Quantum Fisher information of a teleported state in Heisenberg XXX chain with x-component of Dzyaloshinskii-Moriya interaction

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Abstract

This article investigates the quantum Fisher information associated with a teleported state in a Heisenberg XXX chain with the x-component of the Dzyaloshinskii-Moriya interaction. The correspondence between the output state and the fidelity of entanglement teleportation expressions was constructed utilizing physical variables associated with the selected system. Our findings suggest that temperature, the spin coupling constant $J$, and the x-components $D_x$ may all contribute to the degree of intricacy between states and, therefore, to the possibilities of teleportation protocols. Additionally, these results suggest that the states’ separability needs either a high-temperature regime, strong spin-orbit coupling through the Dzyaloshinskii-Moriya interaction, or a ferromagnetic chain. Entanglement-based teleportation is improbable. The system states grow increasingly entangled even with an antiferromagnetic chain, weak spin-orbit coupling, or low temperature. As a consequence, the channel becomes entangled, making the teleportation protocol conceivable and feasible.

PACS numbers: 03.67.Lx, 03.67.Hk, 75.10.Jm

Keywords: entanglement, concurrence, Heisenberg chain, Dzyaloshinskii-Moriya coupling, quantum Fisher information, teleportation, thermal state.

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1 Introduction

The phenomenon of quantum teleportation is, as the name suggests, utterly fascinating. In itself, teleportation [1] is a striking manifestation of what is commonly referred to as the non-locality of quantum mechanics. It allows quantum information to be transmitted directly from port A to port B without any intermediate point. From a technological point of view, it is essential for everything related to quantum communications. One can imagine it at the heart of several quantum cryptography protocols and as a necessary component of the future quantum computer. Quantum cryptography makes it possible, among other things, to establish, in theory, and conditionally to the validity of quantum mechanics, communication channels (classical or quantum) whose eavesdropping is impossible [2]. The quantum computer, on the other hand, can solve some unsolvable problems in a reasonable time for the classical computer, the one we use today. Shor’s algorithm for factoring large prime numbers is a powerful example [3]. Still, the quantum computer should go beyond that: if we are to believe Church-Turing-Deutsch’s thesis, the quantum computer would effectively simulate any physical phenomenon [4,5]. This fundamental task is currently impossible for the classical computer since affecting a quantum system requires classical exponential resources to the best of current knowledge. It is essential to mention that while quantum mechanics is very unsettling at first glance, it is a physical theory with an excellent experimental basis. For example, Bell’s inequalities [6], which constitute a demonstration of the entanglement phenomenon, have been verified and rechecked over and over again [7–9], perhaps precisely because of their counterintuitive character. In recent years, quantum teleportation has also fallen out of the ranks of theoretical curiosities: it was carried out, to give only a few examples, between photons over distances of 600 m below the Danube [10] and 143 km in the Canary Islands [12], as well as over a distance of 3 m between sets of nitrogen-vacancy centres of two diamonds [13].

A point common to all physical implementations of quantum teleportation is the fatality of noise. By this is meant that any concrete realization of theoretical protocols necessarily suffers from imperfections. In quantum computing, this difficulty is compounded by the fact that one cannot clone quantum information. This is very fragile and degrades during any interaction with its environment. The problem is such that before the invention of quantum corrective codes [14], some scientists doubted that the quantum computer could ever be built. Hope has since returned, but as evidenced by the current lack of a quantum computer, the fight against noise has yet to be won and much remains to be learned.

A variety of entanglement measurements have been proposed and studied [5,15,16], with an entanglement of formation being the most relevant to this study since it is meant to quantify the resources required to establish a particular entangled state [15]. Wootters has shown that, in the case of a two-qubit system, the entanglement of a formation may be produced directly by applying the concurrence formula of the state to the system [16]. The spin chain is one of the most obvious choices for the realization of entanglement. The Heisenberg model is the most straightforward way of examining and researching the behavior of spin chains. Heisenberg XXX-chain entanglement was explored for the first time by Nielsen [17], who was the first person to do so. He demonstrated that entanglement in such systems happens only in the antiferromagnetic situation when the temperature is below a certain threshold. Several studies have been conducted after Nielsen’s work on entanglement in two-qubit sys-
tems such as the two-qubit XXX, XXZ, and XY systems in the presence of both a homogeneous and an inhomogeneous magnetic field [18–23]. Anisotropy owing to spin coupling has also been investigated in the x, y, and z directions, as has been done in several previous studies, [24–26]. Using XYZ Heisenberg systems, Yang et al. Shown in ref [27] that an inhomogeneous external magnetic field may improve the critical temperature and increase the amount of entangled states. The interaction of spin-orbits causes another sort of anisotropy (SO) [28–34]. The influence of (SO) interaction on thermal entanglement in a XXX system with two qubits, in the absence of a magnetic field, was published in [35]. Entanglement in the XYZ Heisenberg system with (SO) interaction in an inhomogeneous magnetic field, on the other hand, has not been addressed in any detail yet. We study the impact of (SO) contact on entanglement and entanglement teleportation in a thermally stable two-qubit system in this work as a result of this.

