Maps and twists relating $U(sl(2))$ and the nonstandard $U_h(sl(2))$: unified construction

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Abstract

A general construction is given for a class of invertible maps between the classical $U(sl(2))$ and the Jordanian $U_h(sl(2))$ algebras. Different maps are directly useful in different contexts. Similarity transformations connecting them, in so far as they can be explicitly constructed, enable us to translate results obtained in terms of one to the other cases. Here the role of the maps is studied in the context of construction of twist operators between the cocommutative and noncocommutative coproducts of the $U(sl(2))$ and $U_h(sl(2))$ algebras respectively. It is shown that a particular map called the ‘minimal twist map’ implements the simplest twist given directly by the factorized form of the $R_h$-matrix of Ballesteros-Herranz. For other maps the twist has an additional factor obtainable in terms of the similarity transformation relating the map in question to the minimal one. The series in powers of $h$ for the operator performing this transformation may be obtained up to some desired order, relatively easily. An explicit example is given for one particularly interesting case. Similarly the classical and the Jordanian antipode maps may be interrelated by a similarity transformation. For the ‘minimal twist map’ the transforming operator is determined in a closed form.

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1 Introduction

In the previous articles [1]-[3] we have introduced two different sets of invertible maps between the classical $sl(2)$ algebra

\[ [J_0, J_+] = \pm 2J_+ \quad [J_+, J_-] = J_0 \] (1.1)

and the corresponding nonstandard Jordanian $h$-deformed $U_h(sl(2))$ algebra [4]

\[ [H, X] = \frac{2}{\hbar} \sinh \hbar X, \quad [H, Y] = -Y(\cosh \hbar X) - (\cosh \hbar X)Y, \quad [X, Y] = H, \] (1.2)

where obviously in the $\hbar \to 0$ limit, we have $(X, Y, H) \to (J_+, J_-, J_0)$. The map presented in [1], and subsequently generalized in [2] to the Jordanian deformation of the $so(4)$ algebra, maintains the diagonalization of the generator $H$:

\[ H = J_0, \quad X = \frac{2}{\hbar} \text{arctanh}(\frac{\hbar}{2} J_+), \quad Y = (1 - (\frac{\hbar J_+}{2})^2)^{\frac{1}{2}} J_- (1 - (\frac{\hbar J_+}{2})^2)^{\frac{1}{2}}. \] (1.3)

The inverse of the map (1.3) is easily obtained [1]. The diagonalization of $H$ postulated above leads to attractive properties and easy construction of the irreducible representations of $U_h(sl(2))$ algebra and its finite dimensional $R_h$-matrices [1]. The second map [3]

\[ e^{\pm \hbar X} = \pm \hbar J_+ + (1 + (\hbar J_+)^2)^{\frac{1}{2}}, \quad H = (1 + (\hbar J_+)^2)^{\frac{1}{2}} J_0, \quad Y = J_+ + \frac{\hbar^2}{4} J_0 (1 - J_0^2) \] (1.4)

plays a key role in the construction of the Jordanian $R_h$ matrices of the $U_h(sl(2))$ algebra as contraction limits of the $R_q$ matrices of the standard $q$-deformed $U_q(sl(2))$ algebra [5]. The standard $R_q$ matrices were transformed [3] by the operator

\[ E_q(\eta J_+) \otimes E_q(\eta J_+), \] (1.5)

where $(J_+, J_0)$ are the generators of the $q$-deformed $U_q(sl(2))$ algebra [5] and $\eta = \frac{\hbar}{(q-1)}$. The deformed exponential in (1.5) reads

\[ E_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!}, \quad [n] \equiv \frac{q^n - q^{-n}}{q - q^{-1}}. \] (1.6)

The factor $\eta$ is singular in the $q \to 1$ limit, but the singularities systematically cancel rendering the transformed $R_q$-matrices finite. In the $q \to 1$ limit, the transformed $R_q$ matrices, on account of the map (1.4), reduce to the finite dimensional representations of the Ballesteros-Herranz (B-H) form [6] of the universal $R_h$ matrix of the Jordanian algebra:

\[ R_h = \exp(-hX \otimes e^{hX} H) \exp(e^{hX} \otimes hX). \] (1.7)
The universal $R_h$ matrix in (1.7) satisfies $\sigma \circ R_h = (R_h)^{-1}$; and consequently is triangular in nature. The flip operator $\sigma$ mentioned above permutes vector spaces as follows: $\sigma(x \otimes y) = y \otimes x$. In [3] the case $\frac{1}{2} \otimes j$ was treated fully. It was pointed out that the treatment may be generalised not only to the general case $j_1 \otimes j_2$ but also to higher dimensional algebras. The last point was illustrated with the $U_q(sl(3))$.

