WELL-POSEDNESS ISSUES FOR SOME CRITICAL COUPLED NON-LINEAR KLEIN-GORDON EQUATIONS

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Abstract. The initial value problem for some coupled non-linear wave equations is investigated. In the defocusing case, global well-posedness and ill-posedness results are obtained. In the focusing sign, the existence of global and non global solutions are discussed via the potential-well theory. Finally, strong instability of standing waves are established.

1. Introduction. This manuscript is concerned with the initial value problem for a coupled wave system with power-type nonlinearities

\[
\begin{aligned}
\partial_t^2 u_j - \Delta u_j + \delta^* u_j + \mu \left( \sum_{k=1}^{m} a_{jk} |u_k|^p \right) |u_j|^p - 2 u_j &= 0; \\
 u_j(0,.) = \psi_j, \quad \partial_t u_j(0,.) = \phi_j.
\end{aligned}
\]  

(1)

Here and hereafter $u_j : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ for $j \in [1,m]$, $\Delta := \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator in $\mathbb{R}^N$, $\mu = \pm 1$, $\delta^* \in \{0,1\}$ and $a_{jk} = a_{kj}$ are positive real numbers.

A solution $u := (u_1, \ldots, u_m)$ to (1) formally satisfies the conservation of energy

\[
E(t) := E(u(t), \partial_t u(t))
\]

\[
= E_0(u(t), \partial_t u(t)) + \frac{\mu}{2\rho} \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |u_j(t,x)|^p |u_k(t,x)|^p \, dx
\]

\[
= E(0),
\]

where the kinetic energy defined by

\[
E_0(u(t), \partial_t u(t)) := \frac{1}{2} \sum_{j=1}^{m} \int_{\mathbb{R}^N} \left( |\nabla u_j|^2 + \delta^* |u_j|^2 + |\partial_t u_j|^2 \right) \, dx.
\]

If $\mu = 1$, the energy is always positive and the problem (1) is said to be defocusing, otherwise a control of the $(H^1)^m$ norm with the energy is no longer possible and (1) is focusing.

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In order to motivate this note, let us present some related works. Concerning the non-linear coupled wave system, in [26], Segal proposed the following problem

\[
\begin{align*}
\partial_t^2 u - \Delta u + \alpha^2 u + g^2 v^2 u &= 0; \\
\partial_t^2 v - \Delta v + \beta^2 v + h^2 u^2 v &= 0,
\end{align*}
\]

(2)

where \(\alpha, \beta, g\) and \(h\) are non zero real constants. This system arises in quantum fields theory which describes the motion of charged mesons in an electromagnetic field.

Later on, many authors treated this problem and several results concerning global existence and blow-up of solutions were established. Medeiros and Menzala [19] studied such a system and found the existence of weak solutions for a mixed problem in a bounded domain. This result was generalized by Miranda and Medeiros in [15, 16] to a non-linear terms of the form \(|v|^{p+2}|u|^p u\). Decay of solutions to the system (2) was established in [5] by Ferreira and Menzala. The existence of global/non-global solutions for a non-linear coupled wave system was discussed by many authors [30, 31, 32]. Recently, a coupled non-linear Klein-Gordon equation with damping term was treated by Pilipkin and Korpusov [12, 23, 24].

Before going further, let us recall some existing results about the Cauchy problem for the scalar wave equation which is a particular case of (1) for \(m = 1\). The defocusing semilinear wave equation with power \(p > 1\) reads

\[
(NLW)_p \quad \partial_t^2 u - \Delta u + u|u|^{p-1} = 0, \quad u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}.
\]

This problem has been widely investigated and there is a large literature dealing with the well-posedness theory of \((NLW)_p\) in the scale of Sobolev spaces \(H^s\). For the global solvability in the energy space \(H^1 \times L^2\), there are mainly three cases. In the subcritical case \(p < p_c := \frac{N+2}{N+4}\), Ginibre and Velo [6] proved that problem \((NLW)_p\) has a unique solution in the energy space. In the critical case \(p = p_c\), the global existence was first proved by Struwe in the radially symmetric case [27], then by Grillakis [7] in the general case and later on by Shatah-Struwe [29] in other dimensions. In the supercritical case \(p > p_c\), local well-posedness was recently solved by Kenig-Merle [10] for initial data in the homogeneous Sobolev spaces \(\dot{H}^{s_p} \times \dot{H}^{s_p-1}\) with \(1 < s_p < \frac{3}{4}\). Except for some partial results about weak ill-posedness [4, 13, 14], well-posedness in the energy space is still an open problem.

In two space dimensions, any polynomial nonlinearity is subcritical with respect to \(H^1\) norm and the critical one is of exponential growth. Nakamura and Ozawa [20, 21] showed the global well posedness in the defocusing case for small data. Atallah [3] treated the radial case, then Ibrahim, Majdoub and Masmoudi [8] gave a more clear critical threshold. Struwe [28] constructed global solutions for smooth data. Similar results without any smallness condition hold [17, 18].

The purpose of this manuscript is two-fold. First, using some Strichartz type estimates, global well-posedness of (1) in the subcritical defocusing and instability in the energy supercritical case are obtained. Second, in the focusing case, using the potential well method, global and non global existence of solutions are discussed via the existence of ground state. Moreover, strong instability of standing waves is proved.

The rest of the paper is organized as follows. The next section contains the main results and some technical tools needed in the sequel. The third and fourth sections are devoted to prove well-posedness of (1). In section five, an ill-posedness result is proved. The section six, existence of critical ground state is established. The seventh section is devoted to discuss global and non global existence of solutions via
the potential-well theory. The last section is devoted to obtaining strong instability of standing waves.

Define the product space
\[ H := H^1(\mathbb{R}^N) \times \cdots \times H^1(\mathbb{R}^N) = [H^1(\mathbb{R}^N)]^m, \]
where \( H^1(\mathbb{R}^N) \) is the usual Sobolev space endowed with the complete norm
\[ \| \cdot \|_{H^1(\mathbb{R}^N)} := \left( \| \cdot \|^2_{L^2(\mathbb{R}^N)} + \| \nabla \cdot \|^2_{L^2(\mathbb{R}^N)} \right)^{\frac{1}{2}}. \]

Denote the real numbers \( \mathbb{R} \)
\[ T \]
We mention that \( \mathbb{R} \) are nonnegative real numbers, \( A \leq B \) means that \( A \leq CB \). For \( 1 \leq r \leq \infty \) and \( (s, T) \in [1, \infty) \times (0, \infty) \), we denote the Lebesgue space \( L^r := L^r(\mathbb{R}^N) \) with the usual norm \( \| \cdot \|_r := \| \cdot \|_{L^r} \) and
\[ \| u \|_{L^r_t(L^s)} := \left( \int_0^T \| u(t) \|^r_r dt \right)^{\frac{1}{r}}, \quad \| u \|_{L^r(L^s)} := \left( \int_0^{+\infty} \| u(t) \|^r_r dt \right)^{\frac{1}{r}}. \]

For simplicity, denote the usual Sobolev Space \( W^{s,p} := W^{s,p}(\mathbb{R}^N) \) and \( H^s := W^{s,2} \).
If \( X \) is an abstract space \( C_\mathbb{R}(X) := C([0, T], X) \) stands for the set of continuous functions valued in \( X \) and \( X_{rd} \) is the set of radial elements in \( X \), moreover for an eventual solution to (1), we denote \( T^* > 0 \) it’s lifespan.

2. Main results and background.

In what follows, the main results and some estimates needed in the sequel are collected.

2.1. Main results. First, we deal with local well-posedness of the wave problem (1).

Theorem 2.1. Let \( 1 \leq N \leq 4 \), \( p \geq 2 \) and \( ((\psi_1, \ldots, \psi_m), (\phi_1, \ldots, \phi_m)) \in (H^1 \times L^2)^{(m)} \). Assume that \( p < \infty \) if \( 1 \leq N \leq 2 \) and \( p \leq p^* \) if \( 3 \leq N \leq 4 \). Then, there exist \( T^* > 0 \) and a unique maximal solution to (1),
\[ \mathbf{u} \in \left( C([0, T^*], H^1) \cap C([0, T^*], L^2) \right)^{(m)}. \]

Moreover,
1. \( \mathbf{u} \) satisfies conservation of the energy;
2. \( \mathbf{u} \) is global in the subcritical defocusing case \( (p < p^*, \mu = 1) \).

Remark 2.1. The unnatural condition \( p \geq 2 \) seems to be technical and gives some restriction on the dimension.

In the critical case, global existence and scattering hold in the energy space, for small data.

