APPLICATIONS OF TERNARY RINGS TO \( C^* \)-ALGEBRAS

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Abstract. We show that there is a functor from the category of positive admissible ternary rings to the category of \( * \)-algebras, which induces an isomorphism of partially ordered sets between the families of \( C^* \)-norms on the ternary ring and its corresponding \( * \)-algebra. We apply this functor to obtain Morita-Rieffel equivalence results between cross sectional \( C^* \)-algebras of Fell bundles, and to extend the theory of tensor products of \( C^* \)-algebras to the larger category of full Hilbert \( C^* \)-modules. We prove that, like in the case of \( C^* \)-algebras, there exist maximal and minimal tensor products. As applications we give simple proofs of the invariance of nuclearity and exactness under Morita-Rieffel equivalence of \( C^* \)-algebras.

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1. Introduction

An important tool in the study of \( C^* \)-algebras is Morita-Rieffel equivalence. When two \( C^* \)-algebras are Morita-Rieffel equivalent, they are related by a certain type of bimodule, from which one can see that these algebras share many properties. A Morita-Rieffel equivalence between two \( C^* \)-algebras implies that these algebras have many characteristics in common: they have the same K-theory, their spectra and primitive ideal spaces are homeomorphic, etc. In [12], Zettl introduced and studied \( C^* \)-ternary rings, and showed that these objects are essentially Morita-Rieffel equivalence bimodules. In fact, given a \( C^* \)-ternary ring \( E \), there

Date: November 9, 2018.
Key words and phrases. ternary rings, Morita-Rieffel equivalence, nuclear, exact.
exists essentially a unique structure of Morita-Rieffel equivalence bimodule on $E$ compatible with its structure of ternary ring (perhaps after a minor change on the ternary product).

On the other hand, when dealing with constructions such as tensor products or any sort of crossed products of $C^*$-algebras, in general one has to follow two steps: first one defines some $*$-algebra, and then one takes the completion of that algebra with respect to a $C^*$-norm. A situation that appears frequently is that there is more than one reasonable $C^*$-norm to perform this second step. In many cases, for instance in several imprimitivity theorems, one is interested in finding a Morita-Rieffel equivalence between different $C^*$-completions of a given pair of $*$-algebras which are related by a certain bimodule. This is the situation we study in the present paper, adopting a viewpoint similar to that in Zettl's work, but starting from a more algebraic level.

More precisely, suppose $E$ is an $A - B$ bimodule, where $A$ and $B$ are $*$-algebras, $\langle \cdot, \cdot \rangle_A : E \times E \to A$ and $\langle \cdot, \cdot \rangle_B : E \times E \to B$ satisfy all the algebraic properties of Hilbert bimodule inner products. In particular $\langle x, y \rangle_A z = x \langle y, z \rangle_B$, $\forall x, y, z \in E$. Then we can endow $E$ with a $*$-ternary ring structure by defining a ternary product $(\cdot, \cdot, \cdot) : E \times E \times E \to E$ such that $(x, y, z) = x \langle y, z \rangle_B$. We show that, under certain conditions, the partially ordered sets of $C^*$-norms on $E$ and on the $*$-algebras $A$ and $B$ are isomorphic to each other, in such a way that the completions with respect to corresponding $C^*$-norms under these isomorphisms yields a Morita-Rieffel equivalence bimodule.

We think that the best way to do it is by using the above mentioned abstract characterization of equivalence bimodules given by Zettl in [12], under the name of $C^*$-ternary rings. Such an object is a Banach space with a ternary product on it, which implicitly carries all the structure of an equivalence bimodule. Natural morphisms between $C^*$-ternary rings are linear maps that preserve ternary products. With such morphisms, one obtains a $C^*$-category, which is very convenient for the study of properties invariant under Morita-Rieffel equivalence.

The structure of the paper is as follows. In the next section, working in a pure algebraic level, we define the category of admissible $*$-ternary rings, and we show there is a functor from this category to the category of $*$-algebras or, more precisely, to the category of right basic triples (see Definition 2.6). In Section 3, given an admissible ternary ring $E$ with associated basic triple $(E, A, \langle \cdot, \cdot \rangle_A)$, we consider the lattice of $C^*$-seminorms on $A$ that satisfy the Cauchy-Schwarz inequality $\|\langle x, y \rangle_A\|^2 \leq \|\langle x, x \rangle\| \|\langle y, y \rangle\|$, $\forall x, y \in E$. Then we prove that this lattice is isomorphic to the lattice of $C^*$-seminorms on $E$. In passing we obtain some of the results of [12] and [1] regarding $C^*$-ternary rings. Besides, since there is also a functor to the category of left basic triples, we obtain a fortiori an isomorphism between the lattices of $C^*$-seminorms (satisfying the Cauchy-Schwarz property) on the $*$-algebras associated to the left and to the right sides. The Hausdorff completions of corresponding $C^*$-seminorms under this isomorphism turn out to be Morita-Rieffel equivalent. In the last part of Section 3 we consider positive ternary rings, for which the $C^*$-seminorms on the associated $*$-algebras automatically satisfy the Cauchy-Schwarz inequality. In Section 4 we briefly study the case of $C^*$-ternary rings, in which basic triples are replaced by $C^*$-basic triples, that is, Hilbert modules, and the functors from $C^*$-ternary rings to $C^*$-basic triples are shown to be exact. Finally, Section 5 is devoted to applications. We first refine a result from
concerning cross sectional algebras of Fell bundles over groups. Then we consider tensor products of $C^*$-ternary rings, which is essentially the same as tensor products of Hilbert modules. We show that the theory of tensor products of $C^*$-algebras extends to this larger category, in the sense that there exist a maximal and a minimal tensor products. By using this theory we obtain easy and natural proofs of the known results of the Morita-Rieffel invariance of nuclearity and exactness of $C^*$-algebras.

2. Ternary rings

2.1. Ternary rings.

Definition 2.1. A $*$-ternary ring is a complex linear space $E$ with a map $\mu : E \times E \times E \to E$, called $*$-ternary product on $E$, which is linear in the odd variables and conjugate linear in the second one, and such that:

$$\mu(\mu(x, y, z), u, v) = \mu(x, \mu(u, z, y), v) = \mu(x, y, \mu(z, u, v)), \forall x, y, z, u, v \in E$$

A homomorphism of $*$-ternary rings is a linear map $\phi : (E, \mu) \to (F, \nu)$ such that

$$\nu(\phi(x), \phi(y), \phi(z)) = \phi(\mu(x, y, z)), \forall x, y, z \in E.$$ Sometimes we will write $(x, y, z)_E$ or $(x, y, z)_F$ instead of $\mu(x, y, z)$, and we will use the expression $*$-tring instead of $*$-ternary ring.

There is an inclusion of the category of $*$-algebras into the category of $*$-trings: if $A$ is a $*$-algebra, then $(x, y, z) \mapsto xy^*z$ is a ternary product on $A$, and if $\pi : A \to A'$ is a homomorphism of $*$-algebras, then so is of $*$-trings.

Definition 2.2. If a subspace $F$ of a $*$-tring $E$ is invariant under the ternary product, we say that it is a sub-$*$-tring of $E$, or just a subring of $E$. A subring $F$ is said to be hermetic in $E$ if for $x \in E$ we have $(x, x, x) \in F \iff x \in F$.

Definition 2.3. A $*$-tring $E$ will be called admissible if $\{0\}$ is hermetic in $E$. A $*$-algebra $A$ will be called admissible if it is admissible as a $*$-tring.

Note that a $*$-algebra $A$ is admissible if and only if the condition $a^*a = 0$ implies $a = 0$.

Definition 2.4. Let $E$ be a $*$-tring and $F \subseteq E$ a subspace. We say that $F$ is an ideal of $E$ if $(E, E, F) + (E, F, E) + (F, E, E) \subseteq F$.

If $\pi : E \to F$ is a homomorphism into an admissible $*$-tring $F$, then $\ker \pi$ is an hermetic ideal of $E$:

$$\pi((x, x, x)) = 0 \iff (\pi(x), \pi(x), \pi(x)) = 0 \iff \pi(x) = 0$$

In case $F$ is an ideal of $E$, then $E/F$ has an obvious structure of $*$-tring for which the canonical map $q : E \to E/F$ is a homomorphism of $*$-trings. Note that $E/F$ is admissible whenever $F$ is hermetic. In particular if $\pi : E \to F$ is a homomorphism into an admissible $*$-tring $F$, then $E/\ker \pi$ is admissible.

Suppose $E$ is a complex vector space, and let $E^*$ denote its complex conjugate linear space. If $(E, \mu)$ is a $*$-tring, then $\mu^* : E^* \times E^* \times E^* \to E^*$ given by $\mu^*(x, y, z) = \mu(z, y, x), \forall x, y, z \in E^*$, is a $*$-ternary product on $E^*$. We call $(E^*, \mu^*)$ the adjoint or reverse $*$-tring of $(E, \mu)$. If $\pi : E \to F$ is a homomorphism, then $\pi$ remains a homomorphism $E^* \to F^*$, so it is clear that reversion is an autofunctor of order two of the category of $*$-trings, which moreover sends admissible $*$-trings
into admissible ∗-trings. If A is a ∗-algebra considered as a ∗-tring as above, then its reverse ∗-tring A∗ is the conjugate linear space of Aop considered as a ∗-tring.

**Example 2.5 (Basic triples).** Suppose \((E, A, \langle \cdot, \cdot \rangle)\) is a triple consisting of a \(\mathbb{C}\)-vector space \(E\), a ∗-algebra \(A\) over which \(E\) is a right module, and a sesquilinear map \(\langle \cdot, \cdot \rangle : E \times E \to A\) (conjugate linear in the first variable), such that \(\langle x, y \rangle a = \langle x, ya \rangle\) and \(\langle x, y \rangle^* = \langle y, x \rangle\), ∀\(x, y \in E\), \(a \in A\). Then \((\cdot, \cdot) : E \times E \times E \to E\) given by \(\langle x, y, z \rangle \mapsto x\langle y, z \rangle\) is a ternary product. We will say that \((E, (\cdot, \cdot))\) is the ternary ring associated with \((E, A, (\cdot, \cdot))\).

**Definition 2.6.** Triples as in Example 2.5 will be referred to as (right) basic triples. A basic triple \((E, A, \langle \cdot, \cdot \rangle)\) will be called admissible whenever \(A\) is admissible, and full if \(\text{span}\{\langle x, y \rangle_A : x, y \in E\} = A\). By a homomorphism from the basic triple \((E, A, \langle \cdot, \cdot \rangle)\) into the basic triple \((F, B, \langle \cdot, \cdot \rangle)\) we mean a pair \((\varphi, \psi)\) of maps such that \(\varphi : E \to F\) is linear, \(\psi : A \to B\) is a homomorphism of ∗-algebras, and \(\varphi(xa) = \varphi(x)\psi(a)\), ∀\(x \in E\), \(a \in A\).

Similarly we can define left basic triples, using left instead of right \(A\)-modules.

We will see soon that any admissible ∗-tring can be described in terms of basic triples as in Example 2.5.

**Proposition 2.7.** Let \((E, A, \langle \cdot, \cdot \rangle)\) be a basic triple.

1. If \(A\) is admissible, and \(\langle x, x \rangle = 0\) implies \(x = 0\), then the ∗-tring \(E\) is admissible as well.
2. If \((E, A, \langle \cdot, \cdot \rangle)\) is admissible and full, then \(E\) is faithful as an \(A\)-module.

**Proof.** If \(x \in E\) is such that \(\langle x, x \rangle = 0\), then
\[
\langle x, x \rangle^* \langle x, x \rangle = \langle x, x \rangle \langle x, x \rangle = \langle x, x \rangle = 0.
\]
Now, if \(A\) is admissible, the latter equality implies \(\langle x, x \rangle = 0\), so \(x = 0\). As for the second statement suppose \((E, A, \langle \cdot, \cdot \rangle)\) is admissible and full, and \(a \in A\) is such that \(a = \sum_{j=1}^n \langle y_j, z_j \rangle\) and \(ya = 0\), ∀\(y \in F\). Then we have \(a^*a = \sum_{j=1}^n \langle y_j, z_j \rangle a = 0\), so \(a = 0\). Then \(E\) is a faithful \(A\)-module. \(\square\)

**Lemma 2.8.** Suppose that \((E, A, \langle \cdot, \cdot \rangle)\) and \((F, B, \langle \cdot, \cdot \rangle)\) are basic triples, with the former full, and \(F\) admissible as ∗-tring and faithful as a \(B\)-module. Then, if \(\varphi : (E, (\cdot, \cdot)) \to (F, (\cdot, \cdot))\) is a homomorphism between their associated ∗-trings, there exists a unique homomorphism of ∗-algebras \(\psi : A \to B\) such that \(\psi((x, y)_A) = \langle \varphi(x), \varphi(y) \rangle_B\), ∀\(x, y \in E\). Besides we have \(\varphi(xa) = \varphi(x)\psi(a)\), ∀\(x \in E\), \(a \in A\), and
\[
\ker \psi \subseteq \{a \in A : Ea \subseteq \ker \varphi \} \subseteq \{a \in A : \psi(a)^* \psi(a) = 0\},
\]
both inclusions being equalities if \(B\) is admissible. If \(E\) is also a faithful \(A\)-module and \(\varphi\) is injective, then so is \(\psi\).

