A new proof of reversibility of $SLE_\kappa$ for $\kappa \leq 4$

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Abstract
We give a new proof of the reversibility of the Schramm Loewner evolution for $\kappa \leq 4$. The main ideas used in the proof are similar to those used in the original proof of this result, given by Zhan.

1 Introduction
The Schramm Loewner evolution ($SLE_\kappa$) is a one parameter family of probability measures on curves in the plane parametrized by $\kappa = 2/a > 0$. It gives the only candidate for scaling limits of discrete lattice models exhibiting conformal invariance in the continuum limit. We recall the definition; see [2] for more detail. Let $\mathbb{H}$ denote the upper half plane in $\mathbb{C}$. If $\gamma: [0, \infty) \to \mathbb{H}$ is a curve, we write $\gamma_t$ for $\gamma[0,t]$ and $D_t$ for the unbounded component of $\mathbb{H} \setminus \gamma_t$. Chordal $SLE_\kappa$ from 0 to $\infty$ is defined (up to a choice of parametrization) as a random curve $\gamma: [0, \infty) \to \mathbb{H}$ with $\gamma(0) = 0$ such that the following holds. Let $g_t$ be the “mapping-out function”, that is, the unique conformal transformation $g_t: D_t \to \mathbb{H}$ with $g_t(z) = z + o(1)$ as $z \to \infty$. Then $g_t$ satisfies
\begin{equation}
\dot{g}_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z
\end{equation}
Under this parameterization, $\text{hcap}(\gamma_t) = at$, where $\text{hcap}$ denotes the half-plane capacity. In other words, we have the expansion
$g_t(z) = z + \frac{at}{z} + O(|z|^{-2}), \quad z \to \infty.$

If $z \in \mathbb{C} \setminus 0$, the solution to [1] holds for all $t < T_z$ where
$T_z = \sup \{ \min \{|g_s(z) - U_s|: 0 \leq s \leq t\} > 0 \}.$

To get chordal $SLE_\kappa$ connecting distinct boundary points in a simply connected domain $D$, one takes the conformal image of this under a conformal map. Again this is defined up to a change of parametrization.

While $SLE$ is a model for curves in equilibrium, the definition uses conditional probabilities given the path up to a certain time and hence adds an artificial dynamic. One disadvantage is that some properties that are expected of the limit curve, in particular reversibility, do not follow immediately. Zhan showed this to be true [6] for $\kappa \leq 4$, while Miller and Sheffield were able to extend these results to $\kappa \in (0, 8)$ [3, 4, 5] by realizing $SLE_\kappa$ curves as flow lines of the Gaussian free field.

The purpose of this note is to give a new proof of reversibility for $\kappa \leq 4$; we hope in future work to extend this to $4 < \kappa < 8$ to give a proof that does not make use of the tools of the Gaussian free field. While we say that it is a new proof, the basic idea of the proof is the same as that given by Zhan. Our hope is that we our argument simplifies some of the details. We write $SLE$ for $SLE_\kappa$.

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We compare \( \text{SLE} \) from 0 to \( x \) in \( \mathbb{H} \) to \( \text{SLE} \) from \( x \) to 0. These are probability measures on bounded curves \( \gamma \) with \( \text{hcap}(\gamma) < \infty \). While \( \text{hcap}(\gamma) \) is a random quantity, it is almost immediate from the definition that the distribution of \( \text{hcap}(\gamma) \) is the same for \( \text{SLE} \) in both directions.

We view \( \text{SLE} \) connecting two points in \( \mathbb{R} \) as a probability measure on the final mapping-out functions \( g_{\gamma} \).

We then focus on \( \text{SLE} \) from 0 to \( x \) and \( x \) to 0 conditioned to have a specific half-plane capacity. We show that these two probability measures agree on the conformal maps \( g_{\gamma} \) for each value of \( \text{hcap}[\gamma] \). By scaling it suffices to prove this for all \( x \) assuming \( \text{hcap}[\gamma] = a \).

For each \( r \in [0, 1] \) we consider the probability measure \( \mu_r \) which corresponds to the following:

- Take \( \text{SLE} \) from 0 to \( x \) conditioned to have \( \text{hcap} = ra \) stopped at time \( r \), that is, when \( \text{hcap} = ra \) giving \( \gamma^1 \).
- Given \( \gamma^1 \), let \( \gamma^2 \) be \( \text{SLE} \) from \( x \) to \( \gamma^1(r) \) in \( \mathbb{H} \setminus \gamma^1 \) conditioned so that \( \text{hcap}(\gamma^1 \cup \gamma^2) = a \).
- Output \( g_{\gamma} \) where \( \gamma = \gamma^1 \oplus \gamma^2 \) where \( \gamma^2 \) is the reversal of \( \gamma^2 \).

This gives a probability measure on transformations \( g_{\gamma} \) with \( \text{hcap}[\gamma] = a \) which we denote by \( \mu_r \).

We consider this as a measure on continuous functions on a fixed closed ball \( K = K_h \subset \mathbb{H} \) where \( h \) is large enough so that \( \text{Im}[g_{\gamma}(z)] \geq a \) for all \( z \in K \) and \( \text{hcap}[\gamma] = a \). We show that the Prokhorov distance between \( \mu_r \) and \( \mu_s \) is less than \( c|s-r|^{1+\delta} \) for some \( \delta > 0 \). We conclude that \( \mu_s \) is a constant function of \( s \). In particular, \( \mu_0 = \mu_1 \) which is the main result.

The main local commutation relation which is similar to the relations in [6] and [11] is expressed in terms of Radon-Nikodym derivatives of independent \( \text{SLE} \) paths tilted by a Brownian loop term. This relation is nicest for \( \kappa \leq 4 \), but we discuss the \( \kappa < 8 \) case here in order to prepare for future work.

The paper is organized as follows. In Section 2 we review \( \text{SLE} \) connecting two points on the boundary, together with some other basic notation, and then we state the main theorem of this paper. In Section 3 we describe the commutation relation, and show explicitly that the measures under consideration have the same Radon-Nikodym derivative with respect to a particular measure. In Section 4 we prove the main theorem in a sequence of steps, relying on a few Loewner chain estimates. Finally, in Section 5 we give the (delayed) proof of a basic Bessel process fact.

