On decomposition numbers with Jantzen filtration of cyclotomic $q$-Schur algebras

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Abstract. Let $\mathcal{S}(\Lambda)$ be the cyclotomic $q$-Schur algebra associated to the Ariki-Koike algebra $H_{n,r}$, introduced by Dipper-James-Mathas. In this paper, we consider $v$-decomposition numbers of $\mathcal{S}(\Lambda)$, namely decomposition numbers with respect to the Jantzen filtrations of Weyl modules. We prove, as a $v$-analogue of the result obtained by Shoji-Wada, a product formula for $v$-decomposition numbers of $\mathcal{S}(\Lambda)$, which asserts that certain $v$-decomposition numbers are expressed as a product of $v$-decomposition numbers for various cyclotomic $q$-Schur algebras associated to Ariki-Koike algebras $H_{n, r_i}$ of smaller rank. Moreover we prove a similar formula for $v$-decomposition numbers of $H_{n, r}$ by using a Schur functor.

0. Introduction

Let $\mathcal{H} = H_{n,r}$ be the Ariki-Koike algebra over an integral domain $R$ associated to the complex reflection group $S_n \rtimes (\mathbb{Z}/r\mathbb{Z})^n$. Dipper, James and Mathas [DJM] introduced the cyclotomic $q$-Schur algebra $\mathcal{S}(\Lambda)$ associated to the Ariki-Koike algebra $\mathcal{H}$, and they showed that $\mathcal{H}$ and $\mathcal{S}(\Lambda)$ are cellular algebras in the sense of Graham and Lehrer [GL], by constructing the cellular basis respectively. It is a fundamental problem for the representation theory to determine the decomposition numbers of $\mathcal{H}$ and $\mathcal{S}(\Lambda)$. It is well-known that the decomposition matrix of $\mathcal{H}$ coincides with the submatrix of that of $\mathcal{S}(\Lambda)$ by the Schur functor.

In the case where $\mathcal{H}$ is the Iwahori-Hecke algebra $H_n$ of type A, Lascoux, Leclerc and Thibon [LLT] conjectured that the decomposition numbers of $H_n$ can be described by using the canonical basis of a certain irreducible $U_v(\hat{\mathfrak{sl}}_e)$-module, and gave the algorithm to compute this canonical basis. The conjecture has been solved by Ariki [A1], by extending to the case of Ariki-Koike algebras.

In the case of the $q$-Schur algebra associated to $H_n$, Leclerc and Thibon [LT] conjectured that the decomposition matrix coincides with the transition matrix between the canonical basis and the standard basis of the Fock space of level 1 equipped with the $U_v(\hat{\mathfrak{sl}}_e)$-module structure, and gave the algorithm to compute the transition matrix. This conjecture has been solved by Varagnolo and Vasserot in [VV].

More generally, in the case of the cyclotomic $q$-Schur algebra $\mathcal{S}$, Yvonne [Y] has conjectured that the decomposition matrix coincides with the transition matrix between the canonical basis and the standard basis of the higher-level Fock space. This canonical basis was constructed by Uglov [U] and the algorithm to compute...
the transition matrix was also given there. Yvonne’s conjecture is still open. We
remark that Ariki’s theorem, Varagnolo-Vasserot’s theorem and Yvonne’s conjecture
are concerned with the situation where $R$ is a complex number field and parameters
are roots of unity.

In order to study the decomposition numbers of $\mathcal{F}$, we constructed in [SW]
some subalgebras $\mathcal{F}^p$ of $\mathcal{F}(\Lambda)$ and their quotients $\mathcal{F}^p$, and showed that $\mathcal{F}^p$ is a
standardly based algebra in the sense of Du and Rui [DR], and that $\mathcal{F}^p$ is a cellular
algebra. Hence, one can consider the decomposition numbers of $\mathcal{F}$, $\mathcal{F}^p$ and $\mathcal{F}^p$ also.

We denote the decomposition numbers of $\mathcal{F}$, $\mathcal{F}^p$ and $\mathcal{F}^p$ by $d_{\lambda \mu}$, $d^{(\lambda,0)}_{\lambda \mu}$ and $\overline{d}_{\lambda \mu}$ respectively, where $d_{\lambda \mu}$ is a decomposition number of the irreducible module $L^\mu$ in
the Weyl module $W^\lambda$ of $\mathcal{F}$ for $r$-partitions $\lambda, \mu$, and $d^{(\lambda,0)}_{\lambda \mu}$, $\overline{d}_{\lambda \mu}$ are defined similarly
for $\mathcal{F}^p$ and $\mathcal{F}^p$ (see Section 1 for details). It is proved in [SW, Theorem 3.13] that

\begin{equation}
\overline{d}_{\lambda \mu} = d^{(\lambda,0)}_{\lambda \mu} = d_{\lambda \mu}
\end{equation}

whenever $\lambda, \mu$ satisfy a certain condition $\alpha_p(\lambda) = \alpha_p(\mu)$. Moreover for such $\lambda, \mu$, the product formula for $\overline{d}_{\lambda \mu},$

\begin{equation}
\overline{d}_{\lambda \mu} = \prod_{k=1}^g d^{(\lambda,0)}_{\lambda \mu[k]},
\end{equation}

was proved in [SW, Theorem 4.17], where $d^{(\lambda[k],\mu[k])}_{\lambda \mu[k]}$ for $k = 1, \ldots, g$ is the decompo-
sition number of the cyclotomic $q$-Schur algebra associated to a certain Ariki-Koike
algebra $\mathcal{H}_{n_k, r_k}$.

Related to the above conjectures on Fock spaces, Leclerc-Thibon and Yvonne
give a more precise conjecture concerning the $v$-decomposition numbers defined by
using Jantzen filtrations of Weyl modules. (For definition of $v$-decomposition num-
bers, see [P2]) We remark that decomposition numbers coincide with $v$-decomposition
numbers at $v = 1$. Thus we regard $v$-decomposition numbers as a $v$-analogue of de-
composition numbers. The conjecture for $v$-decomposition numbers has been still
open even in the case of the $q$-Schur algebra of type A.

In this paper, we show that similar formula as (1) and (2) also hold for $v$-
decomposition numbers. We denote the $v$-decomposition numbers of $\mathcal{F}(\Lambda)$, $\mathcal{F}^p(\Lambda)$
and $\mathcal{F}^p(\Lambda)$ by $d_{\lambda \mu}(v)$, $d^{(\lambda,0)}_{\lambda \mu}(v)$ and $\overline{d}_{\lambda \mu}(v)$ respectively. Then for $r$-partitions $\lambda, \mu$
such that $\alpha_p(\lambda) = \alpha_p(\mu)$, we have (Theorem 2.8)

\begin{equation}
\overline{d}_{\lambda \mu}(v) = d^{(\lambda,0)}_{\lambda \mu}(v) = d_{\lambda \mu}(v),
\end{equation}

and (Theorem 2.14)

\begin{equation}
d_{\lambda \mu}(v) = \overline{d}_{\lambda \mu}(v) = \prod_{k=1}^g d^{(\lambda[k],\mu[k])}_{\lambda \mu[k]}(v),
\end{equation}

where $d^{(\lambda[k],\mu[k])}_{\lambda \mu[k]}(v)$ is the $v$-decomposition number of the cyclotomic $q$-Schur algebra
appeared in (2).
We note that our result is a $v$-analogue of (1), (2), and it reduces to them by taking $v \mapsto 1$. Moreover, for a certain $v$-decomposition number $d^v_{\mu}(v)$ of the Ariki-Koike algebra, we also have the following product formula (Theorem 3.5).

$$d^v_{\mu}(v) = \prod_{k=1}^g d^v_{\lambda^k|\mu|k}(v),$$

where $d^v_{\lambda^k|\mu|k}(v)$ is the $v$-decomposition number of the certain Ariki-Koike algebra $\mathcal{H}_{n,k}$.

We remark that our results hold for any parameters and any modular system, even for the case where the base field has non-zero characteristic, though Yvonne’s conjecture is formulated under certain restrictions for parameters and modular systems.

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1. A review of known results

1.1. Throughout the paper, we follow the notation in [SW]. Here we review some of them. We fix positive integers $r$, $n$ and an $r$-tuple $m = (m_1, \cdots, m_r) \in \mathbb{Z}^r_{>0}$. A composition $\lambda = (\lambda_1, \lambda_2, \cdots)$ is a finite sequence of non-negative integers, and $|\lambda| = \sum_i \lambda_i$ is called the size of $\lambda$. If $\lambda_i \neq 0$ and $\lambda_k = 0$ for any $k > l$, then $l$ is called the length of $\lambda$. If the composition $\lambda$ is a weakly decreasing sequence, $\lambda$ is called a partition. An $r$-tuple $\mu = (\mu^{(1)}, \cdots, \mu^{(r)})$ of compositions is called the $r$-composition, and size $|\mu|$ of $\mu$ is defined by $\sum_{i=1}^r |\mu^{(i)}|$. In particular, if all $\mu^{(i)}$ are partitions, $\mu$ is called an $r$-partition. We denote by $\Lambda = P_{n,r}(m)$ the set of $r$-compositions $\mu = (\mu^{(1)}, \cdots, \mu^{(r)})$ such that $|\mu| = n$ and that the length of $\mu^{(k)}$ is smaller than $m_k$ for $k = 1, \cdots, r$. We define $\Lambda^+ = P_{n,r}(m)$ as the subset of $\Lambda$ consisting of $r$-partitions.

We define the partial order, the so-called “dominance order”, on $\Lambda$ by $\mu \succeq \nu$ if and only if

$$\sum_{i=1}^l |\mu^{(i)}| + \sum_{j=1}^k \nu_j^{(l)} \geq \sum_{i=1}^l |\mu^{(i)}| + \sum_{j=1}^k \nu_j^{(l)}$$

for any $1 \leq l \leq r$, $1 \leq k \leq m_i$. If $\mu \succeq \nu$ and $\mu \neq \nu$, we write it as $\mu \triangleright \nu$.

For $\lambda \in \Lambda^+$, we denote by $\text{Std}(\lambda)$ the set of standard tableau of shape $\lambda$. For $\lambda \in \Lambda^+$ and $\mu \in \Lambda$, we denote by $T_0(\lambda, \mu)$ the set of semistandard $\lambda$-tableau of type $\mu$. Moreover we set $T_0(\lambda) = \cup_{\mu \in \Lambda} T_0(\lambda, \mu)$. For definitions of standard tableau and semistandard tableau, see [SW] or [DJM].

