Explicit solutions for a $(2 + 1)$-dimensional Toda-like chain

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Received 15 July 2012, in final form 27 November 2012
Published 17 January 2013
Online at stacks.iop.org/JPhysA/46/055202

Abstract

We consider a $(2 + 1)$-dimensional Toda-like chain which can be viewed as a two-dimensional generalization of the Wu–Geng model and which is closely related to the two-dimensional Volterra, two-dimensional Toda and relativistic Toda lattices. In the framework of the Hirota direct approach, we present equations describing this model as a system of bilinear equations that belongs to the Ablowitz–Ladik hierarchy. Using the Jacobi-like determinantal identities and the Fay identity for the theta-functions, we derive its Toeplitz, dark-soliton and quasiperiodic solutions as well as the similar set of solutions for the two-dimensional Volterra chain.

PACS numbers: 02.30.Ik, 05.45.Yv, 02.10.Yn, 02.30.Gp
Mathematics Subject Classification: 37J35, 35Q51, 37K10, 11C20, 37K20, 14K25

1. Introduction

In this paper, we consider a $(2 + 1)$-dimensional Toda-like chain

$$\Delta u_n = (\nabla u_n, \nabla u_n) \left( \frac{1}{u_n - u_{n+1}} + \frac{1}{u_n - u_{n-1}} \right),$$

(1.1)

where $u_n = u_n(x, y)$, $\Delta$ and $\nabla$ are the two-dimensional Laplacian and gradient. This equation appears in the literature in various contexts. For example, it is known to describe the Bäcklund transformations of a Heisenberg-like magnetics (see [1]) and Laplace transformations of hydrodynamic-type systems in Riemann invariants [2]. In a recent paper [3], this system was shown to describe $O(3)$ $\sigma$-fields (three-component vectors $\sigma_n$ of unit length) coupled by a nearest-neighbour Heisenberg-like interaction such as, e.g., graphite-like magnetics when the spins inside one layer are governed by the Landau–Lifshitz theory with an effective Heisenberg interaction between adjacent layers: $\mathcal{H} = \mathcal{H}_{\text{LL}} + \mathcal{H}_{\text{H}}$ where $\mathcal{H}_{\text{LL}} = \sum_n \mathcal{E}_n$ with

$$\mathcal{E}_n = \int_{\mathbb{R}^2} dx \, dy \, (\nabla \sigma_n, \nabla \sigma_n)$$

(1.2)
and $H_{11} = \frac{1}{2} \sum_n \sum_{p=q+1} U_{np}$ with

$$U_{np} = g^2 \int_{\mathbb{R}^2} \text{d}x \text{d}y \ln(1 + (\sigma_n, \sigma_p)).$$  (1.3)

A remarkable feature of the model (1.1), which we will write in terms of the complex variables $z = x + iy$ and $\bar{z} = x - iy$,

$$\partial \bar{\partial} u_n = (\partial u_n)(\bar{\partial} u_n) \left( \frac{1}{u_n - u_{n+1}} + \frac{1}{u_n - u_{n-1}} \right),$$  (1.4)

where $\partial$ and $\bar{\partial}$ stand for $\partial/\partial z$ and $\partial/\partial \bar{z}$, is that it can be viewed as a connecting link between almost all Toda-like chains. In the following section, we present its relationships with the Wu–Geng chain (WGC) [4], the ‘two-dimensional Volterra equation’ (2DVE) [5, 6], the two-dimensional Toda lattice (2DTL) [7] and the relativistic Toda chain (RTC) [8–11].

The main goal of this work is to obtain some explicit solutions for (1.4). We will not elaborate the inverse scattering transform or the algebro-geometric approach from scratch. Instead, we use the links between our equation and the 2DTL together with the results [12] that give us the possibility of bilinearizing (1.4) and reducing it in section 3 to a system that belongs to the Ablowitz–Ladik hierarchy (ALH) [13]. Starting from the structure of the already known solutions for the ALH, we derive in sections 4 and 5 Toeplitz and dark-soliton solutions directly from some determinantal identities. In section 6, we use the Fay identity for the $\theta$-functions to derive the quasiperiodic solutions. Finally, in section 7 the results of sections 4–6 are used to obtain the Toeplitz, dark-soliton and periodic solutions for the 2DVE.

2. WGC, 2DVE, 2DTL and RTC

The main equations of this paper can be viewed as a straightforward generalization of the WGC to $(2 + 1)$ dimensions. Indeed, from equations (2) of [4],

$$u_{nt} = \frac{1}{v_{n+1}} - \frac{1}{v_n}, \quad v_{nt} = \frac{1}{u_n} - \frac{1}{u_{n-1}},$$  (2.1)

where $u_{nt}$ stands for $du_n/dt$, one can obtain that functions $w_n$ defined by $v_n = w_n - w_{n-1}$ satisfy

$$w_{nt} = \frac{1}{u_n} + C, \quad C = \text{constant}$$  (2.2)

which leads, after setting $C = 0$, to

$$w_{ntt} = w_{nt}^2 \left( \frac{w_{n+1} - 2w_n + w_{n-1}}{(w_{n+1} - w_n)(w_n - w_{n-1})} \right).$$  (2.3)

Clearly, this equation coincides with (1.4) after replacing $\partial$ and $\bar{\partial}$ with $d/dt$.

On the other hand, rewriting equations (1.4) in terms of the variables

$$a_n = \frac{i\partial u_n}{u_{n+1} - u_n}, \quad b_n = \frac{-i\bar{\partial} u_n}{u_n - u_{n-1}},$$  (2.4)

one arrives at the system

$$\begin{cases} \partial \bar{\partial} a_n = a_n(b_{n+1} - b_n) \\ \partial \bar{\partial} b_n = b_n(a_{n-1} - a_n) \end{cases}$$  (2.5)

which is closely related to the 2DTL: the quantities $f_n$ defined by

$$f_n = \ln a_nb_n$$  (2.6)
satisfy
\[ \partial \bar{f}_n = e^{f_{n+1}} - 2e^{f_n} + e^{f_{n-1}}. \quad (2.7) \]

The system (2.5) is known since the works of Leznov, Savel’ev and Smirnov [5, 6] where the authors demonstrated that this system, which was named the ‘two-dimensional Volterra equation’, represents the B"acklund transformations for the 2DTL and constructed its general solutions in the finite case using the results of [14, 15].

Another model that we would like to discuss is the RTC [8–11] which can be presented as a Hamiltonian system
\[ i\partial q_n = \partial H/\partial p_n, \quad i\partial p_n = -\partial H/\partial q_n \quad (2.8) \]
with
\[ H = \sum_n e^{p_n} (e^{q_n+q_{n+1}+1} - 1). \quad (2.9) \]
Equations (2.8)
\[ \begin{cases} i\partial q_n = e^{p_n} (e^{q_n+q_{n+1}+1} - 1) \\ i\partial p_n = e^{p_n+q_{n+1}+q_{n+2} - e^{p_n+q_{n+1}+q_{n+2}}} \end{cases} \quad (2.10) \]
rewritten in terms of the variables
\[ a_n = e^{p_n+q_{n+1}+q_{n+2}}, \quad b_n = e^{-p_n} \quad (2.11) \]
are
\[ \begin{cases} i\partial \ln a_n = a_{n+1} - a_{n-1} - 1/b_{n+1} + 1/b_n \\ i\partial \ln b_n = a_{n+1} - a_n \end{cases} \quad (2.12) \]
One can easily note that the second of the above equations coincides with the second equation of (2.5). In a similar manner, starting from the Hamiltonian system
\[ i\partial q_n = \partial \bar{H}/\partial p_n, \quad i\partial p_n = -\partial \bar{H}/\partial q_n \quad (2.13) \]
with
\[ \bar{H} = \sum_n e^{-p_n} (e^{q_n+q_{n+1}+1} - 1) \quad (2.14) \]
that has the form
\[ \begin{cases} i\partial q_n = e^{-p_n} (1 - e^{q_n+q_{n+1}+1}) \\ i\partial p_n = e^{-p_n+q_{n+1}+q_{n+2} - e^{-p_n+q_{n+1}+q_{n+2}}} \end{cases} \quad (2.15) \]
one arrives, after using substitution (2.11), at the system
\[ \begin{cases} i\partial \ln a_n = b_{n+1} - b_n \\ i\partial \ln b_n = a_{n+1} - a_n - b_n b_{n+1} \end{cases} \quad (2.16) \]
whose first equation is the first one between the two RTCs, (2.12) and (2.16), and our system (2.5) becomes transparent if one considers RTCs from the zero-curvature viewpoint based on the scattering problem [9–11]
\[ a_n \psi_{n+1} - b_{n+1}^{-1} \psi_n = \Lambda (\psi_n - \psi_{n-1}). \quad (2.17) \]
The system (2.12) is the compatibility condition of (2.17) and
\[ i\partial \psi_n = a_n (\psi_{n+1} - \psi_n) \quad (2.18) \]
while (2.16) plays the same role for (2.17) and
\[ i\partial \psi_n = b_n (\psi_{n-1} - \psi_n). \quad (2.19) \]
As to our system, (2.5), it ensures the consistency of (2.18) taken together with (2.19),
\[
\begin{align*}
\dot{\psi}_n &= a_n (\psi_{n+1} - \psi_n) \\
\dot{\bar{\psi}}_n &= b_n (\psi_{n-1} - \psi_n).
\end{align*}
\tag{2.20}
\]
To summarize, equations that are the subject of this paper can be viewed as describing the commutativity of two relativistic Toda flows. It should be noted that this commutativity leads, again, to the 2DTL. Indeed, considering equations (2.10) combined with (2.15), one can obtain by direct calculations that functions \(q_n\) satisfy
\[
\partial \bar{\partial} q_n = \exp(q_{n+1} - q_n) - \exp(q_n - q_{n-1})
\tag{2.21}
\]
which is another form of the 2DTL.

