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**Witt vectors and K-theory of automorphisms via noncommutative motives**

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**Abstract.** We prove that the functor ring-of-rational-Witt-vectors $W_0(\cdot)$ becomes co-representable in the category of noncommutative motives. As an application, we obtain an immediate extension of $W_0(\cdot)$ from commutative rings to schemes. Then, making use of the theory of noncommutative motives, we classify all natural transformations of the functor $K$-theory-of-automorphisms.

**1. Introduction and statement of results**

The theory of **noncommutative motives**, envisioned by Kontsevich [9] in his seminal talk, was initiated in a series of articles [4,5,10,11,15,16]. Among other applications, this theory allowed the first conceptual characterization of algebraic $K$-theory since Quillen’s pioneering work, a streamlined construction of all the higher Chern characters, a universal characterization of Drinfeld’s DG quotient, etc; see the survey [17]. Here we further the study of noncommutative motives by developing its interactions with Witt vectors and $K$-theory of automorphisms.

**Witt vectors**

Witt vectors were introduced in the thirties by Witt [21] in his work on algebraic number theory. Given a commutative ring $A$, the *Witt ring* $W(A)$ of $A$ is the abelian group of all power series of the form $1 + a_1 t + a_2 t^2 + \cdots$, with $a_i \in A$, endowed with the multiplication $\ast$ determined by the equality $(1 - a_1 t) \ast (1 - a_2 t) = (1 - a_1 a_2 t)$. The *rational* Witt ring $W_0(A)$ of $A$ consists of the elements of the form

$$\left\{ \frac{1 + a_1 t + \cdots + a_i t^i + \cdots + a_n t^n}{1 + b_1 t + \cdots + b_j t^j + \cdots + b_m t^m} \mid a_i, b_j \in A \right\} \subset W(A);$$

consult [7] for details. Although widely used in several branches of mathematics the rings $W(R)$ and $W_0(R)$ remain rather mysterious. Our co-representability Theorem 1.2 below offers a new viewpoint on the subject.

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Recall from [16] the construction of the universal additive invariant

\[ U: \text{dgcat} \longrightarrow \text{Hmo}_0. \]

Roughly speaking, \( U \) is the universal functor (defined on dg categories) with values in an idempotent complete additive category which inverts Morita equivalences and satisfies additivity; see Sect. 4. Examples of additive invariants include algebraic \( K \)-theory, Hochschild homology, cyclic homology (and all its variants), and even topological Hochschild homology. Because of its universal property, which is a reminiscence of motives, \( \text{Hmo}_0 \) is called the category of noncommutative motives.

The tensor product of rings extends naturally to dg categories and, as proved in Proposition 4.3, induces in a universal way a symmetric monoidal structure on \( \text{Hmo}_0 \) making the functor \( U \) symmetric monoidal. The ring homomorphisms

\[
\begin{align*}
\Delta : \mathbb{Z}[r] &\rightarrow r \otimes r, \\
\epsilon : \mathbb{Z}[r] &\rightarrow \mathbb{Z}
\end{align*}
\]  

(1.1)

endow the ring of polynomials \( \mathbb{Z}[r] \) with a co-unital co-associative co-algebra structure. Since \( U \) is symmetric monoidal the noncommutative motive \( U(\mathbb{Z}[r]) \) becomes then a co-unital co-associative co-monoid in \( \text{Hmo}_0 \). As a consequence, given any unital associative monoid \( O \in \text{Hmo}_0 \), the symmetric monoidal structure of \( \text{Hmo}_0 \) gives rise to a unital associative convolution ring structure on \( \text{Hom}_{\text{Hmo}_0}(U(\mathbb{Z}[r]), O) \). Our first main result is the following:

**Theorem 1.2.** The natural homomorphisms \( \mathbb{Z} \rightarrow \mathbb{Z}[r] \) and \( \mathbb{Z}[r] \rightarrow \mathbb{Z} \) give rise to a direct sum decomposition \( U(\mathbb{Z}[r]) \simeq U(\mathbb{Z}) \oplus W_0 \) of co-unital co-associative co-monoids in \( \text{Hmo}_0 \).

For every (flat) commutative ring \( A \) we have a ring isomorphism

\[
\text{Hom}_{\text{Hmo}_0}(W_0, U(A)) \simeq W_0(A).
\]  

(1.3)

Note that the isomorphism (1.3) furnish us a conceptual characterization of the ring of rational Witt vectors since \( \text{Hom}_{\text{Hmo}_0}(W_0, U(A)) \) is defined solely in terms of precise universal properties. Roughly speaking, the noncommutative motive associated to the “affine line” decomposes into a “point” plus a “formal variable” which co-represents the ring of rational Witt vectors.