1.2. Let $\mathcal{H} = \mathcal{H}_{n,r}$ be the Ariki-Koike algebra over an integral domain $R$ with parameters $q, Q_1, \cdots, Q_r$ with defining relations in [SW] §1.1. It is known by [DJM] that $\mathcal{H}$ has a structure of the cellular algebra with a cellular basis $\{m_{\lambda|\mu|k}^{g}| \lambda, \mu, k \in \text{Std}(\lambda) \text{ for some } \lambda \in \Lambda^+\}$. Then the general theory of a cellular algebra by [GL] implies the following results. There exists an anti-automorphism $h \mapsto h^*$ of $\mathcal{H}$ such
that \( m^*_t = m_{ts} \). For \( \lambda \in \mathbb{A}^+ \), let \( \mathbb{H}^{\lambda} \) be the \( R \)-submodule of \( \mathbb{H} \) spanned by \( m_{st} \), where \( s, t \in \text{Std}(\mu) \) for some \( \mu \in \mathbb{A}^+ \) such that \( \mu \triangleright \lambda \). Then \( \mathbb{H}^{\lambda} \) is an ideal of \( \mathbb{H} \).

One can construct the standard (right) \( \mathbb{H} \)-module \( S^\lambda \), called a Specht module, with the \( R \)-free basis \( \{ m_t \mid t \in \text{Std}(\lambda) \} \). We define the bilinear form \( \langle , \rangle_{\mathbb{H}} \) on \( S^\lambda \) by

\[
\langle m_s, m_t \rangle_{\mathbb{H}} m_{uv} \equiv m_{us} m_{tv} \mod \mathbb{H}^{\lambda} \quad (s, t \in \text{Std}(\lambda)),
\]

where \( u, v \in \text{Std}(\lambda) \), and the scalar \( \langle m_s, m_t \rangle_{\mathbb{H}} \) does not depend on the choice of \( u, v \in \text{Std}(\lambda) \). The bilinear form \( \langle , \rangle_{\mathbb{H}} \) is associative, namely we have

\[
(1.2.1) \quad \langle x h, y \rangle_{\mathbb{H}} = \langle x, y h^* \rangle_{\mathbb{H}} \quad \text{for} \ x, y \in S^\lambda, \ h \in \mathbb{H}.
\]

Let \( \text{rad } S^\lambda = \{ x \in S^\lambda \mid \langle x, y \rangle_{\mathbb{H}} = 0 \text{ for any } y \in S^\lambda \} \). Then \( \text{rad } S^\lambda \) is the \( \mathbb{H} \)-submodule of \( S^\lambda \) by the associativity of the bilinear form. Put \( D^\lambda = S^\lambda / \text{rad } S^\lambda \).

Assume that \( R \) is a field. Then \( D^\lambda \) is an absolutely irreducible module or zero, and the set \( \{ D^\lambda \mid \lambda \in \mathbb{A}^+ \text{ such that } D^\lambda \neq 0 \} \) gives a complete set of non-isomorphic irreducible \( \mathbb{H} \)-modules.

1.3. Let \( \mathcal{I} = \mathcal{I}(A) \) be the cyclotomic \( q \)-Schur algebra introduced by [DJM], associated to the Ariki-Koike algebra \( \mathbb{H} \) with respect to the set \( A \). It is known by [DJM] that \( \mathcal{I} \) is a cellular algebra with a cellular basis \( \{ \phi_{ST} \mid S, T \in \mathcal{T}_0(\lambda) \text{ for some } \lambda \in \mathbb{A}^+ \} \). Again by the general theory of a cellular algebra, the following results hold.

There exists the anti-automorphism \( x \mapsto x^* \) of \( \mathcal{I} \) such that \( \phi_{ST}^* = \phi_{TS} \). For \( \lambda \in \mathbb{A}^+ \), let \( \mathcal{I}^{\lambda} \) be the \( R \)-submodule spanned by \( \phi_{ST} \), where \( S, T \in \mathcal{T}_0(\mu) \) for some \( \mu \in \mathbb{A}^+ \) such that \( \mu \triangleright \lambda \). Then \( \mathcal{I}^{\lambda} \) is an ideal of \( \mathcal{I} \). One can construct the standard (right) \( \mathcal{I} \)-module \( W^\lambda (\lambda \in \mathbb{A}^+) \), called a Weyl module, with the \( R \)-free basis \( \{ \phi_T \mid T \in \mathcal{T}_0(\lambda) \} \). We define a bilinear form \( \langle , \rangle \) on \( W^\lambda \) by

\[
\langle \phi_S, \phi_T \rangle_{\mathcal{I}} \phi_{UV} \equiv \phi_{US} \phi_{TV} \mod \mathcal{I}^{\lambda} \quad (S, T \in \mathcal{T}_0(\lambda)),
\]

where \( U, V \in \mathcal{T}_0(\lambda) \), and the scalar \( \langle \phi_S, \phi_T \rangle \) does not depend on a choice of \( U, V \in \mathcal{T}_0(\lambda) \). The bilinear form \( \langle , \rangle \) is associative, namely we have

\[
(1.3.1) \quad \langle x \varphi, y \rangle = \langle x, y \varphi^* \rangle \quad \text{for} \ x, y \in W^\lambda, \ \varphi \in \mathcal{I}.
\]

Let \( \text{rad } W^\lambda = \{ x \in W^\lambda \mid \langle x, y \rangle = 0 \text{ for any } y \in W^\lambda \} \). Then \( \text{rad } W^\lambda \) is the \( \mathcal{I} \)-submodule of \( W^\lambda \). Put \( L^\lambda = W^\lambda / \text{rad } W^\lambda \). Then it is known by [DJM] that \( L^\lambda \neq 0 \) for any \( \lambda \in \mathbb{A}^+ \). Assume that \( R \) is a field. Then \( L^\lambda \) is an absolutely irreducible module, and the set \( \{ L^\lambda \mid \lambda \in \mathbb{A}^+ \} \) gives a complete set of non-isomorphic irreducible \( \mathcal{I} \)-modules.

1.4. We recall some definitions and results in [SW]. We fix a positive integer \( g \leq r \) and \( p = (r_1, \ldots, r_g) \in \mathbb{Z}_{>0}^g \) such that \( r_1 + \cdots + r_g = r \), and set \( p_i = 0 \), \( p_i = \sum_{j=1}^{i-1} r_j \) for \( i = 2, \ldots, g \). For \( \mu = (\mu(1), \ldots, \mu(r)) \in \Lambda \), we define \( \alpha_p(\mu) = (n_1, \ldots, n_g) \) and \( \alpha_p(\mu) = (a_1, \ldots, a_g) \), where \( n_k = \sum_{i=1}^k |\mu(p_i + 1)| \) and \( a_k = \sum_{i=1}^{k-1} n_i \) for \( k = 1, \ldots, g \) with \( a_1 = 0 \). We define a partial order on \( \mathbb{Z}_{>0}^g \) by \( a = (a_1, \ldots, a_g) \geq b = (b_1, \ldots, b_g) \)
if $a_i \geq b_i$ for any $i = 1, \ldots, g$ and we write $a > b$ if $a \geq b$ and $a \neq b$. Later we consider the partial order on $\{a_\mu(\mu) \mid \mu \in A\}$ by this order.

For $\lambda \in A^+$ and $\mu \in A$, we set $T_0^p(\lambda, \mu) = T_0(\lambda, \mu)$ if $\alpha(\lambda) = \alpha(p, \mu)$, and is empty otherwise. Moreover we set $T_0^p(\lambda) = \bigcup_{\mu \in A} T_0^p(\lambda, \mu)$. We set

$$\Sigma^p = (A^+ \times \{0, 1\}) \setminus \left\{(\lambda, 1) \in A^+ \times \{0, 1\} \mid T_0(\lambda, \mu) = \phi\right\},$$

for any $\mu \in A$ such that $a_\mu(\lambda) > a_\mu(\mu)$,

and define a partial order $\geq$ on $\Sigma^p$ by $(\lambda_1, \varepsilon_1) > (\lambda_2, \varepsilon_2)$ if $\lambda_1 > \lambda_2$ or if $\lambda_1 = \lambda_2$ and $\varepsilon_1 > \varepsilon_2$. For $\eta = (\lambda, \varepsilon) \in \Sigma^p$, we set

$$I(\eta) = \begin{cases} T_0^p(\lambda) & \text{if } \varepsilon = 0, \\ \bigcup_{\mu \in A, a_\mu(\lambda) > a_\mu(\mu)} T_0(\lambda, \mu) & \text{if } \varepsilon = 1, \end{cases}$$

$$J(\eta) = \begin{cases} T_0^p(\lambda) & \text{if } \varepsilon = 0, \\ T_0(\lambda) & \text{if } \varepsilon = 1, \end{cases}$$

$$C^p(\eta) = \{\varphi_{ST} \mid (S, T) \in I(\eta) \times J(\eta)\}$$

for $\eta \in \Sigma^p$, and

$$C^p = \bigcup_{\eta \in \Sigma^p} C^p(\eta).$$

Let $\mathcal{F}^p = \mathcal{F}^p(A)$ be the $R$-submodule of $\mathcal{F}(A)$ spanned by $C^p$. We also define $(\mathcal{F}^p)^{\vee \eta}$ as the $R$-submodule of $\mathcal{F}^p$ spanned by

$$\{\varphi_{UV} \mid (U, V) \in I(\eta') \times J(\eta') \text{ for some } \eta' \in \Sigma^p \text{ such that } \eta' > \eta\}.$$ 

It is known by [SW, Theorem 2.6] that $\mathcal{F}^p$ is a standardly based algebra with the standard basis $C^p$ in the sense of [DR].

