The Jacobi equation and Poisson geometry on $\mathbb{R}^4$

Rubén Flores-Espinoza
Departamento de Matemáticas,
Universidad de Sonora, México.

Abstract

This paper is devoted to the study of solutions of the Jacobi equation in Euclidean four dimensional space $\mathbb{R}^4$. Each one of such solutions define a Poisson tensor. By using the elementary vector calculus operations, we give explicit formulas for the main geometric objects associated to the solutions of the Jacobi equation, including its characteristic foliation, their symmetries and its generators, normal forms and some useful decomposition results for the solutions. In particular we study the classes of Poisson tensors of constant rank and those preserving a volume form.

1 Introduction.

The Jacobi equation is an integrability condition for generalized vector field distributions defined in a manifold by contravariant antisymmetric 2-tensors. When the Jacobi equation is satisfied, each 2-tensor solution is called a Poisson tensor. The study of Poisson tensors has been in the last fifty years an important field of research in differential geometry and mathematical physics for its relevance in the mathematical foundations of classical and quantum mechanics. For low dimensional vector spaces we can find in the literature several papers devoted to the study of Jacobi equations, Poisson geometry and Hamiltonian dynamics. For the tridimensional case, see for example [1], [2], [3]. In dimension four, we have the classification of linear and quadratic Poisson tensors in [17] and [5]. In this article, we study from a general point of view the Jacobi equation on the four dimensional euclidean space $\mathbb{R}^4$. Using the trace operator associated to the volume form and the elementary operations of vector calculus, we describe the different concepts and geometric quantities associated to Poisson geometry on $\mathbb{R}^4$. Using the trace operator associated to the volume form and the elementary operations of vector calculus, we describe the different concepts and geometric quantities associated to Poisson geometry on $\mathbb{R}^4$. With this approach, we can show explicit formulas for the different quantities and structures associated to Poisson geometry on $\mathbb{R}^4$. In particular, we consider various important classes of solutions of Jacobi equation as those of constant rank and those preserving a given volume form. We include a discussion on automorphisms and
infinitesimal symmetries for a solution of Jacobi equation emphasizing on the modular Poisson vector field and its relevance on the preserving volume properties of Hamiltonian vector fields. We present various decomposition formulas for Poisson tensors extending some given results in the literature used to classify linear and quadratic Poisson tensors. We include several examples and remarks.

The paper is organized as follows: Section 2 is devoted to to fix notation and introduce some general facts on solutions of the Jacobi equation and the geometry of Poisson tensors. We express the Jacobi equation using global coordinates in \( \mathbb{R}^4 \) and the usual operators of three dimensional vector calculus. We include a description of the characteristic foliation associated to a Poisson tensor and a global decomposition theorem for Poisson tensors in \( \mathbb{R}^4 \). In section 2, we introduce the trace operator associated to the canonical volume form on \( \mathbb{R}^4 \) and give explicit formulas for its action on the different contravariant antisymmetric tensors in \( \mathbb{R}^4 \). In section 3, we study symmetries of a solution of the Jacobi equation making focus on the infinitesimal generators (Poisson vector fields) tangent and transversal to the characteristic foliation and particulary in those symmetries preserving the canonical volume form. Finally in section 4, we discuss the class of regular Poisson structures and give conditions for the existence of transversal Poisson vector fields.

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2 The Jacobi equation in \( \mathbb{R}^4 \)

On \( \mathbb{R}^4 \) with global coordinates \( (x, y) = (x_1, x_2, x_3, y) \), antisymmetric contravariant 2-tensor \( \Lambda \) has the following form

\[
\Lambda = \Psi_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + \Psi_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + \Psi_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \Phi_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y} + \Phi_2 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y} + \Phi_3 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial y} \tag{1}
\]

where \( \Psi_i, \Phi_i \) for \( i = 1, 2, 3 \) are smooth functions on \( \mathbb{R}^4 \). Considering the vector functions \( \Psi = (\Psi_1, \Psi_2, \Psi_3) \) and \( \Phi = (\Phi_1, \Phi_2, \Phi_3) \), we can identify the 2-tensor \( \Lambda \) with the pair \( (\Psi, \Phi) \) and write \( \Pi \) in the simplified notation

\[
\Lambda = \Psi \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} + \Phi \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \tag{2}
\]

In the sequel, we use the operations \( \cdot, \times, \nabla, \text{div}, \text{rot} \) of elementary vector calculus on the three variables \( x = (x_1, x_2, x_3) \) to write down the quantities and main results of Poisson geometry on \( \mathbb{R}^4 \).
Each antisymmetric contravariant 2-tensor \( \Lambda \) establishes a morphism \( \Lambda^\# : \Lambda^1(\mathbb{R}^4) \to \mathcal{X}(\mathbb{R}^4) \) between the space of differential 1-forms and the space of vector fields on \( \mathbb{R}^4 \). The value of \( \Lambda^\# \) on a 1-form \( \alpha = gdf \) with \( g \) and \( f \) smooth functions on \( \mathbb{R}^4 \) is the vector field

\[
\Lambda^\# (gdf) = g \left( \Psi \times \nabla f - \frac{\partial f}{\partial y} \Phi \right) \frac{\partial}{\partial x} + \nabla f \cdot \Phi \frac{\partial}{\partial y} \quad (3)
\]

Moreover, the contravariant 2-tensor \( \Lambda \) define a bilinear operation on the space of smooth functions on \( \mathbb{R}^4 \) with values on the same space of smooth functions given by the bracket

\[
\{f, g\} = dg(\Lambda^\#(df))
\]

In terms of the vector functions \( \Psi \) and \( \Phi \) we have

\[
\{f, g\} = \Psi \cdot \nabla f \times \nabla g + \Phi \cdot \left( \frac{\partial f}{\partial y} \nabla g - \frac{\partial g}{\partial y} \nabla f \right) \quad (4)
\]

A 2-tensor \( \Lambda \) is called a Poisson tensor on \( \mathbb{R}^4 \) if the bracket \( \{\cdot, \cdot\} \) satisfies the Jacobi identity

\[
\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0 \quad (5)
\]

for any smooth functions \( f, g, h \) on \( \mathbb{R}^4 \). In this case, the bracket \( \{\cdot, \cdot\} \) is called the Poisson bracket associated to the Poisson tensor \( \Lambda \). If a 2-tensor \( \Lambda \) satisfies the Jacobi identity \( (6) \), the vector functions \( (\Psi, \Phi) \) are solutions of the Jacobi equation

\[
\Psi \cdot \left( \text{rot} \Psi + \frac{\partial \Phi}{\partial y} \right) = \frac{\partial}{\partial y} (\Psi \cdot \Phi),
\]

\[
\Phi \times \left( \text{rot} \Psi + \frac{\partial \Phi}{\partial y} \right) + (\text{div} \Phi) \Psi = \nabla (\Psi \cdot \Phi),
\]

and we identify the Poisson tensors on \( \mathbb{R}^4 \) with solutions of \( (7) \)-\( (8) \).

