ON THE $\Sigma$-INVARIENTS OF WREATH PRODUCTS

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Abstract. We present a full description of the Bieri-Neumann-Strebel invariant of restricted permutational wreath products of groups. We also give partial results about the 2-dimensional homotopical invariant of such groups. These results may be turned into a full picture of these invariants when the abelianization of the basis group is infinite. We apply these descriptions to the study of the Reidemeister number of automorphisms of wreath products in some specific cases.

1. Introduction

In this paper we study the so called $\Sigma$-invariants of restricted permutational wreath products of groups. The $\Sigma$-invariants of a group are some subsets of its character sphere and contain a lot of information on finiteness properties of its subgroups. Their definitions and most general results appeared in a series of papers by Bieri, Neumann, Strebel, Renz ([6],[7],[8]) and others.

Let $\Gamma$ be a finitely generated group. The character sphere $S(\Gamma)$ is the set of non-zero homomorphisms $\chi : \Gamma \to \mathbb{R}$ (these homomorphisms are called characters) modulo the equivalence relation given by $\chi_1 \sim \chi_2$ if there is some $r \in \mathbb{R}_{>0}$ such that $\chi_2 = r\chi_1$. The class of $\chi$ will be denoted by $[\chi]$. The character sphere may be seen as the $(n-1)$-sphere in the vector space $Hom(\Gamma, \mathbb{R}) \cong \mathbb{R}^n$, where $n$ is the torsion-free rank of the abelianization of $\Gamma$.

In this paper we deal with the homotopical invariants in low dimension, that is, those denoted by $\Sigma^1(\Gamma)$ and $\Sigma^2(\Gamma)$, the second defined when $\Gamma$ is finitely presented. Their most important feature is that they classify the properties of being finitely generated and being finitely presented for subgroups of $\Gamma$ containing the derived subgroup $[\Gamma, \Gamma]$ (see Theorem 2.1).

Recall that a group $\Gamma$ is of type $F_n$ if there is a $K(\Gamma, 1)$-complex with compact $n$-skeleton. A group is of type $F_1$ (resp. $F_2$) if and only if it is finitely generated (resp. finitely presented). The homological version of the property $F_n$ is the property $FP_n$: a group $\Gamma$ is of type $FP_n$ if the trivial $\mathbb{Z}\Gamma$-module $\mathbb{Z}$ admits a projective resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

with $P_j$ finitely generated for all $j \leq n$. Again a group is of type $FP_1$ if and only if it is finitely generated, but the properties $F_n$ are in general stronger then $FP_n$. In particular, $FP_2$ is strictly weaker then finite presentability ([2],[3]).

There are some higher homotopical invariants, denoted by $\Sigma^n(\Gamma)$, which are defined for groups of type $F_n$ and fit in a decreasing sequence

$$S(\Gamma) \supseteq \Sigma^1(\Gamma) \supseteq \Sigma^2(\Gamma) \supseteq \cdots \supseteq \Sigma^n(\Gamma) \supseteq \cdots$$

whenever defined. They classify the property $F_n$ for subgroups above the derived subgroup. Similarly, the homological invariants $\Sigma^n(\Gamma; \mathbb{Z})$ are defined for groups of type $FP_n$ and classify this same property for subgroups containing the derived
In general $\Sigma^1(\Gamma) = \Sigma^1(\Gamma; \mathbb{Z})$ if $\Gamma$ is finitely generated and $\Sigma^n(\Gamma) \subseteq \Sigma^n(\Gamma; \mathbb{Z})$ if $\Gamma$ has type $F_n$ \cite{17}.

All these invariants are in general hard to describe for specific groups, and has been done only for a few classes of groups. For right-angled Artin groups, for example, the invariant $\Sigma^1$ was computed first by Meier and VanWyk \cite{23} and then generalized for higher dimensions (for both homotopical and homological versions) by the same authors and Meinert \cite{21}. This is connected with the existence of subgroups of these groups having a wide variety of finiteness properties, as shown by Bestvina and Brady \cite{2}. Another line of generalization was followed by Meinert, who computed the invariants in dimension 1 for graph products \cite{24}.

Another interesting group for which the invariants are known is Thompson’s group $F$. Both homological and homotopical invariants have been computed in all dimensions by Bieri, Geoghegan and Kochloukova \cite{5}. The $\Sigma^2$-invariants of the generalized Thompson groups $F_{n,\infty}$ were then computed by Kochloukova \cite{19} and recently Zaremsky extended it to higher dimensions \cite{31}.

We considered the homotopical invariants $\Sigma^1$ and $\Sigma^2$ of wreath products. Recall that given $H$ and $G$ groups and a $G$-set $X$, the wreath product $H \wr_X G$ is defined as the semi-direct product $M \times G$, where $M = \bigoplus_{x \in X} H_x$ is the direct sum (that is, the restricted direct product) of copies of $H$ indexed by $X$ and $G$ acts by permuting this copies according to the action on $X$. We shall always assume that $X \neq \emptyset$ and $H \neq 1$, to avoid trivial cases. The finiteness properties for these groups were studied by Cornulier in \cite{10} (finite generation and finite presentability) and more recently by Bartholdi, Cornulier and Kochloukova \cite{1} (properties $F_P$ studied by Cornulier in \cite{10} (finite generation and finite presentability) and more recently by Bartholdi, Cornulier and Kochloukova \cite{1} (properties $F_P$).

**Remark 1.1.** By $H_x$ we always mean the copy of $H$ associated to the element $x \in X$. On the other hand, $G_x$ denotes the stabilizer of $x \in X$ in the action of $G$. To avoid confusion, we will always denote by $G$ the group that acts.

Our first result is the full description of $\Sigma^1$.

**Theorem A.** Let $\Gamma = H \wr_X G$ be a finitely generated wreath product and let $\chi : \Gamma \to \mathbb{R}$ be a non-trivial character. We set $M = \bigoplus_{x \in X} H_x \leq \Gamma$.

1. If $\chi|M = 0$, then $[x] \in \Sigma^1(\Gamma)$ if and only if $[\chi|G] \in \Sigma^1(G)$ and $\chi|G_x \neq 0$ for all $x \in X$.
2. If $\chi|M \neq 0$, then $[x] \in \Sigma^1(\Gamma)$ if and only if at least one of the following conditions holds:
   (a) There exist $x, y \in X$ with $x \neq y$, $\chi|H_x \neq 0$ and $\chi|H_y \neq 0$;
   (b) There exists $x \in X$ with $\chi|H_x \neq 0$ and $[\chi|H_x] \in \Sigma^1(H)$ or
   (c) $\chi|G \neq 0$.

Part (1) of the above theorem generalizes Theorem 8.1 in \cite{11} in dimension 1, where $H$ has infinite abelianization by hypothesis. For regular wreath products, that is, $H = H \wr_G G$, the action being by multiplication on the left, the $\Sigma^1$-invariant was already computed by Strebel in Proposition C1.18 in \cite{29}.

For the invariant $\Sigma^2$ we consider two cases, the same as in the theorem above. For characters $\chi : H \wr_X G \to \mathbb{R}$ such that $\chi|M \neq 0$ the criteria developed by Renz \cite{27} are especially powerful, and have allowed us to prove part (2) of Theorem A.

**Theorem B.** Let $\Gamma = H \wr_X G$ be a finitely presented wreath product and let $\chi : \Gamma \to \mathbb{R}$ be a non-trivial character. If the set

$$T = \{x \in X \mid \chi|H_x \neq 0\}$$

has at least 3 elements, then $[x] \in \Sigma^2(\Gamma)$. 

The cases where $T$ is non-empty but has less than 3 elements can be dealt with using the direct product formula (see Theorem 2.2) and the results on the $\Sigma^1$-invariant (see Theorem 6.5 and the comment right before it).

For the characters $\chi : \Gamma \to \mathbb{R}$ with $\chi|_M = 0$ we were not able to obtain a complete result, by lack of a general method to study necessary conditions for $[\chi] \in \Sigma^2(\Gamma)$. By the results of Bartholdi, Cornulier and Kochloukova on homological invariants, the most general theorem we can enunciate is the following, where $G_{(x,y)}$ denotes the stabilizer subgroup associated to an element $(x, y)$ of $X^2$, which is equipped with the diagonal $G$-action.

**Theorem C.** Let $\Gamma = H \wr X G$ be a finitely presented wreath product and let $\chi : \Gamma \to \mathbb{R}$ be a non-zero character such that $\chi|_M = 0$. Then $[\chi] \in \Sigma^2(\Gamma)$ if all three conditions below hold

1. $[\chi|_G] \in \Sigma^2(G)$;
2. $[\chi|_{G_x}] \in \Sigma^1(G_x)$ for all $x \in X$ and
3. $\chi|_{G_{(x,y)}} \neq 0$ for all $(x, y) \in X^2$.

In general, conditions (1) and (3) are necessary for $[\chi] \in \Sigma^2(\Gamma)$. If we assume further that the abelianization of $H$ is infinite, then condition (2) is necessary as well.

Restrictions on the abelianization of the basis group $H$ have been recurrent in the study of finiteness properties of wreath products and related constructions. Besides appearing in the work of Bartholdi, Cornulier and Kochloukova [1], they also pop up in the paper by Kropholler and Martino [20], which deals with the wider class of graph-wreath products (see Section 5) from a more homotopical point of view.

Finally, we consider some applications to twisted conjugacy. Recall that given an automorphism $\varphi$ of a group $G$, the Reidemeister number $R(\varphi)$ is defined as the number of orbits of the twisted conjugacy action, which is given by $g \cdot h := gh\varphi(g^{-1})$, for $g, h \in G$.

Exploring the connections between $\Sigma$-theory and Reidemeister numbers, as found out by Koban and Wong [17] and Gonçalves and Kochloukova [14], we obtain some results about the Reidemeister numbers of automorphisms contained in some subgroups of finite index of $Aut(H \wr X G)$, under some relatively strong restrictions. For precise statements, see Corollaries 9.3 and 9.5.

2. Background on the $\Sigma$-invariants

Let us start by recalling the definition of the invariant $\Sigma^1$. For a finitely generated group $\Gamma$ and a finite generating set $\mathcal{X} \subseteq \Gamma$, we consider the Cayley graph $Cay(\Gamma; \mathcal{X})$. Its vertex set is $\Gamma$ and two vertices $\gamma_1$ and $\gamma_2$ are connected by an edge if and only if there is some $x \in \mathcal{X}^{\pm 1}$ such that $\gamma_2 = \gamma_1 x$ (therefore $\Gamma$ acts on the left). This graph is always connected. Given a non-zero character $\chi : \Gamma \to \mathbb{R}$ we can define the submonoid $\Gamma_{\chi} = \{ \gamma \in \Gamma \mid \chi(\gamma) \geq 0 \}$.

