GLOBAL STRICHARTZ ESTIMATES FOR NONTRAPPING PERTURBATIONS OF THE LAPLACIAN

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1. Introduction

The purpose of this paper is to establish estimates of Strichartz type, globally in space and time, for solutions to certain nontrapping, spatially compact perturbations of the Minkowski wave equation. Precisely, we consider the following wave equation on the exterior domain $\Omega$ to a compact obstacle, where the spatial dimension is an odd integer $n \geq 3$.

\[
\begin{aligned}
\partial_t^2 u(t, x) - \Delta_g u(t, x) &= F(t, x), \quad (t, x) \in \mathbb{R} \times \Omega, \\
u(0, x) &= f(x) \in \dot{H}^\gamma_D(\Omega), \\
\partial_t u(0, x) &= g(x) \in \dot{H}^{\gamma-1}_D(\Omega), \\
u(t, x) &= 0, \quad x \in \partial\Omega.
\end{aligned}
\] (1.1)

The operator $\Delta_g$ is assumed to be the Laplace-Beltrami operator associated to a smooth, time-independent Riemannian metric $g(x)$, such that $g_{ij}(x) = \delta_{ij}$ for $|x| \geq R$. The set $\Omega$ is assumed to be the complement in $\mathbb{R}^n$ of a smoothly bounded, compact subset of $|x| < R$, such that $\Omega$ is strictly geodesically concave with respect to $g$. The case that $\Omega = \mathbb{R}^n$ is also permitted. Finally, the
geodesic flow on $\Omega$ with respect to $g$ and normal reflection on $\partial \Omega$ is assumed to be nontrapping, in the sense that all geodesics exit the set $|x| \leq R$ within some fixed, finite time.

The proof consists of showing that exponential energy decay bounds for (1.1) (for the case of compactly supported data) allow one to deduce a global Strichartz type estimate for (1.1) from knowledge of the same estimate locally in space-time, together with the global estimate for solutions of the Minkowski wave equation on $\mathbb{R}^{1+n}$. The local Strichartz estimates for (1.1) were established, in the homogeneous case $F = 0$, by the authors in [13]. By a special case of a lemma of Christ and Kiselev [2], this allows one to deduce local Strichartz estimates for the inhomogeneous problem (1.1). We thank T. Tao for pointing out this fact to us, and we include the details in this paper for completeness.

We say that $1 \leq r, s \leq 2 \leq p, q \leq \infty$ and $\gamma$ are admissible if the following two mixed norm estimates hold.

**Local Strichartz estimates.** For data $f, g, F$ supported in $|x| \leq R$, for solutions to (1.1) the following holds,

$$
\|u\|_{L_t^p L_x^q([0,1] \times \Omega)} + \sup_{0 \leq t \leq 1} \|u\|_{H^\gamma_t(\Omega)} + \sup_{0 \leq t \leq 1} \|\partial_t u\|_{H^{\gamma-1}_t(\Omega)} \\
\leq C \left( \|f\|_{H^\gamma_t(\Omega)} + \|g\|_{H^{\gamma-1}_t(\Omega)} + \|F\|_{L_t^r L_x^s([0,1] \times \Omega)} \right). \tag{1.2}
$$

**Global Minkowski Strichartz estimates.** For solutions to (1.1), in the case of $\Omega = \mathbb{R}^n$ and $g_{ij}(x) = \delta_{ij}$, the following holds,

$$
\|u\|_{L_t^p L_x^q(\mathbb{R}^{1+n})} \leq C \left( \|f\|_{H^\gamma(\mathbb{R}^n)} + \|g\|_{H^{\gamma-1}(\mathbb{R}^n)} + \|F\|_{L_t^r L_x^s(\mathbb{R}^{1+n})} \right). \tag{1.3}
$$
Additionally, for technical reasons we need assume $p > r$, and $\gamma \leq (n - 1)/2$. For such a set of indices, we show that the same estimate holds (with different $C$) for solutions to the perturbed equation.

