Invariants of mixed representations of quivers II: defining relations and applications

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Introduction

In this article we give the complete answer for the problem 1 stated in the [Zub7]. We recall necessary definitions and notations (see also [Gab, Don1, PrB1, PrB2, Zub4]).

A quiver is a quadruple $Q = (V, A, h, t)$, where $V$ is a vertex set, $A$ is an arrow set of $Q$, and the maps $i, t : A \to V$ associate to each arrow $a \in A$ its origin $i(a) \in V$ and its end $t(a) \in V$. We enumerate elements of the vertex set as $V = \{1, \ldots, n\}$.

We consider a collection of vector spaces $E_1, \ldots, E_n$ over an algebraically closed field $K$. Let $\dim E_1 = d_1, \ldots, \dim E_n = d_n$. Denote by $d$ the vector $(d_1, \ldots, d_n)$. This vector is called a dimension vector. For two dimension vectors $d(1), d(2)$ we write $d(1) \geq d(2)$ iff $\forall i \in V, d(1)_i \geq d(2)_i$. Denote by $GL(d)$ the group $GL(E_1) \times \ldots \times GL(E_n) = GL(d_1) \times \ldots \times GL(d_n)$. The representation space of the quiver $Q$ of dimension $d$ is $R(Q, d) = \prod_{a \in A} \text{Hom}_K(E_{i(a)}, E_{t(a)})$. The group $GL(d)$ acts on $R(Q, d)$ by the rule:

$$(y_a^g)_{a \in A} = (g_{i(a)}y_ag_{i(a)}^{-1})_{a \in A}, g = (g_1, \ldots, g_n) \in GL(d), (y_a)_{a \in A} \in R(Q, d).$$

For example, if our quiver $Q$ has one vertex and $m$ loops which are necessarily incident to this vertex then the $d = (d)$-representations space of this quiver is isomorphic to the space of $m d \times d$-matrices with respect to the diagonal action of the group $GL(d)$ by conjugation.

The coordinate ring of the affine variety $R(Q, d)$ is isomorphic to $K[y_{ij}(a) \mid 1 \leq j \leq d_{i(a)}, 1 \leq i \leq d_{t(a)}, a \in A]$. For any $a \in A$ denote by $Y_a(d)$ the general matrix $(y_{ij}(a))_{1 \leq j \leq d_{i(a)}, 1 \leq i \leq d_{t(a)}}$. The action of $GL(d)$ on $R(Q, d)$ induces the action on the coordinate ring by the rule $Y_a(d) \mapsto g_{i(a)}^{-1}Y_a(d)g_{i(a)}, a \in A$. We omit the lower index $d$ if it does not lead to confusion. For example, we write just $Y(a)$ instead of $Y_a(d)$.

In [Zub7] the concept of a representation of a quiver was generalized as follows. We partition the vertex set of a given quiver $Q$ into several disjoint subsets. To be
precise, let $V = V_{ord} \sqcup (\bigsqcup_{q \in \Omega} V_q)$. The vertices from $V_{ord}$ are said to be **ordinary.** Moreover, all subsets $V_q$ have cardinality two, that is for any $q \in \Omega$, $V_q = \{i_q, j_q\}$.

A dimension vector $d$ is said to be **compatible** with this partition of $V$ if for any $q \in \Omega$, $d_{i_q} = d_{j_q} = d_q$. The next step is to replace all $E_{j_q}, q \in \Omega$ by their duals. To indicate that some vertices correspond to the duals of vector spaces we introduce a new dimension vector $t = (t_1, \ldots, t_l)$, where $t_i = d_i$ iff we assign to $i$ the space $E_i$, otherwise $t_i = d_i^*$. We call $d$ the vector underlying $t$. These notations will be used throughout. By definition, the $t$-dimensional representation space of the quiver $Q$ is equal to the space $R(Q, t) = \prod_{a \in A} \text{Hom}_K(W_{i(a)}, W_{t(a)})$, where $W_i = E_i$ iff $t_i = d_i$, otherwise $W_i = E_i^*$. The space $R(Q, t)$ is a $G = GL(d)$-module under the same action:

$$(ya)_a^{g} = (g_{i(a)}y_{a}g^{-1}_{i(a)})_a \in A, g = (g_1, \ldots, g_l) \in G, (ya)_a^{g} \in R(Q, t).$$

If $\Omega = \emptyset$ then $t = d$ and $R(Q, t) = R(Q, d)$. Without loss of generality one can identify the coordinate algebras $K[R(Q, t)]$ and $K[R(Q, d)]$. Finally, replacing all subfactors $GL(E_{i_q}) \times GL(E_{j_q}) = GL(d_q) \times GL(d_q)$ of the group $G = GL(d)$ by their diagonal subgroups we obtain a new group $H(t)$. The space $R(Q, t)$ with respect to the action of the group $H(t)$ is called a **mixed** representation space of the quiver $Q$ of dimension $t$ relative to the partition $V = V_{ord} \sqcup (\bigsqcup_{q \in \Omega} V_q)$. In [Zub7] author formulated the following:

**Problem 1** What are the generators and the defining relations of the algebra $J(Q, t) = K[R(Q, t)]^{H(t)}$?

In [Zub7] some necessary definitions and notations were introduced as well as some auxiliary results were proved. We remind them since they are necessary to understand this article. We start with the notion of the **doubled** quiver $Q^{(d)}$. This quiver is constructed with respect to the partition of the vertex set of $Q$ into ordinary vertices and couples $\{i_q, j_q\}, q \in \Omega$. More precisely, the vertex set $V^{(d)}$ of $Q^{(d)}$ is equal to $V \sqcup V_{ord}^*$, where $V_{ord}^* = \{i^* \mid i \in V_{ord}\}$. Respectively, the arrow set $A^{(d)}$ of $Q^{(d)}$ is equal to $A \sqcup \overrightarrow{A}$, where $\overrightarrow{A} = \{\overrightarrow{a} \mid a \in A\}$.

Further, if $i(a), t(a) \in V_{ord}$ then $i(\overrightarrow{a}) = t(a)^*, t(\overrightarrow{a}) = i(a)^*$ but if $i(a)$ or $t(a)$ lies in some $V_q, q \in \Omega$, then

$$i(\overrightarrow{a}) = \begin{cases} j_q, t(a) = i_q, \\ i_q, t(a) = j_q, \end{cases}$$

and symmetrically

$$t(\overrightarrow{a}) = \begin{cases} j_q, i(a) = i_q, \\ i_q, i(a) = j_q. \end{cases}$$

Finally, for any $a \in A^{(d)}$ we suppose $Z(a) = Y(a)$ if $a \in A$, otherwise $a = \overrightarrow{b}, b \in A$, and $Z(a) = \overrightarrow{Y(b)}$, where $\overrightarrow{Y(b)}$ is the transpose of $Y(b)$.  

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A product $Z(a_r) \ldots Z(a_1)$ is said to be admissible if $a_r \ldots a_1$ is a closed path in $Q^{(d)}$, that is if $t(a_i) = i(a_{i+1}), i = 1, \ldots, m - 1,$ and $i(a_1) = t(a_m)$.

The main result of [Zub7] is

**Theorem 1** The algebra $J(Q, t)$ is generated by the elements $\sigma_j(p)$, where $p$ is an admissible product, $\sigma_j$ is $j$-th coefficient of characteristic polynomial, $1 \leq j \leq \max_{1 \leq i \leq n} \{d_i\}$.

Moreover, it was proved that for any $d(1) \geq d(2)$ there is a natural epimorphism $\phi_{t(1), t(2)} : J(Q, t(1)) \to J(Q, t(2))$ which is an isomorphism on homogeneous components of fixed degree whenever $t(1), t(2)$ are sufficiently large. For example, if this degree is $r$ then $t(1), t(2)$ are sufficiently large if all coordinates of their underlying dimension vectors are not less than $r$ [Zub7]. Since all epimorphisms $\phi_{t(1), t(2)}$ are homogeneous we have the countable set of spectrums:

$$\{J(Q, t)(r), \phi_{t(1), t(2)} | d(1) \geq d(2)\}, r = 0, 1, 2, \ldots$$

The inverse limit of $r$-th spectrum denote by $J(Q)(r)$. It is clear that $J(Q) = \bigoplus_{r \geq 0} J(Q)(r)$ can be endowed with a graded algebra structure in obvious way. The algebra $J(Q)$ is said to be a free invariant algebra of mixed representations of the quiver $Q$. One can prove that the algebra $J(Q)$ is a polynomial algebra on infinite many variables (see [Zub4]).

By definition there is a natural epimorphism $J(Q) \to J(Q, t)$. Denote its kernel by $T(Q, t)$. It is clear that this ideal is a $T$-ideal with respect to some specific set of substitutions on formal variables corresponding to arrows of $Q^{(d)}$. More precisely, consider the non-unital path algebra $K \prec Q^{(d)} >$ of the quiver $Q$. This algebra is generated by the elements $Z(a), a \in A^{(d)}$. The defining relations between these generators are $Z(a)Z(b) = 0$ iff $i(a) \neq t(a), a, b \in A^{(d)}$. Let $C(Q) = J(Q) < Q^{(d)} > = J(Q) \otimes K \prec Q^{(d)} >$.

Notice that $C(Q)$ has an $J(Q)$-algebra involution $\iota$ which is uniquely defined by $\iota : Z(a) \mapsto Z(\bar{a})$. It is defined correctly since we suppose that $\bar{a} = a, a \in A$. It is obvious that $C(Q)$ is generated by the elements $Y(a), a \in A$, as a $J(Q)[\iota]$-algebra.

Any non-zero monomial $m = Z(a_k) \ldots Z(a_1)$ corresponds to the path $p = a_k \ldots a_1$ in $Q^{(d)}$. Therefore, one can define its origin $i(m) = i(p) = i(a_1)$ and end $t(m) = t(p) = t(a_k)$. An element $f \in C(Q)$ is said to be incident to $i \in V^{(d)}$ if all monomials belonging to $f$ are closed paths in $Q^{(d)}$ starting with $i$. In the same way, we say that $f \in C(Q)$ is passing from $i$ to $j$, $i, j \in V^{(d)}$, if all monomials belonging to $f$ are passing from $i$ to $j$. The substitution $Y(a) \mapsto f_a \in C(Q), a \in A$, is said to be admissible iff all monomials belonging to $f_a$ has non-zero degree and pass from $i(a)$ to $t(a)$ [Zub1, Zub4].

Specializing all variables $Y(a)$ to $Y(a) \in J(Q, t)$ we see that all admissible substitutions induce endomorphisms of this algebra. Moreover, these endomorphisms are compatible with the epimorphisms $\phi_{t(1), t(2)}$. In particular, all admissible substitutions induce endomorphisms of $J(Q)$ and all ideals $T(Q, t)$ are stable under these
endomorphisms. Therefore, it is possible to regard the ideals \( T(Q, t) \) as \( T \)-ideals and set the second part of Problem 1 as follows.

**Problem 2** What are the generators of \( T(Q, t) \) as a \( T \)-ideal?

In this article we give complete solution of Problem 2. To formulate our main result one need more definitions and results from [Zub7]. Decompose the arrow set \( A \) into three subsets \( A_i, i = 1, 2, 3 \), where

\[
A_1 = \{ a \in A \mid W_{i(a)} = E_{i(a)}, W_{t(a)} = E_{t(a)} \},
A_2 = \{ a \in A \mid W_{i(a)} = E_{i(a)}, W_{t(a)} = E^*_{t(a)} \},
A_3 = \{ a \in A \mid W_{i(a)} = E^*_{i(a)}, W_{t(a)} = E_{t(a)} \}.
\]

In other words, \( A_1 = \{ a \in A \mid i(a), t(a) \in V_{ord} \}, A_2 = \{ a \in A \mid t(a) = j_q, q \in \Omega \}, A_3 = \{ a \in A \mid i(a) = j_q, q \in \Omega \}. \) Remark that the case \( W_{i(a)} = E^*_{i(a)}, W_{t(a)} = E^*_{t(a)} \) can be easily eliminated so we do not consider it at all (see [Zub7]).

Fix a multidegree \( \bar{r} = (r_a)_{a \in A} \) and denote \( \sum_{a \in A} r_a \) by \( r \). Denote by \( J(Q, t)(\bar{r}) \) the homogeneous component of the algebra \( J(Q, t) \) of degree \( r_a \) in \( Y(a), a \in A \). It was proved in [Zub7] that \( J(Q, t)(\bar{r}) \neq 0 \) iff \( \sum_{a \in A_2} r_a = \sum_{a \in A_3} r_a = s. \) As in [Zub7] denote by \( t = r - 2s \).

We extend the set of matrix variables \( \{ Y(a) \mid a \in A \} \) in the following way. Replace each \( Y(a) \) by some new set of matrices having the same size as \( Y(a) \). The cardinality of this set is equal to \( r_a \). The same procedure is possible for formal variables \( Y(a), a \in A \). Simultaneously, we replace each arrow \( a \) by \( r_a \) new arrows with the same origin and end as \( a \) and set them in one-to-one correspondence with these new matrices. So we get a new quiver \( \hat{Q} \). The vertex set of \( \hat{Q} \) coincides with \( \tilde{V} \) but the arrow set \( \hat{A} \) can be different from \( A \).

