Abstract

Lattice gauge theories are considered with a partial axial gauge fixing along one direction only. This leaves a residual gauge symmetry that is still local in three directions but now global in one. It is found that this $N^{d-1}$ fold symmetry (on an $N^d$ lattice) breaks spontaneously at weak coupling with the gauge field elements on links averaged over 1-d chains along the gauge-fixing direction as order parameters. This phase transition is observed with Monte-Carlo simulations for both 3-d $\mathbb{Z}_2$ and 4-d $SU(2)$ pure gauge theories and appears to be coincident with the deconfinement transition. This work calls into question the equivalence of different gauges in certain circumstances.

1 Introduction

It is well known that Elitzur’s theorem[1] prevents the spontaneous breaking of a local gauge symmetry. However, once the gauge is fixed with a suitable gauge-fixing term in the Lagrangian, then the remaining global gauge symmetry can be spontaneously broken, such as by the Higgs field in the standard model. In this letter the possibility that the global symmetry (or partially-global gauge symmetry that results from partial gauge fixing) might already be spontaneously broken by the gauge fields themselves is explored. This is known to take place in continuum quantum electrodynamics, where the spontaneous breaking of a residual gauge symmetry left over after formulating in the Lorentz gauge has been shown to account for the masslessness of photons. In this picture the photons are seen as Goldstone bosons associated with the spontaneously broken symmetries[2]. The situation for non-abelian continuum theories is not as clear.

Earlier papers explored the sub-maximal axial gauge, defined by omitting the last line of links from the usual maximal axial gauge-fixing tree, leaving
an N-fold gauge symmetry still local in one direction\cite{3,4}. Here it was found
that these residual gauge symmetries break spontaneously at weak coupling
and are unbroken at strong coupling. These regions are separated by a phase
transition which appears to be coincident with the deconfinement transition
in both the 3-d $Z_2$ and the 4-d SU(2) theories. The order parameters for
these transitions are the N-th direction pointing links (gauge elements) averaged
over N-1 dimensional layers perpendicular to the Nth direction. These
systems have a large vacuum degeneracy since each layer can lie in a dif-
ferent broken-symmetry direction. This can have an effect on the critical
behavior at the phase transition because the sudden loss in ergodicity across
the phase transition will result in a change in the entropy (because some sec-
tors are no longer being visited, the number of microstates changes). In a
more or less restrictive gauge the amount of entropy change at the phase
transition will be different because the vacuum will have a different multi-
plicity change at the transition (see below). The usual argument that gauge
fixing is irrelevant because it multiplies the partition function by the same
“overcounting” factor breaks down because this factor is different on the two
sides of the transition. If a symmetry is spontaneously broken the multiple
vacua are already only being single-counted due to normal loss of ergodicity
- the other vacuum sectors are excluded because they cannot be tunneled to.
Choosing a more-restrictive gauge may select a particular vacuum, but does
not change the size of the accessible ensemble. However, in the unbroken
confining phase, which is ergodic, gauge fixing does change multiplicities by
restricting the space of available states. The more-restrictive gauges will
change this factor by a larger amount then the less-restrictive ones. There-
fore it would appear that different gauges will have different results at such
a phase transition.

In a previous paper it was shown that the Fradkin-Shenker proof\cite{5} of the
lack of a phase transition between Higgs and confinement phases in gauge-
Higgs theories is only valid in fully-fixed unitary gauge\cite{4}. When the gauge
condition is relaxed to allow for the symmetry to be local in one direction
then these regions are separated by a symmetry-breaking phase transition.
The layered gauge symmetry is spontaneously broken in the Higgs phase and
unbroken in the confinement phase. Thus the critical behavior is drastically
different in the different gauges. By locking down the gauge symmetry
completely, the unitary gauge explicitly breaks all of the gauge symmetries
allowing no room for spontaneous breaking to occur.

