Many-body problem in Kaluza-Klein models with toroidal compactification

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Received: date / Accepted: date

Abstract

In this paper, we consider a system of gravitating bodies in Kaluza-Klein models with toroidal compactification of extra dimensions. To simulate the astrophysical objects (e.g., our Sun and pulsars) with energy density much greater than pressure, we suppose that these bodies are pressureless in the external/our space. At the same time, they may have nonzero parameters $\omega(\alpha-3)$ of the equations of state in the extra dimensions. We construct the Lagrange function of this many-body system for any value of $\Sigma = \sum_{\alpha} \omega(\alpha-3)$. Moreover, the gravitational tests (PPN parameters, perihelion and periastron advances) require negligible deviation from the latent soliton value $\Sigma = -(D-3)/2$. However, the presence of pressure/tension in the internal space results necessarily in the smearing of the gravitating masses over the internal space and in the absence of the KK modes. This looks very unnatural from the point of quantum physics.

Keywords: extra dimensions · Kaluza-Klein models · toroidal compactification · tension · black strings · black branes · many-body problem

PACS: 04.25.Nx · 04.50.Cd · 04.80.Cc · 11.25.Mj

1 Introduction

The idea of multidimensionality of our Universe demanded by the theories of unification of the fundamental interactions is one of the most breathtaking ideas of theoretical physics. It takes its origin from the pioneering papers by Th. Kaluza and O. Klein \cite{1}, and now the most self-consistent modern theories of unification such as superstrings, supergravity and M-theory are constructed in spacetimes with extra dimensions (see, e.g., \cite{2}). Different aspects of the idea of multidimensionality are intensively used in numerous modern articles.

Therefore, it is important to find experimental evidence for the existence of the extra dimensions. For example, one of the aims of Large Hadronic Collider consists in detecting of Kaluza-Klein (KK) particles which correspond to excitations of the internal spaces (see, e.g., \cite{3}). Such excitations were investigated in a lot of articles (see, e.g., the classical papers \cite{4,5,6}). Quite recently, KK particles were considered, e.g., in the papers \cite{7,8}.

On the other hand, if we can show that the existence of the extra dimensions is contrary to observations, then these theories are prohibited.

Much work was done in this direction including the models with toroidal compactification. Obviously, any gravitational theory modified with respect to the General Relativity (GR) can result in some observable deviations from GR. A number of papers were devoted to the search of such deviations. For example, the nonrelativistic gravitational potentials in these theories can be different from the Newtonian potentials \cite{9,10,11,12,13,14,15}. In principle, this difference can be experimentally observed \cite{16}. Parameterized Post-Newtonian formalism is a powerful tool for the determination of gravitational theories consistent with experiments \cite{17,18}.

The relation with particle physics is another important point of KK models. It was shown that multidimensional models can give a reasonable explanation of the hierarchy problem \cite{9,10}. Then, it was indicated...
that such framework can be embedded in the string theory [19]. On the other hand, the interaction between KK states and ordinary matter can result in new observable channels of reactions [9,10,19,20,21,22,23].

In our previous papers [18,24,25] devoted to KK models with toroidal compactification of the extra dimensions, we have shown that gravitating masses should have tension in the internal space to be in agreement with gravitational experiments in the Solar system. For example, black strings/branes with the parameter $\omega = -1/2$ of the equation of state in the internal space satisfy this condition. For this value of $\omega$, the variations of the internal space volume are absent [26]. In the dust-like case with $\omega = 0$, such variations generate the fifth force, that leads to contradictions with the experimental data.

It is worth noting that black strings/branes generalize the known Schwarzschild solution to the multidimensional case (see, e.g., [27,28,29,30] and the corresponding literature therein). Obviously, any multidimensional theory should have such solutions, as they must correspond to the observed astrophysical objects. Black strings/branes have toroidal compactification of the internal spaces. This compactification type is the simplest among the possible ones. However, it makes sense to investigate such models because they may help to reveal new important properties for more physically reliable multidimensional models. The ADD model [9] presents a good example of it. Even if the authors use the localization of the Standard model fields on a brane, they explore the toroidal compactification of the internal space to get the relation between the multidimensional and four-dimensional gravitational constants [10]. That gives a possibility to solve the hierarchy problem and to introduce the notion of large extra dimensions. We will not use the brane approach for our model remaining within the standard Kaluza-Klein theory. However, even in this case the large extra dimensions can be achieved for KK models with toroidal compactification [15].

The main purpose of this paper is to construct the Lagrange function for a many-body system in the case of models with toroidal compactification. We need such theory e.g. to calculate the formula for advance of periastron in the case of a binary system. The measurement of this advance for the pulsar PSR B1913+16 was performed with very high accuracy. Therefore, such measurements can be a very good test for gravitational theories. From our previous papers [18,24,25] we know that gravitating bodies should have pressure/tension in the extra dimensions to satisfy the observable data for the deflection of light and the experimental restrictions for the parameterized post-Newtonian parameter (PPN) $\gamma$.

