NEON\textsuperscript{\textdagger}: Accelerated Gradient Methods for Extracting Negative Curvature for Non-Convex Optimization\textsuperscript{*}

Yi Xu\textsuperscript{\textast} 
Rong Jin\textsuperscript{\dagger} 
Tianbao Yang\textsuperscript{\textdagger} 

\textsuperscript{\textast} Department of Computer Science, The University of Iowa, Iowa City, IA 52242 
\textsuperscript{\dagger} Alibaba Group, Bellevue, WA 98004 

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Abstract

Accelerated gradient (AG) methods are breakthroughs in convex optimization, improving the convergence rate of the gradient descent method for optimization with smooth functions. However, the analysis of AG methods for non-convex optimization is still limited. It remains an open question whether AG methods from convex optimization can accelerate the convergence of the gradient descent method for finding local minimum of non-convex optimization problems. This paper provides an affirmative answer to this question. In particular, we analyze Nesterov’s Accelerated Gradient method for extracting the negative curvature from random noise, which is central to escaping from saddle points. By leveraging the proposed NAG methods for extracting the negative curvature, we present a new AG algorithm with double loops for non-convex optimization, which converges to second-order stationary point $x$ such that $\|\nabla f(x)\| \leq \epsilon$ and $\nabla^2 f(x) \geq -\sqrt{\epsilon}I$ with $\tilde{O}(1/\epsilon^{1.75})$ iteration complexity, improving that of gradient descent method by a factor of $\epsilon^{-0.25}$ and matching the best iteration complexity of second-order Hessian-free methods for non-convex optimization.

1. Introduction

We consider the following optimization problem in this paper:

$$
\min_{x \in \mathbb{R}^d} f(x),
$$

where $f(x)$ is a twice differentiable non-convex smooth function, whose Hessian is Lipschitz continuous. Recently, this problem has received increasing interests in the machine learning community due to that many learning problems are non-convex. A renowned method in the machine learning community is gradient descent (GD) method, which updates the solution according to the following equation:

$$
\mathbf{x}_{t+1} = \mathbf{x}_t - \eta \nabla f(\mathbf{x}_t),
$$

\textsuperscript{*} The main result of extracting negative curvature by NEON\textsuperscript{\textdagger} in this manuscript is merged into our earlier manuscript “First-order Stochastic Algorithms for Escaping From Saddle Points in Almost Linear Time” (Xu et al., 2017b).

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where $\eta$ is a constant step size. It is not difficult to show that GD converges to an $\epsilon$-critical point $x$, i.e., $\|\nabla f(x)\| \leq \epsilon$, with an iteration complexity of $O(1/\epsilon^2)$. However, such a critical point could be a saddle point, which could be far from a local minimum and can harm the performance of prediction in machine learning.

To address this issue, one solution is to design an optimization algorithm that can converge to an $(\epsilon, \gamma)$-second-order stationary point (SSP) $x$ such that

$$\|\nabla f(x)\| \leq \epsilon, \quad \nabla^2 f(x) \geq -\gamma I.$$ 

When the objective function is non-degenerate (i.e., the Hessian matrix at all saddle points has negative eigen-values), an $(\epsilon, \gamma)$-SSP $x$ is guaranteed to be close to a local minimum and even a global minimum for certain problems.

Despite the popularity of GD, second-order methods have emerged to provide second-order convergence guarantee for non-convex optimization, which utilize the Hessian matrix or the Hessian-vector product for updating the solution. Starting from the seminal work by (Nesterov and Polyak, 2006), a wave of studies have been devoted to designing efficient second-order optimization algorithms with fast convergence to a SSP (Agarwal et al., 2017; Carmon et al., 2016; Xu et al., 2017a; Cartis et al., 2011a,b; Royer and Wright, 2017; Liu and Yang, 2017; Carmon and Duchi, 2016). A state-of-the-art result for second-order optimization algorithms is to use Hessian-vector products for finding an $(\epsilon, \sqrt{\epsilon})$-SSP with an $\tilde{O}(T_h/\epsilon^{1.75})$ time complexity where $T_h$ denotes the runtime of the Hessian-vector product.

