Approximate stress-energy tensor of the massless spin-1/2 field in
Schwarzschild spacetime

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Abstract

The approximate stress-energy tensor of the conformally invariant massless spin-1/2 field in the
Hartle-Hawking state in the Schwarzschild spacetime is constructed. It is shown that by solving the
conservation equation in conformal space and utilizing the regularity conditions in a physical metric
one obtains the stress-energy tensor that is in a good agreement with the numerical calculations.
The back reaction of the quantized field upon the spacetime metric is briefly discussed.

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Recently, the renormalized stress-energy tensor of the quantized conformally invariant massless spin-1/2 field in the Schwarzschild and Reissner-Nordström spacetimes has been evaluated numerically [1, 2]. This important and long awaited result completes our knowledge of the behaviour of three basic quantum test fields in the Hartle-Hawking state. However, obtained as a by-product of the calculations its analytical approximation is, contrary to the spin 0 and spin 1 cases, very poor near the event horizon (although it is accurate at large distances).

It has been shown [3, 4] that the stress-energy tensor of the massless fields in the Schwarzschild geometry decomposes naturally as

\[ \langle T_{\mu \nu} \rangle_{\text{ren}} = \langle T_{\mu \nu} \rangle_{\text{analytic}} + \Delta_{\mu \nu}^{\nu}, \]

where \( \langle T_{\mu \nu} \rangle_{\text{analytic}} \) is the analytical approximation of the stress-energy tensor and \( \Delta_{\mu \nu}^{\nu} \) is a traceless and covariantly conserved tensor that must be calculated numerically. For the conformally invariant massless scalar field the analytical part coincides with the Page approximation [5] (and the approximations constructed in [6, 7, 8]), whereas for the vector fields such approximation has been constructed in Refs. [4, 8]. It has been shown that for both scalar and vector fields the approximation is reasonable. (Analytical approximation of tensor \( \Delta_{\mu \nu}^{\nu} \) of the scalar field has been constructed in [9]).

Unfortunately, as has been pointed out in Ref. [2], \( \langle T_{\mu \nu} \rangle_{\text{analytic}} \) for the spin-1/2 field gives a wrong sign of the energy density at the event horizon, invalidating thus any prospect applications. On the other hand, however, the expectation value of the stress-energy tensor of the conformally invariant massless fields in the Hartle-Hawking state in the Schwarzschild geometry is known to possess some general features [10]. The asymptotic behavior of tangential and radial components of \( \langle T_{\mu \nu} \rangle_{\text{ren}} \), the regularity conditions on the event horizon and the trace anomaly are quite restrictive, and allow construction of a class of approximate tensors. Further, a piece of the numerical data, such as the exact value of one of the components of the stress-energy tensor, say \( \langle T_{\theta \theta} \rangle_{\text{ren}} \), on the event horizon may be used in the final determination of the model. It should be noted that calculations of the horizon value of the stress-energy tensor could be regarded as a relatively simple task since it is sufficient to retain only the two smallest frequencies in the mode sums [2, 11]. In this sense this information could be regarded as independent of the numerical calculations of the stress-energy tensor for \( r > 2M \).
The idea of reconstructing \( \langle T^\mu_\nu \rangle_{\text{ren}} \) from the knowledge of its asymptotic behaviour is not new and belongs to Christensen and Fulling [10]. It has been subsequently elaborated in Refs. [12, 13, 14, 15]. In this note we shall show that, contrary to the widespread opinion, it is possible to construct the stress-energy tensor of massless spin 1/2 field in the Schwarzschild spacetime, which satisfactorily approximates the ‘exact’ \( \langle T^\theta_\theta \rangle_{\text{ren}} \). The method is similar to that of Ref [10] with the one reservation: here we consider the four component spinors rather than the two component ones. Moreover, we shall explicitly demonstrate that the Frolov-Zel’nikov method [7] also yields reasonable results and subsequently compare the stress-energy tensors constructed within the frameworks of both methods.

