Fermions and noncommutative emergent gravity II: Curved branes in extra dimensions

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Abstract

We study fermions coupled to Yang-Mills matrix models from the point of view of emergent gravity. The matrix model Dirac operator provides an appropriate coupling for fermions to the effective gravitational metric for general branes nontrivial embedding, albeit with a non-standard spin connection. This generalizes previous results for 4-dimensional matrix models. Integrating out the fermions in a nontrivial geometrical background induces indeed the Einstein-Hilbert action of the effective geometrical metric, as well as additional terms which couple the Poisson tensor to the Riemann tensor, and a dilaton-like term.

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1 Introduction

Quantum field theory (QFT) and general relativity (GR) provide the basis for our present understanding of fundamental physics. However, these two theories together imply that classical space-time loses its meaning in the small. It is expected that the conventional concepts of space and time will no longer hold at the Planck scale and instead some kind of quantum structure of space-time should take over in this regime. One way of
describing such a structure is obtained by taking a noncommutative algebra for spacetime coordinates. The basic idea is that the classical space-time $\mathbb{R}^4$ is replaced by a space where the coordinate functions $x^\mu$ satisfy Heisenberg-like commutation relations,

$$[x^\mu, x^\nu] = i \theta^{\mu\nu}. \quad (1)$$

The situation is quite analogous to quantum mechanics. At the semi-classical level, these commutation relations reduce to a Poisson structure $\theta^{\mu\nu}(x)$ on space-time. One can then write down so-called noncommutative quantum field theories which incorporate quantum fluctuations of spacetime coordinates naturally, see [1]. It was conjectured that these fluctuations should then in some way be linked to gravity. In recent years a specific realization of this idea was developed under the name of “emergent noncommutative gravity”, see [2–8].

In this series of papers it was understood that matrix models of Yang-Mills type

$$S_{YM} = -\text{Tr}[X^a, X^b][X^a', X^b']\eta_{aa'}\eta_{bb'} \quad (2)$$

as known from noncommutative (NC) gauge theory and string theory not only describe dynamical NC spaces, in fact they also incorporate gravity. These models contain an effective metric

$$\tilde{G}^{\mu\nu}(x) = e^{-\sigma} \theta^{\mu\alpha}(x)\theta^{\nu\beta}(x)g_{\alpha\beta}(x), \quad (3)$$

which couples to all types of fields. Both the metric responsible for gravity as well as space-time are not fundamental objects of the theory, but rather they are determined by the Poisson structure $\theta(x)$ and the background metric $g_{\mu\nu}(x)$.

Fermions can be naturally included in the matrix model. The fermionic action is similar to the standard action for fermions on curved backgrounds coupled to the effective metric $\tilde{G}^{\mu\nu}$, however with a non-standard spin connection. More precisely, the spin connection vanishes in the special (but unobservable) matrix coordinates. This implies that the induced effective action due to integrating out the fermions does not quite have the standard form in terms of an induced Einstein-Hilbert action. We computed this induced action in [4] in the special case of branes with flat embedding but general $\theta^{\mu\nu}(x)$, showing that the expected Einstein-Hilbert action is indeed induced however with a non-standard factor, along with an extra scalar term.

In this work, we generalize these results to the general case of non-trivially embedded branes in a higher-dimensional matrix model. We study the quantization of fermions in the matrix model, and compute the effective gravitational action obtained by integrating out the fermions. Due to the non-standard coupling of the fermions to gravity we cannot apply the standard results for the one-loop effective action that can be found in the literature, e.g. [3, 11]. Instead we have to evaluate it directly. We are able to cast the effective action into a covariant geometrical form. It turns out that it contains indeed expected Einstein-Hilbert term $R(\tilde{G})$, plus additional terms which involve the curvature
tensor coupled to $\theta^{\mu\nu}$. Due to technical complications we focus on two special cases: 1) “on-shell geometries” as determined by the semi-classical equations of motion of the matrix model, and 2) the class of geometries where the effective metric $\tilde{G}^{\mu\nu}$ coincides with the induced metric $g^{\mu\nu}$. The latter class seems to be general enough for a large class of physical situations [7, 8]. Our main result is the effective action (86), (94) in these two cases which have essentially the same structure, and alternative covariant expressions (58) and (59) for the novel terms.

This paper is quite technical and involves lengthy computations. This is necessary because of the non-standard Dirac operator, and the composite nature of the effective metric which involves the Poisson tensor as well as the embedding metric given in terms of scalar fields. Even the demonstration that the induced effective action is a well-defined geometrical quantity is non-trivial and requires a lot of work. In order to make the paper as readable as possible we have delegated much of the computations to the Appendices.

2 The matrix model in higher dimensions

As a starting point consider the matrix model

$$S_{\text{YM}} = -(2\pi)^n \text{Tr} \left( \frac{1}{4} [X^a, X^b] [X^{a'}, X^{b'}] \eta_{aa'} \eta_{bb'} + \frac{1}{2} \gamma_a [X^a, \Psi] \right),$$

(4)

where the $X^a$ for $a = 1, \ldots, D$ are infinite dimensional hermitian matrices or operators acting on some Hilbert space $\mathcal{H}$ and

$$\eta_{aa'} = \text{diag}(1, 1, \ldots, 1) \quad \text{or} \quad \eta_{aa'} = \text{diag}(-1, 1, \ldots, 1)$$

(5)

is an (unphysical) background metric that fixes the signature of the theory, Euclidean or Minkowski space, respectively. $\gamma_a$ generate the Clifford algebra in $D$ dimensions, and $\Psi$ are spinors consisting of Grassmann-valued matrices. This model can be obtained e.g. as dimensional reduction of large-N super-Yang-Mills theory to 0 dimensions. A particularly important case is the IKKT model [11], which was first proposed in the context of string theory.

**Quantization.** The basic assumption of the present approach is that space-time carries a Poisson structure $\{x^\mu, x^\nu\} = \theta^{\mu\nu}(x)$. Space-time is then considered to be the quantization of such a Poisson manifold. It is well-known [12] that a Poisson manifold $(\mathcal{M}, \theta^{\mu\nu}(x))$ can be quantized and that there exists a quantization map

$$\mathcal{C}(\mathcal{M}) \to \mathcal{A} \subset L(\mathcal{H})$$

$$f(x) \mapsto \hat{f}(X)$$

$$i \{f, g\} \mapsto [\hat{f}, \hat{g}] + O(\theta^2).$$

(6)
\( \mathcal{C}(\mathcal{M}) \) denotes some space of functions on \( \mathcal{M} \), and \( \mathcal{A} \) is interpreted as quantized algebra of functions on \( \mathcal{M} \). The matrices \( X^\mu \) are interpreted as quantization of the coordinate function \( x^\mu \). Moreover, for the sake of simplicity we will consider only the semi-classical limit of such a quantum space, i.e. we keep only terms linear in \( \theta \). Then we have

\[
[X^\mu, X^\nu] \sim i \theta^{\mu\nu}(x) \tag{7}
\]

\[
[X^\mu, f(X)] \sim i \theta^{\mu\nu} \frac{\partial}{\partial x^\nu} f(x). \tag{8}
\]

The trace is replaced by an integral where the appropriate density factor is given by the symplectic volume,

\[
(2\pi)^n \text{Tr} \hat{f}(X) \sim \int d^{2n}x \, \rho(x) f(x) \tag{9}
\]

\[
\rho(x) = (\det \theta^{-1})^{1/2}. \tag{10}
\]

\( \theta^{\mu\nu} \) is assumed to be non-degenerate and \( \det \theta^{\mu\nu} > 0 \).

**Embedding.** We want to study 2\( n \)-dimensional NC spaces \( \mathcal{M}^{2n} \subset \mathbb{R}^D \), which we interpret as space-time manifold embedded in \( D \) dimensions. To realize this we split the matrices as

\[
X^a = (X^\mu, \phi^i), \quad \mu = 1, \ldots, 2n, \quad i = 1, \ldots, D - 2n, \tag{11}
\]

where the “scalar fields” \( \phi^i = \phi^i(X^\mu) \) are assumed to be functions of \( X^\mu \) which determine the embedding of a 2\( n \)-dimensional submanifold \( \mathcal{M}^{2n} \) in \( \mathbb{R}^D \). \( \mathcal{M}^{2n} \) carries then the induced metric

\[
g_{\mu\nu}(x) = \eta_{\mu\nu} + (\partial_\mu \phi^i)(\partial_\nu \phi^j) \delta_{ij} = (\partial_\mu x^a)(\partial_\nu x^b) \eta_{ab}. \tag{12}
\]

Note that the background metric \( g_{\mu\nu}(x) \) is *not* the metric responsible for the gravitational coupling in the action, since there \( g_{\mu\nu} \) will enter only implicitly. Moreover, all fields that couple to such a background will live on the brane \( \mathcal{M}^{2n} \) only. In contrast to braneworld-scenarios, in this model there is no higher-dimensional “bulk” that could carry any physical degrees of freedom.

**Matrix model coordinates.** Throughout this work we work with so called “matrix coordinates” which are preferred coordinates \( x^\mu \sim X^\mu \) in the model. They are such that in the case \( D = 4 \) the background metric is given by \( g_{\mu\nu} = \eta_{\mu\nu} \) resp. \( g_{\mu\nu} = \delta_{\mu\nu} \). In the general case of extra dimensions the model allows a \( SO(D - 1) \) resp. \( SO(D) \) rotation such that at some given point \( p \in \mathcal{M} \) the background metric is again \( g_{\mu\nu} = \eta_{\mu\nu} \) resp. \( g_{\mu\nu} = \delta_{\mu\nu} \), see Sect. [3]
Geometry arises dynamically. In the above matrix model Eq. (4) a priori there is no geometry, all we have is matrices. The geometry of this model arises dynamically. The matrix model is a theory of space-time itself, in the sense that the physically realized geometry has to fulfill the equations of motion (e.o.m.) of the theory which are given by

\[ [X^a, [X^b, X^c]] g_{aa'} = 0. \] (13)

In the semi-classical limit these are given in matrix coordinates by

\[
\begin{align*}
\theta^{\mu\alpha}(\partial_\mu \theta^{\nu\beta}) g_{\alpha\beta} &= -\theta^{\mu\alpha} \theta^{\nu\beta}(\partial_\mu g_{\alpha\beta}), \\
\Delta \tilde{G} \phi^i(x) &= 0.
\end{align*}
\] (14)

Here \( \tilde{G} \) is given by Eq. (3) and \( \Delta \tilde{G} \) is the Laplace-Beltrami operator. Note that at the semi-classical level we have an harmonic embedding condition for the embedding scalar fields.

The most prominent example for such a space-time is the 4-dimensional Moyal-Weyl plane, which is a flat manifold with

\[
\begin{align*}
[X^\mu, X^\nu] &= i \tilde{\theta}^{\mu\nu}, \quad \mu, \nu = 0, \ldots, 3 \\
\phi^i(X) &= 0, \quad i = 1, \ldots, D - 4,
\end{align*}
\] (16)

where \( \tilde{\theta} \) is constant. However, in general the solutions will fulfill

\[
\begin{align*}
[X^\mu, X^\nu] &= i \theta^{\mu\nu}(X), \\
\phi^i &= \phi^i(X).
\end{align*}
\] (17)

They describe a dynamical, noncommutative and non-flat 4-dimensional manifold with nontrivial embedding in 10 dimensions.

Effective metric. To understand the effective geometry on \( \mathcal{M} \) we couple a scalar field as a particle on \( \mathcal{M} \) to the matrix model. The only reasonable kinetic term is

\[
S[\varphi] = -(2\pi)^n \text{Tr}[X^a, \varphi][X^b, \varphi] \eta_{ab},
\] (18)

which becomes in the semi-classical limit

\[
S[\varphi] \sim \int d^{2n}x |\tilde{G}_{\mu\nu}|^{1/2} \tilde{G}^{\mu\nu}(\partial_\mu \varphi)(\partial_\nu \varphi).
\] (19)

Now we can see that in fact it is the effective metric

\[
\tilde{G}^{\mu\nu}(x) = e^{-\sigma} \theta^{\mu\alpha}(x) \theta^{\nu\beta}(x) g_{\alpha\beta}(x)
\] (20)
which is responsible for the gravitational coupling. \( g_{\alpha\beta}(x) \) is the metric (12) induced on \( M \subset \mathbb{R}^D \) via pull-back on \( g_{ab} \) and

\[
 e^{-\sigma} = \rho(x) |g_{\mu\nu}|^{-1/2},
\]

where \( \rho(x) \) is stated in Eq. (3). Note that the matrix model action Eq. (4) with \( \Psi = \bar{\Psi} = 0 \) can be written in the semi-classical limit as

\[
 S_{YM} = -(2\pi)^n \text{Tr} \left( \frac{1}{4} [X^a, X^a'] [X^b, X^b'] \eta_{ab} \eta_{ab'} \right) \sim \int d^2 n \rho(x) \eta(x),
\]

where

\[
 \eta(x) = \frac{1}{4} e^{\sigma} \tilde{G}^{\mu\nu} g_{\mu\nu}.
\]

**Self-dual solutions and** \( \tilde{G}_{\mu\nu} = g_{\mu\nu} \). Using (15), the e.o.m. (14) can be written in covariant form [5],

\[
 \tilde{G}^{\mu\nu} \nabla_\mu \left( e^\sigma \theta^{-1}_{\nu\rho} \right) = e^{-\sigma} \tilde{G}_{\rho\mu} \theta^{\mu\nu} \partial_\nu \left( \eta(x) \right).
\]

A simple but important class of solutions of this equation is given by 2-forms \( \theta^{-1}_{\mu\nu} \) satisfying

\[
 \tilde{G}_{\mu\nu} = g_{\mu\nu}.
\]

In this case, (24) simplifies to

\[
 g_{\mu\nu} \nabla_\mu \theta^{-1}_{\nu\alpha} = 0.
\]

