A COMPUTATIONAL HISTORY OF PRIME NUMBERS AND Riemann Zeros

PIETER MOREE, IZABELA PETRYKIEWICZ, AND ALISA SEDUNOVA

Abstract. We give an informal survey of the historical development of computations related to prime number distribution and zeros of the Riemann zeta function.

The fundamental quantity in the study of prime numbers is the prime counting function $\pi(x)$, which counts the number of primes not exceeding $x$; in mathematical notation we have

$$\pi(x) = \sum_{p \leq x} 1.$$ 

The first mathematicians to investigate the growth of $\pi(x)$ had of course to start with collecting data. They did this by painfully setting up tables of consecutive prime numbers, e.g., Krüger in 1746 and Vega in 1797 (primes up to 100,000 and 400,031 respectively). The most celebrated of these prime table computers was Gauss. In 1791, when he was 14 years old, he noticed that as one gets to larger and larger numbers the primes thin out, but that locally their distribution appears to be quite erratic. He based himself on a prime number table contained in a booklet with tables of logarithms he had received as a prize, and went on to conjecture that the “probability that an arbitrary integer $n$ is actually a prime number should equal $1/\log n$”. Thus Gauss conjectured that

$$\pi(x) \approx \sum_{2 \leq n \leq x} \frac{1}{\log n} \approx \text{Li}(x),$$

with

$$\text{Li}(x) = \int_2^x \frac{dt}{\log t},$$

the logarithmic integral. Since by partial integration it is easily seen that $\text{Li}(x) \sim x/\log x$, the conjecture of Gauss implies that asymptotically

$$\pi(x) \sim \frac{x}{\log x}.$$
a conjecture that was proved much later, in 1896, by Hadamard and de la Vallée-Poussin independently. This asymptotic for $\pi(x)$ is called the **Prime Number Theorem** (PNT).

Gauss kept a life long interest in primes and what he did was to count primes in blocks of 1 000 (a Chiliade). As he wrote in a letter to Bessel, he would use an idle quarter of hour here and there to deal with a further block. By the end of his life he would extend the tables up to 3 000 000. After Gauss, number theorists kept extending the existing prime number tables. Thus in 1856 Crelle published a table of primes up to 6 000 000, and a few years later Dase extended this to 9 000 000. The most impressive feat in this regard is due to Kulik, who spent 20 years preparing a factor table of the numbers coprime to 30 up to 1 000 330 200 (he did so in eight manuscript volumes, totalling 4 212 pages).

The holy grail in computational prime number theory is to find sharp estimates of $\pi(x)$. These estimates should be in terms of elementary functions.

An early attempt is by Legendre, who claimed (1808) that $x/(\log x − 1.0836)$ should approximate $\pi(x)$ well. We now know that this is a reasonable estimate (the estimate $x/(\log x − 1)$ is actually better). A much more recent and rigorous example is provided by the estimates

$$
\frac{x}{\log x} \left(1 + \frac{1}{2\log x}\right) < \pi(x) < \frac{x}{\log x} \left(1 + \frac{3}{2\log x}\right), \quad x \geq 59.
$$

due to Rosser and Schoenfeld. Some further examples can be found in Section 8.2.

The reason why sharp estimates of $\pi(x)$ and of related prime counting functions are so important is that many problems in number theory use them as input. There are plenty of number theoretical problems where one comes to a solution only on assuming that a sharp estimate for $\pi(x)$ is available, an estimate we cannot currently prove, but which we could if we knew that the **Riemann Hypothesis** (RH) holds true (we will come back to this shortly). Under RH it can be shown that for every $x > 2 657$ we have

$$
|\pi(x) − \text{Li}(x)| < \frac{1}{8\pi} \sqrt{x} \log x.
$$

(1)

This is a sharp inequality as the estimate $\pi(x) = \text{Li}(x) + O(\sqrt{x}/\log x)$ does not hold. The latter result says that the primes do behave irregularly to some extent. The importance of the RH is that if true it would imply that we can very well approximate

---

4Jacques Salomon Hadamard (1865 Versailles, France – 1963 in Paris, France), French mathematician, professor at the University of Bordeaux, Collège de France, École Polytechnique and École Centrale.

5Charles Jean Gustave Nicolas Baron de la Vallée-Poussin (1866 Leuven, Belgium – 1962 Leuven, Belgium), Belgian mathematician, professor at Catholic University of Leuven, Collège de France and Sorbonne.

6August Leopold Crelle (1780 Eichwerder, Germany – 1855 Berlin, Germany), German mathematician, founder of *Journal für die reine und angewandte Mathematik*.

7Johann Martin Zacharias Dase (1824 Hamburg, Germany – 1861 Hamburg, Germany), German mathematician, having great calculating skills, but little mathematical knowledge.

8Jakob Philipp Kulik (1793 Lemberg, Austrian Empire – 1863 Prague, Bohemia), Polish-Austrian mathematician, professor at the Charles University of Prague.

9Adrien-Marie Legendre (1752 Paris, France – 1833 Paris, France), French mathematician and author of an influential number theory book.
\( \pi(x) \) by the simple function \( \text{Li}(x) \), which makes proving results involving primes in general much easier.

The sharpest estimates to date for \( \pi(x) \) are obtained by using properties of the so-called **Riemann zeta function**, defined by

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},
\]

with \( s \) a complex number having real part \( \text{Re} s > 1 \), see \(^10\). The function converges for all complex numbers \( s \) such that \( \text{Re} s > 1 \). In 1859 a renowned Göttingen professor, Riemann\(^11\), published “Über die Anzahl der Primzahlen unter einer gegebenen Grösse”. This paper\(^12\) is without doubt the most important paper ever written in analytic number theory; indeed it is foundational as Riemann makes essential use of \( s \) being a complex variable, whereas a century earlier Euler\(^13\) only considered \( \zeta(s) \) for real values of \( s \). That Riemann considered his zeta function as an analytic function comes as no surprise as the development of complex analysis was one of his central preoccupations. He brought the study of \( \pi(x) \) to a completely new level, but actually proved little as the tool he used, complex function theory, did not have on a firm theoretical foundation at the time. It took about 40 years and a lot of preliminary work, mainly by Cahen, Halphen and Phragmén (see, e.g., \(^{51}\) for more details), before the PNT could be finally proved using methods of complex function theory. A tremendous amount of work was carried out by Landau\(^15\) who went meticulously through all earlier relevant work on this subject, checked its correctness, simplified it\(^16\) and wrote a standard work on prime number theory in 1909 \(^{44, 45}\). He himself proved many important results as well, such as the prime ideal theorem (see Section \(^9.2\)).

A lot of effort was put into proving results about prime numbers without using complex analysis. The most celebrated results were obtained by Selberg\(^17\) (and, more or less, independently) by Erdős. They based themselves on the identity, now called **Selberg’s**
symmetry formula,
\[ \sum_{p \leq x} \log^2 p + \sum_{p,q \leq x} \log p \cdot \log q = 2x \log x + O(x), \]
and used it to obtain an **elementary proof** of the PNT. Until 1950 it was widely believed (e.g., by Landau) that no such elementary proof could be developed and so this result greatly impressed the contemporaries. Later Selberg extended this combinatorial technique to show the PNT for primes in arithmetic progression, see [68]. For a nice survey see Diamond [18]. Unfortunately, the high hopes placed in new insights coming from finding an elementary proof of PNT were thwarted.

The level of insight that Riemann had reached was finally surpassed by Hardy and Littlewood in the 1920’s. As far as his zeta function is concerned, Riemann was certainly more than half a century ahead of his time!

Using partial integration it is easy to deduce that for \( \text{Re} \, s > 0 \) an analytic continuation of \( \zeta(s) \) is given by
\[ \zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{t - [t]}{t^{1+s}} \, dt, \] (3)
with \([t]\) being the floor function. Another analytic continuation is obtained on noting that for \( \text{Re} \, s > 1 \) we have
\[ (1 - 2^{1-s}) \zeta(s) = \sum_{n=1}^\infty (-1)^n n^{-s}. \] (4)
Since the right hand side actually converges for \( \text{Re} \, s > 0 \), the identity furnishes an analytic continuation for all \( \text{Re} \, s > 0 \).

In his paper, Riemann showed that the zeta function actually has an analytic continuation to the whole complex plane, except for a simple pole at \( s = 1 \), and that it vanishes at all negative even integers. The **trivial zeros** of \( \zeta(s) \) are the ones at negative even integers, the **non-trivial zeros** come from complex numbers \( s = \sigma + it \) with \( 0 \leq \sigma \leq 1 \). The zeta function satisfies the **functional equation**
\[ \Gamma \left( \frac{s}{2} \right) \pi^{-s/2} \zeta(s) = \Gamma \left( \frac{(1-s)/2} \right) \pi^{(s-1)/2} \zeta(1-s), \] (5)
where \( \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt \) for \( \text{Re} \, z > 0 \) denotes the **Gamma function** (a continuous extension of the **factorial function**\textsuperscript{19}). This function is never equal to zero, and is holomorphic everywhere except at the points \( 0, -1, -2, \ldots \), where it has simple poles.

It turns out that \( \Gamma(s/2) \pi^{-s/2} \zeta(s) \) has a simple pole at both \( s = 0 \) and \( s = 1 \). This suggests that we should multiply it by \( s(s-1) \). In this way we obtain the **Riemann \( \xi \)-function**. It is an entire function whose zeros \( \rho \) are the non-trivial zeros of \( \zeta(s) \). Note that we can rewrite (5) as
\[ \xi(s) = \xi(1-s). \]

\textsuperscript{19}If \( n \) is a non negative integer, then, e.g., \( \Gamma(n+1) = n! \). For this reason some authors write \( \Gamma(z+1) \) instead of \( \Gamma(z) \).
We have the following Hadamard factorisation of $\xi$:

$$\xi(s) = s(s-1)\Gamma(s/2)\pi^{-s/2}\zeta(s) = \xi(0) \prod_{\rho} (1 - s/\rho) e^{s/\rho},$$

where here (and in the sequel) the zeros $\rho$ are counted with their own multiplicities, e.g., a double zero is counted twice.

Riemann gave two beautiful proofs of the functional equation; one is related to the theory of modular forms and makes use of the transformation property $\Omega(x) = x^{-1/2}\Omega(x^{-1})$, valid for all positive $x$, with

$$\Omega(x) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi x},$$

while the second proof uses the integral representation (18) from Section 4. Note that if $\rho$ is a non-trivial zero, then so is $1 - \rho$ by the functional equation. Moreover, since $\zeta(s) = \zeta(\overline{s})$, we deduce that $\overline{\rho}$ and $1 - \rho$ are also zeros. Thus the zeros are symmetrically arranged about the half line (also called the critical line) given by $\text{Re } s = 1/2$ and also about the real axis. Therefore we often only calculate the zeros in the upper half plane. Riemann shows that the non-trivial zeros lie in the critical strip, the strip $0 \leq \text{Re } s \leq 1$, and, furthermore, are not real. Moreover, he wrote that probably all its non-trivial zeros lie on the half line and continues: “Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation.”