According to C. Bennet et al. [1], teleportation may be accomplished by combining two entangled but geographically distant particles. Theoretically, they claimed, less entangled states may still be used for teleportation. Afterwards, S. Popescu demonstrated, using the hidden variable concept, that teleportation of a quantum state through pure classical transmission cannot be accomplished with a fidelity more than $\frac{2}{3}$ [1]. To transport quantum information with fidelity more than $\frac{2}{3}$, mixed quantum channels with fidelity greater than $\frac{2}{3}$ are beneficial. Horodecki et al. [36] estimated the best fidelity of teleportation for bipartite states by using the isomorphism between quantum channels and a class of bipartite states, as well as whirling operations. When the channel is quantum mechanically correlated, J. Lee and M.S. Kim demonstrate that quantum teleportation retains the nature of quantum correlation in the unknown entangled state. The following study entanglement teleportation using two copies of Werner states [37].

In the current article, we chose to examine the problem of Quantum Fisher information of a teleported state in Heisenberg XXX chain with x-component of Dzyaloshinskii-Moriya interaction (DM). We reduce our system’s Hamiltonian mathematically to identify the expressions of output concurrence $C_{\text{out}}$ and fidelity of teleportation protocol $F$. To accomplish this, we diagonalize our system to discover the solutions to the energy spectrum, which lets us find the density matrix, which is important to identify the output concurrence and fidelity expressions. The following is the structure of the present paper. We quickly provide the model Hamiltonian and the characteristics of the ground states and the thermal states, which are given in section 2. Section 3 custom to the thermal entanglement teleportation by determining the concurrence of output state and the fidelity of entanglement teleportation. In the section 4 we integrate numerical results and interpretations to draw conclusions and understand our study’s aspect. Afterwards, Section 5, we give a summary of the results.

## 2 Theoretical model

A one-half isotropic Heisenberg XXX chain with an x-component of the DM interaction is considered in this study, and the $N$ spins of the chain are taken into account. The Hamiltonian is written in the following form

$$H = J \sum_n \left( s_n^x s_{n+1}^x + s_n^y s_{n+1}^y + s_n^z s_{n+1}^z \right) + D_x \left( s_n^y s_{n+1}^z - s_n^z s_{n+1}^y \right)$$

$s^{x,y,z}$ are the typical Pauli matrices, and $J$ is the real coupling constant of the spin interaction. Antiferromagnetic chains have values greater than zero, while ferromagnetic chains have values less than zero.
The symbol $D_x$ denotes the x-component of the DM interaction. Using the conventional computing base $|00>, |01>, |10>, |11>$, the Hamiltonian equation (1) for two qubits $N = 2$ may be expressed using its matrix form by

$$
H = \begin{pmatrix}
J & iD_x & -iD_x & 0 \\
-iD_x & -J & 2J & iD_x \\
iD_x & 2J & -J & -iD_x \\
0 & -iD_x & iD_x & J
\end{pmatrix}
$$

(2)

