Numerical Approximations for the Fractional Fokker–Planck Equation with Two-Scale Diffusion

Jing Sun · Weihua Deng · Daxin Nie

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Abstract
Fractional Fokker–Planck equation plays an important role in describing anomalous dynamics. To the best of our knowledge, the existing numerical discussions mainly focus on this kind of equation involving one diffusion operator. In this paper, we first derive the fractional Fokker–Planck equation with two-scale diffusion from the Lévy process framework, and then the fully discrete scheme is built by using the $L_1$ scheme for time discretization and finite element method for space. With the help of the sharp regularity estimate of the solution, we optimally get the spatial and temporal error estimates. Finally, we validate the effectiveness of the provided algorithm by extensive numerical experiments.

Keywords Fractional Fokker–Planck equation · Two-scale diffusion · Finite element · $L_1$ scheme · Error estimates

1 Introduction
We provide the numerical methods for the fractional Fokker–Planck equation with two-scale diffusion, i.e.,

$$ \begin{cases} 0 \partial_t^\alpha (u - u_0) + (-\Delta)u + (-\Delta)^s u = f, & (x, t) \in \Omega \times (0, T], \\ u = 0, & (x, t) \in \Omega^c \times (0, T], \\ u(0) = u_0, & x \in \Omega, \end{cases} \tag{1.1} $$

where $\Omega \subset \mathbb{R}^n$ ($n = 1, 2, 3$) is a bounded domain with smooth boundary and $\Omega^c$ means its complement set; $f$ is a given source term; $T$ is the fixed terminal time; $\Delta$ denotes Laplace operator; $(-\Delta)^s$ is the fractional Laplacian defined by

$$ (-\Delta)^s u(x) = c_{n,s} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \quad s \in (0, 1) $$

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Weihua Deng
dengwh@lzu.edu.cn

1 School of Mathematics and Statistics, Gansu Key Laboratory of Applied Mathematics and Complex Systems, Lanzhou University, Lanzhou 730000, People’s Republic of China
with $c_{n,s} = \frac{2^{2s+1} \Gamma(n/2 + s)}{\pi^{n/2} \Gamma(1-s)}$ and P.V. being the principal value integral; $0^\alpha \partial_t^\alpha u$ is the Riemann-Liouville fractional derivative defined by
\[
0^\alpha \partial_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t (t - \xi)^{-\alpha} u(\xi) d\xi
\]
with $\alpha \in (0, 1)$.

Now we briefly state how to derive Eq. (1.1) from the framework of the Lévy process [4]. As we all know, Lévy process is one of the important stochastic processes with stationary and independent increments and it is thought to be an efficient model to approximate the non-Gaussian process [4]. Let $L(t)$ be a Lévy process with Fourier exponent $\eta(w)$ and take
\[
\eta(w) = -w^T w + \int_{\mathbb{R}^n/\{0\}} (e^{iw^T y} - 1 - iw^T y)\nu(dy)
\]
with $i$ denoting the imaginary unit, $w^T$ being the transpose of $w$ and $\nu$ a sigma-finite Lévy measure on $\mathbb{R}^n$, i.e.,
\[
\nu(dy) = c_{n,s}|y|^{-1-2s} dy
\]
and $c_{n,s} = \frac{2^{2s+1} \Gamma(n/2 + s)}{\pi^{n/2} \Gamma(1-s)}$. Further let $T(t)$ be a strictly increasing subordinator independent of $L(t)$ with Laplace exponent $\Psi(z) = z^\alpha$, where $\alpha \in (0, 1)$. Define the inverse subordinator $E(t) = \inf\{r > 0 : T(r) > t\}$, which has the probability density function in Laplace space as
\[
\tilde{p}_E(r, z) = z^{\alpha-1} e^{-rz^\alpha}.
\]
Denote $p_X(x, t)$ as the probability density function of the stochastic process $X(t) = L(E(t))$. Introduce $p_L(x, t)$ as the probability density function of $L(t)$. Using the following formula
\[
p_X(x, t) = \int_0^\infty p_L(x, r) p_E(r, t) dr,
\]
we can get the Fourier–Laplace transforms of $p_X(x, t)$, i.e.,
\[
\mathcal{F}(\tilde{p}_X(x, t))(w, z) = \int_0^\infty e^{-ct} \int_{\mathbb{R}^n} e^{iw^T L(E(t))} p_X(L(E(t)), t) dL(E(t)) dt
\]
\[
= \int_0^\infty e^{-ct} \int_{\mathbb{R}^n} e^{iw^T L(E(t))} p_L(L(E(t)), E(t)) dL(E(t)) p_E(E(t), t) dE(t) dt
\]
\[
= \int_0^\infty e^{-ct} \int_{\mathbb{R}^n} e^{E(t)\eta(w)} p_E(E(t), t) dE(t) dt
\]
\[
= \int_0^\infty \frac{z^{\alpha-1}}{z^\alpha - \eta(w)} e^{-E(t)z^\alpha} dE(t)
\]
where $\mathcal{F}$ means Fourier transform and ‘−’ denotes Laplace transform. Thus $p_X(x, t)$ satisfies
\[
0^\alpha \partial_t^\alpha (p_X(x, t) - p_X(x, 0)) + (-\Delta) p_X(x, t) + (-\Delta)^s p_X(x, t) = 0,
\]
\[
(1.2)
\]
Fractional Fokker–Planck equations have attracted many attentions in recent years and the corresponding numerical schemes are extensively proposed [1, 2, 6, 8, 11, 12, 15, 17, 18, 21]. But these numerical methods are mainly constructed for the fractional Fokker–Planck equation with one diffusion operator. To the best of our knowledge, the relative numerical discussions are few for this kind of equation involving two-scale diffusion. In this paper, we
discuss the numerical method for Eq. (1.1). To be specific, we first provide a sharp regularity
estimate for Eq. (1.1), in which we treat \((-\Delta)^s u\) as a “source term” to overcome the difficulties
caused by diffusion operators with different scales in regularity analysis; with the help of the
elliptic regularity of \((-\Delta)\) and the equivalence of different fractional Sobolev norms, we show
that \(u \in \hat{H}^{2\min\{1, \frac{3}{2} - s - \epsilon\}}(\Omega)\) when \(u_0, f(0) \in L^2(\Omega)\) and \(\int_0^t \|f'(r)\|_{L^2(\Omega)} dr < \infty\); the
detailed proofs can refer to Theorem 4.1. Then the finite element method is used to discretize
spatial operator and an auxiliary equation (see (4.8)) is introduced to help us derive the spatial
error estimates. At the same time, the \(L_1\) scheme [15] is used to approximate the temporal
derivative and an \(O(\tau)\) convergence is obtained.

The rest of the paper is organized as follows. In Sect. 2, we give some notations and function
spaces. The fully discrete scheme is built based on \(L_1\) discretization in time and finite element
method in space in Sect. 3. In Sect. 4, we first provide the regularity estimate for the solution,
and then present the complete error estimates for spatial semi-discrete scheme and the fully
discrete scheme, respectively. Some numerical examples, in Sect. 5, are proposed to support
the theory. We conclude the paper with some discussions in the last section. Throughout the
paper, \(C\) is a generic positive constant, whose value may vary at different places and \(\epsilon > 0\)
is arbitrarily small.

2 Preliminaries

2.1 Notations

Let \(A = (-\Delta)\) and \(\mathcal{A}^s = (-\Delta)^s\) with zero Dirichlet boundary conditions and \(\{ (\lambda_i, \phi_i)\}_{i=1}^\infty\)
be the eigenvalues and eigenfunctions of \(A\), where \(\{ \phi_i\}_{i=1}^\infty\) are orthonormal bases in \(L^2(\Omega)\)
and \(\lambda_i \leq \lambda_{i+1}\) for \(i \geq 1\). Introduce \(A^\sigma\) for \(\sigma \in \mathbb{R}\) as

\[
A^\sigma u = \sum_{i=1}^\infty \lambda_i^\sigma (u, \phi_i) \phi_i.
\]

Let \(\hat{H}^\sigma(\Omega) = \mathcal{D}(A^\sigma)\) equipped with \(\|u\|_{\hat{H}^\sigma(\Omega)} = \|A^\sigma u\|_{L^2(\Omega)}\) [20], where \(\mathcal{D}(A^\sigma)\) means
the domain of \(A^\sigma\). It is easy to verify that \(\hat{H}^0(\Omega) = L^2(\Omega), \hat{H}^1(\Omega) = H^1_0(\Omega),\) and
\(\hat{H}^2(\Omega) = H^1_0(\Omega) \cap H^2(\Omega).\) In the following, we denote \(\| \cdot \|\) as the operator norm from
\(L^2(\Omega)\) to \(L^2(\Omega)\) and the notation ‘\(\sim\)’ as Laplace transform.

For \(\kappa > 0\) and \(\pi/2 < \theta < \pi\), the definitions of sectors \(\Sigma_\theta\) and \(\Sigma_{\theta, \kappa}\) in the complex plane
\(\mathbb{C}\) are

\[
\Sigma_\theta = \{ z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \theta \},
\]

\[
\Sigma_{\theta, \kappa} = \{ z \in \mathbb{C} : |z| \geq \kappa, |\arg z| \leq \theta \},
\]

and the contour \(\Gamma_{\theta, \kappa}\) is defined by

\[
\Gamma_{\theta, \kappa} = \{ z \in \mathbb{C} : |z| = \kappa, |\arg z| \leq \theta \} \cup \{ z \in \mathbb{C} : z = re^{\pm i\theta} : r \geq \kappa \},
\]

oriented with an increasing imaginary part, where \(i^2 = -1\).
2.2 Function Spaces

Following [1, 2, 6, 8, 9], for \( s \in (0, 1) \) and \( \Omega \subset \mathbb{R}^n \), the fractional Sobolev space can be defined by

\[
H^s(\Omega) = \left\{ w \in L^2(\Omega) : |w|_{H^s(\Omega)}^2 = \int_{\Omega} \int_{\Omega} \frac{|w(x) - w(y)|^2}{|x - y|^{n+2s}} \, dx \, dy < \infty \right\}
\]

and the corresponding norm is \( \| \cdot \|_{H^s(\Omega)} = \| \cdot \|_{L^2(\Omega)} + \| \cdot \|_{H^s(\Omega)} ; \) for \( s > 1 \) and \( s \notin \mathbb{N} \), we define the fractional Sobolev space \( H^s(\Omega) \) as

\[
H^s(\Omega) = \left\{ w \in H_{\text{loc}}^{s,1}(\Omega) : |D^\alpha w|_{H^s(\Omega)} < \infty \text{ for all } \alpha \text{ s.t. } |\alpha| = |s| \right\}
\]

with \( \sigma = s - \lfloor s \rfloor \) and \( \lfloor s \rfloor \) being the biggest integer not larger than \( s \). On the other hand, a crucial subspace of \( H^s(\mathbb{R}^n) \) with \( s \in (0, 2) \) is defined by

\[
H^s_0(\Omega) = \left\{ w \in H^s(\mathbb{R}^n) : \text{supp } w \subset \Omega \right\}
\]

and its norm \( \| \cdot \|_{H^s_0(\Omega)} = \| \cdot \|_{H^s(\mathbb{R}^n)} ; \) in particular, for \( s \in (0, 1) \), its norm can also be defined by [1, 2, 6, 8, 9]

\[
\|u\|_{H^s_0(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \langle u, u \rangle_s,
\]

where

\[
\langle u, v \rangle_s = \frac{c_{n,s}}{2} \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy.
\]

Moreover, denote \( H^{-s}(\Omega) \) as the dual space of \( H^s_0(\Omega) \) with \( s > 0 \).

