FOURIER MAJORANTS THAT MATCH NORMS

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Abstract. Denote the coefficients in the complex form of the Fourier series of a function \( f \) on the interval \([-\pi, \pi)\) by \( \hat{f}(n) \). It is known that if \( p = 2j/(2j-1) \) for some integer \( j > 0 \), then for each function \( f \) in \( L^p \) there exists another function \( F \) in \( L^p \) that majorizes \( f \) in the sense that \( \hat{F}(n) \geq |\hat{f}(n)| \) for all \( n \), but that also satisfies \( \|F\|_p \leq \|f\|_p \). Rescaling \( F \) suitably then gives a majorant with the same \( L^p \) norm as \( f \). We show how that majorant comes from a variant in \( L^{2j} \) of the notion of exact majorant in \( L^2 \).

1. Introduction

Thus \( \hat{f}(n) = \int_{-\pi}^{\pi} f(\theta)e^{-in\theta} \, d\theta/2\pi \). Call \( F \) a majorant of \( f \), and \( f \) a minorant of \( F \), when \( |\hat{f}(n)| \leq \hat{F}(n) \) for all integers \( n \). In that case, \( \|f\|_2 \leq \|F\|_2 \); also, if \( j \) is an integer greater than 1, then \( F^j \) majorizes \( f^j \), and hence

\[
(\|f\|_2^j)^2 = (\|f^j\|_2)^2 \leq (\|F^j\|_2)^2 = (\|F\|_2^j)^2.
\]

Finally, \( \|f\|_\infty \leq \|F\|_\infty \) when \( F \) majorizes \( f \).

This pattern does not persist for other exponents. Hardy and Littlewood [9] considered the upper majorant property, asserting that there is a constant \( U(p) \) so that

\[
\|f\|_p \leq U(p)\|F\|_p
\]

whenever \( F \) majorizes \( f \). They gave an example showing that if this property holds in \( L^3 \) then the constant \( U(3) \) must be strictly larger than 1. Other work [2, 1] revealed that the property fails for the exponents \( p \) in the interval \((0, \infty)\) that are not even integers. See [13, pp. 131–134], [8, 7, 12, 3] and [5] for refinements of these results, complements to them and connections with other questions.

Date: 23 May 2021.

2010 Mathematics Subject Classification. Primary 42A32, 43A15; Secondary 47B10.

Results partially announced at the June 2001 meeting of the Canadian Mathematical Society.

Research partially supported by NSERC grant 4822.
Here we consider the lower majorant property, also introduced in \cite{9}. It holds when there is a constant $L(p)$ so that each function $f$ in $L^p$ has a majorant $F$ with

\begin{equation}
\|F\|_p \leq L(p)\|f\|_p.
\end{equation}

This is clearly true when $p = 2$ with $L(2) = 1$, with $F$ equal to the exact majorant of $f$ given by $\hat{F} = |\hat{f}|$. It also holds when $p = 1$, since one can write a given function $f$ in $L^1$ as the square of a function $g$ in $L^2$, form the exact majorant $G$ of $g$, and let $F = G^2$. Then

\[ \|F\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |G(t)|^2 \, dt = (\|G\|_2)^2 = (\|g\|_2)^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |g(t)|^2 \, dt = \|f\|_1. \]

The sequence $\hat{F}$ is equal to the convolution of two copies of the sequence $\hat{G}$, and similarly for $\hat{f}$ and $\hat{g}$. Since there is no cancelation in the convolution giving $F$, the fact that $\hat{G} = |\hat{g}|$ implies that $\hat{F} \geq |\hat{f}|$.

When $p \in [1, \infty)$, a simple duality argument \cite[Section 3]{2} shows that if $L^p$ has the lower majorant property, then its dual space $L^{p'}$ has the upper majorant property with $U(p') \leq L(p)$. By the work cited earlier, this can only happen when $p' = \infty$ or $p'$ is an even integer, thus ruling out all exponents $p$ in the interval $(1, \infty)$ except for $p = 2$ and the special exponents with $p = 2j/(2j - 1)$, where $j$ is an integer strictly greater than 1. When $1 < p < \infty$, a less simple duality argument \cite{9,11} shows that the upper majorant property for $L^{p'}$ implies the lower majorant property for $L^p$, with $L(p) \leq U(p')$. In particular, it holds, with $L(p) = 1$, for the special exponents.

