Derivative with two fractional orders: New door of investigation toward revolution in fractional calculus

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Abstract: In order to describe more complex problem using the concept of fractional derivative, we introduce in this paper a derivative with two fractional orders. Each order therefore plays an important rule while modeling for instance problems with two layers with different properties. This is the case for instance in thermal science for a reaction diffusion within a media with two different layers with different properties. Another case of this study can be used in the study of groundwater flowing with an aquifer where the ground has layers with different properties. The paper presents some basic definitions and properties.

Keywords: New concept of fractional derivative, bi-order, generalized Mittag-Leffler function.

1 Introduction

The concept of fractional calculus was introduced by mathematicians to solve problems that could not be handled by local derivative, the fractional order alpha that appears in the concept of fractional derivative can be used to represent some physical parameters. Nevertheless it is not possible for the fractional derivative to be used in order to describe the movement of for instance heat via material with different layers where each layer possesses different material from each other. Riemann-Liouville introduced the fractional derivative as a derivative with convolution of a given function and the power law function [1-5]. Caputo introduced the fractional derivative as convolution of derivative of a given function with power law function. Caputo and Fabrizio introduced the fractional derivative as convolution of derivative of a given function with exponential [2-4]. Atangana and Goufo independently with Caputo and Fabrizio introduced another one as derivative of a convolution of a given function with the exponential function. Atangana and Dumitru introduced fractional derivative as a both convolution of a derivative of a given function and the Mittag-Leffler function [4-5]. These concepts of fractional derivative so far has been used with great success and misused also. Nevertheless they are not able to handle the concept of heterogeneity with great success. The concept of Brownian motion was introduced using fractional derivative to describe to choice that a particle
has when moving via heterogeneous media, however, this concept failed due to the fact that, the particle does not choose where to move within a given heterogeneous media, but it condition by the properties of the media [6-7]. For instance, the drop of water is moving via a given geological formation under the influence of the properties of the soul. It is therefore important to introduce a suitable derivative that can used to handle this type of physical problem. In this paper, we introduce the concept of fractional order with two orders alpha and beta.

2 New concept of fractional derivatives

In this section, some new definitions of fractional derivatives with bi-order are presented.

**Definition 1**: Let $f(x)$ defined in $[a, X]$ necessary differentiable such that for all $x \in [a, X]$ the convolution of $x^{1-\alpha-n}E_\beta\left\{ -\frac{\beta}{1-\beta}x^{\beta+\alpha}\right\}$ with the function $\frac{d^n f(x)}{dx^n}$ exists, then the fractional derivative of the function $f$ with order $\alpha, \beta \in (0, 1)$ is given as:

$$A^{\alpha,\beta}_a D_x^\alpha f(x) = A(\beta) \frac{1}{1-\beta \Gamma\{1-\alpha\}} \int_a^x \frac{d^n f(t)}{dt^n} (x-t)^{1-\alpha-n}E_\beta\left\{ -\frac{\beta}{1-\beta}(x-t)^{\beta+\alpha}\right\} dt,$$

$0 < \beta < 1, \quad n-1 < \alpha < n.$

**Definition 2**: Let $f(x)$ defined in $[a, X]$ not necessary differentiable such that for all $x \in [a, X]$ the convolution of $x^{1-\alpha-n}E_\beta\left\{ -\frac{\beta}{1-\beta}x^{\beta+\alpha}\right\}$ with the function $f(x)$ exists, then the fractional derivative of the function $f$ with order $\alpha, \beta \in (0, 1)$ is given as:

$$A^{\alpha,\beta}_a D_x^\alpha f(x) = A(\beta) \frac{1}{1-\beta \Gamma\{1-\alpha\}} \int_a^x f(t)(x-t)^{1-\alpha-n}E_\beta\left\{ -\frac{\beta}{1-\beta}(x-t)^{\beta+\alpha}\right\} dt,$$

