The analytic classification of plane curves with two branches

A. Hefez · M. E. Hernandes · M. E. Rodrigues Hernandes

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Abstract In this paper we solve effectively the problem of local analytic classification of
plane curves singularities with two branches. We present normal forms for these singularities
and show how to reduce them to their normal forms. This is accomplished by introducing
a new analytic invariant that relates vectors in the tangent space to the orbits under analytic
equivalence in a given equisingularity class to Kähler differentials on the curve.

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1 Introduction

Our objects of study are germs of reduced plane analytic curves \((f) : f = 0\), where \(f\) is
a reduced element in \(\mathbb{C}\{X, Y\}\), the ring of power series in two variables over the complex
numbers which are convergent close to the origin of \(\mathbb{C}^2\). Our problem is the classification
of these objects under contact equivalence, when \(f\) and \(g\) are considered equivalent, writing
\((f) \sim (g)\), if and only if there exist \(\Phi \in Aut(\mathbb{C}\{X, Y\})\) and a unit \(u\) in \(\mathbb{C}\{X, Y\}\) such
that \(\Phi(f) = ug\). This is what is usually meant by analytic classification of plane curve
singularities.

The aim of this work is to continue the program of analytic classification of germs of plane
curves, initiated by Ebey [4] and Zariski [11] in the sixtieth of last century and solved by

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the first two authors in the irreducible case in [8]. In the reducible (but reduced) case, some results in very special situations were already known: Kolgushkin and Sadykov [9] obtained normal forms for stably simple reducible curves, Genzmer and Paul [7] present normal forms for curves with smooth and transversal branches. Here we solve the classification problem for general curves with two branches. A major step for passing from the irreducible to the reducible case is the introduction, for a plane curve \((f)\) with \(r\) branches, of the set \(\Lambda \subset N^r\) whose elements have components equal to the values of a Kähler differential on \((f)\) with respect to the valuations given by each of its branches. This set is an analytic invariant for plane curves with \(r\) branches and we relate it with tangent vectors to orbits of multigerms as we will explain in due time. We believe that the general case will follow along these lines after we overcome some non-trivial combinatorial difficulties.

From now on, we will assume that \(f\) has two irreducible components \(f_1\) and \(f_2\). Each branch \((f_i)\) admits a parametrization \(\phi_i: (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)\). We will use coordinates \(t_1\) and \(t_2\) on \((\mathbb{C}, 0)\) (one for each \(\phi_i\)) and coordinates \(x, y\) on \((\mathbb{C}^2, 0)\) (the same for both). Now, because each branch is invariant by changes of coordinates in the source of the \(\phi_i\), and the curve is analytically invariant by any automorphism of \((\mathbb{C}^2, 0)\) (the same automorphism for both branches), we easily conclude that contact equivalence for curves \((f)\) is translated into permutation of the two branches and changes of analytic coordinates in the source and in the target, i.e., \(S_2 \times A\)-equivalence on the space \(B\) of bigerm\(s\) \(\phi = [\phi_1, \phi_2]\), where \(A\) is the group \(Aut(\mathbb{C}[t_1]) \times Aut(\mathbb{C}[t_2]) \times Aut(\mathbb{C}[X, Y])\) acting on \(B\) as follows:

\[
(\rho_1, \rho_2, \sigma) \cdot \phi = \left[\sigma \circ \phi_1 \circ \rho_1^{-1}, \sigma \circ \phi_2 \circ \rho_2^{-1}\right].
\]

Our analysis will be split into two cases, namely, whether the two components of \((f)\) have distinct tangents or equal tangents. In what follows, we will denote by \(m_i\) the multiplicity of \(f_i, i = 1, 2\).

Case (1) Distinct tangents. In this case, by \(A\)-equivalence, we may assume that the tangent of the first component is \((Y)\) and of the second one is \((X)\), so that

\[
\phi_i = (x(t_i), y(t_i)), \text{ where } \ord_{t_i}x(t_1) < \ord_{t_1}y(t_1) \text{ and } \ord_{t_2}x(t_2) > \ord_{t_2}y(t_2).
\]

Case (2) Same tangent. In this case, by \(A\)-equivalence, we may assume that the common tangent is \((Y)\), in which case, \(\phi_i = (x(t_i), y(t_i))\) with \(\ord_{t_i}x(t_i) < \ord_{t_i}y(t_i), i = 1, 2\).

The main result of this work is Theorem 8 that describes normal forms of bigerms with respect to \(A\)-action with a given analytic invariant \(\Lambda\), which we summarize below.

**Theorem** Any bigerm \(\phi\) with given analytic invariant \(\Lambda\) is \(A\)-equivalent to one in the following form:

**Distinct tangents case**

(a) \(\left[\left(t_1^{m_1}, t_1^{j_1}\right) + t_1^k + \sum_{j \in J} a_{1j} t_1^j, \left(\sum_{l \in L} a_{2l} t_2^l, t_2^{m_2}\right)\right]\);

(b) \(\left[\left(t_1^{m_1}, t_1^{j_1}\right), \left(t_2^{j_2}, t_2^{m_2}\right)\right]\);

(c) \(\left[\left(t_1^{m_1}, t_1^{j_1}\right), (0, t_2)\right]\);

(d) \(\left[(t_1, 0), (0, t_2)\right]\).