Set any linear order on \( A \). Denote this order by usual symbol \( < \). We enumerate arrows of the quiver \( \hat{Q} \) by the integers \( 1, \ldots, r \). One can assume that for any \( a \in A \) the corresponding set of new arrows is enumerated by the integers from the segment \( [a, a] = [\sum_{b < a} r_b + 1, \sum_{b < a} r_b] \). We obtain some arrow specialization \( f : [1, r] = \hat{A} \to A \) by \( f(j) = a \) iff \( j \in [a, a], a \in A \). The specialization of matrix variables \( Y(j) \mapsto Y(a) \) (or formal variables \( Y(j) \mapsto Y(a) \)) iff \( j \in [a, a], a \in A \), denote by the same symbol \( f \).

Without loss of generality it can be assumed that \( \forall a \in \hat{A}_1, b \in \hat{A}_2, c \in \hat{A}_3, a < b < c \). Thus it follows that \( \hat{A}_1 = [1, t], \hat{A}_2 = [t + 1, t + s], \hat{A}_3 = [t + s + 1, r] \) and \( f([1, t]) = A_1, f([t + 1, s + t]) = A_2, f([s + t + 1, r]) = A_3. \) It is clear that \( i(j) = i \) or \( t(j) = i \) iff \( i(f(j)) = i \) or \( t(f(j)) = i \) respectively, \( j \in \hat{A} = [1, \ldots, r], i \in V \).

We set

\[
T(i) = \{ j \in \hat{A} \mid t(j) = i \}, I(i) = \{ j \in \hat{A} \mid i(j) = i \}, i \in V_{ord},
\]

\[
T(i_q) = \{ j \in \hat{A} \mid t(j) = i_q \}, I(i_q) = \{ j \in \hat{A} \mid i(j) = i_q \},
\]
\[ T(j_q) = \{ j \in \hat{A} \mid i(j) = j_q \}, I(j_q) = \{ j \in \hat{A} \mid t(j) = j_q \}, q \in \Omega. \]

Denote by \( L(Q) \) the subset of the group \( S_r \) consisting of all permutations \( \sigma \) which satisfy the conditions:

1. \( \forall i \in \text{ord}, \sigma((\mathcal{I}(i) \cap \hat{A}_1) \cup (\mathcal{I}(i) \cap \hat{A}_3 - s)) = (\mathcal{I}(i) \cap \hat{A}_1) \cup (\mathcal{I}(i) \cap \hat{A}_2 + s); \)
2. \( \forall q \in \Omega, \sigma((\mathcal{T}(i_q) \cap \hat{A}_1) \cup (\mathcal{T}(i_q) \cap \hat{A}_3 - s) \cup T(j_q)) = (\mathcal{T}(i_q) \cap \hat{A}_1) \cup (\mathcal{T}(i_q) \cap \hat{A}_2 + s) \cup I(j_q). \)

These conditions will be called \textit{admissibility} conditions. As in [Zub7] denote the right hand side sets of these equations by \( \mathcal{I}(i), \mathcal{T}(q) \). Denote by \( \mathcal{T}(i), \mathcal{T}(q) \) the left hand side sets, that is the arguments of the permutation \( \sigma \).

Any element \( z \in J(\hat{Q})((1^r)) \) is a \( K \)-linear combination of products of traces \( \text{tr}(\mathcal{Z}(a) \ldots \mathcal{Z}(b)) \), where \( a \ldots b \) is a closed path in \( \hat{Q} \) and \( (1^r) = (1, \ldots, 1) \). As in [Zub7] we omit the functional symbol \( \text{tr} \) in a record of any \( z \) if it does not lead to confusion. The same convention works in \( J(Q) \). For example, if \( z = (a \ldots b) \ldots (c \ldots d) \) then the specialization \( f \) takes \( z \) into \( f(z) = (f(a) \ldots f(b)) \ldots (f(c) \ldots f(d)) \). An equation \( (p) = (q) \) for given closed paths \( p, q \) in \( Q^{(d)} \) \( (\hat{Q}^{(d)}) \) takes a place iff these paths are the same up to a cyclic permutation or involution \( \iota \). We get an equivalence on the set of closed paths in \( Q^{(d)} \) \( (\hat{Q}^{(d)}) \). Any equivalence class is said to be a \textit{cycle}. The given cycle \( p \) is called \textit{primitive} if it is not a proper power [Don2].

There is a \( K \)-linear isomorphism of vector spaces \( \text{tr}^* : K[L(Q)] \to J(\hat{Q})((1^r)) \) which can be defined as follows. For any \( \sigma \in L(Q) \) we have \( \text{tr}^*(\sigma) = (a \ldots b) \ldots (c \ldots d) \), where all symbols \( a, \ldots b, \ldots, c, \ldots, d \) lie in the set \( [1, r] \cup \hat{[1, r]}, [\hat{1}, \hat{r}] = \iota([1, r]) \), and the set \( \{a, \ldots b, \ldots, c, \ldots, d\} \) has cardinality \( r \). Notice that this record is uniquely defined up to the equivalence mentioned above. To describe the computation of \( \text{tr}^*(\sigma) \) one has to define the right hand side neighbor of any symbol \( j \in [1, r] \cup \hat{[1, r]} \) in a record of \( \text{tr}^*(\sigma) \). We list all possibilities for \( j \) as follows (see Proposition 2.5 [Zub7]).

Let \( j \in [1, r] \) then we have

1. If \( j \in \hat{A}_1 \) then \( (\ldots jk \ldots) \), where

\[
k = \begin{cases} \sigma^{-1}(j), \sigma^{-1}(j) \in \hat{A}_1, \\ \sigma^{-1}(j) + s, \sigma^{-1}(j) \in \hat{A}_2, \\ \sigma^{-1}(j), \sigma^{-1}(j) \in \hat{A}_3. \end{cases}
\]

2. If \( j \in \hat{A}_2 \) then \( (\ldots jk \ldots) \), where

\[
k = \begin{cases} \sigma^{-1}(j + s), \sigma^{-1}(j + s) \in \hat{A}_1, \\ \sigma^{-1}(j + s) + s, \sigma^{-1}(j + s) \in \hat{A}_2, \\ \sigma^{-1}(j + s), \sigma^{-1}(j + s) \in \hat{A}_3. \end{cases}
\]
3. If \( j \in \hat{A}_3 \) then \((\ldots jk\ldots)\), where

\[
k = \begin{cases} 
\sigma(j), \; \sigma(j) \in \hat{A}_1, \\
\sigma(j), \; \sigma(j) \in \hat{A}_2, \\
\sigma(j) - s, \; \sigma(j) \in \hat{A}_3.
\end{cases}
\]

If \( j = \bar{l} \) then we have the following rules:

1. If \( l \in \hat{A}_1 \) then \((\ldots jk\ldots)\), where

\[
k = \begin{cases} 
\sigma(l), \; \sigma(l) \in \hat{A}_1, \\
\sigma(l), \; \sigma(l) \in \hat{A}_2, \\
\sigma(l) - s, \; \sigma(l) \in \hat{A}_3.
\end{cases}
\]

2. If \( l \in \hat{A}_2 \) then \((\ldots jk\ldots)\), where

\[
k = \begin{cases} 
\sigma^{-1}(l), \; \sigma^{-1}(l) \in \hat{A}_1, \\
\sigma^{-1}(l) + s, \; \sigma^{-1}(l) \in \hat{A}_2, \\
\sigma^{-1}(l), \; \sigma^{-1}(l) \in \hat{A}_3.
\end{cases}
\]

3. If \( l \in \hat{A}_3 \) then \((\ldots jk\ldots)\), where

\[
k = \begin{cases} 
\sigma(l - s), \; \sigma(l - s) \in \hat{A}_1, \\
\sigma(l - s), \; \sigma(l - s) \in \hat{A}_2, \\
\sigma(l - s) - s, \; \sigma(l - s) \in \hat{A}_3.
\end{cases}
\]

Following [Zub7] we call these rules \textit{contracting}. For other way to compute \( tr^*(\sigma) \) see Lemma 3.5 from [Zub7]. Denote \( f(tr^*(\sigma)) \) by \( tr^*(\sigma, f) \).

Consider the case when \( \Omega = \{q\} \) and \( V_{ord} = \{i_q\} \). It is clear that \( A_1 \) is the set of loops necessary incident to \( i_q \). We suppose additionally that \( |A_1| = |A_2| = |A_3| = 1 \) and \( X, Y, Z \) correspond to the single arrows from \( A_1, A_2 \) and \( A_3 \) respectively. For given \( r, s, 2s \leq r \), we define the element

\[
\sigma_{r,s}(X, Y, Z) = \frac{1}{t!(s!)^2} \sum_{\sigma \in S_t} (-1)^{tr^*(\sigma, f)}
\]

Here \( f([1, t]) = X, f([t + 1, t + s]) = Y, f([t + s + 1, r]) = Z \). Notice that in the case \( s = 0 \) this element coincides with \( \sigma_r(X) \). Finally, we formulate the main result of this article.

**Theorem 2** For any dimension \( t \) the ideal \( T(Q, t) \) is generated as a \( T \)-ideal by the elements \( \sigma_r(f), \sigma_{r,s}(f_1, f_2, f_3) \), where \( f, f_1, f_2, f_3 \in C(Q) \), \( f \) is incident to some \( i \in V \) and \( r > d_i \), \( f_1 \) is incident to \( i \in V_{ord} \) or to \( i_q \), \( q \in \Omega \), and \( f_2(f_3) \) are passing from \( i \) or \( i_q \) (from \( i^* \) or \( j_q \)) to \( i^* \) or \( j_q \) (to \( i \) or \( i_q \)) correspondingly. Moreover, \( r > d_i \) and \( r > d_q \) respectively.
Remark 1 If $t = d$ then we get the main result of [Zub4]. Notice that this result was formulated in [Zub4] incorrectly. Indeed, it was claimed that $f$ in a relation $\sigma_r(f)$ is a monomial incident to $i$. It is true if there is a loop $p$ incident to $i$ and then one can replace all $\sigma_r(f)$ by $\sigma_r(p)$ up to the obvious substitutions. The same remark is for concomitants.

In the last section we give some applications for orthogonal and symplectic invariants of several matrices. More precisely, we describe some approach to the problem of computation of defining relations. Notice that in the characteristic zero case it has been done in [Pr]. I hope to get a complete solution of this problem in the next article.

1 Preliminaries

1.1 Specializations

For given $d(1) \geq d(2)$ define an epimorphism

$$p_{t(1),t(2)} : K[R(Q,t(1))] \to K[R(Q,t(2))]$$

by the following rule. Take any arrow $a \in A$. Let $i(a) = i$ and $t(a) = j$. For the sake of simplicity denote $d_i(f)$ and $d_j(f)$ by $m_f$ and $l_f$ respectively, $f = 1, 2$. We know that $m_1 \geq m_2$ and $l_1 \geq l_2$. If $m_f \neq l_f, f = 1, 2$, then our epimorphism takes $y_{sr}(a)$ to zero iff either $s > l_2$ or $r > m_2$. On the remaining variables our epimorphism is the identical map. If $m_f = l_f, f = 1, 2$ then one can define our epimorphism on the coefficients of $Y_{t(1)}(a)$ in the other way taking $y_{ss}(a)$ to unit iff $s > m_2 = l_2$.

We admit both ways and say that $p_{t(1),t(2)}$ is standard if the second way does not happen at all. Otherwise, $p_{t(1),t(2)}$ is called non-standard on arrow $a$ if we define $p_{t(1),t(2)}$ on $Y(a)$ by the second way. For example, the epimorphisms $\phi_{t(1),t(2)}$ from the introduction are just restrictions of the standard epimorphisms $p_{t(1),t(2)}$.

On the other hand, one can define an isomorphism $i_{t(2),t(1)}$ of the variety $R(Q,t(2))$ onto a closed subvariety of $R(Q,t(1))$ by the dual rule, that is the epimorphism defined above is the comorphism $i^*_{t(2),t(1)}$.

By the same way one can define the isomorphism $j_{t(2),t(1)}$ of $H(t(2))$ onto a closed subgroup $H(t(1))$ just bordering any invertible $d_i(2) \times d_i(2)$ matrix by the $d_i(1) - d_i(2)$ additional rows and columns which are zero outside of the diagonal tail of length $d_i(1) - d_i(2)$. The entries on this diagonal tail must all be 1’s.

It is not hard to check that $i_{t(2),t(1)}(\phi g) = i_{t(2),t(1)}(\phi) j_{t(2),t(1)}(g)$ for any $g \in H(t(2))$ and $\phi \in R(Q,t(2))$. The analogous equation is valid for the epimorphism $p_{t(1),t(2)}$.

1.2 Young subgroups

Decompose the interval $[1, k] = \{1, \ldots , k\}$ into some disjoint subsets, say $[1, k] = \bigsqcup_{1 \leq j \leq m} T_j$. Define the Young subgroup $S_T = S_{T_1} \times \ldots \times S_{T_m}$ of $S_k$ as the subgroup
consisting of all permutations $\sigma \in S_k$ such that $\sigma(T_j) = T_j, 1 \leq j \leq m$. By definition, $S_T = \{\sigma \in S_k \mid \sigma(T) = T, \forall j \notin T, \sigma(j) = j\}$ for arbitrary subset $T$. The subsets $T_1, \ldots, T_m$ are said to be the layers of $S_T$ [Zub1, Zub4].

