Gauss-Bonnet Theorem in Sub-Riemannian Heisenberg Space $\mathbb{H}^1$

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Abstract We prove a version of the Gauss-Bonnet theorem in sub-Riemannian Heisenberg space $\mathbb{H}^1$. The sub-Riemannian distance makes $\mathbb{H}^1$ a metric space that consequently has a spherical Hausdorff measure. Using this measure, we define a Gaussian curvature at points of a surface $S$ where the sub-Riemannian distribution is transverse to the tangent space of $S$. If all points of $S$ have this property, we prove a Gauss-Bonnet formula and, for compact surfaces (which are topologically a torus), we obtain $\int_S K = 0$.

Keywords Sub-Riemannian geometry · Heisenberg space · Gauss-Bonnet Theorem · Gaussian curvature

Mathematics Subject Classification (2010) 53C17

The principal result of this paper is a Gauss-Bonnet type theorem for surfaces in the Heisenberg group $\mathbb{H}^1$:

Theorem 2 (Gauss-Bonnet formula) Let $U$ be a region contained in a coordinate domain of $S$ such that $T_p S \neq D_p$ for all $p \in U$, let the bounding curve $\gamma$ of $U$ be a simple closed...
transverse curve, and let $c_1, \ldots, c_r$ be the exterior corner areas of $\gamma$. Then

$$
\int_{|\gamma|} k + \sum_{j=1}^{r} c_j + \int_{U} K = 0,
$$

where $k$ is the curvature function on $\gamma$ and $K$ is the Gaussian curvature function on $U$.

If the surface $S$ is compact and oriented, $S$ is diffeomorphic to a torus. In this case, we obtain:

**Corollary 3** Suppose $S$ is a differentiable compact surface in $\mathbb{H}^1$ such that $T_pS \neq D_p$ for all $p \in S$. Then

$$
\int_{S} K = 0.
$$

This paper is organized in seven sections. In first section, we introduce the Heisenberg group $\mathbb{H}^1$ and basic information on this group. Section 2 presents the adapted covariant derivative for $\mathbb{H}^1$. In Section 3, we present an original Gauss map for a transversal surface $S$ and use this map to introduce a sub-Riemannian Gaussian curvature for $S$ in $\mathbb{H}^1$. Section 4 presents the induced covariant derivative on $S$ and the proof that the curvature tensor of this derivative is essentially the Gaussian curvature of $S$. Section 5 introduces the second fundamental form associated to $S$, but this section is not necessary to prove the Gauss-Bonnet Theorem and can be skipped. In Section 6, we define the curvature of a transverse curve on $S$. Finally, in Section 7, we define the concept of corner area between two transverse vectors on $S$ and prove the Gauss-Bonnet Theorem and the Corollary 3.

In this introduction, we present our results briefly, but all the notions will be stated carefully and the proofs given in the posterior sections.

In $\mathbb{H}^1$, we consider a distribution $D$ generated by vector fields

$$
e_1 = \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial z}, \quad \ne_2 = \frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial z},
$$

and a scalar product in $D$ such that $\ne_1, \ne_2$ are orthonormal. Complete these vector fields to a basis of left invariant vector fields in $\mathbb{H}^1$, introducing

$$
\ne_0 = [\ne_1, \ne_2] = \frac{\partial}{\partial z}.
$$

Therefore, if $\ne^0, \ne^1, \ne^2$ are dual forms to $\ne_0, \ne_1, \ne_2$, then the volume element invariant by the group action is $dV = \ne^0 \wedge \ne^1 \wedge \ne^2$.

Using the scalar product in $D$, consider the distance between two points as the infimum of the lengths of curves tangent to $D$ that connect them. With this distance, $\mathbb{H}^1$ is a metric space with Hausdorff dimension four and the differentiable surfaces have dimension three. At points on a surface $S$ where the distribution $D$ does not coincide with $TS$, the intersection $D \cap TS$ has dimension one, and we obtain a direction called the characteristic at this point of $S$. We suppose every point of surface $S$ has a characteristic direction and that there exists a global unitary vector field $\eta$ in $D$ orthogonal to the characteristic direction $TS \cap D$. The vector field $\eta$ is the normal horizontal of $S$. Given a compact set $Q \subset S$, the 3-dimensional (spherical) Hausdorff measure of $Q$ is given by $\int_{Q} i(\eta)dV$. A curve transverse to $D$ has Hausdorff dimension two, and its (spherical) Hausdorff measure is given by $\int_{\gamma} \ne^0$. For more details, see [6, 8–10].