Finally, comparing (2.4) with (2.20) one can conclude that solutions for our system, \(u_n\), are nothing but solutions for the auxiliary linear problems for the 2DVE; in other words, the \(u_n\)-model is dual to the \((a_n, b_n)\)-model.

3. Bilinearization

To bilinearize our equation, which can be written as
\[
\begin{align*}
\partial \bar{\partial} u_n &= p_n (u_{n+1} - 2u_n + u_{n-1}) \\
(\partial u_n)(\bar{\partial} u_n) &= p_n (u_{n+1} - u_n)(u_n - u_{n-1}),
\end{align*}
\tag{3.1}
\]
we start with the 2DTL, assuming
\[
p_n = \frac{\tau_{n-1} \tau_{n+1}}{\tau_n^2}
\tag{3.2}
\]
where \(\tau_n\) is a solution for
\[
\partial \bar{\partial} \ln \tau_n = \frac{\tau_{n-1} \tau_{n+1}}{\tau_n^2}
\tag{3.3}
\]
or, in the bilinear form,
\[
D D \tau_n \cdot \tau_n = 2 \tau_{n-1} \tau_{n+1},
\tag{3.4}
\]
where \(D\) and \(\bar{D}\) are the Hirota operators corresponding to \(\partial\) and \(\bar{\partial}\): \(D a \cdot b = (\partial a) b - a (\bar{\partial} b)\), \(\bar{D} a \cdot b = (\bar{\partial} a) b - a (\bar{\partial} b)\).

One can easily obtain a wide range of solutions for the first equation of (3.1) by differentiating the 2DTL tau-function \(\tau_n\) with respect to any parameter,
\[
u_n = \frac{\partial}{\partial \xi} \ln \tau_n.
\tag{3.5}
\]
Since \(\tau_n\) can be considered as a solution for all equations of the 2DTL hierarchy, i.e. as a function of an infinite number of ‘times’, \(\tau_n = \tau_n(t_1, t_2, \ldots, \bar{t}_1, \bar{t}_2, \ldots), t_1 = \bar{z}, \bar{t}_1 = \bar{\bar{z}},\) the last formula can be generalized as follows:
\[
u_n = \sum_{j=1}^{\infty} \left( c_j \frac{\partial \ln \tau_n}{\partial t_j} + \bar{c}_j \frac{\partial \ln \tau_n}{\partial \bar{t}_j} \right)
\tag{3.6}
\]
with arbitrary constants \(c_j\) and \(\bar{c}_j\). However, this apparently most straightforward approach is rather hard to implement because the second equation of (3.1), which plays the role of the restriction for the set \(c_j\) and \(\bar{c}_j\), is a bilinear system that we cannot solve. That is why we will use another way to deal with our equations. The key point, which will be proved below, is that \(u_n\) can be built of two solutions for the 2DTL equation:
\[
u_n = \frac{\alpha_n}{\tau_n},
\tag{3.7}
\]
\[ \frac{1}{\tau_n^2} \partial \omega_n \cdot \tau_n = \frac{u_n}{\tau_n} \partial \tau_n \cdot \tau_n, \]  
and noting that the rhs of the second equation of (3) imply as a consequence the first equation of (3) \( \lambda \) is a unimportant constant that can be eliminated by multiplying \( \omega_n \) and \( \tau_n \) by \( \exp(-\lambda z \bar{z}) \) without modifying \( u_n \). Calculating the first derivatives of \( u_n \),

\[ \partial u_n = \frac{1}{\tau_n^2} \partial \omega_n \cdot \tau_n, \quad \hat{\partial} u_n = \frac{1}{\tau_n^2} \hat{\partial} \omega_n \cdot \tau_n, \]  
and its Laplacian,

\[ \partial \hat{\partial} u_n = \frac{1}{\tau_n^2} \partial \hat{\partial} \omega_n \cdot \tau_n - \frac{u_n}{\tau_n^2} \partial \omega_n \cdot \tau_n, \]  
one can rewrite (3.1) as

\[ \begin{cases} 
(D \hat{\partial} - 2\lambda) \omega_n \cdot \tau_n = \tau_{n-1} \omega_{n+1} + \tau_{n+1} \omega_{n-1} \\
(D \omega_n \cdot \tau_n)(D \omega_n \cdot \tau_n) = (\tau_{n-1} \omega_n - \tau_n \omega_{n-1})(\tau_{n+1} \omega_{n+1} - \tau_n \omega_n).
\end{cases} \]  

The next step is to split the second equation of the above system in two bilinear equations. Introducing new \( \tau \)-functions \( \hat{\tau}_n, \omega_n, \hat{\tau}_n \) and \( \omega_n \) by

\[ iD \omega_n \cdot \tau_n = \hat{\tau}_n \hat{\omega}_n \]  
and noting that the rhs of the second equation of (3.11) can be presented as \( X_n X_{n-\delta} \) where

\[ X_n = \tau_n \omega_{n+\delta} - \tau_{n+\delta} \omega_n, \quad \delta = \pm 1, \]  

it is possible to achieve our goal by setting

\[ \hat{\omega}_n = \omega_{n-\delta}, \quad \hat{\tau}_n = \tau_{n+\delta} \]  
which leads to

\[ \begin{cases} 
\tau_n \omega_{n+\delta} - \tau_{n+\delta} \omega_n = \hat{\tau}_n \hat{\omega}_n \\
iD \omega_n \cdot \tau_n = \hat{\tau}_{n-\delta} \hat{\omega}_n \\
-iD \omega_n \cdot \tau_n = \hat{\tau}_n \hat{\omega}_{n-\delta}.
\end{cases} \]  

Now we have to close this system adding the equations describing the dependence of the new \( \tau \)-functions on the variables \( z \) and \( \bar{z} \). These equations should be (1) bilinear, (2) compatible and (3) imply as a consequence the first equation of (3.11). The resulting system can be written as the union of the ‘positive’ part,

\[ \begin{cases} 
0 = iD \omega_n \cdot \tau_n - \hat{\tau}_{n-\delta} \hat{\omega}_n \\
0 = iD \hat{\tau}_n \cdot \tau_n + \hat{\tau}_{n-\delta} \tau_{n+\delta} \\
0 = iD \omega_n \cdot \hat{\omega}_{n-\delta} + \omega_{n-\delta} \hat{\omega}_n,
\end{cases} \]  
the ‘negative’ one,

\[ \begin{cases} 
0 = i\hat{D} \omega_n \cdot \tau_n + \hat{\tau}_{n+\delta} \hat{\omega}_n \\
0 = i\hat{D} \hat{\tau}_{n-\delta} \cdot \tau_n + \hat{\tau}_n \tau_{n-\delta} \\
0 = i\hat{D} \omega_n \cdot \hat{\omega}_n + \omega_{n+\delta} \hat{\omega}_{n-\delta},
\end{cases} \]  
combined with the restriction

\[ 0 = \tau_n \omega_{n+\delta} - \tau_{n+\delta} \omega_n - \hat{\tau}_n \hat{\omega}_n. \]
One can verify that these bilinear equations indeed lead to the solution of our problem. As to their compatibility (which can be checked directly, but after rather long and tedious calculations), we would like to note that they are a part of the extended ALH [16]. Actually, we have passed from $\tau$-functions of the 2DTL to ones of the ALH using the results of [12]. In other words, the above proceeding can be viewed as an explanation of the following fact, which can be verified by straightforward algebra: the compatible system (3.16)--(3.18) that belongs to the extended ALH provides solutions for the system (3.1). The reference to the ALH not only resolves the question of compatibility, but also gives us the possibility of using the already known solutions for the Ablowitz–Ladik equations to build ones for the problem we are dealing with.

Definitions (2.4) and the above equations yield the following expressions for solutions of the 2DVE:

$$\delta = 1 : \quad a_n = \frac{\tau_{n-1}\tau_{n+1}}{\tau_n}, \quad b_n = \frac{\tau_{n-1}\tau_n}{\tau_{n-1}\tau_n}$$

$$\delta = -1 : \quad a_n = -\frac{\tau_{n-1}\tau_{n+1}}{\tau_n}, \quad b_n = -\frac{\tau_{n-1}\tau_{n+1}}{\tau_n}$$

that will be used in section 7.