That duality proof and alternatives \cite{14,4,11} to it do not include a description of a suitable majorant of a given function for these values of $p$. For various good reasons, those arguments covered general exponents $p$ and $p'$, but the duality is now known to mainly have impact for the special values of $p$.

In \cite{6}, this hindsight was exploited by showing in those cases that a certain product of $2j - 1$ functions in $L^{2j}$ gives a majorant with minimal norm in $L^{2j/(2j - 1)}$. Suitably rescaling that product yields a majorant with the same $L^{2j/(2j - 1)}$ norm as the original function.

In this paper, we offer a more direct description of that particular majorant. We state our conclusions about it in the next section, and prove those conclusions in the following section. Then we reformulate them as statements about convolution on the integers, and comment on
how they extend to all discrete groups, abelian or not. In an appendix, we discuss the modifications needed on groups like $\mathbb{R}^n$.

2. Dual maximizers

We suppose in the rest of this paper that $1 < p < \infty$. In this paragraph, we summarize some of the reasoning in [9]. First, if $f \in L^p$ and $g \in L^{p'}$, then the series $\sum_n \hat{f}(n)\overline{\hat{g}(n)}$ converges to $\int_{-\pi}^\pi f(t)\overline{g(t)}dt/2\pi$. By Hölder’s inequality,

$$\left| \sum_n \hat{f}(n)\overline{\hat{g}(n)} \right| \leq \|f\|_p \|g\|_{p'}.$$  

In the special case where $p'$ is even and $\hat{G} \geq 0$, it is also true that

$$\sum_n |\hat{f}(n)|\hat{G}(n) \leq \|f\|_p \|G\|_{p'},$$

because each term $|\hat{f}(n)|\hat{G}(n)$ can be rewritten as $\hat{f}(n)\overline{\hat{g}(n)}$ for a function $g$ that is majorized by $G$.

Recall that a complex function-valued function $f$ factors as $|f|\text{sgn}(f)$, where $\text{sgn}(f)$ vanishes off the support of $f$. If $f \in L^p$, then letting $g = |f|^{p-1}\text{sgn}(f)$ puts $g$ in $L^{p'}$, and makes $f = |g|^{p'-1}\text{sgn}(g)$.

In the special cases where $p' = 2j$ for some positive integer $j$, this simplifies to

$$f = |g|^{2j-1}\text{sgn}(g) = |g|^{2j-2}|g|\text{sgn}(g) = |g|^{2(j-1)}g = (gg)^{j-1}g.$$  

Moreover, $|f|^{2j/(2j-1)} = |g|^{2j}$, and $\|f\|_{2j/(2j-1)} = \|g\|_{2j}^{2j-1}$.

Denote the set of functions $G$ in $L^{2j}$ for which $\hat{G} \geq 0$ by $PD^{2j}$. Also consider its closed subset $PD^{2j}_f$ where $\hat{G}$ vanishes off the support of $\hat{f}$, and the closed subset $PD^{2j}_{(g,f)}$ of $PD^{2j}_f$ where $G$ has the same norm in $L^{2j}$ as $g$. Let $\Phi_f$ be the function on $PD^{2j}_f$ that maps $G$ to $\sum_n |\hat{f}(n)|\hat{G}(n)$, which is finite by inequality (2.2).

**Theorem 2.1.** There is a unique function $G$ in the set $PD^{2j}(f,g)$ for which $\Phi_f(G)$ is maximal in that set. The product $F := (GG)^{j-1}G$ then has the same norm in $L^{2j/(2j-1)}$ as $f$, and $F$ majorizes $f$.

**Remark 2.2.** When $j = 1$, the two functions $f$ and $g$ coincide, as do $G$ and $F$, which are their exact majorants. When $j > 1$, it may be reasonable to say that the pair $(G,F)$ specified above is the exact majorant in $L^{2j} \times L^{2j/(2j-1)}$ of the pair $(g,f)$. 
Remark 2.3. As in [1], the function $f$ above has a unique majorant, $F'$ say, of minimal $L^{2j/(2j-1)}$ norm. That majorant coincides with the one in Theorem 2.1 if and only if $\|F'\|_{2j/(2j-1)} = \|f\|_{2j/(2j-1)}$. It is shown in [3] §5 that those two norms agree if and only if forming the exact majorant in $L^2$ of the function $g$ above does not change its $L^{2j}$ norm.

