A commutative $\mathbb{P}^1$-spectrum representing motivic cohomology over Dedekind domains

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Abstract

We construct a motivic Eilenberg-MacLane spectrum with a highly structured multiplication over smooth schemes over Dedekind domains which represents Levine’s motivic cohomology. The latter is defined via Bloch’s cycle complexes. Our method is by gluing $p$-completed and rational parts along an arithmetic square. Hereby the finite coefficient spectra are obtained by truncated étale sheaves (relying on the now proven Bloch-Kato conjecture) and a variant of Geisser’s version of syntomic cohomology, and the rational spectra are the ones which represent Beilinson motivic cohomology.

As an application the arithmetic motivic cohomology groups can be realized as Ext-groups in a triangulated category of Tate sheaves with integral coefficients. These can be modelled as representations of derived fundamental groups.

Our spectrum is compatible with base change giving rise to a formalism of six functors for triangulated categories of motivic sheaves over general base schemes including the localization triangle.

Further applications include a generalization of the Hopkins-Morel isomorphism and a structure result for the dual motivic Steenrod algebra in the case where the coefficient characteristic is invertible on the base scheme.

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1 Introduction

This paper furnishes the construction of a motivic Eilenberg-MacLane spectrum in mixed characteristic. Our main purpose is to construct from that spectrum triangulated categories of motivic sheaves with integral (and thus also arbitrary) coefficients over general base schemes which satisfy properties which are reminiscent of properties of triangulated categories of motives which have already been constructed. In [6] a theory of Beilinson motives is developed yielding a satisfying theory of motives with rational coefficients over general base schemes. Voevodsky constructed triangulated categories of motives over a (perfect) field ([11], [31]) in which motivic cohomology of smooth schemes is represented. In the Cisinski-Deglise category over a regular base the resulting motivic cohomology are Adams-graded pieces of rationalized $K$-theory, which fits with the envisioned theory of Beilinson. In [35] modules over the motivic Eilenberg-MacLane spectrum over a field are considered and it is proved that those are equivalent to Voevodsky’s triangulated categories of motives in the characteristic 0 case. This result has recently been generalized to perfect fields [21] where one has to invert the characteristic of the base field in the coefficients. Étale motives are developed in [1] and [5].

We build upon these works and construct motivic categories using motivic stable homotopy theory. More precisely we define objects with a (coherent) multiplication in the category of $\mathbb{P}^1$-spectra over base schemes and consider as in [35] their module categories. The resulting homotopy categories are defined to be the categories of motivic sheaves.

This family of commutative ring spectra is cartesian, i.e. for any map between base schemes $X \to Y$ the pullback of the ring spectrum over $Y$ compares via an isomorphism (in the the homotopy category of ring spectra) to the ring spectrum over $X$. This is equivalent to saying that all spectra pull back from $\text{Spec}(\mathbb{Z})$.

To ensure good behavior of our construction our spectra have to satisfy a list of desired properties. Over fields the spectra coincide with the usual motivic Eilenberg-MacLane spectra (this ensures that over fields usual motivic cohomology is represented in our categories of motivic sheaves). Rationally we recover the theory of Beilinson motives, because the rationalizations of our spectra are isomorphic to the respective Beilinson spectra, and there is a relationship to Levine’s motivic cohomology defined using Bloch’s cycle complexes in mixed characteristic ([20]).

To ensure all of that we first construct a spectrum over any Dedekind domain $D$ of mixed characteristic satisfying the following properties: It represents Bloch-Levine’s motivic cohomology of smooth schemes over $D$ (Corollary [7,19], it pulls back to the
usual motivic Eilenberg-MacLane spectrum with respect to maps from spectra of fields to the spectrum of $D$ (Theorem 9.16) and it is an $E_\infty$-ring spectrum. (We remark that such an $E_\infty$-structure can always be strictified to a strict commutative monoid in symmetric $\mathbb{P}^1$-spectra by results of [17].)

The latter property makes it possible to consider the category of highly structured modules over pullbacks of the spectrum from the terminal scheme (the spectrum of the integers), thus defining triangulated categories of motivic sheaves $\text{DM}(X)$ over general base schemes $X$ such that over smooth schemes over Dedekind domains of mixed characteristic the Ext-groups compute Bloch-Levine’s motivic cohomology (Corollary 7.20). For general base schemes we define motivic cohomology to be represented by our spectrum, i.e.

$$H^i_{\text{mot}}(X, \mathbb{Z}(n)) \coloneqq \text{Hom}_{\text{SH}(X)}(1, \Sigma^{i,n} f^* \mathcal{M}_{\text{Spec} \mathbb{Z}}) \cong \text{Hom}_{\text{DM}(X)}(\mathbb{Z}(0), \mathbb{Z}(n)[i]).$$

Here $f : X \to \text{Spec} \mathbb{Z}$ is the structure morphism, $\mathcal{M}_{\text{Spec} \mathbb{Z}}$ is our spectrum over the integers and $1$ is the sphere spectrum (the unit with respect to the smash product) in the stable motivic homotopy category $\text{SH}(X)$. By the base change property these cohomology groups coincide with Voevodsky’s motivic cohomology if $X$ is smooth over a field. We note that the ring structure on our Eilenberg-MacLane spectrum gives the (bi-graded) motivic cohomology groups a (graded commutative) ring structure, a property which was (to the knowledge of the author) missing for Levine’s motivic cohomology.

By the work of Ayoub [2] the base change property enables one to get a full six functor formalism for these categories of motivic sheaves including the localization triangle (Theorem 10.1).

We remark that the spectrum we obtain gives rise to motivic complexes over any base scheme $X$. More precisely one can extract objects $\mathbb{Z}(n)^X$ in the derived category of Zariski sheaves on the category of smooth schemes over $X$ representing our motivic cohomology. There are unital, associative and commutative multiplication maps $\mathbb{Z}(n)^X \otimes^L \mathbb{Z}(m)^X \to \mathbb{Z}(n+m)^X$ inducing the multiplication on motivic cohomology. (These multiplications are in fact part of a graded $E_\infty$-structure (which follows from the existence of the strong periodization, see section 8), but we do not make this explicit since we have no application for this enhanced structure.) If $X$ is a smooth scheme over a Dedekind domain of mixed characteristic or over a field we have isomorphisms $\mathbb{Z}(0)^X \cong \mathbb{Z}$ and $\mathbb{Z}(1)^X \cong \mathcal{O}^*_X[-1]$ (for the latter isomorphism see Theorem 7.10).

One can also extract motivic Eilenberg-MacLane spaces. If $X$ is as above there are isomorphisms $K(\mathbb{Z}(1), 2)_X \cong \mathbb{P}_X^\infty$ (Proposition 11.7) and $K(\mathbb{Z}/n(1), 1)_X \cong W_{X,n}$ (Proposition 11.8) in the motivic pointed homotopy category $\mathcal{H}_*(X)$ of $X$. Here $W_{X,n}$
is the total space of the line bundle $\mathcal{O}_{\mathbb{P}^\infty_X}(-n)$ on $\mathbb{P}^\infty_X$ with the zero section removed (a motivic lens space).

Among our applications is a generalization of the Hopkins-Morel isomorphism (Theorem 11.3), relying on the recent work of Hoyois ([20], which in turn relies on work of Hoyois-Kelly-Østvær [21]). In certain cases it follows that the Eilenberg-MacLane spectrum is cellular (Corollary 11.4). We obtain a description of the dual motivic Steenrod algebra over base schemes over which the coefficient characteristic is invertible (Theorem 11.24). We note that one can ask if the statement of this Theorem is valid over any base scheme (thus asking for a description of the smash product of the mod-$p$ motivic Eilenberg-MacLane spectrum with itself in characteristic $p$).

The outline of this paper is as follows. In section 3 we define motivic complexes over small sites and describe their main properties, most notably the localization sequence due to Levine (Theorem 3.1) and the relation to étale sheaves (where the Bloch-Kato conjecture enters) (Theorem 3.9).

In section 4 an $E_\infty$-spectrum $M\mathbb{Z}$ is constructed with the main property that it represents motivic cohomology with finite coefficients (which follows from Corollary 4.1.2) and is rationally isomorphic to the Beilinson spectrum.

For the definition we use an arithmetic square, i.e. we first define $p$-completed spectra for all prime numbers $p$ and glue their product along the rationalization of this product to the Beilinson spectrum (Definition 4.27).

The spectra with finite $p$-power coefficients which define the $p$-completed parts are constructed using truncated étale sheaves outside characteristic $p$ and logarithmic de Rham-Witt sheaves at characteristic $p$.

Our spectrum is constructed in the world of complexes of sheaves of abelian groups and spectrum objects therein. By transfer of structure this also defines ($E_\infty$- or commutative ring) spectra in the world of $\mathbb{P}^1$-spectra in motivic spaces.

In order to prove that $M\mathbb{Z}$ represents integrally Bloch-Levine’s motivic cohomology we define in section 5 a second motivic spectrum $\mathcal{M}$ which by definition represents Bloch-Levine’s integral motivic cohomology (and which will finally be isomorphic to $M\mathbb{Z}$). To do that we introduce a strictification process for Bloch-Levine’s cycle complexes to get a strict presheaf on smooth schemes over a Dedekind domain. Hereby we rely heavily on a moving Lemma due to Levine (Theorem 5.8). Using a localization sequence for the pair $(\mathbb{A}^1, \mathbb{G}_m)$ we obtain bonding maps arranging the motivic complexes into a $\mathbb{G}_m$-spectrum (see section 5.3). This section also contains the construction of an étale cycle class map (inspired by the construction in [26]) which is compatible with certain localization sequences (Proposition 5.2.3).
After treating motivic complexes over a field (section 6) we give our comparison statements in section 7. First we compute the exceptional inverse image of \( \mathcal{M} \) with respect to the inclusion of a closed point into our Dedekind scheme (Theorem 7.1). Theorem 7.14 states that the rationalization \( \mathcal{M}_Q \) is just the Beilinson spectrum. Our main comparison statement is Theorem 7.18 which asserts a canonical isomorphism between \( \mathcal{M} \) and \( \mathcal{M}_Z \) as spectra.

Our motivic Eilenberg-MacLane spectrum is strongly periodizable in the sense of \( \cite{40} \) (Theorem 8.2, Remark 11.2). This shows that geometric mixed Tate sheaves with integral coefficients over a number ring or similar bases which satisfy a weak version of the Beilinson-Soulé vanishing conjecture can be modelled as representations of an affine derived group scheme along the lines of \( \cite{38} \) (Corollary 8.4).

Section 9 discusses base change. Here the Bloch-Kato filtration on \( p \)-adic vanishing cycles plays a key role to obtain the part of base change where the characteristics of the base field and of the coefficients coincide.

We treat the motivic functor formalism in section 10. Section 11 contains the applications to the Hopkins-Morel isomorphism and the dual motivic Steenrod algebra.

Two appendices discuss (semi) model structures on sheaf categories and algebra objects therein and definitions and properties of pullbacks of algebraic cycles.

We finally remark that it should be possible to generalize our strictification process in section 5 to define a homotopy coniveau tower over Dedekind domains as in \( \cite{28} \). We will come back to this question in future work.

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2 Preliminaries and Notation

For a site $S$ and a category $C$ we denote by $\text{Sh}(S,C)$ the category of sheaves on $S$ with values in $C$. If $R$ is a commutative ring we set $\text{Sh}(S,R) := \text{Sh}(S,\text{Mod}_R)$, where $\text{Mod}_R$ denotes the category of $R$-modules.

For a Noetherian separated base scheme $S$ of finite Krull dimension we denote by $\text{Sch}_S$ the category of separated schemes of finite type over $S$ and by $\text{Sm}_S$ the full subcategory of $\text{Sch}_S$ of smooth schemes over $S$.

For $t \in \{\text{Zar}, \text{Nis}, \text{étel}\}$ we denote by $\text{Sm}_{S,t}$ the site $\text{Sm}_S$ equipped with the topology $t$.

For $S$ and $t$ as above we denote by $S_t$ the site consisting of the full subcategory of $\text{Sm}_S$ of étale schemes over $S$ equipped with the topology $t$.

If $m$ is invertible on $S$ we write $\mathbb{Z}/m(r)^S$ for the sheaf $\mu_m^r$ on $S_{ét}$. If it is clear from the context we also write $\mathbb{Z}/m(r)$.

We let $\epsilon: \text{Sm}_{S,ét} \to \text{Sm}_{S,\text{Zar}}$ and $\epsilon: S_{ét} \to S_{\text{Zar}}$ be the canonical maps of sites.

If $X$ is a presheaf of sets on $\text{Sm}_S$ we let $R[X]_t$ be the sheaf of $R$-modules on $\text{Sm}_{S,t}$ freely generated by $X$. If $Y \hookrightarrow X$ is a monomorphism we let $R[X,Y]_t := R[X]_t/R[Y]_t$.

For sections 3 through 8 of the paper we fix a Dedekind domain $D$ of mixed characteristic and set $S := \text{Spec}(D)$. For a prime $p$ we let $S[\frac{1}{p}] := \text{Spec}(D[\frac{1}{p}])$ and $Z_p \subset S$ the closed complement of $S[\frac{1}{p}]$ with the reduced scheme structure. Then $Z_p$ is a finite union of spectra of fields of characteristic $p$.

For $S'$ the spectrum of a Dedekind domain we let $\text{Sm}'_{S'}$ be the full subcategory of $\text{Sch}_{S'}$ of schemes $X$ over $S'$ such that each connected component of $X$ is either smooth over $S'$ or smooth over a closed point of $S'$.

For an $\mathbb{F}_p$-scheme $Y$ we let $W_n\Omega^*_Y$ be the De Rham-Witt complex of $Y$. It is a complex of sheaves on $Y_{ét}$ with a multiplication. These complexes assemble to a complex of sheaves on the category of all $\mathbb{F}_p$-schemes. There are canonical epimorphisms $W_{n+1}\Omega^*_Y \twoheadrightarrow W_n\Omega^*_Y$ respecting the multiplication.

For $Y$ as above let $\text{dlog}: \mathcal{O}_Y^* \to W_n\Omega^1_Y$ be defined by $x \mapsto \frac{dx}{x}$, where $x = (x,0,0,\ldots)$ is the Teichmüller representative of $x$.

The logarithmic De Rham-Witt sheaf $W_n\Omega^r_{Y,\log}$ is defined to be the subsheaf of $W_n\Omega^r_Y$ generated étale locally by sections of the form $\text{dlog}x_1\ldots\text{dlog}x_r$. Also $W_n\Omega^0_{Y,\log}$ is the constant sheaf on $\mathbb{Z}/p^n$.

These sheaves assemble to a subcomplex $W_n\Omega^r_{Y,\log}$ of $W_n\Omega^*_Y$.

The $W_n\Omega^r_{Y,\log}$ assemble to a sheaf $\nu'_n$ on the category of all $\mathbb{F}_p$-schemes. We set $\nu'_n = 0$ for $r < 0$. There are natural epimorphisms $\nu'_n \twoheadrightarrow \nu'_n$. 

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We will also denote restrictions of $\nu_r$ to certain sites, e.g. to $Y_{Zar}$ or $\text{Sm}_{k,t}$, $k$ some field of characteristic $p$, by $\nu_r$.  

If $\mathcal{A}$ is an abelian category we denote by $D(\mathcal{A})$ its derived category. We denote by $D^A_1(\text{Sh}(\text{Sm}_{S,t}, R))$ the $\mathbb{A}_1$-localization of $D(\text{Sh}(\text{Sm}_{S,t}, R))$.

We let $\mathcal{SH}(S)$ be the stable motivic homotopy category and $\mathcal{H}_*(S)$ the pointed $\mathbb{A}_1$-homotopy category of $S$.

We sometimes use the notation $f_*, f^*$ for a (non-derived or derived) push forward or pullback between sheaf categories corresponding to sites induced by a scheme morphism $f$. The precise sites which are used can always be read off from the source and target categories.

$E_\infty$-structures are understood with respect to (the image of) the linear isometries operad.

3 Motivic complexes I

Let $S'$ be the spectrum of a Dedekind domain. For $X \in \text{Sm}'_{S'}$ and $r \geq 0$ we denote by $\mathcal{M}^X(r) \in D(\text{Sh}(X_{Zar}, \mathbb{Z}))$ Levine’s cycle complex. A representative is the complex with $z^r(\_ 2r - i)$ in cohomological degree $i$, see [13, §3], [27]. For $r < 0$ we set $\mathcal{M}^X(r) = 0$.

When it is clear from the context which $X$ is meant we also write $\mathcal{M}(r)$. We also write $\mathcal{M}^X_{\text{ét}}(r)$ for $\mathcal{M}^X(r)$ and $\mathcal{M}^X(r)/m$ for $\mathcal{M}(r) \otimes \mathbb{Z}/m$.

Theorem 3.1: (Levine) Let $i: Z \to X$ be a closed inclusion in $\text{Sm}'_{S'}$ of codimension $c$ and $j: U \to X$ the complementary open inclusion. Then there is an exact triangle in $D(\text{Sh}(X_{Zar}, \mathbb{Z}))$

\[ \mathbb{R}i_* \mathcal{M}^Z(r - c)[-2c] \to \mathcal{M}^X(r) \to \mathbb{R}j_* \mathcal{M}^U(r) \to \mathbb{R}i_* \mathcal{M}^Z(r - c)[-2c + 1]. \]  

(1)

Proof. This is [27] Theorem 1.7.

Corollary 3.2: Let $i: Z \to X$ be a closed inclusion in $\text{Sm}'_{S'}$ of codimension $c$. Then there is a canonical isomorphism

\[ \mathbb{R}i^! \mathcal{M}^X(r) \cong \mathcal{M}^Z(r - c)[-2c] \]

in $D(\text{Sh}(X_{Zar}, \mathbb{Z}))$.

Theorem 3.3: For $X \in \text{Sm}'_{S'}$ we have $\mathcal{H}^k(\mathcal{M}^X(r)) = 0$ for $k > r$.

Proof. This is [13] Corollary 4.4.
Theorem 3.4: Suppose $X \in \text{Sm}_S'$ is of characteristic $p$. Then there is an isomorphism

$$\mathcal{M}^X(r)/p^n \cong \nu_n^r[-r]$$

in $D(\text{Sh}(X_{\text{Zar}}, \mathbb{Z}/p^n))$.

Proof. If $X$ is smooth over a perfect field this is [14, Theorem 8.3]. The general case follows by a colimit argument (using [22, I. (1.10.1)]). \hfill \Box

Corollary 3.5: Let $p$ be a prime, $X \in \text{Sm}_S$ and $\pi: X \to S$ the structure morphism. Let $i: Z := \pi^{-1}(Z_p) \to X$ be the closed and $j: U := \pi^{-1}(S[1/\frac{1}{p}]) \to X$ be the open inclusion. Then $H^k(\mathbb{R} j_* \mathcal{M}^U(r) \otimes^L \mathbb{Z}/p^n) = 0$ for $k > r$ and the natural map

$$H^r(\mathbb{R} j_* (\mathcal{M}^U(r)/p^n)) \to i_* \nu_n^{r-1}$$

induced by the triangle (1) and the isomorphism (2) is an epimorphism.

Proof. This follows from Theorem 3.3, the exactness of $i_*$ and the long exact sequence of cohomology sheaves induced by the exact triangle (1). \hfill \Box

Lemma 3.6: Suppose $X \in \text{Sm}_S'$ is of characteristic $p$. Then the diagram

$$\begin{array}{ccc}
\mathcal{M}^X(r)/p^{n+1} & \xrightarrow{=} & \nu_n^{r+1}[-r] \\
\downarrow & & \downarrow \\
\mathcal{M}^X(r)/p^n & \xrightarrow{=} & \nu_n^r[-r]
\end{array}$$

in $D(\text{Sh}(X_{\text{Zar}}, \mathbb{Z}/p^{n+1}))$ commutes.

Suppose $m$ is invertible on $X \in \text{Sm}_S'$. Then there is a cycle class map

$$\mathcal{M}^X(r)/m \to \mathbb{R} \epsilon_* \mathbb{Z}/m(r).$$

For a definition see the proof of Theorem 3.12. The étale sheafification of the cycle class map is an isomorphism in $D(\text{Sh}(X_{\text{ét}}, \mathbb{Z}/m))$, see [13, Theorem 1.2. 4].

Let $f: Y \to X$ be a flat morphism of schemes for which the motivic cycle complexes are defined. Then there is a flat pullback $f^* \mathcal{M}^X(r) \to \mathcal{M}^Y(r)$.
Lemma 3.7: Let $f: Y \to X$ be a flat morphism of schemes for which the motivic cycle complexes are defined. Suppose $m$ is invertible on $X$. Then the diagram

$$
\begin{array}{ccc}
\mathcal{M}^X(r)/m & \xrightarrow{f^*} & f^*\mathcal{R}_e\mathbb{Z}/m(r) \\
\downarrow & & \downarrow \\
\mathcal{M}^Y(r)/m & \xrightarrow{\mathcal{R}_e\mathbb{Z}/m(r)} & \mathcal{R}_e\mathbb{Z}/m(r)
\end{array}
$$

commutes.

Proof. This follows from the definition of the étale cycle class map. $\Box$

Lemma 3.8: Let $X \in \text{Sm}_{/S}'$ and suppose $m$ is invertible on $X$. Let $m'|m$. Then the diagram

$$
\begin{array}{ccc}
\mathcal{M}^X(r)/m & \xrightarrow{\mathcal{R}_e\mathbb{Z}/m(r)} \\
\downarrow & & \downarrow \\
\mathcal{M}^X(r)/m' & \xrightarrow{\mathcal{R}_e\mathbb{Z}/m'(r)}
\end{array}
$$

in $\mathcal{D}(\text{Sh}(X_{\text{Zar}}, \mathbb{Z}/m))$ commutes.

Proof. This follows from the definition of the étale cycle class map. $\Box$

Theorem 3.9: Let $X \in \text{Sm}_{/S}'$ and suppose $m$ is invertible on $X$. Then there is an isomorphism

$$
\mathcal{M}^X(r)/m \cong \tau_{\leq r}(\mathbb{R}_e\mathbb{Z}/m(r))
$$

in $\mathcal{D}(\text{Sh}(X_{\text{Zar}}, \mathbb{Z}/m))$ induced by the cycle class map.

Proof. By [13, Theorem 1.2. 2.] (which we can apply since the Bloch-Kato conjecture is proven, [44]) we have

$$
\mathcal{M}^X(r) \cong \tau_{\leq r}(\mathbb{R}_e\mathbb{Z}/m(r)).
$$

By Theorem 3.3 it follows that $\mathbb{R}^{r+1}_{\text{et}}\mathcal{M}^X(r) = 0$. Thus

$$
\mathcal{M}^X(r)/m \cong \tau_{\leq r}(\mathbb{R}_e\mathcal{M}^X(r)/m).
$$

But by [13, Theorem 1.2. 4.] we have

$$
\mathcal{M}^X_{\text{et}}(r)/m \cong \mathbb{Z}/m(r)
$$

induced by the cycle class map (see the proof of [13, Theorem 1.2. 4.]). This shows the claim. $\Box$
**Theorem 3.10:** Let $i: Z \to X$ be a closed inclusion in $\text{Sm}'_S$ of codimension $c$ and suppose $m$ is invertible on $X$. Then there is a canonical isomorphism

$$\mathbb{R}i^!\mathbb{Z}/m(r) \cong \mathbb{Z}/m(r - c)[-2c]$$

in $D(\text{Sh}(Z_{\text{et}}, \mathbb{Z}/m))$.

**Proof.** This is contained in [33]. □

A consequence is the localization/Gysin exact triangle for étale cohomology.

**Corollary 3.11:** Let $i: Z \to X$ be a closed inclusion in $\text{Sm}'_S$ of codimension $c$ and $j: U \to X$ the complementary open inclusion. Suppose $m$ is invertible on $X$. Then there is an exact triangle

$$i_*\mathbb{Z}/m(r - c)[-2c] \to \mathbb{Z}/m(r) \to \mathbb{R}j_*\mathbb{Z}/m(r) \to i_*\mathbb{Z}/m(r - c)[-2c + 1]$$

in $D(\text{Sh}(X_{\text{et}}, \mathbb{Z}/m))$.

**Proof.** This follows from Theorem 3.10 and the corresponding exact triangle involving $\mathbb{R}i^!\mathbb{Z}/m(r)$.

**Theorem 3.12:** Let $i: Z \to X$ be a closed inclusion in $\text{Sm}'_S$ of codimension $c$ and suppose $m$ is invertible on $X$. Then the diagram

$$\mathbb{R}i^!\mathcal{M}^X(m)(r)/m \xrightarrow{\cong} \mathcal{M}^Z(m)(r - c)/m[-2c]$$

$$\mathbb{R}i^!\mathbb{R}\epsilon_*\mathbb{Z}/m(r) \xrightarrow{\cong} \mathbb{R}\epsilon_*\mathbb{Z}/m(r - c)[-2c]$$

commutes.

**Proof.** Let $U = X \smallsetminus Z$. For $V \in \text{Sm}'_S$ we denote by $c^r(V,n)$ the set of cycles (closed integral subschemes) of $V \times \Delta^n$ which intersect all $V \times Y$ with $Y$ a face of $\Delta^n$ properly.

Let $\mu^r_m \to \mathcal{G}$ be an injectively fibrant replacement in $\text{Cpx}(\text{Sh}(\text{Sm}_{X_{\text{et}}}, \mathbb{Z}/m))$. 

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Let $V \in \text{Sm}_X$. For $W$ a closed subset of $X$ such that each irreducible component has codimension greater or equal to $r$ set $\mathcal{G}^W(V) := \ker(\mathcal{G}(V) \to \mathcal{G}(V \setminus W))$.

As in [26, 12.3] there is a canonical isomorphism of $H^{2r}(\mathcal{G}^W(V))$ with the free $\mathbb{Z}/m$-module on the irreducible components of $W$ of codimension $r$ and the map $\tau_{\leq 2r} \mathcal{G}^W(V) \to H^{2r}(\mathcal{G}^W(V))[-2r]$ is a quasi isomorphism.

For $V \in X_{\text{\acute{e}t}}$ denote by $\mathcal{G}^r(V,n)$ the colimit of the $\mathcal{G}^W(V \times \Delta^n)$ where $W$ runs through the finite unions of elements of $c^r(V,n)$. The simplicial complex of $\mathbb{Z}/m$-modules $\tau_{\leq 2r} \mathcal{G}^r(V,\bullet)$ augments to the simplicial abelian group $z^r(V,\bullet)/m[-2r]$. This augmentation is a levelwise quasi isomorphism. We denote by $\mathcal{G}^r(V)$ the total complex associated to the double complex which is the normalized complex associated to $\tau_{\leq 2r} \mathcal{G}^r(V,\bullet)$. Thus we get a quasi isomorphism $\mathcal{G}^r(X) \to z^r(X)/m[-2r]$. Here for $V \in \text{Sm}'_{\text{\acute{e}t}}$ the complex $z^r(V)$ is defined to be the normalized complex associated to the simplicial abelian group $z^r(V,\bullet)$.

On the other hand for $V \in X_{\text{\acute{e}t}}$ we have a canonical map $\mathcal{G}^r(V, n) \to \mathcal{G}(V \times \Delta^n)$ compatible with the simplicial structure. We denote by $\mathcal{G}'(V)$ the total complex associated to the double complex which is the normalized complex associated to $\mathcal{G}(V \times \Delta^*)$. We have a canonical quasi isomorphism $\mathcal{G}(V) \to \mathcal{G}'(V)$ and a canonical map $\mathcal{G}^r(V) \to \mathcal{G}'(V)$. The above groups and maps are functorial in $V \in X_{\text{\acute{e}t}}$.

Thus we get a map

$$z^r(\bullet)/m[-2r] \cong \mathcal{G}^r \to \mathcal{G}' \cong \mathcal{G}$$

in $\text{D}(\text{Sh}(X_{\text{\acute{e}t}}, \mathbb{Z}/m))$. This is (the adjoint of) the cycle class map.

Denote by $\tilde{\mathcal{G}}$, $\tilde{\mathcal{G}}'$, $\tilde{\mathcal{G}}^{r,c}$ the analogous objects defined for $Z$ instead for $X$, so we have a diagram

$$z^{r-c}(\bullet)/m[-2(r-c)] \leftarrow \tilde{\mathcal{G}}^{r,c} \rightarrow \tilde{\mathcal{G}}' \leftarrow \tilde{\mathcal{G}}$$

in $\text{Cpx}(\text{Sh}(Z_{\text{\acute{e}t}}, \mathbb{Z}/m))$.

For $V \in \text{Sm}_X$ set $\mathcal{G}_Z(V) := \ker(\mathcal{G}(V) \to \mathcal{G}(V|_U))$. Thus $\mathcal{G}_Z \in \text{Cpx}(\text{Sh}(\text{Sm}_{X,\text{\acute{e}t}}, \mathbb{Z}/m))$ computes $i_* \mathbb{R}^e \mu^\text{\acute{e}t}_m$.

There is an absolute purity isomorphism $\mathcal{G}_Z \cong i_* \tilde{\mathcal{G}}[-2c]$ in $\text{D}(\text{Sh}((\text{Sm}_{X,\text{\acute{e}t}}, \mathbb{Z}/m))$. Choose a representative $\varphi : \mathcal{G}_Z \to i_* \tilde{\mathcal{G}}[-2c]$ in $\text{Cpx}(\text{Sh}(\text{Sm}_{X,\text{\acute{e}t}}, \mathbb{Z}/m))$ of this isomorphism. This exists since $i_* \tilde{\mathcal{G}}[-2c]$ is injectively fibrant.

For $V \in X_{\text{\acute{e}t}}$ denote by $\mathcal{G}'_Z(V)$ the total complex associated to the double complex which is the normalized complex associated to $\mathcal{G}_Z(V \times \Delta^*)$. Moreover let $\mathcal{G}'_Z(V,n)$ be the colimit of the $\mathcal{G}^W(V \times \Delta^n)$ where $W$ runs through the finite unions of elements of $c^{r-c}(V|_Z,n)$. Denote by $\mathcal{G}^r_Z(V)$ the total complex associated to the double complex
which is the normalized complex associated to $\tau_{\leq 2} G^r_Z(V, \mathord{\bullet})$. Denote by $z^r_Z(V)$ the complex $z^{r-c}(V|_Z)$.

Set $G_U(V) := G(V|_U)$, $G'_U(V) := G'(V|_U)$, $G^r_U(V) := G^r(V|_U)$ and $z^r_U(V) := z^r(V|_U)$.

We have the diagram

\[
\begin{array}{cccc}
& i_* \tilde{G}[-2c] & \sim & G_Z \\
& \downarrow \sim & & \downarrow \sim \\
& i_* \tilde{G}'[-2c] & \sim & G'_Z \\
& \downarrow \sim & & \downarrow \sim \\
i_* \tilde{G}^{r-c}[-2c] & \sim & G'^r_Z & \sim & G^r \\
& \downarrow \sim & & \downarrow \sim \\
i_* z^{r-c}(\mathord{\bullet})[-2r] & \sim & z^r_Z(\mathord{\bullet})[-2r] & \sim & z^r_U(\mathord{\bullet})[-2r].
\end{array}
\]

The upper three left most horizontal maps are induced by $\varphi$. The lower left square commutes by the naturality of the purity maps in étale cohomology. All other squares commute by construction. The last two arrows in each horizontal line compose to 0 and constitute an exact triangle, thus the second vertical line computes $i_* \mathbb{R} i^!$ of the third vertical line. The claim follows.

\[\square\]

**Corollary 3.13:** Let $i: Z \to X$ be a closed inclusion in $\text{Sm}_{S'}$ of codimension $c$ and $j: U \to X$ the complementary open inclusion. Suppose $m$ is invertible on $X$. Then the diagram

\[
\begin{array}{c}
i_* \mathcal{M}^Z(r-c)/m[-2c] \to \mathcal{M}^X(r)/m \to \mathbb{R} j_* \mathcal{M}^U(r)/m \to i_* \mathcal{M}^Z(r-c)/m[-2c+1] \\
i_* \mathcal{R} \epsilon_* Z/m(r-c)[-2c] \to \mathbb{R} j_* \mathcal{R} \epsilon_* Z/m(r) \to i_* \mathcal{R} \epsilon_* Z/m(r-c)[-2c+1] \\
\mathbb{R} \epsilon_* i_* Z/m(r-c)[-2c] \to \mathbb{R} \epsilon_* Z/m(r) \to \mathbb{R} \epsilon_* j_* Z/m(r) \to \mathbb{R} \epsilon_* i_* Z/m(r-c)[-2c+1]
\end{array}
\]

commutes.
Proof. The diagram

\[
\begin{array}{cccc}
i_*R^i\mathcal{M}^X(r/m) & \longrightarrow & \mathcal{M}^X(r/m) & \longrightarrow & \mathbb{R}j_*\mathcal{M}^U(r/m) & \longrightarrow & i_*R^i\mathcal{M}^X(r/m)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
i_*R^i\mathbb{R}\epsilon_*\mathbb{Z}/m & \longrightarrow & Rj_*\mathbb{R}\epsilon_*\mathbb{Z}/m(r) & \longrightarrow & i_*R^i\mathbb{R}\epsilon_*\mathbb{Z}/m[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
i_*\mathbb{R}\epsilon_*R^i\mathbb{Z}/m(r) & \longrightarrow & \mathbb{R}j_*\mathbb{R}\epsilon_*\mathbb{Z}/m(r) & \longrightarrow & i_*\mathbb{R}\epsilon_*R^i\mathbb{Z}/m(r)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{R}\epsilon_*i_*R^i\mathbb{Z}/m(r) & \longrightarrow & \mathbb{R}\epsilon_*\mathbb{Z}/m(r) & \longrightarrow & \mathbb{R}\epsilon_*\mathbb{R}j_*\mathbb{Z}/m(r) & \longrightarrow & \mathbb{R}\epsilon_*i_*R^i\mathbb{Z}/m(r)[1] \\
\end{array}
\]
commutes. Thus the claim follows from Theorem 3.12.

**Theorem 3.14:** Let \( X \in \text{Sm}_{S'} \). Let \( q: \mathbb{A}_X^1 \to X \) be the projection. Then the canonical map

\[
\mathcal{M}^X(r) \to \mathbb{R}q_*\mathcal{M}^X_{\mathbb{A}_X}(r)
\]

is an isomorphism in \( D(\text{Sh}(X_{\text{Zar}}, \mathbb{Z})) \).

Proof. This is [13 Corollary 3.5].

## 4 The construction

### 4.1 The \( p \)-parts

#### 4.1.1 Finite coefficients

We fix a prime \( p \) and set \( U := S[\frac{1}{p}] \), \( Z := \mathbb{Z}_p \), \( i: Z \hookrightarrow S \) the closed and \( j: U \hookrightarrow S \) the open inclusion.

For a scheme \( X \) for which the motivic complexes are defined we set \( \mathcal{M}^X_n(r) := \mathcal{M}^X(r)/p^n \).

For \( n \geq 1 \) and \( r \in \mathbb{Z} \) let \( L_n(r) := \mu^\otimes_{p^n} \) viewed as sheaf of \( \mathbb{Z}/p^n \)-modules on \( \text{Sm}_{U,\text{et}} \).

The pullback \( j^{-1}: \text{Sm}_S \to \text{Sm}_U \), \( X \mapsto X \times_S U \), induces a push forward

\[
j_*: \text{Sh}(\text{Sm}_{U,\text{Zar}}, \mathbb{Z}/p^n) \to \text{Sh}(\text{Sm}_{S,\text{Zar}}, \mathbb{Z}/p^n)
\]
(we suppress the dependence on $n$ of the functor $j_\ast$). The same is true for étale sheaves.

Similarly, we have the pullback $i^{-1}\colon \text{Sm}_S \to \text{Sm}_Z$, $X \mapsto X \times_S Z$, inducing also a push forward on sheaf categories.

Let $QL_n(1) \to L_n(1)$ be a cofibrant replacement in $\text{Cpx}(\text{Sh}(\text{Sm}_{U,\acute{e}t},\mathbb{Z}/p^n))$ (the latter category is equipped with the local projective model structure, see Appendix A) and let $QL_n(1) \to RQL_n(1)$ be a fibrant replacement via a cofibration. Thus $\mathcal{T} := RQL_n(1)[1]$ is both fibrant and cofibrant.

Recall the decomposition

$$\mathbb{R}\text{Hom}_D(\text{Sh}(\text{Sm}_{U,\acute{e}t},\mathbb{Z}/p^n))(G_{m,U}, L_n(1)[1]) = L_n(1)[1] \oplus L_n(0). \tag{4}$$

The first summand splits off because the projection $G_{m,U} \to U$ has the section $\{1\}$. To define the isomorphism of the remaining summand with $L_n(0)$ we use the Gysin sequence for the situation

$$G_{m,U} \to A^1_S \leftarrow \{0\}.$$

Let $\iota \colon \mathbb{Z}/p^n[G_{m,U}, \{1\}]_{\acute{e}t} \to \mathcal{T}$ be a map which classifies the canonical element $1 \in H^1_{\acute{e}t}(G_{m,U}, L_n(1))$ under the above decomposition (here the source of $\iota$ is the chain complex having the indicated object in degree 0). Note that $\mathbb{Z}/p^n[G_{m,U}, \{1\}]_{\acute{e}t}$ is cofibrant.