By the general theory of standardly based algebra due to [DR], we have the following results. For $\eta \in \Sigma^p$, one can consider the standard left $\mathcal{F}^p$-modules $\mathring{\mathcal{F}}^{n\eta}$ with the basis $\{\varphi^n_T \mid T \in I(\eta)\}$ and the standard right $\mathcal{F}^p$-module $Z^n$ with the basis $\{\varphi^n_T \mid T \in J(\eta)\}$. We call them Weyl modules of $\mathcal{F}^p$. We define the bilinear form $\beta_\eta : \mathring{\mathcal{F}}^{n\eta} \times Z^n \to R$ by

$$\beta_\eta(\varphi^n_S, \varphi^n_T) \varphi_{UV} = \varphi_{UT} \varphi_{SV} \mod (\mathcal{F}^p)^{\vee \eta} \quad (S \in I(\eta), T \in J(\eta)),$$

where $\beta_\eta$ is determined independent of the choice of $U \in I(\eta)$ and $V \in J(\eta)$. The bilinear form $\beta_\eta$ is associative, namely we have

\begin{equation}
\beta_\eta(\varphi x, y) = \beta_\eta(x, y \varphi) \quad \text{for } x \in \mathring{\mathcal{F}}^{n\eta}, y \in Z^n, \varphi \in \mathcal{F}^p. \tag{1.4.1}
\end{equation}
Let \( \text{rad } Z^n = \{ x \in Z^n \mid \beta_n(y, x) = 0 \text{ for any } y \in Z^n \} \). Then \( \text{rad } Z^n \) is a \( \mathcal{P} \)-submodule of \( Z^n \) by associativity of \( \beta_n \). Put \( L^n = Z^n/\text{rad } Z^n \). Assume that \( R \) is a field. Then \( L^n \) is an absolutely irreducible module or zero, and the set \( \{ L^n \mid \eta \in \Sigma^p \text{ such that } \beta_n \neq 0 \} \) is a complete set of non-isomorphic irreducible (right) \( \mathcal{P} \)-modules.

Later we shall only consider the Weyl modules \( Z^n \) and irreducible modules \( L^n \) of \( \mathcal{P} \) for \( \eta \) of the form \( (\lambda, 0) \). Note that the composition factors of \( Z^{(\lambda, 0)} \) are isomorphic to \( L^{(\mu, 0)} \) for some \( \mu \in \Lambda^+ \) by [SW, Proposition 3.3 (i)].

1.5. Let \( \widehat{\mathcal{P}} \) be the \( R \)-submodule of \( \mathcal{P} \) spanned by

\[
\mathcal{C} \setminus \{ \varphi_{ST} \mid S, T \in T_0^p(\lambda) \text{ for some } \lambda \in \Lambda^+ \}.
\]

It is known by [SW] that \( \widehat{\mathcal{P}} \) is a two-sided ideal of \( \mathcal{P} \). Thus, we can define the quotient algebra

\[
\mathcal{P} = \mathcal{P} / \widehat{\mathcal{P}}.
\]

We denote by \( \overline{\varphi} \) the image of \( \varphi \in \mathcal{P} \) under the natural surjection \( \pi : \mathcal{P} \to \mathcal{P} \), and set

\[
\overline{\mathcal{C}} = \{ \overline{\varphi}_{ST} \mid S, T \in T_0^p(\lambda) \text{ for some } \lambda \in \Lambda^+ \}.
\]

Then \( \overline{\mathcal{C}} \) is a free \( R \)-basis of \( \overline{\mathcal{P}} \). By [SW] Theorem 2.13, \( \overline{\mathcal{P}} \) turns out to be a cellular algebra with the cellular basis \( \overline{\mathcal{C}} \). Hence by the general theory of cellular algebra, the following results hold. For \( \lambda \in \Lambda^+ \), we can consider the standard (right) \( \mathcal{P} \)-module \( \overline{Z}^\lambda \) with the free \( R \)-basis \( \{ \overline{\varphi}_{ST} \mid T \in T_0^p(\lambda) \} \). We call it a Weyl module of \( \mathcal{P} \). We define the bilinear form \( \langle \cdot, \cdot \rangle_p : \overline{Z}^\lambda \times \overline{Z}^\lambda \to R \) by

\[
\langle \overline{\varphi}_{S}, \overline{\varphi}_{T} \rangle_p \overline{\varphi}_{UV} \equiv \overline{\varphi}_{US} \overline{\varphi}_{TV} \mod (\overline{\mathcal{P}})^{\overline{\lambda}} \quad (S, T \in T_0^p(\lambda)),
\]

where \( \langle \cdot, \cdot \rangle_p \) is determined independent of the choice \( U, V \in T_0^p(\lambda) \), and \( (\overline{\mathcal{P}})^{\overline{\lambda}} \) is the \( R \)-submodule of \( \overline{\mathcal{P}} \) spanned by

\[
\{ \overline{\varphi}_{ST} \mid S, T \in T_0^p(\lambda) \text{ for some } \lambda' \in \Lambda^+ \text{ such that } \lambda' \triangleright \lambda \}.
\]

The bilinear form \( \langle \cdot, \cdot \rangle_p \) is associative, namely we have

\[
\langle \overline{x}, \overline{y} \rangle_p = \langle \overline{x}, \overline{y} \overline{\varphi}^* \rangle_p \quad \text{for any } \overline{x}, \overline{y} \in \overline{Z}^\lambda, \overline{\varphi} \in \overline{\mathcal{P}}.
\]

Let \( \text{rad } \overline{Z}^\lambda = \{ \overline{x} \in \overline{Z}^\lambda \mid \langle \overline{x}, \overline{y} \rangle_p = 0 \text{ for any } \overline{y} \in \overline{Z}^\lambda \} \), then \( \text{rad } \overline{Z}^\lambda \) is an \( \overline{\mathcal{P}} \)-submodule of \( \overline{Z}^\lambda \). Put \( \overline{T}^\lambda = \overline{Z}^\lambda / \text{rad } \overline{Z}^\lambda \). Assume that \( R \) is a field. Then \( \overline{T}^\lambda \) is an absolutely irreducible module, and the set \( \{ \overline{T}^\lambda \mid \lambda \in \Lambda^+ \} \) is a complete set of non-isomorphic irreducible (right) \( \overline{\mathcal{P}} \)-modules.
1.6. Assuming that $R$ is a field, we set, for $\lambda, \mu \in \Lambda^+$,
\[
\begin{align*}
d_{\lambda \mu} &= [W^\lambda : L^\mu], \\
d_{\lambda \mu}^{(\lambda,0)} &= [Z^{(\lambda,0)} : L^{(\mu,0)}], \\
\overline{d}_{\lambda \mu} &= [\overline{Z}^{\lambda} : \overline{T}^\mu],
\end{align*}
\]
where $[W^\lambda : L^\mu]$ is the decomposition number of $L^\mu$ in $W^\lambda$, and similarly for $\mathcal{F}^p$ and $\overline{\mathcal{F}}^p$. The following theorem was proved in [SW].

Theorem 1.7. [SW, Theorem 3.13] Assume that $R$ is a field. For $\lambda, \mu \in \Lambda^+$ such that $\alpha_p(\lambda) = \alpha_p(\mu)$, we have
\[
\overline{d}_{\lambda \mu} = d_{\lambda \mu}^{(\lambda,0)} = d_{\lambda \mu}
\]

1.8. For $\mu = (\mu^{(1)}, \cdots, \mu^{(r)}) \in \Lambda$, we write it in the form $\mu = (\mu^{[1]}, \cdots, \mu^{[g]})$, where $\mu^{[i]} = (\mu^{(p_i+1)}, \cdots, \mu^{(p_i+r_i)})$. According to the expression of $\mu$ as above, $T = (T^{(1)}, \cdots, T^{(r)}) \in T_0(\lambda)$ can be expressed as $T = (T^{[1]}, \cdots, T^{[g]})$ with $T^{[i]} = (T^{(p_i+1)}, \cdots, T^{(p_i+r_i)})$. By [SW, Lemma 4.3 (iii)], we have a bijection $T_0^p(\lambda, \mu) \simeq \Delta_0(\lambda^{[1]}, \mu^{[1]}) \times \cdots \times \Delta_0(\lambda^{[g]}, \mu^{[g]})$ given by the map $T \mapsto (T^{[1]}, \cdots, T^{[g]})$. Thus we have a bijection $T_0^p(\lambda) \simeq T_0(\lambda^{[1]}) \times \cdots \times T_0(\lambda^{[g]})$.

We write $m = (m_1, \cdots, m_r)$ in the form $m = (m^{[1]}, \cdots, m^{[g]})$, where $m^{[k]} = (m_{p_k+1}, \cdots, m_{p_k+r_k})$. For each $n_k \in \mathbb{Z}_{\geq 0}$, put $A_{n_k} = \mathcal{P}_{n_k,r_k}(m^{[k]})$, and $A_{n_k}^+ = \mathcal{P}_{n_k,r_k}(m^{[k]})$. $(A_{n_k}$ or $A_{n_k}^+$ is regarded as the empty set if $n_k = 0$.) Let $\mathcal{F}(A_{n_k})$ be the cyclotomic $q$-Schur algebra associated to the Ariki-Koike algebra $\mathcal{H}_{n_k,r_k}$ with parameters $q, Q_{p_k+1}, \cdots, Q_{p_k+r_k}$. Let $\Delta_{n,g}$ be the set of $(n_1, \cdots, n_g) \in \mathbb{Z}_{\geq 0}$ such that $n_1 + \cdots + n_g = n$. Then we have the following decomposition theorem of $\overline{\mathcal{F}}^p$ by [SW, Theorem 4.15].

\[
(1.8.1) \quad \overline{\mathcal{F}}^p(\Lambda) \cong \bigoplus_{(n_1, \cdots, n_g) \in \Delta_{n,g}} \mathcal{F}(A_{n_1}) \otimes \cdots \otimes \mathcal{F}(A_{n_g}) \quad \text{as } R\text{-algebra},
\]

under the isomorphism given by
\[
(1.8.2) \quad \overline{\varphi}_{ST} \mapsto \varphi_{S^{[1]}T^{[1]}} \otimes \cdots \otimes \varphi_{S^{[g]}T^{[g]}} \quad \text{for } S, T \in T_0^p(\lambda).
\]

Assuming that $R$ is a field, for $\lambda^{[k]} \in A_{n_k}$, let $W^\lambda^{[k]}$ be the Weyl module of $\mathcal{F}(A_{n_k})$, and $L^\lambda^{[k]} = W^\lambda^{[k]} / \text{rad } W^\lambda^{[k]}$ be the irreducible module. By [SW, Corollary 4.16], the following properties hold. Under the isomorphism in (1.8.1), we have, for
\[ \lambda, \mu \in \Lambda^+, \]

\[ (1.8.3) \quad Z^\lambda \cong W^{\lambda[1]} \otimes \cdots \otimes W^{\lambda[g]}, \]

\[ (1.8.4) \quad L^\mu \cong L^{\mu[1]} \otimes \cdots \otimes L^{\mu[g]}, \]

\[ (1.8.5) \quad [Z^\lambda : L^\mu] = \begin{cases} \prod_{k=1}^g [W^{\lambda[k]} : L^{\mu[k]}] & \text{if } \alpha_p(\lambda) = \alpha_p(\mu), \\ 0 & \text{otherwise} \end{cases}. \]

Under the isomorphism in (1.8.3), a bilinear form \( \langle \cdot, \cdot \rangle_p \) on \( Z^\lambda \) decomposes to a product of bilinear forms on \( W^{\lambda[k]} \) for \( k = 1, \ldots, g \), namely we have the following lemma.

