BCOV RINGS ON ELLIPTIC CURVES AND ETA FUNCTION

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Abstract. Associated Legendre functions of the first kind give a family of BCOV rings on elliptic curves. We prove that the family is parametrized by \( q \)-exponents of the eta function \( \eta(q^{24}) \). Our method involves a classification of rational solutions of a Riccati equation under some constraints.

1. Introduction

In this paper, we parametrize a family of BCOV rings on elliptic curves by the eta function. As such, this paper can be seen as a step forward on understanding meromorphic ambiguity on BCOV theory [BCOV] by a modular form.

BCOV rings [Hos] have been introduced to study BCOV holomorphic anomaly equations of Bershadsky, Cecotti, Ooguri and Vafa [BCOV]. BCOV theory has gained much interest in mathematics and physics [YamYau, Ali, AliLan, KanZho].

A major challenge of BCOV theory is meromorphic ambiguity to compute Gromov-Witten potentials. For this, let us take \( \Gamma = \langle \left( \begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array} \right), \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \rangle \subset \text{SL}_2(\mathbb{Z}) \) and recall a finitely-generated \( \Gamma \)-invariant BCOV ring \( \mathcal{R}^\Gamma_{BCOV} \) on elliptic curves. This ring is fundamental in BCOV rings. To define each \( \mathcal{R}^\Gamma_{BCOV} \), we need to choose \( r(x) \in \mathbb{C}(x) \) that solves the following Riccati equation. This choice corresponds to the meromorphic ambiguity of the BCOV theory. For \( x \in \mathbb{P}^1, \lambda \in \mathbb{Q} \), and the Griffith-Yukawa coupling \( C_x = \frac{1}{1-432x} \), there is the Riccati equation:

\[
(1.1) \quad r'(x) + C_x r^2(x) - 60 = \lambda C_x.
\]

For Legendre associated functions of the first and second kinds \( P^\beta_\alpha(x) \) and \( Q^\beta_\alpha(x) \) and \( C \in \mathbb{P}^1 \), Equation (1.1) admits the general solution:

\[
r(x, \lambda, C) = \frac{1}{12} \left( 5 + 4320x + (-5 + 12\sqrt{\lambda}) \frac{CP^2\sqrt{\lambda}(-1 + 864x) + Q^2\sqrt{\lambda}(-1 + 864x)}{CP^2\sqrt{\lambda}(-1 + 864x) + Q^2\sqrt{\lambda}(-1 + 864x)} \right).
\]

By taking \( C \to \infty \), set

\[
r(x, \lambda) = \frac{1}{12} \left( -5 + 4320x + (-5 + 12\sqrt{\lambda}) \frac{P^2\sqrt{\lambda}(-1 + 864x)}{P^2\sqrt{\lambda}(-1 + 864x)} \right).
\]

Let \( R_\infty \) be the family of \( \mathcal{R}^\Gamma_{BCOV} \) for all \( r(x, \lambda) \in \mathbb{C}(x) \). Let \( \chi(n) \) be the Dirichlet character of mod 12 such that \( \chi(\pm 1) = 1 \) and \( \chi(\pm 5) = -1 \). We prove the following.

Theorem 1.1. The family \( R_\infty \) of finitely-generated \( \Gamma \)-invariant BCOV rings on elliptic curves is parametrized by the \( q \)-exponents of the eta function \( \eta(q^{24}) = \sum_{n=1}^{\infty} \chi(n) q^{n^2} = q - q^{25} - q^{49} + q^{121} + q^{169} + \cdots \). Namely, \( r(x, \lambda) \in \mathbb{C}(x) \) if and only if \( 144\lambda = 1, 25, 49, 121, 169, \cdots \), squares of numbers prime to 6.
2. Proofs

To study when \( r(x, \lambda) \in \mathbb{C}(x) \), let us first consider \( r(x, \lambda) \) at a fixed singularity of Equation 1.1.

**Lemma 2.1.** If \( r(x, \lambda) \in \mathbb{C}(x) \), then \( 144\lambda = 1, 25, 49, 121, 169, \cdots \).  

**Proof.** At \( x = \infty \), unless \( \frac{1}{6} - 2\sqrt{\lambda} \in \mathbb{Z}_{\leq 0} \), the Laurent expansion of \( r(x, \lambda) \) is

\[
72x - \frac{6 \cdot 2^\frac{3}{2} \cdot \sqrt{\pi} \cdot \Gamma\left(\frac{5}{6} - 2\sqrt{\lambda}\right)}{\Gamma(\frac{1}{3}) \cdot \Gamma(\frac{1}{3} - 2\sqrt{\lambda})} x^\frac{1}{2} + O\left(\frac{1}{x}\right)
\]

Since \( r(x, \lambda) \in \mathbb{C}(x) \), \( \frac{1}{6} - 2\sqrt{\lambda} \in \mathbb{Z}_{\leq 0} \). \( \square \)

For \( n, m \in \mathbb{R} \), let us study \( f(n, m, x) = \frac{P_{m+1}^n(x)}{P_m^n(x)} \) when \( m \) increases.

