Optimal Order of Uniform Convergence for Finite Element Method on Bakhvalov-Type Meshes

Jin Zhang · Xiaowei Liu

Abstract
We propose a new analysis of convergence for a $k$th order ($k \geq 1$) finite element method, which is applied on Bakhvalov-type meshes to a singularly perturbed two-point boundary value problem. A novel interpolant is introduced, which has a simple structure and is easy to generalize. By means of this interpolant, we prove an optimal order of uniform convergence with respect to the perturbation parameter. Numerical experiments illustrate these theoretical results.

Keywords
Singular perturbation · Convection–diffusion equation · Finite element method · Bakhvalov-type mesh · Uniform convergence

Mathematics Subject Classification
65N30 · 65N50

1 Introduction
We consider the two-point boundary value problem

$$Lu := -\varepsilon u'' - b(x)u' + c(x)u = f(x) \quad \text{in } \Omega := (0, 1), \quad u(0) = u(1) = 0,$$

(1)

where $\varepsilon$ is a positive parameter, $b$, $c$ and $f$ are sufficiently smooth functions such that $b(x) \geq \beta > 1$ on $\bar{\Omega}$ and

$$c(x) + \frac{1}{2}b'(x) \geq \gamma > 0 \quad \text{on } \bar{\Omega}$$

(2)

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✉ Jin Zhang
jinzhangalex@hotmail.com

Xiaowei Liu
xwliuvivi@hotmail.com

1 School of Mathematics and Statistics, Shandong Normal University, Jinan 250014, China

2 School of Mathematics and Statistics, Qilu University of Technology (Shandong Academy of Sciences), Jinan 250353, China
with some constants $\beta$ and $\gamma$. The condition (2) ensures that the boundary value problem has a unique solution. In the cases of interest the diffusion parameter $\varepsilon$ can be arbitrarily small and satisfies $0 < \varepsilon \ll 1$. Thus this problem is *singularly perturbed* and its solution typically features a boundary layer of width $O(\varepsilon \ln(1/\varepsilon))$ at $x = 0$ (see [13]).

Solutions to singularly perturbed problems are characterized by the presence of boundary or interior *layers*, where solutions change rapidly. Numerical solutions of these problems are of significant mathematical interest. Classical numerical methods are often inappropriate, because in practice it is very unlikely that layers are fully resolved by common meshes. Hence specialised numerical methods are designed to compute accurate approximate solutions in an efficient way. For example, standard numerical methods on layer-adapted meshes, which are fine in layer regions and standard outside, are commonly used; see [11,13] and many references therein. On these meshes, classical numerical methods are *uniformly* convergent with respect to the singular perturbation parameter; see [9]. Among them, there are two kinds of popular grids: Bakhvalov-type meshes (B-meshes) and Shishkin-type meshes (S-meshes); see [9].

The accuracy of finite difference methods on these locally refined meshes has been extensively studied and sharp error estimations have been derived (see [6,9,11]). For instance, in [9] the author presented convergence rates of $O(N^{-1})$ and $O(N^{-1} \ln N)$ for a first-order upwind difference scheme on Bakhvalov grid [1] and Shishkin grid [17], respectively, where $N$ is the number of mesh intervals in each coordinate direction. Usually, the performance of B-meshes is superior to that of S-meshes. This advantage is more and more obvious when higher-order schemes are used. Besides, the width of the mesh subdomain used to resolve the layer is $O(\varepsilon \ln(1/\varepsilon))$ for B-meshes and $O(\varepsilon \ln N)$ for S-meshes. The former is independent of the mesh parameter $N$ and this property will be important under certain circumstances.

For finite element methods, the development of numerical theories on B-meshes is completely different from one on S-meshes. On standard Shishkin meshes Stynes and O’Riordan [18] derived a sharp uniform convergence in the energy norm for finite element method. Henceforward numerous articles deal with uniform convergence of finite element methods on S-meshes; see e.g. [10,13,15,16,19–22] and the references therein. However, it is still open for the optimal uniform convergence of finite element methods on B-meshes (see [15, Question 4.1] for more details).

This dilemma arises from the fact that the standard Lagrange interpolant does not work for uniform convergence of finite element methods on B-meshes. More specifically, the Lagrange interpolant cannot provide enough stability in $L^2$ norm on a special mesh interval, which lies in the fine part and is adjacent to the coarse part of B-meshes. In [14] and [3] a quasi-interpolant is used and provides enough stability for the optimal uniform convergence. Unfortunately, in both articles the analysis is limited to one dimension and linear finite element. It is hard to extend the analysis to higher dimensions or higher-order finite elements for singularly perturbed problems.

In this contribution we will study the optimal uniform convergence of a finite element method of any fixed order $k \in \mathbb{N}$ on Bakhvalov-type meshes including the original Bakhvalov mesh. A novel interpolant is constructed by redefining the standard Lagrange interpolant to the solution. This interpolant has a simple structure and it can also be applied to higher-dimensional problems in a straightforward way. By means of this novel function, we prove the optimal order of uniform convergence in a standard way.

The rest of the paper is organized as follows. In Sect. 2 we describe our regularity on the solution $u$ to (1), introduce the Bakhvalov-type mesh and define the finite element method. Some preliminary results for the subsequent analysis are also derived in this section. In Sect. 3 we construct and analyze an interpolant $\Pi u$ for the uniform convergence on B-
meshes. In Sect. 4 uniform convergence is obtained by means of the interpolant \( \Pi u \) and
careful derivations of the convective term in the bilinear form. In Sect. 5, numerical results
illustrate our theoretical bounds.

