MODULI SPACES OF FRAMED SYMPLECTIC AND ORTHOGONAL BUNDLES ON $\mathbb{P}^2$ AND THE K-THEORETIC NEKRASOV PARTITION FUNCTIONS

JAEYOO CHOY

Dedicated to the late Professor Kentaro Nagao

Abstract. Let $K$ be the compact Lie group $\text{USp}(N/2)$ or $\text{SO}(N,\mathbb{R})$. Let $\mathcal{M}^K_n$ be the moduli space of framed $K$-instantons over $S^4$ with the instanton number $n$. By [12], $\mathcal{M}^K_n$ is endowed with a natural scheme structure. It is a Zariski open subset of a GIT quotient of $\mu^{-1}(0)$, where $\mu$ is a holomorphic moment map such that $\mu^{-1}(0)$ consists of the ADHM data.

The purpose of the paper is to study the geometric properties of $\mu^{-1}(0)$ and its GIT quotient, such as complete intersection, irreducibility, reducedness and normality. If $K = \text{USp}(N/2)$ then $\mu$ is flat and $\mu^{-1}(0)$ is an irreducible normal variety for any $n$ and even $N$. If $K = \text{SO}(N,\mathbb{R})$ the similar results are proven for low $n$ and $N$.

As an application one can obtain a mathematical interpretation of the K-theoretic Nekrasov partition function of [36].

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1. Introduction

1.1. The Nekrasov partition function was formulated by Nekrasov [33] in the 4-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory in physics, especially relevant to the Seiberg-Witten prepotential [38]. It is defined as a generating function of the equivariant integration of the trivial cohomology class 1 over Uhlenbeck partial compactification of the (framed) instanton moduli space on $\mathbb{R}^4$ for all the instanton numbers. Its logarithm turned out to contain the Seiberg-Witten prepotential as the coefficient of the lowest degree (with respect to two variables from the torus action on $\mathbb{R}^4$), which is the remarkable result proven by Nakajima-Yoshioka [31], Nekrasov-Okounkov [35] (both for SU($N$)) and Braverman-Etingof [5] (for any gauge group) in completely independent methods. The Nekrasov partition function is an equivariant version of Donaldson-type invariants for $\mathbb{R}^4$, where the ordinary Donaldson invariants are integrals over Uhlenbeck compactifications of instanton moduli spaces on compact 4-manifolds. More generally, a Nekrasov partition function for the theory with matters is defined via the equivariant integration of a cohomology class other than the trivial class. For instance, in the works of Göttsche-Nakajima-Yoshioka [15][17], such partition functions are used to express wallcrossing terms of Donaldson invariants and a relation between them and Seiberg-Witten invariants.

Our interest in this paper is the K-theoretic Nekrasov partition function. It arose from the 5-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory proposed by Nekrasov [34]. Mathematically, one change in its definition is made [33]: the equivariant integration of elements in the equivariant K-theory of coherent sheaves instead of cohomology classes. There is a technical, but a subtle problem here. We need a scheme structure on the Uhlenbeck space for the definition, but there are several choices (see [5]). Therefore it is not clear what is the correct definition. For type A, one can use framed moduli spaces of coherent sheaves instead, which are smooth. The K-theoretic Nekrasov partition functions also appeared in mathematics literatures as K-theoretic Donaldson invariants studied in [32][16] in which the K-theory version of Nekrasov conjecture and blowup equations are proven for type A.

Nekrasov-Shadchin [36] took a different approach when the gauge group is of classical type. They defined the partition function using the K-theory class of the Koszul complex which defines the ADHM data of the instanton moduli spaces given in [1].

The main purpose of this paper is to show that Nekrasov-Shadchin’s definition coincides with a generating function of the coordinate rings of the instanton moduli spaces for the classical gauge groups. The answer for the gauge group SU($N$) has been already known by a general result of Crawley-Boevey on quiver varieties [11]. (See §1.6 for the precise motivating questions which we pursue). Our study for the gauge groups USp($N/2$) and SO($N, \mathbb{R}$) informs of some geometry of the moduli spaces in algebraic geometry, as the instanton moduli space for USp($N/2$) and SO($N, \mathbb{R}$) is isomorphic to the moduli space of (framed) vector bundles with symplectic and orthogonal structures on $\mathbb{P}^2$ respectively. (See §1.4 for explanation on the scheme structures of moduli spaces.)

1.2. Let us fix the notation and explain earlier results in order to state our result precisely. Let $K$ be a compact connected simple Lie group. Let $P_K$ be a principal $K$-bundle over the 4-sphere $S^4$. Since $\pi_3(K) \cong \mathbb{Z}$, an integer $n$ uniquely determines the topological
Let $P_K$. Let $\mathcal{M}_n^K$ be the quotient of the space of ASD connections (instantons) by the group of the gauge transformations trivial at infinity $\infty \in S^4 \ [12] \ \S 5.1.1$. By \cite[Table 8,1]{} $\mathcal{M}_n^K$ is a $C^\infty$-manifold with $\dim \mathcal{M}_n^K = 4n h^\vee$, where $h^\vee$ is the dual Coxeter number of $K$. In this paper, we will also consider the case $K = SO(4, \mathbb{R})$. Its universal cover $\text{Spin}(4)$ is the product $SU(2) \times SU(2)$, and hence $SO(4, \mathbb{R})$-bundles over $S^4$ are classified by pairs of integers $(n_L, n_R)$. We have $\mathcal{M}_n^K \cong \mathcal{M}_{n_L}^{SU(2)} \times \mathcal{M}_{n_R}^{SU(2)}$, and hence $\dim \mathcal{M}_n^K = 8(n_L + n_R)$.

1.3. Let $K = SU(N)$. Donaldson \cite{12} showed that $\mathcal{M}_n^K$ is naturally isomorphic to the moduli space of framed holomorphic vector bundles $E$ with $c_2(E) = n$ over $\mathbb{P}^2$, where a framing is a trivialization of $E$ over the line $l_\infty$ at infinity. This was proved by using the ADHM description \cite{1} and the relation between moment maps and geometric invariant theory (GIT). Let $K = USp(N/2)$ or $SO(N, \mathbb{R})$ and $G$ be its complexification. Since $G$-bundles can be identified with rank $N$ vector bundles with symplectic or orthogonal structures, we have the following description of $\mathcal{M}_n^K$.

Let $G'_k := O(k)$ (resp. $Sp(k/2)$) if $G = Sp(N/2)$ (resp. $SO(N)$). Let $p(\mathbb{C}^k)$ be the space of symmetric endomorphisms of $\mathbb{C}^k$ (see \cite{2} for the precise definition). The ADHM data are elements $x \in \mathbb{N}$ satisfying $\mu(x) = 0$, where $\mathbb{N} := p(\mathbb{C}^k)^{\otimes 2} \oplus \text{Hom}(\mathbb{C}^N, \mathbb{C}^k)$ and $\mu$ is the holomorphic moment map defined on $\mathbb{N}$. The $G'_k$-action on $\mu^{-1}(0)$ induces a $G'_k$-action on $\mu^{-1}(0)$. Now $\mathcal{M}_n^K$ is the image of the regular locus $\mu^{-1}(0)|^{\text{reg}}$ of the GIT quotient $\mu^{-1}(0) \rightarrow \mu^{-1}(0)/G'_k$. Here the second Chern number $k$ of associated vector bundles is determined from $n$ by the argument in \cite{2} \S 10 as

$$
(1.1) \quad k = \begin{cases} 
\quad n & \text{if } G = Sp(N/2), \\
2n & \text{if } G = SO(N), \text{ where } N \geq 5, \\
\quad 4n & \text{if } G = SO(3).
\end{cases}
$$

The GIT quotient $\mu^{-1}(0)/G'_k$ endows the Donaldson-Uhlenbeck partial compactification of $\mathcal{M}_n^K$ with a scheme structure except the case $G = SO(3)$ (cf. \cite{2,6}). We call $\mu^{-1}(0)/G'_k$ the scheme-theoretic Donaldson-Uhlenbeck partial compactification of $\mathcal{M}_n^K$ unless $G = SO(3)$. As we will see in \cite{16} the K-theoretic Nekrasov partition function coincides with the generating function of the coordinate rings of algebraic functions on these Donaldson-Uhlenbeck spaces if $G = Sp(N/2)$.

1.4. The scheme structure of $\mathcal{M}_n^K$ is given by $\mu^{-1}(0)^{\text{reg}}/G'_k$. Since $\mu^{-1}(0)^{\text{reg}}/G'_k$ is a Zariski-open subset of $\mu^{-1}(0)/G'_k$, $\mathcal{M}_n^K$ is a quasi-affine scheme. We notice that $\mathcal{M}_n^K$ may have two scheme structures because it is possible that $\text{Lie}(K) = \text{Lie}(K')$ for a different classical group $K'$ (hence $\mathcal{M}_n^K = \mathcal{M}_n^{K'}$) but $\mathcal{M}_n^K$ and $\mathcal{M}_n^{K'}$ have different ADHM descriptions. Such pairs of $(K, K')$ are $(SU(2), USp(1))$, $(SU(2), SO(3, \mathbb{R}))$, $(USp(2), SO(5, \mathbb{R}))$ and $(SU(4), SO(6, \mathbb{R}))$. A scheme-theoretic isomorphism $\mathcal{M}_n^K \cong \mathcal{M}_n^{K'}$ is induced by sending the associated vector bundles $E$ to itself, $\text{ad}E$, $(\Lambda^2 E)_0$ and $\Lambda^2 E$ respectively. The notations in the above are as follows: $\text{ad}E$ is the trace-free part of $\text{End}(E)$, and $(\Lambda^2 E)_0$ is the kernel of the symplectic form $\Lambda^2 E \rightarrow \mathcal{O}$.

In the above isomorphism we used the following two assertions: First by Donaldson’s theorem \cite{12}, $\mathcal{M}_n^K$ is canonically isomorphic to the moduli space of framed rank $N$ vector
bundles $E$ with $c_2(E) = k$ and $G$-structure. Then the latter space is endowed with a (smooth) scheme structure and the canonical isomorphism is scheme-theoretic. See \$3\.

1.5. There is a canonical $G \times G'_k$-action on $\mathcal{N}$. The canonical $G$-action on $\mathcal{N}$ induces a $G$-action on $\mathcal{M}^K_n$. We define a $(\mathbb{C}^*)^2$-action on $\mathcal{N}$ by $(q_1, q_2).(B_1, B_2, i) = (q_1B_1, q_2B_2, \sqrt{q_1q_2}i)$. More precisely this action is well-defined only on a double cover of $\mathbb{C}^* \times \mathbb{C}^*$, but we follow the convention in physics. It commutes with the $G$-action on $\mathcal{N}$, so that the origin $0$ is the unique fixed point in $\mathcal{N}$. Let $T_G$ be a maximal torus of $G$. We identify $T_G = (\mathbb{C}^*)^l$, where $l = \text{rank}(G)$. Let $T := T_G \times (\mathbb{C}^*)^2 = (\mathbb{C}^*)^{l+2}$. Then $T$ acts on $\mu^{-1}(0)/G'_k$ and $\mathcal{M}^K_n$. Nekrasov-Shadchin \cite{[30]} defined the $K$-theoretic instanton partition function $Z^K$ for the gauge group $K$ by an explicit integral formula. Their formula can be interpreted as follows.

Fix the instanton number $n$. Let us consider $\mu$ as a section of the trivial vector bundle $E = \text{Lie} G'_k \times \mathcal{N}$ over $\mathcal{N}$. We endow $E$ with a $G'_k \times T$-equivariant structure so that $\mu$ is a $T$-equivariant section. Then $\mu$ defines a Koszul complex

\begin{equation}
\Lambda^{\text{rank}E}E^\vee \to \cdots \to \Lambda^2E^\vee \to E^\vee \to \Lambda^0E^\vee = \mathcal{O}_\mathcal{N}.
\end{equation}

The alternating sum $\sum_i (-1)^i \Lambda^iE^\vee$ of terms gives an element in $K^{T \times G'_k}(\mathcal{N})$, the Grothendieck group of $T \times G'_k$-equivariant vector bundles over $\mathcal{N}$. We then take a pushforward of the class with respect to the obvious map $p: \mathcal{N} \to \text{pt}$. This is not a class in the representation ring $R(T \times G'_k)$ because $\mathcal{N}$ is not proper. However, it is a well-defined class in $\text{Frac}(R(T \times G'_k))$, the fractional field of $R(T \times G'_k)$ by equivariant integration: Check first that the origin $0$ is the unique fixed point in $\mathcal{N}$ with respect to $T \times T'_k$, where $T'_k$ is a maximal torus of $G'_k$. (See \cite{[A]} Therefore the pushforward homomorphism $p_*$ can be defined as the inverse of $i_*$ thanks to the fixed point theorem of the equivariant K-theory, where $i_*: \{0\} \to \mathcal{N}$ is the inclusion. In practice, we take the Koszul resolution of the skyscraper sheaf at the origin as above, replacing $E$ by $\mathcal{N}$, and then divide the class by $\sum_i (-1)^i \Lambda^i\mathcal{N}^\vee$.

In our situation, $\mathcal{N}$ has only positive weights with respect to the factor $(\mathbb{C}^*)^2$ of $T$. (See \cite{[A]} Therefore the rational function can be expanded as an element in a completed ring of $T \times G'_k$-characters

$$\tilde{R}(T \times G'_k) := R(T_G \times G'_k)[[q_1^{-1}, q_2^{-1}]],$$

where $q_m$ denote the $T$-characters of the 1-dimensional representations for the second factor $(\mathbb{C}^*)^2$ of $T = T_G \times (\mathbb{C}^*)^2$. The expansion is considered as a formal character, as the sum of dimensions of weight spaces: Each weight space is finite dimensional and its dimension is given by the coefficient of $f \in \tilde{R}(T \times G'_k)$ of the monomial corresponding to the weight.

Next we take the $G'_k$-invariant part. This is given by an integral over the maximal compact subgroup of $T'_k$ by Weyl’s integration formula. Finally taking the generating function for all $n \in \mathbb{Z}_{\geq 0}$ with the formal variable $q$, we define the instanton partition function

$$Z^K := \sum_{n \geq 0} q^n \sum_i (-1)^i p_*(\Lambda^iE^\vee)^{G'_k} \in \tilde{R}(T)[[q]],$$

where $\tilde{R}(T) := R(T_G)[[q_1^{-1}, q_2^{-1}]]$. 
Explicit computation of $Z^K$ has been performed for small instanton numbers in physics literature. See [41, 21] for instance.

1.6. Nekrasov-Shadchin’s definition of $Z^K$ is mathematically rigorous, but depends on the ADHM description, hence is not intrinsic in the instanton moduli space $\mathcal{M}_n^K$. We can remedy this problem under the following geometric assumptions:

1. $\mu$ is a flat morphism;
2. $\mu^{-1}(0)/G'_k$ is a normal variety and $\mathcal{M}_n^K$ has complement of codimension $\geq 2$ in $\mu^{-1}(0)/G'_k$.

Under these assumptions, $\sum_i (-1)^i p_* (\Lambda^i E^\vee)^{G'_k}$ is the formal character of the ring $\mathbb{C}[\mathcal{M}_n^K]$ of regular functions on $\mathcal{M}_n^K$: First, (1) asserts $\mu$ gives a regular sequence so that the Koszul complex (1.2) is a resolution of the coordinate ring $\mathbb{C}[\mu^{-1}(0)]$ of $\mu^{-1}(0)$. Secondly (2) asserts $\mathbb{C}[\mathcal{M}_n^K] = \mathbb{C}[\mu^{-1}(0)//G'_k] = \mathbb{C}[\mu^{-1}(0)]^{G'_k}$ by extension of regular functions on $\mathcal{M}_n^K = \mu^{-1}(0)/G'_k$ to $\mu^{-1}(0)/G'_k$. As explained in [11, 14] the scheme structure of $\mathcal{M}_n^K$ is independent of the choice of the ADHM description.

When $K = \text{SU}(N)$, (1, 2) are proved by Crawley-Boevey [10] in a general context of quiver varieties. Moreover the Gieseker space $\widetilde{\mathcal{M}}_n^N$ of framed rank $N$ torsion free sheaves $E$ on $\mathbb{P}^2$ with $c_2(E) = n$ gives a resolution of singularities of $\mu^{-1}(0)/G'_k$. The higher direct image sheaves $R^l p_* \mathcal{O}_{\widetilde{\mathcal{M}}_n^N} = 0$ for $l \geq 1$, where $p: \widetilde{\mathcal{M}}_n^N \to \text{pt}$ is the obvious map. For, $\widetilde{\mathcal{M}}_n^N$ is symplectic and thus the canonical sheaf $K_{\widetilde{\mathcal{M}}_n^N}$ is isomorphic to $\mathcal{O}_{\widetilde{\mathcal{M}}_n^N}$. By the Grauert-Riemenschneider vanishing theorem, $R^l \pi_* K_{\widetilde{\mathcal{M}}_n^N} = 0$ for $l \geq 1$, where $\pi$ denotes the resolution of singularities. Now the vanishing of higher direct image sheaves comes from the $E_2$-degeneration of the spectral sequence $R^l p_* R^m \pi_* \Rightarrow R^{l+m} p_* \pi_*$, where $p: \mu^{-1}(0)/G'_k \to \text{pt}$ is the obvious affine map. Therefore $\mathbb{C}[\mathcal{M}_n^N] \cong \mathbb{C}[\mu^{-1}(0)/G'_k]$. The $T$-action lifts to $\widetilde{\mathcal{M}}_n^N$, where fixed points are parametrized by $N$-tuples of Young diagrams corresponding to direct sums of monomial ideal sheaves. Therefore the character of $\mathbb{C}[\mathcal{M}_n^N]$ is given by a sum over $N$-tuples of Young diagrams, which is the original definition of the instanton partition function in [33]. See [32] for more detail on the latter half of this argument.

1.7. A goal of this paper is to prove (1, 2) for $K = \text{USp}(N/2)$. (See Theorem 3.11.) A key of the proof is a result of Panyushev [37], which gives the flatness of $\mu$ for $N = 0$.

We also study $\mu^{-1}(0)$ for $K = \text{SO}(N, \mathbb{R})$ for $(N, k) = (2, k), (N, 2)$ or $(3, 4)$. (See Theorems 4.2, 4.1, 4.3 respectively.) The case $N = 2$ is less interesting since there are no $\text{SO}(2, \mathbb{R})$-instantons except for $k = 0$. But $\mu^{-1}(0)$ does make sense, hence the study of its properties is a mathematically meaningful question. We show that $\mu$ is not flat, i.e., (1) is not true in this case. Similarly there is no instanton for $(N, k) = (3, 2)$. We will see that $\mu$ is flat, but $\mu^{-1}(0)/G'_k$ with reduced scheme structure is isomorphic to $\mathbb{C}^2$, so (2) is false in this case. In the case $(N, 2)$ with $N \geq 4$, we prove that $\mu^{-1}(0)/G'_k$ is isomorphic to the product of $\mathbb{C}^2$ and the closure $P$ of the minimal nilpotent $O(N)$-orbit. When $(N, k) = (3, 4)$, we show that $\mu$ is flat and $\mu^{-1}(0)/G'_k$ is a union of two copies of $\mathbb{C}^2 \times P$ meeting along $\mathbb{C}^2 \times \{0\}$. Here $P$ is the minimal (= regular) nilpotent $O(3)$-orbit closure, which is isomorphic to $\mathbb{C}^2/\mathbb{Z}_2$. In particular, (2) is false. The isomorphism $\mathcal{M}_n^{\text{SU}(2)} \cong \mathcal{M}_n^{\text{SO}(3)}$
implies that both \( \mathcal{M}^{SU(2)}_n \) and \( \mathcal{M}^{SO(3)}_n \) are \( \mathbb{C}^2 \times (\mathbb{P} \setminus \{0\}) \) for \( n = 1 \). However their ADHM descriptions are different, as it is \( \mathbb{C}^2 \times \mathbb{P} \) for \( SU(2) \). This phenomenon happens only for \( SO(3) \). See Theorem 2.6. These examples show that the definition of \( Z^K \) depends on the ADHM description, and hence must be studied with care.

