PENCILS OF GENUS TWO CURVES ON RATIONAL SURFACES

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Abstract

We consider relatively minimal fibrations of curves of genus two on rational surfaces whose Picard numbers are not maximal. By birational morphisms, such fibred surfaces are interpreted as pencils of plane curves. We show that only four are canonical, among a variety of possible models. For each canonical pencil, we give an example with trivial Mordell-Weil group.

1 Introduction

The theory of the Mordell-Weil lattices are sufficiently developed by Oguiso and Shioda in [12] for minimal elliptic rational surfaces. In their work, the even unimodular root lattice $E_8$ of rank eight played very important role as the predominant frame. For example, it was shown that the Mordell-Weil group is trivial if and only if there exists a singular fibre of type II* in the sense of Kodaira [9] whose dual graph contains $E_8$ as a subgraph. The lattice $E_8$ also appears in another application by Shioda [14] to describe a hierarchy of deformations of rational double points.

In this paper, we consider fibred rational surfaces of genus two over $\mathbb{C}$ in order to look for right candidates for the “frame lattices” in this case. Here, a fibred rational surface of genus two means a smooth projective rational surface $X$ together with a relatively minimal fibration $f : X \to \mathbb{P}^1$ whose general fibre $F$ is a smooth projective curve of genus two. It is shown in Proposition 4.1 that $f$ has a section. Hence we can always associate to $f$ the Mordell-Weil group (or lattice) by applying the general machinery of Shioda [15].

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We regard a section of \( f \) as a curve on \( X \), and call it a \((-1)\)-section of \( f \) if the self-intersection number is equal to \((-1)\). To clarify the structure of the Mordell-Weil lattice, we choose a ruling on \( X \) and its relatively minimal model \( \Sigma_d \) carefully so that we get a natural \( \mathbb{Z} \)-basis of \( \text{NS}(X) \) (the Néron-Severi group) which gives us a simple presentation of \( F \). This is done by choosing a birational morphism \( X \to \Sigma_d \) which contracts step by step a \((-1)\)-curve whose intersection number with \( F \) is the smallest among all \((-1)\)-curves.

Let \( \rho(X) \) denote the Picard number of \( X \). Then it can be shown that \( \rho(X) \leq 14 \). When \( \rho(X) = 14 \) and the Mordell-Weil group of \( f \) is non-trivial, as an above procedure, it is not so hard to see that there are twelve \((-1)\)-curves each of which meets \( F \) at one point and one \((-1)\)-curve meeting \( F \) at two points such that, by contracting them all, we get a pencil of plane quartic curves whose base points are resolved to get \( |F| \). If the group is trivial, then \( X \) has a unique ruling and we can only take \( \Sigma_2 \) or \( \Sigma_3 \) as its minimal model, while there actually exists some \( f \)'s not admitting such a model when the Mordell-Weil group is non-trivial (cf. [6]). In other words, \( \mathbb{P}^2 \) with images become such models of more \( f \)'s than \( \Sigma_2 \) and \( \Sigma_3 \).

When \( \rho(X) \leq 13 \), by choosing birational morphisms like the one above, \( \mathbb{P}^2 \) with images of \( |F| \)'s become such models for all \( f \)'s, while \( \Sigma_d \) with any images can not be models of some \( f \)'s for \( d \neq 1 \) (cf. Proposition 3.7 and Theorem 3.8). Furthermore, the main theorem describes the such models on \( \mathbb{P}^2 \) as follows:

**Theorem 1.1** (cf. Theorem 3.1). Let \( f : X \to \mathbb{P}^1 \) be a fibred rational surface of genus two. Assume that \( \rho(X) \neq 14 \). Then there exists a birational morphism \( \nu_0 : X \to \mathbb{P}^2 \) such that \( \nu_0(F) \) is one of the following:

(A) In the case where \( \rho(X) = 13 \) and \( f \) has a \((-1)\)-section, \( \deg \nu_0(F) = 6 \) and singularities of \( \nu_0(F) \) are eight double points.

(B1) In the case where \( \rho(X) = 12 \) and \( f \) has no \((-1)\)-section, \( \deg \nu_0(F) = 7 \) and singularities of \( \nu_0(F) \) are one triple point and ten double points.

(B2) In the case where \( \rho(X) = 12 \) and \( f \) has a \((-1)\)-section, \( \deg \nu_0(F) = 9 \) and singularities of \( \nu_0(F) \) are eight triple points and two double points.

(C) In the case where \( \rho(X) = 11 \) and \( f \) has no \((-1)\)-section, \( \deg \nu_0(F) = 13 \) and singularities of \( \nu_0(F) \) are one quintuple point and nine quadruple points.

Furthermore, \( \deg \nu'_0(F) \geq \deg \nu_0(F) \), for any birational morphism \( \nu'_0 : X \to \mathbb{P}^2 \). If the equality sign holds, then the types of singularities of \( \nu'_0(F) \) are the same as \( \nu_0(F) \)'s.
The last two statements in Theorem 1.1 imply that four pencils in (A), (B1), (B2) and (C) are canonical. Furthermore, we expect the followings to be the frame lattices.

**Theorem 1.2 (cf. [4]).** For fibred surfaces in cases (A), (B1), (B2) and (C) respectively, Mordell-Weil lattices of the maximal rank \(2(\rho(X) - 8)\) are isomorphic to a unimodular integral lattice whose extended Dynkin diagram is given by Figures 1, 2, 3 and 4. Here the numbers in the circles denote the self-pairings of elements except for roots,

![Figure 1](image1.png)

Figure 1.

![Figure 2](image2.png)

Figure 2.

![Figure 3](image3.png)

Figure 3.

and a line between two circles shows that the pairing of the corresponding two elements is equal to \((-1)\).

Furthermore, any other Mordell-Weil lattice of a fibred surface as above is isomorphic to the dual lattice of at most a sublattice of the maximal one.

In particular, the maximal Mordell-Weil lattice is \(E_8\) in the case (B2). On the other hand, that in the case (B1), or as in Figure 2 is an odd lattice. Therefore, the existence of a \((-1)\)-section affects the structure of \(f\) essentially. In the last of this paper, for each case in Theorem 1.1 we describe an example which is extremal in the sense that Mordell-Weil group of \(f\) is trivial. The sum of all dual graphs of reducible fibres of \(f\) in the example, or as in Figures 6, 7, 8 and 9 respectively contains the extended Dynkin diagram as in Figures 1, 2, 3 and 4. These are the same as in the elliptic case, and we are interested in singularities obtained from the reducible fibres by contracting irreducible components.

A sketch of the proof of Theorem 1.1 is as follows: For a fibred rational surface \(f : X \to \mathbb{P}^1\) of genus two with \(\rho(X) \leq 13\), we consider a \#-minimal model of the reduction of \((X, F)\). The method as in [5, §2] leaves five numerical possibilities for the
#-minimal models. However, we can exclude one of them in the course of the study of the branch divisor, associated with the relative canonical map classified by Horikawa [3]. Now, a procedure taking the #-minimal model gives a birational morphism $X \to \mathbb{P}^2$ naturally.

We pay a special attention to $\rho(X)$ and the existence of a $(-1)$-section of $f$. Although a #-minimal model is not unique in general, the four cases in Theorem 1.1 correspond in a one-to-one manner to the four types of #-minimal models. By comparing base-point-free pencils of rational curves on $X$ with a minimal one, we have the last two statements.

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2 Preliminaries

Let $X$ be a smooth projective rational surface defined over $\mathbb{C}$ and $f : X \to \mathbb{P}^1$ a relatively minimal fibration whose general fibre $F$ is a smooth projective curve of genus $g \geq 2$. Then $K_X + F$ is nef and it follows that the self-intersection number of a section of $f$ is negative. Furthermore, from Noether’s formula and the genus formula, the Picard number $\rho(X)$ of $X$ is as follows:

$$\rho(X) = 4g + 6 - (K_X + F)^2 \leq 4g + 6. \quad (2.1)$$

Put $\Sigma_d = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d)) \to \mathbb{P}^1$. Let $\Delta_0$ be a section of $\Sigma_d$ with $\Delta_0^2 = -d$ and $\Gamma$ a fibre. When $\rho(X) = 4g + 6$, that is $(K_X + F)^2 = 0$, we obtain $f : X \to \mathbb{P}^1$ from a suitable subpencil $\Lambda_f$ of $|2\Delta_0 + (d + g + 1)\Gamma|$ with $0 \leq d \leq g + 1$ by blowing $\Sigma_d$ up at the $(4g + 4)$ base points (cf. [13] Theorem 4.1 and [11] Theorem 2.4, see also [6]). In particular, $f$ always has a $(-1)$-section $E$, i.e., a $(-1)$-curve with $F.E = 1$.

Assume that $(K_X + F)^2 > 0$. We briefly review basic notation and results of such fibred surfaces $f : X \to \mathbb{P}^1$ according to [8] and [3].

Suppose that there exists a $(-1)$-curve $E$ with $(K_X + F).E = 0$ and let $\mu_1 : X \to X_1$ be its contraction. Since $F.E = 1$, $F_1 := (\mu_1)_*F$ is smooth on $X_1$. Furthermore, we have $\mu_1^*(K_{X_1} + F_1) = K_X + F$. If there exists a $(-1)$-curve $E_1$ with $(K_{X_1} + F_1).E_1 = 0$, then, by contracting it, we get the pair $(X_2, F_2)$ with $F_2$ smooth and $K_{X_2} + F_2$ pulls back to
$K_X + F$. We can continue the procedure until we arrive at a pair $(X_n, F_n)$ such that we cannot find a $(-1)$-curve $E_n$ with $(K_{X_n} + F_n).E_n = 0$. We put $Y := X_n$ and $G := F_n$. If $\mu : X \to Y$ denotes the natural map, then $\mu^*(K_Y + G) = K_X + F$ and $G = \mu_*F$ is a smooth curve isomorphic to $F$. The original fibration $f : X \to \mathbb{P}^1$ corresponds to a pencil $\Lambda_f \subset |G|$ with at most simple (but not necessarily transversal) base points. Remark that

(2.2) \[ G^2 = \#\text{Bs}\Lambda_f \geq 0, \quad K_X^2 = K_Y^2 - G^2. \]

Since $K_X + F$ is nef and big, $Y$ is the minimal resolution of singularities of the surface $\text{Proj}(R(X, K_X + F))$, which has at most rational double points from \[8\, \text{Lemma 1.2}], where $R(X, K_X + F) = \bigoplus_{n \geq 0} H^0(X, n(K_X + F))$. Therefore, such a model is uniquely determined. We call the pair $(Y, G)$ the reduction of $(X, F)$.

If $Y = \mathbb{P}^2$, then $G$ is a smooth plane curve of degree $b$ with $b \geq 4$. In this case, we have $g = (b - 1)(b - 2)/2$ and

(2.3) \[ (K_X + F)^2 = (K_Y + G)^2 = (b - 3)^2 = 4g + 5 - b^2 \leq 4g - 11. \]

In the case of $Y = \Sigma_d$, from \[5\, \text{Lemma 2.5}], we have

(2.4) \[ (K_X + F)^2 = 2c(g - c - 1)/(c + 1) < 2g - 2 \]

with the Clifford index $c$ of $F$, which is a non-negative integer.

