Extensions of the finite nonperiodic Toda lattices with indefinite metrics

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Abstract

In this paper, we firstly construct a weakly coupled Toda lattices with indefinite metrics which consist of $2N$ different coupled Hamiltonian systems. Afterwards, we consider the iso-spectral manifolds of extended tridiagonal Hessenberg matrix with indefinite metrics what is an extension of a strict tridiagonal matrix with indefinite metrics. For the initial value problem of the extended symmetric Toda hierarchy with indefinite metrics, we introduce the inverse scattering procedure in terms of eigenvalues by using the Kodama’s method. In this article, according to the orthogonalization procedure of Szegő, the relationship between the $\tau$-function and the given Lax matrix is also discussed. We can verify the results derived from the orthogonalization procedure with a simple example. After that, we construct a strongly coupled Toda lattices with indefinite metrics and derive its tau structures. At last, we generalize the weakly coupled Toda lattices with indefinite metrics to the $\mathbb{Z}_n$-Toda lattices with indefinite metrics.

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1 Introduction

Toda system is one of the most important integrable systems in mathematical physics. Many mathematicians have made significant contributions to the Toda equations and its generalization [1], such as M. Toda, Y. Kodama, etc. In recent years, some senior mathematicians have used different methods to study the Toda equations, such as symmetry and bilinear method, etc. This paper aims to extend the Toda equations via an extended algebra group [2], and the solutions can be pasted together to constitute a compact manifold. On the basis of [3,4], the Toda lattices are produced by semisimple Lie algebra. In the process of solving the extended Toda equations, we use the inverse scattering method, it also promotes the development of mathematical physics, integrable systems and Lie algebras [5,6]. According to the Hamiltonian systems of \(2N\) particles which described by the finite nonperiodic Toda lattice hierarchy [7], then we introduce a pair of Hamiltonian systems \((H, \hat{H})\) given by

\[
\begin{aligned}
H &= \frac{1}{2} \sum_{i=1}^{N} y_{i}^2 + \sum_{i=1}^{N} \exp(x_{i} - x_{i+1}), \\
\hat{H} &= \sum_{i=1}^{N} y_{i} \hat{y}_{i} + \sum_{i=1}^{N} (\hat{x}_{i} - \hat{x}_{i+1}) \exp(x_{i} - x_{i+1}).
\end{aligned}
\tag{1.1}
\]

The topology of an iso-spectral set of tridiagonal Hessenberg matrices was considered [7], and it has distinct real eigenvalues in the following form,

\[
L_{H} = \begin{pmatrix}
\alpha_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
\beta_1 & \alpha_2 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & \alpha_{N-1} & 1 \\
0 & 0 & 0 & \cdots & \beta_{N-1} & \alpha_N
\end{pmatrix},
\tag{1.2}
\]

\[
\hat{L}_{H} = \begin{pmatrix}
\hat{\alpha}_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
\hat{\beta}_1 & \hat{\alpha}_2 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & \hat{\alpha}_{N-1} & 1 \\
0 & 0 & 0 & \cdots & \hat{\beta}_{N-1} & \hat{\alpha}_N
\end{pmatrix},
\tag{1.3}
\]

where the variables \(\hat{\alpha}_k\) and \(\hat{\beta}_k\) in \(L_{H}\) and \(\hat{L}_{H}\) are expressed as \(s_k \hat{\alpha}_k = -\frac{1}{2} \hat{y}_k\) and \(\hat{b}_k = \frac{\hat{x}_{k} - \hat{x}_{k+1}}{4} \exp(\frac{\hat{x}_{k} - \hat{x}_{k+1}}{2})\) with \(s_i = \pm 1\). while different signs of \(s_i\) may create different systems.

The initial value problem of Toda equations were studied by applying the inverse scattering method in [8], we generalize the above method and get general results on the basis of the method mentioned in the references. With the help of the defined inner product in this paper, the elements of \(L\) and \(\hat{L}\) can be expressed in a simple way. This paper is arranged as follows. In Section 2, we construct the weakly coupled Toda equations via the transformation [9], by calculating the equations (2.10), the elements of \(L_{H}\) and \(\hat{L}_{H}\) can be expressed through an inner product and the initial value of the weakly coupled Toda lattices. Finally, we briefly describe the relationships between the elements of \(L_{H}\), \(\hat{L}_{H}\) and \(\tau\)-functions, and analyse specific relationships between \(\hat{\tau}_i\) and \(\hat{D}_i\). In section 3, we give a proof that the wave functions of the weakly coupled Toda lattices can be solved by the inverse scattering method with the Gram-Schmidt’s orthogonalization. In Section 4, we illustrate these results with a specific example, and some properties of the elements in the example are discussed. In section 5, we introduce the strongly coupled Toda lattices with indefinite metrics, and give some different conclusions compare with the weakly coupled Toda lattices. In section 6, first we give a definition of the \(Z_n\)-Toda equations by the algebraic transformation, and the solutions of the \(Z_n\)-Toda equations are obtained according to the initial value.
2 Weakly coupled Toda lattices with indefinite metrics

In this section, we define a weakly coupled Toda lattices with indefinite metrics. For the Hamiltonian (1.1), a transformation of variables will be introduced similarly as the one from Flaschka [9], which is about the classical Toda lattices with indefinite metrics:

\[
\begin{aligned}
\frac{da_k}{dt} &= \frac{1}{4} (s_{k+1}b_k^2 - s_{k-1}b_{k-1}^2), \\
\frac{db_k}{dt} &= s_{k+1}b_k - s_{k-1}b_{k-1};
\end{aligned}
\tag{2.3}
\]

Then the extended Toda equations are written in this form with \( b_0 = \hat{b}_0 = b_N = \hat{b}_N = 0, \)

\[
\begin{aligned}
\frac{db_k}{dt} &= \frac{1}{4} b_k (s_{k+1}a_{k+1} - s_{k}a_k), \\
\frac{db_k}{dt} &= \frac{1}{4} [s_{k}\hat{a}_k - s_{k+1}\hat{a}_{k+1}]b_k + (s_{k}a_k - s_{k+1}a_{k+1})\hat{b}_k].
\end{aligned}
\tag{2.4}
\]

The equations (2.3) and (2.4) can also be expressed as the following Lax equations:

\[
\begin{aligned}
\frac{d}{dt}L &= [B, L], \\
\frac{d}{dt}\hat{L} &= [\hat{B}, L] + [B, \hat{L}],
\end{aligned}
\tag{2.5}
\]

where \( L \) and \( \hat{L} \) are a \( N \times N \) tridiagonal matrix with real entries,

\[
L = \\
\begin{pmatrix}
s_{1}a_{1} & s_{2}b_{1} & \cdots & 0 \\
s_{1}b_{1} & s_{2}a_{2} & \cdots & 0 \\
\ddots & \ddots & \ddots & \ddots \\
0 & \cdots & s_{N-1}a_{N-1} & s_{N}b_{N-1} \\
0 & \cdots & s_{N-1}b_{N-1} & s_{N}a_{N}
\end{pmatrix},
\tag{2.6}
\]

\[
\hat{L} = \\
\begin{pmatrix}
s_{1}\hat{a}_{1} & s_{2}\hat{b}_{1} & \cdots & 0 \\
s_{1}\hat{b}_{1} & s_{2}\hat{a}_{2} & \cdots & 0 \\
\ddots & \ddots & \ddots & \ddots \\
0 & \cdots & s_{N-1}\hat{a}_{N-1} & s_{N}\hat{b}_{N-1} \\
0 & \cdots & s_{N-1}\hat{b}_{N-1} & s_{N}\hat{a}_{N}
\end{pmatrix},
\tag{2.7}
\]

\( B \) and \( \hat{B} \) are the projection of \( L, \hat{L} \) given by

\[
\begin{aligned}
B := \frac{1}{4} ([L] > 0 - (L) < 0], \\
\hat{B} := \frac{1}{4} ([\hat{L}] > 0 - (\hat{L}) < 0].
\end{aligned}
\tag{2.8}
\]

Note that, \( LS^{-1} \) and \( \hat{L}S^{-1} \) are symmetric tridiagonal matrix and \( S \) is a diagonal matrix \( S = \text{diag}(s_1, s_2, ..., s_N) \). In the this section, the extended Hamilton equations (1.1) can be
expressed by Lax equations (2.5) and the matrices (1.2), (1.3). In fact, the variables in (1.2) and (1.3) are given by

$$\begin{cases}
\alpha_k = s_k \alpha_k, \\
\hat{\alpha}_k = s_k \hat{\alpha}_k,
\end{cases}$$

and there is no doubt that they are equivalent. In order to solve the problem of the Lax equations (2.5), we can use the inverse scattering method to construct a specific formula from [8]. There are four linear equations that are contained in (2.5).

$$\begin{pmatrix}
L \Phi = \Phi \Lambda, \\
\hat{L} \Phi + L \hat{\Phi} = \hat{\Phi} \Lambda, \\
\frac{d}{dt}\Phi = B \Phi, \\
\frac{d}{dt}\hat{\Phi} = \hat{B} \Phi + B \hat{\Phi},
\end{pmatrix}$$

(2.10)

where $\Phi$ is the eigenmatrix of $L$, and $\hat{\Phi}$ is the eigenmatrix of $\hat{L}$, $\Lambda = \text{diag}(\lambda_1, ..., \lambda_{N-1}, \lambda_N)$ is a diagonal matrix, and $\Phi, \hat{\Phi}$ also satisfy the following relationship:

$$\begin{pmatrix}
\Phi^T S \Phi = S, \\
\Phi^T S \hat{\Phi} + \hat{\Phi}^T S \Phi = 0, \\
\Phi S^{-1} \Phi^T = S^{-1}, \\
\Phi S^{-1} \hat{\Phi}^T + \hat{\Phi} S^{-1} \Phi^T = 0.
\end{pmatrix}$$

(2.11)

