Topical Review

Ground state, collective mode, phase soliton and vortex in multiband superconductors

Shi-Zeng Lin

Theoretical Division, Los Alamos National Laboratory, Los Alamos, NM 87545, USA
E-mail: szl@lanl.gov

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Abstract
This article reviews theoretical and experimental work on the novel physics in multiband superconductors. Multiband superconductors are characterized by multiple superconducting energy gaps in different bands with interaction between Cooper pairs in these bands. The discovery of prominent multiband superconductors MgB₂ and later iron-based superconductors, has triggered enormous interest in multiband superconductors. The most recently discovered superconductors exhibit multiband features. The multiband superconductors possess novel properties that are not shared with their single-band counterpart. Examples include: the time-reversal symmetry broken state in multiband superconductors with frustrated interband couplings; the collective oscillation of number of Cooper pairs between different bands, known as the Leggett mode; and the phase soliton and fractional vortex, which are the main focus of this review. This review presents a survey of a wide range of theoretical exploratory and experimental investigations of novel physics in multiband superconductors. A vast amount of information derived from these studies is shown to highlight unusual and unique properties of multiband superconductors and to reveal the challenges and opportunities in the research on the multiband superconductivity.

Keywords: multiband superconductivity, Leggett mode, phase soliton, fractional vortex, time-reversal symmetry breaking

(Some figures may appear in colour only in the online journal)

1. Introduction

The advent of the BCS theory (Bardeen et al 1957) provides a solid theoretical framework to understand various physical properties of superconductors. According to this theory, electrons near Fermi surface form Cooper pairs and condense into a macroscopic quantum state. Real superconducting materials usually involve multiple Fermi surfaces. It is possible that electrons/holes in each Fermi surface form superconducting condensate with interaction between them as a result of electron/hole hopping between different bands: see figure 1 for an example. These multiband superconductors, which were mostly found in transition metals in the 20th century, can be described by a multiband BCS theory (Moskalenko 1959, Suhl et al 1959), which was proposed shortly after the BCS theory. The research on multiband superconductors was reinvigorated by the discovery of MgB₂ with pronounced multiband characteristics in 2001 (Nagamatsu et al 2001). The discovery of multiband iron-based superconductors in 2008 added momentum to the study on multiband superconductors (Kamihara et al 2008). With advances in crystal growth, experimental measurements and theoretical modeling, many superconductors originally labeled as single-band superconductors were rediscovered as multiband superconductors. The discovery of these multiband superconductors has added a new dimension to the superconductivity research.

The physical properties of multiband superconductors deviate significantly from their single-band counterpart. This
deviation is usually a signature of multiband behavior in experiments. One line of research is to calculate physical properties with a more realistic multiband model by taking details of band structure, interactions and crystal structure into account. Moreover, multiband superconductors possess novel physics that are not shared by single-band superconductors. One famous example is the collective oscillation of number of Cooper pairs between different bands, known as the Leggett mode (Leggett 1966). Caution must be taken for temperatures close to superconducting transition temperature $T_c$. According to the Landau argument, it is sufficient to describe a symmetry broken phase with one order parameter near $T_c$ where only one symmetry is broken. This means that multiband superconductors with interband couplings behave as single-band superconductors for temperatures sufficiently close to $T_c$ (Geilikman et al. 1967, Kogan and Schmalian 2011). Nevertheless, there are pronounced multiband characteristics at low temperatures, as revealed by various measurements. In short, multiband superconductivity at low temperatures is not a straightforward extension of single-band superconductivity. Instead, new physics appear due to the multiband nature, which makes multiband superconductors interesting and promising for applications.

The present review is intended to give an overview of the novel physics in multiband superconductors. For the purpose of demonstration, we adopt a minimal model by focusing on isotropic $s$-wave multiband superconductors. Such an approach is legitimate, as we mainly focus on the qualitative new features in multiband superconductors. It is also very interesting when superconducting condensates in different bands have different pairing symmetries (Balatsky et al. 2000, Lee et al. 2009). In the following, we will review the ground state, collective excitation, phase soliton and vortex in multiband superconductors. As the phase soliton and vortex are topological objects, their overall properties should not depend on the microscopic details of the Hamiltonian. A review emphasizing the thermodynamic properties and materials realizations can be found in Zehetmayer (2013).

Superconductivity in each band can be described by a complex gap function $\Psi_j = \Delta_j \exp(i\phi_j)$. The phase differences between different bands are determined by the interband coupling. The interband coupling can be either attractive, which favors the same superconducting phase, or repulsive, which favors a $\pi$ phase shift. Frustration may arise in three or more bands superconductors. Without frustration, the phase difference is either 0 or $\pi$. With a strong frustration, it is possible for superconductors to break the $Z_2$ time reversal symmetry in addition to the $U(1)$ symmetry. In this case, the phase differences can take a value neither 0 nor $\pi$. This state without time-reversal symmetry was first considered by Agterberg et al. (1999) and later by Stanev and Tešanović (2010) in the context of iron-based superconductors. As a consequence of the time-reversal symmetry breaking, new phenomena such as the appearance of spontaneous magnetic fields in the presence of non-magnetic defects (Lin and Hu 2012b, Garaud and Babaev 2014), existence of a gapless Leggett mode (Lin and Hu 2012a) and phase solitons between two distinct time-reversal symmetry broken systems emerge (Garaud et al. 2011, Lin and Hu 2012b).

Superconductivity as a consequence of symmetry breaking allows for the existence of several collective modes (Kulik et al. 1981, Littlewood and Varma 1982). One is the Goldstone mode associated with the breaking of $U(1)$ continuous symmetry and in the context of superconductors is known as the Bogoliubov–Anderson–Goldstone boson (Anderson 1958, Bogoliubov 1959). This mode becomes a gapped plasma mode when coupled to electromagnetic fields due to the Anderson–Higgs mechanism (Anderson 1963, Higgs 1964). Near $T_c$, the Bogoliubov–Anderson–Goldstone mode with electromagnetic fields is not pushed up to the plasma frequency because the conversion rate between normal current and supercurrent is slow. In the two-fluid model picture, the normal current and supercurrent oscillate in order to maintain charge neutrality. This mode is called the Carlson–Goldman mode (Artemenko and Volkov 1975, Schmid and Schön 1975) and has been observed in Al thin film near $T_c$ (Carlson and Goldman 1975). The collective oscillation of the amplitude of the order parameter $\Delta$ is known as the Schmid mode (Schmid 1968). It has a gap of $2\Delta$ and can be regarded as a Higgs boson. The amplitude mode has been observed in the $2H$-NbSe$_2$ superconductor (Sooryakumar and Klein 1980, Littlewood and Varma 1981). For multiband superconductors, besides the modes mentioned above, it hosts another collective excitation associated with oscillation of a number of Cooper pairs between different bands due to the interband coupling, known as the Leggett mode. This mode corresponds to a small out-of-phase oscillation of the phase mode in different bands and was first highlighted by Leggett (1966). The Leggett mode is gapped. The Leggett mode has been observed experimentally in MgB$_2$ (Blumberg et al. 2007). The multiband superconductors without time-reversal symmetry have...
significant effects on the collective modes. In the time-reversal symmetry broken state, the phase mode hybridizes with the amplitude mode and forms a phase-amplitude composite mode (Stanev 2012). At the time-reversal symmetry breaking transition, one of the Leggett modes becomes gapless (Lin and Hu 2012a).

Besides the small out-of-phase oscillation in multiband superconductors, there exist phase soliton excitations in the superconducting phase difference between different condensates. For a Josephson-like interband coupling, there are energy degenerate ground states for the phase difference; it also allows for the phase soliton between any pair of the degenerate ground states. Such a phase soliton unique to multiband superconductors was first discussed by Tanaka (2001), and has been observed experimentally in an artificial multiband superconductor (Bluhm et al. 2006). In multiband superconductors without time-reversal symmetry, one can also have phase solitons between two time-reversal symmetry broken states, similar to the domain walls in ferromagnets. The phase soliton can only be stable in 1D systems, but it could be stabilized by defects or vortices in higher dimensions (Garaud et al. 2011, Garaud and Babaev 2014). Inside the phase soliton, the superconducting phase differences are neither 0 or π and therefore the time-reversal symmetry is broken locally. Spontaneous magnetic fields can exist in the phase soliton region under proper conditions (Lin and Hu 2012b, Garaud and Babaev 2014). As with the Leggett mode, the phase soliton is neutral and does not couple to magnetic fields. However, the phase soliton can be excited dynamically by an electric field in nonequilibrium region (Gurevich and Vinokur 2003).

Another hallmark of the U(1) symmetry breaking in superconductors is the existence of vortex carrying quantized magnetic flux \( n \Phi_0 = n \hbar c/(2e) \), with the integer \( n \) being the phase winding number. In multiband superconductors, the phase winding number for superconducting condensates in different bands may be different, which results in vortex carrying fractional quantum flux. This fractional vortex was first studied by Babaev (2002). Fractional vortices with the same polarization in the same band interact repulsively due to the magnetic interaction. The fractional vortices in different condensates also repel each other due to the exchange of massive photon. In addition, they attract each other due to the coupling to the same gauge field (Lin and Bulaevskii 2013). They also attract because of the interband coupling in superconducting channel. The attraction outweighs the repulsion and the net interaction of fractional vortices in different condensates is attractive. Therefore, fractional vortices in different condensates in the ground state bind together with their normal cores locked together to form a composite vortex with the standard integer quantum flux \( \Phi_0 \). In the flux flow region with a high current drive, the composite vortex lattice can dissociate into fractional vortex lattices with different velocities because of the disparity in the vortex viscosity and magnetic flux of the fractional vortices in different bands (Lin and Bulaevskii 2013). After turning off the current, the fractional vortices can be trapped when pinning centers are present, which therefore results in metastable fractional vortices (Lin and Reichhardt 2013). The fractional vortices can also be stabilized in a mesoscopic multiband superconductor (Chibotaru et al. 2007, Chibotaru and Dao 2010, Geurts et al. 2010, Pereira et al. 2011, Piña et al. 2012, Gillis et al. 2014). Because of the possible existence of distinct length scales for condensates in different bands at lower temperatures, vortex may interact repulsively at short distance, attractively at intermediate distance and repulsively at large distance, due to the demagnetization effect. The existence of nonmonotonic inter-vortex interaction in multiband superconductors was first discussed by Babaev and Speight (2005). This kind of vortex interaction leads to unusual magnetic response in multiband superconductors, which differs from the conventional type I and type II superconductors.

The remainder of this review is organized as follows. In section 2 we will introduce the models, discuss the ground state properties, behavior near \( T_c \) and material realizations. Section 3 is devoted to the Leggett modes. In section 4 the phase solitons are discussed and in section 5 we will review vortices in multiband superconductors. The paper is concluded by discussions in section 6.

2. Model and ground state

In this section, we will introduce the isotropic Ginzburg–Landau free energy functional and BCS Hamiltonian. Their relation will be discussed. We then will show that multiband superconductors undergoing a single \( U(1) \) symmetry breaking at \( T_c \) behave as a single-band superconductor near \( T_c \). We will present a zero magnetic field phase diagram for multiband superconductors, focusing on three-band superconductors with frustrated interband couplings where time-reversal symmetry may be broken inside the superconducting phase. Finally, material realizations of multiband superconductivity will be reviewed.

2.1. Model

Here we introduce models to describe multiband superconductors. A phenomenological description can be obtained by generalizing the single-band Ginzburg–Landau free energy functional to multiband case. Such as a phenomenological theory may be broken inside the superconducting phase. Finally, material realizations of multiband superconductivity will be reviewed.

\[
\mathcal{F} = \sum_j \left[ \alpha_j |\Psi_j|^2 + \frac{\beta_j}{2} |\Psi_j|^4 + \frac{1}{2m_j} \left| \left( -i \hbar \nabla - \frac{2e}{c} A \right) \Psi_j \right|^2 \right] + \frac{1}{8\pi} (\nabla \times A)^2 + \sum_{i<j} \gamma_{ij} (\Psi_i^\dagger \Psi_j^\dagger + \Psi_i \Psi_j). \tag{1}
\]
Here $\alpha_j$ depends on temperature, while $\beta_j$ and the interband Josephson-like coupling $\gamma_{ij}$ are temperature-independent. Here $m_j$ is the electron effective mass. The supercurrent density is

\[
J_s = \sum_j \left[ -\frac{i e}{m_j} \left( \nabla \psi_j - \psi_j \nabla \psi_j^* \right) - \frac{4e^2}{m_j^* c} |\psi_j|^2 \mathbf{A} \right]. \tag{2}
\]

We have assumed a Josephson-like interband coupling. Other forms of coupling such as coupling between superfluid density, $|\psi_j|^2$, can also exist (Gurevich 2007). In addition to the interband Josephson coupling, the superconducting phases in different bands couple to the same gauge field $\mathbf{A}$. When interband coupling is absent $\gamma_{ij} = 0$, we can define a coherence length $\xi_i$ for each band,

\[
\xi_i = \frac{\hbar^2}{2m_i |\alpha_i|}. \tag{3}
\]

Since only one gauge field is involved in equation (1), there is only one London penetration depth

\[
\lambda^{-2} = \sum_i \lambda_i^{-2}, \tag{4}
\]

with the parameter $\lambda_i = \sqrt{m_i c^2 / (16\pi e^2 \psi_{i0}^2)}$ and the uniform amplitude of the order parameter $\psi_{i0} = \sqrt{|\alpha_i|/|\beta_i|}$. The interband coupling mixes different condensates and $\xi_i$, $\psi_{i0}$ need to be redefined. In the strong coupling limit, $|\gamma_{ij}| \gg |\alpha_i|$, $\xi_i$ for different bands becomes the same.

The dynamics of superconductivity can be described by the time-dependent Ginzburg–Landau theory

\[
\frac{\hbar^2}{2m_j D_j} \left( \partial_t + \frac{2e}{\hbar} \nabla \psi_j \right) \Psi_j = -\frac{\delta F}{\delta \Psi_j^*}, \tag{5}
\]

\[
\frac{\sigma}{c} \left( \frac{1}{c} \nabla A + \nabla \psi \right) = -\frac{\delta F}{\delta A}, \tag{6}
\]

with $D_j$ the diffusion constant, $\sigma$ the normal conductivity and $\psi$ the electric potential. The time-dependent Ginzburg–Landau equation can be derived from a microscopic theory near $T_c$.

The generalized BCS model for multiband superconductors has the form Suhl et al (1959) and Moskalenko (1959)

\[
\mathcal{H} = \sum_{l,\sigma} \int d^3 r \psi_{l\sigma}^\dagger(r) \left( \epsilon_l - \mu \right) \psi_{l\sigma}(r)
- \sum_{j,l} \int d^3 r \psi_{j\sigma}^\dagger(r) \psi_{j\sigma}(r) V_{jl} \psi_{l\sigma}(r) \psi_{l\sigma}(r), \tag{7}
\]

where $\psi_{l\sigma}^\dagger$ ($\psi_{l\sigma}$) is the electron creation (annihilation) operator in the $l$-th band with the dispersion $\epsilon_l(k)$ and the chemical potential $\mu$ and spin index $\sigma$. We consider a parabolic dispersion for electrons $\epsilon_l(k) = \hbar^2 k^2 / 2m_l$ with an electron mass $m_l$. $V_{jl}$ is the intraband for $l = j$ and interband for $l \neq j$ scattering respectively, which can be either repulsive or attractive depending, for instance, on the strength of the Coulomb and electron–phonon interaction. Here we have assumed a contact interaction for $V_{jl}$. Equation (7) reduces to equation (1) at a temperature close to $T_c$ in the clean limit, i.e. $(T_c - T) / T_c \ll 1$ (Tilley 1964, Zhitomirsky and Dao 2004). In the dirty limit, the interband impurity scattering induces additional coupling between different bands, other than the Josephson coupling in equation (1) (Gurevich 2007). The applicability region of equation (1) near $T_c$ depends on materials. It was argued in Koshelev and Golubov (2004) and Koshelev et al (2005) that the applicability region of equation (1) shrinks practically to zero for MgB$_2$.

2.2. Behavior of multiband superconductors at temperatures close to $T_c$

Here we discuss the behavior of multiband superconductors near $T_c$ when interband Josephson couplings are present. We restrict discussion to the case in which there is only a single continuous phase transition associated with the breaking of $U(1)$ symmetry at $T_c$. We will show that in this region the multiband superconductors behave as single-band superconductors, i.e. there exists only one coherence length for all superconducting condensates. This was realized a long time ago by Geilikman et al (1967) and later independently by Kogan and Schmalian (2011). We follow the derivation in Kogan and Schmalian (2011) and Geyer et al (2010). As an example, we consider a two-band isotropic Ginzburg–Landau free energy functional in equation (1).

