An Experimental Study of Formula Embeddings for Automated Theorem Proving in First-Order Logic

Ibrahim Abdelaziz†∗, Veronika Thost‡†, Maxwell Crouse§, Achille Fokoue†
†IBM Research
‡MIT-IBM Watson AI Lab
§Northwestern University, Department of Computer Science

Abstract
Automated theorem proving in first-order logic is an active research area which is successfully supported by machine learning. While there have been various proposals for encoding logical formulas into numerical vectors – from simple strings to much more involved graph-based embeddings –, little is known about how these different encodings compare. In this paper, we study and experimentally compare pattern-based embeddings that are applied in current systems with popular graph-based encodings, most of which have not been considered in the theorem proving context before. Our experiments show that some graph-based encodings help finding much shorter proofs and may yield better performance in terms of number of completed proofs. However, as expected, a detailed analysis shows the trade-offs in terms of runtime.

1 Introduction
First-order logic (FOL) theorem proving is important in many application domains. State-of-the-art automated theorem provers (ATPs) excel at finding complex proofs in restricted domains, but they have difficulty when reasoning in broader contexts; for example, with common sense knowledge and domains, but they have difficulty when reasoning in broader contexts; for example, with common sense knowledge and large mathematical libraries. Recently, the latter have become available in the form of logical theories (i.e., collections of axioms), and thus for reasoning [Grabowski et al., 2010]. The challenge is now to extend traditional automated theorem provers to cope with the computational challenges inherent to reasoning at scale.

The ATP problem is as follows: given a set of axioms, formulas known to be true, and a conjecture formula, the goal is to find a proof, if existent, that shows that the latter is true as well. Classical algorithms for automated theorem proving usually rely on custom, manually designed heuristics based on analyses of formulas [Sekar et al., 2001]. Several machine-learning based techniques have been shown recently to outperform or achieve competitive performance when compared to traditional heuristic-based methods [Alama et al., 2014], but they still depend on carefully selected manual features. Current research therefore focuses on the development of neural approaches [Bansal et al., 2019; Chvalovský et al., 2019; Crouse et al., 2019b], which are independent of the latter; but, next to the hardness of the proof search, the highly structural and semantic nature of logical formulas makes this development challenging.

The formula representations proposed in literature vary greatly: from rather simple approaches based on strings or sub-terms [Alemi et al., 2016], over more complex patterns [Jakubuv and Urban, 2017; Crouse et al., 2019b], to encodings based on Tree LSTMs [Loos et al., 2017] and graph neural networks [Crouse et al., 2019a; Palwali et al., 2019]. The different encodings also come with different advantages and disadvantages. Consider, the following example formula: \( \forall A, B, C. \ r(A, B) \land (p(A) \lor \neg q(B, f(A)) \lor q(C, f(A))) \), which is already in conjunctive normal form (CNF). The most simple encoding approaches consider the formula as a sequence of characters and use one-hot encodings followed by standard sequence encodings like LSTMs [Alemi et al., 2016]. This encoding does not capture much logical information, not even syntactically. For example, the fact that \( f(A) \) occurs twice in different contexts represents two different constraints on the interpretation of \( A \), which should be reflected in the embedding of \( A \). [Alemi et al., 2016] therefore also propose a word-level encoding based on an iterative combination of the former embeddings. Still, simple logical properties like the commutativity of \( \lor \), meaning that the order of the literals \( \neg q(B, f(A)) \) and \( q(C, f(A)) \) is not relevant, cannot be captured by a sequence-based approach.

As is shown in Figure 1 (left), the formula and its subexpressions are actually trees, and subsequent works [Jakubuv and Urban, 2017; Chvalovský et al., 2019; Crouse et al., 2019b] have taken this into account by developing patterns able to capture such structures. For instance, directed node paths oriented from the root, of length 3, and where variables are replaced by a placeholder symbol \( * \), called term walks, are used as features in [Jakubuv and Urban, 2017; Chvalovský et al., 2019]. More specifically, the embedding vector contains the counts of the feature occurrences so that, for our example, it would contain a 2 for the path \( (q, f, *) \) from \( q \) to \( A \). Also Tree LSTMs can be used for encoding the parse tree and they even allow for capturing it entirely, but they still only focus on the syntactic structure.

For the actual logical interpretation (i.e., the seman-
and to evaluate the usefulness in the context of one exemplary coding of FOL formulas for automated theorem proving.