Notice that $\Gamma_{\chi_1} = \Gamma_{\chi_2}$ if and only if $\chi_1$ and $\chi_2$ represent the same class in the character sphere $S(\Gamma)$. The full subgraph spanned by $\Gamma_{\chi}$, which we denote by $Cay(\Gamma; \mathcal{X})_{\chi}$, may not be connected. We put:

$$\Sigma^1(\Gamma) = \{ [\chi] \in S(\Gamma) \mid Cay(\Gamma; \mathcal{X})_{\chi} \text{ is connected} \}.$$  

It can be shown that this definition does not depend on the (finite) generating set $\mathcal{X}$. This invariant is known as the Bieri-Neumann-Strebel invariant (or simply BNS-invariant), in reference to the authors who studied it first [6].
The invariant $\Sigma^2$ is defined similarly. If $\Gamma$ is finitely presented and $(\mathcal{F}; \mathcal{P})$ is a finite presentation, we consider the Cayley complex $\text{Cay}(\Gamma; (\mathcal{F}; \mathcal{P}))$. This complex is obtained from the Cayley graph by gluing 2-dimension cells with boundary determined by the loops defined by the relations $r \in R$, for each base point in $\Gamma$. The resulting complex is always 1-connected. Again we define $\text{Cay}(\Gamma; (\mathcal{F}; \mathcal{P}))_\chi$ to be the full subcomplex spanned by $\Gamma_\chi$. The 1-connectedness of this complex depends on the choice of the presentation. We define $\Sigma^2(\Gamma)$ as the subset of $S(\Gamma)$ containing exactly all the classes $[\chi]$ of characters such that $\text{Cay}(\Gamma; (\mathcal{F}; \mathcal{P}))_\chi$ is 1-connected for some finite presentation $(\mathcal{F}; \mathcal{P})$ of $\Gamma$. More details on these definitions may be found in [25].

The main feature of these invariants is that they classify the related finiteness properties for subgroups containing the derived subgroup. For the invariants $\Sigma^1$ and $\Sigma^2$, this can be stated as follows.

**Theorem 2.1** ([6], [28]). Suppose that $\Gamma$ is finitely generated and let $N \subseteq \Gamma$ be a subgroup such that $[\Gamma, \Gamma] \subseteq N$. Then $N$ is finitely generated if and only if

$$\Sigma^1(\Gamma) \supseteq \{[\chi] \in S(\Gamma) \mid \chi|_N = 0\}.$$

Similarly, if $\Gamma$ is further finitely presented then $N$ is finitely presented if and only if

$$\Sigma^2(\Gamma) \supseteq \{[\chi] \in S(\Gamma) \mid \chi|_N = 0\}.$$

The homological invariants can be defined by means of the monoid ring $\mathbb{Z} \Gamma_\chi$. This is of course the subring of $\mathbb{Z} \Gamma$ containing exactly all elements $\sum a_\gamma \gamma \in \mathbb{Z} \Gamma$ such that $a_\gamma \neq 0$ only if $\gamma \in \Gamma_\chi$. We put

$$\Sigma^m(\Gamma; \mathbb{Z}) = \{[\chi] \in S(\Gamma) \mid \mathbb{Z} \text{ is of type } FP_m \text{ over } \mathbb{Z} \Gamma_\chi\}.$$

As observed by Bieri and Renz [7] if $\Sigma^m(\Gamma; \mathbb{Z}) \neq 0$ then $\Gamma$ is of type $FP_m$. All we need about these homological invariants is that $\Sigma^2(\Gamma) \subseteq \Sigma^2(\Gamma; \mathbb{Z})$ whenever $\Gamma$ is finitely presented. Details may be found in [7] and [28].

Some of the general results we will need about these invariants concern direct products of groups, subgroups of finite index and retracts.

**Theorem 2.2** (Direct product formulas, [13]). Let $G_1$ and $G_2$ be finitely generated groups and let $\chi = (\chi_1, \chi_2) : G_1 \times G_2 \to \mathbb{R}$ be a non-zero character. Then $[\chi] \in \Sigma^1(G_1 \times G_2)$ if and only if at least one of the following conditions holds:

1. $\chi_i \neq 0$ for $i = 1, 2$ or
2. $[\chi_i] \in \Sigma^1(G_i)$ for some $i \in \{1, 2\}$.

Similarly, if $G_1$ and $G_2$ are finitely presented, then $[\chi] \in \Sigma^2(G_1 \times G_2)$ if and only if at least one of the following conditions holds:

1. $[\chi_1] \in \Sigma^1(G_1)$ and $\chi_2 \neq 0$;
2. $[\chi_2] \in \Sigma^1(G_2)$ and $\chi_1 \neq 0$ or
3. $[\chi_i] \in \Sigma^2(G_i)$ for some $i \in \{1, 2\}$.

There was a conjecture suggesting how to compute the $\Sigma$-invariants of direct products in higher dimensions, but it turned out to be false. Counterexamples were found by Meier, Meinert and VanWyk [24] for the homotopical invariants and by Schütz [28] in the homological case. For precise statements see [4], which also brings a proof of the homological conjecture if coefficients are taken in a field (rather than $\mathbb{Z}$).

**Theorem 2.3** (Finite index subgroups, [25]). Let $G$ be a finitely presented group and let $H \leq G$ be a subgroup of finite index. Let $\chi : G \to \mathbb{R}$ be a non-zero character and denote by $\chi_0$ its restriction to $H$. Then $[\chi] \in \Sigma^2(G)$ if and only if $[\chi_0] \in \Sigma^2(H)$.
Theorem 2.4 (Retracts, [25]). Let $G$ be a finitely presented group and suppose that $H$ is a retract, that is, there are homomorphisms $p : G \to H$ and $j : H \to G$ such that $p \circ j = id_H$. Suppose that $\chi : H \to \mathbb{R}$ is a non-zero character. Then 

$[\chi \circ p] \in \Sigma^3(H) \Rightarrow [\chi] \in \Sigma^2(G).$

Theorem 2.5 (Theorem C, [18]). Suppose that $G$ is a group of type $FP_n$ (resp. $F_n$) and $N$ is a normal subgroup of $G$ that is locally nilpotent-by-finite. Then 

$\{[\chi] \in S(G) \mid \chi(N) \neq 0\} \subseteq \Sigma^m(G; \mathbb{Z})$ (resp. $\Sigma^m(G)$).

As pointed out to me by D. Kochloukova, in [18] the result is stated for $N$ locally polycyclic-by-finite, but actually the proof works for nilpotent-by-finite. We will use it with $N$ being abelian. The case $m = 1$, with $N$ abelian, can also be found as Lemma C1.20 in Strebel’s notes [29].

3. The $\Sigma^1$-invariant of wreath products

Let $\Gamma = H \wr_X G$ be a finitely generated wreath product. As shown by Cornulier [10], both $G$ and $H$ are finitely generated and $G$ acts on $X$ with finitely many orbits. Denote $M = \oplus_{x \in X} H_x \leq \Gamma$. We start working with the characters $\chi : \Gamma \to \mathbb{R}$ such that $\chi|_{M} = 0$, for which there are some partial results by Bartholdi, Cornulier and Kochloukova. We quote their result in its most general form, which deals with the higher homological invariants.

Theorem 3.1 ([1], Theorem 8.1). Let $\Gamma = H \wr_X G$ be a wreath product of type $FP_m$ and let $M = \oplus_{x \in X} H_x \leq \Gamma$. Let $\chi : \Gamma \to \mathbb{R}$ be a non-zero character such that $\chi|_{M} = 0$. The following conditions are sufficient for $[\chi] \in \Sigma^m(\Gamma; \mathbb{Z})$:

1. $[\chi|_G] \in \Sigma^m(G; \mathbb{Z})$;
2. $[\chi|_{G_x}] \in \Sigma^{m-1}(G_x; \mathbb{Z})$ for all stabilizers $G_x$ of the diagonal action of $G$ on $X$ and for all $1 \leq i \leq m$.

Moreover, if the abelianization of $H$ is infinite, then such conditions are also necessary.

Notice that item 2 contains a statement about invariants in dimension 0. For any finitely generated group $V$ and $\chi : V \to \mathbb{R}$, the condition $[\chi] \in \Sigma^0(V; \mathbb{Z})$ amounts to saying that $\chi$ is a non-zero homomorphism.

Recall that the homological and homotopical invariants coincide in dimension 1, that is, $\Sigma^1(V; \mathbb{Z}) = \Sigma^1(V)$ whenever $V$ is a finitely generated group (see Corollary C1.5, [29], for instance). It is worth mentioning that if we consider the original definitions of the invariants in [9] and [17], we get that actually the sets $\Sigma^1(V)$ and $\Sigma^1(V; \mathbb{Z})$ are antipodal in $S(V)$, that is, $\Sigma^1(V; \mathbb{Z}) = -\Sigma^1(V)$. This happens because in [17] the authors chose to work with left group actions, while in [9] the actions are on the right. The sign disappears if the choice is consistent.

We can now extract from Theorem 3.1 a set of sufficient conditions for $[\chi] \in \Sigma^1(\Gamma)$. Namely:

Proposition 3.2. Let $\Gamma = H \wr_X G$ be a finitely generated wreath product and let $\chi : \Gamma \to \mathbb{R}$ be a non-zero character such that $\chi|_{M} = 0$. If $[\chi|_G] \in \Sigma^1(G)$ and if $\chi|_{G_x} \neq 0$ for all stabilizers $G_x$ of the action of $G$ on $X$, then $[\chi] \in \Sigma^1(\Gamma)$.

Remark 3.3. This conditions could also be obtained by considering an action of $\Gamma$ on a sufficiently nice complex. We shall apply this reasoning in the study of the invariant $\Sigma^2(H \wr_X G)$.

This set of conditions is in fact necessary. First, if $\chi : \Gamma \to \mathbb{R}$ and $M \subseteq ker(\chi)$, then 

$[\chi] \in \Sigma^1(\Gamma) \Rightarrow [\chi|_G] \in \Sigma^1(G), \quad [\chi]\}
since \( \chi|_G \) coincides with the character \( \tilde{\chi} \) induced on the quotient \( \Gamma/M \simeq G \) (see [29] Proposition A4.5).