**Remark 4.1:** The map $\mathbb{Z}/p^n[G_{m,U}]_{\acute{e}t} \to \mathcal{T}$ induced by $\iota$ represents the map induced by the last map of the exact triangle

$$L_n(1) \to G_{m,U} \xrightarrow{\iota} G_{m,U} \to L_n(1)[1]$$

in $D(\text{Sh}(\text{Sm}_{U,\acute{e}t},\mathbb{Z}))$. This follows from the construction of the Gysin isomorphism.

We get a map

$$\text{Sym}(\iota) \colon \text{Sym}(\mathbb{Z}/p^n[G_{m,U}, \{1\}]_{\acute{e}t}) \to \text{Sym}(\mathcal{T})$$

of commutative monoids in symmetric sequences in $\text{Cpx}(\text{Sh}(\text{Sm}_{U,\acute{e}t},\mathbb{Z}/p^n))$, in other words $\text{Sym}(\mathcal{T})$ is a commutative monoid in the category of symmetric $\mathbb{Z}/p^n[G_{m,U}, \{1\}]_{\acute{e}t}$-spectra $\text{Sp}_{\mathbb{Z}/p^n[G_{m,U}, \{1\}]_{\acute{e}t}}$. In particular it gives rise to an $E_\infty$-object in $\text{Sp}_{\mathbb{Z}/p^n[G_{m,U}, \{1\}]_{\acute{e}t}}$.

Let $Q\text{Sym}(\mathcal{T}) \to \text{Sym}(\mathcal{T})$ be a cofibrant replacement via a trivial fibration and $Q\text{Sym}(\mathcal{T}) \to RQ\text{Sym}(\mathcal{T})$ a fibrant resolution of $Q\text{Sym}(\mathcal{T})$ in $E_\infty(\text{Sp}_{\mathbb{Z}/p^n[G_{m,U}, \{1\}]_{\acute{e}t}})$ (here $E_\infty(\text{Sp}_{\mathbb{Z}/p^n[G_{m,U}, \{1\}]_{\acute{e}t}})$ is equipped with the transferred semi model structure, see Appendix A in particular $RQ\text{Sym}(\mathcal{T})$ is underlying levelwise fibrant for the local projective model structure and is therefore suitable to compute the derived push forward along $\iota$).
Lemma 4.2: The map $QSym(T) \rightarrow RQSym(T)$ is a level equivalence, i.e. $Sym(T)$ is an $\Omega$-spectrum.

Proof. This follows from the fact that we have chosen the map $\iota$ in such a way that the derived adjoints of the structure maps of $Sym(T)$ give rise to the isomorphism $\mathbb{R}Hom((G_{m,U}, \{1\}), L_n(r)[r]) \cong L_n(r-1)[r-1]$.

Set $A := \epsilon_*(RQSym(T))$, so the spectrum $A$ is $RQSym(T)$ viewed as $E_\infty$-algebra in $\mathbb{Z}/p^n[G_{m,U}, \{1\}]_{\text{Zar}}$-spectra in $\text{Cpx}(\text{Sh}(\text{Sm}_{U,\text{Zar}}, \mathbb{Z}/p^n))$.

We denote by $A_r$ the $r$-th level of $A$. Thus $A_r \cong \mathbb{R}\epsilon_*(L_n(r)[r])$.

Set $A'_r := \tau_{\leq 0}(A_r)$, where $\tau_{\leq 0}$ denotes the good truncation at degree 0, i.e. the complex $A'_r$ equals $A_r$ in (cohomological) degrees $< 0$, consists of the cycles in degree 0 and is 0 in positive degree.

Thus by Theorem 3.9 there is for every $X \in \text{Sm}_U$ an isomorphism

$$A'_r|_{X_{\text{Zar}}} \cong M_n^X(r)[r]$$

in $D(\text{Sh}(X_{\text{Zar}}, \mathbb{Z}/p^n))$, where $A'_r|_{X_{\text{Zar}}}$ denotes the restriction of $A'_r$ to $X_{\text{Zar}}$.

Lemma 4.3: The complexes $A'_r$ assemble to a $\mathbb{Z}/p^n[G_{m,U}, \{1\}]_{\text{Zar}}$-spectrum $A'$. This spectrum is equipped with an $E_\infty$-structure together with a map of $E_\infty$-algebras $A' \rightarrow A$ which is levelwise the canonical map $A'_r \rightarrow A_r$.

Proof. This follows from the fact that the truncation $\tau_{\leq 0}$ is right adjoint to the symmetric monoidal inclusion of (cohomologically) non-positively graded complexes into all complexes and that $\mathbb{Z}/p^n[G_{m,U}, \{1\}]_{\text{Zar}}$ lies in this subcategory of non-positively graded complexes.

Let $QA' \rightarrow A'$ be a cofibrant replacement via a trivial fibration in $\mathbb{Z}/p^n[G_{m,U}, \{1\}]_{\text{Zar}}$-spectra in $\text{Cpx}^{SO}(\text{Sh}(\text{Sm}_{U,\text{Zar}}, \mathbb{Z}/p^n))$ (so $QA'$ is also cofibrant viewed as spectrum in unbounded complexes) and $QA' \rightarrow RQA'$ be a fibrant resolution (as $E_\infty$-algebras in $\mathbb{Z}/p^n[G_{m,U}, \{1\}]_{\text{Zar}}$-spectra in $\text{Cpx}(\text{Sh}(\text{Sm}_{U,\text{Zar}}, \mathbb{Z}/p^n))$).

Proposition 4.4: The map $QA' \rightarrow RQA'$ is a level equivalence, i.e. $A'$ and $QA'$ are $\Omega$-spectra.

Proof. Set $m := p^n$. Let $X \in \text{Sm}_U$. Let $\tilde{i}: \{0\} \rightarrow \mathbb{A}^1_X$ be the closed, $\tilde{j}: G_{m,X} \rightarrow \mathbb{A}^1_X$ the open inclusion and $q: \mathbb{A}^1_X \rightarrow X$ the projection. By Corollary 3.11 we have an exact triangle

$$\tilde{i}_*\mathbb{Z}/m(r-1)[-2] \rightarrow \mathbb{Z}/m(r)^{\mathbb{A}^1_X} \rightarrow \mathbb{R}\tilde{j}_*\mathbb{Z}/m(r) \rightarrow \tilde{i}_*\mathbb{Z}/m(r-1)[-1].$$
Note
\[ \mathbb{R}q_\ast \mathbb{R}j_\ast Z/m(r) \cong \mathbb{R}Hom_{\mathcal{D}(\mathcal{Sm}_{U, \{1\}})}(\mathcal{G}_{m,U}, L_n(r))|_{X_{\text{et}}} \]
and that \( \mathbb{R}q_\ast \) applied to the last map in the sequence gives the projection to the second summand in our decomposition (4). Thus by construction of the map \( \iota \) this map also gives the inverse of the adjoint of the structure map in \( R\text{Sym}(T) \).

By Theorem 4 there is an exact triangle
\[ \tilde{i}_\ast \mathcal{M}_n(r - 1)[-2] \to \mathcal{M}_{n}^{A_1}(r) \to \mathbb{R}j_\ast \mathcal{M}_n(r) \to \tilde{i}_\ast \mathcal{M}_n(r - 1)[-1]. \]

Hence by Theorem 3.3 the canonical map
\[ \tau_{\text{et}}(\mathbb{R}j_\ast \mathcal{M}_n(r)) \to \mathbb{R}j_\ast \mathcal{M}_n(r) \]
is an isomorphism. Thus in view of Theorem 3.9 the same truncation property holds for \( \mathbb{R}j_\ast \tau_{\text{et}} \mathbb{R}\varepsilon_\ast Z/m(r) \). Thus the map
\[ \mathbb{R}j_\ast \tau_{\text{et}} \mathbb{R}\varepsilon_\ast Z/m(r) \to \mathbb{R}\varepsilon_\ast \tilde{i}_\ast Z/m(r - 1)[-1] \]
factors through \( \tau_{\text{et}}(\mathbb{R}\varepsilon_\ast \tilde{i}_\ast Z/m(r - 1)[-1]) \).

Moreover the map
\[ \mathbb{R}j_\ast \tau_{\text{et}} \mathbb{R}\varepsilon_\ast Z/m(r) \cong \tau_{\text{et}}(\mathbb{R}j_\ast \mathbb{R}\varepsilon_\ast Z/m(r)) \to \tau_{\text{et}}(\mathbb{R}j_\ast \mathbb{R}\varepsilon_\ast Z/m(r)) \]
is an isomorphism, thus we have a canonical map
\[ \tau_{\text{et}} \mathbb{R}\varepsilon_\ast Z/m(r)^{A_1} \to \mathbb{R}j_\ast \tau_{\text{et}} \mathbb{R}\varepsilon_\ast Z/m(r). \]

Using Corollary 3.13 these maps fit into the commutative diagram
\[
\begin{array}{ccc}
\mathcal{M}_{n}^{A_1}(r) & \xrightarrow{=} & \mathbb{R}j_\ast \mathcal{M}_n(r) \\
| & & | \\
\tau_{\text{et}} \mathbb{R}\varepsilon_\ast Z/m(r)^{A_1} & \xrightarrow{=} & \mathbb{R}j_\ast \tau_{\text{et}} \mathbb{R}\varepsilon_\ast Z/m(r) \\
\tau_{\text{et}}(\mathbb{R}j_\ast \mathbb{R}\varepsilon_\ast Z/m(r - 1)[-1]) & \xrightarrow{=} & \tau_{\text{et}}(\tilde{i}_\ast \mathbb{R}\varepsilon_\ast Z/m(r - 1)[-1])
\end{array}
\]
where the top row is part of the triangle given by Theorem 3.1. The composition
\[ A'_{r-1}[-r]|_{X_{\text{zar}}} \to \mathbb{R}Hom(Z/m[\mathcal{G}_{m,U}, \{1\}]_{\text{zar}}, A'_{r}[-r])|_{X_{\text{zar}}} \to \mathbb{R}Hom(Z/m[\mathcal{G}_{m,U}]_{\text{zar}}, \tau_{\text{et}} \mathbb{R}\varepsilon_\ast L_n(r))|_{X_{\text{zar}}} \]

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\[ \mathbb{R}q_s \mathbb{R}j_\ast \tau_{\leq r} \mathbb{R}^e_s \mathbb{Z}/m(r) \to \tau_{\leq r} (\mathbb{R}^e_s \mathbb{Z}/m(r-1)[-1]) \cong A'_{r-1}[-r]_{X \text{zar}} \]

is the identity.

By Theorem 3.14 \( \mathbb{R}q_s \mathcal{M}^X_n(r) \) identifies with \( \mathcal{M}^X_n(r) \), thus \( \mathbb{R}q_s \) applied to the left bottom arrow in (6) is an isomorphism to the trivial summand and \( \mathbb{R}q_s \) of the bottom row splits. Thus also \( \mathbb{R}q_s \) of the top row splits. This shows that in fact

\[ \mathbb{R}q_s \tilde{i}_\ast \mathcal{M}_n(r-1)[-1] \cong \mathcal{M}^X_n(r-1)[-1] \]

is via the right vertical isomorphism and the right lower map in the diagram isomorphic to the non-trivial summand in \( \mathbb{R}q_s \mathbb{R}j_\ast \tau_{\leq r} \mathbb{R}^e_s \mathbb{Z}/m(r) \). Since this holds over every \( X \in \text{Sm}_U \) we are done.

Thus \( B := j_\ast (RQA') \) is a \( \mathbb{Z}/p^n[G_{m,S}, \{1\}]_{\text{Zar}} \)-spectrum and computes also levelwise the derived push forward of \( A' \) along \( j \). (Note that to compute the levelwise push forward we also could have used the levelwise model structure.)

By (5) for every \( X \in \text{Sm}_S \) we have

\[ B_r|_{X \text{zar}} \cong \mathbb{R}(j_X)_\ast (\mathcal{M}^X_n(r))[r] \]

in \( \mathcal{D}(\text{Sh}(X_{\text{Zar}}, \mathbb{Z}/p^n)) \) (here \( X_U = X \times_S U \) and \( j_X \) denotes the inclusion \( X_U \to X \)).

Thus by Corollary 3.5 the map \( B'_r := \tau_{\leq 0} B_r \to B_r \) is a quasi-isomorphism.

As in Lemma 4.3 the \( B'_r \) assemble to an \( E_\infty \)-algebra \( B' \), and the natural map \( B' \to B \) is an equivalence.

By the following Lemma we could have used \( j_\ast A' \) instead of \( B \) and \( B' \).

**Lemma 4.5:** The natural maps \( j_\ast (QA') \to j_\ast A' \) and \( j_\ast (QA') \to B \) are level equivalences.

**Proof.** Note first that each \( A'_r \) is fibrant in \( \text{Cpx}^{\leq 0}(\text{Sh}(\text{Sm}_U, \mathbb{Z}/p^n)) \), hence so are the \( (QA')_r \), thus the \( j_\ast A' \) and \( j_\ast (QA') \) are the derived push forwards to the homotopy category of \( \text{Cpx}^{\leq 0}(\text{Sh}(\text{Sm}_S, \mathbb{Z}/p^n)) \). But truncation commutes with derived push forward (both are right adjoints), so the claim follows from the fact that \( B' \to B \) is an equivalence.

**Corollary 4.6:** There is a natural isomorphism

\[ \mathbb{R}^r j_\ast L_n(r) \cong e^* \mathcal{H}^0(B'_r) = \mathcal{H}^0(B'_r)_{\text{et}} \]

in \( \text{Sh}(\text{Sm}_S, \text{et}, \mathbb{Z}/p^n) \).
Proof. We have $j_{\ast}A' = \epsilon_{\ast}\tau_{\leq 0}j_{\ast}(RQSym(T))$, thus

$$R^1j_{\ast}L_n(r) \cong H^0(j_{\ast}((RQSym(T))_r)) = H^0(\tau_{\leq 0}j_{\ast}((RQSym(T))_r)) = \epsilon^*H^0(\tau_{\leq 0}j_{\ast}(A')) = \epsilon^*H^0(B'_r).$$

At the end we used Lemma 4.5.

By (7) and Corollary 3.5 we have for every $X \in Sm_S$ a natural epimorphism

$$s_X: H^0(B'_r|_{X_{\text{et}}}) \twoheadrightarrow (i_X)_*\nu_n^{-1},$$

(8)

where $i_X$ is the inclusion $X \times_S Z \hookrightarrow X$.

**Proposition 4.7:** The maps $s_X$ assemble to an epimorphism

$$s: H^0(B'_r) \twoheadrightarrow i_*\nu_n^{-1}.$$

In order to prove this Proposition we describe the maps $s_X$ in a way Geisser used to define his version of syntomic cohomology in [13, §1.6].

Let $X \in Sm_S$. We first give a construction of a map

$$b_X: (i_X)_*\mathcal{M}_{r,X_{\text{et}}} \to (i_X)_*\nu_n^{-1}$$

in $\text{Sh}((XZ)_{\text{et}}, \mathbb{Z}/p^n)$). Over a complete discrete valuation ring of mixed characteristic such a map was constructed in [4, §(6.6)], see also [13, §6].

We fix a point $p \in Z$ and let $\Lambda$ be the completion of the discrete valuation ring $D_p$. Set $T := \text{Spec}(\Lambda)$. Let $\eta$ be the generic point of $T$. Let $X_T := X \times_S T$, and let $X_p$ be the special fiber and $X_\eta$ the generic fiber of $X_T$.

We let $j_{X_T}: X_\eta \to X_T$ and $i_{X_T}: X_p \to X_T$ be the canonical inclusions.

Then the map

$$b_{X_T}: M^r_{n,X_T} := (i_{X_T})_*R^r(j_{X_T})_*(\mathbb{Z}/p^n) \to \nu_n^{-1}$$

in [4, §(6.6)] is defined as follows (recall $\mathbb{Z}/p^n = \mu_{p^n}$):

By [4, Corollary (6.1.1)] the sheaf $M^r_{n,X_T}$ is (étale) locally generated by symbols $\{x_1, \ldots, x_r\}$, $x_i \in (i_{X_T})_*\mathcal{O}^*_{X_p}$ (for the definition of symbol see [4, §(1.2)]).

Then for any $f_1, \ldots, f_r \in (i_{X_T})_*\mathcal{O}^*_{X_T}$ the map $b_{X_T}$ sends the symbol $\{f_1, \ldots, f_r\}$ to 0 and the symbol $\{f_1, \ldots, f_{r-1}, \pi\}$ ($\pi$ a uniformizer of $\Lambda$) to $\text{dlog} \widetilde{f_1} \cdots \text{dlog} \widetilde{f_{r-1}}$, where $\widetilde{f_i}$ is the reduction of $f_i$ to $\mathcal{O}^*_{X_p}$.

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By multilinearity and the fact that \( \{x, -x\} = 0 \) for \( x \in (i_{X_T})^*(j_{X_T})_*O_{X_0}^* \) this characterizes \( b_{X_T} \) uniquely.

The base change morphism for the square

\[
\begin{array}{ccc}
X_n & \xrightarrow{f_{X_U}} & X_U \\
\downarrow{j_{X_T}} & & \downarrow{j_X} \\
X_T & \xrightarrow{f_X} & X \\
\end{array}
\]

applied to the sheaf \( \mathbb{Z}/p^n(r) \) on \( (X_U)_{\text{ét}} \) yields

\[
(f_X)^* \mathbb{R}^n(j_X)_* \mathbb{Z}/p^n(r) \to \mathbb{R}^n(j_{X_T})_* \mathbb{Z}/p^n(r)
\]

(note that \( (f_{X_U})^* \mathbb{Z}/p^n(r) = \mathbb{Z}/p^n(r) \)). Applying \( (i_{X_T})^* \) and noting that \( (i_{X_T})^*(f_X)^* = (i_p)^* \) where \( i_p \) is the inclusion \( i_p: X_p \to X \) we get a map

\[
(i_p)^* \mathbb{R}^n(j_X)_* \mathbb{Z}/p^n(r) \to M_{n,X_T}^r.
\]

Composing with \( b_{X_T} \) gives a map

\[
(i_p)^* \mathbb{R}^n(j_X)_* \mathbb{Z}/p^n(r) \to \nu_n^{r-1}.
\]

Taking the disjoint union over all points in \( Z \) we finally get the map

\[
b_{X_T} : (i_X)^* \mathbb{R}^n(j_X)_* \mathbb{Z}/p^n(r) \to \nu_n^{r-1},
\]

the adjoint of which is a map

\[
b'_{X_T} : \mathbb{R}^n(j_X)_* \mathbb{Z}/p^n(r) \to (i_X)_* \nu_n^{r-1}.
\]

Together with the isomorphism of Corollary 4.6 we get the composition

\[
s'_X : \mathbb{H}^0(B'_r)|_{X_{\text{Zar}}} \to \epsilon_* \mathbb{H}^0(B_r)|_{X_{\text{Zar}}} \cong \epsilon_* \mathbb{R}^n(j_X)_* \mathbb{Z}/p^n(r) \to (i_X)_* \nu_n^{r-1}
\]

(by our convention \( \nu_n^{r-1} \) also denotes the logarithmic De Rham-Witt sheaf on \( (X_Z)_{\text{Zar}} \)).

**Proposition 4.8:** With the notation as above we have \( s_X = s'_X \).

**Proof.** We keep the local completed situation at a point \( p \) of \( Z \) from above.

We have a natural map induced by flat pullback \( (f_{X_U})^* \mathcal{M}_n^{X_U}(r) \to \mathcal{M}_n^{X}(r) \), whence we get a base change morphism

\[
f^*_X \mathbb{R}^n(j_X)_* \mathcal{M}_n^{X_U}(r) \to \mathbb{R}^n(j_{X_T})_* \mathcal{M}_n^{X}(r).
\]
We get a diagram

\[
\begin{array}{ccc}
\mathcal{M}_n^{X^n} (r) & \xrightarrow{f_X^* \mathbb{R}(j_X)_*} & \mathcal{M}_n^{X^n} (r) \\
\downarrow & & \downarrow \\
\epsilon_* \mathcal{M}_n^{X^n} (r) & \xrightarrow{f_X^* \epsilon_* \mathbb{R}(j_X)_*} & \epsilon_* \mathcal{M}_n^{X^n} (r) \\
\downarrow & & \downarrow \\
\nu_n^{-1} & \xrightarrow{(i_X)_*} & (i_X)_* \nu_n^{-1}.
\end{array}
\]

The left and middle vertical maps are induced by the isomorphism of Corollary [1.6] and [7]. The left lower horizontal map is induced by the transformation \( f_X^* \epsilon_* \to \epsilon_* f_X^* \).

The upper right horizontal arrow is part of the localization sequence for the motivic complexes. The lower right horizontal map is induced by \( b_{X_T} \).

The claim of the Proposition follows from the commutativity of the outer square. Indeed, a map from the left upper corner to the right lower corner is adjoint to a map

\[
(i)_{p}^* \mathbb{R}(j_X)_* \mathcal{M}_n^{X^n} (r) = (i_{X_T})_* f^* \mathbb{R}(j_X)_* \mathcal{M}_n^{X^n} (r) \to \nu_n^{-1}.
\]

The assertion that the outside compositions are the same implies that the adjoints of \( s_X \) and \( s'_X \) coincide over the point \( p \). Since this is true for all points in \( Z \) the claim follows.

The left square of the above square commutes by naturality of the cycle class map, Lemma [3.7].

So we are left to prove the commutativity of the right hand square.

Since the right lower corner is an étale sheaf we can also sheafify this square in the étale topology to test commutativity.

The resulting square is adjoint to a square

\[
\begin{array}{ccc}
(i_X)_* \epsilon_* \mathcal{M}_n^{X^n} (r) & \xrightarrow{(i_X)_* \epsilon_*} & (i_X)_* \epsilon_* \mathcal{M}_n^{X^n} (r) \\
\downarrow & & \downarrow \\
(i_X)_* \epsilon_* \mathcal{M}_n^{X^n} (r) & \xrightarrow{(i_X)_* \epsilon_*} & (i_X)_* \epsilon_* \mathcal{M}_n^{X^n} (r) \\
\downarrow & & \downarrow \\
\nu_n^{-1} & \xrightarrow{(i_X)_*} & \nu_n^{-1}
\end{array}
\]

(the left vertical map is an isomorphism by Corollary [1.6]). This commutativity would follow from the commutativity of the right hand square in the first diagram in the proof of [13, Theorem 1.3]. This commutativity is not explicitly stated in loc. cit., but the proof in loc. cit. that \( \kappa \circ \alpha \circ c = 0 \) shows the commutativity of our diagram.

As in loc. cit. let \( R \) be the strictly henselian local ring of a point in the closed fiber \( X_p \) of \( X_T \), let \( L \) be the field of quotients of \( R \), \( F \) the field of quotients of \( R/\pi \), \( V = R(\pi) \), \( V^h \) the henselization of \( V \) and \( L^h \) the quotient field of \( V^h \).
We have to show the commutativity of
\[
H^r(R[\frac{1}{\pi}], \mathcal{M}_n(r)) \longrightarrow H^{r-1}(R/\pi, \mathcal{M}_n(r-1)) \cong \\
H^r_\text{ét}(R[\frac{1}{\pi}], \mathbb{Z}/p^n(r)) \longrightarrow \nu_n^{-1}(R/\pi).
\]

The map \(\nu_n^{-1}(R/\pi) \to \nu_n^{-1}(F)\) is injective (see the proof of [13, Theorem 1.3], where it is attributed to [16, Corollary 1.6]).

Thus by the naturality of the localization sequence for motivic complexes and the fact that the \(b_{x_z}\) are sheaf maps it is enough to show commutativity of the square which one gets from the last square by replacing \(R[\frac{1}{\pi}]\) with \(L\) and \(R/\pi\) with \(F\). But this square factors as
\[
H^r(L, \mathcal{M}_n(r)) \longrightarrow H^r(L^h, \mathcal{M}_n(r)) \longrightarrow H^{r-1}(F, \mathcal{M}_n(r-1)) \cong \\
H^r_\text{ét}(L, \mathbb{Z}/p^n(r)) \longrightarrow H^r_\text{ét}(L^h, \mathbb{Z}/p^n(r)) \longrightarrow \nu_n^{-1}(F).
\]

The right upper horizontal map is induced from the localization sequence of the motivic complexes for \(V^h\), its generic and its closed point.

The left hand square commutes by naturality of the cycle class map, and the commutativity of the right hand square is shown in the proof of [13, Theorem 1.3] in the paragraph before the last paragraph. This finishes the proof. \(\square\)

We next discuss functoriality of the construction of the morphisms \(s'_X\). So let \(g: Y \to X\) be a morphism in \(\text{Sm}_S\). We still keep the local completed situation from above. We let \(g_z\), \(g_T\), \(g_\eta\) and \(g_p\) be the base changes of \(g\) (over \(S\)) to \(Z\), \(T\), \(\eta\) and \(p\).

Consider the diagram
\[
\begin{array}{ccc}
Y_\eta & \xrightarrow{g_\eta} & X_\eta \\
\downarrow j_{YT} & & \downarrow j_{XT} \\
Y_T & \xrightarrow{g_T} & X_T \\
\downarrow i_{YT} & & \downarrow i_{XT} \\
Y_p & \xrightarrow{g_p} & X_p.
\end{array}
\]

A base change morphism gives us
\[
(g_T)^*\mathbb{R}^r(j_{XT})_*(\mathbb{Z}/p^n(r)) \to \mathbb{R}^r(j_{YT})_*(\mathbb{Z}/p^n(r)).
\]
Applying $(i_{Y_T})^*$ and using $(i_{Y_T})^*(g_T)^* \simeq (g_p)^*(i_{X_T})^*$ gives

$$(g_p)^* M_{n,X_T}^r \to M_{n,Y_T}^r.$$ 

**Lemma 4.9:** The diagram

$$
\begin{array}{ccc}
(g_p)^* M_{n,X_T}^r & \xrightarrow{(g_p)^*(b_{X_T})} & (g_p)^* \nu_{r-1}^n \\
\downarrow & & \downarrow \\
M_{n,Y_T}^r & \xrightarrow{b_{Y_T}} & \nu_{r-1}^n 
\end{array}
$$

commutes.

**Proof.** This follows from the definition of the morphisms $b_{X_T}$ and $b_{Y_T}$ in terms of symbols and the functoriality of the symbols.

As above for $X$ let $f_Y$ be the map $Y_T \to Y$.

**Lemma 4.10:** The diagram

$$
\begin{array}{ccc}
g_T^* f_X^* \mathbb{R}^r(j_X)_* \mathbb{Z}/p^n(r) & \xrightarrow{g_T^* \mathbb{R}^r(j_{X_T})_* \mathbb{Z}/p^n(r)} & g_T^* \mathbb{R}^r(j_{X_T})_* \mathbb{Z}/p^n(r) \\
\downarrow & & \downarrow \\
f_Y^* \mathbb{R}^r(j_Y)_* \mathbb{Z}/p^n(r) & \xrightarrow{\mathbb{R}^r(j_{Y_T})_* \mathbb{Z}/p^n(r)} & \mathbb{R}^r(j_{Y_T})_* \mathbb{Z}/p^n(r),
\end{array}
$$

where all maps are induced by base change morphisms, commutes.

**Proof.** This follows by the naturality of the base change morphisms.

**Corollary 4.11:** The diagram

$$
\begin{array}{ccc}
(g_Z)^* (i_X)^* \mathbb{R}^r(j_X)_* \mathbb{Z}/p^n(r) & \xrightarrow{(g_Z)^*(b_{X_T})} & (g_Z)^* \nu_{r-1}^n \\
\downarrow & & \downarrow \\
(i_Y)^* \mathbb{R}^r(j_Y)_* \mathbb{Z}/p^n(r) & \xrightarrow{by} & \nu_{r-1}^n,
\end{array}
$$

where the left vertical map is induced by a base change morphism, commutes.

**Proof.** This follows by combining Lemmas 4.9 and 4.10.

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Corollary 4.12: The diagram

\[
g^*R^r(j_X)_*Z/p^n(r) \rightarrow g^*(\nu_X^\ast) \rightarrow g^*(i_X)_*\nu_n^{-1} \\
\downarrow \downarrow \downarrow \\
R^r(j_Y)_*Z/p^n(r) \rightarrow (i_Y)_*\nu_n^{-1}
\]

commutes.

Proof. We check that the adjoints with respect to the pair \((i_Y)_\ast, (i_Y)_\ast\) of the two compositions are the two compositions of Corollary 4.11. For the composition via the left lower corner this is immediate. For the other composition one uses a compatibility between adjoints and pullbacks. \(\square\)

Corollary 4.13: The maps \(s'_X\) assemble to a map of sheaves \(\mathcal{H}^0(B'_r) \rightarrow i_*\nu_n^{-1}\).

Proof. This follows directly from Corollary 4.12. \(\square\)

Proof of Proposition 4.7: The assertion follows by combining Proposition 4.8 and Corollary 4.13. \(\square\)

Let \(C_r\) be the kernel of the composition

\[
B'_r \rightarrow \mathcal{H}^0(B'_r) \xrightarrow{s} i_*\nu_n^{-1}.
\]

Then by construction of the maps \(s_X\) we have for any \(X \in \text{Sm}_S\) an isomorphism

\[
C_r|_{\text{X zar}} \cong \mathcal{M}_n^X(r)[r]
\]

in \(\text{D}(\text{Sh}(X_{\text{zar}}, \mathbb{Z}/p^n))\) since both objects appear as (shifted) homotopy fibers of the map

\[
\mathbb{R}(j_X)_*\mathcal{M}_n^{X_U}(r) \rightarrow \mathbb{R}(i_X)_*\mathcal{M}_n^{X_Z}(r-1)[-1].
\]

This isomorphism is even uniquely determined since there are no non-trivial maps \(\mathcal{M}_n^X(r) \rightarrow (i_X)_*\nu_n^{-1}[-r-1]\) in \(\text{D}(\text{Sh}(X_{\text{zar}}, \mathbb{Z}/p^n))\).
Lemma 4.14: Let $R$ be a commutative ring, $T \in \text{Sh}(\text{Sm}_{S,\text{Zar}}, R)$ and $E$ an $E_\infty$-algebra in symmetric $T$-spectra in $\text{Cpx}^{\leq 0}(\text{Sh}(\text{Sm}_{S,\text{Zar}}, R))$. Let $E_r$ be the levels of $E$. Let for any $r > 0$ an epimorphism $H^0(E_r) \to e_r$ in $\text{Sh}(\text{Sm}_{S,\text{Zar}}, R)[\Sigma_r]$ be given. Let $E'_r$ be the kernel of the induced map $E_r \to e_r$ and set $E'_0 := E_0$. Suppose the canonical map $\varphi: T \to E_1$ (which is the composition $T \cong R \otimes T \xrightarrow{u \otimes \text{id}} E_0 \otimes T \to E_1$ ($u$ abbreviates unit)) factors through $E'_1$ and that for any $r, r' \geq 0$ the composition in $\text{Sh}(\text{Sm}_{S,\text{Zar}}, R)$ induced by the $E_\infty$-multiplication on $H^0(E'_r) \otimes H^0(E'_r) \to H^0(E_r) \otimes H^0(E_r)$ is the zero map. Then there is an induced structure of an $E_\infty$-algebra $E'_r$ in symmetric $T$-spectra on the collection of the $E'_r$ together with a map of $E_\infty$-algebras $E'_r \to E_r$.

Proof. The condition implies that we have natural maps

$$\phi_{r,r'}: H^0(E'_r) \otimes H^0(E'_r) \to H^0(E_r) \otimes H^0(E_r') \to H^0(E_{r+r'}) \to e_{r+r'}$$

(the tensor products are over $R$) is the zero map. Then there is an induced structure of an $E_\infty$-algebra $E'$ in symmetric $T$-spectra on the collection of the $E'_r$ together with a map of $E_\infty$-algebras $E' \to E$ which is levelwise the canonical map $E'_r \to E_r$.

To show that the composition

$$\mathcal{O}(k) \otimes E'_{r_1} \otimes \cdots \otimes E'_{r_k} \to \mathcal{O}(k) \otimes E_{r_1} \otimes \cdots \otimes E_{r_k} \to E_r$$

factors through $E'_r$, it is sufficient to show that the induced map on $H^0$ factors through $H^0(E'_r)$. But since $\mathcal{O}$ is $E_\infty$ the map on $H^0$ is a map

$$H^0(E'_{r_1}) \otimes \cdots \otimes H^0(E'_{r_k}) \to H^0(E_r)$$

and the conditions to be $E_\infty$ imply that this map is an iteration of the maps $\phi_{r',r''}$. Thus we get the factorization.
To handle the case of the $T$-spectrum structure maps it is again sufficient to show that the composition
\[
\psi : \mathcal{H}^0(E'_r) \otimes T \rightarrow \mathcal{H}^0(E_r) \otimes T \rightarrow \mathcal{H}^0(E_{r+1})
\]
factors through $\mathcal{H}^0(E'_{r+1})$. But the commutativity of the diagram

\[
\begin{array}{ccc}
\mathcal{O}(2) \otimes E_r \otimes T & \xrightarrow{\psi} & \mathcal{O}(1) \otimes E_r \otimes T \\
\xrightarrow{\text{id} \otimes \varphi} & & \xleftarrow{\text{id} \otimes u \otimes \text{id}} \\
\mathcal{O}(2) \otimes E_r \otimes R \otimes T & \rightarrow & \mathcal{O}(1) \otimes E_r \otimes T \\
\xrightarrow{\text{id} \otimes u \otimes \text{id}} & & \xleftarrow{\text{id} \otimes u \otimes \text{id}} \\
\mathcal{O}(2) \otimes E_r \otimes E_0 \otimes T & \rightarrow & \mathcal{O}(1) \otimes E_r \otimes T \\
\xrightarrow{\text{id} \otimes s} & & \xleftarrow{\text{id} \otimes s} \\
\mathcal{O}(2) \otimes E_r \otimes E_1 & \rightarrow & \mathcal{O}(1) \otimes E_r \otimes E_1 \\
\xrightarrow{\text{id} \otimes s} & & \xleftarrow{\text{id} \otimes s} \\
E_r \otimes T & \rightarrow & E_r \otimes T \\
\xrightarrow{\text{id} \otimes s} & & \xleftarrow{\text{id} \otimes s}
\end{array}
\]

(the only horizontal arrow is a structure map of the operad using $R \cong \mathcal{O}(0)$) implies that $\psi$ is the composition
\[
\mathcal{H}^0(E'_r) \otimes T \rightarrow \mathcal{H}^0(E'_r) \otimes \mathcal{H}^0(E'_1) \rightarrow \mathcal{H}^0(E_{r+1})
\]
which factors through $\mathcal{H}^0(E'_{r+1})$ by assumption. This finishes the proof.

We want to apply Lemma 4.14 with $T = \mathbb{Z}/p^n[\mathbb{G}_{m,S}, \{1\}]_{\text{Zar}}, E = B'$ and $e_r = i_*\nu_n^{r-1}$. Then we have $E'_r = C_r$.

**Lemma 4.15:** The $\Sigma_r$-action on $\mathcal{H}^0(B'_r)$ is the sign representation.

**Proof.** This follows from the fact that there is a zig zag of quasi-isomorphisms between $T^{\otimes r}$ and $(L_n(1)[1])^{\otimes r}$, and on the latter the $\Sigma_r$-action is strictly the sign representation.

So if we equip $\nu_n^{r-1}$ with the sign representation of $\Sigma_r$ the map $\mathcal{H}^0(B'_r) \rightarrow i_*\nu_n^{r-1}$ is $\Sigma_r$-equivariant.

The exact sequence
\[
0 \rightarrow L_n(1) \rightarrow \mathbb{G}_{m,U} \rightarrow \mathbb{G}_{m,U} \rightarrow 0
\]
on Sm\(_{U,\text{ét}}\) induces a boundary homomorphism
\[
\beta: j_* G_{m,U} \to R^1 j_* L_n(1)
\]
of sheaves on Sm\(_{S,\text{ét}}\). We denote the precomposition of \(\beta\) with the canonical map \(G_{m,S} \to j_* G_{m,U}\) by \(\beta'\).

**Lemma 4.16:** The composition
\[
G_{m,S} \to \mathbb{Z}/p^n[G_{m,S},\{1\}]_{\text{Zar}} \xrightarrow{\varphi} B'_1 \to \mathcal{H}^0(B'_1) \to \mathcal{H}^0(B'_1)_{\text{ét}} \cong R^1 j_* L_n(1)
\]
equals \(\beta'\).

