Narrow exotic tetraquark mesons in large-\(N_c\) QCD

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To provide the theoretical understanding of exotic tetraquark mesons, many candidates for which have been reported in the recent years (see \([1,2]\)), it is conventional to refer to QCD with a large number of colors \(N_c\) [i.e., \(SU(N_c)\) gauge theory for large \(N_c\)] with a simultaneously decreasing coupling \(\alpha_s \sim 1/N_c\) [\([3,4]\): at \(N_c\)-leading order, large-\(N_c\) QCD Green functions have only non-interacting mesons as intermediate states; tetraquark bound states may emerge only in \(N_c\)-subleading diagrams \([5]\). For many years, this fact was believed to provide the theoretical explanation of the non-existence of exotic tetraquarks. However, as emphasized in \([6]\), even if the exotic tetraquark bound states appear only in subleading diagrams, the crucial question is the width of these objects: if narrow, they might be well observed in nature. The conclusion of \([6]\) was that, if tetraquark states exist, they may be as narrow as the ordinary mesons, i.e., have a width \(\sim 1/N_c\). This issue has been further addressed in \([7]\), discussing the dependence of the width of tetraquark mesons on their flavor structure. Finally, \([8]\) reported for the cryptoexotic tetraquarks an even smaller width of order \(N_c^{-3}\).

Before drawing a conclusion about the width of a potential tetraquark pole in large-\(N_c\) QCD, it is mandatory to formulate rigorous criteria for selecting diagrams appropriate for the analysis of potential tetraquark poles. We find that both flavor-exotic and cryptoexotic (i.e., flavor-nonexotic) tetraquarks, if such poles exist, have a width of order \(O(1/N_c^2)\), so they are parametrically narrower compared to the ordinary \(q\bar{q}\) mesons, which have a width of order \(O(1/N_c)\). Moreover, for flavor-exotic states, the consistency of the large-\(N_c\) behavior of “direct” and “recombination” Green functions requires two narrow flavor-exotic states, each coupling dominantly to one specific meson-meson channel.

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1. MOTIVATION

To provide the theoretical understanding of exotic tetraquark mesons, many candidates for which have been reported in the recent years (see \([1,2]\)), it is conventional to refer to QCD with a large number of colors \(N_c\) [i.e., \(SU(N_c)\) gauge theory for large \(N_c\)] with a simultaneously decreasing coupling \(\alpha_s \sim 1/N_c\) [\([3,4]\): at \(N_c\)-leading order, large-\(N_c\) QCD Green functions have only non-interacting mesons as intermediate states; tetraquark bound states may emerge only in \(N_c\)-subleading diagrams \([5]\). For many years, this fact was believed to provide the theoretical explanation of the non-existence of exotic tetraquarks. However, as emphasized in \([6]\), even if the exotic tetraquark bound states appear only in subleading diagrams, the crucial question is the width of these objects: if narrow, they might be well observed in nature. The conclusion of \([6]\) was that, if tetraquark states exist, they may be as narrow as the ordinary mesons, i.e., have a width \(\sim 1/N_c\). This issue has been further addressed in \([7]\), discussing the dependence of the width of tetraquark mesons on their flavor structure. Finally, \([8]\) reported for the cryptoexotic tetraquarks an even smaller width of order \(N_c^{-3}\).

Before drawing a conclusion about the width of a potential tetraquark pole in large-\(N_c\) QCD, it is mandatory to formulate rigorous criteria for selecting those QCD diagrams that may lead to the appearance of this pole: the crucial property of such sequence of diagrams is the presence of four-quark intermediate states and the corresponding cuts in the variable \(s\) if one expects to observe a tetraquark pole in \(s\) \([9]\). The presence of such a four-particle \(s\)-cut should be established on the basis of the Landau equations \([10]\). It turns out that some of the diagrams attributed to the tetraquark pole in previous analyses are lacking the necessary four-particle cut and thus may not be related to the tetraquark properties. According to our findings, the tetraquark width at large \(N_c\) does not depend on its flavor structure; both flavor-exotic and flavor-nonexotic tetraquarks have the same width of order \(1/N_c^2\).