In this regard, the question arises about the possibility of building a many-body Lagrange function in the presence of pressure/tension in the extra dimensions. To answer this question, we need the metrics components $g_{00}$ up to $O(1/c^4)$, $g_{0\alpha}$ up to $O(1/c^3)$ and $g_{\alpha\beta}$ up to $O(1/c^2)$. It is worth noting that for the expressions of the deflection of light and PPN parameter $\gamma$, it is sufficient to calculate the metrics coefficients up to $O(1/c^2)$. Obviously, the agreement with observations up to $O(1/c^2)$ does not guarantee the agreement up to $O(1/c^4)$. Hence, we calculate the metrics coefficients in the required orders $1/c$. We demonstrate that the many-body Lagrange function can be constructed for any value of $\Sigma$ where $\Sigma$ is a sum of the parameters of the equations of state in the extra dimensions. We demonstrate that the gravitational tests (PPN parameter $\gamma$, and perihelion/periastron advance) allow very small deviation from the latent soliton value $\Sigma = -(D-3)/2 \neq 0$. We prove that nonzero $\Sigma$ leads necessarily to the uniform smearing of the gravitating masses over the internal space. However, uniformly smeared gravitating bodies cannot have excited KK states (KK particles). As we mentioned above, KK particles were recently considered in the papers [7,8]. Here, the metric and form-field perturbations are studied without taking into account the reason of such fluctuations. Our present analysis clearly shows that the inclusion of the matter sources, being responsible for the perturbations, imposes strong restrictions on the model, e.g., leading to the absence of KK particles. Until now, KK particles were not detected in experiments at LHC. So, it looks tempting to interpret their absence in the light of our paper (i.e. due to the smearing of the gravitating particles over the internal space). However, the absence of KK particles looks rather unnatural from the point of quantum mechanics and statistical physics (see below). Therefore, in our opinion, this is a big disadvantage of the Kaluza-Klein models with the toroidal compactification.

The paper is structured as follows. In Sec. 2, we obtain the $1/c^2, 1/c^3$ and $1/c^4$ correction terms to the metric coefficients for the considered many-body system. In Sec. 3, we demonstrate that gauge conditions lead to the uniform smearing of the gravitating bodies over the extra dimensions. The Lagrange function for the many-body system is constructed in the Sec. 4. The formulas for PPN parameters $\beta, \gamma$ and perihelion and periastron advances are calculated in Section 5. These formulas allow us to obtain experimental constraints on the parameters of the model. The main results are summarized in concluding Sec. 6.
2 Metric coefficients in the weak field approximation

To construct the Lagrange function of a system of $N$ massive bodies in $(D + 1)$-dimensional spacetime, we define first the nonrelativistic gravitational field created by this system. To do it, we need to get the metric coefficients in the weak field limit. The general form of the multidimensional metrics is

$$ds^2 = g_{ik} dx^i dx^k = g_{00} (dx^0)^2 + 2g_{0i} dx^0 dx^i + g_{ij} dx^i dx^j,$$

(1)

where the Latin indices $i, k = 0, 1, \ldots, D$ and the Greek indices $\mu, \nu = 1, \ldots, D$. We make the natural assumption that in the case of the absence of matter sources the spacetime is Minkowski spacetime: $g_{00} = \eta_{00} = 1, g_{0\mu} = \eta_{0\mu} = 0, g_{\mu\nu} = \eta_{\mu\nu} = -\delta_{\mu\nu}$. In our paper, we consider in detail the case where the extra dimensions have the topology of tori. In the presence of matter, the metrics is not the Minkowskian one, and we investigate it in the weak field limit. It means that the gravitational field is weak and velocities of test bodies are small compared with the speed of light $c$. In the weak field limit the metrics is only slightly perturbed from its flat spacetime value. We will define the metrics (11) up to 1/$c^2$ correction terms. Because the coordinate $x^0 = ct$, the metric coefficients can be expressed as follows:

$$g_{00} \approx 1 + h_{00} + f_{00}, \quad g_{0\mu} \approx h_{0\mu} + f_{0\mu}, \quad g_{\mu\nu} \approx -\delta_{\mu\nu} + h_{\mu\nu},$$

(2)

where $h_{ik} \sim O(1/c^2), f_{00} \sim O(1/c^4)$ and $f_{0\mu} \sim O(1/c^3)$. In particular, $h_{00} \equiv 2\varphi/c^2$ where $\varphi$ is the nonrelativistic gravitational potential. To get these correction terms, we should solve (in the corresponding orders of 1/c) the multidimensional Einstein equation

$$R_{ik} = \frac{2S_D \tilde{G}_D}{c^4} \left( T_{ik} - \frac{1}{D-1} g_{ik} T \right),$$

(3)

where $S_D = 2\pi^{D/2}/\Gamma(D/2)$ is the total solid angle (the surface area of the $(D - 1)$-dimensional sphere of the unit radius), $\tilde{G}_D$ is the gravitational constant in the $(D = D + 1)$-dimensional spacetime. We consider a system of $N$ discrete massive (with rest masses $m_p, p = 1, \ldots, N$) bodies. We suppose that the pressure of these bodies in the external three-dimensional space is much less than their energy density. This is a natural approximation for ordinary astrophysical objects such as our Sun. For example, in general relativity, this approach works well for calculating the gravitational experiments in the Solar system [31]. In the case of pulsars, pressure is not small but still much less than the energy density, and the pressureless approach was used in General Relativity to get the formula of the periastron advance [32]. Therefore, the gravitating bodies are pressureless in the external/our space. On the other hand, we suppose that they may have pressure in the extra dimensions. Therefore, nonzero components of the energy-momentum tensor of the system can be written in the following form:

$$T^{ik} = \rho c^2 u^i u^k, \quad i, k = 0, \ldots, 3,$$

$$T^{\alpha\beta} = \rho c^2 u^\alpha u^\beta, \quad \alpha = 0, \ldots, 3; \quad \beta = 4, \ldots, D,$$

(4)

and we introduced the rest-mass density $\rho(x) = \sum_{p=1}^{N} m_p \delta(x - x_p)$.

