The Entropy of the BTZ Black Hole and AdS/CFT Correspondence

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Abstract

We construct an action, which governs the dynamics of the Bañados-Teitelboim-Zanelli (BTZ) black hole and perform the canonical quantization. The quantum action is given by a $SL(2,R)$ Wess-Zumino-Witten model on the boundary coupled to the classical anti-de Sitter background, representing a massless BTZ black hole. The coupling, determined by a one-cocycle condition, is found to give dominant contribution to the central charge of Virasoro algebra. The entropy of the BTZ black hole is discussed from the point view of the AdS/CFT correspondence and an explanation is given to the puzzle of black hole entropy in the BTZ case. The BTZ black hole is a quantum object and the BTZ black hole with finite mass should be considered as a quantum excitation of the massless one.

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I. INTRODUCTION

The statistical interpretation of Bekenstein-Hawking entropy [1] for the black hole has been one of the most outstanding problems in quantum gravity, since it may reveal what are the dynamical degrees of freedom one should count for in quantum gravity. Recent studies on the black holes in the string theory brought us one step closer to the heart of the question. In the string theory one can construct the D-brane configurations corresponding to the extremal black holes in five [2] and four dimensions [3]. Given that the number of BPS states is a topological invariant, one can obtain the black hole entropy by counting the microstates in the weak coupling regime, which are the BPS bound states of D-brane configuration. This was one of the remarkable achievements in the recent development of string theory. There also have been some attempts [4] to extend this microstate counting to the non-extremal cases. However, the evaluation of the entropy based on counting the BPS states in the weak coupling regime does not provide a satisfactory answer to the question of what are the microstates of the black hole, i.e., the dynamical degrees of gravity in the strong coupling regime. And it is not directly applicable to the non-extremal black hole.

More accurate understanding of the microstates of the black hole follows from the recent discovery of the relationship of the BTZ black hole [5,6] in (2+1) dimensions and those higher dimensional black holes in string theory. The intimate relationship becomes clear as we observe that the higher dimensional black holes can be transformed into the BTZ black hole configuration [7] by some dual transformations of the M-theory and the near horizon geometry of the higher dimensional black holes [8] is described by the BTZ black hole. The advantage of utilizing this equivalence is that we can get more direct statistical interpretation of the black hole and this approach is not limited to the extremal cases, since for the case of BTZ black hole it may be possible to count explicitly the number of the microstates of the black hole regardless of whether it is extremal or not. Thus, recently the BTZ black hole has stepped into the limelight in the string theory [9].

In a recent work [10] Strominger suggested that the entropy of BTZ black hole can be
obtained by counting the number of the microstates in the conformal theory induced on the boundary of spatial infinity. His evaluation of the entropy of the BTZ black hole is based on the work of Brown and Henneaux [11] where the generators for the diffeomorphism at asymptotic region are found to form Virasoro algebra with central charge $c = \frac{3l}{2G}$ where $-1/l^2$ is the cosmological constant and $G$ denotes the Newton constant. Then, identifying the mass and angular momentum of the black hole in terms of Virasoro generators and employing the asymptotic growth [12] of the number of states in the corresponding conformal field theory, he obtains the Bekenstein-Hawking entropy of the BTZ black hole. The statistical derivation of the entropy for the BTZ black hole also has been discussed by Bañados, Brotz and Ortiz [13] in their construction of the canonical partition function for the BTZ black hole. However, there seems some discrepancy between these recent works [10,13] and the previous ones [14]. In a recent paper [15] Carlip summarized the open problems associated with the known approaches to the entropy of the BTZ black hole.

The purpose of the present work is to clarify some of the issues raised in the literature on the BTZ black hole and to provide an improved quantum theory of the BTZ black hole. In the present paper we will derive the conformal field theory on the boundary of spatial infinity, which describes the dynamical degrees of freedom for the black hole, in the framework of the Chern-Simons theory defined on the space-time with boundary. The strategy to be employed is the one-cocycle construction of Faddeev and Shatashvili [16]. The dynamics of the BTZ black hole is described by the Chern-Simons gauge fields in the bulk and two sets of $SL(2, R)$ chiral WZW models coupled to the Chern-Simons gauge fields on the boundary. The coupling of the chiral WZW boson fields to the gauge fields is dictated by the gauge invariance. It will be pointed out that the coupling of the WZW model to the Chern-Simons gauge fields on the boundary plays an important role, giving the most dominant contribution to the central charge. The importance of the coupling term was overlooked in the previous studies but was introduced later by hand in the form of Wilson line to produce the correct entropy in the construction of the grand-canonical partition function [13].

The main result of the present study is that in order to get the correct entropy for the
BTZ black hole one must take into account the contribution of the gauge fields as well as the WZW chiral bosons. In fact the former dominates over the latter. The BTZ black hole is a quantum object, i.e., the BTZ black hole with finite mass should be considered as a quantum excitation of the massless black hole and the Bekenstein-Hawking entropy counts the density of quantum states with respect to the vacuum state, being the massless black hole.