**Proof.** We will suppose that \((F, B, (\cdot, \cdot))\) is full: otherwise we just replace \(B\) by \(\text{span}(F, B)_B\). We concentrate in showing the existence of the map \(\psi\), because its uniqueness is obvious. To this end suppose that \(x_1, \ldots, x_n\) and \(y_1, \ldots, y_n\) are elements in \(E\) such that \(\sum_{j=1}^n \langle x_j, y_j \rangle_A = 0\), and therefore also \(\sum_{j=1}^n \langle y_j, x_j \rangle_A = 0\). Consider the element \(c := \sum_{j=1}^n \langle \varphi(x_j), \varphi(y_j) \rangle_B\) of \(B\). All we have to do is to show
that $c = 0$. Now, if $x \in E$ and $u \in F$ we have
\[
(\varphi(x), uc, uc) = \sum_k (\varphi(x), (u, \varphi(x_k), \varphi(y_k)), uc)
\]
\[
= \sum_k (\varphi(x), \varphi(y_k), \varphi(x_k)), uc) = (\varphi(x \sum_k (y_k, x_k)_A), u, uc) = 0.
\]
Hence, if $u \in F$:
\[
(uc, uc, uc) = \sum_j ((u, \varphi(x_j), \varphi(y_j)), uc, uc) = \sum_j (u, \varphi(x_j), (\varphi(y_j), uc, uc)) = 0
\]
Since $F$ is admissible, it follows that $uc = 0$, $\forall u \in F$, so $c = 0$ because $F$ is a faithful $B$-module.

Suppose now that $a \in \ker \psi$. Then $\varphi(xa) = \varphi(x)\psi(a) = 0$, so $Ea \subseteq \ker \varphi$. On the other hand, if the element $a = \sum_j (x_j, y_j)_A$ is such that $Ea \subseteq \ker \varphi$, then
\[
\psi(a^*)a = \psi(\sum_i (y_i,x_i,y_i)_A) = \sum_i ((\varphi(y_i), \varphi(x_i), \varphi(y_i), y_i)_A) = 0
\]
because $\varphi(x_i,a) = 0 \forall i$. In case $B$ is admissible we have $\psi(a^*)a = 0$ if and only if $a \in \ker \psi$, so in this case the three considered sets agree. Finally, when $E$ is faithful and $\ker \varphi = 0$, we have $\{a \in A : Ea \subseteq \ker \varphi\} = 0$, so $\ker \psi = 0$. 

Given two modules $E$ and $F$ over a ring $R$, we denote by $\text{Hom}_R(E,F)$ the abelian group of $R$-linear maps from $E$ into $F$, and just by $\text{End}_R(E)$ in case $E = F$. Let $E$ be an admissible $*$-tring, and suppose $T \in \text{End}_C(E)$ is such that there exists $S \in \text{End}_C(E)$ that satisfies $(x,Ty,z) = (Sx,y,z), \forall x,y,z \in E$. Since $\{0\}$ is hermetic in $E$, given $T \in \text{End}_C(E)$, there exists at most one such endomorphism $S$; in this case we say that $S$ is the adjoint of $T$ to the left, and we denote it by $T^*$. The set $\mathcal{L}(E)$ of $C$-linear endomorphisms of $E$ that have an adjoint to the left is clearly a unital subalgebra of $\text{End}_C(E)$. Every pair of elements $y,z \in E$ gives rise to an endomorphism $\theta_{y,z} : E \to E$ given by $\theta_{y,z}(x) := (x,y,z)$. It is readily checked that $\theta_{y,z}$ is adjointable with adjoint $\theta_{z,y}$.

**Proposition 2.9.** Let $E$ be an admissible $*$-tring. Then the map $* : \mathcal{L}(E) \to \mathcal{L}(E)$, given by taking the adjoint, is an involution in $\mathcal{L}(E)$. Moreover, the $*$-algebra $\mathcal{L}(E)$ is an admissible $*$-tring, and $\text{span}\{\theta_{y,z} : y,z \in E\}$ is a twosided ideal of $\mathcal{L}(E)$, which is essential in the sense that $T \theta_{y,z} = 0 \forall y,z \in E$ or $\theta_{y,z}T = 0 \forall y,z \in E$ implies $T = 0$.

**Proof.** It is clear that the map $T \mapsto T^*$ is conjugate linear and antiprimitive. On the other hand, if $T \in \mathcal{L}(E)$:
\[
(u, T(x,y,z), u) = (u, z, (y, T(x), u)) = (u, z, (T^*(y), x, u)) = (u, (x, T^*(y), z), u)
\]
\forall x,y,z,u \in E$ and $T \in \mathcal{L}(E)$, which shows that $T^{**} = T$. Now, if $x \in E$, and $T \in \mathcal{L}(E)$ is such that $T^*T = 0$: $(Tx,Tx,Tx) = (x, T^*T(x), x) = 0$, so $T(x) = 0$, and therefore $T = 0$. Finally, if $T \in \mathcal{L}(E)$ and $x,y,z \in E: \theta_{y,z}T(x) = (x, T^*(y), z) = \theta_{y,z}(x)$. Thus $T \theta_{y,z} = (\theta_{z,y}T^*)^* = \theta_{y,z}T$. This shows that $\text{span}\{\theta_{y,z} : y,z \in E\}$ is an ideal of $\mathcal{L}(E)$. If $\theta_{y,z}T = 0 \forall y,z \in E$, then $0 = \theta_{T(x),T(x)}T(x) = (Tx,Tx,Tx), \forall x \in E$. Then $Tx = 0 \forall x \in E$ because $E$ is admissible, so $T = 0$. 

\qed
The next result shows that any admissible $*$-tring $E$ gives rise to an admissible and full right basic triple $(E, E^r_0, \langle \cdot, \cdot \rangle_r)$. In the same way one could show that $E$ also defines a left basic triple $(E, E'^l_0, \langle \cdot, \cdot \rangle_l)$.

**Theorem 2.10.** Let $E$ and $F$ be admissible $*$-trings. Then:

1. There exists a pair $(E^r_0, \langle \cdot, \cdot \rangle_r)$ such that $(E, E^r_0, \langle \cdot, \cdot \rangle_r)$ is an admissible and full basic triple, whose associated $*$-tring is $E$.
2. If $\pi : E \to F$ is a homomorphism of $*$-trings, and $(E^r_0, \langle \cdot, \cdot \rangle_r)$ and $(F^r_0, \langle \cdot, \cdot \rangle_r)$ are pairs like above for $E$ and $F$ respectively, there exists a unique homomorphism of $*$-algebras $\pi^*_0 : E^r_0 \to F^r_0$ such that $\pi^*_0((x,y)_r) = (\pi(x), \pi(y))_r, \forall x, y \in E$.

Moreover, $\pi(xb) = \pi(x)\pi(b), \forall x \in E, b \in E^r_0$, that is, the pair $(\pi, \pi^*_0)$ is a homomorphism of basic triples.

3. The pair $(E^r_0, \langle \cdot, \cdot \rangle_r)$ is the unique (up to canonical isomorphisms) such that the triple $(E, E^r_0, \langle \cdot, \cdot \rangle_r)$ is a full and admissible with $E$ as associated $*$-tring.

**Proof.** Note that $E$ is a faithful right $\mathcal{L}(E)^{\text{op}}$-module with $xT := T(x)$. Consider the ideal $E^r_0 := \text{span}\{\theta_{y,z} : y,z \in E\}$ of $\mathcal{L}(E)^{\text{op}}$ and let $\langle \cdot, \cdot \rangle_r : E \times E \to E^r_0$ be given by $(x,y)_r := \theta_{x,y}$. It is routine to verify that $(E, E^r_0, \langle \cdot, \cdot \rangle_r)$ is a full and admissible basic triple whose associated $*$-tring is $E$. The second statement follows at once from 2.8 and 2.7 while the last assertion of the theorem follows immediately from the second one.

**Corollary 2.11.** The assignment

$$(E \cong F) \mapsto (E, E^r_0, \langle \cdot, \cdot \rangle_r) \xrightarrow{\pi, \pi^*_0} (F, F^r_0, \langle \cdot, \cdot \rangle_r)$$

defines a functor from the category of admissible $*$-trings into the category of admissible and full basic triples.

**Corollary 2.12.** Let $(E, A, \langle \cdot, \cdot \rangle_A)$ be a basic triple such that $E$ is faithful as an $A$-module and $E$ is admissible as a $*$-tring. Then there exists a unique homomorphism $\psi : E^r_0 \to A$ such that $\langle x, y \rangle_r = \langle x, y \rangle_A, \forall x, y \in E$. The homomorphism $\psi$ is injective, and it is an isomorphism if $(E, A, \langle \cdot, \cdot \rangle_A)$ is full.

**Proof.** Let $(E, E^r_0, \langle \cdot, \cdot \rangle_r)$ be the full and admissible basic triple provided by Theorem 2.10. The identity map on $E$ is an injective homomorphism of $*$-trings, so by 2.8 there exists a unique homomorphism $\psi : E^r_0 \to A$ such that $\langle x, y \rangle_r = \langle x, y \rangle_A, \forall x, y \in E$, which is injective because $E$ is faithful as $E^r_0$-module. It is clear that $\psi$ is also surjective when the given basic triple is full.

**Corollary 2.13.** Let $(E, A, \langle \cdot, \cdot \rangle_A)$ be a full basic triple such that $E$ is faithful as an $A$-module. Then $A$ is admissible if $E$ is admissible.

**Proof.** Just note that if $E$ is admissible, then $E^r_0 \cong A$ by 2.12 and $E^r_0$ is admissible according to 2.10.

**Corollary 2.14.** Let $F$ be an ideal of the admissible $*$-tring $E$, $(E, E^r_0, \langle \cdot, \cdot \rangle_E)$ and $(F, F^r_0, \langle \cdot, \cdot \rangle_F)$ the full and admissible basic triples associated, respectively, with $E$ and $F$ (given by Theorem 2.10). If $A := \text{span}\{\langle x, y \rangle_E : x, y \in F\}$, then $A$ is a $*$-ideal of $E^r_0$, and the basic triples $(F, F^r_0, \langle \cdot, \cdot \rangle_F)$ and $(F, A, \langle \cdot, \cdot \rangle_E)$ are isomorphic.
Proof. The triple $(F, A, \langle \cdot, \cdot \rangle_r)$ is admissible and full, with $F$ as induced $*$-tring. Then $F$ is a faithful $A$-module by 2.7. According to 2.12, there exists a unique map $\psi : F_r^0 \to A$ such that $(id, \psi)$ is a homomorphism from $(F, F_0^r, \langle \cdot, \cdot \rangle_F)$ to $(F, A, \langle \cdot, \cdot \rangle_E)$, and $\psi$ is an isomorphism of $*$-algebras. It follows that $(id, \psi)$ is the inverse homomorphism of $(id, \psi)$. \hfill \Box

From now on if $F$ is an ideal in the admissible $*$-tring $E$, we will think of $F_0^r$ as a $*$-ideal of $E_0^r$ via the identification provided by 2.14:

$$F_0^r \cong \text{span}\{\langle x, y \rangle_E : x, y \in F\}. \tag{2.2}$$

For the next result recall that an ideal $F$ of the $*$-tring $E$ is hermetic if and only if $E/F$ is admissible.

**Proposition 2.15.** Let $\pi : E \to F$ be a homomorphism between the admissible $*$-trings $E$ and $F$, such that $\ker \pi$ is hermetic. If $I_{\ker \pi} := \{a \in E_0^r : Ea \subseteq \ker \pi\}$, then:

$$(\ker \pi)_0^r \subseteq \ker \pi^r \subseteq I_{\ker \pi}$$

**Proof.** Taking into account (2.2) above and the second part of 2.10, the first inclusion is clear. The second inclusion follows from the admissibility of $E/\ker \pi$ and (2.1) in Lemma 2.8. \hfill \Box

**Remark 2.16.** Suppose $F$ is an hermetic ideal of the admissible $*$-tring $E$. Let $q : E \to E/F$ be the quotient map, $I_F := \{a \in E_0^r : Ea \subseteq F\}$, $p : E_0^r \to E_0^r/I_F$ the canonical projection and $q_0^r : E_0^r/I_F \to (E/F)_0^r$ the isomorphism induced by $q_0^r$, so the following diagram commutes:

$$
\begin{array}{ccc}
E_0^r & \xrightarrow{q_0^r} & (E/F)_0^r \\
p \downarrow & & \downarrow q_0^r \\
E_0^r/I_F & \xrightarrow{\overline{q}_0^r} & \overline{(E/F)_0^r}
\end{array}
$$

Then:

$$\overline{q}_0^r(p(\langle x, y \rangle_E)) = q_0^r(\langle x, y \rangle_E) = \langle q(x), q(y) \rangle_{E/F}, \quad \forall x, y \in E.$$  

Therefore the pair $((E/F)_0^r, \langle \cdot, \cdot \rangle_{E/F})$ associated with $E/F$ in Theorem 2.10 may be replaced by the pair $(E_0^r/I_F, \langle \cdot, \cdot \rangle_{E/F})$, where $\langle q(x), q(y) \rangle_{E/F} = p(\langle x, y \rangle_E), \forall x, y \in E$ and the action of $E_0^r/I_F$ on $E/F$ is given by $q(x)p(a) = q(xa), \forall x \in E, a \in A$.