Remark 2.4. These are the cases where the maximizing function $G$ above is the exact majorant of $g$ in $L^2$. In particular, $\hat{G} = |\hat{g}|$ if the support of $\hat{g}$ is a Sidon $B_j$ set, that is each number in the sum of $j$ copies of the set has a unique representation modulo permutation of the $j$ summands.

Remark 2.5. We will comment at end of the next section on the cases where $\|F'\|_{2j/(2j-1)} < \|f\|_{2j/(2j-1)}$.

3. More details

Proof of Theorem 2.1. Let $L^2_j$ comprise all functions $h$ in $L^2_j$ whose coefficients vanish off the support of $\hat{f}$. When $f$ is a trigonometric polynomial, $L^2_j$ is finite dimensional, and its subset $PD^{2j}_{(g,f)}$ is compact. Now $\Phi_f$ is obviously continuous on $PD^{2j}_{(g,f)}$ in this case; so there is a function $G$ in that set at which $\Phi_f$ is maximal. The existence of such a maximizer $G$ for other functions $f$ will be established at the end of this proof.

Let $M = \Phi_f(G)$. If $\Phi_f(H) = M$ for another function $H$ in $PD^{2j}_{(g,f)}$, let $G' = (G + H)/2$. Then $\Phi_f(G') = M$ too, but $\|G'\|_{2j} < \|g\|_{2j}$ by the strict convexity of the $L^{2j}$ norm. Rescaling $G'$ to have the same $L^{2j}$ norm as $g$ would then give a function in $PD^{2j}_{(g,f)}$ where $\Phi_f$ takes a value larger than $M$, contrary to the maximality of $M$. So $G$ is unique.

As noted earlier, $\|f\|_{2j/(2j-1)} = \|g\|_{2j}^{2j-1}$. The corresponding statement about $F$ and $G$ holds for the same reasons. It then follows that $\|F\|_{2j/(2j-1)} = \|f\|_{2j/(2j-1)}$ because $\|G\|_{2j} = \|g\|_{2j}$. By inequality (2.2) and the fact that $\|F\|_{2j/(2j-1)} = \|G\|_{2j}^{2j-1}$, we have

$$\Phi_f(G) \leq \|G\|_{2j}^{2j}.$$  

(3.1)

Given an integer $n$ in the support of $\hat{f}$, let $z^n$ be the function mapping $\theta$ to $e^{in\theta}$. Another useful property of $(\|\cdot\|_{2j})^{2j}$ is that, for a real parameter $t$,

$$\left. \frac{d}{dt} \left( \|G + t z^n\|_{2j}^{2j} \right) \right|_{t=0} = 2j \hat{F}(n).$$  

(3.2)

As in [9], this follows from differentiation inside an integral sign, with $2j$ replaced by any exponent in the interval $(1, \infty)$. For the $L^2_{2j}$ norm,
however, it is enough to expand \((G + tz^n)^j(G + tz^n)^j\) as a polynomial in powers of \(t\), integrate each term with respect to \(\theta\), and examine the resulting coefficient of \(t\).

The conclusions of the theorem are trivial when \(\|f\|_{2j/(2j-1)} = 0\). In the remaining cases, as \(H\) runs through \(PD^{2j}(g, f)\), the \(L^{2j}\) norm of \(H\) is constant, and \(\Phi_f(H) \neq 0\). So the fraction \(\left[\frac{\|H\|_{2j}}{\Phi_f(H)}\right]^{2j}\) is minimized when \(H = G\). That fraction extends to the nontrivial part of \(PD^j\), that is to the union of rays of the form

\[
\{sH : s > 0, H \in PD^{2j}(g, f)\}.
\]

The fraction is constant on each such ray, and minimal on the one where \(H = G\). Form the quotient

\[
Q_n(t) := \frac{(\|G + tz^n\|_{2j})^{2j}}{\Phi_f(G + tz^n)^{2j}}
\]

when \(t \geq 0\). The derivative of \(Q_n(t)\) at \(t = 0\) exists and is equal to

\[
\frac{2j\Phi_f(G)^{2j-1}}{\Phi_f(G)^{4j}} \left[\Phi_f(G) \hat{F}(n) - (\|G\|_{2j})^{2j} \left|\hat{f}(n)\right|\right].
\]

Since \(Q_n(t)\) is minimal at \(t = 0\), this derivative is nonnegative. So

\[
(3.3) \quad \hat{F}(n) \geq \left(\frac{\|G\|_{2j}^{2j}}{\Phi_f(G)}\right) \left|\hat{f}(n)\right| \geq \left|\hat{f}(n)\right|,
\]

by inequality (3.1).

