Vortex bound states of charge and magnetic fluctuations-induced topological superconductors in heterostructures

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The helical electron states on the surface of topological insulators or elemental Bismuth become unstable toward superconducting pairing formation when coupled to the charge or magnetic fluctuations. The latter gives rise to pairing instability in chiral channels $d_{xy} \pm id_{x-y}$, as has been observed recently in epitaxial Bi/Ni bilayer system at relatively high temperature, while the former favors a pairing with zero total angular momentum. Motivated by this observation we study the vortex bound states in these superconducting states. We consider a minimal model describing the superconductivity in the presence of a vortex in the superconducting order parameter. We show that zero-energy states appear in the spectrum of the vortex core for all pairing symmetries. Our findings may facilitate the observation of Majorana modes bounded to the vortices in heterostructures with no need for a proximity-induced superconductivity.

I. INTRODUCTION

For many years it was a common belief that quantum mechanical particles are either bosons or fermions whose wave functions take a plus or minus sign, respectively, upon the exchange of two identical particles. But in the past forty years it has been realized that this picture for point-like particles is correct only for a space with dimensions equal or greater than three. There are other possibilities in two-dimensional (2D) space. In 2D systems, particles' wave functions can in general get a complex phase when one particle turns around another one, the so-called Abelian anyons. The first physical realization of Abelian anyons occurred in systems exhibiting the fractional quantum Hall effect. For the non-Abelian anyons, on the other hand, the multi-component wave function lives in a degenerate subspace, and the interchange of particles brings about a unitary evolution matrix within the subspace. The non-commutative structure of matrices promises a platform for fault-tolerant topological quantum computations.

Majorana fermions, a special class of non-Abelian anyons, discovered first by theoretical particle physicist Ettore Majorana in 1937, have the property that they are their own anti-particles. If $\gamma_i$ and $\gamma_i^\dagger$ are donated as the annihilation and creation operators for a Majorana fermion in a quantum state $|i\rangle$ then $\gamma_i = \gamma_i^\dagger$ and $\{\gamma_i, \gamma_j\} = 2\delta_{ij}$. Majorana fermions were not observed in elementary particle physics, since these particles are their own anti-particles and therefore they cannot carry any quantum charge. But the developed concepts were traced in condensed matter physics years after the theoretical discovery. In the seminal work by Reed and Green, it is shown that a boundary between a 2D topological $p + ip$ superconductor in the strong coupling limit and a trivial one in the weak coupling limit hosts a single Majorana mode. In a vortex the zero mode is confined to the vortex core and a Majorana bound state (MBS) is formed. A prime example of a system hosting MBS was introduced by Kitaev. The model is a chain of spin-less fermions with superconducting $p$-wave order parameter. Under certain conditions, where the bulk of the system becomes topologically non-trivial, two unpaired MBSs appear at the two ends of an open chain.

The discovery of topological insulators (TI) provided yet another boost to the realization of Majorana fermions in solid state systems. In a celebrated work by Fu and Kane in Ref. [15], it is shown that the MBSs appear in the vortex cores of a conventional $s$-wave superconductor proximitized to the surface of TIs. This arises due to the spin-momentum locked structure of the surface states of a TI and the underlying Berry phase, which makes the $s$-wave pairing formally like a $p$-wave pairing upon projection onto the surface states. Ma-
jorana states bounded to vortices can even be realized in doped TIs.\textsuperscript{16,17} Moving along the same direction, it is shown that the surface of TIs can be replaced with more conventional semiconductors with strong Rashba spin-orbit coupling. The latter coupling lifts the spin degeneracy and creates multiple spin-momentum locked Fermi surfaces. An external magnetic or Zeeman field is required to tune a quantum phase transition from an induced $s$-wave superconductor to a topological superconductor with Majorana fermions dispersing along the edges of 2D systems\textsuperscript{18–21} or bound to the end points of semiconductor nano-wires\textsuperscript{22}.

