GLOBAL DYNAMICS FOR A CLASS OF REACTION-DIFFUSION EQUATIONS WITH DISTRIBUTED DELAY AND NEUMANN CONDITION

TARIK MOHAMMED TOUAOULA
Département de Mathématiques, Faculté des Sciences, Université de Tlemcen
Laboratoire d’Analyse Non Linéaire et Mathématiques Appliquées
Tlemcen, BP 119, 13000, ALGERIA

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Abstract. In this paper, we investigate a class of non-monotone reaction-diffusion equations with distributed delay and a homogeneous Neumann boundary condition. The main concern is the global attractivity of the unique positive steady state. To achieve this, we use an argument based on sub and super-solutions combined with the fluctuation method. We also give a condition under which the exponential stability of the positive steady state is reached. As particular examples, we apply our results to the diffusive Nicholson blowfly equation and the diffusive Mackey-Glass equation with distributed delay. We obtain some new results on exponential stability of the positive steady state for these models.

1. Introduction. In this paper, we study the initial boundary value problem

\[
\begin{aligned}
& u_t(x, t) - \Delta u(x, t) = -f(u(x, t)) + \int_0^\tau h(a)g(u(x, t - a))da, \quad t > 0, x \in \Omega \\
& \frac{\partial u}{\partial n}(x, t) = 0, \quad x \in \partial \Omega, t > 0, \\
& u(x, t) = \phi(x, t), \quad (x, t) \in \Omega \times [-\tau, 0],
\end{aligned}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \) and \( \frac{\partial u}{\partial n} \) denotes the derivative along the outward normal direction on the boundary of \( \Omega \).

Recently, an increasing attention has been paid to local and non-local delay reaction-diffusion equations in bounded and unbounded domains. This class of equations has a great range of applications, particularly in population dynamics, see for instance [1, 5, 27, 24, 31, 32, 33, 34, 35] and references therein.

Previous results on the asymptotic behavior of solutions for this class of problems have been obtained by different methods, (e.g. [1, 27, 30, 33, 34, 38]); however, most of them suppose that, either \( f \) is linear or \( g \) is monotone. If \( f \) is linear, Yi et al. [33] established a relationship between the convergence of solutions of (1) to the positive steady state and the convergence of the sequence defined by \( x_{n+1} = g(x_n) \).

The key condition for delay independent stability is that the map \( g \) does not have 2-periodic points. For functional differential equations this idea has been already

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used by many authors, see [2, 7, 10, 11, 15, 16, 17, 22, 33] and the references therein. We particularly mention the leading works [18, 19, 20].

In this paper we analyze the global dynamics of the solutions of (1) when $g$ is not necessarily monotone and $f$ is not necessarily linear. To reach it, we will use an argument based on the theory of sub and supersolutions to (1). With the help of comparison principles, which require some careful construction of monotone deformations of the delayed term $g$, we will prove the existence of an interval that attracts all solutions of (1). From then, the global attractivity and exponential stability of the unique positive steady state are established by using the fluctuation method.

Our result about exponential stability leads to some new results when applied to the Nicholson blowfly and the Mackey-Glass models.

The method used here does not treat the case when $g$ is a bistable nonlinearity, in this direction see for example [9, 14] and the references therein.

Our work is organized as follows. In the next section, we establish criteria of existence and uniqueness of positive solution for the problem (1). Some useful estimates are also proved based on the theory of sub and supersolutions. Moreover, we prove that the unique positive steady state is a global attractor when the delayed term $g$ is nondecreasing. Section 3 is devoted to the non-monotone case. We will principally show the existence of closed attractive intervals for the solutions of (1). In Section 4 we will present some theorems related to the global attractivity and the exponential stability of the positive steady state. Finally some examples are given to illustrate our theorems.

Throughout this paper, we will make the following assumptions: the function $h$ is positive and

\[ \int_0^\tau h(a)da = 1. \]

(T1) $f$ and $g$ are Lipschitz continuous in each compact interval with $f(0) = g(0)$.

(T2) $g(s) > g(0)$ for all $s > 0$ and there exists a constant $B > 0$ such that $\max_{v \in [0,s]} \{ g(v) \} < f(s)$ for all $s > B$.

Let $C = C(\bar{\Omega}, \mathbb{R})$ and $X = C(\bar{\Omega} \times [-\tau, 0], \mathbb{R})$ be the spaces of continuous functions equipped with the usual supremum norm $||.||$. Also, let $C_+ = C(\bar{\Omega}, \mathbb{R}^+)$ and $X_+ = C(\bar{\Omega} \times [-\tau, 0], \mathbb{R}^+)$. For any $\phi, \psi \in X$, we write $\phi \geq \psi$ if $\phi - \psi \in X_+$; $\phi > \psi$ if $\phi - \psi \in X_+ \setminus \{0\}$.

We define the ordered intervals

\[ [\phi, \psi]_Y := \{ \xi \in Y; \phi \leq \xi \leq \psi \}, \]

with $Y$ a Banach space. For any $\chi \in \mathbb{R}$, we write $\chi^*$ for the element of $X$ satisfying $\chi^*(x, \theta) = \chi$ for all $(x, \theta) \in \bar{\Omega} \times [-\tau, 0]$. The segment $u_t \in X$ of a function $u$ is defined by the relation $u_t(x, \theta) = u(x, t + \theta)$ for $x \in \bar{\Omega}$ and $\theta \in [-\tau, 0]$. The family of maps

\[ U : \mathbb{R}^+ \times X_+ \to X_+, \]

such that

\[ (t, \phi) \to u_t(\phi) \]

defines a continuous semiflow on $X_+$, for more details see [29]. The map $U(t, \cdot)$ is defined from $X_+$ to $X_+$ as the semiflow $U_t$, denoted by

\[ U_t(\phi) = U(t, \phi). \]
The set of equilibria of the semiflow generated by (1) is given by
\[ E = \{ \chi^* \in X_+ ; \chi \in \mathbb{R} \text{ and } g(\chi) = f(\chi) \} . \]

2. Preliminary results. Let \( T(t) \ (t \geq 0) \) be the strongly continuous semigroup of bounded linear operators on \( C \) generated by the Laplace operator \( \Delta \) under the homogenous Neumann conditions. It is well known that \( T(t) \ (t \geq 0) \) is analytic, compact and strongly positive semigroup on \( C \) see for instance [29]. Define \( F : X \to C \) by

\[ F(\phi)(x) = -f(\phi(x,0)) + \int_0^T h(a)g(\phi(x,-a))da, \quad \text{for all } x \in \tilde{\Omega}. \]  

We consider the integral equation with the given initial data

\[ \begin{cases} u(t) = T(t)\phi(\cdot, 0) + \int_0^t T(t-s)F(u_s)ds, & t \geq 0, \\ u_0 = \phi \in X. \end{cases} \]  

For each \( \phi \in X \), \( u(\cdot, t) \) with values in \( C \) on its maximum interval \([0, \sigma_\phi)\), is called a mild solution of (1), see for instance [6, 12, 13, 29], and it is called a classical solution if \( u(\cdot, t) \) is \( C^2 \) in \( x \) and \( C^1 \) in \( t \).