The group $S_T$ can be defined in other way. In fact, let $f$ be a map from $[1, k]$ onto $[1, m]$ defined by the rule $f(T_j) = j, j = 1, \ldots, m$. Then $S_T = \{\sigma \in S_k \mid f \circ \sigma = f\}$. Sometimes we will denote the group $S_T$ by $S_f$.

For any group $G$ and its subgroup $H$ we denote by $G/H$ some fixed representative set of the left $H$ cosets if it does not lead to confusion.

For a given superpartition $\lambda = (\lambda_1, \ldots, \lambda_n)$ which is composed from ordinary (not necessarily ordered) partitions $\lambda_i = (\lambda_{i1}, \ldots, \lambda_{is_i}), i = 1, \ldots, n$, denote by $S_\lambda$ the Young subgroup of $S_{[\lambda]}$ corresponding to the decomposition of $[1, | \lambda |]$ into sequential subintervals of lengths $\lambda_1, \lambda_{1s_1}, \lambda_2, \ldots, \lambda_2s_2, \ldots$.

For example, fix some multidegree $\bar{r}$ and consider the superpartition $\Theta = (\lambda_1, \mu_1, \gamma_{A_1})$, where $\lambda_{A_1} = (\lambda_a)_{a \in A_1}, \mu_{A_2} = (\mu_a)_{a \in A_2}, \gamma_{A_2} = (\gamma_a)_{a \in A_2}$, $\lambda_a, \mu_b, \gamma_c$ are partitions such that $\forall a \in A_1, \forall b \in A_2, \forall c \in A_3$, $| \lambda_a | = r_a, | \mu_b | = r_b, | \gamma_c | = r_c$. If $f$ is the specialization from the introduction then $S = S_\Theta \leq S_f$ [Zub7]. Another important Young subgroup of $S_r$ is $S_0 = S_{A_1} \times S_{A_2} \times S_{A_3}$. Any $\pi \in S_0$ can be decomposed as $\pi = \pi_1\pi_2\pi_3$, where $\pi_i \in S_{A_i}, i = 1, 2, 3$. Moreover, any $\pi \in S_{A_i}, i = 2, 3$, has a double $\pi^x$, where $x = \prod_{i \in \bar{A}_2}(i + s)$. As in [Zub7] we denote $\pi^x$ by $\pi + s$ if $\pi \in S_{\bar{A}_2}$, otherwise by $\pi - s$. In [Zub7] two shift homomorphisms $\rho_i : S_0 \to S_r, i = 1, 2$, were defined by the rule $\rho_1(\pi) = \pi_1\pi_2(\pi_2 + s), \rho_2(\pi) = \pi_1(\pi_3 - s)\pi_3, \pi \in S_0$. The sets $T(x)$ (or $T(x)$), $x \in V_{ord} \cup \Omega$, form a decomposition of the interval $[1, r]$. Therefore, we have two Young subgroups $S_T, S_I$. It is not hard to prove that $S_f = \rho_1^{-1}(S_I) \cap \rho_2^{-1}(S_T)$.

Divide each $T(z), z \in V_{ord} \cup \Omega$, into some sublayers in a monotonic way. In other words, let $T(z) = \bigcup_{1 \leq j \leq z} \beta_j$, where $\max \beta_{zj_1} < \min \beta_{zj_2}$ as soon as $j_1 < j_2$, and $\max(\min)\beta_{zj}$ means the maximal (minimal) integer from this sublayer. Joining over all indices $z$ we obtain a decomposition of the segment $[1, r]$. Denote by $S_\beta$ the Young subgroup corresponding to this decomposition. As in [Zub1, Zub7] this subgroup will be called the base group. We say that $S_\beta$ is sufficiently large for the dimension $t$ if there is some $\beta_{ij}$ or $\beta_{ij}$ such that $| \beta_{ij} | > d_i$ or $| \beta_{ij} | > d_q$ respectively, $i \in V_{ord}, q \in \Omega$.

1.3 Suitable generators

For a given base group $S_\beta$ and elements $\sigma_1 \in L(Q), \sigma_2 \in S_T$ one can define the so-called suitable generator

$$z = \frac{1}{|S_f|} tr^*(\sum_{\tau \in S_\beta} \sum_{\pi \in S_f/(\rho_1^{-1}(S_\beta^1) \cap \rho_2^{-1}(S_\beta^2)) \cap S_\beta} (-1)^{r} \rho_1(\pi) \sigma_1 \tau \sigma_2^{-1} \rho_2(\pi)^{-1}, f).$$

It was proved in [Zub7] that the ideal $T(Q, t)$ is generated as a vector space by those suitable generators which belong to sufficiently large base groups $S_\beta$ for $t$. 
Without loss of generality one can suppose that \( S = S_f = \rho_1^{-1}(S_{\beta}^{\sigma_1}) \cap \rho_2^{-1}(S_{\beta}^{\sigma_2}) \) and \( \sigma_2 = 1 \) [Zub7]. This assumption is correct because \( S_f \geq \rho_1^{-1}(S_{\beta}^{\sigma_1}) \cap \rho_2^{-1}(S_{\beta}^{\sigma_2}) \) for any base group \( S_{\beta}^{\sigma_2} \leq S_r \). If we replace \( S_{\beta}^{\sigma_2} \) by \( S_{\beta}^{\sigma_2} \) it means that we make possible any partitions of the sets \( T \).

Sometimes we will denote an element \( \frac{1}{|S|^{n}} \) if \( f \) is a good filtration such that \( H \) satisfies the following properties:

1. If \( g.f.d.(V) \leq n, g.f.d.(U) \leq m \) then \( g.f.d.(V \otimes U) \leq n + m \).

2. If \( 0 \to V \to X_0 \to \ldots \to X_n \to 0 \) is an exact sequence of \( G \)-modules then \( g.f.d.(V) \leq \max\{g.f.d(X_i) + i\} \).

3. If \( 0 \to X_n \to \ldots \to X_0 \to V \to 0 \) is an exact sequence of \( G \)-modules then \( g.f.d.(V) \leq \max\{g.f.d(X_i) - i\} \).

Suppose that for another algebraic group \( H \) and an \( H \)-module \( W \) we have a homomorphism of pairs \((i, p) : (G, V) \to (H, W)\) [Weiss], that is a homomorphism \( i : H \to G \) and a linear map \( p : V \to W \) such that \( xp(v) = p(i(x)v), x \in H, v \in V \). We call \( p \) the linear part of \((i, p)\). Using Hochschild complexes [Jan] it is easy to see that the map \( p \) induces a map of cohomology groups \( H^n(G, V) \to H^n(H, W), n = 0, 1, 2, \ldots \). Moreover, we get the following

**Lemma 1.2** If we have a diagram

\[
\begin{array}{ccccccc}
0 & \to & A & \to & B & \to & C & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & A' & \to & B' & \to & C' & \to & 0
\end{array}
\]

with exact top and bottom rows of \( G \) and \( H \)-modules respectively such that the vertical arrows are homomorphisms of pairs whose linear parts commute with morhisms in rows, then we obtain a long commutative diagram

\[
\begin{array}{ccccccc}
\ldots & \to & H^n(G, A) & \to & H^n(G, B) & \to & H^n(G, C) & \to & H^{n+1}(G, A) & \to & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ldots & \to & H^n(H, A') & \to & H^n(H, B') & \to & H^n(H, C') & \to & H^{n+1}(H, A') & \to & \ldots
\end{array}
\]
Here the top and bottom sequences are standard long exact sequences of cohomology groups.

Proof. The Lemma is proved by a routine verification in the three-dimensional diagram composed from Hochschild complexes of all modules. For more details see [Weiss], Theorem 2-1-9.

2 Proof of the main theorem

We have to show that any suitable generator \( z = c(\sum_{x \in S_d}(-1)^x \sigma, f) \) is a linear combination of elements from Theorem 2 possibly multiplied by several \( \sigma_j(p), p \in Q(d) \), with integral coefficients.

The layers of the group \( S_f = \rho_1^{-1}(S_{f/2}) \cap \rho_2^{-1}(S_{f/2}) \) are \( A_1 \)-layers \( \sigma(\beta_{uv}) \cap \beta_{fg} \cap \hat{A}_1 \), \( A_2 \)-layers \( (\sigma(\beta_{uv}) \cap A_3 - s) \cap \sigma(\beta_{fg}) \cap A_2 \) or \( A_3 \)-layers \( (\beta_{uv} \cap A_2 + s) \cap \beta_{fg} \cap A_3 \).

It is clear that \( \sigma \) can be chosen such that its any non-trivial cycle does not contain two integers from the same layer of \( S_{\beta} \). Therefore, the \( A_1 \)-layers can be divided into passive and active layers [Zub1, Zub4]. More precisely, the passive layers are \( (\beta_{uv} \setminus (\bigcup_{f \neq u} \sigma(\beta_{fg})) \cap A_1 = \{ i \in \beta_{uv} \cap A_1 \mid \sigma(i) = i \} \) and all other \( A_1 \)-layers are active.

Let \( B \) be a union of all passive layers. By the definition, \( B \subseteq \hat{A}_1 \). Consider an element \( \tau \in S_r \). The cyclic decomposition of \( \tau^{-1} \) has the form \( (B_1 i_1 \ldots B_k i_k) \ldots (B_l i_l \ldots B_s i_s) \), where \( B_1, \ldots, B_s \) are some fragments of this decomposition consisting of integers from \( B \) and all \( i_1, \ldots, i_s \) are not contained in \( B \). Notice that some of these fragments can be empty or coincide with cycles containing them.

Lemma 2.1 The element \( \text{tr}^*(\tau) \) is uniquely defined by the integers \( i_1, \ldots, i_s \).

Proof. We have \( \text{tr}^*(\tau) = (D_1 j_1 \ldots D_l j_l) \ldots (D_m j_m \ldots D_l j_l) \), where any \( D_u \) coincides either with some \( B_v \) or with its transposed \( \iota(B_v) \). More precisely, we consider the blocks \( i_k B_{i_1} \ldots, i_{k-1} B_{i_k} \ldots, i_s B_{i_s} \ldots, i_{s-1} B_{i_s} \). By the contracting rules one has to replace each \( l_u B_v j_v \) by \( C_v = l'_v B_{v'} j'_{v'} \), where

\[
\begin{align*}
l'_v &= \begin{cases} l_v, & \text{if } l_v \in \hat{A}_1, \\
\tilde{l}_v, & \text{if } l_v \in \hat{A}_2, \\
l_v - s, & \text{if } l_v \in \hat{A}_3,\end{cases} \\
\end{align*}
\]

\[
\begin{align*}
\tilde{j}'_v &= \begin{cases} j_v, & \text{if } j_v \in \hat{A}_1, \\
j_v + s, & \text{if } j_v \in \hat{A}_2, \\
\tilde{j}_v, & \text{if } j_v \in \hat{A}_3.\end{cases} \\
\end{align*}
\]

The next step is to join all these blocks by same ends. For example, if \( C_v = l'_v B_v j'_v, C_w = l'_w B_w j'_w \) then they can be joined if \( j'_v = l'_v \) or \( j'_w = l'_w \) as follows: \( l'_v B_w j'_w B_v j'_v \) or \( l'_v B_v j'_v B_w j'_w \) respectively. If \( \tilde{j}'_v = j'_w = l'_w \) or \( \tilde{j}'_w = l'_v \) then one has to transpose one of these blocks and join them in the obvious way. Continuing this process we will get all cycles of \( \text{tr}^*(\tau) \) step by step like growing crystalles.
Example 2.1  Let $r = 7, s = 2, t = 3, \tau = (145)(267), B = \{2,3\}$. Then $\tau^{-1}$ has the following blocks: $1(0)5, 5(0)4, 4(0)1, 7(0)6, 6(2)7, 3$. In other words, $B_1 = B_2 = B_3 = B_4 = \emptyset, B_5 = 2, B_6 = 3$. We have $C_1 = 1(0)7, C_2 = \overline{5}(0)6, C_3 = \overline{1}(0)1, C_4 = 5(0)6, C_5 = 4(2)7, C_6 = 3$. By Lemma 2.1 we obtain that $\text{tr}^*(\tau) = (1724)(56)(3)$. It can easily be checked that the contracting rules give the same result.

Remark 2.1 Using the obvious rule $(CiBj)(ij) = (Ci)(Bj)$, where $C, B$ are fragments of given cycles, we see that $\text{tr}^*(\sigma\tau)$ can be computed by the following way. Deleting all integers from $B$ in the cyclic decomposition of $\tau$ we get some permutation $\bar{\tau}$ which acts on the set $\{1, \ldots, r\}$ such that $\bar{\tau}^{-1} = (i_1 \ldots i_k) \ldots (i_1 \ldots i_s)$. Then $\text{tr}^*(\sigma\tau)$ is obtained from $\text{tr}^*(\bar{\sigma}\bar{\tau})$ by substituting all passive fragments (or their transposed). More precisely, let $(\sigma\bar{\tau}) = (j_1 \ldots j_m) \ldots (j_n \ldots j_s)$. For given fragment, say $B_v$, one can find an integer $x$ such that $j_v = j_x$ and define $C_v = j'_{x-1}B_vj'_x$. It remains to join all $C_v$ in the way described above.