To prove a Gauss-Bonnet theorem, we need a concept of curvature of surfaces. The image under the left transport of the normal horizontal in a neighborhood of a point in $S$ is
contained in $S^1 \subset T_0 H^1$, and therefore the normal horizontal does not suit as a Gauss map. However, we can consider the 1-form $\eta^*$ defined on $S$ by $\eta^*(\eta) = 1$ and $\eta^*|_{TS} = 0$. The analogous Gauss application is

$$g := \exp \circ L^* \circ \eta^* : S \rightarrow H^1$$

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Since $\mathbb{H}^1$ is nilpotent, the exponential map $\exp : H^1 \to \mathbb{H}^1$ is a diffeomorphism. Let $e_1, e_2$ be a basis of $V_1$, with $e_0 = [e_1, e_2] \in V_2$. By applying the Baker-Campbell-Hausdorff formula, we have

$$\exp^{-1}(\exp(X) \exp(Y)) = X + Y + \frac{1}{2}[X, Y].$$

Since $[e_1, e_2] = e_0$, writing $X = x_1e_1 + y_1e_2 + z_1e_0$ and $Y = x_2e_1 + y_2e_2 + z_2e_0$, we obtain

$$X + Y + \frac{1}{2}[X, Y] = (x_1 + x_2)e_1 + (y_1 + y_2)e_2 + (z_1 + z_2 + \frac{1}{2}(x_1y_2 - x_2y_1))e_0.$$  

We identify $\mathbb{H}^1$ with $\mathbb{R}^3$ by identifying $(x, y, z)$ with $\exp(xe_1 + ye_2 + ze_0)$, which are known as the canonical coordinates of first kind or exponential coordinates. In these coordinates, the group operation is

$$(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(x_1y_2 - x_2y_1)).$$

the exponential is

$$\exp(xe_1 + ye_2 + ze_0) = (x, y, z),$$

and the left invariant vector fields $e_1, e_2, e_0$ are given by

$$\begin{align*}
  e_1 & = \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial z}, \\
  e_2 & = \frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial z}, \\
  e_0 & = \frac{\partial}{\partial z},
\end{align*}$$

with brackets $[e_1, e_2] = e_0$, and $[e_0, e_1] = [e_0, e_2] = 0$. The dual basis is

$$\begin{align*}
  e^1 & = dx, \\
  e^2 & = dy, \\
  e^0 & = dz + \frac{1}{2}(ydx - xdy),
\end{align*}$$

with $de^0 = -e^1 \wedge e^2, de^1 = de^2 = 0$. For more details, see [3].

We identify naturally $T\mathbb{H}^1$ with $T^*\mathbb{H}^1$ by identifying $ae_1 + be_2 + ce_0$ with $ae^1 + be^2 + ce^0$, and through this identification we identify $H^1$ with $(H^1)^*$. Therefore, we can define the exponential map on the dual by

$$\exp : (H^1)^* \to \mathbb{H}^1$$

$$xe^1 + ye^2 + ze^0 \mapsto (x, y, z).$$

The left translation is defined by

$$L_{(x, y, z)}(x_1, y_1, z_1) = (x, y, z)(x_1, y_1, z_1),$$

and

$$L^{-1}_{(x, y, z)} = L(-x, -y, -z).$$

Let $D \subset T\mathbb{H}^1$ be the two-dimensional distribution generated by the vector fields $e_1, e_2$, so that $D$ is the null space of $e^0$. On $D$, we define a scalar product $\langle , \rangle$, such that $\{e_1, e_2\}$ is an orthonormal basis of $D$. An operator $J : D \to D$ is well-defined by

$$J(ae_1 + be_2) = -be_1 + ae_2.$$  

The element of volume $dV$ in $\mathbb{H}^1$ is $dV = e^0 \wedge e^1 \wedge e^2 = dx \wedge dy \wedge dz$. A differentiable curve $\gamma : [a, b] \subset \mathbb{R} \to \mathbb{H}^1$ is transverse if $e^0(\gamma'(t)) \neq 0$ for every $t \in [a, b]$, and a transverse curve $\gamma$ is unitarily parametrized if $|e^0(\gamma'(t))| = 1$ for every $t \in [a, b]$. 

\footnote{Springer}
2 The Adapted Covariant Derivative

If \( X, Y \) are vector fields on \( \mathbb{H}^1 \), we define the adapted covariant derivative introduced in [5] by:

\[
\nabla_X Y = \sum_{i=0}^{2} d b_j (X) e_j,
\]

where \( Y = b_0 e_0 + b_1 e_1 + b_2 e_2 \). Then \( \nabla \) is null in left invariant vector fields on \( \mathbb{H}^1 \).