At the heart of these settings for generating a topological phase with MBSs lies the existence of three conventional ingredients: a semiconductor quantum well with strong Rashba coupling, an $s$-wave superconductor, and a ferromagnetic insulator. The Zeeman field produced by the ferromagnetic layer should be strong enough to remove the extra Fermi surfaces, and simultaneously should be weak to survive the induced superconducting pairing amplitude, a condition which severely restricts the choice of materials. Therefore, it is highly demanding to look for heterostructures with less impeding ingredients.

The recently discovered superconductivity in epitaxial Bismuth/Nickel (Bi/ Ni) bilayer heterostructure with relatively large transition temperature $T_c \approx 4.2$K may provide an example of an intrinsic topological superconductor with chiral $d_{xy} \pm id_{x^2-y^2}$ order parameter.\textsuperscript{23} The nodeless structure of the proposed gap function is consistent with the measurements of frequency-dependent optical conductivity in the time-domain THz spectroscopy.\textsuperscript{24} A schematic representation of the bilayer system is shown in Fig. 1. The main advantage of the latter system over the quantum well structures discussed above is that here the superconductivity is intrinsically driven by the ferromagnetic fluctuations circumventing the proximity to an extra $s$-wave superconductor layer.

Now an important question arises: what are the vortex bound states of intrinsic $d_{xy} \pm id_{x^2-y^2}$ topological superconductors in Bi/ Ni bilayer system? The aim of our paper is to answer this question. We show that the Berry phase effects change the angular momentum of the order parameter by one giving rise to an odd parity with zero-energy states. Note that our system is distinct from the case of $d_{x^2-y^2}$ superconductor proximitized to an electron gas studied in Ref. [25], where an external Zeeman field is applied to the system while in our work the time-reversal symmetry is already broken by the nature of chiral pairing.

The paper is organized as follows. In Sec. II we derive the single-particle Hamiltonian describing the helical electron states near the Fermi surface, then in Sec. III we derive the pairing correlations driven by magnetic fluctuations. In Sec. IV we study the vortex bound states for various cases and the existence of zero-energy states, and finally Sec. V concludes.

II. THE REAL SPACE PROJECTED NON-INTERACTING HAMILTONIAN

We begin with the electronic structure of the surface of Bi exposed to the vacuum as shown in Fig. 1. Most likely, the interface adjacent to Ni doesn’t contribute to superconductivity due to strong pair breaking effects, which is consistent with the thickness dependency of $T_c$.\textsuperscript{25,26} The strong Rashba coupling spits the surface electron states to spin-momentum locked states with the largest pocket centered around the center of surface of Brillouin zone. For simplicity we only consider this pocket and, hence, the Bi thin film in our setup in Fig. 1 can be replaced with a surface of Ti as well. Therefore the helical electron states are described by the following action in Euclidean-time formalism:

$$S_c = \int d\tau dr \bar{\Psi}(\partial_r + \tilde{H}_0) \Psi.$$  \hspace{1cm} (1)

The non-interacting Hamiltonian $\tilde{H}_0$ reads as

$$\tilde{H}_0 = \int dr \, \Psi^\dagger(r) (v_F |\sigma \times p|_z - \mu) \Psi(r).$$ \hspace{1cm} (2)

where $\Psi = (\psi_\uparrow, \psi_\downarrow)^T$ with $\psi_\pm$ as electron annihilation operators, $v$ is the magnitude of spin-orbit interaction or equivalently the Fermi velocity of electrons at the surface of Ti, $\mu$ is the chemical potential, $\sigma$ is a vector of Pauli matrices, and $p = -i \nabla$. We set $\hbar = k_B = 1$ throughout.

