Quantum metrology at the Heisenberg limit with the presence of independent dephasing

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The Heisenberg limit is the superior precision available by entanglement sensors. However, entanglement is fragile against dephasing, and there is no known quantum metrology protocol that can achieve Heisenberg limited sensitivity with the presence of independent dephasing. Here, we show that the Heisenberg limit is attainable under the effect of independent dephasing under conditions where the probe qubits decohere due to both target fields and local environments. To detect the target fields, we exploit the entanglement properties to decay much faster than the classical states due to collective noise while most of the previous schemes use a coherent phase shift from the target fields. Actually, if the temporally fluctuating target fields behave as Markovian collective dephasing, we can estimate the collective dephasing rate with a sensitivity at the Heisenberg limit under the effect of independent dephasing. Our work opens the possibility for robust Heisenberg-limited metrology.

Quantum metrology is a field where one attempts to improve the performance of the sensors measuring target fields by using quantum properties [1,2]. Qubits typically play the role as a probe to measure target fields when those qubits interact with the fields we want to sense. When one prepares the qubits in a superposition state, there are interference terms (non-diagonal elements) in the density matrix where the information of the target fields can be encoded. Moreover, entanglement is considered a resource to enhance this sensitivity [7–12]. If we use a separable state composed of L qubits to estimate our target fields with a parameter θ, the uncertainty in this estimation scales as δθ = O(L^{−1/2}). This is known as the standard quantum limit (SQL). On the other hand, it is in principle possible without noise to obtain a scaling of δθ = O(L^{−1}) by using an L-qubit entangled state [13]. Such a scaling is called the Heisenberg limit (HL) [9,11].

One of the major obstacles of quantum metrology is the fragility of the entangled probe state against decoherence and especially dephasing [13–15]. While these entangled probe states can be strongly coupled with the target fields, these entangled states are highly sensitive to environmental noise. It is crucial in the field of quantum metrology to construct a robust entanglement sensor under the effect of realistic decoherence [16–18]. Actually, there are many proposals to improve the sensitivity of the quantum sensors with realistic noise by using the quantum Zeno effect [19–23], quantum error correction [24–29], strong interaction between qubits [30], qubit motion [31–33], and even adaptive control [34,35].

Dephasing (or parallel noise) is considered a major challenge that needs to be overcome for the robust quantum metrology [13,36–38]. Metrologically useful entanglement is typically designed to have a large non-diagonal terms where such target field information is encoded. Environmental dephasing parallel to the target fields induces a rapid decay of the non-diagonal terms where our target field information is encoded. Such a decay significantly degrades the performance of the quantum sensors. To recover the performance of the entanglement sensor, there is in principle a scheme to utilize the spatial correlation within the environment that induces the dephasing [39]. In such a case, one can achieve the HL scaling only if the form of the environmental spatial correlation satisfies very specific conditions [39]. Currently however there is no known metrological protocol that achieves the HL scaling under the effect of independent dephasing where each local environment independently couples with the probe qubits. It is generally thought that, under the independent Markovian dephasing, an entanglement based sensor is metrologically equivalent to the classical sensors as the entanglement sensors cannot beat the SQL [13–15]. If the environment has a finite correlation time, the dephasing becomes non-Markovian, and one achieves a sensitivity of δθ = O(L^{−3/4}), which beats the SQL but does not reach the HL [13,23].

In this letter, we present a sensing scheme that achieves the HL under the effect of independent dephasing. Consider that our L probe qubits are affected by independent dephasing due to local environments, and that we want to use these probe qubits to measure a property of the target fields. Previous schemes typically wanted to measure the amplitude of the time-independent target field using the probe qubits [13–14,20–23]. On the other hand, we can consider the situation where the target fields are temporally fluctuating and inducing collective Markovian dephasing on those probe qubits. In this situation our purpose is to estimate the dephasing rate of this collective noise. We show that it is possible to estimate the collective dephasing rate with HL sensitivity even under the effect of the independent dephasing.

