[
\mathbb{P}^{2g-1} \to \mathbb{P}((\mathcal{E} \oplus \mathcal{E})^\vee) \to J
\]
\[
\mathbb{P}^{2g-1} \to \mathbb{P}((\mathcal{E} \oplus \mathcal{E})^\vee) \to J
\]

Remark 4. Note that \( \mathcal{E}_x = R^1\pi_*\mathcal{O}(\mathcal{L}) \otimes \mathcal{L}^2 = R^1\pi_*\mathcal{O}_{\mathbb{P}^1}(\mathcal{L}^2 \otimes \Lambda^{-1}) = H^0(\mathcal{O}_{\mathbb{P}^1}(1) \otimes R^1\pi_*\mathcal{L}^2 \otimes \Lambda^{-1}) = H^0(\mathcal{O}_{\mathbb{P}^1}(1) \otimes \mathcal{E} \cong \mathcal{E} \oplus \mathcal{E}). \) So \( \mathfrak{M} = \mathbb{P}((\mathcal{E} \oplus \mathcal{E})^\vee) \), canonically.

Proposition 5. The map \( i : N \to M_{\Sigma} \) induces a map \( i_* : \mathcal{N}_L \to \mathcal{M}_A \). The composition \( \mathcal{N}_L \to \mathcal{M}_A \to \mathfrak{M} \) is the natural inclusion of (7).
Proof. The first assertion is clear as \( i \) is a holomorphic map. For the second, consider the universal sheaf on \( \Sigma \times N \),

\[
0 \to \mathcal{L} \otimes U \to E \to \Lambda \otimes \mathcal{L}^{-1} \to 0,
\]

where \( U = \mathcal{O}_N(1) \) is the tautological bundle of the fibre bundle \( \mathbb{P}^{g-1} \to N \to J \). Any element in \( \mathcal{N}_L \) is a line \( \mathbb{P}^1 \hookrightarrow N \), which must lie inside a single fibre \( \mathbb{P}^{g-1} \). Restricting (8) to this line, we have an extension

\[
0 \to L \otimes \mathcal{O}_{\mathbb{P}^1}(1) \to E \to \Lambda \otimes \mathcal{L}^{-1} \to 0
\]
on \( S = \Sigma \times \mathbb{P}^1 \), which is the image of the given element in \( \mathcal{M} \) (here \( L \) is the line bundle corresponding to the fibre in which \( \mathbb{P}^1 \) sits). Now it is easy to check that the map \( \mathcal{N}_L \to \mathcal{M} \) is the inclusion of (7).

Corollary 6. \( i_* \) is an isomorphism.

Proof. By proposition 5, \( i_* \) has to be an open immersion. The group \( PGL(2, \mathbb{C}) \) acts on both spaces \( \mathcal{N}_L \) and \( \mathcal{M} \), and \( i_* \) is equivariant. The quotient \( \mathcal{N}_L/PGL(2, \mathbb{C}) \) is compact, being a fibration over the Jacobian with all the fibres the Grassmannian \( \text{Gr}(\mathbb{C}^2, \mathbb{C}^{g-1}) \), hence irreducible. As a consequence \( i_* \) is an isomorphism.

Remark 7. Notice that the lines in \( M_\Sigma \) are all contained in the image of \( N \), which is of dimension \( 4g-2 \) against \( 6g-6 = \dim M_\Sigma \). They do not fill all of \( M_\Sigma \) as one would naively expect.

4. Computation of \( \Psi^M_\Sigma \)

The manifold \( N \) is positive with \( \pi_2(N) = \mathbb{Z} \) and \( L \in \pi_2(N) \) is the positive generator. Under the map \( i : N \to M_\Sigma \), we have \( i_*L = A \). Now \( \dim N = 4g-2 \) and \( c_1(N)[L] = c_1(\mathbb{P}^{g-1})[L] = g \). So quantum cohomology of \( N \), \( QH^*(N) \), is well-defined and \( \mathbb{Z}/2g\mathbb{Z} \)-graded. From corollary 6, it is straightforward to prove

**Lemma 8.** For any \( \alpha_i \in H^{p_i}(M_\Sigma), 1 \leq i \leq r \), such that \( p_1 + \cdots + p_r = 6g-2 \), it is \( \Psi^M_\Sigma(\alpha_1, \ldots, \alpha_r) = \Psi^N_\Sigma(i^*\alpha_1, \ldots, i^*\alpha_r) \). □