Theorem 2.2. Take \( N \in \{3, 4\} \) and \( p = p^* \). There exist \( \epsilon_0 > 0 \) such that if \( ((\psi_1, \ldots, \psi_m), (\phi_1, \ldots, \phi_m)) \in (H^1 \times L^2)^{(m)} \) satisfies \( \sum_{j=1}^m \int_{\mathbb{R}^N} (|\nabla \psi_j|^2 + |\phi_j|^2) dx \leq \epsilon_0 \), the system (1) possesses a unique global solution \( \mathbf{u} \in \left( C(\mathbb{R}, H^1) \cap C^1(\mathbb{R}, L^2) \right)^{(m)} \) which scatters.
In the defocusing case, we obtain an ill-posedness result, precisely we prove that the flow map is discontinuous.

**Theorem 2.3.** Assume that \( N \geq 3 \) and \( p > p^* \). There exist a sequence \( \varphi_k := (\varphi_k^1, \ldots, \varphi_k^m) \) in \( (\dot{H}^1)^{(m)} \) and a sequence \( (t_k) \) satisfying
\[
\|\nabla \varphi_k\|_{(L^2)^{(m)}} \to 0, \quad t_k \to 0, \quad \sup_k E(\varphi_k, 0) < \infty,
\]
and such that any weak solution \( u_k \) to (1) with initial data \( (\varphi_k, 0) \), satisfies
\[
\liminf_{k \to \infty} \|\partial_t u_k(t_k)\|_{(L^2)^{(m)}} \geq 1.
\]
In particular, the Cauchy problem (1) is ill-posed in \( (H^1 \times L^2)^{(m)} \).

Now, we are interested on the focusing sign in (1). Let us recall few results about the existence of a ground state solution to the stationary problem associated to (1). Define for \( u := (u_1, \ldots, u_m) \in H \), the action
\[
S^{\delta^*}(u) := \frac{1}{2} \sum_{j=1}^m (|\nabla u_j|^2 + \delta^* |u_j|^2) - \frac{1}{2p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p \, dx.
\]
If \( \alpha, \beta \in \mathbb{R} \), we call constraint
\[
2K^{\delta^*}_{\alpha,\beta}(u) := \sum_{j=1}^m \left((2\alpha + (N-2)\beta)\|\nabla u_j\|^2 + (2\alpha + N\beta)\delta^* |u_j|^2\right)
- \frac{1}{p} \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} (2p\alpha + N\beta)|u_j u_k|^p \, dx.
\]
Take the minimization problem
\[
m^{\delta^*}_{\alpha,\beta} := \inf_{\Psi \neq 0} \{ S^{\delta^*}(u) \quad \text{s.t.} \quad K^{\delta^*}_{\alpha,\beta}(u) = 0 \}. \tag{3}
\]

**Definition 2.2.** We call a ground state any solution to
\[
-\Delta \psi_j + \delta^* \psi_j = \sum_{k=1}^m a_{jk} |\psi_k|^p |\psi_j|^{p-2} \psi_j, \quad 0 \neq \Psi := (\psi_1, \ldots, \psi_m) \in H, \quad m^{\delta^*}_{\alpha,\beta} = S^{\delta^*}(\Psi). \tag{4}
\]

In the critical case, the situation is as follows.

**Proposition 2.3.** Take a couple of real numbers \( (\alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R}_+ \cup \{(1, -\frac{2}{N})\} \) and \( p = p^* \). Then,
(1) \( m^0 := m^0_{\alpha,\beta} \) is nonzero and independent of \( (\alpha, \beta) \);
(2) there is a minimizer of (3) in the following meaning
\[
0 \neq \Psi \in H_{rd} \quad \text{and} \quad m^0 = E^0(\Psi). \tag{5}
\]

**Remark 2.4.** We don’t prove that \( \Psi \) is a solution to (4) because uniqueness of such a solution is not clear in general. Despite, we will call \( \Psi \) as ground state.

The next sets are invariant under the flow of (1).
\[
A^{+,\delta^*}_{\alpha,\beta} := \{(u, v) \in (H^1 \times L^2)^{(m)} \quad \text{s.t.} \quad E^{\delta^*}(u, v) < m \quad \text{and} \quad K^{\delta^*}_{\alpha,\beta}(u) \geq 0 \};
A^{-,\delta^*}_{\alpha,\beta} := \{(u, v) \in (H^1 \times L^2)^{(m)} \quad \text{s.t.} \quad E^{\delta^*}(u, v) < m \quad \text{and} \quad K^{\delta^*}_{\alpha,\beta}(u) < 0 \}.
\]
Theorem 2.4. Take solutions to the focusing problem (1). 

For easy notation, denote

\[ S := S^1, K_{\alpha, \beta} := K^1_{\alpha, \beta}, m_{\alpha, \beta} := m^1_{\alpha, \beta}, A^{+}_{\alpha, \beta} := A^{+1}_{\alpha, \beta}, \text{ and } A^{-}_{\alpha, \beta} := A^{-1}_{\alpha, \beta}. \]

Using the potential well method, we discuss the existence of global and non global solutions to the focusing problem (1).

Theorem 2.5. Take two wave-admissible pairs \((q, r)\). Then, for any \(\varepsilon > 0\), there exist \(u_0 \in H, T_\varepsilon > 0\) such that

\[ \|u_0 - \Psi\|_H < \varepsilon \quad \text{and} \quad \limsup_{T \to T_\varepsilon} \|u(t)\|_H = \infty \]

where \(u \in \left( C_T(H^1) \cap C^1_T(L^2) \right)^{(m)} \) is the solution to (1) with data \((u_0, 0)\).

In the next subsection, we give some standard estimates needed in the paper.

2.2. Tools. First, recall the so-called Strichartz estimate [9, 22].

Definition 2.6. A pair \((q, r)\) of positive real numbers is said to be wave-admissible or \(\frac{N-1}{2}\) admissible if

\[ 2 \leq q < \infty, \quad 2 \leq r < \infty \quad \text{and} \quad \frac{1}{q} \leq \frac{N-1}{2} \left( \frac{1}{2} - \frac{1}{r} \right). \]

Proposition 2.7. Take two wave-admissible pairs \((q, r)\) and \((a, b)\). Then, for any \(T > 0\),

\[ \| (u, \partial_t u) \|_{C_T(H^s) \cap L^q_x(L^r)} \times C_T(H^{s-1}) \]

\[ \leq \| (u(0), \partial_t u(0)) \|_{H^s \times H^{s-1}} + \| \nabla^\rho (\partial_t^2 u - \Delta u) \|_{L^q_t(L^r)}, \]

whenever

\[ \frac{1}{q} + \frac{N}{r} = \frac{N}{2} - s = \frac{1}{a'} + \frac{N}{b'} - 2 - \rho. \]

Remark 2.8. The Cauchy problem

\[ \partial_t^2 u - \Delta u + k = 0, \quad (u(0, \cdot), \partial_t u(0, \cdot)) = (u_0, u_1), \]

will be treated taking account of the integral form

\[ u(t) = \partial_t R(t) * u_0 + R(t) * u_1 - \int_0^t \left( R(t-s) * k \right) ds, \]

where

\[ \mathcal{F} R(t) = \frac{\sin(t|\xi|^2)}{|\xi|^2}. \]

In the subcritical case, existence of ground state holds [25].
Proposition 1. Take $N \geq 2$, $p_* < p < p^*$ and two real numbers $(0, 0) \neq (\alpha, \beta) \in \mathbb{R}_+^2 \cup \{1, -\frac{2}{N}\}$. Then

1. $m := m_{\alpha, \beta}$ is nonzero and independent of $(\alpha, \beta)$;
2. there is a minimizer of (3), which is some nontrivial solution to (4).

The following Gagliardo-Nirenberg inequality will be useful.

Proposition 2. Let $2 \leq p \leq p^*$, for any $(u_1, \ldots, u_m) \in H$ we have

$$
\sum_{j,k=1}^m \int_{\mathbb{R}^N} |u_j u_k|^p \, dx \leq C_{N,p} \left( \sum_{j=1}^m \|\nabla u_j\|^2 \right)^{(p-1)N} \left( \sum_{j=1}^m \|u_j\|^2 \right)^{\frac{N-p(N-2)}{2}}.
$$

Recall some useful Sobolev embeddings [1].

Proposition 3. The continuous injections hold

1. $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ whenever $1 < p < q < \infty$, $s > 0$ and $\frac{1}{p} \leq \frac{1}{q} + \frac{s}{N}$;
2. $W^{s,p_1}(\mathbb{R}^N) \hookrightarrow W^{s-N\left(\frac{1}{p_1} - \frac{1}{p_2}\right),p_2}(\mathbb{R}^N)$ whenever $1 \leq p_1 \leq p_2 \leq \infty$.

