The centralizer of an $I$-matrix in $M_2(R/I)$, $R$ a UFD

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Abstract. The concept of an $I$-matrix in the full $2 \times 2$ matrix ring $M_2(R/I)$, where $R$ is an arbitrary UFD and $I$ is a nonzero ideal in $R$, is introduced. We obtain a concrete description of the centralizer of an $I$-matrix $\hat{B}$ in $M_2(R/I)$ as the sum of two subrings $S_1$ and $S_2$ of $M_2(R/I)$, where $S_1$ is the image (under the natural epimorphism from $M_2(R)$ to $M_2(R/I)$) of the centralizer in $M_2(R)$ of a pre-image of $\hat{B}$, and where the entries in $S_2$ are intersections of certain annihilators of elements arising from the entries of $\hat{B}$. It turns out that if $R$ is a PID, then every matrix in $M_2(R/I)$ is an $I$-matrix. However, this is not the case if $R$ is a UFD in general. Moreover, for every factor ring $R/I$ with zero divisors and every $n \geq 3$ there is a matrix for which the mentioned concrete description is not valid.

1. Introduction

We denote the centralizer of an element $s$ in an arbitrary ring $S$ by $\text{Cen}_S(s)$. Knowing that $M_n(R)$, the full $n \times n$ matrix ring over a commutative ring $R$, is a prime example of a non-commutative ring, it is surprising that a concrete description of $\text{Cen}_{M_n(R)}(B)$ for an arbitrary $B \in M_n(R)$ has not yet been found. If $R[x]$ is the polynomial ring in the variable $x$ over $R$, then

\begin{equation}
\{ f(B) \mid f(x) \in R[x] \} \subseteq \text{Cen}_{M_n(R)}(B).
\end{equation}

In fact, it is known that (see [2])

\[ \{ f(B) \mid f(x) \in R[x] \} = \text{Cen}_{M_n(R)}(\text{Cen}_{M_n(R)}(B)). \]
The most progress, finding a concrete description of $\text{Cent}_{M_n(R)}(B)$, has been made for the case when the underlying ring $R$ is a field (see [1], [3], [4], [5] and [7]). The following well-known result in this case provides a necessary and sufficient condition for equality in (1).

**Theorem 1.1.** If $B$ is an $n \times n$ matrix over a field $F$, then

$$\text{Cent}_{M_n(F)}(B) = \{ f(B) \mid f(x) \in F[x] \}$$

if and only if the minimum polynomial of $B$ coincides with the characteristic polynomial of $B$.

In this paper we consider the centralizer of a so-called $I$-matrix in $M_2(R/I)$, with $R/I$ a factor ring of a UFD $R$ and $I$ a nonzero ideal in $R$.

In Section 2 we obtain an explicit description of the centralizer of a $2 \times 2$ matrix over a field or over a unique factorization domain. Section 2 also contains other preliminary results concerning the centralizer of an $n \times n$ matrix that will be used in the subsequent sections, including Proposition 2.6 which may be considered as the inspiration behind this paper. In this proposition we show that the centralizer of an $n \times n$ matrix $\hat{B}$ over a homomorphic image of a commutative ring $R$ contains the sum of two subrings $S_1$ and $S_2$ of $M_2(S)$, where $S_1$ is the image of the centralizer in $M_2(R)$ of a pre-image of $\hat{B}$, and where the entries in $S_2$ are intersections of certain annihilators of elements arising from the entries of $\hat{B}$.

In Section 3 we introduce the concepts of $I$-invertibility in a factor ring $R/I$ of a UFD $R$ (Definition 3.3) and of an $I$-matrix in $M_2(R/I)$ (Definition 3.23). We show in Corollaries 3.9 and 3.29 that if $R$ is a PID, then every element in $R/I$ is $I$-invertible and every matrix in $M_2(R/I)$ is an $I$-matrix. Examples 3.22 and 3.30(b) show that this is not true for UFD’s in general, not even if $I$ is a principal ideal.

Section 4 contains the main result of the paper, namely Theorem 4.1, which provides a concrete description of the centralizer of an $I$-matrix in $M_2(R/I)$ as the sum of the above mentioned two subrings, where $R$ is a UFD and $I$ is a nonzero ideal in $R$.

Since every $2 \times 2$ matrix over a factor ring of a PID is an $I$-matrix, Theorem 4.1 applies to all $2 \times 2$ matrices over factor rings of PID’s. In Example 4.4, we exhibit a UFD $R$, which is not a PID, a finitely generated ideal $I$ and a matrix in $M_2(R)$, which is not an $I$-matrix, for which Theorem 4.1 does not hold. In Example 4.5, we show that if $R$ is a UFD and $R/I$ is such that $R/I$ is not an integral domain, then for every $n \geq 3$ there is a matrix in $M_n(R)$ for which we do not have equality in Proposition 2.6.
2. Preliminary Results

Since the minimum polynomial and characteristic polynomial of any \(2 \times 2\) non-scalar matrix over a field coincide, the following corollary follows from Theorem 1.1.

**Corollary 2.1.** If \(B\) is a \(2 \times 2\) matrix over a field \(F\), then

\[
\text{Cen}_{M_2(F)}(B) = \begin{cases} 
M_2(F), & \text{if } B \text{ is a scalar matrix} \\
\{f(B) \mid f(x) \in F[x]\}, & \text{if } B \text{ is a non-scalar matrix.}
\end{cases}
\]

In this paper we denote the identity matrix by \(E\).

**Remark 2.2.** Let \(B = \begin{bmatrix} e & f \\
g & h \end{bmatrix} \in M_2(R),\) \(R\) a commutative ring. Elementary matrix multiplication shows that

\[
\begin{align*}
A &= \begin{bmatrix} a & b \\
c & d \end{bmatrix} \in \text{Cen}_{M_2(R)}(B) \\
\end{align*}
\]

if and only if

\[
(a - d)f = b(e - h), \quad bg = cf, \quad c(e - h) = (a - d)g
\]

if and only if \(A' + vE\) and \(B\) commute if and only if \(A' + vE\) and \(B' + wE\) commute if and only if \(A'\) and \(B'\) commute, where

\[
A' = \begin{bmatrix} a - d & b \\
c & 0 \end{bmatrix} \quad \text{and} \quad B' = \begin{bmatrix} e - h & f \\
g & 0 \end{bmatrix}.
\]

Throughout the sequel, for \(R\) a UFD and for a nonempty set \(X \subset R\), we mean by \(\gcd(X)\) an arbitrary greatest common divisor of \(X\) in \(R\).

The following result is an extension of Corollary 2.1 to UFD's.

**Corollary 2.3.** Let \(B = \begin{bmatrix} e & f \\
g & h \end{bmatrix} \in M_2(R),\) \(R\) a UFD. Then \(\text{Cen}_{M_2(R)}(B)\)

\[
= \begin{cases} 
M_2(R), & \text{if } e = h, f = 0 \text{ and } g = 0 \text{ (i.e. } B \text{ is a scalar matrix)} \\
\{m^{-1}w \begin{bmatrix} e - h & f \\
g & 0 \end{bmatrix} + vE \mid v, w \in R\}, & \text{if at least one of } e - h, f, g \text{ is nonzero,}
\end{cases}
\]

where \(m^{-1}\) is the inverse of \(m := \gcd(e - h, f, g)\) in the quotient field of \(R\).

