Simulation of the FPT for Multi-Dimensional Wiener Process with Dynamic Thresholds

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Abstract: Reliability modeling and analysis has become an essential issue in complex system management. Degradation system reliability modeling is an important part of the reliability field due to that the failure data is more and more difficult to obtain. Degradation data can well describe the failure process of a system or product. In the stochastic process, the Wiener process has been widely used in literature due to its good properties. Multidimensional diffusion Wiener process describes the multidimensional degradation process of products. First passage time is the lifetime of a system to reach a threshold value. In this paper, a generalized model is developed for evaluating the reliability of the multidimensional degradation process. Meanwhile, we use the Monte Carlo method for the solution of the FPT with dynamic threshold value. The comparisons show that the results of the proposed model agree well with the Monte Carlo results. The research can support decision making in the reliability improving.

1. Introduction
With the development of the technology for design and production, the reliability of products become higher and higher. Failure Data is difficult to obtain now. Multidimensional diffusion Wiener process describes the multidimensional degradation process of products. Degradation analysis is becoming an important method to analyze the performance of products.

The first passage time (FPT) problem for one multidimensional Wiener process through fixed boundaries is applied in many fields, such as economics[1], engineering, neuroscience[2], physics, psychology and reliability theory[3]. Another important application is in finance, such as pricing barrier options or lookback options, which involve crossing certain levels[4], or pricing American options[5], which entail evaluating the first passage time density for a time varying boundary. But except for some special cases, the analytical solution of the stochastic process is difficult to obtain. So the Monte Carlo simulation becomes an effective method to get the solution of FPT.

2. System Description and Assumptions
Let us consider a complete probability space \((\Omega, F, P)\) and an information filtration \(F_t\), and suppose that \(X(t)\) is a Markov process in some state space \(D \subseteq \mathbb{R}^n\), the multidimensional diffusion Wiener process can be described as

\[
dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dB(t)
\]
where $B(t)$ is a standard Brownian motion in $\mathbb{R}^n$, and $\mu(X(t),t): D \rightarrow \mathbb{R}^n$, $\sigma(X(t),t): D \rightarrow \mathbb{R}^{n\times n}$. We assume that all the conditions on existence and uniqueness of the solution are satisfied with [6].

For simplicity, we consider the following diffusion Wiener process

$$dX(t) = \mu(t)dt + \sigma(t)dB(t)$$

where $X(0) = (X_1(0), X_2(0), \ldots, X_n(0))$, and the assumptions here is same as (1).

2.1 One Dimensional Performance Degradation Model

When there are only one performance degradation parameters of the system, stochastic differential equations can be rewrite as

$$dX(t) = \mu(t)dt + \sigma(t)dB(t) \quad \text{or} \quad X(t) = \mu(t) + \sigma B(t) + x$$

where $X(0) = x$, now the multidimensional problems can be transformed into the one dimensional problem.

2.1.1 Constant failure threshold value. We assume that the constant failure threshold value is $l$ , the lifetime of the system is $T$ , then the lifetime satisfies $T = \inf \{t : X(t) = l \mid X(0) = x < l \}$. According to [7], the distribution of the first passage time of the Wiener process is Inverse Gaussian distribution. The reliability of system is

$$R(t) = \Phi\left(\frac{l - \mu(t)}{\sigma \sqrt{t}}\right) - \exp\left(\frac{\mu(t) - x}{\sigma^2 t} \right) \Phi\left(\frac{l - \mu(t) + 2x}{\sigma \sqrt{t}}\right)$$

(4)

2.1.2 Dynamic failure threshold value. When the dynamic threshold value satisfies that $l(t) = kt + a$, the problem can be converted into the traditional constant threshold problem. The new Wiener process can be rewrite as

$$X(t) = \mu(t) - kt + \sigma B(t)$$

(5)

Where $X(0) = x$. Then according to the above conclusion, the distribution of the first passage time is

$$P[T_t \leq t] = \Phi\left(\frac{-a + \mu(t) + kt}{\sigma \sqrt{t}}\right) + \exp\left(\frac{\mu(t) - kt - x}{\sigma^2 t} \right) \Phi\left(\frac{-a - \mu(t) + kt + 2x}{\sigma \sqrt{t}}\right)$$

