GLOBAL WELL-POSEDNESS FOR THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION IN $H^\frac{1}{2}(\mathbb{R})$

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Abstract. We prove that the derivative nonlinear Schrödinger equation is globally well-posed in $H^\frac{1}{2}(\mathbb{R})$ when the mass of initial data is strictly less than $4\pi$.

1. Introduction. In this paper, we study the Cauchy problem to the derivative nonlinear Schrödinger equation (DNLS):

$$\begin{cases}
i\partial_t u + \partial_x^2 u = i\partial_x(|u|^2 u), & t \in \mathbb{R}, \ x \in \mathbb{R}, \\
u(0, x) = u_0(x). &
\end{cases}$$

(1)

This equation was derived by [12, 13] for studying the propagation of the circular polarised nonlinear Alfvén waves in plasma, and has been extensively studied since then. It is well-known that (1) is completely integrable (see [8, 9, 17]), and thus has infinite number of conservation laws. In particular, in this paper we will use the following three conservation laws: if $u$ is a $H^1$-solution of (1) then

$$
M_D(u) := \int_\mathbb{R} |u|^2 \, dx = M_D(u_0),
$$

$$
E_D(u) := \int_\mathbb{R} (|u_x|^2 + \frac{3}{2} |u|^2 \text{Im}(u\bar{u}_x) + \frac{1}{2} |u|^6) \, dx = E_D(u_0),
$$

$$
P_D(u) := \int_\mathbb{R} \left( \text{Im}(\bar{u}u_x) - \frac{1}{2} |u|^4 \right) \, dx = P_D(u_0).
$$

Equation (1) has been extensively studied. On the well-posedness, Hayashi and Ozawa [5, 6, 7, 14] proved local well-posedness in $H^\frac{1}{2}(\mathbb{R})$, and moreover global well-posedness for initial data in $H^1$ satisfying

$$
\int_\mathbb{R} |u_0(x)|^2 \, dx < 2\pi.
$$

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The condition above appears naturally in the sharp Gagliardo-Nirenberg inequality to ensure an apriori estimate of $H^1$-norm by mass and energy conservation. Later, local well-posedness in $H^s$ for $s \geq 1/2$ was obtained by Takaoka [15], and this result is sharp in the sense that the solution map fails to be uniformly continuous in a ball of $H^s$ if $s < 1/2$. Low regularity global well-posedness was also studied, for example, global well-posedness in $H^s(\mathbb{R})$ under (2) was obtained in [16, 2, 3] for $s > 1/2$, and finally in [11] for $s = 1/2$. On the long-time behavior and modified scattering theory, see [4] and references therein.

A natural question is whether blowup occurs for (1). To the authors' knowledge, this problem is still open. See [10] for a numerical blowup analysis on a class of DNLS. Recently, the second author [19] showed the global well-posedness in $H^1(\mathbb{R})$ under a weaker condition

$$\int_{\mathbb{R}} |u_0(x)|^2 \, dx < 4\pi,$$

improving his previous result [15]. This result shows a striking difference between DNLS and other mass critical equations like focusing generalized KdV and quintic focusing nonlinear Schrödinger equation.

The purpose of this paper is to prove the low-regularity global well-posedness under (3). The main result is

**Theorem 1.1.** The Cauchy problem (1) is global well-posed in $H^{1/2}(\mathbb{R})$ under (3).

We explain the ideas of the proof of the theorem. Inspired by [19], we derive directly an apriori estimate using the conservation laws of mass, momentum and energy as well as the sharp Gagliardo-Nirenberg inequality, and thus provide a simplified proof of the result of [19]. We do not prove by contradiction and can get a clear bound of $H^1$-norm. Then we combine it with the I-method to prove the theorem.