It suffices then to analyze the restriction of \( \chi \) to the stabilizer subgroups under the hypothesis that \( \{x\} \in \Sigma^1(\Gamma) \).

Proposition 3.4. If \( \{x\} \in \Sigma^1(\Gamma) \) and \( \chi|_M = 0 \), then \( \chi|_{G_x} \neq 0 \) for all \( x \in X \).

Proof. Let \( X = G \cdot x_1 \cup \ldots \cup G \cdot x_n \). We only need to show that \( \chi|_{G_{x_i}} \neq 0 \) for all \( i \).

By taking the quotient by \( M' = \bigoplus_{x \in X \setminus G \cdot x_i} H_x \), we may assume that \( n = 1 \), that is, we consider wreath products of the form \( \Gamma = H \wr X G \) with \( X = G \cdot x_1 \).

Let \( Y \) and \( Z \) be finite generating sets for \( H \) and \( G \), respectively. Since \( X = G \cdot x_1 \) it is clear that \( Y \cup Z \) is a finite generating set for \( \Gamma \) (we see \( Y \) as a subset of the copy \( H_{x_1} \)). Then \( \text{Cay}(\Gamma; Y \cup Z)_x \) must be connected, since \( \{x\} \in \Sigma^1(\Gamma) \) by hypothesis.

First, we show that \( M \) can be generated by the left conjugates of elements of \( Y^{\pm 1} \) by elements of \( G_x \). Indeed if \( m \in M \) there is a path in \( \text{Cay}(\Gamma; Y \cup Z)_x \) connecting \( 1 \) to \( m \), since \( m \in M \subseteq \ker(\chi) \subseteq \Gamma_x \). Such a path has as label a word with letters in \( Y^{\pm 1} \cup Z^{\pm 1} \), so we can write:

\[
m = w_1 v_1 w_2 v_2 \cdots w_k v_k,
\]

where each \( w_j \) is a word in \( Y^{\pm 1} \) and each \( v_j \) is a word in \( Z^{\pm 1} \) (possibly trivial). We rewrite:

\[
m = w_1(v_1 w_2)(v_1 v_2 w_3) \cdots (v_1 \cdots v_{k-1} w_k)(v_1 \cdots v_k).
\]

Now, \( w_1(v_1 w_2)(v_1 v_2 w_3) \cdots (v_1 \cdots v_{k-1} w_k) \in M \) and \( v_1 \cdots v_k \in G \). But \( m \in M \) and \( \Gamma = M \rtimes G \), so \( v_1 \cdots v_k = 1_G \). Moreover, since \( \chi|M = 0 \), it is clear that \( \chi(v_1 \cdots v_j) \geq 0 \) for all \( 1 \leq j \leq k \), so:

\[
m = w_1(v_1 w_2)(v_1 v_2 w_3) \cdots (v_1 \cdots v_{k-1} w_k) \in \langle G_x (Y^{\pm 1}) \rangle,
\]

as we wanted.

But then

\[
M = \langle G_x (Y) \rangle \subseteq \langle G_x (H_{x_1}) \rangle = \bigoplus_{x \in G_x \cdot x_1} H_{x_1},
\]

that is, \( X = G_x \cdot x_1 \). Finally, as \( \chi|_G \neq 0 \) there is some \( g_1 \in G \) such that \( \chi(g_1) < 0 \). On the other hand, there must be some \( g_0 \in G_x \) such that \( g_0 : x_1 = g_1 \cdot x_1 \). It follows that \( g_1^{-1} g_0 \in G_{x_1} \), with \( \chi(g_1^{-1} g_0) = -\chi(g_1) + \chi(g_0) > 0 \), hence \( \chi|_{G_{x_1}} \neq 0 \).

We obtain part (1) of Theorem A by combining Propositions 3.2 and 3.3.

4. The \( \Sigma^1 \)-invariant and Renz’s criterion

We shall use the results of Renz [27] to consider the characters \( \chi : H \wr X G \to \mathbb{R} \) such that \( \chi|_M \neq 0 \). Let \( \Gamma \) be any finitely generated group and let \( \mathcal{X} \subseteq \Gamma \) be a finite generating set. For a non-zero character \( \chi : \Gamma \to \mathbb{R} \) and for any word \( w = x_1 \cdots x_n \), with \( x_i \in \mathcal{X}^{\pm 1} \), we denote:

\[
v_\chi(w) := \min\{\chi(x_1 \cdots x_j) \mid 1 \leq j \leq n\}.
\]

Theorem 4.1 ([27], Theorem 1). With the notations above, \( \{x\} \in \Sigma^1(\Gamma) \) if and only if there exists \( t \in \mathcal{X}^{\pm 1} \) with \( \chi(t) > 0 \) and such that for all \( x \in \mathcal{X}^{\pm 1} \setminus \{t, t^{-1}\} \) the conjugate \( t^{-1} x t \) can be represented by a word \( w_x \) in \( \mathcal{X}^{\pm 1} \) such that:

\[
v_\chi(t^{-1} x t) < v_\chi(w_x).
\]

Proposition 4.2. Let \( \Gamma = H \wr X G \) be a finitely generated wreath product and let \( \{x\} \in S(\Gamma) \). Suppose that there is some \( x_1 \in X \) such that \( G \cdot x_1 \neq \{x_1\} \) and \( \chi|_{H_{x_1}} \neq 0 \). Then \( \{x\} \in \Sigma^1(\Gamma) \).
Proof. Let $Y$ and $Z$ be finite generating sets for $H$ and $G$, respectively, and choose $x_1, \ldots, x_n \in X$ such that $X = \bigsqcup_{j=1}^n G \cdot x_j$ (the element $x_1$ is already chosen to satisfy the hypotheses). For each $1 \leq j \leq n$ let $Y_j$ be a copy of $Y$ inside $H_{x_j}$. It is clear that $\Gamma$ is generated by $Y_1 \cup \ldots \cup Y_n \cup Z$.

Now, since $G \cdot x_1 \neq \{x_1\}$ we can choose $g_1 \in G$ such that $g_1 \cdot x_1 \neq x_1$. Furthermore, since $\chi|_{H_{x_1}} \neq 0$, we can choose a generator $h \in Y_1$ such that $\chi(h) \neq 0$. We may assume without loss of generality that $\chi(h) > 0$. Define $t := g_1 h \in H_{g_1 \cdot x_1}$. We take $\mathcal{X} = Y_1 \cup \ldots \cup Y_n \cup Z \cup \{t\}$ as a generating set for $\Gamma$ and we show that the conditions of Theorem 4.4 are satisfied.

If $y \in (Y_1 \cup \ldots \cup Y_n)^{\pm 1}$ then $t$ and $y$ commute in $\Gamma$, hence $w_y := y$ is word that represents $t^{-1}yt$. Also, $v_\chi(w_y) = \chi(y)$ and

$$v_\chi(t^{-1}yt) \leq \chi(t^{-1}y) = \chi(y) - \chi(t) < \chi(y),$$

so $v_\chi(t^{-1}yt) < v_\chi(w_y)$.

If $z \in Z^{\pm 1}$, there are two cases: $z \in G_{g_1 \cdot x_1}$ or $z \notin G_{g_1 \cdot x_1}$. In the first case $z$ and $t$ commute in $\Gamma$, so we may proceed as in the previous paragraph: we take the word $w_z := z$, which represents $t^{-1}zt$ and satisfies $v_\chi(t^{-1}zt) < v_\chi(w_z)$. If $z \notin G_{g_1 \cdot x_1}$ notice that $zt$ and $t^{-1}$ lie in different copies of $H$ in $\Gamma$, therefore they commute, so:

$$t^{-1}zt = t^{-1}(zt) = (zt)t^{-1}z = zt^{-1}t^{-1}z.$$  

In this case define $w_z := zt^{-1}t^{-1}z$. Observe that $v_\chi(w_z) = \min\{0, \chi(z)\}$. If this minimum is 0 then $\chi(z) \geq 0$, and so $v_\chi(t^{-1}zt) = -\chi(t) < 0$. Otherwise $v_\chi(w_z) = \chi(z) < 0$ and so $v_\chi(t^{-1}zt) \leq \chi(t^{-1}z) = \chi(z) - \chi(t) < \chi(z)$. In both cases $v_\chi(t^{-1}zt) < v_\chi(w_z)$. Thus $[\chi] \in \Sigma^1(\Gamma)$ by Theorem 4.4. □

In order to complete the proof of Theorem 4.3 we only need to consider the cases where the restriction of $\chi$ to the copies of $H$ is non-zero only for copies associated to orbits that are composed by only one element, and this is done by use of the direct product formula, as follows.

**Theorem 4.3.** Let $\Gamma = H \wr_X G$ be a finitely generated wreath product and set $M = \oplus_{x \in X} H_x \subseteq \Gamma$. Let $\chi : \Gamma \to \mathbb{R}$ be a non-zero character such that $\chi|_M \neq 0$. Then $[\chi] \in \Sigma^1(\Gamma)$ if and only if at least one of the following conditions holds:

1. The set $T = \{x \in X \mid \chi|_{H_x} \neq 0\}$ has at least two elements;
2. $T = \{x_1\}$ and $\chi|_{G} \neq 0$;
3. $T = \{x_1\}$ and $[\chi|_{H_{x_1}}] \in \Sigma^1(H)$.

**Proof.** By Proposition 4.4 it is enough to consider the case where $G \cdot x = \{x\}$ for all $x \in T$. Notice that in this case $T$ must be finite, since each of its elements is an entire orbit of $G$ on $X$ and there are finitely many of those. Let $P = \prod_{x \in T} H_x$ and $X' = X \setminus T$. Then

$$\Gamma = H \wr_X G \simeq P \times (H \wr_{X'} G).$$

If $T$ has at least two elements, then $[\chi|_P] \in \Sigma^1(P)$ and hence $[\chi] \in \Sigma^1(\Gamma)$, by two applications of the direct product formula for $\Sigma^1$. If $T = \{x_1\}$, the formula gives us exactly that $[\chi] \in \Sigma^1(\Gamma)$ if and only if one of conditions (2) or (3) holds, since $\chi|_G \neq 0$ if and only if $[\chi|_{H_{x_1} \cdot G}] \neq 0$. □

5. **Graph-wreath products**

We now digress a bit and obtain a generalization of the results of Section 3 to a wider class of groups. Besides being interesting in its own right, this will be useful in the analysis of the $\Sigma^2$-invariants of wreath products.
Given two groups $G$ and $H$ and a $G$-graph, the graph-wreath product $H \wr_K G$ is defined by Kropholler and Martino [29] as the semi-direct product $H^{(K)} \rtimes G$, where $H^{(K)}$ is the graph product of $H$ with respect to the graph $K$ (that is, $H$ is the group associated to every vertex of $K$). The action of $G$ is given by permutation of the copies of $H$ according to the $G$-action on the vertex set of $K$. When $K$ is the complete graph $H \wr_K G$ is simply $H \wr_X G$, where $X$ is the vertex set of $K$.