**Lemma 1.9.** For \( S, T \in T_0^p(\lambda) \), we have

\[ \langle \varphi_S, \varphi_T \rangle_p = \langle \varphi_{S[1]}, \varphi_{T[1]} \rangle \cdots \langle \varphi_{S[g]}, \varphi_{T[g]} \rangle, \]

where \( \langle \varphi_{S[k]}, \varphi_{T[k]} \rangle \) denotes the bilinear form on \( W^{\lambda[k]} \) for \( k = 1, \ldots, g \).

**Proof.** Fix \( U, V \in T_0^p(\lambda) \). Then by (1.8.2) and the definition of the bilinear form on \( W^{\lambda[k]} \), we have

\[ \varphi_{US \varphi_{TV}} = (\varphi_{U[1]S[1]} \otimes \cdots \otimes \varphi_{U[g]S[g]})(\varphi_{T[1]V[1]} \otimes \cdots \otimes \varphi_{T[g]V[g]}) \]

\[ = \varphi_{U[1]S[1]} \varphi_{T[1]V[1]} \otimes \cdots \otimes \varphi_{U[g]S[g]} \varphi_{T[g]V[g]} \]

\[ \equiv \langle \varphi_{S[1]}, \varphi_{T[1]} \rangle \varphi_{U[1]V[1]} \otimes \cdots \otimes \langle \varphi_{S[g]}, \varphi_{T[g]} \rangle \varphi_{U[g]V[g]} \]

\[ \mod \mathcal{J}(A_{n_1})^{\lambda[1]} \otimes \cdots \otimes \mathcal{J}(A_{n_k})^{\lambda[g]} \]

\[ = \langle \varphi_{S[1]}, \varphi_{T[1]} \rangle \cdots \langle \varphi_{S[g]}, \varphi_{T[g]} \rangle \varphi_{U[1]V[1]} \otimes \cdots \otimes \varphi_{U[g]V[g]} \]

\[ = \langle \varphi_{S[1]}, \varphi_{T[1]} \rangle \cdots \langle \varphi_{S[g]}, \varphi_{T[g]} \rangle \varphi_{UV}. \]

Since \( \mathcal{J}(A_{n_1})^{\lambda[1]} \otimes \cdots \otimes \mathcal{J}(A_{n_k})^{\lambda[g]} \subset (\mathcal{J}^p)^{\lambda} \), we see that

\[ \langle \varphi_S, \varphi_T \rangle_p \varphi_{UV} \equiv \varphi_{US \varphi_{TV}} \equiv \langle \varphi_{S[1]}, \varphi_{T[1]} \rangle \cdots \langle \varphi_{S[g]}, \varphi_{T[g]} \rangle \varphi_{UV} \mod (\mathcal{J}^p)^{\lambda}. \]

The lemma is proved. \( \square \)

**Remark 1.10.** For the isomorphism in (1.8.3), we do not need to assume that \( R \) is a field. But for (1.8.4) and (1.8.5), we need that \( R \) is a field.

**Theorem 1.11.** [SW, Theorem 4.17] Assume that \( R \) is a field. For \( \lambda, \mu \in \Lambda^+ \) such that \( \alpha_p(\lambda) = \alpha_p(\mu) \), we have the following.

\[ d_{\lambda \mu} = \overline{d}_{\lambda \mu} = \prod_{k=1}^g d_{\lambda[k] \mu[k]}, \]

where \( d_{\lambda[k] \mu[k]} = [W^{\lambda[k]} : L^{\mu[k]}] \).
2. Decomposition numbers with Jantzen filtration

2.1. In the rest of this paper, we assume that $R$ is a discrete valuation ring. Let $\wp$ be a unique maximal ideal of $R$ and $F = R/\wp$ be the residue filed. Fix $\widehat{q}, \widehat{Q}_1, \cdots, \widehat{Q}_r$ in $R$ and let $q = \widehat{q} + \wp, Q_1 = \widehat{Q}_1 + \wp, \cdots, Q_r = \widehat{Q}_r + \wp$ be their canonical images in $F$. Moreover let $K$ be the quotient field of $R$. Then $(K, R, F)$ is a modular system with parameters. Let $\mathcal{I}_R = \mathcal{I}_R(\Lambda)$ be the cyclotomic $\widehat{q}$-Schur algebra over $R$ with parameters $\widehat{q}, \widehat{Q}_1, \cdots, \widehat{Q}_r$ and $\mathcal{I} = \mathcal{I}(\Lambda)$ be the cyclotomic $q$-Schur algebra over $F$ with parameters $q, Q_1, \cdots, Q_r$. Then $\mathcal{I} = (\mathcal{I}_R + q\mathcal{I}_F)/q\mathcal{I}_R$.

We consider the subalgebra $\mathcal{I}_R^p$ (resp. $\mathcal{I}_p$) of $\mathcal{I}_R$ (resp. $\mathcal{I}$) and its quotient $\mathcal{I}_R^p$ (resp. $\mathcal{I}^p$) as in the previous section with the notation there. Note that the subscript $R$ is used to indicate the objects related to $R$.

For $\lambda \in \Lambda^+$, let $W_R^\lambda$ be the Weyl module of $\mathcal{I}_R$. For $i \in \mathbb{Z}_{\geq 0}$, we set

$$W_R^\lambda(i) = \{x \in W_R^\lambda | \langle x, y \rangle \in \wp^i \text{ for any } y \in W_R^\lambda\}$$

and define

$$W^\lambda(i) = (W_R^\lambda(i) + \wp W_R^\lambda)/\wp W_R^\lambda.$$ 

Then $W^\lambda = W^\lambda(0)$ is the Weyl module of $\mathcal{I}$, and we have the Jantzen filtration of $W^\lambda$,

$$W^\lambda = W^\lambda(0) \supset W^\lambda(1) \supset W^\lambda(2) \supset \cdots.$$ 

Similarly, by using the bilinear form $\langle , \rangle_p$ on $\mathcal{I}_R^p$, one can define the Jantzen filtration of $\mathcal{I}_R^p$-module $\mathcal{Z}^\lambda$

$$\mathcal{Z}^\lambda = \mathcal{Z}^\lambda(0) \supset \mathcal{Z}^\lambda(1) \supset \mathcal{Z}^\lambda(2) \supset \cdots.$$ 

Moreover for the Weyl module $Z_R^{(\lambda, 0)}$ of $\mathcal{I}_R^p$, we set

$$Z_R^{(\lambda, 0)}(i) = \{x \in Z_R^{(\lambda, 0)} | \beta_\lambda(y, x) \in \wp^i \text{ for any } y \in \mathcal{Z}_R^{(\lambda, 0)}\}$$

and define

$$Z^{(\lambda, 0)}(i) = (Z_R^{(\lambda, 0)}(i) + \wp Z_R^{(\lambda, 0)})/\wp Z_R^{(\lambda, 0)}.$$ 

Then we have the Jantzen filtration of $Z^{(\lambda, 0)}$

$$Z^{(\lambda, 0)} = Z^{(\lambda, 0)}(0) \supset Z^{(\lambda, 0)}(1) \supset Z^{(\lambda, 0)}(2) \supset \cdots.$$ 

Since $W^\lambda$ is a finite dimensional $F$-vector space, one can find a positive integer $k$ such that $W^\lambda(k') = W^\lambda(k)$ for any $k' > k$. We choose a minimal $k$ in such numbers and set $W^\lambda(k+1) = 0$. Then the Jantzen filtration of $W^\lambda$ becomes a finite sequence. Similarly, Jantzen filtrations of $Z^{(\lambda, 0)}$ and $\mathcal{Z}^\lambda$ also become finite sequences.

We can easily see that $W^\lambda(i)$ (resp. $Z^{(\lambda, 0)}(i)$) is a $\mathcal{I}$-submodule of $W^\lambda$ (resp. $\mathcal{I}_R^p$-submodule of $Z^{(\lambda, 0)}$, $\mathcal{I}_p$-submodule of $\mathcal{Z}^\lambda$) by associativity of the bilinear form $(1.3.1)$ (resp. $(1.4.1)$, $(1.5.1)$).

2.2. Take $\lambda, \mu \in \Lambda^+$, and $W^\lambda = W^\lambda(0) \supset W^\lambda(1) \supset \cdots$ be the Jantzen filtration of $W^\lambda$. Let $[W^\lambda(i)/W^\lambda(i+1) : L^\mu]$ be the composition multiplicity of $L^\mu$ in
\[ W^\lambda(i)/W^\lambda(i + 1). \] Let \( v \) be an indeterminate. We define a polynomial \( d_{\lambda\mu}(v) \) by
\[
d_{\lambda\mu}(v) = \sum_{i \geq 0} \left[ W^\lambda(i)/W^\lambda(i + 1) : L^\mu \right] \cdot v^i.
\]
Similarly we define, for \( Z^{(\lambda,0)} \) and \( \overline{Z}\lambda \)
\[
d_{\lambda\mu}^{(\lambda,0)}(v) = \sum_{i \geq 0} \left[ Z^{(\lambda,0)}(i)/Z^{(\lambda,0)}(i + 1) : L^{(\mu,0)} \right] \cdot v^i,
\]
\[
\overline{d}_{\lambda\mu}(v) = \sum_{i \geq 0} \left[ \overline{Z}\lambda(i)/\overline{Z}\lambda(i + 1) : \overline{L}\mu \right] \cdot v^i.
\]
Thus \( d_{\lambda\mu}(v) \), \( d_{\lambda\mu}^{(\lambda,0)}(v) \) and \( \overline{d}_{\lambda\mu}(v) \) are polynomials whose coefficients are non-negative integers. Note that since the Jantzen filtration of \( W^\lambda \), etc. are finite sequences, these summations are finite sums. We call \( d_{\lambda\mu}(v) \) (resp. \( d_{\lambda\mu}^{(\lambda,0)}(v), \overline{d}_{\lambda\mu}(v) \)) decomposition number with Jantzen filtration of \( \mathcal{S} \) (resp. \( \mathcal{S}^p, \overline{\mathcal{S}}^p \)). We also call them \( v \)-decomposition numbers as they coincide at \( v = 1 \) with decomposition numbers given in \([\text{L6}]\).