In spite of the theoretical promise of second-order optimization algorithms, first-order (or gradient-based) algorithms are still the first choice in practice due to their simplicity. A recent breakthrough for gradient-based non-convex optimization methods is due to Jin et al. (2017a), who proposed a gradient-based method converging to an $(\epsilon, \sqrt{\epsilon})$-SSP with an iteration complexity of $\tilde{O}(1/\epsilon^2)$. Although it promotes GD for finding a SSP, it is still worse by a factor of $\epsilon^{-0.25}$ than the state-of-the-art second-order optimization algorithms. Given the dramatic success of accelerated gradient methods for convex optimization, an interesting question is:

*Can we use accelerated gradient methods from convex optimization to accelerate the convergence of non-convex optimization for finding a SSP?*

This paper gives an affirmative answer to this question. Our main contribution is summarized below:

- We analyze Nesterov’s Accelerated Gradient (NAG) method (Nesterov, 2004) for extracting the negative curvature of a Hessian matrix, which is central to algorithms escaping from (non-degenerate) saddle points for non-convex optimization. We refer to the proposed procedure for extracting the negative curvature of a Hessian matrix as NEON$^+$ since it is an accelerated variant of its predecessor NEON (Xu et al., 2017b).

- By combining the proposed AG methods for extracting the negative curvature of a Hessian matrix and an existing AG method for minimizing a regularized almost-convex function (Carmon et al., 2016), we present a new AG algorithm with double loops (dubbed NEAG) for finding a SSP to a general non-convex optimization problem in (1).
• The proposed NEAG algorithm enjoys an iteration complexity of $\tilde{O}(1/\epsilon^{1.75})$ for finding an $(\epsilon, \sqrt{\epsilon})$-SSP, matching the state-of-the-art result of second-order Hessian-free methods.

2. Related Work

Although there are extensive studies about AG methods for convex optimization, the analysis of AG for non-convex optimization is still limited. Ghadimi and Lan (2016) analyzed a variant of AG for minimizing non-convex smooth functions. However, its rate of convergence to a critical point is the same as standard GD method. Li and Lin (2015) analyzed variants of accelerated proximal gradient (APG) methods for minimizing a family of non-convex functions consisting of a smooth component and a non-smooth component. For general functions in this family, they only proved the asymptotic convergence to a critical point. Asymptotic results with explicit convergence rates are established for functions that satisfy the Kurdyka - Lojasiewicz (KL) property. Yang et al. (2016) analyzed two variants of AG methods in a stochastic setting for non-convex optimization under a unified framework. Again, their convergence analysis are only for finding critical points and the convergence rates of the analyzed two AG methods in the stochastic setting is the same as stochastic gradient method.

Recently, O’Neill and Wright (2017) proved that the Polyak’s heavy-ball method does not converge to critical points that do not satisfy second-order necessary conditions. They also analyzed the divergence rate of two accelerated gradient methods (including Polyak’s heavy-ball method and Nesterov’s AG method) from a (non-degenerate) saddle point of a non-convex quadratic function, showing that both methods can diverge from this point more rapidly than GD. Nevertheless, their analysis does not provide any guarantee on permanent escaping from a (non-degenerate) saddle point. In addition, no explicit convergence rate of AG methods to a SSP was established for a general non-convex optimization problem (1). Carmon et al. (2016) analyzed an AG method for minimizing an almost-convex function, which achieves faster convergence to a critical point than GD method. They also developed an accelerated method for finding a SSP of a general non-convex optimization problem by combining the AG method for minimizing an almost-convex function and a second-order method (e.g., the Lanczos method) for extracting the negative curvature. For finding an $(\epsilon, \sqrt{\epsilon})$-SSP, their accelerated method achieves the best iteration complexity of $\tilde{O}(1/\epsilon^{1.75})$ among existing second-order Hessian-free methods. However, they do not address the question raised before, i.e., finding a SSP with an AG method. Carmon et al. (2017) proposed the first AG algorithm with provable acceleration over GD for non-convex optimization. Their algorithm can converge to a first-order stationary point with $\tilde{O}(1/\epsilon^{1.75})$ iterations. However, it is not clear whether their method can guarantee finding a SSP with an iteration complexity of $\tilde{O}(1/\epsilon^{1.75})$.