II. BTZ BLACK HOLE

The BTZ black hole with mass $M$ and angular momentum $J$ is described in the (2+1) dimensional gravity [17] by the following metric [5]

$$ds_{BTZ}^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^2 dt + d\phi)^2,$$

$$N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2},$$

$$N^\phi(r) = -\frac{J}{2r^2}.$$ (1)

$N(r)$ has two roots, $r_+ > r_-$, which set the inner and outer horizons of the black hole respectively. If one introduces $SL(2, R) \otimes SL(2, R) \simeq SO(2, 2)$ Lie algebra valued gauged fields $A = A^a J_a$, $\bar{A} = \bar{A}^a \bar{J}_a$ which can be written in terms of the dreibein and the spin connection as follows; $A_\mu^a = \omega_\mu^a + \frac{e_\mu^a}{l}$, $\bar{A}_\mu^a = \omega_\mu^a - \frac{e_\mu^a}{l}$, we can describe the gravity in the presence of a cosmological constant by a Chern-Simons action [18]

$$I_{CS}(A, \bar{A}) = \frac{k}{4\pi} \int_M tr \left( A dA + \frac{2}{3} AAA \right) - \frac{k}{4\pi} \int_M tr \left( \bar{A} d\bar{A} + \frac{2}{3} \bar{A} \bar{A} \bar{A} \right)$$ (2)

where $k = \frac{-l}{4G} < 0$. The equations of motion obtained from the Chern-Simons action are satisfied by the BTZ black hole solution.

The BTZ black hole solution has two boundaries: the horizon at $r = r_+$ and the boundary of spatial infinity. Here we will mainly concern the boundary of spatial infinity. If we confine ourselves to the region $r \geq r_+$, it is convenient to introduce a coordinates $(\tau, \rho, \phi)$ where
\[ \tau = t/l \text{ and } r^2 = r_+^2 \cosh^2 \rho - r_-^2 \sinh^2 \rho. \] In these new coordinate system the BTZ solution reads as follows

\[
\begin{pmatrix}
A_R = 2\Delta R (\sinh \rho J_0 + \cosh \rho J_2), \\
A_L = 0, \\
A_\rho = J_1,
\end{pmatrix}
\]

\begin{align}
A_R &= 0 \\
\bar{A}_L &= -2R (\sinh \rho \bar{J}_0 - \cosh \rho \bar{J}_2), \\
\bar{A}_\rho &= -\bar{J}_1.
\end{align}

\text{(3a)}

\text{(3b)}

where

\[
\begin{align*}
AR/L &= A_t \pm A_\phi, \\
\bar{A}R/L &= \bar{A}_t \pm \bar{A}_\phi,
\end{align*}
\]

\[
\Delta R = \frac{(r_+ - r_-)}{l}, \quad R = \frac{(r_+ + r_-)}{l}.
\]

III. CANONICAL QUANTIZATION

In the presence of a boundary, we should impose appropriate boundary conditions and supplement some boundary terms. We may choose the following boundary conditions, compatible with the BTZ black hole solution, \(A_L = 0, \bar{A}_R = 0\) and accordingly, the boundary terms, which do not pose a unitarity problem

\[
I_B = -\frac{k}{4\pi} \int_{\partial M} \text{tr}(A_t - A_\phi)A_\phi + \frac{k}{4\pi} \int_{\partial M} \text{tr}(\bar{A}_t + \bar{A}_\phi)\bar{A}_\phi
\]

where \(\partial M\) denotes the boundary of spatial infinity. As is well known, the gauge invariance is broken in the Chern-Simons theory if the space-time has boundaries. As a consequence, the degrees of freedom of the gauge fields corresponding to the broken symmetry become dynamical. The action for these "would be" gauge degrees of freedom has been constructed as a one-cocycle in \([19]\) by adopting the Faddeev-Shatashvili procedure \([13]\): \n
\[
\alpha_G[A, \bar{A}, g, \bar{g}] = I_G(A^g, \bar{A}^\bar{g}) + I_B(A^g, \bar{A}^\bar{g}) - I_G(A, \bar{A}) - I_B(A, \bar{A})
\]

\text{(5)}

\[
A^g = g^{-1}dg + g^{-1}Ag
\]

\[
\bar{A}^\bar{g} = \bar{g}^{-1}d\bar{g} + \bar{g}^{-1}\bar{A}\bar{g}.
\]
The explicit expression for the one-cocycle is given as

\[ \alpha_G(A, \bar{A}, g, \bar{g}) = \alpha_1(A, g) + \bar{\alpha}_1(\bar{A}, \bar{g}), \] (6)

\[ \alpha_1(A, g) = -\Gamma^L[g] - \frac{k}{2\pi} \int_{\partial M} \text{tr}(\partial_\phi g g^{-1}) A_L, \]

\[ \bar{\alpha}_1(\bar{A}, \bar{g}) = \Gamma^R[\bar{g}] + \frac{k}{2\pi} \int_{\partial M} \text{tr}(\partial_\phi \bar{g} \bar{g}^{-1}) \bar{A}_R, \]

\[ \Gamma^L[g] = \frac{k}{4\pi} \int_{\partial M} (g^{-1} \partial_- g)(g^{-1} \partial_\phi g) + \frac{k}{12\pi} \int_M \text{tr}(g^{-1} dg)^3, \]

\[ \Gamma^R[\bar{g}] = \frac{k}{4\pi} \int_{\partial M} (\bar{g}^{-1} \partial_+ \bar{g})(\bar{g}^{-1} \partial_\phi \bar{g}) + \frac{k}{12\pi} \int_M \text{tr}(\bar{g}^{-1} d\bar{g})^3 \]

where \( \partial_\pm = \partial_\tau \pm \partial_\phi \). (If \( \partial M \) is the boundary of the horizon, one would replace \( k \) by \( -k \) in the action except for the coefficients of the WZ terms.) The induced action on \( \partial M \) is given by a direct sum of two chiral WZW actions; one with left moving chiral boson fields only and the other with right moving chiral boson fields only.

