REMARKS ON MULTI-DIMENSIONAL GENERALIZED BROWNIAN MOTIONS

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Abstract. We consider certain questions pertaining to generalized Brownian motions with multiple processes. We establish a framework for generalized Brownian motion with multiple processes similar to that defined by Gută and prove multi-dimensional analogs of some theorems of Gută and Maassen. We then consider examples of processes are indexed by a two-element set and characterize the function on I-indexed pair partitions associated via the I-indexed generalized Brownian motion construction to certain pairs of representations connected to certain spherical representations of infinite symmetric groups. In doing so, we generalize the notion (introduced by Bożejko and Gută) of the cycle decomposition of a pair partition. We then introduce a generalization of Gută’s q-product of generalized Brownian motions to a product corresponding to a (possibly infinite) matrix $q_{ij}$ and show that this product satisfies a central limit theorem.

1. Introduction

Bożejko and Speicher initiated the study of generalized Brownian motions, introducing operators satisfying an interpolation between Fermionic and Bosonic commutation relations [3]. Specifically, for $q \in [-1, 1]$ and a complex Hilbert space $\mathcal{H}$, they constructed a $q$-twisted Fock space $\mathcal{F}_q(\mathcal{H})$ with creation operators $c^*(f)$ and annihilation operators $c(f)$ for $f \in \mathcal{H}$ satisfying the relations

$$c(f)c^*(g) - qc^*(g)c(f) = \langle f, g \rangle \cdot 1.$$ 

Subsequently, they developed a broader framework of generalized Brownian motion which incorporated this example [5]. In this general framework, one considers the algebra obtained by applying the GNS construction to the free tensor algebra of a real Hilbert space $\mathcal{H}$ with certain states, called Gaussian states, associated to a class of functions, called positive definite, on pair partitions via a pairing prescription.

Gută and Maassen further explored this notion of generalized Brownian motion [7, 8]. They showed that Gaussian states $\rho_t$ can be alternatively characterized by sequences of complex Hilbert spaces $(V_n)_{n=1}^{\infty}$ with densely defined maps $j_n : V_n \to V_{n+1}$ and representations $U_n$ of the symmetric group $S_n$ on $V_n$ satisfying $j_n \cdot U(\pi) = U(i_n(\pi)) \cdot j_n$ where $i_n : S_n \to S_{n+1}$ is the inclusion arising from the map $\{1, \ldots, n\} \hookrightarrow \{1, \ldots, n + 1\}$, data which give rise to a symmetric Fock space with creation and annihilation operators. They also provided an algebraic characterization of the notion of positive definiteness for a function $t$ on pair partitions and characterized the functions $t$ which give rise to analogs of the Gaussian functor. Separately [8], they examined a class of Brownian motions arising from the combinatorial notion of species of structure.

Bożejko and Gută [2] considered a special case of the generalized Brownian motion of Gută and Maassen arising from $II_1$-factor representations of the infinite symmetric group $S_\infty$ constructed by Vershik and Kerov [14]. Lehner [9] considered these generalized Brownian motions in the context of exchangeability systems, which generalize various notions...
of independence and give rise to cumulants analogous to the well-known free and classical cumulants. Recent work of Avsec and Junge [1] offers another point of view on the subject of noncommutative Brownian motion.

In [6] Gută extended the notion of generalized Brownian motion to multiple processes indexed by some set $\mathcal{I}$. He went on to define for $-1 \leq q \leq 1$ a $q$-product of generalized Brownian motions interpolating between the graded tensor product previously considered by Mingo and Nica [10] ($q = -1$), the reduced free product [15] ($q = 0$) and the usual tensor product ($q = 1$). He also showed that this $q$-product obeys a central limit theorem as the size of the index set $\mathcal{I}$ grows.

In this paper, we explore certain additional questions pertaining to the $\mathcal{I}$-indexed generalized Brownian motions. As a warmup, we begin with the very simple case of a generalized Brownian motion arising from a tensor product of representations of the infinite symmetric group $S_\infty$. We compute the functions on pair partitions associated to the Gaussian states in this context. We then proceed to consider the generalized Brownian motions associated to spherical representations of the Gelfand pair $(S_\infty \times S_\infty, S_\infty)$. Here again we give a combinatorial formula for the function on pair partitions arising from the associated Gaussian states, and in the course of doing so we generalize the notion of a cycle decomposition of a pair partition introduced by Bożejko and Gută [2]. We also generalize Gută’s $q$-product of generalized Brownian motions to a $q_{ij}$ product, where $i, j \in \mathcal{I}$ and show that a central limit theorem holds when $q_{ij} = q_{ji}$ and the $q_{ij}$ are periodic in both $i$ and $j$.

The paper has four sections, excluding this introduction. In Section 2 we expand upon the notion of generalized Brownian motion with multiple processes established by Gută [6], proving analogs of some results of Gută and Maassen [7]. We also review Vershik and Kerov’s factor representations of symmetric groups [14] and Bożejko and Gută’s work on generalized Brownian motions with one process associated to the infinite symmetric group [2]. In Section 3 we move on to consider generalized Brownian motions indexed by a two-element set associated to tensor products of factor representations of the infinite symmetric group $S_\infty$. In Section 4 we consider generalized Brownian motions associated to spherical representations of $(S_\infty \times S_\infty, S_\infty)$. In Section 5 we generalize Gută’s $q$-product to a $q_{ij}$ product, where $i, j \in \mathcal{I}$.

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2. Preliminaries

2.1. Generalized Brownian motion. We begin by reviewing the notion of a multiple generalized Brownian motion. The multiple generalized Brownian motion described here is slightly more general than that defined in [6]. However everything in this section is in the spirit of results found in [6] and [7]. Throughout the section, we assume that $\mathcal{I}$ is some fixed index set.
Notation 1. We will make extensive use of notations for integer intervals, which are very common in the combinatorial literature:

\[ [n] = \{1, 2, \ldots, n-1, n\} \]
\[ [m, n] = \{m, m+1, \ldots, n-1, n\} \]

for \( m, n \in \mathbb{N} \).

Notation 2. For a real Hilbert space \( \mathcal{K} \), \( \mathcal{A}^f(\mathcal{K}) \) will denote the free unital \(*\)-algebra with generators \( \omega_i(h) \) for \( h \in \mathcal{K} \) and \( i \in \mathcal{I} \) with relations

\[ \omega_i(cf + dg) = c\omega_i(f) + d\omega_i(g), \quad \omega_i(f) = \omega_i(f)^* \]

for all \( f, g \in \mathcal{K}, i \in \mathcal{I} \) and \( c, d \in \mathbb{R} \).

Notation 3. If \( \mathcal{H} \) is a complex Hilbert space, \( \mathcal{C}^f(\mathcal{H}) \) denotes the free unital \(*\)-algebra with generators \( a_i(h) \) and \( a_i^*(h) \) for all \( h \in \mathcal{H} \) and \( i \in \mathcal{I} \) with relations

\[ a_i^*(\lambda f + \mu g) = \lambda a_i^*(f) + \mu a_i^*(g), \quad a_i^*(f) = a_i(f)^* \]

for all \( f, g \in \mathcal{H}, i \in \mathcal{I} \), and \( \lambda, \mu \in \mathbb{C} \). We will also use the notations \( a_i^1(h) := a_i(h) \) and \( a_i^2(h) := a_i^*(h) \).

Definition 1. If \( P \) is a finite ordered set, let \( \mathcal{P}_2(P) \) be the set of pair partitions of \( P \). That is,

\[ \mathcal{P}_2(P) := \left\{ ((l_1, r_1), \ldots, (l_n, r_n)) : l_k < r_k, \bigcup_{k=1}^{n} \{l_k, r_k\} = P, \{l_p, r_p\} \cap \{l_q, r_q\} = \emptyset \text{ if } p \neq q \right\}. \]

The set of \( \mathcal{I} \)-indexed pair partitions, \( \mathcal{P}^\mathcal{I}_2(P) \) is the set of pairs \((\mathcal{V}, c)\) with \( \mathcal{V} \in \mathcal{P}_2(P) \) and \( c : \mathcal{V} \to \mathcal{I} \). We will sometimes refer to the elements of \( \mathcal{I} \) as colors and the function \( c \) as the coloring function. If \( P' \) is another finite ordered set and \( \alpha : P \to P' \) is an order-preserving bijection, then \( \alpha \) induces a bijection \( \mathcal{P}^\mathcal{I}_2(P) \to \mathcal{P}^\mathcal{I}_2(P') \). Considering all order-preserving bijections gives an equivalence relation on the union of \( \mathcal{P}^\mathcal{I}_2(P) \) over sets of cardinality \( 2n \). Let \( \mathcal{P}^\mathcal{I}_2(2n) \) be the set of equivalence classes under this relation, and let \( \mathcal{P}^\mathcal{I}_2(\infty) := \bigcup_{n=1}^{\infty} \mathcal{P}^\mathcal{I}_2(2n) \).

We can represent an \( \mathcal{I} \)-colored pair partition visually by connecting the pairs \((l_j, r_j)\) by a piecewise linear path and labeling that path with the color \( c((l_j, r_j)) \). When the number of colors is small, we may find it convenient to use different line styles to indicate colors, instead of an explicit label. Figure \([1]\) gives the diagram for a simple example with \( \mathcal{I} = \{-1, 1\} \).
It is clear that the coloring function \( c : \mathcal{V} \to \mathcal{I} \) defines a function \( c : [2n] \to \mathcal{I} \). It should not create confusion to denote this function by the same name. Note that \( c(l) = c(r) \) when \((l, r) \in \mathcal{V}\). We will use these two descriptions of the coloring function \( c \) interchangeably.

**Definition 2.** A Fock state on the algebra \( \mathcal{C}^\mathcal{I}(\mathcal{H}) \) is a positive unital linear functional \( \rho_t : \mathcal{C}^\mathcal{I}(\mathcal{H}) \to \mathbb{C} \) satisfying

\[
\rho_t \left( \prod_{k=1}^{m} a_{i_k}^{\pm}(f_k) \right) = \sum_{(\mathcal{V}, c) \in \mathcal{P}_2^\mathcal{I}(m)} t_{\mathcal{V}, j}((\mathcal{V}, c)) \prod\limits_{(l, r) \in \mathcal{V}} \langle f_l, f_r \rangle \delta_{i_l, i_r} B_{e_l e_r},
\]

where the \( e_i \) are chosen from \( \{1, 2\} \) and

\( B := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \).

**Definition 3.** A Gaussian state on the algebra \( \mathcal{A}^\mathcal{I}(\mathcal{K}) \) is a positive unital linear functional \( \tilde{\rho}_t : \mathcal{A}^\mathcal{I}(\mathcal{H}) \to \mathbb{C} \) of the form

\[
\tilde{\rho}_t \left( \prod_{k=1}^{m} \omega_{i_k}(f_k) \right) = \sum_{(\mathcal{V}, c) \in \mathcal{P}_2^\mathcal{I}(m)} t_{\mathcal{V}, j}((\mathcal{V}, c)) \prod\limits_{(l, r) \in \mathcal{V}} \langle f_l, f_r \rangle \delta_{i_l, i_r},
\]

for any \( f_k \in \mathcal{H} \).

**Remark 1.** If \( \mathcal{K} \) is a real Hilbert space, then there is a canonical injection \( \mathcal{A}^\mathcal{I}(\mathcal{K}) \to \mathcal{C}^\mathcal{I}(\mathcal{K}_\mathbb{C}) \) given by

\[
\omega_i(h) \mapsto a_i(h) + a_i^*(h),
\]

where \( \mathcal{K}_\mathbb{C} \) denotes the complexification of \( \mathcal{K} \). Considering \( \mathcal{C}^\mathcal{I}(\mathcal{K}_\mathbb{C}) \) as a subalgebra of \( \mathcal{A}^\mathcal{I}(\mathcal{K}) \), the restriction of a Fock state \( \rho_t \) on \( \mathcal{C}^\mathcal{I}(\mathcal{K}_\mathbb{C}) \) to \( \mathcal{A}^\mathcal{I}(\mathcal{K}) \) is a Gaussian state.

While we can use (5) and (3) to define linear functionals \( \tilde{\rho}_t \) and \( \rho_t \) for any choice of \( t : \mathcal{P}_2^\mathcal{I}(\infty) \to \mathbb{C} \), these linear functionals are not always positive. This leads to the following definition.

**Definition 4.** A function \( t : \mathcal{P}_2^\mathcal{I}(\infty) \to \mathbb{C} \) is positive definite if \( \rho_t \) is a positive linear functional on \( \mathcal{C}^\mathcal{I}(\mathcal{K}) \) for any complex Hilbert space \( \mathcal{K} \).

**Remark 2.** Our definition of a positive definite function \( t : \mathcal{P}_2^\mathcal{I}(\infty) \to \mathbb{C} \) is not the same as the definition in [8]. However, we will see in Theorem 3 that the definitions are equivalent.

Suppose that for each \( n : \mathcal{I} \to \mathbb{N} \cup \{0\} \), \( V_n \) is a complex Hilbert space with unitary representation \( U_n \) of the direct product group

\[
S_n := \prod_{a \in \mathcal{I}} S_{n(a)}.
\]

If \( \mathcal{H} \) is a complex Hilbert space, then the Fock-like space is given by

\[
\mathcal{F}_\mathcal{V}(\mathcal{H}) := \bigoplus_{n} \frac{1}{n!} V_n \bigotimes_{a \in \mathcal{I}} \mathcal{H}^{\otimes n(a)}.
\]

Here \( n! \) means \( \prod_{a \in \mathcal{I}} n(a) \) and the factor \( \frac{1}{n!} \) refers to the inner product. Also, we set \( \mathcal{H}^{\otimes 0} = \mathbb{C}\Omega \) for some distinguished unit vector \( \Omega \). The notation \( \bigotimes_{a} \) denotes the subspace of vectors which are invariant under the action of \( S_n \) given by \( U \otimes \bar{U} \), where \( \bar{U} \) is the natural action.
of $S_n$ on $\bigotimes_{a \in I} \mathcal{H}^{\otimes n(a)}$. That is, $\tilde{U}(\pi)$ permutes the vectors according to $\pi$. The projection onto the subspace $\frac{1}{n!} V_n \otimes_s \bigotimes_{a \in I} \mathcal{H}^{\otimes n(a)}$ is given by

$$P_n := \frac{1}{n!} \sum_{\sigma \in S_n} U(\sigma) \otimes \tilde{U}(\sigma).$$

For $v \in V_n$ and $f \in \bigotimes_{a \in I} \mathcal{H}^{n(a)}$ we denote by $v \otimes_s f$ the vector $P_n(v \otimes f)$.

