Revisiting non-perturbative effects in the jet broadenings

Yu.L. Dokshitzer, G. Marchesini, G.P. Salam

Dipartimento di Fisica, Università di Milano-Bicocca
and INFN, Sezione di Milano, Italy

Abstract

We show that taking into account the interplay between perturbative and non-perturbative effects, the power-suppressed shift to the broadening distributions becomes $B$ dependent, and the non-perturbative contribution to the mean values becomes proportional to $1/(Q \sqrt{\alpha_s(Q)})$. The new theoretical treatment greatly improves the consistency of the phenomenology with the notion of the universality of confinement effects in jet shapes.

---

*This work was supported in part by the EU Fourth Framework Programme ‘Training and Mobility of Researchers’, Network ‘Quantum Chromodynamics and the Deep Structure of Elementary Particles’, contract FMRX-CT98-0194 (DG 12-MIHT).

† On leave from St. Petersburg Nuclear Institute, Gatchina, St. Petersburg 188350, Russia
1 Introduction

There is mounting evidence in favour of the universal pattern of leading $1/Q$ non-perturbative power corrections to collinear/infrared safe (CIS) observables \[1\]. They include jet-shape observables such as the thrust $T$, the $C$-parameter, squared jet masses and the jet broadening $B$ (definitions can be found, for example, in \[2\]). $1/Q$ power effects are expected also in the energy-energy correlation function and $\sigma_L$ in $e^+e^-$ annihilation. The $1/Q$ power terms are seen both in the means and the distributions. For instance, the distribution of events in $T$ in the 2-jet region $1 \gg 1 - T \gg \Lambda/Q$ was predicted \[3, 4\] to be given by a simple shift of the corresponding perturbative QCD prediction by a constant value inversely proportional to $Q$. The same pattern was expected to hold for other jet shapes as well, with the true distribution in the kinematical region $1 \gg V \gg \Lambda/Q$ being related to its perturbative (PT) counterpart by a shift,

$$\frac{d\sigma}{dV}(V) = \frac{d\sigma^{(PT)}}{dV}(V - \Delta V), \quad \Delta V = c_V P,$$

with $c_V$ an observable dependent, perturbatively calculable, numerical coefficient which for the thrust, $C$-parameter and total and heavy-jet masses, e.g., is

| $V$ | $1-T$ | $C$ | $M_T^2/Q^2$ | $M_H^2/Q^2$ |
|-----|-------|-----|-------------|-------------|
| $c_V$ | $\frac{1}{2}$ | $3\pi$ | $2$ | $1$ |

The parameter $P \propto 1/Q$ is the non-perturbative (NP) quantity which effectively “measures” the intensity of the QCD interaction over the infrared momentum region. Its magnitude can be interpreted as being related to a mean value of the QCD coupling over the infrared region, say, $k \leq \mu_I = 2$ GeV,

$$\alpha_0(\mu_I) = \mu_I^{-1} \int_0^{\mu_I} dk \, \alpha_s(k).$$

A proper definition of $P$ includes an infrared matching scale $\mu_I$ which is necessary for merging, in a renormalon-free manner, the PT and NP contributions to a given observable. It also calls for a two-loop analysis of the non-perturbative contribution \[4\], without which the magnitude of the power correction cannot be quantified better than up to a factor of order unity.

Experimentally, NP effects in the thrust distribution have been found to be consistent with the shift rule \[4\], with $\alpha_0 \simeq 0.5$. The same value was experimentally extracted from the $Q$-dependence of $\langle 1 - T \rangle$ \[5, 6\]. The $C$-parameter has also been studied (both the distribution and the mean) and found to be consistent, with a similar value of $\alpha_0$ \[8\].

Power effects in the broadening $B$ were also expected to shift the distribution, but with $\Delta B$ logarithmically enhanced ($\Delta B \propto \ln Q/Q$) \[2\]. The H1 collaboration however stated that the data was not consistent with a $\ln Q$ enhancement \[7\]. Most recently the JADE collaboration studied the discrepancy and showed that relative to the perturbative distribution, the experimental distribution is not only shifted, but also “squeezed” \[10\].

In this paper we revisit the broadening. We show that the coefficient of the power correction shift, $Q\Delta_B$, is neither proportional to $\ln Q$, nor a constant, but rather is a function
logarithmically depending on $B$. The reason is the following. The non-perturbative contribution to $V$ comes from the emission of *gluers* (gluons with finite transverse momenta, of the order of the QCD scale $\Lambda$, with respect to the quark direction \(\Pi\)). For instance, in the soft approximation, the contributions to the thrust, $C$-parameter and broadening from a secondary parton $i$ can be expressed as

\[ (1 - T)_i = \frac{k_{ti}}{Q} e^{-|\eta_i|}, \quad C_i = \frac{3k_{ti}}{Q \cosh \eta_i}, \quad 2B_i = \frac{k_{ti}}{Q}, \]  

(1.4)

where the transverse momentum $k_{ti}$ and the rapidity $\eta_i$ are measured with respect to the thrust axis. The feature that $1 - T$ and $C$ have in common is that the dominant non-perturbative contribution to these and similar shapes is determined by the radiation of soft gluers at large angles. This radiation is insensitive to a tiny mismatch, $\langle \Theta_q \rangle = \mathcal{O}(\alpha_s)$, between the quark and thrust axis directions which is due to perturbative gluon radiation. Therefore the quark momentum direction can be identified with the thrust axis in the \(\text{PT}\) and \(\text{NP}\) analysis of $T$ and $C$.

The broadening, on the contrary, accumulates contributions which do not depend on rapidity, so that the mismatch between the quark and the thrust axis could be important. As we shall see later, for the \(\text{PT}\) analysis of broadening this mismatch is irrelevant, to next-to-leading accuracy. However, the mismatch plays a crucial rôle for the \(\text{NP}\) effects in the broadening, both in the means and the distributions.

If one naively assumes that the quark direction can be approximated by that of the thrust axis, as is the case for the perturbative contribution, then $B$ accumulates \(\text{NP}\) contributions from gluons with rapidities up to the kinematically allowed value $\eta_i \leq \eta_{\text{max}} \simeq \ln(Q/k_{ti})$. In this case one would find the shift in the $B$ spectrum to be logarithmically enhanced,

\[ \Delta B = c_B \mathcal{P} \cdot \ln \frac{Q}{Q_B}, \]  

(1.5)

where $c_B = 1(\frac{1}{2})$ for the total (single-jet; wide-jet) broadening \([2]\). What this overlooks is the fact that the *uniform* distribution in $\eta_i$ (defined with respect to the thrust axis) holds only for gluon rapidities not exceeding $|\ln \Theta_q|$. Hard gluons with energies $k_{0i} > k_{ti}/\Theta_q$ are collinear to the quark direction rather than to that of the thrust axis and therefore do not contribute essentially to $B$. As a result, for a given quark angle, the \(\text{NP}\) contribution $\delta B$ to the broadening $B$ comes out proportional to the quark rapidity (we note the distinction between the \(\text{NP}\) contribution $\delta B$ to a given perturbative configuration, and the \(\text{NP}\) shift $\Delta B$ to the perturbative distribution, where the latter is integrated over all perturbative configurations leading to particular value of $B$). For the single-jet broadening one has

\[ \delta B \simeq c_1 \mathcal{P} \cdot \ln \frac{1}{\Theta_q}. \]  

(1.6)

Let us describe, semi-quantitatively, how (1.6) affects the \(\text{NP}\) corrections to $\langle B \rangle$ and to the $B$ distribution. The power correction to the mean single-jet broadening $\langle B \rangle_1$ is obtained by evaluating the perturbative average of $\delta B_1$ in (1.6),

\[ \langle B \rangle_1^{(\text{NP})} \equiv \langle B \rangle_1 - \langle B \rangle_1^{(\text{PT})} \simeq c_1 \mathcal{P} \cdot \left\langle \ln \frac{1}{\Theta_q} \right\rangle. \]  

(1.7)
The $\tau\tau$-distribution in $\Theta_q$ at the Born level is singular at $\Theta_q = 0$. In high orders this singularity is damped by the double-logarithmic Sudakov form factor. As a result, the NP component of $\langle B \rangle_1$ gets enhanced by

$$\left\langle \ln \frac{1}{\Theta_q} \right\rangle \simeq \frac{\pi}{2\sqrt{C_F\alpha_s(Q)}}.$$  \hfill (1.8)

For the mean wide-jet broadening $\langle B \rangle_W$ the result has the same structure with the replacement $C_F \to 2C_F$ due to the fact that now it is radiation off two jets which determines the $\Theta_q$ distribution.

The shift in the single-jet (or wide-jet) broadening can be expressed as

$$\Delta_1(B) \simeq c_1\mathcal{P} \cdot \left\langle \ln \frac{1}{\Theta_q} \right\rangle_B,$$  \hfill (1.9)

where the average is taken over the perturbative distribution in the quark angle $\Theta_q$ while keeping the value of $B$ fixed. Since $\Theta_q$ is kinematically proportional to $B$, the log-enhancement of the shift in the $B$ spectrum becomes

$$\Delta_1(B) \simeq c_1\mathcal{P} \cdot \ln \frac{B_0}{B},$$  \hfill (1.10)

with $B_0 = B_0(\alpha_s \ln B)$ a calculable function slowly depending on $B$. Thus, the shift in the $B_1(B_W)$ distribution becomes logarithmically dependent on $B$.

Finally, the shift in the total two-jet broadening distribution $\Delta_T(B)$ will also be derived. It has a somewhat more complicated $B$ dependence. In the kinematical region where the multiplicity of gluon radiation is small, $\alpha_s \ln^2 B \ll 1$, one of the two jets is responsible for the whole $\tau\tau$-component of the event broadening, while the second is “empty”. That “empty” jet contributes the most to the shift: in the absence of perturbative radiation the direction of the quark momentum in this jet stays closer to the thrust axis. This results in

$$\Delta_T(B) \simeq \Delta_1(B) + \langle B \rangle_1^{(NP)} \simeq c_1\mathcal{P} \left( \ln \frac{1}{B} + \frac{\pi}{2\sqrt{C_F\alpha_s}} + \mathcal{O}(1) \right).$$  \hfill (1.11)

In these circumstances the $B$ dependence of the total shift practically coincides with that of a single jet. In the opposite regime of well developed $\tau\tau$ radiation, $\alpha_s \ln^2 B \gg 1$, the jets are forced to share $B$ equally, and we have

$$\Delta_T(B) \simeq 2 \cdot \Delta_1(B/2) \simeq 2 \cdot c_1\mathcal{P} \ln \frac{1}{B}.$$  \hfill (1.12)

In the present paper we derive the correspondingly modified predictions for the $B$ dependent $1/Q$ shifts in the broadening spectra and the $1/Q$ corrections to the means, which supersede the earlier results of [2]. The paper is organised as follows.

Section 2 introduces the quantities to be studied, and recollects the next-to-leading-logarithmic perturbative result for the single-jet, wide-jet and total distributions.

In section 3 we derive the new results for the power corrections to the means and distributions.
Section 4 presents a comparison of these new results with experimental data.

In the concluding section 5 we discuss our results in the context of the interplay between PT and NP effects.

The remainder of the paper consists of appendices which contain technical details of the derivation and a Monte Carlo study which illustrates some important features of the analytical results. In the last Appendix 6 we give the full list of formulas to be used to describe the leading power correction to the broadening means and distributions.

2 Broadening distribution

2.1 Distributions and means

Broadening in $e^+e^-$ annihilation with c.m.s. energy $Q$ is defined as a sum of transverse momenta of final particles with respect to the thrust axis,

$$2Q \cdot B_X = \sum_{i \in X} |\vec{p}_i| .$$

(2.1)

The symbol $X$ here marks the set of selected final particles. The three options under discussion are

1. single-jet broadening, $B_1$, with the sum running over particles in one hemisphere (right, $B_1 = B_R$, or left, $B_1 = B_L$),

2. total broadening of the event $B_T = B_R + B_L$, and

3. wide-jet broadening which is the larger of the two, $B_W = \max\{B_R, B_L\}$ on an event by event basis.

We define the integrated single-jet distribution as

$$\Sigma_1(B) = \int_0^B dB \sigma^{-1} \frac{d\sigma}{dB} , \quad \Sigma'_1(B) = \sigma^{-1} \frac{d\sigma}{dB} .$$

(2.2)

For the purposes of the present paper it suffices to treat the double differential distribution in $B_R, B_L$ as the product of two independent jets,

$$\sigma^{-1} \frac{d^2\sigma}{dB_R dB_L} = \sigma^{-1} \frac{d\sigma}{dB_R} \sigma^{-1} \frac{d\sigma}{dB_L} = \Sigma'_1(B_R) \cdot \Sigma'_1(B_L) .$$

(2.3)

In this approximation the total and the wide broadening distributions are related to $\Sigma_1$ through

$$\frac{d\Sigma_T(B)}{dB} = \int_0^B dB_1 \Sigma'_1(B_1) \Sigma_1(B - B_1) ,$$

(2.4)

$$\frac{d\Sigma_W(B)}{dB} = 2 \cdot \Sigma'_1(B) \int_0^B dB_N \Sigma_1(B_N) ; \quad \Sigma_W(B) = \Sigma_1^2(B) .$$

(2.5)
**Single-jet distribution.** The Mellin image of the single-jet $B$ distribution, $\sigma(\nu)$, is defined as

$$
\Sigma'_1(B) = \int \frac{d\nu}{\pi i} e^{2B\nu} \sigma(\nu),
$$

$$
\Sigma_1(B) = \int \frac{d\nu}{2\pi i \nu} e^{2B\nu} \sigma(\nu),
$$

(2.6)

where the contour runs parallel to the imaginary axis (in the second case we place the contour at $\text{Re}\,\nu > 0$ in order to satisfy the normalisation condition (2.8), see below). $\sigma(\nu)$ is a regular function of $\nu$ in the entire complex plane. It is limited (decreases) at $\text{Re}\,\nu \to +\infty$ which results in

$$
\Sigma'_1(B < 0) = \Sigma_1(B < 0) = 0.
$$

In the left half-plane it increases exponentially,

$$
\sigma(\nu) \propto e^{-2B_m\nu}, \quad \text{Re}\,\nu \to -\infty,
$$

(2.7)

with $B_m$ the maximal value of single-jet broadening. This ensures the kinematical constraints,

$$
\Sigma_1(B \geq B_m) = \sigma(0) \equiv 1,
$$

(2.8)

$$
\Sigma'_1(B \geq B_m) = 0.
$$

(2.9)

**Total broadening distribution.** The Mellin representation for the integrated total broadening distribution in the factorisation approximation follows from (2.4):

$$
\Sigma_T(B) = \int_{c-i\infty}^{c+i\infty} \frac{d\nu}{2\pi i \nu} e^{2B\nu} \sigma^2(\nu).
$$

(2.10)

**Mean single-jet broadening.** To calculate $\langle B \rangle$ we average $B$ with the differential distribution to write

$$
\langle B \rangle_1 \equiv \int_0^{B_m} BdB \Sigma'_1(B).
$$

(2.11)

The property (2.9) allows us to extend the $B$ integration in (2.11) to infinity, provided that the $\nu$-contour has been shifted to run to the left of the imaginary axis, $\text{Re}\,\nu < 0$:

$$
\langle B \rangle_1 = \int_0^\infty BdB \int_{c-i\infty}^{c+i\infty} \frac{d\nu}{\pi i} e^{2B\nu} \sigma(\nu) = \int_{c-i\infty}^{c+i\infty} \frac{d\nu}{4\pi i \nu^2} \frac{\sigma(\nu)}{\nu^2} = -\frac{1}{2} \sigma'(0).
$$

(2.12)

**Mean total broadening.** The representation (2.10) immediately leads to

$$
\langle B \rangle_T = -\frac{1}{2} \left. \frac{d}{d\nu} \sigma^2(\nu) \right|_{\nu=0} = -\sigma'(0) = 2 \langle B \rangle_1.
$$

(2.13)
Mean wide-jet broadening. The mean $B_W$, according to \(2.5\), is given by the integral
\[
\langle B \rangle_W = \int_0^{B_m} B dB \frac{d\Sigma_1^2(B)}{dB} = - \int_0^{B_m} dB \left\{ 2[\Sigma_1(B) - 1] + [\Sigma_1(B) - 1]^2 \right\}
\]
\[
= \int \frac{d\nu}{2\pi i} \frac{\sigma(\nu)}{\nu^2} \left\{ 1 + \frac{1}{2} \int \frac{d\nu_1}{2\pi i} \frac{\nu \sigma(\nu_1)}{\nu_1(\nu + \nu_1)} \right\}
\]
\[
= \int \frac{d\nu}{2\pi i} \frac{\sigma(\nu)}{\nu^2} \left[ 1 + \sigma(-\nu) \right] = -\frac{1}{2} \sigma'(0) + \int_{-\infty}^{\infty} \frac{d\nu}{2\pi i} \frac{\sigma(\nu)\sigma(-\nu)}{\nu^2}.
\]
(2.14)
The latter integral cannot be evaluated by residues, since the integrand exponentially increases in both directions, $\sigma(\nu)\sigma(-\nu) \propto \exp(\Re\nu|B_m|)$.