It is interesting to consider how far one can go in the program of loosening
the gauge fixing so as to allow for a greater multiplicity of gauge symmetries
and still see spontaneous breaking. In this paper it is shown that even if
the gauge fixing is limited to a single direction, leaving the gauge symmetry
still local in N-1 directions and global in only one, symmetry breaking still
takes place in both the discrete and continuous cases. This is somewhat surprising at first, in that the order parameters are now the average of gauge-links along only one-dimensional chains lying in the gauge-fixing direction. Spontaneous symmetry breaking is known not to occur in one-dimensional systems at non-zero temperature. However, these chains interact with other chains through the gauge couplings. The action is still N-dimensional, so the arguments preventing 1-d symmetry breaking are not valid. There are now $N^{d-1}$ separate symmetries breaking, resulting in an even much greater degree of vacuum degeneracy. If one’s intent is to find a gauge-fixed system that has the same behavior as the unfixed system, then it is important not to fix beyond the level one needs to invalidate Elitzur’s theorem to allow for symmetry breaking. In other words, one should not fix symmetries that are already spontaneously broken, because that will introduce different counting weights on the two sides of the transition than exist in the unfixed case. Since Elitzur’s theorem requires an infinite number of fields to be involved in a symmetry transformation for it to be able to break spontaneously, it would appear we have essentially reached this limit with the infinite chain. One could perhaps leave out one more gauge-fixing on a single link in each chain leaving two semi-infinite chains, but probably this is too small an addition to the vacuum degeneracy to affect anything in the infinite volume limit. Therefore the results in this minimal axial gauge involving only links in a single direction should be comparable to those of the unfixed case. The reason for fixing the gauge at all is to allow the transition to be visible through the definition of the abovementioned order parameters. The same transitions are in-principle observable in the unfixed case by using non-local order parameters, as will be detailed below. It would appear, therefore that our previous unmasking of the layered gauge symmetry breaking did not go far enough. The number of broken symmetries is much larger than $N$-fold — it is $N^{d-1}$ fold. The picture that emerges for the weak-coupling phase is one of intersecting bundles of spin chains locked into a high-degeneracy broken-symmetry pattern.

2 Monte Carlo simulations

Simulations were performed in both the 3-d Z2 pure lattice gauge theory (i.e. with no matter fields) and also the SU(2) pure-gauge theory in four dimensions on symmetric lattices with periodic boundary conditions. The abovementioned minimal axial gauge was implemented by setting all links in the 1-direction equal to unity except forward-pointing links attached to sites with $x_1 = 0$. The remaining links in the 1-direction cannot be set to unity by a gauge transformation if one is using periodic boundary conditions. These remaining 1-links are gauge-covariant and are equal to the Polyakov loops in
that direction. Normally, one completes the gauge fixing by building a tree of fixed links in other directions in the $x_1 = 0$ (hyper)surface. However, we leave these other links unfixed. This leaves a large residual gauge group unfixed. Any gauge transformation which is independent of $x_1$ is still allowed. Thus the residual symmetry is global in the 1-direction but still local in all of the other directions.