We have the following relation between \( d_{\lambda\mu}^{(\lambda,0)}(v) \) and \( \overline{d}_{\lambda\mu}(v) \).

**Proposition 2.3.** For \( \lambda, \mu \in \Lambda \), we have

(i) If \( \alpha_p(\lambda) \neq \alpha_p(\mu) \), then \( \overline{d}_{\lambda\mu}(v) = d_{\lambda\mu}^{(\lambda,0)}(v) = 0 \).

(ii) \( \left[ \overline{Z}\lambda(i)/\overline{Z}\lambda(i + 1) : \overline{L}\mu \right] = \left[ Z^{(\lambda,0)}(i)/Z^{(\lambda,0)}(i + 1) : L^{(\mu,0)} \right] \) for any \( i \geq 0 \).

Hence we have \( \overline{d}_{\lambda\mu}(v) = d_{\lambda\mu}^{(\lambda,0)}(v) \).

**Proof.** (i) is clear since \( d_{\lambda\mu}^{(\lambda,0)} = \overline{d}_{\lambda\mu} = 0 \) by \([\text{SW}]\) Proposition 3.3).

Recall that \( \overline{Z}\lambda \cong Z^{(\lambda,0)} \) and \( \overline{L}\mu \cong L^{(\mu,0)} \) as \( \mathcal{S}^p \)-modules by \([\text{SW}]\) Lemma 3.2.

By definition, we have \( \beta_{\lambda}(\varphi_S^{(\lambda,0)}, \varphi_T^{(\lambda,0)}) = \langle \varphi_S, \varphi_T \rangle_p \) for any \( S, T \in T_0^p(\lambda) \). Then under the isomorphism \( \overline{Z}\lambda \cong Z^{(\lambda,0)} \), the Jantzen filtration of \( \overline{Z}\lambda \) coincides with that of \( Z^{(\lambda,0)} \). So (ii) is proved. \( \square \)

2.4. Next, we consider the relation between \( d_{\lambda\mu}^{(\lambda,0)}(v) \) and \( d_{\lambda\mu}(v) \). In order to see this we prepare two lemmas. Recall that there exists an injective \( \mathcal{S}^p \)-homomorphism \( f_\lambda : Z^{(\lambda,0)} \hookrightarrow W^\lambda \) such that \( f_\lambda(\varphi_T^{(\lambda,0)}) = \varphi_T \) for \( T \in T_0^p(\lambda) \) by \([\text{SW}]\) Lemma 3.5] and that \( Z^{(\lambda,0)} \otimes_{\mathcal{S}^p} \mathcal{S} \cong W^\lambda \) as \( \mathcal{S} \)-module by \([\text{SW}]\) Proposition 3.6. Let \( \iota_i : Z^{(\lambda,0)}(i) \hookrightarrow Z^{(\lambda,0)} \) be an inclusion map. Then \( (\iota_i \otimes \text{id}_{\mathcal{S}}) (Z^{(\lambda,0)}(i) \otimes_{\mathcal{S}^p} \mathcal{S}) \) is the \( \mathcal{S} \)-submodule of \( Z^{(\lambda,0)} \otimes_{\mathcal{S}^p} \mathcal{S} \). Similar results hold also for \( R \). We have the following.

**Lemma 2.5.** Let \( \lambda \in \Lambda^+ \). For any \( i \geq 0 \), we have
\[
f_\lambda^{-1}(W^\lambda(i)) = Z^{(\lambda,0)}(i)
\]

**Proof.** By definition, we see that \( \beta(\varphi_T^{(\lambda,0)}, \varphi_S^{(\lambda,0)}) = \langle \varphi_S, \varphi_T \rangle \) for any \( S, T \in T_0^p(\lambda) \), and that \( \langle \varphi_S, \varphi_T \rangle = 0 \) if \( S \in T_0^p(\lambda), T \in T_0(\lambda) \setminus T_0^p(\lambda) \). Then for \( x \in Z^{(\lambda,0)}_R \), we
have
\[ x \in Z^{(\lambda,0)}_R(i) \iff \beta_\lambda(\varphi^{(\lambda,0)}_T, x) \in \wp^i \text{ for any } T \in T_0^p(\lambda) \]
\[ \iff \langle f_\lambda(x), \varphi_T \rangle \in \wp^i \text{ for any } T \in T_0(\lambda) \]
\[ \iff f_\lambda(x) \in W_R^\lambda(i) \]

By taking the quotient, we obtain the lemma.

\[ \square \]

**Lemma 2.6.** Let \( \lambda \in \Lambda^+ \). For any \( i \geq 0 \), we have
\[ (t_i \otimes \text{id}_\mathcal{P})(Z^{(\lambda,0)}(i) \otimes \mathcal{P}) \subset W^\lambda(i) \]

under the isomorphism \( Z^{(\lambda,0)}(0) \otimes \mathcal{P} \mathcal{P} \cong W^\lambda \).

**Proof.** Recall that any element of \( Z^{(\lambda,0)}_R \) can be written in the form \( \varphi^{(\lambda,0)}_T \cdot \psi \) with \( \psi \in \mathcal{P} \mathcal{P} \). Moreover it follows from [SW Proposition 3.6] that, under the isomorphism \( g_\lambda : Z^{(\lambda,0)}_R \otimes \mathcal{P} \mathcal{P} \cong W_R^\lambda \), we have \( g_\lambda(\varphi^{(\lambda,0)}_T \cdot \psi) = \varphi_T \cdot \psi \cdot \varphi \) for \( \psi \in \mathcal{P} \mathcal{P}, \varphi \in \mathcal{P} \).

This is true also for \( Z^{(\lambda,0)}_R, W^\lambda \). Thus in order to show the lemma, it is enough to prove the following.

(2.6.1) Suppose that \( \varphi^{(\lambda,0)}_T \psi \in Z^{(\lambda,0)}_R(i) \) for \( \psi \in \mathcal{P} \mathcal{P} \). Then we have \( \varphi^{(\lambda,0)}_T \psi \in W^\lambda(i) \) for any \( \varphi \in \mathcal{P} \).

Now take \( \varphi^{(\lambda,0)}_T \psi \in Z^{(\lambda,0)}_R(i) \). If \( \varphi^{(\lambda,0)}_T \psi \in Z^{(\lambda,0)}_R(i) \), then \( \beta_\lambda(x, \varphi^{(\lambda,0)}_T \psi) \in \wp^i \) for any \( x \in \wp Z^{(\lambda,0)}_R(i) \). This implies that \( \langle \varphi^{(\lambda,0)}_T \psi, y \rangle \in \wp^i \) for any \( y \in W_R^\lambda \) by a similar argument as the proof of Lemma 2.5.

Since \( \langle \varphi^{(\lambda,0)}_T \psi, y \rangle = \langle \varphi^{(\lambda,0)}_T \psi, y \varphi^* \rangle \) for any \( y \in W_R^\lambda \) and any \( \varphi \in \mathcal{P} \mathcal{P} \), we see that \( \varphi^{(\lambda,0)}_T \psi \in Z^{(\lambda,0)}_R(i) \) implies that \( \varphi^{(\lambda,0)}_T \psi \in W_R^\lambda(i) \). By taking the quotient, we obtain (2.6.1). Thus the lemma is proved.

These two lemmas imply the following proposition about the relation between \( d^{(\lambda,0)}_\mu(v) \) and \( d^{(\lambda,0)}_\mu(v) \).

**Proposition 2.7.** Let \( \lambda, \mu \in \Lambda^+ \) be such that \( \alpha_\mu(\lambda) = \alpha_\mu(\mu) \). Then for any \( i \geq 0 \), we have
\[ [Z^{(\lambda,0)}(i) / Z^{(\lambda,0)}(i + 1) : L^{(\mu,0)}] = [W^\lambda(i) / W^\lambda(i + 1) : L^\mu] . \]
Hence we have \( d^{(\lambda,0)}_\mu(v) = d^{(\lambda,0)}_\mu(v) \) if \( \alpha_\mu(\lambda) = \alpha_\mu(\mu) \).

**Proof.** Fix \( \lambda, \mu \in \Lambda^+ \) such that \( \alpha_\mu(\lambda) = \alpha_\mu(\mu) \), and an integer \( i \geq 0 \). Thanks to Lemma 2.5, we have the following result by similar arguments as in the proof of [SW Proposition 3.12].

(2.7.1) \[ [Z^{(\lambda,0)}(i) / Z^{(\lambda,0)}(i + 1) : L^{(\mu,0)}] \geq [W^\lambda(i) / W^\lambda(i + 1) : L^\mu] . \]

Conversely, thanks to Lemma 2.6, we have the following result by similar arguments as in the proof of [SW Proposition 3.11].

(2.7.2) \[ [Z^{(\lambda,0)}(i) : L^{(\mu,0)}] \leq [W^\lambda(i) : L^\mu] . \]
We remark that this does not imply nilpotency

\[ Z^{(\lambda,0)}(i)/Z^{(\lambda,0)}(i + 1) : L^{(\mu,0)} \leq [W^{\lambda}(i)/W^{\lambda}(i + 1) : L^{\mu}] \]

since we cannot see whether \((\iota_{i+1} \otimes \text{id}_{\mathcal{S}})(Z^{(\lambda,0)}(i + 1) \otimes_{\mathcal{S}} \mathcal{S}) = W^{\lambda}(i + 1)\) or not. Instead, we argue as follows. Let

\[ \lambda \]

\[ \iota \]

\[ \rho \]

\[ \mathcal{S} \]

In particular,

\[ Z^{(\lambda,0)} = Z^{(\lambda,0)}(0) \supset \cdots \supset Z^{(\lambda,0)}(l) \supset \cdots \supset Z^{(\lambda,0)}(l + 1) = 0 \]

be the Jantzen filtrations of \(W^{\lambda}\) and \(Z^{(\lambda,0)}\) respectively. Then we have

\[ (\iota_{k+1} \otimes \text{id}_{\mathcal{S}})(Z^{(\lambda,0)}(k + 1) \otimes_{\mathcal{S}} \mathcal{S}) \subset W^{\lambda}(k + 1) = 0 \]

by Lemma 2.6. This implies that \(Z^{(\lambda,0)}(k + 1) = 0\) since \((\iota \otimes \text{id}_{\mathcal{S}})(M \otimes_{\mathcal{S}} \mathcal{S}) \neq 0\) for any non-zero submodule \(M\) of \(Z^{(\lambda,0)}\) and the inclusion map \(\iota : M \hookrightarrow Z^{(\lambda,0)}\) by [SW, Lemma 3.8 (ii)]. So we have \(l \leq k\).