Let \( g^+ \text{ and } g^- \) be Lipschitz continuous and nondecreasing functions on \([a, b], b > a > 0 \). Let \( \phi^+ \text{ and } \phi^- \) be nonnegative continuous functions on \( \tilde{\Omega} \times [-\tau, 0] \). Assume that the functions \( g \text{ and } \phi \) in (1) satisfy \( g^- \leq g \leq g^+ \) and \( \phi^- \leq \phi \leq \phi^+ \) on their respective domain of definitions.

A supersolution of (1) is a smooth bounded function \( \bar{u} := \bar{u}(g^+, \phi^+) \) satisfying

\[ \begin{cases} \bar{u}_t(x,t)-\Delta \bar{u}(x,t) \geq -f(\bar{u}(x,t)) + \int_0^T h(a)g^+(\bar{u}(x,t-a))da, &(x,t) \in \Omega \times (0,T), \\ \frac{\partial \bar{u}}{\partial n}(x,t) = 0, &x \in \partial \Omega, t > 0, \\ \bar{u}(x,t) = \phi^+(x,t), &(x,t) \in \Omega \times [-\tau, 0]. \end{cases} \]  

Similarly a subsolution of (1) is a smooth bounded function \( \underline{u} := \underline{u}(g^-, \phi^-) \) satisfying

\[ \begin{cases} \underline{u}_t(x,t)-\Delta \underline{u}(x,t) \leq -f(\underline{u}(x,t)) + \int_0^T h(a)g^-(\underline{u}(x,t-a))da, &(x,t) \in \Omega \times (0,T), \\ \frac{\partial \underline{u}}{\partial n}(x,t) = 0, &x \in \partial \Omega, t > 0, \\ \underline{u}(x,t) = \phi^-(x,t), &(x,t) \in \Omega \times [-\tau, 0]. \end{cases} \]  

Remark 1. If \( u \leq \bar{u} \) on \( \Omega \times [-\tau, T] \) we say that \((u, \bar{u})\) is an ordered pair.

The following proposition is a particular case of Proposition 8.3.4 and Corollary 8.3.5 in [29].

**Proposition 1.** Let \( u, \bar{u} \) be bounded ordered sub and supersolutions of (1). Then (4) has a unique mild solution \( u \) on \([0, \sigma_\phi)\) that satisfies

\[ u(g^-, \phi^-) \leq u(x,t) \leq \bar{u}(g^+, \phi^+), \quad \text{for } (x,t) \in \tilde{\Omega} \times [0, \sigma_\phi), \]

with \( \sigma_\phi \) the largest time for which all these functions are defined.

**Lemma 2.1.** If \( \phi \in X_+ \), then the problem (4) admits a unique solution say \( u \). In addition we have

(i) \( u_t \in X_+ \) for all \( t \in [0, \sigma_\phi) \),

(ii) the solution \( u \) of (1) is bounded and \( \sigma_\phi = \infty \),

(iii) \( u(x,t) \) is a classical solution of (1) for \( x \in \tilde{\Omega} \) and \( t > \tau \).
Proof. Let $L \geq B$. Then for any $\phi \in [0,L] \times \mathbb{R}$, $u = 0$ corresponding to $g^- = g(0)$ in (6), is a subsolution of (1). Also, let $g^+ (s) := \max_{v \in [0,s]} g(v)$. Using (T2) the function $\bar{u}(g^+ , \phi) = L$ is a supersolution of (1). By Proposition 1, the problem (4) admits a unique solution $0 \leq u(x,t) \leq L$ for all $(x,t) \in \Omega \times [0,\tau]$. Since $L$ is arbitrarily large, it follows that $\sigma_\phi = \infty$ for all $\phi \in X_+$. This completes the proof of statements (i) and (ii). The statement (iii) follows from (ii) and Theorem 2.2.6 in [29].

We further have the following results.

Lemma 2.2. If $\phi \in X_+ \setminus \{0\}$, then

(i) $u_t \in \text{Int}(X_+)$ for all $t > 2\tau$, where $\text{Int}(X_+)$ is the interior set of $X_+$.

(ii) The semiflow $U_t$ admits a compact attractor.

Proof. To prove the statement (i) we claim that $u_\tau \in X_+ \setminus \{0\}$. Otherwise, the Lemma 2.1 implies that $u_\tau = 0$ and thus $u(x,t) = 0$ for all $(x,t) \in \Omega \times [0,\tau]$. From (4) we have $\int_0^t T(t-s)F(u_s)ds = 0$ for all $t \in [0,\tau]$. Since $T(.)$ is a strongly positive semigroup we conclude that $F(u_s) = 0$ for all $s \in [0,\tau]$. Moreover, as $u(x,t) = 0$ for all $(x,t) \in \Omega \times [0,\tau]$, and (3), it follows

$$F(u_s) = -f(0) + \int_0^T h(a)g(u(\cdot, s-a))da = 0.$$ 

In view of (T1), we get

$$\int_0^T h(a)\left[g(u(\cdot, s-a)) - g(0)\right]da = 0,$$

for all $s \in [0,\tau]$. This implies that $g(\phi(\cdot, \tau)) = g(0)$. Hence $\phi = 0$ which is absurd. Similarly we may show that $u_{2\tau} \in X_+ \setminus \{0\}$. Thus there exists $(x^*, t^*) \in \Omega \times [\tau, 2\tau]$ such that $u(x^*, t^*) > 0$. It follows from (1), (T1) and (T2) that

$$u_t(x,t) - \Delta u_t(x,t) \geq -f(u(x,t)) + g(0),$$

$$= f(0) - f(u(x,t)),$$

$$\geq -Lu(x,t), \quad \text{in} \quad \Omega \times (t^*, \infty),$$

with $L$ a Lipschitz constant associated to $f$. In addition, $u$ is such that

$$\begin{cases}
\frac{\partial u}{\partial n} = 0 & \text{on} \quad \partial \Omega \times (t^*, \infty), \\
u(x,t^*) \geq 0 & \text{for all} \quad x \in \Omega.
\end{cases}$$

We introduce the problem

$$\begin{cases}
v_t(x,t) - \Delta v(x,t) = -Lv(x,t) & \text{in} \quad (x,t) \in \Omega \times (t^*, \infty), \\
\frac{\partial v}{\partial n} = 0 & \text{on} \quad \partial \Omega \times (t^*, \infty), \\
v(x,t^*) = u(x,t^*) & \text{for} \quad x \in \Omega.
\end{cases}$$

First by applying Theorem 7.3.4 in [23] we obtain that $u(x,t) \geq v(x,t)$ for all $(x,t) \in \Omega \times (t^*, \infty)$. Next by $v(x^*, t^*) > 0$ and Theorem 7.4.1 in [23] we have $v(x,t) > 0$ for all $(x,t) \in \Omega \times (t^*, \infty)$. So $u(x,t) > 0$ for all $(x,t) \in \Omega \times (t^*, \infty)$ and the statement (i) holds. Now we focus on the statement (ii). Let $g^+(s) := \max_{v \in [0,s]} g(v)$ and let $\bar{u}$ be a function verifying the system

$$\begin{cases}
\bar{u}(t) = -f(\bar{u}(t)) + \int_0^t g^+(\bar{u}(t-a))da, \quad t > 0, \\
\bar{u}(t) = \psi(t) := \max_{x \in \Omega} \phi(x,t), \quad t \in [-\tau, 0].
\end{cases}$$

(7)
The existence of $\bar{u}$ can be easily proved, see for instance [8].

