SURJECTIVITY OF LINEAR OPERATORS AND SEMIALGEBRAIC GLOBAL DIFFEOMORPHISMS

FRANCISCO BRAUN∗, LUIS RENATO GONÇALVES DIAS† and JEAN VENATO-SANTOS‡

Abstract. We prove that a $C^\infty$ semialgebraic local diffeomorphism of $\mathbb{R}^n$ with non-properness set having codimension greater than or equal to 2 is a global diffeomorphism if $n-1$ suitable linear partial differential operators are surjective. Then we state a new analytic conjecture for a polynomial local diffeomorphism of $\mathbb{R}^n$. Our conjecture implies a very known conjecture of Z. Jelonek. We further relate the surjectivity of these operators with the fibration concept and state a general global injectivity theorem for semialgebraic mappings which turns out to unify and generalize previous results of the literature.

1. Introduction

Let $F = (f_1, \ldots, f_n) : \mathbb{K}^n \to \mathbb{K}^n$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, be a differentiable mapping. We denote by $J(F)$ the Jacobian determinant of $F$. We say that $F$ is proper at $y \in \mathbb{K}^n$ if there exists a neighborhood $V$ of $y$ such that $F^{-1}(V)$ is compact. We denote by $S_F$ the set of points at which $F$ is not proper. This is the non-properness set of $F$, also known as Jelonek’s set of $F$.

It is known that $S_F$ is a semialgebraic set when $F$ is a semialgebraic mapping, see for instance [16, Theorem 6.4]. Thus, in this case, it makes sense to consider the dimension and the codimension of $S_F$.

The following conjecture was raised by Jelonek in [16]:

Conjecture 1.1 (Jelonek’s conjecture). Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial mapping with nowhere vanishing $J(F)$ and such that $\text{codim} S_F \geq 2$. Then $F$ is a bijective mapping.

In his paper, Jelonek proved that Conjecture [11] is true under the additional assumption $\text{codim}(S_F) \geq 3$, the proof working for the semialgebraic case as well (although the conclusion is just the injectivity of $F$), and also that it is true for $n = 2$, see [13, Proposition 2.13] for a semialgebraic version in $n = 2$. Much more appealing is the fact, proved in the same paper, that Conjecture [11] implies the famous Jacobian conjecture in $\mathbb{C}^n$: “A polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ with $J(F) = 1$ is an automorphism.”

We remark that the Pinchuk [18] mappings $P : \mathbb{R}^2 \to \mathbb{R}^2$ constructed in order to disprove the real Jacobian conjecture, that a polynomial mapping $\mathbb{R}^n \to \mathbb{R}^n$ with...
nowhere zero Jacobian determinant is injective, satisfy codim \((S_F) = 1\) (see yet a rational parametrization of \(S_F\) for a specific Pinchuk mapping in [8]), and so the assumption on codim \((S_F) \geq 2\) is necessary in Conjecture 1.1 and it remains to investigate it in dimension \(n \geq 3\).

On the other hand, for a fixed \(C^\infty\) mapping \(F = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n\) (resp. an entire analytic mapping \(F = (f_1, \ldots, f_n) : \mathbb{C}^n \to \mathbb{C}^n\)) and for an \(i \in \{1, 2, \ldots, n\}\), let \(\Delta^F_i : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)\) (resp. \(\Delta^F_i : E_n \to E_n\)) be the operator defined by

\[
\Delta^F_i(g) = J(f_1, \ldots, f_{i-1}, g, f_{i+1}, \ldots, f_n),
\]

where \(C^k(M)\) denotes the space of \(C^k\) functions defined on a given \(C^k\) manifold \(M\), and \(E_n\) denotes the space of entire analytic functions on \(\mathbb{C}^n\) with the topology of uniform convergence on compact subsets.

Stein [19, 20, 21], for \(n = 2\), and Krasiński and Spodzieja [17], for \(n \geq 2\), proved that for any polynomial mapping \(F : \mathbb{C}^n \to \mathbb{C}^n\) with \(J(F) = 1\), the image \(\Delta^F_i(E_n)\) is dense in \(E_n\) for \(n - 1\) indices \(i \in \{1, \ldots, n\}\) if, and only if, \(F\) is a polynomial automorphism. Actually, for \(n = 2\), it was proved in [21] that \(\Delta^F_i(E_2)\) is closed, hence \(F\) is a polynomial automorphism if and only if \(\Delta^F_i\) is surjective for \(i = 1\) or 2 in the bidimensional case. So the following conjecture, stated for \(n = 2\) in [19], implies the Jacobian conjecture:

**Conjecture 1.2.** Let \(F : \mathbb{C}^n \to \mathbb{C}^n\) be an entire analytic mapping such that \(J(F) = 1\). Then the image \(\Delta^F_i(E_n)\) is dense in \(E_n\) for \(n - 1\) different indices \(i \in \{1, \ldots, n\}\).