Since we are interested in electron states near the Fermi surface, we use momentum helical eigenstates $|k, \pm \rangle = (1, \pm ie^{-i\phi_k})T/\sqrt{2}$, where $H_0(|k|, \pm \rangle = \pm \varepsilon_k |k, \pm \rangle$ with $\varepsilon_k = v_F |k|$. We assume that the Fermi level crosses the “+” band and is located well above the node. We first write the annihilation operators in the spin basis in terms of annihilation operators in the helical “$\pm$” basis:

$$\psi_\uparrow(k) = \frac{1}{\sqrt{2}}(\psi_{k+} + \psi_{k-}), \psi_\downarrow(k) = \frac{ie^{-i\phi_{k'}}}{\sqrt{2}}(\psi_{k+} - \psi_{k-}).$$ \hspace{1cm} (3)

Then in real space they become

$$\psi_\uparrow(r) = \frac{1}{\sqrt{2}}[\psi_+(r) + \psi_-(r)],$$ \hspace{1cm} (4)

$$\psi_\downarrow(r) = \frac{i}{\sqrt{2}} \sum_k e^{-i\phi_k}(\psi_{k+} - \psi_{k-})e^{ikr}. \hspace{1cm} (5)

To perform the momentum sum in Eq. (5) we use the following approximation for the first exponential term

$$e^{-i\phi_k} \approx k_x |k_F| - ik_y |k_F|,$$

which is justified so long as the electron states near the Fermi surface are involved in the physical processes of
interest such as pairing. Here \( k_F = \mu/v_F \) is the Fermi momentum. The field operator \( \psi_{\uparrow}(r) \) then reads as

\[
\psi_{\uparrow}(r) = \frac{1}{\sqrt{2k_F}} (\partial_x - i\partial_y) [\psi_{\uparrow}(r) - \psi_{\downarrow}(r)].
\]

(7)

By projection to the Fermi surface we obtain

\[
\psi_{\uparrow}(r) \approx \frac{1}{\sqrt{v_F}} \psi_{\uparrow}(r), \quad \psi_{\downarrow}(r) \approx \frac{1}{\sqrt{2k_F}} (\partial_x - i\partial_y) \psi_{\uparrow}(r).
\]

(8)

Using these expressions in Eq. (2), the projected form of the non-interacting Hamiltonian reads as follows:

\[
H_0 = -\int dr \psi_{\uparrow}^\dagger(r) \left( \frac{v_F}{k_F} \nabla^2 + \mu \right) \psi_{\uparrow}(r),
\]

(9)

which is not dissimilar to the Hamiltonian of a 2D Fermi gas via the identification \( v_F/k_F = 1/2m \). Hereafter we drop the subindex in the field and write \( \psi_{+}(r) \equiv \psi(r) \).

III. MODEL OF MAGNETIC FLUCTUATIONS AND PAIRING HAMILTONIAN

To make the structure of the paper self-contained, in this section we present the details of a minimal model describing the superconductivity in the bilayer structure shown in Fig. 1 which is largely based on the Ref. [23]. In the regime of interest the superconducting \( T_c \) is much lower than the Curie temperature of the ferromagnetic Ni layer, hence, the ferromagnet is deep inside the ordered phase. We assume that the in-plane moments are aligned along the \( y \) direction (see Fig. 1) and the low-energy fluctuations of the magnetic moments, the spin waves, are described by the vector \( \mathbf{l}(\tau, \mathbf{r}) \) in which \( \mathbf{l} \cdot \mathbf{y} = 0 \). The magnetic fluctuations and their coupling to electrons are described, respectively, by the following actions:

\[
S_{\text{M}} = \frac{\rho_s}{2} \int d\tau d\mathbf{r} \left[ -i(1 \times \partial_\mathbf{r})^1 + \kappa(\nabla l)^2 \right],
\]

(10)

and

\[
S_{eM} = g \int d\tau d\mathbf{r} \psi^\dagger \mathbf{l} \cdot \mathbf{\sigma} \psi,
\]

(11)

where \( \rho_s \) is the density of magnetic moments, \( \kappa \) is the characteristic of spin waves, and \( g \) is the strength of interaction between electron spins and magnetic moments. For simplicity and in the interest of formation of Cooper pairs with zero center-of-mass momentum we only consider the out-of-plane fluctuations denoted by \( l_z \equiv \mathbf{b} \).