We analyse four-point Green functions of bilinear quark currents of various flavor content; any such function depends on six kinematical variables: the four momenta squared of the external currents, \(p_1^2, p_2^2, p_3^2, p_4^2\), \(p = p_1 + p_2 = p_1' + p_2'\), and the two Mandelstam variables \(s = p^2\) and \(t = (p_1 - p_1')^2\). When selecting the diagrams which potentially contribute to the tetraquark pole at \(s = M_T^2\), we apply the following two criteria:

1. The diagram should have a nontrivial (i.e., non-polynomial) dependence on the variable \(s\).

2. The diagram should have four-quark intermediate states and corresponding cuts starting at \(s = (m_1 + m_2 + m_3 + m_4)^2\), where \(m_i\) are the masses of the quarks forming the tetraquark bound state. The presence or absence of this cut is established by solving the Landau equations for the corresponding diagram.

Making use of the Landau equations is an unambiguous way to identify the set of QCD diagrams which have the four-quark cut; this is a necessary (although not a sufficient) condition that these diagrams contribute to the tetraquark pole. The four-quark cut should be present in the individual QCD diagrams that are appropriate for the tetraquark analysis, but of course this cut will be replaced in the complete Green functions by the tetraquark pole (in case it exists), plus the meson continuum, as soon as the infinite set of QCD diagrams is considered.

Not all diagrams in the perturbative expansion of Green functions satisfy the above criteria, so we decompose these diagrams into two sets: diagrams belonging to the first set either do not depend on \(s\) or have no four-quark cut in the \(s\)-channel and thus are not related to tetraquarks; diagrams of the second set satisfy both of the above criteria and thus contribute to the potential tetraquark pole.
With the formulated criteria at hand, we discuss separately two cases: tetraquarks of an exotic flavor content, i.e., built up of quarks of four different flavors, $\bar{q}_1 q_2 \bar{q}_3 q_4$, and cryptoexotic tetraquarks, with flavor content $\bar{q}_1 q_2 \bar{q}_3 q_3$, carrying the same flavor as ordinary mesons. The need for separate treatment of flavor-exotic and cryptoexotic cases arises from the different topologies of the QCD diagrams emerging for these two cases.

2. FLAVOUR-EXOTIC TETRAQUARKS

Let us consider a bilinear quark current $j_{ij} = \bar{q}_i \bar{O} q_j$ producing a meson $M_{ij}$ of flavor content $\bar{q}_i q_j$ from the vacuum, $\langle 0 | j_{ij} | M_{ij} \rangle = f_{M_{ij}}$. Here, $\bar{O}$ is a combination of Dirac matrices corresponding to the meson’s spin and parity. We shall omit all Lorentz structures as they are irrelevant for our analysis. At large $N_c$, the meson decay constants $f_M$ scale as $f_M \sim \sqrt{N_c}$.

In the case of four-point functions of bilinear currents involving quarks of four different flavors, denoted by 1, 2, 3, 4, there are two types of Green functions: the “direct” functions $\Gamma^{(\text{dir})}_I = \langle j_{12}^{\dagger} j_{34} j_{12} j_{34} \rangle$ and $\Gamma^{(\text{dir})}_{II} = \langle j_{14}^{\dagger} j_{32} j_{14} j_{32} \rangle$, and the “recombination” functions $\Gamma^{(\text{rec})} = \langle j_{12}^{\dagger} j_{34} j_{14} j_{32} \rangle$ and $\Gamma^{(\text{rec})\dagger}$.

Figure 1 shows the perturbative expansion of the direct correlator $\Gamma^{(\text{dir})}_I$. Similar diagrams defined by evident flavor rearrangements describe the correlator $\Gamma^{(\text{dir})}_{II}$. Obviously, not all these diagrams satisfy our above criteria for diagrams that potentially contain a tetraquark pole. For instance, the diagrams in Fig. 1(a,b) do not depend on $s$; the leading large-$N_c$ diagram which depends on $s$ and also has a four-quark $s$-cut is given by Fig. 1(c). The diagrams of this type are therefore the leading large-$N_c$ diagrams of interest to us.

![Diagram](image)

$\sim N_c^2$ (a) $\sim N_c^3 \alpha_s$ (b) $\sim N_c^2 \alpha_s^2$ (c)

The analysis of the recombination channel is a bit more involved: among the diagrams in Fig. 2 the first two diagrams (a,b) do depend on $s$; however, in spite of their appearance, they have no four-quark cut. The easiest way to see this is to redraw these diagrams as the usual box diagram (a) and the box diagram with one-gluon exchange (b). The $N_c$-leading diagram exhibiting the four-quark cut is the nonplanar diagram in Fig. 2(c). The four-quark cut with the threshold at $s = (m_1 + m_2 + m_3 + m_4)^2$ may be verified by solving the Landau equations. We thus find the following $N_c$-leading behavior of the Green functions $\Gamma_T$ potentially involving a pole corresponding to some tetraquark $T$:

$$\Gamma^{(\text{dir})}_{1,T} = \langle j_{12}^{\dagger} j_{34} j_{12} j_{34} \rangle = O(N_c^0), \quad \Gamma^{(\text{dir})}_{11,T} = \langle j_{14}^{\dagger} j_{32} j_{14} j_{32} \rangle = O(N_c^0), \quad \Gamma^{(\text{rec})}_T = \langle j_{12}^{\dagger} j_{34} j_{14} j_{32} \rangle = O(N_c^{-1}). \quad (2.1)$$

