Extending quantum control of time-independent systems to time-dependent systems

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We establish that if a scheme can control a time-independent system arbitrarily coupled to a generic finite bath over a short period of time $T$ with control precision $O(T^{N+1})$, it can also realize the control with the same order of precision on smoothly time-dependent systems. This result extends the validity of various universal dynamical control schemes to arbitrary analytically time-dependent systems.

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I. INTRODUCTION

Quantum systems interact with their environments (or baths). This results in errors in controlling evolution of a quantum system, such as decoherence and unwanted dynamics. Inspired by phase-refocusing techniques in magnetic resonance spectroscopy [1,2], various schemes of quantum dynamical control [3,22] have been developed in the context of quantum information processing to average out undesired coupling through fast open-loop modulation on the system evolution. These dynamical control schemes have advantages of correcting errors without measurement, feedback, or redundant encoding [6]. The simplest one is dynamical decoupling of correcting errors without measurement, feedback, or redundant encoding [6]. The simplest one is dynamical decoupling (DD) [3,15], which aims at preservation of system coherence (i.e., quantum memory or NULL quantum control) by achieving a trivial identity evolution. Recent experiments [23–28] have demonstrated the performance of DD. More general aims are to implement non-trivial quantum evolutions [16–22]. Arbitrarily accurate dynamical control can be achieved using a concatenated design [19].

A quantum dynamical control is called universal if it has errors up to an order in short evolution time $T$ for an arbitrary finite bath. Most universal schemes [6,14,15,19] are designed for time-independent systems and their applicability to time-dependent systems is unclear, except an explicit extension [29] of Uhrig dynamical decoupling (UDD) [10,30] to analytically time-dependent systems.

In this paper, we prove that if a dynamical control has errors up to an order in short evolution time $T$ for a generic time-independent system, it will automatically achieve the same order of precision for analytically time-dependent systems. The theorem establishes the validity of universal DD in Refs. [6,14], optimized pulses in Ref. [17], and dynamical quantum error correction [18,19] on non-equilibrium baths. In addition, it greatly simplifies designing new universal dynamical control schemes since we just need to work with time-independent models.

II. CONTROL OF TIME-DEPENDENT SYSTEMS

A. Universal control of time-independent systems

Let us first consider a target system coupled to a bath through a time-independent Hamiltonian

$$H_{SB} = \sum_{\alpha=0}^{D-1} S_{\alpha} \otimes B_{\alpha},$$

where $S_{\alpha}$ and $B_{\alpha}$ are operators of the system and bath, respectively, and in particular, $S_0 \equiv I_S$ is the identity operator and $B_0$ is the bath internal interaction. We assume that $S_{\alpha}$ and $B_{\alpha}$ are bounded in spectrum so that a perturbative expansion of the system-bath propagator driven by $H_{SB}$ converges for a short evolution time $T$. Otherwise, the coupling is generic, that is, the details of $B_{\alpha}$ are unspecified. In Eq. (1), $S \equiv [S_{\alpha}|\alpha = 0, \ldots, D-1]$ does not have to be the basis of the full Lie algebra. For example, in the pure dephasing Hamiltonian of a qubit coupled to a bath, $S = \{I, \sigma_z\}$, which only generates a sub-algebra of a qubit.

Control of the system is implemented by applying a Hamiltonian $V_{c,T}(t)$ on the system. To realize a desired system evolution $Q$ (e.g., a quantum gate) over a given duration of time $T$, $V_{c,T}(t)$ scales with $T$ so that ($h = 1$)

$$U_{c,T}(t) \equiv T \exp \left[-i \int_0^T V_{c,T}(\tau)d\tau \right] = T \exp \left[-i \int_0^T V_c(\theta)d\theta \right] \equiv U_c(t/T),$$

where $T$ denotes the time-ordering operator, and $V_c(\theta) = T V_{c,T}(T \theta)$. We consider the case of perfect control, that is, $U_{c,T}(T) = U_c(1) = Q$ is the desired control of the system. Under influence of the environment, the system-bath propagator reads

