Distribution of ionization and tunneling times in a model of strong field ionization

José T. Lunardi\textsuperscript{1} and Luiz A. Manzoni\textsuperscript{2}

\textsuperscript{1} Department of Mathematics & Statistics, State University of Ponta Grossa, Avenida Carlos Cavalcanti 4748, 84030-900 Ponta Grossa, PR, Brazil
\textsuperscript{2} Department of Physics, Concordia College, 901 8th St. S., Moorhead, MN 56562, USA
E-mail: jttlunardi@upeg.br, manzoni@cord.edu

Abstract. We consider a simple model for strong field ionization and use the Salecker-Wigner-Peres clock to obtain a distribution of (real) times associated with this phenomenon. By coupling the Salecker-Wigner-Peres clock to the particle in different regions, we are able to distinguish between the ionization (exit) time, the tunneling time and the time the particle spends in the well before entering the barrier (the well time). We show that the distribution of ionization/exit times, which are the times usually measured in attoclock experiments, may differ significantly from the distribution of tunneling times, suggesting that the ionization times cannot be generally interpreted as tunneling times.

1. Introduction

Recent developments in ultrafast physics have made it possible to measure quantum tunneling times for the ionization of atoms induced by a strong laser field [1]. This has revitalized the longstanding discussion about quantum tunneling times, which hitherto had been a purely theoretical concern. However, the recent experiments have not been able to eliminate (and, in fact, added to) the controversy surrounding tunneling times, with both finite results [1, 2] and instantaneous tunneling being reported [3]. These disparate results illustrate the difficulties, both theoretical and experimental, to define tunneling times.

From the theoretical standpoint the difficulties are associated with the fact that there exists no self-adjoint time operator in quantum mechanics [4]. Hence, time is not an observable and operational definitions of time must be adopted – which, naturally, are unlikely to be of universal validity. Several such operational definitions have been considered in the literature, such as phase time (group delay) [5], Larmor times [6], dwell time [7], etc. However, the time scale reported as being consistent with the attoclock experiments is given by the Larmor clock [1]. In reality, the Larmor clock always gives at least two possible time scales [6] and the time compatible with the these experiments is better represented by the Salecker-Wigner-Peres (SWP) clock [4, 8], which can be seen as an idealization of the Larmor clock and results in only one (real) time.

In this work, we use a SWP clock and follow [9] to obtain a distribution of times associated with the process of strong field ionization (SFI). We use the same SFI model as in [9], which is a slight modification of the model proposed in [10], and investigate the properties of the distribution of the times obtained when the clock measures the time spent by the particle in three distinct regions: i) the well before entering the barrier (the well time), ii) the classically
forbidden region, or barrier (the tunneling time) and iii) the entire region including the well and the barrier (the exit or ionization time, since it is the time lapse from the application of the strong field and the ionization of the particle). Our results show that the distribution of ionization/exit times may differ significantly from the distribution of tunneling times, since the well time may contribute strongly to the exit time. Since the times measured in attoclock experiments are usually the ionization/exit times, our results suggest that care needs to be exercised when trying to interpret results in those experiments as being tunneling times.

2. The Salecker-Wigner-Peres clock and a distribution of times

The SWP clock is, essentially, a quantum rotor weakly coupled with the particle in the region in which one wants to measure the time [4]. For the stationary case, it can be shown that coupling the particle to a SWP clock is equivalent to introducing a small perturbation \( \eta_m \), in the region of interest, to the particle’s potential (\( \eta_m \) is an eigenvalue of the SWP clock Hamiltonian) [8]. Then, the transmission time corresponding to a wave number \( k \) is given by [8] (we use Rydberg atomic units \( \hbar = 2\mu = 1 \), where \( \mu \) is the particle’s mass)

\[
t_t^T(k) = - \left( \frac{\partial}{\partial \eta_m} \varphi_T^{(m)}(k) \right)_{\eta_m=0}
\]

where \( \varphi_T^{(m)}(k) \) is the phase of the coefficient of transmission in the presence of the perturbation \( \eta_m \).

