Completeness of logics with the transitive closure modality and related logics

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Abstract
We give a sufficient condition for Kripke completeness of modal logics enriched with the transitive closure modality. More precisely, we show that if a logic admits what we call definable filtration (ADF), then such an expansion of the logic is complete; in addition, has the finite model property, and again ADF. This argument can be iterated, and as an application we obtain the finite model property for PDL-like expansions of logics that ADF.

Keywords: Filtration, decidability, finite model property, transitive closure, PDL.

Introduction
This paper makes a contribution to the study of modal logics enriched by the transitive closure modality.

Modal logics that, in addition to the modal operator $\Box$ for a binary relation $R$, also contain the operator $\Box^*$ for the transitive closure of $R$, are quite common. For instance, such are operators ‘everyone knows that’ and the operator of common knowledge in epistemic logic [4]. Other examples include

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the logic Team for collective beliefs and actions [3], \( \text{Log}(\mathbb{N}, \text{succ}, <) \) and the proposition dynamic logic (PDL) [5].

So far, completeness theorems and decidability procedures for such logics have had bespoke proofs, though many of them refer to Segerberg’s [13] and Kozen and Parikh’s arguments for PDL [10].

In this paper we present a toolkit for proving results on completeness, finite model property and decidability for such logics in a general setting. In Section 3 we suggest a sufficient condition for a unimodal logic \( L \) that ensures that axioms for \( L \) together with Segerberg’s axioms for the transitive closure modality are complete for the bimodal logic obtained by enriching the frames for \( L \) with the transitive closure of their accessibility relation. In order to state the condition, we come up with a hierarchy of ‘admits filtration’ properties (thus extending our earlier work [9]) presented in Section 2. In Section 4 we give examples of logics that satisfy these properties and show how our sufficient condition can be iterated for obtaining completeness for ‘PDLizations’ of a family of logics. As a by-product we obtain the finite model property and often decidability (through Harrop’s theorem) for the logics in question.

1 Preliminaries

We assume the reader to be familiar with syntax and semantics of multi-modal logic [1,2], so we only briefly recall some notions and fix notation. Let \( \Sigma \) be a finite alphabet (of indices for modalities). The set \( \text{Fm}(\Sigma) \) of modal formulae (over \( \Sigma \)) is defined from propositional letters \( \text{Var} = \{ p_0, p_1, \ldots \} \) using Boolean connectives and the modalities \([e]\), for \( e \in \Sigma \), according to the syntax:

\[
\varphi ::= \bot \mid p_i \mid \varphi \to \psi \mid [e] \varphi.
\]

We use standard abbreviations (e.g., \( \top, \land \)): in particular, \( (e)\varphi := \neg [e] \neg \varphi \). For a set of formulae \( \Gamma \), by \( \text{Sub}(\Gamma) \) we denote the set of all subformulae of formulae from \( \Gamma \). We say that \( \Gamma \) is \( \text{Sub-closed} \) if \( \text{Sub}(\Gamma) \subseteq \Gamma \).

A \((\Sigma)-\)frame is a pair \( F = (W, (R_e)_{e \in \Sigma}) \), where \( W \neq \emptyset \) and \( R_e \subseteq W \times W \) for \( e \in \Sigma \). A model based on \( F \) is a pair \( M = (F, V) \), where \( V(p) \subseteq W \), for all \( p \in \text{Var} \). The truth relation \( M, x \models \varphi \) is defined in the usual way, e.g.

\[
M, x \models [e] \varphi \iff \text{for all } y \in W, \text{ if } x R_e y \text{ then } M, y \models \varphi.
\]

We write \( M \models \varphi \) if \( M, x \models \varphi \) for all \( x \) in \( M \). A formula \( \varphi \) is valid on \( F \), notation \( F \models \varphi \), if \( M \models \varphi \) for all \( M \) based on \( F \). For a class of frames \( F \), an \( F\)-model is a model based on a frame from \( F \).

A \((\text{normal modal}) \) logic (over \( \Sigma \)) is a set of formulae \( L \) that contains all classical tautologies, the axioms \([e](p \to q) \to ([e]p \to [e]q)\), for each \( e \in \Sigma \), and is closed under the rules of modus ponens, substitution, and necessitation (from \( \varphi \), infer \([e] \varphi \), for each \( e \in \Sigma \)). An \( L\)-frame is a frame \( F \) such that \( F \models L \). The logic of a class of frames \( F \) is the set of all formulae that are valid in \( F \). A logic \( L \) is \( \text{Kripke complete} \) if it is the logic of some class of frames. A logic \( L \) has the \( \text{finite model property} \) (FMP) if it is the logic of some class of finite frames; or equivalently (see, e.g., [1, Th. 3.28]) if, for every formula \( \varphi \notin L \),
there is a finite model $M$ such that $M \models L$ and $M \not\models \varphi$. For a logic $L$, put
\[
\Fr(L) = \{ F \mid F \text{ is a frame and } F \models L \},
\]
\[
\Mod(L) = \{ M \mid M \text{ is a model and } M \models L \}.
\]

Clearly, every $\Fr(L)$-model belongs to $\Mod(L)$. The converse does not hold in general; e.g., the canonical model of a non-canonical logic $L$ is not a $\Fr(L)$-model. But the converse holds in the following special case. A model $M$ is called differentiated if any two points in $M$ can be distinguished by a formula.

**Lemma 1.1** (See e.g. [7, Ex. 4.9]) Let $M = (F, V)$ be a finite differentiated model. If all substitution instances of a formula $\varphi$ are true in $M$, then $F \models \varphi$.

In particular, if $M \models L$, where $L$ is a logic, then $F \models L$.

**Harrop’s theorem.** A finitely axiomatizable logic with the FMP is decidable.

## 2 Filtration

The notion of a filtration we introduce below slightly generalizes the standard one (cf. [1, Def. 2.36], [2, Sect. 5.3]) in the following aspect: given a finite set of formulas $\Gamma$, we define a filtration as a model obtained by factoring a given model through an equivalence relation that we allow to be finer than the one induced by $\Gamma$. This modification seems to first appear in [16]; see also [17].

Let $M = (W, (R_e)_{e \in \Sigma}, V)$ be a model and $\Gamma$ a finite $\text{Sub}$-closed set of $\Sigma$-formulas. An equivalence relation $\sim$ on $W$ is of finite index if the quotient set $W/\sim$ is finite. The equivalence relation induced by $\Gamma$ is defined as follows:

\[
x \sim_\Gamma y \iff \forall \varphi \in \Gamma \ (M, x \models \varphi \iff M, y \models \varphi).
\]

Clearly, $\sim_\Gamma$ is of finite index. We say that an equivalence relation $\sim$ respects $\Gamma$ if $\sim \subseteq \sim_\Gamma$; in other words, if for every class $[x]_\sim \subseteq W$ and every formula $\varphi \in \Gamma$, $\varphi$ is either true in all points of $[x]_\sim$ or false in all points of $[x]_\sim$.

**Definition 2.1 (Filtration)** By a filtration of a model $M$ that respects a set of formulas $\Gamma$ (or a $\Gamma$-filtration of $M$) we call any model $\hat{M} = (\hat{W}, (\hat{R}_e)_{e \in \Sigma}, \hat{V})$ that satisfies the following conditions:

- $\hat{W} = W/\sim$, for some equivalence relation of finite index $\sim$ on $W$;
- the equivalence relation $\sim$ respects $\Gamma$, i.e., $x \sim y$ implies $x \sim_\Gamma y$;
- the valuation $\hat{V}$ is defined on the variables $p \in \Gamma$ canonically: $\hat{x} \models p \iff x \models p$, for all points $x \in W$, where $\hat{x} := [x]_\sim$ denotes the $\sim$-class of a point $x$;
- $R_{\sim, e}^\min \subseteq \hat{R}_e \subseteq R_{\sim, e}^\max$, for each $e \in \Sigma$. Here $R_{\sim, e}^\min$ is the $e$-th minimal filtered relation on $\hat{W}$, and $R_{\sim, e}^\max$ is the $e$-th maximal filtered relation on $\hat{W}$ induced by the set of formulas $\Gamma$; they are defined in the usual way:

\[
\begin{align*}
\hat{x} R_{\sim, e}^\min \hat{y} & \iff \exists x' \sim x \exists y' \sim y : x' R_e y', \\
\hat{x} R_{\sim, e}^\max \hat{y} & \iff \text{for every formula } [e] \varphi \in \Gamma (M, x \models [e] \varphi \Rightarrow M, y \models \varphi).
\end{align*}
\]

If $\sim = \sim_\Phi$ for some finite set of formulas $\Phi$, then we call $\hat{M}$ a definable $\Gamma$-filtration of $M$ (through $\Phi$).
Note that the relations $R_{\omega,e}^{\min}$ and $R_{\omega,e}^{\max}$ are well-defined and $R_{\omega,e}^{\min} \subseteq R_{\omega,e}^{\max}$. The condition $R_{\omega,e}^{\min} \subseteq \hat{R}_e$ is equivalent to that $\forall x, y \in W \ (x R_e y \Rightarrow \hat{x} R_e \hat{y})$. A filtration is always a finite model. If $\sim = \sim_{\Phi} \subseteq \sim_G$, then w.l.o.g. $\Phi \supseteq \Gamma$. The following is the key lemma about filtration (cf. [1, Th. 2.39], [2, Th. 5.23]).

