EXACT AUGMENTED LAGRANGIAN DUALITY FOR MIXED INTEGER CONVEX OPTIMIZATION

AVINASH BHRADWAJ*, VISHNU NARAYANAN*, AND ABHISHEK PARTHAPATI*

Abstract. Augmented Lagrangian dual augments the classical Lagrangian dual with a non-negative non-linear penalty function of the violation of the relaxed/dualized constraints in order to reduce the duality gap. We investigate the cases in which mixed integer convex optimization problems have an exact penalty representation using sharp augmenting functions (norms as augmenting penalty functions). We present a generalizable constructive proof technique for proving existence of exact penalty representations for mixed integer convex programs under specific conditions using the associated value functions. This generalizes the recent results for MILP (Feizollahi, Ahmed and Sun, 2017) and MIQP (Gu, Ahmed and Dey 2020) whilst also providing an alternative proof for the aforementioned along with quantification of the finite penalty parameter in these cases.

Key words. Mixed Integer Convex Optimization, Augmented Lagrangian Duality, Exact Penalty representation

MSC codes. 90C11, 90C46

1. Introduction. Given a polyhedral mixed-integer set \( X \subseteq \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} \) and a real valued function \( f : \mathbb{R}^n \mapsto \mathbb{R} \), consider the following mixed integer programming problem:

\[
(P) \quad z_{IP} = \min \{ f(x) : A x = b, x \in X \}
\]

Solutions to mixed integer programming problems such as \((P)\) are often computationally intractable, and the strong duality doesn’t hold in general. As such, certain constraints of the optimization problem in context may be relaxed by using classical Lagrangian dual (LD) to yield good lower bounds on the optimal objective value. Specifically,

\[
z_{LD} = \sup_{\lambda \in \mathbb{R}^m} \min_{x \in X} (f(x) + \lambda^T (b - A x)) \leq z_{IP}.
\]

In contrast with the convex optimization problems, for nonconvex optimization problems such as \((P)\), classical Lagrangian dual may yield a non-zero duality gap, i.e. \(z_{LD} < z_{IP}\). This duality gap may be avoided if the dual problem could be set up with, instead of affine dual functions, some other class of functions capable of penetrating possible ‘dents’ in the value function [17]. Augmented Lagrangian dual (ALD), as the name suggests, augments the LD with a nonlinear penalty function of the violation of dualized constraints,

\[
z_{ALD}^\rho = \sup_{\lambda \in \mathbb{R}^m} \min_{x \in X} (f(x) + \lambda^T (b - A x) + \rho \psi(b - A x))
\]

where \(\psi(.) > 0\) is the augmenting penalty function and \(\rho > 0\) denotes the penalty parameter. Under certain conditions, a zero duality gap can be reached asymptotically by increasing the penalty parameter, \(\rho\), to infinity [19]. In some cases, the duality gap can be closed with a large enough finite value of the penalty parameter. In such cases, when the duality gap can be closed for a finite value of the penalty parameter \(\rho\),

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the primal problem is termed to have an *exact penalty representation*. Recently, the question of determining whether a class of non-convex optimization problems has an exact penalty representation has garnered quite some interest [6, 13, 14]. Boland and Eberhard [6] use convex, monotone augmenting functions to close the mixed integer linear programming duality gap and also prove that for bounded pure-integer linear programs the gap can be closed for a finite penalty parameter. Additionally, the authors utilize techniques from [9] to prove that duality gap is zero under the assumption that mixed integer linear programming problem attains the solution. Feizollahi et al. [13] and Gu. et al [14] consider level-bounded augmenting functions and prove that these augmenting functions close the duality gap as the penalty parameter goes to infinity for mixed integer linear programming (MILP) and mixed integer quadratic programming (MIQP) problems, respectively. They further prove that one can close the duality gap for a finite penalty parameter for sharp augmented Lagrangians. Burke [10, 11] provides the characterization of the conditions when the duality gap can be closed for finite penalty parameter for augmented (sharp and proximal) Lagrangians. Another stream of research is focused on developing general approaches to solve augmented Lagrangian dual problems. Boland and Eberhard [6] suggest the use of *alternating directions method of multipliers* (ADMM) [7] for solving the MILP ALD problems. Cordova et al. [12] provide a primal dual solution approach in form of a proximal bundle method to solve the ALD problems.

Value (Perturbation) functions of optimization problems provide a key insight into the properties of the augmenting functions that can be utilized for ALD to close the duality gap [17, 10, 11]. The structure of the value functions of MILPs has been extensively discussed in the literature [15, 3, 4, 5, 2, 16]. Ralphs and Hassanzadeh [16] provide an algorithm for the construction of the value function of a MILP and prove the finiteness of this algorithm in certain cases. It is also known that the value functions of both rational mixed-integer linear programs and continuous convex programs are lower semi-continuous [15, 1].

The existence of exact penalty representations for MILPs [6, 13] and MIQPs [14] is well-established. However, the proof techniques used for MILPs and MIQPs are specific to these problem classes and aren’t readily or necessarily generalizable. In the following, we present an alternative proof technique for proving existence of exact penalty representations in these cases using the associated value functions. This new proof technique helps us to generalize and prove the existence of exact penalty representations for mixed integer convex programs (MICPs) under specific conditions. Furthermore, this proof technique is also constructive in nature. Specifically, we provide an analytical form for the finite penalty parameter in case of MILPs and MIQPs and an upper bound in the case of MICPs.

The following discussion is organised in three sections. Section 2 provides necessary definitions and highlights the notation used in the paper. Section 3 outlines our primary results highlighting the developed generalizable constructive proof technique for proving existence of exact penalty representations in the case of MILPs, MIQPs and MICPs and construction of the penalty parameter in the respective cases. Section 4 illustrates the proofs of the results presented in Section 3.
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following discussion we define for some \( \alpha \) convex function continuous variables, respectively. Analogously, for any \( u \) denote the partition of \( u \) by rows indexed by \( U \) and columns indexed by \( V \). We also define \( \left( \begin{array}{c} x \\ y \end{array} \right) \) to denote the index set \( \{ 1, 2, \ldots, n \} \).

Consider, to begin with, the following mixed integer convex programming (MICP) problem,

\[
\begin{align*}
\text{(MICP)} & \quad \text{z}_{IP} = \text{minimize} \quad f(x) \\
\text{subject to} & \quad A x = b \\
& \quad x \in X
\end{align*}
\]

where \( X := \{ x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} : E x \leq f \} \) is a polyhedral mixed integer set, \( f : \mathbb{R}^n \mapsto \mathbb{R} \) is a real valued convex function, \( A, E \) and \( b, f \) are full rank matrices and vectors of appropriate dimensions, respectively. Throughout the remainder of this discussion we assume that the matrices \( A, E \) are rational matrices and \( b, f \) are rational vectors and \( f(\cdot) \) is differentiable. We further assume that both MICP and MICP’s continuous relaxation are feasible and the corresponding optimal solutions exist.

Consider the Lagrangian relaxation of the MICP,

\[
\text{(LR)} \quad z^{LR}(\lambda) = \min_{x \in X} f(x) + \lambda^\top (b - A x).
\]

The corresponding augmented Lagrangian relaxation and augmented Lagrangian dual ([6, 14]) of the (MICP) are defined as,

\[
\begin{align*}
\text{(ALR)} & \quad z^{LR+}(\rho) = \min_{x \in X} f(x) + \lambda^\top (b - A x) + \rho \psi(b - A x) \\
\text{(ALD)} & \quad z^{LD+}(\rho) = \sup_{\lambda \in \mathbb{R}^m} \min_{x \in X} f(x) + \lambda^\top (b - A x) + \rho \psi(b - A x)
\end{align*}
\]

where \( \psi : \mathbb{R}^m \mapsto \mathbb{R} \) is a real valued function. We further designate \( \psi \) to have an exact penalty representation if \( \exists 0 < \rho < \infty \), such that \( z^{LD+}(\rho) = z_{IP} \). In the case \( \psi(\cdot) = ||\cdot|| \) we augmented Lagrangian relaxation is referred to as sharp augmented Lagrangian relaxation. In particular,

\[
\text{(SALR)} \quad z_{SALR}(\rho) = \min_{x \in X} f(x) + \rho ||b - A x||.
\]

It follows that \( z^{LR+}(\lambda) \leq z_{IP} \) and \( z^{LR+}(\rho) = z_{SALR}(\rho) \).

2. Preliminaries. Consider, to begin with, the following mixed integer convex programming (MICP) problem,

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& \quad x \in X
\end{align*}
\]

where \( X := \{ x \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2} : E x \leq f \} \) is a polyhedral mixed integer set, \( f : \mathbb{R}^n \mapsto \mathbb{R} \) is a real valued convex function, \( A, E \) and \( b, f \) are full rank matrices and vectors of appropriate dimensions, respectively. Throughout the remainder of this discussion we assume that the matrices \( A, E \) are rational matrices and \( b, f \) are rational vectors and \( f(\cdot) \) is differentiable. We further assume that both MICP and MICP’s continuous relaxation are feasible and the corresponding optimal solutions exist.

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\]

where \( \psi : \mathbb{R}^m \mapsto \mathbb{R} \) is a real valued function. We further designate \( \psi \) to have an exact penalty representation if \( \exists 0 < \rho < \infty \), such that \( z^{LD+}(\rho) = z_{IP} \). In the case \( \psi(\cdot) = ||\cdot|| \) we augmented Lagrangian relaxation is referred to as sharp augmented Lagrangian relaxation. In particular,

\[
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\]

It follows that \( z^{LR+}(\lambda) \leq z_{IP} \) and \( z^{LR+}(\rho) = z_{SALR}(\rho) \).

2.1. Notation. The sets \( \mathbb{R}^n, \mathbb{Z}^n, \mathbb{Q}^n \) denote the set of real numbers, integers and rational numbers in \( n \)-dimensional vector space, respectively. Additionally, \( \mathbb{R}_+^n, \mathbb{Z}_+^n, \mathbb{Q}_+^n \) denote the non-negative counterparts of the respective sets. We use \( [n] \) to denote the index set \( \{ 1, 2, \ldots, n \} \). For an index set \( E \subset \mathbb{Z}^+ \) and \( y \in \mathbb{Z} \), we define the translation of an index set as \( y + E := \{ x + y : x \in E \} \). For a mixed integer set \( S \subset \mathbb{R}^n \times \mathbb{Z}^m \), we denote by \( S_R \) the continuous relaxation of \( S \). Throughout the following discussion we define \( I \) and \( C \) as the index sets corresponding to integer and continuous variables, respectively. Analogously, for any \( u \in \mathbb{R}^n \), let \( u = (u_I, u_C) \) denote the partition of \( u \) into the integer and continuous variables, respectively. Similarly for a symmetric matrix \( Q \in \mathbb{R}^{n \times n} \), let \( Q_{UV} \) denote the submatrix of \( Q \) formed by rows indexed by \( U \) and columns indexed by \( V \). For \( \delta > 0 \) and \( \bar{x} \in \mathbb{R}^n \), we define by \( N_\delta(\bar{x}) := \{ x \in \mathbb{R}^n : ||x - \bar{x}|| \leq \delta \} \), the ball with center \( \bar{x} \) and radius \( \delta \). The recession cone of a set \( S \) is denoted by \( \text{rec}(S) \). We also define, \( \text{rec}(f) \), recession cone of a convex function \( f : \mathbb{R}^n \mapsto \mathbb{R} \) as the recession cone of a level set \( \{ x \in \mathbb{R}^n : f(x) \leq \alpha \} \), for some \( \alpha \in \mathbb{R} \) ([1], Proposition 2.3.1). For given convex sets \( A \) and \( B \), we say that the sets \( A \) and \( B \) have no common/distinct non-zero directions of recession if
\( \text{rec}(A) \cap \text{rec}(B) \setminus \{0\} = \emptyset \).

