Entanglement generation from thermal spin states via unitary beam splitters

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We suggest a method of generating distillable entanglement form mixed states unitarily, by utilizing the flexibility of dimension of occupied Hilbert space. We present a model of a thermal spin state entering a beam splitter generating entanglement. It is the truncation of the state that allows for entanglement generation. The output entanglement is investigated for different temperatures and it is found that more randomness - in the form of higher temperature - is better for this set up.

I. INTRODUCTION

In quantum information we use quantum mechanics to perform information tasks in ways exceeding the capabilities of classical systems. A natural and important question is then - what is it that gives us the power in quantum information and where does it come from? It has been shown by Knill and Laflamme that it is possible to make computations with exponential speed up over classical algorithms for certain tasks, using only one pure qubit and a resource of entirely mixed states. Bose et al. have proposed a scheme where only one pure state is needed to generate entanglement between an atom in a pure state and a thermal field - no matter what the temperature. Concerning mixed states and nonlocality, Filip et al. used mixed states to violate Bell’s inequalities.

In optics, for example, we know that in the infinite dimensional case of the harmonic oscillator, if a state can be described as a statistical mixture of Glauber states, this state cannot be used to generate entanglement using a beam splitter. A true maximally mixed state cannot be used to generate entanglement via any unitary transformation whatsoever. This is obvious since if the input to an unitary is proportional to identity, so must its output be, in which case the state remains unchanged.

Here we wish to address in part the question of how much mixedness we can deal with and still get entanglement, and how we can do this. We use the simple idea of change in the dimension of occupied space to give an example of how we can get around even maximally mixed inputs to generate entanglement. By expanding the occupied Hilbert space, the overall entropy remains the same, and the state is no longer a maximally mixed state for that space, and thus may have some entanglement. Now, such an operation is not unitary, in the sense of the input Hilbert space. However, if the input state is viewed as a truncated state of a higher-dimensional space on which the unitary acts, and the unitary in question can expand the occupied Hilbert space, we can get unitary generation of entanglement.

To illustrate this we use the model of finite truncations of a thermal state of optical modes entering a beam splitter. Such a state, truncated to dimension \( d \), is exactly analogous to a thermal spin state, with spin \( S = (d - 1)/2 \). We refer to these as thermal spin states. Indeed, in principle, we are not restricted to optics, and can imagine using real spin systems (please see Sec. V). The beam splitter is the example we use of a unitary transformation which increases occupied Hilbert space. It is exactly this, along with truncation, that allows the generation of entanglement. In the case of infinite temperatures this gives unitary entanglement generation from a truncated maximally mixed state.

We note that one must be careful with considering physical applications of this model. It is difficult to imagine a beam splitter type transformation acting on a finite-spin system, and even then, on the restricted space of the input, this is not a unitary transformation. In some cases infinite-dimensional systems can be modelled by considering finite spaces, but this is not what we are interested in here, since it is the truncation of the space that leads to the stark difference between the finite and infinite cases. We present this model simply as in illustration of the ideas using a real unitary that expands the occupied Hilbert space. The problems and possibilities of physical implementations of this model will be looked at in Sec. V.

We begin in the first section by describing the beam splitter transformation and defining the measure for entanglement we use. Next we look at the entanglement generation for a truncated maximally mixed state. We then look at thermal states, again in the truncated sense. Finally we show a simple, if inefficient, protocol proving that the entanglement is distillable for all states considered, then end with discussion and conclusions.

II. BACKGROUND

In optics, the beam splitter is defined by its unitary action on a two mode Fock state, \(|m,n\rangle\), returning the
output state $\tilde{7}$,  
\[ U_{bs}|m,n\rangle \equiv \sum_{M=0}^{(m+n)} f(m,n,M)|M,m+n-M\rangle, \tag{1} \]
where
\[ f(m,n,M) = \sum_{p=m}^{\min(m,M)} n! m! \sqrt{M(M+n-M)!} \rho^{(p-M+n)!/(m-p)!|M-p)! |(M-p)!} \times T^{(p-M+n)} R^{(M-p)} (-1)^{(M-p)}, \tag{2} \]
and where $T$ and $R$ are the complex transition and reflection coefficients respectively, with normalisation $|T|^2 + |R|^2 = 1$. We use this as our definition of the beam splitter on the general number state (i.e. not restricted to optics).