As an application, Theorem 1.2 allows us to immediately extend the functor \( W_0(-) \) from commutative rings to schemes. Let \( X \) be a quasi-compact and quasi-separated scheme. Recall from [12] that the derived category of perfect complexes of \( O_X \)-modules admits a (unique) differential graded enhancement \( D_{\text{perf}}^\alpha(X) \), under which the bi-exact functor \( (E^\bullet, F^\bullet) \mapsto E^\bullet \otimes_{O_X} F^\bullet \) lifts to a dg functor \( D_{\text{perf}}^\alpha(X) \otimes D_{\text{perf}}^\alpha(X) \rightarrow D_{\text{perf}}^\alpha(X) \). The noncommutative motive \( U(X) := U(D_{\text{perf}}^\alpha(X)) \) becomes then a unital associative monoid and so the abelian group \( \text{Hom}_{\text{Hmo}_0}(W_0, U(X)) \) carries a unital associative convolution ring structure. It is then natural to call \( \text{Hom}_{\text{Hmo}_0}(W_0, U(X)) \) the rational Witt ring of \( X \). To the best of the author’s knowledge this invariant of schemes is new in the literature.

**K-theory of automorphisms**

Given a ring \( A \), let \( P(A) \) be the category of finitely generated projective \( A \)-modules and \( \text{Aut}(P(A)) \) the category of automorphisms. This latter category inherits from \( P(A) \) a natural exact structure in the sense of Quillen and so following [1,2] one can consider its
Grothendieck group. This construction extends naturally to dg categories giving rise to a well-defined functor $K_{0}\text{Aut} : \text{dgcat} \to \text{Ab}$ with values in abelian groups; see Sect. 3. Moreover, when $A$ is commutative the tensor product over $A$ endows $K_{0}\text{Aut}(A)$ with a ring structure.

**Theorem 1.4.** The functor $K_{0}\text{Aut}$ becomes co-representable in $\text{Hmo}_{0}$ by the ring $\mathbb{Z}[r, r^{-1}]$ of Laurent polynomials, i.e. for every dg category $A$ there is a group isomorphism

$$\text{Hom}_{\text{Hmo}_{0}}\left(\mathcal{U}(\mathbb{Z}[r, r^{-1}]), \mathcal{U}(A)\right) \simeq K_{0}\text{Aut}(A).$$

Moreover, $\mathbb{Z}[r, r^{-1}]$ inherits naturally from $\mathbb{Z}[r]$ a co-unital co-associative co-algebra structure such that for every (flat) commutative ring $A$ we have a ring isomorphism

$$\text{Hom}_{\text{Hmo}_{0}}\left(\mathcal{U}(\mathbb{Z}[r, r^{-1}]), \mathcal{U}(A)\right) \simeq K_{0}\text{Aut}(A).$$

Informally speaking, Theorem 1.4 shows us that $K_{0}\text{Aut}$ becomes co-represented by the noncommutative motive associated to the “punctured affine line”.

Classically the $K$-theory of automorphisms comes equipped with several natural transformations such as the Frobenius ($F_{n}$) and the Verschiebung ($V_{n}$) operations; consult [1,2]. A fundamental problem in the field is then the classification of all the natural transformations of $K_{0}\text{Aut}$. Making use of noncommutative motives and the above co-representability Theorem 1.4 we solve this problem as follows:

**Theorem 1.7.** There is a natural isomorphism of abelian groups

$$\text{Nat}(K_{0}\text{Aut}, K_{0}\text{Aut}) \simeq K_{0}\text{Aut}\left(\mathbb{Z}[r, r^{-1}]\right),$$

where $\text{Nat}$ stands for the abelian group of natural transformations. Moreover, the quotient of $K_{0}\text{Aut}(\mathbb{Z}[r, r^{-1}])$ by the relations

$$\left\{[(M, \alpha)] + [(M, \beta)] = [(M, \alpha \beta)] \mid M \in P(\mathbb{Z}[r, r^{-1}]), \alpha, \beta \text{ automorphisms}\right\}$$

identifies with $\{-1, 1\} \times \mathbb{Z}$. Furthermore, under the quotient homomorphism

$$\text{Nat}(K_{0}\text{Aut}, K_{0}\text{Aut}) \twoheadrightarrow \{-1, 1\} \times \mathbb{Z},$$

the Frobenius operation $F_{n}$ corresponds to $(1, n)$ and the Verschiebung operation $V_{n}$ to $((-1)^{n+1}, 1)$.

Theorem 1.7 shows us that all the information concerning a natural transformation of $K_{0}\text{Aut}$ can be completely encoded into an element of $K_{0}\text{Aut}(\mathbb{Z}[r, r^{-1}])$. Moreover, by imposing the relations (1.9) this data reduces simply to a parity plus an integer. The Frobenius (resp. Verschiebung) operations become then the simplest ones with respect to the parity (resp. to the integer).