The author plans to study the case \((N \geq 4, k \geq 4)\) in the near future.

1.8. The paper is organized as follows.

In \( \S 2 \) we give the set-up for the entire part of the paper. Specifically we set up linear algebra of vector spaces with nondegenerate forms in \( \S 2.1-2.3 \). In \( \S 2.4 \) we identify the closed \( G'_k \)-orbits in \( \mathbb{N} \). By this identification we see that \( \mathcal{M}^K_n \) is the \( G'_k \)-orbit space of stable-costable representations in \( \mu^{-1}(0) \). In \( \S 2.6 \) we stratify \( \mu^{-1}(0)/G'_k \) with \( \mathcal{M}^K_n \) as a stratum. Each stratum is isomorphic to a product of \( \mathcal{M}^K_{n'} \) and a symmetric product of \( \mathbb{C}^2 \) for some \( n' \leq n \).

In \( \S 3 \) and \( \S 4 \) we state Theorems 3.1, 4.1–4.3 which describes the geometry of \( \mu^{-1}(0) \) when \( K = USp(N/2) \) and \( SO(N, \mathbb{R}) \).

In \( \S 5 \) we explain Kraft-Procesi’s classification theory of nilpotent pairs. In our study of \( \mu^{-1}(0) \), their theory is quite useful to see contribution from the factor \( \text{Hom}(\mathbb{C}^N, \mathbb{C}^k) \) of \( \mathbb{N} \). In the case \( k = 2 \), the geometry of \( \mu^{-1}(0) \) is immediately deduced from Kraft-Procesi’s theory on \( \text{Hom}(\mathbb{C}^N, \mathbb{C}^k) \) since \( p(\mathbb{C}^2) \) consists of the scalars.

In \( \S 6 \) and \( \S 7 \) we will see how Kraft-Procesi’s theory is applied to the cases \( k = 2 \) and \( N = 2 \). This will prove Theorem 4.1 and a part of Theorem 4.2.

In \( \S 8 \) we prove Theorem 4.3 (the case \( (N, k) = (3, 4) \)). The proof involves more than Kraft-Procesi’s theory since the pairs \( (B_1, B_2) \in p(\mathbb{C}^4)^{\oplus 2} \) are no more commuting pairs. Since \([B_1, B_2] \neq 0\) in general, the study of \( \mu^{-1}(0) \) does not solely come from the factor \( \text{Hom}(\mathbb{C}^3, \mathbb{C}^4) \). So we study the commutator map \( p(\mathbb{C}^4)^{\oplus 2} \rightarrow \text{Lie}(\text{Sp}(2)), \ (B_1, B_2) \mapsto [B_1, B_2] \).

In \( \S 9 \) we finish the proof of Theorem 4.2 (the case \( (N, k) = (2, 4) \)) based on the study of the commutator map in the above.

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2. Preliminary

We set up conventions and notation and review basic material for an entire part of the paper.

We are working over \( \mathbb{C} \). Vector spaces are all finite dimensional and schemes are of finite type. We say that a morphism between schemes is irreducible (resp. normal and Cohen-Macaulay) if all the nonempty fibres are irreducible (resp. normal and Cohen-Macaulay).
If $M$ is a scheme then $M^{\text{sm}}$ and $M^{\text{sing}}$ are the smooth locus and the singular locus of $M$ respectively.

Let $G$ be an algebraic group and $\mathfrak{g} := \text{Lie}(G)$ the Lie algebra of $G$. Let $M$ be a $G$-scheme. Let $G^x := \{ g \in G | g.x = x \}$ the stabilizer subgroup of $x \in M$. Let $\mathfrak{g}^x := \text{Lie}(G^x)$.

2.1. The right adjoint. Let $V_1$ and $V_2$ be vector spaces with nondegenerate bilinear forms $(\cdot, \cdot)_{V_1}$ and $(\cdot, \cdot)_{V_2}$ respectively. Then for any $i \in \text{Hom}(V_1, V_2)$, we have the right adjoint $i^* \in \text{Hom}(V_2, V_1)$, i.e. $(v, i^*w)_{V_1} = (iv, w)_{V_2}$, where $v \in V_1$ and $w \in V_2$. The map

$$\ast : \text{Hom}(V_1, V_2) \to \text{Hom}(V_2, V_1), \ i \mapsto i^*$$

is a $C$-linear isomorphism. Further if $V_3$ is a vector space with a nondegenerate bilinear form then for $i \in \text{Hom}(V_1, V_2), \ j \in \text{Hom}(V_2, V_3)$, we have $(ji)^* = i^*j^*$.

2.2. Anti-symmetric and symmetric forms. Let $V$ be a vector space of dimension $k$ with a nondegenerate form $(\cdot, \cdot)_V$. Let $\varepsilon \in \{-1, +1\}$. Let $(\cdot, \cdot)_\varepsilon$ be a nondegenerate bilinear form $(u, v)_\varepsilon = \varepsilon(v, u)_\varepsilon$ on $V$. If $\varepsilon = +1$ (resp. $-1$) then $(\cdot, \cdot)_\varepsilon$ is an orthogonal form (resp. symplectic form). We say $V$ is orthogonal (resp. symplectic) if $\varepsilon = +1$ (resp. $\varepsilon = -1$).

We decompose $\mathfrak{gl}(V) = t(V) \oplus p(V)$ as a vector space, where

$$\begin{align*}
    t(V) & := \{ X \in \mathfrak{gl}(V) | (Xu, v)_\varepsilon = -(u, Xv)_\varepsilon \} = \{ X \in \mathfrak{gl}(V) | X^* = -X \} \\
p(V) & := \{ X \in \mathfrak{gl}(V) | (Xu, v)_\varepsilon = (u, Xv)_\varepsilon \} = \{ X \in \mathfrak{gl}(V) | X^* = X \}.
\end{align*}$$

(2.1)

Let $t := t(V)$ and $p := p(V)$ for short. The followings are immediate to check:

$$[t, t] \subset t, \ [t, p] \subset p, \ [p, p] \subset t.$$

(2.2)

Let $G(V) := \{ g \in \text{GL}(V) | (gv, gv')_\varepsilon = (v, v')_\varepsilon \}$ for all $v, v' \in V$. Then $t = \mathfrak{g}(V)$, where $\mathfrak{g}(V) := \text{Lie}(G(V)$). We have $\dim G(V) = \dim t = \frac{k}{2}k(k - \varepsilon)$. So $\dim p = \frac{k}{2}k(k + \varepsilon).

Remark 2.1. If $V$ is a symplectic vector space of dimension 2 then $p(V)$ consists of scalars.

If $\varepsilon = -1$ then $G(V)$ is denoted by $\text{Sp}(V)$ (called the symplectic group) with the Lie algebra $\mathfrak{sp}(V)$. If $\varepsilon = +1$ then $G(V)$ is denoted by $\text{O}(V)$ (called the orthogonal group) with the Lie algebra $\mathfrak{o}(V)$. Unless no confusion arises, we shorten the notation as $\text{Sp}, \mathfrak{sp}, \text{O}$ and $\mathfrak{o}$.

2.3. It is direct to check that if $(\cdot, \cdot)_{V_1} = (\cdot, \cdot)_\varepsilon$ and $(\cdot, \cdot)_{V_2} = (\cdot, \cdot)_{-\varepsilon}$ for some $\varepsilon \in \{-1, +1\}$ then $** = -\text{Id}$. If both $(\cdot, \cdot)_{V_1}$ and $(\cdot, \cdot)_{V_2}$ are for the same $\varepsilon \in \{-1, +1\}$ then $** = \text{Id}$. In particular if $V$ is symplectic and $W$ is orthogonal then for $i \in \text{Hom}(W, V)$, $ii^* \in \mathfrak{sp}$ by (2.1) since $(ii^*)^* = i^*i^* = -ii^*$. Similarly $i^*i \in \mathfrak{o}$ for $i \in \text{Hom}(W, V)$.

2.4. Orbit-closedness and semisimplicity of quiver representations. Let

$$M := \text{Hom}(V, V)^{\oplus 2} \oplus \text{Hom}(V, W) \oplus \text{Hom}(W, V)$$

where $V$ and $W$ are vector spaces. An element of $M$ is called an ADHM quiver representation. Let $W \cong C^N$ be any isomorphism. Then we have an obvious (GL($V$)-equivariant) linear isomorphism

$$c : M \to \text{Hom}(V, V)^{\oplus 2} \oplus \text{Hom}(V, C)^{\oplus N} \oplus \text{Hom}(V, C)^{\oplus N}.$$
We identify the target space of $c$ with the space of representations of the deframed quiver given in Fig. 1 (cf. Crawley-Boevey’s trick [11, p.57]). The number of arrows from the bottom vertex to the top is $N$, and the number of arrows from the top vertex to the bottom is also $N$.

Let $\varepsilon \in \{-1, +1\}$. In the rest of paper we fix $V$ and $W$ as a $k$-dimensional vector space with $(\,,\,)_\varepsilon$ and an $N$-dimensional vector space with $(\,,\,)_{-\varepsilon}$ respectively. Let

$$\mathcal{N} := \{ (B_1, B_2, i, j) \in \mathcal{M} | B_1 = B_1^*, B_2 = B_2^*, j = i^* \}.$$ 

It is clear that $\mathcal{N}$ is isomorphic to the one we used in §1.3 by the projection.

We use the notation $\perp \ni$ for a direct sum of vector spaces orthogonal with respect to the given nondegenerate forms.

It is well-known that $x \in \mathcal{M}$ has a closed $\text{GL}(V)$-orbit if and only if $c(x)$ is semisimple (cf. [26]). We have the corresponding result for $\text{G}(V)$ as below, whose proof is left as an exercise for the readers.

**Theorem 2.2.** Let $x := (B_1, B_2, i, j) \in \mathcal{N}$. Then the following properties are equivalent.

(a) $c(x)$ is semisimple (i.e. the direct sum of simple quiver representations).

(b) there exists a decomposition

$$V = V^* \perp \bigoplus_a V_a \perp \bigoplus_b (V_b \oplus V'_b)$$

such that

1. $V_b$ and $V'_b$ are dual isotropic subspaces of $V$ for each index $b$;
2. $c(B_1|_{V^*}, B_1|_{V^*}, i, j|_{V^*})$, $(B_1|_{V_a}, B_2|_{V_a}, 0, 0)$, $(B_1|_{V_b}, B_2|_{V_b}, 0, 0)$ and $(B_1|_{V'_b}, B_2|_{V'_b}, 0, 0)$ are simple quiver representations.

(c) $\text{G}(V).x$ is closed in $\mathcal{N}$.

We remark that $(B_1|_{V_b}, B_2|_{V_b}, 0, 0)$ and $(B_1|_{V'_b}, B_2|_{V'_b}, 0, 0)$ in the above statement are dual to each other.

Let us recall the two GIT stability conditions for the $\text{GL}(V)$-actions on $\mathcal{M}$ (cf. [30, Chap. 2]).

**Definition 2.3.** $(B_1, B_2, i, j) \in \mathcal{M}$ is stable (resp. costable) if the following condition holds:
(1) \textit{(stability)} there is no subspace \( S \subseteq V \) such that \( B_1(S) \subset S, B_2(S) \subset S \) and \( \text{Im} i \subset S \),

(2) \textit{(costability)} there is no nonzero subspace \( T \subset V \) such that \( B_1(T) \subset T, B_2(T) \subset T \) and \( T \in \text{Ker}j \).

Note that \( x \in \mathbf{M} \) is stable and costable if and only if \( c(x) \) is simple when \( W \neq 0 \).

Let \( *_M : \mathbf{M} \rightarrow \mathbf{M}, (B_1, B_2, i, j) \mapsto (B_1^*, B_2^*, -j^*, i^*) \). Then \( \mathbf{N} \) is the \( *_M \)-invariant subspace of \( \mathbf{M} \). If \( x \in \mathbf{M} \) is stable (resp. costable) then \( *_M(x) \) is costable (resp. stable). Therefore stability and costability are equivalent on \( \mathbf{N} \). In particular, \( x \in \mathbf{N} \) is stable if and only if \( c(x) \) is simple when \( W \neq 0 \).

2.5. \textbf{ADHM description for framed \( \text{Sp} \)-bundles and \( \text{SO} \)-bundles.} We describe the framed \( \text{Sp} \)-bundles and \( \text{SO} \)-bundles in terms of ADHM data (cf. \cite{12}, see also \cite{8}). Let \( V := \mathbb{C}^k \) with \(( , )_{\varepsilon} \) and \( W := \mathbb{C}^N \) with \(( , )_{-\varepsilon} \), where \( \varepsilon \in \{-1, +1\} \) and \( N \geq 1 \). Let \( \mu : \mathbf{N} \rightarrow \mathfrak{g}(V) \) be the moment map given by \((B_1, B_2, i, j) \mapsto [B_1, B_2] + ij \).

Let \( \mu^{-1}(0)^{\text{reg}} \) be the (regular) locus of stable-costable quiver representations in \( \mu^{-1}(0) \). We have \( \mu^{-1}(0)^{\text{reg}} \subset \mu^{-1}(0)^{\text{sm}} \) since if \( x \in \mathbf{N} \) is stable then the differential \( d\mu_x \) is surjective. By Theorem 2.2 the image of \( \mu^{-1}(0)^{\text{reg}} \) of the GIT quotient map \( \mu^{-1}(0) \rightarrow \mu^{-1}(0) // G(V) \) is a Zariski open subset of \( \mu^{-1}(0) // G(V) \) and it is a \( G(V) \)-orbit space \( \mu^{-1}(0)^{\text{reg}} // G(V) \). In particular \( \mu^{-1}(0)^{\text{reg}} // G(V) \) is a smooth quasi-affine scheme. In fact it is also irreducible by \cite{6} Propositions 2.24 and 2.25.

\textbf{Remark 2.4.} By \cite{11} the scheme structures of \( \mathcal{M}_n^{\text{SO}(N, \mathbb{R})} \) and \( \mathcal{M}_n^{\text{USp}(N/2)} \) are defined as the quasi-affine scheme \( \mu^{-1}(0)^{\text{reg}} // G(V) \), where the relation between \( n \) and \( k \) are given there.

\textbf{Definition 2.5.} Let \( \varepsilon = -1 \) (resp. \( \varepsilon = +1 \)). We say \( x = (B_1, B_2, i, j) \in \mu^{-1}(0) \) is a \( \text{SO}-\text{datum} \) (resp. \( \text{Sp}-\text{datum} \)) if \([B_1, B_2] + ij = 0 \).

2.6. \textbf{Stratification.} Let \( \mu_k : \mathbf{N} \rightarrow \mathfrak{g}(V), (B_1, B_2, i, j) \mapsto [B_1, B_2] + ij \) (the holomorphic moment map). To emphasize \( k = \dim V \), we use \( G_k \) for \( G(V) \). It was denoted by \( G_k \) in \cite{11}. Let \( x := (B_1, B_2, i, j) \in \mu^{-1}(0)_k \). Assume \( G_k x \) is closed in \( \mathbf{N} \). By Theorem 2.2 \((B_1|_{V^*}, B_2|_{V^*}, i|_{V^*}, j|_{V^*}) \) corresponds to a framed vector bundle, and \((B_1|_{V_k}, B_2|_{V_k}, 0, 0) \) and \((B_1|_{V_k'}, B_2|_{V_k'}, 0, 0) \) are all commuting pairs in \( \mathfrak{p}(V) \). Let us focus on commuting pairs here. We simplify the situation: \( B_1, B_2 \in \mathfrak{p}(V) \) with \([B_1, B_2] = 0 \), where \( V \) is orthogonal or symplectic. By the semisimplicity (Theorem 2.2), \( B_1 \) and \( B_2 \) are simultaneously diagonalizable. Therefore all \( V_a, V_b, \) and \( V'_b \) of Theorem 2.2 (b) are 1-dimensional. When \( V \) is symplectic, \( V_a \) does not appear and \( B_1|_{V_a \oplus V'_b}, B_2|_{V_a \oplus V'_b} \) are scalars, as \( \mathfrak{p}(V) = \mathbb{C} \) if \( \dim V = 2 \). When \( V \) is orthogonal, the index set of \( b \) can be absorbed in the index set of \( a \). We have the summary as follows.

\textbf{Theorem 2.6.} Let \( S^n \mathbb{A}^2 \) be the \( n \)th symmetric product of \( \mathbb{A}^2 \).

(1) \textbf{Suppose} \( V \) \textbf{is symplectic.} Then there exists a canonical set-theoretical bijection

\[
\mu^{-1}_k(0) // G_k = \coprod_{0 \leq k' \leq k} \mu^{-1}_k(0)^{\text{reg}} // G_{k'} \times S^{k-k'} \mathbb{A}^2.
\]

(2) \textbf{Suppose} \( V \) \textbf{is orthogonal.} Then there exists a canonical set-theoretical bijection

\[
\mu^{-1}_k(0) // G_k = \coprod_{0 \leq k' \leq k} \mu^{-1}_k(0)^{\text{reg}} // G_{k'} \times S^{k-k'} \mathbb{A}^2.
\]
Note that this stratification is nothing but the one of the Uhlenbeck space except the case $G = \text{SO}(3)$ (cf. [13] p.158, §4.4.1]). For $G = \text{SO}(3)$, $\mu^{-1}(0)/G_k$ is different from the Uhlenbeck space, where the symmetric product is $S^{k-k'}\mathbb{A}^2$ by the formula (1.1).

Since $\mu_k^{-1}(0)\text{reg}/G_k$ is a free quotient (unless $\mu_k^{-1}(0)\text{reg} = \emptyset$), we have $\dim \mu_k^{-1}(0)\text{reg}/G_k = \dim N - 2\dim G_k$. Using $\dim p - \dim t = k$ if $V$ is orthogonal, and $\dim p - \dim t = -k$ if $V$ is symplectic, we have

\[(2.3) \quad \dim \mu_k^{-1}(0)\text{reg}/G_k = \begin{cases} k(N - 2) & \text{if } V \text{ is symplectic}, \\ k(N + 2) & \text{if } V \text{ is orthogonal}, \end{cases}\]

whenever $\mu_k^{-1}(0)\text{reg} \neq \emptyset$. Therefore we get the dimension of the strata. Hence by (1.3) we obtain the following.

**Lemma 2.7.** (1) Assume $V$ is symplectic. Then $\mu_k^{-1}(0)\text{reg}/G_k \times S^{k-k'}\mathbb{A}^2$ is nonempty if and only if either $N = 3$ and $k' \in 4\mathbb{Z}_{\geq 0}$, or $N \geq 4$ and $k' \in 2\mathbb{Z}_{\geq 0}$. If it is nonempty, it is of dimension $k'(N - 2) + (k - k')$

(2) If $V$ is orthogonal then $\mu_k^{-1}(0)\text{reg}/G_k \times S^{k-k'}\mathbb{A}^2$ is nonempty and of dimension $k'(N + 2) + 2(k - k')$. \(\square\)

**Remark 2.8.** (1) Suppose $V$ is symplectic. By $\text{Spin}(3) \cong \text{SU}(2)$, if $N = 3$ and $k \in 4\mathbb{Z}_{\geq 0}$ then $\mu_k^{-1}(0)\text{reg}/G_k$ is irreducible (1.3). Hence, the strata of $\mu^{-1}(0)/G_4$ are $\mu_k^{-1}(0)\text{reg}/G_k$ and $S^2\mathbb{A}^2$, both of which are irreducible varieties of dimension 4. See an alternative proof in Theorem 1.3 (3).