**Proof.** This follows from the defining property of the map \(\iota\).

**Corollary 4.17:** The composition
\[
G_{m,S} \to \mathbb{Z}/p^n[G_{m,S},\{1\}]_{\text{Zar}} \xrightarrow{\varphi} B'_1 \to \mathcal{H}^0(B'_1) \to i_* \nu^{-1}_n
\]
is the constant map to zero.

**Proof.** This follows from Lemma 4.16, the definition of the map \(b_{X_T}\) and the definition of symbol: The symbol \(\{x\}\) for \(x\) an invertible section over a smooth scheme over \(S\) is sent to 0 via \(b_{X_T}\).

Thus the first condition of Lemma 4.14 about the factorization of the map \(\varphi\) is satisfied.

For the second condition we get back to our local completed situation. Let \(X \in \text{Sm}_S, p \in \mathbb{Z}\) and let the notation be as above. By [25, §3, top of p. 277] there is an exact sequence
\[
0 \to U^0 M'_n \to M'_n \to \nu^{-1}_n \to 0 \tag{10}
\]
on \((X_p)_{\text{ét}}\), where \(U^0 M'_n\) is the subsheaf of \(M'_n\) generated étale locally by symbols \(\{x_1,\ldots,x_r\}\) with \(x_i \in (i_{X_T})^* \mathcal{O}_{X_T}^*\). This follows from the exact sequence
\[
0 \to U^1 M'_n \to M'_n \to \nu^{-1}_n \to 0
\]
([25, Theorem (1.4)(i)]), where \(U^1 M'_n\) is generated étale locally by symbols \(\{x_1,\ldots,x_r\}\) with \(x_1 - 1 \in \pi \cdot (i_{X_T})^* \mathcal{O}_{X_T}\). Indeed, given an element in the kernel of \(M'_n \to \nu^{-1}_n\) we can first change it by symbols \(\{x_1,\ldots,x_r\}\) with \(x_i \in (i_{X_T})^* \mathcal{O}_{X_T}^*\) to lie also in the kernel of the map \(M'_n \to \nu^{-1}_n\), and then it lies in \(U^1 M'_n\) which is also generated by symbols (of the indicated type).
Lemma 4.18: Let $r, r' \geq 0$. The composition
\[ \mathcal{H}^0(C_r) \otimes \mathcal{H}^0(C_{r'}) \to \mathcal{H}^0(B'_r) \otimes \mathcal{H}^0(B'_{r'}) \to \mathcal{H}^0(B'_{r+r'}), \]
where the second map is induced by the $E_\infty$-structure on $B'$, factors through $\mathcal{H}^0(C_{r+r'})$.

Proof. Let $y$ be a local section lying in the kernel of $\mathcal{H}^0(B'_r) \to \nu^r_n$, similarly for $y'$. We may view $y$ and $y'$ as local sections of $M^n_r$ and $M^n_{r'}$. They are mapped to 0 by the maps to $\nu^r_n$ and $\nu^{r'}_{n'}$, thus by the exact sequence (10) the sections $y$ and $y'$ can be written locally as linear combinations of symbols of the form $\{x_1, \ldots, x_r\}$ and $\{x'_1, \ldots, x'_{r'}\}$ with $x_i, x'_i \in (i_X)^*O_{X_T}^*$. But the product of such symbols is just the concatenated symbol $\{x_1, \ldots, x_r, x'_1, \ldots, x'_{r'}\}$ which thus also lies in the kernel of the map $M^n_{r+r'} \to \nu^{r+r'-1}_{n}$. This is true over all points $p$ of $Z$, so we see that $y \otimes y'$ is sent to 0 in $i_*\nu^{r+r'-1}_{n}$.

\[ \blacksquare \]

Corollary 4.19: The collection of the $C_r$ forms an $E_\infty$-algebra $C$ in $\mathbb{Z}/p^n[\mathbb{G}_{m,S}, \{1\}]_{\text{Zar}}$-spectra which comes with a map of $E_\infty$-algebras $C \to B'$ which is levelwise the canonical map $C_r \to B'_r$.

Proof. This follows with Corollary 4.17 and Lemma 4.18 from Lemma 4.14.

Thus with (11) we have arranged the motivic complexes $\mathcal{M}^X_n(r)[r]$, $r \geq 0$, into an $E_\infty$-algebra in $\mathbb{Z}/p^n[\mathbb{G}_{m,S}, \{1\}]_{\text{Zar}}$-spectra on $\text{Sm}_{S, \text{Zar}}$.

Proposition 4.20: The algebra $C$ is an $\Omega$-spectrum.

Proof. Set $m := p^n$. Let $X \in \text{Sm}_S$. Let $\tilde{i}: \{0\} \to \mathbb{A}^1_X$ be the closed, $\tilde{j}: \mathbb{G}_{m,X} \to \mathbb{A}^1_X$ the open inclusion and $q: \mathbb{A}^1_X \to X$ the projection.

Since $\mathbb{Z}/m[\mathbb{G}_{m,S}]_{\text{Zar}} \cong \mathbb{Z}/m \oplus \mathbb{Z}/m[\mathbb{G}_{m,S}, \{1\}]_{\text{Zar}}$, we have a decomposition
\[ \mathbb{R}\text{Hom}(\mathbb{G}_{m,S}, C_r) \cong C_r \oplus \mathcal{R}. \]

By Theorem 3.1 we have an exact triangle
\[ \tilde{i}_*\mathcal{M}_n(r-1)[-2] \to \mathcal{M}^X_n(r) \to \mathbb{R}\tilde{j}_*\mathcal{M}_n(r) \to \tilde{i}_*\mathcal{M}_n(r-1)[-1]. \]

The composition
\[ C_r[-r]_{\mathbb{Z}_{\text{zar}}} \cong \mathbb{R}q_*\mathcal{M}^X_n(r) \to \mathbb{R}q_*\mathbb{R}\tilde{j}_*\mathcal{M}_n(r) \]
\[ \cong \mathbb{R}\text{Hom}(\mathbb{G}_{m,S}, C_r[-r])_{\mathbb{Z}_{\text{zar}}} \cong C_r[-r]_{\mathbb{Z}_{\text{zar}}} \oplus \mathcal{R}[-r]_{\mathbb{Z}_{\text{zar}}} \to C_r[-r]_{\mathbb{Z}_{\text{zar}}} \]
is the identity. Thus when we apply $\mathbb{R}q_*$ to the above triangle we obtain a split triangle. Let $\phi: \mathcal{M}^X_n(r-1)[-1] \cong \mathcal{R}[-r]_{\mathbb{Z}_{\text{zar}}}$ be the resulting isomorphism.
We are finished when we prove that the diagram

\[
\begin{array}{c}
C_{r-1}|_{X_{zar}} \\
\downarrow \cong \\
\mathcal{M}_n^X (r-1) \downarrow \cong \\
\phi \\
\downarrow \\
\mathcal{R}|_{X_{zar}}
\end{array}
\]

where the upper horizontal map is the derived adjoint of the structure map of the spectrum \( C \), commutes. To see this it is sufficient to show that the post composition of the two compositions with the map \( \mathcal{R}|_{X_{zar}} \to \mathcal{R}_s \mathcal{R}'|_{X_{zar}} \), where \( \mathcal{R}' \) is defined to be the second summand in the decomposition \( \mathbb{R} \text{Hom}(G_{m,U}, A') \cong A' \oplus \mathcal{R}' \), coincide, since there are no non-trivial maps from \( C_{r-1}|_{X_{zar}} \) to \( (i_X)_* \nu_{n-2} \).

But we have a transformation of diagrams from the above diagram to the diagram

\[
\begin{array}{c}
B'_{r-1}|_{X_{zar}} \\
\downarrow \cong \\
\mathbb{R}(j_X)_* \mathcal{M}_n^{X,U} (r-1) \downarrow \cong \\
\mathbb{R}(j_X)_* \mathcal{R}|_{X_{zar}}
\end{array}
\]

which commutes by the arguments in the proof of Proposition\textsuperscript{14}. So the two prolonged compositions in question are the two compositions in diagram (11) precomposed with the map \( C_{r-1}|_{X_{zar}} \to B'_{r-1}|_{X_{zar}} \), thus they coincide. This finishes the proof. \( \square \)

### 4.1.2 The \( p \)-completed parts

In this section we want to arrange (variants of) the \( C \) for varying \( n \) into a compatible family, such that we can then take the (homotopy) limit of this system.

To start with write \( \mathbb{Z}/p^* \) for the inverse system comprised by the commutative rings \( \mathbb{Z}/p^n \) with the obvious transition maps and Mod\(_{\mathbb{Z}/p^*}\) for the category of modules over this system, i.e. the category whose objects are systems of abelian groups

\[
\cdots \to M_n \to \cdots \to M_2 \to M_1
\]

where each \( M_n \) is annihilated by \( p^n \).

For a site \( \mathcal{S} \) write \( \text{Sh}(\mathcal{S}, \mathbb{Z}/p^*) \) for \( \text{Sh}(\mathcal{S}, \text{Mod}_{\mathbb{Z}/p^*}) \).

The system of the \( L_n(r) \) comprises a natural object \( L_\bullet (r) \) of \( \text{Sh}(\text{Sm}_{U,\text{ét}}, \mathbb{Z}/p^*) \).

Let \( QL_\bullet (1) \to L_\bullet (1) \) be a cofibrant replacement in \( \text{Cpx}(\text{Sh}(\text{Sm}_{U,\text{ét}}, \mathbb{Z}/p^*))) \) (the latter category is equipped with the inverse local projective model structure) and let \( QL_\bullet (1) \to \)
$RQL_\bullet(1)$ be a fibrant replacement via a cofibration. Thus $T := RQL_\bullet(1)[1]$ is both fibrant and cofibrant.

We claim that the maps $\iota$ from section 4.1.1 can be arranged to a map

$$\iota : \mathbb{Z}/p^*[\mathbb{G}_m, U, \{1\}]_{\text{et}} \to T.$$ 

Indeed, suppose we have already defined $\iota$ up to level $n$ in such a way that on each level $k \leq n$ the map represents the canonical element $1 \in H_{\text{et}}^1(\mathbb{G}_m, L_k(1))$. We claim that we can extend the system of maps to level $n + 1$: Choose a representative

$$\iota' : \mathbb{Z}/p^{n+1}[\mathbb{G}_m, U, \{1\}]_{\text{et}} \to T_{n+1}.$$ 

Then the composition with the fibration $T_{n+1} \to T_n$ is homotopic to the map in level $n$. This homotopy can be lifted giving as second endpoint the required lift.

As in section 4.1.1 the map of symmetric sequences $\text{Sym}(\iota)$ gives rise to an $E_{\infty}$-algebra $\text{Sym}(T)$ in $\Omega-\mathbb{Z}/p^*[\mathbb{G}_m, U, \{1\}]_{\text{et}}$-spectra, and we let $Q\text{Sym}(T) \to \text{Sym}(T)$ be a cofibrant replacement via a trivial fibration and $Q\text{Sym}(T) \to RQ\text{Sym}(T)$ be a fibrant resolution.

Set $A := \epsilon_*(RQ\text{Sym}(T))$ and $A' := \tau_{\leq 0}(A)$. As in section 4.1.1 $A'$ is again an $E_{\infty}$-algebra. We set $B := j_* A'$. By Lemma 4.25 the algebra $B$ computes levelwise in the $n$-direction the algebra which was denoted $B'$ in section 4.1.1.

Thus we have for every $n$ and $r$ the epimorphism

$$s_{r,n} : \mathcal{H}^0(B_{r,n}) \to i_* \nu_n^{r-1}$$

of Proposition 4.7.

**Lemma 4.21:** We have a commutative diagram

$$\begin{array}{ccc}
\mathcal{H}^0(B_{r,n+1}) & \xrightarrow{s_{r,n+1}} & i_* \nu_n^{r-1} \\
\downarrow & & \downarrow \\
\mathcal{H}^0(B_{r,n}) & \xrightarrow{s_{r,n}} & \nu_n^{r-1}.
\end{array}$$

**Proof.** We only have to verify that a corresponding diagram involving the maps $s'_{X}$ commutes. This follows by the explicit definition of the maps $b_{X,T}$. \qed

We thus get an epimorphism

$$B_r \to i_* \nu_n^{r-1}.$$
We denote by $C_r$ the kernel of this epimorphism.

As in section 4.1.1 we can apply a variant of Lemma 4.14 (or the Lemma levelwise in the $n$-direction and using functoriality) to see that the collection of the $C_r$ gives rise to an $E_\infty$-algebra $C$ together with a map of $E_\infty$-algebras $C \to B$ which is levelwise (for the $r$-direction) the canonical map $C_r \to B_r$.

Let $X \in \text{Sm}_S$. We want to see that the canonical isomorphisms (9)

$$C_{r,n}|_{X_{\text{Zar}}} \cong \mathcal{M}^X_n(r)[r]$$

are compatible with the reductions $\mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n$.

First by Lemma 3.8 the diagram

$$
\begin{array}{ccc}
\mathcal{M}^X_n(r)[r] & \xrightarrow{=} & A'_{r,n+1}|_{(X_U)_{\text{Zar}}} \\
\downarrow & & \downarrow \\
\mathcal{M}^{X_U}_n(r)[r] & \xrightarrow{=} & A'_{r,n}|_{(X_U)_{\text{Zar}}}
\end{array}
$$

commutes.

This shows that if we compose the two compositions in the square

$$
\begin{array}{ccc}
C_{r,n+1}|_{X_{\text{Zar}}} & \xrightarrow{=} & \mathcal{M}^X_{n+1}(r)[r] \\
\downarrow & & \downarrow \\
C_{r,n}|_{X_{\text{Zar}}} & \xrightarrow{=} & \mathcal{M}^X_n(r)[r]
\end{array}
$$

with the map $\mathcal{M}^X_n(r)[r] \to \mathbb{R}(j_X)_{\ast}\mathcal{M}^{X_U}_n(r)[r]$ the resulting two maps coincide. But $\text{Hom}_{D(\text{Sh}(X_{\text{Zar}}, \mathbb{Z}/p^{n+1}))}(C_{r,n+1}|_{X_{\text{Zar}}}, \nu_n^r[-1] = 0$, hence (12) commutes.

Let $QC \to C$ be a cofibrant and $QC \to C'$ be a fibrant replacement as $E_\infty$-algebras. Then

$$D_p := \lim_n C'_{\ast,n}$$

is an $E_\infty$-algebra in $\mathbb{Z}_p[\mathbb{G}_{m,S}, \{1\}]_{\text{Zar}}$-spectra in $\text{Cpx}(\text{Sh}(\text{Sm}_{S,Zar}, \mathbb{Z}_p))$.

**Corollary 4.22:** For $X \in \text{Sm}_S$ there is an isomorphism

$$D_{p,r}|_{X_{\text{Zar}}} \cong (\mathcal{M}^X(r))^{\wedge_p}[r]$$

in $D(\text{Sh}(X_{\text{Zar}}, \mathbb{Z}_p))$, where $(\mathcal{M}^X(r))^{\wedge_p}$ is the $p$-completion of $\mathcal{M}^X(r)$. 

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Proof. This follows from the commutativity of (12), since the $p$-completion of $\mathcal{M}^X(r)$ is the homotopy limit over all the $\mathcal{M}^X_n(r)$. 

Next we will equip $D_p$ with an orientation.

Denote by $\mathcal{O}_{\mathcal{U}}^*$ the sheaf (in any of the considered topologies) of abelian groups represented by $\mathbb{G}_{m,U}$ over $\text{Sm}_U$, let $\mathcal{O}_{\mathcal{S}}^*$ be defined similarly. For $M$ a sheaf of abelian groups or an object in a triangulated category we set $M/p^n := M \otimes^L \mathbb{Z}/p^n$.

Using the resolution of $\mathcal{O}_{\mathcal{U}}^*$ by the sheaf of meromorphic functions and the sheaf of codimension 1 cycles one sees that $\mathbb{R}^1j_*\mathcal{O}_{\mathcal{U}}^* = 0$ for $i > 0$. Thus we have an exact triangle

$$\mathcal{O}_{\mathcal{S}}^* \rightarrow \mathbb{R}j_*\mathcal{O}_{\mathcal{U}}^* \rightarrow i_*\mathbb{Z} \rightarrow \mathcal{O}_{\mathcal{S}}^*[1]$$

in the Zariski topology, from which we derive an exact triangle

$$\mathcal{O}_{\mathcal{S}}^*/p^n \rightarrow \mathbb{R}j_*\mathcal{O}_{\mathcal{U}}^*/p^n \rightarrow i_*\mathbb{Z}/p^n \rightarrow \mathcal{O}_{\mathcal{S}}^*/p^n[1].$$

(13)

We have a map of exact triangles

$$\begin{array}{cccccc}
\mathcal{O}_{\mathcal{U}}^* & \xrightarrow{p^n} & \mathcal{O}_{\mathcal{U}}^* & \rightarrow & \mathcal{O}_{\mathcal{U}}^*/p^n & \rightarrow & \mathcal{O}_{\mathcal{U}}^*[1] \\
\mathbb{R}\epsilon_*\mathcal{O}_{\mathcal{U}}^* & \xrightarrow{p^n} & \mathbb{R}\epsilon_*\mathcal{O}_{\mathcal{U}}^* & \rightarrow & \mathbb{R}\epsilon_*L_n(1)[1] & \rightarrow & \mathbb{R}\epsilon_*\mathcal{O}_{\mathcal{U}}^*[1].
\end{array}$$

The third vertical map factors uniquely through a map $\mathcal{O}_{\mathcal{U}}^*/p^n \rightarrow \tau_{\leq 0}(\mathbb{R}\epsilon_*L_n(1)[1])$. Since $\mathbb{R}^1\epsilon_*\mathcal{O}_{\mathcal{U}}^* = 0$ we see by the long exact cohomology sheaf sequences associated to these triangles that this map is an isomorphism. Note we have $A'_{1,n} \cong \tau_{\leq 0}(\mathbb{R}\epsilon_*L_n(1)[1])$ in the derived category, and thus $B_{1,n} \cong \mathbb{R}j_*\tau_{\leq 0}(\mathbb{R}\epsilon_*L_n(1)[1]) \cong \mathbb{R}j_*\mathcal{O}_{\mathcal{U}}^*/p^n$.

We note that the diagram

$$\begin{array}{ccc}
\mathbb{R}j_*\mathcal{O}_{\mathcal{U}}^*/p^n & \xrightarrow{i_*\mathbb{Z}/p^n} & i_*\mathbb{Z}/p^n \\
\mathbb{R}j_*\tau_{\leq 0}(\mathbb{R}\epsilon_*L_n(1)[1]) & \xrightarrow{\mathcal{H}^0(B_{1,n})} & i_*\mathbb{Z}/p^n
\end{array}$$

commutes (this follows from the definition of the maps $s_{r,n}$, Proposition 4.8 and the definition of the maps $s'_{X}$). Thus together with the triangle (13) we derive an isomorphism $C_{1,n} \cong \mathcal{O}_{\mathcal{S}}^*/p^n$ in $\mathcal{D}(\text{Sh}({\text{Sm}}_S,\text{Zar},\mathbb{Z}/p^n))$. This isomorphism is moreover unique since there are no non-trivial homomorphisms from $\mathcal{O}_{\mathcal{S}}^*/p^n$ to $i_*\mathbb{Z}/p^n[-1]$. 

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We see that there is an isomorphism $D_{p,1} \cong (O\text{\textsc{i}}_S^*)^{\wedge p}$ in $\mathcal{D}(\text{Sh}(\text{Sm}_{S,Zar}, \mathbb{Z}_p))$. We denote any such isomorphism which is compatible with the projections to $C_{1,n}$ and $O\text{\textsc{i}}_S/p^n$ by $\varphi$.

Since $D_p$ is an $\Omega$-spectrum which satisfies Nisnevich descent and is $A^1$-local the maps $\Sigma^{-2,-1}\Sigma_+^{\infty} \mathbb{P}_c \rightarrow D_p$ in $\text{SH}(S)$ correspond to maps

$$\mathbb{Z}[\mathbb{P}_c]_{\text{Zar}}[-1] \rightarrow D_{p,1}$$

in $\mathcal{D}(\text{Sh}(\text{Sm}_{S,Zar}, \mathbb{Z}))$. We let $\alpha : \Sigma^{-2,-1}\Sigma_+^{\infty} \rightarrow D_p$ correspond to $\mathbb{Z}[\mathbb{P}_c]_{\text{Zar}} \rightarrow O\text{\textsc{i}}_S^*[1] \rightarrow (O\text{\textsc{i}}_S^*)^{\wedge p}[1] \xrightarrow{\varphi^{-1}} D_{p,1}[1]$, where the first map classifies the tautological line bundle $O(-1)$ on $\mathbb{P}_c$.

The definition of the bonding maps in $D_p$ implies that the map $\mathbb{Z}[G_{m,S}, \{1\}]_{\text{Zar}} \rightarrow D_{p,1}$ corresponding to the unit map $\Sigma^{-1,-1}\Sigma_+^{\infty}(G_{m,S}, \{1\}) \cong 1 \rightarrow D_p$ is the map

$$\mathbb{Z}[G_{m,S}, \{1\}]_{\text{Zar}} \rightarrow O\text{\textsc{i}}_S^* \rightarrow (O\text{\textsc{i}}_S^*)^{\wedge p} \xrightarrow{\varphi^{-1}} D_{p,1}.$$ 

Note that this composition is independent of the particular choice of $\varphi$ since we have

$$\text{Hom}_{\mathcal{D}(\text{Sh}(\text{Sm}_{S,Zar}, \mathbb{Z}))}((\mathbb{Z}[G_{m,S}, \{1\}]_{\text{Zar}}, (O\text{\textsc{i}}_S^*)^{\wedge p}) \cong \mathbb{Z}_p$$

$$\cong \lim_n \text{Hom}_{\mathcal{D}(\text{Sh}(\text{Sm}_{S,Zar}, \mathbb{Z}))}((\mathbb{Z}[G_{m,S}, \{1\}]_{\text{Zar}}, O\text{\textsc{i}}_S^*/p^n).$$

Let $\psi : (\mathbb{P}_c, \{\infty\}) \rightarrow G_{m,S} \wedge S^1$ be the canonical isomorphism in $\mathcal{H}_*(S)$ and let $c : \mathcal{H}_*(S) \rightarrow D^{A^1}(\text{Sh}(\text{Sm}_{S,Nis}, \mathbb{Z}))$ be the canonical map. Then the composition

$$\mathbb{Z}[\mathbb{P}_c, \{\infty\}]_{\text{Zar}} \cong c((\mathbb{P}_c, \{\infty\})) \xrightarrow{c(\psi)} c(G_{m,S} \wedge S^1) \cong \mathbb{Z}[G_{m,S}, \{1\}]_{\text{Zar}}[1] \rightarrow O\text{\textsc{i}}_S^*[1]$$

in $D^{A^1}(\text{Sh}(\text{Sm}_{S,Nis}, \mathbb{Z}))$ classifies the tautological line bundle on $\mathbb{P}_c$. We see from these considerations that $o$ is indeed an orientation.

**Proposition 4.23:** The spectrum in $\text{SH}(S)$ associated with $D_p$ is orientable.

### 4.2 The completed part

Set $D := \prod_p D_p$, where the $D_p$ are the algebras from the last section viewed as $E_\infty$-algebras in spectra in $\text{Cpx}(\text{Sh}(\text{Sm}_{S,Zar}, \mathbb{Z}))$ and the product is taken over all primes.

Then for $X \in \text{Sm}_S$ we have

$$D_r|_{X_{\text{Zar}}} \cong (\prod_p (\mathcal{M}^X(r))^{\wedge p})[r]$$

in $\mathcal{D}(\text{Sh}(X_{\text{Zar}}, \mathbb{Z}))$. 

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Corollary 4.24: The spectrum in $\text{SH}(S)$ associated with $D$ is orientable.

Proof. This follows from Proposition 4.23. □

4.3 The rational parts

We denote by $D_Q$ the rationalization of $D$ as an $E_\infty$-spectrum.

We denote by $H_B$ the Beilinson spectrum over $S$, see [6, Definition 13.1.2]. It has a natural $E_\infty$-structure ([6, Corollary 13.2.6]) and is orientable ([6, 13.1.5]).

Theorem 4.25: $H_B$ is the initial $E_\infty$-spectrum among rational orientable $E_\infty$-spectra.

Proof. This is [6, Corollary 13.2.15 (Rv)]. □

Corollary 4.26: There is a canonical map of $E_\infty$-spectra $H_B \to D_Q$.

Proof. This follows from Corollary 4.24 and Theorem 4.25. □

4.4 The definition

Definition 4.27: We denote by $M\mathbb{Z}$ the homotopy pullback in $E_\infty$-spectra of the diagram

$$
\begin{array}{ccc}
D & \longrightarrow & D_Q \\
\downarrow & & \downarrow \\
H_B & \longrightarrow & 
\end{array}
$$

If we want to emphasize the dependence on $S$ we also write $M\mathbb{Z}_S$.

5 Motivic Complexes II

5.1 A strictification

In this section we enlarge the motivic complexes from section 3 to presheaves on all of $\text{Sm}_S$. We need some preparations.

For each $n \in \mathbb{N}$ we define a category $\mathcal{E}_n$ together with a functor $\varphi_n: \mathcal{E}_n \to [n]$, where $[n]$ is the category $0 \to 1 \to \cdots \to n$. The objects of $\mathcal{E}_n$ are triples $(A, B, i)$ where $i \in [n]$ and $A \subset B \subset \{i, \ldots, n\}$ with $i \in A$. There is exactly one morphism from $(A, B, i)$ to $(A', B', j)$ if $i \leq j$, $B \cap \{j, \ldots, n\} \subset B'$ and $A' \subset A$, otherwise there is no such morphism.
The functor \( \varphi_n \) is determined by the fact that \((A, B, i)\) is mapped to \(i\). We declare a map \( f \) in \( \mathcal{E}_n \) to be a weak equivalence if \( \varphi_n(f) \) is an identity.

A category with weak equivalences is a category \( \mathcal{C} \) together with a subcategory \( \mathcal{W} \) of \( \mathcal{C} \) such that every isomorphism in \( \mathcal{C} \) lies in \( \mathcal{W} \). A homotopical category is a category with weak equivalences satisfying the two out of six property, see [9, 8.2].

For a category \( \mathcal{C} \) with weak equivalences \( \mathcal{W} \) we denote by \( L^H_{\mathcal{W}} \mathcal{C} \) its hammock localization, see [8]. If it is clear which weak equivalences are meant we also write \( L^H \mathcal{C} \).

A morphism in \([n]\) is defined to be a weak equivalence if it is an identity. So both \( E_n \) and \([n]\) are homotopical categories. Since \([n]\) is the homotopy category of \( L^H([n]) \) there is a natural simplicial functor \( L^H E_n \to [n] \).

Proposition 5.1: The natural functor \( L^H E_n \to [n] \) is an equivalence of simplicial categories.

Before giving the proof we need some preparations.

For us a direct category is a category with a chosen degree function, see [IN, Definition 5.1.1].

Lemma 5.2: Let \( I \) be a direct category and \( J \subset I \) a full subcategory such that no arrow in \( I \) has a domain which is not in \( J \) and a codomain which is in \( J \). Let \( \mathcal{C} \) be a model category and \( D : I \to \mathcal{C} \) a cofibrant diagram for the projective model structure. Then \( D|_J \) is cofibrant in \( \mathcal{C}^J \).

Proof. The right adjoint \( r \) to the restriction functor \( \mathcal{C}^I \to \mathcal{C}^J \) is a right Quillen functor since for \( i \in I \setminus J \) we have \( r(D)(i) = * \).

Lemma 5.3: Let \( I \) be a direct category and \( J \subset I \) a full subcategory such that no arrow in \( I \) has a domain which is not in \( J \) and a codomain which is in \( J \). Let \( \mathcal{C} \) be a model category and \( D : I \to \mathcal{C} \) a cofibrant diagram for the projective model structure. Then the canonical map \( \text{colim}(D|_J) \to \text{colim}D \) is a cofibration.

Proof. The object \( \text{colim}D \) is obtained from \( \text{colim}(D|_J) \) by successively gluing in the \( D(i) \) for \( i \in I \setminus J \) for increasing degree of \( i \). The domains of the attaching maps are corresponding latching spaces.

For \( i \in [n] \) let \( \mathcal{E}_{n,i} := \varphi^{-1}(i) \) and \( \mathcal{E}_{n,\leq i} \) be the full subcategory of \( \mathcal{E}_n \) of objects \((A, B, j)\) with \( j \leq i \). It is easily seen that \( \mathcal{E}_{n,\leq i} \) can be given the structure of a direct category.
For \( j \leq i \leq n \) let \( \mathcal{E}_{n,[j,i]} := \varphi_n^{-1}(\{j, \ldots, i\}) \).

**Lemma 5.4:** Let \( C \) be a model category and \( D: \mathcal{E}_{n,\leq i} \to C \) be a projectively cofibrant diagram. Then for \( k \leq j \leq i \) the restriction \( D|\mathcal{E}_{n,[k,j]} \) is also cofibrant.

**Proof.** Let \( F: \mathcal{E}_{n,[k,j]} \to \mathcal{E}_{n,\leq i} \) be the inclusion. We claim that the right adjoint to the restriction functor \( C\mathcal{E}_{n,\leq i} \to C\mathcal{E}_{n,[k,j]} \) is a right Quillen functor. This follows from the fact that for \( l < k \) and an object \( (A,B,l) \in \mathcal{E}_{n,\leq i} \) with \( A \cap [k,j] \neq \emptyset \), the category \( (A,B,l)/F \) has the initial object \( (A,B,l) \to (A \cap \{m, \ldots, n\}, B \cap \{m, \ldots, n\}, m\} \), where \( m = \min(A \cap [k,j]) \).

**Lemma 5.5:** Let \( C \) be a model category and \( D: \mathcal{E}_{n,\leq i} \to C \) be a projectively cofibrant diagram such that for any weak equivalence \( f \) in \( \mathcal{E}_{n,\leq i} \) the map \( D(f) \) is also a weak equivalence. Then for any \( X \in \varphi_n^{-1}(i) \) the map \( D(X) \to \text{colim}D \) is a weak equivalence.

**Proof.** We show by descending induction on \( j \), starting with \( j = i \), that for any \( X \in \varphi_n^{-1}(i) \) the map \( D(X) \to \text{colim}D_{\mathcal{E}_{n,[k,j]}} \) is a weak equivalence. For \( j = i \) this follows from the fact \( \mathcal{E}_{n,i} \) has a final object. Let the statement be true for \( 0 < j + 1 \leq i \) and let us show it for \( j \). Let \( J \subset \mathcal{E}_{n,j} \) be the full subcategory on objects \( (A,B,j) \) such that \( A \cap \{j+1, \ldots, i\} \neq \emptyset \). Then we have a pushout diagram

\[
\begin{array}{ccc}
\text{colim}D|_{J} & \longrightarrow & \text{colim}D|_{\mathcal{E}_{n,j}} \\
\downarrow & & \downarrow \\
\text{colim}D|_{\mathcal{E}_{n,[j+1,i]}} & \longrightarrow & \text{colim}D|_{\mathcal{E}_{n,[j,i]}}
\end{array}
\]

First note that by Lemmas 5.4 and 5.2 all objects in this diagram are cofibrant. Furthermore the upper horizontal map is a cofibration by Lemma 5.3. The full subcategory of \( J \) consisting of objects \( (A,B,j) \) with \( B = \{j, \ldots, n\} \) is homotopy right cofinal in \( J \) and contractible (it has an initial object), thus \( J \) is contractible. Since the diagram \( D|_{J} \) is weakly equivalent to a constant diagram it follows that \( D(X) \to \text{colim}D|_{J} \) is a weak equivalence for any \( X \in J \), thus the upper horizontal map in the above diagram is also a weak equivalence and the induction step follows.

**Lemma 5.6:** Let \( C \) be a model category and \( l \) the left adjoint to the pull back functor \( r:C^{[n]} \to C^\mathcal{E}_n \). Let \( D: \mathcal{E}_n \to C \) be (projectively) cofibrant such that for any weak equivalence \( f \) in \( \mathcal{E}_n \) the map \( D(f) \) is a weak equivalence. Then \( D \to r(l(D)) \) is a weak equivalence.

**Proof.** We have \( l(D)(i) = \text{colim}D|_{\mathcal{E}_{n,\leq i}} \). Thus the claim follows from Lemma 5.5.
Lemma 5.7: Let $F: I \to J$ be an essentially surjective functor between small categories and $W \subset I$ a subcategory making $I$ into a category with weak equivalences. Suppose $F$ sends any map in $W$ to an isomorphism. Then the natural map $L^H_{W,I} \to J$ is a weak equivalence between simplicial categories if and only for any projectively cofibrant diagram $D: I \to \text{sSet}$ such that for any map $f$ in $W$ the map $D(f)$ is a weak equivalence the map $D \to r(l(D))$ is a weak equivalence where $l$ is the left adjoint to the pull back functor $r: \text{sSet}^J \to \text{sSet}^I$.

Proof. Let $C$ be the left Bousfield localization of the model category $\text{sSet}^I$ (equipped with the projective model structure) along the maps $\text{Hom}(f,\cdot)$ where $f$ runs through the maps of $W$. Then $(L^H_{W,I})^{\text{op}}$ is weakly equivalent to the full simplicial subcategory of $\text{sSet}^I$ consisting of cofibrant fibrant objects which become isomorphic in $\text{Ho}(\text{sSet}^I)$ to objects in the image of the composed functor $I^{\text{op}} \to \text{Ho}(\text{sSet}^I) \to \text{Ho}C \to \text{Ho}(\text{sSet}^I)$. Similarly (but easier) $J^{\text{op}}$ is weakly equivalent to a full simplicial subcategory of $\text{sSet}^J$.

The functor $(L^H_{W,I})^{\text{op}} \to J^{\text{op}}$ is described via these equivalences by the restriction of the push forward $\text{sSet}^I \to \text{sSet}^J$ followed by a fibrant replacement functor. The claim follows. □

Proof of Proposition 5.1. The claim follows from Lemmas 5.6 and 5.7. □

Let $f: [n] \to [m]$ be a map in $\Delta$. We define a functor $f_*: \mathcal{E}_n \to \mathcal{E}_m$ by setting $f_*((A, B, i)) = (f(A), f(B), f(i))$. One checks that this determines uniquely $f_*$. Thus we get a cosimplicial object $\mathcal{E}: [n] \to \mathcal{E}_n$ in the category of small categories with weak equivalences. Applying the hammock localization yields a cosimplicial simplicial category $L^H \mathcal{E}: [n] \to L^H \mathcal{E}_n$ together with a map from $L^H \mathcal{E}$ to the standard cosimplicial simplicial category $[\bullet]$ which is levelwise a Dwyer-Kan equivalence.

Let $\mathcal{S}$ be the category of triples $(A, n, (a_1, \ldots, a_n))$ where $A$ is a $D$-algebra such that $\text{Spec}(A) \in \text{Sm}_S$ and $a_1, \ldots, a_n \in A$ generate $A$ as a $D$-algebra. Morphisms are morphisms of $D$-algebras with no compatibility of the generators required. Clearly the functor $\mathcal{S}^{\text{op}} \to \text{Sm}_S$, $A \mapsto \text{Spec}(A)$, is an equivalence onto the full subcategory of $\text{Sm}_S$ of affine schemes.

For $X \in \text{Sm}_S$ and $F = \{f_1, \ldots, f_n\}$ a set of closed immersions $f_i: Z_i \hookrightarrow X$ in $\text{Sm}_S$ we denote by $z_F^r(X)$ the normalized chain complex associated to the simplicial abelian group $[n] \to z_F^r(X, n)$ which is the subsimplicial abelian group of $z^r(X, \bullet)$ of cycles in good position with respect to the $Z_i$. We also write $z_F^r(A)$ for $z_F^r(\text{Spec}(A))$. We have the following moving Lemma due to Marc Levine.

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Theorem 5.8: If \( X \in \text{Sm}_S \) is affine then the inclusion of \( z^r_F(X) \) into the normalized chain complex associated to \( z^r(X, \bullet) \) is a quasi isomorphism.

Proof. This is [26, Theorem 4.9].