The critical temperature $T_c$ for equation (1) is given by the condition that the determinant of the coefficient matrix for the quadratic terms $\Psi_i^\dagger \Psi_i$ is zero. For a two-band superconductor, it is given by $\alpha_1(T_c) \alpha_2(T_c) - \gamma_{12}^2 = 0$. We denote $\alpha_1 \equiv \alpha_1(T_c)$ and $\alpha_2 \equiv \alpha_2(T_c)$. The interband Josephson coupling enhances $T_c$ (Kondo 1963), i.e. superconductivity can exist even when $\alpha_i(T) > 0$. In this sense, superconductivity near $T_c$ is induced by interband coupling. This is the reason why condensates in different bands are strongly locked with each other and hence effectively become a single-band superconductor. Using $\delta F / \delta \Psi_i^\dagger = 0$, we obtain

\[
\alpha_1 \Psi_1 + \beta_1 |\Psi_1|^2 \Psi_1 + \frac{1}{2m_1} \left( -i \hbar \nabla - \frac{2e}{c} \mathbf{A} \right)^2 \Psi_1 + \gamma_{12} \Psi_2 = 0, \tag{8}
\]

\[
\alpha_2 \Psi_2 + \beta_2 |\Psi_2|^2 \Psi_2 + \frac{1}{2m_2} \left( -i \hbar \nabla - \frac{2e}{c} \mathbf{A} \right)^2 \Psi_2 + \gamma_{12} \Psi_1 = 0. \tag{9}
\]

In the Ginzburg–Landau approximation, equations (8) and (9) are valid up to the order $\tau^{3/2}$ with $\tau \equiv (T_c - T) / T_c \ll 1$. Keeping terms up to $\tau^{1/2}$, equations (8) and (9) can be rewritten as (Kogan and Schmalian 2011)

\[
(\alpha_1 \alpha_2 - \gamma_{12}^2) \Psi_1 + (\beta_1 \alpha_2 + \beta_2 \alpha_1 \gamma_{12}^2) |\Psi_1|^2 \Psi_1
- \left( \frac{\alpha_1 \hbar^2}{2m_1} + \frac{\alpha_2 \hbar^2}{2m_2} \right) \left( \nabla - i \frac{2e}{c} \mathbf{A} \right)^2 \Psi_1 = 0, \tag{10}
\]

\[
(\alpha_1 \alpha_2 - \gamma_{12}^2) \Psi_2 + (\beta_1 \alpha_2 + \beta_2 \alpha_1 \gamma_{12}^2) |\Psi_2|^2 \Psi_2
- \left( \frac{\alpha_1 \hbar^2}{2m_1} + \frac{\alpha_2 \hbar^2}{2m_2} \right) \left( \nabla - i \frac{2e}{c} \mathbf{A} \right)^2 \Psi_2 = 0. \tag{11}
\]
Close to $T_c$, we expand $\alpha_i(T) = \alpha_i c - \tilde{\alpha}_i \tau$. Equations (10) and (11) become

$$-\tilde{\alpha}_i \Psi_1 + \tilde{\beta}_1 |\Psi_1|^2 \Psi_1 - \tilde{K} \left( \nabla - \frac{2\pi}{\Phi_0} \right)^2 \Psi_1 = 0,$$

$$-\tilde{\alpha}_i \Psi_2 + \tilde{\beta}_2 |\Psi_2|^2 \Psi_2 - \tilde{K} \left( \nabla - \frac{2\pi}{\Phi_0} \right)^2 \Psi_2 = 0. \tag{13}$$

with

$$\tilde{\alpha} = \alpha_1 \alpha_2 + \alpha_2 \alpha_1, \quad \tilde{K} = \frac{\hbar^2 \alpha_1}{2m_2} + \frac{\hbar^2 \alpha_2}{2m_1},$$

$$\tilde{\beta}_1 = \beta_1 \alpha_2 + \beta_2 \alpha_1, \quad \tilde{\beta}_2 = \beta_2 \alpha_2 + \beta_1 \alpha_1 \gamma_1^2.$$

Equation (13) reduces to equation (12) if we replace $\Psi_2(r, T) \rightarrow \Psi_1(r, T) \sqrt{\tilde{\beta}_1/\tilde{\beta}_2}$, i.e. $[\Psi_1(r, T), \Psi_2(r, T)] = [\Psi_1(r, T), \Psi_2(r, T)] = [\Psi_1(r, T), \Psi_2(r, T)]$ is always a solution to equations (8) and (9) near $T_c$. The superconducting order parameters for different bands vary in space with the same conductivity in liquid hydrogen under high pressure (Ashcroft 1968, Jaffe and Ashcroft 1981), the electron and proton superconducting condensates can have two distinct length scales close to $T_c$.

If another symmetry, such as time-reversal symmetry, is broken simultaneously with $U(1)$ symmetry at $T_c$, (see figure 3), the superconductor can have two diverging length scales associated with the breaking of two symmetries. This was demonstrated in Hu and Wang (2012) for three-band superconductors, where the time-reversal symmetry and $U(1)$ symmetry break at the same time at $T_c$.

We cannot rule out the possible existence of many length scales for different condensates at low temperatures, since the Ginzburg–Landau theory cannot be applied to this region. Also, the above derivation is valid for $\tau \ll 1$. In fact, it was shown recently using a microscopic approach that there exists distinct length scales for different condensates at low temperatures (Silaev and Babaev 2011), which thus allows for the emergence of novel properties unique to multiband superconductors.

### 2.3. Ground state and phase diagram

In this subsection we discuss the zero-field phase diagram for equations (1) and (7). For the Ginzburg–Landau theory in equation (1), $\Psi_j$ can be obtained readily using the condition $\delta F/\delta \Psi_j^\dagger = 0$. Here we determine the phase diagram using the BCS Hamiltonian in equation (7) because it is valid at all temperatures. Let us first introduce the general gap equations for multiband superconductors without external magnetic fields.

Introducing the energy gap $\Psi_j$ through the Hubbard–Stratonovich transform and the Nambu spinor operator $\hat{\Theta}_j = (\Psi_j, \Psi_j^\dagger)^T$, we obtain the following action for equation (7) in the imaginary time representation after integrating out the fermionic fields (Alexander and Simons 2010)

$$S = \int d\tau d^3 r \sum_{j,l} M \Psi_j \dot{\Psi}_j - \sum_j \text{Tr} \ln \mathcal{G}_j^{-1}, \tag{14}$$

with $\hat{\Theta} = \hat{\Psi}^{-1}$, $M$ the number of bands and the Gor’kov green function

$$\mathcal{G}_j^{-1} = - \left( \begin{array}{cc} \dot{\Psi}_j + (\epsilon_j - \mu) & -\Psi_j \\ -\Psi_j^\dagger & -\dot{\Psi}_j^\dagger \end{array} \right). \tag{15}$$

Here $\hat{\Psi}$ is a matrix form of interaction in equation (7). The ground state $\Psi_j = \Delta_j \exp(\hat{\theta}_j)$ is given by the condition $\delta S/\delta \Psi_j = 0$, which yields

$$\sum_{j=1}^M \Psi_j \dot{g}_{j,l} = N_j(0) F_j(\Psi_j, T), \tag{16}$$

with

$$F_j(\Psi_j, T) \equiv \int_0^{\omega_{c,j}} \frac{d\xi}{\sqrt{\xi^2 + |\Psi_j|^2}} \tanh \left[ \frac{\sqrt{\xi^2 + |\Psi_j|^2}}{2k_B T} \right]. \tag{17}$$

with $\omega_{c,j}$ a cutoff frequency, which depends on the pairing mechanism. For phonon mediated superconductivity, $\omega_{c,j}$ is the Debye frequency. Here $N_j(0)$ is the density of states at the Fermi surface in normal state.

It is particularly interesting when the interband interactions are frustrated and the system may break time-reversal symmetry in addition to the $U(1)$ symmetry (Agterberg et al 1999, Stanev and Tešanović 2010). We consider a three-band case since it is a minimal model to demonstrate the time-reversal symmetry breaking. We also focus on the case with identical density of state $N_j(0) = N(0)$ and cutoff frequency $\omega_{c,j} = \omega_c$ at $T = 0$. Here $F(\Psi_j, T = 0) = \sinh^{-1}(\omega_{c,j}/|\Psi_j|)$. We also take a set of simplified interband couplings

$$\hat{g} = \frac{1}{V_0} \left( \begin{array}{ccc} \alpha & 1 & 1 \\ 1 & \alpha & \eta \\ 1 & \eta & \alpha \end{array} \right). \tag{18}$$

Here $\alpha > 0$ and $\eta > 0$ correspond to a repulsive interaction. We can always take $\Psi_1 = \Delta_1$ as real by properly choosing the gauge. As $\hat{g}$ is symmetric, the solution for $\Psi_2$ and $\Psi_3$ can be written as $\Psi_2 = \Delta \exp(i\phi) = \Psi_1^\dagger$. For a small $\eta \ll 1$, the repulsion between $\Psi_1$ and $\Psi_2$ (or $\Psi_3$) dominates over the repulsion between $\Psi_2$ and $\Psi_3$ and $\phi = \pi$. As $\eta$ increases, the repulsion between $\Psi_2$ and $\Psi_3$ becomes more important and at a critical $\eta$, $\phi$ starts to deviate from $\pi$, which breaks
time-reversal symmetry. In the state with time-reversal symmetry, $\phi = \pi$ and $\Delta_1$, $\Delta_2$ are given by

$$\frac{\alpha \Delta_1 - 2 \Delta}{N(0)V_0} = \Delta_1 \sinh^{-1} \left( \frac{\omega_c}{\Delta_1} \right)$$

(19)

$$\frac{\Delta_1 - (\alpha + \eta) \Delta}{N(0)V_0} = -\Delta \sinh^{-1} \left( \frac{\omega_c}{\Delta} \right).$$

(20)

In the state without time-reversal symmetry, $\Delta$, $\Delta_1$ are given by

$$\frac{\Delta}{\omega_c} = 1/\sinh \left( \frac{\alpha - \eta}{N(0)V_0} \right);$$

$$\frac{\Delta_1}{\omega_c} = 1/\sinh \left( \frac{\alpha \eta - 1}{\eta N(0)V_0} \right).$$

(21)

The time-reversal symmetry breaking occurs at

$$\cos \tilde{\phi} = -\Delta_1/(2\eta \Delta) = -1.$$  

(22)

The results for $\Psi_j$ as a function of $\eta$ are displayed in figures 2(a) and (b) for $N(0)V_0 = 0.5$ and $\alpha = 2$, where there is a continuous phase transition associated with the breaking of time-reversal symmetry.

The time-reversal symmetry breaking phase transition can also be driven by $N_j(0)V_0$, which can be tuned in experiments by careful chemical doping. As an example, we calculate $\Psi_j$ as a function of $N_j(0)V_0$ for $N_1(0)V_0 = 0.5$, $N_3(0)V_0 = 0.4$, $\alpha = 2$, $\eta = 1$. As displayed in figures 2(c) and (d), the time-reversal symmetry broken phase is stabilized in an intermediate region of $N_j(0)V_0$.

The associated free energy density at $T = 0$ is

$$\mathcal{F} = \sum_{ij} \Psi_i g_{ij} \Psi_j^* - \sum_j N_j(0) |\Psi_j|^2 \left[ \frac{1}{2} + \ln \left( \frac{2|\omega_{ij}|}{|\Psi_j|^2} \right) \right].$$

(23)

and it can be verified that the time-reversal symmetry broken state does indeed have lower free energy, thus is a thermodynamically stable phase.

The phase diagram at $T > 0$ can be obtained numerically. The $T$-$\eta$ phase diagram for a symmetric coupling $g$ in equation (18) with $N_j(0)V_0 = 0.5$ and $\alpha = 2$ is shown in figure 3. For a large $g_3$ = $\eta \gg 1$ at $T > 0$, the superconductivity in the first band is strongly frustrated, resulting in $\Psi_1 = 0$ and the system behaves as a two-band superconductor. The region without time-reversal symmetry...
shrinks when $T$ approaches to $T_c$ and contracts into a point at $\eta^*$ at $T_c$. At $T_c$ and $\eta^*$, the $U(1)$ and $Z_2$ symmetries are broken simultaneously.

One characteristic consequence of time-reversal symmetry breaking in most systems is the appearance of spontaneous magnetic fields. For homogeneous multiband superconductors without time-reversal symmetry, there is interband Josephson current flowing between different bands: $J_{jl} \propto \gamma_{jl} \sin(\phi_j - \phi_l)$ for the state $\Psi \equiv (\Psi_1, \Psi_2, \ldots, \Psi_M)$, while $J_{jl} \propto -\gamma_{jl} \sin(\phi_j - \phi_l)$ for the state $\Psi^* \equiv (\Psi_1^*, \Psi_2^*, \ldots, \Psi_M^*)$, as sketched in figure 4(a). In this sense, the ground is chiral, as demonstrated by the direction of the interband Josephson current. The interband Josephson current occurs in the band space and does not couple to gauge fields. Therefore, the circulation of interband Josephson current does not generate spontaneous magnetic fields for homogeneous multiband superconductors without time-reversal symmetry. When inhomogeneities exist due to non-magnetic impurities, proximity effect, can be used to detect the time-reversal symmetry breaking in multiband superconductors with frustrated interband couplings.

The ground state in two-band isotropic $s$-wave superconductors is very simple. The phase difference between two gap functions $\Psi_1$ and $\Psi_2$ can be either $0$ or $\pi$, depending on the sign of interband coupling $g_{12}$ or $g_{21}$. For an attractive interband coupling $\gamma_{12} < 0$, there is no phase difference between $\Psi_1$ and $\Psi_2$, while for a repulsive interband coupling $\gamma_{12} > 0$, the system favors $s\pm$ pair symmetry, with a $\pi$ phase shift between $\Psi_1$ and $\Psi_2$. A typical dependence of the amplitude of the energy gap for different bands on temperature is shown in the inset of figure 1, where all gaps vanish at the same $T_c$ due to the interband couplings.

Recently, a general classification of the ground states for phase-frustrated multiband superconductors using a graph-theoretical approach was reported by Weston and Babaev (2013).

So far we have adopted the mean-field approximation. The phase diagram for $s$-wave three-band superconductors with frustrated interband couplings was calculated beyond the mean-field approximation by Monte Carlo simulations (Bojesen et al 2013, 2014, Bojesen and Sudbø 2014). A novel phase with $U(1)$ symmetry, but without $Z_2$ (time-reversal symmetry) symmetry, was found. In the $U(1)$ and $Z_2$ symmetry broken phase, the proliferation of vortex and antivortex restores the $U(1)$ symmetry and the proliferation of phase soliton recovers the $Z_2$ symmetry. The former transition belong the $XY$ universality class and the latter belongs to the Ising universality class. It was found in certain parameter space that the energy cost for the vortex proliferation is lower than that for phase soliton proliferation. In this case, the $U(1)$ symmetry is restored prior to $Z_2$ symmetry upon increasing temperatures. Therefore a new dissipative metallic phase with $U(1)$ symmetry, but without time-reversal symmetry, appears.

The multiband nature also has profound effects on the magnetic-field-temperature $H$-$T$ phase diagram. One example is in the case of superconductors with a nonmonotonic inter-vortex interaction, as will be discussed in section 5.6. The upper critical field $H_{\text{c}2}(T)$ as a function of $T$ is particularly interesting from the experimental point of view. The dependence $H_{\text{c}2}(T)$ for multiband superconductors differs from the single-band case (Askerzade et al 2002, Gurevich 2003, 2010, 2011). One can extract microscopic parameters by fitting the measured $H_{\text{c}2}(T)$ to a theoretical model.
2.4. Material realizations of multiband superconductivity

Most superconductors have multiple Fermi surfaces, where electrons/holes form superconducting condensates below $T_c$. Therefore, multiband superconductors are ubiquitous and, strictly speaking, most superconductors can be labeled as multiband superconductors. However, in most cases, superconductivity in these superconductors is dominant by one band and the superconductor behaves as a single-band superconductor. In this subsection, we will present several typical examples of multiband superconductors. The list nevertheless is incomplete; see Zehetmayer (2013) for more discussions.