By $\text{Spin}(4) \cong \text{SU}(2) \times \text{SU}(2)$, if $N = 4$ and $k \in 2\mathbb{Z}_{\geq 0}$ then $\mu_k^{-1}(0)\text{reg}/G_k$ has the $(k/2 + 1)$ irreducible components.

(2) We will prove in Theorem 3.1 that all the strata in Lemma 2.7 (2) are irreducible normal.

### 3. Geometry of the moduli spaces of Sp-data

Let $V$ be an orthogonal vector space of dimension $k$, and $W$ be a symplectic vector space of dimension $N \geq 2$. Let $G := G(V) = \text{O}(V)$.

The main theorem of this section is the following.

**Theorem 3.1.** $\mu$ is normal, irreducible, flat, reduced and surjective. Hence $\mu^{-1}(0)$ is a normal irreducible variety of dimension $k(N + 2) + k(k - 1)/2$. And the Donaldson-Uhlenbeck space $\mu^{-1}(0)/G$ is a normal irreducible variety of dimension $k(N + 2)$, and $(\mu^{-1}(0)/G) \setminus (\mu^{-1}(0)\text{reg}/G)$ is of codimension $\geq 2$ in $\mu^{-1}(0)/G$.

The proof will appear in §3.2.

**Corollary 3.2.** The canonical embedding induces the equality of the ring of regular functions: $\mathbb{C}[\mu^{-1}(0)/G] = \mathbb{C}[\mu^{-1}(0)\text{reg}/G]$.

3.1. Let $m : p \times p \to t$ by $(B_1, B_2) \mapsto [B_1, B_2]$.

**Theorem 3.3.** For any $X \in t$, $m^{-1}(X)$ is an irreducible normal variety and a complete intersection in $p \times p$ of the $\dim t$ hypersurfaces. And the smooth locus of $m^{-1}(X)$ is the locus of $x \in m^{-1}(X)$ such that the differential $dm_x$ is surjective.
Proof. By [37, (3.5)(1)] and [37, Theorem 3.2], $m^{-1}(X)$ is an irreducible complete intersection. See also [7] for a different proof. The normality is due to [37, Corollary 4.4].

For $x \in m^{-1}(X)$, $x$ is a smooth point of $m^{-1}(X)$ if and only if $\dim T_x m^{-1}(X) = \dim m^{-1}(X) (= 2 \dim p - \dim t)$ if and only if $dm_x$ is surjective (cf. [37, the proof of Proposition 4.2]).

For $B \in \mathfrak{gl}$, $\mathfrak{gl}^B = \{ A \in \mathfrak{gl} | [A, B] = 0 \}$. Let $p_B := \mathfrak{gl}^B \cap p$.

Let $gl_t := \{ B \in gl | \dim gl^B = l \}$ and $p_t := \{ B \in p | \dim p^B = l \}$ ($l \in \mathbb{Z}_{\geq 0}$).

Lemma 3.4. ([22, Prop. 5]) For any $B \in p$, $\dim p^B - \dim t^B = k$. □

Lemma 3.5. For any $l \in \mathbb{Z}_{\geq 0}$, $p_{\geq l}$ is a closed subvariety of $p$. And $p_k$ is Zariski open dense in $p$.

Proof. By [29, p.7], for the conjugation action of $GL(V)$ on $\mathfrak{gl}$, the map $A \in \mathfrak{gl} \mapsto \dim GL(V)^A = \dim \mathfrak{gl}^A$ is upper-semicontinuous. Thus $\mathfrak{gl}_t$ is a locally closed subvariety of $\mathfrak{gl}$. By (2.2), for any $B \in p$, $\mathfrak{gl}^B = p^B \oplus t^B$. By Lemma 3.4, for $B \in p$, $B \in p_t$ if and only if $B \in \mathfrak{gl}_{2l-k}$. Since $p$ is a closed subvariety of $\mathfrak{gl}$, $p_t = p \cap \mathfrak{gl}_{2l-k}$ is a locally closed subvariety of $p$.

The second claim comes from the upper-semicontinuity, $p_k = p \cap \mathfrak{gl}_k \neq \emptyset$ and $\mathfrak{gl}_l = \emptyset$ for $l < k$. □

Let $\pi_i : p \times p \to p$ be the $i^{th}$ projection ($i = 1, 2$).

Note that for $X = 0$, the singular locus of $m^{-1}(0)$ is of codimension $\geq 3$ by [7, p.6414, Lemma 1 (cf. [37, Theorem (4.3)])].

Let $M$ be a smooth variety, and $\phi : M \to t$ be a morphism. The fibre-product $(p \times p) \times_t M$ is cut out by the $\dim t$-equations $[B_1, B_2] - \phi(x) = 0$.

Let $S := \{ x \in p \times p | dm_x \text{ is surjective} \}$.

Proposition 3.6. Let $M$ be a smooth irreducible variety and $\phi : M \to t$ be a morphism. Then $(p \times p) \times_t M$ is an irreducible normal variety and a complete intersection in $p \times p \times M$ of $\dim t$ hypersurfaces.

Proof. Let $\tilde{m} : (p \times p) \times_t M \to M$ be the projection. Note that $\tilde{m}$ is surjective since so is $m$ by Theorem 3.3. For each $x \in M$, $\tilde{m}^{-1}(x) \cong m^{-1}(\phi(x))$, which means any fibre dimension of $\tilde{m}$ is $2 \dim p - \dim t$. Therefore $\dim (p \times p) \times_t M = 2 \dim p - \dim t + \dim M$ which proves $(p \times p) \times_t M$ is a complete intersection.

The complete intersection property implies that every irreducible component of $(p \times p) \times_t M$ has the same dimension. Suppose $(p \times p) \times_t M$ is reducible. There are two distinct irreducible components $Z, Z'$. The restrictions to $Z, Z'$ of $\tilde{m}$ are dominant due to the equi-dimensionality of the fibers of $\tilde{m}$ and irreducibility of $M$. For a generic element $x \in M$, $\tilde{m}^{-1}(x) \cap Z$ and $\tilde{m}^{-1}(x) \cap Z'$ are distinct closed subschemes in $\tilde{m}^{-1}(x)$ with the same dimension $2 \dim p - \dim t$. But this contradicts the fact that $m^{-1}(\phi(x)) \cong \tilde{m}^{-1}(x))$ is an irreducible scheme with dimension $2 \dim p - \dim t$ (Theorem 3.3).

By the upper-semicontinuity, $S$ is an open subvariety of $p \times p$. By Theorem 3.3, $m^{-1}(X) \setminus S$ is of codimension $\geq 2$ in $m^{-1}(X)$ for each $X \in t$. Thus $(p \times p) \setminus S$ is of codimension $\geq 2$ in $p \times p$. By the smooth base change ([19, III.10.1 (b)]), $\tilde{m}_{S \times_t M}$ is a smooth morphism.
By [19, III.10.4], $S \times_1 M$ is smooth. By Serre’s criterion ([19, Proposition II.8.23]), we get the normality.  

We have further description of the smooth locus of each fibre of $m$ as follows, which is not necessary in the proof of Theorem 3.1.

**Proposition 3.7.** Let $X \in t$.

1. $m\colon m^{-1}(X) \cap (\pi_1^{-1}(p_k) \cup \pi_2^{-1}(p_k)) \to t$ is a smooth morphism.
2. The codimension of $m^{-1}(X) \setminus (\pi_1^{-1}(p_k) \cup \pi_2^{-1}(p_k))$ in $m^{-1}(X)$ is larger than 1.

**Proof.** (1) Let $L_B: p \to t$ by $A \mapsto [B, A]$, where $B \in p$. If $B \in p_k$ then $L_B$ is surjective since $\dim t + k = \dim p$. Since for $(B_1, B_2) \in (\pi_1^{-1}(p_k) \cup \pi_2^{-1}(p_k))$, the differential $dm_{(B_1, B_2)}: p \times p \to t$ maps $(A_1, A_2)$ to $[B_1, A_2] + [A_1, B_2]$. Since $t = \text{Im}L_{B_1} + \text{Im}L_{B_2} \subset \text{Im}m_{(B_1, B_2)}$, $dm_{(B_1, B_2)}$ is surjective. This proves (1).

(2) will be proven in Appendix C. 

**Remark 3.8.** Let $\tilde{M} := (p \times p) \times_1 M$, $\tilde{S} := S \times_1 M$ and $\tilde{S}' := (\pi_1^{-1}(p_k) \cup \pi_2^{-1}(p_k)) \times_1 M$. Then $\tilde{S}' \subset \tilde{S} \subset \tilde{M}$. By Proposition 3.6 $\tilde{S}$ is a smooth variety such that $\tilde{M} \setminus \tilde{S}$ is of codimension $\geq 2$ in $\tilde{M}$. In fact, we can strengthen the result further as follows: $\tilde{S}'$ is a smooth variety such that $\tilde{M} \setminus \tilde{S}'$ is of codimension $\geq 2$. By Proposition 3.7 (1) and the smooth base change, $\tilde{S}'$ is a smooth variety. By Proposition 3.7 (2), $(p \times p) \setminus (\pi_1^{-1}(p_k) \cup \pi_2^{-1}(p_k))$ is of codimension $\geq 2$ in $p \times p$. Thus $\tilde{M} \setminus \tilde{S}'$ is of codimension $\geq 2$ in $\tilde{M}$.

3.2. **The proof of Theorem 3.1.** Let $M := \text{Hom}(W, V)$ and $\phi: M \to t$, $i \mapsto -\iota^*$. By Proposition 3.6 $\mu^{-1}(0) = (p \times p) \times_1 M$ is an irreducible normal variety and a complete intersection in $p \times p \times M$. By the method of associated cones [23, II.4.2], $\mu$ is normal, irreducible, flat and reduced. Since $m$ is surjective, so is $\mu$.

By [29, p.5], $\mu^{-1}(0)/G$ is an irreducible normal variety. If one can show $\mu^{-1}(0)^{\text{reg}} \neq \emptyset$ then by Lemma 2.7 (2), $\mu^{-1}(0)^{\text{reg}}/G$ is of the complement codimension $\geq 2$, since $N \geq 2$.

It remains to show $\mu^{-1}(0)^{\text{reg}} \neq \emptyset$. Let $i \in \text{Hom}(W, V) \setminus 0$. Let $X := -\iota^*$. Let $B_1 \in p_k$. Let $a_n$ be the eigenvalues of $B_1$ ($n = 1, 2, ..., k$). Let $V_{a_n}$ be the $a_n$-eigenspace. Let $p_n: V \to V_{a_n}$ be the projection. We may take $B_1$ so that $p_n(\text{Im}(i)) \neq 0$ for all $n$.

By Theorem 3.3 for any $B_1 \in p_k$, there exists $B_2 \in p$ such that $[B_1, B_2] = X$. Now, $(B_1, B_2, i, \iota^*) \in \mu^{-1}(0)^{\text{reg}}$, since $V = \sum P(B_1)i(W)$, where $P$ runs over one-variable polynomials.

4. **Geometry of the moduli spaces of SO-data: statements**

We assume that $V$ is symplectic and $W$ is orthogonal in the rest of the paper, except [5.1] where preliminaries from linear algebra will be given. Let $k = \dim V$ and $N = \dim W$.

If $k = 0$ then we have $N = 0$. If $N = 1$ and $k \geq 2$, $\mu^{-1}(0)^{\text{reg}} = \emptyset$ as the right hand side of (2.3) is negative. We will study the next simplest cases, $k = 2$ (Theorem 4.1), $N = 2$ (Theorem 4.2) and $(k, N) = (4, 3)$ (Theorem 4.3). $\mu$ is not flat nor irreducible in general unlike the case of the moduli spaces of Sp-data.

In the following theorems, $\mathcal{P}$ denotes the minimal nilpotent $O(W)$-orbit closure in $\mathfrak{o}(W)$. If $N = 3$ or $N \geq 5$ then $\mathcal{P}$ is irreducible and normal. When $N = 4$, $\mathcal{P}$ has two irreducible components, which are isomorphic by the action of an element in $O(W) \setminus \text{SO}(W)$. Each irreducible component is the closure of a $\text{SO}(W)$-orbit. See [3, Theorems 5.1.4 and 5.1.6].
Theorem 4.1. Let $k = 2$. Then the following properties are verified.
(1) $\mu^{-1}(0)$ is isomorphic to $\mathbb{C}^2 \times \{ i \in \text{Hom}(W, V) | ii^* = 0 \}$ by $(B_1, B_2, i, i^*) \mapsto (\text{tr}B_1, \text{tr}B_2, i)$.
(2) If $N \geq 3$, $\mu$ is flat.
(3) If $N \geq 5$, $\mu^{-1}(0)$ is irreducible and normal. Hence, $\mu$ is irreducible and normal. If $N = 4$, $\mu^{-1}(0)$ is a reduced scheme and a union of two isomorphic irreducible components. If $N = 3$, it is irreducible, but non-reduced.
(4) If $N \leq 3$, $\mu^{-1}(0)_{\text{reg}} = \emptyset$. If $N \geq 4$, $\mu^{-1}(0)_{\text{reg}} = \mu^{-1}(0)_{\text{sm}}$.
(5) If $N \leq 3$, the reduced scheme $\mu^{-1}(0)/\text{Sp}(V)_{\text{red}}$ is isomorphic to $\mathbb{C}^2$.
(6) If $N \geq 4$, $\mu^{-1}(0)/\text{Sp}(V)$ is isomorphic to $\mathbb{C}^2 \times P$. Moreover the isomorphism restricts to $\mu^{-1}(0)_{\text{reg}}/\text{Sp}(V) \cong \mathbb{C}^2 \times (P \setminus 0)$.

Theorem 4.2. (1) Let $N = 2$ and $k \geq 2$. Then $\mu^{-1}(0)$ is not a complete intersection. Hence $\mu$ is not flat.
(2) Let $N = 2$ and $k = 4$. Then $\mu^{-1}(0)_{\text{reg}} = \emptyset$.

It is true that $\mu^{-1}(0)_{\text{reg}} = \emptyset$ for $N = 2$ and any $k \geq 2$, as there is no nontrivial $\text{SO}(2)$-instantons. But the author does not find its proof in terms of the ADHM description.

Theorem 4.3. Let $N = 3$ and $k = 4$. Then we have the following assertions.
(1) $\mu^{-1}(0)$ is a reduced complete intersection and a union of two irreducible components. Hence $\mu$ is flat.
(2) One irreducible component of $\mu^{-1}(0)$ is the closure of $\mu^{-1}(0)_{\text{reg}}$.
(3) $\mu^{-1}(0)/\text{Sp}(V)$ is isomorphic to $\mathbb{C}^2 \times (P \cup_0 P)$. Moreover the isomorphism restricts to $\mu^{-1}(0)_{\text{reg}}/\text{Sp}(V) \cong \mathbb{C}^2 \times (P \setminus 0)$.

Here, $P \cup_0 P := (P \times \{0\}) \cup (\{0\} \times P)$ in $P \times P$. The proofs will appear in the subsequent sections.

Note that the first statement of Theorem 4.1 (4), Theorem 4.2 (2) and the second statement of Theorem 4.3 (3) are either obvious or well-known in the context of instantons on $S^4$ in §1.3 (See also Remark 2.8).

The properties of the moment map $\mu$ in Theorem 4.1 (2), (3), Theorem 4.2 (1) and Theorem 4.3 (1) follow from the corresponding properties of $\mu^{-1}(0)$ by the method of associated cones [23, II.4.2], as in the proof of Theorem 3.1.

The author is planning to study on flatness of $\mu$ for $N \geq 4$ and normality of $\mu$ for $N \geq 5$ in the near future. At least the following is true for small $k$ for a fixed $N$.

Remark 4.4. It is known by [25, Remark 11.3] that $\rho: \text{Hom}(W, V) \to \mathfrak{sp}(V), i \mapsto ii^*$, is flat if $N \geq 2k$. Moreover $\rho$ is normal and irreducible if $N \geq 2k + 1$. By the base change argument used in the proof of Proposition 3.6, the same is true for $\mu$.

5. Kraft-Procesi’s classification theory of nilpotent pairs

In this section we review some geometry of $\text{Hom}(W, V)$ following Kraft-Procesi [25], which will be used in the proofs of the theorems from the previous section.

5.1. Generalized eigenspaces and bilinear forms. Let $V$ be a vector space with $(, ) := (, )_\varepsilon$, where $\varepsilon \in \{-1, +1\}$. 
Definition 5.1. Let $X \in \mathfrak{gl}(V)$. Let $V^n_a := \text{Ker}(X - a\text{Id})^n$ for $n \geq 0$. We call $V_a := \bigcup_{n \geq 1} V^n_a$ the generalized $a$-eigenspace of $X$.

Lemma 5.2. (1) If $X \in \mathfrak{p}(V)$ and $a \neq b$ then $V_a \perp V_b$.
(2) If $X \in \mathfrak{t}(V)$ and $a \neq -b$ then $V_a \perp V_b$.

Proof. (1) Let $m, n \geq 0$. Let $v \in V^n_a$ and $w \in V^m_b$. We will show $(v, w) = 0$. We use the induction on $m, n$. If $m$ or $n = 0$, the assertion is obvious. Suppose the assertion is true for $m - 1, n$ and $m, n - 1$. Let $v' := (X - a)v \in V^{m-1}_a$ and $w' := (X - a)w \in V^{n-1}_b$. We have $(Xv, w) = a(v, w)$ as $(v', w) = 0$ by our assumption. Similarly we also have $(v, Xw) = b(v, w)$. Thus we have $a(v, w) = b(v, w)$ which asserts $(v, w) = 0$.
(2) The proof is same if one uses $(Xv, w) = -(v, Xw)$.

Let $X \in \mathfrak{t}(V)$ or $\mathfrak{p}(V)$. Let us define a bilinear form $| \cdot, \cdot |$ on $\text{Im}(X)$ by $|Xv, Xv'| := (v, Xv')$ (cf. [25, §4.1]).

Lemma 5.3. (1) If $X \in \mathfrak{t}(V)$ then $| \cdot, \cdot |$ on $\text{Im}(X)$ is a nondegenerate bilinear form of type $-\varepsilon$.
(2) If $X \in \mathfrak{p}(V)$ then $| \cdot, \cdot |$ on $\text{Im}(X)$ is a nondegenerate bilinear form of type $\varepsilon$.

Proof. (1) is obvious, as noted in [25, §4.1]. (2) is also clear.

Proposition 5.4. Let $V$ be a 4-dimensional symplectic vector space. Let $X \in \mathfrak{p}(V)$. Then $X$ has an eigenspace of dimension $\geq 2$.

Proof. By Lemma 5.2 either $V = V_a \oplus V_b$ for some $a \neq b$, or $V = V_a$ for some $a$. In the first case, $V_a$ and $V_b$ are 2-dimensional symplectic subspaces of $V$ and thus $X|_{V_a} = a$ and $X|_{V_b} = b$ by Remark 2.1. In the second case, by Lemma 5.3, $\text{Im}(X - a)$ is a symplectic subspace of dimension 0 or 2. If $\dim \text{Im}(X - a) = 0$ then $X = a$. If $\dim \text{Im}(X - a) = 2$ then $X|_{\text{Im}(X - a)} = a$ by Remark 2.1 as $X \in \mathfrak{p}(\text{Im}(X - a))$. We are done.

Let $W$ be a vector space with $(\cdot, \cdot)_{-\varepsilon}$. Recall that for $i \in \text{Hom}(W, V)$, we have $ii^* \in \mathfrak{t}(V)$ and $i^*i \in \mathfrak{t}(W)$ (4.2). Let $V_a$ (resp. $W_a$) be the generalized $a$-eigenspace of $ii^*$ (resp. $i^*i$).

