String quantization:
Fock vs. LQG Representations

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Abstract. We set up a unified framework to compare the quantization of the bosonic string in two approaches: One proposed by Thiemann, based on methods of loop quantum gravity, and the other using the usual Fock space quantization. Both yield a diffeomorphism invariant quantum theory. We discuss why there is no central charge in Thiemann’s approach but a discontinuity characteristic for the loop approach to diffeomorphism invariant theories. Then we show the (un)physical consequences of this discontinuity in the example of the harmonic oscillators such as an unbounded energy spectrum. On the other hand, in the continuous Fock representation, the unitary operators for the diffeomorphisms have to be constructed using the method of Gupta and Bleuler representing the diffeomorphism group up to a phase given by the usual central charge.

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1. Introduction

The most challenging problem in theoretical physics is to find a consistent theory that encompasses General Relativity and quantum physics. Currently, there are two major programs to attack this challenge, string theory and loop quantum gravity. While string theory is rooted in the high energy and particle physics tradition with focus on scattering data and perturbative methods, loop quantum gravity’s emphasis is more on geometrical properties and diffeomorphism and background independence due to its foundations in relativity.

With some notable exceptions, there has not been much interchange of ideas between the two approaches which is partly due to the very different mathematical languages used by the two camps. However recently, there has been a conference at the Albert-Einstein-Institute to bridge this cultural gap and as an effect of this meeting, Thiemann[1] has investigated the bosonic string world-sheet theory using tools of canonical quantization typical of loop quantum gravity.

The world-sheet theory of a string seems to be the ideal toy model and testing ground for approaches to quantum gravity: It is a proper field theory with infinitely many degrees of freedom and it is (at least for the classical theory) reparametrisation and therefore diffeomorphism invariant. Thus it fits into the context of theories to which loop quantum gravity methods apply. Yet, it allows for the “conventional” field theory treatment with Fock space operators. In the string theory literature there are several equivalent formalisms (e.g. light-cone, covariant or Gupta-Bleuler, BRST covariant, see for example [2]) that all lead to the same result: The quantized theory is only consistent in the critical dimension, which is 26 for the bosonic string. Otherwise, Lorentz- or conformal invariance are anomalous or the BRST algebra does not close.

In his paper, Thiemann describes a quantization of bosonic closed string theory that seems to work in any dimension. There are no anomalies and thus no critical dimension. This is surprising from a string theory point of view and it is the aim of this note to shed some light on this result and what went into it.

Thiemann’s treatment deviates in two ways from what is usually found in string theory textbooks: First, there is a different formalism. Instead of unbounded operators like $q$ and $p$ or annihilation and creation operators, Weyl type operators $e^{iq}$ and $e^{ip}$ are used because they are bounded and thus continuous in the Hilbert space sense. Furthermore, this implies that one deals with finite group transformations and not only infinitesimal generators. In addition, care is taken to separate the algebra of observables from its representations as operators on a concrete Hilbert space. This is achieved by first constructing the quantum algebra and then employing the GNS (Gelfand, Naimark, Segal) construction. We should stress that so far the physical content does not differ from the standard treatment except for increased mathematical rigour than usually encountered in physicists’ dealings with functional analysis.

The second difference however is substantial: At some point in the quantization procedure one has to choose an expectation value functional or “state”. While string
theorists usually choose (in most cases implicitly) the state corresponding to the Fock-space vacuum, Thiemann chooses a different, manifestly diffeomorphism invariant vacuum as it is typical for the loop quantum gravity literature. In a similar context it is in fact often argued that this choice is unique if one insists on diffeomorphism invariance. However this so-called “polymer” state has some physically unfavourable properties: Technically speaking, the representation is not (weakly) continuous and practically the state is so singular that for example no momentum operators exist.

Indeed, the Fock-space vacuum as it is is not invariant under the diffeomorphisms obtained from the quantization procedure. However, what is technically required is not an invariant state but only unitary operators on the Hilbert space that implement the action of the diffeomorphisms on the algebra. In a second step (that has to be taken in both approaches), one mods out the Hilbert space by the action of the now unitarily implemented diffeomorphism group to obtain the physical Hilbert space of invariant vectors. We will demonstrate how to construct the unitary operators for the Fock representation in this formalism. In effect, this is done by introducing normal ordering and following the method of Gupta and Bleuler but the presentation will differ significantly from the (often ad hoc) textbook treatment.

The price to pay for this construction of unitary implementers of the diffeomorphisms is that they obey the group relations of the diffeomorphism group only up to a phase. This is the manifestation of the central charge in this formalism. Modding out these unitarities including the phase would lead to an empty physical Hilbert space. To circumvent this problem, we can however use the standard procedure: We will take 26 copies of this theory with central charge one and add to it the theory of a $bc$-ghost system with central charge $-26$ so that in total we end up with a theory without anomalies. Thus, a consistent string theory based on the (continuous) Fock space representation of the quantum algebra has a critical dimension of 26.

In the end we will have learnt not very much about the usual string theorists treatment of string quantization. We will have merely reformulated well known results in a more precise formalism for quantization maybe shedding some new lights on the inner workings of the Gupta-Bleuler procedure; especially, we show that in the end, both the positive and negative energy parts of the generators of symmetry transformation are promoted to unitary operators. However, we can turn Thiemann’s argument around: This paper demonstrates that string theory in the critical dimension in fact provides a highly non-trivial example of a diffeomorphism invariant quantum theory that doesn’t have the unphysical properties of the quite singular “polymer” states usually encountered in the loop quantum gravity literature.