Write some chain rule for fractional derivatives [11].

Lemma 2.6. Let $F \in C^1(\mathbb{R})$ be such that $F(0)$ and $p_1, p_2, p_3, p_4 \in (1, \infty)$. Then, for $0 < \rho < 1$

$$
\|\nabla^\rho (fg)\|_p \lesssim \|f\|_{p_1} \|\nabla^\rho g\|_{p_2} + \|\nabla^\rho f\|_{p_3} \|g\|_{p_4}
$$

$$
\|\nabla^\rho F(f)\|_p \lesssim \|F'(f)\|_{p_1} \|\nabla^\rho f\|_{p_2}.
$$

where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\nabla^\rho := (-\Delta)^{\frac{\rho}{2}}$.

Let us give a classical absorption result.

Lemma 2.7. Let $T > 0$ and $X \in C([0, T], \mathbb{R}_+)$ such that

$$
X \leq a + bX^\theta \text{ on } [0, T],
$$

where $a, b > 0$, $\theta > 1$, $a < (1 - \frac{1}{\theta})\frac{1}{(\theta b)^{\frac{1}{\theta - 1}}}$. Then

$$
X \leq \frac{\theta}{\theta - 1} a \text{ on } [0, T].
$$

Proof. The function $f(x) := bx^\theta - x + a$ is decreasing on $[0, (b\theta)^{-\frac{1}{\theta - 1}}]$ and increasing on $[(b\theta)^{-\frac{1}{\theta - 1}}, \infty)$. The assumptions imply that $f((b\theta)^{-\frac{1}{\theta - 1}}) < 0$ and $f(\frac{\theta}{\theta - 1} a) \leq 0$. As $f(X(t)) \geq 0$, $f(0) > 0$ and $X(0) \leq (b\theta)^{-\frac{1}{\theta - 1}}$, we conclude the proof by a continuity argument.

The following is a classical result about ordinary differential equations [2].

Lemma 2.8. Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be a smooth function and consider the following ODE

$$
\ddot{x}(t) + F'(x(t)) = 0, \quad (x(0), \dot{x}(0)) = (x_0, 0), \quad x_0 > 0.
$$

The ODE has a periodic non constant solution if and only if the function $G : y \mapsto 2\left(F(x_0) - F(y)\right)$ has two distinct simple zeros $\alpha$ and $\beta$ with $\alpha \leq x_0 \leq \beta$ and $G$ has no zero in the interval $[\alpha, \beta]$. The period is then given by

$$
T = 2 \int_\alpha^\beta \frac{dy}{\sqrt{G(y)}} = \sqrt{2} \int_\alpha^\beta \frac{dy}{\sqrt{F(x_0) - F(y)}}.
$$

In addition, $x$ is decreasing on $[0, \frac{T}{4}]$ and $x(\frac{T}{4}) = 0$. 

We close this subsection with some result about ordinary differential equations.

**Lemma 2.9.** Let \( \epsilon > 0 \). There is no real function \( G \in C^2(\mathbb{R}_+) \) satisfying
\[
G(0) > 0, \; G'(0) > 0 \quad \text{and} \quad GG'' - (1 + \epsilon)(G')^2 \geq 0 \quad \text{on} \; \mathbb{R}_+.
\]

**Proof.** Assume with contradiction, the existence of such a function. Then
\[
(G^{-(1+\epsilon)})' \geq 0 \quad \text{and} \quad \frac{G'}{G^{1+\epsilon}} \geq \frac{G'(0)}{G(0)} > 0.
\]
This is a Riccati inequality with blow up time \( T < \frac{1}{\epsilon} \frac{G(0)}{G'(0)} \). This contradiction achieves the proof. \( \square \)

3. **Local well-posedness.** This section is devoted to prove Theorem 2.1. The proof contains three steps. First we prove the existence of a local solution to (1), second we show uniqueness and finally we establish global existence in subcritical case.

3.1. **Local existence.** We use a standard fixed point argument. For \( T > 0 \), we denote the space

\[
E_T := \left( C([0, T], H^1) \cap C^1([0, T], L^2) \cap L^q([0, T], L^r) \right)^{(m)}
\]
endowed with the complete norm

\[
\|u\|_T := \sum_{j=1}^{m} \left( \sup_{[0, T]} \|u_j\|_{H^1} + \sup_{[0, T]} \|\partial_t u_j\|_{L^2} + \|u_j\|_{L^q_r(L^r)} \right),
\]
where \((q, r)\) is the admissible pair given by \( q = \frac{2(2p-1)}{(N-2)+(2p-1)\delta}, \; r = 2(2p-1) \).

Let \( M > 0 \) be a positive real number small enough to fix later and \( B_T(M) \) be the ball in \( E_T \) with center zero and radius \( M \). Let the function be given by

\[
h(u) := \sum_{k=1}^{m} f_{j,k}(u) = \sum_{k=1}^{m} |u_k|^{p-2} u_j.
\]

With the mean value theorem

\[
|f_{j,k}(u) - f_{j,k}(v)| \lesssim \max \{|u_k|^{p-1}|u_j|^{p-1} + |u_k|^p |u_j| |v_k|^p |v_j| |v_k|^{p-2} + |v_k|^{p-1} |v_j|^{p-1}|u - v|.
\]

Define the function

\[
\phi(u)(t) := \left( \partial_t R(t) * \psi_1, \ldots, \partial_t R(t) * \psi_m \right) + \left( R(t) * \phi_1, \ldots, R(t) * \phi_m \right) + \int_{0}^{t} \left( R(t-s) * (\delta^* u_k + \sum_{k=1}^{m} a_{1k} |u_k|^p |u_k|^{p-2} u_1), \ldots, R(t-s) * R(t-s) \right)
\]

\[
* (\delta^* u_m + \sum_{k=1}^{m} a_{mk} |u_k|^p |u_k|^{p-2} u_m) \right) ds
\]

where \((u_0, u_1) := ((\phi_1, \ldots, \phi_m), (\psi_1, \ldots, \psi_m))\). We prove the existence of some small \( T, \delta > 0 \) such that \( \phi \) is a contraction of the ball \( B_T(M) \), with center zero and radius \( M \). Take \( u, v \in E_T(M) \), applying Strichartz estimate (6), we get

\[
\|\phi(u) - \phi(v)\|_T \leq C \sum_{j,k=1}^{m} \|u_k|^{p-2} u_j - |v_k|^p |v_j|^{p-2} v_j\|_{L^q(L^r)} + \delta \|u - v\|_{L^q_r(L^r)}.
\]
Since $p \leq \frac{N}{N-2}$, we have $2p - 1 \leq q$, we let $\alpha \geq 0$ such that $\frac{1}{2p-1} = \frac{1}{q} + \frac{1}{\alpha}$. Denoting the quantity $(I) := \sum_{j,k=1}^{m} \| |u_k|^p |u_j|^{p-2} u_j - |u_k|^p |v_j|^{p-2} v_j \|_{L^p(L^2)}$, by (8) via Hölder inequality, yields

\[
(I) \lesssim \sum_{j,k=1}^{m} \left( \| |u_k|^p |u_j|^{p-1} \|_{L^p(L^2)} + \| |u_k|^p |u_j|^{p-2} \|_{L^p(L^2)} \right) \| \phi \|_{L^p(L^2)}.
\]

Then,

\[
\| \phi(\mathbf{u}) - \phi(\mathbf{v}) \|_T \lesssim T^{\frac{2p-1}{1-p}} \| \mathbf{u} \|_{L^p(L^2)}^{2p-2} \| \mathbf{v} \|_{L^p(L^2)} + CT \| \mathbf{u} - \mathbf{v} \|_{L^p(L^2)}
\]

\[
\lesssim T^{\frac{2p-1}{1-p}} M^{2p-2} \| \mathbf{u} - \mathbf{v} \|_{L^p(L^2)} + CT \| \mathbf{u} - \mathbf{v} \|_{L^p(L^2)}
\]

\[
\lesssim (T^{\frac{2p-1}{1-p}} M^{2p-2} + T) \| \mathbf{u} - \mathbf{v} \|_T.
\]

It remains to prove that $\phi$ maps $B_T(M)$ into itself. Taking in the last inequality $\mathbf{v} = 0$, yields for $M := 2C\sqrt{E_0(\mathbf{u}_0, \mathbf{u}_1)}$,

\[
\| \mathbf{u} \|_T \leq C(\sqrt{E_0(\mathbf{u}_0, \mathbf{u}_1)} + T^{\frac{2p-1}{1-p}} M^{2p-2} + \delta^*T M) < M.
\]

Thanks to a Picard fixed point theorem, $\phi$ has a fixed point in $E_T$. The existence is proved.

