Characters of $\mathcal{U}_q(gl(n))$-reflection equation algebra

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Abstract. We list characters (one-dimensional representations) of the reflection equation algebra associated with the fundamental vector representation of the Drinfeld-Jimbo quantum group $\mathcal{U}_q(gl(n))$.

1. Introduction

Let $V$ be a complex vector space and $S \in \text{End}^{\otimes 2}(V)$ a Yang-Baxter operator, satisfying the braid identity

$$S_{12}S_{23}S_{12} = S_{23}S_{12}S_{23}. $$

Here, $S_{12} = S \otimes 1$ and $S_{23} = 1 \otimes S$. A matrix $A \in \text{End}(V)$ is called a solution to the numerical reflection equation (RE) or a numerical RE matrix if the equality

$$SA_2SA_2 = A_2SA_2S, $$

(1)

where $A_2 = 1 \otimes A$, holds in $\text{End}^{\otimes 2}(V)$. Numerical RE matrices define one-dimensional representations (characters) of the RE algebra, the quotient of the tensor algebra $T(\text{End}^*(V))$ by quadratic relations \[ [\text{KS}], [\text{KS}], \text{RE} \] and related algebraic structures appeared in the theory of integrable models \[ [\text{Cher}, \text{Sk}], [\text{AFS}] \]; they find applications in 3-dimensional topology \[ [\text{K}] \] and problems of covariant quantization on G-spaces \[ [\text{Do1}, \text{Do2}] \]. In the latter case, they are related to representations of quantum groups on $V$; then the operator $S$ is equal to $PR$, the product of the permutation $P$ in $V \otimes V$ and the image $R$ of the universal R-matrix, \[ [\text{Dr}] \]. Numerical solutions to equation (1) are the key ingredient in the character method of covariant quantization on homogeneous spaces developed in \[ [\text{DoM1}, \text{DoM2}] \].

In the present paper we study equation (1) associated with the fundamental vector representation of the quantum group $\mathcal{U}_q(gl(n))$. An example of solution to this equation is the matrix $D_n = \lambda \sum_{i=1}^{n} e_{n+1-i}^i$, $\lambda \in \mathbb{C}$, as well as its similarity transformation by a diagonal invertible matrix, according to \[ [\text{KSS}] \]. Therein, all
the non-degenerate RE matrices were presented in dimensions $n = 2, 3, 4$:

\[
A^{1,1} = \begin{pmatrix} \lambda + \mu & y_1 \\ y_2 & \lambda \end{pmatrix}, \quad A^{2,1} = \begin{pmatrix} \lambda + \mu & y_1 \\ y_3 & \lambda \end{pmatrix},
\]

\[
A^{2,2} = \begin{pmatrix} \lambda + \mu & y_1 \\ \lambda + \mu & y_2 \\ y_3 & \lambda \end{pmatrix}, \quad A^{3,1} = \begin{pmatrix} \lambda + \mu & y_1 \\ \lambda & \lambda \end{pmatrix},
\]

where \( y_i y_{i+1} = -\lambda \mu \neq 0 \).

Here, the parameterization is adapted to our exposition. Besides, in \([\text{DoM1}]\), there were found diagonal numerical RE matrices \( P_k = \lambda \sum_{i=1}^{k} e_i, \ k \leq n \). The above mentioned matrices exhaust all (as far as we can judge) known solutions to \([\mathbb{I}]\) associated to the standard quantum group \( \mathcal{U}_q(gl(n)) \). In the present paper, we list all possible solutions.

2. Result

The image of the universal R-matrix of the Drinfeld-Jimbo quantum group \( \mathcal{U}_q(gl(n)) \) in the fundamental vector representation on \( \mathbb{C}^n \) is, \([\text{FRT}]\),

\[
(2) \quad R = q \sum_{i=1, \ldots, n} e_i^i \otimes e_i^i + \sum_{i,j=1 \atop i < j}^{1, \ldots, n} e_i^i \otimes e_j^j + \omega \sum_{i,k=1 \atop i < k}^{1, \ldots, n} e_i^k \otimes e_k^i.
\]