We say that a function \( g : \mathbb{R}^m \to \mathbb{R} \) is \( L \)-smooth, or alternatively, has \textbf{Lipschitz-continuous} gradients if there exists a Lipschitz constant, \( L < \infty \), such that

\[
\|\nabla g(x) - \nabla g(y)\| \leq L \|x - y\| \quad \forall \ x, y \in \mathbb{R}^m.
\]

Additionally, we say a function \( g : \mathbb{R}^m \to \mathbb{R} \) is \( \mu \)-\textbf{strongly-convex} if there exists \( 0 < \mu < \infty \), such that

\[
\|\nabla g(x) - \nabla g(y)\| \geq \mu \|x - y\| \quad \forall \ x, y \in \mathbb{R}^m.
\]

\( \psi : \mathbb{R}^m \to [0, \infty) \) is termed as a proper, non-negative, lower semi-continuous function, \textbf{level-bounded augmenting function} \( \text{iff} \)

\[
\psi(0) = 0, \ \psi(u) > 0 \ \forall \ u \neq 0, \ \text{and} \quad \text{diam} \{ u : \psi(u) \leq \delta \} < +\infty \ \forall \ \delta > 0.
\]

Moreover, \( \lim_{\delta \downarrow 0} \text{diam} \{ u : \psi(u) \leq \delta \} = 0 \) [13].

Consider the hyperplane \( H_u := \{ x \in \mathbb{R}^n : Ax = b + u \} \), and let the corresponding value function associated with (MICP) be defined as

\[
(2.1) \quad \phi(u) = \min \{ f(x) : x \in X \cap H_u \}.
\]

In particular, \( z_{IP} = \phi(0) = \min_{x \in X \cap H_0} f(x) \). Finally, we define the set \( U \) as the set of all possible perturbation vectors \( u \) such that the feasible set of (2.1) is non-empty. In particular, \( U := \{ u \in \mathbb{R}^m : X \cap H_u \neq \emptyset \} \).

3. Main results. The primary contribution of this work entails a generalizable constructive proof technique for proving existence of exact penalty representations for mixed integer convex programs under specific conditions. We would like to emphasize, in particular, that while the results on existence of exact penalty representations using sharp Lagrangians in the specific case of rational MILPs and bounded integer variable MIQPs have been discussed in literature [13, 14], the proofs use specific properties of MILPs and MIQPs and thus don’t necessarily generalize to MICPs. The proposed proof technique utilizes the properties of the associated value functions. Specifically, the proof utilizes Lemma 4.5 in conjunction with the lower semi-continuity of the value functions of continuous convex optimization problems in both the aforementioned cases. In addition to resolving the existence of an exact penalty representation in the cases discussed, we further provide a quantification of the associated penalty parameter \( \rho \). This quantification, to the best of our knowledge, has not been discussed in literature.

The following theorems formalize this discussion. The proofs of the theorems follow in section 4. It should be noted that in addition to the assumptions stated in section 2, we further assume that \( c = (c_C, c_I) \) is a rational vector and \( Q \) is a rational symmetric positive semi-definite matrix.

**Theorem 3.1.** Consider the following mixed integer linear programming problem,

\[
\text{minimize} \quad c_I^T x_I + c_C^T x_C \\
\text{(MILP) subject to} \quad A_I x_I + A_C x_C = b \\
E_I x_I + E_C x_C \leq f \\
(x_I, x_C) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}
\]
There exists an exact penalty representation for (MILP). Furthermore, the finite penalty parameter $\rho$ depends on $A_C$ and $c_C$ and $c_I$.

**Theorem 3.2.** Consider the following mixed integer quadratic programming problem,

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} x_I^T Q_{II} x_I + \frac{1}{2} x_I^T Q_{IC} x_C + x_I^T Q_{CI} x_C - c_I^T x_I - c_C^T x_C \\
\text{subject to} & \quad A_I x_I + A_C x_C = b \\
& \quad E_I x_I + E_C x_C \leq f \\
& \quad \|x_I\|_\infty \leq M \\
& \quad (x_I, x_C) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}
\end{align*}
\]

There exists an exact penalty representation for (MIQP). Furthermore, the finite penalty parameter $\rho$ depends on $A, Q, c$ and $M$.

Generalizing the discussion on MILPs and MIQPs our next result primarily focuses the discussion on exact penalty representations of MICP’s under specific conditions. We initiate the discussion by proving that the Pure Integer Convex Programs (PICPs) with rational data have an exact penalty representation. We further prove that MICPs where either the objective function is strongly convex or where the recession cone of the epigraph of the objective function and the recession cone of the continuous relaxation of the feasible set have no common non-zero directions of recession the duality gap can be closed with level-bounded functions asymptotically as $\rho \to \infty$. As our concluding result, we establish in **Theorem 3.3** that (a) MICPs where the recession cone of the epigraph of the objective function and the recession cone of the continuous relaxation of the feasible set have no common non-zero directions of recession and (b) MICPs with bounded integer variables, have exact penalty representation when using norms as augmenting functions.

**Theorem 3.3.** Consider the following mixed integer convex programming problem,

\[
\begin{align*}
\text{minimize} & \quad f(x_I, x_C) \\
\text{subject to} & \quad A_I x_I + A_C x_C = b \\
& \quad E_I x_I + E_C x_C \leq f \\
& \quad (x_I, x_C) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}
\end{align*}
\]

Let $F = X \cap H_0$ denote the feasible region of (MICP).

(a) If $\|x_I\|_\infty \leq M, \forall x \in F$ then there exists an exact penalty representation for (MICP).

(b) If the recession cone of $f$ and recession cone of $F_R$, the feasible set of continuous relaxation of (MICP), have no common non-zero directions of recession, then there exists an exact penalty representation for (MICP).

Additionally, if $f$ is $\mu$-strongly convex and $L$ smooth, then the finite penalty parameter $\rho = O \left( \frac{L \beta \gamma}{\mu} \right)$ where $\beta := \max_{B \in \mathcal{B}} \|B^{-1}\|_{F}$, $\mathcal{B}$ being the set of all possible invertible submatrices of $[A_C^T - A_C^T - E_C^T]$ and $\gamma$ can be explicitly computed given $f(0), \|\nabla f(0)\|$ and $f(\bar{x})$ for any $\bar{x} \in F$.

**Remark 3.4.** The parameter $\gamma$ of **Theorem 3.3**(c) can be explicitly computed as $\gamma = 2 \|\nabla f(0)\| + \sqrt{\|\nabla f(0)\|^2 + 2\mu \left( f(\bar{x}) - f(0) \right)}$. Observe that the feasible region $F$ of (MICP) is a rational polyhedron. From Corollary 17.1d of [18], there exists a feasible
\( \mathbf{x} \in F \) whose size is polynomially bounded by the size of \( \mathbf{A}, \mathbf{b}, \mathbf{E}, \) and \( f \). Therefore, the parameter \( \gamma \) of Theorem 3.3 is “small” with respect to the input size if the function value \( f(\cdot) \), and the gradients of \( f(\cdot) \) can be computed efficiently.

4. Proofs of the Theorems.

4.1. Preliminary Lemmas.

**Lemma 4.1.** \( z_{\text{SLR}}(\rho) = \min_{\mathbf{u} \in \mathcal{U}} \phi(\mathbf{u}) + \rho \| \mathbf{u} \| \).

*Proof.* Observe the following inequality which follows immediately from the aforementioned definitions

\[
\begin{align*}
\min_{\mathbf{x} \in X} f(\mathbf{x}) + \rho \| \mathbf{b} - \mathbf{A} \mathbf{x} \| & \leq \min_{\mathbf{x} \in X \cap H_u} f(\mathbf{x}) + \rho \| \mathbf{u} \|, \quad \forall \mathbf{u} \in \mathcal{U} \\
& = \phi(\mathbf{u}) + \rho \| \mathbf{u} \|, \quad \forall \mathbf{u} \in \mathcal{U}.
\end{align*}
\]

(4.1)

Conversely, it holds that \( \forall \mathbf{u} \in \mathcal{U} \)

\[
\begin{align*}
\phi(\mathbf{u}) + \rho \| \mathbf{u} \| & = \min_{\mathbf{x} \in X \cap H_u} f(\mathbf{x}) + \rho \| \mathbf{u} \| \\
& \leq f(\mathbf{x}) + \rho \| \mathbf{Ax} - \mathbf{b} \|, \quad \forall \mathbf{x} \in X \cap H_u.
\end{align*}
\]

This implies that \( \min_{\mathbf{u} \in \mathcal{U}} \phi(\mathbf{u}) + \rho \| \mathbf{u} \| \leq f(\mathbf{x}) + \rho \| \mathbf{Ax} - \mathbf{b} \| \quad \forall \mathbf{x} \in X \cap H_u, \forall \mathbf{u} \in \mathcal{U} \)

which yields that \( \min_{\mathbf{u} \in \mathcal{U}} \phi(\mathbf{u}) + \rho \| \mathbf{u} \| \leq f(\mathbf{x}) + \rho \| \mathbf{Ax} - \mathbf{b} \| \quad \forall \mathbf{x} \in X \). This implies,

\[
\begin{align*}
\min_{\mathbf{u} \in \mathcal{U}} \phi(\mathbf{u}) + \rho \| \mathbf{u} \| & \leq \min_{\mathbf{x} \in X} f(\mathbf{x}) + \rho \| \mathbf{Ax} - \mathbf{b} \|. 
\end{align*}
\]

(4.2)

The result follows from (4.1) and (4.2).

Consider the partition of the feasible set \( X = F_\geq \cup F_\leq, F_\geq \cap F_\leq = \emptyset \) where \( F_\leq = \{ \mathbf{x} \in X : f(\mathbf{x}) \leq z_{IP} \} \) and \( F_\geq = \{ \mathbf{x} \in X : f(\mathbf{x}) > z_{IP} \} \). Observe that for \( \mathbf{x} \in F_\geq \) any positive \( \rho > 0 \) implies \( f(\mathbf{x}) + \rho \| \mathbf{Ax} - \mathbf{b} \| > z_{IP} \). Thus, for MICP to have an exact penalty representation, it suffices to show that \( \exists 0 < \rho < \infty \) such that \( f(\mathbf{x}) + \rho \| \mathbf{b} - \mathbf{Ax} \| \geq z_{IP}, \forall \mathbf{x} \in F_\leq \).