It is therefore important to know the Gromov-Witten invariants of \( N \), i.e. its quantum cohomology. From the universal bundle (8), we can read the first Pontrjagin class \( p_1(g_Z) = -8[\Sigma] \otimes \omega + h^2 - 2[\Sigma] \otimes h + 4h \cdot c_1(L) \in H^4(\Sigma \times N) \), where \( h = c_1(U) \) is the hyperplane class. So on \( N \) we have

\[
\begin{align*}
\alpha &= 2\mu(\Sigma) = 4\omega + h \\
\psi_i &= \mu(\gamma_i^\#) = -h \cdot \phi_i \\
\beta &= -4\mu(x) = h^2
\end{align*}
\]
Let us remark that $h^2$ denotes ordinary cup product in $H^*(N)$, a fact which will prove useful later. Now let us compute the quantum cohomology ring of $N$. The cohomology of $J$ is $H^*(J) = \Lambda H_1$, where $H_1 = H_1(\Sigma)$. Now the fibre bundle description $\mathbb{P}^{2g-1} \to N = \mathbb{P}(\mathcal{E}^\vee) \to J$ implies that the usual cohomology of $N$ is $H^*(N) = \Lambda H_1[h]/ < h^g + c_1 h^{g-1} + \cdots + c_g = 0 >$, where $c_i = c_i(\mathcal{E}) = \frac{4^i}{i!}\omega^i$, from (6). As the quantum cohomology has the same generators as the usual cohomology and the relations are a deformation of the usual relations [18], it must be $h^g + c_1 h^{g-1} + \cdots + c_g = r$ in $QH^*(N)$, with $r \in \mathbb{Q}$. As in [15, example 8.5], $r$ can be computed to be 1. So

$$QH^*(N) = \Lambda H_1[h]/ < h^g + c_1 h^{g-1} + \cdots + c_g = 1 >.$$  

**Lemma 9.** For any $s \in H^{2g-2i}(J), 0 \leq i \leq g$, denote by $s \in H^{2g-2i}(N)$ its pull-back to $N$ under the natural projection. Then the quantum product $h^{2g-1+i}s$ in $QH^*(N)$ has component in $H^{4g-2}(N)$ equal to $\frac{(-8)^i}{4^i}\omega^i \wedge s$.

**Proof.** First note that for $s_1, s_2 \in H^*(J)$ such that their cup product in $J$ is $s_1 s_2 = 0$, then the quantum product $s_1 s_2 \in QH^*(N)$ vanishes. This is so since every rational line in $N$ is contained in a fibre of $\mathbb{P}^{2g-1} \to J \to N$.

Next note that $h^{g-1+i}s$ has component in $H^{4g-2}(N)$ equal to $s_i(\mathcal{E}) \wedge s = \frac{(-4)^i}{4^i}\omega^i \wedge s$. Then multiply the standard relation (9) by $h^{g-1+i}$ and work by induction on $i$. For $i = 0$ we get $h^{2g-1}s = h^g h^{g-1}s$ and the assertion is obvious. For $i > 0$,

$$h^{2g-1+i}s + h^{2g-2+i}c_1s + \cdots + h^{2g-1}c_is = h^{g-1+i}s.$$

So the component of $h^{2g-1+i}s$ in $H^{4g-2}(N)$ is

$$- \sum_{j=1}^{i} \frac{(-8)^{i-j}}{(i-j)!} \omega^{i-j}c_js + \frac{(-4)^i}{i!}\omega^i s = \frac{(-8)^i}{4^i}\omega^i s - \sum_{j=0}^{i}\frac{(-8)^{i-j}4^j}{i!}\omega^j s + \frac{(-4)^i}{i!}\omega^i s = \frac{(-8)^i}{4^i}\omega^i s.$$

\[\square\]

**Lemma 10.** Suppose $g > 2$. Let $\alpha^a \beta^b \psi_{i_1} \cdots \psi_{i_r} \in A(\Sigma)$ have degree $6g - 2$. Then

$$\Psi^{N}_L(\alpha, \beta, \psi_{i_1}, \cdots, \psi_{i_r}) = (4\omega + X)^a(X^2)^b \cdot \psi_{i_1} \cdots \psi_{i_r} X^r, [J],$$

evaluated on $J$, where $X^{2g-1+i} = \frac{(-8)^i}{4^i}\omega^i \in H^*(J)$.