**Proposition 2.17.** Let $\pi : E \to F$ be a homomorphism between admissible $*$-trings. Then:

1. $\pi$ is injective if and only if $\pi_0^r : E_0^r \to F_0^r$ is injective.
2. If $\pi$ is onto, or an isomorphism, then so is $\pi_0^r : E_0^r \to F_0^r$.

**Proof.** Since the second statement is clear we prove only the first one. Now if $\pi_0^r$ is injective and $x \in E$, the admissibility of $E$ and $F$ implies that:

$$\pi(x) = 0 \iff (\pi(x), \pi(x))_r = 0 \iff \pi_0^r(\langle x, x \rangle_r) = 0 \iff x = 0,$$

so $\pi$ is injective as well. On the other hand the injectivity of $\pi$ implies that of $\pi_0^r$ by 2.8. \hfill \Box
3. Correspondence between $C^*$-seminorms.

3.1. $C^*$-seminorms.

**Definition 3.1.** A $C^*$-seminorm on a $*$-tring $(E, \mu)$ is a seminorm such that:

1. $\|\mu(x, y, z)\| \leq \|x\| \|y\| \|z\|$, $\forall x, y, z \in E$.
2. $\|\mu(x, x, x)\| = \|x\|^3$, $\forall x \in E$.

If $\|\cdot\|$ is a norm, we call it a $C^*$-norm, and we say that $(E, \|\cdot\|)$ is a pre-$C^*$-ternary ring. If $(E, \|\cdot\|)$ is also a Banach space, we say that it is a $C^*$-ternary ring, or just a $C^*$-tring.

If $E$ is a $*$-tring, the set of $C^*$-seminorms on $E$ will be denoted by $SN(E)$, and $N(E)$ will denote the set of $C^*$-norms on $E$. The set $SN(E)$ is partially ordered by: $\gamma_1 \leq \gamma_2$ if $\gamma_1(x) \leq \gamma_2(x)$, $\forall x \in E$.

**Definition 3.2.** A $*$-tring $E$ will be called $C^*$-closable, or just closable, in case $N(E) \neq \emptyset$. Similar terminology will be used for $*$-algebras.

Observe that any $C^*$-closable $*$-tring is admissible.

In the next proposition, whose easy proof is left to the reader, we record some basic facts about $*$-trings.

**Proposition 3.3.** Let $E$ be a $*$-tring. Then:

1. $N_\gamma := \{x \in E : \gamma(x) = 0\}$ is an hermetic ideal of $E$, for all $\gamma \in SN(E)$.
2. The intersection of hermetic subrings is also hermetic.
3. The quotient $E/N$ is admissible, where $N := \cap\{N_\gamma : \gamma \in SN(E)\}$ and $N_\gamma$ is as in 1.
4. If $SN(E)$ separates points of $E$, then $E$ is admissible.
5. If $SN(E)$ separates points of $E$ and is bounded, then $E$ is $C^*$-closable.

If $H$ and $K$ are Hilbert spaces and $B(H, K)$ denotes the corresponding space of bounded linear maps, a subspace $E$ of $B(H, K)$ closed under the ternary product $(R, S, T) \mapsto RST \in E$, $\forall R, S, T \in E$, is a $*$-tring with that product. In case $E$ is also closed it is called a ternary ring of operators (TRO). Note that if $(E, \mu)$ is a $C^*$-tring, then $(E, -\mu)$ also is a $C^*$-tring, called the opposite of $(E, \mu)$ and denoted by $E^{\text{op}}$. The opposite of a TRO is called anti-TRO.

New $C^*$-ternary rings can be obtained by direct sums: if $(E, \|\cdot\|, \mu_E)$ and $(F, \|\cdot\|_F, \mu_F)$ are $C^*$-trings, then $(E \oplus F, \max\{\|\cdot\|_E, \|\cdot\|_F\}, \mu_E \oplus \mu_F)$ is a $C^*$-tring. We denote it just by $E \oplus F$.

Suppose that $E$ is a full right Hilbert $A$-module, and define the ternary product on $E$: $\mu_E(x, y, z) := x(y, z)$. Then $(E, \mu_E)$ is a $C^*$-tring with the norm $\|x\| = \sqrt{\langle x, x \rangle}$. Now, if $F$ is a full right Hilbert $B$-module, then $E \oplus F^{\text{op}}$ is also a $C^*$-tring. This is the fundamental example of $C^*$-tring, as shown by Zettl in [12, 3.2] (see also Corollary 3.10 below).

Zettl also showed that there exist unique sub-$C^*$-trings $E_+$ and $E_-$ of $E$ such that $E = E_+ \oplus E_-$, and $E_+$ is isomorphic to a TRO, while $E_-$ is isomorphic to an anti-TRO (see [12]). The decomposition above is called the fundamental decomposition of $E$.

**Definition 3.4.** We say that a $C^*$-tring $E$ is positive (negative) if $E = E_+$ (respectively: if $E = E_-$).
If $E$ is a $C^*$-tring, we define $E_p := E_+ \oplus E^{2p}$. Then $E_p$ is a positive $C^*$-tring.

Let $E^*$ be the reverse $*$-tring of (the $*$-tring) $E$. It is clear that a norm on $E$ is a $C^*$-norm if and only if is a $C^*$-norm on $E^*$. Moreover, $E$ is a (positive) $C^*$-tring if and only if so is $E^*$.

3.2. From pre-$C^*$-trings to pre-$C^*$-algebras. In what follows we will examine an intermediate situation between the $*$-algebraic context of [2,10] and the $C^*$-context originally considered by Zettl.

If $\alpha$ is a seminorm on the vector space $X$, then $N_\alpha := \{x \in X : \alpha(x) = 0\}$ is a closed subspace of $X$, so $X/N_\alpha$ is a normed space with the norm $\tilde{\alpha}$ induced by $\alpha$: $\tilde{\alpha}(x + N_\alpha) = \alpha(x)$. The completion $(X,\tilde{\alpha})$ of $(X/N_\alpha,\tilde{\alpha})$ will be referred to as the Hausdorff completion of the seminormed space $(X,\alpha)$, and the map $x \mapsto x + N_\alpha$ will be called the canonical map.

In case $\gamma$ is a $C^*$-seminorm on the ternary ring $E$, then $E/N_\gamma$ is a pre-$C^*$-tring with the induced norm $\tilde{\gamma}$. Thus the corresponding Hausdorff completion $E_\gamma$ of $E$ is a $C^*$-tring.

**Proposition 3.5.** Suppose $E$ is an admissible $*$-tring and $\gamma \in SN(E)$. Let $\gamma^r : E_\gamma^0 \to [0,\infty)$ be the operator seminorm on $E_\gamma^0$, that is:

$$\gamma^r(a) := \sup\{\gamma(xa) : \gamma(x) \leq 1\}.$$  

Then $\gamma^r \in SN(E_\gamma^0)$, and $\gamma^r \in N(E_\gamma^0) \iff \gamma \in N(E)$. Moreover the following relations hold:

$$\gamma(xa) \leq \gamma(x)\gamma^r(a), \forall x \in E, a \in E_\gamma^0$$  

$$\gamma^r((x,y)_r) \leq \gamma(x)\gamma^r(y), \forall x, y \in E$$  

$$\gamma^r(x)^2 = \gamma^r((x,x)_r), \forall x \in E$$

**Proof.** Given $a = \sum_{i=1}^n (x_i,y_i) \in E_\gamma^0$ the linear map $x \mapsto xa$ is bounded because $\gamma(xa) \leq \gamma(x)\sum_{i=1}^n \gamma(x_i)\gamma(y_i)$. Then (3.2) and (3.3) follow immediately and Definition 3.6 implies (3.4). With $a \in E_\gamma^0$ as before and $x \in E$ we have

$$(xa, xa, xa) = \sum_{i=1}^n ((x_i, y_i), xa, xa) = \sum_{i=1}^n (x_i, x_i, y_i, x_i, x_a),$$

so

$$\gamma^r(xa)^3 = \gamma^r(x, xaa^*), xa \leq \gamma^r(aa^*) \gamma^r(a) \gamma^r(x)^3,$$

from where it follows that $\gamma^r((a)^2) \leq \gamma^r(aa^*) \leq \gamma^r(a)\gamma^r(a)^*$. From the computations above is clear that $\gamma^r \in N(E_\gamma^0) \iff \gamma \in N(E)$. In particular $E_\gamma^0$ is a $C^*$-closable algebra whenever $E$ is a $C^*$-closable tring. 

**Definition 3.6.** Suppose $(E, A, \langle , \rangle_A)$ is a basic triple such that $(E, \gamma)$ is a $C^*$-tring and a Banach module over the $C^*$-algebra $(A, \alpha)$, and that $\langle , \rangle_A : E \times E \to A$ is continuous. Then the triple is said to be a $C^*$-basic triple. We say that it is full if the ideal span $\{x, y \in E \mid x, y \in E\}$ of $A$ is dense in $A$.

The next two results will be useful for studying the relation between a $C^*$-basic triple $(E, A, \langle , \rangle_A)$ and the basic triple $(E, E_\gamma^0, \langle , \rangle_r)$. What we will show first, in 3.4, is that $(E, E_\gamma^0, \langle , \rangle, \gamma)$ can be embedded in $(E, A, \langle , \rangle_A)$.

**Proposition 3.7.** Let $A$ be a Banach $*$-algebra and $I$ a $*$-ideal of $A$, not necessarily closed. Then any $C^*$-seminorm on $I$ can be extended to a $C^*$-seminorm on $A$. If $I$ is dense, such extension is unique.
Proof. Consider $\alpha \in SN(I)$, $\alpha \neq 0$. Let $I_\alpha$ be the Hausdorff completion of $(I, \alpha)$, $p : I \rightarrow I_\alpha$ the canonical map, and let $\pi : I_\alpha \rightarrow B(H)$ be a faithful representation. Now, according to [3, VI-19.11], the representation $\pi p : I \rightarrow B(H)$ can be extended to a representation $\rho$ of $A$. Then $a \mapsto \|\rho(a)\|$ defines a $C^*$-seminorm on $A$ that defines $\alpha$. Note that the continuity of $\rho$ implies the continuity of $\alpha$, from which the uniqueness of the extension follows in case $I$ is dense in $A$. \hfill \Box

Corollary 3.8. Let $I$ be a $*$-ideal of the $C^*$-algebra $A$. Then the unique $C^*$-norm one can define in $I$ is the restriction of $I$ to the norm of $A$.

Proposition 3.9. Let $(E, A, \langle, \rangle_A)$ be a full $C^*$-basic triple, and $\gamma$ and $\alpha$ the corresponding norms of $E$ and $A$. Then $(A, \alpha)$ is the completion of $(E_0, \gamma^r)$, and $(\langle, \rangle_A)$ is the continuous extension of $\langle, \rangle_r$.

Proof. Note that $E$ is admissible for it is a $C^*$-tring. On the other hand $E$ is a faithful $A$-module: if $a \in A$ is such that $xa = 0 \forall x \in E$, then $\langle x, y \rangle_A a = 0 \forall x, y \in E$, so it follows that $ba = 0$ for every $b$ in the dense ideal span$\{\langle x, y \rangle_A : x, y \in E\}$ of $A$, which implies $a = 0$. Thus there exists, by 2.7, a unique homomorphism $\psi : E_0^r \rightarrow A$ such that $\psi(\langle x, y \rangle) = \langle x, y \rangle_A$, $\forall x, y \in E$. Besides $\psi$ is injective and $\psi(E_0^r) = \mathrm{span}\{\langle x, y \rangle_A : x, y \in E\}$ (thus we may suppose $E_0^r$ is a dense ideal of $A$). Now [3,8] implies $\gamma^r_0$ is the restriction of $\alpha$ to $\psi(E_0^r)$, and, since the latter is dense in $A$, we conclude that $A$ is the completion of $E_0^r$.

As a consequence we obtain the following result, due to H. Zettl:

Corollary 3.10 (cf. [12, Proposition 3.2]). Let $(E, \gamma)$ be a $C^*$-tring and $E^r$ the completion of $E_0^r$ with respect to $\gamma^r$. Then $(E, E^r, \langle, \rangle_r)$ is, up to isomorphism, the unique full $C^*$-tring whose first component is $E$.

Proposition 3.11. Let $\pi : E_1 \rightarrow E_2$ be a homomorphism of $*$-trings between the $C^*$-trings $E_1$ and $E_2$. Then there exists a unique homomorphism $\pi^r : E^r_1 \rightarrow E^r_2$ such that $\pi^r(\langle x, y \rangle_{E_2}) = \langle \pi(x), \pi(y) \rangle_{E^r_2}$, $\forall x, y \in E$, and $\pi(xa) = \pi(x)\pi^r(a)$ $\forall x \in E$, $a \in E^r$. Consequently $\pi$ is always contractive, and is isometric if and only if it is injective.