First of all according to Theorem 2-1 in [26] we can show that $\bar{u}$ is uniformly bounded provided $\psi$ is uniformly bounded.

Next observe that the solution of (7) $\bar{u}(t) := \bar{u}(g^+, \psi)$ is a supersolution of (1). Hence Proposition 1 ensures that $u(x, t) \leq \bar{u}(t)$ for all $(x, t) \in \Omega \times [0, \infty)$. We claim that $\limsup_{t \to \infty} \bar{u}(t) \leq B$ for any $\phi \in X_+$. Suppose on the contrary that $\limsup_{t \to \infty} \bar{u}(t) := l > B$. Then in view of (T2) we have

$$- f(l) + g^+(l) < 0. \quad (8)$$

From ([28], Proposition A.22) there exists a sequence $(t_n)_n$, $t_n \to \infty$ such that $\bar{u}(t_n) \to l$ and $\bar{u}'(t_n) \to 0$. Substituting $\bar{u}(t_n)$ in (7)

$$\bar{u}'(t_n) = -f(\bar{u}(t_n)) + \int_0^T h(a)g^+(\bar{u}(t_n - a))da. \quad (9)$$

By the definition of $\limsup \bar{u}(t)$ and the monotonicity of the continuous function $g^+$ we have

$$g^+(\limsup_{n \to \infty} \bar{u}(t_n - a)) \leq g^+(\lim_{n \to \infty} \bar{u}(t_n)) := g^+(l), \quad \forall a \in [0, \tau]. \quad (10)$$

Passing to the limit in (9) and combining with (10) we get

$$0 \leq -f(l) + g^+(l).$$

This provides a contradiction with (8). Therefore the semiflow $U_t : X_+ \to X_+$ is point dissipative and eventually bounded on bounded sets, (these concepts are presented for instance in Definition 2.2.25, [25]). Applying Theorem 2.2.33 in [25], if $U_t$ is point dissipative, eventually bounded and asymptotically smooth then there exists a compact global attractor $K$ which also attracts every bounded set in $X_+$, see also Theorem 3.6. in [22]. We can check that $U_t$ is asymptotically smooth for $t > \tau$ by using point 5 of Theorem 2.2.6 [29], (the concept of asymptotically smooth is introduced, e.g., in definition 2.2.25 [25]). Clearly $K \subset [0^+, B^+]_X$. Hence the statement (ii) is reached. The proof of Lemma 2.1 is completed.

It is well known that under specific conditions on $f$ and $g$ the stationary problem corresponding to (1)

$$\begin{cases}
-\Delta U(x) = -f(U(x)) + g(U(x)), & x \in \Omega, \\
\frac{\partial U}{\partial n}(x) = 0, & x \in \partial \Omega,
\end{cases} \quad (11)$$

admits a unique positive solution.

For the convenience of the reader we give a theorem that ensures existence and uniqueness of positive solution to problem (11).

For a smooth domain $\Omega$, we consider the problem

$$\begin{cases}
-\Delta u = F(u) & \text{in } \Omega, \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega,
\end{cases} \quad (12)$$

with $F(u) = g(u) - f(u)$. Suppose that $F$ satisfies the one-sided Lipschitz condition

$$F(u_1) - F(u_2) \geq -c(u_1 - u_2) \quad \text{for } \underline{u} \leq u_1 \leq u_2 \leq \bar{u}, \quad (13)$$

where $c$ is a nonnegative function and $(\underline{u}, \bar{u})$ is any pair of ordered sub and supersolutions of (12).
Theorem 2.3. Let \( u(x) \) and \( \bar{u}(x) \) be ordered bounded positive sub and supersolutions of (12) respectively. Suppose that (13) holds and that \( \frac{F(u)}{u} \) is a strictly monotone function for \( u \in [\bar{u}, \tilde{u}] \). Then problem (12) admits a unique positive solution in \([\bar{u}, \tilde{u}]\).

Proof. The existence of minimal and maximal positive solutions can be deduced by Theorem 3.2.2 in [21]. Let \( u_1 \) and \( u_2 \) be the minimal and the maximal solution of problem (12) respectively. We have

\[-u_2 \Delta u_1 = u_2 F(u_1)\]
and
\[-u_1 \Delta u_2 = u_1 F(u_2).\]
Subtracting these two equations and integrating the result over \( \Omega \), we get

\[\int_{\Omega} (u_1 \Delta u_2 - u_2 \Delta u_1) dx = \int_{\Omega} (u_2 F(u_1) - u_1 F(u_2)) dx.\]

An integration by parts and the homogenous Newmann conditions give

\[\int_{\Omega} (u_2 F(u_1) - u_1 F(u_2)) dx = 0,\]
so,

\[\int_{\Omega} u_1 u_2 \left( \frac{F(u_1)}{u_1} - \frac{F(u_2)}{u_2} \right) dx = 0.\]

It follows from \( u_1 u_2 > 0, u_1 \leq u_2 \) and the monotone property of \( \frac{F(u)}{u} \) that

\[\frac{F(u_1)}{u_1} = \frac{F(u_2)}{u_2}\]
in \( \Omega \).

Thus \( u_1 = u_2 \). \( \square \)

As a consequence of Theorem 9.3.3 in [29], we have the following asymptotic property of solution of problem (1) in the case where \( g \) is nondecreasing.

Theorem 2.4. Let \( u(x) \) and \( \bar{u}(x) \) be bounded ordered positive sub and supersolutions of (1) respectively (not depending on time \( t \)). Assume also that the problem (11) admits a unique positive solution \( u^* \) in \([\bar{u}, \tilde{u}]\). If \( g \) is a nondecreasing function, then all solutions of (1) converge to the positive steady state \( u^* \), uniformly in \( x \in \bar{\Omega} \).

Suppose now that there exists \( u^* \) such that

\[
\begin{align*}
g(u) > f(u) & \quad \text{for all } 0 \leq u < u^* \\
g(u) < f(u) & \quad \text{for all } u^* < u \leq B.
\end{align*}
\] (14)

Theorem 2.5. Assume that (14) holds and \( g \) is a nondecreasing function. Suppose also that (11) admits a unique positive solution \( u^* \). Then the solution of problem (1) is strongly persistent and converges to \( u^* \), uniformly in \( x \in \bar{\Omega} \) for all initial function \( \phi \in X_+ \setminus \{0\} \).