Now the gauge symmetry is fully restored thanks to the one-cocycle condition satisfied by \( \alpha_G[A, \bar{A}, g, \bar{g}] \)

\[ \delta \alpha_G = \alpha_G[A^h, \bar{A}^\bar{h}, g, \bar{g}] - \alpha_G[A, \bar{A}, h g, \bar{h} g] + \alpha_G[A, \bar{A}, h, \bar{h}] = 0. \] (7)

Once a gauge condition and boundary conditions are chosen appropriately, the classical BTZ black hole solution would be determined uniquely by the equations of motion. Thus, we may replace the gauge fields in the action by the classical BTZ solution in the classical limit. Then the second terms in \( \alpha_1(A, g) \) and \( \bar{\alpha}_1(\bar{A}, \bar{g}) \) depict couplings of the chiral boson fields to the classical BTZ black hole background.

The complete action for the BTZ black hole system consists of the Chern-Simons terms, the boundary terms and the one-cocycle

\[ I_G(A, \bar{A}, g, \bar{g}) = I_{CS}(A, \bar{A}) + I_B(A, \bar{A}) + \alpha_G(A, \bar{A}, g, \bar{g}) \]

\[ = I_G[A, g] + I_G[\bar{A}, \bar{g}]. \] (8)

It would be instructive to rewrite the action in the canonical form

\[ I_G = \frac{k}{4\pi} \int_M \text{tr} e^{ij}(\dot{A}_i A_j + A_i F_{ij}) - \frac{k}{4\pi} \int_{\partial M} \text{tr}(g^{-1} \dot{g})(g^{-1} g') - \frac{k}{12\pi} \int_M \text{tr}(g^{-1} dg)^3 \]
The Gauss' constraints are given as
\[ \rho \Phi = (A_\phi + g' g^{-1})^2 - 2 A_\rho (A_\phi + g' g^{-1}) \],
\[ (\bar{\rho} \bar{\Phi} = (\bar{A}_\phi + \bar{g}' \bar{g}^{-1})^2 - 2 \bar{A}_\rho (\bar{A}_\phi + \bar{g}' \bar{g}^{-1}) \}

where "\( \cdot \)" and "\( \bar{\cdot} \)" denote derivatives with respect to \( \tau \) and \( \phi \) respectively.

The fundamental Poisson brackets can be read as follows
\[ \{ A_i^a(x), A_j^b(x') \} = -\frac{4\pi}{k} \epsilon_{ij} \delta(x - x') \eta^{ab} \]
\[ \{ \bar{A}_i^a(x), \bar{A}_j^b(x') \} = \frac{4\pi}{k} \epsilon_{ij} \delta(x - x') \eta^{ab} \]
\[ \{ A(\phi), B(\phi') \} = \frac{2\pi}{k} \delta(\phi - \phi') \text{tr} ([a,b] g' g^{-1}) + \frac{2\pi}{k} \delta'(\phi - \phi') \text{tr} (ab) \]
\[ \{ \bar{A}(\phi), \bar{B}(\phi') \} = -\frac{2\pi}{k} \delta(\phi - \phi') \text{tr} ([\bar{a},\bar{b}] \bar{g}' \bar{g}^{-1}) - \frac{2\pi}{k} \delta'(\phi - \phi') \text{tr} (\bar{a} \bar{b}) \]

where
\[ A = \text{tr} \left( a g' g^{-1}(\phi) \right), \quad B = \text{tr} \left( b g' g^{-1}(\phi) \right) \]
\[ \bar{A} = \text{tr} \left( \bar{a} \bar{g}' \bar{g}^{-1}(\phi) \right), \quad \bar{B} = \text{tr} \left( \bar{b} \bar{g}' \bar{g}^{-1}(\phi) \right) \]

The Gauss' constraints are given as
\[ \Phi = \frac{1}{2} \epsilon^{ij} F_{ij} - (A_\phi + g' g^{-1}) \delta(\rho - \rho_\infty) = 0, \]
\[ \bar{\Phi} = \frac{1}{2} \epsilon^{ij} \bar{F}_{ij} - (\bar{A}_\phi + \bar{g}' \bar{g}^{-1}) \delta(\rho - \rho_\infty) = 0 \]

where \( \rho_\infty \gg 1 \) is the radius of the boundary of spatial infinity. The Hamiltonian is written as
\[ H = -\frac{k}{4\pi} \int_{\partial M} \text{tr} \left\{ (A_\phi + g' g^{-1})^2 + (\bar{A}_\phi + \bar{g}' \bar{g}^{-1})^2 \right\} \]

and it satisfies \([ H, \Phi^a ] = [ H, \bar{\Phi}^a ] = 0 \).