Our results may influence and guide future development in areas which are rather unexpected. The latest developments for representing logical formulas as vectors have revolved around graph neural networks [Wang et al., 2017; Paliwal et al., 2019; Olšák et al., 2019]. These networks are appealing in the automated theorem proving domain because of the inherent graph structure of logical formulas and the potential for such neural representations to require less expert knowledge to achieve results than more traditional hand-crafted representations. Thus far, they have been applied in both offline tasks [Wang et al., 2017; Crouse et al., 2019a], and online theorem proving [Paliwal et al., 2019; Olšák et al., 2019]. However, [Paliwal et al., 2019] focus on higher-order logic formulas where the corresponding graphs are very different from those for FOL. The latter were only considered very recently in [Olšák et al., 2019] for the first time. [Wang et al., 2017] focus on a graph representation that only slightly extends parse trees by shared constant and vari-

The paper is organized as follows. Section 2 gives an overview of related work, TRAIL is described in Section 3. The embeddings we compare are described in Sections 4 and 5, and the evaluation in Section 6. We assume the reader to be familiar with first-order logic and related concepts; see, e.g., [Taylor and Paul, 1999] for an introduction.

2 Related Work

Most machine learning enhanced large-theory ATP systems extract symbol and structure-based features from their input formulas (e.g., the depth or symbol count of a clause) in addition to hand-designed features known to be relevant to proof search (e.g., the age of a clause, referring to the time point when it was generated). While symbol-based features for a formula are generally just the multiset of symbols found in that formula, structure-based features vary in their design and implementation. Earlier large-theory ATP systems like Flyspeck [Kaliszyk and Urban, 2012] and MaLARea [Urban et al., 2008] derived their features from the multisets of terms and sub-terms present in the formulas. These subsequently inspired the development of term walks (see Section 4) found in Mash [Kühlwein et al., 2013] and Enigma [Jakubuv and Urban, 2017; Chvalovský et al., 2019]. In [Kaliszyk et al., 2015], pattern-based features were introduced from discrimination and substitution trees that could capture the notion of a term matching, unifying, or generalizing another term. In a similar vein, pattern-based features are also used in TRAIL, where the features extracted for a literal were argument-order preserving templates generated from the complete set of paths between the root and each leaf of the literal.

Recently, deep learning methods have demonstrated viability in the setting of real-time theorem proving, with the work of [Loos et al., 2017] being the first to show how a deep neural network could be incorporated into an ATP without incurring insurmountable computational overhead. Since then, there has been a flurry of activity [Chvalovský et al., 2019; Bansal et al., 2019; Olšák et al., 2019; Crouse et al., 2019b] surrounding the applicability of neural networks in the ATP domain. One of the main goals of these approaches is to also learn the formula embeddings in a fully automated way.

The latest developments for representing logical formulas as vectors have revolved around graph neural networks [Wang et al., 2017; Paliwal et al., 2019; Olšák et al., 2019]. The goal is to shed light on the differences between them and to evaluate the usefulness in the context of one exemplary neural ATP; we use the TRAIL system [Crouse et al., 2019b].
able names, while [Crouse et al., 2019a] extend them by special edges for quantification and subexpression sharing (see also Figure 1 (right))1, and [Olšák et al., 2019] propose a special hypergraph representation. All the works use variants of message-massing neural networks [Gilmer et al., 2017] to learn and finally obtain a single numerical vector representation for each formula.

As of yet, little is known about the trade-offs between each of the aforementioned vectorization strategies as they would be used in a neural-guided theorem prover. This is in part due to some features not well lending themselves to the neural-guided theorem proving setting (e.g., features defined for full terms would be far too sparse, which is why recent approaches apply feature hashing [Chvalovský et al., 2019]). The work of [Kaliszyk et al., 2015] provides an extensive comparative analysis of various non-neural vectorization approaches, however, their evaluation focuses on sparse learners and it evaluates features in the setting of offline premise selection (measuring both theorem prover performance and binary classification accuracy) rather than as part of the internal guidance of an ATP system.