We use the standard Sobolev spaces \( W^{m,p}(D) \), \( H^m(D) = W^{m,2}(D) \), \( H_0^m(D) \) for non-negative integers \( m \) and \( 1 \leq p \leq \infty \). Here \( D \) is any measurable subset of \( \Omega \). We denote by \( \| \cdot \|_{W^{m,p}(D)} \) and \( \| \cdot \|_{W^{m,p}(D)} \) the semi-norms and the norms in \( W^{m,p}(D) \), respectively. On \( H^m(D) \), \( \| \cdot \|_{H^m(D)} \) and \( \| \cdot \|_{H^m(D)} \) are the usual Sobolev semi-norm and norm. Denote by \( \| \cdot \|_{L^p(D)} \) the norms in the Lebesgue spaces \( L^p(D) \). We use the notation \( (\cdot, \cdot)_D \) and \( \| \cdot \|_D \) for the \( L^2(D) \)-inner product and the \( L^2(D) \)-norm, respectively. When \( D = \Omega \) we drop the subscript \( D \) from the notation for simplicity. Throughout the article, all constants \( C \) and \( C_i \) are positive and independent of \( \varepsilon \) and the mesh parameter \( N \); unsubscripted constants \( C \) are
generic and may take different values in different formulas while subscripted constants \( C_i \)
are fixed.

2 Regularity, Bakhvalov-Type Mesh and Finite Element Method

2.1 Regularity of the Solution

Information about higher-order derivatives of the solution \( u \) of (1) are usually needed by
uniform convergence of finite element methods. Such estimations appeared in [13, Lemma
1.9] and are reproduced in the following lemma.

**Lemma 1** Assume that (2) holds true and \( b, c, f \) are sufficiently smooth. The solution \( u \) of
(1) can be decomposed into

\[
\begin{align*}
u &= S + E, \\
S^{(i)}(x) &\leq C, \quad |E^{(i)}(x)| \leq C \varepsilon^{-i} \exp \left( -\frac{\beta x}{\varepsilon} \right) \quad \text{for } 0 \leq i \leq k + 1.
\end{align*}
\]

**Remark 1** In [8], Linß shows the minimal regularity requirements for \( b, c, f \), i.e., \( b, c, f \in C^k[0, 1] \), which ensure Lemma 1 holds.

2.2 Bakhvalov-Type Meshes

Bakhvalov mesh first appeared in [1] and is constructed according to layer functions like \( E \)
in Lemma 1. Its mesh generating function is piecewise and belongs to \( C^1 \). Its breakpoint,
which separates the mesh generating function, must be solved by a nonlinear equation and
usually is not explicitly known (see [13, Part I §2.4.1]). Hence many meshes, which arise
from an approximation of Bakhvalov’s mesh generating function, emerge and are usually
called Bakhvalov-type meshes.

In this article, we consider one kind of Bakhvalov-type meshes, which are defined by

\[
x = \varphi(t) = \begin{cases} 
- \frac{\sigma \varepsilon}{\beta} \ln \left( \frac{1}{1 + \kappa \varepsilon} \right) & \text{for } t \in [0, \vartheta), \\
1 - d(1 - t) & \text{for } t \in [\vartheta, 1].
\end{cases}
\]
In (5), \( \sigma > 0 \) and \( q \in (0, 1) \) are some constants specified later, \( \kappa \) and \( \vartheta = q + \theta^* \varepsilon \) are usually specified according to users’ requirements and \( d \) is used to ensure the continuity of \( \varphi(t) \) at \( t = \vartheta \). There \( q \) is roughly the portion of mesh points used to resolve the layer, while \( \sigma \) determines the grading of the mesh inside the layer. The parameters \( \kappa \) and \( \vartheta \) are usually used for the desired transition point \( \varphi(\vartheta) \).

**Remark 2** When we take different values for the parameters in (5), we could obtain different Bakhvalov-type meshes. If we take \( \kappa = 0 \) and \( \sigma = \beta < \theta^* = \theta^*(\varepsilon) < -(1 - q)\beta \) (see [12, Lemma 4]), the original Bakhvalov mesh can be recovered. If we take \( q = 1/2, \kappa = -1 \) and \( \theta^* = 0 \), we obtain Bakhvalov-type mesh in [14]. If we take \( \kappa = 0 \) and \( \theta^* = -C_1 \) with some positive constant \( C_1 \), we have Bakhvalov-type mesh introduced in [6, 7].

**Assumption 1** Assume that \( \varepsilon \leq N^{-1} \) in our analysis, as is not a restriction in practice.

To make sure that (5) is well-defined, we impose some conditions on the parameters

\[
\vartheta + N^{-1} \leq q_1 < 1, \quad \vartheta - N^{-1} \geq q_0 > 0, \\
0 < C_2 \varepsilon \leq 1 - \frac{1 + \kappa \varepsilon}{q} \vartheta \leq C_3 \varepsilon, \\
0 < -\frac{\sigma \varepsilon}{\beta} \ln \left( 1 - \frac{1 + \kappa \varepsilon}{q} \right) \leq v_0 < 1 \quad \text{for} \ t \in [0, \vartheta],
\]

with constants \( q_0, q_1 \) and \( v_0 \) independent of \( \varepsilon \) and \( N^{-1} \). Besides, we require

\[
1/2 \leq 1 + \kappa \varepsilon \leq 2.
\]

Then the breakpoint of Bakhvalov-type mesh (5) is \( O(\varepsilon \ln(1/\varepsilon)) \) from (7). Also, from (8) and (7), we have

\[
C_4 \varepsilon \leq \frac{q}{1 + \kappa \varepsilon} - \vartheta \leq C_5 \varepsilon.
\]

Assume that \( N \geq 4 \) is an integer. Define the mesh points \( x_i = \varphi(i/N) \) for \( i = 0, 1, \ldots, N \). Denote an arbitrary subinterval \([x_i, x_{i+1}]\) by \( I_i \), its length by \( h_i = x_{i+1} - x_i \) and a generic subinterval by \( I \).