2.2 Kinematics of small $B$

Here we introduce the kinematical variables needed to analyse $B$ in the soft limit. We use the Sudakov representation for the momenta of the primary quark and antiquark $p_1$ and $p_2$ and of the accompanying soft partons $k_i$

\[
p_1 = A_1 P + B_1 \bar{P} + p_{t1},
\]
\[
p_2 = A_2 P + B_2 \bar{P} + p_{t2},
\]
\[
k_i = \alpha_i P + \beta_i \bar{P} + k_{ti}.
\]
(2.15)
The two light-like Sudakov vectors $P$ and $\bar{P}$ are taken along the thrust axis and their sum is the total incoming momentum ($2P\bar{P} = Q^2$).

In the soft region all quantities

\[
\alpha_i, \beta_i, (1 - A_1), (1 - B_2), \frac{k_{ti}}{Q}
\]
are small and of the same order while $A_2$ and $B_1$ are much smaller (quadratic in transverse momentum) and will be neglected. As shown in Appendix A, for the two-loop perturbative contribution one can approximate the quark-antiquark direction with that of the thrust axis, i.e. one can neglect $p_{t1}$ and $p_{t2}$. However, as explained above, the non-perturbative contribution is sensitive to the difference between the thrust and quark axis and then we shall take into account $p_{ti}$. We introduce the quark and antiquark angular variables (rescaled quark transverse momenta)

\[
\bar{p} = \frac{1}{A_1} \bar{p}_{t1}, \quad \bar{\bar{p}} = \frac{1}{B_2} \bar{p}_{t2}.
\]
(2.16)
The soft radiation matrix element depends on the invariant

\[
\frac{2(p_1k_i)(p_2k_i)}{(p_1p_2)} \simeq \frac{1}{k_{ti}^2} \left( \vec{k}_{ti} - \alpha_i \bar{p} \right)^2 \left( \vec{k}_{ti} - \beta_i \bar{\bar{p}} \right)^2,
\]
where the first (second) factor is the squared transverse momentum of $k_i$ with respect to the quark (antiquark) direction.
In the following we concentrate on soft partons emitted within the right hemisphere, near the quark direction. The kinematical constraint for such a parton is \( \alpha_i > \beta_i \). Since a massless emitted parton satisfies \( \alpha_i \beta_i = k_{ti}^2/Q^2 \),

\[
\alpha_i > k_{ti}/Q . \tag{2.17}
\]

For such a right-hemisphere parton one can neglect the difference between the thrust and antiquark directions \( (\vec{k}_{ti} - \beta_i\vec{p} \simeq \vec{k}_{ti}) \), so that the matrix element depends only on the transverse momentum with respect to the quark axis

\[
\vec{k}_{ti} \equiv \vec{k}_{ti} - \alpha_i\vec{p}. \tag{2.18}
\]

According to the definition of the thrust axis, the vector sum of the parton transverse momenta in the right (or left) hemisphere is zero and we can write

\[
0 = \vec{p}_{t1} + \sum_{i \in R} \vec{k}_{ti} = A_1\vec{p} + \sum_{i \in R} \vec{k}_{ti} \simeq \vec{p} + \sum_{i \in R} \vec{k}_{ti} . \tag{2.19}
\]

where we have neglected the small \( \alpha \)-components of the left-jet partons \( (1 - A_1 \simeq \sum_{i \in R} \alpha_i) \).

The right-jet broadening \( B_R \) is defined by

\[
2B_RQ = p_{t1} + \sum_{i \in R} k_{ti} = A_1p + \sum_{i \in R} k_{ti} \simeq p + \sum_{i \in R} (k_{ti} - \alpha_ip) , \tag{2.20}
\]

where again we have neglected the small \( \alpha \)-components of the left-jet partons.

Similarly one introduces \( B_L \), the left broadening,

\[
2B_LQ \simeq \vec{p} + \sum_{i \in L} (k_{ti} - \beta_ip) . \tag{2.21}
\]

### 2.3 Resummation and the radiator

Here we recall the resummed expression for the broadening distribution at small \( B \) \([12]\). In this region the distribution is obtained by resumming contributions from the emission of any number of soft partons and can be expressed in terms of a “radiator”. The two-loop analysis of the radiator is presented in detail in Appendix A.

The integrated single-jet broadening distribution (e.g. the right jet) is given by

\[
\Sigma_1(B) = Q^2 \sum_n \int \frac{d\sigma_n}{\sigma} \cdot \delta^2(\vec{p} + \sum_{i \in R} \vec{k}_{ti}) \Theta \left( 2BQ - p - \sum_{i \in R} (k_{ti} - \alpha_ip) \right) , \tag{2.22}
\]

where we have used \( (2.13) \) to represent the transverse momentum constraint in terms of the rescaled quark momentum \( \vec{p} \) \( (2.16) \) and the secondary parton transverse momenta with respect to the quark direction, \( \vec{k}_{ti} \). The factors \( d\sigma_n/\sigma \) are the \( n \)-parton emission distributions which factorise for \( B \ll 1 \), i.e. in the soft limit. To perform the sum over \( n \) one needs to factorise
also the constraints. This is obtained by using the Fourier and Mellin representations for
the delta and theta functions

\[
\delta^2(\vec{p} + \sum_{i \in R} \vec{\kappa}_i) \Theta \left(2BQ - p - \sum_{i \in R} (k_{ti} - \alpha_i p)\right) = \int \frac{d^2b}{(2\pi)^2} e^{-i\vec{b} \cdot \vec{p}} \int \frac{d\nu}{2\pi i \nu} e^{2B\nu} e^{-\nu p/Q} \prod_{i \in R} e^{-\nu (k_{ti} - \alpha_i p) / Q} e^{-i\vec{b} \cdot \vec{\kappa}_i}.
\]

(2.23)

where we have introduced the vector impact parameter \( \vec{b} \) conjugate to the transverse momen-
tum with respect to the quark axis (2.18). One can now resum soft-parton emissions in
the Mellin representation (2.6). The resulting integrated single-jet broadening distribution
is

\[
\Sigma_1(B) = \int \frac{d\nu}{2\pi i \nu} e^{2\nu B} \sigma(\nu), \quad \sigma(\nu) = \int \frac{d^2p}{(2\pi)^2} \frac{d^2b}{(2\pi)^2} e^{-i\vec{b} \cdot \vec{p}} e^{-\nu p/Q} e^{-R(\nu, b; p)},
\]

(2.24)

The exponent \( R \) in (2.24) is the “radiator” analysed in Appendix A. Since soft real and
virtual partons are responsible also for the non-perturbative eff ects, the radiator contains,
together with the PT contribution, also a NP correction,

\[
R(\nu, b; p) = R^{(PT)} + R^{(NP)}.
\]

(2.25)

PT component. At two-loop accuracy, the PT part of the radiator is \( p \)-independent. At
large values of \( \nu \) and/or \( bQ \), it is a function of the combination

\[
\bar{\mu} = e^{\gamma_E} \nu + \sqrt{\nu^2 + (bQ)^2},
\]

(2.26)

and reads

\[
R^{(PT)} = R(\bar{\mu}) = \int_{Q/\bar{\mu}}^{\infty} \frac{dk}{k} R'(Q/k), \quad R'(\bar{\mu}) = \frac{2C_F}{\pi} \alpha_s(Q/\bar{\mu}) \left( \ln \bar{\mu} - \frac{3}{4} \right).
\]

(2.27)

The running coupling is taken in the physical scheme [13]. This expression accommodates all
terms \( \alpha_s^n \ell^{n+1} \) and \( \alpha_s^n \ell^n \) with \( \ell = \ln \bar{\mu} \). We present \( R \) as a function of a single dimensionless
parameter \( \bar{\mu} \) but it is implied that it also contains a \( Q \)-dependence via the running coupling
\( \alpha_s(k_t) \), with \( Q/\bar{\mu} < k_t < Q \).

NP component. According to the procedure developed in [14], the leading power-suppressed
NP contribution to the radiator has the following general form, see Appendix A.2 for details,

\[
R^{(NP)} = M \frac{C_F}{\pi} \int_0^\infty \frac{dm^2}{m^2} \delta \alpha_{\text{eff}}(m^2) \cdot \Omega_0(m^2).
\]

(2.28)

Here \( M \) is the Milan factor which emerges from the two-loop analysis, and \( \delta \alpha_{\text{eff}}(m^2) \) is the
NP component of the effective coupling related to the standard \( \alpha_s \) by the dispersive relation,
with \( m^2 \) the corresponding dispersive variable acting as the gluer’s “mass”.

8
The factor $\Omega_0(m^2)$ is a “trigger function” which is specific to a given observable. A power-behaved NP contribution is determined by the leading non-analytic in $m^2 \to 0$ term in $\Omega_0$. For the broadening measure, as well as for other jet shapes, the leading non-analytic piece, $\delta \Omega_0$, is proportional to $\sqrt{m^2}$ and is given by the following expression,

$$
\delta \Omega_0(m^2) = \nu \frac{\sqrt{m^2}}{Q} \cdot \int_{p/Q}^{p/\sqrt{m^2}} \frac{du}{u} \int_0^{2\pi} \frac{d\psi}{2\pi} \left( \sqrt{1 + u^2 + 2u \cos \psi - u} \right) .
$$

(2.29)

Here

$$u = \frac{\alpha p}{\sqrt{m^2}} = \frac{p}{Q} e^\eta, \quad 0 \leq \eta < \ln \frac{Q}{m},$$

with $\eta$ the gluer rapidity with respect to the thrust axis. The expression in the brackets in (2.29) accounts for the mismatch between the quark and thrust axes. Since, with account of the azimuthal integration, the integrand falls rapidly at $u \to \infty$, the rapidity-integral, in the $m = 0$ limit, has a finite value proportional to $\ln \frac{Q}{\sqrt{m^2}}$. Thus the NP component of the radiator becomes

$$R^{(NP)} = \nu \cdot \mathcal{P} \ln \frac{p_0}{p} , \quad \eta_0 \equiv \ln \frac{p_0}{Q} \simeq -0.6137056 .
$$

(2.30)

The definition of the NP-parameter $\mathcal{P} \propto 1/Q$ introduced in [2] is recalled in Appendices A.2 and F.

We notice that if in (2.29) we formally set $p \equiv 0$, which corresponds to disregarding the perturbative quark recoil, we reconstruct the old wrong answer [2]

$$\delta \Omega_0(m^2) \bigg|_{p=0} \simeq \nu \frac{\sqrt{m^2}}{Q} \int_0^{\ln Q/m} d\eta \Rightarrow \nu \frac{\sqrt{m^2}}{Q} \cdot \left( \ln \frac{Q}{\sqrt{m^2}} - \frac{3}{4} \right) .
$$

(2.31)

where we have restored the $3/4$ “hard correction” coming from the region $\alpha \sim 1$.

In the following we recall the perturbative part of the $B$ distribution, while the non-perturbative one will be treated in the next section.

### 2.4 Recollection of the perturbative result

The $p$-integration giving the perturbative part of the single-jet distribution $\sigma(\nu)$ in (2.24) is explicitly carried out in Appendix A.1, see (A.28),

$$
\sigma^{(PT)}(\nu) = \int \frac{d^2 p d^2 b}{(2\pi)^2} e^{-ibp} e^{-\nu p/Q} e^{-\mathcal{R}(\bar{\mu})} = \int_1^{\infty} \frac{dy}{y^2} e^{-\mathcal{R}(\bar{\mu})}, \quad \bar{\mu} = e^{\gamma_E} \nu \frac{1 + y}{2} .
$$

(2.32)

To evaluate the Mellin integral we introduce the following convenient operator representation which we shall use extensively below,

$$e^{-\mathcal{R}(\bar{\mu})} = e^{-\mathcal{R}(e^{-\alpha})} \left( \bar{\mu} \right)^{-a} \bigg|_{a=0} .
$$

(2.33)

The Mellin image of the integrated single-jet distribution can then be represented as

$$\sigma^{(PT)}(\nu) = e^{-\mathcal{R}(e^{-\alpha})} \left( e^{\gamma_E} \lambda(a) \right)^{-a} \cdot \nu^{-a} \bigg|_{a=0} ,
$$

(2.34)
where we have introduced the function \[12\]
\[
\lambda^{-a} \equiv \int_1^\infty \frac{dy}{y^2} \left( \frac{1 + y}{2} \right)^{-a} = \int_0^1 \frac{dz}{\left( \frac{1 + z}{2z} \right)^{-a}} ; \quad \lambda(0) = 2, \quad \lambda(\infty) = 1. \tag{2.35}
\]
Performing the \(\nu\)-integration we arrive at
\[
\Sigma_1^{(PT)} (B) = \int \frac{d\nu}{2i\pi \nu} e^{2B\nu} \sigma^{(PT)} (\nu) = e^{-\mathcal{R}(e^{-\partial a})} \frac{1}{\Gamma(1 + a)} \left( \frac{e^{\gamma_E} \lambda(a)}{2B} \right)^{-a} \bigg|_{a=0}. \tag{2.36}
\]
Using the identity
\[
e^{-\mathcal{R}(e^{-\partial a})} x^{-a} g(a) \bigg|_{a=0} = e^{-\mathcal{R}(xe^{-\partial a})} g(a) \bigg|_{a=0},
\]
we can absorb the power factor \(x^{-a}\),
\[
x = \frac{\lambda(\mathcal{R}') e^{\gamma_E}}{2B}, \quad \mathcal{R}' = \frac{d\mathcal{R}(x)}{d\ln x}, \tag{2.37}
\]
into rescaling of the argument of the perturbative radiator:
\[
\Sigma_1^{(PT)} (B) = e^{-\mathcal{R}(xe^{-\partial a})} \frac{1}{\Gamma(1 + a)} \left( \frac{\lambda(\mathcal{R}')}{\lambda(a)} \right)^a \bigg|_{a=0}. \tag{2.38}
\]
Performing the logarithmic expansion of the radiator,
\[
-\mathcal{R}(xe^{-\partial a}) = -\mathcal{R}(x) + \mathcal{R}'(x) \partial_a - \frac{1}{2} \mathcal{R}''(x) \partial_a^2 + \ldots,
\]
we conclude that the action of the operator on a function which is \textit{regular} in the origin reduces to substituting \(\mathcal{R}'(x)\) for \(a\), while \(\mathcal{R}''(x) = \mathcal{O}(\alpha_s)\) and higher derivatives produce negligible corrections:
\[
e^{-\mathcal{R}(xe^{-\partial a})} g(a) \bigg|_{a=0} = e^{-\mathcal{R}(x)} g(\mathcal{R}'(x)) \left( 1 + \mathcal{O}(\mathcal{R}''(x)) \right). \tag{2.39}
\]
Thus we can evaluate (2.36) as
\[
\Sigma_1^{(PT)} (B) = \frac{e^{-\mathcal{R}(x)}}{\Gamma(1 + \mathcal{R}'(x))} (1 + \mathcal{O}(\alpha_s)), \tag{2.40}
\]
where the parameter \(x\) is a function of \(B\) implicitly defined by (2.37). This expression can be simplified. To this end we observe that within our accuracy we can substitute \(B^{-1}\) for \(x\) in \(\mathcal{R}'\),
\[
\mathcal{R}'(x) = \mathcal{R}'(B^{-1}) + \mathcal{O}(\alpha_s^{n+1} \ln^n B), \quad n \geq 0; \quad \mathcal{R}'(B^{-1}) \approx \frac{2C_F}{\pi} \alpha_s(QB) \ln B^{-1}.
\]
At the same time, the finite product \(xB\) should be kept in the exponent \(\mathcal{R}(x)\). We finally obtain
\[
\Sigma_1^{(PT)} (B) = \frac{e^{-\mathcal{R}(B^{-1})}}{\Gamma(1 + \mathcal{R}')}, \quad B = \frac{2B}{e^{\gamma_E} \lambda(\mathcal{R}')}, \quad \mathcal{R}' = \mathcal{R}'(B^{-1}). \tag{2.41}
\]
For the wide-jet and total broadening distributions we obtain, in a similar way,
\[
\Sigma_W (B) = \frac{e^{-2\mathcal{R}(B^{-1})}}{\Gamma^2(1 + \mathcal{R}')} , \quad \Sigma_T (B) = \frac{e^{-2\mathcal{R}(B^{-1})}}{\Gamma(1 + 2\mathcal{R}')}. \tag{2.42}
\]
These results are equivalent to those based on the steepest descent evaluation presented in [12]. The resummed \(\nu\)T answers can be given in various forms which are equivalent at the level of the leading \(\alpha_s^n \ln^{n+1} B\) and next-to-leading \(\alpha_s^n \ln^n B\) contributions to the exponent, see, e.g. equations (4.25) and (4.26) of [12]. The specific form which should be used for matching with the exact second order \(\nu\)T answer will be discussed in Appendix F.
3 Non-perturbative correction to the $B$ distribution