2. **Apriori estimate.** To prove the theorem, it suffices to control the $H^{1/2}$-norm of the solution. For convenience, we use the following gauge transformation as [2]. If $u$ is a solution to (1) with $u_0 \in H^{1/2}$, let

$$v(t, x) := e^{-\frac{3}{4} i \int_{-\infty}^x |u(t,y)|^2 \, dy} u(t, x).$$  \hspace{1cm} (4)

Then $v$ solves

$$i\partial_t v + \partial_x^2 v = \frac{i}{2} |v|^2 v_x - \frac{i}{2} v^2 \bar{v}_x - \frac{3}{16} |v|^4 v$$ \hspace{1cm} (5)

with initial data $v(0, x) = v_0(x) := e^{-\frac{3}{4} i \int_{-\infty}^x |u_0|^2 \, dy} u_0$. It’s easy to see the map $u \to v$ is a bijection in $H^{1/2}$. Indeed, by fractional Leibniz rule we get

$$\|D^{1/2} v\|_2 \lesssim \|D^{1/2} u\|_2 + \|u D^{1/2} [e^{-\frac{3}{4} i \int_{-\infty}^x |u(t,y)|^2 \, dy}]\|_2$$

$$\lesssim \|D^{1/2} u\|_2 + \|u\|_4 \|\partial_x [e^{-\frac{3}{4} i \int_{-\infty}^x |u(t,y)|^2 \, dy}]\|_{4/3} \lesssim C(\|u\|_{H^{1/2}}).$$

Here we denote $\| \cdot \|_p = \| \cdot \|_{L^p_x}$ for $1 \leq p \leq \infty$. From now on, we only consider the equation (3) and we need to control the $H^{1/2}$-norm of $v$. 

Under the gauge transformation, the conservation laws reduce to: for solution \( v \) of (5) then

\[
M(v(t)) := \|v(t)\|_{L_x^2}^2 = M(v_0), \quad \text{(mass)} \tag{6}
\]

\[
P(v(t)) := \text{Im} \int \bar{v}(t) v_x(t) \, dx + \frac{1}{4} \int \|v(t)\|^4 \, dx = P(v_0), \quad \text{(momentum)} \tag{7}
\]

\[
E(v(t)) := \|v_x(t)\|_{L_x^2}^2 - \frac{1}{16} \|v(t)\|_{L_x^6}^6 = E(v_0). \quad \text{(energy)} \tag{8}
\]

By the sharp Galiardo-Nirenberg inequality

\[
\|f\|_6^6 \leq \frac{4}{\pi^2} \|f\|_2^4 \|f_x\|_2^2, \quad \tag{9}
\]

then we get

\[
E(v) \geq \|v_x\|_2^2 (1 - \frac{1}{4\pi^2} \|v\|_2^2).
\]

Thus under the condition (2) we can get the apriori bound on \( \|v\|_{H^1} \).

However, as observed in [19] the momentum conservation for (5) plays a significant role. Inspired by [19] we derive directly a priori estimate using the momentum and the following sharp GN inequality (see [1]):

\[
\|f\|_6 \leq C_{GN} \|f\|_{L_x^4}^{8/9} \|f_x\|_{L_x^2}^{1/9}, \quad \tag{10}
\]

where \( C_{GN} = 3^{\frac{7}{6}} (2\pi)^{-\frac{1}{6}} \).

**Lemma 2.1.** If \( v \in H^1(\mathbb{R}) \) and \( v \neq 0 \), then

\[
P(v) \geq \frac{1}{4} \|v\|_4^4 (1 - \frac{1}{2\sqrt{\pi}} \|v\|_2) - \frac{4\sqrt{\pi}E(v)}{\|v\|_4^4} \|v\|_2^2. \quad \tag{11}
\]

**Proof.** Let \( u = e^{i\alpha x} v(t, x) \) with \( \alpha > 0 \) being determined later. Then

\[
|u_x|^2 = |v_x|^2 + \alpha^2 |v|^2 + 2\alpha \text{Im}(v_x \bar{v}),
\]

and thus

\[
\int \text{Im}(v_x \bar{v}) \, dx = - \frac{E(v)}{2\alpha} - \frac{\alpha M(v)}{2} + \frac{E(u)}{2\alpha}.
\]

Now by the sharp GN inequality we have

\[
E(u) = \|u_x\|_2^2 - \frac{1}{16} \|u\|_6^6 \\
\geq C_{GN}^{-18} \|u\|_{L_x^6}^{18} \|u\|_{L_x^4}^{-16} - \frac{1}{16} \|u\|_6^6 \\
= (C_{GN}^{-18} \|v\|_{L_x^4}^{12} \|v\|_{L_x^4}^{-16} - \frac{1}{16} \|v\|_6^6).
\]