**Same tangent case**

(a') \(\left[\left(t_1^{m_1}, t_1^{j_1}\right) + t_1^k + \sum_{j \in J} a_{1j} t_1^j, \left(t_2^{m_2}, \sum_{l \in L} a_{2l} t_2^l\right)\right]\);

(b') \(\left[\left(t_1^{m_1}, t_1^{j_1}\right), \left(t_2^{m_2}, t_2^{j_2}\right)\right]\) with \(m_1 \neq m_2\) or \(j_1 \neq j_2\);
We will perform the analytic classification keeping fixed the tangent cone of \((f)\) which we assume to be \((XY)\) in the case of distinct tangents and \((Y^2)\) in the case of equal tangents. To describe the elements of \(A\) that preserve the tangent cone of the bigerm, as described above, it is convenient to introduce the subgroup \(\mathcal{H}\) of \(A\) of homotheties:

\[
\mathcal{H} = \{(\rho_1, \rho_2, \sigma) \in A; j^1\rho_i = a_i t_i, j^1\sigma = (ax, by), a_i, a, b \in \mathbb{C}, \quad i = 1, 2\},
\]

where \(j^k\xi\) is the \(k\)-th jet of any \(n\)-tuple \(\xi\) of power series.

Now, the elements of \(A\) we are looking for are the compositions \(h \circ g\), where \(h \in \mathcal{H}\) and \(g\) belongs to

\[
\tilde{A}_1 = \{(\rho_1, \rho_2, \sigma) \in A; j^1\rho_i = t_i, i = 1, 2, j^1\sigma = (x + by, y), b \in \mathbb{C}\},
\]

if we have same tangent, or \(g\) belongs to the classical group \(A_1 = \{(\rho_1, \rho_2, \sigma) \in \tilde{A}_1; b = 0\},\)

if we have distinct tangents.

The strategy we use for our classification is to first analyze the action of \(A_1\), or \(\tilde{A}_1\), on the elements of \(B\), according they belong, respectively, to Case 1 (distinct tangents), or to Case 2 (same tangent), and then to take into account the homotheties.

To find distinguished representatives in each case, under the action of the corresponding group, we will use the following version of Complete Transversal Theorem (cf. [1]).

**The Complete Transversal Theorem (CTT)** Let \(G\) be a Lie group acting on an affine space \(k\) with underlying vector space \(V\) and let \(W\) be a subspace of \(V\). Suppose that \(v \in V\) is such that \(T_G(v + w) = T_G(v)\), \(\forall w \in W\), where the notation \(T_G(z)\) means the tangent space at \(z\) of the orbit \(G(z)\), as vector subspace of \(V\). If \(W \subset T_G(v)\), then \(G(v + w) = G(v)\), \(\forall w \in W\).

In order to obtain normal forms we apply the above theorem noting that if \(W \subset T_G(v)\) then \(v + w\) is \(G\)-equivalent to \(v\).

We denote by \(B^k\) the vector space of \(k\)-jets of elements of \(B\) and by \(G^k\) the Lie group of \(k\)-jets of elements of \(G\), where \(G\) is one of the groups \(A_1\) or \(\tilde{A}_1\).

We will show, in the next proposition, that the hypothesis of (CTT) holds for any element \(j^k\phi \in B^k\) which is a bigerm as in Case 2, for \(W = H^k\phi\) the subspace of homogeneous elements of degree \(k\) of \(B^k\) such that the two components, as a bigerm, of the elements in \(j^k\phi + H^k\phi\) have all same multiplicity and same tangent, that is,

\[
H^k\phi = \left\{ [(a_1t_1^k, b_1t_1^k), (a_2t_2^k, b_2t_2^k)] \in B^k; \quad a_i = b_i = 0, \text{ if } k < m_i, \quad i = 1, 2 \right\}.
\]
We describe below the elements of the tangent spaces to the orbit of \( j^k\phi \) in \( B^k \) under the actions of the groups \( A^k_i \) and \( \tilde{A}^k_1 \):

\[
j^k\left[ (\phi_{i1}\epsilon_1 + \eta_1(\phi_1), \phi_{i2}\epsilon_1 + \eta_2(\phi_1)), (\phi_{i1}'\epsilon_2 + \eta_1(\phi_2), \phi_{i2}'\epsilon_2 + \eta_2(\phi_2)) \right],
\]

where \( \phi_i = (\phi_{i1}, \phi_{i2}) \), the (') sign means derivative with respect to the corresponding parameter, \( \epsilon_i \in (t_i)^2\mathbb{C}[t_i] \), \( i = 1, 2 \), \( \eta_2 \in (x, y)^2\mathbb{C}[x, y] \) and

(a) \( \eta_1 \in (x, y)^2\mathbb{C}[x, y] \), in the \( A^k_2 \) case, or

(b) \( \eta_1 \in (x^2, y)^2\mathbb{C}[x, y] \), in the \( \tilde{A}^k_2 \) case.

The case of \( A^k_1 \) is classically known (cf. \([6]\)) and the other one can be computed in a similar way.

**Lemma 1** If \( \phi \in B^k \) as in Case 2, \( (\rho_1, \rho_2, \sigma) \in \tilde{A}^k_1 \), with \( j^1\sigma = (x + by, y) \), and \( \psi \in H^k_\phi \), then

\[
j^k[(\rho_1, \rho_2, \sigma) \cdot (\phi + \psi)] = j^k[(\rho_1, \rho_2, \sigma) \cdot \phi] + \psi + \theta,
\]

where \( \theta = [(bc_{11}t^k_1, 0), (bc_{22}t^k_2, 0)] \), with \( b, c_1, c_2 \in \mathbb{C} \), depending only upon \( \psi \).

The proof is straightforward, following easily from the definitions.