Following Drinfeld’s arguments [7], the classical cocommutative coalgebraic structure (denoted by the subscript $c$) of the $U(sl(2))$ algebra

$$\Delta_c(J_\delta) = J_\delta \otimes 1 + 1 \otimes J_\delta, \quad S_c(J_\delta) = -J_\delta, \quad \epsilon_c(J_\delta) = 0 \quad (\delta = \pm, 0) \quad (1.8)$$

and the noncocommutative coalgebraic structure (denoted by the subscript $n$) of the Jordanian $U_h(sl(2))$ algebra [4]

$$\Delta_n(X) = X \otimes 1 + 1 \otimes X, \quad \Delta_n(Y) = Y \otimes e^{hX} + e^{-hX} \otimes Y,$$
$$\Delta_n(H) = H \otimes e^{hX} + e^{-hX} \otimes H,$$
$$S_n(X) = -X, \quad S_n(Y) = -e^{hX}Y e^{-hX}, \quad S_n(H) = -e^{hX}H e^{-hX},$$
$$\epsilon_n(X) = \epsilon_n(Y) = \epsilon_n(H) = 0,$$

may be related via suitable twist operators corresponding to various maps. Using the maps (1.3) and (1.4) as examples, the relevant twist operators were considered in [3] as series expansions in $h$.

The situation may be envisaged as follows. Different maps may arise naturally in different contexts, and may be particularly useful for different purposes. As these maps relate the same pair of algebras, there exists an equivalence relation between any two. The twist corresponding to different maps may then also be related via the same relation. A unified treatment for a class of maps may enable us to fully exploit the attractive properties of different maps in different situations, and then obtain the results corresponding to others through equivalence relations, in so far as they can be obtained explicitly. This is attempted in the following sections. In particular, we construct explicitly the ‘minimal twist map’ for which the twist operator $F$, relating the classical and the quantum Jordanian coproduct structures, corresponds directly to the factorized B-H form (1.7) of $R_h$: namely,

$$F = \exp(-e^{hX}H \otimes hX). \quad (1.10)$$

Then it is indicated how equivalence transformations can lead to the twists corresponding to other maps. In the context of the above ‘minimal twist map’, the operator interrelating the classical and the Jordanian antipode structures in (1.8) and (1.9) respectively may also be determined in a closed form.

In the following sections we often use the elements $T^{\pm 1} = e^{\pm hX}$ of the $U_h(sl(2))$ algebra for calculational purpose. In terms of these operators, the defining relations (1.2) and (1.9) of $U_h(sl(2))$ algebra read

$$[H, T^{\pm 1}] = T^{\pm 2} - 1, \quad [H, Y] = -\frac{1}{2}(Y(T+T^{-1}) + (T+T^{-1})Y), \quad [T^{\pm 1}, Y] = \pm \frac{h}{2}(HT^{\pm 1} + T^{\pm 1}H) \quad (1.11)$$
and
\[
\triangle_n(T^{\pm 1}) = T^{\pm 1} \otimes T^{\pm 1}, \quad \triangle_n(Y) = Y \otimes T + T^{-1} \otimes Y, \quad \triangle_n(H) = H \otimes T + T^{-1} \otimes H,
\]
\[
S_n(T^{\pm 1}) = T^{\mp 1}, \quad S_n(Y) = -T Y T^{-1}, \quad S_n(H) = -T H T^{-1},
\]
\[
\epsilon_n(T^{\pm 1}) = 1, \quad \epsilon_n(Y) = \epsilon_n(H) = 0.
\] (1.12)

In the constructions to follow all functions of $J_+$ or $X$ may be interpreted as a finite power series on spaces of irreducible representations. This provides well defined operators.