**Proposition 4.2.** Consider the continuous relaxation of MICP, and let the optimal objective value \( z_R = \min_{\mathbf{x} \in H_0} f(\mathbf{x}) \) be attained at \( \mathbf{x}_R \). Let \( \lambda_\mathbf{A} \) and \( \lambda_\mathbf{E} \) be the Lagrange multipliers for the constraints \( \mathbf{Ax} = \mathbf{b} \) and \( \mathbf{Ex} \leq \mathbf{f}, \) respectively.

i) \( \phi(\mathbf{u}) > -\infty \) for all \( \mathbf{u} \in \mathcal{U} \).

ii) If \( \exists \alpha \in X_R \) such that \( f(\alpha) + \rho \| \mathbf{Ax} - \mathbf{b} \| \leq z_{IP} \) and \( \rho > \| \lambda_\mathbf{A} \| \) then

\[
\| \mathbf{Ax} - \mathbf{b} \| \leq \frac{z_{IP} - z_R}{\rho - \| \lambda_\mathbf{A} \|}.
\]

*Proof.* Consider the Lagrangian function for the continuous relaxation of MICP, i.e. for \( \mathbf{x} \in \mathbb{R}^n \)

\[
\mathcal{L}(\mathbf{x}, \lambda_\mathbf{A}, \lambda_\mathbf{E}) = f(\mathbf{x}) - \lambda_\mathbf{A}^\top (\mathbf{Ax} - \mathbf{b}) - \lambda_\mathbf{E}^\top (\mathbf{f} - \mathbf{Ex})
\]

As the relaxed problem has only affine constraints strong duality holds (Chapter 5, Section 5.2.3[8]).
As strong duality holds, the first order necessary (KKT) conditions for \( \mathcal{L}(x, \lambda_A, \lambda_E) \) can be characterized as

\[
\nabla f(x) = A^\top \lambda_A - E^\top \lambda_E
\]

\[
\lambda_E^\top (f - Ex) = 0
\]

\[
\lambda_A^\top (Ax - b) = 0
\]

\[
\lambda_E \geq 0
\]

(4.3)

Observe that \( x_R \) satisfies the system of equations (4.3).

i) To see the result, observe that it suffices to show that \( \phi(u) > -\infty \) for all \( u \in U \) such that \( \phi(0) = z_{IP} \). Further, consider the set \( F_\leq \) defined as,

\[
F_\leq := \{ x \in X : f(x) \leq z_{IP} \}.
\]

We have for all \( x \in F_\leq \), \( z_{IP} \geq f(x) \geq f(x_R) + \nabla f(x_R)(x - x_R) = z_R + \nabla f(x_R)(x - x_R) \). Substituting from (4.5) we obtain, for all \( x \in F_\leq \),

\[
z_{IP} \geq f(x) \geq z_R + \lambda_A^\top (Ax - b) - \lambda_E^\top (Ex - f)
\]

As \( \lambda_E \geq 0 \) and \( x \in X_R \),

\[
z_{IP} \geq f(x) \geq z_R + \lambda_A^\top (Ax - b)
\]

\[
\geq z_R - ||\lambda_A|| ||(Ax - b)||
\]

\[
= z_R - ||\lambda_A|| ||u|| > -\infty.
\]

Since \( f(x) > -\infty \) for all \( x \in F_\leq \), the result follows.

ii) Since we have for \( \alpha \in X_R \), \( f(\alpha) + \rho ||A \alpha - b|| \leq z_{IP} \); convexity of \( f \) yields,

\[
f(x_R) + \nabla f(x_R)^\top (\alpha - x_R) + \rho ||A \alpha - b|| \leq z_{IP}, \ \text{i.e.,}
\]

\[
(4.4) \quad z_R + \nabla f(x_R)^\top (\alpha - x_R) + \rho ||A \alpha - b|| \leq z_{IP}.
\]

It follows from strong duality that

\[
\nabla f(x_R)^\top (x - x_R) = \lambda_A^\top A(x - x_R) - \lambda_E^\top E(x - x_R)
\]

\[
(4.5) \quad = \lambda_A^\top (Ax - b) - \lambda_E^\top (Ex - f).
\]

Substituting in (4.4), we obtain

\[
z_R + \lambda_A^\top (A \alpha - b) + \lambda_E^\top (f - E \alpha) + \rho ||A \alpha - b|| \leq z_{IP}.
\]

As \( \lambda_E \geq 0 \) and \( \alpha \in X_R \), the above inequality can be rewritten as

\[
(4.6) \quad \lambda_A^\top (A \alpha - b) + \rho ||A \alpha - b|| \leq z_{IP} - z_R
\]

Using Cauchy Schwarz inequality in (4.6) yields for all \( x \in X_R \),

\[
- ||\lambda_A|| ||A \alpha - b|| + \rho ||A \alpha - b|| \leq z_{IP} - z_R
\]

\[
(\rho - ||\lambda_A||) ||A \alpha - b|| \leq z_{IP} - z_R
\]

\[
||A \alpha - b|| \leq \frac{z_{IP} - z_R}{\rho - ||\lambda_A||}.
\]

\( \square \)
Corollary 4.3. Consider the set \( U_\rho = \{ u \in U : \phi(u) + \rho \| u \| \leq \phi(0) \} \). If \( \rho > \| \lambda_A \| \) then for all \( u \in U_\rho \),

\[
\| u \| \leq \frac{z_{1P} - z_R}{\rho - \| \lambda_A \|}.
\]

Proof. If for any \( u \in U_\rho \), there exists \( \exists x \in X \cap H_u \) such that \( f(x) + \rho \| Ax - b \| \leq z_{1P} \), the result follows from Proposition 4.2. Alternatively, if \( f(x) + \rho \| Ax - b \| > z_{1P} \forall x \in X \cap H_u \) for some \( u \in U_\rho \).

\[
\phi(u) + \rho(\| u \|) = \inf_{x \cap H_u} f(x) + \rho \| Ax - b \| \geq z_{1P}
\]

If \( \inf_{x \cap H_u} f(x) + \rho \| Ax - b \| > z_{1P} \) then \( u \not\in U_\rho \) leading to a contradiction. If \( \inf_{x \cap H_u} f(x) + \rho \| Ax - b \| = z_{1P} \) and there exists an \( x \in X \cap H_u \) such that \( f(x) + \rho \| Ax - b \| = z_{1P} \) then the result follows from Proposition 4.2. If there doesn’t exist \( x \in X \cap H_u \) such that \( f(x) + \rho \| Ax - b \| = z_{1P} \) then one can find a sequence of \( x_\rho \) such that the follows holds

\[
f(x_\rho) + \rho \| Ax_\rho - b \| \leq z_{1P} + \frac{1}{\rho} \quad \text{(From definition of infimum)}
\]

Let \( x_R \) be the optimal solution of the optimization program \( \inf_{x \in F} f(x) \). Since we are minimizing a convex function over a rational polyhedron the KKT conditions hold as follows.

\[
\nabla f(x_R) = A^T\lambda_A - E^T\lambda_E
\]

\[
\lambda_E(f - Ex_R) = 0
\]

\[
\lambda_A(A x_R - b) = 0
\]

\[
\lambda_E \geq 0
\]

\[
f(x_R) + \nabla f(x_R)^T(x_R - x_R) + \rho \| Ax_R - b \| \leq z_{1P} + \frac{1}{\rho} \quad \text{(Convexity)}
\]

\[
z_R + \lambda_A^T(A x_R - b) + \lambda_E^T(f - Ex_R) + \rho \| Ax_R - b \| \leq z_{1P} + \frac{1}{\rho}
\]

\[
\| u \| \leq \frac{z_{1P} - z_R}{\rho - \| \lambda_A \|} + \frac{1}{p(\rho - \| \lambda_A \|)}
\]

where the last inequality follows from non-negativity of \( \lambda_E \) and \( (f - Ex_R) \). As \( p \) is arbitrary, we have as \( p \to \infty \),

\[
\| u \| \leq \frac{z_{1P} - z_R}{\rho - \| \lambda_A \|} \quad \square
\]

Corollary 4.4. Consider the set \( \overline{U} := \{ u \in U : \phi(u) \geq \phi(0) \} \). For \( \alpha \geq 0 \), \( \phi(0) \leq \phi(u) + \alpha \forall u \in U \) if \( \phi(0) \leq \phi(u) + \alpha \forall u \in \overline{U} \).

Proof. For \( u \in U \setminus \overline{U} \), \( \phi(0) < \phi(u) \leq \phi(u) + \alpha \). The result follows. \( \square \)

Lemma 4.5. If there exist \( \delta, \kappa > 0 \) such that \( \phi(0) \leq \phi(u) + \kappa \| u \| \) for all \( u \in \mathcal{N}_\delta(0) \cap U \), then there exists \( 0 < \rho^* < \infty \) such that \( z_{SALR}(\rho) = z_{1P} \) for all \( \rho > \rho^* \).
\[ z_{SALR}(\rho) = \min_{u \in U} \phi(u) + \rho \|u\|. \]

It follows that,

\[ z_{SALR}(\rho) = \min_{u \in U_{\rho}} \phi(u) + \rho \|u\|. \]

Proposition 4.2 yields that for \( \rho > \|\lambda_A\| \) and \( u \in U_{\rho} \), \( \|u\| \) is bounded. Thus, we have

\[ z_{SALR}(\rho) = \min_{u \in \mathcal{N}_\delta(0) \cap U_{\rho}} \phi(u) + \rho \|u\| \]

where \( \delta = \frac{z_{IP} - z_R}{\rho - \|\lambda_A\|} \). From Proposition 4.2 we see that as \( \rho \to \infty \), \( \mathcal{N}_\delta(0) \cap U_{\rho} \to \{0\} \). If \( \exists\delta > 0 \), and \( 0 < \kappa < \infty \) such that \( \phi(0) \leq \phi(u) + \kappa \|u\| \) for all \( u \in \mathcal{N}_\delta(0) \cap U_{\rho} \), it follows that,

\[ \min_{u \in \mathcal{N}_\delta(0) \cap U_{\rho}} \phi(u) + \kappa \|u\| = \phi(0) = z_{IP} \]

Alternatively, we can increase \( \rho \) such that \( \forall u \in U_{\rho} \), \( \|u\| \leq \frac{z_{IP} - z_R}{\rho - \|\lambda_A\|} < \delta \). Rearranging the terms, we obtain \( \rho > \|\lambda_A\| + \frac{z_{IP} - z_R}{\delta} \). It follows from (4.7) that for any choice of \( \rho > \max \left\{ \kappa, \|\lambda_A\| + \frac{z_{IP} - z_R}{\delta} \right\} \), we have

\[ z_{SALR}(\rho) = \min_{u \in \mathcal{N}_\delta(0) \cap U_{\rho}} \phi(u) + \rho \|u\| = z_{IP}. \]

4.2. Proof of Theorem 3.1.

**Proposition 4.6.** Consider the value function \( \phi(u) \) of a mixed integer linear programming problem,

\[ \phi(u) = \text{minimize} \quad c_I^T x_I + c_C^T x_C \]

subject to

\[ A_I x_I + A_C x_C = b + u \]

\[ x_I \in \mathbb{Z}_{+}^{n_I} \]

\[ x_C \in \mathbb{R}_{+}^{n_C} \]

There exists a \( \delta > 0 \) such that for every \( u \in \mathcal{N}_\delta(0) \cap U \), \( \phi(0) \leq \phi(u) + \Gamma \|u\| \) where \( \Gamma \) is a constant which depends on \( A_C \) and \( c_C \) and \( c_I \).

**Proof.** From Corollary 4.4, it suffices to show the existence of a \( \delta > 0 \) such that for every \( u \in \mathcal{N}_\delta(0) \cap \overline{U} \), \( \phi(0) \leq \phi(u) + \Gamma \|u\| \). Observe that, if there doesn’t exist a limiting sequence to 0 in \( \overline{U} \) then there exists a \( \delta > 0 \) such that \( \overline{U} \cap \mathcal{N}_\delta(0) = \{0\} \). Hence \( \phi(0) \leq \phi(u) \) for all \( u \in \mathcal{N}_\delta(0) \). The result follows in this case.

Conversely, if there does exist a limiting sequence to 0 in \( \overline{U} \) then given \( \phi(u) \leq \phi(0) \), \( \forall u \in \overline{U} \) implies that \( \limsup u \to 0 \phi(u) \leq \phi(0) \).