A fundamental concern in the design of non-convex optimization algorithms for finding a SSP is how to escape from saddle points. The proposed first-order method dubbed NEON$^+$ addresses this concern by extracting the negative curvature from a Hessian matrix with negative eigen-values. It is inspired by a recent work (Xu et al., 2017b), which is the first work that develops a first-order method (named NEON) for extracting negative curvature from a Hessian matrix with negative eigen-values. Their method is a gradient descent
method and suffers from an iteration complexity of $\tilde{O}(1/\gamma)$ for finding a negative curvature for a Hessian matrix whose minimum eigen-value is less than $-\gamma < 0$. In contrast, the proposed $\text{Neon}^+$ is based on PHB or NAG and improves the iteration complexity of $\text{Neon}$ to $\tilde{O}(1/\sqrt{\gamma})$. By utilizing $\text{Neon}^+$ in the framework developed by Carmon et al. (2016), we obtain an AG algorithm for finding an $O(1/\epsilon - 1.75)$ with an iteration complexity of $\tilde{O}(1/\epsilon^{1.75})$. As a byproduct, $\text{Neon}^+$ can be also leveraged in stochastic non-convex optimization to accelerate the convergence for finding a SSP.

It is worth mentioning that, a recent work (Allen-Zhu and Li, 2017) also develops an improved variant of $\text{Neon}$ named $\text{Neon}^2_{\text{det}}$ for finding a negative curvature with an iteration complexity of $\tilde{O}(1/\sqrt{\gamma})$. We emphasize three differences between $\text{Neon}^+$ and $\text{Neon}^2_{\text{det}}$: (i) $\text{Neon}^+$ is based on PHB and NAG, which are more familiar to the machine learning and optimization community; while $\text{Neon}^2_{\text{det}}$ is based on Chebyshev approximation theory; (ii) our analysis of $\text{Neon}^+$ is elementary and self-contained; while the analysis of $\text{Neon}^2_{\text{det}}$ relies on the stability analysis of Chebyshev polynomials; (iii) $\text{Neon}^2_{\text{det}}$ terminates when the resulting vector is within an Euclidean ball with a small radius inversely proportional to a power of the dimensionality, which might cause numerical issues in practice for high-dimensional problems; in contrast $\text{Neon}^+$ terminates when the resulting vector is within an Euclidean ball with a radius almost independent of the dimensionality, rendering it much more viable for high dimensional problems.

It was also brought to our attention that when we prepare this manuscript, an independent work by (Jin et al., 2017b) also analyzed Nesterov's AG method for non-convex optimization. For finding a $(\epsilon, \sqrt{\epsilon})$-SSP, the algorithm (Jin et al., 2017b) and in the present work enjoy the same iteration complexity of $\tilde{O}(1/\epsilon^{1.75})$. The differences between the two work are (i) we focus our analysis on NAG for extracting negative curvature, in contrast they directly analyze NAG for non-convex optimization; (ii) our AG algorithm has a nested loop, while their algorithm is a single loop.

3. Preliminaries

In this section, we present some preliminaries, including some notations, accelerated gradient methods for convex optimization, and the idea of $\text{Neon}$ for non-convex optimization.

Denote by $\| \cdot \|$ the Euclidean norm of a vector and $\| \cdot \|_2$ the spectral norm of a matrix. Let $\lambda_{\min}(\cdot)$ denote the minimum eigen-value of a matrix. We make standard assumptions regarding (1) in order to find a SSP.

Assumption 1

1. $f(x)$ has $L_1$-Lipschitz continuous gradient and $L_2$-Lipschitz continuous Hessian, i.e.,
   \[ \| \nabla f(x) - \nabla f(y) \| \leq L_1 \| x - y \|, \quad \| \nabla^2 f(x) - \nabla^2 f(y) \|_2 \leq L_2 \| x - y \| \] (2)

2. given an initial point $x_0$, assume that there exists $0 < \Delta < \infty$ such that $f(x_0) - \min_{x \in \mathbb{R}^d} f(x) \leq \Delta$.