We find that the generators for the local gauge symmetry can be written as
\[ G(\eta) = \frac{k}{4\pi} \int_M \eta^a \Phi_a = \int_M \eta^a G_a - Q(\eta), \]
\[ Q(\eta) = \frac{k}{2\pi} \int_{\partial M} \text{tr} \eta \left( A_\phi + g' g^{-1} \right) \]
where \( G^a = \frac{k}{8\pi} \epsilon_{ij} F^a_{ij} \). The surface term \( Q \) has been introduced in the previous study \[11,13,20\] to make \( G \) differentiable. Here we note that it arises as a consequence of canonical quantization of the bulk-boundary system. The conserved charges \( Q \) generate the global (asymptotic) symmetries of the theory. With the Poisson brackets Eq.(10), we find that the algebra of these global charges have central extensions

\[
\{G(\eta), G(\lambda)\} = G([\eta, \lambda]) + \frac{k}{4\pi} \int_{\partial M} \eta^a \left( \lambda'_a + \epsilon_{abc} A^b \lambda^c \right),
\]

\[
\{Q(\eta), Q(\lambda)\} = Q([\eta, \lambda]) + \frac{k}{4\pi} \int_{\partial M} \eta^a \left( \lambda'_a + \epsilon_{abc} A^b \lambda^c \right)
\]

(14a)

(14b)

where \( ([\eta, \lambda])^a = \epsilon^a_{bc} \eta^b \lambda^c \).

Similarly, we find the algebra of the global charges for the right moving sector

\[
\bar{G}(\eta) = -\frac{k}{4\pi} \int_M \eta^a \bar{\Phi}_a = \int_M \eta^a \bar{G}_a - \bar{Q}(\eta),
\]

\[
\bar{Q}(\eta) = -\frac{k}{2\pi} \int_{\partial M} \text{tr} \eta \left( \bar{A}_\phi + \bar{g}' \bar{g}^{-1} \right).
\]

(15)

where \( \bar{G}^a = -\frac{k}{8\pi} \epsilon_{ij} \bar{F}^a_{ij} \). They satisfy the following algebra

\[
\{\bar{G}(\eta), \bar{G}(\lambda)\} = \bar{G}([\eta, \lambda]) - \frac{k}{4\pi} \int_{\partial M} \eta^a \left( \lambda'_a + \epsilon_{abc} A^b \lambda^c \right),
\]

\[
\{\bar{Q}(\eta), \bar{Q}(\lambda)\} = \bar{Q}([\eta, \lambda]) - \frac{k}{4\pi} \int_{\partial M} \eta^a \left( \lambda'_a + \epsilon_{abc} A^b \lambda^c \right).
\]

(16a)

(16b)

IV. DIFFEOMORPHISM ON THE BOUNDARY

As discussed in ref. \[18\] the diffeomorphism is equivalent to a gauge transformation with an appropriately chosen gauge function in the Chern-Simons formulation. To be explicit the diffeomorphism generated by a vector

\[
\delta A_\mu^a = -\epsilon^\nu (\partial_\nu A_\mu^a - \partial_\mu A_\nu^a) - \partial_\mu (\epsilon^\nu A_\nu^a)
\]

is equivalent to a gauge transformation with a gauge function \( \eta^a = -\frac{k}{4\pi} \epsilon^\mu A_\mu^a \) on shell. The diffeomorphism on the boundary appears to be the asymptotic symmetry and the corresponding global charges are identified by the Virasoro generators. In order to identify
the conformal algebra let us consider a diffeomorphism generated by a \( \tau \)-independent vector \( e^\mu(\phi) \). Under this surface diffeomorphism the gauge fields of the BTZ black hole solution transform as

\[
\begin{align*}
\delta A_\tau^0 &= -\Delta R \epsilon_1 \cosh \rho, \\
\delta A_\tau^2 &= -\Delta R \epsilon_1 \sinh \rho, \\
\delta A_\phi^0 &= -\Delta R (\epsilon_1 \cosh \rho + \epsilon_+ \sinh \rho), \\
\delta A_\phi^1 &= -\epsilon_1', \\
\delta A_\phi^2 &= -\Delta R (\epsilon_1 \sinh \rho + \epsilon_+ \cosh \rho), \\
\delta A_\tau^1 &= \delta A_\rho^0 = \delta A_\rho^1 = \delta A_\rho^2 = 0
\end{align*}
\]

where \( \epsilon_+ = \epsilon_0 + \epsilon_2 \).

The diffeomorphism on the boundary of spatial infinity is supposed to be the asymptotic symmetry. So it should be required that \( \delta A_i^a = 0 \) at the leading order. From the asymptotic behavior of \( \delta A_\phi^a \),

\[
\delta A_\phi = -\frac{(J^0 + J^2)}{2} \Delta R e^\rho [\epsilon_1 + (\epsilon_+)]' + O(e^{-\rho})
\]

it follows that

\[
\epsilon_1 + (\epsilon_+)' = 0.
\]

Note that it is not necessary to require \( \delta A_\tau^a = 0 \) asymptotically, since \( A_\tau^a \) are Lagrangian multipliers.

In order to transform the system by the two-dimensional diffeomorphism, we need to transform both the gauge fields and the chiral boson fields: The gauge fields transform according to Eq.\((18)\) with the condition Eq.\((20)\) and the chiral boson fields according to the usual diffeomorphism generated by

\[
\delta \tau = \epsilon_0, \quad \delta \phi = \epsilon_2.
\]

If we transform the gauge fields
\[
\delta_1 I_G = -\frac{k}{2\pi} \int_{\partial M} \text{tr} \epsilon_+ ' \left\{ A_\phi (g'g^{-1}) + J^1 (g'g^{-1})' \right\}.
\]

(22)

And if we transform the boson fields,

\[
\delta_2 I_G = -\delta \Gamma^L [g] = -\frac{k}{2\pi} \int_{\partial M} \text{tr} \epsilon_+ (g'g^{-1})^2,
\]

(23)

