Quantum Mechanics and Discrete Time from “Timeless” Classical Dynamics

Hans-Thomas Elze

Instituto de Física, Universidade Federal do Rio de Janeiro
C.P. 68.528, 21941-972 Rio de Janeiro, RJ, Brazil
E-mail: thomas@if.ufrj.br

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Abstract

We study classical Hamiltonian systems in which the intrinsic proper time evolution parameter is related through a probability distribution to the physical time, which is assumed to be discrete.

This is motivated by the “timeless” reparametrization invariant model of a relativistic particle with two compactified extradimensions. In this example, discrete physical time is constructed based on quasi-local observables.

Generally, employing the path-integral formulation of classical mechanics developed by Gozzi et al., we show that these deterministic classical systems can be naturally described as unitary quantum mechanical models. The emergent quantum Hamiltonian is derived from the underlying classical one. It is closely related to the Liouville operator. We demonstrate in several examples the necessity of regularization, in order to arrive at quantum models with bounded spectrum and stable groundstate.
1 Introduction

Recently we have shown that for a particle with time-reparametrization invariant dynamics, both relativistic or nonrelativistic, one can define quasi-local observables which characterize the evolution in a gauge invariant way [1 2].

We insist on quasi-local measurements in describing the evolution, which respect reparametrization invariance of the system. Then, as we have argued, the physical time necessarily becomes discrete, its construction being based on a Poincaré section which reflects ergodic dynamics, by assumption. Most interestingly, due to inaccessibility of globally complete information on trajectories, the evolution of remaining degrees of freedom appears as in a quantum mechanical model when described in relation to the discrete physical time.

While we pointed out in the explicitly ergodic examples of references [1 2] that such emergent discrete time leads to what may, for obvious reasons, be called “stroboscopic” quantization, we report here how this occurs quite generally in classical Hamiltonian systems, if time is discrete and related to the proper time of the equations of motion in a statistical way [3]. In our concluding section, we will briefly comment about extensions, where the prescribed probabilistic mapping of physical onto proper time shall be abandoned in favour of a selfconsistent treatment. A closed system has to include its own “clock”, if it is not entirely static, reflecting the experience of an observer in the Universe.

Previous related work on the “problem of time” has always assumed that global features of the trajectory of the system are accessible to the observer. This makes it possible, in principle, to express the evolution of an arbitrarily selected degree of freedom relationally in terms of others [4 5]. Thereby the Hamiltonian and possibly additional constraints have been eliminated in favour of Rovelli’s “evolving constants of motion” [6]. For a recent development aiming at solving the constraints after discretization see [7]. While appealing by its conceptual clarity, incorporating nonlocal observations seems unrealistic to us in any case.

In distinction, we point out that the emergent discrete time in our approach naturally leads to the “stroboscopic” quantization of the system [1 2 3]. Quantum theory thus appears to originate from “timeless” classical dynamics, due to the lack of globally complete information [3].

Another approach to deterministically induced quantization is proposed in [8], where the consequences of incomplete statistics are analyzed, leading towards Euclidean quantum field theory under very general assumptions. Various other arguments considering quantization as an emergent property of classical systems have recently been proposed, for example, in references [9 10 11 12 13], concerning quantum gravity and dissipation at a fundamental level, “chaotic quantization”, and matrix models, respectively.

Our approach tries to illuminate from a different angle how to arrive at quantum models which describe dynamically evolving systems. In particular, we believe that there may be an intimate connection with how the “problem of time” is resolved for a local observer, namely by counting suitably defined and locally measurable incidents.

We remark that the possibility of a fundamentally discrete time (and possibly other discrete coordinates) has been explored before, ranging from an early realization of Lorentz symmetry in such a case [14] to detailed explorations of its consequences and consistency in classical mechanics, quantum field theory, and general relativity [15 16 17]. However, no detailed models giving rise to such discreteness have been proposed. Quantization, then, is always performed in an additional step, as usual.
In particular, the work by Gambini, Pullin et al. aims at a consistent canonical quantization of gravity via discretization \[7, 17\]. Discretization of time is performed in a static fashion, i.e. independently of the evolution. As shown there, the major advance lies in the possibility to satisfy the constraints, in principle, by suitably choosing the Lagrange multipliers. However, the extraneous discretization is reflected by the persistence of a discrete variable \(n\) after quantization, which apparently has no physical meaning.

Presently, we shall see a vague resemblance to the persistence of proper time \(\tau\) in our approach, as long as the discrete physical time is related through a given probability distribution to proper time. Our formalism, however, is set up in such a way that the “clock” degrees of freedom can be treated dynamically as part of the system. This should help to pinpoint the role of proper time in the resulting model where, in our case, quantization emerges instead of being imposed on the system.

In the following section, we recall the model from \[2\], in order to motivate the emergence of discrete time from quasi-local observations. This is the starting point of our heuristic derivation of a quantum mechanical picture of what appear fundamentally classical systems.

To put our approach into perspective, we remark that there is clearly no need to follow such construction leading to a discrete physical time in ordinary mechanical systems or field theories, where time is an external classical parameter, commonly called “t”. However, assuming for the time being that truly fundamental theories will turn out to be diffeomorphism invariant, adding further the requirement of the observables to be quasi-local (modulo a fundamental length scale), when describing the evolution, then such an approach seems natural, which may lead to quantum mechanics as an emergent description or “effective theory” on the way.

### 2 Discrete Time of a Relativistic Particle with Extradimensions

We consider the (5+1)-dimensional model of a “timeless” relativistic particle (rest mass \(m\)) with the action:

\[
S = \int ds \, L ,
\]

where the Lagrangian is defined by:

\[
L \equiv -\frac{1}{2} (\lambda^{-1} \dot{x}_\mu \dot{x}^\mu + \lambda m^2) .
\]

Here \(\lambda\) stands for an arbitrary “lapse” function of the evolution parameter \(s\), \(\dot{x}^\mu \equiv dx^\mu / ds (\mu = 0, 1, \ldots, 5)\), and the metric is \(g_{\mu\nu} \equiv \text{diag}(1, -1, \ldots, -1)\). Units are such that \(\hbar = c = 1\).

With this form of the Lagrangian, instead of the frequently encountered \(Lds \propto (g_{\mu\nu} dx^\mu dx^\nu)^{1/2}\) which emphasizes the geometric (path length) character of the action, the presence of a constraint is immediately obvious, since there is no \(s\)-derivative of \(\lambda\).

Two spatial coordinates, \(x^{4,5}\) in \([2]\), are toroidally compactified:

\[
x^{4,5} \equiv 2\pi R[\phi] ,
\]

\[
[\phi] \equiv \phi - n , \quad \phi \in [n, n + 1[ ,
\]

for any integer \(n\), i.e. the angular variables are periodically continued; henceforth we set \(R = 1\), for convenience. Alternatively, we can normalize the angular variables to the square \([0, 1[ \times [0, 1[,\)
of which the opposite boundaries are identified, thus describing the surface of a torus with main radii equal to one.

While full Poincaré invariance is broken, as in other currently investigated models with compactified higher dimensions, the usual one remains in four-dimensional Minkowski space together with discrete rotational invariance in the presently two extradimensions; also translational symmetry persists. Furthermore, the internal motion on the torus is ergodic with an uniform asymptotic density for almost all initial conditions, in particular if the ratio of the corresponding initial momenta is an irrational number.

Setting the variations of the action to zero, we obtain:

\[
\frac{\delta S}{\delta \lambda} = \frac{1}{2} (\lambda^{-2} \dot{x}_\mu \dot{x}^\mu - m^2) = 0 ,
\]

\[
\frac{\delta S}{\delta x_\mu} = \frac{d}{ds} (\lambda^{-1} \dot{x}^\mu) = 0 .
\]

In terms of the canonical momenta,

\[
p_\mu \equiv \frac{\partial L}{\partial \dot{x}^\mu} = -\lambda^{-1} \dot{x}_\mu ,
\]

the equations of motion become simply \( \ddot{p}^\mu = 0 \), while \( s \) turns into the mass-shell constraint \( p^2 - m^2 = 0 \).

The equations of motion are solved by:

\[
x^\mu (s) = x^\mu_i - p^\mu \int_0^s ds' \lambda(s') \equiv x^\mu_i + p^\mu \tau(s) ,
\]

where the conserved (initial) momentum \( p^\mu \) is constrained to be on-shell and \( x_i \) denotes the initial position. Here we also defined the fictitious proper time (function) \( \tau \), which allows us to formally eliminate the lapse function \( \lambda \) from \( s \) to \( \dot{x}^\mu (s) \) and \( \dot{x}^\mu (s) = -\lambda(s) \partial_\tau x^\mu (\tau) \).

In order to arrive at a physical space-time description of the motion, the proper time needs to be determined in terms of observables. In the simplest case, the result should be given by functions \( x^\mu \neq 0 (x^0) \), provided there is a physical clock measuring \( x^0 = x^0_i + p^0 \tau \).

