GEODESIC ORBIT METRICS IN A CLASS OF HOMOGENEOUS BUNDLES
OVER REAL AND COMPLEX STIEFEL MANIFOLDS

ANDREAS ARVANITOYEORGOS *, NIKOLAOS PANAGIOTIS SOURIS AND MARINA STATHA

Abstract. Geodesic orbit spaces (or g.o. spaces) are defined as those homogeneous Riemannian spaces \((M = G/H, g)\) whose geodesics are orbits of one-parameter subgroups of \(G\). The corresponding metric \(g\) is called a geodesic orbit metric. We study the geodesic orbit spaces of the form \((G/H, g)\), such that \(G\) is one of the compact classical Lie groups \(\text{SO}(n)\), \(U(n)\), and \(H\) is a diagonally embedded product \(H_1 \times \cdots \times H_s\), where \(H_j\) is of the same type as \(G\). This class includes spheres, Stiefel manifolds, Grassmann manifolds and real flag manifolds. The present work is a contribution to the study of g.o. spaces \((G/H, g)\) with \(H\) semisimple.

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1. Introduction

Geodesic orbit spaces \((M = G/H, g)\) are defined by the simple property that any geodesic \(\gamma\) has the form

\[
\gamma(t) = \exp(tX) \cdot o,
\]

where \(\exp\) is the exponential map on \(G\), \(o = \gamma(0)\) is a point in \(M\) and \(\cdot\) denotes the action of \(G\) on \(M\). These spaces were initially considered in [20], and up to this day they have been extensively studied within various geometric contexts, including the Riemannian ([18]), pseudo-Riemannian ([11]), Finsler ([31]) and affine ([16]) context. The classification of g.o. spaces remains an open problem, whereas several partial classifications have been obtained ([2], [3], [13], [14], [15], [17], [25], [28] to name a few).

There are diverse examples of g.o. spaces, including the classes of symmetric spaces, weakly symmetric spaces ([10], [30]), isotropy irreducible spaces ([29]), \(\delta\)-homogeneous spaces ([7]) and Clifford-Wolf homogeneous spaces ([8]). The most important subclass of g.o. spaces are the naturally reductive spaces, whose complete description is also open (see the recent low-dimensional classifications [1], [27]). Another related topic of recent interest is the study of Einstein metrics that are not g.o. metrics ([12], [23]). For a review about g.o. spaces we refer to [5] and in the introduction of the article [22]. We also point out the recently published book [9].

Determining the g.o. metrics among the \(G\)-invariant metrics on a space \(G/H\) presents some challenges. The main challenge lies in the fact that the space of \(G\)-invariant metrics may have complicated structure, depending on whether the isotropy representation of \(H\) on the tangent space \(T_o(G/H)\) contains pairwise equivalent submodules. To remedy this obstruction, various simplification results for g.o. metrics have been established (e.g. [22], [24]). A general observation is that the existence and the form of the g.o. metrics on \(G/H\) depends to a large extent on the structure of the tangent space \(T_o(G/H)\) induced from the isotropy representation and on the Lie algebraic relations between the corresponding submodules (e.g. [15]).

When \(G\) is compact semisimple, the classification of the g.o. spaces \((G/H, g)\) with \(H\) abelian and \(H\) simple has been obtained in the works [25] and [15] respectively. On the other hand, the
classification of compact g.o. spaces \((G/H, g)\) with \(H\) semisimple remains open, while no general results are known for this case. As a first step towards this direction, in this paper we study the g.o. metrics on a general family of spaces \(G/H\) with \(H\) semisimple, such that the isotropy representation of all of its members has a similar description.

In particular, we consider spaces \(G/H\) where \(G\) is a compact classical Lie group and \(H\) is a diagonally embedded product of Lie groups of the same type as \(G\). More specifically, we study the spaces \(\text{SO}(n)/\text{SO}(n_1) \times \cdots \times \text{SO}(n_s)\) and \(U(n)/U(n_1) \times \cdots \times U(n_s)\) with \(0 < n_1 + \cdots + n_s \leq n\). This class properly includes the spheres \(\text{SO}(n)/\text{SO}(n-1)\) and \(U(n)/U(n-1)\), the Stiefel manifolds \(\text{SO}(n)/\text{SO}(n-k)\) and \(U(n)/U(n-k)\), the Grassmann manifolds \(\text{SO}(n)/\text{SO}(k) \times \text{SO}(n-k)\), \(U(n)/U(k) \times U(n-k)\) as well as the real flag manifolds \(\text{SO}(n)/\text{SO}(n_1) \times \cdots \times \text{SO}(n_s)\) and \(U(n)/U(n_1) \times \cdots \times U(n_s)\) with \(n_1 + \cdots + n_s = n\). If \(n_1 + \cdots + n_s < n\), each of these spaces can be viewed as a total space over a Stiefel manifold, with the fiber being a real flag manifold, e.g.

\[
\text{SO}(m)/\text{SO}(n_1) \times \cdots \times \text{SO}(n_s) \rightarrow \text{SO}(n)/\text{SO}(n_1) \times \cdots \times \text{SO}(n_s) \rightarrow \text{SO}(n)/\text{SO}(m),
\]

with \(m = n_1 + \cdots + n_s\). The first main result is the following.

**Theorem 1.1.** Let \(G/H\) be the space \(\text{SO}(n)/\text{SO}(n_1) \times \cdots \times \text{SO}(n_s)\), where \(0 < n_1 + \cdots + n_s \leq n\), and \(n_j > 1, j = 1, \ldots, s\). A \(G\)-invariant Riemannian metric on \(G/H\) is geodesic orbit if and only if it is a normal metric, i.e. it is induced from an \(\text{Ad}\)-invariant inner product on the Lie algebra \(\mathfrak{so}(n)\) of \(\text{SO}(n)\).

**Remark 1.2.** We remark that if \(n \neq 4\) then \(\mathfrak{so}(n)\) is simple, and thus any \(\text{Ad}\)-invariant inner product is homothetic to the negative of the Killing form \(B(X, Y) = (n-2)\text{Trace}(XY)\). If \(n = 4\) then \(\mathfrak{so}(n) \equiv \mathfrak{so}(3) \oplus \mathfrak{so}(3)\), and thus any \(\text{Ad}\)-invariant inner product is homothetic to the negative of the one-parameter family \(B_1 + \lambda B_2, \lambda > 0\), where \(B_1\) denotes the Killing form of the first simple factor \(\mathfrak{so}(3)\) and \(B_2\) denotes the Killing form of the second simple factor \(\mathfrak{so}(3)\).

As a result of Theorem 1.1 and Proposition 3.9, we obtain the following.

**Corollary 1.3.** Let \(G/H\) be one of the spaces \(\text{O}(n)/\text{O}(n_1) \times \cdots \times \text{O}(n_s)\) or \(\text{SO}(n)/\text{SO}(n_1) \times \cdots \times \text{O}(n_s)\), where \(0 < n_1 + \cdots + n_s \leq n\), \(n_j > 1\). A \(G\)-invariant Riemannian metric on \(G/H\) is geodesic orbit if and only if it is normal.

The second main result is the following.

**Theorem 1.4.** Let \(G/H\) be the space \(U(n)/U(n_1) \times \cdots \times U(n_s)\), where \(n_1 + \cdots + n_s \leq n\), and let \(N_H(G)\) be the normalizer of \(H\) in \(G\). If \(n_1 + \cdots + n_s = n\), then a \(G\)-invariant Riemannian metric on \(G/H\) is geodesic orbit if and only if it is the normal metric induced from the \(\text{Ad}\)-invariant inner product \(B(X, Y) = -\text{Trace}(XY)\) in \(u(n)\). If \(n_1 + \cdots + n_s < n\), then a \(G\)-invariant Riemannian metric \(g\) on \(G/H\) is geodesic orbit if and only if \(g = g_{\mu}\), \(\mu > 0\), where \(g_{\mu}\) denotes a one-parameter family of deformations of the normal metric induced from the inner product \(B\), along the center of the group \(N_H(G)/H\).

The case \(G/H = \text{Sp}(n)/\text{Sp}(n_1) \times \cdots \times \text{Sp}(n_s)\) has been treated in [6].

We note that the metrics in Theorem 1.4 generalize the g.o. metrics on the Berger spheres \(U(n)/U(n-1)\) ([21]) and the g.o. metrics on the complex Stiefel manifolds \(U(n)/U(n-k)\) ([24]). We also note that the g.o. metrics on the related class of real flag manifolds were recently studied in [19]. Among other results, it is shown in [19] that every g.o. metric on the real flag manifold \(\text{SO}(n)/\text{SO}(n_1) \times \cdots \times \text{O}(n_s)\), \(n_1 + \cdots + n_s = n\), is normal, which is a special case of Corollary 1.3.
The paper is structured as follows: In Sections 2 and 3, some preliminary facts for homogeneous spaces and g.o. spaces are given respectively. Theorems 1.1 and 1.4 are proved in Sections 4.2 and 5.2 respectively. To this end, we firstly compute the isotropy representation in terms of a suitable basis for each of the spaces (Sections 4.1 and 5.1 respectively) and then we apply simplification results from [22] and [24] in order to complete the proofs (Sections 4.2 and 5.2 respectively).

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2. Invariant metrics on homogeneous spaces

Let \( G/H \) be a homogeneous space with origin \( o = eH \) and assume that \( G \) is compact. Let \( \mathfrak{g}, \mathfrak{h} \) be the Lie algebras of \( G, H \) respectively. Moreover, let \( \text{Ad} : G \to \text{Aut}(\mathfrak{g}) \) and \( \text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g}) \) be the adjoint representations of \( G \) and \( \mathfrak{g} \) respectively, where \( \text{ad}(X)Y = [X, Y] \). Since \( G \) is compact, there exists an \( \text{Ad} \)-invariant (and hence \( \text{ad} \) skew-symmetric) inner product \( B \) on \( \mathfrak{g} \), which we henceforth fix. In turn, we have a \( B \)-orthogonal reductive decomposition

\[
\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},
\]

where the subspace \( \mathfrak{m} \) is \( \text{Ad}(H) \)-invariant (and \( \text{ad}(\mathfrak{h}) \)-invariant) and is naturally identified with the tangent space of \( G/H \) at the origin.

