Estimating parameter of Rayleigh distribution by using Maximum Likelihood method and Bayes method

Fitri Ardianti¹ and Sutarman²
Department of Mathematics, University of Sumatera Utara, Medan, Indonesia
E-mail: ¹fitriardianti96@gmail.com; ²sutarman@usu.ac.id

Abstract. In this paper, we use Maximum Likelihood estimation and Bayes method under some risk function to estimate parameter of Rayleigh distribution to know the best method. The prior knowledge which used in Bayes method is Jeffrey’s non-informative prior. Maximum likelihood estimation and Bayes method under precautionary loss function, entropy loss function, loss function-L₁ will be compared. We compare these methods by bias and MSE value using R program. After that, the result will be displayed in tables to facilitate the comparisons.

1. Introduction
Rayleigh distribution is a continuous probability distribution family which is used as a model of life time. This distribution has long been considered to have significant applications in many fields such as survival analysis, reliability theory, and especially communication engineering. This distribution is a special case of two parameter Weibull distribution with the shape parameter equal to 2. The Rayleigh distribution was originally derived by Lord Rayleigh [1].

Siddiqui [2] showed that the Rayleigh amplitude distribution (the distribution of the power or amplitude of electronic waves received through a scattering medium) is the asymptotic distribution of a two-dimensional random walk. Several authors have contributed to this model. Howlader and Hossain [3] obtained Bayes estimators for the scale parameter and the reliability function in the case of Type-II censored sampling. Abd Elfattah, et al. [4] studied the efficiency of the maximum likelihood estimates of the parameter under three cases, namely, Type-I, Type-II and progressive Type-II censored sampling schemes. Hendi, et al. [5] obtained Bayes estimators of the scale parameter, reliability function and failure rate by using non-informative prior anda Hartigan prior based on upper record values. Dey and Das [6] obtained Bayesian predictive intervals of the parameter of Rayleigh distribution. Dey [7] also obtained Bayes estimators for the parameter and reliability function of the Rayleigh distribution under different loss function. Dey [7] also studied the Bayes estimators for the parameter and reliability function of Rayleigh distribution based on complete as well as Type-II censored samples, also compared relative risk functions.

Soliman [8], Mostert [9] compared the Bayesian estimators under the linear exponential (LINEX) loss function and squared-error loss (SEL) function. Dey and Dey [10] did similar work for the
complete model by applying Jeffreys prior and a loss function as proposed by Al-Bayyati [11]. Ferreira, et al. [12] extend concepts in the literature for the censored Rayleigh model by considering new loss function, namely the Al-Bayyati loss (ABL) and comparing it to other known result.

Al-Mayali and Al-Shaibani [13] compare some estimators of Rayleigh distribution with simulation. The method which was used in their research were maximum likelihood estimation, Bayes estimation, shrinkage estimation, and Bayesian shrinkage estimation and after that applied the result in simulation study to compare the result, and obtain which method was the best [13]. Singh and Srivastava [14] use Bayesian estimation of Inverse Maxwell distribution under squared error, precautionary, entropy, and another two loss function for using quasi-prior have been obtained.

2. Rayleigh Distribution

The Rayleigh distribution has a wide range of applications including life testing experiments and clinical studies. This distribution has probability density function (pdf):

\[ f(x|\theta) = \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}}, x, \theta > 0 \]  

(1)

3. Maximum Likelihood Estimation

Let \( x_1, x_2, \ldots, x_n \) be random sample of the population with density \( f(x, \theta) \) where \( \theta(\theta_1, \theta_2, \ldots, \theta_k) \) are unknown parameters, the likelihood function is written:

\[ L(\theta_1, \theta_2, \ldots, \theta_3) = \prod_{i=1}^{n} f(x_i; \theta) \]  

(2)