Some binary compounds were found, some time ago, to exhibit prominent multiband superconductivity, such as NbSe$_2$ (Yokoya et al. 2001), V$_2$Si (Nefyodov et al. 2005), ZrB$_{12}$ (Gasparov et al. 2006). It was found from the microwave surface impedance and complex conductivity measurements that interband coupling for V$_2$Si is extremely weak (Yokoya et al. 2001) and V$_2$Si could serve as an arena to observe the decoupling of phases of superconducting order parameters discussed below. The revival of the research on multiband superconductivity can, to some extent, be attributed to the discovery of MgB$_2$ with $T_c \approx 39$ K (Nagamatsu et al. 2001). Superconductivity in MgB$_2$ is mediated by phonons. MgB$_2$ is well characterized due to intensive studies in the past decade (Xi 2008, 2009). Most of its superconducting properties can be described by a two-band $s$-wave superconductor model. The energy gap for the $\sigma$ band is about $\Delta_\sigma = 5.5$–6.5 meV and for the $\pi$ band is $\Delta_\pi = 1.5$–2.2 meV (Szabó et al. 2001, Tsuda et al. 2001, Iavarone et al. 2002, Souma et al. 2003). The interband coupling matrix $V$ has been obtained using the first-principle calculations (Liu et al. 2001, Choi et al. 2002). The reported interband coupling ranges from weak to intermediate coupling. The phases of superconducting order parameters in the two bands are the same, which requires $\gamma_{12} < 0$ in equation (1) and $g_{12} < 0$ in equation (14).

The discovery of iron-based superconductors (Kamihara et al. 2008) has attracted growing interest in the study of multiband superconductivity. There are large families of iron-based superconductors and their Fermi surface topology and pairing symmetry vary: see Ishida et al. (2009), Paglione and Greene (2010), Johnston (2010), Wen and Li (2011), Wang and Lee (2011), Hirschfeld et al. (2011), Stewart (2011) and Dagotto (2013) for a review. Most of them have five bands that contribute to superconductivity (Ding et al. 2008). A simplified two-band model has been proposed to account for superconductivity in these materials (Raghu et al. 2008). Theories predict that the phases of superconducting order parameters change sign between bands with a full gap in each band and the pairing symmetry is denoted $s \pm$ (Kuroki et al. 2008, Mazin et al. 2008). Much experimental evidence supports the $s \pm$ pairing symmetry. This pair symmetry can be modeled by the simplified models in equations (1) and (7) with $\gamma_{12} > 0$ in equation (1) and $g_{12} > 0$ in equation (14).

The compound Ba$_{1-x}$K$_x$Fe$_2$As$_2$ has attracted a lot of attention recently. It was revealed by various measurements near the optimal doping $x = 0.4$ that the superconducting gaps at the two $\Gamma$-centered hole pockets are fully gapped and have the same sign (Christianson et al. 2008, Ding et al. 2008, Khasanov et al. 2009, Luo et al. 2009, Nakayama et al. 2011, Reid et al. 2012a). The gap at the electron pockets
has a $\pi$ phase shift with respect to the gaps at hole pockets. This pairing symmetry is denoted as $s+\pi$. At $x = 1$, it was found from ARPES measurements that only hole pockets exist (Sato et al. 2009, Okazaki et al. 2012). Both the $d$-wave pairing and $s$-wave pairing were proposed for the $x = 1$ case (Thomale et al. 2009, 2011, Maiti et al. 2011, 2012, Suzuki et al. 2011). For the $s$-wave pairing, the energy gaps are largest at the two $\Gamma$-centered hole pockets and they have a $\pi$ phase difference. We denote this pairing symmetry as $s\pm$. The existing experimental data either favor the $d$-wave pairing or $s$-wave pairing (Okazaki et al. 2012, Reid et al. 2012, Abdel-Hafiez et al. 2013, Wang et al. 2014). Maiti and Chubukov assumed the $s$-wave pairing for the $x = 1$ case (Maiti and Chubukov 2013). The pairing symmetry of Ba$_{1-x}$K$_x$Fe$_2$As$_2$ changes from $s \pm \pi$ to $s\pm$ when $x$ is increased. This transition is possible through an intermediate $s + is$ state which breaks the time-reversal symmetry. To describe this transition, a three-band Hamiltonian with frustrated interband coupling is needed. They solved the three-band Hamiltonian and found a phase diagram similar to that in figure 3. Thus Ba$_{1-x}$K$_x$Fe$_2$As$_2$ is a promising arena to test the time-reversal symmetry broken state and the related novel physics.

Several heavy fermion superconductors were shown to exhibit multiband superconductivity, such as UNi$_2$Al$_3$ (Jourdan et al. 2004), PrOs$_4$Sb$_{12}$ (Seyfarth et al. 2005), URu$_2$Si$_2$ (Kasahara et al. 2007), CePt$_3$Si (Mukuda et al. 2009). The recently discovered Bi$_2$S$_3$ based superconductors (Mizuguchi et al. 2012), such as LaO$_{1-x}$F$_x$Bi$_2$S$_2$, were also revealed to exhibit multiband characteristics.

Multiband superconductivity may also exist in the proposed liquid hydrogen under high pressure, where both protons and electrons contribute to superconductivity (Ashcroft 1968, Jaffe and Ashcroft 1981). In this case, the interband Josephson tunneling is absent, $\gamma_{12} = 0$. Moreover, there is a large disparity between the electron and proton superconducting condensate due to the huge mass difference. Methods to identify this hypothetical novel metallic superfluid phase are proposed in Babaev et al. (2005) and tested numerically in Smørbø and Smørvag et al. (2005).

3. Collective mode: the Leggett mode

Having determined the ground states of multiband superconductors, in this section we will investigate the collective excitations in the ground state. We will first present the basic concept of the Leggett mode and give a phenomenological description based on the phase of superconducting order parameters or the number of Cooper pairs. Then we will provide a microscopic derivation of the Leggett mode in a two-band superconductor. In multiband superconductors undergoing time reversal symmetry breaking phase transition, we will show the existence of a gapless Leggett mode at the transition point. At the end of this section, we will discuss the detection of the Leggett mode by Raman spectroscopy and review the experimental observations of the Leggett mode in MgB$_2$. Possible observation of the Leggett mode by measurements of thermodynamical quantities will also be discussed.

3.1. Basic concept

In multiband superconductors, electrons/holes in different bands form superconducting condensate, which can be described by a complex gap function $\Psi_j = \Delta_j \exp(i\phi_j)$. Because electrons/holes can hop between different bands, the number of Cooper pairs in different bands fluctuates. The collective oscillation of the Cooper pairs between different bands was first discussed by Leggett in 1966; this is now known as the Leggett mode (Leggett 1966). The number of Cooper pairs $n_c$ and phase $\phi$ are conjugate variables satisfying the uncertain relation $\Delta n_c, \Delta \phi \geq 1$. The collective oscillation of Cooper pairs between different bands can therefore be described in terms of the superconducting phase difference between different bands $\phi_j - \phi_l$. The dispersion for the Leggett mode can be obtained using a phenomenological approach where each band is described by the Lagrangian (Alexander and Simons 2010),

$$\mathcal{L}_{\phi_j} = \hbar^2 N_j(0)(\partial_t \phi_j)^2 - \frac{\hbar^2}{2m_j} (\nabla \phi_j)^2,$$

where $\Psi_j^2$ is the superfluid density with dimension 1/volume. Here we consider a two-band superconductor without coupling to electromagnetic fields. The interband Josephson coupling is

$$\mathcal{L}_J = -2\gamma_{12} \Psi_{\phi_1} \Psi_{\phi_2} \cos(\phi_1 - \phi_2).$$

The collective modes for equations (25) and (26) in the long wavelength limit are

$$\Omega^2_{\text{BAG}} = \frac{1}{N_1(0) + N_2(0)} \left( \frac{\Psi_{\phi_1}^2}{2m_1} + \frac{\Psi_{\phi_2}^2}{2m_2} \right) q^2,$$

$$\Omega^2_{\text{L}} = \frac{N_1(0)N_2(0)}{N_1(0) + N_2(0)} \left( \frac{\Psi_{\phi_1}^2N_2(0)}{2m_1N_2(0)} + \frac{\Psi_{\phi_2}^2N_1(0)}{2m_2N_2(0)} \right) q^2.$$

The first mode $\Omega_{\text{BAG}}$ is the gapless Bogoliubov–Anderson–Goldstone mode associated with the in-phase oscillations of phases $\phi_1$ and $\phi_2$ (Anderson 1958, Bogoliubov 1959). The second mode is the Leggett mode corresponding to the out of phase oscillation of phase $\phi_1 - \phi_2$ (Leggett 1966). The Leggett mode $\Omega_L$ is gapped with a gap being proportional to the interband coupling.

One can also adopt a hydrodynamic description based on the number of Cooper pairs assuming that the total number of Cooper pairs is conserved, i.e. $\sum_j n_{cj} = \text{constant}$ (Leggett 1966). Accounting for the tunneling of Cooper pairs between bands, we can write a set of equations to describe $n_{cj}$

$$\frac{\partial^2 n_{c1}}{\partial t^2} = \frac{v_F^2}{3} \nabla^2 n_{c1} + T_{12}[N_1(0)n_{c2} - N_2(0)n_{c1}],$$

$$\frac{\partial^2 n_{c2}}{\partial t^2} = \frac{v_F^2}{3} \nabla^2 n_{c2} + T_{12}[N_2(0)n_{c1} - N_1(0)n_{c2}],$$

where $T_{12} > 0$ is the tunneling coefficient. The dispersion for the collective modes in equations (29) and (30) in the long
wavelength limit is
\[ \Omega_{\text{BAG}}^2 = \frac{1}{3} v_1 N_1(0) + \frac{1}{3} v_2 N_2(0) - q^2, \]
\[ \Omega_L^2 = [N_1(0) + N_2(0)]T_12 + \frac{1}{3} \frac{v_1^2 N_2(0) + v_2^2 N_1(0)}{N_1(0) + N_2(0)} q^2 \]  \hfill (32)

Equations (31) and (32) have similar forms to those in equations (27) and (28). If we set \( T_{12} = D_1/2k_F^2 N_1(0)N_2(0) \), equations (31) and (32) coincide with those derived from a microscopic theory, see equations (38) and (39) in section 3.2.

The tunneling of Cooper pairs between different bands shares certain similarities with that in a Josephson junction. The dispersion of the Leggett mode has the same form as the collective excitation in a Josephson junction. In Josephson junctions, the collective mode couples directly to gauge fields and becomes a plasma mode, known as the Josephson plasma (Barone and Paterno 1982). In contrast, the Leggett mode does not respond to gauge fields and is a neutral mode.

### 3.2. Microscopic description of the Leggett mode in two-band superconductors

In this subsection, we present a microscopic description of the Leggett mode in a two-band superconductor using a field theoretical approach (Sharapov et al. 2002). We use the two-band BCS Hamiltonian in equation (7) and consider the \( T = 0 \) case. The ground state \( \Psi_j = \Delta_j \exp(i\phi_j) \) is determined by the two-band version of equations (16) and (17). As we are interested in the low-energy phase fluctuations, we can treat the amplitude of the energy gaps as fixed. We expand the equation in equation (14) up to the second order in the phase fluctuations. This can be done by the following gauge transformation to separate the phase and amplitude of gaps (Sharapov et al. 2002, Loktev 2001)

\[ \begin{align*}
\Psi_j & \to \Delta_j e^{i\theta_j} \quad \text{and} \quad \hat{\Theta}_j(t, r) \\
& \to \left(\begin{array}{c}
e^{i\theta_j/2} \\
e^{-i\theta_j/2}
\end{array}\right) \hat{\Theta}_j(t, r).
\end{align*} \hfill (33)

We then obtain the action for the phase fluctuations

\[ S = \int \mathrm{d}r \mathrm{d}^3r \sum_j \Delta_j g_{ij} \Delta_j e^{i\theta_j} \]
\[- \sum_j \text{Tr} \left[ \ln \left(G_j^{-1} - \Sigma_j \right) \right] \hfill (34)

where

\[ \Sigma_j = -\frac{\hbar^2}{2m_j} \left( \frac{1}{2} \nabla^2 \theta_j + i \nabla \theta_j \nabla \right) \sigma_0 + \left[ i \frac{\partial \theta_j}{2} \right] \sigma_3, \]
\[ + \frac{\hbar^2}{8m_j} \left( \nabla \theta_j \right)^2 \sigma_3, \]

with \( \sigma_j \) being the Pauli matrices and \( \sigma_0 = \text{unit matrix} \) (De Palo et al. 1999, Benfatto et al. 2004). From this action, one can obtain the time-dependent nonlinear Schrödinger Lagrangian for the phase fluctuations (Aitchison et al. 1995, 2000). In \( S \) in equation (34), the most important term for the Leggett mode is the Josephson coupling, \( \Delta_1 \Delta_j \cos(\theta_j - \theta_j) \) which explicitly depends on the relative phase of two condensates \( \theta_1 - \theta_j \). Considering small phase fluctuations around the saddle point \( \theta_j = \theta_j - \phi_j \) and expanding \( S \) up to the second order in \( \theta_j \), we have

\[ S_\theta [\theta_j] = \frac{1}{8} \sum_j \int \mathrm{d}^3q \tilde{\theta}(-\Omega_j, -q) \mathcal{M} \tilde{\theta}^\dagger (\Omega_j, q) \hfill (35) \]

with \( \hat{\theta}(\Omega_j, q) \equiv [\theta_1(\Omega_j, q), \theta_2(\Omega_j, q)]^T \) and

\[ \mathcal{M} = \begin{pmatrix} P_1 - D_1 & D_1 \\ D_1 & P_2 - D_1 \end{pmatrix} \hfill (36) \]

with \( D_1 = 8|g_{12}|\Delta_1 \Delta_2 \). Here \( \Omega_j = 2\pi k_B T \) with \( k_B \) the Boltzmann constant and the excitations are bosons. In the hydrodynamic limit at \( T = 0 \), the dissipation is absent and

\[ P_j = 2N_j(0) - \Omega_j^2 + \frac{1}{3} v_j^2 q^2, \hfill (37) \]

after the analytical continuation \( \Omega_j \leftarrow \Omega + i\omega \), where \( v_j \) is the Fermi velocity. In the calculation of \( P_j \), we have used the random phase approximation

\[ \text{Tr} \left[ \ln \left( G_j^{-1} - \Sigma_j \right) \right] = \text{Tr} \ln G_j^{-1} - \text{Tr} \sum_{n=1}^{N_j} \left( G_j \Sigma_j \right)^n - n, \]

to the second order \( n = 2 \). For details on the evaluation of \( \text{Tr}(G_j \Sigma_j)^n \), please refer to Sharapov et al. (2002).

From \( \text{Det}[\mathcal{M}] = 0 \), we obtain the dispersion relations in the long wavelength limit \( q \ll 1 \)

\[ \Omega_{\text{BAG}}^2 = \frac{1}{3} \frac{1}{N_1(0) + N_2(0)} v_1^2 N_2(0) + \frac{1}{3} \frac{1}{N_1(0) + N_2(0)} v_2^2 N_1(0) - q^2, \hfill (38) \]

\[ \Omega_L^2 = \frac{N_1(0) + N_2(0)}{2\hbar^2 N_1(0) N_2(0)} D_1 + \frac{1}{3} \frac{v_1^2 N_2(0) + v_2^2 N_1(0)}{N_1(0) + N_2(0)} - q^2. \hfill (39) \]

The first mode \( \Omega_{\text{BAG}} \) is the gapless Bogoliubov–Anderson–Goldstone boson (Anderson 1958, Bogoliubov 1959). The second mode \( \Omega_L \) is the neutral gapped Leggett mode (Leggett 1966). The Leggett mode in two-band superconductors does not depend on the sign of the interband Josephson coupling \( g_{12} \). The modes in equations (27) and (28), obtained from a phenomenological Lagrangian, have similar forms as those in equations (38) and (39) if one identifies \( \Psi_j^2 \sim m_j N_j(0) v_j^2 \). To compare quantitatively, one needs to relate the parameters in equations (25) and (26) to those in microscopic theories. When coupled to gauge field \( A \), the Bogoliubov–Anderson–Goldstone mode gains a mass according to the Anderson–Higgs mechanism (Anderson 1963, Higgs 1964). In contrast, the Leggett mode does not couple to the gauge field \( A \).

We have neglected the coulomb repulsion in the above derivation. The effects of Coulomb interaction were studied by Leggett (1966) and by Sharapov et al. (2002). The coulomb interaction does not change the gap of the Leggett mode, but modifies its velocity.

Recently, the Leggett mode in iron-based superconductors was considered in Burrell et al. (2010). Using a strong-coupling two-orbital model that is relevant for iron-based superconductors, it was shown that the Leggett mode lies below the two-particle continuum in certain parameter space.
This could facilitate the experimental observation of the Leggett mode because the damping is weak. At the same time, it is possible to detect the pairing symmetry for the iron-based superconductors by utilizing the Leggett mode because the dispersion of the Leggett mode depends on the pairing symmetry (Burnell et al. 2010). Ota et al studied the Leggett mode in three-band superconductors with time reversal symmetry (Ota et al. 2011). They found that the gap of the Leggett mode is reduced when the Josephson coupling between different bands cancels each other, but it is still larger than zero.