Kropholler and Martino showed that $H \wr_K G$ is finitely generated if and only if $G$ and $H$ are finitely generated and $G$ acts with finitely many orbits of vertices on $K$, that is, $H \wr_K G$ is finitely generated under the same conditions as $H \wr_X G$ is, where $X$ is the vertex set of $K$.

In what follows we fix $\Gamma = H \wr_K G$ and $M = H^{(K)} \subseteq \Gamma$. We assume that $\Gamma$ is finitely generated and we decompose $X$ in orbits as $X = G \circ x_1 \cup \ldots \cup G \circ x_n$. Moreover, we choose finite generating sets $Z$ for $G$ and $Y_1$ for $H_{x_1}$ for $H_{x_1}$ for all $i = 1, \ldots, n$ and we denote $\mathcal{X} = (\cup_{i=1}^n Y_i) \cup Z$, which is seen as a generating set for $\Gamma$.

**Theorem 5.1.** Let $\chi : H \wr_K G \to \mathbb{R}$ be a non-zero character such that $\chi|_M = 0$. Then $[\chi] \in \Sigma^1(H \wr_K G)$ if and only if $[\chi|_G] \in \Sigma^1(G)$ and $\chi|_{G_x} \neq 0$ for all $x \in X$.

**Proof.** Let $N_K$ be the kernel of the obvious homomorphism $M = \oplus_{x \in X} H_x$. Note that $N_K \subseteq \ker(\chi)$ and that $\Gamma := \Gamma/N_K \simeq H \wr_X G$. It follows that $\chi$ induces a character $\bar{\chi} : \bar{\Gamma} \to \mathbb{R}$. For an element $\gamma \in \Gamma$, we denote by $\bar{\gamma}$ its image in $\bar{\Gamma}$.

If $[\chi] \in \Sigma^1(\Gamma)$, then $[\chi] \in \Sigma^1(\bar{\Gamma})$ (again by Proposition A4.5 in [29]). Thus $[\chi|_G] \in \Sigma^1(G)$ and $\chi|_{G_x} \neq 0$ for all $x \in X$ by Theorem A.

Conversely, suppose that $[\chi|_G] \in \Sigma^1(G)$ and that $\chi|_{G_x} \neq 0$ for all $x \in X$. Then $[\chi] \in \Sigma^1(\bar{\Gamma})$. We will show that this implies that $\text{Cay}(\Gamma; \mathcal{X})_\chi$ is connected.

We need to show that for all $\gamma \in \Gamma$, there is a path in $\text{Cay}(\Gamma; \mathcal{X})_\chi$ connecting 1 and $\gamma$. Given such a $\gamma$, notice that $\bar{\gamma} \in \bar{\Gamma}_1$, so there must be a path from 1 to $\bar{\gamma}$ in $\text{Cay}(\Gamma; \mathcal{X})_\chi$. Its obvious lift to $\text{Cay}(\Gamma; \mathcal{X})_\chi$ with 1 as initial vertex is a path in $\text{Cay}(\Gamma; \mathcal{X})_\chi$ that ends at an element of the form $\gamma n$, with $n \in N_K$. If we can connect $\gamma$ to $\gamma n$ inside $\text{Cay}(\Gamma; \mathcal{X})_\chi$ we are done. For that it suffices to find a path in $\text{Cay}(\Gamma; \mathcal{X})_\chi$ connecting 1 and $\gamma n$, and then act with $\gamma$ on the left.

Since $N_K \subseteq M$, each $n \in N_K$ can be written as:

\[(5.1) \quad n = (s_1 h_1) (s_2 h_2) \cdots (s_k h_k),\]

with $h_j \in \cup_{i=1}^n Y_i^{\pm 1}$ and $g_j \in G$ for all $j$. Even more, we may assume that each $\chi(g_j) \geq 0$. Indeed, since $\chi|_{G_x} \neq 0$ for all $x$, we can always pick $t_j \in G$ such that $\chi(t_j) > 0$ and $t_j h_j = h_j$. Then we may change $g_j$ for $g_j t_j^{-1}$ in (5.1), where $k_j$ is some integer such that $k_j \chi(t_j) \geq -\chi(g_j)$.

But if $\chi(g_j) \geq 0$ then $g_j \in G_{\chi|_G}$, and since $[\chi|_G] \in \Sigma^1(G)$, we can choose words $w_j$ in $G^{\pm 1}$ representing $g_j$ and such that $\nu_\chi(w_j) \geq 0$. Finally, the word

\[w = (w_1 h_1 w_1^{-1}) (w_2 h_2 w_2^{-1}) \cdots (w_k h_k w_k^{-1})\]

is the label for a path connecting 1 and $n$ in $\text{Cay}(\Gamma; \mathcal{X})_\chi$, by the choice of each $w_j$ together with the fact that $\chi(h_j) = 0$ for all $j$ by hypothesis.

The above result will be needed only in a special case, namely when $K$ is a graph without edges, so that $\Gamma \simeq (\ast_{x \in X} H_x) \rtimes G$.

6. The $\Sigma^2$-invariant

Renz’s paper [27] also brings a criterion for the invariant $\Sigma^2$. In order to state it, we need to introduce the concept of a diagram over a group presentation, for which we follow [9]. Fix an orientation on $\mathbb{R}^2$. Define a diagram to be a subset $M \subseteq \mathbb{R}^2$ endowed with the structure of a finite combinatorial 2-complex. Thus to each 1-cell of $M$ correspond two opposite directed edges. If $(\mathcal{X}, \mathbb{R})$ is a presentation for a
group \( \Gamma \), a \textit{labeled diagram over} \( (\mathcal{R} | \mathcal{R}) \) is a diagram \( M \) endowed with an edge labeling satisfying:

1. The edges of \( M \) are labeled by elements of \( \mathcal{R}^{\pm 1} \);
2. If an edge \( e \) has label \( x \), then its opposite edge has label \( x^{-1} \);
3. The boundary of each face of \( M \), read as a word in \( \mathcal{R}^{\pm 1} \), beginning at any vertex and proceeding with either orientation, is either a cyclic permutation of some \( r \in \mathcal{R}^{\pm 1} \), or a word of the form \( tt^{-1}t^{-1}t \) for some \( t \in \mathcal{R}^{\pm 1} \).

A labeled diagram \( M \) is said to be \textit{simple} if it is connected and simply connected.

\textbf{Remark 6.1.} This is a weakening of the definition of the usual \textit{van Kampen diagrams}. In fact, a simple diagram \( M \), with a vertex chosen as a base point, differs from a van Kampen diagram only by the fact that it can have what we call \textit{trivial faces}, that is, those labeled by \( tt^{-1}t^{-1}t \) for some \( t \in \mathcal{R}^{\pm 1} \). This weakening has the effect of simplifying the drawing of some diagrams that we will consider in the sequence (see \cite{27}, Subsection 3.3).

Suppose that we are given a simple diagram \( M \) with a base point \( u \) (a vertex in the boundary of \( M \)) and an element \( \gamma \in \Gamma \). Then to each vertex \( u' \) of \( M \) corresponds a unique element of \( \Gamma \), given by \( \gamma \eta \), where \( \eta \) is the image in \( \Gamma \) of the label of any path connecting \( u \) and \( u' \) inside \( M \). In particular, the given group element \( \gamma \) corresponds to the base point \( u \). For any character \( \chi : \Gamma \to \mathbb{R} \) we define the \textit{\( \chi \)-valuation} of \( M \) with respect to \( u \) and \( \gamma \), denoted by \( v_\chi(M) \), to be the minimum value of \( \chi(g) \) when \( g \) runs over the elements of \( \Gamma \) corresponding to the vertices of \( M \).

Now, suppose that \( \Gamma \) is finitely presented (with \( (\mathcal{R} | \mathcal{R}) \) a finite presentation) and assume \( \left[ \chi \right] \in \Sigma^1(\Gamma) \). Then we can distinguish an element \( t \in \mathcal{R}^{\pm 1} \) with \( \chi(t) > 0 \) with which we can apply Renz’s criterion for \( \Sigma^1 \); for each \( x \in \mathcal{R}^{\pm 1} \setminus \{t, t^{-1} \} \) we can associate a word \( w_x \) in \( \mathcal{R}^{\pm 1} \) that represents \( t^{-1}xt \) and for which \( v_\chi(t^{-1}xt) < v_\chi(w_x) \). Additionally, we put \( w_t := t \) and \( w_{t^{-1}} := t^{-1} \). If \( r = x_1 \cdots x_n \in \mathcal{R}^{\pm 1} \), we define:

\[ \hat{r} := w_{x_1} \cdots w_{x_n}. \]

We are now ready to enunciate the criterion for \( \Sigma^2 \).

\textbf{Theorem 6.2} (\cite{27}, Theorem 3). \textit{Let \( \Gamma \), \( \mathcal{R} \) and \( t \) be as above. Suppose that the set \( \mathcal{R} \) of defining relations contains some cyclic permutation of the words \( t^{-1}xtw_x^{-1} \), for all \( x \in \mathcal{R}^{\pm 1} \). Then \( \left[ \chi \right] \in \Sigma^2(\Gamma) \) if and only if for each \( r \in \mathcal{R}^{\pm 1} \) there exist a simple diagram \( M_r \) and vertex \( u \) in its boundary, such that both the following conditions hold:

1. \textit{The boundary path of} \( M_r \), \textit{read from} \( u \), \textit{has as label the word} \( \hat{r} \);
2. \textit{\( v_\chi(r) < v_\chi(M_r) \)}, \textit{where the valuation of} \( M_r \) \textit{is taken with respect to the base point} \( u \) \textit{and the element} \( t \in \Gamma \).

Now, recall that a wreath product \( H \wr X \) is finitely presented if and only if \( G \) and \( H \) are finitely presented, \( G \) acts diagonally on \( X^2 \) with finitely many orbits and the stabilizers of the \( G \)-action on \( X \) are finitely generated. This is the result by Cornulier \cite{10}.