3 The TRAIL Environment

TRAIL [Crouse et al., 2019b] is an automated theorem proving environment in which the proof search is guided by reinforcement learning (RL). The proof search is a sequence of proof steps in which the set of input formulas (i.e., axioms + negated conjecture) is continuously extended by applying an action, which may lead to the derivation of new formulas, and stops if a proof (i.e., a contradiction) is found. An external FOL reasoner (whose proof guidance capabilities are suppressed) is used as the environment in which the learning agent operates. It tracks the state of the search and decides which actions are applicable in a given state. The state encapsulates both the formulas that have been derived or used in the derivation so far and the actions that can be taken by the reasoner at the current step. At each step, this state is passed to the learning agent: an attention-based model [Luong et al., 2015] that predicts a distribution over the actions and uses it to sample one action. This action is then given to the reasoner, which executes it and updates the proof state.

We focus on the representation of the formulas within TRAIL, i.e., in the proof state. All formulas are kept in normalized CNF as sets of clauses, i.e., possibly negated atomic formulas called literals which are connected via ∨. Our example contains the two clauses \((p(A) \lor \neg q(B, f(A)) \lor q(C, f(A)))\) and \(r(A, B)\), and positive and negative literals such as \(p(A)\) and \(\neg q(B, f(A))\). The formula embedding approaches we study transform clauses into numerical vector representations.

4 Pattern-Based Formula Embeddings

Pattern-based formula embeddings are usually based on the parse tree of the formula; see Figure 1 (left) for our example.

We evaluate term walks, which have been used in [Jakubův and Urban, 2017; Goertzel et al., 2018; Chvalovský et al., 2019], and TRAIL patterns, as representatives for patterns that capture entire literals (vs. parts of fixed depth or length).

4.1 Term Walks

As outlined in the introduction, literals in the clauses are considered as trees where all variables and skolem terms are replaced by a special symbol, respectively. Note that the latter helps reflecting structural similarities between literals that are indicative of unifiability. Additionally, a root node is added and labelled by either ⊕ or ⊖, depending on whether the literal appears negated or not. Every directed node path of length 3 in these trees (oriented from the root) represents a feature. For a negative literal such as \(\neg q(B, f(A))\), we would thus count term walks \((\oplus, q, \ast)\), \((\ominus, q, f)\), and \((q, f, \ast)\). The multiset of features for a clause consists of all features of its literals; and the final embedding vector for the clause has the same size as this set, every position is associated to one feature, and contains the multiplicities of the features at the corresponding positions.

4.2 TRAIL Patterns

The idea applied in TRAIL [Crouse et al., 2019b] extends the more simpler patterns of term walks in that the clause embeddings should capture the literals more holistically (i.e., not only patterns of fixed depth), and the relationship between literals and their negations. For example, in the term walks of our example, the first two occurrences of \(A\) are only captured by the term walk \((q, f, \ast)\), so the connection to the contexts is largely lost although it might be rather important since there is a negation in the first one but not in the second. TRAIL patterns captures these features by deconstructing input clauses into sets of TRAIL patterns, where a TRAIL pattern is a chain that begins from a predicate symbol and includes one argument (and its argument position) at each depth level until it reaches a constant or variable. The set of all patterns for a given literal is then simply the set of all linear paths between each predicate and the constants and variables they bottom out with. As with term walks, variables are replaced by a wild-card symbol *, and the latter is similarly used in all argument positions not in focus (i.e., not in the path under consideration). For our example, we obtain patterns such as \(p(\ast), q(\ast, f(\ast)), q(\ast, \ast),\) and \(r(\ast, \ast)\). Note that these patterns do not seem to differ much from term walks, but this changes when considering real-world problem clauses which are often much deeper than our toy example. Again, the set of patterns for a clause consists of all patterns of its literals. A \(d\)-dimensional clause embedding is obtained by hashing the linearization of each pattern \(p\) using MD5 hashes to compute a hash value \(v\), and setting the element at index \(v \mod d\) to the number of occurrences of the pattern \(p\) in the clause under consideration. Further, the difference between patterns and their negations is explicitly encoded by doubling the representation size and hashing them separately, so that the first \(d\) elements encode the patterns of positive literals and the second \(d\) elements encode the negative ones. This hashing approach greatly condenses the representation size compared to one-hot encodings of all patterns.