A Riemannian metric \( g \) on \( G/H \) is called \( G \)-invariant if for any \( x \in G \), the left translations \( \tau_x : G/H \to G/H, pH \mapsto (xp)H \), are isometries of \( (G/H, g) \). The \( G \)-invariant metrics are in one to one correspondence with \( \text{Ad}(H) \)-invariant inner products \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{m} \). Moreover, any such product corresponds to a unique endomorphism \( A : \mathfrak{m} \to \mathfrak{m} \), called the corresponding metric endomorphism, that satisfies

\[
\langle X, Y \rangle = B(AX, Y) \quad \text{for all } \ X, Y \in \mathfrak{m}.
\]

It follows from Equation (2) that the metric endomorphism \( A \) is symmetric with respect \( B \), positive definite and \( \text{Ad}(H) \)-equivariant, that is \( \text{Ad}(h) \circ A(X) = (A \circ \text{Ad}(h))(X) \) for all \( h \in H \) and \( X \in \mathfrak{m} \). Conversely, any endomorphism on \( \mathfrak{m} \) with the above properties determines a unique \( G \)-invariant metric on \( G/H \).

Since \( A \) is diagonalizable, there exists a decomposition \( \mathfrak{m} = \bigoplus_{j=1}^{s} \mathfrak{m}_{\lambda_j} \) into eigenspaces \( \mathfrak{m}_{\lambda_j} \) of \( A \) corresponding to distinct eigenvalues \( \lambda_j \). Each eigenspace \( \mathfrak{m}_{\lambda_j} \) is \( \text{Ad}(H) \)-invariant. When an \( \text{Ad} \)-invariant inner product \( B \) and a \( B \)-orthogonal reductive decomposition (1) have been fixed, we will make no distinction between a \( G \)-invariant metric \( g \) and its corresponding metric endomorphism \( A \).

The form of the \( G \)-invariant metrics on \( G/H \) depends on the isotropy representation \( \text{Ad}^{G/H} : H \to \text{GL}(\mathfrak{m}) \), defined by \( \text{Ad}^{G/H}(h)X := (d\tau_h)_o(X), h \in H, X \in \mathfrak{m} \). We consider a \( B \)-orthogonal decomposition

\[
\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_s,
\]

into \( \text{Ad}^{G/H} \)-invariant and irreducible submodules. We recall that two submodules \( \mathfrak{m}_i \) and \( \mathfrak{m}_j \) are equivalent if there exists an \( \text{Ad}^{G/H} \)-equivariant isomorphism \( \phi : \mathfrak{m}_i \to \mathfrak{m}_j \). The simplest case occurs when all the submodules \( \mathfrak{m}_i \) are pairwise inequivalent. Then any \( G \)-invariant metric \( A \) on \( G/H \) has a diagonal expression with respect to decomposition (3). In particular, \( A|_{\mathfrak{m}_j} = \lambda_j \text{Id}, j = 1, \ldots, s \).
The next proposition is useful for computing the isotropy representation of a reductive homogeneous space.

**Proposition 2.1.** ([4]) Let $G/H$ be a reductive homogeneous space and let $g = h \oplus m$ be a reductive decomposition of $g$. Let $h \in H$, $X \in h$ and $Y \in m$. Then the adjoint representation of $G$ decomposes as $\text{Ad}^G(h)(X + Y) = \text{Ad}^G(h)X + \text{Ad}^{G/H}(h)Y$ that is, the restriction $\text{Ad}^G|_H$ splits into the sum $\text{Ad}^H \oplus \text{Ad}^{G/H}$. We denote by $\chi$ the representation $\text{Ad}^{G/H}$.

The following lemma provides a simple condition for proving that two $\text{Ad}^{G/H}$-submodules are inequivalent.

**Lemma 2.2.** Let $G/H$ be a homogeneous space with reductive decomposition $g = h \oplus m$ and let $m_i, m_j \subseteq m$ be submodules of the isotropy representation $\text{Ad}^{G/H}$. Assume that for any pair of non-zero vectors $X \in m_i$, $Y \in m_j$, there exists a vector $a \in h$ such that $[a, X] = 0$ and $[a, Y] \neq 0$. Then the submodules $m_i, m_j$ are $\text{Ad}^{G/H}$-inequivalent.

**Proof.** If $m_i, m_j$ were equivalent, then there exists an $\text{Ad}^{G/H}$-equivariant isomorphism $\phi : m_i \rightarrow m_j$. Let $X$ be a non-zero vector in $m_i$ and set $Y := \phi(X) \in m_j$. The $\text{Ad}^{G/H}$-equivariance of $\phi$ implies that $\phi$ is $\text{ad}_h$-equivariant, and hence, $\phi([a, X]) = [a, Y]$ for any $a \in h$. However, $\phi$ is an isomorphism, therefore, $[a, X]$ is non-zero if and only if $[a, Y]$ is non-zero, which contradicts the hypothesis of the lemma. Hence, the submodules $m_i, m_j$ are inequivalent. \qed

**Remark 2.3.** For any two $\text{Ad}^{G/H}$-submodules $m_1, m_2$, we denote by $[m_1, m_2]$ the space generated by the vectors $[X_1, X_2]$ where $X_1 \in m_1$ and $X_2 \in m_2$. Similarly, denote by $[h, m_1]$ the space generated by the vectors $[a, X_1]$ where $a \in h$ and $X_1 \in m_1$. If $m_1, m_2$ are $B$-orthogonal then $[m_1, m_2] \subseteq m$. Indeed, $[m_1, m_2]$ is $B$-orthogonal to $h$ because $B([m_1, m_2], h) \subseteq B(m_1, [m_2, h]) \subseteq B(m_1, m_2) = \{0\}$. Moreover, $[m_1, m_2]$ is also an $\text{Ad}^{G/H}$-submodule of $m$ and $[h, m_1]$ is an $\text{Ad}^{G/H}$-submodule of $m_1$.

### 3. Properties of geodesic orbit spaces

**Definition 3.1.** A $G$-invariant metric $g$ on $G/H$ is called a geodesic orbit metric (g.o. metric) if any geodesic of $(G/H, g)$ through $o$ is an orbit of a one parameter subgroup of $G$. Equivalently, $g$ is a geodesic orbit metric if for any geodesic $\gamma$ of $(G/H, g)$ through $o$ there exists a non-zero vector $X \in g$ such that $\gamma(t) = \exp(tX) \cdot o$, $t \in \mathbb{R}$. The space $(G/H, g)$ is called a geodesic orbit space (g.o. space).

Let $G/H$ be a homogeneous space with $G$ compact. We fix an $\text{Ad}$-invariant inner product $B$ on $g$ and consider the $B$-orthogonal reductive decomposition (1). Moreover, identify each $G$-invariant metric on $G/H$ with the corresponding metric endomorphism $A : m \rightarrow m$ satisfying Equation (2). We have the following condition.

**Proposition 3.2.** ([2], [24]) The metric $A$ on $G/H$ is geodesic orbit if and only if for any vector $X \in m$ there exists a vector $a \in h$ such that

$$[a + X, AX] = 0. \tag{4}$$

The following result, which we will call the normalizer lemma, can be used to simplify the necessary form of the g.o. metrics on $G/H$ by using the normalizer $N_G(H^0)$.

**Lemma 3.3.** ([22]) The inner product $\langle , , \rangle$ in (2), generating the metric of a geodesic orbit Riemannian space $(G/H, g)$, is not only $\text{Ad}(H)$-invariant but also $\text{Ad}(N_G(H^0))$-invariant, where $N_G(H^0)$ is the normalizer of the identity component $H^0$ of the group $H$ in $G$. 
As a result of the normalizer lemma, the metric endomorphism \( A \) of a g.o. metric on \( G/H \) is \( \text{Ad}(N_G(H^0)) \)-equivariant. We will now state a complementary result to the normalizer lemma for compact spaces, that characterizes the restriction of a g.o. metric to the compact Lie group \( N_G(H^0)/H^0 \).

**Lemma 3.4.** Let \( G \) be a compact Lie group and let \((G/H, g)\) be a geodesic orbit space with the \( B \)-orthogonal reductive decomposition \( g = h \oplus m \), where \( B \) is an \( \text{Ad} \)-invariant inner product on \( g \). Let \( A : m \to m \) be the corresponding metric endomorphism of \( g \), and let \( n \subseteq m \) be the Lie algebra of the compact Lie group \( N_G(H^0)/H^0 \). Then the restriction of \( A \) to \( n \) defines a bi-invariant metric on \( N_G(H^0)/H^0 \).

**Proof.** We denote by \( n_g(h) \subset g \) the Lie algebra of \( N_G(H^0) \). We have a \( B \)-orthogonal decomposition \( n_g(h) = h \oplus n \), where \( n \) coincides with the Lie algebra of \( N_G(H^0)/H^0 \). Moreover, we have a \( B \)-orthogonal decomposition \( m = n \oplus p \), where \( p \) coincides with the tangent space of \( G/N_G(H^0) \) at the origin. By the normalizer lemma, the restriction of \( A \) on \( p \) defines an invariant metric on \( G/N_G(H^0) \), and thus \( A p \subseteq p \). By taking into account the symmetry of \( A \) with respect to the product \( B \), we deduce that \( B(A n, p) = B(n, A p) \subseteq B(n, p) = \{0\} \). Hence, the image \( A n \) is \( B \)-orthogonal to \( p \) which, along with decomposition \( m = n \oplus p \), yields \( A n \subseteq n \). Therefore, the restriction \( A|_n : n \to n \) defines a left-invariant metric on \( N_G(H^0)/H^0 \). Since \( A \) is a g.o. metric on \( G/H \), Proposition 3.2 implies that for any \( X \in n \) there exists a vector \( a \in h \) such that \( 0 = [a + X, AX] = [a + X, A|_n X] \). Therefore, by the same proposition, \( A|_n \) defines a g.o. metric on \( N_G(H^0)/H^0 \). On the other hand, any left-invariant g.o. metric on a Lie group is necessarily bi-invariant ([3]), and hence \( A|_n \) is a bi-invariant metric on \( N_G(H^0)/H^0 \). \( \blacksquare \)

**Remark 3.5.** As an alternative to the above proof, it was observed by the referee that Lemma 3.4 follows easily from Lemma 3.3, if we take into account that any \( \text{Ad}(N_G(H^0)) \)-invariant Riemannian metric on the Lie group \( N_G(H^0)/H^0 \) is generated by a suitable bi-invariant Riemannian metric on \( N_G(H^0)/H^0 \).