**Proof.** (ii) Suppose that at least one of \(e - h, f, g\) is nonzero. Let \(A'\) and \(B'\) be as in (4). By the symmetry of the system of equations in (3) we may assume that \(e - h \neq 0\). Then, using (3), \(e - h|(a - d)f\) and \(e - h|(a - d)g\) imply that \(e - h|ma(a - d)\). Let \(w \in R\) such that \(m(a - d) = w(e - h)\). Then, again using (3), \((a - d)f = b(e - h)\) and \(c(e - h) = (a - d)g\) imply that \(mb = wf\) and \(mc = wg\). Thus \(mA' = wB'\) and the result follows from Remark 2.2. \(\square\)
Example 2.4. Let $R$ be the UFD $\mathbb{Z}$ of integers, and let $B = \begin{bmatrix} 8 & 3 \\ 6 & 2 \end{bmatrix}$. It follows from Corollary 2.3(ii) that 

$$\text{Cen}_{M_2(\mathbb{Z})}(B) = \left\{ \begin{bmatrix} 2w + v & w \\ 2w & v \end{bmatrix} \mid v, w \in \mathbb{Z} \right\}.$$ 

For the remaining results in this section, let $\theta : R \to S$ be a ring epimorphism and $\Theta : M_n(R) \to M_n(S)$ the induced epimorphism, i.e. $\Theta([b_{ij}]) = [\theta(b_{ij})]$. We denote the annihilator of an element $r$ in a commutative ring $R$ by $\text{ann}_R(r)$. For the sake of notation, we will sometimes denote $\theta(b)$ by $\hat{b}$ and $\Theta(B)$ by $\hat{B}$. Also, if there is no ambiguity, we simply write $\text{Cen}(B)$ instead of $\text{Cen}_{M_2(R)}(B)$ and $\text{Cen}(\hat{B})$ instead of $\text{Cen}_{M_2(S)}(\hat{B})$ for $B \in M_2(R)$, as well as $\text{ann}(\hat{r})$ instead of $\text{ann}_S(\hat{r})$ for $r \in R$. If $r \in R$ and $A \subseteq R$, then $rA$ denotes the set $\{ra \mid a \in A\}$.

Throughout this paper and in particular in Section 4 we use the notation 

$$\begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

to denote the set 

$$\left\{ \begin{bmatrix} b & c \\ d & e \end{bmatrix} \mid b \in B, c \in C, d \in D, e \in E \right\},$$

where $B, C, D$ and $E$ are subsets of a ring $R$.

The following result is straightforward.

Lemma 2.5. Let $S$ be a subring of a ring $T$ and let $s \in S$. Then 

$$\text{Cen}_S(s) = S \cap \text{Cen}_T(s).$$

The following result is the inspiration behind Section 4.

Proposition 2.6. Let $R$ be a commutative ring and let $B = [b_{ij}] \in M_n(R)$. Then 

$$\Theta(\text{Cen}(B)) + [A_{ij}] \subseteq \text{Cen}(\hat{B}),$$

where 

$$A_{ij} = \left( \bigcap_{k, k \neq j} \text{ann}(\hat{b}_{jk}) \right) \bigcap \left( \bigcap_{k, k \neq i} \text{ann}(\hat{b}_{ki}) \right) \bigcap \text{ann}(\hat{b}_{ii} - \hat{b}_{jj}).$$

PROOF. It follows easily that 

(5) $$\Theta(\text{Cen}(B)) \subseteq \text{Cen}(\hat{B}).$$

Now we show that 

(6) $$[A_{ij}] \subseteq \text{Cen}(\hat{B}).$$

Let $[\hat{a}_{ij}] \in [A_{ij}]$. It follows that position $(r, t)$ of $\hat{B}[\hat{a}_{ij}] = [\hat{a}_{ij}][\hat{B}]$ is equal to 

$$\hat{b}_{r1}\hat{a}_{1t} + \cdots + \hat{b}_{r,r-1}\hat{a}_{r-1,t} + \hat{b}_{rr}\hat{a}_{rt} + \hat{b}_{r,r+1}\hat{a}_{r+1,t} + \cdots + \hat{b}_{rn}\hat{a}_{nt} -$$
From here onwards, unless stated otherwise, we assume that $R$ is a UFD, $I$ is a nonzero ideal in $R$ and $k := \gcd(I) \neq 0$. Let $\theta_I : R \to R/I$ and $\Theta_I : M_2(R) \to M_2(R/I)$ be the natural epimorphism and induced epimorphism respectively. We denote the image $\theta_I(b)$ of $b \in R$ by $\hat{b}_I$ and the image $\Theta_I(B)$ of $B \in M_2(R)$ by $\hat{B}_I$.

However, if there is no ambiguity, then we simply write $\theta$, $\Theta$, $\hat{b}$ and $\hat{B}$ respectively.

The following results are trivial.

**Lemma 3.1.** Let $R$ be a UFD. Then an element $\hat{b} = \theta(b) \in R/I$ is a zero divisor if $\gcd(b, k) \neq 1$.

**Lemma 3.2.** Let $R$ be a PID. Then an element $\hat{b} \in R/(k)$, $k \in R$, is invertible if and only if $\gcd(b, k) = 1$.

**Definition 3.3.** An $I$-pre-image of an element $\hat{b} \in R/I$ is a pre-image of $\hat{b}$ in $R$ of the form $r \delta$, where $\gcd(r, k) = 1$ and $(\delta = 0$ or $\delta | k)$. If $\hat{b} = 0$ we define $\delta := 0$. We call $r$ and $\delta$ the relative prime part and divisor part of $r \delta$ respectively. We call $\hat{b}$ $I$-invertible if $\hat{r}$ is invertible in $R/I$ for at least one $I$-pre-image $r \delta$ of $\hat{b}$.

**Remark 3.4.** It follows from Definition 3.3 that if an element $\hat{0} \neq \hat{b} \in R/I$ is $I$-invertible, then there exists a $\hat{c} \in R/I$ such that $\hat{c} \hat{b}$ has a pre-image $\delta \in R$ which is a divisor of $k$.

The converse of the above remark is not in general true. Here follows a counter example.

**Example 3.5.** Let $R = \mathbb{Z}[x]$, let $I = \langle 5x^2 \rangle$ and let $\hat{b}_I = 3x^2$, then $\hat{b}_I$ is not $I$-invertible, but

$$\hat{2}_I \hat{b}_I = \theta_I(6x^2 - 5x^2) = x^2 I.$$

We define the ideal $\delta^{-1} I := \{ \delta^{-1} a | a \in I \} \subset R$. The following result can be easily proved.

**Lemma 3.6.** Let $\delta$ be the divisor part of an $I$-pre-image of $\hat{0} \neq \hat{b}_I \in R/I$. There exists a $\hat{c}_I \in R/I$ such that $\hat{c}_I \hat{b}_I = \hat{\delta}_I$ if and only if $b \delta^{-1} \delta^{-1} I$ is invertible in $R/\delta^{-1} I$, with inverse $\hat{c}_{\delta^{-1} I}$.  

(7) $(a_{r,t} \hat{b}_{tt} + a_{r,t-1} \hat{b}_{t-1,t} + \cdots + a_{r,t-1} \hat{b}_{1,t} + a_{r,t} \hat{b}_{tt} + a_{r,t+1} \hat{b}_{t+1,t} + \cdots + a_{r,t} \hat{b}_{tt})$.