Proposition 1 The covariant derivative \( \nabla \) has the following properties:

1. If \( Y \) is a vector field in \( D \), then \( \nabla_X Y \) is a vector field in \( D \) for any vector field \( X \) in \( T \mathbb{H}^1 \);
2. if \( Y, Z \) are vector fields in \( D \), then
   \[
   \nabla_X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle,
   \]
   for all \( X \) vector fields in \( T \mathbb{H}^1 \);
3. the torsion \( \mathcal{T} \) of \( \nabla \) is
   \[
   \mathcal{T} = -e^1 \wedge e^2 \otimes e_0 = de^0 \otimes e_0;
   \]
4. the curvature tensor \( \mathcal{R} \) of \( \nabla \) is null.

Proof We shall proceed with the proof of 3, the others being similar. If \( X = \sum_{i=0}^{2} a_j e_j \) and \( Y = \sum_{i=0}^{2} b_j e_j \), then

\[
\mathcal{T}(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = -(a_1 b_2 - a_2 b_1) e_0 = -e^1 \wedge e^2 (X, Y) e_0 = de^0 (X, Y) e_0.
\]

Observe that the covariant derivative in the cotangent bundle \( (T \mathbb{H}^1)^* \) satisfies \( \nabla e^i = 0 \) for \( i = 0, 1, 2 \).

3 Surfaces in \( \mathbb{H}^1 \)

Suppose that \( S \) is an oriented differentiable two-dimensional manifold in \( \mathbb{H}^1 \). Note that \( \dim(D \cap T S) \geq 1 \), and, since \( de^0 = e^1 \wedge e^2 \), the set of points where the tangent space of \( S \) coincides with the distribution has empty interior. We denote this set by \( \Sigma \) and its complement on \( S \) by \( S' \):

\[
\Sigma = \{ x \in S : \dim(D_x \cap T_x S) = 2 \} ; \quad S' = S - \Sigma.
\]

The set \( S' \) is open in \( S \). In what follows, we will suppose \( \Sigma = \emptyset \), so \( S = S' \). With this hypothesis on \( S \), the one-dimensional vector subbundle \( D \cap TS \) is well defined. Suppose \( U \subset S \) is an open set such that we can define a unitary vector field \( f_1 \) with values in \( D \cap TS \), so \( \langle f_1, f_1 \rangle = 1 \).

Definition 1 The unitary vector field \( \eta \) in \( D \) defined by

\[
\eta = -J f_1
\]

is the horizontal normal to \( S \).

Then we can define \( \eta^* \in (T \mathbb{H}^1)^*|_S \) by

\[
\eta^*(\eta) = 1 \quad \text{and} \quad \eta^* = 0 \text{ on } TS.
\]
We call $\eta^*$ the *horizontal conormal* to $S$.

We denote by $L_p^*$ the pull-back of the left translation $L_p$, i.e.,

$$L_p^*: T_p^* S \rightarrow T_0^* S,$$

and we have

$$\omega \mapsto L_p^* \omega.$$

**Definition 2** The application

$$g := \exp \circ L^* \circ \eta^* : S \rightarrow T^*_0 S, \quad p \mapsto \exp(L_p^*(\eta^*(p)))$$

is the *Gauss map* of $S$.

Let $f_2 = e_0 - \eta^*(e_0)\eta$. Then $\{f_1, f_2, \eta\}$ is a *special basis* of $T^*_S|S$ on the open set $U$. If

$$\eta = \cos \alpha e_1 + \sin \alpha e_2$$

for some real function $\alpha$ on $U$, reducing $U$ if necessary, then

$$f_1 = -\sin \alpha e_1 + \cos \alpha e_2,$$

and, if we denote $A = -\eta^*(e_0)$, we have

$$f_2 = e_0 + A\eta.$$

The dual basis of $(T^*_S)^*$ on $S$ is

$$\begin{cases}
\eta^* = \cos \alpha e_1 + \sin \alpha e_2 - Ae_0, \\
f^1 = -\sin \alpha e_1 + \cos \alpha e_2, \\
f^2 = e_0.
\end{cases}$$

The inverse relations are

$$\begin{cases}
e_0 = f^2, \\
e^1 = \cos \alpha \eta^* - \sin \alpha f^1 + A \cos \alpha f^2, \\
e^2 = \sin \alpha \eta^* + \cos \alpha f^1 + A \sin \alpha f^2,
\end{cases}$$

and

$$e^1 \wedge e^2 = \eta^* \wedge f^1 - Af^1 \wedge f^2.$$  \hspace{1cm} (1)

Also, it follows that

$$\begin{cases}
df^1 = -d\alpha \wedge \eta^* - A d\alpha \wedge f^2, \\
df^2 = -\eta^* \wedge f^1 + Af^1 \wedge f^2, \\
d\eta^* = (d\alpha + A f^2 + A\eta^*) \wedge f^1 - dA \wedge f^2,
\end{cases}$$

and, since $\eta^* = 0$ on $S$, we have

$$\begin{cases}
df^1 = -A d\alpha \wedge f^2, \\
df^2 = Af^1 \wedge f^2, \\
0 = (d\alpha + A f^2 \wedge f^1 - dA \wedge f^2.
\end{cases}$$