Assume that $Y$ is neither $\mathbb{P}^2$ nor $\Sigma_d$. Then we can find at least one base-point-free pencil of rational curves on $Y$. We choose among them a pencil $|\Gamma_Y|$ of rational curves with $\Gamma_Y^2 = 0$ in such a way that $a := (K_Y + G).\Gamma_Y$ is minimal. Note that $a > 0$ from \[8\, \text{Lemma 1.2}]. We have $G.\Gamma_Y = a + 2$ since $K_Y.\Gamma_Y = -2$. Let $\psi : Y \to \mathbb{P}^1$ be the morphism defined by $|\Gamma_Y|$. We take a relatively minimal model of $Y$ with respect to $\psi$ and consider the image of $G$. Then we perform a succession of elementary transformations (2) at singular points of the image curve to arrive at a particular relatively minimal model $(Y^#, G^#)$, called a $\#$-minimal model in \[4\], enjoying several nice properties which we collect below. The natural map $\nu : (Y, G) \to (Y^#, G^#)$ is a minimal succession of blowing-ups which resolves the singular points of $G^#$. We assume that $Y^# \simeq \Sigma_d$ and $G^# \sim (a + 2)\Delta_0 + b\Gamma$, where the symbol $\sim$ means the linear equivalence of divisors. Let $p_i, 1 \leq i \leq N$, be the singular points of $G^#$ including infinitely near ones, and let $m_i$ be the multiplicity of $G^#$ at $p_i$. Assume for simplicity that $m_1 \geq m_2 \geq \cdots \geq m_N \geq 2$. Since $|\Gamma_Y|$ is chosen so that $(K_Y + G).\Gamma_Y$ is minimal, we can assume that the following are satisfied (see \[2\] and \[4\]):

$(\#1)$ $b \geq (a + 2)d$ when $d > 0$, and $b \geq a + 2$ when $d = 0,$
From a standard calculation, we have

$$\text{(2.5)} \quad (K_Y + G)^2 = 2(a + \bar{b}) - \sum_{i=1}^{N} (m_i - 1)^2,$$

$$\text{(2.6)} \quad g = (a + 1)(a + 1 + \bar{b}) - \frac{1}{2} \sum_{i=1}^{N} m_i (m_i - 1),$$

$$\text{(2.7)} \quad G^2 = 2(a + 2)(a + 2 + \bar{b}) - \sum_{i=1}^{N} m_i^2.$$ 

Compare (2.5) with (2.6) and consider $2(m_i - 1)/m_i = (m_i - 1)^2/((1/2)m_i(m_i - 1))$, which is a monotonically increasing function for $m_i$. If we know only $n$ values of $m_1, \ldots, m_n$ among the multiplicities, then, by mimicking Lemmas 2.6 and 2.8, we have a lower bound of $(K_Y + G)^2$ as follows: If $m_{n+1} \leq m \leq m_n$ for an integer $m$, then

$$(K_Y + G)^2 \geq \frac{2(m - 1)}{m} \left( g - (a + 1)(a + 1 + \bar{b}) + \frac{1}{2} \sum_{i=1}^{n} m_i (m_i - 1) \right)$$

$$+ 2a(a + \bar{b}) - \sum_{i=1}^{n} (m_i - 1)^2.$$ 

When the equality sign holds, $m_{n+1} = \cdots = m_N = m$. Furthermore, we have the following:

**Lemma 2.1.** Keep the notation and assumptions as above. Let $n$ be a non-negative integer and $m$ an integer. If $a$ is even and $m_{n+1} \leq m$, then

$$(K_Y + G)^2 \geq \frac{2(m - 1)}{m} \left( g - (a + 1)(a + 1 + \bar{b}) + \frac{na(a + 2)}{8} \right) + 2a(a + \bar{b}) - \frac{na^2}{4}.$$
When the equality sign holds, \( m_n = (a + 2)/2 \) and \( m_N = m \).

By putting \( n = 0 \) and \( m = (a + 2)/2 \) in Lemma 2.1, we have \((K_Y + G)^2 \geq 2a(g - 1 + \tilde{b})/(a + 2)\) if \( a \) is even. In [3, Lemma 2.6], for another case, we had the following: If \((Y^\#, G^\#)\) is of general type and \( a \) is odd, then

\[(2.8) \quad (K_Y + G)^2 \geq 2g(a - 1)/(a + 1) + 2.\]

When the equality sign holds, \( \tilde{b} = 0 \), \( N \geq 1 \) and \( m_N = (a + 1)/2 \).

Next, we consider the case where \((Y^\#, G^\#)\) is of special type. Let \( m_0 = b - (a + 2) \).

Then

\[G^\# \sim (a + 2)\Delta_0 + (a + 2 + m_0)\Gamma, \quad 2 \leq m_0 < (a + 2)/2\]

and

\[(2.9) \quad G^2 = (a + 2)(a + 2 + 2m_0) - \sum_{i=0}^{N} m_i^2 \geq (a + 2)^2 + 2m_0(a + 2) - Nm_0^2.\]

Furthermore, [3, Lemma 2.7] showed the following: If \( g < a(a + 3)/2 \) and \( a \) is even, then

\[(2.10) \quad (K_Y + G)^2 \geq 2g(a - 2)/a + 4.\]

When the equality sign holds, \( N > 4 + 4/(a - 2) \) and \( m_N = a/2 \). If \( g < (a + 1)(a + 2)/2 \) and \( a \) is odd, then

\[(2.11) \quad (K_Y + G)^2 \geq 2g(a - 1)/(a + 1) + 1.\]

When the equality sign holds, \( N \geq 5 \) and \( m_N = (a + 1)/2 \).

In the last of this section, we give an upper bound of \((K_X + F)^2\) as follows:

**Proposition 2.2.** Let \( X \) be a smooth rational surface and \( f : X \to \mathbb{P}^1 \) a relatively minimal fibration of genus \( g \geq 2 \). Then \((K_X + F)^2 \leq 4g - 5\) holds.

**Proof.** We shall assume that \((K_X + F)^2 > 0\). Let \((Y, G)\) be the reduction of \((X, F)\).

From (2.3) and (2.4), we have that \( Y \) is neither \( \mathbb{P}^2 \) nor \( \Sigma_d \). Furthermore, we take a \#-minimal model \((Y^\#, G^\#)\) is of \((Y, G)\). Remark that \( K_Y^2 = K_{Y^\#}^2 - N = 8 - N \). From (2.2) and the genus formula, we have

\[(2.12) \quad (K_X + F)^2 + G^2 + N = 4g + 4.\]

Hence, we only have to show that \( G^2 + N \geq 9 \). Assume that \( N \leq 8 \). We suppose that \((Y^\#, G^\#)\) is of general type. Then we have \( G^2 \geq (8 - N)(a + 2)^2/4 \geq 8 - N \) from (#2) and (2.7). If \( G^2 = 8 - N \) holds, then \( a \) is even and \((\tilde{b}, m_N, N) = (0, (a + 2)/2, 8)\). However, it implies \( g = 1 \), which is a contradiction. If \((Y^\#, G^\#)\) is of special type, then we have \( G^2 \geq 3a + 4 \geq 13 \) from the assumption \( N \leq 8 \), (#2) and (2.9). \( \Box \)
3 Canonical pencils of genus two curves

From now on, we concentrate on relatively minimal fibrations $f : X \to \mathbb{P}^1$ of curves of genus two on smooth projective rational surfaces with $(K_X + F)^2 > 0$. We recall (2.1). We restate Theorem 1.1 as follows:

**Theorem 3.1.** Let $f : X \to \mathbb{P}^1$ be a fibred rational surface of genus two. Assume that $(K_X + F)^2 > 0$. Then there exists a birational morphism $\nu_0 : X \to \mathbb{P}^2$ such that the linear equivalence class of $F$ is one of the following:

(A) $6\ell - 2 \sum_{i=1}^{8} e_i - \sum_{i=9}^{12} e_i$ with $(K_X + F)^2 = 1$,

(B1) $7\ell - 3e_1 - 2 \sum_{i=2}^{11} e_i$ with $(K_X + F)^2 = 2$,

(B2) $9\ell - 3 \sum_{i=1}^{8} e_i - 2e_9 - 2e_{10} - e_{11}$ with $(K_X + F)^2 = 2$,

(C) $13\ell - 5e_1 - 4 \sum_{i=2}^{10} e_i$ with $(K_X + F)^2 = 3$.

Here $\ell$ is the pull-back to $X$ of a line on $\mathbb{P}^2$ and $e_i$ is that of a $(-1)$-curve contracted by $\nu_0$.

In particular, $f$ has a $(-1)$-section if and only if $\nu_0$ gives (A) or (B2). Furthermore, deg $\nu_0'(F) \geq$ deg $\nu_0(F)$, for any birational morphism $\nu_0' : X \to \mathbb{P}^2$. If the equality sign holds, then the types of singularities of $\nu_0'(F)$ are the same as $\nu_0(F)$’s.

3.1 Proof of Theorem 3.1

Let $X$ be a smooth rational surface and $f : X \to \mathbb{P}^1$ a relatively minimal fibration of genus two. Assume that $(K_X + F)^2 > 0$. Let $(Y, G)$ be the reduction of $(X, F)$. Clearly, $Y$ is not $\mathbb{P}^2$. We also have $Y \neq \Sigma_d$ from [10] or (2.4). We firstly determine numerical possibilities of $\#$-minimal models of $(Y, G)$. Remark that $(K_Y + G)^2 = (K_X + F)^2 \leq 3$ from Proposition 2.2. Keep the same notation as in §2. Then we have the following:

**Lemma 3.2.** Let $f : X \to \mathbb{P}^1$ be a relatively minimal fibration of genus two on a smooth rational surface with $(K_X + F)^2 > 0$ and $(Y, G)$ the reduction of $(X, F)$. Then a $\#$-minimal model $(Y^#, G^#)$ of $(Y, G)$ is of general type. Furthermore, $(Y^#, G^#)$ satisfies
In particular,

\[(3.17)\]  

\[(3.18)\]  

Hence we have \[(3.19)\] and \[(3.20)\]. From now on, we concentrate on the case where \[(3.21)\] holds, Lemma 2.1 and \[(2.6)\] imply that \[(3.22)\] or \[(3.23)\] according as \[(3.24)\] holds. Furthermore, we have \[(3.25)\] or \[(3.26)\] according as \[(3.27)\] holds. Therefore, \[(3.28)\] or \[(3.29)\] according as \[(3.30)\] holds.

Proof. Suppose that \((Y^#, G^#)\) is of special type. From \[(2.10)\] or \[(2.11)\], we have \((K_Y + G)^2 \geq 4\) or \(3 + 2(a - 3)/(a + 1)\) according as \(a\) is even or not. Furthermore, if \((K_Y + G)^2 = 3\) holds, then we have \(a = 3\) and \(m_0 = \cdots = m_N = 2\). However, it contradicts \[(2.9)\] and \[(2.12)\]. Hence, \((Y^#, G^#)\) is of general type.

When \(a\) is odd, from \[(2.8)\], we have \((K_Y + G)^2 \geq 4\), which is a contradiction. Put \(a = 2\). Then \(m_1 = \cdots = m_N = 2\) and \(N = 3\tilde{b} + 7\) from \[(#2)\] and \[(2.6)\]. The assumption that \((K_X + F)^2 \geq 1\) holds, \[(2.2)\] and \[(2.12)\] imply \(N \leq 11\). Hence we have \[(3.13)\] and \[(3.14)\]. From now on, we concentrate on the case where \(a \geq 4\) and \(a\) is even. By putting \(n = 0\) and \(m = (a + 2)/2\) in Lemma \[2.1\] we have

\[(3.18)\]  

\[(K_Y + G)^2 \geq 3 + 2a(\tilde{b} - 1)/(a + 2) + (a - 6)/(a + 2).\]

Hence we have \(a = 4\) or \(6\) when \(\tilde{b} \geq 1\). Since the equality \((K_Y + G)^2 = 3\) in \[(3.18)\] with \(\tilde{b} = 1\) holds, Lemma \[2.1\] and \[(2.6)\] imply that \((Y^#, G^#)\) satisfy \[(3.16)\] when \(a = 6\) and \(\tilde{b} \geq 1\). If \(a = 4\) and \(\tilde{b} \geq 1\), then we also have \(\tilde{b} = 1\) and \((K_Y + G)^2 = 3\), though the equality in \[(3.18)\] does not hold. Furthermore, we have \(N \leq 9\) from \[(2.2)\] and \[(2.12)\]. Then \((#2)\) and \[(2.6)\] imply \(g \geq 30 - 3N \geq 3\), which is absurd.