Thus,

\[
P(v) \geq - \left[ \frac{1}{16} - C_{GN}^{-18} \|v\|_{L_x^6}^{12} \|v\|_{L_x^4}^{-16} \right] \frac{\|v\|_6^6}{2\alpha} + \frac{\|v\|_4^4}{4} - \frac{\alpha \|v\|_2^2}{2} - \frac{E(v)}{2\alpha} \geq - f(\|v\|_6 \|v\|_4^{-8}) \frac{\|v\|_6^8}{2\alpha} + \frac{\|v\|_4^4}{4} - \frac{\alpha \|v\|_2^2}{2} - \frac{E(v)}{2\alpha},
\]

where \( f(x) = (\frac{1}{16} - C_{GN}^{-18} x^2) x \). By calculus we know

\[
\max_x f(x) = f(\frac{C_{GN}^{\frac{9}{4\sqrt{3}}}}{96\sqrt{3}}) = \frac{C_{GN}^{\frac{9}{4\sqrt{3}}}}{96\sqrt{3}} = \frac{1}{64\pi}.
\]
Therefore
\[ P(v) \geq - \frac{\|v\|_4^4}{128\pi\alpha} + \frac{\|v\|_4^2}{4} - \frac{\alpha\|v\|_2^2}{2} - \frac{E(v)}{2\alpha}. \]
Take \( \alpha = \frac{1}{8\sqrt{\pi}\|v\|_2^{-1}} \), then \( P(v) \geq \frac{1}{4}\|v\|_4^4(1 - \frac{1}{2\sqrt{\pi}}\|v\|_2) - \frac{E(v)}{2\alpha}. \)

**Lemma 2.2.** If \( v \in H^1(\mathbb{R}) \), \( v \neq 0 \) and \( \|v\|_2^2 < 4\pi \), then
\[ \|v_x\|_{L^2} \leq 2E(v) + \frac{P(v)^2 + 2\sqrt{\pi}|E(v)|\|v\|_2}{(1 - \frac{1}{2\sqrt{\pi}}\|v\|_2)^2}. \]  
(12)

**Proof.** Let \( x = \|v\|_1^4 \). Then (11) gives an estimate of the form
\[ c \geq ax - \frac{b}{x}. \]
with \( a = \frac{1}{4}(1 - \frac{1}{2\sqrt{\pi}}\|v\|_2), \ b = 4\sqrt{\pi}|E(v)|\|v\|_2, \ c = |P(v)|. \)
(13) implies
\[ ax^2 - cx - b \leq 0. \]
Since \( a > 0 \), we get
\[ x^2 \leq \left( \frac{c + \sqrt{c^2 + 4ab}}{2a} \right)^2 \leq \frac{c^2 + 2ab}{a^2}. \]
Thus we obtain
\[ \|v\|_4^4 \leq 16(1 - \frac{1}{2\sqrt{\pi}}\|v\|_2)^{-2} \left( P(v)^2 + 2(1 - \frac{1}{2\sqrt{\pi}}\|v\|_2)\sqrt{\pi}|E(v)|\|v\|_2 \right). \]
(14)

On the other hand, by (10) and mean value inequality we have
\[ \|v_x\|_{L^2} \leq 2E(v) + 2^{-\frac{4}{3}}\|v\|_4^\frac{8}{3}. \]
(15)

Therefore by (14) and (15) we prove the lemma.

With this lemma, we can get that if \( v \) is a \( H^1 \)-solution of (9) satisfying (3), then \( \|v_x\|_2 \leq C. \) Therefore, global well-posedness of (3) in \( H^1 \) under (3) follows immediately.

3. **Proof of the main theorem.** In this section we prove Theorem 1.1 using the I-method as the previous works [3, 11]. The main difference is that we need to use the momentum conservation.