Particularly, if $S = I$ (the identity matrix), then (2.10) shows that $L$ can be diagonalized by using an orthogonal matrix $O(N)$; if $S = \text{diag}(1, ..., 1, -1, ..., -1)$, the diagonalization can be obtained by using a “orthogonal” matrix $O(p, q)$ with $p + q = N$. From the orthogonality of (2.11), we obtain the eigenmatrix $\Phi(\hat{\Phi})$ of $L(\hat{L})$. Although eigenvalues of $L$ are real, the elements in $\Phi^T S \Phi$ differs from $s_i$. The eigenmatrix $\Phi$ and $\hat{\Phi}$ consist of the eigenvectors of $L$ and $\hat{L}$, and considering the following system of linear equations

$$\begin{pmatrix}
L \phi = \lambda \phi, \\
\hat{L} \phi + L \hat{\phi} = \lambda \hat{\phi},
\end{pmatrix}$$

(2.12)

which the $\phi$ and $\hat{\phi}$ consist of $\Phi, \hat{\Phi}$ are given in the following form,

$$\Phi = \begin{pmatrix}
\phi_1(\lambda_1) & \phi_1(\lambda_2) & \cdots & \phi_1(\lambda_N) \\
\vdots & \ddots & \vdots \\
\phi_N(\lambda_1) & \phi_N(\lambda_2) & \cdots & \phi_N(\lambda_N)
\end{pmatrix},$$

(2.13)

$$\hat{\Phi} = \begin{pmatrix}
\hat{\phi}_1(\lambda_1) & \hat{\phi}_1(\lambda_2) & \cdots & \hat{\phi}_1(\lambda_N) \\
\vdots & \ddots & \vdots \\
\hat{\phi}_N(\lambda_1) & \hat{\phi}_N(\lambda_2) & \cdots & \hat{\phi}_N(\lambda_N)
\end{pmatrix}.$$ 

(2.14)

From the first two equations of (2.11), we get something that looks like an “orthogonality” relationship as follows,

$$\begin{pmatrix}
\sum_{k=1}^N s_k^{-1} s_{ij} \phi_i(\lambda_k) \phi_j(\lambda_k) = \delta_{ij} s_{ii}^{-1}, \\
\sum_{k=1}^N s_k^{-1} [\hat{\phi}_i(\lambda_k) \phi_j(\lambda_k) + \phi_i(\lambda_k) \hat{\phi}_j(\lambda_k)] = 0.
\end{pmatrix}$$

(2.15)%

4
Also, we can get the similarly relationship from the another two equations of \([2.11]\),

\[
\begin{align*}
\sum_{k=1}^{N} s_k \phi_k(\lambda_i)\phi_k(\lambda_j) &= \delta_{ij} s_l, \\
\sum_{k=1}^{N} s_k [\hat{\phi}_k(\lambda_i)\phi_k(\lambda_j) + \phi_k(\lambda_i)\hat{\phi}_k(\lambda_j)] &= 0.
\end{align*}
\]  

(2.16)

According to \([2.15]\), we extend the inner product with four functions of \(\lambda\) from \([2]\),

\[
\begin{align*}
\langle f, g \rangle &:= \sum_{k=1}^{N} s_k^{-1} f(\lambda_k)g(\lambda_k), \\
\langle f, \tilde{g} \rangle &= \langle \tilde{f}, g \rangle := \sum_{k=1}^{N} s_k^{-1} [f(\lambda_k)\tilde{g}(\lambda_k) + \tilde{f}(\lambda_k)g(\lambda_k)],
\end{align*}
\]  

(2.17)

where \(\lambda\) are arbitrary. The elements of \(L\) and \(\hat{L}\) can be expressed:

\[
\begin{align*}
a_{ij} := (L)_{ij} &= s_j < \lambda \phi_i \phi_j >, \\
\hat{a}_{ij} := (\hat{L})_{ij} &= s_j < \lambda \hat{\phi}_i \phi_j > + s_j < \lambda \phi_i \hat{\phi}_j >.
\end{align*}
\]  

(2.18)

Thus, the elements of \(L\) and \(\hat{L}\) can be expressed by the above inner product with \(\phi_i\) and \(\hat{\phi}_i\). In fact, a lot of work has been finished in this area. Not only the \(\Phi\) can be obtained by the orthonormalization procedure of G. Szegö \([10]\), but there is another way to get the orthonormalization procedure, which is introduced by Kodama and McIauclunig \([11]\). The specific forms of \(\phi(t)\) and \(\hat{\phi}(t)\) are given below,

\[
\phi_i(\lambda, t) = \frac{e^{\lambda t}}{\sqrt{D_i(t)D_{i-1}(t)}} \begin{vmatrix}
1 & 2 & \cdots & i-1 & i-1 \\
1 & 2 & \cdots & i-1 & i-1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 2 & \cdots & i-1 & i-1 \\
1 & 2 & \cdots & i-1 & i-1
\end{vmatrix}
\]

\[
\begin{vmatrix}
\phi^0_i(\lambda) \\
\phi^0_i(\lambda) \\
\phi^0_i(\lambda) \\
\phi^0_i(\lambda)
\end{vmatrix},
\]  

(2.19)

and

\[
\hat{\phi}_i(\lambda, t) = \frac{e^{\lambda t}}{[D_i(t)D_{i-1}(t)]^{\frac{1}{2}}} \sum_{k=1}^{i-1} \left( \begin{vmatrix}
1 & 2 & \cdots & i-1 & i-1 \\
1 & 2 & \cdots & i-1 & i-1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 2 & \cdots & i-1 & i-1 \\
1 & 2 & \cdots & i-1 & i-1
\end{vmatrix}
\begin{vmatrix}
\phi^0_i(\lambda) \\
\phi^0_i(\lambda) \\
\phi^0_i(\lambda) \\
\phi^0_i(\lambda)
\end{vmatrix} \right)
\]

\[
\begin{vmatrix}
\hat{\phi}_i^0(\lambda) \\
\hat{\phi}_i^0(\lambda) \\
\hat{\phi}_i^0(\lambda) \\
\hat{\phi}_i^0(\lambda)
\end{vmatrix},
\]  

(2.20)

where \(c_{ij}(t) = < \phi^0_i \phi_j^0 e^{\lambda t} >, \hat{c}_{ij}(t) = < \phi^0_i \phi^0_j e^{\lambda t} > + < \phi^0_i \phi^0_j e^{\lambda t} >\). The \(D_k(t)\) and \(\hat{D}_k(t)\) are expressed by the determinant of the \(k \times k\) matrix with entries \(s_k c_{ij}(t)\) and \(s_k \hat{c}_{ij}(t)\),

\[
D_k(t) = \begin{vmatrix}
s_1 c_{11} & \cdots & s_i c_{i1} \\
\vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots \\
\vdots & \vdots & \ddots \\
s_1 c_{i1} & \cdots & s_i c_{1i}
\end{vmatrix},
\]

(2.21)

\[
\hat{D}_k(t) = \sum_{k=1}^{i} \begin{vmatrix}
s_1 c_{11} & \cdots & s_k \hat{c}_{1k} & \cdots & s_i c_{1i} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
s_1 c_{11} & \cdots & s_k \hat{c}_{1k} & \cdots & s_i c_{1i}
\end{vmatrix}.
\]

(2.22)

Note that \(D_k(0) = 1\) and \(\hat{D}_k(0) = 0\) for any \(k\). With the formulas \([2.19]\) and \([2.20]\), we get the solutions for the problem of \([2.10]\), we can derive the following proposition with the above formulas.
Proposition 1. \(L(t)\) and \(\hat{L}(t)\) blow up to infinity at \(t_0\) while \(\hat{D}_i(t_0) = D_i(t_0) = 0\) with some \(t_0\) and \(i\). For the case of matrices \(L\) and \(\hat{L}\), the \(\hat{D}_i(t)\) and \(D_i(t)\) can be written with the \(\tau\)-functions, and the solutions \(\alpha_i, \hat{\alpha}_i, \beta_i\) and \(\hat{\beta}_i\) can be expressed in the following forms:

\[
\begin{align*}
\alpha_i &= s_i \alpha_i = \frac{d}{dt} \log \frac{\tau_i}{\tau_{i-1}}, \\
\hat{\alpha}_i &= s_i \hat{\alpha}_i = \frac{d}{dt} \log \frac{\tau_i - \hat{\tau}_{i-1}}{\tau_{i-1}}, \\
\beta_i &= s_i s_{i+1} b_i^2 = \frac{\tau_{i+1} \tau_{i-1}}{\tau_i}, \\
\hat{\beta}_i &= 2 s_i s_{i+1} b_i \hat{\beta}_i = \frac{\tau_{i+1} \tau_{i-1} + \hat{\tau}_{i-1} - 2 \tau_{i+1} \tau_{i-1} \hat{\tau}_{i-1}}{\tau_i^2}.
\end{align*}
\]

(2.23) (2.24)

The derivative of the weakly coupled Toda equations (2.3) and (2.4) are expressed in the bilinear form,

\[
\begin{align*}
\tau_i \tau_i'' - (\tau_i')^2 &= \tau_{i+1} \tau_{i-1}, \\
\hat{\tau}_i \tau_i'' + \tau_i \hat{\tau}_i'' - 2 \tau_i' \tau_i' &= \hat{\tau}_{i+1} \tau_{i-1} + \hat{\tau}_{i-1} \tau_{i+1}.
\end{align*}
\]

(2.25)

The \(\tau_i\) and \(\hat{\tau}_i\) can be written as determinants refer to [12][13]:

\[
\tau_i = \begin{vmatrix}
\tau_1 & \tau_1' & \cdots & \tau_1^{(i-1)} \\
\tau_1' & \tau_1'' & \cdots & \tau_1^{(i)} \\
\vdots & \vdots & \ddots & \vdots \\
\tau_1^{(i-1)} & \tau_1^{(i)} & \cdots & \tau_1^{(2i-2)}
\end{vmatrix},
\]

(2.26)

\[
\hat{\tau}_i = \sum_{k=0}^{i-1} \begin{vmatrix}
\tau_1 & \cdots & \tau_1^{(k)} & \cdots & \tau_1^{(i-1)} \\
\tau_1' & \cdots & \tau_1^{(k+1)} & \cdots & \tau_1^{(i)} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\tau_1^{(i-1)} & \cdots & \tau_1^{(k+i-1)} & \cdots & \tau_1^{(2i-2)}
\end{vmatrix},
\]