Proof. It is clear that, if the homomorphism $\pi^r$ exists, it must be an extension of $\pi^r_0 : E_0^r \rightarrow E_0^r$. Let $\rho : F^r \rightarrow B(H)$ be a faithful representation. Then $\rho \pi^r_0$ is a representation of $E_0^r$. Now, since $(E, E^r, \langle, \rangle_r)$ is a $C^*$-tring, $E_0^r$ is a *-ideal in $E^r$. Therefore $\rho \pi^r_0$ can be uniquely extended to a representation $\bar{\rho} : E^r \rightarrow B(H)$ (VI-19.11]). Since $\rho(F^r)$ is closed and $\bar{\rho}(E^r)$ is a subset of the closure of $\rho \pi^r_0(E_0^r)$, we have $\bar{\rho}(E^r) \subseteq \rho(F^r)$. Then take $\pi^r := \rho^{-1} \bar{\rho}$. Note that $\|\pi(x)\| = \|\pi^r(\langle x, x \rangle)\| \leq \|\langle x, x \rangle\| = \|x\|^2$, with equality if $\pi^r$ is injective. This shows that $\pi$ is contractive. Finally, if $\pi$ is injective, so is $\pi^r_0$ and, as in the proof of 3.8, this implies that $\pi^r$ also is injective, thus an isometry. \hfill \Box

Corollary 3.12 (cf. [1] Proposition 4.1). The assignment

$$(E, F) \mapsto (E, E^r, \langle, \rangle_r) \hookrightarrow (F, F^r, \langle, \rangle_r)$$

defines a functor from the category of $C^*$-trings to the category of full $C^*$-basic triples.

It follows from Proposition 3.5 that any $C^*$-seminorm on $E_0^r$ induced by a $C^*$-seminorm on $E$ by means of 3.1 must satisfy the Cauchy-Schwarz condition 3.3.
So it is natural to restrict our attention to the following subsets of $C^*$-seminorms on $E_0^*$:

$$\mathcal{SN}(E_0^*) := \{\alpha \in \mathcal{SN}(E_0^*) : \alpha((x,y)_r)^2 \leq \alpha((x,x)_r)\alpha((y,y)_r)\}$$

$$\mathcal{N}_{cs}(E_0^*) := \mathcal{SN}(E_0^*) \cap \mathcal{N}(E_0^*).$$

In fact it will be convenient to place ourselves in a slightly more general setting:

**Definition 3.13.** Let $(E, A, \langle , \rangle)$ be a basic triple. We define

$$\mathcal{SN}(A) := \{\alpha((x,y)) \leq (\alpha(x)\alpha(y)) \in \mathcal{SN}(A) : \alpha(x)^2 \leq \alpha((x,x))\alpha((y,y))\}, \forall x, y \in E.$$

**Proposition 3.14.** Let $(E, A, \langle , \rangle)$ be a basic triple, and consider $E$ with the *-tring structure induced by $\langle , \rangle$. Given $\alpha \in \mathcal{SN}(A)$, let $\hat{\alpha} : E \to [0,\infty)$ be defined by:

$$\hat{\alpha}(x) := \alpha((x,x))^{1/2} \quad \text{(3.5)}$$

Then

1. $\hat{\alpha}(ax) \leq \hat{\alpha}(a)\alpha(a)$. 
2. $\hat{\alpha} \in \mathcal{SN}(E)$. 
3. If $E$ is a faithful $A$-module and $\hat{\alpha} \in \mathcal{N}(E)$, then $\alpha \in \mathcal{N}_{cs}(A)$. 
4. If $\alpha \in \mathcal{N}_{cs}(A)$ and $\langle x,x \rangle = 0$ implies $x = 0$, then $\hat{\alpha} \in \mathcal{N}(E)$.

**Proof.** Since the Cauchy-Schwarz inequality holds for $\alpha$, it follows as usual that $\hat{\alpha}$ satisfies the triangular inequality and, since homogeneity is obvious, $\hat{\alpha}$ is a seminorm on $E$. On the other hand, since $\alpha$ is a $C^*$-seminorm and satisfies the inequalities, we have, for all $x, y, z \in E, a \in A$:

$$\hat{\alpha}(ax) = \alpha(a^*\langle x,x \rangle a)^{1/2} \leq \alpha(a)\hat{\alpha}(x)$$

$$\hat{\alpha}(x,y,z) = \hat{\alpha}(x,y,z) \leq \hat{\alpha}(x)\alpha(\langle y,z \rangle) \leq \hat{\alpha}(x)\hat{\alpha}(y)\hat{\alpha}(z)$$

$$\hat{\alpha}(\langle x,x \rangle) = \alpha(\langle x,x \rangle)^{3/2} = \alpha((x,x)^3)^{1/2} = \alpha((x,x)^3)^{1/2} = \alpha(x)^3,$$

so $\hat{\alpha}$ is a $C^*$-seminorm on $E$. The first of the above inequalities implies that $\hat{\alpha}^*\hat{\alpha}$ is a norm whenever $\hat{\alpha}$ is and is $E$ is a faithful $A$-module. Finally, if $\alpha$ is a norm, it follows directly from the inequalities that $\hat{\alpha}$ also is a norm when the condition $\langle x,x \rangle = 0$ implies $x = 0$. \qed

**Corollary 3.15.** If $E$ is an admissible *-tring and $\gamma \in \mathcal{SN}(E)$, $\alpha \in \mathcal{SN}_{cs}(E_0^*)$, then $\hat{\gamma}^* = \gamma$ and $\hat{\alpha}^* \leq \alpha$.

**Proof.** The first statement follows immediately from (3.3) and (3.5). As for the second one we have $\hat{\alpha}(a) = \sup\{\hat{\alpha}(ax) : \hat{\alpha}(x) \leq 1\} \leq \alpha(a)$ by 1. of (3.14). \qed

**Corollary 3.16.** Let $(E, A, \langle , \rangle)$ be a full basic triple, and $\alpha \in \mathcal{SN}_{cs}(A)$. If $\hat{\alpha} \in \mathcal{SN}(E)$ is given by (3.5), then $I_{N_{\alpha}} = N_{\alpha}$, where $I_{N_{\alpha}} := \{a \in A : Ea \subseteq N_{\hat{\alpha}}\}$.

**Proof.** The inclusion $N_{\alpha} \subseteq I_{N_{\alpha}}$ is clear because $\hat{\alpha}(ax) \leq \hat{\alpha}(a)\alpha(a)$, $\forall x \in E, a \in A$. Conversely, suppose that $a \in A$ is such that $\hat{\alpha}(ax) = 0, \forall x \in E$. Then $\alpha(a^*\langle x,y \rangle a) = \alpha(\langle xa,ya \rangle) = 0, \forall x, y \in E$. Now, since the triple is full, we can write $aa^* = \sum_j \langle x_j, y_j \rangle$, for certain $x_j, y_j \in E$, so we have:

$$0 \leq \alpha(a)^4 = \alpha(a^*a)^2 \leq \alpha(a^*aa^*a) = \alpha(a^* \sum_j \langle x_j, y_j \rangle a) \leq \sum_j \alpha(a^*\langle x_j, y_j \rangle a) = 0,$$

hence $a \in N_{\alpha}$. \qed
Proposition 3.17. Let \((E, A, \langle \cdot, \cdot \rangle)\) be a full basic triple, and \(\alpha \in SN_{cs}(A)\). Let 
\(\gamma := \bar{\alpha} \in SN(E)\), \(\bar{\alpha}\) given by (6.5). Then \(E_\gamma\) is a \(C^*\)-tring, \((E_\gamma', \bar{\gamma}') = (A_\alpha, \bar{\alpha})\) and \(\bar{\alpha}' = \alpha\).

Proof. Denote by \(q : E \to E/N_\gamma \subseteq E_\gamma\) and \(p : A \to A/N_\alpha \subseteq A_\alpha\) the corresponding canonical maps. We define \(E/N_\gamma \times A/N_\alpha \to E/N_\gamma\) and \([\cdot, \cdot] : E/N_\gamma \times E/N_\gamma \to A_\alpha\) such that \(q(x)p(a) := q(xa)\) and \([q(x), q(y)] := p(\langle x, y \rangle)\) respectively. Let us see that these operations are continuous in the norms \(\bar{\gamma}\) and \(\bar{\alpha}\). The action of \(A_\alpha\) on \(E/N_\gamma\) is continuous, for if \(x, y \in E\) and \(a \in A:\)
\[
\bar{\gamma}(q(x)p(a)) = \bar{\gamma}(q(xa)) = \bar{\gamma}(x)\alpha(a) = \bar{\gamma}(x)\bar{\alpha}(p(a))
\]
And the sesquilinear map \([\cdot, \cdot]_{E/N_\gamma}\) also is continuous, because:
\[
\bar{\alpha}(\langle q(x), q(y) \rangle_{E/N_\gamma}) = \bar{\alpha}(p(\langle x, y \rangle_{E})) = \alpha(\langle x, y \rangle_{E}) \leq \bar{\gamma}(x)\bar{\gamma}(y) = \bar{\gamma}(q(x))\bar{\gamma}(q(y)).
\]
Therefore these operations extend to continuous maps \(E_\gamma \times A_\alpha \to E_\gamma\) and \([\cdot, \cdot] : E_\gamma \times E_\gamma \to A_\alpha\), so we obtain a full \(C^*\)-basic triple \((E_\gamma, A_\alpha, [\cdot, \cdot])\). Therefore \((A_\alpha, \alpha) = (E_\gamma, \bar{\gamma}')\) by (3.9). As for the last assertion, we have to prove that \(\bar{\gamma}' = \alpha\) or, equivalently, that \(\gamma' = \bar{\alpha}\). So it is enough to show that \(\gamma' = \bar{\gamma}'p\). But, if \(a \in A:\)
\[
\gamma'(p(a)) = \sup\{\bar{\gamma}(q(x)p(a)) : \bar{\gamma}(q(x)) \leq 1\} = \sup\{\bar{\gamma}(q(xa)) : \gamma(x) \leq 1\} = \gamma'(a).
\]

Propositions 3.5 and 3.14 allow us to define maps \(\Phi_r : SN(E) \to SN_{cs}(E_0)\) and \(\Psi_r : SN_{cs}(E_0^\gamma) \to SN(E)\) such that \(\Phi_r(\gamma) = \gamma'\), given by (3.1), and \(\Psi_r(\alpha) = \bar{\alpha}\), given by (6.5). We want to show that in fact \(\Phi_r\) and \(\Psi_r\) are mutually inverse maps that preserve the order.

Theorem 3.18. Let \(E\) be an admissible \(*\)-tring. Then the maps \(\Phi_r : SN(E) \to SN_{cs}(E_0)\) and \(\Psi_r : SN_{cs}(E_0^\gamma) \to SN(E)\) are mutually inverse isomorphisms of lattices. Moreover, \(\Phi_r(N(E)) = N_{cs}(E_0)\) and \(\Psi_r(N_{cs}(E_0^\gamma)) = N(E)\).

Proof. By Corollary 3.15 we have \(\Psi_r \Phi_r = Id_{SN(E)}\), and Proposition 3.17 shows that \(\Phi_r \Psi_r = Id_{SN_{cs}(E_0)}\), so the maps \(\Phi_r\) and \(\Psi_r\) are mutually inverse. Besides, it follows from (6.5) that \(\Phi_r(\gamma)\) is a norm if and only if so is \(\gamma\). On the other hand is clear that \(\Psi_r\) preserves the order, thus it remains to be shown that \(\Phi_r\) also preserves the order. To this end consider \(\gamma_1 \leq \gamma_2\) in \(SN(E)\). Since \(id : (E, \gamma_2) \to (E, \gamma_1)\) is continuous, it induces a homomorphism \(\pi : E_{\gamma_2} \to E_{\gamma_1}\), which in turn induces, according with Proposition 3.11 a homomorphism \(\pi^* : E_{\gamma_2}^* \to E_{\gamma_1}^*\), which is necessarily contractive. Thus if \(a \in E_{\gamma_1}^*\), we have:
\[
\gamma_1^*(a) = \bar{\gamma}_1^*(\pi^*(a + N_{\gamma_1}^*)) \leq \bar{\gamma}_2^*(a + N_{\gamma_1}^*) = \gamma_2^*(a),
\]
which shows that \(\gamma_1^* \leq \gamma_2^*\).

All we have done to the right side can be done also to the left side. For example, every admissible \(*\)-tring \(E\) induces a (left) admissible and full basic triple \((E, E_0, \langle \cdot, \cdot \rangle)\), we have an isomorphism of posets \(\Phi_l : SN(E) \to SN_{cs}(E_0)\) with inverse \(\Psi_l : SN_{cs}(E_0) \to SN(E)\), given by \(\Phi_l(\gamma) = \gamma^l\) and \(\Psi_l(\alpha) = \bar{\alpha}\), where \(\gamma^l(a) := \sup\{\gamma(ax) : \gamma(x) \leq 1\}\) and \(\bar{\alpha}(x) := (\alpha(x, x)^{1/2}\), etc. Then we obtain the following consequences:

Corollary 3.19. Let \(E\) be an admissible \(*\)-tring. Then \(\Phi_l \Psi_l : SN_{cs}(E_0) \to SN_{cs}(E_0^\gamma)\) is an isomorphism of lattices such that \(\Phi_l \Psi_l(N_{cs}(E_0)) = N_{cs}(E_0^\gamma)\). The inverse of \(\Phi_l \Psi_l\) is \(\Phi_l \Psi_l\).
As mentioned at the end of 3.1 in [12] Theorem 3.1], Zettl proved that any $C^*$-tring is of the form $E = E_+ \oplus E_-$, where $E_+$ and $E^{op}$ are isomorphic to a TRO. In fact we have $E_+ := \{ x \in E : \langle x, x \rangle_r \text{ is positive} \}$, $E_- := \{ x \in E : -\langle x, x \rangle_r \text{ is positive} \}$, and $E_+$ and $E_-$ are ideals of $E$ such that $\langle E_+, E_- \rangle = 0$. If $E_p := E_+ \oplus E^{op}$, we will have that $E^{op}_p = E$ and $E'_p = E'$, and now $E_p$ is a Morita-Rieffel equivalence between $E'$ and $E$.