Now let \(f\) be any function in \(L^{2j/(2j-1)}\). By the density of trigonometric polynomials in \(L^{2j/(2j-1)}\), one can write \(f\) as an infinite sum of trigonometric polynomials \(f_k\), where

\[
\sum_k \|f_k\|_{2j/(2j-1)} \leq (1 + \varepsilon)\|f\|_{2j/(2j-1)}.
\]

By inequality (2.2), the series \(\sum_k \Phi_f f_k\) converges uniformly to \(\Phi_f\) on bounded subsets of \(PD^{2j}\). It follows that \(\Phi_f\) is continuous on \(PD^{2j}\) and that \(\Phi_f(H) \leq \|f\|_{2j/(2j-1)}\|H\|_{2j}\) for all \(H\) in \(PD^{2j}\).

Since the function \(\Phi_f\) is bounded on the set \(PD^{2j}_{(g, f)}\), there is a sequence \((G_k)\) in \(PD^{2j}_{(g, f)}\) for which \(\Phi_f(G_k)\) converges to the supremum of the values of \(\Phi_f\) in that set. If \((G_k)\) is a Cauchy sequence, then by the continuity of \(\Phi_f\), that supremum is attained at \(G = \lim_{k \to \infty} G_k\).

By the uniform convexity of the \(L^{2j}\) norm, for each positive number \(\varepsilon\) there is a number \(R\) in the interval \([0, 1]\) with the following property. If \(H\) and \(K\) both belong to \(PD^{2j}(g, f)\), and if \(\|(H + K)/2\|_{2j} > R\|g\|_{2j}\), then \(\|H - K\|_{2j} < \varepsilon\). This reduces the task of proving that \((G_k)\) is a
Cauchy sequence to checking when \( R \in [0, 1) \) that if \( k \) and \( k' \) are large enough, then \( \|(G_k + G_{k'})/2\|_{2j} > R\|g\|_{2j} \).

Let \( M \) be the supremum of \( \Phi_f(H) \) as \( H \) runs through \( PD_{(g,f)}^{2j} \). When \( k \) and \( k' \) are large enough, \( \Phi_f(G_k) > RM \), and \( \Phi_f(G_k) > RM \). It follows that \( \Phi_f((G_k + G_{k'})/2) > RM \) too. Rename \( (G_k + G_{k'})/2 \) as \( G' \), and let

\[
H = \frac{\|g\|_{2j}}{\|G'\|_{2j}} G'.
\]

This rescaling puts \( H \) in \( PD_{(g,f)}^{2j} \), so that \( \Phi_f(H) \leq M \). On the other hand,

\[
\Phi_f(H) = \frac{\|g\|_{2j} \Phi_f(G')}{\|G'\|_{2j}}.
\]

Therefore,

\[
RM < \Phi_f(G') = \frac{\|G'\|_{2j}}{\|g\|_{2j}} \Phi_f(H) \leq \frac{\|G'\|_{2j} M}{\|g\|_{2j}},
\]

and \( \|G'\|_{2j} > R\|g\|_{2j} \) as required. \( \square \)

Remark 3.1. Another way to describe \( G \) is that letting \( H = G \) minimizes \( \|H\|_{2j} \) in the part of \( PD_{(g,f)}^{2j} \) where \( \Phi_f(H) = M \). In Appendix A below, this approach is used in \( L^{2j}(\mathbb{R}^n) \).

Remark 3.2. Putting \( H = G/M \) minimizes \( \Phi_f(H) \) subject to the constraint that \( \Phi_f(H) = 1 \). It is shown in [6] §3 that \( \|G/M\|_{2j} \) must then be the reciprocal of the norm of the minimal majorant, \( F' \), say, of \( f \) in \( L^{2j/(2j-1)} \). On the other hand,

\[
(3.4) \quad \frac{\|F\|_{2j/(2j-1)} - \|F'/\|_{2j/(2j-1)}}{\|F'/\|_{2j/(2j-1)}} = \frac{\|G\|_{2j}}{M/\|G\|_{2j}} = \frac{\|G\|_{2j}}{M} = \frac{\|G\|_{2j}}{\Phi_f(G)}.
\]

Denote the last fraction above by \( r \), and rewrite the first inequality in line (3.3) as the statement that \( \hat{F} \geq r|\hat{f}| \). Then \( F/r \) also majorizes \( f \), while \( \|F/r\|_{2j/(2j-1)} = \|F'/\|_{2j/(2j-1)} \) by equation (3.4). The uniqueness of the minimal majorant yields that \( F/r = F' \) and \( F = rF' \).