$$\rho(x) = \sum_{p=1}^{N} m_p \delta(x - x_p).$$

(11)
Then, from the Einstein equation we get

\begin{align}
  h_{00} &= \frac{2\varphi(x)}{c^2}, \quad h_{0\mu} = 0, \\
  h_{\alpha\beta} &= \frac{1 - \Sigma}{D - 2 + \Sigma} \frac{2\varphi(x)}{c^2} \delta_{\alpha\beta}, \\
  h_{\beta\beta} &= \frac{\omega_{(\alpha-3)}(D-1) + 1 - \Sigma}{D - 2 + \Sigma} \frac{2\varphi(x)}{c^2} \delta_{\alpha\beta},
\end{align}

where the function \( \varphi(x) \) satisfies the D-dimensional Poisson equation

\[ \Delta_D \varphi(x) = 2\tilde{S}_D \tilde{G}_D \frac{D - 2 + \Sigma}{D - 1} (\rho(x)). \]

We would remind that \( x \) is a D-dimensional radius-vector. It is worth noting that if \( \omega_{(\alpha-3)} = 0, \forall \alpha \Rightarrow \Sigma = 0 \), then we reproduce the results of the paper \[18\]. On the other hand, if all \( \omega_{(\alpha-3)} = -1/2 \), then \( h_{\alpha\beta} = 0 \) that should take place for black strings/branes \[25\].

Next, we should obtain the \( O(1/c^4) \) and \( O(1/c^3) \) metric correction terms \( f_{00} \) and \( f_{0\mu} \), respectively. In this case, the energy-momentum components read

\begin{align}
  T_{00} &\approx \rho c^2 \left( 1 + \frac{\varphi}{c^2} \right), \\
  T_{0\mu} &\approx -\rho c^2 v^\mu, \\
  T_{\alpha\beta} &\approx \rho v^\alpha v^\beta, \quad T_{\beta\beta} \approx \rho v^\alpha v^\beta,
\end{align}

and the trace

\[ T \approx \rho c^2 (1 - \Sigma) + \rho \varphi \frac{D - \Sigma}{D - 2 + \Sigma} + \rho (\Sigma - 1) \frac{v^2}{2}. \]

Then, from the Einstein equation we get

\begin{align}
  f_{00}(x) &= \frac{2}{c^2} \varphi^2(x) + \frac{2}{c^2} \sum_p \varphi_p(x - x_p) \varphi'(x_p) \\
  &+ \frac{1}{c^2} \frac{D - \Sigma}{D - 2 + \Sigma} \sum_p \varphi_p(x - x_p) v_p^2, \\
  f_{0\mu}(x) &= -\frac{2}{c^2} \frac{D - 1}{D - 2 + \Sigma} \sum_p \varphi_p(x - x_p) v_p^\mu - \frac{1}{c^2} \partial^2 f / \partial x^\mu, \quad \text{and}
\end{align}

where the function \( f \) satisfies the following equation:

\[ \triangle_D f = \varphi(x). \]

3. Gauge conditions and smearig

It should be noted that to calculate the Ricci tensor components in the corresponding orders of \( 1/c \), we used the standard (see, e.g., Eq. (105.10) in [31]) gauge condition

\[ \partial_k \left( h^k_l - \frac{1}{2} h^k_0 \delta^k_l \right) = 0, \quad i, k = 0, 1, \ldots, D, \]

where \( h^k_l = \eta^{km} h_{mi} \). Hence,

\[ h^0_0 = \eta^{00} h_{00} = h_{00}, \quad h^0_\mu = \eta^{0\mu} h_{0\mu} = -h_{\mu 0}. \]

Therefore,

\[ h^0_0 = \frac{2\varphi(x)}{c^2}, \quad h^0_\beta = -\frac{1 - \Sigma}{D - 2 + \Sigma} \frac{2\varphi(x)}{c^2} \delta^0_\beta, \]

\[ h^\beta_\beta = -\frac{\omega_{(\alpha-3)}(D-1) + 1 - \Sigma}{D - 2 + \Sigma} \frac{2\varphi(x)}{c^2} \delta^\beta_\beta, \]

\[ h^\beta_\beta = \frac{2(\Sigma - 1)}{D - 2 + \Sigma} \frac{2\varphi(x)}{c^2}. \]

Let us check that these solutions satisfy the condition \[24\]. For \( i = 0 \), we get immediately

\[ \partial_k \left( h^k_0 - \frac{1}{2} h^k_0 \delta^k_0 \right) = \partial_0 \left( h^0_0 - \frac{1}{2} h^0_0 \right) = 0 + O \left( \frac{1}{c^4} \right). \]

For \( i = \beta \) we have

\[ \partial_k \left( h^k_\beta - \frac{1}{2} h^k_0 \delta^k_\beta \right) = \partial_\beta \left( h^\beta_\beta - \frac{1}{2} h^\beta_\beta \delta^\beta_\beta \right) \]

\[ = \left[ -1 - \frac{1 - \Sigma}{D - 2 + \Sigma} + \frac{1 - \Sigma}{D - 2 + \Sigma} \right] \frac{2}{c^2} \partial_\beta \varphi = 0, \]

that is the condition is automatically satisfied. For \( i = \beta \) we obtain

\[ \partial_k \left( h^k_\beta - \frac{1}{2} h^k_0 \delta^k_\beta \right) = \partial_\beta \left( h^\beta_\beta - \frac{1}{2} h^\beta_\beta \delta^\beta_\beta \right) \]

\[ = -\omega_{(\beta-3)}(D-1) \frac{2}{D - 2 + \Sigma} \frac{2}{c^2} \partial_\beta \varphi = 0. \]