The Jacobi equation \( (7) \)-\( (8) \) are equivalent to the equation

\[
[\Lambda, \Lambda] = 0
\]

where \( [\cdot, \cdot] \) denotes the Schouten bracket on contravariant antisymmetric tensors and \( \Lambda \) a 2-tensor. For complete information on the Schouten bracket calculus see [10] or [9].

Given a Poisson tensor \( \Lambda \) and a smooth function \( H \), the vector field

\[
X_H = \Lambda^\#(dH) = \left( \Psi \times \nabla H - \frac{\partial H}{\partial y} \Phi \right) \frac{\partial}{\partial x} + \nabla H \cdot \Phi \frac{\partial}{\partial y} \quad (9)
\]

is called the Hamiltonian vector field with Hamiltonian function \( H \).

In particular, the Hamiltonian vector fields with Hamiltonian functions \( H_1 = -y \) and \( H_2 = \Phi \cdot \Psi \) are, respectively, \( X_{H_1} = \Phi \frac{\partial}{\partial x} \) and \( X_{H_2} = (\Psi \cdot \Phi)((\text{rot}(\Psi) + \Phi \cdot \Psi) \frac{\partial}{\partial y} \).
\( \frac{\partial \Phi}{\partial y} \frac{\partial}{\partial y} - \text{div}(\Phi) \frac{\partial}{\partial x} \). In the case \( \text{div}(\Phi) = 0 \), the Hamiltonian vector fields \( X_y \) and \( X_{\Psi \cdot \Phi} \) commute. Note that for any smooth functions \( f, g \) we have
\[
\{ f, g \} = L_{X_f} g,
\] (10)
where \( L_X \) is the Lie derivative operator along the vector field \( X \).

The Jacobi identity (6) for the bracket operation (4) endows the space of smooth functions of a Poisson algebra structure homomorphic to the Lie subalgebra of Hamiltonian vector fields
\[
[X_f, X_g] = X_{\{ f, g \}}, \quad \forall f, g \in C^\infty(\mathbb{R}^4)
\] (11)
Then, the space of Hamiltonian vector fields define an integrable generalized distribution of vector fields in the sense of Sussman [12] and the total space \( \mathbb{R}^4 \) is foliated by leaves of possible different even dimensions. This foliation is called the characteristic foliation of \( \Lambda \) and its leaves \( \mathcal{L} \) are submanifolds equipped with a symplectic structure given by the 2-form
\[
\Omega_\mathcal{L}(X_f, X_g) = \{ f, g \} |_{\mathcal{L}}
\] (12)
The characteristic foliation is also called the symplectic foliation of Poisson structure.

Relative to a given Poisson tensor \( \Lambda = (\Psi, \Phi) \), a smooth function \( k \) such that \( \Lambda^\#(dk) = 0 \) is called a Casimir function. In particular \( k \) is a Casimir function if
\[
\left( \Psi \times \nabla k - \frac{\partial k}{\partial y} \Phi \right) = 0 \quad \text{and} \quad \nabla k \cdot \Phi = 0
\]
The Poisson bracket of a Casimir function with any other smooth function vanishes and every Casimir function is a first integral for any vector field \( \Lambda^\#(\omega) \) with \( \omega \) a 1-form in \( \mathbb{R}^4 \).

The rank of the Poisson tensor \( \Lambda = (\Psi, \Phi) \) is constant on the points of each symplectic leaf. The classification of the symplectic leaves according to its dimension is given by the following

**Proposition 1** If \( p = (x, y) \in \mathbb{R}^4 \), the rank of a Poisson tensor \( \Lambda = (\Psi, \Phi) \) (see [17] in \( p \), takes the following values
\[
\text{rank } \Lambda(p) = \begin{cases} 
0 & \text{if and only if} \ (\Phi^2 + \Psi^2)(p) = 0 \\
2 & \text{if and only if} \ (\Phi^2 + \Psi^2)(p) \neq 0 \text{ and } (\Phi \cdot \Psi)(p) = 0 \\
4 & \text{if and only if} \ (\Phi \cdot \Psi)(p) \neq 0 
\end{cases}
\]
Moreover, the characteristic foliation of \( \Lambda \) has the open sets
\[
S^+ = \{(x, u) \in \mathbb{R}^4 \text{ with } \Phi \cdot \Psi > 0 \}
\]
and
\[
S^- = \{(x, u) \in \mathbb{R}^4 \text{ with } \Phi \cdot \Psi < 0 \},
\]
as its 4-dimensional symplectic leaves and its boundary
\[
\partial S = \{(x, u) \in \mathbb{R}^4 \text{ with } \Phi \cdot \Psi = 0 \},
\]
is foliated by 2-dimensional and 0-dimensional symplectic leaves.
Remark 2. The solutions of Jacobi equation with $\Phi = 0$ define parametrized family of three dimensional Poisson structures where the parameter $y$ can be considered as new variable commuting with all functions.

Example 3. The linear solutions $(\Psi, \Phi)$ of the Jacobi equation (7)-(8) take the general form

$$
\Psi(x, y) = Mx + p \times x + y\alpha, \quad (13)
$$
$$
\Phi(x, y) = Nx + q \times x + y\beta, \quad (14)
$$

where $M$ and $N$ are symmetric $3 \times 3$ matrices and $p, q, \alpha, \beta \in \mathbb{R}^3$ satisfy the equations

$$
2Mp = N\alpha - q \times \alpha
$$
$$
2\alpha \cdot p = \alpha \cdot \beta
$$
$$
(2M - \Lambda \circ \beta)(N + \Lambda \circ q) + (N - \Lambda \circ q)(2M + \Lambda \circ \beta) = -2 \text{Tr}(N)M
$$
$$
\text{Tr}(N)\beta = N(2p + \beta) + q \times (2p + \beta)
$$
$$
(\text{Tr} N)\alpha - M\beta - p \times \beta = N\alpha - q \times \alpha
$$

In particular, if we have a nonlinear Poisson tensor $(\Psi, \Phi)$ and $\Psi(p) = \Phi(p) = 0$ on some point $p = (x_0, y_0)$, the linear vector functions approximation given by

$$
A(x, y) = D_x \Psi(p)(x) + \frac{\partial \Psi}{\partial y}(p)(y)
$$
$$
B(x, y) = D_x \Phi(p)(x) + \frac{\partial \Phi}{\partial y}(p)(y)
$$

is linear solution of the Jacobi equation.