From now on we fix some $\beta_{ij}$ or $\beta_{ij}$ such that $|\beta_{ij}| > d_i$ or $|\beta_{ij}| > d_q$ respectively, $i \in V_{ord}, q \in \Omega$. We call it a selected layer but all other layers are called ordinary. Slightly abusing our notations denote the selected layer by $\beta_0$. We prove our theorem by induction on two parameters $(r, r - t)$, where $t$ is the number of ordinary layers. The first step is to eliminate all passive layers of $S_f$ except those which are contained in the selected layer.

Proposition 2.1 Without loss of generality one can assume that the group $S_f$ has not any passive layers except those contained in the selected layer.

We outline the proof and refer for more details to [Zub1, Zub4]. Remark 2.1 is used in the following computations without additional references.

Let $\beta_{ij}$ be an ordinary layer and $\beta_{xj} \cap A_1 = \alpha_1 \cup \ldots \alpha_i \cup \alpha_{i+1}$, where $\alpha_1, \ldots, \alpha_l$ are active layers of $S_f$ but $\alpha_{i+1} \neq \emptyset$ is a passive one. We say that $\tau \in S_f$ has the type $\tilde{m} = (m_1, \ldots, m_d, m_{d+1}, m_{d+2})$, where $d = |\beta_{ij}\backslash \alpha_{i+1}|$ and $m_{d+1} = |\alpha_{i+1}| - \sum_{1 \leq w \leq d} m_w$, if all passive fragments belonging to $\alpha_{i+1}$ in the record of $\text{tr}^*(\tau)$ have lengths $m_1, \ldots, m_d$ up to order. Both $\tau$ and $\sigma\tau$ have the same type. Denote by $I_{\tilde{m}}$ the set consisting of all $\tau \in S_{\tilde{m}}$ of type $\tilde{m}$. The element

$$z = c(\sum_{\tau \in S_{\tilde{m}}} (-1)^r \sigma\tau, f) = \frac{1}{|S_f|} \text{tr}^*(\sum_{\tau \in S_{\tilde{m}}} (-1)^r \sigma\tau, f)$$

can be represented as $\sum_{\tilde{m}} c(g_{\tilde{m}}, f)$, where $g_{\tilde{m}} = \sum_{\tau \in I_{\tilde{m}}} (-1)^r \sigma\tau$.

Notice that any summand $g_{\tilde{m}}$ is $S_f$-invariant with respect to the action $\mu^\pi = \rho_1(\pi)\mu_2(\pi)^{-1}, \pi \in S_f, \mu \in S_r$, and if $\pi \in S_{[1,t]}$ then $\mu^\pi = \pi \mu \pi^{-1}$.

Let $\beta_{xj} \cap \alpha_{i+1} = \{v_1, \ldots, v_d\}$. One can divide each layer $\alpha_j$ into some sublayers in the following way. Integers $v_a$ and $v_b$ from $\alpha_l$ belong to the same new sublayer iff $m_a = m_b$. We obtain a new group $S_{f'}$, where $f'$ corresponds to this new partition.
Denote by $M(\tilde{m})$ the subset of $I_{\tilde{m}}$ consisting of all elements such that each fragment of length $m_t$ is the left hand side neighbor of $v_t$, $1 \leq t \leq d$. The element $g'_{\tilde{m}} = \sum_{\tau \in M(\tilde{m})} (-1)^{\tau} \sigma \tau$ is also $S_f$-invariant and

$$g_{\tilde{m}} = \sum_{x \in S_f/S_{f'}} x g'_{\tilde{m}} x^{-1} = \sum_{x \in S_f/S_{f'}} \rho_1(x) g'_{\tilde{m}} \rho_2(x)^{-1}.$$ 

Using Lemma 3.8 from [Zub7] we see that $c(g'_{\tilde{m}}, f)$ is obtained from $c(g'_{\tilde{m}}, f')$ with the help of some gluing of variables. Choose in $\beta_{x_j}$ some sublayer $\pi$ such that $\pi \subseteq \alpha_{t+1}$ and $|\pi| = m_{d+1}$. Define a new base subgroup $S_{\beta'}$, where $\beta'$ coincides with $\beta$ outside of $\beta_{x_j}$ but $\beta_{x_j}$ is divided into two sublayers $\pi$ and $\beta_{x_j} \setminus \pi$. It is clear that the new group $S_{f''} = \rho_1^{-1}(S_{\beta'}) \cap \rho_2^{-1}(S_{\beta'}) \cap S_f = S_{\beta'} \cap S_f$. In other words, $S_{f''}$ coincides with $S_f'$ outside of $\alpha_{t+1}$ and $S_{f''} \cap S_{\alpha_{t+1}} = S_\pi \times S_{\alpha_{t+1} \setminus \pi}$. For the sake of convenience we represent the group $S_f'$ as $S_g \times S_{\alpha_{t+1}}$. We have

$$g''_{\tilde{m}} = \sum_{x \in S_{f''}/S_{f'}} x g''_{\tilde{m}} x^{-1} = \sum_{x \in S_{\alpha_{t+1}}/(S_\pi \times S_{\alpha_{t+1} \setminus \pi})} x g''_{\tilde{m}} x^{-1}.$$ 

Here $g''_{\tilde{m}} = \sum_{\tau \in S_{\beta'} \cap M(\tilde{m})} (-1)^{\tau} \sigma \tau$. It remains to consider the element $c(g''_{\tilde{m}}, S_{f''})$. We have

$$c(g''_{\tilde{m}}, S_{f''}) = c(\sum_{\tau \in S_{\beta'} \cap M(\tilde{m})} (-1)^{\tau} \sigma \tau, S_g \times S_{\alpha_{t+1} \setminus \pi}) \times c(\sum_{\tau \in S_\pi} (-1)^{\tau} \sigma \tau, S_\pi).$$

Because of $\alpha_{t+1} \subseteq \mathcal{T}(x) \cap \hat{A} = \mathcal{T}(x) \cap \sigma(\mathcal{T}(x)) \cap \hat{A} = \mathcal{T}(x) \cap \mathcal{I}(x) \cap \hat{A}$ we see that for all $j \in \alpha_{t+1}$, $i(j) = t(j) = x(i_j)$, that is the general matrice $Y$ corresponding to this layer is a square matrice. In particular, $c(\sum_{\tau \in S_\pi} (-1)^{\tau} \sigma \tau, S_\pi) = \sigma_{m_{d+1}}(Y)$. If $m_{d+1} > 0$ then induction on $r$ completes our eliminating process.

Let $m_{d+1} = 0$ and $\chi = \hat{\beta} \setminus \alpha_{t+1} = (\ldots, \check{\beta}_{x_j} \setminus \alpha_{t+1}, \ldots)$. Fix a collection of fragments $B_1, \ldots, B_d$ which are contained in $\alpha_{t+1}$, $|B_i| = m_i$, $1 \leq i \leq d$. Let $S(\tilde{m})$ be a set consisting of all $\tau \in M(\tilde{m})$ such that each $B_i$ is the left hand side neighbor of $v_t$, $1 \leq t \leq d$. Then

$$c(g'_{\tilde{m}}, S_f) = \frac{1}{|S_g|} \left( \frac{1}{|S_{\alpha_{t+1}}|} tr^*(\sum_{x \in S_{\alpha_{t+1}}} x(\sum_{\tau \in S(\tilde{m})} (-1)^{\tau} \sigma \tau)x^{-1}, S_{f'}). \right)$$

It can easily be checked that $c(g'_{\tilde{m}}, S_f)$ is obtained from

$$c(\sum_{\tau \in S_{\chi}} (-1)^{\tau} \sigma \tau, S_g)$$

by the substitution $X_{g(v_t)} \rightarrow Y^{m_t}X_{g(v_t)}$, $1 \leq t \leq d$. Arguing as above, we see that all products $Y^{m_t}X_{g(v_t)}$ are defined correctly and $S_g = \rho_1^{-1}(S_\chi) \cap \rho_1^{-1}(S_\chi)$. Since $|\alpha_{t+1}| > 0$ induction on $r$ completes the proof.
We call any preimage $\sigma^{-1}(\beta_{uv}) \cap \beta_{fg} \cap \hat{A}_1) \subseteq \beta_{uv}$ by an $A_1$-prelayer. An $A_2$-prelayer $\gamma$ is uniquely defined by $\sigma(\gamma) = (\sigma(\beta_{uv}) \cap \hat{A}_3 - s) \cap \sigma(\beta_{fg}) \cap \hat{A}_2$ or by $\sigma(\gamma) = \sigma(\beta_{uv}) \cap (\sigma(\beta_{fg}) \cap \hat{A}_2 + s) \cap \hat{A}_3$. Similarly, a $A_3$-prelayer $\gamma$ is uniquely defined by $\gamma = (\beta_{uv} \cap \hat{A}_2 + s) \cap \beta_{fg} \cap \hat{A}_3$ or by $\gamma = \beta_{uv} \cap (\beta_{fg} \cap \hat{A}_3 - s) \cap \hat{A}_2$. A layer containing given prelayer is called overlayer. We call a prelayer ordinary if its overlayer is ordinary. The following proposition plays crucial role in the simplification of suitable generators (see [Zub1, Zub4]).

**Proposition 2.2** Any ordinary $A_i$-prelayer coincides with its overlayer up to induction on the second parameter, $i = 1, 2, 3$. In particular, if $\beta_{uv}$ is an ordinary layer then $\sigma(\beta_{uv})$ belongs to only one set $\hat{A}_1$, $\hat{A}_2$ or $\hat{A}_3$. Analogously, $\beta_{uv}$ belongs to only one of $A_1$, $A_2$ or $A_3$. Furthermore, if additionally $\beta_{uv} \subseteq \hat{A}_1$ then it is covered by only one $A_1$-layer of the group $S_f$.

Proof. The proof is similar to [Zub1, Zub4] up to some specific details which we give below. The proof is step by step splitting of all ordinary layers which do not satisfy at least one condition mentioned in the proposition. To be precise, for any such layer, say $\beta_{uv}$, we extract some proper subset $\gamma \subseteq \beta_{uv}$. Next, denote by $H$ the group $S_{\beta_{uv}} \times S_\gamma \times S_{\beta_{uv}\setminus\gamma}$. The induction step consists of proving that $z$ is a sum of elements $c(S(x), f)$ or $c(S'(x), f)$, where

$$S(x) = (-1)^x \sum_{\tau \in H} \sum_{\pi \in \hat{S}_f/(\rho_1^{-1}(H^x) \cap \rho_2^{-1}(H^x))} (-1)^\tau \rho_1(\pi) \sigma \tau x^{-1} \rho_2(\pi)^{-1}$$

and

$$S'(x) = (-1)^x \sum_{\tau \in H} \sum_{\pi \in \hat{S}_f/(\rho_1^{-1}(H^x) \cap \rho_2^{-1}(H))} (-1)^\tau \rho_1(\pi) \sigma x \tau \rho_2(\pi)^{-1}.$$ 

Here $x$ runs over some subset of $S_{\beta_{uv}} \cap H = S_{\beta_{uv}}/(S_\gamma \times S_{\beta_{uv}\setminus\gamma})$. For a given $x$ denote the groups $\rho_1^{-1}(H^\sigma) \cap \rho_2^{-1}(H^x)$ and $\rho_1^{-1}(H^{\sigma x}) \cap \rho_2^{-1}(H)$ by $S_{f_x}$ and $S'_{f_x}$ correspondingly. It is clear that both $S_{f_x}$, $S'_{f_x}$ are contained in $S_f$.

The set $\sigma S_{\beta}$ is invariant under substitutions $a \mapsto \rho_1(\pi) a \rho_2(\pi)^{-1}$ for all $\pi \in S_f$ and parities of both $a$ and $\rho_1(\pi) a \rho_2(\pi)^{-1}$ are the same. Therefore, all we need is to prove that in any $S(x)(S'(x))$ there are no repeated summands and for any two $S(x), S(y)(S'(x), S'(y))$ either their summands are same or they have no summands in common [Zub1, Zub4].

If $z$ can be represented as a sum of the elements mentioned above, we say that $z$ admits a disjoin reduction. Since the number of ordinary layers of the new base group $H$ is increased, one can apply induction on the second parameter whenever $z$ admits a disjoin reduction. Notice that if some ordinary overlayer coincides with its $A_i$-prelayer, where $i = 1, 2, 3$, then this statement remains true even if we split this overlayer into some sublayers.