With account of the leading NP contribution the integrated single-jet $B$ distribution takes the form (cf. (2.32))

$$
\sigma(\nu) = \int_0^\infty b\,db\,e^{-R(\bar{\mu})} \int_0^\infty p\,dp\,e^{-\nu p/Q} \left( \frac{p}{p_0} \right)^{\nu_P} J_0(bp). \tag{3.1}
$$

Performing the $p$-integration, see (A.39) in Appendix A.2, the distribution assumes the form

$$
\sigma(\nu) = \sigma^{(\text{PT})}(\nu) + \nu P \, f(\nu) + \mathcal{O}(P^2), \tag{3.2}
$$

with the non-perturbative correction given by the following expression

$$
f(\nu) = \int_1^\infty \frac{dy}{y^2} e^{-R(\bar{\mu})} \left( 2 - \gamma_E - \eta_0 + \ln \frac{1+y}{2

\nu y^2} - y \right); \quad \eta \equiv \sqrt{1 + \left( \frac{bQ}{\nu} \right)^2}. \tag{3.3}
$$

We recall that

$$
\bar{\mu} = \frac{1}{2} e^{\gamma_E} \left( \nu + \sqrt{\nu^2 + (bQ)^2} \right) = \nu e^{\gamma_E} \frac{1+y}{2}. \tag{3.4}
$$

The function $f(\nu)$ will determine the NP corrections to the $B$ distributions and to the means. In particular, the NP effects in the distributions are given by the inverse Mellin transform of $f(\nu)$ which we calculate in terms of the operator technique introduced in (2.33). In the case of the PT distribution, the operator $\exp \{-R(xe^{-\partial a})\}$ acted upon a function $g(a)$ regular at $a=0$, see (2.39). In the case of $B_T$, as we shall see later, we need to calculate the operator $\exp \{-R(e^{-\partial a})\}$ acting upon a function $g(a) \propto 1/a$ which is singular in the origin. To this end we introduce and evaluate in Appendix B.2 the function

$$
E(x) = e^{R(x)} e^{-R(xe^{-\partial a})} a^{-1} \bigg|_{a=0} = \int_x^\infty \frac{dz}{z} e^{R(x)-R(z)}. \tag{3.5}
$$

The value $E(1)$ enters into the expression for the power correction to $\langle B \rangle$,

$$
E(1) = \frac{\pi}{2\sqrt{C_F}\alpha_s} + \frac{3}{4} - \frac{\beta_0}{6 C_F} + \mathcal{O}(\sqrt{\alpha_s}). \tag{3.6}
$$

As far as the PT component of the distribution $\sigma^{(\text{PT})}(\nu)$ in (3.2) is concerned, we have derived the resummed expression which is valid for large values of $\nu$ and determines the $B$ distributions in the $B \ll 1$ two-jet kinematics (soft limit). The $B$ distributions at large $B$, as well as the means $\langle B \rangle$, are determined by finite moments $\nu$. The soft resummation programme is irrelevant here, and the exact fixed-order analyses should be carried out instead. At the same time, the NP component $f(\nu)$ in (3.2) which originates from soft gluers remains applicable both for $B \ll 1$ distributions and the means, that is for arbitrary values of $\nu$. Having said that we turn to the calculation of the leading NP effects in mean broadenings.
3.1 Power corrections to means

Substituting the sum of perturbative and non-perturbative components, \(3.2\), into the expressions for the mean broadening \(2.12\) and \(2.13\) we obtain

\[
\langle B \rangle_T - \langle B \rangle_{(PT)}^T = 2 \left( \langle B \rangle_1 - \langle B \rangle_{(PT)}^1 \right) = -\mathcal{P} f(0) .
\]

(3.7)

The value of \(f\) in the origin is related with the function \(E\) \(3.5\) in Appendix \(B\),

\[
f(0) = - (E(1) + \eta_0) \left[ 1 + \mathcal{O}(\alpha_s) \right] .
\]

(3.8)

This results in

\[
\langle B \rangle_1 - \langle B \rangle_{(PT)}^1 \simeq \frac{\mathcal{P}}{2} \left( \frac{\pi}{2\sqrt{C_F} \alpha_s} + \frac{3}{4} - \frac{\beta_0}{6C_F} + \eta_0 \right) .
\]

(3.9)

**Mean wide-jet broadening** is given in terms of the Mellin integral in \(2.14\). Extracting the non-perturbative component we obtain

\[
\langle B \rangle_W = \langle B \rangle_{(PT)}^W + \mathcal{P} \left[ \delta - \frac{1}{2} f(0) \right] + \mathcal{O}(P^2) ,
\]

(3.10)

with

\[
\delta = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi i\nu} \left[ f(\nu)\sigma_{(PT)}(-\nu) - f(-\nu)\sigma_{(PT)}(\nu) \right] .
\]

(3.11)

This integral is evaluated in Appendix \(C\) resulting in

\[
\delta = \frac{1}{2} \left( e^{-2R(e^{-\partial a})} - e^{-R(e^{-\partial a})} \right) a^{-1} \bigg|_{a=0} .
\]

(3.12)

We note that the first operator here corresponds to \(E(1)\) evaluated with \(2R\) substituted for the radiator. This leads to

\[
\langle B \rangle_W - \langle B \rangle_{(PT)}^W = \frac{\mathcal{P}}{2} \left( e^{-2R(e^{-\partial a})} a^{-1} \bigg|_{a=0} + \eta_0 \right) \left[ 1 + \mathcal{O}(\alpha_s) \right]
\]

\[
\simeq \frac{\mathcal{P}}{2} \left( \frac{\pi}{2\sqrt{2C_F} \alpha_s} + \frac{3}{4} - \frac{\beta_0}{12C_F} + \eta_0 \right) .
\]

(3.13)

The result can be obtained from that for the single-jet (total) broadening by the simple substitution \(C_F \rightarrow 2C_F\) in \(3.9\).

3.2 NP shift in the single- and wide-jet distributions

Having obtained the \(1/Q\) power corrections to the means we now turn to the analysis of the \(\text{NP}\) effects in the distributions in the \(B \ll 1\) region. To this end we need to perform the inverse Mellin transform of the distribution \(3.2\) containing the \(\text{NP}\) term \(f(\nu)\) given in \(3.3\).
To perform the integration over $y$ (related to the impact parameter $b$, see (3.3)) we make use of the operator representation (2.33) to write

$$f(\nu) = e^{-R(e^{-\partial_b})} \left[ \frac{1}{\Gamma(1 + a)} + \frac{P}{2B} \left( C(a) + \partial_a - \frac{1}{a} \right) \frac{1}{\Gamma(a)} \right] (2B)^a \bigg|_{a=0}.$$  

The result of the $y$-integration can be represented as follows:

$$f(\nu) = e^{-R(e^{-\partial_b})} (e^{\gamma_E} \lambda)^{-a} \left( 2 - \gamma_E - \ln 2 - \eta_0 + \rho(a) - \chi(a) + \partial_a - \frac{1}{a} \right) \nu^{-a} \bigg|_{a=0},$$

where $\lambda = \lambda(a)$ is defined in (2.35) and $\rho$ and $\chi$ are given by the integrals

$$\rho(a) = \int_1^\infty \frac{dy}{y^2} \left( \frac{1 + y}{2 \lambda} \right)^{-a} \ln \frac{1 + y}{y^2},$$

$$\chi(a) = -\frac{1}{a} + \int_1^\infty \frac{dy}{y} \left( \frac{1 + y}{2 \lambda} \right)^{-a} = 2 \frac{\lambda^a - 1}{a}.$$  

The functions $\rho(a)$ and $\chi(a)$ are regular at $a = 0$ and vary slowly between $\rho(0) = -2 + 2 \ln 2$, $\rho(\infty) = \ln 2$ and $\chi(0) = 2 \ln 2$, $\chi(\infty) = 2$ correspondingly. Introducing the function

$$C(a) = 2 - \gamma_E - \ln 2 - \eta_0 + \rho(a) - \chi(a),$$

the operator expression for the $NP$ component of the distribution takes the form

$$f(\nu) = e^{-R(e^{-\partial_b})} (e^{\gamma_E} \lambda)^{-a} \left( C(a) + \partial_a - \frac{1}{a} \right) \nu^{-a} \bigg|_{a=0}.  \tag{3.15}$$

The $\nu$-integration of (3.2) is now readily performed. For the single-jet distribution we have

$$\Sigma_1(B) = e^{-R(e^{-\partial_b})} (e^{\gamma_E} \lambda)^{-a} \left[ \frac{1}{\Gamma(1 + a)} + \frac{P}{2B} \left( C(a) + \partial_a - \frac{1}{a} \right) \frac{1}{\Gamma(a)} \right] (2B)^a \bigg|_{a=0}.  \tag{3.16}$$

We observe that the potential singularity at $a = 0$ cancels in the combination

$$\left( C(a) + \partial_a - \frac{1}{a} \right) \frac{(2B)^a}{\Gamma(a)} = -a \frac{(2B)^a}{\Gamma(1 + a)} \cdot D(B, a),$$

with

$$-D(B, a) = C(a) + \ln 2B - \psi(1 + a) = \ln B - \eta_0 + 2 + \rho(a) - \chi(a) + \psi(1) - \psi(1 + a)$$

a function regular in the origin, $a = 0$. The function $\psi(x)$ is defined as $\psi(x) = d \ln \Gamma(x)/dx$. We finally arrive at

$$\Sigma_1(B) = e^{-R(e^{-\partial_b})} \frac{(e^{\gamma_E} \lambda(a))^{-a}}{\Gamma(1 + a)} \left[ 1 - a \frac{P}{2B} \tilde{D}(B, a) \right] \bigg|_{a=0}  \tag{3.17}$$

$$= e^{-R(xe^{-\partial_x})} \frac{(\lambda(\B)^{-a})}{\Gamma(1 + a)} \left[ 1 - a \frac{P}{2B} \tilde{D}(B, a) \right] \bigg|_{a=0}. $$
Here we have absorbed the power factor $x^{-\alpha}$ with $x$ defined in (2.37) into a rescaling of the argument of the $\mathrm{PT}$ radiator as we did before. As in the case of the $\mathrm{PT}$ distribution considered above, the action of the operator on a regular function results in substituting $R'(x) \simeq R'(B^{-1})$ for $a$, according to (2.39). We get

$$
\Sigma_1(B) = \frac{e^{-\mathcal{R}(B^{-1})}}{\Gamma(1 + \mathcal{R}')}
\left[ 1 - \mathcal{R}' \cdot \frac{\mathcal{P}}{2\mathcal{B}} \hat{D}(B, \mathcal{R}') \right],
$$

with $B$ defined in (2.41). This correction can be cast as a $B$-dependent $1/Q$ shift of the perturbative distribution, namely

$$
\Sigma_1(B) \simeq \Sigma_1^{(\mathrm{PT})} \left( B - \frac{\mathcal{P}}{2} D_1(B) \right),
$$

with

$$
D_1(B) = \hat{D}(B, \mathcal{R}') = \ln B^{-1} + \eta_{\mathcal{R}} - 2 - \rho(\mathcal{R}') + \chi(\mathcal{R}') + \psi(1 + \mathcal{R}') - \psi(1), \quad \mathcal{R}' = \mathcal{R}'(B^{-1}).
$$

For not too small values of $B$ such that $\mathcal{R}' \ll 1$ the shift $D_1$ assumes a simple form

$$
D_1 \simeq \ln B^{-1} + \eta_{\mathcal{R}}, \quad \mathcal{R}' \ll 1,
$$

while in the opposite limit of extremely small $B$,

$$
D_1 \simeq \ln \frac{\mathcal{R}'}{2\mathcal{B}} + \gamma_{\mathrm{E}} + \eta_{\mathcal{R}}, \quad \mathcal{R}' \gg 1.
$$

The same shift applies also to the wide-jet broadening distribution which, to the needed accuracy, is simply given by the squared single-jet distribution according to (2.5).

### 3.3 NP shift in the total broadening distribution

To obtain the integrated distribution for the total broadening, $\Sigma_T(B)$, we need to perform the inverse Mellin transform of $\sigma^2(\nu)$ in (2.10). Invoking the operator representations (2.34) and (3.15) for $\sigma^{(\mathrm{PT})}(\nu)$ and $f(\nu)$ correspondingly, we construct the product $(a, b \to 0)$

$$
\sigma^2(\nu) = e^{-\mathcal{R}(e^{-\partial_a})} e^{-\mathcal{R}(e^{-\partial_b})} (e^{\gamma_{\mathrm{E}} \lambda(a)})^{-a} (e^{\gamma_{\mathrm{E}} \lambda(b)})^{-b}
\left[ 1 + 2 \cdot \nu \mathcal{P} \left( C(a) + \partial_a - \frac{1}{a} \right) \right] \nu^{-(a+b)} \bigg|_{a=b=0} + \mathcal{O}(\mathcal{P}^2),
$$

where we have made use of the $a \leftrightarrow b$ symmetry. Evaluating the $\nu$-integral we obtain

$$
\Sigma_T(B) = e^{-\mathcal{R}(e^{-\partial_a})} e^{-\mathcal{R}(e^{-\partial_b})} (e^{\gamma_{\mathrm{E}} \lambda(a)})^{-a} (e^{\gamma_{\mathrm{E}} \lambda(b)})^{-b}
\left[ \frac{(2B)^{a+b}}{\Gamma(1 + a + b)} + 2 \cdot \frac{\mathcal{P}}{2\mathcal{B}} \left( C(a) + \partial_a - \frac{1}{a} \right) \frac{(2B)^{a+b}}{\Gamma(a + b)} \bigg|_{a=b=0} \right].
$$
Here the first term gives $\Sigma^{(PT)}(B)$ and the second one accounts for the leading $1/Q$ correction contribution. Recalling the definition of $x$ (2.37) we write $(R' = R'(x))$

$$\Sigma_T(B) = e^{-R(xe^{-\beta_0})} e^{-R(xe^{-\beta_1})} \left( \frac{\lambda(R')}{\lambda(a)} \right)^a \left( \frac{\lambda(R')}{\lambda(b)} \right)^b \Gamma(1 + a + b) \left[ 1 + \frac{P}{B} \Gamma(1 + a + b) \left( C(a) + \ln 2B + \partial_a - \frac{1}{a} \right) \frac{1}{\Gamma(a + b)} \right] \bigg|_{a=b=0}. \quad (3.23)$$

The first factor is identical to that for the perturbative distribution. The second factor can be represented as

$$1 + \frac{P}{B} \left[ (a + b) \left( C(a) + \ln 2B - \psi(1 + a + b) \right) - \frac{b}{a} \right] = 1 - \frac{P}{B} \left[ (a + b) \left( \tilde{D}(B, a) + \psi(1 + a + b) - \psi(1 + a) \right) + \frac{b}{a} \right]. \quad (3.24)$$

We observe that, in contrast to the single-jet case (cf. (3.17)), we get a correction term singular in $a$, which reminds us of the case of $\langle B \rangle$ considered above. With the unity and the non-singular term in (3.24) we proceed as before substituting $R'$ for $a$ and $b$, see (2.39). The factor $a + b$ produces $2R'$ which makes it possible to interpret the correction in terms of a shift. The contribution of the singular piece in (3.24) to the shift is calculated in Appendix D.

The final result reads

$$\Sigma_T(B) = \Sigma^{(PT)}_T \left( B - \frac{P}{2} D_T(B) \right), \quad (3.25a)$$

$$D_T(B) = 2D_1(B) + 2[\psi(1 + 2R') - \psi(1 + R')] + H(B^{-1}) , \quad R' = R(B^{-1}). \quad (3.25b)$$

Here the single-jet shift $D_1(B)$ is given in (3.20), and $B$ defined in (2.41). The $np$ shift in the total broadening distribution, (3.25), includes the function $H(B^{-1})$ which is analysed in Appendix D. The shift has a rather complicated $B$ dependence: it changes from

$$D_T(B) \approx \ln \frac{1}{B} + \text{const} , \quad \text{for} \quad \frac{\alpha_s}{\pi} \ln^2 B < 1,$$

to

$$D_T(B) \approx 2 \cdot \ln \frac{1}{B} , \quad \text{for} \quad \frac{\alpha_s}{\pi} \ln^2 B \gg 1.$$

Indeed, for moderately small values of $B$ such that $\alpha_s \ln^2 B < 1$ ($R' < \sqrt{\alpha_s}$), we have (see (D.3))

$$H(B^{-1}) \approx \ln B + \frac{\pi}{2 \sqrt{C_F} \alpha_s} + \frac{3}{4} - \frac{\beta_0}{6C_F}.$$

Together with (3.21) this results in

$$D_T(B) \approx 2 \cdot (\ln B^{-1} + \eta_0) + H(B^{-1}) \approx D_1(B) + \langle D \rangle . \quad (3.26)$$

Here the first term is the single-jet shift, $D_1 \approx \ln(\rho_0/(BQ)) = \ln B^{-1} + \eta_0$; the second term,

$$\langle D \rangle = \frac{\pi}{2 \sqrt{C_F} \alpha_s} + \frac{3}{4} - \frac{\beta_0}{6C_F} + \eta_0 , \quad (3.27)$$

is nothing but the np correction to the mean single-jet broadening \( (3.9), \)
\[
\langle B \rangle_1 - \langle B \rangle_1^{(PT)} = \frac{p}{2} \langle D \rangle .
\] (3.28)

The physical origin of this result can be simply understood. In the region under consideration, \( (R' < \sqrt{\alpha_s}) \), the multiplicity of perturbative gluon radiation is small. The non-perturbative shift is then dominated by the fluctuations in which one of the two jets is responsible for the perturbative component of the event broadening: \( B_L \ll B_R \approx B_T \), or vice versa. Non-perturbative effects in the narrower jet are better pronounced: in the absence of perturbative radiation the direction of the quark momentum stays closer to the thrust axis thus bringing in a large contribution \( \langle D \rangle \propto \frac{1}{\sqrt{\alpha_s}} \) to the shift \( D_T \). It describes the non-perturbative correction to the mean broadening of the narrower jet. This contribution is practically \( B \) independent: its residual \( B \) dependence emerges at the level of a \( \mathcal{O} \left( \sqrt{\alpha_s} \ln^2 B \right) \) correction. In these circumstances the \( B \) dependence of the total shift coincides with that of a single jet.

