Implementing the fanout gate by a Hamiltonian

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June 8, 2021

Abstract

We show that, for even \( n \), evolving \( n \) qubits according to a simple Hamiltonian can be used to exactly implement an \((n + 1)\)-qubit parity gate, which is equivalent in constant depth to an \((n + 1)\)-qubit fanout gate. We also observe that evolving the Hamiltonian for three qubits results in an inversion-on-three-way-equality gate, which together with single-qubit operations is universal for quantum computation.

1 Introduction

Let \( \mathcal{H} \) be the Hilbert space of \( n + 1 \) qubits. The fanout operator \( F_{n+1}: \mathcal{H} \rightarrow \mathcal{H} \), depicted in Figure 1, copies the (classical) value of a single qubit to \( n \) other qubits.

Fanout gates have been recently shown to be very powerful primitives for making shallow quantum circuits [3, 4, 5, 6]. It has been shown that in the quantum realm, fanout, parity (see below), and Mod\( q \) gates (for any \( q \geq 2 \)) are all equivalent up to constant depth and polynomial size [3, 5]. That is, each gate above can be simulated exactly by a constant-depth, polynomial-size quantum circuit using any of the other gates above, together with standard one- and two-qubit gates (e.g., CNOT, \( H \), and \( T \)). This is not true in the classical case. Furthermore, using fanout gates, in constant depth and polynomial size one can approximate \( n \)-qubit threshold gates, unbounded AND (generalized Toffoli) gates and OR gates, sorting, arithmetical operations, phase estimation, and the quantum Fourier transform [4, 6]. Since long quantum computations may be difficult to maintain due to decoherence, shallow quantum circuits may prove much more realistic, at least in the short term.

On the negative side, fanout gates so far appear hard to implement. There is recent theoretical evidence that fanout gates cannot be simulated in small depth and small width, even if unbounded AND gates are allowed \([2]\). All these results underscore how crucial fanout gates are to providing powerful, small-depth quantum computation.

Rather than trying to implement fanout with a traditional small-depth quantum circuit, we can perhaps take another promising approach: evolve an \( n \)-qubit system according to a (hopefully

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1Any circuit of depth \( d \) using AND and single-qubit gates to compute \( n \)-qubit fanout provably needs at least \( n/2^d \) ancillae.
implementable) Hamiltonian. We show that a simple Hamiltonian, similar to one suggested recently by Chuang [1], does exactly this.

Let $\Sigma Z$ be the $n$-qubit operator $\frac{1}{2} \sum_{i=1}^{n} Z_i$, where $Z_i$ is the $Z$-gate acting on the $i$th qubit. If, for instance, each qubit is represented by a spin-$\frac{1}{2}$ particle, then $\Sigma Z$ is the observable representing the total spin angular momentum in the $z$-direction. Let $J > 0$ be some constant in units of energy. We will show that for even $n$, the $(n+1)$-qubit fanout gate arises naturally by evolving the first $n$ qubits through the Hamiltonian

$$H_n := J(\Sigma Z)^2$$

twice—first for time $\frac{\pi \hbar}{2J}$ then for time $\frac{3\pi \hbar}{2J}$—together with a modest amount of additional processing. We also show that evolving the 3-qubit Hamiltonian $J(\Sigma Z)^2$ results (modulo a global phase factor) in a 3-qubit “inversion on equality” gate $I_3$, which maps $|abc\rangle$ to $(-1)^{\delta_{ab}\delta_{ac}}|abc\rangle$. Applying $I_3$ with one of the three qubits set to $|1\rangle$ results in a controlled $Z$-gate on the other two qubits, which is easily converted to a CNOT gate.

Chuang’s proposed Hamiltonian is closely related to $H_n$. It is

$$K_n := \sum_{1 \leq i < j \leq n} J_{i,j} Z_i Z_j,$$

where the $J_{i,j}$ are energy coefficients, and may be potentially realizable for certain combinations of the $J_{i,j}$, depending on the physical arrangement of the qubits [1]. $K_n$ is the sum of pairwise interactions between the particles. In the special case where all the $J_{i,j}$ are equal to $J/2$, we see that $K_n$ differs from $H_n$ by a multiple of the identity, and so evolving through $H_n$ and evolving through $K_n$ are equivalent up to an overall phase factor.

We will show the $I_3$-gate in Section 2.2. In Section 2.3 we show the implementation of $F_{n+1}$ for even $n$. In the sequel, we will assume for convenience that $\hbar = J/2 = 1$. If $X$ and $Y$ are vectors or operators, we say, “$X \propto Y$” to mean that $X = e^{i\theta} Y$ for some real $\theta$, that is, $X = Y$ up to an overall phase factor. We use the same notation with individual components of $X$ and $Y$, meaning that the phase factor is independent of which component we choose. If $A$ is a set, we let $|A|$ denote the cardinality of $A$. 

![Figure 1: Definition of the fanout gate.](image-url)
2 Main Results

2.1 Evolving $H_n$

Since $H_n$ is represented in the computational basis by a diagonal matrix, it is particularly easy to see how it evolves in time. We only need to find the value of each diagonal element. Let $\vec{x} = x_1 \cdots x_n$ be a vector of $n$ bits. For $1 \leq i, j \leq n$, we see that $Z_i Z_j |\vec{x}\rangle = (-1)^{x_i \oplus x_j} |\vec{x}\rangle$, and thus $Z_i Z_j$ flips the sign iff $x_i \neq x_j$. Suppose $k$ of the $x_i$ are 1 and the rest ($n - k$) are 0. Then,

$$\langle \vec{x} | H_n | \vec{x}\rangle = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{x_i \oplus x_j} = \frac{1}{2} (k^2 + (n - k)^2 - 2k(n - k)) = \frac{n^2}{2} - 2k(n - k).$$

The $n^2/2$ term is independent of $\vec{x}$, so up to addition of a multiple of $I$, the diagonal term is effectively $-2k(n - k)$. We evolve $H_n$ for time $t = \pi/4$. Let $U_n := e^{-iH_n t}$ where $t = \pi/4$. We get

$$\langle \vec{x} | U_n | \vec{x}\rangle \propto e^{i\pi k(n-k)/2} = i^{k(n-k)}. \quad (1)$$

All the off-diagonal matrix elements of $U_n$ are zero.