The solution of the eigenvalue equation generates the eigenvalues listed below

$$
\epsilon_{1,2} = J \\
\epsilon_{3,4} = -J \pm 2\sqrt{D_x^2 + J^2}
$$

as well as the eigenvectors that are connected to them

$$
|\varphi_1\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \\
|\varphi_2\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle \\
|\varphi_3\rangle = -\frac{1}{\sqrt{2}}\sin(\theta_1)|00\rangle - \frac{i}{\sqrt{2}}\cos(\theta_1)|01\rangle + \frac{i}{\sqrt{2}}\cos(\theta_1)|10\rangle + \frac{1}{\sqrt{2}}\sin(\theta_1)|11\rangle \\
|\varphi_4\rangle = -\frac{1}{\sqrt{2}}\sin(\theta_2)|00\rangle + \frac{i}{\sqrt{2}}\cos(\theta_2)|01\rangle - \frac{i}{\sqrt{2}}\cos(\theta_2)|10\rangle + \frac{1}{\sqrt{2}}\sin(\theta_2)|11\rangle
$$

where $\theta_{1,2}$ are determined by the formula

$$
\theta_{1,2} = \arctan \left( \frac{D_x}{\sqrt{D_x^2 + J^2}} \right)
$$

(5)

After establishing the spectrum of our system, getting the density matrix is a straightforward procedure. The density matrix $\rho(T)$ may represent the state of a system at a given temperature $T$ when it is in thermal equilibrium. Indeed, the expression for $\rho(T)$ is as follows:

$$
\rho(T) = \frac{1}{Z} e^{-\beta H}
$$

(6)

where

$$
Z = \text{Tr} e^{-\beta H}
$$

(7)

Where $Z$ denotes the canonical ensemble partition function and $\beta = \frac{1}{k_B T}$ indicates the inverse thermodynamic temperature, where $k_B$ represents the Boltzmann’s constant, which is taken as unity in the following for the sake of simplicity. That will be succeeded by using the spectral decomposition of the Hamiltonian (2), which enables the thermal density matrix $\rho(T)$ to be expressed as

$$
\rho(T) = \frac{1}{Z} \sum_{l=1}^{4} e^{-\beta\epsilon_l} |\varphi_l\rangle \langle \varphi_l|
$$

(8)

It is possible to describe the density matrix of the system in the standard computational basis, as stated before in thermal equilibrium, by putting the equations (3) and (4) into the equation (8) and
getting the result

\[ \rho(T) = \frac{1}{Z} \begin{pmatrix} a & i\mu & i\nu & c \\ -i\mu & b & d & -i\nu \\ -i\nu & d & b & -i\mu \\ c & i\nu & i\mu & a \end{pmatrix} \]  

(9)

the elements matrix is represented by the equations

\[ a = \frac{e^{-\beta\epsilon}}{2} + \frac{1}{2} e^{-\beta\epsilon_3} \sin^2(\theta_1) + \frac{1}{2} e^{-\beta\epsilon_4} \sin^2(\theta_2) \]  

(10)

\[ b = \frac{e^{-\beta\epsilon}}{2} \cos^2(\theta_1) + \frac{1}{2} e^{-\beta\epsilon_3} \cos^2(\theta_2) \]

\[ c = \frac{1}{2} e^{-\beta\epsilon_3} \sin(\theta_1) \cos(\theta_1) - \frac{1}{2} e^{-\beta\epsilon_4} \sin(\theta_2) \cos(\theta_2) \]

\[ d = \frac{1}{2} e^{-\beta\epsilon_3} \cos(\theta_2) - \frac{1}{2} e^{-\beta\epsilon_4} \sin(\theta_1) \cos(\theta_1) \]

Consequently, the partition function is clearly described by

\[ Z = 2e^{-\beta J} + 2e^{\beta J} \cosh \left( 2\beta \sqrt{D_x^2 + J^2} \right) \]  

(11)

After determining all previously required ingredients, the next part will examine thermal entanglement teleportation by analyzing output state concurrence and the fidelity of entanglement teleportation.

3 Thermal Entanglement teleportation

3.1 Concurrence of output state

It is possible to think of the thermal mixed state in the Heisenberg spin chain as a generic depolarizing channel for the entanglement teleportation of a two-qubit system as a general depolarizing channel. In this section, we will investigate Lee and Kim’s two-qubit teleportation protocol \((P_1)\), and we will make use of two copies of the two-qubit thermal state, \(\rho(T) \otimes \rho(T)\), described above as resource \([37]\). Entanglement teleportation for the mixed channel of an entangled input state is similar to regular teleportation. The entangled input state is destroyed, and its replica state emerges at the distant location after applying a local measurement in linear operators. As input, we take a two-qubit in the peculiar pure state and process it. Without losing generality, let us assume that the input state is

\[ |\psi_{in} \rangle = \cos(\frac{\theta}{2})|10\rangle + \sin(\frac{\theta}{2})|01\rangle, \quad (0 \leq \theta \leq \pi). \]  