In the opposite regime, \( \alpha_s \ln^2 B \geq 1 \), multiple perturbative radiation becomes a way of life. Here both jets take responsibility for perturbative broadening, and the coefficient of \( \ln B \) in \( D_T \) grows bigger than unity. It doubles in the limit of extremely small \( B \) values, \( \alpha_s \ln^2 B \gg 1 \ (R' \gg \sqrt{\alpha_s}) \), where the differential spectrum flattens off (and then starts to decrease, when \( R' > 1 \)) because of severe Sudakov suppression. In this regime the jets are forced to share \( B \) equally, and we should expect
\[
D_T(B) \simeq 2D_1 (B/2) \cdot \left[ 1 + \mathcal{O} \left( (\alpha_s \ln^2 B)^{-1} \right) \right], \quad R' \gg \sqrt{\alpha_s}.
\] (3.29)

Indeed, at \( \alpha_s \ln^2 B \geq 1 \ (R' \geq \sqrt{\alpha_s}) \) we have \( H(x) \simeq 1/R' \ll \ln B^{-1} \). Near the maximum of the distribution, \( R' \geq 1 \), the relative size of the \( H \)-contribution becomes as small as \( H/D_T \simeq (R')^{-1}/\ln B^{-1} = \mathcal{O} \left( \alpha_s \right) \). The first two terms in (3.25b) then combine into (3.29).

### 4 Comparison with experimental data

#### 4.1 Means

Using formulas \( (F.12,F.13) \) we perform fits to the data for the mean total and wide-jet broadenings and compare to the values obtained for the thrust, the heavy-jet mass and the \( C \)-parameter (with fixed-order perturbative coefficients taken from \( [3] \)).

The results that we obtain are (fitting to data from \( [6, 7, 15–19] \) and references in \( [20] \), partially based on a pre-existing compilation \( [21] \))

| Variable | \( \alpha_s \) | \( \alpha_0 \) | \( \chi^2/\text{d.o.f.} \) |
|----------|----------------|----------------|-----------------|
| \( B_T \) | 0.1170 ± 0.0023 | 0.4508 ± 0.0225 | 14.9/(23 – 2) |
| \( B_W \) | 0.1189 ± 0.0025 | 0.3911 ± 0.0305 | 12.8/(22 – 2) |
| \( 1 - T \) | 0.1177 ± 0.0013 | 0.4976 ± 0.0087 | 57.0/(40 – 2) |
| \( C \) | 0.1206 ± 0.0021 | 0.4527 ± 0.0110 | 10.7/(10 – 2) |
| \( M_h^2/Q^2 \) | 0.1171 ± 0.0012 | 0.5602 ± 0.0224 | 15.2/(27 – 2) |
Figure 1: Energy dependence of $\langle B \rangle_T$ and $\langle B \rangle_W$. The dotted line is the LO result; the dashed line is LO+NLO and the solid line, LO+NLO+power correction, (3.7) and (3.13) respectively.

Figure 1 shows the a comparison between the data and the fits for $B_T$ and $B_W$.

4.2 Distributions

We consider only the $B_T$ distribution, for the sake of illustration. The relevant equations are (F.4,F.5) for the non-perturbative shift and the perturbative spectrum is as discussed in section F.2. We fit 8 data sets (using log-$R$ matching, with the distribution constrained to go to zero at the kinematic limit) [7, 16–19, 22] and obtain

| $\alpha_s$ | $\alpha_0$ | $\chi^2$/d.o.f. |
|-----------|-----------|-----------------|
| 0.1158 ± 0.0007 | 0.5368 ± 0.0077 | 68.7/(59 − 2) |

Two examples of the distributions are shown in figure 2, together with JADE (35 GeV) and OPAL (91.2 GeV) data.

5 Conclusions/Discussion

In this paper we have demonstrated that the sensitivity of the broadening measure to large parton rapidities has made the non-perturbative shift in the $B$ spectra $B$ dependent. As a result, the $\pi\pi$ distribution gets shifted to larger $B$ values and squeezed on the way, in accord with the recent experimental findings [8]. This result supersedes the previous expectation of
the ln $Q$-enhanced shift \cite{2}. It is worthwhile noticing that the correct result contains a single universal np-parameter $\mathcal{P}$ which puts broadening on an equal footing with other jet shapes $1-T$, $C$, etc. The previous (incorrect) result contained an additional np-parameter, $Q_B$ in (1.9), related to the log-moment of the coupling defined similarly to (1.3) but with an extra factor ln $k$ under the integral.
Figure 2: $B_T$ distributions compared to JADE (35 GeV) \cite{22} and OPAL (91.2 GeV) \cite{18} data. The dashed lines are resummed distributions without the power correction, while the solid lines are the resummed predictions with the power correction.
The reason why the shift in the $B$ spectra has become $B$ dependent is the interplay between $\text{PT}$ and $\text{NP}$ gluon radiation effects. The radiation of gluons with fixed transverse momentum does not depend on the gluon rapidity as defined with respect to the quark direction. At the same time, the broadening accumulates the moduli of transverse momenta of final partons with respect to the thrust axis. The latter, starts to differ from the quark direction when the normal $\text{PT}$ radiation is taken into account. This mismatch does not matter, at next-to-leading accuracy, for the $\text{PT}$ analysis but plays a crucial rôle for the $\text{NP}$ effects both in the $B$ means and distributions.

If one naively assumes that the quark direction coincides with that of the thrust axis then the contributions from gluons (gluers) with finite transverse momenta, $k_i \sim \Lambda_{\text{QCD}}$, and rapidities up to the kinematically allowed value $\eta_i \leq \eta_{\text{max}} \approx \ln(Q/k_t)$ sum up to provide the log $Q$-enhanced $\text{NP}$ shift in the $B$ spectrum, see (2.31). However, gluers with large energies, $k_{0i} \gg k_t/\Theta_q$ are collinear to the quark direction rather than to that of the thrust axis and therefore do not contribute to $B$ because of the CIS nature of the observable (gluons collinear to the quark increase $B$ by exactly the same amount by which the quark contribution to $B$ is reduced due to the longitudinal momentum recoil). As a result, the $\text{NP}$ contribution to $B$ comes out proportional to the quark rapidity, log $1/\Theta_q$, with $\Theta_q$ the quark angle with respect to the thrust axis which is due to radiation of $\text{PT}$ gluons. Since, kinematically, $\Theta_q \propto B$, averaging log $1/\Theta_q$ over the $\text{PT}$ distribution in $\Theta_q$ results in the log $B$ enhancement of the $\text{NP}$ shift.

**NP effects in the presence of PT radiation.** Disregarding the ever-present $\text{PT}$ radiation is known to produce confusing results in a number of cases. For example, the first-order (one-gluon) analysis of the $\text{NP}$ effects in the heavy-jet squared mass produced the wrong expectation: adding a single gluer to the $q\bar{q}$ system in $e^+e^-$ (as the third and only secondary parton) and constructing $M_2^2$ of the $qg$ system we find a $1/Q$ confinement contribution to the squared mass of the heavy jet, the one our gluer belongs to. Meanwhile, the opposite lighter jet acquires neither a $\text{PT}$ nor a $\text{NP}$ contribution to the mass. As a result the $\text{NP}$ correction to $M_2^2$ comes out equal to that for thrust,

$$c_T = c_{M_H^2} = c_{M_L^2} = 0.$$  

(5.1)

In reality there are always normal $\text{PT}$ gluons in the game which are responsible for the bulk of the jet mass: $M_H^2/Q^2 \sim \alpha_s > M_L^2/Q^2 \sim \alpha_s^2 \gg \delta M_{\text{NP}}^2$. In these circumstances it is the $\text{PT}$ radiation to determine which of the two jets is heavier. The gluer(s) contribute equally to both jets,

$$c_T = 2c_{M_H^2} = 2c_{M_L^2}.$$  

(5.2)

It is worthwhile remarking that experimental analyses carried out before 1998 were based on the wrong expectation \textbf{(5.1)}.

Another important example of the interplay between $\text{NP}$ and $\text{PT}$ effects is given by higher moments of jet shapes, e.g. $\langle (1-T)^n \rangle$. For such a quantity one obtains, symbolically,

$$\langle (1-T)^n \rangle \approx \alpha_s + \alpha_s \frac{\Lambda_{\text{QCD}}}{Q} + \cdots \left( \frac{\Lambda_{\text{QCD}}}{Q} \right)^n, \quad n \geq 2.$$
The leading power-suppressed contribution is at the level of $1/Q$. At the same time, the one-gluer analysis for such an observable would formally produce

$$\langle (1-T)^n \rangle_{\text{NP}} \text{ only} \approx \frac{A_n}{Q^n},$$

which NP correction is suppressed as a high power of $1/Q$. In the presence of $p_T$ radiation, however, the leading $1/Q$ contribution is still here, though reduced by the $\alpha_s(Q)$ factor but far more important than the $1/Q^n$ term.

Another “mistake” of this sort brings us closer to the $B$ issue. Consider the transverse momentum broadening of the current-fragmentation jet in Deep Inelastic Scattering (DIS), that is the sum of moduli of transverse momenta of particles in the current jet. Adding a gluer to the Born (parton model) quark scattering picture we get three equal contributions to $B$: two contributions from the quark $p$ which recoils against the gluer $k$ emitted either in the initial (IS) or in the final (FS) state, $|\vec{p}_\perp| = |\vec{k}_\perp|$, and one contribution from the gluer itself when it belongs to FS. Taking into account the $p_T$ radiation, however, the FS quark has already got a non-zero transverse momentum, $p^T_{\perp} \sim \alpha_s \cdot Q$, a substantial amount compared to $k_\perp \sim \Lambda_{\text{QCD}}$. In this environment the direct gluer’s contribution is the only one to survive: the NP recoil upon the quark gets degraded down to a $1/Q^2$ effect after the azimuthal average is performed,

$$\langle |\vec{p}_\perp| \rangle = \left\langle |\vec{p}^T_{\perp} - \vec{k}_\perp| \right\rangle = p^T_{\perp} + \mathcal{O} \left( \frac{\Lambda^2_{\text{QCD}}}{p^T_{\perp}} \right).$$

The true magnitude of the $1/Q$ contribution turns out to be a factor three smaller than that extracted from the one-gluer analysis.

A preliminary health report. Now that the loophole in the theoretical treatment of the jet broadenings has been eliminated, one can return to the much debated issue of the universality of confinement effects in event shapes.

To illustrate the overall consistency of the universality hypothesis we show in Figure 3 2-standard-deviation contours in the $\alpha_s$-$\alpha_0$ plane for a range of means and distributions. The curves labelled “old” are the results to the fits using the old (wrong) formulas. The situation with the total broadening distribution is greatly improved by the updated theoretical treatment. We expect the wide broadening distribution to be equally improved, but this remains to be verified.

The fits for $\alpha_s$ and $\alpha_0$ from the mean values are also generally consistent with each other and with those from the distributions. However, the agreement between different event shapes is still not perfect. In the case of the heavy-jet mass we believe that this may in part be related to the treatment of particle masses, which have more effect on jet masses than on the thrust or the $C$-parameter (which are both defined exclusively in terms of 3-momenta). We leave this question for future consideration.

Outlook. Another important issue is that of the power correction to the jet-broadening in DIS. Formally the extension of our results to the DIS case is quite a non-trivial exercise.
Figure 3: 95% CL contours for jet shape means (dashed) and some distributions (solid). The curves for the $T$, $C$ and old $B_T$ and $B_W$ distributions are taken from [8]. The curves for the means are to be taken as purely indicative since we have not accounted for the correlations between systematic errors (which, where available, are added in quadrature to the statistical errors). Additionally for some observables we may not have found all the available data.

As a first step it would be necessary to carry out a resummed PT calculation for the DIS broadening. This has so far not been done.

The situation for the mean broadening measured with respect to the thrust axis is fairly simple though, since (modulo factors of two associated with the definition of the broadening in DIS [23]) it is equivalent to a single hemisphere in $e^+e^-$:

$$\langle B \rangle_{\text{DIS, thrust}} - \langle B \rangle_{\text{DIS, thrust}}^{(PT)} = \mathcal{P} \left( \frac{\pi}{2\sqrt{2} C_F \alpha_{\text{CMW}}(Q)} + \frac{3}{4} - \frac{\beta_0}{6C_F} + \eta_0 + \mathcal{O} \left( \sqrt{\alpha_s} \right) \right). \quad (5.3)$$

For the mean broadening defined with respect to the photon ($z$) axis the situation is more complicated because of the dependence on perturbative initial-state radiation. To a first approximation, at moderate $x$, one can view the DIS event as a rotated $e^+e^-$ event where the broadening is measured in the right hemisphere with respect to the axis of the quark in the left hemisphere: i.e. the relevant transverse momentum for determining the rapidity available to the NP correction is $p = |\vec{p}_1 + \vec{p}_2|$. Since this is very similar to $\max(p_1, p_2)$ we have a situation like that for the wide-jet broadening, and the leading power correction is suppressed by factor $\sqrt{2}$ compared to (5.3):

$$\langle B \rangle_{\text{DIS, } z} - \langle B \rangle_{\text{DIS, } z}^{(PT)} = \mathcal{P} \left( \frac{\pi}{2\sqrt{2} C_F \alpha_{\text{CMW}}(Q)} + \frac{3}{4} - \frac{\beta_0}{12C_F} + \eta_0 + \mathcal{O} \left( 1 \right) \right). \quad (5.4)$$

Even though we have chosen to include some subleading terms of $\mathcal{O}(1)$, it is likely that there are other terms of $\mathcal{O}(1)$, arising through the $x$ dependence of the problem. In particular,
for the important case of small x which is presently being studies at HERA the process is
dominated by the boson-gluon fusion mechanism and the analogy with the two quark jets in
e+e- gets lost. Parton multiplication in the initial state of the DIS process is more intensive
(t-channel gluon exchange) and increases with ln 1/x leading to increasing characteristic
transverse momentum of the struck quark, which contributes to B_{DIS,z}.

Still more damaging will be x-dependent effects at 1−x ≪ 1 where phase-space restriction
on prtransverse parton momenta may destroy the leading 1/√α_s term through a dependence
on ln(1−x).

So (5.4) can be used only for moderate x, and even then one should remember that it is
subject to subleading corrections of O(1).

Finally let us remark that 1/Q power effects can be envisaged (and should be studied)
also for other jet characteristics such as shape variables of 3-jet events (thrust minor, acoplanar-
nity) as well as, for example, in the distributions of the accompanying E_T flow in hadronic
collisions and DIS.

## Acknowledgements

We are grateful to Siggi Bethke, Othmar Biebel, Mrinal Dasgupta, Pedro Movilla Fernandez,
Klaus Hamacher, Hasko Stenzel, Bryan Webber, Daniel Wicke and Giulia Zanderighi for
helpful discussions and suggestions.

## A Perturbative and non-perturbative radiator

The two-loop radiator in the soft limit is given by (see [2])

\[
R(\nu, b; p) = 4C_F \int \frac{d\alpha d^2\kappa_t}{\alpha \pi \kappa_t^2} \left( \frac{\alpha_s(0)}{4\pi} + \chi(\kappa_t^2) \right) [1 - u(k)]
+ 4C_F \int d\Gamma_2(k_1, k_2) \left( \frac{\alpha_s}{4\pi} \right)^2 \frac{1}{2!} M^2(k_1, k_2) [1 - u(k_1) u(k_2)] ,
\]

where, for the right-broadening distribution, the source u(k) is given by (see [2,23])

\[
1 - u(k) = \left[ 1 - e^{-\nu(k_t - \alpha p)/Q} e^{-i\vec{b}\vec{k}_t} \right] \theta(\alpha - k_t/Q) , \quad \vec{k}_t = \vec{n}_t + \alpha \vec{p}
\]

with the right-hemisphere constraint (2.17) included.