(i) Let $\gamma = \sigma^{-1}(\beta_{uv} \cap \beta_{fg} \cap \hat{A}_1)$ be an ordinary $A_1$-prelayer. We work with sums $S(x)$. The layers of $S_{f_x}$ coincide with layers of $S_f$ except those are contained in
Consider two summands $\rho_1(\pi)\sigma_1 x^{-1} \rho_2(\pi)^{-1}$ and $\rho_1(\pi')\sigma_2 y^{-1} \rho_2(\pi')^{-1}$. If they are same we have $\sigma^{-1} \rho_1(\sigma) = \tau_2 y^{-1} \rho_2(\sigma)^{-1} x \tau_1^{-1}$, where $a = a_1 a_3 = \pi_1^{-1} \pi = \pi_1^{-1} \pi_1 \pi_3^{-1} \pi_3$. As in Lemma 3 from [Zubl] we see that $\sigma^{-1} \rho_1(\sigma) = \sigma^{-1} a_i \sigma \in H$ since there are no any passive layers. In particular, $a \in \rho_1^{-1}(H^\sigma) \cap \rho_2^{-1}(x H y^{-1})$. If $x = y$ then $a = \pi_1^{-1} \pi \in S_{f_1}$, that is the summands of $S(x)$ do not appear twice. The case $x \neq y$ means that we have two equal summands from $S(x)$ and $S(y)$. As above, it follows that $\rho_1(\sigma) = \sigma h_1 \sigma^{-1}$ and $\rho_2(\sigma) = x h_2 y^{-1}, h_1, h_2 \in H, a \in S_f$. For any $\pi \in S_f, \tau \in H$ we obtain

$$\rho_1(\pi)\sigma x^{-1} \rho_2(\pi)^{-1} = \rho_1(\pi)\rho_1(\sigma)\sigma h_1^{-1} \sigma^{-1} \sigma x^{-1} x h_2 y^{-1} \rho_2(\sigma)^{-1} \rho_2(\pi)^{-1} =$$

$$= \rho_1(\pi)\sigma h_1^{-1} \tau h_2 y^{-1} \rho_2(\pi)^{-1}.$$ 

In other words, $S(x)$ and $S(y)$ consist of the same summands. This concludes the proof for ordinary $A_1$-prelayers.

(ii) From now on one can assume that any ordinary layer $\beta_{uv}$ satisfies either $\sigma(\beta_{uv}) \subseteq \hat{A}_1$ or $\sigma(\beta_{uv}) \subseteq \hat{A}_2 \cup \hat{A}_3$.

Consider the case $\sigma(\beta_{uv}) \subseteq \hat{A}_1 \cup \hat{A}_3$. Denote by $\gamma$ the set $\beta_{uv} \cap \hat{A}_1$. We work with sums $S'$. It is clear that all $A_1$ and $A_3$-layers of $S_{f_1}$ and $S_f$ are the same. In particular, any representative $\pi \in S_f/S_{f_1}$ can be choosen in $S_{\hat{A}_2}$ and $\rho_2(\pi) = 1 \in H$. An equation $\rho_1(\pi)\sigma x \tau_1 = \rho_1(\pi')\sigma y \tau_2$ takes place iff $a = \pi_1^{-1} \pi \in S_f \cap S_{\hat{A}_2}$ satisfies $\rho_1(\pi) = \sigma y H(\sigma x)^{-1}$. It remains to repeat the final computations from (i). The same arguments work in the case when $\gamma$ is any $A_3$-prelayer of $\beta_{uv}$. Therefore, one can assume that either $\beta_{uv} \subseteq \hat{A}_1$ or $\beta_{uv}$ coincides with its $A_3$-prelayer.

(iii) We consider $\gamma$ which is a $A_2$-prelayer contained in $\beta_{uv}$ and work with sums $S(x)$. For any $x \in S_{\hat{A}}$ the layers of $S_{f_1}$ coincide with the layers of $S_f$ except those which are contained in $\beta_{uv} + s$ (if $\beta_{uv} \subseteq \hat{A}_2$) or in $\beta_{uv}$ (if $\beta_{uv} \subseteq \hat{A}_3$). In particular, any representative $\pi \in S_f/S_{f_1}$ can be choosen in $S_{\hat{A}_3}$ and $\rho_1(\pi) = 1 \in H$. An equation $\sigma \tau_1 x^{-1} \rho_2(\pi)^{-1} = \sigma \tau_2 y^{-1} \rho_2(\pi')^{-1}$ holds iff the element $a = \pi_1^{-1} \pi' \in S_f \cap S_{\hat{A}_3}$ satisfies $\rho_2(\pi) = x H y^{-1}$. It remains to refer to the final computations from (i) again.

(iv) Now, we consider the case $\beta_{uv} \subseteq \hat{A}_1, \sigma(\beta_{uv}) \subseteq \hat{A}_2 \cup \hat{A}_3$. We extract a $A_2$-prelayer $\gamma \subseteq \beta_{uv}$ and work with sums $S(x)$. It is clear that only some $A_1$-layers of $S_{f_1}$ are different from $A_1$-layers of $S_f$. Moreover, all of them are sublayers of $A_1$-layers of $S_f$ contained in $\beta_{uv}$. Thus all representatives $\pi \in S_f/S_{f_1}$ can be choosen in $S_{\beta_{uv}}$. In particular, $\rho_1(\pi) = \pi, i = 1, 2$. It is obvious that this case is the same as (i).

(v) Finally, let $\sigma(\beta_{uv}) \subseteq \beta_{f_2} \cap \hat{A}_1$. Let $\gamma$ is a $A_1$-layer of $S_f$ or a $A_3$-prelayer belonging to $\beta_{uv}$. We work with sums $S'(x)$. Only some $A_1$-layers of $S_{f_1}$ are different from $A_1$-layers of $S_f$ and all of them are sublayers of $\sigma(\beta_{uv}) \cap \beta_{f_2}$. Thus all representatives $\pi \in S_f/S_{f_1}$ can be choosen in $S_{\beta_{f_2}} \cap S_{\hat{A}_1} \subseteq H$. As above, $\rho_1(\pi) = \pi, i = 1, 2$. An equation $\pi \sigma x \tau_1 \pi^{-1} = \pi' \sigma y \tau_2 \pi'^{-1}$ holds iff $(\sigma y)^{-1} a \sigma x = \tau_2 a \tau_1^{-1} \pi \in H$, where $a = \pi_1^{-1} \pi$. The final computations are already obvious. The proposition is proved.
Now everything is prepared to prove the main theorem. Let \( \beta_{uv} \) be an ordinary layer and \( \sigma(\beta_{uv}) \subseteq A_2 \cup A_3 \). More precisely, suppose that \( \sigma(\beta_{uv}) \subseteq (\sigma(\beta_{fg}) - s) \cap A_2 \). The case \( \sigma(\beta_{uv}) + s \subseteq \sigma(\beta_{fg}) \cap A_3 \) can be checked in the same way. It is possible that \( \beta_{fg} = \beta_0 \). Denote by \( X \) a variable corresponding to the \( A_1 \)-layer \( \sigma(\beta_{uv}) \cap (\sigma(\beta_{fg}) - s) \). Using the contracting rules we see that the right hand side neighbor of \( X \) in the records of all summands \( tr^*(\sigma\tau) \) is either \( Y \) or \( \overline{Y} \), where the variable \( Y \) corresponds to the \( A_1 \)-layer \( \beta_{uv} \) or to the \( A_2 \)-layer \( \beta_{uv} + s \) (\( \beta_{uv} \)). More precisely, when \( \beta_{uv} \subseteq A_2 \) we have a product \( p = X \overline{Y} \) but in other cases \( p = XY \). Notice that the path \( p \) is closed iff \( \beta_{uv} = \beta_{fg} \pm s \).

One can represent \( z \) as \( z = \sum_{I,K} \alpha_{I,K} (\sigma_i(p_1)^{k_1} \ldots \sigma_i(p_l)^{k_l}) \), where \( I = \{i_1, \ldots, i_l\}, K = \{k_1, \ldots, k_l\} \) are collections of indices, \( p_1, \ldots, p_l \) are (not necessary different) primitive cycles (if some \( p_k, p_m \) are equal then we suppose that \( i_k \neq i_m \)) and for any \( I, K \) we have \( i_k k_i | p_1 | + \ldots + i_k k_l | p_l | = r \). Up to possible repetitions among \( p_1, \ldots, p_l \), the set \( \{p_1, \ldots, p_l\} \) coincides with the set of all primitive cycles belonging to at least one summand \( tr^*(\sigma\tau) \) of \( z \) (see [Zub7, Don2]). Since \( p \) is not a proper power, even if it is a cycle, we see that each \( p_i \) either contains \( p \) (it is possible that \( p_i \) contains \( p \) more than one time) or does not contain any arrow belonging to \( p \).

We list all possibilities for origins or ends of \( p \) as follows.

1. If \( \beta_{uv} \subseteq A_1 \) then \( t(p) = a^*, j_q; i(p) = b, i_q; a, b \in V_{ord}, q, q' \in \Omega \).
2. If \( \beta_{uv} \subseteq A_2 \) then \( t(p) = a^*, j_q; i(p) = j_q; a \in V_{ord}, q, q' \in \Omega \).
3. If \( \beta_{uv} \subseteq A_3 \) then \( t(p) = a^*, j_q, i(p) = b^*, j_q; a, b \in V_{ord}, q, q' \in \Omega \).

In the case \( i(p) = b, t(p) = a^*, a, b \in V_{ord} \) we construct a new quiver \( \tilde{Q} \) with the vertex set \( \tilde{V} = V \cup \{a^*\} \) and the arrow set \( \tilde{A} = A \cup \{\tilde{p}\} \), where \( \tilde{p} \) is a new arrow having the same origin and end as the path \( p \). In other words, the difference between \( Q \) and \( \tilde{Q} \) is that the set of couples \( \{i_q, j_q\} \) is completed by the new couple \( \{a, a^*\} \) but \( V_{ord} = V \cap \{a\} \). For the sake of convenience we introduce a symbol \( q_0 \) such that \( a = i_{q_0}, a^* = j_{q_0} \) and \( \tilde{\Omega} = \Omega \cup \{q_0\} \). In all other cases it is not necessary to add new vertices but only new arrows. For example, if \( i(p) = j_q, t(p) = a^*, a \in V_{ord}, q, q' \in \Omega \), then \( \tilde{V} = V, \tilde{A} = A \cup \{\tilde{p}\} \), where \( i(\tilde{p}) = a, t(\tilde{p}) = i_q \). Notice that \( \tilde{V}^{(d)} = V^{(d)} \) but \( \tilde{A}^{(d)} \) is different from \( A^{(d)} \) whenever \( i(p) \in V_{ord}, t(p) \in V_{ord} \).

If \( \tilde{V} = V \) we leave the same dimension vector \( \mathbf{t} \) but if \( \tilde{V} \neq V \) then we replace it by \( \tilde{\mathbf{t}} = (\ldots, d_a, d_a^*, \ldots) \). It is clear that the representation space \( R(\tilde{Q}, \tilde{\mathbf{t}}) \) (\( R(\tilde{Q}, \tilde{\mathbf{t}}) \)) contains the space \( R(Q, \mathbf{t}) \) as a direct summand. Thus \( J(\tilde{Q}, \mathbf{t}) \subseteq J(Q, \mathbf{t}) \) \( J(Q, \mathbf{t}) \subseteq J(Q, \tilde{\mathbf{t}}) \). Replace all occurrences of \( p \) in \( z \) by \( \tilde{p} \). We get some \( \tilde{z} \in J(\tilde{Q}) \). Since matrices \( X_d, Y_d \) appear only in the product \( p \), it is easy to prove that \( z \in T(Q, \mathbf{t}) \) iff \( \tilde{z} \in T(\tilde{Q}, \tilde{\mathbf{t}}) \) \( (\tilde{z} \in T(\tilde{Q}, \tilde{\mathbf{t}})) \). Using the induction hypothesis we obtain

\[
\tilde{z} = \sum_{i \in V_{ord}, u \geq d+1} f_{i,u} \sigma_u(h_{i,u}) + \sum_{i \in V_{ord}, v \geq d+1, 2s \leq v} f_{i,v,s} \sigma_v, s(h_{i,v,s}^{(1)}, h_{i,v,s}^{(2)}, h_{i,v,s}^{(3)}) + \ldots
\]
\[ \sum_{a, b \in s} f_{q,v,s} \sigma_{v,s}(h_{q,v,s}^{(1)}, h_{q,v,s}^{(2)}, h_{q,v,s}^{(3)}) + \sum_{q \in \Omega, v \geq d_q + 1} f_{q,v,s} \sigma_{v,s}(h_{q,v,s}^{(1)}, h_{q,v,s}^{(2)}, h_{q,v,s}^{(3)}). \]

or

\[ \tilde{z} = \sum_{i \in V_{ord}, u \geq d_i + 1} f_{i,u} \sigma_{i,u}(h_{i,u}) + \sum_{i \in V_{ord}, v \geq d_i + 1, 12s \leq v} f_{i,v,s} \sigma_{v,s}(h_{i,v,s}^{(1)}, h_{i,v,s}^{(2)}, h_{i,v,s}^{(3)}) + \sum_{q \in \Omega, v \geq d_q + 1} f_{q,v,s} \sigma_{v,s}(h_{q,v,s}^{(1)}, h_{q,v,s}^{(2)}, h_{q,v,s}^{(3)}). \]

Here \( f_{i,u}, f_{i,v,s}, f_{q,v,s} \) are some monomials from \( J(\hat{Q}) \). The elements \( h_{i,u} \) are incident to \( i \) and \( h_{i,v,s}^{(1)}, h_{i,v,s}^{(2)} \) are incident to \( i \) and \( i_v \) respectively. Furthermore, \( h_{i,v,s}^{(k)}, h_{i,v,s}^{(k)} \) are passing from \( i \) (\( i^* \)) and \( i_q, j_q \) to \( i^* (i) \) and \( j_q (i_q) \) correspondingly, \( k = 2 (k = 3) \).

Replacing all occurrences of \( \hat{p} \) by the product \( p \) we complete the proof.