Throughout this paper we fix \( \kappa = 2/a \in (0, 8) \) and allow constants, both implicit and explicit, to depend on \( \kappa \). We write just \( \text{SLE} \) for \( \text{SLE}_\kappa \). For a number of the results, we need \( \kappa \leq 4 \) and we say that. Let

\[
\nu = \frac{6 - \kappa}{2\kappa} = \frac{3a - 1}{2}
\]

be the boundary scaling exponent.

## 2 \( \text{SLE} \) in \( \mathbb{H} \) from \( x_1 \) to \( x_2 \)

There are several equivalent characterizations of \( \text{SLE} \) connecting two real points; here we will take the perspective of \( \text{SLE} \) from 0 to \( x \in \mathbb{R} \setminus \{0\} \) as \( \text{SLE} \) from 0 to \( \infty \) in \( \mathbb{H} \) tilted by the partition function \( \Psi \). For simply connected domains and locally analytic boundary points \( z, w \), the partition function for \( \text{SLE} \) is

\[
\Psi_D(z, w) = H_{\partial D}(z, w)^b.
\]

Here \( H_{\partial D}(z, w) \) is the boundary Poisson kernel normalized so that \( H_{\partial \mathbb{D}}(0, x) = x^{-2} \). We also define \( \Psi_{\mathbb{H}}(x, \infty) = 1 \) for all \( x \in \mathbb{H} \). The partition function satisfies the scaling rule: if \( f : D \to f(D) \) is a conformal transformation, then

\[
\Psi_D(z, w) = |f'(z)|^b |f'(w)|^b \Psi_{f(D)}(f(z), f(w)).
\]
Although this definition of $\Psi_D(z, w)$ requires that $z, w$ be locally analytic boundary points, ratios of partition functions can often be defined using the scaling rule as we will see below.

Suppose that $g_t$ satisfies \( \Psi \) where $U_t = -B_t$ is a standard Brownian motion defined on a probability space $(\Omega, F, P)$. Let $\gamma(t)$ denote the corresponding $\text{SLE}_\kappa$ curve and we write $\gamma_t = \gamma(0, t)$. Under the measure $P$, $\gamma$ has the distribution of an $\text{SLE}_\kappa$ path from 0 to $\infty$. We will tilt the measure $P$ using an appropriate local martingale to get $\text{SLE}$ from 0 to $\infty$. Suppose $x \in \mathbb{R} \setminus \{0\}$, and let $X_t = g_t(x) - U_t$. Let $T = T_x = \inf\{t > 0 : X_t = 0\}$. For $t < T$, let $D_t$ be the unbounded component of $\mathbb{H} \setminus \gamma_t$, and define the local martingale $M_t$ formally by

$$M_t = x^{1-3\alpha} \frac{\Psi_{H_t}(\gamma(t), x)}{\Psi_{D_t}(\gamma(t), \infty)}, \quad t < T.$$ 

The partition functions on the right-hand side are not well defined but the ratio is well defined using the scaling rule,

$$\frac{\Psi_{D_t}(\gamma(t), x)}{\Psi_{D_t}(\gamma(t), \infty)} = |g'(\gamma_t)|^{b} g'(x)^{b} \frac{\Psi_{D_t}(U_t, g_t(x))}{\Psi_{D_t}(U_t, \infty)} = g'(x)^{b} X_t^{1-3\alpha}.$$ 

While this is formal, this shows that we can define

$$M_t = \left( \frac{X_t}{X_0} \right)^{1-3\alpha} g'(x)^{b}, \quad t < T$$

and one can use Itô’s formula and the Loewner equation to see that $M_t$ is a local martingale satisfying $M_0 = 1$ and

$$dM_t = \frac{1-3\alpha}{X_t} M_t \, dB_t, \quad 0 < t < T.$$ 

(2)

Let $P^*$ be the measure obtained by tilting by $M_t$. More precisely, if $\tau < T$ is a stopping time such that $M_{\tau \wedge T}$ is a martingale and $V$ is an event measurable with respect to $F_{\tau \wedge T}$, then $P^*(V) = E[M_{\tau \wedge T} 1_V]$. The Girsanov theorem states that

$$dB_t = \frac{1-3\alpha}{X_t} dt + dW_t, \quad 0 < t < T,$$

where $W_t$ is a standard Brownian motion with respect to $P^*$ and hence

$$dX_t = \frac{1-2\alpha}{X_t} dt + dW_t, \quad 0 < t < T.$$ 

The following is well known.

**Proposition 2.1.** Suppose $0 < x$ and $g_t$ is the solution to the Loewner equation \( \Psi \) where $U_t = g_t(x) - X_t$ and $X_t$ satisfies

$$dX_t = \frac{1-2\alpha}{X_t} dt + dW_t, \quad X_0 = x, \quad 0 < t < T,$$

(3)

where $W_t$ is a standard Brownian motion and $T = \inf\{ t : X_t = 0 \}$. Then $\gamma(t), 0 \leq t \leq T$ has the distribution of $\text{SLE}_\kappa$ from 0 to $x$ parametrized by half plane capacity from infinity stopped at the time that $\gamma_T$ disconnects $x$ from infinity. In particular, $\text{hcap}[\gamma_T] = aT$.

Indeed to verify this, one needs only check that the conformal image of $\text{SLE}$ from 0 to $\infty$ by a conformal transformation $F : \mathbb{H} \to \mathbb{H}$ with $F(0) = 0, F(\infty) = x$ gives the same distribution on the driving function as \( \Psi \).

Similarly, if $X_t$ satisfies \( \Psi \) and we define $\tilde{U}_t = g_t(0) - X_t$ with corresponding curve $\tilde{\gamma}$, then $\tilde{\gamma}(t), 0 \leq t \leq T$ has the distribution of $\text{SLE}_\kappa$ from $x$ to 0 parametrized by half plane capacity from infinity stopped at the time that $\gamma_T$ disconnects $0$ from infinity.

