The shortest orbital period in scalar hairy kerr black holes

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Abstract

In a very interesting paper, Hod has proven that the equatorial null circular geodesic provides the fastest way to circle a kerr black hole, which is closely related to the Fermat’s principle. In the present paper, we extend the discussion to kerr black holes with scalar field hair. We consider matter fields’ backreaction on the metric and analytically show that the circle with the shortest orbital period is identical to the null circular geodesic. Our analysis also implies that the Hod’s theorem may be a general property in the axially symmetric curved spacetime.

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I. INTRODUCTION

According to the general relativity, there may be null geodesics outside compact objects, such as black holes and regular ultra-compact stars \cite{1, 2}. The null geodesics can reveal significant features of the curved spacetime geometry. In particular, the circular null geodesics provide a path for the massless field to circle compact objects. Due to its important applications in astrophysics and theories, the circular null geodesics have attracted a lot of attentions \cite{3–5}.

The circular null geodesics play an important role in the physics of compact objects. In particular, the circular null geodesic is closely related to strong gravitational effects, such as the lensing, shadow, as well as the gravitational waves \cite{6–13}. From the theoretical aspects, it was found that the null circular orbit is useful in describing hair distributions outside hairy black holes \cite{14–19}. Especially, there are unstable and stable circular null geodesics around the compact objects. It was suggested that the characteristic resonances of black holes can be interpreted as null particles trapped at the unstable circular orbit and slowly leaking out \cite{20–27}. In the regular ultra-compact star spacetime, the existence of stable circular null geodesics could trigger nonlinear instabilities due to that massless fields can pile up on the stable null orbit \cite{28–35}.

An important physical problem is to search for the fast way to orbit a compact objects \cite{36, 37}. It should be pointed out that the circular orbit with the shortest orbital period is distinct from the circular orbit with the smallest radial parameter due to gravitational red-shift effect. We should also consider the dragging of inertial frames by spinning compact objects. Considering the influences of these two interesting physical effects, Hod showed that the null circular geodesic of a black hole spacetime is characterized by the shortest possible orbital period as measured by asymptotic observers \cite{36}. This property is closely related to the Fermat’s principle in flat spacetime that light propagates along the null trajectories of least time \cite{37}. On the other side, Herdeiro and Radu constructed novel Kerr black holes with scalar hair \cite{38, 39}, which are equilibrium states and may play an important role in realistic astrophysical processes \cite{40–45}. In this work, we plan to extend the discussion of orbital period in \cite{36, 37} to scalar hairy kerr black holes and also give some analysis in general axially symmetric spacetimes.

In the next section, we firstly review the gravity model of a kerr black hole coupled to the scalar field. Then we show that the null circular geodesics still coincide with the shortest orbital period circle in scalar hairy kerr black holes. And the last section contains our main conclusions.
II. THE SHORTEST ORBITAL PERIOD IN KERR BLACK HOLES

In Boyer-Lindquist coordinates, the line element of the general axially symmetric spacetime reads

\[ ds^2 = \tilde{g}_{tt}(r, \theta) dt^2 + 2 \tilde{g}_{t\phi}(r, \theta) dtd\phi + \tilde{g}_{rr}(r, \theta) dr^2 + \tilde{g}_{\theta\theta}(r, \theta) d\theta^2 + \tilde{g}_{\phi\phi}(r, \theta) d\phi^2. \]  

The equatorial plane of the black hole is characterized by \( \theta = \frac{\pi}{2} \). The functions can be denoted as \( \tilde{g}_{tt}(r, \frac{\pi}{2}) = g_{tt}(r) \), \( \tilde{g}_{t\phi}(r, \frac{\pi}{2}) = g_{t\phi}(r) \), \( \tilde{g}_{rr}(r, \frac{\pi}{2}) = g_{rr}(r) \), \( \tilde{g}_{\theta\theta}(r, \frac{\pi}{2}) = g_{\theta\theta}(r) \) and \( \tilde{g}_{\phi\phi}(r, \frac{\pi}{2}) = g_{\phi\phi}(r) \).