Let
\[
(A_0, k_0, (a_{0,1}, \ldots, a_{0,k_0})) \to \cdots \to (A_n, k_n, (a_{n,1}, \ldots, a_{n,k_n}))
\]
be a chain of maps in \( S \), i.e. a \( n \)-simplex, which we denote by \( K \), in the nerve of \( S \). Let \( i \in [n] \) and \( B \subseteq \{i, \ldots, n\} \) with \( i \in B \). Set
\[
C_{i,B} := \bigotimes_{j \in B \setminus \{i\}} A_i[T_{t, \ldots, T_{k_j}}] \cong \bigotimes_{j \in B \setminus \{i\}} A_i[T_{j, \ldots, T_{j,k_j}}],
\]
where the tensor products are over \( A_i \). If \( i \leq j \leq n \), \( B' \subseteq \{j, \ldots, n\} \) with \( j \in B' \) and \( B \cap \{j, \ldots, n\} \subseteq B' \) we define a map \( g_{i,B,j,B'} : C_{i,B} \to C_{j,B'} \) over the map \( A_i \to A_j \) by sending a variable \( T_{t,m} \) for \( l > j \) to the respective variable \( T_{t,m} \), and to the image of the element \( a_{t,m} \) in \( A_j \) for \( l \leq j \). If furthermore \( j \leq k \leq n \) and \( B'' \subseteq \{k, \ldots, n\} \) with \( k \in B'' \) and \( B' \cap \{k, \ldots, n\} \subseteq B'' \) then we have
\[
g_{j,B',k,B''} \circ g_{i,B,j,B'} = g_{i,B,k,B''}.
\]

For \( t = (A, B, i) \in \mathcal{E}_n \), we let \( F_t \) be the set of closed subschemes of \( \text{Spec}(C_{i,B}) \) consisting of the \( \text{Spec}(g_{i,B,j,B' \cap \{j, \ldots, n\}}) \) for \( j \in A \setminus \{i\} \). For such \( t \), set \( C_t := C_{i,B} \). Clearly for \( j \in A \) we have a pullback functor \( z^r_F(C_t) \to z^r(C_{j,B' \cap \{j, \ldots, n\}}) \) induced by pullback of cycles (see Appendix [13]), which for \( B' \subseteq \{j, \ldots, n\} \) with \( B \cap \{j, \ldots, n\} \subseteq B' \) we can prolong via smooth pullback to a map to \( z^r(C_{j,B'}) \).

Lemma 5.9: Let \( t \to s \) be a map in \( \mathcal{E}_n \). Then the above map \( z^r_F(C_t) \to z^r(C_s) \) factors through \( z^r_{F_s}(C_s) \).

Proof. Let \( t = (A, B, i) \) and \( s = (A', B', j) \). Set \( s' := (A \cap \{j, \ldots, n\}, B \cap \{j, \ldots, n\}, j) \). Without loss of generality we can assume \( A' = A \cap \{j, \ldots, n\} \). Clearly the map
\[
z^r_{F_t}(C_t) \to z^r(C_{s'})
\]
factors through \( z^r_{F_{s'}}(C_{s'}) \). If \( A' = \{j\} \) we are done, otherwise fix \( k \in A' \setminus \{j\} \). Set \( B'' := (B \cap \{j, \ldots, n\}) \cup (B' \cap \{j, \ldots, k\}) \) and \( s'' := (A', B'', j) \). Then we have a well defined map
\[
z^r_{F_{s'}}(C_{s'}) \to z^r_{g_{j,B'',k,B'' \cup \{k, \ldots, n\}}}(C_{s''})
\]
and
since cycles meet in the correct codimension. Furthermore we have a well defined map
\[ z^r_{(g_j,B''_j,k,B''_j(k,...,n))}(C_{S''}) \to z^r_{(g_j,B''_j,k,B''_j(k,...,n))}(C_s) \]
for the same reason. Altogether we see that cycles meet as claimed.

For \( f : t \to s \) a map in \( E_n \) we let \( \alpha_K(f) : z^r_{F_t}(C_t) \to z^r_{F_s}(C_s) \) be the map constructed above using Lemma 5.9.

**Lemma 5.10:** For \( f : t \to s \) and \( g : s \to r \) two maps in \( E_n \) we have \( \alpha_K(g \circ f) = \alpha_K(g) \circ \alpha_K(f) \).

**Proof.** Let \( t = (A,B,i), s = (A',B',j) \) and \( r = (A'',B'',k) \). Then the map \( \alpha_K(f) \) is defined by pulling back cycles via the map \( \text{Spec}(g_i,B,j,B'') \) and the map \( \alpha_K(g) \) by pull back via \( \text{Spec}(g_j,B',k,B'') \). Thus the claim follows from (14) and Theorem B.3. □

Setting \( \alpha_K(t) := z^r_{F_t}(C_t) \) for \( t \) an object of \( E_n \) and using Lemma 5.10 we get a functor \( \tilde{\alpha}_K : E_n \to \text{Cpx}(\text{Ab}) \).

By restricting everything to opens \( U \) in \( S_{\text{Zar}} \) we get a functor \( \tilde{\alpha}_K : E_n \to \text{Cpx}(\text{Sh}(S_{\text{Zar}}, \mathbb{Z})) \).

**Lemma 5.11:** The functor \( \tilde{\alpha}_K \) sends weak equivalences in \( E_n \) to quasi isomorphisms.

**Proof.** This follows from Theorem 5.8, Theorem 3.14 and the fact that for \( X \in \text{Sm}_{S} \) the push forward of \( (X_{\text{Zar}} \ni Y \mapsto z^r(Y)) \) via the structure morphism \( X \to S \) computes the derived push forward, which follows from [27, Theorem 1.7]. □

**Lemma 5.12:** Let \( f : [m] \to [n] \) be a monomorphism in \( \Delta \) and \( K \) an \( n \)-simplex in the nerve of \( S \). Then the composition \( E_m \xrightarrow{f^*} E_n \xrightarrow{\tilde{\alpha}_K} \text{Cpx}(\text{Sh}(S_{\text{Zar}}, \mathbb{Z})) \) is equal to \( \tilde{\alpha}_{f^*K} \).

**Proof.** We use a superscript \( K \) or \( f^*K \) to distinguish between the objects which are defined above for \( K \) respectively \( f^*K \). We have \( C_t^{f^*K} = C_{f_t}^K \) and \( F_t^{f^*K} = F_{f_t}^K \), for \( t \) an object of \( E_m \). Thus the claim follows on objects. The definitions of the two functors on morphisms also coincide, thus the claim follows. □

For a category \( I \) we let \( \hat{I} \) be the subcategory of \( I \times \mathbb{N} \) (where \( \mathbb{N} \) is a category in the usual way) which has all objects and where a map \( (A,n) \to (B,m) \) belongs to \( \hat{I} \) if and only if the map \( A \to B \) is the identity or if \( m > n \). Note that a composition of non-identity maps is again a non-identity map in \( \hat{I} \).
We let a map \((A, n) \to (B, m)\) in \(\hat{I}\) be a weak equivalence if and only if the map \(A \to B\) is the identity. We have a canonical projection functor \(p: \hat{I} \to I\).

**Proposition 5.13:** For any category \(I\) the canonical functor \(L^H \hat{I} \to I\) is a weak equivalence of simplicial categories.

**Proof.** We use Lemma 5.7. Let \(C\) be a model category and let \(C^\hat{I}\) be equipped with the projective model structure (which exists since \(\hat{I}\) has the structure of a direct category). Let \(D: \hat{I} \to C\) be a cofibrant diagram which preserves weak equivalences. For \(i \in I\) the diagram \(D|_{p/i}\) is also cofibrant by [38, Lemma 4.2] (it is not used here that \(C^I\) also should have a model structure). The full subcategory \(J\) comprised by the \(((i, n), p((i, n)) \xrightarrow{\text{id}} i)\) in \(p/i\) is homotopy right cofinal, thus \(\text{colim}(D|_{p/i}) \simeq \text{hocolim}(D|_j)\) from which it follows that \(D \to r(l(D))\), where \(l\) is the left adjoint to \(r: C^I \to C^\hat{I}\), is a weak equivalence. \(\square\)

Let \(\hat{N}\) be the nerve of \(\hat{S}\) and \(\pi\) the nerve of the map \(p\) from above. For any \(K \in \hat{N}_n\) we let \(f_K: [n] \to [n']\) be the unique epimorphism in \(\Delta\) such that \(K = f_K^*(K')\) with \(K' \in \hat{N}_n'\) non-degenerate. \(K'\) is then also uniquely determined. We let \(\beta_K\) be the composition

\[
\hat{E}_n \xrightarrow{f_K} E_n' \xrightarrow{\alpha_{\pi(K')}} C_{\text{px}}(\text{Sh}(S_{\text{Zar}}, \mathbb{Z})).
\]

The reason for introducing \(\hat{S}\) is the following observation.

**Lemma 5.14:** Let \(h: [m] \to [n]\) be a map in \(\Delta\) and \(K \in \hat{N}_n\). Then the composition \(\hat{E}_m \xrightarrow{h_*} \hat{E}_n \xrightarrow{\beta_K} C_{\text{px}}(\text{Sh}(S_{\text{Zar}}, \mathbb{Z}))\) is equal to \(\beta_{h^*K}\).

**Proof.** Since every composition of non-identity maps in \(\hat{S}\) is a non-identity map we have a commutative diagram

\[
\begin{array}{ccc}
[m] & \xrightarrow{h} & [n] \\
\downarrow & & \downarrow \text{f}_K \\
[m'] & \xrightarrow{\text{f}_{h^*K}} & [n']
\end{array}
\]

where the bottom horizontal map is a monomorphism. Thus the claim follows from Lemma 5.12 and the definition of the maps \(\beta_K\) and \(\beta_{h^*K}\). \(\square\)

Let \(\Gamma: C_{\text{px}}(\text{Sh}(S_{\text{Zar}}, \mathbb{Z})) \to C_{\text{px}}(\text{Ab})\) be a fibrant replacement functor followed by the global sections functor. We denote by \(q_i\) the subcategory of quasi isomorphisms of \(C_{\text{px}}(\text{Ab})\). By Lemma 5.11 we get for any \(K \in \hat{N}_n\) induced functors \(L^H(\Gamma \circ \beta_K): L^H \hat{E}_n \to L^H_{q_i} C_{\text{px}}(\text{Ab})\) which are compatible with maps in \(\Delta\) by Lemma 5.14.
Let \( q_\bullet : Q_\bullet \to L^H \mathcal{E} \) be a map between cosimplicial objects in \( s\text{Cat} \). For \( K \in \mathcal{N}_n \) let \( \gamma_K := L^H (\Gamma \circ \beta_K) \circ q_n \). Then the \( \gamma_K \) are again compatible with maps in \( \Delta \). By a coend construction we can pair a simplicial set \( L \) and any cosimplicial object \( P_\bullet \) in \( s\text{Cat} \) to obtain an object of \( s\text{Cat} \) which we denote by \( \mathcal{D}_{P_\bullet} \). If \( L \) is the nerve of a category \( C \) we let \( \mathcal{D}_{C_\bullet} := \mathcal{D}_{P_\bullet} \). We have \( \mathcal{D}_{[\bullet]} \cong \hat{S} \), thus we have a natural map \( \mathcal{D}_{Q_\bullet} \to \hat{S} \).

**Lemma 5.15:** If \( Q_\bullet \) is Reedy cofibrant and the map \( Q_\bullet \to [\bullet] \) is a weak equivalence then the natural map \( \mathcal{D}_{Q_\bullet} \to \hat{S} \) is a weak equivalence of simplicial categories.

**Proof.** One deduces the result from the analogous statement for the usual adjunction between simplicial sets and simplicial categories involving the simplicial nerve functor, \cite[Theorem 2.2.0.1]{30}.

From now on suppose that \( Q_\bullet \) is Reedy cofibrant and that the map \( Q_\bullet \to [\bullet] \) is a weak equivalence (which can always be achieved by a cofibrant replacement of \( \Gamma \circ \beta_K \) in \( \text{Cat}^{\Delta} \)). The compatible maps \( \gamma_K \) give rise to an induced map \( \gamma : \mathcal{D}_{Q_\bullet} \to L^H Cpx(\text{Ab}) \).

**Lemma 5.16:** The map \( \gamma \) gives rise to a diagram \( \gamma' \in \text{Ho}(Cpx(\text{Ab})^\hat{S}) \) which is well-defined up to canonical isomorphism.

**Proof.** This follows from a strictification result, see \cite[Proposition 4.2.4.4]{30}.

We define the motivic complex \( M(r) \) to be the push forward of \( \gamma'[-2r] \) with respect to the composition \( \text{Ho}(Cpx(\text{Ab})^\hat{S}) \to \text{Ho}(Cpx(\text{Ab})^\hat{S}) \to \text{D}(\text{Sh}(\text{Sm}_S^{\text{Zar}}, \mathbb{Z})) \), where the first map is induced by \( p : \hat{S} \to S \) and the second map is the Zariski localization map.

**5.2 Properties of the motivic complexes**

Let \( C, S' \) be categories and \( I \) a small category. Let \( \mathcal{E}' \) be a cosimplicial object in \( s\text{Cat} \) over \([\bullet]\). Let for any \( n \)-simplex \( K \) of the nerve of \( S' \) be a functor \( \alpha_K : \mathcal{E}'_n \times I \to C \) be given. Suppose these functors are compatible for monomorphisms in \( \Delta \), i.e. that for \( f : [m] \to [n] \) a monomorphism we have \( \alpha_K \circ (f \times \text{id}) = \alpha_{f\circ K} \). Then for \( K \) a \( n \)-simplex of the nerve of \( S' \times I \) we let \( T(\alpha)_{\hat{K}} \) be the composition \( \mathcal{E}'_n \to \mathcal{E}'_n \times I \xrightarrow{\alpha_{\hat{K}}} C \), where the second component of the first map is the composition \( \mathcal{E}'_n \to [n] \to I \) (the second map being the second component of \( \hat{K} \)) and where \( K \) is the first component of \( \hat{K} \). The \( T(\alpha)_{\hat{K}} \) are then again compatible for monomorphisms in \( \Delta \).

Let \( p : S'' \to S' \times I \) be a functor and suppose that the composition in \( S'' \) of two non-identity maps is a non-identity map. Let \( K \) be a \( n \)-simplex of the nerve of \( S'' \).
Let \( f : [n] \to [n'] \) be the unique epimorphism in \( \Delta \) such that \( K = f^*(K') \) for a non-degenerate \( n' \)-simplex \( K' \). Let \( T^p(\alpha)_K \) be the composition \( \mathcal{E}'_{n'} \xrightarrow{f_*} \mathcal{E}'_{n'} \xrightarrow{T(\alpha)_{\tilde{K}}} \mathcal{C} \), where \( \tilde{K} \) is the image of \( K' \) in the nerve of \( S' \times I \). Then the \( T^p(\alpha)_K \) are compatible for all maps in \( \Delta \).

In our applications \( S'' \) will be \( \hat{S}' \times I \).

### 5.2.1 Comparison to flat maps

Let the notation be as in the last section. We denote by \( S^{\text{fl}} \) the subcategory of \( S \) consisting of flat maps.

Let \( K \) be an \( n \)-simplex in the nerve of \( S^{\text{fl}} \). In particular we have a chain \( A_0 \to \cdots \to A_n \) of smooth \( D \)-algebras where each map is flat. We associate to this the functor \( \alpha'_K : [n] \to \text{Cpx} \left( \text{Sh}(S_{Zar}, \mathbb{Z}) \right) \) which sends \( i \) to \( (U \mapsto z^i(\text{Spec}(A_i) \times_S U)) \) and where the maps are induced by flat pullback of cycles. We denote by \( \tilde{\alpha}'_K \) the composition \( \mathcal{E}_n \xrightarrow{\varphi_n} [n] \xrightarrow{\alpha'_K} \text{Cpx} \left( \text{Sh}(S_{Zar}, \mathbb{Z}) \right) \).

Recall the maps \( \tilde{\alpha}_K \). We have a natural transformation \( \tilde{\alpha}'_K \to \tilde{\alpha}_K \) which is induced by the maps \( A_i \to C_t \) for \( t = (A, B, i) \in \mathcal{E}_n \). We note that the cycle conditions given by the \( F_t \) are fulfilled since for a map \( t \to s \) in \( \mathcal{E}_n \) with \( s = (A', B', j) \) the diagram

\[
\begin{array}{ccc}
C_t & \longrightarrow & C_s \\
\downarrow & & \downarrow \\
A_i & \longrightarrow & A_j
\end{array}
\]

commutes.

We denote by \( \alpha_K : \mathcal{E}_n \times [1] \to \text{Cpx} \left( \text{Sh}(S_{Zar}, \mathbb{Z}) \right) \) the functor corresponding to this natural transformation.

Thus as in the beginning of section 5.2 we get a compatible family of maps \( T^p(\alpha)_K \), where \( p \) is the functor \( S^{\text{fl}} \times [1] \to S^{\text{fl}} \times [1] \).

For \( K \) a \( n \)-simplex in the nerve of \( S^{\text{fl}} \times [1] \) let \( \tilde{\gamma}_K := L^H (\Gamma \circ T^p(\alpha)_K) \circ q_n \).

The \( \tilde{\gamma}_K \) glue to give a map

\[
\tilde{\gamma} : D_{Q^*}^{S^{\text{fl}} \times [1]} \to L^H_{q_n} \text{Cpx}(\text{Ab}).
\]

We denote by \( \tilde{\gamma}' \in \text{Ho}(\text{Cpx}(\text{Ab})^{S^{\text{fl}} \times [1]}) \) the diagram canonically associated to \( \tilde{\gamma} \).

**Lemma 5.17:** \( \gamma|_{S^{\text{fl}}} \) and \( \tilde{\gamma}'|_{S^{\text{fl}} \times [1]} \) are canonically isomorphic.
Proof. This follows by construction of $\gamma'$ and $\tilde{\gamma}$.

**Lemma 5.18:** $\gamma|_{\tilde{S}_n \times \{0\}}$ is canonically isomorphic to the diagram on $\tilde{S}$ which associates to an $(A, n, (a_1, \ldots, a_n), m)$ the cycle complex $z^r(A)$.

Proof. This follows by construction of $\tilde{\gamma}$.

Let $\text{Sm}_{\tilde{S}}$ be the subcategory of $\text{Sm}_S$ of flat maps.

**Corollary 5.19:** The complex $\mathcal{M}(r)|_{\text{Sm}_{\tilde{S}}}$ is canonically isomorphic to the diagram on $\tilde{S}$ which associates to an $(A, n, (a_1, \ldots, a_n), m)$ the cycle complex $z^r(A)$.

Proof. This follows from Lemmas 5.17 and 5.18, the fact that the map in $\text{Ho}(\text{Cpx}(\text{Ab})^{\tilde{S}})$ associated to $\tilde{\gamma}$ is an isomorphism, the fact (which follows from these Lemmas) that the push forward of $\gamma'$ with respect to $\text{Ho}(\text{Cpx}(\text{Ab})^{\tilde{S}}) \to \text{Ho}(\text{Cpx}(\text{Ab})^S)$ has Zariski descent and from Proposition 5.13 (or better its proof).

**Corollary 5.20:** For $X \in \text{Sm}_S$ there is a canonical isomorphism $\mathcal{M}^X(r) \cong \mathcal{M}(r)|_{X_Zar}$ in $D(\text{Sh}(X_{Zar}, \mathbb{Z}))$.

Proof. This follows from Corollary 5.19.

### 5.2.2 Some localization triangles

We still keep the notation of section 5.1. Let $A$ be a smooth $D$-algebra. Let $K$ be a $n$-simplex in the nerve of $S$. For $t \in E_n$ set $C_i^A := A \otimes_D C_i$ and $F_i^A := \{\text{Spec}(A) \times_S a | a \in F_i\}$. Then as in section 5.3 we get functors

$$\alpha^K_A : E_n \to \text{Cpx(}\text{Ab}), t \mapsto z^r_{F_i^A}(C_i^A),$$

and

$$\tilde{\alpha}_A^K : E_n \to \text{Cpx(Sh(S_{Zar}, \mathbb{Z}))}.$$ 

Now let $a_1, \ldots, a_k \in A$ be generators of $A$. Set $A := (A, k, (a_1, \ldots, a_k)) \in S$. For $(A', k', (a'_1, \ldots, a'_k)) \in S$ let

$$A \otimes (A', k', (a'_1, \ldots, a'_k)) := (A \otimes_D A', k + k', (a_1 \otimes 1, \ldots, a_k \otimes 1, 1 \otimes a'_1, \ldots, 1 \otimes a'_k)) \in S.$$ 

Similarly for a $n$-simplex $K$ in the nerve of $S$ the $n$-simplex $A \otimes K$ is defined.
For $K$ a $n$-simplex in the nerve of $S$ we have a natural transformation $\tilde{\alpha}^A_K \to \tilde{\alpha}^A_{\Delta \otimes K}$ induced by the obvious inclusion maps of algebras. We denote by 

$$\alpha^A_K : E_n \times [1] \to \text{Cpx}(\text{Sh}(S_{\text{Zar}}, \mathbb{Z}))$$

the functor corresponding to this natural transformation.

For $K$ a $n$-simplex in the nerve of $\hat{S} \times [1]$ we let

$$\gamma^A_K : = L^H(\Gamma \circ T_p(\alpha^A_K) \circ q_n),$$

where $p$ is the functor $\hat{S} \times [1] \to S \times [1]$.

The $\gamma^A_K$ glue to give a map

$$\gamma^A : D_{\hat{S} \times [1]} \to L^H_{q_n} \text{Cpx}(\text{Ab}).$$

We denote by $\gamma'_A \in \text{Ho}(\text{Cpx}(\text{Ab})^{\hat{S} \times [1]})$ the diagram canonically associated to $\gamma^A$.

**Lemma 5.21**: The push forward of $\gamma'_A[-2r][\hat{S} \times \{1\}]$ to $D(\text{Sh}(\text{Sm}_{S, \text{Zar}}, \mathbb{Z}))$ is canonically isomorphic to $\mathbb{R}\text{Hom}(\mathbb{Z}[\text{Spec}(A)]_{\text{Zar}}, M(r))$.

*Proof*. This follows from the definition of $\gamma'_A$. \qed

**Corollary 5.22**: The push forward of $\gamma'_A[-2r][\hat{S} \times \{0\}]$ to $D(\text{Sh}(\text{Sm}_{S, \text{Zar}}, \mathbb{Z}))$ is canonically isomorphic to $\mathbb{R}\text{Hom}(\mathbb{Z}[\text{Spec}(A)]_{\text{Zar}}, M(r))$.

*Proof*. This follows from the fact that the map in $\text{Ho}(\text{Cpx}(\text{Ab})^{\hat{S}})$ associated to $\gamma'_A$ is an isomorphism. \qed

Now let $f : A \to A'$ be a flat map to a smooth $D$-algebra $A'$, let $a'_1, \ldots, a'_{k'} \in A'$ be generators. We have functors

$$a : S \to S, (B, l, (b_1, \ldots, b_l)) \mapsto (A \otimes_D B, k + l, (a_1 \otimes 1, \ldots, a_k \otimes 1, 1 \otimes b_1, \ldots, 1 \otimes b_l))$$

and

$$b : S \to S, (B, l, (b_1, \ldots, b_l)) \mapsto (A' \otimes_D B, k' + l, (a'_1 \otimes 1, \ldots, a'_{k'} \otimes 1, 1 \otimes b_1, \ldots, 1 \otimes b_l))$$

and a natural transformation $a \to b$ induced by $f$. We let $G : S \times [1] \to S$ be the corresponding functor.

Let $K$ be a $n$-simplex in the nerve of $S \times [1]$. Let $\alpha^f_{2,K} : = \tilde{\alpha}^A_{G(K)}$.

We have a natural transformation $\tilde{\alpha}^A_K \to \tilde{\alpha}^{A'}_K$ induced by $f$. Let $\alpha^f_{i,K} : E_n \times [1] \to \text{Cpx}(\text{Sh}(S_{\text{Zar}}, \mathbb{Z}))$ be the corresponding functor.
We have a natural transformation $\alpha^f_{1,K} \to \alpha^f_{2,K}$ induced by the natural inclusion maps of algebras. We denote by $\alpha^f_K: \mathcal{E}_n \times [1]^2 \to \text{Cpx}(\text{Sh}(S_{\text{Zar}}, \mathbb{Z}))$ the corresponding functor.

For $K$ a $n$-simplex in the nerve of $\hat{S} \times [1]^2$ we let $\gamma^f_K := L^H(\Gamma \circ T^p(\alpha^f_K) \circ q_n$, where $p$ is the functor $\hat{S} \times [1]^2 \to S \times [1]^2$.

The $\gamma^f_K$ glue to give a map

$$\gamma^f: D_{Q*}^\hat{S \times [1]^2} \to L^H_q \text{Cpx}(\text{Ab}).$$

We denote by $\gamma^f_f \in \text{Ho}(\text{Cpx}(\text{Ab})^{\hat{S \times [1]^2}})$ the diagram canonically associated to $\gamma^f$.

**Lemma 5.23:** The push forward to $D(\text{Sh}(\text{Sm}_{S_{\text{Zar}}, \mathbb{Z}}))$ of the map in $\text{Ho}(\text{Cpx}(\text{Ab})^{\hat{S}})$ associated to $\gamma^f_f[-2r]_{\hat{S \times [1] \times (0)}}$ is canonically isomorphic to the map

$$\mathcal{R}\text{Hom}(\mathbb{Z}[\text{Spec}(A)]_{\text{Zar}}, \mathcal{M}(r)) \to \mathcal{R}\text{Hom}(\mathbb{Z}[\text{Spec}(A')]_{\text{Zar}}, \mathcal{M}(r)).$$

**Proof.** This follows from the definition of $\gamma^f_f$. \qed

**Corollary 5.24:** The push forward to $D(\text{Sh}(\text{Sm}_{S_{\text{Zar}}, \mathbb{Z}}))$ of the map in $\text{Ho}(\text{Cpx}(\text{Ab})^{\hat{S}})$ associated to $\gamma^f_f[-2r]_{\hat{S \times [1] \times (1)}}$ is canonically isomorphic to the map

$$\mathcal{R}\text{Hom}(\mathbb{Z}[\text{Spec}(A)]_{\text{Zar}}, \mathcal{M}(r)) \to \mathcal{R}\text{Hom}(\mathbb{Z}[\text{Spec}(A')]_{\text{Zar}}, \mathcal{M}(r)).$$

**Proposition 5.25:** Let $i: Z \to X$ be a closed immersion of affine schemes in $\text{Sm}_S$ of codimension 1 with open affine complement $U$. Then there is an exact triangle

$$\mathcal{R}\text{Hom}(\mathbb{Z}[Z]_{\text{Zar}}, \mathcal{M}(r-1))[-2] \to \mathcal{R}\text{Hom}(\mathbb{Z}[X]_{\text{Zar}}, \mathcal{M}(r))$$

$$\to \mathcal{R}\text{Hom}(\mathbb{Z}[U]_{\text{Zar}}, \mathcal{M}(r)) \to \mathcal{R}\text{Hom}(\mathbb{Z}[Z]_{\text{Zar}}, \mathcal{M}(r-1))[-1]$$

in $D(\text{Sh}(\text{Sm}_{S_{\text{Zar}}, \mathbb{Z}}))$, where the second map is induced by the morphism $U \to X$.

**Proof.** Let $A \to A''$ be the map of function algebras corresponding to $i$ and $A \to A'$ the map corresponding to the open inclusion $U \subset X$.

For $K$ a $n$-simplex in the nerve of $\hat{S}$ we define a functor $\alpha^\gamma_K: \mathcal{E}_n \times [1]^2 \to \text{Cpx}(\text{Ab})$ by sending $(t,0,0)$ to $z_{F^{i}A''}[C_{t}^{A''}]$, $(t,1,0)$ to $z_{F^{i}A'}[C_{t}^{A'}]$, $(t,1,1)$ to $z_{F^{i}A'}[C_{t}^{A''}]$ and $(t,0,1)$ to 0. Sheafification on $S$ yields a functor $\alpha^\gamma_K: \mathcal{E}_n \times [1]^2 \to \text{Cpx}(\text{Sh}(S_{\text{Zar}}, \mathbb{Z}))$.

For $K$ a $n$-simplex in the nerve of $\hat{S} \times [1]^2$ let $\gamma^\gamma_K := L^H(\Gamma \circ T^p(\alpha^\gamma_K) \circ q_n$. 

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The $\gamma_K^n$ glue to give a map

$$\gamma^n: D_{Q}^{\Delta \times [1]^2} \to L^{H}_0 \text{Cpx}(\text{Ab}).$$

We denote by $\gamma'_0 \in \text{Ho}(\text{Cpx}(\text{Ab})^{\Delta \times [1]^2})$ the diagram canonically associated to $\gamma^n$.

The square in $D(\text{Sh}(\text{Sm}_{S,\text{Zar}}, \mathbb{Z}))$ associated to the push forward of $\gamma'_0[-2r]$ is exact by [27, Theorem 1.7]. Moreover by Corollary [5.22] the entries in this square in the places $(0,0)$, $(1,0)$ and $(1,1)$ are $\mathbb{R}\text{Hom}(\mathbb{Z}[Z]_{\text{Zar}}, \mathcal{M}(r-1))[-2]$, $\mathbb{R}\text{Hom}(\mathbb{Z}[X]_{\text{Zar}}, \mathcal{M}(r))$ and $\mathbb{R}\text{Hom}(\mathbb{Z}[U]_{\text{Zar}}, \mathcal{M}(r))$, and the map from entry $(1,0)$ to $(1,1)$ is the one induced by the map $U \subset X$ by Corollary [5.24]. Thus by [29, Definition 1.1.2.11] we get the exact triangle as required.

### 5.2.3 The étale cycle class map

For $X \in \text{Sm}_S$ and $F$ a finite set of closed immersions in $\text{Sm}_S$ with target $X$ we denote by $c'_F(X, n)$ the set of cycles (closed integral subschemes) of $X \times \Delta^n$ which intersect all $Z \times Y$ with $Z \in F \cup \{X\}$ and $Y$ a face of $\Delta^n$ properly.

Let $U \subset S$ open. Let $m$ be an integer which is invertible on $U$. Let $\mu^\otimes r \to \mathcal{G}$ be an injectively fibrant replacement in $\text{Cpx}(\text{Sh}(\text{Sm}_{U, \text{ét}}, \mathbb{Z}/m))$.

Let $X \in \text{Sm}_U$. For $W$ a closed subset of $X$ such that each irreducible component has codimension greater or equal to $r$ set $\mathcal{G}^W(X) := \ker(\mathcal{G}(X) \to \mathcal{G}(X \setminus W))$.

As in [20, 12.3] there is a canonical isomorphism of $H^{2r}(\mathcal{G}^W(X))$ with the free $\mathbb{Z}/m$-module on the irreducible components of $W$ of codimension $r$ and the map $\tau_{\leq 2r} \mathcal{G}^W(X) \to H^{2r}(\mathcal{G}^W(X))[-2r]$ is a quasi isomorphism.

For $F$ a finite set of closed immersions in $\text{Sm}_U$ with target $X$ denote by $\mathcal{G}^r_F(X, n)$ the colimit of the $\mathcal{G}^W(X \times \Delta^n)$ where $W$ runs through the finite unions of elements of $c'_F(X, n)$. The simplicial complex of $\mathbb{Z}/m$-modules $\tau_{\leq 2r} \mathcal{G}^r_F(X, \bullet)$ augments to the simplicial abelian group $z^r_F(X, \bullet)/m[-2r]$. This augmentation is a levelwise quasi isomorphism. We denote by $\mathcal{G}^r_F(X)$ the total complex associated to the double complex which is the normalized complex associated to $\tau_{\leq 2r} \mathcal{G}^r_F(X, \bullet)$. Thus we get a quasi isomorphism $\mathcal{G}^r_F(X) \to z^r_F(X)/m[-2r]$.

On the other hand we have a canonical map $\mathcal{G}^r_F(X, n) \to \mathcal{G}(X \times \Delta^n)$ compatible with the simplicial structure. We denote by $\mathcal{G}'(X)$ the total complex associated to the double complex which is the normalized complex associated to $\mathcal{G}(X \times \Delta^n)$. We have a canonical quasi isomorphism $\mathcal{G}(X) \to \mathcal{G}'(X)$ and a canonical map $\mathcal{G}^r_F(X) \to \mathcal{G}'(X)$.

Thus in $D(\mathbb{Z}/m)$ we get a map

$$z^r_F(X)/m[-2r] \cong \mathcal{G}^r_F(X) \to \mathcal{G}'(X) \cong \mathcal{G}(X).$$
Our next aim is to make this assignment functorial in $X$ for all maps in $\text{Sm}_U$. In the following we sometimes insert into the above definitions $A$ instead of $\text{Spec}(A)$. Let $I$ be the category $0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4$. We denote by $\mathcal{S}_m$ the full subcategory of $\mathcal{S}$ such that $m$ is invertible in the algebras belonging to the objects. We use the notation of section 5.1. Let $K$ be a $n$-simplex in the nerve of $\mathcal{S}_m$. We assign to $K$ the following functor $\alpha'_K: \mathcal{E}_n \times I \rightarrow \text{Cpx}(\text{Ab})$: $(t, 0) \mapsto \alpha_K(t)/m[-2r]$, $(t, 1) \mapsto \mathcal{G}'_t(C_t)$, $(t, 2) \mapsto \mathcal{G}'(C_t)$, $(t, 3) \mapsto \mathcal{G}(C_t)$, $(t, 4) \mapsto \mathcal{G}(A_{\varphi_n(t)})$. Sheafifying on $U_{\text{Zar}}$ yields a functor $\tilde{\alpha}'_K: \mathcal{E}_n \times I \rightarrow \text{Cpx}(\text{Sh}(U_{\text{Zar}}, \mathbb{Z}))$. These functors are compatible for monomorphisms in $\Delta$.

For $K$ a $n$-simplex of the nerve of $\tilde{\mathcal{S}}_m \times I$ let $\gamma'_K := L^H(\Gamma \circ T^p(\tilde{\alpha}')_K) \circ q_n$, where $p$ is the functor $\tilde{\mathcal{S}}_m \times I \rightarrow \mathcal{S}_m \times I$. The $\gamma'_K$ glue to give a map

$$\gamma^I: D_{\mathcal{Q}_*}^{\tilde{\mathcal{S}}_m \times I} \rightarrow L^H_{q_i^*}\text{Cpx}(\text{Ab}).$$

We denote by $\gamma^I \in \text{Ho}(\text{Cpx}(\text{Ab})^{\tilde{\mathcal{S}}_m \times I})$ the diagram canonically associated to $\gamma^I$.

The push forward of the $I$-diagram in $\text{Ho}(\text{Cpx}(\text{Ab})^{\tilde{\mathcal{S}}_m})$ corresponding to the diagram $\gamma^I$ to $D(\text{Sh}(\text{Sm}_{U, \text{Zar}}, \mathbb{Z}))$ is an $I$-diagram of the form $(\mathcal{M}(r)/m)|_U \cong \bullet \rightarrow \bullet \rightarrow \bullet \cong \mathbb{R}\epsilon_*\mu_m^{\otimes r}$ which yields the cycle class map.

Next we wish to show the compatibility of this cycle class map with the original cycle class map defined for flat morphisms.

If in the following notation a collection $F$ of closed subschemes is missing we assume that this $F$ is empty. For $K$ a $n$-simplex in the nerve of $\mathcal{S}^\text{fl}_m$ (with the obvious notation) we define a functor $\alpha''_K: \mathcal{E}_n \times I \rightarrow \text{Cpx}(\text{Ab})$ in the following way: $(t, 0) \mapsto z''(A_{\varphi_n(t)})[-2r]$, $(t, 1) \mapsto \mathcal{G}''(A_{\varphi_n(t)})$, $(t, 2) \mapsto \mathcal{G}'(A_{\varphi_n(t)})$, $(t, 3), (t, 4) \mapsto \mathcal{G}(A_{\varphi_n(t)})$. Sheafifying on $U$ yields a functor $\tilde{\alpha}''_K: \mathcal{E}_n \times I \rightarrow \text{Cpx}(\text{Sh}(U_{\text{Zar}}, \mathbb{Z}))$. There is an obvious natural transformation $\tilde{\alpha}''_K \rightarrow \tilde{\alpha}'_K$. We denote by $\overline{\alpha}_K: \mathcal{E}_n \times I \times [1] \rightarrow \text{Cpx}(\text{Sh}(U_{\text{Zar}}, \mathbb{Z}))$ the corresponding functor.

For $K$ a $n$-simplex of the nerve of $\tilde{\mathcal{S}}^\text{fl}_m \times I \times [1]$ let $\gamma_{K}^{I \times [1]} := L^H(\Gamma \circ T^p(\overline{\alpha})_K) \circ q_n$, where $p$ is the functor $\tilde{\mathcal{S}}^\text{fl}_m \times I \times [1] \rightarrow \mathcal{S}^\text{fl}_m \times I \times [1]$. The $\gamma_{K}^{I \times [1]}$ glue to give a map

$$\gamma_{I \times [1]}: D_{\mathcal{Q}_*}^{\tilde{\mathcal{S}}^\text{fl}_m \times I \times [1]} \rightarrow L^H_{q_i^*}\text{Cpx}(\text{Ab}).$$

We denote by $\gamma_{I \times [1]}^{I \times [1]} \in \text{Ho}(\text{Cpx}(\text{Ab})^{\tilde{\mathcal{S}}^\text{fl}_m \times I \times [1]})$ the diagram canonically associated to $\gamma_{I \times [1]}^{I \times [1]}$.

