Regularity of Non-Stationary Multivariate Subdivision

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Abstract

In this paper, we study scalar multivariate non-stationary subdivision schemes with integer dilation matrix $M = mI$, $m \geq 2$, and present a general approach for checking their convergence and for determining their Hölder regularity. The combination of the concepts of asymptotic similarity and approximate sum rules allows us to link stationary and non-stationary settings and to employ recent advances in methods for exact computation of the joint spectral radius. As an application, we prove a recent conjecture on the Hölder regularity of the generalized Daubechies wavelets. We illustrate our results with several examples. We also expose limitations of non-stationary schemes in their capability to reproduce and generate certain function spaces.

Keywords: multivariate non-stationary subdivision schemes, sum rules, approximate sum rules, asymptotic similarity, joint spectral radius, generalized Daubechies wavelets.

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1 Introduction and summary of the results

In this paper we study multivariate non-stationary, i.e. level-dependent, subdivision schemes with the integer dilation matrix $M = mI$, $m \geq 2$. Subdivision schemes are efficient iterative methods for generating smooth curves or surfaces from given initial data $c^{(1)} := \{c^{(1)}(\alpha) \in \mathbb{R}, \alpha \in \mathbb{Z}^s\}$ by means of local refinement rules which are based on the sequence of subdivision...
operators \( \{ S_{a^{(k)}} \}, k \geq 1 \). The subdivision operators \( S_{a^{(k)}} : \ell(\mathbb{Z}^s) \to \ell(\mathbb{Z}^s) \) are linear operators and map coarser sequences \( c^{(k)} \in \ell(\mathbb{Z}^s) \) into finer sequences \( c^{(k+1)} \in \ell(\mathbb{Z}^s) \) via the rules

\[
c^{(k+1)} := S_{a^{(k)}} c^{(k)}, \quad S_{a^{(k)}} c^{(k)}(\alpha) := \sum_{\beta \in \mathbb{Z}^s} a^{(k)}(\alpha - M\beta) c^{(k)}(\beta), \quad k \geq 1, \quad \alpha \in \mathbb{Z}^s. \tag{1}
\]

The masks \( \{ a^{(k)} \}, k \geq 1 \) are sequences of real numbers \( a^{(k)} := \{ a^{(k)}(\alpha) \in \mathbb{R}, \, \alpha \in \mathbb{Z}^s \} \) and are assumed to have bounded supports in \( \{0, \ldots, N\}^s \) with \( N \in \mathbb{N} \). Such schemes are called either level-dependent, or non-stationary or non-homogeneous. Here we use the term non-stationary and denote these type of subdivision schemes by the corresponding collection of subdivision operators \( \{ S_{a^{(k)}} \}, k \geq 1 \). A subdivision scheme whose refinement rules are level independent is said to be stationary (see, \cite{3}, for example) and, for all \( k \geq 1 \), is defined by the same sequence \( \mathbf{a} := \{ a(\alpha) \in \mathbb{R}, \, \alpha \in \mathbb{Z}^s \} \) of refinement coefficients, i.e. \( a^{(k)} = \mathbf{a}, \, k \geq 1 \). The corresponding subdivision scheme is therefore denoted by \( S_{\mathbf{a}} \).

The popularity of stationary and non-stationary subdivision schemes is due to their applications in several different context such as geometric modelling \cite{3, 25}, computer animation \cite{22}, non-stationary multiresolution analysis \cite{2, 10, 29, 33, 13} and, more recently, isogeometric analysis \cite{17}. There is a multitude of results on convergence and regularity of stationary subdivision schemes in the literature (for example see \cite{3, 6, 9, 38, 35, 41} and references therein). These results rely on generation and reproduction properties of subdivision operators. Less is known about the corresponding properties of non-stationary schemes. The main reason for that is a lack of appropriate methods for their regularity analysis. Indeed, the restricted spectral radius and the joint spectral radius approaches are not always applicable, as one cannot expect generation/reproduction of certain polynomial spaces by non-stationary subdivision schemes.

In this paper we provide a general approach for the analysis of convergence and regularity of a vast majority of non-stationary subdivision schemes. We make use of the concepts of approximate sum rules and asymptotic similarity to link stationary and non-stationary settings and show how to employ the joint spectral radius for analysis of non-stationary schemes. We generalize the existing well-known results in \cite{13, 26, 27} that allow us to check convergence and Hölder regularity of special instances of non-stationary schemes.

In fact, the sufficient conditions in \cite{26} are based on the concept of asymptotic equivalence which we recall in the following Definition \( \mathbf{1} \) where \( E := \{ 0, \ldots, m-1 \}^s \) is a set of representatives of \( \mathbb{Z}^s/m\mathbb{Z}^s \).
Definition 1. Let \( \ell \geq 0 \). Two non-stationary schemes \( \{S_{a^{(k)}}, \, k \geq 1\} \) and \( \{S_{b^{(k)}}, \, k \geq 1\} \) are called asymptotically equivalent (of order \( \ell \)), if they satisfy
\[
\sum_{k=1}^{\infty} m^{k\ell} \|S_{a^{(k)}} - S_{b^{(k)}}\|_\infty < \infty, \quad \text{where} \quad \|S_{a^{(k)}}\|_\infty := \max_{\varepsilon \in E} \left\{ \sum_{\alpha \in \mathbb{Z}^*} |a^{(k)}(M\alpha + \varepsilon)| \right\}.
\] (2)

In the case of \( M = 2I \) and under certain additional assumptions on the schemes \( \{S_{a^{(k)}}, \, k \geq 1\} \) and \( \{S_{b^{(k)}}, \, k \geq 1\} \), the method in [26] allows us to determine the regularity of \( \{S_{a^{(k)}}, \, k \geq 1\} \) from the known regularity of the asymptotically equivalent scheme \( \{S_{b^{(k)}}, \, k \geq 1\} \). In [27], in the univariate binary case, the authors relax the condition of asymptotic equivalence. They require that the \( D^j \)-th derivatives of the symbols
\[
a_*^{(k)}(z) := \sum_{\alpha \in \mathbb{Z}} a^{(k)}(\alpha) z^\alpha, \quad z \in \mathbb{C} \setminus \{0\}, \quad k \geq 1,
\]
of the non-stationary scheme \( \{S_{a^{(k)}}, \, k \geq 1\} \) satisfy
\[
|D^j a_*^{(k)}(-1)| \leq C 2^{-((j+1-j)k)}, \quad j = 0, \ldots, \ell, \quad \ell \geq 0, \quad C \geq 0,
\]
and, additionally, assume that the non-stationary scheme is asymptotically equivalent (of order 0) to some stationary scheme. The conditions in (3) can be seen as a generalization of the so-called sum rules in [3]. In the stationary case, sum rules are necessary for smoothness of subdivision, see e.g [2, 3, 34, 37].

Definition 2. Let \( \ell \geq 0 \). The symbol \( a_*(z), \, z \in \mathbb{C} \setminus \{0\} \), satisfies sum rules of order \( \ell + 1 \) if
\[
a_*(1) = m^s \quad \text{and} \quad \max_{|\eta| \leq \ell} \max_{\varepsilon \in \Xi \setminus \{1\}} |D^\eta a_*(\varepsilon)| = 0.
\] (4)
In the above definition, \( \Xi := \{e^{-i \frac{2\pi}{n}} \varepsilon, \, \varepsilon \in E\} \) and \( D^\eta, \, \eta \in \mathbb{N}_0^s \), denotes the \( \eta \)-th directional derivative.

In the spirit of [3], in this paper, we present a generalization of the notion of sum rules which we call approximate sum rules.

Definition 3. Let \( \ell \geq 0 \). The sequence of symbols \( \{a_*^{(k)}(z), \, k \geq 1\} \) satisfies approximate sum rules of order \( \ell + 1 \), if
\[
\mu_k := |a_*^{(k)}(1) - m^s| \quad \text{and} \quad \delta_k := \max_{|\eta| \leq \ell} \max_{\varepsilon \in \Xi \setminus \{1\}} |m^{-k|\eta|} D^\eta a_*^{(k)}(\varepsilon)|
\]
satisfy
\[
\sum_{k=1}^{\infty} \mu_k < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} m^{k\ell} \delta_k < \infty.
\] (6)
We call the sequence \( \{\delta_k, \ k \geq 1\} \) sum rule defects. If the sequences \( \{\mu_k, \ k \geq 1\} \) and \( \{\delta_k, \ k \geq 1\} \) are zero sequences, then the symbols of the corresponding non-stationary scheme satisfy sum rules of order \( \ell + 1 \).

Note that, even in the univariate binary case, the assumption on \( \{\delta_k, \ k \geq 1\} \) in (6), i.e.
\[
\sum_{k=1}^{\infty} 2^{\ell k} \delta_k < \infty, \quad \delta_k := \max_{j \leq \ell} 2^{-k} |D_j^a(k)(-1)|, \tag{7}
\]
is less restrictive, than the decay condition on \( \{\delta_k, \ k \geq 1\} \) in (5).

In Section 4, we show that approximate sum rules are close to being necessary conditions for regularity of non-stationary schemes, see Theorem 1 and Example 1. This resembles the stationary setting and motivates our multivariate convergence and smoothness analysis of non-stationary schemes. Indeed, in the binary univariate case, we show that under assumption of asymptotical similarity (see Definition 5) to a stationary scheme whose basic refinable function is stable, the \( C^{\ell} \)-regularity of the non-stationary scheme implies that the sum rules defects \( \{\delta_k, \ k \geq 1\} \) must decay faster than \( 2^{-\ell k} \). Clearly, there is still a small gap between the corresponding necessary condition \( \lim_{k \to \infty} 2^{-\ell k} \delta_k = 0 \) and one of the sufficient conditions \( \sum_{k \in \mathbb{N}} 2^{-\ell k} \delta_k < \infty \).

In [13], in the univariate binary non-stationary setting, milder sufficient conditions than asymptotical equivalence are essentially derived under the assumptions that the scheme \( \{S_{a(k)}, \ k \geq 1\} \) is asymptotically similar to a suitable non-stationary scheme \( \{S_{b(k)}, \ k \geq 1\} \), i.e. \( \lim_{k \to \infty} \|a^{(k)} - b^{(k)}\|_\infty = 0 \), and both satisfy sum rules of order 1. Here we generalize the notion of asymptotic similarity making use of the following concept of set of limit points of a sequence of masks.

**Definition 4.** For the mask sequence \( \{a^{(k)}, \ k \geq 1\} \) we denote by \( \mathcal{A} \) the set of its limit points, i.e. the set of masks \( a \) such that

\[
a \in \mathcal{A}, \quad \text{if} \quad \exists \{k_n, \ n \in \mathbb{N}\} \text{ such that } \lim_{n \to \infty} a^{(k_n)} = a.
\]

The following definition of asymptotic similarity generalizes the one given in [13]. This notion allows us to relate the properties of non-stationary subdivision schemes to the corresponding properties of the stationary masks in \( \mathcal{A} \).

**Definition 5.** Two non-stationary schemes \( \{S_{a(k)}, \ k \geq 1\} \) and \( \{S_{b(k)}, \ k \geq 1\} \) are called asymptotically similar, if their sets of limit points coincide.
1.1 Summary of the results

For the reader’s convenience, we summarize here the main results presented in this paper. The details are given in Sections 3 and 4.

In the rest of the paper we assume that the symbols \( \{a_*(z), \ k \geq 1\} \) satisfy approximate sum rules and are re-scaled in such a way that \( a_*(1) = m^s, \ k \geq 1 \). In this case \( \mu_k \) in (5) are equal to zero for all \( k \geq 1 \) and do not affect our convergence and regularity analysis. On the contrary, if the sequence \( \{\mu_k, \ k \geq 1\} \) is not summable, then such a re-scaling can change the properties of the scheme, see Example 1.

Our first result states that even in the univariate case approximate sum rules are very close to being necessary for convergence and smoothness of non-stationary subdivision schemes.

**Theorem 1.** Let \( \ell \geq 0 \). Assume that a univariate binary subdivision scheme \( S_a \) is convergent and its basic refinable limit function is stable. Assume, furthermore, that \( a = \lim_{k \to \infty} a^{(k)} \) and the non-stationary subdivision scheme \( \{S^{(k)}_a, \ k \geq 1\} \) converges to \( C^\ell \) limit functions. Then

\[
\lim_{k \to \infty} 2^{\ell k} \delta_k = 0 \quad \text{for} \quad \{\delta_k, \ k \geq 1\} \quad \text{in} \quad (7).
\]

The proof of Theorem 1 is given in Subsection 4.1.

In the stationary case, the Hölder regularity of the subdivision limits, as well as the rate of convergence of the corresponding subdivision scheme \( S_a \), are determined explicitly in terms of the joint spectral radius of the set of certain square matrices which are derived from the subdivision mask \( a \) and depend on the order of sum rules satisfied by \( a_*(z) \). Since, in the non-stationary setting, one cannot assume that all subdivision symbols \( \{a_*(z), \ k \geq 1\} \) satisfy sum rules, see [8, 14], the concept of the joint spectral radius is not directly applicable and has no straightforward generalization. For this reason, in Theorem 2 we establish a link between stationary and non-stationary settings via the set \( A \) of limit points of \( \{a^{(k)}, \ k \geq 1\} \) and provide sufficient conditions for \( C^\ell \)–convergence, \( \ell \geq 0 \), and Hölder regularity of non-stationary schemes. Under \( C^\ell \)–convergence we understand the convergence of subdivision in the norm of \( C^\ell(\mathbb{R}^s) \), see Definition 9 in Section 2. Note that \( C^0 \)–convergence is the usual convergence of subdivision in \( \ell_\infty \) norm and \( C^\ell \)–convergence implies the convergence of the scheme to \( C^\ell \) limit functions, but not vice versa, see Definition 6 in Section 2. As in the stationary setting, each mask in the limit set \( A \) determines a set of transition matrices. We denote the collection of the restrictions of all these transition matrices to a certain finite dimensional difference subspace.
Theorem 2. Let \( \ell \geq 0 \) and \( \{\delta_k, k \geq 1\} \) be defined in (5). Assume that the symbols of \( \{S_{a(k)}, k \geq 1\} \) satisfy approximate sum rules of order \( \ell + 1 \) and \( \rho_A := \rho(T_A|_{V_\ell}) < m^{-\ell} \), where \( A \) is the set of limit points of \( \{a^{(k)}, k \geq 1\} \). Then the non-stationary scheme \( \{S_{a(k)}, k \geq 1\} \) is \( C^\ell \)-convergent and the Hölder exponent \( \alpha \) of its limit functions satisfies
\[
\alpha \geq \min \left\{ -\log_m \rho_A, \limsup_{k \to \infty} \frac{\log_m \delta_k}{k} \right\}.
\]
The proof of Theorem 2 is given in Subsection 3.2.

Note that, as in the stationary case, the order of approximate sum rules satisfied by the symbols of a non-stationary scheme can be much higher than its regularity.

There are several immediate important consequences of Theorem 2 that generalize the corresponding results in [13, 26, 27]. For example the following Corollary extends the results in [13] with respect to the dimension of the space, the regularity of the limit functions and the more general notion of asymptotic similarity given in Definition 5.

Corollary 1. Let \( \ell \geq 0 \). Assume that the symbols of the scheme \( \{S_{a(k)}, k \geq 1\} \) satisfy sum rules of order \( \ell + 1 \) and \( \rho_A := \rho(T_A|_{V_\ell}) < m^{-\ell} \), where \( A \) is the set of limit points of \( \{a^{(k)}, k \geq 1\} \). Then any other asymptotically similar scheme \( \{S_{b(k)}, k \geq 1\} \) whose symbols satisfy sum rules of order \( \ell + 1 \) is \( C^\ell \)-convergent and the Hölder exponent of its limit functions is \( \alpha \geq -\log_m \rho_A \).

Theorem 2 provides a lower bound for the Hölder exponent of the subdivision limits, whereas the next result allows us to determine its exact value, under slightly more restrictive assumptions.

Theorem 3. Let \( \ell \geq 0 \). Assume that a stationary scheme \( S_a \) is \( C^\ell \)-convergent with the stable refinable basic limit function \( \phi \) whose Hölder exponent \( \alpha_\phi \) satisfies \( \ell \leq \alpha_\phi < \ell + 1 \). If the symbols of the scheme \( \{S_{a(k)}, k \geq 1\} \) satisfy approximate sum rules of order \( \ell + 1 \), \( \lim_{k \to \infty} a^{(k)} = a \) and, additionally
\[
\limsup_{k \to \infty} \delta_k^{1/k} < \rho_a := \rho(T_{\varepsilon,a}|_{V_\ell}, \varepsilon \in E),
\]
then the scheme \( \{S_{a(k)}, k \geq 1\} \) is \( C^\ell \)-convergent and the Hölder exponent of its limit functions is also \( \alpha_\phi \).
The proof of Theorem 3 is given in Subsection 3.3. An important special class of non-stationary schemes that satisfy assumptions of Theorem 3 are the schemes whose symbols satisfy sum rules of order $\ell + 1$, see Corollary 4 in Subsection 3.3.