Lemma 5.5. We have $i^*(V_a) \subset W_a$ and $i(W_a) \subset V_a$. Moreover if $a \neq 0$ then $i$ and $i^*$ are isomorphisms between $V_a$ and $W_a$.

Proof. For any $a \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 0}$, we have $i^*(ii^* - a)^n = (i^*i - a)^n i^*$ and $(ii^* - a)^n i = i(i^*i - a)^n$. If $v \in V_a$ then $(i^*i - a)^n i^* v = (i^*i - a)^n v = 0$ for $n \gg 0$. Similarly, if $w \in W_a$ then $(ii^* - a)^n i w = (i^*i - a)^n w = 0$ for $n \gg 0$. Therefore $i^*(V_a) \subset W_a$ and $i(W_a) \subset V_a$.
This proves the first claim.

The restriction of $ii^*$ to $V_a$ is $a\text{Id}_{V_a}$ plus a nilpotent endomorphism of $V_a$. So it is an isomorphism of $V_a$ if $a \neq 0$. Similarly $i^*i$ gives an isomorphism of $W_a$. Therefore $i$ and $i^*$ give isomorphism between $V_a$ and $W_a$.

5.2. Kraft-Procesi's results on nilpotent endomorphisms in $\mathfrak{t}$. Let $\text{Sp} := \text{Sp}(V), O := O(W), \mathfrak{sp} := \mathfrak{sp}(V)$ and $\mathfrak{o} := \mathfrak{o}(W)$, for short. Let $X$ and $Y$ be nilpotent elements of $\mathfrak{sp}(V)$ and $\mathfrak{o}(V)$ respectively.

A nilpotent orbit $\text{Sp}.X$ corresponds uniquely to a partition $\eta = (\eta_1, \eta_2, \ldots)$ such that its transpose $(\tilde{\eta}_1, \tilde{\eta}_2, \ldots)$ is given by $\tilde{\eta}_n := \dim \text{Ker}(X^n)/\text{Ker}(X^{n-1})$. We denote $\eta$ by $\eta_X$. 
We represent \( \eta \) by the Young diagram with the boxes replaced by \( b \) (\( b \)-diagram). For instance, a partition \( (5,3,3,2,2,0,0,...) \) is represented by the \( b \)-diagram:

\[
\begin{array}{ccccccc}
  & b & b & b & b & b \\
  & b & b & b \\
  & b & b \\
  & b & b \\
\end{array}
\]

For a nilpotent orbit \( O.Y \), we use the symbol \( a \) instead of \( b \), and the Young diagram is called an \( a \)-diagram.

**Remark 5.6.** By [39, IV 2.15], for \( \eta_X = (\eta_1, \eta_2, ...) \) (resp. \( \eta_Y = (\eta_1, \eta_2, ...) \)), \( \# \{ n \mid \eta_n = m \} \) is even for any odd \( m \) (resp. for any even \( m \)).

**Proposition 5.7.** ([25, Proposition 2.4]) For \( \eta_X = \eta := (\eta_1, \eta_2, ...) \),
\[
\dim \text{Sp}.X = \frac{1}{2} \left( |\eta|^2 + |\eta| - \sum \hat{\eta}_n^2 - \# \{ n \mid \eta_n \text{ is odd} \} \right).
\]

For \( \eta_Y = \eta := (\eta_1, \eta_2, ...) \),
\[
\dim \text{O}.Y = \frac{1}{2} \left( |\eta|^2 - |\eta| - \sum \hat{\eta}_n^2 + \# \{ n \mid \eta_n \text{ is odd} \} \right).
\]

We say \( \sigma \geq \eta \) for two partitions \( \sigma = (\sigma_1, \sigma_2, ...) \) and \( \eta = (\eta_1, \eta_2, ...) \) with \( |\sigma| = |\eta| \) if \( \sum_{1 \leq i \leq j} \sigma_i \geq \sum_{1 \leq i \leq j} \eta_i \) for any \( j \geq 1 \) (dominance order [28, §I.1])

**Proposition 5.8.** ([20, Theorem 3.10]) Let \( X \) and \( X' \) be nilpotent elements in \( \mathfrak{sp} \). Then \( \eta_X \geq \eta_{X'} \) if and only if \( \text{Sp}.X' \subset \text{Sp}.X \). The same is true for \( \mathfrak{o} \).

**Corollary 5.9.** (1) If \( \dim W \leq 2 \), there is no nonzero nilpotent orbit in \( \mathfrak{o} \).

(2) If \( \dim W = 3 \), there exists a unique nonzero nilpotent orbit, and it is given by the \( a \)-diagram \( aaa \).

(3) If \( \dim W \geq 4 \), the minimal nilpotent orbit is given by the \( a \)-diagram
\[
\begin{array}{ccc}
  a \\
  \vdots \\
  a \\
\end{array}
\]

**Proof.** These assertions follow from Remark 5.6 and Proposition 5.8.

Let us define two maps from \( \text{Hom}(W,V) \)
\[
\pi: \text{Hom}(W,V) \to \mathfrak{o}, \quad i \mapsto i^*i,
\]
\[
\rho: \text{Hom}(W,V) \to \mathfrak{sp}, \quad i \mapsto ii^*.
\]

**Theorem 5.10.** ([25, Theorem 1.2]) \( \pi \) and \( \rho \) are the GIT quotient maps onto the images, by \( \text{Sp} \) and \( O \) respectively.

We review Kraft-Procesi’s classification theory of nilpotent pairs in [24] and [25]. A pair \((i,j) \in \text{Hom}(W,V) \times \text{Hom}(V,W)\) is a nilpotent pair if \( ij \) is a nilpotent endomorphism. As in the case of nilpotent endomorphisms, a Young diagram plays an important role in the classification of the \( \text{GL}(V) \times \text{GL}(W) \)-orbits of nilpotent pairs.
Definition 5.11. ([24, §§4.2–4.3]) By an ab-diagram, we mean a Young diagram whose rows consists of alternating $a$ and $b$. E.g.,

\[
\begin{array}{ccccccc}
   & a & b & a & b & a & b \\
a & b & a & b & a & b & \\
b & a & & & & & \\
& & & & & & \\
\end{array}
\]

(5.2)

An ab-diagram $A$ gives a nilpotent pair as follows. Suppose the number of $a$ (resp. $b$) in $A$ is $k$ (resp. $N$). Let us take any basis $\{b_1, b_2, \ldots, b_k\}$ of $V$ (resp. $\{a_1, a_2, \ldots, a_N\}$ of $W$). We replace all the $a$ and $b$ in $A$ by $a_i$ and $b_i$. We define a nilpotent pair $(i, j)$ such that $i$ maps $a_m$ to $b_n$ in the right adjacent position or 0 if there is no such $b_n$ and $j$ maps $b_m$ to $a_n$ or 0 similarly. In the above example of ab-diagram, we have a nilpotent pair as follows:

\[
\begin{array}{ccccccc}
   & a & b & a & b & a & b \\
a & b & a & b & a & b & \\
b & a & & & & & \\
& & & & & & \\
\end{array}
\]

(5.3)

In the above correspondence, an ab-diagram does not determine uniquely a nilpotent pair as the bases of $V$ and $W$ can be changed. Therefore up to the change of the bases, we have the bijective correspondence ([24, §4.3])

\[
\{\text{ab-diagram with } \#a = \dim V \text{ and } \#b = \dim W\} \
\to \{(\text{GL}(V) \times \text{GL}(W)).(i, j) | (i, j) \text{ is a nilpotent pair}\}.
\]

Suppose $(i, i^*)$ is a nilpotent pair. From the ab-diagram of $(i, i^*)$, the a-diagram of $i^*i$ and the b-diagram of $ii^*$ are obtained by removing $b$ and $a$ respectively. For example,

\[
\begin{array}{ccc}
   bb & ababa & aaa \\
b & aba & aa \\
b & ab & aa \\
b & ba & a \\
b & & a \\
\end{array}
\]

For an ab-diagram we define

(5.4)

\[
\Delta_{ab} := \sum_{n: \text{odd}} (\#\text{rows of length } n \text{ starting with } a) \cdot (\#\text{rows of length } n \text{ starting with } b).
\]

Theorem 5.12. ([25, Theorem 6.5 and Proposition 7.1]) Let $i \in \text{Hom}(W, V)$.

1. The orbits $(\text{Sp} \times O).i$ such that $X := ii^*$ and $Y := i^*i$ are nilpotent, are in 1-1 correspondence with the ab-diagrams whose rows are of one of the types $\alpha, \beta, \gamma, \delta, \epsilon$ from Table [I] with $\#a = \dim W$, $\#b = \dim V$.
2. For the ab-diagram associated to $i$,

(5.5)

\[\dim(\text{Sp} \times O).i = \frac{1}{2}(\dim \text{Sp}.X + \dim O.Y + \dim V. \dim W - \Delta_{ab}).\]
2.1. Thus (where the left-most one actually occurs only when (6.1)\[\text{dim}(\text{Sp} \times 1) = 4\] and \(\text{dim}(\text{Sp} \times O)\).

This section contains the proof of Theorem 4.1.

Let \(V = k = 2\) and \(W = N \geq 2\). Then \(\mathfrak{p} := \mathfrak{p}(V)\) consists of scalars by Remark 2.1. Thus \((B_1, B_2, i, i^*) \in \mu^{-1}(0)\) implies \([B_1, B_2] = ii^* = 0\), and \(\mu^{-1}(0) \cong \mathbb{C}^2 \times \rho^{-1}(0)\).

By the method of associated cones [23, II.4.2], \(\ker(\rho^{-1}(0))\) is normal and irreducible. This proves (1).

Let \(i \in \rho^{-1}(0)\). Since \(ii^* = 0\), \(b\) cannot appear twice in the same row in the ab-diagram of \(i\). Looking at Table 1, we find that the ab-diagram of \(i\) is one of the following:

\[
\begin{align*}
\text{Type} & \quad \alpha_n & \quad \beta_n & \quad \gamma_n & \quad \delta_n & \quad \epsilon_n \\
\text{ab-diagram} & \quad aba \cdots ba & \quad bab \cdots ab & \quad aba \cdots ba & \quad bab \cdots ab & \quad aba \cdots ab \\
\text{\#a} & \quad 2n + 1 & \quad 2n - 1 & \quad 2(n + 1) & \quad 2n & \quad 2n \\
\text{\#b} & \quad 2n & \quad 2n & \quad 2n & \quad 2(n + 1) & \quad 2n \\
\end{align*}
\]

6. Moduli spaces of SO\(^{(N)}\)-data with \(N \geq 2\) and \(k = 2\)

This section contains the proof of Theorem 4.1.

Let \(\dim V = k = 2\) and \(\dim W = N \geq 2\). Then \(\mathfrak{p} := \mathfrak{p}(V)\) consists of scalars by Remark 2.1. Thus \((B_1, B_2, i, i^*) \in \mu^{-1}(0)\) implies \([B_1, B_2] = ii^* = 0\), and \(\mu^{-1}(0) \cong \mathbb{C}^2 \times \rho^{-1}(0)\) by (6.1). By (6.1) any \(\rho^{-1}(0)\) is normal and irreducible. This proves (1).

Let \(i \in \rho^{-1}(0)\). Since \(ii^* = 0\), \(b\) cannot appear twice in the same row in the ab-diagram of \(i\). Looking at Table 1, we find that the ab-diagram of \(i\) is one of the following:

\[
\begin{align*}
\text{Type} & \quad \alpha_n & \quad \beta_n & \quad \gamma_n & \quad \delta_n & \quad \epsilon_n \\
\text{ab-diagram} & \quad aba \cdots ba & \quad bab \cdots ab & \quad aba \cdots ba & \quad bab \cdots ab & \quad aba \cdots ab \\
\text{\#a} & \quad 2n + 1 & \quad 2n - 1 & \quad 2(n + 1) & \quad 2n & \quad 2n \\
\text{\#b} & \quad 2n & \quad 2n & \quad 2n & \quad 2(n + 1) & \quad 2n \\
\end{align*}
\]

where the left-most one actually occurs only when \(N \geq 4\). We denote elements of \(\text{Hom}(W, V)\) corresponding to the above ab-diagrams by \(i_1, i_2\) and \(i_3(= 0)\) respectively, where \(i_1\) does not exist unless \(N \geq 4\). By Theorem 5.12, \(\text{dim}(\text{Sp} \times O)\).

Suppose \(N = 3\). Then \(O = \text{SO} \sqcup \text{SO} \sqcup \text{SO} \sqcup \text{SO}\). Since \(-\text{Id}_V \in \text{Sp}\) and \(-\text{Id}_W \cdot i = -\text{Id}_V \cdot i\) for any \(i \in \text{Hom}(W, V)\), \(\text{Sp} \times O)\).

If \(N = 4\) then by [25, §11.3], \(\rho^{-1}(0)\) is reduced and consists of two irreducible components which are isomorphic by the action of an element of \(O \setminus \text{SO}\).

If \(N \geq 5\), [25, §11.3] asserts \(\rho^{-1}(0)\) is normal and irreducible. This proves (3).

Let \(x := (B_1, B_2, i, i^*) \in \mu^{-1}(0)\). By the ab-diagrams in (6.1), \(\ker(i_2^*)\) and \(\ker(i_3^*)\) are nonempty. Any nonzero vector in \(\ker(i_2^*)\) or \(\ker(i_3^*)\) is a common eigenvector of scalars \(B_1\) and \(B_2\). Thus, if \(i = i_2\) or \(i_3\) then \(x\) is not costable. Since \(\ker(i_1^*) = 0\), if \(i = i_1\) then \(x\) is costable. So \(\mu^{-1}(0)\) is costable if \(N \leq 3\). If \(N \geq 4\), \(\mu^{-1}(0) = 0\) if \(N \leq 3\) and \(\mu^{-1}(0) = (\text{Sp} \times O)\).

We describe the smooth locus of \(\mu^{-1}(0) = \mathbb{C}^2 \times \rho^{-1}(0)\) when \(N \geq 4\). Since \(T_x \rho^{-1}(0) = \ker(dp_x)\) and \(\dim \rho^{-1}(0) = \dim \text{Hom}(W, V) - \dim \text{Sp}\), \(\rho^{-1}(0)\) is the locus of \(x\) such that \(dp_x\) is surjective. Thus it is the locus of \(x\) such that \(\rho\) is smooth at \(x\). By [25, Remark
This proves (4).

Since Sp acts trivially on \( \mathfrak{p} \), we have \( \mu^{-1}(0)/\text{Sp} \cong \mathbb{C}^2 \times (\rho^{-1}(0)/\text{Sp}) \). By Theorem 5.10 and the \( ab \)-diagrams in (6.1), the reduced scheme
\[
\rho^{-1}(0)/\text{Sp}_{\text{red}} \cong \begin{cases} 
0 & \text{if } N \leq 3 \\
\mathbb{P} & \text{if } N \geq 4.
\end{cases}
\]
This proves the statement on \( \mu^{-1}(0)/\text{Sp} \) in (5) and (6). By Theorem 5.10 and the \( a \)-diagrams coming from the \( ab \)-diagrams in (6.1), we have \( (\text{Sp} \times O).i_2/\text{Sp} = 0 \). This proves the second claim of (6).

7. Moduli spaces of SO(2)-data

This section contains the proof of Theorem 4.2 (1). The proof of Theorem 4.2 (2) will appear in [9].

Let \( \dim V = k \geq 2 \) and \( \dim W = N = 2 \). Let \( m: \mathfrak{p} \times \mathfrak{p} \to \mathfrak{t}, (B, B') \mapsto [B, B'] \).

We will show there exists a subvariety \( X \) in \( \mu^{-1}(0) \) of dimension \( > 2 \dim \mathfrak{p} - \dim \mathfrak{t} + \dim \text{Hom}(W, V) \). Let \( X := \{(B, B', i, i^*)| [B, B'] = 0 = i^*\} = \mu^{-1}(0) \times \rho^{-1}(0) \subset \mu^{-1}(0) \).

Let us estimate \( \dim m^{-1}(0) \). Let \( \mathfrak{p}^{(e)} := \{B \in \mathfrak{p}| B \text{ has distinct } e \text{ eigenvalues}\} \). As shown in Appendix [C], \( \mathfrak{p}^{(e)} \) is a closed subvariety of \( \mathfrak{p} \). By Lemma 5.2 (1), if \( e > k/2 \), \( \mathfrak{p}^{(e)} = \emptyset \). With respect to a symplectic basis of \( V \), \( \text{diag}(a_1, a_2, ..., a_{k/2}) \in \mathfrak{p}^{(k/2)} \), where \( a_1, a_2, ..., a_{k/2} \) are all distinct. Therefore \( \mathfrak{p}^{(k/2)} \) is a Zariski dense open subset of \( \mathfrak{p} \). Let \( p: m^{-1}(0) \to \mathfrak{p} \) be the first projection. It is clear that \( p \) is surjective. And \( p^{-1}(B_0) \cong \mathfrak{p}^{B_0} \) for any \( B_0 \in \mathfrak{p} \) (see [3.1] for the notation). If \( B_0 \in \mathfrak{p}^{(k/2)} \) then we claim \( \dim \mathfrak{p}^{B_0} = k/2 \).

We have the (2-dimensional) eigenspace decomposition \( V = \bigoplus_{i=1}^{k/2} V_{a_i} \) of \( B_0 \). Since \( \mathfrak{gl}^{B_0} = \bigoplus_{i=1}^{k/2} \mathfrak{gl}(V_{a_i}) \) we have \( \mathfrak{p}^{B_0} = \bigoplus_{i=1}^{k/2} \mathfrak{p}(V_{a_i}) = \bigoplus_{i=1}^{k/2} \mathbb{C} \).

By the claim we obtain an estimate:
\[
\dim m^{-1}(0) \geq \frac{k}{2} + \dim \mathfrak{p}.
\]

Let us compute \( \dim \rho^{-1}(0) \). Let \( i \in \rho^{-1}(0) \). Since \( \dim W = 2 \) and \( i^*i \) is nilpotent, \( i^*i = 0 \) (Corollary 5.9 (1)). By an argument similar to the one we used in [6], the \( ab \)-diagram of any nonzero \( i \) is
\[
ab \\
ba \\
b \\
\vdots \\
b.
\]
Thus \( \rho^{-1}(0) \) is the Zariski closure of \( (\text{Sp} \times O).i \) and \( \dim \rho^{-1}(0) = k \) by Theorem 5.12 (2).

To sum up, \( \dim X \geq \frac{3}{2}k + \dim \mathfrak{p} > 2 \dim \mathfrak{p} - \dim \mathfrak{t} + 2k \) because \( \dim \mathfrak{t} - \dim \mathfrak{p} = k \), which means \( \mu^{-1}(0) \) is not a complete intersection in \( \mathbb{N} \). ☐

8. Moduli spaces of SO(3)-data with \( k = 4 \)

This section contains the proof of Theorem 4.3.

Let \( \dim V = k = 4 \) and \( \dim W = N = 3 \). Let \( \text{SO} := \text{SO}(W) \) for short.
8.1. **Description of $\mu^{-1}(0)$.** Let $p' := \{D \in p | \text{tr}D = 0\}$. Let $\tilde{X} := \{(B_1, B_2, i, i^*) \in \mu^{-1}(0) | B_1, B_2 \in p'\}$. Then

$$
\mu^{-1}(0) \cong \mathbb{C}^2 \times \tilde{X},
$$

$$(B_1, B_2, i, i^*) \mapsto \left((\text{tr}(B_1), \text{tr}(B_2)), (B_1 - \frac{1}{4} \text{tr}(B_1)\text{Id}_V, B_2 - \frac{1}{4} \text{tr}(B_2)\text{Id}_V, i, i^*)\right).$$

Let $N' := p^{{\oplus}2} \oplus \{(i, i^*) | i \in \text{Hom}(W, V)\}$. Then $\tilde{X}$ is defined by $\mu = 0$ in $N'$. Therefore it is enough to show the corresponding statements for $\tilde{X}$ to prove Theorem 4.3.