The case \( \sigma(\beta_{uv}) \subseteq A_1 \cap \beta_{fg} \) is the same. As above, \( X \) is a variable corresponding to the \( A_1 \)-layer \( \sigma(\beta_{uv}) \cap \beta_{fg} \). It is easy to see that the right hand side neighbor of \( X \) is \( Y \) or \( \overline{Y} \), where \( Y \) corresponds to either an \( A_1 \)-layer \( \beta_{uv} \) or to an \( A_3 \)-layer \( \beta_{uv} + s (\beta_{uv}) \). We leave to the reader to check all details.

It remains to consider the last case when there are no any ordinary layers. It means that either \( V_{ord} = \{1\} \) or \( \Omega = \{q\}, V_{ord} = \{i_q\} \). In both cases \( S_{\beta} = S_{\beta_0} = S_T = S_{T(x)} = S_H = S_{H(x)} = S_r, x = 1, q \). If \( x = 1 \) then our quiver consists of one loop incident to 1 and the element \( z \) is equal to \( \sigma_r(X_r, X, Y, Z) \).

The theorem is proved.

**Remark 2.2** It is possible to give a self-contained proof of this theorem which does not use preliminary description of the free invariant algebra \( J(Q) \) given in [Zub7]. In fact, one has to prove that every time when we replace given suitable generator \( z \) by \( \tilde{z} \), as above, we get a suitable generator again with respect to some other base group and quiver. But there are two reasons why I prefered the way used in this article.

First, it is not obvious that the elements \( \sigma_{r,s}(X, Y, Z) \) can be written as sums \( \sum_{I,K} \alpha_{I,K} \sigma_{i_1}(p_1)^{k_1} \ldots \sigma_{i_t}(p_t)^{k_t} \). Of course, referring to [Zub7] we know that it is true, but how to get this expression for any \( \sigma_{r,s}(X, Y, Z) \) directly?

Second, it is not easy exercise to show that \( \tilde{z} \) is a suitable generator and it requires a lot of case-by-case observations. For example, consider the case \( \sigma(\beta_{uv}) \subseteq (\sigma(\beta_0) - s) \cap A_2, \beta_{uv} = \sigma(\beta_{cd}) \subseteq A_1, t(p) = a^*, i(p) = b, \) where \( \beta_{cd} \) is an ordinary layer and \( a, b \in V_{ord} \). The conditions of admissibility say that \( \beta_{uv} \subseteq T(i_q) \cap A_1, \sigma(\beta_{uv}) \subseteq I(j_q), \mu \subseteq T(a) \) and \( \beta_{cd} \subseteq T(b) \), where \( \mu \subseteq \beta_0, \sigma(\mu) - s = \sigma(\beta_{uv}) \). One can check that

\[ \tilde{z} = c(\sum_{r \in \mathcal{S}_\mu} (-1)^r \sigma^r, S_r). \]
Here $S'_{β} = S_{β|β_{uv}}$, $σ'|_{[1,r]\{β_{uv}\}∪β_{cd}\cupμ} = σ$ but $σ'|_μ = σ - s, σ'|_{β_{cd}} = σ^2 + s$. In other words, if $r'$ is a multidegree of $z$ then $r'_X = r'_Y = 0, r'_p = r_X = |β_{uv}|$, the variable $p$ corresponds to the $A_2$-layer $σ(β_{uv}) = σ'(μ) = σ'(β_{cd}) - s$. Moreover, $S'_0 = S_{β_{uv}}^A \times S_{β_{uv}}^A \times S_{β_{uv}}^A$ and $ρ'_1, ρ'_2 : S'^0 \rightarrow S_{[1,r]\{β_{uv}\}}^{A_1}$ are just restrictions of $ρ_1, ρ_2$. It is clear that the layers of $S'_0 = ρ_1^{-1}(S'_β) \cap ρ^{-1}_2(S'_β)$ are layers of $S$ without $A_1$-layer $β_{uv}$. One also has to check that $σ' ∈ S_{[1,r]\{β_{uv}\}}$. We have $[1,r] \setminus β_{uv} = μ \cup β_{cd} \cup T$. Since $σ'$ acts injectively on the subsets $μ, β_{cd}, T$ it remains to prove that $β_{uv}$ does not intersect the set $σ'([1,r] \setminus β_{uv})$. By definition

$$σ'([1,r] \setminus β_{uv}) = σ'(β_{cd}) \cup σ(μ) \cup σ(T) = (σ(β_{uv}) + s) \cup σ(β_{uv}) \cup σ(T) =$$

$$σ(μ) \cup σ(β_{uv}) \cup σ(T).$$

Now it is obvious because of $β_{uv} = σ(β_{cd})$.

### 3 Applications

The notation of mixed representations of quivers was introduced to generalize Procesi-Razmyslov’s theorem (briefly – PRT) for adjoint action invariants of orthogonal and symplectic groups (see [Zub5, Zub6, Zub7]). In this section modulo the previous theorem we describe some approach to this problem. In fact, the same method works in much more general case of so-called supermixed representations of quivers (see for definitions [Zub7]). The procedure of computation of generating invariants of supermixed representations of any quiver was described in [Zub7]. To be precise, these invariants can be obtained by a specialization of invariants of mixed representations of another quiver (see Section 4 from [Zub7]). The next step is to get all defining relations between them. We demonstrate how to do it in the principal case of the diagonal actions of orthogonal and symplectic groups on several matrices by conjugation. The general case can be reduced to the principal one as in [Zub7].

Consider the quiver $Q$ such that $V = \{1,2\}$ and $A = \{a_1, \ldots, a_m, b, c\}$ with $i(a_j) = t(a_j) = 1, i(b) = t(c) = 1, t(b) = i(c) = 2, 1 ≤ j ≤ m$. Let $V_{ord} = \emptyset, Ω = \{q\}, i_q = 1, j_q = 2$. Any dimension vector $t$ compatible with this partition has the form $(d, d^*)$. It was proved in [Zub5] that for any $d$ we have a short exact sequence

$$0 \rightarrow I_d \rightarrow J(Q, d) \rightarrow S_d \rightarrow 0.$$

Here $J(Q, d) = J(Q, (d, d^*))$, $S_d = K[M(d)^m]^{G_d}, G_d = O(d)$ or $G_d = Sp(d)$. The ideal $I_d$ is equal to $T_d^{GL(d)}$, where $T_d$ is the ideal of $K[R(Q, (d, d^*))] = K[R(Q, d)]$ generated by the coefficients of the matrices $Y(b)Y(c) - E(d), Y(c) - Y(c)$ (in the symplectic case, the last matrix should be replaced by $Y(c) + Y(c)$ if $char K \neq 2$, otherwise one has to fill its zero diagonal by the original coefficients $y_{kk}(c), k = 1, \ldots, d$, $E(d)$ is an $d \times d$ unit matrix. The epimorphism $J(Q, d) \rightarrow S_d \rightarrow 0$ is
induced by the specialization $Y(b), Y(c) \mapsto E(d)$ (by $Y(b) \mapsto J(d), Y(c) \mapsto -J(d)$ in the symplectic group case, where $J(d)$ is a matrix of the skew-symmetric bilinear form defining $Sp(d)$). Recall that in the orthogonal group case we assume that $\text{char} \ K \neq 2$.

First, we consider the orthogonal group case. Fix two integers $N, n, N > n$. For the dimension vectors $\mathbf{N} = (N, N^*)$, $\mathbf{n} = (n, n^*)$ we denote the non-standard (on the arrows $b$ and $c$) specialization $p_{\mathbf{N}, \mathbf{n}}$ just by $p_{\mathbf{N}, \mathbf{n}}$ (respectively – by $j_{\mathbf{N}, \mathbf{n}}$). Denote by $E_d$ the subspace of $K[R(Q, d)]$ which is generated by the polynomials $z_{ij} = \sum_{1 \leq k \leq d} y_{ik}(b)y_{kj}(c) - \delta_{ij}, u_{kl} = y_{kl}(c) - y_{lk}(c), i, j, k, l = 1, \ldots, d, k < l$. It is easy to see that $Z^g = g^{-1}Zg, U^g = g^{-1}Ug^{-1}$, where $Z = (z_{ij}), U = \frac{1}{2}(Y(c) - Y(c)), g \in GL(d)$.

Since the elements $z_{ij}, u_{kl}$ form a regular sequence in $K[R(Q, d)]$, we have the Koszul resolution (see Lemma 1.3(b) [Don7] or [Mats])

$$\ldots \to \Lambda^k(E_d) \otimes K[R(Q, d)] \to \ldots \to E_d \otimes K[R(Q, d)] \to T_d \to 0.$$ 

For any $k$ denote by $\Delta_{d,k}$ the image of $\Lambda^{k+1}(E_d) \otimes K[R(Q, d)]$ in $\Lambda^k(E_d) \otimes K[R(Q, d)]$. Notice that $g.f.d.(S^l(E_d)) = 0$ for any $l \geq 0$ [Kur1, Kur2]. Since $g.f.d(\Lambda^k(E_d) \leq k-1$ ([Don7], Corollary 1.2(d)) we obtain $g.f.d(\Lambda^k(E_d) \otimes K[R(Q, d)]) \leq k-1$ because of $g.f.d.(K[R(Q, d)]) = 0$ [Zub7] and $g.f.d(\Delta_{d,k}) \leq k, k = 1, 2, \ldots$, by Lemma 1.1.

**Proposition 3.1** The specialization $p_{\mathbf{N}, \mathbf{n}}$ induces an epimorphism $I_N \to I_n$.

Proof. It is easy to see that $p_{\mathbf{N}, \mathbf{n}}(T_N) = T_n, p_{\mathbf{N}, \mathbf{n}}(\Lambda^k(E_N) \otimes K[R(Q, N)]) = \Lambda^k(E_n) \otimes K[R(Q, n)]$ and $p_{\mathbf{N}, \mathbf{n}}(I_N) \subseteq I_n$. Moreover, for any $k \geq 1$ we have $p_{\mathbf{N}, \mathbf{n}}(\Delta_{N,k}) \subseteq \Delta_{n,k}$. Applying Lemma 1.2 for the homomorphism of pairs $(p_{\mathbf{N}, \mathbf{n}}, j_{\mathbf{N}, \mathbf{n}})$ one can extract the following commutative fragments of the corresponding long diagrams (we omit the first arguments $GL(N), GL(n)$ from all cohomology groups because of they can be easily recovered by referring to subindices)

$$
(\begin{array}{c}
(E_N \otimes K[R(Q, N)])^{GL(N)} \to I_N \to H^1(\Delta_{N,1}) \to 0 \\
\downarrow \\
(E_n \otimes K[R(Q, n)])^{GL(n)} \to I_n \to H^1(\Delta_{n,1}) \to 0,
\end{array})
$$

and (for $k = 2, 3, \ldots$)

$$
\begin{array}{c}
H^{k-1}(\Lambda^k(E_N) \otimes K[R(Q, N)]) \to H^{k-1}(\Delta_{N,k-1}) \to H^k(\Delta_{N,k}) \to 0 \\
\downarrow \\
H^{k-1}(\Lambda^k(E_n) \otimes K[R(Q, n)]) \to H^{k-1}(\Delta_{n,k-1}) \to H^k(\Delta_{n,k}) \to 0.
\end{array}
$$

Assume that all $H^{k-1}(\Lambda^k(E_N) \otimes K[R(Q, N)]) \to H^{k-1}(\Lambda^k(E_n) \otimes K[R(Q, n)])$ are epimorphisms, $k \geq 1$. Then $I_N \to I_n$ is an epimorphism iff $H^1(\Delta_{N,1}) \to H^1(\Delta_{n,1})$ is. Regarding to the next diagram we see that $H^1(\Delta_{N,1}) \to H^1(\Delta_{n,1})$ is an epimorphism iff $H^2(\Delta_{N,2}) \to H^2(\Delta_{n,2})$ is and so on. But $\Delta_{n,k} = 0$ for sufficiently
large $k$. Therefore, all we need is to prove that $H^{k-1}(\Lambda^k(E_N) \otimes K[R(Q, N)]) \to H^{k-1}(\Lambda^k(E_n) \otimes K[R(Q, n)])$ is an epimorphism for any $k \geq 1$.

Consider some collection of symmetric powers $S^{i_1}(E_d), \ldots, S^{i_s}(E_d), d = N, n$. For the sake of simplicity denote $K[R(Q, d)]$ by $A_d$ and $S^{i_1}(E_d) \otimes \ldots \otimes S^{i_s}(E_d)$ by $B_d$. By the definition of $E_d$ we have

$$B_d = \oplus_{0 \leq i_1 \leq i_1, \ldots, 0 \leq i_s \leq i_s} S^{i_1}(Z_d) \otimes S^{i_1-i_1}(U_d) \otimes \ldots \otimes S^{i_s}(Z_d) \otimes S^{i_s-i_s}(U_d).$$

Here $S^a(Z_d) = K[Z_d](a), S^b(U_d) = K[U_d](b)$ are homogeneous components of the polynomial algebras generated by the coefficients of $Z_d$ and $U_d$ correspondingly.

The following lemma completes the proof of Proposition 3.1.