Proof. In view of (14) there exists \( \varepsilon > 0 \) such that \( g(\varepsilon) \geq f(\varepsilon) \). By Lemma 2.2 (i), \( u(x,t) > 0 \) for all \( t > 2\tau \) and \( x \in \bar{\Omega} \). Now, suppose that \( u(x,t) \geq \varepsilon \) for \( (x,t) \in \bar{\Omega} \times [3\tau, 4\tau] \), then for \( t \geq 3\tau \) the function \( \bar{u}(g, \varepsilon) = \varepsilon \) is a subsolution of (1). Moreover, if \( g^+(u) := \max_{v \in [0,\varepsilon]} g(v) \), the function \( \bar{u}(g^+, \phi) = B \) is a supersolution of (1). According to Proposition 1, the solution of (1) is strongly persistent. The uniform convergence towards the steady state is reached according to Theorem 2.4. \( \square \)
3. The case where the delayed term $g$ is non-monotone.

3.1. **Persistence and estimates of solutions.** The aim of this section is to state some fundamental results, including the strong persistence and the closed attractive intervals for solutions of problem (1) in the case where $g$ is non-monotone.

We make the following assumptions, that will be used from now on.

There exists a positive constant $u^*$ such that

$$
\begin{align*}
\min_{\sigma \in [s, u^*]} g(\sigma) &> f(s), \quad \text{for} \quad 0 < s < u^*, \\
\max_{\sigma \in [u^*, s]} g(\sigma) &< f(s), \quad \text{for} \quad u^* < s \leq B.
\end{align*}
$$

(15)

$f'(0)$ and $g'(0)$ exist and verify $g'(0) > f'(0) > 0$ with $f(s) > f(0), \quad \forall s > 0.$

Due to the continuity of $f$ and $g$ it is clear from (15) that $u^*$ is the unique positive value for which $g(u^*) = f(u^*)$.

Our objective is to prove the strong persistence and to find the attractive intervals of solution of (1). To achieve this we shall construct appropriate sub and supersolutions of (1). We state the following lemmas which were proved in [26]. For the convenience of the reader we shall reproduce the proofs.

**Lemma 3.1.** Suppose that (15) and (16) hold. Then there exists a positive constant $0 < m < u^*$ satisfying

$$
\begin{align*}
g(s) &> f(m) \quad \text{for} \quad m \leq s \leq B, \\
f(s) &< f(m) \quad \text{for} \quad 0 \leq s < m, \\
f(s) &> f(m) \quad \text{for} \quad m < s \leq B.
\end{align*}
$$

(17)

Moreover $f$ and $g$ are strictly increasing over $[0, m]$.

**Proof.** First from (16) there exists $\sigma_0 > 0$ such that $f$ and $g$ are strictly increasing over $[0, \sigma_0]$. Take $\gamma$ as

$$
\gamma := \min_{\sigma \in [\sigma_0, u^*]} \{f(\sigma)\} = f(\sigma_1).
$$

(18)

Since $f(\sigma_1) > f(0)$ there exists $m_1 \in (0, \sigma_0]$ such that $f(m_1) = f(\sigma_1)$. Notice that for $\varepsilon > 0$ small enough the positive constant $(m_1 - \varepsilon)$ satisfies

$$
\begin{align*}
f(s) &< f(m_1 - \varepsilon) \quad \text{for} \quad 0 \leq s < m_1 - \varepsilon, \\
f(s) &> f(m_1 - \varepsilon) \quad \text{for} \quad m_1 - \varepsilon < s \leq B.
\end{align*}
$$

Indeed, since $f$ is strictly increasing over $[0, \sigma_0]$ and $m_1 \leq \sigma_0$ it follows that $f(s) < f(m_1 - \varepsilon)$ for all $s \in [0, m_1 - \varepsilon)$ and $f(s) > f(m_1 - \varepsilon)$ for all $s \in (m_1 - \varepsilon, \sigma_0]$.

Further, for $\sigma_0 \leq s \leq u^*$ and from (18) we get

$$
f(s) \geq f(m_1) = f(\sigma_1).
$$

Thus

$$
f(s) > f(m_1 - \varepsilon).
$$

If $u^* < s \leq B$ by (15)

$$
f(s) > \max_{\sigma \in [u^*, s]} \{g(\sigma)\} \geq g(u^*) = f(u^*) > f(\sigma_1).
$$

From (18) we have

$$
f(s) > f(m_1) > f(m_1 - \varepsilon).
$$
We define \( \alpha := \min_{\sigma \in [m_1 - \varepsilon, B]} \{ g(\sigma) \} \). If \( f(m_1 - \varepsilon) < \alpha \) then \( m = (m_1 - \varepsilon) \) satisfies (17), otherwise there exists \( m_2 \in (0, m_1 - \varepsilon) \) such that \( f(m_2) = \alpha \). Finally using the fact that \( g \) is strictly increasing over \([m_2 - \varepsilon, m_1 - \varepsilon]\) and the first assertion of (15) we see that \( m = (m_2 - \varepsilon) \) satisfies (17). This completes the proof.

Now we consider the function
\[
g_m^B(s) = \begin{cases} 
g(s), & \text{for } 0 < s < \bar{m}, \\
(f(m), & \text{for } \bar{m} < s \leq B, 
\end{cases} 
\]
with \( \bar{m} \) and \( m \) the constants satisfying \( \bar{m} < m \) and \( g(\bar{m}) = f(m) \), (see (17)).

The following result is easily checked.

Lemma 3.2. Assume that (15) and (16) hold. Then \( g_m^B \) defined in (19) is a non-decreasing function over \((0, B)\) satisfying
\[
\begin{align*}
g_m^B(s) &\leq g(s), & 0 \leq s \leq B, \\
g_m^B(s) &> f(s), & 0 < s < m, \\
g_m^B(s) &< f(s), & m < s \leq B.
\end{align*}
\]

Now we are in a position to prove the strong persistence of solutions of (1).

Lemma 3.3. Assume that (15) and (16) hold. Then the solution of problem (1) is strongly persistent provided the corresponding initial data \( \phi \in X_+ \setminus \{0\} \).

Proof. Let \( v \) be the solution of the problem
\[
\begin{align*}
v_t(x,t) - \Delta v(x,t) &= -f(v(x,t)) + \int_0^T h(a)g_m^B(v(x,t-a))da, & t > 0, x \in \Omega \\
\frac{\partial v}{\partial n}(x,t) &= 0, & x \in \partial \Omega, t > 0, \\
v(x,t) &= \phi(x,t), & (x,t) \in \bar{\Omega} \times [-\tau, 0],
\end{align*}
\]
with \( g_m^B \) defined in (19). Since \( g_m^B \) is a nondecreasing function and \( g_m^B \leq g \) then, the function \( v \) can be seen as a subsolution of problem (1), that is \( v := u(\bar{g}_m^B, \phi) \). Thus, in view of Proposition 1 we have \( v(x,t) \leq u(x,t) \) for all \( (x,t) \in \bar{\Omega} \times [0, \infty) \) with \( u \) solution of (1).