The Riemann Hypothesis states that all non-trivial zeros (the zeros in the critical strip) of the zeta function are on the half line. We call the zeros of $\zeta(s)$ on the half line critical zeros. If in the rectangle defined by $0 \leq \text{Re } s \leq 1$ and $0 \leq \text{Im } s \leq T$ there are only critical zeros, we say that RH up to height $T$ holds true.

Solving RH is one of the famous 23 problems posed by Hilbert at the 1900 International Congress of Mathematicians in Paris. These problems held their fascination and influence on the developments through the twentieth century. Resolving RH is one of seven Millenium Prize Problems that were stated by the Clay Mathematics Institute in 2000. A correct solution to any of the problems results in a 1 000 000 dollar prize being awarded by this institute. Many mathematicians regard RH as the biggest open problem in all of mathematics.

The behaviour of $\zeta(s)$ in the critical strip is very closely related to the distributional properties of the primes. The uniqueness of prime factorization finds its analytic counterpart in the identity

$$\zeta(s) = \prod_{\rho} (1 - p^{-s})^{-1}, \text{ Res } > 1,$$

References:

20 In this strip formula (2) for $\zeta(s)$ does not apply, but formula (3) and (4) do.

21 Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung entbehrlich schien. Translation: David R. Wilkins.
which was established by Euler for real $s$. Indeed, to become an inhabitant of the Zeta Zoo (Section 9.2), an analytic function $f(s)$ needs to have a factorization of the form $\prod_p f_p(s)$. A formula of this type is now called an Euler product.

Thus the fact that $\zeta(s)$ tends to infinity if one approaches $s = 1$ from the right over the real axis ensures by (6) that there are infinitely many primes $p$. This was discovered in 1737 by Euler who established the stronger result that $\sum p^{-1}$ is unbounded. It turns out that, in order to prove the PNT, it is enough to show that there are no zeros on the line $\text{Re } s = 1$. The connection between the non-trivial zeros and the prime numbers is actually much closer than this. Indeed, it can be shown that

$$\pi(x) = R(x) - \sum_\rho R(x^{\rho}),$$

where the sum is over all the non-trivial zeros $\rho$ (counted with multiplicities) and

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \frac{1}{\log(x^{1/n})}$$

denotes the Riemann function and $\mu$ the Möbius function (see Section 8.4). Using Gram’s (1893) quickly converging power series

$$R(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{n \zeta(n+1)} \frac{(\log x)^n}{n!},$$

the Riemann function is computable. Two further expressions for $R(x)$ were found by Ramanujan. He independently discovered the importance of $R(x)$ (around 1910) and developed a prime number theory that, as Hardy phrased it, was "what the theory might be if the zeta-function had no complex zeros".

\footnote{Indeed, the Riemann zeta function has a simple pole with residue 1 at $s = 1$.}

\footnote{Gauss in 1796 conjectured, and Mertens proved in 1874 the stronger result that $\sum_{p \leq x} p^{-1} = \log \log x + C + o(1)$, with $C$ a constant.}

| $x$  | $\pi(x)$     | $\text{Li}(x) - \pi(x)$ | $R(x) - \pi(x)$ |
|------|--------------|--------------------------|-----------------|
| $10^8$ | 5 761 455   | 754                      | 97              |
| $10^9$ | 50 847 534  | 1 701                    | −79             |
| $10^{10}$ | 455 052 511 | 3 104                    | −1 828          |
| $10^{11}$ | 4 118 054 813 | 11 588             | −2 318          |
| $10^{12}$ | 37 607 912 018 | 38 263             | −1 476          |
| $10^{13}$ | 346 065 536 839 | 108 971            | −5 773          |
| $10^{14}$ | 3 204 941 750 802 | 314 890           | −19 200         |
| $10^{15}$ | 29 844 570 422 667 | 1 052 619        | 73 218          |
| $10^{16}$ | 279 238 341 033 925 | 3 214 632        | 327 052         |
| $10^{17}$ | 2 623 557 157 654 233 | 7 956 589      | −508 255        |
| $10^{18}$ | 24 739 954 287 740 860 | 21 949 555    | −3 501 366      |

Table 1. The values of $\pi(x)$ compared with values of $\text{Li}(x)$ and $R(x)$. 

The values of $\pi(x)$ compared with values of $\text{Li}(x)$ and $R(x)$.
It turns out that, from a theoretical perspective, it is better to work with certain weighted prime counting functions rather than with $\pi(x)$ (that is, each prime $p$ is counted with a weight $w(p)$). The most well-known of these are $\vartheta(x) = \sum_{p \leq x} \log p$ and $\psi(x) = \sum_{p^m \leq x} \log p$, where the sums are taken over all primes, respectively all prime powers less than $x$. The first function is known as the Chebyshev $\vartheta$-function, the second one as the Chebyshev $\psi$-function. One often sees the $\psi$-function defined as $\psi(x) = \sum_{n \leq x} \Lambda(n)$, where $\Lambda(n)$, the von Mangoldt function, equals $\log p$ if $n$ is a power of a prime $p$ and zero otherwise. This function is more natural than it appears at first sight, since logarithmic differentiation of the Euler identity yields $-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \Lambda(n)n^{-s}$ for $\text{Re} s > 1$.

The quotient $\frac{\zeta'(s)}{\zeta(s)}$ has simple poles in the zeros of $\zeta$ and plays an important role in prime number theory. The complicated explicit formula (7) for $\pi(x)$ takes a much easier form for $\psi(x)$. This result is called the “explicit formula” and due to von Mangoldt. He showed in 1895 that for $x > 1$ and $x$ not a prime power we have

$$\psi(x) = x - \sum_{\zeta(\rho) = 0} \frac{x^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)},$$

where the sum on the right-hand side is over all zeros (also the trivial ones!) of the Riemann zeta function. The explicit von Mangoldt formula was vastly generalized by André Weil. His explicit formula works for a large class of test functions. The summation over the roots involves the test function and this then equals a sum over the primes involving the Fourier transform of the test function. On choosing a suitable test function (depending on the problem one studies), Weil’s explicit formula has actually become an important tool in computational prime number theory.

Note that $|x^\rho| = x^{\text{Re} \rho}$ and consideration of the explicit formula suggests that $\limsup \text{Re} \rho$ determines the error term. Due to the existence of critical zeros, the minimum possible value of the latter quantity is a half. We have $\limsup \text{Re} \rho = 1/2$ if and only if the RH holds true.

Note that if the $\sum \frac{1}{|\rho|}$ were to be bounded, with $\rho$ ranging over all critical zeros, then it would follow from (9) that on RH we have $\psi(x) = x + O(\sqrt{x})$. However, this sum does not converge, but a slightly weaker result is true, namely

$$\sum_{\text{Im} \rho \leq T} \frac{x^\rho}{|\rho|} = O(x^{\alpha} \log^2 T),$$

with $\alpha$ any number such that there are no zeros $\rho$ with $\text{Re} \rho > \alpha$. Using the latter approximation with $\alpha = 1/2$ it can be shown that the RH is equivalent with

$$\psi(x) = x + O(x^{1/2} \log^2 x).$$

24 It is easy to see that $\psi(x)$ is the logarithm of the least common multiple of the integers $1, 2, \ldots, [x]$.

25 Hans Carl Friedrich von Mangoldt (1854 Weimar, Duchy of Saxe-Weimar-Eisenach – 1925 Danzig-Langfuhr, Free City of Danzig), German mathematician, student of Kummer and Weierstrass, professor at Hanover University and Technical University of Aachen.

26 Recall that if $\rho$ is a zero, so is $1 - \rho$. 
By the same argument it can be unconditionally shown that \( \psi(x) = x + O(x^\kappa \log^2 x) \) with \( \kappa = \limsup \Re \rho \).

The analogue of (10) for \( \pi(x) \) is due to von Koch, who showed in 1901 that the RH is equivalent to the formula

\[
\pi(x) = \text{Li}(x) + O(E(x)), \quad E(x) = x^{1/2} \log x.
\] (11)

As in the error term in (10), the \( 1/2 \) in the error term \( E \) is directly related to the half line: if we would have a zero off the half line, it always has a related zero \( s_0 \) having \( \Re s_0 > 1/2 \) and the estimate (11) with \( E(x) = x^{\Re s_0 - \epsilon} \) is false. Indeed, more generally, the larger the area inside the critical strip where we can show there are no zeros, the smaller we can take \( E(x) \) in (11). Quite a bit of computational work was done on determining an explicit zero free region that is as large as possible, e.g., Kadiri \[39\] showed that

\[
\Re s \geq 1 - \frac{1}{R_0 |\Im s|}, \quad |\Im s| \geq 2, \quad R_0 = 5.69693,
\]

is a zero free region. Already in 1899 de la Vallée-Poussin had established the above result with different constants. He showed that his zero free region leads to \( E(x) = \exp \left( -c \sqrt{\log x} \right) \), with some positive constant \( c \). It implies that for every fixed \( r > 1 \) we have \( E(x) = x (\log x)^{-r} \). This function is far from behaving like \( \sqrt{x} \). Indeed, it would be an astounding result if somebody could prove that \( E(x) = x^\alpha \), for some \( \alpha < 1 \). Despite great effort by many number theorists, the above result of de la Vallée-Poussin has not been much improved.

As we have seen a prime number heuristic that works well is to assume that \( n \) is a prime with probability \( 1/\log n \). Riemann’s research leads one to replace this by a more accurate heuristic, namely that the average value of \( \Lambda(n) \) equals 1. This leads us to expect that, e.g., \( \psi(x) \sim x \), which is the PNT in a different guise\[27\]. Indeed, by elementary arguments it can be shown that the assertions

\[
\pi(x) \sim \frac{x}{\log x}, \quad \vartheta(x) \sim x, \quad \psi(x) \sim x,
\]

are all equivalent and so are three different guises of the PNT.

More interesting is to consider \( \Pi(x) = \sum_{n \leq x} \Lambda(n)/\log n \). Here our heuristic suggests that

\[
\Pi(x) = \sum_{k=1}^{\infty} \frac{\pi(x^{1/k})}{k} \approx \sum_{n \leq x} \frac{1}{\log n} \approx \text{Li}(x).
\] (12)

By Möbius inversion we obtain from this the heuristic \( \pi(x) \sim R(x) \), with \( R(x) \) the Riemann function. This approximation is excellent, as shown in Table I.