Since $a_{r,t} \in \text{ann}(\hat{b}_{tt})$ for every $l$ such that $l \neq r$, and $a_{r,q} \in \text{ann}(\hat{b}_{tt})$ for every $q$ such that $q \neq t$, according to the definition of $[\mathcal{A}_{ij}]$, it follows that (7) is equal to

(8) $b_{rr} \hat{a}_{rt} - a_{rt} b_{tt} = \hat{a}_{rt} (b_{rr} - \hat{b}_{tt})$.

Since $a_{r,t} \in \text{ann}(\hat{b}_{rr} - \hat{b}_{tt})$, according to the definition of $[\mathcal{A}_{ij}]$, it follows that (8) is equal to $t$. Thus position $(r, t)$ of $[\hat{a}_{ij}] \hat{B} - \hat{B}[\hat{a}_{ij}]$ is $0$. This proves (8).
Lemma 3.7. An element $\hat{b}_I \in R/I$ is $I$-invertible if and only if there exists an invertible element $\hat{c} \in R/I$ such that $\hat{c}_I\hat{b}_I = \hat{\delta}_I$, where $\delta$ is a divisor part of an $I$-pre-image of $\hat{b}_I$.

Proof. If $\hat{b}_I$ is $I$-invertible then it follows directly from Definition 3.3 that there exists an invertible element $\hat{c}_I$ such that $\hat{c}_I\hat{b}_I = \hat{\delta}_I$. Conversely, suppose there exists an invertible element $\hat{c}_I \in R/I$ such that $\hat{c}_I\hat{b}_I = \hat{\delta}_I$. Since $\hat{c}_I$ is invertible we have that $\hat{b}_I = \hat{c}_I^{-1}\hat{\delta}_I$. Let $c' \in R$ be a pre-image of $\hat{c}_I^{-1}$. Since $\hat{c}_I^{-1}$ is not a zero divisor it follows from Lemma 3.1 that $\gcd(c',k) = 1$. Since $c'\delta$ is an $I$-pre-image of $b_I$ we have the desired result. \[ \square \]

The proof of the next result is constructive.

Lemma 3.8. Every element in $R/I$ has an $I$-pre-image.

Proof. Let $\hat{b} \in R/I$. If $k$ is a unit, then the result follows trivially. Thus suppose $k$ is a nonzero nonunit. Since $R$ is a UFD there exist different primes $p_1, \ldots, p_s$ such that $k = p_1^{m_1} \cdots p_s^{m_s}$, where $m_1, \ldots, m_s \geq 1$. Since $1 \cdot 0$ is an $I$-pre-image of $0$, suppose $\hat{b}$ is nonzero. Let $b$ be a pre-image of $\hat{b}$ in $R$. Again, because $R$ is a UFD, $b$ can be expressed as $r_0p_1^{q_1} \cdots p_s^{q_s}$, where $p_i \nmid r_0$, for $i = 1, \ldots, s$, and $q_1, \ldots, q_s \geq 0$. Therefore $\gcd(r_0, k) = 1$, and
\[
\hat{b} = \hat{r}_0\hat{p}_1^{q_1} \cdots \hat{p}_s^{q_s}.
\]

Suppose we can show that each $\hat{p}_i^{q_i}$ has a pre-image $r_i \cdot p_i^{t_i}$, where $\gcd(r_i, k) = 1$ and $t_i \leq m_i$. Then we have that
\[
\hat{b} = \hat{r}_0(\hat{r}_1p_1^{t_1}) \cdots (\hat{r}_sp_s^{t_s}) = \hat{r}_0\hat{r}_1 \cdots \hat{r}_s(p_1^{t_1} \cdots p_s^{t_s}) = \theta(p_1^{t_1} \cdots p_s^{t_s}),
\]
where $r = r_0r_1 \cdots r_s$. Since $\gcd(r_i, k) = 1$ for $i = 0, 1, \ldots, s$, it follows that $\gcd(r, k) = 1$. Also, since $t_i \leq m_i$ for $i = 1, 2, \ldots, s$, we have that
\[
\delta := p_1^{t_1} \cdots p_s^{t_s} = \frac{p_1^{m_1} \cdots p_s^{m_s}}{k},
\]
implying that $r \cdot \delta$ is an $I$-pre-image of $\hat{b}$ with relative prime part $r$ and divisor part $\delta$.

Let us now prove that each $\hat{p}_i^{q_i}$ has a pre-image $r_i \cdot p_i^{t_i}$, where $\gcd(r_i, k) = 1$ and $t_i \leq m_i$.

If $q_i \leq m_i$ then $p_i^{q_i} = 1 \cdot p_i^{q_i}$, where $t_i = q_i \leq m_i$ and $\gcd(r_i, k) = 1$, with $r_i = 1$. Thus we have the desired result.

Next we consider the case when $m_i < q_i$. Because $p_i^{m_i+1} \nmid k$, it follows that there exist an $a = a'k \in I$ such that $p_i \nmid a'$. Now since
\[
\hat{p}_i^{q_i} = \hat{p}_i^{q_i} + a'k
\]
and
\[ p_i^{q_i} + a'k = p_i^{q_i} + a'p_1^{m_1} \cdots p_s^{m_s} = p_i^{m_i}(p_i^{q_i-m_i} + a'p_1^{m_1} \cdots p_{i-1}^{m_{i-1}}p_{i+1}^{m_{i+1}} \cdots p_s^{m_s}), \]
it follows that \( p_i^{m_i} \cdot r_i = r_i \cdot p_i^{m_i} \) is a pre-image of \( \hat{p}_i^{q_i} \), where
\[ r_i = p_i^{q_i-m_i} + a'p_1^{m_1} \cdots p_{i-1}^{m_{i-1}}p_{i+1}^{m_{i+1}} \cdots p_s^{m_s}. \]
Since
\[ p_i \mid p_i^{q_i-m_i}(q_i > m_i) \quad \text{and} \quad p_i \nmid a'p_1^{m_1} \cdots p_{i-1}^{m_{i-1}}p_{i+1}^{m_{i+1}} \cdots p_s^{m_s}, \]
we have that \( p_i \nmid r_i \). Furthermore, for all \( i \in \{1, \ldots, i-1, i+1, \ldots, s\} \) it follows that
\[ p_i \mid p_i^{q_i-m_i} \quad \text{and} \quad p_i \nmid a'p_1^{m_1} \cdots p_{i-1}^{m_{i-1}}p_{i+1}^{m_{i+1}} \cdots p_s^{m_s} \]
implying that \( p_i \nmid r_i \). Thus \( r_i \) and \( k \) are relatively prime and \( t_i = m_i \leq m_i \).

We will now focus on the \( I \)-invertibility of elements in \( R/I \).
The next result follows directly from Lemma 3.2, Definition 3.3 and Lemma 3.8.

**Corollary 3.9.** If \( R \) is a PID, then every element in \( R/I \) is \( I \)-invertible.

The next example illustrates the constructive proof of Lemma 3.8.