Example 4. The characteristic foliation of the linear Poisson bracket

$$
\Lambda = 2x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + \frac{1}{4} x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y} + \frac{1}{4} x_2 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial y} + \frac{1}{4} x_3 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial y}
$$

consists of:

(a) Two 4-dimensional symplectic leaves: $L_1 = \{(x, y_0) \text{ with } x_1 > 0\}$ and $L_2 = \{(x, y_0) \text{ with } x_1 < 0\}$.

(b) The boundary $\{(x, y_0) \text{ with } x_1^2 = 0\}$ is a submanifold foliated by 2-dimensional symplectic leaves of the form $\{x_1 = 0, \quad bx_2 - cx_3 = 0, \quad bc \neq 0\}$, and zero dimensional leaves given by the points $(0, y)$.

Moreover, the Hamiltonian vector fields take the form

$$
X_f = \left(-2x_1 \frac{\partial f}{\partial x_3} + \frac{1}{4} x_2 \frac{\partial f}{\partial y} \right) \frac{\partial}{\partial x_2} + \left(2x_1 \frac{\partial f}{\partial x_2} + \frac{1}{4} x_3 \frac{\partial f}{\partial y} \right) \frac{\partial}{\partial x_3}
$$

$$
+ \left(\frac{1}{2} x_1 \frac{\partial f}{\partial x_1} + \frac{1}{4} x_2 \frac{\partial f}{\partial x_2} + \frac{1}{4} x_3 \frac{\partial f}{\partial x_3} \right) \frac{\partial}{\partial y}
$$
In particular, the 2-dimensional symplectic leaves are given by intersection of the level set of \( k = \frac{y^2}{x^2} \) with \( x_1^2 = 0 \).

We conclude this section with a global decomposition theorem for Poisson brackets on \( \mathbb{R}^4 \):

**Proposition 5** If for a given a Poisson tensor \( \Lambda = (\Psi, \Phi) \) on \( \mathbb{R}^4 \) we have two independent functions \( f, g \) such that \( \{f, g\} \neq 0 \) on some domain in \( \mathbb{R}^4 \), then \( \Lambda \) has the following decomposition

\[
\Lambda = \frac{1}{\{f, g\}} X_f \wedge X_g - \frac{\Psi \cdot \Phi}{\{f, g\}} S
\]

where

\[
S_{(f,g)} = \left( \frac{\partial g}{\partial y} \nabla f - \frac{\partial f}{\partial y} \nabla g \right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} + (\nabla f \times \nabla g) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}
\]

If \( \Lambda \) is symplectic then \( (\Psi \cdot \Phi)(x,y) \neq 0, \quad \forall (x,y) \).

Remark that in \((15)\), the 2-tensor \( S \) is also a Poisson tensor and \( S^#(df) = S^#(dg) = 0 \). Moreover, around a point \( p \) where \( \{f, g\}(p) \neq 0 \), there exists a function \( h \) with \( \{f, h\} = 1 \), and \( [X_f, X_h] = 0 \) in some neighborhood of \( p \) and we have in the decomposition \((15)\) that the tensor \( X_f \wedge X_h \) is a Poisson tensor. Thus, we have locally a decomposition of \( \Lambda \) as a sum of two Poisson tensors in the form

\[
\Lambda = X_f \wedge X_h - (\Psi \cdot \Phi) S_{(f,h)}.
\]

Notice that in the decomposition formula \((17)\), if \( (\Psi \cdot \Phi)(x,y) = 0 \), the transversal Poisson structure \( S_{(f,h)} \) vanishes in \( p \) and its linear approximation is given by the transversal linear Poisson structure

\[
\Lambda_{\text{lin}}(x,y) = \left( \nabla (\Psi \cdot \Phi)(p) \cdot x + \frac{\partial (\Psi \cdot \Phi)}{\partial y}(p)y \right) S_{(f,h)}(p),
\]

and we recover the classical decomposition theorem by Weinstein \cite{Weinstein}.

For the general theory of Poisson structures the reader can consult the papers by A. Weinstein \cite{Weinstein} and A. Lichnerowicz \cite{Lichnerowicz}, or the books by Karasev and Maslov \cite{Karasev}, I. Vaizman \cite{Vaizman} or Dufour and Zung \cite{DufourZung}.

### 3 The trace operator

In \( \mathbb{R}^4 \), we consider the volume form \( \Omega = dx \wedge dy = dx_1 \wedge dx_2 \wedge dx_3 \wedge dy \) and the trace operator \( D \) (with respect the volume form \( \Omega \)) defined on \( k \)-contravariant tensor fields \( A \) on \( \mathbb{R}^4 \) for \( k = 0, 1, 2, 3, 4 \) as the unique \( (k - 1) \) contravariant tensor \( D(A) \) such that

\[
di_A \Omega = i_{D(A)} \Omega
\]

The trace operator was introduced by Koszul in \cite{Koszul}.

We have the following formulae for \( k \)-contravariant tensor fields:
(a) If \( X = W \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} \) is a vector field on \( \mathbb{R}^4 \), then \( D(X) \) is the smooth function defined through the relation

\[
D(X)\Omega = dx\Omega = L_X\Omega.
\]

In the coordinates \((x, y)\) we have

\[
D(X) = \text{div}(W) + \frac{\partial b}{\partial y}
\]

(19)

Note that in formula (19) we use the symbol \( \text{div}(W) \) to denote the usual divergence operator for three dimensional vector fields. The trace operator is also called the curl operator or divergence operator \[18\].