## 2. Differential graded categories

A differential graded (=dg) category is a category enriched over (unbounded) cochain complexes of abelian groups in such a way that composition fulfills the Leibniz rule: $d(f \circ g) = d(f) \circ g + (-1)^{\deg(f)} f \circ d(g)$. For a survey article, consult Keller’s ICM address [8]. The category of dg categories will be denoted by $\text{dgcat}$. Given a ring $A$, we will still denote by $A$ the dg category with a single object and $A$ as the dg ring of endomorphisms (concentrated in degree zero).
Dg (bi-)modules

Let $\mathcal{A}$ be a dg category. Recall from [8, §3] the construction of the category $\mathcal{C}(\mathcal{A})$ of $\mathcal{A}$-modules. We will denote by $\mathcal{D}(\mathcal{A})$ the derived category of $\mathcal{A}$, i.e. the localization of $\mathcal{C}(\mathcal{A})$ with respect to the class of quasi-isomorphisms. Recall from [8, §2.3] that the tensor product endows $\text{dgcat}$ with a symmetric monoidal structure with $\otimes$-unit $\mathbb{Z}$. Finally, recall that a $\mathcal{A}$-$\mathcal{B}$-bimodule is simply a dg functor $\mathcal{A}^{\text{op}} \otimes \mathcal{B} \to \mathcal{C}_{\text{dg}}(\mathbb{Z})$ with values in the dg category of complexes of abelian groups.

Morita equivalences

A dg functor $F : \mathcal{A} \to \mathcal{B}$ is called a Morita equivalence if the derived extension of scalars functor $L_{F_!} : \mathcal{D}(\mathcal{A}) \sim \to \mathcal{D}(\mathcal{B})$ is an equivalence of (triangulated) categories; see [8, §4.6].

Recall from [8, Thm. 4.10] that $\text{dgcat}$ carries a Quillen model structure, whose weak equivalences are the Morita equivalences. We will denote by $\text{Hmo}$ the homotopy category hence obtained. The tensor product of dg categories can be naturally derived $- \otimes L_{-}$, thus giving rise to a symmetric monoidal structure on $\text{Hmo}$. Given dg categories $\mathcal{A}$ and $\mathcal{B}$, let $\text{rep}(\mathcal{A}, \mathcal{B})$ be the full triangulated subcategory of $\mathcal{D}(\mathcal{A}^{\text{op}} \otimes \mathcal{B})$ spanned by the cofibrant $\mathcal{A}$-$\mathcal{B}$-bimodules $X$ such that for every object $x \in \mathcal{A}$ the associated $\mathcal{B}$-module $X(x, -)$ becomes compact (see [13, Def. 4.2.7]) in $\mathcal{D}(\mathcal{B})$.

3. $K$-theory of automorphisms of dg categories

Given a dg category $\mathcal{A}$, let $\text{perf}(\mathcal{A})$ be the full subcategory of $\mathcal{C}(\mathcal{A})$ consisting of those $\mathcal{A}$-modules which are cofibrant in the projective model structure and which become compact in $\mathcal{D}(\mathcal{A})$. As explained in [6, Example 3.5], $\text{perf}(\mathcal{A})$ carries a Waldhausen structure [19] and so by passing to the Grothendieck group of the associated category $\text{Aut}(\text{perf}(\mathcal{A}))$ of automorphisms we obtain the following composed functor

$$K_0 \text{Aut} : \text{dgcat} \xrightarrow{\text{perf}} \text{Wald} \xrightarrow{\text{Aut}} \text{Wald} \xrightarrow{K_0} \text{Ab} \quad \mathcal{A} \mapsto K_0 \text{Aut}(\text{perf}(\mathcal{A})).$$

Proposition 3.1. (Agreement) For every ring $\mathcal{A}$, the exact functor $\mathbf{P}(\mathcal{A}) \to \text{perf}(\mathcal{A})$ (mapping a $\mathcal{A}$-module to the associated complex of $\mathcal{A}$-modules concentrated in degree zero) gives rise to an abelian group isomorphism

$$K_0 \text{Aut}(\mathbf{P}(\mathcal{A})) \sim \to K_0 \text{Aut}(\text{perf}(\mathcal{A})). \quad (3.2)$$