(2.27)

where \(\tau_1\) and \(\hat{\tau}_1\) are given by \(\tau_1 = c_1 = <\phi_{1}^0 \phi_{1}^0 e^\lambda > = s_1^{-1} D_1\), \(\hat{\tau}_1 = \hat{c}_1 = <2 \phi_{1}^0 \phi_{1}^0 e^\lambda > = s_1^{-1} \hat{D}_1\). From (2.21) and (2.22), we know that there is a certain relationship between \(\tau_i\), \(\hat{\tau}_i\), and \(D_i, \hat{D}_i\),

\[
\begin{align*}
\tau_i &= \frac{1}{s_1} \prod_{k=1}^{i-1} (\beta_k^0)^{i-k} D_i, \\
\hat{\tau}_i &= \frac{1}{s_1} \prod_{k=1}^{i-1} (\beta_k^0)^{i-k} (\beta_k^0)^{i-k} D_i + (\beta_k^0)^{i-k} \hat{D}_i,
\end{align*}
\]

(2.28)

where \(\beta_k^0 = \beta_k(0)\) and \(\hat{\beta}_k^0 = \hat{\beta}_k(0)\). From (2.28), we know that \(\tau_i, \hat{\tau}_i\) turn out to the blow-ups with both \(\alpha_i, \hat{\alpha}_i\) and \(\beta_i, \hat{\beta}_i\) at \(t_0\) for \(D(t_0) = \hat{D}(t_0) = 0\). If there are some \(k\) that make \(\beta_k^0 = \beta_k^0 = 0\), we can not decide \(\beta_k, \hat{\beta}_k\) in the form of (2.23), the reason is that the original system will be split into many small subsystems.

3 The inverse scattering method

Next, we are going to use the inverse scattering method to give the expressions of \(\phi_i(\lambda, t)\) and \(\hat{\phi}_i(\lambda, t)\), which is mainly used in the article of Kodama [2], and the time variable of \(\Phi(t)\) can be obtained by the orthonormalization procedure of Szegö [10]. According to the reference [2], we know that \(B = L - \text{diag}(L) - 2(L)_{<0}\), \(\hat{B} = \hat{L} - \text{diag}(\hat{L}) - 2(\hat{L})_{<0}\).
Due to the first two equations of (2.10), \( L\Phi = \Phi \Lambda \) and \( \hat{L}\Phi + \hat{L}\Phi = \Phi \Lambda \), we get \( B = \Phi \Lambda \Phi^{-1} - \text{diag}(L) - 2(L)_{<0} \), and \( \hat{B} = \Phi \Lambda \hat{\Phi}^{-1} + \hat{\Phi} \Lambda \Phi^{-1} - \text{diag}(\hat{L}) - 2(\hat{L})_{<0} \). Then the latter two equations of (2.10) can be written as:

\[
\begin{align*}
\frac{d}{dt}\Phi &= \Phi \Lambda - [\text{diag}(L) + 2(L)_{<0}]\Phi, \\
\frac{d}{dt}\hat{\Phi} &= \hat{\Phi} \Lambda - [\text{diag}(\hat{L}) + 2(\hat{L})_{<0}]\Phi + [\text{diag}(L) + 2(L)_{<0}]\hat{\Phi}.
\end{align*}
\] (3.1)

The elements of the \( \Phi_t \) and \( \hat{\Phi}_t \) are expressed by the right side of the equations (3.1), and the vectors \( \phi_i(\lambda_k, t) \), \( \hat{\phi}_i(\lambda_k, t) \) \((k = 1, ..., N)\) of the first line in \( \Phi_t \) and \( \hat{\Phi}_t \) are given by

\[
\begin{align*}
\frac{d}{dt} \phi_1(\lambda_k) &= [\lambda_k - s_1 < \lambda \phi_1^2(\lambda) >] \phi_1(\lambda_k), \\
\frac{d}{dt} \hat{\phi}_1(\lambda_k) &= [\lambda_k - s_1 < \lambda \phi_1^2(\lambda) >] \hat{\phi}_1(\lambda_k) - 2s_1 < \lambda \hat{\phi}_1(\lambda) \phi_1(\lambda) > \phi_1(\lambda_k).
\end{align*}
\] (3.2)

Then (3.2) can be readily solved in the form

\[
\begin{align*}
\phi_1(\lambda_k, t) &= \frac{\psi_1(\lambda_k, t)}{\sqrt{s_1 < \psi_1^2(\lambda, t) >}}, \\
\hat{\phi}_1(\lambda_k, t) &= \frac{-\psi_1(\lambda_k, t)}{\sqrt{s_1 < \psi_1^2(\lambda, t) >}}.
\end{align*}
\] (3.3)

with \( \psi_1(\lambda, t) = \phi_0^0(\lambda_k) e^{\lambda t} \) and \( \hat{\psi}_1(\lambda, t) = \hat{\phi}_0^0(\lambda_k) e^{\lambda t} \). The elements of the second line in \( \Phi_t \), \( \hat{\Phi}_t \) are expressed,

\[
\begin{align*}
\frac{d}{dt} \phi_2(\lambda_k) &= [\lambda_k - s_2 < \lambda \phi_2^2(\lambda) >] \phi_2(\lambda_k) - 2s_1 < \lambda \phi_2(\lambda) \phi_1(\lambda) > \phi_1(\lambda_k), \\
\frac{d}{dt} \hat{\phi}_2(\lambda_k) &= (\lambda_k - s_2 < \lambda \phi_2^2(\lambda) >) \hat{\phi}_2(\lambda_k) - 2s_2 < \lambda \hat{\phi}_2(\lambda) \phi_2(\lambda) > \phi_2(\lambda_k) \\
&\quad - 2s_1 < \lambda \phi_2(\lambda) \phi_1(\lambda) > \phi_1(\lambda_k) - 2s_1 < \lambda \phi_2(\lambda) \phi_1(\lambda) > \phi_1(\lambda_k).
\end{align*}
\] (3.4)

Through the integration, the \( \phi_2(\lambda_k) \), \( \hat{\phi}_2(\lambda_k) \) can be written as

\[
\begin{align*}
\phi_2(\lambda_k, t) &= \frac{\psi_2(\lambda, t)}{\sqrt{s_2 < \psi_2^2(\lambda, t) >}}, \\
\hat{\phi}_2(\lambda_k, t) &= \frac{-\psi_2(\lambda, t)}{\sqrt{s_2 < \psi_2^2(\lambda, t) >}}.
\end{align*}
\] (3.5)

where

\[
\begin{align*}
\psi_2(\lambda, t) &= \phi_0^0(\lambda_k) e^{\lambda t} - s_1 < \phi_2^0(\lambda_k) e^{\lambda t} > \phi_1(\lambda, t), \\
\hat{\psi}_2(\lambda, t) &= \hat{\phi}_0^0(\lambda_k) e^{\lambda t} - s_1 < \phi_2^0(\lambda_k) e^{\lambda t} > \hat{\phi}_1(\lambda, t) \\
&\quad + s_1 < \phi_2^0(\lambda) \phi_1(\lambda, t) e^{\lambda t} + \hat{\phi}_2^0(\lambda) \phi_1(\lambda, t) e^{\lambda t} > \phi_1(\lambda, t).
\end{align*}
\] (3.6)

Generally, the ith lines in (3.1) satisfy

\[
\begin{align*}
\frac{d}{dt} \phi_i(\lambda_k) &= [\lambda_k - s_i < \lambda \phi_i^2(\lambda) >] \phi_i(\lambda_k, t) - 2 \sum_{j=1}^{i-1} s_j < \lambda \phi_i(\lambda) \phi_j(\lambda) > \phi_j(\lambda_k, t), \\
\frac{d}{dt} \hat{\phi}_i(\lambda_k) &= [\lambda_k - s_i < \lambda \phi_i^2(\lambda) >] \hat{\phi}_i(\lambda_k, t) - 2s_i < \lambda \hat{\phi}_i(\lambda) \phi_i(\lambda) > \phi_i(\lambda_k, t) \\
&\quad - 2 \sum_{j=1}^{i-1} s_j < \lambda \phi_i(\lambda) \phi_j(\lambda) > \hat{\phi}_j(\lambda_k, t) + < \lambda \hat{\phi}_i(\lambda) \phi_j(\lambda) + \hat{\phi}_i(\lambda) \phi_j(\lambda) > \phi_j(\lambda_k, t).
\end{align*}
\] (3.7)

Then, it is the same with the procedures above,

\[
\begin{align*}
\phi_i(\lambda_k, t) &= \frac{\psi_i(\lambda_k, t)}{\sqrt{s_i < \psi_i^2(\lambda, t) >}}, \\
\hat{\phi}_i(\lambda_k, t) &= \frac{-\psi_i(\lambda_k, t)}{\sqrt{s_i < \psi_i^2(\lambda, t) >}}.
\end{align*}
\] (3.8)
Further, from (3.14), we have
\begin{align*}
\hat{\psi}_i(\lambda_k, t) &= \phi_i^0(\lambda_k)e^{\lambda t} - \sum_{j=1}^{i-1} s_j \phi_i^0(\lambda)\phi_j(\lambda, t)e^{\lambda t} > \phi_j(\lambda_k, t), \\
\hat{\psi}_i(\lambda_k, t) &= \phi_i^0(\lambda_k)e^{\lambda t} - \sum_{j=1}^{i-1} s_j \phi_i^0(\lambda)\phi_j(\lambda, t)e^{\lambda t} > \hat{\psi}_i(\lambda_k, t) \\
&\quad + < \phi_i^0(\lambda)\phi_j(\lambda, t) + \phi_i^0(\lambda)\hat{\phi}_j(\lambda, t) > e^{\lambda t} \phi_j(\lambda_k, t).
\end{align*}

(3.9)