From the papers Riemann left us, it transpires that his computational skills were amazing and that he (being a perfectionist) kept a lot of his findings up his sleeve\[28\]. In a letter to Weierstrass from 1859 Riemann mentioned that he discovered new expansions

\[\text{Recall that } \psi(x) = \sum_{n \leq x} \Lambda(n).\]

\[\text{Unfortunately a lot of his notes were burnt after his death by his housekeeper. A true treasure trove that went up in cinders!}\]
for $\zeta$. At the turn of the 20th century many people tried to find these expansions in the unpublished notes of Riemann in the Göttingen library. Meanwhile in England, Hardy\textsuperscript{29} in the early 1920’s, together with his lifelong collaborator Littlewood\textsuperscript{30} found “an approximate functional equation” \textsuperscript{28, 29}. However, a librarian in Göttingen discovered that this approximate formula was already known to Riemann. Moreover, whereas Hardy and Littlewood had determined the dominating term only, Riemann had a method to estimate the remainder term. In 1926, while in Göttingen, Bessel-Hagen\textsuperscript{31} discovered a previously unknown approximate formula in Riemann’s notes. Siegel\textsuperscript{32} then analysed the notes with these two approximate formulas, working out the details\textsuperscript{33} and in 1932 published them. One of these is now known as the Riemann-Siegel formula. This formula is still used in calculating the zeros of the Riemann zeta function, see Section 4 for more details.

Over time mathematicians calculated non-trivial zeros in the hope of disproving the Riemann Hypothesis (like Turing) or to find evidence for it. At present, more than 100 billion zeros have been verified to be critical; for example, Gourdon in 2004 verified that the first $10^{13}$ zeros are critical (see \textsuperscript{23}); moreover, Odlyzko calculated 10 billion zeros near $10^{22}$ and showed that they all are critical, \textsuperscript{52}. The first fifteen zeros are listed in Table 2. The number of non-trivial zeros computed by the various dramatis personae can be found in Table 4.

The problem of calculating non-trivial zeros can be divided into three challenges. First of all given $s \neq 1$, one wants to be able to compute $\zeta(s)$ with prescribed precision. Secondly one wants to locate the critical zeros, the zeros on the half line, up to a prescribed height $T$. Finally, one wants to show that RH holds true up to a prescribed height $T$. The latter challenge necessitates being able to calculate the total number of zeros in the critical strip

\begin{table}[h]
\centering
\begin{tabular}{|c|c||c|c||c|c|}
\hline
$n$ & $\text{zero}$ & $n$ & $\text{zero}$ & $n$ & $\text{zero}$ \\
\hline
1 & 14.1347 & 6 & 37.5862 & 11 & 52.9703 \\
2 & 21.0220 & 7 & 40.9187 & 12 & 56.4462 \\
3 & 25.0109 & 8 & 43.3271 & 13 & 59.3470 \\
4 & 30.4249 & 9 & 48.0052 & 14 & 60.8318 \\
5 & 32.9351 & 10 & 49.7738 & 15 & 65.1125 \\
\hline
\end{tabular}
\caption{The first fifteen zeros of $\zeta$ rounded to four decimal places.}
\end{table}

\textsuperscript{29}Godfrey Harold Hardy (1877 Cranleigh, England – 1947 Cambridge, England), English mathematician, professor at the University of Cambridge.

\textsuperscript{30}John Edensor Littlewood (1885 Rochester, England – 1977 Cambridge, England), British mathematician, professor at the University of Cambridge.

\textsuperscript{31}Erich Paul Werner Bessel-Hagen (1898 Berlin, German Empire – 1946 Bonn, Germany), German mathematician, student of Carathéodory, professor at the University of Bonn.

\textsuperscript{32}Carl Ludwig Siegel (1896 Berlin, German Empire – 1981 Göttingen, West Germany), German mathematician, student of Landau, professor at the University of Frankfurt and Göttingen.

\textsuperscript{33}Diese Gründe machten eine freie Bearbeitung des Riemannschen Fragmentes notwendig, wie sie im folgenden ausgeführt werden soll.\textsuperscript{69}
up to height $T$ and comparing this with the number of critical zeros found.

These three challenges are addressed in the following sections.

1. **Euler-Maclaurin formula**

The first important result on computing non-trivial zeros dates back to 1903, when Gram published a paper with a list of approximate values of 15 non-trivial zeros and the proof that RH is true up to height 50 \cite{24}. He used the Euler-Maclaurin summation formula, a method originating in the 18th century to evaluate sums, which describes how good an approximation of a sum is obtained by replacing it by the corresponding integral \cite[Chapter 4]{77}. A few years before the publication of his paper, he tried a more elaborate method to compute non-trivial zeros; however, he gave up due to the sheer complexity of the calculations hoping that someone else would discover a better way. Nobody managed to obtain the desired results and after 8 years Gram resumed the work on the numerical estimations of non-trivial zeros. He tried a "na"ive approach", which, to his surprise, worked well. Gram applied the Euler-Maclaurin summation formula to \( \sum_{j=n}^{\infty} j^{-s} \) and, evaluating it at \( s = 1/2 + it \), obtained the (exact), but not absolutely converging, series expansion

\[
\zeta(s) = \sum_{j=1}^{n-1} j^{-s} + \frac{n^{-s}}{2} + \frac{n^{1-s}}{s-1} + \sum_{k=1}^{\infty} R_{k,n}(s),
\]

with

\[
R_{k,n}(s) = \frac{B_{2k}}{(2k)!} n^{1-s-2k} \prod_{j=0}^{2k-2} (s + j),
\]

where \( B_k \) denotes the \( k \)-th Bernoulli number. In order to approximate \( \zeta(s) \) one chooses an appropriate \( m \), computes the \( R_{k,n} \) terms up to this \( m \) and estimates the remainder by

\[
\left| \sum_{k=m+1}^{\infty} R_{k,n}(s) \right| < \left| \frac{s + 2m + 1}{Res + 2m + 1} R_{m+1,n}(s) \right|.
\]

Gram observed that, in order to calculate any zero with \( 0 < \text{Im} s < 50 \) to 7 decimal places, we can take \( n = 20 \). In general, the number \( n \) of terms needed to be calculated in \cite{13} in order to obtain an estimation of \( \zeta(s) \) up to reasonable precision grows linearly with

\(^{34}\)Jørgen Pedersen Gram (1850 Nustrup, Denmark – 1916 Copenhagen, Denmark), Danish mathematician, actuary.

\(^{35}\)Ces difficultés m’ayant paru insurmontables à moins de calculs immenses, j’abandonnai ces recherches en espérant qu’un autre trouverait quelque méthode pouvant servir soit au calcul des coefficients de \( \xi(t) \) soit au calcul direct des racines \( \alpha \). Mais, autant que je sache, aucune méthode de ce genre n’a encore été publiée. \cite{24}

\(^{36}\)Néanmoins l’automne dernier je me suis décidé à faire cet essai, et j’ai été frappé de la facilité avec laquelle il a réussi. \cite{24}

\(^{37}\)Bernoulli numbers can be defined recursively: \( B_0 = 1 \) and \( B_m = -\sum_{k=0}^{m-1} \binom{m}{k} \frac{B_k}{m-k+1} \), or as coefficients of the generating exponential function \( x e^x - 1 = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \).
Ims. Later Backlund provided good bounds for the size of the estimation error for this method in terms of $n$, $m$ and $s$.

This approach is simple and the computational work is proportional to $Ims$. However, in practice, it is only efficient to evaluate zeros with small imaginary part. In Sections 4, 5 and 6 we will present more efficient algorithms to approximate $\zeta(s)$.

2. Gram test

Now that we are able to compute $\zeta(s)$ with arbitrary precision, the next challenge is to locate Riemann’s zeros. Consider the Hardy $Z$-function

$$Z(t) = e^{i\theta(t)}\zeta \left( \frac{1}{2} + it \right),$$

where

$$\theta(t) = \text{Im} \log \Gamma \left( \frac{it}{2} - \frac{3}{4} \right) - \frac{t}{2} \log \pi.$$  \hspace{1cm} (15)

(Recall that $\Gamma$ denotes the Gamma function.) The function $Z(t)$ is a real-valued function and $Z(t) = 0$ if and only if $\zeta \left( \frac{1}{2} + it \right) = 0$. Thus we can detect critical zeros by finding intervals where $Z$ changes sign and then applying Newton’s method. In particular, we can compute $\theta(t)$ and $\zeta \left( \frac{1}{2} + it \right)$ and then look at the sign change. On using well-known asymptotic estimates for the $\Gamma$-function we find the asymptotic formula

$$\theta(t) = \frac{t}{2} \log \left( \frac{t}{2\pi e} \right) - \frac{\pi}{8} + \frac{1}{48t} + O \left( \frac{1}{t^3} \right).$$

Alternatively, we could consider $\text{Re} \zeta$ and $\text{Im} \zeta$ separately. We first observe that $\zeta \left( \frac{1}{2} + it \right) = Z(t) \cos \theta(t) - iZ(t) \sin \theta(t)$. Then we note that $\text{Im} \zeta$ changes sign if either $Z(t)$ or $\sin \theta(t)$ changes sign. The function $\theta(t)$ is increasing for $t \geq 10$, therefore between consecutive zeros of $\sin \theta(t)$ there is exactly one zero of $\cos \theta(t)$ and so $\cos \theta(t)$ changes sign. It follows that, if $\text{Re} \zeta \left( \frac{1}{2} + it \right) = Z(t) \cos \theta(t)$ is positive at two consecutive zeros $z_1 < z_2$ of $\sin \theta(t)$, then $Z$ must change sign in the interval $I = [1/2 + iz_1, 1/2 + iz_2]$ and hence there is a zero of $Z$ (and hence a critical zero) in the interval $I$. Thus it is important to locate those $t$ for which $\sin \theta(t) = 0$. These points are called Gram points. To be precise, the $n$-th Gram point $g_n$ is defined as the unique solution to $\theta(g_n) = n\pi$, $g_n \geq 10$. We conclude that, as long as $\zeta \left( \frac{1}{2} + ig_n \right) > 0$, there is at least one critical zero in the interval $(1/2 + ig_{n-1}, 1/2 + ig_n)$. Checking if $\zeta \left( \frac{1}{2} + ig_n \right) > 0$ at a Gram point $g_n$ is called the Gram test.

In 1925 Hutchinson, using Gram point computations, showed that RH holds up to height $T = 300$. He found two empty Gram intervals, but since the neighbouring Gram interval in each case contained two zeros, he could overcome this defect.