While we may use the same $X_t$ for $\text{SLE}$ in both directions, the distribution of the driving functions $U_t, \tilde{U}_t$ are different. Indeed, $U_0 = x, \tilde{U}_0 = 0$. For this reason, we cannot conclude the reversibility immediately from
Bessel process (3) in a two step process: we first choose a value $T$ at the origin is $T$ measure $P$. Theorem 1. If $g$ gives only the “curve as viewed from infinity”, that is $\gamma$ In particular, for $\kappa$ with $\kappa$ with $1$ this fact. One thing that does follow is that the distribution of the stopping time $T$ is the same for $SLE$ from $0$ to $x$ as for $SLE$ from $x$ to $0$. It is the same as the time to reach the origin for the Bessel process $\tilde{\phi}$. Since $a > 1/4$ the process reaches the origin in finite time.

There is a significant difference between $\kappa \in (0, 4]$ and $\kappa \in (4, 8)$. Let us consider $SLE$ from $0$ to $x$ with $x > 0$ stopped at time $T$. The following statements are with probability one with respect to the tilted measure $P^*$.  

- If $0 < \kappa \leq 4$, then $\gamma(t), 0 \leq t \leq T$ is a simple curve with $\gamma(0) = 0, \gamma(T) = x, \gamma(0, T) \subset \mathbb{H}$.
- If $4 < \kappa < 8$, then $\gamma(T) \in (x, \infty)$. Although the $SLE$ curve continues after time $T$, $D_\infty$, the unbounded connected component of $\mathbb{H} \setminus \gamma$ is the same as the unbounded connected component of $\mathbb{H} \setminus \gamma_T$.

In particular, for $\kappa \leq 4$, the domain $D_\infty$ determines the entire curve while for $4 < \kappa < 8$, the domain $D_\infty$ gives only the “curve as viewed from infinity”, that is $\gamma \cap \overline{D}_\infty$. We will prove reversibility for the domain $D_\infty$. In this paper, we do the $\kappa \leq 4$ case reproving Zhan’s result.

**Theorem 1.** If $\kappa \leq 4$, the distribution of $D_\infty$, is the same for $SLE$ from $x_1$ to $x_2$ and for $SLE$ from $x_2$ to $x_1$. Equivalently, the distribution of the conformal transformation $g_\infty$ is the same.

Our proof is in the same spirit as Zhan’s proof. One novel aspect is that we choose a realization of the Bessel process $\tilde{\phi}$ in a two step process: we first choose a value $T = t_0$ and then given $T$ we run the Bessel process conditioned so that $T = t_0$.

If $X_t$ satisfies (3) where $W_t$ is a $P^*$-Brownian motion, then the transition probability of the process killed at the origin is

$$q_s(x,y) = \frac{y}{x^{2a+1}s^{2a+\frac{1}{2}}} \exp \left\{-\frac{x^2 + y^2}{2s} \right\} h \left(\frac{xy}{s}\right),$$

where $h = h_a$ is an entire function with $h(0) = 0$. The density of $T$ in the measure $P^*$ is a constant times

$$\phi(x,t) := x^{4a-1} t^{-\frac{a}{2}-2a} \exp \left\{-\frac{x^2}{2t}\right\}.$$ (4)

The Bessel process conditioned so that $T = t_0$ is this process tilted by the $P^*$-martingale

$$N_t := \phi(X_t, t_0 - t), \quad 0 \leq t < t_0$$ (5)

which satisfies

$$dN_t = N_t \left[ \frac{4a - 1}{X_t} - \frac{X_t}{t_0 - t} \right] dW_t.$$ (6)

Formally one can write $\phi(X_t, t_0 - t) = E^* [1_{T = t_0} | X_t]$ which can be thought of as a Doob martingale in the measure $P^*$. Otherwise, the unconvinced reader may engage in a brief Itô calculus exercise to derive (6).

Since $M_t$ is a $P$ local martingale and $N_t$ is a $P^*$ local martingale, we can see that $\tilde{M}_t := M_t N_t$ is a $P$ local martingale. Again, one can check this again using Itô calculus. If we let

$$\tilde{M}_t = \tilde{M}_{t,t_0} = M_t N_t = x^{1-3a} X_t^{3a-1} g_t^0(x)^{(3a-1)/2} \phi(X_t, t_0 - t), \quad 0 \leq t < t_0,$$

then using (2) and (5) we see that $\tilde{M}_t, 0 \leq t < t_0$ is a $P$-martingale satisfying

$$d\tilde{M}_t = \left[ \frac{a}{X_t} - \frac{X_t}{t_0 - t} \right] \tilde{M}_t dB_t, \quad 0 \leq t < t_0.$$ (7)

If we tilt in the Girsanov sense as above by $\tilde{M}_t$ giving the new measure $\tilde{P}$ we have

$$dB_t = \left[ \frac{a}{X_t} - \frac{X_t}{t_0 - t} \right] dt + d\tilde{W}_t,$$

$$dX_t = d[g_t^0(x) + B_t] = \left[ \frac{2a}{X_t} - \frac{X_t}{t_0 - t} \right] dt + d\tilde{W}_t,$$

where $\tilde{W}_t$ is a $\tilde{P}$-Brownian motion.

4
**Definition** Suppose $x_1, x_2$ are distinct real numbers, $0 < \kappa < 8$, and $0 < t_0 < \infty$. Then $SLE_\kappa$ from $x_1$ to $x_2$ in $\mathbb{H}$ of time duration $t_0$ is defined to be the solution of (1) where the driving function $U_t = g_t(x_2) - X_t$, and $X_t$ satisfies

\[
\begin{align*}
\frac{dX_t}{X_t} = \left[\frac{2a}{X_t} - \frac{X_t}{t_0 - t}\right] dt + dW_t, & \quad X_0 = x_2 - x_1, \quad (7) \\
U_t = g_t(x_2) - X_t &= x_2 + \int_0^t \frac{a ds}{X_s} - X_t,
\end{align*}
\]

where $W_t$ is a standard Brownian motion.