We assume that \(\tilde{b} = 0\). It follows from \((#2)\) and \[(2.6)\] that \(m_8 \leq a/2\) and \(N \geq 9\). By putting \(n = 7\) and \(m = (a - 2)/2\) in Lemma \[2.1\] we have \((K_Y + G)^2 \geq 4\). Hence \(m_8 = a/2\). If \(m_7 \leq a/2\) holds, then we have \((K_Y + G)^2 \geq 3 + (a - 4)/a\) by putting \(n = 6\) and \(m = a/2\) in Lemma \[2.1\]. Here \((K_Y + G)^2 = 3\) implies that \((a, m_6, m_7) = (4, 3, 2)\) and \(N \leq 9\). In this case, \[(2.6)\] implies \(g = 13 - N \geq 4\), which is absurd. Thus \(m_7 = (a + 2)/2\). Then we have \(G^2 = 3 - \sum_{i=9}^{N} m_i\) from \[(2.6)\] and \[(2.7)\]. Therefore \(N = 9\) and \(m_9 \leq 3\) hold.

From \[(2.6)\], we have \[(3.15)\] or \[(3.17)\] according as \(m_9 = 2\) or \(3\).

We show the last statement in Lemma \[3.2\]. For example, we consider \[(3.16)\]. Then \(G^# \sim 8\Delta_0 + (9 + 4d)\Gamma\) holds and we have \(d \leq 2\) since \(\Delta_0.G^# = 9 - 4d \geq 0\). We concentrate
on the case of $d = 2$. Then, $G^\#$ has no singular points on the minimal section $\Delta_0$, since the intersection number of the strict transforms of $G^\#$ and $\Delta_0$ is also non-negative. We consider the fibre $\Gamma_1$ of $Y^\#$ passing through a singular point $p_1$ of $G^\#$. Perform the elementary transformation $\tau: (Y^\#, G^\#) \rightarrow (Y^\#', G^\#')$ at $p_1$, then the composite of $\nu: (Y, G) \rightarrow (Y^\#, G^\#)$ and $\tau$ is also a birational morphism $\nu': (Y, G) \rightarrow (Y^\#', G^\#')$ contracting the strict transform of $\Gamma_1$ instead of the exceptional curve. Here $Y^\#' = \Sigma_1$ and $(Y^\#', G^\#')$ is a \#-minimal model satisfying (3.16). Reversibly, perform the elementary transformation $\tau^{-1}: (Y^\#', G^\#') \rightarrow (Y^\#, G^\#)$ at the point corresponding to the strict transform of $\Gamma_1$, then the original one is obtained from the \#-minimal model $(Y^\#', G^\#')$. The case of $d = 0$ is similar and simpler.

Next, we observe genus two fibrations in the view of double coverings according to [3]. Remark that the fixed part $Z$ of $|K_X + F|$ is vertical with respect to $f: X \rightarrow \mathbb{P}^1$ and $H^0(X, K_X + F - Z) \simeq H^0(F, K_F)$ by the restriction map (cf. [3, Lemma 1.1]). Consider the rational map $f \times \Phi_{|K_X + F - Z|}: X \rightarrow W^2$ of degree two, where $W^2 \simeq \Sigma_0$. Stein factorization of the morphism obtained by blowing up at the indeterminacy uniquely determines a finite double cover $\pi^\#: X^\# \rightarrow W^2$. Then, by [3, Theorem 1], the branch locus $B^\#$ has only those singularities of types $(0)$, $(I_k)$, $(II_k)$, $(III_k)$, $(IV_k)$ and $(V)$ which are listed in [3, Lemma 6]. Furthermore,

$$(3.19) \quad B^\# \sim 6\Delta_0 + (\epsilon + (K_X + F)^2 + 2)\Gamma, \quad \epsilon = \sum_k \{n(I_k) + n(III_k)\} + n(V)$$

and

$$(3.20) \quad (K_X + F)^2 = \sum_k \{2k - 1)(n(I_k) + n(III_k)) + 2k(n(II_k) + n(IV_k))\} + n(V),$$

where $n(*)$ denotes the number of singularities of type $(*)$, from [3, Theorem 3]. Let $\bar{\pi}: \bar{X} \rightarrow \bar{W}$ be the finite double cover obtained from the canonical resolution (as in [3, §3]) of $\pi^\#: X^\# \rightarrow W^2$. Remark that $\sigma: \bar{W} \rightarrow W^2$ factors through the composite $\sigma^\#: W^\# \rightarrow W^2$ of the blow-ups at exactly $2(K_X + F)^2$ points such that the finite double covering $X^\#$ of $W^\#$ branched along $B^\#$ has at most rational double points as its singularities. Let $E_i$, $1 \leq i \leq 2(K_X + F)^2$ be the pull-back to $W^\#$ of a $(-1)$-curve contracted by $\sigma^\#$. We denote by $\sigma^\#$ the composite of blow-ups such that $\sigma = \sigma^\# \circ \pi^\#. Then \Phi_{|\Gamma|} \circ \sigma \circ \bar{\pi}: \bar{X} \rightarrow \mathbb{P}^1$ is a genus two fibration which contains exactly $(K_X + F)^2$ disjoint $(-1)$-curves in some fibres. In fact, the relatively minimal model is the original fibration $f: X \rightarrow \mathbb{P}^1$. We denote by $\varphi: \tilde{X} \rightarrow X$ the composite of the blow-downs which contract the $(K_X + F)^2$ disjoint $(-1)$-curves. These rational maps gives a commutative diagram in Figure 5. The
pull-back by $\tilde{\pi}$ of the strict transform by $\sigma$ of $\epsilon$ fibres which form $B^2$'s singularities of types $(I_k)$, $(III_k)$ and (V) are $\epsilon$ double $(-1)$-curves on $\tilde{X}$. Let $\mathcal{E}$ be the sum of the $\epsilon$ $(-1)$-curves. In fact, $\varphi(\mathcal{E})$ forms simple base points of $|K_X + F - Z|$. From [3, Lemmas 10 and 13], we have

$$(\sigma^b \circ \tilde{\pi})^*(-K_W^b) \sim \varphi^*(2K_X - (K_X^2)F),$$

and $$(\sigma \circ \tilde{\pi})^* \Delta_0 \sim \varphi^*(K_X + F - Z) - \mathcal{E}$$

and $(\sigma \circ \tilde{\pi})^* \Gamma \sim \varphi^*F$. In fact, from Castelnuovo’s rationality criterion, the branch divisor satisfies

$$(3.21) \quad h^0(W^b, -K_W^b + K_X^2(\sigma^*)^\# \Gamma) = 0.$$  

Conversely, for a desired value of $(K_X + F)^2$ or $K_X^2$, we obtain a relatively minimal fibration $f : X \to \mathbb{P}^1$ of genus two curves from a reducible divisor $B^2$ on $W^2 := \mathbb{P}^1 \times \mathbb{P}^1$ with (3.19) and (3.20) by taking the finite double covering $\pi^\#: X^\# \to W^\#$ whose branch divisor is $B^\#$. Then (3.21) implies that $X$ is rational from Castelnuovo’s rationality criterion.

**Lemma 3.3.** A $\#$-minimal model satisfying (3.17) does not occur.

**Proof.** Suppose that there exists a relatively minimal fibration $f : X \to \mathbb{P}^1$ of genus two on a smooth rational surface such that a $\#$-minimal model $(Y^\#, G^\#)$ of $(X, F)$ satisfies (3.17). Remark that singular points $p_1, \ldots, p_8$ of $G^\#$ are not any infinitely near point of the triple point $p_9$. Let $e_{10}$ be the $(-1)$-curve corresponding by $\nu$ to $p_9$. In particular, the strict transform $\tilde{e}_{10}$ to $\tilde{X}$ of $e_{10}$ is also smooth and irreducible. By applying the projection formula, we have

$$(3.22) \quad (\sigma^b \circ \tilde{\pi})_* \tilde{e}_{10}.(-K_W^b) = \tilde{e}_{10}.\varphi^*(2K_X + F) = e_{10}.(2K_X + F) = 1.$$
We remark that \( e_{10} \) is not a component of any fibre of \( f \). Therefore, we have
\[(\sigma \circ \pi_1)_{\ast}e_{10} \Delta_0 = e_{10} \ast (K_X + F) - e_{10} \ast Z - e_{10} \ast E \leq e_{10} \ast (K_X + F) = 2\]
by applying the projection formula. Furthermore, \((\sigma \circ \pi_1)_{\ast}e_{10} \Gamma = 3\) holds. This implies that \( \sigma \circ \pi_1 \) is birational on \( e_{10} \). Therefore, \((\sigma \circ \pi_1)_{\ast}e_{10} \) is also irreducible and reduced. We also have \((\sigma \circ \pi_1)_{\ast}e_{10} \sim 3\Delta_0 + \Gamma \) or \( 3\Delta_0 + 2\Gamma \). In the former case,
\[(\sigma^b \circ \pi_1)_{\ast}e_{10} \sim 3(\sigma^b)^{\ast}\Delta_0 + (\sigma^b)^{\ast}\Gamma - \sum_{i=1}^{6} m_i E_i\]
with \( m_i = 0 \) or 1. Then \((\sigma^b \circ \pi_1)_{\ast}e_{10} \ast (K_{W^b}) = 8 - \sum_{i=1}^{6} m_i \geq 2\), which contradicts \( (3.22) \). Next, consider the latter case, and put
\[(\sigma^b \circ \pi_1)_{\ast}e_{10} \sim 3(\sigma^b)^{\ast}\Delta_0 + 2(\sigma^b)^{\ast}\Gamma - \sum_{i=1}^{6} m_i E_i\]
with \( 0 \leq m_i \leq 2 \). Since the arithmetic genus of \((\sigma \circ \pi_1)_{\ast}e_{10}\) is two, \#\{\( m_i | m_i = 2 \)\} = 2 holds. This implies \((\sigma^b \circ \pi_1)_{\ast}e_{10} \ast (K_{W^b}) = 10 - \sum_{i=1}^{6} m_i \geq 2\), which is also a contradiction to \( (3.22) \).

For a relatively minimal fibration \( f : X \to \mathbb{P}^1 \) of curves of genus two on a smooth rational surface with \( (K_X + F)^2 > 0 \), we can take a \#-minimal model \((Y^\#, G^\#)\) of the reduction \((Y, G)\) in order that \( Y^\# = \Sigma_1 \) from Lemma 3.2. Let \( \nu_0 : Y \to \mathbb{P}^2 \) be the composite of \( \nu : Y \to Y^\# \) and the blow-down \( Y^\# \to \mathbb{P}^2 \) contracting the minimal section \( \Delta_0 \). We denote by \( \nu_0 : X \to \mathbb{P}^2 \) the composite of the natural morphism \( \mu : X \to Y \) and \( \nu_0 : Y \to \mathbb{P}^2 \). When \((Y^\#, G^\#)\) satisfies \( (3.13), (3.14), (3.15) \) and \( (3.16) \) respectively, \( \nu_0 \) is a birational morphism giving (A), (B1), (B2) and (C) in Theorem 3.1 Conversely, a birational morphism \( \nu_0 : X \to \mathbb{P}^2 \) as in Theorem 3.1 gives a \#-minimal model of the reduction of \((X, F)\) by blowing \((\nu_0(X), \nu_0(F))\) up at a singular point of \( \nu_0(F) \) with a maximal multiplicity.