38Ralf Josef Backlund (1888 Pietarsaari, Finland – 1949 Helsinki, Finland), Finnish mathematician, actuary, student of Lindelöf; shortly after obtaining his PhD, he quit academia to work for an insurance company; he was one of the founders of the Actuarial Society of Finland.

39The first to work with this function was Riemann, not Hardy.

40John Irwin Hutchinson (1867 Bangor, the US – 1935), American mathematician, student of Bolza, professor at Cornell University.
3. Total number of zeros up to a given height

Suppose that we have localized the zeros on the half line for all $0 < \text{Im}(s) < T$. We would like to prove the RH up to height $T$. Already Gram showed that

$$\log \xi \left( \frac{1}{2} + it \right) = \log \xi \left( \frac{1}{2} \right) - t^2 \sum_{\text{Re} \gamma > 0} \gamma^{-2} - \frac{t^4}{2} \sum_{\text{Re} \gamma > 0} \gamma^{-4} - \ldots,$$

where the sum is over the zeros $\rho$ of $\xi$ written in the form $1/2 + i \gamma$. From this identity he could evaluate $\sum \text{Re} \gamma > 0 \gamma^{-2n}$ very precisely. Since $\sum \text{Re} \gamma > 0 \gamma^{-2n}$ is dominated by the first few terms, he compared the estimated term $\sum \gamma^{-10}$ to the finite sum evaluated for the the 15 zeros found on the critical line. He then concluded that RH holds up to height $T = 50$. For more zeros the calculations became too involved.

Let $N(T)$ and $N_0(T)$ denote the number of zeros, respectively critical zeros, of $\zeta(s)$ with $0 < \text{Im}s < T$ counted with multiplicities. Von Mangoldt (1905) showed that

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T). \quad (16)$$

Put

$$S(T) = \frac{1}{\pi} \text{Im} \int_{C_{\varepsilon}} \frac{\zeta'(s)}{\zeta(s)} ds, \quad (17)$$

where $C_{\varepsilon}$ is the path consisting of the segment from $1 + \varepsilon$ to $1 + \varepsilon + iT$ and that from $1 + \varepsilon + iT$ to $1/2 + iT$. Backlund, basing himself on Riemann’s ideas, showed in 1912 that

$$N(T) = \frac{\theta(T)}{\pi} + 1 + S(T).$$

Moreover, if $\text{Re} \zeta$ is not zero on $C_{\varepsilon}$, then $N(T)$ is the nearest integer to $\theta(T)/\pi + 1$, i.e.

$$|S(T)| \leq \frac{1}{2}.$$

Backlund [2, 3] himself proved that $N(200) = 79$ and, using Gram’s approach, showed that all these zeros lie on the half line. Hutchinson also used this method to show that there are no more zeros with $0 < \text{Im}(s) < 300$ other than the ones he found. The problem with the approach of Backlund is to find a $T$ such that $\text{Re} \zeta$ is non-zero on $C$. This is a difficult task. Even worse is the fact that $S(T)$ can get arbitrarily large, so that infinitely often one will pick a $T$ for which $|S(T)| > 1/2$. For these two reasons, nowadays a better method developed by Turing half a century later is used (see Section 5).

We note that the above approach to verify the RH numerically up to a certain height will fail if zeta has zeros that are not simple. It is conjectured that actually the Riemann zeta function has only simple zeros. The reader might regard this as wishful thinking, but Random Matrix Theory, in particular Montgomery’s pair correlation conjecture, see Section 9.4 strongly suggests that all the zeros are simple. It is also a numerical observation that the critical zeros ‘repel’ each other.

We would like to conclude this section by recalling some major results involving $N(T)$,
see Karatsuba [40, Chapter 2] for an introduction. Hardy [26] was the first to show that there are infinitely many critical zeros. Even more, in 1921 he showed with Littlewood that $N_0(T) \geq c_1 T$ with $c_1$ a positive constant and $T \geq 15$. In 1942 Selberg showed that a positive fraction of all non-trivial zeros are critical, that is that $N_0(T) \geq c_2 N(T)$ with $c_2 > 0$. Selberg’s ideas lead to a tiny value of $c_2$ of about $7 \cdot 10^{-8}$ (as was worked out by S. Min in his PhD thesis). In the mid 1970’s Levinson [42] caused a sensation by showing that one can take $c_2 = 0.3474$. It was later improved to 0.4088 by Conrey [9], 0.4105 by Bui, Conrey and Young [7], and 0.4128 by Feng [22].

4. Riemann-Siegel formula

In 1932 Siegel [69] published the results on the Riemann zeta function found in the notes of Riemann stored in the archives of the Göttingen University Library. The following identity was discovered by Riemann. We start by showing the validity of the integral representation

$$
\zeta(s) = \frac{\Gamma(-s)}{2\pi i} \int_{-\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \, dx,
$$

where the contour integral starts at $+\infty$ and descends along the real axis, circulates $0$ in the positive direction and returns to $+\infty$ up to the real axis. We can then evaluate the integral in (18) in two ways: firstly by using the trivial identity

$$
\frac{e^{-Nx}}{e^x - 1} = \sum_{n=N+1}^{\infty} e^{-nx},
$$

and secondly by changing the contour and applying the Cauchy residue theorem. We set $\text{Res} = 1/2$ and arrive at

$$
Z(t) = 2 \sum_{n^2 < t/(2\pi)} \frac{1}{n^{1/2}} \cos(\theta(t) - t \log n) + R(t),
$$

where $Z(t)$ and $\theta(t)$ were defined in (14) and (15) respectively.

The idea of the Riemann-Siegel method [43] is to bound the remainder term $R(t)$. In order to determine the roots of the zeta function, we again use the function $Z$. Riemann, using the saddle point method, found that

$$
R(t) = (-1)^{\lfloor \sqrt{t/(2\pi)} \rfloor - 1} \left( \frac{t}{2\pi} \right)^{-1/4} \left( \sum_{j=0}^{m} C_j t^{-j/2} + R_m(t) \right),
$$

with

$$
C_0 = F(\delta), \quad C_1 = -\frac{F^{(3)}(\delta)}{2^3 \cdot 3}, \quad C_2 = \frac{F^{(2)}(\delta)}{2^4} + \frac{F^{(6)}(\delta)}{2^7 \cdot 3^2},
$$

[42] Norman Levinson (1912 Lynn, the US – 1975 Boston, the US), American mathematician, professor at MIT.

[43] For a derivation and more details see, e.g., [21, Chapter 7].
etc., and
\[ F(x) = \frac{\cos(x^2 + 3\pi/8)}{\cos(\sqrt{2\pi}x)}, \quad \delta = \sqrt{t} + \left(\sqrt{\frac{t}{2\pi}} + \frac{1}{2}\right)\sqrt{2\pi}. \]

The error term \( R_m(t) \) satisfies the estimate \( R_m(t) = O(t^{-(2m+3)/4}) \). Explicitly estimating the error term \( R_m(t) \) is unfortunately quite difficult. Titchmarsh, a former student of Hardy, proved in 1935 \[172\] that if \( t > 250\pi \), then
\[ |R_0(t)| \leq \frac{3}{2} \left( \frac{t}{2\pi} \right)^{-3/4} \]
and used this estimate to verify RH up to height 390.

5. Turing’s method

After Siegel’s publication in 1932 of the Riemann-Siegel formula, Titchmarsh obtained a grant for large scale computation of the non-trivial zeros. Making use of this formula, tabulating machines and “computers” (as the mostly female operators of such machines were called in those days), Titchmarsh established that the 1041 non-trivial zeros up to height 1468 are all critical. Turing, who from his student days onwards was interested in the zeros of the Riemann zeta function, became interested in extending Titchmarsh’s results. In his first paper on this topic \[74\], he considered the Riemann-Siegel method, but improved the estimate of Titchmarsh’s remainder term in such a way that it gives satisfactory results for \( 30 < \text{Im} s < 1000 \). However, the paper was rather technical, and, within some years, better estimates were obtained. Turing designed and even started to build a special purpose analog computer \[46\] in order to verify the RH in the range \( 0 < \text{Im} s < 6000 \). The outbreak of the Second World War prevented him from completing the construction of the machine. However, his attempts to build such a machine had set his mind in motion about building computers and helped him later to build faster his famous Bombes in order to break the Enigma Code. One can thus argue that many people thank their lives to the critical zeros!

After the Second World War Turing returned to the problem of computing non-trivial zeros. In his second paper on the zeros of \( \zeta \), he attempted to calculate the zeros of the Riemann zeta function for large \( \text{Im} s \), that is, for \( 1414 < \text{Im} s < 1540 \), in order to find a counterexample. He writes: The calculations were done in an optimistic hope that a zero would be found off the critical line, and the calculations were directed more towards finding such zeros than proving that none existed \[75\]. During this attempt to disprove the RH he developed what is now called Turing’s method, a method to estimate the number of non-trivial zeros up to a given height, which is still used today. In order to describe the method, we need to go back to 1914, when Littlewood showed that, assuming the RH, the

\[ \text{http://www.turingarchive.org/browse.php/C/2} \]
difference $\pi(x) - \text{Li}(x)$ changes sign infinitely often (see also Section 8.3), thus extending a result of Erhard Schmidt who had proved it, assuming the RH was false \cite{47}. Turing managed to exploit the result of Littlewood in order to average the value of

$$S(t) = N(t) - \left(\frac{\theta(t)}{\pi} + 1\right)$$

over $[0, T]$. He showed that $S(t)$ tends to zero as $t$ tends to infinity. In particular, he proved that, for $h > 0$ and $T > 168\pi$,

$$\left|\int_{T}^{T+h} S(t) dt\right| \leq 2.3 + 0.128 \log \frac{T + h}{2\pi}.$$

If we calculate $S(t)$ using the observed number of zeros $N'(t)$ instead of $N(t)$ and we missed $k$ zeros, i.e., $N'(t) < N(t)$, then we would get that $S(t) \to -k$, which eventually contradicts the above estimate. What the method amounts to is the following. In order to verify RH up to height $T$, one has to find enough critical zeros up to height $T + O(\log T)$.

For more details on Turing’s work on this subject, see for example \cite{4}, \cite[265–279]{13}.