(b) If \( \Lambda = X \wedge Y \) with \( X, Y \) vector fields

\[
D(X \wedge Y) = [Y, X] + D(Y)X - D(X)Y
\]

(20)

(c) If \( A \) is a p-vector field and \( B \) is a q vector field, we have

\[
D(A \wedge B) = (-1)^q D(A) \wedge B + A \wedge D(B) - (-1)^{q+p}[A, B]
\]

(21)

where \([A, B]\) is the Schouten bracket between contravariant antisymmetric tensors \( A \) and \( B \). The trace operator \( D \) satisfies the relation

\[
D^2 = 0
\]

(22)

Remark 6 From (21) it follows that any 2-tensor \( A \), satisfying \([A, L] = nA\), where \( L \) is the Liouville vector field \( X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \) and \( n \) an integer, has a decomposition given by

\[
A = \frac{1}{n+4}(D(A \wedge L) + D(A) \wedge L).
\]

Notice that the tensor \( A \) is written as a sum of a zero-trace tensor and the wedge product of two vector fields. This decomposition has been used in [5] and [17], to classify the linear and quadratic Poisson tensors on \( \mathbb{R}^4 \).

For completeness we include the following formulae in order to calculate the value of the trace operator \( D \) on the different tensors fields on \( \mathbb{R}^4 \):
(a) For each 4-tensor $A = f \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$,
\[ \mathbf{D}A = \frac{\partial f}{\partial y} \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} - \nabla f \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \]

(b) For a 3-tensor $A = g \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} + \Sigma \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$, we have
\[ \mathbf{D}A = \left( \nabla g + \frac{\partial \Sigma}{\partial y} \right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} - \text{rot}(\Sigma) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \]

(c) For 2-tensors $A = \Psi \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} + \Phi \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$, we have
\[ \mathbf{D}(A) = \left( \text{rot}(\Psi) + \frac{\partial \Phi}{\partial y} \right) \frac{\partial}{\partial x} - \text{div}(\Phi) \frac{\partial}{\partial y} \]

From the formula (21) we see that a 2-tensor $\Lambda$ is a Poisson tensor if and only if
\[ \mathbf{D}(\Lambda \wedge \Lambda) = 2\Lambda \wedge \mathbf{D}(\Lambda). \]  

From (21), if $\Lambda$ is a Poisson tensor, the vector field $Z_\Lambda = \mathbf{D}(\Lambda)$ is called the modular vector field and
\[ [\mathbf{D}(\Lambda), \Lambda] = 0. \]  

If $\Lambda$ takes the form (1), the modular vector field $Z_\Lambda$ takes the form
\[ Z_\Lambda = \left( \text{rot}(\Psi) + \frac{\partial \Phi}{\partial y} \right) \frac{\partial}{\partial x} - \text{div}(\Phi) \frac{\partial}{\partial y}. \]

**Remark 7** Notice that the trace operator depends on the volume form $\Omega$. If we consider another volume form $\tilde{\Omega} = f\Omega$ with $f > 0$, the value of the trace operator $\tilde{\mathbf{D}}$ defined by the correspondent formula (18) on a $p$-tensor $A$ is related to value of the initial $\mathbf{D}$ in the following form
\[ \tilde{\mathbf{D}}(A) = \mathbf{D}(A) + (-1)^p[f, A] \]

For more information about the trace operator see [18], [15], [14]. The modular vector field and its meaning for general Poisson structures has been studied by A. Weinstein in [20].

4 Symmetries of Jacobi equation on $\mathbb{R}^4$

Consider a Poisson tensor $\Lambda = (\Psi, \Phi)$ on $\mathbb{R}^4$. A diffeomorphism $F : \mathbb{R}^4 \to \mathbb{R}^4$ with $F(x, y) = (S(x, y), h(x, y))$, where $S : \mathbb{R}^4 \to \mathbb{R}^3$ and $h : \mathbb{R}^4 \to \mathbb{R}$ are smooth functions, is called a Poisson map if it preserves the Poisson tensor or equivalently preserves the Poisson bracket
\[ \{f, g\} \circ F = \{f \circ F, g \circ F\}, \quad \forall \ f, g \in C^\infty(\mathbb{R}^4). \]
The condition \((27)\) in terms of the vector functions \((\Psi, \Phi)\) associated to the Poisson tensor \(\Lambda\) takes the expression

\[
\Psi(S(x, y), h(x, y)) = \det(D_x S)(D_x^{-1} S)^T(\Psi) + D_x S(\Phi) \times \frac{\partial S}{\partial y}, \tag{28}
\]

\[
\Phi(S(x, y), h(x, y)) = -D_x S(\Psi \times \nabla_x h) + \frac{\partial h}{\partial y} D_x S(\Phi) - (\Phi \cdot \nabla_x h) \frac{\partial S}{\partial y}. \tag{29}
\]

Here, \(D_x S\) denotes the differential of \(S\) with respect the variable \(x = (x_1, x_2, x_3)\). Notice that expression \((29)\) is equivalent to

\[
\Phi(S(x, y), h(x, y)) = -DF(X_h), \tag{30}
\]

where \(X_h\) is the Hamiltonian vector field corresponding to function \(h(x, y)\) relative to the Poisson structure \((\Psi, \Phi)\). In any case, the function \(\Phi \cdot \Psi\) is transformed into

\[(\Phi \cdot \Psi) \circ F = (\det DF)(\Phi \cdot \Psi).\]

**Remark 8** If the function \(h\) in the diffeomorphism \(F(x, y) = (S(x, y), h(x, y))\) satisfies \(\Lambda^\#(dh) = 0\), with \(\frac{\partial}{\partial y} h \neq 0\), then the transformed Poisson tensor is a \(y\)-parametrized family of three dimensional Poisson tensors.

Given a smooth function \(k : W \subset \mathbb{R}^4 \to \mathbb{R}\), where \(W\) is an open set, the Poisson tensor fields \(\Lambda\) \(\ref{1}\) having \(k\) as a Casimir function all have the following representation

\[
\Lambda = \left( f \nabla k + \frac{\partial k}{\partial y} A \right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} + (A \times \nabla k) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}, \tag{31}
\]

where \(A(x, y)\) is a smooth vectorial function with values on \(\mathbb{R}^3\) and \(f\) a smooth real function satisfying the Jacobi equation

\[
\left( \nabla f + \frac{\partial A}{\partial y} \right) \times A - f \ \text{rot}(A) \cdot \nabla k - \frac{\partial k}{\partial y} (A \cdot \text{rot}(A) = 0. \tag{32}
\]

**Remark 9** Condition \((32)\) means that the 2-contravariant tensor

\[
\Xi = \left( \nabla f + \frac{\partial A}{\partial y} \right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} - \text{rot}(A) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y},
\]

vanishes when it is valued on the 1-forms \(\nabla k dx + \frac{\partial k}{\partial y} dy\) and \(A dx - f dy\).