Let us start by counting available information. First, it is known that the stress-energy tensor is covariantly conserved and its trace is given by a general formula

\[
\langle T^\mu_\mu \rangle_{\text{ren}} = \alpha (\mathcal{H} + \Box R) + \beta \mathcal{G} + \gamma \Box R, \tag{2}
\]

where

\[
\mathcal{H} = R^\mu_\nu_\rho_\tau R^{\nu_\rho_\tau} - 2R^\mu_\nu R^\nu_\mu + \frac{1}{3} R^2, \tag{3}
\]

\[
\mathcal{G} = R^\mu_\nu_\rho_\tau R^{\nu_\rho_\tau} - 4R^\mu_\nu R^\nu_\mu + R^2 \tag{4}
\]

and the numerical coefficients for the spin-1/2 field are \( \alpha = 18 \lambda, \beta = -11 \lambda, \gamma = 0 \), and \( \lambda = 2^{7/45} \pi^2 \). Further, we observe \( \langle T^\mu_\mu \rangle_{\text{ren}} \) is regular on the past and future event horizon and approaches at large distances the flat spacetime radiation stress-energy tensor. Finally, we expect that the curvature effects enter the stress-energy tensor as \( \sim x^3 \) (\( x = 2M/r \)). This requirement is usually motivated by the observation that far from the event horizon the stress-energy tensor should consist of the red-shifted thermal bath part supplemented by the quantum corrections. This assumption is sometimes referred to as a weak thermal bath hypothesis [17].

The idea is to construct \( \tilde{T}^\mu_\nu \) in the optical metric and subsequently to transform it back to the physical spacetime with the aid of a transformation that relates the stress-energy tensor in conformally related geometries [3, 4, 18]. Now, the independent informations listed above suggest that the tangential component of the stress-energy tensor could be approximated as

\[
\tilde{T}^\theta_\theta = \frac{7}{2} T \left( 1 + \sum_{i}^{N} a_i x^i \right), \tag{5}
\]
with \( N = 6 \) and \( T^{-1} = 90\pi^2(8M)^4 \).

Returning to the Schwarzschild geometry and making use of the regularity conditions in the physical metric, one has

\[
T^\theta_\theta = \frac{7}{2} T \left[ 1 + 2x + 3x^2 + \frac{68}{7}x^3 + \frac{1}{14}(230 + 14a_4)x^4 + \frac{102}{7}x^5 + \frac{87}{7}x^6 \right].
\]  \hspace{1cm} (6)

Remaining components of the approximate stress-energy tensor can be easily obtained from (6) and will not be displayed here. Similar results for the two component spinors have been constructed in Ref. \[16\] to which the reader is referred for the technical details. It should be noted that the logarithmic term \( x^2(a_1 + 2a_2) \ln x \), which appears as the result of integration of the conservation equation survives even if the regularity conditions are satisfied. Only after accepting the weak thermal bath hypothesis the coefficients \( a_1 \) and \( a_2 \) could be equated to zero \[8\]. The parameter \( a_4 \) can be determined form the equation:

\[
T^\nu_\mu(2M) = \langle T^\nu_\mu(2M) \rangle,
\]  \hspace{1cm} (7)

which, by the spherical symmetry, equality of \( T^t_t \) and \( T^r_r \) at the event horizon and the trace anomaly yields in fact only one condition.

Although the horizon values of the components of the stress-energy tensor in the Hartle-Hawking state have never been cited explicitly in literature, and Carlson et al. \[2\] present their results only graphically, we have sufficient informations to construct the model. The horizon value of the trace anomaly in the case at hand is \( 7/7680\pi^2M^4 \), whereas the approximate values of \( T^\theta_\theta \) is \[2\]

\[
T^\theta_\theta(2M) = T^{\phi}_\phi(2M) \approx \frac{105}{90\pi^2(8M)^4}.
\]  \hspace{1cm} (8)

Making use of Eqs. (6) and Eq. (7) with the right hand side given by (8) one finally obtains

\[
T^t_t = -\frac{7}{2} T \left( 3 + 6x + 9x^2 + 12x^3 - \frac{39}{7}x^4 + \frac{186}{7}x^5 - 69x^6 \right),
\]  \hspace{1cm} (9)

\[
T^r_r = \frac{7}{2} T \left( 1 + 2x + 3x^2 - \frac{52}{7}x^3 + \frac{139}{7}x^4 - \frac{18}{7}x^5 + \frac{15}{7}x^6 \right)\]  \hspace{1cm} (10)

and

\[
T^\theta_\theta = \frac{7}{2} T \left( 1 + 2x + 3x^2 + \frac{68}{7}x^3 - \frac{89}{7}x^4 + \frac{102}{7}x^5 + \frac{87}{7}x^6 \right).\]  \hspace{1cm} (11)

The run of the components of \( T^\nu_\mu \) are displayed in Figs. 1-3. The new ‘radial’ coordinate \( \xi \) is defined as \( \xi = (r - r_+)/M \), where \( r_+ \) denotes location of the event horizon.
It should be emphasized that the case of the spin-1/2 fields is special. Indeed, for the electromagnetic field the $T_{\mu}^\nu$ coincides with the analytic part of $\langle T_{\mu}^\nu \rangle$ \[8\], whereas the analogous result constructed for the conformally invariant scalar fields substantially improves the Page approximation.