Remark 2.1 According to [6], when \( s \in [0, \frac{3}{2}] \) and \( \Omega \) is a Lipschitz domain, it holds \( \hat{H}^s(\Omega) = H^s_0(\Omega) \); when \( s \in [0, \frac{1}{2}] \), there holds \( H^s(\Omega) = H^s_0(\Omega) \). Thus, when \( s \in (-\frac{3}{2}, 0] \) and \( \Omega \) is a Lipschitz domain, we have \( \hat{H}^s(\Omega) = H^s(\Omega) \).

3 Numerical Discretizations

In this section, we develop the fully-discrete scheme for (1.1) based on \( L_1 \) discretization in time and finite element approximation in space. Denote \( \mathcal{T}_h \) as a shape regular quasi-uniform partition of the domain \( \Omega \) with mesh size \( h \) and \( X_h \) as the space of continuous piecewise linear functions on \( \mathcal{T}_h \). Let \( \langle \cdot, \cdot \rangle \) be the \( L^2 \) inner product and \( P_h : L^2(\Omega) \rightarrow X_h \) the \( L^2 \) projection defined by

\[
(P_h u, v_h) = \langle u, v_h \rangle \quad \forall v_h \in X_h.
\]

Introduce \( R^s_h : H^s_0(\Omega) \rightarrow X_h \) with \( s \in (0, 1) \) satisfying

\[
\langle u, v_h \rangle_s = \langle R^s_h u, v_h \rangle_s \quad \forall v_h \in X_h.
\]

Using the finite element approximation for the operators \( A \) and \( s \phi^s \) in Eq. (1.1), then the semi-discrete scheme can be written as: find \( u_h \in X_h \) satisfying

\[
(0 \phi^\alpha (u_{h} - u_{0,h}), v_h) + (\nabla u_h, \nabla v_h) + \langle u_h, v_h \rangle_s = (f, v_h) \quad \forall v_h \in X_h,
\]

where \( u_{0,h} = P_h u_0 \).
Introduce two discrete operators $A_h$ and $\mathcal{A}^s_h$ as
\[
(A_h u_h, v_h) = (\nabla u_h, \nabla v_h), \quad (\mathcal{A}^s_h u_h, v_h) = \langle u_h, v_h \rangle_s \quad \forall u_h, v_h \in X_h.
\]
It is easy to verify that \[17\]
\[
\mathcal{A}^s_h R^s_h = P_h \mathcal{A}^s.
\] (3.2)
Then we can rewrite Eq. (3.1) as
\[
\begin{cases}
0 \partial^{\alpha}_t (u_h - u_{0,h}) + A_h u_h + \mathcal{A}^s_h u_h = f_h, \quad (x, t) \in \Omega \times (0, T), \\
u_{0,h} = P_h u_0, \quad x \in \Omega,
\end{cases}
\] (3.3)
where $f_h = P_h f$.
Next, we use $L_1$ scheme introduced in \[15\] to discretize the temporal derivative. Introduce $b^{(\alpha)}_j$ as
\[
b^{(\alpha)}_j = ((j + 1)^{-\alpha} - j^{-\alpha})/\Gamma(2 - \alpha), \quad j = 0, 1, \ldots, n - 1,
\]
and $0 \partial^{\alpha}_t (u(t_n) - u_0)$ with $\alpha \in (0, 1)$ can be approximated by
\[
0 \partial^{\alpha}_t (u(t_n) - u_0) \approx \frac{1}{\tau^{\alpha}} \left( b^{(\alpha)}_0 (u(t_n) - u_0) + \sum_{j=1}^{n-1} (b^{(\alpha)}_j - b^{(\alpha)}_{j-1}) (u(t_{n-j}) - u_0) \right).
\]
Let
\[
d^{(\alpha)}_j = \tau^{-\alpha} \begin{cases}
b^{(\alpha)}_0, & j = 0, \\
b^{(\alpha)}_j - b^{(\alpha)}_{j-1}, & j > 0.
\end{cases}
\] (3.4)
Then we have
\[
0 \partial^{\alpha}_t (u(t_n) - u_0) \approx \sum_{j=0}^{n-1} d^{(\alpha)}_j (u(t_{n-j}) - u_0).
\]
So, the fully discrete scheme of Eq. (1.1) can be written as
\[
\begin{cases}
\sum_{j=0}^{n-1} d^{(\alpha)}_j (u_{h}^{n-j} - u_0^0) + A_h u_n^h + \mathcal{A}^s_h u_n^h = f_n^h, \\
u_0^h = u_{0,h},
\end{cases}
\] (3.5)
where $f_n^h = f_h(t_n)$.

4 Error Analyses

Here we first provide the regularity of the solution of Eq. (1.1), and then we develop the error estimates of the spatial semi-discrete scheme and fully-discrete scheme, respectively.

Below we recall the Grönwall inequality.

Lemma 4.1 ([10, 18]) Let the function $\phi(t) \geq 0$ be continuous for $0 < t \leq T$. If
\[
\phi(t) \leq \sum_{k=1}^{N} a_k t^{-1+\alpha_k} + b \int_0^t (t - r)^{-1+\beta} \phi(r) dr, \quad 0 < t \leq T,
\]
for some positive constants \(\{a_k\}_{k=1}^N\), \(\{\alpha_k\}_{k=1}^N\), \(b\), and \(\beta\), then there exists a positive constant \(C = C(b, T, \{\alpha_k\}_{k=1}^N, \beta)\) such that

\[
\phi(t) \leq C \sum_{k=1}^{N} a_k t^{-1+\alpha_k}, \quad 0 < t \leq T.
\]

### 4.1 Regularity of the Solution

Taking Laplace transform for (1.1), one has

\[
z^{\alpha} \tilde{u} + A \tilde{u} + \mathcal{A}^s \tilde{u} = z^{\alpha-1} u_0 + \tilde{f},
\]

resulting in

\[
\tilde{u} = - \tilde{E}(z) \mathcal{A}^s \tilde{u} + \tilde{E}(z) z^{\alpha-1} u_0 + \tilde{E}(z) \tilde{f},
\]

where

\[
\tilde{E}(z) = (z^{\alpha} + A)^{-1}.
\]

**Theorem 4.1** Let \(u\) be the solution of (1.1). Assuming \(u_0 \in L^2(\Omega)\), \(f(0) \in L^2(\Omega)\), and \(\int_0^t \|f'(r)\|_{L^2(\Omega)}dr < \infty\), we have

\[
\| A^\sigma u(t) \|_{L^2(\Omega)} \leq C t^{-\sigma} \| u_0 \|_{L^2(\Omega)} + C \| f(0) \|_{L^2(\Omega)} + C \int_0^t \| f'(r) \|_{L^2(\Omega)}dr,
\]

where \(\sigma \in [\max\{0, s - \frac{3}{4} + \epsilon\}, \min\{1, \frac{7}{4} - s - \epsilon\}]\).

**Proof** Using (4.1) and \(f(t) = f(0) + \int_0^t f'(r)dr\), we split \(\| A^\sigma u(t) \|_{L^2(\Omega)}\) into four parts, i.e.,

\[
\| A^\sigma u(t) \|_{L^2(\Omega)} \leq C \left\| \int_{\Gamma_0, \kappa} e^{zt} A^\sigma \tilde{E}(z) \mathcal{A}^s \tilde{u}dz \right\|_{L^2(\Omega)} + C \left\| \int_{\Gamma_0, \kappa} e^{zt} A^\sigma \tilde{E}(z) z^{\alpha-1} u_0dz \right\|_{L^2(\Omega)} + C \left\| \int_{\Gamma_0, \kappa} e^{zt} A^\sigma \tilde{E}(z) z^{\alpha-1} f(0)dz \right\|_{L^2(\Omega)} + C \left\| \int_0^t \int_{\Gamma_0, \kappa} e^{z(t-r)} A^\sigma \tilde{E}(z) z^{\alpha-1} f'(r)drdz \right\|_{L^2(\Omega)}.
\]

For the term \(\left\| \int_{\Gamma_0, \kappa} e^{zt} A^\sigma \tilde{E}(z) \mathcal{A}^s \tilde{u}dz \right\|_{L^2(\Omega)}\), denoting ‘\(*\)’ as convolution and using convolution property, we have

\[
\left\| \int_{\Gamma_0, \kappa} e^{zt} A^\sigma \tilde{E}(z) \mathcal{A}^s \tilde{u}dz \right\|_{L^2(\Omega)} \leq C \left\| \left( \int_{\Gamma_0, \kappa} e^{zt} A^{\sigma-\gamma} \tilde{E}(z)dz \ast (A^\gamma \mathcal{A}^s u(t)) \right) (t) \right\|_{L^2(\Omega)} \leq C \left\| \int_0^t \int_{\Gamma_0, \kappa} e^{z(t-r)} A^{\sigma-\gamma} \tilde{E}(z)dz A^\gamma \mathcal{A}^s u(r)dr \right\|_{L^2(\Omega)} \leq C \int_0^t \int_{\Gamma_0, \kappa} |e^{z(t-r)}| \| A^{\sigma-\gamma} \tilde{E}(z) \|dz \| A^\gamma \mathcal{A}^s u \|_{L^2(\Omega)}dr.
\]
By the resolvent estimate \( \|A^\beta (z^\alpha + A)^{-1}\| \leq C |z|^{(\beta - 1)\alpha} \) for \( \beta \in [0, 1] \) [11, 12, 16], Eq. (4.2) can be bounded by
\[
\|A^\sigma u(t)\|_{L^2(\Omega)}^2 \leq C \int_0^t \int_{\Gamma_{\theta,x}} \left| e^{\gamma(t-r)} ||z||^{(\gamma-1)\alpha} |dz| \right| A^{\gamma} \partial^s u(t) \|_{L^2(\Omega)}^2 dr
+ C \int_{\Gamma_{\theta,x}} \left| e^{\gamma(t-r)} ||z||^{(\gamma-1)\alpha} |dz| \right| u_0 \|_{L^2(\Omega)}^2 + C \int_{\Gamma_{\theta,x}} \left| e^{\gamma(t-r)} ||z||^{(\gamma-1)\alpha} |dz| \right| f(0) \|_{L^2(\Omega)}^2 dr
+ C \int_0^t \int_{\Gamma_{\theta,x}} \left| e^{\gamma(t-r)} ||z||^{(\gamma-1)\alpha} |dz| \right| f'(r) \|_{L^2(\Omega)}^2 dr,
\]
(4.3)
where we need to require \( \sigma - \gamma \in [0, 1] \).