**Theorem 1.1.** For admissible $p, q, r, s, \gamma$, the following estimates holds for solutions to the mixed Cauchy problem (1.1)

$$
\| u \|_{L^p_t L^q_x(\mathbb{R} \times \Omega)} \leq C \left( \| f \|_{H^\alpha_\gamma(\Omega)} + \| g \|_{H^{\gamma-1}_\beta(\Omega)} + \| F \|_{L^r_t L^s_x(\mathbb{R} \times \Omega)} \right).
$$

The ingredient that allows us to establish Theorem 1.1 from (1.3) and (1.2) is the following decay estimate for solutions to the homogeneous problem with compact data.

**Exponential energy decay.** For data $f, g$ supported in $|x| \leq R$, and $\beta(x)$ smooth, supported in $|x| \leq R$, there exist $C < \infty$ and $\alpha > 0$ such that for solutions to (1.1) where $F = 0$ the following holds,

$$
\| \beta u(t, \cdot) \|_{H^\alpha_\beta(\mathbb{R}^n)} + \| \beta \partial_t u(t, \cdot) \|_{H^{\gamma-1}_\beta(\Omega)} \leq C e^{-\alpha t} \left( \| f \|_{H^\alpha_\beta(\Omega)} + \| g \|_{H^{\gamma-1}_\beta(\Omega)} \right).
$$

The decay estimate (1.4), which depends on the nontrapping assumption and requires that $n \geq 3$ be odd, was first established in the obstacle framework by Taylor [16]. For a more general discussion of energy decay estimates, we refer to Lax-Philips [8], and Vainberg [17]. It was pointed out to us by M. Zworski that quasimode constructions show that global Strichartz estimates cannot hold if the metric has an elliptic closed geodesic. See, for example, Colin de Verdière [3] and Ralston [12].

There is a long history to establishing the global estimates (1.3) for the Minkowski case, beginning with the original work by Strichartz in [14, 15] for the conformal case $p = q = \frac{2n+2}{n-1}$, $r = s = \frac{2n+2}{n+3}$, and $\gamma = \frac{1}{2}$. We mention here the subsequent work of Genibre-Velo [4], Pecher [11], Kapitanski [6], Lindblad-Sogge [9], Mockenhaupt-Seeger-Sogge [10], and Keel-Tao [7].
Certain local Strichartz estimates (1.2) for the obstacle problem were established by the authors in [13], for the case \( F = 0 \). It will be shown in Theorem 3.2 that these imply the appropriate estimates in the case \( F \neq 0 \), based on an argument of Christ and Kiselev [2]. M. Beals [1] has obtained fixed time \( L^p - L^q \) estimates for the obstacle problem, globally in \( t \), in the case \( g_{ij} = \delta_{ij} \), and for data vanishing near \( \partial \Omega \). It is not known whether the Strichartz estimates hold, even locally, if \( \partial \Omega \) has a point of convexity, but related eigenfunction estimates are known to fail in that case, as observed by Grieser [5], and generalized by the authors in [13]. It also is not known whether (1.3) holds for every set of indices for which (1.2) holds, and we do not address that question in this paper. We do mention that if the following conditions are met, then the indices are admissible in the sense of this paper, provided that \( q, s' < \frac{2(n-1)}{n-3} \),

\[
\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \gamma = \frac{1}{r} + \frac{n}{s} - 2, \quad \frac{1}{p} = \left( \frac{n-1}{2} \right) \left( \frac{1}{2} - \frac{1}{q} \right), \quad \frac{1}{r} = \left( \frac{n-1}{2} \right) \left( \frac{1}{2} - \frac{1}{s'} \right).
\]

We make some remarks concerning the homogeneous Sobolev space \( \dot{H}_D^\gamma(\Omega) \). We always deal with \( \gamma < \frac{n}{2} \), so that smooth cutoffs of functions in \( \dot{H}^\gamma(\mathbb{R}^n) \) are contained in the inhomogeneous Sobolev space \( H^\gamma(\mathbb{R}^n) \), and the two norms are comparable on functions supported in a fixed ball. For simplicity, if \( R \) is large enough so that \( g_{ij}(x) = \delta_{ij} \) when \( |x| > R \), and \( \partial \Omega \subset \{|x| < R\} \), then we fix \( \beta \in C_\infty^\infty \) with \( \beta(x) = 1 \) for \( |x| \leq R \), and define

\[
\|f\|_{\dot{H}_D^\gamma(\Omega)} = \|\beta f\|_{H_D^\gamma(\tilde{\Omega})} + \|(1 - \beta) f\|_{\dot{H}^\gamma(\mathbb{R}^n)},
\]