Let $\tilde{X}_1 := \{(B_1, B_2, i, i^*) \in \tilde{X} | [B_1, B_2] = 0\}$. Then $\tilde{X}_1 = \{(B_1, B_2) \in p' \times p' | [B_1, B_2] = 0\} \times \rho^{-1}(0)$.

**Lemma 8.1.** $\rho^{-1}(0)$ consists of two irreducible strata which are $(\text{Sp} \times \text{O})$-orbits of dimension 5 and 0 respectively. Hence $\rho^{-1}(0)$, as a cone, is irreducible.

**Proof.** By a similar argument as in §6 the possible $ab$-diagrams of $i$ of $\rho^{-1}(0)$ are

$$
\begin{array}{c|c}
ab & b \\
ba & b \\
b & a \\
a & a
\end{array}
$$

(8.1)

By Theorem 5.12, $\rho^{-1}(0)$ consists of two $(\text{Sp} \times \text{O})$-orbits associated to the above $ab$-diagrams of dimension 5 and 0 respectively.

Since $-\text{Id}_W \in O \setminus \text{SO}$ and $-\text{Id}_V \in \text{Sp}$, $(\text{Sp} \times \text{O}).i = (\text{Sp} \times \text{SO}).i$ for any $i \in \text{Hom}(W, V)$. So we have irreducibility.

Let $e_1, e_2, e_3, e_4$ be a basis of $V$ such that $(e_1, e_2)_V = (e_3, e_4)_V = 1$ and $(e_l, e_m)_V = 0$ for other $l, m$ with $l \leq m$. Let $f_1, f_2, f_3$ be an orthogonal basis of $W$ so that $(f_i, f_j) = \delta_{ij}$.

Let $I$ be the $2 \times 2$ identity matrix. Let

$$
J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } Y := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
$$

(8.2)

We identify $\mathfrak{gl}(V) = \text{Mat}_4$ with respect to $e_1, \ldots, e_4$. Then we can write the elements of $t$ and $p$ in matrix forms in $2 \times 2$ subminors:

$$
t := \left\{ \left( \begin{array}{c} P \\ JQ^t \\ S \end{array} \right) \in \mathfrak{gl}(V) \bigg| \text{tr}(P) = \text{tr}(S) = 0 \right\},
$$

$$
p := \left\{ \left( \begin{array}{c} aI \\ -JR^t \\ bI \end{array} \right) \in \mathfrak{gl}(V) \bigg| a, b \in \mathbb{C} \right\}.
$$

(8.3)

where $P, Q, R, S \in \text{Mat}_2$. Note that

$$
\begin{pmatrix} 0 & H \\ -H & 0 \end{pmatrix}, \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & Y \\ -Y & 0 \end{pmatrix}
$$
are elements of \( p' \) by letting \( R = -JH, R = -JX, R = -JY \) and \( R = -J \) respectively. Therefore we obtain a basis of \( p' \) as
\[
(8.4) \quad v_1 := \frac{1}{2} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad v_2 := \begin{pmatrix} 0 & H \\ -H & 0 \end{pmatrix}, \quad v_3 := \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}, \quad v_4 := \begin{pmatrix} 0 & Y \\ -Y & 0 \end{pmatrix}, \quad v_5 := \frac{1}{2} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.
\]
By direct computation, the Lie brackets of pairs of basis elements of \( p' \) are
\[
(8.5) \quad [v_1, v_2] = \begin{pmatrix} 0 & H \\ H & 0 \end{pmatrix}, \quad [v_1, v_3] = \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}, \quad [v_1, v_4] = \begin{pmatrix} 0 & Y \\ Y & 0 \end{pmatrix}, \quad [v_1, v_5] = \frac{1}{2} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},
\]
\[
[v_2, v_3] = -2 \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}, \quad [v_2, v_4] = 2 \begin{pmatrix} Y & 0 \\ 0 & Y \end{pmatrix}, \quad [v_2, v_5] = \begin{pmatrix} H & 0 \\ 0 & -H \end{pmatrix},
\]
\[
[v_3, v_4] = -\begin{pmatrix} H & 0 \\ 0 & H \end{pmatrix}, \quad [v_3, v_5] = \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix}, \quad [v_4, v_5] = \begin{pmatrix} Y & 0 \\ 0 & -Y \end{pmatrix}.
\]
These form a basis of \( t \).

Let us define a linear map \( F: \wedge^2 p' \to t \) by setting \( F(v_i \wedge v_j) := [v_i, v_j] \), where \( 1 \leq i < j \leq 5 \). Then we have a commuting diagram
\[
\begin{array}{ccc}
p' \times p' & \xrightarrow{\wedge^2} & \wedge^2 p' \\
\downarrow & & \downarrow F \\
[\cdot, \cdot] & & t.
\end{array}
\]
By \((8.3)\), \( \text{Im}(F) \) contains the basis of \( t \) and \( \dim t = \dim \wedge^2 p' = 10 \). Thus \( F \) is an isomorphism. In particular, for \( B_1, B_2 \in p' \), \( [B_1, B_2] = 0 \) if and only if \( B_1 \wedge B_2 = 0 \). This proves the following lemma.

**Lemma 8.2.** \( \{(B_1, B_2) \in p' \times p'| [B_1, B_2] = 0\} = \{(aB, bB)| a, b \in \mathbb{C}, B \in p'\} \) and it is irreducible.

**Corollary 8.3.** (1) \( \tilde{X}_1 \) is irreducible of dimension 11.

(2) Any element of \( \tilde{X}_1 \) is an unstable quiver representation.

**Proof.** (1) follows from Lemmas \([5.1] \) and \([5.2] \).

(2) The non-costability amounts to the existence of a common eigenvector \( v \) of \( B_1 \) and \( B_2 \) such that \( i^*(v) = 0 \). From \([5.1] \), we have \( \dim \text{Ker}(i^*) \geq 3 \). Let \( B_1 = a_1B \) and \( B_2 = a_2B \) for some \( a_1, a_2 \in \mathbb{C} \) and \( B \in p' \) (Lemma \([5.2] \)). By Proposition \([5.4] \), we have an eigenspace of \( B \) of dimension \( \geq 2 \). By the dimension reason it has a nonzero vector contained in the eigenspace and \( \text{Ker}(i^*) \).

On the other hand, if \( \sigma \in \text{Im} \omega \) is nonzero then we have
\[
(8.7) \quad \omega^{-1}(\sigma) \cong \text{SL}(2).
\]
since \( \text{Im} \omega \setminus 0 \) is the set of 2-dimensional subspaces \( S \) of \( p' \) with a volume form of \( S \). This isomorphism can be described in more detail as follows. Define an \( \text{SL}(2) \)-action on \( p' \times p' \) as
\[
(8.8) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (B_1, B_2) := (aB_1 + bB_2, cB_1 + dB_2).
\]
The SL(2)-action on $\omega^{-1}(\wedge^2 p' \setminus 0)$ is free. The SL(2)-action on $\wedge^2 p'$ is trivial and $\omega|_{\omega^{-1}(\wedge^2 p' \setminus 0)}$ is SL(2)-equivariant. Now (8.7) is nothing but the identification of a free SL(2)-orbit.

**Definition 8.4.** Let $G := SL(2) \times Sp \times O$. Then we have a $G$-action on $\tilde{X}$ by

$$(g_1, g_2, g_3, ((B_1, B_2), i, i^*)) := (g_1 (g_2 B_1 g_2^{-1}, g_2 B_2 g_2^{-1}), g_2 i g_3^{-1}, (g_2 i g_3^{-1})^*).$$

**Lemma 8.5.** Let $B_1, B_2 \in p'$. Then $([B_1, B_2])^2$ is a scalar endomorphism.

**Proof.** This follows from a tedious, but direct computation.

Let $\tilde{X}_2 := \tilde{X} \setminus \tilde{X}_1 = \{(B_1, B_2, i, i^*) \in \tilde{X} | [B_1, B_2] \neq 0\}$. Let $p: \tilde{X}_2 \to \text{Hom}(W, V)$ be the projection.

**Corollary 8.6.** Let $i \in p(\tilde{X}_2)$. Then $(ii^*)^2 = 0$.

**Proof.** Write $i = p(B_1, B_2, i, i^*)$ for some $B_1, B_2 \in p'$. By Lemma 8.5, $[B_1, B_2]^2 = (ii^*)^2$ is a scalar endomorphism. Since rank $i \leq 3$, the scalar is 0.

**Corollary 8.7.** Let $i \in p(\tilde{X}_2)$. Then the ab-diagram of $i$ is one of the following:

(8.9) (I) \hspace{1cm} (II) \hspace{1cm} (III) \hspace{1cm} (IV) 

\[
\begin{array}{cccc}
\text{bab} & \text{bab} & \text{bab} \\
\text{a} & \text{b} & \text{a} \\
\end{array}
\]

Proof. Since $(ii^*)^2 = 0$ by Corollary 8.6 and $ii^* \neq 0$, the maximal length of rows of the $b$-diagram of $ii^*$ is 2. Thus the $b$-diagram of $ii^*$ is one of the following:

(8.10) 

\[
\begin{array}{cccc}
\text{bb} & \text{bb} \\
\text{bb} & \text{b} \\
\end{array}
\]

From Table I we get the list of ab-diagrams as above.

**Lemma 8.8.** (1) Let $B_1 := v_5$ and $B_2 := v_1 - \frac{1}{2} v_2$. Then $[B_1, B_2]$ is a nilpotent matrix whose $b$-diagram is

(8.11) 

\[
\begin{array}{cccc}
\text{bb} \\
\text{bb} \\
\end{array}
\]

(2) Let $B_1 := v_3$ and $B_2 := \sqrt{-1} v_1 - v_5$. Then $[B_1, B_2]$ is a nilpotent matrix whose $b$-diagram is

(8.12) 

\[
\begin{array}{cccc}
\text{bb} \\
\text{b} \\
\end{array}
\]

Proof. (1) By direct calculation we have

\[
[B_1, B_2] = \frac{1}{2} \begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & -1 & 0 & -1 \\
1 & 0 & -1 & 0 \\
0 & 1 & 0 & 1
\end{pmatrix}
\]

which maps

(8.13) $e_1 \mapsto \frac{1}{2} (e_1 + e_3), \frac{1}{2} (e_1 + e_3) \mapsto 0, e_2 \mapsto -\frac{1}{2} (e_2 - e_4), \frac{1}{2} (e_2 - e_4) \mapsto 0.$
(2) By direct calculation we have

\[ [B_1, B_2] = \begin{pmatrix} 0 & -1 & 0 & -\sqrt{-1} \\ 0 & 0 & 0 & 0 \\ 0 & -\sqrt{-1} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

which maps

\[ (8.14) \quad e_2 \mapsto -(e_1 + \sqrt{-1}e_3), \quad e_1 + \sqrt{-1}e_3 \mapsto 0, \quad e_1 \mapsto 0, \quad e_2 + \sqrt{-1}e_4 \mapsto 0. \]

\[ \square \]

We denote \( B_1, B_2 \) in Lemma 8.8 (1) by \( B_1^{(I)}, B_2^{(I)} \) respectively. We denote \( B_1, B_2 \) in Lemma 8.8 (2) by \( B_1^{(II)}, B_2^{(II)} \) respectively. Let \( B_3^{(iV)} \) and \( B_n^{(iV)} \) be \( B_n^{(II)} \) \((n = 1, 2)\). For each \( A = I, II, III, IV \) there exists \( i^{(A)} \in \text{Hom}(W, V) \) whose \( ab \)-diagram is of type \((A)\) in \( (8.9) \) and \( [B_1^{(A)}, B_2^{(A)}] + i^{(A)}(i^{(A)})^* = 0 \) by Theorem 5.12 (1).

Let us define \( i^{(I)} \) in a matrix form. We take a basis \( \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\} \) of \( V \) by setting \( \tilde{e}_1 := e_1, \tilde{e}_2 := e_2 - e_4, \tilde{e}_3 := e_1 + e_3 \) and \( \tilde{e}_4 := e_4 \). We take a basis \( \{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3\} \) of \( W \) by setting \( \tilde{f}_1 := \frac{1}{\sqrt{2}}(f_1 + \sqrt{-1}f_2), \tilde{f}_2 := \frac{1}{\sqrt{2}}(f_1 - \sqrt{-1}f_2) \) and \( \tilde{f}_3 := f_3 \). Let

\[ i^{(I)} := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

with respect to \( \{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4\} \) and \( \{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3\} \).

Let us define \( i^{(II)}, i^{(III)} \) and \( i^{(IV)} \) in matrix forms respectively. We take a basis \( \{\tilde{e}_1', \tilde{e}_2', \tilde{e}_3', \tilde{e}_4'\} \) of \( V \) by setting \( \tilde{e}_1' := e_1, \tilde{e}_2' := -(e_1 + \sqrt{-1}e_3), \tilde{e}_3' := e_3 \) and \( \tilde{e}_4' := -\sqrt{-1}e_2 + e_4 \). We take a basis \( \{\tilde{f}_1', \tilde{f}_2', \tilde{f}_3'\} \) of \( W \) by setting \( \tilde{f}_1' := \frac{1}{\sqrt{2}}(f_1 + \sqrt{-1}f_3), \tilde{f}_2' := f_2 \) and \( \tilde{f}_3' := \frac{1}{\sqrt{2}}(f_1 - \sqrt{-1}f_3) \). Let

\[ i^{(II)} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad i^{(III)} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad i^{(IV)} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

with respect to \( \{\tilde{e}_1', \tilde{e}_2', \tilde{e}_3', \tilde{e}_4'\} \) and \( \{\tilde{f}_1', \tilde{f}_2', \tilde{f}_3'\} \).

**Corollary 8.9.** \( \tilde{X}_2 \) is the union of four \( G \)-orbits through \( (B_1^{(A)}, B_2^{(A)}, i^{(A)}, i^{(A)*}) \) for \( A = I, II, III, IV \). And each \( G \)-orbit is irreducible of dimension 12, 12, 11 and 9 respectively.

**Proof.** Let \( \tilde{X}'_2 \) be the union of four \( G \)-orbits. Since \( \tilde{X}'_2 \subset \tilde{X}_2 \), we need to show the opposite inclusion. By Corollary 8.7 we have \( p(\tilde{X}_2) \subset \cup_{A=I,II} \text{Sp} \cdot i^{(A)} = p(\tilde{X}'_2) \). Thus \( p(\tilde{X}_2) = \cup_{A=I,II} \text{Sp} \cdot i^{(A)} = p(\tilde{X}'_2) \). On the other hand, \( p^{-1}(p(x)) \cong \text{SL}(2) \) for \( x \in \tilde{X}_2 \) by (8.7). Therefore \( \tilde{X}_2 \subset p^{-1}(p(\tilde{X}_2)) = p^{-1}(p(\tilde{X}'_2)) = \tilde{X}'_2 \).

By Theorem 5.12 (2), the explicit value of \( \dim G.(B_1^{(A)}, B_2^{(A)}, i^{(A)}, i^{(A)*}) = 3 + \dim(\text{Sp} \times \text{O}).i^{(A)} \) for each \( A \), is computed as above.
The irreducibility comes from the fact that the above $G$-orbits are the $(\text{SL}(2) \times \text{Sp} \times \text{SO})$-orbits. To see this we observe that $-\text{Id}_W \in O$ acts on $N$ in the same way as $-\text{Id}_V \in \text{Sp}$. □

**Lemma 8.10.** Let $x \in G.(B_1^{(A)}, B_2^{(A)}, i^{(A)}, i^{(A)^*})$ for $A = I, II, III, IV$. Then we have the following assertions.

1. If $A = I$, $x$ is unstable and $\text{sp}^x$ is trivial.
2. If $A = II$, $x$ is stable (hence $\text{Sp}^x$ is trivial).
3. If $A = III, IV$, $x$ is unstable.
4. $\tilde{X}$ is a reduced complete intersection.

**Proof.** (1) It is direct to check $e_1 + e_3$ is a common eigenvector of $B_1^{(I)}$ and $B_2^{(I)}$. We have $i^{(I)^*}(e_1 + e_3) = i^{(I)^*}(\tilde{e}_3) = 0$. So $e_1 + e_3$ violates the costability of $x$.

We check $\text{sp}^x$ is trivial. Use (8.3). Then we have

$$\text{sp}^{B_1^{(I)}} = \left\{ \left( \begin{array}{cc} P & JQ' \\ JQ & S \end{array} \right) \in \text{sp} \bigg| \left( \begin{array}{cc} P & JQ' \\ JQ & S \end{array} \right), \left( \begin{array}{cc} 0 & I \\ I & 0 \end{array} \right) \right\}$$

$$= \left\{ \left( \begin{array}{cc} P & R \\ R & P \end{array} \right) \bigg| \text{tr}P = 0, R = JQ, Q = Q' \right\}$$

$$\text{sp}^{B_1^{(I)}} \cap \text{sp}^{B_2^{(I)}} = \left\{ \left( \begin{array}{cc} P & R \\ R & P \end{array} \right) \bigg| \left( \begin{array}{cc} P & R \\ R & P \end{array} \right), \left( \begin{array}{cc} I & H \\ -H & -I \end{array} \right) \right\} = 0, \text{tr}P = 0 \right\}$$

$$= \mathbb{C} \left\langle \left( \begin{array}{cc} X & -X \\ -X & X \end{array} \right), \left( \begin{array}{cc} Y & Y \\ Y & Y \end{array} \right), \left( \begin{array}{cc} H & 0 \\ 0 & H \end{array} \right) \right\rangle.$$

On the other hand, $\text{Im}(i^{(I)}) = \mathbb{C}\langle \tilde{e}_2, \tilde{e}_3 \rangle = \mathbb{C}\langle e_1 + e_3, e_2 - e_4 \rangle$. So

$$\text{sp}^{i^{(I)}} = \{ g \in \text{sp} \big| g(e_1 + e_3) = g(e_2 - e_4) = 0 \}.$$

We claim $\text{sp}^{B_1^{(I)}} \cap \text{sp}^{B_2^{(I)}} \cap \text{sp}^{i^{(I)}} = 0$. The solution $a, b, c \in \mathbb{C}$ of the following equations

$$\left( a \left( \begin{array}{cc} X & -X \\ -X & X \end{array} \right) + b \left( \begin{array}{cc} Y & Y \\ Y & Y \end{array} \right) + c \left( \begin{array}{cc} H & 0 \\ 0 & H \end{array} \right) \right) (e_1 + e_3) = 0$$

$$\left( a \left( \begin{array}{cc} X & -X \\ -X & X \end{array} \right) + b \left( \begin{array}{cc} Y & Y \\ Y & Y \end{array} \right) + c \left( \begin{array}{cc} H & 0 \\ 0 & H \end{array} \right) \right) (e_2 - e_4) = 0$$

is trivial. So the claim is proven.

(2) We show there does not exist a common eigenvector $v$ of $B_1^{(II)}$ and $B_2^{(II)}$ such that $i^{(II)^*}(v) = 0$. By direct computation we have $\text{Ker}(B_1^{(II)}) = \mathbb{C}\langle e_1, e_3 \rangle$ and $\text{Ker}(B_2^{(II)}) = \mathbb{C}\langle e_1 + \sqrt{-1}e_3, e_2 + \sqrt{-1}e_4 \rangle$. So we have $\text{Ker}(B_1^{(II)}) \cap \text{Ker}(B_2^{(II)}) = \mathbb{C}\langle e_1 + \sqrt{-1}e_3 \rangle$.

On the other hand $i^{(II)^*}(e_1 + \sqrt{-1}e_3) = -i^{(II)^*}(\tilde{e}_3) \neq 0$. This proves (2).

(3) In (2) we checked that $e_1 + \sqrt{-1}e_3$ is a unique common eigenvector of $B_1^{(A)}$ and $B_2^{(A)}$ for $A = III, IV$ up to constant.