$0 < \beta < 1, \quad n-1 < \alpha < n.$

**Definition 3**: Let $f(x)$ defined in $[X, \infty]$ not necessary differentiable such that for all $x \in [0, \infty]$ the convolution of $x^{1-\alpha-n}E_\beta\left\{ -\frac{\beta}{1-\beta}x^{\beta+\alpha}\right\}$ with the function $\frac{d^n f(x)}{dx^n}$ exists, then the fractional derivative of the function $f$ with order $\alpha, \beta \in (0, 1)$ is given as:

$$A^{\alpha,\beta}_a D_x^\alpha f(x) = A(\beta) \frac{1}{1-\beta \Gamma\{1-\alpha\}} \int_x^\infty f(t)(t-x)^{1-\alpha-n}E_\beta\left\{ -\frac{\beta}{1-\beta}(x-t)^{\beta+\alpha}\right\} dt,$$

$0 < \beta < 1, \quad n-1 < \alpha < n.$

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Definition 4: Let \( f(x) \) defined in \([X, \infty]\) necessary differentiable such that for all \( x \in [0, \infty] \) the convolution of \( x^{1-\alpha-n} E_\beta \left\{ -\frac{\beta}{1-\beta} x^{\beta+\alpha} \right\} \) with the function \( f(x) \) exists, then the fractional derivative of the function \( f \) with order \( \alpha, \beta \in (0, 1) \) is given as:

\[
\frac{AC}{\alpha} D_x^{\alpha, \beta} f(x) = A(\beta) \frac{1}{1 - \beta \Gamma \{1 - \alpha\}} \int_x^\infty \frac{d^n f(t)}{dt^n} (t-x)^{1-\alpha-n} E_\beta \left\{ -\frac{\beta}{1-\beta} (x-t)^{\beta+\alpha} \right\} dt,
\]

\( 0 < \beta < 1, \quad n-1 < \alpha < n. \)

Definition 5: Let \( f(x) \) defined in \([-\infty, X]\) not necessary differentiable such that for all \( x \in [-\infty, X] \) the convolution of \( x^{1-\alpha-n} E_\beta \left\{ -\frac{\beta}{1-\beta} x^{\beta+\alpha} \right\} \) with the function \( f(x) \) exists, then the fractional derivative of the function \( f \) with order \( \alpha, \beta \in (0, 1) \) is given as:

\[
\frac{AR}{\alpha} D_x^{\alpha, \beta} f(x) = A(\beta) \frac{1}{1 - \beta \Gamma \{1 - \alpha\}} \int_{-\infty}^x \frac{d^n f(t)}{dt^n} f(t)(x-t)^{1-\alpha-n} E_\beta \left\{ -\frac{\beta}{1-\beta} (x-t)^{\beta+\alpha} \right\} dt,
\]

\( 0 < \beta < 1, \quad n-1 < \alpha < n. \)

Definition 6: Let \( f(x) \) defined in \([-\infty, X]\) necessary differentiable such that for all \( x \in [-\infty, X] \) the convolution of \( x^{1-\alpha-n} E_\beta \left\{ -\frac{\beta}{1-\beta} x^{\beta+\alpha} \right\} \) with the function \( f(x) \) exists, then the fractional derivative of the function \( f \) with order \( \alpha, \beta \in (0, 1) \) is given as:

\[
\frac{AC}{\alpha} D_x^{\alpha, \beta} f(x) = A(\beta) \frac{1}{1 - \beta \Gamma \{1 - \alpha\}} \int_{-\infty}^x \frac{d^n f(t)}{dt^n} (x-t)^{1-\alpha-n} E_\beta \left\{ -\frac{\beta}{1-\beta} (x-t)^{\beta+\alpha} \right\} dt,
\]

\( 0 < \beta < 1, \quad n-1 < \alpha < n. \)

3 New properties of the A.R and A.C derivatives

Some properties of the A.R and A.C derivatives with two order \( \alpha, \beta \) are given below:

The Laplace transform of A.R:

The Laplace transform of A.R derivative is obtained via Laplace transform
definition below:

\[
L \{A^R D^\alpha_D^\beta_t f(x)\} = \frac{A(\beta)}{1 - \beta \Gamma \{1 - \alpha\}} \int_0^\infty e^{-pt} \frac{d}{dt} \int_0^t f(\tau)(t - \tau)^{-\alpha} E_{\beta} \left\{ -\frac{\beta}{1 - \beta} (t - \tau)^{\beta + \alpha} \right\} d\tau dt
\]