### 2.1 Z2 in three dimensions

For the 3-d Z2 lattice gauge theory, Figs. 1,2 shows the histograms of gauge links lying in other directions averaged along chains in the $x_1$ direction. Each lattice supplies $(d - 1) * N^{(d-1)}$ independent samples for this distribution, leading to high statistics in a relatively small sample. Each separate residual gauge transformation that affects a chain of sites in the $x_1$ direction will transform $2(d - 1)$ attached link-chains. Therefore if the chain magnetizations acquire expectation values then the attached gauge symmetries will be spontaneously broken. Monte Carlo simulations were run for 20,000 sweeps, after 10,000 equilibration sweeps, with measurements taken every 50 sweeps. The Z2 theory has a deconfinement phase transition at $\beta = 0.7613$ on the infinite lattice. This is known because it is dual to the three-dimensional Ising model[6], and has been confirmed with Monte-Carlo simulations[7]. A clear symmetry-breaking is shown in the histograms, with distributions peaked at zero on the confining side of the transition and away from zero in the deconfined phase. In the intermediate region three-peaked histograms with one peak at zero and two symmetrically away from zero give an apparent indication of a first-order phase transition. The modest deepening of valleys when one moves from the $32^3$ to $44^3$ lattice is also indicative of a first-order transition. This seems puzzling, because duality with the 3-d Ising model and previous Monte-carlo simulations would indicate a second-order transition. However, the duality transformation actually relates the 3-d Ising model to the Ising gauge theory in a fully-fixed maximal-tree axial gauge. As seen above, if parts of the gauge symmetry itself break spontaneously, then one actually has good reason to expect different critical behavior in the fully-fixed gauge than in the minimal axial gauge being explored here. In a similar vein, the Higgs-confinement transition, seen in the sub-maximal axial gauge in Ref. [4], is completely absent in the unitary gauge employed in the Fradkin-Shenkar proof of phase continuity between Higgs and confinement regions. Thus phase transitions of this sort do depend on the gauge being used, so different orders in different gauges are not out of the question. Further study, such as a full finite-size scaling analysis will be needed to verify the order of the transition. The primary purpose here is simply to demonstrate the existence of the symmetry breaking coincident with the deconfinement transition and the usefulness of these order param-
eters. Indeed, because of the dual connection to the Ising model, this must be a symmetry-breaking phase transition. However, no broken symmetry or symmetry breaking order parameter had been previously identified for the gauge theory. Therefore it is not surprising to find a hidden symmetry breaking here.

This transition was briefly studied in the sub-maximal axial gauge previously. A transition was verified, but the simulations were plagued with equilibration difficulties. Even after several hundred thousand sweeps there were discrepancies between hot and cold starts when well into the symmetry-broken region. In the present minimal axial gauge no problems with equilibration have been seen. This is probably because now only \(2(d - 1)N\) links are involved with each symmetry transformation, as opposed to \(2(d - 1)N^3\). Needless to say, it is much easier to flip the former than the latter number of links. The histograms for cold-start simulations in the broken phase with only 10,000 equilibration sweeps are already nearly left-right symmetrical, showing the ease of tunneling between vacua (the cold start had all links initially set to unity).

### 2.2 SU(2) in four dimensions

The 4-d SU(2) theory will first be considered in the first-order region of the fundamental-adjoint plane, because the position of the bulk transition is well defined there through the jump in the plaquette. Whether or not this bulk transition is symmetry-breaking is a very important question. Indirect evidence from the scaling of the size of the metastability region with the latent heat strongly suggests it is symmetry breaking\(^8\). The Landau theory makes a clear distinction between first-order transitions which are symmetry-breaking and those which are not, leading to a prediction of quadratic scaling in one case and linear in the other. The data clearly support the case of a symmetry-breaking transition\(^8\). If the transition is symmetry-breaking, then the weak coupling side, which includes the continuum limit, has a different symmetry than the strong coupling (confining) side. For an exact symmetry, these must everywhere be separated by a phase transition. The first-order endpoint in the fundamental-adjoint plane would necessarily be a tricritical point rather than an ordinary liquid-gas type critical point. This means it would not be possible to find an analytic path around this point to connect the confining phase to the continuum limit, as these would be in different symmetry regions.