Remark 3.3. When \( r > 1 \), the majorant \( F \) does not have minimal norm in \( L^{2j/(2j-1)} \), and \( f \) has other majorants with the same norm as \( f \). Indeed, for any nontrivial function, \( H \) say, in \( L^{2j/(2j-1)} \) with nonnegative coefficients, adding a suitably rescaled copy of \( H \) to \( F' \) gives a majorant for \( f \) with the same norm as \( f \). That majorant is not a rescaled copy of \( F' \) unless \( H \) is.

Remark 3.4. In the description of \( G \) above, it is not necessary to insist a priori that \( \hat{G} \) vanish off the support of \( \hat{f} \). Consider minimizing \( \|G\|_{2j} \)
with the constraints that \( \hat{G} \geq 0 \) and \( \Phi_f(G) = M \), for instance. If the solution had some strictly positive coefficient outside the support of \( \hat{f} \), then replacing that coefficient with 0 would decrease the norm of the solution in \( L^2 \) without changing the value of \( \Phi_f \).

4. Convolution on the integers

We reformulate part of Theorem 2.1.

**Theorem 4.1.** Let \( j \) be a positive integer. For each function \( g \) in \( L^{2j} \), there is another function \( G \) in \( L^{2j} \) with the following properties.

1. \( \hat{G} \geq 0 \).
2. \( \hat{G} = 0 \) off the support of the Fourier coefficients of \( (g\overline{g})^{j-1}g \).
3. \( \|G\|_{2j} = \|g\|_{2j} \).
4. \( (G\overline{G})^{j-1}G \) majorizes \( (g\overline{g})^{j-1}g \).

In particular, this holds when \( g \) is a trigonometric polynomial, and then \( G \) is a trigonometric polynomial too. Use the term light version of Theorem 2.1 for that case. Easy approximation arguments using this version of the theorem yield that every function in \( L^{2j/(2j-1)} \) has a majorant with no larger \( L^{2j/(2j-1)} \) norm.

Given a trigonometric polynomial \( g \), let \( a = \hat{g} \), and let \( \tilde{a} \) be the sequence with \( \tilde{a}(n) = a(-n) \) for all \( n \); these are the coefficients of \( \overline{g} \). Fix an integer \( j > 1 \), and let \( (\tilde{a} * a)^{*j} \) denote the \( j \)-th convolution power of the convolution product \( \tilde{a} * a \). The Fourier coefficients of \( |g|^{2j} \) are given by the sequence \( (\tilde{a} * a)^{*j} \), while those of \( g^{j}(\overline{g})^{j-1} \) are given by \( (a * \tilde{a})^{*(j-1)} * a \).

Since \( |g|^{2j} \geq 0 \), its integral with respect to the measure \( d\theta/2\pi \) is equal to its 0-th Fourier coefficient, that is to the value \( [(\tilde{a} * a)^{*j}](0) \) of the sequence \( (\tilde{a} * a)^{*j} \) at the index 0. Similar comments apply to the trigonometric polynomial \( G \), with coefficients \( b \) say. So the light version of Theorem 2.1 is equivalent to the following statement about convolution on the integers.

**Theorem 4.2.** Let \( j \) be a positive integer. Given a finitely-supported function \( a \) on the integers, let \( c = a * (\tilde{a} * a)^{(j-1)} \). Then there is a function \( b \) on the integers with the following properties.

1. \( b \geq 0 \).
2. \( b \) vanishes off the support of \( c \).
3. \( [(\tilde{b} * b)^{*j}](0) = [(\tilde{a} * a)^{*j}](0) \).
4. \( (b * \tilde{b})^{*(j-1)} * b \geq |c| \).
This follows from Theorem 4.1, and can also be proved directly by choosing the function \( b \) to maximize the sum \( \sum_n b(n) |c(n)| \) subject to the first three conditions enumerated above.

**Remark 4.3.** The convolution version of equation (3.2) runs as follows. Let \( \delta_n \) be the sequence that takes value 1 at \( n \), and that vanishes otherwise. Given a real number \( t \), let \( d = b + t\delta_n \), and form \( [(d*d)^\ast]^j(0) \). Then the derivative at \( t = 0 \) of the latter is \( 2j [(b*b)^\ast(j-1) * b](n) \). It is easy to check this by expanding \( [(\tilde{d} * d)^\ast]^j(0) \) in powers of \( t \).