Recently, Hod has investigated the fast circle in the background of a probing kerr black hole [36, 37]. In this paper, we extensively analyze circular trajectories in the background of Kerr black holes with scalar field hair. The asymptotically flat four dimensional deformed scalar hairy Kerr black hole is \[ ds^2 = e^{2F_0} \left( \frac{dr^2}{N} + r^2 d\theta^2 \right) + e^{2F_1} r^2 \sin^2 \theta \left( d\phi - W dt \right)^2 - e^{2F_2} N dt^2, \]

where \( F_0, F_1, F_2 \) and \( W \) are functions of the radial coordinate \( r \). And \( N \) can be expressed as \( N = 1 - \frac{r_H}{r} \) with \( r_H \) as the event horizon. Since the spacetime is asymptotically flat, the functions are characterized by \( F_0 \to 0, F_1 \to 0, F_2 \to 0, W \to 0 \) as approaching the infinity. We also take the usual angular coordinates \( \theta \in [0, \pi] \) and \( \phi \in [0, 2\pi] \).

And we take the ansatz of the scalar field in the form [38, 39]

\[ \psi = R(r, \theta) e^{i(m\phi - \omega t)}, \]

where \( \omega \) is the frequency of the scalar field and \( m = \pm 1, \pm 2 \ldots \) is the azimuthal harmonic index.

Now we would like to search for the circular trajectory with the radius \( r = r_{\text{fast}} \), which corresponds to the shortest orbital period measured by asymptotic observers. We shall consider circular orbits in the black hole equatorial plane characterized by \( \theta = \frac{\pi}{2} \). In order to minimize the orbital period for a given radius \( r \), one should move as close as possible to the speed of light. In this case, the orbital period can be obtained from Eq. (1) with \( ds = dr = d\theta = 0 \) and \( \Delta \phi = 2\pi \) [36, 37]:

\[ T(r) = -\frac{2\pi (g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt} g_{\phi\phi}})}{g_{tt}}. \]

Another case of \( \Delta \phi = -2\pi \) can be obtained by the transformation \( W \to -W \) in the metric (2). The circular trajectory around the central black hole with the shortest orbital period is characterized by

\[ T'(r = r_{\text{fast}}) = 0, \]
where the prime is a derivative with respect to the coordinate $r$. This yields the characteristic equation

$$G(r_{fast}) = T'(r_{fast}) = (g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}) \frac{g_{\phi\phi}}{g_{tt}} - (g_{t\phi} + \frac{g_{tt}g_{\phi\phi} - 2g_{t\phi}g_{\phi\phi} + g_{tt}g_{\phi\phi}}{2\sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}) \frac{1}{g_{tt}} = 0. \quad (6)$$

We should point out that the fast orbital radius may be also obtained at boundaries ($r_H$ and $r = \infty$) or at some points where $T'(r)$ doesn’t exist. In fact, the horizon $r_H$ cannot be the physical orbit since the horizon will swallow any matter fields. The shortest period also cannot be reached at the infinity as the asymptotically flat condition yields that $T(r) \to 2\pi r \to \infty$ at the infinity. $T'(r)$ may doesn’t exist for cases $g_{tt} = 0$ and $g_{t\phi}^2 - g_{tt}g_{\phi\phi} = 0$. In the case of $g_{tt} = 0$, we have $g_{t\phi} = -r^2W e^{2F_2} \neq 0$ and $T(r) = \infty$, otherwise there is $W(r) = 0$ and $N(r) = \frac{2W^2 e^{2F_2}}{e^{2F_0}} = 0$ implying $r = r_H$. Since $g_{t\phi}^2 - g_{tt}g_{\phi\phi} = r^2 N(r) e^{2F_0} + 2F_2 = 0$ implying $r = r_H$, the minimum $T(r)$ also cannot be reached at $g_{t\phi}^2 - g_{tt}g_{\phi\phi} = 0$. In summary, the circle radius with the shortest orbital period can be only obtained from the relation (6).

Now we show that $r_{fast}$ with the shortest orbital period is equal to the radius of the null circular geodesic. In the background of the hairy Kerr black hole, we will derive the relevant equations of the null circular geodesic radius $r = r_\gamma$. The Lagrangian describing the geodesics in the spacetime (1) is given by

$$2L = g_{tt} \dot{t}^2 + 2g_{t\phi} \dot{t} \dot{\phi} + g_{rr} \dot{r}^2 + g_{\phi\phi} \dot{\phi}^2, \quad (7)$$

where a dot denotes the ordinary differentiation with respect to the affine parameter along the geodesic.