\[
= \frac{A(\beta)}{1 - \beta \Gamma \{1 - \alpha\}} L \{t^{-\alpha} E_{\beta} \left\{ -\frac{\beta}{1 - \beta} t^{\beta + \alpha} \right\} \} L \{f(t)\} \tag{7}
\]

\[
= \frac{A(\beta)}{1 - \beta \Gamma \{1 - \alpha\}} L \{t^{1-\alpha-1} E_{\beta} \left\{ -\frac{\beta}{1 - \beta} t^{\beta + \alpha} \right\} \} F(p)
\]

\[
= \frac{A(\beta)}{1 - \beta \Gamma \{1 - \alpha\}} L \{t^{\beta - 1} E_{\beta} \left\{ -\frac{\beta}{1 - \beta} t^{\sigma} \right\} \} F(p)
\]

However, we have from the literature that

\[
L \{t^{\beta-1} E_{\beta, \Psi}(at^\alpha)\} = \frac{s^{-p}}{\Gamma(\delta)} 2\Psi_1 \left[ \begin{array}{ccc} (\delta, 1), (\Psi, \alpha); & \frac{a}{s^\alpha} \\ (\gamma, \beta) \end{array} \right] \tag{8}
\]

for \(\alpha, p, \delta, \beta, \gamma > 0\) thus if we take \(\delta = \Psi = 1\), \(a = -\frac{\beta}{1 - \beta}\), and \(\sigma = \alpha + \beta\) then we obtain the following

\[
L \left\{t^{\beta-1} E_{\beta, \Psi}(at^\alpha)\right\} = \frac{s^{\alpha-1}}{2\Psi_1} \left[ \begin{array}{ccc} (1, 1), (1 - \alpha, \alpha + \beta); & -\frac{\beta}{1 - \beta} \cdot \frac{1}{s^{\alpha + \beta}} \end{array} \right] \tag{9}
\]

Where the function \(p \Psi_q\) is the Wright’s generalized hyper-geometric function defined by means of the similar representation in the following form [8-9]:

\[
p \Psi_q(z) = \sum_{r=0}^{\infty} \left\{ \prod_{j=1}^{p} r(a_j + A_j r) \right\} \frac{z^r}{r!}, \tag{10}
\]

therefore replacing the above we obtain

\[
L \{A^R D^\alpha_D^\beta_t f(t)\} = \frac{A(\beta)}{1 - \beta \Gamma \{1 - \alpha\}} s^\alpha 2\Psi_1 \left[ \begin{array}{ccc} (1, 1), (1 + \alpha + \beta); & -\frac{\beta}{1 - \beta} \cdot \frac{1}{s^{\alpha + \beta}} \end{array} \right] \tag{11}
\]

Existence of A.C:
Theorem 1: Let \( f \) be continuous in \([0, T]\) and \( \alpha, \beta \in (0, 1) \) then the following inequality is obtained if
\[
\| f'(t) \| \leq \Theta \| f(t) \|_C
\]  
(12)
Proof:
\[
\left\| AC_t^\alpha \beta f(t) \right\|_C
= \frac{A(\beta)}{1 - \beta \Gamma \{1 - \alpha\}} \int_0^t \frac{df(\tau)}{d\tau} (t - \tau)^{-\alpha} E_\beta \left\{ -\frac{\beta}{1 - \beta} (t - \tau)^{\beta + \alpha} \right\} d\tau
\]
\[
\leq \frac{A(\beta)}{1 - \beta \Gamma \{1 - \alpha\}} \Theta \| f(t) \|_C \int_0^t (t - \tau)^{-\alpha} E_\beta \left\{ -\frac{\beta}{1 - \beta} (t - \tau)^{\beta + \alpha} \right\} d\tau
\]
\[
\leq \frac{A(\beta)}{1 - \beta \Gamma \{1 - \alpha\}} \Theta \| f(t) \|_C \int_0^t (t - \tau)^{1 - \alpha - 1} E_\beta \left\{ -\frac{\beta}{1 - \beta} (t - \tau)^{\beta + \alpha} \right\} d\tau
\]
\[
\leq \frac{A(\beta)}{1 - \beta \Gamma \{1 - \alpha\}} \Theta \| f(t) \|_C \int_0^t (t - \tau)^{p - 1} E_\beta \left\{ -\frac{\beta}{1 - \beta} (t - \tau)^{\beta + \alpha} \right\} d\tau.
\]
Let \( y = t - \tau \) and \( p = 1 - \alpha \) then
\[
\int_0^t (t - \tau)^{p - 1} E_\beta \left\{ -\frac{\beta}{1 - \beta} (t - \tau)^{\beta + \alpha} \right\} d\tau = \int_0^t y^{p - 1} E_\beta \left\{ -\frac{\beta}{1 - \beta} y^{\beta + \alpha} \right\} dy.
\]
(14)
However
\[
\int_0^t y^{p - 1} E_\beta \left\{ -\frac{\beta}{1 - \beta} y^{\beta + \alpha} \right\} dy = t^p E_{\beta, 2} \left\{ -\frac{\beta}{1 - \beta} t^{\beta + \alpha} \right\}
\]
(15)
then
\[
\left\| AC_t^\alpha \beta f(t) \right\|_C \leq \| f \|_C K,
\]
(16)
\[
K = \frac{A(\beta)}{1 - \beta \Gamma \{1 - \alpha\}} \Theta t^{1 - \alpha} E_{\beta, 2} \left\{ -\frac{\beta}{1 - \beta} T^{\beta + \alpha} \right\}
\]
Test of the Lipschitz continuity:
Let \( f \) and \( g \) be two defined function on \([0, T]\) such that their A.C derivatives exist then