(12)

The density matrix associated with \( |\psi_{in} \rangle \) has the following representation:

\[ \rho_{in} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sin^2(\frac{\theta}{2}) & \frac{1}{2} \sin(\theta) & 0 \\ 0 & \frac{1}{2} \sin(\theta) & \cos^2(\frac{\theta}{2}) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]  

(13)
As a result, the initial state concurrence is

$$C_{in} = \sin(\theta)$$  \hspace{1cm} (14)

A combined measurement and local unitary transformation on the input state $$\rho_{in}$$ yields the output state $$\rho_{out}$$. Consequently, the output state is as follows [38]

$$\rho_{out} = \sum_{n,m} p_{nm}(\sigma^n \otimes \sigma^m)\rho_{in}(\sigma^n \otimes \sigma^m),$$  \hspace{1cm} (15)

which are denoted by the indexes $$n, m = 0, x, y, z \ (\sigma^0 = I)$$, and $$\rho_{channel}$$ indicate the state of the channel that is being utilized for teleportation, $$p_{nm}$$ reflects the probability of successful teleportation as specified by

$$p_{nm} = \text{Tr}[E^n \rho_{channel}]\text{Tr}[E^m \rho_{channel}]$$  \hspace{1cm} (16)

where $$\sum_{n,m} p_{nm} = 1$$, and the quantities $$E^{n,m}$$ are determined by the following:

$$E^0 = |\Psi^-(\cdot)|$$  \hspace{1cm} (17)

$$E^1 = |\Phi^-(\cdot)|$$

$$E^2 = |\Phi^+(\cdot)|$$

$$E^3 = |\Psi^+(\cdot)|$$

where the Bell states that they are

$$|\Psi^\pm\rangle = \frac{(|01\rangle \pm |10\rangle)}{\sqrt{2}}$$  \hspace{1cm} (18)

$$|\Phi^\pm\rangle = \frac{(|00\rangle \pm |11\rangle)}{\sqrt{2}}$$

The quantum channel state of the two-qubit spin system is $$\rho_{channel} = \rho_T$$, which is provided in the equation (9), and therefore one may derive $$\rho_{out}$$ as the output of the quantum channel

$$\rho_{out} = \frac{1}{Z^2}\left(\begin{array}{ccc}
\omega & 0 & 0 \\
0 & A^+ & B \\
\chi & 0 & 0 \\
\end{array}\right)$$  \hspace{1cm} (19)

where

$$\omega = 4ab$$  \hspace{1cm} (20)

$$\chi = 4cd\sin(\theta)$$

$$A^\pm = 2\left(\pm(a^2 - b^2)\cos(\theta) + a^2 + b^2\right)$$

$$B = 2\left(c^2 + d^2\right)\sin(\theta)$$

Following the procedure outlined above, we can evaluate the concurrence of the output state in order to establish the amount of entanglement associated with it by computing the square roots of the output state.

$$R = \rho_{out}S\rho_{out}^S$$  \hspace{1cm} (21)
where $\rho_{\text{out}}$ has a complex conjugate in the form of $\rho^*_{\text{out}}$ and $S$ is calculated using the equation

$$S = \sigma^y \otimes \sigma^y$$

(22)

where $\sigma^y$ indicates the Pauli matrix and the $R$ matrix may be derived by a straightforward computation by

$$R = \frac{1}{Z^4} \begin{pmatrix} 
\chi^2 + \omega^2 & 0 & 0 & 2\chi \omega \\
0 & A^- A^+ + B^2 & 2A^+ B & 0 \\
0 & 2A^- B & A^- A^+ + B^2 & 0 \\
2\chi \omega & 0 & 0 & \chi^2 + \omega^2 
\end{pmatrix}$$

(23)