**Proof.** By definition the left hand side is the component in $H^{4g-2}(N)$ of the quantum product $\alpha^a \beta^b \psi_{i_1} \cdots \psi_{i_r} \in QH^*(N)$. From (9), this quantum product is $(4\omega + h)^a(h^2)^b(-h\phi_{i_1}) \cdots (-h\phi_{i_r})$, upon noting that when $g > 2$, $\beta = h^2$ as a quantum product as there are no quantum corrections because of the degree. Note that $r$ is even, so the statement of the lemma follows from lemma 9. \[\square\]

Now we are in the position of relating the Gromov-Witten invariants $\Psi^{M_\Sigma}_A$ with the Donaldson invariants for $S = \Sigma \times \mathbb{P}^1$ (for definition of Donaldson invariants see [4] [7]).
Theorem 11. Suppose $g > 2$. Let $\alpha^a \beta^b \psi_1 \cdots \psi_r \in \mathbb{A}(\Sigma)$ have degree $6g - 2$. Then
\[ \Psi_{A_2}(\alpha, \ldots, \alpha, \beta, \psi_1, \ldots, \psi_r) = (-1)^{g-1} D_{S,H}^{\mathcal{M}}((2\Sigma)^a(-4pt)^b \gamma_i^# \cdot \cdot \cdot \gamma_r^#), \]
where $D_{S,H}^{\mathcal{M}}$ stands for the Donaldson invariant of $S = \Sigma \times \mathbb{P}^1$ with $w = c_1$ and polarisation $H$.

Proof. By definition, the right hand side is $\epsilon_S(c_1) < \alpha^a \beta^b \psi_1 \cdots \psi_r, [\mathcal{M}] >$, where $\alpha = 2\mu(\Sigma) \in H^2(\mathcal{M}), \beta = -4\mu(x) \in H^1(\mathcal{M}), \psi = \mu(\gamma_i^#) \in H^3(\mathcal{M})$. Here the factor $\epsilon_S(c_1) = (-1)^{\frac{k_S+q+2}{2}} = (-1)^{g-1}$ compares the complex orientation of $\mathcal{M}$ and its natural orientation as a moduli space of anti-self-dual connections [4]. By [9, theorem 10], this is worked out to be $(-1)^{g-1} < (4 \omega + X)^a(X^2)^b \phi_i \cdots \phi_r X^r, [J] >$, where $X^{2g-1+i} = s_i(\mathcal{E}_i) = \frac{(8\mu_i)}{4} \omega^i$. Thus the theorem follows from lemmas 8 and 10. □

Remark 12. The formula in theorem 11 is not right for $g = 2$, as in such case, the quantum product $h^2 \in QH^*(N)$ differs from $\beta$ by a quantum correction.

Remark 13. Suppose $g \geq 2$ and let $\alpha^a \beta^b \psi_1 \cdots \psi_r \in \mathbb{A}(\Sigma)$ have degree $6g - 6$. Then
\[ \Psi_0^{\mathcal{M}_2}(\alpha, \ldots, \alpha, \beta, \psi_1, \ldots, \psi_r) = \epsilon_S(\mathbb{P}^1) < \alpha^a \beta^b \psi_1 \cdots \psi_r, [\mathcal{M}_2] > = -D_{S,H}^{\mathcal{M}_2}((2\Sigma)^a(-4pt)^b \gamma_i^# \cdot \cdot \cdot \gamma_r^#), \]
as the moduli space of anti-self-dual connections on $S$ of dimension $6g - 6$ is $\mathcal{M}_2$.

5. Quantum Cohomology Ring of $\mathcal{M}_2$

The action of the mapping class group on $\mathcal{M}_2$ is symplectic, so the quantum product restricts to the invariant part of the cohomology, thus having defined $QH^*(\mathcal{M}_2)$ (see [16, section 3.1]). To give a description of it, let us define

\[
\begin{aligned}
\hat{\alpha} &= \alpha \\
\hat{\beta} &= \beta + r_g \\
\hat{\gamma} &= -2 \sum \psi_i \psi_i+g
\end{aligned}
\]

where $r_g \in \mathbb{Q}, g \geq 1$ (to be determined shortly), and $\hat{\gamma}$ is given using the quantum product. It might happen that $\hat{\gamma} = \gamma + s_g \alpha$, $s_g \in \mathbb{Q}$. The need of introducing this quantum corrections in the generators was noticed already in [1].