\[
\left\| A^C D_t^{\alpha, \beta} f(t) - A^C D_t^{\alpha, \beta} g(t) \right\|_C \leq \frac{A(\beta)}{1 - \beta} \frac{1}{\Gamma\{1 - \alpha\}} \left\| f - g \right\|_C \int_0^t \frac{d(f - g)(\tau)}{d\tau} (t - \tau)^{-\alpha} E_{\beta} \left\{ -\frac{\beta}{1 - \beta} (t - \tau)^{\beta + \alpha} \right\} d\tau
\]

This implies A.C has Lipschitz continuity condition.

**Theorem 2**: Let \( f \) be a continuous function defined in \([0, T]\) such that \( A^C D_t^{\alpha, \beta} f(t) \) and \( A^R D_t^{\alpha, \beta} f(t) \) exists, then the following relation is established.

\[
A^C D_t^{\alpha, \beta} f(t) = -G(t) + A^R D_t^{\alpha, \beta} f(t).
\]

Proof: We achieve this using the Laplace transform

\[
L \left\{ A^C D_t^{\alpha, \beta} f(t) \right\} = \frac{A(\beta)}{1 - \beta} \frac{1}{\Gamma\{1 - \alpha\}} \left\{ \int_0^t \frac{d(f(\tau))}{d\tau} (t - \tau)^{-\alpha} E_{\beta} \left\{ -\frac{\beta}{1 - \beta} (t - \tau)^{\beta + \alpha} \right\} d\tau \right\}
\]

This implies A.C has Lipschitz continuity condition.
Therefore the inverse Laplace transform applied on the above yields,

\[ \begin{align*}
\mathcal{L}^{-1} \left[ A^C D_t^{\alpha,\beta} f(t) \right] &= \mathcal{L}^{-1} \left[ A^R D_t^{\alpha,\beta} f(t) - \frac{A(\beta)}{1 - \beta \Gamma \{1 - \alpha\}} f(0) t^{-\alpha} E_\beta \left\{ -t^{\beta + \alpha} \right\} \right] \\
&= \mathcal{L}^{-1} \left[ A^R D_t^{\alpha,\beta} f(t) - G(t) \right].
\end{align*} \tag{21} \]

This completes the proof.

**Sumudu Transform of A.C:**

The Sumudu transform of A.C is expressed as follows:

\[ S \left( A^R D_t^{\alpha,\beta} f(t) \right) = \frac{A(\beta)}{1 - \beta \Gamma \{1 - \alpha\}} S \left( \int_0^t f(\tau) (t-\tau)^{-\alpha} E_\beta \left\{ -\frac{\beta}{1 - \beta} (t-\tau)^{\beta + \alpha} \right\} d\tau \right) \]

\[ = \frac{A(\beta)}{1 - \beta \Gamma \{1 - \alpha\}} \frac{1}{u} S \left( \int_0^t f(\tau) (t-\tau)^{-\alpha} E_\beta \left\{ -\frac{\beta}{1 - \beta} (t-\tau)^{\beta + \alpha} \right\} d\tau \right) \]