**Example 3.10.** Let \( R = \mathbb{Z} \) and let \( I = \langle 12 \rangle \). Since \( 12 = 2^2 \cdot 3 \) using the procedure in the proof of Lemma 3.8, it follows that
\begin{itemize}
  \item[(a)] \( \hat{9}_I = \theta_I(2^0 \cdot 3^2) = \theta_I(1 \cdot (3^2 + 12)) = \theta_I(3(7)) = (\hat{7} \cdot 3)I \), where gcd(7, 12) = 1 and 3 \not| 12. Since \( \hat{9}_I \) is invertible in \( \mathbb{Z}_{12} \) it follows that \( \hat{9}_I \) is \( I \)-invertible, as expected from Corollary 3.9.
\end{itemize}

Now, let \( R = \mathbb{Z}[x] \) and let \( I \) be a nonzero, not necessarily finite, ideal, with \( 2^4x^4 \in I \) and \( k := \text{gcd}(I) = 2^3x^3 \).
\begin{itemize}
  \item[(b)] \( 24x^5 + 8x^4 + 4x^2 I = \theta_I(24x^5 + 8x^4 + 4x^2) = \theta_I((6x^3 + 2x^2 + 1)2^2x^2) \),
  \[ \text{where } \text{gcd}(6x^3 + 2x^2 + 1, 2^3x^3) = 1 \text{ and } 2^2x^2|2^3x^3. \] Since \( 6x^3 + 2x^2 + 1 \mid \theta_I((3x+1)2x^2+1) \) is invertible in \( R/I \) by Lemma 3.20, \( 24x^5 + 8x^4 + 4x^2I \) is \( I \)-invertible.
\end{itemize}

We already know from Example 3.5 that Corollary 3.9 does not hold for \( R \) a UFD in general, not even for the case when \( I \) is a principal ideal.

Lemma 3.12, Proposition 3.15, Remark 3.16 and Lemma 3.20 will help us to determine when an element in \( R/I \) is not \( I \)-invertible in case \( R \) is a UFD which is not a PID. In order to conclude that an element \( \hat{b} \in R/I \) is not \( I \)-invertible (using Definition 3.3), we have to show, for every \( I \)-pre-image \( r \hat{o} \) of \( \hat{b} \), that \( \hat{r} \) is not invertible in \( R/I \). However, if \( \hat{b} \) is principal (Definition 3.14), then we will show in
Proposition 3.15 that it suffices to show that \( \hat{r} \) is not invertible in \( R/I \) for at least one \( I \)-pre-image \( r \delta \) of \( \hat{b} \).

We first give a characterization of and establish a relationship between the divisor parts of the \( I \)-pre-images of an element in \( R/I \).

**Lemma 3.11.** Let \( R \) be a UFD and let \( \hat{0} \neq \hat{b} \in R/I \). Then \( \delta \) is a divisor part of an \( I \)-pre-image of \( \hat{b} \) if and only if \( \gcd(b, k) = \delta \), i.e. the divisor parts of the \( I \)-pre-images of \( \hat{b} \) are associates.

**Proof.** Let \( r \delta \) be an \( I \)-pre-image of \( \hat{b} \). Then \( b = r \delta + sk \) for some \( s \in R \). Now, since \( \gcd(r, k) = 1 \), it follows that \( \gcd(b, k) = \gcd(r \delta + sk, k) = \gcd(\delta, k) = \delta \).

For the converse, note that since all the greatest common divisors of \( b \) and \( k \) are associates and every element in \( R/I \) has at least one \( I \)-pre-image, by Lemma 3.8 the result will follow if we can show that for an arbitrary unit \( t \), \( t \delta \) is also a divisor part of some \( I \)-pre-image of \( \hat{b} \). Since \( rt^{-1}t \delta = r \delta = \hat{b}, \gcd(rt^{-1}, k) = 1 \) and \( t \delta | k \), the result follows. \( \square \)

The following result follows trivially from Lemma 3.11.

**Lemma 3.12.** Let \( \hat{0} \neq \hat{b} \in R/I \). If \( \gcd(b, k) = 1 \), then \( \hat{b} \) is \( I \)-invertible if and only if \( \hat{b} \) is invertible in \( R/I \).

**Remark 3.13.** Note that if \( k \) is a unit, it follows from Lemma 3.12 that every \( \hat{0} \neq \hat{b} \in R/I \) is \( I \)-invertible if and only if \( \hat{b} \) is invertible in \( R/I \).

**Definition 3.14.** Let \( R \) be a UFD, let \( k = p_1^{m_1} \cdots p_s^{m_s} \in R \) be a nonunit, with \( p_1, \ldots, p_s \) different primes and \( m_1, \ldots, m_s \geq 1 \), and let \( \hat{b} \in R/I \). If \( \delta := \gcd(b, k) = p_1^{q_1} \cdots p_s^{q_s}, \) where \( 0 \leq q_i < m_i \) for \( i = 1, \ldots, s \), then we call \( \hat{b} \) a principal element of \( R/I \). If \( \delta^{-1}k \) is principal, i.e. \( \delta = p_1^{q_1} \cdots p_s^{q_s}, \) where \( q_i \geq 1 \) for \( i = 1, \ldots, s \), we call \( \hat{b} \) \( q \)-principal.

**Proposition 3.15.** Let \( R \) be a UFD, \( k \) be a nonunit and let \( \hat{0} \neq \hat{b} \in R/I \) be principal, then either \( \hat{r} \) is invertible in \( R/I \) for every \( I \)-pre-image \( r \delta \) of \( \hat{b} \) or no such \( \hat{r} \) is invertible in \( R/I \).

**Proof.** Since, according to Lemma 3.8, there exists a \( I \)-pre-image \( r \delta \) of \( \hat{b} \) in \( R \), with \( \gcd(r, k) = 1 \), all the \( I \)-pre-images, and in particular all the \( I \)-pre-images, of \( r \delta \) are of the form

\[
(9) \quad r \delta + cp_1^{m_1}p_2^{m_2} \cdots p_s^{m_s},
\]

where \( cp_1^{m_1}p_2^{m_2} \cdots p_s^{m_s} \in I \). Because, according to Lemma 3.11, the divisor parts of all the \( I \)-pre-images of \( \hat{b} \) are of the form \( u \delta \), where \( u \) is a unit in \( R \), it follows from (9) that the relative prime parts of all the \( I \)-pre-images of \( \hat{b} \) are of the form

\[
(10) \quad u^{-1}r + cu^{-1}p_1^{m_1-q_1} \cdots p_s^{m_s-q_s},
\]
where \( cp_1^{m_1} p_2^{m_2} \cdots p_s^{m_s} \in I \) and \( u \in R \) is a unit.

Now, suppose \( \hat{r} \) is invertible in \( R/I \) with inverse \( \hat{y} \). In other words

\[
yr = 1 + dp_1^{m_1} p_2^{m_2} \cdots p_s^{m_s},
\]

where \( dp_1^{m_1} p_2^{m_2} \cdots p_s^{m_s} \in I \). If we can show that the image under \( \theta \) of the relative prime part of an arbitrary \( I \)-pre-image of \( \hat{b} \) is invertible, then we are finished.