A function $g(x)$ is $\sigma$-strongly convex ($\sigma > 0$) if for all $x, y$ it holds that
\[ g(x) \geq g(y) + \nabla g(y) ^\top (x - y) + \frac{\sigma}{2} \| x - y \|^2, \quad \forall x, y \]
If the above inequality holds for $\sigma < 0$, then $g(x)$ is called $(-\sigma)$-almost convex.
AG methods for Extracting Negative Curvature From Noise

Algorithm 1 An AG method for minimizing a Smooth and Strongly Convex function: AG-SSC$(g, y_1, \epsilon, L_1, \sigma_1)$

1: Set $\kappa = L_1/\sigma_1$, $z_1 = y_1$, $\zeta = \sqrt{\kappa^2 - 1}/\sqrt{\kappa^2 + 1}$
2: for $j = 1, 2, \ldots$ do
3: if $\|\nabla g(y_j)\| \leq \epsilon$ then
4: return $y_j$
5: end if
6: $y_{j+1} = z_j - \frac{1}{L_1} \nabla g(z_j)$,
7: $z_{j+1} = y_{j+1} + \zeta(y_{j+1} - y_j)$
8: end for

3.1. Accelerated Gradient Methods from Convex Optimization

For minimizing a smooth and (strongly) convex function $g(x)$, AG methods have been proposed with faster convergence rate than GD. Next, we present a variant of AG for minimizing a $L_1$-smooth and $\sigma_1$-strongly convex function since it will be used in our development. A famous variant of AG is Nesterov’s AG (NAG) method (Nesterov, 2004), whose update is given by Step 6 & 7 in Algorithm 1. The iteration complexity of Algorithm 1 for minimizing a $L_1$-smooth and $\sigma_1$-strongly convex function is given by $O(\sqrt{L_1/\sigma_1} \log(1/\epsilon))$.

3.2. NEON for Non-Convex Optimization

NEON (Xu et al., 2017b) is a first-order procedure for extracting the negative curvature from a Hessian matrix $\nabla^2 f(x)$ at a point $x$. In particular, it starts from a random noise vector $u_0$ and iteratively updates $u_r$ according to the following equation:

$$u_{r+1} = u_r - \eta(\nabla f(x + u_r) - \nabla f(x)),$$  

and it terminates when $f(x + u) - f(x) - \nabla f(x)^T u$ is sufficiently small given that $u_r$ resides in an Euclidean ball with a proper radius, or it cannot make $f(x + u) - f(x) - \nabla f(x)^T u$ sufficiently small after a certain number of iterations. In (Xu et al., 2017b), NEON is motivated by using a noisy Power method to compute the negative curvature. In particular, the Lipschitz Hessian condition implies that

$$\|\nabla f(x + u_r) - \nabla f(x) - \nabla^2 f(x) u_r\| \leq \frac{L_2}{2} \|u_r\|^2.$$

Therefore, the sequence in (3) approximates another sequence $u'_{r+1} = (I - \eta\nabla^2 f(x))u'_r$ when $\|u_r\|$ is sufficiently small, which is exactly the sequence generated by applying the Power method to compute the leading eigen-pair of $I - \eta\nabla^2 f(x)$ corresponding to the minimum eigen-pair of $\nabla^2 f(x)$.

On the other hand, we can also consider the sequence (3) as an application of GD to the following objective function:

$$\hat{f}_x(u) = f(x + u) - f(x) - \nabla f(x)^T u.$$  

(4)
Sometimes write $\hat{f}_x(u) = \hat{f}(u)$, where the dependent $x$ should be clear from the context. By the Lipschitz continuous Hessian condition, we have that

$$\frac{1}{2} u^\top \nabla^2 f(x) u - \frac{L_3}{6} \|u\|_3^3 \leq \hat{f}(u) \leq \frac{1}{2} u^\top \nabla^2 f(x) u + \frac{L_3}{6} \|u\|_3^3.$$  

It implies that if $\hat{f}(u)$ is sufficiently less than zero and $\|u\|$ is not too large, then $\frac{u^\top \nabla^2 f(x) u}{\|u\|^2}$ will be sufficiently less than zero.

A key result in (Xu et al., 2017b) is that when $\lambda_{\min}(\nabla^2 f(x)) \leq -\gamma$ Neon can find a negative curvature direction $u$ such that $\frac{u^\top \nabla^2 f(x) u}{\|u\|^2} \leq -\tilde{\Omega}(\gamma)$ with an $\tilde{O}(1/\gamma)$ number of iterations. The main contribution of this paper is to show that both PHB and NAG with an appropriate momentum constant $\zeta$ when applied the function $\hat{f}(u)$ can find a negative curvature much faster than GD.