To highlight the physical features of these polymer states we then discuss the example of the one-dimensional harmonic oscillator: In direct parallel to the case of the bosonic string (which in the end is just an infinite collection of harmonic oscillators) there is the standard Fock space quantization and an inequivalent one, based on the singular polymer state. We find that in the polymer representation only the ground state is stationary, all other states correspond to scattering states. Furthermore, all
other states have contributions of arbitrarily positive and negative energies. This is in direct conflict with the spectral properties of experimentally realised harmonic oscillators showing that at least in the case of quantum mechanics the polymer state is unphysical. As there are no quantum gravity or string theory experiments, the question of which state for the quantum algebra might be realised if at all in nature cannot be decided, but these results for analogous quantum mechanical systems are quite suggestive of what properties (such as weak continuity) one might require in the quantization. These non-regular states however have an application in the treatment of Bloch electrons, see for example [3]. In a periodic potential, the wave function has to be periodic up to a phase. The total Hilbert space is then an orthogonal sum over all possible phases and thus non-separable.

The structure of this paper is as follows: In the next section we reformulate the quantization of the bosonic string in a language that will allow us to compare the similarities and differences of the two approaches. This includes a mathematical formulation of the Gupta Bleuler construction of positive energy representations. Then follows a chapter in which the same quantization procedure is applied to the one dimensional case of the harmonic oscillator (the reader might occasionally want to peek forward to this section as an illustration of the formalism for the string). In this quantum mechanical example we demonstrate that the two quantizations differ observationally‡ in their energy spectrum and thus the Fock space quantization is clearly favoured experimentally. In a final chapter we wrap up with some conclusions.

A final note to the mathematically cautious reader: Although we will here probably employ a higher than usual standard of rigour, our treatment will be purely algebraic. We will not discuss for example convergence of sums that we write down (and most of the time suppress necessary completions of infinite dimensional spaces) in order not to burden the reader with too many notational details. However we are positive that the missing details could be filled in without too much work, for a treatment that contains these details we refer the reader to [6].

2. Canonical quantization of the string

Here we would like to carry out the canonical quantization of the closed bosonic string in all detail.

Ideally, one would like to quantize the theory of unparametrised strings, that is embeddings of $S^1 \times \mathbb{R}$ into target space. However this leads to a much more involved theory (see for example [7]) and we will instead quantize the theory of parametrised strings, that is functions $X(x, \tau)$ and then in the end impose the constraint that physics is invariant of the parametrisation we have chosen on the world-she et. Thus we will deal with a gauge system with unphysical degrees of freedom.

We will only be concerned with strings in flat target space. As in this case all the dimensions of the target decouple we can for the time being just treat an individual

‡ See also the treatment in [5] that comes to a different conclusion.
target coordinate and thus we do not need any target space indices on $X$. Our convention for the size of the closed string follows from

$$X(x + 2\pi, \tau) = X(x, \tau).$$

We will take the Polyakov action

$$S = \int d\tau \int_0^{2\pi} dx \left( -\partial_\tau X \partial_\tau X + \partial_x X \partial_x X \right)$$

as our starting point although this is not essential ([1] considers the Nambu-Goto action as well and ends up with the same symplectic space). The canonical variables $X(x,0)$ and $\partial_\tau X(x,0)$ have the Poisson-bracket

$$\{X(x,0), \partial_\tau X(x',0)\} = \delta(x - x').$$

As $X$ is periodic, all its information is also carried by the currents

$$j^\pm(x) = \partial_\tau X(x,0) \pm \partial_x X(x,0).$$

Their Poisson-brackets

$$\{j^+(x), j^+(x')\} = -\{j^-(x), j^-(x')\} = 2\partial_x \delta(x - x'),
\{j^+(x), j^-(x')\} = 0$$

decouple and therefore we can consider the $+$ and the $-$ components independently. Thus from now we will drop the $\pm$ indices. Finally, in order not to have to worry about distributions, we proceed from the currents to their test-function duals, that is functions $f: S^1 \rightarrow \mathbb{R}$ which we use to smear the currents:

$$f \mapsto j[f] = \frac{1}{\sqrt{2}} \int_0^{2\pi} dx j(x)f(x).$$

This mapping then induces the symplectic structure

$$\sigma(f,g) = \{j[f], j[g]\} = \int_0^{2\pi} dx f(x)\partial_x g(x) = \int f dg. \quad (1)$$

Finally, by going to the rest-frame in target-space, we can assume $f$ to average to zero:

$$\int_0^{2\pi} dx f(x) = 0.$$

We wish then to quantize the space $M$ of real functions on the circle with zero mean and the symplectic structure given by (1). We will not need to specify a basis for this space. However, most of the literature uses a language that corresponds to a choice of basis. Often a Fourier decomposition that is equivalent to a basis of the form $(e^{inx})_{n \in \mathbb{Z}}$ is used. Thiemann however chooses to take characteristic functions of intervals in $S^1$ as his basis as these bear some similarity with holonomy functionals in the loop approach to quantum gravity §. Of course, none of the results depends on the choice of basis.

§ Note however that characteristic functions are discontinuous and the symplectic form (1) is not well-defined on them.
Given any symplectic space \((M, \sigma)\), observables \(F: M \to \mathbb{R}\) are in one-to-one correspondence to Hamiltonian vector fields \(X_F\) via
\[
dF = \sigma(X_F, \cdot).
\]
Already at the classical level the vector fields do not commute, rather they form a Lie algebra
\[
[X_F, X_G] = X_{\{F,G\}}.
\]
To avoid complications with unbounded operators later in the quantum theory, it convenient at this level not to deal with infinitesimal symplectic transformations as given by Hamilton vector fields but with their flows, i.e. finite symplectomorphisms
\[
W(F) = \exp(\text{ad}_{X_F}) = e^{\{F, \cdot\}}.
\]
If \(F\) and \(G\) are canonical coordinates (that is, their Poisson-bracket is a constant function, such as for \(x\) and \(p\)) these maps can be composed as
\[
W(F) \circ W(G) = e^{\frac{\sqrt{2}}{2} \sigma(F,G)} W(F + G)
\]
with the help of the Campbell-Baker-Hausdorff formula. This algebra is easily recognised as the algebra obeyed by Weyl operators. Indeed, quantization of our classical system is the promotion of this classical algebra to a quantum algebra. With our \(f: S^1 \to \mathbb{R}\) as above, the quantum algebra \(\mathcal{W}\) is generated by elements \(W(f)\) with relations
\[
W(f)W(g) = e^{\frac{\sqrt{2}}{2} \sigma(f,g)} W(f + g)
\]
and conjugation \(W(f)^* = W(-f)\) and can be checked to be a \(C^*\)-algebra, the algebra of canonical commutation relations. There is a unique norm on this algebra but we do not use it explicitly in this paper.