**Lemma 2.2.** If \( n \neq m \), we have

\[
x - f(n, m+1, x) = \frac{(n + m + 1)(1 - x^2)}{(n - m + 1)f(n, m, x) - (n + m + 1)x}
\]

**Proof.** Recall the three-term recurrences [DLMF] 14.10.1,14.10.2:

(1.1) \( (1 - x^2)^\frac{1}{2} P_{n+2}^m(x) + 2(m + 1)x P_{n+1}^m(x) = -(n - m)(n + m + 1)(1 - x^2)^\frac{1}{2} P_n^m(x) \),

(2.2) \( (1 - x^2)^\frac{1}{2} P_{n+1}^{m+1}(x) - (n + m + 1) P_n^m(x) = -(n + m + 1)x P^n_m(x) \).

By Equation 2.2, put

(2.3) \( (1 - x^2)^\frac{1}{2} P_{n+2}^m(x) - (n - m) P_{n+1}^{m+1}(x) + (n + m + 2)x P_{n}^{m+1}(x) = 0 \).

Let \( F(n, m, x) = (n-m+1)f(x)-(n+m+1)x \). Then, by \( P_{n+1}^m(x) = P_n^m(x)f(n, m, x) \) and Equation 2.2

(2.4) \( P_{n+1}^{m+1}(x) = (1 - x^2)^{-\frac{1}{2}} P_n^m(x) F(n, m, x) \).

By Equations 2.3 and 2.4

(2.5) \( (1 - x^2)^\frac{1}{2} P_{n+2}^m(x) - (n - m) P_{n+1}^{m+1}(x) = -(n + m + 2)x(1 - x^2)^{-\frac{1}{2}} P_n^m(x) F(n, m, x) \)

Thus, subtracting Equation 2.1 from Equation 2.5 gives

\[
-2(m + 1)x P_{n+1}^m(x) - (n - m) P_{n+1}^{m+1}(x) = P_n^m(x)(-n + m + 2)x(1 - x^2)^{-\frac{1}{2}} F(n, m, x) + (n - m)(n + m + 1)(1 - x^2)^{-\frac{1}{2}}
\]

Thus,

\[
-2(m + 1)x - (n - m) \frac{P_{n+1}^{m+1}(x)}{P_{n+1}^m(x)} = -2(m + 1)x - (n - m)f(n, m + 1, x) = \frac{P_n^m(x)}{P_{n+1}^m(x)} (-(n + m + 2)x(1 - x^2)^{-\frac{1}{2}} F(n, m, x) + (n - m)(n + m + 1)(1 - x^2)^{-\frac{1}{2}})
\]
Again, by Equation 2.4,

\[ -2(m+1)x - (n-m)f(n,m+1,x) = \]
\[ P_m^m(x)(-n + m + 2)x(1 - x^2)^{-\frac{1}{2}} F(n,m,x) + (n-m)(n+m+1)(1-x^2) = \]
\[ (x^2-1)^{-\frac{1}{2}} P_n^m(x)F(n,m,x) = \]
\[ -(n+m+2)x F(n,m,x) + (n-m)(n+m+1)(1-x^2) = \]
\[ F(n,m,x) = \]
\[ -(n+m+2)x + \frac{(n-m)(n+m+1)(1-x^2)}{F(n,m,x)}. \]

Thus, the lemma holds. \(\Box\)

**Proof.** We confirm the converse of Lemma 2.1. Since \( f\left(-\frac{1}{144}, \frac{1}{144}, x\right) = x, r(x, \frac{1}{144}) = -\frac{1}{17} + 72x \in \mathbb{C}(x) \). Thus, by Lemma 2.2 \( r(x, \frac{1}{144}) \in \mathbb{C}(x) \) for \( i = 1, 13, \ldots \). If \( \lambda = \frac{5^2}{144} \), since \(-5 + 12\sqrt{\lambda} = 0, r(x, \lambda) = -\frac{5}{12} + 360x \in \mathbb{C}(x) \). For \( \lambda = \frac{3^2}{144} \) of \( i = 11, 17, 23, \ldots \), \( f\left(-\frac{1}{144}, \frac{1}{144}, x\right) = \frac{1}{x} \) implies \( r(x, \lambda) \in \mathbb{C}(x) \) by Lemma 2.2. Thus, the assertion holds.

**Remark 2.3.** By Lemma 2.1, we do not have to assume \( \lambda \in \mathbb{Q} \) to define \( R_{BCOV}^i \) for \( r(x, \lambda) \). By the proof of the theorem and Lemma 2.2, we observe that \( r(x, \lambda) \in \mathbb{C}(x) \) implies \( r(x, \lambda) \in \mathbb{Q}(x) \). Lemma 2.2 holds for associated Legendre functions of the second kind. But, \( r(x, \frac{1}{144}, C) \notin \mathbb{C}(x) \) unless \( C \to \infty \), since the Laurent expansion of \( r(x, \frac{1}{144}, C) \) at \( x = \infty \) is \( 72x + \frac{4(-2)^{\frac{1}{2}}(144-72x)\pi}{\sqrt{3\pi(2Ci+\pi)}} + O\left(\frac{1}{x}\right) \).

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