**Proposition 2** If \( \phi \in B^k \) as in Case 2 and \( \psi \in H^k_\phi \), then

\[
T\tilde{A}^k_1(\phi + \psi) = T\tilde{A}^k_1(\phi).
\]

**Proof** Recall that \( T\tilde{A}^k_1(\psi) \) of an element \( \psi \in B^k \) is given by the image of the differential at the identity \( I \) of the map \( \Phi_\psi : \tilde{A}^k_1 \to \tilde{A}^k_1(\psi) \), \( \Phi_\psi(g) = g \cdot \psi \).

Therefore, any vector in \( T\tilde{A}^k_1(\psi) \) is of the form \( (\Phi_\psi \circ \lambda)'(0) \), where \( \lambda : (-\alpha, \alpha) \to \tilde{A}^k_1 \), \( \lambda(u) = (\rho_{1u}, \rho_{2u}, \sigma_u) \) is a curve in \( \tilde{A}^k_1 \) such that \( \lambda(0) = I \). Notice that since \( \lambda(u) \in \tilde{A}^k_1 \), then \( j^1\sigma_u = (x + bu(y), y) \).

As a consequence of the above discussion, and from the previous lemma, we have that

\[
(\Phi_\phi \circ \lambda)'(0) = \lim_{u \to 0} \frac{\lambda(u) \cdot (\phi \circ \psi) - \lambda(0) \cdot (\phi \circ \psi)}{u} = \lim_{u \to 0} \frac{\lambda(u) \cdot \phi + \psi + \theta(u) - \phi - \psi}{u} = (\Phi_\phi \circ \lambda)'(0) + \lim_{u \to 0} \frac{\theta(u)}{u} = (\Phi_\phi \circ \lambda)'(0) + \theta'(0),
\]

where \( \theta(u) = [(b(u)c_{11}t^k_1, 0), (b(u)c_{22}t^k_2, 0)] \), with \( b(0) = 0 \) and \( \theta(0) = 0 \), since \( \lambda(0) = I \).

Taking in the description of the tangent spaces to the orbits in \( B^k \) under the \( \tilde{A}^k_1 \)-action, \( \eta_i = 0 \) and \( \epsilon_i(t_i) = b'(0)c_{ii}t^k_i - m_i + 1, i = 1, 2 \), one may easily check that \( \theta'(0) \in T\tilde{A}^k_1(\phi + \psi) \cap T\tilde{A}^k_1(\phi), \forall \psi \in H^k_\phi \). \( \square \)

Notice that our proof may be, without any extra effort, extended to multigerms, and contains as an immediate corollary the result for the \( A^k_2 \)-action (just take \( b = 0 \)). Observe also that the result may be used to make substantial simplifications in the arguments in Section 5 of \([8]\).

Notice that if \( \phi \) and \( \varphi \) are \( G^k \)-equivalent, where \( G \) is \( A_1 \) or \( \tilde{A}_1 \), then \( H^k_\phi = H^k_\varphi \). Moreover if \( \phi \in B \) then for all \( \psi \in H^k_\phi \) we have that \( \phi + \psi \) preserve the tangent cone of \( \phi \).
### 3 $G$-Normal forms

Given an element $\phi \in B$, we are looking for elements $\psi \in H^k_\phi$ such that $j^k(\phi + \psi) = j^{k-1}\phi$ and $\phi + \psi$ is $G^k$-equivalent to $\phi$. So that in this way we will be able to eliminate terms of order $k$ in $\phi$ without changing neither its $k-1$ jet nor its equivalence class. From the (CTT) it is sufficient to verify when an element $\psi \in H^k_\phi$ belongs to the tangent space to the orbit of $\phi$ under the action of the group $G^k$. Similarly, as in the proof of Proposition 2 we get that

$$
\left[ (a_1^k, 0), (0, b_2^k) \right] \in H^k_\phi \cap T A^k_\lambda(\phi) \text{ (distinct tangents case)}
$$

$$
\left[ (a_1^k, 0), (b_2^k, 0) \right] \in H^k_\phi \cap T A^k_\lambda(\phi) \text{ (same tangent case)},
$$

with $a, b \in \mathbb{C}$. With the above considerations, we have that any bigerm is $A$-equivalent to a bigerm $\phi = [\phi_1, \phi_2]$ in Puiseux form, that is, $\phi_1 = (t_1^{m_1}, \sum_{i>m_1} a_it_i^1)$ and

\begin{align*}
\text{Case (1) Distinct tangents: } & \phi_2 = (\sum_{i>m_2} a_{2i}t_2^i, t_2^{m_2}) ; \\
\text{Case (2) Same tangent: } & \phi_2 = (t_2^{m_2}, \sum_{i>m_2} a_{2i}t_2^i).
\end{align*}

The pair $m = (m_1, m_2)$ will be referred to as the multiplicity of the bigerm $\phi$.

In order to get more refined parametrizations for a bigerm we have to impose some restriction on it. This is done by fixing analytic invariants.

As a first invariant we consider the semigroup of values

$$
\Gamma = \{ v(\eta) := (v_1(\eta), v_2(\eta)) ; \; \eta \in \mathbb{C}[x, y] \},
$$

where $v_i(\eta) = ord_{t_i}(\eta \circ \phi_i), i = 1, 2$. This invariant characterizes completely the topological type of the curve as an immersed germ at the origin of the plane (cf. [10] or [5]). Two curves having same $\Gamma$ invariant are called equisingular.