Now, we are going to give a further proof, there are two linear equations given by
\begin{equation}
\begin{cases}
\Phi = T\Psi, \\
\hat{\Phi} = \hat{T}\Psi + T\hat{\Psi},
\end{cases}
\end{equation}

(3.10)

then we give
\begin{equation}
\Psi = \\
\begin{pmatrix}
\psi_1(\lambda_1) & \cdots & \psi_1(\lambda_N) \\
\psi_2(\lambda_1) & \cdots & \psi_2(\lambda_N) \\
\vdots & & \vdots \\
\psi_N(\lambda_1) & \cdots & \psi_N(\lambda_N)
\end{pmatrix},
\end{equation}

(3.11)

\begin{equation}
\hat{\Psi} = \\
\begin{pmatrix}
\hat{\psi}_1(\lambda_1) & \cdots & \hat{\psi}_1(\lambda_N) \\
\hat{\psi}_2(\lambda_1) & \cdots & \hat{\psi}_2(\lambda_N) \\
\vdots & & \vdots \\
\hat{\psi}_N(\lambda_1) & \cdots & \hat{\psi}_N(\lambda_N)
\end{pmatrix},
\end{equation}

(3.12)

and
\begin{equation}
\begin{cases}
T = \text{diag}[s_{i}\langle \psi_i^2 \rangle]^{-\frac{1}{2}}, \quad i = 1, 2, \cdots, N, \\
\hat{T} = \text{diag}\left(-\frac{s_{i}\langle \hat{\psi}_i\hat{\psi}_i \rangle}{s_{i}\langle \psi_i^2 \rangle^{2}}\right), \quad i = 1, 2, \cdots, N.
\end{cases}
\end{equation}

(3.13)

According to (3.10), the expressions of the equations (2.9) can be expressed as
\begin{equation}
\begin{cases}
(T^{-1}LT)\Psi = \Psi\Lambda, \\
(T^{-1}LT)\hat{\Psi} + (\hat{T}^{-1}LT + T^{-1}\hat{L}T + T^{-1}L\hat{T})\Psi = \hat{\Psi}\Lambda, \\
\frac{d}{dt}\Psi = (T^{-1}BT)\Psi - \left(\frac{d}{dt}\log T\right)\Psi, \\
\frac{d}{dt}\hat{\Psi} = (T^{-1}BT)\hat{\Psi} + (\hat{T}^{-1}BT + T^{-1}\hat{B}T + T^{-1}B\hat{T})\Psi - \left(\frac{d}{dt}(\log T\hat{\Psi} + \log T\hat{\Psi})\right).
\end{cases}
\end{equation}

(3.14)

From (3.11), we find
\begin{equation}
\begin{cases}
T^{-1}BT = -2(T^{-1}LT)_{<0} + T^{-1}LT - (T^{-1}\text{diag}(L)T), \\
\hat{T}^{-1}BT + T^{-1}\hat{B}T + T^{-1}\hat{T}B\hat{T} = -2(\hat{T}^{-1}LT + T^{-1}\hat{L}T + T^{-1}L\hat{T})_{<0} + \hat{T}^{-1}LT \\
+ T^{-1}L\hat{T} - (\hat{T}^{-1}\text{diag}(L)T + T^{-1}\text{diag}(\hat{L})T + T^{-1}\text{diag}(L)\hat{T}).
\end{cases}
\end{equation}

(3.15)

Further, from (3.14), we have
\begin{equation}
\begin{cases}
\frac{d}{dt}\psi = -2(T^{-1}LT)_{<0}\psi + \lambda\psi - \left(\text{diag}(L) + \frac{d}{dt}\log T\right)\psi, \\
\frac{d}{dt}\hat{\psi} = -2(\hat{T}^{-1}LT + (TL\hat{T})_{<0}\psi - 2(T^{-1}LT)_{<0}\hat{\psi} \\
+ \lambda\hat{\psi} - \left(\text{diag}(L) + \frac{d}{dt}(\log T)\hat{\psi} - \left(\frac{d}{dt}(T)\hat{\psi}\right)\psi,
\end{cases}
\end{equation}

(3.16)

where the structure of \( \hat{T} \) is similar to \( T \). According to (3.10), the solutions with \( \psi, \hat{\psi} \) expressed as
\begin{equation}
\begin{cases}
\frac{d\psi}{dt} = -2\sum_{j=1}^{i-1} \frac{\langle \lambda\psi_j\psi_i \rangle}{\psi_j} < \psi_j^2 > \psi_j + \lambda\psi_i, \\
\frac{d\hat{\psi}}{dt} = -2\sum_{j=1}^{i-1} < \lambda\psi_j\psi_i > \frac{\psi_j}{< \psi_j^2 >} < \psi_j^2 > \psi_j + \frac{\langle \lambda\psi_i\hat{\psi}_j \rangle}{< \psi_j^2 >} \psi_j + \lambda\hat{\psi}_j,
\end{cases}
\end{equation}

(3.17)
and
\[
\begin{cases}
\frac{1}{2} \frac{d}{dt} \log <\psi_i^2> = s_j <\lambda \phi_i^2> = a_{ij}, \\
\frac{1}{2} \frac{d}{dt} \log <\hat{\psi_i} \hat{\psi_i} >= s_j <\hat{\lambda} \hat{\phi_i} > + <\hat{\lambda} \hat{\phi_i} > = \hat{a}_{ij},
\end{cases}
\] (3.18)

Obviously, (3.18) can be evolved from (3.17). And through simple calculation,
\[
\begin{cases}
\psi(\lambda, t) = Q(t) \phi(\lambda)e^{\lambda t}, \\
\hat{\psi}(\lambda, t) = [\hat{Q}(t) \phi(\lambda) + Q(t) \phi(\lambda)]e^{\lambda t},
\end{cases}
\] (3.19)

where \(Q(t), \hat{Q}(t)\) are lower triangular matrices and \(\phi(\lambda) = \phi(\lambda, 0), \phi(\lambda) = \phi(\lambda, 0)\). In fact, we have to chose the initial conditions of \(\psi(\lambda, t), \hat{\psi}(\lambda, t)\) which satisfy \(\psi(\lambda, 0) = \phi(\lambda)\), \(\hat{\psi}(\lambda, 0) = \phi(\lambda)\), and \(s_i <\psi_i \psi_j >= s_i <\phi_i \phi_j >= \delta_{ij} (t = 0), s_i <\phi_i \phi_j + \phi_i \phi_j >= 0 (t = 0)\). For the “orthogonality” relations of (2.12) and (2.13), it show that the \(<\psi_i \psi_j >= 0, <\hat{\psi_i} \hat{\psi_j} >= 0\) for \(i \neq j\), so the “orthogonality” relations can be written like this:
\[
\begin{cases}
<\psi_i \phi_j e^{\lambda t}> = \sum_{k=1}^N s_k^{-1} <\psi_i(\lambda_k, t) \phi_j(\lambda_k) e^{\lambda t}> = 0, \\
<\hat{\psi_i} \phi_j e^{\lambda t}> + <\phi_i \hat{\phi}_j e^{\lambda t}> = \sum_{k=1}^N (s_k^{-1} <\hat{\psi}_i \phi_j e^{\lambda t}> + s_k^{-1} <\psi_i \phi_j e^{\lambda t}> ) = 0.
\end{cases}
\] (3.20)

The solutions between \(\psi_i(\lambda, t)\) and \(\hat{\psi}_i(\lambda, t)\) of (3.18) are given from [2],
\[
\psi_i(\lambda, t) = \frac{e^{\lambda t}}{D_{i-1}(t)} \begin{vmatrix} s_1 c_{11} & s_2 c_{12} & \cdots & s_i-1 c_{1,i-1} & \phi_i^0(\lambda) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
 s_1 c_{i1} & s_2 c_{i2} & \cdots & s_i-1 c_{i,i-1} & \phi_i^0(\lambda) \end{vmatrix},
\] (3.21)

where \(c_{ij} = <\phi_i \phi_j e^{\lambda t}>\), and the elements of \(D_k(t)\) are \(s_j c_{ij}(t), i = 1, 2, ..., N\),
\[
D_k(t) = |(s_j c_{ij}(t))_{1 \leq i,j \leq k}|.
\] (3.22)

**Lemma 1.** According to (3.17), the \(\hat{\psi}_i\) can be expressed
\[
\hat{\psi}_i(\lambda, t) = \frac{e^{\lambda t}}{D_{i-1}(t)} \sum_{k=1}^{i-1} \begin{vmatrix} s_1 c_{11} & \cdots & s_k \hat{c}_{1,k} & \cdots & \phi_i^0(\lambda) \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
 s_1 c_{i1} & \cdots & s_k \hat{c}_{i,k} & \cdots & \phi_i^0(\lambda) \end{vmatrix} + \frac{e^{\lambda t}}{D_{i-1}(t)} \begin{vmatrix} s_1 c_{11} & \cdots & \phi_i^0(\lambda) \\
\vdots & \ddots & \vdots \\
 s_1 c_{i1} & \cdots & \phi_i^0(\lambda) \end{vmatrix} \\
- \frac{D_{i-1}(t)e^{\lambda t}}{D_{i-1}(t)} \begin{vmatrix} s_1 c_{11} & \cdots & s_{i-1} c_{1,i-1} & \phi_i^0(\lambda) \\
\vdots & \ddots & \vdots & \ddots \\
 s_1 c_{i1} & \cdots & s_{i-1} c_{i,i-1} & \phi_i^0(\lambda) \end{vmatrix},
\] (3.23)

where \(\hat{c}_{ij}(t) = <\hat{\phi}_i \phi_j e^{\lambda t}> + <\phi_i \hat{\phi}_j e^{\lambda t}>\), and \(k\) represents the number of columns in the determinant above,
\[
\hat{D}_k(t) = \sum_{k=1}^{i} \begin{vmatrix} s_1 c_{11} & \cdots & s_k \hat{c}_{1,k} & \cdots & s_i c_{1,i} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
 s_1 c_{i1} & \cdots & s_k \hat{c}_{i,k} & \cdots & s_i c_{i,i} \end{vmatrix}.
\] (3.24)

**Proof.** From (3.20) and (3.19), we have
\[
\begin{cases}
s_j \sum_{k=1}^{i} Q_{ik} k_{kj}(t) = 0, \\
s_j \sum_{k=1}^{i} [\hat{Q}_{ik} k_{kj}(t) + Q_{ik} \hat{k}_{kj}(t)] = 0, \quad 1 \leq j \leq i - 1.
\end{cases}
\] (3.25)
Solving (3.25) for $Q_{ik}$ and $\hat{Q}_{ik}$, we have

\[
\begin{align*}
Q_{ik} &= -\frac{D_{i-1}^k(t)}{D_{i-1}^k(t)}, \\
\hat{Q}_{ik} &= -\frac{\hat{D}_{i-1}^k(t)D_{i-1}^k(t)}{D_{i-1}^k(t)} + \frac{\hat{D}_{i-1}^k(t)D_{i-1}^k(t)}{D_{i-1}^k(t)}.
\end{align*}
\tag{3.26}
\]

In fact, $\hat{D}_{i-1}^k(t)(D_{i-1}^k(t))$ is the substitution of the $k$th and the $i$th row of $\hat{D}_{i-1}^k(t)(D_{i-1}^k(t))$.