We notice that for any input state whose density matrix is diagonal in the two mode Fock basis, the output entanglement is not effected by the phases of $R$ and $T$, since it can be absorbed into a phase on the basis states. In all our states this is the case, so we do not need to consider the phase, and assume, without loss of generality that $R$ and $T$ are real.

As a measure of entanglement we take the logarithmic negativity $E_N$, defined for a given state $\rho$ as,
\[ E_N(\rho) \equiv \log_2 \|\rho^{T_A}\|_1 = \log_2 \sum_i |\mu_i|, \tag{3} \]
where $\mu_i$ are the eigenvalues of $\rho^{T_A}$, the partial transpose of $\rho$ in subspace $A$. For a state on a Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, the partial transpose in subspace $A$ is defined by its matrix elements
\[ \langle i_A,j_B|\rho^{T_A}|k_A,l_B\rangle \equiv \langle k_A,j_B|\rho|i_A,l_B\rangle, \tag{4} \]
where $|i_A,j_B\rangle \equiv |i\rangle_A \otimes |j\rangle_B \in \mathcal{H}_A \otimes \mathcal{H}_B$ form an orthonormal product basis. The logarithmic negativity is a widely used measure of entanglement and is an entanglement monotone [10]. We note that a positive value of this measure does not necessarily imply the existence of useful, i.e. distillable entanglement (however, for our cases we prove distillability by independent means).

III. MAXIMALLY MIXED (THERMAL) STATES

We are now ready to consider unentangled, maximally mixed states, input to our beam splitter,
\[ \rho_{in} = \frac{1}{d} \otimes \frac{1}{d} = \frac{1}{d^2} \sum_{m=0}^{2S} \sum_{n=0}^{2S} |m,n\rangle \langle m,n|, \tag{5} \]
with $d = 2S + 1$. This corresponds to a thermal state at infinite temperature. The output is then,
\[ \rho_{out} = U_{bs}\rho_{in}U_{bs}^\dagger = \frac{1}{d^2} \sum_{m=0}^{2S} \sum_{n=0}^{2S} \sum_{M=0}^{m+n} f(m,n,M) f(m,n,M') \times |M,m+n-M\rangle \langle M',m+n-M'|. \tag{6} \]

For any such input state of finite dimension we get an entangled output. Thus from “truncated” maximally mixed input states we have unitarily derived entanglement. The reason this is allowed is that the unitary operation does not act only on the space of the input; in this sense the beam splitter is not unitary on the truncated Hilbert space of the input.

Figure 1 is a plot of entanglement against $S$ for $R = T = 1/\sqrt{2}$ for truncated maximally mixed input states. We see an initial rise followed by a slow tail. We know that in the infinite $S$ limit the entanglement must fall to zero, since this is a statistical mixture of Glauber states, which does not entangle through a beam splitter (such states are also called “classical” states) [2] [6] [11].

![FIG. 1: Entanglement of two non entangled, maximally mixed states passing through beam splitter against $S$ for $R = T = 1/\sqrt{2}$. Here we see an initial peak followed by drop. In the large $S$ limit this goes to zero - since this is nothing but a “classical” state of the harmonic oscillator.](image-url)
states sent through a beam splitter and thermal states in the one-dimensional Heisenberg Model respectively.

We can also consider optimality of entanglement generation in terms of the beam splitter reflectivity \( R \) (since we are dealing only with real values, \( T \) is set by \( R \) through \( |T|^2 = 1 - |R|^2 \)). Figure 2 shows the entanglement generated by maximally mixed input states against \( S \) and \( |R|^2 \). For low \( S \) there are two maxima, at \( S = 1/2 \) they are around 0.2. As \( S \) increases the peaks get closer and for \( S \) larger than around 3 they merge into one at \( R = 1/\sqrt{2} \). Note, however, that even if we choose \( R \) for each \( S \) to give the maximum entanglement, we will still see an initial peak in entanglement as we go from \( S = 1/2 \) up.