In order to satisfy this condition, we should demand either \( \omega_{(\beta-3)} = 0 \) or \( \partial_\beta \varphi = 0 \). Because we consider the general case \( \omega_{(\beta-3)} \neq 0 \), we must choose the latter condition. Moreover, the gravitational tests require nonzero \( \omega_{(\beta-3)} \) (see Sec. 5). Therefore, the presence of nonzero pressure/tension in the extra dimensions results in the metric coefficients which do not depend on the coordinates of the internal space, i.e. the gravitating masses should be uniformly smeared over the extra dimensions. In this case, the rest mass density \( \tilde{\rho} \) should be rewritten in the form: \( \rho(x) \to \rho(r) = \sum_p m_p \delta(r - \)}
\[ \Delta_3 \varphi(r) = 4\pi G N \sum_p m_p \delta(r - r_p) \] (32)

with the solution

\[ \varphi(r) = -\sum_p G_N m_p \frac{r - r_p}{|r - r_p|} = \sum_p \varphi_p(r - r_p), \] (33)

where \( G_N \) is the Newtonian gravitational constant:

\[ 4\pi G_N = \frac{2S_D(D-2+\Sigma)}{(D-1)\prod_\alpha a_\alpha - 3} \tilde{G}_D. \] (34)

Hereafter, \( r, r_p \) are radius vectors in three-dimensional external/our space.

In the case of the smearing, Eq. (25) has the following solution

\[ f(r) = -\frac{G_N}{2} \sum_p m_p |r - r_p|, \] (35)

where, to get it, we used the well known equation \( \Delta_3 \varphi = 2/r \) in the three-dimensional flat space. Because

\[ \frac{\partial}{\partial t} \left( \frac{\varphi(r)}{|r - r_p|} \right) = \frac{\partial}{\partial t} \left( \frac{x^\alpha - x^\alpha_p}{|r - r_p|^2} \right) = \frac{1}{|r - r_p|^2} \]

\[ \times \left[ -v^\alpha_p |r - r_p| - (x^\alpha - x^\alpha_p)|r - r_p| \sum_\beta (x^\beta - x^\beta_p)(-v^\beta_p) \right], \] (36)

we get for \( f_{\alpha\alpha} \):

\[ f_{\alpha\alpha} = \frac{G_N}{2c^4} \sum_p \frac{m_p}{|r - r_p|} \left( \frac{3D - 2 + \Sigma}{D - 2 + \Sigma} v^\alpha_p + n^\alpha_p (n_p v_p) \right), \] (37)

where we introduce the three-dimensional unit vector in the direction from the \( p \)-th particle to a point with the radius vector \( r \):

\[ n^\alpha_p = \frac{x^\alpha - x^\alpha_p}{|r - r_p|}, \] (38)

and \( (n_p v_p) = \sum_\beta n^\beta_p v^\beta_p \).

It should be noted that, to get the formula (32), we used the following gauge condition:

\[ \frac{\partial f^{\alpha}_{\beta}}{\partial x^\alpha} - \frac{1}{2} \frac{\partial h^{\alpha}_{\beta}}{\partial x^2} = 0, \] (39)

where \( f^{\beta}_{\alpha} = \eta^{\alpha\beta} f_{\beta\mu} = -f_{\alpha\mu} \). In the case of smearing, this condition is reduced to

\[ \frac{\partial f^{\beta}_{\alpha}}{\partial x^\beta} - \frac{1}{2} \frac{\partial h^{\beta}_{\alpha}}{\partial x^2} = 0, \] (40)

where we remind that \( \alpha, \beta = 1, 2, 3 \) and \( \mu, \nu = 1, \ldots, D \).

Taking into account the following auxiliary equations:

\[ \frac{\partial}{\partial x^\alpha} \left( \frac{1}{|r - r_p|^2} n^\alpha_p |r - r_p|^2 \right) = \left( n_p v_p \right), \] (41)

\[ \frac{\partial}{\partial t} \left( \frac{n^\alpha_p |r - r_p|^2}{|r - r_p|^2} \right) = \left( n_p v_p \right), \] (42)

\[ \frac{\partial}{\partial t} \left( \frac{1}{|r - r_p|^2} n^\alpha_p |r - r_p|^2 \right) = \left( n_p v_p \right), \] (43)

we can easily seen that the condition (34) is satisfied:

\[ \frac{\partial f^{\alpha}_{\beta}}{\partial x^\alpha} - \frac{1}{2} \frac{\partial f^{\beta}_{\beta}}{\partial x^2} = \frac{G_N}{2c^4} \left( \frac{3D - 2 - \Sigma}{D - 2 + \Sigma} \sum_p m_p \frac{(n_p v_p)}{|r - r_p|^2} \right) - \frac{2}{D - 2 + \Sigma} \sum_p m_p \frac{(n_p v_p)}{|r - r_p|^2} = 0. \] (44)

Because the presence of pressure/tension in the extra dimensions requires the uniform smearing of the gravitating masses over the internal space, we provide the metric coefficients in this case:

\[ g_{00} \approx 1 + 2\frac{G}{c^2} \left[ \frac{2\varphi(r)}{c^2} + \frac{2\varphi^2(r)}{c^4} \right] \]

\[ + \frac{2G_N^2}{c^4} \sum_p \frac{m_p}{|r - r_p|} \sum_{q \neq p} \frac{m_q}{|r_p - r_q|} \]

\[ - \frac{D - \Sigma}{D - 2 + \Sigma} \frac{G_N}{2c^4} \sum_p \frac{m_p v^2_p}{|r - r_p|}, \] (45)

\[ g_{0\alpha} \approx \frac{3D - 2 - \Sigma}{D - 2 + \Sigma} \frac{G_N}{2c^4} \sum_p \frac{m_p v^\alpha_p}{|r - r_p|} \]

\[ + \frac{G_N}{2c^4} \sum_p \frac{m_p}{|r - r_p|} n^\alpha_p (n_p v_p), \] (46)

\[ g_{\alpha\beta} \approx \left( -1 + \frac{1}{D - 2 + \Sigma} \frac{2\varphi(r)}{c^2} \right) \delta_{\alpha\beta}, \] (47)

\[ g_{\alpha\beta} \approx \left( -1 + \frac{\omega(a - 3)(D - 1) + 1}{D - 2 + \Sigma} \frac{2\varphi(r)}{c^2} \right) \delta_{\alpha\beta}, \] (48)

where the potential \( \varphi(r) \) is given by (34).