where \( \tilde{\Omega} \) is a compact manifold with boundary containing \( \Omega \cap \{|x| = R\} \). For \( 0 \leq \gamma \leq 2 \), the space \( H_D^\gamma(\tilde{\Omega}) \) is the usual Dirichlet space satisfying \( f|_{\partial \tilde{\Omega}} = 0 \) (when this makes sense.) For larger \( \gamma \), the additional compatibility conditions

\[
\Delta_g^j f \in H_D^{\gamma - 2j}(\tilde{\Omega}), \quad 2j \leq \gamma,
\]
must be satisfied to insure that solutions to the Cauchy problem remain in $H_D^\gamma(\tilde{\Omega})$. The spaces with $\gamma < 0$ are defined by duality; for the Strichartz estimates (1.3), always $\gamma - 1 \geq -1$, so that the additional compatibility conditions are irrelevant for negative indices.

2. Global Strichartz Estimates

In this section we provide the proof of Theorem 1.1, based on the assumptions (1.2), (1.3), and (1.4). By dilating, we will take $R = \frac{1}{2}$.

Lemma 2.1. Let $u$ solve the Cauchy problem (1.1) with $F$ replaced by $F + G$, where the data $f, g$ are supported in $\{ |x| \leq 1 \}$, and $F, G$ are supported in $\{ |t| \leq 1 \} \times \{ |x| \leq 1 \}$. Then, for any $\rho < \alpha$, and any admissible $p, q, r, s, \gamma$, there exists $C < \infty$ such that the following holds.

\[
\left\| e^{\rho(|t| - |x|)} u \right\|_{L_t^p L_x^q(\mathbb{R} \times \Omega)} \leq C \left( \| f \|_{H_D^{\gamma}(\Omega)} + \| g \|_{H_D^{\gamma-1}(\Omega)} + \| F \|_{L_t^r L_x^s(\mathbb{R} \times \Omega)} + \int \| G(t, \cdot) \|_{H_D^{\gamma-1}(\Omega)} dt \right).
\]

Proof. We establish the estimate on $t \geq 0$. Observe that, by (1.2) and Duhamel’s principle, the inequality holds for the $L_t^p L_x^q$ norm of $u$ over $[0, 1] \times \Omega$.

Also, by (1.2),

\[
\| u(1, \cdot) \|_{H_D^{\gamma}(\Omega)} + \| \partial_t u(1, \cdot) \|_{H_D^{\gamma-1}(\Omega)} \leq C \left( \| f \|_{H_D^{\gamma}(\mathbb{R}^n)} + \| g \|_{H_D^{\gamma-1}(\mathbb{R}^n)} + \| F \|_{L_t^r L_x^s(\mathbb{R} \times \Omega)} + \int \| G(t, \cdot) \|_{H_D^{\gamma-1}(\Omega)} dt \right).
\]

By considering $t \geq 1$, we may thus take $F = G = 0$, with $f, g$ now supported in $\{ |x| \leq 2 \}$.

We decompose $u = \beta u + (1 - \beta) u$, where $\beta(x) = 1$ for $|x| \leq \frac{1}{2}$, and $\beta(x) = 0$ for $|x| \geq 1$. First consider $\beta u$. We write

\[
\partial_t^2 (\beta u) - \Delta g(\beta u) = \sum_{j=1}^{n} b_j(x) \partial_{x_j} u + c(x) u \equiv \tilde{G}(t, x),
\]
where \( b_j \) and \( c \) are supported in \( \frac{1}{2} \leq |x| \leq 1 \), and in particular vanish near \( \partial \Omega \). By (1.4) the following holds,

\[
\| \tilde{G}(t, \cdot) \|_{H^{\gamma-1}(\Omega)} + \| \beta u(t, \cdot) \|_{H^\gamma(\Omega)} + \| \partial_t(\beta u)(t, \cdot) \|_{H^{\gamma-1}(\Omega)} \leq C e^{-\alpha t} \left( \| f \|_{H^\gamma(\Omega)} + \| g \|_{H^{\gamma-1}(\Omega)} \right). \tag{2.1}
\]

By (1.2) and Duhamel’s principle, it follows that

\[
\| \beta u \|_{L^p_t L^q_x([j,j+1] \times \Omega)} \leq C e^{-\alpha j} \left( \| f \|_{H^\gamma(\Omega)} + \| g \|_{H^{\gamma-1}(\Omega)} \right),
\]

which easily yields the desired estimate for \( \beta u \).