The push forward of the $I \times [1]$-diagram in $\text{Ho}(\text{Cpx}(\text{Ab})^{\tilde{\mathcal{S}}^\text{fl}_m})$ corresponding to the diagram $\gamma_{I \times [1]}^{I \times [1]}$ to $D(\text{Sh}(\text{Sm}^\text{fl}_{U, \text{Zar}}, \mathbb{Z}))$ is an $I \times [1]$-diagram where the subdiagram indexed on $I \times \{0\}$ gives the old cycle class map and the subdiagram indexed on $I \times \{1\}$ the new
cycle class map restricted to flat maps. Thus the two cycle class maps are canonically isomorphic (over flat maps).

**Corollary 5.26:** For $X \in \text{Sm}_U$ the cycle class map $\mathcal{M}^X(r)/m \to \mathbb{R}_\ast \mathbb{Z}/m(r)$ from section 3 is canonically isomorphic to the cycle class map $(\mathcal{M}(r)/m) |_U \to \mathbb{R}_\ast \mu_m^{\otimes r}$ restricted to $X_{Zar}$.

Now we also use the notation of section 5.2.2. We assume $m$ is invertible in $A$. For an $n$-simplex $K$ of the nerve of $S_m$ we define a functor $(\alpha')^A_K: \mathcal{E}_n \times I \to \text{Cpx}(\text{Ab})$ in the following way: $(t,0) \mapsto \alpha^A_K(t)/m[-2r]$, $(t,1) \mapsto G^r_{E^A}(C^A_t)$, $(t,2) \mapsto G'(C^A_t)$, $(t,3) \mapsto G(C^A_t)$. $(t,4) \mapsto G(A \otimes_D A_{\varphi_n(t)})$. Sheafifying on $U_{Zar}$ yields a functor $(\alpha')^A_K: \mathcal{E}_n \times I \to \text{Cpx}(\text{Sh}(U_{Zar},\mathbb{Z}))$. These functors are compatible for monomorphisms in $\Delta$.

We have a natural transformation $(\alpha')^A_K \to \alpha'_{A\otimes K}$ induced by the obvious inclusion maps of algebras. We denote by $\Gamma^A_K: \mathcal{E}_n \times I \to \text{Cpx}(\text{Sh}(U_{Zar},\mathbb{Z}))$ the corresponding functor.

For $K$ a $n$-simplex of the nerve of $S_m \times I \times [1]$ let $\gamma^A_{K, I \times [1]} = L^H(\Gamma \circ T^p(\Gamma^A_K) \circ q_n$, where $p$ is the functor $S_m \times I \times [1] \to S_m \times I \times [1]$. The $\gamma^A_{K, I \times [1]}$ glue to give a map

$$\gamma^A_{K, I \times [1]}: D^\mathcal{S}_m \times I \times [1] \to L^H_{\text{Cpx}(\text{Ab})}.$$ 

We denote by $\gamma^A_{K, I \times [1]} \in \text{Ho}(\text{Cpx}(\text{Ab})^\mathcal{S}_m \times I \times [1])$ the diagram canonically associated to $\gamma^A_{K, I \times [1]}$.

The push forward of the $I \times [1]$-diagram in $\text{Ho}(\text{Cpx}(\text{Ab})^\mathcal{S}_m)$ corresponding to the diagram $\gamma^A_{K, I \times [1]}$ to $D(\text{Sh}(\text{Sm}^U_{Zar},\mathbb{Z}))$ is an $I \times [1]$-diagram where the subdiagram indexed on $I \times \{1\}$ gives the functor $\mathbb{R}_{\text{Hom}}(\text{Spec}(A), -)$ applied to the cycle class map $(\mathcal{M}(r)/m) |_U \to \mathbb{R}_\ast \mu_m^{\otimes r}$.

**Corollary 5.27:** The subdiagram indexed on $I \times \{0\}$ of the above $I \times [1]$ diagram in $D(\text{Sh}(\text{Sm}^U_{Zar},\mathbb{Z}))$ yields a map canonically isomorphic to the map

$$\mathbb{R}_{\text{Hom}}(\text{Spec}(A), (\mathcal{M}(r)/m) |_U) \to \mathbb{R}_{\text{Hom}}(\text{Spec}(A), \mathbb{R}_\ast \mu_m^{\otimes r})$$

induced by the cycle class map.

Let $i: Z \to X$ be a closed immersion of affine schemes in $\text{Sm}_S$ of codimension 1 with open affine complement $V$. The exact triangle

$$\mathbb{R}_{\text{Hom}}(\mathbb{Z}[Z]_{Zar}, \mathcal{M}(r-1))[-2] \to \mathbb{R}_{\text{Hom}}(\mathbb{Z}[X]_{Zar}, \mathcal{M}(r)) \to \mathbb{R}_{\text{Hom}}(\mathbb{Z}[U]_{Zar}, \mathcal{M}(r)) \to \mathbb{R}_{\text{Hom}}(\mathbb{Z}[Z]_{Zar}, \mathcal{M}(r-1))[-1]$$

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in $\text{D}(\text{Sh}(\text{Sm}_{S,\text{Zar}}, \mathbb{Z}))$ from Proposition 5.25 yields an exact triangle

$$\mathbb{R}\text{Hom}(\mathbb{Z}[Z_U]_{\text{Zar}}, (\mathcal{M}(r-1)/m)|_U)[-2] \to \mathbb{R}\text{Hom}(\mathbb{Z}[X_U]_{\text{Zar}}, (\mathcal{M}(r)/m)|_U) \to \mathbb{R}\text{Hom}(\mathbb{Z}[V_U]_{\text{Zar}}, (\mathcal{M}(r)/m)|_U) \to \mathbb{R}\text{Hom}(\mathbb{Z}[Z_U]_{\text{Zar}}, (\mathcal{M}(r-1)/m)|_U)[-1]$$

in $\text{D}(\text{Sh}(\text{Sm}_{U,\text{Zar}}, \mathbb{Z}))$.

**Proposition 5.28:** Let the notation be as above. Then the diagram

$$
\begin{array}{ccc}
\mathbb{R}\text{Hom}(\mathbb{Z}[Z_U]_{\text{Zar}}, (\mathcal{M}(r-1)/m)|_U)[-2] & \longrightarrow & \mathbb{R}\epsilon_*\mathbb{R}\text{Hom}(\mathbb{Z}[Z_U]_{\text{ét}}, \mu_m^{\otimes (r-1)})[-2] \\
\mathbb{R}\text{Hom}(\mathbb{Z}[X_U]_{\text{Zar}}, (\mathcal{M}(r)/m)|_U) & \longrightarrow & \mathbb{R}\epsilon_*\mathbb{R}\text{Hom}(\mathbb{Z}[X_U]_{\text{ét}}, \mu_m^{\otimes r}) \\
\mathbb{R}\text{Hom}(\mathbb{Z}[V_U]_{\text{Zar}}, (\mathcal{M}(r)/m)|_U) & \longrightarrow & \mathbb{R}\epsilon_*\mathbb{R}\text{Hom}(\mathbb{Z}[V_U]_{\text{ét}}, \mu_m^{\otimes r}) \\
\mathbb{R}\text{Hom}(\mathbb{Z}[Z_U]_{\text{Zar}}, (\mathcal{M}(r-1)/m)|_U)[-1] & \longrightarrow & \mathbb{R}\epsilon_*\mathbb{R}\text{Hom}(\mathbb{Z}[Z_U]_{\text{ét}}, \mu_m^{\otimes (r-1)})[-1],
\end{array}
$$

where the first vertical row is the exact triangle from above, the second vertical row is the corresponding exact triangle for étale sheaves and where the horizontal maps are induced by the cycle class maps, commutes.

**Proof.** Let $A \to A''$ be the map of function algebras corresponding to $i$ and $A \to A'$ the map corresponding to the open inclusion $V \to X$. We let $J$ be the category which is defined by gluing the object $(0,0)$ of $[1]^2$ to the object $0$ of $[1]$. We call $c$ the object $1$ of $[1]$ viewed as object of $J$, the other objects are numbered $(k,l), k,l \in \{0,1\}$. Let $\mu_m^{\otimes (r-1)} \to \mathbb{G}$ be an injectively fibrant replacement in $\text{Cpx}(\text{Sh}(\text{Sm}_{U,\text{ét}}, \mathbb{Z}/m))$. We let $\mathcal{H}_t(n)$ be the colimit of the $\mathcal{G}^W((\text{Spec}C^A_t) \times \Delta^n)$, where $W$ runs through the finite unions of elements of $c^{-1}(\text{Spec}C^A_t, n)$. We denote by $\mathcal{H}_t$ the total complex associated to the double complex which is the normalized complex associated to $\tau_{\leq 2r} \mathcal{H}_t(\bullet)$. We have an absolute purity isomorphism $\varphi$ from the sheaf

$$\text{Sm}_U \ni Y \mapsto \ker(\mathcal{G}(Y \times_S X) \to \mathcal{G}(Y \times_X V))$$

to

$$\text{Sm}_U \ni Y \mapsto \mathcal{G}(Y \times_S Z)[-2]$$
5.3 The naive $G_m$-spectrum

**Proposition 5.29:** There is a canonical isomorphism

$$\mathcal{M}(r-1)[-1] \cong \bigoplus \underline{\text{Hom}}(\mathbb{Z}[G_m, S], \mathcal{M}(r))$$

in $D(\text{Sh}(\text{Sm}_{S, \text{Zar}}, \mathbb{Z}))$. 

\[\square\]
Proof. By Proposition 5.25 there is an exact triangle
\[ \mathcal{M}(r-1)[-2] \to \mathbb{R}\text{Hom}(\mathbb{Z}[\Delta^1]_{\text{zar}}, \mathcal{M}(r)) \to \mathbb{R}\text{Hom}(\mathbb{Z}[\mathbb{G}_m, \text{Sar}], \mathcal{M}(r)) \to \mathcal{M}(r-1)[-1]. \]

There is a split \( \mathbb{R}\text{Hom}(\mathbb{Z}[\mathbb{G}_m, \text{Sar}], \mathcal{M}(r)) \to \mathbb{R}\text{Hom}(\mathbb{Z}[\Delta^1]_{\text{zar}}, \mathcal{M}(r)) \) induced by \( \{1\} \subset \mathbb{G}_m, \text{Sar} \) and the \( \mathbb{A}^1 \)-invariance of \( \mathcal{M}(r) \). This induces the required isomorphism. \( \square \)

We thus get a naive \( \mathbb{Z}[\mathbb{G}_m, \text{Sar}, \{1\}]_{\text{zar}} \)-spectrum \( \mathcal{M} \) (in the sense of [32 I 6]) in \( \text{D}(\text{Sh}(\text{Sm}_{k, \text{zar}}, \mathbb{Z})) \) with entry \( \mathcal{M}(r)[r] \) in level \( r \). We also denote a lift of \( \mathcal{M} \) to the homotopy category of \( \mathbb{Z}[\mathbb{G}_m, \text{Sar}, \{1\}]_{\text{zar}} \)-spectra by \( \mathcal{M} \). If we want to emphasize the dependence of \( \mathcal{M} \) on \( S \) we write \( \mathcal{M}_S \) instead of \( \mathcal{M} \).

6 Motivic complexes over a field

We first note that the material from section 5 carries over verbatim to the case of smooth schemes over a field \( k \), except that we do not have to use the functor \( \Gamma \) in the constructions. We denote the resulting motivic complexes in \( \text{D}(\text{Sh}(\text{Sm}_{k, \text{zar}}, \mathbb{Z})) \) by \( \mathcal{M}(r)_k \). The resulting naive \( \mathbb{G}_m \)-spectrum is denoted by \( \mathcal{M}_k \), the same notation is used for a lift to a spectrum. In this section we will use the notation of section 5 (like \( S, C^A_t \) etc.) carried over to the field case.

We let \( \tilde{z}'(X) = C_*(z_{\text{equi}}(\mathbb{A}^r, 0))(X) \) (for notation see e.g. [31]) be the complex introduced by Friedlander and Suslin ([12]), so \( \tilde{z}' \in \text{Cpx}(\text{Sh}(\text{Sm}_{k, \text{zar}}, \mathbb{Z})) \) and the Zariski hypercohomology of \( \tilde{z}' \) computes Bloch’s higher Chow groups (see loc. cit.).

Let \( A = k[T_1, \ldots, T_r] \). For \( K \) a \( n \)-simplex in the nerve of \( S \) (with corresponding chain \( A_0 \to \cdots \to A_n \) of \( k \)-algebras) we define a functor \( \alpha^c_K : \mathcal{E}_n \times [1] \to \text{Cpx}(\text{Ab}) \) by sending \( (t, 0) \) to \( \tilde{z}'(A_{\phi_n(t)}) \) and \( (t, 1) \) to \( z'_{C^A_t}(C^A_t) \).

For \( K \) a \( n \)-simplex in the nerve of \( 
abla \times [1] \) let \( \gamma^c_K := L^H(T^p(\alpha^c_K)) \circ q_n \). The \( \gamma^c_K \) glue to give a map
\[ \gamma^c_n : D^S_{\text{Q}, \bullet} \to L^H_{q_n} \text{Cpx}(\text{Ab}). \]

We denote by \( \gamma^c_n \in \text{Ho}(\text{Cpx}(\text{Ab})^S_{\text{Q}, \bullet}) \) the diagram canonically associated to \( \gamma^c \).

The map in \( \text{D}(\text{Sh}(\text{Sm}_{k, \text{zar}}, \mathbb{Z})) \) associated to the push forward of \( \gamma^c_n[-2r] \) is an isomorphism. Moreover the target is canonically isomorphic to \( \mathcal{M}(r)_k \). We get the

**Proposition 6.1:** The complexes \( \tilde{z}'[-2r] \) and \( \mathcal{M}(r)_k \) are canonically isomorphic in \( \text{D}(\text{Sh}(\text{Sm}_{k, \text{zar}}, \mathbb{Z})) \).
Pairing of cycles gives us pairings $\tilde{z} \otimes \tilde{z}' \rightarrow \tilde{z} + \tilde{z}'$ (involving the Eilenberg-Zilber map), using Proposition 6.1 this gives us pairings

$$\mathcal{M}(r)_k \otimes \mathcal{M}(r')_k \rightarrow \mathcal{M}(r + r')_k$$

(15)

in $D(\text{Sh}(\text{Sm}_k, \text{Zar}, \mathbb{Z}))$ which are unital, associative and commutative. Using the diagonal we obtain for any $X \in \text{Sm}_k$ pairings

$$\mathbb{R}\text{Hom}(\mathbb{Z}[X]_{\text{Zar}}, \mathcal{M}(r)_k) \otimes \mathbb{R}\text{Hom}(\mathbb{Z}[X]_{\text{Zar}}, \mathcal{M}(r')_k) \rightarrow \mathbb{R}\text{Hom}(\mathbb{Z}[X]_{\text{Zar}}, \mathcal{M}(r + r')_k)$$

such that the map

$$\mathcal{M}(\bullet)_k \rightarrow \mathbb{R}\text{Hom}(\mathbb{Z}[X]_{\text{Zar}}, \mathcal{M}(\bullet)_k)$$

is a map of $\mathbb{N}$-graded algebras.

We derive an action

$$\mathcal{M}(r)_k \otimes \mathbb{R}\text{Hom}(\mathbb{Z}[X]_{\text{Zar}}, \mathcal{M}(r')_k) \rightarrow \mathbb{R}\text{Hom}(\mathbb{Z}[X]_{\text{Zar}}, \mathcal{M}(r + r')_k).$$

(16)

In order to achieve a compatibility between the localization triangle (Proposition 5.25) and this action we study an action of $\tilde{z}'$ directly on Bloch’s complexes which appear in the definition of the $\mathcal{M}(r')_k$.

Let $A$ be a smooth $k$-algebra and let $A' := A[T_1, \ldots, T_r]$. For $K$ a $n$-simplex in the nerve of $\mathcal{S}$ (with corresponding chain $A_0 \rightarrow \cdots \rightarrow A_n$ of $k$-algebras) we define a functor $\alpha^a_K : \mathcal{E}_n \times [1] \rightarrow \text{Cpx}(\text{Ab})$ by sending $(t, 0)$ to $\tilde{z}'(A_{\alpha_n(t)}) \otimes \tilde{z}'(C^1_{i})$ and $(t, 1)$ to $\tilde{z}'(C^1_{i})$ (the transition maps from 0 to 1 are induced by pairing of cycles).

For $K$ a $n$-simplex in the nerve of $\mathcal{S} \times [1]$ let $\gamma^a_K := L^H(T^p(\alpha^a)_K) \circ q_n$. The $\gamma^a_K$ glue to give a map

$$\gamma^a : D^{\mathcal{S}_n \times [1]} \rightarrow L^H q_n \text{Cpx}(\text{Ab}).$$

We denote by $\gamma'_a \in \text{Ho}(\text{Cpx}(\text{Ab})^{\mathcal{S}_n \times [1]})$ the diagram canonically associated to $\gamma^a$.

The map in $D(\text{Sh}(\text{Sm}_k, \text{Zar}, \mathbb{Z}))$ associated to the push forward of $\gamma'_a[-2r - 2r']$ yields an action map

$$\mathcal{M}(r)_k \otimes \mathbb{R}\text{Hom}(\mathbb{Z}[X]_{\text{Zar}}, \mathcal{M}(r')_k) \rightarrow \mathbb{R}\text{Hom}(\mathbb{Z}[X]_{\text{Zar}}, \mathcal{M}(r + r')_k),$$

(17)

where $X = \text{Spec}(A)$. We will show in Lemma 6.3 that the action maps (16) and (17) coincide.

**Lemma 6.2:** The pairing (17) for $A = k$ coincides with the pairing (15).
Proof. Let $A := k[T_1, \ldots, T_r]$ and $A' := A[T_1, \ldots, T_r]$. For $K$ a $n$-simplex in the nerve of $S$ (with corresponding chain $A_0 \to \cdots \to A_n$ of $k$-algebras) we define a functor $\alpha^x_K: \mathcal{E}_n \times [1]^2 \to \text{Cpx}(\text{Ab})$ by sending $(t, 0, 0)$ to $\tilde{z}^r(A_{\varphi_0}) \otimes \tilde{z}^r(A_{\varphi_1}), (t, 0, 1)$ to $\tilde{z}^r(A_{\varphi_0}) \otimes \tilde{z}^r(A_{\varphi_1})$, $(t, 1, 0)$ to $\tilde{z}^r(A_{\varphi_0}) \otimes \tilde{z}^r(A_{\varphi_1})$ and $(t, 1, 1)$ to $\tilde{z}^r(A_{\varphi_0}) \otimes \tilde{z}^r(A_{\varphi_1})$.

For $K$ a $n$-simplex in the nerve of $\hat{S} \times [1]^2$ let $\gamma^{c}_K := L^H(T^p(\alpha^c)_K) \circ q_n$. The $\gamma^{c}_K$ glue to give a map

$$\gamma^{c}: D_{\hat{S} \times [1]^2} \to L^H_{q_n} \text{Cpx}(\text{Ab}).$$

We denote by $\gamma^{c}_K \in \text{Ho}(\text{Cpx}(\text{Ab}))^{\hat{S} \times [1]^2}$ the diagram canonically associated to $\gamma^{c}$.

The commutativity of the square in $\text{D}(\text{Sh}(\text{Sm}_k, \text{Zar}, \mathbb{Z}))$ associated to the push-forward of $\gamma^{c}_K[-2r-2r']$ shows the claim for the $A$ under consideration. The claim for $A = k$ is shown in a similar manner. □

Lemma 6.3: For affine $X$ the action maps (16) and (17) coincide.

Proof. Let $X = \text{Spec}(A)$ and let $A' := A[T_1, \ldots, T_r]$. For $K$ a $n$-simplex in the nerve of $S$ (with corresponding chain $A_0 \to \cdots \to A_n$ of $k$-algebras) we define a functor $\alpha^{g}_K: \mathcal{E}_n \times [2] \to \text{Cpx}(\text{Ab})$ by sending $(t, 0)$ to $\tilde{z}^r(A_{\varphi_0}) \otimes \tilde{z}^r_1(A_{\varphi_1})$, $(t, 1)$ to $\tilde{z}^r(A_{\varphi_0}) \otimes \tilde{z}^r_1(A_{\varphi_1})$ and $(t, 2)$ to $\tilde{z}^r_1(A_{\varphi_1})$.

For $K$ a $n$-simplex in the nerve of $\hat{S} \times [2]$ let $\gamma^{g}_K := L^H(T^p(\alpha^g)_K) \circ q_n$. The $\gamma^{g}_K$ glue to give a map

$$\gamma^{g}: D_{\hat{S} \times [2]} \to L^H_{q_n} \text{Cpx}(\text{Ab}).$$

We denote by $\gamma^{g}_K \in \text{Ho}(\text{Cpx}(\text{Ab}))^{\hat{S} \times [2]}$ the diagram canonically associated to $\gamma^{g}$.

Using methods as in the beginning of section 5.2.2 one shows that the second map in the diagram $[2] \to \text{Ho}(\text{Cpx}(\text{Ab})^S)$ associated to $\gamma^{g}_K$ is $\mathbb{R}
\text{Hom}([X], f)$, where $f$ is the map induced by $\gamma^{g}_K$ with $A$ the $k$-algebra $k$. The composite map associated to $\gamma^{g}_K$ gives the action map (17), thus the claim follows from Lemma 6.2. □

Proposition 6.4: Let $i: Z \to X$ be a closed immersion of affine schemes in $\text{Sm}_k$ of
codimension 1 with open affine complement \( U \). Then the diagram

\[
\begin{array}{ccc}
\mathcal{M}(r)_k \otimes^L \mathbb{R} \text{Hom}([Z]_{\text{Zar}}, \mathcal{M}(r')_k)[-1] & \longrightarrow & \mathbb{R} \text{Hom}([Z]_{\text{Zar}}, \mathcal{M}(r + r' - 1)_k)[-1] \\
\downarrow & & \downarrow \\
\mathcal{M}(r)_k \otimes^L \mathbb{R} \text{Hom}([X]_{\text{Zar}}, \mathcal{M}(r')_k) & \longrightarrow & \mathbb{R} \text{Hom}([X]_{\text{Zar}}, \mathcal{M}(r + r')_k) \\
\downarrow & & \downarrow \\
\mathcal{M}(r)_k \otimes^L \mathbb{R} \text{Hom}([U]_{\text{Zar}}, \mathcal{M}(r')_k) & \longrightarrow & \mathbb{R} \text{Hom}([U]_{\text{Zar}}, \mathcal{M}(r + r')_k) \\
\downarrow & & \downarrow \\
\mathcal{M}(r)_k \otimes^L \mathbb{R} \text{Hom}([Z]_{\text{Zar}}, \mathcal{M}(r')_k)[-1] & \longrightarrow & \mathbb{R} \text{Hom}([Z]_{\text{Zar}}, \mathcal{M}(r + r' - 1)_k)[-1],
\end{array}
\]

where the horizontal maps are the above action maps and the columns are the triangles from Proposition 5.25, commutes.

**Proof.** For \( K \) a \( n \)-simplex in the nerve of \( S \) one defines a functor \( \mathcal{E}_n \times [1] \times [1]^2 \rightarrow \text{Cpx}(\text{Ab}) \) combining the action maps from above and the functors used in the proof of Proposition 5.25. The commutativity of the diagram associated to the resulting functor \( D_{Q*}^{\mathcal{E}[1]\times[1]^2} \rightarrow L_{q_!}^{\text{R}} \text{Cpx}(\text{Ab}) \) shows the claim.

The isomorphism

\[
\mathbb{Z}[-1] \cong \mathcal{M}(0)_k[-1] \cong \mathbb{R} \text{Hom}(\mathbb{Z}[\mathbb{G}_{m,k}, \{1\}]_{\text{Zar}}, \mathcal{M}(1)_k)
\]

in \( \text{D(Sh(\text{Sm}_k,Zar,Z))} \) from Proposition 5.29 induces a map

\[
\iota_1 : \mathbb{Z}[\mathbb{G}_{m,k}, \{1\}]_{\text{Zar}}[-1] \rightarrow \mathcal{M}(1)_k.
\]

**Lemma 6.5:** The composition

\[
\mathbb{Z}[\mathbb{G}_{m,k}, \{1\}]_{\text{Zar}}[-1] \otimes^L \mathcal{M}(r - 1)_k \rightarrow \mathcal{M}(1)_k \otimes^L \mathcal{M}(r - 1)_k \rightarrow \mathcal{M}(r)_k,
\]

where the first map is induced by \( \iota_1 \) and the second map is the above multiplication, is adjoint to the isomorphism from Proposition 5.29.

**Proof.** This follows from Proposition 6.4.

Recall the isomorphisms

\[
\mathbb{Z}[-2r] \cong C_*(\mathbb{Z}_{\text{tr}}((\mathbb{G}_{m,k}, \{1\})^{n^r}))[-r]
\]
in $\text{D}(\text{Sh}(\text{Sm}_{k, \text{Zar}}, \mathbb{Z}))$ constructed in [42]. We get a natural map

$$
iota_2: Z[G_{m,k}, \{1\}]_{\text{Zar}}[-1] \to C_* (Z_{tr} ((G_{m,k}, \{1\})))[-1] \cong \tilde{z}^1 [-2] \cong \mathcal{M}(1)_k.$$

Lemma 6.6: The maps $\iota_1$ and $\iota_2$ agree.

Proof. Let $i_1: G_{m,k} \to \mathbb{A}^1_k$ be the natural inclusion and $i_2: G_{m,k} \to \mathbb{A}^1_k$ the inversion followed by the natural inclusion. Let $Q$ be the sheaf cokernel of the map

$$C_* (Z_{tr} ((G_{m,k}, \{1\}))) \overset{i_1 \oplus i_2}{\longrightarrow} C_* (Z_{tr} ((\mathbb{A}^1_k, \{1\}))) \oplus C_* (Z_{tr} ((\mathbb{A}^1_k, \{1\}))).$$

Since this map is injective and the target is acyclic $Q$ is a representative of the shifted complex $C_* (Z_{tr} ((G_{m,k}, \{1\}))) [1]$. For $X \in \text{Sm}_{k}$ and maps $f, g: X \to \mathbb{A}^1_k$ let $h(f, g)$ be the map $X \times \Delta^1 \to \mathbb{A}^1_k$ given by $sf + (1-s)g$, where $s$ is the standard coordinate on the algebraic 1-simplex $\Delta^1$.

Let $c: G_{m,k} \to \mathbb{A}^1_k$ be the constant map to 1. Let $\varphi \in C_1 (Z_{tr} ((\mathbb{A}^1_k, \{1\}))) (G_{m,k})$ be given by $h(i_1, c)$, in a similar manner let $\psi$ be given by $-h(i_2, c)$. Then

$$\partial (\varphi, \psi) \in C_0 (Z_{tr} ((\mathbb{A}^1_k, \{1\}))) (G_{m,k}) \oplus C_0 (Z_{tr} ((\mathbb{A}^1_k, \{1\}))) (G_{m,k})$$

is the image of $\partial_{G_{m,k}} \in C_0 (Z_{tr} ((G_{m,k}, \{1\}))) (G_{m,k})$ with respect to the map $i_1 \oplus i_2$. Thus the canonical map $Z[G_{m,k}, \{1\}]_{\text{Zar}}[1] \to Q$ is represented by the image of $(\varphi, \psi)$ in $Q_1 (G_{m,k})$.

Note there is a canonical map $Q \to C_* (Z_{tr} ((\mathbb{P}^1_k, \{1\})))$ which is induced by the two canonical covering maps $\mathbb{A}^1_k \to \mathbb{P}^1_k$. We denote the image of $(\varphi, \psi)$ in the group $C_1 (Z_{tr} ((\mathbb{P}^1_k, \{1\}))) (G_{m,k})$ by $\eta$. Thus $\eta$ induces a map

$$Z[G_{m,k}, \{1\}]_{\text{Zar}}[1] \to C_* (Z_{tr} ((\mathbb{P}^1_k, \{1\}))). \tag{19}$$

The comparison isomorphism [18] is constructed using the natural map

$$C_* (Z_{tr} ((\mathbb{P}^1_k, \{1\}))) \to C_* (z_{\text{equi}} (\mathbb{P}^1_k \setminus \{1\}, 0)) \cong C_* (z_{\text{equi}} (\mathbb{A}^1_k, 0)),$$

and precomposition with [19] gives the map $\iota_2$ (modulo the identification $\tilde{z}^1 [-2] \cong \mathcal{M}(1)_k$ and a shift). Let us denote the image of $\eta$ in $z^1 (G_{m,k} \times_k \mathbb{A}^1_k, 1)$ by $\eta'$. The cycle (in the sense of homological algebra) $\eta'$ is a sum $\varphi' + \psi'$ of chains (where each summand is a chain constituted by an algebraic cycle or the negative thereof). Here $\varphi'$ (resp. $\psi'$) is the image of $\varphi$ (resp. $\psi$). We want to compute the boundary of $\eta'$ for the triangle defined by the sequence

$$z^0 (\{0\} \times_k \mathbb{A}^1_k) \to z^1 (\mathbb{A}^1_k \times_k \mathbb{A}^1_k) \to z^1 (G_{m,k} \times_k \mathbb{A}^1_k). \tag{20}$$
Therefore we lift \( \eta' \) to the middle complex, take the boundary and view it as an element of the left complex. We first give a lift of \( \varphi' \).

Let \( c':\mathbb{A}^1_k \to \mathbb{A}^1_k \) be the constant map to \( \{1\} \). We let \( \tilde{\varphi} \in C_1(\mathbb{Z}_{tr}((\mathbb{A}^1_k, \{1\}))(\mathbb{A}^1_k)) \) be given by \( h(\text{id}_{\mathbb{A}^1_k}, c') \) and \( \tilde{\varphi}' \) be the image of \( \tilde{\varphi} \) with respect to the composition

\[
C_1(\mathbb{Z}_{tr}((\mathbb{A}^1_k, \{1\}))(\mathbb{A}^1_k)) \to C_1(\mathbb{Z}_{tr}((\mathbb{P}^1_k, \{1\}))(\mathbb{A}^1_k)) \to \mathbb{Z}^1(\mathbb{A}^1_k, \mathbb{A}^1_k, 1).
\]

Then \( \tilde{\varphi}' \) is a lift of \( \varphi' \). The boundary of the image of \( \tilde{\varphi} \) in \( C_1(\mathbb{Z}_{tr}((\mathbb{P}^1_k, \{1\}))(\mathbb{A}^1_k)) \) is the graph of the canonical embedding \( \mathbb{A}^1_k \to \mathbb{P}^1_k \).

We let \( t \) be the standard coordinate on \( \mathbb{G}_{m,k} \), \( s \) the standard coordinate on \( \Delta^1 \) and \([x_0 : x_1]\) homogeneous coordinates on \( \mathbb{P}^1_k \). Then the effective cycle corresponding to the image of \( -\psi \) in \( C_1(\mathbb{Z}_{tr}((\mathbb{P}^1_k, \{1\}))(\mathbb{G}_{m,k})) \) is given by the homogeneous equation

\[
sx_0 + t(1-s)x_0 = tx_1.
\]

The closure \( Z \) in \( \mathbb{A}^1_k \times \Delta^1_k \times \mathbb{P}^1_k \) is given by the same equation. Intersecting with \( s = 0 \) (resp. \( s = 1 \)) gives the closed subscheme with equation \( t(x_0 - x_1) = 0 \) (resp. \( x_0 = tx_1 \)). This shows that the intersections of this closure with the faces of \( \Delta^1 \) are proper. We view the restriction of \( Z \) to \( \mathbb{P}^1_k \setminus \{1\} \) as a cycle in \( \mathbb{Z}^1(\mathbb{A}^1_k \times \mathbb{A}^1_k, 1) \) and denote its negative by \( \tilde{\psi}' \). Thus \( \tilde{\psi}' \) is a lift of \( \psi' \).

We see that the boundary of \( \tilde{\varphi}' \) cancels with the contribution of the boundary of \( \tilde{\psi}' \) for \( s = 1 \). Thus the boundary of \( \varphi' + \tilde{\psi}' \) is given by the equation \( t = 0 \). It follows that the boundary of \( \eta' \) for the triangle defined by \([20]\) corresponds to 1. The claim follows. \( \square \)

**Theorem 6.7:** The spectrum \( \mathcal{M}_k \) is isomorphic to the motivic Eilenberg-MacLane spectrum \( \mathcal{M}_{\mathbb{Z}k} \) over \( k \).

**Proof.** The isomorphisms \([18]\) are compatible with the product structures, see \([24]\) Proposition 3.3 for the case of a perfect ground field and \([23]\) for the general case. Thus the claim follows from Lemmas 6.5 and 6.6. \( \square \)

7 Comparisons

7.1 The exceptional inverse image of \( \mathcal{M} \)

We let \( x \) be a closed point of \( S \), \( k \) its residue field and \( i: \text{Spec}(k) \to S \) the corresponding closed inclusion. Set \( U := S \setminus \{x\}, U = \text{Spec}(D') \), and let \( j \) be the open inclusion \( U \to S \).

We view \( k \) as a \( D \)-algebra in the canonical way.
We use the notation of section 5. Let $K$ be a $n$-simplex in the nerve of $S$. For $t \in E_n$ set $C'_t := D' \otimes_D C_t$ and $F'_t := \{U \times_S a | a \in F_t\}$. Also set $C''_t := k \otimes_D C_t$ and $F''_t := \{\{x \times_S a | a \in F_t\} \}$. Sheafification on $S$ yields a functor $\tilde{\alpha!}_K : E_n \times [1]^2 \to \text{Cpx}(\text{Sh}(S_{Zar}, \mathbb{Z}))$.

For $K$ a $n$-simplex in the nerve of $\tilde{S} \times [1]^2$ let $\gamma_K^! := L^H(\Gamma \circ T^p(\tilde{\alpha}^1)_K) \circ q_n$, where $p$ is the functor $\tilde{S} \times [1]^2 \to S \times [1]^2$.

The $\gamma_K^!$ glue to give a map $\gamma^!: D^{\tilde{S} \times [1]^2}_{\text{Q},*} \to L^H_{qi} \text{Cpx}(\text{Ab})$.

We denote by $\gamma'_K \in \text{Ho}(\text{Cpx}(\text{Ab})^{\tilde{S} \times [1]^2})$ the diagram canonically associated to $\gamma^!$.

The square in $D(\text{Sh}(\text{Sm}_{S, Zar}, \mathbb{Z}))$ associated to the push forward of $\gamma'_K[-2r]$ is exact.

We thus obtain the

**Proposition 7.1:** There is an exact triangle

$$i_! M(r-1)_k[-2] \to M(r) \to j_* j^! M(r) \to i_! M(r-1)_k[-1]$$

in $D^{A_1}(\text{Sh}(\text{Sm}_{S, Nis}, \mathbb{Z}))$.

**Corollary 7.2:** There is a canonical isomorphism

$$i_!^! M(r) \cong M(r-1)_k[-2]$$

in $D^{A_1}(\text{Sh}(\text{Sm}_{k, Nis}, \mathbb{Z}))$.

**Corollary 7.3:** There is a canonical isomorphism of naive $\mathbb{G}_m$-spectra

$$i_!^! M \cong M_k(-1)[-2]$$

and also such an isomorphism of spectra.

**Proof.** The bonding maps are the same. □

**Theorem 7.4:** There is an isomorphism of spectra

$$i_!^! M \cong \mathbb{M}Z_k(-1)[-2].$$

**Proof.** This follows from Corollary 7.3 and Theorem 6.7. □
7.2 Pullback to the generic point

Let $K$ be the fraction field of $D$ and $f: \text{Spec}(K) \to S$ the canonical morphism.

Lemma 7.5: There is an isomorphism $f^* \mathcal{M}(r) \cong \mathcal{M}(r)_K$ in $\mathcal{D}(\text{Sh}(\text{Sm}_S, \text{Zar}, \mathbb{Z}))$.

Theorem 7.6: There is an isomorphism $f^* \mathcal{M} \cong M_K$ in $\text{SH}(K)$.

Proof. There is an isomorphism $f^* \mathcal{M} \cong M_K$. The result now follows from Theorem 6.7.

7.3 Weight 1 motivic complexes

We keep the notation of the last section.

Proposition 7.7: Let $k$ be a field, $\mathbb{Z}(1) = C_* (\mathbb{Z}_{\text{tr}}(\mathbb{G}_{m,k}, \{1\}))[-1]$ be the motivic complex of weight 1 (in the notation of [42] or [31]). Then there is an isomorphism $\mathbb{Z}(1) \cong O^*[-1]$ in $\mathcal{D}(\text{Sh}(\text{Sm}_k, \text{Zar}, \mathbb{Z}))$. Moreover the map $\mathbb{Z}[\mathbb{G}_{m,k}, \{1\}]_{\text{Zar}} \to O^*$ induced by this map is the canonical one.

Proof. The first part is [31, Theorem 4.1], the second part is contained in the proof of [31, Lemma 4.4].

We denote by $S^{(1)}$ the set of codimension 1 points of $S$, and for each $p \in S^{(1)}$ we let $\kappa(p)$ be the residue field of $p$ and $i_p$ the corresponding inclusion $\text{Spec}(\kappa(p)) \to S$.