It is not hard to see that in 4 dimensions, (25) is equivalent to self-dual \( \theta^{\mu\nu} \) [8]. Such solutions are of great interest in this framework because they correlate to the cosmological constant problem. In that case the bare matrix model action Eq. (22) becomes

\[
 S_{YM} = \int d^4 x \rho(x) \eta(x) = \int d^4 x \rho e^{\sigma} = \int d^4 x \sqrt{|g_{\mu\nu}|},
\]

which is precisely the form of the induced vacuum energy interpreted as cosmological constant in GR. Now the variation of this term

\[
 \delta \int d^4 x \sqrt{|g|} \sim \frac{1}{4} \int d^4 x \sqrt{|g|} g^{\mu\nu} \delta g_{\mu\nu} \sim \frac{1}{4} \int d^4 x \sqrt{|g|} \delta \phi^i \Delta g^{ij} \delta \phi^j
\]

vanishes for harmonic embedding Eq. (15). Then the coefficient of this term is irrelevant, and harmonically embedded branes are protected from the cosmological constant problem. Therefore the term \( \int d^4 x \sqrt{|g|} \) should not be interpreted as a cosmological constant, but as a brane tension.
3 Fermions

The obvious way to include fermions into the matrix model is through the action (4). This not only provides the appropriate coupling to the metric \( \tilde{G}^{\mu\nu} \), it is also dictated by supersymmetry [11]. The quantization of such fermions has been studied in the 4-dimensional model in [4] from the point of view of noncommutative emergent gravity. There it was shown that the action

\[
S[\Psi] = (2\pi)^2 \text{Tr} \bar{\Psi} \gamma_\mu [X^\mu, \Psi]
\]

indeed induces the Einstein-Hilbert action at the quantum level, along with a dilaton-like term. The purpose of the present work is to show how the results in [4] can be generalized to the case of extra dimensions. We split the action according to Section 2

\[
S[\Psi] = (2\pi)^n \text{Tr} \bar{\Psi} \gamma_a [X^a, \Psi] = (2\pi)^n \text{Tr} (\bar{\Psi} \gamma_\mu [X^\mu, \Psi] + \bar{\Psi} \gamma_i [X^i, \Psi])
\]

\[
\sim \int d^n x \rho(x) \bar{\Psi} i (\gamma_\mu + \gamma_{3+i} \partial_\mu \phi^i) \theta^{\mu\nu}(x) \partial_\nu \Psi
\]

where \( \gamma_a \) denotes the \( D \)-dimensional Euclidean Clifford algebra. We have introduced the “tangential” Clifford algebra associated with the background metric \( g_{\mu\nu}(x) \) on \( \mathcal{M} \) whose elements are denoted by

\[
\tilde{\gamma}_\mu(x) = (\gamma_\mu + \gamma_{3+i} \partial_\mu \phi^i),
\]

and which satisfies

\[
\{\tilde{\gamma}_\mu, \tilde{\gamma}_\nu\} = 2 \left( \eta_{\rho\sigma} + 2(\partial_\rho \phi^i)(\partial_\sigma \phi^j) \delta_{ij} \right) = 2g_{\mu\nu}(x).
\]

Notice that the \( \tilde{\gamma}(x) \)-matrices are functions of \( x \), and related to \( \gamma_\mu \) via some vielbein relating the tangent space to the ambient \( \mathbb{R}^D \); in particular, \( \tilde{\gamma}_\mu(x) = \gamma_\mu \) in normal coordinates (60). The (matrix) Dirac operator is then given by

\[
\hat{D} \Psi = \gamma_\alpha [Y^\alpha, \Psi] \sim i \left( \gamma_\mu + \gamma_{3+i} \left( \partial_\mu \phi^i \right) \right) \theta^{\mu\nu}(y) \partial_\nu \Psi \equiv i \tilde{\gamma}_\mu \theta^{\mu\nu} \partial_\nu \Psi.
\]

As it has already been pointed out in [13] the above result does not quite match with the standard covariant derivative for spinors [13]

\[
\hat{D}_{\text{comm}} \Psi = i \gamma^\mu e^\mu_{\alpha} (\partial_\mu + \Sigma_{bc} \omega_{\mu}^{bc}) \Psi,
\]

where

\[
\omega^{ab}_\mu = \frac{i}{2} \epsilon^{a\nu} \nabla_\mu e^b_\nu
\]
is the usual spin connection, and $\Sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$. While the explicit derivative term is essentially the same, the spin connection vanishes in matrix coordinates $x^\mu$. This means that fermions (as long as they can be considered as point particles) move along geodesics of $\tilde{G}^{\mu\nu}$ as expected, however with a non-standard gravitational “spin-dragging”. A further remark is in order. In contrast to [4], in the case of extra dimensions the Poisson structure $\theta^{\mu
u}$ does not play the sole role of a vielbein, rather it is part of a vielbein structure composed of $\theta^{\mu\nu}$ and $\partial_\mu \phi^i$.

It is easy to show that the corresponding effective metric for fermions

$$\tilde{G}_\tau^{\mu\nu}(x) = e^{-\tau} \theta^{\mu\rho} \theta^{\nu\sigma} g_{\rho\sigma}(x),$$

comes with an unusual scaling factor $e^{-\tau}$,

$$e^{-\tau} = |G_{\mu\nu}|^{1/6} |g_{\mu\nu}|^{-1/6} = e^{-\frac{2}{3} \sigma},$$

where

$$G^{\mu\nu}(x) = \theta^{\mu\alpha}(x) \theta^{\nu\beta}(x) g_{\alpha\beta}(x).$$

The scaling factor is such that it gives the correct density factor $\sqrt{|\tilde{G}_\tau^{\mu\nu}|}$ in the action (30).

### 4 Quantization

Starting from the action

$$S[\Psi] = (2\pi)^n \text{Tr} \Psi \gamma_a [X^a, \Psi]$$

we want to study the quantization of the above matrix model via a path integral over all $\Psi$,

$$e^{-\Gamma_\Psi} = \int d\Psi d\bar{\Psi} e^{-S[\Psi]}$$

$$= \exp(\ln \det(D)) = \exp \left( \frac{1}{2} \log \det(D)^2 \right)$$

$$= \exp \left( \frac{1}{2} \text{Tr} \log(D)^2 \right).$$

which gives the effective action $\Gamma_\Psi$

$$\Gamma_\Psi = -\frac{1}{2} \text{Tr} \log D^2.$$
Let us consider the Euclidean case for the sake of rigor. The square of the Dirac operator takes the following form
\[
\mathcal{D}^2 \Psi = -\tilde{\gamma}_\mu \tilde{\gamma}_\rho \theta^\mu\theta^\rho \partial_\nu \partial_\sigma \Psi - \tilde{\gamma}_\mu \tilde{\gamma}_\rho \theta^\mu \theta^\rho (\partial_\nu \theta^\sigma) (\partial_\sigma \Psi) - \tilde{\gamma}_\mu (\partial_\nu \tilde{\gamma}_\rho) \theta^\mu \theta^\rho \partial_\sigma \Psi
\]
(42)
where \(a^\mu\) is the term linear in the partial derivatives
\[
a^\sigma = \tilde{\gamma}_\mu \tilde{\gamma}_\rho \theta^\mu \theta^\rho (\partial_\nu \tilde{\gamma}_\rho) + \tilde{\gamma}_\mu (\partial_\nu \tilde{\gamma}_\rho) \theta^\mu \theta^\rho.
\]
(43)
The last term in the above equation is new in comparison to [4], where the background \(g_{\mu\nu}\) was flat and the associated Clifford algebra elements were the usual constant Dirac matrices. To proceed, we note that \(\mathcal{D}^2\) defines the quadratic form
\[
S_{\text{square}} := (2\pi)^n \text{Tr} \Psi^\dagger \mathcal{D}^2 \Psi \sim \int d^2 x \rho(x) \Psi^\dagger \mathcal{D}^2 \Psi
\]
(44)
which has the appropriate covariant form in terms of the metric \(\tilde{G}\),
\[
e^{-\sigma} = \rho(x) |g_{\mu\nu}(x)|^{-1/2} = |G_{\mu\nu}|^{1/4} |g_{\mu\nu}|^{-1/4}
\]
\[
\tilde{G}^{\mu\nu} = e^{-\sigma} G^{\mu\nu}
\]
\[
\tilde{D}^2 = -\tilde{G}^{\mu\nu} \partial_\mu \partial_\nu - e^{-\sigma} a^\mu \partial_\mu.
\]
(45)
In order to compute the effective action we can use the following integral representation of the functional determinant
\[
\frac{1}{2} \text{Tr} \left( \log \mathcal{D}^2 \right) = \frac{1}{2} \text{Tr} \int_0^\infty \frac{d\alpha}{\alpha} \left( e^{-\alpha \mathcal{D}^2} \right)
\]
\[
= -\frac{1}{2} \text{Tr} \int_0^\infty \frac{d\alpha}{\alpha} \left( e^{-\alpha \mathcal{D}^2} \right) e^{-\frac{1}{2} \alpha},
\]
(46)
where we have introduced a cutoff \(\Lambda^2\) which regularizes the divergence of \(\mathcal{D}^2\) for small \(\alpha\). Now we can apply the heat kernel expansion [3,4]
\[
\text{Tr} e^{-\alpha \mathcal{D}^2} = \sum_{n \geq 0} \alpha^{n-1} \int_M d^4 x a_n(x, \mathcal{D}^2)
\]
(47)
where the Seeley-de Witt coefficients \(a_n(y, \mathcal{D}^2)\) are given by
\[
a_0(x) = \frac{1}{16 \pi^2} \text{tr} \mathbf{1},
\]
\[
a_2(x) = \frac{1}{16 \pi^2} \text{tr} \left( \frac{\text{R} [\tilde{G}]}{6} + \mathcal{E} \right).
\]
(48)
Here tr denotes the trace over the spinorial matrices and
\[
\mathcal{E} = -\tilde{G}^{\mu\nu} \left( \partial_\mu \Omega_\nu + \Omega_\mu \Omega_\nu - \tilde{\Gamma}_\mu^\rho \Omega_\rho \right),
\]
\[
\Omega_\mu = \frac{1}{2} \tilde{G}_{\mu\nu} (e^{-\sigma} a^\nu + \tilde{G}^{\rho\sigma} \tilde{\Gamma}_{\rho\sigma});
\]
note that this expression is valid only in the matrix coordinates \( x^\mu \). This gives rise to the effective action
\[
\Gamma_\Psi = \frac{1}{16\pi^2} \int d^{2n} x \sqrt{|\tilde{G}|} \left( 2\text{tr}(1) \Lambda^4 + \text{tr} \left( \frac{R[\tilde{G}]}{6} + \mathcal{E} \right) \Lambda^2 + \mathcal{O}(\log \Lambda) \right).
\]
This is the idea of emergent gravity observed first by Sakharov [14]. For the standard coupling of Dirac fermions to gravity on commutative spaces, one has [10]
\[
\text{tr}\mathcal{E}_{\text{comm}} = -R.
\]
In our case tr\(\mathcal{E}\) is modified due to the non-standard spin connection. Therefore we cannot use the standard results, and the geometrical meaning of (49) is unclear since this expression is not covariant and valid only in matrix coordinates. The purpose of the present work is to evaluate the quantity tr\(\mathcal{E}\) and see whether it gives indeed the Ricci scalar \( R[\tilde{G}] \) in order to obtain the correct induced Einstein-Hilbert action. We will show that tr\(\mathcal{E}\) contains as expected the appropriate curvature scalar, plus three additional terms. This will be discussed in detail in the following sections.

5 Evaluation of tr\(\mathcal{E}\)

We will now determine explicitly the second Seeley-de Witt coefficient for the squared Dirac operator Eq. (12). In order to do so we compute the Ricci scalar as well as the quantity tr\(\mathcal{E}\) explicitly in terms of the Poisson structure, and then compare those two. First we have to compute the following expression
\[
\text{tr}\mathcal{E} = -\text{tr} \left\{ \tilde{G}^{\mu\nu} \Omega_\mu \Omega_\nu + \tilde{G}^{\mu\nu} \partial_\mu \Omega_\nu - \tilde{\Gamma}_{\mu}^{\rho} \Omega_\rho \right\}
\]
\[
= -\text{tr} \left( \frac{1}{4} \tilde{G}_{\mu\nu} \tilde{a}^{\mu} \tilde{a}^{\nu} - \frac{1}{4} \tilde{G}_{\mu\nu} \tilde{\Gamma}^{\mu \nu} + \frac{1}{2} \tilde{G}^{\mu\nu} \partial_\mu (\tilde{G}_{\nu\rho} \tilde{a}^{\rho} + \tilde{G}_{\nu\rho} \tilde{\Gamma}_{\rho}) \right),
\]
where $\bar{\Gamma}^{\mu\nu} = \tilde{G}^{\rho\sigma} \bar{\Gamma}^{\mu\rho}_{\rho\sigma}$. The explicit evaluation of $\text{tr}\mathcal{E}$ is given in Appendix A. The result Eq. (112) is

$$\text{tr}\mathcal{E} = -\text{tr} \left( \frac{1}{4} \tilde{G}_{\mu\nu} \tilde{a}^\mu \tilde{a}^\nu - \frac{1}{4} \tilde{G}_{\mu\nu} \bar{\Gamma}^\mu \bar{\Gamma}^\nu \right)$$

$$= -e^{-\sigma} \frac{k}{4} \left\{ -G^{\mu\nu} G^{\rho(\partial_\mu \theta^{-1}_{\rho\sigma})(\partial_\nu \theta^{-1}_{\sigma\beta}) g^{\alpha\beta} + G^{\mu\nu} G^{\rho\sigma} (\partial_\mu \theta^{-1}_{\rho\alpha})(\partial_\nu \theta^{-1}_{\nu\beta}) g^{\alpha\beta} \\
+ G^{\mu\rho} G^{\sigma\alpha}(\partial_\rho g_{\alpha\beta})(\partial_\sigma g_{\nu\beta}) g^{\mu\nu} + G^{\mu\rho} G^{\sigma\alpha} (\partial_\rho g_{\alpha\beta})(\partial_\nu g_{\sigma\beta}) g^{\mu\nu} \\
- G^{\mu\rho} G^{\sigma\alpha}(\partial_\rho g_{\alpha\beta})(\partial_\nu \theta^{-1}_{\nu\beta}) g^{\mu\nu} + G^{\mu\rho} G^{\sigma\alpha} (\partial_\rho g_{\alpha\beta})(\partial_\nu \theta^{-1}_{\nu\beta}) g^{\mu\nu} \\
- \frac{1}{2} G^{\mu\nu}(g_{\nu\beta} g_{\mu\beta} - 2G^{\rho\delta} \partial_\rho \partial_\delta g_{\sigma\beta} + g^{\mu\nu} G^{\rho\sigma} \partial_\rho \partial_\sigma g_{\rho\beta}) \\
+ \frac{1}{2} g^{\mu\nu}(\partial_\mu g_{\nu\beta}) (\Delta \tilde{G}^{\phi^i} + \tilde{\Gamma}^{\mu} \partial_\mu \phi^i) \delta_{ij} + \frac{k}{4} \tilde{G}_{\mu\nu} \bar{\Gamma}^\mu \bar{\Gamma}^\nu, \right\}$$

where

$$k = \text{rank}(\gamma).$$

$k$ is the rank of the representation of $D$-dimensional Clifford algebra, depending on the number of extra dimensions.