For the time-dependent case of a localized particle described by a wave packet, it was shown in [11] that, in addition to the pre-selection (preparation) of the initial state, one must perform a post-selection of the final, transmitted (or ionized), asymptotic state to obtain a correct description of the tunneling process (or ionization, in the present case). Then, after tracing out the particle’s degrees of freedom, one can define the “expectation value” (or average) clock transmission time by \( \left\langle t_t^T \right\rangle = \int dk \rho(k)t_t^T(k) \) [11], where the spectral probability density of transmission (i.e., the probability density of finding the component \( k \) in the transmitted wave packet) is \( \rho(k) = |A(k)T(k)|^2 / \int dk |A(k)T(k)|^2 \), with \( T(k) \) indicating the transmission coefficient for the stationary problem in the absence of the clock, and \( A(k) \) is the (normalized) spectral distribution of the initial state.

Even though the average \( \left\langle t_t^T \right\rangle \) in terms of the spectral density highlights the probabilistic nature of the tunneling phenomenon, it does not lend itself to an easy interpretation, since a localized particle is represented by a wave packet with infinite spectral components, all of which in general take different times to tunnel. Therefore, an average weighed by a distribution of the (real) tunneling times is desirable. Such a representation can be obtained by a simple transformation between two random variables in probability theory, which allows us write the average transmission time as [9]

\[
\left\langle t_t^T \right\rangle = \int_0^\infty d\tau \rho_t(\tau) , \quad \rho_t(\tau) = \int \rho(k)\delta \left( \tau - t_t^T(k) \right) dk = \sum_j \frac{\rho(k_j(\tau))}{t_t^T(k_j(\tau))} ,
\]

where \( \rho_t(\tau) \) is the probability density for observing a particular tunneling time \( \tau \) for the asymptotically transmitted wave packet given by [9]. In the above, \( t_t^T(k) \) stands for the derivative of \( t_t^T(k) \) with respect to \( k \) and \( \{k_j(\tau)\} \) indicates the set of zeros of the function \( q(k) = t_t^T(k) - \tau \).

The above expression for \( \rho_t(\tau) \) is valid whenever the initial state of the particle is localized and there is a post-selection of an asymptotic transmitted/ionized state. Consequently, (2) are valid for any time scales for which a spectral average can be obtained. The distribution of times \( \rho_t(\tau) \) was investigated in [9] for tunneling through a square potential barrier as well as for a simple model of SFI, which will be further examined below.
3. Tunneling and ionization times for a model of strong field ionization

We consider the simple model of SFI investigated in [9], which is a slight modification of the model proposed in [10]. In this model, for \( t < 0 \) the particle is in an eigenstate of a semi-infinite square-well potential \( V_1(x) \) and, therefore, cannot decay by tunneling [9, 10]. Then, at \( t = 0 \) the potential is suddenly deformed to a potential \( V_2(x) \) which allows the possibility of tunneling. Specifically, the potentials \( V_1(x) \) and \( V_2(x) \) are

\[
V_1(x) = \begin{cases} 
+\infty, & x < 0 \\
0, & 0 \leq x \leq a \\
V_0, & x > a 
\end{cases} \quad V_2(x) = \begin{cases} 
+\infty, & x < 0 \\
0, & 0 \leq x \leq a \\
V_0, & a < x < b \\
0, & x \geq b 
\end{cases} \tag{3}
\]

The initial state of the system is taken to be the ground state, \( \phi_0(x) \), with energy \( E_0 \), of the Hamiltonian with potential \( V_1(x) \). Then, assuming that the wave function does not change during the sudden change of the potential, for \( t \geq 0 \) the particle is no longer in an eigenstate, but rather in a superposition of the energy eigenstates, \( \psi_k(x) \) \( (k = \sqrt{E}, \text{where } E \text{ is the particle’s energy}) \), for the potential \( V_2(x) \), satisfying [10, 9]

\[
\psi(x, t = 0) = \phi_0(x) = \int_0^\infty S(k) \psi_k(x) dk, \quad \text{with} \quad S(k) = \int_0^\infty \phi_0(x) \psi_k(x) dx, \tag{4}
\]

It is then possible to show that, for the case considered here of a decaying particle which is eventually transmitted with probability one, the spectral density of the asymptotically ionized wave packet is \( \rho(k) = |S(k)|^2 \). From the spectral density we can obtain the density of times by using a Monte Carlo routine (for details, see [9]).

We consider here the SWP clock coupled to the particle in three distinct regions, leading to three distinct distributions of times. Namely, the SWP clock is coupled to the particle in the entire region \( 0 \leq x \leq b \), which leads to a distribution of the ionization or exit times [12]; the clock is coupled to the particle strictly in the classically forbidden region \( a < x < b \) – this is the distribution of tunneling times, considered in [9]; finally, we assume that the SWP clock is coupled to the particle only in the region of the well, \( 0 \leq x \leq a \), which gives the distribution of times that the particle spends in the well in front of the barrier, before it begins to tunnel, which we call the well time.