For this theory the SU(2) links in a given direction are averaged along chains, again lying in the 1-direction which is the gauge-fixing direction. This average link is, of course, no longer a normalized SU(2) gauge element. It is reinterpreted as an average spin in a 4-d space, with the components being the coefficients of the unit matrix and three Pauli matrices. Indi-
individual SU(2) gauge elements are unit vectors in this space. For this O(4) order parameter one must take into account the geometrical factor (from solid angle) that biases the distribution toward larger magnitudes. In the unbroken phase, the distribution of magnetization moduli, $m$, is expected to be a factor of $m^3$ times a Gaussian, $\exp(-m^2/2\sigma^2_m)$. To more easily see the Gaussian behavior, the probability distribution $P(m)$ is obtained by histogramming, and the quantity $P(m)/m^3$ is plotted. Figs. 44 show the these distributions just below the strong first-order transition, which occurs for $\beta_{\text{adj}} = 1.5$ at $\beta = 1.04 \pm 0.02$, and just above it. Gauge links are first averaged along each chain in the gauge-fixing direction, and then the modulus is taken. The value of $m$ for each bin is not taken at the center, but at a value that would produce a flat histogram in an $m^3$ distribution, regardless of bin-size choice. This is

$$m_{\text{bin}}^3 = \frac{1}{4} \frac{(m_2^4 - m_1^4)}{(m_2 - m_1)}.$$  

where $m_2$ and $m_1$ are the bin edges. This detail affects only the first couple of bins in the histograms shown. At this value of the adjoint coupling the average plaquette jumps about 0.27 at the transition. One can see from the histograms that a definite symmetry breaking takes place. The widening of distributions for the $20^4$ lattice over the $16^4$ seems atypical for a first-order transition, however. This will be discussed below.

Fig. 5 shows the time history of a quench. For a lattice equilibrated at $\beta = 1.15$, $\beta_{\text{adj}} = 1.5$, $\beta$ was suddenly changed to 1.0 at the beginning of Monte-Carlo time shown in the figure. One can see that the plaquette, link magnetization, and Polyakov loop all appear to tunnel coincidently. This, together with the energy scaling evidence given in 8 strongly supports the idea that this is a single integrated bulk transition which is both symmetry-breaking and deconfining. Previously there had been speculation that a finite-temperature deconfining transition could be lying on top of an unrelated bulk transition. An important point to be made here is that the new order parameter (link magnetization in chains) is the average of a local quantity and thus a bulk order parameter. This contrasts with the Polyakov loop which is a global order parameter that exists only for periodic boundary conditions. In addition, when the link magnetization order parameters acquire non-zero expectation values both the residual gauge symmetry and the Polyakov loop symmetry are spontaneously broken. The Polyakov loop symmetry, which multiplies all links in a particular direction in a single perpendicular hyperlayer by -1, also flips the magnetization of all spin-chains made of that-direction links in that hyperlayer. Once the Polyakov loop symmetry is broken, there is nothing to protect the Polyakov loops from gaining an expectation value, which they are seen to do. Thus the residual gauge symmetry breaking naturally carries with it the Polyakov loop...
symmetry breaking and thus deconfinement.

Moving on to the Wilson axis, the histograms in Figs. 6,7 also show a symmetric phase in the confining region, and a symmetry-broken phase at weak coupling. In the intermediate regions a flat-topped distribution is seen, suggestive of a higher-order transition in this case. The connected susceptibilities for the link magnetizations averaged on chains are shown in Fig. 8. Broad peaks are seen which are definitely growing with lattice size. It is not surprising that the transition is broad, because the ”system size” for each individual order parameter is only $N$, i.e. only $N$ links are being averaged as opposed to $N^4$ for an ordinary global symmetry breaking.