Here, \( \{e_i^j\} \) is the standard basis in \( \text{End}(\mathbb{C}^n) \) with the multiplication \( e_i^j e_k^l = \delta_{jk} e_i^l \) expressed through the Kronecker symbols \( \delta_{jk} \); \( \omega \) stands for \( q-q^{-1} \). It is convenient to represent the corresponding braid matrix in the form

\[
(3) \quad S = \sum_{i,k=1, \ldots, n} s_{ik} e_i^i \otimes e_k^k + \sum_{i,j=1 \atop i < j}^{1, \ldots, n} e_i^j \otimes e_j^i, \quad \text{where} \quad s_{ik} = \begin{cases} \omega, & i < k, \\ q, & i = k, \\ 0, & i > k, \end{cases}
\]

We shall deal with subsets in \( \mathbb{Z} \) and denote by \([a, b]\) the intervals \( \{k \in \mathbb{Z} | a \leq k \leq b\} \).

Instead of the square brackets, we use parentheses for the intervals defined by strict inequalities. To formulate the classification theorem, we need the following data.

**Definition 2.1.** An admissible pair \((Y, \sigma)\) consists of an ordered subset \(Y \subset I = [1, n]\) and a decreasing injective map \(\sigma : Y \to I\) without stable points.

Clearly, the map \(\sigma\) is determined by its image \(\sigma(Y)\). In the set \(Y\), we distinguish two non-intersecting subsets \(Y_+ = \{i \in Y | i > \sigma(i)\}\) and \(Y_- = \{i \in Y | i < \sigma(i)\}\); obviously \(Y = Y_- \cup Y_+\). Denote \(b_- = \max\{Y_- \cup \sigma(Y_-)\}\) and \(b_+ = \min\{Y_+ \cup \sigma(Y_+)\}\).
Because the injection $\sigma$ is decreasing, $b_- < b_+$. We adopt the convention $b_- = 0$ and $b_+ = n + 1$ if $Y = \emptyset$.

**Theorem 2.2 (Classification).** General solution to equation (1) with the braid matrix (3) has the form

$$A = \sum_{i \in I} x_i e_i^\alpha + \sum_{j \in Y} u_j e^{\sigma(j)}_j,$$

where

- $(Y, \sigma)$ is an admissible pair,

$$Y = [1, b_-] \cup [b_+, b_+ + b_- - 1],$$

$$\sigma(i) = b_+ + b_- - i$$

for $b_-, b_+ \in \mathbb{N}$ subject to the conditions $b_- < b_+$, $b_- + b_+ \leq n + 1$ and

$$x_i = \begin{cases} 
\lambda + \mu, & i \in [1, b_-], \\
\lambda, & i \in (b_-, b_+), \\
0, & i \in [b_+, n], 
\end{cases}$$

$$y_{i, \sigma(i)} = -\lambda \mu \neq 0$$

for $\lambda, \mu \in \mathbb{C}$;

- $(Y, \sigma)$ is an admissible pair such that $Y \cap \sigma(Y) = \emptyset$ and

$$x_i = \begin{cases} 
\lambda, & i \in [1, b], \\
0, & i \in (b, n], 
\end{cases}$$

$$y_i \neq 0$$

for $b \in [b_-, b_+]$ and $\lambda \in \mathbb{C}$.

**Remark 2.3.** As follows from the theorem, there are two classes of numerical RE matrices, those corresponding to $\sigma(Y) = Y$ and $\sigma(Y) \cap Y = \emptyset$. We call them solutions of Type 1 and 2, respectively.

**3. Proof of the classification theorem**

The proof of Theorem (2.2) formulated in the previous section is combinatorial. It is a result of the direct analysis of equation (1) organized into a sequence of lemmas. Putting $A = \sum_{\alpha, \beta = 1}^n A_\beta^\alpha e_\alpha^\beta$, we compute

$$A_2 S A_2 = \sum_{i, j, \alpha, \beta} s_{i, j, \alpha, \beta} A_\beta^\alpha A_\alpha^\nu e_i^\nu \otimes e_\alpha^\beta + \sum_{i, j, \alpha, \beta} A_i^\alpha A_j^\beta e_i^\alpha \otimes e_\beta^\beta.$$
Substituting this into the left- and right-hand sides of (13), we rewrite it in a more explicit form:

\[
SA_2SA_2 = \sum_{i,\beta,\alpha,\nu} s_{i\beta} s_{i\nu} A_{i\beta}^\nu A_{i\nu}^\alpha e_i^\alpha \otimes e_i^\beta + \sum_{i,j,\beta,\alpha} s_{i\beta} A_{i\alpha}^j e_j^\alpha \otimes e_j^\beta
\]

\[
+ \sum_{i,\beta,\alpha,\nu} s_{i\beta} A_{i\nu}^\alpha e_i^\beta \otimes e_i^\alpha + \sum_{i,j,\beta,\alpha} A_{i\beta}^\alpha e_j^\beta \otimes e_j^\alpha,
\]

