ON LINEAR HYPERSINGULAR BOLTZMANN TRANSPORT EQUATION AND ITS VARIATIONAL FORMULATION

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ABSTRACT. For charged particle transport the linear Boltzmann transport equation (BTE) turns out to be a partial hyper-singular integro-differential operator. This is due to the fact that the related differential cross-sections $\sigma(x, \omega', \omega, E', E)$ may have hyper-singularities. In these cases the energy integral appearing in the collision terms must be interpreted as the Hadamard finite part integral leading to hyper-singular integral operators. The article considers a refined expression for the exact transport operator and related variational formulations of the inflow initial boundary value problem for one particle equation containing hyper-singularities. We find that the exact BTE contains the first-order partial derivatives with respect to energy combined by partial Hadamard (first-order) singular integral operators. In addition, it contains the second-order partial derivatives with respect to angle and some mixed terms. The analysis will be carried out only for the so called Møller-type interaction (scattering) which is a kind of prototype of hyper-singular interactions. The generalizations to other type of collisions, such as to Bremsstrahlung, go analogously. Moreover, we expose a weak form (the variational formulation) of the hyper-singular transport problem. Another variant variational formulation decreases the level of singularities in the integration (appearing in the due bilinear form) containing only singularities of order one that is, singularities like $\frac{1}{E'-E} dE'dE$. The variational formulation is an essential step in order to show the existence of generalized solutions e.g. by Lions-Lax-Milgram Theorem based methods (proceedings for solution spaces and existence theory are omitted here). It also gives a platform needed for Galerkin finite element methods.

Keywords: Linear Boltzmann equation, hyper-singular integral operators, variational formulation, charged particle transport

AMS-Classification: 35Q20, 45E99, 35R09

1. INTRODUCTION

The Boltzmann transport equation (BTE) models changes of the number density of particles in phase space whose variables are position, velocity direction (angle) and energy. For general theory of linear BTEs with relevant boundary conditions we refer to [9] and [11]. In [1], [5], [11], [26] the subject is considered from a more physical point of view. Some more recent issues (including spectral and certain inverse problems) is exposed in [23], and general non-linear transport theory e.g. in [35], [2]. A mathematical survey of non-linear collision theory of particle transport is given in [36].

In the case of charged particle transport the linear BTE turns out to be a so called partial hyper-singular integro-differential operator. For theoretical and computational reasons it has been approximated by more simple models, including e.g. in the Continuous Slowing Down Approximation (CSDA) (e.g. [37], [14], [31]) and the linear Fokker-Plank approximation (e.g. [27]). In these approximations the resulting operator is a pure partial integro-differential operator without hyper-singular integrals. In [33] or more extensively in [31] we gave a (non-conventional) partial integro-pseudo-differential approximation for certain kind of charged particle transports. In this paper we investigate more closely the exact partial hyper-singular integral operator related to the so called Møller scattering.
The transport of relevant particles (among others in dose calculation for radiation therapy, [21]) can be formally modelled by the following linear coupled system of BTEs
\[
(\omega \cdot \nabla_x \psi_j)(x,\omega,E) + \Sigma_j(x,\omega,E)\psi_j(x,\omega,E) - (K_j\psi)(x,\omega,E) = f_j(x,\omega,E), \quad (x,\omega,E) \in G \times S \times I
\]
for \(j = 1,\ldots,N\), combined with an inflow boundary condition
\[
\psi_j|_{\Gamma_-} = g_j, \quad j = 1,\ldots,N
\]
where,
\[
(K_j\psi)(x,\omega,E) = \sum_{k=1}^{N} \int_{S_{n-1} \times I} \sigma_{kj}(x,\omega',\omega,E,E') \psi_k(x,\omega',E')d\omega'dE'.
\]
Here \(G \subset \mathbb{R}^n\) is the spatial domain. In applications \(n\) is typically 1, 2 or 3. \(S_{n-1} \subset \mathbb{R}^n\) is the unit sphere and \(I = [E_0,E_m]\) is the energy interval. There are some benefits to choose \(G \times S \times I\) for the state space instead of \(G \times V\) where \(V\) is the velocity space. Nevertheless, the below analysis could be carried out in \(G \times V\) as well. The set \(\Gamma_- \) is “the inflow boundary part of \(\partial G\)" (see section 2). \(N\) assigns the number of different particles. Above \((\omega',E')\) refers to the angle and energy of the incoming particle whereas \((\omega,E)\) refers to the angle and energy of the leaving particle. On the right in (1), the functions \(f_j\) represent (internal) sources and in (2) \(g_j\) are (inflow) boundary sources. The system is coupled through the integral operators \(K_j\). The solution \(\psi = (\psi_1,\ldots,\psi_N)\) of the problem (1)-(2) is a vector-valued function whose components describe the radiation fluxes of various particles under consideration. The equation (1) is a steady state counterpart of its dynamical (time-dependent) equation. In many applications it is sufficient to consider only the steady state equations because the flux \(\psi\) reaches the steady state nearly instantly. For simplicity we restrict ourselves to the case \(n = 3\) and we denote \(S = S_2\).

The differential cross-sections \(\sigma_{kj}(x,\omega',\omega,E',E')\) may have singularities, or even hyper-singularities, and in these cases the integral \(\int_P\) appearing in the collision terms \(K_j\) must be interpreted as the Hadamard finite part integral which we denote by \(\text{p.f.} \int_P\). Moreover, differential cross sections may contain Dirac’s \(\delta\)-distributions with respect to \(\omega\). In [31] we presented some details of real, physical collision operators. In particular, we found that certain differential cross sections may contain hyper-singularities like \(\frac{1}{p-1}dE'\). These kind of singularities lead to extra pseudo-differential like (or approximately partial differential) terms in the transport equation (3.1), section 2.3. We also remark that in these cases additionally an initial condition (or conditions) with respect to \(E\) must be imposed to obtain mathematically (and physically) well-posed problems. The analysis presented in [31] revealed the exact form of transport operators for certain interactions (collisions).

The existence of solutions for the problem (1), (2), (as well as for the time-dependent problem with the due initial condition) has been studied for coupled systems in [30] (the results of which remaining valid, after slight modifications, for any \(1 \leq p < \infty\)) and for single equations e.g. in [12], [9], [1]. In these references it is assumed that the collision operator \(K\) satisfies the so-called \((partial)\) Schur criterion (for boundedness) which is not valid for all species of particle interactions. In [32], [33] (or more comprehensively in [31]) we studied systematically the existence of solutions for the CSDA-system which is an approximation of the hyper-singular BTE-equation. It extended the results of [14], where a spatially homogeneous stopping power was assumed. In this article we derive a refined expression for the exact transport operator and the related variational formulation of the inflow initial boundary value problem for one particle equation (\(N = 1\)) containing hyper-singularities. The analysis will be carried out only for the Möller-type scattering which is a kind of prototype of hyper-singular interactions. The generalizations to other type of collisions (such as to Bremsstrahlung) go analogously.

For the first instance, in section 3.1 we consider shortly the Möller scattering. This interaction models the electron’s (and positron’s) inelastic collisions in particle transport. We expose the
hyper-singular partial integral expression and its pseudo-differential like expression obtained in \cite{31} for the corresponding collision operator $K$ (and transport operator). These expressions are still quite implicit and the refined expression of the transport operator, say $T$, will be given in section 4.2. It turns out that $T$ is of the form (Theorem 4.16)

\[
(T\psi)(x, \omega, E) = -\frac{\partial}{\partial E}\left(\mathcal{H}_1((\mathcal{K}_2\psi)(x, \omega, \cdot, E))(E)\right) - 2\pi \tilde{\sigma}_2(x, E, E)\frac{\partial \psi}{\partial E}(x, \omega, E)
\]

\[-\tilde{\sigma}_2(x, E, E) \sum_{|\alpha| \leq 2} a_\alpha(E, \omega)(\partial_\alpha^\omega \psi)(x, \omega, E)
\]

\[+
\text{p.f.} \int_\mathcal{E} \frac{1}{E' - E} \tilde{\sigma}_2(x, E', E) \int_0^{2\pi} \left\langle \nabla_s \psi(x, \gamma(E', E, \omega)(s), E'), \frac{\partial \gamma}{\partial E}(E', E, \omega)(s) \right\rangle ds dE'
\]

\[+
\text{p.f.} \int_\mathcal{E} \frac{1}{E' - E} \frac{\partial \tilde{\sigma}_2}{\partial E}(x, E', E) \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E') ds dE'
\]

\[-\mathcal{H}_1((\mathcal{K}_1\psi)(x, \omega, \cdot, E))(E) - 2\pi \frac{\partial \tilde{\sigma}_2}{\partial E}(x, E, E)\psi(x, \omega, E)
\]

\[+ \omega \cdot \nabla_x \psi + \Sigma(x, \omega, E)\psi - (K_r\psi)(x, \omega, E)
\]

where for $j = 1, 2$ the (first-order) Hadamard finite part integral operator is defined by

\[
\mathcal{H}_1((\mathcal{K}_j\psi)(x, \omega, \cdot, E))(E) := \text{p.f.} \int_\mathcal{E} \frac{1}{E' - E} \tilde{\sigma}_j(x, E', E) \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E') ds dE'.
\]

The operator $K_r$ above is the so called restricted collision operator. Roughly speaking it is the residual when the singular part is separated from the collision operator. In \cite{33} we showed that $K_r$ is a bounded operator in $L^2(G \times S \times I)$. The result is based on the Schur criterion. In addition, we showed that $\Sigma - K_r$ is coercive (accretive) operator in $L^2(G \times S \times I)$ under relevant assumptions. The operator \cite{4} reminds of the CSDA-Fokker-Plank operator but it contains additionally the Hadamard finite part integral operators which can be considered as pseudo-differential-like operators. The expression \cite{4} of $T$ gives a solid background for the use of CSDA-Fokker-Plank operator as an approximation.

The relevant transport problem requires besides of the operator equation $T\psi = f$ the due initial and inflow boundary conditions and therefore the total transport problem is

\[T\psi = f, \quad \psi|_{\Gamma_-} = g, \quad \psi(\cdot, \cdot, E_m) = 0\]

(5)

where $E_m$ is the so called cut-off energy. In section 5 we expose a weak form of the hyper-singular transport problem. The obtained weak form is the so called variational equation

\[B(\psi, v) = Fv, \quad v \in \mathcal{D}.
\]

for the transport problem. Here $\mathcal{D}$ is a relevant space of test functions and $B(\cdot, \cdot)$ and $F$ are the due bilinear and linear forms, respectively. The variational formulation given in section 5.2 decreases the level of singularities in the integration and it contains only singularities of order one that is, singularities like $\frac{1}{E} dE'dE$. The variational formulation is an essential step in order to define generalized solutions and to show the existence of solutions e.g. by Lions-Lax-Milgram Theorem based methods. It also gives a platform needed for Galerkin finite element methods. The consideration of singular integral operators by element methods is well known in the field of boundary element methods (BEM), e.g. \cite{8}. So in principle, no approximations (such as CSDA or Fokker-Plank) are needed for numerical solutions. In section 4.3 we compute some related formal adjoint operators which are beneficial to existence theory as well. In this paper we, however, omit proceedings for solution spaces and existence theory.
2. Preliminaries

2.1. Basic notations. We assume that $G$ is an open bounded set in $\mathbb{R}^3$ such that $\overline{G}$ is a $C^1$-manifold with boundary (as a submanifold of $\mathbb{R}^3$; cf. [19]). In particular, it follows from this definition that $G$ lies on one side of its boundary. The unit outward (with respect to $G$) pointing normal on $\partial G$ is denoted by $\nu$ and the surface measure (induced by the Lebesgue measure $dx$) on $\partial G$ is denoted as $d\sigma$. We let $S = S_2$ be the unit sphere in $\mathbb{R}^3$. Furthermore, let $I = [E_0, E_m]$ where $0 \leq E_0 < E_m < \infty$. We could replace $I$ by $I = [E_0, \infty]$ but we neglect this case here. We shall denote by $I^0$ the interior of $I$. The interval $I$ is equipped with the Lebesgue measure $dE$.

All functions considered in this paper are real-valued, and all linear (Hilbert, Banach) spaces are real.

For $(x, \omega) \in G \times S$ the escape time (in the direction $\omega$) $t(x, \omega) = t_-(x, \omega)$ is defined by $t(x, \omega) := \inf\{s > 0 \mid x - s\omega \notin G\} = \sup\{T > 0 \mid x - s\omega \in G\}$ for all $0 < s < T$. We define

$$\Gamma' := (\partial G) \times S, \quad \Gamma := \Gamma' \times I,$$

and their subsets

$$\Gamma_0 := \{(y, \omega) \in \Gamma' \mid \omega \cdot \nu(y) = 0\}, \quad \Gamma_0 := \Gamma_0 \times I,$$

$$\Gamma^- := \{(y, \omega) \in \Gamma' \mid \omega \cdot \nu(y) < 0\}, \quad \Gamma^- := \Gamma^- \times I,$$

$$\Gamma^+ := \{(y, \omega) \in \Gamma' \mid \omega \cdot \nu(y) > 0\}, \quad \Gamma^+ := \Gamma^+ \times I.$$ 

Note that $\Gamma = \Gamma_0 \cup \Gamma^- \cup \Gamma^+$.

In the sequel we denote for $k \in \mathbb{N}_0$,

$$C^k(\overline{G} \times S \times I) := \{\psi \in C^k(G \times S \times I) \mid \psi = f|_{G \times S \times I^0}, \; f \in C^k_0(\mathbb{R}^3 \times S \times \mathbb{R})\},$$

where for a $C^k$-manifold $M$ without boundary, the set $C^k_0(M)$ denotes the set of all $C^k$-functions on $M$ with compact support. Define the (Sobolev) space $W^2(G \times S \times I)$ by

$$W^2(G \times S \times I) := \{\psi \in L^2(G \times S \times I) \mid \omega \cdot \nabla_x \psi \in L^2(G \times S \times I)\}.$$

The space $W^2(G \times S \times I)$ is a Hilbert space when equipped with the inner product

$$\langle \psi, v \rangle_{W^2(G \times S \times I)} := \langle \psi, v \rangle_{L^2(G \times S \times I)} + \langle \omega \cdot \nabla_x \psi, \omega \cdot \nabla_x v \rangle_{L^2(G \times S \times I)}.$$ 

The space $C^1(\overline{G} \times S \times I)$ is a dense subspace of $W^2(G \times S \times I)$ (e.g. [13]).

Let $T^2(\Gamma)$ be the weighted Lebesgue space $L^2(\Gamma, |\omega \cdot \nu|d\sigma d\omega dE)$. The trace $\gamma(\psi) := \psi|_{\Gamma}$ (for a detailed study of inflow trace theory see e.g. [6] and [31], section 2.2) is well-defined in the space

$$\overline{W}^2(G \times S \times I) := \{\psi \in W^2(G \times S \times I) \mid \gamma(\psi) \in T^2(\Gamma)\},$$

which is a Hilbert space when equipped with the inner product

$$\langle \psi, v \rangle_{\overline{W}^2(G \times S \times I)} := \langle \psi, v \rangle_{W^2(G \times S \times I)} + \langle \gamma(\psi), \gamma(v) \rangle_{T^2(\Gamma)},$$

where

$$\langle h_1, h_2 \rangle_{T^2(\Gamma)} := \int_{\Gamma} h_1(x, \omega, E)h_2(x, \omega, E)|\omega \cdot \nu|d\sigma d\omega dE.$$ 

We recall that the Green’s formula ([9], p. 225)

$$\int_{G \times S \times I} (\omega \cdot \nabla_x \psi)v d\sigma d\omega dE + \int_{G \times S \times I} (\omega \cdot \nabla_x v)\psi d\sigma d\omega dE = \int_{\partial G \times S \times I} (\omega \cdot \nu)v \psi d\sigma d\omega dE, \quad (6)$$

is valid for every $\psi, \; v \in \overline{W}^2(G \times S \times I)$.
2.2. Some tools from analysis. We recall the following standard concepts from analysis which we shall frequently need. The Taylor’s expansion (of order \( r \) on a open set \( U \subset \mathbb{R}^N \) is given by

\[
f(x) = \sum_{|\alpha| \leq r} \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial x^\alpha}(x_0)(x-x_0)^\alpha + \sum_{|\alpha|=r+1} R_\alpha(x)(x-x_0)^\alpha
\]

where the residual term (one of its variant forms) is

\[
R_\alpha(x) := \frac{|\alpha|}{\alpha!} \int_0^1 (1-t)^{|\alpha|-1} \frac{\partial^\alpha f}{\partial x^\alpha}(x_0 + t(x-x_0))dt.
\]

Recall also the definitions of Hadamard finite part integrals for discontinuous functions \( f : [a, b] \to \mathbb{R} \) by \([22]\), pp. 5 and 32, formulas (14) and (32) therein or \([29, p. 104]\). Applying these definitions (for a fixed \( x \)) to the function \( F_x(t) := \chi_{[x,b]}(t)f(t) \), where \( f \in C([a,b]) \) and \( \chi_{[x,b]}(t) \) is the characteristic function of the interval \([x,b]\), we have

\[
p.f. \int_a^b \frac{F_x(t)}{t-x} dt = \text{p.f.} \int_a^b \frac{f(t)}{t-x} dt = \lim_{\varepsilon \to 0} \left( \int_{x+\varepsilon}^b \frac{f(t)}{t-x} dt + f(x^+) \ln(\varepsilon) \right)
\]

and

\[
p.f. \int_a^b \frac{F_x(t)}{(t-x)^2} dt = \text{p.f.} \int_a^b \frac{f(t)}{(t-x)^2} dt = \lim_{\varepsilon \to 0} \left( \int_{x+\varepsilon}^b \frac{f(t)}{(t-x)^2} dt + f'(x^+) \ln(\varepsilon) - \frac{1}{\varepsilon} f(x^+) \right).
\]

Analogously we define

\[
p.f. \int_a^x \frac{f(t)}{t-x} dt = \lim_{\varepsilon \to 0} \left( \int_a^{x-\varepsilon} \frac{f(t)}{t-x} dt - f(x^-) \ln(\varepsilon) \right)
\]

or equivalently

\[
p.f. \int_a^x \frac{f(t)}{x-t} dt = \lim_{\varepsilon \to 0} \left( \int_a^{x-\varepsilon} \frac{f(t)}{x-t} dt + f(x^-) \ln(\varepsilon) \right)
\]

and

\[
p.f. \int_a^x \frac{f(t)}{(t-x)^2} dt = \lim_{\varepsilon \to 0} \left( \int_a^{x-\varepsilon} \frac{f(t)}{(t-x)^2} dt - f'(x^-) \ln(\varepsilon) - \frac{1}{\varepsilon} f(x^-) \right)
\]

In particular, these formulas give

\[
p.f. \int_a^b \frac{1}{t-x} dt = \ln(b-x),
\]

\[
p.f. \int_a^b \frac{1}{(t-x)^2} dt = -\frac{1}{b-x}.
\]

Note that \( p.f. \int_x^b \frac{f(t)}{t-x} dt \) is well-defined (at least) for all \( f \in C^\alpha([a,b]), \alpha > 0 \) and (cf. \([24]\))

\[
p.f. \int_x^b \frac{f(t)}{t-x} dt = \int_x^b \frac{f(t)-f(x)}{t-x} dt + f(x) \ln(b-x). \tag{14}
\]

We recall from \([31], \text{Lemma 3.2}\)

**Lemma 2.1** Suppose that \( f \in C^2([a,b] \times [a,b]) \). Then for \( x \in [a,b] \)

\[
\frac{d}{dx} \left( p.f. \int_x^b \frac{f(x,t)}{t-x} dt \right) = p.f. \int_x^b \frac{f(x,t)}{(t-x)^2} dt + p.f. \int_x^b \frac{\partial f(x,t)}{t-x} dt - \frac{\partial f}{\partial t}(x,x) \tag{15}
\]
and
\[
\frac{d}{dx} \left( \text{p.f.} \int_a^x \frac{f(x, t)}{x-t} dt \right) = -\text{p.f.} \int_a^x \frac{f(x, t)}{(t-x)^2} dt + \text{p.f.} \int_a^x \frac{\partial f(x, t)}{x-t} dt - \frac{\partial f}{\partial t}(x, x). 
\]

(16)

We remark that the derivation result of the previous lemma \[2.1\] is valid for more general \( f \) and that under appropriate assumptions \( Hf \in W^{1,p}([a, b]) \) where
\[
(Hf)(x) := \text{p.f.} \int_x^b \frac{f(t)}{t-x} dt.
\]

For example, we have

**Lemma 2.2** Suppose that \( f \in C^{1+\alpha}([a, b] \times [a, b]), \ \alpha > 0 \) and let
\[
F_1(x) := \frac{f(x, x)}{b-x}, \quad F_2(x) := \ln(b-x) \frac{\partial f}{\partial t}(x, x), \quad F_3(x) := \ln(b-x) \frac{\partial f}{\partial x}(x, x).
\]
Furthermore, suppose that
\[
F_j \in L^1([a, b]), \quad j = 1, 2, 3.
\]

Then \( Hf \in W^{1,1}([a, b]) \) and
\[
(Hf)'(x) = \text{p.f.} \int_x^b \frac{f(x, t)}{(t-x)^2} dt + \text{p.f.} \int_x^b \frac{\partial f(x, t)}{t-x} dt - \frac{\partial f}{\partial t}(x, x).
\]

(18)

**Proof.** By the assumptions one sees (by the Taylor’s formula) that the function
\[
h(x) := \text{p.f.} \int_x^b \frac{f(x, t)}{(t-x)^2} dt + \text{p.f.} \int_x^b \frac{\partial f(x, t)}{t-x} dt - \frac{\partial f}{\partial t}(x, x)
\]
is in \( L^1([a, b]) \). Applying the techniques used in Lemmas \[4.1\] and \[4.2\] below one can verify that for \( \varphi \in C_0^\infty([a, b]) \)
\[
\int_a^b (Hf)(x) \varphi'(x) dx = -\int_a^b h(x) \varphi(x) dx.
\]

We omit the details of the proof. \[\square\]

The assumption \( f \in C^{1+\alpha}([a, b] \times [a, b]), \ \alpha > 0 \) in the above lemma can be replaced with the weaker condition:
\[ f \in C^1([a, b] \times [a, b]), \ \alpha > 0 \ \text{such that} \]
\[
\left| \frac{\partial f}{\partial t}(x, t') - \frac{\partial f}{\partial t}(x, t) \right| \leq C|t' - t|^\alpha, \quad t' \geq t.
\]

We finally mention that generally for \( f \in L^2([a, b]) \) the Hadamard finite part integrals
\[
(H_j f)(x) := \text{p.f.} \int_x^b \frac{f(t)}{(t-x)^j} dt, \quad j = 1, 2
\]
are interpreted as distributions defined by
\[
(H_j f)(\varphi) := \int_a^b f(t) \left( \text{p.f.} \int_a^t \frac{\varphi(x)}{(t-x)^j} dx \right) dt \quad \text{for} \ \varphi \in C_0^\infty([a, b]).
\]

(19)

Using the Taylor’s expansion
\[
\varphi(x) = \varphi(t) + \varphi'(t)(x-t) + \int_0^1 (1-s) \varphi''(t+s(t-x)) ds \cdot (x-t)^2
\]
one sees that \( H_j f \) defined by \[19\] is really a distribution (and similarly \( H_1 f \)). Note that this generalization does not work for more general \( f = f(x, t) \).
We need additionally the Taylor’s expansion for a sufficiently smooth function \( f : S \to \mathbb{R} \). Since \( S \) is a manifold the expansion requires some explanation. The detailed presentation of the subject is outside of this paper and so we give only some essential technicalities. For some additional formulations see e.g. [24], pp. 185-186.