By taking Fourier transform to momentum space, the Eqs. (10-11) become

\[
S_{\text{M}} = \frac{T}{2} \sum_q D^{-1}(q)b_q^\dagger b_q,
\]

(12)

and

\[
S_{eM} = \frac{gT}{2} \sum_{k,q,\sigma\beta} (b_q \psi_{kq\sigma}^\dagger \psi_{k+q,\beta} + \text{h.c.}),
\]

(13)

where \( T \) is the temperature, \( q = (q, 2n\pi T), k = (k, (2n+1)\pi T) \) with \( n \) as an integer, and \( D(q) = 1/(\kappa \rho_s |q|^2 + \zeta) \) is the magnon propagator with a small gap \( \zeta \) due to anisotropy. Upon integrating out the bosonic field \( b \) and subsequent projection of electron fields \( \psi_{kq} \) onto the Fermi surface described by effective spinless fermion operators \( \psi_{k} \) in Eq. (3), we obtain the following effective interaction between electrons in the Cooper channel:

\[
S_c = \frac{T}{2A} \sum_{k,k'} U(k,k') e^{-i(\phi_k - \phi_{k'})} \psi_{k\uparrow}^\dagger \psi_{k\downarrow}^\dagger \psi_{-k\downarrow} \psi_{-k\uparrow},
\]

(14)

where \( \mathcal{A} \) is the area of the system. Here \( U(k,k') \) is an even function of its arguments and can be expanded in angular harmonics as

\[
U(k,k') = \sum_{l=\text{even}} U_l e^{i l(\phi_k - \phi_{k'})}.
\]

(15)

Therefore the even angular momentum components of the interaction matrix contribute to the odd component of the condensate \( f = \langle \psi_{-k} \psi_{k} \rangle \). This is a direct result of the non-trivial topology of Dirac Fermions. The effective angular momentum of the condensate \( f \) is decreased by one due to the Berry phase. Inserting Eq. (15) in Eq. (14) and decoupling the interaction in the Cooper channels \( f \), we obtain the mean-field BCS Hamiltonian as follows:

\[
H_\Delta = \sum_k \Delta(|k|) e^{i(l-1)\phi_k} \psi_{k\uparrow}^\dagger \psi_{-k\downarrow} + \text{h.c.}
\]

(16)

In this work we only consider the channels with the lowest angular momenta \( l = 0, \pm 2 \). The superconducting instability in channels \( l = \pm 2 \) is driven by the magnetic fluctuations being relevant to Bi/Ni bilayer system, while the instability with \( l = 0 \) arises from phonons or charge fluctuations, i.e. \( \sigma^z \rightarrow 1 \) in Eq. (11). In our formalism below we study all cases.

IV. SPECTRUM OF VORTEX BOUND STATES

The formulation and arguments presented in preceding sections provide a minimal superconducting Hamiltonian from Eqs. (9) and (16), i.e. \( H = H_0 + H_\Delta \). In the following subsections we first derive the corresponding Bogoliubov-de Gennes (BdG) equations for each channel \( l \) and then study the spectrum of vortex bound states.
A. The cases with $l = 2$ and $l = 0$

By inspection we see that for both of these cases the phases of the pairing in Eq. (16) are simply complex conjugate of each other, and thus, they can be treated within the same formalism. We present the details of calculations for $l = 2$ and will briefly discuss the $l = 0$ case at the end of this subsection. For the former case the pairing term $H_{\Delta}$ in Eq. (16) is written as

$$H_{\Delta} = \sum_{k} \frac{\Delta(|k|)}{k_{F}} (k_{x} + ik_{y}) \tilde{\psi}_{k}^\dagger \tilde{\psi}_{-k} + \text{h.c.},$$  