To show that the solution \( v \) of problem (20) is strongly persistent we use Theorem 2.5. Finally, since \( v \leq u \), \( u \) is also strongly persistent.

In the following we focus on functions \( g \) having a maximum. More precisely, assume that the function \( g \) satisfies:

There exists a positive constant \( M \) such that
\[
g(M) = \max_{s \in \mathbb{R}^+} g(s). 
\]

We will investigate two cases, namely \( u^* \leq M \) and \( u^* > M \).

In order to state our next result we need the following lemma.

Lemma 3.4. Under the hypotheses (15) and (21) and the fact that \( u^* \in [0^*, M^*] \), Then the interval \([0^*, M^*] \) attracts every solution \( u \) of problem (1), equivalently there exists \( T > 0 \) such that
\[
0 \leq u(x,t) \leq M, \quad \forall x \in \bar{\Omega}, \quad t \geq T.
\]
Proof. Let \( g^+(s) = \max_{\sigma \in [0, s]} \{g(\sigma)\} \). Consider the problem
\[
\begin{cases}
  v'(t) = -f(v(t)) + \int_0^r h(a) g^+(v(t - a)) da, & \text{for } t > 0, \\
  v(t) = \psi(t) := \max_{x \in \Omega} \phi(x, t) & \text{for } -\tau \leq t \leq 0.
\end{cases}
\] (22)

\( \bar{u}(g^+, \psi) \) is a supersolution of (1) with \( \bar{u} = v \). Proposition 1 leads to \( u(x, t) \leq v(t) \) for all \((x, t) \in \Omega \times [0, \infty)\). We will prove now that there exists \( T > 0 \) such that \( v(T) \leq M \). Indeed, suppose by contradiction that \( v(t) > M \) for all \( t > 0 \). Then, from the second assertion of (15) and the fact that \( u^* < M < v \) we have
\[
g^+(v(t)) = \max_{s \in [0, v(t)]} g(s) = g(M) = \max_{s \in [u^*, v(t)]} g(s) < f(v(t)).
\]

For all \( t > \tau \) (22) gives
\[
v'(t) = -f(v(t)) + g(M) < 0.
\]

Since \( v'(t) < 0 \) and \( v(t) > M \) for all \( t > \tau \), \( v(t) \) admits a limit \( l \geq M \) as \( t \) tends to infinity. From this, there exists a sequence \((t_n)\), \( t_n \to \infty \) such that \( v(t_n) \to l \) and \( v'(t_n) \to 0 \) as \( t_n \to \infty \). By substituting \( v(t_n) \) in the equation of (22) and passing to the limit as \( t_n \to \infty \) we obtain
\[
f(l) = g^+(l).
\]

From the second assertion of (15) and as \( l \geq M > u^* \) we get
\[
f(l) = g^+(l) := \max_{s \in [0, l]} \{g(s)\} = g(M) = \max_{s \in [u^*, l]} \{g(s)\} < f(l).
\]

This is a contradiction. Therefore there exists \( T > 0 \) such that \( v(T) \leq M \). Next we claim that \( v(t) \leq M \) for all \( t > T \). Indeed, suppose on the contrary that there exists a positive constant \( \bar{t} \geq 0 \) such that \( v(\bar{t}) = M \) and \( v'(\bar{t}) \geq 0 \). By (15) and \( u^* \leq M \) we have
\[
g(M) \leq f(M). \tag{23}
\]

First suppose that \( u^* < M \) so
\[
g(M) < f(M). \tag{24}
\]

Next by substituting \( \bar{t} \) in the equation of (22) we obtain
\[
0 \leq -f(M) + \int_0^\tau h(a) g^+(v(\bar{t} - a)) da,
\]

since \( g^+(s) := \max_{\sigma \in [0, s]} \{g(\sigma)\} \leq g(M) \) for all \( s \in \mathbb{R}^+ \) we arrive at
\[
f(M) \leq g(M).
\]

This is a contradiction with (24). The claim is proved.

Now if \( u^* = M \) it follows that
\[
\max_{v \in [0, s]} \{g(v)\} = g(u^*), \quad \forall s > u^*. \tag{25}
\]

Observe that from the second assertion of (15) we get \( g(u^*) < f(s) \) for all \( u^* < s \leq B \). Thus combining this with (25) we conclude that
\[
\max_{v \in [0, s]} \{g(v)\} < f(s), \quad \forall s > u^*. \tag{26}
\]

Therefore according to Lemma 2.2 (ii) (substituting in the hypothesis (T2), \( B \) by \( u^* \)) we show that \( \limsup_{t \to \infty} v(t) \leq u^* \). Moreover, using the same arguments as above we can show that \( [0^*, M^*]_C \) is positively invariant. Finally, since the semiflow
$U_t$, defined in (2) for $t > \tau$ admits a compact global attractor $K$ which also attracts every bounded set in $X_\cdot$. Then $K \subset [0^*, M^*]$ see also the end of the proof of Lemma 2.2.

In the rest of this section we focus on $u^* > M$. We impose an additional hypothesis on $f$ and $g$.

Assume that

$$
\begin{cases}
  f(s) < f(M) & \text{for } 0 \leq s < M, \\
  f(s) > f(M) & \text{for } M < s \leq B.
\end{cases}
$$

Setting

$$
\mathbb{D} \equiv \{ s \in [0, M] : g(s) = f(M) \}.
$$

Observe that $\mathbb{D} \neq \emptyset$ since $g(0) = f(0) < f(M) < g(M)$. In the same way we define the set

$$
\mathbb{D} \equiv \{ s \in [M, B] : g(s) = f(M) \}.
$$

The rest of this subsection is devoted to estimate the solutions of (1) when either $\mathbb{D} = \emptyset$ or $\min \mathbb{D}$ exists. The following lemma deals with the first case.

**Lemma 3.5.** Assume that $\mathbb{D} = \emptyset$ and $u^* > M$. We also suppose that (15), (16), (27) hold. Then the interval $[M^*, B^*] \chi$ attracts every solution $u$ of problem (1).

**Proof.** It is clear that $\mathbb{D} = \emptyset$ implies

$$
g(s) > f(M) \quad \text{for all } s \in [M, B].
$$

First from (15) there exists $\bar{m} \in (0, M)$ such that

$$
g(\bar{m}) = f(M) \quad \text{and} \quad \bar{m} = \max \mathbb{D}.
$$

In addition since $g(M) > f(M)$ the value $\bar{m}$ satisfies

$$
\min_{\sigma \in [\bar{m}, M]} \{ g(\sigma) \} = f(M),
$$

otherwise $g(\bar{\sigma}) := \min_{\sigma \in [\bar{m}, M]} \{ g(\sigma) \} < f(M)$ with $\bar{\sigma} \in (\bar{m}, M)$. Further since $g(M) > f(M)$ there exists $\bar{\sigma} \in (\bar{\sigma}, M)$ such that $g(\bar{\sigma}) = f(M)$, a contradiction with (29).