The following analogous statement in the real \(C^\infty\) case was recently proved false in [4] for \(n \geq 3\) (in \(\mathbb{R}^2\) it is true: see for instance [5]):

Let \(F : \mathbb{R}^n \to \mathbb{R}^n\) be a \(C^\infty\) mapping with nowhere vanishing \(J(F)\).

If \(\Delta^F_i(C^\infty(\mathbb{R}^n)) = C^\infty(\mathbb{R}^n)\) for \(n - 1\) indices \(i \in \{1, \ldots, n\}\), then \(F\) is injective.

The first main result of this paper is to consider the real \(C^\infty\) case patching together assumptions on the operators \(\Delta^F_i\) and on the set \(S_F\) in the following theorem. In order to consider \(S_F\) we assume \(F\) to be semialgebraic.

**Theorem 1.3.** Let \(F : \mathbb{R}^n \to \mathbb{R}^n\) be a \(C^\infty\) semialgebraic mapping with nowhere vanishing \(J(F)\) and such that codim \(S_F \geq 2\). Then \(F\) is a bijective mapping if and only if \(\Delta^F_i(C^\infty(\mathbb{R}^n)) = C^\infty(\mathbb{R}^n)\) for \(n - 1\) indices \(i \in \{1, \ldots, n\}\).

We remark that in the preceding analogous polynomial case in \(\mathbb{C}^n\), the set \(S_F\) is empty or a hypersurface, according to [15, Theorem 3.8], hence its real codimension is greater than or equal to 2. Thus our assumption in Theorem 1.3 is the natural real semialgebraic counterpart of the preceding complex results.

We also observe that the counterexample constructed in [4] is not semialgebraic, but we can (and will) construct a semialgebraic one, see Example 3.1. This supports the assumption on \(S_F\) in Theorem 1.3.

A similar result than the above ones in the polynomial case is Theorem 4.1 of [17] that for any polynomial mapping \(F : \mathbb{K}^n \to \mathbb{K}^n\), \(\mathbb{K} = \mathbb{R}\) or \(\mathbb{C}\), such that \(J(F) = 1\), then \(F\) is a polynomial automorphism if, and only if, \(\Delta^F_i(\mathbb{K}[x_1, \ldots, x_n]) = \mathbb{K}[x_1, \ldots, x_n]\) for \(n - 1\) different indices \(i \in \{1, \ldots, n\}\). The bijective mapping of \(\mathbb{R}^2\) given by \(F(x, y) = (x(x^2 + 1), y(y^2 + 1))\) does not satisfy the hypothesis of this result but does satisfy the ones of Theorem 1.3.

Related to the above conjectures, we state the following
Conjecture 1.4. Let $F: \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial mapping with nowhere vanishing $J(F)$ and such that codim $S_F \geq 2$. Then $\Delta_i^F(\mathcal{C}^\infty(\mathbb{R}^n)) = \mathcal{C}^\infty(\mathbb{R}^n)$ for $n-1$ indices $i \in \{1, \ldots, n\}$.

By Theorem 1.3 this conjecture implies the Jelonek’s one and so the Jacobian conjecture as well.

Our second main result is Theorem 1.5 below. It provides topological properties that certain leaves of foliations related to $F$ must satisfy when $\Delta_i^F$ is surjective. This theorem is also the main tool for the proof of Theorem 1.3.

Theorem 1.5. Let $F = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^\infty$ mapping with nowhere vanishing $J(F)$ and such that there exists $i \in \{1, \ldots, n\}$ with $\Delta_i^F(\mathcal{C}^\infty(\mathbb{R}^n)) = \mathcal{C}^\infty(\mathbb{R}^n)$. Let $j \in \{1, \ldots, n\} \setminus \{i\}$ fixed. Then for any nonempty connected component $L$ of $\bigcap_{i \in \{1, \ldots, n\} \setminus \{i, j\}} f_i^{-1}(c_i)$ it holds:

(i) The non-empty fibers of the restriction $f_j|_L : L \to \mathbb{R}$ are connected.

(ii) $L$ is diffeomorphic to $\mathbb{R}^2$.

Remark 1.6. It is clear that the operator $\Delta_i^F$ as defined in (1) does not depend on the component $f_i$ of the mapping $F = (f_1, \ldots, f_n)$. Actually, by considering $F_i = (f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_n)$, we can denote by $\Delta_i^{F_i}$ the same operator defined in (1). Further, we do not need to assume that the submersion $F_i : \mathbb{R}^n \to \mathbb{R}^{n-1}$ is part of a local diffeomorphism $F$ in order to state Theorem 1.5. But the assumption on the surjectivity of $\Delta_i^{F_i}$ immediately guarantees the existence of a function $f_i \in \mathcal{C}_n^\infty$ such that $J(f_1, \ldots, f_{i-1}, f_i, f_{i+1}, \ldots, f_n) \equiv 1$, for instance. So we do not miss generality in the statement of Theorem 1.5.