(11) \hspace{1cm} A_2SA_2S = \sum_{i,\beta,\alpha,\nu} s_{i\alpha} s_{i\beta} A_{i\alpha}^\nu A_{i\beta}^\nu e_i^\alpha \otimes e_i^\beta + \sum_{i,j,\beta,\alpha} s_{j\beta} A_{i\alpha}^j e_j^\beta \otimes e_j^\alpha

\[
+ \sum_{i,\beta,\alpha,\nu} s_{i\beta} A_{i\nu}^\alpha e_i^\alpha \otimes e_i^\beta + \sum_{i,j,\beta,\alpha} A_{i\beta}^\alpha e_j^\beta \otimes e_j^\alpha.
\]

(12) Comparison of (11) with (12) gives rise to the system of quadratic equations on the matrix elements \( A_{ij} \).

**Lemma 3.1.** Equation (11) is equivalent to the following system of equations:

\[
\begin{align*}
& A_{ij}^\mu A_{ij}^\nu = 0, & m \neq n \neq i \neq j, \\
& A_{ij}^\mu A_{ij}^\nu = 0, & \{j \neq m \neq n \neq i, (m-i)(n-j) < 0, \\
& A_{ij}^\mu A_{ij}^\nu = 0, & j \neq m \neq n \neq i, (m-i)(n-j) < 0, \\
& (q - s_{im}) A_{ij}^\mu A_{ij}^\nu = \sum_{\nu} s_{i\nu} A_{i\nu}^\mu A_{i\nu}^\nu, & i \neq m, \\
& (q - s_{im}) A_{ij}^\mu A_{ij}^\nu = \sum_{\nu} s_{i\nu} A_{i\nu}^\mu A_{i\nu}^\nu, & i \neq m, \\
& 0 = \sum_{\nu} s_{i\nu} A_{i\nu}^\mu A_{i\nu}^\nu, & (m-i)(n-j) < 0, \\
& A_{ij}^\mu A_{ij}^\nu - \sum_{\nu} s_{i\nu} A_{i\nu}^\mu A_{i\nu}^\nu = (s_{nj} - s_{im}) A_{ij}^\mu A_{ij}^\nu, & m \neq i \neq n \neq m, \\
& \omega A_{ij}^\mu A_{ij}^\nu = \sum_{\nu} s_{i\nu} A_{i\nu}^\mu A_{i\nu}^\nu - \sum_{\nu} s_{m\nu} A_{i\nu}^\mu A_{i\nu}^\nu, & i < m
\end{align*}
\]

(13) (14) (15) (16) (17)

**Proof.** This statement is verified by the direct analysis.

The next lemma accounts for equations (13) and (14).

**Lemma 3.2.** If \( A \) a solution to equation (11), then it can be represented in the form (4), where \( (Y, \sigma) \) is an admissible pair, \( x_i = A_{ij}^\mu \) for \( i \in I \), and \( y_i = A_{ij}^{\sigma(i)} \neq 0 \) for \( i \in Y \).

**Proof.** By virtue of equations (13), the matrix \( A \) has at most one non-zero off-diagonal entry in every row and every column. Indices of such rows form a subset \( Y \) in \( I \), and the off-diagonal entries may be written as \( y_i = A_{ij}^{\sigma(i)} \neq 0, i \in Y \), for some bijection \( \sigma: Y \rightarrow I \) with no stable points. This is equivalent to equation (13). The map \( \sigma \) is decreasing; that is encoded in equation (14).
Lemma 3.3. The equations of system (13) are equivalent to

$$\sum_{\alpha \geq \max(i,m)} A_i^\alpha A_m^\alpha = 0, \quad i \neq m,$$

by virtue of (14). They imply (18).