Similarly, the Poisson structures \(\Lambda\) having 2 independent global Casimir functions \(k_1, k_2\) on some domain, take the form

\[
\Lambda = f \left( \frac{\partial k_1}{\partial y} \nabla k_1 - \frac{\partial k_1}{\partial x} \nabla k_2 \right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} + f(\nabla k_1 \times \nabla k_2) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}, \tag{33}
\]

where \(f\) is an arbitrary smooth function on \(\mathbb{R}^4\).
Example 10  The linear Poisson tensors $\Lambda$ having as Casimir function an homogeneous polynomial of degree two $k_1 = \frac{1}{2}x^T M x + (\alpha \cdot x)y + \frac{1}{2}by^2$, with $M = M^T$, $\alpha \in \mathbb{R}^3$ and $b \in \mathbb{R}$, take the form

$$\Lambda = \left( c(Mx + y\alpha) + (\alpha \cdot x + by)A \right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} + (A \times (Mx + y\alpha)) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$$

where $A \in \mathbb{R}^3$, $c \in \mathbb{R}$. In this case, we have another global Casimir function $k_2$ given by the linear function

$$k_2 = A \cdot x - cy$$

In general, applying the transformation formulae \cite{[28], [29]}, the Poisson tensors with one and two global Casimir function take the following normal form:

**Proposition 11**  Any Poisson tensor $\Lambda = (\Psi, \Phi)$ on $\mathbb{R}^4$ having a global Casimir function $k$ with $\frac{\partial k}{\partial y} \neq 0$ is transformed under the diffeomorphism $F : \mathbb{R}^4 \to \mathbb{R}^4$ with $F(x, y) = (x, k(x, y))$ into the normal form

$$\tilde{\Lambda} = \tilde{\Psi}(x, y) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x}. \quad (34)$$

The constructions of Poisson brackets on smooth manifolds with prescribed Casimir functions has been studied in \cite{[19]}.

Given a solution $\Lambda$ of Jacobi equation (1), a vector field $X = W \frac{\partial}{\partial x} + b \frac{\partial}{\partial y}$ whose flow is a 1-parameter group of Poisson maps, is called an infinitesimal automorphism of $\Lambda$ or Poisson vector field. In terms of the Poisson bracket, $W$ is a Poisson vector field if

$$L_X \{ f, g \} = \{ L_X f, g \} + \{ f, L_X g \}, \quad (35)$$

for any $f, g$ smooth functions. The condition (35) for Poisson tensors $(\Psi, \Phi)$ and vector fields $X$ on $\mathbb{R}^4$ takes the form

$$D(\Lambda \wedge X) + D(\Lambda) \wedge X - D(X) \wedge \Lambda = 0,$$

and in global coordinates $(x, y)$

$$\nabla(\Psi \cdot W) = (\text{div } W)\Psi - W \times \text{rot } \Psi + b \frac{\partial \Psi}{\partial y} + \frac{\partial W}{\partial y} \times \Phi = 0, \quad (36)$$

$$\text{rot}(\Phi \times W) - \text{div}(W)\Phi + \text{div}(\Phi)W + \Psi \times \nabla b - \frac{\partial b}{\partial y} \Phi + b \frac{\partial \Phi}{\partial y} = 0. \quad (37)$$

The space of Poisson vector fields is a Lie algebra of vector fields.

The Poisson vector fields can be tangent or transversal to the symplectic leaves. The tangent Poisson vector fields take the form

$$W = (\Psi \times \alpha - g\Phi) \frac{\partial}{\partial x} + \alpha \cdot \Phi \frac{\partial}{\partial y}. \quad (38)$$
where $\alpha$ is a vector function and $g$ a smooth function such that

$$
\left( -\nabla g + \frac{\partial \alpha}{\partial y} \right) \cdot \Phi - \Psi \cdot \text{rot}(\alpha) \right) \Psi = (\Phi \cdot \Psi) \left( -\nabla g + \frac{\partial \alpha}{\partial y} \right). \tag{39} $$

$$
\left( -\nabla g + \frac{\partial \alpha}{\partial y} \right) \cdot \Phi - \Psi \cdot \text{rot}(\alpha) \right) \Phi = - (\Phi \cdot \Psi) \text{rot}(\alpha). \tag{40} $$

The conditions (39) (40) means that the 1-form $\omega = \alpha dx + g dy$ is closed on vector fields tangent to the symplectic leaves.

Given a symplectic Poisson tensor $\Lambda$ and a pair of smooth functions $f, g$ with $\{f, g\} \neq 0$, the 2-tensor

$$
\Delta = \Lambda - \frac{1}{\{f, g\}} X_f \wedge X_g, \tag{41} $$

is also a Poisson tensor called the Dirac Poisson tensor defined by the pair of smooth functions $f$ and $g$. The smooth functions $f$ and $g$ are Casimir functions and the vector fields

$$
W_1 = \frac{1}{\{f, g\}} X_f \\
W_2 = \frac{1}{\{f, g\}} X_g
$$

are independent Poisson vector fields transversal to the symplectic leaves of (41). The regular Poisson structures having a maximal number of transversal independent Poisson vector fields are called transversally constant Poisson structures [10].

**Proposition 12** The modular tensor field $Z_\Lambda$ (25), associated to a Poisson tensor $\Lambda$ on $\mathbb{R}^4$ has the following properties:

1. $Z_\Lambda$ is a Poisson vector field with zero trace and tangent to the level sets of $\Phi \cdot \Psi$.

2. For each Hamiltonian vector field $X_H$ we have:

$$
D(X_H) = -L_{Z_\Lambda} H, \tag{42} $$

$$
[X_H, Z_\Lambda] = -\Lambda^\#(d(D(X_H))), \tag{43} $$

where $L_{Z_\Lambda} H$ denotes the Lie derivative of $H$ along $Z_\Lambda$.

3. The commutator of each Poisson vector field $W$ with the modular vector field $Z_\Lambda$ is a Hamiltonian vector field with Hamiltonian function given by the trace function of $W$

$$
[W, Z_\Lambda] = -\Lambda^\#(d(D(W))). \tag{44} $$

From Proposition 12 we deduce the following facts:
(a) If the modular Poisson vector field vanishes, then each Hamiltonian vector field has zero trace. Moreover if all Hamiltonian vector fields have zero trace, then the modular vector field vanishes.