The deconfinement transition in SU(2) is normally considered to be a finite-temperature transition, one which occurs on lattices finite in at least one direction, and which disappears in the symmetric infinite-volume limit. However, the transition observed here seems more likely to be a bulk transition, because the order parameter is the average of a local density, and symmetries being broken exist for any boundary conditions. The Polyakov loop, usually invoked as the finite-temperature deconfinement order parameter, and the finite temperature interpretation itself exist only for periodic boundary conditions. To test this further, simulations were run with open boundary conditions in all directions. Sample results are shown in Fig. 9. Chains were included in the measurements only if they were more than two lattice spacings away from any boundary and the chains themselves were terminated two spacings before the boundary. The symmetry breaking here appears similar to the case of periodic boundary conditions. This strongly supports the idea of a bulk transition, one which will continue to exist on the infinite lattice. It is true that the apparent critical point does shift an unusual amount with lattice size, which is a primary motivation for the finite-temperature interpretation of the phase transition. This, however, could be due in part to the small effective size of the system. It seems possible the shift observed can simply be interpreted as the ordinary finite-size shift of the bulk critical point. The existence of an order parameter with $\chi$ peaks allows for a finite-size scaling analysis which should shed light on the critical exponents, and may allow a determination of the infinite lattice critical point. This will probably require several much-larger lattices and higher statistics for definitive results. Binder fourth-order cumulant crossings would cement the existence of a finite order transition. For the O(4) order parameter, the Binder cumulant, defined here as

$$U = 1 - \frac{<m^4>}{(3<m^2>^2)},$$

varies from 1/2 in the full unbroken phase to 2/3 in the fully broken limit. Data taken to date show Binder cumulants for the different lattice sizes merging at weak coupling rather than crossing (Fig. 10), similar to that
seen for the sub-maximal axial gauge. Merging would be the expected behavior for a Berezinskii-Kosterlitz-Thouless (BKT) transition. Three other features also favor a BKT transition. One is the behavior of the specific heat. The SU(2) theory shows a large peak around $\beta = 2.2$ which does not vary significantly with lattice size. The BKT transition in the 2-d XY model has a specific heat curve that looks very similar to this and is also independent of lattice size. The actual infinite order singularity is very soft and not visible in numerical specific heat data. It lies at a weaker coupling, near where the specific heat begins to rise. Eventually a large peak, not much dependent on lattice size, grows far inside the strong coupling phase at the point where vortex unbinding finally dominates. BKT transitions also exhibit an unusually large finite-lattice shift in apparent critical point, because the shift is logarithmic rather than a power law. Finally, histograms for BKT transitions near the critical point in the broken phase tend to show broad highly-asymmetric distributions that extend all the way to zero, not unlike those seen here, and also for the high-adjoint coupling case above. Even though the transition is first order there, the weak coupling phase would have to be the same as seen on the Wilson axis, with similar properties.

3 Continuum Limit

If the confining and weak-coupling phases are separated by a symmetry-breaking bulk phase transition, then the continuum theory, which is the weak coupling limit, will not lie in the confining phase. It will be in a Coulomb-like phase. This is contrary to usual expectations for the non-abelian case, where the confining phase is generally assumed to extend all the way to $\beta \to \infty$ on the infinite lattice. The situation proposed here would instead be comparable to the abelian case where confinement is strictly a strong-coupling lattice artifact, separated from the Coulomb phase of the continuum theory by a phase transition. Although in this case the continuum pure gauge non-abelian theory would not be confining, when light quarks are added, one could still have chiral symmetry breaking (CSB). Physical confinement could result as a byproduct of the CSB, or through the closely related Gribov scenario.

The existence of a bulk transition at finite $\beta$ could be proven analytically if the residual gauge symmetry in the minimal axial gauge could be shown to be broken in some small region around $\beta = \infty$ (i.e. a finite region in $1/\beta$ around zero). This is because it is clearly unbroken in the strong coupling limit ($\beta \to 0$), due to the completely random nature of configurations there, and the lack of any energy barriers to tunneling whatsoever. The strong coupling expansion preserves this symmetry and it undoubtedly persists
throughout the confining phase. One can see that the symmetry is broken at $\beta = \infty$ from the following argument. With the 1-direction links set to unity, the 1-2, 1-3, and 1-4 plaquettes all behave as spin couplings, i.e. if the gauge links are written as

$$s_0 + i \sum_{j=1}^{3} s_j \tau_j,$$  \hspace{1cm} (3)