Fixing the semigroup of values, which determines the intersection index of the two branches of the curve, we are fixing the contact order of their parametrizations. This will imply the coincidence of the coefficients of the Puiseux expansions of the branches up to the order of contact minus 1. On the other hand, since $\Gamma$ has a conductor $(c_1, c_2)$, we may eliminate analytically all terms in both parametrizations with order greater than $c - 1$, where $c = \max\{c_1, c_2\}$, without affecting the preceding terms (cf. [5]). This tells us that we have simultaneous finite determinacy of both parametrizations and gives us a finite dimensional space of parameters $\Sigma_\Gamma$ for a complete set of analytic representatives in the equisingularity class determined by $\Gamma$.

With the semigroup $\Gamma$, we get only a rough normal form for bigerms. In order to refine this normal form, we will use the finer analytic invariant

$$
\Lambda = \{ v(\omega) := (v_1(\omega), v_2(\omega)) ; \; \omega \in \mathbb{C}[x, y]dx + \mathbb{C}[x, y]dy \},
$$

where $\omega = \eta_1dx + \eta_2dy$ with $\eta_i \in \mathbb{C}[x, y], i = 1, 2$, we define $v_i(\omega) := ord_{t_i}(\omega \circ \phi_i) + 1 = ord_{t_i}(\eta_1(\phi_i)\phi_i^1 + \eta_2(\phi_i)\phi_i^2) + 1$.

The fact that $\Lambda$ is an analytic invariant is clear since by its definition it is independent from reparametrizations of the branches and change of coordinates in $\mathbb{C}^2$. From the definition it also follows that $\Gamma \setminus \{(0, 0)\} \subset \Lambda$.

It is easy to check that the set $\Lambda$ has the following properties:

(A) If $(a_1, a_2), (b_1, b_2) \in \Lambda$, are such that $a_1 < b_1$ and $a_2 > b_2$, then $(a_1, b_2) \in \Lambda$. 

\[ \text{Springer} \]
(B) If \((a_1, a_2), (a_1, b_2) \in \Lambda\), then there exists \((a, \min\{a_2, b_2\}) \in \Lambda\) with \(a > a_1\). The same

This is sufficient to guarantee that \(\Lambda\) behaves combinatorially as \(\Gamma\), except that it is not

In \(\Lambda\) there is a finite subset \(M\) of points \((k_1, k_2)\), called the maximal points of \(\Lambda\), contained in the rectangle with sides parallel to the axes and opposite vertexes the origin

and respectively called the vertical and the horizontal fibers of \((a_1, a_2)\).

In particular, the set \(\Lambda\) is determined by the sets of values of differentials \(\Lambda_1, \Lambda_2\) of the branches of the curve and the maximal points of \(\Lambda\) (cf. [5] or [3], in the case of the set \(\Gamma\)).

This implies that there are finitely many possibilities for sets of values of differentials \(\Lambda\) for each equisingularity class of curves.

There is a tight connection between the tangent space to the orbit of a bigerm \(\phi\) under the action of the group \(G\) (\(A_1\) or \(A_1\)) and the set

\[\Lambda_G = \{v(\omega) - m; \omega \in \Omega_G\} \subset \Lambda - m,\]

where \(m = (m_1, m_2)\) is the multiplicity of the bigerm \(\phi\),

\[\Omega_{A_1} = \{\eta_1 dx + \eta_2 dy; \eta_1, \eta_2 \in (x, y)^2 \mathbb{C}[x, y]\}\]

and

\[\Omega_{A_1} = \{\eta_1 dx + (\beta y + \eta_2) dy; \eta_1, \eta_2 \in (x, y)^2 \mathbb{C}[x, y], \beta \in \mathbb{C}\}.\]

The same argument used for \(\Lambda\) shows that the set \(\Lambda_G\) is an invariant with respect to the action

of the group \(G\). For each fixed semigroup of values \(\Gamma\) there exist finitely many possibilities

for \(\Lambda_G\). The finite dimensional space that parametrizes the bigerms in Puiseux form with

fixed \(\Gamma\) and \(\Lambda_G\) will be denoted by \(\Sigma_{\Gamma, \Lambda_G}\), which we will identify with the set of the bigerms

that they determine.

**Proposition 3** Let \(\phi = [\phi_1, \phi_2] \in \Sigma_{\Gamma, \Lambda_G}\). Given \(h_i \in \mathbb{C}[t_i], i = 1, 2,\) we have that

1. \([0, h_1], (-h_2, 0)] \in T.A_1^k(\phi);\)
2. \([0, h_1], (0, h_2)] \in T.A_1^k(\phi),\]

if and only if there exists \(\omega \in \Omega_G\) such that \(h_i = j^k \frac{\omega(\phi)}{m_i t_i^{m_i - 1}}\).

**Proof** We prove the result only for \(G = A_1\), since the other case, i.e., \(G = A_2\) is similar.

If \([0, h_1], (0, h_2)] \in T.A_1^k(\phi),\) then from Eq. (1) there exist \(\epsilon_i \in (t_i^2)\mathbb{C}[t_i], i = 1, 2, \eta, \eta_2 \in (x, y)^2 \mathbb{C}[x, y], \eta_1 = \beta y + \eta\) with \(\beta \in \mathbb{C}\) such that

\[0 = \phi_{11}' \cdot \epsilon_1 + \eta_1(\phi_1) \mod t_1^{k+1},\]

\[h_1 = \phi_{12}' \cdot \epsilon_1 + \eta_2(\phi_1) \mod t_1^{k+1},\]

\[0 = \phi_{21}' \cdot \epsilon_2 + \eta_1(\phi_2) \mod t_2^{k+1},\]

\[h_2 = \phi_{22}' \cdot \epsilon_2 + \eta_2(\phi_2) \mod t_2^{k+1},\]