Further, using Remark 2.1 leads to
\[
\|A^\gamma \partial^s u(t)\|_{L^2(\Omega)} \leq C \|\partial^s u(t)\|_{\mathcal{H}_{0}^{\gamma}(\Omega)} \left( \text{we need to require } 2\gamma \in \left( -\frac{3}{2}, \frac{3}{2} \right) \right)
\leq C \|\partial^s u(t)\|_{\mathcal{H}^{2\gamma}(\Omega)} \left( \text{we need to require } 2\gamma < \frac{1}{2} \right)
\leq C \|u(t)\|_{\mathcal{H}^{2s+2\gamma}(\mathbb{R}^n)} \left( \text{we need to require } 2s + 2\gamma < \frac{3}{2} \right).
\]
(4.4)
Thus choosing a suitable \( \gamma \) such that \( s + \gamma \leq \sigma \), we have
\[
\|A^\sigma u(t)\|_{L^2(\Omega)} \leq C \int_0^t \int_{\Gamma_{\theta,x}} \left| e^{\gamma(t-r)} ||z||^{(\gamma-1)\alpha} |dz| \right| u(r) \|_{\mathcal{H}^{2s+2\gamma}(\Omega)}^2 dr
+ C \int_{\Gamma_{\theta,x}} \left| e^{\gamma(t-r)} ||z||^{(\gamma-1)\alpha} |dz| \right| u_0 \|_{L^2(\Omega)}^2 + C \int_{\Gamma_{\theta,x}} \left| e^{\gamma(t-r)} ||z||^{(\gamma-1)\alpha} |dz| \right| f(0) \|_{L^2(\Omega)}^2 dr
+ C \int_0^t \int_{\Gamma_{\theta,x}} \left| e^{\gamma(t-r)} ||z||^{(\gamma-1)\alpha} |dz| \right| f'(r) \|_{L^2(\Omega)}^2 dr,
\]
where \( \gamma \in (\max\{\sigma - 1 - \epsilon, -\frac{3}{4}\}, \min\{\frac{3}{4} - s, \sigma - s + \epsilon, \frac{1}{4}\}) \) and \( \sigma \in [\max\{0, s - \frac{3}{4} + \epsilon\}, \min\{1, \frac{7}{4} - s - \epsilon\}] \).

Doing simple calculations and taking \( r = |z| \), one has
\[
\int_{\Gamma_{\theta,x}} \left| e^{\gamma(t-r)} ||z||^{\beta} |dz| \right| \leq \int_{\kappa}^\infty e^{\epsilon \cos(\theta)\eta} \rho^{\beta} d\eta + \kappa^{1+\beta} \int_{\theta}^\pi e^{\kappa \cos(\eta)\eta} d\eta \leq Ct^{\beta - 1} + C\kappa^{1+\beta}.
\]
When \( \beta \geq -1 \), the fact \( T/t \geq 1 \) gives \( \int_{\Gamma_{\theta,x}} \left| e^{\gamma(t-r)} ||z||^{\beta} |dz| \right| \leq Ct^{\beta - 1} \); as for \( \beta < -1 \), the same estimate can be obtained by taking \( \kappa > 1/t \). Thus
\[
\|A^\sigma u(t)\|_{L^2(\Omega)} \leq C \int_0^t \int_{\Gamma_{\theta,x}} \left| e^{\gamma(t-r)} ||z||^{(\gamma-1)\alpha} |dz| \right| u(r) \|_{\mathcal{H}^{2s+2\gamma}(\Omega)}^2 dr
+ C t^{\sigma\alpha} \|u_0\|_{L^2(\Omega)}^2 + C t^{(1-s)\alpha} \|f(0)\|_{L^2(\Omega)}^2
+ C \int_0^t \int_{\Gamma_{\theta,x}} \left| e^{\gamma(t-r)} ||z||^{(\gamma-1)\alpha} |dz| \right| f'(r) \|_{L^2(\Omega)}^2 dr.
\]
Combining Lemma 4.1 and \( s + \gamma \leq \sigma \), the desired results are obtained. \( \Box \)
Remark 4.1 Alternatively, we can also get the above regularity estimate by combining the elliptic regularity of the operator $A + s\partial^\alpha$ (see Proposition 3.4 in [7]) and resolvent estimate of the operator $A + s\partial^\alpha$ (see Eq. (1.6) in [16]).

Remark 4.2 From Theorem 4.1, we find the optimal regularity of $u$ is $u \in \dot{H}^2(\Omega)$ for $s \in (0, \frac{3}{4})$, but for $s \in [\frac{3}{4}, 1)$, it only behaves as $u \in \dot{H}^{\frac{7}{2} - 2s - 2\varepsilon}(\Omega)$. The main reason is as follows. For a function $w \in C^{0,\gamma}_{\infty}(\Omega) \cap H_0^1(\Omega)$, only has $w \in H^1(\Omega)$ with $\bar{w}$ being the extension of $w$ with zero values outside $\Omega$. Thus in (4.4), we must require $2\gamma < \frac{3}{4} - 2s$. On the other hand, to preserve the boundedness of $\|A^\sigma u(t)\|_{L^2(\Omega)}$, we need to require $\gamma \geq \sigma - 1$ in (4.3). Combining the above requirements, we can obtain $\sigma < \frac{7}{4} - s$, i.e., the optimal regularity of $u$ is $u \in \dot{H}^{\frac{7}{2} - 2s - 2\varepsilon}(\Omega)$ for $s \in [\frac{3}{4}, 1)$.

### 4.2 Spatial Error Estimate

To get the spatial error estimate, we first provide the expression of $u_h$ from (3.3). By Laplace transform, there holds

$$\tilde{u}_h = -\tilde{E}_h(z)s_h^\alpha \tilde{u}_h + \tilde{z}^{\alpha - 1}\tilde{E}_h(z)P_h u_0 + \tilde{E}_h(z)P_h \tilde{f},$$

where

$$\tilde{E}_h(z) = (z^\alpha + A_h)^{-1}.$$

Then, we provide some useful lemmas.

**Lemma 4.2** ([5, 20]) Let $v \in L^2(\Omega)$ and $z \in \Sigma_\theta$. Denote $w = (z^\alpha + A)^{-1} v$ and $w_h = (z^\alpha + A_h)^{-1} P_h v$. Then one has

$$\|w - w_h\|_{L^2(\Omega)} + h\|w - w_h\|_{H^1(\Omega)} \leq Ch^2\|v\|_{L^2(\Omega)}.$$

Similar to the proofs of Lemma 3.4 in [5], we have

**Lemma 4.3** Let $v \in \dot{H}^{-\sigma}(\Omega)$ with $\sigma \in [0, 1]$ and $z \in \Sigma_\theta$. Denote $w = (z^\alpha + A)^{-1} v$ and $w_h = (z^\alpha + A_h)^{-1} P_h v$. Then it holds

$$\|w - w_h\|_{L^2(\Omega)} + h\|w - w_h\|_{H^1(\Omega)} \leq Ch^{2 - \sigma}\|v\|_{\dot{H}^{-\sigma}(\Omega)}.$$

By using Lemmas 4.2 and 4.3, the boundedness of $\|A_h^\alpha P_h A^\sigma\|$ with $s \in [0, \frac{1}{2}]$ (one can refer to Eq. (4.7) in [13] or the second formula at page 354 of [20]) and the stability of $L^2$ projection $P_h$ [20], it holds

$$\|\tilde{E}(z) - \tilde{E}_h(z)P_h\| \leq \begin{cases} Ch^2 & \text{for } z \in \Sigma_\theta, \\ C|z|^{-\alpha} & \text{for } z \in \Sigma_\theta; \end{cases}$$

and

$$\|\tilde{E}(z) - \tilde{E}_h(z)P_h\|_{L^2(\Omega) \rightarrow H^1(\Omega)} \leq Ch \quad \text{for } z \in \Sigma_\theta;$$

$$\|\tilde{E}(z) - \tilde{E}_h(z)P_h\|_{H^{-1}(\Omega) \rightarrow L^2(\Omega)} \leq \begin{cases} Ch & \text{for } z \in \Sigma_\theta, \\ C|z|^{-\alpha/2} & \text{for } z \in \Sigma_\theta. \end{cases}$$

In what follows, introduce $\tilde{u}_h(t) : (0, T] \rightarrow X_h$ as the solution of the following auxiliary equation to help to obtain the spatial error estimate, i.e.,

$$(0\partial_t^\alpha (\tilde{u}_h - u_{0,h}), v_h) + (\nabla \tilde{u}_h, \nabla v_h) + \langle u, v_h \rangle_s = (f, v_h) \quad \forall v_h \in X_h,$$
where \( u \) is the exact solution of (1.1) and \( u_{0,h} = P_h u_0 \). Equation (4.8) can be rewritten as

\[
\begin{align*}
0 & \partial_t^\alpha (\tilde{u}_h - u_{0,h}) + A_h \tilde{u}_h + P_h \mathcal{A}^{\xi} u = f_h \quad (x, t) \in \Omega \times (0, T], \\
\tilde{u}_{0,h} &= P_h u_0 \quad x \in \Omega.
\end{align*}
\]

(4.9)

Taking Laplace transform for (4.9) gives

\[
\tilde{u}_h = -\tilde{E}_h(z) P_h \mathcal{A}^{\xi} \tilde{u} + z^{\alpha - 1} \tilde{E}_h(z) P_h u_0 + \tilde{E}_h(z) P_h \tilde{f}.
\]

(4.10)

**Lemma 4.4** Let \( u(t) \) and \( \tilde{u}_h(t) \) be the solutions of (1.1) and (4.9), respectively. Assume \( u_0 \in L^2(\Omega), f(0) \in L^2(\Omega), \) and \( \int_0^t \| f'(r) \|_{L^2(\Omega)} dr < \infty \). Then the following estimates hold

\[
\| u(t) - \tilde{u}_h(t) \|_{\tilde{H}^s(\Omega)} \leq \begin{cases} 
Ch^{2-\sigma} t^{\alpha} \| u_0 \|_{L^2(\Omega)} + Ch^{2-\sigma} \| f(0) \|_{L^2(\Omega)} \\
+ Ch^{2-\sigma} \int_0^1 \| f'(r) \|_{L^2(\Omega)} dr, & s \in \left(0, \frac{3}{4}\right), \\
Ch^{2.5-2s-\epsilon - \sigma} t^{\alpha} \| u_0 \|_{L^2(\Omega)} + Ch^{2.5-2s-\epsilon - \sigma} \| f(0) \|_{L^2(\Omega)} \\
+ Ch^{3.5-2s-\epsilon - \sigma} \int_0^1 \| f'(r) \|_{L^2(\Omega)} dr, & s \in \left[\frac{3}{4}, 1\right)
\end{cases}
\]

with \( \sigma \in [0, 1] \).