3.2. **Uniqueness.** In what follows, we prove the uniqueness of solutions to the equation (1) in the energy space. Let $T > 0$, $\mathbf{u}$, $\mathbf{v}$ two solutions of (1) in $(C_T(H^1) \cap C_T^1(L^2))^m$ and $\mathbf{w} := \mathbf{u} - \mathbf{v}$. Then

\[
\partial_t^2 w_j - \Delta w_j + \delta w_j = h(\mathbf{u}) - h(\mathbf{v}), \quad (w_j(0, .), \partial_t w_j(0, .)) = (0, 0).
\]

Taking $T > 0$ small enough, with a continuity argument, we may assume that

\[
\max_{j=1, \ldots, m} \| w_j \|_{L^p(H^1)} \leq 1.
\]

Taking account of the precedent computation, we have

\[
\| \mathbf{w} \|_T \lesssim T^{\frac{2p-1}{1-p}} \| \mathbf{u} \|_{L^p(L^2)}^{2p-2} \| \mathbf{v} \|_{L^p(L^2)} + C\delta^* T \| \mathbf{w} \|_{L^p(L^2)}
\]

\[
\lesssim C(T^{\frac{2p-1}{1-p}} \| \mathbf{u} \|_{L^p(L^2)} + \delta^* T) \| \mathbf{w} \|_T.
\]

The following Lemma concludes the uniqueness proof.

**Lemma 3.1.** Let $\delta > 0$, $(\mathbf{u}_0, \mathbf{u}_1) \in (H^1 \times L^2)^m$ and $\mathbf{u} \in \left( C_T(H^1) \cap C_T^1(L^2) \right)^m$ a solution to (1). Then, there exist $T_\delta > 0$ such that

\[
\| \mathbf{u} \|_{L^p(L^2)}(m) \leq \delta, \quad \text{for all} \quad T \leq T_\delta.
\]
3.3. Global existence in the subcritical case. We prove that the maximal solution of (1) is global in the subcritical defocusing case. The global existence is a consequence of energy conservation and previous calculations. Let \( u \in (C([0, T^*), H^1) \cap C([0, T^*), L^2])^{(m)} \) be the unique maximal solution of (1). By contradiction, suppose that \( T^* < \infty \). Consider for \( 0 < s < T^* \), the problem

\[
(P_s) \begin{cases}
\partial_t^2 v_j + \Delta v_j + v_j + \left( \sum_{k=1}^m a_{jk}|v_k|^p \right)|v_j|^{p-2}v_j = 0; \\
v_j(s, .) = u_j(s, .), \partial_t v_j(s, .) = \partial_t u_j(s, .).
\end{cases}
\]

By the same arguments used in the local existence proof, we can find a real number \( \tau > 0 \) and a solution \( v = (v_1, \ldots, v_m) \) to \((P_s)\) on \( C([s, s + \tau], H)\). Using the conservation of energy we see that \( \tau \) does not depend on \( s \). Thus, letting \( s \) be close to \( T^* \) such that \( T^* < s + \tau \), this contradicts the maximality of \( T^* \) and finishes the proof.

4. Global existence in the critical case. In this section \( N \in \{3, 4\}, p = p^*, \ s_2 \in [\frac{1}{2}, 1], \rho_1 := 1 - s_2 \) and \( T(t) \) denotes the free operator associated to (1). Letting \( I \subset \mathbb{R} \) a time slab, we define the norms

\[
\|u\|_{M(I)} := \|u\|_{L^\frac{2(N+1)}{N-2-p} \cap W^{\rho_1, \frac{2(N+1)}{N-2-p}}}, \quad \|u\|_{S(I)} := \|u\|_{L^\frac{2(N+1)}{N-2-p} \cap L^\frac{2(N+1)}{N-2-p}}.
\]

Let us start with an auxiliary result.

**Proposition 4.** Take \((u_0, u_1) \in (\dot{H}^1 \times L^2)^{(m)}, A := \|(u_0, u_1)\|_{(\dot{H}^1 \times L^2)^{(m)}}\) and an interval \( I = [0, T] \). There exist \( \delta := \delta(A) > 0 \) such that if

\[
\|T(t)(u_0, u_1)\|_{(S(I))^{(m)}} < \delta,
\]
then there is a unique solution \( u \in (C(I, H^1) \cap C^1(I, L^2))^{(m)} \) of (1) which satisfies
\[
\sum_{j=1}^{m} \| u_j \|_{S(I)} \leq 2\delta \quad \text{and} \quad \sum_{j=1}^{m} \| u_j \|_{M(I)} < \infty.
\]

Proof. The proposition follows from a contraction mapping argument, we let the function \( \phi(u) \) given by
\[
\phi(u)(t) := T(t)(u_0, u_1) + \int_0^t T(t-s) \left( \sum_{k=1}^{m} a_{1k} |u_k|^{\frac{N}{N-2}} |u_1|^{\frac{4-N}{N-2}} u_1, \ldots, \sum_{k=1}^{m} a_{mk} |u_k|^{\frac{N}{N-2}} |u_m|^{\frac{4-N}{N-2}} u_m \right) ds,
\]
Define the set
\[
X_{a,b} := \left\{ u \in (M(I))^{(m)}; \sum_{j=1}^{m} \| u_j \|_{S(I)} \leq a, \sum_{j=1}^{m} \| u_j \|_{M(I)} \leq b \right\}
\]
defined with the complete distance
\[
d(u, v) := \sum_{j=1}^{m} \| u_j - v_j \|_{L^{\frac{2(N+1)}{N-2}}(I, L^{\frac{2(N+1)}{N-2}})}.
\]
Using Strichartz estimate with the couple \((s, \rho) = (s_2, 0),\) it follows that
\[
d(\phi(u), \phi(v)) \lesssim \sum_{j,k=1}^{m} \| f_{j,k}(u) - f_{j,k}(v) \|_{L^{\frac{2(N+1)}{N-2}}(I, L^{\frac{2(N+1)}{N-2}})}.
\]
Define the set
\[
X_{a,b} := \left\{ u \in (M(I))^{(m)}; \sum_{j=1}^{m} \| u_j \|_{S(I)} \leq a, \sum_{j=1}^{m} \| u_j \|_{M(I)} \leq b \right\}
\]
defined with the complete distance
\[
d(u, v) := \sum_{j=1}^{m} \| u_j - v_j \|_{L^{\frac{2(N+1)}{N-2}}(I, L^{\frac{2(N+1)}{N-2}})}.
\]
Using Strichartz estimate with the couple \((s, \rho) = (s_2, 0),\) it follows that
\[
d(\phi(u), \phi(v)) \lesssim \sum_{j,k=1}^{m} \| f_{j,k}(u) - f_{j,k}(v) \|_{L^{\frac{2(N+1)}{N-2}}(I, L^{\frac{2(N+1)}{N-2}})}.
\]

Moreover, taking in the previous inequality \( v = 0,\) we get
\[
\| \phi(u) \|_{L^{\frac{2(N+1)}{N-2}}(I, L^{\frac{2(N+1)}{N-2}})} \leq C A + C b a^{\frac{4}{N-2}}.
\]
Choose \( b = 2 AC \) and \( a \) so that \( C a^{\frac{4}{N-2}} \leq \frac{1}{2}.\)
Thanks to Strichartz estimate with the couple \((s, \rho) = (s_2, 0)\) and applying to \( \nabla^{\rho_1},\)
\[
\| u \|_{M(I)} \lesssim \| u_0 \|_{\dot{H}^{1}} + \| u_1 \| + \sum_{j,k=1}^{m} \| \nabla^{\rho_1} f_{j,k}(u) \|_{L^{\frac{2(N+1)}{N-2}}(I, L^{\frac{2(N+1)}{N-2}})}.
\]
Then, using the fractional chain rule 2.6, it follows that
\[
\|u\|_{M(I)} \lesssim A + \left\| \nabla^{\rho_1} (f_{j,k}(u)) \right\|_{L^\infty_{t,x}} \left( L^\infty_{t,x} \right)
\]
\[
\lesssim A + \left( \|u_k\|_{L^\infty_{x,t}} \right)^{\frac{1}{2(N+1)}} \left( L^\infty_{x,t} \right) \left( L^\infty_{x,t} \right) \\
+ \|u_k\|_{L^\infty_{x,t}} \left( L^\infty_{x,t} \right) \left( L^\infty_{x,t} \right) \left( L^\infty_{x,t} \right) \left( L^\infty_{x,t} \right) \left( L^\infty_{x,t} \right)
\]
\[
\lesssim A + \|u\|_{S(I)}^{\frac{1}{2}} \|u\|_{M(I)} \leq CA + Ca^{\frac{4}{N+1}} b.
\]