$$U(T) = T \exp \left[-i \int_0^T [H_{SB} + V_{c,T}(t)] dt \right].$$

The errors induced by $H_{SB}$ can be isolated in the interaction picture by writing $U(T) = QU_E(T)$, where the error propagator

$$U_E(T) \equiv T \exp \left[-i \int_0^T U_{c,T}(t) H_{SB} U_{c,T}(t) dt \right] = T \exp \left[-iT \int_0^1 U_c^\dagger(\theta) H_{SB} U_c(\theta)d\theta \right].$$

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We suppose that the control $V_{c,t}(t)$ has been designed to suppress the errors due to $H_{SB}$ up to the $N$th order of the evolution time $T$, which is assumed short, that is,

$$U_E(T) = U_\Omega \left[ 1 + O(T^{N+1}) \right],$$  \hspace{1cm} (5)

where $U_\Omega$ is an operator commuting with a certain set of system operators $\Omega$. A control $V_{c,t}(t)$ is universal if Eq. (5) holds for arbitrary time-independent $B_{\alpha}$. Ref. [19] shows that $V_{c,t}(t)$ can be designed to achieve Eq. (5) with arbitrary $N$ and $\Omega \neq I_S$ for a general time-independent model [Eq. (1)]. For the special case of dynamical decoupling, $Q = I_S$ and all the operators in $\Omega$ are preserved; if $\Omega$ spans the full algebra of the system, $U_\Omega$ is a pure bath operator and any system states (hence quantum correlations) will be protected [14].

### B. Generalization to time-dependent systems

A time-dependent version of Eq. (1) reads

$$H'_{SB}(t) = \sum_{\alpha=0}^{D-1} S_\alpha \otimes B'_\alpha(t),$$ \hspace{1cm} (6)

where the generic bath operators are assumed analytic in time:

$$B'_\alpha(t) = \sum_{p=0}^{\infty} B^{(p)}_{\alpha} \frac{1}{p!} t^p.$$ \hspace{1cm} (7)

We also assume that the bath operators $B^{(p)}_{\alpha}$ are bounded in spectrum. We are to prove the following theorem.

**Theorem.** If $V_{c,t}(t)$ realizes Eq. (5) for an arbitrary time-independent Hamiltonian in Eq. (1), it will realize the control with the same order of precision for an arbitrary time-dependent Hamiltonian in Eq. (6), that is, the system-bath propagator commutes with the system operator set $\Omega$ up to an error $O(T^{N+1}),$

$$U'(T) \equiv T \exp \left( -i \int_0^T [H'_{SB}(t) + V_{c,t}(t)] dt \right)$$

$$\equiv Q U'_\Omega(T) = Q U'_\Omega \left[ 1 + O(T^{N+1}) \right],$$ \hspace{1cm} (8)

where $U'_\Omega$ commutes with the operator set $\Omega$.

**Proof.** We write

$$U_E(T) = e^{-i B_0 T} \hat{U}_E(T),$$ \hspace{1cm} (9)

and

$$\hat{U}_E(T) = T \exp \left[ -i T \int_0^T U'_\alpha(\theta) B_\alpha(\theta) U_\alpha(\theta) d\theta \right],$$ \hspace{1cm} (10)

$$B_\alpha(\theta) = e^{i B_0 T} B_\alpha e^{-i B_0 T} = \sum_{k=0}^{\infty} \left[ i B_0, B_\alpha \right]_k \frac{(T \theta)^k}{k!},$$ \hspace{1cm} (11)

with $[i B_0, B_\alpha]_{k+1} = [i B_0, [i B_0, B_\alpha]]_k$ and $[i B_0, B_\alpha]_0 \equiv B_\alpha$. The perturbative expansion of $\hat{U}_E(T)$ in short time $T$ reads