On the other hand $i^{(A)^*}(e_1 + \sqrt{-1}e_3) = -i^{(A)^*}(\tilde{e}_3) = 0$ for $A = III, IV$. So $e_1 + \sqrt{-1}e_3$ violates the costability of $x$.

(4) By Corollaries (8.3, 8.9) and the dimension reason, $\tilde{X}$ is a complete intersection of dimension 12. By (1) and (2), $\tilde{X}$ is smooth along the two 12-dimensional $G$-orbits since
$\mathfrak{sp}^x = 0$ implies that $d\mu_x : T_x \mathcal{N}' \to \mathfrak{sp}$ is surjective. Therefore $\tilde{X}$ is reduced along the two 12-dimensional $G$-orbits. Hence $\tilde{X}$ itself is reduced by [18, Prop. 5.8.5]. □

8.2. Description of $\mu^{-1}(0)//Sp$. In the previous subsection we proved

1. $\tilde{X}$ is a reduced complete intersection with two irreducible components $\overline{S}$ and $\overline{U}$, where $S$ is the stable locus and $U$ is a $G$-orbit $G.(B_1^{(I)}, B_2^{(I)}, i(i), i(i^*))$.

2. $S$ is a $G$-orbit $G.(B_1^{(II)}, B_2^{(II)}, i_4, i_4^*)$.

3. The Sp-action on $U$ is locally free (i.e., $\mathfrak{sp}^x = 0$ for any $x \in U$).

4. $\tilde{X}//Sp = \overline{S}//Sp \cup \overline{U}//Sp$ as varieties (as $\tilde{X}$ is reduced).

From now on we fix an orthogonal basis of $W$. Then $\mathfrak{o}$ is the set of anti-symmetric matrices. Let us identify

\begin{align}
\mathfrak{o} \cong \mathbb{C}^3, \quad \begin{pmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{pmatrix} \mapsto (e, f, g).
\end{align}

The characteristic polynomial of $A := \begin{pmatrix} 0 & e & f \\ -e & 0 & g \\ -f & -g & 0 \end{pmatrix}$ in $t$ is $t^3 + (e^2 + f^2 + g^2)t$. Therefore $A$ is nilpotent if and only if $e^2 + f^2 + g^2 = 0$. Since any nonzero nilpotent element $x$ in $\mathfrak{o}$ has the $a$-diagram $aaa$, the minimal nilpotent orbit is $O.x$. Hence, $P$ is the quadric surface in $\mathbb{C}^3$ defined by $e^2 + f^2 + g^2 = 0$, which also equals the nilpotent variety.

**Lemma 8.11.** The map $[(x, y)] \mapsto (x^2, \sqrt{-1}xy, y^2)$ gives an isomorphism $\mathbb{C}^2/\mathbb{Z}_2 \cong P$, and hence $P$ is an irreducible normal variety. Moreover, $P^{\text{rank}2} := \{ A \in P | \text{rank} A = 2 \} = P \setminus 0 \cong (\mathbb{C}^2 \setminus 0)/\mathbb{Z}_2$.

**Proof.** The first isomorphism is well-known in invariant theory. Since $\mathbb{C}^2$ is irreducible and normal, so is $\mathbb{C}^2/\mathbb{Z}_2$.

We prove the second assertion. The $a$-diagram of a nilpotent matrix $A \in \mathfrak{o}$ is either

\begin{align}
a \\
a \\
a \quad \text{or} \quad aaa
\end{align}

Therefore $A \neq 0$ means $\text{rank} A = 2$. □

**Lemma 8.12.** $\Phi_{\tilde{X}_1} : \tilde{X}_1 \to P, (B_1, B_2, i, i^*) \mapsto (\text{tr}(B_1^2), \sqrt{-1}\text{tr}(B_1B_2), \text{tr}(B_2^2))$, is the GIT quotient by $Sp$.

**Proof.** Let $x := (B_1, B_2, i, i^*) \in \tilde{X}_1$. By Corollary 8.3 (2), $x$ is unstable. Suppose $Sp.x$ is closed in $N$. Since $x \notin \mu^{-1}(0)_{\text{reg}}$, we have $x^* = 0$ by Theorem 2.2 and Theorem 4.1 (4). Thus $i = 0$. Let $T := \{(aB, bB, 0, 0) | B \in \mathfrak{p}', a, b \in \mathbb{C} \}$. Let $\phi := \Phi_{\tilde{X}_1}|_T$. It is enough to show that $\phi$ is the GIT quotient by $Sp$ by Lemma 8.2.

Since the Sp-action on $P$ is trivial and $P$ is normal, we need to show that $\phi//Sp : T//Sp \to P$ is bijective by Zariski’s main theorem.

Since $\text{tr}(v_1^2) \neq 0$, $\phi$ is surjective and thus so is $\phi//Sp$.

To show injectivity, it is enough to show that $\phi^{-1}(c)$ is an Sp-orbit for any $c \in P \setminus 0$ and that 0 is the unique closed Sp-orbit in $\phi^{-1}(0)$. 


Lemma 8.13. \Phi: S \to P, (B_1, B_2, i, i^*) \mapsto i^*i, is the GIT quotient by Sp.

Proof. Let \( x := (B_1, B_2, i, i^*) \in S \). By the ab-diagram (8.9) of i, we have \( i^*i \in P^{\text{rank}2} \). So \( \Phi(S) = P^{\text{rank}2} \) and \( \Phi \) is well-defined.

We claim that \( \Phi^{-1}(a) \) is an Sp-orbit for any \( a \in P \setminus 0 = P^{\text{rank}2} \). Note that since \( S \) is irreducible, so is \( \Phi^{-1}(a) \). Since \( x \) is stable, \( S^x \) is trivial. Thus \( \dim S \times x = 10 \). On the other hand, \( \Phi^{-1}(i^*i) = (\text{SL}(2) \times \text{Sp}) \times x \) by Theorem 5.10 and (8.7). Thus \( \Phi^{-1}(i^*i) \) is an irreducible 10-dimensional variety. If \( S \times x \subseteq (\text{SL}(2) \times \text{Sp}) \times x \) then for \( y \in (\text{SL}(2) \times \text{Sp}) \times x \setminus S \times x \), we have \( \dim S \times y = 10 \). But then \( \Phi^{-1}(i^*i) \) contains two disjoint locally closed subvarieties \( S \times x \) and \( S \times y \) of dimension 10. This contradicts the irreducibility of \( \Phi^{-1}(i^*i) \). Therefore \( \dim S \times i^*i = \dim \Phi^{-1}(i^*i) \) as desired.

Since \( S \) consists of Sp-closed orbits (Theorem 2.2), \( S/Sp \) is Zariski open in \( \bar{S}/Sp \). By Luna’s slice theorem [27, S/Sp is a smooth variety. By the above claim and Zariski’s main theorem, \( \Phi_S|_{S/Sp} : S/Sp \to P^{\text{rank}2} \) is an isomorphism, where \( \Phi_S : S/Sp \to P \) is the induced morphism. Let us finish the proof of the lemma. Let \( f \in C(S|_{Sp}) \). Then \( f|_S \in \Phi \Gamma(O_{P \times 0}) \). By the normality of \( P, \Gamma(O_{P \times 0}) = C(P) \). Thus \( f \in \Phi_S \Gamma(P) \). This means \( C(S|_{Sp}) = \Phi_S \Gamma(P) \), equivalently \( \Phi \) is the GIT quotient by Sp as desired.

Lemma 8.14. \( \bar{X}_1 \subset \bar{U} \) and \( \bar{X}_1/Sp = \bar{U}/Sp \).

Proof. Since \( U \) is an irreducible reduced locally free G-orbit of dimension 12, we have \( \bar{U}/Sp \) is an irreducible reduced variety of dimension \( \leq 2 \). Now the second assertion follows from the first by Lemma 8.12.

Let us prove the first assertion. Suppose \( \bar{X}_1 \not\subseteq \bar{U} \). Then \( \bar{X}_1 \subset \bar{S} \) since \( \bar{X}_1 \) is irreducible by Corollary 8.3 (1). Let \( T := \{(aB, bB) \in \bar{P} \times \bar{P} | a, b \in \mathbb{C}, B \in \bar{P} \} \). By Lemma 8.1 and (8.1), \( \bar{X}_1 \) is the closure of \( T \times (Sp \times 0) \). for a nonzero \( i \in \text{Hom}(W, V) \) with \( i^*i = 0 \). By Lemma 8.13 \( \Phi_{\bar{S}}(\bar{X}_1) = 0 \). This contradicts Lemma 8.12.

Definition 8.15. Define \( \Phi : \bar{X} \to (P \times 0) \cup (0 \times P) (\subset P \times P), (B_1, B_2, i, i^*) \mapsto ((\text{tr}(B_1^2), \sqrt{-1} \text{tr}(B_1B_2), \text{tr}(B_2^2)), i^*i) \).

To see \( \Phi \) is well-defined morphism one notices that \( i^{(1)}(i^{*}) = i^{(II)}(i^{*}) = i^{(III)}(i^{*}) = 0 \) and \( \text{tr}(B_1^{(II)}) = \sqrt{-1} \text{tr}(B_1^{(II)}B_2^{(II)}) = \text{tr}(B_2^{(II)}) = 0 \), which come from Corollary 8.7 and the direct computation respectively.
Theorem 8.16. \( \Phi \) is the GIT quotient by \( \text{Sp} \) onto \( (P \times 0) \cup (0 \times P) \). Hence, \( \mu^{-1}(0)/\text{Sp} \cong \mathbb{C}^2 \times ((P \times 0) \cup (0 \times P)) \).

Proof. By Lemmas 8.12, 8.13 and 8.14, \( \Phi \) factors through \( \tilde{X}/\text{Sp} \). The canonical morphism given by the composite

\[
(P \times 0) \cup (0 \times P) \xrightarrow{f} \tilde{X}/\text{Sp} \xrightarrow{\text{Sp}} \tilde{X}/\text{Sp}
\]

is the inverse of \( \Phi/\text{Sp} \), where \( f := (\Phi|_{\tilde{X}}/\text{Sp})^{-1} \prod_{[0]}(\Phi|_{\text{Sp}}/\text{Sp})^{-1} \) and \( \Phi/\text{Sp}: \tilde{X}/\text{Sp} \rightarrow (P \times 0) \times (0 \times P) \) is the induced morphism. \( \square \)

9. Moduli spaces of SO(2)-data with \( k = 4 \)

This section will be devoted to the proof of Theorem 4.2 (2).

Let \( \dim V = k = 4 \) and \( \dim W = N = 2 \). Let \( p^i := \{ B \in p | \text{tr}(B) = 0 \} \). Let \( \tilde{X} := \{(B_1, B_2, i, i^*) \in \mu^{-1}(0)|B_1, B_2 \in p^i \} \). Then as in \( \S 8.1 \) \( \mu^{-1}(0) \cong \mathbb{C}^2 \times \tilde{X} \). Let \( p: \tilde{X} \rightarrow \text{Hom}(W, V) \) be the projection.

Lemma 9.1. Let \( i \in \text{Im}(p) \). Then \( ii^* \) is nilpotent and the ab-diagram of \( i \) is one of the following:

\[
\begin{array}{cccc}
ab & b & & \text{bab} \\
ba & b & bab & b \\
b & b & bab & b \\
b & a & & a
\end{array}
\]

(9.1)

Proof. Let \( (B_1, B_2, i, i^*) \in \tilde{X} \). By Lemma 8.5, \( ([B_1, B_2])^2 = (i^*)^2 \) is a scalar. Since rank \( i \leq 2 \) and \( \dim V = 4 \), the scalar is 0. Since there is no nontrivial nilpotent element in \( a \), we have \( i^*i = 0 \). By a similar argument as in \( \S 6 \) the ab-diagram of \( i \) is one of (9.1) from Table \( \S 1 \). \( \square \)

Let \( \tilde{X}_1 := \{(B_1, B_2) \in p^i \times p'|[B_1, B_2] = 0 \} \times \{ i \in \text{Hom}(W, V) | ii^* = 0 \} \). Let \( \tilde{X}_2 := \tilde{X} \setminus \tilde{X}_1 \).

Theorem 9.2. \( \mu^{-1}(0)_{\text{reg}} = \emptyset \).

Proof. Let \( x := (B_1, B_2, i, i^*) \in \tilde{X} \). We will prove that \( x \) is not costable.

Suppose \( x \in \tilde{X}_1 \). The ab-diagram of \( i \) is either the first or the second in (9.1). Therefore \( \dim \ker(i^*) \geq 3 \). As in the proof of Corollary 8.3 (2), we see that \( x \) is not costable.

Suppose \( x \in \tilde{X}_2 \). As in the proofs of Lemma 8.10 (1) and (3), we deduce that \( x \) is not costable. \( \square \)

Appendix A. Finite dimensionality of weight spaces

The main purpose of this section is to prove that each weight space of \( \mathbb{C}[\mu^{-1}(0)]^G(V) \) with respect to \( T \) is finite-dimensional.

Let \( \epsilon(m) = 0 \) (resp. \( \epsilon(m) = 1 \)) if \( m \) is even integer (resp. odd integer). If \( \epsilon = -1 \) then using a symplectic basis of \( V \), we identify \( V = \mathbb{C}^k \). If \( \epsilon = +1 \) then using the basis

\[
\{ f_1 \pm \sqrt{-1}f_2, f_3 \pm \sqrt{-1}f_4, \ldots, f_{k-\epsilon(k)-1} \pm \sqrt{-1}f_{k-\epsilon(k)}, (f_k) \}
\]
we identify \( V = \mathbb{C}^k \); where \( \{f_1, f_2, \ldots, f_k\} \) is an orthogonal basis of \( V \). Here the notation \((f_k)\) denotes \( f_k \) only when \( k \) is odd (vacuous otherwise). By a similar way we identify \( W = \mathbb{C}^N \). Let \( [a] \) be the maximal integer in \( \mathbb{Z}_{\leq a} \), where \( a \in \mathbb{R} \). We fix maximal tori of \( G(V) \) and \( G(W) \) as

\[
T_{G(V)} = \{ \text{diag}(z_1, z_2, \ldots, z_{k/2}) \} \oplus \text{diag}(z_{1/2}^{-1}, z_{2/2}^{-1}, \ldots, z_{k/2}^{-1}) \} |_{z_1, z_2, \ldots, z_{k/2} \in \mathbb{C}^*} ,
\]

\[
T_{G(W)} = \{ \text{diag}(t_1, t_2, \ldots, t_{N/2}) \} \oplus \text{diag}(t_{1/2}^{-1}, t_{2/2}^{-1}, \ldots, t_{N/2}^{-1}) \} |_{t_1, t_2, \ldots, t_{N/2} \in \mathbb{C}^*} \]

respectively.

Now we identify the rings of characters of \( T_{G(V)}, T_{G(W)} \) and \((\mathbb{C}^*)^2\)

\[
R(T_{G(V)}) = \mathbb{Z}[z_1^{\pm 1}, z_2^{\pm 1}, \ldots, z_{k/2}^{\pm 1}],
\]

\[
R(T_{G(W)}) = \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_{N/2}^{\pm 1}]
\]

\[
R((\mathbb{C}^*)^2) = \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]
\]

respectively.

Let us prove

(A.1) \[ \mathcal{X}_{C(\mu^{-1}(0))G(V)} \in \hat{R}(T) \] := \( R(T_{G(W)})[[q_1^{-1}, q_2^{-1}]] \]

where \( T = T_{G(W)} \times (\mathbb{C}^*)^2 \) and \( \mathcal{X}_{C(\mu^{-1}(0))G(V)} \) is the formal \( T \)-character. The idea is to use the following two:

1. \( \mu^{-1}(0) \setminus G(V) \) is a closed \((\mathbb{C}^*)^2 \times G(W)\)-subscheme of \( N \setminus G(V) \);
2. the (surjective) GIT quotient \( N \setminus G(V) \) is \((\mathbb{C}^*)^2 \times G(W)\)-equivariant.

So each weight space of \( C(\mu^{-1}(0))G(V) \) is a subquotient of the weight space of the same weight of \( C[N] \). So the proof will be done if we show a stronger claim:

\[ \mathcal{X}_{C[N]} \in \hat{R}(T). \]

The claim will follow from an even stronger claim:

\[ \mathcal{X}_{C[N]}^{(\mathbb{C}^*)^2} \in \mathbb{Z}[[q_1^{-\frac{1}{2}}, q_2^{-\frac{1}{2}}]] \]

where \( \mathcal{X}_{C[N]}^{(\mathbb{C}^*)^2} \) denotes the formal \((\mathbb{C}^*)^2\)-character of \( C[N] \). This is because all the monomials in \( \mathcal{X}_{C[N]}^{(\mathbb{C}^*)^2} \) have nonnegative integer coefficients.

Let us check the last claim. The decomposition \( N = p(V) \oplus p(V) \oplus \text{Hom}(W, V) \) is the weight decomposition with respect to \((\mathbb{C}^*)^2\). The first two direct summands \( p(V) \) of \( N \) are of weight \( q_1 \) and \( q_2 \) respectively. The last summand \( \text{Hom}(W, V) \) is of weight \( (q_1 q_2)^{\frac{1}{2}} \). Since \( C[N] = S(p(V)^{\vee}) \otimes S(p(V)^{\vee}) \otimes S(\text{Hom}(W, V)^{\vee}) \), where \( S \) denotes the symmetric product, we have

\[
\mathcal{X}_{C[N]}^{(\mathbb{C}^*)^2} = \left( \sum_{n \geq 0} q_1^n \right)^{\frac{1}{2}k(k+\epsilon)} \left( \sum_{n \geq 0} q_2^{-n} \right)^{\frac{1}{2}k(k+\epsilon)} \left( \sum_{n \geq 0} (q_1 q_2)^{-\frac{n}{2}} \right)^{kN}. 
\]

This proves the last claim.

To complete the proof of (A.1) we recall from (1.3) that \( i \in \text{Hom}(W, V) \) always appears together with \( i^* \) in \( C[N]^{G(W)} \). Hence any monomials with non-integer exponents in \( \mathcal{X}_{C(\mu^{-1}(0))G(W)} \) have coefficient 0. This proves (A.1).
Remark A.1. (1) $\hat{R}(T)$ is a ring.

(2) The origin 0 is the unique $(\mathbb{C}^*)^2$-fixed point of $N$.

Appendix B. Scheme structure of moduli space of framed vector bundles with symplectic and orthogonal structure

Let $V, W$ be vector spaces of dimension $k, N$ with $(\ , \ )_\varepsilon, (\ , \ )_{-\varepsilon}$ respectively, where $\varepsilon = \pm 1$. The above pairings give isomorphisms $a_\varepsilon: V \to V^{\vee}$ and $b_{-\varepsilon}: W \to W^{\vee}$ given by $v \mapsto (v, \bullet)_\varepsilon$ and $w \mapsto (w, \bullet)_{-\varepsilon}$ respectively. Then, $a_\varepsilon = \varepsilon a_{-\varepsilon}$ and $b_{-\varepsilon} = -\varepsilon b_{-\varepsilon}$.

We use the notations $M$ and $*_M$ in §2.3 Let $*: M$ and $x* := *(x)$ for simplicity. Let $\mu_M: M \to \mathfrak{gl}(V)$ be the moment map given by $(B_1, B_2, i, j) \mapsto [B_1, B_2] + ij$. It is obvious that $\mu = \mu_M|_N$.

We define an involution $\overline{\ }: M^{reg}/GL(V) \to M^{reg}/GL(V)$ by $GL(V).x \mapsto GL(V).x^\ast$. Then the fixed locus $M^{reg}/GL(V)\overline{\ }$ (resp. $\mu_M^{-1}(0)^{reg}/GL(V)\overline{\ }$) is a smooth subscheme of $M^{reg}/GL(V)$ (resp. $\mu_M^{-1}(0)^{reg}/GL(V)$).