As a consequence of the description of $H^*(\mathcal{M}_2)$ given in section 2, the results in [18] imply that there is a presentation
\[ QH^*(\mathcal{M}_2) = \mathbb{Q}[\hat{\alpha}, \hat{\beta}, \hat{\gamma}]/J_g, \]
where the ideal $J_g$ is generated by three elements $Q_g^1$, $Q_g^2$ and $Q_g^3$ which are deformations graded mod 4 of $q_g^1$, $q_g^2$ and $q_g^3$, respectively (see [16]). This means that $J_g = (Q_g^1, Q_g^2, Q_g^3)$ for

$$Q_g^i = \sum_{j \geq 0} Q_{g,j}^i,$$

where $\deg(Q_{g,j}^i) = \deg(q_{g}^i) - 4j$, $j \geq 0$, and $Q_{g,0}^i = q_g^i$. There is still one source of possible ambiguity coming from adding a scalar multiple of $Q_g^i$ to $Q_g^3$. To avoid this, we require the coefficient of $\tilde{g}^q$ in $Q_g^3$ to be zero. Recall the main result from [10].

**Theorem 14 ([10]).** Define $R_g^1 = 1$, $R_g^2 = 0$, $R_g^3 = 0$ and then recursively, for all $g \geq 1$,

$$\begin{align*}
R_{g+1}^1 &= \alpha R_g^1 + g^2 R_g^2 \\
R_{g+1}^2 &= (\beta + (-1)^{g+1}8)R_g^1 + \frac{2g}{g+1} R_g^3 \\
R_{g+1}^3 &= \gamma R_g^1
\end{align*}$$

Then the invariant part of the instanton Floer cohomology of $\Sigma \times S^1$ is $HF^*(\Sigma \times S^1)_I = \mathbb{Q}[\alpha, \beta, \gamma]/(R_g^1, R_g^2, R_g^3)$. $R_g^1$, $R_g^2$, $R_g^3$ are uniquely determined by the conditions that the leading term of $R_g^i$ is $q_g^i$, $i = 1, 2, 3$ and that the coefficient of $\alpha^q$ in $R_g^3$ is zero.

Accounting for the difference in signs between theorem 11 and remark 13, the prospective generators of $J_g$ are defined as follows.

**Definition 15.** For $i = 1, 2, 3$ and $g \geq 1$, set

$$\hat{R}_g^i(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) = (\sqrt{-1})^{-\deg(q_g^i)} R_g^i(\sqrt{-1}^g \tilde{\alpha}, \sqrt{-1}^{2g} \tilde{\beta}, \sqrt{-1}^{3g} \tilde{\gamma}),$$

i.e. when $g$ is even, $\hat{R}_g^i = R_g^i$ and when $g$ is odd, $\hat{R}_g^i$ is obtained from $R_g^i$ by changing the sign of the homogeneous components of degrees $\deg(q_g^i) - 4 - 8j$, $j \geq 0$.

Thus we expect that $Q_g^i = \hat{R}_g^i$, $i = 1, 2, 3$ (i.e. $J_g = (\hat{R}_g^1, \hat{R}_g^2, \hat{R}_g^3)$), for $g \geq 1$ (see [1] [16]). Let us review the known cases.

**Example 16.** For $g = 1$, we set $r_1 = -8$, so that $\hat{\alpha} = \alpha$, $\hat{\beta} = \beta - 8$, $\hat{\gamma} = \gamma$. Then the ideal of relations is generated by $\hat{R}_1^1 = \hat{\alpha}$, $\hat{R}_1^2 = \hat{\beta} + 8$ and $\hat{R}_1^3 = \hat{\gamma}$. The correction $r_1 = -8$ is arranged in such a way that things work.

