ON FINITE NON-DEGENERATE BRAIDED TENSOR CATEGORIES WITH A LAGRANGIAN SUBCATEGORY

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Abstract. Let $W$ be a finite dimensional purely odd supervector space over $\mathbb{C}$, and let $\text{sRep}(W)$ be the finite symmetric tensor category of finite dimensional superrepresentations of the finite supergroup $W$. We show that the set of equivalence classes of finite non-degenerate braided tensor categories $\mathcal{C}$ containing $\text{sRep}(W)$ as a Lagrangian subcategory is a torsor over the cyclic group $\mathbb{Z}/16\mathbb{Z}$. In particular, we obtain that there are 8 non-equivalent such braided tensor categories $\mathcal{C}$ which are integral and 8 which are non-integral.

1. Introduction

There are two types of finite braided tensor categories, which in a sense are opposite to each other. On one extreme we have the symmetric tensor categories, i.e., braided tensor categories which coincide with their Müger centralizer, while on the other extreme we have the non-degenerate braided tensor categories, i.e., braided tensor categories with trivial Müger centralizer.

On one hand, finite symmetric tensor categories over $\mathbb{C}$ are completely classified in supergroup-theoretical terms. Namely, by a theorem of Deligne [D], every such category is braided tensor equivalent to the category $\text{sRep}(G \rtimes W, u)$ of finite dimensional superrepresentations of a unique (up to isomorphism) finite supergroup $G \rtimes W$ with a central element $u \in G$ of order $\leq 2$ acting via the parity automorphism (see [EG1, EGNO]).

On the other hand, the classification of non-degenerate braided fusion categories $\mathcal{C}$ which contain a Tannakian subcategory $\mathcal{E} := \text{Rep}(G)$ as a Lagrangian subcategory (i.e., $\mathcal{E}$ coincides with its Müger centralizer inside $\mathcal{C}$) is known too. Namely, such categories $\mathcal{C}$ are precisely the centers $\mathcal{Z}(\text{Vec}^\omega(G))$ of pointed fusion categories $\text{Vec}^\omega(G)$ [DGNO1, DGNO2].

Date: September 16, 2018.
Key words and phrases. non-degenerate braided tensor category; symmetric tensor category; Lagrangian subcategory; finite supergroup.
Furthermore, the classification of non-degenerate braided fusion categories $\mathcal{B}$ which contain sVec as a Lagrangian subcategory is also known \cite{DGNO1, DGNO2}. Namely, it is known that there are exactly 16 such categories $\mathcal{B}$, up to braided tensor equivalence. Moreover, the categories $\mathcal{B}$ form a group $\mathcal{B}$ isomorphic to $\mathbb{Z}/16\mathbb{Z}$ with respect to a certain modified Deligne tensor product $\tilde{\boxtimes}$ [DMNO, DNO] (see Section 4 for more details).

Our purpose in this paper is to take the first step in an attempt to extend the classification of \cite{DGNO1, DGNO2} to finite (non-semisimple) non-degenerate braided tensor categories containing a Lagrangian subcategory. Namely, to classify finite non-degenerate braided tensor categories $\mathcal{C}$ containing $\mathcal{E} := \text{sRep}(W)$ as a Lagrangian subcategory (i.e., the case $G = \langle u \rangle$). Observe that the center $\mathcal{Z}(\mathcal{E})$ of $\mathcal{E}$ is an example of such category $\mathcal{C}$ (see Theorem 2.6).

More precisely, first we prove in Theorem 3.5 that every finite non-degenerate braided tensor category, containing a Lagrangian subcategory $\text{sRep}(W)$, admits a natural $\mathbb{Z}/2\mathbb{Z}$-faithful grading, and has 2 invertible objects and exactly 1 or 2 more simple objects (noninvertible, if $W \neq 0$).

We then show in Theorem 4.4 that the group $\mathcal{B} \cong \mathbb{Z}/16\mathbb{Z}$ acts freely on the set of equivalence classes of finite non-degenerate braided tensor categories $\mathcal{C}$ containing $\text{sRep}(W)$ as a Lagrangian subcategory.

Finally, we use Theorems 3.5, 4.4 to prove the following theorem, which is the main result of this paper.

**Theorem 1.1.** The following hold:

1. The action of the group $\mathcal{B} \cong \mathbb{Z}/16\mathbb{Z}$ on the set of equivalence classes of finite non-degenerate braided tensor categories containing $\text{sRep}(W)$ as a Lagrangian subcategory, is free and transitive.

2. There are 8 equivalence classes of finite non-degenerate braided integral tensor categories containing $\text{sRep}(W)$ as a Lagrangian subcategory, and 8 equivalence classes of finite non-degenerate braided non-integral tensor categories containing $\text{sRep}(W)$ as a Lagrangian subcategory.

In particular, Theorem 1.1 yields precise information on the number and projectivity of the simple objects in a finite non-degenerate braided tensor category which contains a Lagrangian subcategory $\text{sRep}(W)$ (see Corollary 5.5).

The structure of this paper is as follows. Section 2 is devoted to some preliminaries on finite (braided) tensor categories and their exact module categories, Hopf superalgebras, and the finite non-degenerate...
braided tensor category $Z(sRep(W))$. Section 3 is devoted to the proof of Theorem 3.5. Section 4 is devoted to the group $B$, and to the proof that it acts on the set of equivalence classes of finite non-degenerate braided tensor categories which contain the same $sRep(W)$ as a Lagrangian subcategory, freely. Section 5 is devoted to the proof of Theorem 1.1. In Section 6 we classify finite degenerate braided tensor categories containing a Lagrangian subcategory $sRep(W)$. In Section 7 we relate Theorem 1.1 with the works of Davydov and Runkel [DR1, DR2, DR3].

**Remark 1.2.** In a future publication, we plan to extend the results of this paper to finite non-degenerate braided tensor categories which contain a Lagrangian subcategory $E := sRep(G \ltimes W, u)$ of the most general form.

**Acknowledgements.** We are grateful to Alexei Davydov, Pavel Etingof, Dmitri Nikshych, Victor Ostrik and Ingo Runkel for very useful discussions and helpful comments.

This work was done under the supervision of the first author, as part of the second author’s Ph.D dissertation.

Part of this work was done while the first author was visiting the Department of Mathematics at the University of Michigan in Ann Arbor; he is grateful for their warm hospitality.

This work was partially supported by the Israel Science Foundation (grant no. 561/12).

## 2. Preliminaries

Throughout this paper, the ground field will be the field $\mathbb{C}$ of complex numbers, and all categories will be assumed to be $\mathbb{C}$-linear abelian. We refer the reader to the book [EGNO] for a general background on finite tensor categories.

### 2.1. Finite tensor categories

Let $\mathcal{C}$ be a finite tensor category over $\mathbb{C}$, and let $\text{Gr}(\mathcal{C})$ be the Grothendieck ring of $\mathcal{C}$. Recall [EO, Subsection 2.4] that we have a character $\text{FPdim} : \text{Gr}(\mathcal{C}) \to \mathbb{R}$, attaching to $X \in \mathcal{C}$ the Frobenius-Perron dimension of $X$. Following [EO, Subsection 2.4], we set $\text{FPdim}(\mathcal{C}) := \sum_{X \in \text{Irr}(\mathcal{C})} \text{FPdim}(X)\text{FPdim}(P(X))$, where $\text{Irr}(\mathcal{C})$ is the (finite) set of isomorphism classes of simple objects of $\mathcal{C}$, and $P(X)$ is the projective cover of $X$.

We will need the following (straightforward) extension of [GN, Theorem 3.10] to the non-semisimple case.
Lemma 2.1. Let $\mathcal{C}$ be a finite tensor category with simples $X_i$, projective covers $P_i$, and $\text{FPdim}(X_i) = d_i$. Suppose $\text{FPdim}(\mathcal{C})$ is an integer. If $\text{Hom}(P_i, P_j) \neq 0$ (i.e., $X_i$ occurs in $P_j$) then $d_id_j$ is an integer. In particular, $d_i^2$ is an integer, so $d_i$ is the square root of a positive integer.

Proof. Let $\text{FPdim}(\mathcal{C}) = d$. Let $N_{ij}$ be the multiplicity of $X_i$ in $P_j$. Then $\sum_{i,j} N_{ij}d_id_j = d$. Hence for any $g \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, $\sum_{i,j} N_{ij}g(d_id_j) = d$. Also, the numbers $d_i$ are algebraic integers largest in absolute value in their Galois orbits. Hence $|g(d_id_j)| \leq d_id_j$ for all $i, j$. This means that $g(d_id_j) = d_id_j$, (i.e., $d_id_j$ is an integer) whenever $N_{ij} \neq 0$, as desired. □

Corollary 2.2. Let $\mathcal{C}$ be a weakly integral finite tensor category (i.e., $\text{FPdim}(\mathcal{C})$ is an integer). Then there is an elementary abelian 2-group $E$, a set of distinct square free positive integers $n_x$, $x \in E$, with $n_0 = 1$, and a faithful grading $\mathcal{C} = \bigoplus_{x \in E} \mathcal{C}_x$ such that $\text{FPdim}(X) \in \mathbb{Z}\sqrt{n_x}$ for each $X \in \mathcal{C}_x$. In particular, any tensor subcategory of $\mathcal{C}$ is weakly integral.