Since \( \phi(u) \) is lower semi-continuous[15], we have \( \liminf u \to 0 \phi(u) \geq \phi(0) \). It follows that \( \lim u \to 0 \phi(u) = \phi(0) \).
Further define, where \( \text{lcm} \) stands for least common multiple. Observe that the quantities \( \epsilon_\kappa \) highlighted earlier, is an integer. If this term is non-zero then

\[
\left| \phi(u) - \phi(0) \right| < \epsilon \quad \forall \ u \in \mathcal{N}_\delta(0) \cap \overline{U}
\]

Substituting for \( \phi(u) \) from (4.8),

\[
\left| c_I^u x_I^u + c_B u A_B^{-1}(b + u - A_I x_I^u) - \phi(0) \right| < \epsilon
\]

(4.9)

\[
\left| c_I^u x_I^u + c_B u A_B^{-1}(b - A_I x_I^u) - \phi(0) + c_B u A_B^{-1} u \right| < \epsilon
\]

Since \( A \) and \( b \) are rational, one can assume without loss of generality that \( A \) and \( b \) are integral. This implies that \( A_B \) is integral. Consequently, \( \det(A_B) \) and \( \text{Adj}(A_B) \) are integral as well, where \( \det(M) \) and \( \text{Adj}(M) \) denote the determinant and adjugate matrix of \( M \), respectively. Consider the function \( \text{denom} : \mathbb{Q} \to \mathbb{Z} \),

\[
\text{denom}(r) = \begin{cases} |q| & r = \frac{p}{q} \text{ such that } \gcd(p, q) = 1, r \neq 0, \\ 1 & r = 0 \end{cases}
\]

Further define, \( Q := \{ \text{denom}(c_i) \}_{i=1}^n \) where \( c_i \in (c_I, c_C), B \) as the set of all possible bases of \( A_C \) such that \( A_B, B \in B \) is invertible and

\[
\kappa = \text{lcm}(\text{lcm}(Q), |\text{lcm}_{B \in B}(\det(B))|, \text{denom}(\phi(0))),
\]

where \( \text{lcm} \) stands for least common multiple. Observe that the quantities \( \kappa^2 c_B u A_B^{-1}, \kappa^2 \phi(0), \) and \( \kappa^2 c_I \) are all integers. Therefore multiplying (4.9) with \( \kappa^2 \) we obtain

\[
|\kappa^2 c_I^u x_I^u + \kappa^2 c_B u A_B^{-1}(b - A_I x_I^u) - \kappa^2 \phi(0) + \kappa^2 c_B u A_B^{-1} u| < \kappa^2 \epsilon
\]

Let \( \tilde{\epsilon} = \kappa^2 \epsilon \).

\[
\tilde{\epsilon} > |\kappa^2 c_I^u x_I^u + \kappa^2 c_B u A_B^{-1}(b - A_I x_I^u) - \kappa^2 \phi(0) + \kappa^2 c_B u A_B^{-1} u| \\
\geq |\kappa^2 c_I^u x_I^u + \kappa^2 c_B u A_B^{-1}(b - A_I x_I^u) - \kappa^2 \phi(0) - |\kappa^2 c_B u A_B^{-1} u||
\]

(4.10)

There are two terms in (4.10) only one of which involves \( u \), which can be upper bounded as,

\[
|\kappa^2 c_B u A_B^{-1} u| \leq \kappa^2 \max_{B \in B} \| A_B^{-1} \| \| u \| < \kappa^2 \max_{B} \| c_C \| \| A_B^{-1} \| \| u \|.
\]

Letting \( \beta := \max_{B \in B} \| A_B^{-1} \| \) and \( K := \max(\kappa^2 \beta \| c_C \|, 1) \) yields,

\[
|\kappa^2 c_B u A_B^{-1} u| < K \delta
\]

Now the remaining term from (4.10), \( \kappa^2 c_I^u x_I^u + \kappa^2 c_B u A_B^{-1}(b - A_I x_I^u) - \kappa^2 \phi(0), \) as highlighted earlier, is an integer. If this term is non-zero then

\[
|\kappa^2 c_I^u x_I^u + \kappa^2 c_B u A_B^{-1}(b - A_I x_I^u) - \kappa^2 \phi(0)| \geq 1.
\]
Letting \( \delta = \min(\bar{\epsilon}, \delta) \) and \( \bar{\epsilon} < \frac{1}{2K} \), we have \( \delta < \frac{1}{2K} \) implying that \( K \delta < 1/2 \). Now, we have

\[
(4.12) \quad |\kappa^2 c_{B_u} A_{B_u} u| < K \delta < 1/2
\]

Combining the inequalities (4.11) and (4.12) yields,

\[
(4.13) \quad \left|\kappa^2 c_j x_j^u + \kappa^2 c_{B_u} A_{B_u}^{-1} (b - A_j x_j^u) - \kappa^2 \phi(0)\right| - |\kappa^2 c_{B_u} A_{B_u} u| > 1/2
\]

If \( |\kappa^2 c_j x_j^u + \kappa^2 c_{B_u} A_{B_u}^{-1} (b - A_j x_j^u)| \) is non-zero for any \( u \in \mathcal{N}_s(0) \cap U \) then from (4.10) and (4.13) one has

\[
1/2 < |\kappa^2 c_j x_j^u + \kappa^2 c_{B_u} A_{B_u}^{-1} (b - A_j x_j^u) - \kappa^2 \phi(0) - |\kappa^2 c_{B_u} A_{B_u} u| |
\]

\[
\leq |\kappa^2 c_j x_j^u + \kappa^2 c_{B_u} A_{B_u}^{-1} (b - A_j x_j^u) - \kappa^2 \phi(0)| - \kappa^2 c_{B_u} A_{B_u} u| |
\]

\[
< 1/(2K)
\]

which presents a contradiction, since \( K \geq 1 \).

This implies that for every \( u \in \mathcal{N}_s(0) \cap U \) where \( 0 < \delta < 1/(2K) \) we have

\[
k^2 c_j x_j^u + \kappa^2 c_{B_u} A_{B_u}^{-1} (b - A_j x_j^u) - \kappa^2 \phi(0) = 0.
\]

It follows from (4.8) that \( \exists \delta > 0 \) such that \( \forall u \in \mathcal{N}_s(0) \cap U \) we have \( \phi(u) - \phi(0) = c_{B_u} A_{B_u}^{-1} u \). It follows,

\[
|\phi(u) - \phi(0)| = |c_{B_u} A_{B_u}^{-1} u| \leq |c_{B_u}| \cdot |A_{B_u}^{-1}| \cdot |u| \leq \beta ||c_C|| ||u||
\]

As a result,

\[
|\phi(u) - \phi(0)| \leq \Gamma ||u|| \quad \text{where} \quad \Gamma = \beta ||c_C||
\]

**Corollary 4.7.** Consider the value function \( \phi(u) \) of a mixed integer linear programming problem,

\[
\phi(u) = \text{minimize} \quad c_j^T x_j + c_C^T x_C
\]

subject to

\[
A_j x_j + A_C x_C = b + u \]

\[
E_j x_j + E_C x_C \leq f \]

\[
x_j \in \mathbb{Z}^{n_1} \]

\[
x_C \in \mathbb{R}^{n_2}
\]

There exists a \( \delta > 0 \) such that for every \( u \in \mathcal{N}_s(0) \cap U \), we have \( \phi(0) \leq \phi(u) + \Gamma ||u|| \)

where \( \Gamma \) is a constant which depends on \( A_C \) and \( c_C \) and \( c_j \).

**Proof.** Without loss of generality, we can represent the given mixed integer linear program in the following form,

\[
\phi(u') = \text{minimize} \quad c_j'^T x_j' + c_C'^T x_C'
\]

subject to

\[
A_j' x_j' + A_C' x_C' = b' + u' \]

\[
x_j' \in \mathbb{Z}^{n_1} \]

\[
x_C' \in \mathbb{R}^{n_2}
\]
where $c_i^T = (c_i, -c_i)$, $c_C^T = (c_C, -c_C, 0)$ and

$$A_i' = \begin{pmatrix} A_i & -A_i \\ E_i & -E_i \end{pmatrix}, A_C' = \begin{pmatrix} A_i & -A_C \\ E_i & -E_i \end{pmatrix}, x_I' = \begin{pmatrix} x_I \\ x_C \end{pmatrix}, x_C' = \begin{pmatrix} x_C^- \\ x_C^+ \\ x_C \end{pmatrix},$$

$$b' = \begin{pmatrix} b \\ f \end{pmatrix}, u' = \begin{pmatrix} u \\ 0 \end{pmatrix}$$

and $\mathbb{0}, \mathbb{I}$ denote the matrix of all zeros and identity matrix of appropriate dimensions. The result follows from Proposition 4.6.

4.3. Proof of Theorem 3.2.

Proposition 4.8. Consider the value function $\phi(u)$ of a mixed integer quadratic programming problem,

$$\phi(u) = \minimize \quad \frac{1}{2} x_I^T Q_{II} x_I + \frac{1}{2} x_I^T Q_{IP} x_I + x_I^T Q_{IC} x_C - c_I^T x_I - c_C^T x_C$$

subject to

$$A_I x_I + A_C x_C = b + u$$

$$\|x_I\|_\infty \leq M$$

$$x_I \in \mathbb{Z}^{n_1}$$

$$x_C \in \mathbb{R}^{n_2+}$$

There exists a $\delta > 0$ such that for every $u \in N_\delta(0) \cap U$, we have $\phi(0) \leq \phi(u) + K_1 \|u\|^2 + K_2 \|u\|$ for some $K_1, K_2$ which depend on $A, Q, c$ and $M$.

Proof. As the set $S_I = \{x_I \in \mathbb{Z}^{n_1} : \|x_I\|_\infty \leq M\}$ is finite, let $S_I = \{x_I^{(i)} \}_{i \in [k]}$ for some natural number $k$. We now characterize the value functions of the continuous restrictions, parameterized in $x_I^{(i)}$, $i \in [k]$ as,

$$\Phi(u, x_I^{(i)}) = \minimize \quad \frac{1}{2} x_I^{(i)}^T Q_{II} x_I^{(i)} + \frac{1}{2} x_C^T Q_{CC} x_C + x_I^{(i)}^T Q_{IC} x_C - c_I^T x_I^{(i)} - c_C^T x_C$$

$$\text{(QP}_u^{(i)}\text{)}$$ subject to

$$A_C x_C = b + u - A_I x_I^{(i)}$$

$$x_C \in \mathbb{R}^{n_2+}$$

and $\phi(u) = \min_{1 \leq i \leq k} \Phi(u, x_I^{(i)})$. We can assume without loss of generality that $(\text{QP}_u^{(i)})$ is feasible for all $((u, x_I^{(i)})) \in U \times S_I$, since, if $\exists ((u, x_I^{(i)})) \in U \times S_I$, such that $(\text{QP}_u^{(i)})$ is infeasible, we can assign $\Phi(u, x_I^{(i)}) = \infty$. Observe that continuous quadratic programs are lower semi-continuous at $u = 0$ ([1], Proposition 6.5.2) and the minimum of a finite number of lower semi-continuous functions is lower semi-continuous as well. This implies that $\phi(u)$ is lower semi-continuous at $u = 0$ and consequently $\lim_{u \to 0} \phi(u) = \phi(0)$ where $u \in \overline{U}$. The first order necessary conditions
for optimality (KKT conditions) for \((QP^u(i))\) can be expressed as,

\[
\begin{bmatrix}
Q_{CC} & -A_C^T & A_C^T & -I \\
A_C & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x_C \\
\lambda_+ \\
\lambda_- \\
\pi
\end{bmatrix} =
\begin{bmatrix}
c_C - Q_{CI} x_I^{(i)} \\
b + u - A_I x_I^{(i)}
\end{bmatrix}
\tag{4.14}
\]

\[\pi^T x_C = 0\]
\[x_C, \lambda_+, \lambda_-, \pi \geq 0\]