**Lemma 2.2 (Filtration lemma)** Let $\Gamma$ be a finite $\Sub$-closed set of formulas and $\bar{M}$ is a $\Gamma$-filtration of a model $M$. Then, for all points $x \in W$ and all formulas $\varphi \in \Gamma$, we have: $M, x \models \varphi \Leftrightarrow \bar{M}, \hat{x} \models \varphi$.

### 2.1 Admissibility of filtration

**Definition 2.3 (ADF for classes of frames)** We say that a class of frames $\mathcal{F}$ admits (definable) filtration if, for any finite $\Sub$-closed set of formulas $\Gamma$ and an $\mathcal{F}$-model $M$, there exists an $\mathcal{F}$-model that is a (definable) $\Gamma$-filtration of $M$.

**Lemma 2.4 (AF for frames implies FMP)**

If a logic $L$ is Kripke complete and $\Fr(L)$ admits filtration, then $L$ has the FMP.

**Definition 2.5 (ADF for classes of models)** We say that a class of models $\mathcal{M}$ admits (definable) filtration if, for any finite $\Sub$-closed set of formulas $\Gamma$ and a model $M \in \mathcal{M}$, there a model in $\mathcal{M}$ that is a (definable) $\Gamma$-filtration of $M$.

**Lemma 2.6 (AF for models implies FMP)**

If $\Mod(L)$ admits filtration, then the logic $L$ has the FMP and hence is complete.

Note that this lemma does not involve the completeness assumption, since any normal logic $L$ is complete w.r.t. the class of its models $\Mod(L)$.

We have two variants of the notion “a logic $L$ admits (definable) filtration”:

(I) the class of frames $\Fr(L)$ admits (definable) filtration (Definition 2.3);

(II) the class of models $\Mod(L)$ admits (definable) filtration (Definition 2.5).

In both variants, we filtrate a model $M = (F, V)$ into a model $\bar{M} = (\hat{F}, \hat{V})$. The precondition ($F \models L$) in (I) is stronger than that ($M \models L$) in (II). The postcondition ($\hat{F} \models L$) in (I) is stronger than ($\hat{M} \models L$) in (II), too. However, we can always make sure that the finite model $\bar{M}$ is differentiated. Then $\hat{M} \models L$ if and only if $\hat{F} \models L$. Thus, (II) is stronger than (I), as the following lemma states.

**Lemma 2.7 (ADF for models implies ADF for frames)**

For any logic $L$, if $\Mod(L)$ admits (definable) filtration, then so does $\Fr(L)$.

**Proof.** Take any finite $\Sub$-closed set of formulas $\Gamma$ and a model $M = (F, V)$ with $F \models L$. Then $M \in \Mod(L)$. Since $\Mod(L)$ A(D)F, the model $M$ has a (definable) $\Gamma$-filtration $\bar{M} = (\hat{F}, \hat{V})$ with $\bar{M} \models L$. The model $\hat{M}$ is finite and, without loss of generality, differentiated, by Lemma A.2 (in Appendix). Then $\hat{F} \models L$, by Lemma 1.1. Thus, $\Fr(L)$ admits (definable) filtration. □

Next we prove that, for the canonical logics, the notions (I) and (II) coincide, if we consider definable filtration. To simplify notation, we work with the unimodal case. Recall that one can build the canonical frame $F_T = (W_T, R_T)$ and model $M_T = (F_T, V_T)$ not only for a (consistent) normal logic, but more generally for a normal theory $T$ (which contains all theorems of $K$ and is
closed under monus ponens and necessitation). Any point \( x \in W_T \) is a consistent (never \( A, \neg A \in x \)) complete (always \( A \in x \) or \( \neg A \in x \)) theory (i.e., closed under modus ponens) containing \( T \). A logic \( L \) is called canonical if \( F_L \models L \). The following well-known fact is useful.

**Lemma 2.8 (Canonical generated submodel)** If \( T \subseteq T' \) are consistent normal theories, then \( M_{T'} \) is a generated submodel of \( M_T \). Similarly for frames.

**Proof.** Assume \( x \in W_{T'}, y \in W_T \), and \( x R_T y \). To prove that \( y \in W_T \), i.e., \( T' \subseteq y \), take any formula \( A \in T' \). By normality \( \Box A \in T' \). Since \( T' \subseteq x \), we have \( \Box A \in x \). By definition of \( R_T \), we obtain \( A \in y \). \( \square \)

A typical example of a normal theory is the theory of a model \( T = \text{Th}(M) \).

For a model \( M = (W, R, V) \), consider the canonical model \( M_T \) of its theory and the canonical mapping \( t \) from \( M \) to \( M_T \) defined, for \( a \in W \), by

\[
t(a) = \text{Th}(M, a) \in W_T.
\]

It is monotonic (\( a R b \Rightarrow t(a) R_T t(b) \)), but in general it is neither surjective, nor a \( p \)-morphism. The next lemma shows what happens to the canonical mapping if we filtrate both \( M \) and \( M_T \) through a finite set of formulas \( \Phi \).

**Lemma 2.9** Under the above conditions, any finite set of formulas \( \Phi \) induces a bijection between the quotient sets \( W/\sim_\Phi \) and \( W_T/\sim_\Phi \) defined, for \( a \in W \), by

\[
f([a]_{\sim_\Phi}) := [t(a)]_{\sim_\Phi}.
\]

**Proof.** This technical proof is put in Appendix, see Lemma A.1. \( \square \)

**Theorem 2.10 (ADF for frames implies ADF for models)** If \( L \) is a canonical logic \( L \), then \( \text{Fr}(L) \) admits definable filtration iff so does \( \text{Mod}(L) \).

**Proof.** (\( \Rightarrow \)) By Lemma 2.7. (\( \Rightarrow \)) IDEA: in order to filtrate a model \( M \models L \), we filtrate the canonical model \( M_T \) of its theory \( T = \text{Th}(M) \) and then use the bijection from Lemma 2.9 to transfer the filtration back to \( M \).

Take a finite \( \text{Sub}-\)closed set of formulas \( \Gamma \) and a model \( M = (W, R, V) \) with \( M \models L \). Its theory \( T = \text{Th}(M) \) contains \( L \), hence \( F_T \) is a generated subframe of \( F_\Gamma \), by Lemma 2.8. Since \( L \) is canonical, we have \( F_L \models L \) and so \( F_T \models L \). Thus, \( M_T \) is a \( \text{Fr}(L) \)-model and, by assumption, we can filtrate it.

Therefore, the model \( M_T \) has a \( \Gamma \)-filtration \( \tilde{M}_T = (W_T, \tilde{R}_T, \tilde{V}_T) \) (through some finite set of formulas \( \Phi \supseteq \Gamma \)) with \( \tilde{F}_T \models L \). By Lemma 2.9, there is a bijection \( f \) between the finite sets \( \tilde{W} = (W/\sim_\Phi) \) and \( \tilde{W}_T = (W_T/\sim_\Phi) \). Now we build a model \( \widehat{M} = (\tilde{W}, \hat{R}, \hat{V}) \) isomorphic to \( \tilde{M}_T \), by putting, for all \( a, b \in W \):

\[
\hat{a} \hat{R} \hat{b} \text{ iff } f(\tilde{a}) \tilde{R}_T f(\tilde{b}); \quad \hat{a} \models p \text{ iff } f(\tilde{a}) \models p, \text{ for all variables } p \in \Gamma.
\]

Since the frames \( \tilde{F} \) and \( \tilde{F}_T \) are isomorphic and \( \tilde{F}_T \models L \), we have \( \tilde{F} \models L \). It remains to prove that \( \hat{M} \) is a \( \Gamma \)-filtration (through \( \Phi \)) of \( M \). Below, we denote \( x = t(\tilde{a}) = \text{Th}(M, a) \) and \( y = t(\tilde{b}) = \text{Th}(M, b) \), so that \( f(\tilde{a}) = \hat{x} \) and \( f(\tilde{b}) = \hat{y} \).

