Localization and dynamic persistent currents in long cylinders

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Abstract

A dynamic response to a magnetic field in a long disordered cylinder is considered. We show that, although at high frequencies conduction is classical in all directions, the low frequency behavior corresponds to localization in the longitudinal direction and to a diamagnetic dynamic persistent current in the transversal one. The current density does not vanish even in the limit of the infinitely long cylinder. Being of a purely dynamic origin the current can be destroyed by inelastic scattering but at low temperatures the decay time can be very large.

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The recent measurement of the magnetization of an array of $10^7$ isolated Cu-rings attracted a lot of attention to the problem of persistent currents in disordered systems. The circumference of these rings was much larger than the elastic mean free path and therefore at short distances electron motion was diffusive. Considerable effort has been taken in order to build a proper theory of the observed effect in the diffusive regime. Calculation of the persistent current by averaging over disorder at fixed Fermi energy gives in this regime an exponentially small amplitude. In order to obtain a non vanishing effect one should either take into consideration electron-electron interactions or calculate with a fixed number of particles. In both cases it is essential that all the sizes of the sample are small. It is clear that for a macroscopic object the difference between results for the canonical and grand canonical ensembles cannot be important and therefore one should not hope to have a thermodynamically stable persistent current in such a case.

However, does it mean that there is no possibility to have a long living macroscopic current in a macroscopic piece of a disordered metal? Of course, if such a current did not correspond to the minimum of the free energy different types of inelastic scattering would kill the effect. Nevertheless, the corresponding decay times would be very large and strongly dependent on temperature which makes a discussion of such a possibility quite interesting.

In order to understand better what type of effect we want to consider below let us imagine first a very long clean metallic cylinder with the length $L_z$, the circumference $L$ and the thickness of the walls $d$. Applying a magnetic field $H$ along the cylinder one induces currents. In the ballistic regime the cylinder is an ideal diamagnet and the induced total current $I_d$ is equal to

$$I_d = dL_zLK_dH/4\pi c, \quad K_d = -ne^2/m. \quad (1)$$

In Eq. (1) $m$ is the electron mass, $e$ is the electron charge, $n$ is the electron density, $c$ is the light velocity. The current $I_d$ can exist at very low temperatures for a very long time but it does not correspond to the free energy minimum and must finally decay. Experimentally, the current can be obtained first cooling down the cylinder and then applying the magnetic field.
The inverse procedure would lead to a thermodynamic current which can be much smaller. The averaged thermodynamic persistent current \( I_{\text{th}} \) for clean cylinders was calculated in ref. [6] and can be estimated as

\[
I_{\text{th}} \sim \frac{|I_d|}{(p_0 L)^{1/2}},
\]

(2)

where \( p_0 \) is the Fermi momentum. Eq.(2) shows that the thermodynamic persistent current can be much smaller than the dynamic one which, of course, is not unexpected. The difference between the dynamic and thermodynamic currents is due to existence of degenerate states. If one represents energy levels by functions of the magnetic field the degeneracies correspond to level crossing. In the diffusive regime \( p_0^{-1} \ll l \ll L \), where \( l \) is the mean free path, the thermodynamic current is proportional to \( \exp(-L/l) \) [6] and is negligibly small.

In this Letter we calculate the corresponding dynamic current and show that it can be quite noticeable. For simplicity we consider a standard model of non interacting electrons moving in a random potential. Calculating the dynamic current we use standard formulae for linear response. We assume that the external magnetic field \( \tilde{H} \) has the form

\[
\tilde{H} = H + H_\omega \cos(\omega t),
\]

(3)

where \( H \) is the static component which can be arbitrary. The second term in Eq.(3) describes the oscillating part of the magnetic field which induces an electric field directed along the circumference and, correspondingly, an oscillating current. We assume that the amplitude \( H_\omega \) of the oscillating part of the magnetic field is small and calculate the induced current using the conventional linear response theory. The density of the oscillating current \( j_\omega \) can be written as

\[
j_\omega = c^{-1} K(\omega) A_\omega,
\]