Thus we have:

**Corollary 3.20.** Let $E$ be an admissible *-tring and $\gamma \in SN(E)$. Then $E^I_\gamma$ and $E^*_\gamma$ are Morita-Rieffel equivalent $C^*$-algebras.

In general we will have to deal with algebras that strictly contain $E^*_0$, but whose $C^*$-seminorms are essentially the same, as the following results show.

**Proposition 3.21.** Let $I$ be a selfadjoint ideal of a *-algebra $A$, and suppose that $\alpha \in SN(I)$. Let $\alpha' : A \to [0, \infty]$ be given by $\alpha'(a) := \sup\{ \alpha(ax) : x \in I, \alpha(x) \leq 1 \}$. For every $a \in A$ consider $L_a : I \to I$, such that $L_a(x) = ax$, $\forall x \in I$. Then the following statements are equivalent:

1. $\alpha'(a) < \infty$, $\forall a \in A$.
2. $L_a$ is bounded, $\forall a \in A$.
3. $\alpha$ can be extended to a $C^*$-seminorm on $A$.

Suppose that one of the conditions above holds true. Then:

(a) $\alpha'$ is a $C^*$-seminorm on $A$, and $\alpha' \leq \beta$ for every $\beta \in SN(A)$ that extends $\alpha$.
(b) If $\alpha$ is a norm, then $\alpha'$ is a norm if and only if $\text{Ann}_A(I) = 0$, where $\text{Ann}_A(I) := \{ a \in A : ax = 0, \forall x \in I \}$.

**Proof.** Since $\|L_a\| = \alpha'(a)$, we have that conditions 1. and 2. are equivalent. It is also clear that 3. $\Rightarrow$ 1. Suppose now that $\alpha'(a) < \infty$, $\forall a \in A$. Let show that $\alpha'$ is a $C^*$-seminorm on $A$ that extends $\alpha$. It is easy to check that $\alpha'(ab) \leq \alpha'(a)\alpha'(b)$, $\forall a, b \in A$. Moreover:

$$\alpha'(a^*a) = \sup\{\alpha(a^*ax) : x \in I, \alpha(x) \leq 1 \} \geq \sup\{\alpha(x^*a^*ax) : x \in I, \alpha(x) \leq 1 \}$$

$$\geq \sup\{\alpha(ax)^2 : x \in I, \alpha(x) \leq 1 \} = \alpha'(a)^2.$$  

Therefore $\alpha' \in SN(A)$. The fact that $\alpha'$ extends $\alpha$, as well as assertion (a), are consequences of the fact that for every $C^*$-seminorm $\beta$ on $A$ one has that $\beta(a) = \sup\{\beta(ab) : \beta(b) \leq 1 \}$. Finally, suppose that $\alpha$ is a norm on $I$. Then $\alpha'(a) = 0 \iff \alpha(ax) = 0, \forall x \in I$, that is $\alpha'(a) = 0 \iff a \in \text{Ann}_A(I)$. \qed

**Theorem 3.22.** Let $(E, A, \langle , , \rangle)$ be an admissible basic triple, with $E$ a faithful $A$-module, and admissible as *-tring. Suppose that any $C^*$-seminorm on $E^*_0$ can be extended in a unique way to a $C^*$-seminorm on $A$ (recall Corollary 2.12). Then the lattices $SN(E)$ and $SN^{(0)}_{cs}(A)$ are isomorphic. If in addition $\text{Ann}_A(E^*_0) = 0$, the posets $N(E)$ and $N^{(0)}_{cs}(A)$ are isomorphic as well.

**Proof.** Since any $C^*$-seminorm on $E^*_0$ can be uniquely extended to a $C^*$-seminorm on $A$, we are allowed to identify $SN(A)$ and $SN(E^*_0)$ as lattices, and it is clear that this yields also an identification between $SN^{(0)}_{cs}(A)$ and $SN_{cs}(E^*_0)$, and the latter is isomorphic to $SN(E)$ by [3]. If moreover $\text{Ann}_A(E^*_0) = 0$, the same argument applies to $N(E)$ and $N_{cs}(A)$. \qed

In case $A$ is a Banach *-algebra, any $C^*$-seminorm on a *-ideal can be extended to a $C^*$-seminorm defined on the whole algebra. Moreover we have:
Proposition 3.23. Let $A$ be an admissible Banach $*$-algebra and $I$ a dense $*$-ideal of $A$, not necessarily closed. Then any $C^*$-norm on $I$ can be uniquely extended to a $C^*$-norm on $A$.

Proof. Let $\alpha \in \mathcal{N}(I)$. By $[3.21]$ $\alpha$ has a unique extension to a $C^*$-seminorm on $A$, and by $[3.22]$ (this extension must be $\alpha'$ such that $\alpha'(a) = \sup \{\alpha(ax) : x \in I, \alpha(x) \leq 1\}$.

Suppose $a \in \text{Ann}_A(I)$. Then $aa^* = 0$, because $I$ is dense in $A$ and $ax = 0$, $\forall x \in I$. Thus $a = 0$ for $A$ is admissible. Then $\alpha'$ is a norm by $[3.21]$. □

Corollary 3.24. Let $(E, A, \langle \cdot, \cdot \rangle_E)$ be an admissible basic triple with $A$ a Banach $*$-algebra and $E$ a faithful $A$-module. Suppose in addition that $E$ is an admissible $*$-triple such that $E^+_0$ is a dense ideal of $A$ (recall Corollary 2.12). Then the lattices $SN(E)$ and $SN_{cs}(A)$ are isomorphic, as well as the partially ordered sets $\mathcal{N}(E)$ and $\mathcal{N}_{cs}(A)$.

Proof. Just combine Theorem 3.22 with Proposition 3.21 and Proposition 3.23. □

Corollary 3.25. Let $(E, A, \langle \cdot, \cdot \rangle_A)$ and $(E, B, \langle \cdot, \cdot \rangle_B)$ be respectively left and right admissible basic triples, with $A$ and $B$ Banach $*$-algebras such that $E$ is an $(A-B)$-bimodule with the given structure, and $\langle x, y \rangle_A z = x \langle y, z \rangle_B$, $\forall x, y, z \in E$. If $E$ is faithful as a left $A$-module and as a right $B$-module, and $E^0_0$ and $E^0_1$ are dense in $A$ and $B$ respectively, then there is an isomorphism of lattices between $SN\langle \cdot, \cdot \rangle_A(A)$ and $SN\langle \cdot, \cdot \rangle_B(B)$, that restricts to an isomorphism between the posets $\mathcal{N}_{cs} \langle \cdot, \cdot \rangle_A(A)$ and $\mathcal{N}_{cs} \langle \cdot, \cdot \rangle_B(B)$.

3.3. Positive modules. In general is not a simple task to decide if a given $C^*$-seminorm satisfies the Cauchy-Schwarz property with respect to a certain sesquilinear map. However this is always the case for the positive modules we introduce next.

Let $\alpha$ be a $C^*$-seminorm on the $*$-algebra $A$, and let $p_\alpha : A \rightarrow A_\alpha$ be the canonical map, where $A_\alpha$ is the Hausdorff completion of $A$. If $\Lambda \subseteq SN(A)$, then $A_\Lambda^+ := \cap_{\alpha \in \Lambda} p_\alpha^{-1}(A_\alpha^+)$ is a cone. When $\Lambda = SN(A)$, we write $A^+$ instead of $A_\Lambda^+$. Therefore $A^+$ is the set of elements of $A$ that are positive in any $C^*$-Hausdorff completion of $A$. Of course the map $A \rightarrow A_+^\alpha$ is order reversing.

Definition 3.26. Given $\Lambda \subseteq SN(A)$, we say that $a \in A$ is positive in $(A, \Lambda)$, or that it is $\Lambda$-positive, if $a \in A_\Lambda^+$. The elements of $A^+$ are just called the positive elements of $A$.

It is clear that $A^+$ contains the cone $C_A := \{\sum_{i,j=1}^n a_i^* a_j : n \in \mathbb{N}, a_i \in A, i = 1, \ldots, n\}$, and that $p_\alpha(C_A)$ is dense in $A_\alpha^+$, $\forall \alpha \in SN(A)$. Also note that if $\phi : A \rightarrow B$ is a homomorphism between $*$-algebras, then $\phi(A^+) \subseteq B^+$ and $\phi(C_A) \subseteq C_B$.

If $SN(A)$ is bounded, with $\alpha := \max SN(A)$, then $a$ is positive in $A$ if and only if $a$ is positive in $(A, \alpha)$. In particular, if $A$ is a Banach $*$-algebra, then $a \in A^+$ if and only if $i(a) \in C^*(A)^+$, where $i : A \rightarrow C^*(A)$ is the natural map of $A$ into its $C^*$-enveloping algebra $C^*(A)$.

Lemma 3.27. Let $A$ be $C^*$-closable. Then $A^+ = \bigcap \{p_\alpha^{-1}(A_\alpha^+) : \alpha \in N(A)\}$.

Proof. Clearly we have that $A^+ \subseteq \bigcap \{p_\alpha^{-1}(A_\alpha^+) : \alpha \in N(A)\}$. Let $\beta \in SN(A)$. Since the maximum of two $C^*$-seminorms is again a $C^*$-seminorm, and since $A$ is $C^*$-closable, we may pick $\beta' \in N(A)$ such that $\beta' \geq \beta$. Then the identity map on $A$ induces a homomorphism $\phi : A_{\beta'} \rightarrow A_{\beta}$, determined by $\phi(p_{\beta'}(a)) = p_{\beta}(a)$, $\forall a \in A$. \[\square\]
If \( a \in \bigcap \{ p_\alpha^{-1}(A)_\beta^+ : \alpha \in \mathcal{N}(A) \} \) then \( p_\beta(a) \in A_\beta^+ \), and therefore \( p_\beta(a) \in A_\beta^+ \). This proves the converse inclusion.

Once we have a cone of positive elements on a \( * \)-algebra \( A \), we are able to define a notion similar to that of Hilbert module.

**Definition 3.28.** Let \( A \) be a \( * \)-algebra, \( E \) a right \( A \)-module, and \( \Lambda \subseteq S\mathcal{N}(A) \). We say that a map \( \langle \cdot, \cdot \rangle : E \times E \to \Lambda \) is a \( \Lambda \)-semi-pre-inner product on \( E \) if:

1. \( \langle x, \lambda_1 y + \lambda_2 z \rangle = \lambda_1 \langle x, y \rangle + \lambda_2 \langle x, z \rangle, \forall x, y, z \in E, \lambda_1, \lambda_2 \in \mathbb{C} \).
2. \( \langle x, ya \rangle = \langle x, y \rangle a, \forall x, y \in E, a \in A \).
3. \( \langle y, x \rangle = \langle x, y \rangle^*, \forall x, y \in E \).
4. \( \langle x, x \rangle \) is \( \Lambda \)-positive, \( \forall x \in E \).

The pair \( (E, \langle \cdot, \cdot \rangle) \) is then called a right positive \( \Lambda \)-module. In case \( \Lambda = S\mathcal{N}(A) \) we say that \( (E, \langle \cdot, \cdot \rangle) \) is a right positive \( A \)-module.

Similarly we define left semi-pre-inner-products and left positive modules.

**Definition 3.29.** An admissible \( * \)-tring \( E \) is right (left) positive if \( (E, \langle \cdot, \cdot \rangle_r) \) is a positive \( E_0^+ \)-module (respectively: \( (E, \langle \cdot, \cdot \rangle_l) \) is a positive \( E_0^- \)-module). It is said positive if it is both left and right positive.

Observe that if \( E \) is a \( C^* \)-tring, which is positive as an admissible \( * \)-tring, then it is obviously a positive \( C^* \)-tring. Conversely, it is readily checked that any positive \( C^* \)-tring is a positive admissible \( * \)-tring.