## 2 A class of maps

To construct a class of maps important for our purpose, we take an ansatz
\[
J_+ = f_1, \quad J_0 = f_2 H, \quad J_- = f_3 Y + u + v H + w H^2,
\] (2.1)
where we introduce $(f_1, f_2, f_3; u, v, w)$ as functions of $T$ only. We start with (2.1) rather than with (2.14) to follow since, in the context of the former, the map crucial for our determination of the twist operator assumes a simple form. An additive function $f(T)$ in the expression for $J_0$ may be absorbed by a similarity transformation. To ensure correct classical limits, the introduced functions are required to satisfy the limiting properties
\[
(f_1, f_2, f_3; u, v, w) \rightarrow (X, 1, 1; 0, 0, 0)
\] (2.2)
as $h \rightarrow 0$. We will, moreover, be interested in invertible maps. For any function $f(T)$, we denote $f' = \frac{df}{dT}$. Using the identities
\[
[H, f] = (T^2 - 1)f',
\]
\[
[f, Y] = \frac{h}{2} ((T f') H + H (T f')) = h(T f') H + \frac{h}{2} (T^2 - 1)(T f')'
\] (2.3)
and the algebraic constraints (1.1) and (1.11) systematically, we, for a given $f_1$, obtain a set of seven coupled nonlinear equations (one being of second order in $f_2$) for the five unknown functions $(f_2, f_3; u, v, w)$:
\[
(T^2 - 1)f_2 f_1' = 2 f_1, \quad h T f_3 f_1' - 2 w (T^2 - 1) f_1' = f_2,
\]
\[
\frac{h}{2} f_3 (T f_1')' - v f_1' - w ((T^2 - 1) f_1')' = 0, \quad f_2 \left( (T^2 - 1) f_3' - (T + T^{-1}) f_3 \right) = -2 f_3,
\]
\[
(T^2 - 1)f_2 \left( \frac{h}{4} (1 + T^{-2}) f_3 + u' \right) = -2 u, \quad h T f_3 f_2' + (T^2 - 1)(f_2 w' - 2 w f_2') = -2 w,
\]
\[
(T^2 - 1) \left( \frac{h}{2} f_3 (f_2 + T (T f_2') T^{-1} - (v f_2' - f_2 v') - w ((T^2 - 1) f_2')' \right) = -2 v.
\] (2.4)
These equations may then be solved consistently and the limiting behaviour (2.2) as $h \to 0$ may be implemented through proper choice of integration constants. This leads unambiguously to the following solution

$$f_2 = \frac{2}{(T^2 - 1)f_1}, \quad f_3 = \frac{1}{2} (T - T^{-1}), \quad u = -\frac{(T - T^{-1})^2}{16f_1},$$

$$v = -\frac{1}{2} \left( \frac{f'_2}{f_1} - \frac{f_2}{f_1} + \frac{T + T^{-1}}{2f_1} \right), \quad w = \frac{1}{4f_1}(1 - f_2^2).$$

(2.5)

In the solution (2.5) the unknown functions $(f_2, f_3; u, v, w)$ may be fully expressed in terms of the function $f_1$ and its derivatives, assumed to be known. Thus a suitably chosen $f_1$ and its properties completely determine the map. The simplest choice

$$f_1 = \pm \frac{1}{h}(T^{\pm 1} - 1)$$

(2.6)
do not seem to lead to particularly interesting properties. This is true for our present main concern, namely construction of twists. But (2.6) probably deserves further study in other contexts. The inverse of the map (1.3) corresponds to the choice $f_1 = \frac{2}{h} \left( \frac{T^{-1}}{T^{-1} + 1} \right)$. The unknown functions now, through the solution (2.5), assume the form

$$f_2 = 1, \quad f_3 = \left( \frac{T^{\pm} + T^{-\frac{1}{2}}}{2} \right)^2, \quad u = -\frac{h}{8} \left( \frac{T^{\pm} + T^{-\frac{1}{2}}}{2} \right)^2 (T - T^{-1}),$$

$$v = -\frac{h}{8}(T - T^{-1}), \quad w = 0.$$  

(2.7)

