ABSTRACT

We investigate how the Hausdorff dimension and measure of a self-similar set $K \subseteq \mathbb{R}^d$ behave under linear images. This depends on the nature of the group $\mathcal{T}$ generated by the orthogonal parts of the defining maps of $K$. We show that if $\mathcal{T}$ is finite, then every linear image of $K$ is a graph directed attractor and there exists at least one projection of $K$ such that the dimension drops under the image of the projection. In general, with no restrictions on $\mathcal{T}$ we establish that $\mathcal{H}^t(L \circ O(K)) = \mathcal{H}^t(L(K))$ for every element $O$ of the closure of $\mathcal{T}$, where $L$ is a linear map and $t = \dim_H K$. We also prove that for disjoint subsets $A$ and $B$ of $K$ we have that $\mathcal{H}^t(L(A) \cap L(B)) = 0$. Hochman and Shmerkin showed that if $\mathcal{T}$ is dense in $SO(d, \mathbb{R})$ and the strong separation condition is satisfied, then $\dim_H (g(K)) = \min \{\dim_H K, l\}$ where $g$ is a continuously differentiable map of rank $l$. We deduce the same result without any separation condition and we generalize a result of Eroğlu by obtaining that $\mathcal{H}^t(g(K)) = 0$.

1. Introduction

1.1. Overview. Studying the Hausdorff dimension and measure of orthogonal projections and linear images of sets has a long history. The most fundamental
result is that for an analytic subset $K$ of $\mathbb{R}^d$,
\[ \dim_H \Pi_M(K) = \min \{ l, \dim_H(K) \} \]
for almost all $l$-dimensional subspaces $M$, where $\dim_H$ denotes the Hausdorff dimension and $\Pi_M : \mathbb{R}^d \to M$ denotes the orthogonal projection onto $M$. If $\dim_H(K) > l$, then
\[ \mathcal{H}^l(\Pi_M(K)) > 0 \]
for almost all $l$-dimensional subspaces $M$, where $\mathcal{H}^s$ denotes the $s$-dimensional Hausdorff measure. These were proved in the case $d = 2, l = 1$ by Marstrand [21], and generalized to higher dimensions by Mattila [23]. We call a set $K$ an $s$-set if $0 < \mathcal{H}^s(K) < \infty$. If $l$ is an integer, then we call an $l$-set $K$ irregular if $\mathcal{H}^l(K \cap M) = 0$ for every differentiable $l$-manifold $M$. It was shown by Besicovitch [2] in the planar case and by Federer [12] in the higher dimensional case that for an $l$-set $K$ where $l$ is an integer,
\[ \mathcal{H}^l(\Pi_M(K)) = 0 \]
for almost all $l$-dimensional subspaces $M$ if and only if $K$ is irregular. If $K$ is not irregular, then $\mathcal{H}^l(\Pi_M(K)) > 0$ for almost all $l$-dimensional subspaces $M$.

While the results above provide information about generic projections they do not give any information about an individual projection or linear image of the set. There are examples that show that the ‘exceptional set’ for which the conclusions do not hold can be ‘big’ [21]. Analyzing the image of a set under a particular linear map is more difficult even in simple cases; see, for example, Kenyon [19] and Hochman [16, Theorem 1.6] who consider the 1-dimensional Sierpinski gasket. Hence we restrict the attention to a certain family of sets, namely we assume $K$ to be a self-similar set.

While studying self-similar sets the ‘open set condition’ is a convenient assumption that makes the proofs significantly simpler. That is why the case when the open set condition is satisfied is quite well-understood but we know much less in the general situation when no separation condition is assumed. The results in this paper include this general situation. Recent results of Hochman were a major breakthrough in studying overlapping self-similar sets. A folklore conjecture is that for a self-similar set $K$ on the line $\dim_H K < \min \{ 1, s \}$ if and only if exact overlapping occurs among the cylinder sets where $s$ denotes the similarity dimension of $K$. Hochman [16, Theorem 1.5] proves this conjecture when only algebraic parameters occur in the defining maps of $K$. In Example