(17)

where we use Eq. (6), i.e. pairing occurs near the Fermi surface. To introduce a vortex in the order parameter, we assume that the space profile of pairing gap in the polar coordinate is $\Delta(r) = \Delta(r)e^{in\theta}$, where $r$ is measured from the center of vortex and $n$ denotes the winding of the vortex, the degree of vorticity. Thus, the full mean-field Hamiltonian of this system in real space can be recast as

$$H = \int dr \left[ -\tilde{\psi}^\dagger \left( \frac{v_{F}}{k_{F}} \nabla^{2} + \mu \right) \tilde{\psi} + \frac{i \Delta(r)}{2k_{F}} \tilde{\psi}^\dagger \{ e^{in\theta}, \partial_{x} \} \tilde{\psi} + i \frac{\Delta(r)}{2k_{F}} \tilde{\psi}^\dagger \{ e^{in\theta}, \partial_{x} - i\partial_{y} \} \tilde{\psi} \right],$$  

(18)

where $n' = n + 1$. We use a pseudo-rotation operator defined by the unitary transformation $U(\theta) = e^{-i(m+\frac{2}{2}\tau^{z})\theta}$, where $\tau^{z}$ is the Pauli matrix acting in particle-hole space, to remove the phase dependency of the pairing gap$^{19}$. That is, we write the wave function $\varphi(r)$ in the form $\varphi(r, \theta) = e^{i(m+\frac{2}{2}\tau^{z})\theta}\varphi(r)$. The possible values for $m$ are determined by the single-valued condition of wave function implying that $m$ is an integer (half-integer) for even (odd) $n'$. Using this transformation the eigenvalue problem turns into a set of differential equations for $v(r)$ and $u(r)$ as

$$\partial_{r}^{2}u + \frac{1}{r} \partial_{r}u - \frac{m_{z}^{2}}{r^{2}}u + \frac{k_{F}\mu}{v_{F}}u - i \frac{\Delta(r)}{v_{F}} \partial_{r}v - \frac{2m - 1}{2} v = \frac{k_{F}E}{v_{F}}u, \tag{23}$$

$$\partial_{r}^{2}v + \frac{1}{r} \partial_{r}v - \frac{m_{z}^{2}}{r^{2}}v + \frac{k_{F}\mu}{v_{F}}v + i \frac{\Delta(r)}{v_{F}} \partial_{r}u + \frac{2m + 1}{2} u = \frac{k_{F}E}{v_{F}}v, \tag{24}$$

where $m_{z} = (2m \pm n')/2$. We see that the equations are not symmetric under $n \rightarrow -n$. Note that the shift in $n'$ relative to winding $n$ by one comes from the $\tau$-Berry phase of the electron states on the Fermi surface.
The latter phase shifts the relative angular momentum of pairs by one. Therefore the bound states of cores with opposite phase winding around the vortex would have different energy spectra.

A set of equations similar to those quoted in Eqs. (23-24) is presented for a p-wave superconductor, where the kinetic terms are usually in the long wavelength limit and it’s assumed that the chemical potential is negative in the vortex core (the strong coupling phase) and positive outside (the weak coupling phase) with \( \Delta_0(r) \) as a constant.

Following Ref. [9] and Ref. [31], we assume that the wave function takes the following form:

\[
\begin{align*}
\left( \begin{array}{c} u \\ v \\ \end{array} \right) &= 
\left( \begin{array}{c} f_+ \\ g_+ \\ \end{array} \right) H_q^1(x) + 
\left( \begin{array}{c} f_- \\ g_- \\ \end{array} \right) H_q^2(x),
\end{align*}
\]

where \( H_q^1 \) and \( H_q^2 \) are the Hankel functions of first and second kinds, respectively, and \( f \) and \( g \) are slowly varying functions. We insert the above ansatz in Eqs. (25-26) and neglect the second derivatives of \( f_\pm \) and \( g_\pm \). Using the asymptotic behavior of Hankel functions, we obtain the following differential equations governing \( f_\pm \) and \( g_\pm \):

\[
\begin{align*}
\frac{df_\pm}{dx} - i \frac{\Delta_0}{2} g_\pm &= \pm i \left( \frac{\tilde{E}}{2} - \frac{n'm}{2x^2} \right) f_\pm + \frac{\Delta m}{2x} g_\pm, \\
\frac{dg_\pm}{dx} + i \frac{\Delta_0}{2} f_\pm &= \mp i \left( \frac{\tilde{E}}{2} - \frac{n'm}{2x^2} \right) g_\pm + \frac{\Delta m}{2x} f_\pm.
\end{align*}
\]