(4)

where \( K(\omega) \) is the current-current correlation function and \( A_\omega \) is the vector potential corresponding to the magnetic field \( H_\omega \) in Eq.(3). We use the gauge with the vector potential directed along the circumference. Using the retarded \( G^R_{\epsilon}(r, r') \) and advanced \( G^A_{\epsilon}(r, r') \) Green functions we can write the function \( K(\omega) \) as follows.
\[ K(\omega) = K_1(\omega) + K_2(\omega), \quad (5) \]

\[ K_1(\omega) = (e^2/i\pi) \int [n(\epsilon - \mu) - n(\epsilon - \omega - \mu)] \langle \hat{v}_r G^R_\epsilon (r, r') \hat{v}_r' G^A_{\epsilon - \omega} (r', r) \rangle dr' d\epsilon, \quad (6) \]

\[ K_2(\omega) = (e^2/i\pi) \int n(\epsilon - \mu) \left[ \langle [\hat{v}_r G^R_\epsilon (r, r') \hat{v}_r' G^R_\epsilon (r', r) - \hat{v}_r G^A_\epsilon (r, r') \hat{v}_r' G^A_{\epsilon - \omega} (r', r)] \rangle \right] dr' d\epsilon + \]

\[ (e^2/i\pi m) \int n(\epsilon - \mu) \left[ \langle [G^R_\epsilon (r, r) - G^A_\epsilon (r, r)] \rangle \right] d\epsilon, \quad (7) \]

where \( \hat{v}_r = (1/m) [-i \nabla - (e/c) A] \) is the component of the velocity operator perpendicular to the axis of the cylinder, \( A \) is the vector potential corresponding to the static component \( H \) of the magnetic field, \( n(\epsilon) \) is the Fermi distribution, and the angular brackets stand for averaging over impurities. The terms \( K_1(\omega) \) and \( K_2(\omega) \) have completely different origins.

In the limit \( \omega \to 0 \) the term \( K_2(\omega) \) can be obtained from the free energy by differentiating at a fixed chemical potential over the vector potential and, as we have mentioned, can be much smaller than the first one. In the diffusive regime the average over impurities of a function containing a product of only retarded or only advanced Green functions can be substituted by the product of the corresponding averaged Green functions \( \langle G^{R,A} \rangle \). In the momentum representation this is given by

\[ \langle G^{R,A}_\epsilon \rangle = (\epsilon - \epsilon (\hat{p}) \pm i/2\tau)^{-1}, \quad (8) \]

where \( \tau \) is the mean free time. Substituting Eq.(8) into Eq.(7) and integrating the first term in Eq.(7) over momentum by parts one can see that in the limit \( \omega \to 0 \) it cancels the second one. Here, we consider the situation when the magnetic field is concentrated in the cavity of the cylinder. Otherwise, we would have to sum over the Landau levels and would obtain that the quantity \( K_2 \) gives additionally the Landau diamagnetism.

In the following we concentrate on calculation of the quantity \( K_1(\omega) \). Before carrying out explicit calculations in the diffusive regime let us demonstrate how one can calculate this quantity in the ballistic one. Using the representation
\[ G^{R,A}_\epsilon (r,r') = \sum_\alpha \frac{\varphi_\alpha (r) \varphi_\alpha^* (r')}{\epsilon - \epsilon_\alpha \pm i\delta}. \] (9)

and taking the limit \( \omega \to 0 \) we obtain from Eq.(9)

\[ \lim_{\omega \to 0} K_1 (\omega) = -(2e^2/V) \sum_{\epsilon_\alpha = \epsilon_\beta} |\hat{v}_{\alpha\beta}|^2 \delta (\epsilon_\alpha - \mu), \] (10)

where \( \mu \) is the chemical potential, \( V \) is the volume.