Next consider \((1-\beta)u\). Since \( \Delta_b = \Delta \) on the support of \((1-\beta)u\), we have

\[
\partial_t^2 u - \Delta u = -\tilde{G},
\]

and by Duhamel’s principle we have

\[
u(t, x) = u_0(t, x) + \int_0^\infty u_s(t, x) \, ds,
\]

where \( u_0 \) is the solution of the Minkowski wave equation on \( \mathbb{R}^{1+n} \) with initial data \( (1-\beta)f, (1-\beta)g \), and where \( u_s(t, x) \) is the solution of the Minkowski wave equation on the set \( t > s \) with Cauchy data \( (0, \tilde{G}(s, \cdot)) \) on the surface \( t = s \). (Recall that \( \tilde{G} \) and \( (1-\beta) \) vanish near \( \partial \Omega \).) By Huygen’s principle, on the support of \( u_s(t, x) \) we have \( t \geq s \) and \( t - |x| \in [s-1, s+1] \), so that by (1.3) and (2.1) we have

\[
\left\| e^{\rho(|t| - |x|)} u_s \right\|_{L^p_t L^q_x([R^n \times \Omega] \times \Omega)} \leq C e^{(\rho - \alpha)s} \left( \| f \|_{H^\gamma(\Omega)} + \| g \|_{H^{\gamma-1}(\Omega)} \right),
\]

which leads to the desired estimate for \( u \).

\[\square\]

**Lemma 2.2.** Let \( \beta(x) \) be smooth and supported in \( \{|x| \leq 1\} \), and \( 2\gamma \leq n-1 \). Then the following holds

\[
\int_{-\infty}^{\infty} \left\| \beta(\cdot) \left( e^{it|D|} f \right)(t, \cdot) \right\|_{H^\gamma(\mathbb{R}^n)}^2 dt \leq C_{n,\gamma,\beta} \| f \|_{H^\gamma(\mathbb{R}^n)}^2.
\]
Proof. By Plancherel’s theorem over \( t, x \), the left hand side can be written as
\[
\int_0^\infty \int \left| \hat{\beta}(\xi - \eta) \hat{f}(\eta) \delta(\tau - |\eta|) \right| \left( 1 + |\xi|^2 \right)^\gamma d\xi d\tau.
\]
We next apply the Schwarz inequality over \( \eta \) to bound this by
\[
\int_0^\infty \int \left[ \int \left| \hat{\beta}(\xi - \eta) \delta(\tau - |\eta|) \right| d\eta \right] \left[ \int \left| \hat{\beta}(\xi - \eta) \right|^2 \delta(\tau - |\eta|) d\eta \right] \times (1 + |\xi|^2)^\gamma d\xi d\tau.
\]
From the fact that
\[
\sup_\xi (1 + |\xi|^2)^\gamma \left[ \int \left| \hat{\beta}(\xi - \eta) \right| \delta(\tau - |\eta|) d\eta \right] \leq C_{n, \gamma, \beta} \min \left[ \tau^{n-1}, (1 + \tau^2)^\gamma \right]
\]
this is in turn bounded by
\[
C_{n, \gamma, \beta} \int \left| \hat{f}(\eta) \right|^2 \min \left[ |\eta|^{n-1}, (1 + |\eta|^2)^\gamma \right] d\eta \leq C_{n, \gamma, \beta} \| f \|^2_{H^\gamma(\mathbb{R}^n)}. \qed
\]

**Corollary 2.3.** Let \( \beta \) be a smooth function, supported in \( \{ |x| \leq 1 \} \). Suppose that the global Minkowski Strichartz estimate (1.3) holds, and that \( 2\gamma \leq n-1 \). Let \( u \) solve the Cauchy problem for the Minkowski wave equation, with data \( f, g, F \). Then the following holds,
\[
\sup_{|\alpha| \leq 1} \int_{-\infty}^\infty \left\| \beta \partial_{t,x}^\alpha u(t, \cdot) \right\|_{H^{\gamma-1}(\mathbb{R}^n)}^2 dt \leq C \left( \| f \|_{H^\gamma(\mathbb{R}^n)} + \| g \|_{H^{\gamma-1}(\mathbb{R}^n)} + \| F \|_{L^r_t L^s_x(\mathbb{R}^1 \times \mathbb{R}^n)} \right)^2.
\]