(\( \text{var} \)) Let us check that \( \hat{M}, \hat{a} \models p \) iff \( M, a \models p \), for all \( p \in \Gamma \). We have:

\[
\hat{M}, \hat{a} \models p \iff \hat{M}_T, \hat{x} \models p \iff M_T, x \models p \iff p \in x \iff M, a \models p.
\]
(min) We check that \( R_{\leq}^{\min} \subseteq \tilde{R} \), i.e., \( \forall a, b \in W \ (a R b \Rightarrow \tilde{a} \tilde{R} \tilde{b}) \).
We use the monotonicity of \( t(\cdot) \) and the condition \( (\text{min}) \) for \( \tilde{R}_T \):
\[
a R b \implies t(a) R_T t(b) \iff x R_T y \implies \tilde{x} \tilde{R}_T \tilde{y} \iff \tilde{a} \tilde{R} \tilde{b}.
\]

(max) We check that \( \tilde{R} \subseteq \tilde{R}_T^{\max} \). Assume \( \tilde{a} \tilde{R} \tilde{b} \). Then \( \tilde{x} \tilde{R}_T \tilde{y} \).
By the condition \( (\text{max}) \) for \( \tilde{R}_T \), we have \( \tilde{x} \tilde{(}((\tilde{R}_T)^{\max})\tilde{)} \).
We need to show that \( \tilde{a} \tilde{R}_T^{\max} \tilde{b} \). For any formula \( \square A \in \Gamma \), we have:
\[
M, a \models \square A \iff \square A \in x \iff M_T, x \models \square A \iff M_T, y \models A \iff A \in y \iff M, b \models A.
\]
This completes the proof of the theorem. \( \square \)

3 Logics with the transitive closure modality

In this section, \( L \subseteq \text{Fr}(\square) \) is a normal unimodal logic. Let \( L^{\square} \subseteq \text{Fr}(\square, \Box) \) be the minimal normal logic that extends \( L \) with the following axioms describing the interaction between the modality \( \square \) and the transitive closure modality \( \Box \):

\[
\begin{align*}
&A1) \Box p \rightarrow \square p, \\
&A2) \Box p \rightarrow \square \Box p, \\
&A3) \Box (p \rightarrow \square p) \rightarrow (\square p \rightarrow \Box p).
\end{align*}
\]

Segerberg [13] (see also [14, 15]) and later Kozen and Parikh [10] proved that the logic \( K^{\square} \) (and even PDL) is complete and has the FMP; in other words, it is the logic of the class of finite frames of the form \((W, R, R^+)\); hence it is decidable (more exactly, \( \text{ExpTime}\)-complete). Note that the logic \( K^{\square} \) is not canonical: indeed, canonical logics are compact; however, the set of formulas \( \Gamma = \{ \square^n p \mid n \geq 1 \} \cup \{ \neg \square p \} \) is unsatisfiable (in the class of \( K^{\square}\)-frames), although each finite subset of \( \Gamma \) is satisfiable. So even for simple logics we cannot use canonical models as a method of obtaining completeness.

To the best of our knowledge, up to now, there were no general results on the completeness and decidability for the \( \Box\)-companions of logics other than \( K \). Here we obtain one such result. We give a condition for \( L \) sufficient for the completeness of \( L^{\square} \). The condition is strong enough and guarantees not only the completeness, but the FMP of \( L^{\square} \); this limits the scope of our approach.

For simplicity, in this section we assume that \( L \) is unimodal. The results transfer easily to multi-modal logics. Given a unimodal frame \( F = (W, R) \), we denote \( F^+ = (W, R, R^+) \). Given a class of unimodal frames \( \mathcal{F} \), we denote \( \mathcal{F}^+ = \{ F^+ \mid F \in \mathcal{F} \} \). Similarly for a model \( M^+ \) and a class of models \( \mathcal{M}^+ \).

Lemma 3.1 \((W, R, S) \models \{(A1), (A2), (A3)\}) \iff R_T^+ = S.\)

Proof. This is a known fact. Lemma A.4 (in Appendix) gives more details. \( \square \)

Lemma 3.2 (a) \( \text{Mod}(L) \subseteq \text{Mod}(L^{\square}) \). (b) \( \text{Fr}(L) = \text{Fr}(L^{\square}) \).

Proof. Any frame of the form \((W, R, R^+)\) validates \((A1), (A2), (A3)\). \( \square \)

Lemma 3.3 (Conservativity) For any consistent normal logic \( L \), the logic \( L^{\square} \) is a conservative extension of \( L \): if \( A \in \text{Fr}(\square) \) and \( L \vdash A \), then \( L^{\square} \vdash A \).

Proof. If \( L \not\vdash A \), then \( M_L \not\models A \) and \( M_L^+ \not\models A \). But \( M_L^+ \models L^{\square} \). So \( L^{\square} \not\vdash A \). \( \square \)
3.1 Completeness for logics with the transitive closure modality

In the proof of the main result, we will need to modify a valuation definably. By $A^\sigma$ we denote the application of the valuation to formula $A$.

**Definition 3.4** By a (modally) definable variant of a model $M = (F,V)$ we mean a model of the form $M^\sigma = (F,V^\sigma)$, for some substitution $\sigma$, where the valuation $V^\sigma$ is defined by $V^\sigma(p) = V(p^\sigma)$, for every variable $p$.

In other words, $M^\sigma,x \models p$ iff $M,x \models p^\sigma$. By induction one can easily prove:

**Lemma 3.5** $M^\sigma,x \models A$ if and only if $M,x \models A^\sigma$, for all formulas $A$.

Since a logic is closed under substitutions, we obtain the following fact.

**Lemma 3.6** If $L$ is a logic and $M \models L$, then $M^\sigma \models L$, for any substitution $\sigma$.

Recall that the axioms (A1) and (A2) are canonical. In particular, they are valid in the canonical frame of the logic $L^\mathbb{B}$. For (A3), this is not the case. However, in order to obtain our completeness result, we do not necessarily need its validity in the canonical frame. Instead, we only need that after taking a definable filtration of a model $M \models L^\mathbb{B}$ (in particular, of $M_2$, or any other model in which all substitution instances of (A3) hold) into a finite model $\hat{M}$, the resulting frame $\hat{F}$ satisfies the inclusion $\hat{S} \subseteq (\hat{R})^+$. The following key lemma states that this is indeed the case for the minimal filtration.

Let us write $M \models A^*$ if we have $M \models A^\sigma$ for all substitutions $\sigma$.

**Lemma 3.7 (Induction axiom and minimal filtration)**

Let $M = (W,R,S,V) \models (A3)^*$ and let $\Phi \subseteq \text{Fm}$ be finite. Then $S_{\min}^\Phi \subseteq (R_{\min}^\Phi)^+$.

**Proof.** Denote $r := R_{\min}^\Phi$ and $s := S_{\min}^\Phi$. To prove $s \subseteq r^+$, assume $\hat{x} s \hat{y}$. By definition of $R_{\min}^\Phi$, without loss of generality, $x Sy$. Consider $Y := r^+(\hat{x}) \subseteq W$. We need to show that $\hat{y} \in Y$.

Since $\Phi$ is finite, every $\sim_\Phi$-equivalence class $\hat{z} \subseteq W$ is a definable (by some formula) subset of $W$. Since $Y$ is a finite collection of such subsets, their union $\bigcup Y \subseteq W$ is also a definable subset of $W$. So, there is a formula $\varphi$ such that, for all $z \in W$, we have: $M,z \models \varphi$ iff $z \in \bigcup Y$ iff $\hat{z} \in Y$.

Firstly, $M \models \varphi \rightarrow \Box \varphi$. Indeed, if $M,a \models \varphi$, then $\hat{a} \in Y$. Hence $\hat{a} \in Y$ and $M,b \models \varphi$. Therefore, $M \models \Box (\varphi \rightarrow \Box \varphi)$.

Secondly, $M,x \models \Box \varphi$. Indeed, if $M,x \models Rz$ then $\hat{x} \in Y$, hence $\hat{z} \in Y$ and $M,z \models \varphi$.

Now we use that $M \models \Box (\varphi \rightarrow \Box \varphi) \rightarrow (\Box \varphi \rightarrow \Box \varphi)$. Thus, $M,x \models \Box \varphi$. Now recall that $x Sy$. Then $M,y \models \varphi$, hence $\hat{y} \in Y$. \qed

In Appendix (Lemma A.5) we strengthen the above lemma.

Now we come to the main technical tool of our paper.

**Theorem 3.8 (Transfer of ADF to logics with transitive closure)**

If the class $\text{Mod}(L)$ admits definable filtration, then so does the class $\text{Mod}(L^\mathbb{B})$.

**Proof.** Idea: in order to filtrate a model $M = (W,R,S,V) \models L^\mathbb{B}$ for $\Gamma \subseteq \text{Fm}(\Box,\Pi)$, we build a special set of formulas $\Delta \subseteq \text{Fm}(\Box)$ and $\Delta$-filtrate the...
reduce $N = (W, R, V) \models L$ of $M$ into a finite model $\hat{N} = (\hat{W}, \hat{R}, \hat{V}) \models L$. Then we show that $\hat{N}^+ = (\hat{W}, \hat{R}, (\hat{R})^+, \hat{V}) \models L^\equiv$ is a $\Gamma$-filtration of $M$. A subtlety is that we first take a modified valuation $V^\sigma$ and actually filtrate $N^\sigma$, not $N$.