Let us describe our scheme. Suppose that the target fields to interact with the L probe qubits are fluctuating which induces decoherence. In this case, we can adopt a spin-boson model to describe the interaction between the probe qubits and target fields [40,41] where each qubit is affected by its local environment. We define operators where \( \hat{N}_z = \sum_{j=1}^{L} \hat{\sigma}_z^{(j)} \) denotes the collective operator of the qubits, \( \hat{\sigma}_z^{(j)} = |1\rangle_j \langle 1| - |0\rangle_j \langle 0| \) denotes the Pauli operator, \( \hat{b}_k, (\hat{b}_k^\dagger) \) denotes the annihilation (creation) operator of the modes of the target fields, \( \hat{c}_{j,k'}, (\hat{c}_{j,k'}^\dagger) \) denotes the annihilation (creation) operator of the local environmental modes coupled with a qubit at \( j \)-th site. We assume \[ \{\hat{b}_k, \hat{b}_{k'}\} = \delta_{k,k'}, \{\hat{c}_{j,k'}, \hat{c}_{j,k'}^\dagger\} = \delta_{j,j'} \delta_{k,k'} . \] The
Hamiltonian is as following

\[
H = H_S + H_{1}^{\text{(ST)}} + H_{1}^{\text{(SE)}} + H_T + H_E
\]

\[
H_S = \frac{\hbar \omega}{2} \hat{M}_z
\]

\[
H_{1}^{\text{(ST)}} = \sum_k h g_k \hat{M}_z (\hat{b}_k^\dagger + \hat{b}_k)
\]

\[
H_{1}^{\text{(SE)}} = \sum_{j=1}^L \sum_{k'} \hat{h} \hat{g}_{j,k'} \hat{\sigma}_z^{(j)} (\hat{c}_{j,k'}^\dagger + \hat{c}_{j,k'})
\]

\[
H_T = \sum_k \hbar \omega_k \hat{b}_k^\dagger \hat{b}_k
\]

\[
H_E = \sum_{j,k'} \hbar \omega_{j,k'} \hat{c}_{j,k'}^\dagger \hat{c}_{j,k'}
\]

(1)

where \(\omega\) denotes the qubit frequency, \(g_k\) denotes the interaction strength between the qubits and the modes of the target, \(\hat{h}\) denotes the interaction strength between the qubit and the modes of the environment at \(j\)-th site, \(\omega_k\) denotes the frequency of the modes of the target fields, and \(\omega_{j,k'}\) denotes the frequency of the modes of the environment at \(j\)-th site. It is worth mentioning that, if a non-linear interaction among qubits such as \(H_1 = \hbar g \hat{M}_z^2\) is available, a super Heisenberg-limit is attainable to estimate the value of \(g\) [22-40]. However, here, we consider a linear interaction \(H_{1}^{\text{(ST)}}\) where the HL is considered to be the ultimate precision [9-11]. In the interaction picture, the Hamiltonian is

\[
H_1(t) = \left( \sum_{j=1}^L \hat{a}_z^{(j)} \right) \sum_k h g_k (\hat{b}_k^\dagger e^{i \omega_k t} + \hat{b}_k e^{-i \omega_k t})
\]

\[
+ \sum_{j=1}^L \sum_{k'} \hat{h} \hat{g}_{j,k'} \hat{\sigma}_z^{(j)} (\hat{c}_{j,k'}^\dagger e^{i \omega_{j,k'} t} + \hat{c}_{j,k'} e^{-i \omega_{j,k'} t})
\]

(2)

To characterize the property of the target fields (environment), we define a power spectral density for the modes as \(J(\omega) = \sum_k h^2 |g_k|^2 \delta(\omega - \omega_k) (J'_{f}(\omega) = \sum_k \hbar^2 |\hat{g}_{j,k'}|^2 \delta(\omega - \omega_{j,k'})\). Although our main interest is to measure the collective dephasing rate with Markovian properties (that corresponds to a frequency-independent power spectral density), we adopt a more general setup of a Lorentzian spectral density for the modes of the target fields (environment) such as \(J(\omega) = \frac{1}{\pi \tau_c \omega_c} \left( J'_{f}(\omega) = \frac{1}{\pi \tau'_{c} \omega_c} \right)\) where \(a(a')\) denotes the amplitude and \(\tau_c, \tau'_{c}\) (\(\tau'_{c}\)) denotes the correlation time of the modes of the target fields (environment). It is worth mentioning that, by taking a limit of a small correlation time on the power spectral density, we can consider the Markovian behavior as a special case in this model. We assume that the probe qubits, the target fields, and the local environments are separable at \(t = 0\). The initial state of the modes of target fields (environment) is a thermal equilibrium state such as \(\rho_T = \frac{1}{2} e^{-\frac{\hbar \omega}{k_B T}} (\rho_E = \frac{1}{2} e^{-\frac{\hbar g_k}{k_B T}})\) where \(T\) denotes the temperature, \(k_B\) denotes the Boltzmann factor respectively.