Now, if \(L^{\mu}\) is a composition factor of \(W^{\lambda}(i)/W^{\lambda}(i + 1)\), then we have

\[ 1 \leq [W^{\lambda}(i)/W^{\lambda}(i + 1) : L^{\mu}] \leq [Z^{(\lambda,0)}(i)/Z^{(\lambda,0)}(i + 1) : L^{(\mu,0)}] \]

by (2.7.1). Hence, we have \(Z^{(\lambda,0)}(i) \neq 0\). This implies that \(i \leq l\) if \(L^{\mu}\) is a composition factor of \(W^{\lambda}(i)\). In particular, \([W^{\lambda}(l + 1) : L^{\mu}] = 0\). Thus we have

\[ Z^{(\lambda,0)}(l)/Z^{(\lambda,0)}(l + 1) : L^{(\mu,0)} = [Z^{(\lambda,0)}(l) : L^{(\mu,0)}] \]

\[ W^{\lambda}(l)/W^{\lambda}(l + 1) : L^{\mu} = [W^{\lambda}(l) : L^{\mu}] \]

Combining these equalities with (2.7.1) and (2.7.2), we have

\[ Z^{(\lambda,0)}(l)/Z^{(\lambda,0)}(l + 1) : L^{(\mu,0)} = [W^{\lambda}(l)/W^{\lambda}(l + 1) : L^{\mu}] \]

and so

\[ Z^{(\lambda,0)}(l) : L^{(\mu,0)} = [W^{\lambda}(l) : L^{\mu}] \]

(2.7.3)

Next we consider the case where \(i = l - 1\). Note that

\[ Z^{(\lambda,0)}(l - 1)/Z^{(\lambda,0)}(l) : L^{(\mu,0)} = [Z^{(\lambda,0)}(l - 1) : L^{(\mu,0)}] - [Z^{(\lambda,0)}(l) : L^{(\mu,0)}] \]

\[ W^{\lambda}(l - 1)/W^{\lambda}(l) : L^{\mu} = [W^{\lambda}(l - 1) : L^{\mu}] - [W^{\lambda}(l) : L^{\mu}] \]

Combined with (2.7.1), (2.7.2) and (2.7.3), we have

\[ Z^{(\lambda,0)}(l - 1)/Z^{(\lambda,0)}(l) : L^{(\mu,0)} = [W^{\lambda}(l - 1)/W^{\lambda}(l) : L^{\mu}] \]
and so $[Z^{(\lambda,0)}(l-1) : L^{(\mu,0)}] = [W^{(\lambda,0)}(l-1) : L^{\mu}]$. Therefore by backward induction on $l$, we obtain the proposition. ∎

Combining Proposition 2.8 and Proposition 2.7, we have the following theorem.

**Theorem 2.8.** For any $\lambda, \mu \in \Lambda^+$ such that $\alpha_{\varphi}(\lambda) = \alpha_{\varphi}(\mu)$, we have

$$d_{\lambda\mu}(v) = d_{\lambda\mu}^{(\lambda,0)}(v) = d_{\lambda\mu}(v).$$

If we specialize $v = 1$, the theorem reduces to Theorem 1.7

2.9. For later use, we shall consider the basis of $W_{\mathcal{P}_G}$ (2.9.1)

$$P = \text{diag}(d_{S_1}, d_{S_2}, \ldots, d_{S_N})$$

where $d \in \mathcal{P}_G$. Since $R$ is a PID, there exist $P, Q \in \text{GL}_N(R)$ (where $N = |\mathcal{P}_G|$) such that $P\mathcal{G}^\lambda Q = \text{diag}(d_{S_1}, d_{S_2}, \ldots, d_{S_N})$, where $d_{S_k} \in R$ and $\{S_1, \ldots, S_N\} = \mathcal{P}_G$. Let $P = (p_{ST})_{S,T \in \mathcal{P}_G}$, $Q = (q_{ST})_{S,T \in \mathcal{P}_G}$ and we define, for $S, T \in \mathcal{P}_G$,

$$f_S = \sum_{S' \in \mathcal{P}_G} p_{SS'} \varphi_{S'} , \quad g_T = \sum_{T' \in \mathcal{P}_G} q_{T'T} \varphi_{T'}.$$

Since both $P$ and $Q$ are regular matrices, $\{f_S | S \in \mathcal{P}_G\}$ and $\{g_T | T \in \mathcal{P}_G\}$ are basis of $W_{\mathcal{P}_G}$ respectively. Moreover we have $\text{diag}(d_{S_1}, \ldots, d_{S_N}) = P\mathcal{G}^\lambda Q = ( \langle f_S, g_T \rangle )_{S,T \in \mathcal{P}_G}$ by definition. Thus we have

$$(2.9.1) \quad \langle f_S, g_T \rangle = \delta_{ST} d_S \quad (S, T \in \mathcal{P}_G)$$

where $\delta_{ST} = 1$ if $S = T$ and $\delta_{ST} = 0$ otherwise. For $x = \sum_{S \in \mathcal{P}_G} r_S f_S \in W_{\mathcal{P}_G}$ ($r_S \in R$), we have

$$x \in W_{\mathcal{P}_G}^\lambda(i) \iff \langle x, g_T \rangle \in \varphi^i \quad \text{for any } T \in \mathcal{P}_G$$

$$\iff r_T d_T \in \varphi^i \quad \text{for any } T \in \mathcal{P}_G \quad \text{(by (2.9.1))}$$

$$\iff \nu_{\varphi}(r_T d_T) = \nu_{\varphi}(r_T) + \nu_{\varphi}(d_T) \geq i \quad \text{for any } T \in \mathcal{P}_G.$$ 

It follows from this that $W_{\mathcal{P}_G}^\lambda(i)$ is a free $R$-module with basis

$$(2.9.2) \quad \{f_T \mid T \in \mathcal{P}_G, \nu_{\varphi}(d_T) \geq i\} \cup \{\pi^{i-\nu_{\varphi}(d_T)} f_T \mid T \in \mathcal{P}_G, \nu_{\varphi}(d_T) < i\}.$$ 

2.10. We consider the Jantzen filtration of $W_{\mathcal{P}_G}^\lambda[k]$ ($1 \leq k \leq g$) as in the case of $W_{\mathcal{P}_G}^\lambda$ and use the notation similar to the case of $W_{\mathcal{P}_G}^\lambda$. Since we see that $W_{\mathcal{P}_G}^{\lambda[k]}(i_k)$ ($i_k \geq 0$)

is a free $R$-module (see 2.9), $W_{\mathcal{P}_G}^{\lambda[i_1]}(i_1) \otimes \cdots \otimes W_{\mathcal{P}_G}^{\lambda[i_g]}(i_g)$ ($(i_1, \ldots, i_g) \in Z^g_\geq 0$) becomes the submodule of $W_{\mathcal{P}_G}^{\lambda[i_1]} \otimes \cdots \otimes W_{\mathcal{P}_G}^{\lambda[i_g]}$.

For $1 \leq k \leq g$, let $\{f_{S[i]} \mid S[i] \in \mathcal{P}_G(\lambda[i])\}$ and $\{g_{T[i]} \mid T[i] \in \mathcal{P}_G(\lambda[i])\}$ be the bases of $W_{\mathcal{P}_G}^{\lambda[i]}$ as of $W_{\mathcal{P}_G}^\lambda$ in 2.9. For $S, T \in \mathcal{P}_G^p(\lambda)$, we define $\bar{f}_S := f_{S[i]} \otimes \cdots \otimes f_{S[i]}$ and $\bar{g}_T := g_{T[i]} \otimes \cdots \otimes g_{T[i]}$. Then $\{\bar{f}_S \mid S \in \mathcal{P}_G(\lambda)\}$ and $\{\bar{g}_T \mid T \in \mathcal{P}_G^p(\lambda)\}$ turn
out to be the bases of $\mathbb{Z}^\lambda_R$. By Lemma 1.9 and (2.9.1), we have

\begin{equation}
\langle T_S, g_T \rangle_p = \delta_{ST} d_{T[i]} \cdots d_{T[g]} \quad \text{for} \ S, T \in T_0^p(\lambda).
\end{equation}

We set $d_T = d_{T[i]} \cdots d_{T[g]}$. Then we have the following result by a similar argument as in 2.9. $\mathbb{Z}^\lambda_R(i)$ is a free $R$-module with basis

\begin{equation}
\begin{aligned}
\{ f_T \mid T \in T_0^p(\lambda), \nu_p(d_T) \geq i \} & \cup \{ \pi^{-\nu_p(d_T)} f_T \mid T \in T_0^p(\lambda), \nu_p(d_T) < i \}. \\
\end{aligned}
\end{equation}

Recall that $\Delta_{i,g}$ is the set of $(i_1, \ldots, i_g) \in \mathbb{Z}_{\geq 0}^g$ such that $i_1 + \cdots + i_g = i$. Then we have the following proposition.

**Proposition 2.11.** Let $\lambda \in \Lambda^+$ and $i \geq 0$. Under the isomorphism $\mathbb{Z}^\lambda_R \cong W^\lambda R[i] \otimes \cdots \otimes W^\lambda R[g]$, we have

\begin{equation}
\mathbb{Z}^\lambda_R(i) = \sum_{(i_1, \ldots, i_g) \in \Delta_{i,g}} W^\lambda R[i_1] \otimes \cdots \otimes W^\lambda R[i_g].
\end{equation}

**Proof.** First we show that the right hand side is contained in the left hand side. Take $x = x[i_1] \otimes \cdots \otimes x[i_g] \in W^\lambda R[i_1] \otimes \cdots \otimes W^\lambda R[i_g]$ such that $i_1 + \cdots + i_g = i$. By Lemma 1.9 we have

\begin{align*}
\langle x, f_T \rangle_p &= \langle x[i_1], \varphi_T[i_1] \rangle \cdots \langle x[i_g], \varphi_T[i_g] \rangle \\
&\in \varphi^{i_1} \cdots \varphi^{i_g} = \varphi^i \quad \text{for any} \ T \in T_0^p(\lambda).
\end{align*}

Thus we have $x \in \mathbb{Z}^\lambda_R(i)$.