4. **NEON**: Accelerated Gradient Methods for Extracting Negative Curvature

In this section, we will analyze NAG methods. We first present the updates of NEON and discuss the underlying intuition why NAG can find a negative curvature faster than GD.

The updates of NAG method applied to the function $\hat{f}(u)$ at a given point $x$ is given by

$$y_{\tau+1} = u_\tau - \eta \nabla \hat{f}(u_\tau)$$

$$u_{\tau+1} = y_{\tau+1} + \zeta (y_{\tau+1} - y_\tau)$$

(5)

where the term $\zeta (y_{\tau+1} - y_\tau)$ is the momentum term, and $\zeta \in (0, 1)$ is the momentum parameter. The proposed algorithm based on the NAG method (referred to as NEON) for extracting NC of a Hessian matrix $\nabla^2 f(x)$ is presented in Algorithm 2, where

$$\Delta_x(y_\tau, u_\tau) = \hat{f}_x(y_\tau) - \hat{f}_x(u_\tau) - \nabla \hat{f}_x(u_\tau)^\top (y_\tau - u_\tau),$$

and NCFind is a procedure that returns a NC by searching over the history $y_{0:\tau}, u_{0:\tau}$ shown in Algorithm 3. The condition check in Step 4 is to detect easy cases such that NCFind can easily find a NC in historical solutions without continuing the update. It is notable that NCFind is similar to a procedure called Negative Curvature Exploitation (NCE) in (Jin et al., 2017b). However, the difference is that NCFind is tailored to finding a negative curvature, while NCE in (Jin et al., 2017b) is for ensuring a decrease on a modified objective.

Before presenting our main result and formal analysis, we first present an informal analysis about why NEON is faster than Neon. This analysis is from a perspective of Power method for computing dominating eigen-vectors of a matrix in light of our goal is to compute a negative curvature of a Hessian matrix corresponding to negative eigen-values. In particular, by ignoring the error of $\nabla \hat{f}(u_\tau) = \nabla f(x + u_\tau) - \nabla f(x)$ for approximating $H = \nabla^2 f(x)$, the update in (5) can be written as

$$\hat{u}_{\tau+1} = \left[ \begin{array}{cc} (1 + \zeta)(I - \eta H) & -\zeta(I - \eta H) \\ I & 0 \end{array} \right] \hat{u}_{\tau-1}.$$  

(6)
Algorithm 2 Accelerated Gradient methods for Extracting NC from Noise: \( \text{NEON}^+(f, x, t, F, U, \zeta, r) \)

1: **Input:** \( f, x, t, F, U, \zeta, r \)
2: Generate \( y_0 = u_0 \) randomly from the sphere of an Euclidean ball of radius \( r \)
3: for \( \tau = 0, \ldots, t \) do
4: if \( \Delta_x(y_{\tau}, u_{\tau}) < -\frac{\gamma}{2}\|y_{\tau} - u_{\tau}\|^2 \) then
5: return \( v = \text{NCFind}(y_{0:\tau}, u_{0:\tau}) \)
6: end if
7: compute \( (y_{\tau+1}, u_{\tau+1}) \) by (5)
8: end for
9: if \( \min_{\|y\| \leq U} \hat{f}_x(y_{\tau}) \leq -2F \) then
10: let \( \tau' = \arg \min_{\|y\| \leq U} \hat{f}_x(y_{\tau}) \)
11: return \( y_{\tau'} \)
12: else
13: return 0
14: end if

Algorithm 3 \( \text{NCFind}(y_{0:\tau}, u_{0:\tau}) \)

1: if \( \min_{j=0, \ldots, \tau} \|y_j - u_j\| \geq \zeta \sqrt{6\eta F} \) then
2: return \( y_j \),
3: where \( j = \min\{j' : \|y_{j'} - u_{j'}\| \geq \zeta \sqrt{6\eta F}\} \)
4: else
5: return \( y_{\tau} - u_{\tau} \)
6: end if

where

\[ \hat{u}_{\tau+1} = \begin{bmatrix} u_{\tau+1} \\ u_{\tau} \end{bmatrix}, \]

The above sequence can be considered as an application of the Power method to an augmented matrix:

\[ A = \begin{bmatrix} (1 + \zeta)(I - \eta H) & -\zeta(I - \eta H) \\ I & 0 \end{bmatrix}. \]