**Proof** According to (4.1) and (4.10) and using (4.6), one can get

\[
\| u(t) - \tilde{u}_h(t) \|_{L^2(\Omega)} \leq \begin{align*}
C & \left\| \int_{\Gamma_{t,x}} e^{zt} (\tilde{E}(z) - \tilde{E}_h(z) P_h) \mathcal{A}^{\xi} \tilde{u}dz \right\|_{L^2(\Omega)} \\
+ C & \left\| \int_{\Gamma_{t,x}} e^{zt} z^{\alpha - 1} (\tilde{E}(z) - \tilde{E}_h(z) P_h) u_0 dz \right\|_{L^2(\Omega)} \\
+ C & \left\| \int_{\Gamma_{t,x}} e^{zt} z^{-1} (\tilde{E}(z) - \tilde{E}_h(z) P_h) f(0) dz \right\|_{L^2(\Omega)} \\
+ C & \left\| \int_{\Gamma_{t,x}} e^{zt} z^{-1} (\tilde{E}(z) - \tilde{E}_h(z) P_h) \tilde{f} dz \right\|_{L^2(\Omega)} \\
\leq C & \left\| \int_{\Gamma_{t,x}} e^{zt} (\tilde{E}(z) - \tilde{E}_h(z) P_h) \mathcal{A}^{\xi} \tilde{u}dz \right\|_{L^2(\Omega)} \\
+ Ch^{2-\sigma} \| u_0 \|_{L^2(\Omega)} + Ch^2 \| f(0) \|_{L^2(\Omega)} + Ch^2 \int_0^t \| f'(s) \|_{L^2(\Omega)} ds.
\end{align*}
\]
Using (4.1), there holds
\[
\left\| \int_{T_{0,k}} e^{iz} (\tilde{E}(z) - \tilde{E}_h(z) P_h) \cdot \nabla^s \tilde{u} dz \right\|_{L^2(\Omega)} \\
\leq C \left\| \int_{T_{0,k}} e^{iz} (\tilde{E}(z) - \tilde{E}_h(z) P_h) \cdot \nabla^s \tilde{E}(z) \cdot \nabla^s \tilde{u} dz \right\|_{L^2(\Omega)} \\
+ C \left\| \int_{T_{0,k}} e^{iz} (\tilde{E}(z) - \tilde{E}_h(z) P_h) \cdot \nabla^s \tilde{E}(z) z^{s-1} u_0 dz \right\|_{L^2(\Omega)} \\
+ C \left\| \int_{T_{0,k}} e^{iz} (\tilde{E}(z) - \tilde{E}_h(z) P_h) \cdot \nabla^s \tilde{E}(z) z^{-1} f(0) dz \right\|_{L^2(\Omega)} \\
+ C \left\| \int_{T_{0,k}} e^{iz} (\tilde{E}(z) - \tilde{E}_h(z) P_h) \cdot \nabla^s \tilde{E}(z) z^{-1} f'(r) dz \right\|_{L^2(\Omega)} \\
\leq I + II + III + IV.
\]
For \( s \in (0, \frac{3}{4}) \), using the property of \( \cdot \nabla^s \), we have \( \| \cdot \nabla^s \tilde{E}(z) \| \leq C \| \tilde{E}(z) \| \rightarrow \hat{H}^{2s}(\Omega) \leq C |z|^{(s-1)\alpha} \), thus combining (4.6) and convolution property, one can get
\[
I \leq C \int_0^t \int_{T_{0,k}} |e^{z(t-r)}||\tilde{E}(z) - \tilde{E}_h(z) P_h|| \cdot |\nabla^s \tilde{E}(z)||dz||\nabla^s u(r)||_{L^2(\Omega)} dr \\
\leq Ch^2 \int_0^t \int_{T_{0,k}} |e^{z(t-r)}| |z|^{(s-1)\alpha} |dz||u(r)||_{\hat{H}^{2s}(\Omega)} dr \\
\leq Ch^2 \int_0^t (t - r)^{(1-s)\alpha-1} ||u(r)||_{\hat{H}^{2s}(\Omega)} dr.
\]
Using embedding theorem, Theorem 4.1, and \( T/t \geq 1 \), one has
\[
I \leq Ch^2 t^{-\alpha} ||u_0||_{L^2(\Omega)} + Ch^2 ||f(0)||_{L^2(\Omega)} + Ch^2 \int_0^t ||f'(r)||_{L^2(\Omega)} dr.
\]
Similarly, we have
\[
II + III + IV \leq Ch^2 t^{-\alpha} ||u_0||_{L^2(\Omega)} + Ch^2 ||f(0)||_{L^2(\Omega)} + Ch^2 \int_0^t ||f'(r)||_{L^2(\Omega)} dr.
\]
Thus
\[
\|u(t) - \tilde{u}_h(t)\|_{L^2(\Omega)} \leq Ch^2 t^{-\alpha} ||u_0||_{L^2(\Omega)} + Ch^2 ||f(0)||_{L^2(\Omega)} + Ch^2 \int_0^t ||f'(r)||_{L^2(\Omega)} dr.
\]
Similarly, it holds
\[
\|u(t) - \tilde{u}_h(t)\|_{\hat{H}^{1}(\Omega)} \leq Ch^{-\alpha} ||u_0||_{L^2(\Omega)} + Ch \|f\|_{L^2(\Omega)} + Ch \int_0^t \|f'(r)\|_{L^2(\Omega)} dr.
\]
Combining interpolation property [3], we obtain
\[
\|u(t) - \tilde{u}_h(t)\|_{\hat{H}^\alpha(\Omega)} \leq Ch^{2-\sigma} t^{-\sigma} ||u_0||_{L^2(\Omega)} + Ch^{2-\sigma} \|f\|_{L^2(\Omega)} + Ch^{2-\sigma} \int_0^t \|f'(r)\|_{L^2(\Omega)} dr
\]
with \( \sigma \in [0, 1] \).
Remark 4.3 In fact, when \( s \in \left[ \frac{3}{4}, 1 \right) \), using convolution property, interpolation property, and Eqs. (4.6) and (4.7), one can obtain
\[
I \leq C \int_0^t \int_{\Gamma_{h,k}} |e^{z(t-r)}| \| \tilde{E}(z) - \tilde{E}(z) P_h \| A^s \frac{3-\varepsilon}{2} + \frac{3}{2} \| A^s \frac{3-\varepsilon}{2} \sigma \| \| u(r) \| L^2(\Omega) dr
\]

\[
\leq Ch^{3.5-2s-\varepsilon} \int_0^t \int_{\Gamma_{h,k}} |e^{z(t-r)}| |z|^{(s-1)a} |d\tilde{z}| \| u(r) \| L^2(\Omega) dr
\]

\[
\leq Ch^{3.5-2s-\varepsilon} \int_0^t (t-r)^{(1-s)a-1} \| u(r) \| L^2(\Omega) dr
\]

\[
\leq Ch^{3.5-2s-\varepsilon} \| u_0 \| L^2(\Omega) + Ch^{3.5-2s-\varepsilon} \| f(0) \| L^2(\Omega) + Ch^{3.5-2s-\varepsilon} \int_0^t \| f'(r) \| L^2(\Omega) dr.
\]

Thus, there holds
\[
\| u(t) - \tilde{u}(t) \| L^2(\Omega) \leq Ch^{3.5-2s-\varepsilon} \| u_0 \| L^2(\Omega)
\]

\[
+ Ch^{3.5-2s-\varepsilon} \| f(0) \| L^2(\Omega) + Ch^{3.5-2s-\varepsilon} \int_0^t \| f'(r) \| L^2(\Omega) dr.
\]

As for the \( \tilde{H}^1 \)-norm error estimate, we can obtain analogously
\[
\| u(t) - \tilde{u}(t) \| \tilde{H}^1(\Omega) \leq Ch^{2.5-2s-\varepsilon} \| u_0 \| L^2(\Omega)
\]

\[
+ Ch^{2.5-2s-\varepsilon} \| f(0) \| L^2(\Omega) + Ch^{2.5-2s-\varepsilon} \int_0^t \| f'(r) \| L^2(\Omega) dr.
\]

The interpolation property gives
\[
\| u(t) - \tilde{u}(t) \| \tilde{H}^\sigma(\Omega) \leq Ch^{3.5-2s-\varepsilon-\sigma} \| u_0 \| L^2(\Omega)
\]

\[
+ Ch^{3.5-2s-\varepsilon-\sigma} \| f(0) \| L^2(\Omega) + Ch^{3.5-2s-\varepsilon-\sigma} \int_0^t \| f'(r) \| L^2(\Omega) dr.
\]

with \( \sigma \in [0, 1] \). The proof is completed. \( \square \)

Remark 4.3 In fact, when \( s \in \left[ \frac{3}{4}, 1 \right) \), the estimate of \( \| u(t) - \tilde{u}(t) \| \tilde{H}^\sigma(\Omega) \) with \( \sigma \in [0, 1] \) should be
\[
\| u(t) - \tilde{u}(t) \| \tilde{H}^\sigma(\Omega) \leq Ch^{3.5-2s-\varepsilon-\sigma} \| u_0 \| L^2(\Omega)
\]

\[
+ Ch^{3.5-2s-\varepsilon-\sigma} \| f(0) \| L^2(\Omega) + Ch^{3.5-2s-\varepsilon-\sigma} \int_0^t \| f'(r) \| L^2(\Omega) dr
\]

\[
+ Ch^{2-\sigma} \| u_0 \| L^2(\Omega) + Ch^{2-\sigma} \| f(0) \| L^2(\Omega) + Ch^{2-\sigma} \int_0^t \| f'(r) \| L^2(\Omega) dr.
\]

Due to the index \( 2 > 3.5 - 2s - \varepsilon \), we omit the last three terms in (4.11). But from the numerical experiments, we find these terms may have influence on the convergence rates because the mesh size \( h \) cannot be arbitrarily small constrained by memory.
Below we consider the estimate of $\tilde{u}_h(t) - u_h(t)$ in some specific space $V$ ($V$ is $L^2(\Omega)$ or $H^{2s-1}(\Omega)$). According to Eqs. (3.2), (4.5), and (4.10), one can split it into the following two parts

$$
\|\tilde{u}_h(t) - u_h(t)\|_V \leq C \left\| \int_{\Gamma_{h,s}} e^{zt} (\tilde{E}_h(z) P_h \mathcal{A}^s \tilde{u} - \tilde{E}_h(z) \mathcal{A}^s \tilde{u}_h) dz \right\|_V \\
\leq C \left\| \int_{\Gamma_{h,s}} e^{zt} (\tilde{E}_h(z) P_h \mathcal{A}^s \tilde{u} - \tilde{E}_h(z) \mathcal{A}^s \tilde{u}_h) dz \right\|_V \\
\leq C \left\| \int_{\Gamma_{h,s}} e^{zt} \tilde{E}_h(z) P_h \mathcal{A}^s (\tilde{u} - \tilde{u}_h) dz \right\|_V \\
\leq C \left\| \int_{\Gamma_{h,s}} e^{zt} \tilde{E}_h(z) P_h \mathcal{A}^s (\tilde{u} - \tilde{u}_h) dz \right\|_V \\
+ C \left\| \int_{\Gamma_{h,s}} e^{zt} \tilde{E}_h(z) P_h \mathcal{A}^s (\tilde{u} - \tilde{u}_h) dz \right\|_V \\
\leq C \|I(t)\|_V + C \|I(t)\|_V.
$$

(4.12)

where we use the definition of $R^s_h$. 

Next, we consider the estimates of $I(t)$ and $II(t)$ in different spaces.