From this last relation, we obtain

$$d\alpha(f_2) = -(dA(f_1) + A^2).$$

**Definition 3** The *element of area* in $S$ is

$$i(\eta)dV.$$
Since \( dV = \eta^* \wedge f^1 \wedge f^2 \), then \( dS = f^1 \wedge f^2 \). We now find the area of \( g(U) \) for a region \( U \subset S \) such that \( g|_U \) is injective. Observe that, for all \( p \in S \),

\[
g(p) = (\cos \alpha(p), \sin \alpha(p), -A(p)).
\]

Then \( g(U) \) is contained on the cylinder \( C = \{(x, y, z) : x^2 + y^2 = 1\} \). The tangent space \( TC \) is generated by

\[
\begin{cases}
-ye_1 + xe_2 + \frac{1}{2}e_0 \\
e_0
\end{cases}
\]

\[
(= -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y})
\]

If \( \{\tilde{f}_1, \tilde{f}_2, \tilde{\eta}\} \) is the special basis of \( T^\Sigma_1|_C \), it follows that

\[
\begin{cases}
\tilde{f}_1 = -ye_1 + xe_2 \\
\tilde{f}_2 = e_0
\end{cases},
\]

so

\[
\tilde{\eta} = -J(\tilde{f}_1) = xe_1 + ye_2.
\]

The element of area on \( C \) is \( d\tilde{S} = \tilde{f}_1 \wedge \tilde{f}_2 \). Then

\[
\text{Area}(g(U)) = \int_{g(U)} d\tilde{S} = \int g(U) \tilde{f}_1 \wedge \tilde{f}_2 = \int U g^*(\tilde{f}_1 \wedge \tilde{f}_2)
\]

\[
= \int U (\tilde{f}_1 \wedge \tilde{f}_2)(g_\ast f_1, g_\ast f_2) dS.
\]

Now,

\[
dg = -\sin \alpha d\alpha \otimes \frac{\partial}{\partial x} + \cos \alpha d\alpha \otimes \frac{\partial}{\partial y} - dA \otimes \frac{\partial}{\partial z}
\]

\[
= -\sin \alpha d\alpha \otimes (e_1 + \frac{1}{2} \sin \alpha e_0) + \cos \alpha d\alpha \otimes (e_2 - \frac{1}{2} \cos \alpha e_0) - dA \otimes e_0
\]

\[
= d\alpha \otimes \tilde{f}_1 - \frac{1}{2} (d\alpha + dA) \otimes \tilde{f}_2,
\]

and so

\[
(\tilde{f}_1 \wedge \tilde{f}_2)(g_\ast f_1, g_\ast f_2) = d\alpha(f_1) \left(\frac{1}{2} d\alpha(f_2) + dA(f_2)\right)
\]

\[
+ d\alpha(f_2) \left(\frac{1}{2} d\alpha(f_1) + dA(f_1)\right)
\]

\[
= -d\alpha \wedge dA(f_1, f_2).
\]

We have proved that

\[
\text{Area}(g(U)) = \int_U -d\alpha \wedge dA(f_1, f_2) dS. \tag{2}
\]

**Definition 4** Let \( S \) be a surface in \( \mathbb{H}^1 \) (with \( \Sigma = \emptyset \)) and \( p \in S \). We define the **Gaussian curvature** of surface \( S \) at point \( p \) by

\[
K(p) = \lim_{U \to \{p\}} \frac{\text{Area}(g(U))}{\text{Area}(U)} = \lim_{U \to \{p\}} \frac{\int_{g(U)} i(\tilde{\eta}) dV}{\int_U i(\tilde{\eta}) dV}. \tag{3}
\]

where \( U \) is a neighborhood of \( p \) which converges to \( p \) in the usual sense.

Since \( \text{Area}(U) = \int_U dS \), we obtain from (2) and (3), the following
Proposition 2 The Gaussian curvature $K$ of $S$ is given by
\[ K = -d\alpha \wedge dA(f_1, f_2). \]

4 The Projection of $\nabla$ by $\eta^*$

Given that $X, Y$ are vector fields in $TS$, we define
\[ \nabla_X Y = \nabla_X Y - \eta^*(\nabla_X Y)\eta. \]

Proposition 3 The operator $\nabla$ is a covariant derivative in $TS$, and satisfies:
1. $\nabla f_1 = 0$;
2. $\nabla f_2 = A\, d\alpha \otimes f_1$;
3. $\nabla f^1 = -A\, d\alpha \otimes f^2$;
4. $\nabla f^2 = 0$.