First we recall the definition of \( I \)-operator. Let \( N \gg 1 \) be fixed, and the Fourier multiplier operator \( I_N \) be defined as
\[ \hat{I_N}f(\xi) = m_N(\xi)\hat{f}(\xi). \]
(16)

Here \( m_N(\xi) \) is a smooth, radially decreasing function satisfying \( 0 < m_N(\xi) \leq 1 \) and
\[ m_N(\xi) = \begin{cases} 1, & |\xi| \leq N, \\ N^\frac{1}{2}|\xi|^{-\frac{1}{2}}, & |\xi| > 2N. \end{cases} \]
(17)

For simplicity we denote \( I_N \) by \( I \) and \( m_N \) by \( m \) if there is no confusion. \( I_N \) maps \( H^\frac{1}{2} \) to \( H^1 \), moreover, we have the following estimates,
\[ \|f\|_{H^\frac{1}{2}} \lesssim \|I_Nf\|_{H^1} \lesssim N^\frac{1}{2}\|f\|_{H^\frac{1}{2}}, \]
(18)
where the implicit constants are independent on \( N \).

Next we use the rescaling. For \( v_0 \in H^\frac{1}{2} \), let \( v \) be the solution to (5). For \( \lambda > 0 \), let
\[ v_\lambda = \lambda^{-\frac{1}{4}}v\left( \frac{x}{\lambda}, \frac{t}{\lambda^2} \right) \quad \text{and} \quad v_{0,\lambda} = \lambda^{-\frac{1}{4}}v_0\left( \frac{x}{\lambda} \right). \]
Then $v_\lambda$ is a solution of (5) with the initial data \( v_\lambda(0) = v_{0,\lambda}(x) \). Meanwhile, $v_\lambda$ exists on $[0,T]$ if and only if $v$ exists on $[0,\lambda^{-2}T]$. We have
\[
\| Iv_{0,\lambda} \|_2 \leq \| v_{0,\lambda} \|_2 = \| v_0 \|_2
\]
and
\[
\| \partial_x Iv_{0,\lambda} \|_2 \lesssim N^{1/2} \lambda^{-1/2} \| v_0 \|_{\dot{H}^{1/2}}.
\]
Thus choosing
\[
\lambda \sim N,
\]
we can make
\[
\| \partial_x Iv_{0,\lambda} \|_2 \leq \varepsilon_0 \ll 1,
\]
where $\varepsilon_0$ will be determined later.

We recall a variant local well-posedness obtained in [11].

**Lemma 3.1.** The Cauchy problem (5) is locally well-posed for the initial data $v_0$ satisfying $Iv_0 \in H^1(\mathbb{R})$. Moreover, the solution exists on the interval $[0,\delta]$ with the lifetime
\[
\delta \sim \| Iv_0 \|_{\dot{H}^{1/2}}^{-\mu}
\]
for some $\mu > 0$, where the implicit constant is independent of $N$. Furthermore, the solution satisfies the estimate
\[
\| Iv \|_{L^\infty([0,\delta] ; H^1)} \leq 2 \| Iv_0 \|_{H^1}.
\]

By the above lemma, we need to control the growth of $\| Iv_\lambda(t) \|_{H^1}$. By mass conservation we have $\| Iv_\lambda(t) \|_{L^2} \leq \| v_\lambda \|_{L^2} \leq C$. It suffices to control $\| \partial_x Iv_\lambda \|_2$. We will use (12) since $\| Iv_\lambda \|_2^2 \leq \| v_\lambda \|_2^2 = \| v_0 \|_2^2 < 4\pi$. We define the modified momentum and energy as follows
\[
P_I(v_\lambda) := P(Iv_\lambda), \quad E_I(v_\lambda) := E(Iv_\lambda).
\]
Then by (21), Hölder’s and Sobolev’s inequalities, we have
\[
P_I(v_0,\lambda) \lesssim 1; \quad E_I(v_0,\lambda) \lesssim 1
\]
Moreover,
\[
P(v_0,\lambda) = \frac{1}{\lambda} P(v_0) \sim N^{-1} P(v_0).
\]
If $N \to \infty$, $I_N$ tends to the identity operator. Thus $P_I(v_\lambda)$ and $E_I(v_\lambda)$ increase slowly in $t$ if $N$ is large enough. Indeed, in the previous works the growth of $E_I(v_\lambda)$ was already studied. Collecting the results obtained in [11] (see Section 7), we have