Similarly as in the nonrelativistic example studied in [1], the lapse function introduces a gauge degree of freedom into the dynamics, which is related to the reparametrization of the evolution parameter \( s \). In fact, the action, \( s \), is invariant under the set of gauge transformations:

\[
s \equiv f(s'), \quad x^\mu (s) \equiv x^\mu (s'), \quad \lambda(s) \frac{ds}{ds'} \equiv \lambda' (s') .
\]

It can be shown that the corresponding infinitesimal transformations actually generate the evolution of the system. This is the basis of statements that there is no time in systems where dynamics is pure gauge, i.e. of the “problem of time”. We refer to [1] for further discussion.

Instead, with an evolution obviously taking place in such systems, we conclude from these remarks that the space-time description of motion requires a gauge invariant construction of a suitable time, replacing the fictitious proper time \( \tau \). To this we add the important requirement that such construction should be based on quasi-local measurements, since global information (such as invariant path length) is generally not accessible to an observer in more realistic, typically nonlinear or higher-dimensional theories.
2.1 “Timing” Through an Extradimensional Window

Our construction of a physical time is based on the assumption that an observer in (3+1)-dimensional Minkowski space can perform measurements on full (5+1)-dimensional trajectories, however, only within a quasi-local window to the two extradimensions. In particular, the observer records the incidents (“units of change”) when the full trajectory hits an idealized detector which covers a small convex area element on the torus (compactified coordinates $x^{4,5}$).\(^{1}\)

Thus, our aim is to construct time as an emergent quantity related to the increasing number of incidents measured by the reparametrization invariant incident number:

$$I \equiv \int_{s_i}^{s_f} ds' \lambda(s')D(x^4(s'), x^5(s'))$$,

where $x^{4,5}$ describe the trajectory of the particle in the extradimensions, the integral is taken over the interval which corresponds to a given invariant path $x^\mu_i \rightarrow x^\mu_f$, and the function $D$ represents the detector features. Operationally it is not necessary to know the invariant path, in order to count the incidents.

In the following examples we choose for $D$ the characteristic function of a small square of area $d^2$, $D(x^4, x^5) \equiv C_d(x^4)C_d(x^5)$, with $C_d(x) \equiv \Theta(x)(1 - \Theta(x - d))$, which could be placed arbitrarily. Our results will not depend on the detailed shape of this idealized detector, if it is sufficiently small. More precisely, an incident is recorded only when, for example, the trajectory either leaves or enters the detector, or according to some other analogous restriction which could be incorporated into the definition of $D$. Furthermore, in order not to undo records, we have to restrict the lapse function $\lambda$ to be (strictly) positive, which also avoids trajectories which trace themselves backwards (or stall). The records correspond to a uniquely ordered series of events in Minkowski space, which are counted, and only their increasing total number is recorded, which is the Lorentz invariant incident number.

Considering particularly the free motion on the torus, solution (8) yields:

$$\vec{\phi}(\tau) = \vec{\phi}_0 + \vec{\pi} \tau$$,

where $\vec{\phi}$ is the vector formed of the angles $\phi^{4,5}$, and correspondingly $\vec{\pi}$, with $\pi^{4,5} \equiv p^{4,5}/2\pi R$; the quantities in (11) are periodically continued, as before, see (3)-(4). Without loss of generality we choose $\vec{\phi}_0 = 0$ and $\pi^5 > \pi^4 > 0$, and place the detector next to the origin with edges aligned to the positive coordinate axes for simplicity.

Since here we are not interested in what happens between the incidents, we reduce the description of the internal motion to coupled maps. For proper time intervals $\Delta\tau$ with $\pi^4 \cdot \Delta\tau = 1$, the $\phi^4$-motion is replaced by the map $m \rightarrow m + 1$, where $m$ is a nonnegative integer, while:

$$[\phi^5] = [Pm]$$,

with $P \equiv \pi^5/\pi^4 > 1$. Then, also the detector response counting incidents can be represented as a map:

$$I(m + 1) = I(m) + \Theta(\delta - [\phi^5]) + \Theta([\phi^5] - (1 - P\delta))$$,

\(^{1}\)One could invoke a popular distinction between brane and bulk matter as in string theory inspired higher-dimensional cosmology, in order to construct more realistic models involving local interactions.
with \( I(0) \equiv 1 \), and where \( \delta \equiv d/2\pi R \) corresponds to the detector edge length \( d \), assumed to be sufficiently small, \( P\delta \ll 1 \). The two \( \Theta \)-function contributions account for the two different edges through which the trajectory can enter the detector in the present configuration.

The nonlinear two-parameter map \((13)\) has surprising universal features, some of which we explored in \([2]\). Here, first of all, following the reparametrization invariant construction up to this point, we identify the physical time \( T \) in terms of the incident number \( I \) from \((13)\):

\[
T \equiv \frac{I}{\delta(\pi^4 + \pi^5)}. \tag{14}
\]

A statistical argument for the scaling factor \( \delta^{-1}(\pi^4 + \pi^5)^{-1} \), based on ergodicity, has been given in \([1]\), which applies here similarly.

![Figure 1: The physical time \( T \) as a function of proper time \( \tau \) with detector parameter \( \delta = .005 \) and ratio of initial internal momenta \( P = \sqrt{31} \) (top), \( \sqrt{2} \), \( \pi \) (bottom) (see main text); upper two curves displaced upwards by +10 and +20 units, respectively, for better visibility (from \([2]\)).](image)

We show in Fig. 1 how the physical time \( T \) typically is correlated with the fictitious proper time \( \tau \). The proper time is extracted as those \( m \)-values when incidents happen: \( \tau = m + 1 \iff I(m + 1) - I(m) = 1 \), corresponding to a particularly simple specification of the detector response. For a sufficiently small detector other such specifications yield the same results as described here. This is achieved by always rounding the extracted proper time values to integers, involving negligible errors of order \( \delta, \delta/P \ll 1 \).

We find that the time \( T \) does not run smoothly. This is due to the coarse-grained description of the internal motion: as if we were reading an analog clock under a stroboscopic light. In our construction, it is caused by the reduction of the full motion to a map (Poincaré section), corresponding to the recording of the physical incidents by the quasi-local detector.
Furthermore, already after a short while, i.e. at low incident numbers, the constructed time approximates well the proper time \( \tau \) on average.\(^2\) The fluctuations on top of the observed linear dependence result in the discreteness of the constructed time.

While in reference \([2]\) we further analyzed the statistical properties of the map considered in this example, we take from here only the result that a physical time can be constructed based on suitable localized observations. Furthermore, after embarking on some useful formal developments in the next section, we will incorporate the resulting probabilistic mapping between discrete physical time and proper time of the equations of motion in Section 4.

### 3 Classical Mechanics in Path-integral Form

Classical mechanics can be cast into path-integral form, as originally developed by Gozzi, Reuter and Thacker \([18]\), and with recent addenda reported in \([19]\). While the original motivation has been to provide a better understanding of geometrical aspects of quantization, we presently use it as a convenient tool. We refer the interested reader to the cited references for details, on the originally resulting extended (BRST type) symmetry in particular. We suitably incorporate time-reparametrization invariance, assuming equations of motion written in terms of proper time.

Let us begin with a \((2n)\)-dimensional classical phase space \( \mathcal{M} \) with coordinates denoted collectively by \( \varphi^a \equiv (q^1, \ldots, q^n; p^1, \ldots, p^n) \), \( a = 1, \ldots, 2n \), where \( q, p \) stand for the usual coordinates and conjugate momenta. Given the proper-time independent Hamiltonian \( H(\varphi) \), the equations of motion are:

\[
\frac{\partial}{\partial \tau} \varphi^a = \omega^{ab} \frac{\partial}{\partial \varphi^b} H(\varphi) \ , \tag{15}
\]

where \( \omega^{ab} \) is the standard symplectic matrix and \( \tau \) denotes the proper time; summation over indices appearing twice is understood.

To the equation of motion we add the (weak) Hamiltonian constraint, \( C_H \equiv H(\varphi) - \epsilon \simeq 0 \), with \( \epsilon \) a suitably chosen parameter. This constraint has to be satisfied by the solutions of the equations of motion. Generally, it arises in reparametrization invariant models, similarly as the mass-shell constraint in the case of the relativistic particle \([2]\). It is necessary when the Lagrangian time parameter is replaced by the proper time in the equations of motion. In this way, an arbitrary “lapse function” is eliminated, which otherwise acts as a Lagrange multiplier for this constraint.

We remark that field theories can be treated analogously, considering indices \( a, b \), etc. as continuous variables.