This paper is organized as follows. In Section 2 we summarize important known fact about stationary and non-stationary subdivision schemes. The proofs of the results stated above in Subsection 1.1 are given in Sections 3 and 4. In particular, in Subsection 3.1 we provide sufficient conditions for convergence of non-stationary subdivision schemes whose symbols satisfy assumptions of Theorem 2 with $\ell = 0$. The argument in the proof of the corresponding Theorem 4 is actually independent of the choice of the dilation matrix $M$. For that reason we give a separate proof of convergence and, then, in Subsection 3.2 present the proof of the more general statement of Theorem 2. In Subsection 3.3 we give the proof of Theorem 3. We illustrate our convergence and regularity results with several examples in Subsection 3.4. There we also prove the conjecture formulated in [24] about the regularity of generalized Daubechies wavelets. Next, in Section 4 we present the analysis of further properties of non-stationary subdivision: Subsection 4.1 is devoted to the proof of the necessary conditions stated in Theorem 1. In Subsection 4.2 we expose the limitations of non-stationary schemes in their capability to reproduce and generate certain function spaces.

2 Background and preliminary definitions

In this section we recall well-known properties of subdivision schemes.

We start by defining convergence and Hölder regularity of non-stationary and, thus, also of stationary subdivision schemes. We would like to distinguish between the following two different types of convergence, both being investigated in the literature on stationary and non-stationary subdivision schemes. We denote by $\ell_\infty(\mathbb{Z}^s)$ the space of all scalar sequences $c = \{c(\alpha), \alpha \in \mathbb{Z}^s\}$ indexed by $\mathbb{Z}^s$ and such that

$$
\|c\|_{\ell_\infty} := \sup_{\alpha \in \mathbb{Z}^s} |c(\alpha)| < \infty.
$$

**Definition 6.** We say that a subdivision scheme $\{S_{\alpha(k)}, \ k \geq 1\}$ converges to $C^\ell$ limit functions, if for any initial sequence $c \in \ell_\infty(\mathbb{Z}^s)$, there exists the limit function $g_c \in C^\ell(\mathbb{R}^s)$ (which is nonzero for at least one nonzero sequence $c$) such that

$$
\lim_{k \to \infty} \|g_c(M^{-k}\alpha) - S_{\alpha(k)}S_{\alpha(k-1)} \cdots S_{\alpha(1)}c(\alpha)\|_{\ell_\infty} = 0. \tag{8}
$$
In the next Definition 9 we consider a stronger type of convergence, the so-called $C^\ell$—convergence of subdivision. Note that both types of convergence coincide in the case $\ell = 0$. In Definition 9 we make use of the concept of a test function (see, for example [18]). To define this concept we need to recall the following properties of the test functions.

**Definition 7.** Let $\ell \geq 0$. We say that a compactly supported summable function $f$ satisfies Strang-Fix conditions of order $\ell + 1$, if its Fourier transform $\hat{f}$ satisfies

$$\hat{f}(0) = 1, \quad D^\mu \hat{f}(\alpha) = 0, \quad \alpha \in \mathbb{Z}^s \setminus \{0\}, \quad \mu \in \mathbb{N}_0^s, \quad |\mu| < \ell + 1.$$ 

**Definition 8.** We say that a compactly supported $f \in L^\infty(\mathbb{R}^s)$ is stable, if there exists $0 < C_1 \leq C_2 < \infty$ such that

$$C_1 \|c\|_{\ell^\infty} \leq \left\| \sum_{\alpha \in \mathbb{Z}^s} c(\alpha) f(\cdot - \alpha) \right\|_{\ell^\infty} \leq C_2 \|c\|_{\ell^\infty}$$

for all $c \in \ell^\infty(\mathbb{Z}^s)$.

By [3, p.24], this type of stability is equivalent to $\ell^\infty$ linear independence of integer shifts of $f$.

The function $f$ is called a test function, if it is sufficiently smooth, compactly supported, stable and satisfies Strang-Fix conditions of order $\ell + 1$. Possible examples of the test functions $f$ are tensor-product box splines.

**Definition 9.** We say that a subdivision scheme $\{S_{a_k}, k \geq 1\}$ is $C^\ell$-convergent, if for any initial sequence $c \in \ell^\infty(\mathbb{Z}^s)$ there exists the limit function $g_c \in C^\ell(\mathbb{R}^s)$ such that for any test function $f \in C^\ell(\mathbb{R}^s)$

$$\lim_{k \to \infty} \left\| g_c(\cdot) - \sum_{\alpha \in \mathbb{Z}^s} S_{a_k} S_{a_{k-1}} \cdots S_{a_1} c(\alpha) f(M_k \cdot - \alpha) \right\|_{C^\ell} = 0. \quad (9)$$

Note that, for $C^\ell$—convergence, it suffices to check (9) just for one such test function $f$.

In this paper, we also investigate the Hölder regularity of subdivision limits.

**Definition 10.** We say that the Hölder regularity of the $C^0$—convergent scheme $\{S_{a_k}, k \geq 1\}$ is $\alpha = \ell + \zeta$, if $\ell$ is the largest integer such that $g_c \in C^\ell(\mathbb{R}^s)$ and $\zeta$ is the supremum of $\nu \in [0,1]$ such that

$$\max_{\mu \in \mathbb{N}_0^s, |\mu| = \ell} |D^\mu g_c(x) - D^\mu g_c(y)| \leq |x - y|^\nu, \quad x, y \in \mathbb{R}^s.$$

We call $\alpha$ the Hölder exponent of the limit functions of $\{S_{a_k}, k \geq 1\}$. 
Instead of studying the regularity of all limit functions of a $C^0$–convergent subdivision scheme, one usually restricts the analysis to the so-called basic limit functions, which are defined as follows. Let $\delta := \{\delta(\alpha) = \delta_{0,\alpha}, \ \alpha \in \mathbb{Z}^s\}$, where $\delta_{0,\alpha}, \ \alpha \in \mathbb{Z}^s$, is the Kroneker delta symbol, i.e., $\delta_{0,0} = 1$ and zero otherwise. The compactly supported basic limit functions $\phi_k$ generated from the initial sequence $\delta$ are given by

$$\phi_k := \lim_{\ell \to \infty} S_{a^{(k+\ell)}}S_{a^{(k+1)}} \cdots S_{a^{(k)}}\delta, \ k \geq 1.$$  

An interesting fact about convergent non-stationary schemes is that the compactly supported basic limit functions $\phi_k$ are mutually refinable, i.e., they satisfy the functional equations

$$\phi_k = \sum_{\alpha \in \mathbb{Z}^s} a^{(k)}(\alpha)\phi_{k+1}(M \cdot -\alpha), \ k \geq 1, \quad (10)$$

where $\{a^{(k)}(\alpha), \ \alpha \in \mathbb{Z}^s\}$ is the $k$-level subdivision mask. We remark that, without loss of generality, to study convergence and regularity of a non-stationary subdivision scheme it suffices to study the continuity and the Hölder regularity of the function $\phi_1$. This fact is shown in the next lemma (see also [43]).

**Lemma 1.** Let $\alpha_{\phi_k}$ be the Hölder exponent of $\phi_k$, $k \geq 1$. If $a^{(k)}(0) \neq 0$, then $\alpha_{\phi_1} = \alpha_{\phi_k}, \ k \geq 1$.

**Proof.** Let $k \geq 1$. Due to [10] and the compact support of the mask $a^{(k)}$, we have $\alpha_{\phi_k} \geq \alpha_{\phi_{k+1}}$ and it suffices to show that $\alpha_{\phi_{k+1}} \geq \alpha_{\phi_k}$. To do that we show that, for any compactly supported function $h$, the operator $g = \Phi h = \sum_{\alpha \in \mathbb{Z}^s} a^{(k)}(\alpha)h(\cdot - \alpha)$ preserves the regularity of $h$. Note that, due to $a^{(k)}(0) \neq 0$, its symbol satisfies $a^{(k)}_*(z) \neq 0$ in the neighborhood of zero. Thus, the meromorphic function $b^{(k)}_*(z) = 1/a^{(k)}_*(z), \ z \in \mathbb{C}^s \setminus \{0\}$, has the Taylor expansion $b^{(k)}_*(z) = \sum_{\beta \in \mathbb{N}^s_0} b^{(k)}(\beta)z^\beta$ in the neighborhood of zero. Then, due to $a^{(k)}_*(z)b^{(k)}_*(z) = 1$ and by the Cauchy product formula, we get

$$\sum_{\beta \in \mathbb{N}^s_0} b^{(k)}(\beta)g(\cdot - \beta) = \sum_{\beta \in \mathbb{N}^s_0} \sum_{\alpha \in \{0, \ldots, N\}^s} b^{(k)}(\beta - \alpha)a^{(k)}(\alpha)h(\cdot - \beta) = b^{(k)}(0)a^{(k)}(0)h = h.$$  

Therefore, for the Hölder exponents of $g$ and $h$ we get $\alpha_h \geq \alpha_g$ and, thus, also $\alpha_{\phi_{k+1}} \geq \alpha_{\phi_k}$. \qed

For our analysis, for the sequence of masks $\{a^{(k)}, \ k \geq 1\}$ supported on $\{0, \ldots, N\}^s$, we define the so-called transition matrices $T^{(k)}_\varepsilon, \ k \geq 1, \ \varepsilon \in E$, as follows. Firstly, as in e.g. [10], we set

$$K := \sum_{r=1}^{\infty} M^{-r}G, \textrm{ where } G := \{0, \ldots, N\}^s - E + \{-1, 1\}^s, \quad (11)$$
and denote by $\ell(K)$ the linear space of all sequences supported in $K$ of cardinality $|K|$. Due to the choice of $K$, the linear operators

$$
\mathcal{L}^{(k)}_\varepsilon \mathbf{v} := \sum_{\beta \in \mathbb{Z}^s} v(\beta)a^{(k)}(\varepsilon + M \cdot -\beta), \quad \varepsilon \in E, \quad k \geq 1,
$$

map $\mathbf{v} \in \ell(K)$ into $\ell(K)$ and, thus, possess $|K| \times |K|$ matrix representations

$$
T^{(k)}_{\varepsilon,a^{(k)}} := \left[a^{(k)}(\varepsilon + M \alpha - \beta)\right]_{\alpha,\beta \in K}, \quad \varepsilon \in E.
$$

(13)

For simplicity we write $T^{(k)}_{\varepsilon}$ instead of $T^{(k)}_{\varepsilon,a^{(k)}}$. If the symbols $\{a^{(k)}(z), \ k \geq 1\}$ satisfy sum rules of order $\ell + 1$, then the operators $\mathcal{L}^{(k)}_\varepsilon$ have common invariant finite dimensional difference subspaces $V_j \subset \ell(K)$, $j = 0, \ldots, \ell$. This is definitely the case for $C^\ell$-convergent stationary subdivision schemes and is indeed used for analysis of convergence and regularity in the stationary setting. We refer the reader for example to the papers [3, 2, 5, 9, 35, 38] for details on the structure of $V_j$ and for characterizations of regularity of stationary subdivision schemes in terms of spectral properties of the matrices $T^{(k)}_{\varepsilon,a^{(k)}}|_{V_j}, \varepsilon \in E$. Similarly to (13), these matrices are derived from the stationary mask $a$ as follows: define

$$
T^{(k)}_{\varepsilon,a} := \left[a(\varepsilon + M \alpha - \beta)\right]_{\alpha,\beta \in K}, \quad \varepsilon \in E,
$$

and determine their restrictions $T^{(k)}_{\varepsilon,a}|_{V_j}$ to the subspace $V_j$. Since, in general, in the non-stationary setting, the existence of such invariant subspaces is not guaranteed by the regularity of the limit functions, in this paper we study non-stationary schemes $\{S^{(k)}_a, \ k \geq 1\}$ whose sequences of masks possess sets $\mathcal{A}$ of limit points, see Definition 4. This allows us, similarly to the stationary setting, to establish a link between the regularity of non-stationary schemes and the spectral properties of the collection of square $|K| \times |K|$ matrices $T^{(k)}_{\varepsilon,a}$ restricted to $V_j$, $j = 0, \ldots, \ell$. This collection we denote by $T_{\mathcal{A}}|_{V_j} := \{T^{(k)}_{\varepsilon,a}|_{V_j}, \varepsilon \in E, \ a \in \mathcal{A}\}$.

We conclude this section by recalling the notion of the joint spectral radius of a set of square matrices, see [47].

**Definition 11.** The joint spectral radius (JSR) of a compact collection of square matrices $\mathcal{M}$ is defined by

$$
\rho(\mathcal{M}) := \lim_{n \to \infty} \max_{M_j \in \mathcal{M}} \left\| \prod_{j=1}^n M_j \right\|^{1/n}.
$$

Note that it is easily proved that $\rho(\mathcal{M})$ is independent of the choice of the matrix norm $\| \cdot \|$. 

10
3 Convergence and Hölder regularity of non-stationary schemes

In this section we derive sufficient conditions for convergence and regularity of a certain big class of non-stationary subdivision schemes. Namely, in Subsection 3.1 we show that a non-stationary subdivision scheme \( \{ S_{a}^{(k)}, \ k \geq 1 \} \) is convergent if its symbols satisfy approximate sum rules of order 1 and, in addition, its sequence of masks \( \{ a^{(k)}, \ k \geq 1 \} \) possesses the set of limit points \( \mathcal{A} \) such that \( \rho (T_{\mathcal{A}}|_{V_0}) < 1 \). Note that the proof of this convergence result is also valid in the case of a general integer dilation matrix \( M \). In Subsection 3.2 we analyze the Hölder regularity of the basic limit function \( \phi_1 \) under the assumptions of approximate sum rules of order \( \ell + 1, \ \ell \geq 0, \) and \( \rho (T_{\mathcal{A}}|_{V_{\ell}}) < m^{-\ell} \). In Subsection 3.3 we prove Theorem \( \ref{thm:convergence} \) and show that under a certain stability assumption the quantity \( \rho (T_{\mathcal{A}}|_{V_{\ell}}) \) determines the exact Hölder exponent of the subdivision limits. This result allows us to prove in Subsection 3.4 a recent conjecture on regularity of Daubechies wavelets stated in \( \cite{24} \). In Subsection 3.4 we also illustrate our results with several examples.

We start by stating important properties of the set \( \mathcal{A} \).

**Proposition 1.** Let \( \ell \geq 0 \). Let \( \mathcal{A} \) be the set of limit points of \( \{ a^{(k)}, \ k \geq 1 \} \). Assume that \( \{ a_{z}^{(k)}(z), \ k \geq 1 \} \) satisfy approximate sum rules of order \( \ell + 1 \). Then, the symbols associated with the masks in \( \mathcal{A} \) satisfy sum rules of order \( \ell + 1 \).

**Proof.** The proof follows from Definition \( \ref{def:approximate} \) and the fact that approximate sum rules imply that \( \lim_{k \to \infty} \delta_k = \lim_{k \to \infty} \mu_k = 0 \). \qed

Next, we would like to remark that the class of the non-stationary schemes we analyze is not empty.

**Remark 1.** In general, for an arbitrary compact set \( \mathcal{A} \) of masks, there exists a non-stationary subdivision scheme \( \{ S_{a}^{(k)}, \ k \geq 1 \} \) with the set of limit points \( \mathcal{A} \). One possible way of constructing \( \{ S_{a}^{(k)}, \ k \geq 1 \} \) from a given set \( \mathcal{A} \) is presented in Example 2.

In our proofs we make use of the special structure of the matrices \( T_{\varepsilon,a}, \ a \in \mathcal{A}, \) and the matrices \( T_{\varepsilon}^{(k)} \) associated with a sequence of masks \( \{ a^{(k)}, \ k \geq 1 \} \). This structure is ensured after a suitable change of basis, which we discuss in the following remark. In the rest of the paper, we call such a basis a **transformation basis**.
Remark 2. Let $\ell \geq 0$ and $\Pi_j$ be the spaces of polynomials of total degree less than or equal to $j = 0, \ldots, \ell$. If the symbols of the masks $a \in A$ satisfy sum rules of order $\ell + 1$, then the corresponding stationary subdivision operators $S_a$ posses certain, possibly different for different $a$, polynomial eigensequences $\{p_a(\alpha), \alpha \in \mathbb{Z}^s\}$, $p_a \in \Pi_j, j = 0, \ldots, \ell$. For each $j = 0, \ldots, \ell$, the number $d_{j+1}$ of such eigensequences is equal to the number of monomials $x^\eta, \eta \in \mathbb{N}_0^s$, of total degree $|\eta| = j$, see [34, 37]. These eigensequences, written in a vector form with ordering of the entries as in (13), become common left-eigenvectors of the corresponding matrices $T_{\varepsilon,a}$. There are at least two different ways of constructing the so-called transformation basis of $\mathbb{R}^{[K]}$. The approach in [2] makes use of the eigensequences of the stationary subdivision operator. We cannot do that as the eigensequences of $S_a, a \in A$, possibly differ for different $a$. For that reason, we follow the approach in [20, 43], which makes use of the elements in the common invariant subspaces $V_j$ of $T_{\varepsilon,a}, a \in A$. For $K$ given in (11), the subspaces $V_j$ are usually defined by

$$V_j = \{ v \in \ell(K) : \sum_{\alpha \in K} v(\alpha)p(-\alpha) = 0, \quad p \in \Pi_j \}, \quad j = 0, \ldots, \ell,$$

see e.g. [2, 9, 38]. Then the transformation basis can be constructed as follows: Take the first unit vector of $\mathbb{R}^{[K]}$ and extend it to a basis of $\mathbb{R}^{[K]}$ by choosing appropriate $d_{j+2}$ sequences from some basis of $V_j, j = 0, \ldots, \ell - 1$, and a complete basis of $V_\ell$; write these sequences in the vector form, respecting the ordering in (13). Note that, by definition of $V_j$, any sequence from $V_j$ annihilates the polynomial eigensequences $\{p_a(\alpha), \alpha \in \mathbb{Z}^s\}, p_a \in \Pi_i, i = 0, \ldots, j$. We choose $d_{j+2}$ sequences, say $v_{j,\eta}$, from $V_j$ in such a way that each one of them annihilates all but one sequence $\{\alpha^\eta, \alpha \in \mathbb{Z}^s\}$ for the corresponding $\eta \in \mathbb{N}_0^s$ with $|\eta| = j + 1$ and transforms that one particular sequence into a constant sequence.