Suppose that we also have densely defined operators $j_a : V_n \rightarrow V_{n+\delta_a}$ (where $\delta_a(b) = \delta_{a,b}$) satisfying the following intertwining relations:

$$j_a \cdot U_n(\sigma) = U_{n+\delta_a}(i_n^{(a)}(\sigma)) \cdot j_a,$$

where $i_n^{(a)}$ is the natural embedding of $S_n$ into $S_{n+\delta_a}$. Note that we have used the same notation for the map on $V_n$ for different $n$, but confusion should not result, as the choice of $n$ should be clear from context. Then we can define creation and annihilation operators on the Fock-like space $\mathcal{F}_V(\mathcal{H})$ for each $a \in I$ and each $h \in \mathcal{H}$. Let $\left(r_b^{(n)}\right)^* (h)$ be the operator

$$\left(r_b^{(n)}\right)^* (h) : \bigotimes_{a \in I} \mathcal{H}^{\otimes n(a)} \rightarrow \bigotimes_{a \in I} \mathcal{H}^{\otimes n(a)+\delta_{a,b}}$$

which acts as right creation operator on $\mathcal{H}^{\otimes n(b)}$ and identity on $\mathcal{H}^{\otimes n(a)}$ for $a \neq b$. Let $r_b^{(n)}(h)$ be the adjoint of $\left(r_b^{(n)}\right)^* (h)$. The annihilation operator $a_b^{V,j}(f)$ is defined on the level $n$ component of the Fock-like space by

$$a_b^{V,j}(f) : V_n \otimes_s \bigotimes_{v' \in I} \mathcal{H}^{\otimes n(v')} \rightarrow V_{n-\delta_b} \otimes_s \bigotimes_{v' \in I} \mathcal{H}^{\otimes n(v')-\delta_{v',b}}$$

$$a_b^{V,j}(f) : \phi \mapsto (j_b^* \otimes r_b^{(n)}(f))\phi.$$  

The creation operator $(a_b^{V,j})^*(h)$ is the adjoint of $a_b^{V,j}(h)$, and its action on a vector $v \otimes_s f$ is given by

$$(a_b^{V,j})^*(h)v_n \otimes_s f = n(b)(j_b v_n) \otimes_s (r_b^{(n)})^*(h)f.$$  

We denote by $C_{V,j}(\mathcal{H})$ the $*$-algebra generated by the operators $a_b^{V,j}(f)$ and $(a_b^{V,j})^*(f)$ for $f \in \mathcal{H}$, and $b \in I$.

There is a representation $\mu_{V,j}$ of $C^2(\mathcal{H})$ on the Fock-like space $\mathcal{F}_V(\mathcal{H})$ satisfying

$$\mu_{V,j} : a_b(f) \mapsto a_b^{V,j}(f) \quad \text{and} \quad \left(a_b^{V,j}\right)^*(f) \mapsto (a_b^{V,j})^*(f)$$

for all $b \in I$ and $f \in \mathcal{H}$.

When it will not cause confusion, we will sometimes write $a_b(f)$ and $a_b^*(f)$ for the annihilation and creation operators $a_b^{V,j}(f)$ and $(a_b^{V,j})^*(f)$. We will also use the notation $a_b^{V,j,1}(f)$ or simply $a_1^j(f)$ for $a_b^{V,j}(f)$ and likewise $a_b^{V,j,2}(f)$ or simply $a_2^j(f)$ for $(a_b^{V,j})^*(f)$.

The following is an $I$-indexed generalization of Theorem 2.6 of [7].

**Theorem 1.** Let $I$ be an index set and $\mathcal{H}$ a complex Hilbert space. Let $(U_n, V_n)$ be representations of $S_n$ with maps $j_b : V_n \rightarrow V_{n+\delta_b}$ satisfying the intertwining relation (10). Let $\xi_V \in V_0$ be a distinguished unit vector. The state $\rho_{V,j}$ on $C^2(\mathcal{H})$ defined by

$$\rho_{V,j}(X) = \langle \xi_V \otimes_s \Omega, \mu_{V,j}(X)(\xi_V \otimes_s \Omega) \rangle$$

is a Fock state. That is, there is a positive definite function $t$ such that $\rho_{V,j} = \rho_t$. 

The proof is very similar to the proof of Theorem 2.6 of [7], but we include it for completeness.

**Proof.** Let \( \mathcal{H} \) be an infinite-dimensional complex Hilbert space, and let \( \{ f_k \}_{k=1}^{\infty} \) be an orthonormal basis for \( \mathcal{H} \). Also let \( \mathcal{V} = \{(l_k, r_k) : k \in [n]\} \) with \( l_k < r_k \) for \( 1 \leq k \leq n \) and \( l_k < l_{k'} \) for \( k < k' \).

Define
\[
\rho_{\mathcal{V}, j} = \rho_{\mathcal{V}, j} \left( \prod_{k=1}^{n} a_{b_k}^{c_k}(f_k) \right)
\]
(16)
where \( b_k \) and \( e_k \) are chosen as follows. Each \( k \) is in one pair \((l_i, r_i) \in \mathcal{V} \) for some \( i \). If \( k = l_i \), we let \( e_k = 1 \), and if \( k = r_i \) we let \( e_k = 2 \). In either case, we let \( b_k = c((l_i, r_i)) \).

For \( A \in \mathcal{B}(\mathcal{H}) \) and \( b \in \mathcal{I} \), define the operator \( d\Gamma_{\mathcal{V}}^b(A) \) on \( \mathcal{F}_{\mathcal{V}}(\mathcal{H}) \) by
\[
d\Gamma_{\mathcal{V}}^b(A) : v \otimes_s \otimes_{b \neq b'} h_{b', 1} \otimes \cdots \otimes h_{b', m'_b} \mapsto \sum_{k=1}^{m_b} v \otimes_s \left( \otimes_{b' \neq b} h_{b', 1} \otimes \cdots \otimes h_{b', m'_b} \right) \otimes (h_{b, 1} \otimes \cdots \otimes Ah_{b, k} \otimes \cdots \otimes h_{b, m_b}).
\]
(17)
The operators \( d\Gamma_{\mathcal{V}}^b(A) \) satisfies the commutation relations
\[
[a_{b}^{V,j}(f), d\Gamma_{\mathcal{V}}^b(A)] = \delta_{b,b'} a_{b'}^{V,j}(A^* f) \quad \text{and} \quad [d\Gamma_{\mathcal{V}}^b(A), (a_{b}^{V,j})^* (f)] = \delta_{b,b'} a_{b'}^{V,j}(Af).
\]
(18)
We write \( a_{b, k}^e \) for \( a_{b}^{V,j,e}(f_k) \), and denote by \( |f_{i_0}\rangle \langle f_i| \) the rank-one operator \( X \) on \( \mathcal{H} \) which is 0 on the orthogonal complement of \( f_i \) and such that \( Xf_i = f_{i_0} \). Applying (18) with \( A = |f_{i_0}\rangle \langle f_i| \), we have when \( i_k \neq i_0 \)
\[
[d\Gamma_{\mathcal{V}}^b(|f_{i_0}\rangle \langle f_i|), a_{b, i_k}^e] = \delta_{i_k, 1} \cdot a_{b, i_0}^e
\]
(19)
Consider a vector of the form \( \phi = \prod_{k=1}^{n} a_{b_k, i_k}^e(\xi_{\mathcal{V}} \otimes_s \Omega) \). Choose \( i_0 \) different from \( i_1, \ldots, i_n \), so that \( a_{b, i_0}^e \phi = 0 \) for any \( b \in \mathcal{I} \). Applying (19),
\[
a_{b, i_0}^e \phi = [d\Gamma_{\mathcal{V}}^b(|f_{i_0}\rangle \langle f_i|), a_{b, i_0}^e] \phi = a_{b, i_0}^e \Gamma_{\mathcal{V}}^b(|f_{i_0}\rangle \langle f_i|) \phi
\]
(20)
Applying (19) repeatedly now yields
\[
a_{b, i_0}^e \phi = \sum_{k=1}^{n} \delta_{i_k, 1} \cdot a_{b, i_0}^e \left( \prod_{r=1}^{k-1} a_{b_r, i_r}^e \right) \cdot a_{b, i_0}^e \left( \prod_{r=k+1}^{n} a_{b_r, i_r}^e \right)(\xi_{\mathcal{V}} \otimes_s \Omega)
\]
(21)
Considering a monomial in creation and annihilation operators \( \prod_{k=1}^{n} a_{b_k, i_k}^e \), the theorem follows by applying (21) for each annihilation operator in the product with a new index \( i_0 \).

There is also the following partial converse.

**Theorem 2.** Let \( \mathcal{H} \) be a separable, infinite-dimensional complex Hilbert space, and let \( \mathcal{I} \) be a countable index set. Let \( t \) be a positive definite function on \( \mathcal{I} \)-indexed pair partitions. Then there exist Hilbert spaces \( V_n \) with representations \( U_n \) of \( S_n \) and maps \( j_a : V_n \rightarrow V_{n+\delta_a} \).
satisfying (10) and a unit vector $\xi_V \in V_0$ such that the GNS representation of $(\mathcal{C}^*(\mathcal{H}), \rho_t)$ is unitarily equivalent to $(\mathcal{F}_V(\mathcal{H}), \mathcal{C}_V,j(\mathcal{H}), \xi_V \otimes_s \Omega)$.

Remark 3. If we are given complex Hilbert spaces $V_n$ with representations $U_n$ of $S_n$ and maps $j_{\alpha} : V_n \to V_{n+\delta_n}$, then Theorem 1 gives a corresponding positive definite function $t : P_2(\infty) \to \mathbb{C}$. Applying Theorem 2 gives complex Hilbert spaces $V'_n$ with representations $U'_n$ of $S_n$ and maps $j'_{\alpha} : V'_n \to V'_{n+\delta_n}$. We will see in Example 1 that the Hilbert spaces $V'_n$ need not be the same as the original Hilbert spaces $V_n$.

The proof of Theorem 2 is very similar to the proof of Theorem 2.7 in [7] (though we are only able to prove it for infinite-dimensional $\mathcal{H}$), but we include the proof of the $I$-indexed theorem here for the sake of completeness.

Proof of Theorem 2 Choose an orthonormal basis $\{f_{k,b}\}_{k \in \mathbb{N}, b \in I}$ for $\mathcal{H}$. Let $\mathcal{F}_t(\mathcal{H})$, $C_t(\mathcal{H})$, and $\Omega_t$ be the complex Hilbert space, operator algebra, and distinguished unit vector of the GNS construction of $\mathcal{C}^*(\mathcal{H})$ with respect to the state $\rho_t$. We denote the image of $a_{b}(f)$ in $C_t(\mathcal{H})$ by $a_{b}^{t}(f)$ and the image of $a_{b}(f)$ in $C_t(\mathcal{H})$ by $(a_{b}^{t})^*(f)$. We will use the notation $a_{b}^{t,e}(f)$ to mean $(a^{t})_{b,e}(f)$ for $e = 2$ and $a_{b}(f)$ for $e = 1$. We will construct the complex Hilbert spaces $V_n$ as subspaces of $\mathcal{F}_t(\mathcal{H})$.

Suppose that for each function $\mathfrak{n} : I \to \mathbb{N} \cup \{0\}$ with only finitely many non-zero values and for each $i \in I$, we have an injective function $\alpha_{\mathfrak{n},i} : [\mathfrak{n}(i)] \to \mathbb{N}$ and that $\alpha_{\mathfrak{n},i}(j) = \alpha_{\mathfrak{n}',i}(j)$ when $j < \mathfrak{n}(i), \mathfrak{n}'(i)$.

We define $R_{\mathfrak{n}}^{\alpha}$ as the set of the vectors of the form

$$
(22) \quad \left( \prod_{k=1}^{2p+|\mathfrak{n}|} a_{b_{k}}^{t,\mathfrak{n},i_{k}}(f_{b_{k},i_{k}}) \right) \Omega_t
$$

where $|\mathfrak{n}| = \sum_{i \in I} \mathfrak{n}(i)$ and the following conditions are satisfied:

1. In the product $\prod_{k=1}^{2p+|\mathfrak{n}|} a_{b_{k}}^{t,\mathfrak{n},i_{k}}(f_{b_{k},i_{k}})$, a creation operator $a_{b}^{t,2}(f_{b,\alpha_{\mathfrak{n}}(j)})$ appears exactly once provided that $1 \leq j \leq \mathfrak{n}(b)$.

2. Among the remaining $2p$ operators in the product, there are $p$ creation operators $(a_{b}^{t,2}(f_{b,i_{q}}))_{q=1}^{p}$ and $p$ annihilation operators $(a_{b}^{t,1}(f_{b,i_{q}}))_{q=1}^{p}$. Moreover, each annihilation operator appears to the left of the corresponding creation operator in the product.

We also let $V_{\mathfrak{n}}^{\alpha}$ be the span of the vectors in $R_{\mathfrak{n}}^{\alpha}$. We define the map $j_{\mathfrak{n},i}^{\alpha} : V_n \to V_{n+\delta_n}$ by restricting the creation operator $a_{b}^{t,2}(f_{b,\alpha_{\mathfrak{n}}(\mathfrak{n}(\mathfrak{n}'))})$ to the subspace $V_{\mathfrak{n}}^{\alpha}$ of $\mathcal{F}_t(\mathcal{H})$. It follows immediately from the definition of $V_{\mathfrak{n}}^{\alpha}$ that the image of this restriction lies in $V_{\mathfrak{n}+\delta_n}^{\alpha}$.

We define a unitary representation $U_{\mathfrak{n}}^{\alpha}$ of $S_n$ on $V_{\mathfrak{n}}^{\alpha}$. Since $\rho_t$ is a Fock state, it is invariant under unitary transformations $U$ on $\mathcal{H}$ in the sense that

$$
(23) \quad \rho_t \left( \prod_{k=1}^{n} a_{b_{k}}^{\alpha}(f_{b_{k},i_{k}}) \right) = \rho_t \left( \prod_{k=1}^{n} a_{b_{k}}^{\alpha}(U f_{b_{k},i_{k}}) \right).
$$

Therefore, there is a unitary map $\mathcal{F}_t(U)$ given by

$$
(24) \quad \mathcal{F}_t(U) : \prod_{k=1}^{n} a_{b_{k}}^{t,\mathfrak{n},i_{k}}(f_{b_{k},i_{k}})\Omega_t \mapsto \prod_{k=1}^{n} a_{b_{k}}^{\alpha}(U f_{b_{k},i_{k}}).
$$
The map $F_t(U)$ induces an automorphism on the algebra of creation and annihilation operators by

$$F_t(U)\alpha_{b}(\xi) F_t(U^*) = \alpha_{b}(\xi).$$

For $\sigma \in S_n$, let $U_\sigma^\alpha$ be the unitary operator on $\mathcal{H}$ which for each $b \in \mathcal{I}$ acts by permuting \{$f_{b,\alpha_n,b(1)}, \ldots, f_{b,\alpha_n,b(n(b))}$\} according to $\sigma$ and fixes $f_{b,r}$ when $r > n(b)$. The map $U_\sigma^\alpha : \sigma \mapsto U_\sigma^\alpha$ is a unitary representation of $S_n$ on $V_n^\alpha$.

Define $\iota_{n,i} : \mathbb{N} \to \mathbb{N}$ by $\iota_{n,i}(j) = j$, and let $R_n := R_n^\iota$, $V_n := V_n^\iota$, $j_b := j_b^\iota$, and let $U_n := U_n^\iota$. It follows from the definitions that these data satisfy the intertwining property $[\mathbb{1}]$. We also define $\xi_V := \Omega_t \in V_0$.

We now show that $(F_V(\mathcal{H}), C_{V,j}(\mathcal{H}), \xi_V \otimes_s \Omega)$ is unitarily equivalent to the GNS representation of $(\mathcal{C}_F^Z(\mathcal{H}), \rho_t)$. We will begin by showing that $\rho_t = \rho_{V,j}$. By Theorem $[\mathbb{2}]$, $\rho_{V,j}$ is a Fock state associated to some positive definite function $t' : P^2_\mathbb{F}(\infty) \to \mathbb{C}$, so it will suffice to show that $t' = t$.