6. Odlyzko-Schönhage method

In 1988 Odlyzko\cite{47} and Schönha\cite{48} developed a different approach to evaluate the Riemann zeta function. In \cite{53} they observed that the problem of evaluating sums of the form $\sum_{k=1}^{M} d_k k^{-it}$ at an evenly spaced set of $t$’s can be transformed into a problem of evaluating a rational function of the form $\sum_{k=1}^{n} a_k (z - b_k)^{-1}$ at all $n$-th roots of unity using the fast Fourier transform. The algorithm they designed rapidly evaluates such functions at multiple values. In particular, they showed that, for any positive constants $\delta$, $\sigma$, $c_1$, there exists an effectively computable constant $c_2$ and an algorithm which, for any $T$, computes $\zeta(\sigma + it), T \leq t \leq T + T^{1/2}$ to $T^{-c_1}$ precision in less than $c_2 T^\delta$ operations (addition, subtraction, multiplication, division). The algorithm has no advantage over the Riemann-Siegel method when a single value is evaluated, but it significantly improves the verification time of the RH for the first $N$ zeros as shown in Table 3.

---

\textsuperscript{47} Andrew Michael Odlyzko (1949 Tarnów, Poland), Polish-American mathematician, student of Stark, professor at the University of Minnesota.

\textsuperscript{48} Arnold Schönha (1934 Lockhausen, the Free State of Lippe), German mathematician, computer scientist, student of Hoheisel, professor at the University of Bonn, the University of Tübingen and the University of Konstanz.
Table 4. Record history of calculating critical zeros.

| Year | Number of computed zeros | Author |
|------|--------------------------|--------|
| 1859 | $\geq 2$                 | Riemann|
| 1903 | 15                       | Gram   |
| 1914 | 79                       | Backlund|
| 1925 | 138                      | Hutchinson|
| 1935 | 1,041                    | Titchmarsh|
| 1953 | 1,104                    | Turing |
| 1956 | 25,000                   | Lehmer |
| 1966 | 250,000                  | Lehman |
| 1968 | 3,500,000                | Rosser, Yohe, Schoenfeld |
| 1979 | 81,000,001               | Brent |
| 1982 | 200,000,001              | Brent, van de Lune, te Riele, Winter |
| 1983 | 300,000,001              | van de Lune, te Riele |
| 1986 | 1,500,000,001            | van de Lune, te Riele, Winter |
| 2004 | 900,000,000,000          | Wedeniwski |
| 2004 | 10,000,000,000,000       | Demichel, Gourdon |

7. A BRIEF HISTORY OF NON-TRIVIAL ZEROS CALCULATIONS

Riemann seems to have computed only a few non-trivial zeros. He certainly found zeros at approximately $1/2 + i14.1386$ and at $1/2 + i25.31$, and very likely computed more. He derived and tried to use the wonderful identity

$$\sum_{\text{Im} \rho > 0} \left( \frac{1}{\rho} + \frac{1}{1 - \rho} \right) = 1 + \frac{\gamma}{2} - \frac{\log \pi}{2} - \log 2 = 0.02309570896612103381\ldots$$

to prove that the root at $14.1$ is the first root.$^{49}$ The latter identity can be used to infer that $\text{Re} \rho > 10$ for the first non-trivial zero. The decimals Riemann gave for the two roots are slightly off, but the 20 decimals above he computed correctly!

In the earlier sections we reported about the computations and innovations due to Gram, Backlund, Hutchinson, Titchmarsh and Turing. After Turing computers with every increasing performance played a major role. At the beginning of the 21st century, large scale computations were performed. Between 2001 and 2005 the project ZetaGrid, led by Wedeniwski, was established. It involved distributed computation of the non-trivial zeros. It ran on more than 10,000 computers in over 70 countries and was based on software developed by van de Lune, te Riele and Winter$^{76}$ who after years of work eventually computed the first $1.5 \cdot 10^9$ Riemann zeros. More than $9 \cdot 10^{11}$ first zeros were verified to lie on the critical line. In 2004 Demichel and Gourdon,$^{23}$ performed the calculation of the zeros using the Odlyzko-Schönhage method to verify that the first $10^{13}$ lie on the critical line. The calculation was not repeated though. Unfortunately, neither of the results was published in a mathematical journal.

The above history is summarized in Table 4.

$^{49}$The identity can be derived using the Hadamard factorization for $\zeta$. Here $\gamma$ denotes Euler’s constant.
8. Applications

In this section we give some applications of being able to determine many non-trivial zeros with high precision, e.g., the determination of \( \pi(x) \) for very large \( x \). The three subsections that follow are about prime number inequalities; the first one about inequalities that hold true, and the other two about famous inequalities conjectured in the 19th century that turn out to be too good to be true. The final subsection is about the ternary Goldbach conjecture, which was recently resolved using very extensive zero calculations of Dirichlet \(^{50}\) \( L \)-functions (that behave like \( \zeta(s) \) in many respects).

8.1. The exact value of \( \pi(x) \). The record values for which \( \pi(x) \) has been exactly computed are a good indicator for the progress in computational prime number theory. The obvious way of computing \( \pi(x) \) is, of course, by counting all primes \( p \leq x \). For large values of \( x \) this is quite inefficient. In 1871 Meissel devised an ingenious method of computing \( \pi(x) \) without computing all primes \( p \leq x \). This method requires only the knowledge of the primes \( p \leq \sqrt{x} \), as well as the values of \( \pi(y) \) for some values of \( y \leq \frac{x^{2/3}}{3} \).

In 1885 Meissel determined \( \pi(10^9) \) (albeit not quite accurately). The algorithm was steadily improved and, e.g., in 2007 Oliveira e Silva used it to compute \( \pi(10^{23}) \). Assuming RH in 2010 Büthe, Franke, Jost and Kleinjung announced a value of \( \pi(10^{24}) \). Very recently by a different method Platt \[^{57}\] based on an explicit formula of Riemann for the quantity on the left hand side of \[^{12}\], managed to show unconditionally that \( \pi(10^{24}) = 18,435,599,767,349,200,867,866 \), in agreement with the value of Büthe et al. This computation rests on the first 69,778,732,700 critical zeros computed with 25 decimal accuracy. For the values of \( \pi(10^k) \) with \( 8 \leq k \leq 18 \) the reader is referred to Table 1.

8.2. Explicit prime number bounds. One of the most often quoted papers in computational number theory is the one by Rosser and Schoenfeld \[^{62}\]. In this paper, among other things, they prove the explicit bounds of \( \pi(x) \) that we mentioned in the introduction. The importance of this paper is that their proof was based on verifying the RH up to a certain height. Then using the fact that the first 3,502,500 zeros of \( \zeta \) lie on the critical line, the authors together with Yohe found explicit bounds for the \( \vartheta \)-function \[^{63}\]. Let \( p_n \) denote the \( n \)th prime. Rosser and Schoenfeld obtained the estimates

\[
 n (\log n + \log \log n - 3/2) < p_n < n (\log n + \log \log n - 1/2),
\]

valid for every \( n \geq 21 \). From this we deduce that \( p_n > n \log n \) for every \( n \geq 1 \), a result that had been obtained already in 1939 by Rosser \[^{61}\]. Note that Table 1 suggests that \( \pi(x) \geq [x/\log x] \) for all \( x \) large enough. Indeed, Rosser and Schoenfeld in 1962 using a delicate analysis established the truth of this inequality for \( x \geq 17 \). Meanwhile most explicit estimates of Rosser and Schoenfeld have been sharpened, e.g., by Dusart who exploited the verification of the RH for the first 1.5 \( \cdot \) 10\(^9\) zeros \[^{20}\].

\[^{50}\]Johann Peter Gustav Lejeune Dirichlet, German mathematician (1805–1859), who gave his name to the Dirichlet series; the Dirichlet theorem on primes in arithmetic progressions precedes by almost 70 years the prime number theorem. He was the advisor of Bernhard Riemann.
A recent result from B"uthe shows explicitly the connection between a partial RH and a sharp prime number estimate. He proved that if the RH holds true up to height $T$, then
\[ \pi(x) = \text{Li}(x) - \frac{\text{Li}(\sqrt{x})}{2} - \sum_{\rho} \text{Li}(x^\rho) + W(x), \]
where $W(x)$ relatively to the three earlier summands is of lower order. Riemann’s writings suggest (he was rather vague about it) that he thought that the inequality
\[ \pi(x) < \text{Li}(x) \] (19)
is always satisfied. Gauss and Goldschmidt had established the validity up to $x = 10^5$. Today we know that it even holds up to $x = 10^{14}$, cf. also Table 1. However, in 1914 Littlewood proved the spectacular result that the difference $\pi(x) - \text{Li}(x)$ changes sign infinitely often. In the mid-1930s Ingham showed that this result follows from knowledge of some initial non-trivial zeros. His proof was both simpler and more explicit than Littlewood’s, but also more computational. Let $x_0$ be the smallest integer for which $\pi(x_0) > \text{Li}(x_0)$. Skewes showed that
\[ x_0 < 10^{10^{10^34}} \text{ (1933, on RH)}, \quad x_0 < 10^{10^{10^964}} \text{ (1955, unconditionally)}. \]
For a long time these bounds of Skewes were considered to be the largest “naturally” occurring numbers in mathematics.

Using tables of non-trivial zeros accurate to 28 digits for the first 15 000 zeros and to 14 digits for the next 35 000 zeros, te Riele showed that (19) is false for at least 10 180 successive integers in $[6.627 \cdot 10^{370}, 6.687 \cdot 10^{370}]$. He made use of an earlier result of Lehman, which allows one to put bounds on $\pi(x) - \text{Li}(x)$, assuming that we have found all non-trivial zeros to height $T$, and that RH is checked up to height $T$ [46]. Lehman himself had obtained $x_0 < 10^{1166}$ using this result.

We now know that once (19) holds the wait until the inequality is reversed grows again, on average, as a function of the starting $x$. Thus we should perhaps not be surprised that the average of $\chi(t)$ with $\chi(t) = 1$ if $\pi(t) < \text{Li}(t)$ and $\chi(t) = 0$ otherwise, does not exist. However, under various plausible conjectures the density
\[ \delta = \lim_{x \to \infty} \frac{1}{\log x} \int_1^x \frac{\chi(t)}{t} dt \]
does exist and satisfies $\delta = 0.99999973 \ldots$ (see [65]). This conjecturally quantifies the dominance of $\text{Li}(x)$ over $\pi(x)$. The strong bias towards $\text{Li}(x)$ is an example of an interesting phenomenon that is called Chebyshev’s bias (see Section 9.2).

\[ ^{51}\text{Subsequently, Turing whose initial interest was in improving the results of Skewes, turned his interest to computing non-trivial zeros.} \]
8.4. The Mertens conjecture. The Mertens function $M(x)$ denotes the difference between the number of squarefree integers $n \leq x$ having an even number of prime factors and those having an odd number of prime factors. More formally, we have

$$M(x) = \sum_{n \leq x} \mu(n),$$

where $\mu(n) = 0$ if a square exceeding one divides $n$ and $\mu(n) = (-1)^m$ with $m$ being the number of different prime factors of $n$ otherwise. The function $\mu$ is called the Möbius function and outside number theory arises very frequently in combinatorial counting (inclusion-exclusion). It is not difficult to show that

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = s \int_1^{\infty} \frac{M(t)}{t^{1+s}} dt,$$

from which one can infer that the RH is equivalent with $|M(t)| = O(t^{1/2+\varepsilon})$. The PNT is equivalent to $M(x) = o(x)$. Most number theorists envision the Mertens function as something like a random walk, with the $\pm 1$ contributions from $\mu$ to $M(x)$ being close to a random coin flip. It is known by the probability theory that for the summatory function $w(x) = \sum_{n \leq x} w_n$, with $w_n = \pm 1$, randomly and independently, we have

$$\lim sup_{x \to \infty} \frac{w(x)}{\sqrt{(x/2) \log \log x}} = 1.$$  

This suggests that if $\mu$ is "sufficiently" random, then the ratio $M(x)/\sqrt{x}$ is expected to be unbounded.