We will apply Theorem 6.2 to show that if \( \Gamma = H \wr X \) is finitely presented and if \( \chi : \Gamma \to \mathbb{R} \) is a character such that \( \chi|_{H_{x_1}} \neq 0 \) for some \( x_1 \in X \) with \( |G : x_1| = \infty \), then \( \left[ \chi \right] \in \Sigma^2(\Gamma) \).

We start by assuming that \( G \) acts transitively on \( X \), with \( X = G \cdot x_1 \). Let \( \langle Y | R \rangle \) and \( \langle Z | S \rangle \) be finite presentations for \( H \) and \( G \), respectively. We may assume that \( Z \) contains a generating set \( E \) for the stabilizer subgroup \( G_{x_1} \), and a set \( J \) of representatives for the non-trivial double cosets of \( (G_{x_1}, G_{x_1}) \) in \( G \), since both \( E \) and \( J \) can be taken to be finite by the proof of the main theorem in \cite{10}.
We think of $\Gamma = H \cup G$ with the presentation considered by Cornulier. So $\Gamma$ is generated by the set $Y \cup Z$, subject to the following defining relations:

(1) $r$, for all $r \in R$ (defining relations for $H$);
(2) $s$, for all $s \in S$ (defining relations for $G$);
(3) $[g_1, y_1 2]$, for $g_1 \in J$, $y_1, y_2 \in Y$;
(4) $[e, y]$, for $e \in E$ and $y \in Y$.

Let us adapt a bit this presentation. We are under the hypothesis that $\chi|_{\mu_{t_i}} \neq 0$ and $|G \cdot x_1| = \infty$. We may assume without loss of generality that $\chi(h) > 0$ for some $h \in Y$. Choose $g_i \in Z$, for $1 \leq i \leq 5$, such that \{ $x_1 \} \cup \{ g_i \cdot x_1 \mid 1 \leq i \leq 5 \}$ is a set with exactly six elements (of course we may assume that $Z$ contains elements $g_i$ with this property). Define

$$t_i := g_i h,$$

for $i = 1, \ldots, 5$. Then $\Gamma$ is generated by $Y \cup Z \cup \{ t_i \mid 1 \leq i \leq 5 \}$, subject to the following defining relations:

(1) $r$, for all $r \in R$ (defining relations for $H$);
(2) $s$, for all $s \in S$ (defining relations for $G$);
(3) $[g_1, y_1 \cdot y_2]$, for all $g_1, y_2 \in Y \cup \{ t_i \mid 1 \leq i \leq 5 \}$ and $g, g' \in Z \cup \{ 1 \}$ whenever the commutator $[g_1, y_1 \cdot y_2]$ is indeed a relation in $\Gamma$;
(4) $[e, y]$, for $e \in E$ and $y \in Y$ and $[z, t_i]$, for $z \in Z \cap G_{g_i \cdot x_1}$;
(5) $g_i g_i^{-1} = 1$, for $1 \leq i \leq 5$.

Remark 6.3. We could write the conditions of item (3) in a more precise way, but it would require writing many cases. If $y_1 \in Y$ and $y_2 = t_1$, for example, then $[g_1, y_1 \cdot y_2]$ is a defining relation if $g \cdot x_1 \neq (g' g_1) \cdot x_1$.

Note that we have added a few relations of the types (3) and (4), but clearly they are consequences of the others. Furthermore, the set of relations is clearly still finite.

Set $t = t_1$. We will continue using the notation of Proposition 6.2. Thus for $y \in Y^{\pm 1}$ we have chosen $w_y = y$. If $z \in Z^{\pm 1}$, then $w_z = z$ if $z \in G_{g_1 \cdot x_1}$, and $w_z = z t \cdot z^{-1} t^{-1}$ otherwise. Moreover, since $t_i$ and $t$ commute in $\Gamma$ for all $1 \leq i \leq 5$, we can define $w_{t_i} := t_i$ and $w_{t_i} := t_i^{-1}$.

Let us check that the chosen presentation satisfies the conditions of Theorem 6.2. First, the set of defining relations contains the relations $t^{-1} x t w_y^{-1}$. Indeed if $y \in Y^{\pm 1} \cup \{ t_i \mid 1 \leq i \leq 5 \}$, then $t^{-1} y t w_y$ is a relation of type (3), since $w_y = y$. If $z \in Z^{\pm 1} \cap G_{g_1 \cdot x_1}$, then $w_z = z$ and $t^{-1} z t z^{-1}$ is a relation of type (4). Finally, if $z \in Z^{\pm 1}$ but $z \notin G_{g_1 \cdot x_1}$, then $w_z = z t z^{-1} t^{-1}$ and

$$t^{-1} z t w_y^{-1} = t^{-1} z t z^{-1} t z^{-1} t^{-1} = t^{-1} (t z t^{-1}) t^{-1},$$

which is a cyclic permutation of $[z, t]$, a relation of type (3).

According to Theorem 6.2 now we need to apply the transformation $r \mapsto \hat{r}$ to each defining relation and then find a simple diagram $M_\hat{r}$ satisfying the stated conditions. The following subsections are devoted to the verification of the existence of these diagrams. Observe that we do not need to consider the inverses of defining relations, since any simple diagram for $\hat{r}$ is a simple diagram for the inverse of $\hat{r}$ if we read its boundary backwards.

6.1. Relations of type (1). Note that the relations of type (1) involve only generators in $Y^{\pm 1}$. But $w_y = y$ for all $y \in Y^{\pm 1}$, so $\hat{r} = r$ whenever $r$ is a relation of type (1). Thus the one-faced diagram $M$ that represents the relation $r$, with base point corresponding to $t$, is already a choice for $M_\hat{r}$, since its $\chi$-value is increased by $\chi(t) > 0$. 

In Figure 1 we represent the diagram $M_r$, for $r = \hat{r} = y_1y_2y_3y_4$, as the internal square of the figure. The external boundary represents the path beginning at the base point $1 \in \Gamma$ and with label the original relation $r$. The edges labeled by $t$ indicate that $\hat{r}$ is obtained from $r$ by conjugation by $t$.

6.2. Relations of type (2). Since $\chi(h) \neq 0$, the order of $h$ in $\Gamma$ is infinite. Consider the subgroup

$$\Gamma_0:= \langle h, G \rangle \leq \Gamma.$$ Notice $\Gamma_0 \simeq \mathbb{Z} \wr \chi G$. Let $\chi_0$ be the restriction of $\chi$ to $\Gamma_0$. The group $\Gamma_0$ is an extension of an abelian group $A = \oplus_{x \in \chi} \mathbb{Z}$ by $G$, so it follows from Theorem 2.5 that $[\chi_0] \in \Sigma^2(\Gamma)$, as $\chi_0|A \neq 0$. Now choose a presentation for $\Gamma_0$ that is compatible with the chosen presentation for $\Gamma$: write the same presentation with $Y = \{h\}$ and discard the relations of type (1). Naturally, this presentation satisfies the hypothesis of Theorem 6.2.

Let $r = z_1 \cdots z_n$ be a relation of type (2). We can see $r$ as a relation in $\Gamma_0$. By Theorem 6.2 there is a simple diagram $M_\hat{r}$, with respect to the presentation of $\Gamma_0$, whose base point corresponds to $t$ and such that $\partial M_r = \hat{r}$ and $v_\chi(r) < v_\chi(M_\hat{r})$. But all the relations in the chosen presentation of $\Gamma_0$ are also relations in the original presentation of $\Gamma$, after identifying the generating sets. Then $M_\hat{r}$, if seen as a diagram over the presentation of $\Gamma$, is the diagram we wanted.

6.3. Relations of the types (4) or (5). All cases are similar: we can obtain simple diagrams whose only vertices are those of the boundary, that is, those that are defined by the word $\hat{r}$. In this case, the diagram automatically satisfies the hypothesis about its $\chi$-value, exactly as in the case of the relations of type (1). See the diagram for the relation $g_1h_1g_1^{-1}t^{-1}$ in Figure 2. As before, the external boundary represents $r = g_1h_1g_1^{-1}t^{-1}$, and the diagram $M_\hat{r}$ itself is the internal diagram composed by the five squares. Again, the edges labeled by $t$ and with origin at some point in the external path indicate conjugation by $t$ and represent the growth of $v_\chi$ from $r$ to $\hat{r}$.

Figure 2 is an illustration of the case when $g = g_1 \notin G_{g_1,x}$, when the word $w_g$ is more complicated. If the letter representing an element of $G$ is an element of $G_{g_1,x}$, the argument is simpler: all the letters involved in the relation commute with $t$, so $r = \hat{r}$ and the argument follows as in the case of the relations of type (1).

6.4. Relations of type (3). Let $g \in Y \cup \{t_i \mid 1 \leq i \leq 5\}$ and $g \in Z$. Let $\eta_{g,y}$ be the word obtained from $gyg^{-1}$ by applying the transformation that takes each letter $\alpha$ to $w_\alpha$:

$$\eta_{g,y} = (y)t^{-1}(y)t(y)t^{-1}, \quad (6.1)$$
if $g \notin G_{g_1 \cdot x_1}$, or

\[ (6.2) \quad \eta_{g, y} = g y. \]

if $g \in G_{g_1 \cdot x_1}$. If $g = 1$, put $\eta_{1, y} := y$. In all cases we see that $\eta_{g, y}$ is a product of subwords representing elements of at most 3 copies of $H$ in $\Gamma$. Indeed, $^g t$ and $(^g t)^{-1}$ are elements of $H_{g \cdot g_1 \cdot x_1}$, while $t$ and $t^{-1}$ are elements of $H_{g_1 \cdot x_1}$ and, finally, $^g y$ is an element of $H_{g \cdot x_1}$ or $H_{g_1 \cdot x_1}$ for some $1 \leq i \leq 5$, depending on $y$.

Consider $r = [^g y_1, ^g y_2]$, a relation of type (3). The word $^h \tilde{r} = [\eta_{g, y_1}, \eta_{g', y_2}]$ is a relation in $\Gamma$, so we can always find a simple diagram $M_1$ with some base point corresponding to $t$ and such that $\partial M_1 = \tilde{r}$. If $v_{\chi}(M_1) > v_{\chi}(r)$ we are done. Otherwise there is a vertex $p$ of $M_1$ such that $\chi(p) \leq v_{\chi}(r)$. Notice that this vertex can not lie on the boundary of $M_1$, since $v_{\chi}(\tilde{r}) > v_{\chi}(t^{-1}r t)$.