![Figure 2: Entanglement of two maximally mixed states passing through two arms of a beam splitter against \( S \) and \( |R|^2 \).](image)

**IV. THERMAL STATES OF DIFFERENT TEMPERATURES**

We now consider a general thermal state of our truncated system. We assume the Hamiltonian

\[
H = \sum_{n=0}^{2S} n\hbar \omega \langle n | n \rangle,
\]

(7)

 giving the equilibrium thermal density matrix

\[
\sigma_T = \frac{\sum_{n=0}^{2S} e^{-n\hbar \omega / KT} |n\rangle \langle n|}{Z},
\]

(8)

where \( Z = \sum_{n=0}^{2S} e^{-n\hbar \omega / KT} \) is the partition function.

For two such thermal states incident on a beam splitter at different temperatures,

\[
\rho_{in} = \sigma_{T_1} \otimes \sigma_{T_2}
\]

\[
= \frac{1}{Z_1 Z_2} \sum_{m=0}^{2S} \sum_{n=0}^{2S} e^{(-\frac{m}{\hbar \omega} - \frac{n}{\hbar \omega})h\omega} |m, n\rangle \langle m, n|,
\]

(9)

where \( Z_i = \sum_{n=0}^{2S} e^{-n\hbar \omega / KT_i} \), the output state after the beam splitter transformation is then,

\[
\rho_{out} = U_{bs} \rho_{in} U_{bs}^\dagger
\]

\[
= \frac{1}{Z_1 Z_2} \sum_{m=0}^{2S} \sum_{n=0}^{2S} e^{(-\frac{m}{\hbar \omega} - \frac{n}{\hbar \omega})h\omega}
\]

\[
\sum_{M=0}^{2S} \sum_{M'=0}^{2S} f(m, n, M) f(m, n, M')
\]

\[
\times |M, m + n - M\rangle \langle M', m + n - M'|.
\]

(10)

Figure 3 shows some plots for various different input temperatures, both for the case of a thermal state entering one port and the other remaining empty (i.e. vacuum state, we can also think of this as zero temperature), and for thermal states entering both ports. We see that the general trends are the same for all temperatures, that of an initial peak followed by a slow decline. All plots are for reflectivity \( R = T = 1/\sqrt{2} \). This is chosen since, when plots similar to figure 2 were done for the different temperature inputs, the same trends were found, thus the overall trends are not affected.

In general the case where the vacuum enters one port gives more entanglement - this can be explained as resulting from the fact the two beams do not interfere with one another, which destroys entanglement.

Starting from the maximally mixed state, which represents an infinite temperature thermal state, the height
of the peak is less for lower temperatures and the peak occurs earlier. We might expect the entanglement to be greater for lower temperatures, and that the maximally mixed state gives the lowest entanglement, since these are the most disordered and classical like states. The observed trend can be explained by noticing that for lower temperatures the higher dimensional states are not as populated, restricting the possible entanglement. For the zero temperature we have the ground state which offers no entanglement. We also see a sharper fall after the peak for lower temperatures.

V. DISCUSSION

For all the output states mentioned here we can devise a protocol to distill a minimal amount of entanglement. From equations (11) and (6) we can see that if local projections are made on both arms, onto the subspace spanned by the states $|0\rangle|0\rangle$ and $|4S\rangle|4S\rangle$, the remaining state is an entangled state of the form $\rho = F|0\rangle|0\rangle + (1-F)(|0\rangle|4S\rangle + |4S\rangle|0\rangle)$, which has entanglement for any finite $F$. For finite $S$, $F$ is also finite (hence we have entanglement). This can be considered as a two dimensional state and it can be easily shown that it has negative partial transpose, which, for a two dimensional, bipartite state implies distillable entanglement [4]. As $S$ goes to infinity, $F$ goes to one and no entanglement can be found in this way as we expect. All measurements where the projection falls onto the remaining space are discarded. This scheme shows that for any finite $S$ we indeed have distillable entanglement for all the thermal states, though it is not necessarily inefficient and it destroys at least some, of not most entanglement.