Thus, we may identify the generators for the two-dimensional diffeomorphism as follows

\[
\delta I_G (g) = -\frac{k}{2\pi} \int_{\partial M} \epsilon_+ ' \text{tr} \left\{ (g'g^{-1})^2 + A_\phi (g'g^{-1}) + (g'g^{-1})'J^1 \right\}
\]

(24)

\[
\sum_{-\infty}^{\infty} L_n \epsilon_n^{in\phi} = -\frac{k}{2} \text{tr} \left\{ (g'g^{-1})^2 + A_\phi (g'g^{-1}) + (g'g^{-1})'J^1 \right\}.
\]

Applying the same procedure we also find the variation of the right moving part of the action. Under the surface diffeomorphism, the gauge fields \( \bar{A} \) transform as

\[
\delta \bar{A}_r^0 = R \bar{\epsilon}_1 \cosh \rho,
\]

\[
\delta \bar{A}_r^2 = -R \bar{\epsilon}_1 \sinh \rho,
\]

\[
\delta \bar{A}_\phi^0 = R (-\bar{\epsilon}_1 \cosh \rho + \bar{\epsilon}_- ' \sinh \rho),
\]

\[
\delta \bar{A}_\phi^1 = \bar{\epsilon}_1 ',
\]

\[
\delta \bar{A}_\phi^2 = R (\bar{\epsilon}_1 \sinh \rho - \bar{\epsilon}_- ' \cosh \rho),
\]

\[
\delta \bar{A}_r^1 = \delta \bar{A}_\rho^0 = \delta \bar{A}_\rho^1 = \delta \bar{A}_r^2 = 0
\]

(25)

where \( \bar{\epsilon}_- = \bar{\epsilon}_0 - \bar{\epsilon}_2 \).

From the leading behavior of \( \delta \bar{A}_i \) in the asymptotic region,

\[
\delta \bar{A}_\phi = -\frac{(J^0 - J^2)}{2} R \text{e}^\rho \left[ \bar{\epsilon}_1 - (\bar{\epsilon}_-) ' \right] + \mathcal{O} (\text{e}^{-\rho}),
\]

(26)

we obtain the condition for the surface diffeomorphism,

\[
\bar{\epsilon}_1 - (\bar{\epsilon}_-) ' = 0.
\]

(27)

Under this surface diffeomorphism the gauge fields \( \bar{A} \) contributes to the variation of the action through the coupling term as follows,
\[ \delta_1 I_G = \frac{k}{2\pi} \int_{\partial M} \text{tr} \bar{\epsilon}' \left\{ \bar{A}_\phi (\bar{g}' \bar{g}^{-1}) - \bar{J}^1 (\bar{g}' \bar{g}^{-1})' \right\}. \]  
\text{(28)}

As we transform the chiral boson fields \( \bar{g} \) by the usual surface diffeomorphism generated by Eq.\( (21) \)

\[ \delta_2 I_G = \delta \Gamma^R_{\bar{g}} = \frac{k}{2\pi} \int_{\partial M} \text{tr} \bar{\epsilon}' (\bar{g}' \bar{g}^{-1})^2. \]  
\text{(29)}

Taking into account transformation of both gauge fields \( \bar{A} \) and chiral boson fields \( \bar{g} \), we have

\[ \delta \bar{I}_G(\bar{g}) = \frac{k}{2\pi} \int_{\partial M} \bar{\epsilon}' \text{tr} \left\{ (\bar{g}' \bar{g}^{-1})^2 + \bar{A}_\phi (\bar{g}' \bar{g}^{-1}) - \bar{J}^1 (\bar{g}' \bar{g}^{-1})' \right\}. \]  
\text{(30)}

Rewriting it in terms of Fourier modes, we get the generators of the two-dimensional diffeomorphism for the right moving chiral bosons

\[ \sum_{-\infty}^{\infty} L_n e^{i n \phi} = -\frac{k}{2} \text{tr} \left\{ (\bar{g}' \bar{g}^{-1})^2 + \bar{A}_\phi (\bar{g}' \bar{g}^{-1}) - \bar{J}^1 (\bar{g}' \bar{g}^{-1})' \right\}. \]  
\text{(31)}

V. ENTROPY OF THE BTZ BLACK HOLE

According to the Bekenstein-Hawking formula the thermodynamic entropy of a black hole is proportional to the area of the event horizon \( \square \): \( S = \frac{A}{4G} \) where \( A \) is the area of the horizon and \( G \) is the Newton constant. The Bekenstein-Hawking entropy suggests that it has a statistical interpretation. It also implies that the statistical interpretation must be of classical origin, since the quantum corrections contribute to the entropy only at the order of 1. On the other hand, it contradicts the no-hair theorem, which asserts that the classical state of the black hole can be uniquely determined by its conserved charges measured asymptotically. Since the seminal works of Bekenstein and Hawking the statistical interpretation of the black hole entropy has been one of the outstanding problems in theoretical physics. It has been believed long that understanding the black hole entropy and related problems may provide clue to the consistent quantum theory of gravity. Since the Chern-Simons gravity is regarded as a consistent quantum theory for gravity in (2+1) dimensions, it is supposed to provide
a solution to this conundrum. Here in this section we will discuss how the puzzle can be resolved in the framework of the Chern-Simons gravity.