**Lemma 3.4.** When \( \nu_0 : X \to \mathbb{P}^2 \) gives (B1) and (C) in Theorem 3.1 respectively, \( F.C \geq 2 \) and 4 for all \((-1)\)-curves \( C \) on \( X \). In particular, \( f : X \to \mathbb{P}^1 \) has a \((-1)\)-section if and only if \( \nu_0 \) gives (A) or (B2) in Theorem 3.1 Furthermore, for a given \( f \), all of \#-minimal models of the reduction satisfy one of the four \( (3.13), (3.14), (3.15) \) and \( (3.16) \).

**Proof.** Assume that \( \nu_0 : X \to \mathbb{P}^2 \) gives (C) in Theorem 3.1 Since \( \nu_0(e_1) \) is not any infinitely near point of another singular point of \( \nu_0(F) \), we have a base-point-free pencil.
\(|\ell - e_1|\), where \(\ell\) and \(e_1\) denote the same in Theorem 3.1. It implies that \((\ell - e_1).C \geq 0\) for all \((-1)\)-curves \(C\) on \(X\). Hence, we have \(F.C = ((\ell - e_1) - 4K_X).C \geq -4K_X.C = 4\). Therefore, \(f\) indeed have no \((-1)\)-section. The case of (B1) in Theorem 3.1 is the quite same argument. When \(v_0\) gives (B2) in Theorem 3.1, \(e_{11}\) must be irreducible, since \(F.(e_{11} - e_i) < 0\) for all \(i < 11\). Thus \(e_{11}\) is a \((-1)\)-section of \(f\). In the same way, \(f\) has at least one \((-1)\)-section when \(v_0\) gives (A) in Theorem 3.1.

Consider \(\rho(X)\). From Lemmas 3.2 and 3.3, all of \#-minimal models of the reduction of \((X,F)\) with \((K_X + F)^2 = 1\) and 3 satisfy (3.13) and (3.16), respectively. Furthermore, in the case of \((K_X + F)^2 = 2\), a \#-minimal model satisfies (3.15) or (3.14) according as \(f\) has a \((-1)\)-section or not.

Remark that a \#-minimal model of the reduction of \((X,F)\) is not unique in general for a relatively minimal fibration \(f : X \to \mathbb{P}^1\) of genus two on a rational surface with \((K_X + F)^2 > 0\). Hence a birational morphism \(v_0 : X \to \mathbb{P}^2\) as before Lemma 3.4 is also not unique. However, from Lemma 3.1 the linear equivalence class of \(F\) is one of the four in Theorem 3.1 by any birational morphism \(v_0 : X \to \mathbb{P}^2\) which gives a \#-minimal model of the reduction of \((X,F)\).

Let \(\Gamma_X\) be the pull-back of \(\Gamma_Y\) to \(X\). In fact, \(|\Gamma_X|\) is that of \(|\Gamma|\) on \(Y^\#\). Indeed, the base-point-free pencil \(|\Gamma_X|\) of rational curves has a minimality for \(F\) as follows:

**Lemma 3.5.** Keep the notation as above. Let \(R\) be a general member of a base-point-free pencil of rational curves on \(X\). Then \(F.R \geq F.\Gamma_X\). Furthermore, \(F.R = F.\Gamma_X\) if and only if \(R \sim \Gamma_X\) when \(v_0\) gives (B1) or (C) in Theorem 3.1.

**Proof.** Let \(v_0 : X \to \mathbb{P}^2\) be a birational morphism giving (C) in Theorem 3.1. From the assumption \(R^2 = 0\), we have \(F.R = (-4K_X + \Gamma_X).R = 8 + \Gamma_X.R \geq 8 = F.\Gamma_X\). Here \(F.R = F.\Gamma_X\) if and only if \(\Gamma_X.R = 0\), which implies that a fibre of the ruling defined by \(|\Gamma_X|\) contains \(R\). Hence we have \(R \sim \Gamma_X\). We see the case of (B1) in Theorem 3.1 by the quite same argument. The other cases are also similar.

We finally show the last two statements in Theorem 3.1 as follows: Let \(v_0 : X \to \mathbb{P}^2\) be a birational morphism as in Theorem 3.1. We restrict ourselves to the case of (C), since the other cases are similar and simpler. At first, we suppose that there exists a birational morphism \(v'_0 : X \to \mathbb{P}^2\) causing \(\deg v'_0(F) \leq 12\). Remark that a linear projection whose centre is a singular point of \(v'_0(F)\) induces a ruling on \(X\). Therefore, multiplicities of singular points of \(v'_0(F)\) are at most \((\deg v'_0(F) - 8)\) from Lemma 3.3. We have a \#-minimal model which is different from (3.16) by blowing \((v'_0(X), v'_0(F))\) up at a singular point of \(v'_0(F)\). It contradicts Lemma 3.4.
Next, we assume that there exists another birational morphism \( \psi' : X \to \mathbb{P}^2 \) making \( \deg \psi'(F) = 13 \). In the same way as above, multiplicities of singular points of \( \psi'(F) \) are at most five. On the other hand, from Lemma 3.4, they are at least four. Furthermore, \( \rho(X) = 11 \) implies that the number of singular points of \( \psi'(F) \) is also exactly ten. Since the genus of \( F \) is two, singularities of \( \psi'(F) \) must be the same as \( \psi_0(F) \)'s by considering (2.6).

This completes the proof of Theorem 3.1. \( \square \)

Remark 3.6. Let \( f : X \to \mathbb{P}^1 \) be a fibred rational surface of genus two and \( \tilde{X} \) as before Lemma 3.3, which is uniquely determined by the relative canonical map. Assume that \( \rho(X) \neq 14 \) or Mordell-Weil group of \( f \) is non-trivial. Then, from Theorem 3.1 and [6], \( \tilde{X} \) can be obtained by blowing \( \mathbb{P}^2 \) up at thirteen points.

3.2 #-minimal models

Let \( f : X \to \mathbb{P}^1 \) be as in Theorem 3.1 and \((Y, G)\) the reduction of \((X, F)\). By contracting a \((-1)\)-curve whose intersection number with \( G \) is the smallest among all of \((-1)\)-curves on \( Y \), we have a pair \((Y_1, G_1)\) as the image of \((Y, G)\). By performing the same procedure for \((Y_1, G_1)\), we get a pair \((Y_2, G_2)\). In this way, \( Y_{(\rho(Y) - 2)} \) becomes a relatively minimal model \( \Sigma_d \) of a ruling on \( X \) such that the image of \( F \) is simple in the view of its singular points, though there exist many rulings on \( X \) and relatively minimal models in general.

In fact, such a model is essentially a #-minimal model of \((Y, G)\) as follows: Let \( \nu : Y \to \Sigma_d \) be the birational morphism giving the above model. When \( d = 0 \) and \( \Gamma, \nu_*G > \Delta_0, \nu_*G \), for simplicity, change the considered ruling for the other one. This is non-essential since contracted \((-1)\)-curves are kept. First of all, we remark that \( \nu_*G \) has singularities. Let \( p_i \) be the singular points of \( \nu_*G \) including infinitely near ones, and let \( m_i \) be the multiplicity of \( \nu_*G \) at \( p_i \). We may assume that \( m_i \geq m_{i+1} \) for simplicity. Put \( a = \nu^* \Gamma(K_Y + G) = \Gamma(K_{\Sigma_d} + \nu_*G) \) and \( b = (a + 2)d + \Delta_0, \nu_*G \). Then ( \#1 \) is satisfied. We suppose that \( (\Gamma, \nu_*G)/2 < m_1 \), which induces a contradiction as follows: We consider the fibre \( \Gamma_1 \) of \( \Sigma_d \) passing through a singular point \( p_1 \) of \( \nu_*G \). Furthermore, we perform the elementary transformation \( \tau : \Sigma_d \dashrightarrow \Sigma_{d+1} \) or \( \Sigma_{d-1} \) at \( p_1 \) according as \( p_1 \) lies on \( \Delta_0 \) or not. Then the composite \( \tau \circ \nu : Y \to \Sigma_{d+1} \) or \( \Sigma_{d-1} \) is also a birational morphism contracting the strict transform \( \widehat{\Gamma_1} \) of \( \Gamma_1 \) instead of the exceptional curve. The intersection number of \( \widehat{\Gamma_1} \) and the image of \( G \) is \((\nu_*G, \Gamma - m_1)\), which contradicts the procedure of \( \nu \). Similarly, we have \( m_1 \leq b - (a + 2) \) when \( d = 1 \). Hence, \((\Sigma_d, \nu_*G)\) must be a #-minimal model of \((Y, G)\). From Lemmas 3.2, 3.3 and 3.4, it satisfies one of the four (3.13), (3.14), (3.15).
and (3.16). In fact, the converse holds as follows:

**Proposition 3.7.** Let \( f : X \to \mathbb{P}^1 \) denote a fibred rational surface of genus two. Assume that \((K_X + F)^2 > 0\). Let \((Y, G)\) be the reduction of \((X, F)\) and \(\nu : Y \to \Sigma_d\) a birational morphism. When \(d = 0\) and \(\Gamma.\nu_*G > \Delta_0.\nu_*G\), change the considered ruling for the other one. Then, \(\nu\) is associated with the procedure as at the start of §3.2 from Lemma 3.4. When \(\nu_0\) gives (A) or (B2), the definition of the reduction implies that \(C.G \geq 2\) for any \((-1)\)-curve \(C\) on \(Y\). Hence, \(\nu_0\) giving (A) is associated with the procedure. When \(\nu_0\) gives (B2), two \((-1)\)-curves \(e_{10}\) and \(e_9\) as in Theorem 3.1 corresponds to double points of \(\nu_0(F)\). Let \((Z, H)\) denote the image of \((Y, G)\) by contracting only the two. From the genus formula, \(D.H \geq 3\) for any \((-1)\)-curve \(D\) on \(Z\) since \(H \sim -3K_Z\). Thus it also does so.

Furthermore, by comparing all of \#-minimal models of the reductions, we develop Lemma 3.2 as follows:

**Theorem 3.8.** Let \( f : X \to \mathbb{P}^1 \) denote a fibred rational surface of genus two. Assume that \(\rho(X) \leq 13\). Let \((Y^#, G^#)\) be a \#-minimal model of the reduction \((Y, G)\) of \((X, F)\) and \(d\) the degree of \(Y^#\) as a relatively minimal model \(\Sigma_d\). Then \(d \leq 2\), and the following holds.

1. In the cases (B1) and (C) in Theorem 1.1, any \( f : X \to \mathbb{P}^1 \) admits \(d = 0\). Otherwise, there exists \( f : X \to \mathbb{P}^1 \) not admitting \(d = 0\).
2. All of \( f : X \to \mathbb{P}^1 \)'s admit \(d = 1\).
3. For each case in Theorem 1.1, there exists \( f : X \to \mathbb{P}^1 \) not admitting \(d = 2\).

In order to show the cases (B1) and (C) for \(d = 2\), we consider a \((-2)\)-section of \(f\), i.e., a \((-2)\)-curve whose intersection number with \(F\) is equal to one. As a sufficient condition of the existence of it, we have the following:
Lemma 3.9. Let \( f : X \to \mathbb{P}^1 \) be a fibred rational surface which has a birational morphism \( \nu_0 : X \to \mathbb{P}^2 \) giving (B1) or (C) in Theorem 3.1. If \((X, F)\) admits a \#-minimal model \((\Sigma_2, \nu_*F)\), then the pull-back to \( X \) of the minimal section \( \Delta_0 \) of \( \Sigma_2 \) is a \((-2)\)-section of \( f \).