Now, the commutator between the words $\eta_{g, y_1}$ and $\eta_{g', y_2}$ is a product of elements of the form $z^2 y$, with $z \in Z \cup \{1\}$ and $y \in Y^{\pm 1} \cup \{t_1, \ldots, t_5\}^{\pm 1}$. By the remarks above, these elements lie in at most most five different copies of $H$, one of them being indexed by $g_1 \cdot x_1$ when five copies do pop up. It follows that for some $u \in \{h, t_2, t_3, t_4, t_5\}$, the words $[^z y, u]$ are defining relations for all the subwords $z^2 y$ appearing in $^h \tilde{r} = [\eta_{g, y_1}, \eta_{g', y_2}]$ (we consider the subwords $z^2 y$ that appear when $\eta_{g, y_1}$ and $\eta_{g', y_2}$ are written exactly as in (6.1) or (6.2)). Observe that $\chi(u) = \chi(h) > 0$ in all cases. So we can build a diagram $M_2$ by surrounding $M_1$ with faces representing the commutators $[^z y, u]$, for all these subwords $z^2 y$.

Clearly the boundary of $M_2$ is also labeled by $^h \tilde{r}$. If we set as base point the vertex on the new boundary corresponding to the base point of $M_1$ (that is, the one joined to it by an edge with label $u$), then the $\chi$-value of the interior points (including $p$) is increased by $\chi(u) > 0$, so that $v_{\chi}(M_2) > v_{\chi}(M_1)$. Repeating this process finitely many times, we obtain a simple diagram $M_n$ satisfying the conditions of the theorem.

We record what we have proved in the following proposition.

**Proposition 6.4.** Let $\Gamma = H \wr_X G$ be a finitely presented wreath product and let $M = \bigoplus_{x \in X} H_x \subseteq \Gamma$. Suppose that $G$ acts transitively on the infinite set $X$. If $\chi : \Gamma \to \mathbb{R}$ is a character with $\chi|_M \neq 0$, then $|\chi| \in \Sigma^2(\Gamma)$.

The arguments above essentially contain what we need when $G \cdot x$ is infinite for some $x \in X$ such that $\chi|_{H_x} \neq 0$ (but the $G$-action on $X$ is not necessarily transitive), so we will only indicate in the proof of the following theorem how to deal with this case.

\[ Figure 2. Diagram for relations of type (5) \]
Recall that we denote by $T$ the set of elements $x \in X$ such that $\chi|_{H_x} \neq 0$. Notice that if $T = \{x_1\}$, then $\Gamma$ is a direct product
\[ \Gamma \simeq H_{x_1} \times (H \wr X,G), \]
where $X' = X \setminus \{x_1\}$. Then the direct product formula and the results on the $\Sigma^1$-invariants of wreath products already contain all the information we need. The remaining cases are all part of the following theorem, which includes Theorem \[13\]

**Theorem 6.5.** Let $\Gamma = H \wr X G$ be a finitely presented wreath product and let $M = \bigoplus_{x \in X} H_x \subseteq \Gamma$. Suppose that the set
\[ T = \{x \in X \mid \chi|_{H_x} \neq 0\}. \]
has at least two elements. Then $[\chi] \in \Sigma^2(\Gamma)$ if and only if at least one of the following conditions holds:
1. $[\chi|_{H_x}] \in \Sigma^1(H)$ for some $x \in T$;
2. $\chi|_{G} \neq 0$;
3. $T$ has at least three elements.

**Proof.** Suppose first that $T$ is a finite set and consider the subgroup $B = \bigcap_{x \in T} G_x \leq G$. It is of finite index in $G$, so $\Gamma_1 = H \wr X B$ is of finite index in $\Gamma$. Notice that
\[ \Gamma_1 \simeq (\prod_{x \in T} H_x) \times (H \wr X B). \]
Denote $P = \prod_{x \in T} H_x$ and $Q = H \wr X B$. The fact that $T$ has at least two elements implies that $[\chi|_P] \in \Sigma^1(P)$, by the direct product formula. By applying the formula again, now to the product $\Gamma_1 = P \times Q$, we get that $[\chi|_{\Gamma_1}] \in \Sigma^2(\Gamma_1)$ if and only if $[\chi|_P] \in \Sigma^2(P)$ or $[\chi|_Q] \neq 0$. The former happens if and only if at least one of conditions (1) or (3) is satisfied (once again, by the direct product formula), while the latter clearly happens if and only if $\chi|_B \neq 0$, which in turn is equivalent with $\chi|_G \neq 0$, since $B$ is a subgroup of finite index. Finally, since the index of $\Gamma_1$ in $\Gamma$ is finite, we are done by Theorem \[2,3\].

We are left with the case where $T$ is infinite and we want to show that $[\chi] \in \Sigma^2(\Gamma)$. Since $G$ acts on $X$ with finitely many orbits, there must be some $x_1 \in T$ such that $[G \cdot x_1] = \infty$. We adapt the proof of Proposition \[6,4\] putting the orbit of $x_1$ in a distinguished position.

Choose $x_2, \ldots, x_n \in X$ such that $X = \bigsqcup_{i=1}^n G \cdot x_i$. For each $j$ choose a finite generating set $E_j$ for the stabilizer subgroup $G_{x_j}$. For each pair par $(i,j)$, with $1 \leq i,j \leq n$, choose a finite set $J_{ij}$ of representatives of the non-trivial double cosets of $(G_{x_i},G_{x_j})$ in $G$. Finally, choose finite presentations $(Y|R)$ and $(Z|S)$ for $H$ and $G$ respectively. We may assume that $Z$ contains $E_j$ and $J_{ij}$ for all $1 \leq i,j \leq n$.

A finite presentation for $\Gamma$, adapted from the presentation given by Cornulier \[10\], can be given as follows. For each $1 \leq i \leq n$ we associate a copy $(Y_i|R_i)$ of the presentation for $H$ and, as before, we define $t_i := g_i^h$ for some $g_i \in Z$ and $h \in Y_1$ with $\chi(h) > 0$ and $\{x_i\} \cup \{g_i \cdot x_1 | 1 \leq i \leq 5\} = 6$. We think of $\Gamma$ as generated by $(\bigcup_{i=1}^n Y_i) \cup Z \cup \{t_i | 1 \leq i \leq 5\}$ and subject to the defining relations given by:
1. $r$, for all $r \in \bigcup_{i=1}^n R_i$ (defining relations for the copies of $H$);
2. $s$, for all $s \in S$ (defining relations for $G$);
3. $[y_1,g^h y_2]$, for $y_1,y_2 \in (\bigcup_{i=1}^n Y_i) \cup \{t_i | 1 \leq i \leq 5\}$ and $g,g^h \in Z \cup \{1\}$ whenever $[y_1,g^h y_2]$ is indeed a relation in $\Gamma$;
4. $[e_1,y_i]$, for all $e_i \in E_1$, $y_i \in Y_i$ and $1 \leq i \leq n$ and $[z,t_1]$, for all $z \in Z \cap G_{g_1 \cdot x_1}$;
5. $g_i g_i^{-1} t_i^{-1}$, for $1 \leq i \leq 5$. 
Set \( t = t_1 \). We use again the notation of Proposition \( \ref{proposition} \). So we use the same words \( w_2 \) if \( z \in Z^\pm 1 \), and \( w_y = y \) for all other generators \( y \). It is clear that the set of defining relations above still satisfies the hypothesis of Theorem \( \ref{theorem} \). The construction of the diagrams associated to each defining relation can be done exactly as in the case where the action is transitive, as we will argue below. The key fact is that the generators coming from copies of \( H \) associated to all other orbits of \( G \) (other than \( G \cdot x_1 \)) commute with \( t = t_1 \).

First, notice that the construction of the diagrams associated to the relations of types \( (1) \) or \( (4) \) in the case of a transitive action depends only on the fact that \( [t, y] \) is a defining relation for all \( y \in Y = Y_1 \). But \( t_1 \) commutes also with all elements of \( Y_2 \cup \ldots \cup Y_n \), so the construction can be carried out in the same way. For the case of relations of type \( (3) \), it was only necessary that for any generators \( g, g' \in Z \cup \{1\} \) and \( y, y' \in Y_1 \), we could find some \( u \in \{h, t_2, \ldots, t_5\} \) that commutes with all the following elements: \( t, gt, g't, gy \) and \( g' y' \). If we allow \( y \) to be an element of \( Y_2 \cup \ldots \cup Y_n \), then any \( u \) that commutes with \( t, gt, g't \) and \( g' y' \) will do it, since \( y \) commutes any choice of \( u \). Thus the five options for \( u \), coming from different copies of \( H \), are enough to let us repeat the argument. Similar considerations cover the cases where either only \( y' \), or both \( y \) and \( y' \) are elements of \( Y_2 \cup \ldots \cup Y_n \).

This is all we needed to check, since relations of types \( (2) \) and \( (5) \) do not involve any of the new generators. \( \square \)

7. Some observations about \( \Sigma^2 \)

Let \( \Gamma \) be a finitely presented group and let \([\chi] \in S(\Gamma)\). Let \( \langle \mathcal{D} | \mathcal{R} \rangle \) be a finite presentation for \( \Gamma \). Denote by \( C = \text{Cay}(\Gamma; \langle \mathcal{D} | \mathcal{R} \rangle) \) the associated Cayley complex and by \( C_\chi \) the full subcomplex of \( C \) spanned by \( \Gamma_{\chi} \). The canonical action of \( \Gamma \) on \( C \) restricts to an action by the monoid \( \Gamma_{\chi} \) on \( C_\chi \).

Remark 7.1. If a monoid \( K \) acts on some set \( X \) we still say that the sets \( K \cdot x \) are orbits. By \( "K \) has finitely many orbits on \( X" \) we mean that there are elements \( x_1, \ldots, x_n \in X \) such that \( X = \bigcup_{i=1}^n K \cdot x_j \).

The following lemma can be found in Renz’s thesis \( \cite{renz} \).

Lemma 7.2. \( C_\chi \) has finitely many \( \Gamma_{\chi} \)-orbits of \( k \)-cells for \( k \leq 2 \).