Let \( u^{-1} r + cu^{-1} p_1^{m_1-q_1} p_2^{m_2-q_2} \cdots p_s^{m_s-q_s} \) be the relative prime part of an arbitrary \( I \)-pre-image of \( \hat{b} \). Furthermore, let \( l \in \mathbb{N} \) such that

\[
(11) \quad 2^l > \max \left\{ \frac{m_i}{m_i - q_i} \mid i \in \{1, \ldots, s\} \right\} > 0.
\]

For the sake of notation, let

\[
v = dp_1^{q_1} \cdots p_s^{q_s} + cy \quad \text{and} \quad w = p_1^{m_1-q_1} p_2^{m_2-q_2} \cdots p_s^{m_s-q_s}.
\]

Then

\[
(u^{-1}r + cu^{-1} p_1^{m_1-q_1} p_2^{m_2-q_2} \cdots p_s^{m_s-q_s})yu(1 - vw)(1 + (vw)^{2^l})
\]

\[
(1 + (vw)^{2^l})(1 + (vw)^{2^{2^l}})(1 + (vw)^{2^{2^l}}) \cdots (1 + (vw)^{2^{2^l - 1}})
\]

\[
= (1 + dp_1^{m_1} p_2^{m_2} \cdots p_s^{m_s} + cy p_1^{m_1-q_1} p_2^{m_2-q_2} \cdots p_s^{m_s-q_s})(1 - vw)
\]

\[
(1 + (vw)^{2^l})(1 + (vw)^{2^{2^l}}) \cdots (1 + (vw)^{2^{2^l - 1}})
\]

\[
= 1 - (vw)^{2^l}.
\]

Let \( 1 \leq i \leq s \). Since \( m_i > q_i \), it follows from (11) that

\[
2^l(m_i - q_i) > \frac{m_i}{m_i - q_i}(m_i - q_i) = m_i,
\]

and so

\[
w^{2^l} = ap_1^{m_1} p_2^{m_2} \cdots p_s^{m_s}
\]

for some \( a \in R \). Since \( dp_1^{m_1} \cdots p_s^{m_s} \in I \) and \( cp_1^{m_1} \cdots p_s^{m_s} \in I \) imply that \( vw^{2^l} \in I \), it follows that \( (vw)^{2^l} \in I \). Therefore

\[
\theta \left( (u^{-1}r + cu^{-1} p_1^{m_1-q_1} p_2^{m_2-q_2} \cdots p_s^{m_s-q_s})yu(1 - vw)
\right)
\]

\[
(1 + (vw)^{2^l})(1 + (vw)^{2^{2^l}}) \cdots (1 + (vw)^{2^{2^l - 1}})
\]

\[
= \theta \left( 1 - (vw)^{2^l} \right)
\]

\[
= \hat{1}.
\]

Hence, we conclude that

\[
\theta \left( yu(1 - vw)(1 + (vw)^{2^l})(1 + (vw)^{2^{2^l}}) \cdots (1 + (vw)^{2^{2^l - 1}}) \right)
\]

is the inverse of the image under \( \theta \) of the relative prime part of the arbitrary chosen \( I \)-pre-image of \( \hat{b} \). \( \square \)
Remark 3.16. Note that if $I = \langle p^n \rangle$, for a prime $p \in R$ and $n > 0$, then every $\hat{0} \neq \hat{b} \in R/I$ is principal. Thus Proposition 3.15 is applicable to all nonzero elements in $R/I$. Furthermore, it is helpful to notice that every pre-image of $\hat{0} \neq \hat{b}$ is an $I$-pre-image.

Next we show that Proposition 3.15 does not hold in general if $q_i = m_i$ for some $i$.

Example 3.17. Let $R = \mathbb{Z}[x]$, let $k = 2x$ (with 2 and $x$ primes in $\mathbb{Z}[x]$) and let $I = \langle 2x \rangle$. Consider $\hat{0} \neq \hat{x} \in \mathbb{Z}[x]/\langle 2x \rangle$. Then $1 \cdot \hat{x}$ and $3 \cdot \hat{x}$ are $\langle 2x \rangle$-pre-images of $\hat{x}$ with relative prime parts 1 and 3 respectively, and $\hat{1}$ is invertible in $\mathbb{Z}[x]/\langle 2x \rangle$, but $\hat{3}$ is not.

Lemma 3.18. Let $k$ be a nonunit and let $\hat{0}_I \neq \hat{b}_I \in R/I$ be principal with $\delta$ the divisor part of an $I$-pre-image of $\hat{b}$. Then there exists a $\hat{c}_I \in R/I$ such that $\hat{c}_I \hat{b}_I = \hat{\delta}_I$ if and only if $\hat{b}_I$ is $I$-invertible.

Proof. Let $\hat{r}_I$ be an $I$-pre-image of $\hat{b}_I$ and suppose that there exists a $\hat{c}_I \in R/I$ such that $\hat{c}_I \hat{b}_I = \hat{\delta}_I$. Then it follows from Lemma 3.6 that $cr = c\delta^{-1} = 1 + \gamma \delta^{-1}k$, for some $\gamma \in R$ such that $\gamma k \in I$. Suppose $k = p_1^{q_1} \cdots p_s^{q_s}$, for $p_1, \ldots, p_s$ prime and $q_1, \ldots, q_s \geq 1$. Since $\hat{b}_I$ is principal, it follows that $k\delta^{-1}$ is of the form $w = p_1^{v_1} \cdots p_s^{v_s}$, where $v_1, \ldots, v_s \geq 1$. Now let $l \in \mathbb{N}$ such that $2^l > \max\{q_1, \ldots, q_s\}$. Then

$$(1 + \gamma \delta^{-1}k)(1 - \gamma \delta^{-1}k)(1 + (\gamma \delta^{-1}k)^2) \cdots (1 + (\gamma \delta^{-1}k)^{2^{l-1}}) = 1 - (\gamma \delta^{-1}k)^{2^l}$$

which implies that

$$cr(1 - \gamma \delta^{-1}k)(1 + (\gamma \delta^{-1}k)^2) \cdots (1 + (\gamma \delta^{-1}k)^{2^{l-1}}) = 1 - (\gamma \delta^{-1}k)^{2^l},$$

where $k|\delta^{-1}k)^{2^l}$. Since $\gamma k \in I$, it follows that $(\gamma \delta^{-1}k)^{2^l} \in I$. Hence $\hat{r}_I$ is invertible in $R/I$ and we can conclude that $\hat{b}_I$ is $I$-invertible in $R/I$. The converse follows from Remark 3.3. \hfill \Box

Remark 3.19. Let $k$ be a nonunit and let $\hat{0}_I \neq \hat{b}_I \in R/I$ be principal. Using, Lemma 3.7 and Lemma 3.18 it is only necessary to consider invertible elements in $R/I$ to determine whether there exists a $\hat{c}_I$ in $R/I$ such that $\hat{b}_I \hat{c}_I = \hat{\delta}_I$, where $\delta$ is a divisor part of an $I$-pre-image of $\hat{b}_I$.

The following result will help us to determine whether an image of a relative prime part of an $I$-pre-image of an element is invertible in $R/I$ and can be proved by a similar method than the method in the proof of Lemma 3.18.

Lemma 3.20. Let $k \in R$ be a nonzero nonunit. If $\hat{b} \in R/I$ has a pre-image of the form $b' + 1$, where $b'$ is $q$-principal and $b'k \in I$, then $\hat{b}$ is invertible in $R/I$ (see Example 3.10(b)).
Remark 3.21. The converse of Lemma 3.20 is not in general true. For example \(\hat{3} = (2 + 1)\) is invertible in \(\mathbb{Z}_5\), although \(5 \nmid 2\).

Example 3.22. Let \(R = F[x, y]\) and let \(I := \langle y^5 \rangle\). Since \(\hat{x^3} I\) is not invertible in \(F[x, y]/\langle y^5 \rangle\), we conclude from Remark 3.16 that \(\hat{x^3} I\) is not \(I\)-invertible. Because \(\gcd(x^5, y^5) = 1\) we could also concluded from Lemma 3.12 that \(x^5 I\) is not \(I\)-invertible.