**Remark 4.4.** In this context, inequality (2.1) is only required when \( g \) is a trigonometric polynomial, and \( f \) is a suitable product of such polynomials. The corresponding statement for convolution is that if \( a \) and \( k \) are finitely supported functions on the integers, then

\[
\left| \left[ \tilde{k} * (a * \tilde{a})^{(j-1)} * a \right]^j(0) \right| \leq \left| \left[ (\tilde{k} * k)^j(0) \right]^{1/2j} \left[ (\tilde{a} * a)^j(0) \right]^{(2j-1)/2j} \right| \cdot
\]

This follows from Hölder’s inequality on the unit circle. It can also be verified in the style of the proof of Theorem 2.1. Just rescale in nontrivial cases to make \( (\tilde{a} * a)^j(0) \) equal to 1, and then minimize

\[
\frac{(\tilde{k} * k)^j(0)}{\left[ \left[ \tilde{k} * (a * \tilde{a})^{(j-1)} * a \right]^j(0) \right]^{2j}}.
\]

The fact that positive bounded operators on Hilbert spaces have unique positive \( j \)-th roots is useful here.

**Remark 4.5.** Theorem 4.2 and the two remarks above extend to all discrete groups, abelian or not. Hölder’s inequality extends to noncommutative \( L^p \) spaces as in [17]. The counterpart of the instance in Remark 4.4 can also be proved by the method outlined there.

**Remark 4.6.** Similar comments apply in the context of trace ideals [16].

**Appendix A. Majorization on nondiscrete duals**

We now explain how some of our methods extend to the spaces \( L^p(\mathbb{R}^n) \), where comparisons between transforms are made on a dual copy of \( \mathbb{R}^n \), which is not discrete. When \( 1 < p \leq 2 \), transforms of \( L^p \) functions on \( \mathbb{R}^n \) can be identified with (equivalence classes of) functions on the dual copy. When \( p' > 2 \), functions in \( L^{p'}(\mathbb{R}^n) \) have transforms in the sense of tempered distributions, but in many cases those transforms are not functions.

Duality arguments in [15, 10] used summability on the group \( \mathbb{R}^n \) and its dual copy to reduce the study of majorant properties in \( L^p \) spaces to
instances where \( f \) and \( \hat{f} \) are both functions. Again \( \text{[10][15]} \) the upper majorant property only holds when \( p \) is infinite or even, and the lower majorant property only holds when \( p = 1 \) or \( p' \) is an even integer. Here, we offer a different proof that the upper majorant property implies the lower majorant property for the dual exponent when \( p \) is even; again, we get more information about the forms of some majorants.

Recall that a distribution is said to be nonnegative if it maps each nonnegative test function to a nonnegative number, and then the distribution acts by integration against a nonnegative Borel measure. When the distributions acts by integration against a function, the distribution is nonnegative if and only if that function is nonnegative almost everywhere.

When two functions \( f \) and \( F \) both belong to \( L^p \), where \( 1 \leq p \leq 2 \), call \( F \) a majorant of \( f \) if \( \hat{F} \geq |\hat{f}| \) almost everywhere on the dual copy of \( \mathbb{R}^n \). More generally, for two distributions \( \Phi \) and \( \Psi \), call \( \Phi \) a majorant of \( \Psi \) if the transform \( \hat{\Phi} \) is represented by a nonnegative measure, while \( \hat{\Psi} \) is represented by a measure with the property that \( |\hat{\Psi}(S)| \leq \hat{\Phi}(S) \) for all Borel sets \( S \).

If \( p \) is even, then the upper majorant property holds with constant 1. That is, if \( \Phi \in L^p \), then \( \Psi \in L^p \) and \( \|\Psi\|_p \leq \|\Phi\|_p \). The only cases of this needed here are those where \( \hat{\Phi} \) and \( \hat{\Psi} \) are represented by functions in \( L^1 \cap L^\infty \). Then those transforms also belong to \( L^2 \), and it follows that \( \Phi \) and \( \Psi \) belong to \( L^2 \cap L^\infty \). One can then compare the \( L^2 \) norms of \( \Phi \) and \( \Psi \) by considering the \( j \)-th convolution powers of \( \hat{\Phi} \) and \( \hat{\Psi} \) and comparing their \( L^2 \) norms.