If $q_t(x, y)$ denotes the transition probability for a Bessel process satisfying (5), killed upon reaching the origin, then the density for a process satisfying (7) is

\[
\psi_t(x, y; t_0) = q_t(x, y) \frac{\phi(y, t_0 - t)}{\phi(x, t_0)}.
\]

We will need one very believable fact about this process. The proof uses standard techniques but we delay the proof to Section 5. This estimate is not optimal but will be more than sufficient for our purposes.

**Proposition 2.2.** For every $0 < \kappa < 8$, there exists $c < \infty, u > 0$ such that if $X_t$ satisfies (7), then for all $r > 0$,

\[
P\left\{ \max_{0 \leq t \leq t_0} |U_t - x_1| \geq \sqrt{r} \left(|x_2 - x_1| + r^2 \right) \right\} \leq c e^{-ur}.
\]

We denote the corresponding probability measure on paths (modulo reparametrization) by $\mu^\#(x_1, x_2; t_0)$. Assuming $x_1 < x_2$, we have the following:

- $\gamma(0) = x_1$, $T = t_0$;
- $h_{\operatorname{cap}}[\gamma] = at$, $0 \leq t \leq T$;
- If $\kappa \leq 4$, then $\gamma_T$ is a simple curve with $\gamma(0, t_0) \subset \mathbb{H}$ and $\gamma(t_0) = x_2$. Moreover, $\partial D_\infty \cap \mathbb{H} = \gamma(0, t_0)$;
- If $4 < \kappa < 8$, then
  \[
  \gamma(T) = x_+ := \max\{y \in \mathbb{R} : y \in \gamma_{t_0}\} > x_2, \\
  x_- := \min\{y \in \mathbb{R} : y \in \gamma_{t_0}\} < x_1.
  \]

Indeed, $\partial D_\infty \cap \mathbb{H}$ is a curve connecting $x_-$ to $x_+$.

To prove Theorem 1 it suffices to prove the following.

**Theorem 2.** If $\kappa \leq 4$ then for every $t_0 > 0$ and $x_1 < x_2$, the measure $\mu^\#(x_1, x_2; t_0)$ is the same as $\mu^\#(x_2, x_1; t_0)$ if considered as probability measures on the conformal transformation $g = g_{t_0}$.

By scaling and translation invariance it suffices to prove this with $x_1 = 0, x_2 = x > 0$ and $t_0 = 1$.

Fix $x > 0$ and consider the measure $\mu_r = \mu_{r,x}$, $0 \leq r \leq 1$ obtained as follows:

- Grow the curve $\gamma$ under the measure $\mu^\#(0, x; 1)$ until time $r$ giving curve $\gamma_r$ and corresponding map $g_r$. Let $z_1 = g_r(\gamma(0)), w_1 = g_r(1)$.
- Given $\gamma_r$, let $\tilde{\gamma}$ be $SLE$ from $x$ to $\gamma(r)$ in $\mathbb{H} \setminus \gamma_r$ conditioned so that $h_{\operatorname{cap}}[\gamma_r \cup \tilde{\gamma}] = a$. Equivalently, let $\eta$ be chosen from $\mu^\#(w_1, z_1; 1 - r)$ and let $\tilde{\gamma} = g_r^{-1} \circ \eta$. Let $h = g_\eta$ and $g = h \circ g_r$.

Note that $\mu_0 = \mu^\#(0, x; 1), \mu_1 = \mu^\#(x, 0; 1)$. We will prove the following stronger result,

**Proposition 2.3.** If $\kappa \leq 4$ and $x > 0$, then for all $0 \leq r \leq 1$, $\mu_r = \mu_0$. 

Since $\text{hcp}[\gamma_1] = a$, we know that $\gamma_1 \subset \{z : \text{Im}(z)^2 \leq 2a\}$ and hence with probability one for each $\mu_r$, $D_\infty \supset \{z : \text{Im}(z)^2 > 2a\}$. Let $\mathcal{I} = \{z : |z - (\sqrt{8a} + 1)i| \leq 1\}$. Let $S$ denote the set of continuous functions from $\mathcal{I}$ to $\mathbb{C}$ endowed with the supremum norm $\|\cdot\|$. We also write $\rho$ for the corresponding Prokhorov metric on probability measures on $S$. Since the conformal map $g$ is determined by its values on $\mathcal{I}$, it suffices to prove that for every $\epsilon > 0$ and $0 \leq r < s \leq 1$, $\rho(\mu_r, \mu_s) < \epsilon$. We will show the following.

**Proposition 2.4.** For every $K < \infty$, there exists $c, \delta$ such that if $0 < x \leq K$ and $0 \leq r \leq s \leq 1$, we can couple $(g, \tilde{g})$ on the same probability space such that $g$ has distribution $\mu_r$, $\tilde{g}$ has distribution $\mu_s$ and

\[
\mathbb{P}\{\|g - \tilde{g}\| \geq c(s - r)^{1+\delta}\} \leq c(s - r)^{\delta},
\]

\[
\mathbb{P}\{\|g - \tilde{g}\| \geq c(s - r)\} \leq c(s - r)^{1+\delta}.
\]

We state it this way in preparation for later work in the $4 < \kappa < 8$ case. For $0 < \kappa \leq 4$, we do significantly better by giving a coupling that satisfies $\|g - \tilde{g}\| \leq c(s - r)$ for all $(g, \tilde{g})$ and such that $\mathbb{P}\{\|g - \tilde{g}\| \geq (s - r)^{5/4}\}$ decays faster than every power of $s - r$.

Note that Proposition 2.4 implies that there exist $c, \delta$

\[\rho(\mu_r, \mu_s) \leq c(s - r)^{1+\delta}.\]

This shows that $\mu_r$ is Hölder continuous of order $1 + \delta$ in $r$ and a standard argument shows that this means that $\mu_r$ is a constant function of $r$ and hence Proposition 2.3 holds.