The function \( \chi(\kappa_t^2) \) is the virtual correction to one-gluon emission. In the physical scheme
which defines the coupling as the intensity of soft-gluon radiation [13], it can be written in terms of the dispersive integral

\[
\chi(\kappa_t^2) = \int_0^\infty \frac{d\mu^2 \kappa_t^2}{\mu^2(\kappa_t^2 + \mu^2)} \left( \frac{\alpha_s}{4\pi} \right)^2 \left\{ -2C_A \ln \frac{\kappa_t^2(\kappa_t^2 + \mu^2)}{\mu^4} \right\}.
\]
The collinear divergence in $\chi$ is compensated by that of the real two-parton contribution described by the matrix element $M(k_1, k_2)$. Taken together with an ill-defined $\alpha_s(0)$ of one-gluon emission, they participate in forming a finite running coupling.

The two-parton phase space in \(\text{(A.1)}\) is

$$d\Gamma_2(k_1, k_2) = dm^2 \frac{d^2\kappa_t}{\pi} \frac{d\alpha}{\alpha} \cdot dz \frac{d\phi}{2\pi}, \quad \text{(A.4)}$$

The first three variables $\alpha = \alpha_1 + \alpha_2$, $\kappa_t = \kappa_{t1} + \kappa_{t2}$ and $m^2 = (k_1 + k_2)^2$ are those of the parent gluon $k$, while $z$, $1-z$ and $\phi$ are the momentum fractions and the relative azimuth of the two secondary partons, $q\bar{q}$ or gluons:

$$\alpha_1 = z\alpha, \quad \alpha_2 = (1-z)\alpha, \quad m^2 = z(1-z)q_t^2, \quad \vec{q}_t = \frac{\vec{k}_{t1}}{z} - \frac{\vec{k}_{t2}}{1-z}. \quad \text{(A.5)}$$

Hereafter we choose $\phi$ as the angle between 2-vectors $\vec{q}_t$ and $\vec{\kappa}_t$.

The probing functions $u(k_1)$, $u(k_2)$ depend on all the parton variables. In order to extract the contribution responsible for the running coupling we split

$$1 - u(k_1)u(k_2) = [1 - u(k)] + [u(k) - u(k_1)u(k_2)], \quad \text{(A.6)}$$

where we have introduced a “probing function” $u(k)$ for the parent gluon. There are various ways to define the source $u(k)$ if $k$ is massive. The prescription we choose consists in substituting $\kappa_t^2 \rightarrow \kappa_t^2 + m^2$ in the massless expression \(\text{(A.2)}\). In particular, the transverse momentum with respect to the thrust axis, $k_t$,

$$k_t^2 = \kappa_t^2 + (\alpha p)^2 + 2\alpha p k_t \cos \psi_{kp},$$

gets replaced by

$$k_t' = \kappa_t^2 + m^2 + (\alpha p)^2 + 2\alpha p \sqrt{\kappa_t^2 + m^2} \cos \psi_{kp}. \quad \text{(A.7)}$$

Thus we define the inclusive probe (first term in \(\text{(A.3)}\)) by

$$1 - u(k) = \left(1 - e^{-\nu(k_t' - \alpha p)/Q} e^{-i\beta_0 / \sqrt{\kappa_t^2 + m^2} \cos \psi_{kp}} \right) \cdot \partial(\alpha - k_t'/Q), \quad \text{(A.8)}$$

The inclusive contribution depends only on the parent gluon variables, so that the two-parton matrix element $M^2(k_1, k_2)$ can be integrated over $z$ and $\phi$ to give (see \[2\])

$$\int dz \frac{d\phi}{2\pi} 2! M^2(k_1, k_2) = \frac{1}{m^2(m^2 + \kappa_t^2)} \left\{ -\beta_0 + 2C_A \ln \left( \frac{\kappa_t^2 + m^2}{m_4} \right) \right\}, \quad \text{(A.9)}$$

with $\beta_0 = \frac{11}{3} C_A - \frac{2}{3} n_f$ the first beta function coefficient. We treat the $\beta_0$ term of this equation together with $\alpha_s(0)$ to form the so-called naive contribution, the logarithmic term together with $\chi(\kappa_t^2)$ (inclusive correction), and the second term of the trigger function \(\text{(A.6)}\) as the non-inclusive correction.

At the perturbative level, the naive term provides the dominant contribution, while the inclusive and non-inclusive terms stay at the level of the next-to-next-to-leading two-loop correction. As far as the non-perturbative $1/Q$ correction to the perturbative radiator is concerned, all three give comparable contributions, the latter two providing the so-called rescaling Milan factor to the naive one.
Naive contribution. The naive contribution reads

\[
R_0(\nu, b; p) \equiv 4C_F \int \frac{dm^2 d\kappa_t^2}{\kappa_t^2 + m^2} \left\{ \alpha_s(0)\delta(m^2) - \frac{\beta_0}{m^2} \left( \frac{\alpha_s}{4\pi} \right)^2 \right\} \cdot \Omega_0 \left( \kappa_t^2 + m^2 \right). \tag{A.10}
\]

Here we have introduced the “naive trigger function” representing the \([1 - u(k)]\) factor integrated over \(\alpha\) and \(\psi_{kp}\):

\[
\Omega_0 \left( \kappa_t^2 + m^2 \right) = \frac{d\alpha}{\alpha} \int_{-\pi}^{\pi} \frac{d\psi_{kp}}{2\pi} \left[ 1 - e^{-\nu(k_t - \alpha p)/Q} e^{-ib\sqrt{\kappa_t^2 + m^2} \cos \psi_{kp}} \right], \tag{A.11}
\]

with \(k_t'\) defined in (A.7). The lower limit of the logarithmic \(\alpha\)-integration, corresponding to separation between the two hemispheres, is actually \(k_t'/Q\). However for \(p \ll Q\), this limit can be approximated by the \(p\)-independent value given here.

Inclusive correction. The inclusive correction can be represented in terms of a difference between the naive trigger functions for \(m \neq 0\) (real) and \(m = 0\) (virtual contribution) as

\[
R_{in}(\nu, b; p) = 8C_F C_A \int \frac{dm^2 d\kappa_t^2}{\kappa_t^2 + m^2} \left( \frac{\alpha_s}{4\pi} \right)^2 \int \frac{d\kappa_t^2}{\kappa_t^2 + m^2} \ln \left( \frac{\kappa_t^2 + m^2}{m^4} \right) \left( \Omega_0(\kappa_t^2 + m^2) - \Omega_0(\kappa_t^2) \right). \tag{A.12}
\]

Non-inclusive correction. Finally, the non-inclusive correction describes the mismatch between the actual contribution to \(B_R\) from two partons and that of their parent:

\[
R_{ni}(\nu, b; p) = 4C_F \int dm^2 d\kappa_t^2 d\phi \int_{-\pi}^{\pi} \frac{dz}{2\pi} \left( \frac{\alpha_s}{4\pi} \right)^2 \frac{1}{2!} M^2(m^2, \kappa_t^2, z, \phi) \cdot \Omega_{ni}, \tag{A.13}
\]

with the non-inclusive “trigger function”

\[
\Omega_{ni} \equiv \int_0^1 \frac{d\alpha}{\alpha} \int \frac{d\psi}{2\pi} \left[ u(k_1 + k_2) - u(k_1)u(k_2) \right]. \tag{A.14}
\]

A.1 PT contribution to the radiator

In the perturbative evaluation of the naive contribution we neglect \(m^2 \ll \kappa_t^2 \ll Q^2\) in the trigger function and use the dispersive relation for \(\alpha_s\),

\[
\int_0^{Q^2} \frac{dm^2 \kappa_t^2}{m^2 + \kappa_t^2} \left\{ \alpha_s(0)\delta(m^2) - \frac{\beta_0}{m^2} \left( \frac{\alpha_s}{4\pi} \right)^2 \right\} = \alpha_s(\kappa_t) \cdot \left( 1 + \mathcal{O} \left( \frac{\alpha_s^2}{\kappa_t^2} \right) \right), \quad \tag{A.15}
\]

to obtain

\[
R_0^{(PT)}(\nu, b; p) = \frac{C_F}{\pi} \int \frac{d\kappa_t^2}{\kappa_t^2} \alpha_s(\kappa_t) \cdot \Omega_0(\kappa_t^2). \quad \tag{A.16}
\]

The inclusive and the non-inclusive terms do not contribute to the PT radiator at two loops. This is due to the fact that, in spite of the singular behaviour of the matrix element, \(M^2 \propto \).
1/m^2, the m^2-integrals in (A.12) and (A.13) converge, because the trigger functions \( \Omega_0(\kappa_t^2 + m^2) - \Omega_0(\kappa_t^2) \) and \( \Omega_{ni} \) vanish in the collinear parton limit. As a result, these contributions are determined by the non-logarithmic integration regions \( \kappa_t \sim m \sim Q/\nu \) and provide negligible corrections to the radiator of the order of \( \alpha_s^2 \ln \nu \implies \alpha_s^2 \ln B \), with a single-logarithmic enhancement factor originating from the \( \alpha_s \) (rapidity-) integration.

The PT expression (A.16) can be greatly simplified, to our accuracy, by observing that one can neglect the difference between quark and thrust axis and set \( p = 0 \) in the trigger function (A.11). Moreover, for large \( \nu \) and \( b \) one can substitute the trigger function \( \Omega_0 \) by a suitably chosen cutoff in the phase space integration.

**Negligible \( p \neq 0 \) effect.** As we shall see later, the effects of quark recoil \( (p \neq 0) \) modifies the structure of the answer for the non-perturbative contribution. However, at the perturbative level a small departure of the quark direction from that of the thrust axis proves to be negligible, producing a \( \mathcal{O}(\alpha_s^2 \ln B) \) subleading correction which lies beyond the scope of the approximation, i.e. keeping under control terms \( \mathcal{O}(\alpha_s^n \ln^m B) \) with \( m \geq n \) [12, 24].

To verify this we observe that the difference between the PT radiator (A.16), (A.11) and its \( p = 0 \) value,

\[
R^{(\text{PT})}(b, \nu; 0) = \frac{C_F}{\pi} \int_0^{Q^2} \frac{d\kappa_t^2}{\kappa_t^2} \alpha_s(\kappa_t) \ln \frac{Q}{\kappa_t} \left[ 1 - e^{-\nu \kappa_t/Q} J_0(b \kappa_t) \right],
\]

in the large-\( \nu \) limit is a function of the ratios \( \nu p/Q \) and \( b Q/\nu \),

\[
R^{(\text{PT})}(b, \nu; p) - R^{(\text{PT})}(b, \nu; 0) = f \left( \frac{\nu p}{Q}, \frac{b Q}{\nu} \right) + \mathcal{O} \left( \frac{1}{\nu} \right).
\]

To verify this we introduce the rescaled dimensionless variables \( q = \frac{\nu \kappa_t}{\alpha_Q} \) and \( \tilde{u} = \frac{b Q}{\nu} \) to write

\[
R^{\text{PT}}(p) - R^{\text{PT}}(0) = \frac{C_F}{2\pi} \int_0^1 \frac{d \alpha}{\alpha} \alpha_s \int_0^\nu \frac{d^2 q}{\pi} \left\{ 1 - e^{-\alpha(q-\nu p/Q)} e^{-i \alpha \tilde{u} (\tilde{q} - \nu \tilde{p})/Q} (\tilde{q} - \nu \tilde{p})^2 - \frac{1 - e^{-\alpha q} e^{-i \alpha \tilde{u} \tilde{q}}}{\tilde{q}^2} \right\}.
\]

Eq. (A.18) follows from the fact that the \( q \)-integral converges and can be extended to run up to \( q = \infty \) instead of \( q = \nu \). As a result, it does not depend explicitly on \( \nu \) except through the combinations \( \nu p/Q \) and \( b Q/\nu \). In the essential region, \( b Q \sim \nu \sim Q/p \gg 1 \), these combinations are of the order one, so that

\[
f \left( \frac{\nu p}{Q}, \frac{b Q}{\nu} \right) = \alpha_s(Q) \cdot \mathcal{O}(1).
\]

**Radiator as a function of a single variable.** The final simplification of (A.17) for large \( \nu \) and \( b \) is obtained by the replacement [12]

\[
[1 - e^{-\nu \kappa_t/Q} J_0(b \kappa_t)] \Rightarrow \Theta(\kappa_t - Q/\bar{\mu}), \quad \bar{\mu} \equiv e^{\gamma_E} \frac{\nu + \sqrt{\nu^2 + (bQ)^2}}{2},
\]
which introduces a negligible error $O(\alpha_s)$ to the radiator and gives, in the next-to-leading logarithmic approximation,

$$ R^{(PT)}(\bar{\mu}) = \frac{2C_F}{\pi} \int_{Q/\bar{\mu}}^Q \frac{dk_t}{k_t} \alpha_s(k_t) \left( \ln \frac{Q}{k_t} - \frac{3}{4} \right). \quad (A.20) $$

Here we have replaced the soft $d\alpha/\alpha$ factor in the trigger function (A.11) by the exact $q \rightarrow qg$ splitting probability to account for a “hard” subleading correction originating from the region $\alpha \sim 1$. The result is the $-3/4$ term in the integrand. While necessary for achieving single-logarithmic accuracy in the perturbative treatment, the hard part of the kernel is irrelevant, as we shall see below, for the non-perturbative contribution which is dominated by the region $\alpha \sim \kappa_t \sim m \ll Q$.

We give expressions for the radiator for three forms of the coupling:

- For fixed coupling
  
  $$ R^{(PT)}(\bar{\mu}) = \frac{C_F \alpha_s}{\pi} \left( \ell^2 - \frac{3}{4} \ell \right), \quad \ell \equiv \ln \bar{\mu}. \quad (A.21) $$

- Taking into account the one-loop running of $\alpha_s$ we obtain
  
  $$ R^{(PT)}(\bar{\mu}) = \frac{4C_F}{\beta_0} \int_0^\ell d\ell' \frac{\ell'}{L - \ell'} = -\frac{4C_F}{\beta_0} \left[ \left( L - \frac{3}{4} \right) \ln \left( 1 - \frac{\ell}{L} \right) + \ell \right], \quad L \equiv \ln \frac{Q}{\Lambda}. \quad (A.22) $$

- To evaluate the radiator with the two-loop running coupling we use
  
  $$ \alpha_s(Q) = \frac{2\pi}{\beta_0(L + \gamma \ln L)}, \quad \beta_1 = \frac{19}{3}n_f, \quad (A.23) $$

  to obtain

  $$ R^{(PT)}(\bar{\mu}) = \frac{4C_F}{\beta_0} \left\{ \left( T - \frac{3}{4} \right) \ln \frac{T}{T_{\bar{\mu}}} - (T - T_{\bar{\mu}}) - \frac{1}{2} \gamma \ln^2 \frac{T}{T_{\bar{\mu}}} \right\}. \quad (A.24) $$

Here

$$ T \equiv \frac{2\pi}{\beta_0 \alpha_s(Q)} + \gamma, \quad T_{\bar{\mu}} \equiv \frac{2\pi}{\beta_0 \alpha_s(Q/\bar{\mu})} + \gamma, \quad (A.25) $$

and $\alpha_s$ is the running coupling in the so-called physical scheme [13] which can be related to the standard $\overline{MS}$ coupling by (F.7).

The radiators with the two-loop and one-loop $\alpha_s$ deviate at the level of a $O(\alpha_s^3 \ln^3 \bar{\mu})$ term which contribution is under control and should be kept in the $\cal PT$ distributions. At the same time, for the sake of simplicity we use the (two-loop) radiator with the one-loop coupling [A.22] for evaluating the $\cal NP$ contributions to the means and spectra. For example, in the means we know that $\beta_0$ appears as a constant term, suppressed with respect to the leading term by a factor of $\sqrt{\alpha_s}$, see (3.9). It is clear therefore that a $\beta_1$ contribution will enter at the level of a term of $O(\alpha_s)$, which is beyond our control.

In conclusion, the $\cal PT$ part of the Mellin transform of the single-jet broadening in (2.24) reads

$$ \sigma^{(PT)}(\nu) = \int_0^\infty bdb \ e^{-R^{(PT)}(\bar{\mu})} \int_0^\infty pdp \ e^{-\nu p/Q} J_0(bp). \quad (A.26) $$
The $p$-integration gives
\[ \int_0^{\infty} \frac{p dp}{Q^2} e^{-\nu p/Q} J_0(bp) = \frac{1}{\nu^2 y^3}, \quad y \equiv \frac{\sqrt{\nu^2 + (bQ)^2}}{\nu}, \quad (A.27) \]
which results in
\[ \sigma^{(\text{PT})}(\nu) = \int_1^{\infty} \frac{dy}{y^2} e^{-\bar{\rho}(\text{PT})} \; \bar{\mu} = \frac{1}{2} e^{\gamma_E} (1 + y) \nu. \quad (A.28) \]

### A.2 Non-perturbative contribution of the radiator

To extract the power-suppressed contribution to the radiator we use the procedure developed in [14]. It is based on introducing the effective coupling $\alpha_{\text{eff}}(m^2)$ related to the usual coupling constant $\alpha_s$ via the dispersive relation
\[ \frac{\alpha_s(k)}{k^2} = \int_0^{\infty} \frac{dm^2}{m^2} \frac{\alpha_{\text{eff}}(m^2)}{(m^2 + k^2)^2}. \quad (A.29) \]
We then substitute the non-perturbative “effective coupling modification” $\delta \alpha_{\text{eff}}$ for $\alpha_{\text{eff}}$ and look for the leading non-analytic in $m^2$ term in the $m^2 \to 0$ limit.