The likelihood function is a function of an unknown parameter. In the application \( L(\theta) \) denotes the joint probability density function of the random sample. If \( S \) parameter space is an open interval and \( L(\theta) \) is a function that can be derived and assumed to be maximum at \( S \). Then the equation of maximum likelihood is:

\[ \frac{d}{d\theta} L(\theta) = 0 \]  

(3)

The parameters are estimated with solving the equation (3)

4. Bayes Method

Let \( f(y|\theta); \theta \in \Theta \) be the probability density function of lifetime distribution of a component or an animate, where the parameter space \( \Theta \) is known but the true value of \( \theta \) is unknown. Let \( g(\theta) \) be the prior density function of the random variable \( \theta \). Let \( y = (y_1, \ldots, y_n) \) be an independent observations from \( f(y; \theta) \). Then using Bayes’s theorem (1763) the posterior distribution \( f(\theta|y) \) of \( \theta \) is given by:

\[ f(\theta|y) = \frac{f(y|\theta)g(\theta)}{\int_{\theta} f(y|\theta)g(\theta) d\theta} \]  

(4)

where \( f(y|\theta) \) is the joint probability density function of \( y = (y_1, \ldots, y_n) \). For given sample \( y \), the posterior p.d.f \( f(\theta|y) \) is the basis for most types of Bayesian inference [14].
5. Risk Function

Let $L(\hat{\theta}, \theta)$ be the loss function associated with an estimate of the parameter $\theta$. Let $g_\theta(\theta|W = w)$ be the posterior distribution of the random variable $\Theta$. Then the risk associated with $\hat{\theta}$ is the expected value of the loss function with the respect to the posterior distribution of $\theta$ [14]. The risk function is:

$$risk = \int_\theta L(\hat{\theta}, \theta)g_\theta(\theta|W = w)d\theta$$

(5)

The fundamental problems in Bayesian analysis is that of the choice of prior distribution and loss function $L(\theta, \theta)$ which may be appropriate for the situation at hand. The Bayes estimation $\hat{\theta}$ of $\theta$ is the course optimal relative to the loss function chosen. A commonly used loss function is the squared error loss function (SELF)

$L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2$

(6)

The Bayes estimator under the above loss function, say $\hat{\theta}$ the posterior mean, i.e.

$\hat{\theta} = E(\theta)$

(7)

The risk function is given by:

$$R(\hat{\theta}) = E(\hat{\theta})^2 - 2\theta E(\hat{\theta}) + \theta^2$$

(8)

(b) Precautionary Loss Function

The posterior expectation of loss function is:

$$E[L(\hat{\theta} - \theta)] = E\left(\frac{\theta^2}{\theta}\right) + E(\hat{\theta}) - E(\theta)$$

(9)

The value of $\hat{\theta}$ that minimize (9), denoted by $\hat{\theta}_p$, is obtained by solving the following equation

$$\frac{d}{d\theta}E[L(\hat{\theta} - \theta)] = 0$$

(10)

$$\bar{\theta}_p = [E(\theta^2)]^{\frac{1}{2}}$$

(11)

(c) Entropy Loss Function

The Bayes estimator of entropy loss function is:

$$\bar{\theta}_e = [E(\frac{1}{\theta})]^{-1}$$

(12)

(d) Loss function- $L_1$

Consider the loss function given by:

$L_1(\hat{\theta}, \theta) = (\frac{\hat{\theta}}{\theta} - 1)^2$

(13)
The Bayes estimator of under loss function-$L_1$ is:

$$\hat{\theta}_1 = \frac{E(\theta)}{E(\theta)^2}$$  (14)

(e) Loss function-$L_2$

Consider the loss function given by:

$$L_2(\hat{\theta}, \theta) = \left(\frac{\theta}{\hat{\theta}} - 1\right)^2$$  (15)

The Bayes estimator of under loss function-$L_2$ is:

$$\bar{\theta}_2 = \frac{E(\theta^2)}{E(\theta)}$$  (16)