4. Toeplitz solutions

These types of solutions can be constructed of the Toeplitz determinants

$$A_k^n = \det |\alpha_k+\alpha_{n+a}|_{a,b=1, \ldots ,\ell}. \quad (4.1)$$

The main idea is that among various determinantal identities one can find ones that are similar to the equations we want to solve. In the appendix, we derive the necessary formulae by applying the Jacobi identity to the determinants (4.1) and the framed ones,

$$F_{\ell+1}^k(\xi) = \begin{vmatrix} 1 & \xi & \xi^2 & \ldots & \xi^{\ell} \\ \alpha_{k-1} & \alpha_k & \alpha_{k+1} & \ldots & \alpha_{k+\ell-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{vmatrix} \quad (4.2)$$

(see (A.28)--(A.33)). These formulae can then be rewritten in terms of the $\ell$-order determinants $B_{\ell}^k$, similar to (4.1), instead of the $(\ell + 1)$-order ones, $F_{\ell+1}^k(\xi)$,

$$F_{\ell+1}^k(\xi) = (-\xi)^\ell B_{\ell}^k(\xi), \quad (4.3)$$

where

$$B_{\ell}^k(\xi) = \det |\beta_{k+a+b}(\xi)|_{a,b=1, \ldots ,\ell} \quad (4.4)$$

with

$$\beta_k(\xi) = \alpha_k - \xi^{-1}\alpha_{k+1}. \quad (4.5)$$

The set of identities that we need to solve our equations is

$$0 = D_1B_{\ell}^k+A_{\ell+1}^k-\xi^{-1}A_{\ell+1}^{k+1}B_{\ell-1}^k \quad (4.6a)$$

$$0 = D_2 B_{\ell}^k + A_{\ell+1}^k + A_{\ell+1}^{k+1}B_{\ell-1}^k \quad (4.6b)$$

$$0 = D_z B_{\ell}^k + A_{\ell+1}^{k+1} - A_{\ell+1}^{k+1}B_{\ell-1}^k \quad (4.6c)$$
Thus, our

\[ 0 = D_+ B_+^k \cdot A_+^k + \zeta^{-1} A_+^{k+1} B_+^{k-1} \]  

(4.7a)

\[ 0 = D_+ B_+^k \cdot A_+^{k+1} + A_+^{k+1} B_+^{k-1} \]  

(4.7b)

\[ 0 = D_+ B_+^k \cdot A_+^{k+1} + A_+^{k+1} B_+^{k-1} \]  

(4.7c)

together with

\[ 0 = A_+^{k+1} B_+^k - A_+^k B_+^{k+1} + \zeta^{-1} A_+^{k+1} |B_+^{k-1}. \]  

(4.8)

Here, \( D_\pm \) stand for the Hirota operators corresponding to \( \alpha \) and \( \alpha \), \( \omega \pm \) and \( \omega \pm \), \( \tau \pm \) and \( \tau \pm \), \( \hat{\omega} \pm \) and \( \hat{\omega} \pm \). Below, we obtain three families of solutions: infinite (with respect to \( n \)), semi-infinite and finite ones.

### 4.1. Infinite chain

These types of solutions correspond to the following choice of the \( \tau \)-functions:

\[ \tau_n = \tau_0 A_0^n, \quad \omega_n = \omega_0 B_0^n \]  

(4.9)

and

\[ \hat{\tau}_n = \hat{\tau}_0 B_0^n, \quad \hat{\omega}_n = \hat{\omega}_0 A_0^n \]  

(4.10)

with constant \( \tau_0, \omega_0, \hat{\tau}_0, \hat{\omega}_0 \). From (4.8), one can immediately derive that

\[ 0 = \tau_0 \partial \omega_0 = \tau_0 \partial \hat{\omega}_0 = \zeta \hat{\tau}_0 \hat{\omega}_0 \]  

(4.11)

provided

\[ \tau_n \omega_n = \zeta \hat{\tau}_n \hat{\omega}_n. \]  

(4.12)

Thus, our \( \tau \)-functions solve (3.18) with \( \delta = 1 \). Furthermore, equations (4.6a)–(4.7c) lead to

\[ 0 = D_+ \omega_n \cdot \tau_n = \hat{\tau}_{n-1} \hat{\omega}_n \]  

(4.13a)

\[ 0 = D_+ \hat{\tau}_n \cdot \tau_n + \hat{\tau}_{n-1} \tau_{n+1} \]  

(4.13b)

\[ 0 = D_+ \omega_n \cdot \hat{\omega}_{n+1} + \omega_{n-1} \hat{\omega}_n \]  

(4.13c)

and

\[ 0 = D_- \omega_n \cdot \tau_n + \hat{\omega}_{n-1} \hat{\tau}_n \]  

(4.14a)

\[ 0 = D_- \hat{\tau}_{n-1} \cdot \tau_n + \tau_{n-1} \hat{\tau}_n \]  

(4.14b)

\[ 0 = D_- \omega_n \cdot \hat{\omega}_n + \omega_{n-1} \hat{\omega}_{n+1} \]  

(4.14c)

It is clear that to complete solution for (3.16)–(3.17) one has to meet

\[ \partial_+ = i\partial, \quad \partial_- = i\bar{\partial} \]  

(4.15)

or, to take \( \alpha_k \) to be solutions for the linear system

\[ \begin{cases} i\partial \alpha_k = \alpha_{k+1} \\ i\bar{\partial} \alpha_k = \alpha_{k-1} \end{cases} \]  

(4.16)

The ‘symmetric’ set of solutions can be obtained by using, instead of (4.3), another representation of the determinants \( F_{\ell+1}^k(\zeta) \):

\[ F_{\ell+1}^k(\zeta) = C_{\ell+1}^k(\zeta). \]  

(4.17)
where

$$C_l^j(\zeta) = \det \left| \gamma_{k+a-b}(\zeta) \right|_{a,b=1,\ldots,l}$$  \hspace{1cm} (4.18)

with

$$\gamma_k(\zeta) = \alpha_k - \zeta \alpha_{k-1}.$$  \hspace{1cm} (4.19)

The calculations similar to the ones presented above demonstrate that \(r\)-functions defined by

$$\tau_n = \tau_\zeta \mathcal{A}_n, \quad \omega_n = \omega_\zeta \mathcal{C}_n$$  \hspace{1cm} (4.20)

and

$$\bar{\tau}_n = \bar{\tau}_\zeta \mathcal{C}^{*}_{n+1}, \quad \bar{\omega}_n = \bar{\omega}_\zeta \mathcal{A}^{*}_{n+1}$$  \hspace{1cm} (4.21)

with

$$\zeta \tau_\zeta \omega_\zeta = -\bar{\tau}_\zeta \bar{\omega}_\zeta$$  \hspace{1cm} (4.22)

solve (3.16)–(3.18) with \(\delta = 1\).

Both these sets of solutions can be written as

$$u_\zeta(z, \bar{z}) = \frac{u_n}{\det \left| \alpha_{k+a-b}(\zeta, \bar{\zeta}) \right|_{a,b=1,\ldots,l}}$$  \hspace{1cm} (4.23)

where \(l\) is an arbitrary positive integer and the elements of the determinants are given by

$$\alpha_k(z, \bar{z}) = \int_\Gamma dh \hat{\alpha}(h) h^k \exp[-i\Theta_h(z, \bar{z})]$$  \hspace{1cm} (4.24a)

$$\alpha_k^+(z, \bar{z}) = \int_\Gamma dh \hat{\alpha}(h)[1 - (h/\zeta)^{\pm 1}]h^k \exp[-i\Theta_h(z, \bar{z})]$$  \hspace{1cm} (4.24b)

(we replaced \(\beta_\zeta\) with \(\alpha_k^+\) with arbitrary contour \(\Gamma\), function \(\hat{\alpha}(h)\) and constant \(u_n\). The 'dispersion law' \(\Theta_h(z, \bar{z})\) is given by

$$\Theta_h(z, \bar{z}) = hz + h^{-1}\bar{z}.$$  \hspace{1cm} (4.25)

As an example, let us consider one of the simplest solutions of (4.16). Noting that system (4.16) leads to the Helmholtz equation, \(\hat{\partial}^2 \alpha_k + \alpha_k = 0\), and rewriting the latter using the polar coordinates, \(z = re^{i\theta}\), one can obtain in a standard way

$$\alpha_k = \exp[-i(\theta + \pi/2)]k\!J_k(2r),$$  \hspace{1cm} (4.26)

where \(J_k\) is the \(k\)th Bessel function. These solutions correspond to (4.24a) and (4.24b) with \(\hat{\alpha}(h) = (2\pi i\hbar)^{-1}\) and \(\Gamma\) being the unit circumference: \(\Gamma = \{h : |h| = 1\}\). In the 'elementary' case of \(l = 1\), expression (4.23) can be written as

$$u_n = u_* + u^*_a J_{n+\pm}(2r) e^{\mp i\theta},$$  \hspace{1cm} (4.27)

where \(u_*\) and \(u_a\) \((u_a = \pm iu_\zeta \bar{\zeta}^{\mp 1})\) are arbitrary constants. It is easy to see that solutions we have obtained are complex and singular, which, however, does not mean that they are non-physical. Say in the case of [3], which was discussed in the introduction, the spin components, \(\sigma_n = (\sigma_n^{(1)}, \sigma_n^{(2)}, \sigma_n^{(3)})\), are related to \(u_\zeta\) by \(\sigma_n^{(1)} + i\sigma_n^{(2)} = 2u_\zeta/(1 + |u_\zeta|^2)\) and \(\sigma_n^{(3)} = (1 - |u_\zeta|^2)/(1 + |u_\zeta|^2)\). Thus, in the general case \((\sigma_n^{(2)} \neq 0)\) \(u_\zeta\) is complex and singularities of \(u_\zeta\) correspond to the vertical (southward) orientation of \(\sigma_n\): \(u_\zeta = \infty \leftrightarrow \sigma_n^{(1)} = \sigma_n^{(2)} = 0\) and \(\sigma_n^{(3)} = -1\).
4.2. Semi-infinite chain