Starting point for our following considerations is the classical generating functional,

\[
Z[J] \equiv \int_H \mathcal{D}\varphi \, \delta[\varphi^a(\tau) - \varphi^a_{cl}(\tau)] \exp(i \int d\tau \, J_a \varphi^a) \ , \tag{16}
\]

where \( J \equiv \{ J_a = 1, \ldots, 2n \} \) is an arbitrary external source, \( \delta[\cdot] \) denotes a Dirac \( \delta \)-functional, and \( \varphi_{cl} \) stands for a solution of the classical equations of motion satisfying the Hamiltonian constraint; its presence is indicated by the subscript “\( H \)” on the functional integral. The relevant boundary conditions shall be discussed in the following section. It is important to realize that \( Z[0] \) gives

\(^2\)The apparent excursion for the parameter value \( P = \pi \) in Fig. 1, does not persist for longer times, as shown in \([2]\).
weight 1 to a classical path satisfying the constraint and zero otherwise, integrating over all initial conditions.

Using the functional equivalent of \( \delta(f(x)) = |df/dx|_{x_0}^{-1} \cdot \delta(x - x_0) \), the \( \delta \)-functional under the integral for \( Z \) can be replaced according to:

\[
\delta[\varphi^a(\tau) - \varphi_{a1}^a(\tau)] \Rightarrow \delta[\partial_\tau \varphi^a - \omega^{ab} \partial_b H] \det[\delta^b_a \partial_\tau - \omega^{ac} \partial_c \partial_b H]
\]

slightly simplifying the notation, e.g. \( \partial_b \equiv \partial/\partial \varphi^b \). Here the modulus of the functional determinant has been dropped [18, 19].

Finally, the \( \delta \)-functionals and determinant are exponentiated, using the functional Fourier representation and ghost variables, respectively. Thus, we obtain the generating functional in the convenient form:

\[
Z[J] = \int_H D\varphi D\lambda D\bar{c} Dc \exp \left( i \int d\tau (L + J_a \varphi^a) \right)
\]

which we abbreviate as \( Z[J] = \int_H D\Phi \exp(i \int d\tau L_J) \). The enlarged phase space is \((8n)\)-dimensional, consisting of points described by the coordinates \((\varphi^a, \lambda_a, c^a, \bar{c}_a)\). The effective Lagrangian is now given by [18, 19]:

\[
L \equiv \lambda_a \left( \partial_\tau \varphi^a - \omega^{ab} \partial_b H \right) + i \bar{c}_a \left( \delta^b_a \partial_\tau - \omega^{ac} \partial_c \partial_b H \right) c^b
\]

where \( c^a, \bar{c}_a \) are anticommuting Grassmann variables. We remark that an entirely bosonic version of the path-integral exists [19].

This completes our brief review of how to put (reparametrization invariant) classical mechanics into path-integral form.

4 From Discrete Time to “States”

We recall from our previous example that the discrete physical time \( t \) has been obtained by counting suitably defined incidents, i.e., coincidences of points of the trajectory of the system with appropriate detectors [1, 2]. Thus, it is given by a nonnegative integer multiple of some unit time, \( t \equiv nT \). Then, we would like to express the proper time \( \tau \) which parametrizes the evolution in terms of \( t \).

Here we assume instead that the physical time \( t \) is mapped onto a normalized probability distribution \( P \) of proper time values \( \tau \):

\[
P(\tau; t) \equiv \exp \left( -S(\tau; t) \right), \quad \int d\tau \: P(\tau; t) = 1.
\]

For uniqueness, we require that if \( S(\tau; t_1) \) and \( S(\tau; t_2) \), for \( t_1 \neq t_2 \), have overlapping support, then they should coincide in this region.

Thus, we describe the idealized case that the system can be separated into degrees of freedom which are employed in the construction of a physical “clock”, yielding the values of \( t \), and remaining degrees of freedom evolving in proper time. Neglecting the interaction between both components, and the details of the clock in particular, we describe the relation between physical and proper time by a probability distribution. In this situation, the Hamiltonian constraint only applies to the remaining degrees of freedom, while generally the system will be constrained as a whole. These aspects were exemplified in detail in the simple models of [1, 2].
Correspondingly, we introduce the modified generating functional:

\[
Z[J] \equiv \int_H d\tau_i d\tau_f \int \mathcal{D}\Phi \exp \left( i \int_{\tau_i}^{\tau_f} d\tau \ L_J - S(\tau_i; t_i) - S(\tau_f; t_f) \right),
\]

instead of (13), using the condensed notation introduced there. In the present case, \(Z[0]\) sums over all classical paths satisfying the constraint with weight \(P(\tau; t_i) \cdot P(\tau; t_f)\), depending on their initial and final proper times, while all other paths get weight zero. In this way, the distributions of proper time values \(\tau_{i,f}\) associated with the initial and final physical times, \(t_i\) and \(t_f\), respectively, are incorporated.

Next, we insert \(1 = \int d\tau P(\tau; t)\) into the expression for \(Z\), with an arbitrarily chosen physical time \(t > t_0\), and with \(t_i = t_f \equiv t_0\). We require the two sets of trajectories created in this way to present branches of forward ("\(>\)") and backward ("\(<\)"") motion. This leads us to factorize the path-integral into two connected ones:

\[
Z[J] = \int d\tau \ P(\tau; t) \cdot \int d\tau_f \int_H \mathcal{D}\Phi < \exp \left( i \int_{\tau}^{\tau_f} d\tau' \ L_J^\tau - S(\tau_f; t_0) \right) \]
\[
\cdot \int d\tau_i \int_H \mathcal{D}\Phi > \exp \left( i \int_{\tau_i}^{\tau} d\tau'' \ L_J^{\tau_i} - S(\tau_i; t_0) \right) \]
\[
\cdot \prod_a \delta(\varphi^a_\tau(\tau) - \epsilon(a)\varphi^a_\tau(t_0)) ,
\]

where \(\epsilon(a \leq n) \equiv 1\), \(\epsilon(a \geq n + 1) \equiv -1\), and \(J \equiv J_>, J_<\), depending on the branch. The ordinary \(\delta\)-functions assure continuity of the classical paths in terms of the coordinates \(q^a\), \(a = 1, \ldots, n\), and reflect the momenta \(p^a\), \(a = 1, \ldots, n\), at proper time \(\tau\).

We observe that the generating functional will only be independent of the physical time \(t\), in the absence of an external source, if we assume that the probability distribution \(P\) does not explicitly depend on time, \(-\log P(\tau; t) = S(\tau; t) \equiv S(\tau - t)\), and if we suitably specify the boundary conditions. We set:

\[
\varphi^a_\tau(\tau_i) = \epsilon(a)\varphi^a_\tau(\tau_f) \equiv \phi^a(t_0) , \ a = 1, \ldots, 2n .
\]

Note that the boundary conditions are defined at the physical time \(t_0\), to which correspond the distributed values of the proper times \(\tau_{i,f}\). This establishes a one-to-one correspondence between both sets of trajectories. They could be viewed as closed loops with reflecting boundary conditions at both ends, \(t_0\) and \(t\), and fixed initial condition at \(t_0\).

Exponentiating the \(\delta\)-functions via Fourier transformation, the generating functional can be recognized indeed as a scalar product of a “state” and its adjoint. We define the normalized states by the path-integral:

\[
|\tau, \pi_a; t\rangle \equiv Z[J]^{-1/2} \int d\tau_i \int_H \mathcal{D}\Phi \exp \left( i \int_{\tau_i}^{\tau+t} d\tau' \ L_J - S(\tau_i; t_0) + i\pi_a\varphi^a(\tau + t) \right),
\]

and, similarly, the adjoint states:

\[
\langle \tau, \pi_a; t | \equiv Z[J]^{-1/2} \int d\tau_f \int_H \mathcal{D}\Phi \exp \left( i \int_{\tau+t}^{\tau_f} d\tau' \ L_J - S(\tau_f; t_0) - i\pi_a\varphi^a(\tau + t) \right),
\]

where the paths are forward and backward going as indicated by the integral boundaries in the exponent respectively, dropping “\(>\), \(<\)”; note that the summation is to be read as \(\sum_a \epsilon(a)\pi_a\varphi^a\) in (25).
The redundancy in designating the states, which depend on the sum of proper and physical time, only arises here, since the probability distribution $P$ is assumed not to be explicitly depending on the physical time, for simplicity.

The scalar product of two such states is now defined, and calculated, as follows:

$$\langle t_2|t_1 \rangle \equiv \int d\tau d\pi P(\tau, \pi; t_2|\tau, \pi; t_1) = \delta_{t_2,t_1},$$

(26)

with $d\pi \equiv \prod_a (d\pi_a/2\pi)$. In particular, we have $\langle t|t \rangle = 1$, which corresponds to (22), using definitions (24) and (25). Furthermore, for $t_1 \neq t_2$, we find that states are orthogonal, $\langle t_2|t_1 \rangle = 0$, by the symmetry of the motion on the forward and backward branches, for a correspondingly symmetric source $J$, and by uniqueness of the Hamiltonian flow generating the paths.

A remark is in order here concerning the integration over $d\pi$ above, which originates from exponentiating the $\delta$-functions of (22). Due to the presence of the Hamiltonian constraints on both branches of a trajectory, one of the $\delta$-functions is redundant. We absorb the resulting $\delta(0)$ in the normalization of the states.