The choice $f_1 = \frac{1}{2h}(T - T^{-1})$ reproduces the inverse of the map (1.4). The solution (2.5) now restricts the unknown functions as

$$f_2 = \frac{2}{T + T^{-1}}, \quad f_3 = 1, \quad u = -\frac{h}{8}(T - T^{-1}),$$

$$v = -\frac{h}{2} (\frac{T - T^{-1}}{T + T^{-1}})^3, \quad w = \frac{h}{2}(T - T^{-1}).$$

(2.8)

The special interest of (2.7) and (2.8) have already been indicated. Other choices of $f_1$ may prove interesting. For a purpose of this article, namely finding the map corresponding to the form (1.10) of the twist operator, the pertinent choice is

$$f_1 = \frac{1}{2h}(1 - T^{-2}).$$

(2.9)
The solution (2.5) now yields

$$f_2 = f_3 = T, \quad u = -\frac{h}{8}(T^2 - 1), \quad v = -\frac{h}{2}(T^2 - 1)T, \quad w = -\frac{h}{2}T^2.$$  

(2.10)
This is the ‘minimal twist map’ resulting in the twist operator (1.10) as will be shown in a
subsequent section.

Even for invertible maps it is interesting to construct the general solution starting from
the other end. So let us consider directly the class of maps given by

\[ T = g_1, \ H = g_2 J_0, \ Y = g_3 J_+ + a + b J_0 + c J_0^2, \]  

(2.11)

where \((g_1, g_2, g_3; a, b, c)\) are functions of \(J_+\) only. The expansion

\[ g_1 = 1 + h J_+ + O(h^2). \]  

(2.12)

conforms to the correct limiting properties as \(h \to 0\); and, in the same limit, the other
functions behave as

\[ (g_2, g_3; a, b, c) \to (1, 1; 0, 0, 0). \]  

(2.13)

For any function \(g(J_+)\), we denote its derivative as \(g' = \frac{dg}{dJ_+}\). The following identities

\[ [J_0, g] = 2 J_+ g', \quad [g, J_-] = \frac{1}{2} (J_0 g' + g' J_0) = J_+ g'' + g' J_0 \]  

(2.14)

and a systematic use of the algebraic properties (1.1) and (1.11) yield, as before, a set of
seven coupled nonlinear equations. These equations, after taking into account the limiting
properties (2.11) and (2.12) for determination of the constants of integrations, may be solved
unambiguously. The discussion following (2.3) is also relevant here. The unknown functions
read

\[ g_2 = \frac{(g_1^2 - 1)}{2 J_+ g_1'}, \quad g_3 = 2h \frac{J_+ g_1}{(g_1^2 - 1)}, \quad a = \frac{h}{8} (g_1 - g_1^{-1}), \]

\[ b = -\left(2c + \frac{h}{2} \frac{g_1'}{g_1^2} g_2\right), \quad c = \frac{h g_1 (1 - g_2^2)}{2(g_1^2 - 1)}. \]  

(2.15)

Thus again, as is to be expected, the choice of \(g_1\) and the limiting properties as \(h \to 0\)
completely determine the map. The three particular invertible solutions introduced before
well illustrate the situation.

3 Equivalence of the maps

As emphasized before, different maps play useful roles in specific contexts. These maps are,
however, equivalent to one another through appropriate similarity transformations. Under-
standing the equivalence properties precisely is of importance as they allow us to carry over
the results easily derived using one map in the context of another one. This will be illus-
trated in the construction of twists. The class of maps (2.11) discussed above, may be related
through similarity transformations by operator of the form \(exp(\lambda(J_+) J_0)\). Alternately, the
class of inverse maps (2.1) may be interrelated in a parallel way by the operator \( \exp(\mu(T)H) \).

In the classical \( h \to 0 \) limit, the generating functions \( \lambda(J_+) \) and \( \mu(T) \) introduced above behave as \( (\lambda, \mu) \to 0 \). It is usually difficult to obtain the exact, closed form expressions of the generating functions \( \lambda \) and \( \mu \). But the constant coefficients in the series for, say,

\[
\lambda(J_+) = c_1(hJ_+) + c_2(hJ_+)^2 + \cdots + c_n(hJ_+)^n + \cdots, \tag{3.1}
\]

may be obtained iteratively in a systematic fashion. We will demonstrate this in the following. A similar series in \( (hX) \) for the function \( \mu(T) \) may also be constructed.