**Remark 3.6.** Let \( p \subset m \) be the tangent space of \( G/N_G(H^0) \), let \( n \subset m \) be the Lie algebra of \( N_G(H^0)/H^0 \), and consider the decomposition \( m = n \oplus p \). By combining Lemmas 3.3 and 3.4, we conclude that the metric endomorphism \( A \) corresponding to a g.o. metric on \( G/H \) has the block-diagonal form

\[
A = \begin{pmatrix}
A|_n & 0 \\
0 & A|_p
\end{pmatrix}.
\]

The next lemma describes the invariant g.o. metrics on compact Lie groups, in terms of their metric endomorphism with respect to an \( \text{Ad} \)-invariant inner product \( B \). Recall that the Lie algebra \( g \) of a compact Lie group \( G \) has the (Lie algebra) direct sum decomposition \( g = g_1 \oplus \cdots \oplus g_k \oplus z \), where \( g_j \) are the simple ideals of \( g \) and \( z \) is its center.

**Lemma 3.7.** ([3], [24]) Let \( G \) be a compact Lie group with Lie algebra \( g = g_1 \oplus \cdots \oplus g_k \oplus z \). A left invariant metric \( A \) on \( G \) is a g.o. metric if and only if it is bi-invariant. In particular, \( A \) is a g.o. metric if and only if

\[
A = \begin{pmatrix}
\lambda_1 \text{Id}|_{g_1} & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \lambda_k \text{Id}|_{g_k} & 0 \\
0 & \cdots & 0 & A|_z
\end{pmatrix}, \quad \lambda_j > 0.
\]

We set some notation. For a subspace \( W \) of a vector space \( V \) we write \( V = W \oplus W^\perp \) with respect to some inner product \( B \) on \( V \). Then, for \( v \in V \) it is \( v = w + w' \), where \( w \in W \) and \( w' \in W^\perp \). We
say that \(v\) has non zero projection on \(W\) if \(w \neq 0\). Moreover, we say that a subset \(S\) of \(V\) has non zero projection on \(W\) if there exists a vector \(v \in S\) that has non zero projection on \(W\).

The following lemma is useful, since it will enable us to equate some of the eigenvalues of a g.o. metric.

**Lemma 3.8.** ([24]) Let \((G/H, g)\) be a g.o. space with \(G\) compact and with corresponding metric endomorphism \(A\) with respect to an \(\text{Ad}\)-invariant inner product \(B\). Let \(m\) be the \(B\)-orthogonal complement of \(h\) in \(g\).

1. Assume that \(m_1, m_2\) are \(\text{ad}(h)\)-invariant, pairwise \(B\)-orthogonal subspaces of \(m\) such that \([m_1, m_2]\) has non-zero projection on \((m_1 \oplus m_2)\). Let \(\lambda_1, \lambda_2\) be eigenvalues of \(A\) such that \(A\) is normal (resp. standard). Moreover, the converse is true if \(H\) is connected.

2. Assume that \(m_1, m_2, m_3\) are \(\text{ad}(h)\)-invariant, pairwise \(B\)-orthogonal subspaces of \(m\) such that \([m_1, m_2]\) has non-zero projection on \(m_3\). Let \(\lambda_1, \lambda_2, \lambda_3\) be eigenvalues of \(A\) such that \(A\) is normal (resp. standard). Moreover, the converse is true if \(H\) is connected.

Finally, the following result will be used in relating g.o. metrics on a space \(G/H\) to g.o. metrics on its universal covering space \(\tilde{G}/\tilde{H}\) (see also [26]). Recall that a \(G\)-invariant metric is called standard if it is induced by the negative of the Killing form on \(g\).

**Proposition 3.9.** Let \(G/H, \tilde{G}/\tilde{H}\) be homogeneous spaces with \(G\) compact and \(\tilde{H}\) connected, such that the Lie algebras of \(G\) and \(\tilde{G}\) coincide and the Lie algebras of \(H\) and \(\tilde{H}\) coincide. If every \((\tilde{G}\)-invariant) g.o. metric on \(\tilde{G}/\tilde{H}\) is normal (resp. standard), then every \((G\)-invariant) g.o. metric on \(G/H\) is also normal (resp. standard). Moreover, the converse is true if \(H\) is connected.

**Proof.** Let \(g\) denote the Lie algebra of the groups \(G\) and \(\tilde{G}\), and let \(h\) denote the Lie algebra of the groups \(H\) and \(\tilde{H}\). Let \(B\) be an \(\text{Ad}\)-invariant inner product on \(g\) and consider the \(B\)-orthogonal decomposition \(g = h \oplus m_B\). Then \(m_B\) can be identified with the tangent spaces \(T_o(G/H)\) and \(T_0(\tilde{G}/\tilde{H})\). Let \(g\) be a \(G\)-invariant g.o. metric on \(G/H\). We will prove that \(g\) is normal. The proof will be completed in three steps.

**Step 1.** The \(G\)-invariant metric \(g\) on \(G/H\) induces a \(\tilde{G}\)-invariant metric \(\tilde{g}\) on \(\tilde{G}/\tilde{H}\). Indeed, let \(A_B : m_B \rightarrow m_B\) be the corresponding metric endomorphism of \(g\). The endomorphism \(A_B\) is \(\text{Ad}_{\tilde{H}}\)-invariant and hence \(\text{ad}_\tilde{h}\)-equivariant. Given that \(\tilde{H}\) is connected, the \(\text{ad}_\tilde{h}\)-equivariance of \(A_B\) yields its \(\text{Ad}_{\tilde{G}}\)-equivariance. Therefore, \(A_B\) defines a \(\tilde{G}\)-invariant metric \(\tilde{g}\) on \(\tilde{G}/\tilde{H}\).

**Step 2.** The metric \(\tilde{g}\) is also a g.o. metric on \(\tilde{G}/\tilde{H}\). Indeed, since \(g\) is a g.o. metric then the corresponding metric endomorphism \(A_B\) satisfies Proposition 3.2. Since \(A_B\) is also the metric endomorphism of \(\tilde{g}\), Proposition 3.2 implies that \(\tilde{g}\) is a g.o. metric on \(\tilde{G}/\tilde{H}\).

**Step 3.** The metric \(\tilde{g}\) is normal. Indeed, since \(\tilde{g}\) is a g.o. metric on \(\tilde{G}/\tilde{H}\), by hypothesis it is also normal. Hence, there exists an \(\text{Ad}\)-invariant inner product \(B'\) on \(g\) and a \(B'\)-orthogonal decomposition \(g = h \oplus m_{B'}\) such that the corresponding metric endomorphism \(A_{B'} : m_{B'} \rightarrow m_{B'}\) of \(\tilde{g}\) satisfies \(A_{B'} = \lambda \text{Id}\), \(\lambda > 0\). Since \(A_{B'}\) coincides with the metric endomorphism of \(g\), we conclude that \(g\) is also normal. \(\square\)

**Corollary 3.10.** If every \((\tilde{G}\)-invariant) g.o. metric on the universal cover \(\tilde{G}/\tilde{H}\) of \(G/H\) is normal (resp. standard), then every \((G\)-invariant) g.o. metric on \(G/H\) is also normal (resp. standard).

4. The space \(M = G/H = \text{SO}(n)/\text{SO}(n_1) \times \cdots \times \text{SO}(n_s), n_1 + \cdots + n_s \leq n, n_j > 1\).

4.1. Isotropy representation of \(G/H = \text{SO}(n)/\text{SO}(n_1) \times \cdots \times \text{SO}(n_s)\). We set

\[
n_0 := n - (n_1 + \cdots + n_s).
\]
We view $H = \text{SO}(n_1) \times \cdots \times \text{SO}(n_s)$ embedded diagonally in $\text{SO}(n)$, so that $H \cong \begin{pmatrix} \text{Id}_{n_0} & 0 \\ 0 & H \end{pmatrix}$.

Hence
\[
\mathfrak{h} = \begin{pmatrix} 0_{n_0} & \mathfrak{so}(n_1) \\ \vdots & \ddots \\ 0 & \mathfrak{so}(n_s) \end{pmatrix},
\]

where $0_{n_0}$ is the $n_0 \times n_0$ zero matrix. We remark that the above embedding of $H$ is equivalent (via conjugation in $\text{SO}(n)$) to any block-diagonal embedding of the factors $\text{SO}(n_j)$.

Recall that if $\pi : G \to \text{Aut}(V)$, $\pi' : G \to \text{Aut}(W)$ are two representations of $G$, then for the second exterior power the following identity is valid: $\wedge^2(\pi \oplus \pi') = \wedge^2\pi \oplus \wedge^2\pi' \oplus (\pi \otimes \pi')$.

Denote by $\lambda_n : \text{SO}(n) \to \text{Aut}(\mathbb{R}^n)$ the standard representation of $\text{SO}(n)$. Then the adjoint representation $\text{Ad}^\mathcal{O}(n)$ of $\text{SO}(n)$ (and $\text{Ad}^\geq(n)$ of $O(n)$) is equivalent to $\wedge^2\lambda_n$.