If \((x_C, x_I^{(i)})\) is a solution to the KKT conditions then it satisfies the complementary slackness conditions. Let \(J = \{n_2\}\) and consider the partition of \(J\) for \(i \in [k]\), \(J(i) = \{J_1^{(i)}, J_2^{(i)}\}\) such that \(J_1^{(i)} = \{j \in J : \pi^{(i)} = 0\}\) and \(J_2^{(i)} = J \backslash J_1^{(i)}\). It follows that \(\forall j \in J_2^{(i)}, x_C^{(i)} = 0\). Define,

\[
A_{aug} = \begin{bmatrix}
Q_{CC} & -A_C^T & A_C^T & -I \\
A_C & 0 & 0 & 0
\end{bmatrix}
\]

and let the columns of \(A_{aug}\) be indexed by \([2n_2 + 2m]\). Define \(A_{aug}^{J(i)}\) as a sub-matrix \(A_{aug}\) which has all the columns of \(A_{aug}\) except the columns \(j \in J_2^{(i)} \cup ((n_2 + 2m) + J_1^{(i)})\). Consider the set of solutions to \((4.14)\),

\[
P_{J(i)} := \left\{ (x_C, \lambda_+, \lambda_-, \pi) \in \mathbb{R}^{n_2}_+ \times \mathbb{R}^m_+ \times \mathbb{R}^m_+ \times \mathbb{R}^{n_2}_+ : \right\}
\]

\[
A_{aug}^{J(i)} \begin{bmatrix}
x_C \\
\lambda_+ \\
\lambda_- \\
\pi
\end{bmatrix} = \begin{bmatrix}
c_C - Q_{CI} x_I^{(i)} \\
b + u - A_I x_I^{(i)}
\end{bmatrix}
\]

Observe that as the polyhedron \(P_{J(i)}\) doesn’t contain a line, it must have an extreme point. Consider an extreme point of \(P_J\) corresponding to the basis \(B_u^{(i)}\) of columns of \(A_{aug}^{J(i)}\). Let \(A_{B_u}^{(i)}\) be the sub-matrix formed by the columns corresponding to basis \(B_u^{(i)}\).

In any solution to \((4.14)\) we have,

\[
\Phi(u, x_I^{(i)}) = c_{u1}^T u + \frac{1}{2} u^T C_2^u Q_{CC} C_2^u u + \Theta(b, x_I^{(i)}, C_1^u, C_2^u)
\]

where \(C_1^u\) and \(C_2^u\) are submatrices of \(A_{B_u}^{(i)}\). Substituting \(x_C\) from \((4.15)\) in the objective function of \((QP^u(i))\), we obtain

\[
\Phi(u, x_I^{(i)}) = c_{u1}^T u + \frac{1}{2} u^T C_2^u Q_{CC} C_2^u u + \Theta(b, x_I^{(i)}, C_1^u, C_2^u)
\]

where,

\[
c_{u1} = 2 \zeta u^T Q_{CC} C_2^u + x_I^{(i)^T} Q_{CI} C_2^u + C_1^u C_2^u,
\]

\[
\Theta(b, x_I^{(i)}, C_1^u, C_2^u) = \frac{1}{2} x_I^{(i)^T} Q_{II} x_I^{(i)} + u^T C_1^u u + \frac{1}{2} \zeta u^T Q_{CC} C_2^u + x_I^{(i)^T} Q_{CI} C_2^u, \quad \text{and} \]

\[
\zeta u = \left( C_1^u \left( c_C - Q_{CI} x_I^{(i)} \right) + C_2^u \left( b - A_I x_I^{(i)} \right) \right)
\]
Now, for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|\phi(u) - \phi(0)| < \epsilon \quad \forall \mathcal{N}_\delta(0) \cap \mathcal{U}$$

It follows that $\forall \|u\| < \delta$, we have

$$\left|c_{u_1}^\top u + \frac{1}{2} u^\top C_2 u + \Theta(b, x_{i}^{(i)}, C_1, C_2) - \phi(0)\right| < \epsilon, \quad \text{i.e.}$$

$$\left|\Theta(b, x_{i}^{(i)}, C_1, C_2) - \phi(0)\right| - \left|c_{u_1}^\top u + \frac{1}{2} u^\top C_2 u + \Theta(b, x_{i}^{(i)}, C_1, C_2) - \phi(0)\right| < \epsilon$$

Let $\mathbf{A}_B^* = \arg \max \{||\mathbf{A}_B^{-1}|| : \mathbf{B} \in \mathcal{B}\}$ where $\mathcal{B}$ is the set of all possible bases of $\mathbf{A}_{aug}$. Since $\left|\mathbf{A}_B^{(i)-1}\right|_F \leq \left|\mathbf{A}_B^{-1}\right|_F$ and $C_1 u$ and $C_2 u$ are submatrices of $\mathbf{A}_B^{(i)-1}$ it follows that $\left|C_{1u}\right|_F \leq \left|\mathbf{A}_B^{(i)-1}\right|_F$ and $\left|C_{2u}\right|_F \leq \left|\mathbf{A}_B^{(i)-1}\right|_F$. We can bound the second term in inequality (4.17) as

$$\left|c_{u_1}^\top u + \frac{1}{2} u^\top C_2 u + \Theta(b, x_{i}^{(i)}, C_1, C_2) - \phi(0)\right| - \left|c_{u_1}^\top u + \frac{1}{2} u^\top C_2 u + \Theta(b, x_{i}^{(i)}, C_1, C_2) - \phi(0)\right| < \epsilon$$

where $K_1 = ||Q||_F \left|\mathbf{A}_B^{-1}\right|_F^2$, and

$$||c_{u_1}|| = \left|2c_{u_1}^\top Q_{CC} C_{2u} + x_{i}^{(i)} Q_{CI} C_{2u} + C_{1u}^\top C_{2u}\right|$$

$$\leq 2 \left(||C_{1u}||_F \left|c_{C} - Q_{CI} x_{i}^{(i)}\right| + ||C_{2u}||_F \left|b - A_{I} x_{i}^{(i)}\right|\right) \left(||Q_{CC}||_F \left|C_{2u}\right|_F\right)$$

$$+ ||x_{i}^{(i)}|| \left|Q_{CI}\right|_F \left|C_{2u}\right|_F + ||C_{C}|| \left|C_{2u}\right|_F$$

$$\leq 2 \left(||c|| + M \left|Q\right|_F + ||b|| + M \left|A_{I}\right|_F\right) \left|Q\right|_F \left|\mathbf{A}_B^{-1}\right|_F^2$$

$$+ M \left|Q\right|_F \left|\mathbf{A}_B^{-1}\right|_F^2 + ||c|| \left|\mathbf{A}_B^{-1}\right|_F = K_2.$$
This yields a contradiction. Hence $\psi = 0$, which further yields $\Theta(b, x_i(t), C_i^u, C_{2i}^u) = \phi(0)$. Substituting in (4.16), it follows from (4.18) that
\[ \phi(0) \leq \phi(u) + K_1||u||^2 + K_2||u|| \]
\[ \square \]

**Corollary 4.9.** Consider the value function $\phi(u)$ of a mixed integer quadratic programming problem,
\[
\phi(u) = \min \left\{ \frac{1}{2} x_C^\top Q_{CC} x_C + \frac{1}{2} x_i^\top Q_{II} x_I + x_i^\top Q_{IC} x_C - c_i^\top x_I - c_C^\top x_C \right\}
\text{subject to}
A_I x_I + A_C x_C = b + u
E_I x_I + E_C x_C \leq f
||x_I||_\infty \leq M
x_I \in \mathbb{Z}^{n_1}
x_C \in \mathbb{R}^{n_2}
\]
There exists a $\delta > 0$ such that $\forall u \in N_\delta(0) \cap U$, $\phi(0) \leq \phi(u) + K_1||u||^2 + K_2||u||$ for some $K_1, K_2$ which depend on $A, Q, c$ and $M$.

**Proof.** Without loss of generality, we can represent the given mixed integer quadratic program in the following form,
\[
\min \left\{ \frac{1}{2} x_C^\top Q'_{CC} x_C + \frac{1}{2} x_i^\top Q_{II} x_I + x_i^\top Q'_{IC} x_C - c_i^\top x_I - c_C^\top x_C' \right\}
\text{subject to}
A_I x_I + A'C_C x_C = b' + u'
||x_I||_\infty \leq M
x_I \in \mathbb{Z}^{n_1}
x_C \in \mathbb{R}^{n_2}
\]
where
\[
Q'_{CC} = \begin{pmatrix} Q_{CC} & -Q_{CC} & 0 \\ -Q_{CC} & Q_{CC} & 0 \\ 0 & 0 & 0 \end{pmatrix},
Q'_{IC} = \begin{pmatrix} Q_{IC} & -Q_{IC} & 0 \end{pmatrix},
\]
\[
A'_C = \begin{pmatrix} A_I & -A_C & \emptyset \\ E_I & -E_I & \mathbb{I} \end{pmatrix},
x'_C = \begin{pmatrix} x_C^+ \\ x_C^- \\ s \end{pmatrix},
c'_C = \begin{pmatrix} c_C \\ -c_C \\ 0 \end{pmatrix},
b' = \begin{pmatrix} b \\ f \end{pmatrix},
u' = \begin{pmatrix} u \\ 0 \end{pmatrix}
\]
and $\emptyset, \mathbb{I}$ denote the matrix of all zeros and identity matrix of appropriate dimensions. The result follows from Proposition 4.8. \[ \square \]

**Theorem 3.2** follows immediately.

**4.4. Proof of Theorem 3.3.** Consider the following mixed integer convex programming problem.
\[
\hat{\phi}(u) = \min \left\{ f(x_I, x_C) \right\}
\text{(MICP)}_u
\text{subject to}
A_I x_I + A_C x_C = b + u
E_I x_I + E_C x_C \leq f
||x_I||_\infty \leq M
(x_I, x_C) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}
\]
where \(0 \leq M < \infty\). Denote by \(F_u := X \cap H_u \cap S_I\) the feasible region of \((MICP)_u\), where the set \(S_I = \{x_I \in \mathbb{Z}^{n_1} : ||x_I||_\infty \leq M\}\) is finite. As before, we let \(S_I = \{x_I^{(1)}, x_I^{(2)}, \ldots, x_I^{(k)}\}\). Using this enumeration we characterize the value functions of the continuous restrictions, parameterized in \(x_I^{(i)}\), \(i \in [k]\) as,
\[
\Phi(u, x_I^{(i)}) = \text{minimize } f(x_I^{(i)}, x_C) \quad \text{subject to } \quad A_C x_C = b + u - A_I x_I^{(i)} \\
E_C x_C \leq f - E_I x_I^{(i)} \\
x_C \in \mathbb{R}^{n_2}
\]
where \(\Phi(u, x_I^{(i)})\) is the value function of \((CP_{(i)}^u)\), a continuous convex optimization problem. Since the constraints of \((CP_{(i)}^u)\) are linear, Slater’s condition holds and therefore the value function is lower semi-continuous.

**Lemma 4.10.** If the integer variables are bounded, then the value function \(\hat{\phi}(u) = \min_{i \in [k]} \Phi(u, x_I^{(i)})\) is lower semi-continuous at 0.