Actually, for $n = 3$, we can characterize the surjectivity of $\Delta_i^F$ as

$\mathbf{Corollary 1.7.}$ Let $F = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ be a $\mathcal{C}^\infty$ mapping with nowhere vanishing $J(F)$ and let $i \in \{1, 2, 3\}$. Then $\Delta_i^F(\mathcal{C}^\infty(\mathbb{R}^3)) = \mathcal{C}^\infty(\mathbb{R}^3)$ if and only if for $j, k \in \{1, 2, 3\} \setminus \{i\}$, $j \neq k$, and for any connected component $L_j$ and $L_k$ of nonempty fibers $f_j$ and $f_k$, respectively, the sets $f_k^{-1}(c) \cap L_j$ and $f_j^{-1}(c) \cap L_k$ are connected for any $c \in \mathbb{R}$.

The sufficient condition above is Theorem 2 of [4]. That this condition is necessary follows directly from (i) of Theorem 1.3. See also an $n$-dimensional version of this in [5].

The proofs of theorems 1.3 and 1.5 are presented in Section 2.

We end the paper in Section 3 bringing the concept of local fibration to the subject of the paper. In Section 4 we discuss how this topological concept is related to the surjectivity of the operators defined in (1). Also, motivated by the works of Byrnes and Lindquist [6] and Campbell [7], we show that if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a semialgebraic local homeomorphism such that either $F$ or $F_i$ (as above defined) is a local fibration on its range, then $F$ is injective, see Theorem 4.2.

2. Proof of Theorems 1.3 and 1.5

For a given mapping $F = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ and for any positive integer $i \leq n$, we will denote by $F_i : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the mapping with coordinate functions $(f_1, f_{i-1}, f_{i+1}, \ldots, f_n)$.

We begin gathering the following straightforward properties:
Lemma 2.1. Let $F = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a local homeomorphism. Let $i \in \{1, \ldots, n\}$ fixed and $R$ be a connected component of a nonempty fiber of $F_i : \mathbb{R}^n \to \mathbb{R}^{n-1}$. Then

(a) $R$ is homeomorphic (diffeomorphic if $F$ is $C^1$) to $\mathbb{R}$ and its both ends are unbounded.

(b) The function $f_i$ is strictly monotone along $R$.

(c) If $F$ is $C^\infty$, the integral curves of $\Delta^F_i$ are the connected components of the nonempty fibers of $F_i$.

In particular, for any positive integer $k \leq n$, the non-empty connected components of $\bigcap_{j=1}^k f_{\sigma(j)}^{-1}(c_j)$ are unbounded for any injective mapping $\sigma : \{1, \ldots, k\} \to \{1, \ldots, n\}$. We remark that this could not be the case if $(f_{\sigma(1)}, \ldots, f_{\sigma(k)}) : \mathbb{R}^n \to \mathbb{R}^k$ is just a submersion (not coming from coordinate functions of a local homeomorphism) with $n \geq 3$, see for instance [11, Section II]: there exists a submersion $g : \mathbb{R}^3 \to \mathbb{R}^2$ with a fiber having one connected component diffeomorphic to $S^1$.

A key point in our paper is the following characterization of surjectivity of vector fields given in [12].

Lemma 2.2 (Part of Theorem 6.4.2 of [12]). Let $M$ be a $C^\infty$ manifold and $X : C^\infty(M) \to C^\infty(M)$ be a vector field. Then $X(C^\infty(M)) = C^\infty(M)$, if and only if

(i) No integral curve of $X$ is contained in a compact subset of $M$.

(ii) For all compact $K \subset M$, there exists a compact $K' \subset M$ such that every compact interval on an integral curve of $X$ with endpoints in $K$ is contained in $K'$.

We first provide the

Proof of Statement (i) of Theorem 155. The proof presented here is an adaptation to a bidimensional submanifold $(L)$ of the proof given in [13, Proposition 1.4] in the plane. Lemma 2.1 will be used throughout the proof. Observe that $L$ is invariant by $\Delta^F_i$: the integral curves of $\Delta^F_i$ in $L$ are given by the connected components of the fibers of $f_i|_L$. This is a regular foliation of $L$. Below, when referring to the flow of $\Delta^F_i$ we mean this foliation, i.e. the flow of $\Delta^F_i$ restricted to $L$.

In the proof we will denote by $\gamma_p = \gamma_p(t)$ the integral curve of $\Delta^F_i$ passing through $p \in L$, i.e., such that $\gamma_p(0) = p$. Further, we will say that a continuous path $\sigma$ in $L$ is tangent to a given integral curve $\gamma$ at a point $q$ when $\sigma$ is not transversal to $\gamma$ at $q$. Finally, we will say that a continuous injective path $\sigma$ in $L$ is ideal if it has at most finitely many points of tangency with the flow of $\Delta^F_i$.

We begin proving that any two points of $L$ can be joined by an ideal path. Indeed, for a given $p \in L$, we define $Z(p) = \{q \in L \mid \text{there exists an ideal path from } p \text{ to } q\}$. It is enough to prove that $Z(p) = L$.