### 3 Local commutation relation

In this section we will state the basic “commutation” relation that we will use. In order to state the relation precisely we will set up some notation. Although we only use it for $\kappa \leq 4$ in this paper, we will also give a result that holds for all $4 < \kappa < 8$. We fix $x_1 \neq x_2$ and $t_0 > 0$. Suppose $\gamma : [0, t_0] \to \mathbb{H}$ is a non-crossing curve parametrized by capacity from $x_1$ to $x_2$ in $\mathbb{H}$. Let $\tilde{\gamma}_R$ denote the reversed curve from $x_2$ to $x_1$ defined by $\tilde{\gamma}(t) = \gamma(t_0 - t)$, $0 \leq t \leq t_0$. Although $\tilde{\gamma}_R$ is not parametrized by capacity, we can reparametrize it $\gamma_R(t) = \tilde{\gamma}_R(\sigma(t))$ so that for each $t$, $\text{hcp}[\gamma_R] = at$. The total time duration of $\gamma_R$ is the same as that of $\gamma$, $t_0$.

If $0 < s_1 < s_2 < t_0$, we can write

\[\gamma = \gamma_{s_1} \oplus \gamma[s_1, s_2] \oplus \gamma[s_2, t_0],\]

Let us write $\gamma^1$ for $\gamma_{s_1}$ and $\gamma^2$ for the reversal of $\gamma[s_2, t_0]$, so that we have

\[\gamma = \gamma^1 \oplus \eta \oplus (\gamma^2)_R.\]

Let us view this at the moment as a decomposition modulo reparametrization but still remember that $\text{hcp}[\gamma] = at_0$ and we assume that

\[\text{hcp}[\gamma^1 \cup (\gamma^2)_R] = \text{hcp}[\gamma^1 \cup \gamma^2] < at_0.\]

We will also assume that

\[\gamma^1 \cap \gamma^2 = \emptyset.\]

If $\kappa \leq 4$, this will happen with probability one since $SLE_\kappa$ is supported on simple curves, but for $\kappa > 4$ this is a nontrivial constraint.

Suppose $r_1 + r_2 \leq t_0$, $V_1, V_2$ fixed subsets of $C$, and $\tau_1, \tau_2$ are stopping times for $\gamma^1, \gamma^2$ of the form

\[\tau_j = \min\{s : \text{hcp}[\gamma^j] = a r_j \text{ or } \gamma^j \notin \bar{V_j}\}.\]
Proof. Without loss of generality, we assume \( m \) corresponds to time \( r \). We then start \( SLE \) from \( x \) to \( \gamma_1(r) \) stopped before its hcap reaches \( a(1 - r) \). The difference in the construction of the measures \( \mathbb{P}_1^\kappa \) and \( \mathbb{P}_2^\kappa \) comes in the middle piece, which we may construct in two ways.

We view probability measures on curves from \( x_1 \) to \( x_2 \) of half-plane capacity \( at_0 \) as probability measures on ordered pairs \( \gamma = (\gamma^1, \gamma^2) := (\gamma^1, \gamma^2) \). Here \( \gamma^1, \gamma^2 \) are parametrized by capacity, that is, hcap[\( \gamma^2 \)] = \( a \). Note that if \( \gamma^1, \gamma^2 \) are nontrivial, then

\[
\text{hcap}[\gamma^1 \cup \gamma^2] < \text{hcap}[\gamma^1] + \text{hcap}[\gamma^2] \leq a(r_1 + r_2) = t_0,
\]

and hence the \( \eta \) in (8) is nontrivial. We will also assume that the stopping time is such that with probability one, \( \gamma^1 \cap \gamma^2 = \emptyset \). If \( \kappa \leq 4 \), \( t \) since \( \eta \) is not trivial. For \( \kappa > 4 \), we will guarantee it by choosing stopping times such that \( \gamma^1 \subset V_1, \gamma^2 \subset V_2 \) for some deterministic \( V_1, V_2 \) with \( V_1 \cap V_2 = \emptyset \). We now let \( \mathbb{P}_j^\kappa \) be the probability measure on \( \gamma \) given by

- Choose \( \gamma^j \) from \( SLE_\kappa \) from \( x_j \) to \( x_{3-j} \), conditioned to have total capacity \( at_0 \), stopped at time \( \tau_j \). Let \( z_j = \gamma^j(\tau_j) \).
- Given \( \gamma^j \), choose \( \gamma^{3-j} \) from \( SLE_\kappa \) from \( x_{3-j} \) to \( z_j \) in \( \mathbb{H} \setminus \gamma^j \), conditioned to that the total capacity of the union of the curve and \( \gamma^j \) is \( at_0 \), stopped at time \( \tau_{3-j} \).

The commutation result is that \( \mathbb{P}_1^\kappa = \mathbb{P}_2^\kappa \). We sketch the proof by giving the Radon-Nikodym derivative of each of the measures with respect to \( \mathbb{P} \), the measure obtained from independent \( SLE_\kappa \) paths. To state this we give some notation. Let \( D^j = \mathbb{H} \setminus \gamma^j, D = \mathbb{H} \setminus \gamma \). Let \( g^j, g \) be the corresponding conformal maps; let \( z_j = \gamma^j(\tau_j), U^j = g^j(z_j) \) and define \( h_2, h_1 \) by \( h_2 \circ g^j = g = h_1 \circ g_2 \).

**Proposition 3.1.** The Radon-Nikodym derivative of \( \mathbb{P}_j^\kappa \) with respect to \( \mathbb{P} \), the measure obtained from independent \( SLE \) paths from \( 0 \) to infinity stopped at times \( \tau_1, \tau_2 \), is given by

\[
\frac{d\mathbb{P}_j^\kappa}{d\mathbb{P}}(\gamma) = h_1^*(U^2) h_2^*(U^1) \exp \left\{ \frac{c}{2} \mathbb{m}_\mathbb{H}(\gamma^1, \gamma^2) \right\} \left| g(z_2) - g(z_1) \right|^{2b} \frac{\phi(|U^2 - U^1|, t_0 - \tau_1 + \tau_2)}{|x_2 - x_1|^{2b}} \frac{\phi(|x_2 - x_1|, 1)}{\phi(|x_2 - x_1|, 1)}.
\]

Here \( b = (6 - \kappa)/2\kappa \) is the boundary scaling exponent, \( c = (6 - \kappa)(3\kappa - 8)/2\kappa \) is the central charge, and \( m_\mathbb{H}(\gamma^1, \gamma^2) \) denotes the Brownian loop measure of loops in \( \mathbb{H} \) that intersect both \( \gamma^1 \) and \( \gamma^2 \) and \( \phi \) is as in (3). In particular, \( \mathbb{P}_1^\kappa = \mathbb{P}_2^\kappa \).