Proof. If \( F = 0 \), this is a direct consequence of Lemma 2.2 above. If \( f = g = 0 \), then the Strichartz estimate (1.3), duality, and Huygen’s principle imply the following (for \( t > 0 \))
\[
\sup_{|\alpha| \leq 1} \left\| \beta \partial_{t,x}^\alpha u(t, \cdot) \right\|_{H^{\gamma-1}(\mathbb{R}^n)}^2 \leq C \| F \|^2_{L^r_t L^s_x(\Gamma_t)},
\]
where
\[
\Gamma_t = \{ (t', x) : t' \geq 0, t' + |x| \in [t-1, t+1] \}.
\]
Since \( r, s \leq 2 \), the following holds
\[
\int_0^\infty \| F \|^2_{L^r_t L^s_x(\Gamma_t)} \leq 2 \| F \|^2_{L^r_t L^s_x(\mathbb{R}^1 \times \mathbb{R}^n)}. \qed
\]
Proof of Theorem 1.1. By Lemma 2.1, we may without loss of generality assume that \( f \) and \( g \) vanish for \( |x| \leq 1 \). We write

\[
u = u_0 - v = (1 - \beta)u_0 + \beta u_0 - v,
\]

where \( u_0 \) solves the Cauchy problem for the Minkowski wave equation, with data \( f, g, F \), where we set \( F = 0 \) on \( \mathbb{R}^n \setminus \Omega \). By (1.3), we can restrict attention to \( \beta u_0 - v \). We write

\[
\left( \partial_t^2 - \Delta_g \right) (\beta u_0 - v) = \beta F + G
\]

where \( G(t, x) = \sum_{j=1}^n b_j(x)\partial_x u_0(t, x) + c(x)u_0(t, x) \) vanishes for \( |x| \geq 1 \), and satisfies

\[
\int_{-\infty}^{\infty} \| G(t, \cdot) \|_{H^{\gamma - 1}_D(\Omega)}^2 \, dt \leq C \left( \| f \|_{H^{\gamma}(\mathbb{R}^n)} + \| g \|_{H^{\gamma - 1}(\mathbb{R}^n)} + \| F \|_{L^r_tL^s_x(\mathbb{R}^n)} \right)^2,
\]

by Corollary 2.3. Note that the initial data of \( \beta u_0 - v \) vanishes. Let \( F_j, G_j \) denote the restrictions of \( F, G \) to the set \( t \in [j, j+1] \), and write (for \( t > 0 \))

\[
\beta u_0 - v = \sum_{j=0}^{\infty} u_j(t, x),
\]

where \( u_j(t, x) \) is the forward solution to \( \partial_t^2 u_j - \Delta_g u_j = \beta F_j + G_j \).

By Lemma 2.1, the following holds

\[
\| e^{\rho(t-j-|x|)} u_j \|_{L^r_tL^s_x(\mathbb{R} \times \Omega)} \leq C \left( \| \beta F_j \|_{L^r_tL^s_x(\mathbb{R} \times \Omega)} + \int_j^{j+1} \| G(t, \cdot) \|_{H^{\gamma - 1}_D(\Omega)} \, dt \right) .
\]

Furthermore, \( u_j(t, x) \) is supported in the region \( t - j - |x| \geq -1 \). Consequently, we have

\[
\| u \|_{L^r_tL^s_x(\mathbb{R} \times \Omega)} \leq C \sum_{j=0}^{\infty} \| e^{\rho(t-j-|x|)} u_j \|_{L^r_tL^s_x(\mathbb{R} \times \Omega)}^2 \leq C \sum_{j=0}^{\infty} \| F_j \|_{L^r_tL^s_x(\mathbb{R} \times \Omega)}^2 + C \sum_{j=0}^{\infty} \left( \int_j^{j+1} \| G(t, \cdot) \|_{H^{\gamma - 1}_D(\Omega)} \, dt \right)^2 \leq C \| F \|_{L^r_tL^s_x(\mathbb{R} \times \Omega)}^2 + C \int_0^{\infty} \| G(t, \cdot) \|_{H^{\gamma - 1}_D(\Omega)}^2 \, dt ,
\]
where we use the fact that \( p, q \geq 2 \geq r, s \).