Lemma 7.8: There is an exact triangle

$$\bigoplus_{p \in S^{(1)}} i_{p,*} \mathcal{M}(r-1)_{\kappa(p)}[-2] \to \mathcal{M}(r) \to f_* \mathcal{M}(r)_K \to \bigoplus_{p \in S^{(1)}} i_{p,*} \mathcal{M}(r-1)_{\kappa(p)}[-1]$$

in $\mathcal{D}(\text{Sh}(\text{Sm}_S, \text{Zar}, \mathbb{Z}))$.

Proof. This follows from Corollary 7.2 and Lemma 7.5.

Corollary 7.9: We have $\mathcal{H}^i(\mathbb{Z}(1)) \cong 0$ for $i \neq 1$ and there is an exact sequence

$$0 \to \mathcal{H}^1(\mathbb{Z}(1)) \to f_* O^*_K \to \bigoplus_{p \in S^{(1)}} i_{p,*} \mathbb{Z} \to 0$$

in $\text{Sh}(\text{Sm}_S, \text{Zar}, \mathbb{Z})$.

Proof. This follows from Lemma 7.8 and Proposition 7.7.
Theorem 7.10: There is a canonical isomorphism
\[ \mathcal{M}(1) \cong \mathcal{O}_S^*[1] \]
in \( \mathcal{D}(\text{Sh}(\text{Sm}_S, \text{Zar}, \mathbb{Z})) \).

Proof. We have a canonical map \( \mathbb{Z}[\mathbb{G}_{m,S}]_{\text{Zar}} \to H^1(\mathcal{M}(1)) \) whose composition with the map \( H^1(\mathcal{M}(1)) \to f_! \mathcal{O}_{/K}^* \) is the canonical map by Proposition 7.7. The image of this canonical map \( \mathbb{Z}[\mathbb{G}_{m,S}]_{\text{Zar}} \to f_! \mathcal{O}_{/K}^* \) is \( \mathcal{O}_S^* \). The claim follows now from Corollary 7.9.

Let \( H_{B,1} \in \mathcal{D}(\text{Sh}(\text{Sm}_S, \text{Zar}, \mathbb{Z})) \) be the first \( A^1 \)- and Nisnevich-local space in a \( \Omega \)-\( \mathbb{G}_{m,S} \)-spectrum model of \( H_B \).

Theorem 7.11: There is a canonical isomorphism
\[ H_{B,1} \cong (\mathcal{O}_S^*)_\mathbb{Q} \]
in \( \mathcal{D}(\text{Sh}(\text{Sm}_S, \text{Zar}, \mathbb{Z})) \).

Proof. The proof is similar to the proof of Theorem 7.10.

7.4 Rational spectra

We keep the notation of the last sections.

Corollary 7.12: There is an exact triangle
\[ \bigoplus_{p \in \mathbb{S}\{1\}} i_{p,*} \mathbb{M} \mathbb{Z}_{\kappa(p)}(-1)[-1] \to \mathcal{M} \to f_! \mathbb{M} \mathbb{Z}_K \to \bigoplus_{p \in \mathbb{S}\{1\}} i_{p,*} \mathbb{M} \mathbb{Z}_{\kappa(p)}(-1)[-1] \]
in \( \mathcal{S}(\mathbb{S}) \).

Proof. This follows from Theorems 7.4 and 7.6.

We call a spectrum \( E \in \mathcal{S}(\mathbb{S}) \) a Beilinson motive if it is \( H_B \)-local (compare with [6 Definition 13.2.1]). This is the case if and only if the canonical map \( E \to H_B \wedge E \) is an isomorphism (6 Corollary 13.2.15)).

Corollary 7.13: The rationalization \( \mathcal{M}_{\mathbb{Q}} \) is a Beilinson motive.
Proof. The rational motivic Eilenberg-MacLane spectra $\mathbb{M}_Q^{\kappa(p)}$ for $p \in S^{(1)}$ and $\mathbb{M}_Q^K$ are orientable, thus their push forwards to $S$ are Beilinson motives \([6\text{ Corollary \ref{cor:orientability}}](\text{Ri})\). Now the claim follows from Corollary \ref{cor:orientability}.

By construction we have $\mathcal{M}_{0,0} = \mathbb{Z}$. By Corollary \ref{cor:orientability} the elements of $(\mathcal{M}_Q)_{0,0}$ correspond bijectively to maps $H_B \to \mathcal{M}_Q$, and we let $u : H_B \to \mathcal{M}_Q$ be the map corresponding to $1 \in \mathbb{Q} = (\mathcal{M}_Q)_{0,0}$.

**Theorem 7.14:** The map $u$ is an isomorphism.

**Lemma 7.15:** For $p \in S^{(1)}$ the map

$$i^*_p H_B \xrightarrow{i^*_p u} i^*_p \mathcal{M}_Q$$

is an isomorphism.

Proof. The definition of $u$ is in such a way that the map $u_1 : H_{B,1} \to \mathcal{M}(1)_Q[1]$ induced by $u$ is compatible with the natural maps from $\mathbb{Q}[\mathbb{G}_{m,S}]_{zar}$ to $H_{B,1}$ and $\mathcal{M}(1)_Q[1]$. Since these latter maps are surjections it follows that the composition

$$(\mathcal{O}_S^*)_Q \cong H_{B,1} \xrightarrow{u_1} \mathcal{M}(1)_Q[1] \cong (\mathcal{O}_S^*)_Q,$$

where the first resp. third map is the identification from Theorem \ref{thm:identification} resp. Theorem \ref{thm:identification} is the identity. It follows that the map

$$\varphi : H_{B,\kappa(p)}(-1)[-2] \cong i^*_p H_B \xrightarrow{i^*_p u} i^*_p \mathcal{M}_Q \cong \mathcal{M}_Q^{\kappa(p)}(-1)[-2],$$

where the first isomorphism is \([6\text{ Theorem \ref{thm:13.4.1}}](\text{Ri})\), induces an isomorphism on zeroth spaces of $\Omega$-$\mathbb{G}_{m,\kappa(p)}$-spectra, thus $\varphi(1)[2]$ corresponds to a nonzero element in $\mathbb{Q} = \text{Hom}_{\mathcal{SH}(\kappa(p))}(H_{B,\kappa(p)}, \mathbb{M}_Q^{\kappa(p)})$. The claim follows.

Proof of Theorem \ref{thm:identification} The map $f^* u$ is an isomorphism. Now the claim follows from Lemma \ref{lem:isomorphism} and the map between triangles of the form as in Corollary \ref{cor:orientability} induced by $u$.

\section{The isomorphism between $\mathcal{M}Z$ and $\mathcal{M}$}

First we recast the definition of $\mathcal{M}Z$ working purely in triangulated categories. We use the notation of section \ref{sec:triangulated_categories}.
The canonical triangle
\[ L_n(r-1)[-2] \to \mathbb{R}\text{Hom}(\mathbb{Z}/p^n[\mathbb{A}^1], L_n(r)) \to \mathbb{R}\text{Hom}(\mathbb{Z}/p^n[\mathbb{G}_m, U], L_n(r)) \to L_n(r-1)[-1] \]
in \( \text{D}(\text{Sh}(\text{Sm}_U, \etale, \mathbb{Z}/p^n)) \) induces a canonical isomorphism
\[ L_n(r-1)[-1] \cong \mathbb{R}\text{Hom}(\mathbb{Z}/p^n[\mathbb{G}_m, U], \mathbb{1}) \etale, L_n(r)) \]
We thus get a naive \( \mathbb{Z}/p^n[\mathbb{G}_m, U], \mathbb{1} \) \etale-spectrum \( L_n \) with entry \( L_n(r)[r] \) in level \( r \).
We also have canonical isomorphisms
\[ \mathbb{R}\text{Hom}(\mathbb{Z}/p^n[\mathbb{G}_m, U], \mathbb{1}) \etale, L_n(r)) \]
we get a naive \( \mathbb{Z}/p^n[\mathbb{G}_m, U], \mathbb{1} \) \etale-spectrum \( \mathbb{R}\epsilon_\ast L_n \). The naive prespectrum (with the obvious definition of naive prespectrum) \( \tau\leq 0 \mathbb{R}\epsilon_\ast L_n \) with entries \( \tau\leq 0 \mathbb{R}\epsilon_\ast L_n,r \) is a naive spectrum by Proposition 4.4.
We also have canonical isomorphisms
\[ \mathbb{R}\text{Hom}(\mathbb{Z}/p^n[\mathbb{G}_m, S], \mathbb{1}) \etale, \mathbb{R}\epsilon_\ast L_n,r) \]
thus \( H_n := \mathbb{R}j_\ast \tau\leq 0 \mathbb{R}\epsilon_\ast L_n \) is a naive \( T_n := \mathbb{Z}/p^n[\mathbb{G}_m, S], \mathbb{1} \) \etale-spectrum.
By construction the underlying naive \( T_n \)-spectrum of \( B' \) (see section 4.1.1) equals \( H_n \).
Proposition 4.7 furnishes canonical maps
\[ H_{n,r} \to i_\ast \nu_n^{r-1} \]
whose homotopy fibers we denote by \( F_{n,r} \).
As after equation (9) it follows that the \( F_{n,r} \) are determined up to canonical isomorphisms.
Using the structure maps of \( H_n \) we get maps
\[ T_n \otimes F_{n,r} \to T_n \otimes H_{n,r} \to H_{n,r+1} \to i_\ast \nu_n^{r} \]
which are 0 since we know already from section 4.1.1 that the \( F_{n,r} \) organize themselves into a naive \( T \)-spectrum such that the maps \( F_{n,r} \to H_{n,r} \) form a map of naive \( T \)-spectra.
Thus in turn the maps $T_n \otimes F_{n,r} \to H_{n,r+1}$ factorize through $F_{n,r+1}$. These factorizations are unique since there are no non-trivial maps $T_n \otimes F_{n,r} \to i_*\nu_{n\mathbf{1}}[-1]$.

Thus we see that the $F_{n,r}$ assemble in a unique way into a naive $T_n$-spectrum $F_n$ together with a map of naive $T_n$-spectra $F_n \to H_n$.

Of course the underlying naive $T_n$-spectrum of $C$ (see section 4.1.1) is $F_n$.

Set $M_n(r) = \mathbb{Z}[\mathbb{G}_{m,S},\{1\}]_{zar}$ and $M_n := \mathbb{M}/p^n$. We have étale cycle class maps

$$M_n(r)|_U \to \mathbb{R}^e(L_n(r))$$

(see section 5.2.3).

Proposition 5.28 (applied with $X = \mathbb{A}^1_S$ and $Z = \{1\}$) implies that these cycle class maps combine to give a map of naive $\mathbb{M}$-spectra $M_n \to H_n$ which factors as

$$M_n \to \mathbb{R}^e(j_*j^*M_n) \cong H_n.$$

We have commutative diagrams

$$M_n(r)[r] \xrightarrow{\mathbb{R}^e} \mathbb{R}^eM_n(r)[r] \xrightarrow{i_*M_nZ(r-1)[r-1]} M_n(r)[r+1]$$

$$H_{n,r} \xrightarrow{=} i_*\nu_{n\mathbf{1}}^{r-1}$$

(the vertical maps being isomorphisms) with an exact triangle as upper row: The exact triangle is induced by a variant of Proposition 7.1. The right vertical isomorphism is a global version of Theorem 3.4. The diagram commutes by a global version of Proposition 3.4.

Thus we get a unique factorization $M_n \to F_n$ which is an isomorphism.

Set $T := \mathbb{Z}[\mathbb{G}_{m,S},\{1\}]_{zar}$. By adjointness we get a map of naive $T$-spectra $M \to F_n$.

For $F,G \in D(Sh(Sm_{S,zar},\mathbb{Z}/p^n))$ we denote by map$_{\mathbb{Z}/p^n}(F,G) \in HosSet$ the mapping space between $F$ and $G$. We use the same notation for étale sheaves and also for spectra. If the coefficients are $\mathbb{Z}$ we use the notation map.

Lemma 7.16: For any $r \geq 0$ we have map$_{\mathbb{Z}/p^n}(F_{n,r},F_{n,r}) \cong \mathbb{Z}/p^n$.

Proof. We have

$$\text{map}_{\mathbb{Z}/p^n}(F_{n,r},\mathbb{R}^e(j_*\tau_{r\mathbb{R}^eL_n,r}))$$

$$\cong \text{map}_{\mathbb{Z}/p^n}(j^*F_{n,r},\tau_{r\mathbb{R}^eL_n,r})$$

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We have a long exact sequence

\[ \cdots \rightarrow \text{Hom}(F_{n,r}[i+1], i_* \nu_n^{r-1}) \rightarrow \text{Hom}(F_{n,r}[i], F_{n,r}) \rightarrow \]

\[ \rightarrow \text{Hom}(F_{n,r}[i], \mathbb{R} j_* \tau_{\leq 0} \mathbb{R} \epsilon_* \mathcal{L}_{n,r}) \rightarrow \text{Hom}(F_{n,r}[i], i_* \nu_n^{r-1}) \rightarrow \cdots \]

Thus the maps

\[ \text{Hom}(F_{n,r}[i], F_{n,r}) \rightarrow \text{Hom}(F_{n,r}[i], \mathbb{R} j_* \tau_{\leq 0} \mathbb{R} \epsilon_* \mathcal{L}_{n,r}) \]

are isomorphisms for \( i > 0 \) and injective for \( i = 0 \). But since \( \text{Hom}(F_{n,r}, \mathbb{R} j_* \tau_{\leq 0} \mathbb{R} \epsilon_* \mathcal{L}_{n,r}) \cong \mathbb{Z}/p^n \) the map for \( i = 0 \) is also surjective, so the claim follows.

**Corollary 7.17:** The naive spectra \( F_n \) have lifts to spectra which are unique up to canonical isomorphism. Denoting these lifts also by \( F_n \) we have

\[ \text{map}_{\mathbb{Z}/p^n}(F_n, F_n) \cong \mathbb{Z}/p^n. \]

Moreover we have \( \text{map}(\mathcal{M}, F_n) \cong \mathbb{Z}/p^n \) (the latter mapping space is computed in \( T \)-spectra).

Clearly we have \( F_{n+1} \otimes_{\mathbb{Z}/p^{n+1}} \mathbb{Z}/p^n \cong F_n \).

Thus we get compatible maps \( \mathcal{M} \rightarrow F_n \) for all \( n \) in the homotopy category of \( T \)-spectra. This furnishes a map \( \mathcal{M} \rightarrow \text{holim}_n F_n \) which is the \( p \)-completion map. Note that the homotopy limit is uniquely determined up to canonical isomorphism.

We have a canonical isomorphism \( \text{holim}_n F_n \cong D_p \), where we use the notation of section 4.1.2. Thus we get a canonical map \( \mathcal{M} \rightarrow D \) (notation from section 4.2) in the homotopy category of \( T \)-spectra.

Moreover the diagram

\[
\begin{array}{ccc}
\mathcal{M} & \longrightarrow & D \\
\downarrow_{u^{-1} \circ f} & & \downarrow \\
H_B & \longrightarrow & D_Q,
\end{array}
\]

where \( f : \mathcal{M} \rightarrow \mathcal{M}_Q \) is the rationalization map and and \( u \) is from section 7.4 commutes (maps out of \( \mathcal{M}_Q \cong H_B \) into Beilinson motives correspond to elements in \( \pi_{0,0} \)). Thus we
obtain a map \( M \to \mathbb{M} \) (for the latter see definition 4.27). This map is an isomorphism since \( u \) is an isomorphism (Theorem 7.14) and each map \( M \to D_p \) is the \( p \)-completion map.

We have shown

**Theorem 7.18:** There is a canonical isomorphism \( M \cong \mathbb{M} \). We have map \( (M, M) \cong \mathbb{Z} \), where map can denote the mapping space in \( T \)-spectra or in \( \mathsf{SH}(S) \).

We leave the last assertion as an exercise to the reader.

**Corollary 7.19:** For \( X \in \mathsf{Sm}_S \) there is a canonical isomorphism

\[
\hom_{\mathsf{SH}(S)}(\Sigma^\infty X_+, \mathbb{M}(n)[i]) \cong \mathit{H}^i_{\text{mot}}(X, n),
\]

where the latter group denotes Levine’s motivic cohomology.

For \( X \in \mathsf{Sm}_S \) we denote by \( \mathsf{DM}(X) \) the homotopy category of the category of \( f^*\mathbb{M} \)-modules, where \( f \) is the structural morphism of \( X \).

**Corollary 7.20:** For \( X \in \mathsf{Sm}_S \) there is a canonical isomorphism

\[
\hom_{\mathsf{DM}(X)}(\mathbb{Z}, \mathbb{Z}(n)[i]) \cong \mathit{H}^i_{\text{mot}}(X, n).
\]

### 8 The periodization of \( \mathbb{M} \)

We set ourselves in the situation of section [4.1.2] before the definition of \( A \). Since \( L_\bullet(r) = L_\bullet(1)^{\otimes r} \) for any \( r \in \mathbb{Z} \) the collection of the \( L_\bullet(r)[2r] \) gives rise to a strictly commutative algebra \( L_\bullet(*)[2*] \) in \( \mathsf{Cpx}(\mathsf{Sh}(\mathsf{Sm}_U, \mathsf{\acute{e}t}, \mathbb{Z}/\text{slash.left} p \bullet \mathbb{G}_m, \mathbb{1})) \). We denote by \( e \) the embedding

\[
\mathsf{Cpx}(\mathsf{Sh}(\mathsf{Sm}_U, \mathsf{\acute{e}t}, \mathbb{Z}/p^\bullet)) \to \mathsf{Cpx}(\mathsf{Sh}(\mathsf{Sm}_U, \mathsf{\acute{e}t}, \mathbb{Z}/p^\bullet))^{\mathbb{Z}}
\]

which sets everything into outer degree 0 and by the same symbol the induced embedding of \( \mathbb{Z}/p^\bullet[G_{m, U}, \mathbb{1}]_{\text{\acute{e}t}} \)-spectra into \( e(\mathbb{Z}/p^\bullet[G_{m, U}, \mathbb{1}]_{\text{\acute{e}t}}) \)-spectra. The tensor product in \( e(\mathbb{Z}/p^\bullet[G_{m, U}, \mathbb{1}]_{\text{\acute{e}t}}) \)-spectra of \( e(\mathsf{Sym}(\mathcal{T})) \) with the suspension spectrum of \( L_\bullet(*)[2*] \) can be written as the outer tensor product \( \mathsf{Sym}(\mathcal{T}) \otimes L_\bullet(*)[2*] \). We let

\[
\mathsf{Sym}(\mathcal{T}) \otimes L_\bullet(*)[2*] \to R(\mathsf{Sym}(\mathcal{T}) \otimes L_\bullet(*)[2*])
\]

be a fibrant replacement in \( E_\infty \)-algebras in \( e(\mathbb{Z}/p^\bullet[G_{m, U}, \mathbb{1}]_{\text{\acute{e}t}}) \)-spectra (i.e. in the semi model category \( E_\infty((\mathsf{Sp}_{\mathbb{Z}/p^\bullet[G_{m, U}, \mathbb{1}]_{\text{\acute{e}t}}})^{\mathbb{Z}}) \)). Set \( A := e_*(R(\mathsf{Sym}(\mathcal{T}) \otimes L_\bullet(*)[2*])) \).
For $k \in \mathbb{Z}$ we denote by $A_k$ the contribution of $A$ in outer $\mathbb{Z}$-degree $k$, so $A_k$ is a $\mathbb{Z}/p^\bullet[\mathbb{G}_m, U, \{1\}]$-spectrum. We set $A'_{k} := \tau_{\leq -k}(A_k)$. The $A'_k$ assemble to an $E_\infty$-algebra $A' \in E_\infty((\text{Sp}_{\mathbb{Z}/p^\bullet[\mathbb{G}_m, U, \{1\}], \text{Zar}})^\mathbb{Z})$. We set $B := j_* A'$.

As in section 4.1.2 we have canonical epimorphisms $B_{k, r} \to i_* \nu_{k+r-1}[k]$. We denote by $C_{k, r}$ the kernels of these epimorphisms. A variant of Lemma 4.14 implies that the collection of the $C_{k, r}$ gives rise to an $E_\infty$-algebra $C \in E_\infty((\text{Sp}_{\Sigma\mathbb{Z}/p^\bullet[\mathbb{G}_m, S, \{1\}], \text{Zar}})^\mathbb{Z})$.

Let $C \to C'$ be a fibrant replacement in the latter semi model category and set $D_p := \lim_n C'_{n, \bullet, n} \in E_\infty((\text{Sp}_{\Sigma p[\mathbb{G}_m, S, \{1\}], \text{Zar}})^\mathbb{Z})$.

Set $D := \prod_p D_p \in E_\infty((\text{Sp}_{\Sigma p[\mathbb{G}_m, S, \{1\}], \text{Zar}})^\mathbb{Z})$. We let $PH_B$ be the periodic version of $H_B$, then there is a canonical $E_\infty$-map $PH_B \to D_Q$.

**Definition 8.1:** We let $PM_Z$ denote the homotopy pullback in $E_\infty$-spectra of the diagram

$$
\begin{array}{ccc}
D & \to & D_Q \\
\downarrow & & \downarrow \\
PH_B & \longrightarrow & D_Q.
\end{array}
$$

Clearly we have

**Theorem 8.2:** The $E_\infty$-spectrum $PM_Z$ is a strong periodization of $M_Z$ in the sense of [40].

For $X \in Sm_S$ let $DMT(X)$ be the full localizing triangulated subcategory of $DM(X)$ spanned by the $\mathbb{Z}(n)$, $n \in \mathbb{Z}$. We denote by $DMT_{gm}(X)$ the full subcategory of $DMT(X)$ of compact objects.

**Corollary 8.3:** For $X \in Sm_S$ there is a $E_\infty$-algebra $A$ in $Cpx(\text{Ab})^\mathbb{Z}$ and a tensor triangulated equivalence $DMT(X) \simeq D(A)$.

**Proof.** This follows now from [40] Theorem 4.3.

**Corollary 8.4:** Let $X \in Sm_S$ such that for any $n$ we have $H^i_{\text{mot}}(X, n)_Q = 0$ for $i \ll 0$ (for example $X = \text{Spec}(R)$, $R$ the localization of a number ring, or $X = \mathbb{P}^1_R \setminus \{0, 1, \infty\}$). Then there is an affine derived group scheme $G$ over $\mathbb{Z}$ such that $\text{Perf}(G)$, the (derived) category of perfect representations of $G$, is tensor triangulated equivalent to $DMT_{gm}(X)$.

**Proof.** This follows from [35] Theorem 6.21.
9 Base change

**Proposition 9.1:** Let \( f \colon T \to S \) be a morphism of schemes, \( R \) a commutative ring and \( t \in \{ \text{Zar}, \text{Nis}, \text{ét} \} \). Let \( F \in D(\text{Sh}(\text{Sm}_S, R)) \). For each \( X \in \text{Sm}_S \) let \( f_X \) be the map \( X_T := T \times_S X \to X \). Suppose that for each \( X \in \text{Sm}_S \) the object \( f^*_X(F|_{X_t}) \in D(\text{Sh}(X_{T,t}, R)) \) is zero. Then \( Lf^*F \in D(\text{Sh}(\text{Sm}_{T,t}, R)) \) is zero.

**Proof.** We use the language of \( \infty \)-categories. For any scheme \( U \) let \( \theta(U) \) be the functor on \( \text{Sm}_U^{\text{op}} \) which associates to any \( X \in \text{Sm}_U \) the \( \infty \)-category associated to the model category \( \text{Cpx}(\text{Sh}(X_t, R)) \). Let \( \text{Sect}(\theta(U)) \) be the category of sections of \( \theta(U) \) and \( \text{Sect}(\theta(U))_{\text{ét}-\text{cart}} \) the full subcategory which consists of objects which are cartesian for étale morphisms in \( \text{Sm}_U \). Then \( \text{Sect}(\theta(U))_{\text{ét}-\text{cart}} \) is canonically equivalent to the \( \infty \)-category associated to \( \text{Cpx}(\text{Sh}(\text{Sm}_U, R)) \). Note that the inclusion

\[
\text{Sect}(\theta(U))_{\text{ét}-\text{cart}} \hookrightarrow \text{Sect}(\theta(U))
\]

preserves limits and colimits, so it has both a left and a right adjoint. For a morphism \( g \colon V \to U \) of schemes one has a base change left adjoint \( g^* : \text{Sect}(\theta(U)) \to \text{Sect}(\theta(V)) \), and the base change \( Lf^* : \text{Cpx}(\text{Sh}(\text{Sm}_{U,t}, R)) \to \text{Cpx}(\text{Sh}(\text{Sm}_{V,t}, R)) \) is modelled by the composition

\[
\text{Sect}(\theta(U))_{\text{ét}-\text{cart}} \hookrightarrow \text{Sect}(\theta(U)) \to \text{Sect}(\theta(V)) \to \text{Sect}(\theta(V))_{\text{ét}-\text{cart}},
\]

where the last morphism is the left adjoint to the inclusion.

The assumption on \( F \) implies that the composition

\[
\text{Sect}(\theta(S))_{\text{ét}-\text{cart}} \hookrightarrow \text{Sect}(\theta(S)) \to \text{Sect}(\theta(T))
\]

applied to \( F \) gives zero, hence the claim. \( \square \)

**Proposition 9.2:** Let \( f : Y \to X \) be a morphism of schemes which induces isomorphisms on residue fields and \( R \) a commutative ring. Let \( \varepsilon \) denote the maps of sites \( X_\text{ét} \to X_\text{Nis} \) and \( Y_\text{ét} \to Y_\text{Nis} \). Let \( F \in D(\text{Sh}(X_\text{ét}, R)) \). Then the canonical map \( f^* \mathbb{R}\varepsilon_* F \to \mathbb{R}\varepsilon_* f^* F \) is an isomorphism in \( D(\text{Sh}(Y_\text{Nis}, R)) \).

**Proof.** Note first that for a scheme \( U, F \in D(\text{Sh}(U_\text{ét}, R)) \), field \( K \) and map \( x : \text{Spec}K \to U \) which is an isomorphism on residue fields we have

\[
(x^*(\mathbb{R}\varepsilon_* F))(\text{Spec}K) \cong \mathbb{R}\Gamma(\text{Spec}K, x^* F),
\]

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\[ \varepsilon \text{ the map of sites } U_{\text{et}} \to U_{\text{Nis}}. \text{ Thus for a map } x: \text{Spec} K \to Y \text{ inducing an isomorphism on residue fields we have} \]

\[ (x^* (R\varepsilon_*(f^* F)))(\text{Spec} K) \cong R\Gamma(\text{Spec} K, x^* f^* F) \cong (x^* f^* R\varepsilon_*) (\text{Spec} K). \]

This shows the claim. \[ \square \]

**Proposition 9.3:** Let \( i: Z \to X \) be a closed immersion between separated Noetherian schemes of finite Krull dimension and \( R \) a commutative ring. Let \( F \in D(\text{Sh}(\text{Sm}_{X,Nis}, R)) \), \( G \in D(\text{Sh}(\text{Sm}_{Z,Nis}, R)) \) and \( \varphi: L \to \xi F \to G \) be a map. Suppose that for any \( Y \in \text{Sm}_X \) the map \( i_Z^*(F|_{\text{Nis}}) \to G|_{\text{Nis}} \) induced by \( \varphi \) is an isomorphism in \( D(\text{Sh}(Y_{Z,\text{Nis}}, R)) \). Then \( \varphi \) is an \( \mathbb{A}^1 \)-weak equivalence.

**Proof.** The cofiber of the adjoint \( F \to R_\ast G \) of \( \varphi \) satisfies the assumption of Proposition 9.1 thus the map \( R_\ast F \to R_\ast R_\ast G \) is an isomorphism. But the map \( i_\ast R_\ast G \to G \) is an \( \mathbb{A}^1 \)-weak equivalence, because \( R_\ast \) preserves \( \mathbb{A}^1 \)-weak equivalences (since \( i \) is finite) and the composition

\[ D^A(\text{Sh}(\text{Sm}_{Z,Nis}, R)) \to D^A(\text{Sh}(\text{Sm}_{X,Nis}, R)) \to D^A(\text{Sh}(\text{Sm}_{Z,Nis}, R)) \]

is naturally equivalent to the identity. \[ \square \]

Let now \( U \) be the spectrum of a Dedekind domain of mixed characteristic and \( p \) a prime which is invertible on \( U \). Let \( x \in U \) be a closed point of positive residue characteristic and \( \kappa := \kappa(x) \). We denote by \( i \) the closed inclusion \( \{ x \} \to U \). We let \( L_{U,n}(r) = \mu_{p^n r}^{\text{tr}} \) viewed as object of \( D(\text{Sh}(\text{Sm}_{U,\text{et}}, \mathbb{Z}/p^n)) \), similarly \( L_{\kappa,n}(r) = \mu_{p^n r}^{\text{tr}} \) viewed as object of \( D(\text{Sh}(\text{Sm}_{\kappa,\text{et}}, \mathbb{Z}/p^n)) \). We have a natural map \( \varphi: L_{U,n}(r) \to R_\ast L_{\kappa,n}(r) \). Let \( \varepsilon \) denote the maps of sites \( \text{Sm}_{U,\text{et}} \to \text{Sm}_{U,\text{Nis}} \) and \( \text{Sm}_{\kappa,\text{et}} \to \text{Sm}_{\kappa,\text{Nis}} \). The adjoint of \( \varphi \) induces the second map in the composition

\[ L_i^\ast R\varepsilon_! L_{U,n}(r) \to R\varepsilon_! L_i^\ast L_{U,n}(r) \to R\varepsilon_! L_{\kappa,n}(r). \]

Applying \( \tau_{\leq r} \) to this composition yields the second map in the composition

\[ g_{n,r}: L_i^\ast \tau_{\leq r} R\varepsilon_! L_{U,n}(r) \to \tau_{\leq r} L_i^\ast R\varepsilon_! L_{U,n}(r) \to \tau_{\leq r} R\varepsilon_! L_{\kappa,n}(r), \]

whereas the first map canonically exists since \( L_i^\ast \) preserves \((-r)\)-connected objects.

The \( (\tau_{\leq r} R\varepsilon_! L_{U,n}(r))[r] \) assemble into a naive \( \mathbb{Z}/p^n[\mathbb{G}_{m,U}, \{ 1 \}]_{\text{Nis}} \)-spectrum \( F_n \), and the \( (\tau_{\leq r} R\varepsilon_! L_{\kappa,n}(r))[r] \) into a naive \( \mathbb{Z}/p^n[\mathbb{G}_{m,\kappa}, \{ 1 \}]_{\text{Nis}} \)-spectrum \( G_n \), and the \( g_{n,r} \) give a map of naive prespectra \( g_n: L_i^\ast F_n \to G_n \).

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Proposition 9.4: The maps $g_n$ are levelwise $\mathbb{A}^1$-weak equivalences.

Proof. Let $X \in \text{Sm}_U$, $X_\kappa$ the inverse image of $x$ and $i_X: X_\kappa \to X$ the closed inclusion. Proposition 9.2 implies that the canonical map

$$i_X^* \mathbb{R}\varepsilon_* \mu_p^{\otimes r} \to \mathbb{R}\varepsilon_* \mu_p^{\otimes r}$$

is an isomorphism in $D(\text{Sh}(X_\kappa, \text{Nis}, \mathbb{Z}/\mathbb{p}^n))$ (here the first $\mu_p^{\otimes r}$ denotes an object in $D(\text{Sh}(X_\kappa, \text{ét}, \mathbb{Z}/\mathbb{p}^n))$, whereas the second an object in $D(\text{Sh}(X_\kappa, \text{ét}, \mathbb{Z}/\mathbb{p}^n))$). By the exactness of $i_X^*$ the canonical map

$$i_X^* \tau_\delta \mathbb{R}\varepsilon_* \mu_p^{\otimes r} \to \tau_\delta \mathbb{R}\varepsilon_* \mu_p^{\otimes r}$$

is thus also an isomorphism (in the same category). Thus the claim follows from Proposition 9.3.

Note that map $\mathbb{Z}/\mathbb{p}^n(\mathbb{G}_n,r, \mathbb{G}_n,r)$ is (homotopy) discrete, so that $\mathbb{E}_n$ has a canonical model as spectrum which we also denote by $\mathbb{E}_n$. Moreover $E := \text{holim}_n \mathbb{E}_n$ is well-defined up to canonical isomorphism, and the canonical map $\mathbb{M}_\kappa \to E$ is the $p$-completion map.

We denote by $H_{B,\kappa}$ the Beilinson spectrum over $\kappa$. There is a canonical map

$$H_{B,\kappa} \to (E \times \prod_{p \neq \text{char}(\kappa)} D_p)_{\mathbb{Q}}.$$
and the canonical diagram

\[
\begin{array}{ccc}
\mathcal{M}_\kappa & \xrightarrow{\cong} & E \times \prod_{p=\text{char}(\kappa)} D_p \\
\downarrow & & \downarrow \\
\mathcal{H}_{B,\kappa} & \xrightarrow{\cong} & (E \times \prod_{p=\text{char}(\kappa)} D_p)_{\mathbb{Q}}
\end{array}
\]

(21)

is homotopy cartesian.

Suppose now that the \( U \) from above is the spectrum of a complete discrete valuation ring \( \Lambda \) and \( x \) is the closed point of \( U \). Let \( p = \text{char}(\kappa) \). Above we have for any prime \( l \neq p \) constructed maps of naive spectra

\[
\Lambda^i_* \mathbb{M}_U | l^n \cong \Lambda^i_* F_n \to G_n
\]

(here the dependence of \( G_n \) and \( F_n \) on \( l \) is suppressed) which are isomorphisms by Proposition 9.4 (here we view the naive spectra taking values in the \( \mathbb{A}^1 \)-local categories). Thus these maps lift uniquely to isomorphisms between the corresponding spectra. We get canonical maps \( \Lambda^i_* \mathbb{M}_U \to D_l \) for all primes \( l \neq p \).

Let \( \eta \) be the complement of \( \{x\} \) in \( U \) and \( j: \eta \to U \) the open inclusion. We again have the objects \( L_{\eta,n}(r) := p^{\otimes r}_n \in D(\text{Sh}(\text{Sm}_{\eta,\mathbb{A}^1}, \mathbb{Z}/p^n)) \) and the map of sites \( \varepsilon: \text{Sm}_{\eta,\mathbb{A}^1} \to \text{Sm}_{\eta,\mathbb{A}^1} \). By Lemma 4.3 we get the first isomorphism in the chain of isomorphisms

\[
\mathbb{R}j_* \tau_{\mathbb{A}^1} \mathbb{R}\varepsilon_* L_{\eta,n}(r) \cong \tau_{\mathbb{A}^1} \mathbb{R}j_* \mathbb{R}\varepsilon_* L_{\eta,n}(r) \cong \tau_{\mathbb{A}^1} \mathbb{R}\varepsilon_* \mathbb{R}j_* L_{\eta,n}(r)
\]

in \( D(\text{Sh}(\text{Sm}_{U,\mathbb{A}^1}, \mathbb{Z}/p^n)) \) (the second isomorphism is obvious).

Mapping to \( \mathcal{H}' \) we get a map

\[
\mathbb{R}j_* \tau_{\mathbb{A}^1} \mathbb{R}\varepsilon_* L_{\eta,n}(r) \to \mathcal{H}'(\mathbb{R}\varepsilon_* \mathbb{R}j_* L_{\eta,n}(r))[-r]
\]

\[
\to \varepsilon_* \mathcal{H}'(\mathbb{R}\varepsilon_* \mathbb{R}j_* L_{\eta,n}(r))[-r] \cong \varepsilon_* \mathbb{R}^r j_* L_{\eta,n}(r)[-r].
\]

For any \( X \in \text{Sm}_U \) we have a map

\[
(i_X)^* \mathbb{R}^r j_{X,*} (L_{\eta,n}(r)|_{X_{\eta,\mathbb{A}^1}}) \to \nu_n^r \oplus \nu_n^{-1}
\]

in \( \text{Sh}(X_{\kappa,\mathbb{A}^1}, \mathbb{Z}/p^n) \) constructed in [4, §(6.6)]. The second projection of this map was already described in section [4.1.1] the first projection is similar: it sends a symbol \( \{f_1, \ldots, f_r\}, f_1, \ldots, f_r \in (i_X)^* \mathcal{O}_X^r \), to \( d\log f_1 \cdots d\log f_r \) and a symbol \( \{f_1, \ldots, f_{r-1}, \pi\}, \pi \) a fixed uniformizer of \( \Lambda \), to 0.
As in section 4.1.1 these maps glue to give a map
\[ \mathbb{R}^r j_* L_{\eta,n}(r) \xrightarrow{\varphi \oplus \psi} i_* \nu_n^r \oplus i_* \nu_n^{r-1}. \]

As in section 7.5 we denote by \( F_{n,r} \) the homotopy fiber of the composition
\[ \mathbb{R} j_* \tau_{\leq r} \mathbb{R} \varepsilon_* L_{\eta,n}(r) \xrightarrow{\varepsilon_*} \mathbb{R} j_* L_{\eta,n}(r) \xrightarrow{i_*} \nu_n^r \oplus i_* \nu_n^{r-1} \xrightarrow{i_*} \nu_n^{r-1}, \]
and the \( F_{n,r} \) assemble to a naive spectrum \( F_n \).