Since $N$ is the fixed locus of $M$, we have a canonical embedding $\iota: \mu^{-1}(0)^{reg}/G(V) \to \mu_M^{-1}(0)^{reg}/GL(V)\overline{\ }$. We will see $\iota$ is surjective.

Let $G$ be the vector bundle locus of the Gieseker moduli scheme of framed torsion-free sheaves $E$ with rank $N$ and $c_2(E) = k$. By Barth’s correspondence [3], there is an isomorphism $F^\prime: \mu_M^{-1}(0)^{reg}/GL(V) \to G$ (cf. [30, §2]). We denote by the same notation $\overline{\ }$ the induced involution on $G$ via $F^\prime$. Thus we have the isomorphism between the fixed loci $F: \mu_M^{-1}(0)^{reg}/GL(V)\overline{\ } \to G\overline{\ }$ by restriction.

Donaldson’s argument [12] asserts that the image of the composite $F\iota$ is the set of the isomorphism classes of framed vector bundles $E$ which admits an isomorphism $\phi: E \to E^\vee$ with $\phi^\vee = -\varepsilon \phi$ and $\phi|_x = b_{-\varepsilon}$ (after identifying $E|_x = W$ via the given framing for any $x \in l_\infty$). See also [8].

We claim that the image of $F$ is contained in that of $F\iota$ given as above. So our claim will assert that $\iota$ is surjective. The proof itself goes along Donaldson’s argument, so will be sketchy. Let $x \in \mu_M^{-1}(0)^{reg}$ such that $g.x = x^\ast$ for some $g \in GL(V)$. By taking $*$ to the both sides of $g.x = x^\ast$, we obtain $g.x = g^\ast.x$. By the stability of $x$, we have $g = g^\ast$. The cohomology sheaf of the monad associated to $x^\ast$ is isomorphic to $E^\vee$. The induced maps by $g, g^\ast$ between the monads are explicitly written in terms of linear maps in $\text{End}(V^\oplus 2 \oplus W)$. By diagram-chasing and passing to the cohomology sheaves, the constraint $g = g^\ast$ gives an isomorphism $\overline{\gamma}: E \to E^\vee$ with $\overline{\gamma}^\vee = -\varepsilon \overline{\gamma}$ and $\overline{\gamma}|_x = b_{-\varepsilon}$. This finishes the proof of the claim.

By Zariski’s main theorem we obtain the isomorphism $M^K_n \cong G\overline{\ }$, which was used in §1.4.

Appendix C. Proof of Proposition 3.7 (2)

We prove Proposition 3.7 (2) in this section: the codimension of $m^{-1}(X) \setminus (\pi_1^{-1}(p_k) \cup \pi_2^{-1}(p_k))$ in $m^{-1}(X)$ is larger than 1.

Lemma C.1. ([14, Theorem XI.4]) Let $B \in p$. Then $O(V).B = p \cap GL(V).B$, where $O(V).B$ and $GL(V).B$ are the orbits by conjugation. \hfill $\square$
Let $E_{\text{gl}}: \mathfrak{gl} \to S^k \mathbb{C}$ be the morphism mapping $B$ to the unordered set of eigenvalues of $B$. Here, $S_k$ is the symmetric group of $k$-letters acting on $\mathbb{C}^k$ by permutation of coordinates, so that $S^k \mathbb{C} := \mathbb{C}^k/S_k$. Let $E := E_{\text{gl}}|_p$ and $E_t := E|_{p_t}$. To construct $E_{\text{gl}}$ explicitly, let $P: S^k \mathbb{C} \to \mathbb{C}^k$ be the isomorphism by $[(a_1, ..., a_k)] \mapsto (p_1(a), ..., p_k(a))$, where $a := (a_1, ..., a_k)$ and $p_i(a) = a_i + \cdots + a_k$ (the $i$th power sum). Let $E_{\text{gl}}': \mathfrak{gl} \to \mathbb{C}^k$ by $B \mapsto (\text{tr}B, \text{tr}B^2, \ldots, \text{tr}B^k)$. Let $E_{\text{gl}} := P^{-1} \circ E_{\text{gl}}'$.

Let $p_{l}^{(i)} := \{B \in p_t|B \text{ has } e \text{ distinct eigenvalues}\}$. Let $p_{l}^{(\leq e)} := \bigsqcup_{e \leq e} p_{l}^{(e)}$. Then $p_{l}^{(\leq e)}$ is a closed subvariety of $\mathfrak{gl}$. Indeed, let $\Delta^{(e)} \subset S^k \mathbb{C}$ be the locus of all the unordered sets of $e$ distinct points. Let $\Delta^{(\leq e)} := \bigsqcup_{e \leq e} \Delta^{(e)}$. Then $\Delta^{(\leq e)}$ is a closed subvariety of $S^k \mathbb{C}$. Therefore $E^{-1}(\Delta^{(\leq e)})$ is a closed subvariety of $\mathfrak{gl}$. In particular, $p_{l}^{(e)} = p_l \cap E^{-1}(\Delta^{(e)})$ is locally closed in $\mathfrak{p}$ by Lemma 3.5. It is manifest that if $e > k$ then for any $l$, $p_{l}^{(e)} = \emptyset$ and that if $e = k$ then $p_{l}^{(e)} \neq \emptyset$ if and only if $l = k$.

**Lemma C.2.** (1) If $p_{l}^{(e)} \neq \emptyset$ then $\dim p_{l}^{(e)} \leq \dim \mathfrak{p} - l + e$. In particular, if $l > k$ then $\dim p_{l}^{(e)} \leq \dim \mathfrak{p} - 2$.

(2) If $e = k - 1$ then $p_{l}^{(e)} \neq \emptyset$ if and only if $l \in \{k, k + 1\}$.

(3) $p_{l}^{(k-1)}$ consists of $B \in \mathfrak{p}$ conjugate by $O(V)$ to $\text{diag}(a_1, a_1, a_2, \ldots, a_{k-1})$, where $a_1, a_2, \ldots, a_{k-1}$ are distinct in $\mathbb{C}$.

**Proof.** (1) The image $E_{\text{gl}}: p_{l}^{(e)} \to S^k \mathbb{C}$ is contained in $\Delta^{(e)}$. Any nonempty fibre of $E_t$ is a union of $O(V).B$ for finitely many $B \in p_{l}^{(e)}$ (Lemma C.1), so that its dimension is $\dim O(V).B = \dim O(V) - \dim O(V)^B = \dim t - \dim t^B = \dim \mathfrak{p} - \mathfrak{p}^B = \dim \mathfrak{p} - l$, where the third identity comes from Lemma 3.4. Therefore $\dim p_{l}^{(e)} \leq \dim \mathfrak{p} - l + \dim \Delta^{(e)}$. Since $\dim \Delta^{(e)} = e$, we have proven (1).

(2) Let $e = k - 1$. Let $B \in p_{l}^{(e)}$. Let $a_1, \ldots, a_{k-1}$ be the (distinct) eigenvalues of $B$. We may assume that only the $a_1$-eigenspace of $B$ is 2-dimensional while the other ones are all 1-dimensional. The Jordan normal form of $B$ is either $\begin{pmatrix} a_1 & 1 \\ 0 & a_1 \end{pmatrix} \oplus \text{diag}(a_2, \ldots, a_{k-1})$ or $\text{diag}(a_1, a_1, \ldots, a_{k-1})$. Both cases actually happen, since $\begin{pmatrix} a_1 & 1 \\ 0 & a_1 \end{pmatrix}$ is conjugate by $\text{GL}(2)$ to a symmetric matrix $\begin{pmatrix} a_1 + \sqrt{-1} & 1 \\ 1 & a_1 - \sqrt{-1} \end{pmatrix}$. We have $\dim \mathfrak{gl}^B = k$ and $k + 2$ respectively. Using $\mathfrak{gl}^B = \mathfrak{p}^B \oplus t^B$ (by (2.2)) and Lemma 3.4 we have $\dim \mathfrak{p}^B = k$ and $k + 1$ respectively. This proves (2).

(3) follows from Lemma C.1.

**Lemma C.3.** Let $X \in \mathfrak{t}$. Let $i \in \{1, 2\}$. Suppose $\pi_i^{-1}(p_{l}^{(e)}) \cap \mathfrak{m}^{-1}(X) \neq \emptyset$. Then $\dim \pi_i^{-1}(p_{l}^{(e)}) \cap \mathfrak{m}^{-1}(X) \leq \dim \mathfrak{p} + e$. In particular, if $e \leq k - 2$ then $\dim \pi_i^{-1}(p_{l}^{(e)}) \cap \mathfrak{m}^{-1}(X) \leq \dim \mathfrak{p} + k - 2 = \dim \mathfrak{m}^{-1}(X) - 2$.

**Proof.** We claim that any nonempty fibre of $\pi_i|: \pi_i^{-1}(p_{l}^{(e)}) \cap \mathfrak{m}^{-1}(X) \to p_{l}^{(e)}$ is of dimension $l$. Take $B \in p_{l}^{(e)}$ and identify $\pi_i^{-1}(B)$ with $p$. Then $\pi_i^{-1}(B) \cap \mathfrak{m}^{-1}(X)$, unless empty, is an
affine space isomorphic to $p^B$, since for any $B' \in \pi^{-1}_i(B) \cap m^{-1}(X)$, $B' - B \in p^B$. The base dimension $\dim_{\mathbb{C}}p_i^{(e)}$ is estimated in Lemma [C.2] Thus the lemma is proven. □

Now we are ready to estimate the codimension of $m^{-1}(X) \setminus (\pi^{-1}(p_k) \cup \pi^{-1}(p_{k+1}))$ in $m^{-1}(X)$. By Lemma [C.3] and Lemma [C.2] (2), to check the codimension $\geq 2$, it suffices to check that so is the codimension of $\pi^{-1}(p_{k+1}) \cap \pi^{-1}(p_k) \cap m^{-1}(X)$ in $m^{-1}(X)$, whenever $\{i, j\} = \{1, 2\}$. By Lemma [C.3], $\pi^{-1}(p_{k+1}) \cap m^{-1}(X)$ is of codimension $\geq 1$ in $m^{-1}(X)$. It remains to prove $\pi^{-1}(p_{k+1}) \cap \pi^{-1}(p_k) \cap m^{-1}(X)$ is Zariski (open) dense in $\pi^{-1}(p_{k+1}) \cap m^{-1}(X)$. Let $B_1 \in p_{k+1}$. This is reduced to check that

\[(C.1) \quad \pi^{-1}(B_1) \cap \pi^{-1}(p_k) \cap m^{-1}(X) \neq \emptyset \text{ provided } \pi^{-1}(B_1) \cap m^{-1}(X) \neq \emptyset\]

since $\pi^{-1}(B_1) \cap m^{-1}(X) \cong p^{B_1}$ irreducible (see the proof of Lemma [C.3]). Let $B_0 \in \pi^{-1}(B_1) \cap m^{-1}(X) \subset p$, where $\pi^{-1}(B_1)$ are canonically identified with $p$. Let us write $B_1 = g.\text{diag}(a_1, a_1, a_2, ..., a_{k-1})$, where $g \in O(V)$ and $a_1, ..., a_{k-1}$ are distinct (Lemma [C.1] and Lemma [C.2] (3)). Let $B_2 := g.\text{diag}(b_1, ..., b_k)$, where $b_1, ..., b_k$ are distinct so that $B_2 \in p_k$. By the Zariski openness of $p_k$ in $p$, there exists $u \in \mathbb{C} \setminus \{1\}$ such that $(1 - u)B_0 + uB_2 \in p_k$ since for $u = 1$, $B_2 \in p_k$. Therefore $B_0 + \frac{u}{1 - u}B_2 \in p_k$. Now we have $(B_1, B_0 + \frac{u}{1 - u}B_2)$ or $(B_0 + \frac{u}{1 - u}B_2, B_1) \in \pi^{-1}(B_1) \cap \pi^{-1}(p_k) \cap m^{-1}(X)$, which shows (C.1). This completes the proof of Proposition 3.7 (2).

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Dept. Math., Kyungpook Nat’l Univ., Sangyuk-dong, Buk-gu, Daegu 702-701, Korea

E-mail address: choy@knu.ac.kr
CORRIGENDUM AND ADDENDUM TO “MODULI SPACES OF FRAMED SYMPLECTIC AND ORTHOGONAL BUNDLES ON $\mathbb{P}^2$ AND THE K-THEORETIC NEKRASOV PARTITION FUNCTIONS”

JAEYOO CHOY

ABSTRACT. We give a scheme-theoretic description of $\mu^{-1}(0)$ and $\mu^{-1}(0)/\text{Sp}(2)$, where $\mu$ is the moment map defining the space of ADHM data associated to framed $\text{SO}(3, \mathbb{R})$-instantons with instanton number 1. The irreducible component in $\mu^{-1}(0)/\text{Sp}(2)$, other than the Zariski closure of the instanton moduli space, is non-reduced, which is a corrigendum to [2, Theorem 4.3]. We further describe this component using the factorization property [1].

1. Corrigendum

The Hamiltonian reduction $\mu^{-1}(0)/\text{Sp}(2)$ associated to the framed $\text{SO}(3, \mathbb{R})$-instanton moduli space with instanton number 4 is not reduced. Hence the statement of [2, Theorem 4.3] should be corrected as:

\textbf{Theorem.} Let $N = 3$ and $k = 4$. Then we have the following properties.

1. $\mu^{-1}(0)$ is a non-reduced complete intersection and a union of two irreducible components. Hence $\mu$ is flat.
2. One irreducible component of $\mu^{-1}(0)$ is the Zariski closure of $\mu^{-1}(0)_{\text{reg}}$.
3. The reduced scheme $(\mu^{-1}(0)/\text{Sp}(V))_{\text{red}}$ is isomorphic to $\mathbb{C}^2 \times (\mathbb{P} \sqcup_0 \mathbb{P})$. Moreover the isomorphism restricts to $\mu^{-1}(0)_{\text{reg}}/\text{Sp}(V) \cong \mathbb{C}^2 \times (\mathbb{P} \setminus 0)$.

The (corrected) proof will appear in Section 1.2 after correction of [2, Lemma 8.10 (1), (4)]. In Sections 2, 3, we describe further the non-reduced locus of $\mu^{-1}(0)$ and its GIT quotient.

1.1. Correction of [2, Lemma 8.10 (1), (4)]. This erratum is due to the author’s wrong computation of the dimension of the stabilizers in [2, Lemma 8.10 (1)]. This statement is corrected as follows: If $A = I$, then $x$ is unstable and $\dim \mathfrak{sp}^x = 2$. We give the correct computation of $\mathfrak{sp}^x$, where $x = (B_1^{(1)}, B_1^{(2)}, i^{(1)}, i^{(2)*})$. Using the matrix expression of $t = \mathfrak{sp}$
(\[2, (8.3)]\)), we have
\[
\sp_{B_1} = \left\{ \begin{pmatrix} P & JQ^t \\ JQ & S \end{pmatrix} \in \sp \left| \begin{pmatrix} P & JQ^t \\ JQ & S \end{pmatrix}, \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right| = 0 \right\} 
\]
\[
\sp_{B_1} = \left\{ \begin{pmatrix} P & R \\ R & P \end{pmatrix} \left| \begin{pmatrix} P & R \\ R & P \end{pmatrix}, \begin{pmatrix} I & -H \\ H & -I \end{pmatrix} \right| = 0, \tr P = 0 \right\},
\]
\[
\sp_{B_1} \cap \sp_{B_2} = \mathbb{C}\left( \begin{pmatrix} X & X \\ X & X \end{pmatrix}, \begin{pmatrix} Y & -Y \\ -Y & Y \end{pmatrix} \right),
\]
where \( \{ H, X, Y \} \) is the standard \( \mathfrak{sl}_2 \)-triple. (The original computation in the proof of \[2, \text{Lemma 8.10 (1)} \] is wrong because the matrices \( H, -H \) in the above were interchanged mistakenly). On the other hand \( \text{Im}(i^{(4)}) = \mathbb{C}\langle \tilde{e}_2, \tilde{e}_3 \rangle = \mathbb{C}\langle e_1 + e_3, e_2 - e_4 \rangle \) and this is contained in the kernel of any endomorphisms in the above \( \sp_{B_1} \cap \sp_{B_2} \). Thus we have
\[
\sp^x = \sp_{B_1} \cap \sp_{B_2} = \mathbb{C}\left( \begin{pmatrix} X & X \\ X & X \end{pmatrix}, \begin{pmatrix} Y & -Y \\ -Y & Y \end{pmatrix} \right).
\]
(See Remark 2.4 for \( \sp^x \).

The statement of \[2, \text{Lemma 8.10 (4)} \] should be corrected as follows: \( \tilde{X} = \{ (B_1, B_2, i, i^*) \mid \tr(B_1) = \tr(B_2) = 0 \} \) is a non-reduced complete intersection with two irreducible components. Moreover the Zariski closure of \( G.x \) is the only non-reduced component, where \( G = \text{SL}(2) \times \text{Sp} \times O \) and \( x \) is of type I as in the above computation.

The verification is as follows: Since \( \sp^x \neq 0 \), the differential \( d\mu_x \) of the moment map \( \mu \) at \( x \) is not surjective, which means that \( \tilde{X} \) has non-reduced scheme structure along the stratum \( G.x \) (cf. the proof of \[2, \text{Theorem 3.3} \]).

1.2. Proof of Theorem. Since \( \sp^x \) is not trivial, we need to correct every relevant statement in \[2, \text{Section 8.2} \]: \( \tilde{X} \) should be replaced by \( (\tilde{X})_{\text{red}} \), and the \( G \)-action on \( U = G.x \) is not locally free (\[2, \text{p.300}, \text{the item (3)} \] and \[2, \text{proof of Lemma 8.14} \]).

Theorem immediately follows from \[2, \text{Lemma 8.14} \] once we assure \( \dim U/\text{Sp} \leq 2 \) in the proof of the lemma (without assuming the local freeness of the \( G \)-action on \( U \)). We use the decomposition \( \mu^{-1}(0)/\text{Sp} = \mu^{-1}(0)_{\text{reg}}/\text{Sp} \sqcup S^2 \mathbb{A}^2 \) (\[2, \text{Theorem 2.6 (1)} \]). Here we used the fact that there is no \( \text{SO}(3, \mathbb{R}) \)-instanton with instanton number 2, i.e. \( \mu_{k'}^{-1}(0) = \emptyset \), \( k' = 2 \). Thus the image of \( U \) in the GIT quotient is contained in the stratum \( S^2 \mathbb{A}^2 \), which implies \( \dim U/\text{Sp} \leq 2 \). This completes the proof of Theorem.

2. Further description on the scheme structure of the non-reduced locus of \( \tilde{X} \)

We describe further details of the scheme structure of \( \tilde{X} \) along the orbit \( G.x \), where \( x \) is given in the previous section. Let
\[
\tilde{X}_0 := \left\{ (B_1, B_2, i) \in \tilde{X} \mid B_1 \text{ has two distinct eigenvalues} \right\}
\]
(an affine open subset of \( \tilde{X} \)). Here we identified \( \tilde{X} \) with the image of a projection so that \( \tilde{X} \) consists of triples rather than quadruples. Let \( \rho: \text{Hom}(W, \mathbb{C}^2) \to \mathfrak{sp}(1), \ i \mapsto ii^* \), where \( W(= \mathbb{C}^3), \mathbb{C}^2 \) are the orthogonal and symplectic vector spaces respectively. We used this map in the proof of [2, Theorem 4.1].

**Proposition 2.1.** (1) There is a surjective smooth morphism from \( \text{Sp} \times \mathbb{C}^* \times \mathbb{C} \times (\rho^{-1}(0))^2 \) to \( \tilde{X}_0 \).

