A note on commuting graphs of matrix rings over fields.

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Abstract

We will give a short proof of the fact that if the algebraic closure of a field $F$ is a finite extension, then for $n \geq 3$ the commuting graph $\Gamma(M_n(F))$ is connected and its diameter is four.

Keywords: Commuting graph; Diameter; Full matrix ring; Jacobson radical.

1 Introduction

Let $F$ be a field and $M_n(F)$ be the algebra of all $n \times n$ matrices over $F$. The commuting graph of $M_n(F)$, denoted by $\Gamma(M_n(F))$, is the graph whose vertices are all non-central matrices and two distinct vertices $A$ and $B$ are adjacent if and only if $AB = BA$.

The connectedness and diameter of the commuting $\Gamma(M_n(F))$ have been studied extensively; see [1, 4, 5, 6]. Note that for $n = 2$ the commuting graph $\Gamma(M_n(F))$ is always disconnected; see [3, remark 8]. So, from now on we assume that $n$ is an integer greater than 2. If the field $F$ is algebraically closed, then the commuting graph $\Gamma(M_n(F))$ is connected and its diameter is always equal to four; see [1]. If the field $F$ is not algebraically closed, then the commuting graph $\Gamma(M_n(F))$ may be disconnected for an arbitrarily large integer $n$; see [2]. However, in [1] S. Akbari et al. proved that for any field $F$ if the graph $\Gamma(M_n(F))$ is connected, then its diameter is between four and six. Also in [1] the authors conjectured that the upper bounded six can be improved to five. Recently, Yaroslav Shitov in [10] gave an example of a field $F$ such that $\Gamma(M_{38}(F))$ is connected with diameter six, disproving the conjecture of S. Akbari et al.

The aim of this note is to give a short proof that if the algebraic closure $\overline{F}$ of the field $F$ is a finite extension of $F$, then the commuting graph $\Gamma(M_n(F))$ is connected and its diameter is four. We will prove the following:

**Theorem 1.1.** If the algebraic closure $\overline{F}$ of a field $F$ is a finite extension of $F$, then the commuting graph $\Gamma(M_n(F))$ is connected and its diameter is four.
Notice that, the well-known fact that for matrix rings over algebraically closed fields, if the commuting graph is connected, then its diameter is four, is a particular case of theorem 1. Moreover, the results proved in [9] and [7] for the field \( \mathbb{R} \) of real numbers can also be deduced from theorem 1.

2 Preliminaries.

Our proof involves looking at the idempotents and nilpotent matrices and apply results from the structure theory of rings. Recall that in a ring \( R \) both the zero and the identity are always idempotent. An indempotent which is different from these is called non trivial idempotent. Note that, For a field \( F \) the central idempotent in the matrix ring \( M_n(F) \) are exactly the trivial idempotent.

**Lemma 2.1.** Let \( F \) be field and assume that all the irreducible polynomials in \( F[x] \) have degree < \( n \). Then, every matrix \( A \in M_n(F) \) commute with a non zero nilpotent matrix or with a nontrivial idempotent matrix.

**Proof.** First observe that, if the matrix \( A \) is derogatory, then by the rational canonical form we can find a nonsingular matrix \( S \in M_n(F) \) such that

\[
A = S^{-1}(C_1 \oplus \ldots \oplus C_k)S,
\]

where \( k \geq 2 \) and \( C_i \), for \( i = 1, \ldots, k \), is a companion matrix of a monic polynomial over \( F \). Hence, \( A \) commutes with the non trivial idempotent

\[
E = S^{-1}(I_{C_1} \oplus 0_{C_2} \oplus \ldots \oplus 0_{C_k})S,
\]

where \( I_{C_1} \) is the identity matrix whose order equals the order of \( C_1 \) and \( 0_{C_i} \), for \( i = 2, \ldots, k \), is the zero matrix whose order equals the order of \( C_i \).