By the flow box theorem, [17, Lemma 3, pag 69], it follows that $Z(p)$ is non-empty. $Z(p)$ is open: Indeed, let $x \in Z(p)$ and denote by $\sigma_1$ an ideal path joining $p$ to $x$. Applying the flow box theorem to the point $x$, we get a neighborhood of $x$ such that any point $y$ of it can be joined to $x$ by an ideal path $\sigma_2$. Clearly the concatenation of $\sigma_1$ and $\sigma_2$ is an ideal path connecting $p$ to $y$. $Z(p)$ is closed: Indeed, let $\{x_n\}$ be a sequence in $Z(p)$ converging to $x$. Similarly as above, for an $n$ big enough, we manage to construct an ideal path from $p$ to $x$, passing by $x_n$, and so $x \in Z(p)$. So $Z(p) = L$ as we wanted.
Now we pass to the proof of the result. Acting by contradiction, we suppose that for some $c \in \mathbb{R}$ the set $f_j | L^{-1}(c)$ is not connected. We let $R_1$ and $R_2$ be two distinct connected components of $f_j | L^{-1}(c)$ and we consider $p_1 \in R_1$ and $p_2 \in R_2$. So $R_k = \gamma_{p_k}$, $k = 1, 2$. We denote by $\Omega(p_1, p_2)$ the (non-empty) set of ideal paths $\sigma : [0, 1] \to L$ such that $\sigma(0) = p_1$ and $\sigma(1) = p_2$. We then fix a path $\sigma \in \Omega(p_1, p_2)$ that minimizes the number of tangencies with the flow of $\Delta^F$.

We claim that $R_1 \cap \sigma([0, 1]) = \{ p_1 \}$ and $R_2 \cap \sigma([0, 1]) = \{ p_2 \}$. If, for instance, $R_1 \cap \sigma([0, 1]) \neq \{ p_1 \}$, let $t_1 \in (0, 1]$ be the greatest time $t$ such that $\sigma(t) \in R_1$. We consider $\lambda$ to be the new path formed by the concatenation of $R_1$ from $p_1$ to $\sigma(t_1)$ and by $\sigma([t_1, 1])$. Then by applying the flow box theorem in this interval of $R_1$, we can modify $\lambda$ to a new path $\overline{\sigma} \in \Omega(p_1, p_2)$ without tangency points from $0$ to $\sigma(t)$, for a $t > t_1$ close enough of $t_1$. Since $\sigma$ must have a tangency point in $(0, t_1)$, it follows that $\overline{\sigma}$ has less tangencies than $\sigma$, a contradiction with the choice of $\sigma$. The proof for $R_2$ is analogous. The claim is proved. So, in the reasoning below, we can fix our attention to the open interval $(0, 1)$.

Since $f_j \circ \sigma$ has a maximum or a minimum point in $(0, 1)$, it follows that the finite set

$$\{ s \in (0, 1) \mid \sigma \text{ is tangent to } \gamma_{\sigma(s)} \text{ at some point of } \sigma([0, 1]) \}$$

is not empty and we can let $s_0 > 0$ be its minimum. We have two possibilities: (i) $\sigma$ is tangent to $\gamma_{\sigma(s_0)}$ at $\sigma(s_0)$ or (ii) $\sigma$ is transversal to $\gamma_{\sigma(s_0)}$ at $\sigma(s_0)$.

Below we will show that assuming either (i) or (ii) leads to a contradiction. This will finish the proof.

First we assume (i) and let $T := \{ s \in (0, s_0) \mid \#(\gamma_{\sigma(s)} \cap \sigma([0, 1])) \geq 2 \}$, where as usual $\#Z$ denotes the cardinality of a set $Z$. By continuous dependence, it follows that $s \in T$ for all $s < s_0$ close enough to $s_0$. Thus $T \neq \emptyset$ and we may consider $\tilde{s} = \inf T$.

If $\tilde{s} = 0$, then $\tilde{s} \notin T$. If $\tilde{s} > 0$, we have that $\gamma_{\sigma(\tilde{s})}$ is transversal to $\sigma((0, 1])$ by the choice of $s_0$. Then, it follows by the flow box theorem that $\tilde{s} \notin T$. In any case, so, we have that $\tilde{s} \notin T$.

We are going to apply Lemma 2.2 to show that $\Delta^F_i$ is not surjective, the aimed contradiction. Let the compact set $K := \sigma([0, 1])$ and let $K'$ be any given compact set. The integral curve $\gamma_{\sigma(s)}$ is not bounded in any direction by Lemma 2.1. So we consider a tubular neighborhood $N$ of it escaping from $K'$ in both directions. Then for any $t$ such that $\sigma(t) \in N$ it follows that the integral curve $\gamma_{\sigma(t)}$ stays in $N$ before it can eventually return to $K$. In particular, by the definition of $\tilde{s}$, there does exist $s \in T$ close enough to $\tilde{s}$ such that the orbit $\gamma_{\sigma(s)}$ cuts at least twice the compact $K$ but in between it escapes $K'$. By Lemma 2.2 it follows that $\Delta^F_i$ is not surjective. So (i) cannot be true.