Proof. By Lemma 2.1 every simple object of $\mathcal{C}$ has dimension $\sqrt{n}$ for some positive integer $n$. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the tensor subcategory generated by all simple objects of integer FP dimension. Then all objects of $\mathcal{C}_0$ have integer FP dimension, since an integer can equal a sum of square roots of integers only if all the summands are integers. A similar argument shows that for each square free positive integer $n$ the simple objects of $\mathcal{C}$ whose dimension is in $\mathbb{Z}\sqrt{n}$ generate a $\mathcal{C}_0$-subbimodule category $\mathcal{C}_n$ of $\mathcal{C}$. Moreover, it follows from Lemma 2.1 that for every $n$, $\mathcal{C}_n \subseteq \mathcal{C}$ is a Serre subcategory (see [EGNO, Definition 4.14.1]).

Let $E := \{n$ is square free $| \exists X \in \mathcal{C}$, such that $\text{FPdim}(X) \in \mathbb{Z}\sqrt{n}\}$. It is clear that for $X \in \mathcal{C}_n$ and $Y \in \mathcal{C}_m$, $X \otimes Y$ is in $\mathcal{C}_{(nm)'}$ where $l'$ denotes the square free part of $l$. This defines a commutative group operation on $E$ and a faithful grading on $\mathcal{C}$. Since the order of every element in $E$ is at most 2, $E$ is an elementary abelian 2-group.

Finally, let $\mathcal{D} \subseteq \mathcal{C}$ be a tensor subcategory. Since $\mathcal{C}_x$ are Serre subcategories, for any simple $X \in \mathcal{C}_x$ the projective cover $P_X(X)$ of $X$ in $\mathcal{D}$ is also in $\mathcal{C}_x$ since $P(X)$ surjects onto $P_X(X)$ by the proof of [EGNO, Proposition 6.3.3], so $\text{FPdim}(X)\text{FPdim}(P_X(X))$ is an integer. Hence the sum of all these numbers over all the simples $X \in \mathcal{D}$, which is $\text{FPdim}(\mathcal{D})$, is also an integer. □

Recall from [EO, Section 3] that a left $\mathcal{C}$-module category $\mathcal{N}$ is said to be indecomposable if it is not a direct sum of two nonzero module
categories, and is called exact if \( P \otimes N \) is projective for any projective \( P \in \mathcal{C} \) and any \( N \in \mathcal{N} \). The same definition applies to right module categories.

2.2. Centralizers and Lagrangian subcategories. Let \( \mathcal{C} \) be a finite braided tensor category over \( \mathbb{C} \) with braiding \( c \), and let \( s \) be the squared braiding, i.e., \( s_{X,Y} := c_{Y,X} \circ c_{X,Y} \) for every \( X, Y \in \mathcal{C} \). Recall that two objects \( X, Y \in \mathcal{C} \) centralize each other if \( s_{X,Y} = \text{id}_{X \otimes Y} \), and that the (Müger) centralizer \( \mathcal{D}' \) of a full tensor subcategory \( \mathcal{D} \subseteq \mathcal{C} \) is the full subcategory of \( \mathcal{C} \) consisting of all objects which centralize every object of \( \mathcal{D} \) (see, e.g., [DGNO1]). Clearly, \( \mathcal{D} \) is symmetric if and only if \( \mathcal{D}' = \mathcal{D} \). If \( \mathcal{D}' = \mathcal{D} \) then \( \mathcal{D} \) is called Lagrangian. A Lagrangian subcategory of \( \mathcal{C} \) is a maximal full symmetric tensor subcategory of \( \mathcal{C} \). The category \( \mathcal{C} \) is called non-degenerate if \( \mathcal{C}' = \text{Vec} \), slightly degenerate if \( \mathcal{C}' = s\text{Vec} \), and degenerate otherwise.

By [Sh, Theorem 4.9], we have

\[
(1) \quad \text{FPdim}(\mathcal{D}) \text{FPdim}(\mathcal{D}') = \text{FPdim}(\mathcal{C}) \text{FPdim}(\mathcal{C}' \cap \mathcal{D})
\]

and

\[
(2) \quad \mathcal{D}'' = \mathcal{D} \lor \mathcal{C}'.
\]

We shall say that two objects \( X, Y \in \mathcal{C} \) anti-centralize each other if \( s_{X,Y} = -\text{id}_{X \otimes Y} \), and that the anti-centralizer of a full tensor subcategory \( \mathcal{D} \subseteq \mathcal{C} \) is the full subcategory of \( \mathcal{C} \) consisting of all objects which anti-centralize every object of \( \mathcal{D} \).

2.3. Exact commutative algebras. Let \( \mathcal{C} \) be a finite braided tensor category over \( \mathbb{C} \), and let \( A \) be an exact commutative algebra object in \( \mathcal{C} \) such that \( \dim \mathcal{C}(\text{Hom}_\mathcal{C}(1, A)) = 1 \) (see, e.g., [EGNO] Section 8.8). Let \( \mathcal{A} := \text{Mod}_\mathcal{C}(A) \) be the category of right \( A \)-modules in \( \mathcal{C} \). Then \( \mathcal{A} \) is an exact indecomposable module category over \( \mathcal{C} \). Moreover, using the braiding on \( \mathcal{C} \) and its inverse one can define on every \( M \in \mathcal{A} \) two structures \( M_+ \), \( M_- \) of a left \( A \)-module:

\[
A \otimes M_+ \xrightarrow{c_{A,M}} M_+ \otimes A \rightarrow M_+ \quad \text{and} \quad A \otimes M_- \xrightarrow{c^{-1}_{M,A}} M_- \otimes A \rightarrow M_-(\text{see [DGNO2], [EGNO], Exercise 8.8.3)}).
\]

Both structures turn \( M \) into an \( A \)-bimodule, so \( \mathcal{A} \) is fully embedded in the finite tensor category \( \text{Bimod}_\mathcal{C}(A) \), and hence inherits from it a structure of a finite tensor category with tensor product \( \otimes_A \) [EGNO Section 8.8].

Recall [EGNO, Proposition 8.8.10] that the free module functor

\[
(3) \quad F : \mathcal{C} \rightarrow \mathcal{A}, \ X \mapsto X \otimes A,
\]
is a surjective (i.e., any \( Y \in \mathcal{A} \) is a subquotient of \( F(X) \) for some \( X \in \mathcal{C} \)) tensor functor.

**Lemma 2.3.** We have

\[
\text{FPdim}(\mathcal{A}) = \frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}_C(A)},
\]

**Proof.** Let \( I : \mathcal{A} \to \mathcal{C} \) be the forgetful functor. Then \( I \) is the right adjoint to \( F \), and \( I(1_A) = I(A) = A \). Thus the claim follows from [EGNO, Lemma 6.2.4]. □

Recall from [EGNO, Proposition 8.8.10] that the free module functor \( \mathcal{B} \) has a central structure, i.e., it lifts to a braided tensor functor

\[\tilde{\mathcal{F}} : \mathcal{C} \to \mathcal{Z}(\mathcal{A})\]

in such a way that \( F \) is the composition of \( \tilde{\mathcal{F}} \) and the forgetful functor \( \mathcal{Z}(\mathcal{A}) \to \mathcal{A} \).

The following results were proved for fusion categories in [DGNO2, Proposition 4.2] and [DMNO, Corollary 3.32]. Thanks to [Sh, Theorem 1.1], which states that \( \mathcal{C} \) is non-degenerate if and only if it is factorizable, and Equation (1), the proof in the non-semisimple case is parallel.

**Theorem 2.4.** Suppose \( \mathcal{C} \) is non-degenerate. Then the following hold:

1. The braided tensor functor \( \tilde{\mathcal{F}} : \mathcal{C} \to \mathcal{Z}(\mathcal{A}) \) in (4) is injective.
2. There is a braided tensor equivalence \( \mathcal{C} \boxtimes \mathcal{C}' \cong \mathcal{Z}(\mathcal{A}) \), where \( \mathcal{C}' \) is the centralizer of \( \mathcal{C} \) in \( \mathcal{Z}(\mathcal{A}) \). In particular, \( \mathcal{C}' \) is non-degenerate.

If the braided tensor functor (4) is an equivalence, the algebra \( A \) is called Lagrangian.

### 2.4. Equivariantization and de-equivariantization

Let \( \mathcal{B} \) be a finite tensor category with an action of a finite group \( G \), and let \( \mathcal{B}^G \) be the \( G \)-equivariantization of \( \mathcal{B} \). Recall that \( \mathcal{B}^G \) is a finite tensor category which contains \( \text{Rep}(G) \) as a full Tannakian subcategory, and \( \text{FPdim}(\mathcal{B}^G) = |G| \text{FPdim}(\mathcal{B}) \). (For more details see, e.g., [EGNO, Section 4.15].)