In the clean macroscopic cylinder under consideration the eigenstates are plane waves

\[ \text{and the matrix elements in Eq.(10) can easily be calculated. All sums are to be substituted} \]

\[ \text{by an integral over momenta and we can see that } \lim_{\omega \to 0} K_1 (\omega) = K_d. \]

The classical limit in the diffusive regime corresponds to high frequencies \( \omega \). In this limit

\[ \text{one can substitute the Green functions in Eq.(6) by their averages Eq.(8) and then calculate} \]

\[ \text{the integral over momenta. As the result one has} \]

\[ K_1 (\omega) = -i\omega \sigma_0, \] (11)

where \( \sigma_0 = e^2 \tau_n / m \) is the Drude conductivity.

The limit of low frequencies requires more sophisticated methods of calculation. In

\[ \text{the diffusive regime the supersymmetry method developed by one of the authors} \]

\[ \text{is by now the only possibility to obtain results analytically. This method has been used} \]

\[ \text{to study localization in long disordered wires} \] (7,8). As we will see below the problem of

\[ \text{localization is closely related to} \] (7,10) the problem of the dynamic persistent currents. The

\[ \text{responses both in the longitudinal and transversal directions can be reduced to calculation} \]

\[ \text{of functional integrals over supermatrices} \] (7,8). Calculating a correlation function for a model

\[ \text{of a wire corresponding to a longitudinal response for the cylinders it was shown in Refs.} \]

\[ \text{that the system has dielectric properties in the longitudinal direction. Here, we present} \]

\[ \text{a calculation of the transversal response describing the persistent currents. In Refs.} \]

\[ \text{a general expression for the function} \] (7,10) \( K_1 (\omega) \) has been presented in terms of integrals over

\[ \text{supermatrices} \] (7,8) with a free energy functional of the non linear supersymmetric \( \sigma \)-model. This expression has the following structure
\[ K_1(\omega) = -i\omega \langle B(Q) \rangle_Q , \] (12)

where \( B(Q) \) is a sum of products of elements of the supermatrix \( Q \).

In Eq. (12) the notation \( \langle \ldots \rangle_Q \) stands for the functional average

\[ \langle \ldots \rangle_Q = \int \langle \ldots \rangle \exp(-F\left[Q\right])DQ, \] (13)

where \( F\left[Q\right] \) is the free energy functional of the non-linear supersymmetric \( \sigma \)-model.

The form of the function \( R(\omega) \) which denoted in Refs. [9,10] the average \( \langle B(Q) \rangle_Q \) remains unchanged for the system studied, here. While the dynamic response for the problem of persistent currents in mesoscopic rings is calculated using the zero dimensional version of the \( \sigma \)-model [9,10], for the problem of localization in wires one has to take the one dimensional version of the \( \sigma \)-model [7,8]. Although for the cylinder in the diffusive regime the one dimensional \( \sigma \)-model is valid, let us consider a more general model of a stack of small metallic rings. Changing the probability of tunneling from ring to ring we can describe the crossover from the case of the homogeneously weakly disordered cylinder to the case of isolated rings. For such a model the derivatives in the longitudinal coordinate are substituted by finite differences. Then the free energy \( F\left[Q\right] \) takes the form

\[ F[Q] = -\sum_{ij} J_{ij} ST r\left[ Q_i Q_j \right] + \sum_i F_0\left[Q_i\right], \] (14)

\[ F_0\left[Q\right] = \frac{\pi \nu Sh}{8} \sum_i ST r \left[ -D_0 \left( \frac{e}{c} A\left[Q_i, \tau_3\right] \right)^2 + 2i\omega \Lambda Q_i \right], \] (15)

where \( S = Ld \) is the cross section of the cylinder, \( h \) is the height of each ring, \( i \) and \( j \) enumerate the rings in the stack, \( D_0 \) is the classical diffusion coefficient. (We remind that \( Q^2 = 1 \)). The function \( F_0\left[Q\right] \) in Eqs. (14,15) describes electron motion inside the rings, the first term in Eq.(14) stands for coupling between the rings. We assume that only nearest neighbors interact, so that \( J_{ij} = J \) for the neighbors, and \( J_{ij} = 0 \) otherwise. The limit \( J = 0 \) corresponds to the stack of the isolated rings. In the limit \( J \gg 1 \) the model on the lattice Eq.(14) can be substituted by the continuous one and we return to the one dimensional
σ-model. The notation \( STr \) stands for the Supertrace, the definition of the matrices \( \tau_3 \) and \( \Lambda \) can be found in previous works \[7,9,10\].