Finally, the symmetry between the states and the adjoint states, given the stated assumptions, is perfect. We find:

$$\langle \tau, \pi; t \rangle = |\tau, \pi; t \rangle^*,$$

(27)

which is a desirable property of states in a Hilbert space (in “$\tau, \pi$-representation”). However, from our heuristic discussion this appears as a restriction which could be relaxed, resulting in a less familiar relation between the vector space and its dual.$^3$

### 5 Unitary Evolution

Following the same approach which led to the definition of states in (24), (25), we consider the time evolution of states in the absence of a source, $J = 0$. Suitably inserting “1”, as before, and splitting the path-integral, we obtain:

$$|\tau', \pi'; t' \rangle = \int d\tau d\pi P(\tau)U(\tau', \pi'; t'|\tau, \pi; t)|\tau, \pi; t \rangle,$$

(28)

with the kernel:

$$U(\tau', \pi'; t'|\tau, \pi; t) \equiv \int D\Phi \exp \left(i \int_{\tau+t}^{\tau'+t'} d\tau'' L + i\pi' \cdot \varphi(\tau' + t') - i\pi \cdot \varphi(\tau + t) \right),$$

(29)

where the integral is over all paths running between $\tau + t$ and $\tau' + t'$, subject to the constraint; here we abbreviate $\pi \cdot \varphi \equiv \pi_a \varphi^a$. We interpret this as a matrix element of the evolution operator $\hat{U}(t'|t)$.

Then, it is straightforward to establish the following composition rule:

$$\hat{U}(t''|t') \cdot \hat{U}(t'|t) = \hat{U}(t''|t),$$

(30)

where integration over the intermediate variables, say $\tau', \pi'$, with appropriate weight factor $P(\tau')$, is understood. These integrations effectively remove the “1”, which is inserted when

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$^3$One might consider only forward going paths, for example, with boundary conditions on the coordinates set at $\pm t_0$. In this case, however, it is not obvious how to obtain the correspondent of (27).
factorizing path-integrals, and link the endpoint coordinates of one classical path to the initial of another.

Since the Hamiltonian constraint is a constant of motion, there is no need to constrain the path integral representing the evolution operator. Integrating over intermediate variables removes all contributions violating the Hamiltonian constraint, provided we work with properly constrained states. This will be further discussed in the following section.

The physical-time dependence of the evolution operator amounts to translations of proper time variables. Therefore, we may study further properties of $\hat{U}$ without explicitly keeping it. This simplicity, of course, is related to the analogous property of the states, which we mentioned.

We begin by rewriting the functional integral of (29):

$$U(\tau', \pi'; \tau, \pi) = \int D\varphi \delta[\varphi^a(\tau) - \varphi^a_\text{cl}(\tau)] \exp \left( i\pi' \cdot \varphi(\tau') - i\pi \cdot \varphi(\tau) \right),$$

(31)

cf. Section 3, where the paths run between $\tau$ and $\tau'$, integrating over all initial conditions. Fixing the initial condition of a classical path, we can pull the exponential factors out of the integral, due to the $\delta$-functional, and integrate over all initial conditions in the end:

$$U(\tau', \pi'; \tau, \pi) = \int d\varphi_i(\tau) \exp \left( i\pi' \cdot \varphi_i(\tau') - i\pi \cdot \varphi_i(\tau) \right) \int D\varphi \delta[\varphi^a(\tau) - \varphi^a_\text{cl}(\tau)],$$

(32)

where $\varphi_i(\tau')$ denotes the endpoint of the path singled out by the particular initial condition $\varphi_i(\tau)$. The functional integral equals one. Then, we obtain the simple but central result:

$$U(\tau', \pi'; \tau, \pi) = \int d\varphi \exp \left( i\pi' \cdot \varphi(\tau') - i\pi \cdot \varphi \right) \equiv \mathcal{E}(\pi', \pi; \tau' - \tau),$$

(33)

(34)

where $d\varphi \equiv \prod_a d\varphi^a$, and with the Liouville operator:

$$\hat{\mathcal{L}} \equiv -\frac{\partial H}{\partial \varphi} \cdot \omega \cdot \frac{\partial}{\partial \varphi},$$

(35)

which is employed in order to propagate the classical solution from the initial condition at $\tau$ to proper time $\tau'$; $\omega$ is the symplectic matrix.

Using (33), one readily confirms (30) once again. In particular, then $\tilde{U}(t|t') \cdot \tilde{U}(t'|t) = \tilde{U}(t|t)$, which is not diagonal, in general, in this $\tau, \pi$-representation. We have: $U(\tau', \pi'; t|\tau, \pi; t) = \mathcal{E}(\pi', \pi; \tau' - \tau)$, as defined in (34).

In order to proceed, we consider the time dependence of the evolution kernel $\mathcal{E}$. It is determined by the equation:

$$i\partial_\tau \mathcal{E}(\pi', \pi; \tau) = -\int d\varphi \exp \left( i\pi' \cdot \varphi(\tau) - \pi \cdot \varphi \right) \pi' \cdot \omega \cdot \frac{\partial}{\partial \varphi} H(\varphi(\tau))$$

$$= \tilde{\mathcal{H}}(\pi', -i\partial_\tau) \mathcal{E}(\pi', \pi; \tau),$$

(36)

with the effective *Hamilton operator*:

$$\tilde{\mathcal{H}}(\pi, -i\partial_\tau) \equiv -\pi \cdot \omega \cdot \frac{\partial}{\partial \varphi} H(\varphi)|_{\varphi=-i\partial_\tau}.$$
Here we used (33)–(34), together with the equation of motion (15). The initial condition is:

\[ \mathcal{E}(\pi', \pi; 0) = (2\pi)^{2n} \delta^{2n}(\pi' - \pi) , \]

as read off from (33).

Using (36), we finally obtain the Schrödinger equation which describes the evolution of the states in physical time:

\[
\begin{align*}
    i\partial_t \langle \tau, \pi | \Psi(t) \rangle &= \int d\tau' d\pi' P(\tau') i\partial_\pi \mathcal{E}(\pi, \pi' \tau + t - \tau') \langle \tau', \pi' | \Psi(0) \rangle \\
    &= \hat{H}(\pi, -i\partial_\pi) \langle \tau, \pi | \Psi(t) \rangle .
\end{align*}
\]

Here we implicitly employed the relation \( \langle \tau, \pi | \Psi(t) \rangle = \langle \tau + t, \pi | \Psi(0) \rangle = \langle 0, \pi | \Psi(t + \tau) \rangle \), in order to analytically continue to real values of \( t \) and to perform the derivative, despite that the physical time is discrete.

Clearly, the hermitian Hamiltonian must be incorporated in a unitary transfer matrix, in order to describe the evolution through one discrete physical time step. It is plausible that presently the need for regularization of the Hamilton operator, demonstrated in subsequent sections, arises here. It is also conceivable that in a more realistic situation, with clock degrees of freedom forming dynamically part of the system, this particular complication is alleviated.

Furthermore, considering stationary states, we have:

\[
\begin{align*}
    \langle \tau, \pi | \Psi_E(t) \rangle &= \exp(-iEt) \langle \tau, \pi | \Psi(0) \rangle = \exp(-iE(t + \tau)) \langle 0, \pi | \Psi(0) \rangle \\
    &= \exp(-iE(t + \tau)) \langle \pi | \Psi_E \rangle ,
\end{align*}
\]

due to the previously discussed additivity of proper and physical time in the present context. Similarly, the Hamiltonian \( \hat{H} \) is independent of the probability distribution \( P \), mapping physical to proper time, since in the presently idealized situation the clock is decoupled from the system.

Note that there is \( \bar{\hbar} \) in our equations. If introduced, it would merely act as a conversion factor of units. On the other hand, there is an intrinsic scale corresponding to the clock’s unit time interval \( T \), which could be analyzed in a more complete treatment where clock and mechanical system are part of the Universe and interact.

Before we will illustrate in some examples the type of quantum Hamiltonians that one obtains, we have to first address the classical observables and their place in the emergent quantum theory, in particular we need to implement the classical Hamiltonian constraint. We recall that in a reparametrization invariant classical theory the Hamiltonian constraint is an essential ingredient related to the gauge symmetry one is dealing with.