For the sake of simplicity and manageability we will sometimes use the equations of motions, Eq. (14) and (15) and work with on-shell geometries. Then the contracted Christoffel symbols vanish [5],

$$\bar{\Gamma}^{\mu} = -\partial_\rho \tilde{G}^{\rho\mu} - \frac{1}{2} \tilde{G}^{\rho\mu} (\tilde{G} \partial_\nu \tilde{G}^{-1})$$

$$= e^{-\sigma} \left\{ - (\partial_\mu \theta^{-1}_{\rho\beta}) \theta^{\rho\alpha} g_{\alpha\beta} - \theta^{\mu\beta} \theta^{\rho\alpha} (\partial_\rho g_{\alpha\beta}) \right\}$$

$$\text{e.o.m.} = 0.$$  (55)

Due to Eq. (55) also the harmonic embedding condition simplifies as

$$\Delta \tilde{G}^{\phi} = \left( \tilde{G}^{\mu\nu} \partial_\mu \partial_\nu - \bar{\Gamma}^{\mu} \partial_\mu \right) \phi$$

$$= \tilde{G}^{\mu\nu} \partial_\mu \partial_\nu \phi$$

$$= 0.$$  (56)

These handy features simplify our calculations a lot since $\text{tr}\mathcal{E}$ is then determined by a single term,

$$\text{tr}\mathcal{E} = -\frac{e^{-\sigma}}{4} \text{tr} \left( G_{\mu \alpha} a^\mu a^\nu \right).$$  (57)
In principle one could now go on and compute the Ricci scalar in terms of the Poisson tensor $\theta^{\mu\nu}$ and compare the two quantities. This strategy was pursued in [4]. However, it turns out that this procedure is too complicated and seems to be not feasible in the case of extra dimensions. Hence we simplify our computations by going to normal coordinates, i.e. coordinates where first order derivatives in the embedding scalar fields, $\partial_\mu \phi(x)$, vanish. But to do that, we should first show that $\text{tr}\mathcal{E}$ is a covariant expression. This is done in Appendix B, and we only quote the result here. For on-shell geometries which satisfy (14), (15), we find

$$
\text{tr}\mathcal{E} = -\frac{e^{-\sigma}}{4} k \left( G^{\mu\nu} (\nabla_\mu \theta^{-1}_\rho\sigma) g^{\alpha\beta} - G^{\mu\nu} (\nabla_\rho \theta^{-1}_\sigma\beta) g^{\alpha\beta} \right)
$$

see also Eq. (135) of Appendix B. In the special case of identical background and effective metric $\tilde{G}_{\mu\nu} = g_{\mu\nu}$ the use of on-shell geometries is not necessary. Then we have

$$
\text{tr}\mathcal{E} = -\frac{e^{-\sigma}}{4} \text{tr} G_{\mu\nu} a^\mu a^\nu + \frac{e^{-\sigma}}{4} \text{tr} g_{\mu\nu} \Gamma^\mu \Gamma^\nu - \frac{k}{4} \tilde{G}_{\mu\nu} R_{\mu\nu}[g],
$$

as stated in Eq. (157). This expression has a clear geometrical meaning (taking into account extrinsic geometry in the last term) and can thus be considered as covariant. For on-shell geometries, the last term vanishes and (59) agrees with (58) using the Bianci identity (150).

A short remark regarding notation. We have to distinguish between the effective metric $\tilde{G}_{\mu\nu}$ and the background metric $g_{\mu\nu}$. Covariant derivatives and Christoffel symbols with respect to the effective metric $\tilde{G}_{\mu\nu}$ in NC emergent gravity are usually denoted as $\tilde{\nabla}_\mu$ and $\tilde{\Gamma}^\mu_{\rho\sigma}$, respectively. Covariant derivatives and Christoffel symbols with respect to the background metric $g_{\mu\nu}$ as they appear in Eq.(58) are written as $\nabla_\mu$ and $\Gamma^\mu_{\rho\sigma}$.

6 Going to a normal embedding coordinate system

6.1 $\text{tr}\mathcal{E}$ in normal coordinates.

Since the matrix model action is invariant under $SO(D)$ resp. $SO(1, D - 1)$ rotations as well as translations, one can choose for any given point $p \in \mathcal{M}$ adapted coordinates such that the brane is tangential to the plane spanned by the first $2n$ components. Then we
have at this point
\[ \partial_{\mu} \phi^i |_{p} = 0, \]  
\[ \partial_{\mu} g_{\rho \sigma} |_{p} = 0. \]  

We denote such coordinates as “normal embedding coordinates” or simply “normal coordinates”. They are still matrix coordinates \(x^a \sim X^a\) and thus the e.o.m. Eq. (14) and (15) still hold. We can now take our result of Eq. (53) for \( \text{tr} \mathcal{E} \) and write it in normal coordinates by simply omitting all terms with first partial derivatives of the background metric \( g_{\mu \nu} \). Since the covariance of \( \text{tr} \mathcal{E} \) of Eq. (58) for general \( \tilde{G} \) is only established by making use of the e.o.m., we have to work with on-shell geometries. \( \text{tr} \mathcal{E} \) in normal coordinates is thus given by

\[
\text{tr} \mathcal{E} = -\frac{e^{-\sigma}}{4} \text{Tr} \left\{ G_{\mu \nu} a^\mu a^\nu \right\}
\]

\[
= e^{-\sigma} k \left\{ \frac{1}{4} G^{\mu \nu} G^{\rho \sigma} (\partial_{\mu} \theta_{\rho \alpha}^{-1})(\partial_{\nu} \theta_{\sigma \beta}^{-1}) \eta_{\alpha \beta} - \frac{1}{4} G^{\mu \nu} G^{\rho \sigma} (\partial_{\rho} \theta_{\mu \alpha}^{-1})(\partial_{\sigma} \theta_{\nu \beta}^{-1}) \eta_{\alpha \beta} + \frac{1}{8} (G^{\nu \sigma} (g^{\rho \sigma} \partial_{\mu} g_{\rho \sigma}) - 2 G^{\rho \sigma} \gamma_{\rho}^{\beta} \partial_{\mu} \gamma_{\sigma}^{\delta} + g^{\mu \nu} G^{\mu \sigma} \partial_{\nu} g_{\rho \sigma}) \right\}
\]

\[
+ \frac{k}{4} g^{\mu \nu} (\partial_{\mu} \partial_{\nu} \phi^i) \left( \Delta_{G^i} \phi^j + \tilde{\Gamma}^\mu (\partial_{\mu} \phi^j) \right) + \frac{k}{4} \tilde{G}_{\mu \nu} \tilde{\Gamma}^\mu \tilde{\Gamma}^\nu.
\]

We make use of the relation Eq. (113) of Appendix A

\[ (\partial_{\lambda} \phi^i)(\partial_{\mu} \partial_{\nu} \phi^j) \delta_{ij} = \frac{1}{2} (\partial_{\mu} g_{\nu \lambda} + \partial_{\nu} g_{\mu \lambda} - \partial_{\lambda} g_{\mu \nu}). \]  

(63)

Differentiating once more gives

\[ (\partial_{\rho} \partial_{\sigma} \phi^i)(\partial_{\mu} \partial_{\nu} \phi^j) \delta_{ij} = \frac{1}{2} (\partial_{\mu} \partial_{\rho} g_{\nu \sigma} + \partial_{\nu} \partial_{\sigma} g_{\mu \rho} - \partial_{\rho} \partial_{\sigma} g_{\mu \nu}) \]

(64)

where the superscript “nc” stands for normal coordinates. In normal coordinates we have

\[
g^{\rho \sigma} G^{\mu \nu} (\partial_{\rho} \partial_{\mu} g_{\nu \sigma}) = g^{\rho \sigma} (\partial_{\rho} \partial_{\sigma} \phi^i) G^{\mu \nu} (\partial_{\mu} \partial_{\nu} \phi^j) \delta_{ij} + \frac{1}{2} g^{\mu \nu} G^{\mu \sigma} (\partial_{\rho} \partial_{\sigma} g_{\nu \mu})
\]

\[
= e^{\sigma} g^{\rho \sigma} (\partial_{\rho} \partial_{\sigma} \phi^i) \left( \Delta_{G^i} \phi^j + \tilde{\Gamma}^\mu (\partial_{\mu} \phi^j) \right) \delta_{ij} + \frac{1}{2} g^{\mu \nu} G^{\mu \sigma} (\partial_{\rho} \partial_{\sigma} g_{\nu \mu}).
\]

Our final result for \( \text{tr} \mathcal{E} \) in normal coordinates is then

\[
\text{tr} \mathcal{E} = e^{-\sigma} k \left\{ \frac{1}{4} G^{\mu \nu} G^{\rho \sigma} (\partial_{\mu} \theta_{\rho \alpha}^{-1})(\partial_{\nu} \theta_{\sigma \beta}^{-1}) g^{\alpha \beta} - G^{\mu \nu} G^{\rho \sigma} (\partial_{\rho} \theta_{\mu \alpha}^{-1})(\partial_{\sigma} \theta_{\nu \beta}^{-1}) g^{\alpha \beta} + \frac{1}{8} (G^{\nu \sigma} (g^{\rho \sigma} \partial_{\mu} g_{\rho \sigma}) - 2 G^{\rho \sigma} \gamma_{\rho}^{\beta} \partial_{\mu} \gamma_{\sigma}^{\delta} + g^{\mu \nu} G^{\mu \sigma} \partial_{\nu} g_{\rho \sigma}) \right\}
\]

\[
+ \frac{k}{4} g^{\mu \nu} (\partial_{\mu} \partial_{\nu} \phi^i) \left( \Delta_{G^i} \phi^j + \tilde{\Gamma}^\mu (\partial_{\mu} \phi^j) \right) + \frac{k}{4} \tilde{G}_{\mu \nu} \tilde{\Gamma}^\mu \tilde{\Gamma}^\nu.
\]

(66)

\[
\text{e.o.m.} = -\frac{e^{-\sigma}}{4} \text{Tr} \left\{ G_{\mu \nu} a^\mu a^\nu \right\}
\]

(60)

\[
\frac{\partial_{\mu} G_{\rho \sigma}}{p} = 0.
\]

(61)
6.2 $\text{tr}\mathcal{E}$ in normal coordinates for $\tilde{G} = g$

Since it was shown in the last section that $\text{tr}\mathcal{E}$ for $\tilde{G} = g$ is a covariant expression even for off-shell geometries, we will not use the e.o.m. here. The term $\frac{k}{4} g_{\mu\nu} \Gamma^\mu \Gamma^\nu$ now vanishes due to the normal coordinate system. Since $\theta^{-1}_{\mu\nu}$ fulfills the Jacobi identity the following equation

$$2(\partial_\mu \theta^{-1}_{\rho\alpha})(\partial_\nu \theta^{-1}_{\beta}) g^{\rho\sigma} g^{\alpha\beta} = (\partial_\mu \theta^{-1}_{\rho\alpha})(\partial_\nu \theta^{-1}_{\alpha\beta}) g^{\rho\sigma} g^{\alpha\beta}$$

(67)

holds for $\tilde{G} = g$, see also Appendix B. This simplifies $\text{tr}\mathcal{E}$ to

$$\text{tr}\mathcal{E} = \frac{k}{4} g^{\mu\nu} g^{\rho\sigma}(\partial_\mu \theta^{-1}_{\rho\alpha})(\partial_\nu \theta^{-1}_{\alpha\beta}) g^{\alpha\beta} + \frac{k}{8} g^{\mu\nu}(g^{\rho\sigma} \partial_\mu \partial_\nu g_{\rho\sigma}).$$

(68)

6.3 The Ricci scalar $R[\tilde{G}]$ in normal coordinates.

Let us now study the Ricci scalar $R[\tilde{G}]$ in normal coordinates. The curvature tensor and the Ricci scalar are given as usual by

$$R_{\mu\nu\rho}^\sigma[\tilde{G}] = \partial_\sigma \tilde{\Gamma}_{\mu\nu}^\rho - \partial_\rho \tilde{\Gamma}_{\mu\nu}^\sigma + \tilde{\Gamma}_{\lambda\rho}^\mu \tilde{\Gamma}_{\mu\nu}^\lambda - \tilde{\Gamma}_{\lambda\nu}^\mu \tilde{\Gamma}_{\mu\rho}^\lambda,$$

$$R[\tilde{G}] = \tilde{G}^\mu_{\rho} R_{\mu\nu\rho}^\nu.$$  

(69)