Proof. Setting $i = j$ in (14) and substituting it into the equations of system (15), we reduce them to (18). Setting $m = \sigma(i) < i \in Y_+$ in (18) leads to $x_i y_i = 0$ and therefore $x_i = 0$. Similarly, the assumption $m = \sigma(i) > i \in Y_-$ reduces (18) to $y_i x_{\sigma(i)} = 0$ and hence $x_{\sigma(i)} = 0$. So $x_i = 0$ for all $i \in Y_+ \cup \sigma(Y_-)$ and $i = b_+$ in particular.

Lemma 3.4. Lemma 3.3 taken into account, equation (16) is equivalent to the following two assertions.

1. For any $m \in Y$, either $\sigma^2(m) = m$ or $\sigma(m) \notin Y$.
2. $x_i = x_{b_-}$ and $x_j = 0$ whenever $i \leq b_-$ and $j \geq b_+$.

Proof. First note that if $m \notin Y$, equation (16) holds identically. So we can assume $m \in Y$ and rewrite (16) as

$$A_i^m A_m^i - s_i m A_m^i A_m^m - s_i \sigma(m) A_m^\sigma(m) A_m^\sigma(m) = (s_{ni} - s_{im}) A_i^m A_m^m,$$

$m \neq i \neq n \neq m$.

Supposing $n \neq \sigma(m)$ we find $A_i^m A_m^i - s_i \sigma(m) A_m^\sigma(m) A_m^\sigma(m) = 0$. If $i = \sigma(m)$, then, having in mind $s_{ii} = q \neq 1$ and $A_m^\sigma(m) = y_m \neq 0$, we obtain $A_m^\sigma(m) = 0$. Since $n \neq \sigma(m) = i$ and $n \neq m$, this means either $\sigma(m) \notin Y$ or $\sigma^2(m) = m$. Assuming $i \neq \sigma(m)$, we come to the equation $s_i \sigma(m) A_m^\sigma(m) A_m^\sigma(m) = 0$, which is fulfilled as well, due to $A_m^\sigma(m) = 0$.

It remains to study the case $n = \sigma(m)$. Under this hypothesis, the term $A_i^m A_m^i$ vanishes. Indeed, since $i \neq m$, one has $A_m^i \neq 0 \Rightarrow i = \sigma(m)$. But this contradicts the condition $i \neq n = \sigma(m)$. In terms of the variables $x_i$ and $y_i$, equation (19) reads

$$-s_i m x_m y_m - s_i \sigma(m) y_m x_{\sigma(m)} = (s_{\sigma(m)i} - s_{im}) x_i y_m,$$

$m \neq i \neq \sigma(m)$.
Depending on allocation of the indices $i$, $m$, and $\sigma(m)$, this equation splits into the following four implications.

\[(20) \quad i < m \text{ and } i < \sigma(m) \implies x_i = x_m + x_{\sigma(m)},\]
\[(21) \quad i < m \text{ and } i > \sigma(m) \implies x_m = 0,\]
\[(22) \quad i > m \text{ and } i < \sigma(m) \implies x_{\sigma(m)} = 0,\]
\[(23) \quad i > m \text{ and } i > \sigma(m) \implies x_i = 0.\]

Recall that the index $m$ is assumed to be from $Y$. Equation (23) is equivalent to $x_i = 0$ for $i > b_+$. By Lemma 3.3, this is also true for $i \geq b_+$. Equations (21) and (22) are thus satisfied as well. Equation (20) states $x_i = x_{b_+} + x_{\sigma(b_+)}$ if $i \in (b_-, b_+)$ or $x_i = x_{b_-} + x_{\sigma^{-1}(b_-)}$ if $i < b_- \in \sigma(Y_+)$. Applying Lemma 3.3, we find $x_i = x_{b_-}$, in either cases.

**Remark 3.5.** We used Lemma 3.3 in the proof of Lemma 3.4 in order to find the values $x_{b_{\pm}}$ on the boundaries of the intervals $[1, b_-]$ and $[b_+, n]$. If $(b_-, b_+) \neq \emptyset$, equations (21) and (22) are enough for that purpose, because we can put $i \in (b_-, b_+)$ there and come to the same result.