To recover the case of the homogeneously weakly disordered cylinder one should relate \( J \) to \( D_0 \) as

\[
J = \frac{\pi \nu SD_0}{8h}.
\]  

The main contribution for the transversal current when calculating the integral Eqs.(12, 13) comes from terms containing products of two matrix elements of \( Q \) taken at the same point (terms \( R_1(\omega) \) and \( R_3(\omega) \) in Refs. \[9,10\] and the substitution of the weakly homogeneous cylinder by the stack of the rings does not change the form of these terms.

Due to the one dimensionality of the model Eq.(14) one can use the transfer matrix technique. Corresponding partial differential equations have been written in Refs. \[7,8\] for the continuous σ-model. For the model on the lattice, Eq.(14), the corresponding recurrence equation is an integral. Analogous integral equations were written in Refs. \[11,12\] for the model on the Bethe lattice and also for one dimensional chains. Repeating the main steps of these works we reduce calculation of integrals over \( Q_i \) for all cites \( i \) to one integral over \( Q \). The corresponding changes can be done substituting Eq.(13) by

\[
\langle \ldots \rangle_Q = \int \langle \ldots \rangle \Psi^2(Q) \exp(-F_0(Q)) dQ.
\]  

The function \( \Psi(Q) \) in Eq.(17) satisfies the equation

\[
\Psi(Q) = \int \exp(2JSTrQQ') \exp(-F_0(Q')) \Psi(Q') dQ'.
\]  

The solution of the Eq.(18) depends on the vector potential \( A \) entering \( F_0 \) Eq.(15). In principle, it can be solved, at least numerically, for arbitrary magnetic fields using the parameterization proposed recently \[13\] but we will present results only in the limits of zero and high magnetic fields. These limits correspond to the orthogonal and unitary ensembles. In both cases one can omit the first term in Eq.(13) because in the unitary case the supermatrix \( Q \) commutes with the matrix \( \tau_3 \) \[7\]. Due to the symmetry of the free energy the
function $\Psi (Q)$ depends only on the variables $\lambda$ and $\lambda_{1,2}$ parameterizing the supermatrix $Q$. Therefore, in the integral over $Q$ in Eq. (12) one can integrate first over all other variables reducing thus the integral to an integral over $\lambda$ and $\lambda_{1,2}$. At high frequencies deviations of the supermatrix $Q$ from $\Lambda$ are small and one comes to Eq. (11). In this limit the conduction is classical in all directions.

Calculations for arbitrary frequency are most simple for the unitary ensemble. Corresponding calculations are not very different from those performed in Refs. [9,10] and we obtain

$$K_{unit}^{1} (\omega) = -i\omega \sigma_0 \left\{ 1 + \frac{1}{2} \int_{-1}^{1} \int_{1}^{\infty} \exp \left[ (i\pi \omega / \Delta - \delta) (\lambda_{1} - \lambda) \right] \Psi^2 (\lambda, \lambda_{1}) d\lambda d\lambda_{1} \right\},$$

(19)

where $\Delta = (\nu S h)^{-1}$ is the mean level spacing in one ring in the stack and $\delta \to 0$.

