How significant are the known collision and element distinctness quantum algorithms?

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Quantum search is a technique for searching \( N \) possibilities in only \( O(\sqrt{N}) \) steps. It has been applied in the design of quantum algorithms for several structured problems. Many of these algorithms require significant amount of quantum hardware. In this paper we observe that if an algorithm requires \( O(P) \) hardware, it should be considered significant if and only if it produces a speedup of at least \( O(\sqrt{P}) \) over a simple quantum search algorithm. This is because a speedup of \( O(\sqrt{P}) \) can be trivially obtained by dividing the search space into \( O(P) \) separate parts and handing the problem to independent processors that do a quantum search. We argue that the known algorithms for collision and element distinctness fail to be non-trivial in this sense.

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BACKGROUND

The quantum search algorithm gave a means of searching \( N \) items in only \( \sqrt{N} \) steps \(^1\). Unlike most computer science applications, this did not require the problem under consideration to have any structure that the algorithm could make use of.

It is easy to see that any classical algorithm, whether probabilistic or deterministic, would need \( O(N) \) oracle queries for unstructured searching - it had generally been assumed that \( O(N) \) steps would be required by any algorithm. However, quantum mechanical systems can be in multiple states simultaneously and there is no clearly defined bound on how rapidly they can search. It was proven through subtle properties of unitary transformations that any quantum computer would need at least \( O(\sqrt{N}) \) queries to search \( N \) items \(^2\). Subsequently it was shown that the number of queries required by the algorithm was optimal; it can not be improved even by one \(^3\).

The technique behind the algorithm is very general and through the amplitude amplification principle \(^4,5\), the algorithm has been applied to a number of different structured problems, where it has yielded the best known algorithms. In many of these settings the algorithm requires additional hardware in the form of memory registers.

PARALLELIZED QUANTUM SEARCHING

If we have to search \( N \) items for a target state, the quantum search algorithm takes \( O(\sqrt{N}) \) operations. Alternatively, we could divide the \( N \) items into \( P \) groups of \( \frac{N}{P} \) items each and hand each group to an independent quantum processor each of which runs an independent quantum search in \( O\left(\sqrt{\frac{N}{P}}\right) \) steps. This division gives a speedup of \( \sqrt{P} \) over the quantum search algorithm by using \( P \) processors. Zalka proved that this was the best possible speedup for the quantum search algorithm using parallelization. His proof was for unstructured problems. It leaves open the possibility for better parallel speedups for structured problems. However, as we show in this critique, many well known algorithms fail to meet this simple benchmark, i.e. the speedup they get is \( \leq \sqrt{P} \).

QUANTUM HARDWARE: PROCESSORS & MEMORY

In traditional classical computing, there are considerable differences between the requirements that information processors and information storers (memory) necessarily satisfy. As such, the physical realizations of these components can be quite distinct - e.g. transistors make good processors, oriented magnetic domains make good memory. Memory is normally “cheaper”. Consequently, it is common to treat these components as completely different resources within classical computer science.

Within quantum computing, however, the distinction between the two types of component is much more blurred. A qubit register that must act as quantum memory (to hold the output of some computation say) is generally required to remain coherent with the other systems comprising the quantum computer. In fact, within the standard quantum computational model they must not merely remain coherent - they must be capable of dynamically coupling to other quantum systems within the computer via coherent unitary evolution. A distinction between “memory” and “computer” qubits could perhaps be artificially imposed by dictating that memory qubits can only undergo controlled-NOT or Toffoli gates - this does not, however, seem pragmatically justifiable. Most realizations of a quantum computer are more easily capable of single qubit unitary evolution than these two or three qubit gates, and thus such memory qubits could be trivially extended to processing qubits.

Thus it seems clear to us that analysis of the space/time complexity of quantum algorithms is best
served by simply treating all required qubits as available for running any aspect of the algorithm under consideration.

ELEMENT DISTINCTNESS & COLLISION PROBLEMS

Problem description

Two problems that contain some structure, and that therefore could potentially be solved on a quantum computer better than by exhaustive searching, are the collision and element distinctness problems. We focus here on the simplest versions of these problems.

In the (two-to-one) collision problem, we are given a (black box) function \( F(x) \) with a domain of (even) cardinality \( N \), and we are asked to determine whether \( F \) is one-to-one or two-to-one. That is, we know that either every item in the domain maps onto a unique point or exactly one pair of items map onto every point in the range - we need to know which it is. This is an important problem - it is used quite widely in cryptography. Also a logarithmic time algorithm for this would solve the graph isomorphism problem.

The element distinctness problem is similar to the collision problem, except that now we have to determine whether there is any pair of inputs \( x, y \) to the function, such that \( F(x) = F(y) \).

Algorithms & bounds in terms of query complexity

Collision:

A well known classical algorithm (based on the birthday paradox) can find a collision in \( O(\sqrt{N}) \) steps, and in space \( O(\sqrt{N}) \). This is because with a high probability there will be at least one collision if we examine \( O(\sqrt{N}) \) random items. Naive quantum searching requires the same amount of time, but can reduce the space complexity to a constant factor. To see this, note that we could search every pair of points in the domain for a possible collision, there are \( N^2 \) items to be searched for \( N \) target items where there could be a collision. Quantum searching would require \( O\left(\sqrt{N^2} \right) \) which is \( O\left(\sqrt{N} \right) \) steps. This is the same amount of time as it would take classically, but the space required has been reduced to a constant.

The first lower bound for the collision problem was obtained by Aaronson \(^8\), refinements by Shihi \(^9\), Kutin \(^8\) and Ambainis \(^8\) have shown that there is a \( \Omega(N^{1/3}) \) lower bound on the number of queries (calls to \( F \)) for any quantum algorithm. This bound matches the algorithm of \(^10\), which is discussed in detail below.