These diagrams potentially contain the tetraquark pole, although the actual existence of this pole is still a conjecture. Now, let us assume that narrow resonances (i.e., resonances with widths vanishing for large $N_c$) show up at the lowest possible $1/N_c$ order and that the resonance mass $M_T$ remains finite at large $N_c$. The fact that direct and recombination amplitudes have different behaviors in $N_c$ leads us to the conclusion that a single pole is not sufficient and we need at least two exotic poles, denoted by $T_A$ and $T_B$: $T_A$ couples stronger to the $M_{12}M_{34}$ channel, while $T_B$ couples stronger to the $M_{14}M_{32}$ channel.
In contrast to the flavor-exotic case, now both the direct and the recombination Green functions have the same leading behavior at large $N_c$. The couplings of this state to the meson-meson channels are restricted by the dominance of the tetraquark states. For the direct Green functions, the situation changes qualitatively: the new diagrams in Fig. 2(b) determine the behavior of $\Gamma^{(\text{dir})}_{T}$. The only new ingredient is the appearance of these diagrams.

So far we have ignored the mixing between $\Gamma^{(\text{dir})}_{I,T}$ and $\Gamma^{(\text{rec})}_{T}$. Introducing their mixing parameter $g_{AB}$, we get additional contributions to the above Green functions. Most restrictive for $g_{AB}$ is the recombination function, for which mixing provides the additional contribution

$$
\Gamma^{(\text{rec})}_{T} = O(N_c^{-1}) = f_4^4 \frac{A(T_A \to M_{12}M_{34})A(T_A \to M_{14}M_{32})}{p^2 - M_{T_A}^2},
$$

Taking into account that $f_4 \sim \sqrt{N_c}$ and that we are seeking tetraquarks with finite mass at large $N_c$, these equations have the following solution:

$$
A(T_A \to M_{12}M_{34}) = O(N_c^{-1}), \quad A(T_A \to M_{14}M_{32}) = O(N_c^{-2}),
$$

$$
A(T_B \to M_{12}M_{34}) = O(N_c^{-1}), \quad A(T_B \to M_{14}M_{32}) = O(N_c^{-1}).
$$

The widths $\Gamma(T_{A,B})$ of the states $T_A$ and $T_B$ are determined by the dominant channel, which yields $\Gamma(T_{A,B}) = O(N_c^{-2})$.

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\Gamma^{(\text{rec})}_{T} = O(N_c^{-1}) = f_4^4 \frac{A(T_A \to M_{12}M_{34})A(T_B \to M_{14}M_{32})}{p^2 - M_{T_A}^2},
$$

Equations (2.3) and (2.4) restrict the behavior of the mixing parameter to $g_{AB} \leq O(N_c^{-1})$. Thus, the two flavor-exotic tetraquarks of the same flavor content do not mix at large $N_c$.

### 3. CRYPTOEXOTIC TETRAQUARKS

We now turn to tetraquarks with nonexotic flavor content, i.e., having the same flavor as the ordinary mesons. The analysis proceeds along the same line as for the exotic states. The only new ingredient is the appearance of diagrams of new topologies compared to the flavor-exotic case. For the dominant Green functions $\Gamma^{(\text{dir})}_{I,I,T}$ (Fig. 3), the new diagrams do not change the leading large-$N_c$ behavior compared to the diagrams of the same topology in the flavor-exotic case. For the recombination functions, however, the situation changes qualitatively: the new diagram, Fig. 3(b), dominates at large $N_c$ and thus modifies the leading large-$N_c$ behavior of $\Gamma^{(\text{rec})}_{T}$. We thus find

$$
\Gamma^{(\text{dir})}_{I,I,T} = \langle j_{12}^4 j_{23}^4 j_{12} j_{23} \rangle = O(N_c^0), \quad \Gamma^{(\text{dir})}_{I,I,T} = \langle j_{13}^4 j_{22}^4 j_{13} j_{22} \rangle = O(N_c^0), \quad \Gamma^{(\text{rec})}_{T} = \langle j_{12}^4 j_{23}^4 j_{13} j_{22} \rangle = O(N_c^0).
$$