This type of solution appears if one identifies the index $n$ with the size of the determinants $A^k_n$, $B^k_n$, $C^k_n$. Consider the functions $T_n$, $W_n$, $\tilde{T}_n$, $\tilde{W}_n$ given by

$$T_n = T_a A^k_n, \quad W_n = W_a B^k_n$$

(4.28)

and

$$\tilde{T}_n = \tilde{T}_a B^{k-1}_n, \quad \tilde{W}_n = \tilde{W}_a A^{k+1}_{n+1}$$

(4.29)

with constants $T_a$, $W_a$, $\tilde{T}_a$, $\tilde{W}_a$ being related by

$$T_n W_n = -\zeta \tilde{T}_n \tilde{W}_n.$$  

(4.30)

It is straightforward to verify that they solve

$$0 = T_n W_{n+1} - T_{n+1} W_n - \tilde{T}_n \tilde{W}_n = 0$$

(4.31)

as well as

$$0 = D_+ W_n \cdot T_n + \tilde{T}_{n-1} \tilde{W}_n$$

$$0 = D_- \tilde{T}_n \cdot T_n - \tilde{T}_{n-1} T_{n+1}$$

$$0 = D_+ W_n \cdot \tilde{W}_{n-1} - W_{n+1} \tilde{W}_n$$

(4.32)

and

$$0 = (D_+ + \zeta^{-1}) W_n \cdot T_n + \tilde{W}_{n-1} \tilde{T}_n$$

$$0 = (D_- + \zeta^{-1}) \tilde{T}_{n-1} \cdot T_n + T_{n+1} \tilde{T}_n$$

$$0 = (D_+ + \zeta^{-1}) W_n \cdot \tilde{W}_{n-1} + \tilde{W}_{n+1} W_n$$

(4.33)

that almost coincide with (3.16)–(3.18) for $\delta = 1$ after identifying

$$\bar{\partial} = i \partial_+ \quad \tilde{\partial} = -i \partial_-.$$  

(4.34)

The extra terms can be eliminated by exp($i\tilde{\zeta}/\zeta$) factor and one arrives at the following solutions:

$$\tau_n = \tau_a \exp(i\tilde{\zeta}/\zeta) A^k_n, \quad \omega_n = \omega_a B^k_n$$

(4.35)

and

$$\tilde{\tau}_n = \tilde{\tau}_a B^{k-1}_n, \quad \tilde{\omega}_n = \tilde{\omega}_a \exp(i\tilde{\zeta}/\zeta) A^{k+1}_{n+1},$$

(4.36)

where $\tau_a$, $\omega_a$, $\tilde{\tau}_a$, $\tilde{\omega}_a$ are arbitrary constants and

$$\zeta = -\frac{\tau_a \omega_a}{\tilde{\tau}_a \tilde{\omega}_a}.$$  

(4.37)

In a similar way, one can derive the ‘symmetric’ set of solutions using the determinants $C^k_n$. These two sets of the Toeplitz solutions $u_n$ can be written as

$$u_n(z, \bar{z}) = u_a e^{i\theta_a(z, \bar{z})} \det |\alpha^k_{a+b-h}(z, \bar{z})|_{a, b=1, \ldots, n}$$

(4.38)

with an arbitrary positive integer $k$. The elements of the determinants are given again by (4.24a) and (4.24b) with the ‘dispersion law’

$$\Theta_h(z, \bar{z}) = -h\bar{z} + h^{-1} \bar{z},$$

(4.39)

while the phases $\phi_h$ are given by

$$\phi_+(z, \bar{z}) = -\bar{z}/\zeta, \quad \phi_-(z, \bar{z}) = \zeta.$$  

(4.40)
The above formulae for the Toeplitz solutions demonstrate a well-known fact: solutions for the nonlinear equations are determinants of matrices that satisfy linear ones. This rule in our case can be extended as follows: the relationships between the $\tau$-functions $\tau_n$ and $\omega_n$ (which are, recall, solutions for the 2DTL used to construct $u_0$) become linear when rewritten in terms of the ‘inside-determinant’ objects, $\alpha_k$ and $\alpha_k^\pm$.

As in the case of the infinite chain, let us consider one of the simplest solutions for (4.34). Using the polar coordinates $z = re^{i\theta}$, one can obtain

$$\alpha_k = \exp[-i(\theta + \pi/2)]I_k(2r),$$

(4.41)

where $I_k$ are the modified Bessel functions. This leads, together with definition (4.40) of $\phi_{ik}$,

$$\phi_{ik}(r, \theta) = \mp r\tilde{r}^1 e^{\mp\theta},$$

(4.42)

to

$$\alpha_k^\pm = \exp[-i(\theta + \pi/2)]k \left[ I_k(2r) + \frac{\phi_{ik}(r, \theta)}{ir}I_{k\pm1}(2r) \right]$$

(4.43)

and consequently to

$$u_n = u_0 e^{i\phi_{ik}(z, \tilde{z})} \frac{\det |I_{k+n-b}(2r) + \frac{\phi_{ik}(r, \theta)}{ir}I_{k+n-b\pm1}(2r)|_{a, b=1,\ldots,n}}{\det |I_{k+n-b}(2r)|_{a, b=1,\ldots,n}}.$$  

(4.44)

In the context of the Heisenberg-like model [3], these solutions describe circular magnetic domain structures.

4.3. Finite chain

The semi-infinite solutions derived in the previous subsection can be easily modified to provide the finite ones with $u_n \neq 0$ for $n = 0, 1, \ldots, N$ only. To this end one has to define

$$u_n(z, \tilde{z}) = u_0 e^{i\phi_{ik}(z, \tilde{z})},$$

(4.45)

where $\phi_{ik}$ are defined by (4.40), and to replace the integrals in (4.24$\alpha$) and (4.24$\beta$) with finite sums,

$$\alpha_k(z, \tilde{z}) = \sum_{p=1}^{N} \hat{\alpha}_p h^p_p \exp[-i\Theta_p(z, \tilde{z})]$$

(4.46$\alpha$)

$$\alpha_k^\pm(z, \tilde{z}) = \sum_{p=1}^{N} \hat{\alpha}_p [1 - (\hbar_p/\tilde{\zeta})^{\pm1}] h^p_p \exp[-i\Theta_p(z, \tilde{z})]$$

(4.46$\beta$)

where

$$\Theta_p(z, \tilde{z}) = -h_p z + h_p^{-1} \tilde{z}.$$  

(4.47)

These modifications do not change the fact that $i\partial \alpha_k = -\alpha_{k+1}$ and $i\tilde{\partial} \alpha_k = \alpha_{k-1}$ which implies that $u_0$ and $u_n$ given by (4.38) still solve (1.4) for $n = 1, \ldots, N - 1$. Thus, it remains to prove that they solve (1.4) for $n = 0, N$ as well. The case $n = 0$ is trivial because for both choices of $\phi_{ik}$,

$$\tilde{\partial}u_0 = (\partial u_0)(\tilde{\partial}u_0) = 0,$$

(4.48)

converting equation (1.4) with $n = 0$ into a trivial one. Considering the right-hand equation, one can show, using the identity

$$\det \sum_{p=1}^{N} a_p x_p^{a_p-b_p} \prod_{a, b=1,\ldots,N}^{(N-1)} a_p x_p^{a_p-b_p} \prod_{1 \leq p < q \leq N} (x_p - x_q)^2,$$

(4.49)
Hence, with the matrices $H$ that ensure a solution of (1)

$$u' = u_0 \prod_{p=1}^{N} \left[ 1 - (h_p/\xi)^{\pm 1} \right].$$

(4.51)

Hence,

$$\tilde{\partial} \tilde{u}_N = (\tilde{\partial} u_N) (\tilde{\partial} u_N) = 0,$$

(4.52)

ensuring a solution of (1.4) for $n = N$.

To summarize, $N + 1$ functions, $u_0$ given by (4.45) and

$$u_n(z, \tilde{z}) = u_0(z, \tilde{z}) \frac{\det |a_{k+a-b}^\pm(z, \tilde{z})|_{a, b=1, ..., n}}{\det |a_{k+a-b}^\pm(z, \tilde{z})|_{a, b=1, ..., n}}, \quad n = 1, \ldots, N,$$

(4.53)

with (4.46a)–(4.47) solve the system of $N + 1$ equations (1.4) for $n = 0, \ldots, N$.