This choice of the transformation basis guarantees that the transformed matrices $T_{\varepsilon,a} \in \mathbb{R}^{[K] \times [K]}$, $\varepsilon \in E$, are block-lower triangular and of the form

$$\begin{pmatrix}
1 & & & \\
B_2 & & & \\
\vdots & & & \\
0 & b_{1,\varepsilon,a} & b_{2,\varepsilon,a} & \cdots & B_{\ell+1} & b_{\ell+1,\varepsilon,a}
\end{pmatrix},$$

(15)

where the $d_j \times d_j$ matrices $B_j$ are diagonal with diagonal entries equal to $m_{-j+1}$; the matrices
are of size $(|K| - \sum_{i=1}^{j} d_i) \times d_j$. Moreover, if $\{a^{(k)}_*(z), k \geq 1\}$ satisfy approximate sum rules of order $\ell + 1$, then, after the same change of basis, the matrices $T^{(k)}_\varepsilon \in \mathbb{R}^{|K| \times |K|}$, $\varepsilon \in E$, $k \geq 1$, are sums of a block-lower and a block-upper triangular matrices

$$
\tilde{T}^{(k)}_\varepsilon + \Delta^{(k)}_\varepsilon := \begin{pmatrix} 1 & B_2 & 0 \\ b_{1,\varepsilon}^{(k)} & \ddots & \ddots \\ b_{2,\varepsilon}^{(k)} & B_{\ell+1} & \ddots \\ b_{\ell,\varepsilon}^{(k)} & \ddots & \ddots \end{pmatrix} + \begin{pmatrix} c_{1,\varepsilon}^{(k)} & \cdots & \cdots \\ \vdots & \ddots & \ddots \\ \vdots & \cdots & c_{\ell,\varepsilon}^{(k)} \end{pmatrix}, \quad (16)
$$

where $b_{j,\varepsilon}^{(k)}$ are of size $(|K| - \sum_{i=1}^{j} d_i) \times d_j$; the matrices $c_{j,\varepsilon}^{(k)}$ are of size $d_j \times (|K| - \sum_{i=1}^{j-1} d_i)$; $O$ is the zero matrix of the same size as $Q^{(k)}_\varepsilon$.

The following example illustrates two important facts about approximate sum rules stated in Definition 3. Firstly, the re-scaling of all symbols of a non-stationary subdivision masks to ensure that $\mu_k = 0$, $k \geq 1$, can change the properties of the non-stationary scheme if the sequence $\{\mu_k, k \geq 1\}$ is not summable. In other words, on the contrary to the stationary case, the properties of $a^{(k)}_*(1)$, $k \geq 1$, are crucial for convergence and regularity analysis of non-stationary schemes. Secondly, even in the univariate case, the existence of the factor $(1 + z)$ for all non-stationary symbols $a^{(k)}_*(z)$ and the contractivity of the corresponding difference schemes do not guarantee the convergence of the associated non-stationary scheme, if $\{\mu_k, k \geq 1\}$ is not summable.

Example 1. Let $s = 1$, $M = 2$. It is well-known that the convergence of $S_a$ in the stationary case is equivalent to the fact that the difference scheme $S_b$ with the symbol $b_*(z)$ such that

$$
a_*(z) = (1 + z) b_*(z), \quad z \in \mathbb{C} \setminus \{0\},
$$

is zero convergent, i.e, for every $v \in \mathbb{R}^{|K|}$ orthogonal to a constant vector and $\varepsilon_1, \ldots, \varepsilon_k \in \{0, 1\}$, the norm $\|T^{(k)}_{\varepsilon_1, a} \cdots T^{(k)}_{\varepsilon_k, a} v\|$ goes to zero as $k$ goes to $\infty$. In the non-stationary case, this characterization is no longer valid. Consider the non-stationary scheme with the masks

$$
a^{(k)} := \left(1 + \frac{1}{k}\right) a, \quad k \geq 1. \quad (17)
$$
Note that \( \mu_k = \frac{2^k}{k} \), \( \delta_k = 0 \) and, thus, we can conclude that the non-stationary scheme \( \{S_{a(k)}, k \geq 1\} \) does not satisfy approximate sum rules. However, it is asymptotically similar to \( S_a \) and the associated symbols satisfy

\[
 a_{a}^{(k)}(z) := (1 + z) \left( 1 + \frac{1}{k} \right) b_{a}(z), \quad k \geq 1, \quad z \in \mathbb{C} \setminus \{0\}.
\]

We show next that the zero convergence of the associated difference schemes with symbols \((1 + \frac{1}{k}) b_{a}(z)\) does not imply the convergence of the corresponding non-stationary scheme. Indeed, for \( \varepsilon_j \in \{0, 1\} \), we get

\[
\|T_{\varepsilon_1}^{(1)} \cdots T_{\varepsilon_k}^{(k)} v\| = \prod_{j=1}^{k} \left( 1 + \frac{1}{j} \right) \|T_{\varepsilon_1,a} \cdots T_{\varepsilon_k,a} v\| = (k + 1) \|T_{\varepsilon_1,a} \cdots T_{\varepsilon_k,a} v\|.
\]

The convergence of \( S_a \) implies the existence of an operator norm such that

\[
\|T_{\varepsilon_1,a} \cdots T_{\varepsilon_k,a} v\| \leq C \gamma^k, \quad C > 0, \quad \gamma < 1.
\]

Therefore, the norm \( \|T_{\varepsilon_1}^{(1)} \cdots T_{\varepsilon_k}^{(k)} v\| \) goes to zero as \( k \) goes to \( \infty \), but the corresponding non-stationary scheme is not convergent. Otherwise, the Fourier-transform of its basic limit function \( \phi_1 \) would satisfy

\[
\hat{\phi}_1(\omega) = \prod_{j=1}^{\infty} a_{a}^{(j)}(e^{-i2\pi 2^{-j}\omega}), \quad \omega \in \mathbb{R},
\]

but

\[
\hat{\phi}_1(0) = \lim_{k \to \infty} 2 \prod_{j=1}^{k} \left( 1 + \frac{1}{j} \right) b_{a}(1) = \lim_{k \to \infty} 2(k + 1)b_{a}(1) = \infty.
\]

Note that, if we rescale the masks so that all \( \mu_k = 0, k \geq 1 \), we get back the convergent stationary scheme \( S_a \).

### 3.1 Convergence

We start by recalling that, in the stationary case, for convergence analysis via the joint spectral radius approach one uses the subspace

\[
V_0 := \{ v \in \ell(K) : \sum_{\alpha \in K} v(\alpha) = 0 \}, \tag{18}
\]

where \( K \) is defined in \([11]\). This subspace also plays an important role in the proof of the following Theorem that provides sufficient conditions for convergence of a certain big class of
non-stationary schemes. In the case $M = mI$, $m \geq 2$, Theorem 4 is an instance of Theorem 2 with $\ell = 0$. Note though that in the proof of Theorem 4 we do not assume that $M = mI$, $m \geq 2$, and, thus, we need a more general definition of approximate sum rules of order 1.

**Definition 12.** Let $\Xi = \{ e^{2\pi i M^{-T}} \xi : \xi \text{ is a coset representative of } \mathbb{Z}^s/M^T \mathbb{Z}^s \}$. The symbols $\{ a^{(k)}_e(z), \ k \geq 1 \}$ satisfy approximate sum rules of order 1, if the sequences $\{ \mu_k, \ k \geq 1 \}$ and $\{ \delta_k, \ k \geq 1 \}$ with

$$
\mu_k := \left| a^{(k)}_e(1) - |\det(M)| \right| \quad \text{and} \quad \delta_k := \max_{\epsilon \in \Xi \setminus \{1\}} | a^{(k)}_e(\epsilon) | \tag{19}
$$

are summable.

In the case of a general dilation matrix, the set $E$ is the set of coset representatives $E \simeq \mathbb{Z}^s/M \mathbb{Z}^s$.

**Theorem 4.** Assume that the sequence of symbols $\{ a^{(k)}_e(z), \ k \geq 1 \}$ satisfies approximate sum rules of order 1 and $\rho (T_A|_{V_0}) < 1$, where $A$ is the set of limit points of $\{ a^{(k)}, \ k \geq 1 \}$. Then the non-stationary scheme $\{ S_{a^{(k)}}, \ k \geq 1 \}$ is $C^0$-convergent.

**Proof.** By [29], the convergence of a non-stationary scheme is equivalent to the convergence of the associated cascade algorithm. Thus, to prove the convergence of the non-stationary scheme $\{ S_{a^{(k)}}, \ k \geq 1 \}$, we show, for $v \in \mathbb{R}^{|K|}$, that the vector-sequence with the elements

$$
T^{(1)}_{\epsilon_1} \cdots T^{(k)}_{\epsilon_k} v, \quad k \geq 1,
$$

converges as $k$ goes to infinity for every choice of $\epsilon_1, \ldots, \epsilon_k, \in E$.

Due to Proposition 1 each $a \in A$ satisfies sum rules of order 1. Therefore, by Remark 2, the vector $(1 \ 0 \ \ldots \ 0)$ is a common left eigenvector of all matrices

$$
T_{\epsilon, a} = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
{b}_{\epsilon, a} & T_{\epsilon, a}|_{V_0}
\end{pmatrix}, \quad \epsilon \in E, \quad a \in A.
$$

Due to the assumption of approximate sum rules of order 1, by Remark 2 we have

$$
T^{(k)}_{\epsilon} = \tilde{T}^{(k)}_{\epsilon} + \Delta^{(k)}_{\epsilon}, \quad \epsilon \in E, \quad k \geq 1, \tag{20}
$$

15
with

\[ \widetilde{T}^{(k)}_\varepsilon = \begin{pmatrix} 1 & 0 \cdots 0 \\ \epsilon (k) b_\varepsilon & \epsilon (k) Q_\varepsilon \end{pmatrix}, \quad \Delta^{(k)}_\varepsilon = \begin{pmatrix} \epsilon (k) c_\varepsilon \\ 0 \end{pmatrix}. \]

Thus, the canonical row unit vector \((1 \ 0 \ldots 0)\) is a quasi-common left-eigenvector of the operators \(T^{(k)}_\varepsilon, \varepsilon \in E\), i.e.

\[ (1 \ 0 \ldots 0) T^{(k)}_\varepsilon = (1 \ 0 \ldots 0) + \epsilon (k) c_\varepsilon, \]

where the row vector \(c_\varepsilon\) vanishes as \(k\) tends to infinity and the corresponding sequence of norms \(\{\|\Delta^{(k)}_\varepsilon\|, k \geq 1\}\) is summable in any norm. Moreover, \(b_\varepsilon\) and \(Q_\varepsilon\) converge by subsequences as \(k\) goes to infinity to \(b_\varepsilon, a\) and \(T^{(k)}_\varepsilon, a \mid V_0\) for some \(a \in A\), respectively.

By assumption \(\rho (\{T^{(k)}_\varepsilon, a \mid V_0, \varepsilon \in E, a \in A\}) < 1\). Thus, the existence of the operator norm of \(\{T^{(k)}_\varepsilon, a \mid V_0, \varepsilon \in E, a \in A\}\) and the continuity of the joint spectral radius imply that there exists \(\bar{k}\) such that \(\rho \left( \{Q^{(k)}_\varepsilon, \varepsilon \in E, k \geq \bar{k}\} \right) < 1\). This implies that for all vectors \(v \in \mathbb{R}^{\lfloor K \rfloor}\), the product \(\widetilde{T}^{(1)}_\varepsilon \cdots \widetilde{T}^{(k)}_\varepsilon v\) converges as \(k\) goes to infinity for every choice of \(\varepsilon_1, \ldots, \varepsilon_k \in E\).

By well-known results on the joint spectral radius of block triangular families of matrices (see e.g. [1]), we obtain that \(\rho \left( \{\widetilde{T}^{(k)}_\varepsilon, \varepsilon \in E, k \geq \bar{k}\} \right) = 1\). Moreover, the family of matrices \(\{\widetilde{T}^{(k)}_\varepsilon, \varepsilon \in E, k \geq \bar{k}\}\) is non-defective (see e.g. [31]), thus by [1, 47], there exists an operator norm \(\| \cdot \|\) such that \(\|\widetilde{T}^{(k)}_\varepsilon\| \leq 1\) for all \(\varepsilon \in E, k \geq \bar{k}\).

By assumption of approximate sum rules of order 1 we also have

\[ \|\Delta^{(k)}_\varepsilon\| \leq C \delta_k \text{ where } \sum_{k=1}^{\infty} \delta_k < \infty, \quad (22) \]

and \(C\) is a constant which does not depend on \(k\).

Next, for \(n, \ell \in \mathbb{N}\), we observe that

\[ T^{(n)}_{\varepsilon_n} \cdots T^{(n+\ell)}_{\varepsilon_{n+\ell}} = \left( \widetilde{T}^{(n)}_{\varepsilon_n} + \Delta^{(n)}_{\varepsilon_n} \right) \cdots \left( \widetilde{T}^{(n+\ell)}_{\varepsilon_{n+\ell}} + \Delta^{(n+\ell)}_{\varepsilon_{n+\ell}} \right) = \widetilde{T}^{(n)}_{\varepsilon_n} \cdots \widetilde{T}^{(n+\ell)}_{\varepsilon_{n+\ell}} + R_{n,\ell}, \]

where \(R_{n,\ell}\) is obtained by expanding all the products. From \(21\), \(22\) we get \(\lim_{n \to \infty} R_{n,\infty} = 0\).
implying convergence of \( \prod_{j=1}^{k} T_{\varepsilon_j}^{(j)} v \) as \( k \to \infty \). The reasoning for \( \lim_{n \to \infty} R_{n,\infty} = 0 \) is as follows

\[
\| R_{n,\infty} \| \leq \sum_{j=1}^{\infty} \left( \sum_{k=n}^{\infty} \delta_k \right)^{j} = \sum_{j=0}^{\infty} \left( \sum_{k=n}^{\infty} \delta_k \right)^{j} - 1 = \frac{\sum_{k=n}^{\infty} \delta_k}{1 - \sum_{k=n}^{\infty} \delta_k}.
\]

\[\square\]

**Corollary 2.** Let \( \{S_{a(k)}, \ k \geq 1\} \) be a \( C^0 \)-convergent subdivision scheme with the set of limit points \( A \) such that \( \rho(T_A|V_0) < 1 \). Then any other asymptotically similar non-stationary scheme \( \{S_{b(k)}, \ k \geq 1\} \) satisfying approximate sum rules of order 1 is \( C^0 \)-convergent.

We would like to remark that Theorem 4 generalizes [13, Theorem 10] dealing with the binary univariate case under the assumption that the non-stationary scheme reproduces constants. Theorem 4 is also a generalization of the corresponding results in [26, 27] that require that stationary and non-stationary schemes are asymptotically equivalent.

### 3.2 \( C^\ell \)-convergence and Hölder regularity

In this section we prove Theorem 2 stated in the Introduction, i.e. we derive sufficient conditions for Hölder regularity of non-stationary multivariate subdivision schemes. Note that Theorem 2 with \( \ell = 0 \) also implies the convergence of the corresponding non-stationary scheme. We, nevertheless, gave the convergence proof separately in Theorem 4 see Subsection 3.1 to emphasize that it is not affected by the choice of the dilation matrix \( M \), whereas the proof of Theorem 2 does depend on the choice of \( M = mI \), \( m \geq 2 \).

The proof of Theorem 2 is long, thus, in Subsection 3.2.1 we present several crucial auxiliary results and then prove Theorem 2 in the above Subsection 3.2.2.

#### 3.2.1 Auxiliary results

In the proof of Theorem 2 we make use of the summable sequence \( \{\eta_k, \ k \geq 0\} \) which we define next. Note first that under assumption \( \rho(T_A|V_0) < m^{-\ell} \) of Theorem 2 there exist \( 0 < \gamma < m^{-\ell} \) and \( \bar{k} \) such that

\[
\| Q_{\varepsilon}^{(k)} \| < \gamma < m^{-\ell}, \quad \varepsilon \in E, \quad k \geq \bar{k},
\]

(23)
where $Q_{\varepsilon}^{(k)}$ are sub-matrices of the matrices $\tilde{T}_{\varepsilon}^{(k)}$ in (14). This property of $Q_{\varepsilon}^{(k)}$ is guaranteed by the same line of reasoning as in the proof of Theorem 4. Furthermore, by approximate sum rules of order $\ell + 1$ (Definition 3), the sequence $\{\sigma_0 := 1, \sigma_k = m^k \delta_k, k \geq 1\}$ is summable and so is the sequence $\{\eta_k, k \geq 0\}$ with

$$\eta_k := \sum_{j=0}^{k} \sigma_j q^{k-j}, \quad q := m^\ell \gamma.$$  \hfill (24)

Indeed, since $q < 1$, we have

$$\sum_{k=0}^{\infty} \eta_k = \sum_{j=0}^{\infty} \sigma_j \sum_{n=0}^{\infty} q^n = \frac{1}{1-q} \sum_{j=0}^{\infty} \sigma_j < \infty. \hfill (25)$$

In the following Lemma 2 we estimate the asymptotic behavior of the matrix products

$$P_k = R_1 \cdots R_k, \quad k \geq 1,$$  \hfill (26)

where, for some non-negative real number $c$, the $(\ell + 2) \times (\ell + 2)$ matrices $R_j$ are defined by

$$R_j := \begin{pmatrix}
1 + \sigma_j m^{-\ell j} & \sigma_j m^{-\ell j} & \sigma_j m^{-\ell j} & \ldots & \sigma_j m^{-\ell j} & \sigma_j m^{-\ell j} \\
\vdots & m^{-1} + \sigma_j m^{-(\ell-1)j} & \sigma_j m^{-(\ell-1)j} & \ldots & \sigma_j m^{-(\ell-1)j} & \sigma_j m^{-(\ell-1)j} \\
c & c & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
c & \ldots & c & \sigma_j m^{-j} & \sigma_j m^{-j} \\
c & \ldots & c & m^{-\ell} + \sigma_j & \sigma_j \\
c & \ldots & c & \ldots & c & \gamma
\end{pmatrix} \hfill (27)$$

In particular, in Lemma 2 we show that, in any matrix norm, the $r$-th column of the matrix product $P_k, P_k e_r$, with $e_r$ being the standard $r$-th unit vector, is bounded uniformly over $k \geq 1$.