Let $\sigma_{n_i} : \text{SO}(n_1) \times \cdots \times \text{SO}(n_s) \to \text{SO}(n_i)$ be the projection onto the $i$-factor and $p_i = \lambda_{n_i} \circ \sigma_{n_i}$ be the standard representation of $H$, i.e.
\[
\text{SO}(n_1) \times \cdots \times \text{SO}(n_s) \xrightarrow{\sigma_{n_i}} \text{SO}(n_i) \xrightarrow{\lambda_{n_i}} \text{Aut}(\mathbb{R}^{n_i}).
\]

Then we have:
\[
\text{Ad}^G \big|_H = \wedge^2\lambda_n \big|_H = \wedge^2(p_1 \oplus \cdots \oplus p_s \oplus 1_{n_0}) = \wedge^2p_1 \oplus \wedge^2p_2 \oplus \cdots \oplus \wedge^2p_s \oplus \wedge^21_{n_0}
\oplus \{(p_1 \otimes p_2) \oplus \cdots \oplus (p_1 \otimes p_s)\} \oplus \{(p_2 \otimes p_3) \oplus \cdots \oplus (p_2 \otimes p_s)\} \oplus \cdots \oplus (p_{s-1} \otimes p_s)
\oplus (p_1 \otimes 1_{n_0}) \oplus (p_2 \otimes 1_{n_0}) \oplus \cdots \oplus (p_s \otimes 1_{n_0}),
\]

where $\wedge^21_{n_0}$ is the sum of $\binom{n_0}{2}$ trivial representations.

The dimension of the representation $\wedge^2p_1 \oplus \wedge^2p_2 \oplus \cdots \oplus \wedge^2p_s$ is $\binom{n_1}{2} + \binom{n_2}{2} + \cdots + \binom{n_s}{2}$ and is equal to the dimension of the adjoint representation of $H = \text{SO}(n_1) \times \cdots \times \text{SO}(n_s)$, that is $\text{Ad}^H = \wedge^2p_1 \oplus \wedge^2p_2 \oplus \cdots \oplus \wedge^2p_s$. Therefore, by Proposition 2.1 the isotropy representation of $G/H$ is given by
\[
\chi = \wedge^21_{n_0} \oplus (p_1 \otimes p_2) \oplus \cdots \oplus (p_1 \otimes p_s) \oplus (p_2 \otimes p_3) \oplus \cdots \oplus (p_2 \otimes p_s) \oplus \cdots \oplus (p_{s-1} \otimes p_s)
\oplus (p_1 \otimes 1_{n_0}) \oplus (p_2 \otimes 1_{n_0}) \oplus \cdots \oplus (p_s \otimes 1_{n_0}).
\]

The dimension of each of $p_i \otimes p_j$ is $n_i \times n_j$ and each of $p_i \otimes 1_{n_0}$, $i = 1, 2, \ldots, s$, contains $n_0$ equivalent representations of dimension $n_j$.

**Remark 4.1.** Each of the representations $p_i \otimes p_j$ corresponds to the isotropy representation of the Grassmannian $\text{SO}(n_1 + n_j)/\text{SO}(n_i) \times \text{SO}(n_j)$ and is irreducible unless $n_i = n_j = 2$. If $n_i = n_j = 2$, then the summand $p_i \otimes p_j$ in (6) reduces into two 2-dimensional irreducible non-equivalent representations. This summand corresponds to the isotropy representation of the Grassmannian $\text{SO}(4)/\text{SO}(2) \times \text{SO}(2)$. However, the reducibility does not carry to the Grassmannian $O(4)/O(2) \times O(2)$, which is isotropy irreducible. Therefore, even if the isotropy subgroup $O(n_1) \times \cdots \times O(n_s)$ contains at least two $O(2)$-factors, the isotropy representation of $O(n)/O(n_1) \times \cdots \times O(n_s)$ is still (6).

Expression (6) induces a decomposition of the tangent space $\mathfrak{m}$ of $G/H$ as
\[
\mathfrak{m} = n_1 \oplus \cdots \oplus n_0 \bigg|_{n_j} \oplus \mathfrak{m}_{ij} \oplus \mathfrak{m}_{0j},
\]

where $\mathfrak{m}_{ij} = \bigoplus_{1 \leq i < j \leq s} \mathfrak{m}_{ij}$.
where \( \dim(n_i) = 1 \), \( m_{ij} = m_1^j \oplus m_2^j \oplus \cdots \oplus m_{n_0}^j \) with \( m_\alpha^j \cong m_\beta^j \), \( \alpha \neq \beta \) and \( \dim(m_\ell^j) = n_j \), \( \ell = 1, 2, \ldots, n_0 \). Note that \( n_1 \oplus \cdots \oplus n_{n_0} \cong \mathfrak{so}(n_0) \). Moreover, if \( n_0 = 0 \) or \( n_0 = 1 \) then there are no trivial submodules and hence \( \mathfrak{so}(n_0) = \{0\} \).

We now give explicit matrix representations of the modules \( m_{ij} \) and \( m_{0j} \). We consider the Ad(\( SO(n) \))-invariant inner product \( B : \mathfrak{so}(n) \times \mathfrak{so}(n) \to \mathbb{R} \) given by

\[
B(X, Y) = -\text{Trace}(XY), \quad X, Y \in \mathfrak{so}(n),
\]

and obtain a \( B \)-orthogonal decomposition \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \), where \( \mathfrak{h} = \mathfrak{so}(n_1) \oplus \cdots \oplus \mathfrak{so}(n_s) \) and \( \mathfrak{m} \cong T_o(G/H) \). We consider a basis of \( \mathfrak{g} = \mathfrak{so}(n) \) as follows:

Let \( M_n \mathbb{R} \) be the set of real \( n \times n \) matrices and let \( E_{ab} \in M_n \mathbb{R} \) be the matrix with 1 in the \((a, b)\)-entry and zero elsewhere. For \( 1 \leq a < b \leq n \) we set

\[
e_{ab} := E_{ab} - E_{ba}.
\]

Note that \( e_{ab} = -e_{ba} \). The set

\[
\mathcal{B} := \{e_{ab} : 1 \leq a < b \leq n\},
\]

constitutes a basis of \( \mathfrak{so}(n) \), which is orthogonal with respect to \( B \). The proof of the following lemma is immediate.

**Lemma 4.2.** The only non zero bracket relations among the vectors \((9)\) are \([e_{ab}, e_{bc}] = e_{ac} \), for \( a, b, c \) distinct.

A choice for the modules in the decomposition \((7)\) is the following:

\[
m_{ij} = \text{span}\{e_{ab} : n_0 + n_1 + \cdots + n_{i-1} + 1 \leq a \leq n_0 + n_1 + \cdots + n_i,
\]

\[
\quad n_0 + n_1 + \cdots + n_{j-1} + 1 \leq b \leq n_0 + n_1 + \cdots + n_j \}, \quad 1 \leq i < j \leq s.
\]

\[
m_{0j} = \text{span}\{e_{ab} : 1 \leq a \leq n_0, \ n_0 + n_1 + \cdots + n_{j-1} + 1 \leq b \leq n_0 + n_1 + \cdots + n_j \}, \quad 1 \leq j \leq s.
\]

\[
\mathfrak{so}(n_0) = \text{span}\{e_{ab} : 1 \leq a < b \leq n_0\}.
\]

The equivalent modules in the decomposition of \( m_{0j} \) are given by

\[
m_\ell^j = \text{span}\{e_{tb} : n_0 + n_1 + \cdots + n_{j-1} + 1 \leq b \leq n_0 + n_1 + \cdots + n_j \}, \quad \ell = 1, \ldots, n_0.
\]

Also,

\[
\mathfrak{so}(n_j) = \text{span}\{e_{ab} : n_0 + n_1 + \cdots + n_{j-1} + 1 \leq a < b \leq n_0 + n_1 + \cdots + n_j \}, \quad j = 1, \ldots, s.
\]

Hence, we obtain the \( B \)-orthogonal decomposition

\[
m = n \oplus p,
\]

where

\[
n = \mathfrak{so}(n_0), \quad p = \bigoplus_{1 \leq i < j \leq s} m_{ij} \bigoplus_{j=1}^s m_{0j}.
\]

The above decomposition can be depicted in the following matrix, which shows the upper triangular part of \( \mathfrak{so}(n) \):

\[
\begin{pmatrix}
\mathfrak{so}(n_0) & m_{01} & m_{03} & \cdots & m_{0s} \\
0_{n_1} & m_{12} & m_{13} & \cdots & m_{1s} \\
0_{n_2} & m_{23} & \cdots & m_{2s} \\
\vdots & \ddots & \ddots & \vdots \\
\ast & \cdots & \ast & 0_{n_s} \\
\end{pmatrix}
\]
The matrices $m_{0j}$ are of size $n_0 \times n_j$, the matrices $m_{ij}$ are of size $n_i \times n_j$, and the matrices $so(n_i)$ of size $n_i \times n_i$. We remark that if $n_0 = 0$ or $n_0 = 1$, then $n = \{0\}$. In the former case, the submodules $m_{0j}$ are zero while in the latter case they are non zero and irreducible.

Moreover, using Lemma 4.2, we observe that

$$[so(n_i), m_{lm}] = \begin{cases} m_{lm}, & \text{if } i = l \text{ or } i = m \\ \{0\}, & \text{otherwise} \end{cases}, \quad 0 \leq i \leq s, \quad 0 \leq l < m \leq s,$$

and

$$[m_{ij}, m_{ij}] = m_{id} \quad \text{for all } \ 0 \leq i < j < l \leq s. \quad (13)$$

**Remark 4.3.** In view of Remark 4.1, if the isotropy subgroup $H = SO(n_1) \times \cdots \times SO(n_s)$ contains at least two $SO(2)$-factors, say $n_i = n_j = 2$, then the module $m_{ij}$ splits into two Ad($H$)-irreducible non equivalent summands each of dimension 2. More precisely, if we set $I = n_0 + n_1 + \cdots + n_{i-1} + 1$, $J = n_0 + n_1 + \cdots + n_{j-1} + 1$, $K = n_0 + n_1 + \cdots + n_{i-1} + 2$, $L = n_0 + n_1 + \cdots + n_{j-1} + 2$, then $m_{ij} = V_{ij}^1 \oplus V_{ij}^2$, where $V_{ij}^1 = \text{span}\{e_{IJ} - e_{KL}, e_{IL} + e_{KJ}\}$, $V_{ij}^2 = \text{span}\{e_{IJ} + e_{KL}, e_{IL} - e_{KJ}\}$.