Proof It is clear that, if $X, Y \in TS$, then $\nabla_X Y \in TS$, $\nabla_X Y$ is linear on $X$ and additive on $Y$. Furthermore, if $f$ is a real function on $S$, we have
\[ \nabla_X fY = df(X)Y + f\nabla_X Y - \eta^*(df(X)Y + f\nabla_X Y)\eta = df(X)Y + f\nabla_X Y. \]

Finally,
1. $\nabla_X f_1 = \nabla_X (-\sin \alpha e_1 + \cos \alpha e_2) = d\alpha(X)(-\cos \alpha e_1 - \sin \alpha e_2) = -d\alpha(X)\eta$.
2. $\nabla_X f_2 = \nabla_X (e_0 + A\eta) = dA(X)\eta + A\nabla_X (\cos \alpha e_1 + \sin \alpha e_2)
   = dA(X)\eta + A\, d\alpha(X)(-\sin \alpha e_1 + \cos \alpha e_2)
   = dA(X)\eta + A\, d\alpha(X)f_1$.
3. $(\nabla_X f^1)(f_1) = -f^1(\nabla_X f_1) = 0$.
   $(\nabla_X f^1)(f_2) = -f^1(\nabla_X f_2) = -A\, d\alpha(X)$.
4. $(\nabla_X f^2)(f_1) = -f^2(\nabla_X f_1) = 0$.
   $(\nabla_X f^2)(f_2) = -f^2(\nabla_X f_2) = 0.$

It follows from this proof that, for $X \in TS$, $\nabla_X \eta = \nabla_X \eta = d\alpha(X)f_1$ and $\nabla_X \eta^* = d\alpha(X)f^1 - dA(X)f^2$.

Definition 5 The covariant derivative $\nabla$ is the adapted covariant derivative on $S$.

Proposition 4 The torsion $T$ of $\nabla$ is given by
\[ T = Af^1 \wedge f^2 \otimes f_2. \]

Proof We have, for $X, Y \in TS$,
\[ T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \]
\[ = \nabla_X Y - \nabla_Y X - [X, Y] - \eta^*(\nabla_X Y - \nabla_Y X - [X, Y])\eta \]
\[ = \overline{T}(X, Y) - \eta^*(\overline{T}(X, Y))\eta, \]
so, from Proposition 1 and (1), we get
\[ T = -e^1 \wedge e^2 \otimes (e_0 - \eta^*(e_0)) = Af^1 \wedge f^2 \otimes f_2. \]

**Proposition 5** The curvature tensor \( R \) of \( \nabla \) is \( R = dA \wedge d\alpha \otimes f^2 \otimes f_1 \).

**Proof** Clearly, \( R(X,Y)f_1 = 0 \), and
\[
R(X,Y)f_2 = \nabla_X \nabla_Y f_2 - \nabla_Y \nabla_X f_2 - \nabla_{[X,Y]} f_2
\]
\[
= \nabla_X (\text{Ad}_{\alpha}(Y)f_1) - \nabla_Y (\text{Ad}_{\alpha}(X)f_1) - \text{Ad}_{\alpha}([X,Y])f_1
\]
\[
= (X(\text{Ad}_{\alpha}(Y)) - Y(\text{Ad}_{\alpha}(X)) - \text{Ad}_{\alpha}([X,Y]))f_1
\]
\[
= d(\text{Ad}_{\alpha})(X,Y)f_1.
\]

The Proposition 5 allows us to recover the Gaussian curvature \( K \) from the tensor of curvature \( R \) of connection \( \nabla \):

**Corollary 1** The Gaussian curvature \( K \) is given by
\[
K = \langle R(f_1, f_2)f_2, f_1 \rangle = dA \wedge d\alpha(f_1, f_2).
\] (4)

**5 The Second Fundamental Form**

This section is not strictly necessary to prove Gauss-Bonnet Theorem, but shows that we can introduce a second fundamental form analogous to Riemannian case, write similar Gauss and Codazzi equations and give an expression for Gaussian curvature in terms of the second fundamental form and the real function \( \alpha \).

From the equation
\[
\nabla_X Y = \nabla_X Y + \eta^*(\nabla_X Y)\eta = \nabla_X Y - (\nabla_X \eta^*)(Y)\eta
\]
for \( X, Y \) vector fields in \( TS \), we define a bilinear form \( V : TS \times TS \rightarrow \mathbb{R} \):

**Definition 6** The bilinear form \( V : TS \times TS \rightarrow \mathbb{R} \) defined by
\[
V(X,Y) = -(\nabla_X \eta^*)(Y)
\]
is the second fundamental form associated to \( S \).