On the other hand, if (ii) is in force, let $s_1 > s_0$ be such that $\sigma(s_1) \in \gamma_{\sigma(s_0)}$ and $\sigma$ is tangent to $\gamma_{\sigma(s_0)}$ at $\sigma(s_1)$. We assume that $s_1$ is the smallest $s \in (s_0, 1)$ with this property.

We define now the continuous path $\overline{\sigma}$ being the concatenation of the the following three paths: the path $\sigma((0, s_0))$, the path $\gamma_{\sigma(s_0)}$ from $\sigma(s_0)$ to $\sigma(s_1)$, and the path $\sigma([s_1, 1])$. By using once more the flow box theorem in a tube around this interval of $\gamma_{\sigma(s_0)}$, we can approximate the path $\overline{\sigma}$ to a path $\overline{\sigma} \in \Omega(p_1, p_2)$ with less tangencies with the flow of $\Delta^F_i$ than the path $\sigma$, a contradiction with the choice of $\sigma$.

For our next proof we recall the following consequence of 2.2 Proposition 2.7:
Proposition 2.3. Let $M \subset \mathbb{R}^n$ be a $C^\infty$ submanifold of dimension $m+1$ and $g: M \to \mathbb{R}^m$ be a $C^\infty$ submersion. If the fibers of $g$ are diffeomorphic to $\mathbb{R}$ and closed in $\mathbb{R}^n$, then $M$ is diffeomorphic to $\mathbb{R}^{m+1}$.

Proof of Statement (ii) of Theorem 1.3. By Statement (i) of the theorem, together with (a) of Lemma 2.1, it follows that the non-empty fibers of the restriction $f_j|_L : L \to \mathbb{R}$ are diffeomorphic to $\mathbb{R}$. Also, they are clearly closed in $\mathbb{R}^n$. Since $f_j|_L (L)$ is an interval (and so diffeomorphic to $\mathbb{R}$), it follows from Proposition 2.3 that $L$ is diffeomorphic to $\mathbb{R}^2$.

The following result of [14] will be used in the proof of Theorem 1.3.

Theorem 2.4 (Theorem 1.1 of [14]). Let $F = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^2$ semialgebraic mapping with nowhere vanishing $J(F)$ and such that $\text{codim} S_F \geq 2$. If for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$, any nonempty connected component of $\bigcap_{l \in \{1, \ldots, n\} \setminus \{i, j\}} f_l^{-1}(c_l)$ is diffeomorphic to $\mathbb{R}^2$, then $F$ is bijective.

Proof of Theorem 1.3. Assume that $F$ is a global diffeomorphism and let any $i \in \{1, \ldots, n\}$. For a given $\psi \in C^\infty(\mathbb{R}^n)$, let $\ell(x) = \int_0^x \psi \circ F^{-1}(z) J(F^{-1})(z) dz$, with $z = (x_1, \ldots, x_{i-1}, 8, x_{i+1}, \ldots, x_n)$. Since for any $g \in C^\infty(\mathbb{R}^n)$ it holds
\[
\partial_i (g \circ F^{-1})(x) = J((f_1, \ldots, f_{i-1}, g, f_{i+1}, \ldots, f_n) \circ F^{-1})(x) = \Delta_i^F(g)(F^{-1}(x)) J(F^{-1})(x),
\]
it follows that defining $g = \ell \circ F \in C^\infty(\mathbb{R}^n)$, we have $\Delta_i^F(g) = \psi$. This proves the surjectivity of $\Delta_i^F$.

On the other hand, we now assume that $\Delta_i^F(C^\infty(\mathbb{R}^n)) = C^\infty(\mathbb{R}^n)$ for $n-1$ indices $i \in \{1, \ldots, n\}$ and let distinct $j, k \in \{1, \ldots, n\}$. It follows by (ii) of Theorem 1.5 that any nonempty connected component of $\bigcap_{l \in \{1, \ldots, n\} \setminus \{i, j, k\}} f_l^{-1}(c_l)$ is diffeomorphic to $\mathbb{R}^2$. Now the result follows by Theorem 2.4.

3. A SEMIALGEBRAIC COUNTEREXAMPLE

Example 3.1. The example we are going to construct here is strongly based on the construction presented in [3]. The idea is to consider $C^\infty$ semialgebraic functions in place of the $C^\infty$ ones appearing in that paper. The most difficult part is that we have to modify also the functions $g_h$ of [3]: this is a family of $C^\infty$ functions satisfying a list of properties. What happens is that properties (c) and (d) in this list implies that the function $g_h^{(1)}$ is flat at the point 1 without being identically zero. This is impossible if $g_h$ is $C^\infty$ and semialgebraic, see for instance [3] Proposition 2.9.5. Since the proofs in [3] use the specific $g_h$, when we modify them, we have to provide new proofs.