**Lemma 3.6.** Equation (18) is fulfilled by virtue of Lemma 3.4.

**Proof.** Consider the case $m < i$. Equation (18) holds if $i \notin Y$, because the sum turns into $A_i^1A_m^m$. So we may assume $i \in Y$ and distinguish two cases: $i \in Y_-$ and $i \in Y_+$. Assumption $i < \sigma(i)$ leads to $A_i^1A_m^m + A_i^{\sigma(i)}A_m^{\sigma(i)} = 0$. The equality $m = \sigma(i)$ is impossible, since otherwise $m = \sigma(i) < i < \sigma(i)$. Therefore the first term vanishes and we come to $A_m^m = 0$, $m \neq i$. This condition is satisfied, by Lemma 3.4, Statement 1. The case $i \in Y_+$ results in $x_i A_m^m = 0$. Setting $m = \sigma(i)$, we come to $x_i = 0$. Once $i \in Y_+ \implies i \geq b_+$, we encounter a particular case of Lemma 3.4, Statement 2.

We should study the situation $i < m$. Equation (18) evidently holds if $\sigma(i) < m$. We may think that $m \leq \sigma(i)$; then

\[
\begin{align*}
  i < m = \sigma(i) & \implies A_m^m = 0 \implies x_{\sigma(i)} = 0, \\
i < m < \sigma(i) & \implies A_{\sigma(i)}^m = 0 \implies \sigma(i) \notin Y.
\end{align*}
\]

These requirements are fulfilled, by Lemma 3.4.

It remains to satisfy equation (17), to complete the proof of Theorem 2.2. Let $Y_0$ be the subset in $Y$, such that $\sigma$ restricted to $Y_0$ is involutive. By Lemma 3.4.
either $\sigma(i) \not\in Y$ or $i$ and $\sigma(i)$ belong to $Y_0$ simultaneously. Equation (14) falls into the four equations

\begin{align}
(24) \quad x_m(x_i - x_m) &= 0, \quad i \notin Y_0, m \notin Y_0, \\
(25) \quad \omega x_m(x_i - x_m) &= s_{i \sigma(m)} y_m y_{\sigma(m)}, \quad i \notin Y_0, m \in Y_0, \\
(26) \quad \omega x_m(x_i - x_m) &= -s_{m \sigma(i)} y_i y_{\sigma(i)}, \quad i \in Y_0, m \notin Y_0, \\
(27) \quad \omega x_m(x_i - x_m) &= s_{i \sigma(m)} y_m y_{\sigma(m)} - s_{m \sigma(i)} y_i y_{\sigma(i)}, \quad i \in Y_0, m \in Y_0.
\end{align}

Everywhere $i < m$, see (17).

**Lemma 3.7.** Suppose $Y_0 \neq \emptyset$. Then $Y = Y_0$ and, moreover,

\begin{align}
(28) \quad Y_- &= \{1, \ldots, b_-\}, \quad Y_+ = \{b_+, \ldots, b_+ + b_- - 1\}, \\
(29) \quad \sigma(i) &= b_+ + b_- - i.
\end{align}

**Proof.** Let $k \in Y_0 \cap Y_-$ and $\sigma(k) \in Y_0 \cap Y_+$. Suppose either $l \in Y_- \setminus Y_0$ or $l < \min(Y_-)$. The assumption $l < k$ contradicts equation (25) if one sets $i = l$, $m = k$. The inequality $k < l$ does not agree with (26) if one sets $i = k$, $m = l$. In both cases one uses Lemma 3.4. Statement 2, and gets $0 = y_k y_{\sigma(k)}$; that is impossible since $y_k \neq 0$ for all $k \in Y$, by definition of $Y$. Therefore $Y_- \subset Y_0$ and $\min(Y_-) = 1$. Assume now $l \in Y_+ \setminus Y_0$. Then, $l$ cannot exceed $\sigma(k)$, because otherwise $\sigma(l) < \sigma^2(k) = k \Rightarrow \sigma(l) \in Y_0 \Rightarrow l \in Y_0$, the absurdity. The only possibility is $l < \sigma(k)$. But this again contradicts equation (26) if one sets $i = k$ and $m = l$.