### 2.3 The Finite Element Method

The weak form of problem (1) is to find \( u \in H^1_0(\Omega) \) such that

\[
a(u, v) = (f, v) \quad \forall v \in H^1_0(\Omega),
\]

where \( a(u, v) := \varepsilon(u', v') - (bu', v) + (cu, v) \). Note that the variational formulation (10) has a unique solution by means of the Lax-Milgram lemma.

Define the \( C^0 \) finite element space on the meshes

\[
V^N = \{ w \in C(\Omega) : w(0) = w(1) = 0, \ w|_{I_i} \in P_k(I_i) \ \text{for} \ i = 0, \ldots, N - 1 \}.
\]

The finite element method for (10) reads as: find \( u^N \in V^N \) such that

\[
a(u^N, v^N) = (f, v^N) \quad \forall v^N \in V^N.
\]

The natural norm associated with \( a(\cdot, \cdot) \) is defined by

\[
\|v\|_\varepsilon := \left\{ \varepsilon |v|^2 + \|v\|^2 \right\}^{1/2} \quad \forall v \in H^1(\Omega).
\]
Using (2), it is easy to see that one has the coercivity
\[ a(v^N, v^N) \geq C \|v^N\|_\varepsilon^2 \] for all \( v^N \in V^N \). \hspace{1cm} (12)

It follows that \( u^N \) is well defined by (11) (see [4] and references therein).

### 2.4 Preliminary Results of Bakhvalov-Type Mesh

In this subsection, we present some important properties of the Bakhvalov-type mesh (5) and the layer function \( E \), which are necessary for our uniform convergence.

From now on, we set \( \sigma \geq k + 1 \) in (5). Assume
\[ i^* N^1 \leq \vartheta < (i^* + 1) N^{-1}, \] \hspace{1cm} (13)

with \( 1 \leq i^* \leq N - 2 \) and define
\[ \delta(\vartheta) = \begin{cases} 
0 & \text{if } \vartheta = i^* N^{-1}, \\
1 & \text{if } i^* N^{-1} < \vartheta < (i^* + 1) N^{-1}.
\end{cases} \]

We present some properties about the step sizes of Bakhvalov-type mesh as follows.

**Lemma 2** Let Assumption 1 hold. For Bakhvalov-type mesh (5), one has
\[ h_0 \leq h_1 \leq \ldots \leq h_{i^* - 2}, \] \hspace{1cm} (14)
\[ C_6 \varepsilon \leq h_{i^* - 2} \leq C_7 \varepsilon, \] \hspace{1cm} (15)
\[ C \varepsilon \leq h_{i^* - 1}, h_{i^* - 1 + \delta(\vartheta)} \leq C N^{-1}, \] \hspace{1cm} (16)
\[ C_8 N^{-1} \leq h_i \leq C_9 N^{-1} \ i^* + \delta(\vartheta) \leq i \leq N - 1. \] \hspace{1cm} (17)

For mesh (5), one has
\[ x_i \geq C \sigma \varepsilon \ln N \ i = i^* - 1, i^* - 1 + \delta(\vartheta), \] \hspace{1cm} \( x_{i^* + \delta(\vartheta)} \geq C \sigma \varepsilon \ln \varepsilon \), \hspace{1cm} (18)

and
\[ h_i^\mu e^{-\beta x_i/\varepsilon} \leq C \varepsilon^\mu N^{-\mu} \text{ for } 0 \leq i \leq i^* - 2 \text{ and } 0 \leq \mu \leq \sigma. \] \hspace{1cm} (19)

**Proof** From (6) and (13), one obtains (17) directly.

For \( 0 \leq i \leq i^* \), for mesh (5) one has
\[ h_i = x_{i+1} - x_i = \int_{x_i}^{x_{i+1}} \frac{\sigma \varepsilon}{\beta} \frac{1}{q} - t \ dt \]

and
\[ \frac{\sigma \varepsilon}{\beta} \frac{1}{q + \kappa \varepsilon} - i N^{-1} N^{-1} \leq h_i \leq \frac{\sigma \varepsilon}{\beta} \frac{1}{q + \kappa \varepsilon} - (i + 1) N^{-1} N^{-1}. \] \hspace{1cm} (20)

From (20), we prove (14), (15) and (16) easily. Here we just present analysis for \( h_{i^* - 1} \) and \( h_{i^* - 1 + \delta(\vartheta)} \) in the case of \( \delta(\vartheta) = 1 \). The bound (15) can be proved in a similar way.

Now we are going to prove (16). From (9) and (13), we have
\[ \frac{q}{1 + \kappa \varepsilon} - i^* N^{-1} \geq \vartheta + C \varepsilon - i^* N^{-1} \geq C \varepsilon, \]
\[ \frac{q}{1 + \kappa \varepsilon} - (i^* - 1) N^{-1} \leq \vartheta + C \varepsilon - i^* N^{-1} + N^{-1} \leq C \varepsilon + 2 N^{-1}. \] \hspace{1cm} (21)
\[ C \varepsilon \leq h_{i^*-1} \leq C N^{-1}. \]