Let \( \mathcal{C} \) be a finite braided tensor category containing \( \text{Rep}(G) \) as a full Tannakian subcategory. Let \( A := \text{Fun}(G) \) be the algebra of functions on \( G \) (= regular algebra). Then \( A \) is a commutative algebra in \( \text{Rep}(G) \), and hence in \( \mathcal{C} \). Recall that the de-equivariantization \( \mathcal{C}_G \) of \( \mathcal{C} \) is the finite tensor category \( \text{Mod}_C(A) \) of right \( A \)-modules in \( \mathcal{C} \) (see Subsection 2.1), and that \( \text{FPdim}(\mathcal{C}_G) = \text{FPdim}(\mathcal{C})/|G| \). (For more details see, e.g., [EGNO, Section 8.23].)
Let $\mathcal{E} \subseteq \mathcal{A}$ be finite tensor categories. Recall from [GNN] that the relative center $\mathcal{Z}_\mathcal{E}(\mathcal{A})$ is the category of exact $\mathcal{E}$-bimodule functors from $\mathcal{E}$ to $\mathcal{A}$. By [Sh, Lemma 4.5], $\mathcal{Z}_\mathcal{E}(\mathcal{A}) = \mathcal{Z}(\mathcal{E}; \mathcal{A})$ is a finite tensor category. The following result was proved for fusion categories in [GNN, Theorem 3.5]. The proof in the non-semisimple case is parallel.

**Theorem 2.5.** Let $\mathcal{A} = \bigoplus G A_g$ be a finite tensor category, faithfully graded by a finite group $G$, with identity component $A_1 = \mathcal{E}$. Then the following hold:

1. There is a braided tensor equivalence $\mathcal{Z}(\mathcal{A}) \cong (\mathcal{Z}_\mathcal{E}(\mathcal{A}))^G$. In particular $\mathcal{Z}(\mathcal{A})$ contains a Tannakian subcategory $\mathcal{T} := \text{Rep}(G)$.
2. The forgetful functor $\mathcal{Z}(\mathcal{A}) \to \mathcal{A}$ maps $\mathcal{T}$ and $\mathcal{T}'$ to $\text{Vec}$ and $\mathcal{E}$, respectively.
3. There is a braided tensor equivalence $(\mathcal{T}')^G \cong \mathcal{Z}(\mathcal{E})$.

\[ \square \]

2.5. **Superalgebras and supermodules.** Recall that a supervector space is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $V = V_0 \oplus V_1$. The elements in $V_0 \cup V_1$ are called homogeneous, the elements in $V_0$ are called even, and the elements in $V_1$ are called odd. The parity of a homogeneous element $v$ is denoted by $|v|$.

A linear map $f : V \to W$ between two supervector spaces is called even if $f(V_i) \subseteq W_i$ for $i = 0, 1$. The category with finite dimensional supervector spaces as objects and even linear maps between them as morphisms is denoted by sVec.

A superalgebra $A$ is a supervector space $A = A_0 \oplus A_1$, together with an even linear map $m : A \otimes A \to A$ such that $(A, m)$ is an ordinary associative algebra. One defines supercoalgebras, Hopf superalgebras, etc..., similarly.

A Hopf superalgebra $A = A_0 \oplus A_1$ is called supercommutative if for every homogeneous elements $a, b \in A$, $ba = (-1)^{|a||b|}ab$, and is called supercocommutative if for every homogeneous element $a \in A$, $\Delta(a) = \sum a_1 \otimes a_2 = \sum (-1)^{|a_1||a_2|}a_2 \otimes a_1$.

A left supermodule over a superalgebra $A$ is a supervector space $V = V_0 \oplus V_1$ together with an even linear map $\mu : A \otimes V \to V$ such that $(V, \mu)$ is an ordinary left module over the ordinary algebra $A$. We let $\text{sRep}(A)$ denote the category of finite dimensional (left) supermodules over $A$ with even morphisms. Then $\text{sRep}(A)$ is an abelian category. If moreover $A$ is a finite dimensional Hopf superalgebra then $\text{sRep}(A)$ is a finite tensor category.

2.6. **The non-degenerate braided tensor category $\mathcal{Z}(\text{sRep}(W))$.** Let $W$ be a finite dimensional purely odd supervector space. Then the exterior algebra $\wedge W$ is a supercommutative and supercocommutative
Hopf superalgebra such that \( \Delta(w) = 1 \otimes w + w \otimes 1, \varepsilon(w) = 0, \) and \( S(w) = -w \) for every \( w \in W \). It follows that \( \text{sRep}(W) := \text{sRep}(\wedge W) \) is a finite tensor category, which depends only on \( \dim(W) \) (up to tensor equivalence).

Furthermore, the triangular structure \( 1 \otimes 1 \) on \( \wedge W \) induces a symmetric structure on \( \text{sRep}(W) \). Namely, for \( X, Y \in \text{sRep}(W) \), the isomorphism \( X \otimes Y \cong Y \otimes X \) is given on homogeneous elements by \( x \otimes y \mapsto (-1)^{|x||y|} y \otimes x \). In this paper, whenever we refer to \( \text{sRep}(W) \) as a symmetric tensor category, it will be with this symmetric structure (unless otherwise explicitly stated).

The category \( \text{sRep}(W) \) has exactly two nonisomorphic simple objects, \( 1 := C^1|0 \) and \( S := C^0|1 \), both of which are invertible. We have \( P(1) = \wedge W, P(S) = S \otimes P(1) \), and

\[
\text{FPdim}(\text{sRep}(W)) = 2\text{FPdim}(P(1)) = 2\dim(\wedge W) = 2^{\dim(W)+1}.
\]

Note that if \( \dim(W) \) is even then \( P(1) \) and \( P(S) \) are self dual, while if \( \dim(W) \) is odd then \( P(1) \) and \( P(S) \) are dual to each other.

For every integer \( n \geq 1 \), let \( H = H(n) \) be the Nichols’ Hopf algebra associated with an \( n \)-dimensional purely odd supervector space \( W \) (it does not depend on the choice of \( W \), up to Hopf algebra isomorphism). Namely, as an algebra \( H = \mathbb{C}[\mathbb{Z}/2\mathbb{Z}] \rtimes \wedge W \), where \( \mathbb{Z}/2\mathbb{Z} = \langle u \rangle \) is the group of grouplike elements of \( H \) and \( W \) is the space of \((1, u)\)-skew-primitive elements of \( H \). It is known that \( (H, R_u) \) is triangular, where \( R_u := \frac{1}{2}(1 \otimes 1 \otimes u + u \otimes 1 - u \otimes u) \), and that \( \text{sRep}(W) \cong \text{Rep}(H) \) as symmetric tensor categories [AEG, Theorem 3.1.1].

Recall that the center \( Z(\text{sRep}(W)) \) of \( \text{sRep}(W) \) is a finite braided tensor category. By the above, \( Z(\text{sRep}(W)) \cong \text{Rep}(D(H)) \) as braided tensor categories, where \( D(H) \) is the Drinfeld double of \( H = H(n) \).

**Theorem 2.6.** Let \( \mathcal{E} := \text{sRep}(W) \). Then \( Z(\mathcal{E}) \) is a finite non-degenerate braided integral tensor category containing \( \mathcal{E} \) as a Lagrangian subcategory.

*Proof. * By [EGNO, Proposition 8.6.3], \( Z(\mathcal{E}) \) is factorizable. Hence by [Sh, Theorem 1.1], \( Z(\mathcal{E}) \) is non-degenerate. Since \( \mathcal{E} \) is braided we have a canonical injective tensor functor \( \mathcal{E} \hookrightarrow Z(\mathcal{E}) \), so we may view \( \mathcal{E} \) as a symmetric tensor subcategory of \( Z(\mathcal{E}) \).

Now by Equation (1), \( \text{FPdim}(\mathcal{E})\text{FPdim}(\mathcal{E}') = \text{FPdim}(Z(\mathcal{E})) \). Therefore, since \( \text{FPdim}(Z(\mathcal{E})) = \text{FPdim}(\mathcal{E})^2 \) and \( \mathcal{E} \subseteq \mathcal{E}' \), we obtain that \( \mathcal{E} = \mathcal{E}' \), as desired.

**Remark 2.7.** The categories \( \text{Rep}(D(H(n))) \) were recently studied by Bontea and Nikshych in [BN1], where they describe their varieties of Lagrangian subcategories.
3. The $\mathbb{Z}/2\mathbb{Z}$-faithful grading

Let $\mathcal{C}$ be a finite non-degenerate braided tensor category containing a Lagrangian subcategory $\mathcal{E} := \text{sRep}(W) \subset \mathcal{C}$.