1The work actually also introduces edges from quantifiers to variables which we do not show in Figure 1 since, in our setting, we encode clauses and thus ignore these edges.
5 Graph-Based Formula Embeddings

As outlined in the introduction, graph-based embeddings of logical formulas seem to better suit their actual semantics. However, such embeddings have been considered in the ATP context only very recently and focusing on the subtask of premise selection [Crouse et al., 2019a; Olsák et al., 2019]. We consider the approach from [Crouse et al., 2019a], which focuses on graphs as presented in Figure 1 (right) (i.e., parse trees extended by subexpression sharing), applies message-passing neural networks (MPNNs), and proposes a pooling technique to obtain the final graph encoding that has been specifically designed for formula graphs. In addition, we evaluate variants of the simpler, but equally popular, graph convolutional neural networks (GCNs) in the ATP context. For the latter, we also use a relatively simple graph representation of formulas, depicted in Figure 2, which only slightly extends the parse trees: in that variable and constant names are shared as suggested in [Wang et al., 2017] (vs. arbitrary subexpressions); note that we introduce name nodes for this.

5.1 GCNs

The originally proposed graph convolutional neural networks (GCNs) [Kipf and Welling, 2017] compute node embeddings by iteratively aggregating the embeddings of neighbor nodes, and a graph embedding (i.e., a clause embedding) is obtained by a subsequent pooling operation, like max or min pooling. The node embedding for a node $u$ is formalized as follows, assuming initial node embeddings $h_u^0$ are given:

$$h_u^{t+1} = \rho \left( \sum_{v \in \mathcal{N}_u} c_u W^t v \cdot h_v^t \right)$$

where $\mathcal{N}_u$ is the set of neighbors of node $u$, $c_u$ is a normalization constant, $W^t$ is a learnable weight matrix, and $\rho$ is a non-linear activation function.

The initial node embedding can be obtained in various ways, and arbitrary initialization represents a common and easy solution. We additionally experimented with bag-of-character features (BoC), extracted without using any learning; and character features learned via a character convolutional neural network [Zhang et al., 2015]. The idea behind these embeddings is to consider the names of the symbols in addition to the structural features of the formulas, which are encoded by the GCN. Moreover, we do not want to rely upon a fixed token vocabulary, but to instead capture the overall shape of symbols which sometimes may encode important characteristics (e.g., in many datasets, variables or function symbols start with a fixed letter which is numbered). However, since the results for the character convolutional neural network turned out to be not competitive at all, we will omit them in our analysis later.

5.2 Relational GCNs

Relational GCNs (R-GCNs) [Schlichtkrull et al., 2018] extend GCNs in that they distinguish different types of relations for computing node embeddings. Specifically, they learn different weight matrices for each edge type in the graph:

$$h_u^{t+1} = \rho \left( \sum_{r \in R} \sum_{v \in \mathcal{N}_{u,r}} c_{u,r} W_{r}^t h_v^t \right)$$

Here, $R$ is the set of edge types; $\mathcal{N}_{u,r}$ is the set of neighbors connected to node $u$ through the edge type $r$; $c_{u,r}$ is a normalization constant; $W_{r}^t$ are the learnable weight matrices, one per $r \in R$; and $\rho$ is a non-linear activation function.

5.3 MPNNs

Message-passing graph neural networks (MPNNs) [Gilmer et al., 2017] extend GCNs in that the aggregation of information from the local neighborhood includes edge embeddings:

$$m_u^{t+1} = \sum_{v \in \mathcal{N}_u} F_{r}^t (h_u^t; h_v^t; c_{uv})$$

$$h_u^{t+1} = h_u^t + F_{U}^t ([h_u^t; m_u^{t+1}])$$

$F_{r}^t$ and $F_{U}^t$ are feed-forward neural networks and $[\cdot; \cdot; \cdot]$ denotes vector concatenation. The initial node embeddings $h_u^0$ and the edge embeddings $c_{uv}$ are assumed to be given for all nodes $u$ and $v$. The vectors $m_u^t$ are messages to be passed to $h_u$. As above, all the node embeddings from the last iteration are passed through a subsequent pooling layer, which computes the embedding for the whole graph.