From (5) and (13), one has in the case of \( \delta(\vartheta) = 1 \)
\[
h_{i^*-1+\delta(\vartheta)} = \int_{i^* N^{-1}}^{\vartheta} \frac{\sigma \varepsilon}{\beta} \frac{1}{q + \kappa \varepsilon} - t \, dt + \int_{\vartheta}^{(i^* + 1) N^{-1}} d \, dt,
\]
and
\[
\frac{\sigma \varepsilon}{\beta} \frac{\vartheta - i^* N^{-1}}{q + \kappa \varepsilon} \leq h_{i^*-1+\delta(\vartheta)} - d((i^* + 1) N^{-1} - \vartheta) \leq \frac{\sigma \varepsilon}{\beta} \frac{\vartheta - i^* N^{-1}}{q + \kappa \varepsilon}.
\]

From (9) and (13) again, we have
\[
\frac{q}{1 + \kappa \varepsilon} - i^* N^{-1} = \left( \frac{q}{1 + \kappa \varepsilon} - \vartheta \right) + (\vartheta - i^* N^{-1}) \leq C \varepsilon + N^{-1}, \tag{22}
\]
and
\[
C \varepsilon \leq h_{i^*-1+\delta(\vartheta)} \leq C N^{-1}.
\]

From (9), (21) and (22), we can easily prove (18).

Let \( 0 \leq i \leq i^* - 1 \). From (5) one has
\[
-\frac{\sigma \varepsilon}{\beta} \ln \left( 1 - \frac{1 + \kappa \varepsilon}{q} i N^{-1} \right) = x_i \leq x \leq x_{i+1} = -\frac{\sigma \varepsilon}{\beta} \ln \left( 1 - \frac{1 + \kappa \varepsilon}{q} (i + 1) N^{-1} \right),
\]
and for \( x \in [x_i, x_{i+1}] \)
\[
\left( 1 - \frac{1 + \kappa \varepsilon}{q} (i + 1) N^{-1} \right)^{\sigma} \leq e^{-\beta x/q} \leq e^{-\beta x_{i}/q} = \left( 1 - \frac{1 + \kappa \varepsilon}{q} i N^{-1} \right)^{\sigma}. \tag{23}
\]

From (20) and (23), we have
\[
h_i^{\mu} \max_{x_i \leq x \leq x_{i+1}} e^{-\beta x/q} \leq C_i e^{\mu} N^{-\mu} \left( \frac{q}{1 + \kappa \varepsilon} - i N^{-1} \right)^{\sigma} \left( \frac{q}{1 + \kappa \varepsilon} - (i + 1) N^{-1} \right)^{-\mu}
\leq C_i e^{\mu} N^{-\mu} \left( \frac{q}{1 + \kappa \varepsilon} - i N^{-1} \right)^{\sigma-\mu} \left( \frac{q}{1 + \kappa \varepsilon} - (i + 1) N^{-1} \right)^{\mu}
\leq C_i^* C_2 e^{\mu} N^{-\mu},
\]
where \( C_i^* = (\sigma/\beta)^{\mu} \left( \frac{1 + \kappa \varepsilon}{q} \right)^{\sigma} \leq (2/q)^{\sigma} (\sigma/\beta)^{\mu}, C_2 = \left( \frac{q}{1 + \kappa \varepsilon} - i N^{-1} \right)^{\sigma-\mu} \leq (2q)^{\sigma-\mu} \) and for \( 0 \leq i \leq i^* - 2 \)
\[
C_3^* = \left( \frac{q}{1 + \kappa \varepsilon} - i N^{-1} \right)^{\mu} \leq 2^\mu.
\]

Thus (19) is proved. \( \square \)

We collect some bounds of the layer function \( E \) on the meshes.
Lemma 3. Let Assumption 1 hold. On mesh (5), one has

\[ |E(x_i)| \leq C N^{-\sigma}, \quad i = i^*-1, i^*-1 + \delta(\vartheta) \]

\[ |E(x_i+\delta(\vartheta))| \leq C \epsilon^{\sigma}, \quad i^*-1, i^*+\delta(\vartheta) \]

\[ \leq C N^{-\sigma} - \sigma, \quad i = i^*-1 \]

\[ \leq C N^{-\sigma} + \delta(\vartheta) \]

\[ \|E\|_{[x_i, x_i+\delta(\vartheta)]} + \epsilon \|E\|_{[x_i-1, x_i+\delta(\vartheta)]} \leq C \epsilon^{1/2} N^{-\sigma}, \quad (24) \]

\[ \|E\|_{[x_i+\delta(\vartheta), x_N]} \leq C \epsilon^{\sigma-1/2}. \quad (25) \]

Proof. From (4) and (18), we could prove (24), (25) and (26) directly. \( \square \)

3 Interpolation Operator and Interpolation Errors

Now a new interpolation operator is introduced, which is used for our uniform convergence. Let

\[ x_s^i := x_i + (s/k) h_i \quad \text{for} \quad i = 0, 1, \ldots, N-1 \quad \text{and} \quad s = 0, \ldots, k-1. \]

For the consistency of notation, set

\[ x_0^N := x_N. \]

Let \( \theta_s^i(x) \) be the standard nodal basis functions with respect to the nodes \( x_s^i \) in the finite element space \( V_N \). For any \( v \in C^0(\Omega) \) its Lagrange interpolant \( v^I \in V_N \) on each mesh is defined by

\[ v^I(x) = \sum_{i=0}^{N} v(x_0^i) \theta_0^i(x) + \sum_{i=0}^{N-1} \sum_{s=1}^{k-1} v(x_s^i) \theta_s^i(x). \]

Let

\[ A = \{ i^*-1, i^*+\delta(\vartheta) \}. \]

For the solution \( u \) to (1), recall (3) in Lemma 1 and define the interpolant \( \Pi u \) by

\[ \Pi u = S^I + \pi E, \quad (27) \]

where \( S^I \) is the Lagrange interpolant to \( S \) and

\[ (\pi E)(x) = \sum_{i=0, j \notin A}^{N} E(x_0^i) \theta_0^i(x) + \sum_{i=0, j \notin A}^{N-1} \sum_{j=1}^{k-1} E(x_j^i) \theta_j^i(x). \]