$$\hat{U}_E(T) = 1 + T \sum_{n=1}^{\infty} \sum_{\alpha=0}^{D-1} \sum_{\beta=0}^{D-1} T_n^{\alpha\beta} \frac{\partial^{n+\beta} \hat{U}_E}{\partial B_\alpha^{n+\beta}},$$ \hspace{1cm} (12)

with short-hand notations $T_n^{\alpha\beta} = \sum_{k=1}^{\infty} \sum_{\alpha=0}^{D-1} \sum_{\beta=0}^{D-1} \sum_{p=0}^{\infty} \frac{1}{p!} T^p B^{(p)}_{\alpha} B^{(p)}_{\beta}$, and $\hat{U}_E \equiv \sum_{\alpha=0}^{D-1} B_\alpha \hat{U}_E$, where the bath and system operators are,

$$B^{(p)}_{\alpha} = \frac{[i B_0, B_{\alpha_1}]_{p_1}}{p_1!} \cdots \frac{[i B_0, B_{\alpha_n}]_{p_n}}{p_n!} = \frac{[i B_0, B_{\alpha_1}]_{p_1}}{p_1!} \cdots \frac{[i B_0, B_{\alpha_n}]_{p_n}}{p_n!},$$ \hspace{1cm} (13)

$$S_n^{\alpha\beta} = \int_0^T \int_0^T U^{(1)}_{\alpha_1}(\theta_1) S_{\alpha_1} U^{(1)}_{\beta_1}(\theta_1) \int_0^T \int_0^T U^{(2)}_{\alpha_2}(\theta_2) S_{\alpha_2} U^{(2)}_{\beta_2}(\theta_2) \int_0^T \int_0^T \cdots \int_0^T U^{(n)}_{\alpha_n}(\theta_n) S_{\alpha_n} U^{(n)}_{\beta_n}(\theta_n) d\theta_1 d\theta_2 \cdots d\theta_n,$$ \hspace{1cm} (14)

respectively.

For the time-dependent Hamiltonian $H'_{SB}(t)$, the expansion of $\hat{U}'_E(T)$ has a similar form

$$\hat{U}'_E(T) = 1 + T \sum_{n=1}^{\infty} \sum_{\alpha=0}^{D-1} \sum_{\beta=0}^{D-1} T_n^{\alpha\beta} S_n^{\alpha\beta} \frac{\partial^{n+\beta} \hat{U}'_E}{\partial B_\alpha^{n+\beta}},$$ \hspace{1cm} (15)

where $S_n^{\alpha\beta} \equiv \left( B^{(p_1)}_{\alpha_1} / p_1! \right) \cdots \left( B^{(p_n)}_{\alpha_n} / p_n! \right) / B^{(p_0)}_{\alpha_0}$ is defined by the expansion of $B'_\alpha(t)$ in the interaction picture,

$$\left( T e^{-i \tilde{H}_S(t) dt} \right) \left( T e^{-i \tilde{H}_B(t) dt} \right) = \sum_{n=0}^{\infty} B^{(n)}_{\alpha_0} / p_n!.$$

By assumption, $V_{c,t}(t)$ realizes Eq. (5) for a generic time-independent $H_{SB}$. In Appendix, we give an explicit construction of $B_\alpha$, for which the set of bath operators $B^{(n)}_{\alpha_0} / p_n!$ is linear independent. Eqs. (5) and (9) indicate that $U'_E(T)$ commutes with $\Omega$ up to the $N$th order in $T$. If the bath operators $B^{(n)}_{\alpha_0} / p_n!$ are linear independent (non-zero, of course), $S_n^{\alpha\beta}$ must commute with the system operator set $\Omega$ for $n + |\beta| \leq N$. Since Eq. (15) is also an expansion of $S_n^{\alpha\beta}$, $\hat{U}'_E(T)$ and hence $U'_E(T)$ commute with the system operator set $\Omega$ up to an error $O(T^{N+1})$. \hfill $\Box$