Normally one might assume that from a maximally mixed state, $1/d \otimes 1/d$, no entanglement can be extracted via unitary operations. We see here that this is not the case and that when considering finite dimensions, there is an extra subtlety allowed by expanding of contracting the occupied Hilbert space in some sense we can consider the operation as expanding the “mixedness” of the inputs into two, individually less mixed outputs with some entanglement. More precisely we have the input output entropy relation $S(\rho_{in1}) + S(\rho_{in2}) \leq S(\rho_{out1}) + S(\rho_{out2})$, where the inequality comes from the entropy that has gone to create entanglement (the overall entropy is conserved since the operation is unitary). From this perspective it is not so surprising that more entanglement is generated for the more mixed (higher temperature) thermal input states. Conversely we can think of the inverse operation as contracting all the mixedness into a separable mixed state of lower dimension. This must be true of many unitaries other than the beam splitter, indeed we may consider the power of a unitary in terms of expanding or contracting the occupied Hilbert spaces. For example we can imagine easily a unitary in a six dimensional space acting on maximally mixed state of two qubits to create entanglement. The beam splitter simply represents a simple, well known and physical example of this type of unitary operation. The scheme by Browne et al. [15] represents an application of a beam splitter used to extend the accessed Hilbert space to distill gaussian states.

The morel we use here, as mentioned in the introduction, is seemingly not readily very implementable. This model is intended as an illustration of the ideas using a real physical unitary that expands the occupied Hilbert space, rather than a proposal for an experiment. However, it is hoped that that it could be useful as a basis for more sophisticated models.

For this, one can think of the input being either a truncation of an infinite space, or a real thermal state of a finite system. With infinite dimension, for these thermal states to be physical, such a truncation should be physically imposed - this could happen for example, for a finite number of photons in a mode, or atoms in a Bose-Einstein condensate (BEC). A BEC gas with attractive interactions, for example, has a natural limit on the number of particles, imposed by the interactions [16]. The beam splitter interaction also exists for BEC [17]. A finite maximally mixed state for example, could be envisaged as one of a pair of finite atom-number BECs in a maximally entangled state. In principle, then, this could offer an implementation of the scheme.

One naive possibility may be the projection of a real optical thermal state onto a finite-dimensional subspace and then passing through a beam splitter, giving entanglement. In such a case, the problem of entanglement generation becomes one of truncation or projection. This is related to the idea of measurement-induced entanglement (see e.g. [18, 19]). In these schemes a measurement is made at the end of the protocol which gives guaranteed entanglement conditioned on certain outcomes - it is the measurement that generates entanglement (whereas we would measure first and generate entanglement after). These schemes too implement projection, which does reduce the size of the occupied space. The difference is that such schemes are concerned with guaranteed entanglement generation, rather than exploiting the dimension of the occupied Hilbert space to deal with mixed state inputs.

Another possibility might be to map a mixed state of some finite dimension to a higher-dimensional space, and perform a unitary analogous to the beam splitter in expanding the occupied Hilbert space, creating entanglement. This could be, for example, a spin system in a thermal state mapped to the photon number states in a cavity, then passed through a beam splitter. Feasible mapping of spin systems onto light has been studied recently (e.g. [15, 16]). In this sense we could begin with a genuine thermal spin state (even infinite temperature) and extract entanglement via unitary transformation.
VI. CONCLUSION

In this paper we have given an example of distillable entanglement generation from a maximally mixed state via unitary operation. This result is explained by the fact that the state is in fact a truncated maximally entangled state, or equivalently, that the operation of the beam splitter is not unitary in the finite dimensional Hilbert space of the input. We have then looked at the entanglement generation for different thermal state inputs. Again against intuition, we found that the higher the input temperature, the more the resulting entanglement. This was put down to the fact that for lower temperature the population of the higher dimensional states is lower and hence the possibility for entanglement generation reduced.

The source of the entanglement, or “quantumness” for this scenario can either be thought of as the truncation of the input state or the power of the unitary to expand the occupied Hilbert space. There are many natural questions which arise, such as the minimal unitary way to extract entanglement from a maximally mixed state (as mentioned, the beam splitter is probably not the optimal unitary). More comparisons of these finite systems to optics should lead to more understanding and intuition. As experiments get closer to high dimensional, finite systems, (e.g. [21]), these questions become more and more valid and useful.

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