\[
Z = \text{Tr}[e^{-\frac{\hbar \omega}{k_B T}}] (Z' = \text{Tr}[e^{-\frac{\hbar g_k}{k_B T}}])\] denotes the renormalization factor. As an initial probe state, we choose the GHZ states \(|\psi_{\text{GHZ}}\rangle = \frac{1}{\sqrt{2}} (|00\cdots0\rangle + |11\cdots1\rangle\). By tracing out the modes of the target fields and the environments with Born approximation, the decoherence dynamics of the probe qubits by the master equation in the Schrodinger picture is described by

\[
\frac{d\rho}{dt} = -i\omega [\hat{M}_z, \rho] - \sum_k |g_k|^2 \frac{\sin \omega_k t}{\omega_k} \coth(\frac{\omega_k}{2k_B T}) [\hat{M}_z, [\hat{M}_z, \rho]]
\]

\[
- \sum_{j=1}^L \sum_{k'} |g_{j,k'}|^2 \frac{\sin \omega_{j,k'} t}{\omega_{j,k'}} \coth(\frac{\omega_{j,k'}}{2k_B T}) [\hat{\sigma}_z^{(j)}, [\hat{\sigma}_z^{(j)}, \rho]]
\]

(3)

For the zero temperature of \(T = 0\), we can solve the master equation to obtain

\[
\rho(t) = \frac{1}{2} (|00\cdots0\rangle\langle 00\cdots0| + |11\cdots1\rangle\langle 11\cdots1|)
\]

\[
+ \frac{e^{-i \omega t - L \tau_{MC} - L \gamma t}}{2} (|11\cdots1\rangle\langle 00\cdots0| + |00\cdots0\rangle\langle 11\cdots1|)
\]

where \(\Gamma_t = \frac{2 \alpha \tau_{c}^2}{\pi} (-1 + e^{-\frac{\omega}{\omega_c}} + \frac{1}{\omega_c})\) denotes the time-dependent collective dephasing rate and \(\gamma_t = \frac{2 \alpha \tau_{c}^2}{\pi} (-1 + e^{-\frac{\omega}{\omega_c}} + \frac{1}{\omega_c})\) denotes a time-dependent dephasing rate of the local environments [21, 40, 41]. If the correlation time is much shorter than the typical time of the dynamics which we call Markovian approximation, the dephasing rate becomes time-independent. We define the Markovian dephasing rate of the target fields (environment) as \(\Gamma_{MC} = 2 \alpha \tau_{c} \gamma_{MC} \equiv 2 \alpha \tau_{c}^2 \). On the other hand, in the limit of a long target-fields (environmental) correlation time, \(\Gamma_t (\gamma_t)\) increases linearly against time. In this regime, we obtain \(\Gamma_t \approx \alpha t = \frac{\gamma_t}{2} \) (\(\gamma_t \approx \alpha t = \frac{\gamma_{MC}}{2} \)). We define \(\Gamma_{NMC} \equiv \sqrt{\frac{\Gamma_{MC}}{2 \pi}} (\gamma_{NMC} \equiv \sqrt{\frac{\gamma_{MC}}{2 \pi}})\) as a non-Markovian dephasing rate of the target fields.