Definition 3.23. We call a matrix \[
\begin{bmatrix}
\hat{e}_I & \hat{f}_I \\
\hat{g}_I & \hat{h}_I
\end{bmatrix}
\in M_2(R/I)
\]
a \(I\)-matrix if \(\langle \hat{e}_I - \hat{h}_I, \hat{f}_I \rangle = \langle \hat{t}_I \rangle\) or \(\langle \hat{e}_I - \hat{h}_I, \hat{g}_I \rangle = \langle \hat{t}_I \rangle\), where \(t|k\).

The following result is easy to prove.

Lemma 3.24. Let \(\hat{a}_I, \hat{b}_I \in R/I\). If \(\langle \hat{a}_I, \hat{b}_I \rangle = \langle \hat{t}_I \rangle\), where \(t|k\), then \(t = \gcd(a, b, k)\).

The following results can be used to determine whether a matrix is an \(I\)-matrix.

Lemma 3.25. A matrix is an \(I\)-matrix if it satisfies the following conditions:

(i) For at least one of the three elements \(\hat{e}_I - \hat{h}_I, \hat{f}_I\) and \(\hat{g}_I\), say \(\hat{a}_I\), there exists a \(\hat{c}_I \in R/I\) such that \(\hat{c}_I \hat{a}_I = \hat{t}_I\), where \(\hat{r}\) is an \(I\)-pre-image of \(\hat{a}_I\) that has divisor part \(\hat{t}\); pick such an element, and call the remaining two elements \(\hat{a}_I\) and \(\hat{b}_I\), say.

(ii) For at least one of the elements \(\hat{a}_I/\delta\) and \(\hat{b}_I/\delta\), say \(\hat{a}_I/\delta\), there exists a \(\hat{d}_I/\delta \in R/\langle \delta \rangle\) such that \(\hat{d}_I/\delta \hat{b}_I/\delta = \hat{t}_I/\delta\), where \(t|\delta\).

Remark 3.26. Note that if Lemma 3.25(i) is satisfied, with \(\delta\) a unit, then Lemma 3.25(ii) is always satisfied.

The following result is in some cases helpful to determine when a matrix is not an \(I\)-matrix.

Lemma 3.27. Let \(\hat{a}_I, \hat{b}_I \in R/I\) and suppose that there exists a \(\hat{c}_I\) such that \(\hat{c}_I \hat{a}_I = \hat{t}_I\), with \(\delta\) a divisor part of an \(I\)-pre-image of \(\hat{a}_I\). Then \(\langle \hat{a}_I, \hat{b}_I \rangle = \langle \hat{t}_I \rangle\), where \(t|k\), if and only if there exists a \(\hat{d}_I/\delta\) such that \(\hat{d}_I/\delta \hat{b}_I/\delta = \hat{t}_I/\delta\).

Proof. Suppose there exists a \(\hat{c}_I \in R/I\) such that \(\hat{c}_I \hat{a}_I = \hat{t}_I\), with \(\delta\) a divisor part of an \(I\)-pre-image of \(\hat{a}_I\).

Using Lemma 3.24 suppose that \(\langle \hat{a}_I, \hat{b}_I \rangle = \langle \hat{t}_I, \hat{b}_I \rangle = \langle \hat{t}_I \rangle\), where \(t = \gcd(\delta, b, k) = \gcd(\delta, b)\). Then, since \(t|\delta|k\), \(\alpha \delta + \beta b \equiv t + I\), for some \(\alpha, \beta \in R\), implies that \(\alpha \delta + \beta b = t + \gamma \delta\), for some \(\gamma \in R\), and so \(\beta b = t + (\gamma - \alpha) \delta\). The converse follows trivially.
Lemma 3.28. If \( \hat{e} - \hat{h} \), \( \hat{f} \) or \( \hat{g} \) is invertible in \( R/I \) then \( \begin{bmatrix} \hat{e} & \hat{f} \\ \hat{g} & \hat{h} \end{bmatrix} \in M_2(R/I) \) is an \( I \)-matrix.

Proof. Suppose \( \hat{c}_I \in \{ \hat{e}_I - \hat{h}_I, \hat{f}_I, \hat{g}_I \} \) is invertible in \( R/I \). Then it follows from Lemma 3.12 that \( \hat{c}_I \) is \( I \)-invertible with an \( I \)-pre-image \( c \cdot 1 \) that has divisor part \( 1 \), and so the result follows from Remark 3.24, Lemma 3.25 and Remark 3.26.

The following result follows directly from Corollary 3.29, Remark 3.24 and Lemma 3.25.

Corollary 3.29. If \( R \) is a PID, then every matrix in \( M_2(R/I) \) is an \( I \)-matrix.

We show that Corollary 3.29 does not hold for UFD’s in general.

Example 3.30. Let \( R = \mathbb{Z}[x] \) and let \( I \) be a nonzero ideal in \( R \), with \( 2^4x^4 \in I \) and \( k = 2^3x^3 \). We exhibit (a) a matrix which is an \( I \)-matrix and (b) a matrix which is not an \( I \)-matrix.

(a) Let \( \hat{B}_I = \begin{bmatrix} 7x^2 & 24x^3 + 8x^4 + 4x^2 \\ 14x \\ 0 \end{bmatrix} \in M_2(R/I) \).

We have already seen in Example 3.10(b) that \( 2^4x^4 + 8x^4 + 4x^2 \) is \( I \)-invertible with divisor part \( 2^2x^2 \). Since \( 7x^2(\delta) = -1x^2(\delta) \), it follows that \( 7x^2(\delta) \) is \( (\delta) \)-invertible and therefore, using Remark 3.24 and Lemma 3.25, \( \hat{B}_I \) is an \( I \)-matrix.

(b) Let \( \hat{B}_I = \begin{bmatrix} 3 & 24x^3 + 8x^4 + 4x^2 \\ 14x \\ 0 \end{bmatrix} \in M_2(R/I) \).

We first consider the ideals \( (3, 24x^3 + 8x^4 + 4x^2) \) and \( (14x, 24x^3 + 8x^4 + 4x^2) \). We have already seen in Example 3.10(b) that \( 24x^3 + 8x^4 + 4x^2 \) is \( I \)-invertible with divisor part \( 2^2x^2 \). Since \( 3(\delta) \) and \( 14x(\delta) = 7(\delta)2x(\delta) \) are both principal, it follows from Proposition 3.15 that \( 3(\delta) \) and \( 14x(\delta) \) are both not \( (\delta) \)-invertible. Therefore it follows from Lemma 3.18, Lemma 3.27 and Lemma 3.24 that \( \hat{B}_I \) is an \( I \)-matrix if and only if \( (3, 14x) = R/I \). Since this is not the case \( \hat{B}_I \) is a non-\( I \)-matrix.

4. The centralizer of an \( I \)-matrix

The purpose of this section is to obtain a concrete description of the centralizer of an \( I \)-matrix in \( M_2(R/I) \), \( R \) a UFD and a nonzero ideal \( I \) in \( R \), with \( k := \gcd(I) \), by showing that the converse containments \( \supseteq \) hold in Proposition 2.6. We also provide an example of a UFD, which is not a PID, and a non-\( I \)-matrix in \( M_2(R/I) \) for which the mentioned converse containment does not hold. We conclude with
an example where we show that if \( R \) is a UFD and \( R/I \) is such that \( R/I \) is not an integral domain, then for every \( n \geq 3 \) there is a matrix in \( M_n(R) \) for which we do not have equality in Proposition \( 2.6 \). Note that we still assume that \( \theta_t : R \to R/I \) and \( \Theta_I : M_2(R) \to M_2(R/I) \) are the natural and induced epimorphism respectively.