Formally: take a model $M = (W, R, S, V)$ such that $M \models L^\equiv$ and a finite Sub-closed set of formulas $\Gamma \subseteq \text{Fm}(\square, \Box)$. For each formula $\varphi \in \Gamma$, fix a fresh (not occurring in $\Gamma$) variable $q_\varphi$. Consider a substitution $\sigma : \text{Var} \rightarrow \text{Fm}(\square, \Box)$ defined by $\sigma(q_\varphi) = \varphi$ for all $\varphi \in \Gamma$ and $\sigma(p) = p$ for all other variables $p$. In the definable variant $M^\sigma = (W, R, S, V^\sigma)$ of $M$ we have: $M^\sigma \models q_\varphi \leftrightarrow \varphi$ for all $\varphi \in \Gamma$ (since $\varphi^\sigma = \varphi$), hence $M^\sigma \models \Box q_\varphi \leftrightarrow \Box \varphi$ and even $M^\sigma \models A \leftrightarrow A^\sigma$, for any formula $A \in \text{Fm}(\Box)$. We also have $M^\sigma \models L^\equiv$ by Lemma 3.6.

Now consider the reduct $N^\sigma = (W, R, V^\sigma)$ of $M^\sigma$. Clearly, $N^\sigma \models L$. Consider the following finite Sub-closed set of $\Box$-formulas:

$\Delta := \{ q_\varphi, \Box q_\varphi \mid \varphi \in \Gamma \} \subseteq \text{Fm}(\Box)$.

$\text{Mod}(L)$ admits definable filtration, so there is a $\Delta$-filtration $\hat{N}^\sigma = (\hat{W}, \hat{R}, \hat{V}^\sigma)$ of $N^\sigma$ through some finite set $\Phi \subseteq \text{Fm}(\Box)$ with $\Delta \subseteq \Phi$ such that $\hat{N}^\sigma \models L$. Let us change $\hat{V}^\sigma$ on the variables $p \in \text{Var}(\Gamma)$ by putting:

$$\hat{x} \models p \Rightarrow \hat{x} \models q_p.$$  

Remark. Since we will have several models on the same set of points, we need a more subtle notation. In particular, we have $\hat{W} = W/\sim_{\hat{\Phi}}$, this notation shows explicitly in which models we consider the $\sim_{\hat{\Phi}}$-equivalence of points.

It remains to prove the following statement.

Claim. The model $\hat{M} := (\hat{W}, \hat{R}, (\hat{R})^+, \hat{V}^\sigma)$ is a $\Gamma$-filtration (through $\Phi^\sigma$) of $M$.

(1) We show that $\hat{W} = W/\sim_{\hat{\Phi}}$. For any $x \in W$ and $A \in \text{Fm}(\Box)$, we have:

$N^\sigma, x \models A \iff M^\sigma, x \models A \iff M, x \models A^\sigma.$

Therefore, for all $x, y \in W$, we have: $(x \sim_{\hat{\Phi}}^\sigma y) \iff (x \sim_{\hat{\Phi}}^\sigma y)$.

This allows us to introduce a simpler notation $\sim$ for $\sim_{\hat{\Phi}}^\sigma$ and $\sim_{M}^\sigma$.

Since $\hat{N}^\sigma$ is a $\Delta$-filtration of $N^\sigma$, we have: $R_{\text{min}}^\sigma \subseteq \hat{R} \subseteq R_{\text{max}}^\sigma$. (*).

(2) The relation $\sim$ respects $\Gamma$. Indeed, $\Phi \supseteq \Gamma' \supseteq \{ q_\varphi \mid \varphi \in \Gamma \}$, hence $\Phi^\sigma \supseteq \Gamma$.

(3) Let us show that $M, x \models p \iff \hat{M}, \hat{x} \models p$, for all $x \in W$ and $p \in \text{Var}(\Gamma)$.

$M, x \models p \iff M^\sigma, x \models q_p \iff N^\sigma, x \models q_p$ (§)

$\hat{M}, \hat{x} \models p \iff \hat{M}, \hat{x} \models q_p \iff \hat{N}^\sigma, \hat{x} \models q_p$

(4) $R_{\text{min}}^\sigma \subseteq \hat{R}$. This holds by (*).
(5) \( S_{\min}^{\text{min}} \subseteq \hat{S} \), where \( \hat{S} := (\hat{R})^+ \). Using (4) and Lemma 3.7, we obtain:
\[
S_{\min}^{\text{min}} \subseteq (R_{\min})^+ \subseteq (\hat{R})^+ = \hat{S}.
\]
(6) \( \hat{R} \subseteq R_{\min}^{\max} \). Due to (5), it suffices to prove that \( R_{\min}^{\max} \subseteq R_{\min}^{\max} \).
Assume that \( \hat{x} (R_{\min}^{\max}) \hat{y} \). To show that \( \hat{x} (R_{\min}^{\max}) \hat{y} \), take any \( \square \varphi \in \Gamma \). Then:
\[
M, x \models \square \varphi \quad \Rightarrow \quad M^\sigma, x \models \square q_\varphi \quad \Rightarrow \quad N^\sigma, x \models \square q_\varphi
\]
\[
M, y \models \varphi \quad \Rightarrow \quad M^\sigma, y \models q_\varphi \quad \Rightarrow \quad N^\sigma, y \models q_\varphi
\]
We used: (a) Lemma 3.5; (b) \( q_\varphi, \square q_\varphi \in \text{Fm} (\square) \); (c) \( \square q_\varphi \in \Delta \) and \( \hat{x} (R_{\min}^{\max}) \hat{y} \).

(7) \( \hat{S} \subseteq S_{\max}^{\max} \). Due to (5), it suffices to prove that \( (R_{\min}^{\max})^+ \subseteq S_{\min}^{\max} \).
Let us denote \( r := R_{\min}^{\max} \) and \( s := S_{\min}^{\max} \). In order to prove that \( r^+ \subseteq s \), it suffices to prove two inclusions: \( r \subseteq s \) and \( r \circ s \subseteq s \).

(7a) Proof of \( r \subseteq s \). We will use the axiom \( \square p \rightarrow \square p \).
Assume \( \hat{x} R_{\min}^{\max} \hat{y} \). To prove that \( \hat{x} S_{\min}^{\max} \hat{y} \), take any \( \exists \varphi \in \Gamma \). Then:
\[
M, x \models \exists \varphi \quad \Rightarrow \quad M, x \models \square \varphi \quad \Rightarrow \quad M^\sigma, x \models \square q_\varphi \quad \Rightarrow \quad N^\sigma, x \models \square q_\varphi \quad \Rightarrow \quad N^\sigma, y \models q_\varphi
\]
(d) holds since \( M \models \exists \varphi \rightarrow \square \varphi \). The explanations of \( a, b, c \) are the same.

(7b) Proof of \( r \circ s \subseteq s \). We will use the axiom \( \exists \varphi \rightarrow \square \exists \varphi \).
Assume \( \hat{x} R_{\min}^{\max} \hat{y} S_{\min}^{\max} \hat{z} \). To prove that \( \hat{x} S_{\min}^{\max} \hat{z} \), take any \( \exists \varphi \in \Gamma \). Then:
\[
M, x \models \exists \varphi \quad \Rightarrow \quad M, x \models \exists \exists \varphi \quad \Rightarrow \quad M^\sigma, x \models \exists q_\exists \varphi \quad \Rightarrow \quad N^\sigma, x \models \exists q_\exists \varphi
\]
\[
M, y \models \varphi \quad \Rightarrow \quad M^\sigma, y \models q_\varphi \quad \Rightarrow \quad N^\sigma, y \models q_\varphi
\]
We used: (e) \( M \models \exists \varphi \rightarrow \exists \exists \varphi \); (a) Lemma 3.5; (b) \( \square q_\exists \varphi \in \text{Fm} (\square) \);
(c) \( \square q_\exists \varphi \in \Delta \) and \( \hat{x} (R_{\min}^{\max}) \hat{y} ; (g) \) \( \exists \varphi \in \Gamma \) and \( \hat{y} S_{\min}^{\max} \hat{z} \).

This completes the proof of theorem. \( \square \)

Note that in (7a) and (7b) we proved inclusions that involve maximal relations, and these inclusions resemble the axioms (A1) and (A2). This is not a coincidence. In Lemma 4.3 of our paper [9], we already made this observation for any right-linear grammar axiom and both (A1) and (A2) are right-linear.

Let us summarize the main result on logics with transitive closure. We give two versions. The first uses the (rather unusual) property that the models of \( L \) admit filtration. The second uses the filtration of frames of \( L \), but has an additional requirement of canonicity.

Theorem 3.9 (Main result, version 1) Assume that the class of models \( \text{Mod}(L) \) of a logic \( L \) admits definable filtration. Then:

1. the class of models \( \text{Mod}(L^{\exists}) \) admits definable filtration;
2. hence the logic \( L^{\exists} \) has the finite model property;
3. hence the logic \( L^{\exists} \) is Kripke complete.
The above theorem allows us to ‘iterate’ the ADF property, see Section 4.1.