5. Soliton solutions

The dark-soliton solutions for our equations can be constructed of the determinants

$$\Omega(A) = \det |1 + A|$$

(5.1)

of the $N \times N$ matrices $A$ that satisfy the ‘almost rank-1’ condition

$$LA - AR = |\ell \rangle \langle a|.$$

(5.2)

Here, 1 is the $N \times N$ unit matrix, $L$ and $R$ are the constant diagonal matrices, $|\ell \rangle$ is the constant $N$-component column, $|\ell \rangle = (\ell_1, \ldots, \ell_N)^T$ and $|a\rangle$ is the $N$-component row depending on the coordinates, $|a(z, \tilde{z})\rangle = (a_1(z, \tilde{z}), \ldots, a_N(z, \tilde{z}))$. The second part of the ‘solitonic ansatz’, except for (5.2), is that the dependence of $A$ on all coordinates ($z$, $\tilde{z}$ and $n$) can be built by means of the shifts

$$\bar{T}_z \Omega = \Omega(A \bar{H}_z)$$

(5.3)

with the matrices $H_z$ being defined by

$$H_z = (L - \xi)(R - \zeta)^{-1}$$

(5.4)

(we do not write the unit matrix explicitly, so $(L - \xi)$ stands for $(L - \xi 1)$ etc).

The remarkable property of the above matrices is that the determinants

$$\Omega_\xi = \bar{T}_z \Omega, \quad \Omega_\eta = \bar{T}_z \Omega \bar{T}_\eta \Omega$$

(5.5)

satisfy the Fay-like identity

$$(\xi - \eta) \Omega_\xi \Omega_\eta + (\eta - \zeta) \Omega_\zeta \Omega_\eta + (\zeta - \xi) \Omega_\xi \Omega_\eta = 0$$

(5.6)

(see, e.g., the appendix of [17]). Introducing the differential operators $\partial_\ell$ defined by

$$\bar{T}^{-1}_z T^{z+\ell} \Omega = \Omega + i \delta \partial_\ell \Omega + O(\delta^3)$$

(5.7)

one can derive from (5.6) the differential Fay identities

$$(\xi - \alpha) i D_\xi \Omega_\alpha \cdot \Omega = (\bar{T}^{-1}_z \Omega_\alpha) (\bar{T}_z \Omega) - \Omega_\alpha \Omega$$

(5.8)

and

$$(\xi - \alpha)(\xi - \beta) i D_\xi \Omega_\alpha \cdot \Omega_\beta = (\alpha - \beta) \left[ (\bar{T}^{-1}_z \Omega_\alpha) (\bar{T}_z \Omega) - \Omega_\alpha \Omega_\beta \right]$$

(5.9)
Applying these identities two times, one arrives at
\[ \frac{1}{2} (\xi - \eta)^2 \partial_\xi \partial_\eta \Omega \cdot \Omega = \Omega^2 - (T_\xi^{-1} \partial_\eta) (T_\xi^{-1} T_\eta \Omega) \]  
(5.10)
from which it is easy to obtain solutions for the 2DTL by associating the \( \partial_\xi \) and \( \partial_\eta \) (with fixed \( \xi \) and \( \eta \)) and introducing the \( n \)-variable by means of the powers of \( T_\nu \)
\[ \Omega_n = T_\nu^n \Omega \]  
(5.11)
with
\[ H_\xi = H_\eta H_\nu. \]  
(5.12)
From the Fay identities (5.6) and (5.9), one can derive after straightforward calculations that the functions
\[ T_n = T_\nu h^a_\alpha T_\nu^n \Omega_\alpha \]  
(5.13a)
\[ W_n = W_\nu h^a_\beta T_\nu^n \Omega_\beta \]  
(5.13b)
\[ \tilde{T}_n = \tilde{T}_\nu \tilde{h}^\alpha T_\nu^{-1} \Omega_{\alpha \beta} \]  
(5.13c)
\[ \tilde{W}_n = \tilde{W}_\nu \tilde{h}^\alpha T_\nu^{-1} \Omega_\xi , \]  
(5.13d)
where the \( h \)-factors are defined by
\[ h_\alpha = \frac{\xi - \alpha}{\eta - \alpha}, \quad h_\beta = \frac{\xi - \beta}{\eta - \beta}, \]  
(5.14)
\[ \tilde{h} = \mu h_\alpha h_\beta (\xi - \eta), \quad \tilde{\mu} = \tilde{\mu} (\xi - \eta) \]  
(5.15)
(with constants \( \mu \) and \( \tilde{\mu} \) satisfying \( \mu \tilde{\mu} = (\xi - \eta)^{-2} \)) and the constants \( T_\nu, W_\nu, \tilde{T}_\nu, \tilde{W}_\nu \) are related by
\[ \tilde{T}_\nu \tilde{W}_\nu = T_\nu W_\nu \frac{(\beta - \alpha)(\xi - \eta)}{(\alpha - \eta)(\beta - \eta)} \]  
(5.16)
satisfy the following equations:
\[ [iD_\xi + \lambda_\xi (\beta, \alpha)] W_n \cdot T_n = \mu \tilde{T}_{n-1} \tilde{W}_n \]  
(5.17a)
\[ [iD_\xi + \lambda_\xi (\beta, \eta)] T_n \cdot T_n = -\mu \tilde{T}_{n-1} T_{n+1} \]  
(5.17b)
\[ [iD_\xi + \lambda_\xi (\beta, \eta)] W_n \cdot \tilde{W}_{n-1} = -\mu W_{n-1} \tilde{W}_n \]  
(5.17c)
and
\[ [iD_\eta + \lambda_\eta (\beta, \alpha)] W_n \cdot T_n = \tilde{\mu} \tilde{W}_{n-1} \tilde{T}_n \]  
(5.18a)
\[ [iD_\eta + \lambda_\eta (\beta, \xi)] \tilde{T}_n \cdot T_n = \tilde{\mu} T_{n-1} \tilde{T}_n \]  
(5.18b)
\[ [iD_\eta + \lambda_\eta (\beta, \xi)] W_n \cdot \tilde{W}_{n-1} = \tilde{\mu} \tilde{W}_{n-1} W_{n+1} \]  
(5.18c)
Here
\[ \lambda_\xi (\alpha, \beta) = \lambda_\xi (\alpha) - \lambda_\xi (\beta) \]  
(5.19)
with
\[ \lambda_\xi (\gamma) = \frac{1}{\xi - \gamma} \]  
(5.20)
Comparing the above equations with (3.16) and (3.17), it is easy to conclude that to obtain solutions for our problem one has to identify

$$\hat{\vartheta} = \mu^{-1} \vartheta_{\xi}, \quad \hat{\varphi} = -\hat{\mu}^{-1} \varphi_{\eta}$$  \hspace{1cm} (5.21)

and to introduce some phases, which are linear functions of \(z\) and \(\bar{z}\), to eliminate extra \(\lambda\)-terms in (5.17a)–(5.18c). This leads to the following expression for \(u_n\):

$$u_n(z, \bar{z}) = u_n \exp(i \phi(z, \bar{z}) \hbar) \frac{\det|1 + A_n(z, \bar{z}) H_{\bar{\rho}}|}{\det|1 + A_n(z, \bar{z}) H_{\rho}|}$$  \hspace{1cm} (5.22)

where

$$h = \frac{h_{\rho}}{h_{\bar{\rho}}},$$  \hspace{1cm} (5.23)

$$A_n(z, \bar{z}) = A(z, \bar{z}) H_{\bar{\rho}}^n$$  \hspace{1cm} (5.24)

and

$$\phi = \mu^{-1} \lambda_{\xi}(\alpha, \beta)z + \hat{\mu}^{-1} \lambda_{\eta}(\beta, \alpha)\bar{z} + \text{constant}.$$  \hspace{1cm} (5.25)

Finally, it remains to resolve the restriction (5.12) and to write down explicitly the dependence of \(A\) on the coordinates, which can be done by calculating the limit in (5.7),

$$i A^{-1} \partial_{\zeta} A = (L - R)(L - \zeta)^{-1}(R - \zeta)^{-1}$$  \hspace{1cm} (5.26)

(for any \(\zeta\)).

The restriction (5.12) implies that the matrices \(L\) and \(R\) are not independent,

$$(L - \xi)(R - \xi) = (\nu - \xi)(\eta - \xi) \cdot 1.$$  \hspace{1cm} (5.27)

Introducing matrices \(\hat{L}\) and \(\hat{R}\),

$$\hat{L} = \frac{1}{\xi - \nu} (L - \nu), \quad \hat{R} = \frac{1}{\xi - \nu} (R - \nu)$$  \hspace{1cm} (5.28)

satisfying

$$(\hat{L} - 1)(\hat{R} - 1) = \frac{\xi - \eta}{\xi - \nu} \cdot 1$$  \hspace{1cm} (5.29)

and parameter \(f\), instead of \(\mu\) and \(\hat{\mu}\),

$$f = \frac{1}{\mu(\xi - \eta)} = \hat{\mu}(\xi - \eta),$$  \hspace{1cm} (5.30)

the \(z\)- and \(\bar{z}\)-dependences of \(A\) and \(\phi\) can be presented in a symmetric form as

$$A(z, \bar{z}) = A_{\ast} \exp\{i M(z, \bar{z})\},$$  \hspace{1cm} (5.31)

where \(A_{\ast}\) is a constant matrix,

$$M(z, \bar{z}) = f (\hat{R} - \hat{L}) z + f^{-1} (\hat{R}^{-1} - \hat{L}^{-1}) \bar{z},$$  \hspace{1cm} (5.32)

and

$$\phi(z, \bar{z}) = f \left(\hbar_{\rho}^{-1} - h_{\bar{\rho}}^{-1}\right) z + f^{-1} \left(\hbar_{\rho} - h_{\bar{\rho}}\right) \bar{z}.$$  \hspace{1cm} (5.33)

These formulae together with (5.22) describe the \(N\)-dark-soliton solutions for our problem.