It is important to realise that although the classical symplectic maps \(W(F)\) exist for all functions on phase space, at first we promote only the elementary ones (i.e. linear functionals on \(M\); in the case of mechanics, these are \(x\) and \(p\) but not powers of them) to quantum operators that obey the Weyl algebra (3).

Noether’s theorem says that classically, all one-parameter groups of symplectomorphisms are inner, i.e. they can be written as \(W(F)\) where \(F\) is the related conserved charge. However this does not automatically lead to an inner transformation in the quantum theory. This is because there is not necessarily an operator that represents \(F\) on the Hilbert space as in general \(F\) is not one of the elementary observables for which we defined quantum operators in the beginning. In fact, the rest of this section is concerned with constructing such operators for two Hilbert space realisations of the quantum algebra (3).

As in Lie algebra theory where one first studies the abstract algebras and then in a second step their representations in order to separate properties of the algebra from properties of specific representations, the same can be done in quantum theory: First we analyse the abstract \(C^*\)-algebra of observables and only then study its representations in terms of linear operators on a Hilbert space. This split is usually not considered
necessary for quantum mechanics (and therefore omitted in most textbooks) as the Stone-von Neumann theorem guaranties that for the Weyl algebra of a finite number of degrees of freedom the usual Schrödinger representation is the only one possible. We will however later have to reconsider this short-cut as one might want to drop one of the assumptions of this theorem (see section 3). The situation is however different in quantum field theory where the C*-algebras involved admit inequivalent representations, giving rise to the theory of superselection sectors.

In the case of the bosonic string, again at the level of abstract C*-algebras, the usual textbook treatment and the approach of [1] agree, it is only at the level of representations where they differ. As we will see later, this difference could already be made for quantum mechanical systems such as the one-dimensional harmonic oscillator.

It is a classic result by Gelfand, Naimark, and Segal (for an extended discussion see for example [8]) that representations of C*-algebras \( \mathcal{A} \) arise from states via the construction named after them: A state is a linear functional \( \omega : \mathcal{A} \to \mathbb{C} \) that should be thought of as assigning an expectation value to each observable. One requires that, for an algebra with a unit \( 1 \in \mathcal{A} \), the state is normalised to \( \omega(1) = 1 \), and that it is positive for positive elements of the algebra:

\[
\forall A \in \mathcal{A}: \quad \omega(A^*A) \geq 0.
\]

Given such a state, we can define \( \mathcal{J} = \{ A \in \mathcal{A} | \omega(A^*A) = 0 \} \) (which in the cases we will be interested in can be checked to be trivial, \( \mathcal{J} = \{0\} \)) and verify with the help of the Cauchy-Schwarz inequality that it is an ideal. Then we can define a vector space \( \mathcal{H} = \mathcal{A}/\mathcal{J} \). This carries a representation

\[
\rho(A)|B\rangle = |AB\rangle
\]

where we use \(|B\rangle\) to indicate the class of \( B \) in \( \mathcal{A}/\mathcal{J} \). To make \( \mathcal{H} \) into a Hilbert space we also need a scalar product and this is given by

\[
\langle A|B \rangle = \omega(A^*B).
\]

This however, is not the whole story, as the the algebra of the \( W(f) \) is that of parametrised strings but we want our final quantum theory to be invariant under reparametrisations. So let \( S : S^1 \to S^1 \) be a reparametrisation of the circle. It can be pulled back to the \( f \) via

\[
(Sf)(x) = f(S(x))
\]

and furthermore to the Weyl operators

\[
S : W(f) \mapsto \alpha_S(W(f)) = W(Sf).
\]

From the second form of the symplectic structure (1) it follows that \( S \) induces a symplectomorphism \( \sigma(Sf, Sg) = \sigma(f, g) \). As it respects the Weyl relation (3), \( \alpha_S \) is an automorphism of the quantum algebra. Moreover, diffeomorphism of the circle can be composed and form a group. Obviously, the map \( \alpha : Diff(S^1) \to Aut(W) \) that maps \( S \) to \( \alpha_S \) is a group homomorphism:

\[
\alpha_{S_1} \circ \alpha_{S_2} = \alpha_{S_1 \circ S_2}.
\]
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The crucial task will turn out to be to give a map from the group of C*-algebra automorphisms $\alpha_S$ to the group of unitary operators $U(S)$ on the representation Hilbert space $\mathcal{H}$, such that
\[ U(S)\rho(A)U(S)^{-1} = \rho(\alpha_S(A)) \]
and then mod out by these unitary transformations to obtain the physical Hilbert space. Ideally, one would want the map $U : \alpha_S \to U(S)$ to be a group homomorphism, so that $U(S)U(S') = U(S \circ S')$, but we will see that this property in general can not be satisfied: This will be the place where the central charge appears in this description. But we should note that the central charge is really a property of the representation and not of the quantum algebra!