Let $L > 0$ and fix $A : \mathbb{R} \to (-L, L)$ and $E : \mathbb{R} \to (0, \infty)$ two $C^\infty$ semialgebraic diffeomorphisms. For instance, we can take $A(x) = Lx/\sqrt{1+x^2}$ and $E(x) = x + \sqrt{1+x^2}$. Further, for any $h > L$, we define the polynomial
\[
g_h(z) = \frac{h}{50}z(2z+3)(2z-3)(3z^2+7).
\]
Then for $h_2 > h_1 > L$, we define the $C^\infty$ semialgebraic mapping $F : \mathbb{R}^3 \to \mathbb{R}^3$ by
\[
F(x, y, z) = ((A(x) + g_{h_1}(z))E(y), (A(x) + g_{h_2}(z))E(y), (1 - z^2)E(y)).
\]
We claim that (i) $J(F)$ is nowhere zero, (ii) $F$ is not injective and (iii) $\Delta_1^F$ and $\Delta_2^F$ are surjective. Indeed, straightforward calculations give

$$J(F)(x, y, z) = \frac{h_1 - h_2}{50} E(y)^2 E'(y) A'(x) \left(36z^6 - 59z^4 + 60z^2 + 63\right),$$

which is nowhere zero as the polynomial $m(z) = 36z^6 - 59z^4 + 60z^2 + 63$ is positive (to see this, take $z^2 = t$ and observe that the resulting degree 3 polynomial has derivative strictly positive, and so it has only one real zero, that must be negative as the leading and the constant coefficients of $m(z)$ are positive). Moreover, $F(x, y, 3/2) = F(x, y, -3/2)$ for any $x, y \in \mathbb{R}$. So it remains to prove (iii). From Corollary 1.7, by defining

$$f_h(x, y, z) = (A(x) + g_h(z)) E(y),$$

with $h > L$, it is enough to prove that for any connected component $L_i$ of $f_i^{-1}(c_i)$, $i = h, 3$, it holds that $L_h \cap f_3^{-1}(c_3)$ and $L_3 \cap f_h^{-1}(c_h)$ are connected sets.

This is the most involving part. We begin by calculating the connected components of $f_h^{-1}(c)$ and of $f_3^{-1}(c)$ for any $c \in \mathbb{R}$: see the complete frame in Table 1 where, similarly as in [4], when we write, for instance, $A(x) = -g_h(z), z < -1$, we mean the largest interval contained in $z < -1$ where this expression makes sense.

| $c < 0$ | $f_h^{-1}(c)$ | $f_3^{-1}(c)$ |
|---------|---------------|---------------|
| 2 components | $A(x) = cE^{-1}(y) - g_h(z), z < 1$ | 2 components: $E^{-1}(y) = (1 - z^2)/c, z < -1$ |
| 3 components: | $A(x) = -g_h(z), z < -1$ | 2 components: $E^{-1}(y) = (1 - z^2)/c, z > 1$ |
| | $A(x) = -g_h(z), -1 < z < 1$ | $z = -1$ |
| | $A(x) = -g_h(z), z > 1$ | $z = 1$ |
| $c = 0$ | 2 components: | 1 component: |
| | $A(x) = cE^{-1}(y) - g_h(z), z < -1$ | $E^{-1}(y) = (1 - z^2)/c, -1 < z < 1$ |
| | $A(x) = cE^{-1}(y) - g_h(z), z > -1$ |
| $c > 0$ |

Table 1. Connected components of fibers

Clearly the intersection of each of the two connected components of $f_3^{-1}(0)$ with the fibers of $f_h$ are connected or empty (use the fact that the function $y \rightarrow cE^{-1}(y) + cte$ is injective). Analogously, since $z = -1$ and $z = 1$ do not occur both in the same connected component of $f_h^{-1}(c)$, it follows that $f_3^{-1}(0)$ intersected with any of them is also connected.

Each one of the other required intersections is a solution of the system of equations

$$E^{-1}(y) = (1 - z^2)/c_3, \quad A(x) = c_hE^{-1}(y) - g_h(z),$$

for $z$ in only one of the intervals

$$I_1 = (-\infty, -1), \quad I_2 = (-1, 1), \quad I_3 = (1, \infty).$$

They are all connected by
Lemma 3.2. Let $\alpha \in \mathbb{R}$ and let $k_\alpha : \mathbb{R} \to \mathbb{R}$ be defined by $k_\alpha(z) = \alpha(1 - z^2) - g_\alpha(z)$. Then the sets $k_\alpha^{-1}(-L, L) \cap I_i$, $i = 1, 2, 3$ are connected sets. In particular, for each $c_3 \neq 0$, the solution of the system $E^{-1}(y) = (1 - z^2)/c_3$, $A(x) = k_\alpha(z)$, $z \in I_i$, form a connected set of $\mathbb{R}^3$ for $i = 1, 2, 3$.