Proof. The following assignment

$$(M^\bullet, \alpha) \mapsto \sum_{n \in \mathbb{Z}} (-1)^n \left[ \left( H^n(M^\bullet), H^n(\alpha) \right) \right]$$

gives rise to the inverse to (3.2). □
4. Noncommutative motives

For a survey article on noncommutative motives we invite the reader to consult [17]. Recall from [16] the construction of the category \( \mathrm{Hmo}_0 \) of noncommutative motives. It is defined as the pseudo-abelian envelope of the category whose objects are the dg categories and whose abelian groups of morphisms are given by

\[
\text{Hom}_{\text{Hmo}_0}(A, B) := K_0 \text{rep}(A, B).
\]

Composition is induced by the derived tensor product of bimodules. The category \( \text{Hmo}_0 \) is additive, idempotent complete, and there is a natural functor

\[
U : \text{dgcat} \to \text{Hmo}_0,
\]

sending a dg functor \( F : A \to B \) to the class in the Grothendieck group \( K_0 \text{rep}(A, B) \) of the bimodule in \( \text{rep}(A, B) \) naturally associated to \( F \). Recall also that a functor \( \text{dgcat} \to D \), with values in an idempotent complete additive category, is called an additive invariant if it inverts Morita equivalences and satisfies additivity. As explained in [17, §4], the above functor (4.1) is the universal additive invariant, i.e. given any idempotent complete additive category \( D \) we have an induced equivalence of categories

\[
U^* : \text{Fun}_{\text{add}}^{\otimes}(\text{Hmo}_0, D) \sim \to \text{Fun}^{\otimes}_A(\text{dgcat}, D),
\]

where \( \text{Fun}_{\text{add}}(\text{Hmo}_0, D) \) denotes the category of additive functors and \( \text{Fun}^{\otimes}_A(\text{dgcat}, D) \) the category of additive invariants.

Now, recall from [16, Lemma 4.3] that the derived tensor product \( - \otimes^L - \) on \( \text{Hmo} \) (see Sect. 2) induces a symmetric monoidal structure on \( \text{Hmo}_0 \) making the functor \( U \) symmetric monoidal. This symmetric monoidal structure is characterized by the following universal property.

**Proposition 4.3.** Given any idempotent complete symmetric monoidal additive category \( D \), the above equivalence (4.2) admits the following monoidal refinement

\[
U^* : \text{Fun}_{\text{add}}^{\otimes}(\text{Hmo}_0, D) \sim \to \text{Fun}^{\otimes}_A(\text{dgcat}, D),
\]

where \( \text{Fun}_{\text{add}}^{\otimes}(\text{Hmo}_0, D) \) denotes the category of symmetric monoidal additive functors and \( \text{Fun}^{\otimes}_A(\text{dgcat}, D) \) the category of additive invariants which are moreover symmetric monoidal.

**Proof.** Let \( E \) be an object of \( \text{Fun}_{\text{add}}^{\otimes}(\text{Hmo}_0, D) \). Since \( U \) is symmetric monoidal the composite \( E \circ U \) is also symmetric monoidal and so by equivalence (4.2) we conclude that it belongs to \( \text{Fun}^{\otimes}_A(\text{dgcat}, D) \). Now, let \( H \) be an object of \( \text{Fun}^{\otimes}_A(\text{dgcat}, D) \). By equivalence (4.2) it factors uniquely through \( U \) giving rise to an additive invariant \( \overline{H} : \text{Hmo}_0 \to D \). By construction of \( \text{Hmo}_0 \) the functor \( \overline{H} \) remains symmetric monoidal and so it belongs to \( \text{Fun}_{\text{add}}^{\otimes}(\text{Hmo}_0, D) \). This achieves the proof. \( \square \)

5. Proofs

**Proof of Theorem 1.2**

Let us start by showing that the natural ring homomorphisms \( \iota : \mathbb{Z} \to \mathbb{Z}[r] \) and \( \pi : \mathbb{Z}[r] \overset{r=0}{\twoheadrightarrow} \mathbb{Z} \) give rise to a direct sum decomposition \( \mathcal{U}(\mathbb{Z}[r]) \simeq \mathcal{U}(\mathbb{Z}) \oplus \mathcal{W}_0 \) of co-unital co-associative
co-monoids in \( \text{Hmo}_0 \). Since \( \pi \circ \iota = \text{id}_2 \) the composite \( \iota \circ \pi \) is an idempotent endomorphism of \( \mathbb{Z}[r] \) and hence \( U(\iota \circ \pi) \) is an idempotent endomorphism of \( U(\mathbb{Z}[r]) \). By construction the category \( \text{Hmo}_0 \) is idempotent complete and so we obtain a direct sum decomposition \( U(\mathbb{Z}) \oplus \mathcal{W}_0 \). The noncommutative motive \( \mathcal{W}_0 \) identifies then with the kernel of the map 
\[ U(\pi) : U(\mathbb{Z}[r]) \to U(\mathbb{Z}). \]

The following commutative diagram

\[
\begin{array}{ccc}
\mathbb{Z}[r] & \xrightarrow{\Delta} & \mathbb{Z}[r] \otimes \mathbb{Z}[r] \\
\pi & \downarrow & \pi \otimes \pi \\
\mathbb{Z} & \xrightarrow{1 \otimes 1} & \mathbb{Z} \otimes \mathbb{Z}
\end{array}
\]

allows us then to conclude that the co-monoid structure on \( U(\mathbb{Z}[r]) \) restricts to \( \mathcal{W}_0 \). Its co-unit is given by the composition \( \mathcal{W}_0 \hookrightarrow U(\mathbb{Z}[r]) \xrightarrow{U(\epsilon)} U(\mathbb{Z}) \).