The theorem extends the validity of universal dynamical control to analytically time-dependent systems. Note that in Eq. (1), $S = \{S_\alpha | \alpha = 0, \ldots, D-1 \}$ does not have to span the full algebra of the system but should contain the identity operator $I_S$. Therefore the theorem does not rely on the specific algebra generated by $S$ and is general for any system-bath interactions provided that the total generic Hamiltonian includes a free bath term $I_S \otimes B_0$. It should be stressed that because the details of the bath operators $B'_\alpha(t)$ are unspecified, the dynamical control is still valid if we do the following variation:

$$H'_{SB}(t) \rightarrow H_S(t) \otimes I_B + \sum_{\alpha=0}^{D-1} S_\alpha \otimes B'_\alpha,$$ \hspace{1cm} (16)

where $H_S(t)$ is a bounded system operator in the space spanned by $S$ and analytic in time and $I_B$ is the identity operator of the bath. Actually, the first term in the right-hand
side of Eq. 16 can be readily absorbed into the system-bath coupling. The drift errors introduced by the system’s internal Hamiltonian $H_S(t)$ are also eliminated.

The operator $H'_{SB}(t)$ is required to be bounded for any $t \in [0, T]$. However, to implement a system evolution $U_{c,T}(T) = U_c(1) = Q \neq I_S$, the control $V_{c,T}(t)$ must scale as $\sim 1/T$; this scaling induces a faster evolution on the system as $T \to 0$ in the limit of instantaneous pulses and this is the requirement of any dynamical control schemes to suppress errors: the evolution of the system driven by the control field $V_{c,T}(t)$ needs to be faster than the bath evolution induced by $H_{SB} = H'_{SB}(t)$.

III. CONCLUSIONS

We have proved that a universal dynamical control which implements a quantum evolution of a system up to an error $O(T^{N+1})$ in total evolution time $T$ for a generic time-independent system-bath Hamiltonian automatically suppresses errors to $O(T^{N+1})$ for any analytically time-dependent Hamiltonians. The extension of various universal dynamical control schemes to arbitrary analytically time-dependent systems is therefore established. This result also simplifies the design of other universal control schemes. The current research raises an interesting question for future study: Are there minimal models for which a control scheme works with a certain degree of precision will work with the same order of precision for arbitrary systems, time-independent or not?

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Appendix

Here we give an explicit construction of $\{B_n\}$, so that the set of bath operators $\{B_n^\alpha|n + |\tilde{p}| \leq N\}$ in Eq. 12 are linear independent. For that purpose we prove the following lemma.

Lemma. For a finite number $R$, there exists a construction of $K'$ Hermitian operators $\{O_{k}|k=1,\ldots,K'\}$, such that all the operator products $O_{k1}O_{k_2}\cdots O_{k_r}$ for different sequences $(k_1k_2\cdots k_r)$ with $1 \leq k_i \leq K'$ and $1 \leq r_j \leq R$ are linear independent.

Proof. Let $K = K' + 1$ and $|\alpha| = 0,\ldots, (KR + 1)K$ be an orthonormal basis. We construct the following Hermitian operators in the Hilbert space expanded by this basis

$$O_k = \sum_{l=0}^{K^r} |l\rangle \langle Kl + k| + \text{h.c.}$$

Examination of the projection $\langle 0|0\rangle O_k O_{k_1} \cdots O_{k_r}$ shows that the operator $O_k O_{k_2} \cdots O_{k_r}$ contains one and only one component of the form $\langle 0|0\rangle |k_1k_2\cdots k_r\rangle$, where

$$(k_1k_2\cdots k_r)_{K} \equiv K^{r-1}k_1 + \cdots + Kk_{r-1} + k_r,$$  \hspace{1cm} (A.2)

is a number of base $K$ with $1 \leq k_i \leq K - 1$. Therefore all the operator products $O_{k1}O_{k_2}\cdots O_{k_r}$ for different sequences $(k_1k_2\cdots k_r)$ with $1 \leq k_i \leq K'$ and $1 \leq r_j \leq R$ are linear independent. $\square$