Proof. Denote by \( D \) and \( D_\chi \) the sets of \( k \)-cells of \( C \) and \( C_\chi \), respectively (for a fixed \( k \leq 2 \)). We know that \( \Gamma \) acts on \( D \) with finitely many orbits. Choose representatives \( d_1, \ldots, d_n \) for these orbits so that \( d_j \in D_\chi \) but \( \gamma \cdot d_j \notin D_\chi \) for all \( j \) and for all \( \gamma \in \Gamma \) with \( \chi(\gamma) < 0 \). For this it suffices to take any representatives \( \tilde{d}_1, \ldots, \tilde{d}_n \) and then put \( d_j := \gamma_j^{-1} \cdot \tilde{d}_j \), where \( \gamma_j \in \Gamma \) is the vertex of \( \tilde{d}_j \) with lowest \( \chi \)-value. Thus if \( d \in D_\chi \), then \( d = \gamma \cdot \tilde{d}_j \) for some \( j \) and, by choice of \( \tilde{d}_j \), we have that \( \chi(\gamma) \geq 0 \). So \( D_\chi = \bigcup_{j=1}^n \Gamma_{\chi} \cdot \tilde{d}_j \). \( \square \)

Denote by \( F(\mathcal{D}, \chi) \) the submonoid of \( F(\mathcal{G}) \) consisting of the classes of reduced words \( w \) with \( v_\chi(w) \geq 0 \). Note that \( F(\mathcal{D}, \chi) \) is indeed closed under the product, since \( w_1, w_2 \in F(\mathcal{D}, \chi) \) implies \( v_\chi(w_1 w_2) \geq 0 \), and this property is preserved by elementary reductions (that is, canceling out terms of the form \( x^{-1} r x \) or \( x^{-1} r \)).

Let \( R(\chi) \) be the subgroup of \( F(\mathcal{G}) \) consisting of the classes of reduced words \( w \) that represent relations (that is, \( w \in \langle \mathcal{D} \rangle F(\mathcal{G}) \)) and such that \( v_\chi(w) \geq 0 \). Observe that \( R(\chi) \subseteq F(\mathcal{D}, \chi) \) and notice that \( R(\chi) \) is indeed a subgroup, since \( v_\chi(w) \geq 0 \) implies \( v_\chi(w^{-1}) \geq 0 \) whenever \( w \) is a relation. Finally, observe that \( R(\chi) \) admits an action by the monoid \( F(\mathcal{D}, \chi) \) via left conjugation.

Now, let \( r \) be a reduced word in \( \mathcal{D}^\pm 1 \) representing a relation in \( \Gamma \), that is, \( r \in \langle \mathcal{D} \rangle F(\mathcal{G}) \). Suppose that \( M \) is a van Kampen diagram over \( \langle \mathcal{D} | \mathcal{R} \rangle \) whose
boundary, read in some orientation from some base point p, is exactly r. Then it holds in \( F(\mathcal{X}) \) that
\[
\begin{equation}
 r = w_1 r_1 \cdots w_n r_n,
\end{equation}
\]
where each \( r_i \) is a word read on the boundary of some face of \( M \) and \( w_i \) is the label for a path in \( M \) connecting \( p \) to a base point of the face associated to \( r_i \). Both the facts that such a diagram exists and that \( r \) can be written as above are consequences of van Kampen’s lemma (see Proposition 4.1.2 and Theorem 4.2.2 in [9], for instance).

**Lemma 7.3.** If \( \chi : \Gamma \to \mathbb{R} \) is a character such that \( C_{\chi} = \text{Cay}(G, \langle \mathcal{X} | \mathcal{R} \rangle)_{\chi} \) is 1-connected, then \( R(\chi) \) is finitely generated over \( F(\mathcal{X}, \chi) \).

**Remark 7.4.** By “\( R(\chi) \) is finitely generated over \( F(\mathcal{X}, \chi) \)” we mean that every element of \( R(\chi) \) can be written as a product of elements of the form \( w s \), where \( w \in F(\mathcal{X}, \chi) \) and \( s \in S \) for some finite set \( S \subseteq R(\chi) \).

**Proof.** Let \( r \in R(\chi) \) and consider the path \( \rho \) in \( C \) beginning at 1 and with label \( r \).
Notice that this path runs inside \( C_{\chi} \), since \( \nu_{\chi}(r) \geq 0 \). Also, \( \rho \) is clearly a loop and it must be nullhomotopic in \( C_{\chi} \), since \( C_{\chi} \) is 1-connected. A homotopy from \( \rho \) to the trivial path can then be realized by a van Kampen diagram \( M \) with \( \nu_{\chi}(M) \geq 0 \) (the valuation is taken with respect to 1, seen both as base point in \( C \) and group element). This is made precise by Theorem 2 in [27].

Write \( r \) as in (2.1). Thus \( r \) is a product of relations corresponding to the faces of \( M \) conjugated on the left by elements of \( \text{Stab}(\Gamma) \). Since \( \nu_{\chi}(M) \geq 0 \), such faces are faces of \( C_{\chi} \), so by Lemma 7.2 and using that every element of \( \Gamma_{\chi} \) can be written as a word in \( F(\mathcal{X}, \chi) \), each \( w_j r_j \) can be rewritten as \( u_j s_j \) where \( u_j \in F(\mathcal{X}, \chi) \) and each \( s_j \) is a word read on the boundary of a face in a finite set \( S \) of representatives of \( \Gamma_{\chi} \)-orbits of faces of \( C_{\chi} \). It follows that \( S \) is a finite generating set for \( R(\chi) \).

\[ \square \]

8. \( \Sigma^2 \) for characters with \( \chi|M = 0 \)

We get back to a finitely presented wreath product \( \Gamma = H \wr X G = M \rtimes G \). We consider now the non-zero characters \( \chi : \Gamma \to \mathbb{R} \) such that \( \chi|M = 0 \).

In order to find sufficient conditions for \( |\chi| \in \Sigma^2(\Gamma) \), we consider a nice action of \( \Gamma \) on a complex. We will briefly describe the construction in the proof of Theorem A in [20], with the simplifications allowed by the fact that our situation is less general than what is considered in that paper.

We are assuming that \( \Gamma = H \wr X G \) is finitely presented, so \( H \) is also finitely presented. Choose a \( K(H, 1) \)-complex \( Y \), with base point *, having a single 0-cell and finitely many 1-cells and 2-cells. Let \( Z = \bigoplus_{x \in X} Y_x \) be the \textit{finite} product of copies of \( Y \) indexed by \( X \), that is, \( Z \) is the subset of the cartesian product \( \prod_{x \in X} Y_x \) consisting on the families \( (y_x)_{x \in X} \) such that \( y_x \) is not the base point * only for finitely many indices \( x \in X \). It follows by the results in [11] and [12] that \( Z \) is an Eilenberg-MacLane space for \( M = \bigoplus_{x \in X} H_x \). Notice that \( Z \) has a natural cell structure. There is a single 0-cell, given by the family \( (y_x)_{x \in X} \) with \( y_x = * \) for all \( x \). For \( n \geq 1 \), an \( n \)-cell can be seen as a product \( c_1 \times \cdots \times c_k \) of cells of \( Y \), supported by some tuple \( (x_1, \ldots, x_k) \in X^k \), such that \( \text{dim}(c_1) + \cdots + \text{dim}(c_k) = n \).

There is an obvious action of \( G \) on \( Z \). On the other hand, \( M \) acts freely on the universal cover \( E \) of \( Z \). By putting together these two actions, we get an action of \( \Gamma = M \rtimes G \) on \( E \). Clearly the 2-skeleton of \( E \) has finitely many \( \Gamma \)-orbits of cells. Moreover, since the action of \( M \) is free, the stabilizer subgroups are all conjugate to subgroups of \( G \), and can be described as follows:

1. The stabilizer subgroup of any 0-cell is a conjugate of \( G \);
Proposition 8.2. Let $\chi : \Gamma \to \mathbb{R}$ be a non-zero character such that $[\chi]_c \in \Sigma^{2-\dim(c)}(G_c)$ for all cells $c$ in $E^\prime$, then $[\chi] \in \Sigma^2(\Gamma)$.

We apply the theorem above with $E^\prime$ being the 2-skeleton of $E$. We obtain:

Proposition 8.4. Let $\Gamma = H \wr X G$ be a finitely presented wreath product and let $\chi : \Gamma \to \mathbb{R}$ be a non-zero character such that $[\chi]_c = 0$. Suppose also that all properties below hold:

1. $[\chi|_G] \in \Sigma^2(G)$;
2. $[\chi|_{G_2}] \in \Sigma^1(G_2)$ for all $x \in X$ and
3. $[\chi|_{G(x,y)}] \neq 0$ for all $(x,y) \in X^2$.

Then $[\chi] \in \Sigma^2(\Gamma)$.

The fact that we can state the proposition above with reference only to the stabilizers contained in $G$ follows from the invariance of the $\Sigma^1$-invariants under isomorphisms (Proposition B.1.5 in [29]). It is also clear that item (3) is equivalent with asking that the restriction of $\chi$ to the actual stabilizers is non zero.

By Theorem 8.1, if $[\chi] \in \Sigma^2(\Gamma)$ and $[\chi]_M = 0$, then $[\chi|_G] \in \Sigma^2(G)$. We can also show that condition (3) of Proposition 8.2 is necessary.

Lemma 8.3. If $[\chi|_{G(x,y)}] = 0$, then the monoid $G_\chi$ can not have finitely many orbits on $G \cdot (x,y)$.

Proof. Suppose that $G \cdot (x,y) = \bigcup_{j=1}^n G_j \cdot (x_j,y_j)$ and choose $g_1, \ldots, g_n \in G$ such that $(x_j,y_j) = g_j \cdot (x,y)$. Choose $g \in G$ such that

$$\chi(g) < \min \{ \chi(g_j) \mid 1 \leq j \leq n \}.$$ 

Since $g \cdot (x,y) \in \bigcup_{j=1}^n G_j \cdot (x_j,y_j)$, there must be some $g_0 \in G_\chi$ and $1 \leq j \leq n$ such that $g \cdot (x,y) = g_0 \cdot (x_j,y_j) = g_0 g_j (x,y)$. But then $g^{-1} g_0 g_j \in G_{(x,y)}$, with:

$$\chi(g^{-1} g_0 g_j) = \chi(g_0) + (\chi(g_j) - \chi(g)) > 0,$$

so $[\chi|_{G(x,y)}] \neq 0$. \hfill \ensuremath{\Box}

Proposition 8.4. Let $\Gamma = H \wr X G$ be a finitely presented wreath product and let $\chi : \Gamma \to \mathbb{R}$ be a non-zero character. Let $M = \bigoplus_{x \in X} H_x \subseteq \Gamma$ and suppose that $[\chi]_M = 0$. If $[\chi|_{G(x,y)}] = 0$ for some $(x,y) \in X^2$, then $[\chi] \not\in \Sigma^2(\Gamma)$.