Proof. When \( \nu_0 \) gives (B1) and (C) respectively, \((\Sigma_2, \nu_*F)\) satisfies (3.14) and (3.16) from Lemmas 3.2, 3.3 and 3.4. Since \( \Delta_0 \).\( \nu_*F = 1 \), any singular point of \( \nu_*F \) is not on \( \Delta_0 \). Therefore, the pull-back to \( X \) of \( \Delta_0 \) is a \((-2)\)-section of \( f \).

A necessary condition of the existence of a \((-2)\)-section of \( f \) is as follows:

Lemma 3.10. Let \( f : X \to \mathbb{P}^1 \) be a fibred rational surface of genus two and \( D \) a \((-2)\)-section of \( f \). Then the image of \( D \) by \( f \times \Phi_{|K_X + F - Z|} : X \to W^3 \) as before Lemma 3.3 is a section of \( W^3 \) which is linearly equivalent to \( \Delta_0 + \Gamma \) or \( \Delta_0 \) according as \( D \) meets the fixed locus of \( |K_X + F| \) or not. Furthermore, for each case, among the \( 2(K_X + F)^2 \) points which correspond to \((-1)\)-curves contracted by \( \sigma^2 : W^3 \to W^3 \), the number of points lying on the image of \( D \) is the following:

- When \( (K_X + F)^2 = 1 \), it passes through one of the two points in the case where it is linearly equivalent to \( \Delta_0 + \Gamma \), and the latter case does not occur.
- When \( (K_X + F)^2 = 2 \), it passes through two of the four points in the former case, and it does not pass through any point of them in the latter case.
- When \( (K_X + F)^2 = 3 \), it is three and one respectively, in the former and latter case.

Proof. Let \( D \) be a \((-2)\)-section of \( f \). The genus formula implies \( K_X.D = 0 \). We use the notation as before Lemma 3.3. Remark that the strict transform \( \hat{D} \) to \( \tilde{X} \) of \( D \) is also smooth and irreducible. In the way similar to the proof of Lemma 3.3, we have that

\[
(\sigma \circ \pi)_*\hat{D}.\Delta_0 = D.(K_X + F) - D.Z - \hat{D}.\mathcal{E} \leq D.(K_X + F) = 1
\]

and \( (\sigma \circ \pi)_*\hat{D}.\Gamma = D.F = 1 \). Thus \( (\sigma \circ \pi)_*\hat{D} \sim \Delta_0 \) or \( \Delta_0 + \Gamma \). Furthermore,

\[
(\sigma^b \circ \pi)_*\hat{D}.(-K_{W^3}) = D.(2K_X - (K_X^2)F) = -K_X^2
\]

implies the statements for the \( 2(K_X + F)^2 \) points which correspond to the contracted \((-1)\)-curves.

Similarly, we have the following:

Lemma 3.11. Let \( f : X \to \mathbb{P}^1 \) be a fibred rational surface of genus two. Assume that \( (K_X + F)^2 = 2 \). Let \( f \times \Phi_{|K_X + F - Z|} : X \to W^3 \) denote the double cover branched along
Then, there exists a birational morphism \( v_0 : X \to \mathbb{P}^2 \) such that \( v_0(F) \) is as in (B2) in Theorem 1.1 if and only if \( B^3 \) include exactly one minimal section of \( W^3 \), which induces the unique \((-1)\)-section of \( f \).

**Proof.** Let \( f : X \to \mathbb{P}^1 \) be a fibred rational surface of genus two. Assume that \((K_X + F)^2 = 2\). Then the branch divisor \( B^3 \) includes at most one minimal section of \( W^3 \), which induces the unique \((-1)\)-section of \( f \).

Firstly, we assume that there exists \( v_0 : X \to \mathbb{P}^2 \) giving (B2). Remark that the strict transform \( \hat{e}_{11} \) to \( \widetilde{X} \) of the unique \((-1)\)-section \( e_{11} \) of \( f \) is smooth and irreducible. By the quite same argument in Lemma 3.10, \((\sigma \circ \pi)_* \hat{e}_{11}\) is a minimal section \( \Delta_\infty \) of \( W^3 \), and \( \Delta_\infty \) passes through two of the four points which correspond to \((-1)\)-curves contracted by \( \sigma^3 \). Consider singularities of \( B^3 \) on \( \Delta_\infty \). Then \( B^3 \) must include \( \Delta_\infty \) from (3.19) and (3.20).

Next, we return the first situation and assume that \( B^3 \) includes a minimal section \( \Delta_\infty \) of \( W^3 \). We only have to show that the strict transform of \( \Delta_\infty \) by \( \sigma^3 : W^3 \to W^3 \) is an isolated component whose self-intersection number is \((-2)\) of \( B^3 \). Thus a standard calculation for a double cover leads the desired statement.

A genus two fibration on a rational surface is obtained from a finite double cover of \( W^3 = \mathbb{P}^1 \times \mathbb{P}^1 \). In particular, a branch divisor \( B^3 \) on \( W^3 \) with (3.19), (3.20) and (3.21) uniquely determines a fibred rational surface \( f : X \to \mathbb{P}^1 \) of genus two in the way as before Lemma 3.3. We also use the notation as before Lemma 3.3. Let \( \Gamma_p \) be the fibre of \( p \in \mathbb{P}^1 \) by the first projection \( pr_1 : W^3 \to \mathbb{P}^1 \) and \( \Delta_q \) the fibre of \( q \in \mathbb{P}^1 \) by the second projection \( pr_2 \). We take \((t, x)\) as an affine coordinate on \( W^2 \setminus (\Delta_\infty \cup \Gamma_\infty) \).

**Example 3.12.** Let \( A \) be Zariski closure on \( W^3 \) of the divisor defined by
\[
x^4 - 2x^2t - 2x^3t + x^4t + t^2 + 2xt^2 - x^2t^2 + t^3 = 0,
\]
which is irreducible. The singularities of \( A \) are on \((0, 0)\) and on \((\infty, \infty)\). Take the section \( C \) of \( pr_2 \) defined by \( x^2 - t - xt = 0 \). The four-fold double point \((0, 0)\) of \( A \) has a contact of order eight with \( C \). Put \( B^3 = A + C \), which is linearly equivalent to \( 6\Delta_0 + 4\Gamma \). In fact, the singularities of \( B^3 \) are of type (IV_1) on \( \Gamma_0 \) and of type \((0)\) on \( \Gamma_\infty \). We take the finite double
cover of $W^2$ branched along $B^2$ and the canonical resolution. Then we also see (3.21) since $B^2$ does not include any minimal section of $pr_1$. Furthermore, from Lemma [3.11] we have a fibred rational surface $f : X \to \mathbb{P}^1$ which have a birational morphism $\nu_0 : X \to \mathbb{P}^2$ giving (B1).

Any section of $pr_1$ which is linearly equivalent to $\Delta_0 + \Gamma$ passes through at most one of the four points which correspond to the $(-1)$-curves contracted by $\sigma^2 : W^b \to W^2$. Since minimal sections of $pr_1$ not passing through any of the four points must meet $C$ at a smooth point of $B^2$, the strict transforms to $\tilde{X}$ are irreducible. In particular, any of them is not a section of $f$. Since $\Delta_\infty$ meets $A$ at the smooth point $(-1, \infty)$ of $B^2$, the strict transform to $\tilde{X}$ is also irreducible. Therefore, $f$ has no $(-2)$-section from Lemma 3.10.

**Example 3.13.** Let $A$ be Zariski closure on $W^2$ of the divisor defined by

$$x^4 + 2x^2t + 4x^3t - 4x^4t + t^2 + 4xt^2 - 4x^2t^2 - 4t^3 = 0,$$

which is irreducible. The singularities of $A$ are on $(0,0)$ and on $(\infty, \infty)$. $A$ is tangent to $\Gamma_1$ at $(1,1)$, and passes through $(1/4, 0)$ and $(1/4, \infty)$. The bisection $C$ of $pr_1$ or of $pr_2$ defined by $x^2 + t + 2xt - 4t^2 = 0$ has a double point at $(\infty, \infty)$. Furthermore, $C$ passes through $(1/4, 0)$ and $(1,1)$. In addition, the four-fold double point $(0, 0)$ of $A$ has a contact of order eight with $C$. Put $B^2 = A + C + \Gamma_\infty$, which is linearly equivalent to $6\Delta_0 + 6\Gamma$.

In fact, the singularities of $B^2$ are of type $(IV_1)$ on $\Gamma_0$, of type (V) on $\Gamma_\infty$, of type (0) on $\Gamma_{1/4}$ and of type (0) on $\Gamma_1$. If a section of $pr_2$ which is linearly equivalent to $2\Delta_0 + \Gamma$ passes through five points $(0, 0)$, $(\infty, \infty)$ and the three infinitely near points of $(0, 0)$ corresponding to double points of $A$, then it is unique and is defined by $x^2 + 2xt + t = 0$. The section of $pr_2$ meets $\Gamma_\infty$ transversally. Therefore, (3.21) also holds when we take the finite double cover of $W^2$ branched along $B^2$ and the canonical resolution. Hence, we obtain a fibred rational surface $f : X \to \mathbb{P}^1$ of genus two with $(K_X + F)^2 = 3$.

Any section of $pr_1$ which is linearly equivalent to $\Delta_0 + \Gamma$ passes through at most two of the six points which correspond to the $(-1)$-curves contracted by $\sigma^2 : W^b \to W^2$. $\Delta_0$ meets $B^2$ at a smooth point $(\infty, 0)$ of $B^2$ transversally. $\Delta_\infty$ also meets $B^2$ at a smooth point $(1/4, \infty)$ transversally. Therefore, the strict transforms to $\tilde{X}$ are irreducible. In particular, they meet $\varphi^*F$ at two points. Thus, $f$ has no $(-2)$-section from Lemma 3.10.

From Lemma 3.9 Examples 3.12 and 3.13 we have the following:

**Lemma 3.14.** In each of the cases (B1) and (C) in Theorem 1.1 there exists a fibred rational surface $f : X \to \mathbb{P}^1$ not admitting $\Sigma_2$ as the surface of a #-minimal model of $(X, F)$. 

18
Lemma 3.15. Let \( f : X \to \mathbb{P}^1 \) be a fibred rational surface which has a birational morphism \( v_0 : X \to \mathbb{P}^2 \) giving (B1) or (C) in Theorem 3.11. Then \( \Sigma_0 \) appears as the surface of a \( \#\)-minimal model of \((X,F)\).

Proof. Let \( f : X \to \mathbb{P}^1 \) be a fibred rational surface of genus two. We may assume that there exists a birational morphism \( v_0 : X \to \mathbb{P}^2 \) giving (B1) in Theorem 3.11 since the other case is similar and simpler. Let \( p_{i-1} \) be the point which corresponds to \( e_i \) as in Theorem 3.11 by contracting it. For simplicity, we assume that \( i < j \) if \( p_j \) is an infinitely near point of \( p_i \). Furthermore, we can take the composite \( v_i : X \to X_i \) of the blow-downs which contracts \( e_{11}, e_{10}, \ldots, e_{i+1} \) with \( v_i(e_{i+1}) = p_i \). If \( v_i(e_i) \) is not on the minimal section \((v_1)_*e_1 \) of \( X_1 \) for some \( i \geq 2 \), then, in a way similar to the proof of the last statement in Lemma 3.2, \( \Sigma_0 \) appears as the surface of a \( \#\)-minimal model of \((X,F)\) by performing the elemental transformation of \( X_1 \) at \( v_i(e_i) \). Assume that \( p_{i+1} \) is an infinitely near point of \( p_i \) for all \( i \). Then \((e_1 - e_2)\) is a \((-2)\)-section of \( f \), and \((e_i - e_{i+1}), i = 2,3, \ldots, 10\) are \((-2)\)-curves contained in a fibre since they are connected.