Element distinctness:

Classically, it is possible to check whether or not every item in the domain maps onto a unique item, by sorting the items according to their function values and then checking adjacent items. This sorting and checking would take \( O(N) \) steps (up to logarithmic factors) and \( O(N) \) memory. If we were to use the quantum search algorithm naively, we could search every pair of points in the domain to check whether or not the function assumed distinct values, there are \( N^2 \) items to be searched for a single target item. Quantum searching would require \( O\left(\sqrt{NC_2} \right) \) which is \( O\left(N \right) \) steps. This is the same amount of time as it would take classically, but the space required has been reduced to a constant. The algorithm of \(^11\), discussed in detail below, achieves a complexity \( O(N^{3/4}) \), while Ambainis has recently discovered an algorithm which needs \( O(N^{2/3}) \) steps. Quantum mechanically, the best known lower bound is \( \Omega(N^{2/3}) \) on the number of queries any quantum algorithm must make; this is matched by Ambainis’ algorithm.

We see that for both problems the lower bounds on the number of queries required are matched by known algorithms - this may lead to the belief that both problems are effectively closed.

THREE ALGORITHMS

We now discuss three well known algorithms for the above problems. As we will show, these algorithms fail to achieve more than a square-root factor speedup over the number of available processing qubits. Thus they fail the simple criteria we propose for determining whether an algorithm makes meaningful use of problem structure.

Collision algorithm

In 1997, in one of the first significant applications of the search algorithm, Brassard et al discovered an \( O(N^{1/3}) \) step algorithm for the collision problem \(^10\). The algorithm selects \( O(N^{1/3}) \) random items, evaluates \( F \) and sorts the outputs in \( O(N^{1/3}) \) memory. Then it randomly selects \( O(N^{2/3}) \) items from the remainder. It may be shown that with a high probability these selected items will have at least one collision with the sorted items. If the quantum search algorithm is run on these \( O(N^{2/3}) \) items, it will find the collision in \( O(N^{1/3}) \) queries. Each query takes only a logarithmic number of time steps since the \( O(N^{1/3}) \) items have been sorted. Thus the total number of time steps required by the algorithm is \( O(N^{1/3}) \) steps to do the sorting plus \( O(N^{1/3}) \) steps to do the searching which is \( O(N^{1/3}) \) steps in all.
This algorithm achieves an $O\left(\frac{N^{1/2}}{\sqrt{N}}\right) = O\left(N^{1/6}\right)$ speedup over what a simple quantum search would take, but at the cost of using $O\left(N^{1/3}\right)$ quantum hardware in the form of memory registers. This is exactly the same speedup that would be obtained by using parallel processors to run standard quantum searching.

Element distinctness algorithms

Algorithm (i)

A quantum algorithm that took only $O\left(N^{3/4}\right)$ time steps is given in [11]. This used a two level quantum search. At the top level, it divided the $N$ items into $\sqrt{N}$ groups each and it ran a quantum search on the $\sqrt{N}$ groups which took $O\left(N^{1/4}\right)$ queries. In each query, the algorithm sorts the $\sqrt{N}$ items in the group and then runs a quantum search on all $N$ items to check if any of the $\sqrt{N}$ items has the same function value as any of the $N$ items. Since the $\sqrt{N}$ items have been sorted, each check takes only a logarithmic number of steps. Therefore each top level query takes $O\left(N^{1/2}\right)$ steps to do the sorting plus $O\left(N^{1/2}\right)$ steps to do the searching which is $O\left(N^{1/2}\right)$ steps in all.

This algorithm achieves an $O\left(\frac{N}{\sqrt{N^{3/4}}}\right) = O\left(N^{1/4}\right)$ speedup over what a simple quantum search would take, but at the cost of using $O\left(N^{1/2}\right)$ quantum hardware in the form of memory registers. This is the same speedup that would be obtained by using parallel processors to run standard quantum searching.

Algorithm (ii)

Recently Ambainis has discovered an algorithm for element distinctness which we believe takes $O\left(N^{2/3}\right)$ time steps and requires $O\left(N^{2/3}\right)$ memory registers [12]. We do not know the details of this, however based on these parameters this algorithm achieves an $O\left(\frac{N}{N^{2/3}}\right) = O\left(N^{1/3}\right)$ speedup over what a simple quantum search would take but at the cost of using $O\left(N^{2/3}\right)$ quantum hardware in the form of memory registers. Once again, this is exactly the same speedup that would be obtained by using parallel processors to run quantum searching as discussed above.

Optimality of the algorithms?

How fast could these algorithms possibly run? As indicated previously, there are lower bounds known on how many time steps are required. These are derived by lower bounding the number of queries that an algorithm would need. This is because the number of queries is generally considered the most convenient parameter to use for analyzing the behavior of an algorithm. However, query complexity is simply one way of characterizing the algorithmic difficulty of a problem - the bottom line is the time it takes and the hardware it uses. The algorithms above saturate the known query complexity lower bounds for the particular problems. As we have seen, if we characterize these algorithm in terms of more general space/time tradeoffs, they demonstrate no advantage over parallel quantum searching.

CONCLUSION

We have argued that three well known algorithms for the collision and element distinctness problems do not, in any meaningful way, make algorithmic use of the problem structure to go beyond the standard quantum searching paradigm. We therefore leave the reader with the following:

Challenge 1: Find an algorithm for collision and/or element distinctness which gives a searching speedup greater than merely a square-root factor over the number of available processing qubits.

Challenge 2: Find ‘physically significant’ lower bounds for collision and/or element distinctness - i.e. lower bounds in terms of the total hardware/total time required as opposed to the less meaningful bounds in terms of number of queries.

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