For the proof, we will also need to define for a unitary map $U$ on $\mathcal{H}$,

$$F_V(U) : F_V(\mathcal{H}) \to F_V(\mathcal{H})$$

$$v \otimes_s \left(\bigotimes_{b \in \mathcal{I}} h_{b,1} \otimes \cdots \otimes h_{b,n(b)}\right) \mapsto v \otimes_s \left(\bigotimes_{b \in \mathcal{I}} Uh_{b,1} \otimes \cdots \otimes Uh_{b,n(b)}\right)$$

for all $v \in V_n$. This induces an action on the creation and annihilation operators satisfying

$$F_V(U)\alpha_{V,j}^\iota(f) F_V(U^*) = \alpha_{V,j}^\iota(Uf)$$

For $\alpha_{n,j}$ as before, define

$$V_n^\alpha := \text{span}\{v \otimes_s \bigotimes_{b \in \mathcal{I}} f_{b,\alpha(1)}(1) \otimes \cdots \otimes f_{b,\alpha(n(b))}\}.\]$$

Define an isometry

$$T_n : V_n \to F_{V,j}(\mathcal{H})$$

$$v \mapsto v \otimes_s \bigotimes_{b \in \mathcal{I}} f_{b,1} \otimes \cdots \otimes f_{b,n(b)}$$

Let $U_{\alpha,n}$ be a unitary map on $\mathcal{H}$ which permutes the basis vectors $f_{b,j}$ such that $U_{\alpha} f_{n,j} = f_{n,\alpha(j)}$ whenever $1 \leq j \leq n(b)$. Define a map $T^\alpha_n : V_n^\alpha \to V_n^\alpha$ by

$$T^\alpha_n := F_V(U_{\alpha,n}) T_n F_V(U_{\alpha,n}^*)$$

This map does not depend on the choice of $U_{\alpha,n}$ permuting the basis vectors according to $\alpha$. It follows immediately from the definitions that the diagram

$$V_n \xrightarrow{T_n} \tilde{V}_n$$

$$V_n \otimes_s f_{b,\alpha(n(b)+1)} \xrightarrow{(a^\iota_{b,j})^* f_{b,\alpha(n(b))+1}} \tilde{V}_n$$
is commutative, whence the diagram

\[
\begin{array}{ccc}
V_n^\alpha & \xrightarrow{T_n^\alpha} & \tilde{V}_n^\alpha \\
\downarrow j_n^\alpha & & \downarrow (a_{b,\alpha}^{V,j} \star (f_{k,b,n,b}(\alpha(b)+1))) \\
V_{n+\delta_b}^\alpha & \xrightarrow{T_{n+\delta_b}^\alpha} & \tilde{V}_{n+\delta_b}
\end{array}
\]

(31)

also commutes. A similar argument gives a corresponding commutative diagram for the annihilation operators, and this implies the equality of the states \( \rho_k \) and \( \rho_{V,j} \), which implies \( t = \tilde{t} \).

Finally, we must prove that the vacuum vector \( \Omega_V := \xi_V \otimes \Omega \) is cyclic for \( C^T(\mathcal{H}) \). It will suffice to show that for any \( n \), any \( v \in R_n \) and any vectors \( h_1, \ldots, h_n \in \mathcal{H} \), there is some \( X \in C^T(\mathcal{H}) \) with

\[
X \Omega_V = v \otimes_s \bigotimes_{b \in I} h_{b,1} \otimes \cdots \otimes h_{b,n(b)}.
\]

(32)

By the definition of \( R_n \), we can write

\[
v = \left( \prod_{k=1}^{2p+|n|} a_{b_k}^{t,e_k}(f_{b,k,i_k}) \right) \Omega_t
\]

(33)

where a creation operator \( a_{b_k}^{t,2}(f_{b,j}) \) appears exactly once for \( 1 \leq j \leq n(b) \), and among the remaining \( 2p \) operators in the product, there are \( p \) creation operators \( (a_{b_q}^{t,2}(f_{b_q,t_q}))_{q=1}^p \) and \( p \) annihilation operators \( a_{b_q}^{t,1}(f_{b_q,t_q})_{q=1}^p \), with each annihilation operator appearing to the left of the corresponding creation operator in the product. We need simply choose \( X \) of the form

\[
X := \frac{1}{n!} \cdot \prod_{k=1}^{2p+|n|} a_{b_k}^{t,e_k}(g_k),
\]

(34)

where the \( g_k \) satisfy:

\[
g_k := \begin{cases} h_{b_k,r_k}, & \text{if } 1 \leq i_k \leq n(b_k) \\ h_{l_k}', & \text{otherwise} \end{cases}
\]

(35)

where \( r_k \) is defined so that \( k \) is the \( r_k \)-th smallest element of the set \( \{ u : b_u = b_k, 1 \leq u \leq n(b_k) \} \), the \( (h')_i^{\infty} \) is an orthonormal sequence of vectors which are orthogonal to each \( h_{b,k} \), and \( l_k = l_k' \) if and only if \( b_k = b_{k'} \) and \( i_k = i_{k'} \). It follows from the definitions that this \( X \) satisfies (32), so the proof is complete. \( \Box \)

We now pursue an algebraic characterization of positive definiteness for functions on \( I \)-indexed pair partitions. This will involve Gută’s \( \ast \)-semigroup of \( I \)-indexed broken pair partitions [6].

**Definition 5.** Let \( X \) be an arbitrary finite ordered set and \( (L_a, P_a, R_a)_{a \in I} \) a disjoint partition of \( X \) into triples of subsets indexed by elements of \( I \). Suppose that for each \( a \in I \), we have a triple \( (\mathcal{V}_a, f_a^{(l)}, f_a^{(r)}) \) where \( \mathcal{V}_a \in \mathcal{P}_2(P_a) \) and

\[
f_a^{(l)} : L_a \rightarrow \{1, \ldots, |L_a|\} \quad \text{and} \quad f_a^{(r)} : R_a \rightarrow \{1, \ldots, |R_a|\}
\]

(36)
broken pair partition as just defined, we write the elements of the base set
with the index \(a\) to the numbers 1, . . . , \(|I_a|\). For \(I_a \neq \emptyset\), we write the numbers 1, . . . , \(|I_a|\) in order on the left side and connect each \(y \in I_a\) to the number \(f_a^{(l)}(y)\). Likewise, for each color \(a \in I\) such that \(R_a \neq \emptyset\), we write the numbers 1, . . . , \(|R_a|\) in order on the left side and connect each \(y \in R_a\) to the number \(f_a^{(r)}(y)\). When \(|I|\) is small, we may also use different line styles (e.g. dotted and solid lines) to indicate the different colors \(a \in I\). Figure 2 gives two examples of these diagrams.

The diagrammatic representations of the \(I\)-colored broken pair partitions inspires some additional terminology. Namely, we call the functions \(f_a^{(l)}\) and \(f_a^{(r)}\) the left and right leg functions for the color \(a\). Moreover, we call the piecewise-linear paths from the domains of \(f_a^{(l)}\) and \(f_a^{(r)}\) to the numbers \(f_a^{(l)}(y)\) and \(f_a^{(r)}(y)\) the left and right legs of the \(I\)-colored broken pair partitions. This terminology will be useful in describing the semigroup structure on \(BP^I_2(\infty)\).

In the case that \(|I| = 1\), we recover the (uncolored) broken pair partitions of Guță and Maassen [7]. Moreover, each \(d \in BP^I_2(\infty)\) gives for each \(a \in I\) a broken pair partition \(d_a\) in the sense of [7]. However, all but finitely many of the \(d_a\) are the unique broken pair partition on the empty set.

The space \(BP^I_2(\infty)\) can be given the structure of a semigroup with involution, similar to the \(*\)-semigroup of broken pair partitions of [7]. In terms of the diagrams, multiplication of two \(I\)-colored broken pair partitions corresponds to concatenation of diagrams. Right legs
of the first diagram are joined with left legs of the second diagram of the same color to form pairs. In the event that the second diagram has more left legs of some color $a$ than the first diagram has right legs of color $a$, we join the right legs of the first diagram with the largest-numbered left legs of the second diagram, and the remaining left legs of the second diagram are extended to become low-numbered left legs in the product. An analogous rule is used when the first diagram has more right legs of some color $a$ than the second diagram has left legs of color $a$.

The precise definition of the product on $B\mathcal{P}_2(\infty)$ is as follows. For $i = 1, 2$, let $d_i = (V_{a,i}, f_{a,i}^{(l)}, f_{a,i}^{(r)})_{a \in \mathcal{I}}$ be an $\mathcal{I}$-colored broken pair partition on the ordered base set $X_i$. The product is a broken pair partition on the base set $X := X_1 \coprod X_2$ with the order relation $x < x'$ if either $x < x'$ in $X_1$ or $x \in X_1$ and $x \in X_2$. For each $a \in \mathcal{I}$, define $M_a = \min(|R_{a,1}|, |L_{a,2}|)$. Following [6], we define

\[
d_1 \cdot d_2 = (V_a, f_a^{(l)}, f_a^{(r)})_{a \in \mathcal{I}},
\]

where

\[
V_a = V_{a,1} \cup V_{a,2} \cup \left\{ (f_{a,1}^{(r)})^{-1}([|R_{a,1}| - j]), (f_{a,2}^{(l)})^{-1}([|L_{a,2}| - j]) \right\} : j \in [M_a]
\]

and $f_a^{(l)}$ is defined on the disjoint union of $L_{a,1}$ and $(f_{a,2}^{(l)})^{-1}([|L_{a,2}| - M_a])$ by

\[
f_a^{(l)}(i) = \begin{cases} f_{a,1}^{(l)}(i), & \text{if } i \in L_{a,1} \\ f_{a,2}^{(l)}(i) + |L_{a,1}| - M_a, & \text{if } i \in (f_{a,2}^{(l)})^{-1}([|L_{a,2}| - M_a]) \end{cases}.
\]

The function of right legs, $f_a^{(r)}$ is defined on the disjoint union of $R_{a,2}$ and $(f_{a,1}^{(r)})^{-1}([|R_{a,1}| - M_a])$ by

\[
f_a^{(r)}(i) = \begin{cases} f_{a,2}^{(r)}(i), & \text{if } i \in R_{a,2} \\ f_{a,1}^{(r)}(i) + R_{a,2} - M_a, & \text{if } i \in (f_{a,1}^{(r)})^{-1}([|R_{a,1}| - M_a]) \end{cases}.
\]

An example of multiplication of $\mathcal{I}$-colored broken pair partitions is illustrated in Figure 3

The involution is given by mirror reflection of the $\mathcal{I}$-colored broken pair partitions. Formally, if $d = (V_a, f_a^{(l)}, f_a^{(r)})_{a \in \mathcal{I}}$ with underlying set $X$ then $d^* = (V_a^*, f_a^{(r)}, f_a^{(l)})_{a \in \mathcal{I}}$ is an $\mathcal{I}$-colored broken pair partition with underlying set $X^*$, the same as $X$ but with the order reversed, where $V_a^* = \{ (i, j) : (j, i) \in V_a \}$. The involution is illustrated in Figure 4

\footnote{The multiplication for $B\mathcal{P}_2(\infty)$ stated here differs slightly from that stated in [6]. We believe that the rule stated here is the one intended by the author of that work, as it ensures that condition (10) is satisfied. However, we do not believe that this discrepancy is consequential for Gut\'a’s results.}
Figure 4. The involution of the \{-1,1\}-colored broken pair partition $\tilde{d}_2$ defined in Figure 1.

Figure 5. The standard form of the pair partition $(V,c) \in P_2^I(10)$ with $I = \{-1,1\}$ and $V = \{(1,5), (2,8), (3,6), (4,10), (7,9)\}$ and $c((1,5)) = c((2,8)) = c((7,9)) = -1$ and $c((3,6)) = c((4,10)) = 1$. The solid lines represent the “color” $-1$ and the dotted lines represent the “color” $1$.

For each function $n : I \to \mathbb{N}$ which is zero except on finitely many elements of $I$, let $BP_2^I(n,0)$ be the subset of $BP_2^I(\infty)$ consisting of elements having exactly $n(a)$ left legs of color $a$ and no right legs.

Let $d_a$ be the unique element of $BP_2^I(\infty)$ with no right legs, no pairs, and only one left leg, colored $a \in I$. We call $d_a$ the $a$-colored left hook and $\tilde{d}_a$ the $a$-colored right hook.

An $I$-colored broken-pair partition can be written as a sequence of left hooks, followed by permutations acting on the legs of the same color, followed by right hooks connecting with left legs of the same index, possibly followed by additional sequences of left hooks, permutations, and right hooks. The “standard form” of an element of $P_2^I(\infty)$ is the sequence of this form such that if two like-colored pairs cross, they do so in the rightmost permutation possible. A diagram showing the standard form of one example is shown in Figure 5.

As in [6], we can use the standard form of $(V,c)$ for $V \in P_2(2n)$ to compute the value of $t((V,c))$ as follows. Consider $c$ to be a function $[2n] \to I$ taking the same value on points belonging to the same pair of $V$. Partition $[2n]$ into $2m$ blocks $B_i^{(r)}$, $B_i^{(l)}$ for $i = 1, \ldots, n$ such that the $B_j^{(r)}$ contain left legs of the pairs of $V$ and the $B_j^{(l)}$ contain right legs of the pairs of $V$. Write $B_i^{(r)} := \{k_{i-1}, \ldots, k_i\}$ and $B_j^{(l)} := \{p_{i+1}, \ldots, k_i\}$ with $k_0 = 1$ and $k_r = 2n$.

Then there are permutations $\pi_j$ such that

$$ (\mathcal{V},c) = \prod_{l=1}^{p_1} d_{c^{(l)}} \prod_{l=p_1+1}^{k_1} d_{c^{(l)}} \cdots \prod_{l=p_m+1} U_{n_l}(\pi_1) \prod_{l=p_{m+1}+1} U_{n_l}(\pi_r) \prod_{l=p_{m+1}+1} d_{c^{(l)}} $$

The function on pair partitions can then be calculated as

$$ t_{V,j}((\mathcal{V},c)) = \left\langle \xi_{\mathcal{V}} \prod_{l=1}^{p_1} j_{c^{(l)}} \prod_{l=p_1+1}^{k_l} j_{c^{(l)}} \cdots \prod_{l=p_m+1}^{k_r} \prod_{l=p_m+1}^{2n} j_{c^{(l)}} \xi_{\mathcal{V}} \right\rangle. $$
An $\mathcal{I}$-colored pair partition $\mathcal{V}$ can be considered as an element of $\mathcal{BP}_2^\mathcal{I}(\infty)$ having no left or right legs in the obvious way. A function $\mathbf{t}: \mathcal{P}_2^\mathcal{I}(\infty) \to \mathbb{C}$ thus extends to a function $\hat{\mathbf{t}}: \mathcal{BP}_2^\mathcal{I}(\infty) \to \mathbb{C}$ by

$$
\hat{\mathbf{t}}(\mathbf{d}) = \begin{cases} 
\mathbf{t}(\mathbf{d}), & \text{if } \mathbf{d} \in \mathcal{P}_2^\mathcal{I}(\infty), \\
0, & \text{otherwise}
\end{cases}
$$

(44)

The following is an $\mathcal{I}$-indexed generalization of Theorem 3.2 of [7].