Let us consider two maps starting respectively with

\[
T = g_1(J_+) \quad \text{and} \quad \hat{T} = \hat{g}_1(J_+). \tag{3.2}
\]

To avoid confusion we distinguish between the two sets of generators \( (T, H, Y) \) and \( (\hat{T}, \hat{H}, \hat{Y}) \), both satisfying (1.11) corresponding to the two maps. Our task now is to construct the generating function \( \lambda(J_+) \) that transforms one map to another as

\[
e^{-\lambda(J_+)J_0}g_1(J_+)e^{\lambda(J_+)J_0} = \hat{g}_1(J_+). \tag{3.3}
\]

It is important to realize that it is sufficient to ensure (3.3). The required equivalence of the pairs \( (H, \hat{H}) \) and \( (Y, \hat{Y}) \) follow once that of \( (T, \hat{T}) \) is assured. This is an evident consequence of the following facts:

1. A well defined similarity transformation conserves the algebra.
2. The transforming operator in (3.3) preserves the form of the ansatz (2.11) altering only the functions of \( J_+ \).
3. For our maps and limiting \( (h \to 0) \) constraints, a choice of the function \( g_1(\hat{g}_1) \) determines the remaining functions unambiguously.

The equivalence relation (3.3) may be recast in the form

\[
g_1 \left( e^{-\lambda(J_+)J_0} J_+ e^{\lambda(J_+)J_0} \right) = \hat{g}_1(J_+). \tag{3.4}
\]

The invertibility of our maps, in conjunction with (3.4), now yield

\[
e^{-\lambda(J_+)J_0} J_+ e^{\lambda(J_+)J_0} = f_1(\hat{g}_1(J_+)). \tag{3.5}
\]

The transforming relation (3.5) may be systematically used to generate the series (3.1), where the coefficients \( \{c_i, |i = 1, 2, \ldots\} \) may be evaluated iteratively. This is best illustrated through a relatively simple but particularly interesting example. We consider the maps (1.4) and (2.9) along with their inverses. The relation (3.5) now implies

\[
e^{-\lambda(J_+)J_0} (hJ_+)e^{\lambda(J_+)J_0} = (hJ_+) \left( -(hJ_+) + \left(1 + (hJ_+)^2\right)^{1/2} \right). \tag{3.6}
\]
Using standard expansion scheme in (3.6), we now obtain the generating function $\lambda(J_+)$ up to $O(h^5)$:

$$\lambda(J_+) = \frac{1}{2}(hJ_+) + \frac{1}{4}(hJ_+)^2 + \frac{1}{8}(hJ_+)^3 + \frac{1}{24}(hJ_+)^4 - \frac{1}{96}(hJ_+)^5 + \cdots$$  \hspace{1cm} (3.7)

The higher order terms may be computed similarly. Using our maps we may express the transforming operator $\exp(\lambda(J_+)J_0)$ as $\exp(\mu(T)H)$ and vice versa. An application of such equivalence will be indicated in the following section. Unlike the general solutions (2.5) and (2.15) for the class of maps considered here, we are, however, unable to present an exact general solution for (3.3) in an explicit form.

4 Twists: the role of maps

The classical algebra (1.1) with the cocommutative coalgebra structure (1.8) may be assumed to have a trivial universal $R$ matrix:

$$R_c = 1 \otimes 1.$$  \hspace{1cm} (4.1)

On the other hand, the factorized B-H form [6] of the universal $R_h$ matrix (1.7) of the $U_h(sl(2))$ algebra may be recast as

$$R_h = (\sigma \circ V)V^{-1},$$  \hspace{1cm} (4.2)

where $V^{\pm 1} = \exp(\mp e^{hX} \otimes hX) = \exp(\mp TH \otimes hX)$. The flip operator $\sigma$ has been introduced earlier following equation (1.7). Following [7] it may be observed that the triangular universal $R_h$ matrix (1.7) may, by a suitable twist $F \in U_h(sl(2)) \otimes 2$, be brought into the classical form (4.1). Such twists, relating the quantum comultiplication (1.9) with classical cocommutative coproduct (1.8), may be called factorizing twist as they factorize [7] the universal $R_h$ matrix as

$$R_h = (\sigma \circ F) R_c F^{-1} = (\sigma \circ F) F^{-1}.$$  \hspace{1cm} (4.3)