3. Homogeneous estimates imply inhomogeneous estimates

In this section, we provide an elementary proof that mixed-norm estimates for the homogeneous Cauchy problem imply the appropriate estimates for the inhomogeneous Cauchy problem. The proof is quite general, but we present it here in the context of the obstacle problem. The main ingredient is a special case of a lemma of Christ and Kiselev [2], the proof of which we present here for completeness. We thank T. Tao for pointing out the relevance of this lemma to inhomogeneous estimates for the wave equation.

**Lemma 3.1.** Let \( X \) and \( Y \) be Banach spaces and assume that \( K(t, s) \) is a continuous function taking its values in \( B(X, Y) \), the space of bounded linear mappings from \( X \) to \( Y \). Suppose that \( -\infty \leq a < b \leq \infty \), and set

\[
T f(t) = \int_{a}^{b} K(t, s) f(s) \, ds.
\]

Assume that

\[
\| T f \|_{L^{q}([a,b], Y)} \leq C \| f \|_{L^{p}([a,b], X)}. \tag{3.1}
\]

Set

\[
W f(t) = \int_{a}^{t} K(t, s) f(s) \, ds.
\]

Then, if \( 1 \leq p < q \leq \infty \),

\[
\| W f \|_{L^{q}([a,b], Y)} \leq \frac{2^{-2(1/p-1/q)} \cdot 2C}{1 - 2^{-1/p-1/q}} \| f \|_{L^{p}([a,b], X)}.
\]

As remarked in [2], if \( K(s, t) = 1/(t-s) \), then Lemma 3.1 does not hold in the case \( p = q \in (1, \infty) \).

Using this lemma, we establish the following.
Theorem 3.2. Suppose that the following estimates hold for solutions to the Cauchy problem (1.1), where $F = 0$, and the Cauchy data $f, g$ are supported in the set $|x| \leq R$.

$$
\|u\|_{L^p_tL^q_x([0,1] \times \Omega)} \leq C \left( \|f\|_{H^\gamma_0(\Omega)} + \|g\|_{H^{-\gamma}_0(\Omega)} \right),
$$

$$
\|u\|_{L^p_tL^q_x([0,1] \times \Omega)} \leq C \left( \|f\|_{H^{1-\gamma}_0(\Omega)} + \|g\|_{H^{\gamma}_0(\Omega)} + \|F\|_{L^r_tL^s_x([0,1] \times \Omega)} \right).
$$

Then the following estimate holds for solutions to the inhomogeneous Cauchy problem (1.1), provided that $f, g$ and $F$ are supported in the set $|x| \leq R$.

$$
\|u\|_{L^p_tL^q_x([0,1] \times \Omega)} \leq C \left( \|f\|_{H^\gamma_0(\Omega)} + \|g\|_{H^{-\gamma}_0(\Omega)} + \|F\|_{L^r_tL^s_x([0,1] \times \Omega)} \right).
$$

Proof of Theorem 3.2. By finite propagation velocity, we may replace $\Omega$ by a compact manifold $\tilde{\Omega}$ with boundary. Let $\Lambda = \sqrt{-\Delta_g}$, so that the spectrum of $\Lambda$ is bounded below. By the assumptions of the theorem and duality, the following hold.

$$
f \rightarrow \Lambda^{-\gamma} e^{\pm it\Lambda} f \ : \ L^2(\tilde{\Omega}) \rightarrow L^p_tL^q_x([0,1] \times \tilde{\Omega}),
$$

$$
F \rightarrow \Lambda^{\gamma-1} \int_0^1 e^{\pm it\Lambda} F(s, \cdot) \, ds \ : \ L^p_tL^q_x([0,1] \times \tilde{\Omega}) \rightarrow L^2(\tilde{\Omega}).
$$

Let $K(t, s) = \Lambda^{-1} \sin((t - s)\Lambda)$. An application of the addition formula to $\sin((t - s)\Lambda)$ yields the following.

$$
\int_0^1 K(t, s) F(s, \cdot) \, ds \ : \ L^p_tL^q_x([0,1] \times \tilde{\Omega}) \rightarrow L^p_tL^q_x([0,1] \times \tilde{\Omega}).
$$

By Duhamel’s principle and Lemma 3.1, the result follows. (To satisfy the conditions of Theorem 3.2, we should properly consider a smooth truncation of the wave group to finite frequencies. The estimates are then uniform, and thus hold in the limit.)