Therefore, in this section we have shown that, to be compatible with the gravitational tests, the gravitating masses should be uniformly smeared over the internal space. This conclusion has the following important effect. Suppose that we have solved for the considered particle the multidimensional quantum Schrödinger equation and found its wave function \( \Psi(x) \). In general, this function depends on all spatial coordinates \( x = (r, y) \),
where \( \mathbf{y} \) are the coordinates in the internal space, and we can expand it in appropriate eigenfunctions of the compact internal space, i.e. in the Kaluza-Klein modes. The ground state corresponds to the absence of these particles. In this state the wave function may depend only on the coordinates \( \mathbf{r} \) of the external space. The classical rest-mass density is proportional to the probability density \( |\Psi|^2 \). Therefore, the demand that the rest-mass density depends only on the coordinates of the external space means that the particle can be only in the ground quantum state, and KK excitations are absent. This looks very unnatural from the point of quantum and statistical physics, because the nonzero temperature must result in excitations.

### 4 Lagrange function for a many-body system

Let us construct now the Lagrange function of the many-body system described above. To perform it, we will follow the procedure described in [31] (see §106). The Lagrange function of a particle \( p \) with the mass \( m_p \) in the gravitational field created by the other bodies is given by the expression

\[
L_p = -m_pc^2 \frac{ds_p}{dt} = -m_pc^2 \left( g_{00} + 2 \sum_\mu g_{0\mu} \frac{v^\mu_p}{c} + \sum_{\mu\nu} g_{\mu\nu} \frac{v^\mu_p v^\nu_p}{c^2} \right)^{1/2},
\]

where the metric coefficients are taken at \( \mathbf{r} = \mathbf{r}_p \). We should keep in mind that in the case of the smeared (over the extra dimensions) gravitating masses, the components of the velocity in the extra dimensions are equal to zero: \( v_p^{\mu} = (v_p^0, 0) \). It is convenient to rewrite the metric coefficients in the following form:

\[
g_{00} \approx 1 + \frac{1}{c^2} \gamma_1^{(1)} + \frac{1}{c^2} \gamma_0^{(2)}, \quad g_{0\alpha} \approx \frac{1}{c^2} \gamma_0^{\alpha},
\]

\[
g_{\alpha\beta} \approx -1 + \frac{1}{c^2} \gamma_0^{(\alpha)} \delta_{\alpha\beta},
\]

where the meaning of the functions \( \gamma \) is evident. Then, we get

\[
\frac{ds_p}{dt} \approx c \left\{ 1 + \frac{1}{c^2} \gamma_0^{(1)} - v^2_p \right\} \\
+ \frac{1}{2c^4} \left[ \frac{\gamma_0^{(2)}}{c^4} + 2 \sum_\alpha \gamma_0^{\alpha} v^\alpha_p + \sum_{\alpha\beta} \gamma_0^{(\alpha)} \delta_{\alpha\beta} v^\alpha_p v^\beta_p \right] \\
- \frac{1}{8c^4} \left[ \gamma_0^{(1)} - v^2_p \right]^2.
\]

Substituting the explicit form of the metric coefficients \([45]-[47]\), we obtain

\[
L_p = -m_pc^2 + \frac{m_pv^2_p}{2} + \frac{m_pv^4_p}{8c^2} + G_N \sum_s \frac{m_pm_s}{|\mathbf{r} - \mathbf{r}_s|} \\
- \frac{1}{2c^2} G_N^2 \sum_s \sum_q \frac{m_pm_q m_s}{|\mathbf{r} - \mathbf{r}_s||\mathbf{r} - \mathbf{r}_q|} \\
- \frac{1}{c^2} G_N^2 \sum_s \sum_{q \neq s} \frac{m_pm_q m_s}{|\mathbf{r} - \mathbf{r}_s||\mathbf{r}_s - \mathbf{r}_q|} \\
+ \frac{1}{2c^2} G_N \sum_s \frac{m_pm_s}{|\mathbf{r} - \mathbf{r}_s|} \left[ a(D, \Sigma) v^2_s + 2 \left( b(D, \Sigma) + \frac{1}{2} \right) v^2_p \right. \\
- \left. c(D, \Sigma)(v_s v_p) - (n_s v_s)(n_s v_p) \right].
\]

Here, we use the following abbreviations:

\[
a(D, \Sigma) \equiv \frac{D - \Sigma}{D - 2 + \Sigma}, \quad b(D, \Sigma) \equiv \frac{1 - \Sigma}{D - 2 + \Sigma},
\]

\[
c(D, \Sigma) \equiv \frac{3D - 2 - \Sigma}{D - 2 + \Sigma}.
\]