If in general one has a C*-algebra $A$ and an automorphism $\alpha$ acting on it, there is no canonical way to promote it to a unitary operator on the representations. However, if the state $\omega$ that GNS-constructs this representation is invariant ($\omega(\alpha(A)) = \omega(A)$ for all $A$), then there is a unitary right at hand: We can define $U$ to act as $U|A\rangle = |\alpha(A)\rangle$. This is unitary as invariance of the state induces invariance of the scalar product:
\[ (U|A\rangle)(U|B\rangle) = \langle \alpha(A)| \alpha(B) \rangle = \omega(\alpha(A)^* \alpha(B)) = \omega(A^*B) = \langle A|B\rangle. \]
As $U$ arises from a pull-back of $S$, it preserves the group structure of the symplectomorphisms $S$, that is it is a group homomorphism from the group of $S$’s to the unitary group of $\mathcal{H}$ and thus unitarities constructed this way do not give rise to central charges. However, if the state is not invariant, this construction of unitary implementers is not available and the group property is not automatic.

So far, we have only discussed the abstract algebra and have not yet decided to proceed to the representation theory, that is we have not decided for a (“vacuum”) state or, equivalently, a Hilbert space on which the algebra acts. Up to this point, the only difference between the standard treatment and Thiemann’s is in the language used not in the content. However, the two approaches differ in their choice of state. Thiemann makes this explicit in his paper whereas this choice is only implicit in the usual textbook treatments.

Thiemann chooses a state of “polymer type” similar to the state that is used in LQG, specifically, he takes
\[ \omega_P(W(f)) = \begin{cases} 1 & \text{if } f = 0 \\ 0 & \text{else.} \end{cases} \]
This choice has the obvious advantage of being invariant under diffeomorphisms of the circle as a reparametrisation maps non-zeros functions to non-zero functions. Thus one can apply the above construction for the unitary implementers on the Hilbert space. In fact, in the context of gravity this appears to be the unique diffeomorphism invariant state[9, 1]. However, for an invariant theory, we only need the unitary operators (and their kernels) and an invariant state provides those canonically but as we will see below, in other states they can be defined as well.
On the other side, this state has one unusual property: It is not continuous in the argument $f$! This implies that we cannot take the derivative of the Weyl operators $\rho(W(f))$ to obtain operators for the field ("position operator" in mechanics) or its momentum. The action of the Weyl operators on this Hilbert space is so singular, that the field and the field-momentum cannot be defined. Furthermore, the Hilbert space constructed from $\omega_P$ is huge in the sense that it is not separable. Thus it does not have a (countable) orthonormal system. Rather, its basis is labelled by (the continuum) of functions $f$ and the overlap $\langle W(f)|W(g)\rangle$ vanishes unless $f = g$.

Even in quantum mechanics, (weak) continuity is an assumption of the Stone-von Neumann theorem that guarantees the choice of position and momentum operators to be unique (up to unitary equivalence such as change from position to momentum representation). If that assumption is dropped there are non-standard quantizations of "polymer"-type with non-separable Hilbert-spaces as well and we will study the physical consequences of such a quantization of the harmonic oscillator in section 3.

It should be said however, that this non-separable Hilbert space is only an intermediary. In [10, 11] it is showed that once the constraints have been modded out (at least in the case of 3+1 gravity) the physical Hilbert space is again separable.

The usual Fock representation in contrast comes from a state that is continuous. To define it we need some more input. Namely, as the Fock representation is a positive energy representation (and negative energy modes of all the fields annihilate the vacuum) we have to introduce a way to distinguish positive and negative energy modes. At this point, most textbooks now introduce the Fock vacuum (that is annihilated by negative modes) in an ad hoc way and then later proceed to impose (under the names of Gupta and Bleuler) only the positive energy half of the constraints for "quantum consistency". Here, we will spend some more time on the details of this procedure and show how it fits into the general framework of deformation quantization and GNS-construction. Our treatment follows along the lines of [6].

The distinction between positive and negative modes can be encoded in the definition of a complex structure $J$ for the functions $f$ that turns the symplectic space into the complex one-particle Hilbert space. Specifically, we require $J$ to square to $-1$ and to be skew with respect to the symplectic structure:

$$\sigma(Jf, g) = -\sigma(f, Jg)$$

for all $f$ and $g$. In addition we require

$$\sigma(f, Jf) \geq 0$$

for all $f$. The conventional choice amounts to

$$(Jf)(x) = \frac{1}{2\pi} \int_0^{2\pi} dy f(y) \cot \frac{1}{2}(y - x)$$

where the integral is evaluated using the principal value prescription. This form becomes more familiar in a Fourier basis. Defining

$$f_n = \frac{1}{2\pi} \int_0^{2\pi} dx f(x) e^{-inx}$$

(4)
the complex structure acts as

\[ J : f_n \mapsto \text{sgn}(n) i f_n. \]

So the positive energy modes are in the eigenspace of \( J \) with eigenvalue \(+ i\). Now, we can complexify the space of \( f \)'s and with the scalar product

\[ \langle f | g \rangle = \sigma(f, Jg) + i \sigma(f, g) \] (5)

(which is sesquilinear because of the properties of \( J \)) it becomes the one particle Hilbert space. After all these preparations, second quantization amounts simply to give the state that corresponds to the Fock vacuum:

\[ \omega_F(W(f)) = e^{-\frac{1}{4} \langle f | f \rangle}. \] (6)

Obviously, this choice is continuous in \( f \) and, using the CBH formula, the reader can easily convince herself for example in the Fourier basis that is just the standard vacuum expectation value of the operator \( \exp(\sum_n f_n a_n) \). As the Fock state is weakly continuous, we also take derivatives of the Weyl operators and define the usual hermitian field operators as

\[ \pi(f) = \left. \frac{d}{d\lambda} \right|_{\lambda=0} \rho(W(\lambda f)) \] (7)

and creation and annihilation operators as

\[ a^*(f) = \frac{1}{\sqrt{2}}(\pi(f) - i \pi(Jf)), \quad a(f) = \frac{1}{\sqrt{2}}(\pi(f) + i \pi(Jf)). \] (8)

If we write the vacuum vector \( |W(0)\rangle = |1\rangle \) in the usual way as \( |0\rangle \), they act in the usual way on vectors of the form

\[ a^*(f_1) \vee \cdots \vee a^*(f_n) |0\rangle. \]

(\( \vee \) indicates symmetrisation) with commutation relations

\[ [a(f), a^*(g)] = \langle f | g \rangle \cdot 1. \]

as follows from (3), (7) and (8).