that is, \(j^k \epsilon_1 = -j^k \frac{\eta_1(\phi_1)}{\phi_{11}'}\) and \(j^k \epsilon_2 = -j^k \frac{\eta_1(\phi_2)}{\phi_{21}'}\). So,

\[h_1 = \frac{\eta_2(\phi_1) \phi_{11}' - \eta_1(\phi_1) \phi_{12}'}{m_1 t_1^{m_1 - 1}} \mod t_1^{k+1}\]

and

\[h_2 = \frac{\eta_2(\phi_2) \phi_{21}' - \eta_1(\phi_2) \phi_{22}'}{m_2 t_2^{m_2 - 1}} \mod t_2^{k+1}.\]
Defining \( \omega = \eta_2 dx - \eta_1 dy \in \Omega_{\tilde{A}_1} \), we have that \( h_i = j^k_{(\omega(\phi)_i)/m_i} \), \( i = 1, 2 \).

Conversely, given \( \omega = g_2 dx + g_1 dy \in \Omega_{\tilde{A}_1} \) where \( g_1 = \beta y + h \) with \( h, g_2 \in (x, y)^2 \mathbb{C}[x, y] \) and \( \beta \in \mathbb{C} \), consider \( \eta_1 = -g_1, \eta_2 = g_2, \epsilon_1 = \frac{g_1(\phi)}{m_i} \in (t_1)^2 \mathbb{C}[t_1] \) and \( \epsilon_2 = \frac{g_1(\phi)}{m_2 t_2^2} \). So, from Eq. (1), we have that \([0, (h_1), (0, h_2)] \in T\tilde{A}_1^k(\phi)\), where \( h_i = j^k_{(\omega(\phi)_i)/m_i} \), \( i = 1, 2 \).

In the sequel we will need the notions of fibers \( F_i \) and the set \( M \) of maximal points of the sets \( \Lambda_G \), which are defined in a similar way as for \( \Lambda \). We will also use the notation \( k = (k, k) \in \mathbb{N}^2 \).

**Corollary 4** Let \( \phi = [\phi_1, \phi_2] \in \Sigma_{\Gamma, \Lambda_G} \) and \( k \in \mathbb{N} \).

(a) If \( k > m_1 \) then \( F_1(k) \not= \emptyset \) if and only if \([0, t_1^k), (0, 0)] \in T\mathcal{G}^k(\phi)\);
(b) If \( k > m_2 \) then \( F_2(k) \not= \emptyset \) if and only if \([0, 0), (-t_2^k, 0)] \in T\tilde{A}_1^k(\phi) \) or \([0, 0), (0, t_2^k)] \in T\tilde{A}_1^k(\phi) \);
(c) If \( k \in M \) then there exist \( a, b \in \mathbb{C}^* \), such that

\[
[(0, at_1^k), (-bt_2^k, 0)] \in T\tilde{A}_1^k(\phi) \quad \text{or} \quad [(0, at_1^k), (0, bt_2^k)] \in T\tilde{A}_1^k(\phi).
\]

**Proof** We have that \( (\gamma_1, \gamma_2) \in \Lambda_{\tilde{A}_1} \) if and only if there exists \( \omega \in \Omega_{\tilde{A}_1} \) such that

\[
ord_{\tilde{A}_1^k(\phi)}(\omega(\phi)_1) = \gamma_1.
\]

This, in turn, is equivalent, from the preceding result, to

\[
[(0, h_1), (0, h_2)] \in T\tilde{A}_1^k(\phi), \tag{2}
\]

where \( h_i = j^k_{(\omega(\phi)_i)/m_i} \).

Now, suppose that \( k > m_1 \). Then \( F_1(k) \not= \emptyset \) if and only if there exists \( (k, \gamma) \in \Lambda_{\tilde{A}_1} \) with \( \gamma > k \). The last condition, from Eq. (2), is equivalent to the condition \([0, t_1^k), (0, 0)] \in T\tilde{A}_1^k(\phi) \), proving in this way (a). The proof of (b) is analogous.

Now, if \( k \in M \), then from Eq. (2) we have that \([0, at_1^k), (0, bt_2^k)] \in T\tilde{A}_1^k(\phi) \).

For \( \mathcal{G} = \mathcal{A}_1 \) the proof is similar.

The next result will give us the normal forms of bigerms under the action of the group \( \mathcal{G} \).

**Proposition 5** Let \( \phi = [\phi_1, \phi_2] \in \Sigma_{\Gamma, \Lambda_G} \). If \( F_i(k) \not= \emptyset \) and \( k > m_i \) for some \( i \in \{1, 2\} \) (respectively \( k \in M \)), then there exists \( \varphi = [\varphi_1, \varphi_2] \in \Sigma_{\Gamma, \Lambda_G} \) such that \( \varphi \) is \( \mathcal{G} \)-equivalent to \( \phi \) with \( j^{k-1} \varphi = j^{k-1} \phi \) and \( j^k \varphi_i = j^{k-1} \phi_i \) (respectively \( j^k \varphi_1 = j^{k-1} \phi_1 \) or \( j^k \varphi_2 = j^{k-1} \phi_2 \)).

**Proof** From Corollary 4(a), if \( F_1(k) \not= \emptyset \) and \( k > m_1 \), then \([0, t_1^k), (0, 0)] \in H_\phi^k \cap T\mathcal{G}^k(\phi) \).