We remind that in the expression \([51]\) \( \mathbf{r} = \mathbf{r}_p \) and all infinite terms should be cast out. For our purposes, it is sufficient to consider the case of two particles. Then, for the particle “1”, we have the following expression:

\[
L_1 = f(v_1^2) + G_N \frac{m_1 m_2}{|\mathbf{r} - \mathbf{r}_2|} \frac{1}{2c^2} G_N^2 \frac{m_1 m_2^2}{|\mathbf{r} - \mathbf{r}_2|^2} \\
- \frac{1}{c^2} G_N^2 \frac{m_1^2 m_2}{|\mathbf{r} - \mathbf{r}_2||\mathbf{r}_1 - \mathbf{r}_2|} + \frac{1}{2c^2} G_N m_1 m_2 \frac{a(D, \Sigma)}{|\mathbf{r} - \mathbf{r}_2|} \frac{v_2^2}{2} \\
- \frac{1}{c^2} G_N m_1 m_2 \left[ b(D, \Sigma) + \frac{1}{2} \right] v^2_1 \\
- c(D, \Sigma)(v_1 v_2) - (n_1 v_1)(n_2 v_2),
\]

where \( f(v^2) \equiv m_1 v^2_1/2 + m_1 v^2_1/(8c^2) \) and we drop the term \(-m_1 v^2_1 \).

The total Lagrange function of the two-body system should be constructed so that it leads to the correct values of the forces \( \partial L_p / \partial \mathbf{r}_p \), acting on each of the bodies for given motion of the others \([31]\). To achieve it, we, first, will differentiate \( L_1 \) with respect to \( \mathbf{r} \), setting \( \mathbf{r} = \mathbf{r}_1 \) after that. Then, we should integrate this expression with respect to \( \mathbf{r}_1 \). Following this prescription and taking into account a useful auxiliary relation

\[
\left( -\frac{1}{2c^2} G_N^2 m_1 m_2 \frac{\partial}{\partial \mathbf{r} |\mathbf{r} - \mathbf{r}_2|^2} \right)_{\mathbf{r} = \mathbf{r}_1} \\
= -\frac{1}{2c^2} G_N^2 m_1 m_2 (m_1 + m_2) \frac{\partial}{\partial \mathbf{r}_1 |\mathbf{r}_1 - \mathbf{r}_2|^2},
\]

we get
we obtain from \( (59) \) the two-body Lagrange function
\[
L_1^{(2)} = \tilde{v}_1^2 \cdot \tilde{v}_2^2 + \frac{G_{N} m_1 m_2}{r_{12}} - \frac{G_{N} m_1 m_2 (m_1 + m_2)}{2c^2 r_{12}}
\]
\[
+ \frac{G_{N} m_1 m_2}{2c^2 r_{12}} \left[ a(D, \Sigma) v_1^2 + (2b(D, \Sigma) + 1) v_2^2 - c(D, \Sigma)(v_1 v_2) - (n_{12} v_1)(n_{12} v_2) \right],
\]
and find the experimental limitations on \( \varepsilon \).

**PPN parameters**

To get the parameterized post-Newtonian parameters (PPN) \( \beta \) and \( \gamma \), we consider the case of one particle at rest. Then, we can easily obtain from Eqs. \( (15) \) and \( (17) \) that
\[
\beta = 1, \quad \gamma = \frac{1 - \Sigma}{D - 2 + \Sigma},
\]
i.e. PPN parameter \( \beta \) exactly coincides with the value in the General Relativity. There are strong experimental restrictions on the value of \( \gamma \). The tightest constraint on \( \gamma \) comes from the Shapiro time-delay experiment using the Cassini spacecraft, namely: \( \gamma - 1 = (2.1 \pm 2.3) \times 10^{-5} \) \([32,33,34]\). In our case
\[
\gamma - 1 \approx - \frac{4 \varepsilon}{D - 1}.
\]
Therefore, the Shapiro time-delay experiment results in the following limitation:
\[
|\varepsilon| \lesssim \frac{D - 1}{2} \times 10^{-5}.
\]

**Perihelion shift of the Mercury**

For a test body orbiting around the gravitating mass \( m \), the perihelion shift for one period is given by the formula \([32,35]\)
\[
\delta \psi = \frac{1}{3} (2 + 2\gamma - \beta) \frac{6\pi G_{N} m}{c^2 a(1 - e^2)} \equiv \frac{1}{3} (2 + 2\gamma - \beta) \delta \psi_{GR},
\]
with \( a \) and \( e \) being the semi-major axis and the eccentricity of the ellipse, respectively. \( \delta \psi_{GR} \) is the value for General Relativity. In the case of Mercury this calculated value is equal to 42.98 arcsec per century \([32,36]\). This predicted relativistic advance agrees with the observations to about 0.1% \([32]\). Substituting the PPN parameters \([59]\) in this formula, we obtain the advance in our case:
\[
\delta \psi = \frac{1}{3} \frac{D - \Sigma}{3D - 2 + \Sigma} \delta \psi_{GR} \approx \left( 1 - \frac{8}{3(D - 1)} \varepsilon \right) \delta \psi_{GR}.
\]
Obviously, to be in agreement with the observation no worse than General Relativity, the parameter \( \varepsilon \) should satisfy the condition
\[
|\varepsilon| \lesssim \frac{3(D - 1)}{8} \times 10^{-3}.
\]
Therefore, this limitation is less strong than \([61]\).