Let $S \in \mathcal{E}$ be the unique nontrivial invertible object of $\mathcal{E}$. We have inclusions of braided tensor categories $\text{sVec} = \langle 1, S \rangle \subseteq \mathcal{E} \subset \mathcal{C}$. Let $\mathcal{C}_0 := (\text{sVec})'$ be the centralizer of $\text{sVec}$ inside $\mathcal{C}$, and let $\mathcal{C}_1$ be the anti-centralizer of $\text{sVec}$ inside $\mathcal{C}$. Clearly, $\mathcal{E} \subseteq \mathcal{C}_0$. Also since $\mathcal{C}$ is non-degenerate it follows that $\mathcal{C}_0$ is strictly contained in $\mathcal{C}$, and by Equation (2), $\mathcal{C}_0' = \text{sVec}$ (i.e., $\mathcal{C}_0$ is slightly degenerate).

Lemma 3.1. If $X \notin \mathcal{C}_0$ is simple in $\mathcal{C}$ then $X$ belongs to $\mathcal{C}_1$.

Proof. By Schur’s lemma, $s_{S,X} = \lambda \cdot \text{id}_{S \otimes X}$ for some $\lambda \in \mathbb{C}^\times$. Hence

$$\text{id}_X = s_{S \otimes S,X} = c_{X,S \otimes S} \circ c_{S \otimes S,X}$$

$$= ((\text{id}_S \otimes c_{X,S}) \circ (c_{X,S} \otimes \text{id}_S)) \circ ((c_{S,X} \otimes \text{id}_S) \circ (\text{id}_S \otimes c_{S,X}))$$

$$= \lambda^2 \cdot \text{id}_{S \otimes S \otimes X} = \lambda^2 \cdot \text{id}_X,$$

which implies that $\lambda^2 = 1$. Since $X \notin \mathcal{C}_0 = (\text{sVec})'$, it follows that $\lambda = -1$, as claimed. □

It follows from Lemma 3.1 that $\mathcal{C}_1 \neq 0$ is a full abelian subcategory of $\mathcal{C}$, and that $\mathcal{C}$ admits a $\mathbb{Z}/2\mathbb{Z}$-faithful grading

$$\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1,$$

with $\mathcal{C}_0$ being the identity component. In particular, $\mathcal{C}_1$ is an invertible $\mathcal{C}_0$-bimodule category of order 2, $\text{FPdim}(\mathcal{C}) = 2\text{FPdim}(\mathcal{C}_0)$, and the projective covers $P(1)$ and $P(S)$ belong to $\mathcal{C}_0$.

Proposition 3.2. The objects 1 and $S$ are the unique simple objects of $\mathcal{C}_0$.

Proof. The statement is clear for $W = 0$.

Assume $\dim(W) = 1$, and suppose $\mathcal{C}_0$ has a simple object not isomorphic to 1 or $S$. Let $\mathcal{D} \subset \mathcal{C}_0$ be the Serre tensor closure of $\mathcal{E}$. Note that by our assumption, $\mathcal{D}$ is strictly contained in $\mathcal{C}_0$. By Corollary 2.2 $\mathcal{D}$ has an integer FP dimension (as $\text{FPdim}(\mathcal{C}_0) = 8$). Since $\mathcal{D}$ contains $\mathcal{E}$ we have $4 \leq \text{FPdim}(\mathcal{D}) < 8$, and hence since $\text{FPdim}(\mathcal{D})$ divides 8 we have $\mathcal{D} = \mathcal{E}$. This implies that the projective cover $P(1)$ in $\mathcal{C}_0$ (and hence in $\mathcal{C}$) coincides with the projective cover $P(1)$ of 1 in $\mathcal{E}$. But $\mathcal{C}$ (and hence $\mathcal{C}_0$) is unimodular by [EGNO, Proposition 8.10.10] and [Sh, Theorem 1.1], while $\mathcal{E}$ is not, a contradiction.

From now on we assume that $\dim(W) \geq 2$.

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1The assumption that $\mathcal{C}$ is non-degenerate is not needed in this proposition.

2We are grateful to Pavel Etingof for his help with the proof.
Note that by \[\text{[EGNO, Proposition 5.11.9]}\], \(\mathcal{E} = \text{sRep}(W) \subset \mathcal{C}_0\) is tensor generated by indecomposable 2-dimensional objects (as the corresponding Nichols Hopf algebra is generated in degree 1).

Let \(V\) be a simple object of \(\mathcal{C}_0\) not isomorphic to \(1\) or \(S\). Then there exists a nontrivial extension \(E\) of \(1\) by \(S\) in \(\mathcal{E}\) such that \(V\) does not centralize \(E\). Indeed, otherwise \(V\) would centralize \(\mathcal{E}\) (as it is generated by indecomposable 2-dimensional objects, which are all extensions of \(1\) by \(S\) and their duals), so \(V \in \mathcal{E}\), a contradiction.

This implies that \(S \otimes V \cong V\). Indeed, otherwise \(E \otimes V\) has a 2-step filtration with quotients \(V', V\) with \(V' \neq V\). Let \(s := s_{E,V}\). Then \(s - \text{id}\) is strictly upper triangular with respect to the above filtration, so it maps \(V\) to \(V'\), i.e., is zero, a contradiction.

Now let \(U\) be the universal extension of \(1\) by a multiple of \(S\) in \(\mathcal{E}\).

It is included in the exact sequence

\[0 \to W \otimes S \to U \to 1 \to 0\]

since \(\text{Ext}(1, S) = W^*\) in \(\mathcal{E}\). Consider the endomorphism \(s - \text{id}\) of \(U \otimes V\), where \(s := s_{U,V}\). The object \(U \otimes V\) has a 2-step filtration with quotients \(W \otimes V, V\), and \(s - \text{id}\) is strictly upper triangular under this filtration, i.e., defines a morphism \(V \to W \otimes V\), i.e., a vector \(w\) in \(W\). This vector is well defined up to scaling, since it rescales when we rescale the isomorphism \(V \cong S \otimes V\). Also \(w \neq 0\), since all extensions of \(1\) by \(S\) are quotients of \(U\), and there exists one not centralizing with \(V\) as shown above. Thus, we obtain a well defined line \(L_V\) in \(W\) spanned by \(w\).

Since the number of simple objects \(V\) is finite, there exist distinct codimension 1 subspaces \(W_1, W_2\) in \(W\) which do not contain \(L_V\) for any \(V\). Consider the subcategories \(\mathcal{D}_i := \text{sRep}(W/W_i)\) in \(\text{sRep}(W)\), \(i = 1, 2\), and let \(\mathcal{D}'_i\) be the centralizers of \(\mathcal{D}_i\) inside \(\mathcal{C}\) (equivalently, inside \(\mathcal{C}_0\)). Then \(\mathcal{D}'_i\) cannot contain any simple objects \(V\) not isomorphic to \(1, S\) (as \(V\) does not centralize \(\mathcal{D}_i\), since \(L_V\) is not contained in \(W_i\)).

Since \(\text{FPdim}(\mathcal{D}_i) = 4, i = 1, 2\), we have \(\text{FPdim}(\mathcal{D}'_i) = \text{FPdim}(\mathcal{C}_0)/2\) for each \(i\). Consider the tensor subcategory \(\mathcal{D}\) of \(\mathcal{C}_0\) generated by \(\mathcal{D}'_1\) and \(\mathcal{D}'_2\). It has integer FP dimension by Corollary \([2.2]\), which divides \(\text{FPdim}(\mathcal{C}_0)\) (by \([\text{EO, Theorem 3.47}]\)). Also \(\mathcal{D}\) is bigger than \(\mathcal{D}'_1, \mathcal{D}'_2\) (as \(\mathcal{D}_1 \neq \mathcal{D}_2\) as subcategories of \(\mathcal{C}_0\)), hence \(\text{FPdim}(\mathcal{D}) > \text{FPdim}(\mathcal{C}_0)/2\). Hence \(\text{FPdim}(\mathcal{D}) = \text{FPdim}(\mathcal{C}_0)\), i.e., \(\mathcal{D} = \mathcal{C}_0\). But \(\mathcal{D}\) has no simple objects other than \(1, S\) (as \(\mathcal{D}'_1, \mathcal{D}'_2\) do not have such objects). The proposition is proved.
Corollary 3.3. There is a tensor equivalence $C_0 \cong s\text{Rep}(W \oplus W^*)$ such that the tensor embedding $\mathcal{E} \hookrightarrow C_0$ is induced by the linear projection $W \oplus W^* \to W$.

Proof. By Proposition 3.2, $C_0$ is a finite tensor category with radical $\langle 1, S \rangle \cong \text{Rep}(\mathbb{Z}/2\mathbb{Z})$ (with trivial associativity) of codimension 2. Hence by [EG2, Corollary 3.5], $C_0$ is tensor equivalent to $s\text{Rep}(V)$ for some finite dimensional purely odd supervector space $V$.

Now, on one hand, we have
\[
\text{FPdim}(C_0) = 2\text{FPdim}(P(1)) = 2^{\dim(V)+1}
\]
and, on the other hand, we have
\[
\text{FPdim}(C_0) = \frac{\text{FPdim}(C)}{2} = \frac{\text{FPdim}(\mathcal{E})^2}{2} = 2^{\dim(W)+1}.
\]
Hence, $\dim(V) = 2 \dim(W)$, and the rest of the claim follows.

