On a Generalisation of the Marčenko-Pastur Problem

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September 2020

Abstract

We study the spectrum of generalized Wishart matrices, defined as $F = \frac{(XY^\top + YX^\top)}{2}$, where $X$ and $Y$ are $N \times T$ matrices with zero mean, unit variance IID entries and such that $\mathbb{E}[x_i y_j] = c \delta_{i,j}$. The limit $c = 1$ corresponds to the Marčenko-Pastur problem. For a general $c$, we show that the Stieltjes transform of $F$ is the solution of a cubic equation. In the limit $c = 0$, $T \gg N$ the density of eigenvalues converges to the Wigner semi-circle.

The celebrated Marčenko-Pastur problem concerns the eigenvalue spectrum of random covariance matrices in the large dimension limit. More precisely, consider an $N$-dimensional time series, $x^t_i$, where $i = 1, \cdots, N$ and $t = 1, \cdots, T$. Suppose that the $x$’s are IID random variables, of zero mean and unit variance. Then the empirical (or sample) covariance matrix is defined as $E$.

$$E := \frac{1}{T} \sum_{t=1}^T x'_t x'_t = \frac{1}{T} XX^\top,$$

where $X$ is the $N \times T$ matrix defined by $X_i = x'_i$. $E$ is called a (white) Wishart matrix.

Since our assumption is that the $x$’s are IID, the “true” covariance matrix is simply the identity matrix, which is the result obtained for $E$ in the limit $T \to \infty$, for a fixed value of $N$. But there is another limit, which is very relevant in many applications, where $T$ and $N$ are both large; more precisely, where $N, T \to \infty$ with a fixed ratio $q = N/T$.

What is the spectrum of $E$ in that regime? The answer was provided by Marčenko and Pastur in 1967 [1], and is a classic result in Random Matrix Theory. When $q < 1$, the result for the density of eigenvalues $\rho(\lambda)$ is:

$$\rho(\lambda) = \frac{\sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}}{2\pi q \lambda}, \quad \lambda_- \leq \lambda \leq \lambda_+,$$

with $\lambda_\pm = (1 \pm \sqrt{q})^2$. As expected $\lambda_+ = \lambda_- = 1$ when $q \to 0$, i.e. when $T \gg N$, $E$ becomes the identity matrix.

The problem we want to consider in this note is defined by the following symmetric cross-correlation matrix:

$$F := \frac{1}{2T} (XY^\top + YX^\top)$$

where $X$ and $Y$ are two $N \times T$ rectangular matrices of unit variance IID random variables, such that $\mathbb{E}[x'_i y'_j] = c \delta_{i,j}$, where $c \in [-1, 1]$ is the correlation coefficient between $x$’s and $y$’s. Clearly, when $c = 1$,

\footnote{The singular value spectrum of the un-symmetrized matrix $XY^\top/T$ was considered for $c = 0$ in [2, 3].}
\(X = Y\) and one recovers the Marčenko-Pastur problem. The aim of this work is to determine the spectrum of \(F\).

The most efficient way to approach this problem is by using the tools of free random matrix theory (see e.g. [4, 5]). In the large dimension limit, \(F\) can be seen as the free addition of \(T\) two-dimensional projectors:

\[
F = \sum_t \mathbf{P}_t, \quad (\mathbf{P}_t)_{ij} := \frac{1}{2T} \left(x_i^j y_j^i + y_j^i x_i^j\right).
\]  

In the large \(N\) limit, the spectrum of \(\mathbf{P}_t\) becomes independent of \(t\) and is composed of \(N-2\) zero eigenvalues and two non-zero eigenvalues, equal to \(q(c \pm 1)/2\). The corresponding Stieltjes function is thus:

\[
G(z) = \frac{N-2}{N} \frac{1}{z} + \frac{1}{N} \left(\frac{1}{z - q(c + 1)/2} + \frac{1}{z - q(c - 1)/2}\right),
\]  

from which we deduce the functional inverse \(z(u)\) defined as \(z(G(z)) = z\). To leading order in \(1/N\) one finds:

\[
z(u) = \frac{1}{u} + \frac{4c+2q^2 u(1-c^2)}{N (2-cqu)^2 - q^2 u^2} + O(N^{-2}).
\]  

Free matrix addition means that the \(R\)-transform, defined as \(R(u) = z(u) - u^{-1}\) is additive. Hence the \(R\)-transform of \(F\) is given by

\[
R_F(u) = \frac{4c + 2qu(1-c^2)}{(2-cqu)^2 - q^2 u^2}, \quad N \to \infty.
\]  

From \(R_F(u)\) one backtracks to get \(z_F(u)\) and finally \(G_F(z)\), the Stieltjes transform of \(F\) that contains all the information about the density of eigenvalues of \(F\). Finally, \(G_F(z)\) is given by the appropriate solution of the following cubic equation:

\[
q^2(1-c^2)^2 z G_F^3 + \left[ \frac{q(2-q)(1-c^2)}{4} + cz^2 \right] G_F^2 (z + c(q - 1)) G_F + 1 = 0.
\]  

This is the main result of the present paper. One can check that when \(c = 1\) one recovers the Stieltjes transform of a Wishart matrix, solution of the following quadratic equation:

\[
qz G_W^2 - (z + q - 1) G_W + 1 = 0,
\]  

Figure 1: Analytical densities of eigenvalues, \(\rho(\lambda)\), for \(c = 0\), different \(q\)'s (left) and \(q = 0.5\), different \(c\)'s (right). The density for \(c = 0, q = 1\) (green curve left) is known as the “Tetilla law” as its shape is similar to that of a Galician cheese.
from which the Marčenko-Pastur result immediately follows. In Fig. 1 we show the spectrum of $F$ for some representative values of $c$ and $q$, obtained as usual from the imaginary value of $G_F$ when $\text{Im}(z) \to 0$. Note that when $c \neq 1$, a fraction of the eigenvalues are negative. In fact, the mean of the density of eigenvalues is equal to $c$. The variance can be read-off the R-transform using $\sigma^2 = R'(0) = q(1 + c^2)/2$.