**Lemma 2.** For every $k \in \mathbb{N}$

$$\|P_k e_r\|_1 = \begin{cases} 
O(m^{-(r-1)k}), & r = 1, \ldots, \ell + 1, \\
O(m^{\ell+1}m^{-tk}), & r = \ell + 2.
\end{cases}$$

**Proof.** For simplicity of presentation we consider the case of $M = 2I$, i.e $m = 2$. Let $C_1$ be the smallest constant such that for each $r = 1, \ldots, \ell + 1$, the $r$-th column of $R_1$ does not exceed $C_1 2^{-(r-1)}$, and the $(\ell + 2)$nd column does not exceed $C_1 2^{\ell} \eta_1$. We show by induction on $k$
that the sum of the $r$–th column entries of $P_k$ does not exceed $C_k 2^{-(r-1)k}$, $r = 1, \ldots, \ell + 1$, or $C_k 2^{(\ell+1)2^{-\ell}k} \eta_k$, $r = \ell + 2$, where

$$ C_k = C_1 \prod_{j=2}^{k} \left( 1 + 2^{\ell+1} \sigma_j + c 2^{\ell+1} 2^{-j} + c 2^{2\ell+1} \eta_{j-1} \right), \quad k \geq 2. $$

Due to

$$ C_k = C_{k-1} \left( 1 + 2^{\ell+1} \sigma_k + c 2^{\ell+1} 2^{-k} + c 2^{2\ell+1} \eta_{k-1} \right), \quad k \geq 2, \tag{28} $$

the sequence $\{C_k, \ k \geq 1\}$ increases and converges to

$$ \check{C} := C_1 \prod_{j=2}^{\infty} \left( 1 + 2^{\ell+1} \sigma_j + c 2^{\ell+1} 2^{-j} + c 2^{2\ell+1} \eta_{j-1} \right). \tag{29} $$

Since the sums $\sum_{j=1}^{\infty} \sigma_j$, $\sum_{j=1}^{\infty} 2^{-j}$, and $\sum_{j=1}^{\infty} \eta_{j-1}$ are all finite, the infinite product in (29) converges.

By induction assumption, we have $\|P_{k-1} e_j\|_1 \leq C_{k-1} 2^{-(j-1)(k-1)}$, for $j = 1, \ldots, \ell + 1$, and $\|P_{k-1} e_{\ell+2}\|_1 \leq C_{k-1} 2^{\ell+1} 2^{-\ell(k-1)} \eta_{k-1}$. Since the $r$–th column of $P_k$ is $P_k e_r$, where $e_r$ is the $r$–th basis vector of $\mathbb{R}^{\ell+2}$, we have

$$ \|P_k e_r\|_1 = \|P_{k-1} R_k e_r\|_1 \leq \sum_{j=1}^{\ell+2} \|P_{k-1} e_j\|_1 \|R_k\|_{jr}. $$

Thus,

$$ \|P_k e_r\|_1 \leq \sum_{j=1}^{\ell+2} \|P_{k-1} e_j\|_1 \|R_k\|_{jr}. \tag{30} $$

Next we consider the following three cases.

Case 1: $r = 1$. The first column of the matrix $R_k$ is $(1 + \sigma_k 2^{-\ell k}, c, \ldots, c)^T$. By induction assumption and due to $\sum_{j=2}^{\ell+1} 2^{-(j-1)(k-1)} = 0$ for $\ell = 0$ and (28), the estimate (30) yields

$$ \|P_k e_1\|_1 \leq C_{k-1} \left( 1 + \sigma_k 2^{-\ell k} + c \sum_{j=2}^{\ell+1} 2^{-(j-1)(k-1)} + c 2^{\ell+1} 2^{-\ell(k-1)} \eta_{k-1} \right) \leq C_{k-1} \left( 1 + \sigma_k 2^{-\ell k} + 2 c 2^{-(k-1)} + c 2^{\ell+1} \eta_{k-1} \right) \leq C_k. $$

Case 2: $2 \leq r \leq \ell + 1$. The $r$–th column of the matrix $R_k$ is

$$ (R_k)_r = (\sigma_k 2^{-\ell k}, \ldots, \sigma_k 2^{-(\ell-r+2)k}, 2^{-(r-1)} + \sigma_k 2^{-(\ell-(r-1))k}, c, \ldots, c)^T. $$
By induction assumption the estimate in (30) becomes

\[
\begin{align*}
\|P_k e_r\|_1 & \leq C_{k-1} \left( \sigma_k \sum_{j=1}^{r-1} 2^{-(\ell+1-j)k} 2^{-(j-1)(k-1)} + (2^{-(r-1)} + \sigma_k 2^{-(\ell+1-r)k}) 2^{-(r-1)(k-1)} \\
& \quad + c \sum_{j=r+1}^{\ell+1} 2^{-(j-1)(k-1)} + c 2^{\ell+1} 2^{-(\ell-1)(k-1)} \eta_{k-1} \right) \\
& \leq C_{k-1} \left( \sigma_k 2^{\ell k} \sum_{j=1}^{r} 2^{j-1} + 2^{-(r-1)k} + 2 c 2^{-(r-1)k} + c 2^{\ell+1} 2^{-(r-1)(k-1)} \eta_{k-1} \right) \\
& \leq C_{k-1} \left( \sigma_k 2^{\ell k} 2^r + 2^{-(r-1)k} + 2^{r+1} 2^{-(r-1)k} 2^{-k} + c 2^{-(r-1)k} 2^{\ell+r} \eta_{k-1} \right) \\
& \leq C_{k-1} 2^{-(r-1)k} \left( 1 + 2^{\ell+1} \eta_k + c 2^{\ell} 2^{-k} + c 2^{\ell+1} \eta_{k-1} \right) \leq C_k 2^{-(r-1)k}.
\end{align*}
\]

Case 3: \( r = \ell + 2 \). The last column of the matrix \( R_k \) is \((\sigma_k 2^{-\ell k}, \ldots, \sigma_k 2^{-k}, \sigma_k, \gamma)^T\). Note that by definition of \( \eta_k \) we have \( \eta_k = \sigma_k = \eta_{k-1} \), and recall that \( \gamma < 2^{-\ell} \). Then, by induction assumption, we get

\[
\begin{align*}
\|P_k e_{\ell+2}\|_1 & \leq C_{k-1} \left( \sigma_k \sum_{j=1}^{\ell+1} 2^{-(\ell+1-j)k} 2^{-(j-1)(k-1)} + 2^{\ell+1} 2^{-(\ell-1)(k-1)} \gamma \eta_{k-1} \right) \\
& = C_{k-1} \left( \sigma_k 2^{-\ell k} \sum_{j=1}^{\ell+1} 2^{j-1} + 2^{\ell+1} 2^{-\ell k} (\eta_k - \sigma_k) \right) \\
& \leq C_{k-1} \left( \sigma_k 2^{-\ell k} 2^{\ell+1} + 2^{\ell+1} 2^{-\ell k} \eta_k - 2^{\ell+1} 2^{-\ell k} \sigma_k \right) \\
& = C_{k-1} 2^{\ell+1} 2^{-\ell k} \eta_k \leq C_k 2^{\ell+1} 2^{-\ell k} \eta_k.
\end{align*}
\]

The estimates in Lemma 2 allow us to estimate the norms of the columns of the matrix products \( T_{\varepsilon_1}^{(1)} \cdots T_{\varepsilon_k}^{(k)} \), \( \varepsilon_1, \ldots, \varepsilon_k \in E \).

**Lemma 3.** Let \( \varepsilon_1, \ldots, \varepsilon_k \in E \), \( \ell \geq 0 \). Assume that the symbols of \( \{S_{\alpha(k)}, k \geq 1\} \) satisfy approximate sum rules of order \( \ell + 1 \) and \( \rho(T_{\alpha}|V_\ell) < m^{-\ell} \). Then the norms of the columns of \( T_{\varepsilon_1}^{(1)} \cdots T_{\varepsilon_k}^{(k)} \) with indices \( 1 + \sum_{j=1}^{r-1} d_j, \ldots, \sum_{j=1}^{r} d_j \) are equal to \( O(m^{-(r-1)k}) \) for \( r = 1, \ldots, \ell + 1 \). The norms of the other columns of this matrix product are equal to \( O(m^{-\ell k}\eta_k) \).
Proof. Let \( \varepsilon \in E \). Under assumptions of Theorem 2, the matrices \( \tilde{T}_\varepsilon^{(k)} \) in (16) have the following properties: the matrix sequences \( \{b_{j,\varepsilon,a}^{(k)}, \, k \geq 1\} \) and \( \{Q_{\varepsilon,a}^{(k)}, \, k \geq 1\} \) converge by subsequences as \( k \) goes to \( \infty \), respectively, to \( b_{j,\varepsilon,a} \) and \( T_{\varepsilon,a} \) for some \( a \in A \); there exists \( c > 0 \) such that all the norms \( \|b_{j,\varepsilon,a}\|_\infty \leq c < \infty \); the estimate in (23) holds for \( 0 < \gamma < m^{-\ell} \) and for some matrix norm \( \|\cdot\|_{\text{ext}} \). Furthermore, approximate sum rules of order \( \ell+1 \) and the definition of \( \sigma_k \) imply that the entries of the matrices \( c_{j,\varepsilon}^{(k)}, \, j = 1, \ldots, \ell+1 \), are bounded by \( \sigma_k m^{-\ell-j}k \).

Let \( L_0 = 0 \) and \( L_i = \sum_{j=1}^{i} d_j, \, i = 1, \ldots, \ell+1 \), with \( d_j \) defined in Remark 2. Set \( L = L_{\ell+1} \) and write a vector \( v = (v_1, \ldots, v_{|K|})^T \in \mathbb{R}^{|K|} \) as

\[
v = (v[1], v[2], \ldots, v[\ell+1], v[\ell+2])
\]

with

\[
v[i] := (v_{L_i-1+1}, \ldots, v_{L_i})^T, \quad i = 1, \ldots, \ell+1, \quad v[\ell+2] := (v_{L+1}, \ldots, v_{|K|})^T.
\]

Consider the vector norm

\[
\|v\| := \sum_{i=1}^{\ell+1} \|v[i]\|_\infty + \|v[\ell+2]\|_{\text{ext}}, \quad v \in \mathbb{R}^{|K|}.
\]

Then

\[
\|T_\varepsilon^{(k)}v\| \leq \|R_k \tilde{v}\|, \quad \tilde{v} = (\|v[1]\|_\infty, \ldots, \|v[\ell+1]\|_\infty, \|v[\ell+2]\|_{\text{ext}}) \in \mathbb{R}^{\ell+2},
\]

where \( R_k \) is given in (27). Analogously, we get

\[
\|T_\varepsilon^{(1)} \cdots T_\varepsilon^{(k)}v\| \leq \|R_1 \cdots R_k \tilde{v}\|.
\]

The claim follows by Lemma 2.

3.2.2 Proof of Theorem 2

The proof of Theorem 2 is long, so we split it into two parts: Proposition 2 and Proposition 3. In the first part of the proof, given in Proposition 2, we show that the assumptions of Theorem 2 are indeed sufficient for the \( C^\ell \) convergence of non-stationary schemes. In particular, we let \( f \in C^\ell(\mathbb{R}^s) \) be compactly supported, stable and refinable with respect to the dilation matrix \( M = mI \) and the mask \( e \in \ell_0(\mathbb{Z}^s) \). Then, for every \( j = 0, \ldots, \ell \) and for every \( \nu \in \mathbb{N}_0^s, \, |\nu| = j \), we consider the sequence \( \{D^{\nu} f_k, \, k \geq 1\} \), where

\[
D^{\nu} f_k = m^{jk} T^{(1)} \cdots T^{(k)} D^{\nu} f, \quad f_k := T^{(1)} \cdots T^{(k)} f, \quad T^{(k)} f = \sum_{\alpha \in \mathbb{Z}^s} a^{(k)}(\alpha) f(M \cdot -\alpha), \quad (31)
\]
i.e. $T^{(k)}$ is the transition operator associated with the mask $a^{(k)}$, and show that $\{D^\nu f_k, \ k \geq 1\}$ converges uniformly to the $\nu$-th partial derivative of $\phi_1$.

**Proposition 2.** Let $\ell \geq 0$. Assume that the symbols of $\{S_{a^{(k)}}, \ k \geq 1\}$ satisfy approximate sum rules of order $\ell + 1$ and $\rho(T_A|V_\ell) < m^{-\ell}$. Then, for every $j = 0, \ldots, \ell$ and for every $\nu \in \mathbb{N}_0^s$, $|\nu| = j$, the sequence $\{D^\nu f_k, \ k \geq 1\}$ in (31) converges uniformly to the $\nu$-th partial derivative of $\phi_1$. Moreover, there exists a constant $C > 0$ independent of $k$ such that for $\eta_n$ as in (24) we have

$$
\|D^\nu f_k - D^\nu \phi_1\|_\infty \leq C \sum_{n=k}^{\infty} \eta_n, \quad |\nu| = \ell, \quad k \geq 1.
$$

(32)

**Proof.** Note that, by [9, p.137], the function $f$ is an appropriate starting function for the cascade algorithm. Moreover, by [36, Theorem 6.3], the assumptions on $f$ imply that $f$ satisfies Strang-Fix conditions of order $\ell + 1$, i.e. its Fourier transform $\hat{f}$ satisfies

$$
\hat{f}(0) = 1, \quad D^\mu \hat{f}(\alpha) = 0, \quad \alpha \in \mathbb{Z}^s \setminus \{0\}, \quad \mu \in \mathbb{N}_0^s, \quad |\mu| < \ell + 1.
$$

Consequently, its derivatives $D^\nu f, \ \nu \in \mathbb{N}_0^s, \ |\nu| = j, \ j = 1, \ldots, \ell$, satisfy

$$
D^\mu(D^\nu f)(\alpha) = 0, \quad \alpha \in \mathbb{Z}^s, \quad \mu \in \mathbb{N}_0^s, \quad |\mu| < j.
$$

Thus, by Poisson summation formula, we get

$$
\sum_{\alpha \in \mathbb{Z}^s} p(\alpha)D^\nu f(x - \alpha) = 0, \quad x \in \mathbb{R}^s, \quad (33)
$$

for all polynomial sequences $\{p(\alpha), \ \alpha \in \mathbb{Z}^s\}, \ p \in \Pi_j$. Note that we can chose $f$ such that $\text{supp} f \cap \mathbb{Z}^s \subset K$. Then, the properties (33) of $D^\nu f$ imply that, after the transformation discussed in Remark 2, the first $\sum_{i=1}^{\ell} d_i$ entries of the vectors

$$
v(x) := (D^\nu f(x + \alpha))_{\alpha \in K}, \quad x \in [0, 1]^s, \quad |\nu| = j, \quad j = 1, \ldots, \ell,
$$

are equal to zero. Note that the ordering of the entries in $v(x)$ corresponds to the ordering of the columns of $T^{(k)}_\varepsilon$ defined in (13). By Theorem 4, the limit functions of the non-stationary scheme are $C^0(\mathbb{R}^s)$, i.e. the sequence $\{f_k, \ k \geq 1\}$ is a uniformly convergent Cauchy sequence. Similarly to the stationary case, to show that the non-stationary scheme is $C^j$-convergent, $j = 1, \ldots, \ell$, we need to study the uniform convergence of the sequences $\{D^\nu f_k, \ k \geq 1\}$ for all
\( \nu \in \mathbb{N}_0^s, |\nu| = j \). Equivalently, for every choice of \( \varepsilon_1, \ldots, \varepsilon_k \in E \), need to study the convergence of the vector-sequences \( \{ m^{jk} T_{\varepsilon_1}^{(1)} \cdots T_{\varepsilon_{k}}^{(k)} w, \ k \geq 1 \} \), where \( T_{\varepsilon_i}^{(k)} \) are defined from \( \{ a^{(k)}, \ k \geq 1 \} \) and the vector \( w \in \mathbb{R}^{|K|} \) is arbitrary and such that its first \( \sum_{i=1}^{j} d_i \) entries are zero. Lemma \( \Box \) the structure of \( w \) and the summability of \( \{ \eta_k, \ k \geq 1 \} \) imply the convergence of the vector-sequences \( \{ m^{jk} T_{\varepsilon_1}^{(1)} \cdots T_{\varepsilon_{k}}^{(k)} w, \ k \geq 1 \} \) for \( j = 1, \ldots, \ell \). Thus, the non-stationary scheme is \( C^\ell \)-convergent.