### 6 Observables

It follows from our introduction of states in Section 4, see particularly (21)–(25), how the classical observables of the underlying mechanical system can be determined. Considering observables which are function(al)s of the phase space variables \( \varphi \), the definition of their expectation value at physical time \( t \) is obvious:

\[
\langle O[\varphi]; t \rangle \equiv \int d\tau P(\tau; t) O[-i\frac{\delta}{\delta J(\tau)}] \log Z[J]|_{J=0}
\]

(42)
\[ \int d\tau d\pi \ P(\tau - t) \langle \tau, \pi; 0 | O[\varphi(\tau)] | \tau, \pi; 0 \rangle \]  
\[ = \int d\tau d\pi \ P(\tau) \langle \tau, \pi; t | O[\varphi(\tau + t)] | \tau, \pi; t \rangle \]  
\[ = \int d\tau d\pi \ P(\tau) \langle \tau, \pi; t | O[-i\partial_\pi] | \tau, \pi; t \rangle \]  
\[ = \langle \Psi(t) | \hat{O}[\varphi] | \Psi(t) \rangle \]  

where:
\[ \hat{O}[\varphi] \equiv O[\hat{\varphi}] \]  
\[ \hat{\varphi} \equiv -i\partial_\pi \]  

in \( \tau, \pi \)-representation. In \textbf{(13)} - \textbf{(14)} the notation is symbolical, since the observable should be properly included in the functional integral defining the ket state, for example.

Thus, a classical observable is represented by the corresponding function(al) of a suitably defined \textit{momentum} operator. Furthermore, its expectation value at physical time \( t \) is represented by the effective quantum mechanical expectation value of the corresponding operator with respect to the physical-time dependent state under consideration, which incorporates the weighted average over the proper times \( \tau \), according to the distribution \( P \). Not quite surprisingly, the evaluation of expectation values involves an integration over the whole \( \tau \)-parametrized “history” of the states.

Furthermore, making use of the evolution operator \( \hat{U} \) of Section 5, in order to refer observables at different proper times \( \tau_1, \tau_2, \ldots \) to a common reference point \( \tau \), one can construct \textit{correlation functions} of observables as well, similarly as in \textbf{[8]}, for example.

The most important observable for our present purposes is the classical Hamiltonian, \( H(\varphi) \), which enters the Hamiltonian constraint of a classical reparametrization invariant system. It is, by assumption, a constant of the classical motion. However, it is easy to see that also its quantum descendant, \( \hat{H}(\varphi) \equiv \hat{H}(\hat{\varphi}) \), is conserved, since it commutes with the effective Hamiltonian of \textbf{(37)}:

\[
[\hat{H}, \hat{H}] = H(-i\partial_\pi) \pi \cdot \omega \cdot \frac{\partial}{\partial \varphi} H(\varphi)|_{\varphi=-i\partial_\pi} \pi \cdot \omega \cdot \frac{\partial}{\partial \varphi} H(\varphi)|_{\varphi=-i\partial_\pi} H(-i\partial_\pi)
\]
\[
= \frac{\partial}{\partial \varphi} H(\varphi)|_{\varphi=-i\partial_\pi} \cdot \omega \cdot \frac{\partial}{\partial \varphi} H(\varphi)|_{\varphi=-i\partial_\pi} = 0 \]  

due to the antisymmetric character of the symplectic matrix. Therefore, it suffices to implement the Hamiltonian constraint at an arbitrary time.

Then, the constraint of the form \( C_H \equiv H[\varphi] - \epsilon \sim 0 \) may be incorporated into the definition of the states in \textbf{(24)} by including an extra factor \( \delta(C_H) \) into the functional integral, and analogously for the adjoint states. Exponentiating the \( \delta \)-function, we pull the exponential out of the functional integral, as before. Thus, we find the following operator representing the constraint:

\[ \hat{C} \equiv \int d\lambda \ \exp \left( i\lambda(\hat{H}(\varphi) - \epsilon) \right) = \delta(\hat{C}_H) \]  

which acts on states as a projector. Of course, a corresponding number of projectors should be included into the definition of the generating functional, see \textbf{(22)}, for appropriate normalization of the states.
Supplementing (42)–(46) by the insertion of the Hamiltonian constraint, the properly constrained expectation values of observables should be calculated according to:

$$\langle O[\varphi]; t \rangle_H \equiv \langle \Psi(t) | \hat{O}[\varphi] \hat{C} | \Psi(t) \rangle \ ,$$  

which will deviate from the results of the previous definition.

Finally, also the eigenvalue problem of stationary states, see (40)–(41), should be studied in the projected subspace:

$$\hat{H} \hat{C} | \Psi \rangle = E \hat{C} | \Psi \rangle \ ,$$

to which we shall return in the following examples.

### 7 Examples of Emergent Quantum Systems

#### 7.1 Quantum Harmonic Oscillator from Classical One Beneath

All integrable models can be presented as collections of independent harmonic oscillators. Therefore, we begin with the harmonic oscillator of unit mass and of frequency $\Omega$. The action is:

$$S \equiv \int dt \left( \frac{1}{2\lambda} (\partial_t q)^2 - \frac{\lambda \Omega^2}{2} (q^2 - 2\epsilon) \right) \ ,$$

where $\lambda$ denotes the arbitrary lapse function, i.e. Lagrange multiplier for the Hamiltonian constraint, and $\epsilon > 0$ is the parameter fixing the energy presented by this constraint.

Introducing the proper time, $\tau \equiv \int dt \lambda$, the Hamiltonian equations of motion and Hamiltonian constraint for the oscillator are:

$$\partial_\tau q = p \ , \quad \partial_\tau p = -\Omega^2 q \ ,$$

$$\frac{1}{2}(p^2 + \Omega^2 q^2) - \epsilon = 0 \ ,$$

respectively.

Comparing the general structure of the equations of motion (15) with the ones obtained here, we identify the effective Hamilton operator (37), while the constraint operator follows from (49):

$$\hat{H} = \Omega \hat{L}_z = -i\Omega \partial_\phi \ ,$$

$$\hat{C} = \delta(\Delta_2 + 2\epsilon) = \delta(\partial^2_{\rho^2} + \rho^{-1} \partial_\rho + \rho^{-2} \partial_\phi^2 + 2\epsilon) \ ,$$

where $\hat{L}_z$ denotes the $z$-component of the usual angular momentum operator and $\Delta_2$ the Laplacian in two dimensions.
We observe that the eigenfunctions of the eigenvalue problem posed here factorize into a radial and an angular part. The radial eigenfunction, a cylinder function, is important for the calculation of expectation values of certain operators and the overall normalization of the resulting wave functions. However, it does not influence the most interesting spectrum of the Hamiltonian.

In the absence of the full angular momentum algebra, we discretize the angular derivative. Then, the energy eigenvalue problem consists in:

\[ \hat{H} \psi(\phi_n) = -i(\Omega N/2\pi)(\psi(\phi_{n+1}) - \psi(\phi_n)) = E \psi(\phi_n) , \]  

with \( \phi_n \equiv 2\pi n/N , 1 \leq n \leq N \), and the continuum limit will be considered momentarily.

A complete orthonormal set of eigenfunctions and the eigenvalues are:

\[ \psi_m(\phi_n) = N^{-1/2} \exp[i(m + \delta)\phi_n] , \quad 1 \leq m \leq N , \]  
\[ E_m = i(\Omega N/2\pi) \left(1 - \exp[2\pi i(m + \delta)/N]\right) \]  
\[ \overset{N \to \infty}{\longrightarrow} \Omega(m + \delta) , \quad m \in \mathbb{N} , \]

where \( \delta \) is an arbitrary real constant.

Obviously, the freedom in choosing the constant \( \delta \), which arises from the regularization of the Hamilton operator, is very wellcome. Choosing \( \delta \equiv -1/2 \), we arrive at the quantum harmonic oscillator, starting from the corresponding classical system. Thus, we recover in a straightforward way 't Hooft’s result, derived from an equivalent cellular automaton \[9\]. See also \[2\] for the completion of a similar quantum model. In the following example we will encounter one more model of this kind and demonstrate its solution in detail.

Here, and similarly in following examples, the eigenvalues are complex, with the real spectrum only obtained in the continuum limit. This is due to the fact that we discretize first-order derivatives most simply, i.e. asymmetrically. It can be avoided easily by employing a symmetric discretization, if necessary.

We find it interesting that our general Hamilton operator \[37\] does not allow for the direct addition of a constant energy term, while the regularization performed here does.

### 7.2 Quantum System with Classical Relativistic Particle Beneath

Introducing proper time as in Section 2, but leaving the extradimensions for now, the equations of motion and the Hamiltonian constraint of the reparametrization invariant kinematics of a classical relativistic particle of mass \( m \) are given by:

\[ \partial_\tau q^\mu = m^{-1} p^\mu , \quad \partial_\tau p^\mu = 0 , \]  
\[ p \cdot p - m^2 = 0 , \]

respectively. Here we have \( \varphi^a \equiv (q^0, \ldots, q^3; p^0, \ldots, p^3) , \quad a = 1, \ldots, 8 \); four-vector products involve the Minkowski metric, \( g_{\mu\nu} \equiv \text{diag}(1, -1, -1, -1) \).

Proceeding as before, we identify the effective Hamilton operator:

\[ \hat{H} = -m^{-1} \pi_q \cdot \varphi_p = -m^{-1} \pi_q \cdot (-i\partial_{\pi_p}) , \]
corresponding to \(37\); the notation is as introduced after \(56\), however, involving four-vectors. Furthermore, the Hamiltonian constraint is represented by the operator:

\[
\hat{C} = \delta(\hat{\varphi}_p^2 - m^2) = \delta(\partial_{\pi_p}^2 + m^2)
\]

following from \(19\).