Let us now construct the ring isomorphism (1.3). Recall from [18, Thm. 1.1] that for every ring \( A \) we have a natural isomorphism

\[
\text{Hom}_{\text{Hmo}_0}(U(\mathbb{Z}[r]), U(A)) \simeq K_0\text{End}(A),
\]

where \( K_0\text{End}(A) \) denotes the Grothendieck group of endomorphisms of \( A \). We start by proving that (5.1) is a ring isomorphism whenever \( A \) is commutative. The isomorphism (5.1) is obtained by composing

\[
\text{Hom}_{\text{Hmo}_0}(U(\mathbb{Z}[r]), U(A)) := K_0\text{rep}(\mathbb{Z}[r], A) \xrightarrow{\sim} K_0\text{End}(\text{perf}(A))
\]

with \( K_0\text{End}(\text{perf}(A)) \simeq K_0\text{End}(\text{P}(A)) \). The tensor product of perfect \( A \)-modules induces naturally a ring structure on \( K_0\text{End}(\text{perf}(A)) \) making the isomorphism \( K_0\text{End}(\text{perf}(A)) \simeq K_0\text{End}(\text{P}(A)) \) into a ring isomorphism. Hence, it suffices to show that under the above group isomorphism (5.2) the convolution multiplication (on the left-hand-side) identifies with the one given by the tensor product (on the right-hand-side). Let \( X, Y \in \text{rep}(\mathbb{Z}[r], A) \). Since the underlying abelian group of \( \mathbb{Z}[r] \) is torsionfree, [20, Corollary 3.1.5] implies that the ring \( \mathbb{Z}[r] \oplus A \) (and hence the dg category \( \mathbb{Z}[r] \)) is flat. As a consequence, \( \mathbb{Z}[r] \otimes A \simeq \mathbb{Z}[r] \otimes A \) and so \( \text{rep}(\mathbb{Z}[r], A) \) identifies with the full triangulated subcategory of \( D(\mathbb{Z}[r]^{\text{op}} \otimes A) \) spanned by those \( \mathbb{Z}[r] \)-\( A \)-bimodules \( X \) such that the \( A \)-module \( X \) belongs to \( \text{perf}(A) \), where \( * \) is the unique object of \( \mathbb{Z}[r] \). Let us denote by \( [X, \alpha] \) and \( [Y, \beta] \) the images of \( [X] \) and \( [Y] \) under the isomorphism (5.2), where \( \alpha \) and \( \beta \) are the endomorphisms associated to the left action of \( \mathbb{Z}[r] \) on \( X \) and \( Y \). The multiplication of \( [X, \alpha] \) with \( [Y, \beta] \) is then the element

\[
[(X \otimes_A Y), \alpha \otimes_A \beta] \in K_0\text{End}(\text{perf}(A)).
\]

Note that by definition of the category \( \text{perf}(A) \) there is no need to derive the tensor product over \( A \). Now, let us analyze the convolution multiplication. Since by hypothesis \( A \) is commutative, its multiplication \( m : A \otimes A \to A \) is a ring homomorphism. Hence, we have a well-defined dg functor \( m : A \otimes A \to A \). Let us denote by \( mA \) the associated bimodule and by \( [mA] \) its class in \( K_0\text{rep}(A \otimes A, A) \). Note that since by hypothesis \( A \) is flat, the bimodule \( mA \) belongs to \( \text{rep}(A \otimes A, A) \). Similarly, let us denote by \( \Delta(\mathbb{Z}[r] \otimes \mathbb{Z}[r]) \) the bimodule associated to \( \Delta \) (see (1.1)) and by \( [\Delta(\mathbb{Z}[r] \otimes \mathbb{Z}[r])] \) its class in \( K_0\text{rep}(\mathbb{Z}[r], \mathbb{Z}[r] \otimes \mathbb{Z}[r]) \). Note that the (derived) tensor product of \( X \) with \( Y \) is an element of \( \text{rep}(\mathbb{Z}[r] \otimes \mathbb{Z}[r], A \otimes A) \). The convolution multiplication of \( [X] \) with \( [Y] \) is then the following composition