At first, we suppose that \( p_0, p_1 \) and \( p_2 \) are colinear. Then the reducible fibre also contains the \((-2)\)-curve \((\ell - e_1 - e_2 - e_3)\), since \((\ell - e_1 - e_2 - e_3).e_3 - e_4 = 1\). Furthermore, only the irreducible components \((\ell - e_1 - e_2 - e_3)\) and \((e_i - e_{i+1}), i = 2,3, \ldots, 10\) do not generate \( F \) by considering the class in \( \text{NS}(X) = \mathbb{Z}\ell \oplus \bigoplus_{i=1}^{11} \mathbb{Z}e_i \). In particular, the number of irreducible components of the reducible fibre is at least eleven. Now, for Jacobian surface of the generic fibre of \( f \), we regard \((e_1 - e_2)\) as an origin, and compute the rank of the Mordell-Weil group from the formula [15, (3) in Theorem 3]. In fact, a reducible fibre of \( f \) is unique and the number of the irreducible components is exactly eleven. On the other hand, the unique reducible fibre is of type \((\Pi_1)\) or \((\IV_1)\) in the sense of Horikawa [3] from (3.20). However, any singular fibre of type \((\Pi_1)\) or \((\IV_1)\) whose dual graph contains Dynkin diagram of the root lattice \( D_{10} \) of rank ten has more than eleven irreducible components, which is a contradiction.

Next, we assume that \( p_0, p_1 \) and \( p_2 \) are not colinear. We consider \((e_2 - e_3), (e_3 - e_4)\) and a \((-1)\)-bisection \((\ell - e_1 - e_2)\) of \( f \). Remark that they are disjoint with \((e_5 - e_6), (e_6 - e_7), \ldots, (e_{10} - e_{11})\) and \( e_{11} \). Let \( v'_3 : X \to X'_3 \) be the composite of \( v_4 : X \to X_4 \) and the blow-down \( X_4 \to X'_4 \) contracting \((v_4)_*(\ell - e_1 - e_2)\). Then \((v'_3)_*(e_2 - e_3).F = 2 \) and \((v'_3)_*(e_2 - e_3)\) is a \((-1)\)-curve. We take the composite \( v'_2 : X \to X'_2 \) of \( v'_3 \) and the blow-down \( X'_3 \to X'_2 \) contracting \((v'_3)_*(e_2 - e_3)\). Similarly, we can define a birational morphism \( v'_1 : X \to X'_1 \) as the composite of \( v'_2 \) and the blow-down \( X'_2 \to X'_1 \) contracting \((v'_2)_*(e_3 - e_4)\). We regard \( X'_1 \) as \( \Sigma_0 \) so that \((v'_1)_*(e_1 - e_2)\) is a minimal section of \( X'_1 \) and \((v'_1)_*(e_4 - e_5)\) is a fibre. Then \((X'_1, (v'_1)_*F)\) is a \#-minimal model of \((X,F)\).
The statements in Theorem 3.8 for (B1) and (C) follow from Lemmas 3.14 and 3.15. When \( v_0 : X \to \mathbb{P}^2 \) gives (A) and (B2) respectively, \(|F + K_X|\) and \(|F + 2K_X|\) induces an elliptic fibration with a section as follows:

**Theorem 3.16** (cf. [7]). Let \( f : X \to \mathbb{P}^1 \) be a fibred rational surface which has a birational morphism \( v_0 : X \to \mathbb{P}^2 \) giving (A) as in Theorem 3.1 and \((Y,G)\) the reduction of \((X,F)\). Then \(|−K_Y|\) is a pencil of elliptic curves with one base point, and \(G \sim −2K_Y\). Conversely, any minimal elliptic rational surface with a section induces a fibred rational surface of genus two as the above.

**Theorem 3.17** (cf. [7]). Let \( f : X \to \mathbb{P}^1 \) be a fibred rational surface which has a birational morphism \( v_0 : X \to \mathbb{P}^2 \) giving (B2) as in Theorem 3.1. Then the sum \((e_{10} + e_9)\) as in Theorem 3.1 is unique. In particular, it is independent of a choice of \(v_0\). Furthermore, \(e_{10}\) and \(e_9\) are disjoint from the \((-1)\)-section \(e_{11}\) of \(f\). Let \(S\) be the surface obtained from \(X\) by contracting \(e_{10}\) and \(e_9\). Then \(\Phi_{|−K_S|} : S \to \mathbb{P}^1\) is a minimal elliptic rational surface and the image of \(e_{11}\) on \(S\) is a section of \(\Phi_{|−K_S|}\). Conversely, any minimal elliptic rational surface with a section induces a fibred rational surface of genus two as the above.

Let \( \epsilon : S \to \mathbb{P}^1 \) be any minimal elliptic surface with a section \((O)\). From Theorem 3.17 we can take a subpencil of \(|−3K_S + 2(O)|\) whose general members have just two double points as its singularities, which are exactly the base points. Furthermore, a fibred rational surface \(f : X \to \mathbb{P}^1\) which has a birational morphism \(v_0 : X \to \mathbb{P}^2\) giving (B2) as in Theorem 3.1 is obtained from the subpencil by blowing \(S\) up at the two base points. Then the strict transform of \((O)\) to \(X\) is the \((-1)\)-section \(e_{11}\) of \(f\). Let \((Z,H)\) be the image of \((X,F)\) by contracting \(e_{11}, e_{10}\) and \(e_9\). Then any \#-minimal model of the reduction \((Y,G)\) is obtained from \((Z,H)\) by contracting step by step a \((-1)\)-curve whose intersection number with \(H\) is three. In particular, we remark that \(Z\) is obtained from \(S\) by contracting \(e_{11}\).

**Corollary 3.18.** For each of \(d = 0\) and \(2\), when a birational morphism \(v_0 : X \to \mathbb{P}^2\) as in Theorem 3.1 gives (A) or (B2), there exists a fibred rational surface \(f : X \to \mathbb{P}^1\) not admitting \(\Sigma_d\) as the surface of a \#-minimal model of the reduction.

**Proof.** Keep the situation and the notation as before Corollary 3.18. At first, assume that \(\epsilon : S \to \mathbb{P}^1\) is a general one, or any fibre of \(\epsilon\) is irreducible. Then \(C^2 \geq −1\) for all of irreducible curves \(C\) on \(S\) from the genus formula. Hence, there exists no birational morphism from \(S\) to \(\Sigma_d\) for \(d \geq 2\). Thus \(\Sigma_2\) does not appear as the surface of any \#-minimal model of \((Y,G)\).
Next, we assume that $\epsilon$ has a singular fibre of type $\Pi^*$ in the sense of Kodaira [9]. Then a ruling on $S$ is unique, and we can only take $\Sigma_1$ or $\Sigma_2$ as its relatively minimal model. In particular, there exists no birational morphism from $S$ to $\Sigma_0$. Therefore, $f : X \to \mathbb{P}^1$ does not admit $\Sigma_0$ as such a model.

The case of (A) is similar and simpler by applying Theorem 3.16. □

This completes the proof of Theorem 3.8. □

4 Trivial Mordell-Weil groups

Tseng's theorem saw that any ruled surface has a section. By mimicking the proof in [1], we have the following.

**Proposition 4.1.** Every genus two fibration on a smooth projective surface whose geometric genus is zero has a section.

**Proof.** Let $X$ be a smooth projective surface whose geometric genus is zero, $B$ a smooth projective curve and $f : X \to B$ a relatively minimal fibration whose general fibre $F$ is a genus two curve. $\text{NS}(X)$ coincides with $H^2(X,\mathbb{Z})$, since the geometric genus is zero. As $D$ runs through $H^2(X,\mathbb{Z})$, the set of integers $D.F$ is an ideal in $\mathbb{Z}$, of the form $n\mathbb{Z}$ with $n \geq 1$. Consider the map $D \mapsto (1/n)(D.F)$ is a linear form on $H^2(X,\mathbb{Z})$. From Poincaré duality, there exists a divisor $F' \in H^2(X,\mathbb{Z})$ such that $D.F' = (1/n)(D.F)$ for all $D \in H^2(X,\mathbb{Z})$, so that $F$ is numerically equivalent to $nF'$. Remark that $F'^2 = 0$ from $n^2F'^2 = F'^2 = 0$. Furthermore, from the genus formula, we have $nF'.K_X = F.K_X + F^2 = 2$ and $F'.K_X = F'.(K_X + F')$ is even. These imply $n = 1$. Hence, there exists a divisor $D_0 \in H^2(X,\mathbb{Z})$ such that $D_0.F = 1$. Take a sufficiently ample divisor $L$ on $B$, then we have an effective divisor $E \in |D_0 + f^*L|$. Thus, there exists a component $C$ of $E$ such that $C.F = 1$. □

A relatively minimal fibration $f : X \to \mathbb{P}^1$ of genus two on a smooth rational surface always has a section from Proposition 4.1. Furthermore, we are interested in Mordell-Weil group and lattice of $f$ introduced by Shioda [15].

**Theorem 4.2.** For each (A), (B1), (B2) or (C) in Theorem 3.1, there exists a genus two fibration on a rational surface whose Mordell-Weil group is trivial.

As a proof of Theorem 4.2, we give an example for each (A), (B1), (B2) or (C) in Theorem 3.1 with a concrete description of $(-1)$-curves contracted by a birational
morphism \( \nu_0 : X \to \mathbb{P}^2 \). We also use the notation and the setup as before Example 3.12 throughout the following four examples. In addition, we denote by \( F_p \) the fibre of \( p \in \mathbb{P}^1 \) by \( f \).

At first, we describe an example of (A) in Theorem 3.1 with trivial Mordell-Weil group.

**Example 4.3.** Let \( A \) be Zariski closure on \( W \) of the divisor defined by \( x^5 + t^3 + t^2 x = 0 \), which is irreducible. The singularities of \( A \) are on \((0, 0)\) and on \((\infty, \infty)\). Put \( B := A + \Delta_0 + \Gamma_0 \), which is linearly equivalent to \( 6\Delta_0 + 4\Gamma \). In fact, the singularities of \( B \) are of type (V) on \( \Gamma_0 \) and of type (0) on \( \Gamma_\infty \). Take the finite double cover of \( W \) branched along \( B \) and the canonical resolution. Then \( (\sigma^*)^*\Delta_0 + (-K_W - 3(\sigma^*)^*\Gamma) = -1 \) implies (3.21). Thus we have a relatively minimal fibration \( f : X \to \mathbb{P}^1 \) of genus two on a rational surface with \( (K_X + F)^2 = 1 \).