Also recall that a distribution is said to vanish on an open set if the distribution annihilates every test function whose support is a compact subset of the open set. Then there is a largest such open set, and the support of the distribution is defined to be the complement of that largest open set. When the distribution acts by integration against a function, that support is the smallest closed set outside which the function takes the value 0 almost everywhere.

**Theorem A.1.** Let \( p = 2j/(2j - 1) \) for some integer \( j > 1 \), let \( f \) be a function in \( L^p(\mathbb{R}^n) \), and let \( S \) be the support of the distribution \( \hat{f} \). Then \( f \) has a majorant of the form \( G^j(\hat{G})^{j-1} \), where \( G \in L^p(\mathbb{R}^n) \), and \( \hat{G} \) is a nonnegative distribution whose support is included in \( S \). The majorant of minimal \( L^p \) norm has this form, and \( \|G\|_{p'} \leq (\|f\|_p)^{1/(2j-1)} \) in that case. The support of the transform of the minimal majorant is included in the closure of the algebraic sum of \( j \) copies of \( S \) and \( j - 1 \) copies of \( -S \).
Proof. Modify the approach in Remark \[3.1\] replacing sums with integrals, and replacing pointwise inequalities with ones that hold almost everywhere. For now, only require that \(1 < p < 2\).

Given a measurable function \(c \in L^p'\) on the dual copy of \(\mathbb{R}^n\), let \(R(c)\) be the set of distributions \(w\) for which \(\hat{w}\) can be identified with a nonnegative, bounded, measurable function with bounded support on the dual copy of \(\mathbb{R}^n\), and for which

\[
(A.1) \quad \int_{\mathbb{R}^n} |c| \hat{w} = 1.
\]

Then \(R(c)\) is nonempty when \(c\) is nontrivial, because there are bounded sets of positive measure on which the values of \(|c|\) are bounded away from 0 and \(\infty\). Suitably rescaling the indicator function of such a set gives a function \(\hat{w}\) satisfying condition \((A.1)\).

The inverse transform of that function belongs to both \(L^\infty(\mathbb{R}^n)\) and \(L^2(\mathbb{R}^n)\), and hence to \(L^{p'}(\mathbb{R}^n)\). Let \(K_p(c)\) be the infimum of \(L^{p'}\) norms of members of the nonempty convex set \(R(c)\). Since the \(L^{p'}\) norm on is uniformly convex, the closure of \(R(c)\) in \(L^{p'}\) has a unique element of smallest norm, which must be \(K_p(c)\).

Say that a function \(F \in L^p(\mathbb{R}^n)\) is a partial majorant of \(\hat{c}\) if \(\hat{F} \geq |c|\) almost everywhere in the set where \(c \neq 0\). The set of partial majorants is convex, and it is closed in \(L^p\). If this set is nonempty, then it has a unique element of minimal \(L^p\) norm, by uniform convexity again.

**Lemma A.2.** Let \(1 < p < 2\), and let \(c \in L^{p'}(\mathbb{R}^n)\). If \(\|c\|_{p'} \neq 0\), then \(\hat{c}\) has a partial majorant in \(L^p\) if and only if \(K_p(c) > 0\). In that case, the minimal \(L^p\) norm of partial majorants of \(\hat{c}\) is equal to \(1/K_p(c)\). The partial majorant of minimal norm is a rescaled copy of \(h|h|^{p'-2}\), where \(h\) has minimal \(L^{p'}\) norm in the closure of \(R(c)\) in \(L^{p'}\). Finally, the distributional support of the transform of \(h\) is included in the distributional support of \(c\).

To prove this, start with the fact that if \(\hat{c}\) has a partial majorant \(F\) in \(L^p\), and if \(w \in R(c)\), then

\[
(A.2) \quad 1 = \int_{\mathbb{R}^n} \hat{w}|c| \leq \int_{\mathbb{R}^n} \hat{w}\hat{F} = \int_{\mathbb{R}^n} \overline{w}F \leq \|w\|_{p'}\|F\|_p.
\]

The second equality above is the instance of the Parseval relation that makes \(\int_{\mathbb{R}^n} \overline{h}\hat{f} = \int_{\mathbb{R}^n} \overline{h'}f\) when \(f \in L^p\) and \(h\) is the inverse transform of a bounded function with bounded support.