Since the projective objects of $C_0$ are projective in $C$ (as they are direct sums of copies of $P(1)$ and $P(S)$), $C$ and $C_1$ are exact module categories over $C_0$ via the tensor product in $C$.

Proposition 3.4. The exact $C_0$-module category $C_1$ is indecomposable.

In particular, $C_1$ has at most two nonisomorphic simple objects.

Proof. By Corollary 3.3, $C$ has more than two simple objects. If $C_1$ has exactly one simple object, there is nothing to prove. Let us therefore assume that $C_1$ has at least two nonisomorphic simple objects.

Decompose $C_1$ into a direct sum $C_1 = \bigoplus_{i=1}^n M_i$ of exact indecomposable module categories over $C_0$. By Proposition 3.3 and [EO, Example 4.7], each module category $M_i$ has at most two simple objects.

Suppose that $M_1 = M_2 = 0$, and let $X \in M_1$, $Y \in M_2$ be simple objects. Since $C$ is generated by $X$ as a module category over itself there exists an object $Z \in C$ such that $\text{Hom}_C(Z \otimes X, Y) \neq 0$. Clearly $Z \otimes X \notin M_1$, and hence $Z \notin C_0$. Also, since
\[
\text{Hom}_C(X \otimes^* Y,^* Z) \cong \text{Hom}_C(Z \otimes X, Y) \neq 0,
\]
we can choose a nonzero morphism $g : X \otimes^* Y \to^* Z$. Let $W$ be a simple quotient of $\text{Im}(g)$. Then $\text{Hom}_C(W^* \otimes X, Y) \cong \text{Hom}_C(X \otimes^* Y, W) \neq 0$. Thus we may assume that $Z$ is simple (replacing it by $W^*$, if necessary). But then it follows that $Z \in C_1$, and hence $Z \otimes X \in C_0$, a contradiction. Hence, $C_1$ is indecomposable. Since by Corollary 3.3 and [EO, Example 4.7], $C_1$ has at most two simple objects it follows that $C_1$ has exactly two nonisomorphic simple objects, as desired.

To summarize, we have proved the following theorem.
Theorem 3.5. Let $W$ be a purely odd supervector space, and let $\mathcal{C}$ be a finite non-degenerate braided tensor category containing $\mathcal{E} := \text{sRep}(W)$ as a Lagrangian subcategory. Let $\mathcal{C}_0$ and $\mathcal{C}_1$ be the centralizer and anti-centralizer of $\text{sVec} \subseteq \mathcal{E}$ inside $\mathcal{C}$, respectively. Then $\mathcal{C}$ admits a $\mathbb{Z}/2\mathbb{Z}$-faithful grading $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$, with $\mathcal{C}_0$ as the identity component, such that the following hold:

1. $\mathcal{C}_0$ is tensor equivalent to $\text{sRep}(W \oplus W^*)$, contains $\mathcal{E}$ as a symmetric tensor subcategory, and is slightly degenerate.
2. $\mathcal{C}_1$ is an exact indecomposable $\mathcal{C}_0$-module category with at most two nonisomorphic simple objects. □

Remark 3.6. In [BN2, Section 8] the authors show in particular that $\mathcal{Z}(\mathcal{E})$ is nilpotent of nilpotency class 2.

Example 3.7. Let $\mathcal{C} := \mathcal{Z}(\text{sVec})$. Then $\mathcal{C}_0 = \text{sVec}$ and $\mathcal{C}_1$ has exactly two invertible objects of order 2.

Indeed, let $A \in \mathcal{C}_0$ be the commutative algebra corresponding to the forgetful functor $\mathcal{C}_0 \to \mathcal{E}$. It is easy to see that the Lagrangian algebra $B \in \mathcal{Z}(\mathcal{E})$, corresponding to the forgetful functor $F : \mathcal{Z}(\mathcal{E}) \to \mathcal{E}$, has the form $B = A \oplus p$, for some $p \in \mathcal{C}_1$. In particular, we have

$$\text{FPdim}(p) = \text{FPdim}(A) = \text{FPdim}(\mathcal{E})/2.$$ 

Now, since $F(P(p))$ is projective in $\mathcal{E}$, and

$$\text{Hom}_\mathcal{E}(F(P(p)), 1) = \text{Hom}_{\mathcal{Z}(\mathcal{E})}(P(p), B) = \text{Hom}_{\mathcal{Z}(\mathcal{E})}(P(p), p) \neq 0,$$

it follows that $P_\mathcal{E}(1)$ projects onto $F(P(p))$. Hence, we have

$$\text{FPdim}(\mathcal{E})/2 \leq \text{FPdim}(P(p)) \leq \text{FPdim}(P_\mathcal{E}(1)) = \text{FPdim}(\mathcal{E})/2,$$

which implies that $P(p) = p$. Thus, $p$ is projective. Therefore, $F(p)$ is projective in $\mathcal{E}$. Since

$$\text{Hom}_\mathcal{E}(F(p), 1) = \text{Hom}_{\mathcal{Z}(\mathcal{E})}(p, B) = \text{Hom}_{\mathcal{Z}(\mathcal{E})}(p, p) \neq 0,$$

it follows that if the dimension of $\text{Hom}_{\mathcal{Z}(\mathcal{E})}(p, p)$ was $\geq 2$, so would be the dimension of $\text{Hom}_\mathcal{E}(P_\mathcal{E}(1), 1)$ (as $P_\mathcal{E}(1)$ projects onto $F(p)$), which is not the case. Hence, $p$ is also simple.
Now, since \( p \otimes p^* \in \mathcal{C}_0 \) is projective and \( \text{Hom}_\mathcal{C}(p \otimes p^*, 1) \) is 1-dimensional, we have \( p \otimes p^* \cong P(1) \). In particular, it follows that \( \text{Hom}_\mathcal{C}(p, p \otimes S) = \text{Hom}_\mathcal{C}(p \otimes p^*, S) = 0 \). Hence, \( q := p \otimes S \not\cong p \) is another simple projective object in \( \mathcal{C}_1 \), as desired.

Moreover, we have

\[
2\text{FPdim}(\mathcal{C}_0) = \text{FPdim}(\mathcal{C}) = \text{FPdim}(\mathcal{C}_0) + 2\text{FPdim}(p)^2,
\]

which implies that \( \text{FPdim}(p)^2 = \text{FPdim}(\mathcal{C}_0)/2 = \text{FPdim}(\mathcal{E})^2/4 \). We have thus established (1) and (2).

Finally, by [EGNO] Theorem 6.1.16 the forgetful functor \( Z(\mathcal{E}) \to \mathcal{E} \) maps \( p \) and \( q \) to projective objects, and since the Frobenius-Perron dimensions of \( p \) and \( q \) are equal to \( \text{FPdim}(P_\mathcal{E}(1)) = \text{FPdim}(P_\mathcal{E}(S)) \), it follows that \( F \) must map \( p \) to \( P_\mathcal{E}(1) \) or \( P_\mathcal{E}(S) \), and vice versa for \( q \). Thus (3) from the fact that \( P_\mathcal{E}(1) \) and \( P_\mathcal{E}(S) \) are self dual if and only if \( \text{dim}(W) \) is even (equivalently, \( P_\mathcal{E}(1) \) and \( P_\mathcal{E}(S) \) are dual to each other if and only if \( \text{dim}(W) \) is odd).

4. THE ACTION OF THE GROUP \( B \)

Recall from [DGNO1] Lemma A.11 that there are exactly 8 non-degenerate braided pointed fusion categories of Frobenius-Perron dimension 4, which contain \( s\text{Vec} \) as a Lagrangian subcategory (up to braided tensor equivalence), and that 4 of them are supported on the group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), while the other 4 are supported on the group \( \mathbb{Z}/4\mathbb{Z} \).

Recall also from [DGNO1] Corollary B.16 that there are exactly 8 non-degenerate braided non-integral fusion categories of Frobenius-Perron dimension 4, which contain \( s\text{Vec} \) as a Lagrangian subcategory (up to braided tensor equivalence). These non-integral categories are called Ising categories.

Thus, all together, there are exactly 16 non-degenerate braided fusion categories of Frobenius-Perron dimension 4, which contain \( s\text{Vec} \) as a Lagrangian subcategory (up to braided tensor equivalence). Let us denote this set by \( B \).