From (3.19), we have

\[
\hat{\psi}_i = e^{\lambda t} \sum_{k=1}^{i} (Q_{ik}\hat{\phi}_k^0 + \hat{Q}_{ik}\phi_k^0)
\tag{3.27}
\]

\[
e^{-\lambda t} \sum_{k=1}^{i-1} \left( \frac{\hat{D}_{i-1}^k(t)D_{i-1}^k(t) - D_{i-1}^k(t)}{D_{i-1}^k(t)} \phi_k^0 + e^{\lambda t} \frac{D_{i-1}^k(t)}{D_{i-1}^k(t)} \hat{\phi}_k^0 \right) + e^{\lambda t} \frac{D_{i-1}^k(t)}{D_{i-1}^k(t)} \hat{\phi}_i^0
\tag{3.28}
\]

\[
e^{-\lambda t} \sum_{k=1}^{i-1} \left( (-1)^{i+k+k} \hat{\phi}_k^0 \right) \left| \begin{array}{cccc}
s_1c_{i1} & \cdots & s_1c_{ii} & \cdots & s_1c_{i,i-1} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
s_{i-1-i-1} & \cdots & s_{i-1-i-1} & \cdots & s_{i-1-i-1}
\end{array} \right| + \hat{\phi}_i^0 \frac{D_{i-1}^k(t)}{D_{i-1}^k(t)}
\tag{3.29}
\]

\[
- e^{-\lambda t} \sum_{k=1}^{i-1} \frac{\hat{D}_{i-1}^k(t)D_{i-1}^k(t) - D_{i-1}^k(t)}{D_{i-1}^k(t)} \phi_k^0,
\tag{3.30}
\]

which is just (3.23). From (3.21) and (3.23), $<\psi_i^2>$, $<2\hat{\psi}_i\psi_i>$ can be expressed with $D_i$ and $\hat{D}_i$

\[
\begin{align*}
<\psi_i^2> &= \frac{D_i}{s_iD_i}, \\
<2\hat{\psi}_i\psi_i> &= \frac{D_iD_i - \hat{D}_iD_i}{s_iD_i}.
\end{align*}
\tag{3.31}
\]

Then we can obtain the formulas that we have mentioned above,

\[
\phi_t(\lambda, t) = \frac{e^{\lambda t}}{\sqrt{D_i(t)D_{i-1}(t)}} \left| \begin{array}{cccc}
s_1c_{i1} & s_2c_{i2} & \cdots & s_{i-1}c_{i,i-1} & \phi_0^0(\lambda) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
s_1c_{i1} & s_2c_{i2} & \cdots & s_{i-1}c_{i,i-1} & \phi_0^0(\lambda)
\end{array} \right|
\tag{3.32}
\]

\[
\hat{\phi}_t(\lambda, t) = \frac{e^{\lambda t}}{[D_i(t)D_{i-1}(t)]^{1/2}} \left( \sum_{k=1}^{i-1} \left| \begin{array}{cccc}
s_1c_{i,i-1} & \cdots & s_kc_{i,k} & \cdots & s_{i-1}c_{i,i-1} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
s_1c_{i,i-1} & \cdots & s_kc_{i,k} & \cdots & s_{i-1}c_{i,i-1}
\end{array} \right| + \\
\frac{[\hat{D}_i(t)D_{i-1}(t) + D_i(t)\hat{D}_{i-1}(t)]e^{\lambda t}}{2[D_i(t)D_{i-1}(t)]^{1/2}} \left| \begin{array}{cccc}
s_1c_{i1} & \cdots & \phi_0^0(\lambda) \\
\vdots & \ddots & \vdots & \vdots \\
s_1c_{i1} & \cdots & \phi_0^0(\lambda)
\end{array} \right| \right)
\tag{3.33}
\]

By using the formulas (3.33) and (3.32), we get the solutions for (2.10). The method above is similar to szegö [10], and it is also equivalent to the procedure of Gram-Schmidt [7].
3.1 Example

Next, we give a simple example to verify our results, and talk about the properties of the solutions. Let $L_2, \hat{L}_2$ be a $2 \times 2$ matrix with $S = \text{diag}(1, -1)$, and $L_2, \hat{L}_2$ are given by

$$L_2 = \begin{pmatrix} a_1 & -b_1 \\ b_1 & -a_2 \end{pmatrix}, \quad (3.34)$$

$$\hat{L}_2 = \begin{pmatrix} \hat{a}_1 & -\hat{b}_1 \\ \hat{b}_1 & -\hat{a}_2 \end{pmatrix}. \quad (3.35)$$

According to (2.3) and (2.4), we have

$$\begin{aligned}
\frac{da_1}{dt} &= -\frac{1}{2}b_1^2 - \frac{1}{2}\hat{b}_1^2, \\
\frac{d\hat{a}_1}{dt} &= -b_1\hat{a}_1, \\
\frac{db_1}{dt} &= -\frac{1}{4}(a_1 + a_2)b_1 + (\hat{a}_1 + \hat{a}_2)b_1, \\
\frac{d\hat{b}_1}{dt} &= -\frac{1}{4}b_1^2, \\
\frac{da_2}{dt} &= -\frac{1}{2}b_1^2, \\
\frac{d\hat{a}_2}{dt} &= -b_1\hat{a}_2 - \frac{1}{2}\hat{b}_2^2,
\end{aligned} \quad (3.36)$$

note that, $b_2 = \hat{b}_2 = 0$. From (3.36), if the initial value of $(a_1 + a_2)$ is positive, one can find $(a_1 + a_2) \to \infty$, then $b_1, \hat{b}_1$ increase bigger and faster, and the corresponding result is that $\hat{a}_1 \to \infty$. On the contrary, if the initial value of $a_1 + a_2 \to -\infty$, then $b_1$ and $\hat{b}_1 \to 0$.

The following we give some special values to $a_i, \hat{a}_i, b_i$ and $\hat{b}_i$ for $(i=1, 2)$, and keep one parameter $m$ in $L$, then we discuss its eigenvalues and eigenvectors of $L$ and their properties at the same time. While $a_1, \hat{a}_1$ and $\hat{b}_1$ are equal to 0, take $b_1$ and $\hat{a}_2$ are equal to 1, and take $a_2$ as a parameter, then we have

$$L_2 = \begin{pmatrix} 0 & -1 \\ 1 & -m \end{pmatrix}, \quad (3.37)$$

$$\hat{L}_2 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.38)$$

And what is more, we can get the eigenvalues and eigenvectors of the particular matrix, which the specific forms are given as follow,

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}(\sqrt{m^2 - 4} - m), \\
\lambda_2 &= \frac{1}{2}(\sqrt{m^2 - 4} + m), \\
\hat{\lambda}_1 &= 0, \\
\hat{\lambda}_2 &= -1,
\end{aligned} \quad (3.39)$$

$$\Phi_2^0 = \begin{pmatrix} \lambda_1 & \lambda_2 \\ -1 & -1 \end{pmatrix}, \quad (3.40)$$

$$\hat{\Phi}_2^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (3.41)$$

where $\lambda_i$ and $\hat{\lambda}_i (i = 1, 2)$ are characteristic value. Then we discuss the value of $m$ below, different values of $m$ produce different results: (1) when $m \geq 2$, then $0 \geq \lambda_1 \geq \lambda_2$; (2)
when \(|m| < 2\), then \(\lambda_1\) and \(\lambda_2\) are complex; (3) when \(m \leq -2\), then \(\lambda_0 > \lambda_1 > 0, \lambda_2\) and \(\lambda_1\) are real, that is exactly the cases we need. According to (3.39), (3.40) and (3.41), we can get some specific results:

\[
\hat{\Phi}_2(t) = \frac{1}{e^{(\lambda_2+2\lambda_1)t}} \begin{pmatrix}
\lambda_1 e^{\lambda_1 t} & \lambda_2 e^{\lambda_2 t} \\
-e^{\lambda_1 t} & -e^{\lambda_2 t}
\end{pmatrix},
\]

(3.42)

\[
\hat{\Phi}_2(t) = \left( \frac{2\lambda_1+1}{e^{\frac{\lambda_1}{2}} t^\frac{1}{2}} + \frac{2\lambda_1+\lambda_0 e^{\lambda_1 t}}{(2e^{\lambda_1 t})^\frac{1}{2}} \right) - \left( \frac{2\lambda_1+1}{e^{\frac{\lambda_1}{2}} t^\frac{1}{2}} + \frac{2\lambda_1+\lambda_0 e^{\lambda_1 t}}{(2e^{\lambda_1 t})^\frac{1}{2}} \right) e^t
\]

\[
\Phi_2(t) = \left( \frac{2\lambda_1+1}{e^{\frac{\lambda_1}{2}} t^\frac{1}{2}} + \frac{2\lambda_1+\lambda_0 e^{\lambda_1 t}}{(2e^{\lambda_1 t})^\frac{1}{2}} \right) - \left( \frac{2\lambda_1+1}{e^{\frac{\lambda_1}{2}} t^\frac{1}{2}} + \frac{2\lambda_1+\lambda_0 e^{\lambda_1 t}}{(2e^{\lambda_1 t})^\frac{1}{2}} \right) e^t
\]