Define

\[ (\mathcal{P} E)(x) = \sum_{i \in A}^{N} E(x_0^i) \theta_0^i(x) + \sum_{i \in A}^{N-1} \sum_{j=1}^{k-1} E(x_j^i) \theta_j^i(x), \]

and clearly we have

\[ \pi E = E^I - \mathcal{P} E, \quad \Pi u = u^I - \mathcal{P} E, \]

\[ \pi E|_{[x_0, x_0+\vartheta)} \cup [x_0+\vartheta, x_N] = E^I|_{[x_0, x_0+\vartheta)} \cup [x_0+\vartheta, x_N], \]

\[ \Pi u \in V_N. \]

Interpolation theories in Sobolev spaces [5, Theorem 3.1.4] tell us that

\[ \|v - v^I\|_{W^{l,q}(I_i)} \leq Ch_i^{k+1-l+1/q-1/p} \|v\|_{W^{k+1,p}(I_i)}, \]

for all \( v \in W^{k+1,p}(I_i) \), where \( i = 0, 1, \ldots, N-1, l = 0, 1 \) and \( 1 \leq p, q \leq \infty. \)
Lemma 4 Let Assumption 1 hold. On mesh $(5)$ one has

\[
\|E - E^l\|_{L^\infty(\Omega)} + \|S - S^l\|_{L^\infty(\Omega)} + \|u - u^l\|_{L^\infty(\Omega)} \leq CN^{-(k+1)}, \tag{33}
\]

\[
\|E - E^l\| + \|S - S^l\| + \|u - u^l\| \leq CN^{-(k+1)}, \tag{34}
\]

\[
\|E^l\|_{I_i} \leq C h_i^{1/2} N^{-\sigma} \quad \forall i \in A, \tag{35}
\]

\[
\|E^l\|[x_i, x_{i+1}) \leq C \varepsilon^\sigma, \tag{36}
\]

\[
\|E - E^l\|_\varepsilon + \|u - u^l\|_\varepsilon + \|(S^l - S)^l\| \leq CN^{-k}, \tag{37}
\]

\[
\|\mathcal{P}E\|_\varepsilon \leq CN^{-\sigma}, \tag{38}
\]

\[
\|E - \pi E\| \leq CN^{-(k+1)}, \tag{39}
\]

where $\mathcal{P}E$ is defined in (29).

Proof From (32) and (4), for $0 \leq i \leq i^* - 2$ one has

\[
\|E - E^l\|_{L^\infty(I_i)} \leq C h_i^{k+1} |E|_{W^{k+1, \infty}(I_i)} \\
\leq C \varepsilon^{-(k+1)} h_i^{k+1} e^{-\beta x_i/\varepsilon} \leq CN^{-(k+1)}, \tag{40}
\]

where we have used $(19)$ with $\mu = k + 1$ and $\sigma \geq k + 1$. For $i^* - 1 \leq i \leq N - 1$ we have

\[
\|E - E^l\|_{L^\infty(I_i)} \leq \|E\|_{L^\infty(I_i)} + \|E^l\|_{L^\infty(I_i)} \leq C \varepsilon^{-\beta x_i/\varepsilon} \leq CN^{-\sigma}. \tag{41}
\]

Collecting (40), (41) and noting $\sigma \geq k + 1$, we prove $\|E - E^l\|_{L^\infty(\Omega)} \leq CN^{-(k+1)}$. Lemma 2, (32) and (4) yield $\|S - S^l\|_{L^\infty(\Omega)} \leq CN^{-(k+1)}$. From (3) we prove (33). The bound (34) can be easily obtained from (33) and Hölder inequalities.

From (4), (24) and direct calculations one can easily prove (35) and (36).

Now we are ready to analyze $\|E - E^l\|_\varepsilon$. First we decompose $\varepsilon \|(E - E^l)^l\|^2$ into the following two parts

\[
\varepsilon \|(E - E^l)^l\|^2 = \varepsilon \sum_{i=0}^{i^*-2} \|(E - E^l)^l\|_{I_i}^2 + \varepsilon \|(E - E^l)^l\|[x_{i^*-1}, x_N]^2 \tag{42}
\]

\[
=: S_1 + S_2.
\]

From (32), (4), (19) with $\mu = (2k + 1)/2$ and $\sigma \geq k + 1$, we have

\[
S_1 \leq C \varepsilon \sum_{i=0}^{i^*-2} h_i^{2k} |E|_{k+1, I_i}^2 \leq C \varepsilon \sum_{i=0}^{i^*-2} h_i^{2k} \int_{x_i}^{x_{i+1}} e^{-2(k+1)} e^{-2\beta x_i/\varepsilon} dx \\
\leq C \varepsilon h_i^{2k} e^{-2(k+1)} e^{-2\beta x_i/\varepsilon} h_i \leq C \varepsilon \sum_{i=0}^{i^*-2} e^{-2(k+1)} \left(h_i^{2k+1/2} e^{-\beta x_i/\varepsilon}\right)^2 \tag{43}
\]

\[
\leq C \varepsilon e^{-2(k+1)} e^{2k+1} N^{-(2k+1)} \leq CN^{-2k}.
\]
From a triangle inequality, (16), (17), (25), (26), (35), (36) and inverse inequality [5, Theorem 3.2.6], one has

\[ S_2 \leq C \varepsilon \left( \| E' \|_{x_{i-1},x_N}^2 + \sum_{i \in \Lambda} \| (E')' \|_{I_i}^2 + \| (E')' \|_{[x_{i+1}(\beta),x_N]}^2 \right) \]

\[ \leq C \varepsilon (\varepsilon^{-1} N^{-2\sigma} + \sum_{i \in \Lambda} h_i^{-2} \| E \|_{I_i}^2 + N^2 \| E \|_{[x_{i+1}(\beta),x_N]}^2) \]

\[ \leq C N^{-2\sigma} + C \varepsilon^{2\sigma+1} N^2. \]