**Theorem 3.** A function $\mathbf{t}: \mathcal{P}_2^\mathcal{I}(\infty) \to \mathbb{C}$ is positive definite if $\hat{\mathbf{t}}: \mathcal{BP}_2^\mathcal{I}(\infty) \to \mathbb{C}$ is positive definite in the usual sense of positive definiteness for a function on a semigroup with involution.

The proof is very similar to the proof of Theorem 3.2 of [7], but we include it here in the interest of completeness.

**Proof.** Suppose that $\hat{\mathbf{t}}$ is positive definite on the $*$-semigroup $\mathcal{BP}_2^\mathcal{I}(\infty)$. Then there is a representation $\chi$ of $\mathcal{BP}_2^\mathcal{I}(\infty)$ on a complex Hilbert space $V$ having cyclic vector $\xi \in V$ satisfying

$$
\langle \xi, \chi(d)\xi \rangle = \hat{\mathbf{t}}(d)
$$

(45)

for all $d \in \mathcal{BP}_2^\mathcal{I}(\infty)$.

The complex Hilbert space $V$ is expressible as a direct sum

$$
V = \bigoplus_n V_n,
$$

(46)

where the sum is over functions $\mathbf{n}: \mathcal{I} \to \mathbb{N}$ with only finitely many nonzero values and

$$
V_\mathbf{n} = \overline{\text{span}\{\chi_t(\mathbf{n})\xi: d \in \mathcal{BP}_2^\mathcal{I}(\mathbf{n}, 0)\}}.
$$

(47)

The action of $S_\mathbf{n}$ on $\mathcal{BP}_2^\mathcal{I}(\mathbf{n}, 0)$ (by permutation of the left legs) gives a unitary representation $U_\mathbf{n}$ of $S_\mathbf{n}$ on $V_\mathbf{n}$. Restriction of $j_\mathbf{b}$ := $\chi(d_\mathbf{b})$ (where, as before $d_\mathbf{b}$ is the $\mathbf{b}$-colored left hook) gives a map $j_\mathbf{b}: V_\mathbf{n} \to V_{\mathbf{n}+\delta_\mathbf{b}}$ satisfying (10). Choose a unit vector $\xi_\mathbf{V} \in V_0$.

Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space. Using the $U_\mathbf{n}$, $V_\mathbf{n}$ and $j_\mathbf{b}$, we can construct the Fock space $\mathcal{F}_V(\mathcal{H})$ and the algebra $\mathcal{C}_{\mathcal{V}, j}(\mathcal{H})$ with vacuum vector $\Omega_V$. By Theorem [1] the vacuum state is a Fock state arising from some positive definite function $\mathbf{t}' : \mathcal{BP}_2^\mathcal{I}(\infty) \to \mathbb{C}$. It will suffice to show that $\hat{\mathbf{t}}' = \mathbf{t}$, whence it will follow that $\mathbf{t}$ is positive definite. In fact, this follows from Theorem 2.3 of [6], but we also provide a proof for completeness.

Given an $\mathcal{I}$-indexed pair partition $(\mathcal{V}, c) \in \mathcal{P}_2^\mathcal{I}(2n)$ with $\mathcal{V} = \{(l_1, r_1), \ldots, (l_n, r_n)\}$, let the standard form of $(\mathcal{V}, c)$ be

$$
(\mathcal{V}, c) = \prod_{l=1}^{p_1} d_{c(l)}^* U_{\mathbf{n}_1}(\pi_{1}) \prod_{l=p_1+1}^{k_1} d_{c(l)} U_{\mathbf{n}_r}(\pi_{r}) \prod_{l=p_m+1}^{2n} d_{c(l)}.
$$

(48)

Let $\mathcal{H} = \ell^2(\mathbb{Z})$ have an orthonormal basis $(f_k)_{k=1}^{\infty}$, and choose a monomial

$$
M := \prod_{k=1}^{2n} a_{b_k}(f_k),
$$

(49)
where $i_k$ is chosen such that either $k = l_{i_k}$ or $k = r_{i_k}$, and $b_k = c((l_{i_k}, r_{i_k}))$, and $e_k = 1$ if $k = l_{i_k}$ and $e_k = 2$ if $k = r_{i_k}$. From the definition of a Fock state,
\begin{equation}
(50)
t'(\mathcal{V}) = \langle \Omega_V, M\Omega_V \rangle.
\end{equation}

Using the definition of the creation operator and \eqref{eq:10}, we get
\begin{equation}
(51)
t'(\mathcal{V}) = \prod_{l=1}^{p_1} d_{c(l)}^* U_{n_1}(\pi_1) \prod_{l=p_1+1}^{k_1} d_{c(l)} U_{n_r}(\pi_r) \prod_{l=p_{m+1}}^{2n} d_{c(l)}.
\end{equation}

This completes the proof that $t : \mathcal{P}_2^f(\infty) \to \mathbb{C}$ is positive definite.

Suppose now that $t : \mathcal{P}_2^f(\infty) \to \mathbb{C}$ is positive definite. Applying Theorem 2 we get complex Hilbert spaces $V_n$ with representations $U_n$ of $S_n$ and densely defined maps $j_b : V_n \to V_{n+\delta_b}$ satisfying the intertwining relation \eqref{eq:10}. Let
\begin{equation}
(52)\quad V = \bigoplus_n V_n
\end{equation}

Since the left hooks $\{d_b : b \in \mathcal{I}\}$ and the actions $\lambda_n$ of the symmetric groups $S_n$ on $\mathcal{B}_2^f(n, 0)$ generate $\mathcal{B}_2^f(\infty)$, we have a representation $\chi$ of $\mathcal{B}_2^f(\infty)$, and it is easily verified that $\langle \xi, \chi(d)\xi \rangle = t(d)$ for any unit vector $\xi \in V_0$ and any $d \in \mathcal{B}_2^f(\infty)$, whence $t$ is positive definite. \hfill \square

Guţă \cite{Gut2} considered the somewhat less general case in which the $V_n$ and the $j_a$ are defined as follows. Let $t : \mathcal{P}_2^f(\infty) \to \mathbb{C}$ be a positive definite function. As before, denote by $\mathcal{B}_2^f(n, 0)$ the set of $d \in \mathcal{B}_2^f(\infty)$ having $|R_a| = 0$ and $|L_a| = n(a)$ for each $a \in \mathcal{I}$. Consider the GNS representation $(\chi_t, V, \xi_t)$ of $\mathcal{B}_2^f(n, 0)$ with respect to $\hat{t}$, characterized by
\begin{equation}
(53)\quad \langle \chi_t(d_1)\xi_t, \chi_t(d_2)\xi_t \rangle_V = \hat{t}(d_1d_2).
\end{equation}

The complex Hilbert space $V$ is given by
\begin{equation}
(54)\quad V := \bigoplus_n V_n \quad \text{where} \quad V_n = \overline{\text{span}\{\chi_t(d)\xi_t : d \in \mathcal{B}_2^f(n, 0)\}}.
\end{equation}

Each $V_n$ has a representation of $S_n$ with $S_n(a)$ acting by permuting the left legs of $\mathcal{B}_2^f(n, 0)$ of color $a$. That is for $\pi = (\pi_a)_{a \in \mathcal{I}} \in S_n$ and $(\mathcal{V}_a, f_a^{(l)}, f_a^{(r)})_{a \in \mathcal{I}} \in \mathcal{B}_2^f(n, 0)$,
\begin{equation}
(55)\quad U_n(\pi)(\mathcal{V}_a, f_a^{(l)}, f_a^{(r)})_{a \in \mathcal{I}} = (\mathcal{V}_a, \pi_a^{-1} f_a^{(l)}, f_a^{(r)})_{a \in \mathcal{I}}.
\end{equation}

We denote by $\mathcal{F}_t(\mathcal{H})$ the Fock-like space arising from using these $V_n$ in \eqref{eq:8}
\begin{equation}
(56)\quad \mathcal{F}_t(\mathcal{H}) := \bigoplus_n \frac{1}{n!} V_n \otimes_s \mathcal{H}^\otimes n(a).
\end{equation}

The creation and annihilation operators on $\mathcal{F}_t(\mathcal{H})$ will be assumed to be those associated to the following operators $j_a$. For $a \in \mathcal{I}$, denote by $j_a$ the operator $\chi_t(d_a)$, where $d_a$ is the broken pair partition with no pairs, no right legs, and one $a$-colored left leg.

We can now state the following theorem of Guţă \cite{Gut2}.

\begin{proof}...
\end{proof}
**Theorem 4.** Let \( f_1, \ldots, f_n \) be vectors in a complex Hilbert space \( \mathcal{H} \). Then the expectation values with respect to the vacuum state \( \rho_t \) of the monomials in creation and annihilation operators on the Fock space \( \mathcal{F}_t(\mathcal{H}) \) have the expression

\[
(57) \quad \rho_t \left( \prod_{i=1}^{m} a^*_{b_i}(f_i) \right) = \sum_{(\mathcal{V},c)\in P_2(n)} t((\mathcal{V},c)) \prod_{(i,j)\in \mathcal{V}} \langle f_i, f_j \rangle \delta_{b_i,b_j} B_{e_i,e_j},
\]

where the \( e_i \) are chosen from \( \{1, 2\} \) and

\[
(58) \quad B := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

2.2. **Factor representations of \( S_\infty \).** In Sections 3 and 4, we will take an interest in generalized Brownian motions which are connected to factor representations of group \( S_\infty \) of permutations of \( \mathbb{N} \) which fix all but finitely many points. Here we provide briefly recall the relevant background information concerning those representations.

The finite factor representations of a group are determined by the group’s characters, that is, the positive, normalized indecomposable functions which are constant on conjugacy classes. In the case of \( S_\infty \), the characters are given by the following famous result.

**Theorem 5** (Thoma’s Theorem [13]). The normalized finite characters of \( S_\infty \) are given by the formula

\[
(59) \quad \phi_{\alpha,\beta}(\sigma) = \prod_{m \geq 2} \left( \sum_{i=1}^{\infty} \alpha_i^m + (-1)^{m+1} \sum_{i=1}^{\infty} \beta_i^m \right) \rho_m(\sigma)
\]

where \( \rho_m(\sigma) \) is the number of cycles of length \( m \) in the permutation \( \sigma \), and \( (\alpha_i)_{i=1}^{\infty} \) and \( (\beta_i)_{i=1}^{\infty} \) are decreasing sequences of positive real numbers such that \( \sum_i \alpha_i + \sum_i \beta_i \leq 1 \).

The pairs of sequences \( (\alpha_i)_{i=1}^{\infty} \) and \( (\beta_i)_{i=1}^{\infty} \) satisfying the conditions in Theorem 5 are commonly called Thoma parameters.

We now recall Vershik and Kerov’s representation of the symmetric group \( S_n \) (for \( n \in \{0, 1, 2, \ldots, \infty\} \) [14].

**Notation 4.** Fix sequences \( (\alpha_i)_{i=1}^{\infty} \) and \( (\beta_i)_{i=1}^{\infty} \), and let \( \gamma = 1 - \sum_i \alpha_i - \sum_i \beta_i \) and let \( \mathcal{N}_+ \) and \( \mathcal{N}_- \) be two copies of the set \( \{1, 2, \ldots\} \). Let \( Q := \mathcal{N}_+ \cup \mathcal{N}_- \cup [0, \gamma] \), and define a measure \( \mu \) on \( Q \) as the Lebesgue measure on \( [0, \gamma] \), \( \mu(i) = \alpha_i \) for \( i \in \mathcal{N}_+ \) and \( \mu(j) = \beta_j \) for \( j \in \mathcal{N}_- \). Let \( \mathcal{X}_n \) denote the \( n \)-fold Cartesian product of \( Q \) with the product measure \( m_n = \prod_1^n \mu \), and let \( S_n \) act on \( \mathcal{X}_n \) by \( \sigma(x_1, \ldots, x_n) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}) \). For \( x, y \in \mathcal{X}_n \), say that \( x \sim y \) if there exists \( \sigma \in S_n \) such that \( x = \sigma y \). Let \( \mathcal{X}_n = \{(x, y) \in \mathcal{X}_n \times \mathcal{X}_n : x \sim y\} \). The complex Hilbert space \( \mathcal{V}_n^{(\alpha,\beta)} \) defined by

\[
(60) \quad \mathcal{V}_n^{(\alpha,\beta)} := \left\{ f : \mathcal{X}_n \to \mathbb{C} | \infty > \|f\|^2 = \int_{\mathcal{X}_n} \sum_{x \sim y} |f(x,y)|^2 dm_n^{(\alpha,\beta)}(x) \right\}
\]

carries a unitary representation \( U_n^{(\alpha,\beta)} \) of \( S(n) \) given by

\[
(61) \quad (U_n^{(\alpha,\beta)}(\sigma)h)(x, y) = (-1)^{i(\sigma,x)} h(\sigma^{-1}x, y),
\]

where \( i(\sigma, x) \) is the number of inversions in the sequence \( (\sigma_1(x), \sigma_2(x), \ldots) \) of indices \( i_r(x) \) for which \( \sigma x_i \in \mathcal{N}_- \). Denote by \( 1_n \) the indicator function of the diagonal \( \{(x, x)\} \subset \mathcal{X}_n \).
Vershik and Kerov showed the following.

**Theorem 6 ([13]).** On $V_n^{(\alpha,\beta)}$,

\begin{equation}
\langle U_n^{(\alpha,\beta)}(\sigma)1_n,1_n \rangle = \phi_{\alpha,\beta}(\sigma).
\end{equation}

For $n = \infty$ we get the representation of $S_{\infty}$ associated to $\phi_{\alpha,\beta}$ in the convex hull of $1_{\infty}$.

There is an isometry $j_n : V_n^{(\alpha,\beta)} \to V_{n+1}^{(\alpha,\beta)}$ defined by

\begin{equation}
(j_n h)(x,y) = \delta_{x_{n+1},y_{n+1}} h((x_1,\ldots,x_n),(y_1,\ldots,y_n))
\end{equation}

### 2.3. Generalized Brownian motions arising from factor representations of $S_{\infty}$.

Before considering multi-dimensional generalized Brownian motions associated to representations of $S_{\infty}$, we review some of the work of [2] on Brownian motions with only one process. The Vershik-Kerov factor representations of the symmetric groups $S_n$ give all the data needed for a 1-dimensional generalized Brownian motion. Bożejko and Gută [2] were able to characterize the function on pair partitions arising from Theorem 2.6 of [7] (the one-dimensional version of Theorem 1). Before we can state their result, we will need some additional terminology.

**Definition 6 ([2]).** Let $\mathcal{V} \in \mathcal{P}_2(2n)$, and denote by $\hat{\mathcal{V}}$ the unique noncrossing pair partition such that the set of left points of $\mathcal{V}$ and $\hat{\mathcal{V}}$ coincide. A cycle in $\mathcal{V}$ is a sequence of pairs $((l_1,r_1),\ldots,(l_m,r_m))$ of $\mathcal{V}$ such that the pairs $(l_1,r_2),(l_2,r_3),\ldots,(l_m,r_1)$ belong to $\hat{\mathcal{V}}$. (In the case that $m = 1$ we interpret this condition as $(l_1,r_1) \in \hat{\mathcal{V}}$.). The number $m$ is called the length of the cycle. Denote by $\rho_m(\mathcal{V})$ the number of cycles of length $m$ in the pair partition $\mathcal{V}$.