The strict transform by \( \sigma \circ \bar{\pi} : \tilde{X} \to W \) of \( \Delta_0 \) is the pull-back by \( \varphi : \tilde{X} \to X \) of a double \((-1)\)-section of \( f \). Let \((O)\) be the \((-1)\)-section of \( f \). Reducible fibres of \( f \) are \( F_0 \) and \( F_\infty \). The irreducible components \( \Theta_0, \Theta_1, \ldots, \Theta_{12} \) satisfy the following: Firstly, the irreducible decompositions are

\[
F_0 = \Theta_{11} + \Theta_9 + 2\Theta_{10} + 2\Theta_{12}
\]
\[
F_\infty = \Theta_0 + 4\Theta_1 + 7\Theta_2 + 5\Theta_8 + 2 \sum_{i=3}^{7} (8 - i)\Theta_i.
\]

Secondary, \( \Theta_0 \) is a \((-4)\)-curve and \( \Theta_{12} \) is a \((-1)\)-elliptic curve, i.e., an elliptic curve whose self-intersection number is \((-1)\). The others are \((-2)\)-curves. Furthermore, \( \Theta_{11} \) and \( \Theta_0 \) intersect with \((O)\). Finally, the dual graphs are as in Figure 6. Here the numbers of the vertices corresponds to the suffixes of irreducible components. In particular, a double circle means that the corresponding component is an elliptic curve. The strict transform by \( \sigma \circ \bar{\pi} : \tilde{X} \to W \) of \( \Delta_\infty \) is the pull-back by \( \varphi : \tilde{X} \to X \) of a \((-1)\)-curve. We denote by \( e_8 \) the \((-1)\)-curve. Remark that \( e_8, \Theta_{12} = e_8, \Theta_7 = 1 \) and \( e_8.F = 2 \).

Let \( \nu_{11} : X \to X_{11} \) be the blow-down contracting \((O)\). Then \( (\nu_{11})_* \Theta_{11} \) is a \((-1)\)-curve. We take the composite \( \nu_{10} : X \to X_{10} \) of \( \nu_{11} \) and the blow-down \( X_{11} \to X_{10} \) contracting \( \Theta_{11} \). In this way, we have a birational morphism \( \nu_8 : X \to X_8 \) contracting \((O), \Theta_{11}, \Theta_{10} \) and \( \Theta_9 \), where \( (X_8, (\nu_8)_* F) \) consists
with the reduction of \((X, F)\). Remark that \((O), \Theta_{11}, \Theta_{10}\) and \(\Theta_9\) do not meet \(e_8\). Let \(v_7 : X \to X_7\) be the composite of \(v_8\) and the blow-down \(X_8 \to X_7\) contracting \((v_8)_*e_8\). Then \((v_7)_*\Theta_7, (v_7)_*F = 2\) and \((v_7)_*\Theta_7\) is a \((-1)\)-curve. Similarly, we define a birational morphism \(v_i : X \to X_i\) as the composite of \(v_{i+1}\) and the blow-down \(X_{i+1} \to X_i\) contracting \((v_{i+1})_*\Theta_{i+1}\) for \(i = 6, 5, \ldots, 0\). Then \(v_0 : X \to X_0 = \mathbb{P}^2\) is a birational morphism giving \((A)\) in Theorem 3.1. Furthermore, we have

\[
(O) = e_{12}, \quad \Theta_0 = \ell - e_1 - e_9 - e_{10} - e_{11} - e_{12}, \quad \Theta_8 = \ell - e_1 - e_2 - e_3, \\
\Theta_{12} = 3\ell - e_1 - e_2 - \cdots - e_{10}, \quad \Theta_i = e_i - e_{i+1}, \quad i = 1, 2, \ldots, 7, 9, 10, 11,
\]

and \(\text{NS}(X) = \mathbb{Z}\ell \oplus (\bigoplus_{i=1}^{12} \mathbb{Z}e_i)\), where \(\ell\) and \(e_i\) denote the same in Theorem 3.1. In addition, an orthogonal decomposition of the Néron-Severi lattice \(\mathfrak{NS}(X)\), that is, \(\text{NS}(X)\) equipped with the bilinear form which is \((-1)\) times of the intersection form, is as follows:

\[
\mathfrak{NS}(X) = (\mathbb{Z}F \oplus \mathbb{Z}(O)) \oplus 1 (\mathbb{Z}\Theta_0 \oplus \mathbb{Z}\Theta_{10} \oplus \mathbb{Z}\Theta_{12}) \oplus 1 \left( \bigoplus_{i=1}^{8} \mathbb{Z}\Theta_i \right).
\]

Here \(\mathcal{L} \oplus 1 \mathfrak{M} \oplus 1 \cdots \oplus 1 \mathfrak{N}\) means that lattices \(\mathcal{L}, \mathfrak{M}, \ldots, \mathfrak{N}\) are orthogonal each other. Thus, Mordell-Weil group of \(f\) is trivial from [15, Theorem 1].

Secondly, we take an example of (B1) in Theorem 3.1 with trivial Mordell-Weil group.

**Example 4.4.** Let \(A\) be Zariski closure on \(W^3\) of the divisor defined by

\[
x^5 + tx^4 + tx^3 + t^2x + t^3 = 0,
\]

which is irreducible. The singularities of \(A\) are on \((0, 0)\) and on \((\infty, \infty)\). Take the section \(C\) of \(pr_1\) defined by \(x + t = 0\). Both simple triple points \((0, 0)\) and \((\infty, \infty)\) of \(A\) have contacts of order four with \(C\). Put \(B^3 = A + C + \Gamma_0 + \Gamma_\infty\), which is linearly equivalent to \(6\Delta_0 + 6\Gamma\). In fact, the singularities of \(B^3\) are of type \((V)\) on \(\Gamma_0\) and of same type on \(\Gamma_\infty\).

We take the finite double cover of \(W^3\) branched along \(B^3\) and the canonical resolution. Then we also have (3.2) since \(B^3\) does not include any minimal section of \(pr_1\). Therefore, we obtain a relatively minimal fibration \(f : X \to \mathbb{P}^1\) of genus two on a rational surface with \((K_X + F)^2 = 2\).

The strict transform by \(\sigma \circ \pi : \tilde{X} \to W^3\) of \(C\) is the pull-back by \(\varphi : \tilde{X} \to X\) of a double \((-2)\)-section of \(f\). Let \((O)\) be the \((-2)\)-section of \(f\). Reducible fibres of \(f\) are \(F_0\) and \(F_\infty\). Furthermore, the irreducible components \(\Theta_0, \Theta_1, \ldots, \Theta_{11}\) satisfy the following: Firstly, the irreducible decompositions are

\[
F_0 = \Theta_0 + \Theta_7 + 2 \sum_{i=8}^{11} \Theta_i, \quad F_\infty = \Theta_1 + \Theta_2 + 2 \sum_{i=3}^{6} \Theta_i.
\]
Secondary, $\Theta_6$ and $\Theta_{11}$ are $(-1)$-elliptic curves. The others are $(-2)$-curves. Furthermore, $\Theta_0$ and $\Theta_2$ intersect with $(O)$. Finally, the dual graphs are as in Figure 7. Here the numbers of the vertices corresponds to the suffixes of irreducible components. The strict transform by $\sigma \circ \tilde{\pi} : \tilde{X} \to W^2$ of $\Delta_0$ is the pull-back by $\varphi : \tilde{X} \to X$ of a $(-1)$-curve. We denote by $e_{11}$ the $(-1)$-curve. That of $\Delta_\infty$ is the pull-back by $\varphi : \tilde{X} \to X$ of a $(-1)$-curve. Let $e_6$ be the $(-1)$-curve. Remark that $e_6, \Theta_6 = e_{11}, \Theta_{10} = 1$ and $e_6, F = e_{11}, F = 2$.

Let $v_{10} : X \to X_{10}$ be the blow-down contracting $e_{11}$. Then $(v_{10})_*\Theta_{10}, (v_{10})_*F = 2$ and $(v_{10})_*\Theta_{10}$ is a $(-1)$-curve. We take the composite $v_9 : X \to X_9$ of $v_{10}$ and the blow-down $X_{10} \to X_9$ contracting $(v_{10})_*\Theta_{10}$. Similarly, we define a birational morphism $v_i : X \to X_i$ as the composite of $v_{i+1}$ and the blow-down $X_{i+1} \to X_i$ contracting $(v_{i+1})_*\Theta_{i+1}$ for $i = 8, 7, 6$. Remark that $e_{11}, \Theta_6, \Theta_9, \Theta_8$ and $\Theta_7$ do not meet $e_6$. Let $v_5 : X \to X_5$ be the composite of $v_6$ and the blow-down $X_6 \to X_5$ contracting $(v_6)_*e_6$. Furthermore, we also define a birational morphism $v_i : X \to X_i$ as the composite of $v_{i+1}$ and the blow-down $X_{i+1} \to X_i$ contracting $(v_{i+1})_*\Theta_{i+1}$ for $i = 4, 3, 2, 1$. Then $(v_1)_*(O), (v_1)_*F = 3$ and $(v_1)_*(O)$ is a $(-1)$-curve. Hence, the composite $v_0 : X \to \mathbb{P}^2$ of $v_1$ and the blow-down $X_1 \to \mathbb{P}^2$ contracting $(v_1)_*(O)$ is a birational morphism giving (B1) in Theorem 3.1. Furthermore, we have

$$
\Theta_6 = \ell - e_1 - e_7 - e_8, \quad \Theta_1 = \ell - e_1 - e_2 - e_3, \quad \Theta_6 = 3\ell - \sum_{i=1}^{5} e_i - \sum_{i=7}^{11} e_i, \\
\Theta_i = e_i - e_{i+1}, \quad i = 2, 3, 4, 5, 7, 8, 9, 10, \quad \Theta_{11} = 3\ell - \sum_{i=1}^{10} e_i, \quad (O) = e_1 - e_2,
$$

where $\ell$ and $e_i$ denote the same in Theorem 3.1. In addition, we have an orthogonal decomposition

$$
\mathfrak{M}(X) = (ZF \oplus Z(O)) \oplus \left( \bigoplus_{i=7}^{11} Z\Theta_i \right) \oplus \left( Z\Theta_1 \oplus \bigoplus_{i=3}^{6} Z\Theta_i \right).
$$

Thus, Mordell-Weil group of $f$ is trivial from \([15, \text{Theorem 1}].\)

Next, we show an example of (B2) in Theorem 3.1 with trivial Mordell-Weil group.
Example 4.5. Let $A$ be Zariski closure on $W^2$ of the divisor defined by
\[
t^2x^5 - 3t^4x^4 - 3t^2x^4 + 6t^3x^3 + 3tx^3 + 3t^4x^2 - 3t^2x^2 - x^2 - 3tx - t^4 = 0,
\]
which is irreducible. The singularities of $A$ are on $(0,0)$ and on $(0,\infty)$. Let $C$ be the
section of $pr_1$ defined by $tx - 1 = 0$. The three-fold triple point $(0, \infty)$ of $A$ has a contact
of order nine with $C$. Put $B^3 = A + \Delta_0$, which is linearly equivalent to $6\Delta_0 + 4\Gamma$. In
fact, the singularities of $B^3$ are of type $(\Pi_1)$ on $F_0$. We take the finite double cover of
$W^2$ branched along $B^3$ and the canonical resolution. Since $B^3$ includes just one minimal
section $\Delta_0$ of $pr_1$, we also have $(3, 2, 1)$. Thus we obtain a relatively minimal fibration
$f: X \to \mathbb{P}^1$ of genus two on a rational surface with $(K_X + F)^2 = 2$.