(b) From (44), the commutator of $Z_\Lambda$ with any other Poisson vector field $W$ is a Hamiltonian vector field. The trace of any Poisson vector field commuting with the modular vector field $Z_\Lambda$ is a Casimir function.

Notice that $\Lambda^\#(d(\Phi \cdot \Psi)) = (\Phi \cdot \Psi)Z_\Lambda$ and for each Hamiltonian vector field with Hamiltonian function $H$, we have

\[
(\Phi \cdot \Psi)L_{Z_\Lambda}H = -L_X H(\Phi \cdot \Psi),
\]

then, if $H$ is a first integral of the modular vector field $Z_\Lambda$, the Hamiltonian vector field $X_H$ is tangent to the level set of the function $\Phi \cdot \Psi$. In the contrary, at any point $p \in \mathbb{R}^4$ where $(\Phi \cdot \Psi)(p) \neq 0$ and $L_{Z_\Lambda} H(p) \neq 0$, the Hamiltonian vector field $X_H$ is transversal to the level set of $\Phi \cdot \Psi$.

Example 13 The quadratic Poisson tensor \[5\]

\[
\Lambda = -x_1 x_2 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + x_1 x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} - y x_1 \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial x_1} - y x_2 \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial x_2} + y x_3 \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial x_3}
\]

has as modular Poisson vector field given by

\[
Z_\Lambda = -3 x_1 \frac{\partial}{\partial x_1} + 2 x_3 \frac{\partial}{\partial x_3} + y \frac{\partial}{\partial y}.
\]

In this case, we have two open 4-dimensional symplectic leaves given by $\mathcal{L} = \{(x,y) \text{ with } x_1 x_2 x_3 y > 0(< 0)\}$. The set of points $(x,y)$ where rank $\Lambda(x,y) = 2$ is $\{(x,y) \text{ with } x_1^2 + y^2 \neq 0, \ x_2 x_3 = 0, \ x_2 \neq x_3\}$, and the zero dimensional leaves are given by the points $\{(0, x_2, x_3, 0) \text{ or } (0, y) \text{ with } y \neq 0\}$. The modular vector field (46) is in general transversal to the 2-dimensional leaves.

For any Poisson tensor we have:

**Proposition 14 (Decomposition of Poisson tensors)** Each Poisson tensor \[1\] can be written in the following form

\[
(d \Phi) \Lambda = Z_\Lambda \wedge \Phi \frac{\partial}{\partial x} + \nabla(\Phi \cdot \Psi) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x}
\]

Moreover, $\nabla(\Phi \cdot \Psi) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x}$ is also a Poisson tensor. If $d\Phi = c$ with $\mathbb{R} \ni c \neq 0$, the 2-tensor $Z_\Lambda \wedge \Phi \frac{\partial}{\partial x}$ is also a Poisson tensor and $\Lambda$ is the sum of two Poisson tensors. If $d\Phi \neq 0$, we have the decomposition

\[
\Lambda = Z_\Lambda \wedge \left( \frac{\Phi}{d\Phi} \right) \frac{\partial}{\partial x} + \Lambda_0,
\]

with

\[
\Lambda_0 = \frac{1}{d\Phi} \nabla(\Phi \cdot \Psi) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x}.
\]
Remark 15 If the Poisson tensor is linear with \((\Psi, \Phi)\) given by (13), (14) and \(\text{div} \Phi \neq 0\), then the decomposition formula (47) can be rewritten as

\[
\Lambda = Z_\Lambda \wedge \Sigma \frac{\partial}{\partial x} + \nabla (\Sigma \cdot \Psi) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} \nabla (\Sigma \cdot \Phi) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} (48)
\]

with \(\Sigma = \frac{1}{\text{div} \Phi} \Phi\). In this case we obtain the decomposition formula given in [10].

A Poisson tensor \(\Lambda\) is called a Poisson-Liouville tensor if the modular vector field vanishes \(Z_\Lambda = 0\) [18]. In simply connected domains of \(\mathbb{R}^4\), the Poisson-Liouville tensors \(\Lambda\) take the form

\[
\Lambda = \left( \nabla f + \frac{\partial \Sigma}{\partial y} \right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} - \text{rot}(\Sigma) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \quad (49)
\]

for some vector function \(\Sigma\) and a smooth function \(f\) satisfying

\[
(\nabla f + \frac{\partial \Sigma}{\partial y}) \cdot \text{rot}(\Sigma) = c = \text{constant} \quad (50)
\]

If the constant \(c\) in (50) is non-zero, the Poisson-Liouville tensor (49) is symplectic and if \(c = 0\), then \(\Lambda\) is a regular Poisson tensor with rank 2. From (49) we have that Poisson-Liouville tensors on \(\mathbb{R}^4\) are parametrized by a smooth function \(f\) and a vector function \(\Sigma\) satisfying (50).

The Poisson-Liouville tensors have the following properties: a) The Poisson vector fields tangent to the symplectic foliation have zero trace; b) The trace of a Poisson vector field transversal to symplectic leaves is a Casimir function; c) For any 2-tensor \(\Theta\) commuting under the Schouten bracket with the Poisson-Liouville tensor \(\Lambda\), the vector field \(D(\Theta)\) is a Poisson vector field.

We notice that for a Poisson-Liouville tensor, the flow of Hamiltonian vector fields preserve the volume form. Moreover, from (44), for any Poisson-Liouville tensor the trace of a Poisson vector field is a Casimir function. An interesting example of Liouville-Poisson tensors are those in \(\mathbb{R}^4\) of the form

\[
\Lambda = \nabla G(x,y) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x}
\]

Here, we have two global Casimir functions \(k_1(x,y) = y\) and \(k_2 = G(x,y)\).

Example 16 The linear Poisson-Liouville tensors on \(\mathbb{R}^4\) are given by linear vector functions \((\Psi, \Phi)\) as in (13), (14) satisfying:

(a) \(2p + \beta = 0\),
(b) \(\text{Tr}(N) = 0\),
(c) \(2Mp = N\alpha - q \times \alpha\),
(d) \((M + \Lambda \circ p)(N + \Lambda \circ q) + (N - \Lambda \circ q)(M + \Lambda \circ p) = 0\).