We now state the result of Bożejko and Gută [2].

**Theorem 7.** Let $(\alpha_i)_{i=1}^{\infty}$ and $(\beta_i)_{i=1}^{\infty}$ be decreasing sequences of positive real numbers such that $\sum_i \alpha_i + \sum_i \beta_i \leq 1$. Let $V_n^{(\alpha,\beta)}$ be the complex Hilbert space of the Vershik-Kerov representation of $S_n$, and let $j_n : V_n^{(\alpha,\beta)} \to V_{n+1}^{(\alpha,\beta)}$ be the natural isometry. Let $\xi_{V_n^{(\alpha,\beta)}} = 1_n$. Denote the function on $\mathcal{P}_2(\infty)$ associated to these representations by Theorem 7 by $t_{\alpha,\beta}$. Then

\begin{equation}
t_{\alpha,\beta}(\mathcal{V}) = \prod_{m \geq 2} \left( \sum_{i=1}^{\infty} \alpha_i^m + (-1)^{m+1} \sum_{i=1}^{\infty} \beta_i^m \right)^{\rho_m(\mathcal{V})}.
\end{equation}

**Remark 4.** There is also another equivalent characterization of the cycle decomposition of a pair partition. As in Definition 6 let $\mathcal{V} = \{(a_1,z_1),\ldots,(a_n,z_n)\} \in \mathcal{P}_2(2n)$ and let $\mathcal{V}$ be the noncrossing pair partition whose left points coincide with those of $\mathcal{V}$. Let $\sigma \in S_n$ be the permutation such that $\hat{\mathcal{V}} = \{(a_1,z_{\sigma^{-1}(1)}),\ldots,(a_n,z_{\sigma^{-1}(n)})\}$. If the cycles of $\sigma$ are $\tau_i = (b_{i1} \cdots b_{ir_i}) \in S_n$ ($1 \leq i \leq m$), then the cycles of $\mathcal{V}$ are $\{(a_{b_{i1}},z_{b_{i1}}),\ldots,(a_{b_{ir_i}},z_{b_{ir_i}})\}$. Moreover, Theorem 7 says that $t_{\alpha,\beta}(\mathcal{V}) = \phi_{\alpha,\beta}(\sigma)$.

We are now in a position to show that the framework for multi-dimensional generalized Brownian motion presented here is more general than that presented in [6]. More precisely, we will exhibit complex Hilbert spaces $V_n$ with representations $U_n$ of $S_n$ and maps $j_n : V_n \to V_{n+\delta_n}$ and $V'_n$ with representations $U'_n$ of $S_n$ and maps $j'_n : V'_n \to V'_{n+\delta'_n}$ such that both sets of data give rise to the same function on pair partitions according to Theorem 1.
Example 1. We work with the index set \( \mathcal{I} = \{1\} \), which places us in the setting of the generalized Brownian motion with only one process, developed by Guță and Maassen in [7]. Fix an integer \( N \) with \( |N| > 1 \) and let \( \mathcal{H} \) be a complex Hilbert space. We will consider generalized Brownian motions associated to the character \( \phi_N \) of \( S_\infty \) given by the sequences

\[
\alpha_n = \begin{cases} 
\frac{1}{N}, & \text{if } 1 \leq n \leq N \\
0, & \text{otherwise}
\end{cases} \quad \text{and} \quad \beta_n = \begin{cases} 
\frac{1}{N}, & \text{if } 1 \leq n \leq -N \\
0, & \text{otherwise}
\end{cases}
\]

For each \( n \), let \( V_n^{(N)} \) be the complex Hilbert space of the Vershik-Kerov representation of \( S_n \), and let \( j^{(N)} : V_n^{(N)} \to V_{n+1}^{(N)} \) be the natural isometry. Let \( \xi_{V^{(N)}} = 1_0 \) be the indicator function of the diagonal. Denote the function on \( \mathcal{P}_2(\infty) \) associated to these representations by Theorem [1] by \( t_N \). It was shown in [2] that

\[
t_N(V) = \left( \frac{1}{N} \right)^{n - \rho(V)}.
\]

We will exhibit another sequence of complex Hilbert spaces \( \hat{V}_n^{(N)} \) with unitary representations \( \hat{U}_n^{(N)} \) which gives rise to the same positive function on pair partitions. Since the character \( \phi_N : S_\infty \to \mathbb{C} \) restricts to a positive definite function on \( S_n \), there is a representation \( \hat{j}_n^{(N)} \) of \( S_n \) on a complex Hilbert space \( \hat{V}_n^{(N)} \) with a cyclic vector \( \xi_n \) such that

\[
\left\langle \xi_n, \hat{U}_n^{(N)}(\pi)\xi_n \right\rangle = \phi_N(\pi).
\]

for every \( \pi \in S_n \). There is also a natural inclusion \( \hat{j}_n^{(N)} : \hat{V}_n^{(N)} \to \hat{V}_{n+1}^{(N)} \) satisfying

\[
\hat{j}_n^{(N)}(\hat{U}_n^{(N)}(\pi)\xi_n) = \hat{U}_{n+1}^{(N)}(\iota_n\pi)\xi_{n+1},
\]

where \( \iota_n \) is the inclusion \( S_n \to S_{n+1} \) induced by the natural inclusion \( [n] \subset [n+1] \). By construction, the maps \( \hat{j}_n^{(N)} \) and representations \( \hat{V}_n^{(N)} \) satisfy the intertwining relation (10), so we can construct the Fock space \( \mathcal{F}_{\hat{V}^{(N)},j^{(N)}} \) with creation and annihilation operators \( a^*_n \hat{V}_n^{(N)}j^{(N)}(f) \) and \( \phi_{\hat{V}^{(N)}j^{(N)}}(f) \).

The action of the algebra \( \mathcal{C}_{\hat{V}^{(N)},j^{(N)}}(\mathcal{H}) \) on \( \mathcal{F}_{\hat{V}^{(N)},j^{(N)}}(\mathcal{H}) \) is unitarily equivalent to the action of the algebra creation and annihilation operators on the following deformed Fock space. Let \( \mathcal{F}^{(alg)}(\mathcal{H}) = \bigoplus_n \mathcal{H}^{\otimes n} \). Define a sesquilinear form on \( \mathcal{F}^{(alg)}(\mathcal{H}) \) by sesquilinear extension of

\[
\langle f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_m \rangle_N = \delta_{mn} \sum_{\pi \in S_n} \phi_N(\pi) \langle f_1, g_{\pi(1)} \rangle \cdots \langle f_n, g_{\pi(n)} \rangle.
\]

This form is positive definite and thus gives an inner product on \( \mathcal{F}^{(alg)}(\mathcal{H}) \). Let \( \mathcal{F}_N(\mathcal{H}) \) be the completion of \( \mathcal{F}^{(alg)}(\mathcal{H}) \) with respect to the inner product \( \langle \cdot, \cdot \rangle_N \). Let \( D_N \) be the operator in \( \mathcal{F}^{(alg)}(\mathcal{H}) \) whose restriction to \( \mathcal{H}^{\otimes n} \) is given by

\[
D_N^{(n)} := \begin{cases} 
1 + \frac{1}{N} \sum_{k=2}^{n} \tilde{U}_n(\tau_{1,k}), & \text{if } n > 0 \\
1, & \text{otherwise},
\end{cases}
\]

where \( \tau_{i,k} \in S_n \) is the permutation transposing \( i \) and \( k \) and fixing all other elements of \( [n] \) and \( \tilde{U}_n \) is the representation of \( S_n \) such that \( \tilde{U}_n(\pi) \) permutes the tensors in \( \mathcal{H}^{\otimes n} \) according to \( \pi \). For \( f \in \mathcal{H} \) let \( l(f) \) and \( l^*(f) \) denote the left annihilation and creation (respectively)
operators on the free Fock space over $\mathcal{H}$. We define annihilation and creation operators on $\mathcal{F}^{(alg)}(\mathcal{H})$ by
\begin{align}
a_N(f) &= l(f)D_N \\
a_N^*(f) &= l^*(f).
\end{align}
It was shown in [2] that these operators are bounded with respect to $\langle \cdot, \cdot \rangle_N$ and thus extend to bounded linear operators on $\mathcal{F}_N(\mathcal{H})$.

The map
\begin{equation}
\mathcal{F}_N(\mathcal{H}) \mapsto \mathcal{F}_{\hat{V}_N}^{(alg)}(\mathcal{H}) \\
v_1 \otimes \cdots \otimes v_n &\mapsto \xi_n \otimes_s v_n \otimes \cdots \otimes v_1
\end{equation}
is unitary. It was shown in [2] that the vacuum state on the algebra of creation and annihilation operators on $\mathcal{F}_N(\mathcal{H})$ is the Fock state associated to the function $t_N(V) = \frac{1}{N}^{n-\rho(V)}$. This shows that $t_{V^{(N)},j^{(N)}} = t_{\hat{V}^{(N)},j^{(N)}}$ even though dim $V^{(N)} < \dim \hat{V}^{(N)}$ for $n \geq 1$.

3. Generalized Brownian motions associated to tensor products of representations of $S_\infty$

In this section, we are interested in the case where $\mathcal{I} = \{1, 2\}$ and the $V_n$ arise from unitary representations of the group $S_\infty$ of permutations of $\mathbb{N}$ fixing all but finitely many points.

**Notation 5.** When $\mathcal{I} = \{1, 2\}$, we will represent a function $n : \mathcal{I} \rightarrow \mathbb{N}$ by the pair $n(1), n(2)$, so we write $V_{r,s}$ for $V_n$ where $n(1) = r$ and $n(2) = s$. We will also represent an element $(\mathcal{V}, c)$ of $\mathcal{P}_2^\infty(\mathcal{V})$ as $(\mathcal{V}_1, \mathcal{V}_2)$, where $\mathcal{V}_i = c^{-1}(i)$.

One of the simplest such cases is that arising from the tensor product of two unitary representations of $S_\infty$. In this setting, we can prove the following.

**Proposition 1.** Let $(U^{(i)}, V^{(i)})$ be unitary representations of $S_\infty$ for $i = 1, 2$. Suppose that each $V^{(i)}_n$ is a subspace of $V^{(i)}$ carrying a unitary representation $U^{(i)}_n$ of $S_n$ with $j^{(i)}_1 : V^{(i)}_n \rightarrow V^{(i)}_{n+1}$ an isometry. Assume that we have distinguished unit vectors $\xi_{V^{(i)}} \in V^{(i)}_0$ and let $\xi_V = \xi_{V^{(1)}} \otimes \xi_{V^{(2)}}$. Let $V_{m,n} = V^{(1)}_m \otimes V^{(2)}_n$, $j_1 = j^{(1)} \otimes 1$, and $j_2 = 1 \otimes j^{(2)}$. Then
\begin{equation}
t_{V^{(j)}_1, j^{(j)}_2}((\mathcal{V}, c)) = t_{V^{(1)}, j^{(1)}}(\mathcal{V}_1) \cdot t_{V^{(2)}, j^{(2)}}(\mathcal{V}_2).
\end{equation}

**Proof.** From the definitions, it is clear that for any $v_1 \in V_m$ and $v_2 \in V_n$,
\begin{align}
&j_1 j_2(v_1 \otimes v_2) = j_2 j_1(v_1 \otimes v_2) \\
j_1^* j_2^*(v_1 \otimes v_2) &= j_2^* j_1^*(v_1 \otimes v_2) \\
j_1^* j_2(v_1 \otimes v_2) &= j_2 j_1^*(v_1 \otimes v_2) \\
j_1^* j_2^*(v_1 \otimes v_2) &= j_2^* j_1(v_1 \otimes v_2).
\end{align}
Consequently, if $b \neq b'$ the operators $a^{V^{(j)}, e}_b(f)$ and $a^{V^{(j)}, e'}_{b'}(f')$ commute for all $e, e' \in \mathcal{I}$ and all $f, f' \in \mathcal{H}$.

Assume that $\mathcal{V} := \{(l_1, r_1), \ldots, (l_n, r_n)\}$ with $l_k < r_k$ and $l_k < l_{k+1}$ for all $k$. Let $H$ be $\ell^2(\mathbb{N})$ with orthonormal basis $\{h_k\}_{k=1}^\infty$. We can compute $t_{V^{(j)}_1, j^{(j)}_2}((\mathcal{V}, c))$ as
\begin{equation}
t_{V^{(j)}_1, j^{(j)}_2}((\mathcal{V}, c)) = \left(\prod_{p=1}^{2n} a^{V^{(j)}, e}_p(h_{kp})\right) \xi_V \otimes_s \Omega, \xi_V \otimes_s \Omega.
\end{equation}
where \( k_p \) is the unique \( k \in [n] \) such that \( p \) is an element of the \( k \)-th pair of \( \mathcal{V} \) and \( e_p = 2 \) if \( p \) is a right point in \( \mathcal{V} \) and \( e_p = 1 \) if \( p \) is a right point.

Now, using the fact that \( a_{c(p)}^{e_p}(h_{k_p}) \) commutes with \( a_{c(p')}(h_{k_p}) \) when \( c(p) \neq c(p') \), we have

\[
\mathbf{t}_{V,j}((\mathcal{V}, c)) = \left( \prod_{c(p)=1} a_{1}^{j_{c(p)}(h_{k_p})} \right) \left( \prod_{c(p)=2} a_{2}^{j_{c(p)}(h_{k_p})} \right) \xi_V \otimes_s \Omega, \xi_V \otimes_s \Omega
\]

\[= 2 \left( \prod_{b=1}^{2} \left( \prod_{c(p)=b} a_{b}^{j_{c(p)}(h_{k_p})} \right) \xi_V \otimes_s \Omega, \xi_V \otimes_s \Omega \right) \]

\[= \mathbf{t}_{V(1),j(1)}(\mathcal{V}_1) \cdot \mathbf{t}_{V(2),j(2)}(\mathcal{V}_2).\]

This completes the proof. \(\square\)

Combining Theorem 7 with our Proposition 1 immediately gives the following.

**Corollary 1.** Let \( \mathcal{I} = \{1, 2\} \). Fix \((\alpha_i)_{i=1}^\infty\) and \((\beta_i)_{i=1}^\infty\) decreasing sequences of positive real numbers such that \( \sum_i \alpha_i + \sum_i \beta_i \leq 1 \) and let \( V_n^{(1)} = V_n^{(2)} = V_n^{(\alpha, \beta)} \) with the Vershik-Kerov representation of \( S_n \). For \( i \in \{1, 2\} \), let \( j^{(i)} : V_n^{(i)} \rightarrow V_{n+1}^{(i)} \) be the natural isometry. Let \( \xi_{V(i)} = 1_0 \) and let \( \xi_V = \xi_{V(1)} \otimes \xi_{V(2)} \). Let \( V_{m,n} = V_m^{(1)} \otimes V_n^{(2)}, j_1 = j^{(1)} \otimes 1, \) and \( j_2 = 1 \otimes j^{(2)} \). Then for \((\mathcal{V}, c) \in \mathcal{P}_2^{(\infty)}\),

\[
(76) \quad \mathbf{t}_{V,j}((\mathcal{V}, c)) = \prod_{m \geq 2} \left( \sum_{i=1}^\infty \alpha_i^m + (-1)^{m-1} \sum_{i=1}^\infty \beta_i^m \right)^{\rho_m(\mathcal{V}_1) + \rho_m(\mathcal{V}_2)}
\]

4. **Generalized Brownian motions associated to spherical representations of \((S_\infty \times S_\infty, S_\infty)\)**

In this section, we fix the index set \( \mathcal{I} = \{-1, 1\} \). Of course, we could equivalently take \( \mathcal{I} \) to be any two-element set, but we have chosen \( \{-1, 1\} \) for the reason that if \( a \in \mathcal{I} \) is a color then the other color can be concisely referenced as \(-a\).