It follows from (CTT) that \( j^{k-1} \phi \) is \( \mathcal{G}^k \)-equivalent to \([j^{k-1} \phi_1, j^{k-1} \phi_2]\) and therefore there exists \( \varphi \) which is \( \mathcal{G} \)-equivalent to \( \phi \) such that \( j^k \varphi_1 = j^{k-1} \phi_1 \) and \( j^k \varphi_2 = j^{k-1} \phi_2 \). The case \( F_2(k) \not= \emptyset \) and \( k > m_2 \) is analogous.

If \( k \in M \), then \( F_1(k) = F_2(k) = \emptyset \) and, from Corollary 4(c), the element given by \([0, at_1^k), (-bt_2^k, 0)] \) with well determined \( a, b \in \mathbb{C}^* \) and arbitrary \( d \in \mathbb{C} \) belongs to \( H_\phi^k \cap T\tilde{A}_1^k(\phi) \) if we have distinct tangents or \([0, at_1^k), (0, bt_2^k)] \) belongs to \( H_\phi^k \cap T\tilde{A}_1^k(\phi) \) for the same tangent case. Choosing \( d \) conveniently, it follows, as we argued before, that there exists a bigerm \( \psi \) which is \( \mathcal{G} \)-equivalent to \( \phi \) such that \( j^{k-1} \varphi = j^{k-1} \phi \) with \( j^k \varphi_1 = j^{k-1} \phi_1 \) or \( j^k \varphi_2 = j^{k-1} \phi_2 \) according to the choice of \( d \).
Since there are two different choices to be made in this process when $k$ is in $M$, given \( \phi = [\phi_1, \phi_2] \in \Sigma_{\Gamma, \Lambda_G} \) and $k \in M$ we will choose a $G$-equivalent $\varphi$ to $\phi$ such that $j^{k-1}\varphi = j^{k-1}\phi$ and $j^k \varphi_1 = j^k \phi_1$. In this way, we have the following description of the normal forms for bigerms in $\Sigma_{\Gamma, \Lambda_G}$:

**Theorem 6** ($G$-NORMAL FORM) A bigerm $\phi = [\phi_1, \phi_2] \in \Sigma_{\Gamma, \Lambda_G}$ is always $G$-equivalent to a unique $\varphi = [\varphi_1, \varphi_2]$ such that

\[
\varphi_1 = \left( t_{1}^{m_1}, \sum_{j \in M, F_1(j) = 0} a_{1j} t_1^j \right) \quad \varphi_2 = \begin{cases} \left( \sum_{F_2(j) = 0} a_{2j} t_2^j, t_2^{m_2} \right) & (G = A_1, i.e, distinct tangents) \\ \left( t_2^{m_2}, \sum_{F_2(j) = 0} a_{2j} t_2^j \right) & (G = \tilde{A}_1, i.e, same tangent). \end{cases}
\]

(3)

It remains to show the uniqueness part of the theorem which will be done by arguments similar to those used in [8] and will occupy the rest of this section.

The set

\[ N = \{ \varphi \in \Sigma_{\Gamma, \Lambda_G}; \quad \varphi \text{ as given in } (3) \} \]

is an open set in some affine space of finite dimension. Denoting by $N^k$ the space $j^k(N)$, we have the following lemma:

**Lemma 7** If $\phi = [\phi_1, \phi_2] \in N$, then for all $k > \min \{m_1, m_2\}$, we have

\[ N^k \cap \{ j^k \phi + TG^k( j^k \phi) \} = \{ j^k \phi \}. \]

**Proof** Suppose the assertion not true. Take $k$ minimal with the following property:

\[ N^k \cap \{ j^k \phi + TG^k( j^k \phi) \} \neq \{ j^k \phi \}. \]

So, there exists $\psi \in N^k \cap \{ j^k \phi + TG^k( j^k \phi) \}$ such that $\psi \neq j^k \phi$ and $j^{k-1} \psi = j^{k-1} \phi$ because $k$ is minimal. Therefore, there exist $b_1, b_2 \in C$ with $b_1 \neq 0$ or $b_2 \neq 0$ such that

\[ \psi - j^k \phi = \left[ (0, b_1 t_1^i), (b_2 t_2^j, 0) \right] \in T\dot{A}_1^k( j^k \phi) \quad \text{or} \]

\[ \psi - j^k \phi = \left[ (0, b_1 t_1^i), (0, b_2 t_2^j) \right] \in T\dot{A}_1^k( j^k \phi). \]

If $F_1(k) \neq \emptyset$ for some $i = 1, 2$, then we have a contradiction, since $\psi, j^k \phi \in N^k$ are given as in (3). So, we have $k \in M$. But since $\psi, j^k \phi \in N^k$ we have $b_1 = 0$, then $b_2 \neq 0$. In this way, $\psi - j^k \phi = [(0, 0), (b_2 t_2^j, 0)] \in T\dot{A}_1^k( j^k \phi)$ or $\psi - j^k \phi = [(0, 0), (0, b_2 t_2^j)] \in T\dot{A}_1^k( j^k \phi)$, and $F_2(k) \neq \emptyset$ which is again a contradiction. \[ \square \]

We now conclude the proof of the uniqueness of $G$-normal forms.

Let $\phi = [\phi_1, \phi_2] \in \Sigma_{\Gamma, \Lambda_G}$. Observe that our bigerms are finitely determined up to order $c$ (as defined at the beginning of this section), that is, $\phi$ is $G$-equivalent to $j^c \phi$. We have to prove that $N^c \cap G^c( j^c \phi) = \{ j^c \phi \}$. Suppose that $\varphi \in N^c \cap G^c( j^c \phi)$, with $\varphi \neq j^c \phi$. Since $G^c( j^c \phi)$ is arcwise connected, there exists an arc in $G^c( j^c \phi)$ joining $j^c \phi$ to $\varphi$. Since the reduction process to the normal form is continuous, it follows that $j^c \phi$ is not an isolated point in $N^c \cap G^c( j^c \phi)$. This is a contradiction because of Lemma 7.