Proof. The polynomial function

$$k_\alpha(z) = \alpha(1 - z^2) - g_\alpha(z) = \frac{6h}{25}z^5 + \frac{h}{50}z^3 - \frac{63h}{50}z + \alpha(1 - z^2)$$

is such that $k''''_\alpha(z) = \frac{72h}{25}z^2 + \frac{2h}{50} > 0$, $\forall z$, implying that $k''''_\alpha(z)$ is strictly increasing. Since $k''''_\alpha$ is a polynomial of degree 3, it has exactly one real root, say $q_0$. Therefore, $k''_\alpha(z) = \frac{6h}{25}z^4 + \frac{2h}{50}z^2 - 2\alpha z - \frac{63h}{50}$ is strictly decreasing in $(-\infty, q_0)$ and strictly increasing in $(q_0, \infty)$. So, since this is a polynomial of degree 4 with positive leader coefficient and $k'(0) = -\frac{63h}{50} < 0$, it follows that $k''_\alpha(z)$ has exactly two simple zeros, being one negative, say $p^-$, and the other one positive, say $p^+$. As a consequence, $k_\alpha(z)$ has a local maximum at $p^-$ and a local minimum at $p^+$, being strictly increasing in $(-\infty, p^-) \cup (p^+, \infty)$ and strictly decreasing in $(p^-, p^+)$. Therefore, since $h > L$ and $k_\alpha(\pm 1) = \mp h$, it follows that $k_\alpha^{-1}(-L, L) \cap I_i$, $i = 1, 2, 3$, are connected sets.

4. Fibrations and geometric properties of the operators $\Delta_i^F$  

Let $M$ and $N$ be topological spaces and $f : M \to N$ be a continuous mapping. We recall that $f$ is said to be a local fibration at $t_0 \in N$ if there exist a neighborhood $V$ of $t_0$ and a homeomorphism $h : f^{-1}(t_0) \times V \to f^{-1}(V)$ such that $f \circ h = \pi_2$, where $\pi_2 : f^{-1}(t_0) \times V \to V$ is the canonical projection. The set of points of $N$ where $f$ is not a local fibration is called the bifurcation set of $f$ and it is denoted by $B(f)$. We say that $f$ is a local fibration if $B(f) = \emptyset$. We further say that $f$ is a local fibration on its range if $B(f) \cap f(M) = \emptyset$. Finally, when it is possible to take $V = N$ for some $t_0 \in N$ above, we say that $f$ is a trivial fibration.

Next result provides a relation between the local fibration concept and the surjectivity of the operators $\Delta_i^F$ associated to a given $C^\infty$ local diffeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$.

Proposition 4.1. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^\infty$ mapping with nowhere vanishing $J(F)$ and $i \in \{1, \ldots, n\}$. If $F_i$ is a local fibration on its range then $\Delta_i^F (C^\infty(\mathbb{R}^n)) = C^\infty(\mathbb{R}^n)$.

Proof. We denote by $\gamma_x$ the integral curve of $\Delta_i^F$ passing through $x$. We suppose on the contrary that $\Delta_i^F (C^\infty(\mathbb{R}^n)) \neq C^\infty(\mathbb{R}^n)$. Since from \ref{a} and \ref{c} of Lemma 2.1 the integral curves of $\Delta_i^F$ are unbounded, it follows from Lemma 2.2 that there exist $p_1, p_2 \in \mathbb{R}^n$ and sequences $\{x_k\} \subset \mathbb{R}^n$ and $\{s_k\}, \{t_k\} \subset \mathbb{R}$, with $0 < s_k < t_k$, such that

$$x_k \to p_1, \; \gamma_{x_k}(t_k) \to p_2 \quad \text{and} \quad |\gamma_{x_k}(s_k)| \to \infty.$$  

By using the flow box theorem together with \ref{2} (consider also \ref{b} of Lemma 2.1 it follows that $p_1$ and $p_2$ are in distinct integral curves of $\Delta_i^F$, say $\gamma_{p_1}$ and $\gamma_{p_2}$, respectively. Setting $c = F_i(p_1)$ it follows by continuity and by Lemma 2.1 that $\gamma_{p_1}$ and $\gamma_{p_2}$ are distinct connected components of $F_i^{-1}(c)$.

Since $F_i$ is a local fibration at $c$, there exist an open neighborhood $V$ of $c$ and a homeomorphism $h : F_i^{-1}(c) \times V \to F_i^{-1}(V)$ such that $F_i \circ h = \pi_2$. The open sets
Lemma 4.1 is not true. Let, for instance, the polynomial mapping \( F(x, y, z) = (P(x, y), z) \), where \( P : \mathbb{R}^2 \to \mathbb{R} \) is the specific above-mentioned Pinchuk mapping considered in [8]. The operator \( \Delta_F^x = J(F)\partial_3 \) is clearly surjective, but \( F_3 \) cannot be a fibration at its range because there are points in \( P(\mathbb{R}^2) \) with different number of pre-images, according to [8].