After a Fourier transformation, which replaces the variable \(\pi_q\) by a derivative (four-vector) \(+i\partial_x\), and with \(\pi_p \equiv \bar{x}\), the Hamiltonian and constraint operators become:

\[
\hat{H} = -m^{-1}\partial_x \cdot \partial_x , \quad \hat{C} = \delta(\partial_x^2 + m^2)
\]

respectively.

Similarly as in the harmonic oscillator case, the eigenvalue problem is properly defined by solved by discretizing the system on a hypercubic lattice of volume \(L^8\) (lattice spacing \(l \equiv L/N\)) with periodic boundary conditions, for example. Here we obtain the eigenfunctions:

\[
\psi_{k_x,k_x}(x_n, \bar{x}_n) = N^{-1} \exp[i(k_x + \delta_x) \cdot x_n + i(k_x + \delta_x) \cdot \bar{x}_n]
\]

with coordinates \(x_n^\mu \equiv x n^\mu\) and momenta \(k_n^\mu \equiv 2\pi k n^\mu / L\), with \(1 \leq n^\mu, k^\mu \leq N\), and where \(\delta^\mu_x\) are arbitrary real constants, for all \(\mu = 0, \ldots, 3\) (analogously \(\bar{x}_n^\mu, k_n^\mu, \delta^\mu_x\)).

The energy eigenvalues are:

\[
E_{k_x,k_x} = -m^{-1}l^{-2} \left( \left( \exp[il(k_x + \delta_x)^0] - 1 \right) \left( \exp[il(k_x + \delta_x)^0] - 1 \right) \right)
\]

\[
- \sum_{j=1}^{3} \left( \left( \exp[il(k_x + \delta_x)^j] - 1 \right) \left( \exp[il(k_x + \delta_x)^j] - 1 \right) \right)
\]

\[
= m^{-1}(k_x + \delta_x) \cdot (k_x + \delta_x) + O(l)
\]

where is \(L\) is kept constant in the continuum limit, \(l \to 0\). Furthermore, in this limit, one finds that the Hamiltonian constraint requires timelike “on-shell” vectors \(k_x\), obeying \((k_x + \delta_x)^2 = m^2\), while leaving \(k_x\) unconstrained.

Continuing, we perform also the infinite volume limit, \(L \to \infty\), which results in a continuous energy spectrum in \(70\). We observe that no matter how we choose the constants \(\delta_x, \delta_{\bar{x}}\), the spectrum will not be positive definite. Thus, the emergent model is not acceptable, since it does not lead to a stable groundstate.

However, let us proceed more carefully with the various limits involved and show that indeed a well-defined quantum model can be obtained. For simplicity, considering \((1+1)\)-dimensional Minkowski space and anticipating the massless limit, we rewrite \(70\) explicitly:

\[
E_{k,k} = -\left( \frac{2\pi}{\sqrt{mL}} \right)^2 (\bar{k}^1 + \bar{\delta}^1) \left( (k^0 + \delta^0) + (k^1 + \delta^1) \right) + O(m)
\]

where we suitably rescaled and renamed the constants and the momenta, which run in the range \(1 \leq \bar{k}^1, k^{0,1} \leq N \equiv 2s + 1\). Furthermore, we incorporated the Hamiltonian (on-shell) constraint, such that only the positive root contributes: \(\bar{k}^0 + \delta^0 = |\bar{k}^1 + \delta^1| + O(m^2) = -(\bar{k}^1 + \delta^1) + O(m^2)\). This can be achieved by suitably choosing \(\delta^{0,1}\).

In fact, just as in the previous harmonic oscillator case, the choice of the constants is crucial in defining the quantum model. Here we set:

\[
\delta^0 \equiv \frac{1}{2}, \quad \delta^1 \equiv \frac{1}{2} - 2s - 3, \quad \delta^{0,1} \equiv 0
\]
This results in the manifestly positive definite spectrum:

\[
E(\bar{s}_z, s_z^{0.1}) = \left( \frac{2\pi}{\sqrt{mL}} \right)^2 \left( (\bar{s}_z + s + \frac{1}{2}) + 1 \right) \left( (s_z^0 + s + \frac{1}{2}) + (s_z^1 + s + \frac{1}{2}) + 1 \right) + O(m) ,
\]  

(73)

with (half)integer quantum numbers \( \bar{s}_z, s_z^{0.1} \), all in the range \(-s \leq s_z \leq s\), replacing \( \bar{k}^1, k^{0.1} \).

Recalling the algebra of the \( SU(2) \) generators, with \( S_z |s_z⟩ = s_z |s_z⟩ \) in particular, we are led to consider the generic operator:

\[
h \equiv S_z + s + \frac{1}{2} ,
\]  

(74)
i.e., diagonal with respect to \( |s_z⟩ \)-states of the (half)integer representations determined by \( s \).

In terms of such operators, we obtain the regularized Hamiltonian corresponding to (73):

\[
\hat{H} = \left( \frac{2\pi}{\sqrt{mL}} \right)^2 \left( \bar{h} + h_0 + h_1 + \bar{h}(h_0 + h_1) \right) + O(m) ,
\]  

(75)

which will turn out to be equivalent to three harmonic oscillators, including a coupling term plus an additional contribution to the vacuum energy.

A Hamiltonian of the type of \( h \) has been the starting point of ’t Hooft’s analysis \[9\], which we adapt for our purposes in the following.

Continuing with standard notation, we have \( S^2 \equiv S_x^2 + S_y^2 + S_z^2 = s(s+1) \), which suffices to obtain the following identity:

\[
h = \frac{1}{2s+1} \left( S_x^2 + S_y^2 + \frac{1}{4} + h^2 \right) .
\]  

(76)

Furthermore, using \( S_\pm \equiv S_x \pm iS_y \), we define coordinate and conjugate momentum operators:

\[
\hat{q} \equiv \frac{1}{2} (a S_- + a^* S_+) , \quad \hat{p} \equiv \frac{1}{2} (b S_- + b^* S_+) ,
\]  

(77)

where \( a \) and \( b \) are complex coefficients. Calculating the basic commutator with the help of \([S_+, S_-] = 2S_z\) and using (74), we obtain:

\[
[\hat{q}, \hat{p}] = i(1 - \frac{2}{2s+1}h) ,
\]  

(78)

provided we set \( \Im(a^*b) \equiv -2/(2s+1) \). Incorporating this, we calculate:

\[
S_x^2 + S_y^2 = \frac{(2s+1)^2}{4} \left( |a|^2 \hat{p}^2 + |b|^2 \hat{q}^2 - (\Im a \cdot \Im b + \Re a \cdot \Re b) \{\hat{q}, \hat{p}\} \right) .
\]  

(79)

In order to obtain a reasonable Hamiltonian in the continuum limit, we set:

\[
a \equiv i \frac{\Omega^{-1/2}}{\sqrt{s+1/2}} , \quad b \equiv \frac{\Omega^{1/2}}{\sqrt{s+1/2}} , \quad \Omega \equiv \left( \frac{2\pi}{\sqrt{mL}} \right)^2 .
\]  

(80)

Then, the previous (73) becomes:

\[
\Omega h = \frac{1}{2} \hat{p}^2 + \frac{1}{2} \Omega^2 \hat{q}^2 + \frac{1}{(2s+1)\Omega} \left( \frac{1}{4} \hat{p}^2 + (\Omega h) \right) ,
\]  

(81)
reveiling a nonlinearly modified harmonic oscillator Hamiltonian, similarly as in [2, 9].

Now it is safe to consider the continuum limit, \( 2s + 1 = N \to \infty \), keeping \( \sqrt{mL} \) and \( \Omega \) finite. This produces the usual \( \hat{q}, \hat{p} \)-commutator in (78) for states with limited energy and the standard harmonic oscillator Hamiltonian in (81).

Using these results in (75), the Hamilton operator of the emergent quantum model is obtained:

\[
\hat{H} = \Omega + \frac{1}{2} \sum_{j=1,0,1} \left( \hat{p}_j^2 + \Omega^2 \hat{q}_j^2 \right) + \frac{1}{4\Omega} \left( \hat{p}_1^2 + \Omega^2 \hat{q}_1^2 \right) \sum_{j=0,1} \left( \hat{p}_j^2 + \Omega^2 \hat{q}_j^2 \right),
\]

(82)

where the massless limit together with the infinite volume limit is carried out, \( m \to 0, L \to \infty \), in such a way that \( \Omega \) remains finite.

The resulting Hamiltonian here is well defined in terms of continuous operators \( \hat{q} \) and \( \hat{p} \), as usual, and has a positive definite spectrum. The coupling term might appear slightly less unfamiliar, if the oscillator algebra is realized in terms of bosonic creation and annihilation operators.