We prove next the estimate \( (32) \). Let \( \nu \in \mathbb{N}_0^s, |\nu| = \ell \). Due to \( \phi_1 = \lim_{k \to \infty} T^{(1)} \cdots T^{(k)} f \) and by the assumption of refinability of \( f \), i.e. \( f = T f = \sum_{\alpha \in \mathbb{Z}^s} e(\alpha) f(\alpha) \), we have

\[
\| D^\nu f_k - D^\nu \phi_1 \|_\infty \leq \sum_{n=k}^{\infty} \| D^\nu f_{n+1} - D^\nu f_n \|_\infty = \sum_{n=k}^{\infty} m^{\ell(n+1)}\| T^{(1)} \cdots T^{(n)} (T^{(n+1)} - T) (D^\nu f)(M^{-(n+1)})) \|_\infty.
\]

As above, to estimate the norms \( \| T^{(1)} \cdots T^{(n)} (T^{(n+1)} - T) (D^\nu f)(M^{-(n+1)})) \|_\infty \), we need to estimate the vector-norms of

\[
T^{(1)}_{\varepsilon_1} \cdots T^{(n)}_{\varepsilon_n} (T^{(n+1)}_{\varepsilon_{n+1}} - T^{(n+1)}_{\varepsilon_{n+1}, e}) w,
\]

where \( |K| \times |K| \) matrices \( T_{\varepsilon, e}, \varepsilon \in E \), are derived from the mask \( e \), see \( [15] \), and the first \( \sum_{j=1}^{\ell+1} d_j \) entries of the vector \( w \in \mathbb{R}^{|K|} \) are zero. By assumption, there exists a constant \( \beta > 0 \) such that the entries of all \( b_{j,\varepsilon, e} \) and \( b_{j,\varepsilon, e}^{(k)} \) are less than \( \beta \) in the absolute value. The approximate sum rules of order \( \ell + 1 \) imply that the absolute values of the entries of the vectors \( (T^{(n+1)}_{\varepsilon_{n+1}} - T^{(n+1)}_{\varepsilon_{n+1}, e}) w \) with indices \( 1 + \sum_{j=1}^r d_j, \ldots, \sum_{j=1}^{r+1} d_j, r = 0, \ldots, \ell \), are bounded respectively by \( \sigma_{n+1} m^{-(\ell-r)(n+1)} \). All other entries are bounded by \( 2\beta \). Thus, by Lemma \( \Box \) we get that the entries of the vectors \( m^{\ell(n+1)}T^{(1)}_{\varepsilon_1} \cdots T^{(n)}_{\varepsilon_n} (T^{(n+1)}_{\varepsilon_{n+1}} - T^{(n+1)}_{\varepsilon_{n+1}, e}) w \) with indices \( 1 + \sum_{j=1}^r d_j, \ldots, \sum_{j=1}^{r+1} d_j, r = 0, \ldots, \ell \), are equal to \( O(\sigma_{n+1}) \), all other entries are equal to \( O(\eta_n) \). Therefore, by definition of \( \{ \eta_k, \ k \geq 1 \} \) in \( [21] \), we get

\[
\| D^\nu f_k - D^\nu \phi_1 \|_\infty \leq C \sum_{n=k}^{\infty} \eta_n, \ k \geq 1,
\]

for some \( C > 0 \) independent of \( k \). \( \Box \)
The second part of the proof of Theorem 2 is given in Proposition 3 which yields the desired estimate for the Hölder regularity $\alpha$ of the scheme $\{S_{a(k)}, \ k \geq 1\}$.

**Proposition 3.** Let $k \geq 1$, $h \in \mathbb{R}^s$, $m^{-(k+1)} < \|h\|_{\infty} \leq m^{-k}$ and $\ell \geq 0$. Assume that the symbols of $\{S_{a(k)}, \ k \geq 1\}$ satisfy approximate sum rules of order $\ell + 1$ and $\rho(T_A|V_\ell) < m^{-\ell}$.

Then there exists a constant $C > 0$ independent of $k$ such that, for $\eta_n$ as in (24), we have

$$\|D^\nu \phi_1(\cdot + h) - D^\nu \phi_1(\cdot)\|_{\infty} \leq C \sum_{n=k}^{\infty} \eta_n, \quad \nu \in \mathbb{N}_0^s, \ |\nu| = \ell. \quad (34)$$

Moreover, the Hölder exponent $\alpha$ of $\phi_1 \in C^\ell(\mathbb{R}^s)$ satisfies

$$\alpha \geq \min \left\{ -\log m \rho_A, -\limsup k \frac{\log m}{k} \right\}. \quad (35)$$

**Proof.** Let $k \geq 1$, $|\nu| = \ell$, and $h \in \mathbb{R}^s$ satisfy $m^{-(k+1)} < \|h\|_{\infty} \leq m^{-k}$. To derive the estimate in (34), we use the triangular inequality

$$\|D^\nu \phi_1(\cdot + h) - D^\nu \phi_1(\cdot)\|_{\infty} \leq \|D^\mu \phi_1(\cdot + h) - D^\mu f_k(\cdot + h)\|_{\infty} + \|D^\mu \phi_1 - D^\mu f_k\|_{\infty} \quad (36)$$

where $\{f_k, \ k \geq 1\}$ are defined in (31), and estimate each of the summands on the right hand side. Note that, for $\Delta_h f_k := f_k(\cdot + h) - f_k(\cdot)$, we have

$$\Delta_h D^\mu f_k = m^{\ell k} T^{(1)} \cdots T^{(k)} \Delta_m h D^\nu f.$$

Due to $\|m^k h\|_{\infty} \leq 1$ and by the definition of $\Delta_h$, we have

$$\text{supp} \Delta_m h D^\nu f \subset \text{supp} f + [-1, 1]^s,$$

where without loss of generality we assume that $(\text{supp} f + [-1, 1]^s) \cap \mathbb{Z}^s \subseteq K$. Define the vector-valued function

$$v(x) := (\Delta_m h D^\nu f(x + \alpha))_{\alpha \in K}, \quad x \in [0, 1]^s.$$

By the same argument as in the proof of Proposition 2 and by the definition of the operator $\Delta_h$, the first $\sum_{j=1}^{\ell+1} d_j$ components of $v$ are zero for all $x \in \mathbb{R}^s$. Therefore, by Lemma 3, we get

$$\|T^{(1)}_{\varepsilon_1} \cdots T^{(k)}_{\varepsilon_k} v(x)\| \leq C_1 m^{-\ell k} \eta_k \|v(x)\| \leq C_1 m^{-\ell k} \eta_k 2 C_2 |K|, \quad x \in \mathbb{R}^s,$$

24
where \( \|v(x)\| \leq 2C_2|K|, x \in \mathbb{R}^s \), due to \( \max_{|\nu|=\ell} \|D^{\nu}f\|_\infty \leq C_2 \). Thus,

\[
\|D^{\nu}f_k(\cdot + h) - D^{\nu}f_k(\cdot)\|_\infty \leq C_3 \eta_k, \quad C_3 := C_1 m^{-\ell k} 2C_2 |K|.
\]

The estimates for the two remaining terms in (36) and, thus, the estimate (34) follow by (32).

Next, we derive the lower bound for the H"older exponent \( \alpha \) of \( \phi_1 \). Note that, by definition of \( \sigma_k \), we have the equivalence

\[
\limsup_{k \to \infty} \frac{\sigma_1}{k} \geq 1 \iff \ell + \limsup_{k \to \infty} \frac{\log m \delta_k}{k} \geq 0.
\]

Thus, if \( \limsup_{k \to \infty} \sigma_1/k \geq 1 \), then

\[
\min \left\{ -\log m \rho_A, -\limsup \log m \delta_k \right\} \leq \ell
\]

and the estimate (33) holds, since \( \phi_1 \in C^\ell(\mathbb{R}^s) \) and, thus, \( \alpha \geq \ell \). Otherwise, if \( \limsup_{k \to \infty} \sigma_1/k < 1 \), then there exists \( \theta \) such that

\[
\limsup_{k \to \infty} \sigma_1/k < \theta < 1
\]

and, thus, a constant \( C_0 > 0 \) such that \( \sigma_k \leq C_0 \theta^k, k \geq 1 \). Therefore, by definition of \( \eta_k \) and using the estimate (33), we get

\[
\|\Delta_h D^{\nu} \phi_1\|_\infty \leq C \left( \eta_k + \sum_{n=k+1}^{\infty} \eta_n \right) = C \left( \sum_{j=0}^{k} \sigma_j q^{k-j} + \frac{1}{1-q} \sum_{n=k+1}^{\infty} \sigma_n \right)
\]

\[
\leq CC_0 \left( \sum_{j=0}^{k} \theta^j q^{k-j} + \frac{1}{1-q} \sum_{n=k+1}^{\infty} \theta^n \right)
\]

\[
\leq CC_0 \left( \sum_{j=0}^{k} \max \{\theta, q\}^j \max \{\theta, q\}^{k-j} + \frac{\theta^{k+1}}{1-q} \sum_{n=k+1}^{\infty} \theta^{n-(k+1)} \right)
\]

\[
\leq CC_0 \left( (k+1) \max \{\theta, q\}^k + \frac{\theta^{k+1}}{(1-q)(1-\theta)} \right).
\]

Therefore, due to \( 0 \leq 1 - q < 1 \), we get

\[
\|\Delta_h D^{\nu} \phi_1\|_\infty \leq C_4 (k+1) \max \{\theta, q\}^k, \quad C_4 := \max \left\{ 1, \frac{\theta}{1-\theta} \right\} \frac{CC_0}{1-q}.
\]

Moreover, due to \( \|h\|_\infty \leq m^{-k} \), we have

\[
\max \{\theta, q\}^k = m^{k \log_m \max \{\theta, q\}} \leq \|h\|_\infty^{-\log_m \max \{\theta, q\}}
\]
and, from $\frac{1}{m} \| h \|_{\infty} \leq m^{-(k+1)}$, we get $(k + 1) \leq \log_m \left( \frac{m}{\| h \|_{\infty}} \right)$. Thus,

$$\| \Delta_h D^{\nu} \phi_1 \|_{\infty} \leq C_1 \log_m \left( \frac{m}{\| h \|_{\infty}} \right) \| h \|_{\infty}^{-\log_m \max\{\theta, q\}}.$$ 

Note that for any $\epsilon \in (0, 1)$, due to the fact that $-\log(t)$ is bounded by $t^{-\epsilon}$ for sufficiently small $t$, we get, for small $\| h \|_{\infty}$,

$$\| \Delta_h D^{\nu} \phi_1 \|_{\infty} \leq C_1 \| h \|_{\infty}^{-\log_m \max\{\theta, q\} - \epsilon}.$$ 

By (23) and (24), we have $q > m^{\ell} \rho_A$. Thus, since $\theta > \limsup_{k \to \infty} \sigma_k^{1/k}$, we finally get

$$\alpha \geq \ell - \log_m \max\{\theta, q\} = \ell - \max\{\ell + \log_m \rho_A, \ell + \limsup_{k \to \infty} \frac{\log_m \delta_k}{k}\} = \min\{-\log_m \rho_A, -\limsup_{k \to \infty} \frac{\log_m \delta_k}{k}\}.$$ 

Combining Proposition 2 and Proposition 3, we complete the proof of Theorem 2.

### 3.3 Rapidly vanishing approximate sum rules defects

The following immediate consequence of Theorem 2 states that, if the sequence of defects $\{\delta_k, k \geq 1\}$ of the approximate sum rules decays fast, then the lower bound on the Hölder exponent $\alpha$ of $\phi_1$ only depends on the joint spectral radius $\rho_A$ of the set $T_A|_{V_\ell}$.

**Corollary 3.** Assume that the symbols of $\{S_{a^{(k)}}, k \geq 1\}$ satisfy approximate sum rules of order $\ell + 1$ and $\rho(T_A|_{V_\ell}) < m^{-\ell}$. If $\limsup_{k \to \infty} \delta_k^{1/k} < \rho_A$, then $\alpha \geq -\log_m \rho_A$.

Next, in this subsection we prove Theorem 3 stated in the Introduction. It shows that the inequality $\alpha \geq -\log_m \rho_A$ in Corollary 3 becomes equality, if the set $A$ of the limit points of the sequence $\{a^{(k)}, k \geq 1\}$ consists only of a single element $a$ and the corresponding refinable limit function of $S_a$ is stable.

Note that Theorem 3 is a generalization of a well-known fact about the exact Hölder regularity of stationary schemes in the stable case.

In the proof of Theorem 3 we make use of several auxiliary facts on long matrix products. The first one of them is stated in the following lemma which is a special case of [44, Proposition 2].
Lemma 4. Let $\mathcal{M}$ be a compact set of $d \times d$ matrices and $y \in \mathbb{R}^d$. If $\rho(\mathcal{M}) > 1$ and $y$ does not belong to a common invariant subspace of the matrices in $\mathcal{M}$, then the sequence $\left\{ \max_{P_n \in \mathcal{M}^n} \| P_n y \|, \ n \geq 1 \right\}$ goes to $\infty$ as $n \to \infty$.

Lemma 4 and the definition of the sequence $\left\{ \max_{P_n \in \mathcal{M}^n} \| P_n y \|, \ n \geq 1 \right\}$ yield the following.

Lemma 5. Let $\mathcal{M}$ be a compact set of $d \times d$ matrices and $y \in \mathbb{R}^d$. If $\rho(\mathcal{M}) > 1$ and $y$ does not belong to a common invariant subspace of the matrices in $\mathcal{M}$, then there exists $n \geq L$ such that

$$\| M_1 \cdots M_n y \| > \| y \| \quad \text{and} \quad \| M_1 \cdots M_n y \| > \| M_{n-i} \cdots M_n y \|, \ i = 0, \ldots, n - 2, \ M_j \in \mathcal{M}.$$ 

Proof. Let $L \in \mathbb{N}$ and $C_L = \max \{ \| P_j y \| \mid P_j \in \mathcal{M}^j, j \leq L \}$. Then the shortest product $P_n \in \mathcal{M}^n$ such that $\| P_n y \| > C_L$ (the set of such products is nonempty by Lemma 4) possesses the desired property and has its length bigger than $m$.

Next, we adapt Lemma 5 to the non-stationary setting. The proof of the following result is similar to the proof of Lemma 5 and we omit it.

Lemma 6. Let $\mathcal{M}$ and $\mathcal{M}^{(k)}$, $k \geq 1$, be compact sets of $d \times d$ matrices and $y \in \mathbb{R}^d$. Assume that $\rho(\mathcal{M}) > 1$, the sequence $\{ \mathcal{M}^{(k)}, \ k \geq 1 \}$ converges to $\mathcal{M}$ and $y$ does not belong to a common invariant subspace of the matrices in $\mathcal{M}$. Then there exists $L \in \mathbb{N}$ and $C > 0$ such that for any $\tilde{L} \geq L$ there exists $n \geq \tilde{L}$ such that, for $M_j \in \mathcal{M}^{(j+L-1)}$,

$$\| M_1 \cdots M_n y \| > C \| y \| \quad \text{and} \quad \| M_1 \cdots M_n y \| > C \| M_{n-i} \cdots M_n y \|, \ i = 0, \ldots, n - 2.$$ 

We are ready to prove Theorem 3.

Proof of Theorem 3: Due to Corollary 3, we only need to show that $\alpha \leq -\log_m \rho_\alpha$. Furthermore, by Lemma 1, it suffices to show that $\alpha = \alpha_{\phi_n} \leq -\log_m \rho_\alpha$ for some $n \geq 1$. We choose an appropriate $n$ in the following way. Firstly, $n$ should be such that

$$\rho(\{ Q_{\varepsilon}^{(k)}; \ \varepsilon \in E, \ k \geq n \}) < \rho_\alpha.$$ 

(See Remark 2 for the definition of the matrices $Q_{\varepsilon}^{(k)}$.) Secondly, since by assumption, there exists $\beta > 0$ such that

$$\limsup_{k \to \infty} \frac{\delta_k^{1/k}}{\frac{\delta_k^{1/k}}{\beta} < \rho_\alpha,$$
thus, by definition of \( \limsup \), we can choose \( n \) such that for any constant \( C_0 > 0 \) we have \( \delta_k < C_0 \beta^k \) for \( k \geq n \). At the end of the proof we specify the particular constant \( C_0 \) needed for our argument. Next, define
\[
v(x) = (D^\nu \phi_n(x + \alpha))_{\alpha \in K}, \quad x \in [0, 1]^s, \quad \nu \in \mathbb{N}_0^s, \quad |\nu| = \ell.
\]
Let \( k \geq 1 \). By definition of \( \phi_n \), for \( x = \sum_{j=1}^k \varepsilon_j m^{-j}, \varepsilon_j \in E \), and \( \|h\|_\infty \leq m^{-1} \), we have
\[
\Delta_{m^{-k}h}v(x) = m^k T_{\varepsilon_1}^{(n)} \Delta_{m^{-k+1}h}v \left( \sum_{j=2}^k \varepsilon_j m^{-j} \right) = m^k T_{\varepsilon_1}^{(n)} \cdots T_{\varepsilon_k}^{(n)} \Delta_h v(0).
\]
(37)

By the same argument as in Proposition 2, the first \( L = \sum_{j=1}^{\ell+1} d_j \) components of the vector \( y := \Delta_h v(0) \) are zero. Denote by \( \tilde{y} := (y_{L+1}, \ldots, y_{|K|})^T \) the non-zero components of \( y \). W.l.o.g. we can assume that the vector \( \tilde{y} \) does not belong to any common invariant subspace of the matrices in \( \{T_{\varepsilon,a}|_{V_\ell} : \varepsilon \in E\} \). Otherwise, due to the stability of \( \phi \) we have
\[
\alpha_\phi = -\log_m \rho_a = -\log_m \rho\{T_{\varepsilon,a}|W : \varepsilon \in E\},
\]
where \( W \) is the smallest subspace of \( V_\ell \) such that it is invariant under all operators in \( \{T_{\varepsilon,a} : \varepsilon \in E\} \) and such that \( T_{\varepsilon,a}|W, \varepsilon \in E \), do not have any common invariant subspace. For simplicity, we assume that \( W = V_\ell \), but the same argument we give below would apply, if \( W \) is a proper subspace of \( V_\ell \).