The second equality in (4.3) follows from (4.1). The consistency of the results (4.2) and (4.3) requires

$$F = VF_S,$$  \hspace{1cm} (4.4)

where $F_S$ is symmetric under permutation: $\sigma \circ F_S = F_S$. In the structure (4.4) of the twist operator $F$ the factor $V$ is fixed; whereas different choices of $F_S$ imply different maps between the relevant algebras. For an invertible map

$$m : (T^{\pm 1}, H, Y) \rightarrow (J_{\pm}, J_0), \quad m^{-1} : (J_{\pm}, J_0) \rightarrow (T^{\pm 1}, H, Y)$$  \hspace{1cm} (4.5)

the coproducts given by (1.8) and (1.9) respectively may be related via the twist operator $F$ as

$$F^{-1}(\Delta_n(\Phi))F = \tilde{\Delta}_c(\Phi), \quad \forall \Phi \in U_h(sl(2)),$$  \hspace{1cm} (4.6)
where, \( \tilde{\Delta}_c = (m^{-1} \otimes m^{-1}) \circ \Delta_c \circ m \). As a consistency check of our factorization scheme (4.4) for the twist operator \( F \), we utilize (4.6) to obtain the identity

\[
F_S(\tilde{\Delta}_c(\Phi))F_S^{-1} = V^{-1}(\Delta_n(\Phi))V
\]

\[
= \frac{1}{2} V^{-1} \left( \Delta_n(\Phi) + R_h^{-1}(\sigma \circ \Delta_n(\Phi))R_h \right) V
\]

\[
= \frac{1}{2} \left( V^{-1}(\Delta_n(\Phi))V + \sigma \circ \left( V^{-1}(\Delta_n(\Phi))V \right) \right),
\]

(4.7)

where we have used the standard properties of the universal \( R_h \) matrix and its factorized form (4.2). As the operator \( \tilde{\Delta}_c(\Phi) \) and \( F_S \) on the lhs of (4.7) are both symmetric under the permutation, the identity explicitly shows the consistency of our factorization (4.4) of the twist operator \( F \) with the defining relation (4.6). At this stage it is natural to ask the following question: Is it possible to consistently set \( F_S = 1 \otimes 1 \)? And if so, what does it imply? An affirmative answer to the above question, in view of (4.4), leads to the following elegant and useful construction of the twist operator:

\[
F = V \left( = \exp(-e^{hX} H \otimes hX) \right).
\]

(4.8)

We show that (4.8) can indeed be realized and that it implies the implementation of a specific map. This map will be constructed explicitly, step by step, by exploring the action of \( V \) on suitable operators. The following results are crucial. Setting \( V^{-1} = \exp(-\Lambda) \), we assume the construction (4.8) to be valid and thereby evaluate the lhs of (4.6) using the standard Campbell-Hausdorff series expansion:

\[
V^{-1}(\Delta_n(\Phi))V = \Delta_n(\Phi) + [\Lambda, \Delta_n(\Phi)] + \frac{1}{2} [\Lambda, [\Lambda, \Delta_n(\Phi)]] + \cdots.
\]

(4.9)

Our task is now to make appropriate choices for \( \Phi \) to realize the map, thus validating the construction (4.8) of the twist operator \( F \). For the choice \( \Phi = (1 - T^{-2}) \), the coproduct rules (1.12) yield

\[
\Delta_n(\Phi) = 1 \otimes 1 - T^{-2} \otimes T^{-2}.
\]

(4.10)

Using (4.8)-(4.10) we obtain a remarkable result

\[
V^{-1} \left( \Delta_n \left( 1 - T^{-2} \right) \right) V = \left( 1 - T^{-2} \right) \otimes 1 + 1 \otimes \left( 1 - T^{-2} \right).
\]

(4.11)

Similarly the following result

\[
\Delta_n(TH) = TH \otimes T^2 + 1 \otimes TH
\]