Note that the maps \( \varphi \) give rise to maps \( F_{n,r} \rightarrow i_* \nu_n^r \), thus to maps
\[ \alpha_{n,r} : Li^* F_{n,r} \rightarrow \nu_n^r = E_{n,r}. \]

**Lemma 9.5:** The maps \( \alpha_{n,r} \) assemble to a map \( \alpha_n : Li^* F_n \rightarrow E_n \) of naive prespectra.

**Proof.** We leave the verification to the reader. \( \square \)

Our next goal is to show that the \( \alpha_{n,r} \) are \( \mathbb{A}^1 \)-weak equivalences.

**Proposition 9.6:** Let \( X \) be a scheme of characteristic \( p \) and \( F \in \text{Sh}(X_{\text{ét}}, \mathbb{Z}) \) a \( p \)-torsion sheaf. Then \( \mathbb{R}^i \varepsilon_* F \in \text{Sh}(X_{\text{Nis}}, \mathbb{Z}) \) is zero for \( i > 1 \).

**Proof.** Let \( K \) be a field and \( x : \text{Spec} K \rightarrow X \) a map inducing an isomorphism on residue fields. Then
\[ \Gamma(\text{Spec} K, x^* \mathbb{R} \varepsilon_* F) \cong \mathbb{R} \Gamma(\text{Spec} K, x^* F) \]
in \( \text{D}(\text{Ab}) \). The latter complex (which is Galois cohomology) vanishes in cohomological degrees \( > 1 \) by [37] II, Prop. 3]. \( \square \)

**Proposition 9.7:** Let \( X \) be a scheme and \( F \) a quasi coherent sheaf on \( X_{\text{ét}} \). Then \( \mathbb{R}^i \varepsilon_* F \in \text{Sh}(X_{\text{Nis}}, \mathbb{Z}) \) is zero for \( i > 0 \).

**Proof.** Let \( K \) be a field and \( x : \text{Spec} K \rightarrow X \) a map inducing an isomorphism on residue fields. Then
\[ \Gamma(\text{Spec} K, x^* \mathbb{R} \varepsilon_* F) \cong \mathbb{R} \Gamma(\text{Spec} K, x^* F) \]
in \( \text{D}(\text{Ab}) \). But for a finite Galois extension \( L/K \) the object \((x^* F)(\text{Spec} L)\) is an induced Gal\((L/K)\)-module, thus its cohomology vanishes in degrees \( > 0 \). \( \square \)
For $X \in \text{Sm}_k$ let $\Omega^1_X$ be the sheaf on $X_{\acute{e}t}$ (and thus also on $X_{\text{Nis}}$ and $X_{\text{Zar}}$) of absolute Kähler differentials on $X$. It is quasi coherent. Let $\Omega^\bullet_X$ be the exterior algebra over $\mathcal{O}_X$ of $\Omega^1_X$. Define subsheaves

$$B^i_X := \text{im}(d: \Omega^{i-1}_X \to \Omega^i_X)$$

and

$$Z^j_X := \ker(d: \Omega^j_X \to \Omega^{j+1}_X)$$
of $\Omega^j_X$ on $X_{\acute{e}t}$.

**Lemma 9.8:** For $X \in \text{Sm}_k$ we have $\mathbb{R}^i \varepsilon_* \Omega^j_X = \mathbb{R}^i \varepsilon_* B^j_X = \mathbb{R}^i \varepsilon_* Z^j_X = 0$ in $\text{Sh}(X_{\text{Nis}}, \mathbb{F}_p)$ for $j \geq 0$ and $i > 0$.

**Proof.** We have $\mathbb{R}^i \varepsilon_* \Omega^j_X = 0$ for $j \geq 0$ and $i > 0$ by Proposition 9.7. We have isomorphisms

$$\Omega^j_X \cong Z^j_X/B^j_X$$
given by the inverse Cartier operator, see [4, top of p. 112], thus the claim follows for $j = 0$. Suppose by induction the claim for $j$. The exact sequence

$$0 \to Z^j_X \to \Omega^j_X \to B^{j+1}_X \to 0$$
shows the claim for $B^{j+1}_X$, the above isomorphism for $j + 1$ shows the claim for $Z^{j+1}_X$. This finishes the proof. \hfill \square

Let $L_\eta(r) := L_{\eta,1}(r)$. For any $X \in \text{Sm}_k$ we have the following isomorphisms

$$i_X^* (\mathbb{R} j_\ast \tau_{\leq r} \mathbb{R} \varepsilon_* L_\eta(r))|_{X_{\text{Nis}}} \cong i_X^* (\tau_{\leq r} \mathbb{R} \varepsilon_* j_\ast L_\eta(r)|_{X_{\text{Nis}}})$$

$$\cong \tau_{\leq r} i_X^* (\mathbb{R} \varepsilon_* j_\ast L_\eta(r)|_{X_{\text{Nis}}}) \cong \tau_{\leq r} \mathbb{R} \varepsilon_* i_X^* \tau_{\leq r} (\mathbb{R} j_\ast L_\eta(r)|_{X_{\acute{e}t}}).$$

The first isomorphism uses (22), the second the exactness of $i_X^*$ and the third also Proposition 9.2 (strictly speaking we do not need the third isomorphism, but we give it for motivation).

Set $K_{X,0} := i_X^* \tau_{\leq r} (\mathbb{R} j_\ast L_\eta(r)|_{X_{\acute{e}t}})$. We will define filtrations on the $\mathcal{H}^k(K_{X,0})$, $0 \leq k \leq r$ (compare with [4]). We start with $k = r$.

For $m \geq 1$ let $U^m \mathcal{H}^r(K_{X,0})$ be the subsheaf of $\mathcal{H}^r(K_{X,0})$ generated étale locally by sections of the form \{ $x_1, \ldots, x_r$ \}, $x_i \in i_X^* j_{X,r} \mathcal{O}^*_X$, such that $x_1 - 1 \in \pi^m i_X^* \mathcal{O}^*_X$, see [4] p. 111. We define $U^0 \mathcal{H}^r(K_{X,0}) := \mathcal{H}^r(K_{X,0})$.

Let $e$ be the absolute ramification index of $\Lambda$ and $e' := \frac{ep}{p-1}$.
The 0-th graded piece of the filtration $U^\bullet$ on $\mathcal{H}^r(K_{X,0})$ is $\nu_1^r \ast \nu_1^{-1}$, the $m$-th graded piece for $m \geq e'$ is 0, and for $1 \leq m < e'$ and $m$ prime to $p$ is $\Omega_{X,1}^{-1}$ and for $p \mid m$ is $B_{X,m}^r \ast B_{X,n}^{-1}$, see [1] Cor. (1.4.1). We denote these graded pieces by $Q^m_X$. These sheaves glue to sheaves $Q^m$ on $\text{Sm}_{k,\ell^t}$.

To define the filtrations for $k < r$ we have to adjoin a $p$-th root of unity and descend a filtration upstairs.

Let $\bar{\Lambda} := \Lambda[\zeta_p]$ be the integral closure of $\Lambda$ in $\bar{K} := K(\zeta_p)$, where $K$ is the quotient field of $\Lambda$ and $\zeta_p$ is a primitive $p$-th root of unity. Let $d$ be the degree of $\bar{K}$ over $K$ and $G := \text{Gal}(\bar{K}/K)$. We have $d \mid p - 1$. The group $G$ is canonically identified with a subgroup of $\mathbb{F}_p^* \cong \mu_{p-1} \subset \mathbb{K}^*$.

There exists a uniformizer $\pi$ of $\bar{\Lambda}$ such that $\pi^d \in \Lambda$, since this is true in the case $\Lambda = \mathbb{Z}_p$ (take $\pi = r^{\sqrt{-p}}$) (use Kummer theory).

Let $\bar{U} := \text{Spec} \bar{\Lambda}$, $\bar{\eta}$ the generic point of $\bar{U}$, $\bar{\kappa}$ the residue field of $\bar{\Lambda}$.

For $X \in \text{Sm}_U$ denote by $\bar{X}$ the base change to $\bar{U}$.

The notations $\bar{X}_\eta$, $\bar{X}_k$, $\bar{j}_X$ and $\bar{i}_X$ explain themselves.

We fix now $X \in \text{Sm}_U$.

We have the commutative diagram

$$
\begin{array}{ccc}
\bar{X}_k & \xrightarrow{i_X} & \bar{X} \\
\downarrow f'' & & \downarrow f \\
X_k & \xrightarrow{i_X} & X
\end{array}
$$

We denote by $\bar{L}(k)$ the sheaf $\mu_p^{\otimes k}$ viewed as object of $D(\text{Sh}(\bar{X}_\eta, \text{Sm}_p))$ and make the same definition without tildas. We have a canonical isomorphism $L(r) \cong (\mathbb{R}i'_X \bar{L}(r))^G$, where $(-)^G$ denotes homotopy fixed points. Since the order of $G$ is prime to $p$ the cohomology sheaves of the homotopy fixed points are the fixed points of the cohomology sheaves. Also homotopy fixed points commutes with $\mathbb{R}j_{X,*}$ and $i_X^*$, thus we have

$$i_X^* \mathbb{R}j_{X,*}L(r) \cong (i_X^* \mathbb{R}j_{X,*} \mathbb{R}f'_X \bar{L}(r))^G \cong (i_X^* \mathbb{R}f_* \mathbb{R}j_{X,*} \bar{L}(r))^G \cong (\mathbb{R}f'_X i_X^* \mathbb{R}j_{X,*} \bar{L}(r))^G.$$

For the last isomorphism we have used the proper base change formula.

So we have

$$\mathcal{H}^k(i_X^* \mathbb{R}j_{X,*}L(r)) \cong (f''_X \mathcal{H}^k(i_X^* \mathbb{R}j_{X,*} \bar{L}(r)))^G.$$

For an object $X$ with $\mathbb{F}_p$-coefficients and $G$-action we denote by $X\{k\}$ the same object with the $G$-action twisted by the $k$-fold tensor power of the canonical action of $G$ on $\mathbb{F}_p$.
We have an isomorphism $\overline{L}(0) \cong \overline{L}(1)$ sending 1 to $\zeta_p$. This gives us an isomorphism $\overline{L}(0) \cong \overline{L}(k)$ for any $k$. We get the isomorphism

$$\overline{L}(r) \cong \overline{L}(k) \otimes \overline{L}(r-k) \cong \overline{L}(k) \otimes \overline{L}(0) \cong \overline{L}(k).$$

We thus have

$$f''_*\mathcal{H}^k(i^*_X \mathcal{R}^{j_0}j_{X,*} \overline{L}(r)) \cong (f''_*\mathcal{H}^k(i^*_X \mathcal{R}^{j_0}j_{X,*} \overline{L}(k)))\{r-k\}$$

as $G$-objects.

We have a filtration on $\mathcal{H}^k(i^*_X \mathcal{R}^{j_0}j_{X,*} \overline{L}(k))$ by the $U^m\mathcal{H}^k(i^*_X \mathcal{R}^{j_0}j_{X,*} \overline{L}(k))$, where the latter subsheaf is generated as above by sections of the form $\{x_1, \ldots, x_r\}$ such that $x_1 - 1 \in \pi^m\pi^X_{X,*}(m \geq 1$, for $m = 0$ we again take the whole sheaf). Clearly the $U^m\mathcal{H}^k(i^*_X \mathcal{R}^{j_0}j_{X,*} \overline{L}(k))$ are invariant under the $G$-action, thus the

$$U^k, m := ((f''_*U^m\mathcal{H}^k(i^*_X \mathcal{R}^{j_0}j_{X,*} \overline{L}(k)))\{r-k\})^G$$

filter $\mathcal{H}^k(i^*_X \mathcal{R}^{j_0}j_{X,*} \overline{L}(r)) \cong \mathcal{H}^k(K_{X,0})$.

Let $\check{e}$ be the absolute ramification index of $\overline{X}$ and $\check{e}' := \frac{\check{e}}{p-1}$.

The 0-th graded piece of the filtration $U^k \mathcal{H}^k(i^*_X \mathcal{R}^{j_0}j_{X,*} \overline{L}(k))$ is $\nu_0^k \ast \nu^{k-1}_0$, the $m$-th graded piece for $m \geq \check{e}'$ is 0, and for $1 \leq m < \check{e}'$ and $m$ prime to $p$ is $\Omega^{k-1}_{X,0}$ and for $p \mid m$ is $B^k_{X,0} \oplus B^{k-1}_{X,0}$, see [X Cor. (1.4.1)].

Let us denote these graded pieces by $\overline{Q}_{X,m}^k$. Let us equip the $P^k, m := f''_*\overline{Q}_{X,m}^k$ with $G$-actions coming from $\overline{L}(k)$.

On the $f''_*(\nu_0^k \ast \nu^{k-1}_0)$, $f''_*(\nu_0^k \ast \nu^{k-1}_0)$ and $f''_*(B^k_{X,0} \oplus B^{k-1}_{X,0})$ there is a canonical $\text{Gal}(\overline{\kappa}/\kappa)$-action, thus a canonical $G$-action. These also induce $G$-actions on the $f''_*\overline{Q}_{X,m}^k$. We denote these $G$-objects by $R^k_{X,m}$. The formulas in [X (4.3)] and the definition of the map to the 0-graaded part show that there are isomorphisms

$$P^k, m \cong B^k_{X,0} \{m\}$$

as $G$-objects.

We let $Q^k, m := \text{gr}^m(U^k_X) \cong (P^k, m)\{r-k\}^G$.

The considerations made show the following

**Proposition 9.9:** The sheaves $Q^k, m_X$ on $X_{\kappa, \text{ét}}$ only depend on $X_\kappa$ and glue to a sheaf $Q^k, m$ on $\text{Sm}_{\kappa, \text{ét}}$.

Define inductively objects $K_{X,m} \in \text{D}(\text{Sh}(X_{\kappa, \text{ét}}, \mathbb{F}_p))$ in the following way. For $m = 0$ we have already defined the object. We define $K_{X,1}$ to be the homotopy fiber of the composition

$$K_{X,0} \to \mathcal{H}^r(K_{X,0})[-r] \to Q^0_X[-r].$$
There is a canonical map $K_{X,1} \to Q_{X}^{1}[-r]$. Suppose $K_{X,m}$ together with a map

$$K_{X,m} \to Q_{X}^{m}[-r]$$

is already defined for $m < e'$. Define then $K_{X,m+1}$ to be the homotopy fiber of this last map. If $m + 1 < e'$ there is a map $K_{X,m+1} \to Q_{X}^{m+1}[-r]$. If $m + 1 \geq e'$ we have $K_{X,m+1} \cong \tau_{\leq(r-1)}K_{X,0}$ and there is a map

$$K_{X,m+1} \to Q_{X}^{r-1,0}[-r+1].$$

Keep going this way splitting off successively the $Q_{X}^{r-1,0}[-r+1], Q_{X}^{r-1,1}[-r+1], \ldots, Q_{X}^{r-k,m}[-k], \ldots, Q_{X}^{0,0}$ (where for this $m$ we require $0 \leq m < e'$) obtaining the $K_{X,m+2}, \ldots, K_{X,N} = 0$.

By construction we have triangles

$$K_{X,m+1} \to K_{X,m} \to Q_{X}^{k(m),m'(m)}[-k(m)] \to K_{X,m+1}[1],$$

where $k(m)$ and $m'(m)$ depend in a way on $m$ which we do not make explicit.

Set $H_{0} := \tau_{\leq r}R_{s}L_{\eta}(r)$. The maps $H_{0}|_{X_{\text{et}}} \to i_{X,*}Q_{X}^{\emptyset}[-r]$ glue to a map of sheaves $H_{0} \to Q_{X}^{\emptyset}[-r]$. We let $H_{1}$ be the homotopy fiber of this last map. We have a map $i_{X}^{*}H_{1}|_{X_{\text{et}}} \to K_{X,0}$ which factors uniquely through $K_{X,1}$, thus we get a map $i_{X}^{*}H_{1}|_{X_{\text{et}}} \to Q_{X}^{1}$ with adjoint $H_{1}|_{X_{\text{et}}} \to i_{X,*}Q_{X}^{1}$. These maps glue to a map $H_{1} \to i_{*}Q_{X}^{1}$ whose homotopy fiber we denote by $H_{2}$. Inductively one constructs objects $H_{m} \in D(\text{Sh}(\text{Sm}_{U}, \text{et}, \mathbb{F}_{p}))$, $0 \leq m \leq N$, with maps

$$i_{X}^{*}H_{m}|_{X_{\text{et}}} \to K_{X,m} \to Q_{X}^{k(m),m'(m)}[-k(m)]$$

(here we suppose $m > e'$, the other case is similar) whose adjoints glue to a map

$$H_{m} \to \mathcal{H}^{k(m)}(H_{m})[-k(m)] \to i_{*}Q^{k(m),m'(m)}[-k(m)].$$

$H_{m+1}$ is then defined to be the homotopy fiber of this map. By construction we have triangles

$$H_{m+1} \to H_{m} \to i_{*}Q^{k(m),m'(m)}[-k(m)] \to H_{m+1}[1].$$

Moreover for $X \in \text{Sm}_{U}$ we have

$$i_{X}^{*}(H_{m}|_{X_{\text{et}}}) \cong K_{X,m}.$$
Note that we have $\mathbb{R}^j\varepsilon_* Q^m \simeq i_* \mathbb{R}^j\varepsilon_* Q^m = 0$ for $j > 0$ and $m \geq 1$ by Lemma 9.8. Similarly we have $\mathbb{R}^j\varepsilon_* Q^{km} = 0$ for $j > 1$ by Proposition 9.6. Thus the canonical maps

$$\tau_\leq \mathbb{R}_* H_m \to \mathbb{R}_* H_m$$

are isomorphisms for $m \geq 1$. Since $\mathcal{H}^{r+1}(\mathbb{R}_* H_1) = 0$ it also follows that the map

$$\mathcal{H}^r(\mathbb{R}_* H_0) \to \varepsilon_* i_* Q^0 \simeq i_* \nu_1^r \oplus i_* \nu_1^{r-1}$$

is an epimorphism. It follows that we have an exact triangle

$$\mathbb{R}_* H_1 \to \tau_\leq \mathbb{R}_* H_0 \to \varepsilon_* i_* Q^0 \to \mathbb{R}_* H_1[1].$$

(23)

**Lemma 9.10:** For any $F \in D(\text{Sh}(\text{Sm}, \text{ét}, F_p))$ we have $\mathbb{R}_* F \in D(\text{Sh}(\text{Sm}_U, \text{Nis}, F_p))$ is $\mathbb{A}^1$-weakly contractible.

*Proof.* Tensoring $F$ with the Artin-Schreier exact triangle $F_p \to G_a \to G_a \to F_p[1]$ shows that it is sufficient to show that $\mathbb{R}_* (F \otimes G_a)$ is $\mathbb{A}^1$-contractible. The standard $\mathbb{A}^1$-contraction of $G_a$ does the job. \qed

**Proposition 9.11:** The homotopy cofiber of the map $\mathbb{R}_* H_N \to \mathbb{R}_* H_1$ is $\mathbb{A}^1$-weakly contractible in $D(\text{Sh}(\text{Sm}_U, \text{Nis}, F_p))$.

*Proof.* This homotopy cofiber is filtered with graded pieces the $\mathbb{R}_* i_* Q^m \simeq i_* \mathbb{R}_* Q^m$, $m \geq 1$, and the $\mathbb{R}_* i_* Q^{km} \simeq i_* \mathbb{R}_* Q^{km}$, so it is sufficient to show that these are $\mathbb{A}^1$-weakly contractible. But $i_*$ preserves $\mathbb{A}^1$-weak equivalences since it is finite, so the claim follows from Lemma 9.10. \qed

With Proposition 9.2 for any $X \in \text{Sm}_U$ we have $i^*_X(\mathbb{R}_* H_N)|_{X_{\text{Nis}}} \simeq \mathbb{R}_* i^*_X(H_N|_{\text{ét}}) \simeq \mathbb{R}_* K_{X,N} \simeq 0$, thus with Proposition 9.1 we get $\mathbb{L}^i \mathbb{R}_* H_N \simeq 0$. With Proposition 9.11 we get that $\mathbb{L}^i \mathbb{R}_* H_1$ is $\mathbb{A}^1$-weakly contractible. With (23) it follows that

$$\mathbb{L}^i \tau_\leq \mathbb{R}_* H_0 \to \mathbb{L}^i \varepsilon_* i_* Q^0$$

is an $\mathbb{A}^1$-weak equivalence.

We get

**Theorem 9.12:** The maps $\alpha_{n,r}$ defined before Lemma 9.5 are $\mathbb{A}^1$-weak equivalences.
So we have isomorphisms of naive spectra

\[ \mathbb{L} \iota^* \mathbb{M} \mathbb{Z}_U / p^n \cong \mathbb{L} \iota^* F_n \cong E_n \]

which lift uniquely to isomorphisms of spectra. The induced map

\[ \mathbb{L} \iota^* \mathbb{M} \mathbb{Z} \to E \]

is the \( p \)-completion map.

We get that \( \mathbb{L} \iota^* \mathbb{M} \mathbb{Z}_U \) sits in the same homotopy cartesian square as \( \mathcal{M}_\kappa \) (see diagram (21)), whence (using Theorem 6.7)

**Corollary 9.13:** There are canonical isomorphisms \( \mathbb{L} \iota^* \mathbb{M} \mathbb{Z}_U \cong \mathcal{M}_\kappa \cong \mathbb{M} \mathbb{Z}_\kappa \).

We are next going to construct natural comparison maps for our spectra for morphisms between Dedekind domains of mixed characteristic.

So let \( D \to \tilde{D} \) be a map of Dedekind domains of mixed characterisitc. We use the notation of section 4.1.2 without tildas for the situation over \( S = \text{Spec}(D) \) and with tildas for the situation over \( \tilde{S} = \text{Spec}(\tilde{D}) \). Note that for the various categories of complexes of sheaves we have Quillen adjunctions between the categories attached to \( S \) and \( \tilde{S} \). We let \( f \) denote the various maps from the situation with tildas to the situation without, e.g. \( f: \tilde{S} \to S \) or \( f: \tilde{U} \to U \).

We have an isomorphism \( \varphi: f^* \mathbb{L}_\bullet(r) \cong \tilde{\mathbb{L}}_\bullet(r) \). Choose map \( \psi: f^* \mathcal{T} \to \mathcal{T} \) lifting the image of \( \varphi \) in the homotopy category. Thus we get a map

\[ \psi': f^* \text{Sym}(\mathcal{T}) \to \text{Sym}(\mathcal{T}) \]

of commutative monoids in symmetric \( \mathbb{Z}/p^*[\mathbb{G}_{m,\tilde{U}},\{1\}]_{\acute{e}t} \)-spectra. Using lifting arguments one gets a map

\[ \psi'': f^* \text{RQSym}(\mathcal{T}) \to \text{RQSym}(\mathcal{T}) \].

One gets induced maps \( f^* A \to \tilde{A}, f^* A' \to \tilde{A}', f^* B \to \tilde{B}, f^* C \to \tilde{C}, f^* C' \to \tilde{C}' \) and \( f^* D_p \to \tilde{D}_p \).

By the definition of \( \mathbb{M} \mathbb{Z}_S \) and \( \mathbb{M} \mathbb{Z}_{\tilde{S}} \) it is then clear that we get the comparison map

\[ \Phi_f: \mathbb{L}f^* \mathbb{M} \mathbb{Z}_S \to \mathbb{M} \mathbb{Z}_{\tilde{S}} \]

which is a map of \( E_\infty \)-spectra.

**Lemma 9.14:** If \( \tilde{D} \) is a filtered colimit of smooth \( D \)-algebras then the comparison map \( \Phi_f \) is an isomorphism.

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Proof. Let $\tilde{D} = \colim_\alpha D_\alpha$, where each $D_\alpha$ is a smooth $D$-algebra and set $S_\alpha := \text{Spec}(D_\alpha)$. Let $f_\alpha : S_\alpha \to S$ be the canonical maps. We show that for any $X \in \text{Sm}_S$ and integers $p, q$ the induced map

$$\text{Hom}_{\text{SH}(\tilde{S})}(\Sigma^p \Sigma^\infty_+ X, f^* M_{\mathbb{Z}}) \to \text{Hom}_{\text{SH}(\tilde{S})}(\Sigma^p \Sigma^\infty_+ X, M_{\mathbb{Z}})$$

(24)

is an isomorphism. By the the remarks after [20, Definition A.1] we can write $X = \lim_\alpha X_\alpha$, where each $X_\alpha$ is a smooth and separated $S_\alpha$-scheme of finite type. By [20, Lemma A.7.(1)] the left side of (24) can be written as

$$\colim_\alpha \text{Hom}_{\text{SH}(S_\alpha)}(\Sigma^p \Sigma^\infty_+ X_\alpha, f_\alpha^* M_{\mathbb{Z}}).$$

A similar formula holds for the right hand side, using the isomorphism $M_{\mathbb{Z}} \tilde{S} \cong M_{\mathbb{Z}}$ and observing that the cycle complexes defining $M_{\mathbb{Z}}$ evaluated over $X$ are a colimit of the cycle complexes over the $X_\alpha$.

\[\Box\]

Corollary 9.15: Suppose $\tilde{D}$ is the completion of a local ring of $S$ at a closed point of positive residue characteristic. Then $\Phi_f$ is an isomorphism.

Proof. It is known that then $\tilde{D}$ is a filtered colimit of smooth $D$-algebras, hence Lemma 9.14 applies. \[\Box\]

Theorem 9.16: Let $S = \text{Spec}(D)$, $D$ a Dedekind domain of mixed characteristic, $x \in S$ a closed point of positive residue characteristic and $i : \{x\} \to S$ the inclusion. Then there is a canonical isomorphism $L i^* M_{\mathbb{Z}} \cong M_{\mathbb{Z}(x)}$ which respects the $E_\infty$-structures.

Proof. The isomorphism as spectra follows now from Corollary 9.14 and Corollary 9.15. The isomorphism can be made to respect the $E_\infty$-structures by the uniqueness of $E_\infty$-structures on $M_{\mathbb{Z}(x)}$, which holds since this spectrum is the zero-slice of the sphere spectrum.

\[\Box\]

Lemma 9.17: Let $g : k \to k'$ be a field extension. Then the natural map $g^* M_{\mathbb{Z}}_k \to M_{\mathbb{Z}}_{k'}$ is an isomorphism.

Proof. This follows from [20, Theorem 4.18] (taking $U$ to be the spectrum of the prime field contained in $k$). \[\Box\]

Lemma 9.18: Let $\tilde{S} = \text{Spec}(\tilde{D})$, $\tilde{D}$ a Dedekind domain, and let $\varphi : E \to F$ be any map in $\text{SH}(\tilde{S})$. Suppose for any $x \in \tilde{S}$ that $L i_x^* \varphi$ is an isomorphism, where $i_x$ denotes the inclusion $\{x\} \to \tilde{S}$. Then $\varphi$ is an isomorphism.
Proof. This follows from [6, Proposition 4.3.9] and localization. □

**Theorem 9.19**: For any $f$ as above the comparison map $\Phi_f$ is an isomorphism.

Proof. This follows from Theorem [9.16](#) Lemma [9.17](#) and Lemma [9.18](#) □

### 10 The motivic functor formalism

For any Noetherian separated scheme $X$ of finite Krull dimension (we call such schemes from now on base schemes) we let $\text{MZ}_X := f^*\text{MZ}_{\text{Spec}(\mathbb{Z})}$, where $f: X \to \text{Spec}(\mathbb{Z})$ is the structure morphism. We let $\text{MZ}_X – \text{Mod}$ be the model category of highly structured $\text{MZ}_X$-module spectra and set $\text{DM}(X) := \text{Ho}(\text{MZ}_X – \text{Mod})$. This is done e.g. along the lines of [35](#). For any map of base schemes $f: X \to Y$ we get an adjunction

$$f^*: \text{DM}(Y) \leftrightarrow \text{DM}(X): f_*$.$$

The categories $\text{DM}(X)$ are closed tensor triangulated and the functors $f^*$ are symmetric monoidal.

If $f$ is smooth the functor $f^*$ has a left adjoint $f_!$.

Note that all these functors commute with the forgetful functors

$$\text{DM}(X) \to \text{SH}(X).$$

It follows that the assignment

$$X \mapsto \text{DM}(X)$$

has the structure of a stable homotopy functor in the sense of [2](#).

Thus the main results of loc. cit. are valid for this assignment.

In particular for a morphism of finite type $f: X \to Y$ we have an adjoint pair

$$f!: \text{DM}(X) \leftrightarrow \text{DM}(Y): f^!.$$

Moreover the projective base change theorem holds.

We also have the

**Theorem 10.1**: Let $i: Z \hookrightarrow X$ be a closed inclusion of base schemes and $j: U \to X$ the open complement. Then for any $F \in \text{DM}(X)$ there is an exact triangle

$$j_!j^*F \to F \to i_*i^*F \to j_!j^*F[1]$$

in $\text{DM}(X)$.
11 Further applications

11.1 The Hopkins-Morel isomorphism

We first equip $\mathbb{M}Z$ with an orientation.

**Proposition 11.1:** Let $X$ be a smooth scheme over Dedekind domain of mixed characteristic. Then there is a unique orientation on $\mathbb{M}Z_X$. The corresponding formal group law is the additive one.

**Proof.** Let $S$ be the spectrum of a Dedekind domain of mixed characteristic. Let $P \in \text{D}(\text{Sh}(\text{Sm}_{S,\text{zar}},\mathbb{Z}))$ be the first $\mathbb{A}^1$- and Nisnevich-local space of an $\Omega$-$\mathbb{P}^1$-spectrum model of $\mathbb{M}Z_S$. Then by Theorem 7.10 there is a canonical isomorphism $P \cong \mathcal{O}^*_S[1]$. Moreover by the proof of this theorem the canonical map $\mathbb{Z}[\mathbb{P}^1,\{\infty\}]_{\text{zar}} \to P$ induced by the first bonding map is induced by the suspension of the canonical map $\mathbb{Z}[G_{m,S},\{1\}]_{\text{zar}} \to \mathcal{O}^*_S$, using the canonical isomorphism $\mathbb{P}^1 \cong \mathbb{G}_{m,S} \wedge S^1$ in $\mathcal{H}_*(S)$. Thus our map $\mathbb{Z}[\mathbb{P}^1,\{\infty\}]_{\text{zar}} \rightarrow \mathcal{O}^*_S[1]$ classifies the line bundle $\mathcal{O}(-1)$. So the map $\Sigma^{-2,-1}\mathbb{P}^\infty \rightarrow \mathbb{M}Z_S$ corresponding to the map $\mathbb{Z}[\mathbb{P}^\infty]_{\text{zar}} \to \mathcal{O}^*_S[1]$ which classifies the tautological line bundle is an orientation of $\mathbb{M}Z_S$. Pulling back to any smooth scheme $X$ over $S$ gives an orientation of $\mathbb{M}Z_X$. Since motivic cohomology of $X$ with negative weight vanishes this orientation is unique and the corresponding formal group law is the additive one.

By pulling back the unique orientation of $\mathbb{M}Z_{\text{Spec}(\mathbb{Z})}$ to any base scheme $X$ we see that $\mathbb{M}Z_X$ has a canonical additive orientation.

**Remark 11.2:** We note that over smooth schemes $X$ over Dedekind domains of mixed characteristic or over fields the orientation map $\mathbb{M}GL_X \to \mathbb{M}Z_X$ has a unique structure of an $E_\infty$-map. This $E_\infty$-map is achieved as the composition of the $E_\infty$-maps

$$\mathbb{M}GL_X \to s_0\mathbb{M}GL_X \cong s_01 \to s_0\mathbb{M}Z_X \to \mathbb{M}Z_X,$$

where the last map exists since the map $f_0\mathbb{M}Z_X \to s_0\mathbb{M}Z_X$ is an isomorphism since $\mathbb{M}Z_X$ is 0-truncated.

Thus for any base scheme $X$ the orientation $\mathbb{M}GL_X \to \mathbb{M}Z_X$ has a canonical $E_\infty$-structure. Since $\mathbb{M}GL_X$ has a strong periodization this gives a second proof that $\mathbb{M}Z_X$ is strongly periodizable.

We see that we can factor the orientation map $\mathbb{M}GL \to \mathbb{M}Z_{\text{Spec}(\mathbb{Z})}$ through the quotient $\mathbb{M}GL/(x_1,x_2,\ldots)\mathbb{M}GL$, where the $x_i$ are images of generators of $\mathbb{M}U_n$ with respect
to the natural map $\text{MU}_* \to \text{MGL}_{2*}$. Pulling back this factorization to any base scheme $X$ we get a map $\Phi_X: \text{MGL}_X/(x_1, x_2, \ldots) \to \text{MZ}_X$.

**Theorem 11.3:** Let $R$ be a commutative ring and $X$ a base scheme whose positive residue characteristics are all invertible in $R$. Then $\Phi_X \wedge M_R$, where $M_R$ denotes the Moore spectrum on $R$, is an isomorphism.

**Proof.** We only have to show this statement for $X$ being equal to the spectrum of a localization of $\mathbb{Z}$. Then it follows by pullback to the points of $X$ and [20, Theorem 7.12] using Theorem 9.16 and Lemma 9.18. □

**Corollary 11.4:** Let $R$ be a commutative ring and $X$ a base scheme whose positive residue characteristics are all invertible in $R$. Then $MR_X$ (which denotes $MZ_X$ with $R$-coefficients) is cellular.

### 11.2 The motivic dual Steenrod algebra

In the whole section we fix a prime $l$.

For a base scheme $S$ we denote by $\text{Pic}_S$ a strictification of the 2-functor which assigns to any $X \in \text{Sm}_S$ the Picard groupoid of line bundles on $X$. We denote by $\nu \text{Pic}_S$ the motivic space which assigns to any $X \in \text{Sm}_S$ the nerve of $\text{Pic}_S(X)$.

**Proposition 11.5:** Let $S$ be a regular base scheme and let $f: \mathbb{P}_S^{\infty} \to \nu \text{Pic}_S$ be a map classifying a $\mathbb{G}_m$-torsor $P$ on $\mathbb{P}_S^{\infty}$. Then there is an $A^1$-fiber sequence

$$P \to \mathbb{P}_S^{\infty} \to \nu \text{Pic}_S$$

of motivic spaces.

**Proof.** The sequence is a fiber sequence in simplicial presheaves equipped with a model structure with objectwise weak equivalences. Thus the claim follows from the $A^1$- and Nisnevich-locality of $\nu \text{Pic}_S$ (and e.g. right properness of motivic model structures). □

For a base scheme $S$ we let $W_{S,n,k}$ be the $\mathbb{G}_m$-torsor on $\mathbb{P}_S^k$ corresponding to the line bundle $\mathcal{O}_{\mathbb{P}_S^k}(-n)$. We let $W_{S,n} := \colim_k W_{S,n,k}$ be the corresponding $\mathbb{G}_m$-torsor on $\mathbb{P}_S^{\infty}$.

We are going to compute the motivic cohomology of $W_{S,n}$ with $\mathbb{Z}/m$-coefficients for $m|n$ relative to the motivic cohomology of the base $S$. We orient ourselves along the lines of [43, §6].
We have a cofibration sequence

\[ W_{S,n,+} \to \mathcal{O}_{\mathbb{P}^\infty_S}(-n)_+ \to \text{Th}(\mathcal{O}_{\mathbb{P}^\infty_S}(-n)). \]  

(25)

For a motivic space \( \mathcal{X} \) over \( S \) let

\[ H^{p,q}(\mathcal{X}) := \text{Hom}_{\text{SH}(S)}(\Sigma^\infty \mathcal{X}_+, \Sigma^{p,q} \mathbb{M} \mathbb{Z}) \]

be the motivic cohomology of \( \mathcal{X} \). More generally for an abelian group \( A \) we set

\[ H^{p,q}(\mathcal{X}, A) := \text{Hom}_{\text{SH}(S)}(\Sigma^\infty \mathcal{X}_+, \Sigma^{p,q} A) \]

We denote the respective reduced motivic cohomology groups of pointed motivic spaces by \( \tilde{H} \).

Then (25) gives a long exact sequence

\[ \cdots \to H^{s-2,\ast-1}(S)[[\sigma]] \xrightarrow{m\sigma} H^{\ast,\ast}(S)[[\sigma]] \to H^{\ast,\ast}(W_{S,n}) \to H^{s-1,\ast-1}(S)[[\sigma]] \to \cdots. \]  

(26)

Here \( \sigma \) is the class of \( \mathcal{O}_{\mathbb{P}^\infty_S}(-1) \) in \( H^{2,1}(\mathbb{P}^\infty_S) \).