\[ = \frac{A(\beta)}{1 - \beta \Gamma \{1 - \alpha\}} \frac{1}{u} S (f(t)).S \left( -t^{-\alpha} E_\beta \left\{ -\frac{\beta}{1 - \beta} t^{\beta + \alpha} \right\} \right) \tag{22} \]

where

\[ S \left( t^{-\alpha} E_\beta \left\{ -\frac{\beta}{1 - \beta} t^{\beta + \alpha} \right\} \right) = \int_0^\infty e^{-u} t^{-\alpha} E_\beta \left\{ -\frac{\beta}{1 - \beta} (ut)^{\beta + \alpha} \right\} du \tag{23} \]

\[ = u^{-\alpha} \int_0^\infty e^{-t} t^{-\alpha} E_\beta \left\{ -\lambda t^{\beta + \alpha} \right\} dt, \]

where \( \lambda = \frac{\beta}{1 - \beta} u^{\alpha + \beta} \).

\[ S \left( t^{-\alpha} E_\beta \left\{ -\frac{\beta}{1 - \beta} t^{\beta + \alpha} \right\} \right) = u^{-\alpha} \int_0^\infty e^{-t} t^{-\alpha} E_\beta \left\{ -\lambda t^{\beta + \alpha} \right\} dt \tag{24} \]

\[ = u^{-\alpha} \int_0^\infty e^{-t} t^{-\alpha} \sum_{j=0}^{\infty} \left\{ -\lambda t^{\beta + \alpha} \right\}^j \frac{1}{\Gamma (\alpha j + 1)} \]
where \( \lambda = \frac{\beta}{1-\beta} u^{\alpha+\beta} \).

\[
\begin{align*}
\lambda &= \beta u^{\alpha+\beta} \\
&= u^{-\alpha} \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{\Gamma(\alpha j + 1)} \int_{0}^{\infty} e^{-t^{\alpha j + \beta j - \alpha - 1}} dt \\
&= u^{-\alpha} \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{\Gamma(\alpha j + 1)} \int_{0}^{\infty} e^{-t^{\Theta-1}} dt
\end{align*}
\]

where \( \Theta = \alpha j + \beta j - \alpha + 1 \).

\[
\begin{align*}
\Theta &= \alpha j + \beta j - \alpha + 1 \\
&= u^{-\alpha} \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{\Gamma(\alpha j + 1)} \Gamma(\Theta) \\
&= u^{-\alpha} \sum_{j=0}^{\infty} \frac{(-\lambda)^j}{\Gamma(\alpha j + 1)} \Gamma(\alpha j + \beta j - \alpha + 1) \\
&= u^{-\alpha} \sum_{j=0}^{\infty} \left( -\frac{\beta}{1-\beta} u^{\beta+\alpha} \right)^j \Gamma(\alpha j + \beta j - \alpha + 1)
\end{align*}
\]

Therefore the Sumudu transform of \( A^R D_t^{\alpha,\beta} f(t) \) is given as

\[
\begin{align*}
S \left( A^R D_t^{\alpha,\beta} f(t) \right) &= A(\beta) \frac{1}{1-\beta} u^{-\alpha-1} \sum_{j=0}^{\infty} \left( -\frac{\beta}{1-\beta} u^{\beta+\alpha} \right)^j \Gamma(\alpha j + \beta j - \alpha + 1) \\
\end{align*}
\]

### 4 Partial derivative with A.C and A.R

Since one can model many natural occurrence using the space and time, it is therefore important to propose in this section the partial A.C and A.R fractional derivative with order \( \alpha, \beta \).