5.4 DAG LSTM Pooling

In order to incorporate more of the information in the formula into its embedding, [Crouse et al., 2019a] suggest to combine MPNNs with a pooling based on DAG LSTMs. These LSTMs aggregate node and edge embeddings using techniques from LSTMs: input gates to decide which information is updated; tanh layers for creating the candidate updates; memory cells for storing intermediate computations; forget gates modulating the flow of information from individual arguments into a node’s computed state; and output gates similar modulating the flow of information, but on a higher level. Given initial node embeddings from the MPNN, and edge embeddings, new node embeddings are computed in topological order, starting from the leaves, as depicted in Figure 3. In this way, node embeddings are directly pooled and the embedding of the clause’s root node represents the clause embedding. For lack of space, we refer to the original paper for the exact computing equations.
6 Evaluation

In this section, we report the evaluation results of the embedding approaches described above. We also compare these approaches against Beagle [Baumgartner et al., 2015] (i.e., used inside TRAIL w/o suppressing its proving capabilities); a theorem proving system using manually designed heuristics which provides competitive performance on ATP datasets.

6.1 Network Configurations and Training

We generally constructed embedding vectors of size 64 per clause with all approaches. The most important parameters for the individual approaches are described below.

For the GCNs, we used the parameters suggested in [Kipf and Welling, 2017] (see Eq. (2) and Sec. 3.1): (symmetric) normalized Laplacian as a normalization constant, ReLU activation, two convolutional layers in total, and Xavier initialization for the weights. The output is passed through an additional linear layer and then pooled using summation, as suggested in [Xu et al., 2019]. We consider one GCN with arbitrary initial node embeddings (denoted GCN in the tables) and one based on initial bag-of-character embeddings (BoGCN). The R-GCN implementation differs from the GCN in that we consider three edge types: edges to name nodes, edges from commutative operators to operands, all others.

The MPNN configurations were taken from [Crouse et al., 2019a]. We considered one MPNN with max-pooling (MPNN) and one using the DAG LSTM (GLSTM-MPNN).

All our models were constructed in PyTorch\(^3\) and trained with the Adam Optimizer [Kingma and Ba, 2014] with default settings. The loss function optimized for was binary cross-entropy. We trained each model for 5 epochs per iteration on all datasets. Validation performance was measured after each epoch and the final model used for the test data was then the version from the epoch with best performance.

6.2 Datasets and Experimental Setup

We considered the standard Mizar\(^3\) [Grabowski et al., 2010] and the Thousands of Problems for Theorem Provers (TPTP)\(^4\)

datasets. Mizar is a well-known and large mathematical library of formalized and mechanically-verified mathematical problems. TPTP is the definitive benchmarking library for theorem provers, designed to test ATP performance across a wide range of problem domains. From each dataset, 500 problems were drawn randomly. We used a 50/15/35 split for train/valid/test, and set a time limit of 100 seconds per problem solving attempt for each vectorization approach, thereafter the proof attempt was stopped and the problem declared unsolved. For training, we ran the models for 30 iterations over the training sets.

We consider the metrics: 1) completion rate: proportion of problems solved within the specified time limit; 2) average proof length relative to Beagle: number of steps of Beagle number of steps taken to find a proof (a value greater than one indicates a shorter proof compared to Beagle’s); 3) runtimes: for different phases of problem solving; and 4) useless steps: percentage of proof steps taken that generated inferred facts not used directly in the derivation of the final contradiction.

6.3 Results and Discussion

Completion Rate. Table 1 shows that there is a relatively large gap between the completion rate of Beagle and the approaches we evaluate. Note that this gap is larger than in other ATP evaluations, however, many of the latter include symbol or structure-based features (while we solely consider the encodings learned) or compare to ATPs instead of manually-designed systems. The numbers on TPTP are smaller than those on Mizar, but the overall order of results does not change greatly. Specifically, the TRAIL patterns, which capture the formulas more holistically than term walks turn out to largely outperform the latter. The GCN variants, which are combined with a more simple graph representation of formulas, generally perform worse that the pattern-based encodings. On the other hand, the message-passing neural networks outperform the term walks. In our experiments, the GLSTM pooling does not provide benefits over a standard max-pooling with the message-passing neural network. This confirms the results from [Crouse et al., 2019a] for FOL. MPNN slightly outperforms the TRAIL patterns on Mizar, but it is the opposite on TPTP.

Proof Length. We also report in Table 1 the average proof length of each approach since it is an indicator for efficiency.