In the discussions so far we have neglected quasiparticles, which is valid when the gap of the Leggett mode lies below the superconducting gaps $\Delta_i$. When the gap of the Leggett mode is above one of the superconducting energy gaps, the Leggett mode is damped by transferring energy into quasiparticles, a process called the Landau damping. When the damping is strong, the lifetime of the Leggett mode is short and the Leggett mode becomes ill-defined collective excitations. On the other hand, when the gap of the Leggett mode is below the superconducting energy gap, the damping due to quasiparticles is weak. The damping of the Leggett boson can also arise due to the interaction between the Leggett bosons when the amplitude of the Leggett mode is strong. The Leggett mode can lose energy to other bosonic degrees of freedom, such as phonons.

Here we have considered the Leggett modes in clean multiband superconductors. The collective modes in dirty multiband superconductors were investigated by Anishchanka et al. (2007), where the interplay between the Leggett mode and the Carlson–Goldman mode was studied. Finally, an alternative derivation of the dispersion of the Leggett mode and the Carlson–Goldman mode was studied (Det$\mathbf{M} = 0$, we obtain the dispersion relations for the phase fluctuations in the case of an identical density of state and Fermi velocity for the three bands, i.e. $N_j(0) = 0$ and $v_j = v_f$).

$$\Omega_{\text{BAG}}^2 = \frac{1}{3} q^2 v_f^2,$$

$$\Omega_{s-}^2 = -\frac{D_1 + 2 D_2}{2 h^2 N_0} + \frac{1}{3} q^2 v_f^2,$$

$$\Omega_{s+}^2 = -\frac{3 D_1}{2 h^2 N_0} + \frac{1}{3} q^2 v_f^2.$$ 

The first mode is the gapless Bogoliubov–Anderson–Goldstone mode, as displayed in the right of figure 5. The second and third are the Leggett modes $\Omega_{s-}$ and $\Omega_{s+}$ in the three-band superconductors. Importantly, as depicted in the left of figure 5, the mode $\Omega_{s-}$ corresponds to the oscillations of the relative phase $\phi_{23}$ between the gaps of $\Psi_2$ and $\Psi_3$ and becomes gapless at the time-reversal symmetry breaking phase transition depicted in figure 6. One may regard $\phi_{23}$ as the order parameter for the time-reversal symmetry: it increases continuously from 0 at the transition and therefore, the associated fluctuations become gapless at the transition. In stark contrast to the example for conventional symmetry breaking phase transition in equations (40) and (41), there exist stable gapped Leggett modes both before and after time-reversal symmetry breaking transition, as shown in figure 6, because the relative phase between different condensates is fixed in the states with and without time-reversal symmetry.
The coupling between superconductors and the gauge field can be introduced into $S_0$ through the standard replacement $\nabla \varphi \rightarrow \nabla \varphi - 2\pi A/\Phi_0$. In this case, it is more convenient to write the phase fluctuations in terms of $\theta_1$, $\theta_{12} \equiv \theta_1 - \theta_2$, and $\theta_{13} \equiv \theta_1 - \theta_3$. $\theta_1$ corresponds to the Bogoliubov–Anderson–Goldstone mode, while $\theta_{12}$ and $\theta_{13}$ describe the Leggett modes. The gauge fields couple with $\theta_1$ in the form $(\nabla \varphi - 2\pi A/\Phi_0) \cdot \mathbf{J}$. After integrating out $\theta_1$, the gapless Bogoliubov–Anderson–Goldstone mode becomes the gapped plasma mode due to the Anderson–Higgs mechanism (Anderson 1963, Higgs 1964).

In contrast, one of the Leggett modes remains gapless at the time-reversal symmetry breaking phase transition, since the phase differences $\theta_{12}$ and $\theta_{13}$ do not couple with gauge field $A$.

In the static region, the gapless Leggett mode manifests itself as a new divergent length scale (Carlström et al 2011b, Hu and Wang 2012). When approaching the time-reversal symmetry breaking transition from the state without time-reversal symmetry, this new divergent length is associated with the spatial variation of the amplitude and phase of the superconducting order parameters. On the other hand, this new divergent length corresponds to the spatial variation of the phase of the superconducting order parameters if we approach the time-reversal symmetry breaking transition from the state with time-reversal symmetry.

Kobayashi et al studied the Leggett mode in multiband superconductors with frustrated interband coupling by mapping the multiband tight-binding Hamiltonian with pair-hopping interaction into a frustrated spin Hamiltonian (Kobayashi et al 2013a, 2013b). For three-band superconductors, they also found that the Leggett mode becomes gapless at the time-reversal symmetry breaking transition, consistent with the results in Lin and Hu (2012a). However, for four-band superconductors, they revealed the existence of a gapless Leggett mode in a wider phase region, which is not limited to the time-reversal symmetry breaking transition point because of the degeneracy in the ground states. The gap value of Leggett modes can be used to characterize the time-reversal symmetry in multiband superconductors with frustrated interband coupling. It was suggested in Maiti and Chubukov (2013) and Marciani et al (2013) that the possible time-reversal symmetry broken state $s + is$ in the doped $\text{Ba}_1\text{−}x\text{K}_x\text{Fe}_2\text{As}_2$ should be verified by checking the existence of the gapless Leggett mode.

The Leggett modes can couple to other neutral modes, such as phonons. This coupling may modify the dispersion of the Leggett mode, such as the gap and group velocity. As far as the time-reversal symmetry breaking transition remains continuous, one of the Leggett modes is always gapless at the transition point. However, it is also possible that the coupling with other neutral modes results in a first order phase transition and this case requires further study.

### 3.4. Experimental observation of the Leggett mode

In this subsection, we will discuss the possible experimental observation of the Leggett mode. One very useful technique is the Raman spectroscopy. We will first derive the Raman response due to the presence of the Leggett mode, taking the three-band case as an example. The experimental observation of the Leggett mode in MgB$_2$ will be reviewed. In the second part, we will discuss the thermodynamical signatures of the Leggett mode. The possible observation of the gapless Leggett mode in iron-based superconductors will be discussed.

The Leggett modes can be probed indirectly by electric fields through the coupling to the charge density. Therefore, the Leggett modes can be detected by the Raman spectroscopy through the inelastic scattering of photon with the charge density (Abrikosov and Falkovskyi...
The interaction between the incident photon and the charge density can be modeled as

\[ \hat{\rho}(\tau, q) = \sum_{j=1}^{3} \sum_{k,\sigma} \hat{y}_j(k) \psi^\dagger_{j\sigma}(\tau, k + \frac{q}{2}) \psi_{j\sigma}(\tau, k - \frac{q}{2}), \]  

(47)

Here \( \hat{y}_j(k) \) is the scattering coefficient, which is determined by the polarization of the incident and scattered photon. For the non-resonant electronic Raman scattering, the coefficients \( \hat{y}_j(k) \) reads

\[ \hat{y}_j(k) = \sum_{\alpha_x, \rho_x} e^{\dagger}_{\alpha_x} \frac{\partial^2 \epsilon_{\alpha}}{\partial k_{\alpha_x} \partial k_{\beta_x}} e_{\beta_x}, \]  

(48)

where \( e^{\dagger}_{\alpha_x} \) and \( e_{\beta_x} \) are the polarization vectors of the incoming and outgoing photon respectively. \( \alpha_x, \rho_x \) denote the coordinates perpendicular to the photon momentum and \( \epsilon_{\alpha} \) is the electron energy. The electronic Raman cross section is proportional to the dynamical structure factor \( S_f(\omega, q \to 0) \), which is related to the retarded correlation function \( \chi_{\hat{p}\hat{p}} \) in the following way

\[ S_f(\omega, q) = [1 + n_B(\omega)] \left[ -\frac{1}{\pi} \text{Im} \chi_{\hat{p}\hat{p}}^R(\omega, q) \right], \]  

(49)

where \( n_B(\omega) \) is the Bose distribution function. We need to calculate the correlation function

\[ \chi_{\hat{p}\hat{p}}(\tau - \tau', q) = -\langle \hat{T}_R(\tau, q) \hat{\rho}(\tau', -q) \rangle, \]

with \( \hat{T}_R \) being time-ordering operator. To compute \( \chi_{\hat{p}\hat{p}} \), we add a new term in equation (34), \( S_j(\tau) = -\sum_q \hat{\rho}(\tau, q) J_j(\tau, -q) \) with \( J \) an external field. Then \( \chi_{\hat{p}\hat{p}} \) can be computed by using the linear response theory with respect to \( J \). Here \( S_j \) in the Nambu space can be written as

\[ S_j = \sum_{j=1}^{3} \sum_{k,\sigma} \psi^\dagger_{j\sigma}(\tau, k + \frac{q}{2}) G^{-1}_{j,j}(\tau; k, q) \psi_{j\sigma}(\tau, k - \frac{q}{2}), \]  

(50)

with \( G^{-1}_{j,j}(\tau; k, q) = -\hat{y}_j(k) J_j(\tau, -q) \sigma_3 \). The effective action with incident photons after integrating out the fermionic fields \( \psi_j \) becomes

\[ S = \int \mathrm{d}r \mathrm{d}^3r \sum_{i,j} \psi^\dagger_{j\sigma} g_{ij} \psi_{i\sigma} + \sum_{i} \text{Tr} \ln \left( G^{-1}_{j,j} + G_{j,j} \right). \]  

(51)

We may neglect the fluctuations of the amplitudes of the order parameters when the incident wave is weak. The fluctuations for the phase of superconducting order parameters acquire a form \( S = S_\theta + S_\chi \), with

\[ S_\chi = \frac{1}{2} \sum_{j,q} \left[ J(j)Z_j(q)\theta_j^\dagger(-q) + J(-q)\bar{Z}_j(q)\theta_j(q) + J(q)\bar{J}(-q)\Pi^{\gamma\gamma}_{j,33} \right]. \]

(52)

Figure 7. Sketch of the Raman response in three-band superconductors with time-reversal symmetry breaking transition. The finite line-width of peaks is caused by damping of the Leggett boson. The background at energy larger than \( 2\Delta_j \) is due to the quasiparticle excitations. Reprinted with permission from Lin and Hu (2012a). Copyright 2012 by the American Physical Society.

\[ Z_j(q) = \Psi_j[ -\sin \phi_j \Pi^\gamma_{j,31}(q) - \cos \phi_j \Pi^\gamma_{j,32}(q)], \]

\[ \bar{Z}_j(q) = \Psi_j[ -\sin \phi_j \Pi^\gamma_{j,13}(q) - \cos \phi_j \Pi^\gamma_{j,12}(q)], \]

with the polarization functions

\[ \Pi^\gamma_{j,ml} = 1/(L^3\beta) \sum_n \int \mathrm{d}^3k Y_{j,ml}(k + \frac{q}{2}) Y_{j,-l}(k - \frac{q}{2}). \]

We then obtain the correlation function after integrating out the phase fluctuations \( \theta_j \)

\[ \chi_{\hat{p}\hat{p}}(\Omega, q = 0) = \sum_j \left[ \Pi^\gamma_{j,31} - Z_j|\mathbf{M}^{-1}|_{jj}\bar{Z}_j^T \right], \]

(53)

with the matrix \( \mathbf{M} \) being given in equation (43). The first term accounts for the resonance with quasiparticles at \( \Omega = 2\Delta_j \). The second term gives the resonant scattering with the Leggett modes, as displayed in figure 7. \( \mathbf{M}^{-1} \) becomes singular and gives delta peaks in the spectroscopy when the energy difference between the incident and scattered photons matches the gap of the Leggett modes. The delta-function peaks are rounded in reality by the damping effect arising from the interactions between the Leggett bosons when the oscillations of the Leggett modes become strong, or interaction with other bosonic degrees of freedom or thermal fluctuations, which are neglected in our treatment. The response of a genuinely gapless Leggett mode is hidden into the elastic scatterings. The gapless Leggett mode can be traced out clearly if one can tune the gap of the Leggett mode through changing \( n \) systematically by electron/hole doping because the interband scattering is renormalized by the density of state, as in equation (16).

The Leggett modes have been observed in MgB2 using polarized Raman scattering measurements in the excellent experiments by Blumberg et al (2007). The main results are summarized in figure 8. The Raman response in the \( E_2g \) channel starts to appear at a threshold Raman shift 4.6 meV, which is assigned as the smaller superconducting energy...
Figure 8. The Raman response spectra of an MgB$_2$ crystal in the normal (red) and superconducting (blue) states for the $E_{2g}$ (top row) and $A_{1g}$ (bottom row) scattering channels. The columns are arranged in the order of increasing excitation energy $\Omega_{ex}$. Solid lines are fitted to the data points. The data in the superconducting state is decomposed into a sum of a gapped normal state continuum with temperature broadened $2\Delta_0 = 4.6$ meV gap cutoff, the superconducting coherence peak at $2\Delta_1 = 13.5$ meV (shaded in violet) and the collective modes at $\omega_{LR} = 9.4$ meV and $\omega_{LR2} = 13.2$ meV (shaded in dark and light green). The solid hairline is the sum of both modes. Panels (d) and (h) also show the high energy part of spectra for respective symmetries. The broad $E_{2g}$ band at 79 meV is the boron stretching mode, the only phonon that exhibits renormalization below $T_c$ (Mialitsin et al. 2007). For the $A_{1g}$ channel, the spectra are dominated by two-phonon scattering. Reprinted with permission from Blumberg et al. (2007). Copyright 2007 by the American Physical Society.

The Leggett mode also manifests itself in several thermodynamic behaviors of $s$-wave superconductors, such as specific heat. For fully gapped superconductors, the quasiparticle contribution to the specific heat at $T \ll T_c$ depends exponentially on temperature $C_v \propto (\Delta/k_B T)^{3/2} \exp(-\Delta/k_B T)$. The contribution of the Leggett modes to the specific heat can be obtained analytically by treating the Leggett bosons as free quantum gas. For the gapped Leggett mode with a gap $\hbar\omega_0$, the specific heat due to the Leggett mode is $C_L \propto (\hbar\omega_0/k_B T) \exp(-\hbar\omega_0/k_B T)$ for $k_B T \ll \hbar\omega_0$ and is $C_L \propto (k_B T/\hbar\omega_0)^3$ for $k_B T \gg \hbar\omega_0$. For the gapless Leggett mode, the dependence of the specific heat originated from the Leggett mode on $T$ is $T^3$. Thus it is possible to detect the Leggett mode by measuring the electronic specific heat.

Several experiments reported a $T^3$ dependence of the electronic specific heat $C_v$ in iron-base superconductors after subtracting the residue electronic contribution (linear in $T$) and phonon contribution (also $T^3$ dependence) (Kim et al. 2010, Gofryk et al. 2011, Zeng et al. 2011). This $T^3$ dependence could also result from a line node in the gap function. This possibility was excluded from the measurements of the dependence of $C_v$ on magnetic fields, which suggests fully gapped order parameters. The authors of Gofryk et al. (2011) suggested that the additional $T^3$ contribution might originate from some bosonic modes. The existence of the gapless Leggett mode can explain these experimental observations naturally. Such an explanation is quite plausible as regards the possible time-reversal symmetry breaking transition suggested for the iron-based superconductors (Maiti and Chubukov 2013). The samples used in Kim et al. (2010), Gofryk et al. (2011) and Zeng et al. (2011) may well be in the vicinity of the
time-reversal symmetry breaking transition and more measurements, such as the Raman spectroscopy, are much anticipated.

4. Phase soliton

Multiband superconductors with interband Josephson coupling allow for the phase kink or phase soliton excitation due to the degenerate energy minima in the Josephson coupling. For multiband superconductors without time-reversal symmetry, it supports another type of phase soliton between two symmetry-broken domains, similar to the domain walls in ferromagnets. In this section, we will review these two types of phase solitons and discuss the difference between them and their stability. We will also discuss methods to excite phase solitons. In the phase kink region, the time-reversal symmetry is violated locally and under certain conditions spontaneous magnetic fields appear in the phase kink region. These can serve as experimental signatures of the existence of phase solitons. At the end of the section, the experimental detection of phase solitons will be reviewed.