In terms of the metric (now with respect to the effective metric $\tilde{G}$) and its derivatives the Ricci scalar is given by

$$R[\tilde{G}] = -\tilde{G}_{\mu\nu}(\partial_\rho \tilde{G}^\rho_{\mu\nu})(\partial_\sigma \tilde{G}^\sigma_{\nu}) + \tilde{G}^\mu_{\rho} \tilde{G}^\rho_{\sigma} (\partial_\mu \partial_\rho \tilde{G}_{\nu\sigma})$$

$$- \tilde{G}_{\mu\nu} \tilde{G}^{\rho\sigma} \partial_\rho \partial_\sigma \tilde{G}_{\mu\nu} - (\partial_\rho \tilde{G}^{\rho\sigma})(\tilde{G}_{\mu\nu} \partial_\sigma \tilde{G}_{\mu\nu})$$

$$- \frac{3}{4} \tilde{G}^{\mu}_{\nu}(\partial_\rho \tilde{G}^{\rho\sigma})(\partial_\sigma \tilde{G}_{\mu\nu}) + \frac{1}{2} \tilde{G}^{\rho\sigma}(\partial_\rho \tilde{G}^{\mu\nu})(\partial_\nu \tilde{G}_{\mu\nu})$$

$$- \frac{1}{4} \tilde{G}^{\mu\nu}(\tilde{G}^{\rho\sigma}\partial_\mu \tilde{G}_{\rho\sigma})(\tilde{G}^{\kappa\lambda}\partial_\nu \tilde{G}_{\kappa\lambda}).$$

(70)

$R[\tilde{G}]$ for on-shell geometries $\tilde{G} \neq g$ using e.o.m. See Appendix C for the evaluation of the Ricci scalar in normal coordinates. The result is found to be

$$R[\tilde{G}] = \text{tr} \sum_{\alpha\beta} \{ \frac{1}{2} \eta_{\alpha\beta}(\partial_\mu \theta^{\mu\alpha})(\partial_\nu \theta^{\nu\beta}) \}.$$

(71)
we find the following relation between the Ricci scalar and $\text{tr} E$

Hence, in that special case we have

Let us write this result again as a covariant expression. In order to do so, we notice that

using the e.o.m and Eq. (64). A consequence of the above relation is then

\[ (\partial_{\mu} \theta^{\mu \alpha})(\partial_{\nu} \theta^{\nu \beta})g_{\alpha \beta} + (\partial_{\mu} \theta^{\mu \alpha})(\partial_{\nu} \theta^{\nu \beta})g_{\alpha \beta} = e^{2 \sigma} g^{\mu \nu} g^{\rho \sigma} (\partial_{\mu} \theta^{\nu -1})(\partial_{\nu} \theta^{\rho -1})g^{\alpha \beta} \]

\[ = e^{2 \sigma} g^{\mu \nu} g^{\rho \sigma} (\partial_{\mu} \theta^{\nu -1})(\partial_{\nu} \theta^{\rho -1})g^{\alpha \beta}. \]

6.4 A comparison of $\text{tr} E$ & $R[g]$

Let us finally compare our results for $\text{tr} E$ and the Ricci scalar $R[\tilde{G}]$. In normal coordinates we find the following relation between the Ricci scalar and $\text{tr} E$.

\[ \text{tr} E = -\frac{k}{2} R[\tilde{G}] - \frac{k}{8} G^{\mu \nu} (g^{\rho \sigma} \partial_{\mu} \partial_{\nu} g_{\rho \sigma}) \]

\[ + \frac{k}{4} e^{-\sigma}(\partial_{\mu} \theta^{\mu \alpha})(\partial_{\nu} \theta^{\nu \beta})g_{\alpha \beta} + \frac{k}{4} e^{-\sigma}(\partial_{\mu} \theta^{\mu \alpha})(\partial_{\nu} \theta^{\nu \beta})g_{\alpha \beta} \]

Let us write this result again as a covariant expression. In order to do so, we notice that

using the e.o.m and Eq. (64). A consequence of the above relation is then

\[ (\nabla_{\mu} \theta^{\mu \alpha})(\nabla_{\nu} \theta^{\nu \beta})g_{\alpha \beta} = -\theta^{\mu \alpha}(\nabla_{\nu} \nabla_{\mu} \theta^{\nu \beta})g_{\alpha \beta}. \]

In normal coordinates this is

\[ (\partial_{\mu} \theta^{\mu \alpha})(\partial_{\nu} \theta^{\nu \beta})g_{\alpha \beta} = -\theta^{\mu \alpha}(\partial_{\mu} \partial_{\nu} \theta^{\nu \beta})g_{\alpha \beta} - \theta^{\mu \alpha} \theta^{\nu \beta} \partial_{\mu} \partial_{\nu} g_{\alpha \beta}, \]

which can also be derived from the equation of motion. Now remember that

\[ \theta^{\mu \alpha}(\nabla_{\nu} \nabla_{\mu} \theta^{\nu \beta})g_{\alpha \beta} = -\theta^{\mu \alpha}(\nabla_{\mu} \nabla_{\nu} \theta^{\nu \beta})g_{\alpha \beta} + R^{\mu}_{\lambda \nu \alpha} G^{\lambda \nu} + R^{\beta}_{\lambda \mu \alpha} \theta^{\mu \lambda} \theta^{\nu \alpha} g_{\alpha \beta} \]

\[ = -\theta^{\mu \alpha}(\nabla_{\mu} \nabla_{\nu} \theta^{\nu \beta})g_{\alpha \beta} + G^{\mu \nu} R[g]_{\mu \nu} - \frac{1}{2} R[g]_{\mu \nu \sigma \rho} \theta^{\mu \nu} \theta^{\rho \sigma}. \]
Next consider the Ricci tensor in normal coordinates.

\[ R_{\mu\nu}^{nc} = \partial_\rho \Gamma_\mu^\rho - \partial_\mu \Gamma_\rho^\nu = \frac{1}{2} g^{\rho\lambda} \left( 2 \partial_\mu \partial_\rho g_{\lambda\nu} + \partial_\nu \partial_\rho g_{\mu\lambda} - \partial_\rho \partial_\lambda g_{\mu\nu} + \partial_\mu \partial_\nu g_{\rho\lambda} - \partial_\mu \partial_\lambda g_{\nu\rho} \right) \]  

(79)

Due to Eq. (65) we find then

\[ G_{\mu\nu} R^g_{\mu\nu} = -\frac{1}{2} G_{\mu\nu} \partial_\mu \partial_\nu \sigma = G_{\mu\nu} \nabla_\mu \nabla_\nu \sigma \]  

(80)

Using also

\[ \theta^{\mu\alpha} (\nabla_\mu \nabla_\nu \theta^{\nu\beta}) g_{\alpha\beta} = G_{\mu\nu} \partial_\mu \partial_\nu \sigma = G_{\mu\nu} \nabla_\mu \nabla_\nu \sigma \]  

(81)

as well as

\[ (\nabla_\mu \theta^{\mu\alpha})(\nabla_\nu \theta^{\nu\beta}) g_{\alpha\beta} \]

(82)

we obtain the following covariant form of \( \text{tr} \mathcal{E} \),

\[ \text{tr} \mathcal{E} = -\frac{k}{2} R[\tilde{G}] + k \frac{1}{4} \tilde{G}^{\mu\nu} (\nabla_\mu \sigma) (\nabla_\nu \sigma) + \frac{k}{4} G^{\mu\nu} \nabla_\mu \nabla_\nu \sigma + \frac{k}{8} e^{-\sigma} R[g]_{\mu\nu\rho\sigma} \theta^{\mu\nu} \theta^{\rho\sigma} \]  

(83)

or

\[ \text{tr} \mathcal{E} = -\frac{k}{2} R[\tilde{G}] + \frac{k}{4} e^{-\sigma} \Delta \tilde{G} e^{\sigma} + \frac{k}{8} e^{-\sigma} R[g]_{\mu\nu\rho\sigma} \theta^{\mu\nu} \theta^{\rho\sigma} \]  

(84)

As a check, we can compare this with the result of [4], where the case of a 4-dimensional manifold with flat background metric was studied. \( \text{tr} \mathcal{E} \) for on-shell geometries was shown to be

\[ \int \text{d}^4 x \ \text{tr} \mathcal{E} = \int \text{d}^4 x k_{4d} \left( -\frac{1}{2} R[\tilde{G}] + G^{\mu\nu} (\partial_\mu \sigma) (\partial_\nu \sigma) \right), \]  

(85)

where the rank \( k_{4d} \) of the 4-dimensional representation of the Clifforda algebra is four. Hence the two results are consistent if the background metric is flat and the number of dimensions equals four.

Finally for on-shell geometries we find the following one-loop effective action

\[ \Gamma_\psi = \frac{k}{16 \pi^2} \int \text{d}^{2n} x \sqrt{|\tilde{G}|} \left( 2 \Lambda^4 + \left( -\frac{1}{3} R[\tilde{G}] + \frac{1}{4} e^{-\sigma} \Delta \tilde{G} e^{\sigma} + \frac{1}{8} e^{-\sigma} R[g]_{\mu\nu\rho\sigma} \theta^{\mu\nu} \theta^{\rho\sigma} \right) \Lambda^2 + \mathcal{O}(\log \Lambda) \right), \]  

(86)
A comparison of $\text{tr} \mathcal{E}$ and $R$ for $\tilde{G} = g$ without using the e.o.m.

\[
\text{tr} \mathcal{E} \equiv \frac{k}{4} e^{-\sigma} \left( (\partial_\mu \theta^{\mu\alpha})(\partial_\nu \theta^{\nu\beta})g_{\alpha\beta} + (\partial_\mu \theta^{\mu\alpha})(\partial_\nu \theta^{\mu\beta})g_{\alpha\beta} \right) - \frac{k}{4} R[g] \tag{87}
\]

\[
+ \frac{k}{4} (\Delta_g \phi^i)(\Delta_g \phi^j)\delta_{ij}. \tag{88}
\]

Recall that

\[
\theta^{\mu\alpha}(\nabla_\mu \theta^{\nu\beta})g_{\alpha\beta} = \theta^{\mu\alpha}(\partial_\mu \theta^{\nu\beta})g_{\alpha\beta} + G^{\mu\rho} \Gamma_{\mu \rho} + \theta^{\mu\alpha} \theta^{\nu\beta}(\partial_\nu g_{\alpha\beta})
\]

\[
\tilde{G} = -e^\sigma \Gamma^\nu + e^\sigma \Gamma_\nu = 0 \tag{89}
\]

is true also for off-shell geometries in the case of $\tilde{G} = g$. With the help of Eq. (65) we see that

\[
(\nabla_\mu \theta^{\mu\alpha})(\nabla_\nu \theta^{\nu\beta})g_{\alpha\beta} = -\theta^{\mu\alpha}(\nabla_\nu \nabla_\mu \theta^{\nu\beta})g_{\alpha\beta} \tag{90}
\]

Using Eq. (151) of Appendix B we have

\[
\theta^{\mu\alpha}(\nabla_\nu \nabla_\mu \theta^{\nu\beta})g_{\alpha\beta} = \theta^{\mu\alpha}(\partial_\nu \partial_\mu \theta^{\nu\beta})g_{\alpha\beta} + \frac{e^\sigma}{2} g^{\mu\nu} g^{\rho\sigma} \partial_\mu \partial_\nu \partial_\sigma \left(\theta^{\mu\alpha} \partial_\mu \theta^{\nu\beta}g_{\alpha\beta} \right) \tag{91}
\]

\[
\text{tr} \mathcal{E} = -\frac{k}{2} R[g] + \frac{k}{4} e^{-\sigma} g^{\mu\nu} \nabla_\mu \nabla_\nu e^\sigma + \frac{k}{8} e^{-\sigma} R[g]_{\mu\nu\rho\sigma} \theta^{\mu\nu} \theta^{\rho\sigma} + \frac{k}{4} (\Delta_g x^a)(\Delta_g x^b)\eta_{ab}. \tag{93}
\]

The total one-loop effective action is thus

\[
\Gamma_\Psi = \frac{k}{16\pi^2} \int \! d^2n \sqrt{|g|} \left( 2\Lambda^4 + \left( -\frac{1}{3} R[g] + \frac{1}{4} e^{-\sigma} \Delta_g e^\sigma \right. \right.
\]

\[
\left. + \frac{1}{8} e^{-\sigma} R[g]_{\mu\nu\rho\sigma} \theta^{\mu\nu} \theta^{\rho\sigma} + \frac{1}{4} (\Delta_g x^a)(\Delta_g x^b)\eta_{ab} \right) \Lambda^2 + O(\log \Lambda). \tag{94}
\]

We find that for this off-shell but self-dual case the result agrees with Eq. (86) plus a term depending on the extrinsic geometry.
7 Conclusion

In this work, fermions are studied in the framework of emergent noncommutative gravity for the general case of branes embedded in higher dimensions. The model is realized via a matrix model of Yang-Mills type. The fermionic term in the matrix model action leads to a specific coupling to geometry determined by a nontrivial effective metric $\tilde{G}_{\mu\nu}$. It goes along with a vanishing spin connection in the preferred coordinates associated with the matrix model. In the case of extra dimensions the vielbein is not given entirely by the Poisson tensor $\theta_{\mu\nu}$ as in [4], rather the Poisson tensor corresponds to a vielbein which relates the effective metric to the “tangential” embedding metric which in turn is non-trivial. This is responsible for the difference with the standard case.