For example, for $M = G/H = SO(14)/SO(3) \times SO(2) \times SO(3) \times SO(2) \times SO(2)$ we have $n_0 = 2, n_1 = 3, n_2 = 2, n_3 = 3, n_4 = 2, n_5 = 2$, and

$$m = n \oplus \mathfrak{p} = \mathfrak{so}(2) \bigoplus_{1 \leq i < j \leq 5} m_{ij} \bigoplus m_{0j}.$$ 

The submodules $m_{24}, m_{25}$ and $m_{45}$ split as above. For instance, $m_{24} = V_{24}^1 \oplus V_{24}^2$, where $V_{24}^1 = \text{span}\{e_{6,11} - e_{7,12}, e_{6,12} + e_{7,11}\}$, $V_{24}^2 = \{e_{6,11} + e_{7,12}, e_{6,12} - e_{7,11}\}$.

### 4.2. Proof of Theorem 1.1.

Any normal metric on $G/H$ induced from an Ad-invariant inner product $B$ on $\mathfrak{g}$ is a g.o. metric, and hence the sufficiency part of the theorem holds trivially. For the necessity part, assume initially that $n_1 = \cdots = n_s = 2$ so that $H$ is abelian. Since $G$ is semisimple, the main results in [25] imply that any g.o. metric on $G/H$ is normal, and hence Theorem 1.1 follows in this case.

Now assume that $n_j \neq 2$ for some $j = 1, \ldots, s$. Let $g$ be a $G$-invariant g.o. metric on $G/H$. Assume that $A : m \to m$ is the corresponding metric endomorphism satisfying Equation (2). Recall the spaces $n = so(n_0)$ and $\mathfrak{p} = \bigoplus_{0 \leq i < j \leq s} m_{ij}$ defined in Section 4.1, and the decomposition $m = n \oplus \mathfrak{p}$. Since $H$ is connected, we have $H^0 = H$. The Lie algebra of the normalizer $N_G(H^0) = N_G(H)$ coincides with $n_\mathfrak{g}(\mathfrak{h}) = \{Y \in \mathfrak{g} : [Y, \mathfrak{h}] \subseteq \mathfrak{h}\}$. In our case, $\mathfrak{h} = so(n_1) \oplus \cdots \oplus so(n_s)$ and it is not hard to show using the results in Section 4.1 that

$$n_\mathfrak{g}(\mathfrak{h}) = so(n_0) \oplus so(n_1) \oplus \cdots \oplus so(n_s) = n \oplus \mathfrak{h}.$$ 

Therefore, the tangent space of $G/N_G(H)$ coincides with $\mathfrak{p}$. The normalizer Lemma 3.3 then implies that $A_{\mathfrak{p}}$ defines a $G$-invariant g.o. metric on $G/N_G(H)$.

By taking into account Lemma 4.2 and the expressions of the subspaces $m_{ij}, so(n_j), so(n_0)$ in terms of the basis $\mathcal{B}$, we deduce that the submodules $m_{ij}, 0 \leq i < j \leq s$ are ad($n_\mathfrak{g}(\mathfrak{h})$)-invariant. If $n_i \neq 2$ or $n_j \neq 2$ then the submodules $m_{ij}$ are ad($n_\mathfrak{g}(\mathfrak{h})$)-irreducible. On the other hand, in view of Remark 4.3, $m_{ij} = V_{ij}^1 \oplus V_{ij}^2$ if $n_i = n_j = 2$.

We now claim that the ad($n_\mathfrak{g}(\mathfrak{h})$)-submodules $m_{ij}, 0 \leq i < j \leq s$, are pairwise inequivalent. Indeed, if $m_{ij}$ and $m_{im}$, with $0 \leq i < j \leq s, 0 \leq l < m \leq s$, are two distinct ad($n_\mathfrak{g}(\mathfrak{h})$)-submodules, then there exists an index $i_0$ such that one of the following happens:

**Case 1.** $i_0 = i$ or $i_0 = j$ and $i_0 \neq l, m$.  **Case 2.** $i_0 = l$ or $i_0 = m$ and $i_0 \neq i, j$.

For **Case 1**, we take into account relation (12) and obtain that $[so(n_{i_0}), m_{ij}] = m_{ij}$ and $[so(n_{i_0}), m_{lm}] = \{0\}$. For **Case 2** we have that $[so(n_{i_0}), m_{ij}] = \{0\}$ and $[so(n_{i_0}), m_{lm}] = m_{lm}$. By virtue of Lemma
2.2, we deduce that the submodules \( m_{ij} \) and \( m_{lm} \) are inequivalent in both cases, which proves the claim.

Therefore, the restriction of the g.o. metric \( A|_p \) of \( G/N_G(H) \) to each of the \( \text{ad}(n_0(\mathfrak{h})) \)-submodules \( m_{ij}, 0 \leq i < j < l \leq s \), has the diagonal form

\[
A|_{m_{ij}} = \begin{cases} 
\lambda_{ij} \text{Id}, & \text{if } n_i \neq 2 \text{ or } n_j \neq 2 \\
\left( \begin{array}{cc}
\lambda_{ij}^1 \text{Id}|_{V_{ij}^1} & 0 \\
0 & \lambda_{ij}^2 \text{Id}|_{V_{ij}^2}
\end{array} \right), & \text{if } n_i = n_j = 2.
\end{cases}
\]  
(14)

In view of the decomposition \( m = n \oplus p \) and Remark 3.6, Theorem 1.1 now follows from the following two propositions, whose proofs we present in the next subsection.

**Proposition 4.4.** \( A|_p = \lambda \text{Id} \), where \( \lambda > 0 \).

**Proposition 4.5.** \( A|_n = \lambda \text{Id} \), where \( \lambda \) is given by Proposition 4.4. \( \square \)

4.3. Proof of Propositions 4.4 and 4.5. To prove Proposition 4.4 we need the following lemmas.

**Lemma 4.6.** Let \( A \) be the metric endomorphism of a g.o. metric on \( M = \text{SO}(n)/\text{SO}(n_1) \times \cdots \times \text{SO}(n_s) \). Then for any \( r = 1, \ldots, s \), the restriction of \( A \) to the tangent space of \( \hat{M} := \text{SO}(n_r + \cdots + n_s)/\text{SO}(n_r) \times \cdots \times \text{SO}(n_s) \) defines a \( \text{SO}(n_r + \cdots + n_s) \)-invariant g.o. metric on \( \hat{M} \).

**Proof.** Let \( \mathfrak{g} := \mathfrak{so}(n_r + \cdots + n_s) \) and \( \mathfrak{h} := \mathfrak{so}(n_r) \oplus \cdots \oplus \mathfrak{so}(n_s) \) be the Lie algebras of \( \text{SO}(n_r + \cdots + n_s) \) and \( \text{SO}(n_r) \times \cdots \times \text{SO}(n_s) \) respectively. Taking into account the notation and results of section 4.1, the tangent space of \( M \) coincides with the B-orthogonal complement \( \hat{m} := \bigoplus_{i+j \leq s} m_{ij} \) of \( \mathfrak{h} \) in \( \mathfrak{g} \). Relation (14) implies that \( A|_{\hat{m}} \) defines an endomorphism of \( \hat{m} \), which is \( \text{ad}(\mathfrak{h}) \)-equivariant and hence \( \text{ad}(\mathfrak{h}) \)-equivariant. Therefore, \( A|_{\hat{m}} \) defines an \( \text{SO}(n_r + \cdots + n_s) \)-invariant metric on \( \hat{M} \). Since \( A \) defines a g.o. metric on \( G/H \), Proposition 3.2 implies that for any \( X \in \hat{m} \), there exists an \( a \in \mathfrak{h} \) such that

\[
0 = [a + X, AX] = [a + X, A|_{\hat{m}} X].
\]  
(15)

On the other hand, relation (12) along with the definition of \( \hat{m} \) imply that \( [\mathfrak{h}, \hat{m}] = [\mathfrak{h}, \hat{m}] \). Therefore, we may assume that the vector \( a \) in expression (15) lies in \( \mathfrak{h} \). By Proposition 3.2 we conclude that \( A|_{\hat{m}} \) defines a g.o. metric on \( \hat{M} \). \( \square \)

**Lemma 4.7.** Let \( R_s = \{\lambda_{ij} : 0 \leq i < j \leq s\} \) be a set such that \( \lambda_{ij} = \lambda_{jk} = \lambda_{ik} \) for all \( 0 \leq i < j < k \leq s \). Then \( R_s \) is a singleton.

**Proof.** We will proceed by induction on \( s \geq 1 \). If \( s = 1 \) then \( R_1 = \{\lambda_{01}\} \) and the result holds trivially. Assume that the lemma holds for \( s = N \) and let \( s = N + 1 \). The assumption that \( \lambda_{ij} = \lambda_{jk} = \lambda_{ik} \) for all \( 0 \leq i < j < k \leq s = N + 1 \) implies that \( \lambda_{ij} = \lambda_{jk} = \lambda_{ik} \) for all \( 0 \leq i < j < k \leq N \). By the induction hypothesis, the set \( R_N = \{\lambda_{ij} : 0 \leq i < j \leq N\} \) is a singleton, i.e. \( R_N = \{\lambda\} \).

To prove that \( R_{N+1} \) is a singleton, it remains to show that

\[
\lambda_{0,N+1} = \lambda_{1,N+1} = \cdots = \lambda_{N,N+1} = \lambda.
\]  
(16)

To this end, consider two arbitrary elements \( \lambda_{i,N+1}, \lambda_{j,N+1} \). Without loss of generality, assume that \( i < j \). Since \( 0 \leq i < j < N \), we have \( \lambda_{ij} \in R_N \) and hence \( \lambda_{ij} = \lambda \). On the other hand, the assumption of the lemma yields \( \lambda = \lambda_{ij} = \lambda_{j,N+1} = \lambda_{i,N+1} \). Since the choice of \( i, j \) is arbitrary, Equation (16) is true and thus \( R_{N+1} = \{\lambda\} \), which concludes the induction. \( \square \)
Proof of Proposition 4.4. The proof consists of two steps.

Step 1. Prove that both quantities $\lambda_{ij}^1$, $\lambda_{ij}^2$ in relation (14) are equal for all $i, j$, which will imply that

$$A|_{m_{ij}} = \lambda_{ij} \text{Id}. \quad (17)$$

Step 2. Prove that all $\lambda_{ij}$ in relation (17) are equal.