The first order Taylor’s expansion of a function \( f : S \to \mathbb{R} \) which is \( C^2 \)-function around \( \omega \in S \) is of the form

\[
f(\omega') = f(\omega) + \langle (\nabla_S f)(\omega), \zeta \rangle + (R_\omega f)(\omega')(\zeta, \zeta), \quad \zeta \in T_\omega (S)
\]

where \( \nabla_S \) is the gradient on \( S \) and \( R_\omega f \) is the residual. The inner product \( \langle \cdot, \cdot \rangle \) is the Riemannian inner product on \( T_\omega(S) \) induced by the euclidean inner product on \( \mathbb{R}^3 \). In addition, there exists a constant \( C \geq 0 \) such that

\[
\|\zeta\| \leq C \|\omega' - \omega\|.
\]

Leaving the residue \( R_\omega f \) away we get Taylor’s approximations for \( f(\omega') \) near \( \omega \).

The basic principle in deriving (20) is to apply an appropriate pull-back \( H_\omega : V \to U_\omega \) where \( U_\omega \subset S \) and \( V \subset T_\omega(S) \) are open neighbourhoods such that \( \omega \in U_\omega \), \( 0 \in V \) and \( H_\omega(0) = \omega \). One assumes that \( H_\omega \) is a sufficiently smooth diffeomorphism and so the (smooth) inverse mapping \( H_\omega^{-1} : U_\omega \to V \) exists. Let \( \omega' \in U_\omega \) and let

\[
H_\omega^{-1}(\omega') = \zeta = \xi_1 \overline{\Omega}_1 + \xi_2 \overline{\Omega}_2
\]

where \( \overline{\Omega}_1 = \overline{\Omega}_1(\omega') \), \( \overline{\Omega}_2 = \overline{\Omega}_2(\omega') \). Here \( \overline{\Omega}_1 \), \( \overline{\Omega}_2 \) are the (locally defined) canonical tangent vectors of \( S \) at \( \omega \in S \) that is,

\[
\overline{\Omega}_1 = \frac{1}{\sqrt{\omega_1^2 + \omega_2^2}}(-\omega_1, \omega_2, 0),
\]

\[
\overline{\Omega}_2 = \left(\frac{\omega_1 \omega_3}{\sqrt{\omega_1^2 + \omega_2^2}}, \frac{\omega_2 \omega_3}{\sqrt{\omega_1^2 + \omega_2^2}}, -\sqrt{1 - \omega_3^2}\right).
\]

One often chooses \( H_\omega \) to be the (differential geometry’s) exponential mapping \( H_\omega = \exp_\omega \) (for exponential mapping see [10] and the Example 2.4 below). The pull-back obeys (at least locally)

\[
\|H_\omega^{-1}(\omega')\| = \|\zeta\| \leq C \|\omega' - \omega\|.
\]

The tangent space \( T_\omega(S) \) can be isomorphically (and isometrically) identified with \( \mathbb{R}^2 \) by

\[
\zeta = \xi_1 \overline{\Omega}_1 + \xi_2 \overline{\Omega}_2 \sim (\xi_1, \xi_2) =: \xi.
\]

In fact we can define

\[
J(\zeta) = J(\xi_1 \overline{\Omega}_1 + \xi_2 \overline{\Omega}_2) = (\xi_1, \xi_2).
\]

Let \( J(V) = V' \subset \mathbb{R}^2 \). Using this identification we find that the mapping \( f \circ H_\omega : V' \to \mathbb{R} \) is well-defined (and as smooth as \( f \)). Note that actually

\[
(f \circ H_\omega)(\xi) = (f \circ H_\omega)((\xi_1 \overline{\Omega}_1 + \xi_2 \overline{\Omega}_2)).
\]

Hence we are able to write the Taylor’s expansion (the expansion (7) with \( r = 1 \)) near \( 0 \in \mathbb{R}^2 \)

\[
(f \circ H_\omega)(\zeta) = (f \circ H_\omega)(\xi) = (f \circ H_\omega)(0) + \sum_{j=1}^2 \partial_j (f \circ H_\omega)(0) \xi_j
\]

\[
+ \sum_{|\alpha|=2} \frac{|\alpha|}{\alpha!} \int_0^1 (1-t) \partial^\alpha_t (f \circ H_\omega)(t\xi) dt \cdot \xi^\alpha.
\]

By (22) for \( \zeta = H_\omega^{-1}(\omega') \)

\[
\|\xi\| = \|\zeta\| \leq C \|\omega' - \omega\|.
\]
Finally, one has
\[ \partial_j(f \circ H_\omega)(0) = (\partial_{\omega_j}f)(\omega) \] (25)
and
\[ \sum_{j=1}^{2} \partial_j(f \circ H_\omega)(0) \xi_j = \sum_{|\alpha|=1} \frac{1}{\alpha!} \partial_\xi^\alpha (f \circ H_\omega)(0) \xi^\alpha = \langle (\nabla_S f)(\omega), \zeta \rangle. \] (26)
and so (20) can be seen.

The gradient $\nabla_S f$ on sphere $S$ can be shortly depicted as follows (for $n = 3$). Suppose that $f$ is defined and smooth in a neighbourhood of $S \subset \mathbb{R}^3$. Then
\[ \nabla_S f = \langle \nabla f, \overline{\Omega}_1 \rangle \overline{\Omega}_1 + \langle \nabla f, \overline{\Omega}_2 \rangle \overline{\Omega}_2 \] (27)
where $\nabla f$ is the gradient of $f$ in the ambient space $\mathbb{R}^3$. Note that the right hand side of (27) is the projection of $\nabla f$ onto the tangent space. Hence
\[ \omega \cdot \nabla_S f = 0. \] (28)

The first order Taylor’s approximation of $\psi$ with respect to $\omega$ around $\omega$ is
\[ \psi(x, \omega', E) \approx \psi(x, \omega, E) + \langle (\nabla_S \psi)(x, \omega, E), \zeta \rangle \] (29)
where $\zeta = H_\omega^{-1}(\omega') \in T_\omega(S)$ and it satisfies by (24)
\[ ||\zeta|| \leq C ||\omega' - \omega||. \] (30)
The residual, for example, for the approximation (29) is by (28)
\[ R_\omega(\psi(x, ., E))(\omega')(\zeta, \zeta) = \sum_{|\alpha|=2} \frac{|\alpha|}{\alpha!} \int_0^1 (1-t) \partial_\xi^\alpha (\psi(x, ., E) \circ H_\omega)(t\xi) dt \cdot \xi^\alpha. \] (31)
where $\xi = J(\zeta)$. \hfill (31)

**Remark 2.3** Let $\langle ., . \rangle_r$ be the Riemannian inner product on $T(S)$ that is,
\[ \langle \xi_1, \xi_2 \rangle_r = \xi_1^1 \xi_2^1 + \xi_1^2 \xi_2^2 \]
for $\xi_j = \xi_j^1 \overline{\Omega}_1 + \xi_j^2 \overline{\Omega}_2 \in T_\omega(S)$. Furthermore, let $a_{pr,\omega'}, \omega' \in S$ be the projection of a vector $a \in \mathbb{R}^3$ onto $T_{\omega'}(S)$ that is,
\[ a_{pr,\omega'} := \sum_{j=1}^{2} \langle a, \overline{\Omega}_j(\omega') \rangle \overline{\Omega}_j(\omega'). \]
Then that for $w \in T_{\omega'}(S)$ and $a \in \mathbb{R}^3$
\[ \langle w, a_{pr,\omega'} \rangle_r = \langle w, a \rangle_{\mathbb{R}^3} = \langle w, a \rangle. \] (32)
In the sequel the inner product $\langle w, a \rangle$ is interpreted by (32).

**Example 2.4** Recall that the geodesics of the sphere $S = S_2$ are great circles. Let $\omega \in S$ and let $\zeta = \xi_1 \overline{\Omega}_1 + \xi_2 \overline{\Omega}_1 \in T_\omega(S)$ be the tangent vector of $S$ at $\omega$. Then there exists a geodesic $\gamma_\omega : ]-2, 2[ \to S$ which satisfies the initial conditions
\[ \gamma_\omega(0) = \omega, \quad \gamma'_\omega(0) = \zeta. \]
In fact in the case of $S$
\[ \gamma_\omega(t) = \cos(\|\zeta\| t) \omega + \sin(\|\zeta\| t) \frac{\zeta}{\|\zeta\|}. \] (33)
The differential geometry's exponential mapping $\exp_\omega : T_\omega(S) \to S$ at $\omega$ is defined by

$$\exp_\omega(\zeta) := \gamma_\omega(1).$$

Note that the exponential map is dependent on the parametrization $\gamma_\omega$ of the geodesic. Its basic properties are, however independent of $\gamma_\omega$. It can be shown that there exist open neighbourhoods $U_\omega \subset S$ and $V \subset T_\omega(S)$ such that $\omega \in U_\omega$, $0 \in V$ and for which

$$\exp_\omega : V \to U_\omega$$

is a diffeomorphism. Hence the (smooth) inverse mapping $\exp_\omega^{-1} : U_\omega \to V$ exists. Let $\omega' \in U_\omega$ and let

$$\exp_\omega^{-1}(\omega') = \zeta = \xi_1\Omega_1 + \xi_2\Omega_2.$$

From (33) we immediately get that for the sphere the exponential map using the above parametrization is

$$\exp_\omega(\zeta) = \gamma_\omega(1) = \cos(\|\zeta\|)\omega + \sin(\|\zeta\|)\frac{\zeta}{\|\zeta\|}. \quad (34)$$

To see that $\exp_\omega(\zeta) = \exp_\omega(\xi_1\Omega_1 + \xi_2\Omega_2)$ has the needed differentiability properties, apply Taylor’s expansions

$$\cos(\|\zeta\|) = 1 + \frac{1}{2!}\|\zeta\|^2 + \cdots \quad \text{and} \quad \sin(\|\zeta\|) = \|\zeta\| - \frac{1}{3!}\|\zeta\|^3 + \cdots.$$

The inverse $\exp_\omega^{-1}$ can be computed explicitly. In fact, let $\exp_\omega(\zeta) = \omega'$ that is,

$$\cos(\|\zeta\|)\omega + \sin(\|\zeta\|)\frac{\zeta}{\|\zeta\|} = \omega'. \quad (35)$$

Since $\omega \perp \zeta$ we find that

$$\cos(\|\zeta\|) = \langle \omega', \omega \rangle, \quad \sin(\|\zeta\|) = \sqrt{1 - \langle \omega', \omega \rangle^2} \quad (36)$$

and then

$$\|\zeta\| = \arccos(\langle \omega', \omega \rangle). \quad (37)$$

From (35), (36) and (37) we obtain

$$\exp_\omega^{-1}(\omega') = \zeta = \frac{\arccos(\langle \omega', \omega \rangle)}{\sqrt{1 - \langle \omega', \omega \rangle^2}}(\omega' - \langle \omega', \omega \rangle\omega). \quad (38)$$

We find that there exists a constant $C > 0$ such that

$$\|\exp_\omega^{-1}(\omega')\| = \|\zeta\| \leq C \|\omega' - \omega\|. \quad (39)$$

Actually, from the Hospital’s rule it follows that

$$\lim_{x \to 1} \frac{\arccos(x)}{\sqrt{1 - x^2}} = 1$$

and so (since $S \times S$ is compact) there exists a constant $C''$ such that

$$\left| \frac{\arccos(\langle \omega', \omega \rangle)}{\sqrt{1 - \langle \omega', \omega \rangle^2}} \right| \leq C'' \quad (40)$$

Furthermore, we have

$$\omega' - \langle \omega', \omega \rangle \omega = \omega' - \langle \omega' - \omega, \omega \rangle \omega.$$

Hence the assertion follows from (35), (36), (37).
3. On hyper-singular collision operators related to charged particle transport

The differential cross-sections may have singularities, or even hyper-singularities, which would lead to extra pseudo-differential-like terms in the transport equation. In the case where \( \sigma(x, \omega', \omega, E', E) \) has hyper-singularities (like in the case of Möller-type differential cross sections analysed below) the integral \( \int_P \) occurring in the collision operator must be understood in the sense of Cauchy principal value p.v. \( \int_P \) or more generally in the sense of Hadamard finite part integral p.f. \( \int_P \) (see [31], section 5). Hyper-singular integral operators form a subclass of pseudo-differential operators ([18], Chapter 7, [20], Chapter 3). Moreover, the \( (\omega', \omega) \)-dependence in differential cross-sections typically contain Dirac’s \( \delta \)-distributions (on \( \mathbb{R} \)). More precisely, in \( \sigma(x, \omega' \omega, E', E) \) there may occur terms like \( \delta(\omega \cdot \omega' - \mu(E', E)) \) or \( \delta(E - E') \) which require special treatment.

Consider the following partial singular integral operator,

\[
(K \psi)(x, \omega, E) = \text{p.f.} \int_P \int_{S'} \sigma(x, \omega', \omega, E', E) \psi(x, \omega', E') d\omega' dE'.
\] (42)

The simplest is the case where \( \sigma \) has at most a so-called weak singularity with respect to energy. This means that \( \sigma = \sigma_0(x, \omega', \omega, E', E) \) is a measurable non-negative function \( G \times S \times S \times (I \times I \setminus D) \rightarrow \mathbb{R} \), where \( D = \{(E, E) \mid E \in I \} \) is the diagonal of \( I \times I \), obeying for \( E \neq E' \) the estimates

\[
\text{ess sup}_{(x, \omega)} \int_{S'} \sigma_0(x, \omega', \omega, E', E) d\omega' \leq \frac{C}{|E - E'|^{\kappa}},
\] (43)

\[
\text{ess sup}_{(x, \omega)} \int_{S'} \sigma_0(x, \omega, \omega', E', E') d\omega' \leq \frac{C}{|E - E'|^{\kappa}},
\] (44)

where \( \kappa < 1 \). The corresponding collision operator

\[
(K \psi)(x, \omega, E) = \int_P \int_{S'} \sigma_0(x, \omega, \omega', E', E) \psi(x, \omega', E') d\omega' dE',
\] (45)

is the usual partial Schur integral operator that is, \( \sigma_0(x, \omega, \omega', E', E) \) satisfies the Schur criterion for the boundedness ([17], p. 22) and so \( K \) is a bounded operator \( L^2(G \times S \times I) \rightarrow L^2(G \times S \times I) \) (see [31], section 5).

Nevertheless, the collision operator \( K \) is not generally of the above form. \( (E', E) \)-dependence in differential cross section \( \sigma(x, \omega', \omega, E', E) \) may contain hyper-singularities of higher order, such as \( \frac{1}{(E' - E)^j} \), for \( j = 1, 2 \). Below we shall consider in more detail the Möller scattering. We remark that the analysis e.g. for Bremsstrahlung goes quite similarly (but it is more simple).

3.1. Möller scattering as a prototype for hyper-singular collision operator. So called Möller scattering is a kind of prototype of interactions of charged particles leading to hyper-singular integral operators. Hence we express our analysis in the frames of it. In [31] we verified that the cross section \( \sigma \) for the Möller interaction is of the form

\[
\sigma(x, \omega', \omega, E', E) = \chi(E', E) \left( \frac{1}{(E' - E)^2} \sigma_2(x, \omega', \omega, E', E) + \frac{1}{E' - E} \sigma_1(x, \omega', \omega, E', E) + \sigma_0(x, \omega', \omega, E', E) \right)
\] (46)

where

\[
\chi(E', E) := \chi_{\mathbb{R}_+}(E - E_0) \chi_{\mathbb{R}_+}(E_m - E) \chi_{\mathbb{R}_+}(E' - E).
\]
Here each of \( \sigma_j(x, \omega', \omega, E', E) \), \( j = 0, 1, 2 \) may contain the above mentioned \( \delta \)-distributions, and hence they are not necessarily measurable functions on \( G \times S \times I \times I \). Denote for \( j = 0, 1, 2 \),

\[
(K_j \psi)(x, \omega, E', E) := \int_{S'} \sigma_j(x, \omega', \omega, E', E) \psi(x, \omega', E')d\omega',
\]

\[
(\hat{K}_j \psi)(x, \omega, E', E) := \chi(E', E) (K_j \psi)(x, \omega, E', E).
\]

Here the integral \( \int_{S'} \) is originally interpreted as a distribution. However, it can be shown \([31]\), section 3.2 that \( K_j \) is of the form

\[
(\bar{K}_j \psi)(x, \omega, E', E) = \hat{\sigma}_j(x, E', E) \int_{S'} \delta(\omega' \cdot \omega - \mu(E, E')) \psi(x, \omega', E')d\omega'
\]

\[
= \hat{\sigma}_j(x, E', E) \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E')ds,
\]

where \( \mu(E', E) := \sqrt{\frac{E(E' + 2)}{E(1 + 2)}} \) and where \( \gamma = \gamma(E', E, \omega) : [0, 2\pi] \to S \) is a parametrization of the curve

\[
\Gamma(E', E, \omega) = \{ \omega' \in S \mid \omega' \cdot \omega - \mu(E', E) = 0 \}.
\]

For example, we can choose

\[
\gamma(E', E, \omega)(s) = R(\omega)(\sqrt{1 - \mu^2} \cos(s), \sqrt{1 - \mu^2} \sin(s), \mu), \quad s \in [0, 2\pi],
\]

where \( \mu = \mu(E', E) \), and \( R(\omega) \) is any rotation (unitary) matrix which maps the vector \( e_3 = (0, 0, 1) \) into \( \omega \). We choose

\[
R(\omega) = \begin{pmatrix}
\omega_1 \omega_2 \\
\omega_2 \omega_3 \\
\omega_3 \omega_1
\end{pmatrix} = \begin{pmatrix}
\Omega_2 & \Omega_1 & \omega
\end{pmatrix}.
\]

Combining the above treatments we found in \([31]\), section 3.2 that \( K \) is of the form

\[
(K \psi)(x, \omega, E) = \mathcal{H}_2((\bar{K}_2 \psi)(x, \omega, \cdot, E))(E)
\]

\[
+ \mathcal{H}_1((\bar{K}_1 \psi)(x, \omega, \cdot, E))(E) + \int_{I} (\bar{K}_0 \psi)(x, \omega, E', E)dE',
\]

where \( \mathcal{H}_j, j = 1, 2, \) are the Hadamard finite part integral operators with respect to \( E' \)-variable defined by

\[
(\mathcal{H}_j u)(E) := \text{p.f.} \int_{E}^{E_m} \frac{1}{(E' - E)^{j}} u(E')dE'.
\]

The expression \((49)\) is the hyper-singular integral operator form of \( K \).