According to existing analysis of the Power method for computing top eigen-vectors in the top-\( k \) eigen-space of the matrix \( A \), the iteration complexity depends on the eigen-gap \( \Delta_k \) of \( A \) in an order of \( \tilde{O}(1/\Delta_k) \), where the eigen-gap is defined as the difference between the \( k \)-th largest eigen-value of \( A \) and the \( (k + 1) \)-th largest eigen-value of \( A \). The following result exhibits that by appropriately choosing the momentum constant \( \zeta \), the eigen-gap \( \Delta_k \) of \( A \) matrix involved in NAG method scales as \( \sqrt{\gamma} \) for a Hessian matrix \( H \) whose negative eigen-values are less than \(-\gamma \). Following by the similar analysis in (O’Neill and Wright, 2017), we can state the result formally in the following lemma.

**Lemma 1** Assume the eigen-values of \( H \) satisfy \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq -\gamma < 0 \leq \lambda_{k+1} \leq \lambda_d \) and \( \eta \leq 1/L_1 \) is sufficiently small. Let \( e_1, \ldots, e_d \) denote the corresponding eigen-values of
H. Then the top-$k$ eigen-pairs of $A$ are $(\lambda_i^h(A), \tilde{e}_i)$, $i = 1, \ldots, k$, where $\tilde{e}_i = \left[ \frac{e_i}{1/\lambda_i^h(A)} \right]$ and
\[
\lambda_i^h(A) = \frac{1}{2} \left[ (1 + \zeta)(1 - \eta \lambda_i) + \sqrt{(1 + \zeta)^2 (1 - \eta \lambda_i)^2 - 4 \zeta (1 - \eta \lambda_i)} \right].
\]
By choosing $\zeta = 1 - \sqrt{\eta \gamma} \in (0, 1)$, the eigen-gap $\Delta_k$ of $A$ corresponding to its top-$k$ eigenspace is at least $\sqrt{\eta \gamma}/2$.

The proof of above lemma can be found in (Xu et al., 2017b). We emphasize that the above informal analysis is not enough for proving our main result. Nevertheless, a formal analysis yields the following result.

**Theorem 2** For any $\gamma \in (0, 1)$ and a sufficiently small $\delta \in (0, 1)$, let $x$ be a point such that $\lambda_{\min}(\nabla^2 f(x)) \leq -\gamma$. For any constant $\hat{c} \geq 43$, there exists a constant $c_{\max}$ that depends on $\hat{c}$, such that if $\text{NEON}^+$ is called with $t = \sqrt{2 \log(dL_1/(\gamma \delta))}$, $F = \eta \gamma^3 L_1 L_2^{-2} \log^3(dL_1/(\gamma \delta))$, $r = \sqrt{\eta \gamma^2 L_1^{-1/2} L_2^{-1} \log^2(dL_1/(\gamma \delta))}$, $U = 12 \hat{c} \sqrt{\eta L_1 F / L_2} / \gamma^2$, a small constant $\eta \leq c_{\max}/L_1$, and a momentum parameter $\zeta = 1 - \sqrt{\eta \gamma}$, then with high probability $1 - \delta$ it returns a vector $u$ such that $\frac{u^T \nabla^2 f(x) u}{\|u\|^2} \leq -\gamma_2 \frac{1}{\log(dL_1/(\gamma \delta))} \leq -\tilde{\Omega}(\gamma)$. If $\text{NEON}^+$ returns $u \neq 0$, then the above inequality must hold; if $\text{NEON}^+$ returns $0$, we can conclude that $\lambda_{\min}(\nabla^2 f(x)) \geq -\gamma$ with high probability $1 - O(\delta)$.

**Remark.** The proof of above theorem can be found in (Xu et al., 2017b).

5. Applications

5.1. An AG Algorithm for Non-Convex Optimization

Built upon $\text{NEON}^+$ and an existing framework proposed in (Carmon et al., 2016), we can obtain an accelerated algorithm using accelerated gradient methods from convex optimization for non-convex optimization. The proposed AG algorithm is based on two building blocks: (i) an AG method based on $\text{NEON}^+$ conducting negative curvature descent for driving the solution to reach a point where the objective function $f(\cdot)$ is locally almost-convex (i.e., the Hessian matrix at the point has all eigen-values larger than $-\gamma$); (ii) an AG method for minimizing a regularized almost-convex function. The first building block is presented in Algorithm 4 and the second building block is proposed in (Carmon et al., 2016). For completeness, we also present it in Algorithm 5. The proposed Algorithm (named NEAG) is presented in Algorithm 6 and its complexity result is presented in the following theorem.