4. Results and conclusion

Figure 1 shows histograms, obtained by a Monte Carlo procedure, for the distributions of the three times considered. The peaks of the distribution \( \rho_\tau(\tau) \) occur at the stationary points of \( t_\tau^l(k) \) for which the spectral density \( \rho(k) \) is non-negligible, according to the last equation in (2). Therefore, since \( \rho(k) \) is strongly peaked at \( k_0 = \sqrt{E_0} \) \( (E_0 \text{ is the energy of the initial bound state}) \) and chosen such that \( \rho(k) \approx 0 \) for non-tunneling values of \( k \), the distributions \( \rho_\tau(\tau) \) are “U”-shaped with peaks at the local maximum and minimum of \( t_\tau^l(k) \) corresponding to tunneling values of \( k \) [9].

![Figure 1](image-url)

**Figure 1.** Histograms for \( \rho_\tau(\tau) \) with \( a = 1, \ b = 2.5, \ V_0 = 7 \) and \( E_0 = 4.7 \) (in Rydberg a.u.). Vertical lines indicate the percentiles 1%, 5%, 10%,...90%, 95% and 99%. (a) Exit times; (b) tunneling times; (c) well times.
It is clear from Fig. 1(a) that the distribution of exit times allows for much longer times and, in fact, that the maximum exit time is roughly a sum of the maximum tunneling and dwell times, Figs. 1(b) and (c). This is, of course, not surprising, since for a fixed $k$ we have the identity $t_{c}^{w+b}(k) = t_{c}^{w}(k) + t_{c}^{b}(k)$, where the indexes $w$ and $b$ stand respectively for the well and barrier stationary clock times. It can also be proven that the following identity holds: \[ \langle t_{c}^{w+b} \rangle = \langle t_{c}^{w} \rangle + \langle t_{c}^{b} \rangle. \]

However, for a given ionization time $\tau$, the contributions of the barrier and the well time do not have definite values, since different values of $k$ contribute to a fixed ionization time $\tau$, and the well and barrier contributions change for different $k$.

The above shows unequivocally that to obtain the tunneling time one must consider exclusively the time spent by the particle in the classically forbidden region. This, of course, suggests that the times measured in attoclock experiments, which are usually exit times (truncated by the natural period of the laser field applied), cannot be promptly interpreted as tunneling times, unless the well time contribution is negligible.

Figure 2 shows $\rho(t)$ in the region of small times for the exit and tunneling distributions. The probability of having superluminal times is extremely small for the parameters used. However, the distributions $\rho(t)$ are strongly affected by the portion of the wave packet already within or to the right of the barrier at $t = 0$, and this fact may have an important role in explaining the emergence of superluminal or zero tunneling and exit times in real experiments.

Summarizing, the above simple model reveals important characteristics of the distributions of times and, in particular, it indicates that care must be exercised when interpreting the time measurements in attoclock experiments as tunneling times. A more detailed and quantitative comparison with results of actual attoclock experiments will require the use of a potential that simulates the experimental situation more closely.

Acknowledgements

JTL thanks Fundação Araucária (State of Parana, Brazil) for partial support.

References

[1] Landsman A S, Weger M, Maurer J, Boge R, Ludwig A, Heuser S, Cirelli C, Gallmann L and Keller U 2014 Optica 1 343-349
[2] Hofmann C, Landsman A S and Keller U 2019 J. Mod. Optics 66 1052-1070
[3] Sainadh U S et al. 2019 Nature 568 75-77
[4] Peres A 1980 Am. J. Phys. 48 552-557
[5] Wigner E P 1955 Phys. Rev. 98 145-147
[6] Büttiker 1983 Phys. Rev. B 27 6178-6188
[7] Smith F T 1960 Phys. Rev. 118 349-356
[8] Calçada M, Lunardi J T and Manzoni L A 2009 Phys. Rev. A 79 012110
[9] Lunardi J T and Manzoni L A 2018 Adv. High En. Phys. 1372359
[10] Ban Y, Sherman E Ya, Muga J G and Büttiker M 2010 Phys. Rev. A 82 062121
[11] Lunardi J T, Manzoni L A and Nystrom A T 2011 Phys. Lett. A 375 415-421
[12] Teeny N, Yakaboylu E, Bauke H and Keitel C H 2016 Phys. Rev. Lett. 116 063003