4.1. Phase soliton in multiband superconductors with time-reversal symmetry

For Josephson coupled multiband superconductors, the Josephson coupling $\gamma_{jl} \cos(\phi_j - \phi_l)$ has multiple degenerate energy minimal at $\phi_j - \phi_l = 2n\pi$ for $\gamma_{jl} < 0$. Therefore phase kink can be formed between these energy minimums, which corresponds to the homotopy class $\pi_1(S^0)$. The kink solution was first considered by Tanaka (2001). To illustrate the kink solution or phase soliton in multiband superconductors, let us consider a two-band superconductor in one dimension with a free energy functional given by equation (1). As will be discussed later, the phase soliton solution is only stable in one dimension. In one dimension we can take $A = 0$. Without loss of generality, we consider the $\gamma_{12} < 0$ case and $\phi_1 = \phi_2$ is constant in the ground state. We also assume that the amplitude of the order parameters $|\Psi_j| = |\Psi_{j0}|$ is constant in space; the validity of this approximation will be made clear later. Minimizing equation (1) with respect to $\phi_j$, we obtain

$$\frac{\hbar^2}{m_1} \Psi_{10}^2 \nabla^2 \phi_1 + \frac{\hbar^2}{m_2} \Psi_{20}^2 \nabla^2 \phi_2 = 0,$$

$$\lambda_k^2 \partial_x \phi_{12} + \sin(\phi_{12}) = 0,$$

where $\phi_{12} \equiv \phi_1 - \phi_2$ and the kink width or soliton size $\lambda_k$ is

$$\lambda_k^2 = \frac{\hbar^2}{2|\gamma_{12}| \Psi_{10} \Psi_{20}} \left( \frac{m_1}{\Psi_{10}^2} + \frac{m_2}{\Psi_{20}^2} \right)^{-1}.$$

Equation (55) is the well-known sine-Gordon equation and it supports soliton solutions. One of such soliton solutions is $\phi_{12} = 4 \tan^{-1}[\exp(x/\lambda_k)]$. From equation (54) with the boundary condition $\partial_x \phi_1 = \partial_x \phi_2 = 0$ and $\phi_1 = \phi_2$ away from the soliton at $x = \pm \infty$, we obtain the profile of $\phi_1$ and $\phi_2$ for the soliton solution

$$\phi_1 = 4 \tan^{-1}[\exp(x/\lambda_k)] \bar{g},$$

$$\phi_2 = 4 \tan^{-1}[\exp(x/\lambda_k)](\bar{g} - 1),$$

with $\bar{g} = \left( \frac{m_2 \Psi_{10}^2 + m_1 \Psi_{20}^2}{\bar{m}_1 \Psi_{10}^2 + \bar{m}_2 \Psi_{20}^2} \right)^{-1}$. One typical configuration of the kink solution is schematically shown in the middle of figure 9. The time-reversal symmetry is broken locally in the kink region because $\phi_{12} \neq 0$ or $\phi_{12} \neq \pi$, while it is preserved in the region far away from the kink.

The approximation of constant $\Psi_{10}$ and $\Psi_{20}$ in space is valid when the kink width is much larger than the superconducting coherence length, $\lambda_k \gg \xi$. At low temperatures, this approximation is valid for a weak interband coupling $|\lambda_{12}| \ll |\alpha_i|$. However, as the temperature approaches $T_c$, $\lambda_k$ becomes temperature-independent, while $\xi \propto 1/\sqrt{(T_c - T)/T_c}$ diverges. The constant $\Psi_{10}$ and $\Psi_{20}$ approximation is then no longer valid and superconductivity at the kink region is suppressed significantly.

The phase soliton in 1D wire does not carry magnetic flux. However, if we wrap the wire into a ring, then the phase soliton has fractional magnetic flux (Tanaka 2001). The supercurrent for a constant $|\Psi_j|$ in the ring is given by

$$J_j = \sum_i \frac{2e\hbar}{m_j} \left( \nabla \phi_j - \frac{2\pi}{\Phi_0} A \right).$$

We integrate along a closed loop in the outer region of the ring where $J_j = 0$ because the magnetic field is fully screened for a ring width much larger than $\lambda$. Moreover, the phase $\phi_i$ of superconducting order parameter can only change by $2n_\pi$ if we move around the ring. The integration yields a total magnetic flux $\Phi$ enclosed by the ring

$$\Phi = \frac{|\Psi_{10}^2|}{|m_1| + |\Psi_{20}^2|/|m_2|} \Phi_0,$$

where $n_j = \int \nabla \phi_j \cdot dl/(2\pi)$ is the winding number. Here $\Phi$ is integer quantized only when $n_1 = n_2$. The existence of a

Figure 9. Domain structure in multiband superconductors due to the presence of phase solitons (left); a phase soliton in a two-band superconductor (middle); and that in a three-band superconductor, with each domain corresponding to distinct time-reversal symmetry broken states (right). Arrows denote the phase of superconducting order parameter in different bands. Reproduced with permission from Lin and Hu (2012b).
phase soliton requires that $n_1 \neq n_2$, hence the phase soliton in a ring carries fractional quantum flux.

The discussions so far are based on the Ginzburg–Landau approach. Samokhin studied the phase soliton with a microscopic approach and calculated the quasiparticle spectrum in the presence of a phase soliton (Samokhin 2012). He found the existence of quasiparticle bound states localized near the soliton, with energies being nonuniversal fractions of the bulk superconducting gaps. Such bound states can be measured in tunneling experiments.

The kink solution or phase soliton also appears in Josephson junctions, where the gauge invariant phase difference is also governed by the sine-Gordon equation (55) (Barone and Paterno 1982). In Josephson junctions, the phase difference is coupled with gauge fields, therefore, a phase soliton carries $\Phi_0$ flux and it can be created by applying magnetic fields or driven by currents. The motion of soliton is resonant with the Josephson plasma oscillation, which yields current steps for certain voltages; see Hu and Lin (2010) for a recent review. In contrast, the phase soliton in a two-band superconductor does not couple with gauge fields and it is neutral. Thus it does not respond to magnetic fields or currents. Nevertheless, the phase soliton can be created by an electric field in nonequilibrium, which will be discussed in section 4.4.

4.2. Phase soliton in multiband superconductors without time-reversal symmetry

In three or more band superconductors with frustrated interband coupling, time reversal symmetry may be broken. In this case, we have two distinct domains with degenerate energy, which is very similar to domains in ferromagnets. In the time-reversal symmetry broken state, these two domains have order parameter $\Psi = (\Psi_1, \Psi_2, \Psi_3, \ldots)$ or $\Psi^* = (\Psi_1^*, \Psi_2^*, \Psi_3^*, \ldots)$ with $\Psi \neq \Psi^* \exp(i\theta)$, i.e. one cannot obtain one domain from the other by global rotation of phase. Therefore, there can be stable kink solution between two domains $\Psi$ and $\Psi^*$ in one dimension (Guraud et al 2011, Lin and Hu 2012b, Tanaka and Yanagisawa 2010), which is quite different from the kink solution discussed in multiband superconductors with time-reversal symmetry in section 4.1. Note that multiband superconductors without time-reversal symmetry still support kink solution with one domain $\Psi$ or $\Psi^*$, which is similar to that in multiband superconductors with time-reversal symmetry.

To illustrate the idea, we consider a minimal model with three identical bands $\alpha_j = \alpha, \gamma_j = \gamma > 0$ and $m_j = m$. In this case, time-reversal symmetry is violated and there are two degenerate ground states with $\Psi = \Delta(1, e^{i\pi/3}, e^{i4\pi/3})$ and $\Psi^* = \Delta(1, e^{-i\pi/3}, e^{i4\pi/3})$, which are sketched in figure 9 (right). The phase kink is described by Lin and Hu (2012b)

$$\partial_t \Phi_1 = 0,$$
$$\partial_t (\Phi_{12} + \Phi_{13}) = 0,$$
$$\frac{1}{2m\gamma^2} \partial^2 \Phi_{12} + \sin \Phi_{12} + \sin (2\Phi_{12}) = 0,$$

for constant amplitudes of order parameters valid at $\gamma \ll |\alpha|$. The same double sine-Gordon equation in equation (62) was also derived by Yanagisawa et al (2012). The potential corresponding to equation (62) is $V_p = \cos \Phi_{12} + \cos(\Phi_{12})/2$, which has many degenerate energy minima at $\Phi_{12,\pm} = \pm2\pi/3 + 2n\pi$. A phase kink can be constructed between any pair of the energy minima with qualitatively the same physical properties and stability. Using the Bogomolny inequality (Manton and Sutcliffe 2004), we find a phase kink solution analytically

$$\Phi_{12} = 2 \arctan \left[ \sqrt{3} \tanh \left( \frac{-\sqrt{3} m \gamma}{2} \right) \right].$$

with an energy

$$E_k = \frac{4}{3} \sqrt{m \gamma} \left( 3\sqrt{3} - \pi \right).$$

4.3. Stability of phase soliton

In this subsection, we study the stability of the phase soliton in equation (55) for multiband superconductors with time-reversal symmetry by accounting for the suppression of the amplitude of order parameters. The presence of the phase soliton suppresses the amplitudes of the order parameters, which depends on the ratio of the width of the phase kink $\lambda_k$ to coherence length $\xi_c$. The amplitudes of order parameters are greatly depressed when $T$ is tuned to $T_c$ because $\xi$ increases while $\lambda_k$ hardly changes. Thus at a threshold $T$, the phase soliton becomes unstable and the system evolves into a uniform state with $\Phi_0 = 0$. The dynamics of the instability transition can be detected in experiments by measuring the voltage in the phase soliton. This is because the change of the phases of superconducting order parameter results in a voltage according to the ac Josephson relation. This process can be regarded as a new type of phase slip, which differs from that in single-band superconductors driven by quantum or thermal fluctuations (Tinkham 1996).

The above discussions are borne out by numerical calculations of the time-dependent Ginzburg–Landau in equations (5) and (6) for a two-band superconductor with identical bands $\alpha_j = \alpha, \beta_j = \beta, m_j = m$ in one dimension. We initially put a phase soliton at the center of a superconducting wire. We then increased temperature by changing $\alpha$ and obtained a stable configuration of superconducting order parameters. As displayed in figure 10(a), superconductivity is greatly suppressed in the phase kink region and it becomes weaker upon increasing $T$. The phase soliton becomes unstable at a critical $\alpha_c$ (symbols in figure 10(b)). Then the system transits into the uniform state and a voltage pulse is generated during this process. As shown in figure 10(b), the phase soliton is stable in a small temperature (a) window for a strong interband coupling $|\gamma_1|$. The sign of $\gamma_1$ does not matter here. Therefore, the phase solitons in multiband superconductors with time-reversal symmetry are more stable for weak interband couplings.

A multiband superconducting wire with a phase soliton can be regarded as a Josephson junction because of the weakened superconductivity near the soliton region. In the ground state, the phase differences between two
domains separated by the phase soliton is nonzero according to equation (57). The wire with a phase soliton thus realizes a \( \phi \)-junction (Buzdin 2008), or \( \pi \)-junction (Bulaevskii et al. 1977) if the two bands are identical. We then investigate the effect of an external current on the stability of the phase soliton. The external current is introduced by twisting the phase of superconducting order parameter at the two ends of the wire. The phase kink is deformed in the presence of current, as depicted in figure 10(c). At a threshold current, the deformation renders the phase kink unstable and this threshold current can be regarded as the critical current for the Josephson junction. The dependence of the critical current on \( |\gamma_{12}| \) is present in figure 10(d) and it decreases with \( |\gamma_{12}| \). At the instability current when the system evolves from the kink state to the uniform state, a phase slip occurs associated with a voltage pulse similar to the case with increasing temperature.

The phase kink in equation (63) between two time-reversal symmetry broken states \( \hat{\Psi}_1 \) and \( \hat{\Psi}_1^* \) is different to that in superconductors with time-reversal symmetry. In the former case, to remove the kink, one needs to change \( \hat{\Psi}^* \) into \( \hat{\Psi} \) or vice versa, which requires one to overcome a huge energy barrier proportional to the volume of domains. Thus, the phase soliton in this case is topologically protected as a result of breaking \( Z_2 \) symmetry. In the latter case, the domains separated by the phase soliton are essentially the same, except for a common phase factor, as depicted in figure 9 (middle). One can remove the phase soliton by rotating the phase of a domain without costing energy in the domain. Energy costs only occur in the kink region and do not depend on the size of domains.

4.4. Creation of phase soliton

The phase solitons are stable topological excitations, which generally are not present in the ground state. In this subsection, we discuss possible ways to create the phase solitons. Since the phase soliton does not couple directly with gauge fields, one cannot create them by applying magnetic fields. First, let us consider the dynamical excitation of phase kink in two-band superconductors with time-reversal symmetry by electric fields in nonequilibrium region, following the arguments of Gurevich and Vinokur (2003). They considered a two-band superconducting wire attached to a normal electrode, from which a current is applied. For a large enough current, electric field penetrates into the superconductor. However, the electric field cannot exist uniformly in the superconductor, otherwise Cooper pairs are accelerated indefinitely and superconductivity would be destroyed. The electric field is localized in
a finite region, where phase slips occur according to the ac Josephson effect. For multiband superconductors with different relaxation time in different bands, the rate of phase slip for different bands is different, therefore the phase difference between different bands increases linearly with time. As a consequence, the phase solitons are nucleated at the edge and then are pushed towards the center of the wire.

We adopt the time-dependent Ginzburg–Landau theory in equations (5) and (6) to describe the nonequilibrium dynamics of superconducting order parameters. In one dimension, we can put

\[
\frac{\hbar^2}{2m_1 D_1} \left( \partial_t \phi_1 + \frac{2e}{\hbar} \phi_1 \right) = \frac{\hbar^2}{2m_1} \nabla^2 \phi_1 - \gamma_1 \phi_1 |\Psi_{10}|^2 \sin(\phi_2 - \phi_1) \tag{65}
\]

\[
\frac{\hbar^2}{2m_2 D_2} \left( \partial_t \phi_2 + \frac{2e}{\hbar} \phi_2 \right) = \frac{\hbar^2}{2m_2} \nabla^2 \phi_2 + \gamma_1 \phi_1 |\Psi_{10}|^2 \sin(\phi_2 - \phi_1) \tag{66}
\]

Multiplying equations (65) and (66) by proper factors and subtracting equation (66) from equation (65), we obtain an equation for the phase \(\phi_j\) in a two-band superconductor,

\[
\tau_2 \partial_t \phi_2 = \frac{\hbar^2}{2m_2} \nabla^2 \phi_2 - \gamma_1 \phi_1 |\Psi_{10}|^2 \sin(\phi_2 - \phi_1) \tag{67}
\]

with

\[
\tau_2 = \frac{\hbar^2 |\Psi_{10}|^2}{2(m_2 |\Psi_{10}|^2 + m_1 |\Psi_{20}|^2)|\gamma_{12}|}, \tag{68}
\]

\[
\lambda_2^2 = \frac{\hbar^2 |\Psi_{10}|^2}{2(m_2 |\Psi_{10}|^2 + m_1 |\Psi_{20}|^2)|\gamma_{12}|}, \tag{69}
\]

\[
\sigma_2 = \frac{\hbar^2 |\Psi_{10}|^2 (D_1 - D_2)}{4e|\gamma_{12}|(m_2 |\Psi_{10}|^2 + m_1 |\Psi_{20}|^2)} \left( \frac{|\Psi_{20}|^2 D_1}{m_2} + \frac{|\Psi_{10}|^2 D_2}{m_1} \right)^{-1}. \tag{70}
\]

We have used the expression for supercurrent during the derivation,

\[
J_0 = \frac{2e\hbar}{m_1} |\Psi_{10}|^2 \nabla(\phi_1 - \phi_2) + \left( \frac{2e\hbar}{m_1} |\Psi_{10}|^2 + \frac{2e\hbar}{m_2} |\Psi_{20}|^2 \right) \nabla \phi_2, \tag{71}
\]

and \(\nabla \cdot J_0 = -\sigma \nabla \cdot E\) because of the current conservation in one dimension,

\[
J_x + \sigma E = J_{ext}, \tag{72}
\]

with \(J_{ext}\) being the bias current.

Adding equation (66) to equation (65) and using equations (71) and (72), we obtain an equation for \(E\)

\[
-\alpha_e \partial_t \nabla \phi_2 + \tau_e (\partial_t J_{ext}/\sigma - \partial_t E) = -\lambda_e^2 \nabla (\nabla \cdot E), \tag{73}
\]

with

\[
\tau_e = \frac{\sigma}{4e^2} \left( \frac{|\Psi_{10}|^2}{m_1} + \frac{|\Psi_{20}|^2}{m_2} \right)^{-1}, \tag{74}
\]

\[
\lambda_e^2 = \frac{\sigma}{4e^2} \left( \frac{|\Psi_{10}|^2 D_1}{m_1} + \frac{|\Psi_{20}|^2 D_2}{m_2} \right)^{-1}, \tag{75}
\]

\[
\alpha_e = 2|\gamma_{12}| |\Psi_{10}|^2 \alpha_{12}. \tag{76}
\]
tends to align the phases of superconducting order parameter in the $s\pm$ superconductor, while the $s\pm$ pairing symmetry favors a $\pi$ phase shift in the phase of superconducting order parameters. For a strong proximity effect, the formation energy of phase soliton can be reduced even to a negative value, thus rendering the phase soliton thermodynamically stable. The reduction of the formation energy of phase soliton does not occur for the $s++$ pairing symmetry. Thus in this way, one may be able to confirm the pairing symmetry of a two-band superconductor by exploiting the proximity effect to a $s$-wave superconductor. The authors proposed to measure the magnetization in a ring, which is made of a two-band superconductor with the $s\pm$ pairing symmetry in proximity to a patch of $s$-wave superconductor, to observe the phase soliton, because the phase soliton carries magnetic flux in a ring geometry.