The simplest (one-soliton) solution can be written as

$$u_n(z, \bar{z}) = u_n \exp(2 i \nu(z, \bar{z}) \{1 + \tanh \rho \ \tanh U_n(z, \bar{z})\},$$  \hspace{1cm} (5.34)

where

$$\rho = \frac{1}{2} \ln \frac{1 - h_{\rho} \hat{L}}{1 - h_{\rho} \hat{R}} - \ln \frac{1 - h_{\rho} \hat{L}}{1 - h_{\rho} \hat{R}}$$  \hspace{1cm} (5.35)
(note that now $\hat{L}$ and $\hat{R}$ are just complex numbers) the phase $\varphi_n$ is given by
$$
\varphi_n(z, \bar{z}) = \xi_0 z + \eta_0 \bar{z} + \zeta_0 n
$$
(5.36)
with
$$
\xi_0 = \frac{f}{2} (h^{-1}_\alpha - h^{-1}_\beta), \quad \eta_0 = \frac{1}{2f} (h_\beta - h_\alpha)
$$
(5.37)
and
$$
\sin^2 \zeta_0 = \xi_0 \eta_0.
$$
(5.38)
while $U_n$ is given by
$$
U_n(z, \bar{z}) = \xi_s z + \eta_s \bar{z} + \zeta_s n
$$
(5.39)
with
$$
\xi_s = \frac{if}{2} (\hat{R} - \hat{L}), \quad \eta_s = \frac{i}{2f} (\hat{R}^{-1} - \hat{L}^{-1})
$$
(5.40)
and
$$
\sinh^2 \zeta_s = \xi_s \eta_s.
$$
(5.41)

To conclude, we would like to note that in the soliton case the link between $\tau_n$ and $\omega_n$ is linear in the terms of the matrices $A_n$: the matrices that appear in the determinants in (5.22) are related by the constant diagonal matrix $H_\beta H_\alpha^{-1}$.

6. Quasiperiodic solutions

In this section, we derive the periodic solutions for our equation proceeding in a similar way to the one used in the previous section. The main difference is in the starting point: instead of identity (5.6) we use the original Fay trisecant identity (see (6.7) below).

The solutions that we derive below are combinations of the $\theta$-functions defined over a compact Riemann surface $\Gamma$ of the genus $g$ for which one can choose in a standard way a set of closed contours (cycles) $\{a_i, b_i\}_{i=1,\ldots,g}$ with the intersection indices
$$
a_i \circ a_j = b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij} \quad i, j = 1, \ldots, g
$$
(6.1)
and $g$ independent holomorphic differentials $\sigma_k$ satisfying the normalization conditions
$$
\oint_{a_k} \sigma_k = \delta_{ik}, \quad i, k = 1, \ldots, g.
$$
(6.2)
The matrix of the $b$-periods,
$$
\Omega_{ik} = \oint_{b_i} \sigma_k, \quad i, k = 1, \ldots, g.
$$
(6.3)
determines the so-called period lattice, $L_\Omega = \{m + \Omega n : m, n \in \mathbb{Z}^g\}$, and the Abel mapping from $\Gamma$ to the $2g$ torus $\mathbb{C}^g/L_\Omega$ (the Jacobian of this surface),
$$
P \rightarrow \int_{P_0}^P \omega, \quad \omega = \sum_{i=1}^g \sigma_i \in \mathbb{C}^g
$$
(6.4)
where $P$ is a point of $\Gamma$, $\omega$ is the $g$-vector of the 1-forms, $\omega = (\sigma_1, \ldots, \sigma_g)^T$ and $P_0$ is some fixed point of $\Gamma$.

The $\theta$-function, $\theta(\xi) = \theta(\xi, \Omega)$, is defined by
$$
\theta(\xi) = \sum_{n \in \mathbb{Z}^g} \exp \{ \pi i (\xi, \Omega n) + 2\pi i (\xi, \zeta) \},
$$
(6.5)
where \((n, \zeta)\) stands for \(\sum_{i=1}^{g} n_i \zeta_i\). This is a quasiperiodic function on \(\mathbb{C}^g\):

\[\theta(\zeta + n) = \theta(\zeta) \tag{6.6a}\]

\[\theta(\zeta + \Omega n) = \exp \{-\pi i (n, \Omega n) - 2\pi i (n, \zeta)\} \theta(\zeta) \tag{6.6b}\]

for any \(n \in \mathbb{Z}^g\).

The calculations presented below are based on the famous Fay trisecant formula \([18, 19]\) that can be written as

\[\varepsilon^P \varepsilon^Q \theta^P \theta^Q - \varepsilon^P \varepsilon^Q \theta^P \theta^Q + \varepsilon^P \varepsilon^Q \theta^P \theta^Q = 0. \tag{6.7}\]

Here

\[\theta(P_1 \cdots Q_n) = \theta\left(\zeta + \sum_{i=1}^{m} \int_{P_i}^{Q_i} \omega\right) \tag{6.8}\]

and the skew-symmetric function \(\varepsilon^P, \varepsilon^Q\) is given by

\[\varepsilon^P = \theta\left(\epsilon + \int_{P}^{Q} \omega\right), \tag{6.9}\]

where \(\epsilon\) is a zero of the \(\theta\)-function: \(\theta(\epsilon) = 0\).

Now let us define the differential operators \(\partial_X\) by

\[\theta^P = \theta + \varepsilon^X \partial_X \theta + o(\varepsilon^X) \tag{6.10}\]

In what follows, we use \(\partial_A\) and \(\partial_B\) defined near two points \(A\) and \(B\) that are fixed. One of them will play the role of \(\partial\) and another of \(\partial\). Taking the limit in (6.7) one can obtain the differential Fay identity

\[\left[D_X + \lambda_X(P,Q)\right] \theta^P = \gamma_X(P,Q) \theta^Q, \tag{6.11}\]

where \(D_X\) is the Hirota operator corresponding to \(\partial_X\) and the functions \(\lambda_X(P)\) and \(\gamma_X(P,Q)\) are defined by

\[\lambda_X(P,Q) = \lambda_X(P) - \lambda_X(Q) \tag{6.12}\]

with

\[\lambda_X(P) = \lim_{Y \to X} \frac{\varepsilon_Y - \varepsilon_X}{\varepsilon_Y - \varepsilon_X} \tag{6.13}\]

and

\[\gamma_X(P,Q) = \frac{\varepsilon_X^P}{\varepsilon_X^Q} \varepsilon_X^Q \tag{6.14}\]

Consider the functions

\[T_n = T[h(P)] \theta \left(\zeta_n + \int_{A}^{P} \omega\right) \tag{6.15a}\]

\[W_n = W[h(Q)] \theta \left(\zeta_n + \int_{A}^{Q} \omega\right) \tag{6.15b}\]

\[\tilde{T}_n = \tilde{T} \hat{h}^n \theta(\zeta_n) \tag{6.15c}\]

\[\tilde{W}_n = \tilde{W} \hat{h}^n \theta(\zeta_n) \tag{6.15d}\]
where

\[ h(P) = \frac{\varepsilon_p^A}{\varepsilon_p^B} \]  

and the \( n \)-dependence is given by

\[ \tilde{\zeta}_n = \zeta_n + \int_B^A \omega. \]  

It follows from (6.7) that if one imposes the restrictions

\[ \tilde{T} \cdot \hat{W}_n = T \cdot W_n \]  

and

\[ \tilde{\hat{h}} = h(P)h(Q), \]  

then these functions satisfy

\[ T_nW_{n+1} - T_{n+1}W_n = \tilde{T}_n\hat{W}_n. \]  

Furthermore, by taking

\[ \tilde{\hat{h}} = \mu_A \varepsilon^B_A \frac{\varepsilon^P_A}{\varepsilon^B_B}, \quad \hat{h} = \frac{1}{\mu_A \varepsilon^B_A} \]

where \( \mu_A \) and \( \mu_B \) are two constants related by

\[ \mu_A \mu_B (\varepsilon^B_A)^2 = 1. \]  

one can obtain from (6.11) the set of identities which will be associated with the \( \tilde{\partial} \)-flows:

\[ [D_A + \lambda_A(Q, P)]W_n \cdot T_n = -\mu_A \tilde{T}_n W_n \]  

\[ [D_A + \lambda_A(Q, B)]\tilde{T}_n \cdot T_n = \mu_A \tilde{T}_n \tilde{T}_{n+1} \]  

\[ [D_A + \lambda_A(Q, B)]W_n \cdot \hat{W}_{n+1} = \mu_A W_{n+1} \hat{W}_n \]  

and another one,

\[ [D_B + \lambda_B(Q, P)]W_n \cdot T_n = -\mu_B \hat{W}_{n-1} \tilde{T}_n \]  

\[ [D_B + \lambda_B(Q, A)]\hat{T}_{n-1} \cdot T_n = -\mu_B T_{n-1} \tilde{T}_n \]  

\[ [D_B + \lambda_B(Q, A)]W_n \cdot \hat{W}_n = -\mu_B \tilde{W}_{n-1} W_{n+1} \]

associated with the \( \tilde{\partial} \)-equations.