**Theorem 3.10 (Main result, version 2)** Assume that a logic \( L \) is canonical and the class of its frames \( \text{Fr}(L) \) admits definable filtration. Then:

1. the class \( \text{Mod}(L^\Box) \) admits definable filtration;
2. hence the logic \( L^\Box \) has the finite model property;
3. hence the logic \( L^\Box \) is Kripke complete.

**4 PDLization of logics that admit filtration**

### 4.1 Main corollary

Now we apply Theorem 3.9 to show that if \( \text{Mod}(L) \) admits definable filtrations, then the following PDL-like expansions of \( L \) have the finite model property.

**Definition 4.1** For an alphabet \( \Sigma \), let \( \Sigma^\sharp = \Sigma \cup \{(e \circ f), (e \lor f), e^+ | e, f \in \Sigma\} \), assuming that the added symbols are not in \( \Sigma \). Put \( \Sigma^{(0)} = \Sigma, \Sigma^{(n+1)} = (\Sigma^{(n)})^\sharp \).

For a frame \( F = (W, (R_e)_{e \in \Sigma}) \), put \( F^\sharp = (W; (R_e)_{e \in \Sigma^\sharp}) \), where for \( e, c \in \Sigma, \)

\[
R_{ec} = R_e \circ R_c, \quad R_{ec^\dagger} = R_e \cup R_c, \quad R_e^+ = (R_e)^+.
\]

Put \( F^{(0)} = F, F^{(n+1)} = (F^{(n)})^\sharp \).

For a model \( M = (F, V) \), we put \( M^\sharp = (F^\sharp, V) \) and \( M^{(n)} = (F^{(n)}, V) \).

For a logic \( L \) over \( \Sigma \), let \( L^\sharp \) be the smallest (normal) logic over \( \Sigma^\sharp \) that contains \( L \) and the following PDL-like axioms, for all \( e, c \in \Sigma: \)

\[
[e \lor c]p \leftrightarrow [e]p \land [c]p,
[e \circ c]p \leftrightarrow [e][c]p,
[e^+]p \rightarrow [e]p, \quad (e^+)p \rightarrow [e][e^+]p, \quad (e^+)p \rightarrow (c)p \rightarrow (e^+)p).
\]

We put \( L^{(0)} = L, L^{(n+1)} = (L^{(n)})^\sharp \).

The following is a simple analogue of Lemma 3.2.

**Lemma 4.2** (a) \( M \models L \) implies \( M^\sharp \models L^\sharp \). \hspace{1cm} (b) \( F \models L \iff F^\sharp \models L^\sharp \).

By an easy induction on \( n \), we obtain

**Proposition 4.3** For a frame \( F \) and \( n < \omega, F \models L \iff F^{(n)} \models L^{(n)} \).

**Proposition 4.4** For a logic \( L \) and \( n < \omega, L^{(n)} \) is conservative over \( L \).

**Proof.** As in Lemma 3.3, using \( M^{(n)} \) instead of \( M^\Box \) and Lemma 4.2(a). \( \Box \)

**Lemma 4.5** Let \( L \) be a logic over \( \Sigma, e, c \in \Sigma \). Let \( L_1 \) and \( L_2 \) be the logics over \( \Sigma \cup \{g\} \), where \( g \notin \Sigma \), such that

\[
L_1 \text{ extends } L \text{ with the axiom } [g]p \leftrightarrow [e]p \land [c]p,
L_2 \text{ extends } L \text{ with the axiom } [g]p \leftrightarrow [e][c]p.
\]

If \( \text{Mod}(L) \) admits definable filtration, then so do \( \text{Mod}(L_1) \) and \( \text{Mod}(L_2) \).

**Proof.** Straightforward. Details can be reconstructed from the proof of Lemma 2.3 in [9], which is the analog of our lemma for the classes of frames. \( \Box \)

**Theorem 4.6** Let \( L \) be a logic over a finite alphabet \( \Sigma \). If the class of its models \( \text{Mod}(L) \) admits definable filtration, then, for every \( n < \omega \), we have:
(i) $\text{Mod}(L^{(n)})$ admits definable filtration.

(ii) $L^{(n)}$ has the finite model property; a fortiori, $L^{(n)}$ is Kripke complete.

(iii) If $L$ is finitely axiomatizable, then $L^{(n)}$ is decidable.

(iv) If the class of finite frames of $L$ is decidable, then $L^{(n)}$ is co-recursively enumerable.

Proof. (i) By Theorem 3.8 and Lemma 4.5, if $\text{Mod}(L)$ admits definable filtration, then so does $\text{Mod}(L^*)$. So, (i) follows by induction on $n$.

(ii) By Lemma 2.6.

(iii) Note that if $L$ is finitely axiomatizable, then so is $L^{(n)}$. The claim then follows from Harrop’s Theorem (see Section 1).

(iv) If the class of finite frames of $L$ is decidable, then the class of finite frames of $L^{(n)}$ is decidable, too. In this case $L^{(n)}$ is co-recursively enumerable, since $L^{(n)}$ is the logic of its finite frames. $\Box$

4.2 Expansions of locally finite logics

For $k \leq \omega$, a $k$-formula is a formula in proposition letters $p_i$, $i < k$. Recall that a logic is called locally finite (or locally tabular), if, for every $k < \omega$, there exist only finitely many $k$-formulas up to the equivalence in $L$.

Theorem 4.7 If $L$ is locally tabular, then $\text{Mod}(L)$ admits definable filtration.

Proof. Let $M$ be a model, $M \models L$, $\Gamma$ a finite $\text{Sub}$-closed set of formulas. For some $k < \omega$, every formula in $\Gamma$ is a $k$-formula. Let $\Phi$ be the set of all $k$-formulas. Consider the maximal filtration $\widehat{M}$ of $M$ through $\Phi$ (in [18], such filtrations are called canonical). Since $L$ is locally tabular, $\widehat{M}$ is finite. In this case $\widehat{M}$ is a p-morphic image of $M$ (for details, see [18, Proposition 2.32]). Hence $\widehat{M} \models L$, as required. $\Box$

Corollary 4.8 If $L$ is locally tabular, then $L^{(n)}$ has the FMP, for every $n < \omega$.

4.3 Agents that admit filtration

Here we consider a special kind of definable filtration, called strict filtration.

Definition 4.9 If, in terms of Definition 2.1, $\sim = \sim_\Gamma$, then we call the filtration $\widehat{M}$ strict. The corresponding notions “a class of frames (or models) admits strict filtration” are introduced in the obvious way.

Strict filtration is the most standard variant of filtration; it is well-known that the classes of frames of the logics $K, T, K4, S4, S5$ admit strict filtration (for the logics $K$ and $T$, even the minimal strict filtration works; for $K4, S4, S5$, strict filtration is obtained by taking the transitive closure of the minimal filtered relation [12]).

Let us recall the notion of the fusion of logics. Let $L_1, \ldots, L_k$ be logics over finite alphabets $\Sigma_1, \ldots, \Sigma_k$. Without loss of generality we assume that these alphabets are disjoint. The fusion $L_1 \ast \ldots \ast L_k$ of these logics is the smallest normal logic over the alphabet $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_k$ that contains $L_1 \cup \ldots \cup L_k$. 
It is well-known that the fusion operation preserves Kripke completeness, the finite model property, and decidability [11]. We observe that it also preserves the admits strict filtration property.

**Theorem 4.10 (Fusion and strict filtration)** If classes of frames \( Fr(L_i) \), \( 1 \leq i \leq k \), admit strict filtration, then \( Fr(L_1 \ast \ldots \ast L_k) \) admits strict filtration.

**Proof.** The idea is the same as in the proof of Theorem 3.8. To simplify notation, we consider the case of unimodal logics. Let \( L = L_1 \ast \ldots \ast L_k, M = (F, V) \) be a model on an \( L \)-frame \( F = (W, R_1, \ldots, R_k) \), \( \Gamma \subseteq \text{FM}(\Box_i, \ldots, \Box_k) \) be finite and Sub-closed. For \( \varphi \in \Gamma \), we take fresh variables \( q_{\varphi} \), and consider a model \( M' = (F, V') \) such that

\[
M, x \models \varphi \text{ iff } M', x \models \varphi \text{ iff } M', x \models q_{\varphi}
\]

for all \( x \) in \( M \). For \( 1 \leq i \leq k \), we put:

\[
\Gamma_i = \{ q_{\varphi} \mid \varphi \in \Gamma \} \cup \{ \Box q_{\varphi} \mid \Box_i \varphi \in \Gamma \};
\]

remark that \( \Gamma_i \subseteq \text{FM}(\Box) \). Let \( \sim_i \) be the equivalence induced by \( \Gamma_i \) in the model \( M_i = (W, R_i, V') \), and \( \sim_\Gamma \) the equivalence induced by \( \Gamma \) in \( M \). Observe that

\[
M_i, x \models \Box q_{\varphi} \text{ iff } M, x \models \Box_i \varphi \text{ for all } \varphi \in \Gamma.
\]

Therefore, one can see that \( \sim_i = \sim_\Gamma \) for all \( i \). Put \( \hat{W} = W/\sim_\Gamma \). For each \( i \), there exists a filtration \( \hat{M}_i = (\hat{W}, \hat{R}_i, \hat{V}_i) \) of \( M_i \) through \( \Gamma_i \) such that \( (\hat{W}, \hat{R}_i) \models L_i \).