Consider that $R$ measures the degree of equality between $\rho$ and $\rho^*$, which in turn indicates how closely $\rho$ approximates a mixture of generalized Bell states. Understand the meaning of $R$; we can consider the $\text{Tr} R$, which ranges from 0 to 1, measures the degree of equality between $\rho$ and $\rho^*$, which in turn indicates how closely $\rho$ approximates a mixture. The eigenvalues of $R$ are also invariant under local unitary transformations of the individual qubits, qualifying them for inclusion in an entanglement formula, which must itself be invariant under such transformations for entanglement to exist. Using the (20) and (23) functions, one can quickly verify that the following equations give the eigenvalues of the matrix $R$:

$$\lambda_{1,2} = \frac{16 (ab \pm cd C_{1m})^2}{Z^4}$$

(24)

$$\lambda_{3,4} = \frac{4 \left( \sqrt{(a^2 - b^2)^2 C_{1m}^2 + 4a^2 b^2} \pm (c^2 + d^2) C_{1m} \right)^2}{Z^4}$$

(25)

When trying to measure the degree of entanglement connected with $\rho$, we take into account the concurrence [15,16], which may be defined as follows:

$$C = \max \left[ 0, 2 \max (\lambda_1, \lambda_2, \lambda_3, \lambda_4) - \sum_{i=1}^{4} \lambda_i \right]$$

(26)

Consequently, we have obtained the concurrence expression for output state, which is implicitly reliant on three variables: the coupling constant $J$, the DM interaction’s x-components, and the temperature $T$. As a result, we now have all of the components necessary to investigate the behavior of our proposed system concerning the previously specified quantities.

3.2 The Fidelity of entanglement teleportation

The degree to which $\rho_{\text{in}}$ and $\rho_{\text{out}}$ are compatible describes the quality of the teleported state $\rho_{\text{out}}$. Whenever the input state is pure, the idea of fidelity may be used as a helpful indication of the teleportation performance of a quantum channel [39,40]. The maximal fidelity of $\rho_{\text{in}}$ and $\rho_{\text{out}}$ is defined as

$$F(\rho_{\text{in}}, \rho_{\text{out}}) = \text{Tr} \left[ \sqrt{(\rho_{\text{in}})^{\frac{1}{2}} \rho_{\text{out}} (\rho_{\text{in}})^{\frac{1}{2}}} \right]^2 = |\psi_{\text{in}}\rangle \rho_{\text{out}} |\psi_{\text{in}}\rangle.$$
By inserting $\rho_{\text{in}}$ and $\rho_{\text{out}}$ from the previous equation, we get

$$F(\rho_{\text{in}}, \rho_{\text{out}}) = \frac{2(a^2 - b^2 + c^2 + d^2)C_{\text{in}}^2 + 4b^2}{Z^2}$$

(28)

In order to make the preceding formula more understandable, we may write that the maximum fidelity $F(\rho_{\text{in}}, \rho_{\text{out}})$ is dependent on the initial entanglement $C_{\text{in}}$:

$$F(\rho_{\text{in}}, \rho_{\text{out}}) = h_1 + h_2C_{\text{in}}^2$$

(29)

where $h_1 = \frac{4b^2}{Z^2}$ and $h_2 = \frac{2(a^2 - b^2 + c^2 + d^2)}{Z^2}$. The coefficients $h_1$ and $h_2$ are simply related to the entanglement of the channel, although $h_1$ may be a positive value for Heisenberg chains, notwithstanding the Werner states. As a result of this, it is possible to send and receive more entangled starting states with greater accuracy across the same channel. The assertion is not constructive because when we choose settings for the channel such that $h_1 > 0$, $h_2$ decreases, and finally, $F(\rho_{\text{in}}, \rho_{\text{out}})$ becomes less than $\frac{2}{3}$, which suggests that the entanglement teleportation of a mixed state is inferior to classical communication. Consequently, the larger-entangled channel is necessary to provide the same fidelity as a smaller-entangled channel.

After establishing the concurrence of output state and fidelity, we will demonstrate the overall performance of the proposed model by devoting the following section to a numerical assessment of the $C_{\text{out}}$ and $F$, as detailed below. Following that, we provided a few plots in appropriate circumstances, and we will continue to discuss them.

4 Numerical results

This part of the study will quantitatively examine several properties of entanglement teleportation in a two-qubit Heisenberg XXX chain with an $x$-component of DM interaction. Indeed, we will investigate the output concurrence and fidelity as a function of the channel parameters under appropriate circumstances. Following that, the values $k_B = h = 1$ will be assumed for simplicity.