We previously calculated the matrix elements of operators \( \hat{q}, \hat{p} \) with respect to the SU(2) basis of primordial states in a similar case, showing that localization of the quantum oscillator has little to do with localization in the classical model beneath [1, 2].

Finally, we remark that had we chosen \( \bar{\delta}_0, \bar{\delta}_1 = \delta_0, \delta_1 \equiv 1/2 \), instead of (72), then a relative sign between terms would remain, originating from the Minkowski metric, and this would yield the Hamiltonian \( \hat{H} \propto (1 + \bar{h})(h_0 - h_1) \), which is not positive definite. Similarly, any symmetric choice, \( \bar{\delta}_0, \bar{\delta}_1 = \delta_0, \delta_1 \equiv \delta \) would suffer from this problem.

This raises the important issue of the role of canonical transformations, and of symmetries in particular. It is conceivable that symmetries will play a role in restricting the present arbitrariness of the regularization defining a quantum model. We will address further aspects of this in the following section.

8 Remarks on (Non)Integrable Interactions

We resume our discussion of general features of the emergent quantum mechanics. Specifically, let us consider a classical system with \( n \) degrees of freedom, for example, a chain of particles with harmonic coupling and anharmonic potentials. Denoting the phase space variables by \( \varphi \equiv (Q, P) \), where \( Q, P \) are \( n \)-component vectors, we assume for definiteness a Hamiltonian of the form, \( H(\varphi) \equiv (1/2)P^2 + V(Q) \), i.e. with a kinetic term which is simply quadratic in the momenta.

In this case, following (37) and (49), and with:

\[
\hat{Q} \equiv -i\partial_{\pi Q}, \quad \hat{P} \equiv -i\partial_{\pi P}, \quad (83)
\]

the Hamiltonian and constraint operators, respectively, are given by:

\[
\hat{H} = -\pi_Q \cdot \hat{P} + \pi_P \cdot V'(\hat{Q}), \quad (84)
\]

\[
\hat{C} = \delta(1/2 \hat{P}^2 + V(\hat{Q}) - \epsilon), \quad (85)
\]

where, of course, \( V'(Q) \equiv \nabla_Q V(Q) \), and the wave function is considered as a function \( \psi(\pi_P, \pi_Q) \) of the indicated vectors.
The previous oscillator example suggests to perform a Fourier transformation to variables $x, y$, such that the eigenvalue problem becomes:

$$
\hat{H}\psi(x, y) = \left(x \cdot (-i\partial_y) - V'(y) \cdot (-i\partial_x)\right)\psi(x, y) = E\psi(x, y),
$$

while the constraint operator equation turns into an algebraic constraint:

$$
\hat{C}\psi(x, y) = \delta\left(\frac{1}{2}x^2 + V(y) - \epsilon\right)\psi(x, y) = 0.
$$

In agreement with the general result (86), we easily confirm here that $\hat{H}\hat{C} = \hat{C}\hat{H}$. The constraint equation then simply states that the phase space variables are constrained to a constant energy surface of the underlying classical system.

The first order quasi-linear partial differential equation (86) can be studied by the method of characteristics [20]. Thus, one finds one equation taking care of the inhomogeneity (right-hand side), which can be trivially integrated. Furthermore, the remaining $2n$ equations for the characteristics present nothing but the classical Hamiltonian equations of motion.

It follows that integrable classical models can (in principle) be decoupled in this context of the characteristic equations by canonical transformations. This assumes that we can apply them freely at the pre-quantum level, which might not be the case. It would lead us essentially to the collection of harmonic oscillators mentioned at the beginning of Section 7.1, and corresponding quantum harmonic oscillators as studied there.

Classical crystal-like models with only harmonic forces, or free field theories, respectively, will thus give rise to corresponding free quantum mechanical systems here. These are constructed in a different way in [27]. Presumably, the (fixing of a large class of) gauge transformations invoked there can be related to the existence of integrals of motion implied by integrability here. In any case, we conclude that in the present framework truly interacting quantum (field) theories might be connected with nonintegrable deterministic systems beneath.

Furthermore, we emphasize that the Hamiltonian equations of motion preclude motion into classically forbidden regions of the underlying system. Nevertheless, quantum mechanical tunneling is an intrinsic property of the quantum oscillator models that we obtained, as well as of the anharmonic oscillator example considered in the following. Similarly, spreading of wave packets is to be expected in the latter case.

In order to demonstrate additional features of the eigenvalue problem of (86), we concentrate on one degree of freedom with phase space coordinates $p, q$ and with a generic anharmonic potential.

Since the potential depends only on $q$, by locally stretching or squeezing the coordinate, i.e. by an “oscillator transformation” $q \equiv f(\bar{q})$, we can bring it into oscillator form, such that $V(f(\bar{q})) = (1/2)\bar{q}^2$. Implementing this type of transformation, the equations for one degree of freedom are:

$$
-\frac{i}{f'(\bar{q})} \left(p\partial_{\bar{q}} - \bar{q}\partial_p\right)\psi(p, \bar{q}) = E\psi(p, \bar{q}) ,
$$

$$
\delta\left(\frac{1}{2}(p^2 + \bar{q}^2) - \epsilon\right)\psi(p, \bar{q}) = 0 ,
$$

with $f'$ denoting the derivative of $f$. This is very much oscillator-like indeed and, once more employing polar coordinates, we obtain:

$$
-\frac{i}{f'(\rho \sin \phi)} \partial_{\phi}\psi(\rho, \phi) = E\psi(\rho, \phi) ,
$$
\[ \delta (\rho^2 - \epsilon) \psi (\rho, \phi) = 0 \, . \]  

The eigenvalue problem seems underdetermined. As it stands, it would give rise to an unbound continuous spectrum, with no groundstate in particular.

This apparent defect persists for any number of degrees of freedom. However, as we have seen already, sense can be made of the Hamilton operator by a suitable regularization, especially by discretizing the phase space coordinates. The principles of such regularization we still do not know, other than either preserving or intentionally breaking symmetries.

### 8.1 An Anharmonic Oscillator

It is worth while to consider one more example, a onedimensional system with Hamiltonian:

\[ H \equiv \frac{1}{2} p^2 + V_0 |q| \, , \quad (92) \]

in order to demonstrate the subtleties associated with regularization. For the linear potential, the coordinate dependence of the operator on the left-hand side of (90) is mild, since \( f'(\rho \sin \phi) = V_0^{-1} \rho \sin \phi \), and the eigenvalues of its discretized counterpart can be found as follows.

Conveniently discretizing the angular variable as \( \phi_n \equiv (2\pi n/N) + (3\pi/2) \), \( 1 \leq n \leq N \), the eigenvalue equation becomes:

\[ \prod_{n=1}^{N} \left( 1 - \lambda \cos (2\pi n/N) \right) = 1 \, , \quad \lambda \equiv 2 \pi i \sqrt{\epsilon E/NV_0} \, . \quad (93) \]

Setting \( \lambda \equiv 2z/(1 + z^2) \), and employing a known identity for the finite product arising here, the eigenvalue equation can be transformed into: \( (1 + z^N)^2 = (1 + z^2)^N \). With hindsight, we choose \( N \equiv 2(4N' + 1) \) and set \( z \equiv +\sqrt{u - 1} \), to obtain:

\[ u^{N/2} + (1 - u)^{N/2} = 1 \, . \quad (94) \]

This equation has the nice property that, if \( u \) is a solution, then so is \( 1/u \). The location of the solutions \( u = 0, 1, \infty \) suggests to look for further solutions in the form of \( u \equiv \exp 2i\alpha \). Thus, combining the equations for \( u \) and \( 1/u \), we arrive at the transcendental equation:

\[ 2 \sin(\alpha N/2) = (2 \sin \alpha)^{N/2} \, . \quad (95) \]

From the multitude of its solutions, due to periodicity, we need to find \( N \) solutions of (93).

Closer inspection shows that, in the limit of large \( N \), solutions of (95) consist essentially of those zeros of \( \sin(\alpha N/2) \) which lie inside the intervals \([n\pi - \pi/6, n\pi + \pi/6]\), with integer \( n \). Thus, positive energy solutions will be obtained momentarily from:

\[ E_{\pm} = NV_0 \exp(i \frac{\pi}{4} \pm \frac{3}{2} i\alpha) \left( (2/\epsilon) \sin(\pm \alpha) \right)^{1/2} \, , \quad (96) \]

where either “+” or “−” has to be chosen consistently, corresponding to the solutions coming in pairs \( \exp \pm 2i\alpha \).

A remark is in order here. Considering only the positive root above, \( z \equiv +\sqrt{u - 1} \), we avoided negative energy solutions. However, there is a price to pay: careful counting reveals
that the positive energy spectrum is doubly degenerate. The finite positive part is obtained from (96), incorporating $\alpha = 2\pi m/N$:

$$E = NV_0 \exp(3i\pi m/N)\left((2/\epsilon)\sin(2\pi m/N)\right)^{1/2}, \quad 0 \leq m_0 \leq m \leq N/12,$$

$$N \to \infty, \quad \tilde{V}_0(m + m_0 - 1)^{1/2}, \quad m \in \mathbb{N},$$

(97)  \hspace{1cm}  (98)

where $m_0$ is an arbitrary constant, within the allowed range, which defines the zeropoint energy of the emergent quantum model. The continuum limit is to be taken such that $\tilde{V}_0 \equiv V_0(4\pi N/\epsilon)^{1/2}$ stays finite. The additional solutions can be chosen in a way that their real parts move to $+\infty$, as $N \to \infty$.