In the semi-classical region, where the cosmological constant is small, \( k \gg 1 \) the entropy of the black hole must be given by the Bekenstein-Hawking formula \( S = \frac{A}{4G} \) where \( A \) is the area of the horizon. For the BTZ black hole it reads as

\[
S = \frac{2\pi r_+}{4G} = \sqrt{2\pi} \left( M + \sqrt{M^2 - (J/l)^2} \right)^{1/2}.
\]  

(32)

Since the BTZ black hole is described by a \( SL(2, R) \) WZW model as discussed in the previous sections, it is suggestive that the entropy may be obtained by counting the microstates in the conformal field theory on the boundary. Given \( n_L \) and \( n_R \), the eigenvalues of the Virasoro generators \( L_0 \) and \( \bar{L}_0 \), the asymptotic growth of the states is given by the Cardy’s formula \[12\] when \( n_L \gg c_L \) and \( n_R \gg c_R \). Then, the entropy may be given as

\[
S = 2\pi \sqrt{\frac{c n_L}{6}} + 2\pi \sqrt{\frac{c n_R}{6}}.
\]  

(33)

In ref. \[10\] Strominger derived the entropy of the BTZ black hole, employing the Cardy’s formula and the previous study on (2+1) dimensional AdS space by Brown and Henneaux. Adopting the analysis of ref. \[11\] on (2+1) dimensional gravity where the generators for the diffeomorphism at asymptotic region are found to form a Virasoro algebra with central charge \( c = \frac{3\ell^2}{2G} = -6k \), identifying the mass and angular momentum of the black hole as follows

\[
M = \frac{8G}{\ell} (L_0 + \bar{L}_0), \quad J = L_0 - \bar{L}_0,
\]  

(34)

he found that the Cardy’s formula yields the Bekenstein-Hawking entropy of the BTZ black hole exactly.

Given a conformal field theory for the BTZ black hole at hand, it looks straightforward to give a concrete form to the Strominger’s derivation. However, one may readily realize that the generators for the two-dimensional diffeomorphism in Eqs.(24,31) do not form the Virasoro algebra contrary to expectations except when \( A_{\phi} = 0 \) on \( \partial M \). Moreover,
the expectation values of the Virasoro generators $L_0$ and $\bar{L}_0$ do not satisfy the Eq.(34). However, if we discern that the system faithfully respects the AdS/CFT (anti-de Sitter space/conformal field theory) correspondence \[8,21\], we can find a resolution.

Note that since the Guass' law $F = \bar{F} = 0$ is imposed for the gauge field in the bulk, we may write the gauge fields as $A = (u_0 u)^{-1} d(u_0 u)$, $\bar{A} = (\bar{u}_0 \bar{u})^{-1} d(\bar{u}_0 \bar{u})$ with some multi-valued gauge functions $u$ and $\bar{u}$ \[22\]. Here the gauge field configuration, \{ $A_0 = u_0^{-1} du_0$, $\bar{A}_0 = \bar{u}_0^{-1} d\bar{u}_0$ \}, depicts the BTZ black hole with zero mass. If we substitute it for the gauge fields in the action Eq.(9),

$$ I_G[A, g] = -\Gamma^L[u_0 u] - \Gamma^L[g] - \frac{k}{2\pi} \int_{\partial M} \text{tr}(g' g^{-1})(u^{-1} \partial_- u), \quad (35a) $$

$$ \bar{I}_G[\bar{A}, \bar{g}] = \Gamma^R[\bar{u}_0 \bar{u}] + \Gamma^R[\bar{g}] + \frac{k}{2\pi} \int_{\partial M} \text{tr}(\bar{g}' \bar{g}^{-1})(\bar{u}^{-1} \partial_+ \bar{u}). \quad (35b) $$

Making use of the Polyakov-Wiegman identity,

$$ \Gamma^{R/L}[ug] = \Gamma^{R/L}[u] + \Gamma^{R/L}[g] + \frac{k}{2\pi} \int_{\partial M} \text{tr}(g' g^{-1})(u^{-1} \partial_\pm u), \quad (36) $$

or the one-cocycle condition Eq.(7), we find

$$ I_G[A, g] = -\Gamma^L[u_0] - \Gamma^L[ug], \quad \bar{I}_G[\bar{A}, \bar{g}] = \Gamma^R[\bar{u}_0] + \Gamma^R[\bar{u} \bar{g}]. \quad (37) $$

This equation implies that the system can be equivalently described by the boundary fields $h = ug$ and $\bar{h} = \bar{u} \bar{g}$, which coupled to a trivial background, i.e., the massless BTZ black hole, but may have non-trivial holonomies in contrast to $g$ and $\bar{g}$. Also it reveals that the generating functional defined as

$$ Z[A], \bar{A}] = \int D[A] D[\bar{A}] D[g] D\bar{g} \exp i \left[ I_G[A, g] + \bar{I}_G[\bar{A}, \bar{g}] \right], \quad (38) $$

does not change under small variation of the boundary values of the gauge fields $A|$, $\bar{A}|$ and the bulk degrees of freedom perfectly match the boundary degrees of freedom. This may be considered as the simplest realization of the AdS/CFT correspondence \[23\].