**Theorem 4.1.** Let \( R \) be a UFD, \( I \) a nonzero ideal in \( R \), and let \( \hat{B}_I = \begin{bmatrix} \hat{\epsilon}_I & \hat{f}_I \\ \hat{g}_I & \hat{h}_I \end{bmatrix} \in M_2(R/I) \) be an I-matrix, then

\[
\begin{align*}
\text{Cen}(\hat{B}) &= \Theta(\text{Cen}(B)) + \begin{bmatrix} \hat{0} & \text{ann}(\hat{g}) \cap \text{ann}(\hat{e} - \hat{h}) \\ \text{ann}(\hat{f}) \cap \text{ann}(\hat{e} - \hat{h}) & \text{ann}(\hat{f}) \cap \text{ann}(\hat{g}) \end{bmatrix} \\
&= \Theta(\text{Cen}(B)) + \begin{bmatrix} \text{ann}(\hat{f}) \cap \text{ann}(\hat{g}) & \text{ann}(\hat{g}) \cap \text{ann}(\hat{e} - \hat{h}) \\ \text{ann}(\hat{f}) \cap \text{ann}(\hat{e} - \hat{h}) & \hat{0} \end{bmatrix} \\
&= \Theta(\text{Cen}(B)) + \begin{bmatrix} \text{ann}(\hat{f}) \cap \text{ann}(\hat{g}) & \text{ann}(\hat{g}) \cap \text{ann}(\hat{e} - \hat{h}) \\ \text{ann}(\hat{f}) \cap \text{ann}(\hat{e} - \hat{h}) & \text{ann}(\hat{f}) \cap \text{ann}(\hat{g}) \end{bmatrix}.
\end{align*}
\]

**Proof.** By the symmetry in (3) it is sufficient to consider the case where \( \langle \hat{f}, \hat{g} \rangle = \langle \hat{t} \rangle, \hat{t} = \text{gcd}(f, g, k) \) by Lemma 3.2. Suppose \( \hat{A} = \begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{bmatrix} \in M_2(R/I) \) such that \( \hat{A}B = B\hat{A}, \) i.e. \( A \in M_2(R) \) such that \( AB = BA + I. \) Since \( c(e - h) = (a - d)g + I \) and \( t|g, k \) it follows that \( t|c(e - h). \) Let \( m = \text{gcd}(e - h, f, g, k), \) then \( \text{gcd}(e - h, t) = m, \) which implies that \( t|m. \) Similarly \( (a - d)f \equiv b(e - h) + I \) yields \( t|bm. \) Since \( \langle \hat{f}, \hat{g} \rangle = \langle \hat{t} \rangle, \) there exists an \( \hat{a}, \hat{b} \in R/I \) such that \( \hat{t} = \hat{a}\hat{f} + \hat{b}\hat{g}, \) i.e. \( t \equiv af + bg + I. \) Let \( w \in R \) such that \( w = \hat{a}b + \hat{b}c, \) then \( t|wm. \) Let \( v \in R \) such that \( vt = um. \) It follows from (3), using the notation of Remark 2.2 that

\[
fA' \equiv bB' + I \quad gA' \equiv cB' + I \quad \text{and so} \quad wB' \equiv tA' + I.
\]

Write \( B' \) as \( mB'' \), then \( vtB'' = wmB'' = wB' \equiv tA' + I. \) Let \( \hat{K} = \begin{bmatrix} \hat{e}' & \hat{f}' \\ \hat{g}' & \hat{h}' \end{bmatrix} \) be the image of \( A' - vB'' \) in \( M_2(R/I) \) and \( L = vB'', \) then \( L \in \text{Cen}(B) \), by Lemma 2.3,

\[
t\hat{K} = \hat{0} \quad \text{and} \quad \hat{A}' = \hat{L} + \hat{K}.
\]

Here \( \hat{K} \) commutes with \( \hat{B}' \), and hence with \( \hat{B}, \) and therefore \( (\hat{e}' - \hat{h}')\hat{f} = \hat{f}'(\hat{e} - \hat{h}), \) \( \hat{f}'\hat{g} = \hat{g}'\hat{f} \) and \( \hat{g}'(\hat{e} - \hat{h}) = (\hat{e}' - \hat{h}')\hat{g}. \) But \( (\hat{e}' - \hat{h}')\hat{f} = \hat{0}, \) since \( (\hat{e}' - \hat{h}')\hat{t} = \hat{0} \) and \( t|f. \) Similarly \( (\hat{e}' - \hat{h}')\hat{g} = \hat{0}, \) \( \hat{f}'\hat{g} = \hat{0}, \) \( \hat{f}'(\hat{e} - \hat{h}) = \hat{0}, \) \( \hat{g}'\hat{f} = \hat{0} \) and \( \hat{g}'(\hat{e} - \hat{h}) = \hat{0}. \) Hence

\[
\hat{K} \subseteq \begin{bmatrix} \text{ann}(\hat{f}) \cap \text{ann}(\hat{g}) & \text{ann}(\hat{g}) \cap \text{ann}(\hat{e} - \hat{h}) \\ \text{ann}(\hat{f}) \cap \text{ann}(\hat{e} - \hat{h}) & \text{ann}(\hat{f}) \cap \text{ann}(\hat{g}) \end{bmatrix}.
\]

Since \( \hat{A} = \hat{A}' + d\hat{E}, \) we have the containment \( \subseteq \) in (14). The converse follows from Proposition 2.6. \( \square \)
Example 4.2. Consider \( \hat{B}_I = \begin{bmatrix} \hat{x}^2 I & 2\hat{x}^3 I + 8\hat{x}^2 I + 4\hat{x}^2 I \\ 14x I & 0_I \end{bmatrix} \in M_2(R/I) \) in Example 3.30(a), with \( I = \langle 5 \cdot 2^3 x^2, 2^4 x^4 \rangle \). We use Theorem 4.1, (13), to obtain \( \text{Cen}(\hat{B}) \). According to Corollary 2.3(ii), Example 4.4.

Consider Example 4.2.