The low-energy spectrum of bound states in the vortex lies within the superconducting bulk gap. Therefore a natural assumption is to assume \( \tilde{E} < \Delta_0 \) in the equations above, otherwise there would be no bound states at the vortex core. Physically the bound states result from the Andreev reflections of quasiparticles in the vortex core. We also assume that \( x \gg 1 \) which means that we are considering the long distance behavior of the system. Then by treating the expressions on the right-hand side as perturbations, we obtain the following expressions for \( f \) and \( g \) up to first order:

\[
\begin{align*}
\left( \begin{array}{c} f_1 \\ g_1 \\ \end{array} \right) &= A_1 \left\{ \left( \begin{array}{c} 1 \\ i \\ \end{array} \right) e^{-\chi(x)} - \left( \begin{array}{c} i \\ 1 \\ \end{array} \right) e^{\chi(x)} \right\} \int_x^\infty \left( \frac{\Delta m}{2x'} + \frac{n'm}{2x'^2} - \frac{\tilde{E}}{2} \right) e^{-2\chi(x')} dx', \\
\left( \begin{array}{c} f_2 \\ g_2 \\ \end{array} \right) &= A_2 \left\{ \left( \begin{array}{c} 1 \\ i \\ \end{array} \right) e^{-\chi(x)} + \left( \begin{array}{c} i \\ 1 \\ \end{array} \right) e^{\chi(x)} \right\} \int_x^\infty \left( \frac{\Delta m}{2x'} + \frac{n'm}{2x'^2} + \frac{\tilde{E}}{2} \right) e^{-2\chi(x')} dx',
\end{align*}
\]

where \( \chi(x) = \int_0^x \frac{\Delta_0(x')}{2} dx' \). In order to avoid the singularity of Hankel functions at the origin we should take \( A_1 = A_2 \) and the second terms should vanish as \( x \to 0 \). These boundary conditions eventually lead to the following expression for the energy spectrum of vortex bound states:

\[
\tilde{E} = m \int_0^\infty \frac{\Delta_0(x') + \frac{n'm}{2x'}}{\int_0^\infty e^{-2\chi(x')} dx'}
\]

Within the approximations used it turns out that the vortices with \( n' \neq 0 \) would have very large energy if \( m \neq 0 \) simultaneously. Let us consider a vortex with the lowest value of vorticity \( n' = 0 \) corresponding to \( n = -1 \) as shown schematically in Fig. 1. The condition \( U(2\pi) = 1 \) implies that the \( m \) has to be an integer number with \( m = 0 \) corresponding to a zero-energy state. Therefore the vortex core hosts a Majorana fermion. In Appendix A we use an alternative approach similar to Ref. [19] and study (25) and (26) for the explicit case of \( \tilde{E} = 0 \) confirming the existence of zero-energy mode.
As we mentioned at the beginning of this subsection the cases with \( l = 2 \) and \( l = 0 \) can be treated on equal footing, since the corresponding equations are the same. The latter case, \( l = 0 \), yields \( n' = n - 1 \) and a vortex with lowest winding number will have \( n = 1 \). Again the vortex can host a zero-energy state.

### B. The Case with \( l = -2 \)

In this case the BdG equations become third order and an analytical solution for them is not tractable. To circumvent this problem, we use the semi-classical approximation used in the analysis of the Andreev bond states in superconductors\(^{33}\) and closely follow Refs.\(^{32}\) and \(^{34}\). The BdG equation in this case is:

\[
H_{BdG} = (h_0 - \mu) \tau_3 + i \frac{\Delta(r)}{2k_F} \left\{ e^{i\theta}, (\partial_x + i\partial_y)^3 \right\} \tau_+ \\
+ i \frac{\Delta(r)}{2k_F} \left\{ e^{-i\theta}, (\partial_x - i\partial_y)^3 \right\} \tau_-, \tag{33}
\]