Moreover, we in \([31]\) verified that \((19)\) can be equivalently given in the ”pseudo-differential operator-like form” by

\[
(K \psi)(x, \omega, E) = \frac{\partial}{\partial E} \left( \mathcal{H}_1((\bar{K}_2 \psi)(x, \omega, \cdot, E))(E) \right) - \mathcal{H}_1(\left( \frac{\partial(\bar{K}_2 \psi)}{\partial E'} \right)(x, \omega, \cdot, E))(E)
\]

\[
+ \frac{\partial(\bar{K}_2 \psi)}{\partial E'}(x, \omega, E', E)|_{E' = E}
\]

\[
+ \mathcal{H}_1((\bar{K}_1 \psi)(x, \omega, \cdot, E))(E) + \int_{I} (\bar{K}_0 \psi)(x, \omega, E', E)dE'
\]

where only the ”first-order” Hadamard finite part integral operator \( \mathcal{H}_1 \) appears. We neglect the details but we recall that the derivation of \((50)\) founded on the use of Lemma \([21]\).
As a conclusion we see that some interactions produce the first-order partial derivatives with respect to energy $E$ combined with the Hadamard part operator (which is a pseudo-differential-like operator; see Remark 3.3 in [31]). For instance, in dose calculation (radiation therapy) these problematic interactions are the primary electron-electron, primary positron-positron collisions and Bremsstrahlung. The exact transport operator for Møller scattering is

$$(T\psi)(x,\omega,E) := -\mathcal{H}_2((\mathcal{K}_2\psi)(x,\omega,\cdot,E))(E) - \mathcal{H}_1((\mathcal{K}_1\psi)(x,\omega,\cdot,E))(E)$$

where

$$(K_r\psi)(x,\omega,E) := \int_{E'} (\mathcal{K}_0\psi)(x,\omega,E',E)dE'.$$

$\mathcal{K}_r$ is called a restricted collision operator which (by the Schur criterion) is a bounded operator $L^2(G \times S \times I) \rightarrow L^2(G \times S \times I)$. Basic properties of (more general) restricted collision operators are exposed in [32] and more widely in [31], section 5.4. Equivalently, in virtue of [50] $T$ can be given by

$$(T\psi)(x,\omega,E) := -\frac{\partial}{\partial E} \left( \mathcal{H}_1((\mathcal{K}_2\psi)(x,\omega,\cdot,E))(E) \right) + \mathcal{H}_1((\frac{\partial(\mathcal{K}_2\psi)}{\partial E}(x,\omega,\cdot,E))(E)$$

$$- \frac{\partial(\mathcal{K}_2\psi)}{\partial E'}(x,\omega,E',E)|_{E'=E}$$

+ $\omega \cdot \nabla_x \psi + \Sigma(x,\omega,E)\psi - \mathcal{H}_1((\mathcal{K}_1\psi)(x,\omega,\cdot,E))(E) - (K_r\psi)(x,\omega,E).$$

We finally mention that the pseudo-differential-like parts can be approximated by partial differential operators ([31], section 4) to obtain pure partial integro-differential operator approximations, so called continuous slowing down approximations (CSDA).

4. Exact transport operator and formal adjoints

We shall give more details about the transport operator (51) or (52). The obtained refined expression reveals that hyper-singular transport operators generate partial differential terms with respect to angle and energy variables as well. We start by verifying some essential tools for Hadamard finite part integrals. The below Lemmas are needed frequently in the rest of the paper.

4.1. Auxiliary lemmas for Hadamard finite part integrals. Suppose that $f \in C(\mathcal{G}, C^\alpha(I^2))$ where $\alpha > 0$. Then we find that

$$\text{p.f.} \int_{E}^{E_m} \frac{1}{E' - E} f(x, E', E)dE' = \int_{E}^{E_m} \frac{f(x, E', E) - f(x, E, E)}{E' - E}dE' + f(x, E, E) \ln(E_m - E)$$

since p.f. $\int_{E}^{E_m} \frac{1}{E' - E}dE' = \ln(E_m - E)$. Noting that $\int_I |\ln(E_m - E)|dE < \infty$ and that

$$\left| \int_{E}^{E_m} \frac{f(x, E', E) - f(x, E, E)}{E' - E}dE' \right| \leq \int_{E}^{E_m} C_\alpha |E' - E|^{\alpha}dE' = C_\alpha \frac{1}{\alpha} (E_m - E)^{\alpha}$$

where $C_\alpha := \|f\|_{C(\mathcal{G}, C^\alpha(I^2))}$ we see that the function

$$E \rightarrow \text{p.f.} \int_{E}^{E_m} \frac{1}{E' - E} f(x, E', E)dE'$$

is integrable on $I$ (in the sense of ordinary improper Riemann integrals). Analogously we see that the function

$$E' \rightarrow \text{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} f(x, E', E)dE$$

is integrable on $I'$.
We start with the next Fubin-type lemma

**Lemma 4.1** For \( f \in C(\mathbb{R}, C^\alpha(I^2)) \), \( \alpha > 0 \) we have

\[
\int_I \left( \int_E^{E_m} \frac{1}{E' - E} f(x, E', E) dE' \right) dE = \int_I \left( \int_{E_\epsilon}^{E_m} \frac{1}{E' - E} f(x, E', E) dE' \right) dE' \tag{53}
\]

**Proof.** Define integrals

\[
I_1^\epsilon := \int_I \left( \int_{E_\epsilon}^{E_m} \frac{1}{E' - E} f(x, E', E) dE' \right) dE + f(x, E, E) \ln(\epsilon) dE,
\]

\[
I_2^\epsilon := \int_I \left( \int_{E_\epsilon}^{E_m} \frac{1}{E' - E} f(x, E', E) dE + f(x, E', E') \ln(\epsilon) \right) dE'.
\]

By the Fubin’s Theorem

\[
I_1^\epsilon := \int_I \left( \int_{E_\epsilon}^{E_m} \frac{1}{E' - E} f(x, E', E) dE' \right) dE + f(x, E, E) \ln(\epsilon) dE
\]

\[
= \int_{E_\epsilon}^{E_m} \left( \int_I \frac{1}{E' - E} f(x, E', E) dE \right) dE + \int_I f(x, E, E) \ln(\epsilon) dE
\]

\[
= \int_{E_\epsilon}^{E_m} \left( \int_{E_\epsilon}^{E_m} \frac{1}{E' - E} f(x, E', E) dE' \right) dE' + \int_I f(x, E', E') \ln(\epsilon) dE' := I_2^\epsilon.
\]

Let

\[
F_1^\epsilon(x, E) := \int_{E_\epsilon}^{E_m} \frac{1}{E' - E} f(x, E', E) dE' + f(x, E, E) \ln(\epsilon).
\]

Then we have

\[
F_1^\epsilon(x, E) := \int_{E_\epsilon}^{E_m} \frac{f(x, E', E) - f(x, E, E)}{E' - E} dE' + \int_{E_\epsilon}^{E_m} \frac{f(x, E, E)}{E' - E} dE' + f(x, E, E) \ln(\epsilon)
\]

\[
= \int_{E_\epsilon}^{E_m} \frac{f(x, E', E) - f(x, E, E)}{E' - E} dE' + f(x, E, E) \ln(\epsilon)
\]

since \( \int_{E_\epsilon}^{E_m} \frac{1}{E' - E} dE' = \ln(E_m - E) - \ln(\epsilon) \). Noting that (here \( C_\alpha \) is as above)

\[
\left| \frac{f(x, E', E) - f(x, E, E)}{E' - E} \right| \leq C_\alpha |E' - E|^{\alpha - 1}
\]

we find that

\[
|F_1^\epsilon(x, E)| \leq \int_{E_\epsilon}^{E_m} C_\alpha |E' - E|^{\alpha - 1} dE' + |f(x, E, E) \ln(E_m - E)|
\]

\[
= C_\alpha \frac{1}{\alpha} ((E_m - E)^\alpha - \epsilon^\alpha) + |f(x, E, E) \ln(E_m - E)|. \tag{55}
\]

Hence the sequence \( \{F_1^\epsilon(x, E)\} \) is bounded by an integrable function. By the definition of the Hadamard finite part integrals

\[
\lim_{\epsilon \to 0} F_1^\epsilon(x, E) = \text{p.f.} \int_E^{E_m} \frac{1}{E' - E} f(x, E', E) dE'
\]

Similarly we see that the sequence of functions

\[
F_2^\epsilon(x, E') := \int_{E_\epsilon}^{E_m} \frac{1}{E' - E} f(x, E', E) dE + f(x, E', E') \ln(\epsilon)
\]
is bounded by an integrable function and

$$\lim_{\epsilon \to 0} F^2_{\epsilon}(x, E') = \text{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} f(x, E', E) dE.$$  \hspace{1cm} (57)

In virtue of \((54)\)

$$\int_I F^1_{\epsilon}(x, E) dE = I^1_{\epsilon} = I^2_{\epsilon} = \int_{I'} F^2_{\epsilon}(x, E') dE'.$$  \hspace{1cm} (58)

The Lebesgue Dominated Convergence Theorem implies by \((56), (57), (58)\) that

$$\int_{I'} \lim_{\epsilon \to 0} F^1_{\epsilon}(x, E') dE' = \int_{I'} F^2_{\epsilon}(x, E') dE'.$$  \hspace{1cm} (59)

which completes the proof. \(\square\)

We prove also the next partial integration-like results.

**Lemma 4.2** Suppose that \(f \in C(\overline{G} \times I, C^\alpha(I'))\), \(\alpha > 0\). Then

$$\text{p.f.} \int_{E}^{E_m} \frac{1}{E' - E} f(x, E', E) dE'$

$$= - \int_{E}^{E_m} \ln(E' - E) \frac{\partial f}{\partial E'}(x, E', E) dE' + \ln(E_m - E) f(x, E_m, E).$$  \hspace{1cm} (60)

**Proof.** Using partial integration we have

$$\int_{E + \epsilon}^{E_m} \frac{1}{E' - E} f(x, E', E) dE' + \ln(\epsilon) f(x, E, E)$$

$$= \left|_{E + \epsilon}^{E_m} \ln(E' - E) f(x, E', E) - \int_{E + \epsilon}^{E_m} \ln(E' - E) \frac{\partial f}{\partial E'}(x, E', E) dE' + \ln(\epsilon) f(x, E, E) \right|$$

$$= - \int_{E + \epsilon}^{E_m} \ln(E' - E) \frac{\partial f}{\partial E'}(x, E', E) dE' + \ln(E_m - E) f(x, E_m, E)$$  \hspace{1cm} (61)

from which the assertion follows by letting \(\epsilon \to 0\). \(\square\)

**Lemma 4.3** Suppose that \(f \in C(\overline{G} \times I, C^{1+\alpha}(I'))\), \(\alpha > 0\). Then

$$\text{p.f.} \int_{E}^{E_m} \frac{1}{(E' - E)^2} f(x, E', E) dE'$$

$$= \text{p.f.} \int_{E}^{E_m} \frac{\partial f}{E' - E} (x, E', E) dE' + \frac{\partial f}{E'(x, E, E)_+} - \frac{1}{E_m - E} f(x, E_m, E)$$  \hspace{1cm} (62)

where \(\frac{\partial f}{\partial E'}(x, E, E)_+\) denotes the right hand partial derivative (actually here \(\frac{\partial f}{\partial E'}(x, E, E)_+ = \frac{\partial f}{\partial E'}(x, E, E)\)).
Proof. Using partial integration we have for $\epsilon > 0$

$$
\int_{E+\epsilon}^{E_m} \frac{1}{(E' - E)^2} f(x, E', E) dE' - \frac{1}{\epsilon} f(x, E, E) + \ln(\epsilon) \frac{\partial f}{\partial E'}(x, E, E)
$$

$$
= \left|_{E+\epsilon}^{E_m} \frac{1}{E' - E} f(x, E', E) + \int_{E+\epsilon}^{E_m} \frac{1}{E' - E} \frac{\partial f}{\partial E'}(x, E', E) dE' - \frac{1}{\epsilon} f(x, E, E) + \ln(\epsilon) \frac{\partial f}{\partial E'}(x, E, E) \right|
$$

$$
\left|_{E+\epsilon}^{E_m} \frac{1}{E' - E} f(x, E', E) - \frac{1}{E_m - E} f(x, E_m, E) \right|
$$

from which the assertion follows by letting $\epsilon \to 0^+$.

Similarly we have

**Lemma 4.4** Suppose that $f \in C(G \times I', C^{1+\alpha}(I))$, $\alpha > 0$. Then

$$
p.f. \int_{E_0}^{E'} \frac{1}{(E' - E)^2} f(x, E', E) dE = -p.f. \int_{E_0}^{E'} \frac{\partial f}{\partial E'}(x, E', E) dE - \frac{1}{E' - E_0} f(x, E', E_0)
$$

where $\frac{\partial f}{\partial E'}(x, E', E_0)$ denotes the left hand partial derivative.

The previous lemma 4.3 gives immediately

**Lemma 4.5** Suppose that $f \in C(G, C^{1+\alpha}(I^2))$. Then

$$
\int_I \left( p.f. \int_E^{E_m} \frac{1}{(E' - E)^2} f(x, E', E) dE' \right) dE = \int_I' \left( p.f. \int_{E_0}^{E'} \frac{1}{E' - E} \frac{\partial f}{\partial E'}(x, E', E) dE' \right) dE'
$$

$$
+ \int_I' \frac{\partial f}{\partial E'}(x, E', E')_+ dE' - \int_I' \frac{1}{E_m - E'} f(x, E_m, E') dE' \quad (65)
$$

when the integral $\int_I' \frac{1}{E_m - E'} f(x, E_m, E') dE'$ exists.

Proof. In virtue of Lemma 4.3

$$
\int_I \left( p.f. \int_E^{E_m} \frac{1}{(E' - E)^2} f(x, E', E) dE' \right) dE = \int_I \left( p.f. \int_E^{E_m} \frac{1}{E' - E} \frac{\partial f}{\partial E'}(x, E', E) dE' \right) dE
$$

$$
+ \int_I \frac{\partial f}{\partial E'}(x, E, E)_+ dE - \int_I \frac{1}{E_m - E} f(x, E_m, E) dE.
$$

Hence the assertion follows from Lemma 4.3

Moreover, we have the following Fubini-type theorem
Lemma 4.6 For \( f \in C(\overline{G} \times S \times I, C^{1+\alpha}(I')) \), \( \alpha > 0 \) and for \( E \neq E_m \)
\[
\int_S \left( \text{p.f.} \int_E^{E_m} \frac{1}{(E' - E)^2} f(x, E', E, \omega) dE' \right) d\omega = \text{p.f.} \int_E^{E_m} \frac{1}{(E' - E)^2} \left( \int_S f(x, E', E, \omega) d\omega \right) dE'.
\] (67)

Proof. By Fubin’s Theorem we have
\[
\int_S \left[ \int_{E+\epsilon}^{E_m} \frac{1}{(E' - E)^2} f(x, E', E, \omega) dE' - \frac{1}{\epsilon} f(x, E, E, \omega) + \ln(\epsilon) \frac{\partial f}{\partial E'}(x, E, E, \omega) \right] d\omega
\]
\[
= \int_{E+\epsilon}^{E_m} \frac{1}{(E' - E)^2} \left( \int_S f(x, E', E, \omega) d\omega \right) dE'
\]
\[
- \frac{1}{\epsilon} \int_S f(x, E, E, \omega) d\omega + \ln(\epsilon) \int_S \frac{\partial f}{\partial E'}(x, E, E, \omega) d\omega
\]
\[
= \int_{E+\epsilon}^{E_m} \frac{1}{(E' - E)^2} \left( \int_S f(x, E', E, \omega) d\omega \right) dE'
\]
\[
- \frac{1}{\epsilon} \int_S f(x, E, E, \omega) d\omega + \ln(\epsilon) \frac{\partial f}{\partial E'} \left( \int_S f(x, E, \omega) d\omega \right)\bigg|_{E'=E}.
\] (68)

Denote
\[
F_\epsilon(x, E, \omega) := \int_{E+\epsilon}^{E_m} \frac{1}{(E' - E)^2} f(x, E', E, \omega) dE' - \frac{1}{\epsilon} f(x, E, E, \omega) + \ln(\epsilon) \frac{\partial f}{\partial E'}(x, E, E, \omega).
\]

Applying the Taylor’s formula
\[
f(x, E', E, \omega) = f(x, E, E, \omega) + \int_0^1 \frac{\partial f}{\partial E'}(x, E + t(E' - E), E, \omega) dt \cdot (E' - E)
\]
and the identity
\[
\int_0^1 \frac{1}{E' - E} \frac{\partial f}{\partial E'}(x, E + t(E' - E), E, \omega) dt = \frac{1}{E' - E} \frac{\partial f}{\partial E'}(x, E, E, \omega)
\]
\[
+ \int_0^1 \frac{1}{E' - E} \left( \frac{\partial f}{\partial E'}(x, E + t(E' - E), E, \omega) - \frac{\partial f}{\partial E'}(x, E, E, \omega) \right) dt
\]
we see analogously to the proof of Lemma 4.1 that by the Lebesgue Dominated Convergence Theorem
\[
\int_S \lim_{\epsilon \to 0} F_\epsilon(x, E, \omega) d\omega = \lim_{\epsilon \to 0} \int_S F_\epsilon(x, E, \omega) d\omega.
\] (69)

In virtue of definition (9) and (68), (69)
\[
\int_S \left( \text{p.f.} \int_E^{E_m} \frac{1}{(E' - E)^2} f(x, E', E, \omega) dE' \right) d\omega
\]
\[
= \int_S \lim_{\epsilon \to 0} \left( \int_E^{E_m} \frac{1}{(E' - E)^2} f(x, E', E, \omega) dE' - \frac{1}{\epsilon} f(x, E, E, \omega) + \ln(\epsilon) \frac{\partial f}{\partial E'}(x, E, E, \omega) \right) d\omega
\]
\[
= \int_S F_\epsilon(x, E, \omega) d\omega = \lim_{\epsilon \to 0} \int_S F_\epsilon(x, E, \omega) d\omega
\]
\[
= \text{p.f.} \int_E^{E_m} \frac{1}{(E' - E)^2} \left( \int_S f(x, E', E, \omega) d\omega \right) dE'
\]
which completes the proof. \(\square\)
In the same way (but more easily) we find that for \( f \in C(\overline{G} \times S \times I, C^n(I')) \), \( \alpha > 0 \) and \( E \neq E_m \)
\[
\int_S \left( \text{p.f.} \int_{E_m}^{E} \frac{1}{E' - E} f(x, E', E, \omega) dE' \right) d\omega = \text{p.f.} \int_{E_m}^{E} \frac{1}{E' - E} \left( \int_S f(x, E', E, \omega) d\omega \right) dE'.
\]

(70)

The integral \( \int_S \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E') v(x, \omega, E) ds d\omega \) below (in sections 4.3 and 5) emerging from Møller collision operator needs a special treatment which is yielded by the next lemma.

**Lemma 4.7** Let \(-1 < t < 1\), and let for any \( \omega \in S \)
\[
\Gamma_t^\omega = \{ \omega' \in S \mid \omega \cdot \omega' = t \}.
\]
Then for any \( f \in L^1(S \times S') \) one has
\[
\int_S \int_{\Gamma_t^\omega} f(\omega, \omega') d\ell(\omega') d\omega = \int_{S'} \int_{\Gamma_t^\omega} f(\omega, \omega') d\ell(\omega) d\omega'
\]
where \( \int_{\Gamma_t^\omega} h(\omega') d\ell(\omega') \) is the path integral of \( h \) along \( \Gamma_t^\omega \).

**Proof.** We apply the idea given in [31], Lemma 5.4. Let \(-1 < t_0 < t < 1\). For every \( \omega \in S \), the usual surface measure \( d\mu_S(\omega) = d\omega \) on \( S \) disintegrates into a family of measures \((1 - r^2)^{-1/2} d\ell_r \otimes dr\), \(-1 < r < 1\), where \( \ell_r \) is the path length measure along the curve \( \Gamma_r^\omega \). Let \( S_{\omega, [t_0, t]} \) be the spherical zone
\[
S_{\omega, [t_0, t]} := \{ \omega' \in S \mid t_0 \leq \omega' \cdot \omega \leq t \}.
\]
Then we have by Fubin’s Theorem
\[
\int_S \int_{S_{\omega, [t_0, t]}} f(\omega, \omega') d\omega' d\omega = \int_S \int_{t_0}^t \int_{\Gamma_r^\omega} f(\omega, \omega')(1 - r^2)^{-1/2} d\ell_r(\omega') dr d\omega
\]
\[
= \int_{t_0}^t (1 - r^2)^{-1/2} \int_{S'} \int_{\Gamma_r^\omega} f(\omega, \omega') d\ell_r(\omega') d\omega dr.
\]

(72)

We use the parametrization
\[
r \rightarrow R(\omega)(\sqrt{1 - r^2} \cos(s), \sqrt{1 - r^2} \sin(s), r)
\]
for \( \Gamma_r^\omega \).

Furthermore, let \( \chi_{\mathbb{R}_+} \) be the characteristic function of \( \mathbb{R}_+ \). Then again by the Fubinis’s Theorem
\[
\int_S \int_{S_{\omega, [t_0, t]}} f(\omega, \omega') d\omega' d\omega
\]
\[
= \int_S \int_{S'} \chi_{\mathbb{R}_+}(\omega' \cdot \omega - t_0) \chi_{\mathbb{R}_+}(t - \omega' \cdot \omega) f(\omega, \omega') d\omega' d\omega
\]
\[
= \int_{S'} \int_S \chi_{\mathbb{R}_+}(\omega' \cdot \omega - t_0) \chi_{\mathbb{R}_+}(t - \omega' \cdot \omega) f(\omega, \omega') d\omega d\omega'
\]
\[
= \int_{S'} \int_{S_{\omega', [t_0, t]}} f(\omega, \omega') d\omega' d\omega'.
\]

Thus by (72)
\[
\int_{t_0}^t (1 - r^2)^{-1/2} \int_{S'} \int_{\Gamma_r^\omega} f(\omega, \omega') d\ell_r(\omega') d\omega' dr = \int_{t_0}^t (1 - r^2)^{-1/2} \int_{S'} \int_{\Gamma_r^\omega} f(\omega, \omega') d\ell_r(\omega') d\omega' dr.
\]

(73)
Noting that the function
\[ r \mapsto \int_S \int_\Gamma \omega f(\omega, \omega') d\ell(\omega') d\omega = \int_S \int_0^{2\pi} f(\omega, R(\omega)(\sqrt{1-r^2 \cos(s)}, \sqrt{1-r^2 \sin(s)}, r)) dr d\omega \]
is continuous we get the assertion by taking (in formula (73)) the derivative with respect to \( t \) on each side.