The valuations \( \hat{V}_i \) coincide on the variables \( q_{\varphi} \). W.l.o.g., they also coincide on other variables (since they do no occur in \( \Gamma_i \)), and that \( \hat{M}, \hat{x} \models p \text{ if } M, x \models p \) for each variable \( p \in \Gamma \). The resulting valuation on \( \hat{W} \) is denoted by \( \hat{V} \).

Consider the model \( M = (\hat{W}, \hat{R}_1, \ldots, \hat{R}_k, \hat{V}) \). Note that its frame validates the fusion \( L \). We claim that \( \hat{M} \) is a filtration of \( M \) through \( \Gamma \). Clearly, \( \hat{R}_i \) contains the \( i \)-th minimal filtered relation. To check that \( \hat{R}_i \) is contained in the \( i \)-th maximal filtered relation, assume that \( \hat{x}_i \hat{R}_i \hat{y}_i, M, x \models \Box \varphi_i, \text{ and } \Box_i \varphi_i \in \Gamma \). Then \( \hat{M}_i, \hat{x}_i \models \Box q_{\varphi_i} \text{ by } (*) \). Since \( \hat{M}_i \) is a filtration of \( M_i \) through \( \Gamma_i \), and \( \Box q_{\varphi_i} \in \Gamma_i \), we have \( \hat{M}_i, \hat{y}_i \models q_{\varphi_i} \). By Filtration lemma, \( \hat{M}_i, \hat{y}_i \models q_{\varphi_i} \). Hence, \( M', y \models q_{\varphi_i} \) and we conclude that \( M, y \models \varphi_i \), as required.

**Theorem 4.11** Let \( L_1, \ldots, L_k \) be canonical logics and their classes of frames \( Fr(L_i) \), \( 1 \leq i \leq k \), admit strict filtration. Then, for every \( n < \omega \), the logic \( (L_1 \ast \ldots \ast L_k)^{(n)} \) has the finite model property.

**Proof.** The fusion \( L = L_1 \ast \ldots \ast L_k \) is canonical. By Theorem 4.10, the class \( Fr(L) \) admits strict filtration. Hence \( Mod(L) \) admits definable filtration, by Theorem 2.10. Finally, \( (L)^{(n)} \) has the FMP, by Theorem 4.6.

### 4.4 A class of formulas that admit strict filtration

We present a collection of modal formulas that admit strict (and so definable) filtration. The obvious candidates are modal formulas whose first-order equivalents belong to a certain FO fragment we call MFP.\(^6\) We define it inductively as the minimal set of FO formulas satisfying the following conditions:

---

\(^6\) The abbreviation stems from "preserved under minimal filtration".
• if \( x \) and \( y \) are variables, \( R \) is a binary relation symbol, then \( R(x,y) \in \text{MFP} \) and \( x = y \in \text{MFP} \);
• if \( A \) and \( B \) are in \( \text{MFP} \), then \( (A \land B) \) and \( (A \lor B) \) are in \( \text{MFP} \);
• if \( A \in \text{MFP} \), and \( v \) is a variable, then \( \forall v A \) and \( \exists v A \) are in \( \text{MFP} \);
• if \( x \) and \( y \) are variables, \( R \) is a binary relation symbol, and \( A \in \text{MFP} \), then \( \forall x \forall y (R(x,y) \rightarrow A) \) and \( \forall x \forall y (x = y \rightarrow A) \) are in \( \text{MFP} \).

This definition is the restriction of the fragment \( \text{POS} + \forall \text{G} \) from [6] to the first-order language with only binary predicates. Examples of \( \text{MFP} \)-sentences are reflexivity \( \forall x R(x,x) \), symmetry \( \forall x \forall y (R(x,y) \rightarrow R(y,x)) \), and density \( \forall x \forall y (R(x,y) \rightarrow \exists z (R(x,z) \land R(z,y))) \), but not transitivity.

\( \text{FO} \) counterparts of minimal filtrations are strong onto homomorphisms.

**Definition 4.12** Given two frames \( F = (W,R) \) and \( F' = (W',R') \), a map \( h: W \rightarrow W' \) is a strong onto homomorphism if the following conditions hold:

1. \( h \) is onto;
2. for all \( x,y \in W \), if \( x \sim y \) then \( h(x) \sim h(y) \) (monotonicity);
3. for all \( x',y' \in W' \), if \( x' \sim y' \), then there exist \( x,y \in W \) such that \( h(x) = x' \), \( h(y) = y' \), and \( x \sim y \) (weak lifting).

Note that a strong homomorphism \( h \) from \( F \) onto \( F' \) induces an equivalence \( \sim \) on \( W \) defined by \( x \sim y \) iff \( h(x) = h(y) \), and then \( F' \) is isomorphic to the minimal filtrated frame \( F_{\text{min}} = (W/\sim, R_{\text{min}}) \). Conversely, if a model \( M \) is a minimal filtration of a model \( M' \), then the filtration map \( h(x) = \hat{x} \) (this term is also used in [18, Definition 2.27]) is a strong homomorphism from \( F \) onto \( F' \).

In [6, Proposition 5.2], it was shown that \( \text{MFP} \)-formulas are preserved under strong onto homomorphisms. Moreover, any \( \text{FO} \) formula that is preserved under strong onto homomorphisms is equivalent to some \( \text{MFP} \)-formula [8].

**Definition 4.13** A modal formula \( \varphi \) is called a modal \( \text{MFP} \)-formula if it has a \( \text{FO} \) equivalent (on frames) from \( \text{MFP} \).

Expressions of the the form \( p \land \square q \rightarrow \psi \), where \( \psi \) is a positive modal formula, are typical examples of modal \( \text{MFP} \)-formulas. Note that these examples are Sahlqvist formulas, and hence canonical.

**Theorem 4.14** For any set \( \Phi \) of modal \( \text{MFP} \)-formulas over a finite alphabet \( \Sigma \), the class of frames \( \text{Fr}(\mathbf{K}_\Sigma + \Phi) \) admits strict filtration.

**Proof.** Denote \( \mathcal{F} = \text{Fr}(\mathbf{K}_\Sigma + \Phi) \). Let \( M = (F,V) \) be an \( \mathcal{F} \)-model and \( \Gamma \) a finite Sub-closed set of formulas. Take the minimal filtration \( \hat{M} = (\hat{F}, \hat{V}) \) of \( M \) through \( \Gamma \); note that this filtration is strict. Then the filtration map \( h(x) = \hat{x} \) is a strong homomorphism from \( F \) onto \( \hat{F} \). Since the set \( \Phi^* \) of \( \text{MFP} \) first-order equivalents of \( \Phi \) is true in \( F \), it is also true in \( \hat{F} \). Hence \( \hat{M} \) is an \( \mathcal{F} \)-model. \( \square \)

From Theorem 4.11, we obtain:

**Corollary 4.15** Let each \( L_1, \ldots, L_k \) be any of the logics \( \mathbf{K}, \mathbf{T}, \mathbf{K}4, \mathbf{S}4, \mathbf{S}5 \), or a logic axiomatized by canonical modal \( \text{MFP} \)-formulas. Then, for any \( n < \omega \),
the logic \((L_1 \ast \ldots \ast L_k)^{(n)}\) has the finite model property.

5 Conclusions and further research

We proved that if \(L\) is a canonical logic, and the class of its models \(\text{Fr}(L)\) admits definable filtration, then the logic \(L^\square\) is Kripke complete and, moreover, has the FMP (and is decidable, if \(L\) was finitely axiomatizable). The first problem we pose is whether we can weaken the pre-conditions and obtain completeness of \(L^\square\) without obtaining the FMP.

**Problem 1.** If \(L\) is canonical, then is \(L^\square\) Kripke complete?

The second problem deals with the possible weakening the ‘canonicity’ condition to just ‘completeness’ in our result.

**Problem 2.** If the logic \(L\) is complete and the class of its frames \(\text{Fr}(L)\) admits definable filtration, then does the same holds for the logic \(L^\square\)?

The following questions have a more technical character.

**Question 1:** If we replace in the logic \(K^\square\) the axiom \((A2) \Diamond p \to \square \Diamond p\) with \((A2') \Diamond p \to \square \Diamond p\), then will the resulting logic be complete? Clearly, the frames of this logic are the same as for \(K^\square\), so if it is complete, it must coincide with \(K^\square\). More concrete, does the logic with \((A2')\) derive \((A2)\)?