From
\[
\nabla \eta^* = \nabla (\cos \alpha e^1 + \sin \alpha e^2 - Ae^0)
\]
\[
= d\alpha \otimes (-\sin \alpha e^1 + \cos \alpha e^2) - dA \otimes e^0,
\]
we have
\[
V(X,Y) = -d\alpha(X)f^1(Y) + dA(X)f^2(Y).
\]
The second fundamental form is not symmetric in general. For \( X, Y \in TS \), taking account that \( \eta^*(X) = \eta^*(Y) = \eta^*([X,Y]) = 0 \),
\[
V(X,Y) - V(Y,X) = -(\nabla_X \eta^*)(Y) + (\nabla_Y \eta^*)(X) = \eta^*(\nabla_X Y) - \eta^*(\nabla_Y X)
\]
\[
= \eta^*(T(X,Y)) = d\theta^0(X,Y)\eta^*(e_0) = -A d f^2(X,Y)
\]
\[
= -A^2 f^1 \wedge f^2(X,Y).
\]
Theorem 1  The curvature tensor $R$ and the second fundamental form $V$ satisfy:

1. (Gauss equation) $R(X,Y)Z = (-d\alpha(X)V(Y,Z) + d\alpha(Y)V(X,Z))f_1$;
2. (Codazzi equation) $\nabla_X V(Y,Z) - \nabla_Y V(X,Z) + V(T(X,Y),Z) = 0$.

Proof  By applying the definition of curvature, we obtain

$$R(X,Y)Z = \nabla_X (\nabla_Y Z - V(Y,Z)\eta) - V(X,\nabla_Y Z)\eta - \nabla_Y (\nabla_X Z - V(X,Z)\eta) + V(Y,\nabla_X Z)\eta
- \nabla_{[X,Y]}Z + V([X,Y],Z)\eta$$

$$= \overline{R}(X,Y)Z - X(V(Y,Z))\eta - Y(V(X,Z))\eta + X(V(Y,Z))\eta + Y(V(X,Z))\eta
+ V(Y,\nabla_X Z)\eta + V([X,Y],Z)\eta$$

$$= (\nabla_X V(Y,Z) - V(\nabla_X Y,Z)
+ \nabla_Y V(X,Z) + V(\nabla_Y X,Z) + V([X,Y],Z))\eta
+ X(V(Y,Z))\eta + Y(V(X,Z))\eta)
f_1$$

Since $R(X,Y)Z \in TS$, we obtain

$$R(X,Y)Z = - (V(Y,Z)\eta - V(X,Z)d\alpha(Y))f_1,$$

and

$$\nabla_X V(Y,Z) - \nabla_Y V(X,Z) + V(T(X,Y),Z) = 0.$$ 

As a consequence of this theorem, we recover the Gaussian curvature of a surface from the second fundamental form and $\alpha$:

Corollary 2  The Gaussian curvature $K$ of $S$ satisfies

$$K = -d\alpha(f_1)V(f_2,f_2) + d\alpha(f_2)V(f_1,f_2)$$

Proof  It follows easily from equation (4) and the Gauss equation.

6  Curvature of Transverse Curves in the Surface $S$

Let $\gamma : [a, b] \subset \mathbb{R} \rightarrow S$ be a differentiable curve such that $\gamma'(s) \neq 0$, that is, $f^2(\gamma'(s)) \neq 0$ for all $s \in [a, b]$. Let $t$ be defined by

$$t(s) = \frac{1}{|f^2(\gamma'(s))|}\gamma'(s),$$
the unitary tangent field along $\gamma$. As $f^2(t(s)) = \pm 1$, we have
\[ \nabla_t f^2(t) + f^2(\nabla_t t) = 0, \]
and as $\nabla f^2 = 0$, we know that $\nabla_t t$ is a multiple of $f_1$. We write
\[ \nabla_t t = k n, \]
where the vector field $n = \epsilon f_1$ on $\gamma$, and $\epsilon = +1$ if $\{t, f_1\}$ is positively oriented, otherwise $\epsilon = -1$. Observe that $\epsilon f^2(t) < 0$. The function $k : [a, b] \to \mathbb{R}$ is the \textit{curvature} of $\gamma$.

**Definition 7** The function $k = \langle \nabla_t t, n \rangle$ is the \textit{curvature} of the transverse curve $\gamma$.