6. Result

(a) Estimating parameter by using Maximum Likelihood method

In this paper, we will estimate one parameter of Rayleigh distribution by using maximum likelihood estimation. The probability of density function (pdf) of this distribution is:

$$f(x|\sigma) = \frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}, x, \sigma > 0$$  (17)

and From the equation (14), we will find the maximum likelihood function:

$$L(x, \sigma) = \prod_{i=1}^{n}x_i\left(\frac{1}{\sigma^2}\right)^n\exp\left(-\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^2\right)$$  (18)

After that, we will log the equation (18) and the last differential partial of the equation, so we will find the estimator of rayleigh distribution by using maximum likelihood function. It will showed:

$$\ln[L(x, \sigma)] = \ln[\prod_{i=1}^{n}x_i\left(\frac{1}{\sigma^2}\right)^n\exp\left(-\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^2\right)]$$  (19)

$$\hat{\sigma}_{MLE} = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}x_i^2}$$  (20)

(b) Estimating parameter by using Bayes method

The first step to estimate the parameter by using bayes method is we have to find the prior distribution, because bayes method is a method which calculate the role of prior. In this paper, the prior distribution which used is non-informatif prior. The non-informative prior which is used in this paper is Jeffrey’s method. The non-infomatif prior of the rayleigh distribution is:

$$f(\sigma) = \sqrt{I(\sigma^2)} \propto \frac{1}{\sigma}$$  (21)

by the equation (21), we will find the posterior distribution:

$$f(\sigma|x) \propto \frac{c}{\sigma^{2n+1}}\prod_{i=1}^{n}x_i\exp\left(-\frac{1}{2n}\sum_{i=1}^{n}\left(\frac{x_i}{\sigma}\right)^2\right)$$  (22)
from the equation (22), we will find the posterior density function:

$$
\pi^*(\sigma|x_i) = \frac{\left(\sum_{i=1}^{n} x_i^2\right)^n exp\left[-\frac{1}{2} \sum_{i=1}^{n} \left(\frac{x_i}{\sigma}\right)^2\right]}{\sigma^{2n+1} 2^n \Gamma(n + \frac{3}{2})}.
$$

(23)

The Bayes estimator is an estimator that minimizes the risk function by representing the expected of loss function. There are some loss function that can be applied to the bayes metod such as squared error loss function, precautionary loss function, entropy loss function-L_1 and loss function-L_2 [14].

In this paper, the author use three loss functions to estimate rayleigh distribution in bayesian approach. Namely, precautionary loss function, entropy loss function and loss function-L_1.

i. Precautionary loss function

The Bayes estimator under precautionary loss function:

$$
\sigma_p = [E(\sigma^2)]^{\frac{1}{2}}
$$

(24)

$$
= \sqrt{\int_{0}^{\infty} \sigma^2 \pi* (\sigma|x_i) d\sigma}
$$

(25)

$$
= \sqrt{\frac{\Gamma(n + \frac{1}{2}) \sum_{i=1}^{n} x_i^2}{2 \Gamma(n + \frac{3}{2})}}
$$

(26)

ii. Entropy loss function

The Bayes estimator under Entropy loss function:

$$
\sigma_e = [E(\frac{1}{\sigma})]^{-1}
$$

(27)

$$
= \left[ \int_{0}^{\infty} \frac{1}{\sigma} \pi* (\sigma|x_i) d\sigma \right]^{-1}
$$

(28)

$$
= \frac{\Gamma(n + \frac{3}{2}) \sqrt{\sum_{i=1}^{n} x_i^2}}{\sqrt{2} \Gamma(n + 2)}
$$

(29)

iii. Loss function-L_1

The Bayes estimator under Loss function-L_1:

$$
\sigma_1 = \frac{E(\frac{1}{\sigma})}{E(\frac{1}{\sigma^2})}
$$

(30)

$$
= \sqrt{\frac{2 \Gamma(n + 2)}{2 \Gamma(n + \frac{5}{2})}} \left(\sum_{i=1}^{n} x_i^2\right)
$$