Proof of Lemma 3.1. Since the argument for $q = \infty$ is similar to the case $q < \infty$, we assume for simplicity that $q$ is finite.
We can normalize $f$ so that
\[ \|f\|_{L^p([a,b],X)} = 1. \]
We may also assume without loss of generality that $f(s)$ is a continuous function (with values in $X$) and that if
\[ F(t) = \int_a^t \|f(s)\|_X^p \, ds, \]
then $F : \mathbb{R} \to [0,1]$ is a bijection. Note then that if $I \subset [0,1]$ is an interval, then
\[ \|\chi_{F^{-1}(I)}(s)f(s)\|_{L^p(\mathbb{R},X)} = |I|^{1/p}. \quad (3.2) \]

Next consider the set of all dyadic subintervals of $[0,1]$. If $I$ and $J$ are two such subintervals, we say that $I \sim J$ if the following hold. First, $I$ and $J$ must have the same length, and $I$ must lie to the left of $J$. We also require that $I$ and $J$ be non-adjacent but have adjacent parents (i.e., $I \subset I_0$ and $J \subset J_0$ where $I_0$ and $J_0$ are adjacent dyadic intervals of twice the length).

Note then that if $J$ is fixed there are only two intervals with $I \sim J$. Moreover, for almost every $(x,y) \in [0,1]^2$ with $x < y$ there is a unique pair $I$, $J$ with $I \sim J$ and $x \in I$ and $y \in J$.

If we apply this fact using the variables $x = F(s)$ and $y = F(t)$ then we conclude that the following identity holds almost everywhere:
\[
\chi_{\{(s,t) \in [a,b]^2 : s < t\}}(s,t) = \chi_{\{(x,y) \in [0,1]^2 : x < y\}}(x,y)
\]
\[
= \sum_{\{I,J : I \sim J\}} \chi_I(x)\chi_J(y)
\]
\[
= \sum_{\{I,J : I \sim J\}} \chi_{F^{-1}(I)}(s)\chi_{F^{-1}(J)}(t).
\]
Consequently,
\[ Wf = \sum_{\{I, J: I \sim J\}} \chi_{F^{-1}(J)} T(\chi_{F^{-1}(I)} f) . \]

From this we conclude that
\[ \| Wf \|_{L^q([a,b], Y)} \leq \sum_{j=2}^{\infty} \left\| \sum_{\{I, J: I \sim J, |I| = 2^{-j}\}} \chi_{F^{-1}(J)} T(\chi_{F^{-1}(I)} f) \right\|_{L^q([a,b], Y)} . \]  \hspace{1cm} (3.3)

Since for every \( J \) there are at most two \( I \) with \( I \sim J \), and the \( J \) with \( |J| = 2^{-j} \) are disjoint, we conclude that
\[ \left\| \sum_{\{I, J: I \sim J, |I| = 2^{-j}\}} \chi_{F^{-1}(J)} T(\chi_{F^{-1}(I)} f) \right\|_{L^q([a,b], Y)} \leq 2 \left( \sum_{\{I: |I| = 2^{-j}\}} \| T(\chi_{F^{-1}(I)} f) \|_{L^q([a,b], Y)}^q \right)^{1/q} . \]

By (3.1) and (3.2), it follows that this quantity is bounded by
\[ 2C \left( \sum_{\{I: |I| = 2^{-j}\}} \| \chi_{F^{-1}(I)} f \|_{L^p([a,b], X)}^q \right)^{1/q} \leq 2C \left( \sum_{\{I: |I| = 2^{-j}\}} 2^{-jq/p} \right)^{1/q} = 2^{-j(1/p-1/q)} \cdot 2C . \]

The lemma follows by (3.3) after summing over \( j \), where we use the fact that \( p < q \).

\[ \square \]

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