Let us return to the boundary conformal field theory. We may develop the quantum theory in terms of the $h$ and $\bar{h}$ in parallel with the previous sections; as a result we may simply
replace $g$ with $h$ and the gauge fields $A$ with the those of massless black hole solution. Here the vacuum state corresponds to the massless BTZ black hole. In the canonical quantization the global charges become

$$Q(\eta) = \frac{k}{2\pi} \int_{\partial M} \text{tr} \left( \eta h' h^{-1} \right), \quad \bar{Q}(\eta) = -\frac{k}{2\pi} \int_{\partial M} \text{tr} \left( \eta \bar{h}' \bar{h}^{-1} \right)$$

and satisfy the usual Kac-Moody algebra

$$\{ Q(\eta), Q(\lambda) \} = Q([\eta, \lambda]) + \frac{k}{4\pi} \int_{\partial M} \eta^a \lambda'_a$$

and

$$\{ \bar{Q}(\eta), \bar{Q}(\lambda) \} = \bar{Q}([\eta, \lambda]) - \frac{k}{4\pi} \int_{\partial M} \eta^a \lambda'_a.$$ 

The algebra of the global charges can be obtained also by applying the symplectic reduction method and the Noether procedures [24]. Expanding it in terms of Fourier modes, we have

$$\{ T^a_n, T^b_m \} = \epsilon^{ab} e^c_{\partial} T^c_{n+m} + \frac{k}{2} \eta^{ab} n \delta(n + m),$$

$$\{ \bar{T}^a_n, \bar{T}^b_m \} = \epsilon^{ab} e^c_{\partial} \bar{T}^c_{n+m} - \frac{k}{2} \eta^{ab} n \delta(n + m),$$

$$T^a_n = \frac{k}{2\pi} \int_{\partial M} \text{tr} J^a (h' h^{-1}) e^{-in\phi} d\phi,$$

$$\bar{T}^a_n = -\frac{k}{2\pi} \int_{\partial M} \bar{J}^a (\bar{h}' \bar{h}^{-1}) e^{-in\phi} d\phi.$$

Now the action transforms under the two-dimensional diffeomorphism as

$$\delta I_G(h) = -\frac{k}{2 \pi} \int_{\partial M} e^+ \text{tr} \left\{ \left( h' h^{-1} \right)^2 + (h' h^{-1})' J^1 \right\},$$

$$\delta \bar{I}_G(\bar{h}) = \frac{k}{2 \pi} \int_{\partial M} \bar{e}^+ \text{tr} \left\{ \left( \bar{h}' \bar{h}^{-1} \right)^2 - (\bar{h}' \bar{h}^{-1})' \bar{J}^1 \right\}$$

and the generators for the two-dimensional diffeomorphism read as

$$\sum_{-\infty}^{\infty} L_n e^{in\phi} = -\frac{k}{2} \text{tr} \left\{ \left( h' h^{-1} \right)^2 + (h' h^{-1})' J^1 \right\},$$

$$\sum_{-\infty}^{\infty} \bar{L}_n e^{in\phi} = -\frac{k}{2} \text{tr} \left\{ \left( \bar{h}' \bar{h}^{-1} \right)^2 - (\bar{h}' \bar{h}^{-1})' \bar{J}^1 \right\}.$$

Note that if the one-loop radiative corrections of the $SL(2, R)$ Chern-Simons theory [25] is taken into consideration, $k$ in Eqs.(41,43) would be shifted by 2. (This radiative correction, however can be ignored in the large $k$ limit.)
Rewriting the diffeomorphism generators in terms of the $SL(2,R)$ currents, $T^a_n$ and $\bar{T}^a_n$,

\[ L_n = -\frac{1}{k+2} \sum_m T^a_{n-m} T^a_m - \frac{i}{2} n T^1_n, \quad (44a) \]

\[ \bar{L}_n = -\frac{1}{k+2} \sum_m \bar{T}^a_{n-m} \bar{T}^a_m - \frac{i}{2} n \bar{T}^1_n, \quad (44b) \]

we find that they form the Virasoro algebra

\[ [L_n, L_m] = (n-m)L_{n+m} + \frac{c_L}{12} n(n^2 - 1) \delta(n+m), \quad (45a) \]

\[ [L_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{c_R}{12} n(n^2 - 1) \delta(n+m) \quad (45b) \]

with central charges, $c_L = c_R = \frac{3k}{k+2} - 6k$. Here the shift in $k$ may be considered as a consequence of the the radiative correction or the Sugawara construction of the energy-momentum tensor. It is interesting to observe that the algebraic structure of the conformal theory for the BTZ black hole is identical to that of the two-dimensional gravity \[ \text{[26]} \]. In the large $k$ limit the central charges are approximated by $-6k$.

The evaluation of the entropy would be completed by taking expectation values of $L_0$ and $\bar{L}_0$. Observe that

\[ \text{tr}(h' h^{-1})^2 = \text{tr} \left( u^{-1} u' + g' g^{-1} \right)^2, \]

\[ \text{tr}(\bar{h}' \bar{h}^{-1})^2 = \text{tr} \left( \bar{u}^{-1} \bar{u}' + \bar{g}' \bar{g}^{-1} \right)^2. \]

We take the expectation values of $L_0$ and $\bar{L}_0$ with respect to the quantum state corresponding to the BTZ black hole with mass $M$ and angular momentum $J$. For the BTZ black hole state the expectation values of $(u^{-1} u')$ and $(\bar{u}^{-1} \bar{u}')$ take $(A_{BTZ})_\phi$ and $(\bar{A}_{BTZ})_\phi$ of the classical BTZ solution Eq.(3). If neglecting the excitation in $g$, we have

\[ n_L = \langle BTZ | L_0 | BTZ \rangle = -\frac{k}{2} \text{tr} \left[ (A_{BTZ})_\phi \right]^2 = -\frac{k}{4} \left( M - \frac{J}{l} \right), \quad (46a) \]

\[ n_R = \langle BTZ | \bar{L}_0 | BTZ \rangle = -\frac{k}{2} \text{tr} \left[ (\bar{A}_{BTZ})_\phi \right]^2 = -\frac{k}{4} \left( M + \frac{J}{l} \right). \quad (46b) \]