(31)

7. Simulation

Estimate by using Maximum likelihood and Bayes method will be compared using simulation. Data simulations were performed by generating various types of data conditions involving four different sample sizes n=10, n=25, n=50, and n=100. The data obtained were analyzed to estimate the parameters using Maximum Likelihood and Bayes methods. The next step is calculate the bias and MSE (Mean Square Error) values of both methods. The simulation in this paper is done with R program.
From Table 7.1 it can be seen that the bias value of both methods shows the smaller the bias value with the larger the sample size. The bias value of the Bayes method with loss function-$L_1$ shows smaller numbers than Maximum Likelihood method and Bayes method with precautionary loss function loss function, entropy loss function, and loss function-$L_1$.

For the MSE value, from Table 7.2 shows the smaller error value with the condition of the larger the sample size. The MSE value in the Maximum Likelihood method shows smaller numbers than the Bayes method with the loss function of precautionary loss function, entropy loss function, and loss function-$L_1$. From the data simulation, results in the table above shows that Bayes method is not always better in estimating the parameters compared to Maximum Likelihood method.

References

[1] Johnson N L, Kotz S, and Balakrishman S 1994 *Continuous Univariate Distributions*, 456
[2] Siddiqui M M (1962) *Some problems connected with Rayleigh distributions*, 60D, 167-174.
[3] Howlader H A and Hussain A 1995 *On Bayesian estimation and prediction from Rayleigh distribution based on type-II censored data*, 24(9), 2249-2259.
[4] Abd Elfattah A M, Hassan A S and Ziedan D M (2006) *Efficiency of Maximum Likelihood Estimators under Different Censored Sampling Schemes for Rayleigh Distribution*
[5] Hendi M L, Abu S E and Alraddadi A A 2007 *A Bayesian Analysis of Record Statistics from Rayleigh Model*, International Mathematical Forum, 2 (13), 619-631.
[6] Dey S and Das M K (2007) *A Note on Prediction Inverval for a Rayleigh Distribution: Bayesian Approach*, American Journal of Mathematical and Management Science, 1&2, 43-28.
[7] Dey S 2009 *Comparison of Bayes Estimators of the parameter and reliability function for Rayleigh distribution under different loss function*, Malaysian Journal of Mathematical Science, 3, 217-261.
[8] Soliman A A 2000 *Comparison of LINEX and quadratic Bayes estimators for the Rayleigh distribution*, Communications in Statistics - Theory and Methods, 29(1), 95-107.
[9] Mostert P J *A Bayesian method to analyze cancer lifetimes using Rayleigh models*, Unpublished PhD thesis, University of South Africa.
[10] Dey S and Dey T 2011 *Rayleigh distribution revisited via an extension of Jeffreys prior information and a new loss function*, Revstat, 9(3), 213-226.
[11] Al-Bayyati H N *Comparing methods of estimating Weibull failure models using simulation*, Unpublished PhD thesis, College of Administration and Economics, Baghdad University, Iraq.
[12] Ferreira J T, Bekker A, and Arashi M 2015 *Objective bayesian estimators for the right censored rayleigh distribution: evaluating The Al-Bayyati loss function*, Revstat, 14 433-454.
[13] Al-Mayali and Al-Shaibani 2013 *A Comparison for some of the estimators of rayleigh distribution with simulation*, Revstat, 11(4).
[14] Singh K L and Srivastava R S (2012) *Bayesian estimation of parameter of inverse maxwell distribution via size-biased sampling*, International Journal of Science and Research(IJSR), 3.358, 2319-7064.