**Proof.** Without loss of generality, we assume \( t_0 = 1 \). We will prove the result for \( j = 1 \).

- We start by choosing \( \gamma^1 \) using \( SLE_\kappa \) from \( x_1 \) to \( x_2 \) stopped at time \( \tau_1 \). Here we are not conditioning on the total time duration of the path. The Radon-Nikodym derivative of this with respect to \( SLE \)
Figure 2: The maps $g^1, g^2, h^1$ and $h^2$ exhibit a commutative relation.

from 0 to infinity, restricted to the event that the total time duration is greater than $\tau_1$ is

$$
\frac{g'_1(x_2)^b \left| g_1(x_2) - U^1_2 \right|^{2b}}{|x_2 - x_1|^{2b}}.
$$

Let $\eta_2 = g_1 \circ \gamma^2$.

- Given $\gamma^1$, we will choose $\gamma^2$ using SLE from $x_2$ to $z_1$ in the domain $D_1$. We will do this in two steps.

- We first choose $\gamma^2$ using SLE from $x_2$ to infinity in $D_1$. Using the basic martingale of the restriction property this gives Radon-Nikodym derivative

$$
\exp \left\{ \frac{c}{2} m_D(\gamma^1, \gamma^2) \right\} \frac{h'_2(U^1)^b}{g'_1(x_2)^b}.
$$

Note that $\eta_2 := g_1 \circ \gamma_2$ is an SLE from $g_1(x_2)$ to infinity.

- We now tilt again so that $\eta_2 := g_1 \circ \gamma_2$ is an SLE from $g_1(x_2)$ to $U^1$. This gives a Radon-Nikodym derivative

$$
\frac{h'_2(U^1)^b \left| h_2(\eta_2(\tau_2)) - h_2(U_1) \right|^{2b}}{|g_1(x_2) - U|^{2b}} = h'_2(U^1)^b \frac{\left| g(z_2) - g(z_1) \right|^{2b}}{|g_1(x_2) - U|^{2b}}.
$$

- Multiplying the last two gives

$$
\exp \left\{ \frac{c}{2} m_D(\gamma^1, \gamma^2) \right\} \frac{h'_1(U^2)^b h'_2(U^1)^b}{g'_1(x_2)^b} \frac{\left| g(z_2) - g(z_1) \right|^{2b}}{|x_2 - x_1|^{2b}}.
$$

- We thus have that the Radon-Nikodym derivative restricted to the event that the total time duration is greater than $\tau_1 + \tau_2$ is given by:

$$
\frac{h'_1(U^2)^b h'_2(U^1)^b \exp \left\{ \frac{c}{2} m_D(\gamma^1, \gamma^2) \right\} \left| g(z_2) - g(z_1) \right|^{2b}}{|x_2 - x_1|^{2b}}.
$$
Figure 3: The difference in the construction comes in the curves $\eta$ and $\tilde{\eta}$. Given this, we may sample from $\mu_r$ and $\mu_{r+\epsilon}$ respectively, allowing us to conclude using basic facts about the Loewner equation.

- If we now condition so that the total time duration is one we get

$$h'_2(U^1)^b h'_2(U^2)^b \exp \left\{ \frac{c}{2} m_D(\gamma^1, \gamma^2) \right\} \frac{|g(z_2) - g(z_1)|^{2b}}{|x_2 - x_1|^{2b}} \phi(|U^2 - U^1|, 1 - (r_1 + r_2)) \phi(|x_2 - x_1|, 1).$$

4 Proof of main Theorem

We will use some basic facts about the Loewner equation.

Proposition 4.1. [Proposition 3.46] There exists $c < \infty$ such that if $D = \mathbb{H} \setminus K$ is a simply connected domain with $r = \sup \{|z| : z \in K\}$ and $h = \text{hcap}(K)$, then the corresponding conformal map $g : D \to \mathbb{H}$ satisfies for $|z| \geq 2r$,

$$|g_D(z) - z - \frac{h}{z}| \leq \frac{crh}{|z|^2}.$$  

In particular, if $K, \tilde{K}$ are two such hulls with $h = \tilde{h}$, then for $|z| \geq 2(r \wedge \tilde{r})$,

$$|g(z) - \tilde{g}(z)| \leq \frac{c(r + \tilde{r})h}{|z|^2}.$$  

Proposition 4.2. [Proposition 4.13] There exists $c < \infty$, such that if $U_t$ is a driving function with $U_0 = 0$ and $\gamma_t$ is the corresponding curve, then

$$\text{diam}[\gamma_t] \leq c \left[ \sqrt{t} + \max_{0 \leq s \leq t} |U_s| \right].$$  

We also need some easy estimates about our Bessel process conditioned to reach the origin at a given time.
Lemma 4.3. If $K < \infty$, there exists $\epsilon_0 > 0$ such that if $X_t$ satisfies the with $t_0 = 1$ and $|x_2 - x_1| \leq K$, then as $\epsilon \to 0$,

$$\mathbb{P}\{|X_{1-\epsilon}| \geq \sqrt{\epsilon} \log(1/\epsilon)\}$$

decays faster than every power of $\epsilon$.