We explain our protocol for the sensing by using \(L\) probe qubits for a given total time \(T\). Assume that we can prepare and readout the probe qubits with a time scale much faster than the coherence time of the probe qubits. First, we prepare the GHZ state of the \(L\) probe qubits. Second, we let the probe qubits evolves for a time \(t\) according to the master equation in the Eq. 3. Third, we then perform a measurement with a projective operator of \(\mathcal{P} = |\psi_{\text{read}}\rangle\langle \psi_{\text{read}}|\). Finally, we repeat these process \(N \approx T / t\) times. The uncertainty to estimate a parameter \(\theta\) of the target is described as \(\delta \theta = \frac{\sqrt{F(1-F)}}{N} \frac{1}{\sqrt{N}}\) [13] where \(P = \text{Tr}[\rho (t) \mathcal{P}]\) denotes a probability distribution and \(\rho(t)\) denotes a density matrix of the probe qubits at a time \(t\). Since our model is general, our results include previously studied schemes [13, 20, 21, 22] as special cases.

Let us review the previous quantum metrology to measure the amplitude of time-independent target fields [13, 20, 21].
Table I: Performance of our sensing scheme where $L$ probe qubits interacts with both target fields and local environments. The target fields are temporally fluctuating which induces collective dephasing on the probe qubits. Surprisingly, under the effect of independent dephasing due to the local environments, we can achieve a Heisenberg limit scaling when the target fields have a Markovian (or time local) nature. On the other hand, if the target fields have a memory effect, the property becomes non-Markovian (or time non-local), and we cannot even beat the standard quantum limit under the effect of independent dephasing.

\begin{tabular}{|c|c|c|}
\hline
 & Markovian independent dephasing environment & Non-Markovian independent dephasing environment \\
\hline
Markovian collective dephasing fields & $\delta \Gamma_{MC} = O(L^{-1})$ & $\delta \Gamma_{MC} = O(L^{-1})$ \\
\hline
Non-Markovian collective dephasing fields & $\delta \Gamma_{NMC} = O(L^{-1/2})$ & $\delta \Gamma_{NMC} = O(L^{-1/2})$ \\
\hline
\end{tabular}

We assume the amplitude of the target fields has a linear relationship with the frequency $\omega$ and this amplitude is weak. The aim in these research is to estimate the value of $\omega$. Also, in these calculations [13, 20, 21, 23], the collective dephasing is not considered, and so we set $a = 0$. The uncertainty of the estimation is given as $\delta \omega = \frac{\exp(\frac{L\gamma t}{\sqrt{\Gamma t}})}{\sqrt{\Gamma t}}$, where we choose $|\psi_{\text{read}}\rangle = \frac{1}{\sqrt{2}}(\ket{0}\cdots\ket{0} + i\ket{1}\cdots\ket{1})$. For the independent Markovian environment with a short $\tau_0$, we obtain $\delta \omega = \frac{\exp(\frac{L\gamma t}{\sqrt{\Gamma t}})}{\sqrt{\Gamma t}}$, and this scales as $\delta \omega = O(L^{-1/2})$ by taking an optimized interaction time as $t = O(L^{-1})$ [13]. On the other hand, for the independent non-Markovian environment with a long $\tau_0$, we obtain $\delta \omega = \frac{\exp(\frac{L\gamma t}{\sqrt{\Gamma t}})}{\sqrt{\Gamma t}}$, which scales as $\delta \omega = O(L^{-3/4})$ by taking for an optimized interaction time as $t = O(L^{-1/2})$ [19, 21, 23]. To estimate the amplitude of the time-independent target fields, the non-Markovian properties of dephasing contribute to improve the sensitivity of the entanglement sensor. However, in either case, we cannot achieve the HL under the effect of the independent dephasing.

We can show that, for the estimation of the Markovian collective dephasing rate due to the temporally fluctuating target fields, we can obtain the HL under the effect of independent Markovian dephasing. More specifically, we can calculate the uncertainty of the estimation of with a white noise power spectral density $J(\omega) = \frac{\delta \omega^2}{\omega^2}$ where we take a limit of a small correlation time for the Lorentzian power spectral density. Since we assume the that the qubit frequency $\omega$ is known for this estimation, we can ignore this effect. Now, let us discuss the case of using a separable state of the $L$ probe qubits for the estimation of $\Gamma_{MC}$. For a single qubit sensor with an initial state of $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, we obtain $\delta \Gamma_{MC} = \frac{\sqrt{1 - \exp(-2\gamma t - 2T \Gamma_{MC} t)}}{L \sqrt{T t \exp(-\gamma t - T \Gamma_{MC} t)}} = O(L^0)$. By using $L$ qubits in parallel as a separable state, the sensitivity can be enhanced by a factor of $\sqrt{L}$ due to a central limit theorem, and so the uncertainty of the separable sensor is $\delta \Gamma_{MC} = O(L^{-1/2})$, which is bounded by the SQL. Next, we can calculate the uncertainty with the GHZ states composed of $L$ probe qubits as