Since the $A$-action on bigerms is the composition of the $G$-action with homotheties, the $A$-action on the $G$-normal forms reduces to the action of the group of homotheties.
4 Homothety action

We will consider initially the case of bigerms with transversal components.

In this case, we may write

\[ \phi = [\phi_1, \phi_2] = \left[ \left( t_1^{m_1}, \sum_{j=j_1}^{c} a_{1j} t_1^j \right), \left( \sum_{j=j_2}^{c} a_{2j} t_2^j, t_2^{m_2} \right) \right] \] with \( m_i > j_i \) for \( i = 1, 2 \).

In order to preserve the above form, we have to consider the following particular homotheties: \( (\rho_1, \rho_2, \sigma) \in \mathcal{H} \) with \( \sigma(x, y) = (\alpha_1 x, \alpha_2 y) \) and \( \rho_i(t_i) = \alpha_i^{-1} t_i, \alpha_i \in \mathbb{C}^*, i = 1, 2 \).

In this way we get

\[ (\rho_1, \rho_2, \sigma) \cdot \phi = \left[ \left( t_1^{m_1}, \sum_{j=j_1}^{c} \alpha_1^{-1} t_1^j \alpha_2 a_{1j} t_1^j \right), \left( \sum_{j=j_2}^{c} \alpha_2^{-1} \alpha_1 a_{2j} t_2^j, t_2^{m_2} \right) \right]. \]

In this situation, with a convenient choice of \( \alpha_1 \) and \( \alpha_2 \) we may reduce two any non-zero coefficients in the above sums to 1. We will always choose to apply this reduction to the coefficients of the terms lower order of \( \phi_1 \), if they exist. If not, we continue in the same way the reduction on the terms of \( \phi_2 \).

Similarly, when the components of \( \phi \) have same tangent, that is, when

\[ \phi = [\phi_1, \phi_2] = \left[ \left( t_1^{m_1}, \sum_{j=j_1}^{c} a_{1j} t_1^j \right), \left( t_2^{m_2}, \sum_{j=j_2}^{c} a_{2j} t_2^j \right) \right], \]

we have to consider \( \sigma(x, y) = (\alpha_1 x, \alpha_2 y) \) and \( \rho_i(t_i) = \alpha_i^{-1} t_i \) with \( \alpha_i \in \mathbb{C}^*, i = 1, 2 \). In this case, we get

\[ \sigma \circ \phi_i \circ \rho_i^{-1}(t_i) = \left( t_1^{m_1}, \sum_{j=j_1}^{c} \alpha_1^{-1} \alpha_2 a_{1j} t_1^j \right), \quad i = 1, 2. \]

In the same way as above, we may reduce to 1 any two coefficients in the above sums, unless both components of \( \phi_1 \) and \( \phi_2 \) are monomials with \( m_1 = m_2 \) and \( j_1 = j_2 \). In this case, we may reduce to 1 only one of the coefficients.

The above discussion may be summarized in the following theorem:

**Theorem 8** Any \( \phi \in \Sigma_{\Gamma, A, \phi} \) is \( A \)-equivalent to one in the following form:

**Distinct tangents case**

- **(a)** \[ \left[ \left( t_1^{m_1}, t_1^{j_1} + t_1^k, \sum_{\substack{j \not\in M \\cap \ F_1(j) = \emptyset}} a_{1j} t_1^j \right), \left( \sum_{F_2(l) = \emptyset} a_{2l} t_2^l, t_2^{m_2} \right) \right] : \]

- **(b)** \[ \left[ \left( t_1^{m_1}, t_1^{j_1}, (t_2^j, t_2^{m_2}) \right), (t_2^j, t_2^{m_2}) \right]; \]

- **(c)** \[ \left( t_1^{m_1}, t_1^{j_1}, (0, t_2) \right); \]

- **(d)** \[ \left( (t_1, 0), (0, t_2) \right). \]

\[ \text{Springer} \]
Same tangent case

\[(a') \left[ \left( t_1^{m_1}, t_1^{j_1} + t_1^k + \sum_{\substack{l \neq M, \\ F_1(j) = \emptyset}} a_{1j} t_1^l \right), \left( t_2^{m_2}, \sum_{F_2(l) = \emptyset} a_{2l} t_2^l \right) \right] ;
\]

\[(b') \left[ \left( t_1^{m_1}, t_1^{j_1} \right), \left( t_2^{m_2}, t_2^{j_2} \right) \right] \text{ with } m_1 \neq m_2 \text{ or } j_1 \neq j_2;
\]

\[(c') \left[ \left( t_1^{m_1}, t_1^{j_1} \right), \left( t_2^{m_2}, a t_2^{j_2} \right) \right], \text{ with } a \notin \{0, 1\};
\]

\[(d') \left[ \left( t_1^{m_1}, t_1^{j_1} \right), (t_2, 0) \right],
\]

with \(m_1 < j_1 < k < j \) and \(m_2 < j_2 < l \) for all \(j\) and \(l\).

Let us observe that two bigerms in the above list with distinct normal forms are not \(A\) equivalent since their corresponding sets \(\Lambda\) are not equal.