Non zero persistent currents are possible if at low frequencies $\omega$ the second term in the brackets in Eq. (19) is proportional to $1/\omega$. In the most interesting limit $\omega \to 0$ all calculations when solving Eq. (18) become more simple because the main contribution in all integrals over $\lambda_{1}$ comes from $\lambda_{1} \sim 1/\omega$. Then the function $\Psi (Q)$ depends only on one variable $z = 2\omega \lambda_{1}$ and we obtain for the response $K_{1} (\omega)$

$$K_{unit}^{1} (0) = -\sigma_0 \Delta_{eff} (J) / \pi,$$

(20)

where

$$\Delta_{eff} (J) = \Delta \int_{0}^{\infty} \exp (-z) \Psi^2_{J} (z) dz,$$

(21)

is a non trivial function of the coupling $J$ between the rings. This function is known numerically for arbitrary $J$. [12] In the limit $J = 0$ the function $\Psi = 1$ and

$$\Delta_{eff} (0) = \Delta.$$

(22)

In the opposite limit $J \gg 1$ the function $\Psi (z)$ is the solution of the differential equation

$$zd^2 \Psi / dz^2 - 16J \Psi = 0.$$

(23)

Solving Eq. (23) and calculating the integral Eq. (21) we find
\[ \Delta_{\text{eff}}(J) = \Delta / (96J). \]  

(24)

As we have mentioned above one has localization in the longitudinal direction. In the unitary ensemble the localization length \( L_c \) was calculated in Ref. \[7\] and can be related with the help of Eq. (16) to the coupling \( J \) as \( L_c = 16Jh \). In this limit the function \( \Delta_{\text{eff}}(J) \) can be rewritten as

\[ \Delta_{\text{eff}}(J \gg 1) = (6\nu SL_c)^{-1}. \]  

(25)

We see from Eq. (23) that in the limit of large \( J \) the response \( K^{\text{unit}}_1(0) \) Eq. (20) looks as if the cylinder consisted of rings with the height of the order of \( L_c \). Comparing Eqs. (11 and 20) we see that the characteristic frequency of the crossover from the quantum to the classical regime is of the order \( \Delta_{\text{eff}} \). In the model of the weakly homogeneously disordered cylinder we can see with the help of Eq. (16) that the response \( K^{\text{unit}}_1(0) \) does not depend on the disorder. In this limit it is very small. Decreasing the coupling \( J \) makes the localization length shorter which leads to a larger value of the response.

Analogous calculations can be carried out for the orthogonal ensemble corresponding to the zero static component \( H \) of the magnetic field Eq. (3). In this case we obtain as in Refs. 9,10

\[ K^{\text{orth}}_1(0) = 0. \]  

(26)

Changing the static component \( H \) of the magnetic field we can have a crossover between Eqs. (20) and (20). The characteristic flux \( \phi_c \) of this crossover is of the order of \( \phi_0(\Delta_{\text{eff}}/E_c)^{1/2} \), where \( E_c = \pi^2D_0/L^2 \) is the Thouless energy, \( \phi_0 \) is the flux quantum. The whole dependence of the response on the flux \( \phi \) is periodic with the period \( \phi_0/2 \).

Adding magnetic or spin-orbit impurities changes Eqs. (20, 20). The system with the magnetic impurities corresponds to the unitary ensemble. One can use as before Eq. (20) provided \( \Delta_{\text{eff}}(J) \) is substituted by \( \Delta_{\text{eff}}(2J)/2 \). If the magnetic impurities are absent spin-orbit ones lead to a different function \( \tilde{\Delta}_{\text{eff}}(J) \). However, this difference is only numerical and does not change the sign of the response which is in all cases diamagnetic.
In conclusion, we showed that a magnetic field applied to a disordered cylinder parallel to the axis induces a macroscopic diamagnetic current. In the absence of inelastic scattering which can correspond to low temperatures this current can live for a very long time. The current density remains finite even in the limit of an infinitely long cylinder. At the same time the longitudinal response shows a dielectric behavior usual for disordered wires. In fact, the shorter the localization length in the longitudinal direction is the larger is the current in the transversal one. In the limit of high frequencies or short inelastic times the transport in the model under consideration is classical in all directions and is described by the conventional Ohm law.

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