Next we introduce the function

$$
g^B_M(s) = \begin{cases}
  \min_{\sigma \in [s, M]} \{ g(\sigma) \}, & \text{for } 0 < s < \bar{m}, \\
  f(M), & \text{for } \bar{m} < s \leq B.
\end{cases}
$$

We claim that the function $g^B_M$ is nondecreasing and satisfies

$$
\begin{cases}
  g^B_M(s) \leq g(s), & \text{for } 0 \leq s \leq B, \\
  g^B_M(s) > f(s), & \text{for } 0 < s < M, \\
  g^B_M(s) < f(s), & \text{for } M < s \leq B.
\end{cases}
$$

In fact from (28), (30) and (31) it is easily checked that $g^B_M$ is nondecreasing and $g^B_M(s) \leq g(s)$ for all $s \in [0, B]$. Further, taking $0 < s < \bar{m}$ and using the fact that $M < u^*$ we have

$$
g^B_M(s) = \min_{\sigma \in [s, M]} \{ g(\sigma) \} \geq \min_{\sigma \in [s, u^*]} \{ g(\sigma) \}.
$$

In view of (15) we get

$$
g^B_M(s) > f(s).
$$
For \( \bar{m} \leq s < M \), the hypothesis (27) implies that
\[
g^B_M(s) = f(M) > f(s).
\]
For \( M < s \leq B \), using again (27) we obtain
\[
g^B_M(s) = f(M) < f(s).
\]
Next, we consider the problem
\[
\begin{cases}
  v'(t) = -f(v(t)) + \int_\tau^t h(a)g^B_M(v(t-a))da, & t > 0, \\
v(t) = \psi(t) := \min_{x \in \bar{\Omega}} \phi(x,t), & -\tau \leq t \leq 0.
\end{cases}
\]
(33)

Then \( u(g^-,\psi) \) is a subsolution of (1) with \( u = v \) and \( g^- = g^B_M \). By Proposition 1 we get \( v(t) \leq u(x,t) \leq B \) for all \((x,t) \in \Omega \times [T,\infty)\). Moreover, since \( g^B_M \) is a nondecreasing function and the problem (33) admits only \( M \) as constant positive steady state, the function \( v(t) \) goes to \( M \) as \( t \) tends to infinity by Theorem 2.4. As a conclusion
\[
\lim_{t \to \infty} u(x,t) \geq M \quad \text{for all } x \in \bar{\Omega}.
\]
Observe also that \( [M^*,B^*]_X \) is an invariant closed interval for the system (1), that is for \( \phi \in [M^*, B^*]_X \) we have \( u \in [M^*, B^*]_X \) (this result is a direct consequence of Proposition 1 and the fact that \( u(g^-,M) \) with \( u = M \) and \( g^-(\cdot) = g^B_M(\cdot) \) is a subsolution of (1)). Finally, the proof is completed by Lemma 2.2 (ii). \( \square \)

Now, we will focus on the case where \( \min \mathbb{D} \) exists.

In this context and throughout the rest of the paper we define the constant \( A \) as
\[
A := \min \{ s \in (M, B], \ g(s) = f(M) \},
\]
(34)
and \( M \) is defined in (21). The constant \( A \) plays a crucial role in estimating the solution of problem (1).

We suppose that
\[
\begin{align*}
  f(s) &< f(A) &\text{for } u^* \leq s < A, \\
  f(s) &> f(A) &\text{for } A < s \leq B.
\end{align*}
\]
(35)
The following lemma proves the existence of an attractive interval of solutions of (1).

**Lemma 3.6.** Suppose that (15), (16), (27), (35) are fulfilled. Assume also that \( u^* > M \) and
\[
f(A) \geq g(M).
\]
(36)
Then the interval \([M^*,A^*]_X\) attracts every solution \( u \) of problem (1).

**Proof.** We claim that \( u^* < A \). On the contrary suppose \( u^* \geq A \). Then, either \( u^* > A \) or \( u^* = A \). First, \( u^* \) cannot be greater than \( A \). For, it follows from (15) and \( M < A \) that
\[
f(M) := g(A) \geq \min_{\sigma \in [M,u^*]} \{ g(\sigma) \} > f(M).
\]
This is a contradiction. Next, \( u^* \neq A \), otherwise, by (15) again,
\[
f(M) = g(A) = f(A) = f(u^*).
\]
However \( u^* > M \). The second assertion in (27) leads to a contradiction.
With the help of (35) and (36) we can construct a nondecreasing function $g^+$ over $[0, B]$ satisfying
\[
\begin{cases}
g^+(s) \geq g(s), & \text{for } 0 \leq s \leq B, \\
g^+(s) > f(s), & \text{for } 0 < s < A, \\
g^+(s) < f(s), & \text{for } A < s \leq B.
\end{cases}
\] (37)

Now, let $\bar{v}$ be the solution of the problem
\[
\begin{aligned}
v_t(x,t) - \Delta v(x,t) &= -f(v(x,t)) + \int_0^T h(a)g^+(v(x,t,a))da, \quad (x,t) \in \Omega \times [0,T], \\
\frac{\partial v}{\partial n}(x,t) &= 0, \quad x \in \partial \Omega, \quad t > 0, \\
v(x,0) &= \phi(x), \quad (x,t) \in \bar{\Omega} \times [-\tau,0].
\end{aligned}
\] (38)

Then $\bar{u}(g^+, \phi) = \bar{v}$ is a supersolution of (1). It follows from Proposition 1 that $u(x,t) \leq \bar{v}(x,t)$ for all $(x,t) \in \Omega \times [0, \infty)$. Therefore, Theorem 2.4 combined with (37) give
\[
\limsup_{i \to \infty} u(x,t) \leq \lim_{i \to \infty} \bar{v}(x,t) = A, \quad \forall x \in \bar{\Omega}.
\]
Notice that $\bar{u}(g^+, A) = A$ is a supersolution of (1). Then $[0^*, A^*]_X$ is an invariant closed interval for (1). The result follows from Lemma 2.2 (ii) . There exists $T > 0$ such that $0 \leq u(x,t) \leq A$ for all $(x,t) \in \Omega \times [T, \infty)$.

To find a positive lower bound for $u$ we use (27), (34). Substitute $A$ for $B$ in (31) we obtain the nondecreasing function $g^+_M$ that satisfies (32). Replacing $g^+$ by $g^+_M$ we obtain a solution $\bar{v}$ of (38). Then $u(g^+_M, \phi) = \bar{v}$ is a subsolution of (1), so that, $\bar{v}(x,t) \leq u(x,t)$ for all $(x,t) \in \Omega \times [0, \infty)$. Thus $\lim_{i \to \infty} u(x,t) \geq \lim_{i \to \infty} \bar{u}(x,t) = M$ for all $x \in \bar{\Omega}$. Finally $[M^*, A^*]_X$ is an invariant closed interval for (1). This completes the proof.

\section*{4. Global attractivity and exponential stability of the positive steady state.}

In this section we prove the global attractivity of the positive steady state $u^*$. In order to achieve this we introduce the following assumption.