(3.43)

where \(\hat{\Phi}_{i,j}(t)(1 \leq i, j \leq 2)\) are obtained from (3.43). The solutions of the extended Toda equations are obtained from (2.19) and (2.20),

\[
L_2(t) = \begin{pmatrix}
\lambda_2 e^{2(\lambda_1+\lambda_2)t} + \lambda_1 e^{4\lambda_1 t} & -2(\lambda_1+\lambda_2)e^{4\lambda_1 t} - e^{4\lambda_1 t} \\
-e^{2(\lambda_1+\lambda_2)t} - e^{4\lambda_1 t} & \lambda_1 e^{2(\lambda_1+\lambda_2)t} + \lambda_2 e^{4\lambda_1 t}
\end{pmatrix},
\]

(3.44)

\[
\tilde{L}_2(t) = \frac{1}{e^{(\lambda_2+2\lambda_1)t}} \begin{pmatrix}
\tilde{a}_{1,1}(t) & \tilde{a}_{1,2}(t) \\
\tilde{a}_{2,1}(t) & \tilde{a}_{2,2}(t)
\end{pmatrix},
\]

(3.45)

where

\[
\tilde{a}_{1,1}(t) = [\lambda_1 e^{(3\lambda_1+2\lambda_2)t} + \lambda_1 e^{4\lambda_1 t}]\hat{\Phi}_{2,1}(t) - [e^{(3\lambda_1+2\lambda_2)t} + \lambda_1^2 e^{5\lambda_1 t}]\hat{\Phi}_{1,1}(t) + [\lambda_2 e^{(3\lambda_1+2\lambda_2)t} + \lambda_1 e^{5\lambda_1 t}]\hat{\Phi}_{1,2}(t)
\]

\[
\tilde{a}_{1,2}(t) = [\lambda_2 e^{(3\lambda_2+2\lambda_1)t} + \lambda_2 e^{4\lambda_1 t}]\hat{\Phi}_{2,1}(t) - [e^{(3\lambda_2+2\lambda_1)t} + \lambda_2^2 e^{5\lambda_1 t}]\hat{\Phi}_{1,1}(t) + [\lambda_2 e^{(3\lambda_2+2\lambda_1)t} + \lambda_2 e^{4\lambda_1 t}]\hat{\Phi}_{1,2}(t)
\]

\[
\tilde{a}_{2,1}(t) = [\lambda_1 e^{(3\lambda_1+2\lambda_2)t} + \lambda_1 e^{5\lambda_1 t}]\hat{\Phi}_{2,1}(t) - [e^{(3\lambda_1+2\lambda_2)t} + \lambda_1^2 e^{5\lambda_1 t}]\hat{\Phi}_{1,1}(t) + [\lambda_1 e^{(3\lambda_1+2\lambda_2)t} + \lambda_1 e^{5\lambda_1 t}]\hat{\Phi}_{1,2}(t)
\]

\[
\tilde{a}_{2,2}(t) = [\lambda_2 e^{(3\lambda_2+2\lambda_1)t} + \lambda_2 e^{4\lambda_1 t}]\hat{\Phi}_{2,1}(t) - [e^{(3\lambda_2+2\lambda_1)t} + \lambda_2^2 e^{5\lambda_1 t}]\hat{\Phi}_{1,1}(t) + [\lambda_2 e^{(3\lambda_2+2\lambda_1)t} + \lambda_2 e^{4\lambda_1 t}]\hat{\Phi}_{1,2}(t),
\]

(3.46)

In addition, according to (3.43), there is a situation that is \(m = \pm 2\), one of the things to pay attention is that \(L_2(t) \mapsto \pm \text{diag}(1,1)\).

## 4 Strongly coupled Toda lattices with indefinite metrics

In this section, we introduce a new strongly coupled Toda lattices with indefinite metrics. For the Hamiltonian (1.1), we give a extended transformation of variables,

\[
\begin{cases}
\forall_k \theta_k = \frac{\theta_k}{2}, \\
\forall_k \theta_k = -\frac{\theta_k}{2}, k = 1, \ldots, N,
\end{cases}
\]

(1.1)

\[
\begin{cases}
b_k = \frac{1}{2} \exp\left(\frac{x_k-x_{k+1}}{2}\right) \cosh\left(\frac{x_k-x_{k+1}}{2}\right), \\
b_k = \frac{1}{2} \exp\left(\frac{x_k-x_{k+1}}{2}\right) \sinh\left(\frac{x_k-x_{k+1}}{2}\right), k = 1, \ldots, N - 1.
\end{cases}
\]

(4.2)
In addition, the strongly coupled Toda lattices with indefinite metrics can be expressed as:

\[
\begin{aligned}
\frac{da_k}{dt} &= \left[ s_{k+1} (b_k^2 + \tilde{b}_k^2) - s_{k-1} (\tilde{b}_{k-1}^2 + \tilde{b}_{k-1}^2) \right], \\
\frac{db_k}{dt} &= 2s_{k+1}b_kb_k + 2s_{k-1}\tilde{b}_{k-1}b_{k-1},
\end{aligned}
\] (4.3)

\[
\begin{aligned}
\frac{d\tilde{a}_k}{dt} &= \frac{1}{2} [s_{k+1} (b_k\tilde{a}_{k+1} + \tilde{b}_k\tilde{a}_{k+1}) - s_k (b_k\tilde{a}_k + \tilde{b}_k\tilde{a}_k)], \\
\frac{d\tilde{b}_k}{dt} &= \frac{1}{2} [s_{k+1} (\tilde{b}_k\tilde{a}_{k+1} + b_k\tilde{a}_{k+1}) - s_k (\tilde{b}_k\tilde{a}_k + b_k\tilde{a}_k)],
\end{aligned}
\] (4.4)

where \(b_0 = \tilde{b}_0 = b_N = \tilde{b}_N = 0\). According to the strongly coupled Toda lattices with indefinite metrics above, we can use Lax pair to express it as following,

\[
\begin{aligned}
\frac{d}{dt} L &= [B, L] + [\tilde{B}, \tilde{L}], \\
\frac{d}{dt} \tilde{L} &= [\tilde{B}, L] + [B, \tilde{L}],
\end{aligned}
\] (4.5)

where \(L, \tilde{L}\) have the following form:

\[
L = \begin{pmatrix}
  s_1a_1 & 0 & s_2b_1 & 0 & \cdots & 0 & 0 \\
  0 & s_1a_1 & 0 & s_2b_1 & 0 & \cdots & 0 \\
  s_1\tilde{a}_1 & 0 & s_2\tilde{b}_1 & 0 & \cdots & 0 & 0 \\
  0 & s_1\tilde{a}_1 & 0 & s_2\tilde{b}_1 & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & \cdots & \cdots & \cdots & \cdots & s_{N-1}a_{N-1} & 0 \\
  0 & \cdots & \cdots & \cdots & \cdots & 0 & s_{N-1}a_N
\end{pmatrix},
\] (4.6)

\[
\tilde{L} = \begin{pmatrix}
  0 & s_1\tilde{a}_1 & 0 & s_2\tilde{b}_1 & 0 & \cdots & 0 \\
  s_1\tilde{a}_1 & 0 & s_2\tilde{b}_1 & 0 & \cdots & 0 & 0 \\
  0 & s_1\tilde{a}_1 & 0 & s_2\tilde{b}_1 & 0 & \cdots & 0 \\
  s_1b_1 & 0 & s_2\tilde{a}_2 & 0 & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & \cdots & \cdots & \cdots & \cdots & s_{N-1}\tilde{b}_{N-1} & 0 \\
  0 & \cdots & \cdots & \cdots & \cdots & 0 & s_{N-1}\tilde{a}_N
\end{pmatrix},
\] (4.7)

and \(B, \tilde{B}\) are given by

\[
\begin{aligned}
B := \frac{1}{2} [(L)_{>0} - (L)_{<0}], \\
\tilde{B} := \frac{1}{2} [(\tilde{L})_{>0} - (\tilde{L})_{<0}].
\end{aligned}
\] (4.8)

For Lax equations (4.5), we get some linear equations,

\[
\begin{aligned}
L\Phi + \tilde{L}\bar{\Phi} &= \Phi\Lambda, \\
\tilde{L}\Phi + L\bar{\Phi} &= \bar{\Phi}\Lambda, \\
\frac{d}{dt}\Phi &= B\Phi + \tilde{B}\bar{\Phi}, \\
\frac{d}{dt}\bar{\Phi} &= \tilde{B}\Phi + B\bar{\Phi},
\end{aligned}
\] (4.9)

where \(\Phi\) is the eigenmatrix of \(L\), \(\bar{\Phi}\) is the eigenmatrix of \(\tilde{L}\), and \(\Lambda\) is a diagonal matrix. Thus, \(\Phi\) and \(\bar{\Phi}\) satisfy some relationships:

\[
\begin{aligned}
\Phi S^{-1}\Phi^T + \bar{\Phi} S^{-1}\bar{\Phi}^T &= S^{-1}, \\
\bar{\Phi} S^{-1}\Phi^T + \Phi S^{-1}\bar{\Phi}^T &= 0, \\
\Phi^T S\Phi + \bar{\Phi}^T S\bar{\Phi} &= S, \\
\bar{\Phi}^T S\Phi + \Phi^T S\bar{\Phi} &= 0,
\end{aligned}
\] (4.10)
where
\[
\Phi = [\phi_i(\lambda_j)]_{1 \leq i,j \leq N}, \quad \Phi = [\phi_i(\lambda_j)]_{1 \leq i,j \leq N}.
\]