where \( h_0 = -\frac{\hbar v_F}{k_F} \nabla^2 \) is the kinetic energy. For solving BdG equations we use an ansatz for the wave function as \( \Psi = \varphi(r)e^{i\mathbf{q} \cdot \mathbf{r}} \) with an approximation that the momentum \( \mathbf{q} \) is restricted to the Fermi surface, i.e. \( \mathbf{q} = k_F (\cos \phi, \sin \phi) \) known as the momentum of a quasi-particle in the Andreev approximation. Using \( \Psi \) in Eq. (33), we obtain

\[
H = -iv \cdot \nabla \tau_3 + \Delta(r) \cos(\theta') \tau_1 + \Delta(r) \sin(\theta') \tau_2, \tag{34}
\]

which acts only on \( \varphi(r) \). Here we defined \( v = (2v_F/k_F)\mathbf{q} \) and \( \theta' = n\theta + 3\phi \). We rotate the coordinates such that the new \( x \)-axis becomes parallel to \( \mathbf{q} \):

\[
H = -iv \partial_x \tau_3 + \Delta(r) \cos(n\theta + (3 - n)\phi) \tau_1 \\
+ \Delta(r) \sin(n\theta + (3 - n)\phi) \tau_2 \tag{35}
\]

Then the \( \phi \) dependence in the Hamiltonian can be removed using the transformation \( \varphi \rightarrow e^{i(n-3)\phi/2} \varphi' \):

\[
H = -i v \partial_x \tau_3 + \Delta(r) \cos(n\theta) \tau_1 + \Delta(r) \sin(n\theta) \tau_2. \tag{36}
\]

This is a quasi one dimensional problem derived in Ref.\(^{32}\) (see Eq. (3.10) in the latter reference with the replacement \( \theta \rightarrow -n\theta \)). To proceed we define an impact parameter as \( b = r \sin \theta \) which measures the minimum distance of the quasi particle trajectory from the origin of the vortex core. For the pairing gap we use a profile as \( \Delta(r) = \Delta \Theta(r - R) \), where \( \Theta(x) \) is the usual step function and \( R \) is the radius of the vortex. The latter is of order of \( R \approx v_F/\Delta \). With these assumptions the quasi one-dimensional model Eq.(36) can be solved to obtain the energy spectrum of the bound states. For small values of \( bk_F \ll 1 \), corresponding to trajectories passing near the origin, the spectrum reads as

\[
E_m = \omega_0 \left(-n\pi + 2\pi \left(m + \frac{1}{2}\right)\right), \tag{37}
\]

where \( \omega_0 = v_F/2R \) is the angular velocity of the superfluid\(^{32}\). Now it is clearly seen that for \( n = 1 \) the spectrum becomes \( E_m = 2\pi \omega_0 m \) and a zero mode corresponds to \( m = 0 \). Therefore the vortex bound states for the \( l = -2 \) case also contain a zero-energy mode.

### V. CONCLUSIONS

This work is mainly motivated by the efforts conducted in recent years to find Majorana bound states in the vortex core of the superconducting states. We proposed the superconducting epitaxial Bi/Ni bilayer as a platform to create an manipulate the Majorana states. The system has a superiority over the heterostructures proposed in the literatures in that the chiral superconducting states are created intrinsically due to the magnetic fluctuations of the ferromagnetic layer circumventing the need for a proximized superconducting layer. The heterostructure here can be replaced by other materials combinations, e. g., a thin film of topological insulator Bi\(_2\)Te\(_3\) deposited on the magnetic insulator layer FeTe\(^{35,36}\), or superconducting states in oxide interfaces\(^{37}\), making our proposal for creating and manipulating of zero modes experimentally feasible.