Now that we have defined the Fock state and understand the structure of the corresponding Hilbert space we can investigate the action of diffeomorphisms. Given a diffeomorphism \( S \) acting on \( f \), we would like to define a corresponding unitary operator \( U(S) \) on the Fock space. As with the Fock space derivatives exist, we can as well work with the infinitesimal version by writing \( S = e^A \). As a diffeomorphism \( S \) leaves the symplectic structure invariant, it follows that \( A \) is skew \( (\sigma(Af, g) = -\sigma(f, Ag)) \). From (6), it follows that if \( A \) anti-commutes with \( J \) the norm \( \langle f | f \rangle = \sigma(f, Jf) \) is annihilated by \( A \) and thus the state \( \omega_F \) is invariant under \( S \).

Unfortunately, this is not always the case. \( S \) in general leads to a Bogoliubov transformation and those do not leave the vacuum invariant. In the general situation, we can split \( A = A_1 + A_2 \) as

\[ A_2 = \frac{1}{2}(A \mp JAJ) \]
such that $A_1$ anti-commutes and $A_2$ commutes with $J$. Physically, this means that $A_1$ maps positive to positive and negative to negative energy modes whereas $A_2$ mixes the two. As we saw, $A_1$ leaves $\omega_F$ invariant so we can find a unitary $U(e^{A_1})$ (and its derivative $dU(A_1)$) in the canonical way. Concretely, it acts as

$$U(e^{A_1})(a^+(f_1) \vee \cdots \vee a^+(f_n)|0\rangle) = a^+(e^{A_1} f_1) \vee \cdots \vee a^+(e^{A_1} f_n)|0\rangle.$$  

The complex anti-linear part $A_2$ also has a uniquely defined[12] action in the Fock space, which is, however, slightly more involved. It turns out [6] that in our case $A_2$ is a self-adjoint Hilbert-Schmidt operator. Thus there exist orthogonal sets $\{u_i\}_{i \in \mathbb{N}}$ and $\{v_i\}_{i \in \mathbb{N}}$ that span the range of $A_2$ such that the action of $A_2$ can be written as

$$A_2 f = \sum_{i \in \mathbb{N}} \langle f|v_i\rangle u_i = \sum_{i \in \mathbb{N}} \langle f|u_i\rangle v_i  \tag{9}$$

where the second equation follows from self-adjointness. This representation leads us to the definition of a creation operator $a^*(A_2)$ acting as

$$a^*(A_2)(a^+(f_1) \vee \cdots \vee a^+(f_n)|0\rangle) = \sum_{i \in \mathbb{N}} a^*(u_i) \vee a^*(v_i) \vee a^+(f_1) \vee \cdots \vee a^+(f_n)|0\rangle$$

and an annihilation operator as its adjoint $a(A_2) = (a^*(A_2))^*$. Altogether, we obtain an anti-hermitian operator on the Fock space by

$$dU(A_2) = \frac{1}{2}(a(A_2) - a^*(A_2))$$

and finally $dU(A) = dU(A_1) + dU(A_2)$, that can be exponentiated to yield the unitary operator

$$U(e^A) = \exp(dU(A)).$$

From these definitions it follows that

$$dU(A)|0\rangle = -\frac{1}{2}a^*(A_2)|0\rangle$$

as the $A_1$-part leaves the vacuum invariant and the annihilation part of $dU(A_2)$ vanishes on the vacuum. Thus we have $\langle 0|dU(A)|0\rangle = 0$ and

$$\langle 0|dU(A)dU(B)|0\rangle = -\frac{1}{4} \langle 0|a(A_2)a^*(B_2)|0\rangle = -\frac{1}{2} \text{Tr}(B_2 A_2)$$

where the trace exists because the anti-linear parts $A_2$ and $B_2$ are Hilbert-Schmidt and the last equality follows from the spectral form (9). Finally, it is easy to check that $dU(A)$ actually implements $A$ on the Fock space, i. e.

$$[dU(A), \pi(f)] = \pi(Af)$$

which is the infinitesimal form of

$$U(S)^{-1} W(f) U(S) = W(S^{-1} f).$$
Obviously, this equation would specify $U(S)$ only up to a phase and thus the uniqueness of the Weyl algebra implies that the group relations from compositions of diffeomorphisms hold in the Fock space up to a phase. Infinitesimally, this means

$$[dU(A), dU(B)] = dU([A, B]) + \phi(A, B)1$$

(10)

where $\phi$ is purely imaginary. Taking the vacuum expectation value of this equation then gives us

$$\phi(A, B) = \frac{1}{2} \text{Tr}([A_2, B_2]).$$

(11)

For the trace of a commutator not to vanish it is necessary that the one particle Hilbert space in which $A_2$ and $B_2$ act is infinite dimensional. So we find that in the Fock representation of the Weyl algebra $\phi$ is an obstruction to the unitary implementation of the classical diffeomorphism group.