Thus, employing the Cardy’s formula, we obtain the correct entropy of the BTZ black hole

\[ S = \pi |k| \left( \sqrt{M - \frac{J}{l}} + \sqrt{M + \frac{J}{l}} \right) \quad (47) \]

\[ = \frac{2\pi r_+}{4G}. \]
(When we employ the Cardy’s formula, we assume that the conformal field theory is unitary. However, $SL(2, R)$ group does not have a finite unitary representation. In order to get finite unitary representations, one may gauge a $U(1)$ subgroup \[19\]. This results in $SL(2, R)/U(1)$ WZW model for the BTZ black hole, and shift of the central charges by 1. But this shift is negligible in the large $k$ limit.)

The Virasoro algebra of the surface diffeomorphism was first derived by analyzing the asymptotic symmetry group of the AdS space in the ADM formulation \[11\] and later by constructing the global charges corresponding to the diffeomorphism in the Chern-Simons formulation of the (2+1) dimensional gravity \[20\]. In the present paper we give a direct derivation, compatible with the AdS/CFT correspondence, by examining the variation of the action under the surface diffeomorphism. The analysis given in the present paper points out that the surface diffeomorphism should be considered in the context of the three dimensional diffeomorphism. So when we take the transformation on the boundary, $\delta \tau = \epsilon_0$, $\delta \phi = \epsilon_2$, we should also transform the boundary values of the gauge fields consistently; otherwise the central charge of the Virasoro algebra would be obtained as $\frac{3k}{k+2}$. Thus, the coupling of the bulk field degrees of freedom to the boundary ones plays an important role, and moreover, it gives dominant contribution to the central charge in the semi-classical regime. The analysis given in this section also clearly exhibit that one must consider the gauge field degrees of freedom, which often ignored, as a part of the dynamical degrees of freedom of the black hole. In fact the entropy of the black hole is mainly comprised of the contribution from the gauge field degrees of freedom.

VI. CONCLUSIONS

Understanding the Bekenstein-Hawking entropy of the black hole from the first principles has been the crux of gravity, and is believed to guide us eventually to a consistent quantum theory of gravity. Since in (2+1) dimensions the gravity can be formulated by a Chern-Simons theory, which is supposed to be finite, this open problem may be treated in a
tractable manner in (2+1) dimensions. Recent development of the string duality and the M-theory points out that the three dimensional BTZ black hole is extremely important to study the higher dimensional black holes in the string theory in that many of them contain the BTZ black hole as a part of their near-horizon geometries. In the present paper we discuss the entropy of the BTZ black hole in the Chern-Simons formulation and identify the relevant dynamical degrees of freedom of the black hole. The derivation of the BTZ black hole entropy given in this work is consistent with the AdS/CFT correspondence and clarifies the issues raised in the literature.

We summarize the present work and conclude with discussions on the puzzle of the black hole entropy. We construct an action, which governs the dynamics of the BTZ black hole and perform the canonical quantization. Then we identify the global charges, which generate the asymptotic symmetry of the system, in the framework of the canonical quantization and find that the algebra of these charges have central extensions. The generators of the surface diffeomorphism are directly obtained from the variation of the action under the diffeomorphism, which comprises the transformation of the chiral boson fields and the gauge fields on the boundary. The coupling between the chiral boson fields and the gauge fields on the boundary is found to play an important role, giving dominant contribution to the central charge.

In the presence of the coupling term, we can apply the Polyakov-Wiegman identity to the action, and find that the system can be equivalently described by chiral boson fields with non-trivial holonomies, which coupled to the background given as the massless black hole. In this new setting, the global charges satisfy the Kac-Moody algebra with central extension at the classical level and the diffeomorphism generators form a Virasoro algebra as desired. From this point of view, the appropriate vacuum state for the quantum theory is the massless black hole rather than the classical BTZ black hole configuration with finite mass: The state of massless black hole is the eigenstate of $L_0$ and $\bar{L}_0$ with zero eigenvalues. Accordingly, the BTZ black hole with finite mass should be understood as a quantum excitation of the massless black hole. And the Bekenstein-Hawking entropy counts the density of quantum
states at the BTZ state with respect to the vacuum state of the massless black hole.

The present derivation also provides an appropriate explanation to the puzzle of the black hole entropy, which briefly mentioned at the beginning of the section V. Although the Bekenstein-Hawking entropy is proportional to $1/\hbar$, it is of quantum mechanical origin. In other words the mass and angular momentum of the black hole are quantum hairs, which unlike the small fluctuation of $g$, have large macroscopic expectation values. Hence, the classical no-hair theorem does not apply. The relevant dynamical degrees of freedom, which give main contribution to the entropy are not the chiral boson fields $g$, which only describe small fluctuations, but the gravity, here the Chern-Simons gauge fields: The gauge fields of the classical solution should be interpreted as expectation values of the gauge fields in the quantum theory. This interpretation also gives a proper answer to the question of where the relevant degrees of freedom of the black hole are located. The true meaning of the Bekenstein-Hawking black hole entropy can be found only when the gravity is fully quantized. We hope that this work also sheds some light on the study of the D-brane dynamics.

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