Now, let \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) be two finite non-degenerate braided tensor categories containing \( s\text{Vec}(W_1) \) and \( s\text{Vec}(W_2) \) as a Lagrangian subcategory, respectively. Consider the finite non-degenerate braided tensor category \( \mathcal{D}_1 \boxtimes \mathcal{D}_2 \). Then \( \mathcal{D}_1 \boxtimes \mathcal{D}_2 \) contains \( s\text{Vec} \boxtimes s\text{Vec} \) as a braided tensor category, and hence contains a Tannakian subcategory \( \mathcal{T} := \text{Rep}(\mathbb{Z}/2\mathbb{Z}) \). Let \( \mathcal{T}' \) be the centralizer of \( \mathcal{T} \) inside \( \mathcal{D}_1 \boxtimes \mathcal{D}_2 \), and let

\[
\mathcal{D}_1 \boxtimes \mathcal{D}_2 := (\mathcal{T}')_{\mathbb{Z}/2\mathbb{Z}}.
\]
Proposition 4.1. Let $\mathcal{C}, \mathcal{D}_1$ and $\mathcal{D}_2$ be finite non-degenerate braided tensor categories containing a Lagrangian subcategory $\mathcal{E} := \text{sRep}(W)$, $\mathcal{E}_1 := \text{sRep}(W_1)$ and $\mathcal{E}_2 := \text{sRep}(W_2)$, respectively. Then the following hold:

1. $\mathcal{D}_1 \boxtimes \mathcal{D}_2$ is a finite non-degenerate braided tensor category containing $\mathcal{E}_3 := \text{sVec}(W_1 \oplus W_2)$ as a Lagrangian subcategory.

2. For every $\mathcal{B} \in \mathcal{B}$, $\mathcal{C} \boxtimes \mathcal{B}$ is a finite non-degenerate braided tensor category containing $\mathcal{E}$ as a Lagrangian subcategory.

Proof. (1) The first claim follows from Equation (2). As for the second claim, it is clear that $\mathcal{D}_1 \boxtimes \mathcal{D}_2$ contains the centralizer of $\mathcal{T}$ inside $\mathcal{E}_1 \boxtimes \mathcal{E}_2$, and that the later category is braided tensor equivalent to $\mathcal{E}_3$. The fact that $\mathcal{E}_3$ is Lagrangian in $\mathcal{D}_1 \boxtimes \mathcal{D}_2$ follows from dimension considerations.

(2) Follows from Part (1). □

In particular for $W = 0$, Proposition 4.1(2) says that $\mathcal{B}$ forms a group under the product $\boxtimes$, with unit element $Z(\text{sVec})$. It is well known that $\mathcal{B} \cong \mathbb{Z}/16\mathbb{Z}$ [DNO] (see also [BGHNPRW, LKW]). Also, the pointed categories in $\mathcal{B}$ form a subgroup of index 2.

Moreover, Proposition 4.1(1) states that $\mathcal{B} \cong \mathbb{Z}/16\mathbb{Z}$ acts on the set of equivalence classes of finite non-degenerate braided tensor categories containing a Lagrangian subcategory $\mathcal{E}$ of the same Frobenius-Perron dimension. Hence, by Proposition 2.6, we have the following result.

Corollary 4.2. For every $\mathcal{B} \in \mathcal{B}$, $\mathcal{Z}(\mathcal{E}) \boxtimes \mathcal{B}$ is a finite non-degenerate braided tensor category containing $\mathcal{E}$ as a Lagrangian subcategory. □

Now for $\mathcal{B} \in \mathcal{B}$, let $F : \mathcal{T}' \to \mathcal{Z}(\mathcal{E}) \boxtimes \mathcal{B}$, $Z \mapsto Z \otimes A$, be the free module functor. Recall that $F$ is a surjective braided tensor functor.

Theorem 4.3. Let $\mathcal{B} \in \mathcal{B}$, and set $\mathcal{C} := \mathcal{Z}(\mathcal{E}) \boxtimes \mathcal{B}$. The following hold:

1. If $\mathcal{B}$ is not pointed then $\mathcal{C}$ is not integral. In this case, $\mathcal{C}$ has exactly 2 invertible (non-projective, if $W \neq 0$) objects $1$ and $S$, and exactly 1 non-invertible simple projective object $X$. We have, $X \cong X^* \cong S \otimes X$, $X \otimes X = P(1) \oplus P(S)$, and $FPdim(X) = FPdim(\mathcal{E})/\sqrt{2}$.

2. If $\mathcal{B}$ is pointed then $\mathcal{C}$ is integral. In this case, $\mathcal{C}$ has exactly 2 invertible (non-projective, if $W \neq 0$) objects $1$ and $S$, and exactly 2 simple projective (non-invertible, if $W \neq 0$) objects $P$ and $Q \cong P \otimes S$. We have, $P \otimes P^* = P(1)$ and $FPdim(P) = FPdim(\mathcal{E})/2$.

Proof. Let $p$ and $q$ be the simple projective objects of $\mathcal{Z}(\mathcal{E})$ (see Example 3.7).
(1) Let $Z$ be the unique noninvertible simple object of $\mathcal{B}$. We have $Z \cong Z^* \cong S \otimes Z$ and $Z \otimes Z = 1 \oplus S$.

By Theorem 3.5, $p \boxtimes Z$ is in $\mathcal{T}'$, and we have that $X := F(p \boxtimes Z)$ is simple projective in $\mathcal{C}$. Thus the claim follows from Example 3.7 and the properties of $F$.

(2) Let $h \in \mathcal{B}$ be as in the proof of Theorem 4.4. By Theorem 3.5, $p \boxtimes h$ and $q \boxtimes h$ are in $\mathcal{T}'$, and we have that $P := F(p \boxtimes h)$ and $Q := F(q \boxtimes h)$ are simple projective in $\mathcal{C}$. Thus the claim follows from Example 3.7 and the properties of $F$. □

We conclude this section by proving that the action of $\mathcal{B}$ is free.

**Theorem 4.4.** The action of $\mathcal{B}$ on the set of equivalence classes of finite non-degenerate braided tensor categories containing $\mathcal{E}$ as a Lagrangian subcategory, is free.

**Proof.** Fix a finite non-degenerate braided tensor category $\mathcal{C}$ containing $\mathcal{E}$ as a Lagrangian subcategory. It is sufficient to show that the categories $\mathcal{C} \boxtimes \mathcal{B}$, $\mathcal{B} \in \mathcal{B}$ is pointed, are pairwise non-equivalent braided tensor categories.

Let $g \in \mathcal{B}$ be such that $S \boxtimes g$ is the nontrivial object of $\mathcal{T}$. Then $g$ has order 2, and the braiding $c(g, g)$ on $g^2 = 1$ in $\mathcal{B}$ is equal to $-\text{id}_1$. Let $A := 1 \boxtimes 1 \oplus S \boxtimes g$ be the regular algebra of $\mathcal{T}$. Also, pick a simple object $h \neq 1, g$ in $\mathcal{B}$. We have $c(h, g)c(g, h) = -\text{id}_{gh}$.

Pick an object $Z \in \mathcal{C}$ such that $s_{S, Z} = -\text{id}_{S \otimes Z}$ (such an object exists by Theorem 3.5). Then $Z \boxtimes h \in \mathcal{T}'$. Consider the free braided tensor functor $F : \mathcal{T}' \to \mathcal{C} \boxtimes \mathcal{B}$, and let $z := F(Z \boxtimes h) = (Z \boxtimes h) \otimes A$. Since $F$ is braided, we have the following commutative diagram

$$
\begin{array}{ccc}
F(Z \boxtimes h) \otimes_A F(Z \boxtimes h) & \xrightarrow{\tilde{c}_{z, z}} & F(Z \boxtimes h) \otimes_A F(Z \boxtimes h) \\
\cong & & \cong \\
F((Z \otimes Z) \boxtimes h^2) & \xrightarrow{F(c_{ZZ\boxtimes h, ZZ\boxtimes h})} & F((Z \otimes Z) \boxtimes h^2),
\end{array}
$$

where $\tilde{c}_{z, z}$ is the braiding on $z \otimes_A z$ in $\mathcal{C} \boxtimes \mathcal{B}$. But,

$$
c_{ZZ\boxtimes h, ZZ\boxtimes h} = c_{Z, Z} \boxtimes c(h, h)\text{id}_{h^2},
$$

which implies that $c(h, h)$ is determined by the braided tensor category $\mathcal{C} \boxtimes \mathcal{B}$. Since $c(h, h)$ determines $\mathcal{B}$ (see, e.g., [DGNO1, Lemma A.11]), we are done. □

5. **The proof of Theorem 1.1**

We already proved in Theorem 4.4 that the action of $\mathcal{B}$ is free, so it remains to show it is transitive.
Let \( \mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1 \) be a finite non-degenerate braided tensor category containing \( \mathcal{E} \) as a Lagrangian subcategory (see Theorem 3.5).

Let \( F : \mathcal{C}_0 \to \mathcal{E} \) be the forgetful tensor functor induced by the embedding \( W \to W \oplus W^* \), and let \( I \) be the right adjoint functor to \( F \). It is known that \( A := I(1) \) has a canonical structure of an associative algebra object in \( \mathcal{C}_0 \). We have \( \text{FPdim}(A) = \text{FPdim}(\mathcal{E})/2 \) (see Lemma 2.3).

Lemma 5.1. The algebra object \( A \) is commutative, that is, we have \( m = m \circ c_{A,A} \), where \( m : A \otimes A \to A \) is the multiplication map on \( A \).