Proof. By a coupling argument, the probability on the left restricted to $|x_2 - x_1| \leq K$ is maximized when $x_2 - x_1 = K$. In this case, we can look at the transition probability. \hfill \Box

We write $\epsilon = s - r$. We decompose a simple path $\gamma$ from 0 to $x$ with $\text{hcp}[\gamma] = a$ as

$$\gamma = \gamma^1 \oplus \eta \oplus (\gamma')^R \oplus (\gamma^2)^R,$$

where the decomposition is defined by

$$\text{hcp}[\gamma^1] = ra, \quad \text{hcp}[\gamma^1 \cup \eta] = sa, \quad \text{hcp}[\gamma^1 \cup \gamma^2] = (t_0 - \epsilon)a.$$

Using the definition and the conformal Markov property, we can see that when we sampling from $\mu_{s}$ we choose the paths in order $\gamma^1, \eta, \gamma^2, \gamma'$. When we sample from $\mu_{r}$ we use the order $\gamma^1, \gamma^2, \eta', \gamma^R$. In each case the distribution is $\text{SLE}$ to the endpoint of the other curve in the domain slit by the curves at that point, conditioned to have the appropriate total half-plane capacity and stopped as specified above.

We now use Proposition 3.1 to say that another way to sample from $\mu_{s}$ is to choose the paths in order $\gamma^1, \gamma^2, \eta, \eta'$. Hence we can write the sampling as follows. Steps 1 and 2 are the same for both sampling methods. Step 3a is used for $\mu_{s}$ and Step 3b is used for $\mu_{r}$.

- **Step 1:** Choose $\gamma^1$ from $\text{SLE}$ from 0 to $x_0$ conditioned to have total half-plane capacity $a$ stopped at time $r$, that is, stopped when $\text{hcp}[\gamma^1] = ar$. Let $z_1 = \gamma(r)$, let $\tilde{g} : \mathbb{H} \setminus \gamma^1 \to \mathbb{H}$ be the corresponding transformation, and let $y_1 = \tilde{g}(z_1)$, $x_1 = \tilde{g}(x_0)$.

- **Step 2:** Choose $\eta$ from $\text{SLE}$ from $x_1$ to $y_1$ conditioned to have total half-plane capacity $a(1-r)$ stopped at time $1 - s$, that is, stopped when $\text{hcp}[\eta] = a(1-s)$. Let $h : \mathbb{H} \setminus \eta \to \mathbb{H}$ be the corresponding transformation, and let $y_2 = h(y_1), x_2 = h(\eta(1-s))$. Let $\gamma^2 = \tilde{g}^{-1} \circ \eta$ and $w_1 = g^{-1}(\eta(1-s))$. Let $\hat{h} = h \circ \tilde{g}$ and note that $\hat{h} : \mathbb{H} \setminus (\gamma^1 \cup \gamma^2) \to \mathbb{H}$ is the corresponding conformal transformation which satisfies $\hat{h}(z_1) = y_2, \hat{h}(w_1) = x_2$.

- **Step 3a:** Choose $\omega^1$ from $\text{SLE}$ from $y_2$ to $x_2$ conditioned to have total half-plane capacity $a \epsilon$ stopped at the first time that

$$\text{hcp}[h^{-1} \circ \omega^1] = a \epsilon.$$

This is the same as the first time that

$$\text{hcp}[\gamma^1 \cup \hat{h}^{-1} \circ \omega^1] = as.$$

Let this time be $u$ and let $\phi : \mathbb{H} \setminus \omega^1 \to \mathbb{H}$ be the corresponding transformation with $y_3 = \phi(\omega^1), x_3 = \phi(x_2)$. Let $\tilde{\omega}^2$ be chosen from $\text{SLE}$ from $x_3$ to $y_3$ conditioned to have half-plane capacity $\epsilon(u)$ giving conformal map $\phi$ and let $\omega^2 = \phi^{-1} \circ \tilde{\omega}^2$ and

$$\omega = \omega^1 \oplus [\omega^2]^R.$$

Let $\psi : \mathbb{H} \setminus \omega \to \mathbb{H}$ be the corresponding conformal transformation.

- **Step 3b** Choose $\omega^*$ from $\text{SLE}$ from $x_2$ to $y_2$ conditioned to have total half-plane capacity $a \epsilon$ and set

$$\hat{\omega} = [\omega^*]^R.$$

Let $\hat{\psi} : \mathbb{H} \setminus \hat{\omega} \to \mathbb{H}$ be the corresponding conformal transformation.
Figure 4: A schematic showing the full picture, though not drawn to scale (in particular, the yellow and blue segments ought not to have comparable lengths). The dotted arrows on the right correspond to the commutation relation, and together with some Loewner estimates, we may conclude that the laws of the measures obtained, regardless of the path one chooses in the schematic, are the same.
In our coupling we use the complete coupling for steps 1 and 2. Hence we write

\[ g = \psi \circ h, \quad \tilde{g} = \tilde{\psi} \circ h, \]

where \( h \) is the same in both cases. If \( z \in \mathcal{I} \), then \( \text{Im}(h(z)) \geq \sqrt{4a} \). Except for an event of probability that decays faster than every power of \( \epsilon \), we have \( x_2 - y_2 \leq \epsilon^{1/2} \log(1/\epsilon) \). Using this, we see that in step 3a and in step 3b we get a curve with the same initial and terminal points, of half plane capacity \( a\epsilon \) and such that, except for an event of probability that decays faster than every power of \( \epsilon \), has diameter bounded by \( \epsilon^{1/2} \log^2 \epsilon \). Let \( \psi, \tilde{\psi} \) be the conformal transformations. Then if \( \text{Im}(z) \geq \sqrt{a} \) we have

\[ |\psi(z) - \tilde{\psi}(z)| \leq c \epsilon, \]

and, except for an event of probability that decays faster than every power of \( \epsilon \),

\[ |\psi(z) - \tilde{\psi}(z)| \leq \epsilon^{5/4}. \]

Therefore, in this coupling, with probability one \( \|g - \tilde{g}\| \leq c \epsilon \) and

\[ \mathbb{P}\{\|g - \tilde{g}\| \geq \epsilon^{5/4}\} \leq c \epsilon^3. \]

5 Proof of Lemma 2.2

We fix \( a > 1/4 \) and allow constants to depend on \( a \). We assume that \( X_t \) satisfies (7). For ease we will assume \( x > 0 \) but the proof with \( x < 0 \) is essentially the same.