(4.12)

and the similarity transformation (4.9) yield

\[
V^{-1} \Delta_n(TH)V = TH \otimes 1 + 1 \otimes TH.
\]

(4.13)
The ‘classical’ aspect of the r.h.s. of (4.11) and (4.13) is not an accident. The map (2.9) and (2.10) may be written after a regrouping of the terms for $J_-$ as

$$J_+ = \frac{1}{2\hbar} \left( 1 - T^{-2} \right), \quad J_0 = TH, \quad J_- = TY - \frac{\hbar}{2} (TH)^2 - \frac{\hbar}{8} (T^2 - 1). \quad (4.14)$$

Indeed it may be demonstrated by direct computation that the expression for $J_-$ satisfies the requirement (4.6):

$$V^{-1} (\triangle_n(J_-)) V = J_- \otimes 1 + 1 \otimes J_-(= \tilde{\Delta}_c(J_-)) \quad (4.15)$$

The equation (20) of [6] is quite close to this result. But the authors, their context being different, did not construct the map completely. The above results (4.11)-(4.15) may be combined as

$$V^{-1} (\triangle_n(J_\delta)) V = J_\delta \otimes 1 + 1 \otimes J_\delta(= \tilde{\Delta}_c(J_\delta)), \quad (\delta = \pm, 0). \quad (4.16)$$

In (4.15) and (4.16) the classical generator $J_\delta$ stand for the expressions in the rhs of inverse map (4.14). Comparing (4.16) with (4.7), we now obtain

$$F_S(\tilde{\Delta}_c(J_\delta)) F_S^{-1} = \tilde{\Delta}_c(J_\delta). \quad (4.17)$$

This verifies our previously announced structure of the twist operator corresponding to the inverse map (4.14):

$$F_S = 1 \otimes 1 \quad \Rightarrow \quad F = V. \quad (4.18)$$

From this point of view we call (4.14) the ‘minimal twist map’. The twist operator $V$ in (4.8) satisfies the cocycle condition:

$$((\triangle_n \otimes 1)V) (V \otimes 1) = ((1 \otimes \triangle_n)V) (1 \otimes V). \quad (4.19)$$

Starting with the ‘minimal twist map’, $F_S$ corresponding to another one may be expressed in terms of the similarity transformation relating them. Let $J_\delta$ ($\delta = \pm, 0$) correspond to the ‘minimal twist map’ (4.14) and $\hat{J}_\delta$ to another map related to the former by a transformation $U$ of the class discussed before in the context of (3.2)-(3.7), such that

$$J_\delta = U \hat{J}_\delta U^{-1}. \quad (4.20)$$

The equivalence property (4.20) and the following identity,

$$F_S(\tilde{\Delta}_c(\hat{J}_\delta)) F_S^{-1} = V^{-1} (\triangle_n(\hat{J}_\delta)) V \quad (4.21)$$

obtained à la (4.7) now yield

$$F_S \left( \tilde{\Delta}_c U \right)^{-1} \left( \tilde{\Delta}_c J_\delta \right) \left( \tilde{\Delta}_c U \right) F_S^{-1} = \left( V^{-1} (\triangle_n U^{-1}) V \right) \left( V^{-1} (\triangle_n J_\delta) V \right) \left( V^{-1} (\triangle_n U) V \right). \quad (4.22)$$
The property (4.16) of the ‘minimal twist map’ $J_\delta$ now imply

$$F_S = \left(V^{-1}(\Delta_n(U))V\right)^{-1}\left(\tilde{\Delta}_c(U)\right), \quad (4.23)$$

where not only the factor $\left(\tilde{\Delta}_c(U)\right)$ on the rhs but first factor, as follows from (4.7), is also symmetric under permutation. This assures consistency with the postulated symmetry of $F_S$. Finally, using (4.4) and (4.23), we obtain the general structure of the twist operator for arbitrary maps of the class (2.1):

$$F = (\Delta_n(U))^{-1}V\left(\tilde{\Delta}_c(U)\right). \quad (4.24)$$

In (4.20)-(4.24) the equivalence operator $U$ is of the form $U = \exp(-\mu(T)H)$ discussed earlier in section 3. The twist operators corresponding to any two maps may now be related in an evident way.