G. Olshanski initiated the study of a broad class of representations of the infinite symmetric group \[12\], and this study was further developed by Okounkov \[11\]. In this framework, one considers unitary representations of a pair of groups \( K \subset G \) forming a Gelfand pair. Two groups \((G, K)\) form a Gelfand pair if for every unitary representation \((T, \mathcal{H})\) of \( G \), the operators \( P_K T(g) P_K \) commute with each other as \( g \) ranges over \( G \). Here \( P_K \) denotes the orthogonal projection of \( \mathcal{H} \) onto the subspace of \( K \)-invariant vectors for the representation \( T \).

Of interest to us are the spherical representations, which are defined as irreducible unitary representations of \( G \) with a non-zero \( K \)-fixed vector \( \xi \). If \( T \) is a spherical representation of the pair \((G, K)\), then the function \( g \mapsto \langle \xi, T(g) \xi \rangle \) is called a spherical function of \((G, K)\). Here we consider the case where \( G = S_\infty \times S_\infty \) and \( K = S_\infty \) is the diagonal subgroup. It is well-known (c.f. \[12\]) that the finite factor representations of a discrete group \( G \) are in natural bijective correspondence with the spherical representations of the Gelfand pair \((G \times G, G)\), where \( G \) is a subgroup of \( G \times G \) by the diagonal embedding.
In light of Thoma’s Theorem (Theorem 5) this means that the spherical functions of $(S_\infty \times S_\infty, S_\infty)$ are parametrized by the Thoma parameters and given by the formula

$$
\chi_{\alpha, \beta}(\pi, \pi') = \phi_{\alpha, \beta}(\pi' \pi^{-1}) = \prod_{m \geq 2} \left( \sum_{i=1}^{\infty} \alpha_i^m + (-1)^{m+1} \sum_{i=1}^{\infty} \beta_i^m \right) \rho_m(\pi' \pi^{-1}) .
$$

For the generalized Brownian motion construction, we can consider the following data. Let $(\alpha_i)_{i=1}^\infty$ and $(\beta_i)_{i=1}^\infty$ be a Thoma parameter. That is, let $(\alpha_i)_{i=1}^\infty$ and $(\beta_i)_{i=1}^\infty$ be decreasing sequences of positive real numbers such that $\sum_i \alpha_i + \sum_i \beta_i \leq 1$. Given $n_1, n_1 \in \mathbb{N} \cup \{0\}$ let $n = \max(n_1, n_1)$ and define

$$
V_{n_1, n_1} = V_n(\alpha, \beta),
$$

where $V_n(\alpha, \beta)$ is as in (60). Then $V_{n_1, n_1}$ carries a natural representation of $S_n \times S_n$ defined by

$$
(U_n(\alpha, \beta)(\pi, \pi')h)(x, y) = (-1)^{i(\sigma, x) + i(\pi, y)}h(\sigma^{-1}x, \pi^{-1}y),
$$

and thus a representation of $S_{n_1} \times S_{n_1}$ by restriction. It is easy to see that the indicator function of the diagonal is fixed by the diagonal subgroup.

Moreover, we define the map $j_{-1} : V_{n_1, n_1} \rightarrow V_{n_1+1, n_1}$ to be the natural embedding. For the case $\sum \alpha_i + \sum \beta_i = 1$, this means that $j_{-1}$ is given by

$$
j_{-1} \delta_{x, y} = \begin{cases} 
\delta_{x, y}, & \text{if } n_1 > n_1, \\
\sum_{z \in G} \delta_{(x, z), (y, z)}, & \text{otherwise}.
\end{cases}
$$

where $(x, z)$ means $(x_1, \ldots, x_{n_1}, z) \in \tilde{X}_{n+1}$. Likewise, we define the map $j_1 : V_{n_1, n_1} \rightarrow V_{n_1, n_1+1}$ to be the natural embedding.

We will also need to make use of the maps $j_a^\ast$ for $a \in I$. These maps are given by

$$
j_{-1}^\ast \delta_{x, y} = \begin{cases} 
\delta_{x, y}, & \text{if } n_1 \geq n_1, \\
\mu(x_{n_1}) \delta_{x_{n_1}, y_1} \delta_{(x_{n_1-1}, y_{n_1-1})}, & \text{otherwise}.
\end{cases}
$$

Here $x^{(n_1-1)}$ refers to the first $n_1 - 1$ terms of the $n_1$-tuple $(x_1, \ldots, x_{n_1-1})$ and the measure $\mu$ is as in Notation 4.

To motivate our results in the 2-colored case, we will consider another interpretation of the cycle decomposition of a pair partition. This interpretation involves some graph theory. We assume that a reader is familiar with the basic definitions from that field. We will sometimes need to consider graphs which have 2 edges with the same endpoints, so in this paper the word “graph” should be interpreted to mean “multigraph.” For a subgraph $H$ of $G$, we write $V(H)$ to mean the vertex set of $H$ and $E(H)$ to mean the edge set of $H$.

Given a pair partition $\mathcal{V} \in \mathcal{P}_2(2n)$, let $G_\mathcal{V}$ be the graph with vertex set $[2n]$ and edges $\mathcal{V} \cup \hat{\mathcal{V}}$. The graph $G_\mathcal{V}$ is the disjoint union of cycles (in the graph-theoretic sense), and the cycles of the graph $G_\mathcal{V}$ give the cycles of the pair partition $\mathcal{V}$. More precisely, if $C$ is a cycle of $G_\mathcal{V}$ then $E(C) \cap \mathcal{V}$ is a cycle of $\mathcal{V}$. In particular, this means that $\rho_m(\mathcal{V})$ is the number of cycles of $G_\mathcal{V}$ of length $2m$.

We can define a color function on $E(G_\mathcal{V})$ by

$$
c(\{l, r\}) = \begin{cases} 
-1, & \text{if } (l, r) \in \mathcal{V} \\
1, & \text{if } (l, r) \in \hat{\mathcal{V}}.
\end{cases}
$$
Traversing a cycle, we alternate between edges of color $-1$ and edges of color $1$. Thus, the graph has no monochrome paths longer than one edge, and the length of a cycle is the same as the number of maximal monochrome paths in that cycle.

Combining with Theorem [7]

\begin{equation}
(83) \quad t_{\alpha,\beta}(G) = \prod_{m \geq 2} \left( \sum_{i=1}^{\infty} \alpha_i^m + (-1)^{m+1} \sum_{i=1}^{\infty} \beta_i^m \right)^{\gamma_m(G)}.
\end{equation}

where $\gamma_m(G)$ denotes the number of cycles of $G$ having $2m$ maximal monochrome paths.

The case of a 2-colored pair partition is naturally more complicated. As in the uncolored case, our function on 2-colored pair partitions $(V, c)$ will be calculated with the aid of the cycle decomposition of a graph (denoted $G(V)$) with a coloring function on its edges, but the construction of a graph from a 2-colored pair partition will be rather more involved. However, in the case that the coloring function is the constant function $c((l, r)) = -1$, the graph $G(V)$ will be identical to the graph $G_V$ just described.

Suppose that $V = \{(l_1, r_1), \ldots, (l_n, r_n)\}$, and let $c : V \rightarrow \{-1, 1\}$ be a coloring function. To define the graph $G(V)$ we will need the functions

\begin{equation}
(84) \quad p^a_{(V, c)} : [0, 2|V|] \rightarrow \mathbb{N} \cup \{0\}
\end{equation}

\begin{equation*}
p^a_{(V, c)}(u) = \begin{cases} \left| \{ j \in [n] : l_j \leq u < r_j, c((l_j, r_j)) = a \} \right|, & \text{if } u \in [2|V|], \\ 0, & \text{if } u = 0. \end{cases}
\end{equation*}

for $a \in I = \{-1, 1\}$. We also define

\begin{equation}
(85) \quad p_{(V, c)}(u) = \max\{p_{(V, c)}^{-1}(u), p_{(V, c)}^1(u)\}.
\end{equation}

In terms of the diagrams (e.g. Figure [1]), $p^a_{(V, c)}(u)$ is the number of $a$-colored paths intersecting the vertical line drawn between $m$ and $m + 1$. We also define

\begin{equation}
(86) \quad S^a_{(V, c)}(u) = \{ u \in [0, 2|V|] : p^a_{(V, c)}(u) \geq m \}.
\end{equation}

The following properties are immediate consequences of the definitions and will be used frequently.

**Proposition 2.** Suppose that $(V, c) \in \mathcal{BP}_{\mathcal{F}}^\infty$, $V = \{(l_1, r_1), \ldots, (l_n, r_n)\}$, $L_V = \{l_1, \ldots, l_n\}$, $R_V = \{r_1, \ldots, r_n\}$, $u \in [0, 2n]$ and $a \in I = \{-1, 1\}$.

1. If $p^a_{(V, c)}(u) > p^a_{(V, c)}(u - 1)$ then $c(u) = a$ and $u \in L_V$.
2. If $p^a_{(V, c)}(u) < p^a_{(V, c)}(u - 1)$ then $c(u) = a$ and $u \in R_V$.
3. $p^a_{(V, c)}(u) - p^a_{(V, c)}(u - 1) \in \{-1, 0, 1\}$.

**Definition 7.** If $(V, c) \in \mathcal{P}_{\mathcal{F}}^\infty$, the multigraph $G(V)$ is the multi-graph with vertices $[2|V|]$ and edges defined as follows. Let $V = \{(l_1, r_1), \ldots, (l_n, r_n)\}$, we let $R_V = \{r_1, \ldots, r_n\}$
and $L_V = \{l_1, \ldots, l_n\}$. Let
\[
E_{(V,c)}^{(0)} = \{\{l, r\} : (l, r) \in V\}
\]
\[
E_{(V,c)}^{(1,l)} = \{\{l, l'\} \subseteq L_V : c(l) = -c(l'), l < l', [l, l'] \cap S^c_{(V,c)}(p_{(V,c)}^c(l)) = \{l'\}\}
\]
\[
E_{(V,c)}^{(1,r)} = \{\{r, r'\} \subseteq R_V : c(r) = -c(r'), r > r', [r, r'] \cap S^c_{(V,c)}(p_{(V,c)}^c(r)) = \{r'\}\}
\]
\[
E_{(V,c)}^{(2)} = \{\{r, l\} : r \in R_V, l \in L_V, r < l, c(r) = c(l), p_{(V,c)}^c(r) - 1 = p_{(V,c)}^c(l),
[r, l - 1] \subseteq S^{-c}_{(V,c)}(p_{(V,c)}^c(l)), [r, l - 1] \cap S^{-c}_{(V,c)}(p_{(V,c)}^c(l)) = \emptyset\}
\]
\[
E_{(V,c)}^{(3)} = \{\{l, r\} : r \in R_V, l \in L_V, l < r, c(r) = c(l), p_{(V,c)}^c(r) - 1 = p_{(V,c)}^c(l),
[l, r - 1] \subseteq S^c_{(V,c)}(p_{(V,c)}^c(l) + 1), [l, r - 1] \cap S^{-c}_{(V,c)}(p_{(V,c)}^c(l) + 1) = \emptyset\}
\]

The graph $G_{(V,c)}$ has edge set
\[
E_{(V,c)} := E_{(V,c)}^{(0)} \bigsqcup E_{(V,c)}^{(1,l)} \bigsqcup E_{(V,c)}^{(1,r)} \bigsqcup E_{(V,c)}^{(2)} \bigsqcup E_{(V,c)}^{(3)}
\]

Note that $E_{(V,c)}^{(0)}$ and $E_{(V,c)}^{(0)}$ may have nonempty intersection. We want to allow for the possibility that two vertices be connected by two distinct edges, which is why we must use a disjoint union. An example of the graph $G_{(V,c)}$ is depicted in Figure 4.

Lemma 1. Let $(V, c) \in P_2^T(\infty)$ be a $\{-1, 1\}$-colored pair partition with $|V| = n$. Then the graph $G_{(V,c)}$ is the disjoint union of cycles.

Proof. It will suffice to show that each vertex has degree 2. Evidently, each vertex is an endpoint of one edge in $E_{(V,c)}^{(0)}$. It therefore suffices to show that each vertex is an endpoint of exactly one other edge in $E_{(V,c)}^{(1,l)} \bigsqcup E_{(V,c)}^{(1,r)} \bigsqcup E_{(V,c)}^{(2)} \bigsqcup E_{(V,c)}^{(3)}$. We will first show that each vertex is an endpoint of at least one edge in that set. Then we will show that none of those edges share an endpoint.

Let $L_V$ denote the set of left points of $V$ and $R_V$ denote the set of right points of $V$. Consider the case of a vertex $l \in L_V$. First suppose that $p_{(V,c)}^c(l) > p_{(V,c)}^c(l)$. Define
\[
\hat{r} = \min \{r \in R_{(V,c)} : r > l, c(r) = c(l), p_{(V,c)}^c(r) - 1 = p_{(V,c)}^c(l)\}
\]
and let $T = [l + 1, \hat{r} - 1] \cap S_{(V,c)}^{-c(l)}(p_{(V,c)}^{c(l)}(l))$. If $T \neq \emptyset$, let $l' = \min T$. We claim that 

$\{l, l'\} \in E_{(V,c)}^{(1,l)}$. By Proposition 2, $l' \in L_V$ and $c(l') = -c(l)$. The conditions $l < l'$ and $[l, l'] \cap S_{(V,c)}^{-c(l)}(p_{(V,c)}^{c(l)}(l)) = \{l'\}$ follow from the choice of $l'$. If $T = \emptyset$, then it follows from the choice of $\hat{r}$ and Proposition 2 that $\{l, \hat{r}\} \in E_{(V,c)}^{(3)}$.

Next consider the case in which $p_{(V,c)}^{c(l)}(l) \leq p_{(V,c)}^{c(l)}(l)$. Let

$$l' = \max \left\{ u \in L_V \cap [1, \ldots, l - 1] : p_{(V,c)}^{c(l)}(u) = p_{(V,c)}^{c(l)}(l) \right\}$$

and

$$U = [l', l - 1] \cap S_{(V,c)}^{c(l)}(p_{(V,c)}^{c(l)}(l)).$$

We claim that if $U = \emptyset$ then $\{l', l\} \in E_{(V,c)}^{(1,l)}$. By Proposition 2, $l' \in L_V$ and $c(l) = -c(l')$, and the other conditions follow immediately from the definitions. If $U \neq \emptyset$, take $r = \max U$.

Then $\{r, l\} \in E_{(V,c)}^{(2)}$, again by using Proposition 2 and the choice of the point $r$.