Substituting (43), (44) into (42) and recalling \( \varepsilon \leq N^{-1} \) and \( \sigma \geq k+1 \), we obtain

\[ \varepsilon \| (E - E')' \|_\varepsilon^2 \leq C N^{-2k} \]

and prove \( \| E - E' \|_\varepsilon \leq C N^{-k} \) from (34). From (32) and Lemma 2, one can easily prove \( \| (S' - S') \|_\varepsilon \leq C N^{-k} \) and \( \| S' - S' \|_\varepsilon \leq C (\varepsilon^{1/2} N^{-k} + N^{-(k+1)}) \). A triangle inequality yields \( \| u - u' \|_\varepsilon \leq C N^{-k} \). Thus (37) is proved.

Now we consider (38). Direct calculations yield

\[ \| \mathcal{P} E \|_\varepsilon \leq \sum_{i \in \Lambda} |E(x_i^0)\| \| \theta_i^0(x)\|_\varepsilon + \sum_{i \in \Lambda} \sum_{j=1}^{k-1} |E(x_i^j)\| \| \theta_i^j(x)\|_\varepsilon \]

\[ \leq C N^{-\sigma} \sum_{i \in \Lambda} \sum_{j=0}^{k-1} \| \theta_i^j(x)\|_\varepsilon \leq C N^{-\sigma}, \]

where we have used (29), (24), (16) and (17). The triangle inequality, (30), (34) and (38) yield (39). \( \square \)

From (30), Lemma 4 and the triangle inequality \( \| \Pi u - u \|_\varepsilon \leq \| u' - u \|_\varepsilon + \| \mathcal{P} E \|_\varepsilon \), we obtain the following result.

**Corollary 1** Let Assumption 1 hold. On mesh (5) one has

\[ \| \Pi u - u \|_\varepsilon \leq C N^{-k}. \]

## 4 Uniform Convergence

Introduce \( \chi := \Pi u - u^N \). From (12), the Galerkin orthogonality, (3), (27), (30) and integration by parts for \( \int_0^1 b(\pi E - E)' \chi \ dx \), one has

\[ C \| \chi \|_\varepsilon^2 \leq a(\chi, \chi) = a(\Pi u - u, \chi) \]

\[ = \varepsilon \int_0^1 (u' - u)' \chi' \ dx - \varepsilon \int_0^1 (\mathcal{P} E)' \chi' \ dx \]

\[ + \int_0^1 b(\pi E - E)' \chi' \ dx - \int_0^1 b(S' - S)' \chi' \ dx \]

\[ + \int_0^1 b'(\pi E - E) \chi' \ dx + \int_0^1 c(u' - u) \chi \ dx - \int_0^1 c(\mathcal{P} E) \chi \ dx \]

\[ =: I + II + III + IV + V + VI + VII. \]
In the following we will analyze the terms in the right-hand side of (45). The Cauchy-Schwarz inequality yields

\[(I + VI) + (II + VII) \leq C\|u - u^I\|_e \|\chi\|_e + C\|\mathcal{P}E\|_e \|\chi\|_e \leq CN^{-k} \|\chi\|_e,\]  

(46)

where (37) and (38) have been used. From (37) and (39), we obtain

\[IV + V \leq C(\|(S^I - S)\| + \|\pi E - E\|) \|\chi\| \leq CN^{-k} \|\chi\|.\]  

(47)

We put the arguments for III in the following lemma.

**Lemma 5** Let Assumption 1 hold. Let the mesh \(\{x_i\}\) be mesh (5). Let \(\pi E\) be defined in (28). Then one has

\[|III| = \left| \int_0^1 b(\pi E - E)\chi' dx \right| \leq CN^{-k} \|\chi\|_e.\]  

(48)

**Proof** According to (31), the term \((b(\pi E - E), \chi')\) is separated into three parts as follows:

\[
\int_0^1 b(\pi E - E)\chi' dx = \int_{x_0}^{x_{i+2}} b(E^I - E)\chi' dx + \int_{x_{i+2}}^{x_{i+3} + \delta(\vartheta)} b(\pi E - E)\chi' dx + \int_{x_{i+3} + \delta(\vartheta)}^{x_N} b(E^I - E)\chi' dx
\]

\[\quad =: I_1 + I_2 + I_3.\]  

(49)

From Hölder inequalities, (32), (19) with \(\mu = k + 1\) and \(\sigma \geq k + 1\), we obtain

\[|I_1| \leq C \sum_{i=0}^{i^* - 3} \int_{x_i}^{x_{i+1}} \|E^I - E\| \|\chi'\| dx
\]

\[\leq C \sum_{i=0}^{i^* - 3} \|E^I - E\|_{L^\infty(I_i)} \|\chi'\|_{L^1(I_i)}
\]

\[\leq C \sum_{i=0}^{i^* - 3} h_i^{k+1} e^{-(k+1)} \epsilon^{-\beta x_i/\epsilon} : h_i^{1/2} \|\chi'\|_{L^1(I_i)} \leq C \epsilon^{1/2} \sum_{i=0}^{i^* - 3} N^{-(k+1)} \|\chi'\|_{L^1(I_i)}
\]

\[\leq C \epsilon^{1/2} \left( \sum_{i=0}^{i^* - 3} N^{-2(k+1)} \right)^{1/2} \left( \sum_{i=0}^{i^* - 3} \|\chi'\|_{L^1(I_i)}^2 \right)^{1/2}
\]

\[\leq CN^{-(k+1/2)} \|\chi\|_e,
\]

(50)

where (14) and (15) have been used.