On the basis of numerical work Stieltjes in 1885 and independently Mertens in 1897, conjectured that $|M(x)| < \sqrt{x}$, a conjecture that is now known as the Mertens conjecture. Daublebsky von Sterneck earlier had made the stronger claim that $|M(x)| < \sqrt{x}/2$ for $x > 200$. That conjecture was disproved in 1963 by Neubauer. We now know that 7725038629 is the minimal integer $> 200$ for which the Daublebsky von Sterneck bound does not hold. In 1983 Odlyzko and te Riele caused a sensation by disproving the Mertens conjecture. They showed that

$$\lim sup_{x \to \infty} \frac{M(x)}{\sqrt{x}} > 1.06, \quad \lim inf_{x \to \infty} \frac{M(x)}{\sqrt{x}} < -1.009,$$
without actually giving a specific integer \(x_0\) with \(|M(x_0)| \geq \sqrt{x_0}\). Later Kotnik and te Riele \[43\] using very extensive computer calculations showed that there is an \(x_0 < e^{1.59 \cdot 10^{40}}\) for which the Mertens conjecture fails.

The first step in disproving the Mertens conjecture is to find the analogue of (9) for \(M(x)/\sqrt{x}\). Assuming RH and that all critical zeros are simple this can be done. It involves a sum having terms of the form \(e^{\gamma y}/(\rho \zeta' (\rho))\), where \(\rho = 1/2 + i\gamma\), \(x = e^y\). Next, one finds an upper bound for the error made on cutting this formula at a given height \(T\) up to which one has verified RH numerically and computed the zeros with sufficient numerical precision. In order to obtain the desired result, one needs to verify RH and the simplicity of the zeros up to a height large enough. Secondly, we want each of the terms to be close to its maximum, which happens when \(\gamma y\) is close to an even integer. The latter leads to a problem in simultaneous Diophantine approximation. Now the key factor that allowed Odlyzko and te Riele to progress beyond earlier failed attempts to disprove the Mertens conjecture, was using the at the time recently developed Lenstra-Lenstra-Lovász algorithm, or \textbf{LLL-algorithm} for short. With the use of this new algorithm less extensive computing was needed to reach rather stronger results than before. Indeed only the first 2000 non-trivial zeros calculated with circa 100 significant decimal places, were used in the disproof.

### 8.5. The ternary Goldbach conjecture

The \textbf{Goldbach conjecture} (formulated in 1742 in a letter to Euler) states that every even number exceeding 2 can be represented as a sum of two primes. This is a very famous conjecture that remains unsolved. A weaker variant is the \textbf{ternary Goldbach conjecture}, also known as odd or weak Goldbach conjecture. It says that every odd number greater than 7 can be expressed as the sum of three odd primes.

Hardy and Littlewood showed in 1923 that on GRH\[57\] the odd Goldbach conjecture is true for all sufficiently large odd numbers. In 1937, Vinogradov \[78\] established this result unconditionally. Vinogradov used the \textbf{circle method}, which involves both minor and major arc estimates. His student Borozdkin in his PhD thesis (unpublished), made the sufficiently large explicit, yielding a huge number \(e^{e^{41.96}}\) and further published a result with the smaller bound \(e^{e^{16.038}}\), see \[5\]. In 1997, Deshouillers, Effinger, te Riele and Zinoviev published a result showing that GRH implies Goldbach’s weak conjecture \[16\]. This required checking all integers \(\leq 10^{20}\) as for all larger integers they could establish the result by theoretical means. Without GRH it was known in 2002 that the \(10^{20}\) has to be replaced by \(10^{1347}\). In 2012 and 2013, Helfgott released a series of preprints improving the major and minor arc estimates sufficiently to unconditionally prove the weak Goldbach conjecture (see \[31\], \[32\], \[33\] and \[34\]). Helfgott made use of a result of Platt \[58\] who had rigorously verified, using interval arithmetic, the RH for all Dirichlet \(L\)-functions for modulus \(q \leq 400\,000\) up to height around \(10^8/q\). The binary Goldbach conjecture is numerically verified up to \(4 \cdot 10^{18}\) by Oliveira e Silva, Herzog and Pardi \[55\].

The Goldbach conjecture is also studied in the field of \textbf{additive number theory}, where

\[\text{Grand Riemann Hypothesis, see Section [9.2].}\]
one considers a special set \( A \) (primes, squares, etc.) and then wonders what the sumset \( A + A = \{ a + b : a, b \in A \} \) looks like (similarly, with \( A + A + A \), etc.). For a nice introduction to additive number theory see the book by Tao and Vu \[71\].

9. MAJOR RECENT DEVELOPMENTS

In this section we discuss some major more recent developments related to the Riemann zeta function.

9.1. Complex analytic number theory. Riemann’s paper and, in its wake, the proof of the PNT were a major achievement of 19th century mathematics and gave rise to complex analytic number theory.

A function \( f \) from the natural numbers to the complex numbers is called an arithmetic function. An important subclass are the multiplicative functions that satisfy \( f(1) = 1 \) and \( f(mn) = f(m)f(n) \) for arbitrary coprime natural numbers \( m \) and \( n \) (the Möbius function is an example of a multiplicative function). The behaviour of arithmetic functions is usually very erratic. For that reason it makes sense to investigate related quantities that show more regular behaviour, for example the summatory function \( \sum_{n \leq x} f(n) \). The zeta function reflex leads one to consider \( F(s) = \sum_{n=1}^{\infty} f(n)n^{-s} \), which is called a Dirichlet series. Using the Perron integral and assuming that \( F(s) \) converges absolutely for some \( \Re s > \sigma \) one then finds that

\[
\sum_{n \leq x} f(n) = \int_{c-i\infty}^{c+i\infty} F(s) x^s \frac{ds}{s},
\]

with \( c > \sigma \) arbitrary and tries to estimate the integral. For this it is particularly important to be able to shift the line of integration as far as possible to the left. Here it becomes relevant to have an analytic continuation of \( F(s) \). The more one knows about \( F(s) \) the more information one can deduce about \( f \). However, often \( F(s) \) is not so well-behaved and one tries to consider a closely related function \( f^* \) instead, with the property that its Dirichlet series \( F^*(s) \) behaves in a nicer way\[58\]. The results one obtains in this way about \( f^* \) one tries to relate back to \( f \).

The attentive reader sees of course immediately that this whole approach is patterned on Riemann’s 1859 paper.

9.2. The Zeta Zoo. The study of the Riemann zeta-function has proved so extraordinarily successful in deepening our understanding of the primes, that it has become standard in number theory to try to associate zeta type functions to arithmetic structures (some kind of Pavlov reflex). Indeed, these days there is an enormous zoo of zeta functions. The alpha animal in the Zeta Zoo was and still is zeta. It codifies the behaviour of the integers and their atomic constituents: the primes. An important species of zeta functions are the Dirichlet \( L \)-functions. These were introduced by Dirichlet in order to understand the behaviour of primes in arithmetic progression. Like zeta, they satisfy a product expansion, a functional equation and it is conjectured that their zeros are also on the half line. The

\[58\text{E.g., } \sum p^{-s} \text{ is not nicely behaved, but } \sum \Lambda(n)n^{-s} \text{ is.}\]
latter hypothesis is called the **Extended Riemann Hypothesis** (ERH). Again a lot of computational number theory was done to verify RH for individual Dirichlet $L$-series up to a certain height and this was used to derive explicit bounds for $\pi(x; d, a)$, with $a$ and $d$ being coprime integers. These computations played an important role in the recent proof of the ternary Goldbach conjecture by Helfgott (see Section 8.5). De la Vallée Poussin (1897) proved that every of the arithmetic progressions $a \pmod{d}$ with $1 \leq a < d$ and $a$ and $d$ coprime (of which there are $\varphi(d)$) gets asymptotically its fair share of the primes, i.e., that asymptotically

$$\pi(x; d, a) \sim \frac{\pi(x)}{\varphi(d)}.$$ 

In 1837 Dirichlet in a ground breaking paper (where he introduced characters in number theory) had proved a weaker version of this result. Although the primes are asymptotically equidistributed, they have some positive bias towards progressions modulo $d$ where $a$ is a non-square modulo $d$. This was noted in 1853 by Chebyshev and is now known as **Chebyshev’s bias**. E.g., Bays and Hudson found in 1979 that 608,981,813,029 is the smallest prime for which $\pi(x; 3, 2) > \pi(x; 3, 1)$. For a nice introduction to this phenomenon see Granville and Martin [25]. As in the $\pi(x)$ versus $\mathrm{Li}(x)$ problem, under various assumptions there is a computable logarithmic measure for how often $\pi(x; d, a_1) > \pi(x; d, a_2)$.

Another important class of zeta functions are the **Dedekind zeta functions**. This is the analogue of the Riemann zeta function for a number field and can be treated by similar methods. E.g., Landau in 1903 showed the **prime ideal theorem**, stating that in a given number field the number of prime ideals of norm $\leq x$ grows asymptotically as $x/\log x$. The hypothesis that every Dedekind zeta function has its non-trivial zeros on the critical line is called the **Grand Riemann Hypothesis** (GRH).

The reader might wonder about a more stringent definition of a zeta function. Here is what Selberg, with a life time of experience with zeta functions, thought about this.

A zeta function $F(s)$ is a function of a complex variable $s$ that satisfies the following properties.

1. **Dirichlet series:** for $\text{Re} \ s > 1$, one can write $F(s) = \sum_{n=1}^{\infty} a_n n^{-s}$.

2. **Ramanujan hypothesis:** the growth of the coefficients $a_n$ has to be modest, in essence like that of the divisor function $\sum_{d|n} 1$.

3. **Analytic continuation:** $F(s)$ extends to a meromorphic function.

4. **Functional equation:** there is ”a connection” between $F(s)$ and $F(1 - s)$.