\[
\begin{array}{ccc}
\mathbb{Z}[r] & \xrightarrow{[\Delta(\mathbb{Z}[r] \otimes \mathbb{Z}[r])]} & \mathbb{Z}[r] \otimes \mathbb{Z}[r] \\
& \xrightarrow{[X \otimes Y]} & A \otimes A \\
& \xrightarrow{[mA]} & A
\end{array}
\]
in the category $\text{Hmo}_0$. By definition of $\text{Hmo}_0$ this composition corresponds to the class

$$\Delta([\mathbb{Z}[r] \otimes \mathbb{Z}[r]) \otimes_{[\mathbb{Z}[r] \otimes [\mathbb{Z}[r]} (X \otimes Y) \otimes_A m A],$$

which naturally identifies with $[X \otimes_A Y]$. Since the left action of $\mathbb{Z}[r]$ on $X \otimes_A Y$ is the diagonal one, we conclude finally that the image of $[X \otimes_A Y]$ under the above isomorphism (5.2) is precisely the element (5.3). This shows that (5.1) is a ring isomorphism.

Now, note that the exact functors $P(A) \rightarrow \text{End}(P(A)), M \mapsto (M, 0)$, and $\text{End}(P(A)) \rightarrow P(A), (M, \alpha) \mapsto M$, induce a splitting $K_0\text{End}(P(A)) \simeq K_0(A) \oplus \tilde{K}_0\text{End}(P(A))$ of abelian groups which is moreover compatible with the ring structure. In order to simplify the exposition let us denote the ring $\tilde{K}_0\text{End}(P(A))$ by $\tilde{K}_0\text{End}(A)$. The map

$$\text{Hom}_{\text{Hmo}_0}(\mathcal{U}(\mathbb{Z}[r]), \mathcal{U}(A)) \simeq K_0\text{End}(A) \overset{\mathcal{U}(\iota)^*}{\longrightarrow} K_0(A) \simeq \text{Hom}_{\text{Hmo}_0}(\mathcal{U}(\mathbb{Z}), \mathcal{U}(A))$$

induced by $\mathcal{U}(\iota)$ identifies with the homomorphism

$$K_0\text{rep}(\mathbb{Z}[r], A) \longrightarrow K_0\text{rep}(\mathbb{Z}, A) \quad [(M, \alpha)] \mapsto [M].$$

As a consequence, the abelian group $\text{Hom}_{\text{Hmo}_0}(\mathcal{W}_0, \mathcal{U}(A))$ identifies with the kernel of $\mathcal{U}(\iota)^*$. This kernel is by construction $\tilde{K}_0\text{End}(A)$ and so we obtain the following abelian group isomorphism

$$\text{Hom}_{\text{Hmo}_0}(\mathcal{W}_0, \mathcal{U}(A)) \simeq \tilde{K}_0\text{End}(A). \tag{5.4}$$

From the above ring isomorphism (5.1) and the decomposition of $K_0\text{End}(A)$ one concludes that (5.4) preserves moreover the ring structure.

Now, recall from [2] that we have the following ring isomorphism

$$\tilde{K}_0\text{End}(A) \longrightarrow W_0(A) \quad (M, \alpha) \mapsto \det(\text{Id} + \alpha t).$$

By combining it with (5.4) we obtain then the searched ring isomorphism (1.3).

**Proof of Theorem 1.4**

Recall that by construction we have

$$\text{Hom}_{\text{Hmo}_0}(\mathcal{U}(\mathbb{Z}[r, r^{-1}]), \mathcal{U}(A)) := K_0\text{rep}(\mathbb{Z}[r, r^{-1}], A).$$

Similarly to the proof of Theorem 1.2, the ring $\mathbb{Z}[r, r^{-1}]$ is flat and so $\text{rep}(\mathbb{Z}[r, r^{-1}], A)$ identifies with the full triangulated subcategory of $\mathcal{D}(\mathbb{Z}[r, r^{-1}]^{\text{op}} \otimes A)$ spanned by the $\mathbb{Z}[r]$.-A-bimodules $X$ such that the $A$-module $X(\cdot, \ast)$ belongs to $\text{perf}(A)$. Such a bimodule consists precisely on the same data as an object of $\text{Aut}(\text{perf}(A))$. Hence, we have an equivalence $\text{rep}(\mathbb{Z}[r, r^{-1}], A) \simeq \text{Aut}(\text{perf}(A))[w^{-1}]$, where $w$ denotes the class of quasi-isomorphisms. Since $K_0\text{Aut}(\text{perf}(A)) \simeq K_0\text{Aut}(\text{perf}(A)[w^{-1}])$ we obtain then the searched abelian group isomorphism (1.5).