**Question 2:** Is the logic \(K.2^\square\) Kripke complete? (We conjecture: yes.)

Recall that the logic \(K.2\) extends \(K\) with the formula \(\Diamond \square p \to \square \Diamond p\). It is canonical and hence complete with respect to the class of frames \((W, R)\) that satisfy the following first-order convergence (or Church–Rosser) condition:

\[
\forall x, y, z (x R y \land x R z \Rightarrow \exists w (y R w \land z R w)).
\]

Our main result is not applicable to this logic, since the class of its frames \(\text{Fr}(K.2)\) does not admit filtration, as we shown in [9, Theorem 5.4].

One can easily see that if the relation \(R\) is convergent, then so is its transitive closure \(R^+\). So it is natural to attempt to derive the formula \(\Diamond \Diamond p \to \square \Diamond p\) in \(K.2^\square\). We succeeded in deriving it (see Lemma A.3 in Appendix).

**Question 3.** In Lemma A.5, the bimodal formula \(\Diamond (p \to \square p) \to (\square p \to \square p)\) is shown to have the following property crucial for our main result: if all its substitution instances are true in some model \(M = (F, V)\), then this formula is valid on the frame of every definable minimal filtration: if \(M \models A^*\) then \(F_{\Phi}^\text{min} \models A\), for any finite set of formulas \(\Phi\). Are there any other examples of such formulas? How is this property related to the admissibility of filtration, completeness, decidability of a logic axiomatized by such formulas?

References

[1] Blackburn, P., M. de Rijke and Y. Venema, “Modal Logic.” Cambridge Tracts in Theoretical Computer Science **53**, Cambridge University Press, 2002.
Appendix

A.1 On filtration of the canonical model of a theory of a model

Lemma A.1 (Filtration and canonical mapping) Let $M = (W, R, V)$ be a model, $M_T = (W_T, R_T, V_T)$ the canonical model of its theory $T = \text{Th}(M)$, and $t: M \to M_T$ the canonical mapping: $t(a) = \text{Th}(M, a) \in W_T$, for $a \in W$.

Then, for any finite set of formulas $\Phi$, we have a bijection between the (finite) quotient sets $W/\sim_\Phi$ and $W_T/\sim_\Phi$ defined, for $a \in W$, by

$$f([a]_{\sim_\Phi}) := [t(a)]_{\sim_\Phi}.$$  

Proof. We denote $\hat{a} := [a]_{\sim_\Phi}$. Note that $\hat{x} = \hat{y}$ iff $x \cap \Phi = y \cap \Phi$, for all $x, y \in W_T$. Hence, by definition of $f$, for all $a \in W$ and $x \in W_T$, we have

$$f(\hat{a}) = \hat{x} \iff t(a) \cap \Phi = x \cap \Phi.$$  

First, let us show that $f$ is well-defined and injective: for all $a, b \in W$:

$$\hat{a} = \hat{b} \iff a \sim \Phi b \iff \text{Th}(M, a) \cap \Phi = \text{Th}(M, b) \cap \Phi \iff [t(a)]_{\sim_\Phi} = [t(b)]_{\sim_\Phi}.$$  

To prove that $f$ is surjective, take any $\hat{x} \in (W_T/\sim_\Phi)$. Denote $A := \bigwedge (x \cap \Phi')$, where $\Phi' = \Phi \cup \{\neg B \mid B \in \Phi\}$. Clearly, $A \in x$. Now $M \not\models \neg A$, for otherwise $\neg A \in \text{Th}(M) = T \subseteq x$ and $x$ is inconsistent.

Thus, $A$ is satisfiable in $M$, so $M, a \models A$ for some $a \in W$. We claim that $f(\hat{a}) = \hat{x}$, i.e., for all $B \in \Phi$, we have $M, a \models B$ iff $B \in x$. If $B \in x$, then $B \in (x \cap \Phi')$, so $M, a \models B$. If $B \notin x$, then $\neg B \in (x \cap \Phi')$, so $M, a \models \neg B$. \qed

A.2 On differentiated filtration

Lemma A.2 Assume that $\text{Mod}(L)$ admits (definable) filtration. Then for every finite sub-closed set of formulas $\Gamma$ and every model $M \in \text{Mod}(L)$, there exists a (definable) $\Gamma$-filtration $\hat{M} \in M$ of $M$ that is a differentiated model.

Proof. Idea: first, build a $\Gamma$-filtration $M_1$ of $M$, then a $\text{Fm}$-filtration $M_2$ of $M_1$; finally, build a differentiated filtration $\hat{M}$ of $M$ that is isomorphic to $M_2$.

Formally, let $M = (W, R, V)$, $M \models L$, and let $\Gamma$ be as stated above.

1. Since $\text{Mod}(L)$ admits filtration, there is a $\Gamma$-filtration $M_1 = (W_1, R_1, V_1)$ of $M$ with $M_1 \models L$. So, $W_1 = W/\sim$ for some equivalence relation $\sim$ of finite index, $\sim$ respects $\Gamma$, $R_1^\text{min} \subseteq R_1 \subseteq R_1^\text{max}$, $V_1$ is defined canonically on $\Var(\Gamma)$.

2. Let $M_2 = (W_2, R_2, V_2)$ be a filtration of $M_1$ through the set of all formulas. So, $W_2 = W_1/\equiv$, where $\equiv$ is the modal equivalence relation; $V_2$ is canonical on all variables. By the Filtration lemma 2.2, $M_1 \equiv M_2$, so $M_2 \models L$.

3. Now we build a model $\hat{M} = (\hat{W}, \hat{R}, \hat{V})$ isomorphic to $M_2$ as follows. Put $\hat{W} := W/\sim$, where, for all $x, y \in W$, we define an equivalence relation $\sim$ by

$$x \sim y \overset{\text{def}}{\iff} (M_1, [x]_\sim) \equiv (M_1, [y]_\sim) \iff [[x]_\equiv]_\equiv = [[y]_\equiv]_\equiv.$$  

Claim 1. The function $h([x]_\equiv) = [[x]_\equiv]$ is a bijection between $\hat{W}$ and $W_2$.

Proof. Easy. This does not rely on the fact that $\equiv$ and $\equiv$ are of finite index.

---

7 In fact, if a filtration through the set of all formulas is finite, then it is unique, i.e., the minimal and the maximal relations coincide. But here we do not need this fact.
From now on, we denote $\hat{x} = [x]_\sim$.

**Claim 2.** The equivalence relation $\sim$ on $W$ respects $\Gamma$: if $x \sim y$, then $x \sim \Gamma y$.

**Proof.** If $x, y \in W$ and $x \sim y$ then, by the Filtration lemma 2.2, we have:

$$(M, x) \sim \Gamma (M_1, [x]_{\approx}) \sim_{Fm} (M_1, [y]_{\approx}) \sim \Gamma (M, y).$$

Using the bijection $h$, we transfer $R_2$ and $V_2$ to $\hat{M}$ in the obvious way:

$$(\hat{x} \hat{R} \hat{y} \iff h(\hat{x}) R_2 h(\hat{y}); \quad \hat{x} \models q \iff M_2, h(\hat{x}) \models q, \text{ for all } q \in \text{Var}.)$$

Since the models $\hat{M}$ and $M_2$ are isomorphic, we have $\hat{M} = L$.

**Claim 3.** $\hat{V}$ is canonical on each $p \in \text{Var}(\Gamma)$: $M, x \models p \iff \hat{M}, \hat{x} \models p$.

**Proof.** Indeed: $(M, x) \sim \Gamma (M_1, [x]_{\approx}) \sim_{Fm} (M_2, [x]_{\approx}) \sim_{\text{Var}} (\hat{M}, \hat{x})$.

**Claim 4.** The inclusions $R_{\text{min}}^{\sim} \subseteq \hat{R} \subseteq R_{\text{max}}^{\sim}$ hold.

**Proof.** (min) Clearly, $x R y \Rightarrow [x]_{\approx} = R_1 [y]_{\approx} \Rightarrow [x]_{\approx} = R_2 [y]_{\approx} \Rightarrow \hat{x} \hat{R} \hat{y}$.

(max) If $\hat{x} \hat{R} \hat{y}$, then $[x]_{\approx} = R_2 [y]_{\approx}$. But $R_2 \subseteq (R_1)_{\text{max}}^{\approx}$. So, for $\hat{R} \in \Gamma$, $M, x \models \square A \Leftrightarrow M_1, [x]_{\approx} \models \square A \Leftrightarrow M_1, [y]_{\approx} \models A \Leftrightarrow M, y \models A$.

**Claim 5.** If $M_1$ is a definable filtration of $M$, then $\hat{M}$ is definable too.

**Proof.** We use the following **Fact.** Let $M$ be a model and $\sim$ an equivalence relation on $W$ of finite index. Then $\sim$ is of the form $\sim_\Phi$ for some finite set of formulas $\Phi$ if and only if each equivalence class $[x]_\sim$ is defined in $M$ by some formula $\Phi$.