5. **Euler product:** one should be able to write $F(s) = \prod_p F_p(s)$.

9.3. **Correlation of pairs of non-trivial zeros.** So far we exclusively focused on finding and counting non-trivial zeros. A refinement is to ask about the distribution of the zeros, e.g., how are the differences between (consecutive) zeros distributed? Here the first

---

59 A profound database can be found at [http://www.lmfdb.org/](http://www.lmfdb.org/)
60 Patterned after Euler’s product formula [6].
theoretical work is due to Montgomery\textsuperscript{61}. He assumed the RH and wrote the zeros as $1/2 + i\gamma_i$ with $0 < \gamma_1 \leq \gamma_2 \leq \ldots$. To account for the increase of the density of the zeros as one goes up the critical strip and rescales them as

$$\overline{\gamma}_i = \gamma_i \frac{\log(\gamma_i/(2\pi e))}{2\pi}.$$ 

By the asymptotic (16) for $N(T)$ it then follows that the mean spacing between two rescaled consecutive zeros is 1. The \textbf{pair correlation conjecture} of Montgomery states that, with $0 \leq \alpha < \beta$, we have, as $M$ tends to infinity,

$$\frac{1}{M} | \{ 1 \leq i < j \leq M : \overline{\gamma}_j - \overline{\gamma}_i \in [\alpha, \beta) \} | \sim \int_{\alpha}^{\beta} \left( 1 - \left( \frac{\sin(\pi t)}{\pi t} \right)^2 \right) dt.$$ 

The integrand here is small when $t$ is close to zero, suggesting that the non-trivial zeros repel each other. Montgomery proved a smoothened version of his conjecture and used it to show that on the RH more than two thirds of the non-trivial zeros are simple.

Odlyzko numerically tested both this conjecture and the RH, beginning in the late 1980s. In the 1990s this led to monumental computations where billions of zeros were computed high up the critical strip. For a graphical demonstration of the computations, see Figure 1.

9.4. \textbf{Random matrix theory}. In 1972 Montgomery discussed his pair correlation conjecture with the physicist Dyson\textsuperscript{62}. Dyson immediately saw that the statistical distribution found by Montgomery appeared to be the same as the pair correlation distribution for the eigenvalues of a random Hermitian matrix that he had discovered a decade earlier.

\textbf{Random matrix theory} (RMT) was proposed by the physicist Eugene Wigner\textsuperscript{63} in 1951 to describe nuclear physics. The quantum mechanics of a heavy nucleus is complex and not well understood. Wigner made the bold conjecture that the statistics of the energy levels can be captured by random matrices.

RMT turns out to be a powerful tool in making conjectures involving the Riemann zeta and related functions. These conjectures lie typically way beyond the reach of current theoretical tools. Since this is the case, doing numerical checks on the conjectures is very important. These checks are often very computationally intensive.

As a very important example let us consider the problem of determining the even moments of the Riemann zeta function on the half line. We define

$$I_k(T) = \frac{1}{T} \int_0^T |\zeta(1/2 + it)|^{2k} dt.$$ 

Progress on determining the moments was slow. Hardy and Littlewood\textsuperscript{27} determined in 1916 the asymptotic behaviour of $I_1(T)$. Ingham\textsuperscript{36} improved this in 1926 in two ways.

\textsuperscript{61}Hugh Lowell Montgomery (1944 Muncie, the US), American mathematician, student of Davenport, professor at Michigan University.

\textsuperscript{62}Freeman John Dyson (1923 Crowthorne, England), American physicist, professor at Princeton University.

\textsuperscript{63}Eugene Paul Wigner (1902 Budapest, Hungary – 1995 Princeton, the US), Hungarian-American physicist and mathematician, Laureate of the Nobel Prize in Physics (1963).
Figure 1. Odlyzko’s pair correlation plot for $2 \cdot 10^8$ non-trivial zeros near the $10^{23}$th zero. In the displayed interval, the data agrees with the pair correlation conjecture to within about 0.002.

calculating the full asymptotic expansion when $k = 1$ and determining the leading term for $k = 2$ (the lower-order terms for $k = 2$ were determined by Heath-Brown [30] in 1979).
It was conjectured for general \( k \) that
\[
I_k(T) \sim \frac{f(k)a(k)}{k^{2!}}T(\log T)^{k^2},
\]
with \( a(k) \) a certain infinite product over all primes and \( f(k) \) an integer. Hard grinding analytic number theory led to the conjecture that \( f(3) = 42 \) [11] and \( f(4) = 24024 \) [12]. Progress beyond this seemed very challenging. Using RMT and modelling the zeros up to height \( T \) with \( N \) by \( N \) matrices with \( N \) around \( \log(T/2\pi) \), Keating and Snaith came with a conjectural integer value for \( f(k) \) for all \( k \). Their conjecture is consistent with all earlier results and the conjectural values of \( f(3) \) and \( f(4) \) derived using hardcore analytic number theory. Extensive numerical work corroborates the Keating and Snaith conjecture.

The RMT method offers a dictionary that allows one to translate number theoretical problems into random matrix problems that usually can be solved. However, there is no proof whatsoever that the dictionary always works. As long as this is the case, computational number theoretical work will play a very important role. The work of Odlyzko [52] on the zeros near \( 10^{22} \) is of enormous importance here, as the Riemann zeta function starts showing “its true face” only for very large values of \( T \).

The RMT method also works in the setting of function fields. These share many similarities with the number field setting, but often are easier to deal with. E.g., for them the Riemann Hypothesis is proved! In this setting Katz and Sarnak actually managed to prove various important results suggested by the RMT dictionary (published in their book [41] and surveyed in [42]).

The quest for an explanation of the RMT connection is ongoing and has led to active research at the intersection of number theory, mathematical physics, probability and statistics.

10. Words of warning

We hope that we have whetted the appetite of the reader to do computations in analytic number theory him or herself. A word of warning is, however, not amiss. Asymptotic estimates in analytic number theory often involve repeated logarithms. These are very difficult to detect by numerical computation. Thus a function that grows, e.g., like \( \log \log \log x \), looks on a computer like a bounded function. For this reason it is highly dangerous to make conjectures based on numerics alone (e.g., the Mertens conjecture), without some theoretical and heuristic considerations supporting the truth of the conjecture.

11. Further reading

Nice popular introductions to prime number theory are Sabbagh [66] and du Sautoy [19]. Halfway between a popular and more mathematical treatment is an interesting collection of prime number records and results modelled after the Guinness book of records [59]. For more mathematical introductions see, e.g., [10, 15, 21, 37, 56]. In the book of Edwards [64] there is a detailed explanation of the method used by Gram, Backlund and Hutchinson to compute

\[ \text{Warning: Edwards writes } \Gamma(s+1) \text{ instead of } \Gamma(s). \]
the smallest 300 non-trivial zeros. More advanced books are Ivić [38] and Titchmarsh [73]. The contribution of Hejhal and Odlyzko [13, 265–279] on the work of Turing on the Riemann zeta function is very informative. Snaith [70] wrote a nice overview of the connections between $L$-functions and random matrix theory, focusing on the example of the Riemann zeta function. A very readable conference proceedings on this matter is [49]. Rubinstein’s beautiful article [64, pp. 633–679] also discusses RMT, but with main focus on the influence of Riemann.

This survey owes a lot to the book of Narkiewicz [51] on the development of prime number theory that provides a nice mix of mathematical ideas and historical material. Also the book of Crandall and Pomerance [14] on computational prime number theory turned out to be quite helpful.

Finally, in tune with the dictum of Edwards “that one should read the masters and beware of secondary sources”[65] we would like to point out [6], which has many articles by the Riemann zeta masters.

12. Acknowledgement

The authors are enormously grateful to Alexandru Ciolan for his TEXnical assistance and his excellent proofreading of the many earlier versions. Also they thank Alex Weisse for his TEXnical assistance. Special thanks are due to Jan Büthe, Andreas Decker, Tomoko Kitagawa (alias Kate Kattegat), Władysław Narkiewicz, Olivier Ramaré, Michael Rubinstein, Kannan Soundararajan, Caroline Turnage-Butterbaugh and Tim Trudgian.

References

[1] R. Ayoub. Euler and the zeta function. Amer. Math. Monthly, 81:1067–1086, 1974.
[2] R. J. Backlund. Einige numerische Rechnungen die Nullprodukte der Riemannschen $\zeta$-Funktion betreffend. Öfvs. Finska Vet. Soc. Frh., 54(3):1–7, 1912.
[3] R. J. Backlund. Sur les zéros de la fonction $\zeta(s)$ de Riemann. Comp. Rend. Acad. Sci., 158:1979–1981, 1914.
[4] A. R. Booker. Turing and the Riemann Hypothesis. Notices Amer. Math. Soc., 53(10):1208–1211, 2006.
[5] K. G. Borodzkin. On the problem of i. m. vinogradovs constant (in russian). Proc. Third All-Union Math. Conf, 1, 1956.
[6] P. Borwein, S. Choi, B. Rooney, and A. Weirathmueller, editors. The Riemann hypothesis. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York, 2008. A resource for the afficionado and virtuoso alike.
[7] H. M. Bui, B. Conrey, and M. P. Young. More than 41% of the zeros of the zeta function are on the critical line. Acta Arith., 150(1):35–64, 2011.
[8] J. Carlson, A. Jaffe, and A. Wiles, editors. The Millennium Prize Problems. Clay Mathematics Institute, Cambridge, MA; American Mathematical Society, Providence, RI, 2006.
[9] J. B. Conrey. More than two fifths of the zeros of the Riemann zeta function are on the critical line. J. Reine Angew. Math., 399:1–26, 1989.
[10] J. B. Conrey. The Riemann Hypothesis. Notices Amer. Math. Soc., 50(3):341–353, 2003.
[11] J. B. Conrey and A. Ghosh. A conjecture for the sixth power moment of the Riemann zeta-function. Internat. Math. Res. Notices, (15):775–780, 1998.

[65] Such as the present survey written by non-masters.
[12] J. B. Conrey and S. M. Gonek. High moments of the Riemann zeta-function. *Duke Math. J.*, 107(3):577–604, 2001.

[13] S. Cooper and J. van Leeuwen. *Alan Turing: His Work and Impact*. Elsevier Science, 2013.

[14] R. Crandall and C. Pomerance. *Prime numbers*. Springer, New York, second edition, 2005. A computational perspective.

[15] H. Davenport. *Multiplicative number theory*, volume 74 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2000. Revised and with a preface by Hugh L. Montgomery.

[16] J.-M. Deshouillers, G. Effinger, H. te Riele, and D. Zinoviev. A complete Vinogradov 3-primes theorem under the Riemann hypothesis. *Electron. Res. Announc. Amer. Math. Soc.*, 3:99–104, 1997.