The chiral superconducting states in Bi/Ni bilayer are characterized by total angular momentum \( l = \pm 2 \) corresponding to \( d_{x^2-y^2} \pm d_{z^2-x^2} \), which break the time-reversal symmetry. We showed that the underlying strong spin-orbit coupling alter the bound state spectrum in the vortex core. In particular we demonstrate that a zero-energy state corresponding to Majorana bound state appears at the vortex core for both cases of the pairing wave function. We also showed that the case with total angular momentum \( l = 0 \), which intrinsically doesn’t break the time-reversal symmetry and may be induced by charge fluctuations, can also support a zero-energy state. The set-up studied in our work, as shown in Fig. 1, should be contrasted with proposals in the literatures where the superconductivity is induced by proximity. Our findings may motivate the search for Majorana zero modes in vortex in heterostructures with no need for proximity to an extra superconducting layer.

### VI. ACKNOWLEDGMENTS

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Appendix A: The Explicit Study of Zero Energy Mode

Here we consider (25) and (26) and follow Ref.[19] for $E = 0$. We assume that the vortex boundary is at $x = x_0$ and take $\Delta(x)$ to be zero for $x < x_0$ and a constant value for $x > x_0$. For the region inside the vortex the equations become a set of decoupled Bessel equations with the general analytical solution:

$$
\begin{align*}
\begin{bmatrix}
u(x) \\
\phi(x)
\end{bmatrix} &= \begin{bmatrix} A_1 J_{m_+}(x) \\
A_2 J_{m_-}(x)
\end{bmatrix}
\end{align*}
$$

(A1)

Where $J_m(x)$ is the Bessel function of the first kind. A closed solution of equations for $x > x_0$ is not tractable. Instead we try to find the asymptotic solution of equations in the limit $x \gg 1$. In this limit (25) and (26) become:

$$
\begin{align*}
\partial_x^2 u + u - i\Delta \partial_x v &= 0 \\
\partial_x^2 v + v + i\Delta \partial_x u &= 0
\end{align*}
$$

(A2)

(A3)

Looking for a decaying solution of the form $\begin{bmatrix} u \\
v
\end{bmatrix} = \begin{bmatrix} u_0 \\
v_0
\end{bmatrix} e^{\pm \kappa x}$ one gets:

$$
\begin{align*}
\begin{bmatrix} u(x) \\
v(x)
\end{bmatrix} &= A_3 \begin{bmatrix} 1 \\
-i
\end{bmatrix} f_+(x) + A_4 \begin{bmatrix} 1 \\
-i
\end{bmatrix} f_-(x)
\end{align*}
$$

(A4)

Where $f_\pm(x)$ are decaying functions with the asymptotic form $f_\pm(x) \rightarrow e^{-\kappa_{\pm} x}$ and $\kappa_{\pm} = \frac{|\Delta|}{2} \pm \sqrt{(\frac{|\Delta|}{2})^2 - 1}$. We have to match the solutions for inside and outside of the vortex at the vortex boundary. The condition $\varphi(x_0) = \varphi(x_0^-)$ gives:

$$
\begin{align*}
A_1 J_{m_+}(x_0) &= A_3 f_+(x_0) + A_4 f_-(x_0) \\
A_2 J_{m_-}(x_0) &= -i(A_3 f_+(x_0) + A_4 f_-(x_0))
\end{align*}
$$

(A5)

(A6)
\[ \phi'(x_0^-) = \phi'(x_0^+) \] yields:

\[ A_1 J'_{m_+}(x_0) = A_3 f'_+(x_0) + A_4 f'_-(x_0) \quad (A7) \]

\[ A_2 J'_{m_-}(x_0) = -i(A_3 f'_+(x_0) + A_4 f'_-(x_0)) \quad (A8) \]

These equations alongside with the normalization condition

\[ \int (u^2(r) + v^2(r))dr = 1 \]

should be satisfied in order to have a solution. In general, it is not possible to satisfy all of these conditions by adjusting only 4 unknowns \( A_1-A_4 \) and the problem is over-constrained and a zero mode solution does not exist. Nevertheless, in the special case of \( n' = 0 \) where \( m_+ = m_- \) from (A5) and (A6) we have \( A_2 = -iA_1 \) which makes (A7) and (A8) identical and therefore there are only 3 independent boundary conditions which alongside with the normalization condition assign a unique value to \( A_1-A_4 \). This is compatible with (32) in which a zero energy state exists only if \( n' = 0 \).