Comparing the above equations with (3.16) and (3.17), one arrives at

\[ \tilde{\partial} = \frac{i}{\mu_A} \partial_A, \quad \hat{\partial} = \frac{1}{i\mu_B} \partial_B \]  

which determines the dependence on \( z \) and \( \tilde{z} \),

\[ \zeta_n(z, \tilde{z}) = \zeta_n + za + \tilde{z}b + nc \]  

where \( c \) was defined above,

\[ c = \int_B^A \omega. \]
which leads to

\[ a = \lim_{P \to A} \int_{A}^{P} \omega \quad \text{(6.28a)} \]

\[ b = \lim_{P \to B} \int_{B}^{P} \omega, \quad \text{(6.28b)} \]

where we have introduced the constant \( f \) instead of \( \mu_A \) and \( \mu_B \),

\[ f = \frac{1}{\mu_A e_B^A} = \frac{\mu_B e_B^A}{\lambda}. \quad \text{(6.29)} \]

Again, the \( \lambda \)-terms in (6.23a)–(6.24c) can be eliminated by adding linear in \( z \) and \( \bar{z} \) phases, which leads to

\[ u_n = \frac{W_n}{T_n} e^{i \phi}, \quad \text{(6.30)} \]

where

\[ \phi(z, \bar{z}) = f e_B^A \phi_A(P, Q) z + f^{-1} e_B^A \phi_B(P, Q) \bar{z} + \text{constant}. \quad \text{(6.31)} \]

Using the definitions of \( W_n \) and \( T_n \), and introducing \( h = h(Q)/h(P) \),

\[ h = e_{P}^A e_{B}^A, \quad \text{(6.32)} \]

we can write the final expression for the quasiperiodic solutions as

\[ u_n(z, \bar{z}) = u_n e^{i \phi_n(z, \bar{z})} \frac{\theta'\xi_n(z, \bar{z}) + f_{1}^Q \omega}{\theta(\xi_n(z, \bar{z}) + f_{1}^P \omega)}. \quad \text{(6.33)} \]

This time, the link between \( \omega_n \) and \( \tau_n \) is linear in terms of \( \xi_n \); the transformation \( \tau_n \to \omega_n \) is achieved by \( \xi_n \to \xi_n + f_{1}^Q \omega \).

The simplest of the quasiperiodic solutions (6.33), after some slight modifications, can be rewritten as a cnoidal wave:

\[ u_n(z, \bar{z}) = u_n e^{i \phi_n(z, \bar{z})} \text{sn} \xi_n(z, \bar{z}). \quad \text{(6.34)} \]

where the phase \( \phi_n \) and the function \( \xi_n \) are given by

\[ \phi_n(z, \bar{z}) = \xi_0 z + \eta_0 \bar{z} + \delta_0 n, \quad \text{(6.35a)} \]

\[ \xi_n(z, \bar{z}) = \xi_p z + \eta_p \bar{z} + \delta_p n \quad \text{(6.35b)} \]

and \( \text{sn} z = \text{sn}(z, k) \) is the elliptic sine. The parameters \( \xi_{0, p}, \eta_{0, p}, \delta_{0, p} \) and the elliptic modulus \( k \) are related by

\[ \xi_p \eta_p = \text{sn}^2 \delta_p \quad \text{(6.36a)} \]

\[ \xi_0 \eta_0 + \xi_p \eta_p = 2 \sin \delta_0 \text{ sn} \delta_p \quad \text{(6.36b)} \]

and

\[ \xi_0 \eta_0 = | \text{dn} \delta_p - e^{i \delta_0} \text{cn} \delta_p |^2 \quad \text{(6.36c)} \]

for real \( \delta_0 \) and \( \delta_p \).

7. Solutions of the two-dimensional Volterra equation

As was mentioned in section 2, the authors of \([5, 6]\) derived general solutions of the 2DVE in the case of a finite chain. Here we would like to present several other classes of solutions for this system, namely ones that can be obtained from the results presented in this paper using (3.19a) and (3.19b).
7.1. Toeplitz solutions

As follows from (3.19a) and (3.19b), the constants $\tau$ and $\bar{\tau}$ as well as the phases $\tilde{\tau}/\xi$ (that we defined in (4.9), (4.10) and (4.35), (4.36)) disappear from the final formulae for $a_n$ and $b_n$, that can be written as

$$a_n = \frac{A_n^{c+1}B_n^{-1}}{A_n^{c}B_n^{-1}} , \quad b_n = \frac{A_n^{c-1}B_n^{-1}}{A_n^{c}B_n^{-1}} \quad (7.1)$$

in the infinite case ($-\infty < n < \infty$) and

$$a_n = \frac{A_n^{k+1}B_n^{-1}}{A_n^{k}B_n^{-1}} , \quad b_n = \frac{A_n^{k-1}B_n^{-1}}{A_n^{k}B_n^{-1}} \quad (7.2)$$

in the semi-infinite case ($1 \leq n < \infty$). One can easily obtain similar solutions with $\mathcal{C}^{c}_{\ell}$- and $\mathcal{C}^{k}_{\ell}$-determinants which we do not write here.

7.2. Soliton solutions

Making the shift $A_n \rightarrow A_n H_{\xi}^{-1}$ and introducing the matrix $B_n$,

$$B_n = A_n H_{\xi}, \quad (7.3)$$

one can present soliton solutions for (2.5) in the following form:

$$a_n = c_s \frac{\det [1 + A_n H_{\xi} H_n] \det [1 + B_n]}{\det [1 + A_n H_{\xi}] \det [1 + B_n]} \quad (7.4a)$$

$$b_n = \frac{1}{c_s} \frac{\det [1 + A_n H_n] \det [1 + B_n H_n]}{\det [1 + A_n H_n] \det [1 + B_n]} \quad (7.4b)$$

with $c_s = f/h_{\beta}$ and matrices $A_n = A_n(\xi, \bar{\xi})$ and $H_{\xi}$ defined in section 5. In these expressions one can find a symmetry $1/b_n = a_n(\xi \rightarrow \eta)$ which is a manifestation of many symmetries inherent in the 2DVE.

7.3. Periodic solutions

It follows from (3.19a) and (3.19b) that the phases that one has to introduce to eliminate the $\lambda$-terms in (6.23a)–(6.24c) cancel themselves and formulae (6.15a) and (6.15c) yield periodic solutions for the 2DVE. After the shift $\xi_n \rightarrow \xi_n - \int_{\lambda}^{P} \omega$ one can write these solutions as

$$a_n = c_s \frac{\theta(\xi_n + \int_{\lambda}^{\omega} \theta(\xi_n + \int_{\lambda}^{\omega}) \theta(\xi_n + \int_{\lambda}^{\omega})}{\theta(\xi_n) \theta(\xi_n + \int_{\lambda}^{\omega})} \quad (7.5a)$$

$$b_n = \frac{1}{c_s} \frac{\theta(\xi_n + \int_{\lambda}^{\omega}) \theta(\xi_n + \int_{\lambda}^{\omega})}{\theta(\xi_n) \theta(\xi_n + \int_{\lambda}^{\omega})} \quad (7.5b)$$

where $c_s = f/\varepsilon^{P}_{\xi}$ with $\xi_n(\xi, \bar{\xi})$ and $f$ defined in section 6. One can see that of two points that parametrize solutions (6.33), $P$ and $Q$, only one is left and, again, one can find the symmetry linking $a_n$ and $b_n$: $b_n \propto a_n(\lambda \leftrightarrow P)$.
Applying the Jacobi identity

\[ \Delta \Delta_{k_1 k_2} = \Delta_{k_1} \Delta_{k_2} - \Delta_{k_2} \Delta_{k_1}, \]

where \( \Delta \) is the determinant of a matrix, \( \Delta_{k_1} \) is the determinant of the same matrix with the \( j \)th row and \( l \)th column being excluded, etc., to \( F^{k}_{\ell+1}(\zeta) \) for \( (j_1, j_2; k_1, k_2) \) being equal to \( (1, \ell + 1; 1, \ell + 1), (1, 2; 1, \ell + 1), (1, 3; 1, \ell + 1) \) and \( (1, \ell; 1, \ell + 1) \), one can obtain

\[ X^k_\ell := A^k_{\ell-1} F^{k+1}_{\ell+1}(\zeta) - A^k_{\ell} F^{k+1}_\ell(\zeta) + \zeta A^k_{\ell} F^{k+1}_\ell(\zeta) = 0 \]  

\[ Z^k_\ell := A^k_{\ell-1} F^{k+1}_{\ell+1}(\zeta) - A^k_{\ell} F^{k+1}_\ell(\zeta) + \zeta A^k_{\ell} F^{k+1}_\ell(\zeta) = 0 \]

\[ I^k_\ell := F^{k+1}_{\ell+1}(\zeta) \partial_{\zeta} A^k_{\ell-1} - A^k_{\ell} \partial_{\zeta} F^{k+1}_\ell(\zeta) + \zeta A^k_{\ell} \partial_{\zeta} F^{k+1}_\ell(\zeta) = 0 \]

\[ \bar{I}^k_\ell := F^{k+1}_{\ell+1}(\zeta) \partial_{\zeta} A^k_{\ell-1} - A^k_{\ell} \partial_{\zeta} F^{k+1}_\ell(\zeta) + \zeta A^k_{\ell} \partial_{\zeta} F^{k+1}_\ell(\zeta) = 0 \]

correspondingly. These identities, combined with

\[ \Delta^k_\ell := (A^k_\ell)^2 - A^k_{\ell-1} A^k_{\ell+1} - A^k_{\ell} A^{k+1}_\ell = 0, \]
lead to

\[ W^\ell_\ell := A^\ell_k F_{\ell+1} - A^\ell_{k+1} F^\ell_\ell - A^{k+1}_\ell F^\ell_{\ell+1} = 0 \]  
(A.7)

\[ Y^\ell_\ell := A^\ell_k F_{\ell+1} - A^\ell_{k+1} F^\ell_\ell + \zeta A^{k+1}_\ell F^\ell_{\ell+1} = 0. \]  
(A.8)