Corollary 4.8 For \( \psi, v \in C(\bar{G} \times S \times I) \)
\[ \int_S \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E') v(x, \omega, E) ds d\omega = \int_S \int_0^{2\pi} \psi(x, \omega', E') v(x, \gamma(E', E, \omega')(s), E) ds d\omega' \]
where \( \gamma(E', E, \omega) \) is as in section 3.1.

Proof. Let for fixed \( x, E' \)
\[ f(\omega, \omega') := \psi(x, \omega', E') v(x, \omega, E). \]
Recall that \( \gamma(E', E, \omega) \) is a parametrization of the curve \( \Gamma_\mu := \{ \omega' \in S \mid \omega \cdot \omega' = \mu(E', E) \} \) and \( \| \gamma(E', E, \omega')(s) \| = \sqrt{1-\mu(E', E)^2} \). Hence we have by Lemma 4.7
\[ \int_S \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E') v(x, \omega, E) ds d\omega = \frac{1}{\sqrt{1-\mu(E', E)^2}} \int_S \int_\Gamma f(\omega, \omega') d\ell(\omega') d\omega \]
\[ = \frac{1}{\sqrt{1-\mu(E', E)^2}} \int_S \int_\Gamma f(\omega, \omega') d\ell(\omega') d\omega' \]
\[ = \int_S \int_0^{2\pi} \psi(x, \omega', E') v(x, \gamma(E', E, \omega')(s), E) ds d\omega', \]
as desired.

Finally we prove the following result.

Theorem 4.9 Let \( \psi \in C(\bar{G}, C^2(I, C^3(S))) \). Then for fixed \( x, \omega, E \) the mapping
\[ F(E') := \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E') ds \]
is differentiable and
(1) for \( E' \neq E \)
\[ F'(E') = \frac{\partial}{\partial E'} \left( \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E') ds \right) \]
\[ = \int_0^{2\pi} \left( (\nabla_S \psi)(x, \gamma(E', E, \omega)(s), E'), \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) \right) ds \]
\[ + \int_0^{2\pi} \frac{\partial \psi}{\partial E}(x, \gamma(E', E, \omega)(s), E') ds \] (76)
We proceed as follows.

Above we have to prove that $\gamma$

Since (recall that $\eta$ from section 2.2) and $\xi$

The term $E$ in (76) requires further study for $E$

show that the limit $\lim_{T} E$

Because the mapping $\mu \in S$

Therefore formula (76) follows from the chain rule on manifolds.

Hence formula (76) follows from the chain rule on manifolds.

where $\partial j = \partial \xi$, and where the limit is given in Lemma 4.10 below.

Above $\xi(E', E, \omega, s) := J(\zeta(E', E, \omega, s))$, $\zeta(E', E, \omega, s) = H^{-1}_E(\gamma(E', E, \omega)(s))$ (recall $J$ and $\zeta$ from section 2.2) and $\eta(E', E, \omega, s) := \frac{\partial \xi}{\partial E'}(E', E, \omega)(s)$.

Proof. A. Note that the mapping $E' \rightarrow \gamma(E', E, \omega)(s)$ is a curve on $S$ and so $\frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) \in T_{\gamma(E', E, \omega)(s)}(S)$. Therefore formula (76) follows from the chain rule on manifolds.

B. Writing $\mu = \mu(E', E)$, $\frac{\partial \mu}{\partial E'}(E', E)$ we find that

The term

in (76) requires further study for $E' = E$ since the partial derivative $\frac{\partial \gamma}{\partial E'}(E', E, \omega)(s)$ does not exist for $E' = E$.

Because the mapping $F$ is continuous it suffices (by the consequence of L'Hospital's rule) to show that the limit $\lim_{E' \rightarrow E} F'(E')$ exists and that

Since (recall that $\gamma(E', E, \omega)(s) = \omega$

we have to prove that

We proceed as follows.
In virtue of the Taylor’s formula (recall \( \eta = \frac{\partial \mu}{\partial E'}(E', E, \omega)(s) \))

\[
\left\langle (\nabla_S \psi)(x, \gamma(E', E, \omega)(s), E'), \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) \right\rangle = \left\langle (\nabla_S \psi)(x, \omega, E'), \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) \right\rangle + \sum_{j=1}^{2} \partial_j \left( \langle (\nabla_S \psi \circ H_\omega)(x, \cdot, E'), \eta(E', E, \omega, s) \rangle \right) (0) \xi_j(E', E, \omega, s) + (R_\omega \psi(x, \cdot, E'))(\omega') \left( \zeta(E', E, \omega, s), \zeta(E', E, \omega, \eta(E', E, \omega, s)) \right)
\]

(82)

where the residual is

\[
R_\omega(\psi(x, \cdot, E'))(\omega') \left( \zeta(E', E, \omega, s), \zeta(E', E, \omega, s), \eta(E', E, \omega, s) \right) = \sum_{|\alpha|=2} [\alpha]! \int_0^1 (1 - t) \xi^\alpha \left( \langle (\nabla_S \psi \circ H_\omega)(x, \cdot, E'), \eta(E', E, \omega, s) \rangle \right) (t \xi) dt \cdot \xi^\alpha.
\]

(83)

We have for \( E' \neq E \)

\[
\int_0^{2\pi} \left\langle (\nabla_S \psi)(x, \omega, E'), \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) \right\rangle ds = \left\langle (\nabla_S \psi)(x, \omega, E'), \int_0^{2\pi} \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) ds \right\rangle
\]

= \( 2\pi (\partial_{E'} \mu)(E', E)(\omega \cdot \nabla_S \psi)(x, \omega, E') \)

(84)

where we recalled that \( R(\omega)e_3 = \omega \), and so

\[
\lim_{E' \to E} \left( \int_0^{2\pi} \left\langle (\nabla_S \psi)(x, \omega, E'), \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) \right\rangle ds \right)
\]

= \( 2\pi (\partial_{E'} \mu)(E, E)(\omega \cdot \nabla_S \psi)(x, \omega, E) \).

(85)

For the residual we have

\[
\lim_{E' \to E} \int_0^{2\pi} (R_\omega \psi(x, \cdot, E'))(\omega') \left( \zeta(E', E, \omega, s), \zeta(E', E, \omega, s), \eta(E', E, \omega, s) \right) ds = 0
\]

(86)

This is seen as follows. By (24)

\[
\| \zeta(E', E, \omega, s) \| \leq C \| \gamma(E', E, \omega)(s) - \omega \|.
\]

(87)

Letting \( \gamma_0(E', E)(s) := (\sqrt{1 - \mu^2} \cos(s), \sqrt{1 - \mu^2} \sin(s), \mu) \) and recalling that \( \omega = R(\omega)e_3 \) we obtain

\[
\| \gamma(E', E, \omega)(s) - \omega \|^2 = \| R(\omega)\gamma_0(E', E)(s) - \omega \|^2 = \| \gamma_0(E', E)(s) - R(\omega)^{-1} \omega \|^2
\]

= \( \| \gamma_0(E', E)(s) - e_3 \|^2 = 2(1 - \mu(E', E)) \).

(88)

Furthermore, noting that

\[
\| \eta(E', E, \omega, s) \| = \left\| \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) \right\|, \quad \| \xi(E', E, \omega, s) \| = \| \zeta(E', E, \omega, s) \|
\]
we get
\[ \| (R_\omega \psi(x, \cdot, E')(\omega')) \left( \zeta(E', E, \omega, s), \zeta(E', E, \omega, s), \eta(E', E, \omega, s) \right) \| \]
\[ \leq \sum_{|\alpha|=2} |\alpha| \int_0^1 (1 - t) \| \partial^\alpha_x (\nabla S \psi \circ H_\omega)(x, t \xi, E') \| \| \eta(E', E, \omega, s) \| \| \xi \|^{\alpha} \]
\[ \leq C_1 \| \psi \|_{L^\infty(G \times I, C^3(S))} \| \zeta(E', E, \omega, s) \|^2 \| \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) \|
\[ \leq C_1 C^2 \| \psi \|_{L^\infty(G \times I, C^3(S))} \| \gamma(E', E, \omega)(s) - \omega \|^2 \| \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) \|
\[ \leq 2C_1 C^2 \| \psi \|_{L^\infty(G \times I, C^3(S))} (1 - \mu) \frac{|\partial E' \mu|}{\sqrt{1 - \mu^2}} = 2C_1 C^2 \| \psi \|_{L^\infty(G \times I, C^3(S))} \frac{|\partial E' \mu|}{\sqrt{1 + \mu}} \] (89)

In (89) the right hand side converges to zero since \( \mu \to 1 \) as \( E' \to E \). Hence the assertion (11) follows by combining (82), (85), (86).

\[ \square \]

**Lemma 4.10**  Let \( \xi = J(\zeta), \zeta := H^{-1}_\omega(\gamma(E', E, \omega)(s)) \in T_\omega(S) \) and \( \eta = \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) \). Then
\[ \lim_{E' \to E} \int_0^{2\pi} 2 \sum_{j=1}^2 \partial_j \left( \langle (\nabla S \psi \circ H_\omega)(x, \cdot, E'), \eta(E', E, \omega, s) \rangle \right)(0) \xi_j((E', E, \omega, s)) ds \]
\[ = -(\partial E' \mu)(E, E) \int_0^{2\pi} 2 \sum_{j=1}^2 \partial_j \left( \langle (\nabla S \psi \circ H_\omega)(x, \cdot, E), R(\omega)(\cos(s), \sin(s), 0) \rangle \right)(0) \]
\[ \cdot \langle R(\omega)(\cos(s), \sin(s), 0), \Omega_j \rangle ds \] (90)

**Proof.** We choose \( H_\omega = \exp_\omega \) given in Example 2.4. Denote \( \omega' := \gamma(E', E, \omega)(s) \) and \( \mu = \mu(E', E) \). In due to (35) we find that
\[ \xi(E', E, \omega, s) = \zeta(E', E, \omega, s) = \exp_\omega^{-1}(\omega') = \frac{\arccos(\langle \omega', \omega \rangle)}{\sqrt{1 - \langle \omega', \omega \rangle^2}} (\omega' - \langle \omega', \omega \rangle \omega) \]
\[ = \frac{\arccos(\langle \gamma(E', E, \omega)(s), \omega \rangle)}{\sqrt{1 - \langle \gamma(E', E, \omega)(s), \omega \rangle^2}} (\gamma(E', E, \omega)(s) - \langle \gamma(E', E, \omega)(s), \omega \rangle \omega). \] (91)

Furthermore, we find that
\[ \langle \gamma(E', E, \omega)(s), \omega \rangle = \langle R(\omega)\gamma_0(E', E)(s), \omega \rangle = \langle \gamma_0(E', E)(s), R(\omega)^* \omega \rangle \]
\[ = \langle \gamma_0(E', E)(s), e_3 \rangle = \mu(E', E) \]
where we noticed that \( R(\omega)^* \omega = R(\omega)^{-1} \omega = e_3 \). Hence
\[ \xi(E', E, \omega, s) = \frac{\arccos(\mu)}{\sqrt{1 - \mu^2}} (\gamma(E', E, \omega)(s) - \mu \omega) \]
\[ = \frac{\arccos(\mu)}{\sqrt{1 - \mu^2}} R(\omega)(\gamma_0(E', E)(s) - (0, 0, \mu)) \] (92)

since \( R(\omega)(0, 0, \mu) = \mu \omega \) (recall \( R(\omega) \) from section 3.1).

In virtue of Example 2.4
\[ \lim_{E' \to E} \frac{\arccos(\mu(E', E))}{\sqrt{1 - \mu^2}} = 1. \]
In addition,
\[ R(\omega)(\gamma_0(E', E)(s) - (0, 0, \mu)) = R(\omega)(\sqrt{1 - \mu^2 \cos(s)}, \sqrt{1 - \mu^2 \sin(s)}, 0). \]

Hence (because \(\mu(E, E) = 1\))
\[
\lim_{E' \to E} \int_0^{2\pi} \sum_{j=1}^2 \partial_j \left( \langle \nabla_S \psi \circ H_\omega(x, ., E') \rangle R(\omega) \left( \frac{(\partial_{E'}\mu)\mu}{\sqrt{1 - \mu^2}} \cos(s), -\frac{(\partial_{E'}\mu)\mu}{\sqrt{1 - \mu^2}} \sin(s), \partial_{E'}\mu \right) \right) (0)
\]
\[
\cdot \left( R(\omega)(\sqrt{1 - \mu^2 \cos(s)}, \sqrt{1 - \mu^2 \sin(s)}, 0) \right) ds = \lim_{E' \to E} \int_0^{2\pi} \sum_{j=1}^2 \partial_j \left( \langle \nabla_S \psi \circ H_\omega(x, ., E') \rangle R(\omega) \left( \cos(s), \sin(s), 0 \right) \right) (0)
\]
\[
\cdot \left( R(\omega)(\cos(s), \sin(s), 0) \right) ds
\]
\[
= - (\partial_{E'}\mu)(E, E) \int_0^{2\pi} \sum_{j=1}^2 \partial_j \left( \langle \nabla_S \psi \circ H_\omega(x, ., E) \rangle R(\omega)(\cos(s), \sin(s), 0) \right) (0)
\]
\[
\cdot \left( R(\omega)(\cos(s), \sin(s), 0) \right) ds = 0.
\]

Since
\[
\langle R(\omega)(\cos(s), \sin(s), 0), \omega \rangle = \langle \langle \cos(s), \sin(s), 0 \rangle, R_1(\omega) \rangle = \langle \langle \cos(s), \sin(s), 0 \rangle, e_3 \rangle = 0
\]
the vector \(R(\omega)(\cos(s), \sin(s), 0)\) lies in \(T_\omega(S)\) (as it should be). In addition (since \(\overline{\Omega}_1, \overline{\Omega}_2\) are orthogonal),
\[
R(\omega)(\cos(s), \sin(s), 0) = R(\omega)(\cos(s), \sin(s), 0, \overline{\Omega}_j), \quad j = 1, 2.
\]
Hence we conclude the assertion from (93).

\[ \Box \]

**Remark 4.11** A. The result of the limit in (90) can be further computed but we omit computations here.

Note that by using the matrix partition \(R(\omega) = \begin{pmatrix} \overline{\Omega}_2 & \overline{\Omega}_1 & \omega \end{pmatrix}\) we find that
\[
\langle R(\omega)(\cos(s), \sin(s), 0), \overline{\Omega}_j \rangle = \langle \langle \cos(s), \sin(s), 0 \rangle, R_1(\omega) \overline{\Omega}_j \rangle = \langle \langle \cos(s), \sin(s), 0 \rangle, \begin{pmatrix} \overline{\Omega}_j \\ \overline{\Omega}_1 \\ \omega \end{pmatrix} \rangle
\]
\[
= \langle \langle \cos(s), \sin(s), 0 \rangle, \langle \overline{\Omega}_2, \overline{\Omega}_j \rangle, \langle \overline{\Omega}_1, \overline{\Omega}_j \rangle, 0 \rangle.
\]

B. We remark also that by (28) in (77)
\[
2\pi (\partial_{E'}\mu)(E, E)(\omega \cdot \nabla_S \psi)(x, \omega, E) = 0.
\]

4.2. **A refined form of the exact equation.** We derive a refined form for the exact transport operator under consideration. Recall that for \(j = 1, 2\)
\[
(\mathcal{R}_j \psi)(x, \omega, E') = \tilde{\sigma}_j(x, E', E') \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E') ds,
\]
and so
\[
\mathcal{H}_j((\mathcal{R}_j \psi)(x, \omega, \cdot, E))(E) = \text{p.f.} \int_{E} \frac{1}{(E' - E)} \tilde{\sigma}_j(x, E', E) \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E') ds dE'
\]

We begin with
Lemma 4.12 Suppose that \( f \in C(\overline{G} \times S \times I, C^\alpha(I')) \), \( \alpha > 0 \). Then
\[
p.f. \int_E^{E_m} \frac{f(x, \omega, E, E')}{E' - E} dE' = \ln(E_m - E)f(x, \omega, E, E) + u(x, \omega, E)
\] (97)
where \( u \in C(\overline{G} \times S \times I) \).

Proof. Recalling that \( \gamma(E, E, \omega)(s) = \omega \) we have
\[
p.f. \int_E^{E_m} \frac{f(x; \omega, E, E')}{E' - E} dE' = p.f. \int_E^{E_m} \frac{f(x; \omega, E, E)}{E' - E} dE'
\]
\[
+ \int_E^{E_m} \frac{f(x, \omega, E, E') - f(x, \omega, E, E)}{E' - E} dE' = \ln(E_m - E)f(x, \omega, E, E) + u(x, \omega, E)
\]
where
\[
u(x, \omega, E) := \int_E^{E_m} \frac{f(x, \omega, E, E') - f(x, \omega, E, E)}{E' - E} dE'.
\]
We find that
\[
u(x, \omega, E) = \int_I \chi_{R_+}(E' - E) \frac{f(x, \omega, E, E') - f(x, \omega, E, E)}{E' - E} dE'
\]
where
\[
\left| \chi_{R_+}(E' - E) \frac{f(x, \omega, E, E') - f(x, \omega, E, E)}{E' - E} \right| \leq C_\alpha \chi_{R_+}(E' - E)|E' - E|^{\alpha - 1}.
\]
In addition,
\[
\int_I \chi_{R_+}(E' - E)|E' - E|^{\alpha - 1} dE' = \frac{1}{\alpha} (E_m - E)^\alpha \leq \frac{1}{\alpha} E_1^\alpha
\]
and so the continuity of \( u \) follows from the Lebesgue Dominated Convergence Theorem. This implies the assertion.

□

Remark 4.13 Actually, under the assumptions of the previous lemma \( 4.12 \) \( u \in C(\overline{G} \times S, C^\beta(I)) \) for \( 0 < \beta < \alpha \).

The next Lemmas give information about the domains of the Hadamard finite part integral terms \( \mathcal{H}_1((\overline{K}_j\psi)(x, \omega, \cdot, E))(E) \), \( j = 1, 2 \).

Lemma 4.14 Suppose that \( \hat{\sigma}_1 \in C(\overline{G} \times I, C^1(I')) \) and that \( 0 < \alpha < 1 \), \( 0 < \beta < 1 \). Then for \( \psi \in C(\overline{G} \times I, C^\alpha(S)) \cap C(\overline{G} \times S, C^\beta(I)) \)
\[
\mathcal{H}_1((\overline{K}_i\psi)(x, \omega, \cdot, E))(E) = 2\pi \ln(E_m - E)\hat{\sigma}_1(x, E, E)\psi(x, \omega, E) + u(x, \omega, E)
\] (98)
where \( u \in C(\overline{G} \times S \times I) \).

Proof. We have
\[
\mathcal{H}_1((\overline{K}_i\psi)(x, \omega, \cdot, E))(E) = p.f. \int_E^{E_m} \frac{1}{E' - E} \hat{\sigma}_1(x, E', E) \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E') ds dE'
\]
\[
= p.f. \int_E^{E_m} \frac{1}{E' - E} f(x, \omega, E, E') dE'
\] (99)
where
\[
f(x, \omega, E, E') := \hat{\sigma}_1(x, E', E) \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E') ds.
\]
We show that \( f \in C(\overline{G} \times S \times I, C^\delta(I')) \) where \( 0 < \delta \leq \min\{\frac{\alpha}{2}, \beta\} \).
Hence the assertion is a consequence of Lemma 4.12.

We have for $E', E'' \in I$

\[
\begin{align*}
&h(x, \omega, E, E') - h(x, \omega, E, E'') \\
&= \int_0^{2\pi} [\psi(x, \gamma(E', E, \omega)(s), E') - \psi(x, \gamma(E', E, \omega)(s), E'')]ds \\
&\quad + \int_0^{2\pi} [\psi(x, \gamma(E', E, \omega)(s), E'') - \psi(x, \gamma(E'', E, \omega)(s), E'')]ds =: I_1 + I_2. \quad (100)
\end{align*}
\]

Since $\psi \in C(\overline{G} \times I, C^\beta(I))$ we obtain

\[
|I_1| \leq \int_0^{2\pi} |\psi(x, \gamma(E', E, \omega)(s), E') - \psi(x, \gamma(E', E, \omega)(s), E'')|ds \leq C_\beta|E' - E''|^\beta. \quad (101)
\]

Similarly since $\psi \in C(\overline{G} \times I, C^\alpha(S)))$ we have

\[
|I_2| \leq \int_0^{2\pi} |\psi(x, \gamma(E', E, \omega)(s), E'') - \psi(x, \gamma(E'', E, \omega)(s), E'')|ds \leq C_\alpha \|\gamma(E', E, \omega)(s) - \gamma(E'', E, \omega)(s)\|^\alpha \leq C_\alpha C|E' - E''|^{\frac{\beta}{2}}. \quad (102)
\]

Here we used the estimate

\[
\|\gamma(E', E, \omega)(s) - \gamma(E'', E, \omega)(s)\|^2 = (\sqrt{1 - \mu(E', E)^2} - \sqrt{1 - \mu(E'', E)^2})^2 + (\mu(E', E) - \mu(E'', E))^2 \leq C|E' - E''| \quad (103)
\]

which can be seen by elementary computations.

Combining (100), (101) and (102) we get

\[
|h(x, \omega, E, E') - h(x, \omega, E, E'')| \leq |I_1| + |I_2| \leq C_\beta|E' - E''|^\beta + C_\alpha C|E' - E''|^{\frac{\beta}{2}}
\]

from which (since $I$ is bounded) it follows that

\[
|h(x, \omega, E, E') - h(x, \omega, E, E'')| \leq C_\delta|E' - E''|^\delta
\]

Hence the assertion is a consequence of Lemma 4.12.