![Figure 1](https://example.com/figure1.png)

Figure 1 – (Color online) Entanglement of output state $C_{\text{out}}$ vs. the channel’s parameters and $C_{\text{in}}$. (a) $C_{\text{out}}$ vs. $C_{\text{in}}$ and $J$ where $D_x = T = 1$, (b) $C_{\text{out}}$ vs. $C_{\text{in}}$ and $D_x$ where $J = T = 1$, (c) $C_{\text{out}}$ vs. $C_{\text{in}}$ and $T$ for $J = D_x = 1$.

Plot 1 displays the behavior of $C_{\text{out}}$ for the channel parameters and $C_{\text{in}}$. In Fig. 1(a), we plotted $C_{\text{out}}$ versus $C_{\text{in}}$, and $J$ were $D_x = T = 1$. Under the constraint, $J > 0$, replicate state $C_{\text{out}}$ entanglement rises linearly as $C_{\text{in}}$ increases. $J$ designates the degree of this rise. However, when $J < 0$,
the concurrence $C_{out}$ is zero for small values of $J$. Fig. 1(b) indicates that $C_{out}$ is symmetrical for the value $D_x = 0$ for any value of $C_{in}$. Still, on the other hand, $C_{out}$ is greatest when the spin-orbit coupling through DM interaction is zero for $C_{in} = 1$, which generates the more entangled states of the system. Otherwise, when $|D_x|$ starts to rise, $C_{out}$ falls, and the impact of $C_{in}$ stays minimal and vice versa. However, for high values of the spin-orbit coupling $C_{out}$ goes towards zero; therefore, we can conclude in this instance that the impact of the spin-orbit coupling through DM interaction is more dominating than the concurrence of the input state. Fig. 1(c) demonstrates that for $T = 0$ and $C_{in} = 1$, the concurrence of output state is most significant, which renders the states of the system more entangled, so when there is a rise in the temperature or deterioration of $C_{in}$ we notify a decrease of $C_{out}$. Furthermore, when $C_{in} \to 0$ for whichever value of $T$ or $T \to \infty$ and any $C_{in}$, $C_{out}$ tending towards zero, i.e., the states in this circumstance become separable and not entangled. To summarize, the separability of the states needs a high-temperature regime or a significant spin-orbit coupling through DM interaction or a ferromagnetic chain while obeying the characteristics mentioned above, such that the channel becomes disentangled. Entanglement teleportation is impractical. But even for an antiferromagnetic chain, weak spin-orbit coupling through DM interaction, or low-temperature, the system states get more entangled. Consequently, the channel becomes entangled. Entanglement teleportation is achievable in this circumstance.

**Figure 2** – (Color online) (a) Entanglement of output state $C_{out}$ vs. the channel’s parameters and $C_{in}$. $C_{out}$ vs. $J$ and $D_x$ where $C_{in} = T = 1$, (b) $C_{out}$ vs. $T$ and $D_x$ where $C_{in} = J = 1$, (c) $C_{out}$ vs. $J$ and $T$ for $C_{in} = D_x = 1$.

Fig. 2 displays the behavior of $C_{out}$ in response to the channel parameters. In Fig. 2(a) we plot the concurrence of output state as a function of $J$ and $D_x$ for the values $C_{in} = T = 1$, which is noted that for $J < 0$, $C_{out}$ stays zero in interval $-1.5 < D_x < 1.5$, although for large values of $|D_x|$, $C_{out}$ stays small, but in the other hand for $J > 0$, the concurrence $C_{out}$ grows as $J$ increases and diminishes when $D_x$ raises. The Fig. 2(b) the concurrence of output state is plotted as a function of $J$ and $D_x$, for $C_{in} = J = 1$, in these conditions, we find that $C_{out}$ stays zero of the large values of $D_x$ or $T$, which illustrates the separability states in this region. However, at the value $D$ and low temperatures, $C_{out}$ becomes maximal, and the system’s states become entangled. In Fig 2(c), we displayed the concurrence of the out state as a function of $J$ and $T$ by setting $C_{in} = D_x = 1$, so for the value $D_x = 0$ and at low temperature, $C_{out}$ reaches maximal, and the states of the system become entangled. We have seen that $C_{out}$ stays zero for large values of $T$ and $J = 0$, and that the system gets increasingly entangled.
at low temperatures and as \(|J|\) grows. Then we may deduce that the DM interaction and temperature control the state’s system. Large values of \(D_x\) and high temperatures make the system less entangled. The channel gets disentangled, and entanglement teleportation is infeasible.