**Lemma 3.1** The homomorphism

$$H^{k-1}(\Lambda^k(E_N) \otimes A_N \otimes B_N) \to H^{k-1}(\Lambda^k(E_n) \otimes A_n \otimes B_n)$$

induced by $p_{N,n}$ is an epimorphism for any $k \geq 1$.

Proof. The space $(E_N \otimes A_N \otimes B_N)^{GL(N)}$ can be regarded as a sum of homogeneous components of the invariant algebra of supermixed representations of the other quiver $Q'$ with the same vertex set as $Q$ but with $s + 1$ new loops incident to the vertex 1 and $s + 1$ new arrows passing from the vertex 2 to the vertex 1. The spaces of the linear maps which belong to the last arrows are the spaces of skew-symmetric matrices corresponding to $s + 1$ copies of $U$. The spaces belonging to the new loops are the spaces of square matrices corresponding to $s + 1$ copies of $Z$. For more detailed explanations we refer to [Zub7]. Finally, the restriction $p_{N,n}$ on $(E_N \otimes A_N \otimes B_N)^{GL(N)}$ can be identified with the standard specialization of the corresponding invariant algebras which is a homogeneous epimorphism (see [Zub7], Section 4). Thus yields the case $k = 1$.

Let $k > 1$. We have an exact sequence (it is also a partial case of the Koszul resolution [Don7, Mats, Bur])

$$0 \to \Lambda^k(E_d) \to \Lambda^{k-1}(E_d) \otimes S^1(E_d) \to \ldots \to \Lambda^1(E_d) \otimes S^{k-1}(E_d) \to S^k(E_d) \to 0.$$

Tensoring by $C_d = A_d \otimes B_d$ we get an exact sequence

$$0 \to \Lambda^k(E_d) \otimes C_d \to \Lambda^{k-1}(E_d) \otimes S^1(E_d) \otimes C_d \to \ldots \to \Lambda^1(E_d) \otimes S^{k-1}(E_d) \otimes C_d \to S^k(E_d) \otimes C_d \to 0.$$

As above, denote by $\Delta_{d,i}$ the image of $\Lambda^{i+1}(E_d) \otimes S^{k-i-1}(E_d) \otimes C_d$ in $\Lambda^i(E_d) \otimes S^{k-i}(E_d) \otimes C_d$. Repeating the previous arguments we obtain a collection of commutative diagrams with exact top and bottom rows.
\[ H^{k-2}(\Lambda^{k-1}(E_N) \otimes S^1(E_N) \otimes C_N) \rightarrow H^{k-2}(\Delta_{N,k-2}) \rightarrow H^{k-1}(\Lambda^k(E_N) \otimes C_N) \rightarrow 0 \]

\[ H^{k-2}(\Lambda^{k-1}(E_n) \otimes S^1(E_n) \otimes C_n) \rightarrow H^{k-2}(\Delta_{n,k-2}) \rightarrow H^{k-1}(\Lambda^k(E_n) \otimes C_n) \rightarrow 0, \]

and (for \( i = 3, \ldots \))

\[ H^{k-i}(\Lambda^{k-i+1}(E_N) \otimes S^{i-1}(E_N) \otimes C_N) \rightarrow H^{k-i}(\Delta_{N,k-i}) \rightarrow H^{k-i+1}(\Delta_{N,k-i+1}) \rightarrow 0 \]

\[ H^{k-i}(\Lambda^{k-i+1}(E_n) \otimes S^{i-1}(E_n) \otimes C_n) \rightarrow H^{k-i}(\Delta_{n,k-i}) \rightarrow H^{k-i+1}(\Delta_{n,k-i+1}) \rightarrow 0. \]

By induction hypothesis \( H^{k-2}(\Lambda^{k-1}(E_N) \otimes S^1(E_N) \otimes C_N) \rightarrow H^{k-2}(\Lambda^{k-1}(E_n) \otimes S^1(E_n) \otimes C_n) \) is an epimorphism. Thus \( H^{k-1}(\Lambda^k(E_N) \otimes C_N) \rightarrow H^{k-1}(\Lambda^k(E_n) \otimes C_n) \) is an epimorphism if \( H^{k-2}(\Delta_{N,k-2}) \rightarrow H^{k-2}(\Delta_{n,k-2}) \) is. It is clear that the next typical step is to show that \( H^{k-i+1}(\Delta_{N,k-i+1}) \rightarrow H^{k-i+1}(\Delta_{n,k-i+1}) \) is an epimorphism iff \( H^{k-i}(\Delta_{N,k-i}) \rightarrow H^{k-i}(\Delta_{n,k-i+1}) \) is. But for \( i = k \) it is obviously the base of induction. The lemma and the proposition are proved.

Denote by \( \tilde{p}_{N,n} \) the standard specialization \( J(Q,N) \rightarrow J(Q,n) \). Using Amit-sur’s formulae [Am] we see that \( p_{N,n} \) takes any \( \sigma_j(m) \) to either \( \sigma_j(\tilde{p}_{N,n}(m)) \) or to \( \sum_{0 \leq j \leq \ell} \sigma_j(\tilde{p}_{N,n}(m))\sigma_{j-\ell}(E(N,n)) \), where \( m \in Q^\ell \) (as above we identify monomials with paths in \( Q^\ell \)) and \( E(N,n) \) is a \( N \times N \) matrix with 1-s on the diagonal except the first \( n \) places and 0-s on all other places. In particular, we get

**Lemma 3.2** For any \( f \in J(Q,N)(r) \) \( p_{N,n}(f) = \tilde{p}_{N,n}(f) + (\text{summands of degree } < r) \).

If \( A \) is a graded algebra we denote by \( A^{(r)} \) the sum \( \oplus_{0 \leq l \leq r} A(i) \). So the algebra \( A \) turns to a filtered algebra with filtration \( A^{(0)} \subseteq A^{(1)} \subseteq \ldots \subseteq A^{(r)} \subseteq \ldots \). For any ideal \( I \) of \( A \) denote \( I \cap A^{(r)} \) by \( I^{(r)}, r = 0, 1, 2, \ldots \).

**Lemma 3.3** For any \( r \geq 0 \) we have \( p_{N,n}(J(Q,N)^{(r)}) = J(Q,n)^{(r)} \). Moreover, if \( N' \geq N \geq r \) then \( p_{N',N} |_{J(Q,N')(r)} \) is an isomorphism.

Proof. Use induction on \( r \) and Theorem 2.1 from [Zub7]. If \( f \in J(Q,N')^{(l)} \setminus J(Q,N')^{(l-1)}, l \leq r \), then by Lemma 3.2 \( f = f_l + (\text{summands of degree } < l) \), where \( f_l \) is the non-zero \( l \)-th homogeneous component of \( f \). Thus \( p_{N',N}(f) = \tilde{p}_{N',N}(f_l) + (\text{summands of degree } < l) \) and \( \tilde{p}_{N',N}(f_l) \neq 0 \) [Zub7] (see the remark after Theorem 1 or Proposition 2.2 from [Zub7]). The lemma is proved.

**Remark 3.1** By [Zub5] the algebra \( S_d \) is generated by the elements \( \sigma_j(p), 1 \leq j \leq d \), where \( p \) is an arbitrary product of matrices \( Y(a_i, Y(a_i)), 1 \leq i \leq m \). Thus it obviously follows that \( p_{N,n}(S_N(r)) = S_n(r) \) for any \( N > n, r \geq 0 \). Moreover, it is not hard to prove that the space \( S_n^{(r)} \) is covered by \( J(Q,n)^{(2r)} \). Summarizing all previous statements one can say that for any \( N \geq n \) in the commutative diagram
Lemma 3.4 For a fixed $r$ and $N' \geq N \geq r$, $p_{N', N} |_{S_{N'}(r)}$ is an isomorphism.

Proof. It is equivalent to prove that $\dim S_{N'}(r) = \dim S_N(r)$. Since $K[M(n)^m](r)$ is an $O(n)$-module with good filtration for any $n, r$ [Zub3] we see that $\dim S_n(r) = \dim K[M(n)^m](r)^{O(n)}$ equals the multiplicity of a trivial module and does not depend on the characteristic of the ground field $K$. In fact, the formal character of $K[M(n)^m](r)$ as well as its representation as a sum of formal characters of induced modules does not depend on the characteristic of $K$ (see [Jan, Don2, Don3, Don4, Don5, Don6, Zub1, Zub2, Zub3] for more explanations). In particular, one can suppose that $\text{char} K = 0$. If $S_{N'}(r) \to S_N(r) \to 0$ is not an isomorphism that there is $f \in S_{N'}(r) \setminus 0$ such that $p_{N', N}(f) = 0$. It remains to replace $f$ by its complete linearization and refer to [Pr]. The lemma is proved.

We have the countable set of spectrums $\{J(Q, n)^{(r)}, p_{N, n} | N \geq n\}, r = 0, 1, 2, \ldots$. Denote by $J'(Q)^{(r)}$ the inverse limit of $r$-th spectrum. By Lemma 3.3 one can identify $J'(Q)^{(r)}$ with $J(Q, N)^{(r)}$ for sufficiently large $N$. In particular, we have an inclusion $J'(Q)^{(r)} \to J'(Q)^{(r')}$ for any $r' \geq r$. Denote by $J'(Q)$ the direct limit of the spectrum consisting of all $J'(Q)^{(r)}$ and inclusions defined above.

Lemma 3.5 The graded algebra $\text{gr} J'(Q) = \oplus_{r \geq 0} J'(Q)^{(r+1)}/J'(Q)^{(r)}$ is isomorphic to $J(Q)$.

Proof. It is sufficient to notice that the following diagram

$$
\begin{array}{ccc}
J(Q, N')^{(r+1)}/J(Q, N')^{(r)} & \to & J(Q', N')^{(r+1)}/J(Q, N')^{(r)} = J(Q, N')(r+1) \\
\downarrow p_{N', N} & & \downarrow \hat{p}_{N', N} \\
J(Q, N)^{(r+1)}/J(Q, N)^{(r)} & \to & J(Q, N)^{(r+1)}/J(Q, N)^{(r)} = J(Q, N)(r+1)
\end{array}
$$

is commutative for any $N' \geq N \geq r+1$. Here the first two vertical maps are induced by $p_{N', N}$ and $\hat{p}_{N', N}$ respectively. The horizontal maps are natural identifications. We leave checking of all rest details to the reader.

Remark 3.2 As it was noticed in the introduction one can prove that $J(Q)$ is a polynomial algebra with homogeneous free generators. They can be choosed as $\sigma_j(p)$, where $p$ runs over all primitive cycles. In particular, $J'(Q) \cong J(Q)$!

By Lemma 3.3 we have an epimorphism $J'(Q) \to J(Q, n)$. Denote by $T'(Q, n)$ the kernel of this epimorphism. Since $p_{N', N}$ coincides with $\hat{p}_{N', N}$ on $S_N$, the definitions of a free algebra of orthogonal invariants as a filtred or graded algebra are
the same. We denote this algebra by $S$. As above any homogeneous component $S(r)$ can be naturally identified with $S_N(r)$ for sufficiently large $N$ and we have an epimorphism $S \to S_n$ with a kernel $K_n$.

Now our aim is to describe the generators of $K_n$ as we declared at the beginning of this section. Let $f \in K_n^{(r)}$. Without loss of generality one can assume that $f \in S_N^{(r)}$, $N \gg r$.

**Lemma 3.6** For sufficiently large $N, r', N \geq r' \geq r$, there is $t \in T'(Q,n)^{(r')} = T'(Q,n) \cap J(Q,N)^{(r')}$ such that the epimorphism $J(Q,N) \to S_N$ takes $t$ to $f$.

Proof. As we noticed in Remark 3.1 there is $f' \in J(Q,N)^{(2r)}$ such that $f$ is the image of $f'$. Thus the image of $f'' = p_{N,n}(f')$ in $S_n$ equals zero, that is $f'' \in I_n^{(2r)}$. By Proposition 3.1 for sufficiently large $r' \geq 2r$ there is $f''' \in I_N^{(r')}$ such that $p_{N,n}(f''') = f''$. In particular, $p_{N,n}(f' - f''') = 0$. Increasing $N$ one can assume that $N \geq r'$. It remains to take a preimage of $f' - f'''$, say $t$, in $J(Q,N)^{(r')}$. The lemma is proved.

Let $f \in T(Q,n)^{(r)}$, that is $f \in J(Q,N)^{(r)}, N \gg r$. By Lemma 3.2 $p_{N,n}(f) = g' \in J(Q,n)^{(r-1)}$ and by Lemma 3.3 one can choose an element $g \in J(Q,N)^{(r-1)}$ such that $p_{N,n}(g) = g'$. Let $g_{r-1}$ be a $(r-1)$-th homogeneous component of $g$. Again, by Lemma 3.2 $p_{N,n}(f - g_{r-1}) = h' \in J(Q,n)^{(r-2)}$ and one can repeat the previous step. After $r$ steps like above we obtain some $\tilde{f} \in T'(Q,n)^{(r)}$ such that it has $r$-th homogeneous component $\tilde{f}_r$ coincided with $f_r - r$-th homogeneous component of $f$. Notice that if $f$ does not depend on $Y(b),Y(c)$ then $\tilde{f} = f$.

**Lemma 3.7** The ideal $T'(Q,n)$ is generated by the elements $\tilde{f}$, where $f$ runs over the set of generators of $T(Q,n)$ from Theorem 2.