For Step 1, recall the assumption that at least one of the $n_j$, $j = 1, \ldots, s$ is not equal to 2. Recall also from definition (5) that $n_0 = n - (n_1 + \cdots + n_s)$. Without any loss of generality, we may apply a conjugation $\phi \in \text{Aut} (SO(n))$, permuting the position of the diagonal blocks $SO(n_j)$, $j = 1, \ldots, s$, in the embedding of $H$ in $G$, so that

$$n_j \neq 2 \text{ for } j = 0, \ldots, r \text{ and } n_{r+1} = n_{r+2} = \cdots = n_s = 2, \text{ i.e.} \quad (18)$$

$$n_\mathfrak{g}(h) = \mathfrak{so}(n_0) \times \cdots \times \mathfrak{so}(n_r) \times \mathfrak{so}(2) \times \cdots \times \mathfrak{so}(2), \quad r = 0, \ldots, s.$$

For example, any diagonal embedding of $H = \text{SO}(2) \times \text{SO}(3) \times \text{SO}(3)$ in $\text{SO}(10)$ is conjugate to

$$\begin{pmatrix}
\text{SO}(3) & 0 & 0 \\
0 & \text{SO}(3) & 0 \\
0 & 0 & \text{SO}(2) \\
0 & 0 & \text{Id}_2
\end{pmatrix}, \text{ in which case } n_\mathfrak{g}(h) = \mathfrak{so}(3) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(2).$$

Such a conjugation leaves the space $\mathfrak{p}$ invariant by permuting the $\text{ad}(n_\mathfrak{g}(h))$-submodules $m_{ij}$, $0 \leq i < j \leq s$. Under this conjugation and in the case where the number $s - r$ of $\text{SO}(2)$-blocks is greater than 2, relation (14) becomes

$$A|_{m_{ij}} = \begin{cases} 
\lambda_{ij} \text{Id}, & \text{if } 0 \leq i < j \leq r + 1 \\
\left( \begin{array}{cc}
\lambda_{ij}^1 \text{Id}|_{V_{ij}^1} & 0 \\
0 & \lambda_{ij}^2 \text{Id}|_{V_{ij}^2}
\end{array} \right), & \text{if } r + 1 \leq i < j \leq s.
\end{cases} \quad (19)$$

If $s - r < 2$ then relation (17) is trivially true.

We now assume that $s - r \geq 2$ and consider the space $\tilde{M} := \text{SO}(2(s - r))/ \text{SO}(2) \times \cdots \times \text{SO}(2)$, whose tangent space is $\tilde{\mathfrak{m}} := \bigoplus_{r+1 \leq i < j \leq s} m_{ij} \subset \mathfrak{p}$. By Lemma 4.6 the restriction $A|_{\tilde{\mathfrak{m}}}$ defines a $\text{SO}(2(s - r))$-invariant g.o. metric on $\tilde{M}$.

We consider the following cases:

**Case I** $s - r > 2$ and **Case II** $s - r = 2$.

In **Case I**, it is $\text{SO}(2(s - r)) \neq \text{SO}(4)$ and hence $\text{SO}(2(s - r))$ is simple. On the other hand, the isotropy subgroup $\text{SO}(2) \times \cdots \times \text{SO}(2)$ of $\tilde{M}$ is abelian. By the main theorem in [25], the g.o. metric $A|_{\tilde{\mathfrak{m}}}$ is standard and thus $A|_{\tilde{\mathfrak{m}}} = \lambda \text{Id}|_{\tilde{\mathfrak{m}}}$. In particular, the last relation implies that the quantities $\lambda_{ij}^1$, $\lambda_{ij}^2$ in relation (14) are equal for all $i, j$ with $r + 1 \leq i < j \leq s$, and thus relation (17) is true for **Case I**.

In **Case II**, we have $\tilde{M} = \text{SO}(4)/ \text{SO}(2) \times \text{SO}(2)$, $\tilde{\mathfrak{m}} = m_{s-1,s} = V_{s-1,s}^1 \oplus V_{s-1,s}^2$ and relation (19) yields $A|_{\tilde{\mathfrak{m}}} = \begin{pmatrix}
\lambda_{s-1,s}^1 \text{Id}|_{V_{s-1,s}^1} & 0 \\
0 & \lambda_{s-1,s}^2 \text{Id}|_{V_{s-1,s}^2}
\end{pmatrix}$. We consider the $\text{ad}(n_\mathfrak{g}(h))$-irreducible submodules $m_{1,s-1}$, and $V_{s-1,s}^1$. 
Relation (13) yields \([m_{1,s-1}, V_{s-1,s}^1] \subseteq [m_{1,s-1}, m_{s-1,s}] = m_{1,s}\). More importantly, along with the description of the submodules \(V_{s}^1\) in Remark 4.3, we deduce that \([m_{1,s-1}, V_{s-1,s}^1] \subseteq m_{1,s} \setminus \{0\}\). As a result, the space \([m_{1,s-1}, V_{s-1,s}^1] \) has non-zero projection on \((m_{1,s-1} \oplus V_{s-1,s}^1)^\perp\). By using the first part of Lemma 3.8, the last relation, along with the facts that \(A|_{m_{1,s-1}} = \lambda_{1,s-1} Id|_{m_{1,s-1}}\) and \(A|_{V_{s-1,s}^1} = \lambda_{s-1,s} Id|_{V_{s-1,s}^1}\), yield

\[
\lambda_{1,s-1} = \lambda_{s-1,s}^1.
\]

By using the same argument for \(V_{s-1,s}^2\) we deduce that

\[
\lambda_{1,s-1} = \lambda_{s-1,s}^2.
\]

Equations (20) and (21) yield \(\lambda_{s-1,s}^1 = \lambda_{s-1,s}^2\) and thus relation (17) is also true for Case II. This concludes Step 1.

For Step 2 we proceed as follows: Due to relation (13), along with Equation (17), the \(\text{ad}(h)\)-invariance of \(m_{ij}, m_{jl}\) and their \(B\)-orthogonality, part 2. of Lemma 3.8 yields

\[
\lambda_{ij} = \lambda_{jl} = \lambda_{il} \quad \text{for all } 0 \leq i < j < l \leq s.
\]

By Lemma 4.7 we deduce that the set \(R_s = \{\lambda_{ij} : 0 \leq i < j \leq s\}\) of the eigenvalues of \(A|_p\) has only one element, and thus Step 2 is concluded. \(\square\)

Before we proceed to the proof of the second proposition, we note that if \(n_0 = 0\) or \(n_0 = 1\), then \(n = \{0\}\) and \(m = p\), and thus Theorem 1.1 follows directly from Proposition 4.4.

**Proof of Proposition 4.5.** We recall the spaces \(m_{0j} = m_1^j \oplus \cdots \oplus m_{n_0}^j\) defined in Section 4.1. By Proposition 4.4 we have

\[
A|_{m_j^i} = \lambda \text{Id}, \quad i = 1, \ldots, n_0.
\]

The Lie algebra \(n = \mathfrak{so}(n_0)\) coincides with the Lie algebra of \(N_G(H^0)/H^0 = N_G(H)/H\). By Lemma 3.4, \(A|_n\) defines a bi-invariant metric on \(N_G(H)/H\), which in turn corresponds to an \(\text{Ad}\)-invariant inner product on \(\mathfrak{so}(n_0)\). For \(n_0 = 2\), \(n\) is one-dimensional and thus \(A|_n = \mu \text{Id}\). For \(2 < n_0 \neq 4\), \(\mathfrak{so}(n_0)\) is simple and the only \(\text{Ad}\)-invariant inner product is a scalar multiple of the Killing form. Therefore, if \(2 < n_0 \neq 4\) we also have \(A|_n = \mu \text{Id}\). For both cases, choose the vectors \(e_{12} \in n\) and \(e_{1,n_0+1} \in m_1^1\). We have

\[
[e_{12}, e_{1,n_0+1}] = -e_{2,n_0+1} \in m_1^2.
\]

Therefore, \([n, m_1]\) has non-zero projection on \((n \oplus m_1)^\perp\). Along with the \(\text{ad}(h)\)-invariance of \(n\) and \(m_1\) and the facts that \(A|_{m_1^i} = \lambda \text{Id}\) and \(A|_n = \mu \text{Id}\), Lemma 3.8 yields \(\lambda = \mu\). We conclude that \(A|_{\mathfrak{so}(n_0)} = \lambda \text{Id}\) if \(2 \leq n_0 \neq 4\).

If \(n_0 = 4\), \(n\) decomposes into two simple ideals as

\[
n = \mathfrak{so}(4) = n_1 \oplus n_2,
\]

where \(n_1 = \text{span}\{e_{12} + e_{34}, -e_{13} + e_{24}, e_{23} + e_{14}\} \approx \mathfrak{so}(3)\) and \(n_2 = \text{span}\{e_{12} - e_{34}, -e_{13} - e_{24}, e_{23} - e_{14}\} \approx \mathfrak{so}(3)\). By Lemma 3.7 we have

\[
A|_{n_1} = \mu_1 \text{Id} \quad \text{and} \quad A|_{n_2} = \mu_2 \text{Id}.
\]

It remains to show that \(\mu_1 = \mu_2 = \lambda\). To this end, choose the vectors \(e_{12} + e_{34} \in n_1, e_{12} - e_{34} \in n_2\) and \(e_{15} \in m_1^1\). The submodules \(n_1, n_2\) and \(m_1^1\) are \(\text{ad}(h)\)-invariant. Moreover, we have

\[
[e_{12} + e_{34}, e_{15}] = -e_{25} \in m_2^1 \subset (n_1 \oplus m_1^1)^\perp \quad \text{and} \quad [e_{12} - e_{34}, e_{15}] = -e_{25} \in m_2^1 \subset (n_2 \oplus m_1^1)^\perp.
\]
Hence, \([n_1, m_1^1]\) and \([n_2, m_1^1]\) have non zero projections on \((n_1 \oplus m_1^1)\) and \((n_2 \oplus m_1^1)\) respectively. Along with Equations (22) and (23), part 1. of Lemma 3.8 yields \(\mu_1 = \lambda = \mu_2\), which concludes the proof. \(\square\)

5. The space \(M = G/H = U(n)/U(n_1) \times \cdots \times U(n_s)\)

5.1. Isotropy representation of \(G/H = U(n)/U(n_1) \times \cdots \times U(n_s)\). Denote by \(\mu_n\) the standard representation of \(U(n)\) in \(\mathbb{C}^n\). Then the complexified adjoint representation of \(U(n)\) is

\[
\text{Ad}^{U(n)} \otimes \mathbb{C} = \mu_n \otimes \mathbb{C} \mu_n.
\] (24)

By taking into account the assumption that \(H\) is embedded diagonally in \(G\), we can identify \(H\) with the subgroup

\[
H = \begin{pmatrix}
\text{Id}_{n_0} & 0 \\
U(n_1) & \ddots \\
0 & \cdots & U(n_s)
\end{pmatrix},
\]

of \(G\), where \(n_0 := n - (n_1 + \cdots + n_s)\).