Let for $E' \neq E$

\[
\begin{align*}
&h(x, \omega, E', E) := \int_0^{2\pi} \left\langle \nabla_S \psi(x, \gamma(E', E, \omega)(s), E'), \frac{\partial \gamma}{\partial E}(E', E, \omega)(s) \right\rangle ds.
\end{align*}
\]

By the proof of Theorem 4.9 $h$ is defined also for $E' = E$ and

\[
\begin{align*}
&h(x, \omega, E, E) = \lim_{E' \to E} h(x, \omega, E', E) = \sum_{|\alpha| \leq 2} a_\alpha(E, \omega)(\partial^\alpha \psi)(x, \omega, E) \\
&= \lim_{E' \to E} \int_0^{2\pi} 2 \sum_{j=1}^2 \partial_j \left( \langle (\nabla_S \psi \circ H_\omega)(x, .., E), \eta(E', E, \omega, s) \rangle \right)(0) \xi_j(E', E, \omega, s)ds
\end{align*}
\]

where now $\eta = \frac{\partial \gamma}{\partial E}(E', E, \omega)(s)$ and where we used that

\[
2\pi (\partial_E \mu)(E, E)(\omega \cdot \nabla_S \psi)(x, \omega, E) = 0.
\]

We have the following technical lemma
Lemma 4.15 Suppose that $\psi \in C(\overline{G}, C^1(I, C^1(S)))$. Then the mapping $h$ obeys

$$|h(x, \omega, E', E) - h(x, \omega, E, E)| \leq C(E' - E)^{\frac{1}{2}}, \ E' \geq E. \quad (104)$$

Proof. A. We have for $E' \neq E$

$$h(x, \omega, E', E) = \int_0^{2\pi} \left\langle \nabla S \psi(x, \gamma(E', E, \omega)(s), E'), \frac{\partial \gamma}{\partial E}(E', E, \omega)(s) \right\rangle ds$$

$$= \int_0^{2\pi} \left\langle \nabla S \psi(x, \gamma(E', E, \omega)(s), E), \frac{\partial \gamma}{\partial E}(E', E, \omega)(s) \right\rangle ds$$

$$+ \int_0^{2\pi} \left\langle \left( \nabla S \psi(x, \gamma(E', E, \omega)(s), E') - \nabla S \psi(x, \gamma(E', E, \omega)(s), E), \frac{\partial \gamma}{\partial E}(E', E, \omega)(s) \right) \right\rangle ds$$

$$=: h_1(x, \omega, E', E) + h_2(x, \omega, E', E). \quad (105)$$

Since $\psi \in C(\overline{G}, C^1(I, C^1(S)))$

$$|h_2(x, \omega, E', E)| \leq C_1|E' - E| \int_0^{2\pi} \left\| \frac{\partial \gamma}{\partial E}(E', E, \omega)(s) \right\| ds$$

$$= 2\pi C_1|E' - E| \left\| (\partial_E \mu)(E', E) \right\| \frac{1}{\sqrt{1 - \mu(E', E)^2}} \leq C_1' |E' - E|^{\frac{1}{2}} \quad (106)$$

where we noticed that $1 - \mu(E', E)^2 = 1 - \frac{(E' + E^2)}{E(E + 2)} = \frac{2(E' - E)}{E(E + 2)}$. Note that

$$h(x, \omega, E', E) - h(x, \omega, E, E) = h_1(x, \omega, E', E) - h(x, \omega, E, E) + h_2(x, \omega, E', E) \quad (107)$$

and so it suffices to consider the term $h_1(x, \omega, E', E) - h(x, \omega, E, E)$. This is done in the next paragraph.

B. Let $\zeta = \xi(E', E, \omega, s) = H^{-1}_{\omega}(\omega') = \xi(E', E, \omega, s) = \xi, \ \omega' = \gamma(E', E, \omega)(s)$ and $\hat{\eta} = \hat{\eta}(E', E, \omega, s) = \frac{\partial \gamma}{\partial E}(E', E, \omega)(s)$. In virtue of the Taylor’s formula (cf. (23)) we have for $E' \neq E$

$$h_1(x, \omega, E', E)$$

$$= \int_0^{2\pi} \left\langle \nabla S \psi(x, \gamma(E', E, \omega)(s), E', \omega), \frac{\partial \gamma}{\partial E}(E', E, \omega)(s) \right\rangle ds$$

$$= \int_0^{2\pi} \left\langle \nabla S \psi \circ H_\omega(x, E, \omega), \frac{\partial \gamma}{\partial E}(E', E, \omega)(s) \right\rangle ds$$

$$= \int_0^{2\pi} \left\langle \nabla S \psi(x, \omega, E), \frac{\partial \gamma}{\partial E}(E', E, \omega)(s) \right\rangle ds$$

$$+ \sum_{j=1}^{2} \int_0^{2\pi} \partial_j \left( \left\langle (\nabla S \psi \circ H_\omega)(x, E, \omega), \hat{\eta}(E', E, \omega, s) \right\rangle \right) (0) \xi_j(E', E, \omega, s) ds$$

$$+ \int_0^{2\pi} (R_\omega \psi(x, \cdot, E))(\omega') \left( \zeta(E', E, \omega, s), \zeta(E', E, \omega, s), \hat{\eta}(E', E, \omega, s) \right) ds \quad (108)$$

where the residual $(R_\omega \psi(x, \cdot, E))(\omega')$ is

$$R_\omega(\psi(x, \cdot, E))(\omega')(\zeta(E', E, \omega, s), \zeta(E', E, \omega, s), \hat{\eta}(E', E, \omega, s)$$

$$= \sum_{|\alpha| = 2} \frac{|\alpha|}{\alpha!} \int_0^1 (1 - t) \partial_t^\alpha \left( \left\langle (\nabla S \psi)(x, E, E, \omega, s) \circ H_\omega(t\xi, \hat{\eta}) \right\rangle dt \cdot \xi^\alpha. \quad (109)$$
Hence
\begin{align*}
  h_1(x, \omega, E', E) - h(x, \omega, E, E) &= \int_0^{2\pi} \left\langle (\nabla_S \psi)(x, \omega, E), \frac{\partial\gamma}{\partial E}(E', E, \omega)(s) \right\rangle \, ds \\
  &\quad + \sum_{j=1}^2 \int_0^{2\pi} \partial_j \left( \langle (\nabla_S \psi \circ H_\omega)(x, \omega, E), \hat{\eta}(E', E, \omega, s) \rangle \right) (0) \xi_j(E', E, \omega, s) \, ds \\
  &\quad - \lim_{E' \to E} \sum_{j=1}^2 \int_0^{2\pi} \partial_j \left( \langle (\nabla_S \psi \circ H_\omega)(x, \omega, E), \hat{\eta}(E', E, \omega, s) \rangle \right) (0) \xi_j(E', E, \omega, s) \, ds \\
  &\quad + \int_0^{2\pi} R_\omega(\psi(x, \omega))(\omega')(\xi(E', E, \omega, s), \xi(E', E, \omega, s), \hat{\eta}(E', E, \omega, s)) \, ds.
\end{align*}

Similarly as in the proof of Theorem 4.9 we find that
\begin{equation}
  \left| \int_0^{2\pi} R_\omega(\psi(x, \omega))(\omega')(\xi(E', E, \omega, s), \xi(E', E, \omega, s), \hat{\eta}(E', E, \omega, s)) \, ds \right| 
  \leq C \sqrt{1 - \mu(E', E)} \leq C' |E' - E|^\frac{1}{2}
\end{equation}
and that
\begin{equation}
  \int_0^{2\pi} \left\langle (\nabla_S \psi)(x, \omega, E), \frac{\partial\gamma}{\partial E}(E', E, \omega)(s) \right\rangle \, ds = 0.
\end{equation}

Consider the term
\begin{align*}
  &\sum_{j=1}^2 \int_0^{2\pi} \partial_j \left( \langle (\nabla_S \psi \circ H_\omega)(x, \omega, E), \hat{\eta}(E', E, \omega, s) \rangle \right) (0) \xi_j(E', E, \omega, s) \, ds \\
  &\quad - \lim_{E' \to E} \sum_{j=1}^2 \int_0^{2\pi} \partial_j \left( \langle (\nabla_S \psi \circ H_\omega)(x, \omega, E), \hat{\eta}(E', E, \omega, s) \rangle \right) (0) \xi_j(E', E, \omega, s) \, ds.
\end{align*}

In due to proof of Lemma 4.10 (here we again denote shortly $\mu = \mu(E', E)$)
\begin{align*}
  &\int_0^{2\pi} \partial_j \left( \langle (\nabla_S \psi \circ H_\omega)(x, \omega, E), \hat{\eta}(E', E, \omega, s) \rangle \right) (0) \xi_j(E', E, \omega, s) \, ds = \frac{\arccos(\mu)}{\sqrt{1 - \mu^2}} \int_0^{2\pi} \\
  &\quad \partial_j \left( \langle (\nabla_S \psi \circ H_\omega)(x, \omega, E), R(\omega)(- (\partial E \mu) \cos(s), -(\partial E \mu) \sin(s), \sqrt{1 - \mu^2} \partial E \mu) \rangle \right) (0) \, ds \\
  &\quad \cdot \left( R(\omega)(\cos(s), \sin(s), 0) \right) \, ds \\
  &\quad = \frac{\arccos(\mu)}{\sqrt{1 - \mu^2}} \int_0^{2\pi} b_j(\omega, s) q_j(E', E, \omega, s) \, ds
\end{align*}

where
\begin{align*}
  b_j(\omega, s) := \langle R(\omega)(\cos(s), \sin(s), 0) \rangle &= \langle R(\omega)(\cos(s), \sin(s), 0), \overline{\gamma}_j \rangle
\end{align*}
and
\begin{align*}
  q_j(E', E, \omega, s) := \partial_j \left( \langle (\nabla_S \psi \circ H_\omega)(x, \omega, E), R(\omega)(- (\partial E \mu) \cos(s), -(\partial E \mu) \sin(s), \sqrt{1 - \mu^2} \partial E \mu) \rangle \right) (0)
\end{align*}
Note that
\[ \lim_{E' \to E} \frac{1}{2\pi} \sum_{j=1}^{2\pi} \partial_j \left( \langle (\nabla_S \psi \circ H_\omega)(x,.,E), \hat{\eta}(E',E,\omega,s) \rangle \right)(0) \xi_j(E',E,\omega,s)ds \]
\[ = \int_0^{2\pi} b_j(\omega,s)q_j(E,E,\omega,s)ds. \]

Hence we have
\[
\left| \int_0^{2\pi} \partial_j \left( \langle (\nabla_S \psi \circ H_\omega)(x,.,E), \hat{\eta}(E',E,\omega,s) \rangle \right)(0) \xi_j(E',E,\omega,s)ds \right|
\]
\[ - \lim_{E' \to E} \frac{1}{2\pi} \sum_{j=1}^{2\pi} \partial_j \left( \langle (\nabla_S \psi \circ H_\omega)(x,.,E), \hat{\eta}(E',E,\omega,s) \rangle \right)(0) \xi_j(E',E,\omega,s)ds \]
\[ = \left| \frac{\arccos(\mu(E',E))}{\sqrt{1 - \mu(E',E)^2}} \int_0^{2\pi} b_j(\omega,s)q_j(E',E,\omega,s)ds - \int_0^{2\pi} b_j(\omega,s)q_j(E,E,\omega,s)ds \right| \]
\[ \leq \left| \frac{\arccos(\mu(E',E))}{\sqrt{1 - \mu(E',E)^2}} - 1 \right| \left| \int_0^{2\pi} b_j(\omega,s)q_j(E',E,\omega,s)ds \right| \]
\[ + \left| \int_0^{2\pi} b_j(\omega,s)q_j(E',E,\omega,s)ds - \int_0^{2\pi} b_j(\omega,s)q_j(E,E,\omega,s)ds \right|. \]  

(114)

Recall that \( \arccos(\mu) = \arcsin(\sqrt{1 - \mu^2}) \) and
\[ \arcsin(x) = x + \frac{1}{2}x^3 + \frac{1}{2 \cdot 4}x^5 + \ldots, \quad |x| < 1. \]

Hence we find that
\[
\left| \frac{\arccos(\mu(E',E))}{\sqrt{1 - \mu(E',E)^2}} - 1 \right| \leq C_4(1 - \mu(E',E)^2) \leq C_4|E' - E|. \]  

(115)

Since \( \psi \in C(\bar{G}, C^1(I, C^3(S))) \)
\[
\left| \int_0^{2\pi} b_j(\omega,s)q_j(E',E,\omega,s)ds \right| \leq C. \]

Moreover, we have
\[
\left| \int_0^{2\pi} b_j(\omega,s)q_j(E',E,\omega,s)ds - \int_0^{2\pi} b_j(\omega,s)q_j(E,E,\omega,s)ds \right|
\]
\[ \leq \int_0^{2\pi} |b_i(\omega,s)| \left( |(\partial_E \mu)(E',E)\mu(E',E) - (\partial_E \mu)(E,E)|^2 \right.
\]
\[ + \left| (\partial_E \mu)(E',E)\mu(E',E) - (\partial_E \mu)(E,E)|^2 + (1 - \mu(E',E)^2) \right| \frac{1}{2} ds. \]  

(116)

Since \( E_0 > 0 \) we see that \( \mu \in C^2(I^2) \) and so
\[
|(\partial_E \mu)(E',E)\mu(E',E) - (\partial_E \mu)(E,E)| = |(\partial_E \mu \cdot \mu)(E',E) - (\partial_E \mu \cdot \mu)(E,E)| \leq C_5|E' - E|. \]  

(117)
Then the exact transport operator can be expressed by

\[ T \psi = -\mathcal{H}_2((\mathcal{K}_2 \psi)(x, \omega, \cdot, E))(E) - \mathcal{H}_1((\mathcal{K}_1 \psi)(x, \omega, \cdot, E))(E) + \omega \cdot \nabla_x \psi + \Sigma \psi - K_r \psi \]  

(119)

This can be written as

\[ (T \psi)(x, \omega, E) = -\frac{\partial}{\partial E} \left( \mathcal{H}_1((\mathcal{K}_2 \psi)(x, \omega, \cdot, E))(E) \right) - 2\pi \hat{\sigma}_2(x, E, E) \frac{\partial \psi}{\partial E}(x, \omega, E) \]

\[ - \hat{\sigma}_2(x, E, E) \sum_{|\alpha| \leq 2} a_{\alpha}(E, \omega)(\partial^\omega_2 \psi)(x, \omega, E) \]

\[ + \text{p.f.} \int_{E}^{E_m} \frac{1}{E' - E} \hat{\sigma}_2(x, E', E) \int_{0}^{2\pi} \left< \nabla_s \psi(x, \gamma(E', E, \omega)(s), E'), \frac{\partial \gamma}{\partial E}(E', E, \omega)(s) \right> ds dE' \]

\[ + \text{p.f.} \int_{E}^{E_m} \frac{1}{E' - E} \hat{\sigma}_2(x, E', E) \int_{0}^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E') ds dE' \]

\[ - \mathcal{H}_1((\mathcal{K}_1 \psi)(x, \omega, \cdot, E))(E) - 2\pi \frac{\partial \hat{\sigma}_2}{\partial E'}(x, E, E) \psi(x, \omega, E) \]

\[ + \omega \cdot \nabla_x \psi + \Sigma(x, \omega, E) \psi - (K_r \psi)(x, \omega, E) \]  

(120)

where the term \( \sum_{|\alpha| \leq 2} a_{\alpha}(E, \omega)(\partial^\omega_2 \psi)(x, \omega, E) \) is as above in (75).

Proof. A. At first we show that (cf. [31], section 3.2)

\[ \mathcal{H}_2((\mathcal{K}_2 \psi)(x, \omega, \cdot, E))(E) = \frac{\partial}{\partial E} \left( \mathcal{H}_1((\mathcal{K}_2 \psi)(x, \omega, \cdot, E))(E) \right) - \mathcal{H}_1 \left( \frac{\partial (\mathcal{K}_2 \psi)}{\partial E'}(x, \omega, \cdot, E) \right)(E) \]

\[ + \frac{\partial (\mathcal{K}_2 \psi)}{\partial E'}(x, \omega, E', E)|_{E' = E} \]  

(121)

Let

\[ f(x, \omega, E', E) := (\mathcal{K}_2 \psi)(x, \omega, E', E). \]
Then
\[
\frac{\partial f}{\partial E'}(x, E', E) = \frac{\partial \hat{\sigma}^2}{\partial E'}(x, E', E, \omega)(s) ds
\]
\[
+ \hat{\sigma}^2(x, E', E) \int_0^{2\pi} \left\langle \nabla S \psi(x, \gamma(E', E, \omega)(s), E'), \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) \right\rangle ds
\]
\[
+ \hat{\sigma}^2(x, E', E) \int_0^{2\pi} \frac{\partial \psi}{\partial E'}(x, \gamma(E', E, \omega)(s), E') ds.
\] (122)

Since \(\hat{\sigma}^2 \in C(G, C^{1+\alpha}(I' \times I))\) and \(\psi \in C(G, C^2(I, C^3(S)))\) we find in due to Theorem 4.9 and Lemma 4.14 that \(f \in C(G \times S, C^1(I' \times I))\) and for \(E' > E\)
\[
\left| \frac{\partial f}{\partial E'}(x, \omega, E', E) - \frac{\partial f}{\partial E'}(x, \omega, E, E) \right| \leq C(E' - E)^{\frac{\alpha}{2}}
\]
(we omit here further details). Thus the relation (121) is valid (see the note after Lemma 2.2).

B. We have for \(E' > E\)
\[
\frac{\partial (K_2 \psi)}{\partial E}(x, \omega, E', E) = \frac{\partial \hat{\sigma}^2}{\partial E}(x, E', E) \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E') ds
\]
\[
+ \hat{\sigma}^2(x, E', E) \int_0^{2\pi} \left\langle \nabla S \psi(x, \gamma(E', E, \omega)(s), E'), \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) \right\rangle ds
\] (123)
and similarly recalling that \(\gamma(E, E, \omega)(s) = \omega\)
\[
\frac{\partial (K_2 \psi)}{\partial E'}(x, \omega, E', E) \bigg|_{E' = E} = 2\pi \frac{\partial \hat{\sigma}^2}{\partial E}(x, E, E) \psi(x, \omega, E')
\]
\[
+ \hat{\sigma}^2(x, E, E) \left( \int_0^{2\pi} \left\langle \nabla S \psi(x, \gamma(E', E, \omega)(s), E'), \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) \right\rangle ds \right) \bigg|_{E' = E}
\]
\[
+ 2\pi \hat{\sigma}^2(x, E, E) \frac{\partial \psi}{\partial E}(x, \omega, E).
\] (124)

In virtue of (81)
\[
\left( \int_0^{2\pi} \left\langle \nabla S \psi(x, \gamma(E', E, \omega)(s), E'), \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) \right\rangle ds \right) \bigg|_{E' = E}
= 2\pi (\partial_{E'} \mu)(E, E)(\omega \cdot \nabla S \psi)(x, \omega, E) + \sum_{|\alpha| \leq 2} a_\alpha(E, \omega)(\partial^\alpha \psi)(x, \omega, E).
\] (125)

Recall that by (28)
\[
(\omega \cdot \nabla S \psi)(x, \omega, E) = 0.
\]

Hence the assertion follows by combining (121), (123), (124) and (125). □
Remark 4.17 We unveil an additional expression for the Møller-like transport operator. Recall again that \( \gamma(E, E, \omega)(s) = \omega \). Hence in virtue of the Taylor’s expansion with respect to \( E' \)

\[
(\mathcal{K}_1 \psi)(x, \omega, E', E) \\
= (\mathcal{K}_1 \psi)(x, \omega, E, E) + \int_0^1 \frac{\partial (\mathcal{K}_1 \psi)}{\partial E'}(x, \omega, E', E)|_{E' = E + t(E' - E)} dt \cdot (E' - E) \\
= 2\pi \hat{\sigma}_1(x, E, E) \psi(x, \omega, E) \\
+ \int_0^1 \frac{\partial \hat{\sigma}_1}{\partial E'}(x, E + t(E' - E), E) \\
\cdot \int_0^{2\pi} \psi(x, \gamma(E + t(E' - E), E, \omega)(s), E + t(E' - E)) dt ds \cdot (E' - E) \\
+ \int_0^1 \hat{\sigma}_1(x, E + t(E' - E), E) \\
\cdot \int_0^{2\pi} \frac{\partial \psi}{\partial E}(x, \gamma(E + t(E' - E), E, \omega)(s), E + t(E' - E)) dt ds \cdot (E' - E) \\
\tag{126}
\]

and so

\[
\mathcal{H}_1((\mathcal{K}_1 \psi)(x, \omega, \cdot, E))(E) \\
= 2\pi \ln(E_m - E) \hat{\sigma}_1(x, E, E) \psi(x, \omega, E) \\
+ \int_{E_m}^{E} \int_0^1 \frac{\partial \hat{\sigma}_1}{\partial E'}(x, E + t(E' - E), E) \\
\cdot \int_0^{2\pi} \psi(x, \gamma(E + t(E' - E), E, \omega)(s), E + t(E' - E)) dt ds dE' \\
+ \int_{E_m}^{E} \int_0^1 \hat{\sigma}_1(x, E + t(E' - E), E) \\
\cdot \int_0^{2\pi} \gamma(E + t(E' - E), E, \omega)(s), E + t(E' - E)) dt ds dE' \\
\cdot \frac{\partial \gamma}{\partial E'}(E + t(E' - E), E, \omega)(s), E + t(E' - E)) \} ds dt dE' \\
+ \int_{E_m}^{E} \int_0^1 \hat{\sigma}_1(x, E + t(E' - E), E) \\
\cdot \int_0^{2\pi} \frac{\partial \psi}{\partial E}(x, \gamma(E + t(E' - E), E, \omega)(s), E + t(E' - E)) dt ds dE'. \\
\tag{127}
\]
Furthermore, for $E' \neq E$

\[
\frac{\partial}{\partial E'} \left( \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E') ds \right) \\
= \int_0^{2\pi} \left( \left( \nabla_S \psi \right)(x, \gamma(E', E, \omega)(s), E'), \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) \right) ds \\
+ \int_0^{2\pi} \frac{\partial \psi}{\partial E}(x, \gamma(E', E, \omega)(s), E') ds
\]

and by (125)

\[
\left. \left( \int_0^{2\pi} \left( \nabla_S \psi(x, \gamma(E', E, \omega)(s), E') \right) ds \right) \right|_{E'=E} \\
= 2\pi \left( \partial_{E'} \mu \right)(E, E)(\omega \cdot \nabla_S \psi)(x, \omega, E) + \sum_{|\alpha| \leq 2} a_{\alpha}(E, \omega)(\partial_{\alpha}^{\omega} \psi)(x, \omega, E).
\]