From Hölder inequalities and inverse inequalities, one has

\[|I_3| \leq C \|E^I - E\|_{[x_{i+3} + \delta(\vartheta), x_N]} \|\chi'\|_{[x_{i+3} + \delta(\vartheta), x_N]}
\]

\[\leq CN^{-(k+1)} \cdot N \|\chi\|_{[x_{i+3} + \delta(\vartheta), x_N]} \leq CN^{-k} \|\chi\|.
\]

(51)

where (34) has been used.
Now we analyze the term $\mathcal{I}_2$. Note $\pi E = E^I - E(x_{i^*-1})\theta^0_{i^*-1}(x)$ on $[x_{i^*-2}, x_{i^*-1}]$ and one has

$$
\left| \int_{x_{i^*-2}}^{x_{i^*-1}} b(\pi E - E) \chi' \, dx \right| 
\leq C \int_{x_{i^*-2}}^{x_{i^*-1}} |E^I - E| |\chi'| \, dx + C |E(x_{i^*-1})| \int_{x_{i^*-2}}^{x_{i^*-1}} |\theta^0_{i^*-1}| \, dx
$$

(52)

$$
\leq C \left( \|E^I - E\|_{L^\infty(I_{i^*-2})} + |E(x_{i^*-1})| \right) \|\chi'\|_{L^1(I_{i^*-2})}
\leq C \left( h_{i^*-2}^{k+1} \epsilon^{-k+1} + N^{-\sigma} \right) \|I_{i^*-2}\| \|\chi'\|_{I_{i^*-2}}
\leq C \left( N^{-k+1} + N^{-\sigma} \right) \|\chi\|_{H^1(I_{i^*-2})}
$$

where Hölder inequalities, (24), (19) with $\mu = k + 1$ and $\sigma \geq k + 1$, (15) have been used. On $[x_{i^*-1}, x_{i^*+\delta(\theta)}]$, we have $\pi E = E(x_{i^*+\delta(\theta)})\theta^0_{i^*+\delta(\theta)}(x)$ from (28) and

$$
\left| \int_{x_{i^*-1}}^{x_{i^*+\delta(\theta)}} b(\pi E - E) \chi' \, dx \right| 
\leq C |E(x_{i^*+\delta(\theta)})| \int_{x_{i^*-1}}^{x_{i^*+\delta(\theta)}} |\theta^0_{i^*+\delta(\theta)}| |\chi'| \, dx + C \int_{x_{i^*-1}}^{x_{i^*+\delta(\theta)}} |E| |\chi'| \, dx
$$

(53)

$$
\leq C \left( \epsilon^\sigma \|\theta^0_{i^*+\delta(\theta)}\|_{H^1(I_{i^*-1}, x_{i^*+\delta(\theta)})} + \|E\|_{H^1(I_{i^*-1}, x_{i^*+\delta(\theta)})} \right) \|\chi'\|_{H^1(I_{i^*-1}, x_{i^*+\delta(\theta)})}
\leq C \left( \epsilon^\sigma h_{i^*-1}^{1/2} + \epsilon^{1/2} N^{-\sigma} \right) \|\chi'\|_{H^1(I_{i^*-1}, x_{i^*+\delta(\theta)})}
\leq C \left( \epsilon^\sigma + 1/2 N^{-1/2} + N^{-\sigma} \right) \|\chi\|_{H^1(I_{i^*-1}, x_{i^*+\delta(\theta)})}
$$

where Hölder inequalities, (24), (25) and (16) have been used. From (52) and (53) we prove

$$
|\mathcal{I}_2| \leq C N^{-(k+1)} \|\chi\|_{H^1},
$$

(54)

where $\epsilon \leq N^{-1}$ and $\sigma \geq k + 1$ have been used. Substituting (50), (51) and (54) into (49), we are done.

Now we are in a position to present the main result.

**Theorem 1** Let Assumption 1 hold. Let the mesh $\{x_i\}$ be mesh (5). Let $u$ and $u^N$ be the solutions of (1) and (11), respectively. Then one has

$$
\|u - u^N\|_{H^1} \leq C N^{-k}.
$$

**Proof** Substituting (46), (47) and (48) into (45), we obtain $\|\Pi u - u^N\|_{H^1} \leq C N^{-k}$. From a triangle inequality and Corollary 1, one has

$$
\|u - u^N\|_{H^1} = \|\Pi u - u^N\|_{H^1} + \|u - \Pi u\|_{H^1} \leq C N^{-k}.
$$

Thus we are done.

**5 Numerical Experiments**

We now present the results of some numerical experiments in order to illustrate the conclusions of Theorem 1, and to check if they are sharp. All calculations were carried out by using
Table 1 Errors and convergence rates for problem (55)