**Example 17.** For $g = 2$, the quantum cohomology ring $QH^*(M_\Sigma)$ has been computed by Donaldson [3], using an explicit description of $M_\Sigma$ as the intersection of two quadrics in $\mathbb{P}^5$. Let $h_2$, $h_4$ and $h_6$ be the integral generators of $QH^2(M_\Sigma)$, $QH^4(M_\Sigma)$ and $QH^6(M_\Sigma)$, respectively. Then, with our notations, $\alpha = h_2$, $\beta = -4h_4$ and $\gamma = 4h_6$ (see [1]). The computations in [3] yield $\hat{\gamma} = \gamma - 4\alpha$. Now we set $r_2 = 4$, i.e. $\hat{\beta} = \beta + 4$. It is now easy to check that the relations found in [3] can be translated to the relations $\hat{R}_2^1 = \hat{\alpha}^2 + \hat{\beta} - 8$, $\hat{R}_2^2 = (\hat{\beta} + 8)\hat{\alpha} + \hat{\gamma}$ and $\hat{R}_2^3 = \hat{\alpha}\hat{\gamma}$ for $QH^1_1(M_\Sigma)$. 
The artificially introduced term $r_2 = 4$ is due to the same phenomenon which causes the failure of lemma 10 for $g = 2$, i.e. the quantum product $h^2$ differs from $\beta$ in (9) (defined with the cup product) because of a quantum correction in $QH^*(N)$ which appears when $g = 2$.

In the general case we have

**Theorem 18.** Let $g \geq 3$. Put $r_g = 0$, so that $\hat{\alpha} = \alpha$ and $\hat{\beta} = \beta$. Then we can write

$$
\begin{align*}
Q_g^1 &= \hat{R}_g^1 + f_g^1 \\
Q_g^2 &= \hat{R}_g^2 + f_g^2 \\
Q_g^3 &= \hat{R}_g^3 + f_g^3
\end{align*}
$$

where $\deg(f_g^i) \leq \deg(q_g^i) - 8$, $i = 1, 2, 3$.

**Proof.** As in equation (12), we can write $\hat{R}_g^i = \sum_{j \geq 0} \hat{R}_g^{i,j}$, where $\deg(\hat{R}_g^{i,j}) = \deg(q_g^i) - 4j$, $j \geq 0$ (and analogously for $R_g^i$). Clearly, $\hat{R}_g^{i,0} = q_g^i = Q_{g,0}^i$, for $i = 1, 2, 3$. We want to check that $Q_g^1 = \hat{R}_g^{1,1} = (-1)^g R_g^{1,1}$. Pick any $z = \alpha^a \beta^b \psi_{i_1} \cdots \psi_{i_r} \in \mathbb{A}(\Sigma)$ of degree $6g - 2 - \deg(q_g^1)$. By theorem 14, $\phi^w(\Sigma \times D^2, R_g^i) = 0$ (see [10] for notations), so $D_{S,H}^{\varphi^w}(R_g^i z) = 0$, i.e.

$$D_{S,H}^{\varphi^w}(R_g^{1,1} z) + D_{S,H}^{\varphi^w}(R_g^{0,0} z) = 0.$$ 

From theorem 11 and remark 13 this is translated as the component in $H^{6g-6}(M_\Sigma)$ of the quantum product $-\hat{R}_g^{1,1} z + (-1)^g R_g^{1,1} z \in QH^*(M_\Sigma)$ vanishing. On the other hand, by definition $Q_g^1 z = 0 \in QH^*(M_\Sigma)$, so the component in $H^{6g-6}(M_\Sigma)$ of the quantum product $Q_g^1 z + Q_g^{1,0} z \in QH^*(M_\Sigma)$ is zero. Thus $< Q_g^{1,1}, z > = (-1)^g R_g^{1,1} z$, for any $z$ of degree $6g - 2 - \deg(q_g^1)$, and hence $Q_g^{1,1} \equiv (-1)^g R_g^{1,1} (\mod I_g)$ ($I_g$ is the ideal defined in section 2). This gives the required equality $Q_g^{1,1} = \hat{R}_g^{1,1}$ (in the case $i = 3$ we have to use the vanishing of the coefficient of $\hat{\alpha}^g$ for both $Q_g^3$ and $R_g^3$).

---

6. **The case of genus $g = 3$**

It is natural to ask to what extent the first quantum correction determine the full structure of the quantum cohomology of $M_\Sigma$. In [3], Donaldson finds the first quantum correction for $M_\Sigma$ when the genus of $\Sigma$ is $g = 2$ and proves that this determines the quantum product. In this section we are going to check that this also happens for $g = 3$, finding thus the quantum cohomology of the moduli space of stable bundles over a Riemann surface of genus $g = 3$. 