If, we choose \( b = 2AC \), so that \( Ca^{\frac{4}{N+1}} \leq \frac{1}{4} \). Then,
\[
\|u\|_{M(I)} \leq b.
\]

Similarly, using Strichartz estimate with the couple \((s, \rho) = (1, \rho_1)\), it follows that
\[
\|u\|_{S(I)} \lesssim \|u_0\|_{H^s} + \|u_1\| + \sum_{j,k=1}^m \|\nabla^{\rho_1} f_{j,k}(u)\|_{L^\infty_{x,t}} \left( L^\infty_{x,t} \right) \left( L^\infty_{x,t} \right)
\]
\[
\lesssim \delta + \|u\|_{S(I)}^{\frac{4}{4}} \|u\|_{M(I)} \leq C\delta + Ca^{\frac{4}{N+1}} b.
\]

For small \( a = 2\delta, b > 0 \), we obtain
\[
\|u\|_{S(I)} \leq 2\delta.
\]

With a classical Picard argument, there exist \( u \in X_{a,b} \) a solution to (1). Finally, in order to prove that the solution belongs to the energy space, it is sufficient to apply Strichartz estimate via previous computations.

We are ready to prove Theorem 2.2.

**Proof of Theorem 2.2.** Thanks to Strichartz estimate,
\[
T(t)(u_0, u_1) \lesssim \xi(u, \partial_t u) := \sum_{j=1}^m \int_{\mathbb{R}^N} (|\nabla u_j|^2 + |\partial_t u_j|^2) \, dx.
\]

So, taking account of the previous Proposition, it suffices to prove that \( \xi(u, \partial_t u) \) remains small on the whole interval of existence of \( u \). By conservation of energy and Sobolev’s inequality, write
\[
E(u(t), \partial_t u(t)) = E(u_0, u_1) \leq C(\xi(u_0, u_1) + \xi(u_0, u_1)^{\frac{N}{N-2}}).
\]

Global existence in the defocusing case follows from the last inequality since
\[
\xi(u(t), \partial_t u(t)) \leq E(u(t), \partial_t u(t)).
\]

In the focusing case, with conservation of the energy and Sobolev’s inequality, we have
\[
\xi(u, \partial_t u) \leq 2E(u_0, u_1) + \left( \frac{N - 2}{N} \right) \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |u_j(x, t)|^{\frac{N}{N-2}} |u_k(x, t)|^{\frac{N}{N-2}} \, dx
\]
\[
\leq C(\xi(u_0, u_1) + \xi(u_0, u_1)^{\frac{N}{N-2}}) + C\left( \sum_{j=1}^m \|\nabla u_j\|^{\frac{2N}{N-2}} \right)^{\frac{N}{N-2}}
\]
\[
\leq C(\xi(u_0, u_1) + \xi(u_0, u_1)^{\frac{N}{N-2}}) + C\xi(u, \partial_t u)^{\frac{2N}{N-2}}.
\]

So by Lemma 2.7, if \( \xi(u_0, u_1) \) is sufficiently small, then \( \xi(u, \partial_t u) \) stays small.
Now, we prove scattering. Using classical arguments [6], it is sufficient to prove that \( u \in S(\mathbb{R}) \). Arguing as previously,

\[
\| u \|_{M(\mathbb{R}) \cap S(\mathbb{R})} \leq \delta + \frac{2}{L} \left( \| u_k \|_{L^{2(N+1)}(\mathbb{R} \times \mathbb{R}, \mathbb{R})} \right)^{2(N+1)}
\]

On the other hand, denoting \( \phi = a \int_{\mathbb{R}} \frac{2}{L} \left( \| u_j \|_{L^{2(N+1)}(\mathbb{R} \times \mathbb{R}, \mathbb{R})} \right)^{2(N+1)} \)

In this section we prove Theorem 2.3. The proof uses the finite sequence,

\[
\| \nabla \phi k \|_2^2 \geq \delta_1^2 \| \phi k \|_2^2 \int \frac{1}{\epsilon k} \left( |x| \right)^{-2N-1} \, dx
\]

where \( a(\epsilon k) = \frac{2}{\epsilon k} \) is chosen such that \( \phi k \) is continuous. By an easy computation, we have

\[
\int_{|x| \leq \frac{1}{2}} G(x) \, dx = \frac{a^2}{2p} \sum_{i,j=1}^m a_{ij} |u_i u_j|^p
\]

On the other hand, denoting \( F(u) := \frac{1}{2p} \sum_{i,j=1}^m a_{ij} |u_i u_j|^p \) and \( G(x) := F(x, 0, \ldots, 0) \)

Write

\[
\int_{\mathbb{R}^N} \left( \frac{1}{2} |u_k|^2 \right) \, dx = \sum_{j=0}^2 C_j^2 \left( \frac{1}{2} \right)^{2p-j} \int_{\mathbb{R}^N} \left( \frac{1}{2} \right)^{(2-N)j} \, dx
\]

\[
\approx \sum_{j=0}^2 C_j^2 \left( \frac{1}{2} \right)^{2p-j} \left( \frac{\epsilon k}{k} \right)^{(2-N)j} \, dx
\]

5. **Ill-posedness.** In this section we prove Theorem 2.3. The proof uses the finite speed of propagation and a quantitative study of the associated ODE.

Construction of \( \phi k \). For \( k \geq 1 \) and \( \epsilon k := k^{1-\frac{4}{2p+2}} > 0 \), consider the following sequence, \( \phi k := (\phi k^1, 0, \ldots, 0) \) such that

\[
\phi k^1(x) = \begin{cases} 0 & \text{if } |x| \geq 1, \\ a(\epsilon k) |x|^{2-N} - 1 & \text{if } \frac{2}{\epsilon k} < |x| \leq 1, \\ \frac{2}{\epsilon k} & \text{if } |x| \leq \frac{1}{2}, \end{cases}
\]

where \( a(\epsilon k) = \frac{2}{\epsilon k} \) is chosen such that \( \phi k \) is continuous. By an easy computation, we have

\[
\int_{\mathbb{R}^N} \left( \frac{1}{2} |u_k|^2 \right) \, dx = \frac{a^2}{2p} \sum_{i,j=1}^m a_{ij} |u_i u_j|^p
\]

On the other hand, denoting \( F(u) := \frac{1}{2p} \sum_{i,j=1}^m a_{ij} |u_i u_j|^p \) and \( G(x) := F(x, 0, \ldots, 0) \)

Write

\[
\int_{\mathbb{R}^N} \left( \frac{1}{2} |u_k|^2 \right) \, dx = \sum_{j=0}^2 C_j^2 \left( \frac{1}{2} \right)^{2p-j} \int_{\mathbb{R}^N} \left( \frac{1}{2} \right)^{(2-N)j} \, dx
\]

\[
\approx \sum_{j=0}^2 C_j^2 \left( \frac{1}{2} \right)^{2p-j} \left( \frac{\epsilon k}{k} \right)^{(2-N)j} \, dx
\]
Thus,
\[ a(k, \epsilon_k)^{2p} \int_{\frac{1}{k}}^{1} (r^{2-N}-1)^{2p}r^{N-1} \, dr \simeq \frac{k^{\frac{N-2}{p}}}{(k^{\frac{N-2}{p}}-k^{2-N})^{2p}(\frac{\epsilon_k}{k}^{2-N})^{N-2p}} \]
\[ \simeq k^{(N-2)p(1+2\frac{p}{p^*})-N\frac{p}{p^*}} \]
\[ \simeq k^{(N-2)p-\frac{1}{p^*}N} \simeq k. \]

Then,
\[ \sup_k E(\varphi_k, 0) < \infty. \]

Construction of \( t_k \). Consider the ordinary differential equation associated to \eqref{eq:1}.
\[ \ddot{\varphi}_k + G'(\varphi_k) = 0 \quad (\varphi_k(0), \dot{\varphi}_k(0)) = (k^{\frac{N-2}{2}}, 0). \] (11)