**Lemma 4.5** Let $I(t)$ be defined in (4.12), $u_0 \in L^2(\Omega)$, $f(0) \in L^2(\Omega)$, and $\int_0^t \|f'(r)\|_{L^2(\Omega)} dr < \infty$. Then the following estimates hold

$$
\|I(t)\|_{L^2(\Omega)} \leq \begin{cases} 
Ch^{(2-2s)(1-\epsilon)} \int_0^t (t - r)^{(1-s)\alpha - 1} \|u(r) - u_h(r)\|_{L^2(\Omega)} dr, & s \in \left(0, \frac{1}{2}\right], \\
Ch^{-\epsilon} \int_0^t (t - r) \frac{\alpha - 1}{2} \|u(r) - u_h(r)\|_{\tilde{H}^{2s-1}(\Omega)} dr, & s \in \left(\frac{1}{2}, 1\right), 
\end{cases}
$$

and

$$
\|I(t)\|_{\tilde{H}^{2s-1}(\Omega)} \leq C \int_0^t (t - r)^{(1-s)\alpha - 1} \|u(r) - u_h(r)\|_{\tilde{H}^{2s-1}(\Omega)} dr, & s \in \left(\frac{1}{2}, 1\right).
$$

**Proof** For $s \in (0, \frac{1}{2}]$, we have

$$
\|I(t)\|_{L^2(\Omega)} \leq C \left\| \int_{\Gamma_{h,s}} e^{zt} (\tilde{E}_h(z) P_h - \tilde{E}(z)) A^s A^{-s} \mathcal{A}^s (\tilde{u} - \tilde{u}_h) dz \right\|_{L^2(\Omega)} \\
\leq C \left\| \int_{\Gamma_{h,s}} |e^{zt(\epsilon)}| \|((\tilde{E}_h(z) P_h - \tilde{E}(z)) A^s A^{-s} \mathcal{A}^s (\tilde{u} - \tilde{u}_h)) \|dz| \right\|_{L^2(\Omega)} \\
\times \left\| \|A^{-s} \mathcal{A}^s (u(r) - u_h(r)) \|_{L^2(\Omega)} dr. 
$$

Using (4.6), (4.7), and interpolation properties, we have

$$
\|((\tilde{E}_h(z) P_h - \tilde{E}(z)) A^s \| \leq Ch^{(2-2s)(1-\epsilon)} |z|^{(s-1)\alpha \epsilon},
$$

which yields

$$
\|I(t)\|_{L^2(\Omega)} \leq Ch^{(2-2s)(1-\epsilon)} \int_0^t (t - r)^{(1-s)\alpha - 1} \|u(r) - u_h(r)\|_{L^2(\Omega)} dr.
$$
Similarly, for \( s \in \left( \frac{1}{2}, 1 \right) \), one can get
\[
\| I(t) \|_{L^2(\Omega)} \leq C \left\| \int_{I_{0,x}} e^{zt} (\tilde{E}_h(z) P_h - \tilde{E}(z)) A^\frac{1}{2} A^{-\frac{1}{2} \partial^s} (\tilde{u} - \tilde{u}_h) dz \right\|_{L^2(\Omega)} \\
\leq C \int_{0}^{t} \int_{I_{0,x}} |e^{z(t-r)}| \| (\tilde{E}_h(z) P_h - \tilde{E}(z)) A^\frac{1}{2} \| |dz| \\
\times \left\| A^{-\frac{1}{2} \partial^s} (u(r) - u_h(r)) \right\|_{L^2(\Omega)} \, dr \\
\leq Ch^{1-s} \int_{0}^{t} (t - r)^{\frac{2s - 1}{2}} \| u(r) - u_h(r) \|_{\dot{H}^{2s-1}(\Omega)} \, dr.
\]

On the other hand, for \( s \in \left( \frac{1}{2}, 1 \right) \), there holds
\[
\| I(t) \|_{\dot{H}^{2s-1}(\Omega)} \leq C \left\| \int_{I_{0,x}} e^{zt} A^s \left( \tilde{E}_h(z) P_h - \tilde{E}(z) \right) A^{\frac{1}{2}} A^{-\frac{1}{2} \partial^s} (\tilde{u} - \tilde{u}_h) dz \right\|_{L^2(\Omega)} \\
\leq C \int_{0}^{t} \int_{I_{0,x}} |e^{z(t-r)}| \| A^{\frac{1}{2}} (\tilde{E}_h(z) P_h - \tilde{E}(z)) A^{\frac{1}{2}} \| |dz| \\
\times \left\| A^{-\frac{1}{2} \partial^s} (u(r) - u_h(r)) \right\|_{L^2(\Omega)} \, dr \\
\leq C \int_{0}^{t} (t - r)^{(1-s)\alpha - 1} \| u(r) - u_h(r) \|_{\dot{H}^{2s-1}(\Omega)} \, dr,
\]
where we have used Remark 2.1 and the fact \( \partial^s : H^1(\mathbb{R}^n) \to H^{1-2s}(\mathbb{R}^n) \). \( \square \)

**Lemma 4.6** Let \( I(t) \) be defined in (4.12). If \( u_0 \in L^2(\Omega) \), \( f(0) \in L^2(\Omega) \), and \( \int_{0}^{t} \| f'(r) \|_{L^2(\Omega)} \, dr < \infty \), then we have
\[
\| I(t) \|_{L^2(\Omega)} \leq C \int_{0}^{t} (t - r)^{(1-s)\alpha - 1} \| u(r) - u_h(r) \|_{L^2(\Omega)} \, dr, \quad s \in \left( 0, \frac{3}{4} \right)
\]
and
\[
\| I(t) \|_{\dot{H}^{2s-1}(\Omega)} \leq C \int_{0}^{t} (t - r)^{(1-s)\alpha - 1} \| u(r) - u_h(r) \|_{\dot{H}^{2s-1}(\Omega)} \, dr, \quad s \in \left( \frac{1}{2}, 1 \right).\]

**Proof** For \( s \in \left( 0, \frac{3}{4} \right) \), using resolvent estimate [11, 12, 16] and Remark 2.1, one has
\[
\| I(t) \|_{L^2(\Omega)} \leq C \left\| \int_{I_{0,x}} e^{zt} \tilde{E}(z) A^s A^{-s} \partial^s (\tilde{u} - \tilde{u}_h) dz \right\|_{L^2(\Omega)} \\
\leq C \int_{0}^{t} \int_{I_{0,x}} |e^{z(t-r)}| \| \tilde{E}(z) A^s \| |dz| \| A^{-s} \partial^s (u(r) - u_h(r)) \|_{L^2(\Omega)} \, dr \\
\leq C \int_{0}^{t} (t - r)^{(1-s)\alpha - 1} \| u(r) - u_h(r) \|_{L^2(\Omega)} \, dr.
\]
Moreover, for $s \in (\frac{1}{2}, 1)$, simple calculations lead to

$$
\| I I (t) \|_{\bar{H}^{2s-1}(\Omega)} \leq C \left\| \int_{I_{\bar{v}, s}} e^{z t} A^{4 - \frac{1}{2}} \tilde{E}(z) A^{\frac{1}{2}} A^{-\frac{1}{2}} g f(s)(\bar{u} - \bar{u}_h) dz \right\|_{L^2(\Omega)} \\
\leq C \int_0^t (t - r)^{(1-s)\alpha - 1} \| u(r) - u_h(r) \|_{\bar{H}^{2s-1}(\Omega)} dr.
$$

The proof is completed. \(\square\)

Thanks to the above lemmas, we can provide the following error estimate for the spatial semi-discrete scheme.

**Theorem 4.2** Let $u(t)$ and $u_h(t)$ be the solutions of Eqs. (1.1) and (3.3), respectively. Assuming $u_0 \in L^2(\Omega) \cap H^1(\Omega)$, $f(0) \in L^2(\Omega)$, and $\int_0^t \| f'(r) \|_{L^2(\Omega)} dr < \infty$, we have

$$
\| u(t) - u_h(t) \|_{L^2(\Omega)} \leq C t^{-\alpha} h^{3-2s} \| u_0 \|_{L^2(\Omega)} + C h^{3-2s} \| f(0) \|_{L^2(\Omega)} \\
+ C h^{4.5-4s-\epsilon} \| u_0 \|_{L^2(\Omega)} + C h^{4.5-4s-\epsilon} \| f(0) \|_{L^2(\Omega)} \\
+ C h^{4.5-4s-\epsilon} \int_0^t \| f'(r) \|_{L^2(\Omega)} dr, \quad s \in \left(0, \frac{3}{4}\right),
$$

and

$$
\| u(t) - u_h(t) \|_{\bar{H}^{2s-1}(\Omega)} \leq \begin{cases} 
C t^{-\alpha} h^{3-2s} \| u_0 \|_{L^2(\Omega)} + C h^{3-2s} \| f(0) \|_{L^2(\Omega)} \\
+ C h^{4.5-4s-\epsilon} \| u_0 \|_{L^2(\Omega)} + C h^{4.5-4s-\epsilon} \| f(0) \|_{L^2(\Omega)} \\
+ C h^{4.5-4s-\epsilon} \int_0^t \| f'(r) \|_{L^2(\Omega)} dr, \quad s \in \left[\frac{3}{4}, 1\right).
\end{cases}
$$

**Proof** For $s \in (0, \frac{1}{2}]$, Lemmas 4.4, 4.5, and 4.6 give

$$
\| u(t) - u_h(t) \|_{L^2(\Omega)} \leq \| u(t) - \bar{u}_h(t) \|_{L^2(\Omega)} + \| \bar{u}_h(t) - u_h(t) \|_{L^2(\Omega)} \\
\leq C h^{2-\alpha} \| u_0 \|_{L^2(\Omega)} + C h^{2} \| f(0) \|_{L^2(\Omega)} + C h^{2} \int_0^t \| f'(r) \|_{L^2(\Omega)} dr \\
+ C h^{2} \int_0^t (t - r)^{(1-s)\alpha - 1} \| u(r) - u_h(r) \|_{L^2(\Omega)} dr \\
+ C \int_0^t (t - r)^{(1-s)\alpha - 1} \| u(r) - u_h(r) \|_{L^2(\Omega)} dr \\
\leq C h^{2-\alpha} \| u_0 \|_{L^2(\Omega)} + C h^{2} \| f(0) \|_{L^2(\Omega)} + C h^{2} \int_0^t \| f'(r) \|_{L^2(\Omega)} dr \\
+ C \int_0^t (t - r)^{(1-s)\alpha - 1} \| u(r) - u_h(r) \|_{L^2(\Omega)} dr.
$$

Thus, by Lemma 4.1, one has

$$
\| u(t) - u_h(t) \|_{L^2(\Omega)} \leq C h^{2-\alpha} \| u_0 \|_{L^2(\Omega)} + C h^{2} \| f(0) \|_{L^2(\Omega)} + C h^{2} \int_0^t \| f'(r) \|_{L^2(\Omega)} dr.
$$
Combining the above estimate, we get while to desire by Lemma 4.1.

Thus one has

\[ \|u(t) - u_h(t)\|_{\tilde{H}^{2s-1}(\Omega)} \]

\[ \leq C t^{3-2s} T^{-\alpha} \|u_0\|_{L^2(\Omega)} + C t^2 \|f(0)\|_{L^2(\Omega)} + C t^{3-2s} \int_0^T \|f'(r)\|_{L^2(\Omega)} \] 

\[ + C \int_0^T (t-r)^{(1-s)\alpha-1} \|u(r) - u_h(r)\|_{\tilde{H}^{2s-1}(\Omega)} \, dr. \]

Combining the above estimate, we get

\[ \|u(t) - u_h(t)\|_{L^2(\Omega)} \]

\[ \leq \|u(t) - \bar{u}_h(t)\|_{L^2(\Omega)} + \|\bar{u}_h(t) - u_h(t)\|_{L^2(\Omega)} \]

\[ \leq C t^{-\alpha} h^2 \|u_0\|_{L^2(\Omega)} + C t^2 \|f(0)\|_{L^2(\Omega)} + C t^{3-2s} \int_0^T \|f'(r)\|_{L^2(\Omega)} \] 

\[ + C \int_0^T (t-r)^{(1-s)\alpha-1} \|u(r) - u_h(r)\|_{\tilde{H}^{2s-1}(\Omega)} \, dr \]

\[ \leq C h^2 \|u_0\|_{L^2(\Omega)} + C t^2 \|f(0)\|_{L^2(\Omega)} + C t^{3-2s} \int_0^T \|f'(r)\|_{L^2(\Omega)} \] 

\[ + C \int_0^T (t-r)^{(1-s)\alpha-1} \|u(r) - u_h(r)\|_{L^2(\Omega)} \, dr, \]

which leads to the desired result by Lemma 4.1.