Let us compute this obstruction in the concrete case of the closed string. Diffeomorphisms are generated by vector fields

$$L_k = e^{ikx} \frac{d}{dx}$$

where for simplicity we complexify from the very beginning. They act on the Fourier components (4) as $L_k f_n = -nf_{n+k}$. In this basis, the scalar product is given by

$$\langle f | g \rangle = \sum_{n>0} n\bar{f}_n g_n.$$

First we compute

$$(L_k)_2 f_n = \frac{1}{2}(L_k f_n + JL_k Jf_n)$$

$$= \frac{1}{2}(-nf_{n+k} + isgn(n)JL_k f_n)$$

$$= \frac{n}{2}(sgn(n)sgn(n + k) - 1)f_{n+k},$$

finding it non-zero only if $n$ and $n + k$ have different signs. Setting $d(n, k) = sgn(n)sgn(k) - 1$ we finally compute

$$\text{Tr} \left( [(L_n)_2, (L_k)_2] \right) = \sum_{j>0} \frac{1}{j} \langle f_j | [(L_n)_2, (L_k)_2] | f_j \rangle$$

$$= \sum_{j>0} \langle f_j | (- (L_n)_2 d(k, j) | f_{j+k} \rangle + (L_k)_2 d(n, j) | f_{j+n} \rangle$$

$$= \sum_{j>0} \delta_{k, -n}((j + k)d(k, j)d(-k, k + j)$$

$$- (j - k)d(-k, j)d(k, -k + j)).$$

This sum has only $|k| - 2$ non-vanishing terms and yields

$$\phi(L_n, L_k) = \frac{1}{12} n(n^2 - 1)\delta_{k, -n}$$

which reflects the well known fact that the free boson has central charge 1. Note that $L_n$ and $L_{-n}$ are adjoint to each other so that the anti-hermitian combinations $\frac{1}{2}(L_n - L_{-n})$
and $\frac{1}{2}(L_n + L_{-n})$ that are exponentiated to give the unitary operators actually have a purely imaginary central component in their commutator. Note that our calculation was finite at all stages and did not require any regularisation.

The Hilbert spaces we have constructed so far are by themselves not physical: The diffeomorphisms of the string do not act trivially on them and they would describe the quantum theory of parametrised strings. What we have done in this section is to construct unitary operators on these kinematical Hilbert spaces that implement the action of the diffeomorphisms. In a final step, these have to be divided out and only the vectors that are left invariant by the $U(S)$ are the physical states.

The appearance of the central charge in the Fock space implying that multiples of the unit operator are in the symmetry algebra is deadly for this second step: Invariance would require that one of the physical state conditions would be

$$e^{i\phi}\langle \psi | = | \psi \rangle$$

where the central $\phi$ is an imaginary number. This condition would imply that there are no physical states. The way out of this problem however is well known: The quantization of only one $X$ is not consistent, one has to add further fields such that the total central charge vanishes. In the case of the bosonic string this is done by taking 26 copies of the theory we have studied here and add a similarly quantized $bc$ ghost system that provides a central charge of $-26$ so the total sum of the symmetry implementers $L_n$ do not have a central charge and thus no multiples of the unit operator in their algebra.

In this section we have discussed the two inequivalent Hilbert space representation of the quantum C*-algebra of the bosonic string, arising from the polymer state as suggested by Thiemann and the Fock state that underlies the usually treatment of the theory. We mentioned that Thiemann’s $\omega_p$ is not (weakly) continuous and therefore many of the usual operators do not exist on the polymer Hilbert space. In the next section we will discuss the physical consequences of this fact in the simpler (and of course experimentally accessible) but completely analogous case of the harmonic oscillator.

We have made no attempt here to investigate whether there are further inequivalent states on the algebra of the $W(f)$. It is however well known[6, 13] that if one insists on continuity there is a theorem similar to the one by Stone and von Neumann in the quantum mechanical case, that guarantees the uniqueness of the Fock representation. So even the apparent choice of complex structure $J$ we have had to make above has no room for alternatives that lead to inequivalent Hilbert spaces.

3. The harmonic oscillator as a testing ground

It has been argued[14], that as nobody has so far observed a quantum string in an experiment, there is no empirical data on which to base the decision for one or the other state leading to different physical properties of the quantized string. Especially, the fact that the diffeomorphism group acts on the polymer Hilbert space without a central charge (as the state $\omega_P$ on which it is based is invariant under diffeomorphisms)
seems to make this quantization far more generally applicable. In this paper, we have only discussed a single $X$ and therefore target space symmetries (especially Poincaré invariance) did not play a rôle here, but Thiemann describes in his paper that as he lets these symmetries act trivially on the internal theory of the string, the oscillations of the string carry no momentum in his theory and there are strings of all rest-masses like it is the case for point-particles. In particular, he can find a tachyon-free bosonic string. We will not comment here on this property of Thiemann’s string but just say that all these features seem to make his model far more attractive as a physical theory than the ordinary string.

Again, up to the foreseeable future, there are no experiments in sight that directly test fundamental string theory or quantum gravity so one might think there are no empirical preferences for the choices one has to make during the quantization procedure. However, we will argue in this section that one has exactly the same choices in the quantization of ordinary quantum mechanical systems like the harmonic oscillator or the hydrogen atom that are experimentally tested on a daily basis and that there the choice for the polymer state leads to unphysical consequences.

Before we start, let us however warn the reader that there is one significant difference between the string and quantum gravity on one side and the mechanical systems on the other: The later are not gauge systems with redundant degrees of freedom that have to be modded out to turn the kinematical Hilbert space into the physical one. We make no attempt here to understand the structure of the physical Hilbert space (in the Fock space case it is known to be generated by DDF states for a recent development, see [15]) and physical criteria should strictly only be applied to what is left after the gauge freedom has been removed but we still think it is useful to illustrate the physical consequences of non-continuous representations in the case of the harmonic oscillator.

In the case of gravity, it has been argued that polymer-type states are the only states on the quantum algebra of observables that are invariant under diffeomorphisms. But it is one of the main points of this paper to emphasise that invariant states are not necessarily needed and that at least in the case of the bosonic string there are other ways to construct unitary implementers of the symmetry group that act on a much less singular Hilbert space than the one obtained from $\omega_P$.