| N   | σ   | $\hat{\sigma}_{MLE}$ | $\hat{\sigma}_{BSP}$ | $\hat{\sigma}_{BSe}$ | $\hat{\sigma}_{BS1}$ | Bias$_{MLE}$ | Bias$_{BSP}$ | Bias$_{BSe}$ | Bias$_{BS1}$ |
|-----|-----|-----------------------|-----------------------|-----------------------|-----------------------|--------------|--------------|--------------|--------------|
| 10  | 0.5 | 0.7496453             | 0.7315789             | 0.7066852             | 0.691493             | -0.2846832   | -0.2898723   | -0.2970224   | -0.3013859   |
|     | 1   | 0.7496453             | 0.7315789             | 0.7066852             | 0.691493             | -0.7846832   | -0.7898723   | -0.7970224   | -0.8013859   |
|     | 1.5 | 0.7496453             | 0.7315789             | 0.7066852             | 0.691493             | -1.2846832   | -1.2898723   | -1.2970224   | -1.3013859   |
|     | 2   | 0.7496453             | 0.7315789             | 0.7066852             | 0.691493             | -1.7846832   | -1.7898723   | -1.7970224   | -1.8013859   |
|     | 2.5 | 0.7496453             | 0.7315789             | 0.7066852             | 0.691493             | -2.2846832   | -2.2898723   | -2.2970224   | -2.3013859   |
| 25  | 0.5 | 0.6645003             | 0.6579534             | 0.6484712             | 0.6423827             | -0.3881442   | -0.3892462   | -0.3908424   | -0.3918672   |
|     | 1   | 0.6645003             | 0.6579534             | 0.6484712             | 0.6423827             | -0.8881442   | -0.8892462   | -0.8908424   | -0.8918672   |
|     | 1.5 | 0.6645003             | 0.6579534             | 0.6484712             | 0.6423827             | -1.3881442   | -1.3892462   | -1.3908424   | -1.3918672   |
|     | 2   | 0.6645003             | 0.6579534             | 0.6484712             | 0.6423827             | -1.8881442   | -1.8892462   | -1.8908424   | -1.8918672   |
|     | 2.5 | 0.6645003             | 0.6579534             | 0.6484712             | 0.6423827             | -2.3881442   | -2.3892462   | -2.3908424   | -2.3918672   |
| 50  | 0.5 | 0.7435811             | 0.7398909             | 0.7344527             | 0.7308961             | -0.4095031   | -0.4099522   | -0.4106141   | -0.4110469   |
|     | 1   | 0.7435811             | 0.7398909             | 0.7344527             | 0.7308961             | -0.9095031   | -0.9099522   | -0.9106141   | -0.9110469   |
|     | 1.5 | 0.7435811             | 0.7398909             | 0.7344527             | 0.7308961             | -1.4095031   | -1.4099522   | -1.4106141   | -1.411047    |
|     | 2   | 0.7435811             | 0.7398909             | 0.7344527             | 0.7308961             | -1.9095031   | -1.9099522   | -1.9106141   | -1.911047    |
|     | 2.5 | 0.7435811             | 0.7398909             | 0.7344527             | 0.7308961             | -2.4095031   | -2.4099522   | -2.4106141   | -2.411047    |
| 100 | 0.5 | 0.7096232             | 0.7078558             | 0.7052282             | 0.7034933             | -0.4382613   | -0.438415    | -0.4386436   | -0.4387946   |
|     | 1   | 0.7096232             | 0.7078558             | 0.7052282             | 0.7034933             | -0.9382613   | -0.938415    | -0.9386436   | -0.9387946   |
|     | 1.5 | 0.7096232             | 0.7078558             | 0.7052282             | 0.7034933             | -1.4382613   | -1.438415    | -1.4386436   | -1.4387946   |
|     | 2   | 0.7096232             | 0.7078558             | 0.7052282             | 0.7034933             | -1.938261    | -1.938415    | -1.9386436   | -1.9387946   |