A different method of calculating the stress-energy tensor in static spacetimes has been proposed by Frolov and Zel’nikov in Ref. \[7\]. It has been shown that it is possible to construct a family of expressions describing the approximate stress-energy tensor $T_{\mu}^{(FZ)\nu}$ solely form the curvature, the Killing vector and their covariant derivatives up to some given order. By construction the Frolov-Zel’nikov tensors have a correct trace and a proper behaviour under the scale transformations. Upon imposing appropriate regularity conditions at the event horizon of the Schwarzschild black hole one obtains a one parameter family of the approximate stress-energy tensor. As expected, the resulting expressions are simple polynomials in $x$. If for the conformally invariant scalar fields the free parameter is set to zero the result coincides with the Page approximation. It should be noted, however, that there are no a priori reasons to accept such a choice. Indeed, for a vector field such a choice leads to evidently wrong results. On the other hand the free parameter may be easily adjusted employing Eq. (7).

It could be easily shown that

$$T_{\mu}^{(FZ)\nu} = T_{\mu}^\nu + D_{\mu}^\nu,$$

(12)

where $T_{\mu}^\nu$ is the approximate stress-energy tensor \[9\], and the conserved and traceless tensor $D_{\mu}^\nu$ is given by

$$D_t^t = -\frac{7}{2} T \left( \frac{783 x^4}{35} - \frac{174 x^5}{35} - \frac{87 x^6}{5} \right),$$

(13)

$$D_r^r = \frac{7}{2} T \left( \frac{232 x^3}{35} - \frac{667 x^4}{35} + \frac{174 x^5}{35} + \frac{261 x^6}{35} \right),$$

(14)

and

$$D_\theta^\theta = -\frac{7}{2} T \left( \frac{116 x^3}{35} - \frac{145 x^4}{7} + \frac{174 x^5}{35} + \frac{87 x^6}{7} \right).$$

(15)

The components of $D_{\mu}^\nu$ are displayed in Figs. 1-3 (right panel).

Now we can compare the approximate tensors \[9\] to \[12\] and to the numerical results presented in Ref. \[2\]. By construction the tensors are exact at infinity and very close to
the exact value at the event horizon. Inspection of similar figures presented in Ref. [2] shows
that our approximation (9-11) is in good agreement with the exact numerical calculations.
The Frolov-Zel’nikov approximation is slightly worse but still reasonable. Unfortunately,
we are unable to provide detailed comparison between the exact numerical results and the
approximations as the former were presented only graphically.

The run of $T_{\mu}^{\nu}$ and $T_{\mu}^{(FZ)\nu}$ constructed for the massless spinor field qualitatively resembles
behaviour of the approximate stress-energy tensor of the electromagnetic field [4]. Indeed,
for the electromagnetic field the Frolov-Zel’nikov approximation is also less accurate than
the analytic part of $\langle T_{\mu}^{\nu} \rangle$.

\begin{equation}
G_{\mu}^{\nu}[g] = 8\pi \left( T_{\mu}^{(cl)\nu}[g] + \langle T_{\mu}^{\nu}[g] \rangle \right).
\end{equation}

The first term in the right hand side of the above equation comes from the classical source
whereas the second one is due to the contribution of the quantized fields. Unfortunately,
FIG. 2: This graph shows the radial dependence of the rescaled component $T_r^r [\lambda = 90(8M)^4\pi^2]$ of the approximate stress-energy tensor of the massless spin 1/2 field in Schwarzschild spacetime and the rescaled component $D_r^r$ (right panel).

FIG. 3: This graph shows the radial dependence of the rescaled component $T_\theta^\theta [\lambda = 90(8M)^4\pi^2]$ of the approximate stress-energy tensor of the massless spin 1/2 field in Schwarzschild spacetime and the rescaled component $D_\theta^\theta$ (right panel).

all our current understanding of the stress-energy tensor of the quantized massless fields is limited to the static black holes described by the electric charge and the mass. Our results, however, may be of use in the linearized back reaction calculations [19, 20]: the curved spacetime leads to the stress-energy tensor approximated by (9 -11), which, in turn, modifies the background geometry.