Let \( r \in (\beta, \rho_a) \) be a real number. The sets
\[
\mathcal{M} = \{r^{-1}T_{\varepsilon,a}|_{V_\ell}, \varepsilon \in E\} \quad \text{and} \quad \mathcal{M}^{(k)} = \{r^{-1}Q_{\varepsilon}^{(n+k-1)}, \varepsilon \in E\}, \quad k \geq 1,
\]
and the vector \( \tilde{y} \) satisfy the assumptions of Lemma 6. Thus, we can appropriately modify \( n \) chosen above to get
\[
\|Q_{\varepsilon_1}^{(n)} \cdots Q_{\varepsilon_k}^{(n+k-1)} \tilde{y}\| > C r^k \|\tilde{y}\| \quad \text{and}
\]
\[
\|Q_{\varepsilon_1}^{(n)} \cdots Q_{\varepsilon_k}^{(n+k-1)} \tilde{y}\| > C r^{k-i} \|Q_{\varepsilon_{k-i+1}}^{(n+k-1)} \cdots Q_{\varepsilon_k}^{(n+k-1)} \tilde{y}\|, \quad i = 1, \ldots, k - 1.
\]
(38)

Denote by \( H_{j}^{(n+k-1)} \in \mathbb{R}^{1 \times |K|} \) the \( j \)-th row of the matrix \( T_{\varepsilon}^{(n+k-1)} \), \( \varepsilon \in E \). Define \( y_0 := y \), the we have
\[
T_{\varepsilon_k}^{(n+k-1)} y_0 = \left( \begin{array}{c} 0 \\ \vdots \\ 0 \\ Q_{\varepsilon_k}^{(n+k-1)} \tilde{y} \end{array} \right) + \sum_{j=1}^{L} \langle H_{j}^{(n+k-1)}, y_0 \rangle e_j,
\]
(39)

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where \( e_j, j = 1, \ldots, L \), are the standard first \( L \) unit vectors of \( \mathbb{R}^{|K|} \) and \( \langle H_j^{(n+k-1)}, y_0 \rangle \) is the scalar product of the vectors \( H_j^{(n+k-1)} \) and \( y_0 \). Define \( y_1 := \left( 0 \ldots 0 \ Q_{e_k}^{(n+k-1)} \ y_0 \right)^T \). Then, applying \( T_{\varepsilon_1}^{(n)} \ldots T_{\varepsilon_{k-1}}^{(n+k-2)} \) to both sides of (38), we get

\[
T_{\varepsilon_1}^{(n)} \ldots T_{\varepsilon_{k-1}}^{(n+k-2)} y_1 = T_{\varepsilon_1}^{(n)} \ldots T_{\varepsilon_{k-1}}^{(n+k-2)} y_0 - \sum_{j=1}^{L} \langle H_j^{(n+k-1)}, y_0 \rangle T_{\varepsilon_1}^{(n)} \ldots T_{\varepsilon_{k-1}}^{(n+k-2)} e_j ,
\]

and, thus, by triangular inequality,

\[
\| T_{\varepsilon_1}^{(n)} \ldots T_{\varepsilon_{k}}^{(n+k-1)} y_0 \| \geq \| T_{\varepsilon_1}^{(n)} \ldots T_{\varepsilon_{k-1}}^{(n+k-2)} y_1 \| - \sum_{j=1}^{L} \| \langle H_j^{(n+k-1)}, y_0 \rangle \| T_{\varepsilon_1}^{(n)} \ldots T_{\varepsilon_{k-1}}^{(n+k-2)} e_j \| .
\]

Note that \( n \) is such that, for any \( n + k - i \geq n \), the matrix \( T_{\varepsilon}^{(n+k-i)} \) is bounded by the matrix \( R_{n+k-i} \), in the sense of Lemma 3. Then, due to the structure of \( y_0 \), we have \( \| \langle H_j^{(n+k-1)}, y_0 \rangle \| = \mathcal{O}(m^{-(l-j+1)(n+k-1)} \sigma_{n+k-1}) \| y_0 \|. \) By Lemma 3 we also obtain the estimate \( \| T_{\varepsilon_1}^{(n)} \ldots T_{\varepsilon_{k-1}}^{(n+k-2)} e_j \| = \mathcal{O}(m^{-(j-1)k}), j = 1, \ldots, L \). And, thus,

\[
\sum_{j=1}^{L} \| \langle H_j^{(n+k-1)}, y_0 \rangle \| T_{\varepsilon_1}^{(n)} \ldots T_{\varepsilon_{k-1}}^{(n+k-2)} e_j \| = \sum_{j=1}^{L} \mathcal{O}(m^{-(l-j+1)(n+k-1)} \sigma_{n+k-1} m^{-(j-1)k}) \| y_0 \| = \mathcal{O}(m^{-l(n+k-1)} \sigma_{n+k-1}) \| y_0 \| .
\]

The definition of \( \sigma_k \) and the choice of \( \beta \) yield \( \mathcal{O}(m^{-l(n+k-1)} \sigma_{n+k-1}) = \mathcal{O}(\delta_{n+k-1}) < \hat{C} C_0 \beta^{n+k-1}, \) \( \hat{C} > 0 \). Therefore,

\[
\| T_{\varepsilon_1}^{(n)} \ldots T_{\varepsilon_{k}}^{(n+k-1)} y_0 \| \geq \| T_{\varepsilon_1}^{(n)} \ldots T_{\varepsilon_{k-1}}^{(n+k-2)} y_1 \| - \hat{C} C_0 \beta^{n+k-1} \| y_0 \| .
\]

Set \( y_i := \left( 0 \ldots 0 \ Q_{e_k}^{(n+k-i)} \ y_i \right)^T, i = 2, \ldots, k \). Then, analogous, successive argument for \( \| T_{\varepsilon_1}^{(n)} \ldots T_{\varepsilon_{k-i}}^{(n+k-i)} y_{i-1} \|, i = 2, \ldots, k \), yields

\[
\| T_{\varepsilon_1}^{(n)} \ldots T_{\varepsilon_{k}}^{(n+k-1)} y_0 \| \geq \| y_k \| - \hat{C} C_0 \sum_{i=0}^{k-1} \beta^{n+k-i-1} \| y_i \| .
\]

From (38) we get \( \| y_k \| < r^{-k+i} C^{-1} \| y_k \|, i = 0, \ldots, k - 1 \), which implies

\[
\| T_{\varepsilon_1}^{(n)} \ldots T_{\varepsilon_{k}}^{(n+k-1)} y_0 \| > \left( 1 - \frac{\hat{C} C_0 \beta^{-1}}{C} \sum_{i=0}^{k-1} \left( \frac{\beta}{r} \right)^{k-i} \right) \| y_k \| > \left( 1 - \frac{\hat{C} C_0 \beta^{-1}}{C(1 - \frac{2}{r})} \right) \| y_k \| .
\]

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In the second estimate above we used the fact that \( \beta < r \). Choose \( 0 < C_0 < \frac{C(1-\frac{\beta}{r})}{C(\beta n - 1)} \) and define \( C_1 := 1 - \frac{\hat{C}_\beta \beta n}{C(\beta n - 1)} > 0 \). Therefore, by (38), we have \( \|y_k\| > C r^k \|y_0\| \) and, thus,

\[
\|T_{\varepsilon_1}^{(n)} \cdots T_{\varepsilon_k}^{(n+k-1)} y_0\| > C_1 \|y_k\| > C_1 C r^k \|y_0\|, \quad k \geq 1.
\]

Finally, this estimate and (37) yield \( \|\Delta_m^{-1} v(x)\| > C_1 C r^k m^{\ell k} \|y_0\|, \quad k \geq 1 \). Therefore, the Hölder exponents of all \( D^\nu \phi_n, \nu \in N_0^s, |\nu| = \ell \), are bounded from above by \( -\ell - \log_m r \) and, thus, \( \alpha = \alpha \phi_n \leq -\log_m r \). Taking the limit as \( r \) goes to \( \rho_a \), we obtain the desired estimate \( \alpha \leq -\log_m \rho_a \).

If the symbols of the scheme \( \{S_a^{(k)}, \ k \geq 1\} \) satisfy sum rules of order \( \ell + 1 \), then we get the following immediate consequence of Theorem 3.

**Corollary 4.** Let \( \ell \geq 0 \). Assume that the stationary scheme \( S_a \) is \( C^\ell \)-convergent with the stable refinable basic limit function \( \phi \) whose Hölder exponent \( \alpha \phi \) is \( \ell \leq \alpha \phi < \ell + 1 \). If the symbols of the scheme \( \{S_a^{(k)}, \ k \geq 1\} \) satisfy sum rules of order \( \ell + 1 \) and \( \lim_{k \to \infty} a^{(k)} = a \), then \( \{S_a^{(k)}, \ k \geq 1\} \) is \( C^\ell \)-convergent and the Hölder exponent of its limit functions is also \( \alpha \phi \).

### 3.4 Applications and examples

In this section, see Subsection 3.4.1, we prove the conjecture formulated in [24], which stipulates the Hölder regularity of the generalized Daubechies wavelets. The proof of this conjecture is a direct consequence of Theorem 3. We also determine the exact Hölder regularity of some of such generalized Daubechies wavelets. Moreover, in Subsection 3.4.2, we illustrate our theoretical convergence and Hölder regularity results with several examples for which neither the results of [13] nor the ones in [26, 27] are applicable.

Note that, in this section, we use the techniques from [32] that allow for exact computation of the joint spectral radius of the corresponding matrix sets. The method in [32] determines the so-called spectrum maximizing product of such sets, which yields the exact value of the joint spectral radius.

**Definition 13.** Let \( \mathcal{M} \) be a compact collection of square matrices. The product \( P := M_1 \cdots M_m, \ M_j \in \mathcal{M}, \) is spectrum maximizing, if \( \rho(\mathcal{M}) = \rho(P)^{1/m} \), where \( \rho(P) \) is the spectral radius of \( P \).
3.4.1 Exact Hölder regularity of generalized Daubechies wavelets

The non-stationary Daubechies wavelets are defined and studied in [24] and are obtained from Daubechies wavelets in [19] by suitable perturbation of the roots of the stationary symbols. Let \( n \geq 2 \). To an arbitrary set \( \Lambda_n := \{\lambda_0, \ldots, \lambda_{n-1}\} \) of real numbers \( \lambda_j, j = 0, \ldots, n - 1 \), the authors in [24] associate the generalized Daubechies wavelet function \( \psi_{\Lambda_n} \). The corresponding refinable function

\[
\phi^{\Lambda_n} := \lim_{k \to \infty} S_{a^{(k)}} S_{a^{(k-1)}} \cdots S_{a^{(1)}} \delta
\]

is the limit function of a non-stationary subdivision scheme \( \{S_{a^{(k)}}, k \geq 1\} \) reproducing exponential polynomials, i.e., solutions of the ODE of order \( n \) with constant coefficients and with spectrum \( \Lambda_n \). The interested reader can find more details on the construction and properties of these wavelets \( \psi^{\Lambda_n}, n \geq 2 \), in [24].

Next we would like to mention the following two properties of these masks \( \{a^{(k)}, k \geq 1\} \):

(i) the sequence of masks \( \{a^{(k)}, k \geq 1\} \) converges to the mask \( m_n \) of the classical \( n \)-th Daubechies refinable function

\[
\varphi_n := \lim_{k \to \infty} S^k_{m_n} \delta;
\]

(ii) the corresponding symbols \( \{a^{(k)}_*(z), k \geq 1\} \) satisfy approximate sum rules of order \( n \) with \( \delta_k = O(2^{-nk}), k \geq 1 \).

The following conjecture was formulated in [24].

**Conjecture A.** Let \( n \geq 2 \). For every set \( \Lambda_n = \{\lambda_0, \ldots, \lambda_{n-1}\} \), the Hölder regularity of the generalized Daubechies type wavelet \( \psi^{\Lambda_n} \) is equal to the Hölder regularity of the classical Daubechies wavelet \( \psi_n \) derived from \( \varphi_n \).

We prove this conjecture using Theorem 3.

**Theorem 5.** Conjecture A holds true.

*Proof.* Since a compactly supported wavelet function has the same regularity as the corresponding refinable function, we need to show that the functions \( \phi^{\Lambda_n} \) and \( \varphi_n \) have the same regularity. The non-stationary subdivision scheme \( \{S_{a^{(k)}}, k \geq 1\} \) generating \( \phi^{\Lambda_n} \) satisfies the assumptions of Theorem 2 with \( \ell = n - 1 \) and \( A = \{m_n\} \). Indeed, the masks of the scheme \( \{S_{a^{(k)}}, k \geq 1\} \) are constructed in [24] in such a way that they converge to the mask \( m_n \). The Daubechies refinable function \( \varphi_n \) is stable and, hence, its Hölder exponent is \( \alpha_{\varphi_n} = -\log_2 \rho_A \). It
is well-known that $\alpha_{\varphi_n} < n$, therefore $\rho_A > 2^{-n}$. Thus, by $(ii)$ we have $\limsup_{k \to \infty} \delta^{1/k}_k \leq 2^{-n} < \rho_A$. Therefore, all assumptions of Theorem 3 are satisfied and the Hölder exponent $\alpha$ of $\phi^\Lambda_n$ satisfies $\alpha = \alpha_{\varphi_n} = -\log_2 \rho_A$. 

In [24] the Hölder exponent $\alpha$ is estimated by the rate of decay of the Fourier transform $\hat{\phi}^\Lambda_n$ of $\phi^\Lambda_n$. It is well-known that for any continuous, compactly supported function $f$, its Hölder exponent $\alpha_f$ satisfies

$$\eta(f) - 1 \leq \alpha_f \leq \eta(f), \quad \eta(f) = \sup \{ \beta \geq 0 : |\hat{f}(\omega)| \leq C(1 + |\omega|)^{-\beta}, \omega \in \mathbb{R} \},$$

and this gap of length 1 is, in general, unavoidable [48]. In [24, Theorem 29] the authors show that $\eta(\phi^\Lambda_n) \geq \eta(\varphi_n)$, which, thus, implies the following lower bound for the Hölder exponent $\alpha$ of $\phi^\Lambda_n$

$$\alpha \geq \eta(\varphi_n) - 1.$$

Using lower bounds for the values $\eta(\varphi_n)$ known from the literature, one can estimate the regularity of the generalized Daubechies wavelets. Table 1 compares those rough bounds given in [19] (computed by the method of invariant cycles) with the exact values of $\alpha = -\log_2 \rho_A$, which we compute using the techniques in [32].

| $n$ | $\eta(\varphi_n) - 1$ | $\alpha = -\log_2 \rho_A$ |
|-----|-------------------|-------------------|
| 2   | 0.339             | 0.5500            |
| 3   | 0.636             | 1.0878            |
| 4   | 0.913             | 1.6179            |
| 5   | 1.177             | 1.9690            |
| 6   | 1.432             | 2.1891            |
| 7   | 1.682             | 2.4604            |
| 8   | 1.927             | 2.7608            |
| 9   | 2.168             | 3.0736            |
| 10  | 2.406             | 3.3614            |

(40)

### 3.4.2 Further examples

In this subsection we apply our convergent and regularity results to several non-stationary subdivision schemes whose analysis was impossible so far.
**Example 2:** We start with a non-stationary subdivision scheme with a general dilation matrix $M$ and masks which are level dependent convex combination of two multivariate masks $a, b \in \ell_0(\mathbb{Z}^s)$. We assume that $a$ defines a (stationary) convergent subdivision scheme and that $b$ satisfies sum rules of order 1. Convex combinations of such subdivision masks were also investigated in [7, 12]. In particular, we define the non-stationary subdivision scheme \( \{ S_{a(k)}, k \geq 1 \} \) by

\[
a^{(k)} := \left( 1 - \frac{1}{k} \right) a + \frac{1}{k} b, \quad k \geq 1.
\]

(41)

This non-stationary scheme does not satisfy the conditions in (2) for \( \ell = 0 \), since

\[
\sum_{k \in \mathbb{N}} \max_{\varepsilon \in E} \left\{ \sum_{\alpha \in \mathbb{Z}^s} |a^{(k)}(\varepsilon + M\alpha) - a(\varepsilon + M\alpha)| \right\} = \max_{\varepsilon \in E} \left\{ \sum_{\alpha \in \mathbb{Z}^s} |b(\varepsilon + M\alpha) - a(\varepsilon + M\alpha)| \right\} \cdot \sum_{k \in \mathbb{N}} \frac{1}{k}.
\]

Nevertheless, \( \{ S_{a(k)}, k \geq 1 \} \) satisfies the assumptions of Theorem 4 since by construction, all symbols satisfy approximate sum rules of order 1 and \( \lim_{k \to \infty} a^{(k)} = a \). Therefore, we are able to conclude that the scheme is at least \( C^0 \)-convergent. Moreover, in the case \( M = mI \), the assumptions that \( S_a \) is \( C^\ell \)-convergent and that \( b \) satisfies sum rules of order \( \ell + 1 \), imply, by Theorem 2, that the Hölder regularity of the scheme in (41) is at least as high as for \( S_a \). Indeed, for \( s = 2 \) and \( M = 2I \), let \( a \) be the mask of the butterfly scheme (30) with \( \omega = 1/16 \) and \( b \) be the mask of the Courant element \( B_{111} \) [21]. Then, using the method in [32], we compute \( \rho(T_A|V_0) = 1/4 \) and, thus, the scheme \( \{ S_{a(k)}, k \geq 1 \} \) is \( C^1 \)-convergent and its Hölder exponent is \( \alpha = 2 \).