The rank of a linear Liouville-Poisson structure is constant.
5 Regular Poisson tensors

An important class of 4-dimensional Poisson tensors on $\mathbb{R}^4$ are those whose symplectic foliation is a regular foliation in the sense of Frobenius. We have two cases: one is the symplectic case, when the rank of the Poisson tensor is 4, and the other one, when each symplectic leaf has dimension 2. Here, we will consider Poisson tensors defined in open dense sets of $\mathbb{R}^4$.

A 2-contravariant tensor $\Lambda = (\Psi, \Phi)$ defines a symplectic structure on a domain of $\mathbb{R}^4$ if and only if

\begin{align}
(\Phi \cdot \Psi) & \neq 0, \\
\text{div}(\frac{\Phi}{\Phi \cdot \Psi}) & = 0, \\
\frac{\partial}{\partial y} (\frac{\Phi}{\Phi \cdot \Psi}) + \text{rot}(\frac{\Psi}{\Phi \cdot \Psi}) & = 0
\end{align}

If the domain is simply connected, from (52) we conclude that $\Phi = (\Phi \cdot \Psi) \text{rot}(\Xi)$ for some vector function $\Xi$ and from (53), there exists a vector function $\Xi$ and a smooth function $h$ such that

\begin{align}
\frac{\Phi}{\Phi \cdot \Psi} & = \text{rot}(\Xi), \\
\frac{\Psi}{\Phi \cdot \Psi} & = \nabla h - \frac{\partial \Xi}{\partial y}.
\end{align}

Hence, a Poisson tensor $\Lambda$ define a symplectic structure on a simply connected domain of $\mathbb{R}^4$ if and only if it has the form

\[ \Lambda = f \left( \nabla h - \frac{\partial \Xi}{\partial y} \right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} + f \left( \text{rot} \ (\Xi) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \right) \]

with $\Xi$ a smooth vector function and $f$ a smooth scalar function such that $f((\nabla h - \frac{\partial \Xi}{\partial y}) \cdot \text{rot}(\Xi)) = 1$. In this case the symplectic 2-form $\Omega$ takes the form

\[ \Omega = - \text{rot}(\Xi)d\mathbf{x} \wedge d\mathbf{x} - \left( \nabla h - \frac{\partial \Xi}{\partial y} \right) d\mathbf{x} \wedge dy. \]

In the symplectic case, the modular vector field $Z_\Lambda$ is a Hamiltonian vector field with Hamiltonian function $H = - \ln |f|$.

Now, let us consider the case of Poisson tensors whose characteristic foliation is regular of dimension 2 in some open dense set $O$. The Poisson tensors in this case take the forms:

\[ \Lambda = (\Phi \times \Sigma) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} + \Phi \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \]

\[ = \Phi \frac{\partial}{\partial x} \wedge \left( \Sigma \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) \]

14
with Φ ≠ 0 on \( O \) or
\[
\Lambda = \Psi(x,y) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x},
\]
(55)
with \( \Psi \neq 0 \) on \( O \).

In the first case (54), the relations (7)(8) coming from the Jacobi equation reduce to single
\[
\Phi \times \left( [\Sigma, \Phi] + \frac{\partial \Phi}{\partial y} \right) = 0,
\]
or equivalently to
\[
\Phi \times \left( \text{rot}(\Phi \times \Sigma) + \text{div}(\Phi) \Sigma + \frac{\partial \Phi}{\partial y} \right) = 0.
\]
(57)
In the second case (55), the Jacobi equation has the form
\[
\Psi \cdot \text{rot}(\Psi) = 0.
\]
(58)
Note that if the vector functions \( \Phi \) and \( \Sigma \) satisfies (56), the same holds for \( \lambda \Phi \) and \( \Sigma \) for any smooth scalar function \( \lambda \). Each Poisson tensor \( \Lambda \) of rank 2 of the form (54) can be written as
\[
\Lambda = \frac{1}{\{f, g\}} X_f \wedge X_g,
\]
where \( f, g \) are smooth functions such that \( \{f, g\} \neq 0 \). In particular, in the open set \( \Phi \times x \neq 0 \) we have
\[
\Lambda = \frac{-1}{\Phi \times x} \Lambda\# \left( \frac{1}{2} dx^2 \right) \wedge \Lambda\#(dy).
\]
(60)

Example 17 The linear Poisson tensors of rank 2 can be written as
\[
\Lambda = (A x + p \times x + y) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x},
\]
with \( A p = 0 \), \( p \cdot \alpha = 0 \), or
\[
\Lambda = (m \cdot x + y b)(A \times B) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} + (m \cdot x + y b) A \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}
\]

Remark 18 From the expression (56) and (63) we have that the Poisson-Liouville tensors \( \Lambda \) on \( \mathbb{R}^4 \) with rank = 2 are of the form (74) with \( \text{div}(\Phi \frac{\partial}{\partial x}) = 0 \). In this case, we have
\[
\Sigma \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \Phi \frac{\partial}{\partial x} = - \text{div}(\Sigma) \Phi \frac{\partial}{\partial x},
\]
and locally \( \Lambda \) take the form
\[
\Lambda = \left( \frac{\partial k_2}{\partial y} \right) \nabla k_1 - \left( \frac{\partial k_1}{\partial y} \right) \nabla k_2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} + (\nabla k_1 \times \nabla k_2) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y},
\]
(61)
where \( k_1, k_2 \) local Casimir functions. Note that if \( \text{div}(\Sigma) = 0 \), then the symplectic leaves are torii.
The relations (56) is an integrability condition for the distribution generated by the vector fields $\Sigma \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and $\Phi \frac{\partial}{\partial x}$. In this case, the Hamiltonian vector field $X_H$, at each point, belongs to the distribution generated by the vector fields $\Phi \frac{\partial}{\partial x}$ and $\Sigma \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and

$$X_H = - \left( L_{\Sigma \frac{\partial}{\partial x} + \frac{\partial}{\partial y}} H \right) \Phi \frac{\partial}{\partial x} + \left( L_{\Phi \frac{\partial}{\partial x}} H \right) \left( \Sigma \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right).$$

(62)

Moreover, the modular vector field $Z_{\Lambda}$ (25) takes the form

$$Z_{\Lambda} = \left( \left[ \Sigma \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \Phi \frac{\partial}{\partial x} \right] - \text{div}(\Phi) \left( \Sigma \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) + \text{div}((\Sigma) \frac{\partial}{\partial x}) \right),$$

(63)

and taking into account (56) we verify that $Z_{\Lambda}$ in (63) is always a tangent Poisson vector field to the symplectic leaves. Note that the relation (63) allows us to write the Poisson tensor (54) as the wedge product between a Poisson vector field and a Hamiltonian vector field modulus the smooth function $\text{div}(\Phi)$.

$$\text{div}(\Phi) \Lambda = Z_{\Lambda} \wedge \left( \Phi \frac{\partial}{\partial x} \right).$$

(64)

**Proposition 19** A regular four dimensional Poisson tensor of the form (54), has the following properties:

1. The symplectic foliation is spanned by the vector fields $\Phi \frac{\partial}{\partial x}$ and $\Sigma \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$.