**Proposition 6** The curvature $k$ is given by
\[ k = \frac{\epsilon}{f^2(\gamma')} \left( \frac{d}{ds} f^1(\gamma') + A d\alpha(\gamma') \right). \]

**Proof** It follows from the definition that $k = \epsilon f^1(\nabla_t t)$, so
\[ k = \frac{\epsilon}{|f^2(\gamma')|} f^1(\nabla_{\gamma'}(\frac{1}{|f^2(\gamma')|} \gamma')) = \frac{\epsilon}{|f^2(\gamma')|} \left( \nabla_{\gamma'}(f^1(\frac{1}{|f^2(\gamma')|} \gamma')) - (\nabla_{\gamma'} f^1)(\frac{1}{|f^2(\gamma')|} \gamma') \right), \]
and the proposition follows. \hfill $\square$

Let be $\{t^*, n^*\}$ the dual basis of $\{t, n\}$ along $\gamma$.

**Definition 8** If $\gamma : [a, b] \to S$ is a tranverse curve with support $|\gamma| = \gamma([a, b])$, we define for a continuous function $h : |\gamma| \to \mathbb{R}$,
\[ \int_{|\gamma|} h = \int_{\gamma} h t^* = \int_{a}^{b} h(\gamma(s)) t^*(\gamma'(s)) ds. \]

It follows from
\[ t^*(\gamma') = |f^2(\gamma')| = -\epsilon f^2(\gamma') \]
that
\[ \int_{|\gamma|} h = -\epsilon \int_{a}^{b} h(\gamma(s)) f^2(\gamma'(s)) ds. \quad (5) \]

### 7 Gauss-Bonnet Theorem

In this section, we prove the Gauss-Bonnet Formula and, for compact surfaces which are topologically a torus, we show that $\int_S K = 0$.

Following the presentation of Hicks’ book ([7, Chapter 7]), let $U \subset S$ be a fundamental set and $c$ be a fundamental 2-chain such that its support $|c| = U$. That means $c = \sigma_1 + \ldots + \sigma_k$ is a finite formal linear combination of $C^\infty$ 2-cubes with real coefficients, such that
(1) each $\sigma_i : [0, 1] \times [0, 1] \to S$ is a 2-cube that is a orientation preserving diffeomorphism onto its image;
(2) $\text{Int}(|\sigma_i|) \cap \text{Int}(|\sigma_j|)$ is empty for $i \neq j$.

where $|\sigma_i| = \sigma_i([0, 1] \times [0, 1])$ and $|c| = |\sigma_1| \cup \ldots \cup |\sigma_k|$ are the supports of $\sigma_i$ and $c$, respectively.

**Definition 9** If $c$ is a fundamental 2-chain with support $|c| = U \subset S$, we define for a real continuous function $h$ on $U$
$$
\int_U h = \int_c hf^1 \wedge f^2.
$$

To establish the Gauss-Bonnet Theorem, we need a similar concept of exterior corner angle in Riemannian surfaces:

**Definition 10** The *corner area* between two tangent vectors of $S$ at a point is
$$
\text{ca}(v, w) = \frac{dV(\eta, v, w)}{e^0(v)e^0(w)}.
$$

The oriented curve $\gamma = \partial c$ is the bounding curve of $U$, where $\partial$ denotes the boundary map on chains. The curve $\gamma$ is piecewise differentiable and composed of differentiable curves $\gamma_j : [s_j, s_{j+1}] \to S$, $j = 1, \ldots, r$, with $\gamma_j(s_{j+1}) = \gamma_{j+1}(s_{j+1})$ for $j = 1, \ldots, r-1$, and $\gamma_1(s_1) = \gamma_r(s_{r+1})$. We define the corner area at the vertices $\gamma_j(s_{j+1})$ as
$$
\text{ca}_j = \text{ca}(\gamma'_j(s_{j+1}), \gamma'_{j+1}(s_{j+1})), \quad j = 1, \ldots, r - 1
$$
and
$$
\text{ca}_r = \text{ca}(\gamma'_r(s_{r+1}), \gamma'_1(s_1)).
$$

**Theorem 2** *(Gauss-Bonnet formula)* Let $U$ be a fundamental set contained in a coordinate domain of $S$, let the positively oriented bounding curve $\gamma$ of $U$ be a simple closed transverse curve, and let $\text{ca}_1, \ldots, \text{ca}_r$ be the exterior corner areas of $\gamma$. Then
$$
\int_{|\gamma|} k + \sum_{j=1}^r \text{ca}_j + \int_U K = 0,
$$
where $k$ is the curvature function on $\gamma$ and $K$ is the Gaussian curvature on $U$.