There exists a unique positive constant $M$ such that,
\[
g(M) = \max_{s \in \mathbb{R}^+} \{g(s)\}, \quad (39)
\]
the function $g$ is nondecreasing over $(0, M)$ and nonincreasing over $(M, B)$.

In case $u^* \leq M$ we have

\begin{corollary}
Assume that (15) and (16) hold. Then the positive steady state $u^*$ of problem (1) is globally attractive provided that $u^* \leq M$ and $\phi \in X_+ \setminus \{0\}$.
\end{corollary}

\begin{proof}
It is a consequence of Lemma 3.4 and Theorem 2.4.
\end{proof}

Now we deal with the situation where $u^* > M$. Following Lemma 3.6 we shall assume without loss of generality that $\phi \in [M^*, A^*]_X$.

Let $\hat{f}$ be the restriction of $f$ to $[M, A]$. In case $\hat{f}$ is strictly increasing over $[M, A]$ we let $G(s) := \hat{f}^{-1}(g(s))$ for $s \in [M, A]$.

\begin{theorem}
Under the assumptions of Lemma 3.6, the positive steady state of problem (1) is globally attractive provided that one of the following conditions holds
\begin{enumerate}
\item[(H1)] $(f(s) - f(0))(g(s) - g(0))$ is a nondecreasing function over $[M, A]$.
\item[(H2)] $f + g$ is a nondecreasing function over $[M, A]$.
\end{enumerate}
\end{theorem}
(H3) \((f(s) - f(0))(g(s) - g(0)) + f + g\) is a nondecreasing function over \([M, A]\).
(H4) \(f\) is strictly increasing over \([M, A]\) and \(\frac{(GoG)(s)}{s}\) is a nonincreasing function over \([M, x^*]\).
(H5) \(f\) is strictly increasing over \([M, A]\) and \(\frac{(GoG)(s)}{s}\) is a nonincreasing function over \([x^*, A]\).

Proof. 1 Non-oscillatory case. We first suppose that for all solutions \(u\) of (1) there exists \(T > 0\) such that \(u(x, t) \leq u^*\) for all \(x \in \Omega\) and \(t > T\). Let \(g^-\) be defined as
\[
g^-(s) = \begin{cases} 
\min_{\sigma \in [s, u^*]} \{g(\sigma)\} & \text{for } 0 < s < u^*, \\
g(u^*) & \text{for } u^* \leq s \leq A.
\end{cases}
\]
The function \(u(g^-, \phi)\) is a subsolution of (1), it follows from Proposition 1 that \(u(x, t) \leq u^*\) for all \((x, t) \in \Omega \times [0, \infty)\). Further, in view of Theorem 2.5, \(u(x, t)\) converges to \(u^*\) as \(t\) tends to infinity, uniformly in \(x \in \Omega\). This implies that each solution of (1) converges to the positive steady state \(u^*\), uniformly in \(x \in \Omega\).

Next, if \(u(x, t) \geq u^*\) for all \(t \geq T\), then define \(g^+\) by
\[
g^+(s) = \begin{cases} 
\min_{\sigma \in [s, u^*]} \{g(\sigma)\} & \text{for } 0 < s \leq u^*, \\
\max_{\sigma \in [u^*, A]} \{g(\sigma)\} & \text{for } u^* < s \leq A.
\end{cases}
\]
Thus \(\tilde{u}(g^+, \phi)\) is a supersolution of (1), so that all solutions \(u(x, t)\) of (1) satisfy \(u(x, t) \leq \tilde{u}(x, t)\) for all \((x, t) \in \Omega \times [0, \infty)\). Moreover, \(\tilde{u}(x, t)\) converges to \(u^*\) as \(t\) goes to infinity, uniformly for all \(x \in \Omega\).

2 Oscillatory case. We claim that this situation is not possible. Assume by contradiction that for each \(x \in \Omega\) the solution \(u(x, .)\) oscillates infinitely around the positive steady state \(u^*\). Using the fluctuation method developed in [24], (see also [38] and the references therein) we will arrive at the desired contradiction.

Let
\[
u(x) := \limsup_{t \to \infty} u(x, t), \quad \nu(x) := \liminf_{t \to \infty} u(x, t),
\]
and
\[
u(x) := \max_{x \in \Omega} u(x), \quad \nu(x) := \min_{x \in \Omega} u(x).
\]
Lemma 3.6 implies that
\[
M \leq \nu(x) \leq \nu(x) \leq A \quad \forall x \in \Omega,
\]
and
\[
M \leq \nu(x) < \nu(x) \leq A.
\]
Write \(f(u) = -\alpha u + (\alpha u + f(0) - f(u)) - f(0)\). Since \(u\) is uniformly bounded we can choose a constant \(\alpha > 0\) such that \(\alpha u + f(0) - f(u)\) is increasing in \(u\). Using the Green's function \(\Gamma\) associated with \(w = \Delta w\) subject to homogenous Neumann boundary conditions and the fact that \(f(0) = g(0)\) we have
\[
u(x, t) = e^{-\alpha t} \int_{\Omega} \Gamma(x, y, t) u(y, 0) dy
\]
\[
+ \int_{0}^{t} e^{-\alpha s} \int_{\Omega} \Gamma(x, y, s) \left(\alpha u(y, t - s) + f(0) - f(u(y, t - s))\right) dy ds,
\]
\[
+ \int_{0}^{t} h(a) \left(g(u(y, t - s) - g(0)) da\right) dy ds,
\]
where $u$ is solution of problem (1).

Since $g$ is nonincreasing over $[M, A]$ Fatou’s Lemma implies that
\[ u^\infty(x) \leq \int_0^\infty e^{-\alpha s} \int_{\Omega} \Gamma(x, y, s) \left( \alpha u^\infty(y) - f(u^\infty(y)) + g(u^\infty(y)) \right) dy ds. \]

It is well known that
\[ \int_{\Omega} \Gamma(x, y, s) dy = 1, \quad \forall s > 0, \quad x \in \Omega. \]

Hence
\[ f(u^\infty) \leq g(u^\infty). \quad (41) \]

Similarly we obtain
\[ f(u^\infty) \geq g(u^\infty). \quad (42) \]

Multiplying the expression (43) by $(g(u^\infty) - g(0))$ and combining with (44) we obtain
\[ (f(u^\infty) - f(0))(g(u^\infty) - g(0)) \leq (f(u^\infty) - f(0))(g(u^\infty) - g(0)). \]

This fact together with (H1) give $u^\infty \leq u^\infty$. We reach a contradiction. Arguing as before we may conclude the results using (H2) and (H3). Now suppose that (H4) holds. First recall that $\hat{f}$ is strictly increasing over $[M, A]$. Since $g$ is nonincreasing over $[M, A]$ we have $g(A) \leq g(s) \leq g(M)$ for all $s \in [M, A]$. Taking into account (34) and (36) we obtain
\[ \hat{f}(M) \leq g(s) \leq \hat{f}(A) \quad \text{for all} \quad s \in [M, A]. \]

As a conclusion, the function $G$ is nonincreasing and maps $[M, A]$ to $[M, A]$.