We now show that no left point has degree larger than two. First suppose we have $l, l', l'' \in L_V$ such that $\{l, l'\}, \{l, l''\} \in E_{(V,c)}^{(1,l)}$ and $l < l'$. From the definition, $l < l'$ if and only if

$$p_{(V,c)}^{c(l)}(l) > p_{(V,c)}^{c(l')}(l) = p_{(V,c)}^{c(l)}(l) = p_{(V,c)}^{c(l'')}(l).$$

Thus $l < l'$ if and only if $l < l''$. If $l < l' < l''$ then

$$[l, l'] \cap S_{(V,c)}^{c(l)}(p_{(V,c)}^{c(l)}(l)) = \{l'\} \quad \text{and} \quad [l, l''] \cap S_{(V,c)}^{c(l)}(p_{(V,c)}^{c(l)}(l)) = \{l''\}.$$  

However, since $c(l') = c(l'')$ and $l' \in [l, l'']$, this is a contradiction. Suppose now that $l' < l'' < l$. Then

$$p_{(V,c)}^{c(l')}(l') = p_{(V,c)}^{c(l)}(l) = p_{(V,c)}^{c(l'')}(l'') = p_{(V,c)}^{c(l')}(l'').$$

But $l'' \in [l', l]$, contradicting

$$[l', l] \cap S_{(V,c)}^{c(l)}(p_{(V,c)}^{c(l')}(l')) = \{l\}.$$ 

This shows that no two edges in $E_{(V,c)}^{(1,l)}$ share an endpoint.

Since each edge in $E_{(V,c)}^{(1,l)}$ has endpoints in $L_V$ and each edge in $E_{(V,c)}^{(1,r)}$ has endpoints in $R_V$, it is clear that the edges in $E_{(V,c)}^{(1,l)}$ and $E_{(V,c)}^{(1,r)}$ have no endpoints in common.

Now suppose that $\{l, l'\} \in E_{(V,c)}^{(1,l)}$ and $\{r, l\} \in E_{(V,c)}^{(3)}$. If $l < l' < r$ then $p_{(V,c)}^{c(l')}(l') = p_{(V,c)}^{c(l)}(l)$, which conflicts with the assumption that $[l, r - 1] \subset S_{(V,c)}^{c(l)}(p_{(V,c)}^{c(l)}(l) + 1)$. If on the other hand, $l < r < l'$ then $p_{(V,c)}^{c(l)}(r - 1) \geq p_{(V,c)}^{c(l)}(l)$ by the definition of $E_{(V,c)}^{(3)}$, contradicting

$$[l, l'] \cap S_{(V,c)}^{c(l)}(p_{(V,c)}^{c(l)}(l)) = \{l'\}.$$ 

Now consider the case $l' < l < r$. In this case, we have $p_{(V,c)}^{c(l'}(l') = p_{(V,c)}^{c(l)}(l) \text{ since } \{l', l\} \in E_{(V,c)}^{(1,l)}$ but $p_{(V,c)}^{c(l)}(l) < p_{(V,c)}^{c(l)}(l') \text{ since } \{l, r\} \in E_{(V,c)}^{(3)}$. Thus an edge in $E_{(V,c)}^{(1,l)}$ cannot share an endpoint with an edge in $E_{(V,c)}^{(3)}$. The proof that an edge in $E_{(V,c)}^{(1,l)}$ cannot share an endpoint with an edge in $E_{(V,c)}^{(2)}$ is very similar.
Suppose that there are two edges \((r, l), (r', l) \in E^{(2)}_{(V, c)}\) with \(r \neq r'\). We can assume without loss of generality that \(r < r' < l\). Then
\[
P^{c(l)}_{(V, c)}(r') = p^{c(l)}_{(V, c)}(r) = p^{c(l)}_{(V, c)}(l).
\]
But this means that \(r' \in S^{c(l)}(p^{c(l)}_{(V, c)}(l))\), which contradicts \([l, r - 1] \cap S^{c(l)}(p^{c(l)}_{(V, c)}(l)) = \emptyset\). A similar argument shows that no two edges in \(E^{(3)}_{(V, c)}\) can share a left point.

Suppose now that for some \(r, r'\) we have \(\{r, l\} \in E^{(2)}_{(V, c)}\) and \(\{l, r\} \in E^{(3)}_{(V, c)}\). Evidently we must have \(r < l < r'\) and \(c(r) = c(l) = c(r')\). Then \(l - 1 \in S^{-c(l)}(p^{c(l)}_{(V, c)}(l))\) means that \(p^{-c(l)}_{(V, c)}(l - 1) \geq p^{c(l)}_{(V, c)}(l)\). Since \(p^{-c(l)}_{(V, c)}(l - 1) = p^{-c(l)}_{(V, c)}(l)\), we find that \(p^{-c(l)}_{(V, c)}(l) \geq p^{c(l)}_{(V, c)}(l)\).

However, \(l \in S^{-c(l)}(p^{-c(l)}_{(V, c)}(l)) + 1\) implies that \(p^{c(l)}_{(V, c)}(l) \geq p^{-c(l)}_{(V, c)}(l) + 1\), a contradiction.

This shows that any left point of \(V\) has degree 2 in \(G_{(V, c)}\). The proof for a right point is by similar arguments.

The next lemma, as well as Theorem 8 will rely on an extension of the color function \(c : V \to \{-1, 1\}\) to a function \(c : E(G_{(V, c)}) \to \{-1, 1\}\).

**Definition 8.** If \((V, c) \in \mathcal{P}^I_2(\infty)\), we extend \(c : V \to \mathcal{I}\) to a map \(c : V \to E(G_{(V, c)})\) as follows.
\[
c(e) = \begin{cases} 
c((l, r)), & \text{if } e = (l, r) \in E^{(0)}_{(V, c)} = V, \\
c(l'), & \text{if } e = \{l, l'\} \in E^{(1)}_{(V, c)} \text{ with } l < l' \\
c(r), & \text{if } e = \{r, r'\} \in E^{(1, r)}_{(V, c)} \text{ with } r < r' \\
c(l), & \text{if } e = \{l, r\} \in E^{(2)}_{(V, c)} \\
c(l), & \text{if } e = \{l, r\} \in E^{(3)}_{(V, c)} \\
-c(l), & \text{if } e = \{l, r\} \in E^{(3)}_{(V, c)} \\
\end{cases}
\]

The graph in Figure 8 is colored according to Definition 8.

**Remark 5.** If \(c\) takes only one value on \(V\) then \(E^{(1, r)}_{(V, c)} = E^{(1)}_{(V, c)} = E^{(2)}_{(V, c)} = \emptyset\) and
\[
E^{(3)}_{(V, c)} = \{\{l, r\} : (l, r) \in \hat{V}\}.
\]
Thus \(G_{(V, c)}(e) = G_V(e)\), and the color functions coincide also.

A cycle \(C\) in a graph \(G\) can be decomposed into maximal monochrome paths. That is, there are paths \(P_1, \ldots, P_m\) such that \(c\) is constant on the edges of each \(P_j\), there is no path \(P'_j\) in \(C\) which contains \(P_j\) and \(c\) is constant on the edges of \(P'_j\), and \(E(P_j) \cap E(P_k) = \emptyset\) for \(j \neq k\).

**Lemma 2.** Let \(\mathcal{I} = \{-1, 1\}\) and let \((V, c) \in \mathcal{P}^I_2(2n)\). Define
\[
Y^+_{(V, c)} := \{u \in [0, 2n + 1] : p_{(V, c)}(u - 1) - p_{(V, c)}(u)\} = 1,
\]
\[
Y^-_{(V, c)} := \{u \in [0, 2n + 1] : p_{(V, c)}(u - 1) - p_{(V, c)}(u)\} = -1.
\]
If \(C\) is a cycle of \(G_{(V, c)}\), then each maximal monochrome path of \(C\) has one endpoint in \(Y^+_{(V, c)}\) and one endpoint in \(Y^-_{(V, c)}\), and the vertices internal to the maximal monochrome paths are not in \(Y^+_{(V, c)}\) or \(Y^-_{(V, c)}\). In particular,
\[
|V(C) \cap Y^+_{(V, c)}| = |V(C) \cap Y^-_{(V, c)}| \geq 1.
\]
Proof. Let \( C \) be a cycle of \( G_{(V,c)} \). We first show that any cycle contains edges of both colors in \( \{-1,1\} \). Suppose that some cycle \( C \) contains edges of only one color \( a \in \mathcal{I} \). All of the edges of \( C \) are in \( E^{(0)}_{(V,c)} \) or \( E^{(2)}_{(V,c)} \), because the edges in the other sets by definition must be adjacent to at least one edge of the opposite color. Suppose we start at some vertex \( l_1 \) and walk along an edge \( \{l_1,r_1\} \in E^{(0)}_{(V,c)} \) of \( C \). Then \( l_1 < r_1 \) and the next edge must be \( \{r_1,l_2\} \in E^{(2)}_{(V,c)} \) with \( r_1 < l_2 \). The next edge is necessarily of the form \( \{l_2,r_2\} \in E^{(0)}_{(V,c)} \) with \( l_2 < r_2 \). Continuing in this way, we could construct an arbitrarily long sequence

\[
\tag{101}
l_1 < r_1 < l_2 < r_2 < \cdots < l_N < r_N < \cdots
\]

which is impossible. The same contradiction is reached if we start along an edge in \( E^{(2)}_{(V,c)} \).

Suppose that \( l \in L_V \) is an endpoint of a maximal monochrome path. Then \( l \) is an endpoint of some edge \( \{l,r\} \in E^{(0)}_{(V,c)} \) and some other edge \( \{l,u\} \) with \( u \). Since the edges of \( E^{(2)}_{(V,c)} \) are adjacent to edges of the same color, we cannot have \( \{l,u\} \in E^{(0)}_{(V,c)} \), and since \( l \) is a left point, we cannot have \( \{l,u\} \in E^{(1)}_{(V,c)} \). If \( \{l,u\} \in E^{(1)}_{(V,c)} \) then since \( c(\{l,u\}) = c(\{l,r\}) \), we must have \( l < u \). It is immediate from the definition of \( E^{(1)}_{(V,c)} \) that \( p^{(l)}_{(V,c)}(l) > p^{(u)}_{(V,c)}(l) \). Therefore, \( p_{(V,c)}(l) = p^{(l)}_{(V,c)}(l) \) and \( p_{(V,c)}(l-1) = p^{(l)}_{(V,c)}(l) - 1 \) and so \( l \in Y^+_{(V,c)} \). A very similar argument shows that if \( \{l,u\} \in E^{(3)}_{(V,c)} \) then \( l \in Y^+_{(V,c)} \). In the case that \( r \in R_V \) is the endpoint of a maximal monochrome path, similar arguments show that \( r \in Y^-_{(V,c)} \).

It remains only to show that the internal vertices of a maximal monochrome path do not belong to \( Y^+_{(V,c)} \) or \( Y^-_{(V,c)} \). Suppose that \( l \in L_V \) is such a vertex. Then \( l \) is an endpoint of some edge \( \{l,r\} \in E^{(0)}_{(V,c)} \) and an edge \( \{l,u\} \) with either \( \{l,u\} \in E^{(1)}_{(V,c)} \) or \( \{l,u\} \in E^{(2)}_{(V,c)} \). If \( \{l,u\} \in E^{(1)}_{(V,c)} \) then we must have \( u \in L_V \) and \( u < l \). By the definition of \( E^{(1)}_{(V,c)} \) that \( p^{(u)}_{(V,c)}(l) > p^{(l)}_{(V,c)}(l) \), whence \( p_{(V,c)}(l) = p^{(l)}_{(V,c)}(l) \) and \( p_{(V,c)}(l-1) = p^{(l)}_{(V,c)}(l) \) and so \( l \notin Y^+_{(V,c)} \cup Y^-_{(V,c)} \). A similar argument holds if \( \{l,u\} \in E^{(2)}_{(V,c)} \), as well as for the case of the right point \( r \in R_{(V,c)} \) which is not the endpoint of a maximal monochrome path.

Since the number of maximal monochrome paths is always even, it makes sense to denote by \( \gamma_m(G) \) the number of cycles of a 2-colored graph \( G \) with \( m \) maximal monochrome paths. Alternatively, \( \gamma_m(G) \) is the number of cycles of \( G \) having \( m \) maximal monochrome paths of \( each \) color.

With this established, we can now state the main result of this section.

**Theorem 8.** Let \( (\alpha_i)_{i=1}^{\infty} \) and \( (\beta_i)_{i=1}^{\infty} \) be decreasing sequences of positive real numbers such that \( \sum_i \alpha_i + \sum_i \beta_i \leq 1 \). Let \( V_{n_{(a,b)}} \) be the complex Hilbert space of the Vershik-Kerov representation of \( S_{\max(n-1,n_1)} \) endowed with the representation of \( (0) \), and let \( j_n^{-1}: V_{n_{(a,b)}} \rightarrow V_{n_{(a,b)}} \) and \( j_n^{1}: V_{n_{(a,b)}} \rightarrow V_{n_{(a,b)}} \) be the natural embedding. Let \( \xi_{V_{(a,b)}} = 1_0 \) be the indicator function of the diagonal. Denote by \( t_{a,b} \) the function on \( \mathcal{P}_2(\infty) \) associated to this sequence of representations by Theorem 7.

Let \( (V,c) \) be a \( \{-1,1\} \)-colored pair partition. Then

\[
\tag{102}
t_{a,b}(V,c) = \prod_{m \geq 2} \left( \sum_{i=1}^{\infty} \alpha_i^m + (-1)^{m+1} \sum_{i=1}^{\infty} \beta_i^m \right)^{\gamma_m(G_{(V,c)})}.
\]
Proof. We will focus on the case \( \sum \alpha_i + \sum \beta_i = 1 \) for simplicity. The key ideas of the more general case \( \sum \alpha_i + \sum \beta_i \leq 1 \) are in this case, but this case is slightly simpler in that it allows us to think about discrete sums instead of integrals.

Let \( n = |\mathcal{V}| \) and let \( \mathcal{H} \) be an infinite-dimensional complex Hilbert space with an orthonormal basis \( \{ h_i \}_{i=1}^{\infty} \). By Theorem 1, we can compute \( t_{V,j}((\mathcal{V},c)) \) as

\[
(103) \quad t_{V,j}((\mathcal{V},c)) = \left\langle \left( a_{c(1)}^{e_1}(h_{k_1}) \cdots a_{c(n)}^{e_n}(h_{k_n}) \right), 1_0 \otimes \Omega, 1_0 \otimes \Omega \right\rangle,
\]

where \( k_i \) is the unique \( k \in [n] \) such that \( i \) is an element of the \( k \)-th pair of \( \mathcal{V} \) and \( e_i = 2 \) if \( i \) is a right point in \( \mathcal{V} \) and \( e_i = 1 \) if \( i \) is a right point.