Hence by the Taylor’s expansion

\[
(\mathcal{K}_2 \psi)(x, \omega, E', E) = (\mathcal{K}_2 \psi)(x, \omega, E, E) + \frac{\partial(\mathcal{K}_2 \psi)}{\partial E'}(x, \omega, E, E) \cdot (E' - E) \\
+ \int_0^1 (1 - t) \frac{\partial^2(\mathcal{K}_2 \psi)}{\partial E'^2}(x, \omega, E) \big|_{E'=E+t(E'-E)} dt \cdot (E' - E)^2
\]

where $P(x, \omega, E + t(E' - E), E, \partial_\omega, \partial_E)$ is the second-order partial differential operator which can be computed by (128) (we omit all calculations). That is why

\[
\mathcal{H}_2((\mathcal{K}_2 \psi)(x, \omega, \cdot, E))(E) \\
= -2\pi \frac{1}{E_m - E} \sigma_2(x, E, E) \psi(x, \omega, E) + 2\pi \ln(E_m - E) \frac{\partial \sigma_2}{\partial E'}(x, E, E) \psi(x, \omega, E) \\
+ \ln(E_m - E) \sigma_2(x, E, E) \left( 2\pi \left( \partial_{E'} \mu \right)(E, E)(\omega \cdot \nabla_S \psi)(x, \omega, E) + \sum_{|\alpha| \leq 2} a_{\alpha}(E, \omega)(\partial_{\alpha}^{\omega} \psi)(x, \omega, E) \right) \\
+ 2\pi \ln(E_m - E) \sigma_2(x, E, E) \frac{\partial \psi}{\partial E}(x, \omega, E) \\
+ \int_{E_m}^{E} \int_0^1 (P(x, \omega, E + t(E' - E), E, \partial_\omega, \partial_E) \psi)(x, \omega, E) dt dE'.
\]

In virtue of (51) we immediately obtain an expression for the corresponding transport operator.

The problem in these formulations is that the partial differential operator $P(x, \omega, E + t(E' - E), E, \partial_\omega, \partial_E)$ is involving singularities (in its coefficients) of the form $\frac{1}{(E' - E)^{1/2}}$ and that the residual operators are of the form Schur partial integral operators combined with singular partial differential operators.
4.3. Related formal adjoints.

4.3.1. Formal adjoints of restricted collision and convection-scattering operators. We assume that the restricted collision operator is the sum

\[ K_r = K^1 + K^2 + K^3. \]  

(132)

Here \( K^1 \) is of the form

\[ (K^1 \psi)(x, \omega, E) = \int_{S' \times I'} \sigma^1(x, \omega', \omega, E') \psi(x, \omega', E') d\omega' dE', \]

where \( \sigma^1 : G \times S^2 \times I^2 \to \mathbb{R} \) is a non-negative measurable function such that

\[ \int_{S' \times I'} \sigma^1(x, \omega', \omega, E') d\omega' dE' \leq M_1, \]

\[ \int_{S' \times I'} \sigma^1(x, \omega, \omega', E') d\omega' dE' \leq M_2, \]  

(133)

for a.e. \((x, \omega, E) \in G \times S \times I\). \( K^1 \) is related e.g. to the Bremsstrahlung.

The operator \( K^2 \) is of the form

\[ (K^2 \psi)(x, \omega, E) = \int_{S'} \sigma^2(x, \omega', \omega, E) \psi(x, \omega', E) d\omega', \]

where \( \sigma^2 : G \times S^2 \times I \to \mathbb{R} \) is a non-negative measurable function such that

\[ \int_{S'} \sigma^2(x, \omega', \omega, E) d\omega' \leq M_1, \]

\[ \int_{S'} \sigma^2(x, \omega, \omega', E) d\omega' \leq M_2, \]  

(134)

for a.e. \((x, \omega, E) \in G \times S \times I\). \( K^2 \) models the elastic scattering.

Finally, \( K^3 \) is of the form

\[ (K^3 \psi)(x, \omega, E) = \int_{I'} \int_0^{2\pi} \hat{\sigma}^3(x, E', E) \psi(x, \gamma(E', E, \omega)(s), E') ds dE', \]

where \( \hat{\sigma}^3 : G \times I^2 \to \mathbb{R} \) is a non-negative measurable function such that

\[ \int_{I'} \hat{\sigma}^3(x, E', E) dE' \leq M_1, \]

\[ \int_{I'} \hat{\sigma}^3(x, E, E') dE' \leq M_2, \]  

(135)

for a.e. \((x, E) \in G \times I\). \( K^3 \) relates e.g. to the Møller scattering.

Recall that the restricted collision operator \( K_r \) is a bounded operator \( L^2(G \times S \times I) \to L^2(G \times S \times I) \) ([33], [31]). Furthermore, we have

**Theorem 4.18** The adjoint \( K_r^* \) of \( K_r = K^1 + K^2 + K^3 \) is given by

\[ (K_r^* \psi)(x, \omega, E) = \int_{S' \times I'} \sigma^1(x, \omega, \omega', E, E') \psi(x, \omega', E') d\omega' dE' + \int_{S'} \sigma^2(x, \omega, \omega', E) \psi(x, \omega', E) d\omega' + \int_{I'} \int_0^{2\pi} \hat{\sigma}^3(x, E, E') \psi(x, \gamma(E, E', \omega)(s), E') ds dE'. \]  

(136)
Proof. The adjoint $K^*_r$ is $(K_1^*)^* + (K_2^*)^* + (K_3^*)^*$. We compute $(K_3^*)^*$. The adjoints $(K_1^*)^*$, $(K_2^*)^*$ are similarly computed. We have by Corollary 4.8 for $\psi$ which completes the proof.

\[ \langle K^3, v \rangle_{L^2(G \times S \times I)} = \int_G \int_S \int_I (K^3 \psi)(x, \omega, E)v(x, \omega, E)dE d\omega dx \]

\[ = \int_G \int_S \int_I \int_0^{2\pi} \hat{\delta}^3(x, E', E)\psi(x, \gamma(E', E, \omega)(s), E')v(x, \omega, E)d\sigma dE' dE d\omega dx \]

\[ = \int_G \int_S \int_I \int_0^{2\pi} \hat{\delta}^3(x, E', E)\left(\int_S \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E')v(x, \omega, E)d\omega\right)dE' dE dx \]

\[ = \int_G \int_S \int_I \int_0^{2\pi} \hat{\delta}^3(x, E', E)\left(\int_0^{2\pi} \psi(x, \gamma(E', E, \omega')(s), E')d\omega\right)dE' dE dx \]

\[ = \int_G \int_S \int_I \int_0^{2\pi} \hat{\delta}^3(x, E', E)\left(\int_0^{2\pi} \psi(x, \gamma(E', E, \omega')(s), E')d\omega\right)dE' dE dx \] (137)

which completes the proof. \[\square\]

Let

\[ \Sigma \in L^\infty(G \times S \times I). \] (138)

Define

\[ (A_0 \psi)(x, \omega, E) := \omega \cdot \nabla_x \psi + \Sigma \psi - K_r \psi. \]

Then the formal adjoint of $A_0$ that is,

\[ (A_0^* v)(x, \omega, E) = -\omega \cdot \nabla_x v + \Sigma^* v - K_r^* v \]

where $\Sigma^* = \Sigma$ and $v \in \mathcal{C}_0^1(G \times S \times I^0)$.

4.3.2. Formal adjoints of Hadamard finite part integral operators. Let

\[ (A_1 \psi)(x, \omega, E) := \mathcal{H}_1(\mathcal{K}_1 \psi)(x, \omega, E)(E) \]

\[ = -\text{p.f.} \int_{E_0}^{E_m} \frac{1}{E' - E} \hat{\delta}_1(x, E', E) \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E')d\sigma dE'. \]

Theorem 4.19 Suppose that $\hat{\delta}_1 \in \mathcal{C}(\mathcal{G}, \mathcal{C}^\alpha(I^2))$, $\alpha > 0$. Then the formal adjoint of $A_1$,

\[ (A_1^* v)(x, \omega', E') := -\text{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \hat{\delta}_1(x, E', E) \int_0^{2\pi} \psi(x, \gamma(E', E, \omega')(s), E')d\sigma dE' \] (139)

for $v \in \mathcal{C}(\mathcal{G} \times S, \mathcal{C}^\alpha(I)) \cap \mathcal{C}(\mathcal{G} \times I, \mathcal{C}^\alpha(S))$. 

Proof. By (70) and by Corollary 4.8 we have

\[
\langle A_1 \psi, v \rangle_{L^2(G \times S \times I)} = \int_G \int_S \int_I \left( - \text{p.f.} \int_{E'} \frac{1}{E' - E} \hat{\sigma}_1(x, E', E) \right) \psi(x, \gamma(E', E, \omega)(s), E') v(x, \omega, E) dE d\omega dx
\]

\[
= - \int_G \int_I \left( \text{p.f.} \int_{E'} \frac{1}{E' - E} \hat{\sigma}_1(x, E', E) \right) \psi(x, \gamma(E', E, \omega)(s), E') v(x, \omega, E) dE d\omega dx
\]

\[
= - \int_G \int_I \left( \text{p.f.} \int_{E'} \frac{1}{E' - E} \hat{\sigma}_1(x, E', E) \right) \psi(x, \omega, E') v(x, \gamma(E', E, \omega')(s), E) dE d\omega dx
\]

where \( f(x, E', E) := \int_{S'} \int_0^{2\pi} \psi(x, \omega', E') v(x, \gamma(E', E, \omega')(s), E) d\omega' \).

Hence by Lemma 3.1 and by (70) we obtain

\[
\langle A_1 \psi, v \rangle_{L^2(G \times S \times I)} = - \int_G \int_I \left( \text{p.f.} \int_{E'} \frac{1}{E' - E} \hat{\sigma}_1(x, E', E) f(x, E', E) dE' \right) dE' dx
\]

\[
= - \int_G \int_I \left( \text{p.f.} \int_{E'} \frac{1}{E' - E} \hat{\sigma}_1(x, E', E) \right) \psi(x, \omega', E') v(x, \gamma(E', E, \omega')(s), E) dE d\omega' dx
\]

\[
= \int_G \int_S \int_I \psi(x, \omega', E') \left( - \text{p.f.} \int_{E'} \frac{1}{E' - E} \hat{\sigma}_1(x, E', E) \right) \psi(x, \gamma(E', E, \omega)(s), E) ds dE d\omega dx
\]

where \( A_1^* \) is

\[
(A_1^* v)(x, \omega', E') := -\text{p.f.} \int_{E'} \frac{1}{E' - E} \hat{\sigma}_1(x, E', E) \int_0^{2\pi} v(x, \gamma(E', E, \omega')(s), E) ds dE.
\]

This completes the proof \( \square \)
Finally, define (the rest of the transport operator (120))
\[
(A_2\psi)(x, \omega, E) := -\frac{\partial}{\partial E} \left( \mathcal{H}_1((K_2 \psi)(x, \omega, \cdot, E))(E) \right) - 2\pi \hat{\sigma}_2(x, E, E) \frac{\partial \psi}{\partial E}(x, \omega, E)
\]
\[+ \hat{\sigma}_2(x, E, E) \sum_{|\alpha| \leq 2} a_\alpha(E, \omega)(\partial_\omega^\alpha \psi)(x, \omega, E)
\]
\[+ \text{p.f.} \int_{E}^{E_m} \frac{1}{E' - E} \hat{\sigma}_2(x, E', E) \int_{0}^{2\pi} \left\langle \nabla_S \psi(x, \gamma(E', E), \omega)(s), E' \right\rangle \frac{\partial \gamma}{\partial E}(E', E, \omega)(s) d\omega dE'
\]
\[+ \text{p.f.} \int_{E}^{E_m} \frac{1}{E' - E} \frac{\partial \hat{\sigma}_2}{\partial E}(x, E', E) \int_{0}^{2\pi} \psi(x, \gamma(E', E), \omega)(s), E') d\omega dE'
\]
\[- 2\pi \frac{\partial \hat{\sigma}_2}{\partial E'}(x, E, E) \psi(x, \omega, E').
\] (142)

The formal adjoint \(A^*_2\) can be computed by techniques utilized in the next section 5 and it is (see Theorem 5.2)
\[
(A^*_2\psi)(x, \omega', E')
\]
\[= -\text{p.f.} \int_{E_0}^{E'} \frac{1}{(E' - E)^2} \hat{\sigma}_2(x, E', E) \cdot \int_{0}^{2\pi} v(x, \gamma(E', E), \omega)(s), E) d\omega dE' dx
\]
\[- \hat{\sigma}_2(x, E, E) \sum_{|\alpha| \leq 2} b_\alpha(E, \omega)(\partial_\omega^\alpha v)(x, \omega, E)
\] (143)

for appropriates \(v\). Here
\[
\sum_{|\alpha| \leq 2} b_\alpha(E, \omega)(\partial_\omega^\alpha \psi)(x, \omega, E) := \left( (\partial_{E'} \mu)(E, E) + (\partial_{E}\mu)(E, E) \right)
\]
\[\cdot \int_{0}^{2\pi} \sum_{j=1}^{2} \partial_j \left( \left\langle \nabla_S \psi \circ H_\omega \right\rangle(x, E), R(\omega) \left( \cos(s), \sin(s), 0 \right) \right) (0)
\]
\[\cdot \left\langle R(\omega) \left( \cos(s), \sin(s), 0 \right), \Omega_j \right\rangle d\omega dx.
\] (145)

The formal adjoint of \(T\) is given by
\[T^* = A^*_0 + A^*_1 + A^*_2.
\]

5. Variational formulation of the transport problem

In this section we shall give a weak form, so called variational formulation, of the hyper-singular transport problem for Møller-type interactions. The obtained final weak form (the combination of Theorems 5.2 and 5.4) decreases the level of singularities in integration containing only singularities of order one that is, the singularities of the form \(\frac{1}{E - E'} dE dE'\). The variational formulation is an essential step in order to show the existence of solutions by Lions-Lax-Milgram Theorem based methods. Additionally, it gives a platform needed for Galerkin finite element methods.

We treatise a variational formulation of the initial inflow boundary value transport problem
\[
T \psi = f,
\]
\[
\psi|_{\Gamma_-} = g,
\]
\[
\psi(\cdot, \cdot, E_m) = 0
\]
where \(T\) is given by (119) or equivalently by (120). For some partial integration techniques we additionally require that
\[
\lim_{E \to E_m} \ln(E_m - E) \psi(x, \omega, E) = 0
\] (149)
Theorem 5.2

We prove  Remark 5.1

section 6).

assumed smooth enough. The final weak formulation follows by appropriate extensions (cf. [31],

Recall that 5.1.

part of variational formulations.

transport equations. However, the below computations (for a single equation) form an essential

holds for sufficient regular  which satisfies the conditions (147) and (148). Here  and  are

Since  and  for the solution  which obeys (148). Moreover, we assume that  and  are

\[ \lim_{E \to E_0} \ln(E' - E_0)v(x, \omega, E') = 0 \]  (151)

holds for  . At first, the requirement (150) is imposed in the classical sense that is,  and  are

assumed smooth enough. The final weak formulation follows by appropriate extensions (cf. [31],

section 6).

Remark 5.1 We emphasize that in real problems the modelling often comprises a system of

transport equations. However, the below computations (for a single equation) form an essential

part of variational formulations.

5.1. Basic variational formulation. Let  ( + ) be the negative (positive) part of a function. Recall that

\[ f = f_+ - f_- \quad \text{and} \quad |f| = f_+ + f_- \]  (152)

We prove

Theorem 5.2 Suppose that the assumptions (138), (133), (134), (135) are valid and that  \( \hat{\sigma}_1 \in C(G, C^\alpha(I \times I')) \),  \( \hat{\sigma}_2 \in C(G, C^{1+\alpha}(I \times I')) \),  \( \alpha > 0 \). Furthermore, suppose that  \( \psi \in C^1(G, C^2(I, C^3(S))) \) is a solution of the problem (146), (147), (148). Then for  \( v \in D_{(0)} \)

\[ B(\psi, v) = F(v) \]  (153)

where the bilinear form  \( B(\ldots, \cdot) \) is

\[ B(\psi, v) = B_2(\psi, v) + B_1(\psi, v) + B_0(\psi, v) \]  (154)

and

\[ B_0(\psi, v) := \langle \psi, -\omega \cdot \nabla_x v + \Sigma^* v - K^* v \rangle_{L^2(G \times S \times I)} + \int_{\Gamma_+} (\omega \cdot \nu)_+ \psi v \, d\sigma d\omega dE, \]  (155)

\[ B_1(\psi, v) := -\int_G \int_{S'} \int_{E'} \psi(x, \omega', E') \cdot 
\left( \text{p.f.} \int_{E'} \frac{1}{E' - E} \hat{\sigma}_1(x, E', E) \int_0^{2\pi} v(x, \gamma(E', E, \omega')(s), E) ds dE \right) dE' d\omega' dx, \]  (156)
\[ B_2(\psi, v) \]
\[ := -\int_G\int_S\int_{P'} \psi(x, \omega, E') \left( \text{p.f.} \int_{E_0}^{E'} \frac{1}{(E' - E)^2} \hat{\sigma}_2(x, E', E) \right) \cdot \int_0^{2\pi} v(x, \gamma(E', E, \omega)(s), E) ds dE d\omega dx \]
\[ - \int_G \int_I \int_S \hat{\sigma}_2(x, E, E) \psi(x, \omega, E) \sum_{|\alpha| \leq 2} b_\alpha(\omega, E)(\partial_{\alpha} \psi)(x, \omega, E) dE d\omega dx. \]

(157)

Here
\[ \sum_{|\alpha| \leq 2} b_\alpha(\omega, E)(\partial_{\alpha} \psi)(x, \omega, E) := \left( (\partial_{E'} \mu)(E, E) + (\partial_E \mu)(E, E) \right) \]
\[ \cdot \int_0^{2\pi} \sum_{j=1}^2 \partial_j \left( (\nabla_S \psi \circ H_\omega)(x, \cdot, E), R(\omega)(\cos(s), \sin(s), 0) \right) (0) \cdot \left( R(\omega)(\cos(s), \sin(s), 0), \mathbf{\Pi}_j \right) ds. \]

(158)

The linear form \( F \) is
\[ F(v) := (f, v)_{L^2(G \times S \times I)} + \int_{\Gamma_+} (\omega \cdot v)_+ g v d\sigma d\omega dE. \]

(160)

Proof. A. Utilizing the above notations we have \( T = A_0 + A_1 + A_2 \) and so
\[ (T \psi, v)_{L^2(G \times S \times I)} = (A_0 \psi, v)_{L^2(G \times S \times I)} + (A_1 \psi, v)_{L^2(G \times S \times I)} + (A_2 \psi, v)_{L^2(G \times S \times I)} \]
\[ = B_0(\psi, v) + B_1(\psi, v) + B_2(\psi, v) \]

(161)

where \( B_j(\psi, v) := (A_j \psi, v)_{L^2(G \times S \times I)} \). Note that for \( j = 1, 2 \)
\[ (A_j \psi)(x, \omega, E) = -\mathcal{H}_j(R(\mathcal{K}_j \psi)(x, \omega, \cdot, E))(E). \]

(162)

As in [31], section 6, we see that
\[ B_0(\psi, v) = (A_0 \psi, v)_{L^2(G \times S \times I)} = (\psi, -\omega \cdot \nabla_x v + \Sigma^* v - K_{xx}^* v)_{L^2(G \times S \times I)} \]
\[ + \int_{\Gamma_+} (\omega \cdot v)_+ g v d\sigma d\omega dE - \int_{\Gamma_+} (\omega \cdot v)_- g v d\sigma d\omega dE \]

(163)

By the proof of Theorem 4.4.9 (see [141])
\[ B_1(\psi, v) := (A_1 \psi, v)_{L^2(G \times S \times I)} = (\psi, A_1^* v)_{L^2(G \times S \times I)} \]
\[ = -\int_G\int_S' \int_{P'} \psi(x, \omega', E') \left( \text{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \hat{\sigma}_1(x, E', E) \right) \cdot \int_0^{2\pi} v(x, \gamma(E', E, \omega')(s), E) ds dE d\omega' dx. \]

(164)

We show in Part B below that \( B_2(\psi, v) \) has the expression (157). Hence noting that
\[ (T \psi, v)_{L^2(G \times S \times I)} = (f, v)_{L^2(G \times S \times I)} \]
and combining (161), (163), (164) we obtain the assertion (158).