**Proof.** Observe that for \(u \in U, x_I^{(i)} \in S_I\), \(\Phi(u, x_I^{(i)})\) is lower semi-continuous at \(u\) if and only if strong duality holds at \(u\) [Proposition 6.5.2,1]. If for some \(x_I^{(i)}\), \(\Phi(0, x_I^{(i)}) < +\infty\) then \(\Phi(u, x_I^{(i)})\) is lower semi-continuous at \(u = 0\) due to Slater Condition (in case of affine constraints only feasibility is required). On the contrary, \(\Phi(0, x_I^{(i)}) = +\infty\) implies that the polyhedron \(P = \{x \in \mathbb{R}^{n_2} : \tilde{A}x = \tilde{b}, x \geq 0\}\) is empty, where
\[
\tilde{A} = \begin{pmatrix} A_C & -A_C & 0 \\ E_C & -E_C & I \end{pmatrix}, \quad x = \begin{pmatrix} x_C^+ \\ x_C^- \\ s \end{pmatrix}, \quad \tilde{b} = \begin{pmatrix} b - A_I x_I^{(i)} \\ f - E_I x_I^{(i)} \end{pmatrix}.
\]
Let \(C\) be the closed convex cone spanned by the columns of \(\tilde{A}\). If \(P = \emptyset\) then \(\tilde{b} \not\in C\). Since \(C\) is a closed, \(3 > \delta_i > 0\) such that \(N_{\bar{\delta}_i}(\tilde{b}) \cap C = \emptyset\). Consider \(u \in \mathbb{R}^{n_1} \cap N_{\delta_i}(0)\), then \(\tilde{b} + \begin{pmatrix} u \\ 0 \end{pmatrix} \not\in C\).

For \(J = \{i \in [k] : \Phi(x_I^{(i)}, 0) < \infty\}\), define \(\delta_{\min} := \min_{i \in [k] \setminus J} \delta_i\).

Consider \(y\) such that \(y < \hat{\phi}(0)\), it follows that
\[
y < \hat{\phi}(0) < \min_{i \in J} \Phi(0, x_I^{(i)}) \leq \Phi(0, x_I^{(i)}).
\]
However, since every \(\Phi(u, x_I^{(i)}), i \in J\) is lower semi-continuous at 0, there exists a \(\delta_i > 0\) such that \(\Phi(u, x_I^{(i)}) > y\) for every \(u \in N_{\delta_i}(0), i \in J\). Let \(\delta_{\min} = \min(\delta_i), i \in J\) and \(\delta'' = \min(\delta_{\min}, \delta''_{\min})\). This means that for every \(u \in N_{\delta''}(0)\), \(\Phi(u, x_I^{(i)}) > y \implies \hat{\phi}(u) > y\). The result follows.

Consider \(\overline{U}\) as defined in Corollary 4.4, it follows from Lemma 4.10 that \(\lim_{u \to 0} \hat{\phi}(u) = \hat{\phi}(0)\) where \(u \in \overline{U}\). Now, for every \(\epsilon > 0\) there exists a \(\delta > 0\) such that
\[
|\hat{\phi}(u) - \hat{\phi}(0)| < \epsilon \ \forall u \in N_{\delta}(0) \cap \overline{U}.
\]
Lemma 4.11. Consider the value function \( \hat{\phi}(u) \) as defined earlier for (MICP)\(_u\). There exists a \( \delta > 0 \) such that for all \( u \in N_\delta(0) \cap U \), \( \hat{\phi}(u) = \min_{x_I \in S^u_\delta} \Phi(u, x_I) \) where

\[
S^u_\delta = \left\{ x_I \in S_I : \Phi(0, x_I) = \hat{\phi}(0) \right\}.
\]

Proof. We can partition the index sets corresponding to solutions of (CP\(_i^u\)) into two sets,

\[
S^-_\delta = \left\{ x_I \in S_I : \Phi(0, x_I) > \phi(0) \right\} \quad \text{and} \quad S^u_\delta = \left\{ x_I \in S_I : \Phi(0, x_I) = \phi(0) \right\}.
\]

For each \( x_I^{(i)} \in S^u_\delta \), define \( \overline{U}_i = \left\{ u \in U : \Phi(u, x_I^{(i)}) \leq \Phi(0, x_I^{(i)}) \right\} \). If there is no limiting sequence to \( U \) in \( \overline{U}_i \) then there exists a neighbourhood \( \delta \) such that \( N_\delta(0) \cap \overline{U}_i = \emptyset \). Alternatively, if there does exist a limiting sequence to \( 0 \) and since \( \Phi(u, x_I^{(i)}) \) is lower semi-continuous,

\[
\lim_{u \to 0} \Phi(u, x_I^{(i)}) = \Phi(0, x_I^{(i)})
\]

Thus, for every \( \epsilon_i > 0 \) there exists a \( \delta_i > 0 \) such that

\[
\left| \Phi(u, x_I^{(i)}) - \Phi(0, x_I^{(i)}) \right| < \epsilon_i \quad \forall \ u \in \overline{U}_i \cap N_{\delta_i}(0)
\]

For \( x_I^{(i)} \in S^u_\delta \), let \( \epsilon_i = \Phi(0, x_I^{(i)}) - \phi(0) \). Assuming that \( \Phi(0, x_I^{(i)}) \) is finite, \( \forall \ u \in \overline{U}_i \cap N_{\delta_i}(0) \)

\[
\Phi(0, x_I^{(i)}) - \Phi(u, x_I^{(i)}) < \Phi(0, x_I^{(i)}) - \phi(0)
\]

\[
\phi(0) < \Phi(u, x_I^{(i)})
\]

Thus, for all \( x_I^{(i)} \in S^u_\delta \) there exists a \( \delta_i \) such that \( \Phi(u, x_I^{(i)}) > \phi(0) \). Assigning \( \delta_{\min} = \min_{x_I^{(i)} \in S^u_\delta} \delta_i \) yields \( \hat{\phi}(u) = \min_{x_I \in S^u_\delta} \Phi(u, x_I) \) for every \( u \in N_{\delta_{\min}}(0) \cap U \). \( \square \)

Lemma 4.12. Consider the value function \( \hat{\phi}(u) \) as defined earlier for (MICP)\(_u\). There exists a \( \delta > 0 \) and \( 0 < \Gamma < \infty \) such that for every \( u \in N_\delta(0) \cap U \),

\[
\hat{\phi}(0) \leq \hat{\phi}(u) + \Gamma ||u||.
\]

Proof. Consider \( S^u_\delta \) as defined in Lemma 4.11. Analogously, define

\[
S^-_\delta := \left\{ x_I \in S^-_\delta : \Phi(u, x_I) = \phi(0) \right\}.
\]

For \( x_I^{(i)} \in S^-_\delta \), the first order necessary conditions for optimality (KKT conditions) for (CP\(_i^u\)) can be expressed as,

\[
\nabla_C f(x_I^{(i)}, x_C) = A_C^T \lambda_C^{u(i)} - E_C^T \lambda_{E_C}^{u(i)}
\]

\[
A_C x_C = b + u - A_I x_I^{(i)}
\]

\[
E_C x_C + E_I x_I^{(i)} \leq f
\]

\[
\lambda_{E_C}^{u(i)} (f - E_C x_C - E_I x_I^{(i)}) = 0
\]

\[
\lambda^{u(i)}_A (A_C x_C + A_I x_I^{(i)} - b - u) = 0
\]

\[
\lambda^{u(i)}_E \geq 0
\]
Define \( H_u^{(j)} = \{ x_C \in \mathbb{R}^{n_2} : A_C x_C = b + u - A I x_I^{(j)} \} \). Let \((x_I^{(j)}, x_C) \in X \cap H_u^{(j)}\) where \(x_I^{(j)} \in S^n\). Additionally, let the corresponding Lagrange multipliers at \(u = 0\) for \(x_I^{(j)}\) be \(\lambda_A^{(j)}\) and \(\lambda_{E_C}^{(j)}\).

\[
f(x_I^{(j)}, x_C^{0^*}) & \geq f(x_I^{(j)}, x_C^{0'}) + \nabla_C f(x_I^{(j)}, x_C^{0'})^\top (x_C^{0'_u} - x_C^{0'}) \\
& + \nabla_I f(x_I^{(j)}, x_C^{0'})^\top (x_I^{j'} - x_I^{(j)}) \\
& = f(x_I^{(j)}, x_C^{0'}) + (\lambda_A^{(j)} A_C - \lambda_{E_C}^{(j)} E_C)(x_C^{0'_u} - x_C^{0'}) \\
& = f(x_I^{(j)}, x_C^{0'}) + \lambda_A^{(j)} (A_C x_C^{0'_u} - A_C x_C^{0'}) - \lambda_{E_C}^{(j)} (E_C x_C^{0'_u} - E_C x_C^{0'}) \\
& = f(x_I^{(j)}, x_C^{0'}) + \lambda_A^{(j)} (b + u - A_I x_I^{(j)} - b + A_I x_I^{(j)}) \\
& - \lambda_{E_C}^{(j)} (E_C x_C^{0'_u} - f + E_I x_I^{(j)})
\]

where the equalities follow from the set of equations (4.20). Furthermore, observe that since \(\lambda_{E_C}^{(j)} \geq 0\) and \(E_C x_C^{0'_u} + E x_I^{(j)} \leq f\), it follows that

\[
\inf_{X \cap H_u^{(j)}} f(x_I^{(j)}, x_C^{0'_u}) \geq f(x_I^{(j)}, x_C^{0'}) + \lambda_A^{(j)} (u), \quad \text{i.e.,} \quad \phi(u) \geq \hat{\phi}(u) + \lambda_A^{(j)} (u)
\]

Rearranging the terms in the above inequality, \(-\lambda_A^{(j)} u \geq \hat{\phi}(0) - \hat{\phi}(u)\). As \(\hat{\phi}(0) \geq \hat{\phi}(u) \forall u \in \mathcal{U}\), using Cauchy-Schwarz on the left-hand side of the inequality,

\[
\left\| \lambda_A^{(j)} \right\| \cdot \|u\| \geq \hat{\phi}(0) - \hat{\phi}(u).
\]

Defining \(\Gamma = \max_{u \in \mathcal{U}} \left\| \lambda_A^{(j)} \right\| \) yields

\[
\phi(0) - \hat{\phi}(u) \leq \Gamma \|u\| \forall u \in \mathcal{N}_u(0) \cap \mathcal{U}
\]

where \(\delta = \delta_{\text{min}}\) as defined in the proof of Lemma 4.11. The result follows.

This completes the proof of Theorem 3.3(a). In the following section we present some cases where an equivalence can be established between (MICP) and (MICP) with bounded integer variables. In particular, we highlight that if the objective function \(f\) satisfies \(\text{rec}(f) \cap \text{rec}(F_R) \setminus \{0\} = \emptyset\), then there exists \(M < \infty\) such that the following equivalence holds.

\[
\min \{ f(x) : Ax = b, x \in X \} = \min \{ f(x) : Ax = b, ||x||_\infty \leq M, x \in X \}.
\]

Observe that if \(f\) is \(\mu\)-strongly convex, then \(\text{rec}(f) = \emptyset\), as the level sets of \(f\) are bounded.