**Definition 7**: Let \( f(x, t) \) be a function differentiable in \( x \) or \( t \) direction. Let \( \alpha, \beta \in (0, 1) \) such that the convolution of \( x^{-\alpha} E_{\beta} \left\{ -\frac{\beta}{1-\beta} x^{\beta+\alpha} \right\} \) and \( \frac{\partial f}{\partial x} \) exists than the A.C partial fractional derivative of \( f \) of order \( \alpha, \beta \) is given as:

\[
\begin{align*}
A^C D_x^{\alpha,\beta} f(x, t) &= A(\beta) \frac{1}{1-\beta} \frac{1}{\Gamma(1-\alpha)} \int_{0}^{x} \frac{\partial f(\xi, t)}{\partial \xi} (x - \xi)^{-\alpha} E_{\beta} \left\{ -\frac{\beta}{1-\beta} (x - \xi)^{\beta+\alpha} \right\} d\xi.
\end{align*}
\]
Definition 8: Let \( f(x, t) \) be a function not necessary differentiable in \( x \) or \( t \) direction. Let \( \alpha, \beta \in (0, 1) \) such that the convolution of \( x^{-\alpha} E_{\beta} \left\{ -\frac{\beta}{1-\beta} x^{\beta+\alpha} \right\} \) and \( \frac{\partial f}{\partial x} \) exists than the A.R partial fractional derivative of \( f \) of order \( \alpha, \beta \) is given as:

\[
\begin{align*}
\mathcal{A}_{\mathbb{R}}{}^{\alpha,\beta} D^0_x f(x, t) &= A(\beta) \left[ \frac{1}{1 - \beta \Gamma\{1 - \alpha\}} \frac{\partial}{\partial x} \int_0^x f(\xi, t)(x - \xi)^{-\alpha} E_{\beta} \left\{ -\frac{\beta}{1 - \beta} (x - \xi)^{\beta+\alpha} \right\} d\xi \right].
\end{align*}
\]

Definition 9: Let \( f(x, t) \) be a function such that \( \frac{\partial^2 f}{\partial x \partial t} \) exists. Let \( \alpha, \beta \in (0, 1) \) then A.C partial derivative of \( f \) of order \( \alpha, \beta \) is given as:

\[
\begin{align*}
\mathcal{A}_{\mathbb{C}}{}^{\alpha,\beta} D^0_{x,t} f(x, t) &= A(\beta) \left[ \frac{1}{1 - \beta \Gamma\{1 - \alpha\}} \int_0^t \int_0^t \frac{\partial^2 f(\xi, \lambda)}{\partial \xi \partial \lambda} (x - \xi)^{-\alpha} (t - \lambda)^{-\alpha} \right. \\
&\quad \left. \cdot E_{\beta} \left\{ -\frac{\beta}{1 - \beta} (x - \xi)^{\beta+\alpha} \right\} E_{\beta} \left\{ -\frac{\beta}{1 - \beta} (t - \lambda)^{\beta+\alpha} \right\} d\xi d\lambda. \right]
\end{align*}
\]

However if \( \frac{\partial^2 f}{\partial x \partial t} \) does not exist then

\[
\begin{align*}
\mathcal{A}_{\mathbb{R}}{}^{\alpha,\beta} D^0_{x,t} f(x, t) &= A(\beta) \left[ \frac{1}{1 - \beta \Gamma\{1 - \alpha\}} \int_0^x \int_0^t \frac{\partial^2 f(\xi, \lambda)}{\partial \xi \partial \lambda} (x - \xi)^{-\alpha} (t - \lambda)^{-\alpha} \right. \\
&\quad \left. \cdot E_{\beta} \left\{ -\frac{\beta}{1 - \beta} (x - \xi)^{\beta+\alpha} \right\} E_{\beta} \left\{ -\frac{\beta}{1 - \beta} (t - \lambda)^{\beta+\alpha} \right\} d\xi d\lambda. \right]
\end{align*}
\]

Theorem 3: Let \( f \) be a function such that \( \frac{\partial^2 f}{\partial x \partial t} \) and \( \frac{\partial^2 f}{\partial t \partial x} \) exist and continuous. Let \( \alpha, \beta \in (0, 1) \) then the following equality holds.

\[
\begin{align*}
\mathcal{A}_{\mathbb{C}}{}^{\alpha,\beta} D^0_{x,t} f(x, t) &= \mathcal{A}_{\mathbb{C}}{}^{\alpha,\beta} D^0_{t,x} f(x, t)
\end{align*}
\]