Our quantization procedure for one-dimensional quantum mechanics and the harmonic oscillator specifically will be very similar to the discussion of the bosonic string in the previous section. Notationally, it just consists in the replacement $f \mapsto z$. The only difference is that now our symplectic space has finite dimension. Concretely, we take it to be $\mathbb{C} = \mathbb{R}^2$, and combine position and momentum into real and imaginary parts of a complex number $z$. The usual symplectic form is

$$\sigma(z, z') = \text{Im}(zz').$$

As before, instead of using position and momentum directly, we exponentiate them to $W(z) = \exp(iz)$ to obtain bounded (unitary) Weyl operators after quantization. So,
our quantum algebra is generated by linear combinations of operators $W(z)$ and there is the canonical “commutation” relation

$$W(z_1)W(z_2) = e^{\frac{i}{2}\sigma(z_1,z_2)}W(z_1+z_2).$$

The formal similarity to the bosonic string should not come as a surprise as the latter is a free theory that is formally the sum of infinitely many harmonic oscillators. Hilbert spaces on which this C*-algebra is represented are again obtained from states (expectation value functionals) with the help of the GNS-construction. The fact that usually one does not make a difference between the elements of the abstract algebra and operators on a Hilbert space is justified by the Stone-von Neumann theorem that states that there is only one representation of the Weyl algebra that is continuous in $z$. It is based on the Fock vacuum

$$\omega_F(W(z)) = e^{-|z|^2}.$$

The Hilbert space is then described by acting on the vacuum vector $|0\rangle$ associated to $W(0) = 1$ by operators $\rho(W(z))$ (and taking the completion with respect to the norm coming from the scalar product $\langle z_1|z_2\rangle = \omega_F(W(z_1)^*W(z_2)))$. This Hilbert space is the usual $L_2(\mathbb{R})$ on which for real $x$ the operator $\rho(W(x))$ translates functions by $x$ and the operator $\rho(W(ix))$ multiplies functions by $\exp(ix)$.

Position and momentum operators combined into hermitian and anti-hermitian parts of an operator are then derivatives of Weyl operators

$$\pi(z) = \frac{d}{d\lambda}\bigg|_{\lambda=0} \rho(W(\lambda z))$$

and creation and annihilation operators

$$a^*(z) = \frac{1}{\sqrt{2}}(\pi(z) - i\pi(iz)) \quad a(z) = \frac{1}{2}(\pi(z) + i\pi(iz)).$$

It is easy to check that in fact $a(z)$ annihilates the vacuum $|0\rangle$. Thus the states $|z\rangle = \rho(W(z))|0\rangle$ are the coherent states

$$|z\rangle = e^{-\frac{1}{2}|z|^2}e^{za^*}|0\rangle.$$ 

So far, we described general one dimensional quantum mechanics without reference to a specific system. Classically the dynamics is specified by a Hamilton function, $H = \frac{1}{2}(p^2 + x^2)$. As before, we will however proceed to the integrated flow in phase space rather than the infinitesimal generator (as that might not exist in the quantum theory or at least be an unbounded operator). The time evolution of the harmonic oscillator is just rotation in phase space:

$$U(t): z \mapsto e^{it}z.$$ 

The vacuum state $\omega_F$ is invariant under the corresponding automorphism $\alpha_t(W(z)) = W(e^{it}z)$ of the Weyl algebra. Therefore we directly obtain the unitary implementers on the Hilbert space:

$$U(t)|z\rangle = |e^{it}z\rangle.$$
Using the chain rule, it follows that the eigenstates of this time evolution are just the usual ones:

$$U(t)(a^*(z))^N|0⟩ = e^{iNt}(a^*(z))^N|0⟩$$

This concludes our discussion of the standard Schrödinger quantization of the harmonic oscillator. Next we want to contrast these properties with a quantization that is based on a state that is of the same “polymer” form that Thiemann used in his quantization of the bosonic string and that parallels states used in the loop quantization program of gravity. We define

$$ω_P(W(z)) = \begin{cases} 
1 & \text{if } z = 0 \\
0 & \text{else.}
\end{cases}$$

This is a slightly different choice than the one used in [5] but ours has the advantage of being invariant under the time evolution of the harmonic oscillator, thus making at least the ground state stationary. Again, this state is not continuous in $z$ and thus the derivatives needed to define position and momentum operators $π(z)$ and thus creation and annihilation operators $a^{(*)}(z)$ do not exist.

This choice of state leads to a rather unusual Hilbert space: The states $|z⟩ = W(z)|0⟩$ are all orthogonal as long as their arguments $z$ differ. After taking the completion with respect to the GNS norm, elements of the Hilbert space are functions on $\mathbb{C}$ that are non-zero at at most countably many points $(z_i)_{i \in \mathbb{N}}$

$$φ = \sum_{i \in \mathbb{N}} φ_i |z_i⟩$$

and which are $l_2$-normalisable:

$$\sum_{i \in \mathbb{N}} |φ_i|^2 = \sum_{i \in \mathbb{N}} |φ_i|^2 < ∞.$$  

The scalar product of two such functions $ψ$ and $φ$ is a sum over the points on which both of them are non-zero

$$⟨ψ|φ⟩ = \sum_{i \in \mathbb{N}} ψ(z_i)φ(z_i).$$

It is interesting to note, that this Hilbert space (including its scalar product) does not contain any information about the topology of $\mathbb{C}$ anymore: Any bijective (possibly discontinous) map $\mathbb{C} → \mathbb{C}$ that fixes the origin leaves the state $ω_P$ invariant and leads to unitarily equivalent Hilbert space, that is the Hilbert spaces are “the same”.

It is clear that translation operators $ρ(W(z))$ acting on such functions do not have well defined derivatives and thus position and momentum operators cannot be defined. Furthermore, as the $|z⟩$ are orthogonal for all $z ∈ \mathbb{C}$, this Hilbert space is not separable but we will not dwell on this point as in the case of gravity the physical Hilbert space will again be separable after the constraints have been modded out [10, 11].