We remark that the spectrum of (98) differs from the one obtained for the same potential in standard quantum mechanics, where WKB yields: $E \propto (m - 1/4)^{2/3}$.

Summarizing, the various illustrated features promise to make genuinely interacting models quite difficult to analyze. We hope that more interesting results will be obtained with the help of spectrum generating algebras or some to-be-developed perturbative methods.

## 9 Conclusions

We pursue the view that quantum mechanics is an emergent description of nature, which possibly can be based on classical, pre-quantum concepts.

Our approach is motivated by a construction of a reparametrization-invariant time. In turn, this is based on the observation that “time passes” when there is an observable change, which is localized with the observer. More precisely, necessary are incidents, i.e. observable unit changes, which are recorded, and from which invariant quantities characterizing the change of the evolving system can be derived.

We recall the model of [2], invoking compactified extradimensions in which a particle moves in addition to its relativistic motion in Minkowski space. We employ a window to these extradimensions, i.e., we consider a quasi-local detector which registers the particle trajectory passing by. Counting such incidents, we construct an invariant measure of time.

A basic ingredient is the assumption of ergodicity, such that the system explores dynamically the whole allowed energy surface in phase space. This assures that there are sufficiently frequent observable incidents. They reflect properties of the dynamics with respect to (subsets of) Poincaré sections. Roughly, the passing time corresponds to the observable change there. Then, the particle’s proper time is linearly related to the physical time, however, subject to stochastic fluctuations.

Thus, the reparametrization-invariant time based on quasi-local observables naturally induces stochastic features in the behavior of the external relativistic particle motion. Due to quasi-periodicity (or, generally, more strongly irregular features) of the emerging discrete time, the remaining predictable aspects appear as in unitary quantum mechanical evolution.

In reparametrization-invariant, “timeless” single-particle systems, this idea has been realized in various forms [11, 12]. Presently, this has led us to assume the relation between the constructed physical time $t$ and standard proper time $\tau$ of the evolving system in the form of a statistical distribution, $P(\tau; t) = P(\tau - t)$, cf. (20). Here we assume that the distribution is not explicitly

\footnote{Corresponding eigenfunctions, i.e. $N$-component discrete eigenvectors, are obtained by evaluating products of the kind appearing in (93), with $k - 1 \leq N - 1$ factors for the $k$th component and a constant for $k = 1$.}
time-dependent, which means, the physical clock is decoupled from the system under study. We explore the consequences of this situation for the description of the system.

We have shown how to introduce “states”, eventually building up a Hilbert space, in terms of certain functional integrals, (24)–(25), which arise from the study of a suitable classical generating functional. The latter was introduced earlier in a different context, studying classical mechanics in functional form \[18\ \text{[19]}. We employ this as a convenient tool, and modify it, in order to describe the observables of reparametrization-invariant systems with discrete time (Section 6). Studying the evolution of the states in general (Section 5), we are led to the Schrödinger equation, (39). However, the Hamilton operator (37) has a non-standard first-order form with respect to phase space coordinates.

The choice of boundary conditions of the classical paths contributing to the generating functional plays a crucial role and deserves better understanding.

Furthermore, illustrating the emergent quantum models in various examples, we demonstrate that proper regularization of the continuum Hamilton operator is indispensable, in order that well-defined quantum mechanical systems emerge, with bounded spectra and a stable groundstate, in particular. Most desirable is a deeper understanding of this mapping between the continuum Hamilton operator, which is straightforward to write down, given a classical pre-quantum system, and the effective quantum mechanics, which emerges after proper regularization only. Especially, limitations imposed by symmetries and consistency of the procedure need further study.

It is a common experience that the preservation of continuum symmetries through discretization is difficult, for example, see \[14\ \text{[15, 16, 17]}. We wonder, whether other regularization schemes are conceivable. The possiblity, mentioned after (39), that the need for regularization is an artefact of decoupled clock degrees of freedom deserves further study.

We find that truly interacting quantum (field) theories might be connected to nonintegrable classical models beneath, since otherwise the degrees of freedom represented in the stationary Schrödinger equation here, can principally be decoupled by employing classical canonical transformations.

Finally, we come back to the probabilistic relation between physical time and the evolution parameter figuring in the parameterized classical equations of motion, which is the underlying raison d’être of the presented stroboscopic quantization. One would like to include the clock degrees of freedom consistently into the dynamics, in order to address the closed Universe. This can be achieved by introducing suitable projectors into the generating functional. Their task is to replace a simple quasi-local detector which responds to a particle trajectory passing through in Yes/No fashion; by counting such incidents, an invariant measure of time has been obtained before \[11\ \text{[12}. In a more general setting, this detector/projector has to be defined in terms of observables of the closed system. In this way, typical conditional probabilities can be handled, such as describing “What is the probability of observable \(X\) having a value in a range \(x\) to \(x + \delta x\), when observable \(Y\) has value \(y\)?”. Criteria for selecting the to-be-clock degrees of freedom are still unknown, other than simplicity. Most likely the resulting description of evolution and implicit notion of physical time will correspond to our distribution \(P(\tau; t)\) of (20), however, now evolving explicitly with the system. We leave this for future study.

The stroboscopic quantization emerging from underlying classical dynamics may be questioned in many respects. It might violate one or the other assumption of existing no-go theorems relating to hidden variables theories. However, we believe it is interesting to learn more about working examples, before discussing this. Unitary evolution, tunneling effects, and spreading
of wave packets are recovered in this framework. Interacting theories remain to be explored.

References

[1] H.-T. Elze and O. Schipper: Phys. Rev. D66, 044020 (2002).
[2] H.-T. Elze: Phys. Lett. A310, 110 (2003).
[3] H.-T. Elze: ‘Quantum mechanics emerging from “timeless” classical dynamics’, quant-ph/0306096.
[4] M. Montesinos, C. Rovelli and Th. Thiemann: Phys. Rev. D60, 044009 (1999).
[5] M. Montesinos, Gen. Rel. Grav. 33, 1 (2001).
[6] C. Rovelli: Phys. Rev. D42, 2638 (1990).
[7] R. Gambini, R.A. Porto and J. Pullin, ‘Consistent Discrete Gravity Solution of the Problem of Time: a Model’, gr-qc/0302064.
[8] C. Wetterich: Lecture in this volume; quant-ph/0212031.
[9] G. ‘t Hooft: ‘Quantum Mechanics and Determinism’. In: Particles, Strings and Cosmology, ed. by P. Frampton and J. Ng (Rinton Press, Princeton, 2001), p.275; hep-th/0105105
‘Determinism Beneath Quantum Mechanics’, quant-ph/0212095.
[10] M. Blasone, P. Jizba and G. Vitiello: Lecture in this volume; Phys. Lett. A 287, 205 (2001).
[11] T.S. Biró, B. Müller and S.G. Matinyan: Lecture in this volume;
T.S. Biró, S.G. Matinyan and B. Müller: Found. Phys. Lett. 14, 471 (2001).
[12] L. Smolin: ‘Matrix models as non-local hidden variables theories’, hep-th/0201031.
[13] S.L. Adler: ‘Statistical Dynamics of Global Unitary Invariant Matrix Models as Pre-Quantum Mechanics’, hep-th/0206120.
[14] H.S. Snyder: Phys. Rev. 71, 38 (1947).
[15] T.D. Lee: Phys. Lett. B 122, 217 (1983).
[16] G. Jaroszkiewicz and K. Norton: J. Phys. A 30, 3115 (1997); A 30, 3145 (1997); A 31, 977 (1998).
[17] R. Gambini and J. Pullin: Phys. Rev. Lett. 90, 021301 (2003);
‘Discrete quantum gravity: applications to cosmology’, gr-qc/0212033
C. Di Bartolo, R. Gambini and J. Pullin: Class. Quant. Grav. 19, 5275 (2002).
[18] E. Gozzi, M. Reuter and W.D. Thacker: Phys. Rev. D 40, 3363 (1989); D 46, 757 (1992).
[19] E. Gozzi and M. Regini: Phys. Rev. D62, 067702 (2000);
A.A. Abrikosov (jr.) and E. Gozzi: Nucl. Phys. Proc. Suppl. 66, 369 (2000).
[20] R. Courant and D. Hilbert: Methods Of Mathematical Physics, Vol. II (Interscience Publ., New York, 1962).

[21] D. Levi, P. Tempesta and P. Winternitz: ‘Umbral calculus, difference equations and the discrete Schrödinger equation’, nlin.SI/0305047.