**Lemma 3.2.** Suppose that for $T > 0$
\[
\sup_{t \in [0,T]} \| Iv_\lambda \|_{H^1} \lesssim 1,
\]
then the modified energy $E_I(v_\lambda)$ obeys the following estimate: there exist $C, \alpha > 0$ such that for any $t \in [0,T]$ and any $\varepsilon > 0$
\[
|E_I(v_\lambda(t))| \leq \| \partial_x Iv_{0,\lambda} \|_{L^2}^2 + CN^{-\alpha} \sup_{\tau \in [0,t]} \left( \| Iv_\lambda(\tau) \|_{H^1}^4 + \| Iv_\lambda(\tau) \|_{\dot{H}^1}^6 \right)
+ C\varepsilon N^{-\frac{\alpha}{2} + \varepsilon} \sup_{\tau \in [0,t]} \left( \| Iv_\lambda(\tau) \|_{\dot{H}^1}^6 + \| Iv_\lambda(\tau) \|_{H^1}^6 \right).
\]
Lemma 3.3. We have
\[ |P_I(v_\lambda) - P(v_\lambda)| \lesssim N^{-1}(\|Iv_\lambda\|_{H^1}^2 + \|Iv_\lambda\|_{H^1}^4) \]

Proof. By the definition of momentum, we need to bound
\[ |\text{Im} \int_R (\bar{v}_\lambda \partial_x I v_\lambda - \bar{v}_\partial_x v_\lambda) \, dx| + |\int |Iv_\lambda|^4 \, dx - \int |v_\lambda|^4 \, dx| := I + II. \]

For the first term $I$, since
\[ \text{Im} \int_R \left(I\bar{v}_\lambda \partial_x I v_\lambda - \bar{v}_\partial_x v_\lambda\right) \, dx = \text{Im} \int_R \left(I\partial_x v_\lambda - \partial_x v_\lambda\right)(I\bar{v}_\lambda + \bar{v}_\lambda) \, dx, \]
and $P_{\leq N}(I\partial_x v_\lambda - \partial_x v_\lambda) = 0$, then we get
\[ I \lesssim \|I\partial_x v_\lambda - \partial_x v_\lambda\|_{H^{-\frac{1}{2}}} \left(\|P_{\geq N} I\bar{v}_\lambda\|_{H^{\frac{1}{2}}} + \|P_{\geq N} \bar{v}_\lambda\|_{H^{\frac{1}{2}}}\right). \]

By the definition of $I$-operator, we have
\[ \|I\partial_x v_\lambda - \partial_x v_\lambda\|_{H^{-\frac{1}{2}}} + \|P_{\geq N} I\bar{v}_\lambda\|_{H^{\frac{1}{2}}} + \|P_{\geq N} \bar{v}_\lambda\|_{H^{\frac{1}{2}}} \lesssim N^{-\frac{1}{2}}\|Iv_\lambda\|_{H^1}, \]
and thus
\[ I \lesssim N^{-1}\|Iv_\lambda\|_{H^1}^2. \]

For the second term $II$, we have
\[ \int |Iv_\lambda|^4 \, dx - \int |v_\lambda|^4 \, dx = \int (Iv_\lambda - v_\lambda) P_{\geq N}(v_\lambda^3) \, dx + \text{similar terms.} \quad (27) \]

Using the Hölder inequality, the Sobolev embedding, and the fractional Leibniz inequalities, we get
\[ \left| \int (Iv_\lambda - v_\lambda) P_{\geq N}(v_\lambda^3) \, dx \right| \lesssim \|Iv_\lambda - v_\lambda\|_6 \|P_{\geq N}(v_\lambda^3)\|_{\frac{6}{5}} \]
\[ \lesssim \|Iv_\lambda - v_\lambda\|_{H^{\frac{1}{2}}} N^{-\frac{1}{2}} \|\nabla\|_{\frac{6}{5}}(v_\lambda^3)\|_{\frac{6}{5}} \]
\[ \lesssim N^{-1}\|Iv_\lambda\|_{H^1} \|\nabla\|_{\frac{6}{5}}(v_\lambda^3)\|_{\frac{6}{5}} \]
\[ \lesssim N^{-1}\|Iv_\lambda\|_{H^1}^3 \|v_\lambda\|_{H^{\frac{1}{2}}}^3 \]
\[ \lesssim N^{-1}\|Iv_\lambda\|_{H^1}^3. \]

The similar terms in (27) can be handled in the same way. Thus we prove the lemma. 