the 1-$k$ plaquette can be written as

$$\sum_{j=0}^{3} s_{j,k,x_{j,k,x+1}},$$  \hspace{1cm} (4)

i.e. just the color dot-product of the O(4) spins. Here the first index is a color index the second the link direction (2 to 4) and the third the lattice site label written as a spacetime vector. The symbol $\hat{1}$ represents a unit lattice vector in the 1-direction. This is identical to the interaction of an O(4) spin model. Indeed, the gauge theory can be thought of as a set of O(4) spin chains which are linked together through “sideways” gauge couplings carried by the remaining 2-3, 2-4 and 3-4 plaquettes. At $\beta = \infty$ all links along a given chain must be perfectly aligned. So the chains, even if considered in isolation, spontaneously break the symmetry. (In other words, even one-dimensional spin chains are in the ordered phase at exactly $T = 1/\beta = 0$). The interlinking gauge interactions cause further ordering, so the symmetry must remain broken in the full theory at infinite $\beta$. Therefore a symmetry breaking phase transition must exist - the question is whether it exists at finite coupling or at $\beta = \infty$ itself as in the 1-d spin chain. In the 1-d case, the chain disorders at any non-zero temperature because half of the chain can be flipped costing energy only locally at the position of the flip; but in the current theory flipping a half-chain frustrates sideways plaquettes not only in the flipping region, but all along the semi-infinite flipped half-chain, with an infinite energy penalty. This strongly suggests a behavior more like the higher-dimensional spin-theories, with transitions at finite couplings.

4 Is gauge fixing necessary?

It is interesting to consider whether any signal of the above phase transition can be seen in the completely unfixed theory, where Elitzur's theorem prevents the local gauge symmetries from breaking. One can construct gauge covariant links by multiplying both ends of any given link by a chain of links in the 1-direction, parallel-transporting it back to the $x_1 = 0$ hypersurface. One can then average these covariant links lying along chains in the 1-direction. These objects, when interpreted as O(4) vectors, are identical
to the above link magnetizations defined in the minimal axial gauge. In other words, if an unfixed lattice is transformed to the minimal axial gauge, the resulting link magnetizations will be the same as that computed from the covariant links. This is because the gauge transformation can be accomplished entirely by gauge transformations at sites away from \( x_1 = 0 \). The remaining gauge transformations on \( x_1 = 0 \) are precisely those left unfixed in this gauge. The covariant links, however are constructed to be invariant to gauge transformations away from \( x_1 = 0 \) and covariant to those on \( x_1 = 0 \).

The \( O(4) \) moduli of these chain-averaged covariant links are, in fact, gauge invariant. The signals of spontaneous breaking used previously are all determined from the distributions of these gauge-invariant moduli (since tunneling on a finite lattice prevents actual observation of spontaneous symmetry breaking through symmetry-breaking vacuum expectation values anyway). Therefore it would appear that these phase transitions could be observed, in principle, using gauge invariant objects in the unfixed theory. From a practical point of view, however, it would be expensive to compute all of the gauge-covariant links. The minimal axial gauge simulation would be much faster.