The resulting action in this framework shows some deviations from the standard fermionic coupling. We find an induced gravitational action which includes the expected Einstein-Hilbert term with a modified coefficient, as well as three additional terms. One of these additional terms contains explicitly the Poisson structure $\theta_{\mu\nu}$ coupled to the Riemann tensor, $e^{-\sigma}R_{\mu\nu\rho\sigma}\theta^{\mu\nu}\theta^{\rho\sigma}$. This term leads to some concern because $\theta_{\mu\nu}$ breaks Lorentz invariance. That is irrelevant in the remaining terms of the effective action where $\theta_{\mu\nu}$ enters only implicitly through the effective metric, however $e^{-\sigma}R_{\mu\nu\rho\sigma}\theta^{\mu\nu}\theta^{\rho\sigma}$ corresponds to a direct coupling of $\theta^{\mu\nu}$ on the geometry, which breaks the local Lorentz invariance. Moreover, it is not small due to the $e^{-\sigma}$ factor. While this term vanishes in flat geometries, it will have some impact on gravity. The physical significance of this effect remains to be studied.

The importance of the additional $R\theta\theta$ term depends on the precise mechanism of gravity in the matrix model. As shown in [8], there seem to be 2 possibilities: First, gravity is indeed governed by the induced gravitational terms in the spirit of induced gravity. Then the $R\theta\theta$ term may have important physical implications. Second, gravity is dominated by the deformed harmonic embeddings resp. brane tension, and the cutoff $\Lambda$ (which is given by the scale of $N = 4$ SUSY breaking in the IKKT model) in front of the induced gravitational terms is much smaller than the Planck scale; this scenario is indeed preferred and viable as shown in [8]. In that case, the novel terms $R\theta\theta$ obtained here would have a very small impact on gravity and lead only to minor corrections. The latter scenario is more attractive for a variety of reasons including the cosmological constant problem.

In either case, we found an interesting new term in the one-loop effective action for fermions in the matrix model, as well as the usual Einstein-Hilbert term with a non-standard coefficient. This is a manifestation of the non-standard spin connection in the matrix model. Clearly more work is required before the physical implications of this terms are fully understood.
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8 Appendix A: Evaluation of trE

We want to express trE,

\[
trE = - \text{tr} \left\{ \tilde{G}^{\mu\nu} \Omega_\mu \Omega_\nu + \tilde{G}^{\mu\nu} \partial_\mu \Omega_\nu - \tilde{\Gamma}^\rho_\mu \right\}
\]

\[
= - \text{tr} \left( \frac{1}{4} \tilde{G}_{\mu\nu} a^\mu a^\nu - \frac{1}{4} \tilde{G}_{\mu\nu} \tilde{\Gamma}^\nu_\mu + \frac{1}{2} \tilde{G}^{\mu\nu} \partial_\mu (\tilde{G}_{\nu\rho} a^\rho + \tilde{G}_{\nu\rho} \tilde{\Gamma}^\rho_\nu) \right). 
\]

explicitly by the Poisson tensor \( \theta^{\mu\nu} \) and the background metric \( g_{\mu\nu} \).

To begin with we show that the following relation containing second order partial
derivatives is true.

\[
G^{\rho\mu} (g_{\rho\delta} \partial_\lambda \phi^i) (g_{\nu\sigma} \partial_\nu \phi^j) \delta_{ij} = \frac{1}{2} \left( G^{\rho\mu} \partial_\mu g_{\nu\sigma} + G^{\mu\nu} g_{\rho\sigma} \partial_\nu g_{\rho\lambda} - 2G^{\rho\mu} g_{\nu\lambda} \partial_\nu \partial_\lambda g_{\mu\nu} \right)
\]

\[
+ \epsilon^{\rho\mu} (\partial_\mu \partial_\nu \phi^i) \left( \Delta_{G} \phi^j + \tilde{\Gamma}^\mu_\mu (\partial_\mu \phi^j) \right) \delta_{ij}.
\]

This can be seen by taking

\[
(\partial_\rho \partial_\beta \phi^i)(\partial_\sigma \partial_\delta \phi^j) \delta_{ij} + (\partial_\beta \phi^i)(\partial_\sigma \partial_\delta \partial_\beta \phi^j) \delta_{ij} = \frac{1}{2} (\partial_\rho \partial_\sigma g_{\beta\delta} + \partial_\rho \partial_\delta g_{\beta\sigma} \delta_{ij} - \partial_\rho \partial_\beta g_{\sigma\delta}) \tag{97}
\]

and subtracting from this equation the same equation with the indices \( \rho \) and \( \delta \) inter-
changed. This gives

\[
(\partial_\rho \partial_\beta \phi^i)(\partial_\sigma \partial_\delta \phi^j) \delta_{ij} -(\partial_\delta \phi^i)(\partial_\sigma \partial_\beta \phi^j) \delta_{ij} = \frac{1}{2} (\partial_\rho \partial_\sigma g_{\beta\delta} - \partial_\delta \partial_\gamma g_{\beta\rho} - \partial_\delta \partial_\gamma g_{\rho\sigma} \delta_{ij}) \tag{98}
\]

Hence we have

\[
G^{\rho\sigma} g^{\beta\delta} (\partial_\mu \partial_\beta \phi^i)(\partial_\sigma \partial_\delta \phi^j) \delta_{ij} - g^{\beta\delta} (\partial_\beta \partial_\delta \phi^i) G^{\rho\sigma} (\partial_\mu \partial_\sigma \phi^j) \delta_{ij} = \]

\[
G^{\rho\sigma} g^{\beta\delta} (\partial_\mu \partial_\beta \phi^i)(\partial_\sigma \partial_\delta \phi^j) \delta_{ij} - \epsilon^{\rho\mu} (\partial_\beta \partial_\delta \phi^i) \left( \Delta_{G} \phi^j + \tilde{\Gamma}^\mu_\mu (\partial_\mu \phi^j) \right) \delta_{ij} = \]

\[
\frac{1}{2} G^{\nu\sigma} (g_{\nu\rho} \partial_\nu g^{-1}) - 2G^{\rho\sigma} g^{\beta\delta} \partial_\mu \partial_\beta g_{\sigma\delta} + g^{\mu\nu} G^{\rho\sigma} \partial_\mu \partial_\nu g_{\rho\sigma}. \tag{99}
\]

A simple relation is also

\[
(\partial_\nu \phi^i)(\partial_\mu \partial_\lambda \phi^j) \delta_{ij} \theta^{\mu\nu} = (\partial_\mu g_{\nu\lambda}) \theta^{\mu\nu}. \tag{100}
\]
Computation of tr \((G_{\mu\nu}a^{\mu\nu})\).

\[
\begin{align*}
\text{tr } (G_{\mu\nu}a^{\mu\nu}) &= \text{tr} \left[ \tilde{\gamma}_\alpha \tilde{\gamma}_\beta \theta^{\rho\alpha}(\partial_\rho \theta^{\mu\beta}) + \tilde{\gamma}_\alpha (\partial_\rho \tilde{\gamma}_\beta) \theta^{\rho\alpha} \theta^{\mu\beta} \right] \\
&= \text{tr} \left[ \tilde{\gamma}_\alpha \tilde{\gamma}_\beta \tilde{\gamma}_\gamma \tilde{\gamma}_\delta \theta^{\rho\alpha} \theta^{\sigma\gamma}(\partial_\rho \theta^{\mu\beta})(\partial_\sigma \theta^{\nu\delta}) G_{\mu\nu} \\
&+ 2\tilde{\gamma}_\alpha \tilde{\gamma}_\beta \tilde{\gamma}_\gamma (\partial_\sigma \tilde{\gamma}_\delta) \theta^{\rho\alpha} \theta^{\mu\beta} \theta^{\sigma\gamma} \theta^{\nu\delta} G_{\mu\nu} \\
&+ \tilde{\gamma}_\alpha (\partial_\rho \tilde{\gamma}_\beta) \tilde{\gamma}_\gamma (\partial_\sigma \tilde{\gamma}_\delta) \theta^{\rho\alpha} \theta^{\sigma\gamma} g^{\beta\delta} \right] \\
\end{align*}
\]

(101)

We evaluate the trace of the Gamma matrices \(\tilde{\gamma}\) that appear in the above expression,

\[
\begin{align*}
\text{tr} \tilde{\gamma}_\rho \tilde{\gamma}_\sigma \tilde{\gamma}_\alpha \tilde{\gamma}_\beta &= k \left( (\partial_\rho \phi^i) g_{\alpha \beta} - (\partial_\beta \phi^i) g_{\rho \sigma} + (\partial_\alpha \phi^i) g_{\rho \beta} \right) \delta_{ij}, \\
\text{tr} \tilde{\gamma}_\rho \tilde{\gamma}_\beta \tilde{\gamma}_\alpha \tilde{\gamma}_3 &= k \left( (\partial_\rho \phi^i) g_{\sigma \rho} - (\partial_\sigma \phi^i) g_{\rho \rho} + (\partial_\rho \phi^i) g_{\sigma \sigma} \right) \delta_{ij}, \\
\text{tr} \tilde{\gamma}_\rho \tilde{\gamma}_3 \tilde{\gamma}_3 &= -k \delta_{ij} g_{\rho \sigma} + (\delta_{k \delta} \delta_{ii} + \delta_{k \delta} \delta_{ij})(\partial_\rho \phi^k)(\partial_\sigma \phi^l). \\
\end{align*}
\]

(102)

Here \(k\) is the rank of the representation of the \(\gamma\)-matrices, depending on the number of extra dimensions.

\[
\begin{align*}
\tilde{\gamma}_\alpha \tilde{\gamma}_\beta \tilde{\gamma}_\gamma (\partial_\sigma \tilde{\gamma}_\delta) \theta^{\rho\alpha}(\partial_\rho \theta^{\mu\beta}) \theta^{\sigma\gamma} \theta^{\nu\delta} G_{\mu\nu} &= \text{tr} \left[ \tilde{\gamma}_\alpha \tilde{\gamma}_\beta \tilde{\gamma}_\gamma (\partial_\sigma \tilde{\gamma}_\delta) \theta^{\rho\alpha}(\partial_\rho \theta^{\mu\beta}) \theta^{\sigma\gamma} \theta^{\nu\delta} G_{\mu\nu} \right] \\
&= k \delta_{ij} \left\{ (\partial_\sigma \phi^i) g_{\beta \gamma} - (\partial_\gamma \phi^i) g_{\beta \sigma} + (\partial_\beta \phi^i) g_{\gamma \sigma} \right\} \times \\
&\times (\partial_\sigma \phi^j) \theta^{\rho\alpha}(\partial_\rho \theta^{\mu\beta}) \theta^{\sigma\gamma} \theta^{\nu\delta} G_{\mu\nu} \\
&= \frac{k}{2} \left[ (\partial_\sigma g_{\alpha \delta} + \delta g_{\alpha \sigma} - \partial_\alpha g_{\delta \sigma}) G^{\alpha\beta} \theta^{\rho\alpha}(\partial_\rho \theta^{\mu\beta}) \theta^{\sigma\gamma} \theta^{\nu\delta} G_{\mu\nu} \\
&+ (\partial_\sigma g_{\beta \delta} + \delta g_{\beta \sigma} - \partial_\beta g_{\delta \sigma}) G^{\alpha\beta} \theta^{\rho\alpha}(\partial_\rho \theta^{\mu\beta}) \theta^{\sigma\gamma} \theta^{\nu\delta} G_{\mu\nu} \right] \\
&= \frac{k}{2} \left[ G^{\mu\sigma} \theta^{\rho\alpha}(\partial_\sigma g_{\alpha \delta})(\partial_\rho \theta^{\mu\beta}) g^{\nu\delta} + G^{\mu\sigma} \theta^{\rho\alpha}(\partial_\delta g_{\alpha \rho})(\partial_\rho \theta^{\mu\beta}) g^{\nu\delta} \\
&- G^{\mu\sigma} \theta^{\rho\alpha}(\partial_\rho g_{\alpha \rho})(\partial_\rho \theta^{\mu\beta}) g^{\nu\delta} - G^{\mu\sigma} \theta^{\rho\alpha}(\partial_\rho g_{\alpha \rho})(\partial_\rho \theta^{\mu\beta}) g^{\nu\delta} \\
&- G^{\mu\sigma} \theta^{\rho\alpha}(\partial_\rho g_{\alpha \rho})(\partial_\rho \theta^{\mu\beta}) g^{\nu\delta} + G^{\mu\sigma} \theta^{\rho\alpha}(\partial_\rho g_{\alpha \rho})(\partial_\rho \theta^{\mu\beta}) g^{\nu\delta} \\
&- 2G^{\mu\sigma} (\partial_\rho \theta^{\mu\beta}) G^{\sigma\lambda}(\partial_\rho \theta^{\mu\beta}) g^{\nu\delta} \right] \\
\end{align*}
\]