Now, assume that the matrix \( A \) is non derogatory and denote by \( \langle A \rangle \) the \( F \)-subalgebra of \( M_n(F) \) generated by \( A \). Since the \( F \)-subalgebra \( \langle A \rangle \) is artinian it follows that the Jacobson radical \( J(\langle A \rangle) \) is nilpotent; see [8, p.658]. Consequently, if \( \langle A \rangle \) is not nilsemisimple it necessarily contains a non zero nilpotent matrix \( N \) and we get the result.

We are now reduced to the case where the matrix \( A \) is non derogatory and the \( F \)-algebra \( \langle A \rangle \) is semisimple. Since a semisimple artinian commutative ring is isomorphic to sum of fields; see [8, p.661], if follows that

\[
\langle A \rangle \cong F_1 \oplus \ldots \oplus F_s, \tag{1}
\]

where each \( F_i \), for \( i = 1 \ldots, s \), is a field. If in equation (1) we have \( s \geq 2 \), then \( F_1 \oplus \ldots \oplus F_s \) contains the non-trivial idempotent \((1,0,\ldots,0)\). Let \( \psi : F_1 \oplus \ldots \oplus F_s \to \langle A \rangle \) be an isomorphism. Clearly, a non-trivial idempotent of
F_1 \oplus \ldots \oplus F_s is mapped by \psi to a non-trivial idempotent in \langle A \rangle. As we have noted before, in the matrix ring M_n(F) the central idempotent are exactly the trivial idempotent. Hence, the idempotent \psi(1,0,\ldots,0) is not central and we get the result.

We conclude the proof by showing that in equation (1) we necessary have \( s \geq 2 \). Indeed, if we had \( s = 1 \) in equation (1), then the subalgebra \langle A \rangle is itself a field and by the first isomorphism theorem for rings we obtain

\[ \langle A \rangle \cong F[x]/(p_A), \]

where \((p_A)\) is the ideal, in the polynomial ring \( F[x] \), generated by the minimum polynomial \( p_A \) of \( A \), over the field \( F \). Since the ideal \((p_A)\) is maximal it follows that \( p_A \) is irreducible over \( F \). Now the fact that \( A \) is non derogatory implies that \( p_A \) has degree \( n \), contrary to hypothesis. □

Observe that if a matrix \( A \) commutes with a noncentral nilpotent matrix \( N \) with index of nilpotency \( \alpha \), that is, \( \alpha = \min\{i \in \mathbb{N} : N^i = 0\} \), then the matrix \( A \) also commutes with \( N^{\alpha-1} \). Consequently, if a matrix commutes with a noncentral nilpotent matrix \( N \) we may assume that its index of nilpotency is 2.

3 Proof of the main theorem.

We can now prove our main result.

Proof of Theorem 1.1. Following [1, theorem 11], if \( E_1, E_2 \in M_n(F) \) are two noncentral idempotent matrices, then \( d(E_1, E_2) \leq 2 \) in the commuting graph \( \Gamma(M_n(F)) \). Also by [1, theorem 9] if \( N_1, N_2 \in M_n(F) \) are two non-zero nilpotent matrices, both of index of nilpotence 2, then \( d(N_1, N_2) \leq 2 \) in \( \Gamma(M_n(F)) \). Finally, from [7, proposition 1] if \( E, N \in M_n(F) \) are such that \( E \) is idempotent and \( N \) is nilpotent of index of nilpotence 2, then \( d(E, N) \leq 2 \) in \( \Gamma(M_n(F)) \). By combining these three results with lemma 2.1 we get the result. □

Final Remarks: For a field \( F \) if the commuting graph \( \Gamma(M_n(F)) \) is connected, then there are three possibilities for the diameter namely 4, 5 or 6. According to theorem 1.1 for fields \( F \) with an algebraic closure that is a finite extension the graph \( \Gamma(M_n(F)) \) is connected with diameter 4. On the other hand, Yaroslav Shitov in [10] gave an example of a field \( F \) such that \( \Gamma(M_{38}(F)) \) is connected with diameter 6. However, there is no known example of a connected commuting graph of a matrix ring over a field with diameter 5. The question natural arises whether a field \( F \) exists such that for some integer \( n \) the commuting \( \Gamma(M_n(F)) \) is connected with diameter five?
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