Inequality \((A.2)\) makes \(\|w\|_{p'} \geq 1/\|F\|_p\) for every \(w\) in \(R(c)\), so that \(K_p(c) \geq 1/\|F\|_p\). In particular, \(\|F\|_p \geq 1\) when \(K_p(c) = 1\).
Rescale \( c \) to reduce matters to the latter case, and then let \( h \) be as specified in the statement of the lemma.

Let \( k = h|h|^p - 2 \). Fix a function \( w \) in the set \( R(c) \), and let \( \phi(t) \) be equal to \((\|h + tw\|^p_p)^p\). This has derivative \( p' \int \hat{kw} \) at \( t = 0 \). The quotients \((h + tw)/(1 + t \int |c|\hat{w})\) belong to the closure of \( R(c) \) in \( L^{p'} \) when \( t \geq 0 \). Take \( p' \)-th powers of the \( L^{p'} \) norms of these quotients, and require that the derivatives with respect to \( t \) of these powers be nonnegative at \( t = 0 \). The outcome is that \( \int (\hat{k} - |c|)\hat{w} \geq 0 \) for all \( w \) in \( R(c) \).

It follows that \( \hat{k} \geq |c| \) almost everywhere on the set where \( c \neq 0 \). On the other hand, \( k \) has \( L^p \) norm equal to 1, which is a lower bound for the the norms of partial majorants of \( c \) in \( L^p \). So \( k \) must be the partial majorant of minimal \( L^p \) norm.

By definition, all functions in the set \( R(c) \) have transforms whose distributional supports are included in the distributional support of \( c \). Then \( h \) has this property too. This completes the proof of the lemma.

To deduce the theorem, first check that if \( p = 2j/(2j-1) \) for some integer \( j > 1 \), and \( f \in L^p(\mathbb{R}^n) \), then \( K_p(\hat{f}) \geq 1/\|f\|_p \). To that end, write \( \hat{f} = \varepsilon|\hat{f}| \) for a measurable function \( \varepsilon \) with absolute-value 1. Now let \( h \) be any function in \( R(|\hat{f}|) \), and let \( h' \) be the inverse transform of the product \( \varepsilon h \). Then

\[
1 = \int \hat{h}|\hat{f}| = \int \hat{h'} \hat{f} = \int h' \hat{f}.
\]

By the upper majorant property, \( h' \in L^{2j} \) with \( \|h'\|_{2j} \leq \|h\|_{2j} \). Hölder’s inequality then yields that

\[
1 \leq \|h'\|_{2j} \|f\|_p \leq \|h\|_{2j} \|f\|_p.
\]

This makes \( 1/\|f\|_p \) a lower bound for the \( L^{p'} \) norm of every such function \( h \), and hence for \( K_p(\hat{f}) \). In particular, \( K_p(\hat{f}) > 0 \), and \( f \) has a minimal partial majorant, \( F \) say, in \( L^p \). To see that \( \|F\|_p \leq \|f\|_p \), again rescale to the case where \( K_p(\hat{f}) = 1 \).

In that case, write \( h \) as a limit in \( L^{p'} \) norm of a sequence \((h_n)\) of members of the set \( R(c) \). The transforms of the functions \( h_n \) are nonnegative bounded functions with bounded supports. The functions \( k_n := (h_n)^j(\overline{h_n})^{j-1} \) belong to \( L^p \), and the sequence \((k_n)\) converges in \( L^p \) to \( k \), so that \( (\hat{k_n}) \) converges in \( L^{p'} \) to \( \hat{k} \).

The transform of \( k_n \) is equal to the convolution of \( j \) copies of \( \overline{h_n} \) with \( j - 1 \) copies of \( \overline{h_n} \). It follows that \( \hat{k_n} \) is nonnegative almost everywhere. Moreover, if a test function \( \psi \) vanishes on the closure of the
algebraic sum of $j$ copies of $S$ and $j - 1$ copies of $-S$, then $\int k_n \psi = 0$ for all $n$. So $\int \hat{k} \psi = 0$, and the support of $\hat{k}$ is included in the closure of that sum of sets. Since $\hat{k}$ is nonnegative almost everywhere, it is a full majorant of $\hat{c}$. It must be the one of minimal norm because it has minimal norm among partial majorants.

\[ \blacksquare \]

**Remark A.3.** Suitably rescaling the minimal majorant $G$ again gives one with the same $L^p$ norm as $f$. Proving the existence of such a majorant by the method used to prove Theorem 2.1 seems harder here, because it is not clear that a suitable counterpart of the function $\Phi_f$ is continuous.

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