Using Lemma 2.8, the previous ODE has a unique global periodic solution with period
\[ T_k = 2\sqrt{2} \int_0^{k^{\frac{N-2}{2}}} \frac{d\phi}{\sqrt{G(k^{\frac{N-2}{2}})-G(\phi)}} = 2\sqrt{2} \frac{k^{\frac{N-2}{2}}}{\sqrt{G(k^{\frac{N-2}{2}})}} \int_0^1 \left(1 - \frac{G(vk^{\frac{N-2}{2}})}{G(k^{\frac{N-2}{2}})}\right)^{\frac{1}{2}} \, dv. \]

Because \( p > p^* \),
\[ T_k = 2\sqrt{2} \frac{k^{\frac{N-2}{2}}}{\sqrt{G(k^{\frac{N-2}{2}})}} \int_0^1 \left(1 - v^{\frac{2N}{2}}\right)^{\frac{1}{2}} \, dv \]
\[ \lesssim k^{\frac{N-2}{2}} (G(k^{\frac{N-2}{2}}))^{\frac{1}{2}} \]
\[ \lesssim k^{(N-1)(1-\frac{p}{p^*-2})} \ll \frac{\epsilon_k}{k}. \]

Now we are in position to construct the sequence \( (t_k) \). Recall that by finite speed of propagation, any weak solution \( u^k \) to \eqref{eq:1} with data \((\varphi_k, 0)\) satisfies
\[ u^k(t, x) = (\phi^k_1(t), 0, \ldots, 0) \quad \text{if} \quad 0 < t < \frac{\epsilon_k}{k} \quad \text{and} \quad |x| < \frac{\epsilon_k}{k} - t. \]

Hence
\[ |\partial_t u^k_j(t, x)| = \delta_j^1 |\dot{\phi}^k_1(t)| = \delta_j^1 \sqrt{2(G(k^{\frac{N-2}{2}}) - G(\varphi_k(t)))}. \]

Take \( t_k = \frac{T_k}{4} \), then \( \phi^k_1(t_k) = 0 \) and for \(|x| < \frac{\epsilon_k}{k} - t_k \), we have
\[ |\partial_t u^k_j(t_k, x)| = \delta_j^1 \sqrt{2G(k^{\frac{N-2}{2}})}. \]
So
\[ \| \partial_t u^k(t_k) \|^2 \geq G\left(\frac{\kappa - t_k}{K} \right)^N \]
\[ \geq K^{(N-2)p \left(\frac{\kappa}{K} \right)^N} \]
\[ \geq K^{(N-2)p - \frac{p}{N}} \]
\[ \geq 1. \]

The proof is closed.

6. Existence of critical ground state. In this section we prove the existence of a ground state solution to (5) in the critical case. Precisely, we establish Proposition 2.3. For \((\alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R}_+ \cup \{(1, -\frac{2}{N})\}\) and \(\Psi \in H\), recall the quantity

\[ 2K^0_{\alpha, \beta}(\Psi) := 2L^0_{\alpha, \beta}S^0(\Psi) \]
\[ = \left(2\alpha + (N-2)\beta \sum_{j=1}^m \| \nabla \psi_j \|^2 \right) \omega_{\alpha, \beta}^* \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\psi_j \psi_k|^{p^*} \, dx \]
\[ = \left(2\alpha + \frac{N\beta}{p^*} \right) \left( \sum_{j=1}^m \| \nabla \psi_j \|^2 \right) - \left( \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\psi_j \psi_k|^{p^*} \, dx \right) \]

and the operator

\[ 2H^0_{\alpha, \beta}(\Psi) := (2S^0 - \frac{1}{(2\alpha p^* + N\beta)} K^0_{\alpha, \beta})(\Psi) \]
\[ = \frac{2}{N} \sum_{j=1}^m \| \nabla \psi_j \|^2. \]

Let the real number

\[ d^0_{\alpha, \beta} := \inf_{0 \neq \phi \in H} \left\{ H^0_{\alpha, \beta}(\Psi) \text{ s.t } K^0_{\alpha, \beta}(\Psi) < 0 \right\}. \]

Claim. \( d^0_{\alpha, \beta} = m^0_{\alpha, \beta}. \)

Since \( K^0_{\alpha, \beta} = 0 \) implies that \( S^0 = H^0_{\alpha, \beta} \), it follows that \( m^0_{\alpha, \beta} \geq d^0_{\alpha, \beta}. \)

Conversely, take \( 0 \neq \phi \in H \) such that \( K^0_{\alpha, \beta}(\phi) < 0. \) Thus, when \( 0 < \lambda \rightarrow 0 \), we get

\[ K^0_{\alpha, \beta}(\lambda \Psi) = \left(2\alpha + \frac{N\beta}{p^*} \right) \left( \lambda^2 \sum_{j=1}^m \| \nabla \psi_j \|^2 - \lambda \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\psi_j \psi_k|^{p^*} \, dx \right) \]
\[ \simeq (2\alpha + \frac{N\beta}{p^*}) \lambda^2 \sum_{j=1}^m \| \nabla \psi_j \|^2 > 0. \]

So, there exists \( \lambda \in (0, 1) \) satisfying \( K^0_{\alpha, \beta}(\lambda \Psi) = 0 \) and

\[ m^0_{\alpha, \beta} \leq H^0_{\alpha, \beta}(\Psi) = \lambda^2 H^0_{\alpha, \beta}(\Psi) \leq H^0_{\alpha, \beta}(\Psi). \]

Thus, \( m^0_{\alpha, \beta} \leq d^0_{\alpha, \beta} \) and \( m^0_{\alpha, \beta} = d^0_{\alpha, \beta}. \) The claim is proved.

Because of the definitions of \( K^0_{\alpha, \beta} \) and \( H^0_{\alpha, \beta} \), it is clear that \( m^0_{\alpha, \beta} \) is independent of \((\alpha, \beta)\) and

\[ m := m^0_{\alpha, \beta} = \inf_{0 \neq \Psi \in H} \left\{ \frac{1}{N} \sum_{j=1}^m \| \nabla \psi_j \|^2 \text{ s.t } \sum_{j=1}^m \| \nabla \psi_j \|^2 < \sum_{j,k=1}^m a_{jk} \int_{\mathbb{R}^N} |\psi_j \psi_k|^{p^*} \, dx \right\}. \]
Taking the scaling $\lambda \phi$, yields

\[
m = \inf_{0 \neq \psi \in H} \left\{ \frac{1}{N} \lambda^2 \sum_{j=1}^{m} \| \nabla \psi_j \|^2 \quad \text{s.t.} \quad \lambda^{2-2p'} \sum_{j=1}^{m} \| \nabla \psi_j \|^2 < \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |\psi_j \psi_k|^p \, dx \right\}
\]

\[
= \inf_{0 \neq \psi \in H} \left\{ \frac{1}{N} \sum_{j=1}^{m} \| \nabla \psi_j \|^2 \left( \frac{\sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |\psi_j \psi_k|^p \, dx}{\sum_{j=1}^{m} \| \nabla \psi_j \|^2} \right)^{p'/p} \right\}
\]

\[
= \frac{1}{N} \inf_{0 \neq \psi \in H} \left\{ \left( \frac{\sum_{j=1}^{m} \| \nabla \psi_j \|^2}{\sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |\psi_j \psi_k|^p \, dx} \right)^{1/2} \right\}^N
\]

\[
= \frac{1}{N} (C^*)^{-N}.
\]

Here, $C^*$ denotes the best constant of the Sobolev injection

\[
\left( \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |\psi_j \psi_k|^p \, dx \right)^{1/p} \leq C^* \left( \sum_{j=1}^{m} \| \nabla \psi_j \|^2 \right)^{1/2}.
\]

7. Invariant sets and applications. This section is devoted to obtaining global and non global solutions to the system (1) as claimed in Theorem 2.4. Let us start with a classical result about stable sets under the flow of (1).

Lemma 7.1. The sets $A_{\alpha,\beta}^+$ and $A_{\alpha,\beta}^-$ are invariant under the flow of (1) and independent of $(\alpha, \beta)$.

Proof. 1. Let $(u_0, u_1) \in A_{\alpha,\beta}^+$ and $u \in (C_T(H^1 \times L^2))^{(m)}$ be the maximal solution to (1). Assume that $(u(t_0), \partial_t u(t_0)) \notin A_{\alpha,\beta}^+$ for some $t_0 \in (0, T^*)$, with conservation of the energy, we have $K_{\alpha,\beta}^+(u(t_0)) < 0$. So, with a continuity argument, there exists a positive time $t_1 \in (0, t_0)$ such that $K_{\alpha,\beta}^+(u(t_1)) = 0$ and $E(u, \partial_t u)(t_1) < m$ which is absurd. Similarly $A_{\alpha,\beta}^-$ is invariant under the flow of (1).