The reliability of the system is

$$R(t) = \Phi\left(\frac{a - \mu(t) - kt}{\sigma \sqrt{t}}\right) - \exp\left(\frac{\mu(t) - kt - x}{\sigma^2 t} \right) \Phi\left(\frac{a - \mu(t) + kt + 2x}{\sigma \sqrt{t}}\right)$$

when the threshold satisfies some distribution, we assume the probability density function of the threshold is $f_X(r)$, then the model becomes complex, the cumulative distribution function of the lifetime becomes

$$F(t) = P(T \leq t)$$

$$= P(X(t) \leq R)$$

$$= \int_R P(X(t) \leq r) f_X(r) dr$$

$$= \int_R \left[ \Phi\left(\frac{-r + \mu(t) + kt}{\sigma \sqrt{t}}\right) + \exp\left(\frac{\mu(t) - kt - x}{\sigma^2 t} \right) \Phi\left(\frac{-r - \mu(t) + kt + 2x}{\sigma \sqrt{t}}\right) \right] f_X(r) dr$$
When the threshold follows the Uniform Distribution $R \sim U[a,b]$, then the failure probability is

$$F(t) = \frac{1}{b-a} \int_a^b \Phi \left( \frac{-y + \mu(t) + kt}{\sigma \sqrt{t}} \right) + \exp \left( \frac{\mu(t) - kt - x}{\sigma^2 t} \right) \Phi \left( \frac{-y - \mu(t) + kt + 2x}{\sigma \sqrt{t}} \right) dr$$

and the reliability of the system is

$$R(t) = \frac{1}{b-a} \int_a^b \Phi \left( \frac{y - \mu(t) - kt}{\sigma \sqrt{t}} \right) - \exp \left( \frac{\mu(t) - kt - x}{\sigma^2 t} \right) \Phi \left( \frac{-y - \mu(t) + kt + 2x}{\sigma \sqrt{t}} \right) dr$$

2.2. Multidimensional Performance Degradation Model

Multidimensional diffusion liner Wiener process can be expressed as

$$\begin{bmatrix}
    dX_1(t) \\
    dX_2(t) \\
    \vdots \\
    dX_k(t)
\end{bmatrix} =
\begin{bmatrix}
    \mu_1 \\
    \mu_2 \\
    \vdots \\
    \mu_k
\end{bmatrix} dt + \begin{bmatrix}
    dB_1 \\
    dB_2 \\
    \vdots \\
    dB_k
\end{bmatrix}$$

where $\mu_i (i = 1,2,\ldots,k)$ are constant drift coefficient, $B_i (i = 1,2,\ldots,m)$ are standard Brownian motion and they are independent to each other. $\Omega$ is an constant matrix in $R^k \times R^k$, and the matrix satisfies

$$\Omega \times \Omega = \begin{pmatrix}
    \sigma_1^2 & \rho_{12} \sigma_1 \sigma_2 & \cdots & \rho_{1k} \sigma_1 \sigma_k \\
    \rho_{12} \sigma_1 \sigma_2 & \sigma_2^2 & \cdots & \rho_{2k} \sigma_2 \sigma_k \\
    \vdots & \vdots & \ddots & \vdots \\
    \rho_{1k} \sigma_1 \sigma_k & \rho_{2k} \sigma_2 \sigma_k & \cdots & \sigma_k^2
\end{pmatrix}$$

where $\rho$ is the correlation coefficient, when $\rho = 0$, the model can be converted into a multidimensional independent Wiener process. When $\rho \neq 0$, that means the marginal Wiener process is correlated to each other.

2.2.1 Constant Failure Threshold Value. Assume that $l_1,l_2,\ldots,l_k$ are respectively threshold value of the $k$ marginal Wiener process, the system will fail when any one of the $k$ processes cross the respectively threshold value.