Then in order to show the equality, we have only to show that the basis element of $\mathbb{Z}^\lambda_R(i)$ is contained in the right hand side of (2.11.1). First, we consider $f_T$ such that $\nu_p(d_T) \geq i$. Since $\nu_p(d_T) = \nu_p(d_{T[i]} + \cdots + d_{T[g]})$, one can find $(i_1, \ldots, i_g) \in \mathbb{Z}_{\geq 0}^g$ such that $i_1 + \cdots + i_g = i$ and that $\nu_p(d_{T[k]}) \geq i_k$ for $k = 1, \ldots, g$. Then

\begin{align*}
\bar{f}_T &= f_{T[i_1]} \otimes \cdots \otimes f_{T[i_g]} \in W^\lambda R[i_1] \otimes \cdots \otimes W^\lambda R[i_g],
\end{align*}

and so $\bar{f}_T$ is contained in the right hand side of (2.11.1).

Next we consider $f_T$ such that $\nu_p(d_T) < i$. Then one can find $(i_1, \ldots, i_g)$ such that $i_1 + \cdots + i_g = i$ and that $\nu_p(d_{T[k]}) \leq i_k$ for $k = 1, \ldots, g$. Therefore

\begin{align*}
\pi^{-\nu_p(d_T)} f_T &= \pi^{-\nu_p(d_{T[i_1]})} f_{T[i_1]} \otimes \cdots \otimes \pi^{-\nu_p(d_{T[i_g]})} f_{T[i_g]},
\end{align*}

which is also contained in the right hand side of (2.11.1). The proposition is proved. \hfill \Box

We have the following corollary.

**Corollary 2.12.** For $\lambda \in \Lambda^+$ and $i \geq 0$, under the isomorphism $\mathbb{Z}^\lambda_R \cong W^\lambda R[i] \otimes \cdots \otimes W^\lambda R[g]$, we have

\begin{equation}
\mathbb{Z}^\lambda_R(i) = \sum_{(i_1, \ldots, i_g) \in \Delta_{i,g}} W^\lambda R[i_1] \otimes \cdots \otimes W^\lambda R[i_g].
\end{equation}
Proof. By definition, we have
\[
Z^\lambda(i) = (Z^\lambda_R(i) + \varphi Z^\lambda_R)/\varphi Z^\lambda_R \simeq Z^\lambda_R(i)/(Z^\lambda_R(i) \cap \varphi Z^\lambda_R)
\]
and
\[
W^\lambda[1](i_1) \otimes \cdots \otimes W^\lambda[g](i_g) = \left( (W^\lambda_R[1](i_1) + \varphi W^\lambda_R[1])/\varphi W^\lambda_R \otimes \cdots \otimes (W^\lambda_R[g](i_g) + \varphi W^\lambda_R[g])/\varphi W^\lambda_R \right)
\]
\[
\cong W^\lambda_R[1](i_1) \otimes (W^\lambda_R[1](i_1) \cap \varphi W^\lambda_R[1]) \otimes \cdots \otimes W^\lambda_R[g](i_g) \otimes (W^\lambda_R[g](i_g) \cap \varphi W^\lambda_R[g]).
\]
By Proposition 2.11, we have a surjective map
\[
\Phi : Z^\lambda_R(i) \rightarrow \sum_{(i_1, \ldots, i_g) \in \Delta_{i,g}} W^\lambda[1](i_1) \otimes \cdots \otimes W^\lambda[g](i_g).
\]
We claim that $\ker \Phi = Z^\lambda_R(i) \cap \varphi Z^\lambda_R$. Then the claim implies the corollary. So we shall show the claim. By definition, it is clear that $\ker \Phi$ is contained in $Z^\lambda_R(i) \cap \varphi Z^\lambda_R$. Take $x = \sum_{T \in T^0_p(\lambda)} r_T f_T \in Z^\lambda_R(i) \cap \varphi Z^\lambda_R$. Then $r_T \in \varphi$ for any $T \in T^0_0(\lambda)$. If $\nu_\varphi(d_T) \geq 1$, then $f_T = f_T^{[1]} \otimes \cdots \otimes f_T^{[g]}$ is contained in $W^\lambda_R[1](i_1) \otimes \cdots \otimes W^\lambda_R[g](i_g)$ for some $(i_1, \ldots, i_g) \in \Delta_{i,g}$ by the proof of Proposition 2.11. So we have $r_T f_T \in \ker \Phi$ for $T \in T^0_0(\lambda)$ such that $\nu_\varphi(d_T) \geq 1$. If $\nu_\varphi(d_T) < 1$, then $r_T f_T = r_T^{\pi_1-\nu_\varphi(d_T)} f_T$ for some $r_T \in R$ since $x \in Z^\lambda_R(i)$. By the proof of Proposition 2.11, for some $(i_1, \ldots, i_g) \in \Delta_{i,g}$, $\pi_1-\nu_\varphi(d_T) f_T = (\pi_1^{i_1-\nu_\varphi(d_T)} f_T^{[1]}) \otimes \cdots \otimes (\pi_1^{i_g-\nu_\varphi(d_T)} f_T^{[g]})$ is contained in $W^\lambda_R[1](i_1) \otimes \cdots \otimes W^\lambda_R[g](i_g)$. Moreover one can find at least one $k$ such that $\nu_\varphi(d_T^{[k]}) < i_k$. Then the image of $\pi_1^{i_k-\nu_\varphi(d_T^{[k]})} f_T^{[k]}$ in $W^\lambda_R[1](i_1)/(W^\lambda_R[1](i_1) \cap \varphi W^\lambda_R[1])$ is zero. Hence for $T \in T^0_0$ such that $\nu_\varphi(d_T) < 1$, $r_T f_T$ is also contained in $\ker \Phi$. Now the claim is proved, and the corollary follows.

By using the corollary, we show the following lemma.

Lemma 2.13. Let $\lambda, \mu \in \Lambda^+$ be such that $\alpha_\mu(\lambda) = \alpha_\mu(\mu)$. For any $i \geq 0$, we have
\[
\left[ Z^\lambda(i)/Z^\lambda(i+1) : L^\mu \right] = \sum_{(i_1, \ldots, i_g) \in \Delta_{i,g}} \prod_{k=1}^g \left[ W^\lambda[k](i_k)/W^\lambda[k](i_{k+1}) : L^\mu[k] \right]
\]
Proof. By Corollary 2.12, we have
\[
Z^\lambda(i) / Z^\lambda(i+1) = \left( \sum_{(i_1, \ldots, i_g) \in \Delta_{i,g}} W^\lambda[1](i_1) \otimes \cdots \otimes W^\lambda[g](i_g) \right) / Z^\lambda(i+1)
\]
\[
= \sum_{(i_1, \ldots, i_g) \in \Delta_{i,g}} \left( W^\lambda[1](i_1) \otimes \cdots \otimes W^\lambda[g](i_g) \right) / \left( Z^\lambda(i+1) \cap (W^\lambda[1](i_1) \otimes \cdots \otimes W^\lambda[g](i_g)) \right).
\]
If \((i_1, \cdots, i_g) \neq (j_1, \cdots, j_g)\) such that \(i_1 + \cdots + i_g = j_1 + \cdots + j_g = i\), then
\[
(W^{\lambda^1}(i_1) \otimes \cdots \otimes W^{\lambda^g}(i_g)) \cap (W^{\lambda^1}(j_1) \otimes \cdots \otimes W^{\lambda^g}(j_g)) \subset W^{\lambda^1}(k_1) \otimes \cdots \otimes W^{\lambda^g}(k_g),
\]
where \(k_i = \max\{i_l, j_l\}\). Since \((i_1, \cdots, i_g) \neq (j_1, \cdots, j_g)\), \(k_1 + \cdots + k_g \geq i + 1\). Hence we have
\[
(W^{\lambda^1}(i_1) \otimes \cdots \otimes W^{\lambda^g}(i_g)) \cap (W^{\lambda^1}(j_1) \otimes \cdots \otimes W^{\lambda^g}(j_g)) \subset \mathcal{Z}^\lambda(i + 1).
\]

It follows from this, we see that the sum in (2.13.1) is a direct sum.

For \((i_1, \cdots, i_g) \in \Delta_{i,g}\), we consider a surjective \(\mathcal{F}\)-homomorphism
\[
\Psi : W^{\lambda^1}(i_1) \otimes \cdots \otimes W^{\lambda^g}(i_g) \to W^{\lambda^1}(i_1)/W^{\lambda^1}(i_1 + 1) \otimes \cdots \otimes W^{\lambda^g}(i_g)/W^{\lambda^g}(i_g + 1)
\]
Then we have \(\text{Ker} \, \Psi = \mathcal{Z}^\lambda(i + 1) \cap (W^{\lambda^1}(i_1) \otimes \cdots \otimes W^{\lambda^g}(i_g))\) under the setting in Corollary 2.12. By noting that (2.13.1) is a direct sum, we have
\[
\mathcal{Z}^\lambda(i)/\mathcal{Z}^\lambda(i + 1) \cong \bigoplus_{(i_1, \cdots, i_g) \in \Delta_{i,g}} \left( W^{\lambda^1}(i_1)/W^{\lambda^1}(i_1 + 1) \otimes \cdots \otimes W^{\lambda^g}(i_g)/W^{\lambda^g}(i_g + 1) \right)
\]
Since \(T^\mu \cong L^{\mu^1} \otimes \cdots \otimes L^{\mu^g}\), we have
\[
\left[ \mathcal{Z}^\lambda(i)/\mathcal{Z}^\lambda(i + 1) : T^\nu \right] = \sum_{(i_1, \cdots, i_g) \in \Delta_{i,g}} \left[ W^{\lambda^1}(i_1)/W^{\lambda^1}(i_1 + 1) \otimes \cdots \otimes W^{\lambda^g}(i_g)/W^{\lambda^g}(i_g + 1) : T^\nu \right]
\]
\[
= \sum_{(i_1, \cdots, i_g) \in \Delta_{i,g}} \prod_{k=1}^{g} \left[ W^{\lambda^k}(i_k)/W^{\lambda^k}(i_k + 1) : L^{\mu^k} \right]
\]
The lemma is proved.

We define \(v\)-decomposition numbers of \(\mathcal{S}(A_{n_k})\) for \(k = 1, \cdots, g\) by
\[
d_{\lambda[k],\mu[k]}(v) := \sum_{i_k \geq 0} \left[ W^{\lambda^k}(i_k)/W^{\lambda^k}(i_k + 1) : L^{\mu^k} \right] \cdot v^i
\]
as in the case of \(\mathcal{S}(A)\). Then we have the following theorem.