Similarly, for \( s \in \left(\frac{3}{4}, 1\right) \), there exists

\[ \|u(t) - u_h(t)\|_{\tilde{H}^{2s-1}(\Omega)} \]

\[ \leq C h^{4.5-4s-\epsilon} t^{-\alpha} \|u_0\|_{L^2(\Omega)} + C h^{4.5-4s-\epsilon} \|f(0)\|_{L^2(\Omega)} + C h^{4.5-4s-\epsilon} \int_0^T \|f'(r)\|_{L^2(\Omega)} \]

The proof is completed. \( \square \)

**Remark 4.4** In Theorem 4.2, we mainly provide the optimal estimates of \( u(t) - u_h(t) \) in \( L^2(\Omega) \)-norm for \( s \in (0, \frac{\alpha}{4}) \). As for the the estimate of \( \|u(t) - u_h(t)\|_{L^2(\Omega)} \) for \( s \in \left[\frac{\alpha}{4}, 1\right) \), we have

\[ \|u(t) - u_h(t)\|_{L^2(\Omega)} \leq C t^{-\alpha} h^{4.5-4s-\epsilon} \|u_0\|_{L^2(\Omega)} + C h^{4.5-4s-\epsilon} \|f(0)\|_{L^2(\Omega)} \]

\[ + C h^{4.5-4s-\epsilon} \int_0^T \|f'(r)\|_{L^2(\Omega)} \, dr, \]

but we find this estimate is not optimal from the numerical results. Some new techniques need to be introduced to get the optimal estimate.
4.3 Temporal Error Estimate

Introduce \( \mathcal{L}_h = A_h + \mathcal{A}_h^{\alpha} \) and \( \mu_h^n = u_h^n - u_h^0 \). Then the fully discrete scheme (3.5) can be rewritten as

\[
\sum_{j=0}^{n-1} d_j^{(\alpha)} \mu_h^{n-j} + \mathcal{L}_h \mu_h^n = f_h^n - \mathcal{L}_h u_h^0. \tag{4.13}
\]

Let \( \mu_h = u_h - u_h^0 \). Then the semi-discrete scheme (3.3) can also be represented as

\[
0 \partial_t^\alpha \mu_h + \mathcal{L}_h \mu_h = f_h - \mathcal{L}_h u_h^0. \tag{4.14}
\]

Multiplying \( \xi^n \) on both sides of (4.13) and summing \( n \) from 1 to \( \infty \) show

\[
\sum_{n=1}^{\infty} \sum_{j=0}^{n-1} d_j^{(\alpha)} \mu_h^{n-j} \xi^n + \sum_{n=1}^{\infty} \mathcal{L}_h \mu_h^n \xi^n = \sum_{n=1}^{\infty} f_h^n \xi^n - \mathcal{L}_h \sum_{n=1}^{\infty} u_h^0 \xi^n.
\]

which leads to

\[
\left( \varphi^{(\alpha)}(\xi) + \mathcal{L}_h \right) \sum_{n=1}^{\infty} \mu_h^n \xi^n = \sum_{n=1}^{\infty} f_h^n \xi^n - \mathcal{L}_h \sum_{n=1}^{\infty} u_h^0 \xi^n
\]

with

\[
\varphi^{(\alpha)}(\xi) = \sum_{j=0}^{\infty} d_j^{(\alpha)} \xi^j = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \frac{(1-\xi)^2}{\xi} L_{i\alpha-1}(\xi).
\]

Here, the definitions of \( d_j^{(\alpha)} \) can refer to (3.4) and \( L_{i\rho}(z) \) is defined as [14]

\[
L_{i\rho}(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^\rho}.
\]

Thus, we have

\[
\sum_{n=1}^{\infty} \mu_h^n \xi^n = \left( \varphi^{(\alpha)}(\xi) + \mathcal{L}_h \right)^{-1} \left( \sum_{j=1}^{\infty} f_h^j \xi^j - \mathcal{L}_h \sum_{j=1}^{\infty} u_h^0 \xi^j \right),
\]

which leads to

\[
\mu_h^n = \frac{1}{2\pi i} \int_{|z|=\xi} \xi^{-n-1} \left( \varphi^{(\alpha)}(\xi) + \mathcal{L}_h \right)^{-1} \left( \sum_{j=1}^{\infty} f_h^j \xi^j - \mathcal{L}_h \sum_{j=1}^{\infty} u_h^0 \xi^j \right) d\xi
\]

with \( \xi = e^{-\tau(\kappa+1)} \). Taking \( \xi = e^{-z\tau} \), one obtains

\[
\mu_h^n = \frac{\tau}{2\pi i} \int_{\Gamma^\tau} e^{z\tau n} \left( \varphi^{(\alpha)}(e^{-z\tau}) + \mathcal{L}_h \right)^{-1} \left( \sum_{j=1}^{\infty} f_h^j e^{-ztj} - \mathcal{L}_h \sum_{j=1}^{\infty} u_h^0 e^{-ztj} \right) dz,
\]

where \( \Gamma^\tau = \{ z = \kappa + 1 + iy : y \in \mathbb{R} \text{ and } |y| \leq \pi/\tau \} \). Transforming \( \Gamma^\tau \) to \( \Gamma_{\theta,\kappa}^\tau = \{ z \in \mathbb{C} : \kappa \leq |z| \leq \frac{\pi}{\tau \sin(\theta)} \text{, } |\arg z| = \theta \} \cup \{ z \in \mathbb{C} : |z| = \kappa \text{, } |\arg z| \leq \theta \} \), we have

\[
\mu_h^n = \frac{\tau}{2\pi i} \int_{\Gamma_{\theta,\kappa}^\tau} e^{z\tau n} \left( \varphi^{(\alpha)}(e^{-z\tau}) + \mathcal{L}_h \right)^{-1} \left( \sum_{j=1}^{\infty} f_h^j e^{-ztj} - \mathcal{L}_h \sum_{j=1}^{\infty} u_h^0 e^{-ztj} \right) dz.
\]
Since \( f(t) = f(0) + \int_0^t f'(r) dr \) and taking \( R_h(t) = P_h \int_0^t f'(r) dr \), one has
\[
\mu_h^n = \frac{\tau}{2\pi i} \int_{\Gamma_\theta,\kappa} e^{z\tau} (\varphi^{(\alpha)}(e^{-z\tau}) + L_h)^{-1} \left( \frac{e^{-z\tau}}{1 - e^{-z\tau}} f_h - \frac{e^{-z\tau}}{1 - e^{-z\tau}} L_h u_h^0 \right) dz
\]
\[
+ \frac{\tau}{2\pi i} \int_{\Gamma_\theta,\kappa} e^{z\tau} (\varphi^{(\alpha)}(e^{-z\tau}) + L_h)^{-1} \sum_{j=1}^\infty R_h(t_j)e^{-z\tau} j dz.
\]

Similarly, the solution of (4.14) can also be written as
\[
\mu_h(t_n) = \frac{1}{2\pi i} \int_{\Gamma_\theta,\kappa} e^{z\tau} (z^\alpha + L_h)^{-1} \left( z^{-1} f_h - z^{-1} L_h u_h^0 \right) dz
\]
\[
+ \frac{1}{2\pi i} \int_{\Gamma_\theta,\kappa} e^{z\tau} (z^\alpha + L_h)^{-1} \tilde{R}_h(z) dz,
\]
where \( \tilde{R}_h(z) \) is the Laplace transform of \( R_h(t) \).

**Lemma 4.7** ([11, 21]) Let \( z \in \Gamma_\theta,\kappa \) with \( \theta \in \left( \frac{\pi}{2}, \frac{5\pi}{6} \right) \). Then there hold
\[
|\varphi^{(\alpha)}(e^{-z\tau}) - z^\alpha| \leq C \tau^{2-\alpha}|z|^2
\]
and
\[
|\varphi^{(\alpha)}(e^{-z\tau})| \geq C|z|\tau^{1-\alpha}.
\]

**Theorem 4.3** Let \( u_h(t) \) and \( u_h^n \) be the solutions of (3.3) and (3.5), respectively. Assuming \( u_0 \in L^2(\Omega) \), \( f(0) \in L^2(\Omega) \), and \( \int_0^t ||f'(r)||_{L^2(\Omega)} dr < \infty \), we have
\[
||u_h(t_n) - u_h^n||_{L^2(\Omega)} \leq C \tau t_n^{-1} ||u_h^0||_{L^2(\Omega)} + C \tau t_n^{-1} ||f_h^0||_{L^2(\Omega)}
\]
\[
+ C \tau \int_0^{t_n} (t_n - r)^{\alpha-1} ||f'(r)||_{L^2(\Omega)} dr.
\]

**Proof** According to the definitions of \( \mu_h(t_n) \) and \( \mu_h^n \), it holds
\[
||\mu_h(t_n) - \mu_h^n||_{L^2(\Omega)}
\]
\[
\leq C \left| \int_{\Gamma_\theta,\kappa} e^{z\tau} (z^\alpha + L_h)^{-1} z^{-1} L_h dz \right|
\]
\[
- \int_{\Gamma_\theta,\kappa} e^{z\tau} (\varphi^{(\alpha)}(e^{-z\tau}) + L_h)^{-1} \left( \frac{e^{-z\tau}}{1 - e^{-z\tau}} L_h u_h^0 \right) ||u_h^0||_{L^2(\Omega)}
\]
\[
+ C \left| \int_{\Gamma_\theta,\kappa} e^{z\tau} (z^\alpha + L_h)^{-1} z^{-1} dz \right|
\]
\[
- \int_{\Gamma_\theta,\kappa} e^{z\tau} (\varphi^{(\alpha)}(e^{-z\tau}) + L_h)^{-1} \left( \frac{e^{-z\tau}}{1 - e^{-z\tau}} dz \right) ||f_h^0||_{L^2(\Omega)}
\]
\[
+ C \left| \int_{\Gamma_\theta,\kappa} e^{z\tau} (z^\alpha + L_h)^{-1} \tilde{R}_h dz \right|
\]
\[
- \int_{\Gamma_\theta,\kappa} e^{z\tau} (\varphi^{(\alpha)}(e^{-z\tau}) + L_h)^{-1} \tau \sum_{j=1}^\infty R_h(t_j)e^{-z\tau} j dz ||L^2(\Omega) \leq I + II + III.
\]
For $I$, there holds

$$I \leq C \left( \left\| \int_{\Gamma_{t,x}^I} e^{zt_h} (z^\alpha + \mathcal{L}_h)^{-1} e^{-1} \mathcal{L}_h dz \right\| + \left\| \int_{\Gamma_{t,x}^I} (e^{zt_h} - e^{zt_{n-1}}) (z^\alpha + \mathcal{L}_h)^{-1} e^{-1} \mathcal{L}_h dz \right\| ight. \left. + \left\| \int_{\Gamma_{t,x}^I} e^{zt_{n-1}} (z^\alpha + \mathcal{L}_h)^{-1} e^{-1} \left( \frac{\tau}{1 - e^{-zt}} \right) \mathcal{L}_h dz \right\| \right) \| u^0_h \|_{L^2(\Omega)}.$$