**Proof** Let $\gamma_1, \ldots, \gamma_r$ be the $C^\infty$ pieces of $\gamma$ with $\gamma_j$ defined on the interval $[s_j, s_{j+1}]$, with $\gamma_j(s_{j+1}) = \gamma_{j+1}(s_{j+1})$ for $j = 1, \ldots, r - 1$, and $\gamma_r(s_{r+1}) = \gamma_1(s_1)$. Let $\text{ca}_j = \text{ca}(\gamma'_j(s_j + 1), \gamma'_{j+1}(s_j + 1))$ for $j = 1, \ldots, r - 1$, and $\text{ca}_r = \text{ca}(\gamma'_r(s_{r+1}), \gamma'_1(s_1))$. In each $C^\infty$ piece of $\gamma$, we have the positive orientation $t$ and the curvature $\nabla_{\epsilon} e = \epsilon k f_1$. Then from (4), (5), Stokes Theorem, Proposition 6 and
$$
f_2 = \frac{1}{f^2(\gamma'_j)} f^1(\gamma'_j) - \frac{f^1(\gamma'_j)}{f^2(\gamma'_j)} f_1 = -\epsilon t - \frac{f^1(\gamma'_j)}{f^2(\gamma'_j)} f_1,
$$
since \( \epsilon = -\frac{|f^2(y'_j)|}{f^2(y'_j)} \), we obtain

\[
\int_U K = \int f(K^1 \wedge f^2 = \int f \, dA \wedge d\alpha(f_1, f_2) f^1 \wedge f^2 = \int f \, dA \wedge d\alpha = \int_{\partial \Delta_i} d\alpha.
\]

\[
= \sum_{j=1}^{r} \int_{s_j} \frac{f^1(j)}{f^2(j)} A(y_j(s)) \, d\alpha(y_j(s)) \, ds
\]

\[
= \sum_{j=1}^{r} \int_{s_j} \left( \epsilon f^2(y'_j) \frac{d}{ds} \frac{f^1(y'_j)}{f^2(y'_j)} \right) \, ds
\]

\[
= \int_{\partial \Delta_i} k \epsilon f^2 - \sum_{j=1}^{r} \left( \frac{f^1(y'_j(s_j+1))}{f^2(y'_j(s_j+1))} - \frac{f^1(y'_j(s_j))}{f^2(y'_j(s_j))} \right)
\]

\[
= -\int_{|\gamma|} k - \sum_{j=1}^{r} \frac{f^1(y'_j(s_j+1))}{f^2(y'_j(s_j+1))} + \sum_{j=0}^{r-1} \frac{f^1(y'_j(s_j))}{f^2(y'_j(s_j))}
\]

\[
= -\int_{|\gamma|} k + \sum_{j=1}^{r-1} \frac{f^1(y'_j+1(s_j+1))}{f^2(y'_j+1(s_j+1))} - \frac{f^1(y'_j(s_j+1))}{f^2(y'_j(s_j+1))}
\]

\[
+ \frac{f^1(y'_j(s_1))}{f^2(y'_j(s_1))} - \frac{f^1(y'_j(s_{r+1}))}{f^2(y'_j(s_{r+1}))}
\]

\[
= -\int_{|\gamma|} k + \sum_{j=1}^{r-1} \text{ca}(y'_j(s_j+1), y'_j(s_j+1)) + \text{ca}(y'(s_1), y'(s_{r+1}))
\]

\[
= -\int_{|\gamma|} k - \sum_{j=1}^{r} \text{ca}_j.
\]

\[
\square
\]

If the surface \( S \) is compact and oriented, then there exists a characteristic non-null vector field on \( S \), and therefore, \( S \) is diffeomorphic to a torus. In this case, we obtain the corollary:

**Corollary 3** Suppose \( S \) is a differentiable compact surface in \( \mathbb{H}^1 \) with \( \Sigma = \emptyset \). Then

\[
\int_S K = 0.
\]

**Proof** In fact, we can triangulate \( S \) with a finite number of triangles \( \Delta_i, i = 1, \ldots, s \), such that the boundary of each \( \Delta_i \) is composed of transverse curves. As the triangles are positively oriented,

\[
\int_S K = \sum_{i=1}^{s} \int_{\Delta_i} K = -\sum_{i=1}^{s} \int_{\partial \Delta_i} k - \sum_{i=1}^{s} \sum_{r=1}^{3} \text{ca}_{ir},
\]

where \( \partial \Delta_i \) is the boundary of \( \Delta_i \) positively oriented, and \( \text{ca}_{ir}, r \in \{1, 2, 3\}, \) are the corner areas at each vertex of \( \Delta_i \). If \( \Delta_i \) and \( \Delta_l \) have sides \( \Delta_{ilu} \) and \( \Delta_{ilv} \) in common, they have opposite orientations, so \( \int_{\Delta_{ilu}} k + \int_{\Delta_{ilv}} k = 0 \); therefore, \( \sum_{i=1}^{s} \int_{\partial \Delta_i} k = 0 \). In the same way, at a common vertex, the corner areas sum to zero, so \( \sum_{i=1}^{s} \sum_{r=1}^{3} \text{ca}_{ir} = 0 \), and the corollary is proved. \( \square \)
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