B. Consider the term \( B_2(\psi, v) \). We utilize the expression (162) for \( A_2 \) (alternatively we could use the expression (164)). Denote
\[ f(x, E', E) := \hat{\sigma}_2(x, E', E) \int_S \int_0^{2\pi} \psi(x, \omega, E') v(x, \gamma(E', E, \omega)(s), E) ds d\omega. \]
Applying the formulas (67), (74), (62) we obtain (the existence of the integral \( \int_{G \times S \times I} \) below becomes clear in a course of computations)

\[
B_2(\psi, v) = \langle A_2 \psi, v \rangle_{L^2(G \times S \times I)} = \left\langle -\mathcal{H}_2(\overline{\mathcal{B}} \psi)(x, \omega, \cdot, E)(E), v \right\rangle_{L^2(G \times S \times I)}
\]

\[
= -\int_{G \times S \times I} \left( \text{p.f.} \int_E^{E_m} \frac{1}{(E' - E)^2} \hat{\sigma}_2(x, E', E) \cdot \int_0^{2\pi} \psi(x, E', \omega) v(x, \omega, E) ds \right) dx d\omega dE
\]

\[
= -\int_G \left[ \int_I \left( \text{p.f.} \int_E^{E_m} \frac{1}{(E' - E)^2} \hat{\sigma}_2(x, E', E) \cdot \int_S^{2\pi} \psi(x, \omega, E') v(x, \gamma(E', \omega)(s), E') ds \right) dE' \right] dx
\]

\[
= -\int_G \left[ \int_I \left( \text{p.f.} \int_E^{E_m} \frac{1}{(E' - E)^2} f(x, E', E) dE' \right) \right] dx
\]

\[
= -\int_G \left[ \int_I \left( \text{p.f.} \int_E^{E_m} \frac{1}{E' - E} \frac{\partial f}{\partial E'}(x, E', E) dE' + \frac{\partial f}{\partial E'}(x, E, E) + \frac{1}{E_m - E} f(x, E_m, E) \right) \right] dE dx.
\]

We find that in due to the initial condition \( (148) \)

\[
f(x, E_m, E) = \hat{\sigma}_2(x, E', E) \int_S \int_0^{2\pi} \psi(x, \omega, E_m) v(x, \gamma(E_m, \omega)(s), E) ds d\omega = 0
\]

and so

\[
B_2(\psi, v) = -\int_G \int_I \left( \text{p.f.} \int_E^{E_m} \frac{1}{E' - E} \frac{\partial f}{\partial E'}(x, E', E) dE' \right) dE dx - \int_G \int_I \frac{\partial f}{\partial E'}(x, E, E) + dE dx
\]

\[
= I_1 + I_2.
\]

By Lemma \( (141) \)

\[
I_1 = -\int_G \int_I \left( \text{p.f.} \int_E^{E'} \frac{1}{E' - E} \frac{\partial f}{\partial E'}(x, E', E) dE' \right) dE' dx.
\]

Moreover, for \( E' \neq E \) we have

\[
\frac{\partial f}{\partial E'}(x, E', E) = \frac{\partial \hat{\sigma}_2}{\partial E'}(x, E', E) \int_S \int_0^{2\pi} \psi(x, \omega, E') v(x, \gamma(E', \omega)(s), E) ds d\omega
\]

\[
+ \hat{\sigma}_2(x, E', E) \int_S \int_0^{2\pi} \frac{\partial \psi}{\partial E'}(x, \omega, E') v(x, \gamma(E', \omega)(s), E) ds d\omega
\]

\[
+ \hat{\sigma}_2(x, E', E) \int_S \int_0^{2\pi} \psi(x, \omega, E') \left( (\nabla_s v)(x, \gamma(E', \omega)(s), E), \frac{\partial \gamma}{\partial E'}(E', \omega)(s) \right) ds d\omega
\]

(168)
and so by (70)

\[
I_1 = - \int_G \int_{P'} \left( \mathrm{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \frac{\partial \hat{\sigma}_2}{\partial E'}(x, E', E) \right) dE' \int_{S} \left( \int_0^{2\pi} \psi(x, \omega, E') v(x, \gamma(E', E, \omega)(s), E) d\sigma E' \right) dE' d\omega dx
\]

\[
- \int_G \int_{P'} \left( \mathrm{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \hat{\sigma}_2(x, E', E) \right) \int_{S} \left( \int_0^{2\pi} \psi(x, \omega, E') \frac{\partial \hat{\gamma}}{\partial E'}(x, E', E, \omega)(s) d\sigma E' \right) dE' d\omega dx
\]

\[
- \int_G \int_{P'} \left( \mathrm{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \hat{\sigma}_2(x, E', E) \int_{S} \left( \int_0^{2\pi} \psi(x, \omega, E') \frac{\partial \hat{\gamma}}{\partial E'}(x, E', E, \omega)(s) d\sigma E' \right) dE' d\omega dx \right)
\]

\[
\cdot \left( \int_{E_0}^{E'} \frac{1}{E' - E} \hat{\sigma}_2(x, E', E) \int_{S} \left( \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E) d\sigma E' \right) dE' d\omega dx \right)
\]

\[
= - \int_G \int_{P'} \int_{S} \left( \int_0^{2\pi} \psi(x, \omega, E') \frac{\partial \hat{\sigma}_2}{\partial E'}(x, E', E) \right) dE' \int_{S} \left( \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E) d\sigma E' \right) dE' d\omega dx
\]

\[
- \int_G \int_{P'} \int_{S} \left( \int_0^{2\pi} \psi(x, \omega, E') \frac{\partial \hat{\sigma}_2}{\partial E'}(x, E', E) \right) dE' \int_{S} \left( \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E) d\sigma E' \right) dE' d\omega dx
\]

\[
\cdot \left( \int_{E_0}^{E'} \frac{1}{E' - E} \hat{\sigma}_2(x, E', E) \int_{S} \left( \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E) d\sigma E' \right) dE' d\omega dx \right)
\]

\[
\cdot \left( \int_{E_0}^{E'} \frac{1}{E' - E} \hat{\sigma}_2(x, E', E) \int_{S} \left( \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E) d\sigma E' \right) dE' d\omega dx \right)
\]

\[
=: I_{1,1} + I_{1,2} + I_{1,3}.
\]

The last term

\[
I_{1,3} = - \int_G \int_{P'} \int_{S} \frac{\partial \psi}{\partial E'}(x, \omega, E') \left( \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E) d\sigma E' \right) dE' d\omega dx
\]

\[
\cdot \left( \int_{E_0}^{E'} \frac{1}{E' - E} \hat{\sigma}_2(x, E', E) \int_{S} \left( \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E) d\sigma E' \right) dE' d\omega dx \right)
\]

requires still refining. Let

\[
w(x, \omega, E') := \mathrm{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \hat{\sigma}_2(x, E', E) \int_{S} \left( \int_0^{2\pi} \psi(x, \gamma(E', E, \omega)(s), E) d\sigma E' \right) dE' d\omega dx
\]

Then

\[
I_{1,3} = - \int_G \int_{P'} \int_{S} \frac{\partial \psi}{\partial E'}(x, \omega, E') w(x, \omega, E') dE' d\omega dx
\]

By partial integration we obtain (partial integration is legitimate due to Lemma 4.15; we omit details here)

\[
I_{1,3} = - \int_G \int_{S} \left( \int_{E_0}^{E_m} \psi(x, \omega, E') w(x, \omega, E') - \int_{P'} \psi(x, \omega, E') \frac{\partial w}{\partial E'}(x, \omega, E') dE' \right) d\omega dx
\]
Here we interpret
\[
\left[ E_n \psi(x, \omega, E')w(x, \omega, E') \right] = \psi(x, \omega, E_m)w(x, \omega, E_m) - \lim_{E' \to E_0} \psi(x, \omega, E')w(x, \omega, E') = 0 - 0 = 0
\]
where we used Lemma 5.3 below.

Furthermore, we have by Lemma 2.1
\[
\frac{\partial w}{\partial E'}(x, \omega, E') = \frac{\partial}{\partial E'} \left( \text{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \hat{\sigma}_2(x, E', E) \int_0^{2\pi} v(x, \gamma(E', E, \omega)(s), E) dsdE \right)
\]
\[
= -\text{p.f.} \int_{E_0}^{E'} \frac{1}{(E' - E)^2} \hat{\sigma}_2(x, E', E) \int_0^{2\pi} v(x, \gamma(E', E, \omega)(s), E) dsdE
\]
\[
+ \text{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \frac{\partial \hat{\sigma}_2}{\partial E'}(x, E', E) \int_0^{2\pi} v(x, \gamma(E', E, \omega)(s), E) dsdE
\]
\[
+ \text{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \hat{\sigma}_2(x, E', E) \int_0^{2\pi} \left\langle (\nabla_S v)(x, \gamma(E', E, \omega)(s), E), \frac{\partial \gamma}{\partial E'}(E', E, \omega) \right\rangle dsdE
\]
\[
- \frac{\partial}{\partial E} \left( \hat{\sigma}_2(x, E', E) \int_0^{2\pi} v(x, \gamma(E', E, \omega)(s), E) ds \right) \bigg|_{E=E'} \tag{172}
\]
where
\[
\frac{\partial}{\partial E} \left( \hat{\sigma}_2(x, E', E) \int_0^{2\pi} v(x, \gamma(E', E, \omega)(s), E) ds \right) \bigg|_{E=E'} = 2\pi \frac{\partial \hat{\sigma}_2}{\partial E}(x, E', E')v(x, \omega, E') + \hat{\sigma}_2(x, E', E') \frac{\partial}{\partial E} \left( \int_0^{2\pi} v(x, \gamma(E', E, \omega)(s), E) ds \right) \bigg|_{E=E'}.
\]

Hence we get
\[
I_{1,3} = \int_G \int_{\pi}^{\pi'} \int_0^{2\pi} \psi(x, \omega, E') \left[ -\text{p.f.} \int_{E_0}^{E'} \frac{1}{(E' - E)^2} \hat{\sigma}_2(x, E', E) \int_0^{2\pi} v(x, \gamma(E', E, \omega)(s), E) dsdE \right]
\]
\[
\cdot \int_0^{2\pi} v(x, \gamma(E', E, \omega)(s), E) dsdE \bigg|_{E=E'}
\]
\[
+ \text{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \frac{\partial \hat{\sigma}_2}{\partial E'}(x, E', E) \int_0^{2\pi} v(x, \gamma(E', E, \omega)(s), E) dsdE \bigg|_{E=E'}
\]
\[
+ \text{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \hat{\sigma}_2(x, E', E) \int_0^{2\pi} \left\langle (\nabla_S v)(x, \gamma(E', E, \omega)(s), E), \frac{\partial \gamma}{\partial E'}(E', E, \omega) \right\rangle dsdE \bigg|_{E=E'}
\]
\[
- 2\pi \frac{\partial \hat{\sigma}_2}{\partial E}(x, E', E')v(x, \omega, E')
\]
\[
- \hat{\sigma}_2(x, E', E') \frac{\partial}{\partial E} \left( \int_0^{2\pi} v(x, \gamma(E', E, \omega)(s), E) ds \right) \bigg|_{E=E'} dE' d\omega dx. \tag{173}
\]
Consider the term $I_2$. Since $\gamma(E, E, \omega)(s) = \omega$ we have by (67)

$$
I_2 = - \int G \int_I \int_S \frac{\partial f}{\partial E}(x, E, E) \, dEdx = -2\pi \int G \int_I \int_S \psi(x, \omega, E) \frac{\partial \hat{\sigma}_2}{\partial E}(x, E) v(x, \omega, E) d\omega dEdx \\
- 2\pi \int G \int_I \int_S \frac{\partial \psi}{\partial E}(x, \omega, E) \hat{\sigma}_2(x, E, E) v(x, \omega, E) d\omega dEdx \\
- \int G \int_I \int_S \psi(x, \omega, E) \hat{\sigma}_2(x, E, E) \\
\cdot \left( \int_0^{2\pi} \left( \nabla_S v(x, \gamma(E', E, \omega)(s), E), \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) \right) \, ds \right) \bigg|_{E'=E} d\omega dEdx \tag{174}
$$

where by partial integration

$$
\int G \int_I \int_S \frac{\partial \psi}{\partial E}(x, \omega, E) \hat{\sigma}_2(x, E, E) v(x, \omega, E) d\omega dEdx \\
= \int G \int_I \int_S \left( \left[ \int_{E_m}^{E_0} \hat{\sigma}_2(x, E, E) \psi(x, \omega, E) v(x, \omega, E) \, dE \right] - \int I \psi(x, \omega, E) \frac{\partial (\hat{\sigma}_2(x, E, E) v)}{\partial E}(x, \omega, E) \, dE \right) \, d\omega. \tag{175}
$$

Hence due to $\psi(., ., E_m) = v(., ., E_0) = 0$

$$
I_2 = - \int G \int_I \int_S \psi(x, \omega, E) \left( 2\pi \frac{\partial \hat{\sigma}_2}{\partial E}(x, E) v(x, \omega, E) - 2\pi \frac{\partial (\hat{\sigma}_2(x, E, E) v)}{\partial E}(x, \omega, E) \right) d\omega dEdx \\
- \int G \int_I \int_S \psi(x, \omega, E) \hat{\sigma}_2(x, E, E) \\
\cdot \left( \int_0^{2\pi} \left( \nabla_S v(x, \gamma(E', E, \omega)(s), E), \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) \right) \, ds \right) \bigg|_{E'=E} d\omega dEdx. \tag{176}
$$
Combining (165), (169), (173), (176) we obtain

\[ B_2(\psi, v) = I_{1,1} + I_{1,2} + I_{1,3} + I_2 \]

\[ = - \int_{G} \int_{S} \int_{S^p} \psi(x, \omega, E') \left( \text{p.f.} \int_{E_0}^{E'} \frac{1}{E'-E} \partial \dot{\sigma}_2(x, E', E) \int_{0}^{2\pi} v(x, \gamma(E', E, \omega)(s), E) ds dE' d\omega dx \right) \]

\[ - \int_{G} \int_{S} \int_{S^p} \psi(x, \omega, E') \left( \text{p.f.} \int_{E_0}^{E'} \frac{1}{E'-E} \dot{\sigma}_2(x, E', E) \right) \int_{0}^{2\pi} v(x, \gamma(E', E, \omega)(s), E) ds dE' d\omega dx \]

\[ \cdot \left( \int_{0}^{2\pi} \left( (\nabla_S v)(x, \gamma(E', E, \omega)(s), E), \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) \right) ds dE' d\omega dx \right) \]

\[ - 2\pi \int_{G} \int_{S} \int_{S^p} \psi(x, \omega, E') \frac{\partial \dot{\sigma}_2}{\partial E}(x, E', E) v(x, \omega, E') dE' d\omega dx \]

\[ - \left( \int_{0}^{2\pi} \left( (\nabla_S v)(x, \gamma(E', E, \omega)(s), E), \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) \right) ds \right) \bigg|_{E=E'} \]
\[
\frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) = R(\omega)(- \frac{\mu}{\sqrt{1 - \mu^2}} \cos(s), - \frac{\mu}{\sqrt{1 - \mu^2}} \sin(s), \partial_{E'} \mu),
\]
\[
(\partial_{E'} \mu)(E', E) = \frac{1}{2\mu} \left( \frac{1}{E'(E + 2)^2} (2E' + 4) \right), \quad (\partial_{E'} \mu)(E', E) = \frac{1}{2\mu} \left( \frac{1}{E'^2(E + 2)} (-2E) \right).
\]
Denote
\[
\theta(E', E, \omega)(s) := \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) + \frac{\partial \gamma}{\partial E}(E', E, \omega)(s)
\]
\[
= (\partial_{E'} \mu + \partial_{E} \mu) R(\omega) \left( \frac{-\mu}{\sqrt{1 - \mu^2}} \cos(s), \frac{-\mu}{\sqrt{1 - \mu^2}} \sin(s), 1 \right),
\]
Hence removing the cancelling terms and reorganizing we conclude from (177) that
\[
B_2(\psi, v)
\]
\[
= - \int_G \int_S \int_{P'} \psi(x, \omega, E') \left( \text{p.f.} \int_{E_0}^{E'} \frac{1}{(E' - E)^2} \hat{\sigma}_2(x, E', E) \cdot \int_0^{2\pi} v(x, \gamma(E', E, \omega)(s), E) ds dE \right) \] dE dω dx
\[
- \int_G \int_S \hat{\sigma}_2(x, E, \omega) \psi(x, \omega, E)
\]
\[
\cdot \left( \int_0^{2\pi} \langle (\nabla s v)(x, \gamma(E', E, \omega)(s), E), \theta(E', E') \rangle ds \right) \bigg|_{E' = E} \right) dE d\omega dx. \quad (179)
\]
C. Note that \( \theta(E', E, \omega)(s) = \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) + \frac{\partial \gamma}{\partial E}(E', E, \omega)(s) \) has a singularity of the form \((E' - E)^{-1/2}\) as well. Applying the techniques used in the proof of Lemma 4.10 (cf. (81)) we have
\[
\left( \int_0^{2\pi} \langle (\nabla s v)(x, \gamma(E', E, \omega)(s), E), \theta(E', E') \rangle ds \right) \bigg|_{E' = E} = 2\pi \left( (\partial_{E'} \mu)(E, E) + (\partial_{E'} \mu)(E, E) \right) \langle \omega, \nabla s v \rangle(x, \omega, E) + \sum_{|\alpha| \leq 2 \beta \alpha} b_{\alpha}(E, \omega) (\partial_{s, \alpha} v)(x, \omega, E)
\]
where
\[
\sum_{|\alpha| \leq 2} b_{\alpha}(E, \omega) (\partial_{s, \alpha} v)(x, \omega, E)
\]
\[
= \lim_{E' \to E} \int_0^{2\pi} \sum_{j=1}^2 \partial_j \left( \langle (\nabla s \psi \circ H_s)(x, E', E), \theta(E', E') \rangle \right) (0) \xi_j((E', E, \omega, s)) ds \quad (180)
\]
and \( \theta(E', E, \omega, s) := \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) + \frac{\partial \gamma}{\partial E}(E', E, \omega)(s) \). Now, similarly as in Lemma 4.10
\[
\lim_{E' \to E} \int_0^{2\pi} \sum_{j=1}^2 \partial_j \left( \langle (\nabla s \psi \circ H_s)(x, E', E), \theta(E', E') \rangle \right) (0) \xi_j((E', E, \omega, s)) ds
\]
\[
= \left( (\partial_{E'} \mu)(E, E) + (\partial_{E} \mu)(E, E) \right) \cdot \int_0^{2\pi} \sum_{j=1}^2 \partial_j \left( \langle (\nabla s \psi \circ H_s)(x, E), R(\omega)(\cos(s), \sin(s), 0) \rangle \right) (0)
\]
\[
\cdot \langle R(\omega)(\cos(s), \sin(s), 0), \Omega_j \rangle \right) ds.
\]
Recall that by (28)
\[
(\omega \cdot \nabla s v)(x, \omega, E) = 0.
\]
Then by (181) \[ \lim_{E \to E_0} \psi(x, \omega, E') w(x, \omega, E') = 2\pi \lim_{E \to E_0} \ln(E' - E_0) \hat{\sigma}_2(x, E', E') \psi(x, \omega, E') v(x, \omega, E') = 0. \] (182)

**Proof.** Let

\[ h(x, \omega, E) := \hat{\sigma}_2(x, E', E') \int_0^{2\pi} v(x, \gamma(E', E, \omega)(s), E) ds. \]

Then

\[ w(x, \omega, E') = \text{p.f.} \int_{E_0}^{E'} \frac{h(x, \omega, E)}{E' - E} dE = \text{p.f.} \int_{E_0}^{E'} \frac{h(x, \omega, E')}{E' - E} dE + \int_{E_0}^{E'} \frac{h(x, \omega, E) - h(x, \omega, E')}{E' - E} dE. \]

In virtue of the assumptions on \( \hat{\sigma}_2 \) and \( v \) there exits \( \delta > 0 \) such that (we omit details)

\[ \left| \frac{h(x, \omega, E) - h(x, \omega, E')}{E' - E} \right| \leq C(E' - E)^{\delta - 1} \]

and so

\[ \lim_{E' \to E_0} \int_{E_0}^{E'} \frac{h(x, \omega, E) - h(x, \omega, E')}{E' - E} dE = 0. \]

Moreover,

\[ \text{p.f.} \int_{E_0}^{E'} \frac{h(x, \omega, E')}{E' - E} dE = \ln(E' - E_0) h(x, \omega, E') = 2\pi \ln(E' - E_0) \hat{\sigma}_2(x, E', E') v(x, \omega, E') \]

Hence we get

\[ \psi(x, \omega, E') w(x, \omega, E') = 2\pi \psi(x, \omega, E') \ln(E' - E_0) \hat{\sigma}_2(x, E', E') v(x, \omega, E') \]

\[ + \psi(x, \omega, E') \int_{E_0}^{E'} \frac{h(x, \omega, E') - h(x, \omega, E)}{E' - E} dE \]

\[ \to 2\pi \lim_{E' \to E_0} \psi(x, \omega, E') \ln(E' - E_0) \hat{\sigma}_2(x, E', E') v(x, \omega, E') = 0 \] (183)

since by (151) \( \lim_{E' \to E_0} \ln(E' - E_0) v(x, \omega, E') = 0 \). This completes the proof. □
Note that
\[
\int_{\Gamma^-} (\omega \cdot \nu) - gv \, d\sigma d\omega dE = \langle g, \gamma_-(\nu) \rangle_{T2(\Gamma^-)}
\] (184)
and
\[
\int_{\Gamma^+} (\omega \cdot \nu) + \psi v \, d\sigma d\omega dE = \langle \gamma_+(\psi), \gamma_+(\nu) \rangle_{T2(\Gamma^+)}.
\] (185)

5.2. Lowering the degree of singularities in the bilinear form $B_2(\cdot, \cdot)$. The above bilinear form $B_2(\cdot, \cdot)$ contains singularities of the form $\frac{1}{(E' - E)^2} dEdE'$ (in the first term) which may be problematic e.g. in numerical treatments. The degree of this singularity can be lowered as follows. Consider the term in question

\[I := - \int_G \int_{I'} \left( \text{p.f.} \int_{E_0}^{E'} \frac{1}{(E' - E)^2} \hat{\sigma}_2(x, E', E)\right) \cdot \int_0^{2\pi} \psi(x, \omega, E') v(x, \gamma(E', E, \omega)(s), E) d\sigma dE dE' dx.
\]

Let $f(x, E', E)$ be as above in the proof of Theorem 5.2 Part B that is,

\[f(x, E', E) := \hat{\sigma}_2(x, E', E) \int_S \int_0^{2\pi} \psi(x, \omega, E') v(x, \gamma(E', E, \omega)(s), E) d\sigma d\omega.
\]

Then we have by Lemmas 4.0 and 4.3

\[
I = - \int_G \int_{I'} \left( \text{p.f.} \int_{E_0}^{E'} \frac{1}{(E' - E)^2} \hat{\sigma}_2(x, E', E)\right) \cdot \int_0^{2\pi} \psi(x, \omega, E') v(x, \gamma(E', E, \omega)(s), E) d\sigma dE dE' dx
\]

\[
= - \int_G \int_{I'} \left( \text{p.f.} \int_{E_0}^{E'} \frac{1}{(E' - E)^2} f(x, E', E) dE'\right) dx
\]

\[
= - \int_G \int_{I'} \left( \frac{\partial f}{\partial E} (x, E', E') - \frac{1}{E' - E_0} f(x, E', E_0)\right) dE' dx
\]

\[
= \int_G \int_{I'} \left( \text{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \frac{\partial f}{\partial E} (x, E', E) dE'\right) dE' dx + \int_G \int_{I'} \frac{\partial f}{\partial E} (x, E', E') dE' dx
\]

\[
= \int_G \int_{I'} \left( \text{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \frac{\partial f}{\partial E} (x, E', E) dE'\right) dE' dx + \int_G \int_{I'} \frac{\partial f}{\partial E} (x, E', E') dE' dx
\]

where we assume that the last integral $I_3$ exists.