Graph 3 shows how \(F\) behaves concerning the channel parameters and \(C_{in}\). We plotted fidelity vs. \(C_{in}\) and \(T\) for fixed values of \(J = D_x = 1\). \(3(a)\), it is obvious that \(F\) stays practically constant for certain temperatures, and this constant declines as \(T\) increases. Otherwise, even if the concurrence of the input state changes, the fidelity is optimum for \(T = 0\). Physically, for the value \(T = 0\), the fidelity \(F\) becomes more than 0.6, thus making the channel suitable for performing the teleportation protocol. In figure 3(b), the fidelity \(F\) is shown as a function of \(C_{in}\) and \(J\) for fixed values of \(T = D_x = 1\). Similar statements on the impact of \(C_{in}\) on the fidelity that the previous figure, on the contrary, for \(J > 0\) and particularly for high values of \(J > 0\) the fidelity tends towards 1, which implies that the system is more entangled and the protocol of the teleportation become practical, furthermore, in the scenario where \(J < 0\), the fidelity \(F\) stays around \(0 < C_{in} < 1\), making the system states more separable. As a result, the protocol of teleportation for the proposed system is completely impractical. The fidelity \(F\) is displayed as a function of \(C_{in}\) and \(D_x\) for fixed values of \(T = J = 1\), in Fig. 3(c). We see that the fidelity reaches a maximum of \(D_x = 0\), where the teleportation protocol becomes highly essential. When \(D_x\) is increased, the fidelity decreases to a small amount, and the impact of \(C_{in}\) is insignificant.

Fig. 4 demonstrates how \(F\) behaves to the channel parameters. Fig. 4(a) shows the fidelity in terms of \(J\) and \(D_x\) for fixed values of \(C_{in} = T = 1\). In the plot 4(a), the fidelity \(F\) always trends towards 1 for positive values of \(J\), but when \(J\) turns negative, \(F\) takes a small value on the order of 0.2. On either side, for \(D_x\) farther from zero, there are two observations to make: the first is that for \(J > 0\), the fidelity decreases with rising \(|D_x|\), and the second is that for \(J < 0\), the fidelity grows concurrently with growing \(|D_x|\). Concerning the Figs. 4(b) and 4(c) supplement graphs 2(b) and 2(c) with almost identical findings. In conclusion, the graphs drawn for fidelity confirm all the previous remarks, and there is conformity with all that was found for the concurrence of output state.
Figure 4 – The fidelity \( F \) vs. the channel's parameters and \( C_{in} \). (a) \( F \) vs. \( J \) and \( D_x \) where \( C_{in} = T = 1 \), (b) \( F \) vs. \( T \) and \( D_x \) where \( C_{in} = J = 1 \), (c) \( F \) vs. \( T \) and \( J \) for \( C_{in} = D_x = 1 \).

5 Summary

Concurrence, including entanglement and teleportation protocol fidelity, are explored in a two-qubit Heisenberg XXX chain with x-components of DM interaction. The Hamiltonian model is presented, and the eigenstates of entanglement have been discovered using mathematical calculations, as well as the thermal state at a finite temperature is explicitly derived. After obtaining concurrence and fidelity expressions in terms of the spin’s coupling constant \( J \), the x-components of DM interaction, and the temperature \( T \), the numerical behavior of the concurrence measured entanglements, and we have investigated the fidelity of the protocol teleportation for our study. Additionally, investigations have been conducted on the ferromagnetic and antiferromagnetic phases. We determined in this research that temperature \( T \), the x-component of the DM interactions, and the quality of teleportation may all have a role in defining the degree of intricacy between the states to a greater or lesser extent. Furthermore, it is reasonable to deduce from these findings that the separability of the states requires either a high-temperature regime or a significant spin-orbit coupling through DM interaction or a ferromagnetic chain that exhibits the features above. Teleportation by entanglement is unfeasible. However, even in the presence of an antiferromagnetic chain, weak spin-orbit coupling through DM interaction, or low temperature, the system states become more entangled. As a result, the channel gets entangled, so the teleportation protocol is possible and achievable.

Acknowledgment

Thank you to Mohamed Monkad, director of the Laboratory for Physics of Condensed Matter (LPMC) at Chouaib Doukkali University’s Faculty of Sciences, for his invaluable support.

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