Thus the lifetime of the system can be expresses as

$$T = \inf \left\{ t : X_1(t) > l_1 \text{ or } X_2(t) > l_2 \cdots \text{or } X_k(t) > l_k \right\}$$

The reliability function $R(t)$ of the system is

$$R(t) = P \left\{ X_1(s) < l_1, X_2(s) < l_2,\ldots,X_k(s) < l_k, 0 \leq s \leq t \right\}$$

2.2.2 Dynamic Failure Threshold. In many actual reliability problems, the failure threshold value is not constant but dynamic. Due to the change of environmental factors, the failure threshold can follow a linear change or follow some certain distribution. This may bring us some difficulty in evaluating and calculating the reliability of the system. We have to deal with them according to the different conditions.

For example, if we have three main parameters to describe the performance of the system, we know that $k = 3$ in the above model. $X(1), X(2)$ and $X(3)$ represent the performance degradation parameters of the system respectively. We assume that the dynamic threshold boundary of multidimensional is spherical, and the radius is $R$ which represents a random variable. Its probability
density function is \( f_r(r) \). The system will fail if the sum of squares of the three performance degradation parameters. Then the lifetime of the system is

\[
T_1 = \inf \{t : X_1^2(t) + X_2^2(t) + X_3^2(t) \geq R^2 \} \quad \text{where} \quad R \ \text{obey the uniform distribution}, \quad \text{we denote} \quad R \sim U[a,b].
\]

In this condition, it is difficult to get the explicit expression of the first passage time of the system. Thus we can only adopt the method of Monte Carlo simulation to get the numerical solution of the first passage time. After we generate enough points of the first passage time, we can use the kernel density estimator to acquire the density of the first passage time approximately. The kernel density estimator is based on constructing a kernel function of a fixed width around each generated data point [8]. The function of kernel density estimation is as follows:

\[
f_n(x) = \frac{1}{nh} \sum_{i=1}^{n} K\left(\frac{x - X_i}{h}\right)
\]

(8)

In this example, we use the Gaussian kernel to do the kernel density estimation. Thus we can get the specific expression of the kernel density estimation after we generate \( K(\mu) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \) into the formula.

3. Application Example and Simulation

The following example in this section aims to illustrate the degradation model discussed in previous sections. We supposed that a navigation system whose reliability depend on its navigation accuracy. And the accuracy contains two aspects. They are longitude error and latitude error. When the deviation is too large to exceed the failure threshold value, the navigation system is regarded as failure.

Here we represent the longitude error with \( X_1(t) \) and the latitude error with \( X_2(t) \). We assume that when the Sum of squares exceed the threshold value \( R \), then the navigation system will fail to work, which is described in Figure 1.

\[
X_1(t) = \mu_1 + \sigma_1 \cdot \mathcal{N}(0,1), \quad X_2(t) = \mu_2 + \sigma_2 \cdot \mathcal{N}(0,1)
\]

Figure 1. Degradation track of the system.

First passage time of the navigation system can be expressed as

\[
T_1 = \inf \{t : X_1^2(t) + X_2^2(t) = R^2 \} \quad \text{where} \quad X(0) = (x_1, x_2) \}
\]

We assume that the parameter is as \( \mu_1 = 2, \sigma_1 = 2, \mu_2 = 3, \sigma_2 = 3, \rho = 0.3, x_1 = x_2 = 0 \), and \( R \sim U[4,6] \).

After generating series of first passage time \( T_1 \), we use a kernel density estimator [9] with a Gaussian kernel to estimate the PDF of first passage time. Figure 2 shows the results of PDF of life distribution and the reliability of the system.
4. Conclusion
In this paper, we developed a generalized model for evaluating the reliability of multidimensional diffusion Wiener process with the dynamic threshold value. How to describe the correlation between different dimensions is a difficulty. We use the Covariance matrix to characterize the correlation of them. The bigger the coefficient $\rho$ is, the stronger the correlation between two dimensions is. We obtain cumulative distribution function and reliability function of the system in one dimensional Wiener process. In two dimensional Wiener processes, it is difficult to get the analytical solutions of first passage time. So the Monte Carlo method is proposed to simulate the solution. Finally, we give a numerical example to illustrate the proposed model.

Acknowledgement
This work was financially supported by State Grid Technology Project “Key technologies and applications of data asset valuation based on sharing and trading requirements”.

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