Proof. It is obvious that the injective braided tensor functor \( \mathcal{C}_0 \hookrightarrow \mathcal{Z}(\mathcal{E}) \) determines a central structure on \( F \) ([EGNO, Definition 8.8.6]), hence the claim follows from [EGNO, Proposition 8.8.8]. □

The functor \( F \) defines on \( \mathcal{E} \) a structure of an exact left indecomposable module category over \( \mathcal{C}_0 \), and by [EGNO, Theorem 7.10.1], the functor \( I : \mathcal{E} \xrightarrow{\cong} \text{Mod}_{\mathcal{C}_0}(A) \) induces an equivalence of left module categories over \( \mathcal{C}_0 \).

Lemma 5.2. The functor \( I : \mathcal{E} \xrightarrow{\cong} \text{Mod}_{\mathcal{C}_0}(A) \) induces an equivalence of tensor categories, where \( \text{Mod}_{\mathcal{C}_0}(A) \) is viewed as a tensor subcategory of the finite tensor category \( \text{Bimod}_{\mathcal{C}_0}(A) \) (see Subsection 2.3).

Proof. One shows that \( I \) has a structure of a tensor functor in exactly the same way as one shows that the right adjoint to the forgetful functor \( \mathcal{Z}(\mathcal{E}) \to \mathcal{E} \) has a structure of a tensor functor (see [EGNO, Lemma 8.12.2]). □

Since \( \mathcal{C}_1 \) is an exact invertible \( \mathcal{C}_0 \)-bimodule category, the category \( \text{Mod}_{\mathcal{C}_1}(A) = \mathcal{C}_1 \boxtimes_{\mathcal{C}_0} \text{Mod}_{\mathcal{C}_0}(A) \) of right \( A \)-modules in \( \mathcal{C}_1 \) is naturally a left indecomposable \( \mathcal{C}_0 \)-module category.

Lemma 5.3. The following hold:

1. There is an equivalence \( \text{Mod}_{\mathcal{C}_1}(A) \cong \text{Mod}_{\mathcal{C}_0}(A) \) of left module categories over \( \mathcal{C}_0 \). Thus, \( \text{Mod}_{\mathcal{C}_1}(A) \) is exact over \( \mathcal{C}_0 \).
2. \( \text{Mod}_{\mathcal{C}}(A) \) is an exact left indecomposable module category over \( \mathcal{C} \). Hence, \( \mathcal{A} := \text{Mod}_{\mathcal{C}}(A) \) is a finite tensor subcategory of \( \text{Bimod}_{\mathcal{C}}(A) \) (see Subsection 2.3).
3. The tensor category \( \mathcal{A} \) has a \( \mathbb{Z}/2\mathbb{Z} \)-faithful grading \( \mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \), where \( \mathcal{A}_0 \cong \mathcal{E} \) as tensor categories and \( \mathcal{A}_1 \subseteq \text{Bimod}_{\mathcal{C}_1}(A) \).

---

3If \( W \neq 0 \), \( F \) is not braided!
Proof. (1) & (2) Clearly, $\text{Mod}_C(A) \cong C \boxtimes C_0 \text{Mod}_{C_0}(A)$ is the induced left module category of $\text{Mod}_{C_0}(A)$. Thus by Theorem 3.5(1), we have

$\text{Mod}_C(A) \cong (C_0 \oplus C_1) \boxtimes C_0 \text{Mod}_{C_0}(A) \cong \text{Mod}_{C_0}(A) \oplus \text{Mod}_{C_1}(A)$,

as left $C_0$-module categories. It follows that $\text{Mod}_{C_1}(A)$ and $\text{Mod}_{C_0}(A)$ must be equivalent as left module categories over $C_0$ since $\text{Mod}_C(A)$ is indecomposable over $C$. This implies that $\text{Mod}_{C_1}(A)$ is exact over $C_0$, and hence so is $\text{Mod}_C(A)$. Therefore by [EG2, Corollary 2.5], $\text{Mod}_{C_0}(A)$ is exact over $C_0$. Finally, since $A$ is commutative in $C$, $A$ is a finite tensor subcategory of $\text{Bimod}_C(A)$ (see Subsection 2.3).

(3) The decomposition $\text{Mod}_C(A) = \text{Mod}_{C_0}(A) \oplus \text{Mod}_{C_1}(A)$ of module categories over $C_0$ obtained above clearly induces a $\mathbb{Z}/2\mathbb{Z}$-grading on $A$ with the claimed properties. □

It follows from Proposition 2.4 and Lemma 5.3(2) that we have a decomposition

$C \boxtimes C' \cong \mathcal{Z}(A)$

of braided tensor categories, where $C'$ is the centralizer of $C$ inside $\mathcal{Z}(A)$. Since by Equation (5), $C'$ is non-degenerate with Frobenius-Perron dimension 4, $C'$ is fusion. We have thus obtained the following.

**Proposition 5.4.** There exists an element $B$ in the group $B$ such that $C' \cong B$ as braided tensor categories. Hence, there is a braided tensor equivalence $C \boxtimes B \cong \mathcal{Z}(A)$. □

Finally, it follows from Theorem 2.5 and Proposition 5.4 that there is a braided tensor equivalence $C \boxtimes B \cong (\mathcal{Z}_E(A))^{\mathbb{Z}/2\mathbb{Z}}$. In particular $C \boxtimes B$ contains a Tannakian subcategory $\mathcal{T} := \text{Rep}(\mathbb{Z}/2\mathbb{Z})$, and there is a braided tensor equivalence $(\mathcal{T}')^{\mathbb{Z}/2\mathbb{Z}} \cong \mathcal{Z}(\mathcal{E})$. Hence $C \boxtimes B \cong \mathcal{Z}(\mathcal{E})$, so $C \cong \mathcal{Z}(\mathcal{E}) \boxtimes B^{-1}$, as desired.

The proof of Theorem 1.1 is complete. □

As a corollary of Theorems 1.1, 4.3 we obtain the following result.

**Corollary 5.5.** Let $C$ be finite non-degenerate braided tensor category containing a Lagrangian subcategory $s\text{Rep}(W)$. Then the following hold:

1. If $C$ is integral then $C$ has exactly four nonisomorphic simple objects: two invertible objects (non-projective, if $W \neq 0$), and two simple projective objects (non-invertible, if $W \neq 0$).
2. If $C$ is not integral then $C$ has exactly three nonisomorphic simple objects: two invertible objects (non-projective, if $W \neq 0$), and one simple projective object.
Remark 5.6.  (1) Some of the finite non-degenerate braided integral tensor categories containing a Lagrangian subcategory \( s\text{Rep}(W) \) can be constructed using the interesting method developed by Davydov-Runkel in [DR1, DR2], but not all of them. For example, \( \mathcal{Z}(s\text{Rep}(W)) \) can be constructed using Davydov-Runkel’s method if and only if \( \text{dim}(W) \) is even (see, [DR3, Theorem 1.2]).

(2) In [DR1, Section 5.1] it is shown that all 8 equivalence classes of non-degenerate braided non-integral tensor categories containing a Lagrangian subcategory \( s\text{Rep}\{0\}) = \text{sVec} \) (i.e., Ising categories; see Section 4) can be constructed using the method of Davydov-Runkel.

In the Appendix below we use Theorem [11] and the classification of \( R \)-matrices of the 8-dimensional Nicols’ Hopf algebra \( H(2) \) given in [C], to verify that all 8 equivalence classes of non-degenerate braided non-integral tensor categories containing a Lagrangian subcategory \( s\text{Rep}(\mathbb{C}) \) arise in this way too.

More generally, we expect that Theorem [11] and the classification of \( R \)-matrices of the \( 2n \)-dimensional Nicols’ Hopf algebra \( H(2n) \) given in [PO], can be used in a similar way to verify that all 8 equivalence classes of non-degenerate braided non-integral tensor categories containing a Lagrangian subcategory \( s\text{Rep}(W), \text{dim}(W) = n, \) arise in this way also for every \( n \geq 2. \)

6. DEGENERATE BRAIDED TENSOR CATEGORIES WITH LAGRANGIAN \( s\text{Rep}(W) \)

Let \( \mathcal{C} \) be a finite degenerate braided tensor category over \( \mathbb{C} \) containing \( \mathcal{E} := \text{sRep}(W) \) as a Lagrangian subcategory. Let \( S \in \mathcal{C} \) be the unique nontrivial invertible object of \( \mathcal{E}. \)

**Theorem 6.1.** The following hold:

(1) There exists a purely odd supervector space \( V \) over \( \mathbb{C} \) such that \( \text{dim}(W) \leq \text{dim}(V) \leq 2\text{dim}(W), \) and \( \mathcal{C} \cong \text{sRep}(V) \) as tensor categories.

(2) For every integer \( \text{dim}(W) \leq d \leq 2\text{dim}(W), \) there exist a purely odd supervector space \( V \) of dimension \( d, \) and a finite degenerate braided tensor category \( \mathcal{C}, \) such that \( \mathcal{C} \cong \text{sRep}(V) \) as tensor categories, and \( \mathcal{E} \subseteq \mathcal{C} \subseteq \text{sRep}(W \oplus W^*) \subset \mathcal{Z}(\mathcal{E}) \) as tensor categories.