Now let \( F : \mathbb{K}^n \to \mathbb{K}^n \) be a polynomial mapping for \( \mathbb{K} = \mathbb{C} \) (resp. a rational mapping for \( \mathbb{K} = \mathbb{R} \)) with nowhere vanishing \( J(F) \). It is proved in [8] for \( \mathbb{K} = \mathbb{C} \) (resp. in [8] for \( \mathbb{K} = \mathbb{R} \)) that if \( F \) is proper when restricted to its image, then \( F \) is an invertible mapping. I.e., if \( S_F \cap F(\mathbb{K}^n) = \emptyset \) then \( F \) is invertible. Bearing in mind that in this case \( B(F) = S_F \) and motivated by these results and by the subject of our paper, we state the following:

**Theorem 4.2.** Let \( F : \mathbb{R}^n \to \mathbb{R}^n \) be a semialgebraic local homeomorphism. If either \( B(F) \cap F(\mathbb{R}^n) = \emptyset \) or \( B(F_i) \cap F_i(\mathbb{R}^n) = \emptyset \) for some \( i \in \{1, \ldots, n\} \), then \( F \) is an injective mapping.

Before proving it, we recall a classical equation involving Euler characteristics, see for example [11, Corollary 2.5.5]:

**Lemma 4.3.** Let \( M \) and \( N \) be topological spaces. Let \( f : M \to N \) be a local fibration such that \( N \) and \( R = f^{-1}(t) \), for some \( t \in N \), are homotopy equivalent to finite CW-complexes. Then the three Euler characteristics \( \chi(M) \), \( \chi(N) \) and \( \chi(R) \) are defined and related as follows

\[
\chi(M) = \chi(N)\chi(R).
\]

A semialgebraic set is homeomorphic to a simplicial complex (which is a CW-complex), see for instance [2]. So, since the image of a semialgebraic set by a semialgebraic mapping is again a semialgebraic set, we can apply the above formula and deliver the

**Proof of Theorem 4.2.** For any \( y \in F_i(\mathbb{R}^n) \) (resp. \( y \in F(\mathbb{R}^n) \)), the \( d \) connected components of \( F_i^{-1}(y) \) (resp. \( F^{-1}(y) \)) are homeomorphic to \( \mathbb{R} \) by [8] of Lemma 2.1 (resp. are points). By Proposition 4.3

\[
1 = \chi(F_i(\mathbb{R}^n))d \quad \text{(resp.} \quad 1 = \chi(F(\mathbb{R}^n))d). \]

Since \( \chi(F_i(\mathbb{R}^n)) \in \mathbb{Z} \) (resp. \( \chi(F(\mathbb{R}^n)) \in \mathbb{Z} \)), it follows that \( d = 1 \). This provides the injectivity of \( F \) since the \( i \)-th component of \( F \) is monotone along each nonempty fiber of \( F \), by [10] of Lemma 2.1 (resp. trivially).

The next result follows direct by Proposition 4.2 since an injective polynomial \( f : \mathbb{R}^n \to \mathbb{R}^n \) is bijective, see [1].

**Corollary 4.4.** Let \( F = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n \) be a polynomial mapping with nowhere vanishing \( J(F) \). If either \( B(F) \cap F(\mathbb{R}^n) = \emptyset \) or \( B(F_i) \cap F_i(\mathbb{R}^n) = \emptyset \) for some \( i \in \{1, \ldots, n\} \), then \( F \) is invertible.

In the light of the preceding two results, we finish the paper with an open problem. If we do not assume that \( F \) is semialgebraic, then Proposition 4.2 may be not true, as the non-injective local diffeomorphism \( F(x, y, z) = (e^x \cos y, e^x \sin y, z) \) illustrates: here \( F_3 \) is a local fibration on its range. It is then natural to ask:
Is it true that a given $C^k$ (resp. $C^0$) mapping $F = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$, with nowhere zero $J(F)$ (resp. locally homeomorphism), such that $F_i$ is a local fibration on its range for $n-1$ indices $i \in \{1, \ldots, n\}$ is necessarily injective?

Since $F_i$ being a locally fibration on its range implies the surjectivity of $\Delta F_i$ (in the $C^\infty$ case), this would also provide an alternative to the false claim recalled in the introduction section “If $\Delta F_i(C^\infty_n) = C^\infty_n$ for $n-1$ indices $i \in \{1, \ldots, n\}$, then $F : \mathbb{R}^n \to \mathbb{R}^n$, a $C^\infty$ mapping with nowhere vanishing $J(F)$, is injective.”

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* Departamento de Matemática, Universidade Federal de São Carlos, 13565-905 São Carlos, São Paulo, Brazil
  
  Email address: franciscobraun@dm.ufscar.br

†‡ Faculdade de Matemática, Universidade Federal de Uberlândia, 38408-100 Uberlândia, Minas Gerais, Brazil
  
  Email address: lrgdias@ufu.br
  Email address: jvenatos@ufu.br