When $A = A$, with $A$ a commutative ring, an argument analogous to the one used in the proof of Theorem 1.2 (with $\mathbb{Z}[r]$ replaced by $\mathbb{Z}[r, r^{-1}]$) shows us that (1.5) reduces to the ring isomorphism (1.6). This concludes the proof.
Proof of Theorem 1.7

Let us start by constructing the isomorphism (1.8). The co-representability of the functor $K_0\text{Aut}$ in the category $\text{Hm}_0$ of noncommutative motives (see Theorem 1.4) implies that $K_0\text{Aut}$ is an additive invariant of dg categories. Hence, $K_0\text{Aut}$ belongs to the category $\text{Fun}_A(\text{dgcat, Ab})$. Using the equivalence (4.2) we obtain then an abelian group isomorphism

$$\text{Nat}(K_0\text{Aut}, K_0\text{Aut}) \cong \text{Nat}(K_0\text{Aut}, K_0\text{Aut}) \eta \mapsto \eta.$$  \hfill (5.5)

By Theorem 1.4 the additive functor $K_0\text{Aut}$ is co-represented in $\text{Hm}_0$ by $\mathcal{U}(\mathbb{Z}[r, r^{-1}])$ and so using the enriched Yoneda lemma [3, Thm. 8.3.5] one obtains the following abelian group isomorphism

$$\text{Nat}(K_0\text{Aut}, K_0\text{Aut}) \cong K_0\text{Aut}(\mathcal{U}(\mathbb{Z}[r, r^{-1}])) \eta \mapsto \eta(\mathbb{Z}[r, r^{-1}])([\text{id}_{\mathbb{Z}[r, r^{-1}]}]).$$

By construction of $\text{Hm}_0$ the class $[\text{id}_{\mathbb{Z}[r, r^{-1}]}]$ identifies with $[(\mathbb{Z}[r, r^{-1}], \cdot r)]$. Hence, by combining this latter isomorphism with (5.5) we obtain the isomorphism

$$\text{Nat}(K_0\text{Aut}, K_0\text{Aut}) \cong K_0\text{Aut}(\mathbb{Z}[r, r^{-1}]),$$  \hfill (5.6)

sending a natural transformation $\eta$ to $\eta(\mathbb{Z}[r, r^{-1}])((\mathbb{Z}[r, r^{-1}], \cdot r))$. This is the searched isomorphism (1.8).

Let us now prove the remaining claims of the Theorem. As explained in [14, Def. 3.1.6], the quotient group $K_0\text{Aut}(\mathbb{Z}[r, r^{-1}])/(1.9)$ identifies with the $K_1$-group of the category $\mathcal{P}(\mathbb{Z}[r, r^{-1}])$, which by [14, Thm. 3.1.7] is isomorphic to $K_1(\mathbb{Z}[r, r^{-1}])$. Since $\mathbb{Z}$ is a regular ring we have by Bass-Heller-Swan (see [14, Cor. 3.2.20]) an isomorphism $K_1(\mathbb{Z}[r, r^{-1}]) \simeq K_1(\mathbb{Z}) \oplus K_0(\mathbb{Z})$. Finally, since $K_1(\mathbb{Z}) \simeq \{-1, 1\}$ (see [14, Cor. 2.3.3]) and $K_0(\mathbb{Z}) \simeq \mathbb{Z}$, we conclude that

$$K_0\text{Aut}(\mathbb{Z}[r, r^{-1}])/(1.9) \simeq \{-1, 1\} \times \mathbb{Z}.$$  \hfill (5.7)

Now, let us show that under the quotient homomorphism (1.10) the Frobenius operation $F_n$ corresponds to $(1, n)$ and that the Verschiebung operation $V_n$ corresponds to $((-1)^{n+1}, 1)$. We start with the Frobenius operation

$$F_n : K_0\text{Aut} \to K_0\text{Aut} \quad [(M, \alpha)] \mapsto [(M, \alpha^n)].$$

Its image under the above canonical map (5.6) is the element $[(\mathbb{Z}[r, r^{-1}], \cdot r^n)]$. It suffices then to understand the image of this element under the homomorphism

$$K_0\text{Aut}(\mathbb{Z}[r, r^{-1}]) \to K_1(\mathcal{P}(\mathbb{Z}[r, r^{-1}])), \Phi : K_1(\mathcal{P}(\mathbb{Z})) \oplus K_0(\mathbb{Z}) \sim \{-1, 1\} \times \mathbb{Z},$$