In what follows we will describe the homotheties that preserve the above normal forms. Since in cases (b), (c), (d), (b’), (c’) and (d’), the homotheties act as the identity, we only need to describe such homotheties in the remaining cases a) and a’). In these cases \(\sigma(x, y) = (\alpha x, \alpha^j y), \rho_1(t_1) = \alpha t_1, \) with \(\alpha^{k-j_1} = 1\) and

Case (a) \(\rho_2(t_2) = \alpha^{m_2} t_2\). In this case, two bigerms with coefficients \(a_{ij}\) and \(b_{ij}\) are \(\mathcal{H}\)-equivalent if and only if

\[a_{ij} \alpha^{j_1-j} = b_{1j}, \text{ and } a_{2j} \alpha^{m_2-j_2j} = b_{2j}.\]

Case (a’) \(\rho_2(t_2) = \alpha^{m_2} t_2\). In this case, two bigerms with coefficients \(a_{ij}\) and \(b_{ij}\) are \(\mathcal{H}\)-equivalent if and only if

\[a_{ij} \alpha^{j_1m_1-jm_1} = b_{ij}, \quad i = 1, 2.\]

5 Analytic equivalence

Given any two bigerms \(\phi = [\phi_1, \phi_2]\) and \(\psi = [\psi_1, \psi_2]\), to verify if the associated curves are analytically equivalent one can proceed as follows:

1. Check if, after possibly a permutation of the branches of \(\psi\), one has \(\Gamma_\phi = \Gamma_\psi\) and \(\Lambda_\phi = \Lambda_\psi\). If this is not the case, then the curves are not analytically equivalent.
2. If the above situation occurs, relabel the branches of \(\psi\) in such a way that \(\Gamma_{\phi_i} = \Gamma_{\psi_i}\) and \(\Lambda_{\phi_i} = \Lambda_{\psi_i}\), \(i = 1, 2\).
3. Take representatives of \(\phi\) and \(\psi\) in normal form as in Theorem 8.
4. Verify (this is easy) if one of the homotheties that preserve the normal form transforms \(\phi\) into \(\psi\). In such case, the two curves are analytically equivalent.
5. If \(\Gamma_{\phi_1} \neq \Gamma_{\phi_2}\) or \(\Lambda_{\phi_1} \neq \Lambda_{\phi_2}\), we are done. Otherwise, we have to repeat the same process up to Step 4 for the other permutation of the branches of \(\psi\).

To give an explicit example for a pair of bigerms as in the second part of Step 5, above, consider \(m < j\).

\[\phi = \left[ \left( t_1^m, t_1^j \right), \left( t_2^m, a t_2^j \right) \right] \text{ and } \psi = \left[ \psi_1, \psi_2 \right] = \left[ \left( t_1^m, t_1^j \right), \left( t_2^m, b t_2^j \right) \right], \quad a, b \notin \{0, 1\}\]
which are in the above normal form $c'$.

So, $\phi$ and $\psi$ are $\mathcal{H}$-equivalent if and only if $a = b$. On the other hand, if we permute the branches of $\psi$ and put it in normal form, we get $[\psi_2, \psi_1] = [(t_2^m, t_2^j), (t_1^m, t_1^j)]$. Therefore, $\phi$ and $\psi$ are $\mathcal{A}$-equivalent if and only if $a = b$ or $a = \frac{1}{b}$. This is a generalization of Example 3 of [2].

In the irreducible case, in each equisingularity class determined by semigroups of the form $\mathbb{N}, (2, j)$ with $j \equiv 1 \mod 2$, or $(3, 3 + \alpha)$ with $\alpha = 1, 2$, all curves are analytically equivalent to a monomial curve, that is, for any of these equisingularity classes we have one possible set $\Lambda$, namely, $\Lambda = \Gamma \setminus \{0\}$.

Using the description of the semigroup, the set of maximal points as described in [5] and doing some computations with differentials, we get the following table for bigerms with transversal components and whose semigroups are as described above.

| $(m_1, m_2)$ | Normal form |
|-------------|-------------|
| (1, 1)      | $(t_1, 0), (t_2)$ |
| (1, 2)      | $(t_1^2, t_1^j) (0, t_2) \ j \equiv 1 \mod 2$ |
| (1, 3)      | $(t_1^3, t_1^{3+\alpha}) (0, t_2) \ \alpha = 1, 2$ |
|             | $(t_1^3, t_1^{3+\alpha} + t_1^{3+2\alpha}) (0, t_2) \ \alpha = 1, 2$ |
| (2, 2)      | $(t_2^3, t_2^j) (t_2^j, t_2^j) j \equiv 1 \mod 2, i = 1, 2$ |
| (2, 3)      | $(t_1^3, t_1^{3+\alpha}) (t_2^j, t_2^j) \ \alpha = 1, 2, \ j \equiv 1 \mod 2$ |
|             | $(t_1^3, t_1^{3+\alpha} + t_1^{3+2\alpha}) (t_2^j, t_2^j) \ \alpha = 1, 2, \ a \neq 0, \ j \equiv 1 \mod 2$ |
|             | $(t_1^3, t_1^{3+\alpha} + t_1^{3+\alpha}) (t_2^j, t_2^j) \ \alpha_1, \alpha_2 = 1, 2$ |
| (3, 3)      | $(t_1^3, t_1^{3+\alpha} + t_1^{3+2\alpha}) (t_2^3, t_2^3) \ \alpha_1, \alpha_2 = 1, 2, \ a \neq 0$ |
|             | $(t_1^3, t_1^{3+\alpha} + t_1^{3+2\alpha}) (t_2^3, t_2^3 + bt^2) \ \alpha_1, \alpha_2 = 1, 2, \ a, b \neq 0$ |

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