Since the metric has the time Killing vectors $\partial t$ and the axial Killing vector $\partial \phi$, there are two constants of motion labeled as $E$ and $L$. The generalized momenta can be derived from the Lagrangian as

$$p_t = g_{tt} \dot{t} + g_{t\phi} \dot{\phi} = -E = const, \quad (8)$$

$$p_\phi = g_{t\phi} \dot{t} + g_{\phi\phi} \dot{\phi} = L = const, \quad (9)$$

$$p_r = g_{rr} \dot{r}. \quad (10)$$

The Hamiltonian can be expressed as $\mathcal{H} = p_t \dot{t} + p_r \dot{r} + p_\phi \dot{\phi} - L$, which implies

$$2\mathcal{H} = -E \dot{t} + L \dot{\phi} + g_{rr} \dot{r}^2 = \delta = const. \quad (11)$$

Here we can take the value $\delta = 0$ in the case of null geodesics.

According to (11), we arrive at the relation

$$\dot{r}^2 = \frac{1}{g_{rr}} [E \dot{t} - L \dot{\phi}] \quad (12)$$
for null geodesics.

From relations (8) and (9), we easily get \( \dot{t} \) and \( \dot{\phi} \) in the form

\[
\dot{t} = \frac{E g_{t\phi} + L g_{t\phi}}{g_{t\phi} - g_{tt} g_{\phi\phi}}, \quad \dot{\phi} = \frac{E g_{t\phi} + L g_{tt}}{g_{t\phi} - g_{tt} g_{\phi\phi}},
\]

(13)

Substituting (13) into (12), there is

\[
\dot{r}^2 = \frac{E^2}{g_{rr}} \left[ g_{\phi\phi} + 2 b g_{t\phi} + b^2 g_{tt} \right] - g_{tt} g_{\phi\phi},
\]

(14)

where we introduce a new constant \( b = L/E \).

The requirement \( \dot{r}^2 = 0 \) for a null circular geodesic yields

\[
b_{\pm} = -\frac{g_{r\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt} g_{\phi\phi}}}{g_{tt}}.
\]

(15)

The requirement \( (\dot{r}^2)' = 0 \) and relation (15) yield the equation

\[
\frac{G(r_{\gamma})}{g_{rr} \sqrt{g_{t\phi}^2 - g_{tt} g_{\phi\phi}}} = 0.
\]

(16)

The shortest period circle cannot be on the horizon \( r_{fast} \neq r_H \) since the horizon will absorb all matter fields. So we only focus on the null circular geodesic radius \( r_{\gamma} \neq r_H \). Then there is \( N(r_{\gamma}) \neq 0 \) and \( g_{rr}(r_{\gamma}) = \frac{e^{2F_1}}{N} < \infty \). In addition, there is \( g_{t\phi}^2 - g_{tt} g_{\phi\phi} = r_{\gamma}^2 N(r) e^{2F_0 + 2F_2} < \infty \) at the the null circular geodesic radius \( r_{\gamma} \). The relation (16) can be transformed into

\[
G(r_{\gamma}) = 0 \cdot \left( g_{rr} \sqrt{g_{t\phi}^2 - g_{tt} g_{\phi\phi}} \right)_{r = r_{\gamma}} = 0.
\]

(17)

We can see that Eq. (17) for the null circular geodesic is identical to Eq. (6) for the fastest circular trajectory. Thus it means that the null circular geodesic provides the path with the shortest orbital period as measured by asymptotic observers

\[
r_{fast} = r_{\gamma}.
\]

(18)

It should be emphasized that Eqs. (6) and (17) may have various solutions. In such cases, the circular trajectory with the shortest orbital period still corresponds to one of the null circular geodesics. At last, we emphasize that the metric (2) is needed in the analysis below (6) and above (17). And the similarity between (6) and (16) implies that the Hod’s theorem may be a general property in axially symmetric spacetimes.
III. CONCLUSIONS

We have investigated the equatorial circular orbits in the background of deformed Kerr black holes coupled with scalar fields. In particular, we have obtained the radius of equatorial circular trajectory with the shortest orbital period as measured by asymptotic observers. It was shown that the fast equatorial circular trajectory is identical to the null circular geodesic in the background of scalar hairy Kerr black holes. Our analysis also implied that the Hod’s theorem may be a general property in the axially symmetric curved spacetime. As proposed by Hod, these results are in analogy with the Fermat’s principle in flat spacetime, which asserts that light takes the path with the least traveling time.

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