|     | 2.5 | 0.7096232             | 0.7078558             | 0.7052282             | 0.7034933             | -2.438261    | -2.438415    | -2.4386436   | -2.4387946   |
| N  | σ  | $\hat{\sigma}_{MLE}$ | $\hat{\sigma}_{BSp}$ | $\hat{\sigma}_{BS_e}$ | $\hat{\sigma}_{BSI}$ | $MSE_{MLE}$ | $MSE_{BSp}$ | $MSE_{BS_e}$ | $MSE_{BSI}$ |
|----|----|---------------------|---------------------|---------------------|---------------------|------------|------------|------------|------------|
| 10 | 0.5 | 0.7496453          | 0.7315789          | 0.7066852          | 0.691493           | 0.1299173  | 0.1317209  | 0.1342944  | 0.1359151  |
|    | 1   | 0.7496453          | 0.7315789          | 0.7066852          | 0.691493           | 0.646005   | 0.6715932  | 0.6813168  | 0.687301   |
|    | 1.5 | 0.7496453          | 0.7315789          | 0.7066852          | 0.691493           | 1.699283   | 1.711465   | 1.728338   | 1.738687   |
|    | 2   | 0.7496453          | 0.7315789          | 0.7066852          | 0.691493           | 3.233966   | 3.251337   | 3.27536    | 3.290073   |
|    | 2.5 | 0.7496453          | 0.7315789          | 0.7066852          | 0.691493           | 5.268649   | 5.291209   | 5.322382   | 5.341459   |
| 25 | 0.5 | 0.6645003          | 0.6579534          | 0.6484712          | 0.6423827          | 0.1885944  | 0.1890773  | 0.1897811  | 0.1902356  |
|    | 1   | 0.6645003          | 0.6579534          | 0.6484712          | 0.6423827          | 0.8267386  | 0.8283235  | 0.8306235  | 0.8321028  |
|    | 1.5 | 0.6645003          | 0.6579534          | 0.6484712          | 0.6423827          | 1.964882   | 1.967569   | 1.971465   | 1.973969   |
|    | 2   | 0.6645003          | 0.6579534          | 0.6484712          | 0.6423827          | 3.603026   | 3.606815   | 3.612307   | 3.615836   |
|    | 2.5 | 0.6645003          | 0.6579534          | 0.6484712          | 0.6423827          | 5.74117    | 5.746061   | 5.753149   | 5.757703   |
| 50 | 0.5 | 0.7435811          | 0.7398909          | 0.7344527          | 0.7308961          | 0.196964   | 0.1971867  | 0.1975158  | 0.1977314  |
|    | 1   | 0.7435811          | 0.7398909          | 0.7344527          | 0.7308961          | 0.856471   | 0.8571389  | 0.8581299  | 0.8587783  |
|    | 1.5 | 0.7435811          | 0.7398909          | 0.7344527          | 0.7308961          | 2.01597    | 2.017091   | 2.018744   | 2.019825   |
|    | 2   | 0.7435811          | 0.7398909          | 0.7344527          | 0.7308961          | 3.675473   | 3.677043   | 3.679358   | 3.680872   |
|    | 2.5 | 0.7435811          | 0.7398909          | 0.7344527          | 0.7308961          | 5.834976   | 5.836995   | 5.839972   | 5.841919   |
| 100| 0.5 | 0.7096232          | 0.7078558          | 0.7052282          | 0.7034933          | 0.2095575  | 0.2096487  | 0.2097844  | 0.2098742  |
|    | 1   | 0.7096232          | 0.7078558          | 0.7052282          | 0.7034933          | 0.8978188  | 0.8980637  | 0.898428   | 0.8986688  |
|    | 1.5 | 0.7096232          | 0.7078558          | 0.7052282          | 0.7034933          | 2.086079   | 2.086479   | 2.087073   | 2.087465   |
|    | 2   | 0.7096232          | 0.7078558          | 0.7052282          | 0.7034933          | 3.77434    | 3.774894   | 3.775717   | 3.77626    |
|    | 2.5 | 0.7096232          | 0.7078558          | 0.7052282          | 0.7034933          | 5.962601   | 5.963309   | 5.964361   | 5.965055   |