2.2 Implementing the $I_\pm$-Gate

For the $n = 3$ case, (1) yields

$$U_3 \propto \text{diag} [1 \ -1 \ -1 \ -1 \ -1 \ -1 \ -1 \ 1] \propto I_\pm.$$

No further gates are required.

2.3 Implementing the $F_{n+1}$-Gate

Let $n$ be even. $U_n$ is actually more closely related to the $(n+1)$-qubit (classical) parity gate, shown in Figure 2. We will show how to implement parity with $U_n$. This suffices, because the equality given in Figure 3 was observed in [5, 3], where $H$ is the Hadamard gate.

Suppose for now that $n \equiv 2 \pmod{4}$, and that $\vec{x}$ has $k$ ones and $n - k$ zeros. If $k$ is even, then so is $n - k$, and thus $k(n-k) \equiv 0 \pmod{4}$. If $k$ is odd, then clearly, $k(n-k) \equiv 1 \pmod{4}$. Thus we have, by (1),

$$\langle \vec{x} | U_n | \vec{x}\rangle \propto \begin{cases} 1 & \text{if } \vec{x} \text{ has even parity,} \\ i & \text{if } \vec{x} \text{ has odd parity.} \end{cases} \quad (2)$$

We start with the $(n+1)$-qubit basis state $|x_1 \cdots x_{n-1}rb\rangle$, where $x_1, \ldots, x_{n-1}, r, b \in \{0,1\}$. We prepare the $n$th qubit in the state

$$H|r\rangle = (|0\rangle + (-1)^r|1\rangle)/\sqrt{2},$$
Figure 2: Definition of the parity gate.

\[ x_1 \oplus \cdots \oplus x_n \oplus x_{n+1} \]

Figure 3: Implementing fanout by a parity gate and Hadamard gates.
so that the current state of the first $n$ qubits is $|x_1 \cdots x_{n-1}\rangle(|0\rangle + (-1)^r|1\rangle)/\sqrt{2}$. If we run this state through $U_n$, then by (2) the result is

$$\frac{1}{\sqrt{2}}|x_1 \cdots x_{n-1}\rangle(|0\rangle + i^{1-p}(-1)^r|1\rangle),$$

where $p = (x_1 + \cdots + x_{n-1}) \mod 2$. The two states of the $n$th qubit corresponding to the two values of $p$ are orthogonal, so we just need to rotate the $n$th qubit back to the computational basis. The $n$th qubit state is either $+y$ or $-y$ on the Bloch sphere, depending on $p$ and $r$, so we can use $HS^\dagger$, where $S$ is the phase gate $\text{diag}[1 \ i \ 1]$, to rotate the $y$-axis to the $z$-axis. The final circuit is shown in Figure 4. When the CNOT gate is applied, its control qubit can be seen to be in the state $i^p|p \oplus r\rangle$, unentangled with the other qubits. The rest of the circuit is then needed to uncompute the conditional phase factor $i^p$. We can implement $U_n^\dagger$ by evolving $-H_n$ for time $\pi/4$, or equivalently, by evolving $H_n$ for time $3\pi/4$, since $U_4^\dagger = I$. Of course, if we are willing to keep the phase factor, we can get by without this part of the circuit. In fact, we can get the parity-like gate of Figure 5 with only one use of $U_n$.

As we mentioned before, one gets a fanout gate by applying Hadamards on each qubit on both
sides of a parity gate. So starting with the circuit in Figure 4, after some simplification we get the circuit shown in Figure 6. If \( n \equiv 0 \pmod{4} \), then we get a similar analysis of \( U_n \), except that (2) becomes
\[
⟨\vec{x}|U_n|\vec{x}⟩ \propto \begin{cases} 
1 & \text{if } \vec{x} \text{ has even parity,} \\
-i & \text{if } \vec{x} \text{ has odd parity.}
\end{cases}
\]
It follows that we can swap \( U_n \) with \( U_n^\dagger \) in Figures 4, 5, and 6 above to maintain the equalities.

3 Conclusions and Further Research

A key point in our implementation is that the number of terms in the Hamiltonian \( H_n \) is quadratic in \( n \), which gives a quadratic term in the phase shift. We suspect there is also some way to get parity from the more general \( K_n \) when the \( J_{i,j} \) are not all equal, provided there are still quadratically many terms. We also suspect that this will not work where there are fewer than quadratically many terms, for example, in the case where we just consider interactions between adjacent particles in a ring, i.e., \( J_{i,j} = J > 0 \) if \( j \equiv i + 1 \pmod{n} \), and \( J_{i,j} = 0 \) otherwise. In this case, there are only linearly many terms in the Hamiltonian.

A second Hamiltonian described by Chuang as potentially realizable in the lab [1] is
\[
L_n = \sum_{1 \leq i < j \leq n} J_{i,j}(X_iX_j + Y_iY_j + Z_iZ_j),
\]
which, when all the \( J_{i,j} = 2 \), differs from the squared total spin \( L^2 := (\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2 \) by a multiple of \( I \). We conjecture that \( L^2 \) can also be used to implement parity.

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