The case $c = 0$ is particularly interesting. In this case the spectrum is an even function of $\lambda$, see Fig. 1. After a little work one can establish the following results:

1. The edges of the spectrum, $\lambda_{\pm}$, are given by:

$$\lambda_{\pm} = \frac{1}{2\sqrt{2}} \sqrt{2q^2 + 10q - 1 \pm \sqrt{64q^3 + 48q^2 + 12q + 1}},$$

where $\pm \lambda_+$ are the outer edges and $\pm \lambda_-$ are the inner edges (which only exist for $q > 2$). They are plotted in Fig. 3.

2. In the limit $q \to 0$, the density becomes a Wigner semi-circle of radius $\sqrt{2q}$.

$$\rho_{q\to0,c=0}(\lambda) = \frac{\sqrt{2q - \lambda^2}}{\pi q}.$$
When $q = 0$, one finds that all eigenvalues are zero, as expected since in this case $F \equiv 0.$

3. The case $q = 1, c = 0$ was studied before in the context of the addition of two non-symmetrized Wigner matrices, see [6, 7]. The corresponding distribution of eigenvalues was called the “tetilla” law [7].

The explicit form for the density when $|\lambda| \leq \lambda_+ = 2^{-3/2} \sqrt{11 + 5\sqrt{3}}$ is given by:

$$
\rho_{q=1,c=0}(\lambda) = \frac{1}{2\sqrt{3}\pi|\lambda|} \left(u_+^{1/3} - u_-^{1/3}\right), \quad u_\pm = 1 + 72\lambda^2 \pm 3\sqrt{12\lambda^2 + 528\lambda^4 - 192\lambda^6}.
$$

4. For $q = 2$, we also have a closed form expression for the density supported on $|\lambda| \leq \lambda_+$:

$$
\rho_{q=2,c=0}(\lambda) = \frac{1}{2\pi} \left(\frac{1}{u} - 1\right), \quad u = \left(\frac{\lambda_- - \sqrt{\lambda_+^2 - \lambda^2}}{|\lambda|}\right)^{1/3}, \quad \lambda_+ = \frac{3\sqrt{3}}{2}.
$$

This density has a cubic-root singularity at $\lambda = 0$ (see Fig. 2):

$$
\rho_{q=2,c=0}(\lambda \to 0) \sim \frac{\sqrt{3}}{2\pi|\lambda|^{1/3}}.
$$

5. For $q > 2$, a Dirac mass appears at $\lambda = 0$. This is expected since in that case the $T$ two-dimensional projectors $P_t$ can no longer span the whole $N$-dimensional space.

The above construction does not rely on the fact that the matrices $X$ and $Y$ are real. In fact we can consider $X$ and $Y$ to be with complex or even quaternion entries where the norm of each entry has unit variance, $\mathbb{E}[x_i(y_j^*)^*] = \delta_{i,j}$ and define $F = (XY^T + YX^T)/(2T)$. The eigenvalue spectrum in the complex ($\beta = 2$) and quaternion ($\beta = 4$) cases is then the same as the real case. These are three cases of the same beta-ensemble where as usual the density is independent of beta but not the eigenvalue fluctuations and correlations. The matrix potential of this ensemble satisfies $V'(\lambda) = 2\text{Re} G_{\nu}(\lambda)$ for $|\lambda| \leq \lambda_+$, where $G_{\nu}(z)$ is the correct root of the cubic equation (8), see e.g. [5].

Using $S$-transforms to deal with free products of matrices, we know that the Marčenko-Pastur result can be generalised to the case where the “true” (population) covariance matrix of the $x$’s and of the $y$’s is a general definite positive matrix $C$, i.e.:

$$
\tilde{X} = \sqrt{C}X; \quad \tilde{Y} = \sqrt{C}Y,
$$

with $N^{-1}\text{Tr}C = 1$.

The same trick can be used in the present case as well, with now

$$
\tilde{F} = \frac{1}{2T}(\tilde{X}\tilde{Y}^T + \tilde{Y}\tilde{X}^T) = \frac{1}{2T} \sqrt{C}(XY^T + YX^T) \sqrt{C}.
$$

Interestingly, when $c = 0$, one finds that the spectrum of $\tilde{F}$ is the same as that of $\tilde{F}$ and therefore independent of $C$. Indeed for a trace-less matrix (such as $F$ with $c = 0$) the free product with matrix $C$ is equivalent to a simple scaling of $F$ by $N^{-1}\text{Tr}C = 1$. For $c > 0$, however, this is not true – as it is well known in the Marčenko-Pastur case, see e.g. [5, 8].

In conclusion, we have defined a new class of random correlation matrices with non-positive eigenvalues. We have determined the eigenvalue spectrum, which defines a two-parameter family of distributions that contain the Marčenko-Pastur law in one limit ($c = 1$, $q$ arbitrary) and the Wigner semi-circle in another limit ($c = 0$, $q \to 0$). The $c = 0$ case provides a null-hypothesis to test the existence of cross-correlations between two time series, complementing the results of [3, 9].

We thank F. Benaych-Georges, M. Nowak and D. Savin for useful comments, in particular pointing us to references [6, 7].

2 More generally, when $q = 0$, all eigenvalues are equal to $c$.

3 It is also the distribution of eigenvalues of $XX^T - YY^T$, where $X$ and $Y$ are independent Wigner matrices.
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