Let \(\tau_{n_i} : U(n_1) \times \cdots \times U(n_s) \to U(n_i)\) be the projection onto the \(i\)-factor and \(q_i = \mu_{n_i} \circ \tau_{n_i}\) be the standard representation of \(H\), i.e.

\[
U(n_1) \times \cdots \times U(n_s) \xrightarrow{\tau_{n_i}} U(n_i) \xrightarrow{\mu_{n_i}} \text{Aut}(\mathbb{C}^{n_i}).
\]

By using relation (24), we obtain

\[
\text{Ad}^G \otimes \mathbb{C}|_H = \mu_n \otimes \mathbb{C} \mu_n|_H = (q_1 \oplus \cdots \oplus q_s \oplus 1_{n_0}) \otimes \mathbb{C} (\bar{q}_1 \oplus \cdots \oplus \bar{q}_s \oplus 1_{n_0})
\]

\[
= 1_{n_0}^2 \bigoplus_{i=1}^{s} \{(q_i \otimes \mathbb{C} \bar{q}_i) \otimes \mathbb{C} \bar{q}_i (q_j \otimes \mathbb{C} \bar{q}_j) \otimes \mathbb{C} \bar{q}_j \}_{i \neq j}
\]

\[
\bigoplus_{1 \leq i < j \leq s} \{(q_i \otimes \mathbb{C} \bar{q}_j) \otimes (q_j \otimes \mathbb{C} \bar{q}_i)\}. \tag{25}
\]

The summand \(\bigoplus_{i=1}^{s} \{(q_i \otimes \mathbb{C} \bar{q}_i)\}\) in (25) corresponds to \(\text{Ad}^H \otimes \mathbb{C}\), therefore, by virtue of Proposition 2.1, \(\chi \otimes \mathbb{C}\) is given by

\[
\chi \otimes \mathbb{C} = 1_{n_0}^2 \bigoplus_{j=1}^{s} \{(q_j \otimes \bar{q}_j) \otimes \mathbb{C} \bar{q}_j (q_j \otimes \mathbb{C} \bar{q}_j) \}_{1 \leq i < j \leq s} \bigoplus_{1 \leq i < j \leq s} \{(q_i \otimes \mathbb{C} \bar{q}_j) \otimes (q_j \otimes \mathbb{C} \bar{q}_i)\}. \tag{26}
\]

Expression (26) induces a real decomposition of the tangent space

\[
m = n \oplus p, \tag{27}
\]

where

\[
n = u(n_0), \quad p = \bigoplus_{j=1}^{s} m_{0j} \bigoplus_{1 \leq i < j \leq s} m_{ij}. \tag{28}
\]

In the above decomposition we have \(m_{0j} = m_1^j \oplus m_2^j \oplus \cdots \oplus m_{n_0}^j\) with \(m_{n_0}^j \cong m_{n_0}^j\), \(\alpha \neq \beta\) and \(\dim(m_{n_0}^j) = 2n_j, \ell = 1, 2, \ldots, n_0\).

We now give explicit matrix representations of the modules \(m_{0j}\) and \(m_{ij}\). We consider the \(\text{Ad}(U(n))\)-invariant inner product \(B : u(n) \times u(n) \to \mathbb{R}\) given by

\[
B(X, Y) = -\text{Trace}(XY), \quad X, Y \in u(n). \tag{29}
\]
Then there is a $B$-orthogonal decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},$$

(30)

where $\mathfrak{h} = u(n_1) \oplus \cdots \oplus u(n_s)$ and $\mathfrak{m} \cong T_a(G/H)$. We consider a basis of $\mathfrak{g} = u(n)$ as follows: Let $M_n \mathbb{C}$ be the set of complex $n \times n$ matrices and let $E_{ab} \in M_n \mathbb{C}, a, b = 1, \ldots, n$, be the matrix with $1$ in the $(a, b)$-entry and zero elsewhere. For $a, b = 1, \ldots, n$, we set

$$e_{ab} = E_{ab} - E_{ba}, \quad f_{ab} = \sqrt{-1}(E_{ab} + E_{ba}).$$

(31)

Note that $e_{ab} = -e_{ba}, f_{ab} = f_{ba}$. Then the set

$$\mathcal{B} = \{e_{ab}, f_{cd} : 1 \leq a < b \leq n, \ 1 \leq c < d \leq n\},$$

(32)

constitutes a basis of $u(n)$ which is orthogonal with respect to $B$. We have the following.

**Lemma 5.1.** The non zero bracket relations among the vectors (31) are given by

$$[e_{ab}, e_{cd}] = \delta_{bc}e_{ad} - \delta_{ad}e_{cb} - \delta_{ac}e_{bd} - \delta_{bd}e_{ac},$$

$$[f_{ab}, e_{cd}] = \delta_{bc}f_{ad} - \delta_{ad}f_{cb} + \delta_{ac}f_{bd} - \delta_{bd}f_{ac},$$

$$[f_{ab}, f_{cd}] = -\delta_{bc}e_{ad} + \delta_{ad}e_{cb} - \delta_{ac}e_{bd} + \delta_{bd}e_{ac}.$$  

Proof. We observe that $[E_{ab}, E_{cd}] = \delta_{bc}E_{ad} - \delta_{ad}E_{bc}$ and the lemma follows by direct computation.

Then a choice of the modules in the decomposition (28) is the following:

$$m_{0j} = \text{span} \{e_{ab}, f_{cd} \in \mathcal{B} \text{ : } 1 \leq a, c \leq n_0, \ n_0 + n_1 + \cdots + n_{j-1} + 1 \leq b, d \leq n_0 + n_1 + \cdots + n_j\},$$

$$m_{ij} = \text{span} \{e_{ab}, f_{cd} \in \mathcal{B} \text{ : } n_0 + n_1 + \cdots + n_{i-1} + 1 \leq a, c \leq n_0 + n_1 + \cdots + n_i, \ n_0 + n_1 + \cdots + n_{j-1} + 1 \leq b, d \leq n_0 + n_1 + \cdots + n_j\}, \ 1 \leq i < j \leq s,$$

$$u(n_0) = \text{span} \{e_{ab}, f_{cd} \in \mathcal{B} \text{ : } 1 \leq a < b = n, \ 1 \leq c < d \leq n\}.$$

The equivalent modules in the decomposition of $m_{0j}$ are given by

$$m_{\ell j} = \text{span} \{e_{ab}, f_{cd} \in \mathcal{B} \text{ : } n_0 + n_1 + \cdots + n_{j-1} + 1 \leq b, d \leq n_0 + n_1 + \cdots + n_j\}, \ \ell = 1, \ldots, n_0.$$

Also,

$$u(n_j) = \text{span} \{e_{ab}, f_{cd} \in \mathcal{B} \text{ : } n_0 + n_1 + \cdots + n_{j-1} + 1 \leq a, b, c, d \leq n_0 + n_1 + \cdots + n_j\}, \ j = 1, \ldots, s.$$

Then the $B$-orthogonal decompositions (27) and (28) can be depicted in the following matrix, which shows the upper triangular part of $u(n)$:

$$\begin{pmatrix}
  u(n_0) & m_{01} & m_{02} & m_{03} & \cdots & m_{0s} \\
  0 & m_{12} & m_{13} & \cdots & m_{1s} \\
  0 & 0 & m_{23} & \cdots & m_{2s} \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  0 & \cdots & * & \cdots & 0 \\
  0 & \cdots & 0 & \cdots & 0
\end{pmatrix}$$

The matrices $m_{0j}$ are of size $n_0 \times n_j$, the matrices $m_{ij}$ are of size $n_i \times n_j$, and the matrices $u(n_i)$ of size $n_i \times n_i$. 


5.2. **Proof of Theorem 1.4.** Recall the spaces \( n \) and \( p \) defined in subsection 5.1 and the decompositions (27) and (28). We note that if \( n_0 = 0 \) (i.e. \( n_1 + \cdots + n_s = n \)), then \( n = \{0\} \), and Theorem 1.4 follows directly from Proposition 5.2 below. Assume henceforth that \( n_0 \geq 1 \). To prove Theorem 1.4 we need the following two propositions, which we will prove at the end of the subsection.

**Proposition 5.2.** \( A|_p = \lambda \text{Id} \), where \( \lambda > 0 \).

We further decompose \( n = u(1) \oplus \mathfrak{su}(n_0) = \mathfrak{z}(n) \oplus \mathfrak{su}(n_0) \) into its simple and abelian ideals. With respect to the basis \( B \), we have \( \mathfrak{z}(n) = \text{span}\{f_{11} + \cdots + f_{n_0n_0}\} \) and \( \mathfrak{su}(n_0) \) is the \( B \)-orthogonal complement of \( \mathfrak{z}(n) \) in \( u(n_0) \).