2. Let $(\alpha, \beta)$ and $(\alpha', \beta')$ in $\mathbb{R}_+^2 - \{(0,0)\}$. We denote, for $\delta \geq 0$, the sets

\[
A_{\alpha,\beta}^{+\delta,\delta'} := \{ u \in H \quad \text{s.t.} \quad S^{\delta'}(u) < m - \delta \quad \text{and} \quad K_{\alpha,\beta}^{+\delta}(u) \geq 0\};
\]

\[
A_{\alpha,\beta}^{-\delta,\delta'} := \{ u \in H \quad \text{s.t.} \quad S^{\delta'}(u) < m - \delta \quad \text{and} \quad K_{\alpha,\beta}^{-\delta}(u) < 0\}.
\]

By the first point, the reunion $A_{\alpha,\beta}^{+\delta,\delta'} \cup A_{\alpha,\beta}^{-\delta,\delta'}$ is independent of $(\alpha, \beta)$. Moreover, $(u, v) \in A_{\alpha,\beta}^{+\delta,\delta'}$ if and only if $u \in A_{\alpha,\beta}^{+\delta,\delta'}$. So, it is sufficient to prove that $A_{\alpha,\beta}^{+\delta,\delta'}$ is independent of $(\alpha, \beta)$. If $S(u) < m$ and $K_{\alpha,\beta}(u) = 0$, then $u = 0$. So, $A_{\alpha,\beta}^{+\delta,\delta'}$ is open. The rescaling $v^\lambda := e^{a\lambda}v(e^{-\beta\lambda})$ implies that a neighborhood of zero is in $A_{\alpha,\beta}^{+\delta,\delta'}$. Moreover, this rescaling with $\lambda \to -\infty$ gives that $A_{\alpha,\beta}^{+\delta,\delta'}$ is contracted to zero and so it is connected. Now, write

\[
A_{\alpha,\beta}^{+\delta,\delta'} = A_{\alpha,\beta}^{+\delta,\delta'} \cap (A_{\alpha',\beta'}^{+\delta,\delta'} \cup A_{\alpha',\beta'}^{-\delta,\delta'}) = (A_{\alpha,\beta}^{+\delta,\delta'} \cap A_{\alpha',\beta'}^{+\delta,\delta'}) \cup (A_{\alpha,\beta}^{+\delta,\delta'} \cap A_{\alpha',\beta'}^{-\delta,\delta'}).
\]

Since by the definition, $A_{\alpha,\beta}^{+\delta,\delta'}$ is open and $0 \in A_{\alpha,\beta}^{+\delta,\delta'} \cap A_{\alpha',\beta'}^{+\delta,\delta'}$, using a connectivity argument, we have $A_{\alpha,\beta}^{+\delta,\delta'} = A_{\alpha',\beta'}^{+\delta,\delta'}$. \qed
The next auxiliary result reads.

**Lemma 7.2.** Let \((u_0, u_1) \in A_{\alpha, \beta}^{-\delta^*}\) and \(u \in (C([0, T^*), H^1) \cap C^1([0, T^*), L^2))^{(m)}\) the maximal solution to (1). Then, there exist \(\delta > 0\) such that

\[
K_{1,-\frac{2}{p}}(u) < -\delta \quad \text{on} \quad [0, T^*).
\]

**Proof.** With contradiction. Assume that \(K_{1,-\frac{2}{p}}(u(t_n)) \to 0\) for some sequence of positive real numbers \(t_n \in (0, T^*)\). The next contradiction closes the proof.

\[
m \leq (S^{\delta^*} - K_{1,-\frac{2}{p}})(u(t_n)) \leq E(u_0, u_1) - K_{1,-\frac{2}{p}}(u(t_n)) \to E(u_0, u_1) < m.
\]

\(\square\)

Let us prove the main result of this section.

**Proof of Theorem 2.4.** 1. By absurdity, assume that \(T = \infty\). With a translation argument, we can assume that \(t_0 = 0\). Thus, \(S^{\delta^*}(u_0) \leq E(0) < m\) and with Lemma 7.1, \(u(t) \in A_{\alpha, \beta}^{-\delta^*}\) for any \(t \in [0, T^*)\) and any \((\alpha, \beta) \in \mathbb{R}^2_+ \cup \{(1, -\frac{2}{p})\}\).

Define the real function

\[
y(t) = \sum_{j=1}^{m} \|u_j(t)\|^2.
\]

It is the goal of the rest of this section to show that \(y(t)\) blows up in finite time, precisely

\[
\lim \sup_{t \to T^*} y(t) = \infty.
\]

Using the equation (1), a direct computation gives

\[
y''(t) = 2 \sum_{j=1}^{m} \|\partial_t u_j\|^2 - 2 \sum_{j=1}^{m} \|\nabla u_j\|^2 - 2\delta^* \sum_{j=1}^{m} \|u_j\|^2 + 2 \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |u_k|^p |u_j|^p \, dx.
\]

With Cauchy-Schwarz inequality, for any positive real number \(\gamma\), we have

\[
y(t)y''(t) - \frac{\gamma}{4}(y'(t))^2 \geq y(t)y''(t) - \gamma \left[ \sum_{j=1}^{m} \int_{\mathbb{R}^N} u_j \partial_t u_j \, dx \right]^2
\]

\[
\geq y(t)y''(t) - \gamma \sum_{j=1}^{m} \|\partial_t u_j\|^2 \sum_{j=1}^{m} \|u_j\|^2
\]

\[
= y(t) \left( y''(t) - \gamma \sum_{j=1}^{m} \|\partial_t u_j\|^2 \right).
\]
Denote $J := y''(t) - \gamma \sum_{j=1}^{m} \|\partial_t u_j\|^2$. Letting $\epsilon > 0$ near to zero and $\gamma := 4 + \epsilon$, we compute using (1),

$$J = (2 - \gamma) \sum_{j=1}^{m} \|\partial_t u_j\|^2 - 2 \sum_{j=1}^{m} \|\nabla u_j\|^2 - 2 \delta^* \sum_{j=1}^{m} \|u_j\|^2$$

$$+ 2 \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |u_k|^p |u_j|^p \, dx$$

$$= -(2 + \epsilon) \left( 2E(u, \partial_t u) - 2 \sum_{j=1}^{m} \|\nabla u_j\|^2 - 2 \delta^* \sum_{j=1}^{m} \|u_j\|^2 

+ \frac{1}{p} \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |u_k|^p |u_j|^p \, dx \right) - 2 \sum_{j=1}^{m} \|\nabla u_j\|^2$$

$$- 2 \delta^* \sum_{j=1}^{m} \|u_j\|^2 + 2 \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |u_k|^p |u_j|^p \, dx$$

$$= -2(2 + \epsilon) E(u, \partial_t u) + \epsilon \sum_{j=1}^{m} \|\nabla u_j\|^2 + \epsilon \delta^* \sum_{j=1}^{m} \|u_j\|^2$$

$$+ (2 - \frac{2 + \epsilon}{p}) \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |u_k|^p |u_j|^p \, dx$$

$$= 2(2 + \epsilon)(m - E(u, \partial_t u)) - \epsilon m - 2(2 + \frac{\epsilon}{2}) m$$

$$+ \epsilon \sum_{j=1}^{m} \|\nabla u_j\|^2 + \epsilon \delta^* \sum_{j=1}^{m} \|u_j\|^2 + (2 - \frac{2 + \epsilon}{p}) \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |u_k|^p |u_j|^p \, dx$$

$$> -2(2 + \frac{\epsilon}{2}) m + \epsilon \sum_{j=1}^{m} \|\nabla u_j\|^2 + \epsilon \delta^* \sum_{j=1}^{m} \|u_j\|^2$$

$$+ (2 - \frac{2 + \epsilon}{p}) \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |u_k|^p |u_j|^p \, dx.$$ 

Since we have,

$$m_{\alpha,\beta} = \inf_{0 \neq u \in H} \left\{ H_{\alpha,\beta}(u) := (S^{\delta^*} - \frac{K_{\alpha,\beta}^{\delta^*}}{2\alpha + N\delta^*})(u), \quad \text{s.t.} \quad K_{\alpha,\beta}^{\delta^*}(u) \leq 0 \right\},$$

then for any $\lambda > 0$,

$$H_{1,\lambda}(u) = \frac{1}{2 + N\lambda} \left[ \sum_{j=1}^{m} \lambda \|\nabla u_j\|^2 + (1 - \frac{1}{p}) \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |u_k|^p |u_j|^p \, dx \right] > m.$$
It follows that

\[(I) := -2(2 + \frac{\epsilon}{2})m + \epsilon \sum_{j=1}^{m} \|\nabla u_j\|^2 + (2 - \frac{2 + \epsilon}{p}) \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |u_k|^p |u_j|^p \, dx \]

\[> - \frac{2(2 + \frac{\epsilon}{2})}{2 + N \lambda} \left[ \sum_{j=1}^{m} \lambda \|\nabla u_j\|^2 + (1 - \frac{1}{p}) \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |u_k|^p |u_j|^p \, dx \right] \]

\[+ \epsilon \sum_{j=1}^{m} \|\nabla u_j\|^2 + (2 - \frac{2 + \epsilon}{p}) \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |u_k|^p |u_j|^p \, dx \]

\[> \frac{1}{2 + N \lambda} \left[ (\epsilon(2 + N \lambda) - 2\lambda(2 + \frac{\epsilon}{2})) \sum_{j=1}^{m} \|\nabla u_j\|^2 \right] \]

\[+ ((2 + N \lambda)(2 - \frac{2 + \epsilon}{p}) - 2(2 + \frac{\epsilon}{2})(1 - \frac{1}{p})) \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |u_k|^p |u_j|^p \, dx \].