Lemma 4.7 leads to following fact

$$\left\| \left( \frac{\tau}{1 - e^{-zt}} \right) \mathcal{L}_h \right\| \leq C \tau$$

and combining the resolvent estimate [11, 12, 16], we have

$$I \leq C \tau t_n^{-1} \| u^0_h \|_{L^2(\Omega)}.$$

Similarly, one has

$$II \leq C \tau t_n^{-1} \| f^0_h \|_{L^2(\Omega)}.$$

Using the definition of $R_h(t)$ and doing simple calculations yield

$$\tau \sum_{n=1}^{\infty} R_h(t_n) e^{-zt_n} = \tau \sum_{n=1}^{\infty} \int_{t_n}^{t_{n+1}} P_h f'(r) dr e^{-zt_n}$$

$$= \tau \sum_{n=1}^{\infty} \int_{t_n}^{t_{n+1}} P_h f'(r) dr e^{-zt_n}$$

$$= \tau \sum_{j=1}^{\infty} \left( \int_{t_{j-1}}^{t_j} P_h f'(r) dr \sum_{n=j}^{\infty} e^{-zt_n} \right)$$

$$= \frac{\tau}{1 - e^{-zt}} \sum_{j=1}^{\infty} \left( e^{-zt_j} \int_{t_{j-1}}^{t_j} P_h f'(r) dr \right).$$
which leads to
\[
\left\| \int_0^{t_n} \int_{\Gamma_{\theta,\kappa}} e^{z(t_n-r)} \left( \varphi^{(\alpha)}(e^{-z\tau}) + L_h \right)^{-1} \frac{\tau}{1-e^{-z\tau}} dz P_h f'(r) dr \right\|_{L^2(\Omega)}
\]
\[
- \int_{\Gamma_{\theta,\kappa}} e^{z(t_n-r)} \left( \varphi^{(\alpha)}(e^{-z\tau}) + L_h \right)^{-1} \frac{\tau}{1-e^{-z\tau}} dz P_h f'(r) dr
\]
\[
\leq C \left\| \int_0^{t_n} \int_{\Gamma_{\theta,\kappa}} e^{z(t_n-r)} \left( \varphi^{(\alpha)}(e^{-z\tau}) + L_h \right)^{-1} \frac{\tau}{1-e^{-z\tau}} dz P_h f'(r) dr \right\|_{L^2(\Omega)}
\]
\[
- \sum_{j=1}^n \int_{t_{j-1}}^{t_j} \int_{\Gamma_{\theta,\kappa}} e^{z(t_n-t_j)} \left( \varphi^{(\alpha)}(e^{-z\tau}) + L_h \right)^{-1} \frac{\tau}{1-e^{-z\tau}} dz P_h f'(r) dr \right\|_{L^2(\Omega)}
\]
\[
\leq C \tau \int_0^{t_n} (t_n - r)^{\alpha-1} \| f'(r) \|_{L^2(\Omega)} dr.
\]
Thus
\[
III \leq C \left\| \int_0^{t_n} \left( \int_{\Gamma_{\theta,\kappa}} e^{z(t_n-r)} \left( \varphi^{(\alpha)}(e^{-z\tau}) + L_h \right)^{-1} z^{-1} dz \right) P_h f'(r) dr \right\|_{L^2(\Omega)}
\]
\[
+ C \tau \int_0^{t_n} (t_n - r)^{\alpha-1} \| f'(r) \|_{L^2(\Omega)} dr
\]
\[
\leq C \left\| \int_0^{t_n} \int_{\Gamma_{\theta,\kappa}} e^{z(t_n-r)} \left( \varphi^{(\alpha)}(e^{-z\tau}) + L_h \right)^{-1} \frac{\tau}{1-e^{-z\tau}} dz P_h f'(r) dr \right\|_{L^2(\Omega)}
\]
\[
+ C \left\| \int_0^{t_n} \int_{\Gamma_{\theta,\kappa}} e^{z(t_n-r)} \left( \varphi^{(\alpha)}(e^{-z\tau}) + L_h \right)^{-1} z^{-1} \right\|_{L^2(\Omega)}
\]
\[
- \left( \varphi^{(\alpha)}(e^{-z\tau}) + L_h \right)^{-1} \frac{\tau}{1-e^{-z\tau}} dz P_h f'(r) dr \right\|_{L^2(\Omega)}
\]
\[
+ C \tau \int_0^{t_n} (t_n - r)^{\alpha-1} \| f'(r) \|_{L^2(\Omega)} dr.
\]
Similar to the estimation of I, we have
\[
III \leq C \tau \int_0^{t_n} (t_n - r)^{\alpha-1} \| f'(r) \|_{L^2(\Omega)} dr.
\]
Collecting the above estimates leads to the desired results.

\section{5 Numerical Experiments}

In this section, some numerical examples are presented to validate our theoretical results. Here, we take $\Omega = (0, 1)$. The following four initial values and source terms will be used

(a) $u_0(x) = \chi_{(0.5,1)}(x), \quad f(x, t) = 0;
Table 1 $L^2$ errors and convergence rates with the initial value and source term (a)

| $(\alpha, s)/1/\tau$ | 16   | 32   | 64   | 128  | 256  |
|------------------------|------|------|------|------|------|
| (0.4,0.3)              | 1.722E−04 | 8.360E−05 | 4.116E−05 | 2.041E−05 | 1.015E−05 |
| Rate                   | 1.0425 | 1.0224 | 1.0122 | 1.0068 |
| (0.4,0.7)              | 1.435E−04 | 6.972E−05 | 3.435E−05 | 1.704E−05 | 8.481E−06 |
| Rate                   | 1.0410 | 1.0214 | 1.0115 | 1.0063 |
| (0.8,0.3)              | 1.843E−04 | 8.519E−05 | 4.074E−05 | 1.979E−05 | 9.695E−06 |
| Rate                   | 1.1130 | 1.0642 | 1.0415 | 1.0297 |
| (0.8,0.7)              | 1.396E−04 | 6.519E−05 | 3.137E−05 | 1.531E−05 | 7.522E−06 |
| Rate                   | 1.0982 | 1.0554 | 1.0353 | 1.0248 |

(b) $u_0(x) = 0, \quad f(x, t) = t^{0.1}x^{-0.2};$

(c) $u_0(x) = 1, \quad f(x, t) = 0;$

(d) $u_0(x) = 4x(1-x), \quad f(x, t) = 0,$

where $\chi(a,b)(x)$ means the characteristic function on $(a, b)$. Due to the exact solution is unknown, we use $e_h$ and $e_\tau$ to measure spatial and temporal errors, whose definitions are

$$e_h = u_h - u_h/2, \quad e_\tau = u_\tau - u_\tau/2.$$  

Here $u_h$ and $u_\tau$ denote the numerical solutions under the mesh size $h$ and time step size $\tau$, respectively. Thus the resulting convergence rates in some specific space $\mathcal{V}$ can be calculated by

$$Rate = \frac{\log(\|e_h\|_\mathcal{V}/\|e_h/2\|_\mathcal{V})}{\log(2)}$$

and

$$Rate = \frac{\log(\|e_\tau\|_\mathcal{V}/\|e_\tau/2\|_\mathcal{V})}{\log(2)}.$$  

Example 1 We take $T = 1$ and (a) as the initial value and source term to verify the convergence in temporal direction. Here, to avoid the influence of the spatial errors on temporal errors, we take $h = 1/512$. The corresponding $L^2$ errors and convergence rates are presented in Table 1 and all results agree with Theorem 4.3.

Example 2 Here, we show the $L^2$ errors and convergence rates in temporal direction with the initial value and source term (b) and $T = 1$. We take $h = 1/512$ to decrease the influence caused by spatial discretization. All the corresponding results are shown in Table 2 and in agreement with the theoretical predictions.
Example 3 In this example, we validate the spatial convergence of our scheme with the initial value and source term (c). Here we take $\tau = 1/256$ to avoid the influence on errors caused by temporal discretization and $T = 2$. Tables 3 and 4 show the $L^2$ errors with $s \in (0, \frac{3}{4})$ and $\tilde{H}^{2s-1}$ errors with $s \in (\frac{3}{4}, 1)$, and all convergence rates agree with the predicted rates in Theorem 4.2 except for the case $s = 0.9$. The main reason may be that the error of $\| \int_{\Gamma_{\theta, \kappa}} e^{iz \frac{\partial}{\partial z}} (\tilde{E}(z) - \tilde{E}_h(z) P_h)\frac{\partial u_0}{\partial z} dz \|_{\tilde{H}^{2s-1}(\Omega)}$ (whose convergence rate is $O(h^{3-2s})$ presented in Lemma 4.4 and Remark 4.3) has influence on the convergence rate. So to avoid its influence, we consider the following equation

$$0 \partial_t^\alpha (u - u_0) + (-\Delta)u + a(-\Delta)^s u = f, \quad (x, t) \in \Omega \times (0, T]$$

(5.1)

with a zero Dirichlet boundary condition and taking the initial data and source term (c). Here $a$ is a positive constant. Table 5 presents the errors in $\tilde{H}^{2s-1}(\Omega)$-norm for different $\alpha$, $s$ and $a$, and all the results agree with the predicted results in Theorem 4.2. Moreover, we show the $L^2(\Omega)$ error with $s \in (\frac{3}{4}, 1)$ in Table 6 and the convergence rates are about $O(h^{3-8})$ and $O(h^{1.5})$ when $s = 0.8$ and $s = 0.9$, respectively.

Example 4 Lastly, we validate the spatial convergence with the initial value and source term (b) and choose $\tau = 1/512$ and $T = 1$. We present the $L^2$ errors with $s \in (0, \frac{3}{4})$ and $\tilde{H}^{2s-1}$ errors with $s \in (\frac{3}{4}, 1)$ in Tables 7 and 8; it can be noted that all the convergence rates are consistent with the results in Theorem 4.2 expect for the case $s = 0.9$. It may be caused by the influences of $\| \int_{\Gamma_{\theta, \kappa}} e^{iz \frac{\partial}{\partial z}} (\tilde{E}(z) - \tilde{E}_h(z) P_h) f(0) dz \|_{\tilde{H}^{2s-1}(\Omega)}$ and $\| \int_{\Gamma_{\theta, \kappa}} e^{iz \frac{\partial}{\partial z}} (\tilde{E}(z) - \tilde{E}_h(z) P_h) f(t) dz \|_{\tilde{H}^{2s-1}(\Omega)}$.