$$
\delta \Gamma_{MC} = \sqrt{1 - \exp(-2\gamma t - 2T \Gamma_{MC} t)} \frac{L^2 \sqrt{T t \exp(-\gamma t - T \Gamma_{MC} t)}}{L^2 \sqrt{T t \exp(-\gamma t - T \Gamma_{MC} t)}}.
$$

for $|\psi_{\text{read}}\rangle = \frac{1}{\sqrt{2}}(|0\cdots0\rangle + |1\cdots1\rangle)$. By choosing $t = t_0/L^s$ where $t_0$ denotes a constant time and $s$ denotes a constant value, we obtain $\delta \Gamma_{MC} = \sqrt{1 - \exp(-2\gamma t - 2T \Gamma_{MC} t)} \frac{L^2 \sqrt{T t \exp(-\gamma t - T \Gamma_{MC} t)}}{L^2 \sqrt{T t \exp(-\gamma t - T \Gamma_{MC} t)}}$. The uncertainty becomes $\delta \Gamma_{MC} = \sqrt{1 - \exp(-2\gamma t_0/L - 2T \Gamma_{MC} t_0)} \frac{L^2 \sqrt{T t_0 \exp(-\gamma t_0/L - T \Gamma_{MC} t_0)}}{L^2 \sqrt{T t_0 \exp(-\gamma t_0/L - T \Gamma_{MC} t_0)}}$ for $s = 2$. For a large $L$ the effect of the independent dephasing becomes negligible regardless of the correlation time of the environment, and the uncertainty is approximated as $\delta \Gamma_{MC} \approx \frac{\sqrt{1 - \exp(-2\gamma t_0/L - 2T \Gamma_{MC} t_0)}}{L \sqrt{T t_0 \exp(-\Gamma_{MC} t_0/L)}} = O(L^{-1})$. Therefore, we achieve the HL under the effect of independent dephasing.

We explain intuitive reasons why we can achieve the HL to estimate the collective Markovian dephasing rate by using the entanglement. It is worth mentioning that, if the initial state of the probe qubit is the GHZ state, the collective Markovian dephasing occurs in a time scale of $t = O(L^{-2})$, while independent Markovian (non-Markovian) dephasing occurs in a time scale of $t = O(L^{-1})$ ($t = O(L^{-1/2})$). This means that we can observe the change in the dynamics of the probe qubits due to the collective decay within a time scale of $t = O(L^{-2})$ while the effect of the independent dephasing is negligible within this time scale for a large $L$. Moreover, since it takes a time of $t = O(L^{-2})$ for a single measurement, we can repeat the measurements $N \approx T/t = O(L^2)$ times for a given time $T$. Therefore, we can decrease the uncertainty of the estimation of the collective dephasing rate by $\delta \Gamma_{MC} = O(N^{-1/2}) = O(L^{-1})$, which achieves the HL.

Now for comparison, we calculate the uncertainty to estimate non-Markovian collective dephasing rate $\Gamma_{NMC}$ under the effect of independent dephasing. Here, we take the limit of a long correlation time $\tau_0$ for the target fields. The noise power spectral density is described as $J(\omega) = 2\delta(\omega)$. Similar to the Markovian case, the uncertainty to estimate $\Gamma_{NMC}$ is bounded by the SQL if we use $L$ probe qubits as a separable state. On the other hand, with an entanglement, we obtain

$$
\delta \Gamma_{NMC} = \sqrt{1 - \exp(-2\gamma t - 2T \Gamma_{NMC} t)} \frac{L^2 \sqrt{T t \exp(-\gamma t - T \Gamma_{NMC} t)}}{L^2 \sqrt{T t \exp(-\gamma t - T \Gamma_{NMC} t)}} \frac{L^2 \sqrt{T t \exp(-\gamma t - T \Gamma_{NMC} t)}}{L^2 \sqrt{T t \exp(-\gamma t - T \Gamma_{NMC} t)}}
$$