To create a phase kink between the time-reversal broken pair states $\Psi$ and $\Psi^*$ in a multiband superconducting wire, one may repeat the cooling process for one part of the wire from normal state, while keeping the remaining part in superconducting state (Hu and Wang 2012). In certain circumstances, the cooled part may reach a state that is different from the other part of the wire, provided the cooling process is fast and thus form a kink between the two domains with $\Psi$ and $\Psi^*$. As demonstrated in Garaud and Babaev (2014), the phase kinks can also be created by homogenous fast cooling in bulk superconductors according to the Kibble-Zurek mechanism (Kibble 1976, Zurek 1985). These created kinks are stabilized by random pinning centers or the pre-existing vortices.

4.5. Experimental signatures and observations of phase soliton

In this subsection, we discuss the experimental signatures for the phase solitons. In the phase solitons, the time-reversal symmetry is broken locally and we expect spontaneous magnetic fields under proper conditions. Here we will demonstrate the existence of such magnetic fields due to the presence of phase solitons both in nonequilibrium and equilibrium.

To study the generation of magnetic fields, one has to go beyond one dimension. We first prepare a closed domain wall (phase kink) in two dimensions as initial conditions and solve the time-dependent Ginzburg–Landau equations numerically. During the time evolution in simulations the domain wall organizes itself into a circular shape, regardless of its initial shape, in order to minimize the domain wall energy: see figures 12(a) and (b). There are spontaneous magnetic fields with alternating directions at the domain walls. As displayed in figure 12(a), for the phase kinks in superconductors with time-reversal symmetry (see equation (57)), the induced magnetic field changes polarization in both radial and azimuthal directions. For the kinks between two time-reversal symmetry broken states (see equation (63)), the magnetic field changes polarization only in the azimuthal direction, as shown in figure 12(b). The circular domain wall then shrinks and finally disappears, which results in a uniform state. Therefore the domain walls or phase kinks in dimensions higher than one are intrinsically unstable. The life time of the circular domain wall may be long when the size of domain enclosed by the domain wall is large, which allows for a possible experimental detection by measuring the induced magnetic fields.

We then study the spontaneous magnetic fields produced by the phase kink in equilibrium. We consider a superconducting strip with a phase soliton at its center in proximity to a normal metal, as sketched in figure 13. The time-reversal symmetry is violated at the phase soliton, therefore the spatial variation of amplitude is coupled with the phase of superconducting order parameters. This can be checked by expanding the interband coupling term $\gamma_{ij} \Delta_i \Delta_j \cos(\phi_i - \phi_j)$ to quadratic order in variation of superconducting order parameters. The amplitudes of the superconducting order parameters are modified by the proximity effect and this modification produces supercurrent, hence generates magnetic fields. We solve the time-dependent Ginzburg–Landau equations numerically.
equations with the boundary condition equation (24). As shown in figure 13, magnetic fields are generated at the proximity region when the phase soliton is present. The magnetic fields change sign at the opposite interface and the total magnetic flux over the sample is zero.

We remark that the spontaneous magnetic fields are about $H \sim 10^{-5} H_c$ for typical parameters in simulations, which are strong enough to be measured experimentally by scanning SQUID, Hall, or magnetic force microscopy, etc.

Similar dipolar magnetic fields associated with the phase kinks were also observed in numerical simulations recently in Garaud and Babaev (2014) in superconducting constrictions or bulk superconductors without time-reversal symmetry, which could be used to detect the possible $s + i$ pairing symmetry in Ba$_{1-x}$K$_x$Fe$_2$As$_2$ (Maiti and Chubukov 2013).

There is no experimental observation of the phase kink in bulk multiband superconductors to date. The possible existence of the phase soliton has been inferred from measurements in artificial two-band superconductors by Bluhm et al (2006). They fabricated superconducting aluminum rings of various sizes, deposited under conditions likely to generate a layered structure. They were able to control the number of layers and the coupling between layers by varying the annulus width of the ring. Thus, the ring can behave effectively as a single-band superconductor or two-band superconductor with a tunable interband Josephson coupling. They then measured the current $I$ in the ring as a function of external magnetic flux $\Phi_0$ enclosed in the ring and temperatures. For a narrow annulus width with one superconducting layer, the measured $I-\Phi_0$ curves can be described satisfactorily with a theory for single-band superconductors. For intermediate coupled artificial two-band superconductors, they found metastable states with different winding numbers for different condensates, which was inferred from comparing the measured $I-\Phi_0$ curves to a theory based on the two-band Ginzburg–Landau theory. These observations suggest the possible existence of the phase soliton in these artificial two-band superconductors. In the strong coupling regime, the measured $I-\Phi_0$ signal can again be fitted by a theory for single-band superconductors because the phases of superconducting order parameter for different layers are locked together, which prevents the formation of phase solitons.

5. Vortex

Vortices are well known topological excitations in superconductors, which arise due to the macroscopic quantum nature of superconducting state. As superconductivity is described by a complex wave function, the single value of this wave function requires that the superconducting phase changes by multiple integers of $2\pi$ around a closed loop. When the phase change by $2\pi$ inside superconductors, a vortex emergence appears. For single component $s$-wave superconductors, a vortex has a normal core with a size of the superconducting coherence length $\xi$ and magnetic field region the size of the London penetration depth $\lambda$. The total magnetic flux associated with a vortex is quantized to $\Phi_0 = hc/2e$.

The interaction between normal cores is attractive, while the interaction due to the magnetic region is repulsive, thus the net interaction between vortices is determined by the ratio $\lambda/\xi$ (Kramer 1971, Jacobs and Rebbi 1979).

For type II superconductors when $\lambda/\xi > 1/\sqrt{2}$, vortices repel each other and they are stable; while for type I superconductors with $\lambda/\xi < 1/\sqrt{2}$, there is attraction between vortices and vortices become unstable upon formation of normal domains. At the special point $\lambda/\xi = 1/\sqrt{2}$, vortices do not interact with each other (Bogomol’nyi 1976).

It is possible for the superconducting phase to change by $2n\pi$ along a closed loop with an integer $n > 1$. These vortices with the larger winding number $n$ are called giant vortices carrying $n\Phi_0$ quantum flux. In bulk superconductors, the energy of giant vortices is proportional to $n^2$, thus they are not energetically favorable. However, in mesoscopic superconductors when the size of superconductors is comparable to $\xi$, giant vortices may be stabilized by geometric confinement (Schweigert et al 1998, Baeulos and Peeters 2002, Kanda et al 2004, Cren et al 2011). Vortices are crucial to determine physical properties, such as transport and electromagnetic response, of a superconductor. Many efforts have been made to understand the statics and dynamics of vortices in single component superconductors in the past decades: see Blatter et al (1994) and Brandt (1995) for a review.

Due to the multiband nature, vortices in multiband superconductors possess unique properties that are not shared by single-band superconductors. In this section, we shall explore these novel properties. First, we will present the concept of fractional vortex and study their interaction. Then we will review several theoretical proposal to stabilize fractional vortices. We then proceed to discuss vortices with attraction at large separation and repulsion at short separation. The consequences of the nonmonotonic inter-vortex interaction will be reviewed.

5.1. Fractional vortex

Multiband superconductors have multiple complex gap functions, thus it is possible that the phase associated with these gap functions changes by different integer multiple of $2\pi$ along a closed loop. In this case, the quantum flux associated with this vortex is not an integer multiple of $\Phi_0$ and this vortex is called a fractional vortex (Babaev 2002, Babaev and Ashcroft 2007). Let us consider a two-band
superconductor with the free energy functional in equation (1). We integrate the supercurrent equation (58) along a closed loop far from the vortex core where \( J_1 = 0 \). Then the total flux is given by equation (59). The flux is integer quantized \( \Phi = n \Phi_0 \) only when \( n_1 = n_2 = n \). In other cases, we have a fractional quantized vortex. As vortex energy increases with the winding number, the most important fractional vortices are those with \( n_1 = 0, n_2 = 1 \) or \( n_1 = 1, n_2 = 0 \). The fractional vortex in bulk superconductors however is thermodynamically unstable because its energy diverges logarithmically with system size: see equation (82) below. Therefore, these two fractional vortices always lock together to form a composite vortex with \( n_1 = n_2 = 1 \) in equilibrium. A schematic view of order parameters, supercurrent and magnetic field for a composite vortex is displayed in figure 14. Nevertheless, under certain circumstances, composite vortices with \( n_1 = 1, n_2 = 1 \) can dissociate into fractional vortices, as will be discussed in the next subsections.

Let us compare a fractional vortex to a phase soliton in a superconducting ring discussed in section 4.1. They are the same in terms of the phase winding number and the associated magnetic flux. However, they have an entirely different topological nature. The phase soliton belongs to the homotopy class \( \pi_0(S^1) \) and is stable only in one dimension; while the fractional vortex belongs to the homotopy class \( \pi_1(S^1) \) and can be stable in two or three dimensions.

There is no experimental observation of the fractional vortex in multiband superconductors at the time of writing.

5.2. Interaction between fractional vortices

In this subsection, we formulate the pairwise interaction between fractional vortices in the London limit when \( \xi \ll \lambda \), taking a two-band superconductor as an example. In this case, the normal cores of vortices do not play a role in determining the vortex interaction. There are four sources of interactions. Vortices as magnetic objects, or vortices with the same polarization, repel each other in a short range due to the exchange of massive photons inside superconductors. They can also repel in a long-range due to the exchange of massless photon outside superconductors, which is especially important for thin films. Fractional vortices in different condensates attract because of the coupling to the same gauge field. They attract also as a consequence of the interband coupling.

We consider the London free energy functional for a two-band superconductor with a Josephson-like interband coupling. In contrast to the Ginzburg–Landau free energy functional valid near \( T_c \), the London free energy functional is valid in the whole temperature region. The free energy density is

\[
F_L = \frac{1}{8\pi} \sum_{j=1}^{2} \left[ \frac{1}{\lambda_j^2} \left( A - \frac{\Phi_0}{2\pi} \nabla \phi_j \right)^2 + (\nabla \times A)^2 \right] + 2\gamma_{12} \cos (\phi_1 - \phi_2),
\]

where \( \lambda_j = \sqrt{(m_j \xi_j^2) / (4\pi n_j \xi_j^2)} \) is the London penetration depth for each condensate with superfluid density \( n_j \). The effective penetration depth for the two-band system is \( \lambda^{-2} = \sum_{j=1}^{2} \lambda_j^{-2} \). We can split \( F_L \) into two parts \( F_L = F_m + F_c \) (Silva and Babaev 2011), with the magnetic coupling

\[
F_m = \frac{1}{8\pi} \left[ B^2 + \lambda^2 (\nabla \times B)^2 \right].
\]

\( F_m \) accounts for the magnetic coupling between vortices and is the same as that for single-band superconductors because there is only one gauge field \( A \) in superconductors. The coupling of the superconducting phases in different bands is described by \( F_c \)

\[
F_c = \frac{\Phi_1 \Phi_2}{32\pi^2 \lambda^2} (\nabla (\phi_1 - \phi_2))^2 + 2\gamma_{12} \cos (\phi_1 - \phi_2). \tag{79}
\]

Assuming straight vortex lines, the magnetic field profile for fractional vortices can be obtained by minimizing \( F_L \) with respect to \( A \), which yields the London equation

\[
\lambda^2 \nabla \times \nabla \times B = \sum_{j=1}^{2} \sum_{l} \Phi_j \delta (r - r_{j,l}). \tag{80}
\]

Here \( r_{j,l} = (x_{j,l}, y_{j,l}) \) is the vortex coordinates for the fractional vortex in the \( j \)-th condensate and

\[
\Phi_j = \lambda^2 \Phi_0 / \lambda_j^2. \tag{81}
\]

is its fractional quantum flux. For a fractional vortex where \( \phi_2 \) does not change and \( \phi_1 \) changes by \( 2\pi \) around \( \mathbf{r}_0 \), the self-energy per unit length is

\[
\mathcal{E}_{f_v} = \left( \frac{\Phi_1}{4\pi \lambda} \right)^2 \ln \left( \frac{\lambda}{\xi_1} \right) + \frac{\Phi_1 \Phi_2}{16\pi^2 \lambda^2} \ln \left( \frac{L}{\xi_1} \right) + \left| 2\gamma_{12} \right| \int \mathrm{d}r^2 [1 - \cos(\phi_1)]. \tag{82}
\]

The first term at the right-hand side is due to \( F_m \) and the rest of the terms are contributed from \( F_c \). Here \( L \) is the linear system size. Because of the neutral mode described by the term proportional to \( [\nabla (\phi_1 - \phi_2)]^2 \) in equation (79), \( \mathcal{E}_{f_v} \) diverges in the thermodynamic limit. The energy of the fractional vortex also diverges linearly in \( L \) due to the
Josephson coupling in equation (79). For these reasons, a fractional vortex is thermodynamically unstable in bulk multiband superconductors (Babaev 2002). For a composite vortex with $\phi_1 = \phi_2$, or $\phi_1 = \phi_2 + \pi$, we have $F_c = 0$ and its self-energy is finite.

The equilibrium configuration of $\phi_j$ is obtained by minimizing equation (79) with respect to $\phi_j$

$$\frac{\Phi_1 \Phi_2}{16\pi^2 \lambda^2} \nabla^2 (\phi_1 - \phi_2) + 2 \gamma_{12} \sin (\phi_1 - \phi_2) = 0,$$

(83)

together with the boundary condition accounting for vortex cores

$$\nabla \times (\nabla \phi_j) = 2\pi \sum_{j=1}^{2} \sum_{l} \delta (r - r_{j,l}).$$

(84)

Due to the presence of nonlinear term $\sin (\phi_1 - \phi_2)$, the interaction between vortices is nonlinear and is of many-body interaction. In the presence of a strong magnetic field, such a nonlinear term can be neglected for the following reason. The term $\nabla^2 (\phi_1 - \phi_2)$ is of the order $1/\tilde{a}^2$ with $\tilde{a}$ being the average distance between vortices in the same condensate. For a strong field when $a \ll \lambda_j \equiv \sqrt{\Phi_0 \Phi_2/32\pi^2 \lambda^2 \gamma_{12}}$, the sine term is much smaller than the first term in equation (83) and can be neglected. In this case, $F_c$ is equivalent to the energy for the $XY$ model. For MgB$_2$, $|\gamma_{12}| \approx 75$ J m$^{-3}$ (Gurevich 2003), we can neglect the sine term for fields stronger than 4 T at temperature $T = 0$ K. For $V_2$Si (Kogan et al. 2009) and FeSe$_{1-x}$ (Khasanov et al. 2010), because of the much weaker interband coupling, the required field is smaller.

Neglecting the Josephson coupling in $F_c$, the pairwise interaction between the fractional vortices can be derived analytically because both $F_m$ and $F_c$ are quadratic in $B$ and $\phi_j$. Here $F_m$ describes the short-range interband and intraband magnetic repulsion between vortices with the same polarization. The term proportional to $(\nabla \phi_j)^2$ in $F_c$ accounts for the long-range repulsion between vortices in the same band. The term proportional to $-\nabla \phi_1 \nabla \phi_2$ yields a long-range attraction between vortices in different bands. Putting all of these together, we obtain the repulsion between fractional vortices with flux $\Phi_j$ in the same band separated by a distance $r_{i,j} \equiv r_{i,j} - r_{i,j}$

$$V_{\text{int}}(r_{i,j}) = \frac{\Phi_1^2}{8\pi^2 \lambda^2} K_0 \left( \frac{r_{i,j}}{\lambda} \right) - \frac{\Phi_1 \Phi_2}{8\pi^2 \lambda^2} \ln \left( \frac{r_{i,j}}{\lambda} \right),$$

(85)

where $K_0(r)$ is the modified Bessel function. The attraction between two fractional vortices in different bands with a separation $r_{12,i,j} \equiv r_{i,j} - r_{i,j}$ is

$$V_{\text{int}}(r_{12,i,j}) = \frac{\Phi_1 \Phi_2}{8\pi^2 \lambda^2} \left( K_0 \left( \frac{r_{12,i,j}}{\lambda} \right) + \ln \left( \frac{r_{12,i,j}}{\lambda} \right) \right).$$

(86)

Equations (85) and (86) are valid away from vortex cores.