Moreover, we have

\[
\frac{\partial f}{\partial E}(x, E', E) = \frac{\partial \hat{\sigma}_2}{\partial E}(x, E', E) \int_S \int_0^{2\pi} \psi(x, \omega, E') v(x, \gamma(E', E, \omega)(s), E) d\sigma d\omega
\]

\[
+ \hat{\sigma}_2(x, E', E) \int_S \int_0^{2\pi} \psi(x, \omega, E') \frac{\partial v}{\partial E} (x, \gamma(E', E, \omega)(s), E) d\sigma d\omega
\]

\[
+ \hat{\sigma}_2(x, E', E) \int_S \int_0^{2\pi} \psi(x, \omega, E') \left( \nabla S v \right)(x, \gamma(E', E, \omega)(s), E) \frac{\partial \gamma}{\partial E}(E', E, \omega)(s) d\sigma d\omega
\] (187)
and so by (70)

\[ I_1 = \int_G \int_{I'} \left( \text{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \frac{\partial \hat{\sigma}_2}{\partial E}(x, E', E) \right) \]

\[ \cdot \int_S \int_0^{2\pi} \psi(x, \omega, E')v(x, \gamma(E', E, \omega)(s), E) ds dE' d\omega dx \]

\[ + \int_G \int_{I'} \left( \text{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \hat{\sigma}_2(x, E', E) \right) \]

\[ \cdot \int_S \int_0^{2\pi} \psi(x, \omega, E') \frac{\partial v}{\partial E}(x, \gamma(E', E, \omega)(s), E) ds dE' d\omega dx \]

\[ + \int_G \int_{I'} \left( \text{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \hat{\sigma}_2(x, E', E) \right) \int_S \int_0^{2\pi} \psi(x, \omega, E') \]

\[ \cdot \left( \langle \nabla_{s'} v \rangle(x, \gamma(E', E, \omega)(s), E), \frac{\partial \gamma}{\partial E}(E', E, \omega)(s) \right) ds dE' d\omega dx \]

i.e.

\[ I_1 = \int_G \int_{I'} \int_S \psi(x, \omega, E') \left( \text{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \frac{\partial \hat{\sigma}_2}{\partial E}(x, E', E) \right) \]

\[ \cdot \int_0^{2\pi} v(x, \gamma(E', E, \omega)(s), E) ds d\omega dE' dE' dx \]

\[ + \int_G \int_{I'} \int_S \psi(x, \omega, E') \left( \text{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \hat{\sigma}_2(x, E', E) \right) \]

\[ \cdot \int_0^{2\pi} \left( \langle \nabla_{s'} v \rangle(x, \gamma(E', E, \omega)(s), E), \frac{\partial \gamma}{\partial E}(E', E, \omega)(s) \right) ds dE' d\omega dx \]

\[ + \int_G \int_{I'} \int_S \psi(x, \omega, E') \hat{\sigma}_2(x, E', E) \]

\[ \cdot \left( \int_0^{2\pi} \left( \langle \nabla_{s'} v \rangle(x, \gamma(E', E, \omega)(s), E), \frac{\partial \gamma}{\partial E}(E', E, \omega)(s) \right) ds \right) \bigg|_{E=E'} d\omega dE' dx \]

(188)

Since \( \gamma(E', E', \omega)(s) = \omega \) we have by (187)

\[ I_2 = \int_G \int_{I'} \frac{\partial f}{\partial E}(x, E', E') dE dE' dx \]

\[ = 2\pi \int_G \int_{I'} \int_S \psi(x, \omega, E') \frac{\partial \hat{\sigma}_2}{\partial E}(x, E', E') v(x, \omega, E') d\omega dE' dx \]

\[ + 2\pi \int_G \int_{I'} \int_S \psi(x, \omega, E') \hat{\sigma}_2(x, E', E') \frac{\partial v}{\partial E}(x, \omega, E') d\omega dE' dx \]

\[ + \int_G \int_{I'} \int_S \psi(x, \omega, E') \hat{\sigma}_2(x, E', E') \]

\[ \cdot \left( \int_0^{2\pi} \left( \langle \nabla_{s'} v \rangle(x, \gamma(E', E, \omega)(s), E), \frac{\partial \gamma}{\partial E}(E', E, \omega)(s) \right) ds \right) \bigg|_{E=E'} d\omega dE' dx \]

(189)

In addition,

\[ I_3 = \int_G \int_{I'} \int_S \frac{1}{E' - E_0} \psi(x, \omega, E') \hat{\sigma}_2(x, E', E_0) \int_0^{2\pi} v(x, \gamma(E', E_0, \omega)(s), E_0) ds dE' d\omega = 0. \]

(190)
Substituting $I = I_1 + I_2 + I_3$ to (157) we obtain by (188), (189), (190)

\[ B_2(\psi, v) = \int_G \int_{E'} \int_S \psi(x, \omega, E') \left( \text{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \sigma_2(x, E', E) \right) dx \]

\[ \cdot \int_0^{2\pi} \int_{E'} \psi(x, \gamma(E', E, \omega)(s), E) dE' dx \]

\[ + \int_G \int_{E'} \int_{E'} \psi(x, \omega, E') \left( \text{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \sigma_2(x, E', E) \right) dx \]

\[ \cdot \int_0^{2\pi} \left( (\nabla_S v)(x, \gamma(E', E, \omega)(s), E), \frac{\partial \gamma}{\partial E}(E', E, \omega)(s) \right) dE' dx \]

\[ + \int_G \int_{E'} \int_S \sigma_2(x, E', E) \]

\[ \cdot \int_0^{2\pi} \frac{\partial v}{\partial E}(x, \gamma(E', E, \omega)(s), E) dE' dx \]

\[ + 2\pi \int_G \int_{E'} \int_S \sigma_2(x, E', E') v(x, \omega, E') dE' dx \]

\[ + 2\pi \int_G \int_{E'} \int_S \sigma_2(x, E', E') \frac{\partial v}{\partial E}(x, \omega, E') dE' dx \]

\[ + \int_G \int_{E'} \int_S \sigma_2(x, E', E') \]

\[ \cdot \left( \int_0^{2\pi} \left( (\nabla_S v)(x, \gamma(E', E, \omega)(s), E), \frac{\partial \gamma}{\partial E}(E', E, \omega)(s) \right) ds \right) \bigg|_{E'=E'} dE' dx \]

\[ - \int_G \int_S \psi(x, \omega, E) \sigma_2(x, E, E) \sum_{|\alpha| \leq 2} b_\alpha(E, \omega)(\partial_{E'}^\alpha v)(x, \omega, E) dE d\omega dx. \]

(191)

As in (81) one finds that

\[ \left( \int_0^{2\pi} \left( (\nabla_S v)(x, \gamma(E', E, \omega)(s), E), \frac{\partial \gamma}{\partial E}(E', E, \omega)(s) \right) ds \right) \bigg|_{E'=E'} = 2\pi (\partial_{E'} \mu)(E', E') (\omega \cdot \nabla_S v)(x, \omega, E') + \sum_{|\alpha| \leq 2} c_\alpha(E', \omega)(\partial_{E'}^\alpha v)(x, \omega, E') \]

(192)

where

\[ \sum_{|\alpha| \leq 2} c_\alpha(E', \omega)(\partial_{E'}^\alpha v)(x, \omega, E') \]

\[ := -(\partial_{E'} \mu)(E', E') \cdot \int_0^{2\pi} \sum_{j=1}^2 \partial_j \left( \langle \nabla_S v \circ H_\omega \rangle(x, E', R(\omega)(\cos(s), \sin(s), 0)) \right) (0) \]

\[ \cdot \langle R(\omega)(\cos(s), \sin(s), 0), \nabla_S \rangle ds. \]
Recalling the term \( \sum_{|\alpha| \leq 2} b_\alpha(\omega, E')(\partial^{\alpha} v)(x, \omega, E') \) from (144) we find that

\[
- \int_0^{2\pi} \left( \langle \nabla S v(x, \gamma(E', E, \omega)(s), E), \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) \rangle \right) ds \bigg|_{E = E'} + \sum_{|\alpha| \leq 2} b_\alpha(\omega, E')(\partial^{\alpha} v)(x, \omega, E')
\]

\[
= -2\pi (\partial_E \mu)(E', E')(\omega \cdot \nabla S v)(x, \omega, E') + \sum_{|\alpha| \leq 2} a_\alpha(\omega, E')(\partial^{\alpha} v)(x, \omega, E').
\] (193)

Since by (28)

\[
(\omega \cdot \nabla S v)(x, \omega, E') = 0
\]

we conclude from (192) and (191) (by removing the cancelling terms and rearranging the terms) finally

**Theorem 5.4** Suppose that the assumptions of Theorem 5.2 are valid. Then for \( v \in D(0) \)

\[
B_2(\psi, v) = \int_G \int_{S_0} \int_S \psi(x, \omega, E')(\text{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \delta_2(x, E', E) \cdot \int_0^{2\pi} \frac{\partial v}{\partial E}(x, \gamma(E', E, \omega)(s), E) dsdE) dE'd\omega dx + 2\pi \int_G \int_{S_0} \int_S \psi(x, \omega, E')(\text{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \delta_2(x, E', E) \cdot \int_0^{2\pi} (\nabla S v)(x, \gamma(E', E, \omega)(s), E), \frac{\partial \gamma}{\partial E'}(E', E, \omega)(s) \rangle dsdE) dE'd\omega dx
\]

\[
- \sum_{|\alpha| \leq 2} \int_G \int_{S_0} \int_S \psi(x, \omega, E)\delta_2(x, E, E) a_\alpha(\omega, E) (\partial^{\alpha} v)(x, \omega, E) dEd\omega dx + \int_G \int_{S_0} \int_S \psi(x, \omega, E')(\text{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \delta_2(x, E', E) \cdot \int_0^{2\pi} v(x, \gamma(E', E, \omega)(s), E) dsdE) dE'd\omega dx + 2\pi \int_G \int_{S_0} \int_S \psi(x, \omega, E) \frac{\partial \delta_2}{\partial E}(x, E, E) v(x, \omega, E) dEd\omega dx.
\] (194)

The variational problem (153) becomes

\[
B(\psi, v) = F(v), \ v \in D(0)
\] (195)

The bilinear form \( B(., .) \) is

\[
B(\psi, v) = B_2(\psi, v) + B_1(\psi, v) + B_0(\psi, v)
\] (196)

in which \( B_2(., .) \) is now given by (7.4).
Remark 5.5 The above considerations suggest that the relevant space of test functions is $D_0$. Closer treatments reveal that we can additionally demand that the adjoint inflow boundary condition

$$\gamma_+(v) = 0$$

is valid for $v$ (cf. [31], the proof of Theorem 6.7).

Remark 5.6 We give some preliminary notes concerning for the boundedness of the bilinear forms $B_j(\cdot, \cdot), j = 0, 1, 2$.

**Boundedness of $B_0$.** Define an inner product

$$\langle \psi, v \rangle_{H_0} := \langle \psi, v \rangle_{L^2(G \times S \times I)} + \langle \gamma(\psi), \gamma(v) \rangle_{T^2(\Gamma)}$$

in $C(\overline{G} \times S \times I)$ and an inner product

$$\langle \psi, v \rangle_{\tilde{H}_0} := \langle \psi, v \rangle_{\tilde{W}^2(G \times S \times I)}$$

$$= \langle \psi, v \rangle_{L^2(G \times S \times I)} + \langle \nabla_x \psi, \nabla_x v \rangle_{L^2(G \times S \times I)} + \langle \gamma(\psi), \gamma(v) \rangle_{T^2(\Gamma)}$$

in $C^1(\overline{G}, C(S \times I))$.

The bilinear form $B_0 : C^1(\overline{G}, C(S \times I)) \times C^1(\overline{G}, C(S \times I)) \to \mathbb{R}$ obeys the following boundedness condition (see [31], Theorem 5.4):

Suppose that the assumptions (138), (133), (134), (135) are valid. Then there exists a constant $C_0 > 0$ such that

$$|B_0(\psi, v)| \leq C_0 \|\psi\|_{H_0} \|v\|_{\tilde{H}_0} \quad \forall \, \psi, v \in C^1(\overline{G}, C(S \times I)).$$

(200)

On boundedness of $B_1$. For $0 < \alpha < 1$ we define the subspace $L^2(G \times I, W^{\infty, \alpha}(S))$ of $L^2(G \times S \times I)$ whose elements obey

$$\|\psi(\cdot, \cdot, \cdot) - \psi(\cdot, \cdot, \cdot)\|_{L^2(G \times I)} \leq N_\alpha d_S(\omega', \omega)^\alpha, \quad \text{a.e.} \, \omega, \omega' \in S$$

(201)

where $N_\alpha < \infty$ and where $d_S(\omega', \omega)$ is the geodesic distance of $\omega'$ and $\omega$. As usual, the norm in $L^2(G \times I, W^{\infty, \alpha}(S))$ is $\|\psi\|_{L^2(G \times I, W^{\infty, \alpha}(S))} := \inf\{N_\alpha\}$ for which (201) holds. Note that there exists a constant $C > 0$ such that

$$d_S(\omega', \omega) \leq C \|\omega' - \omega\|$$

(202)

( where $\|\cdot\|$ is the $\mathbb{R}^3$-norm). The space $L^2(G \times S, H^s(I))$, $s \geq 0$ is standardly defined as well and it is equipped with the natural inner product and norm.

Let $0 < \kappa < \frac{1}{2}$, $1 - 2\kappa < \alpha < 1$. Define in $C(\overline{G} \times I, C^\alpha(S)) \cap C(\overline{G} \times S, C^{\frac{1}{2} - \kappa}(I))$ the weighted norm

$$\|\psi\|_{L^2(G \times S \times I, w)} := \left( \|\ln(E - E_0)\|_{L^2}^2 + \|\ln(E - E_m)\|_{L^2}^2 + \|\psi(\cdot, \cdot, E_0)\|_{L^2(G \times S)}^2 + \|\psi(\cdot, \cdot, E_m)\|_{L^2(G \times S)}^2 \right)^{1/2}.$$
On boundedness of $B_2$. The boundedness criteria for $B_2(\cdot, \cdot)$ can retrieved on the basis of the previous paragraph. For example, the first term

$$B_{2,1}(\psi, v) := \int_G \int_{S} \psi(x, \omega, E') \left( \text{p.f.} \int_{E_0}^{E'} \frac{1}{E' - E} \tilde{g}_2(x, E', E) \int_{0}^{2\pi} \frac{\partial v}{\partial E} (x, \gamma(E', E, \omega)(s), E) d\sigma dE \right) dE' d\omega d\sigma$$

appearing in (194) obeys under relevant assumptions

$$|B_{2,1}(\psi, v)| \leq C \|\psi\|_{L^2(G \times S \times I, \mu)} \left\| \frac{\partial v}{\partial E} \right\|_{H_{\alpha, \alpha}}.$$  \hfill (203)

The only term in $B_2(\cdot, \cdot)$ which requires essentially further study is the third one,

$$B_{2,3}(\psi, v) := - \int_G \int_{S} \int_{E_0}^{E'} \psi(x, \omega, E') \left( \text{p.f.} \int_{0}^{2\pi} \left( \langle \nabla_s v \rangle (x, \gamma(E', E, \omega)(s), E), \frac{\partial \gamma}{\partial E}(E', E, \omega)(s) \right) d\sigma dE \right) dE' d\omega d\sigma.$$

We neglect here detailed formulations for $B_2(\cdot, \cdot)$.

Additionally to the boundedness criteria one needs coercivity (accretivity) criteria for $B(\cdot, \cdot)$ and a careful treatise of relevant (generalized) Green and inflow trace theorems. The existence and uniqueness analysis of the non-classical transport problem treated in this paper requires a considerable further research.

6. Discussion

The paper considers an exact linear Boltzmann transport operator related to the charged particle transport. The hyper–singularities in the differential cross-sections of certain charged particle collisions lead to extra singular integral–partial differential terms in kinematic equations. The analysis showed that the terms contain the first-order partial derivatives with respect to energy $E$ combined with Hadamard finite part integral operators which are pseudo-differential-like operators. With respect to angle $\omega$ also second-order partial derivatives appear. Additionally, some mixed terms appear (see [4]). Note especially that in the expression [11] there is the term $\sum_{|a| \leq 2} \partial_a \tilde{\psi}(x, \omega, E) \partial^a_\omega \psi$ which is corresponding to the second order partial differential term with respect to angular variables appearing in Fokker-Plank equation. In many cases this term turns out to be an elliptic operator (on $S$) which helps the analysis and in numerical treatments it stabilizes computations.

The pseudo-differential-like terms can be approximated by pseudo-differential operators ([33] or [31]). Moreover, in [31] we gave a further approximation where the resulting transport operator is containing only partial derivatives and Schur partial integral operators. These approximations are essentially founded on Taylor’s formulas and angular approximations of (new) primary particles. When considering partial differential approximations the resulting approximative transport operator is a partial integro-differential operator of the form

$$T \psi = a(x, E) \frac{\partial \psi}{\partial E} + \sum_{|a| \leq 2} b_a(x, \omega, E) \partial^a_\omega \psi + \omega \cdot \nabla_x \psi + \sum (x, \omega, E) \psi - K_r \psi.$$  \hfill (204)

These approximations are known as Continuous Slowing Down Approximations (CSDA) of BTE. For example, in the dose calculation the scattering events containing hyper–singularities are the primary Möller scattering for electrons and positrons and the Bremsstrahlung. Conventionally these events have needed the CSDA modelling like [20] in numerical computations.

In this paper we additionally exposed the variational formulation for the transport problem where no approximations are applied. In principle, the obtained variational problem can be
numerically solved applying e.g. Galerkin finite element methods (FEM). The numerical computations and approximations of (hyper-)singular integrals has been studied by various methods and for various needs. In particular, for Galerkin methods them are known in the field of boundary element methods (BEM) (see e.g. [3]). BEM considers numerical methods for solutions of the (hyper-)singular integral equations emerging from solutions of certain initial-boundary value problems.

The well-posedness of the transport problem that is, existence and uniqueness of solutions together with pertinent a priori estimates is a central importance. We suggest that the exposed variational formulations together with Lions-Lax-Milgram Theorem give one alternative to investigate well-posedness of the problems where exact operators like (4) are involved in the (coupled) transport system. Other successful methods might be e.g. semigroup-dissipativity-perturbation methods and fixed-point (contraction) methods. In addition, Sobolev regularity (say, on the scales of Sobolev-Slobodevskij spaces) of solutions remains open. It is known that the transport problems have a limited Sobolev regularity but higher order weighted (co-normal) Sobolev regularity can be achieved. The regularity of solutions are needed e.g. in various approximation and convergence (error analysis) treatments. We remark that the initial inflow boundary value problems related to transport problems are so called variable boundary value multiplicity that is, the dimension of the kernel of the boundary operator is not constant (e.g. [25], [28]). This makes the inflow boundary transport problems more subtle independently of the applied methods.

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