The strict transform by $\sigma \circ \tilde{\pi}: \tilde{X} \to W^2$ of $\Delta_0$ is the pull-back by $\varphi: \tilde{X} \to X$ of a
double $(-1)$-section of $f$. Let $(O)$ be the $(-1)$-section of $f$. Now, $F_0$ is a unique reducible
fibre of $f$, and the irreducible components $\Theta_0, \Theta_1, \ldots, \Theta_{10}$ satisfy the following: Firstly,
the irreducible decomposition is
\[
F_0 = \Theta_{10} + \Theta_9 + \sum_{i=3}^{8} (9 - i)\Theta_i + 4\Theta_2 + 2\Theta_1 + 3\Theta_0.
\]
Secondary, $\Theta_8$ is a $(-3)$-curve and $\Theta_{10}$ is a $(-1)$-elliptic curve. The others are $(-2)$-curves.
Furthermore, $\Theta_{10}$ intersects with $(O)$. Finally, the dual graph is as in Figure 8. Here the
numbers of the vertices corresponds to the suffixes of irreducible components. The strict
transform by $\sigma \circ \tilde{\pi}: \tilde{X} \to W^2$ of $\Delta_\infty$ is the pull-back by $\varphi: \tilde{X} \to X$ of a $(-1)$-curve. Let
$e_{10}$ be the $(-1)$-curve. That of $C$ is the pull-back by $\varphi: \tilde{X} \to X$ of a $(-2)$-curve. We
denote by $\hat{e}_8$ the $(-2)$-curve. Remark that $e_{10}.\Theta_9 = e_{10}.\Theta_8 = \hat{e}_8.(O) = \hat{e}_8.\Theta_7 = 1$ and
$e_{10}.F = \hat{e}_8.F = 2$.

Let $v_{10}: X \to X_{10}$ be the blow-down contracting $(O)$. Since $(O)$ do not meet $e_{10}$, we
take the composite $v_9: X \to X_9$ of $v_{10}$ and the blow-down $X_{10} \to X_9$ contracting $(v_{10}).e_{10}$.
Then $(v_9)_*\Theta_9, (v_9)_*F = 2$ and $(v_9)_*\Theta_9$ is a $(-1)$-curve. We denote by $v_8: X \to X_8$
the composite of $v_9$ and the blow-down $X_9 \to X_8$ contracting $(v_9)_*\Theta_9$. Remark that
$(v_8)_*\hat{e}_8, (v_8)_*F = 3$ and $(v_8)_*\hat{e}_8$ is a $(-1)$-curve. Let $v_7: X \to X_7$ be the composite
of $v_8$ and the blow-down $X_8 \to X_7$ contracting $(v_8)_*\hat{e}_8$. Similarly, we define a birational
morphism $v_i: X \to X_i$ as the composite of $v_{i+1}$ and the blow-down $X_{i+1} \to X_i$ contracting
\((v_{i+1})_\ast \Theta_{i+1}\) for \(i = 6, 5, \ldots, 0\). Then \(v_0 : X \to X_0 = \mathbb{P}^2\) is a birational morphism giving (B2) in Theorem 3.1. Furthermore, we have

\[
(O) = e_{11}, \quad \Theta_{10} = 3\ell - \sum_{i=1}^{9} e_i - e_{11}, \quad \Theta_0 = \ell - e_1 - e_2 - e_3,
\]

\[
\Theta_8 = 3\ell - \sum_{i=1}^{7} e_i - 2e_9 - e_{10}, \quad \Theta_i = e_i - e_{i+1}, \quad i = 1, 2, \ldots, 6, 7, 9,
\]

where \(\ell\) and \(e_i\) denote the same in Theorem 3.1. Hence, we have an orthogonal decomposition

\[
\NS(X) = (\mathbb{Z}F \oplus \mathbb{Z}(O)) \oplus \left( \bigoplus_{i=0}^{9} \mathbb{Z}\Theta_i \right).
\]

Thus, Mordell-Weil group of \(f\) is trivial from [15, Theorem 1].

In the last, we see an example of (C) in Theorem 3.1 with trivial Mordell-Weil group.

**Example 4.6.** Let \(A\) be Zariski closure on \(W^3\) of the divisor defined by

\[
x^5 + t^2x^4 - 6tx^4 + 6t^2x^3 + 4tx^3 - 4t^3x^2 - 6t^2x^2 + 6t^3x - t^2x - t^4 = 0,
\]

which is irreducible. The singularities of \(A\) are on \((0, 0)\), on \((1, 1)\) and on \((\infty, \infty)\). The section \(C\) of \(pr_1\) defined by \(x - t = 0\) passes through \((0, 0)\), \((1, 1)\) and \((\infty, \infty)\). Put \(B^3 = A + C + \Gamma_0 + \Gamma_1 + \Gamma_\infty\), which is linearly equivalent to \(6\Delta_0 + 8\Gamma\). In fact, the singularities of \(B^3\) are three of type (V). They are on \(\Gamma_0\), on \(\Gamma_1\) and on \(\Gamma_\infty\). We define sections \(D_8\), \(D_9\) and \(D_{10}\) of \(pr_2\) by the following:

\[
D_8 : x^2 - 2x + t = 0, \quad D_9 : x^2 - t = 0, \quad D_{10} : x^2 - 2tx + t = 0.
\]

They are linearly equivalent to \(2\Delta_0 + \Gamma\), and pass through \((0, 0)\), \((1, 1)\) and \((\infty, \infty)\). Furthermore, \(D_8\) is tangent to \(\Gamma_1\) and to \(\Gamma_\infty\). In fact, \(D_8\) is a unique curve having such properties. However, it meets \(\Gamma_0\) transversally. Therefore, (3.21) also holds when we take the finite double cover of \(W^3\) branched along \(B^3\) and the canonical resolution. Hence, we obtain a relatively minimal fibration \(f : X \to \mathbb{P}^1\) of genus two on a rational surface with \((K_X + F)^2 = 3\). Indeed, \(D_9\) meets \(\Gamma_1\) transversally, though it is also tangent to \(\Gamma_0\) and to \(\Gamma_\infty\). Similarly, \(D_{10}\) is tangent to \(\Gamma_0\) and to \(\Gamma_1\), though it meets \(\Gamma_\infty\) transversally.

The strict transform by \(\sigma \circ \tilde{\pi} : \tilde{X} \to W^3\) of \(C\) is the pull-back by \(\varphi : \tilde{X} \to X\) of a double \((-2)\)-section of \(f\). Let \((O)\) be the \((-2)\)-section of \(f\). Reducible fibres of \(f\)
are \( F_0, F_1 \) and \( F_\infty \). Furthermore, the irreducible components \( \Theta_2, \Theta_3, \ldots, \Theta_{13} \) satisfy the following: Firstly, the irreducible decompositions are

\[
F_0 = \Theta_2 + \Theta_{11} + 2\Theta_5 + 2\Theta_8,
\]
\[
F_1 = \Theta_3 + \Theta_{12} + 2\Theta_6 + 2\Theta_9, \quad F_\infty = \Theta_4 + \Theta_{13} + 2\Theta_7 + 2\Theta_{10}.
\]

Secondary, \( \Theta_8, \Theta_9 \) and \( \Theta_{10} \) are \((-1)\)-elliptic curves. The others are \((-2)\)-curves. Furthermore, \( \Theta_2, \Theta_3 \) and \( \Theta_4 \) intersect with \((O)\). Finally, the dual graphs are as in Figure 9. Here

![Figure 9](image)

the numbers of the vertices corresponds to the suffixes of irreducible components. For \( i = 8, 9, 10 \), the strict transform by \( \sigma \circ \widetilde{\pi} : \widetilde{X} \to W^2 \) of \( D_i \) is the pull-back by \( \varphi : \widetilde{X} \to X \) of a \((-1)\)-curve. We denote by \( e_i \) the \((-1)\)-curve for \( i = 8, 9, 10 \). That of \( A \) is the pull-back by \( \varphi : \widetilde{X} \to X \) of a double \((-1)\)-curve. We denote by \( e_1 \) the \((-1)\)-curve. Remark that

\[
e_8.\Theta_5 = e_9.\Theta_6 = e_{10}.\Theta_7 = e_1.\Theta_{11} = e_1.\Theta_{12} = e_1.\Theta_{13} = 1,
\]
\[
e_1.\Theta_8 = e_1.\Theta_9 = e_1.\Theta_{10} = 2, \quad e_8.F = e_9.F = e_{10}.F = 4
\]

and \( e_1.F = 5 \).

Since \( e_8, e_9 \) and \( e_{10} \) are three disjoint \((-1)\)-curves, we define a birational morphism \( \nu_7 : X \to X_7 \) as the composite of the blow-downs which contract them. Then \( (\nu_7)_*\Theta_7.(\nu_7)_*F = 4 \) and \( (\nu_7)_*\Theta_7 \) is a \((-1)\)-curve. We take the composite \( \nu_6 : X \to X_6 \) of \( \nu_7 \) and the blow-down \( X_7 \to X_6 \) contracting \( (\nu_7)_*\Theta_7 \). Similarly, we define a birational morphism \( \nu_i : X \to X_i \) as the composite of \( \nu_{i+1} \) and the blow-down \( X_{i+1} \to X_i \) contracting \( (\nu_{i+1})_*\Theta_{i+1} \) for \( i = 5, 4, \ldots, 1 \). Remark that \( \Theta_2, \Theta_3, \ldots, \Theta_7, e_8, e_9 \) and \( e_{10} \) do not meet \( e_1 \). Hence, the composite \( \nu_0 : X \to \mathbb{P}^2 \) of \( \nu_1 \) and the blow-down \( X_1 \to \mathbb{P}^2 \) contracting \( (\nu_1)_*e_1 \) is a birational morphism giving \((C)\) in Theorem 3.1. Furthermore, we have

\[
\Theta_i = e_i - e_{i+3}, \quad i = 2, 3, \ldots, 7, \quad \Theta_i = 6\ell - 2\sum_{j=1}^{10} e_j + e_i, \quad i = 8, 9, 10,
\]
\[
\Theta_{i+9} = \ell - e_1 - e_i - e_{i+3}, \quad i = 2, 3, 4, \quad (O) = \ell - e_2 - e_3 - e_4,
\]

where \( \ell \) and \( e_i \) denote the same in Theorem 3.1. Therefore, we have an orthogonal decomposition

\[
\mathfrak{NG}(X) = (\mathbb{Z}F \oplus \mathbb{Z}(O)) \oplus^\perp (\mathbb{Z}\Theta_5 \oplus \mathbb{Z}\Theta_6 \oplus \mathbb{Z}\Theta_{11}) \\
\oplus^\perp (\mathbb{Z}\Theta_6 \oplus \mathbb{Z}\Theta_8 \oplus \mathbb{Z}\Theta_{12}) \oplus^\perp (\mathbb{Z}\Theta_7 \oplus \mathbb{Z}\Theta_{10} \oplus \mathbb{Z}\Theta_{13}).
\]
Thus, Mordell-Weil group of $f$ is trivial from [15, Theorem 1].

This completes the proof of Theorem 4.2. □

**Remark 4.7.** Keep the situation and the notation as in Example 4.6. Remark that $(v_3)_*\Theta_{12}$, $(v_3)_*\Theta_{11}$ and $(v_3)_*(O)$ are three disjoint $(-1)$-curves on $X_3$. Hence, we obtain another birational morphism $v'_0 : X \to \mathbb{P}^2$ from $v_3 : X \to X_3$ by contracting them. Then $\deg v'_0(F) = 13$ and singularities of $v'_0(F)$ are the same as $v_0(F)$’s. In fact, $v'_0 \circ (v_0)^{-1} : \mathbb{P}^2 \to \mathbb{P}^2$ is the Cremona transformation at three points $v_0(e_1)$, $v_0(e_2)$ and $v_0(e_3)$.

**Remark 4.8.** Although the self-intersection number of any section of the genus two fibrations in Examples 3.12 and 3.13 is at most $(-3)$, that of the unique sections of the ones in Examples 4.4 and 4.6 is equal to $(-2)$. In contrast, $f : X \to \mathbb{P}^2$ always has a $(-1)$-section when $v_0 : X \to \mathbb{P}^2$ gives (A) or (B2).

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