**4.4.1. MICPs with implicit integer boundedness.**

**Proposition 4.13.** If \((\text{MICP})\) is feasible \((F \neq \emptyset)\) and bounded (optimal objective value is finite) and the recession cone of \(f\) and recession cone of \(F_R\), the feasible set of continuous relaxation of \((\text{MICP})\), have no common non-zero directions of recession, then the continuous relaxation of \((\text{MICP})\) is bounded.
Proof. Consider \((\text{MICP})_R\), the continuous relaxation of \((\text{MICP})\) and \(F_R\), the feasible set of \((\text{MICP})_R\). Let \(z_R = \min \{ f(x) : x \in F_R \}\) be the optimal objective value of \((\text{MICP})_R\) (If \((\text{MICP})_R\) is unbounded then \(z_R = -\infty\)). Since \(z_R \leq z_{IP}\) and \(-\infty < z_{IP} < \infty\), hence the set

\[
\tilde{F} = \{ x \in \mathbb{R}^n : f(x) \leq z_{IP} \} \neq \emptyset \text{ and } F' = \tilde{F} \cap F_R \neq \emptyset.
\]

The recession cone of \(F'\) can be represented as,

\[
\text{rec}(f) \cap \text{rec}(F_R)
\]

Furthermore, since the recession cone of \(f\) and recession cone of \(F_R\) have no common non-zero directions of recession, \(\text{rec}(f) \cap \text{rec}(F_R) = \{0\}\). This yields that \(F'\) is compact. Since \(F'\) is compact and non-empty, and \(f\) is continuous, \(f\) attains a minimum over \(F'\). The result follows. \(\square\)

Lemma 4.14. If \(f\) is \(\mu\)-strongly convex and \((\text{MICP})\) is feasible and bounded then the continuous relaxation of \((\text{MICP})\) is bounded.

Proof. Let \(F \neq \emptyset\) be the feasible set of \((\text{MICP})\). Since \(f\) is \(\mu\)-strongly convex, it follows that for \(x \in \mathbb{R}^n\)

\[
f(x) \geq \frac{1}{2} \mu \| x - \alpha \|^2 + \nabla f(\alpha)^\top (x - \alpha) + f(\alpha) \quad \text{for some } \alpha \in \mathbb{R}^n
\]

Consider the set \(F'' = \{ x \in \mathbb{R}^n : z_{IP} \geq f(x) \}\). It follows from (4.21), for \(x \in F'\),

\[
z_{IP} \geq \frac{1}{2} \mu \| x - \alpha \|^2 + \nabla f(\alpha)^\top (x - \alpha) + f(\alpha) \quad \text{for some } \alpha \in \mathbb{R}^n
\]

Consider the set \(F'' = \left\{ x \in \mathbb{R}^n : z_{IP} \geq \frac{1}{2} \mu \| x - \alpha \|^2 + \nabla f(\alpha)^\top (x - \alpha) + f(\alpha) \right\}\). It follows that \(F' \subseteq F''\). The recession cone of \(F''\) is,

\[
\text{rec}(F'') := \left\{ x \in \mathbb{R}^n : \exists x = 0, (\nabla f(\alpha) - \mu \alpha)^\top x \leq 0 \right\}
\]

where \(I\) is the \(n \times n\) identity matrix \([1]\). Indeed \(\text{rec}(F'') = \{0\}\). This implies that \(F''\) is compact and consequently \(F'\) is compact. The result follows. \(\square\)

Lemma 4.15. If \(\text{rec}(f)\), recession cone of \(f\) and the recession cone of \(F_R\), the feasible set of continuous relaxation of \((\text{MICP})\), have no common non-trivial directions of recession, then for all \(0 < \rho < \infty\), the set

\[
S := \left\{ x \in \mathbb{R}^n : f(x) + \lambda_A^\top (b - Ax) + \rho \psi(b - Ax) \leq z_{IP}, Ex \leq f \right\}
\]

is compact, where \(\psi\) is a level-bounding function.

Proof. Consider the Lagrangian function for the continuous relaxation of MICP, i.e. for \(x \in \mathbb{R}^n\)

\[
\mathcal{L}(x, \lambda_A, \lambda_E) = f(x) - \lambda_A^\top (Ax - b) - \lambda_E^\top (f - Ex)
\]

As strong duality holds, the first order necessary (KKT) conditions for \(\mathcal{L}(x, \lambda_A, \lambda_E)\) can be characterized as

\[
\nabla f(x) = A^\top \lambda_A - E^\top \lambda_E
\]

\[
\lambda_E^\top (f - Ex) = 0
\]

\[
\lambda_A^\top (Ax - b) = 0
\]

\[
\lambda_E \geq 0
\]
Observe that $x_R$, the solution to the continuous relaxation of (MICP) satisfies the system of equations (4.22).

Consider $x \in S$, i.e. $f(x) + \lambda_A^T(b - Ax) + \rho \psi(b - Ax) \leq z_{IP}$. Using convexity of $f$ we obtain,

$$f(x_R) + \nabla f(x_R)(x - x_R) + \lambda_A^T(b - Ax) + \rho \psi(b - Ax) \leq z_{IP}$$

(4.23)

$$z_R + \nabla f(x_R)(x - x_R) + \lambda_A^T(b - Ax) + \rho \psi(b - Ax) \leq z_{IP}$$

It follows from strong duality that,

$$\nabla f(x_R)(x - x_R) = \lambda_A^T A(x - x_R) - \lambda_E^T E(x - x_R)$$

$$\nabla f(x_R)(x - x_R) = \lambda_A^T (Ax - b) - \lambda_E^T (Ex - f)$$

Substituting in (4.23), we obtain

$$z_R + \lambda_A^T(Ax - b) + \lambda_E^T(f - Ex) + \lambda_A^T(b - Ax) + \rho \psi(b - Ax) \leq z_{IP}$$

(4.24)

$$\lambda_E^T(f - Ex) + \rho \psi(b - Ax) \leq z_{IP} - z_R$$

$$\rho \psi(b - Ax) \leq z_{IP} - z_R$$

where (4.24) follows from the fact that $\lambda_E \geq 0$ and $x \in S$. Since $\psi$ is a level-bounded function, there exists a positive $\kappa_\rho$ such that

$$||b - Ax||_{\infty} \leq \kappa_\rho < \infty \quad \forall \ x \in S.$$  

(4.25)

Additionally, since $f(x) + \lambda_A^T(b - Ax) + \rho \psi(b - Ax) \leq z_{IP}$, $\forall \ x \in S$, $\rho > 0$ and $\psi(u) > 0$ for $u \neq 0$

$$f(x) \leq z_{IP} - \lambda_A^T(b - Ax)$$

$$\leq z_{IP} + ||\lambda_A|| ||b - Ax||$$

$$\leq z_{IP} + \sqrt{n}||\lambda_A|| ||b - Ax||_{\infty}$$

$$\leq z_{IP} + \sqrt{n}||\lambda_A|| \kappa_\rho$$

(4.26)

Consider the set

$$S' = \{x \in \mathbb{R}^n : f(x) \leq z_{IP} + \sqrt{n}||\lambda_A|| \kappa_\rho, ||b - Ax||_{\infty} \leq \kappa_\rho, Ex \leq f\}.$$  

Inequalities (4.25) and (4.26) yield that $S \subseteq S'$. Observe that $S'$ can be expressed as,

$$S' = \left\{ x \in \mathbb{R}^n : \begin{cases} f(x) \leq z_{IP} + \sqrt{n}||\lambda_A|| \kappa_\rho \\ b - \kappa_\rho I \leq Ax \leq b + \kappa_\rho I \\ Ex \leq f \end{cases} \right\}$$

where $I$ is the vector of ones. We have the recession cone of $S'$

$$\text{rec}(S') = \left\{ x \in \mathbb{R}^n : \begin{cases} Ax = 0 \\ Ex \leq 0 \end{cases} \right\} \cap \text{rec}(f)$$

Alternatively,

$$\text{rec}(S') = \text{rec}(F_R) \cap \text{rec}(f)$$

Since $\text{rec}(F_R) \cap \text{rec}(f) = \{0\}$. It follows that $S'$ is a compact set, and consequently $S$ is a compact set.
Proposition 4.16. If $\text{rec}(f)$, recession cone of $f$ and the recession cone of $F_R$, the feasible set of continuous relaxation of (MICP), have no common non-trivial directions of recession, then $\lim_{\rho \to \infty} z^{LD+}_\rho = z_{1P}$.

Proof. Recall that the augmented Lagrangian relaxation of (MICP) is defined as

$$z^{LR+}_\rho(\lambda) = \min_{x \in X} f(x) + \lambda^\top (b - Ax) + \rho \psi(b - Ax)$$

From Lemma 4.15 we have

$$z^{LR+}_\rho = \min_{x \in X} \left\{ f(x) + \lambda^\top_A (b - Ax) + \rho \psi(b - Ax) : ||b - Ax||_\infty \leq \kappa_\rho \right\}$$

It follows that $\theta$ and let $\lambda$ be the consequence of $z^{LR+}_\rho$. Recall that the augmented Lagrangian relaxation of (MICP) is defined as

$$z^{LR+}_\rho(\lambda) = \min_{x \in X} f(x) + \lambda^\top_A (b - Ax) + \rho \psi(b - Ax)$$

Define,

$$\Phi(u, x_i^{(i)}) = \min_{|x|_\infty \leq M, (x_c, x_i^{(i)}) \in X} \left\{ f(x) + \lambda^\top_A (b - Ax) : Ax = b - A_j x_i^{(i)} \right\}$$

and for all $i \in [k]$,

$$\Theta(u, x_i^{(i)}) = \min_{|x|_\infty \leq M, (x_c, x_i^{(i)}) \in X} \left\{ f(x) + \lambda^\top_A (b - Ax) : A_c x_c = b + u - A_j x_i^{(i)} \right\}$$

and let $\phi'(u) = \min_{i \in [k]} \Theta(u, x_i^{(i)})$. Observe that, for $u \in U$ and $i \in [k]$,

$$\Theta(u, x_i^{(i)}) = \Phi'(u, x_i^{(i)}) + \lambda^\top_A u \quad \text{and} \quad \theta(u) = \min_{i \in [k]} \Theta(u, x_i^{(i)}).$$

It follows that $\theta(u) = \min_{i \in [k]} \Phi'(u, x_i^{(i)}) + \lambda^\top_A u = \phi'(u) + \lambda^\top_A u$. Furthermore, as a consequence of (4.27) and (4.28),

$$\liminf_{\rho \to \infty} z^{LR+}_\rho \geq \liminf_{\rho \to \infty} \min_{i \in [k]} \{ \theta(u) : ||u||_\infty \leq \kappa_\rho \}.$$
The second last inequality follows from lower semi-continuity of $\phi'(u)$ at $u = 0$. The result follows.

**Corollary 4.17.** If $f$ is strongly convex or if $F_R$ is compact then

$$\lim_{\rho \to \infty} \rho^{LD+} z_{1P}.$$  

**Lemma 4.18.** Let $\delta > 0$, and let $u \in N_\delta(0) \cap \overline{U}$. Define

$$S^u := \{ x \in X \cap H_u : f(x) \leq z_{1P} \}.$$ 

If (MICP) is feasible ($F \neq \emptyset$) and bounded and the recession cone of $f$ and recession cone of $F_R$, the feasible set of continuous relaxation of (MICP), have no common non-zero directions of recession, then $S^u$ is bounded.

**Proof.** Consider $u \in N_\delta(0) \cap \overline{U}$. Since $\|u\|_\infty \leq \|u\| < \delta$ we have $\|b - Ax\|_\infty < \delta$, $\forall x \in X \cap H_u$. Alternatively, $\forall x \in X \cap H_u$, $-\delta < b - Ax < \delta$. Define, for $u \in N_\delta(0) \cap \overline{U}$

$$S^u := \{ x \in X \cap H_u : f(x) \leq z_{1P} \}$$

Additionally, consider

$$S = \left\{ x \in X_R : \left\{ \begin{array}{l} -\delta \leq b - Ax \leq \delta \\ f(x) \leq z_{1P} \end{array} \right. \right\}$$

Indeed, $S^u \subseteq S$. Furthermore, $\text{rec}(S) = \text{rec}(F_R) \cap \text{rec}(f)$, and since $\text{rec}(F_R) \cap \text{rec}(f) = \{0\}$, it follows that $S$ is compact and consequently $S^u$ is bounded. In particular, $\|x\|_\infty \leq M < \infty$, $\forall x \in S^u$.