![Figure 3: Update staging in graph-based LSTM. Identically colored nodes are updated simultaneously, starting at leaves.](https://example.com/figure3.png)

Table 1: Performance of pattern and GNN-based encodings in terms of completion rate and proof length improvement relative to Beagle.
|                  | Min | Med | Max |                  | Min | Med | Max |                  | Useless Steps (%) |
|------------------|-----|-----|-----|------------------|-----|-----|-----|------------------|-------------------|
| Term Walks       | 0.00| 0.01| 0.51| Action Selection | 0.00| 0.74| 9.98| Reasoning        | 0.18              | 28.3              |
| TRAIL Patterns   | 0.10| 0.04| 2.53|                  | 0.18| 4.32| 71.51|                |                   | 36.5              |
| GCN              | 0.02| 0.68| 78.40|                  | 0.08| 1.31| 8.94|                |                   | 28.5              |
| BoC-GCN          | 0.03| 1.43| 69.65|                  | 0.14| 2.87| 27.27|                |                   | 27.1              |
| R-GCN            | 0.02| 1.05| 79.24|                  | 0.07| 1.39| 40.74|                |                   | 27.9              |
| MPNN             | 0.02| 1.50| 78.81|                  | 0.09| 1.73| 55.65|                |                   | 31.4              |
| GLSTM-MPNN       | 0.06| 2.65| 81.39|                  | 0.10| 2.00| 48.72|                |                   | 27.5              |

Table 2: Average time spent per problem on Mizar for each phase. Minimal and maximal median numbers are in bold, best in completion rate are in gray.

While the length of proofs found should not be considered as a criterion as important as completion rate, it may influence the choice of embedding in specific application scenarios and represents an interesting alternative metric demonstrating that ATP evaluation results may generally change greatly with different measures. On Mizar, most graph-based encodings lead to much shorter proofs than their pattern-based counterparts (for the problems they could solve) with GCNs finding the shortest proofs (confirmed by further experiments) followed by MPNN and others; larger values indicate shorter proofs.

**Runtime.** Table 2 shows the average runtimes for the three main phases of proving a problem: 1) vectorization: time spent in encoding the logical formulas, 2) action selection: time spent in evaluating the RL policy network for selecting the next action, and 3) reasoning: time spent in executing the selected action and producing the next system state. As expected, pattern-based encoders require less time on average than graph-based approaches for encoding the logical formulas and evaluating the policy network, which allows them to spend more time on reasoning. However, TRAIL patterns seem to lead to more useless steps. The otherwise less performant approaches (term walks, GCNs, GLSTM) are better in this regard but hardly a good comparison since they likely solve only simpler problems. Thus, MPNN, which shows similar performance, can be top-rated.

Figure 4 gives an overview of how the completion rate changes with increasing time limit with selected models, using the originally trained models. This confirms the above findings in that this is beneficial primarily with TRAIL patterns, where most time is spent for reasoning.

**Discussion.** The goal of ATP research is to obviate the need of hand-crafted features and patterns, and recent advances seem to agree that GNNs are the way to go next. Generally, our experiments have confirmed the direction taken by the first existing ATP works on GNNs, which focus on MPNNs instead of GCNs. Our MPNNs lack performance in terms of runtime, but the implementations are not fully optimized so that there is indeed room for improvement in this regard.

Nevertheless, to us, it came as surprise that the hand-crafted but rather simple patterns are still competitive given well-known graph neural network embeddings. While the former are very performant in terms of encoding time, they also partially capture properties like argument order and unifiability. We have shown that an increased time limit can further improve their performance, meaning that the MPNNs fall behind. Automated theorem proving is a particularly hard application domain for GNNs and our analysis has shown that straightforward graph representations of formulas are not sufficient to achieve good performance in general. Very recently, there has been an effort to encode more of the logic into the graphs and correspondingly adapt the GNN [Olšák et al., 2019]. Although the latter evaluation only compares to one other ATP system, we believe that this kind of encodings is the direction to take and needs further investigations.

7 Conclusions

So far, little was known about the trade-offs between the different embedding strategies for FOL formulas in automated theorem proving. In this paper, we presented an experimental study comparing the performance of various such strategies in the context of the TRAIL system. We implemented two pattern-based approaches and several variants of graph convolutional and message-passing neural networks, and thus considered a representative set of several popular and recent standard graph embedding methods varying in complexity. As usual, we evaluated them in terms of completion rate, but we also presented a detailed analysis of runtime as well as insights regarding the lengths of the proofs found. Future research has to show if more involved graph embeddings, beyond the standard approaches we focused on and which specifically integrate logical properties, will finally help to obviate the need for hand-crafted features and patterns.
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