Form the equations of (4.10), one can get the relationships,
\[
\begin{align*}
\sum_{k=1}^{N} s_{k}^{-1} [\phi_i(\lambda_k) \phi_j(\lambda_k) + \tilde{\phi}_i(\lambda_k) \tilde{\phi}_j(\lambda_k)] &= \delta_{ij} s_i^{-1}, \\
\sum_{k=1}^{N} s_{k}^{-1} [\tilde{\phi}_i(\lambda_k) \phi_j(\lambda_k) + \phi_i(\lambda_k) \tilde{\phi}_j(\lambda_k)] &= 0, \\
\sum_{k=1}^{N} s_k [\phi_i(\lambda_i) \phi_j(\lambda_j) + \tilde{\phi}_i(\lambda_i) \tilde{\phi}_j(\lambda_i)] &= \delta_{ij} s_i, \\
\sum_{k=1}^{N} s_k [\tilde{\phi}_i(\lambda_i) \phi_j(\lambda_j) + \phi_i(\lambda_i) \tilde{\phi}_j(\lambda_i)] &= 0.
\end{align*}
\]

So, the extended matrices of \(L\) and \(\tilde{L}\) are expressed by
\[
\begin{align*}
a_{ij} := (L)_{ij} &= s_j \langle \lambda \phi_i \phi_j + \lambda \tilde{\phi}_i \tilde{\phi}_j \rangle, \\
\tilde{a}_{ij} := (\tilde{L})_{ij} &= s_j \langle \lambda \phi_i \phi_j + \lambda \phi_i \phi_j \rangle.
\end{align*}
\]

From the inverse scattering method, two new explicit forms of \(\Phi(t)\), \(\tilde{\Phi}(t)\) are given as following
\[
\begin{align*}
\phi_i(\lambda, t) &= M egin{bmatrix} 1 & \cdots & s_{i-1} c_{1,i-1} \phi_0^i \end{bmatrix} + M \sum_{q=0}^{i-1} \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} |s_j c_{k_1}^i| \cdots |s_{i-1} c_{i,i-1}^i| \phi_0^i, \\
\tilde{\phi}_i(\lambda, t) &= \tilde{M} egin{bmatrix} 1 & \cdots & s_{i-1} c_{1,i-1} \phi_0^i \end{bmatrix} + \tilde{M} \sum_{q=0}^{i-1} \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} |s_j c_{k_1}^i| \cdots |s_{i-1} c_{i,i-1}^i| \phi_0^i, \\
\text{where } M &= \frac{H_0 e^{\lambda t}}{H_0 - H_1^2}, \quad \tilde{M} = -\frac{H_1 e^{\lambda t}}{H_0 - H_1^2};
\end{align*}
\]

\[
\begin{align*}
\phi_i(\lambda, t) &= M egin{bmatrix} 1 & \cdots & s_{i-1} c_{1,i-1} \phi_0^i \end{bmatrix} + M \sum_{q=0}^{i-1} \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} |s_j c_{k_1}^i| \cdots |s_{i-1} c_{i,i-1}^i| \phi_0^i, \\
\tilde{\phi}_i(\lambda, t) &= \tilde{M} egin{bmatrix} 1 & \cdots & s_{i-1} c_{1,i-1} \phi_0^i \end{bmatrix} + \tilde{M} \sum_{q=0}^{i-1} \sum_{j=1}^{\lfloor \frac{i}{2} \rfloor} |s_j c_{k_1}^i| \cdots |s_{i-1} c_{i,i-1}^i| \phi_0^i,
\end{align*}
\]

\[
\begin{align*}
\text{where } H_0 &= \frac{\sqrt{a b}}{\sqrt{a + (a^2 - b^2)^{\frac{1}{2}}}}, \quad H_1 = \frac{\sqrt{a + (a^2 - b^2)^{\frac{1}{2}}}}{\sqrt{2}}, \quad a = D_i(t) D_{i-1}(t) + \tilde{D}_i(t) \tilde{D}_{i-1}(t), \quad b = \tilde{D}_i(t) D_{i-1}(t) + D_i(t) \tilde{D}_{i-1}(t),
\end{align*}
\]

\[
\tilde{D}_k(t) = \sum_{q=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor} |s_j c_{k_1}^i| \cdots |s_j c_{k_p}^i| \cdots |s_{i-1} c_{1,i-1}^i|,
\]

(4.16)
\[
D_k(t) = \sum_{q=0}^{[\frac{d}{2}]} \sum_{j=2q} \begin{bmatrix}
\beta_1 c_{11} & \cdots & \beta_p c_{1p} & \cdots & \beta_1 c_{1i} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
\beta_1 c_{j1} & \cdots & \beta_p c_{jp} & \cdots & \beta_1 c_{ji}
\end{bmatrix},
\]
(4.17)

and \(c_{ij}^{[kp]} = c_{ij}, k_p = 0 \) and \(c_{ij}, k_p = 1\). Thus we obtain the solutions (4.14) and (4.15) for the problem (4.9).

In fact, for the matrices \(L_i, \bar{L}_i\), the determinants \(D_i(t), \bar{D}_i(t)\) can be written with \(\tau\)-functions, then \(\alpha_i, \bar{\alpha}_i, \beta_i\) and \(\bar{\beta}_i\) are expressed as

\[
\begin{align*}
\alpha_i &= s_i a_i = \frac{d}{dt} \log \left( \frac{\tau_{i-1,1} - \tau_{i,1}}{\tau_{i-1,1} + \tau_{i,1}} \right), \\
\bar{\alpha}_i &= s_i \bar{a}_i = \frac{1}{2} \frac{d}{dt} \log \left( \frac{\tau_{i+1,1} - \tau_{i,1}}{\tau_{i+1,1} + \tau_{i,1}} \right), \\
\beta_i &= s_i s_{i+1} (b_{i+1}^2 + \bar{b}_{i+1}^2) = \left( \tau_{i+1,1} + \tau_{i+1,1} \right) \left( \tau_{i+1,1} + \tau_{i+1,1} \right) - 2 \left( \tau_{i+1,1} + \tau_{i+1,1} \right) \tau_{i,1}, \\
\bar{\beta}_i &= 2 s_i s_{i+1} b_{i+1} \bar{b}_i = \left( \tau_{i+1,1} + \tau_{i+1,1} \right) \left( \tau_{i+1,1} + \tau_{i+1,1} \right) - 2 \left( \tau_{i+1,1} + \tau_{i+1,1} \right) \tau_{i,1}.
\end{align*}
\]
(4.18)

The strongly coupled Toda equations also can be expressed by

\[
\begin{align*}
\tau_i \tau_i'' + \tau_i \tau_i'' - (\tau_i')^2 + (\tau_i')^2 &= \tau_{i+1,1} + \tau_{i+1,1}, \\
\bar{\tau}_i \bar{\tau}_i'' + \bar{\tau}_i \bar{\tau}_i'' - 2 \bar{\tau}_i \bar{\tau}_i'' &= \tau_{i+1,1} + \tau_{i+1,1}.
\end{align*}
\]
(4.19)

From [12][13], the \(\tau\)-functions are written in the form with a simple structure,

\[
\tau_i = \sum_{q=0}^{[\frac{d}{2}]} \sum_{j=1}^{i} \begin{bmatrix}
\tau_{1,[k_1]} & \cdots & \tau_{1,[k_{i-1}]} \\
\tau_{1,[k_1]} & \cdots & \tau_{1,[k_{i-1}]}
\end{bmatrix},
\]
(4.20)

\[
\bar{\tau}_i = \sum_{q=0}^{[\frac{d}{2}]} \sum_{j=1}^{i} \begin{bmatrix}
\tau_{1,[k_1]} & \cdots & \tau_{1,[k_{i-1}]} \\
\tau_{1,[k_1]} & \cdots & \tau_{1,[k_{i-1}]}
\end{bmatrix},
\]
(4.21)

Similarly, \(\tau_{1,[kp]}^{(i)} = \begin{cases} 
\tau_{1,[kp]}^{(i)}, k_p = 0 \\
\tau_{1,[kp]}^{(i)}, k_p = 1
\end{cases}\)

where \(\tau_1 = c_{11} := < (\phi_0^0)^2 + (\bar{\phi}_0^0)^2 > e^{\lambda t}, \)

\(\bar{\tau}_1 = \bar{c}_{11} := < 2\phi_0^1 \bar{\phi}_0^1 e^{\lambda t} >\), and \(\tau_1^{(i)} = \frac{d^i \tau_1}{dt^i}, \bar{\tau}_1^{(i)} = \frac{d^i \bar{\tau}_1}{dt^i}\). Therefore, the relationships between \(\beta_i, \bar{\beta}_i, D_i\) and \(\bar{D}_i\) are given by

\[
\begin{align*}
\tau_i &= \frac{1}{2s_i} \prod_{k=1}^{i} \left( \beta_k + \bar{\beta}_k \right) - k \left( D_i + \bar{D}_i \right) + \left( \beta_k - \bar{\beta}_k \right) - k \left( D_i - \bar{D}_i \right), \\
\bar{\tau}_i &= \frac{1}{2s_i} \prod_{k=1}^{i} \left( \beta_k + \bar{\beta}_k \right) - k \left( D_i + \bar{D}_i \right) + \left( \beta_k - \bar{\beta}_k \right) - k \left( D_i - \bar{D}_i \right).
\end{align*}
\]
(4.22)

5 Weakly coupled \(Z_n\)-Toda lattices with indefinite metrics

In the next part, we will give a new finite nonperiodic \(Z_n\)-Toda lattices with indefinite metrics as following.
Definition 1. According to (2.3), we define finite nonperiodic $Z_n$-Toda lattice equations with indefinite metrics as:

\[
\begin{align*}
\frac{da_{k,l}}{dt} &= s_{k+1} \sum_{p+q=l+1} b_{k,p} b_{k,q} - s_{k-1} \sum_{p+q=l+1} b_{k-1,p} b_{k-1,q}, \\
\frac{db_{k,l}}{dt} &= \frac{1}{2} (s_{k+1} \sum_{p+q=l+1} b_{k,p} a_{k+1,q} - s_{k} \sum_{p+q=l+1} b_{k,p} b_{k,q}).
\end{align*}
\tag{5.1}
\]