The proof follows from the easy estimate

\[ -X_t \leq U_t \leq \int_0^t \frac{a}{X_s} \, ds \]

and the following two lemmas that handle the two sides of the inequality. For the lower bound, we get a somewhat sharper estimate.

**Lemma 5.1.** There exists \( c < \infty \) such that if \( X_t \) satisfies (7) with \( X_0 = x_0 \sqrt{t_0} > 0 \), then for all \( r > 0 \),

\[ \mathbb{P}\left\{ \max_{0 \leq t \leq t_0} (X_t/\sqrt{t_0}) \geq x_0 + r \right\} \leq c \exp \left\{ -\frac{r^2}{4} \right\}. \]

**Proof.** We may assume that \( r^2 \geq 1 + 4a \) and by scaling we may assume \( t_0 = 1 \). Let \( y = x_0 + r \) and let \( \sigma = \inf\{t : X_t = y\} \). The equation (7) can be obtained by starting with \( X_t \) satisfying (3) where \( W_t \) is a \( \mathbb{P}^* \) Brownian motion and then tilting by the martingale \( N_t \) as in (6) to get the measure \( \mathbb{P}^* \). Hence,

\[ \mathbb{P}\{\sigma < 1\} \leq M_0^{-1} \mathbb{E}^* [M_{\sigma}; \sigma < \infty]. \]

Note that

\[ M_0 = x_0^{4a-1} \exp \left\{ -\frac{x_0^2}{2} \right\}, \]

and if \( \sigma < 1 \),

\[ M_\sigma \leq \max_{0 \leq t \leq 1} y^{4a-1} (1-t)^{-1/2-2a} \exp \left\{ -\frac{y^2}{2(1-t)} \right\} = y^{4a-1} e^{-y^2/2}. \]

The equality uses \( r^2 \geq 1 + 4a \). Therefore,

\[ \frac{M_\sigma}{M_0} \leq \left[ 1 + \frac{\log(1/\epsilon)}{x_0} \right]^{4a-1} \exp \left\{ -x_0 r - \frac{r^2}{2} \right\} \leq c \exp \left\{ -\frac{r^2}{4} \right\}. \]
Lemma 5.2. If \( a > 1/4 \), there exist \( u > 0 \) and \( c < \infty \) such that for any \( x > 0 \) and \( t_0 > 0 \) if \( X_t \) satisfies (7), then for all \( r > 0 \),
\[
\mathbb{P}^x \left\{ \int_0^{t_0} ds \frac{ds}{X_s} dY \geq r \sqrt{t_0} \right\} \leq ce^{-ur}.
\]

Proof. Let
\[
I_n = \int_0^1 ds \frac{ds}{X_s} 1\{2^{-n} \leq X_s < 2^{-n+1}\} ds.
\]

Our first goal is to show that there exists \( c^* < \infty \) such that for all \( x,t_0,n \),
\[
\mathbb{E}^x[I_n] \leq c^* 2^{-n}, \quad \mathbb{P}^x\{I_n \geq c^* 2^{-n+1}\} \leq \frac{1}{2}.
\] (9)

The second follows from the first by the Markov property; by scaling, it suffices to show the first inequality for \( n = 0 \). By the strong Markov property, we may assume that \( 1 \leq x \leq 2 \); otherwise, we first run the process until it reaches \([1,2]\). Also, note that
\[
\mathbb{E}^x[I_0] \leq \int_1^2 \int_0^{t_0} \phi_t(x,y; t_0) dt \, dy.
\]

Using the immediate estimate
\[
\int_1^2 \left[ \int_0^{t_0} \phi_t(x,y; t_0) dt + \int_0^1 \phi_t(x,y; t_0) dt \right] dy \leq 2,
\]
we see that it suffices to show that there exists \( c \) such that for all \( 1 \leq x, y \leq 2 \) and \( t_0 \geq 1 \),
\[
\int_1^{t_0-1} \phi_t(x,y; t_0) dt \leq c.
\]

This can be done in a straightforward way by looking at the transition probability. Indeed, if \( 1 \leq s \leq t_0 - 1 \) and \( 1 \leq x, y \leq 2 \),
\[
\phi_t(x,y; t_0) \leq c \left[ \frac{t_0}{t_0 - t} \right]^{2a+\frac{1}{2}} \frac{1}{t^{2a+\frac{1}{2}}}.
\]

For \( t < t_0/2 \) we estimate this by \( ct^{-(2a+\frac{1}{2})} \) and for \( t \geq t_0/2 \), we estimate this by \( c (t_0 - t)^{-(2a+\frac{1}{2})} \). Provided that \( a > 1/4 \) we see that this integral is uniformly bounded in \( t_0 \). This gives (9).

By scaling it suffices to prove our main result for \( t_0 = 1 \). Note that
\[
\int_0^1 ds \frac{ds}{X_s} ds \leq 1 + \sum_{n=1}^{\infty} I_n,
\]
where
\[
I_n = \int_0^1 ds X_s \{2^{-n} \leq X_s < 2^{-n+1}\} ds.
\]

By iterating (9) using the strong Markov property, we see that for all positive integers \( k \), \( \mathbb{P}\{I_n \geq 2k c^* 2^{-n}\} \leq 2^{-k} \) and hence for all \( r > 0 \),
\[
\mathbb{P}\{I_n \geq r 2^{-n}\} \leq c' e^{-ur},
\]
where \( u = (\log 2)/(2c^*) \), \( c' = e^u \). In particular,
\[
\mathbb{P} \left\{ \sum_{n=1}^{\infty} I_n \geq 2r \right\} \leq \sum_{n=1}^{\infty} \mathbb{P}\{I_n \geq r(2/3)^n\} \leq c' \sum_{n=1}^{\infty} \exp \{-ur(4/3)^n\} \leq c e^{-2u(2r)/3}.
\]
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