By choosing $t = t_0/L^s$, we obtain $\delta \Gamma_{NMC} = \sqrt{1 - \exp(-2\gamma t_0/L - 2T \Gamma_{NMC} t_0)} \frac{L^2 \sqrt{T t_0 \exp(-\gamma t_0/L - T \Gamma_{NMC} t_0)}}{L^2 \sqrt{T t_0 \exp(-\gamma t_0/L - T \Gamma_{NMC} t_0)}}$. This uncertainty is minimized when $N = 1$ such that $\delta \Gamma_{NMC} = \sqrt{1 - \exp(-2\gamma t_0/L - 2T \Gamma_{NMC} t_0)} \frac{L^2 \sqrt{T t_0 \exp(-\gamma t_0/L - T \Gamma_{NMC} t_0)}}{L^2 \sqrt{T t_0 \exp(-\gamma t_0/L - T \Gamma_{NMC} t_0)}} = O(L^{-1/2})$, which is the SQL. Therefore, to estimate the non-Markovian collective dephasing rate, the entanglement sensor does not offer a scaling advantage over the separable sensor.
We explain the reason why we cannot beat the SQL to estimate the non-Markovian collective dephasing rate. Non-Markovian dephasing occurs in a time scale of $t = O(L^{-1})$. This means that it takes a time of $t = O(L^{-1})$ for a single measurement, we can repeat the measurements $N = T/t = O(L)$ times for a given time $T$. So the uncertainty of the estimation of the non-Markovian collective dephasing rate is given $\delta \Gamma_{MC} = O(N^{-1/2}) = O(L^{-1/2})$, which is the SQL.

Our results (summarized in Table I) are essentially different from the previously studied cases of measuring the amplitude of the time-independent fields under the effect of independent dephasing \[13, 20, 21, 23\]. In the previous cases, non-Markovian properties of the local environment let us beat the SQL \[20, 21, 23\], while a Markovian environment made the entanglement sensor metrologically equivalent to the separable ones \[13\]. Non-Markovian properties were important to beat the SQL. On the other hand, Markovian properties of the target fluctuating fields actually helps to achieve the HL in our case, while non-Markovian properties of the target fluctuating fields destroy the advantage of the entanglement sensor.

Let us now calculate the uncertainty of the estimation when we have a finite correlation time $\tau_c$ for the target fields. While we can analytically calculate the uncertainty of the estimation in the limits of short or a long correlation times, will consider the finite $\tau_c$ situation now, and so we numerically plot the uncertainty of the estimation of the collective dephasing rate in the Fig. 1. Here, we choose the interaction time $t$ to minimize the uncertainty, and assume that the local environment is Markovian. We observe a clear transition of the scaling from the HL to the SQL as we increase the number of the probe qubits. This can be understood as follows. For a small number of the qubits, the characteristic time of the collective dephasing is much longer than the correlation time, and so we can use the Markovian assumption. On the other hand, as we increase the number of the qubits, the collective dephasing becomes stronger, and the characteristic time of the collective dephasing will be ultimately shorter than the correlation time. This means that, in the limit of a large $L$, the target fields should show the non-Markovian properties. From the Table II such a change of the property of the target fields clearly affects the uncertainty of the estimation, which induces the transition of the scaling from the HL to the SQL. It is worth mentioning that, although we cannot achieve the HL for a large $L$ with a finite correlation time $\tau_c$, we can still obtain a constant factor improvement with the entanglement sensor over the classical sensors, as shown in the Fig. 1.

In conclusion, we have shown that the Heisenberg limit is attainable in quantum metrology under the effect of independent dephasing. We consider the situation where the probe qubits interacts with both the target fields and local environments. More importantly we were interested in the situation where the target fields are temporally fluctuating which induces Markovian collective dephasing, while the local environment only induces independent dephasing. We find that, when estimating the collective dephasing rate due to the target fields, we can achieve the Heisenberg limited scaling with an entanglement sensor. This in turn paves the way for a future generation of HL sensor measuring fluctuating field. Moreover, our results are essential to understand the ultimate limit of the entanglement sensor with realistic conditions.

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