[17] K. Devlin. *The millennium problems*. Basic Books, New York, 2002. The seven greatest unsolved mathematical puzzles of our time.

[18] H. G. Diamond. Elementary methods in the study of the distribution of prime numbers. *Bull. Amer. Math. Soc. (N.S.)*, 7(3):553–589, 1982.

[19] M. du Sautoy. *The Music of the Primes: Searching to Solve the Greatest Mystery in Mathematics*. Fourth Estate, 2003.

[20] P. Dusart. Inégalités explicites pour $\psi(X)$, $\theta(X)$, $\pi(X)$ et les nombres premiers. *C. R. Math. Acad. Sci. Soc. R. Can.*, 21(2):53–59, 1999.

[21] H. Edwards. *Riemann’s Zeta Function*. Dover books on mathematics. Dover Publications, Mineola, New York, 2001.

[22] S. Feng. Zeros of the Riemann zeta function on the critical line. *J. Number Theory*, 132(4):511–542, 2012.

[23] X. Gourdon. The $10^{13}$ first zeros of the Riemann Zeta function, and zeros computation at very large height. [http://numbers.computation.free.fr/Constants/Miscellaneous/zetazeros1e13-1e24.pdf](http://numbers.computation.free.fr/Constants/Miscellaneous/zetazeros1e13-1e24.pdf) 2004.

[24] J.-P. Gram. Note sur les zéros de la fonction $\zeta(s)$ de Riemann. *Acta Math.*, 27(1):289–304, December 1903.

[25] A. Granville and G. Martin. Prime number races. *Gac. R. Soc. Mat. Esp.*, 8(1):197–240, 2005. With appendices by Giuliana Davidoff and Michael Guy.

[26] G. H. Hardy. Sur les zéros de la fonction $\zeta(s)$ de Riemann. *Comp. Rend. Acad. Sci.*, 158:1012–1014, 1914.

[27] G. H. Hardy and J. E. Littlewood. Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes. *Acta Math.*, 41(1):119–196, 1916.

[28] G. H. Hardy and J. E. Littlewood. The zeros of the Riemann’s Zeta-Function on the critical line. *Math. Zeitschrift*, 10:283–317, 1921.

[29] G. H. Hardy and J. E. Littlewood. The approximate functional equation in the theory of the zeta-function, with applications to the divisor problems of Dirichlet and Piltz. *Proc. London Math. Soc.*, 21(2):39–74, 1922.

[30] D. R. Heath-Brown. The fourth power moment of the Riemann zeta function. *Proc. London Math. Soc. (3)*, 38(3):385–422, 1979.

[31] H. A. Helfgott. Major arcs for goldbachs problem. *Preprint. Available at arXiv:1203.5712*. 1203.5712.

[32] H. A. Helfgott. Minor arcs for goldbachs problem. *Preprint. Available at arXiv:1205.5252*. 1205.5252.

[33] H. A. Helfgott. The ternary goldbach conjecture is true. *Preprint. Available at arXiv:1312.7748*. 1312.7748.

[34] H. A. Helfgott. La conjecture de Goldbach ternaire. *Gaz. Math.*, (140):5–18, 2014. Translated by Margaret Bilu, revised by the author.

[35] J. I. Hutchinson. On the roots of the Riemann zeta function. *Trans. Amer. Math. Soc.*, 27:49–60, 1925.

[36] A. E. Ingham. Mean-Value Theorems in the Theory of the Riemann Zeta-Function. *Proc. London Math. Soc.*, S2-27(1):273, 1928.

[37] A. E. Ingham. *The distribution of prime numbers*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1990. Reprint of the 1932 original, With a foreword by R. C. Vaughan.
[38] A. Ivić. The Riemann zeta-function. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1985. The theory of the Riemann zeta-function with applications.

[39] H. Kadiri. Une région explicite sans zéros pour la fonction ζ de Riemann. Acta Arith., 117(4):303–339, 2005.

[40] A. A. Karatsuba. Complex analysis in number theory. CRC Press, Boca Raton, FL, 1995.

[41] N. M. Katz and P. Sarnak. Random matrices, Frobenius eigenvalues, and monodromy, volume 45 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 1999.

[42] N. M. Katz and P. Sarnak. Zeros of zeta functions and symmetry. Bull. Amer. Math. Soc. (N.S.), 36(1):1–26, 1999.

[43] T. Kotnik and H. te Riele. The Mertens conjecture revisited. In Algorithmic number theory, volume 4076 of Lecture Notes in Comput. Sci., pages 156–167. Springer, Berlin, 2006.

[44] E. Landau. Handbuch der Lehre von der Verteilung der Primzahlen. 2 Bände. Leipzig B.G. Teubner, 1909.

[45] E. Landau. Handbuch der Lehre von der Verteilung der Primzahlen. 2 Bände. Chelsea Publishing Co., New York, 1953. 2d ed, With an appendix by Paul T. Bateman.

[46] R. S. Lehman. On the difference π(x) − li(x). Acta Arith., 11:397–410, 1966.

[47] J. E. Littlewood. Sur la distribution des nombres premiers. Comp. Rend. Acad. Sci., 158:1869–1872, 1914.

[48] F. Mertens. Über eine zahlentheoretische Funktion. Sitzungsberichte Akademie Wissenschaftlicher Wien Mathematik-Naturlich IIa, 106:761–830, 1897.

[49] F. Mezzadri and N. C. Snaith, editors. Recent perspectives in random matrix theory and number theory, volume 322 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2005.

[50] H. L. Montgomery. The pair correlation of zeros of the zeta function. In Analytic number theory (Proc. Sympos. Pure Math., Vol. XXIV, St. Louis Univ., St. Louis, Mo., 1972), pages 181–193. Amer. Math. Soc., Providence, R.I., 1973.

[51] W. Narkiewicz. The development of prime number theory. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000. From Euclid to Hardy and Littlewood.

[52] A. M. Odlyzko. The 10^{22}-nd zero of the Riemann zeta function. In Dynamical, spectral, and arithmetic zeta functions (San Antonio, TX, 1999), volume 290 of Contemp. Math., pages 139–144. Amer. Math. Soc., Providence, RI, 2001.

[53] A. M. Odlyzko and A. Schönhage. Fast algorithms for multiple evaluations of the Riemann zeta function. Trans. Amer. Math. Soc., 309(2):797–809, 1988.

[54] A. M. Odlyzko and H. J. J. te Riele. Disproof of the Mertens conjecture. J. Reine Angew. Math., 357:138–160, 1985.

[55] T. Oliveira e Silva, S. Herzog, and S. Pardi. Empirical verification of the even Goldbach conjecture and computation of prime gaps up to 4 · 10^{18}. Math. Comp., 83(288):2023–2060, 2014.

[56] S. J. Patterson. An introduction to the theory of the Riemann zeta-function, volume 14 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1988.

[57] D. J. Platt. Computing π(x) analytically. Math. Comp., 84(293):1521–1535, 2015.

[58] D. J. Platt. Numerical computations concerning the GRH. Math. Comp., 85(302):3009–3027, 2016.

[59] P. Ribenboim. The new book of prime number records. Springer-Verlag, New York, 1996.

[60] B. Riemann. Über die Anzahl der Primzahlen unter einer gegebenen Grösse. Monatsberichte der Berliner Akademie, pages 671–680, November 1859.

[61] J. B. Rosser. The n-th prime is greater than n log n. Proc. London Math. Soc., 45(2):21–44, 1939.

[62] J. B. Rosser and L. Schoenfeld. Approximate formulas for some functions of prime numbers. Illinois J. Math., 6:64–94, 1962.
[63] J. B. Rosser, J. M. Yohe, and L. Schoenfeld. Rigorous computation and the zeros of the Riemann zeta-function. (With discussion). In Information Processing 68 (Proc. IFIP Congress, Edinburgh, 1968), Vol. 1: Mathematics, Software, pages 70–76. North-Holland, Amsterdam, 1969.

[64] M. Rubinstein. Riemann’s influence in number theory from a computational and experimental perspective. In The legacy of Bernhard Riemann after one hundred and fifty years. Vol. II, volume 35 of Adv. Lect. Math. (ALM), pages 633–679. Int. Press, Somerville, MA, 2016.

[65] M. Rubinstein and P. Sarnak. Chebyshev’s bias. Experiment. Math., 3(3):173–197, 1994.

[66] K. Sabbagh. Dr. Riemann’s Zeros. Atlantic books, 2003. The theory of the Riemann zeta-function with applications.

[67] E. Schmidt. Über die Anzahl der Primzahlen unter gegebener Grenze. Math. Ann., 57(2):195–204, 1903.

[68] A. Selberg. An elementary proof of the prime-number theorem for arithmetic progressions. Canadian J. Math., 2:66–78, 1950.

[69] C. L. Siegel. Über Riemanns Nachlass zur analytischen Zahlentheorie. Quellen und Studien zur Geschichte der Math. Astr. Phys., 2:45–80, 1932.

[70] N. C. Snaith. Riemann zeros and random matrix theory. Milan J. Math., 78(1):135–152, 2010.

[71] T. Tao and V. H. Vu. Additive combinatorics, volume 105 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2010. Paperback edition [of MR2289012].

[72] E. C. Titchmarsh. The zeros of the Riemann zeta-function. Proc. Roy. Soc. London, 151:234–255, 1935.

[73] E. C. Titchmarsh. The theory of the Riemann zeta-function. The Clarendon Press, Oxford University Press, New York, second edition, 1986. Edited and with a preface by D. R. Heath-Brown.

[74] A. M. Turing. A method for the calculation of the zeta-function. Proc. London Math. Soc., 48 (2):180–197, 1945.

[75] A. M. Turing. Some calculations of the Riemann zeta-function. Proc. London Math. Soc., 3 (3):99–117, 1953.

[76] J. van de Lune, H. J. J. te Riele, and D. T. Winter. On the zeros of the Riemann zeta function in the critical strip. IV. Math. Comp., 46(174):667–681, 1986.

[77] V. S. Varadarajan. Euler through time: a new look at old themes. American Mathematical Society, Providence, RI, 2006.

[78] I. M. Vinogradov. Representation of an odd number as a sum of three primes. Dokl. Akad. Nauk. SSR, 15:291–294, 1937.

(P. Moree, A. Sedunova) Max-Planck-Institut für Mathematik, Vivatsgasse 7, D-53111 Bonn, Germany.

E-mail address: moree@mpim-bonn.mpg.de
E-mail address: alisa.sedunova@phystech.edu

(I. Petrykiewicz)
E-mail address: ipetrykiewicz@gmail.com