**5.3. Dissociation of composite vortex lattice in the flux flow region**

In this subsection, we discuss the dissociation of a composite vortex lattice in a two-band superconductor in the flux flow region due to the difference in coherence length and superfluid density for different bands (Lin and Bulaevskii 2013). The higher the superfluid density $\tilde{n}_j$, the larger the magnetic flux $\Phi_j$ for the fraction vortex and hence there is a stronger Lorentz force in the presence of a current. In contrast, the shorter the coherence length $\xi_j$, the bigger the viscosity $\eta_j$ and hence there is a smaller velocity at a given force. As a consequence, the fractional vortex lattice in a certain band tends to move faster. For a small external current, the disparity of vortex motion in different bands can be compensated by the attraction between vortices in different bands. However for a large current, such attraction becomes insufficient to balance the difference in the vortex viscosity and driving force for different bands. As a result, fractional vortex lattices in different bands decouple from each other and they move with different velocities. We will also discuss the consequences of the dissociation of composite vortex lattice, such as the appearance of the Shapiro steps and increase of the flux flow resistivity. We remark that the dissociation transition is possible only in the vortex lattice region. For a single composite vortex, because the attraction between two fractional vortices diverges logarithmically with separation, on which see equation (86), it is impossible to completely decouple these two fractional vortices by applying a current.

We first introduce an equation of motion for fractional vortices. The vortex viscosity results from dissipation, which is caused by motion of normal cores. Employing the Bardeen–Stephen model (Bardeen and Stephen 1965), the viscosity of a fractional vortex in each band is given by

$$\eta_j = \Phi_0^2/(2\pi \rho_j c^2 \xi_j^2),$$

(87)

where $\rho_j$ is the electric resistivity in the $j$-th band. We assume that the vortex structure does not change in the flux flow region and assume an overdamped dynamics for vortices. The equation of motion for fractional vortices is

$$\eta_j \ddot{r}_{j,l} = \frac{1}{8\pi^2 \lambda^2} \sum_i \left[ \Phi_1^2 K_1 \left( \frac{r_{j,l,i}}{\lambda} \right) + \Phi_1 \Phi_2 \frac{K_0 \left( \frac{r_{j,l,i}}{\lambda} \right)}{\lambda} \right]$$

$$+ \frac{\Phi_1 \Phi_2}{8\pi^2 \lambda^2} \sum_i \left[ K_1 \left( \frac{r_{12,l,i}}{\lambda} \right) - \frac{\lambda}{r_{12,l,i}} \right] + J_{\text{ext}} F_j,$$

(88)

with $J_{\text{ext}}$ the external current. The first term at the right-hand side of equation (88) is the repulsion between vortices in the same bands and the second term is the attraction between vortices in different bands. The last term is the Lorentz force. The effects of random pinning centers are not important in the flux flow region because they are quickly averaged out by vortex motion, resulting in an improved lattice order (Koshelev and Vinokur 1994, Besseling et al. 2003). The interaction between fractional vortices in the same bands cancels out in the vortex lattice phase. We assume a square vortex lattice with a lattice constant $\tilde{a}$. By accounting for the dominant wavevector $\mathbf{G} = (\pm 2\pi j/\tilde{a}, 0)$ for the vortex lattice moving along the x direction, the equation of motion for the center of mass of fractional vortex lattice in each band $R_j$ can
be written as

\[ n_2' \partial_t (R_2 - R_1) = -(1 + n_2') \sin (R_2 - R_1) + (\Phi_2' - \eta_2') J_{\text{ext}}. \]  
(89)

\[ \partial_t R_1 + n_2' \partial_t R_2 = (1 + \Phi_2') J_{\text{ext}}. \]  
(90)

We have introduced dimensionless unit: current is in unit of \( cF_d/\Phi_1 \); time is in unit of \( \eta_1 t/(2\pi F_d) \); length is in unit of \( a/(2\pi) \). Here \( \Phi_2' \equiv \Phi_2/\Phi_1 \), \( n_2' \equiv n_2/\eta_1 \) and

\[ F_d = \frac{\Phi_1 \Phi_2 a^4}{64\pi^2 a^4}. \]  
(91)

is the maximal attraction between two fractional vortices in different bands. Equations (89) and (90) are similar to those for the vortex motion in superconducting bilayers (Clem 1974).

The two fractional vortex lattices in different bands move at the same velocity

\[ v_1 = v_2 = (1 + n_1')^{-1} (1 + \Phi_1') J_{\text{ext}} \]  
(92)

for a small current. The composite vortex starts to deform in this region with a separation between the center of mass of these two fractional vortex lattices

\[ a_* = \sin^{-1} \left[ (1 + n_2')^{-1} (\Phi_2' - \eta_2') J_{\text{ext}} \right]. \]  
(93)

These two lattices decouple at a threshold current

\[ J_d = \left| (1 + n_2') (\Phi_2' - \eta_2')^{-1} \right|. \]  
(94)

and they move with different velocities

\[ v_j = (1 + n_j')^{-1} \times \left[ (1 + \Phi_j') J_{\text{ext}} - \eta_j J_{\text{ext}} \sqrt{(\Phi_j' - \eta_j)^2 J_{\text{ext}}^2 - (1 + n_j')^2} \right]. \]  
(95)

In this region, the composite vortex lattice dissociates into two fractional vortex lattices moving at different velocities. For a large current \( J_{\text{ext}} >> J_d \), each lattice moves independently with a velocity \( v_1 = J_{\text{ext}}/n_2' \) and \( v_2 = J_{\text{ext}}/n_1' \). We plot \( v_j \) as a function of \( J_{\text{ext}} \) in figure 15.

The corresponding I-V characteristics can be obtained from the power balance condition

\[ \eta_1 v_1^2 + \eta_2 v_2^2 = J_{\text{ext}} E a^2, \]  
(96)

with \( E \) the electric field. The I-V curve is present in figure 15, where the flux flow resistivity \( dE/dJ_{\text{ext}} \) increases in the decoupled phase. Observation of such enhanced resistivity in experiments might be challenging because the dissociation current \( J_d \) is usually large, where the Larkin–Ovchinnikov instability of vortex lattice may be important (Larkin and Ovchinnikov 1976). The flux flow resistivity also increases in the Larkin–Ovchinnikov instability region. Nevertheless, the dissociation transition can be confirmed unambiguously by measurement of the Shapiro steps in the decoupled phase, as discussed below.

Equation (89) also describes the phase dynamics in an overdamped Josephson junction. When an ac current is added in superposition to the dc current, there will be Shapiro steps when the resonance condition is satisfied. In the decoupled phase, if one takes one lattice as a reference, the other lattice experiences a periodic potential induced by the reference lattice. The oscillation of the moving lattice induced by the ac current may be in resonance with the oscillation due to the periodic potential of the reference lattice, if the period of the ac current matches with the period of the potential. This results in the Shapiro steps in I-V curves. With a current

\[ J_{\text{ext}} = J_{\text{dc}} + \Re \left[ J_{\text{ac}} \exp[i(\omega t + \theta)] \right], \]  

the center of mass of each lattice is

\[ v_j = v_j t + \Re \left[ \tilde{A}_j \exp[i(v_2 - v_1)t] \right] \]  

in the region \( |v_1 - v_2| = \omega \gg 1 \). From equations (89) and (90), we obtain

\[ \tilde{A}_j = \frac{\eta_j [1 - i \Phi_j J_{\text{ac}} \exp(i\theta)]/\Phi_1}{(v_2 - v_1) \eta_1}. \]  

The dc current is

\[ J_{\text{dc}} = (\Phi_2' - \eta_2')^{-1} \times \Re \left[ \eta_j (v_2 - v_1) + (1 + n_2') \left( \tilde{A}_2 - \tilde{A}_1 \right)/2 \right]. \]  

\( \theta \) adjusts correspondingly when one changes \( J_{\text{dc}} \), because \( v_1 - v_2 \) is locked with the driving frequency \( \omega \) and a Shapiro step is traced out. The height of the Shapiro step is

\[ J_{\text{sp}} = \frac{(1 + n_j') (n_j' \Phi_j' - 1)}{(v_2 - v_1) \Phi_j' - n_j'}. \]  
(97)

We solve equations (89) and (90) numerically with \( J_{\text{ext}} = J_{\text{dc}} + 1.2 \sin(\omega t) \) and the results are shown in figure 16. The Shapiro steps appear when \( \omega = v_1 - v_2 \). Generally, the Shapiro steps also occur at \( n \omega = (v_1 - v_2) \) with a much smaller height. Here we only observe the prominent step with \( n = 1 \). The Shapiro steps at \( n \omega = v_j \) (with the reduced units) can also be induced by the periodic passing of vortex lattice through defects (Fiory 1971, Schmid and Hauger 1973). These steps can be separated from those induced by relative motion of two fractional vortex lattices in equation (97) because their resonance condition is different.

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**Figure 15.** Dependence of electric field \( E \) and velocity \( v_1, v_2 \) on the current \( J_{\text{ext}} \), obtained from equations (89), (90) and (95). Reproduced from Lin and Bulaevskii (2013). Copyright 2013 by the American Physical Society.
Generally the decoupling of composite vortex lattice is present in all multiband superconductors and depends on the two parameters $\eta_2/\eta_1$ and $\phi_2/\phi_1$. The dissociation current $J_d$ is high for a small disparity in $\lambda_1$ and $\xi_j$. Here we estimate $J_d$ for MgB$_2$. We use $\lambda_1 = 47.8$ nm, $\lambda_2 = 33.6$ nm, $\xi_1 = 13$ nm and $\xi_2 = 51$ nm at $T = 0$ K (Moschchalkov et al 2009), $\rho_j = 10^{-9}$ $\Omega \cdot m$ (Xi 2008) and $\ell = 40$ nm corresponding to magnetic fields at $B \approx 1$ T. We then estimate $J_d = 5 \times 10^4$ A m$^{-2}$, which is much smaller than the depairing current. The velocity of vortex lattice at the dissociation transition is $v_1 = v_2 \approx 3$ m s$^{-1}$, which is smaller than the typical Larkin–Ovchinnikov instability velocity for vortex lattice (Doettinger et al 1994). In samples with pinning centers, one first needs to overcome the pinning potential to observe the dissociation transition. The effective dissociation current is thus the sum of the depinning current and the dissociation current $J_d$ for clean systems. The dissociation transition may also be observed in the proposed liquid hydrogen two-component superconductor with a small $J_d$ because of the large mass difference between the proton and electron.

The dissociation of composite vortex lattice discussed here is similar to the decoupling of vortex motion in multilayer superconductors (Giaever 1965) and in cuprate superconductors (Busch et al 1992, Wan et al 1993, Safar et al 1994). For the dissociation transition in multiband superconductors, vortices have fractional quantum flux after dissociation and the dissociation takes place in the band (momentum) space. For multilayer superconductors, vortices in different layers carry quantized flux $\Phi_0$ and the decoupling occurs in the real space. This decoupling in multilayer superconductors has been discussed theoretically (Cladis et al 1968, Clem 1974, Uprety and Domínguez 1995) and was observed experimentally (Giaever 1965, Busch et al 1992, Wan et al 1993, Safar et al 1994) some decades ago. The Shapiro steps were also measured in multilayer superconductors in the decoupled phase (Gilbert et al 1994). These observations corroborate the possible experimental observation of the predicted dissociation of composite vortex lattice in multiband superconductors.

5.4. Stabilization of fractional vortex by pinning arrays

In this subsection, we discuss the possibility of stabilizing fractional vortices by pinning arrays (Lin and Reichehardt 2013). When the external current is turned off suddenly in the decoupled phase where two fractional vortex lattices move with different velocities, it is possible for the fractional vortices to become trapped by pinning centers if the density of pinning centers is higher than the vortex density. We will construct a dynamic phase diagram for vortices in a two-band superconductor with pinning arrays. The presence of pinning arrays also yields novel self-induced Shapiro steps in the decoupled phase.

We model the interaction between the pinning site at $r_p$ and the fractional vortex in the $j$-th band at $r_j$ as

$$U_{j,p} (r_j - r_p) = -\Gamma_j \exp \left[ -\left( r_j - r_p \right)^2 / l_j^2 \right],$$

(98)

where $l_j$ is the pinning range and $\Gamma_j$ characterizes the pinning strength. The equation of motion for the fractional vortices is

$$\eta_j \partial_t r_{j,i} = -\nabla_{r_{j,i}} (V_{\text{intra}} + V_{\text{inter}} + U_{j,p}) + J_{\text{ext}} \phi_j / c,$$

(99)

where $V_{\text{intra}}$ and $V_{\text{inter}}$ are given in equations (85) and (86), respectively. Equation (99) is solved numerically using the second order Runge–Kutta method. We use dimensionless units for force: $\Phi_0 \phi_2 / (8 \pi^2 \lambda^2)$; length: $\lambda$; time: $8 \pi^2 \eta_1 \lambda^2 / (\phi_1 \phi_2)$; and current: $c \phi_2 / (8 \pi^2 \lambda^2)$ and we set $\eta_1 = \eta_2$ and $l_1 = 1$.

The attraction between two fractional vortex lattices in different bands is a periodic function of space with a period equal to the lattice constant. The maximal attraction is $F_d$ given in equation (91). When the maximal pinning force $F_{j,p} = -\nabla U_{j,p}$ is much smaller than $F_d$, i.e. $F_{j,p} \ll F_d$, the two fractional vortex lattices depin simultaneously at a current $J_p = (F_{1,p} + F_{2,p}) c / \phi_0$, as shown in the red curve in figure 17(a). They travel with the same velocity after depinning until the current is large enough to decouple them as in the case of clean systems. In this region, the fractional vortices form a composite vortex with deformation. All of these can be seen in the I-V curve, as depicted in figure 17(b). In the other limit when $F_{j,p} \gg F_d$, the two fractional vortex lattices depin at different currents $J_{j,p} = F_{j,p} c / \phi_j$ and they move at different velocities once depinned. The dynamic phase diagram is constructed in figure 18. The depinning current increases with the pinning strength $\Gamma_j$. Meanwhile, the region of flux flow of the composite vortex lattice shrinks and finally disappears. Then the two fractional vortex lattices depin at different currents. At a high current $J_{\text{ext}} \gg 1$, the two fractional vortex lattices travel with different velocities $v_j \approx J_{\text{ext}} \phi_j / (c \eta_j)$. All of these dynamical phase transitions manifest themselves in the I-V curves.

We then study the self-induced Shapiro steps in the decoupled phase where two fractional vortex lattices in bands 1 and 2 are moving with different velocities. The self-induced Shapiro steps are distinct from the conventional Shapiro steps (Shapiro 1963) because they appear without the application of an external ac drive. The appearance of the self-induced Shapiro step is also an experimentally observable signature.

![Figure 16.](image-url)
We use a square pinning array with a lattice constant $a_p$. The Shapiro steps are due to the fast moving lattice in band 2. For simplicity, we have assumed that both the pinning array and the vortex lattice have the same lattice constant $\bar{a}$. In terms of the velocities of the fractional vortex lattices, the Shapiro steps occur when $(j_1 + j_2)v_1 = j_2v_2$. We observe spikes in velocity curves as a consequence of the resonance, as shown in figure 19(a) in the numerical simulations. The resonance can be seen more clearly in the differential resistivity $dE/dJ_{ext}$ plotted in figure 19(b) whenever $v_j$ satisfies the resonance condition.

We proceed to consider the case that the pinning density is twice that of the vortex density $n_p = 2n_v$ to optimize the trapping of fractional vortices by pinning arrays. In the ground state, as depicted in figure 20(a), the composite vortices reside in a checkerboard pattern with every other pinning site occupied. The composite vortex lattice is dissociated into two fractional vortex lattices by applying a large current. The current is then turned off and most of the fractional vortices in bands 1 and 2 get trapped at different pinning sites. This results in a metastable state where every pinning center is occupied by a fractional vortex, as depicted in figure 20(b). The life time of the fractional vortex can be long for a strong pinning potential. The commensuration between vortex configuration and the pinning sites increases the life time further by reducing the fluctuations from vortex–vortex interaction. The trapped fractional vortices could be observed in experiments with various imaging techniques such as a SQUID.