(104)
\[ \tilde{z}_\alpha (\partial_\mu \tilde{z}_\beta) \tilde{z}_\gamma (\partial_\sigma \tilde{z}_\delta) \theta^{\rho \sigma} \gamma^{\rho \sigma} g^{\beta \delta} = \text{tr} \left[ \tilde{z}_\alpha (\partial_\mu \tilde{z}_\beta) \tilde{z}_\gamma (\partial_\sigma \tilde{z}_\delta) \theta^{\rho \sigma} \gamma^{\rho \sigma} g^{\beta \delta} \right] \]

where \( k \) is a constant. The explicit expression for the whole term is then

\[ k \left[ - \delta_{ij} g_{\alpha \gamma} + (\delta_{k_1} \delta_{i_2} + \delta_{k_2} \delta_{i_1}) (\partial_\alpha \phi^{k})(\partial_\gamma \phi^{i}) \right] \times \]

\[ (\partial_\rho \partial_\beta \phi^i)(\partial_\sigma \partial_\delta \phi^j) \theta^{\rho \sigma} \theta^{\beta \delta} g^{\gamma \delta} \]

\[ = k \left[ - (\partial_\rho \partial_\beta \phi^i)(\partial_\sigma \partial_\delta \phi^j) \delta_{ij} G^{\rho \sigma} g^{\beta \delta} + (\partial_\rho g_{\alpha \beta}) \theta^{\rho \sigma} (\partial_\sigma g_{\gamma \delta}) \theta^{\gamma \delta} g^{\beta \delta} \right. \]

\[ + \frac{1}{4} (\partial_\sigma g_{\alpha \delta} + \partial_\delta g_{\alpha \sigma} - \partial_\alpha g_{\sigma \delta})(\partial_\rho g_{\gamma \beta} + \partial_\beta g_{\gamma \rho} - \partial_\gamma g_{\rho \beta}) \theta^{\rho \sigma} \theta^{\gamma \delta} g^{\beta \delta} \]

\[ = k \left[ - \frac{1}{2} (G^{\mu \nu}(g \partial_\mu \partial_\nu g^{-1}) - 2G^{\rho \sigma} g^{\beta \delta} \partial_\rho \partial_\beta g_{\alpha \delta} + g^{\mu \nu} G^{\rho \sigma} \partial_\mu \partial_\nu g_{\rho \sigma}) \right. \]

\[ + G^{\mu \nu}(\partial_\mu \theta^{\rho \sigma} \gamma^{\rho \sigma} (\partial_\nu \theta^{\rho \sigma}) g^{\alpha \beta} \]

\[ + \frac{1}{2} \theta^{\rho \sigma}(\partial_\mu g_{\rho \beta}) (\partial_\nu g_{\sigma \beta}) g^{\alpha \beta} \]

\[ + \frac{1}{2} \theta^{\mu \nu} \theta^{\rho \sigma}(\partial_\mu g_{\rho \sigma}) (\partial_\nu g_{\rho \sigma}) g^{\alpha \beta} \]

\[ - \frac{1}{4} \theta^{\mu \nu} \theta^{\rho \sigma}(\partial_\mu g_{\rho \sigma}) (\partial_\nu g_{\rho \sigma}) g^{\alpha \beta} \]

\[ - e^\sigma g^{\beta \delta}(\partial_\beta \partial_\delta \phi^j) \left( \Delta_G \phi^j + \tilde{\Gamma}^{\mu \nu}(\partial_\mu \phi^j) \right) \delta_{ij} \]

(105)

In the last step we have used Eq.(139). The explicit expression for the whole term is then

\[ \frac{-e^{-\sigma}}{4} \text{tr} \left( G^{\mu \nu} \partial_\mu a_\nu \right) = -k \frac{-e^{-\sigma}}{4} \left\{ G^{\mu \nu}(\partial_\mu \theta^{\rho \sigma} \gamma^{\rho \sigma} (\partial_\nu \theta^{\rho \sigma}) g^{\alpha \beta} \right. \]

\[ - G^{\mu \nu}(g \partial_\mu \partial_\nu g^{-1}) - 2G^{\rho \sigma} g^{\beta \delta} \partial_\rho \partial_\beta g_{\alpha \delta} + g^{\mu \nu} G^{\rho \sigma} \partial_\mu \partial_\nu g_{\rho \sigma} \right. \]

\[ + G^{\mu \nu}(g \partial_\mu \partial_\nu g^{-1}) (\partial_\nu \theta^{\rho \sigma}) g^{\alpha \beta} + G^{\mu \nu} G^{\rho \sigma} (\partial_\rho \theta^{\rho \sigma}) (\partial_\nu \theta^{\rho \sigma}) g^{\alpha \beta} \]

\[ + G^{\mu \nu} \theta^{\rho \sigma} (\partial_\nu g_{\rho \beta}) (\partial_\nu \theta^{\rho \sigma}) g^{\alpha \beta} + G^{\mu \nu} \theta^{\rho \sigma} (\partial_\nu g_{\rho \sigma}) (\partial_\nu \theta^{\rho \sigma}) g^{\alpha \beta} \]

\[ - G^{\rho \sigma}(\partial_\mu \theta^{\rho \sigma} \gamma^{\rho \sigma} (\partial_\nu \theta^{\rho \sigma}) g^{\alpha \beta} \]

\[ + G^{\rho \sigma} (\partial_\mu \theta^{\rho \sigma} \gamma^{\rho \sigma} (\partial_\nu \theta^{\rho \sigma}) g^{\alpha \beta} + \frac{1}{4} \theta^{\mu \nu} \theta^{\rho \sigma} (\partial_\mu g_{\rho \sigma}) (\partial_\nu g_{\rho \sigma}) g^{\alpha \beta} \}

(106)

**Computation of tr\((-G^{\mu \nu}(\partial_\mu G_{\nu \rho}) a^\rho)\) and tr\((\partial_\mu a^\mu)\).** Next we deal with the remaining two terms in tr\(E\). They turn out to cancel each other and moreover both are zero for
on-shell geometries. We evaluate again the trace.

\[
\text{tr}(\partial\tilde{\gamma}_\alpha)(\partial\tilde{\gamma}_\beta) = \text{Tr}[\gamma_{3+i}(\partial\mu\partial_\alpha\phi^i)(\gamma_\beta + \gamma_{3+j}(\partial\beta\phi^j))] \\
= k\delta_{ij}(\partial\mu\partial_\alpha\phi^i)(\partial\beta\phi^j) \\
= k \left( \partial\mu g_{\alpha\beta} + \partial\alpha g_{\mu\beta} - \partial\beta g_{\mu\alpha} \right)
\] (107)

\[
\text{tr}(\partial\tilde{\gamma}_\alpha)(\partial\nu\tilde{\gamma}_\beta)\theta^\mu\alpha\theta^\mu\beta = \text{Tr}\gamma_{3+i}\gamma_{3+j}(\partial\mu\partial_\alpha\phi^i)(\partial\nu\partial_\beta\phi^j)\theta^\nu\alpha\theta^\mu\beta \\
= k\delta_{ij}(\partial\mu\partial_\alpha\phi^i)(\partial\nu\partial_\beta\phi^j)\theta^\nu\alpha\theta^\mu\beta \\
= \frac{1}{2}(\partial\mu\partial\nu g_{\alpha\beta} + \partial\mu\partial\beta g_{\alpha\nu} - \partial\mu\partial\alpha g_{\beta\nu})\theta^\nu\alpha\theta^\mu\beta \\
= \theta^\nu\alpha\theta^\mu\beta \partial\mu\partial\nu g_{\alpha\beta}
\] (108)

First we consider the computation of \(-G^{\mu\nu}(\partial\mu G_{\nu\rho})a^\rho\).

\[
\text{tr}^a = \text{Tr}[\gamma_{\alpha\beta}\theta^\sigma\alpha(\partial\sigma\theta^\rho\beta) + \tilde{\gamma}_\alpha(\partial\sigma\tilde{\gamma}_\beta)\theta^\sigma\alpha\theta^\sigma\beta] \\
= k\left[ g_{\alpha\beta}\theta^\sigma\alpha(\partial\sigma\theta^\rho\beta) + \frac{1}{2}(\partial\sigma g_{\alpha\beta} + \partial\beta g_{\sigma\alpha} - \partial\alpha g_{\sigma\beta})\theta^\sigma\alpha\theta^\sigma\beta \right] \\
= k\left[ \theta^\sigma\alpha(\partial\sigma\theta^\rho\beta)g_{\alpha\beta} + \theta^\sigma\alpha\theta^\rho\alpha(\partial\alpha g_{\beta\sigma}) \right] \\
= -ke^\nu\tilde{\Gamma}^\nu.
\]

The remaining term \(\text{tr}\partial\mu a^\mu\) gives

\[
\text{tr}\partial\mu a^\mu = \text{Tr}\left[ (\partial\nu\tilde{\gamma}_\alpha)(\partial\nu\tilde{\gamma}_\beta)\theta^\mu\nu\alpha(\partial\nu\theta^\mu\beta) \right. \\
+ \tilde{\gamma}_\alpha\tilde{\gamma}_\beta(\partial\nu\theta^\mu\alpha)(\partial\nu\theta^\mu\beta) + \tilde{\gamma}_\alpha\tilde{\gamma}_\beta\theta^\nu\alpha\theta^\nu\beta(\partial\mu g_{\alpha\beta}) \\
+ (\partial\mu\tilde{\gamma}_\alpha)(\partial\nu\tilde{\gamma}_\beta)\theta^\nu\mu\alpha\theta^\nu\mu\beta + \tilde{\gamma}_\alpha(\partial\nu\tilde{\gamma}_\beta)(\partial\mu\theta^\nu\alpha)\theta^\mu\beta \\
+ \tilde{\gamma}_\alpha(\partial\nu\tilde{\gamma}_\beta\theta^\nu\alpha(\partial\mu\theta^\mu\beta)) \right] \\
= k\left\{ \frac{1}{2}(\partial\mu g_{\alpha\beta} + \partial\alpha g_{\mu\beta} - \partial\beta g_{\mu\alpha})\theta^\nu\beta(\partial\nu\theta^\mu\beta) \\
+ (\partial\mu g_{\alpha\beta} + \partial\beta g_{\mu\alpha} - \partial\alpha g_{\mu\beta})\theta^\nu\beta(\partial\nu\theta^\mu\beta) \\
+ 2(\partial\mu\theta^\nu\alpha)(\partial\nu g_{\alpha\beta}) + 2\theta^\nu\alpha(\partial\mu\theta^\nu\beta)g_{\alpha\beta} + 2\theta^\mu\alpha\theta^\nu\beta(\partial\mu\partial\nu g_{\alpha\beta}) \\
+ (\partial\nu g_{\alpha\beta} + \partial\beta g_{\nu g_{\alpha\beta}} - \partial\alpha g_{\nu g_{\beta}})(\partial\mu\theta^\nu\alpha)\theta^\mu\beta \\
+ (\partial\nu g_{\beta\alpha} + \partial\beta g_{\nu g_{\beta\alpha}} - \partial\nu g_{\beta\alpha})(\partial\mu\theta^\nu\alpha)\theta^\mu\beta \right\} \\
= k\left\{ 2\theta^\mu\alpha(\partial\mu\theta^\mu\beta)(\partial\nu g_{\alpha\beta}) + \theta^\mu\alpha(\partial\nu\theta^\mu\beta)(\partial\mu g_{\alpha\beta}) \\
+ (\partial\mu\theta^\mu\alpha)(\partial\nu\theta^\mu\beta)g_{\alpha\beta} + \theta^\mu\alpha(\partial\mu\partial\nu g_{\alpha\beta})g_{\alpha\beta} + \theta^\mu\alpha\theta^\nu\beta(\partial\mu\partial\nu g_{\alpha\beta}) \\
= k\partial\mu\left\{ \theta^\mu\alpha(\partial\nu\theta^\mu\beta)g_{\alpha\beta} + \theta^\mu\alpha\theta^\nu\beta(\partial\nu g_{\alpha\beta}) \right\} \\
= -k\partial\mu \left( e^\nu\tilde{\Gamma}^\nu \right).
\]
Due to Eq. (109) and (110) we find

$$\text{tr} \left( \tilde{G}^{\mu\nu} \partial_\mu (\tilde{G}_\nu \delta^\rho + \tilde{G}_\nu \tilde{\Gamma}^\rho) \right) = 0$$

(111)

and thus

$$\text{tr} \mathcal{E} = -\text{tr} \left( \frac{1}{4} \tilde{G}_{\mu\nu} \tilde{a}^{\mu} \tilde{a}^{\nu} - \frac{1}{4} \tilde{G}_{\mu\nu} \tilde{\Gamma}^{\mu} \tilde{\Gamma}^{\nu} \right)$$

$$= -k \frac{e^{-\sigma}}{4} \left\{ \begin{array}{l}
G^{\mu\nu}(\partial_\mu \theta^{-1} \nu \alpha) G^{\rho\sigma}(\partial_\rho \theta^{-1} \sigma \beta) g^{\alpha \beta} \\
- G^{\mu\nu} G^{\rho\sigma}(\partial_\mu \theta^{-1} \nu \alpha)(\partial_\rho \theta^{-1} \sigma \beta) g^{\alpha \beta} + G^{\rho\mu} G^{\sigma\nu}(\partial_\rho \theta^{-1} \nu \alpha)(\partial_\sigma \theta^{-1} \mu \beta) g^{\alpha \beta} + G^{\mu\nu} \theta^{\alpha \beta}(\partial_\sigma g_{\alpha \beta})(\partial_\rho \theta^{-1} \nu \alpha) g^{\rho \nu} + G^{\mu\nu} \theta^{\alpha \beta}(\partial_\rho g_{\alpha \beta})(\partial_\sigma \theta^{-1} \nu \alpha) g^{\rho \nu} \\
- G^{\mu\nu} \theta^{\alpha \beta}(\partial_\sigma g_{\alpha \beta})(\partial_\rho \theta^{-1} \nu \alpha) g^{\rho \nu} - G^{\mu\nu} \theta^{\alpha \beta}(\partial_\rho g_{\alpha \beta})(\partial_\sigma \theta^{-1} \nu \alpha) g^{\rho \nu} \\
- G^{\mu\nu} \theta^{\alpha \beta}(\partial_\sigma g_{\alpha \beta})(\partial_\rho \theta^{-1} \nu \alpha) g^{\rho \nu} + G^{\mu\nu} \theta^{\alpha \beta}(\partial_\rho g_{\alpha \beta})(\partial_\sigma \theta^{-1} \nu \alpha) g^{\rho \nu} \\
- 2 G^{\mu\nu}(\partial_\rho \theta^{-1} \nu \alpha) G^{\sigma \lambda}(\partial_\sigma \theta^{-1} \lambda \delta) g^{\rho \nu} \\
- \frac{1}{2} \left( G^{\mu\nu}(g_{\rho \nu} g_{\alpha}^{-1} - 2 G^{\rho \sigma} g^{\delta \beta} \partial_\rho \partial_\sigma g_{\gamma \delta} + g^{\mu \nu} G^{\rho \sigma} \partial_\rho g_{\rho \sigma} \right) \\
+ G^{\mu\nu}(\partial_\rho \theta^{-1} \nu \alpha) G^{\rho\sigma}(\partial_\rho \theta^{-1} \nu \alpha) g^{\alpha \beta} + \frac{1}{2} \, \theta^{\rho \alpha}(\partial_\sigma g_{\alpha \beta})(\partial_\rho \theta^{-1} \nu \alpha) g^{\rho \nu} + \frac{1}{2} \, g^{\mu \nu} \theta^{\rho \sigma}(\partial_\alpha g_{\rho \sigma})(\partial_\beta g_{\nu \sigma}) g^{\alpha \beta} \right\} \\
+ \frac{k}{4} g^{\mu \nu}(\partial_\rho \theta^{-1} \nu \alpha) \left( \Delta g^{\rho \nu} + \tilde{G}^{\mu \nu} \delta_{ij} \right) \delta_{ij} \\
+ \frac{k}{4} \tilde{G}_{\mu\nu} \tilde{\Gamma}^{\mu} \tilde{\Gamma}^{\nu}.
\right. \right.$$ 

(112)

That means also that for on-shell geometries $\text{tr} \mathcal{E}$ is given solely by

$$\text{tr} \mathcal{E} = -\frac{e^{-\sigma}}{4} \text{tr} G^{\mu \nu} a_{\mu} a_{\nu}. \quad (113)$$

### 9 Appendix B: Covariance of $\text{tr} \mathcal{E}$

We aim to show that $\text{tr} \mathcal{E}$ can be written in covariant manner. If so, we can change to a normal coordinate system, which will simplify $\text{tr} \mathcal{E}$ and the Ricci scalar enormously. However, notice that now $\text{tr} \mathcal{E}$ should be related to the Ricci scalar directly and not only under the integral, where we would be allowed to use partial integration. Since normal coordinates make sense only at a point, partial integration is not admissible here.