Note that in the above example

\[ \text{ann}(\hat{C}) \cap \text{ann}(14x) = \langle 5 \cdot 2^3 x^2, 2^4 x^4 \rangle, \]
\[ \text{ann}(14x) \cap \text{ann}(24x^5 + 8x^4 + 4x^2) = \langle 5 \cdot 2^2 x^2, 2^3 x^3 \rangle \]
\[ \text{and} \ \text{ann}(\hat{C}) \cap \text{ann}(24x^5 + 8x^4 + 4x^2) = \langle 5 \cdot 2^3 x^2, 2^2 x^3 \rangle \]

and so it follows from (15) and Theorem 4.1, (13), that

\[ \text{Cen}(\hat{B}) = \Theta \left( \begin{bmatrix} h_1 + 7xh_2 & (24x^4 + 8x^3 + 4x)h_2 \\ 14h_2 & h_1 \end{bmatrix} \right| h_1, h_2 \in \mathbb{Z}[x] \right) \]

Furthermore,

\[ \text{ann}(\hat{C}) \cap \text{ann}(14x) = \langle 5 \cdot 2^3 x^2, 2^4 x^4 \rangle, \]
\[ \text{ann}(14x) \cap \text{ann}(24x^5 + 8x^4 + 4x^2) = \langle 5 \cdot 2^2 x^2, 2^3 x^3 \rangle \]
\[ \text{and} \ \text{ann}(\hat{C}) \cap \text{ann}(24x^5 + 8x^4 + 4x^2) = \langle 5 \cdot 2^3 x^2, 2^2 x^3 \rangle \]

Remark 4.3. Note that in the above example

\[ \Theta(\text{Cen}(B)) \nsubseteq \begin{bmatrix} \text{ann}(\hat{f}) \cap \text{ann}(\hat{g}) & \text{ann}(\hat{g}) \cap \text{ann}(\hat{e} - \hat{h}) \\ \text{ann}(\hat{f}) \cap \text{ann}(\hat{e} - \hat{h}) & \text{ann}(\hat{f}) \cap \text{ann}(\hat{g}) \end{bmatrix} \]

and that

\[ \begin{bmatrix} \text{ann}(\hat{f}) \cap \text{ann}(\hat{g}) & \text{ann}(\hat{g}) \cap \text{ann}(\hat{e} - \hat{h}) \\ \text{ann}(\hat{f}) \cap \text{ann}(\hat{e} - \hat{h}) & \text{ann}(\hat{f}) \cap \text{ann}(\hat{g}) \end{bmatrix} \nsubseteq \Theta(\text{Cen}(B)). \]

According to Corollary 3.29 Theorem 4.1 applies to all \( 2 \times 2 \) matrices over factor rings \( R/I \), where \( R \) is a PID. In other words, we have equality in Proposition 2.6 for all \( 2 \times 2 \) matrices over factor rings of PID’s. This is not the case for all \( 2 \times 2 \) matrices over factor rings \( R/I \), where \( R \) is a UFD, as the following example shows.

Example 4.4. Consider \( \hat{B} = \begin{bmatrix} \hat{x} + \hat{y} & \hat{y} \\ \hat{x} & \hat{x} \end{bmatrix} \in M_2(F[x,y]/\langle x^2 \rangle) \). By Corollary 2.3(ii)

\[ \text{Cen}(B) = \left\{ \begin{bmatrix} h_1 & yh_2 \\ xh_2 & h_1 - yh_2 \end{bmatrix} \right| h_1, h_2 \in \mathbb{F}[x,y] \right\} \]
The second term in the righthand side of (13) is
\[
\begin{bmatrix}
\text{ann}(\hat{y}) \cap \text{ann}(\hat{x}) & \text{ann}(\hat{x}) \cap \text{ann}(\hat{y}) \\
\text{ann}(\hat{y}) \cap \text{ann}(\hat{y}) & 0
\end{bmatrix} = \begin{bmatrix}
\hat{0} & \hat{0} \\
\hat{0} & \hat{0}
\end{bmatrix},
\]
because \(\text{ann}(\hat{y}) = \hat{0}\). Therefore the righthand side of (13) is equal to
\[
\left\{ \begin{bmatrix}
\hat{h}_1 & \hat{y}\hat{h}_2 \\
\hat{x}\hat{h}_2 & \hat{h}_1 - \hat{y}\hat{h}_2
\end{bmatrix} \mid \hat{h}_1, \hat{h}_2 \in F[x, y]/\langle x^2 \rangle \right\},
\]
which does not contain the matrix \(\begin{bmatrix}
\hat{x} & \hat{x} \\
\hat{0} & \hat{0}
\end{bmatrix}\). However, direct verification shows that
\[
\begin{bmatrix}
\hat{x} & \hat{x} \\
\hat{0} & \hat{0}
\end{bmatrix} \in \text{Cen}(\hat{B}).
\]

In the following example we will see that for every \(n \geq 3\) and for any UFD \(R\) and ideal \(I\) such that \(R/I\) is a ring with zero divisors, there is a matrix \(B \in M_n(R)\) for which we do not have equality in Proposition 2.6.

**Example 4.5.** Let \(R\) be a UFD and let \(I\) be an ideal in \(R\) such that \(R/I\) has zero divisors. Thus suppose that \(\hat{d}\hat{d}' \in R/I\), \(\hat{d}, \hat{d}' \neq \hat{0}\) and \(\hat{d}\hat{d}' = \hat{0}\). Now let \(B = \begin{bmatrix}
0 & \hat{d} & 1 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} \in M_3(R)\). Note that \(d \neq 0\) since \(\hat{d} \neq \hat{0}\). Because the characteristic polynomial of \(B\) is equal to the minimum polynomial of \(B\) it follows from Theorems 1.1 and Lemma 2.5 that \(\text{Cen}_{M_3(R)}(B) = \left\{ \begin{bmatrix}
a & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix} + b \begin{bmatrix}
0 & d & 1 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix} + c \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \mid a, b, c \text{ are elements of the quotient field of } R. \right\} \cap M_3(R),\)
and so every matrix in \(\Theta(\text{Cen}(B))\) has \(\hat{0}\) in position (2, 1). Furthermore, using the notation in Proposition 2.6 we have
\[
[A_{ij}] = \begin{bmatrix}
\hat{0} & \hat{0} & R/I \\
\hat{0} & \hat{0} & \text{ann}(\hat{d}) \\
\hat{0} & \hat{0} & \hat{0}
\end{bmatrix}.
\]
Hence, every matrix in \(\Theta(\text{Cen}(B)) + [A_{ij}]\) has \(\hat{0}\) in position (2, 1). However, direct multiplication shows that
\[
\begin{bmatrix}
\hat{d}' & \hat{0} & \hat{0} \\
\hat{d}' & \hat{0} & \hat{0} \\
\hat{0} & \hat{0} & \hat{d}'
\end{bmatrix} \in \text{Cen}(\hat{B}),
\]
and so equality in Proposition 2.6 does not hold in this case. Now, again let \( R \) be a UFD and let \( I \) be an ideal in \( R \) such that \( R/I \) has zero divisors. Let us consider the matrix

\[
B' = \begin{bmatrix}
d & 1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix} \in M_n(R).
\]

Then

\[
\text{Cen}(B') \subseteq \begin{bmatrix}
\text{Cen}(B) & R/I \\
R/I & R/I 
\end{bmatrix} \quad \text{and} \quad [A_{ij}] \subseteq \begin{bmatrix}
\hat{0} & \hat{0} & R/I \\
\hat{0} & \hat{0} & \text{ann}(\hat{d}) \\
\hat{0} & \hat{0} & \hat{0} \\
R/I & R/I & R/I
\end{bmatrix}.
\]

Since

\[
\hat{A} := \begin{bmatrix}
\hat{d}' & \hat{0} & \hat{0} & \cdots & \hat{0} \\
\hat{0} & \hat{0} & \hat{0} & \cdots & \hat{0} \\
\hat{0} & \hat{0} & \hat{0} & \cdots & \hat{0} \\
\hat{0} & \hat{0} & \hat{0} & \cdots & \hat{0} \\
\hat{0} & \hat{0} & \hat{0} & \cdots & \hat{0}
\end{bmatrix} \in \text{Cen}(\hat{B}')
\]

but clearly \( \hat{A} \not\in \Theta(\text{Cen}(B')) + [A_{ij}] \), equality in Proposition 2.6 for these cases, does not hold.

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