### 5.5. Other mechanisms to stabilize fractional vortex

It is also possible to stabilize a fractional vortex in equilibrium. Silaev considered vortices near the surface of a two-band superconductor using the London free energy functional in equation (77) by neglecting the interband Josephson coupling (Silaev 2011). He found that fractional vortices are stable near the surface of the superconductor due to the cancellation of the unscreened supercurrent by the image antivortices. He also studied the penetration of fractional vortices through the boundary when the external magnetic field is increased.
For superconductors with different coherence lengths, the fractional vortices with a larger normal core are the first to enter into the superconductors. When the external field is increased further, the fractional vortices with a smaller normal core then enter into the superconductors and they merge with the fractional vortices with a larger normal core to form composite vortices. These composite vortices then proliferate into the bulk superconductor. This two-step penetration process is manifested as two jumps in the magnetization curve as a function of external magnetic fields.

Fractional vortices can also be stabilized in a mesoscopic sized two-band superconductor, where the logarithmic divergence of the fractional vortex self-energy is cut off by the system size (Chibotaru et al 2007, Chibotaru and Dao 2010, Geurts et al 2010, Pereira et al 2011, Piña et al 2012, Gillis et al 2014). In Chibotaru and Dao (2010), the authors minimized numerically the two-band Ginzburg–Landau free energy functional in equation (1) and found a stable fractional vortex configuration, as shown by the red region in figure 21. Superconducting condensates with different coherence length and superfluid density respond to the geometry confinement in a different way. In addition, both superconducting condensates are coupled with the same gauge field and they may also couple via the Josephson coupling, which favors integer quantized vortices because the phases of superconducting order parameter tend to lock with each other. The competition of these two effects gives rise to a plethora of vortex states in mesoscopic superconductors, as plotted in figure 21.

Smørgrav et al considered a two-band superconductor in magnetic fields with a strong disparity in phase stiffness for different superconducting condensates by Monte Carlo simulations (Smørgrav et al 2005). At zero temperature, the fractional vortices in different condensates form a triangular lattice of composite vortex. Upon heating, they found that the sublattice of fractional vortex with lower phase stiffness first melts due to the proliferation of vortex loop driven by thermal fluctuations. This phase transition belongs to the 3D XY universality class. In this temperature region, the fractional vortices in different bands are decoupled. When temperature is increased further, the remaining fractional vortex lattice with higher phase stiffness melts via first order phase transition and the system enters into the vortex liquid phase.

5.6. Vortex with nonmonotonic interaction

In multiband superconductors, it is possible that vortices repel at short distance and attract at large separation, as first pointed out by Babaev and Speight (2005). Here we review briefly the nonmonotonic interaction between vortices and its consequences. Another review is available in Babaev et al (2012). Let us consider a two-band superconductor with \( \xi_2 \ll \lambda \ll \xi_1 \). As depicted in figure 22, when two isolated vortices with quantum flux \( \Phi_0 \) approach each other, the normal core of vortex in the first band \( |\Psi_1| \) first overlaps, which induces an attraction between vortices. As they get closer, the electromagnetic interaction becomes dominant because of the overlapping of magnetic fields and as the vortices repel each other. This results in nonmonotonic interaction between vortices. As the separation is reduced further, the normal core of vortex in the second band starts to overlap. They finally merge into a giant vortex with vorticity equal to two.

The calculation of interaction between vortices as a function of separation poses a challenge to theory since vortices are extended objects. In the London limit, the normal core becomes a point-like object and one can fix the vortex at a
Figure 20. Snapshot of the fractional vortex configuration (a) in the ground state and (b) after a current quench. Open circles are pinning sites; blue and red circles represent the fractional vortices in different bands. Some fractional vortices are trapped at different pinning sites after the current quench. Here $n_p = 2n_v$, $\Gamma_1 = \Gamma_2 = 2.0$, $l_1^2 = l_2^2 = 0.5$, $J_{\text{ext}} = 3.2$ and $\Phi_2/\Phi_1 = 5.0$. We use a rectangular pinning array with a lattice constant $a_{p,x} = 2.0 \phi_0$, $a_{p,y} = 5.0 \phi_0$. Reproduced from Lin and Reichhardt (2013). Copyright 2013 by the American Physical Society.

Figure 21. (a) Phase diagram of a two-band superconducting cylinder with a radius $R$ obtained by numerical minimization of the Ginzburg–Landau free energy functional in equation (1) for $\kappa' \equiv \Phi_2 m_j (\beta_j^2 (2 \pi)^{3/2} / \phi_0) = 10$ (top), $3$ (middle) and $0.5$ (bottom) $(x_{20} \equiv (R/\xi_j)^2 (\alpha_j / |\alpha_j| + \beta_j / \beta_1) = 12, 20$ and $0.5$, respectively) with $\gamma' \equiv -\gamma_2 (\phi_0^2 / 2m_j R^2) = 0.01$. Here $m_j = m_1$ and $\beta_j = \beta_1$. It is divided into domains of superconducting states with no vortex ($V_0$), with a central giant vortex of winding number $n$ in each condensate ($CV_n$) and with $n_j$ separated vortices in the $j$-th condensate ($V_{n_1,n_2}$). Dashed and dotted lines delineate the domains where fractional flux vortices exist as stable phases for $\gamma' = 0.05$ and $0.1$, respectively. The stars on the plots denote points at which current distributions are shown in figure 3 of Chibotaru and Dao (2010). Reprinted with permission from Chibotaru and Dao (2010). Copyright 2010 by the American Physical Society.

desired position $r$, using the boundary condition $\nabla \times \nabla \phi(r) = 2\pi \delta(r - r_i)$. For the general cases explored here, one has to introduce constraints to fix two vortices at a desired separation. One may impose pinning potential to vortices by fixing the amplitude and/or phase of superconducting order parameter in a certain region near the vortex cores (Misko et al. 2003). However, this method may introduce artifacts when two vortices are close to each other, since the order parameters change dramatically near the vortex core. Furthermore, it is sometimes insufficient to pin vortices by imposing the local constraints because the interaction becomes strong when vortices are close. In Lin and Hu (2011), we implemented and generalized the variational method for single-band superconductors (Jacobs and Rebbi 1979) to calculate the inter-vortex interaction in a two-band superconductor. The vortex separation is fixed by choosing proper trial functions for $\Psi_1$, $\Psi_2$ and $A$. By varying the separation $d$ continuously, we obtained nonmonotonic interaction profile between two vortices, as shown in figure 23, where there exists a local minimum. The profiles of $\Psi_j$, magnetic field $B_z$ and supercurrent as a function of separation obtained by variational calculations are displayed in figure 22.

We also introduced two vortices in a square disk with size $L$ through the boundary condition (Doria et al. 1989)

$$A(r + L_\mu) = A(r) + \nabla x_\mu \cdot \Psi_j (r + L_\mu)$$

$$= \Psi_j (r) \exp(i 2\pi x_\mu / \phi_0),$$

(100)

with $\mu = x, y$ and $x_z = H_z L y$ and $x_y = 0$. Here $H_z$ is the applied magnetic field and should obey the vortex quantization...
condition via $\oint \mathbf{d}l \cdot \mathbf{A} = 2n\Phi_0$, which yields $H_a = 2n\Phi_0/L^2$.

By minimizing the two-band Ginzburg–Landau free energy in equation (1) numerically, we found a bound solution with vortex separation independent of the disk size $L$ for a large $L$, which indicates unambiguously an energy minimum in the inter-vortex interaction profile. The vortex separation is consistent with the results in figure 23 obtained by variational calculations.

The presence of nonmonotonic inter-vortex interaction modifies drastically the magnetic response of a superconductor. For superconductors with nonmonotonic interaction between vortices, upon increasing external magnetic fields $H$, clusters of vortex penetrate into superconductors associated with a discontinuous jump in magnetic induction from zero to $B_{c1}$, which may be seen as a hysteresis loop in magnetization curve, as depicted in the inset of figure 24(a). Thus the transition from the pure Meissner state to the vortex cluster phase is of the first order phase transition. It was pointed out that at a low density of vortices, vortex clusters coexist with Meissner state (Babaev and Speight 2005). As the vortex cluster has positive surface energy, these clusters are of circular shape (Lin and Hu 2011). The interaction between the vortex clusters is long-range and repulsive induced by magnetic interaction outside the superconductor and vortex clusters are distributed evenly in clean superconductors (Lin et al 2012). The vortex density then increases gradually with the external magnetic field until $H_{c2}$ at which superconductivity is destroyed completely (see inset of figure 24(a)). Here $H_{c2}$ is the same as that of type I superconductors, which is solely determined by the condensate with the shortest coherence length. The mean-field $H-T$ and the corresponding $B-T$ phase diagram for superconductors with nonmonotonic vortex interaction are depicted in figure 24. Here, the $B$ is the magnetic induction $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$ with $\mathbf{M}$ the magnetization.

The long-range repulsion between vortices due to the dipolar interaction at large separation, the short-range attraction at intermediate separation and strong repulsion at short separation, yield complex vortex configurations in an intermediate vortex density. A low-density clump phase, an intermediate density stripe phase, an anticlump phase and a high-density uniform phase have been observed in simulations (Olson Reichhardt et al 2010, Dao et al 2011, Xu et al 2011, Zhao et al 2012a, 2012b, Drocco et al 2013, Varney et al 2013, Meng et al 2014, Xu et al 2014). As vortices approach each other, the nonlinear effect becomes important such that the interaction between vortices may no longer be pairwise, i.e. many-body interaction such as three-body interaction becomes important (Carlström et al 2011c, Edström 2013). Such a non-pairwise interaction could result in even more complex vortex distributions, such as vortex glassy phases (Sellin and Babaev 2013).
Recently, stripe and gossamer phases of vortex were observed in MgB$_2$ using imaging methods (Moshchalkov et al. 2009, Nishio et al. 2010, Gutierrez et al. 2012). Coexistence of Meissner state and vortex state was observed in Sr$_2$RuO$_4$ using muon-spin rotation measurements (Ray et al. 2014). These observations were interpreted in favor of the existence of nonmonotonic interaction between vortices. However, caution should be taken because at low vortex densities the pinning of vortices by defects unavoidable in superconductors may be dominant over the vortex interaction and produce similar vortex patterns as observed.

The theoretical discussions so far are based on the multiband Ginzburg–Landau free energy functional. When applied to real superconductors, the Ginzburg–Landau theory is valid only for temperatures very close to $T_c$. As discussed in section 2.2, multiband superconductors with interband coupling behave as single-band superconductors near $T_c$. One may extend the applicable region of the Ginzburg–Landau theory by expanding to higher order in $(T - T_c)/T_c$ (Komendová et al. 2011, Shanenko et al. 2011, Vagov et al. 2012). Nevertheless, qualitative features of the vortex interaction may be extracted from the Ginzburg–Landau theory even at low temperatures. To describe the vortex interaction at low temperatures in a rigorous way, a microscopic theory beyond the Ginzburg–Landau theory is required. Such a microscopic theory was developed in Silaev and Babaev (2011) using the two-band Eilenberger formalism, where the authors demonstrated the existence of nonmonotonic inter-vortex interaction for appropriate parameters.

Finally, we would like to remark that the nonmonotonic inter-vortex interaction can also be found in single-band superconductors with $\lambda/\xi$ close to $1/\sqrt{2}$, such as high purity Nb crystal. The nonmonotonic interaction arises from the BCS correction to the Ginzburg–Landau theory. Due to the competing interaction, vortex clusters coexisting with the Meissner phase are stabilized, which was observed in Nb crystals. For details, please refer to Brandt and Das (2011) and references therein.

The physics of vortex with nonmonotonic interaction in multiband superconductors is still under active research. The nonmonotonic interaction between vortices was also found in three-band superconductors with frustrated interband couplings (Carlström et al. 2011b, Takahashi et al. 2013, 2014). The effect of the interband Josephson coupling (Babaev et al. 2010, Geurts et al. 2010, Carlström et al. 2011a) and the condition for the nonmonotonic interaction (Chaves et al. 2011) were studied. The applicability of the Ginzburg–Landau theory to investigate the nonmonotonic inter-vortex interaction was discussed in Kogan and Schmalian (2011), Babaev and Silaev (2012) and Kogan and Schmalian (2012). The comparison between vortex with nonmonotonic interaction in single-band and multiband superconductors was made in Brandt and Das (2011) and Babaev and Silaev (2013).

6. Discussions

It is possible that the vortices, phase solitons and Leggett modes interact with each other. For instance, in three-band superconductors, the presence of a vortex distorts the amplitude and phase of the superconducting order parameters and excites the Leggett mode. This distortion propagates in superconductors and can be felt by another vortex. In this way a mutual interaction is established between vortices (Carlström et al. 2011b). One particularly interesting situation is when the Leggett mode becomes gapless. In this case the interaction between vortices due to the exchange of the Leggett excitation becomes long-range. On the other hand, the motion of vortex excites superconducting amplitude-phase mixed mode or the Leggett mode and thus provides additional viscosity to the vortex motion (Silaev and Babaev 2013). The phase kink in two dimensions forms a circular shape to minimize the energy because the energy cost to excite phase...
kink is positive. For the same reason, the circle shrinks in time and ultimately the kink disappears. Therefore the kink solution in two dimensions is unstable, in accordance with Derrick’s theorem (Derrick 1964). The interaction between the kink and vortex can stabilize the kink solution. Near the kink region, superconductivity is suppressed and therefore vortices are pinned in the kink region. The repulsion between vortices prevents the kink from collapsing and thus stabilizes the kink–vortex’s composite structure (Garaud et al 2011).

There are also many interesting physics arising from the multiband nature of superconductors, which are not discussed in the previous sections. Here we mention them briefly and acknowledge that the list here is rather partial and biased. For further details, readers may consult the original papers. Flux flow and pinning of the vortex was studied in Matsunaga et al (2004). Anomalous flux flow resistivity in MgB$_2$ was observed in Shibata et al (2003). Field dependence of the vortex core size in a multiband superconductor was measured in Callaghan et al (2005). Skyrmions in multiband superconductors were discussed in Garaud et al (2014, 2013) and Agterberg et al (2014). Hidden criticality inside the superconducting state in multiband superconductors was studied in Komendova et al (2012). Entropy-induced and flow-induced superfluid states were proposed in Carlström and Babaev (2014). Thermal fluctuations in multiband superconductors were investigated in Koshelev et al (2005), Berger and Milošević (2011) and Koshelev and Varlamov (2014). Phase slip was studied in Fenchko and Yerin (2012). Magnetic field delocalization and flux inversion in fractional vortices was investigated in Babaev et al (2009). The unusual dependence of superfluid density and specific heat was calculated in Kogan et al (2009). For more discussions on the thermodynamical properties in multiband superconductors, please refer to Zehetmayer (2013) for a review.

Much of the novel physics for multiband superconductors discussed in this review can be realized in Josephson junctions. In junctions, the superconducting electrodes can be regarded as distinct superconducting condensates separated in real space and coupled by the Josephson interaction. Josephson junctions with electrodes made of single-band s-wave superconductors can be regarded as artificial two-band superconductors. The sign of the Josephson coupling can be tuned by using different blocking layers. For instance, one can achieve a π phase shift between superconducting electrodes by using a ferromagnetic blocking layer (Bulaevskii et al 1977), which corresponds to the ±s pairing symmetry in two-band superconductors. One can also use a two-band superconductor as one electron and a single-band superconductor as the other electrode, to realize an artificial three-band superconductor. Frustration can be introduced when the two-band superconductors have s± pairing symmetry. Time-reversal symmetry breaking in these junctions was discussed in Ng and Nagaoasa (2009), Koshelev (2012), Lin (2012) and Tanaka (2001a) and phase solitons with fractional quantum magnetic flux was discussed in Lin (2012). Such configurations were also proposed to detect the s± pairing symmetry (Chen et al 2009, Linder et al 2009, Parker and Mazin 2009, Chen et al 2010, Chen and Zhang 2011, Koshelev and Stanev 2011, Stanev and Koshelev 2012). The main difference between Josephson junctions and multiband superconductors is that the superconducting phase differences in junctions are coupled to gauge fields, while the phase differences between bands in multiband superconductors are not. Josephson junctions using multiband superconductors as electrodes are interesting systems from which novel phenomena emerge (Ota et al 2009, 2010a, 2010b, 2010c; Huang and Hu 2014, Yerin and Omelyanchouk 2014), which deserve a separate investigation. Interested readers may refer to review papers (Brinkman et al 2003, Xi 2009, Seidel 2011) on this topic.

One may obtain qualitative features of the phase solitons and vortices in multiband superconductors using a simplified model. However, to apply to real materials, one has to consider a realistic model derived for a specific material. Multiband superconducting materials with a weak interband coupling can facilitate the experimental observations of the Leggett mode, the phase solitons and fractional vortex. The experimentally observed Leggett mode is unstable while the phase soliton and fractional vortex have not been observed in any bulk multiband superconductor at the time of writing. The interband couplings for the prominent multiband superconductors MgB$_2$ and iron-based superconductors are not weak. It is a significant challenge and opportunity to find multiband superconductors with weak interband couplings.

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