**Notation.** We distinguish between the *effective metric* $\tilde{G}_{\mu\nu}$ and the *background metric* $g_{\mu\nu}$. The covariant derivatives and Christoffel symbols with respect to the background metric $g_{\mu\nu}$ as important in this section are written as $\nabla_\mu$ and $\Gamma^{\mu}_{\rho \sigma}$. 
By using expressions containing derivatives of $g_{\mu\nu}$,

$$\begin{align*}
\partial_\lambda g_{\mu\nu} &= (\partial_\lambda \partial_\mu \phi^i)(\partial_\nu \phi^j)\delta_{ij} + (\partial_\mu \phi^i)(\partial_\lambda \partial_\nu \phi^j)\delta_{ij}, \\
\partial_\nu g_{\lambda\mu} &= (\partial_\nu \partial_\lambda \phi^i)(\partial_\mu \phi^j)\delta_{ij} + (\partial_\lambda \phi^i)(\partial_\nu \partial_\mu \phi^j)\delta_{ij}, \\
\partial_\mu g_{\nu\lambda} &= (\partial_\mu \partial_\nu \phi^i)(\partial_\lambda \phi^j)\delta_{ij} + (\partial_\nu \phi^i)(\partial_\mu \partial_\lambda \phi^j)\delta_{ij},
\end{align*}$$

(114)

we find the following relation

$$(\partial_\lambda \phi^i)(\partial_\nu \partial_\mu \phi^j)\delta_{ij} = \frac{1}{2}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu})$$  

(115)

Since the first term in Eq. (116) does not contain a partial derivative of $g_{\mu\nu}$, we begin
with the second term,

\[ 2 \text{tr} \left( \tilde{\gamma} \tilde{\gamma} \tilde{\gamma}_i (\partial_\sigma \tilde{\gamma}) \theta^{\alpha \alpha} (\partial_\rho \theta^{\mu \beta}) \theta^{\sigma \gamma} \theta^{\mu \delta} G_{\mu \nu} \right) = 2k \delta_{ij} \left( (\partial_\gamma \phi^i) g_{\beta \gamma} - (\partial_\beta \phi^i) g_{\alpha \gamma} + (\partial_\gamma \phi^j) g_{\alpha \beta} \right) \times \]

\[ (\partial_\sigma \delta \phi^i) \theta^{\alpha \alpha} (\partial_\rho \theta^{\mu \beta}) \theta^{\sigma \gamma} \theta^{\mu \delta} G_{\mu \nu} \]

\[ = 2k \left( g_{\alpha \lambda} \Gamma^\lambda_{\sigma \delta} g_{\beta \lambda} \theta^{\alpha \alpha} (\partial_\rho \theta^{\mu \beta}) \theta^{\sigma \gamma} \theta^{\mu \delta} G_{\mu \nu} \right. \]

\[ - g_{\beta \lambda} \Gamma^\lambda_{\sigma \delta} g_{\alpha \gamma} \theta^{\alpha \alpha} (\partial_\rho \theta^{\mu \beta}) \theta^{\sigma \gamma} \theta^{\mu \delta} G_{\mu \nu} \]

\[ + g_{\gamma \lambda} \Gamma^\lambda_{\sigma \delta} g_{\alpha \beta} \theta^{\alpha \alpha} (\partial_\rho \theta^{\mu \beta}) \theta^{\sigma \gamma} \theta^{\mu \delta} G_{\mu \nu} \right) \]

\[ = 2k \left( g_{\alpha \lambda} \Gamma^\lambda_{\sigma \delta} G^{\mu \nu} \theta^{\alpha \alpha} (\partial_\rho \theta^{-1}_{\mu \nu}) g^{\nu \delta} \right) \]

\[ - g_{\alpha \lambda} \Gamma^\lambda_{\sigma \delta} G^{\rho \sigma} (\partial_\rho \theta^{-1}_{\mu \nu}) \theta^{\mu \alpha} g^{\nu \delta} \right) \]

\[ + g_{\alpha \lambda} \Gamma^\lambda_{\sigma \delta} G^{\rho \mu} (\partial_\rho \theta^{-1}_{\mu \nu}) \theta^{\rho \alpha} g^{\nu \delta} \right). \]

(117)

Next we address the third term in Eq. (116). We write

\[ (\partial_\rho \partial_\beta \phi^i)(\partial_\sigma \partial_\delta \phi^j) \delta_{ij} + (\partial_\beta \phi^i)(\partial_\rho \partial_\sigma \partial_\delta \phi^j) \delta_{ij} = (\partial_\rho g_{\beta \lambda})(\Gamma^\lambda_{\sigma \delta} + g_{\beta \lambda}(\partial_\rho \Gamma^\lambda_{\sigma \delta}). \]

(118)

and subtract from this equation the same equation, interchanging this time the indices \( \rho \) and \( \delta \). This gives

\[ (\partial_\rho \partial_\beta \phi^i)(\partial_\sigma \partial_\delta \phi^j) \delta_{ij} G^{\rho \delta} - g^{\delta \beta} (\partial_\delta \partial_\beta \phi^i) G^{\rho \delta} (\partial_\rho \partial_\sigma \phi^j) \delta_{ij} = \]

\[ (\partial_\rho \partial_\beta \phi^i)(\partial_\sigma \partial_\delta \phi^j) \delta_{ij} G^{\rho \sigma} g^{\delta \beta} - \sigma^{\rho \mu} (\partial_\rho \partial_\sigma \phi^i) \left( \Delta_{\rho \sigma} \phi^i + \tilde{\Gamma}^{\rho \sigma} \partial_\rho \phi^i \right) \delta_{ij} = \]

\[ G^{\rho \sigma} g^{\delta \beta} (\partial_\rho g_{\beta \lambda}) \Gamma^\lambda_{\sigma \delta} + G^{\rho \sigma} (\partial_\rho \Gamma^\lambda_{\sigma \delta}) - G^{\rho \sigma} g^{\delta \beta} (\partial_\delta g_{\beta \lambda}) \Gamma^\lambda_{\rho \sigma} - G^{\rho \sigma} (\partial_\lambda \Gamma^\lambda_{\rho \sigma}). \]

(119)

Using

\[ \partial_\rho g_{\beta \lambda} = \Gamma^\eta_{\rho \beta} g_{\eta \lambda} + \Gamma^\eta_{\rho \lambda} g_{\beta \eta} \]

(120)
one finds

\[
(\partial_{\rho}\partial_{\beta}\phi^i)(\partial_{\sigma}\partial_{\delta}\phi^j)G^{\rho\sigma}g^{\delta\beta} = G^{\rho\sigma}g^{\delta3}\Gamma_{\rho\beta}^{\gamma}\Gamma_{\sigma\delta}^{\lambda}g_{\gamma\lambda} + G^{\rho\sigma}\Gamma_{\rho\beta}^{\delta}\Gamma_{\sigma\delta}^{\lambda} + G^{\rho\sigma}(\partial_{\rho}\Gamma_{\sigma\lambda})
- G^{\rho\sigma}g^{\delta3}\Gamma_{\beta\gamma}^{\rho}\Gamma_{\rho\delta}g_{\gamma\lambda} - G^{\rho\sigma}\Gamma_{\eta\lambda}^{\rho}\Gamma_{\rho\sigma} - G^{\rho\sigma}(\partial_{\lambda}\Gamma_{\rho\sigma})
+ e^{g^{\mu\nu}(\partial_{\mu}\partial_{\nu}\phi^i)}(\Delta_{G}\phi^i + \tilde{\Gamma}^{\rho}\partial_{\rho}\phi^j)\delta_{ij}
= G^{\rho\sigma}g^{\delta3}\Gamma_{\rho\beta}^{\gamma}\Gamma_{\sigma\delta}^{\lambda}g_{\gamma\lambda} - G^{\rho\sigma}g^{\delta3}\Gamma_{\delta\beta}^{\rho\sigma}g_{\gamma\lambda}
+ G^{\rho\sigma}{\left\{ \Gamma_{\rho\lambda}^{\delta}\Gamma_{\sigma\lambda}^{\gamma} - \Gamma_{\rho\sigma}^{\delta}\Gamma_{\delta\lambda}^{\gamma} + \partial_{\rho}\Gamma_{\sigma\lambda}^{\lambda} - \partial_{\lambda}\Gamma_{\rho\sigma}^{\lambda} \right\}
+ e^{g^{\mu\nu}(\partial_{\mu}\partial_{\nu}\phi^i)}(\Delta_{G}\phi^i + \tilde{\Gamma}^{\rho}\partial_{\rho}\phi^j)\delta_{ij}
= G^{\rho\sigma}g^{\delta3}\Gamma_{\rho\beta}^{\gamma}\Gamma_{\sigma\delta}^{\lambda}g_{\gamma\lambda} - G^{\rho\sigma}g^{\delta3}\Gamma_{\delta\beta}^{\rho\sigma}g_{\gamma\lambda}
+ e^{g^{\mu\nu}(\partial_{\mu}\partial_{\nu}\phi^i)}(\Delta_{G}\phi^i + \tilde{\Gamma}^{\rho}\partial_{\rho}\phi^j)\delta_{ij}
- G^{\mu\nu}R_{\mu\nu}[g].
\]

We obtain for the third term of \( \text{tr}\mathcal{E} \)

\[
\text{tr}\left( \tilde{\Gamma}_{\alpha}(\partial_{\rho}\tilde{\gamma}_{\beta})(\partial_{\sigma}\tilde{\gamma}_{\delta})\theta^{\rho\alpha}\theta^{\sigma\gamma}g^{\beta\delta} \right) = k\left( -\delta_{ij}g_{\alpha\gamma} + (\delta_{ki}\delta_{ij} + \delta_{kj}\delta_{il}) (\partial_{\alpha}\phi^{k})(\partial_{\gamma}\phi^{l}) \right) \times
(\partial_{\rho}\partial_{\beta}\phi^i)(\partial_{\sigma}\partial_{\delta}\phi^j)\theta^{\rho\alpha}\theta^{\sigma\gamma}g^{\beta\delta}
= k\left( -\underbrace{G^{\rho\sigma}g^{\delta3}\Gamma_{\rho\beta}^{\gamma}\Gamma_{\sigma\delta}^{\lambda}g_{\gamma\lambda}}_{(f)} + G^{\rho\sigma}g^{\delta3}\Gamma_{\delta\beta}^{\rho\sigma}g_{\gamma\lambda}
+ G^{\mu\nu}R_{\mu\nu}[g] - e^{g^{\mu\nu}(\partial_{\mu}\partial_{\nu}\phi^i)}(\Delta_{G}\phi^i + \tilde{\Gamma}^{\rho}\partial_{\rho}\phi^j)\delta_{ij}
+ g_{\alpha\lambda}\Gamma_{\rho\beta}^{\lambda}\Gamma_{\alpha\delta}^{\gamma}\theta^{\rho\alpha}\theta^{\sigma\gamma}g^{\beta\delta} + g_{\alpha\lambda}\Gamma_{\sigma\delta}^{\lambda}\Gamma_{\rho\beta}^{\gamma}\theta^{\rho\alpha}\theta^{\sigma\gamma}g^{\beta\delta} \right).
\]

Let us write the result in an unconventional but simple way.