To obtain the NP contribution to the radiator we substitute
\[ \left\{ \alpha_s(0) \delta(m^2) - \frac{\beta_0}{m^2} \frac{\alpha_s^2}{4\pi} + \cdots \right\} \cdot \alpha_{\text{eff}}(m^2) = -\frac{d}{dm^2}. \quad (A.30) \]
in the naive, inclusive and non-inclusive contributions.

#### Naive contribution.
We obtain
\[ R_0(\nu, b; p) = \frac{C_F}{\pi} \int_0^{Q^2} \frac{dm^2}{m^2} \alpha_{\text{eff}}(m^2) \left( -\frac{d}{dm^2} \right) \int_0^{Q^2} \frac{d\kappa_t^2}{\kappa_t^2 + m^2} \Omega_0(\kappa_t^2 + m^2) \]
\[ \quad = \frac{C_F}{\pi} \int_0^{Q^2} \frac{dm^2}{m^2} \alpha_{\text{eff}}(m^2) \Omega_0(m^2) + O(\alpha_s(Q)). \quad (A.31) \]
At the PT level, this expression is equivalent to (A.16), with the non-logarithmic perturbative $\alpha_s(Q)$ correction coming from the region $m^2 \sim Q^2$.

To trigger the leading power correction in (A.31) we substitute $\delta \alpha_{\text{eff}}$ for $\alpha_{\text{eff}}$ and consider the leading non-analytic in $m^2$ term $\Omega(m^2) \propto \sqrt{m^2}$ which is obtained by linearising $\Omega$ in $m \sim \kappa_t, Q \alpha \ll Q$. In this approximation the NP component of the trigger function, $\delta \Omega$, does not depend on $b$, and we get
\[ \delta \Omega_0(m^2) = \nu \frac{\sqrt{m^2}}{Q} \int_{p/Q}^{p/\sqrt{m^2}} \frac{du}{u} \int_0^{2\pi} \frac{d\psi}{2\pi} \left( \sqrt{1 + u^2 + 2u \cos \psi} - u \right), \quad (A.32) \]
where we have introduced $u = \alpha p/\sqrt{m^2}$ as an integration variable. The $u$-integral converges and is determined by the region $p/Q < u \lesssim 1$. Therefore, in the $m^2 \to 0$ limit we can replace
the actual upper limit, $p/\sqrt{m^2} \gg 1$, by $\infty$ (neglecting the $\mathcal{O}(m^2/p)$ contribution to $\delta\Omega$, which is analytic in $m^2$ and thus does not produce a NP correction). We have then

$$\delta\Omega_0(m^2) = \frac{\sqrt{m^2}}{Q} \cdot \rho(p), \quad \rho \equiv \nu \left( \ln \frac{p_0}{p} + \mathcal{O} \left( \frac{p}{Q} \right) \right),$$

(A.33)

where the integration constant $p_0/Q$ is given by

$$\ln \frac{p_0}{Q} = \int_0^\infty \frac{du}{u} \int_0^{2\pi} \frac{d\psi}{2\pi} \left[ (\sqrt{1 + u^2} + 2u \cos \psi - u) - \vartheta(1 - u) \right] = -0.6137056 \equiv \eta_0.$$

Recalling the definition of the first non-analytic moment of $\delta\alpha_{\text{eff}}$,

$$A_1 = \frac{C_F}{2\pi} \int_0^\infty \frac{dm^2}{m^2} \cdot m \, \delta\alpha_{\text{eff}}(m^2),$$

we finally obtain

$$R^{(\text{NP})}_0(\nu, b; p) = \frac{C_F}{2\pi} \int_0^\infty \frac{dm^2}{m^2} \cdot m \, \delta\alpha_{\text{eff}}(m^2) \cdot \delta\Omega(m^2) = \nu \cdot \frac{2A_1}{Q} \ln \frac{p_0}{p}.$$

(A.34)

**Milan factor.** It is straightforward to verify that the inclusive and non-inclusive trigger functions, in the linear approximation in $\kappa_t \sim m$, are proportional to the same function $\rho(p)$ that determines the naive trigger function $\delta\Omega_0$ given in (A.33). We have

$$\delta\Omega_{in} = \rho \cdot \left( \sqrt{\kappa_t^2 + m^2} - \kappa_t \right),$$

(A.35)

$$\delta\Omega_{ni} = \rho \cdot \left( \kappa_{t1} + \kappa_{t2} - \sqrt{\kappa_t^2 + m^2} \right).$$

(A.36)

Such a structure is typical for $1/Q$ power corrections to various jet shapes and leads to the *universal* rescaling of the naive contribution (A.34) by the so-called Milan factor, for details see [3].

With account of the Milan factor, the full two-loop NP component of the broadening radiator reads

$$R^{(\text{NP})} = R^{(\text{NP})}_0 \cdot \mathcal{M} = \nu \cdot \mathcal{P} \ln \frac{p_0}{p}, \quad \mathcal{P} = \frac{2A_1}{Q} \mathcal{M}.$$

(A.37)

In conclusion, the Mellin transform of the single-jet broadening in (3.1) reads

$$\sigma(\nu) = \int_0^\infty b db e^{-\mathcal{R}(\bar{\mu})} \int_0^\infty p dp \, e^{-\nu p/Q} J_0(bp) \left( \frac{p}{p_0} \right)^{\nu P}.$$

(A.38)

We need to evaluate the $p$, $b$ and $\nu$ integrals. The result of the $p$ integration reads

$$\int_0^\infty p dp \, e^{-\nu p/Q} \left( \frac{p}{p_0} \right)^{\nu P} J_0(bp) = \frac{I_P(\nu, y)}{\nu^2 y^3}, \quad y \equiv \sqrt{\nu^2 + (bQ)^2}.$$

(A.39)

where $I_P(\nu, y)$ is related to Legendre function and has the following small-$\mathcal{P}$ expansion,

$$I_P(\nu, y) = \Gamma(2 + \nu \mathcal{P}) \left( \nu y \frac{p_0}{Q} \right)^{-\nu \mathcal{P}} y P_{1+\nu \mathcal{P}} \left( \frac{1}{y} \right)$$

$$= 1 + \nu \mathcal{P} \left( 2 - \gamma_E - \eta_0 + \ln \frac{1+y}{2y^2} - \ln \nu - y \right) + \mathcal{O}(\mathcal{P}^2) \quad \eta_0 \equiv \ln \frac{p_0}{Q}.$$

(A.40)
B Calculation of $f(0)$

To calculate the non-perturbative correction to $\langle B \rangle_1 = \frac{1}{2} \langle B \rangle_T$ we need to find the value of $f$ in the origin. To this end we split $f(\nu)$ given in (3.3) into two pieces:

$$f(\nu) = f_1(\nu) + f_2(\nu)$$

$$f_1(\nu) = \int_0^\infty \frac{\nu \, b \, db}{(b^2 + \nu^2)^{3/2}} \left( 2 - \gamma_E - \eta_0 + \ln \frac{\nu + \sqrt{b^2 + \nu^2}}{2(b^2 + \nu^2)} \right) e^{-\mathcal{R}(\bar{\mu})}, \quad (B.1)$$

$$f_2(\nu) = - \int_0^\infty \frac{b \, db}{b^2 + \nu^2} e^{-\mathcal{R}(\bar{\mu})}.$$  

Here and in the rest of this section, for simplicity, we measure $b$ in units of $Q$.

After extracting $-\ln \nu$ from the first term, the remaining integral in $y = \sqrt{b^2 + \nu^2}/\nu$ converges and is therefore determined by the region $b \sim \nu \to 0$ where the radiator can be dropped as $\mathcal{O}(\alpha_s)$. We get

$$f_1(\nu \to 0) = 2 - \gamma_E - \eta_0 + \rho(0) - \ln 2\nu + \mathcal{O}(\alpha_s) \simeq \ln 2 - \eta_0 - \gamma_E - \ln \nu. \quad (B.2)$$

The second piece which is logarithmic in $b$ we integrate by parts to write

$$f_2(\nu \to 0) = - \frac{1}{2} \ln \left. (b^2 + \nu^2) \right|_{b=0}^{b=\infty} e^{-\mathcal{R}(\bar{\mu})} + \frac{1}{2} \int_0^\infty d\left( e^{-\mathcal{R}(\bar{\mu})} \right) \ln \left( b^2 + \nu^2 \right)$$

$$= \ln \nu + \int_0^\infty d \left( e^{-\mathcal{R}(\bar{\mu})} \right) \ln b, \quad \bar{\mu} \to \frac{1}{2} e^{\gamma_E} b, \quad \text{with} \ \nu \to 0. \quad (B.3)$$

In the region of finite $b = \mathcal{O}(1)$ the true radiator is $\mathcal{O}(\alpha_s)$ and its contribution to the integral there can be disregarded. To make it more explicit we cut the integral from below at some value $b > b_0 \sim 1$ and integrate by parts back again:

$$\int_{b_0}^\infty d \left( e^{-\mathcal{R}(\gamma_E b/2)} \right) \ln b = - \ln b_0 \cdot (1 + \mathcal{O}(\alpha_s)) - \int_{b_0}^\infty \frac{db}{b} \ e^{-\mathcal{R}(\gamma_E b/2)}. \quad (B.4)$$

The answer to the required accuracy does not depend on $b_0$, which can therefore be chosen arbitrarily. It is convenient to take $b_0 = 2e^{-\gamma_E}$ in order to represent

$$f_2(\nu \to 0) = \ln \nu - (\ln 2 - \gamma_E) - \int_1^\infty \frac{dz}{z} \ e^{-\mathcal{R}(z)}, \quad (B.4)$$

which finally gives

$$f(0) = -\eta_0 - \int_1^\infty \frac{dz}{z} \ e^{-\mathcal{R}(z)} = -\eta_0 - E(1). \quad (B.5)$$

B.1 Mean straight from the distribution (consistency check)

We used the residue of $\sigma(\nu)/\nu^2$ at $\nu = 0$ to calculate $\langle B \rangle$ according to (2.12). Here we verify that the same answer follows, within our accuracy, from integrating the spectrum:

$$\langle B \rangle_1 = \int_0^B dB \frac{d}{dB} (\Sigma_1(B) - 1) = - \int_0^B dB \left( \Sigma_1(B) - 1 \right)$$

$$= \langle B \rangle^{\text{PT}}_1 - \int_0^B dB \left( \Sigma_1(B) - \Sigma_1^{(\text{PT})}(B) \right). \quad (B.6)$$
The NP component of the mean can be obtained directly by integrating (3.16) over $B$ up to the kinematical boundary $B_m \sim 1$: 
\[
\langle B \rangle_1 - \langle B \rangle_{(PT)}_1 = -\frac{P}{2} e^{-\mathcal{R}(e^{-\alpha_s})} \frac{C(a) - \psi(1 + a) + \ln(2B_m) - a^{-1}}{\Gamma(1 + a)} \left( \frac{2B_m}{e^{\gamma_E \lambda}} \right)^a. \tag{B.7}
\]

Setting $a \sim \mathcal{R}' \sim \alpha_s$ to zero in the non-singular pieces we get 
\[
\langle B \rangle_1 - \langle B \rangle_{(PT)}_1 = -\frac{P}{2} \left[ \ln B_m - \eta_0 - e^{-\mathcal{R}(e^{-\alpha_s})} \frac{1}{a} \frac{\langle B_m \rangle^a}{\Gamma(1 + a) e^{\gamma_E}} \right]. \tag{B.8}
\]

In the final term, singular in $a$, the last factor should be expanded to first order in $a$. Using 
\[
\left( \frac{2}{\lambda(a)} \right)^a = 1 + \mathcal{O} \left( a^2 \right), \quad \Gamma(1 + a) e^{\gamma_E} = 1 + \mathcal{O} \left( a^2 \right),
\]
we observe that the dependence on the kinematical boundary value $B_m$ cancels, 
\[
e^{-\mathcal{R}(e^{-\alpha_s})} \frac{\langle B_m \rangle^a}{a} \simeq \ln B_m + e^{-\mathcal{R}(e^{-\alpha_s})} \frac{1}{a},
\]
and the result given in (3.9) is reproduced.

### B.2 Calculation of $E(x)$

We define the function 
\[
E(x) = e^{\mathcal{R}(x)} e^{-\mathcal{R}(xe^{-\alpha_s})} a^{-1} \bigg|_{a=0}. \tag{B.9}
\]
Replacing 
\[
\frac{1}{a} = \int_1^\infty \frac{dy}{y} y^{-a}, \tag{B.10}
\]
we apply the rule (2.33) to absorb the $y^{-a}$ factor into a rescaling of the argument of $\mathcal{R}$, get rid of the differential operator and represent $E$ in terms of the logarithmic integral, 
\[
E(x) = e^{\mathcal{R}(x)} \int_1^\infty \frac{dy}{y} e^{-\mathcal{R}(xy)} = \int_x^\infty \frac{dz}{z} e^{\mathcal{R}(x) - \mathcal{R}(z)}. \tag{B.11}
\]

The perturbative radiator (as a function of a single variable $x > 1$) is defined in (2.27). The first coefficients of the logarithmic expansion of the PT-radiator with the one-loop $\alpha_s$ ($\ell = \ln x$, $L = \ln Q/\Lambda$) are
\[
\mathcal{R}'(x) = \frac{4C_F \ell - \frac{3}{4}}{\beta_0 (L - \ell) \pi} = \frac{2C_F \alpha_s(Q/x)}{\pi} \left( \ln x - \frac{3}{4} \right), \tag{B.12}
\]
\[
\mathcal{R}''(x) = \frac{4C_F}{\beta_0} \frac{L - \frac{3}{4}}{(L - \ell)^2} = \frac{2C_F \alpha_s(Q/x)}{\pi} \frac{\alpha_s(Q/x)}{\alpha_s(Q)} \tag{B.13}
\]
\[
\mathcal{R}'''(x) = \frac{8C_F}{\beta_0} \frac{L - \frac{3}{4}}{(L - \ell)^3} = \frac{\beta_0}{2C_F} \left( \mathcal{R}''(x) \right)^2 \frac{\alpha_s(Q)}{\alpha_s(Q/x)} \tag{B.14}
\]
\[
\mathcal{R}^{(n)}(x) = (2\mathcal{R}''(x))^{\frac{n}{2}} \frac{(n - 1)!}{2} \left( \frac{\beta_0^2 \alpha_s(Q)}{16\pi C_F} \right)^{\frac{n-2}{2}}. \tag{B.15}
\]
with $Q = Qe^{-\frac{3}{4}}$. We also point out the structure of the combination

$$s = \frac{\mathcal{R}'(x)}{\sqrt{2\mathcal{R}''(x)}} = \frac{\sqrt{C_F\alpha_s(Q)}}{\pi} \left(\ln x - \frac{3}{4}\right). \quad (B.16)$$

The next step is to substitute $(\ln x - \partial_a)$ for $\ln x$ to construct the operator

$$\mathcal{R}(x) - \mathcal{R}(xe^{-\partial_a}) = \mathcal{R}'(x)\partial_a - \frac{1}{2}\mathcal{R}''(x)\partial_a^2 + \frac{1}{6}\mathcal{R}'''(x)\partial_a^3 + \ldots \quad (B.17)$$

We start by noticing that

$$F(a; \mathcal{R}'') = \exp \left\{ -\frac{1}{2}\mathcal{R}''(a) \right\} a^{-1} = \sqrt{2\mathcal{R}''} N \left( \frac{a}{\sqrt{2\mathcal{R}''}} \right), \quad (B.18)$$

where the function $N$ is related with the probability integral,

$$N(t) = \frac{2e^{t^2}}{\sqrt{\pi}} \int_t^\infty dx \ e^{-x^2} = e^{t^2}(1 - \Phi(t)) = \frac{2}{\sqrt{\pi}}e^{t^2}\text{Erfc}(t), \quad (B.19)$$

and has the following behaviour:

$$N(t) = 1 - \frac{2t}{\sqrt{\pi}} + t^2 - \frac{4t^3}{3\sqrt{\pi}} + \ldots, \quad t \ll 1, \quad (B.20)$$

$$N(t) = \frac{1}{\sqrt{\pi} t} \left[1 - \frac{1}{2t^2} + \frac{3}{4t^4} + \ldots \right], \quad t \gg 1. \quad (B.21)$$

As a result,

$$F(a; \mathcal{R}'') \simeq a^{-1} \quad \text{for } a \gg \sqrt{\mathcal{R}''}. \quad (B.22)$$

Now we introduce the first derivative to obtain

$$\exp \left\{ \mathcal{R}'\partial_a - \frac{1}{2}\mathcal{R}''\partial_a^2 \right\} a^{-1} = e^{\mathcal{R}'\partial_a} F(a; \mathcal{R}'') = F(a + \mathcal{R}'; \mathcal{R}'') = \sqrt{\frac{2}{\pi}} N \left( \frac{a + \mathcal{R}'}{\sqrt{2\mathcal{R}''}} \right). \quad (B.22)$$

To estimate contributions of higher derivatives, $n \geq 3$, we use (B.13) to derive

$$\frac{\mathcal{R}^{(n)}(\partial_a)^n}{n!} F(a + \mathcal{R}'; \mathcal{R}'') = \left[ \frac{\beta_0}{16C_F n} \frac{\alpha_s(Q)}{\alpha_s(Q/x)} \left( \frac{\beta_0^2 \alpha_s(Q)}{16\pi C_F} \right) \right]^{\frac{n-3}{2}} \left. \frac{d^n N(t)}{dt^n} \right|_{t=a+\sqrt{2\mathcal{R}''}}. \quad (B.23)$$

We conclude that $n = 3$ contributes at the level of $O(1)$, while the contributions of higher derivatives are down by the factor $(\sqrt{\alpha_s})^{n-3} \ll 1$.