Proof. We may assume that $[\chi] \in \Sigma^1(\Gamma)$, otherwise there is nothing to do. Thus $[\chi|_G] \in \Sigma^1(G)$ and $[\chi|_x] \neq 0$ for all $x \in X$ by Proposition 8.3.
Let \( \Gamma_0 = (\star_{x \in X} H_x) \rtimes G \) and let \( \mathcal{X} \subseteq \Gamma_0 \) be a finite generating set. Note that \( \Gamma \) is a quotient of \( \Gamma_0 \), so we can consider the following diagram:

$$
\begin{array}{ccc}
F(\mathcal{X}) & \xrightarrow{\pi} & \Gamma_0 \\
\downarrow & & \downarrow \\
\pi & & \Gamma.
\end{array}
$$

The homomorphism \( \pi \) defines presentations for \( \Gamma \) with generating set \( \mathcal{X} \). We first show that for finite presentations of type \( \Gamma = \langle \mathcal{X} \mid \mathcal{R} \rangle \) (with \( \ker(\pi) = \langle \mathcal{R} \rangle \)) the complex \( \text{Cay}(\Gamma; \langle \mathcal{R} \rangle \langle \mathcal{F} \rangle) \chi \) can not be 1-connected.

Fix \( \langle \mathcal{R} \rangle \mathcal{R} \) such a presentation. We use the notations \( F(\mathcal{X}, \chi) \) and \( R(\chi) \) defined in Section 6.2. We arrive at a contradiction with the first part of the proof, since

$$
\begin{array}{l}
F(\mathcal{X}) \xrightarrow{\pi} \Gamma_0 \\
\pi \downarrow \\
\Gamma.
\end{array}
$$

From what follows that \( \text{Cay}(\Gamma; \langle \mathcal{R} \rangle \langle \mathcal{F} \rangle) \chi \) is not 1-connected anymore, but by Lemma 3 in [27], we can enlarge \( \mathcal{R} \) to a (still finite) set \( \mathcal{R}' \) so that \( \text{Cay}(\Gamma; \langle \mathcal{R}' \rangle \langle \mathcal{F} \rangle) \chi \) is indeed 1-connected. This is done by adding the relations of the form \( t^{-1}xtw^{-1} \), as in Theorem 6.2. We arrive at a contradiction with the first part of the proof, since
$\mathcal{X}'$ satisfies the previous hypothesis, that is, $\mathcal{X}'$ can be lifted to a generating set for $\Gamma_0$.

The above proposition completes the proof of Theorem C as stated in the introduction, since its last assertion (when we assume that $H$ has infinite abelianization) follows from Theorem 3.1.

9. Applications to twisted conjugacy

We now derive some consequences of the previous results to twisted conjugacy, more specifically to the study of Reidemeister numbers of automorphisms of wreath products. For this we start by considering the Koban invariant $\Omega^1$.

Given a finitely generated group $\Gamma$, endow $\text{Hom}(\Gamma, \mathbb{R})$ with an inner product structure, so that it makes sense to talk about angles in $S(\Gamma)$. Denote by $N_{\pi/2}([\chi])$ the open neighborhood of angle $\pi/2$ and centered at $[\chi] \in S(\Gamma)$. Following Koban [16], we can define the invariant $\Omega^1(\Gamma)$ in terms of $\Sigma^1(\Gamma)$:

$$\Omega^1(\Gamma) = \{ [\chi] \in S(\Gamma) \mid N_{\pi/2}([\chi]) \subseteq \Sigma^1(\Gamma) \}.$$ 

A proof of the fact that this does not depend on the inner product can be found in the above-mentioned paper, which contains the original definition of the invariant.

Let $\Gamma = H \wr_X G$ be a finitely generated wreath product. With some restrictions on the action by $G$ on $X$, we can obtain nice descriptions of $\Omega^1(\Gamma)$. Notice that, since the invariant does not depend on the choice of inner product, we can assume that characters $[\chi], [\eta] \in S(\Gamma)$ such that $\chi|_G = 0$ and $\eta|_M = 0$ are always orthogonal, and this will be done in the proposition below.

**Proposition 9.1.** Let $\Gamma = H \wr_X G$ be a finitely generated wreath product. Suppose that

$$\Sigma^1(\Gamma) = \{ [\chi] \in S(\Gamma) \mid \chi|_M \neq 0 \},$$

where $M = \bigoplus_{x \in xH_x} \subseteq \Gamma$. Then

$$\Omega^1(\Gamma) = \{ [\chi] \in S(\Gamma) \mid \chi|_M = 0 \}.$$ 

**Proof.** Let $[\chi] \in S(\Gamma)$ with $\chi|_G = 0$. Clearly $\chi|_M \neq 0$, so $[\chi] \in \Sigma^1(\Gamma)$. Furthermore, if $[\eta] \in N_{\pi/2}([\chi])$, then $\eta|_M \neq 0$, otherwise $\chi$ and $\eta$ would be orthogonal. So $N_{\pi/2}([\chi]) \subseteq \Sigma^1(\Gamma)$ whenever $\chi|_G = 0$. On the other hand, if there were some $[\chi] \in \Omega^1(\Gamma)$ with $\chi|_G \neq 0$, then by taking $\eta : \Gamma \to \mathbb{R}$ defined by $\eta|_M = 0$ and $\eta|_G = \chi|_G$, we would have that $[\eta] \in N_{\pi/2}([\chi])$, but $[\eta] \notin \Sigma^1(\Gamma)$. }

For any group $V$, we denote by $V^{ab}$ its abelianization. By Theorem A if the $G$-action on $X$ does not contain orbits composed by only one element, then many conditions imply the hypothesis on the description of $\Sigma^1(\Gamma)$, such as:

1. $(G_x)^{ab}$ is finite for some $x \in X$, or
2. The set $\{ [\chi] \in \Sigma^1(G) \mid \chi|_{G_x} \neq 0 \}$ is empty for some $x \in X$.

This includes the cases where the $G$-action is free (in particular the regular wreath products $\Gamma = H \wr G$ and the case where $\Sigma^1(G) = \emptyset$).

Recall that the Reidemeister number $R(\varphi)$, for a group isomorphism $\varphi : V \to V$, is defined as the number of orbits of the $\varphi$-twisted conjugacy action of $V$ on itself. A connection between the invariant $\Omega^1$ and Reidemeister numbers was studied by Koban and Wong [17]. Recall that a character $\chi$ is discrete if its image is infinite cyclic.

**Theorem 9.2.** ([17], Theorem 4.3). Let $G$ be a finitely generated group and suppose that $\Omega^1(G)$ contains only discrete characters.

1. If $\Omega^1(G)$ contains only one element, then $G$ is of type $R_\infty$, that is, $R(\varphi) = \infty$ for all $\varphi \in \text{Aut}(G)$.
(2) If $\Omega^1(G)$ has exactly two elements, then there is a subgroup $N \subseteq \text{Aut}(G)$, with $[\text{Aut}(G) : N] = 2$, such that $R(\varphi) = \infty$ for all $\varphi \in N$.

**Corollary 9.3.** Let $\Gamma = H \wr_X G$ be a finitely generated wreath product and suppose that the $G$-action on $X$ is transitive. Suppose further that $\Sigma^1(\Gamma)$ is as described in Proposition 9.2 and that $H^ab$ has torsion-free rank 1. Then there is a subgroup $N \subseteq \text{Aut}(\Gamma)$, with $[\text{Aut}(\Gamma) : N] = 2$, such that $R(\varphi) = \infty$ for all $\varphi \in N$.

**Proof.** By the hypothesis on $H^ab$ we have that

$$\Omega^1(\Gamma) = \{[\nu_1], [\nu_2]\},$$

where $\nu_j(G) = 0$, $\nu_1(h) = 1$ and $\nu_2(h) = -1$ for some lift $h \in H$ of a generator for the infinite cyclic factor of $H^ab$. It suffices then to apply part (2) of Theorem 9.2. \qed

The applications that we keep in mind are the finitely generated regular wreath products of the form $\mathbb{Z} \wr G$.

Gonçalves and Kochloukova [13] exhibited other connections between the $\Sigma$-theory and the property $R_\infty$. Below we denote by $\Sigma^1(G)^c$ the complement of $\Sigma^1(G)$ in $S(G)$, that is, $\Sigma^1(G)^c = S(G) \smallsetminus \Sigma^1(G)$.

**Theorem 9.4 (13, Corollary 3.4).** Let $G$ be a finitely generated group and suppose that

$$\Sigma^1(G)^c = \{[\chi_1], \ldots, [\chi_n]\},$$

where $n \geq 1$ and each $\chi_j$ is a discrete character. Then there is a subgroup of finite index $N \subseteq \text{Aut}(G)$ such that $R(\varphi) = \infty$ for all $\varphi \in N$.

**Corollary 9.5.** Let $\Gamma = H \wr_X G$ be a finitely generated wreath product. Once again, suppose that $\Sigma^1(\Gamma)$ is as described in Proposition 9.2. Suppose further that $G^ab$ has torsion-free rank 1. Then there is a subgroup of finite index $N \subseteq \text{Aut}(\Gamma)$ such that $R(\varphi) = \infty$ for all $\varphi \in N$.

**Proof.** Under the hypothesis above, we have

$$\Sigma^1(\Gamma)^c = \{[\chi_1], [\chi_2]\},$$

where $\chi_j|_M = 0$ and $\chi_1(g) = 1$ and $\chi_2(g) = -1$ for some $g \in G$ whose image in $G^ab$ is a generator of the infinite cyclic factor. Then Theorem 9.3 applies. \qed

This time we can take as an example the regular wreath product $\Gamma = H \wr \mathbb{Z}$.

We note that Gonçalves and Wong [15] and Taback and Wong [30] had already obtained some results about the property $R_\infty$ for regular wreath products of the form $H \wr \mathbb{Z}$, with $H$ abelian or finite. Our results complement theirs in the sense that it considers other basis groups $H$ and non-regular actions, but here we were limited to talk about Reidemeister numbers of automorphisms contained in subgroups of finite index in the automorphism group. In the above-mentioned papers, on the other hand, the authors were able to determine positively the property $R_\infty$ for some choices of $H$.

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