**Theorem 2** With probability at least $1 - \delta$, the Algorithm NEAG returns a vector $\hat{x}_k$ such that $\|\nabla f(\hat{x}_k)\| \leq \epsilon$ and $\lambda_{\min}(\nabla^2 f(\hat{x}_k)) \geq -\gamma$ with a worse-case iteration complexity of $\tilde{O} \left( \frac{1}{\gamma^3 \gamma^{1/2}} + \frac{1}{\gamma^{1/2}} + \frac{\gamma^{1/2}}{\epsilon^2} \right)$.

**Remark:** When $\gamma = \sqrt{\epsilon}$, the iteration complexity of NEAG is $\tilde{O}(1/\epsilon^{1.75})$. 
Algorithm 4 NEON\textsuperscript{+} for Negative Curvature Descent: NEON\textsuperscript{+}-NCD($x_0, \gamma, c, \delta$)

1: **Input:** $x_0, \gamma, c > 0, \delta$
2: $x_1 = x_0, \delta' = \delta / (1 + 12L_2^2 \Delta / \gamma^3)$
3: for $j = 1, 2, \ldots, \delta$
   4: Compute $v_j = \text{NEON\textsuperscript{+}}(f, x_0, t, \mathcal{F}, U, \zeta, r)$
   5: if $v_j \neq 0$ then
      6: $x_{j+1} = x_j - \frac{c \gamma}{L_2} \text{sign}(v_j^\top \nabla f(x_j)) v_j$
   7: else
      8: return $x_j$
   9: end if
10: end for

Algorithm 5 AG for minimizing an almost-convex function: AG-AC($f, z_1, \epsilon, \gamma, L_1$)

1: for $j = 1, 2, \ldots, \delta$
   2: if $\|\nabla f(z_j)\| \leq \epsilon$ then
      3: return $z_j$
   4: end if
   5: Define $g_j(z) = f(z) + \gamma \|z - z_j\|^2$
   6: set $\epsilon' = \epsilon \sqrt{\gamma / 50(L_1 + 2\gamma)}$
   7: $z_{j+1} = \text{AG-SSC}(g_j, z_j, \epsilon', L_1, \gamma)$
8: end for

5.2. Stochastic Non-Convex Optimization

As a byproduct, we can also use NEON\textsuperscript{+} in stochastic non-convex optimization for extracting negative curvature to strengthen first-order stochastic methods for enjoying convergence to a SSP. One can follow the development in (Xu et al., 2017b) to develop new variants of NEON\textsuperscript{+}-SGD, NEON\textsuperscript{+}-SCSG, NEON\textsuperscript{+}-Natasha with improved convergence, which will be omitted here.

Algorithm 6 An AG Algorithm for Non-Convex Minimization: NEAG: ($x_0, \epsilon, \gamma, \delta$)

1: **Input:** $x_0, \epsilon, \gamma, \delta$
2: $K := \left[1 + \Delta \left(\frac{\max(12L_2^2, 2L_1)}{\epsilon_2} + \frac{2\sqrt{10}L_2}{\epsilon_1 \epsilon_2}\right)\right], \delta' := \delta / K$
3: for $k = 1, 2, \ldots, \delta$
   4: $\tilde{x}_k = \text{NEON\textsuperscript{+}}-\text{NCD}(x_k, \gamma, c, \delta')$
   5: if $\|\nabla f(\tilde{x}_k)\| \leq \epsilon_1$ then
      6: return $\tilde{x}_k$
   7: else
      8: Set $f_k(x) = f(x) + L_1 \left(\|x - \tilde{x}_j\| - \epsilon_2 / L_2\right)_+$
      9: $x_{k+1} = \text{AG-AC}(f_k, \tilde{x}_k, \epsilon/2, 3\gamma, 5L_1)$
   10: end if
11: end for
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