where the right-hand-side isomorphism is induced by the determinant and rank assignments. In order to simplify the exposition we will make no notational distinction between the elements of $K_0\text{Aut}(\mathbb{Z}[r, r^{-1}])$ and $K_1(\mathcal{P}(\mathbb{Z}[r, r^{-1}]))$. The relations (1.9) imply that the image of $[(\mathbb{Z}[r, r^{-1}], \cdot r^n)]$ under (5.7) identifies with $n$-times the image of $[(\mathbb{Z}[r, r^{-1}], \cdot r)]$ (corresponding to the Frobenius operation $F_1$). As explained in [14, Prop. 3.2.18], the inverse of the isomorphism $\Phi$ is given by

$$[(M, \alpha)] \mapsto [(\mathbb{Z}[r, r^{-1}] \otimes M, \text{id} \otimes \alpha)] + [(\mathbb{Z}[r, r^{-1}] \otimes M' , r \otimes \text{id})].$$  \hfill (5.8)
Hence, we observe that the element \(((\mathbb{Z}, \text{id})], [\mathbb{Z}])\) corresponds to \(((\mathbb{Z}[r, r^{-1}], \text{id})] + \{([\mathbb{Z}, r^{-1}], r)\} \). Since \(((\mathbb{Z}[r, r^{-1}], \text{id})]\) is the trivial element of \(K_1(\mathbb{P}(\mathbb{Z}[r, r^{-1}]))\) we conclude that the image of \(((\mathbb{Z}[r, r^{-1}], r)\]) under \((5.7)\) is \((1, 1)\). By the previous arguments the image of \(((\mathbb{Z}[r, r^{-1}], r^n)\]) is then \((1, n)\).

Let us now study the Verschiebung operation

\[ V_n : K_0 \text{Aut} \to K_0 \text{Aut} \quad [(M, \alpha)] \mapsto [(M^\oplus, V_n(\alpha))], \]

where

\[ V_n(\alpha) := \begin{bmatrix} 0 & \cdots & 0 & \alpha \\ 1 & \ddots & \vdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}_{(n \times n)}. \]

Its image under \((5.6)\) is the element \(((\mathbb{Z}[r, r^{-1}]^\oplus n), V_n(r))\]) and so it suffices to study the image of this element under \((5.7)\). The matrix equality \(V_n(\alpha) = W_n \cdot I_n(\alpha)\), where

\[ W_n := \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & \ddots & \vdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}_{(n \times n)}, \quad I_n(\alpha) := \begin{bmatrix} 1 & \cdots & 0 & 0 \\ 0 & \ddots & \vdots & 0 \\ 0 & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \alpha \end{bmatrix}_{(n \times n)} \]

combined with the relations \((1.9)\) implies that the image of \(((\mathbb{Z}[r, r^{-1}]^\oplus n), V_n(r))\]) under the homomorphism \((5.7)\) agrees with the sum of the images of \(((\mathbb{Z}[r, r^{-1}]^\oplus n), W_n)\]) and \(((\mathbb{Z}[r, r^{-1}]^\oplus n), I_n(\alpha))\]). Under the assignment \((5.8)\), \(((\mathbb{Z}^\oplus n, W_n), [0])\) corresponds to \(((\mathbb{Z}[r, r^{-1}]^\oplus n), W_n))\]). Since the determinant of the matrix \(W_n\) equals \((-1)^{n+1}\), we then conclude that the image of \(((\mathbb{Z}[r, r^{-1}]^\oplus n), W_n)\]) under \((5.7)\) is exactly \((-1)^{n+1}, 0\). In what concerns the element \(((\mathbb{Z}[r, r^{-1}]^\oplus n), I_n(\alpha))\]) we have the following short exact sequence

\[ 0 \to [(\mathbb{Z}[r, r^{-1}]^\oplus (n-1), \text{Id})] \to [(\mathbb{Z}[r, r^{-1}]^\oplus n, I_n(\alpha))] \to [(\mathbb{Z}[r, r^{-1}], r)] \to 0, \]

where the first map corresponds to the upper-left-block matrix inclusion. Since \(((\mathbb{Z}[r, r^{-1}]^\oplus (n-1), \text{Id})]\) is the trivial element of \(K_1(\mathbb{P}(\mathbb{Z}[r, r^{-1}]))\) we conclude that \(((\mathbb{Z}[r, r^{-1}]^\oplus n), I_n(\alpha))\]) agrees with \(((\mathbb{Z}[r, r^{-1}], r)\]). As explained above the image of \(((\mathbb{Z}[r, r^{-1}], r)\]) under \((5.7)\) is \((1, 1)\). Hence, by adding \((-1)^{n+1}, 0\) with \((1, 1)\) we conclude finally that the image of \(((\mathbb{Z}[r, r^{-1}]^\oplus n), V_n(\alpha))\]) under \((5.7)\) is \((-1)^{n+1}, 1\). This achieves the proof.

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