An explicit construction of $\{B_n\}$ reads

$$B_n = \sum_{r=0}^{N-1} \langle r| \rho \otimes h_\alpha^{(r)}, \quad \alpha \geq 1, \quad (A.3a)$$

$$B_0 = \sum_{r=0}^{N-1} \langle r| \rho + 1 \otimes I_h + \text{h.c.}, \quad (A.3b)$$

where $\langle r| \rho = 0,\ldots, N - 1$ is an $N$-dimensional orthonormal basis with the periodic condition $\langle r + N | \rho = | r \rangle$, $h_\alpha^{(r)}$ is an Hermitian operator, and $I_h$ is the identity operator. Using the Lemma, we have a construction of the operators $\{h_\alpha^{(r)}|0 \leq r \leq N - 1, 1 \leq \alpha \leq D - 1\}$ such that all operator products $h_\alpha^{(r_1)}h_\alpha^{(r_2)}\cdots h_\alpha^{(r_k)}$ are linear independent for different sequences $(\alpha_1,\ldots,\alpha_k)$ and $(r_1,\ldots,r_k)$. We decompose $B_0 = B_+ + B_-$ with $B_+ \equiv \sum_{r=0}^{N-1} \langle r| \rho + 1 \otimes I_h$ and $B_- = (B_+)^\dagger$. Some calculation gives

$$\frac{|iB_+,B_{\tilde{p}}|}{p!} = \sum_{r=0}^{N-1} \langle r| \rho + p | \otimes \left( \sum_{k=0}^{p} |C_{\tilde{p}}^{(k)}|_{r-p-k} h_\alpha^{(r)} \right), \quad (A.4)$$

where $p \geq 0$ and $c_{\tilde{p}}^{(k)} = (-1)^k i^p / [k!(p - k)!]$ is a non-zero coefficient. Therefore

$$\langle 0 \mid G_{n}^{\alpha,\beta} \mid \tilde{p} \rangle = \left( \prod_{l=0}^{n} \frac{|iB_+,B_{\tilde{p}_l}|}{p_l!} \cdots \frac{|iB_+,B_{\tilde{p}_1}|}{p_1!} \right) \langle 0 \mid \tilde{p} \rangle$$

$$= \left( \sum_{l=0}^{p_1} |C_{\tilde{p}_1}^{(l)}|_{p_1-l-k} \right) \left( \sum_{l=0}^{p_2} |C_{\tilde{p}_2}^{(l)}|_{p_2-l-k} \right) \cdots \left( \sum_{l=0}^{p_n} |C_{\tilde{p}_n}^{(l)}|_{p_n-l-k} \right)$$

$$= \left( \sum_{l=0}^{p_1} \sum_{l=0}^{p_2} \cdots \sum_{l=0}^{p_n} \langle n,\alpha,\beta| \right). \quad (A.5)$$

There is one and only one operator product $h_\alpha^{(r_1)}h_\alpha^{(r_2)}\cdots h_\alpha^{(r_k)}$ in $\langle 0 \mid G_{n}^{\alpha,\beta} \mid \tilde{p} \rangle$. If $n + |\tilde{p}| \leq N$, the operators $h_\alpha^{(r_1)}h_\alpha^{(r_2)}\cdots h_\alpha^{(r_k)}$ are linear independent for different $(n,\alpha,\beta)$ according to the Lemma. Thus for $n + |\tilde{p}| \leq N, n \geq 1$, and $|\tilde{p}| \geq 0$, $\langle 0 \mid G_{n}^{\alpha,\beta} \mid \tilde{p} \rangle$ and hence $G_{n}^{\alpha,\beta}$ are linear independent for different $(n,\alpha,\beta)$.
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