**Proposition 3.30.** Let \((A, \alpha) \) be a \( C^* \)-seminormed algebra and \((E, \langle \cdot, \cdot \rangle) \) a right positive \((A, \alpha)\)-module. Let \( \tilde{\alpha} : E \to [0, \infty) \) be given by \( \tilde{\alpha}(x) = \sqrt{\alpha(\langle x, x \rangle)}, \forall x \in E \). Consider \( E \) as a \( * \)-tring with \( \langle x, y, z \rangle := x \langle y, z \rangle, \forall x, y, z \in E \). Then:

1. We have \( \alpha(a) \leq \alpha(b) \) whenever \( a \) and \( b - a \) are positive elements of \( A \).
2. \( \tilde{\alpha}(x)^2 \mu(x) - \langle x, y \rangle^* x, y \rangle \) is positive in \((A, \alpha)\), and \( \alpha(\langle x, y \rangle) \leq \tilde{\alpha}(x)\tilde{\alpha}(y), \forall x, y \in E \) (Cauchy-Schwarz).
3. \( \alpha(x, x) a^* a - a^* x, x \rangle a \geq 0, \forall x \in E, a \in A \).
4. \( \tilde{\alpha}(x a) \leq \tilde{\alpha}(x)\alpha(a), \forall x \in E, a \in A \).
5. \( \tilde{\alpha} \in \mathcal{S}\mathcal{N}(E) \).

**Proof.** Let \( p_a : A \to A/I_a =: A_\alpha \) be the natural map, where \( I_a := \{ \alpha \in A : \alpha(a) = 0 \} \), and let \( \tilde{\alpha} \) be the quotient norm on \( A_\alpha \). Now let \( F := \text{span}\{ x \in E : x \in E, b \in I_a \} \). Then \( E/I_a \subseteq F \), so \( E/F \) is an \( A/I_a \)-module. Moreover, \( (E, F) \subseteq I_a \) and \( (F, E) \subseteq I_a \), so we can consider the map \([,] : E/F \times E/F \to A/I_a \) given by \([q(x), q(y)] = p_\mu([x, y]), \) which satisfies properties 1.-4. of Definition 3.28 above. If \( a \) and \( b - a \) are positive in \( A \), then \( 0 \leq p_\mu(a) \leq p_\mu(b) \) in \( A_\alpha \), and therefore \( \alpha(p_\mu(a)) \leq \alpha(p_\mu(b)) \), that is \( \alpha(a) \leq \alpha(b) \). This proves 1. Now, the first part of the second statement follows from the proof of [6] Proposition 1.1, since \( p_\mu(\tilde{\alpha}(x)^2 \mu(x) - \langle y, x \rangle(x, y)) = \tilde{\alpha}(\langle q(x), q(y) \rangle) \mu(y, q(y)) - \mu(q(y), q(x)) \mu(q(x), q(y)) \) in \( A_\alpha \). The second part of 2. follows from the first one and from 1. To see 3. just observe that by applying \( p_\mu \) to the element \( \alpha(x, x) a^* a - a^* x, x \rangle a \) of \( A \) we get the positive element \( \tilde{\alpha}(x, x) p_\mu(a) - p_\mu(a) a^* x, x \rangle p_\mu(a) \) of \( A_\alpha \). Assertion 4. easily follows from 1. and 3.: by 3. we have \( a^* x, x \rangle a \leq \tilde{\alpha}(x)^2 a^* a, \) then \( \tilde{\alpha}(x a)^2 = \alpha(x a, x a) \) = \( \alpha(a^* x, x)a \), and by 1. this is less or equal to \( \alpha(\tilde{\alpha}(x)^2 a^* a) \) = \( \tilde{\alpha}(x)^2 \alpha(a)^2 \). It is clear that \( \alpha(\lambda x) = |\lambda| \tilde{\alpha}(x), \forall x \in E, \lambda \in \mathbb{C} \), and from the Cauchy-Schwarz inequality just proved it readily follows that \( \tilde{\alpha} \) also
satisfies the triangle inequality, so it is a seminorm on $E$. Now, if $x, y, z \in E$: 
\[ \hat{\alpha}(x, y, z)^2 = \alpha((y, z)^* \langle x, \gamma \rangle \langle y, z \rangle) \] 
Thus, in the case $x = y = z$: 
\[ \hat{\alpha}(x, x, x)^2 = \alpha((x, x)^* \langle x, x \rangle \langle x, x \rangle) = \alpha(x)^3. \]

According to 3, we have $(y, z)^* \langle x, x \rangle \langle y, z \rangle \leq \alpha((x, x)^* \langle y, z \rangle \langle y, z \rangle)$ in $(A, \alpha)$. From this fact, together with 4. and the Cauchy-Schwarz inequality we conclude that 
\[ \hat{\alpha}(x, y, z)^2 \leq \hat{\alpha}(x)^2 \alpha((y, z)^* \langle y, z \rangle) \]
so $\hat{\alpha}$ is a $C^*$-seminorm on $E$. \(\square\)

**Corollary 3.31.** If $E$ is a right positive *-tring, then $SN_{cs}(E_0^r) = SN(E_0^r)$, and $SN(E) \cong SN(E_0^r)$ and $N(E) \cong N(E_0^r)$ as ordered sets.

**Proposition 3.32.** Let $E$ be an admissible *-tring and $\gamma \in SN(E)$. If $E$ is a right positive $(E_0^r, \gamma^*)$-module, then $E$ is also a left positive $(E_0^r, \gamma^*)$-module. Therefore $E$ is right positive if and only if is left positive.

**Proof.** Let $E_\gamma$ be the Hausdorff completion of $(E, \gamma)$. Since $E_\gamma$ is a right Hilbert module over $E_0^r$, it turns out that $E_\gamma$ is a positive $C^*$-tring, and therefore a left Hilbert module over $E_0^r$, so $E$ is a left positive $(E_0^r, \gamma^*)$-module. \(\square\)

**Proposition 3.33.** Let $B$ be an admissible Banach *-algebra and suppose $E$ is a right closed ideal of $B$ such that span${x^* y : x, y \in E}$ is dense in $B$. Let $A$ be the closure in $B$ of span${x y^* : x, y \in E}$. If $x x^*$ is positive in $A$, $\forall x \in E$, then the restriction map $\varphi : SN(B) \to SN(A)$, $\beta \mapsto \beta|_A$, is a lattice isomorphism such that $\varphi(N(B)) = N(A)$, and for each $\beta \in SN(B)$ the Hausdorff completion $B_\beta$ of $B$ is Morita-Rieffel equivalent to the Hausdorff completion $A_{\varphi(\beta)}$ of $A$. In particular, the corresponding enveloping $C^*$-algebras $C^*(B)$ and $C^*(A)$ of $B$ and $A$ are Morita-Rieffel equivalent $C^*$-algebras.

**Proof.** Let $\langle , \rangle_B : E \times E \to B$ and $\langle , \rangle_A : E \times E \to A$ be such that $\langle x, y \rangle_B = x^* y$ and $\langle x, y \rangle_A = x y^*$. Then $E$ is both a positive $B$-module and a positive $A$-module. Since $B$ is admissible, so are $E$ and $A$. Besides $E$ is a faithful $B$-module, for if $x b = 0 \forall x \in E$, then $\sum x^* y b = 0 \forall x, y \in E$, so $b^* b = 0$, and this implies $b = 0$ because $B$ is admissible. Similarly, $E$ is a faithful $A$-module. It follows by 2.12 that we can identify $E_0^r$ with span${x^* y : x, y \in E}$ and $E_0^l$ with span${x y^* : x, y \in E}$. Now the proof ends with an invocation to Corollary 3.25. \(\square\)

### 4. $C^*$-ternary rings

As previously mentioned, Zettl found a unique decomposition $E = E_+ \oplus E_-$ of any $C^*$-tring $E$, $E_+$ being isomorphic to a TRO and $E_-$ being isomorphic to an anti-TRO (see the discussion preceding Corollary 3.20). Of course, because of the uniqueness of the fundamental decomposition, there is a left version of the situation above: $E_+ := \{ x \in E : \langle x, x \rangle \in E_+ \}$, $E_- := \{ x \in E : \langle x, x \rangle \in -E_+ \}$, $\langle x, E_- \rangle = 0$, $E_1^l = E_1^l \oplus E_1^l$, and $(E_+, -, \langle \cdot , \cdot \rangle)$ and $(E_-, -, \langle \cdot , \cdot \rangle)$ are full left Hilbert $E_0^l$ and $E_0^l$ modules respectively. This way, $E$ is an $(E_1^l - E_1^l)$ Banach bimodule that satisfies 
\[ \langle x, y \rangle z = \mu(x, y, z) = \langle x, y \rangle z, \forall x, y, z \in E. \]

If $E$ is a $C^*$-tring, we define $E_p := E_+ \oplus E_{op}$. Then $E_p$ is a positive $C^*$-tring, and $E_1^r = E^r$, $E_1^l = E^l$. Therefore $E_p$ is a $(E_1^l - E_1^r)$-imprimitivity bimodule, so
in particular $E^3$ and $E^r$ are Morita-Rieffel equivalent. Note also that if $\phi : E \to F$ is a homomorphism of $C^*$-trings, then $\phi(E_+) \subseteq F_+$ and $\phi(E_-) \subseteq F_-$, because $(\phi(x), \phi(y)) = \phi((x, y))$. Therefore $\phi : E_p \to F_p$ is also a homomorphism of $C^*$-trings. Thus $E \to E_p$ is a functor.

Let $E^*$ be the reverse $*$-tring of $E$. It is clear that a norm on $E$ is a $C^*$-norm if and only if it is a $C^*$-norm on $E^*$. Moreover, $E$ is a (positive) $C^*$-tring if and only if so is $E^*$, and $E^1 = (E^*)^r$, $E^r = (E^*)^l$. Note that $E$ and $E^*$ are essentially the same object as $C^*$-trings. Thus the properties of $E^r$ and $E^l$ deduced from properties of $E$ will be the same.

**Definition 4.1.** By a left (right) ideal of the $C^*$-ternary ring $E$, we mean a closed subspace $F$ of $E$ such that $(E, E, F) \subseteq F$ (respectively: $(F, E, E) \subseteq F$). An ideal of $E$ is both a left and a right ideal of $E$. We denote by $L(E)$, $R(E)$, and $I(E)$ the families of left, right, and twosided ideals of $E$.

Our definition of ideal, for a closed subspace $F$ of $E$, is equivalent to the definition which just requires the condition $(E, E, F) \subseteq F$ to be satisfied. Note that $E_+$ and $E_-$ are ideals in every $C^*$-tring $E$. Moreover, since $E_+$ and $E_-$ are orthogonal, it easily follows that a closed subspace $F$ of $E$ is an ideal of $E$ if and only if it is an ideal in $E_p$. Thus the ideal structures of $E$ and of $E_p$ are the same.

If $A$ is a $C^*$-algebra, we will denote by $I(A)$ and $H(A)$ respectively the families of (closed) twosided ideals and hereditary $C^*$-subalgebras of $A$.

As in the algebraic case, if $E$ is a $C^*$-tring and $F$ is a sub-$C^*$-tring of $E$, then the subalgebra $\text{span}(F, F)_r$ of $E^r$ may be taken to represent the $C^*$-algebra $F^r$. With this choice of $F^r$ we have the following result:

**Proposition 4.2.** The map $L(E) \to H(E^r)$ given by $F \mapsto F^r$ is a bijection, with inverse given by $A \mapsto E A$. When restricted to $I(E)$, the map $F \mapsto F^r$ is a bijection onto $I(E^r)$. Moreover, all of these maps are lattice isomorphisms.

**Proof.** We prove that the map $L(E) \to H(E^r)$ is a bijection. Recalling that we may replace $E$ by $E_p$ (which can be seen as a full right Hilbert $E^r$-module), the rest of the proof follows from [3, 3.22]. If $A$ is a $C^*$-subalgebra of $E^r$: $(E, E, E) = E(E, E)A = (E, E, E)A \subseteq EA$, so $EA$ is a left ideal in $E$. Conversely, if $F$ is a left ideal in $E$:

\[
(F, F)(E, E)(F, F) = (E(F, F), (E, F, F)) \subseteq (F, F).
\]

Thus, taking the closed linear spans in both sides of the above inclusion we have: $F^r E^r F^r = F^r$, which shows that $F^r$ is hereditary. To see that the correspondences are mutually inverses, note that if $F$ is a $C^*$-tring, then $F = F F^r$. On the other hand, if $A$ is a hereditary $C^*$-subalgebra of $E^r$, then $EA = \text{span}(EA, EA)_r = \text{span}A(E, E)_r A = A E^r A = A$. □

**Corollary 4.3.** Let $\pi : E \to F$ be a homomorphism of $*$-trings, where $E$ and $F$ are $C^*$-trings. Then $(\ker \pi)^r = \ker(\pi^r)$.

**Proof.** It is clear that $\ker \pi \supseteq E \ker \pi^r$, so $(\ker \pi)^r \supseteq \ker \pi^r$. On the other hand $(\ker \pi)^r = \text{span}\{\langle x, y \rangle_r : x, y \in \ker \pi\} \subseteq \ker \pi^r$. □

**Remark 4.4.** By Proposition 2.17 if $\pi : E \to F$ is a surjective homomorphism between $C^*$-trings, then $\pi_0 : E_0 \to F_0$ is also surjective, so also is $\pi^r : E^r \to F^r$ for the image of $\pi^r$ is closed. However the converse is false: consider the Hilbert space inclusion $\mathbb{C} \hookrightarrow \mathbb{C}^2$; then $\iota$ is not onto, although $\iota^r$ is the identity on $\mathbb{C}$. 

For a proof of the next result the reader is referred to [3, 3.25].