Next in view of (41), (42) we get
\[ u^\infty \leq G(u^\infty), \quad (45) \]

and
\[ u^\infty \geq G(u^\infty). \quad (46) \]

Assume that $u^\infty < u^* \leq u^\infty$. Then (45) and (46) become
\[ u^\infty \geq G(u^\infty) \geq (GoG)(u^\infty). \]

This gives
\[ \frac{(GoG)(u^\infty)}{u^\infty} \leq 1 = \frac{(GoG)(u^*)}{u^*}, \quad (47) \]

(H4) shows that $u^* \leq u^\infty$. This is a contradiction. Again, if $u^\infty \leq u^* < u^\infty$ then (47) leads to
\[ u^\infty = u^*. \]

According to (41) we have
\[ f(u^\infty) \leq g(u^\infty) = g(u^*) = f(u^*) < f(u^\infty). \]

The contradiction is also reached. In similar way we prove the result using (H5). Hence the solution cannot be oscillatory around $u^*$. Therefore,
\[ \lim_{t \to \infty} u(x, t) = u^*, \quad (48) \]

uniformly in $x \in \bar{\Omega}$. The Theorem is proved.
Now we investigate the global exponential stability of the positive steady state. For this, we first establish the following lemma.

**Lemma 4.2.** Suppose that there exist two positive constants $\alpha$, $\beta$ and a positive function $w$ such that

$$w_t(x, t) - \Delta w(x, t) \leq -\alpha w(x, t) + \beta \int_0^T h(a)w(x, t - a)da.$$ (49)

If $\alpha > \beta$, then there exists a constant $C > 0$ and $\gamma \in (0, \alpha - \beta)$ such that

$$w(x, t) \leq Ce^{-\gamma t}, \quad \forall (x, t) \in \Omega \times [-\tau, \infty).$$

**Proof.** By the comparison principle, the result will be proved if there exists $\gamma \in (0, \alpha - \beta)$ such that $z(t) = Ce^{-\gamma t}$ is solution of the following problem

$$z'(t) = -\alpha z(t) + \beta \int_0^T h(a)z(t - a)da.$$ (50)

Indeed, by substituting the expression of $z$ in (50) we get

$$\beta \int_0^T h(a)e^{\gamma a}da + \gamma = \alpha.$$

Thus for $F(\gamma) := \beta \int_0^T h(a)e^{\gamma a}da + \gamma$ we have

$$F(\alpha - \beta) = \beta \left( \int_0^T h(a)e^{(\alpha - \beta)a}da - 1 \right) + \alpha > \alpha \quad \text{and} \quad F(0) = \beta < \alpha.$$

The Lemma is proved. \qed

Now we are in a position to present our main result concerning the exponential stability of the steady state.

**Theorem 4.3.** Suppose in addition to the conditions of Theorem 4.1, that $f$ and $g$ are differentiable functions and $f$ is strictly increasing function on $[M, A]$. Then the positive steady state $u^*$ of (1) is globally exponentially stable provided that

$$\inf_{s \in [M, A]} f'(s) > \sup_{s \in [M, A]} |g'(s)|.$$ (51)

**Proof.** We have proved that the oscillatory case is not possible. Set $v(x, t) = (u(x, t) - u^*)$. We suppose that there exists $T > 0$ such that $v(x, t) \geq 0$ for all $x \in \Omega$ and $t \geq T$, (the proof will be the same if we assume that $v(x, t) \leq 0$ for all $x \in \Omega$ and $t \geq T$). Then $v$ satisfies

$$v_t(x, t) - \Delta v(x, t) = -f'(\theta(t))v(x, t) + \int_0^T h(a)g'(\theta_1(t - a))v(x, t - a)da,$$

where

$$u^* \leq \theta(t) \leq u(x, t) \quad \text{for all} \quad x \in \bar{\Omega} \quad \text{and} \quad t > 0(u^* \leq \theta_1(t - a) \leq u(x, t - a)).$$

Since $u(x, t)$ and $u^*$ belong to $[M, A]$ we obtain

$$v_t(x, t) - \Delta v(x, t) \leq -\inf_{s \in [M, A]} f'(s)v(x, t) + \sup_{s \in [M, A]} |g'(s)| \int_0^T h(a)v(x, t - a)da.$$

To complete the proof use (51) and apply Lemma 4.2 for $\alpha = \inf_{s \in [M, A]} f'(s)$ and $\beta = \sup_{s \in [M, A]} |g'(s)|$. \qed
Applications. To illustrate our results we give two examples, the diffusive Blowfly with distributed delay equation and the diffusive Mackey-Glass with distributed delay equation. For a good survey on these models we refer to [3] and references therein. For more details concerning the results of stability of both these models see [26].

First, observe that the condition (15) is verified for both models whenever the positive equilibrium exists. We set \( \int_0^\tau h(a)da = 1 \).

The diffusive Nicholson’s blowfly equation. We consider the diffusive Nicholson’s blowfly equation with distributed delay,

\[
 u_t(x,t) - \Delta u(x,t) = -\delta u(x,t) + \int_0^\tau h(a)u(x,t-a)e^{-u(x,t-a)}da, \quad x \in \Omega, \quad t > 0, \quad (52)
\]

together with homogenous Neumann conditions. As a direct application of Corollary 1, Theorem 4.1 (H4) and Theorem 4.3 we get the following results (for more details see [26]).

**Theorem 4.4.** Suppose that \( \delta < 1 \). Then the unique positive steady state of (52) is globally attractive provided that

\[
 1 < \frac{1}{\delta} \leq e^2.
\]

Moreover, this positive steady state is globally exponentially stable if

\[
e < \frac{1}{\delta} < e^2. \quad (53)
\]

The Mackey-Glass model of hematopoiesis. The diffusive Mackey-Glass equation with distributed delay is

\[
 u_t(x,t) - \Delta u(x,t) = -\delta u(x,t) + \int_0^\tau h(a)u(x,t-a)\frac{u(x,t-a)}{1 + n u(x,t-a)}da. \quad (54)
\]

with Neumann homogenous conditions. For more results about this type of problems see [4] and the references therein.

By the application of our results we obtain

**Theorem 4.5.** Suppose that \( \delta < 1 \). Then the positive steady state of (54) is globally attractive if one of the following conditions is satisfied

\[
 0 < n \leq 2.
\]

\[
 n > 2 \quad \text{and} \quad \frac{1}{\delta} < \frac{n}{n - 2}.
\]

Moreover the positive steady state of (54) is globally exponentially stable if the following condition is satisfied

\[
 n > 1 \quad \text{and} \quad \frac{n}{n - 1} < \frac{1}{\delta} < \frac{4n}{(n - 1)^2}.
\]

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Received October 2018; revised October 2019.
E-mail address: touaoula_tarik@yahoo.fr
E-mail address: tarik.touaoula@mail.univ-tlemcen.dz