When $l = 1$, $a_{k,l}$ and $b_{k,l}$ are equivalent to $a_{k}$ and $b_{k}$ [2]. In fact, before defining finite nonperiodic $Z_n$-Toda lattice equations, we introduce a more general transformation of variables, the specific transformation is given by

\[
\begin{align*}
s_k a_{k,l} &= -\frac{y_{k,l}}{2}, (k = 1, 2, \ldots, N), \\
b_{k,l} &= \frac{1}{2} \sum_{i_1 k_1 + \cdots + i_k k_l = k} \frac{x_{i_1 i_2 \ldots i_k}^{k_1 \ldots k_l}}{k_1! \cdots k_l!} \exp \left( \frac{x_{k,l} - x_{k+1,l}}{2} \right), (k = 1, 2, \ldots, N - 1),
\end{align*}
\tag{5.2}
\]

where $x_{k,l} = \frac{1}{2} (x_{k,l} - x_{k+1,l})$. Meanwhile, the $Z_n$-Hamilton quantity is as

\[
H_k = \frac{1}{2} \sum_{k=1}^{N} \sum_{p+q=k} y_{p,q} y_{k,q} + \sum_{i_1 k_1 + \cdots + i_k k_l = k} \frac{x_{i_1 i_2 \ldots i_k}^{k_1 \ldots k_l}}{k_1! \cdots k_l!} \exp (x_{k,1} - x_{k+1,1}),
\tag{5.3}
\]

where $x_{i,j} = x_{i,j} - x_{i+1,j}$. According to the definition of the finite nonperiodic $Z_n$-Toda lattice equations with indefinite metrics (5.1), we can obtain its Lax equations,

\[
\frac{d}{dt} L_k = \sum_{p+q=k+1} [B_p, L_q],
\tag{5.4}
\]

where

\[
L_k = \begin{pmatrix}
s_1 \sum a_{1,k} \Gamma^{k-1} & s_2 \sum b_{1,k} \Gamma^{k-1} & 0 & \cdots & 0 \\
0 & s_1 \sum b_{1,k} \Gamma^{k-1} & s_2 \sum a_{2,k} \Gamma^{k-1} & \cdots & 0 \\
0 & 0 & \ddots & \ddots & \ddots \\
0 & 0 & \cdots & s_N-1 \sum a_{N-1,k} \Gamma^{k-1} & s_N \sum b_{N-1,k} \Gamma^{k-1} \\
0 & 0 & \cdots & s_N-1 \sum b_{N-1,k} \Gamma^{k-1} & s_N \sum a_{N,k} \Gamma^{k-1}
\end{pmatrix},
\tag{5.5}
\]

and $B_k = \frac{1}{2} [(L_k)_{>0} - (L_k)_{<0}]$. According to the extended general variables, $a_{k,l}$, $b_{k,l}$, $a_{k,l}$ and $b_{k,l}$ are given by

\[
\begin{align*}
\alpha_{k,l} &= s_k a_{k,l}, \\
\beta_{k,l} &= \sum_{p+q=k+1} s_k s_{k+1} b_{p,q} b_{q,l},
\end{align*}
\tag{5.6}
\]

For (5.3), linear equations produced from the inverse scattering method can be expressed as:

\[
\begin{align*}
\frac{\partial}{\partial t} \Phi_k &= B_p \Phi_q, \\
\sum_{p+q=k+1} L_p \Phi_q &= \Lambda \Phi_k,
\end{align*}
\tag{5.7}
\]

where $\Phi_k$ is the eigenmatrix of $L_k$, and $\Phi_k \equiv [\phi^{[k]}(\lambda_1), \cdots, \phi^{[k]}(\lambda_N)] = [\phi^{[k]}_i(\lambda_j)]_{1 \leq i, j \leq N}$. Further more, $\Phi_k$ satisfies

\[
\begin{align*}
\Phi_k^T S \Phi_k &= S, \\
\sum_{p+q=k} \Phi_p^T S \Phi_q &= 0, \quad k = 2, 3, \ldots, N, \\
\Phi_1 S^{-1} \Phi_1^T &= S^{-1}, \\
\sum_{p+q=k} \Phi_p S^{-1} \Phi_q^T &= 0, \quad k = 2, 3, \ldots, N.
\end{align*}
\tag{5.8}
\]
From (5.8), one can get the “orthogonality” relations:

\[
\begin{align*}
&\sum_{k=1}^{N} s_k^{-1} \phi_i^{[1]}(\lambda_k)\phi_j^{[1]}(\lambda_k) = \delta_{ij} s_i^{-1}, \\
&\sum_{k=1}^{N} s_k^{-1} \left( \sum_{p+q=3}^{N} \phi_i^{[p]}(\lambda_k)\phi_j^{[q]}(\lambda_k) \right) = 0, \\
&\sum_{k=1}^{N} s_k \phi_i^{[1]}(\lambda_k)\phi_j^{[1]}(\lambda_j) = \delta_{ij} s_i, \\
&\sum_{k=1}^{N} s_k \left( \sum_{p+q=3}^{N} \phi_i^{[p]}(\lambda_k)\phi_j^{[q]}(\lambda_j) \right) = 0.
\end{align*}
\]

(5.9)

So, the \( L_k \) can be expressed by

\[
L_k = \sum_{p+q=k+1} \Phi_p \Lambda \Phi_q^T.
\]

(5.10)

According to the proof of section 3, the specific form of \( \phi_i^{[k]}(\lambda, t) \) can be given as

\[
\begin{align*}
\phi_i^{[k]}(\lambda, t) &= D_0(t) \sum_{j=1}^{N-1} \left( \sum_{[p_1+\cdots+p_{i-1}-(i-1)]=j} \begin{vmatrix}
\phi_i^{[k_p]}(\lambda) & \cdots & \phi_i^{[k_{p_{i-1}}]}(\lambda) \\
\phi_i^{[k]}(\lambda) & \cdots & \phi_i^{[k]}(\lambda) \\
\ddots & \ddots & \ddots \\
\phi_i^{[k]}(\lambda) & \cdots & \phi_i^{[k]}(\lambda)
\end{vmatrix} \\
+ D_1(t) \sum_{j=1}^{N-1} \left( \sum_{[p_1+\cdots+p_{i-1}-(i-1)]=j} \begin{vmatrix}
\phi_i^{[k_p]}(\lambda) & \cdots & \phi_i^{[k_{p_{i-1}}]}(\lambda) \\
\phi_i^{[k]}(\lambda) & \cdots & \phi_i^{[k]}(\lambda) \\
\ddots & \ddots & \ddots \\
\phi_i^{[k]}(\lambda) & \cdots & \phi_i^{[k]}(\lambda)
\end{vmatrix} \\
+ \cdots + D_{n-1}(t) \begin{vmatrix}
\phi_i^{[k]}(\lambda) & \cdots & \phi_i^{[k]}(\lambda) \\
\phi_i^{[k]}(\lambda) & \cdots & \phi_i^{[k]}(\lambda) \\
\ddots & \ddots & \ddots \\
\phi_i^{[k]}(\lambda) & \cdots & \phi_i^{[k]}(\lambda)
\end{vmatrix}
\right),
\end{align*}
\]

(5.11)

(5.12)

(5.13)

where \( D_i(t) \) is the \( n \)-th order Frobenius form of \( \frac{e^{\lambda t}}{\sqrt{D_i(t)D_{i-1}(t)}} \), and the specific forms are given by \( D_i(t) = \sum_{k_0+k_1+\cdots+k_{i-1}+k_{i}=i-1} \frac{(-1)^i v_{k_0} v_{k_1} \cdots v_{k_{i-1}}}{v_0} e^{\lambda t} \). For \( v_i \) above, we used a variable replacement: \( u_i = \sum_{p+q=i+1} D_{i,p}(t)D_{i-1,q}(t) \), so we can get the relationship between \( v_k \) and \( u_k \) by iterative methods.

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References

[1] C. Z. Li, J. S. He, The extended \( Z_N \)-Toda hierarchy, Theoretical and Mathematical Physics. 185 (2015), 1614-1635.

[2] Y. Kodama, J. Ye, Toda lattices with indefinite metric II: Topology of the iso-spectral manifolds, Physica D. 121 (1998), 89-108.
[3] H. Flaschka, L. Haine, Variétés de drapeaux et réseaux de Toda, Mathematical Z. 208 (1991), 545-556.

[4] H. Flaschka, L. Haine, Toda orbits in G/P, Pacific Journal Mathematics. (1991), 251-292.

[5] Y. Kodama, J. Ye, Iso-spectral deformations of general matrix and their reductions on Lie algebras, Communications in Mathematical Physics. 178 (1996), 765-788.

[6] M. Jimbo, T. Miwa, Solitons and Infinite Dimensional Lie Algebras, Publications of the Research Institute for Mathematical Sciences 19 (1983), 943-1001.

[7] J. Moser, Finitely many mass points on the line under the influence of an exponential potential—an integrable system in dynamical systems, Theory and Applications. Lecture Notes in Physics. 38 (1975), 467-497.

[8] Y. Kodama, J. Ye, Toda lattice with Indefinite metric. Physica D. 91 (1996), 321-339.

[9] H. Flaschka, On the Toda lattice II, Physical Review B. 51 (1974), 703-716.

[10] G. Szegő, Orthogonal Polynomials, AMS Colloquium Publications. 23 (1939).

[11] Y. Kodama and K. Mclaughlin, Explicit integration of the full symmetric toda hierarchy and the sorting property, Letters in Mathematical Physicas. 37 (1995), 37-47.

[12] Yu. M. Berezanskii, Integration of semi-infinite Toda chain by means of the inverse spectral problem, Reports on Mathematical Physics. 24 (1986), 21-47.

[13] Y. Nakamura and Y. Kodama, Moment of Hamburger, hierarchies of interable systems, and the positivity of tau-functions, Acta Applicandae Mathematicae. 39 (1995), 435-443.

[14] Y. Nakamura, A tau-function for the finite nonperiodic Toda lattice, Physics Letters A 195 (1994), 346-350.