An example of $U$ is considered in (3.6) and its series development is indicated in (3.7). If instead of the map (1.4) we consider (1.3), then (3.6) is to be replaced by

$$e^{-\lambda(J_+)}J_0(hJ_+e^{\lambda(J_+)}J_0) = hJ_+ + \left(1+\frac{hJ_+}{2}\right)^2$$

and the coefficients $c_n$ in (3.1) must be evaluated accordingly. For the map (1.3), starting with the coproduct $\Delta_n$ in (1.9), it is possible to write down the induced coproducts for $J_\delta$. An interesting application of these was made [8]-[10] in computing the C.G. coefficients of $U_h(sl(2))$. From our point of view, as per (4.6), such induced coproducts $\Delta_n(J_\delta)$ may be finally expressed in terms of the relevant classical coproduct and the twist operator (4.24). The cocycle condition for an arbitrary twist operator $F$ in (4.24) may be verified by using an appropriate series expansion for $U$.

The antipode structure of Jordanian $U_h(sl(2))$ algebra (1.12) and that of the classical $U_h(sl(2))$ algebra (1.8) may also be related by a similarity transformation as follows:

$$G^{-1}S_h(\Phi)G = m^{-1} \circ S_c \circ m(\Phi) \quad \Leftrightarrow \quad m \circ S_h \circ m^{-1}(\tilde{\Phi}) = \tilde{G}S_c(\tilde{\Phi})\tilde{G}^{-1}, \quad \forall \Phi \in U_h(sl(2)), \quad \tilde{\Phi} \in U(sl(2)), \quad (4.26)$$

where $G \in U_h(sl(2))$ and $\tilde{G} = m \circ G \in U(sl(2))$. For the ‘minimal twist map’ (4.14), the operator $G$ and $\tilde{G}$ performing the similarity transformation (4.26) may be determined in a closed form:

$$G = \exp\left(\frac{1}{2}TH(1-T^{-2})\right) \quad \Leftrightarrow \quad \tilde{G} = \exp(hJ_0J_+). \quad (4.27)$$

For the other maps discussed in section 2, these transforming operators may be determined as power serieses in the deformation parameter $h$. We will not discuss this here. The counit maps in (1.8) and (1.9) readily correspond to each other.
5 Remarks

Contrasting an earlier approach [3] of starting with the ‘classical’ $r$-matrix and then trying to incorporate higher order terms in the construction of a series expansion of the twist operator corresponding to a specific map, here we start from the very simple fact that for a classical algebra the $\mathcal{R}$-matrix is identity. Then we note that the B-H form of $\mathcal{R}_h$ obtained in [6] is already expressed as the twisted form of the identity and the margin for manoeuvre left, as evidenced from (4.4) to (4.7), is a factor $F_S$ symmetric under permutation. We then show that constraining this factor to be unity implies a particular invertible map relating the generators of $U(sl(2))$ and $U_h(sl(2))$ algebras. We call this the ‘minimal twist map’. But other maps are of interest in other contexts. So we give a unified construction of a class of invertible maps, their equivalence relations and the constructions of $F_S$ in terms of the latter ones. The simplest possible twist is undoubtedly given by (4.8), which corresponds to the ‘minimal twist map’ (4.14). For the ‘non-minimal’ cases it is much simpler to compute $U$, the similarity transformation to the minimal case and then $F_S$ in terms of $U$ as in (4.23). An example of series development for $U$ is given in (3.7). Combining the above results in (4.24) we obtain, for an arbitrary map of the class discussed in section 3, the twist operator interrelating the classical and the noncocommutative Jordanian coproducts. The classical and the Jordanian antipode maps may be related through a similarity transformation. The transforming operator corresponding to ‘minimal twist map’ may be obtained in a closed form. For an arbitrary map, the pertinent transforming operator may be obtained as a series in $\hbar$.

Other, quite different looking, constructions for $\mathcal{R}_h$ [11] and the twist operator [4] may be found. We will not attempt to trace their possible relations to the formalism presented here. Finally we just briefly mention that after the first twist leading from the cocommutative to the noncocommutative Hopf structure, it is possible to envisage a second twist leading to a quasi Hopf generalization for the nonstandard case. This will be studied in a following paper.

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