Far more important are the physical properties of the harmonic oscillator in this representation: Again, the polymer state is invariant under the time evolution $α_t(W(z)) = W(e^{it}z)$, therefore there are canonically given unitary operators $U(t)$. 

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However, as again they do not have a derivative with respect to \( t \), there is no Hamilton operator and we cannot directly discuss the energy spectrum.

In [5], this problem is circumvented by introducing an ad hoc scale and defining a Hamilton operator in terms of finite difference operators at that scale instead of derivatives. However, this ad hoc procedure does not connect with the classical time evolution of the harmonic oscillator. In this paper, we take the point of view that the time evolution operators

\[
U(t)|z\rangle = |e^{it}z\rangle
\]

follow from the correspondence principle and we have to investigate their properties (this is in the spirit of deformation quantization, for an introduction, see for example [16]).

First of all, the only vector in the Hilbert space that describes a stationary state and transforms with a phase under this time evolution is \(|0\rangle\). Thus it is fair to say that \(|0\rangle\) is the only bound state of this version of the harmonic oscillator. All other vectors describe "scattering" states.

The closest to what would be a Hamiltonian is the hermitian operator

\[
H_\epsilon = \frac{U(\epsilon) - U(-\epsilon)}{2i\epsilon}
\]

which would converge to the Hamiltonian if the limit \( \epsilon \to 0 \) existed. For fixed \( \epsilon \), \( H_\epsilon \) can have eigenstates. However they will not be eigenstates for all \( U(t) \) as it is the case in the Schrödinger representation. Eigenstates of \( H_\epsilon \) are thus not stationary, they are just periodic up to a phase with period \( 2\epsilon \).

For generic, non-zero \( \epsilon \) the expectation value of \( H_\epsilon \) is zero in any state. However, when acting on a normalised state \( \phi \) as in (12) it produces a state of norm

\[
\|H_\epsilon \phi\|^2 = \frac{1}{2\epsilon^2} \sum_{i \in \mathbb{N}, z_i \neq 0} \bar{\phi}_i \phi_i.
\]

Thus we find that the expectation value of \( H_\epsilon^2 \) diverges as the limit \( \epsilon \to 0 \) is attempted: All states except for the ground state have diverging energy and also the energy spectrum is not bounded from below. This is clearly in conflict with the energy spectra of harmonic oscillators found in nature \( \| \). Therefore we have to conclude that at least for the harmonic oscillator, the quantization based on the polymer state is empirically not correct. The polymer state is too singular and thus gives an unphysical Hilbert space for quantum mechanics.

Of course, it would be too quick to extrapolate this result for the harmonic oscillator directly to the bosonic string or even to the quantization of gravity but it shows that one should have good reasons to depart from the quantization scheme that requires continuity of the states that was successful in the experimentally tested case of quantum mechanics.

|| This result is not surprising in view of the fact that the polymer state can be seen as a thermal state in the limit of infinite temperature, see [3].
4. Conclusions

In this note, we have put both the usual Fock space quantization of the bosonic string and Thiemann’s alternative approach into a common formalism that exposes at which points the two treatments are equivalent and at which points they differ. As long as the algebra of observables is considered as an abstract C*-algebra, both approaches are completely parallel but they differ in the choice of a representation of this algebra as operators on a Hilbert space.

Thiemann chooses a representation based on a polymer state as it is always done in the loop approach to quantum gravity. This state has the advantage that it is invariant under diffeomorphisms and therefore directly leads to unitary operators in the Hilbert space representing the diffeomorphism group. However, this representation is not continuous. Therefore infinitesimal generators like field operators and field momenta cannot be defined as derivatives of Weyl operators. Furthermore, the kinematic Hilbert space constructed this way is not separable.

The usual representation is constructed as a Fock space which has the continuity property that the polymer representation lacks. Therefore the usual creation and annihilation operators can be defined. On the other hand, the Fock vacuum is not invariant under diffeomorphisms as they act as Bogoliubov transformations. Nevertheless, unitary operators for the diffeomorphisms can be defined by a variant of the Gupta Bleuler procedure. Generically, this leads to a representation of the diffeomorphism group up to a phase which is the integrated form of a central charge. However, in the critical dimension this vanishes and a diffeomorphism invariant physical Hilbert space can be obtained.

One could argue on general grounds, that because of Fell’s theorem the choice of a representation is immaterial, as it cannot be determined by any finite number of measurements of finite precision. However, for any two different states one can always find an experiment that distinguishes between them, so the two representations should be regarded as physically inequivalent.

In a later chapter, we discussed that if one drops the requirement of weak continuity and thus circumvents the theorem by Stone and von Neumann, exactly the same choice of representations exists in quantum mechanics and leads to physically inequivalent quantizations of the harmonic oscillator. Especially the energy spectrum in the polymer representation differs significantly from the usual experimentally observed spectrum.

As mentioned in the introduction, this discussion is not only relevant to string theory. The world-sheet theory of the bosonic string appears to be a simple but non-trivial testing ground for the quantization of diffeomorphism invariant theories. As such, the Fock space treatment of string theory text books can be interpreted as providing a canonical quantization of a diffeomorphism invariant theory that differs from the one used in the loop quantum gravity literature. As we tried to argue it has some favourable properties like greater regularity of the representation leading to the existence of derivatives of the Weyl operator and a Hilbert space of smoother functions.
Our construction was based on the fact that the string in a flat target space is a free field theory. However, other exactly solvable conformal field theories should be treatable in exactly the same fashion leading to diffeomorphism invariant interacting theories. However it remains to be seen if similar constructions can also be found for the case of higher dimensional gravity. But at least it is demonstrated that there is a viable alternative to singular representations based on polymer states.

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