In contrast to the flavor-exotic case, now both the direct and the recombination Green functions have the same leading behavior at large $N_c$. As a consequence, one exotic state $T$ suffices to satisfy the expected large-$N_c$ behavior of both Green functions. The couplings of this state to the meson-meson channels are

$$
A(T \to M_{12}M_{23}) = O(N_c^{-1}), \quad A(T \to M_{13}M_{22}) = O(N_c^{-1}).
$$
Thus, the width of this single cryptoexotic state is of order $\Gamma(T) = O(N_c^{-2})$.

If all its quantum numbers allow, $T$ can mix with the ordinary meson $M_{13}$. The restriction on the mixing parameter $g_{TM_{13}}$ may be obtained, e.g., from the direct amplitude

$$\Gamma_{1,T}^{(dir)} = O(N_c^0) = J^M \frac{A(M_{12}M_{34} \rightarrow T)}{p^2 - M_T^2} g_{TM_{13}} \frac{A(M_{13} \rightarrow M_{12}M_{23})}{p^2 - M_{13}^2} + \cdots \quad (3.3)$$

Taking into account that $A(M_{13} \rightarrow M_{12}M_{23}) \sim 1/\sqrt{N_c}$ [3, 4], we obtain $g_{TM_{13}} \leq O(1/\sqrt{N_c})$.

4. CONCLUSIONS

We formulated a set of rigorous criteria for selecting those diagrams that are appropriate for the analysis of potential tetraquark states: In the four-point Green functions, one should take into account only those contributions which have four-quark cuts in the $s$-channel. Using these criteria and requiring that the narrow poles contribute to the appropriate parts of the Green functions at leading large-$N_c$ order, we gained the following insights:

1. The large-$N_c$ behavior of flavor-exotic four-point Green functions (all four quarks of different flavors $\bar{q}_1, \bar{q}_3, q_2, q_4$) is not compatible with merely one flavor-exotic tetraquark but requires two narrow states $T_A$ and $T_B$ with widths $\Gamma(T_A, T_B) = O(N_c^{-2})$. Each of these tetraquarks dominantly couples to one meson-meson channel; its coupling to the other meson-meson channel is suppressed: $A(T_A \rightarrow M_{12}M_{34}) = O(N_c^{-1})$, $A(T_A \rightarrow M_{14}M_{32}) = O(N_c^{-2})$, $A(T_B \rightarrow M_{14}M_{32}) = O(N_c^{-1})$, $A(T_B \rightarrow M_{12}M_{34}) = O(N_c^{-2})$. The parameter describing the mixing between the two tetraquarks vanishes for large $N_c$ at least like $1/N_c$.

2. The large-$N_c$ behavior of four-point Green functions of nonexotic flavor content ($\bar{q}_1, \bar{q}_3, q_2, q_3$) is compatible with the existence of a single narrow cryptoexotic tetraquark, $T$, with width $\Gamma(T) = O(N_c^{-2})$. This tetraquark couples parametrically equally to both two-meson channels: $A(T \rightarrow M_{12}M_{23}) = O(N_c^{-1})$, $A(T \rightarrow M_{13}M_{22}) = O(N_c^{-1})$. If quantum numbers allow, the cryptoexotic tetraquark $T = \bar{q}_1q_3\bar{q}_2q_2$ mixes with the ordinary meson $M_{13} = \bar{q}_1q_3$. The corresponding mixing parameter vanishes like $1/\sqrt{N_c}$, i.e., slower than that of the flavor-exotic tetraquarks.

We would like to mention that, in principle, there is a possibility that narrow tetraquarks do exist but appear only in $N_c$-subleading diagrams with four-quark intermediate states, while they do not contribute to the $N_c$-leading topologies. This possibility seems rather unnatural to us: if such pole exists at all, there should be some special reason, not evident to us, why it does not appear in the set of the appropriate $N_c$-leading diagrams. Nevertheless, also in the $N_c$-subleading
topologies one observes a difference in the large-$N_c$ behavior of the direct and the recombination diagrams of the flavor-exotic case: for any $N_c$-subleading topology, the direct diagrams are $N_c$-even, whereas the recombination diagrams are $N_c$-odd. Accordingly, if the latter scenario is realized in nature, one still encounters the necessity of two flavor-exotic poles, albeit with parametrically smaller widths at large $N_c$.

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