Indeed, if $\Phi$ is finite, then every class $[x]_{\sim_\Phi} \subseteq W$ is defined by the formula

$$\bigwedge \{(\varphi | \varphi \in \Phi \text{ and } M, x \models \varphi) \cup \{\neg \varphi | \varphi \in \Phi \text{ and } M, x \models \neg \varphi\}\}.$$  

Conversely, if $\sim$ partitions $W$ into finitely many classes and each class is defined by some formula $A_i$, $1 \leq i \leq n$, then clearly $\sim = \sim_\Phi$ for $\Phi = \{A_1, \ldots, A_n\}$.

To prove Claim 5, assume $M_1$ is a filtration of $M$ through a finite $\Phi$. Then each class is defined by some formula $A_i$. In $\hat{M}$, each $\sim$-class is obtained as the union of some $\sim_{\Phi_i}$-classes (namely, those that are modally equivalent as points in $M_1$). Hence, each $\sim$-class is defined by the disjunction of some formulas $A_i$. □

### A.3 On the logic of convergent frames

For convenience, we repeat the axioms for the transitive closure modality:

(A1) $\Box p \rightarrow \Box p$,  (A2) $p \rightarrow \Box \Box p$,  (A3) $\Box (p \rightarrow \Box p) \rightarrow (\Box p \rightarrow \Box \Box p)$.

Note that in any logic $L_\Box$, the following inference rule is derivable:

$$\varphi \rightarrow \Box \varphi \quad \varphi \rightarrow \Box \Box \varphi \quad \text{(R}\Box\text{)}$$

Indeed, here is a derivation:

1) $\varphi \rightarrow \Box \varphi$. 2) $\Box (\varphi \rightarrow \Box \varphi)$. 3) $\Box \varphi \rightarrow \Box \Box \varphi$ by (A3). 4) $\varphi \rightarrow \Box \varphi$ from 1 and 3.

Furthermore, in any logic $L_\Box$, the following formula is derivable:

$$\Box p \wedge \Box \Box p \rightarrow \Box p \quad \text{(A}\Box\text{)}$$

since one of its premises, $\Box \Box p$, is stronger than the premise $\Box (p \rightarrow \Box p)$ in (A3).

Recall that the logic **K.2** extends $\text{K}$ with the axiom $\Diamond \Box p \rightarrow \Box \Diamond p$.  

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Lemma A.3 (Convergence for transitive closure) K.2 $\square p \rightarrow \Diamond \Diamond p$.

Proof. The proof is in two stages.

1. We derive $\Diamond \Diamond p \rightarrow \Diamond \Diamond p$, using $\Diamond \Box \varphi \rightarrow \Box \Diamond \varphi$ for $\varphi = p$ and $\varphi = \Box p$:
   \[\Diamond \Diamond p \overset{(A1)}{\rightarrow} \Diamond p \overset{2}{\rightarrow} \Box \Diamond p.\]
   (a)

2. We obtain $\Diamond \Diamond p \overset{(A2)}{\rightarrow} \Diamond \Box \Diamond p \overset{2}{\rightarrow} \Box \Diamond \Diamond p$. Hence:
   \[\Diamond \Diamond p \overset{(A3)}{\rightarrow} \Box \Diamond \Diamond p.\]
   (b)

1’ We obtain $\Diamond \Box p \rightarrow \Box \Diamond p$ by duality from (1).

2 Derive $\Diamond \Box p \rightarrow \Box \Diamond p$ using (1’) similarly (replace $\Diamond$ with $\Box$ above).

Note that the two stages of the derivation in the above lemma correspond to two inductions needed to prove that $R^+$ is convergent, assuming that $R$ is convergent. First, by induction on $m$, one proves:

\[(x R^m y) \rightarrow \exists t: (y R t) \land (z R^m t).\]

Secondly, by induction on $n$ one proves:

\[(x R^m y) \land (x R^n z) \rightarrow \exists t: (y R t) \land (z R^m t).\]

Now, if $y R^+ t$ and $x R^+ z$, then $x R^n y$ and $x R^n z$, for some $m, n$. Then there is $t$ such that $y R^n t$ and $z R^m t$. Hence $y R^+ t$ and $z R^+ t$. So, $R^+$ is convergent.

This additionally justifies the name ‘induction axiom’ for the axiom (A3).

A.4 On the semantics of Segerberg’s axioms

Lemma A.4 Let $F = (W, R, S)$ be a bi-modal frame.

1. $F \models (A1)$, $S \supseteq R$.
2. $F \models (A2)$, $S \supseteq R \circ S$.
3. $F \models (A1) \land (A2)$, the converse does not hold in general.
4. $F \models (A3)$, $S \subseteq R^+$; the converse does not hold in general.
5. $F \models (A1) \land (A2) \land (A3)$ $\iff S = R^+$.

Proof. This is a known fact.

A.5 On induction axiom and minimal filtrated frame

Let us strengthen Lemma 3.7 (recall that $G \models (A3)$ implies $S \subseteq R^+$). Denote the minimal filtered (through $\Phi$) frame by $G_{\sim < \Phi}^{\min} = (W/\sim \Phi, R_{\sim < \Phi}^{\min}, S_{\sim < \Phi}^{\min})$.

Lemma A.5 (Induction axiom and minimal filtrated frame)

Let $M = (W, R, S, V)$, $M \models (A3)^*$ and let $\Phi \subseteq \text{Fm}$ be finite. Then $G_{\sim \Phi}^{\min} \models (A3)$.

Proof. Denote $\tilde{M} := M_{\sim \Phi}^{\min} = (G_{\sim \Phi}^{\min}, \tilde{V})$, where $G_{\sim \Phi}^{\min} = (\tilde{W}, R_{\sim \Phi}^{\min}, S_{\sim \Phi}^{\min})$. Note that $M$ is a $\Phi$-filtration of $M$, since $R_{\sim \Phi}^{\min} \subseteq R_{\Phi}^{\max}$ and similarly for $S$. Therefore, $\tilde{M}$ is a finite differentiated model: indeed, if $x \neq \bar{y}$, then $(M, x) \not\sim \Phi (\tilde{M}, \bar{y})$, hence $(M, \hat{x}) \not\Phi (\tilde{M}, \tilde{y})$, by the Filtration lemma 2.2. Due to Lemma 1.1, in order to prove our lemma, it suffices to show that

$M \models (A3)^*$ implies $\tilde{M} := M_{\sim \Phi}^{\min} \models (A3)^*$.
Assume \( \hat{M} \neq (A3)[p := B] \), for some formula \( B \). Then there is \( \hat{x} \in \hat{W} \) such that (a) \( \hat{x} \models \Box (B \to \Box B) \), (b) \( \hat{x} \models \Box B \), (c) \( \hat{x} \not\models \Box B \). Hence there is \( \hat{y} \in \hat{W} \) such that \( \hat{x} S \hat{y} \) and (d) \( \hat{y} \not\models B \). Since \( \hat{x} S_{\not\hat{y}} \hat{y} \), without loss of generality, \( x S y \).

Consider \( Y := \hat{V}(B) = \{ \hat{z} \in \hat{W} \mid \hat{x} \models B \} \). As in Lemma 3.7, \( Y \) is a finite collection of definable subsets of \( W \), hence their union \( \bigcup Y \) is also a definable subset of \( W \). So, there is a formula \( \varphi \) such that, for all \( z \in W \), we have:

\[
M, z \models \varphi \iff z \in \bigcup Y \iff \hat{z} \in Y \iff \hat{M}, \hat{z} \models B.
\]

Now let us show that \( M, x \not\models (A3)[p := \varphi] \), in contradiction with \( M \models (A3)^* \).

(a’) \( M, x \not\models (A3)[p := \varphi] \). Indeed, take any \( a, b \in W \) such that \( x S a R b \) and \( a \models \varphi \). Then \( \hat{x} S \hat{a} R \hat{b} \) and \( \hat{a} \models B \). Hence \( \hat{b} \models B \) by (a), and so \( b \models \varphi \).

(b’) \( M, x \not\models \Box \varphi \). Indeed, if \( x R z \), then \( \hat{x} R \hat{z} \); hence \( \hat{z} \models B \) by (b), so \( z \models \varphi \).

(d’) \( M, x \not\models \Box \varphi \). Indeed, \( x S y \) and \( M, y \not\models \varphi \), because \( \hat{y} \not\models B \) by (d).

Question: What other formulas transfer from \( M \) to \( \hat{M} \) and back? Maybe, even: \( M \models A[p := \varphi] \) iff \( \hat{M} \models A[p := B] \), for all formulas \( A \) with \( \text{Var}(A) = \{ p \} \).

Another question is: What other modal formulas \( \varphi \) have the property from the above lemma, namely:

\[
\text{if } M \models \varphi^*, \text{ then } \hat{F}_{\varphi^*} \models \varphi,
\]

for any finite set of modal formulas \( \Phi \)?