**Proposition 5.3.** Let \( n = u(1) \oplus \mathfrak{su}(n_0) = \mathfrak{z}(n) \oplus \mathfrak{su}(n_0) \) be the decomposition of \( n = u(n_0) \) into its simple and abelian ideals. If \( n_0 = 1 \) then \( A|_n = \mu \text{Id}|_n \) for some \( \mu > 0 \). If \( n_0 \geq 2 \), then \( A|_n = \begin{pmatrix} \mu \text{Id}|_{\mathfrak{z}(n)} & 0 \\ 0 & \lambda \text{Id}|_{\mathfrak{su}(n_0)} \end{pmatrix} \), where \( \lambda \) is given by Proposition 5.2.

From Propositions 5.2 and 5.3, the decomposition \( m = n \oplus p = u(1) \oplus \mathfrak{su}(n_0) \oplus p \), and after normalizing the metric, we conclude that any g.o. metric on \( G/H \) has necessarily the form (up to homothety)

\[
A = \begin{pmatrix} \mu \text{Id}|_{\mathfrak{z}(n)} & 0 \\ 0 & \text{Id}|_{\mathfrak{su}(n_0) \oplus p} \end{pmatrix}.
\]

(33)

To conclude the proof of Theorem 1.4, it remains to prove that the above form is also sufficient, i.e. the metrics \( A \) are g.o. metrics. Let \( X \in m \). By Proposition 3.2, we need to find a vector \( a \in \mathfrak{h} \) such that

\[
[a + X, AX] = 0.
\]

(34)

Let \( X_p, X_{\mathfrak{su}(n_0)} \) and \( X_{\mathfrak{z}(n)} \) denote the projections of \( X \) on \( p, \mathfrak{su}(n_0) \) and \( \mathfrak{z}(n) \) respectively. Then

\[
X_{\mathfrak{z}(n)} = r \sum_{i=1}^{n_0} f_{ii}, \quad \text{for some} \ r \in \mathbb{R},
\]

and \( X_p = X_1 + X_2 \), where \( X_1 \) is the projection of \( X_p \) on \( m_0 \oplus \cdots \oplus m_0 \) and \( X_2 \) is the projection of \( X_p \) on \( \oplus_{1 \leq i < j \leq s} m_{ij} \). Moreover, we write

\[
X_1 = \sum_{i=1}^{n_0} \sum_{j=n_0+1}^{n} (a_{ij} e_{ij} + b_{ij} f_{ij}), \quad a_{ij}, b_{ij} \in \mathbb{R}.
\]

We also consider the vector

\[
\bar{X}_1 := \sum_{i=1}^{n_0} \sum_{j=n_0+1}^{n} (b_{ij} e_{ij} - a_{ij} f_{ij}).
\]

Finally, we choose the vector

\[
a = r(1 - \mu) \sum_{i=n_0+1}^{n} f_{ii} \in \mathfrak{h} = u(n_1) \oplus \cdots \oplus u(n_s).
\]

By using Lemma 5.1 it is straightforward to check the following relations:

\[
[X_{\mathfrak{z}(n)}, X_1] = -2r \bar{X}_1, \quad [X_{\mathfrak{z}(n)}, X_2] = 0, \quad [a, X_1] = 2r(1 - \mu) \bar{X}_1 \quad \text{and} \quad [a, X_2] = 0.
\]

(35)

More specifically, the last relation can be verified by viewing both vectors \( a, X_2 \) as elements of the Lie algebra \( \mathfrak{k} := u(n_1 + \cdots + n_s) \), embedded diagonally in \( \mathfrak{g} \) as \( \begin{pmatrix} 0_{n_0 \times n_0} & 0 \\ 0 & \mathfrak{k} \end{pmatrix} \), and observing that \( a \)
lies in the center of $\mathfrak{t}$. Finally, since $[\mathfrak{h}, \mathfrak{n}] = 0$ and $[\mathfrak{j}(\mathfrak{n}), \mathfrak{n}] = 0$, we obtain
\[
[a, X_j(n)] = [a, X_{su(n_0)}] = [X_{\mathfrak{j}(\mathfrak{n})}, X_{su(n_0)}] = 0.
\] (36)
We can now verify condition (34). Indeed, by taking into account relations (35) and (36), as well as Equation (33), we obtain
\[
[a + X, AX] = [a + X_p + X_{su(n_0)} + X_{\mathfrak{j}(\mathfrak{n})}, X_p + X_{su(n_0)} + \mu X_{\mathfrak{j}(\mathfrak{n})}] = [a, X_p] + (1 - \mu)[X_{\mathfrak{j}(\mathfrak{n})}, X_p] = [a, X_1 + X_2] + (1 - \mu)[X_{\mathfrak{j}(\mathfrak{n})}, X_1 + X_2] = [a, X_1] + (1 - \mu)[X_{\mathfrak{j}(\mathfrak{n})}, X_1] = 2r(1 - \mu)\bar{X}_1 = 0,
\]
and this concludes the proof of the theorem. □

We now give the proofs of the above propositions.

Proof of Proposition 5.2. Since $H$ is connected, the Lie algebra of the normalizer $N_G(H)$ coincides with $\mathfrak{n}_3(\mathfrak{h}) = \{ Y \in \mathfrak{g} : [Y, \mathfrak{h}] \subseteq \mathfrak{h} \}$. In our case, $\mathfrak{h} = \mathfrak{u}(n_1) \oplus \cdots \oplus \mathfrak{u}(n_s)$ and $\mathfrak{n}_3(\mathfrak{h}) = \mathfrak{u}(n_0) \oplus \mathfrak{u}(n_1) \oplus \cdots \oplus \mathfrak{u}(n_s) = \mathfrak{n} + \mathfrak{h}$.

Therefore, the tangent space of $G/N_G(H)$ coincides with $\mathfrak{p}$. The normalizer Lemma 3.3 then implies that $A|_{\mathfrak{p}}$ defines a $G$-invariant g.o. metric on $G/N_G(H)$.

Similarly to the space $SO(n)/SO(n_1) \times \cdots \times SO(n_s)$, by taking into account Lemma 5.1 and the expressions of the subspaces $\mathfrak{m}_{ij}, \mathfrak{u}(n_j), \mathfrak{u}(n_0)$ in terms of the basis $\mathcal{B}$, we deduce that the submodules $\mathfrak{m}_{ij}, 0 \leq i < j \leq s$ are $\text{ad}(\mathfrak{n}_3(\mathfrak{h}))$-invariant, $\text{ad}(\mathfrak{n}_3(\mathfrak{h}))$-irreducible and pairwise inequivalent. Therefore,
\[
A|_{\mathfrak{m}_{ij}} = \lambda_{ij} \text{Id}.
\] (37)
It remains to prove that all $\lambda_{ij}$ are equal. To this end, we will use similar arguments as in Step 2 in Proposition 4.4. More specifically, Lemma 5.1 yields
\[
[\mathfrak{m}_{ij}, \mathfrak{m}_{jl}] = \mathfrak{m}_{il} \text{ for all } 0 \leq i < j < l \leq s.
\]
Using the above relation, along with Equation (37), the $\text{ad}(\mathfrak{h})$-invariance of $\mathfrak{m}_{ij}, \mathfrak{m}_{jl}$ and their $B$-orthogonality, part 2. of Lemma 3.8 yields that
\[
\lambda_{ij} = \lambda_{jl} = \lambda_{il} \text{ for all } 0 \leq i < j < l \leq s.
\]
By Lemma 4.7, we deduce that the set $R_s = \{ \lambda_{ij} : 0 \leq i < j \leq s \}$ of the eigenvalues of $A|_{\mathfrak{p}}$ has only one element, therefore $A|_{\mathfrak{p}} = \lambda \text{Id}$. □

Proof of Proposition 5.3. The Lie algebra $\mathfrak{n} = \mathfrak{u}(n_0)$ coincides with the Lie algebra of $N_G(H)/H$. By Lemma 3.4, $A|_{\mathfrak{n}}$ defines a bi-invariant (and hence g.o.) metric on $U(n_0)$, which in turn corresponds to an $\text{Ad}$-invariant inner product on $\mathfrak{u}(n_0)$. Since the center $\mathfrak{j}(\mathfrak{n})$ of $\mathfrak{n}$ is one-dimensional, Lemma 3.7 yields
\[
A|_{\mathfrak{n}} = \begin{pmatrix}
\mu \text{Id}|_{\mathfrak{j}(\mathfrak{n})} & 0 \\
0 & \bar{\lambda} \text{Id}|_{su(n_0)}
\end{pmatrix}, \quad \bar{\lambda} > 0.
\]
If $n_0 = 1$, then $\mathfrak{n} = \mathfrak{u}(1) = \mathfrak{j}(\mathfrak{n})$, verifying Proposition 5.3 for this case. Assume that $n_0 \geq 2$. It remains to show that $\bar{\lambda}$ is equal to the eigenvalue $\lambda$ given in Proposition 5.2. We recall the spaces $\mathfrak{m}_0 = \mathfrak{m}_1^j \oplus \cdots \oplus \mathfrak{m}_{n_0}^j$ defined in Section 5.1. By Proposition 5.2, we have
\[
A|_{\mathfrak{m}_i^j} = \lambda \text{Id}.
\]
Choose the vectors $e_{12} \in \mathfrak{su}(n_0)$ and $e_{1,n_0+1} \in m^1_1$. By Lemma 5.1, we have

$$[e_{12}, e_{1,n_0+1}] = -e_{2,n_0+1} \in m^2_1.$$ 

Therefore, $[\mathfrak{su}(n_0), m^1_1]$ has non-zero projection on $(\mathfrak{su}(n_0) \oplus m^1_1)^1$. Along with the $\text{ad}(h)$-invariance of $\mathfrak{su}(n_0)$ and $m^1_1$ and the facts that $A|_{m^1_i} = \lambda \text{Id}$ and $A|_{\mathfrak{su}(n_0)} = \tilde{\lambda} \text{Id}$, Lemma 3.8 yields $\lambda = \tilde{\lambda}$. □

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University of Patras, Department of Mathematics, GR-26500 Rion, Greece
Email address: arvanito@math.upatras.gr

University of Patras, Department of Mathematics, GR-26500 Rion, Greece
Email address: nsouris@upatras.gr

University of Patras, Department of Mathematics, GR-26500 Rion and University of Thessaly, Department of Mathematics, GR-35100 Lamia, Greece
Email address: statha@math.upatras.gr