In the next example we construct non-stationary schemes with sets of limit points \( \mathcal{A} \) of cardinality 2.

**Example 3:** Let \( I \subset \mathbb{N} \) be some infinite set, such that \( \mathbb{N} \setminus I \) is also infinite. We consider the non-stationary scheme with the masks

\[
a^{(k)} := \begin{cases} a, & k \in I, \\ c, & k \in \mathbb{N} \setminus I, \end{cases} \quad k \geq 1.
\]

(42)

We assume that the masks \( a, c \in \ell_0(\mathbb{Z}^s) \) define stationary convergent subdivision schemes with the same dilation matrix \( M \). Moreover, we assume that \( \rho(T_A|V_0) < 1 \), \( \mathcal{A} = \{ a, c \} \). Here the notion of asymptotical equivalence is not applicable, but Theorem 4 allows us to establish \( C^0 \)-convergence of the scheme in (42). If \( M = mI \) and \( a, c \) are such that \( \rho(T_A|V_0) < m^{-\ell} \), Theorem 2 also yields a lower bound for the Hölder regularity of \( \{ S_{a(k)}, k \geq 1 \} \).
For example, for \( s = 1 \) and \( M = 2 \), let

\[
a_*(z) := \frac{1}{8}(1 + z)^4 \quad \text{and} \quad c_*(z) := \frac{1}{16}(-1 + 9z^2 + 16z^3 + 9z^4 - z^6), \quad z \in \mathbb{C} \setminus 0,
\]

be the symbols of the cubic B-spline and the 4-point scheme (with \( \omega = 1/16 \)), respectively. Using the method in [22] we obtain \( \rho(\mathcal{T}_4|V_i) = 0.35385... \). This tells us that the corresponding scheme \( \{S_{a(a)} \}, \ k \geq 1 \} \) has the Hölder exponent \( \alpha \geq 1.49876... \). For the computation of \( \rho(\mathcal{T}_4|V_i) \), we used the set \( \mathcal{T}_4|V_i = \{T_1, T_2, T_3, T_4\} \) with

\[
T_1 := T_{0,e}|V_i = \begin{pmatrix}
\frac{1}{8} & \frac{1}{8} & 0 & 0 \\
-\frac{1}{16} & \frac{3}{8} & -\frac{1}{16} & 0 \\
0 & \frac{1}{8} & \frac{1}{16} & 0 \\
0 & -\frac{1}{16} & -\frac{1}{8} & -\frac{1}{16}
\end{pmatrix}, \quad T_2 := T_{1,e}|V_i = \begin{pmatrix}
-\frac{1}{16} & \frac{3}{8} & -\frac{1}{16} & 0 \\
0 & \frac{1}{8} & -\frac{1}{8} & 0 \\
0 & -\frac{1}{16} & \frac{3}{8} & -\frac{1}{16} \\
0 & 0 & \frac{1}{8} & \frac{1}{8}
\end{pmatrix},
\]

\[
T_3 := T_{0,a}|V_i = \begin{pmatrix}
\frac{1}{8} & 0 & 0 & 0 \\
\frac{1}{8} & \frac{1}{4} & \frac{1}{8} & 0 \\
0 & \frac{1}{8} & \frac{1}{4} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad T_4 := T_{1,a}|V_i = \begin{pmatrix}
\frac{1}{4} & \frac{1}{8} & 0 & 0 \\
0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{8} \\
0 & 0 & 0 & \frac{1}{8} \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

The spectrum maximizing product we obtain is \( T_1(T_1T_3)^{13} \).

If, instead of the mask \( a \) above, we take the mask of the quadratic B-spline, then we obtain \( \rho(\mathcal{T}_4|V_i) = 0.35045... \), which tells us that the corresponding scheme \( \{S_{a(a)} \}, \ k \geq 1 \} \) has Hölder exponent \( \alpha \geq 1.51271... \). For the computation of \( \rho(\mathcal{T}_4|V_i) \) we used the set \( \mathcal{T}_4|V_i = \{T_1, T_2, T_3, T_5\} \) with

\[
T_5 := T_{0,a}|V_i = \begin{pmatrix}
\frac{1}{4} & 0 & 0 & 0 \\
0 & \frac{1}{4} & \frac{1}{4} & 0 \\
0 & 0 & 0 & \frac{1}{4} \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad T_6 := T_{1,a}|V_i = \begin{pmatrix}
\frac{1}{4} & \frac{1}{4} & 0 & 0 \\
0 & \frac{1}{4} & \frac{1}{4} & 0 \\
0 & 0 & 0 & \frac{1}{4} \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

The spectrum maximizing product is \( T_1(T_1T_5)^2 \). Note that \( \rho(\mathcal{T}_4|V_i) \) can be bigger than either \( \rho(\mathcal{T}_a|V_i) \) or \( \rho(\mathcal{T}_c|V_i) \). It is also of interest that \( \rho(\mathcal{T}_4|V_i) \) decreases, if we replace the mask of the cubic B-spline by the mask of the less regular scheme corresponding to the quadratic B-spline.

The next example studies a univariate ternary \( (M = 3) \) non-stationary scheme.
Example 4: We consider the alternating sequence of symbols

\[
\begin{align*}
d^{(k)}_s(z) := & \begin{cases} 
c^{(k)}_s(z) := z^{-6}K^{(k)}_1 (z^2 + z + 1)^2(z + 1)c^{(k)}_s(z), & k \text{ even,} \\
d^{(k)}_s(z) := z^{-6}K^{(k)}_2 (z^2 + z + 1)^2(z + 1)d^{(k)}_s(z), & k \text{ odd,}
\end{cases} \quad k \geq 1, 
\end{align*}
\]

where \(K^{(k)}_1\) and \(K^{(k)}_2\) are suitable normalization constants and the factors \(c^{(k)}_s(z)\) and \(d^{(k)}_s(z)\) are defined by

\[
\begin{align*}
c^{(k)}_s(z) := & \left(z^4 + (4(w^{(k)})^2 - 2)z^3 + (16(w^{(k)})^4 - 16(w^{(k)})^2 + 3)z^2 + (4(w^{(k)})^2 - 2)z + 1\right) \\
& \times \left(16(w^{(k)})^4 + 16(w^{(k)})^3 + 3)z^2 + (-64(w^{(k)})^6 - 64(w^{(k)})^5 + 32(w^{(k)})^4 + 32(w^{(k)})^3 - 12(w^{(k)})^2 - 12w^{(k)} - 6)z + 16(w^{(k)})^4 + 16(w^{(k)})^3 + 3\right),
\end{align*}
\]

and

\[
\begin{align*}
d^{(k)}_s(z) := & \left(z^2 + z + 1\right) \left(w^{(k)}z^4 + 2w^{(k)}z^3 + (4(w^{(k)})^2 - 1)z^2 + 2w^{(k)}z + 1\right),
\end{align*}
\]

with \(w^{(k)} := \frac{1}{2} \left(e^{-3-(k+1)\lambda/2} + e^{-3-(k+1)\lambda/2}\right)\) and \(\lambda \in \mathbb{R}^+ \cup i\mathbb{R}^+\).

The corresponding non-stationary subdivision scheme \(\{S^{(k)}_{\ell}, k \geq 1\}\) was considered in [8, 15]. In [15], the authors investigate the convergence of the sequence of symbols \(\{c^{(k)}_s(z), k \geq 1\}\) to the symbol

\[
c_s(z) = -z^{-6} \frac{1}{1296} (z^2 + z + 1)^4(z + 1)(35z^2 - 94z + 35)
\]

of the ternary dual stationary 4-point Dubuc-Deslaurier scheme, which is known to be at least \(C^2\)-convergent. The sequence of the symbols of the non-stationary subdivision scheme \(\{S^{(k)}_{\ell}, k \geq 1\}\) converges to the symbol

\[
d_s(z) = -z^{-6} \frac{1}{162} (z^2 + z + 1)^5(z + 1),
\]

see [8]. The stationary scheme \(S_{\ell}\) is known to be at least \(C^2\)-convergent. We would like to remark that \(\{S^{(k)}_{\ell}, k \geq 1\}\) and \(\{S^{(k)}_{\ell}, k \geq 1\}\) are both schemes generating/reproducing certain spaces of exponential polynomials, see [14].

Using Theorem 2 with \(A = \{c, d\}\) and the method in [32], we determine a lower bound for the Hölder regularity of the scheme \(\{S^{(k)}_{\ell}, k \geq 1\}\). Since we get \(\rho(T_A|_{V_2}) = 0.04958\ldots\), the corresponding Hölder exponent satisfies \(\alpha \geq 2.73437\ldots\). For the computation of \(\rho(T_A|_{V_2})\) we use the set \(T_A = \{T_1, T_2, T_3, T_4, T_5, T_6\}\) with \(T_j := T_{j-1}\), \(j = 1, 2, 3,\)

\[
T_1 = \frac{1}{1296} \begin{pmatrix} 35 & 0 & 0 \\ -83 & 35 & -24 \\ 0 & 35 & 24 \end{pmatrix}, \quad T_2 = \frac{1}{1296} \begin{pmatrix} -24 & 35 & 0 \\ -24 & -83 & -83 \\ 0 & 0 & 35 \end{pmatrix}, \quad T_3 = \frac{1}{1296} \begin{pmatrix} -83 & -24 & 35 \\ 35 & -24 & -83 \\ 0 & 0 & 0 \end{pmatrix},
\]

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and with $T_j := T_{j-4} a|_{V_2}$, $j = 4, 5, 6$, 

$$T_4 = \frac{1}{162} \begin{pmatrix} 1 & 0 & 0 \\ 5 & 5 & 3 \\ 0 & 1 & 3 \end{pmatrix}, \quad T_5 = \frac{1}{162} \begin{pmatrix} 3 & 1 & 0 \\ 3 & 5 & 5 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_6 = \frac{1}{162} \begin{pmatrix} 5 & 3 & 1 \\ 1 & 3 & 5 \\ 0 & 0 & 0 \end{pmatrix}.$$

The spectrum maximizing product is $T_1 T_3$, which implies that the Hölder exponent $\alpha$ coincides with the Hölder exponent of the scheme $S_c$.

In the next two examples, we construct and analyze the regularity of a univariate and a multivariate non-stationary subdivision schemes obtained by suitable perturbations of the masks of the known stationary subdivision schemes. These non-stationary schemes are not asymptotically equivalent to any stationary scheme and, thus, the results of [26] are not applicable. Note though that these schemes satisfy approximate sum rules of order 2 and the other assumptions of Theorem 2.

**Example 5:** For $s = 1$ and $M = 2$, we consider the sequence of masks $\{a^{(k)}, k \geq 1\}$ with 

$$a^{(k)} := \left\{ \left. \left( \frac{1}{4} - \frac{1}{k} \right), \left( \frac{3}{4} - \frac{1}{k} + 2^{-2k} \right), \left( \frac{3}{4} + \frac{1}{k} \right), \left( \frac{1}{4} + \frac{1}{k} + 2^{-2k} \right) \right\} \right., \quad k \geq 1. \quad (44)$$

Obviously, $\lim_{k \to \infty} a^{(k)} = a$, where $a = \left\{ \frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4} \right\}$ is the mask of the Chaikin subdivision scheme [4]. It is easy to check that the symbols of this non-stationary scheme satisfy 

$$a^{(k)}_s(1) - 2 = 2^{-2k+1}, \quad a^{(k)}_s(-1) = -2^{-2k+1} \quad \text{and} \quad Da^{(k)}_s(-1) = 2^{-2k+2}, \quad k \geq 1,$$

i.e. $\mu_k = \delta_k = 2^{-2k+1}$ and, thus, the symbols satisfy approximate sum rules of order 2. To be able to apply Theorem 2 we need to rescale the masks $a^{(k)}$ so that $\mu_k = 0$, $k \geq 1$. It is easily done by multiplying each of the masks $a^{(k)}$ by the factor $2/(2 + \mu_k)$. After this modification the sequence $\{\delta_k, k \geq 1\}$ is still summable, since 

$$\sum_{k \in \mathbb{N}} \frac{2\delta_k}{2 + \mu_k} < \sum_{k \in \mathbb{N}} \delta_k < \infty.$$ 

Hence, by Theorem 2 and the known fact that $\rho(T_a|_{V_1}) = \frac{1}{4}$, the non-stationary scheme with masks in (44) is $C^1-$convergent with $\alpha = 2$.

**Example 6:** For $s = 2$ and $M = 2I$, we consider the sequence of masks $\{a^{(k)}, k \geq 1\}$ with 

$$a^{(k)} = \frac{1}{16} \begin{pmatrix} 0 & 2 - 2k & 1 + \frac{1}{k} & 2 - \frac{1}{k} & 1 - \frac{1}{k} \\ -\frac{1}{k} & 2 - \frac{k}{k} + 2^{-2k} & 6 + \frac{1}{k} - 2^{-2k} & 6 + \frac{1}{k} + 2^{-2k} & 2 + 2^{-2k} \\ 1 - \frac{1}{k} & 6 + \frac{1}{k} + 2^{-2k} & 10 + \frac{2}{k} & 6 + \frac{1}{k} & 1 - \frac{1}{k} \\ 2 + 2^{-k} & 6 + \frac{1}{k} + 2^{-2k} & 6 + \frac{1}{k} + 2^{-2k} & 2 - \frac{1}{k} + 2^{-2k} & -\frac{1}{k} \\ 1 - \frac{1}{k} & 2 - \frac{1}{k} & 1 + \frac{1}{k} & 2^{-2k} & 0 \end{pmatrix}, \quad k \geq 1. \quad (45)$$
Obviously, \( \lim_{k \to \infty} a^{(k)} = a \), where \( a \) is the mask of the Loop subdivision scheme \([40]\). Note that the symbols of this non-stationary scheme satisfy approximate sum rules of order 2, since, we have \( \mu_k = 5 \cdot 2^{-(2k+4)} \) and \( \delta_k = 6 \cdot 2^{-(2k+4)} \), \( k \geq 1 \). The results in \([28]\) imply that \( \rho(T_a|_{V_1}) = \frac{1}{4} \). Thus, after an appropriate normalization of the masks, by Theorem \([2]\) we get that the non-stationary scheme is \( C^1 \)–convergent with the Hölder exponent \( \alpha = 2 \).

### 4 Further properties of non-stationary schemes

In this section we consider the univariate case only, i.e. \( M = 2 \). In Subsection 4.1 we show that the approximate sum rules in Definition 3 are very close to being necessary conditions for Hölder regularity of such non-stationary schemes. This resembles the stationary setting and motivates our multivariate results in Section 3. For an example of a divergent non-stationary scheme that violates approximate sum rules see Example 1 in Section 3.

In particular, we prove Theorem 1 in the Introduction. It states that, under the certain stability assumption, the \( C^\ell \)–regularity of a non-stationary scheme implies that the sum rules defects

\[
\delta_k := \max_{j \leq \ell} 2^{-k j} |D_j a^{(k)}(-1)|, \quad k \geq 1, \quad \ell \geq 0,
\]

must decay faster than \( 2^{-\ell k} \). It is easy to see that there is still a gap, even in the case \( \ell = 0 \), between this necessary condition \( \lim_{k \to \infty} \delta_k = 0 \) and the sufficient condition \( \sum_{k \in \mathbb{N}} \delta_k < \infty \). Nevertheless, Theorem 1 shows that, even in the simplest, univariate case, the sufficient conditions for convergence and regularity of non-stationary scheme we present in Section 3 cannot be relaxed much further. Thus, it is not to be expected that one can drastically further improve the results in \([13, 26, 27]\).

In the last Subsection 4.2 we show that already in the univariate case the limit functions of convergent non-stationary schemes belong to some special function spaces.

#### 4.1 Necessary conditions for regularity of non-stationary schemes

In this subsection, we prove Theorem 1 stated in Subsection 1.1. Its proof is based on the next Proposition 4 that studies the infinite products of certain trigonometric polynomials. The statement of Proposition 4 involves the following concepts.
Definition 14. A pair of complex numbers \( \{z, -z\} \) is called a pair of symmetric roots of the algebraic polynomial \( q \), if \( q(z) = q(-z) = 0 \).

Let \( \{q_k, k \geq 1\} \) be a sequence of algebraic polynomials of degree \( N \) and define the function

\[
f(x) := \prod_{k=1}^{\infty} p_k(2^{-k}x), \quad p_k(x) := q_k(e^{-2\pi ix}), \quad x \in \mathbb{R}.
\]

By [11], if a sequence of trigonometric polynomials \( \{p_k, k \geq 1\} \) is bounded, then this infinite product converges uniformly on each compact subset of \( \mathbb{R} \), and hence, \( f \) is analytic.