Evaluating the third derivative,

$$\exp \left\{ \frac{\mathcal{R}'''\partial_a^3}{6} \right\} F(a + \mathcal{R}'; \mathcal{R}'') \simeq \left( 1 + \frac{\mathcal{R}'''\partial_a^3}{6} \right) F(a + \mathcal{R}'; \mathcal{R}''), \quad (B.24)$$

we obtain

$$\left. \frac{\mathcal{R}'''\partial_a^3}{6} F(a + \mathcal{R}'; \mathcal{R}'') \right|_{a=0} = -\frac{\mathcal{R}'''}{3(\mathcal{R}'')^2} X(s) = -\frac{\beta_0}{6C_F \alpha_s(Q/x)} \frac{\alpha_s(Q)}{\alpha_s(Q/x)} X(s), \quad (B.25)$$

32
where the function $X$ has been introduced,

$$X(t) = -\frac{\sqrt{\pi}}{8} \frac{d^3 N(t)}{dt^3} = \int_0^\infty zdz e^{-z^2 t^2}. \quad (B.26)$$

It has the asymptotic behaviour

$$X(t) = 1 - \frac{3\sqrt{\pi}}{2} t + 4t^2 + \mathcal{O}(t^3), \quad t \ll 1; \quad (B.27)$$

$$X(t) = \frac{3}{4} t^4 + \mathcal{O}(t^6), \quad t \gg 1. \quad (B.28)$$

For (B.9) we finally obtain

$$E(x) = \sqrt{\frac{\pi}{2}} \frac{\mathcal{R}'}{2 \mathcal{R}''} N \left( \frac{\mathcal{R}'}{\sqrt{2 \mathcal{R}''}} \right) - \frac{\beta_0}{6C_F} X \left( \frac{\mathcal{R}'}{\sqrt{2 \mathcal{R}''}} \right) + \ldots \quad (B.29)$$

The neglected terms in (B.29) are of relative order $\alpha_s$.

For $s \equiv \mathcal{R}'/\sqrt{2 \mathcal{R}''} \ll 1$ (see (B.16)) we substitute for $N$ and $X$ the expansions (B.20), (B.27) and obtain, keeping contributions up to $\mathcal{O}(\sqrt{\alpha_s})$,

$$E(x) \simeq \frac{\pi}{2 \sqrt{C_F} \alpha_s(Q/x)} \left[ \frac{\pi}{\alpha_s(Q/x)} + \frac{3}{4} (\ln x - \frac{3}{4}) \right] - \ln x + \frac{3}{4} - \frac{\beta_0}{6C_F} + \mathcal{O}(\sqrt{\alpha_s} \ln^2 x). \quad (B.30)$$

We get

$$E(x) = \frac{\pi}{2 \sqrt{C_F} \alpha_s(Q)} - \ln x + \frac{3}{4} - \frac{\beta_0}{6C_F} + \mathcal{O}(\sqrt{\alpha_s}). \quad (B.31)$$

For $x = 1$ (the operator entering the expressions for mean broadening) this gives

$$E(1) = \frac{\pi}{2 \sqrt{C_F} \alpha_s(Q)} + \frac{3}{4} - \frac{\beta_0}{6C_F} + \mathcal{O}(\sqrt{\alpha_s}). \quad (B.32)$$

### C Calculation of $\delta$

Here we calculate the Mellin integral necessary to determine $\langle B \rangle_W$,

$$\delta = \frac{1}{2} \int_{-i\infty}^{i\infty} \frac{dv}{2\pi iv} \left[ f(v)\sigma^{(PT)}(-v) - f(-v)\sigma^{(PT)}(v) \right]. \quad (C.1)$$
Invoking the operator representations \( (2.34) \) and \( (3.15) \) for \( \sigma^{(\text{PT})}(\nu) \) and \( f(\nu) \) correspondingly, we obtain

\[
\delta = e^{-\mathcal{R}(z e^{-\delta_0})} e^{-\mathcal{R}(z e^{-\delta_0})} \int_{-i\infty}^{i\infty} \frac{d\nu}{4\pi i\nu} (F(a) - F(b)) \left( \nu \frac{\lambda(a)}{2} \right)^{-a} \left( -\nu \frac{\lambda(b)}{2} \right)^{-b},
\]

with \( z = 2e^{\gamma_E} \) and

\[
F(a) = C(a) + \partial_a - \frac{1}{a}.
\]

It is implied that we have to set \( a = b = 0 \) after applying the differentiations. Now we introduce the real integration variable \( v = -i\nu \), add the negative and positive \( v \)-beams into \( 2\text{Im} \), and start the \( v \)-integration from some finite value \( v_0 = \mathcal{O}(1) \) so as to ensure applicability of the large-\( \nu \) logarithmic expression for the \( \text{PT} \) radiator. This gives

\[
\delta = e^{-\mathcal{R}(z e^{-\delta_0})} e^{-\mathcal{R}(z e^{-\delta_0})} \int_{v_0}^{\infty} \frac{dv}{2\pi v} (F(a) - F(b)) v^{-(a+b)} \sin \frac{\pi(b-a)}{2} \left( \frac{\lambda(a)}{2} \right)^{-a} \left( \frac{\lambda(b)}{2} \right)^{-b}.
\]

Observing that the regular pieces \( C(a), C(b) \) cancel in the difference \( F(a) - F(b) \) at the level of \( \mathcal{O}(\alpha_s) \), and that the ratios \( \lambda/2 \) produce negligible corrections \( \mathcal{O}(a^2 + b^2) \), we are left with

\[
\delta = e^{-\mathcal{R}(z e^{-\delta_0})} e^{-\mathcal{R}(z e^{-\delta_0})} \left\{ \partial_a - \partial_b - \frac{1}{a} + \frac{1}{b} \right\} (v_0)^{-(a+b)} \sin \frac{\pi(b-a)}{2\pi} \bigg|_{a=b=0}
\]

\[
= e^{-\mathcal{R}(v_0 z e^{-\delta_0})} e^{-\mathcal{R}(v_0 z e^{-\delta_0})} \left\{ \frac{1}{a+b} \left( \partial_a - \partial_b - \frac{1}{a} + \frac{1}{b} \right) \right\} \left( \frac{b-a}{4} + \mathcal{O}((a-b)^3) \right) \bigg|_{a=b=0},
\]

where we have absorbed the power of \( v_0 \) into additional rescaling of the arguments of the \( \text{PT} \) radiators \( \mathcal{R} \). Evaluating the derivatives and using the \( a \leftrightarrow b \) symmetry we arrive at

\[
\delta = \frac{1}{2} e^{-\mathcal{R}(v_0 z e^{-\delta_0})} e^{-\mathcal{R}(v_0 z e^{-\delta_0})} \left( \frac{1}{a+b} - \frac{1}{a} \right) \bigg|_{a=b=0} = \frac{1}{2} \left( e^{-2\mathcal{R}(v_0 z e^{-\delta_0})} - e^{-\mathcal{R}(v_0 z e^{-\delta_0})} \right) a^{-1} \bigg|_{a=0}.
\]

The finite rescaling of the argument of the \( \mathcal{R} \) operator by the factor \( v_0 z \) produces, according to \( (3.31) \), a subleading correction \( \ln(v_0 z) = \mathcal{O}(1) \) which is of the relative order \( \sqrt{\alpha_s} \) and should be kept under control. These subleading corrections however are identical for the two terms and cancel in the difference, thus ensuring independence of the result on the arbitrary parameter \( v_0 \), at the \( \mathcal{O}(\alpha_s) \) level. We conclude,

\[
\delta = \frac{1}{2} \left( e^{-2\mathcal{R}(e^{-\delta_0})} - e^{-\mathcal{R}(e^{-\delta_0})} \right) a^{-1} \bigg|_{a=0}.
\]

\[
(C.2)
\]

According to \( (3.10) \), to obtain \( \langle B \rangle_W \) we have to add to \( (C.2) \) half of the \( \text{NP} \) correction to single-jet \( \langle B \rangle \), that is \( -\frac{1}{2} f(0) \) with \( f(0) \) given in \( (3.3) \). The main piece of the latter cancels the subtraction contribution described by the single-jet operator applied to \( 1/a \), and we finally arrive at

\[
\langle B \rangle_W - \langle B \rangle_{W}^{(\text{PT})} = \frac{1}{2} \mathcal{P} \left( e^{-2\mathcal{R}(e^{-\delta_0})} a^{-1} \bigg|_{a=0} + \eta_0 \right)
\]

\[
= \frac{1}{2} \mathcal{P} \left( \frac{\pi}{2\sqrt{2C_F\alpha_s(Q)}} + \frac{3}{4} - \frac{\beta_0}{12C_F} + \eta_0 \right).
\]

\[
(C.3)
\]

This result can be obtained from that for a single-jet (total) broadening by the simple substitution \( C_F \to 2C_F \) in \( (3.31) \).
D Analysis of $D_T$

Consider the singular piece of (3.24),

$$S = e^{-\mathcal{R}(xe^{-\frac{y}{a}})} e^{-\mathcal{R}(xe^{-\frac{y}{b}})} \frac{\left(\frac{\lambda(\mathcal{R}')}{\lambda(a)}\right)^{a} \left(\frac{\lambda(\mathcal{R}')}{\lambda(b)}\right)^{b}}{\Gamma(1+a+b)} \cdot \left(-\frac{b}{a}\right).$$  \hspace{1cm} (D.1)

Using (B.10) we trade the $1/a$ factor for the logarithmic integral of the exponent of the PT radiator with the rescaled argument, as we did in (B.11), to obtain

$$S = \mathcal{R}'(x) e^{-\mathcal{R}(x)} \int_{1}^{\infty} \frac{dy}{y} e^{-\mathcal{R}(xy)} \frac{\lambda(\mathcal{R}'(x))}{\lambda(\mathcal{R}'(xy))} \frac{\mathcal{R}'(xy)}{1+\mathcal{R}'(x) + \mathcal{R}'(xy)} \left[1 + \mathcal{O}(\alpha_s)\right]$$

$$\simeq -\mathcal{R}' \cdot e^{-\mathcal{R}(x)} \int_{x}^{\infty} \frac{dz}{z} e^{-\mathcal{R}(z)} \frac{\lambda(\mathcal{R}'(z))}{\lambda(\mathcal{R}'(z))} \frac{\mathcal{R}'(z)}{1+\mathcal{R}' + \mathcal{R}'(z)}, \quad \mathcal{R}' \equiv \mathcal{R}'(x).$$  \hspace{1cm} (D.2)

The integrand is monotonically decreasing with $z$. Extracting the PT distribution, we write

$$S = -\mathcal{R}' \cdot e^{-2\mathcal{R}(x)} \cdot H(x), \quad H(x) = \int_{x}^{\infty} \frac{dz}{z} e^{\mathcal{R}(x) - \mathcal{R}(z)} \frac{\Gamma(1+2\mathcal{R}')}{\Gamma(1+\mathcal{R}' + \mathcal{R}'(z))},$$  \hspace{1cm} (D.3)

where we have dropped the factor containing the ratio of $\lambda$ functions as it produces a negligible effect. Taken together with the regular piece in (3.24) this leads to the expression for the shift reported in (3.25).

For $\mathcal{R}' \ll 1$ we can expand the ratio of the $\Gamma$ functions in (D.3) to get

$$H(x) = \int_{x}^{\infty} \frac{dz}{z} e^{\mathcal{R}(x) - \mathcal{R}(z)} \left[1 - \gamma_E \mathcal{R}' + \gamma_E \mathcal{R}'(z) + \ldots\right]$$

$$= (1 - \gamma_E \mathcal{R}') \cdot E(x) + \gamma_E + \mathcal{O}(\alpha_s E(x)) .$$  \hspace{1cm} (D.4)

The answer for $H(B^{-1})$ depends on the relation between $\mathcal{R}'$ and $\sqrt{\mathcal{R}'} \sim \sqrt{\alpha_s}$.

For $s \ll 1$ ($\mathcal{R}' \ll \sqrt{\alpha_s}$) (D.4) gives

$$H(B^{-1}) = E(B^{-1}) + \gamma_E + \mathcal{O}(s) \simeq \left(\ln B + \frac{\pi}{2 \sqrt{C_F} \alpha_s} + \frac{3}{4} - \frac{\beta_0}{6 C_F}\right) + \gamma_E$$

$$\simeq \ln B + \frac{\pi}{2 \sqrt{C_F} \alpha_s} + \frac{3}{4} - \frac{\beta_0}{6 C_F},$$  \hspace{1cm} (D.5)

where we have substituted, according to (2.41), $B \simeq B e^{-\gamma_E}$ (for small $\mathcal{R}'$ one has $\lambda(\mathcal{R}')/2 = 1 + \mathcal{O}(\mathcal{R}')$).

For $s \gg 1$ ($\mathcal{R}' \gg \sqrt{\alpha_s}$) we have instead

$$H(B^{-1}) \simeq \mathcal{R}'^{-1},$$

which contribution to the shift becomes negligible,

$$\frac{H}{|D_T|} \sim \frac{1}{\mathcal{R}' \cdot \ln B + \text{const}} \sim \frac{1}{\alpha_s \ln^2 B} \sim s^{-2} \ll 1 .$$

35
For $R' \gg 1$, the contribution of $H$ is even smaller, $H(B^{-1}) \ll (R')^{-1}$, due to additional fall-off of the $\Gamma$-function in the integrand of (D.3).

We conclude that the approximate expression (D.4),

$$H(B^{-1}) \simeq (1 - \gamma_E R') E(B^{-1}) + \gamma_E ,$$

with $B$ given in (2.41), can be used everywhere, for arbitrary $\ln B$ values.

### E Numerical cross-checks and illustrations

In this Appendix we present a numerical approach which allows us to illustrate some important properties of the results and to estimate (some) next-to-next-to leading effects not accounted for by the approximate analytical expressions derived in the paper.

We express the factorised $n$-soft-parton cross section through an equation governing a branching process, which can be implemented as a Monte Carlo “event” generator — where by event one means an ensemble of soft gluons. For the accuracy to which we consider the jet broadening, it will be sufficient to consider just the emissions of gluons from the quark and antiquark, ignoring their subsequent branching [12].

Considering for the time being only a single jet, if the last gluon was produced with transverse momentum $k_{t,i-1}$, the probability distribution for the next gluon’s transverse momentum ($k_{t,i} < k_{t,i-1}$) is given by

$$k_{t,i} \frac{dP}{dk_{t,i}} = 2\bar{\alpha}_s \left( \ln \frac{Q}{k_{t,i}} - \frac{3}{4} \right) \Delta(k_{t,i-1},k_{t,i}),$$

where $\bar{\alpha}_s = \alpha_s C_F / \pi$ and $\Delta$ resums the virtual corrections:

$$\ln \Delta(k_{t,i-1},k_{t,i}) = - \int_{k_{t,i}}^{k_{t,i-1}} \frac{dk_t}{k_t} 2\bar{\alpha}_s \left( \ln \frac{Q}{k_t} - \frac{3}{4} \right).$$

Technically, the branching starts from a “fake” gluon, $i = 0$, with

$$k_{t,0} = Q e^{-3/4},$$

which does not contribute to any of the momentum sums. The azimuthal direction of each $k_{t,i}$ is chosen at random. The quark recoil, $p_t$ is

$$p_t = \sum_{i=1}^{\tilde{k}_{t,i}},$$

and the broadening, as in (2.1), is

$$B = \frac{1}{2Q} \left( p_t + \sum_{i=1}^{\tilde{k}_{t,i}} \right).$$
Implementing the branching as a Monte Carlo generator makes the vector sum extremely simple — and correspondingly so the analysis of the mean \( \ln p_t \) as a function of \( B \), or also averaged over \( B \).