**Proposition 4.5.** Let $F$ be an ideal of a $C^*$-tring $E$, and consider the quotient $E/F$ with its natural structure of $*$-tring. Then $E/F$ is a $C^*$-tring with the quotient norm, and $(E/F)^r = E^r/F^r$.

**Corollary 4.6.** Let $E$ and $G$ be $C^*$-trings, and $\pi : E \to G$ a homomorphism of $*$-trings. Consider $F = \ker(\pi)$, and let $p : E^r \to E^r/F^r$ be the quotient map. Then there exists a unique homomorphism of $C^*$-algebras $\overline{\pi} : E^r/F^r \to G^r$ such that $\overline{\pi}p = \pi^r$. The homomorphism $\overline{\pi}$ is injective. In particular, if $\pi : E \to E/F$ is the quotient map, where $F$ is an ideal of $E$, then $\overline{\pi} : E^r/F^r \to (E/F)^r$ is a natural isomorphism.

**Proof.** Proposition 3.11 provides a unique homomorphism of $C^*$-algebras $\pi^r : E^r \to G^r$ such that $\langle \pi(x), \pi(y) \rangle = \pi^r((x, y))$, $\forall x, y \in E$. The existence and uniqueness of $\overline{\pi}$, as well as its injetivity, follow now from the quotient universal property, together with the fact that $\ker(\pi^r) = F^r$ by Corollary 1.3. Finally, if $F$ is an ideal of $E$, by Proposition 4.5 we have that $E/F$ is a $C^*$-tring, and the projection $\pi : E \to E/F$ is a homomorphism of $*$-trings.

**Corollary 4.7.** The functor $E \mapsto E^r$, $\pi \mapsto \pi^r$, from the category of $C^*$-trings into the category of $C^*$-algebras, is exact. More precisely: if

\[
\begin{array}{cccc}
0 & \longrightarrow & F_1 & \overset{\phi}{\longrightarrow} & F_2 & \overset{\psi}{\longrightarrow} & F_3 & \longrightarrow & 0
\end{array}
\]

is an exact sequence of $C^*$-trings, then the sequence:

\[
\begin{array}{cccc}
0 & \longrightarrow & F_1^r & \overset{\phi^r}{\longrightarrow} & F_2^r & \overset{\psi^r}{\longrightarrow} & F_3^r & \longrightarrow & 0
\end{array}
\]

also is exact.

**Corollary 4.8.** If $\pi : E \to F$ is a homomorphism of $C^*$-trings, then $\pi(E)$ is closed in $F$. The ideals of a $C^*$-tring $E$ are exactly the kernels of the homomorphisms defined on $E$.

5. Applications

5.1. $C^*$-algebras associated with Fell bundles. The proof of Theorem 1.1 of [2] relies on the existence of a certain inner product (see Corollary 5.3 below), although no proof is included there of the fact that such inner product is indeed positive. In the following lines we provide such a proof, and we refine the above mentioned result.

Recall that a right ideal $\mathcal{E} = (E_t)_{t \in G}$ of a Fell bundle $\mathcal{B} = (B_t)_{t \in G}$ is a sub-Banach bundle of $\mathcal{B}$ such that $\mathcal{E}\mathcal{B} \subseteq \mathcal{E}$.

Given a right Hilbert $B$-module $X$, let denote by $D_X$ the cone of finite sums $\sum_i (x_i, x_i) \subseteq B^+$. It is clear that if $\{X_\lambda\}_{\lambda \in \Lambda}$ is a family of right Hilbert $B$-modules and $X := \oplus X_\lambda$ (direct sum of Hilbert modules), then $\sum_\lambda D_{X_\lambda} \subseteq D_X$ -with equality if $\Lambda$ is finite- and $\sum_\lambda D_{X_\lambda}$ is dense in $D_X$.

Similarly, for the right ideal $\mathcal{E}$ of the Fell bundle $\mathcal{B}$, we define $D_\mathcal{E} := \{\sum_{i=1}^n c_i^* c_i : n \in \mathbb{N}, c_i \in \mathcal{E}, \forall i\} \subseteq B_\mathcal{E}^+$. Then we have:
Lemma 5.1. Let $\mathcal{E} = (E_t)_{t \in G}$ be a right ideal of the Fell bundle $\mathcal{B} = (B_t)_{t \in G}$. Then $\text{span}(\mathcal{E}^* \mathcal{E} \cap B_c)$ is dense in $B_c$ if and only if the cone $D_\mathcal{E}$ satisfies the following property:

$$\forall b \in B_c, \; \epsilon > 0, \; \text{there exists } d \in D_\mathcal{E} \text{ such that } \|d\| \leq 1 \text{ and } \|b - bd\| < \epsilon. \quad (5.1)$$

Proof. Suppose that $b \in B_c$ is such that for any $\epsilon > 0$ there exists $d \in D_\mathcal{E}$ such that $\|b - bd\| < \epsilon$. Since $D_\mathcal{E} \subseteq \text{span}(\mathcal{E}^* \mathcal{E} \cap B_c)$ and the latter is an ideal in $B_c$, we conclude that $b \in \overline{\text{span}(\mathcal{E}^* \mathcal{E} \cap B_c)}$. Then $\text{span}(\mathcal{E}^* \mathcal{E} \cap B_c)$ is dense in $B_c$ whenever $D_\mathcal{E}$ satisfies (5.1). Note now that $D_\mathcal{E} = \sum_{t \in G} D_{E_t}$, which is dense in $D_\mathcal{E}$, where $E_\cdot := \oplus_{t \in G} E_t$. Thus $D_\mathcal{E}$ satisfies (5.1) if and only if that property holds for $D_{E_t}$. Assume that $\text{span}(\mathcal{E}^* \mathcal{E} \cap B_c)$ is dense in $B_c$. Then $E$ is a full Hilbert module over $B_c$, and therefore it satisfies (5.1) by [6, (ii) of Lemma 7.2]. □

Lemma 5.2. Let $\mathcal{B} = (B_t)_{t \in G}$ be a Fell bundle over the locally compact group $G$, $A = (A_t)$ a sub-Fell bundle of $\mathcal{B}$, and $\mathcal{E} = (E_t)$ a right ideal of $\mathcal{B}$ such that $A \subseteq \mathcal{E}$, $\mathcal{E}^* \subseteq A$ and $\text{span}(\mathcal{E}^* \mathcal{E} \cap B_c)$ is dense in $B_c$. If $\xi \in L^1(\mathcal{E})$, then $\xi \ast \xi^*$ can be arbitrarily approximated in $L^1(A)$ by a finite sum $\sum_{j=1}^m \eta_j \ast \eta_j^*$, where $\eta_j \in L^1(A)$, $\forall j = 1, \ldots, m$.

Proof. We will suppose that $\xi \in C_c(\mathcal{E})$, which is clearly enough. Since $C_0(\mathcal{E})$ is a nondegenerate right Banach $B_c$-module, given a positive integer $n$ there exists $b_n \in B_c$ such that $\|\xi - \xi b_n\| < 1/n$ and $0 \leq b_n \leq 1$. Then we can find $c_n \in D_\mathcal{E}$ such that $\|b_n^{1/2} - b_n^{1/2} c_n\| < 1/n$. Set $d_n := b_n^{1/2} c_n b_n^{1/2}$ and note that $d_n \in D_\mathcal{E}$ because $\mathcal{E}$ is a right ideal. The continuity of the operations imply $\|b_n - d_n\| \to 0$ and $\|\xi - \xi d_n\| \to 0$. Thus $\|\xi \ast \xi^* - \xi d_n \ast \xi^*\| \to 0$.

Now for every $n$ there exist $u_1, \ldots, u_m \in \mathcal{E}$ such that $d_n = \sum_{j=1}^m u_j \ast u_j$. Thus $\xi d_n \ast \xi^* = \sum_{j=1}^m (\xi u_j \ast u_j) \ast \xi^* = \sum_{j=1}^m (\xi u_j^*) \ast (\xi u_j^*)^*$ and, as $\mathcal{E}$ is a right ideal, $\xi u_j^* \in C_c(A)$. This completes the proof. □

Corollary 5.3. Under the assumptions of Lemma 5.2, let $\|\cdot\|_{L^1(A)} : L^1(A) \to [0, \infty)$ be the maximal $C^*$-norm of $L^1(A)$. Then $L^1(\mathcal{E}) \times L^1(\mathcal{E}) \to L^1(A)$ given by $(\xi, \eta) \mapsto \xi \ast \eta^*$ is an inner product.

Corollary 5.4. Under the assumptions of Lemma 5.2, the map $\varphi : SN(L^1(B)) \to SN(L^1(B))$ given by $\beta \mapsto \beta|_{L^1(A)}$ is an isomorphism of partially ordered sets that sends the maximal and reduced norms on $L^1(B)$ to the maximal and reduced norms on $L^1(A)$ respectively, and such that $\varphi(N(L^1(B))) = N(L^1(A))$. Moreover, the Hausdorff completions of $L^1(B)$ and $L^1(A)$ with respect to $\beta$ and $\varphi(\beta)$ respectively are Morita-Rieffel equivalent.

Proof. We only have to prove the correspondence between the reduced $C^*$-norms, but this is the content of [2]. □

5.2. Tensor products of $C^*$-trings. In the present section we apply the previous results to the study of tensor products of $C^*$-trings. Maximal and minimal tensor product for TROs were constructed in [5] using linking algebras, but we define tensor products of $C^*$-trings $E$ and $F$ using the tensor products of $E^r$ and $F^r$. The main result is Theorem 5.12.

From now on the algebraic tensor product of the $C$-vector spaces $E_1, \ldots, E_n$ will be denoted by $E_1 \bigotimes \ldots \bigotimes E_n$, or just by $\bigotimes_{j=1}^n E_j$. Let $E_{ij}$, $F_{ij}$ be complex vector spaces, $\forall i = 1, \ldots, m$, $j = 1, \ldots, n$, and suppose that $a_{ij} : \prod_{j=1}^n E_{ij} \to $
$F_i$ is a $n$-linear map, for each $i = 1, \ldots, m$. Then it is clear that there exists a unique $n$-linear map $\alpha := a_1 \otimes \cdots \otimes a_m : \prod_{i=1}^n \mathcal{O}_{m_i} \to \mathcal{O}_n$ such that $\alpha(e_1^{(i)}, \ldots, e_n^{(i)}) = \alpha(\gamma(e_1, \ldots, e_n))$. Using this fact we have the following result, whose straightforward proof is left to the reader.

**Proposition 5.8.** If $(E, \mu)$ is a *-ring, then $(E \otimes F, \mu \otimes \nu)$ is also a *-ring. Furthermore, if $(E, A, \langle \cdot, \cdot \rangle_A)$ and $(F, B, \langle \cdot, \cdot \rangle_B)$ are full basic triples associated to $(E, \mu)$ and $(F, \nu)$, respectively, then $(E \otimes F, A \otimes B, \langle \cdot, \cdot \rangle_A \otimes \langle \cdot, \cdot \rangle_B)$ is a full basic triple associated to $(E \otimes F, \mu \otimes \nu)$.

**Definition 6.6.** A $C^*$-tensor product of two *-rings $(E, \mu, \| \|)$ and $(F, \nu, \| \|)$ is a completion of the corresponding algebraic tensor product $(E \otimes F, \mu \otimes \nu)$ with respect to a $C^*$-norm. If $\gamma$ is such a $C^*$-norm, we denote by $E \bigotimes_{\gamma} F$ the corresponding $C^*$-tensor product.

**Definition 6.7.** We say that a $C^*$-ring $E$ is nuclear if for every $C^*$-ring $F$ there exists just one $C^*$-tensor product $E \bigotimes F$.

We will see next that $SN(E \otimes F) = SN(E_p \otimes F)$, which implies, in particular, that a $C^*$-ring $E$ is nuclear if and only if $E_p$ is nuclear.

**Proposition 5.8.** Let $E$ be a *-ring, and $F_1, F_2$ ideals of $E$ such that $E = F_1 \oplus F_2$. If $\gamma \in SN(E)$, and $x = y + z$, with $y \in F_1$ and $z \in F_2$, then $\gamma(x) = \max\{\gamma(y), \gamma(z)\}$.

**Proof.** Since $\gamma(x) = \sup\{\gamma((x, u, v)) : u \in E, \gamma(u) \leq 1\}$, it follows that $\gamma(x) \geq \gamma(z)$, so $\gamma(x) \geq \max\{\gamma(y), \gamma(z)\}$. To prove the converse inequality, let us first introduce the following notation. For $u \in E$ let $u_0 := z$, $u_n := (u_{n-1}, u_{n-1}, u_{n-1})$ if $n \geq 1$. Then we have that $\gamma(u_n) = \gamma(u_{n-1})^3$, $\forall n \geq 1$, so $\gamma(u_n) = \gamma(u)^{3^n}$, $\forall n \geq 0$. Since $(E, F_1, F_2) = 0$, it follows that $x_n = y_n + z_n$. Thus: $\gamma(x) = \gamma(x_n)^{1/3^n} = \gamma(y_n + z_n)^{1/3^n} \leq (\gamma(y_n) + \gamma(z_n))^{1/3^n} = (\gamma(y)^{3^n} + \gamma(z)^{3^n})^{1/3^n} \rightarrow \max\{\gamma(y), \gamma(z)\}$, whence $\gamma(x) \leq \max\{\gamma(y), \gamma(z)\}$. \hfill $\square$

**Corollary 5.9.** Let $E$ and $F$ be $C^*$-rings. Then $SN(E \otimes F) = SN(E_p \otimes F)$ and $N(E \otimes F) = N(E_p \otimes F)$. Consequently a $C^*$-ring $E$ is nuclear if and only if $E_p$ is nuclear.