\( \Sigma = - \frac{D - 3}{2} + \varepsilon \) and find the experimental limitations on \( \varepsilon \).
Periastron shift of the relativistic binary pulsar PSR B1913+16

Much more strong limitation can be found from the measurement of the periastron shift of the relativistic binary pulsar. First, the advance of periastron in these systems in many orders of magnitude bigger than for the Mercury. Second, the measurements are extremely accurate. For example, for the pulsar PSR B1913+16 the shift is 4.226598 ± 0.000005 degree per year [37]. For such system both the pulsar and companion have comparable masses. In the case of General Relativity, a solution for orbital parameters yields mass estimates for the pulsar and its companion, $m_1 = 1.4398 ± 0.0002M_⊙$ and $m_2 = 1.3886 ± 0.0002M_⊙$, respectively. It is worth noting that these are calculated values (not observable!) which are valid for General Relativity. Because two bodies have comparable masses (and one of them cannot be considered as a test body), to get a formula for the advance we need a two-body Lagrangian. Then, following the problem 3 in §106 [31] we get for our two-body Lagrangians [55] and [56] the desired formula in the form of [63] with the well known General Relativity expression

$$\delta \psi_{GR} = \frac{6\pi G_N (m_1 + m_2)}{c^2 a (1 - c^2)}.$$  

(65)

In future, independent measurements of masses $m_1$ and $m_2$ will allow us to obtain a high accuracy restriction on parameter $\varepsilon$.

6 Summary

In this paper, we have constructed the Lagrange function for a two-body system in the case of Kaluza-Klein models with toroidal compactification of the extra dimensions. The case of more than two bodies is straightforward. We supposed that gravitating bodies are pressureless in the external/our space. This is a natural approximation for ordinary astrophysical objects such as our Sun. For example, this approach works well for calculating the gravitational experiments in the Solar system [31]. In the case of pulsars, pressure is not small but still much less than the energy density. Hence, the pressureless approach is used in General Relativity to get the formula [65] which is in very good agreement with the observations of advance of periastron of the pulsar PSR B1913+16.

With respect to the internal space, we supposed that gravitating masses may have nonzero parameters $\omega_{(\alpha - 3)} (\alpha = 4, \ldots, D)$ of the equations of state in the extra dimensions. We have shown that the Lagrange function of this many-body system can be constructed for any value of the parameter $\Sigma = \sum_\alpha \omega_{(\alpha - 3)}$.

To construct the many-body Lagrangian, as well as to get the formulas for the gravitational tests, we obtained the metrics components $g_{00}$ up to $O(1/c^4)$, $g_{0\alpha}$ up to $O(1/c^3)$ and $g_{\alpha\beta}$ up to $O(1/c^2)$. These expressions exactly coincide with the corresponding formulas in General Relativity for the value $\Sigma = \sum_\alpha \omega_{(\alpha - 3)} = -(D - 3)/2$. This is the latent soliton case [25]. Black strings/branes are particular cases of it with all $\omega_{(\alpha - 3)} = -1/2 \varepsilon \alpha$. Obviously, the known gravitational tests (PPN parameters, perihelion/periastron shift) in this case give the same results as for General Relativity. On the other hand, we used these tests to get the restrictions on the deviation from the latent soliton value. At the present, the most strong restriction follows from the time delay of radar echoes (the Cassini spacecraft mission). The two-body Lagrange function allowed us to get the formula for the advance of the periastron. In future, when the masses of the binary pulsar system PSR B1913+16 will be measured (rather than calculated using the formula of General Relativity), the advance of this periastron can be used to get the restriction with very high accuracy. All obtained limitations indicate very small deviation from the latent soliton value. Therefore, the pressureless case $\Sigma = 0$ in the internal space is forbidden, in full agreement with the results of the paper [18]. This conclusion does not depend on the size of extra dimensions. The physical reason of it is that in the case of toroidal compactification, only in the case of latent solitons the variations of the total volume of the internal space are absent [26].

One more important result obtained in this paper is worth noting. As we have shown above (see also [24, 25, 26]), tension in the internal spaces is the necessary condition to satisfy the gravitational experiments in KK models with toroidal compactification. In our paper, we have proven that the presence of pressure/tension in the internal space leads necessarily to the uniform smearing of the gravitating masses over the internal space. For example, black strings/branes have tension in the internal space (see, e.g., [38]). Therefore, they should be smeared. However, uniformly smeared gravitating bodies cannot have excited KK states (KK particles), which looks unnatural from the point of quantum mechanics and statistical physics. In our opinion, this is a big disadvantage of the Kaluza-Klein models with the toroidal compactification. It is of interest to check this property for models with other types of compactification (e.g. Ricci-flat, spherical). This is the subject of our subsequent study.
Acknowledgments

This work was supported in part by the "Cosmomicro-
physics-2" programme of the Physics and Astronomy
Division of the National Academy of Sciences of Ukraine.
The work of M. Eingorn was supported by NSF CREST
award HRD-0833184 and NASA grant NNX09AV07A.

References

1. Th. Kaluza, Zum Unitätsproblem der Physik, Sitzungsber.
d. Preuss. Akad. d. Wiss. (1921) 966; O. Klein, Quanten-
theorie und fünfdimensionale Relativitätstheorie, Zeitschrift
für Physik 37, 895 (1926).

2. J. Polchinski, String Theory, Volume 2: Superstring The-
ory and Beyond (Cambridge University Press, Cambridge,
1998).

3. G. Bhattacharyya, A. Datta, S.K. Majee and A.
Raychaudhuri, Nucl. Phys. B 821, 48 (2009); [hep-
ph/0904.0937].

4. A. Salam and J. Strathdee, On Kaluza-Klein theory
Annals Phys. 141, 316 (1982).

5. P. van Nieuwenhuizen, The complete mass spectrum of D
= 11 supergravity compactified on S(4) and a general mass
formula for arbitrary cosets M(4), Class. Quant. Grav.
2, 1 (1985).

6. H. Kim, L. Romans and P. van Nieuwenhuizen, The mass
spectrum of chiral N = 2 D = 10 supergravity on S(5), Phys.
Rev. D 32, 389 (1985).

7. K. Hinterbichler, J. Levin and C. Zukowski, Kaluza-Klein
towers on general manifolds (2013); [hep-th/1310.6353].

8. A.R. Brown and A. Dahlen, Stability and spectrum
of compactifications on product manifolds (2013); [hep-
th/1310.6360].