**Proposition 19.** Let $\Sigma$ have genus $g = 3$. Then $Q_1^1 = \hat{R}_1^1$, $Q_2^3 = \hat{R}_3^3$ and $Q_3^3 = \hat{R}_3^3$, i.e.

$$QH^*_I(M_\Sigma) = \mathbb{Q}[\hat{\alpha}, \hat{\beta}, \hat{\gamma}]/(\hat{R}_1^1, \hat{R}_3^3, \hat{R}_3^3).$$

**Proof.** Theorem 18 says that

$$\begin{align*}
Q_1^1 &= \hat{R}_1^1 = \hat{\alpha}(\hat{\alpha}^2 + \hat{\beta} + 8) + 4(\hat{\alpha}\hat{\beta} - 8\hat{\alpha} + \hat{\gamma}) \\
Q_2^3 &= \hat{R}_3^3 + x = (\hat{\beta} + 8)(\hat{\alpha}^2 + \hat{\beta} + 8) + \frac{4}{3}\hat{\alpha}\hat{\gamma} + x \\
Q_3^3 &= \hat{R}_3^3 + y\hat{\alpha} = \hat{\gamma}(\hat{\alpha}^2 + \hat{\beta} + 8) + y\hat{\alpha}
\end{align*}$$

where $x, y \in \mathbb{Q}$. The main tool that we shall use is the nilpotency of $\hat{\gamma}$. Actually, from its definition (11), $\hat{\gamma}^4 = 0$ in $QH^*_I(M_\Sigma)$.

Suppose $y \neq 0$. Then the third relation implies $\hat{\alpha}^4 = 0$. Multiplying the first relation by $\hat{\alpha}$ we get $5\hat{\alpha}^2\hat{\beta} + 24\hat{\alpha}^2 + 4\hat{\alpha}\hat{\gamma} = 0$. This is a relation of degree $2g + 2 = 8$. Thus it must be a multiple of $Q_2^3$, which is not. This contradiction implies $y = 0$.

Let us see $x = 0$. It is easy to check that

$$\hat{\gamma}^3 = \frac{\hat{\gamma}^2}{4} \hat{R}_1^1 - \frac{3\hat{\gamma}(\hat{\beta} - 8)}{4} \hat{R}_3^3 = \frac{3(\hat{\beta} + 8)(\hat{\beta} - 8) - \hat{\alpha}\hat{\gamma}}{4} \hat{R}_3^3.$$

The relations above imply that $\hat{\gamma}^3 = -3\hat{\gamma}(\hat{\beta} - 8)x$ in $QH^*_I(M_\Sigma)$. So $\hat{\gamma}^2(\hat{\beta} - 8) = 0$. Also, the third relation (with $y = 0$) gives $\hat{\alpha}^2 = -\hat{\gamma}(\hat{\beta} + 8)$. Thus $\hat{\gamma}^2\hat{\alpha} = -16\hat{\gamma}^2$ and $\hat{\gamma}^2\hat{\beta} = 8\hat{\gamma}^2$. Multiply the first relation by $\hat{\gamma}^2$ to get $\hat{\gamma}^3 = 0$. As $\hat{\gamma}(\hat{\beta} - 8) \neq 0$ (because this is not a multiple of the only relation of degree $2g = 6$), it must be $x = 0$. \[\square\]

**Corollary 20.** Let $\Sigma$ be a Riemann surface of genus $g = 3$. Then

$$QH^*_I(M_\Sigma) = \bigoplus_{k=0}^{g-1} \Lambda^k_0H^3 \otimes \mathbb{Q}[\hat{\alpha}, \hat{\beta}, \hat{\gamma}]/\hat{I}_{g-k}$$

where we put $\hat{I}_r = (R_1^1(\sqrt{-1}^9\hat{\alpha}, \sqrt{-1}^2\hat{\beta}, \sqrt{-1}^{3g}\hat{\gamma}), R_2^3(\sqrt{-1}^9\hat{\alpha}, \sqrt{-1}^2\hat{\beta}, \sqrt{-1}^{3g}\hat{\gamma}), R_3^3(\sqrt{-1}^9\hat{\alpha}, \sqrt{-1}^2\hat{\beta}, \sqrt{-1}^{3g}\hat{\gamma})), 1 \leq r \leq g$.

**Proof.** This is an easy consequence of the former proposition and [10, lemma 7], noting that the hypothesis only has to be checked up to $g = 3$. One has to be careful with the exponents of $\sqrt{-1}$ everywhere. \[\square\]

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