Proposition 4. Assume that the sequence of trigonometric polynomials \( \{p_k, k \geq 1\} \) with \( p_k(0) = 1, k \geq 1 \), converges to a trigonometric polynomial \( p \) that has no symmetric roots on \( \mathbb{R} \). If the function \( f \) in (46) satisfies \( f(x) = o(x^{-\ell}) \) for \( \ell \geq 0 \) and \( x \to +\infty \), then \( \delta_k = o(2^{-\ell k}) \) as \( k \to \infty \), where

\[
\delta_k = \max_{j=0, \ldots, \ell} 2^{-jk} \frac{|D^j p_k(1/2)|}{j!}, \quad k \geq 1.
\]

Proof. By assumption \( f(x) = o(x^{-\ell}) \) for points of the form \( x = 2^{k-1}d + t \), where \( d \) is a fixed natural number, \( t \) is an arbitrary number from \( [0, \sigma] \), \( \sigma > 0 \), and \( k \to \infty \). Next, we choose these parameters \( d \in \mathbb{N} \) and \( \sigma > 0 \) in a special way.

Firstly, we define \( \sigma \). Since \( \{p_k, k \geq 1\} \) converges to \( p \), the sequence \( \{p_k, k \geq 1\} \) is bounded. Moreover, \( p_k(0) = 1, k \geq 1 \), implies that \( f(0) = 1 \). This implies that there are \( \sigma \in (0, 1) \) and \( C_0 > 0 \) such that for every \( r \geq 0 \) and \( R \in \mathbb{N} \cup \{\infty\} \) we have

\[
\left| \prod_{j=1}^{\infty} p_{j+r}(2^{-j}t) \right| \geq C_0, \quad t \in [0, \sigma].
\]

Next we choose the number \( d \). To this end we consider the binary tree defined as follows: the number \( 1/2 \) is at the root, the numbers \( 1/4 \) and \( 3/4 \) are its children, and so on. Every vertex \( \alpha \) has two children \( \alpha/2 \) and \( (\alpha+1)/2 \). For convenience we shall identify a vertex and the corresponding number. Thus, all vertices of the tree are dyadic points from the interval \( (0, 1) \).

Indeed, the \( n \)-th level of the tree (i.e., the set of vertices with the distance to the root equal to \( n \)) consists of points \( 2^{-n-1}j \), where \( j \) is an odd number from \( 1 \) to \( 2^{n+1} - 1 \).

The trigonometric polynomial \( p \) is \( 1 \)-periodic and, thus, has at most \( N \) zeros in \( [0, 1) \), and hence, on the tree. Therefore, there is a number \( q \) such that all roots of \( p \) on the tree are contained on levels \( j \leq q \). Since the polynomial \( p \) has no symmetric roots, at least one of the
two children of any vertex of the tree is not a root of \( p \). Whence, there is a path of length \( q \) along the tree starting at the root (all paths are without backtracking) that does not contain any root of \( p \). Let \( 2^{-q-1}d \) be the final vertex of that path, \( d \) is an odd number, \( 1 \leq d \leq 2^{q+1} - 1 \). Denote as usual by \( \{x\} \) the fractional part of \( x \). Then the sequence \( \{2^{-1}d\}, \ldots, \{2^{-q-1}d\} \) does not contain roots of \( p \). The sequence \( \{2^{-q-2}d\}, \{2^{-q-3}d\}, \ldots \) does not contain them either, because there are no roots of \( p \) on levels bigger than \( q \). Let \( n \) be the smallest natural number such that \( 2^{-q-n-1}d < \sigma/2 \). We have \( p(2^{-1}d) \cdots p(2^{-q-n-1}d) \neq 0 \). Since \( p_k \to p \) as \( k \to \infty \), and all \( p_k \) are equi-continuous on \( \mathbb{R} \), it follows that there is a constant \( C_1 > 0 \) such that

\[
\left| \prod_{j=1}^{q+n} p_{k+j}(2^{-j-1}d + 2^{-k-j}x) \right| \geq C_1, \quad x \in [0, \sigma], \tag{48}
\]

for sufficiently large \( k \). Now we are ready to estimate the value \( f(2^{k-1}d + t) \). We have

\[
\left| f(2^{k-1}d + t) \right| = \left| \prod_{j=1}^{k-1} p_j(2^{k-1-j}d + 2^{-j}t) \right| \times \left| p_k(2^{-1}d + 2^{-k}t) \right| \times \left| \prod_{j=1}^{q+n} p_{k+j}(2^{-j-1}d + 2^{-k-j}t) \right| \times \left| \prod_{j=1}^{\infty} p_{k+q+n+j}(2^{-j}(2^{-q-n-1}d + 2^{-k-q-n}t)) \right| .
\]

To estimate the first factor in this product, we note that \( 2^{k-1-j}d \in \mathbb{Z} \), whenever \( j \leq k - 1 \), and hence \( p_j(2^{k-1-j}d + 2^{-j}t) = p_j(2^{-j}t) \). Thus, the first factor is \( \left| \prod_{j=1}^{k-1} p_j(2^{-j}t) \right| \), which is, by (47), bigger than or equal to \( C_0 \), for every \( t \in [0, \sigma] \).

The third factor \( \left| \prod_{j=1}^{q+n} p_{k+j}(2^{-j-1}d + 2^{-k-j}t) \right| \), by (48), is at least \( C_1 \). Finally, the last factor is bigger than or equal to \( C_0 \). To see this it suffices to use (47) for \( R = \infty, r = k + q + n, x = 2^{-q-n}d + 2^{-k-q-n}t \) and note that \( x < \sigma \) by the choice of \( n \). Thus,

\[
\left| f(2^{k-1}d + t) \right| \geq C_0^2 C_1 \left| p_k(2^{-1}d + 2^{-k}t) \right| .
\]

On the other hand, by assumption, \( f(2^{k-1}d + t) = o(2^{-\ell k}) \) as \( k \to \infty \), consequently \( p_k(2^{-1}d + 2^{-k}t) = o(2^{-\ell k}) \). The number \( d \) is odd, hence, by periodicity, \( p_k(2^{-1}d + 2^{-k}t) = p_k(1/2 + 2^{-k}t) \).

Thus, we arrive at the following asymptotic relation: for every \( t \in [0, \sigma] \) we have

\[
p_k(1/2 + 2^{-k}t) = o(2^{-\ell k}) \quad \text{as } k \to \infty . \tag{49}
\]

This already implies that \( D^j p_k(1/2) = o(2^{(j-\ell)k}) \) as \( k \to \infty \), for every \( j = 0, \ldots, \ell \). Indeed, consider the Tailor expansion of the function \( h(t) = p_k(1/2 + 2^{-k}t) \) at the point 0 with the
remainder in Lagrange form:

\[ h(t) = \sum_{j=0}^{\ell} \frac{D^j h(0)}{j!} t^j + \frac{D^{\ell+1} h(\theta)}{\ell!} t^{\ell+1}, \quad t \in [0, \sigma], \]

where \( \theta = \theta(t) \in [0, t] \). Substituting \( D^j h(0) = 2^{-jk} D^j p_k(1/2) \), we get

\[ p_k(1/2 + 2^{-k} t) = \sum_{j=0}^{\ell} \frac{D^j p_k(1/2)}{j!} 2^{-jk} t^j + \frac{D^{\ell+1} p_k(1/2 + 2^{-k} \theta)}{\ell!} 2^{-(\ell+1)k} t^{\ell+1}, \quad t \in [0, \sigma]. \]

First, we estimate the remainder. Since the sequence of trigonometric polynomials \( \{p_k, \ k \geq 1\} \) is bounded, the norms \( \|D^{\ell+1} p_k\|_{C([0, \sigma])} \) do not exceed some constant \( C_2 \). Therefore,

\[ \left| \frac{D^{\ell+1} p_k(1/2 + 2^{-k} \theta)}{\ell!} 2^{-(\ell+1)k} t^{\ell+1} \right| \leq \frac{C_2}{\ell!} 2^{-(\ell+1)k} \sigma^{\ell+1} = o(2^{-\ell k}) \quad \text{as} \quad k \to \infty. \]

Combining this with (49), we get

\[ \left\| \sum_{j=0}^{\ell} \frac{D^j p_k(1/2)}{j!} 2^{-jk} t^j \right\|_{C([0, \sigma])} = o(2^{-\ell k}) \quad \text{as} \quad k \to \infty. \]  

(50)

Since, in a finite-dimensional space, all norms are equivalent, the norm of an algebraic polynomial of degree \( \ell \) in the space \( C([0, \sigma]) \) is equivalent to its largest coefficient. Whence, (50) implies that

\[ \max_{j=0, \ldots, \ell} 2^{-jk} \left| \frac{D^j p_k(1/2)}{j!} \right| = o(2^{-\ell k}), \quad k \to \infty. \]  

(51)

We are finally ready to prove the main result of this section, Theorem 1.

**Proof of Theorem 1.** Let \( p_k(\omega) := a_{\omega}^{(k)}(e^{-2\pi i \omega}), \ \omega \in \mathbb{R} \), be the symbol of the \( k \)-th mask in the trigonometric form. If the non-stationary scheme converges to a continuous compactly supported refinable function \( \phi \), then its Fourier transform \( \hat{\phi}(\omega) = \int_{\mathbb{R}} \phi(x) e^{-2\pi i x \omega} dx \) is given by

\[ \hat{\phi}(\omega) = \prod_{k=1}^{\infty} p_k(2^{-k} \omega), \quad \omega \in \mathbb{R}. \]  

(52)

If \( \phi \in C^{\ell}(\mathbb{R}) \), then \( \hat{\phi}(\omega) = o(\omega^{-\ell}) \) as \( \omega \to \infty \). Since the refinable function of the limit mask \( a \) is stable, it follows that its symbol \( a_\omega(z) \) has no symmetric roots on the unit circle. The claim follows by Proposition 4. Indeed, by definition of \( p_k \), we get by (51)

\[ \max_{j=0, \ldots, \ell} 2^{-jk} |D^j a_{\omega}^{(k)}(-1)| = o(2^{-\ell k}) \quad \text{as} \quad k \to \infty, \]

which completes the proof. \( \square \)

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4.2 Reproduction and generation properties of non-stationary schemes

It is known that the non-stationary subdivision schemes may generate/reproduce certain spaces of exponential polynomials, see e.g. [8, 14]. In this subsection, we are interested in answering the question: How big is the class of functions that can be generated/reproduced by such schemes?

More precisely, we will show that the zero sets of the Fourier transforms of the limit functions $\phi_k$ of such schemes are unions of the sets

$$\Gamma_r = \{ \omega \in \mathbb{C} : \alpha^{(r)}_k(e^{-i2\pi M^{-r}\omega}) = 0 \}, \quad r \geq k,$$

and that the sets $\Gamma_r$ are such that $\Gamma_r + M^r \mathbb{Z} = \Gamma_r$. Thus, some elementary functions cannot be generated by non-stationary schemes, see Example 8. Also the requirement that

$$\hat{\phi}_k(\omega) = \int_{\mathbb{R}} \phi_k(x)e^{-i2\pi \omega x}dx, \quad \omega \in \mathbb{C}, \quad k \in \mathbb{N},$$

is an entire function, limits the properties of the functions that can be generated by non-stationary subdivision schemes.

**Proposition 5.** Let $\{\phi_k, \ k \in \mathbb{N}\}$ be continuous functions of compact support satisfying

$$\phi_k(x) = \sum_{\alpha \in \mathbb{Z}} a^{(k)}(\alpha)\phi_{k+1}(Mx - \alpha), \quad k \in \mathbb{N}, \quad x \in \mathbb{R}.$$  

Then

$$\{ \omega \in \mathbb{C} : \hat{\phi}_k(\omega) = 0 \} = \bigcup_{r \geq k} \Gamma_r,$$

such that the sets $\Gamma_r$ satisfy

$$\Gamma_r + M^r \mathbb{Z} = \Gamma_r.$$

**Proof.** Let $k \in \mathbb{N}$. By Paley-Wiener theorem, the Fourier transform $\hat{\phi}_k$ defined on $\mathbb{R}$ has an analytic extension

$$\hat{\phi}_k(\omega) = \int_{\mathbb{R}} \phi_k(x)e^{-i2\pi \omega x}dx, \quad \omega \in \mathbb{C},$$

to the whole complex plane $\mathbb{C}$ and $\hat{\phi}_k$ is an entire function. By Weierstrass theorem [16], every entire function can be represented by a product involving its zeroes. Define the sets

$$\Gamma_r := \{ \omega \in \mathbb{C} : \alpha^{(r)}_k(e^{-i2\pi M^{-r}\omega}) = 0 \}, \quad r \in \mathbb{N}.$$
Let \( z_{r,1}, \ldots, z_{r,N} \) be the zeros of the polynomials \( a_s^{(r)}(e^{-i2\pi M^{-r}\omega}) \), counting their multiplicities. Then

\[
\Gamma_r = iM^r \bigcup_{\ell=1}^{N} \text{Ln}(z_{r,\ell}),
\]

where, by the properties of the complex logarithm, each of the sets \( iM^r \text{Ln}(z_{r,\ell}) \) consists of sequences of complex numbers and is \( M^r \)–periodic. Thus, each of the sets \( \Gamma_r \) satisfy

\[
\Gamma_r + M^r \mathbb{Z} = \Gamma_r, \quad r \in \mathbb{N}.
\]

The definition of \( \hat{\phi}_k \) as an infinite product of the trigonometric polynomials \( a_s^{(r)}(e^{-i2\pi M^{-r}\omega}) \), \( r \geq k \), yields the claim.

The following examples illustrate the result of Proposition 5.

**Example 7:** The basic limit function of the simplest stationary scheme is given by \( \phi_1 = \chi_{[0,1)} \). Its Fourier transform is

\[
\hat{\phi}_1(\omega) = \frac{1 - e^{-i2\pi\omega}}{i2\pi}, \quad \{ \omega \in \mathbb{C} : \hat{\phi}_1(\omega) = 0 \} = \mathbb{Z} \setminus \{0\}.
\]

The mask symbol \( a_s(z) = 1 + z \) has a single zero at \( z = -1 \), i.e. \( e^{-i2\pi2^{-r}\omega} = -1 \) for \( \omega = 2^r \left\{ \frac{1}{2} + k : k \in \mathbb{Z} \right\}, \ r \in \mathbb{N}_0 \). In other words, \( \Gamma_1 = \{1 + 2k : k \in \mathbb{Z}\} \) and \( \Gamma_r = 2\Gamma_{r-1} \) for \( r \geq 2 \). Therefore,

\[
\left\{ \omega \in \mathbb{C} : \hat{\phi}_1(\omega) = 0 \right\} = \bigcup_{r \in \mathbb{N}} \Gamma_r.
\]

**Example 8:** The first basic limit function of the simplest non-stationary scheme is given by \( \phi_1(x) = \chi_{[0,1)}(x)e^{\lambda x}, \ \lambda \in \mathbb{C} \). Its Fourier transform is

\[
\hat{\phi}_1(\omega) = \frac{e^{-i2\pi\omega + \lambda} - 1}{-i2\pi\omega + \lambda}, \quad \{ \omega \in \mathbb{C} : \hat{\phi}_1(\omega) = 0 \} = -\frac{i\lambda}{2\pi} + \mathbb{Z} \setminus \{0\}.
\]

The mask symbol \( a_s^{(k)}(z) = 1 + e^{\lambda 2^{-r}} z \) has a single zero at \( z = -e^{-\lambda 2^{-r}} \), i.e. \( e^{-i2\pi2^{-r}\omega} = -e^{-\lambda 2^{-r}} \) for \( \omega = -\frac{i\lambda}{2\pi} + 2^r \left\{ \frac{1}{2} + k : k \in \mathbb{Z} \right\}, \ r \in \mathbb{N} \). Note that \( \Gamma_1 = -\frac{i\lambda}{2\pi} + \{1 + 2k : k \in \mathbb{Z}\} \) and

\[
\bigcup_{r \in \mathbb{N}} 2^r \left\{ \frac{1}{2} + k : k \in \mathbb{Z} \right\} = \mathbb{Z} \setminus \{0\}.
\]
Therefore,
\[ \{ \omega \in \mathbb{C} : \hat{\phi}_1(\omega) = 0 \} = \bigcup_{r \in \mathbb{N}} \Gamma_r. \]

In the next example we identify a compactly supported function that cannot be generated by any non-stationary subdivision scheme.

**Example 9:** Let us consider the compactly supported function
\[ f(x) = \chi_{[-1,1]}(x) \frac{2}{\sqrt{1-x^2}}, \quad x \in \mathbb{R}. \]
It cannot be a limit of any non-stationary subdivision scheme. Indeed, its Fourier transform
\[ J_0(\omega) = \int_{\mathbb{R}} f(x) e^{-ix\omega} dx, \quad \omega \in \mathbb{C}, \tag{53} \]
is the Bessel function $J_0$ of the first kind, which is entire, but has only positive zeros. The lower bound for its zeros $j_{0,s}$, $s \in \mathbb{N}$, is given by $j_{0,s} > \sqrt{(s - \frac{1}{2})^2 \pi^2}$, see [42]. Thus, Proposition 5 implies the claim. Note that, to be consistent with [42], in (53) we used a different definition of the Fourier transform than in the rest of the paper.

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