9. N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys.
Lett. B 429, 263 (1998); [hep-ph/9803315].

10. N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys.
Rev. D 59, 086004 (1999); [hep-ph/9807344].

11. A. Kehagias and K. Sfetsos, Phys. Lett. B 472, 39 (2000);
[hep-th/9905417].

12. A.O. Barvinsky and S.N. Solodukhin, Nucl.Phys. B 675,
159 (2003); [hep-th/0307011].

13. P. Callin and C.P. Burgess, Nucl.Phys. B 752, 60 (2006);
[hep-ph/0511216].

14. M. Eingorn and A. Zhuk, Phys. Rev. D 80, 124037 (2009);
[hep-th/0907.5371].

15. M. Eingorn and A. Zhuk, Class. Quant. Grav. 27, 055002
(2010); [gr-qc/0910.3507].

16. C.D. Hoyle et al, Phys. Rev. D 70, 042004 (2004);
[hep-ph/0405262].

17. P. Xu and Y. Ma, Phys. Lett. B 656, 165 (2007); [gr-
qc/0710.3677].

18. M. Eingorn and A. Zhuk, Class. Quant. Grav. 27, 205014
(2010); [gr-qc/1003.5690].

19. I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G.
Dvali, Phys. Lett. B 456, 257 (1999); [hep-ph/9804398].

20. G.F. Giudice, R. Rattazzi and J.D. Wells, Nucl. Phys. B
544, 3 (1999); [hep-ph/9811291].

21. T. Han, J.D. Lykken and R.-J. Zhang, Phys. Rev. D 59,
105006 (1999); [hep-ph/9811350].

22. J. Lykken and S. Nandi, Phys. Lett. B 485, 224 (2000);
[hep-ph/9908505].

23. S. Hannestad and G. Raffelt, Phys. Rev. Lett. 87, 051301
(2001); [hep-ph/0105201].

24. M. Eingorn and A. Zhuk, Phys. Rev. D 83, 044005 (2011);
[arXiv:1010.5740].

25. M. Eingorn, O. de Medeiros, L. Crispino and A. Zhuk,
Phys. Rev. D 84, 024031 (2011); [gr-qc/1101.3910].

26. M. Eingorn and A. Zhuk, Phys. Lett. B 713, 154 (2012);
[gr-qc/1201.1756].

27. J. Traschen and D. Fox, Class. Quant. Grav. 21, 289
(2004); [gr-qc/0403106].

28. P.K. Townsend and M. Zamaklar, Class. Quant. Grav.
18, 5265 (2001); [hep-th/0107228].

29. T. Harmark and N.A. Obers, JHEP 0405, 043 (2004);
[hep-th/0403103].

30. D. Kastor and J. Traschen, JHEP 0609 (2006) 022-039;
[arXiv:hep-th/0607051].

31. L.D. Landau and E.M. Lifshitz, The Classical Theory of
Fields, Fourth Edition: Volume 2 (Course of Theoretical
Physics Series) (Pergamon Press, Oxford, 2000).

32. C.M. Will, Was Einstein Right? Testing Relativity at the
Centenary. In 100 Years of Relativity: Space-time Structure:
Einstein and Beyond, edited by A. Ashtekar (World
Scientific, Singapore, 2005); [gr-qc/0504080].

33. Bh. Jain and J. Khoury, Cosmological Tests of Gravity
(2010); [astro-ph/1004.3294].

34. B. Bertotti, L. Iess, and P. Tortora, Nature 425, 374
(2003).

35. C.M. Will Theory and Experiment in Gravitational Physics
(Cambridge University Press, Cambridge, 2000).

36. A.M. Nobili and C.M. Will, Nature 320 (1986) 39.

37. J.M. Weisberg, D.J. Nice and J.H. Taylor, Astrophys. J.
722 (2010) 1630; arXiv:astro-ph/1011.0718.

38. D. Kastor and J. Traschen, JHEP 0609, 022 (2006);
[hep-th/0607051].

39. J. Traschen and D. Fox, Class. Quant. Grav. 21, 289
(2004); [gr-qc/0403106].

40. P.K. Townsend and M. Zamaklar, Class. Quant. Grav.
18, 5265 (2001); [hep-th/0107228].

41. T. Harmark and N.A. Obers, JHEP 0405, 043 (2004);
[hep-th/0403103].

42. D. Kastor and J. Traschen, JHEP 0609 (2006) 022-039;
[arXiv:hep-th/0607051].

43. L.D. Landau and E.M. Lifshitz, The Classical Theory of
Fields, Fourth Edition: Volume 2 (Course of Theoretical
Physics Series) (Pergamon Press, Oxford, 2000).

44. C.M. Will, Was Einstein Right? Testing Relativity at the
Centenary. In 100 Years of Relativity: Space-time Structure:
Einstein and Beyond, edited by A. Ashtekar (World
Scientific, Singapore, 2005); [gr-qc/0504080].

45. Bh. Jain and J. Khoury, Cosmological Tests of Gravity
(2010); [astro-ph/1004.3294].

46. B. Bertotti, L. Iess, and P. Tortora, Nature 425, 374
(2003).

47. C.M. Will Theory and Experiment in Gravitational Physics
(Cambridge University Press, Cambridge, 2000).

48. A.M. Nobili and C.M. Will, Nature 320 (1986) 39.

49. J.M. Weisberg, D.J. Nice and J.H. Taylor, Astrophys. J.
722 (2010) 1630; arXiv:astro-ph/1011.0718.

50. D. Kastor and J. Traschen, JHEP 0609, 022 (2006);
[hep-th/0607051].