Expanding the above equations,

\[ W^\ell_\ell = \Delta^\ell_\ell + \zeta \bar{w}^\ell_\ell + \cdots + (-\zeta)^{l-1} w^\ell_\ell \]  
(A.9)

\[ X^\ell_\ell = \zeta \bar{x}^\ell_\ell + \cdots + (-\zeta)^{l-1} x^\ell_\ell \]  
(A.10)

\[ Y^\ell_\ell = \zeta \bar{y}^\ell_\ell + \cdots + (-\zeta)^{l-1} y^\ell_\ell \]  
(A.11)

\[ Z^\ell_\ell = \zeta \bar{z}^\ell_\ell + \cdots + (-\zeta)^{l-1} z^\ell_\ell. \]  
(A.12)

where

\[ w^\ell_\ell = D_+ A^\ell_{\ell-1} \cdot A^\ell_\ell - A^{\ell-1}_\ell A^\ell_{\ell+1} \]  
(A.13)

\[ x^\ell_\ell = D_+ A^\ell_{\ell-1} \cdot A^\ell_\ell - A^{\ell-1}_\ell A^\ell_{\ell+1} \]  
(A.14)

\[ z^\ell_\ell = D_+ A^\ell_\ell \cdot A^\ell_{\ell-1} - A^{\ell-1}_\ell A^\ell_{\ell+1} \]  
(A.15)

with

\[ \bar{w}^\ell_\ell = -A^\ell_\ell \partial w^\ell_\ell + A^\ell_{\ell+1} \partial A^\ell_{\ell-1} + A^{\ell-1}_\ell \partial A^{\ell+1}_\ell \]  
(A.16)

\[ \bar{x}^\ell_\ell = D_- A^\ell_{\ell-1} \cdot A^\ell_\ell + A^{\ell-1}_\ell A^\ell_{\ell+1} \]  
(A.17)

\[ \bar{y}^\ell_\ell = D_- A^\ell_\ell \cdot A^\ell_{\ell+1} + A^{\ell+1}_\ell A^\ell_{\ell-1} \]  
(A.18)

\[ \bar{z}^\ell_\ell = D_- A^\ell_\ell \cdot A^\ell_{\ell-1} + A^{\ell-1}_\ell A^\ell_{\ell+1} \]  
(A.19)

and introducing

\[ J^\ell_\ell := \partial_+ Z^\ell_\ell - I^\ell_\ell = 0 \]  
(A.20)

\[ J^\ell_\ell := \partial_- X^\ell_\ell - I^\ell_\ell = 0 \]  
(A.21)

one can derive by straightforward algebra the following identities:

\[ A^\ell_{\ell-1}(D_+ F_{\ell+1}^\ell, A^\ell_\ell + A^{\ell+1}_\ell F_{\ell+1}^\ell) = A^\ell_\ell J^\ell_\ell - (\partial_+ A^\ell_\ell) Z^\ell_\ell - F^\ell_\ell w^\ell_{\ell+1} \]  
(A.22)

\[ A^{\ell-1}_{\ell-1}(D_+ F_{\ell+1}^\ell, A^\ell_\ell + \zeta A^{\ell+1}_\ell F_{\ell+1}^\ell) = A^\ell_{\ell-1} J^\ell_{\ell-1} - (\partial_+ A^\ell_{\ell-1}) Z^{\ell-1}_{\ell-1} - \zeta F^\ell_{\ell+1} w^\ell_\ell \]  
(A.23)

\[ A^\ell_{\ell-2}(D_+ F_{\ell+1}^\ell, A^\ell_\ell + A^{\ell+1}_\ell F_{\ell+1}^\ell) = A^\ell_{\ell-2} J^\ell_{\ell-2} - (\partial_+ A^\ell_{\ell-2}) Z^{\ell-2}_{\ell-2} + A^{\ell+1}_{\ell} X^{\ell+1}_{\ell-1} - F^\ell_{\ell+1} X^{\ell+1}_{\ell-1} + \zeta F^\ell_{\ell+1} Z^{\ell+1}_{\ell-1} \]  
(A.24)

and

\[ A^{\ell+1}_{\ell-1}(D_+ F_{\ell+1}^\ell, A^\ell_\ell - A^{\ell+1}_\ell F_{\ell+1}^\ell) = A^{\ell+1}_{\ell-1} J_{\ell-1}^{\ell+1} - (\partial_- A^\ell_{\ell-1}) X^{\ell+1}_{\ell-1} - F^{\ell+1}_{\ell+1} Y^{\ell+1}_{\ell} \]  
(A.25)

\[ A^\ell_{\ell-1}(D_+ F_{\ell+1}^\ell, A^\ell_\ell - \zeta A^{\ell+1}_\ell F_{\ell+1}^\ell) = A^\ell_{\ell-1} J^\ell_{\ell-1} - (\partial_- A^\ell_{\ell-1}) X^{\ell+1}_{\ell-1} - \zeta F^{\ell+1}_{\ell+1} Y^{\ell+1}_{\ell} \]  
(A.26)

\[ A^\ell_{\ell-2}(D_+ F_{\ell}^\ell, A^\ell_\ell + A^{\ell-1}_\ell F_{\ell+1}^\ell) = A^\ell_{\ell-2} J^\ell_{\ell-2} + A^{\ell-1}_\ell Z^{\ell-1}_{\ell-1} - (\partial_- A^\ell_{\ell-2}) X^{\ell+1}_{\ell-1} + F^{\ell+1}_{\ell+1} X^{\ell+1}_{\ell-1} - \zeta F^{\ell+1}_{\ell+1} Y^{\ell+1}_{\ell} \]  
(A.27)
which means
\[
D_+ F^k_{\ell+1} \cdot A^k_{\ell} + A^{k+1}_{\ell+1} F^k_{\ell} = 0 \tag{A.28}
\]
\[
D_+ F^k_{\ell+1} \cdot A^k_{\ell} + \zeta A^{k+1}_{\ell+1} F^k_{\ell} = 0 \tag{A.29}
\]
\[
D_+ F^{k+1}_{\ell} \cdot A^k_{\ell} + A^{k+1}_{\ell} F^{k+1}_{\ell} = 0 \tag{A.30}
\]
and
\[
D_- F^k_{\ell+1} \cdot A^k_{\ell} - A^{k-1}_{\ell+1} F^k_{\ell} = 0 \tag{A.31}
\]
\[
D_- F^k_{\ell+1} \cdot A^k_{\ell} - \zeta A^{k-1}_{\ell+1} F^k_{\ell} = 0 \tag{A.32}
\]
\[
D_- F^{k+1}_{\ell} \cdot A^k_{\ell} + A^{k-1}_{\ell+1} F^{k+1}_{\ell} = 0 \tag{A.33}
\]
These equations, when rewritten in terms of \(B^k_\ell\) given by (4.3), are nothing but (4.6a)–(4.7c), while in terms of \(C^k_\ell\) given by (4.17) they become
\[
0 = D_+ C^k_\ell \cdot A^k_{\ell} + \zeta C^{k+1}_{\ell+1} A^k_{\ell+1} \tag{A.34}
\]
\[
0 = D_+ C^{k+1}_\ell \cdot A^k_{\ell+1} + A^{k+1}_{\ell+1} C^k_{\ell-1} \tag{A.35}
\]
\[
0 = D_+ C^{k+1}_\ell \cdot A^k_{\ell+1} + A^{k+1}_{\ell+1} C^k_{\ell-1} \tag{A.36}
\]
and
\[
0 = D_- C^k_\ell \cdot A^k_{\ell} - \zeta C^{k-1}_{\ell+1} A^k_{\ell+1} \tag{A.37}
\]
\[
0 = D_- C^{k+1}_\ell \cdot A^k_{\ell+1} + A^{k-1}_{\ell+1} C^k_{\ell-1} \tag{A.38}
\]
\[
0 = D_- C^{k+1}_\ell \cdot A^k_{\ell+1} - A^{k-1}_{\ell+1} C^k_{\ell-1}. \tag{A.39}
\]
The above formulae are enough to obtain the Toeplitz solutions presented in section 4.

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