If one uses \( \alpha_s(\kappa_t) \) in the CMW, or “physical” scheme [13], then apart from the overall normalisation, the accuracy of the resulting description is the same as that of the resummed expressions of [12, 24], namely next-to-leading-logarithmic in the exponent.

In practice, when comparing with the analytical results given in the main part of the paper it is most informative to consider the results in the limit of fixed \( \alpha_s \) (i.e. setting \( \beta_0 = 0 \)) so as to avoid having to introduce an arbitrary infrared cutoff in the numerical calculation.

We recall that the power correction to the broadenings depends linearly on the logarithm of the quark transverse momentum in each jet. Accordingly, from (3.20) we expect that \( \langle \ln p_t/Q \rangle - \ln B \) in a single jet with broadening \( B \), should be a function only of \( R'(B) \), and we test this by plotting \( \langle \ln p_t/Q \rangle - \ln B \) versus \( R' \). The resulting curves should then be independent of \( \alpha_s \). Figure 4a shows such curves for two values of \( \alpha_s \), compared with the theoretical result. In the Monte Carlo results, there is a small dependence on the value of \( \alpha_s \) — corresponding roughly to a shift of the curves by an amount of order \( \alpha_s \). This is a sublogarithmic effect and so beyond the accuracy of the analytical calculations (and in any case beyond the predictive value of the Monte Carlo). For small values of \( \alpha_s \) there is good agreement between the Monte Carlo and the analytical results.

When considering the total jet broadening, \( B_T \), we just generate two independent single-jet configurations (2.3), and construct the combination \( \langle \ln p_{t,R}/Q + \ln p_{t,L}/Q \rangle - 2 \ln B_T \) which is a function of \( R'(B_T) \) only for large values of \( R' \). For small \( R' \), it goes as \( 1/\sqrt{\alpha_s} \). Accordingly, in figure 4b the curves for different values of \( \alpha_s \) differ at smaller \( R' \). However the analytic predictions and the Monte Carlo results remain in good agreement, except roughly at the level of a shift of order \( \alpha_s \) as in the case of the single jet curves.

While the above figures demonstrate the agreement between the Monte Carlo and the analytical results, they do not illustrate the shifts themselves as a function of \( B \). For the case of a fixed coupling, this is done in figure 5. The main features are the following: \( D_W \) is practically independent of \( \alpha_s \) and almost equals \( \ln 1/B \). For very small \( B \), \( D_T \) is practically twice \( \ln 1/B \). For larger \( B \) one can see that the slope of the \( D_T \) curves tends to that of the \( D_W \) curves. Finally at large \( B \) one sees an offset in \( D_T \) which increases as \( 1/\sqrt{\alpha_s} \).

**F  Collection of final formulas**

We collect here for convenience the final expressions for the broadening distributions and means, which include \( 1/Q \) confinement effects and were used for the phenomenological analysis presented in section 4.
Figure 4: (a) $\langle \ln p_t/Q \rangle - \ln B_1$ as a function of $R'$. The Monte Carlo results are shown for various values of $\alpha_s$ while the analytic curve is independent of $\alpha_s$; (b) $\langle \ln p_{t,R}/Q + \ln p_{t,L}/Q \rangle - 2 \ln B_T$ from the Monte Carlo and from (3.25). The Monte Carlo and theory curves for $\alpha_s = 0.01$ lie on top of each other.
F.1 Expressions for the shifts

The integrated wide-jet broadening distribution:

$$\sigma^{-1} \int_0^B dB_W \frac{d\sigma}{dB_W} \equiv \Sigma_W(B) = \Sigma_W^{(PT)}(B - \frac{1}{2}\mathcal{P}D_1(B))$$ (F.1)

with the $B$-dependent shift $D_1$ given by

$$D_1(B) = \ln B^{-1} + \eta_0 - 2 - \rho(R') + \chi(R') + \psi(1 + R') - \psi(1),$$

$$R' = 2C_F \frac{\alpha_s(BQ)}{\pi} \ln B^{-1} - \frac{3}{4}, \quad \eta_0 = -0.6137056,$$ (F.2)

and $\psi(z)$ the derivative of the logarithm of $\Gamma(z)$. The functions $\rho$ and $\chi$ are

$$\rho(a) = \int_0^1 dz \left( \frac{1 + z}{2z\lambda(a)} \right)^{-a} \ln z (1 + z), \quad \chi(a) = \frac{2}{a} \left( [\lambda(a)]^a - 1 \right),$$

$$[\lambda(a)]^{-a} \equiv \int_0^1 dz \left( \frac{1 + z}{2z} \right)^{-a}.$$ (F.3)

The integrated total broadening distribution:

$$\sigma^{-1} \int_0^B dB_T \frac{d\sigma}{dB_T} \equiv \Sigma_T(B) = \Sigma_T^{(PT)}(B - \frac{1}{2}\mathcal{P}D_T(B))$$ (F.4)

3 We remark that in $R'$ the $3/4$ is beyond the accuracy that we control, however we choose to keep it because it has a clear origin, and is among the next-to-leading corrections to the power-suppressed contribution.
with the $B$-dependent shift $D_T$ given by
\[
D_T(B) = 2D_1(B) + 2[\psi(1 + 2\mathcal{R}') - \psi(1 + \mathcal{R}')] + H(B^{-1}),
\]
\[
H(x) = \int_x^{z_0} \frac{dz}{z} e^{\mathcal{R}(x) - \mathcal{R}(z)} \frac{\Gamma(1 + 2\mathcal{R}')}{\Gamma(1 + \mathcal{R}' + \mathcal{R}'(z))}, \quad B = \frac{2B}{e^{\gamma_E \lambda(\mathcal{R}')}}.
\]
where $z_0$ corresponds to the position of the Landau pole in the perturbative radiator $\mathcal{R}(z)$, where the integrand vanishes. The form that we use for $\mathcal{R}$ is the two-loop radiator with the one-loop coupling, which, in the physical ($\text{CMW}$) scheme has the simple expression
\[
\mathcal{R}(x) = -\frac{4C_F}{\beta_0} \left[ \left(L - \frac{3}{4}\right) \ln \left(1 - \frac{\ln x}{L}\right) + \ln x \right]
\]
where $L = \ln Q/\Lambda = 2\pi/(\beta_0 \alpha_{\text{CMW}}(Q))$, and the physical coupling $\alpha_{\text{CMW}}$ is related to the standard $\alpha_{\text{MS}}$ by
\[
\alpha_{\text{CMW}} = \alpha_{\text{MS}} \left(1 + K \frac{\alpha_{\text{MS}}}{2\pi}\right),
\]
with
\[
K \equiv C_A \left(\frac{67}{18} - \frac{\pi^2}{6}\right) - \frac{5}{9} n_f, \quad \beta_0 = \frac{11C_A}{3} - \frac{2n_f}{3}.
\]
The two-loop radiator with the two-loop coupling is given in Appendix A.1 in \cite{12345}.

### F.2 PT spectra

The resummed expressions (2.42) that we derived for the PT spectra are applicable in the small-$B$ region, and have next-to-leading-logarithmic accuracy. For the purposes of phenomenology, it is necessary to extend the domain of validity of the perturbative spectra towards larger values of $B$, where it is sufficient to use a fixed calculation. Procedures (the $R$ and log-$R$ matching schemes) for combining the fixed order and resummed results are explained in detail in \cite{25} for the thrust and heavy-jet mass distributions, and are directly applicable also to the case of the broadenings. The requirement for the use of these procedures (in particular $R$-matching) is that the resummed perturbative answer have only the following terms
\[
\ln \Sigma(V) = C_1 \alpha_s + \sum_{n=1}^{\infty} G_{n,n+1} \alpha_s^n \ln^{n+1} V + \sum_{n=1}^{\infty} G_{n,n} \alpha_s^n \ln^n V,
\]
and it mustn’t have terms such as $\alpha_s^2 \ln V$. Suitable expressions for the broadenings were presented in \cite{24}, equations (18–22). As was shown in \cite{12} these answers have to be modified by an additional factor (both for the wide and total broadenings):
\[
\Sigma^{(\text{PT})}(B) = \left(\frac{2}{\lambda(\mathcal{R}')}\right)^{2\mathcal{R}'} \Sigma_{\text{CTW}}^{(\text{PT})}(B).
\]
\footnote{For enthusiasts only!}
It is vital that $\mathcal{R}'$ here be taken as:

$$
\mathcal{R}' = \frac{2\alpha_s(Q) C_F}{\pi} \frac{\ln(1/B)}{1 - \frac{\alpha_s(Q) \beta_0}{2\pi} \ln(1/B)}.
$$

We stress that the $3/4$ in $\mathcal{R}'$ is a next-to-next-to-leading effect, and as such is taken care of by the matching procedure\footnote{We note that the $\beta_0$ that we use in this paper differs from that in [24] by a factor of $4\pi$.} For that procedure to remain intact, it must not be included in (F.11).

### F.3 Means

The leading power correction to the mean total broadening is

$$
\langle B \rangle_T - \langle B \rangle_{T}^{(PT)} = \mathcal{P} \left( \frac{\pi}{2\sqrt{C_F \alpha_{CMW}(Q)}} + \frac{3}{4} - \frac{\beta_0}{12C_F} + \eta_0 + \mathcal{O} (\sqrt{\alpha_s}) \right).
$$

and the correction to the mean wide-jet broadening is

$$
\langle B \rangle_W - \langle B \rangle_{W}^{(PT)} = \mathcal{P} \left( \frac{\pi}{2\sqrt{2C_F \alpha_{CMW}(Q)}} + \frac{3}{4} - \frac{\beta_0}{12C_F} + \eta_0 + \mathcal{O} (\sqrt{\alpha_s}) \right).
$$

Here, $\bar{Q} = Q e^{-3/4}$. The use of $\bar{Q}$ rather than $Q$ as the scale for $\alpha_s$, and the choice of $\alpha_{CMW}$ rather than $\alpha_{\overline{MS}}$ both affect the results at the level of a $\mathcal{O} (\sqrt{\alpha_s})$ term, which formally we do not control. However we prefer to keep these corrections since they have clear physical origins (the $e^{-3/4}$ factor in the scale has about a 5% effect on the fitted value of $\alpha_0$, while the change from cmw to $\overline{MS}$ schemes has much less effect).

### F.4 Non-PT parameter

In order to accurately define the non-perturbative parameter $\mathcal{P}$ the problem of merging the PT and NP contributions should be addressed. The relevant procedure was discussed in detail in [2]. It includes introducing an infrared matching scale $\mu_I$ (typically chosen to be $\mu_I = 2$ GeV) and the non-perturbative $\mu_I$-dependent phenomenological parameter $\alpha_0$ (1.3) which quantifies the intensity of QCD interaction over the infrared domain, $k \leq \mu_I$.

An explicit expression for $\mathcal{P}$ depends on the order to which the perturbative contribution is computed, as well as on the scheme. At two-loop level, in the $\overline{MS}$ scheme, we have

$$
\mathcal{P} \equiv \frac{4C_F}{\pi^2} \mathcal{M} \frac{\mu_I}{Q} \left\{ \alpha_0(\mu_I) - \alpha_s - \beta_0 \frac{\alpha_s^2}{2\pi} \left( \frac{\ln \frac{Q}{\mu_I}}{\beta_0} + 1 \right) \right\} ; \quad \alpha_s \equiv \alpha_{\overline{MS}}(Q).
$$

The term proportional to $K$ accounts for mismatch between the $\overline{MS}$ and the physical scheme, with $K$ given above in (F.8). $\mathcal{M}$ in (F.14) is the Milan factor resulting from the two-loop
analysis discussed in Appendix A.2 (see also [3]). This factor is universal for all $1/Q$ jet observables considered in $e^+e^-$ annihilation [2] and DIS processes [23] and reads

$$\mathcal{M} = 1 + \beta_0^{-1}(1.575C_A - 0.104n_f) = 1.490(1.430) \text{ for } n_f = 3 (0).$$

The perturbative terms in (F.14) (proportional to $\alpha_s$ and $\alpha_s^2$) represent the start of the series responsible for subtracting off the infrared renormalon divergence in the perturbative contribution to the observable.

References

[1] A.V. Manohar and M.B. Wise, Phys. Lett. 344B (1995) 407 [hep-ph/9406392]; Yu.L. Dokshitzer and B.R. Webber, Phys. Lett. 352B (1995) 451 [hep-ph/9504219]; R. Akhoury and V.I. Zakharov, Phys. Lett. 357B (1995) 646 [hep-ph/9504248]; G.P. Korchemsky and G. Sterman, in Proc. 30th Rencontres de Moriond, Meribel les Allues, France, 19-25 March 1995, ed. J. Tran Thanh Van, Editions Frontières, 1995 [hep-ph/9505391].

[2] Yu.L. Dokshitzer, A. Lucenti, G. Marchesini and G.P. Salam, JHEP 05 (1998) 003 [hep-ph/9802381] and erratum submitted to J. High Energy Physics.

[3] Yu.L. Dokshitzer and B.R. Webber, Phys. Lett. 404B (1997) 321 [hep-ph/9704298].

[4] G.P. Korchemsky and G. Sterman, Nucl. Phys. B437 (1995) 415 [hep-ph/9411211].

[5] Yu.L. Dokshitzer, A. Lucenti, G. Marchesini and G.P. Salam, Nucl. Phys. B511 (1998) 396 [hep-ph/9707532] and erratum submitted to Nucl. Phys. B.

[6] J.C. Thompson (ALEPH collaboration), contribution 945 to ICHEP-98, Vancouver, Canada, hep-ex/9812004.

[7] DELPHI Collaboration, P. Abreu et al., Zeit. Phys. C73 (1997) 229; D. Wicke, J. Drees, U. Flagmeyer, and K. Hamacher, contribution 137 to ICHEP-98, 1998; J.Drees et al., contribution to “33rd Rencontres de Moriond: QCD and High Energy Hadronic Interactions,” March 1998.

[8] P.A. Movilla Fernández, O. Biebel and S. Bethke, paper contributed to ICHEP-98, Vancouver, Canada, hep-ex/9807007.

[9] H1 Collaboration, C. Adloff et al., Phys. Lett. 406B (1997) 229; contribution 530 to ICHEP July 1998, Vancouver, Canada.

[10] P.A. Movilla Fernández, talk at QCD Euroconference, Montpellier, France, July 1998, hep-ex 9808005.

[11] Yu.L. Dokshitzer, V.A. Khoze, A.H. Mueller and S.I. Troyan, Basics of Perturbative QCD, ed. J. Tran Thanh Van, Editions Frontières, Gif-sur-Yvette, 1991.

---

6The value $\mathcal{M} \approx 1.795$ presented in the original version of [3] and used in the original version of this paper was wrong. The figures shown elsewhere in this paper are still based on the old value of $\mathcal{M}$, but change only slightly with the new, corrected, value.
[12] Yu.L. Dokshitzer, A. Lucenti, G. Marchesini and G.P. Salam, JHEP 01 (1998) 011 [hep-ph/9801324].

[13] S. Catani, G. Marchesini and B.R. Webber, Nucl. Phys. B349 (1991) 635; Yu.L. Dokshitzer, V.A. Khoze and S.I. Troyan, Phys. Rev. D53 (1996) 89 [hep-ph/9506423].

[14] Yu.L. Dokshitzer, G. Marchesini and B.R. Webber, Nucl. Phys. B469 (1996) 93 [hep-ph/9512336].

[15] L3 Collaboration, M. Acciarri et al., Phys. Lett. 411B (1997) 339.

[16] DELPHI Collaboration, P. Abreu et al., Zeit. Phys. C73 (1996) 11.

[17] OPAL Collaboration, K. Ackerstaff et al., Zeit. Phys. C75 (1997) 193.

[18] OPAL Collaboration, P.D. Acton et al., Zeit. Phys. C59 (1993) 1; OPAL Collaboration, G. Alexander et al., Zeit. Phys. C72 (1996) 191.

[19] SLD Collaboration, K. Abe et al., Phys. Rev. D51 (1995) 962 [hep-ex/9501003].

[20] S. Bethke, Presented at the IVth Intern. Symp. on Radiative Corrections, Barcelona, Sept 8-12, 1998; Proc. of QCD Euroconference 97, Montpellier, France, July 1997, Nucl. Phys. Proc. Suppl. 64, 54 (1998) [hep-ex/9710030].

[21] D. Wicke, private communication.

[22] P.A. Movilla Fernández et al., JADE collab., Eur. Phys. J. C1 (1998) 461 [hep-ex/9709003].

[23] M. Dasgupta and B.R. Webber, Eur. Phys. J. C1 (1998) 539 [hep-ph/9704297]; JHEP 10 (1998) 001 [hep-ph/9809247].

[24] S. Catani, G. Turnock and B.R. Webber, Phys. Lett. 205B (1992) 269.

[25] S. Catani, L. Trentadue, G. Turnock and B.R. Webber, Phys. Lett. 263B (1991) 491.