Since the stress-energy tensor is asymptotically constant it is necessary to put the system
in the cavity or spherical box of definite radius, \( r_c \). It should be noted, that presence of the boundary certainly modifies the stress-energy tensor. Therefore the radius \( r_c \) should be chosen to guarantee applicability of the perturbative approach on the one hand and to minimize the error caused by ignoring expected boundary effects on the other.

The geometry of the quantum corrected spherically-symmetric static black hole is generally described by the line element

\[
ds^2 = -e^{2\psi(r)}f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega^2,
\]

where \( f(r) = 1 - 2m(r)/r \) and \( d\Omega^2 \) is the metric on a unit sphere. We shall assume that the functions \( m(r) \) and \( \psi(r) \) can be expanded as

\[
m(r) = M + \varepsilon M_1(r) + \mathcal{O}(\varepsilon^2)
\]

and

\[
\psi(r) = \varepsilon \psi_1(r) + \mathcal{O}(\varepsilon^2).
\]

To keep control of the order of terms we have introduced the dimensionless parameter \( \varepsilon \), which will be set to 1 at the final stage of calculations. Similarly, \( \langle T_{\mu}[g] \rangle \) in the right hand side of Eq. (16) should be substituted by \( \varepsilon \langle T_{\mu}[g] \rangle \).

The solutions of the linearized semi-classical Einstein field equations with a source term given by the stress-energy tensor (9)-(11) reduce to two elementary quadratures:

\[
M_1(r) = -4\pi \int r^2 T_t^t dr + C_1
\]

and

\[
\psi_1(r) = 4\pi \int \frac{r^2 (T_r^r - T_t^t)}{r - 2M} dr + C_2,
\]

where the integration constants \( C_1 \) and \( C_2 \) are to be determined from the boundary conditions. Observe that it is possible to determine the function \( \psi_1(r) \) without prior knowledge of the horizon value of the stress-energy tensor. It is simply because the difference \( T_r^r - T_t^t \) does not depend on the coefficient \( a_4 \).

Our preferred choice of the boundary condition for Eq. (20) is simply \( M_1(r_+) = 0 \), which requires knowledge of the exact location of the event horizon, \( r_+ \). From (18) and (20) one has \( r_+ = 2M \), and hence \( M \) is to be interpreted as the horizon defined mass. For the function \( \psi_1(r) \) we shall adopt the natural condition \( g_{tt}(r_c)g_{rr}(r_c) = -1 \).
Now the equations (20) and (21) can be easily integrated to yield

\[ M_1 (r) = \frac{K}{M} \left( \frac{7}{6} x^3 + \frac{7}{2} x^2 + \frac{21}{2} x - 33 + \frac{13}{2} x - \frac{31}{2} x^2 \right. \\
\left. + \frac{161}{6} x^3 - 14 \ln x \right) \quad (22) \]

and

\[ \psi_1 (r) = \frac{K}{M^2} \left( \frac{7}{6} \left( \frac{1}{x^2} - \frac{1}{x_c^2} \right) + 7 \left( \frac{1}{x} - \frac{1}{x_c} \right) - \frac{50}{3} (x - x_c) \\
- \frac{25}{2} (x^2 - x_c^2) - 13 \left( x^3 - x_c^3 \right) - 14 \ln \frac{x}{x_c} \right) \], \quad (23) \]

where \( K = 3840\pi \) and \( x_c = 2M/r_c \). It is believed that the line element (17) with the metric potentials given by Eqs. (18,19) with (22) and (23) respectively better approximates the physical reality then the original (Schwarzschild) one. Having the quantum corrected geometry of the black hole one may study its properties such as Hawking temperature, trace anomaly and its influence on the motion of test particles. Since the calculations of these effects are elementary we shall not dwell on them here. We only remark that in view of the results of Ref. [2] presented method is the only one, which is able to provide simple and reasonable approximations to the exact stress-energy tensor. This model may be thought of as a minimal one, i. e. requiring only one numerical information: the horizon value of the stress-energy tensor, which, as has been pointed out, is to certain extend independent of the numerical calculations for \( r > 2M \). Of course, more complicated models which approximate the exact \( \langle T_{\mu\nu} \rangle \) better could be easily devised at the expense of the additional numerical informations.

Similar calculations with different asymptotics may be used in construction of the stress-energy tensor of the spin-1/2 field in the Unruh state. We expect that the approximation will be reasonable and the results presented in Refs. [8, 21] suggest that it is a safe anticipation.

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