**Lemma 3.8.** Suppose $Y_0 \neq \emptyset$. Then, there is $a \in \mathbb{C}$ such that $y_i y_{\sigma(i)} = a$ for all $i \in Y$. If $(b_-, b_+) \neq \emptyset$, then $x_m = \lambda$ for all $m \in (b_-, b_+)$, where $\lambda$ is a solution to the quadratic equation $\lambda(\lambda - x_{b_-}) = a$.

**Proof.** By Lemmas 3.4 and 3.7, equation (27) is fulfilled if and only if the product $y_i y_{\sigma(i)}$ does not depend on $i \in Y$ and is equal to some $a \in \mathbb{C}$. If $(b_-, b_+) \neq \emptyset$, equation (26) suggests $x_m(x_m - x_{b_-}) = a$ as soon as $m \in (b_-, b_+)$. It is easy to see that equation (24) is equivalent to $x_m = x_i = \lambda$ for $i, m \in (b_-, b_+)$ and some $\lambda \in \mathbb{C}$.

It is convenient to introduce the parameterization $x_{b_-} = \lambda + \mu$, $a = -\mu \lambda$. Then $x_m = \lambda + \mu$ for $m \leq b_-$, $x_m = \lambda$ for $b_- < m < b_+$, and $y_i y_{\sigma(i)} = -\lambda \mu$. This parameterization makes sense even if $(b_-, b_+) = \emptyset$. The scalars $\lambda$ and $\mu$ will have the meaning of eigenvalues of the matrix $A$. 


Lemma 3.9. Let $A$ be a matrix satisfying equations (13)-(16) and $(Y, \sigma)$ the corresponding admissible pair. Then, equation (17) gives rise to the alternative

- $Y_0 = Y$. $A$ is a solution of Type 1 from Theorem 2.2.
- $Y_0 = \emptyset$. $A$ is a solution of Type 2 from Theorem 2.2.

Proof. If $Y_0 = \emptyset$, then $Y \cap \sigma(Y) = \emptyset$ by Lemma 3.4, Statement 1. Equation (17) is reduced to (24). It is satisfied if and only if $x_m = \lambda$, $i \leq b$, and $x_m = 0$, $b < i$, for some $\lambda \in \mathbb{C}$ and $b \in \mathbb{Z}$. Comparing this with Lemma 3.4, Statement 2, we come to a solution of Type 2 with $b \in [b_-, b_+]$.

Suppose $Y_0 \neq \emptyset$. Then, equation (17) implies Lemmas 3.7 and 3.8. Conversely, these lemmas ensure (24)–(27), which are equivalent to (17). Altogether, this gives a solution of Type 1.

Thus we accomplish the proof of Theorem 2.2 and proceed to the analysis of the solutions obtained.

4. Structure of solutions

The first question about the structure of the numerical RE matrices listed in Theorem 2.2 is what pairs $(Y, \sigma)$ may participate in the classification. The solutions of Type 1 involve two integers $b_-$, $b_+$ subject to the conditions of the theorem. They determine the admissible pair $(Y, \sigma)$. The solutions of Type 2 are labeled with subsets $Y \subset I$ and injective maps $\sigma : Y \to I \setminus Y$. Clearly, $K = \text{card}(Y) \leq \frac{n}{2}$ and, with $Y$ given, there are $C^K_{n-K}$ possibilities for $\sigma$, i.e. the number of subsets in $I \setminus Y$ with card = $K$. Further we describe the properties of solutions to equation (1). We consider the standard basis $\{e^i\}$ in $\mathbb{C}^n$; the action of $\text{End}(\mathbb{C}^n)$ on $\mathbb{C}^n$ being given by $e^j e^k = \delta^k_j e^i$.

Proposition 4.1. Let $(Y, \sigma)$ be an admissible pair corresponding to a numerical RE matrix $A$. The subspaces $V_i^2 = \mathbb{C} e^i \oplus \mathbb{C} e^{\sigma(i)}$, $i \in Y$, and $V_i^1 = \mathbb{C} e^i$, $i \notin Y \cup \sigma(Y)$, are $A$-invariant.

Proof. This follows from the general form (4) of the RE matrices and from Lemma 3.4, Statement 1.

The spectral properties of numerical RE matrices are described by the following proposition.