By Lemma 3.2 and the mass conservation law $\|v_\lambda\|_2 \leq C$ we have under the assumption (25),
\[ E_I(v_\lambda(t)) \leq \|\partial_x I v_{0,\lambda}\|_2^2 + C N^{-\alpha} \sup_{\tau \in [0,t]} \left(\|\partial_x I v_\lambda(\tau)\|_2^4 + \|\partial_x I v_\lambda(\tau)\|_2^6 + 1\right) \]
\[ + C t N^{-\frac{5}{2} + \epsilon} \sup_{\tau \in [0,t]} \left(\|\partial_x I v_\lambda(\tau)\|_2^6 + \|\partial_x I v_\lambda(\tau)\|_2^{10} + 1\right). \quad (28) \]

Note that (21). We will prove by continuity argument that for $T \leq T_0 := N^{\frac{5}{2} - 2\epsilon}$,
\[ \sup_{t \in [0,T]} \|\partial_x I v_\lambda(t)\|_2 \leq 4\gamma_0 \varepsilon_0, \]
where $\gamma_0 = \sqrt{1 + \frac{\sqrt{2\varepsilon_0}}{\sqrt{1 - (\frac{\varepsilon_0}{2\gamma_0})^2}}}$. We choose $\varepsilon_0 \ll 1$ such that $100\gamma_0 \varepsilon_0 < 1$. 

Assuming (29), we get that the solution \( v_{\lambda} \) exists on \([0, T_0]\). Hence, \( v \) exists on \([0, \lambda^{-2}T_0]\). Note that
\[
\lambda^{-2}T_0 \sim N^{-2}N^{\frac{1}{2}-2\epsilon} = N^{\frac{1}{2}-2\epsilon}.
\]
Therefore, we get that \( v \) exists till arbitrarily large \( T \) by choosing sufficient large \( N \), and thus completes the proof of Theorem 1.1.

It remains to prove (29). It is obvious from Lemma 3.1 that (29) holds when \( T = \delta \). We may assume \( \sup_{t \in [0, T]} \| \partial_x \lambda v_{\lambda}(t) \|_2 \leq 8\gamma_0 \varepsilon_0 \ll 1 \). Then the estimate (28) gives
\[
|E_I(v_{\lambda}(t))| \leq \varepsilon_0^2 + CN^{-\epsilon}, \quad t \leq T.
\]
(30)

On the other hand, by Lemma 3.3 we have
\[
|P_I(v_{\lambda}(t))|^2 \leq 2|P_I(v_{\lambda}(t)) - P(v_{\lambda}(t))|^2 + 2|P(v_{\lambda}(t))|^2 \leq CN^{-2}.
\]
(31)

By (12), we have
\[
\| \partial_x \lambda v_{\lambda}(t) \|^2 \leq 2E(Iv_{\lambda}(t)) + \frac{P(Iv_{\lambda}(t))^2 + 2\sqrt{\pi} |E(Iv_{\lambda}(t))| \| v_0 \|_2}{(1 - \frac{1}{2\sqrt{\pi}} \| v_0 \|_2)^2}
\]
\[
\leq 2\gamma_0^2 (\varepsilon_0^2 + CN^{-\epsilon}) + CN^{-2} (1 - \frac{1}{2\sqrt{\pi}} \| v_0 \|_2)^{-2}.
\]
Choosing \( N \) sufficiently large, we get \( \| \partial_x \lambda v_{\lambda}(t) \|_2 \leq 4\gamma_0 \varepsilon_0 \) for \( t \leq T \). By continuity argument we obtain (29) as desired.

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