For any \( m > 0 \) let \( \beta_m : H^{\ast,\ast}(\_, \mathbb{Z}/m) \to H^{\ast+1,\ast}(\_) \) be the Bockstein homomorphism.

Let \( v_n \in H^{2,1}(W_{S,n}) \) be the pullback of \( \sigma \) under the canonical map \( W_{S,n} \to \mathbb{P}^\infty_S \).

Lemma 11.6: For any \( m > 0 \) there is a \( u \in H^{1,1}(W_{S,n}, \mathbb{Z}/m) \) such that the restriction of \( u \) to \( \ast \) is 0 and such that \( \beta_m(u) = \frac{n}{\gcd(m,n)} v_n \). If \( S \) is smooth over a Dedekind domain of mixed characteristic or over a field then this \( u \) is unique with these properties.

Proof. Let \( \tilde{v} \in H^{2,1}(W_{S,n}) \) be any \( m \)-torsion class which restricts to 0 on \( \ast \). Note that \( \frac{n}{\gcd(m,n)} v_n \) is such a class. We will prove that then there is a unique \( \tilde{u} \in H^{1,1}(W_{S,n}, \mathbb{Z}/m) \) which restricts to 0 on \( \ast \) such that \( \beta_m(\tilde{u}) = \tilde{v} \), assuming \( S \) is smooth over a Dedekind domain of mixed characteristic or over a field. The general statement about existence follows then by base change (e.g. from Spec(\( \mathbb{Z} \)) to \( S \)).

Consider the commutative diagram

\[
\begin{array}{ccc}
H^{1,1}(W_{S,n}) & \longrightarrow & H^{1,1}(W_{S,n}, \mathbb{Z}/m) \\
\downarrow & & \downarrow \\
H^{1,1}(S) & \longrightarrow & H^{1,1}(S, \mathbb{Z}/m) \\
\end{array}
\]

with exact rows and where the vertical maps are restriction to \( \ast \) which split the maps on cohomology induced by the structure map \( W_{S,n} \to S \). The exact sequence (25) around \( H^{1,1}(W_{S,n}) \) shows that the first vertical map is an isomorphism. A diagram chase then shows existence and uniqueness of \( \tilde{u} \) with the required properties. \( \square \)
We denote the canonical class in $\tilde{H}^{1,1}(W_{S,n},\mathbb{Z}/m)$ obtained this way by $u_{n,m}$ (by demanding that these classes are compatible with base change). We set $u_n := u_{n,n}$.

We let $K(\mathbb{Z}/n(1),1)_S, K(\mathbb{Z}(1),2)_S \in \mathcal{H}_\bullet(S)$ be the motivic Eilenberg-MacLane spaces which represent the functors $\tilde{H}^{1,1}(\_ ,\mathbb{Z}/n)$ and $\tilde{H}^{2,1}(\_)$ on $\mathcal{H}_\bullet(S)$ respectively.

**Proposition 11.7:** If $S$ is smooth over a Dedekind domain of mixed characteristic or over a field then we have $K(\mathbb{Z}/n(1),1)_S \cong \nu \text{Pic} S \cong B\mathbb{G}_m,S \cong \mathbb{P}^\infty_S$ in $\mathcal{H}_\bullet(S)$.

**Proof.** This follows from the fact the motivic sheaf of weight 1 is in this case $O^*_S$. □

**Proposition 11.8:** If $S$ is smooth over a Dedekind domain of mixed characteristic or over a field then we have $K(\mathbb{Z}/n(1),1)_S \cong W_{S,n}$ in $\mathcal{H}_\bullet(S)$. The isomorphism is given by the class $u_n$.

**Proof.** Let $\mathbb{P}^\infty_S \to \nu \text{Pic}_S$ be the map classifying the line bundle $O_{\mathbb{P}^\infty_S}(-n)$. $W_{S,n}$ is the corresponding $\mathbb{G}_m$-torsor over $\mathbb{P}^\infty_S$. Then the diagram

\[
\begin{array}{ccc}
W_{S,n} & \longrightarrow & \mathbb{P}^\infty_S \\
\biggl\downarrow & & \biggl\downarrow f \\
K(\mathbb{Z}/n(1),1)_S & \longrightarrow & K(\mathbb{Z}(1),2)_S \\
& & n \\
\end{array}
\]

in $\mathcal{H}_\bullet(S)$, where the vertical maps are the canonical identifications, commutes. Moreover the rows are fiber sequences: the first one by Proposition 11.5, the second one by definition. It follows that there is a vertical isomorphism $u':W_{S,n} \to K(\mathbb{Z}/n(1),1)_S$ in $\mathcal{H}_\bullet(S)$ making the whole diagram commutative. The uniqueness clause of Lemma 11.6 shows that $u' = u_n$ finishing the proof. □

For any $X \in \text{Sm}_S$ consider the functor $T_n:\text{Pic}(X) \to \text{Pic}(X)$, $\mathcal{L} \mapsto \mathcal{L} \otimes^n$. Its homotopy fiber is the Picard groupoid $\mathcal{G}_n(X)$ whose objects are pairs $(\mathcal{L},\varphi)$, where $\mathcal{L}$ is a line bundle on $X$ and $\varphi: \mathcal{L} \otimes^n \to O_X$ is an isomorphism, and whose morphisms are isomorphisms of line bundles compatible with the trivializations. Note that we have a fiber sequences

\[
\nu G_n(X) \to \nu \text{Pic}(X) \xrightarrow{\nu T_n} \nu \text{Pic}(X)
\]

functorial in $X$ and that these fiber sequences also make sense for $X \in \text{Set}^{\text{Sm}_S^{op}}$. 82
As in the proof of Proposition 11.8 it follows that we have a canonical equivalence
\[ K(\mathbb{Z}/n(1), 1)_S \cong \nu G_n \] in \( H_* (S) \), provided that \( S \) is smooth over a Dedekind ring of
mixed characteristic or over a field.

Since \( \nu G_n \) is Nisnevich- and \( \mathbb{A}^1 \)-local it follows

**Proposition 11.9:** Suppose \( S \) is smooth over a Dedekind ring of mixed characteristic
or over a field and let \( X \) be in \( \text{Sm}_S \) or \( \text{Set}^{\text{Sm}_S\text{op}} \). There is a canonical group isomorphism
between \( H^{1,1}(X, \mathbb{Z}/n) \) and the group of isomorphism classes of \( G_n(X) \). The boundary
map \( H^{1,1}(X, \mathbb{Z}/n) \to H^{2,1}(X) \) corresponds to the map on groupoids which forgets the
trivialization.

**Lemma 11.10:** Suppose \( S \) is smooth over a Dedekind ring of mixed characteristic
or over a field. The class \( u_{n,m} \) corresponds under the isomorphism of Proposition 11.9 to
the isomorphism class of the object
\[ (p^* \mathcal{O}_{\mathbb{P}^\infty_S} (-\frac{n}{\gcd(m,n)}), (p^* \mathcal{O}_{\mathbb{P}^\infty_S} (-\frac{n}{\gcd(m,n)}))^m \cong p^* \mathcal{O}_{\mathbb{P}^\infty_S} (-\text{lcm}(m,n)) \cong \mathcal{O}_{W_{S,n}}), \]
where the last isomorphism is the \( \frac{m}{\gcd(m,n)} \)-th tensor power of the canonical isomorphism
\( p^* \mathcal{O}_{\mathbb{P}^\infty_S} (-n) \cong \mathcal{O}_{W_{S,n}}. \)

**Proof.** This element clearly satisfies the requirements of Lemma 11.6. \( \square \)

**Lemma 11.11:** The image of the constant function on 1 under the isomorphisms
\[ (\mathbb{Z}/m)^\pi_0(S) \cong H^{0,0}(S, \mathbb{Z}/m) \cong \tilde{H}^{1,1}(G_{m,S}, \mathbb{Z}/m) \]
corresponds under the isomorphism of Proposition 11.9 to the object \((\mathcal{O}_{G_{m,S}}, \varphi)\), where
\( \varphi \) is given by multiplication with the canonical unit in \( \mathcal{O}(G_{m,S}). \)

**Proof.** This unit corresponds to 1 under the map
\[ \mathcal{O}^* (G_{m,S}) \cong H^{1,1}(G_{m,S}) \to \tilde{H}^{1,1}(G_{m,S}) \cong H^{0,0}(S) \cong \mathbb{Z}^{\pi_0(S)}. \]

**Corollary 11.12:** Suppose \( S \) is smooth over a Dedekind ring of mixed characteristic
or over a field. Then the image of \( u_{n,m}|_{W_{S,n,0}} \) under the isomorphisms
\[ \tilde{H}^{1,1}(W_{S,n,0}, \mathbb{Z}/m) \cong \tilde{H}^{1,1}(G_{m,S}, \mathbb{Z}/m) \cong H^{0,0}(S, \mathbb{Z}/m) \cong (\mathbb{Z}/m)^{\pi_0(S)} \]
is the constant function on the class of \( \frac{m}{\gcd(m,n)}. \)
Proof. This follows from Lemmas 11.10 and 11.11.

Definition 11.13: Let $E$ be a motivic ring spectrum (i.e. a commutative monoid in $\text{SH}(S)$) such that $E_{0,0}$ is a $\mathbb{Z}/n$-algebra. A mod-$n$ orientation on $E$ consists of an orientation $c \in E^{2,1}(\mathbb{P}^\infty_S)$ and a class $u \in \widehat{E}^{1,1}(W_{S,n})$ which restricts to 1 under the map $$\widehat{E}^{1,1}(W_{S,n}) \to \widehat{E}^{1,1}(W_{S,n,0}) \cong \widehat{E}^{1,1}(\mathbb{G}_m,S) \cong E_{0,0}.$$ It follows from Corollary 11.12 that the usual orientation of $\mathbb{Z}/m$ together with the class $u_{n,m}$ defines a mod-$n$ orientation on $\mathbb{Z}/m$ provided $m|n$. We call this orientation the canonical mod-$n$ orientation of $\mathbb{Z}/m$.

Note also that any mod-$n$ orientation gives rise to a mod-$n'$ orientation for $n|n'$.

Theorem 11.14: Let $E$ be a motivic ring spectrum such that $E_{0,0}$ is a $\mathbb{Z}/n$-algebra with a mod-$n$ orientation given by classes $c \in E^{2,1}(\mathbb{P}^\infty_S)$ and $u \in E^{1,1}(W_{S,n})$. Let $v \in E^{2,1}(W_{S,n})$ be the pullback of $c$ under the canonical projection $W_{S,n} \to \mathbb{P}^\infty_S$. Let $X$ be a motivic space. Denote by $u$ and $v$ also the pullbacks of $u$ and $v$ to the $E$-cohomology of $X \times W_{S,n}$. Then the elements $v^i$, $wv^i$, $i \geq 0$ form a topological basis of $E^{*,*}(X \times W_{S,n})$ over $E^{*,*}(X)$. More precisely, the elements $v^i$, $wv^i$, $0 \leq i \leq k$, form a basis of $E^{*,*}(X \times W_{S,n,k})$ over $E^{*,*}(X)$, $v^{k+1}$ is zero in $E^{*,*}(X \times W_{S,n,k})$ and the canonical map $$E^{*,*}(X \times W_{S,n}) \to \lim_k E^{*,*}(X \times W_{S,n,k}),$$ where the transition maps are surjective, is an isomorphism.

Proof. By writing $X$ has the homotopy colimit over $\Delta^{op}$ of a diagram with entries disjoint unions of objects from $\text{Sm}_S$ and replacing cohomology groups by mapping spaces we reduce to the case where $X = X \in \text{Sm}_S$. The induced long exact sequences in $E$-cohomology from the cofiber sequence $$W_{X,n,k} \to \mathcal{O}_{\mathcal{P}^k_X}(-n) \to \text{Th}(\mathcal{O}_{\mathcal{P}^k_X}(-n))$$ split into short exact sequences $$0 \to E^{*,*}(X)[\sigma]/(\sigma^{k+1}) \to E^{*,*}(X \times W_{S,n,k}) \to E^{*,*}(X)[\sigma]/(\sigma^{k+1}) \to 0$$ since $E_{0,0}$ is a $\mathbb{Z}/n$-algebra. The image of $u$ in the right group is of the form $1 + \sigma \cdot r$. Using the fact that these sequences are $E^{*,*}(X)[\sigma]/(\sigma^{k+1})$-module sequences the claim follows.

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It follows that every element in \( E^{*,*}(\mathfrak{X} \times W_{S,n}) \) can be uniquely written as a power series

\[
\sum_{i \geq 0} (a_i u^i + b_i v^i)
\]

with \( a_i, b_i \in E^{*,*}(\mathfrak{X}) \). Similar statements are valid for elements in \( E^{*,*}(\mathfrak{X} \times W_{S,n}^j) \). The latter group can be written as the \( j \)-fold completed tensor product over \( E^{*,*}(\mathfrak{X}) \) of copies of \( E^{*,*}(\mathfrak{X} \times W_{S,n}) \).

Note that if \( n \) is odd we have

\[
E^{*,*}(\mathfrak{X} \times W_{S,n}^j) \cong E^{*,*}(\mathfrak{X})[[v_1, \ldots, v_j]](u_1, \ldots, u_j),
\]

but if \( n \) is even there can be more complicated relations for the \( u_i^2 \).

The object \( W_{S,n} \in \mathcal{H}(S) \) is naturally a commutative group object (it represents motivic cohomology over certain \( S \), in particular \( S = \text{Spec}(\mathbb{Z}) \), and pulls back). Moreover it has exponent \( n \). This gives \( E^{*,*}(\mathfrak{X} \times W_{S,n}) \) the structure of a cocommutative Hopf algebra object in a category whose tensor structure is the completed tensor product.

The comultiplication

\[
E^{*,*}(\mathfrak{X} \times W_{S,n}^j) \to E^{*,*}(\mathfrak{X} \times W_{S,n}^2)
\]

is uniquely determined by the images of \( u \) and \( v \) which can be written as power series in \( u_1, u_2, v_1, v_2 \). These power series obey laws which are similar to the familiar formal group laws. We won’t spell out these properties, suffices it to say that they are grouped into unitality, associativity, commutativity, exponent \( n \) and independence of the image of \( v \) of \( u_1, u_2 \).

For \( E = M\mathbb{Z}/m, m | n \), we have the additive law: \( u \mapsto u_1 + u_2, v \mapsto v_1 + v_2 \). This follows from weight reasons for \( S = \text{Spec}(\mathbb{Z}) \) and thus is true in general.

There is the notion of a strict isomorphism of such laws (power series), again given by two power series (in \( u \) and \( v \)) in the target complete ring which start with \( u \) respectively \( v \). Moreover the second power series is independent of \( u \). Caution is required in the case \( n \) is even since then the complete rings in question might not have standard form.

Two mod-\( n \) orientations on a motivic ring spectrum give rise to such a strict isomorphism.

**Proposition 11.15:** Let \( E \) be a motivic ring spectrum such that \( E_{0,0} \) is an \( \mathbb{F}_1 \)-algebra equipped with two additive mod-\( l \) orientations. Then the corresponding strict isomorphism has the form

\[
u \mapsto u + a_0 v + a_1 v^j + \cdots + a_i v^i + \cdots,
\]

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\[ v \mapsto v + b_1 v^i + \cdots + b_i v^i + \cdots. \]

**Proof.** The proof is similar to the case of the additive formal group law over an \( F_t \)-algebra.

**Lemma 11.16:** The suspension spectrum \( \Sigma^\infty_+ W_{S,n,k} \) is finite cellular, in particular dualizable. The suspension spectrum \( \Sigma^\infty_+ W_{S,n} \) is cellular.

**Proof.** This is a standard argument.

In the following let \( T_k := \Sigma^\infty_+ W_{S,n,k} \).

**Lemma 11.17:** Let \( E \) be a mod-\( n \) oriented motivic ring spectrum. For any \( 0 \leq i \leq 1 \), \( 0 \leq j \leq k \) let \( \Sigma^{-2j-i,-j-i} E \to E \wedge T_k^\vee \) be the \( E \)-module map corresponding to the element \( v^i v^j \) in the homotopy of the target spectrum. Then the induced map

\[ \bigoplus_{i,j} \Sigma^{-2j-i,-j-i} E \to E \wedge T_k^\vee \]

is an isomorphism.

**Proof.** Homing out of \( \Sigma^{p,q} \Sigma^\infty_+ X, X \in \text{Sm}_S \), shows the claim.

**Lemma 11.18:** Let \( E \) be a motivic ring spectrum and \( U \) be a dualizable spectrum such that \( E \wedge U \) is a finite sum of shifts of \( E \) as an \( E \)-module. Then \( E \wedge U^\vee \) is the sum over the corresponding negative shifts of \( E \).

**Proof.** For \( F \) a motivic spectrum we have

\[
\text{Hom}(F, E \wedge U^\vee) = \text{Hom}(F \wedge U, E) = \text{Hom}_E(E \wedge U \wedge F, E) = \text{Hom}_E(\bigoplus_{\alpha} \Sigma^{p_{\alpha}, q_{\alpha}} E \wedge F, E) = \text{Hom}(F, \bigoplus_{\alpha} \Sigma^{-p_{\alpha}, -q_{\alpha}} E).
\]

**Lemma 11.19:** Let \( E \) be a mod-\( n \) oriented motivic ring spectrum. We have \( E \)-module isomorphisms

\[ \bigoplus_{i, 0 \leq j \leq k} \Sigma^{2j+i, j+i} E \cong E \wedge T_k \]

and

\[ \bigoplus_{i, 0 \leq j} \Sigma^{2j+i, j+i} E \cong E \wedge T_k, \]

where the corresponding generators are the duals of the \( v^i, uv^j \).

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Proof. Use Lemmas \ref{lem:11.17} and \ref{lem:11.18} (the latter is applied with $U = T_k^\vee$).

Let $E, F$ be mod-$n$ oriented motivic ring spectra.

**Lemma 11.20:** The natural map

$$E_{**}F \otimes_{F_{**}} F_{**}T_k^\vee \to (E \wedge F \wedge T_k^\vee)_{**}$$

is an isomorphism.

**Proof.** This follows from Lemma \ref{lem:11.17}.

From the above Lemma we derive a coaction map

$$E_{**}(W_{S,n,k}) \cong E_{-s,-s}T_k^\vee \to (E \wedge F \wedge T_k^\vee)_{-s,-s} \cong E_{-s,-s}F \otimes_{F_{-s,-s}} F_{**}(W_{S,n,k}),$$

where for the second map we use the unit of $F$. These are compatible for different values of $k$, yielding in the limit a coaction map

$$E_{**}(W_{S,n}) \to E_{-s,-s}F \otimes_{F_{-s,-s}} F_{**}(W_{S,n}).$$

We write the image of $u$ as

$$\sum_{j \geq 0} (\alpha_j \otimes v^j + \beta_j \otimes uv^j),$$

similarly we write the image of $v$ as

$$\sum_{j \geq 0} \gamma_i \otimes v^j.$$

(The latter sum is independent of $u$ since the relation comes already from the projective space. Note also that the $u$’s and $v$’s on both sides are lying in different groups.)

**Proposition 11.21:** The strict isomorphism relating the two mod-$n$ orientations on $E \wedge F$ has the form

$$u_E = \sum_{j \geq 0} (\alpha_j v_F^j + \beta_j u_F v_F^j),$$

$$v_E = \sum_{j \geq 0} \gamma_i v_F^j.$$

Here the $u_E, v_E$ are those generators coming from the orientation on $E$, similarly for $u_F, v_F$. 

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Proof. We leave the verification to the reader. □

Remark 11.22: The coefficients $\alpha_i, \beta_i, \gamma_i$ can also be described by images of canonical homology generators with respect to the maps on $F$-homology of the orientation maps

$$\Sigma^{-2,-1}\Sigma^\infty_{+}F_S \rightarrow E$$

and

$$\Sigma^{-1,-1}\Sigma^\infty W_{S,n} \rightarrow E.$$  

We specialize now to the case $E = F = \mathbb{MF}_l$.

Corollary 11.23: The coaction map

$$H^{**}(W_{S,l}, F_l) \rightarrow (\mathbb{MF}_l \wedge \mathbb{MF}_l)_{-*,-*} \hat{\otimes} (\mathbb{MF}_l)_{-*,*} H^{**}(W_{S,l}, F_l)$$

is given by

$$u \mapsto u + \sum_{i \geq 0} \tau_i \otimes v_i,$$

$$v \mapsto v + \sum_{i \geq 1} \xi_i \otimes v_i$$

with $\tau_i \in (\mathbb{MF}_l \wedge \mathbb{MF}_l)_{2i-1,2i-1}$ and $\xi_i \in (\mathbb{MF}_l \wedge \mathbb{MF}_l)_{2i-1,2i-1}$. 

Proof. This follows from Propositions 11.21 and 11.15. □

Set $A^{**} := (\mathbb{MF}_l \wedge \mathbb{MF}_l)_{**}$. We denote by $B$ the set of sequences $(\epsilon_0, r_1, \epsilon_1, r_2, \ldots)$ with $\epsilon_i \in \{0, 1\}$ and $r_i \geq 0$ with only finitely many non-zero terms. For any $I \in B$ let

$$\omega(I) := \tau_{\epsilon_0} \psi_{r_1} \psi_{\epsilon_1} \psi_{r_2} \ldots \in A^{p(I),q(I)}.$$ 

Theorem 11.24: Suppose $l$ is invertible on $S$. Then the map

$$\bigoplus_{I \in B} \Sigma^{p(I),q(I)}_{+} \mathbb{MF}_l \rightarrow \mathbb{MF}_l \wedge \mathbb{MF}_l,$$

where the map on the summand indexed by $I$ is the $\mathbb{MF}_l$-module map (where we use the right module structure on the target) corresponding to the element $\omega(I)$, is an isomorphism.

Proof. It is sufficient to show the statement for $S = \text{Spec}(\mathbb{Z}[[t]])$. This follows from [21, Theorem 1.1] using Theorem 9.16 and Lemma 9.18. □

Remark 11.25: In the situation of the theorem the pair $(H^{-*,*}(S,F_l), A^{**})$ has the structure of a Hopf algebroid. The operations of $\mathbb{MF}_l$, i.e. $\text{Hom}^{**}(\mathbb{MF}_l, \mathbb{MF}_l)$, are the dual of $A^{**}$. 88
A  (Semi) model structures

**Proposition A.1:** Let $\mathcal{C}$ be a symmetric monoidal cofibrantly generated model category and $I$ an (essentially) small category with 2-fold coproducts. Then the projective model structure on $\mathcal{C}^I$ is symmetric monoidal. If $I$ has an initial object and the tensor unit in $\mathcal{C}$ is cofibrant then the tensor unit in $\mathcal{C}^I$ is also cofibrant.

**Proof.** The assertions follow from the formula

$$(\text{Hom}(i, \_): f) \square (\text{Hom}(j, \_): g) \cong \text{Hom}(i \sqcup j, \_): (f \square g)$$

for maps $f$ and $g$ in $\mathcal{C}$ and objectwise considerations. \qed

**Proposition A.2:** Let $\mathcal{C}$ be a left proper combinatorial model category and $\mathcal{S}$ be an (essentially) small site. Then the projective model structure on $\mathcal{C}^{\mathcal{S}^{\text{op}}}$ can be localized to a local projective model structure where the local objects are presheaves satisfying descent for all hypercovers of $\mathcal{S}$.

**Proof.** We localize at the set of maps

$$\text{hocolim}_{n \in \Delta^{\text{op}}}(U_n \times QA) \to QA,$$

where $U \to X$ runs through a set of dense hypercovers (see [7]) of $\mathcal{S}$ and $A$ through the set of domains and codomains of a set of generating cofibrations of $\mathcal{C}$ ($QA$ denotes a cofibrant replacement of $A$). \qed

**Proposition A.3:** Let $R$ be a commutative ring and $\mathcal{C} = \text{Cpx}_{(\geq 0)}(R)$ be the category of (non-negative) chain complexes of $R$-modules equipped with its standard projective model structure. Let $\mathcal{S}$ be an (essentially) small site with 2-fold products and enough points. Then the local projective model structure on $\mathcal{C}^{\mathcal{S}^{\text{op}}}$ is symmetric monoidal.

**Proof.** The projective model structure is symmetric monoidal by Proposition A.1. It remains to see that the pushout product of a generating cofibration with a trivial cofibration is a weak equivalence. Checking this on stalks does the job (here use the injective model structure on $\mathcal{C}$). \qed

**Remark A.4:** This result is also contained in [7].

**Theorem A.5:** Let $R$ be a commutative ring and $\mathcal{S}$ an (essentially) small site with 2-fold products and enough points. Then $\text{Cpx}_{(\geq 0)}(\text{Sh}(\mathcal{S}, R))$ carries a local projective symmetric monoidal cofibrantly generated model structure transferred from the local model structure on presheaves. The weak equivalences are the quasi isomorphisms.
Proof. One applies the transfer principle (see e.g. [3, §2.5]): One has to check that transfinite compositions of pushouts by images of generating trivial cofibrations are weak equivalences. This follows since the sheafification functor preserves all weak equivalences. The same applies to prove that the model structure is symmetric monoidal.

Let $R$ and $S$ be as in the Theorem above. Then the canonical generating cofibrations of $\text{Cpx}_{(\geq 0)}(\text{Sh}(S, R))$ have cofibrant domain. Thus for a cofibrant $T \in \text{Cpx}_{(\geq 0)}(\text{Sh}(S, R))$ by [19, Theorem 8.11] there is a stable symmetric monoidal model structure on the category $\text{Sp}_T^\Sigma$ of symmetric $T$-spectra in $\text{Cpx}_{(\geq 0)}(\text{Sh}(S, R))$.

It follows from [39, Theorem 4.7] that for a $\Sigma$-cofibrant operad $O$ in $\text{Sp}_T^\Sigma$ the category of $O$-algebras inherits a semi model structure. In particular for the image of the linear isometries operad in $\text{Sp}_T^\Sigma$ we obtain a semi model category $E_\infty(\text{Sp}_T^\Sigma)$ of $E_\infty$-spectra.

B Pullback of cycles

For a regular separated Noetherian scheme $X$ of finite Krull dimension we let $X^{(p)}$ be the set of codimension $p$ points on $X$ and $Z^p(X)$ the free abelian group on $X^{(p)}$. Let $C \in X^{(p)}$, $D \in X^{(q)}$. We say that $C$ and $D$ intersect properly if the scheme theoretic intersection $Z$ of the closures of $C$ and $D$ in $X$ has codimension everywhere $\geq p + q$. If $C$ and $D$ intersect properly then for a point $W$ of $Z$ of codimension $p + q$ in $X$ we set

$$m(W; C, D) := \sum_{i \geq 0} (-1)^i \text{length}_{O_{X, W}}(\text{Tor}_i^{O_{X, W}}(O_{C, W}, O_{D, W})), $$

known as Serre’s intersection multiplicity.

We extend the notion of proper intersection and the intersection multiplicity at an arbitrary $W \in X^{(p+q)}$ in the canonical way to elements of $Z^p(X)$ and $Z^q(X)$.

For $C \in Z^p(X)$ and $D \in Z^q(X)$ which intersect properly we let

$$C \cdot D := \sum_{W \in X^{(p+q)}} m(W; C, D) \cdot W. $$

For a coherent sheaf $F$ on $X$ whose support has everywhere codimension $\geq p$ we let $Z_p(F) \in Z^p(X)$ be given by

$$Z_p(F) := \sum_{W \in X^{(p)}} \text{length}_{O_{X, W}}(F_W) \cdot W. $$

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Proposition B.1: Let \( \mathcal{F} \) and \( \mathcal{G} \) be coherent sheaves on \( X \). Suppose that the supports of \( \mathcal{F} \), \( \mathcal{G} \) and \( \mathcal{F} \otimes O_X \mathcal{G} \) have everywhere at least codimension \( p \), \( q \) and \( p+q \) respectively. Then

\[
Z_p(\mathcal{F}) \cdot Z_q(\mathcal{G}) = \sum_{i \geq 0} (-1)^i Z_{p+q}(\text{Tor}_{i}^{O_X}(\mathcal{F}, \mathcal{G})).
\]

Of course this Proposition is a special case of a statement valid for perfect complexes on \( X \).

Proof. Since the question is local on \( X \) we can assume \( X \) is local and the support of \( \mathcal{F} \otimes O_X \mathcal{G} \) is the closed point of \( X \). Then the proof proceeds as the proof of [36, V.C. Proposition 1], using [34, Theorem 1] or [15] and a filtration argument (using e.g. [10, Proposition 3.7]). \( \square \)

Proposition B.2: Let \( C \in Z^p(X) \), \( D \in Z^q(X) \) and \( E \in Z^r(X) \) such that \( C \cdot D \), \( (C \cdot D) \cdot E \) and \( D \cdot E \) are well defined. Then we have

\[
(C \cdot D) \cdot E = C \cdot (D \cdot E)
\]
in \( Z^{p+q+r}(X) \).

Proof. The proof proceeds as the proof of [36, V.C.3.b) Associativity], using a spectral sequence argument and Proposition B.1. \( \square \)

For a flat map \( X \to Y \) between regular separated Noetherian schemes of finite Krull dimension there is a flat pullback \( f^*: Z^p(Y) \to Z^p(X) \).

Let now \( S \) be a regular separated Noetherian scheme of finite Krull dimension. Let \( f: X \to Y \) be a morphism in \( \text{Sm}_S \) and \( C \in Z^p(Y) \). We say that \( f \) and \( C \) are in good position if for every \( W \in X^{(p)} \) with a non-zero coefficient in \( C \) the scheme theoretic inverse image \( f^{-1}(\overline{W}) \) has everywhere codimension \( \geq p \). If this is the case we define \( f^*(C) \in Z^p(X) \) by

\[
f^*(C) := \Gamma_f \cdot \text{pr}_Y^*(C).
\]

(We view this intersection, which takes place on \( X \times_S Y \), in a canonical way as an element of \( Z^p(X) \). Note also that the graph is not in general of a well defined codimension, but for the definition we can e.g. assume \( X \) and \( Y \) to be connected.)

Theorem B.3: Let \( f: X \to Y \xrightarrow{g} Z \) be maps in \( \text{Sm}_S \). Let \( C \in Z^p(Z) \) and assume \( g \) and \( C \) are in good position and \( f \) and \( g^*(C) \) are in good position. Then \( g \circ f \) and \( C \) are in good position and

\[
(g \circ f)^*(C) = f^*(g^*(C)).
\]
in $Z^p(X)$.

Proof. Let $U := \Gamma_f \times_S Z \subset X \times_S Y \times_S Z$ and $V := X \times_S \Gamma_g \subset X \times_S Y \times_S Z$. Let $pr_Z : X \times_S Y \times_S Z \to Z$ be the projection. The assertion follows from the associativity

$$(U \cdot V) \cdot pr^*_Z(C) = U \cdot (V \cdot pr^*_Z(C))$$

which holds by Proposition B.2. \qed

References

[1] Joseph Ayoub. La réalisation étale et les opérations de Grothendieck. http://user.math.uzh.ch/ayoub/PDF-Files/Realisation-Etale.pdf, to appear in Ann. Sci. Ecole Norm. Sup.

[2] Joseph Ayoub. Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique. I. Astérisque, (314):x+466 pp. (2008), 2007.

[3] Clemens Berger and Ieke Moerdijk. Axiomatic homotopy theory for operads. Comment. Math. Helv., 78(4):805–831, 2003.

[4] Spencer Bloch and Kazuya Kato. $p$-adic étale cohomology. Inst. Hautes Études Sci. Publ. Math., (63):107–152, 1986.

[5] Denis-Charles Cisinski and Frédéric Déglise. Étale motives. arXiv:1305.5361.

[6] Denis-Charles Cisinski and Frédéric Déglise. Triangulated categories of mixed motives. arXiv:0912.2110.

[7] Daniel Dugger, Sharon Hollander, and Daniel C. Isaksen. Hypercovers and simplicial presheaves. Math. Proc. Cambridge Philos. Soc., 136(1):9–51, 2004.

[8] W. G. Dwyer and D. M. Kan. Calculating simplicial localizations. J. Pure Appl. Algebra, 18(1):17–35, 1980.

[9] William G. Dwyer, Philip S. Hirschhorn, Daniel M. Kan, and Jeffrey H. Smith. Homotopy limit functors on model categories and homotopical categories, volume 113 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2004.
[10] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.

[11] Halvard Fausk. T-model structures on chain complexes of presheaves. arXiv:math/0612414.

[12] Eric M. Friedlander and Andrei Suslin. The spectral sequence relating algebraic K-theory to motivic cohomology. *K*-theory server 432.

[13] Thomas Geisser. Motivic cohomology over Dedekind rings. *Math. Z.*, 248(4):773–794, 2004.

[14] Thomas Geisser and Marc Levine. The *K*-theory of fields in characteristic *p*. *Invent. Math.*, 139(3):459–493, 2000.

[15] Henri Gillet and Christophe Soulé. *K*-théorie et nullité des multiplicités d’intersection. *C. R. Acad. Sci. Paris Sér. I Math.*, 300(3):71–74, 1985.

[16] Michel Gros and Noriyuki Suwa. La conjecture de Gersten pour les faisceaux de Hodge-Witt logarithmique. *Duke Math. J.*, 57(2):615–628, 1988.

[17] Jens Hornbostel. Preorientations of the derived motivic multiplicative group. *Algebr. Geom. Topol.*, 13(5):2667–2712, 2013.

[18] Mark Hovey. *Model categories*, volume 63 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[19] Mark Hovey. Spectra and symmetric spectra in general model categories. *J. Pure Appl. Algebra*, 165(1):63–127, 2001.

[20] Marc Hoyois. From algebraic cobordism to motivic cohomology. arXiv:1210.7182.

[21] Marc Hoyois, Shane Kelly, and Paul Arne Østvær. The motivic Steenrod algebra in positive characteristic. arXiv:1305.5690.

[22] Luc Illusie. Complexe de de Rham-Witt et cohomologie cristalline. *Ann. Sci. École Norm. Sup. (4)*, 12(4):501–661, 1979.

[23] Satoshi Kondo and Seidai Yasuda. Letter to the author.

[24] Satoshi Kondo and Seidai Yasuda. Product structures in motivic cohomology and higher Chow groups. *J. Pure Appl. Algebra*, 215(4):511–522, 2011.
[25] Masato Kurihara. A note on $p$-adic étale cohomology. *Proc. Japan Acad. Ser. A Math. Sci.*, 63(7):275–278, 1987.

[26] Marc Levine. K-theory and motivic cohomology of schemes. K-theory archive 336.

[27] Marc Levine. Techniques of localization in the theory of algebraic cycles. *J. Algebraic Geom.*, 10(2):299–363, 2001.

[28] Marc Levine. The homotopy coniveau tower. *J. Topol.*, 1(1):217–267, 2008.

[29] Jacob Lurie. Higher Algebra. available at http://www.math.harvard.edu/~lurie/.

[30] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[31] Carlo Mazza, Vladimir Voevodsky, and Charles Weibel. *Lecture notes on motivic cohomology*, volume 2 of *Clay Mathematics Monographs*. American Mathematical Society, Providence, RI, 2006.

[32] Joël Riou. Opérations sur la K-théorie algébrique et régulateurs via la théorie homotopique des schémas. available at http://www.math.u-psud.fr/~riou/.

[33] Joël Riou. Pureté (d’après Ofer Gabber). available at http://www.math.u-psud.fr/~riou/.

[34] Paul Roberts. The vanishing of intersection multiplicities of perfect complexes. *Bull. Amer. Math. Soc. (N.S.)*, 13(2):127–130, 1985.

[35] Oliver Röndigs and Paul Arne Østvær. Modules over motivic cohomology. *Adv. Math.*, 219(2):689–727, 2008.

[36] Jean-Pierre Serre. *Local algebra*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000. Translated from the French by CheeWhye Chin and revised by the author.

[37] Jean-Pierre Serre. *Galois cohomology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2002. Translated from the French by Patrick Ion and revised by the author.

[38] Markus Spitzweck. Derived fundamental groups for Tate motives. arXiv:1005.2670.
[39] Markus Spitzweck. Operads, Algebras and Modules in Model Categories and Motives. http://www.uni-math.gwdg.de/spitz/diss.pdf.

[40] Markus Spitzweck. Periodizable motivic ring spectra. arXiv:0907.1510.

[41] Vladimir Voevodsky. Triangulated categories of motives over a field. In Cycles, transfers, and motivic homology theories, volume 143 of Ann. of Math. Stud., pages 188–238. Princeton Univ. Press, Princeton, NJ, 2000.

[42] Vladimir Voevodsky. Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic. Int. Math. Res. Not., (7):351–355, 2002.

[43] Vladimir Voevodsky. Reduced power operations in motivic cohomology. Publ. Math. Inst. Hautes Études Sci., (98):1–57, 2003.

[44] Vladimir Voevodsky. On motivic cohomology with $\mathbb{Z}/l$-coefficients. Ann. of Math. (2), 174(1):401–438, 2011.

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