There is one difference between the unfixed and minimal axial gauge simulations. The symmetry the covariant links are sensitive to in the unfixed simulation is a local gauge symmetry at \( x_1 = 0 \). In the minimal axial gauge it is a global symmetry along each chain (independent of \( x_1 \)) and local between chains. Elitzur’s theorem prohibits the symmetry from breaking in the former, but not the latter. Indeed, the direction of each chain-magnetization will drift in the unfixed simulation due to local gauge drift at \( x_1 = 0 \). If these configurations are transformed to minimal axial gauge, different configurations will map into different vacuum sectors of the gauge-fixed theory. Thus the unfixed simulation is akin to a gauge-fixed simulation that also includes a random global/local residual gauge transformation after each sweep. This will jump the simulation around between different vacuum sectors. Because each vacuum sector has identical behavior except for the magnetization direction itself, all moments of the magnetization magnitude from which the critical behavior is derived are still identical. Only the vacuum expectation value of the magnetization itself is erased. This should not be viewed as true tunneling. One could artificially add global flips to an Ising model simulation, which would also erase the expectation value of the order parameter, but nothing else about the phase transition would change. True tunneling involves intermediate lattice configurations which are partially in one vacuum sector and partially in another. If such lattices are energetically allowed, then tunneling can take place. It is the lack of such intermediate lattice configurations that is the true restriction on ergodicity which takes place at a phase transition. Another way of looking at this is that for the
truly infinite lattice, just a single configuration should be sufficient to determine any local or partially local quantity to arbitrary precision through spatial averaging. Ensemble averaging is, in a sense, redundant on the infinite lattice. Any local condition that can exist (in the given vacuum sector) will exist somewhere in space in each configuration. However, for a single infinite configuration, both the gauge-fixed and unfixed theories will exhibit vacuum expectation values of the order parameters. This demonstrates that the gauge-drift in the former is more akin to the artificial tunneling described above than to real tunneling. Consideration of a single infinite configuration and using spatial averaging as opposed to an ensemble average allows one to evade Elitzur’s theorem and observe full symmetry breaking through nonlocal operators in the unfixed theory.

This work shows that over-fixing the gauge beyond the minimal axial gauge is dangerous in the neighborhood of the phase transition. A related question is whether it is equally dangerous within the weak-coupling phase itself. A gauge over-fixing that simply chooses a particular vacuum to work in is not dangerous, if, once in that vacuum, natural fluctuations would not, even in the absence of the extra gauge-fixing, violate it. However some gauge-fixings will be seen by the system as explicit symmetry breaking on top of spontaneous symmetry breaking, which will change observables. For instance, if one imposed the additional constraint that the averages of links along all 1-d spin chains in the gauge-fixing direction lie in the unit-matrix direction (with zero components along the three Pauli-matrix color-directions), that would simply be choosing a vacuum. Natural fluctuations on the infinite lattice would not be able to violate this condition, so the lattice would be “unaware” an extra condition was being imposed. However, if the same residual gauge freedom were used to set single links on particular chains equal to unity (as in the usual maximal-tree axial gauge), then a particular vacuum is not chosen, because all vacua have configurations in which those particular links are unity and others in which they are not. Spontaneous symmetry breaking will still select a vacuum, however, and the new fixed-links will act as explicit symmetry breakings, affecting the natural fluctuations in that vacuum. The fixed link could, for instance, affect Goldstone fluctuations in its vicinity. Thus, only additional gauge fixings which are compatible with the symmetry breaking pattern by being functions of the order parameters are allowed. Others will act as explicit symmetry breakings which could affect the spectrum, such as by giving mass to Goldstone modes. For the infinite lattice, extra gauge fixings on the boundary probably do not affect the bulk properties, but finite volume or finite temperature formulations could be affected. The implications for other popular gauges such as Landau and Coulomb are not immediately clear, and are worth investigating.
5 Conclusion

The concept of symmetry breaking in gauge theories has been a confusing one. On one hand, Elitzur’s theorem prohibits spontaneous symmetry breaking of local symmetries. On the other, the standard model relies on spontaneous breaking of the gauge symmetry to initiate the Higgs mechanism. One way to reconcile these is with partial gauge fixing. If one fixes the gauge enough to make it global in at least one direction, then Elitzur’s theorem no longer applies, allowing the remaining residual symmetries to break spontaneously, if this is energetically favored. This is apparently the case in continuum QED, where it is even possible to interpret the photon as a Goldstone boson. In this paper, the possible breaking of such residual symmetries was explored in pure lattice gauge theories. It was found that the residual gauge symmetries are spontaneously broken at weak coupling in both the 3-d $\mathbb{Z}_2$ and 4-d $\text{SU}(2)$ theories, and that these are separated from the strong-coupling confining region by a phase transition. This may be true for most if not all gauge theories in the minimal axial gauge, where the symmetry is fixed in only one direction, leaving it still local in the others. Since it is presumably dangerous to add explicit symmetry breaking on top of spontaneous symmetry breaking, gauges more restrictive than this could introduce unphysical effects. Thus the partial spontaneous breaking of the gauge symmetry itself appears to violate the concept of gauge-equivalence. This requires further investigation.