2. The vector fields $X = (\Phi \times \Sigma) \frac{\partial}{\partial x}$ and $Y = \Phi \times (\Phi \times \Sigma) \frac{\partial}{\partial x} + (\Phi \times \Sigma)^2 \frac{\partial}{\partial y}$ are transversal to the symplectic foliation and the foliated symplectic $\Omega$ form takes the expression

$$\Omega = \frac{1}{\Phi^2 + (\Phi \times \Sigma)^2} (\Phi dx) \wedge (\Sigma dx + dy)$$

(65)

3. The Casimir functions $k$ are those functions satisfying $\Sigma \cdot \nabla k + \frac{\partial k}{\partial y} = 0$ and $\Phi \cdot \nabla k = 0$ and locally the Poisson structure takes the form

$$\Lambda = f \left[ \left( \frac{\partial k_2}{\partial y} \nabla k_1 - \frac{\partial k_1}{\partial y} \nabla k_2 \right) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} + (\nabla k_1 \times \nabla k_2) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \right],$$

(66)

where $k_1, k_2$ are two independent Casimir functions and $f = f(x,y)$ is some non-zero smooth function.

For regular Poisson tensors with rank 2, the Poisson vector fields tangent to the symplectic leaves take the form

$$W = (\Psi \times \alpha - g \Phi) \frac{\partial}{\partial x} + \alpha \cdot \Phi \frac{\partial}{\partial y}$$

(67)

where $\alpha$ is a vector function and $g$ a smooth function satisfying

$$f = (\nabla g + \frac{\partial \alpha}{\partial y}) \cdot \Phi = \Psi \cdot \text{rot}(\alpha),$$

and we have the following
Proposition 20 For a regular Poisson tensor $\Lambda$ of rank 2, if the tangent vector field $W$ to the characteristic foliation has zero trace, then $W$ is a Poisson vector field if and only if the modular vector field $Z$ annihilates the 1-form $\alpha dx + gdy$.

With respect to the existence of Poisson vector fields transversal to the characteristic foliation for regular Poisson tensors of rank 2, locally we can use its local expression given by (66), and note that the vector fields

$$V_1 = \frac{1}{(\nabla k_1 \times \nabla k_2)^2} \nabla k_2 \times (\nabla k_1 \times \nabla k_2) \frac{\partial}{\partial x},$$  

(68)

$$V_2 = \frac{-1}{(\nabla k_1 \times \nabla k_2)^2} \nabla k_1 \times (\nabla k_1 \times \nabla k_2) \frac{\partial}{\partial x},$$  

(69)

are dual to the Casimir functions $k_1$ and $k_2$ and preserve the characteristic foliation. Moreover, making a direct calculation we can check that

$$[\Lambda, V_i] = (- \text{div}(V_i) + V_i \cdot \nabla (\ln |f|)) \Lambda, \quad i = 1, 2.$$  

(70)

Thus, if $(- \text{div}(V_i) + V_i \cdot \nabla (\ln |f|)) = 0$ for $i = 1$ or $i = 2$, the correspondent vector field $V_i$ is a Poisson vector field transversal to the symplectic leaves and the Poisson-Lichnerowicz Cohomology space $H_1^\Lambda(\mathcal{O})$ can not be trivial.

In the case when the Poisson tensor takes the form (55), we have $k_1(x, y) = y$ as a Casimir function and denote by $k$ the other Casimir. The Poisson tensor takes the form

$$\Lambda = f \nabla k \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x},$$  

(71)

with $f$ a non-zero smooth function. Here, we have two independent transversal vector fields preserving the symplectic foliation

$$V_1 = \frac{\nabla k}{\| \nabla k \|^2} \frac{\partial}{\partial x},$$

$$V_2 = \frac{-\partial k}{\partial y} \frac{\nabla k}{\| \nabla k \|^2} \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

and the preserving foliation vector fields $Z$ take the form

$$Z = \lambda_1(x, y)V_1 + \lambda_2(x, y)V_2 + \Lambda^\#(\alpha)$$

with $\alpha$ a 1-form and $\lambda_1, \lambda_2$ are Casimir functions. From the relation (22) we obtain the following relations

$$[V_1, \Lambda] = (- \text{div}(V_1) + LV_1(\ln |f|)) \Lambda,$$

$$[V_2, \Lambda] = (- \text{div}(V_2) + LV_2(\ln |f|)) \Lambda.$$

Taking into account that for any tangent vector field $T = \nabla k \times \alpha \frac{\partial}{\partial x}$ we have

$$[\Lambda, \nabla k \times \alpha \frac{\partial}{\partial x}] = -\text{div}(\alpha \times \nabla k) \Lambda,$$

then, the vector field $Z$ is a Poisson vector field
for Λ if and only if there exists Casimir functions λ₁, λ₂ and a vector function α satisfying

\[ λ₁(-\text{div}(V₁) + L_{V₁}(ln |f|)) + λ₂(-\text{div}(V₂) + L_{V₂}(ln |f|)) = \text{div}(α × \nabla k). \]

Taking into account the above calculations, we have the following result about the existence of transversal Poisson vector fields for the Poisson tensor (71).

**Proposition 21** If for the Poisson tensor (71) we have

\[-\text{div}(V₁) + L_{V₁}(ln |f|) \neq 0\]

and there exists a smooth vector function α(x, y) such that the scalar function

\[ λ(x, y) = \frac{\nabla k \cdot \text{rot}(α) + \text{div}(V₂) - L_{V₂}(ln |f|)}{-\text{div}(V₁) + L_{V₁}(ln |f|)}, \]

is a Casimir function for (71), then the vector field \( Z = λ(x, y)V₁ + V₂ + Λ(α) \) is a transversal Poisson vector field to the characteristic foliation of Λ.

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