Proof: By definition we have that

\[
\begin{align*}
\mathcal{A}_{\mathbb{C}}{}^{\alpha,\beta} D^0_{x,t} f(x, t) &= A(\beta) \left[ \frac{1}{1 - \beta \Gamma\{1 - \alpha\}} \int_0^x \int_0^t \frac{\partial^2 f(\xi, \lambda)}{\partial \xi \partial \lambda} (x - \xi)^{-\alpha} (t - \lambda)^{-\alpha} \right. \\
&\quad \left. \cdot E_{\beta} \left\{ -\frac{\beta}{1 - \beta} (x - \xi)^{\beta+\alpha} \right\} E_{\beta} \left\{ -\frac{\beta}{1 - \beta} (t - \lambda)^{\beta+\alpha} \right\} d\xi d\lambda. \right]
\end{align*}
\]
Since $\frac{\partial^2 f}{\partial x \partial t}$ and $\frac{\partial^2 f}{\partial t \partial x}$ exist and are continuous then $\frac{\partial^2 f}{\partial x \partial t} = \frac{\partial^2 f}{\partial t \partial x}$ therefore we have

$$AC_{t,x}^\alpha,\beta f(x,t)$$

(34)

$$= \frac{A(\beta)}{1 - \beta \Gamma(1-\alpha)} \int_0^t \int_0^x \frac{\partial^2 f(\xi, \lambda)}{\partial \xi \partial \lambda}(x-\xi)^{-\alpha} (t - \lambda)^{-\alpha}$$

$$\cdot E_\beta \left\{ - \frac{\beta}{1 - \beta}(x-\xi)^{\beta+\alpha} \right\} E_\beta \left\{ - \frac{\beta}{1 - \beta}(t - \lambda)^{\beta+\alpha} \right\} d\xi d\lambda.$$

(35)

Thus

$$AC_{t,x}^\alpha,\beta f(x,t) = AC_{x,t}^\alpha,\beta f(x,t).$$

(36)

This completes the proof.

5 Numerical Approximation

There exist in nature many problems for which mathematical representations are modeled with strong non-linearity. These kind of mathematical equations can mostly handled using numerical approximations [8-10]. Therefore in order to accommodate researchers working in the field of numerical analysis. We present in this section the numerical approximation of the $AC_{t,x}^\alpha,\beta f(t)$.

$$AC_{t}^\alpha,\beta f(t)$$

(37)

$$= \frac{A(\beta)}{1 - \beta \Gamma(1-\alpha)} \int_0^t \frac{df(\tau)}{d\tau} (t - \tau)^{-\alpha} E_\beta \left\{ - \frac{\beta}{1 - \beta}(t - \tau)^{\beta+\alpha} \right\} d\tau$$

$$= \frac{A(\beta)}{1 - \beta \Gamma(1-\alpha)} \int_0^t \frac{f(t) - f(t + \Delta t)}{2\Delta t} (t - \tau)^{-\alpha} E_\beta \left\{ - \frac{\beta}{1 - \beta}(t - \tau)^{\beta+\alpha} \right\} d\tau.$$
\[ \frac{\overset{\mathcal{A}C}{0}}{0} D^{\alpha,\beta}_t f(t) \]
\[ = \frac{A(\beta)}{1 - \beta \Gamma \{1 - \alpha\}} \sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} \frac{f^{i+1} - f^i}{(t_n - \tau)^{\alpha}} \frac{1}{2(\Delta t)} \left\{-\frac{\beta}{1-\beta}(t_n - \tau)^{\beta+\alpha}\right\} \, d\tau \]
\[ = \frac{A(\beta)}{1 - \beta \Gamma \{1 - \alpha\}} \sum_{i=0}^{n} \frac{f^{i+1} - f^i}{2(\Delta t)} \int_{t_i}^{t_{i+1}} (t_n - \tau)^{-\alpha} E_{\beta} \left\{-\frac{\beta}{1-\beta}(t_n - \tau)^{\beta+\alpha}\right\} \, d\tau \]
\[ = \frac{A(\beta)}{1 - \beta \Gamma \{1 - \alpha\}} \sum_{i=0}^{n} \frac{f^{i+1} - f^i}{2(\Delta t)} \int_{t_i}^{t_{i+1}} y^{-\alpha} E_{\beta} \left\{-\frac{\beta}{1-\beta} y^{\beta+\alpha}\right\} \, dy \]
\[ = \frac{A(\beta)}{1 - \beta \Gamma \{1 - \alpha\}} \sum_{i=0}^{n} \frac{f^{i+1} - f^i}{2(\Delta t)} \int_{t_i}^{t_{i+1}} y^{-\alpha} E_{\beta} \left\{-\frac{\beta}{1-\beta} y^{\beta+\alpha}\right\} \, dy \]
\[ = \frac{A(\beta)}{1 - \beta \Gamma \{1 - \alpha\}} \sum_{i=0}^{n} \frac{f^{i+1} - f^i}{2(\Delta t)} \delta_{n,i}. \]