Our aim is to prove that there is an isomorphism between $N(E \otimes F)$ and $N(E^* \otimes F^*)$. The key step is to show that each $C^*$-norm on $E \bigotimes F$ has unique extension to a $C^*$-norm on $E^* \bigotimes F^*$.

**Lemma 5.10.** Let $I$ and $J$ be *-ideals (not necessarily closed) of the $C^*$-algebras $A$ and $B$, respectively. Then the map $\Theta: N(A \otimes B) \to N(I \otimes J)$, $\gamma \mapsto \gamma|_{I \otimes J}$, is an order preserving surjection. If, in addition, $I$ and $J$ are dense in $A$ and $B$, respectively, then $\Theta$ is a bijection.

**Proof.** Clearly $\Theta$ is order preserving. Fix $\delta \in N(I \otimes J)$. Given $a \in A$ and $z = \sum_{j=1}^n x_i \otimes y_j \in I \otimes J$, define $w := \sum_{j=1}^n (|a|^2 - a^*a)^{1/2} x_i \otimes y_j \in A \otimes B$. In case $A$ is unital it is clear that $w \in I \otimes J$. If $A$ is not unital, $I$ is an ideal of the unitization of $A$, so $w \in I \otimes J$ in any case. Then

$$\|a\|^2 z = \left(\sum_{j=1}^n ax_j \otimes y_j\right)^* \left(\sum_{j=1}^n ax_j \otimes y_j\right) = w^* w \in (I \otimes \delta J)^+$$

and $\delta(\sum_{j=1}^n ax_j \otimes y_j) \leq \|a\| \delta(\sum_{j=1}^n x_i \otimes y_j)$. Similarly, if $b \in B$, we also have $\delta(\sum_{j=1}^n x_i \otimes by_j) \leq \|b\| \delta(\sum_{j=1}^n x_i \otimes y_j)$. Thus $\delta((a \otimes b)z) \leq \|a\| \|b\| \delta(z)$, $\forall a \in A$,
Proposition 5.11. Let $E$ and $F$ be positive $C^*$-trings and consider the admissible full basic triples $(E, E_0^f, \langle \cdot, \cdot \rangle^E)$ and $(F, F_0^f, \langle \cdot, \cdot \rangle^F)$ given by Theorem 2.10. Then the full basic triple $(E \otimes F, E_0^f \otimes F_0^f, \langle \cdot, \cdot \rangle^E \otimes \langle \cdot, \cdot \rangle^F)$ is admissible. Furthermore, $E \otimes F$ is positive and

$$SN_\odot(E_0^f \otimes F_0^f) = SN(E_0^f \otimes F_0^f)$$

Proof. To simplify our notation we denote $[\cdot, \cdot]$ the map $\langle \cdot, \cdot \rangle^E \otimes \langle \cdot, \cdot \rangle^F$. Note $E_0^f \otimes F_0^f$-module is admissible because it is a $*$-subalgebra of the $C^*$-closable $*$-algebra $E^r \otimes F^r$. We will show that $E \otimes F$ is a positive $E_0^f \otimes F_0^f$-module. Lemma 5.10 implies there is a maximal $C^*$-norm on $E_0^f \otimes F_0^f$, namely the restriction of the maximal $C^*$-norm of $E^r \otimes F^r$. The comments preceding Lemma 3.27 imply that, to show $E \otimes F$ is positive, it suffices to prove that $[u, u] \geq 0$ in the maximal tensor product $E^r \otimes_{\text{max}} F^r$. Given $u = \sum_{j=1}^{n} x_j \otimes y_j \in E \otimes F$ we have

$$[u, u] = \sum_{j,k=1}^{n} \langle x_j, x_k \rangle^E \odot \langle y_j, y_k \rangle^F.$$  

Then Lemmas 4.2 and 4.3 of [10] give the desired result.

To show $[u, u] = 0$ implies $u = 0$ we use the linking algebras $L(E)$ and $L(F)$ and the linear maps

$$\alpha: E \otimes F \rightarrow L(E) \otimes L(F), \ x \otimes y \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix},$$

$$\beta: L(E) \otimes L(F) \rightarrow E \otimes F, \ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \odot \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \mapsto x_1 \otimes y_1 + x_2 \otimes y_2,$$

$$\gamma: E \otimes F \rightarrow L(E) \otimes L(F), \ a \otimes b \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}.$$  

Then $\alpha(\alpha(u)) = \gamma([u, u]) = 0$, so $\alpha(u) = 0$ and $u = \beta(\alpha(u)) = 0$. 

Theorem 5.12. Let $E$ and $F$ be $C^*$-ternary rings. Then every set among the partially ordered sets $N(E^l \otimes F^l)$, $N(E \otimes F)$ and $N(E^r \otimes F^r)$ is isomorphic to each other. Besides, if $\gamma \in N(E \otimes F)$ and $\gamma^l$ and $\gamma^r$ are the corresponding $C^*$-norms on $N(E^l \otimes F^l)$ and $N(E^r \otimes F^r)$ respectively, then $E \otimes F$ is a Morita-Rieffel equivalence bimodule between $E^l \otimes_{\gamma^l} F^l$ and $E^r \otimes_{\gamma^r} F^r$.

Proof. Proposition 5.11 together with Corollary 3.31 imply $N(E \otimes F)$ is isomorphic (as a partially ordered set) to $N(E \otimes F)_0$. By 5.5 the posets $N(E \otimes F)^r_0$ and $N(E_0^f \otimes F_0^f)$ are isomorphic, and the latter is isomorphic to $N(E^r \otimes F^r)$ by Lemma 5.10. Thus $N(E \otimes F) \cong N(E^r \otimes F^r)$. Similarly we have $N(E \otimes F) \cong N(E^l \otimes F^l)$. 

Corollary 5.13. Let $E$ and $F$ be $C^*$-trings. Then there exist a maximum $C^*$-norm $\| \cdot \|_{\text{max}}$ on $E \otimes F$, and a minimum $C^*$-norm $\| \cdot \|_{\text{min}}$ on $E \otimes F$, and

$$(E \otimes F)^l = E^l \otimes_{\text{max}} F^l, \quad (E \otimes F)^r = E^r \otimes_{\text{max}} F^r,$$

$$(E \otimes F)^l = E^l \otimes_{\text{min}} F^l, \quad (E \otimes F)^r = E^r \otimes_{\text{min}} F^r.$$
Corollary 5.14 (cf. [5] Theorem 6.5). The following assertions are equivalent for a $C^*$-tring $E$:

1. $E$ is a nuclear $C^*$-tring (5.1).
2. $E'$ is a nuclear $C^*$-algebra.
3. $E^r$ is a nuclear $C^*$-algebra.

The equivalence between 2. and 3. in 5.14 is exactly the following well-known result ([3], [11]): if $A$ and $B$ are two Morita-Rieffel equivalent $C^*$-algebras then $A$ is nuclear if and only if so is $B$.

5.3. Exact $C^*$-trings. To end the section we introduce the notion of exact $C^*$-tring, extending the notion of exact TRO of [5], and we prove a result similar to Corollary 5.14. The reader is referred to [9] for the theory of exact $C^*$-algebras.

Suppose that $0 \rightarrow F_1 \xrightarrow{\phi} F_2 \xrightarrow{\psi} F_3 \rightarrow 0$ is an exact sequence of $C^*$-trings, that is, $\phi$ and $\psi$ are homomorphisms of $C^*$-trings, $\phi$ is injective, $\psi$ is surjective, and $\ker \psi = \phi(F_1)$. Let $E$ be a $C^*$-tring. Then the sequence

$$0 \rightarrow E \odot F_1 \xrightarrow{id \otimes \phi} E \odot F_2 \xrightarrow{id \otimes \psi} E \odot F_3 \rightarrow 0$$

also is exact. We have an inclusion

$$(E \otimes F_2)/(E \otimes F_1) \hookrightarrow (E \otimes_{\text{min}} F_2)/(E \otimes_{\text{min}} F_1)$$

and the latter quotient is a $C^*$-tring. Then there exists a $C^*$-norm $\gamma$ on $E \odot F_3$ such that

$$0 \rightarrow E \otimes_{\text{min}} F_1 \xrightarrow{id \otimes \phi} E \otimes_{\text{min}} F_2 \xrightarrow{id \otimes \psi} E \otimes_{\gamma} F_3 \rightarrow 0$$

is exact. Since $\gamma$ is greater or equal to the minimum norm, the identity map on $E \otimes F_3$ extends to a surjective homomorphism $E \otimes_{\gamma} F_3 \rightarrow E \otimes_{\text{min}} F_3$.

Definition 5.15. We say that a $C^*$-tring $E$ is exact if for each exact sequence

$$0 \rightarrow F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow 0$$

of $C^*$-trings we have that

$$0 \rightarrow E \otimes_{\text{min}} F_1 \rightarrow E \otimes_{\text{min}} F_2 \rightarrow E \otimes_{\text{min}} F_3 \rightarrow 0$$

also is exact.

Proposition 5.16. Let $E$ and $F$ be $C^*$-trings, and suppose that $G$ is an ideal of $F$ (Definition 4.1). Then

$$0 \rightarrow E \otimes_{\text{min}} G \rightarrow E \otimes_{\text{min}} F \rightarrow E \otimes_{\text{min}} (F/G) \rightarrow 0$$

is exact if and only if the following sequence is exact:

$$0 \rightarrow E^r \otimes_{\text{min}} G^r \rightarrow E^r \otimes_{\text{min}} F^r \rightarrow E^r \otimes_{\text{min}} (F^r/G^r) \rightarrow 0$$

Proof. Suppose first that the sequence below is exact:

$$0 \rightarrow E \otimes_{\text{min}} G \rightarrow E \otimes_{\text{min}} F \rightarrow E \otimes_{\text{min}} (F/G) \rightarrow 0$$
By Corollaries 5.13 and 4.7 we have the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & (E \otimes_{\min} G)^r & \rightarrow & (E \otimes_{\min} F)^r & \rightarrow & (E \otimes_{\min} (F/G))^r & \rightarrow & 0 \\
\Rightarrow & & \Rightarrow & & \Rightarrow & & \Rightarrow & & \\
0 & \rightarrow & E^r \otimes_{\min} G^r & \rightarrow & E^r \otimes_{\min} F^r & \rightarrow & E^r \otimes_{\min} (F/G)^r & \rightarrow & 0 \\
\Rightarrow & & \Rightarrow & & \Rightarrow & & \Rightarrow & & \\
0 & \rightarrow & E^r \otimes_{\min} G^r & \rightarrow & E^r \otimes_{\min} F^r & \rightarrow & E^r \otimes_{\min} F^r/G^r & \rightarrow & 0 \\
\end{array}
\]

Since the upper two rows are exact, the third one also is exact.

To prove the converse, note first that

\[
\begin{array}{ccccccccc}
0 & \rightarrow & E \otimes_{\min} G & \rightarrow & E \otimes_{\min} F & \rightarrow & (E \otimes_{\min} F)/(E \otimes_{\min} G) & \rightarrow & 0 \\
\end{array}
\]

is exact, and \((E \otimes_{\min} F)/(E \otimes_{\min} G)\) is a \(C^*\)-completion of the ternary ring \(E \otimes (F/G)\). Denoting the corresponding \(C^*\)-norm by \(\gamma\), we have a surjective homomorphism \(\phi: E \otimes \gamma (F/G) \rightarrow E \otimes_{\min} (F/G)\) which extends the identity on \(E \otimes (F/G)\). Now, applying the exact functor \(E \mapsto E^r\) we obtain the commutative diagram with exact rows that follows:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & E^r \otimes_{\min} G^r & \rightarrow & E^r \otimes_{\min} F^r & \rightarrow & E^r \otimes_{\gamma} F^r/G^r & \rightarrow & 0 \\
\Rightarrow & & \Rightarrow & & \Rightarrow & & \Rightarrow & & \\
0 & \rightarrow & E^r \otimes_{\min} G^r & \rightarrow & E^r \otimes_{\min} F^r & \rightarrow & E^r \otimes_{\min} F^r/G^r & \rightarrow & 0 \\
\end{array}
\]

It follows that the homomorphism \(\phi^r\) is an isomorphism. \(\square\)

**Corollary 5.17** (cf. [5, Theorem 6.1]). A \(C^*\)-tring \(E\) is exact (5.15) if and only if \(E^r\) is an exact \(C^*\)-algebra.

**Proof.** Immediate from Proposition [5.10] \(\square\)

As previously for nuclear \(C^*\)-algebras, we easily obtain from 5.17 the following known result([7]): if \(A\) and \(B\) are Morita-Rieffel equivalent \(C^*\)-algebras, then \(A\) is exact if and only if \(B\) is exact.

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