Thus

\[(I) > \frac{1}{2 + N \lambda} \left\{ \left[ 2N \lambda - \frac{\epsilon N}{2} (2 + \lambda(N - 1)) \right] \right. \]

\[\left. \left[ \frac{p - 1}{p} \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |u_k|^p |u_j|^p \, dx - \frac{2}{N} \sum_{j=1}^{m} \|\nabla u_j\|^2 \right] \right. \]

\[+ \left[ (N \lambda + 2) \frac{\epsilon}{2p} ((N - 1)p - (N + 1)) \right] \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |u_k|^p |u_j|^p \, dx \}

\[> \frac{1}{2 + N \lambda} \left\{ - \left[ 2N \lambda - \frac{\epsilon N}{2} (2 + \lambda(N - 1)) \right] \right. \]

\[\left. K_{1, -\frac{\epsilon}{N}}(u) \right. \]

\[+ \left[ (N \lambda + 2) \frac{\epsilon}{2p} ((N - 1)p - (N + 1)) \right] \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |u_k|^p |u_j|^p \, dx \].

Taking \( \epsilon > 0 \) near to zero via the fact that \( K_{1, -\frac{\epsilon}{N}}(u) < -\delta \), we get

\[yy'' - \frac{\gamma}{4}(y')^2 \geq 0.\]

The proof is complete using Lemma 2.2.

2. By Lemma 7.1 \( u(t) \in A_{1, 1}^m \) for any \( t \in [0, T^*) \). So

\[m > E(u, \partial_t u) \geq \left( S - \frac{1}{N + 2} K_{1,1} \right)(u)\]

\[= \frac{1}{N + 2} \left( \sum_{j=1}^{m} \|\nabla u_j\|^2 + (1 - \frac{1}{p}) \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p \, dx \right) \]

\[\geq \frac{1}{N + 2} \sum_{j=1}^{m} \|\nabla u_j\|^2.\]

Thus, \( u \) is bounded in \( (H^1)^m \). Precisely

\[\sup_{0 \leq t \leq T^*} \sum_{j=1}^{m} \|\nabla u_j\|^2 \leq (N + 2)m.\]
Moreover, since the $K_{0,\frac{p}{2}}(u) > 0$, we have

$$2m > 2E(u, \partial_t u) = \frac{2}{N} \sum_{j=1}^{m} \|\nabla u_j\|^2 + \sum_{j=1}^{m} \|\partial_t u_j\|^2 + K_{0,\frac{p}{2}}(u).$$

This implies that

$$\sup_{0 \leq t \leq T^*} \sum_{j=1}^{m} \|\partial_t u_j\|^2 < 2m.$$

Thus,

$$\sum_{j=1}^{m} \|u_j(t)\|^2 = \|u_0\|^2_{L_2^{(\infty)}} + 2 \sum_{j=1}^{m} \int_0^t \int_{\mathbb{R}^N} u_j(s) \partial_t u_j(s) \, dx \, ds \leq \|u_0\|^2_{L_2^{(\infty)}} + 4 \sum_{j=1}^{m} \int_0^t (\|u_j(s)\|^2 + \|\partial_t u_j(s)\|^2) \, ds \leq \|u_0\|^2_{L_2^{(\infty)}} + 8mt + 4 \sum_{j=1}^{m} \int_0^t \|u_j(s)\|^2 \, ds.$$

A Gronwall argument gives

$$\sum_{j=1}^{m} \|u_j(t)\|^2 \leq \phi(t) \sum_{j=1}^{m} \int_0^t \phi(s) \exp(t-s) \, ds, \quad \phi(t) = \|u_0\|^2_{L_2^{(\infty)}} + 8mt.$$

The previous inequality implies that the $L^2$ norm of $u$ does not blow up in finite time. So, $u$ is global.

8. Strong instability. This section is devoted to prove Theorem 2.5 about strong instability of standing waves. For $\lambda > 0$ and $u := (u_1, \ldots, u_m) \in H$ we denote $u_\lambda := \lambda^{\frac{p}{2}} u(\lambda)$ and we recall the quantities

$$S(u) := \frac{1}{2} \sum_{j=1}^{m} \left( \|\nabla u_j\|^2 + \|u_j\|^2 \right) - \frac{1}{2^{p} \lambda} \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p \, dx;$$

$$K_{1,-\frac{p}{2}}(u) := \frac{2}{N} \sum_{j=1}^{m} \|\nabla u_j\|^2 - (1 - \frac{1}{p}) \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p \, dx.$$

The proof of Theorem 2.5 is based on the following Lemmas.

**Lemma 8.1.** Let $u \in H$ such that $K_{1,-\frac{p}{2}}(u) \leq 0$. Then, for any $\lambda > 1$,

1. $\partial_\lambda S(u_\lambda) = \frac{N}{2} K_{1,-\frac{p}{2}}(u_\lambda);$ 
2. $K_{1,-\frac{p}{2}}(u_\lambda) < 0.$

**Proof.** With a direct calculations

$$\partial_\lambda S(u_\lambda) := \partial_\lambda \left( \frac{1}{2} \lambda \sum_{j=1}^{m} \|\nabla u_j\|^2 + \frac{1}{2} \sum_{j=1}^{m} \|u_j\|^2 - \frac{\lambda^{N(p-1)}}{2^{p} \lambda} \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p \, dx \right)$$

$$= \lambda \sum_{j=1}^{m} \|\nabla u_j\|^2 - \frac{N(p-1)}{2^{p} \lambda^{N(p-1)-1}} \sum_{j,k=1}^{m} a_{jk} \int_{\mathbb{R}^N} |u_j u_k|^p \, dx$$

$$= \frac{N}{2} K_{1,-\frac{p}{2}}(u_\lambda).$$
Lemma 8.2. Let $\Psi$ be a ground state solution to (4), $\lambda > 1$ a real number and $u_\lambda$ the solution to (1) with data $\Psi_\lambda = \lambda^{\frac{N}{2}} \Psi(\lambda)$. Then, for any $t \in (0, T^*)$

$$S(u_\lambda(t)) < S(\Psi) \quad \text{and} \quad K_{1,-\frac{N}{2}}(u_\lambda(t)) < 0.$$ 

Proof. By Lemma 8.1, we have

$$S(\Psi_\lambda) < S(\Psi) \quad \text{and} \quad K_{1,-\frac{N}{2}}(\Psi_\lambda) < 0.$$ 

Moreover, by conservation of energy

$$E(u_\lambda(t), 0) = E(\Psi_\lambda, 0) = S(\Psi_\lambda) < S(\Psi).$$

Then $K_{1,-\frac{N}{2}}(u_\lambda(t)) \neq 0$ because $\Psi$ is a ground state. Finally $K_{1,-\frac{N}{2}}(u_\lambda(t)) < 0$ with a continuity argument. \hfill $\square$

Proof of Theorem 2.5: Let $\Psi$ a ground state solution to (4) and $u_\lambda \in (C^r \cdot (H^1) \cap C^m_\lambda(L^2))^m$ the maximal solution to (1) with data $(\Psi_\lambda, 0)$. Using the previous Lemma, we get

$$u_\lambda \in A_{1,-\frac{N}{2}}^- \quad \text{for any} \quad t \in [0, T^*).$$

Then, by Theorem 2.4, it follows that

$$\lim_{t \to T^*} \|u_\lambda(t)\|_H = \infty.$$ 

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