Proposition 4.2. An RE matrix of Type 1 from Theorem 2.2 has eigenvalues $\mu$, $\lambda$, and $0$ of multiplicities $b_-$, $b_+-b_- - 1$, and $n-(b_-+b_+)+1$, respectively. It is semisimple if and only if $\lambda \neq \mu$. 


An RE matrix of Type 2 from Theorem 2.2 has eigenvalues $\lambda$ and $0$ of multiplicities $b$ and $b - K$ respectively. It is semisimple if and only if $\lambda \neq 0$.

Proof. Restricted to $V_i^2$, $i \in [1, b_-]$, an RE matrix $A$ of of Type 1 is equal to

\[
\begin{pmatrix}
\lambda + \mu & y_i \\
y_{\sigma(i)} & 0
\end{pmatrix}
\]

Because $y_i y_{\sigma(i)} = -\lambda \mu$, this $2 \times 2$ matrix has eigenvalues $\mu$ and $\lambda$. On the subspaces $V_i^1$, it acts as multiplication by $\lambda$ when $i \in (b_-, b_+)$, and $0$ when $i \in [b_- + b_+, n]$.

Restricted to $V_i^2$, $i \in Y$, an RE matrix $A$ of of Type 2 is equal to $\begin{pmatrix} \lambda & y_i \\ 0 & 0 \end{pmatrix}$ if $i \in Y^-$ and $\begin{pmatrix} \lambda & 0 \\ y_i & 0 \end{pmatrix}$ if $i \in Y^+$. In either cases, it has eigenvalues $\lambda$ and $0$.

Observe that $Y_-' \cup \sigma(Y_+') \subset [1, b]$ and $Y_- \cap \sigma(Y_+) = \emptyset$; thus $\text{card}(Y_- \cup \sigma(Y_+)) = \text{card}(Y) = K$. Therefore there are $b - K$ one-dimensional subspaces $V_i^1$ where $A$ acts as multiplication by $\lambda$. So the eigenvalue $\lambda$ has multiplicity $b$. The zero eigenvalue enters with multiplicity $n - b$.

Note that when one of the eigenvalues $\lambda, \mu$ tends to zero, the RE matrices of Type 1 turns into RE matrices of Type 2. Non-degenerate numerical RE matrices belong to the first class. This is the case when $b_+ + b_- - 1 = n$ and $\lambda \neq 0 \neq \mu$. The RE matrices from \text{KSS} written out in Introduction are of this kind, and the Type 1 solutions from Theorem 2.2 are their generalization to higher dimensions. Let us illustrate Theorem 2.2 on those examples.

1. The matrix $D_n$ is a solution of Type 1. Here, $\lambda = -\mu = y_i$, $i \in Y$, where the set $Y$ coincides with $I$ if $n = 0 \mod 2$ and $Y = I \setminus \{\frac{n+1}{2}\}$ if $n = 1 \mod 2$.

The similarity transformation by a diagonal matrix changes the elements $y_i$ however preserving the condition $y_i y_{n+1-i} = \lambda^2$.

2. The matrix $A^{1,1}$ has $b_- = 1$, $b_+ = 2$, while for the matrix $A^{2,2}$ these parameters take the values $b_- = 2$ and $b_+ = 3$. These solutions have in common $(b_-, b_+) = \emptyset$. On the contrary, the interval $(b_-, b_+)$ is not empty in the case of matrices $A^{2,1}$ and $A^{3,1}$. For them, one has $b_- = 1$, $b_+ = 3$ and $b_- = 1$, $b_+ = 4$.

3. The diagonal matrix $P_k = \lambda \sum_{i=1}^{b} e_i^i$ is an RE matrix of Type 2 with $Y = \emptyset$ and $k = b \leq n$.

Remark 4.3. Numerical RE matrices play an important role in $\mathcal{U}_q(gl(n))$-covariant quantization on the adjoint orbits in $\text{End}^{\otimes 2}(\mathbb{C}^n)$, \text{DoM1, DoM2}. They allow to explicitly represent the quantized algebras of functions on orbits as quotients of the RE algebra and, simultaneously, as subalgebras of functions on the...
quantum group. Theorem 2.2 classify all such realizations. It implies, in particular, that all the symmetric and bisymmetric orbits (consisting of matrices with two and three different eigenvalues) can be represented in this way.

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