### Table 2 $L^2$ errors and convergence rates with the initial value and source term (b)

| $(\alpha, s)/1/\tau$ | 16     | 32     | 64     | 128    | 256    |
|----------------------|--------|--------|--------|--------|--------|
| (0.3,0.4)            | 1.845E-05 | 8.321E-06 | 3.805E-06 | 1.754E-06 | 8.124E-07 |
| Rate                 | 1.1490  | 1.1287  | 1.1172  | 1.1105  |        |
| (0.3,0.8)            | 1.143E-05 | 5.155E-06 | 2.359E-06 | 1.088E-06 | 5.038E-07 |
| Rate                 | 1.1482  | 1.1281  | 1.1168  | 1.1102  |        |
| (0.7,0.4)            | 3.087E-05 | 1.343E-05 | 5.975E-06 | 2.691E-06 | 1.221E-06 |
| Rate                 | 1.2010  | 1.1682  | 1.1506  | 1.1401  |        |
| (0.7,0.8)            | 1.774E-05 | 7.751E-06 | 3.459E-06 | 1.562E-06 | 7.099E-07 |
| Rate                 | 1.1946  | 1.1640  | 1.1474  | 1.1373  |        |

### Table 3 $L^2$ errors and convergence rates with the initial value and source term (c) and $s \in (0, \frac{3}{4})$

| $(\alpha, s)/1/h$  | 16     | 32     | 64     | 128    | 256    |
|--------------------|--------|--------|--------|--------|--------|
| (0.4,0.3)          | 1.179E-04 | 2.949E-05 | 7.373E-06 | 1.843E-06 | 4.610E-07 |
| Rate               | 1.9995  | 1.9998  | 1.9999  | 1.9999  |        |
| (0.6,0.3)          | 7.273E-05 | 1.819E-05 | 4.547E-06 | 1.137E-06 | 2.843E-07 |
| Rate               | 1.9996  | 1.9998  | 1.9999  | 1.9999  |        |
| (0.4,0.6)          | 1.116E-04 | 2.822E-05 | 7.112E-06 | 1.788E-06 | 4.484E-07 |
| Rate               | 1.9839  | 1.9883  | 1.9918  | 1.9954  |        |
| (0.6,0.6)          | 6.811E-05 | 1.721E-05 | 4.337E-06 | 1.090E-06 | 2.734E-07 |
| Rate               | 1.9845  | 1.9886  | 1.9919  | 1.9956  |        |
Table 4 $\tilde{H}^{2s-1}(\Omega)$ errors and convergence rates with the initial value and source term (c) and $s \in \left(\frac{3}{4}, 1\right)$

| $(\alpha, s)/1/h$ | 16   | 32   | 64   | 128  | 256  |
|-------------------|------|------|------|------|------|
| (0.3,0.8)         | 1.134E−03 | 4.435E−04 | 1.742E−04 | 6.869E−05 | 2.719E−05 |
| Rate              | 1.3549 | 1.3484 | 1.3425 | 1.3370 |      |
| (0.8,0.8)         | 2.403E−04 | 9.391E−05 | 3.689E−05 | 1.455E−05 | 5.763E−06 |
| Rate              | 1.3554 | 1.3482 | 1.3420 | 1.3363 |      |
| (0.3,0.9)         | 2.433E−03 | 1.102E−03 | 5.064E−04 | 2.369E−04 | 1.134E−04 |
| Rate              | 1.1418 | 1.1223 | 1.0959 | 1.0627 |      |
| (0.8,0.9)         | 5.130E−04 | 2.324E−04 | 1.067E−04 | 4.994E−05 | 2.392E−05 |
| Rate              | 1.1426 | 1.1225 | 1.0956 | 1.0619 |      |

Table 5 $\tilde{H}^{2s-1}(\Omega)$ errors and convergence rates with the initial value and source term (c) and $s \in \left(\frac{3}{4}, 1\right)$

| $(\alpha, s, a)/1/h$ | 16   | 32   | 64   | 128  | 256  |
|-----------------------|------|------|------|------|------|
| (0.3,0.8,2)          | 9.868E−04 | 4.054E−04 | 1.677E−04 | 6.980E−05 | 2.915E−05 |
| Rate                 | 1.2836 | 1.2729 | 1.2638 | 1.2587 |      |
| (0.8,0.8,2)          | 2.055E−04 | 8.443E−05 | 3.496E−05 | 1.456E−05 | 6.085E−06 |
| Rate                 | 1.2834 | 1.2720 | 1.2638 | 1.2587 |      |
| (0.3,0.9,3)          | 1.534E−03 | 7.422E−04 | 3.716E−04 | 1.931E−04 | 1.039E−04 |
| Rate                 | 1.0472 | 0.9980 | 0.9442 | 0.8944 |      |
| (0.8,0.9,3)          | 3.153E−04 | 1.525E−04 | 7.641E−05 | 3.974E−05 | 2.139E−05 |
| Rate                 | 1.0473 | 0.9975 | 0.9432 | 0.8933 |      |

Table 6 $L^2(\Omega)$ errors and convergence rates with the initial value and source term (c) and $s \in \left(\frac{3}{4}, 1\right)$

| $(\alpha, s, a)/1/h$ | 16   | 32   | 64   | 128  | 256  |
|----------------------|------|------|------|------|------|
| (0.3,0.8,2)         | 1.071E−04 | 3.020E−05 | 8.643E−06 | 2.502E−06 | 7.296E−07 |
| Rate                | 1.8259 | 1.8047 | 1.7886 | 1.7777 |      |
| (0.8,0.8,2)         | 2.272E−05 | 6.406E−06 | 1.835E−06 | 5.321E−07 | 1.555E−07 |
| Rate                | 1.8263 | 1.8034 | 1.7862 | 1.7747 |      |
| (0.3,0.9,3)         | 6.817E−05 | 1.989E−05 | 6.139E−06 | 2.010E−06 | 6.958E−07 |
| Rate                | 1.7775 | 1.6956 | 1.6106 | 1.5306 |      |
| (0.8,0.9,3)         | 1.420E−05 | 4.140E−06 | 1.279E−06 | 4.198E−07 | 1.457E−07 |
| Rate                | 1.7781 | 1.6941 | 1.6076 | 1.5265 |      |

$\tilde{E}_h(z)P_h \int \tilde{f}dz\| \tilde{H}^{2s-1}(\Omega)$ on convergence rates (whose convergence rates are $O(h^{3-2s})$; see Lemma 4.4 and Remark 4.3). To weaken their influences, we also consider (5.1) with the initial data and the source term (b) and the convergence rates are shown in Table 9, which are consistent with our predicted rates. Furthermore, we provide $L^2(\Omega)$ errors and convergence rates with $s \in \left(\frac{3}{4}, 1\right)$ in Table 10, and the convergence rates are about $O(h^{1.8})$ and $O(h^{1.5})$ when taking $s = 0.8$ and $s = 0.9$, respectively.
Table 7 $L^2$ errors and convergence rates with the initial value and source term (b) and $s \in \left(0, \frac{3}{4}\right)$. 

| $(\alpha, s)/1/h$ | 16   | 32   | 64   | 128  | 256  |
|-------------------|------|------|------|------|------|
| (0.3, 0.2)        | 2.917E-04 | 7.361E-05 | 1.851E-05 | 4.645E-06 | 1.164E-06 |
| Rate              | 1.9867 | 1.9916 | 1.9946 | 1.9966 |       |
| (0.8, 0.2)        | 3.037E-04 | 7.657E-05 | 1.925E-05 | 4.828E-06 | 1.210E-06 |
| Rate              | 1.9878 | 1.9922 | 1.9950 | 1.9968 |       |
| (0.3, 0.6)        | 2.809E-04 | 7.209E-05 | 1.836E-05 | 4.649E-06 | 1.172E-06 |
| Rate              | 1.9622 | 1.9734 | 1.9815 | 1.9880 |       |
| (0.8, 0.6)        | 2.900E-04 | 7.434E-05 | 1.892E-05 | 4.790E-06 | 1.207E-06 |
| Rate              | 1.9636 | 1.9742 | 1.9819 | 1.9884 |       |

Table 8 $\hat{H}^{2s-1}$ errors and convergence rates with the initial value and source term (b) and $s \in \left(\frac{3}{4}, 1\right)$. 

| $(\alpha, s)/1/h$ | 16   | 32   | 64   | 128  | 256  |
|-------------------|------|------|------|------|------|
| (0.4, 0.8)        | 2.387E-03 | 9.462E-04 | 3.751E-04 | 1.488E-04 | 5.906E-05 |
| Rate              | 1.3348 | 1.3348 | 1.3341 | 1.3329 |       |
| (0.6, 0.8)        | 2.403E-03 | 9.527E-04 | 3.777E-04 | 1.498E-04 | 5.947E-05 |
| Rate              | 1.3350 | 1.3349 | 1.3341 | 1.3328 |       |
| (0.4, 0.9)        | 5.050E-03 | 2.322E-03 | 1.078E-03 | 5.079E-04 | 2.436E-04 |
| Rate              | 1.1209 | 1.1066 | 1.0862 | 1.0600 |       |
| (0.6, 0.9)        | 5.083E-03 | 2.337E-03 | 1.085E-03 | 5.112E-04 | 2.452E-04 |
| Rate              | 1.1211 | 1.1067 | 1.0862 | 1.0599 |       |

Table 9 $\hat{H}^{2s-1}$ errors and convergence rates with the initial value and source term (b) and $s \in \left(\frac{3}{4}, 1\right)$. 

| $(\alpha, s, a)/1/h$ | 16   | 32   | 64   | 128  | 256  |
|----------------------|------|------|------|------|------|
| (0.4, 0.8, 2)       | 2.102E-03 | 8.758E-04 | 3.652E-04 | 1.523E-04 | 6.350E-05 |
| Rate                | 1.2634 | 1.2621 | 1.2616 | 1.2621 |       |
| (0.6, 0.8, 2)       | 2.113E-03 | 8.803E-04 | 3.670E-04 | 1.531E-04 | 6.385E-05 |
| Rate                | 1.2634 | 1.2620 | 1.2614 | 1.2619 |       |
| (0.4, 0.9, 3)       | 3.232E-03 | 1.587E-03 | 8.002E-04 | 4.154E-04 | 2.219E-04 |
| Rate                | 1.0260 | 0.9881 | 0.9457 | 0.9049 |       |
| (0.6, 0.9, 3)       | 3.244E-03 | 1.593E-03 | 8.031E-04 | 4.170E-04 | 2.227E-04 |
| Rate                | 1.0261 | 0.9880 | 0.9456 | 0.9047 |       |

**Example 5** In this example, we show the evolution of the solution with the increase of the time under initial data and source term (d). According to Fig. 1, we find the singularity with $\alpha = 0.3$ near to $t = 0$ is stronger than the one with $\alpha = 0.7$, and the diffusion velocity with $s = 0.8$ is higher than the one with $s = 0.2$. 

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Table 10 $L^2$ errors and convergence rates with the initial value and source term (b) and $s \in \left( \frac{3}{4}, 1 \right)$

| $(\alpha, s, a)/1/h$ | 16     | 32     | 64     | 128    | 256  |
|----------------------|--------|--------|--------|--------|------|
| (0.4,0.8,2)          | 2.287E−04 | 6.545E−05 | 1.886E−05 | 5.461E−06 | 1.587E−06 |
| Rate                 | 1.8050 | 1.7953 | 1.7878 | 1.7827 |
| (0.6,0.8,2)          | 2.304E−04 | 6.593E−05 | 1.900E−05 | 5.504E−06 | 1.600E−06 |
| Rate                 | 1.8051 | 1.7950 | 1.7874 | 1.7822 |
| (0.4,0.9,3)          | 1.449E−04 | 4.285E−05 | 1.326E−05 | 4.316E−06 | 1.477E−06 |
| Rate                 | 1.7580 | 1.6918 | 1.6196 | 1.5469 |
| (0.6,0.9,3)          | 1.457E−04 | 4.308E−05 | 1.334E−05 | 4.342E−06 | 1.487E−06 |
| Rate                 | 1.7580 | 1.6915 | 1.6191 | 1.5462 |

Fig. 1 Evolution of the solution of (1.1) with different $\alpha$ and $s$

6 Conclusions

In this paper, the fractional Fokker–Planck equation involving two diffusion operators with different scales is derived from the framework of the Lévy process and this kind of equation can describe the physical phenomena more delicately. We use $L_1$ approximation in time and finite element method in space to get the numerical scheme of the equation. Thanks to the
sharp regularity estimate of the solution, we optimally obtain the spatial and temporal error estimates. Extensive numerical experiments validate the theoretical results.

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