(2) This morphism restricts to a smooth morphism from the open subset \( \text{Sp} \times \mathbb{C}^* \times \mathbb{C} \times (\rho^{-1}(0) \setminus 0)^2 \) to \( G.x \).

The proof and also the precise definition of the above smooth morphism will appear in Section 2.3.

As a corollary, the structures of local analytic singularities of \( \tilde{X}_0 \) and \( (\rho^{-1}(0))^2 \) (resp. \( G.x \) and \( (\rho^{-1}(0) \setminus 0)^2 \)) are equivalent. By [2, Theorem 4.1 (4)], \( \rho^{-1}(0) \) and \( \rho^{-1}(0) \setminus 0 \) are non-reduced and thus so are \( \tilde{X}_0 \) and \( G.x \).

2.1. **Strata of \( \tilde{X}_0 \).** Recall the decomposition \( \tilde{X} = \tilde{X}_1 \sqcup \tilde{X}_2 \), where \( \tilde{X}_1 = \{(B_1, B_2) = 0\} \). Recall also that \( \tilde{X}_2 \) consists of four orbits \( G.x^{(A)} \), \( A = I, II, III, IV \) ([2, Corollary 8.9]).

We notice that \( \tilde{X}_0 \setminus \tilde{X}_1 = G.x (= G.x^{(0)}) \) because \( \tilde{X}_0 \) does not contain the other orbits \( G.x^{(A)}, \ A = II, III, IV \) in \( \tilde{X}_2 \) as

\[
B_1^{(II)} = B_1^{(III)} = B_1^{(IV)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\]

are nilpotent matrices (hence with the only eigenvalue 0). By the dimension reason, \( G.x \) is Zariski open dense in \( \tilde{X}_0 \).

2.2. **Factorization property.** In this subsection we introduce factorization property [1, Section 2.16] as a preliminary step of the proof of Proposition 2.1. This property can be stated more generally, so we assume \( V = \mathbb{C}^k, W = \mathbb{C}^N \) (symplectic and orthogonal vector spaces respectively). The argument here, taken from [3, Section 2.6], originates from the proof of [5, Lemma 12.3.2].

For any given partition \( \eta = (\eta_1, \eta_2, ..., \eta_e) \) of \( k \), we denote by \( S^\eta \mathbb{C} \) the product of symmetric products \( S^{\eta_n} \mathbb{C} \). Let \( (S^\eta \mathbb{C})_0 \) be the Zariski open subset of \( S^\eta \mathbb{C} \) consisting of the \( e \)-tuples \((Z_1, Z_2, ..., Z_e)\) such that the supports \( \text{Supp}(Z_n) \) are mutually disjoint. Let \( \Delta S^\eta \mathbb{C} \) be the Zariski closed subset of \( S^\eta \mathbb{C} \) consisting of \((Z_1, Z_2, ..., Z_e)\) such that \( \text{Supp}(Z_n) \) are one-point sets for all \( n \). Let \( (\Delta S^\eta \mathbb{C})_0 := (S^\eta \mathbb{C})_0 \cap \Delta S^\eta \mathbb{C} \). Let \( A_\eta: S^\eta \mathbb{C} \to S^k \mathbb{C}, (Z_1, Z_2, ..., Z_e) \mapsto \sum_{n=1}^e Z_n \). It is a quasi-finite map and the restriction \( A_\eta|_{(S^\eta \mathbb{C})_0} \) is étale. There is a decomposition of \( S^k \mathbb{C} \) of the form

\[
S^k \mathbb{C} = \bigsqcup_{|\eta|=k} A_\eta((\Delta S^\eta \mathbb{C})_0).
\]

(2.1)

Here \( |\eta| = \sum_n \eta_n \).

Let \( \pi: N_{V,W} \to S^k \mathbb{C}, (B_1, B_2, i) \mapsto E_k(B_1) \). Using \( \pi, A_\eta \), we consider fibre products \( N_{V,W} \times_{S^k \mathbb{C}} (S^\eta \mathbb{C})_0 \) and \( \mu^{-1}(0) \times_{S^k \mathbb{C}} (S^\eta \mathbb{C})_0 \). We notice that unless all \( \eta_n \) are even, these
We know that fibre products are empty, since any generalized eigenspace of $B_1$ is a symplectic subspace ([2, Lemma 5.2 (1)]).

From now on we may assume $\eta_n$ are all even. We fix mutually orthogonal symplectic subspaces $V^0_1, V^0_2, \ldots, V^0_n$ in $V$ with $\dim V^0_n = \eta_n$. Let $\mu_n$ be the moment map with respect to $\text{Sp}(V^0_n)$ on $N_{V^0_n,W}$ for each $n$. Let

$$X^0 := \left(N_{V^0_1,W} \oplus N_{V^0_2,W} \oplus \cdots \oplus N_{V^0_n,W}\right) \times_{S^{\eta_n} C} (S^{\eta_n} C)_0, \quad Y^0 := N_{V,W} \times_{S^{k_n}} (S^{k_n} C)_0$$

for short. Note that $X^0$ is Zariski open in $N_{V,W} \oplus N_{V_2,W} \oplus \cdots \oplus N_{V_n,W}$. Using the orthogonal decomposition $V = V^0_1 \oplus V^0_2 \oplus \cdots \oplus V^0_n$, we define a morphism

$$\pi^0_n : X^0 \to Y^0, \quad (B_{n,1}, B_{n,2}, i_n)_{1 \leq n \leq e} \mapsto (\bigoplus_{n=1}^e B_{n,1}, B_2, \bigoplus_{n=1}^e i_n, Z_1, Z_2, \ldots, Z_e),$$

where the matrix $B_2$ is determined so that its block $B_2^{(m,n)} : V^0_n \to V^0_m$ satisfies

$$B_2^{(m,n)} B_2^{(n,m)} - B_2^{(m,n)} B_2^{(n,m)} = 0 \quad \text{if } m \neq n$$

and $Z_n$ is the set of eigenvalues of $B_1|V^0_n$ counted with multiplicities. Note that $B_2^{(m,n)}$ ($m \neq n$) is determined uniquely by (2.4) as $B_{m,1}$ and $B_{n,1}$ have different eigenvalues. Hence each matrix element of $B_2$ is given as a rational function in the matrix elements of $B_{n,1}, B_{n,2}, i_n$ which is defined on $X^0$. Note that $B_2 \in \mathfrak{p}(V)$. This can be seen by taking the right adjoint to each equation in (2.4). Thus we deduce the following lemma:

**Lemma 2.2.** $\pi^0_n$ is a closed embedding. Moreover $(\pi^0_n)^* : \mathbb{C}[Y^0] \to \mathbb{C}[X^0]$ is a surjective algebra homomorphism with the kernel ideal generated by the second equation of (2.4), where $\mathbb{C}[X^0], \mathbb{C}[Y^0]$ denote the affine coordinate rings of $X^0, Y^0$ respectively.

**Proof.** Note that $X^0, Y^0$ are affine varieties since $(S^{\eta_n} C)_0$ and $A_\eta((S^{\eta} C)_0)$ are affine varieties and $A_\eta(S^{k_n} C)_0$ is a finite morphism. The lemma itself is a direct consequence of the definition of $\pi^0_n$. □

We define closed subschemes of $X^0, Y^0$ as the 0 loci of the moment maps:

$$X := (\mu_1^{-1}(0) \times \mu_2^{-1}(0) \times \cdots \times \mu_e^{-1}(0)) \times_{S^{\eta_n} C} (S^{\eta_n} C)_0, \quad Y := \mu^{-1}(0) \times_{S^{k_n}} (S^{k_n} C)_0.$$

Since $\mu$ becomes the product $\prod_{n=1}^e \mu_n$ on $X^0$ after composition with $\pi^0_n$ and the étale projection $Y^0 \to N_{V,W}$, we have $\pi^0_n(X) \subset Y$. We denote this restriction by $\pi_n : X \to Y$.

Let $\text{Sp} := \text{Sp}(V), \text{Sp}' := \prod_{n=1}^e \text{Sp}(V^0_n)$ with the Lie algebras $\mathfrak{sp}, \mathfrak{sp}'$ respectively. The $\text{Sp}$-action on $N_{V,W}$ naturally lifts to $Y'$ (by giving the trivial $G$-action on the factor $S^{\eta_n} C$). Similarly there is also a natural $\text{Sp}'$-action on $X'$. We consider another $\text{Sp}'$-action on $\text{Sp} \times X'$, defined by $h.(g, x) = (gh^{-1}, h.x)$.

**Lemma 2.3.** (1) $\pi_{\eta}, \pi'_\eta$ are $\text{Sp}'$-equivariant closed embeddings.

(2) The action map $\sigma : \text{Sp} \times X \to Y, (g, x) \mapsto g.\pi_\eta(x)$ is a surjective smooth morphism with fibres isomorphic to $\text{Sp}'$.

**Proof.** (1) By Lemma 2.2 we know that $\pi'_\eta$ is a closed embedding with ideal defined by (2.4). The restriction to $\mu = 0$ gives further equations on $B_1$ and the diagonal blocks $B_{n,1}$.
Taking quotients by the ideals corresponding to \((\pi_\eta')^*\) we obtain the induced surjective homomorphism \((\pi_\eta)^*: \mathbb{C}[Y] \to \mathbb{C}[X]\), so that \(\pi_\eta\) is a closed embedding.

We check that \(\pi_\eta, \pi_\eta'\) are \(Sp^I\)-equivariant. Let \(h = (h_1, h_2, ..., h_e) \in Sp^I\). We need to check the block corresponding to \(\bigoplus_{n=1}^e h_n B_{n,2}\) is \(h_m B_{2(n,m)} h_n^{-1}\). This follows from the uniqueness of the solution of (2.4) after conjugating the first equation (2.4) by \(h_n\) and multiplying the second equation (2.4) by \(h_m\) on the left and \(h_n^{-1}\) on the right. The details are left to the reader (or see the proof of [3, Lemma 2.7]).

(2) We denote the \(Sp^I\)-action map by \(\sigma': Sp \times X' \to Y', (g, x) \mapsto g.\pi_\eta'(x)\). We claim that \(\sigma'\) is a submersion with fibres isomorphic to \(Sp^I\). Let us finish the proof of the lemma assuming this claim. We notice that \(Y\) is a closed subscheme of the image variety of \(\sigma'\). For, \(\text{Im}(\sigma')\) is a subscheme defined only by the second equation of (2.4) while \(Y\) is defined by the additional equations from \(\mu = 0\) on the diagonal blocks. Thus \(\sigma\) satisfies the statement as it comes from \(\sigma'\) via a base change.

We now prove the claim in the rest of the proof. First we describe the fibres of \(\sigma'\). For the second \(Sp^I\)-action on \(Sp \times X'\), let \(O_{(g,x)}\) be the free \(Sp^I\)-orbit of \((g, x)\) in \(Sp \times X'\). Note that set-theoretically

\[
\sigma'^{-1}(\sigma'(g, x)) = O_{(g,x)}.
\]

For, this amounts to \(\sigma'(g, x) = g.\pi_\eta'(x) \in \pi_\eta'(X')\) implies \(g \in Sp^I\). Let us prove this. We write \(x = (B_1, B_2, i, Z_1, Z_2, ..., Z_e) \in \pi_\eta'(X')\). The sum of generalized eigenspaces of \(B_1\) with eigenvalues in \(Z_n\) is \(V_n^0\) for each \(n = 1, 2, ..., e\). The assumption \(g.\pi_\eta'(x) \in \pi_\eta'(X')\) asserts that \(g.B_1\) also has \(V_n^0\) as the sum of generalized eigenspaces with eigenvalues in \(Z_n\) for each \(n\). This implies \(g^{-1}(V_n^0) = V_n^0\) and thus we have \(g \in Sp^I\).

It remains to prove that \(\sigma'\) is submersive: the differential \(d\sigma'\) at any point has constant rank \(\dim Sp + \dim X' - \dim Sp^I\). Equivalently \(\ker(d\sigma'_{(g,x)}) = T_{(g,x)}O_{(g,x)}\). By the \(Sp^I\)-equivariance we may assume \(g\) is the identity element of \(Sp\). A direct computation shows

\[
\ker(d\sigma'_{(g,x)}) = \left\{ (\xi, \delta x) \in sp \oplus \bigoplus_{n=1}^e N_{V_n^0, W} \big| \xi.x + \delta x = 0 \right\} = T_{(g,x)}O_{(g,x)}.
\]

This completes the proof.

If \(\eta = (k)\), we have \(X' = Y', X = Y\) and \(Sp = Sp^I\), so the above lemma gives only the obvious isomorphism. Otherwise the étale open subset of \(\mu^{-1}(0)\) associated to \(\eta\) is isomorphic to a product of strata involving strictly lower instanton numbers. We call this property the factorization property of \(\mu^{-1}(0)\).

2.3. **Proof of Proposition 2.1.** Let \(k = 4, N = 3\) and \(\eta = (2,2)\). Let \(q: Y \to \mu^{-1}(0)\) be the (open étale) projection. We set the morphism in Proposition 2.1 (1) to be the restriction of \(q\sigma\) to \((q\sigma)^{-1}(\tilde{X}_0)\). Due to Lemma 2.3 (2), we need to show that \((q\sigma)^{-1}(\tilde{X}_0) \cong Sp \times \mathbb{C}^* \times \mathbb{C} \times (\rho^{-1}(0))^2\). By [2, Theorem 4.1 (1)], there is a natural identification \(\mu_2^{-1}(0) = \mathbb{C}^* \times \mathbb{C} \times \rho^{-1}(0)\), where \(\mathbb{C}^* = \{\text{tr}(B_1)\} = \mathbb{C}\). By the factorization property, \((q\sigma)^{-1}(\tilde{X}_0)\) is the closed subscheme of \(X = (\mu_2^{-1}(0))^2 \times_{S^0\mathbb{C}} (S^0\mathbb{C})_0\) whose \((\mathbb{C}^*)^2\)- and \(\mathbb{C}^2\)-factors both consist of points of the form \((a, -a), a \in \mathbb{C}\). This proves the first item.
The second item follows from $G.x = \tilde{X}_0 \setminus \tilde{X}_1$, so that $\rho(i) \neq 0$ whenever $i$ comes from $G.x$. This finishes the proof of Proposition 2.1.

2.4. Remark. Note that $B^{(1)} = v_5$ has the $1/2$-eigenspace $\mathbb{C}\langle e_1 + e_3, e_2 + e_4 \rangle$ and the $-1/2$-eigenspace $\mathbb{C}\langle e_1 - e_3, e_2 - e_4 \rangle$. Thus by choosing any $g \in \text{Sp}$ which maps these $\pm 1/2$-eigenspaces to $\mathbb{C}\langle e_1, e_2 \rangle, \mathbb{C}\langle e_3, e_4 \rangle$ respectively, the ADHM datum $g.(B_1^{(1)}, B_2^{(1)}, i^{(1)})$ comes from the product $(\mu^{-1}(0))^{2}$ by the factorization property.

The stabilizer in $\text{Sp}(V)$ of any $x = (B_{1,1}, B_{1,2}, i_1) \oplus (B_{2,1}, B_{2,2}, i_2) \in (\mu^{-1}(0))^{2} \times S^{(2,2)}\mathbb{C}$ is easy to compute, because $p(\mathbb{C}^2) = \mathbb{C}$, and thus $\text{Sp}(V)^{B_{1,1} \oplus B_{2,1}} = \text{Sp}(1)^2$. Therefore $\text{Sp}(V)^x = \text{Sp}(1)^{i_1} \times \text{Sp}(1)^{i_2}$. Since rank$(i_1) = \text{rank}(i_2) = 1$ (as $i_1, i_2 \in \rho^{-1}(0)$), both $\text{Sp}^{i_1}, \text{Sp}^{i_2}$ are isomorphic to the additive group $\mathbb{C}$. This is what we computed in Section 1.1 at the Lie algebra level.

3. Scheme structure of the GIT quotient $\tilde{X}_0//\text{Sp}$

Recall that $\tilde{X}_0$ is the complement of the vanishing locus of a $G$-invariant polynomial, the discriminant of the characteristic polynomial of $B_1$. So it is an affine open subset of $\tilde{X}$ and the GIT quotient $\tilde{X}_0//\text{Sp}$ is also an affine open subset of $\tilde{X}//\text{Sp}$.

In this section we further describe the scheme structure of $\tilde{X}_0//\text{Sp}$.

Lemma 3.1. The GIT quotient $\rho^{-1}(0)//\text{Sp}$ is a non-reduced scheme.

Corollary 3.2. $\tilde{X}_0//\text{Sp}$ is a non-reduced scheme.

3.1. Proof of Corollary 3.2. We first prove the corollary assuming Lemma 3.1.

By Proposition 2.1 we have an isomorphism $\tilde{X}_0 \cong (\text{Sp} \times \mathbb{C}^* \times \mathbb{C} \times (\rho^{-1}(0))^{2})/(\text{Sp}' \times \mathbb{Z}/2\mathbb{Z})$. Here the $\text{Sp}'$-action was introduced before Lemma 2.3, so $\text{Sp}'$ acts nontrivially only on the two factors $\text{Sp}, (\rho^{-1}(0))^{2}$. And $\mathbb{Z}/2\mathbb{Z}$ acts only on the factor $\mathbb{C}$ by $a \mapsto -a$. These two actions commute and the $\text{Sp}' \times \mathbb{Z}/2\mathbb{Z}$-action is free. Considering the quotient by the further $\text{Sp}$-action action on the factor $\text{Sp}$, we obtain an isomorphism $\tilde{X}_0//\text{Sp} \cong (\text{Sp} \times \mathbb{C}^* \times \mathbb{C} \times (\rho^{-1}(0))^{2})/(\text{Sp}' \times \mathbb{Z}/2\mathbb{Z})$. Taking the quotients by $\text{Sp}$ and then by $\text{Sp}' \times \mathbb{Z}/2\mathbb{Z}$, we obtain an isomorphism

$$\tilde{X}_0//\text{Sp} \cong \mathbb{C}^* \times \mathbb{C} \times (\rho^{-1}(0)//\text{Sp}(1))^{2}.$$ 

Since the factor $\rho^{-1}(0)//\text{Sp}(1)$ is non-reduced by Lemma 3.1, our assertion is proven.

3.2. Proof of Lemma 3.1. Recall that from the $ab$-diagram of elements in $\rho^{-1}(0)$ the reduced scheme $\rho^{-1}(0)//\text{Sp}(1)_\text{red}$ is a one-point set ([3, Theorem 4.1 (5)]).

Let us check that $\rho^{-1}(0)//\text{Sp}(1)$ is a non-reduced subscheme of $W^\vee$. On $\text{Hom}(W, V) = W^\vee \otimes V$, the defining equations $\rho(i) = ii^* = 0$ are quadratic in $W^\vee$. Therefore the $\text{Sp}$-invariants in the ideal always have degree $\geq 2$, as the invariants are obtained from the integration over $\text{Sp}$ with respect to the normalized Haar measure (i.e. Reynolds’ operator). So the invariant ideal in $\mathbb{C}[W^\vee]$ is contained in $m_0^2$, where $m_0$ denotes the maximal ideal defining $0 \in W^\vee$. This means that $\rho^{-1}(0)//\text{Sp}(1)$ contains the fat point $\mathbb{C}[x, y, z]/(x, y, z)^2$ as a closed subscheme, hence it is non-reduced.
3.3. **Remark.** In fact $\rho^{-1}(0)/\text{Sp}(1)$ coincides with the above fat point. This can be checked by applying the integration over Sp to the matrix elements of $ii^*$. We do not give the detailed computation here (see [4, Appendix A]).

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Dept. Math., Kyungpook Nat’l Univ., Sangyuk-dong, Buk-gu, Daegu 702-701, Korea
E-mail address: choy@knu.ac.kr