The following result is an immediate implication of Lemma 4.18.

**Corollary 4.19.** If (MICP) is feasible ($F \neq \emptyset$) and bounded and the recession cone of $f$ and recession cone of $F_R$, the feasible set of continuous relaxation of (MICP), have no common non-zero directions of recession, then for $u \in N_\delta(0) \cap \overline{U}$, $\phi(u) = \hat{\phi}(u)$ i.e.

$$\hat{\phi}(u) = \text{minimize} \quad f(x_J, x_C)$$

$$\text{subject to} \quad A_I x_J + A_C x_C = b + u$$

$$E_I x_I + E_C x_C \leq f$$

$$\|x_I\|_\infty \leq M$$

$$(x_I, x_C) \in \mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$$

Furthermore, there exists a $\delta > 0$ and $0 < \Gamma < \infty$ such that for every $u \in N_\delta(0) \cap U$, $\phi(0) \leq \phi(u) + \Gamma \|u\|$. Additionally, if $f$ is $\mu$-strongly convex and $L$ smooth, then $\Gamma = O\left(\frac{L\delta}{\mu}\right)$ where $\beta := \max_{B \in \mathcal{B}} \|B^{-1}\|_\rho$, $\mathcal{B}$ being the set of all possible invertible submatrices of $[A^T_L - A^T_C - E^T_C]$ and $\gamma$ depends on $f(0)$, $\|\nabla f(0)\|$ and $f(\bar{x})$ for any $x \in F$.

**Proof.** Lemma 4.18, along with Lemmas 4.11 and 4.12 readily implies that $\phi(u) = \hat{\phi}(u)$ and there exists a $\delta > 0$ and $0 < \Gamma < \infty$ such that for every $u \in N_\delta(0) \cap U$, $\phi(0) \leq \phi(u) + \Gamma \|u\|$. 

To see the explicit bound on $\Gamma$, recall the stationarity condition from set of equations (4.20) at $u = 0$.

\[
\nabla_C f(x^{(i)}_C, x_C) = A_C^T(\lambda^{(i)}_{A_C} - \lambda^{(i)}_{A_C^c}) - E_C^T \lambda_E
\]

\[
\lambda_{E_C}^T (E x - f) = 0
\]

(4.29)

\[
\lambda_{A_C}^+, \lambda_{A_C^c} \geq 0
\]

\[
\lambda_{E_C} \geq 0
\]

For $(x_C, x^{(i)}_C)$ consider the complementary slackness conditions, that is if for some $j \in [m]$, $E_j x < f$, where $E_j$ is a row of the matrix $E$ then the corresponding Lagrangian multiplier $\lambda_{E_j} = 0$. Partition the set $[m]$ into two sets $J = \{J_c, J_e\}$, where $J_c = \{j \in [m] : E_j x < f\}$ and $J_e = \{j \in [m] : E_j x = f\}$.

Define $A_{aug} = [A_C^T - A_C^T - E_C^T]$ and $A_{aug}^J$ as the sub-matrix which has all the columns of $A_{aug}$ except the columns $2n_2 + J_c$. One possible solution to the system (4.29) is,

\[
(\lambda_{A_C}^+, \lambda_{A_C^c}, \lambda_{E_C}^-) = B^{-1} \nabla_C f(x^{(i)}_C, x_C)
\]

where $B$ is a basis matrix of $A_{aug}^J$.

Furthermore, let $\lambda_{A_C} = \lambda_{A_C^+} - \lambda_{A_C^c}$, this implies that

\[
||\lambda_{A_C}|| \leq ||\lambda_{A_C^+}|| + ||\lambda_{A_C^c}||
\]

Let $x^*$ be an optimal solution to (MICP), i.e. $f(x^*) = 1_{IP}$. Since $f$ is $\mu$-strongly convex, we have

\[
\frac{\mu}{2} ||x^*||^2 + \nabla f(0)^T x^* + f(0) \leq 1_{IP}
\]

\[
\frac{\mu}{2} ||x^*||^2 + \nabla f(0)^T x^* - 1_{IP} + f(0) \leq 0
\]

\[
\frac{\mu}{2} ||x^*||^2 - ||\nabla f(0)|| ||x^*|| - (1_{IP} - f(0)) \leq 0
\]

It follows that,

\[
||x^*|| \leq \frac{||\nabla f(0)|| + \sqrt{||\nabla f(0)||^2 + 2 \mu \cdot (1_{IP} - f(0))}}{\mu}
\]

Since $f$ is L-smooth,

\[
||\nabla f(x^*) - \nabla f(0)|| \leq L ||x^*||
\]

\[
||\nabla f(x^*)|| \leq L ||x^*|| + ||\nabla f(0)||
\]

From (4.30) and (4.31) we obtain

\[
||\lambda_{A_C}|| \leq 2 \left\| B^{-1} \right\|_F ||\nabla f(x^*)||
\]

\[
\leq 2 \left\| B^{-1} \right\|_F (L ||x^*|| + ||\nabla f(0)||)
\]

\[
\leq 2 \beta (L ||x^*|| + ||\nabla f(0)||)
\]
where \( \beta := \max_{B \in \mathcal{B}} \|B^{-1}\|_F \) and \( \mathcal{B} \) is the set of all possible invertible submatrices of \( [A_C^T - A_C^T - E_C^T] \).

\[
\|\lambda_{A_C}\| \leq \frac{2\beta \left( L \left( \|\nabla f(0)\| + \sqrt{\|\nabla f(0)\|^2 + 2\mu \left( z_{1P} - f(0) \right)} \right) + \mu \|\nabla f(0)\| \right)}{\mu}
\]

where the second inequality follows as \( z_{1P} \leq f(\bar{x}), \forall \bar{x} \in F \) and \( L > \mu \). Combining (4.32) with the construction of \( \Gamma \) in Lemma 4.12, yields the result. \( \square \)

Theorem 3.3 follows.

4.5. A Special Case: Pure Integer Convex Programs.

**Proposition 4.20.** Consider the following pure integer convex program (PICP),

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{(PICP)} & \quad \text{subject to} \quad Ax = b \\
& \quad Ex \leq f \\
& \quad x \in \mathbb{Z}^n
\end{align*}
\]

If the continuous relaxation of (PICP) is feasible and bounded then \( \exists \rho < \infty \) such that \( z_{L^D} = z_{1P} \).

**Proof.** Consider the value function of (PICP),

\[
\hat{\phi}(u) = \min f(x)
\]

\[
\text{(PICP)}_u \quad \text{subject to} \quad Ax = b + u \\
& \quad Ex \leq f \\
& \quad x \in \mathbb{Z}^n
\]

Let \( U_\rho = \left\{ u \in U : \hat{\phi}(u) + \rho \|Ax - b\| \leq z_{1P} \right\} \). From Corollary 4.3 it follows that \( \|u\| \) is bounded for all \( u \in U_\rho \). In particular, we have

\[
Ax = b + u \text{ where } \|u\| \leq \frac{z_{1P} - z_R}{\rho - \lambda_A}
\]

Since \( A \) and \( b \) are rational, we can assume, without loss of generality, that \( A \) and \( b \) are integral. It follows that there exists an integral solution to the equation \( Ax = b + u \) only if \( u \) is integral. In particular, for \( 0 < \|u\| < 1 \) there is no integral solution to the system of equations \( Ax = b + u \). It follows from Proposition 4.2 and Corollary 4.4 that the system of equations \( Ax = b + u \) does not have an integral solution for \( u \in U_\rho \) for any \( \rho \) satisfying

\[
\frac{z_{1P} - z_R}{\rho - \|\lambda_A\|} < 1
\]

\[
(z_{1P} - z_R) < \rho - \|\lambda_A\|
\]

Thus for \( \rho^* < \rho < \infty \) there exists a \( \delta_\rho > 0 \) such that \( \forall u \in N_{\delta_\rho}(0) \setminus \{0\}, \ (\text{PICP})_u \) is infeasible. Furthermore for \( u \in U \setminus N_{\delta_\rho}(0) \), \( \hat{\phi}(u) + \rho \|u\| > \hat{\phi}(0) \). The result follows. \( \square \)
REFERENCES

[1] D. Bertsekas, A. Nedic, and A. Ozdaglar, Convex analysis and optimization, vol. 1, Athena Scientific, 2003.
[2] C. Blair, A closed-form representation of mixed-integer program value functions, Mathematical Programming, 71 (1995), pp. 127–136.
[3] C. E. Blair and R. G. Jeroslow, The value function of a mixed integer program: I, Discrete Mathematics, 19 (1977), pp. 121–138.
[4] C. E. Blair and R. G. Jeroslow, The value function of a mixed integer program: II, Discrete Mathematics, 25 (1979), pp. 7–10.
[5] C. E. Blair and R. G. Jeroslow, Constructive characterizations of the value-function of a mixed-integer program I, Discrete Applied Mathematics, 9 (1984), pp. 217–233.
[6] N. L. Boland and A. C. Eberhard, On the augmented Lagrangian dual for integer programming, Mathematical Programming, 150 (2015), pp. 491–509.
[7] S. Boyd, N. Parikh, E. Chu, B. Peleato, J. Eckstein, et al., Distributed optimization and statistical learning via the alternating direction method of multipliers, Foundations and Trends® in Machine learning, 3 (2011), pp. 1–122.
[8] S. P. Boyd and L. Vandenberghe, Convex optimization, Cambridge university press, 2004.
[9] R. S. Burachik and A. Rubinov, On the absence of duality gap for lagrange-type functions, Journal of Industrial & Management Optimization, 1 (2005), p. 33.
[10] J. V. Burke, An exact penalization viewpoint of constrained optimization, tech. report, Argonne National Lab., IL (USA), Mathematics and Computer Science Div., 1987.
[11] J. V. Burke, Calmness and exact penalization, SIAM Journal on control and optimization, 29 (1991), pp. 493–497.
[12] M. Cordova, W. D. Oliveira, and C. Sagastizábal, Revisiting augmented Lagrangian duals, Mathematical Programming, (2021), pp. 1–43.
[13] M. J. Feizollahi, S. Ahmed, and A. Sun, Exact augmented Lagrangian duality for mixed integer linear programming, Mathematical Programming, 161 (2017), pp. 365–387.
[14] X. Gu, S. Ahmed, and S. S. Dey, Exact augmented Lagrangian duality for mixed integer quadratic programming, SIAM Journal on Optimization, 30 (2020), pp. 781–797.
[15] R. R. Meyer, Integer and mixed-integer programming models: general properties, Journal of Optimization Theory and Applications, 16 (1975), pp. 191–206.
[16] T. K. Ralphs and A. Hassanzadeh, On the value function of a mixed integer linear optimization problem and an algorithm for its construction, COR@L Technical Report 14T–004, (2014).
[17] R. T. Rockafellar and R. J.-B. Wets, Variational analysis, vol. 317, Springer Science & Business Media, 2009.
[18] A. Schrijver, Theory of Linear and Integer Programming, John Wiley and Sons, Inc., USA, 1986.
[19] C. Wang, X. Yang, and X. Yang, Nonlinear augmented Lagrangian and duality theory, Mathematics of Operations Research, 38 (2013), pp. 740–760.