We consider the action of the creation and annihilation operators on a vector of the form

\[
(104) \quad \delta_{(x,y)} \otimes \Omega h_{l_1} \otimes \cdots \otimes h_{l_m} \otimes h'_{l_1} \otimes \cdots \otimes h'_{m'}
\]

where \( x, y \in \mathcal{X}_{\text{max}(\mathcal{V})} \) and \( x = \sigma y \) for some \( \sigma \in S_{\text{max}(\mathcal{V})} \). For compactness, we will omit the superscripts \( V^{(\alpha,\beta)}, j^{(\alpha,\beta)} \). A creation operator \( a_{-1}^\ast(h_r) \) applied to this vector gives us

\[
(105) \quad a_{-1}^\ast(h_r) \left( \delta_{(x,y)} \otimes \Omega h_{l_1} \otimes \cdots \otimes h_{l_m} \otimes h'_{l_1} \otimes \cdots \otimes h'_{m'} \right) =
\]

\[
\begin{cases}
\frac{1}{m} U(\tau_{m,k}, 1) \delta_{(x,y)} \otimes \Omega h_{l_1} \otimes \cdots \otimes h_{l_m} \otimes h'_{l_1} \otimes \cdots \otimes h'_{m'}, & \text{if } m < m' \\
\frac{1}{m} \sum_{k \in Q} \sum_{\alpha,\beta} \delta_{(x,y)} \otimes \Omega h_{l_1} \otimes \cdots \otimes h_{l_m} \otimes h'_{l_1} \otimes \cdots \otimes h'_{m'}, & \text{if } m \geq m'
\end{cases}
\]

There is, of course, a corresponding expression for the creation operator \( a_{1}^\ast(h_r) \).

For the annihilation operators, we can take \( r = l_k \) and we may further assume that \( l_k \neq l_k' \) for \( k \neq k' \). In this case,

\[
(106) \quad a_{-1}(h_{l_k}) \left( \delta_{(x,y)} \otimes \Omega h_{l_1} \otimes \cdots \otimes h_{l_m} \otimes h'_{l_1} \otimes \cdots \otimes h'_{m'} \right) =
\]

\[
\begin{cases}
\frac{1}{m} \sum_{k \in Q} \sum_{\alpha,\beta} \delta_{(x,y)} \otimes \Omega h_{l_1} \otimes \cdots \otimes h_{l_m} \otimes h'_{l_1} \otimes \cdots \otimes h'_{m'}, & \text{if } m \geq m'
\end{cases}
\]

where \( \tau_{mn} \) denotes the transposition of \( k \) and \( m \), and \( x^{(k)} \) and \( y^{(m)} \) denote the sequences \( x \) and \( y \) with the \( k \)-th and \( m \)-th terms (respectively) omitted. There is a corresponding expression for the annihilation operator \( a_{1}(h_{l_k}) \) in which permutations are applied to the right sequence instead of the left one.

The creation operators that take a vector from \( V_{\tilde{m}}^{(\alpha,\beta)} \) to something in \( V_{\tilde{m}+1}^{(\alpha,\beta)} \) (for some \( \tilde{m} \)) are precisely those at positions in the sequence of operators in (103) indexed by values in \( Y_{(-c)}^{-} \). Likewise, the annihilation operators that take a vector from \( V_{\tilde{m}}^{(\alpha,\beta)} \) to something in \( V_{\tilde{m}-1}^{(\alpha,\beta)} \) are precisely those at positions indexed by values in \( Y_{(c)}^{+} \). The possible sequences of terms of the form \( \delta_{(x,y)} \in V_{\tilde{m}}^{(\alpha,\beta)} \) produced by the creation operators are in natural correspondence with the labelings of the maximal monochrome paths of the graph \( G_{(c)} \) by elements of \( Q \) such that any two paths sharing the same right-most point have the same label. However, not all of these sequences will survive the action of the annihilation operators. In particular, the annihilation operators in positions numbered at \( Y_{(c)}^{+} \) (that is, the endpoints of the maximal monochrome paths) will remove those terms in which the last element of the words \( x \) and \( y \) do not agree. Thus, the sequences of terms that survive are those corresponding to labelings of the maximal monochrome paths in which any two paths
sharing the same left-most point have the same label. In other words, the labelings of the graph which contribute to the sum are those in which two paths on the same cycle have the same label. Those labelings which do survive contribute a factor of $\mu(x_j)$ for each pair of maximal monochrome paths labeled with $x_j$.

Denoting the set of cycles of $G_{(V, c)}$ by $C(G_{(V, c)})$ and the number of maximal $-1$-colored paths of a cycle $K$ by $M(K)$,

$$t_{\alpha, \beta}((V, c)) = \sum_{x : C(G_{(V, c)}) \to Q} \mu(x(K)) \cdot M(K)$$

$$= \prod_{m \geq 2} \left( \sum_{i=1}^{\infty} \alpha_i^m + (-1)^{m+1} \sum_{i=1}^{\infty} \beta_i^m \right)^{\gamma_m(G_{(V, c)})}.$$ 

\[ \square \]

5. The $q_{ij}$-product of generalized Brownian motions

In this section, we present a generalization of Guță’s $q$-product of generalized Brownian motions.

**Definition 9.** For $V \in \mathcal{P}_2(\infty)$, define the set of crossings of $V$ by

$$\text{cr}(V) = \{(a_1, z_1), (a_2, z_2) \in V \times V : a_1 < a_2 < z_1 < z_2\}.$$ 

If $(V, c) \in \mathcal{P}_2^I(\infty)$, we also define $\text{cr}(V, c) = \text{cr}(V)$. Suppose that for each $i \in I$, a positive-definite function $t_i : \mathcal{P}_2(\infty) \to \mathbb{C}$ are given, and that we have a (possibly infinite) matrix $Q = (q_{ij})_{i, j \in I}$ with $q_{ij} = q_{ji}$ and $q_{ij} \in [-1, 1]$. Then we define the $Q$-product of the $t_i$ to be the function on $\mathcal{P}_2^I(\infty)$ given by

$$\left( *_{a \in I} t_a \right) (V, c) := \prod_{(p, p') \in \text{cr}(V)} q_{c(p), c(p')} \prod_{a \in I} t_a(c^{-1}(a)).$$

**Definition 10.** A function $t : \mathcal{P}_2^I(\infty) \to \mathbb{C}$ is said to be multiplicative if for every $k, l, n \in \mathbb{N}$ with $1 \leq k \leq l \leq n$ and any $I$-colored pair partitions $V_1 \in \mathcal{P}_2^I(\infty)(\{1, \ldots, k, l, \ldots, n\})$ and $V_2 \in \mathcal{P}_2^I(\infty)(\{k + 1, \ldots, l - 1\})$, we have $t(V_1 \cup V_2) = t(V_1) \cdot t(V_2)$.

**Proposition 3.** Suppose that $t_a : \mathcal{P}_2(\infty) \to \mathbb{C}$ $(a \in I)$ are multiplicative positive definite functions such that $t((V, c)) = 1$ whenever $V$ is the element of $\mathcal{P}_2(\infty)$ with only one pair. Suppose also that for each $i, j \in I$, some symmetric $Q = (q_{ij})_{i, j \in I}$ with $q_{ij} \in [-1, 1]$ is given. Then $\left( *_{a \in I} t_a \right) (V, c)$ is a positive definite function on $\mathcal{P}_2^I(\infty)$.

**Remark 6.** In [6], the number of crossings between pairs of different colors was used instead of all crossings. Of course, if we wish to impose this restriction in our framework, we can assume $q_{ii} = 1$ for all $i \in I$.

The proof is essentially the same as the proof of positive definiteness of the $q$-product in [6] but we present the argument again here for completeness.

**Proof.** As a first step, we show that for each $n : I \to \mathbb{N}$, the kernel $k_n$ defined on $\mathcal{B} \mathcal{P}_2^I(n, 0)$ by

$$k_n(d_1, d_2) = \left( *_{a \in I} t_a \right) (d_1 \cdot d_2).$$
is positive definite. Using the definition of the $Q$ product,
\[
k_n(a_1, a_2) = \prod_{(p, p') \in \mathcal{C}(d_1^*, d_2)} q_{a, a'} \prod_{a \in \mathcal{I}} t_a((d_1^* \cdot d_2)_a),
\]
where the subscript $a$ refers to the $a$-colored pair partition. Since the $t_a$ are positive definite and the pointwise product of positive definite kernels is positive definite, if we can show that
\[
k_n(a_1, a_2) = \prod_{(p, p') \in \mathcal{C}(d_1^*, d_2)} q_{a, a'} \prod_{a \in \mathcal{I}} t_a((d_1^* \cdot d_2)_a)
\]
then positive definiteness of $k_n$ will follow. However, positive definiteness of $k_n'$ follows from positivity of the vacuum state on a *-algebra generated by annihilation operators $a_{b,i}$ for $i = 1, \ldots, n(b)$ satisfying the commutation relation
\[
a_{b,i}a_d^{*} - q_{b,c}a_c^{*}a_{b,i} = \delta_{b,c}\delta_{i,j}.
\]
Positivity of that state has already been proven by Bożejko and Speicher in [4].

For each $n$ denote the complex Hilbert space generated by the positive definite kernel $k_n$ by $V_n$ and let $\lambda_n : BP_2^I(n, 0) \to V_n$ be the Gelfand map, i.e. $(\lambda_n(d_1), \lambda_n(d_2)) = k_n(d_1, d_2)$. The natural action of the symmetric group $S(\mathcal{I})$ on $BP_2^I(n, 0)$ preserves $k_n$, and thus gives rise to a unitary representation $U_n$ on $V_n$. On $V := \bigoplus_n V_n$, define the operators $j_a$ (for $a \in \mathcal{I}$) by $j_a \lambda_n(d_1) = \lambda_{n+\delta a}(d_{a,0} \cdot d_1)$. By multiplicativity of $t_a (a \in \mathcal{I})$,
\[
k_n(d_{a,0} \cdot d_1, d_{a,0} \cdot d_2) = k_n(d_1, d_2),
\]
which shows that the definition of $j_a$ makes sense. Since $j_a$ also satisfies the requisite intertwining property, we have a representation of the *-semigroup $BP_2^I(\infty)$ on $V$ with respect to the extension of $(\star_{a \in \mathcal{I}} t_a)$ to the broken pair partitions.

As in the case of [6], we can use this construction to define new positive definite functions on pair partitions provided that our index set $\mathcal{I}$ is finite. Assume that $\mathcal{I}$ is finite and $t_a$ is a multiplicative positive definite function for each $a \in \mathcal{I}$. On the Fock-like space $F(\star_{a \in \mathcal{I}} t_a)(\mathcal{K})$, we can define creation operators
\[
a^*(f) := \frac{1}{\sqrt{|\mathcal{I}|}} \sum_{b \in \mathcal{I}} a_b^*(f)
\]
for $f \in \mathcal{K}$. The restriction of the vacuum state to the *-algebra generated by the $a^*(f)$ is a Fock state, and we denote the associated positive definite function on $BP_2(\infty)$ by $(\star_{a \in \mathcal{I}} t_a)^{(r)}$. Explicitly, this function is given by
\[
(\star_{a \in \mathcal{I}} t_a)^{(r)}(\mathcal{V}) = \frac{1}{|\mathcal{I}|^{\mathcal{V}}} \sum_{c : \mathcal{V} \to \mathcal{I}} \prod_{(p, p') \in \mathcal{C}(\mathcal{V})} q(c(p), c(p')) \prod_{a \in \mathcal{I}} t_a(c^{-1}(a)).
\]
In the case that the functions $t_a$ are all the same, $t_a = t$ for all $a \in \mathcal{I}$, we write $t_t^{\mathcal{I}}$ for $(\star_{a \in \mathcal{I}} t)^{(r)}$, and in the case $\mathcal{I} = [n] := \{1, \ldots, n\}$, we write $t_t^n$. 

Theorem 9 (Central Limit Theorem). Let \( Q \in M_N(\mathbb{R}) \) be an \( N \times N \) real symmetric matrix. Let \( Q_n \) be the \( n \times n \) symmetric matrix with entries \( q_{ij} \) where \( q_{ij} = g_{ij} \) for \( i, j \) such that \( 1 \leq i, j \leq n \) and \( i \equiv i \) (mod \( N \)), \( j \equiv j \) (mod \( N \)). Let \( t : \mathcal{P}_2 \to \mathbb{C} \) be a positive definite multiplicative function such that \( t(V_1) = 1 \) where \( V_1 \) is the pair partition consisting of a single pair. Then \( t_{Q_n}^n \) converges pointwise to \( t_Q \), where

\[
t_Q(V) = N^{-|V|} \sum_{d : V \to [N]} \prod_{(p, p') \in \text{cr}(V)} q_{d(p)d(p')}.
\]

Proof. Fix a pair partition \( V \in \mathcal{P}_2(\infty) \). For a function \( c : V \to [N] \) denote by \( P(c) \) the partition of \( V \) such that two pairs \( p \) and \( p' \) are in the same block if and only if \( c(p) = c(p') \). Then

\[
t_{Q_n}^n(V) = n^{-|V|} \sum_{c : V \to [N]} \prod_{(p, p') \in \text{cr}(V)} q_{c(p), c(p')} \prod_{a \in [n]} t(c^{-1}(a))
\]

\[
= \sum_{\pi \in \Pi(V)} n^{-|V|} \sum_{c : V \to [N]} \prod_{P(c) = \pi} \prod_{(p, p') \in \text{cr}(V)} q_{c(p), c(p')} \prod_{a \in [n]} t(c^{-1}(a)),
\]

where \( \Pi(V) \) is the set of all partitions of \( V \). We will consider the contribution of the various \( \pi \in \Pi(V) \) to the sum as \( n \to \infty \).

First consider the partition \( \pi_1 \) of \( V \) into \( |V| \) blocks of size 1, corresponding (for fixed \( n \)) to injective functions \( c : V \to [n] \). For each such \( c \) and \( a \in [n] \), the pair partition \( c^{-1}(a) \) is empty or a single pair, whence \( \prod_{a \in [n]} t(c^{-1}(a)) = 1 \). Furthermore, for each \( c \),

\[
\prod_{(p, p') \in \text{cr}(V)} q_{c(p), c(p')} = \prod_{(p, p') \in \text{cr}(V)} q_{\tilde{c}(p), \tilde{c}(p')},
\]

where \( \tilde{c} : V \to [N] \) is the map such that \( c(p) \equiv \tilde{c}(p) \) (mod \( N \)) for all \( p \in V \). If \( M \) is the natural number such that \( MN < n \leq (M + 1)N \), then for each function \( d : V \to [N] \), the number \( m_n \) of injective maps \( c : V \to [N] \) such that \( \tilde{c} = d \) is between \( M(M - 1) \cdots (M - |V| + 1) \) and \( (M + 1)(M) \cdots (M - |V|) \). In particular \( m_n/|V| \to N^{-|V|} \) as \( n \to \infty \). Thus, as \( n \to \infty \) the contribution to the sum in (107) by the term corresponding to \( P_1 \) converges to

\[
\frac{1}{N^{|V|}} \sum_{d : V \to [N]} \prod_{(p, p') \in \text{cr}(V)} q_{d(p), d(p')} = t_Q(V).
\]

Now we will show that any other partition \( \pi \neq \pi_1 \) contributes 0 to the sum in (107) in the limit as \( n \to \infty \). Such a partition \( \pi \) has at most \(|V| - 1 \) blocks. For a given \( n \), the number
of maps $c: \mathcal{V} \to [n]$ with $P(c) = \pi$ is

$$n(n-1) \cdots (n-|P|+1) \leq n(n-1) \cdots (n-|\mathcal{V}|+2) < n^{|\mathcal{V}|-1}.$$ 

Thus the contribution of the term indexed by $P$ is indeed 0 in the limit. \qed

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