**Theorem 2.14.** For \(\lambda, \mu \in A^+\) such that \(\alpha_p(\lambda) = \alpha_p(\mu)\), we have
\[
d_{\lambda\mu}(v) = d_{\lambda\mu}(v) = \prod_{k=1}^{g} d_{\lambda[k],\mu[k]}(v).
\]
Proof. The first equality follows from Theorem 2.8. So we prove the second equality. By Lemma 2.13 we have

\[
\bar{d}_{\lambda\mu}(v) = \sum_{i \geq 0} \left[ \sum_{(i_1, \ldots, i_g) \in \Delta_{i,g}} \prod_{k=1}^{g} \left[ W^{\lambda[k]}(i_k)/W^{\lambda[k]}(i_k + 1) : L^{\mu[k]} \right] \right] \cdot v^i
\]

\[
= \sum_{i \geq 0} \left( \sum_{(i_1, \ldots, i_g) \in \Delta_{i,g}} \prod_{k=1}^{g} \left[ W^{\lambda[k]}(i_k)/W^{\lambda[k]}(i_k + 1) : L^{\mu[k]} \right] \right) \cdot v^i
\]

\[
= \sum_{i \geq 0} \left( \sum_{(i_1, \ldots, i_g) \in \Delta_{i,g}} \prod_{k=1}^{g} \left[ W^{\lambda[k]}(i_k)/W^{\lambda[k]}(i_k + 1) : L^{\mu[k]} \right] \right) \cdot v^i
\]

This proves the theorem. \hfill \square

3. \(v\)-DECOMPOSITION NUMBERS FOR ARIKI-KOIKE ALGEBRAS

We keep the notation in the previous section. We consider the \(v\)-decomposition numbers of the Ariki-Koike algebra \(\mathcal{H}\), and show that similar results hold as in the previous section.

3.1. Let \(\omega = (\cdot, \cdots, \cdot, (1^n))\) be the \(r\)-partition and \(T^\omega\) be the \(\omega\)-tableau of type \(\omega\). Since \(\varphi_{T^\omega_{\sim T^\omega}}\) is an identity map on \(M^\omega\) and a zero map \(M^\mu\) for \(\mu \in \Lambda\) such that \(\mu \neq \omega\), \(\varphi_{T^\omega_{\sim T^\omega}}\) is an idempotent in \(\mathfrak{S}\). Moreover we see that \(\varphi_{T^\omega_{\sim T^\omega}}: \mathfrak{S}_{\varphi_{T^\omega_{\sim T^\omega}}} = \text{Hom}_{\mathfrak{S}}(M^\omega, M^\omega) = \text{Hom}_{\mathfrak{S}}(\mathcal{H}, \mathcal{H}) \cong \mathcal{H}\). It is well known that, for an \(\mathfrak{S}\)-module \(M\), \(M\varphi_{T^\omega_{\sim T^\omega}}\) becomes a \(\mathcal{H}\)-module through the isomorphism \(\varphi_{T^\omega_{\sim T^\omega}}: \mathfrak{S}_{\varphi_{T^\omega_{\sim T^\omega}}(\mathcal{H})} \cong \mathcal{H}\). Then we can define a functor, the so-called “Schur functor”, from the category of right \(\mathfrak{S}\)-modules to the category of right \(\mathcal{H}\)-modules by \(M \mapsto M\varphi_{T^\omega_{\sim T^\omega}}\). The following facts are known by [JM] Proposition 2.17.

\[
\begin{align*}
(3.1.1) & \quad W^\lambda \varphi_{T^\omega_{\sim T^\omega}} \cong S^\lambda \quad \text{as \(\mathcal{H}\)-modules} \quad (\lambda \in \Lambda^+) \\
(3.1.2) & \quad L^\mu \varphi_{T^\omega_{\sim T^\omega}} \cong D^\mu \quad \text{as \(\mathcal{H}\)-modules} \quad (\mu \in \Lambda^+) \\
(3.1.3) & \quad [W^\lambda : L^\mu] = [S^\lambda : D^\mu] \quad (\lambda, \mu \in \Lambda^+ \text{ such that } D^\mu \neq 0)
\end{align*}
\]

where \([S^\lambda : D^\mu]\) is the decomposition number of \(D^\mu\) in \(S^\lambda\).

3.2. One can define the Jantzen filtration of the Specht module \(S^\lambda\) in a similar way as in the case of \(W^\lambda\), and we use a similar notation for this case. Then one can
define the \( v \)-decomposition number of \( H \), for \( \lambda, \mu \in \Lambda^+ \) such that \( D^\mu \neq 0 \), by

\[
d_{\lambda \mu}^H(v) := \sum_{i \geq 0} \left[ S^\lambda(i)/S^\lambda(i+1) : D^\mu \right] \cdot v^i.
\]

We have the following lemma.

**Lemma 3.3.** Let \( \lambda \in \Lambda^+ \) and \( i \geq 0 \). Under the isomorphism in (3.1.1), we have

\[
W^\lambda(i) \varphi_{T \omega \tau} = S^\lambda(i).
\]

**Proof.** It is clear that \( W^\lambda(\varphi_{T \omega \tau}) \) has a basis \( \{ \varphi_T | T \in T_0(\lambda, \omega) \} \). We have a bijective correspondence between \( T_0(\lambda, \omega) \) and \( \text{Std}(\lambda) \) by \( T \leftrightarrow t \) such that \( \omega(t) = T \). Moreover, under the isomorphism in (3.1.1), we have

\[
\langle \varphi_T, \varphi_S \rangle = \langle m_t, m_s \rangle_H \quad \text{for } S = \omega(s), T = \omega(t) \in T_0(\lambda, \omega)
\]

by a similar argument as in the proof of \([M2, \text{Theorem 4.18}]\).

First, we show the inclusion \( W^\lambda(i) \varphi_{T \omega \tau} \subseteq S^\lambda(i) \). Take \( x \in W^\lambda_R(i) \). Then \( \langle x, \varphi_T \rangle \in \wp^i \) for any \( T \in T_0(\lambda) \). It follows that

\[
\langle x \cdot \varphi_{T \omega \tau}, \varphi_T \rangle = \langle x, \varphi_T \varphi_{T \omega \tau} \rangle \in \wp^i \quad \text{for any } T \in T_0(\lambda).
\]

This shows that

\[
\langle x \cdot \varphi_{T \omega \tau}, m_t \rangle_H \in \wp^i \quad \text{for any } t \in \text{Std}(\lambda)
\]

by (3.3.1) and (3.3.2). Hence \( x \cdot \varphi_{T \omega \tau} \in S^\lambda(i) \), and the claim follows by taking the quotient.

Next, we show the converse inclusion \( W^\lambda(i) \varphi_{T \omega \tau} \supseteq S^\lambda(i) \). Take \( y \in S^\lambda(i) \). Then we have

\[
\langle y, m_s \rangle_H \in \wp^i \quad \text{for any } s \in \text{Std}(\lambda).
\]

Write \( y = \sum_{t \in \text{Std}(\lambda)} r_t m_t \), and put \( x = \sum_{T \in T_0(\lambda, \omega)} r_T \varphi_T \in W^\lambda \), where \( T \) is the \( \lambda \)-tableau of type \( \omega \) corresponding to \( t \). Then we have \( y = x \cdot \varphi_{T \omega \tau} \), and

\[
\langle x, \varphi_S \rangle \in \wp^i \quad \text{for any } S \in T_0(\lambda, \omega)
\]

by (3.3.1), (3.3.2) and (3.3.3). Since \( \langle \varphi_T, \varphi_S \rangle = 0 \) if the type of \( T \) is not the same as the type of \( S \), we have

\[
\langle x, \varphi_S \rangle \in \wp^i \quad \text{for any } S \in T_0(\lambda).
\]

This shows that \( x \in W^\lambda_R(i) \), and the claim follows. The lemma is proved. \( \square \)
This lemma implies the following proposition.

**Proposition 3.4.** Take $\lambda, \mu \in \Lambda^+$ such that $D^\mu \neq 0$. Then for any $i \geq 0$, we have

$$[W^\lambda(i)/W^\lambda(i + 1) : L^\mu] = [S^\lambda(i)/S^\lambda(i + 1) : D^\mu].$$

In particular, we have $d_{\lambda\mu}(v) = d_{\lambda\mu}^v(v)$.

**Proof.** We consider the $\mathscr{S}$-module filtration

$$W^\lambda(i) = W_0 \supseteq W_1 \supseteq \cdots \supseteq W_k = W^\lambda(i + 1)$$

such that $W_j/W_{j+1} \cong L^\mu$. By applying the Schur functor, together with Lemma 3.3, we have

$$S^\lambda(i) = W^\lambda(i)\varphi_{T^wT^w} \supset W_1\varphi_{T^wT^w} \supset \cdots \supset W_k\varphi_{T^wT^w} = W^\lambda(i + 1)\varphi_{T^wT^w} = S^\lambda(i + 1),$$

where $W_j\varphi_{T^wT^w}/W_{j+1}\varphi_{T^wT^w} \cong (W_j/W_{j+1})\varphi_{T^wT^w} \cong L^\mu i \varphi_{T^wT^w} \cong D^\mu i$ by 3.1.2. The proposition follows from this.

For $\lambda \in \Lambda^+$ such that $\alpha_p(\lambda) = (n_1, \ldots, n_g)$, $\lambda^{[k]}$ is an $r_k$-partition of $n_k$. Then we have the Specht module $S^\lambda$ and its unique quotient $D^\lambda$ for the Ariki-Koike algebra $\mathcal{H}_{n_k,r_k}$. Moreover for $\lambda, \mu \in \Lambda^+$ such that $\alpha_p(\lambda) = \alpha_p(\mu) = (n_1, \ldots, n_g)$, we have the $v$-decomposition number $d_{\lambda\mu}^v(v)$ for $\mathcal{H}_{n_k,r_k}$. Combining Theorem 2.14 with Proposition 3.4, we have the following result.

**Theorem 3.5.** Let $\lambda, \mu \in \Lambda^+$ such that $\alpha_p(\lambda) = \alpha_p(\mu)$. Assume that $D^\mu \neq 0$ and $D^\mu^{[k]} \neq 0$ for any $k = 1, \ldots, g$. Then we have

$$d_{\lambda\mu}^v(v) = \prod_{k=1}^g d_{\lambda^{[k]}\mu^{[k]}}^v(v).$$

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