On the lattice, gauge theories have a resemblance to magnetic spin models. In the axial gauge, this resemblance is strengthened. In two dimensions gauge and spin theories are equivalent, and in three they are sometimes related to each other by duality transformations. In four dimensions half of the plaquette interactions become spin interactions in the axial gauge, leading to a picture of the gauge theory being a system of interacting one-dimensional spin chains. The critical behavior in the minimal axial gauge is especially interesting because it has some features of a four-dimensional system but some of a one-dimensional system. This is due to the order parameters for each of the many broken symmetries being averaged only over each associated 1-d spin chain. When viewed as magnetic systems, it is not surprising to find the spins to be magnetized at weak coupling (low effective 4-d temperature). The disparate behaviors exhibited by different gauge theories may be related to the mode of symmetry breaking, either Nambu-Goldstone, Higgs, or BKT. Clearly, there are many aspects of this phenomenon to be explored.
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Figure 1: Distributions of average link magnetizations on 1-d chains for the 3-d Z2 lattice gauge theory on a $32^3$ lattice for couplings $\beta = (a) 0.69$, (b) 0.73, and (c) 0.76. The infinite-lattice critical point is expected to lie at 0.7613.
Figure 2: Same as Fig. 1 except on a $44^3$ lattice at couplings $\beta = (a) 0.69$, (b) 0.74, and (c) 0.76.
Figure 3: Distributions of the modulus of the average link magnetization on 1-d chains for the 4-d SU(2) lattice gauge theory in the fundamental-adjoint plane with $\beta_{\text{adj}} = 1.5$ on a $16^4$ lattice. Fundamental couplings are $\beta = (a) 1.0$, (b) 1.1, and (c) 1.5. The known first-order transition lies near $\beta = 1.04$. 
Figure 4: Same as Figure 3 except on a $20^4$ lattice.
Figure 5: Time history of a quench starting in the ordered phase of the SU(2) theory near the first-order line in the fundamental-adjoint plane. The evolution of plaquette (upper thin line), average link magnetization (middle thicker line), and average Polyakov loops (lower two lines – right scale) are shown. Only two of the four Polyakov loops are shown for clarity. The other two behave similarly.
Figure 6: Distributions of the modulus of average link magnetization on 1-d chains for the 4-d SU(2) theory on the Wilson axis at (a) $\beta = 2.4$, (b) $\beta = 2.8$, (c) $\beta = 3.2$ and (d) $\beta = 4.0$ on a $16^4$ lattice.
Figure 7: Same as Fig. \ref{fig:6} except on a $20^4$ lattice.
Figure 8: Susceptibility of the link magnetization on 1-d chains for 4-d SU(2) on the Wilson axis for $16^4$ (diamonds), $20^4$ (squares), and $24^4$ (triangles) lattices. Error bars are about one-fifth the size of plotted points.
Figure 9: SU(2) Wilson axis runs with open boundary conditions in all directions on a $16^4$ lattice. (a) $\beta = 2.2$ and (b) $\beta = 3.2$. Open boundary condition runs were also performed in the fundamental-adjoint plane (not shown). These also produced results similar to those obtained with periodic boundary conditions. In general variances are larger than with periodic boundary conditions.
Figure 10: Fourth-order cumulant for link magnetization for the SU(2) theory on the Wilson axis for 16^4 (diamonds), 20^4 (squares), and 24^4 (triangles) lattices. Error bars are less than one-tenth the size of plotted points.