Proof. Fix some $N \gg r$ and consider an element $f \in T'(Q,n) \cap J(Q,N)^{(r)}$. If $f_r$ is the $r$-th homogeneous component of $f$ then by Lemma 3.2 we get $p_{N,n}(f) = \tilde{p}_{N,n}(f_r) + \text{(summands of degree < r)} = 0$. In particular, $\tilde{p}_{N,n}(f_r) = 0$, that is $f_r \in T(Q,n)^{(r)}$ and $f_r$ can be represented as $\sum h_i g_i$, where any $h_i$ is a homogeneous element from $J(Q,N)$ and $g_i$ is a homogeneous component of some generator from Theorem 2. It is clear that an element $t = \sum h_i g_i$ lies in $T'(Q,n)$ and has the same $r$-th homogeneous component as $f$, that is $f - t \in T'(Q,n) \cap J(Q,N)^{(r-1)}$. Induction on $r$ completes the proof.

The symplectic group case can be treated in the same way up to some change in the initial notations. To be precise, in this case $N = 2M, n = 2m$ and $p_{N,n}$ must be redefined as

$$p_{N,n}(y_{ks}(b)) = \begin{cases} y_{ks}(b), & \text{if } M - m + 1 \leq k, s \leq M + m, \\ 1, & \text{if } k + s = N + 1, 1 \leq k \leq M - m, \\ -1, & \text{if } k + s = N + 1, M + m + 1 \leq k \leq N, \\ 0, & \text{otherwise}, \end{cases}$$
\[ p_{N,n}(y_{ks}(c)) = \begin{cases} 
 y_{ks}(c), & \text{if } M - m + 1 \leq k, s \leq M + m, \\
 -1, & \text{if } k + s = N + 1, 1 \leq k \leq M - m, \\
 1, & \text{if } k + s = N + 1, M + m + 1 \leq k \leq N, \\
 0, & \text{otherwise}. 
\]

On the rest variables \( p_{N,n} \) acts by the \textit{symplectically standard} rule

\[ p_{N,n}(y_{ks}(a_i)) = \begin{cases} 
 y_{ks}(a_i), & \text{if } M - m + 1 \leq k, s \leq M + m, \\
 0, & \text{otherwise}, 
\end{cases}, 1 \leq i \leq m. \]

Similarly, \( \tilde{p}_{N,n} \) is symplectically standard on all variables. The homomorphism \( j_{N,n} \) is redefined as

\[ (j_{N,n}(g))_{ks} = \begin{cases} 
 g_{k-M+m,s-M+m}(b), & \text{if } M - m + 1 \leq k, s \leq M + m, \\
 1, & \text{if } k = s, 1 \leq k \leq M - m \text{ or } M - m + 1 \leq k \leq N \\
 0, & \text{otherwise}, 
\end{cases}, g \in GL(n). \]

Denote by the same symbol \( i_{N,n} \) as above the morphism dual to \( p_{N,n} \). A free invariant algebra of symplectic invariants as well as a kernel of an epimorphism of this algebra onto the algebra of symplectic invariants of \( m \times n \) matrices will be denoted by the same symbols \( S, K_n \). It is easy to see that the invariant algebra \( J(Q) \) remains the same even if we replace standard specializations by symplectically standard. The proof of Proposition 3.1 and all consequent lemmas can be word by word repeated. Notice that the correspondence \( f \mapsto \tilde{f} \) from Lemma 3.7 is different from the orthogonal group case because of one has to replace the matrix \( E(N, n) \) by \( J(N, n) = i_{N,n}(0) \). Summarizing we have

**Proposition 3.2** The ideal \( K_n \) (both orthogonal or symplectic) is generated by the images of the elements \( \tilde{f} \) from Lemma 3.7.

### 4 Concluding remarks

Proposition 3.2 gives only some procedure to compute the generators of \( K_n \). Since \( T'(Q, n) \) is not homogeneous ideal they are also not homogeneous. It is not hard exercise to find the elements \( \tilde{f} \) but it is sufficiently difficult problem to describe homogeneous components of their images in \( S \). To illustrate this take an element \( \sigma_r(f) \) from Theorem 2. For the sake of simplicity we consider only orthogonal invariants.

Without loss of generality one can assume that \( f \) is incident to the vertex \( i_q = 1 \). If \( m \) is a monomial belonging to \( f \) then \( p_{N,n}(m) = \tilde{p}_{N,n}(m) \) iff \( m \) contains at least one multiplier \( Y(a_i), 1 \leq i \leq m \), otherwise \( m = (Y(c)Y(b))^l \) or \( m = (\bar{Y}(c)\bar{Y}(b))^l \) and \( p_{N,n}(m) = \tilde{p}_{N,n}(m) + E(N, n) \). Let \( f = f_1 + f_2 \), where \( f_2 \) is a subsum of \( f \) which
contains all monomials of the second type. I claim that there are integer coefficients \( \alpha_k, 0 \leq k \leq r, \alpha_0 = 1 \), such that an element \( z = z(f) = \sum_{0 \leq k \leq r} \alpha_k \sigma_{r-k}(f) \) satisfies \( p_{N,n}(z) = \sigma_r(\tilde{p}_{N,n}(f)) = 0 \). Denote by \( \lambda \) the sum of all coefficients of the monomials belonging to \( f_2 \). We have

\[
p_{N,n}(t) = \sum_{0 \leq j \leq r} \alpha_j \sum_{0 \leq k \leq r-j} C_{N-n}^k \sigma_{r-j-k}(\tilde{p}_{N,n}(f)) = \sum_{0 \leq t \leq r} \sigma_t(\tilde{p}_{N,n}(f)) \sum_{0 \leq j, k+j+k=r-t} C_{N-n}^k \sigma_j \alpha_t.
\]

The required result follows if our coefficients satisfy the equations

\[
\sum_{0 \leq k \leq r-t} C_{N-n}^k \lambda^t \alpha_{r-t-k} = 0, 0 \leq t \leq r - 1.
\]

It is clear that these equations has a unique solution whenever \( \alpha_0 \) is fixed, say \( \alpha_0 = 1 \). One can prove that in this case \( \alpha_j = (-1)^j C_{N-n+j-1}^j \lambda^j, 0 \leq j \leq r \). For the sake of convenience we denote the \( t \)-th equation by Eq. \( t \).

**Lemma 4.1** For any general \( N \times N \) matrix \( X \) and a variable \( y \) we have \( \sigma_k(X + yE(N)) = \sum_{0 \leq s \leq k} C_{N-k+s}^k y^s \alpha_k(X) \).

Proof. Without loss of generality one can assume that \( X \) is a diagonal matrix with diagonal coefficients \( x_1, \ldots, x_N \). Then we have

\[
\sigma_k(X + yE(N)) = \sum_{1 \leq i_1 < \ldots < i_k \leq N} (x_{i_1} + y) \ldots (x_{i_k} + y) = \sum_{0 \leq s \leq k} y^s \sum_{1 \leq r_1 < \ldots < r_{k-s} \leq k} x_{i_{r_1}} \ldots x_{i_{r_{k-s}}}.
\]

It is clear that any summand \( x_{j_1} \ldots x_{j_{k-s}}, j_1 < \ldots < j_{k-s} \), appears as many times as one can choose \( s \) different integers from the set \( \{1, \ldots, N\} \setminus \{j_1, \ldots, j_{k-s}\} \), that is \( C_{N-k+s}^k \). This concludes the proof.

Next, one has to take the matrices \( Y(b), Y(c) \) to \( E(N) \). In particular, \( f_2 \mapsto \lambda E(N) \). We get

\[
z' = z(f_1' + \lambda E(N)) = \sum_{0 \leq k \leq r} \alpha_{r-k} \sum_{0 \leq s \leq k} C_{N-k+s}^k \lambda^s \alpha_{r-s}(f_1') = \sum_{0 \leq t \leq r} \sigma_t(f_1') \sum_{0 \leq s \leq r-t} C_{N-t}^s \lambda^s \alpha_{r-t-s}.
\]

Here \( f_1' \) is just the image of \( f_1 \) under the same specialization \( Y(b), Y(c) \mapsto E(N) \).

I claim that all sums \( \sum_{0 \leq s \leq r-t} C_{N-t}^s \lambda^s \alpha_{r-t-s} \) are equal to zero if \( t \leq n \). More generally, one can prove that the sums \( \sum_{0 \leq s \leq r-t_1} C_{N-t}^s \lambda^s \alpha_{r-t_2} \) are equal to zero for any pair \( (t_1, t_2) \) such that \( 0 \leq t_1 \leq t_2 \leq n \). If \( t_2 = n \) it is just the equation \( Eq_t \) (notice that \( t_1 \leq n \leq r - 1 \)). Let \( n > t_2 \). Using the binomial identity \( C_{n-1}^n + C_{n-1}^{n-1} = C_n^1 \) we have
\[
\sum_{0 \leq s \leq r-t_1} C_{N-t_2}^s \lambda^s \alpha_{r-t_1-s} = \sum_{0 \leq s \leq r-(t_2+1)} C_{N-(t_2+1)}^s \lambda^s \alpha_{r-(t_2+1)-s}. \]

But for pairs \((t_1, t_2 + 1), (t_1 + 1, t_2 + 1)\) the induction hypothesis implies that the both last sums are equal to zero.

These computations show that, up to multipliers, all homogeneous components of \(z'\) are \(\sigma_j(f'_1), n < j \leq r\). It suggests the idea that the defining relations for the orthogonal or symplectic invariants must be very close to the relations from Theorem 2. But, to realize this idea in a complete form one has to investigate the invariants \(\sigma_{r,s}\) more carefully. For example, we need some analog of Amitsur’s formulae for these invariants. By this reason I postpone it for the next article.

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References

[Am] S.A. Amitsur, On the characteristic polynomial of a sum of matrices, *Linear and Mult. Algebra*, 8(1980), 177-182.

[Bur] N. Bourbaki, Algebre homologique, *Masson*, Paris(1980).

[Don1] S. Donkin, Polynomial invariants of representations of quivers, *Comment Math. Helvetic*, 69(1994), 137-141.

[Don2] S. Donkin, Invariant functions on matrices, *Math. Proc. Cambridge Phil. Soc.*, 113, N23(1992), 23-43.

[Don3] S. Donkin, On tilting modules for algebraic groups, *Math. Z.*, 212(1993), 39-60.

[Don4] S. Donkin, Skew modules for reductive groups, *J. Algebra*, 113(1988), 465-479.

[Don5] S. Donkin, Rational representations of algebraic groups: tensor products and filtrations, *Lecture Notes in Math.*, 1140, Springer, 1985.

[Don6] S. Donkin, A filtrations for rational modules, *Math. Z.*, 177(1981), 1-8.
[Don7] S. Donkin, The normality of conjugacy classes of matrices, *Inv. Math.*, 101(1990), 717–736.

[Gab] P. Gabriel, Unzerlegbare Darstellungen I, *Manuscripta Math.*, N6(1972), 71–103.

[Jan] J. Jantzen, Representations of algebraic groups, *Academic Press*, 1987.

[Kur1] K. Kurano, On relations on minors of generic symmetric matrices, *J. Algebra*, 124, N2(1989), 388-413.

[Kur2] K. Kurano, Relations on Pfaffians I: Plethysm formulas, *J. Math. Kyoto Univ.*, 31, N3, 713-731.

[Mats] H. Matsumura, Commutative algebra, *W.A. Benjamin, Inc.*, New York (1970).

[Pr] C. Procesi, The invariant theory of $n \times n$-matrices, *Adv. in Math.*, 19(1976), 306-381.

[PrB1] Lieven Le Bruyn and C. Procesi, Etale local structure of matrix invariants and concomitants, *Lecture Notes Math.*, 127(1987), 143-175.

[PrB2] Lieven Le Bruyn and C. Procesi, Semisimple representations of quivers, *Trans. A.M.S.*, 317(1990), 585-598.

[Parsh] E. M. Friedlander and B. J. Parshall, Cohomology of Lie algebras and algebraic groups, *Amer. J. of Math.*, 108(1986), 235-253.

[Weiss] E. Weiss, Cohomology of groups, *Academic Press*, New York and London, 1969.

[Zub1] A. N. Zubkov, On a generalization of the Procesi-Razmyslov theorem, *Algebra i Logika*, 35, N4(1996), 433-457(russian).

[Zub2] A. N. Zubkov, Endomorphisms of tensor products of exterior powers and Procesi hypothesis, *Commun. in Algebra*, 22, N15(1994), 6385-6399.

[Zub3] A. N. Zubkov, On the procedure of calculation of the invariants of an adjoint action of classical groups, *Commun. in Algebra*, 22, N11(1994), 4457-4474.

[Zub4] A. N. Zubkov, Procesi-Razmyslov’s theorem for quivers, *Fundamental’nyay i Prikladnay Matematika*(russian), 7(2001), N2, 387-421.

[Zub5] A. N. Zubkov, Adjoint action invariants of classical groups, *Algebra i Logika*, 38, N5(1999), 549-584(russian).

[Zub6] A. N. Zubkov, Mixed representations of quivers and relative problems, *Bielefeld University*, SFB 343, Preprint 00-094.
[Zub7] A.N.Zubkov, Invariants of mixed representations of quivers I, submitted to Transformation Groups.