| N   | $k=1$  | $k=2$  |
|-----|--------|--------|
|     | B–R mesh | $e_N^N$ | $r_e^N$ | B–K mesh | $e_N^N$ | $r_e^N$ | B–R mesh | $e_N^N$ | $r_e^N$ | B–K mesh | $e_N^N$ | $r_e^N$ |
|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 8   | 0.206E+00 | 1.01  | 0.205E+00 | 1.00  | 0.436E–01 | 2.19  | 0.437E–01 | 2.19  |
| 16  | 0.102E+00 | 1.00  | 0.102E+00 | 1.00  | 0.957E–02 | 2.06  | 0.958E–02 | 2.06  |
| 32  | 0.511E–01 | 1.00  | 0.511E–01 | 1.00  | 0.230E–02 | 2.02  | 0.230E–02 | 2.02  |
| 64  | 0.255E–01 | 1.00  | 0.255E–01 | 1.00  | 0.569E–03 | 2.02  | 0.569E–03 | 2.00  |
| 128 | 0.128E–01 | 1.00  | 0.128E–01 | 1.00  | 0.230E–03 | 2.02  | 0.230E–03 | 2.00  |
| 256 | 0.638E–02 | 1.00  | 0.638E–02 | 1.00  | 0.355E–04 | 2.00  | 0.355E–04 | 2.00  |
| 512 | 0.319E–02 | 1.00  | 0.319E–02 | 1.00  | 0.887E–05 | 2.00  | 0.887E–05 | 2.00  |
| 1024| 0.159E–02 | 1.00  | 0.159E–02 | 1.00  | 0.222E–05 | 2.00  | 0.222E–05 | 2.00  |
| 2048| 0.797E–03 | –     | 0.797E–03 | –     | 0.554E–06 | –     | 0.554E–06 | –     |

Intel Visual FORTRAN 11 and the discrete problems were solved using GMRES (see, e.g., [2]).

For Bakhvalov-type mesh (5) we take $\sigma = k + 1$, $q = 1/2$ and $\beta = 2$ according to our numerical tests. Furthermore, we consider (5) with $\kappa = -1$ and $\theta^* = 0$, which is Bakhvalov-type mesh in [14] and denoted by B–R mesh. Also we consider (5) with $\kappa = 0$ and $\theta^* = -3\sigma/(4\beta)$, which is Bakhvalov-type mesh in [6,7] and denoted by B–K mesh.

The following boundary value problem is considered

$$
-\varepsilon u'' - (3 - x)u' + u = f(x) \quad \text{in } \Omega = (0, 1),
$$

where the right-hand side $f$ is chosen such that

$$
u(x) = \cos\left(\frac{\pi}{2}\varepsilon x\right)(1 - e^{-2x/\varepsilon}).$$

is the exact solution. The solution (56) exhibits typical boundary layer behavior.

We will consider $\varepsilon = 10^{-4}$, $10^{-5}$, $\ldots$, $10^{-9}$, $k = 1, 2, 3, 4$ and $N = 8, 16, \ldots, 2048$. We estimate the uniform errors for a fixed $N$ by taking the maximum error over a wide range of $\varepsilon$, namely

$$e_N^N := \max_{\varepsilon=10^{-4}, 10^{-5}, \ldots, 10^{-9}} \|u - u^N\|_{\varepsilon}. $$

Rates of convergence $r_e^N$ are computed by means of the formula

$$r_e^N = \log_2(e_N^N/e^2 N).$$

The numerical results are presented in Tables 1 and 2. The errors $e_N^N$ and the convergence rates $r_e^N$ are in accordance with Theorem 1 and illustrate its sharpness. Moreover, in Tables 1 and 2 we can observe that B–R mesh gives almost the same performance as B–K mesh.

Besides, in order to test convergence when a graded mesh is always used for the layer subdomain even for $N^{-1} \leq \varepsilon$, we consider errors $\|u - u^N\|$, $\|u - u^N\|_{L^\infty(\Omega)}$ and $\|u - u^N\|_{\varepsilon}$ on B–R mesh for $\varepsilon = 10^{-3}$, $k = 1$ and $N = 2^3, 2^4, \ldots, 2^{17}$. The data are plotted on log-log chart in Fig. 1. The convergence does not stall as $N \to \infty$. However, the convergence rates of $\|u - u^N\|$ and $\|u - u^N\|_{L^\infty(\Omega)}$ are not optimal when $N^{-1} \geq \varepsilon$. 
Table 2 Errors and convergence rates for problem (55)

| $N$  | $k = 3$                        | $k = 4$                        |
|------|--------------------------------|--------------------------------|
|      | B–R mesh $e_N$ $r_{e_N}$       | B–K mesh $e_N$ $r_{e_N}$       | B–R mesh $e_N$ $r_{e_N}$       | B–K mesh $e_N$ $r_{e_N}$       |
| 8    | 0.697E–02 3.10                 | 0.698E–02 3.11                 | 0.166E–02 4.41                 | 0.166E–02 4.41                 |
| 16   | 0.811E–03 3.03                 | 0.811E–03 3.03                 | 0.783E–04 4.15                 | 0.783E–04 4.15                 |
| 32   | 0.992E–04 3.01                 | 0.992E–04 3.01                 | 0.441E–05 4.04                 | 0.441E–05 4.04                 |
| 64   | 0.123E–04 3.00                 | 0.123E–04 3.00                 | 0.267E–06 4.01                 | 0.267E–06 4.01                 |
| 128  | 0.154E–05 3.00                 | 0.154E–05 3.00                 | 0.166E–07 4.00                 | 0.166E–07 4.00                 |
| 256  | 0.192E–06 3.00                 | 0.192E–06 3.00                 | 0.103E–08 4.00                 | 0.103E–08 4.00                 |
| 512  | 0.240E–07 3.00                 | 0.240E–07 3.00                 | 0.646E–10 4.00                 | 0.646E–10 4.00                 |
| 1024 | 0.300E–08 3.00                 | 0.300E–08 3.00                 | 0.404E–11 4.00                 | 0.404E–11 4.00                 |
| 2048 | 0.375E–09 –                   | 0.375E–09 –                   | 0.252E–12 –                   | 0.252E–12 –                   |

Fig. 1 Errors: $\varepsilon = 10^{-3}$, B–R mesh, $k = 1$

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