Where
\[ \delta_{n,i} = (t_n - t_{i+1})^{1-\alpha} E_{\beta,2-\alpha} \left\{-\frac{\beta}{1-\beta} (t_n - t_{i+1})^{\beta+\alpha}\right\} \]
\[ - (t_n - t_i)^{1-\alpha} E_{\beta,2-\alpha} \left\{-\frac{\beta}{1-\beta} (t_n - t_{i+1})^{\beta+\alpha}\right\}. \] (38)

With the same time of idea we obtain the space approximation as
\[ \frac{\overset{\mathcal{A}C}{0}}{0} D^{\alpha,\beta}_x f(x) = \frac{A(\beta)}{1 - \beta \Gamma \{1 - \alpha\}} \sum_{i=0}^{n} \frac{f^{i+1} - f^i}{2(\Delta x)} \delta_{n,i}. \] (39)

### 6 Conclusion

In this paper we have broadened the scope of fractional calculus by introducing fractional derivatives with two orders. The motivation of this is from the fact that one can find in nature some systems with two different and parallel layers with different properties, as for instance the movement of water through aquifer with parallel layers with different properties. This type of problem can not be portrayed with existing derivatives with fractional order. Some properties connected to this new derivatives are presented. The numerical approximation also presented. This new derivatives will open new doors for PhD thesis, research and new fields.

### References

[1] I. Koca, "A method for solving differential equations of $q$-fractional order", Applied Mathematics and Computation, vo. 266, pp. 1–5, 2015
[2] Ozalp, N. and Koca, I. A fractional order nonlinear dynamical model of interpersonal relationships, Advances in Difference Equations 189, pp.510–544 (2012)

[3] A. Atangana, I. Koca, Chaos in a simple nonlinear system with Atangana-Baleanu derivatives with fractional order, Chaos, Solitons and Fractals: the interdisciplinary journal of Nonlinear Science, and Nonequilibrium and Complex Phenomena, In press, 10.1016/j.chaos.2016.02.012, 2016

[4] A. Atangana, I. Koca, On the new fractional derivative and application to Nonlinear Baggs and Freedman model, J. Nonlinear Sci. Appl. vol. 9, pp. 2467–2480, 2016

[5] I. Koca, N. Ozalp, Analysis of a Fractional-Order Couple Model with Acceleration in Feelings, The Scientific World Journal, Article ID 730736, in press (2013).

[6] Chandrasekhar, S... "Stochastic problems in physics and astronomy". Reviews of Modern Physics 15 (1): 189 (1943)

[7] Morozov, A. N.; Skripkin, A. V.,. "Spherical particle Brownian motion in viscous medium as non-Markovian random process". Physics Letters A 375 (46): 41134115(2011).

[8] Yang Liu, Zhichao Fang, Hong Li, Siriguleng He. A mixed finite element method for a time-fractional fourth-order partial differential equation; Appl. Math. Comput., 243, pp. 703717(2014)

[9] M.M. Meerschaert, C. Tadjeran; Finite difference approximations for fractional advection dispersion equations; J. Comput. Appl. Math., 172, pp. 6577 (2004)

[10] C.M. Chen, F. Liu, I. Turner, V. Anh; A Fourier method for the fractional diffusion equation describing sub-diffusion; J. Comput. Phys., 227, pp. 886897(2007)

[11] Iyanaga, S. and Kawada, Y. (Eds.). "Hypergeometric Functions and Spherical Functions." Appendix A, Table 18 in Encyclopedic Dictionary of Mathematics. Cambridge, MA: MIT Press, pp. 1460-1468, (1980).

[12] Kummer, E. E. "ber die Hypergeometrische Reihe." J. reine angew. Math. 15, 39-83 and 127-172, (1836).