**Proof.** (1) We have \( \text{Vec} \neq \mathcal{C}' \subset \mathcal{E}' = \mathcal{E}, \) and hence \( S \in \mathcal{C}'. \) Arguing now as in the proof of Proposition [3.2] we conclude that \( 1, S \) are the only
simple objects of $\mathcal{C}$. Therefore it follows from [EG2, Theorem 3.1] that $\mathcal{C}$ is tensor equivalent to $\text{sRep}(V)$ for some finite dimensional purely odd supervector space $V$.

Since $\text{sRep}(W) \subseteq \text{sRep}(V)$ as tensor categories, $\dim(W) \leq \dim(V)$. Moreover, by Equation (1),

$$\text{FPdim}(\mathcal{C})\text{FPdim}(\mathcal{C}') = \text{FPdim}(\mathcal{E})\text{FPdim}(\mathcal{E}') = \text{FPdim}(\mathcal{E})^2.$$ 

Therefore,

$$2^{\dim(V)+1} = \text{FPdim}(\mathcal{C}) < \text{FPdim}(\mathcal{E})^2 = 2^{2(\dim(W)+1)},$$

which implies that $\dim(V) \leq 2 \dim(W)$.

(2) By Theorem 2.6, $\mathcal{E}$ is Lagrangian in $\mathcal{Z}(\mathcal{E})$. Clearly, $\mathcal{E}$ is also Lagrangian in every braided tensor subcategory $\mathcal{C}$ of $\mathcal{Z}(\mathcal{E})$ that contains $\mathcal{E}$. Now for every $0 \leq d \leq n$, there is a braided tensor subcategory $\mathcal{C}$ of $\mathcal{Z}(\mathcal{E})$ of Frobenius-Perron dimension $2^{\dim(W)+d+1}$ that contains $\mathcal{E}$. Since by Equation (1), $\text{FPdim}(\mathcal{C})\text{FPdim}(\mathcal{C}' \cap \mathcal{E}) = \text{FPdim}(\mathcal{E})^2$, we see that $\mathcal{C}$ is degenerate.

The proof of the theorem is complete. $\square$

7. APPENDIX

Let $H := H(2)$ be the 8-dimensional Nichols’ Hopf algebra, with grouplike element $u$ and $(1, u)$-skew-primitive elements $x, y$ (see Subsection 2.6). It is well known (and straightforward to check) that the dual Hopf algebra $H^*$ is unimodular (i.e., the distinguished grouplike element $g \in H$ is equal to 1), and that every integral of $H^*$ is of the form $\lambda_s := s(xy)^* + s(uxy)^*$, $s \in \mathbb{C}$.

Let

$$R := R_u - \frac{1}{2} (x \otimes uy + ux \otimes uy + x \otimes y - ux \otimes y)$$

$$+ \frac{1}{2} (y \otimes ux + uy \otimes ux + y \otimes x - uy \otimes x)$$

$$- (xy \otimes xy + uxy \otimes xy + xy \otimes uxy - uxy \otimes uxy).$$

Clearly $R$ is non-degenerate. It is well known [G] that $(H, R)$ is a quasi-triangular Hopf algebra with Drinfeld element $u := u(1 + 2xy)$, and that the finite braided tensor category $\text{Rep}(H, R)$ is slightly-degenerate.
Let
\[ \gamma := R_u + \frac{i}{2} (x \otimes uy - ux \otimes uy + x \otimes y + ux \otimes y) \]
\[ - \frac{i}{2} (y \otimes ux - uy \otimes ux + y \otimes x + uy \otimes x) \]
\[ - \frac{1}{2} (xy \otimes xy + uxy \otimes xy + xy \otimes uxy - uxy \otimes uxy). \]

The following properties of \( \gamma \) can be verified in a straightforward manner.

**Lemma 7.1.** The 2 triples \((\gamma, \lambda_{\pm}, g = 1)\) satisfy all the conditions in [DR2, Theorem 1]. Namely, we have

1. \( \gamma \) is non-degenerate.
2. \((id \otimes \varepsilon)(\gamma) = 1 = (\varepsilon \otimes id)(\gamma)\).
3. \((id \otimes S)(\gamma) = (S \otimes id)(\gamma)\).
4. \((\Delta \otimes id)(\gamma) = \gamma_1 \gamma_{23}, \text{ and } (id \otimes \Delta)(\gamma) = \gamma_{12} \gamma_{13}\).
5. \((\lambda_{\pm} \otimes \lambda_{\pm})(id \otimes S)(\gamma)) = 1\).
6. \((id \otimes S^2)(\gamma) = \gamma_{21}\). \(\square\)

(In the language of [DR2, Theorem 1], (1)-(4) say that \( \gamma \) is a non-degenerate Hopf-copairing.)

Set \( a := (1 - i)/2 \) and \( \zeta := e^{\pi i/4} \). Let
\[ \sigma_+ := (a1 + \bar{a}u)(1 + xy), \sigma_- := u\sigma_+ = \sigma_+ u \in H. \]

It is easy to check that \( \sigma_+, \sigma_- \) are invertible with inverses
\[ \sigma_+^{-1} = (\bar{a}1 + au)(1 - xy), \sigma_-^{-1} = u\sigma_+^{-1} = \sigma_+^{-1} u, \]
and \( \sigma_+^2 = \sigma_-^2 = u \) (hence \( S^2(h) = \sigma_+^2 h \sigma_-^2 \) for every \( h \in H \)).

Now using the above properties of \( \sigma_+, \sigma_- \) and the properties of the \( R \)-matrix \( R \) it is straightforward to verify the following.

**Lemma 7.2.** Each one of the 8 sextuplets
\[ (R, \sigma_\pm, g = 1, \lambda, \gamma, \beta = \pm \zeta) \text{ and } (R, \sigma_\pm, g = 1, \lambda_{\pm}, \gamma, \beta = \pm i \zeta) \]
satisfies all the conditions in [DR2, Theorem 3]. Namely, let \( \sigma = \sigma_\pm \), then we have

1. \( \gamma = (\sigma^{-1} \otimes 1)\Delta(\sigma)(1 \otimes \sigma^{-1}) \).
2. \( \lambda_{\pm}(S(h)) = \lambda_{\pm}(\sigma h \sigma^{-1}) \) for every \( h \in H \).
3. \( \lambda_{\pm}(\sigma) = \beta^2 \) and \( \lambda_{\pm}(\sigma) = \beta^2 \).
4. The map \( H \to H^{\text{cop}}, h \mapsto \sigma h \sigma^{-1} \), is a Hopf algebra isomorphism.
5. \( S(\sigma) = \sigma \).
6. \( \gamma = \sum \sigma \gamma_1 \sigma^{-1} \otimes \sigma^{-1} S(\gamma_2)\sigma, \) where \( \gamma = \sum \gamma_1 \otimes \gamma_2 \). \(\square\)
In conclusion, Lemmas 7.1 and 7.2 establish that each one of the 8 sextuplets
\[(R, \sigma_\pm, g = 1, \lambda_i, \gamma, \beta = \pm \zeta)\] and \[(R, \sigma_\pm, g = 1, \lambda_{-i}, \gamma, \beta = \pm i \zeta)\]
determines a structure of a finite braided tensor category on the category \(\text{Rep}(H) + \text{Vec}\), in the manner prescribed by Davydov-Runkel in [DR1, DR2]. Clearly, these categories are not integral and have Frobenius-Perron dimension 16, and it is straightforward to verify that they are non-degenerate and contain \(\text{sRep}(C)\) as a Lagrangian subcategory.

**Lemma 7.3.** The 8 finite braided tensor categories constructed above are pairwise non-equivalent as braided tensor categories.

**Proof.** Let \(\mathcal{C} = (R, \sigma, g = 1, \lambda, \gamma, \beta)\) be one of the 8 finite braided tensor categories constructed above. Let \(X\) be the unique non-invertible simple object of \(\mathcal{C}\), and let \(\chi\) be the unique non-trivial character of \(H\) (viewed as the unique non-trivial invertible object of \(\mathcal{C}\)). We have, \(\chi(\sigma_\pm) = \mp i\).

Then it is straightforward to verify that the action of the braiding isomorphism \(X \otimes_C X \xrightarrow{\sim} X \otimes_C X\) on the 1-dimensional space \(\text{Hom}_C(1, X \otimes_C X)\) is given by multiplication by \(\beta\).

It is also straightforward to verify that the braiding isomorphism \(\chi \otimes_C X \xrightarrow{\sim} X \otimes_C \chi\) is given by \(\chi(\sigma) \cdot \text{id}_X\).

It thus follows from the above that the braided tensor equivalence class of \(\mathcal{C}\) is determined by \(\sigma\) and \(\beta\), which implies the claim. \(\square\)

It thus follows from the above and Theorem 1.1 that all 8 equivalence classes of finite non-degenerate braided non-integral tensor categories \(\mathcal{C}\), containing a Lagrangian subcategory \(\text{sRep}(C)\), arise from the construction of Davydov-Runkel.

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