\[
-\frac{e^{-\sigma}}{4}\text{tr}G_{\mu\nu}a^\mu a^\nu = -e^{-\sigma}\frac{k}{4}\left\{ G^{\rho\nu}(\partial_{\rho}\theta_{\mu\alpha}^{-1})G^{\sigma\nu}(\partial_{\sigma}\theta_{\nu\beta}^{-1})g^{\alpha\beta} - G^{\rho\nu}G^{\sigma\nu}(\partial_{\rho}\theta_{\mu\alpha}^{-1})(\partial_{\sigma}\theta_{\nu\beta}^{-1})g^{\alpha\beta}
+ G^{\rho\nu}G^{\sigma\nu}(\partial_{\rho}\theta_{\mu\alpha}^{-1})(\partial_{\sigma}\theta_{\nu\beta}^{-1})g^{\alpha\beta} + \sum_{i=a}^{f}(i) + G^{\mu\nu}R_{\mu\nu}[g] + g^{\mu\nu}\Gamma_{\mu\nu}^{\lambda}G^{\rho\lambda}G_{\rho\sigma}g_{\gamma\eta}\right\}
- e^{g^{\mu\nu}(\partial_{\mu}\partial_{\nu}\phi^i)}(\Delta_{G}\phi^i + \tilde{\Gamma}^{\rho}(\partial_{\rho}\phi^j))\delta_{ij},
\]

where the terms \((i), i = a \ldots f\) refer to the terms denoted via curly brace.
Next consider the first term of Eq. (116),

\[
\tilde{\gamma}_\alpha \tilde{\gamma}_\beta \tilde{\gamma}_\delta \theta^\rho \theta^\sigma (\partial_\mu \theta^{\mu 3}) (\partial_\sigma \theta^{\rho \sigma}) G_{\mu \nu} = k \left\{ G^{\mu \nu} (\partial_\mu \theta^{-1}_\nu) G^{\rho \sigma} (\partial_\rho \theta^{-1}_\sigma) g^{\alpha \beta} \right.
\]

\[
- G^{\mu \nu} G^{\rho \sigma} (\partial_\rho \theta^{-1}_\nu) (\partial_\sigma \theta^{-1}_\nu) g^{\alpha \beta}
\]

\[
+ G^{\mu \nu} G^{\sigma \nu} (\partial_\sigma \theta^{-1}_\nu) (\partial_\nu \theta^{-1}_\sigma) g^{\alpha \beta} \right\}.
\]

(124)

In order to write tr\(E\) covariantly we replace every partial derivative by a covariant derivative \(\nabla_\mu\).

\[
G^{\mu \nu} (\nabla_\mu \theta^{-1}_\nu) G^{\rho \sigma} (\nabla_\rho \theta^{-1}_\sigma) g^{\alpha \beta} = G^{\mu \nu} G^{\rho \sigma} (\partial_\mu \theta^{-1}_\nu) - \Gamma^{\lambda \sigma}_\mu \theta^{-1}_\lambda - \Gamma^{\lambda \rho}_\mu \theta^{-1}_\lambda \times
\]

\[
(\partial_\rho \theta^{-1}_\sigma) - \Gamma^{\lambda \sigma}_{\rho \sigma} \theta^{-1}_\lambda \times \Gamma^{\lambda \rho}_{\rho \sigma} \theta^{-1}_\lambda) g^{\alpha \beta}
\]

\[
= G^{\mu \nu} G^{\rho \sigma} (\partial_\mu \theta^{-1}_\nu) (\partial_\rho \theta^{-1}_\sigma) g^{\alpha \beta} - 2 G^{\mu \nu} G^{\rho \sigma} \Gamma^{\lambda \sigma}_{\mu \rho \sigma} \theta^{-1}_\lambda (\partial_\sigma \theta^{-1}_\sigma) g^{\alpha \beta}
\]

\[
+ 2 \Gamma^{\lambda \rho}_{\mu \sigma} g_{\rho \lambda} G^{\rho \sigma} (\partial_\rho \theta^{-1}_\sigma) g^{\alpha \beta} + G^{\mu \nu} G^{\rho \sigma} \Gamma^{\lambda \rho}_{\mu \rho \sigma} G_{\lambda \sigma}
\]

\[
- 2 G^{\rho \sigma} \Gamma^{\lambda \rho}_{\mu \sigma} \Gamma^{\lambda \rho}_{\rho \sigma} g_{\sigma \lambda} \theta^{-1}_\rho g^{\alpha \beta}
\]

(125)

\[
G^{\mu \nu} G^{\rho \sigma} (\nabla_\rho \theta^{-1}_\rho) (\nabla_\sigma \theta^{-1}_\sigma) g^{\alpha \beta} = G^{\mu \nu} G^{\rho \sigma} (\partial_\rho \theta^{-1}_\rho) - \Gamma^{\lambda \rho}_{\mu \rho \rho} \theta^{-1}_\lambda - \Gamma^{\lambda \rho}_{\mu \rho \rho} \theta^{-1}_\lambda \times
\]

\[
(\partial_\sigma \theta^{-1}_\sigma) - \Gamma^{\lambda \rho}_{\rho \sigma} \theta^{-1}_\lambda \times \Gamma^{\lambda \rho}_{\rho \sigma} \theta^{-1}_\lambda) g^{\alpha \beta}
\]

\[
= G^{\mu \nu} G^{\rho \sigma} (\partial_\rho \theta^{-1}_\rho) (\partial_\sigma \theta^{-1}_\sigma) g^{\alpha \beta} - 2 G^{\mu \nu} G^{\rho \sigma} \Gamma^{\lambda \rho}_{\mu \rho \rho} \theta^{-1}_\lambda (\partial_\sigma \theta^{-1}_\sigma) g^{\alpha \beta}
\]

\[
+ 2 G^{\mu \nu} \Gamma^{\lambda \rho}_{\mu \rho \sigma} g_{\rho \lambda} (\partial_\rho \theta^{-1}_\sigma) g^{\alpha \beta} + \Gamma^{\lambda \rho}_{\rho \sigma} G^{\rho \sigma} G_{\lambda \sigma}
\]

\[
- 2 G^{\rho \sigma} \Gamma^{\lambda \rho}_{\mu \sigma} \Gamma^{\lambda \rho}_{\rho \sigma} g_{\sigma \lambda} (\partial_\rho \theta^{-1}_\sigma) g^{\alpha \beta}
\]

(126)

\[
G^{\mu \nu} G^{\rho \sigma} (\nabla_\rho \theta^{-1}_\rho) (\nabla_\sigma \theta^{-1}_\sigma) g^{\alpha \beta} = G^{\mu \nu} G^{\rho \sigma} (\partial_\rho \theta^{-1}_\rho) - \Gamma^{\lambda \rho}_{\mu \rho \rho} \theta^{-1}_\lambda - \Gamma^{\lambda \rho}_{\mu \rho \rho} \theta^{-1}_\lambda \times
\]

\[
(\partial_\sigma \theta^{-1}_\sigma) - \Gamma^{\lambda \rho}_{\rho \sigma} \theta^{-1}_\lambda \times \Gamma^{\lambda \rho}_{\rho \sigma} \theta^{-1}_\lambda) g^{\alpha \beta}
\]

\[
= G^{\mu \nu} G^{\rho \sigma} (\partial_\rho \theta^{-1}_\rho) (\partial_\sigma \theta^{-1}_\sigma) g^{\alpha \beta} - 2 G^{\mu \nu} G^{\rho \sigma} \Gamma^{\lambda \rho}_{\mu \rho \rho} \theta^{-1}_\lambda (\partial_\sigma \theta^{-1}_\sigma) g^{\alpha \beta}
\]

\[
+ 2 G^{\mu \nu} \Gamma^{\lambda \rho}_{\mu \rho \sigma} g_{\rho \lambda} (\partial_\rho \theta^{-1}_\sigma) g^{\alpha \beta} + G^{\mu \nu} \Gamma^{\lambda \rho}_{\rho \sigma} \Gamma^{\lambda \rho}_{\rho \sigma} G_{\lambda \sigma}
\]

\[
- 2 G^{\rho \sigma} \Gamma^{\lambda \rho}_{\mu \sigma} \Gamma^{\lambda \rho}_{\rho \sigma} g_{\sigma \lambda} (\partial_\rho \theta^{-1}_\sigma) g^{\alpha \beta}
\]

(127)
In the above equation two terms cancel due to

\[ G^{\mu\nu} \Gamma^\lambda_{\mu\rho} \theta_{\lambda\alpha}^{-1} (\partial_{\rho\theta_{\sigma\beta}}^{-1}) g^{\alpha\beta} = G^{\mu\nu} G^{\sigma\rho} \Gamma^\lambda_{\mu\rho} \theta_{\lambda\alpha}^{-1} (\partial_{\rho\theta_{\sigma\beta}}^{-1}) g^{\alpha\beta} = G^{\mu\nu} G^{\sigma\rho} \Gamma^\lambda_{\mu\rho} \theta_{\lambda\alpha}^{-1} (\partial_{\rho\theta_{\sigma\beta}}^{-1}) g^{\alpha\beta}. \]
We also exploited the following relations,

\[
G^\mu\nu G^{\rho\sigma} \Gamma^\alpha_{\mu\nu} \theta^\beta_{\lambda\alpha} (\partial_\rho \theta^{-1}_{\sigma\beta}) g^{\alpha\beta} + G^{\rho\sigma} \Gamma^\lambda_{\mu\alpha} \Gamma^\alpha_{\rho\sigma} \theta^\mu\nu g_{\nu\lambda} \theta^{-1}_{\lambda\beta} \delta^\alpha_{\beta} = G^\mu\nu \Gamma^\lambda_{\mu\nu} \theta^{-1}_{\lambda\beta} g^{\alpha\beta} \times \\
\left( G^{\rho\sigma} \left( \partial_\rho \theta^{-1}_{\sigma\beta} \right) + \theta^{\rho\sigma} \Gamma^\eta_{\rho\alpha} g_{\eta\lambda} \right) \\
= -G^\mu\nu \Gamma^\lambda_{\mu\nu} G_{\lambda\sigma} \times \\
\left( \theta^{\rho\sigma} \left( \partial_\rho \theta^{-1}_{\sigma\beta} \right) + \theta^{\rho\sigma} \theta^{\sigma\beta} \left( \partial_\rho g_{\alpha\beta} \right) \right) \\
= \epsilon^{\rho\sigma} G^\mu\nu \Gamma^\lambda_{\mu\nu} G_{\lambda\sigma} \tilde{\Gamma}_{\sigma},
\]

and

\[
\theta^{\rho\sigma} \left( \partial_\rho g_{\sigma\beta} \right) = \Gamma^\lambda_{\rho\alpha} g_{\lambda\sigma} \theta^{\rho\sigma}.
\]

In the end we obtain for \( \text{tr} a^\mu a^\nu G_{\mu\nu} \) the following result,

\[
\text{tr} a^\mu a^\nu G_{\mu\nu} = k \left( G^\mu\nu \left( \nabla_\mu \theta^{-1}_{\rho\nu} \right) G^{\rho\sigma} \left( \nabla_\rho \theta_{\sigma\beta}^{-1} \right) g^{\alpha\beta} - G^\mu\nu G^{\rho\sigma} \left( \nabla_\mu \theta_{\rho\alpha}^{-1} \right) \left( \nabla_\nu \theta_{\sigma\beta}^{-1} \right) g^{\alpha\beta} \right) \\
+ G^\mu\nu G^{\rho\sigma} \left( \nabla_\mu \theta_{\rho\alpha}^{-1} \right) \left( \nabla_\sigma \theta_{\nu\beta}^{-1} \right) g^{\rho\beta} + G^\mu\nu R_{\mu\nu \alpha} + g^{\mu\nu} \Gamma^\lambda_{\mu\nu} G^{\rho\sigma} \Gamma^\eta_{\rho\sigma} g_{\eta\lambda} \\
- \epsilon^\rho\sigma g^{\mu\nu} \left( \partial_\mu \partial_\nu \phi^j \right) \left( \Gamma^\rho_{\alpha} + \tilde{\Gamma}^\rho_{\eta} \left( \partial_\phi \delta^j \right) \right) \delta_{ij} \\
+ 2 G^\mu\nu \Gamma^\lambda_{\mu\nu} \Gamma^\eta_{\lambda\sigma} \tilde{G}^\sigma_{\lambda\eta} \nonumber
\]

\[\text{(132)}\]

**General case: \( \tilde{G} \neq g \) but using e.o.m.** We have

\[
\epsilon^\rho\sigma G^\mu\nu \Gamma^\lambda_{\mu\nu} \Gamma^\eta_{\lambda\sigma} G_{\lambda\eta} = G^{\mu\nu} G^{\rho\sigma} \Gamma^\lambda_{\mu\nu} G_{\lambda\sigma} = G^{\rho\sigma} g^{\delta\beta} \Gamma^\lambda_{\delta\beta} \Gamma^\eta_{\rho\sigma} g_{\eta\lambda} = 0.
\]

(133)

This can be seen by considering on-shell configurations \( \Delta G \phi^j = 0, \tilde{\Gamma}^\mu = 0 \) which imply

\[
G^{\mu\nu} \Gamma^\lambda_{\mu\nu} = G^{\mu\nu} \left( \partial_\rho \phi^j \right) \left( \partial_\mu \partial_\nu \phi^j \right) g^{\rho\lambda} \delta_{ij} \\
= \epsilon^\rho\sigma g^{\mu\nu} \left( \partial_\mu \partial_\nu \phi^j \right) \tilde{G}^{\rho\sigma} \left( \partial_\rho \partial_\nu \phi^j \right) \delta_{ij} \\
= 0.
\]

(134)

We have shown that for on-shell geometries \( \text{tr} \mathcal{E} \) is indeed a covariant expression,

\[
\text{tr} \mathcal{E} = -\frac{e^{-\sigma}}{4} \text{tr} G_{\mu\nu} a^\mu a^\nu \\
= -\frac{e^{-\sigma}}{4} k \left( G^{\mu\nu} \left( \nabla_\mu \theta_{\nu\alpha}^{-1} \right) G^{\rho\sigma} \left( \nabla_\rho \theta_{\sigma\beta}^{-1} \right) g^{\alpha\beta} - G^{\mu\nu} G^{\rho\sigma} \left( \nabla_\mu \theta_{\rho\alpha}^{-1} \right) \left( \nabla_\nu \theta_{\sigma\beta}^{-1} \right) g^{\alpha\beta} \right) \\
+ G^{\mu\nu} G^{\rho\sigma} \left( \nabla_\mu \theta_{\rho\alpha}^{-1} \right) \left( \nabla_\sigma \theta_{\nu\beta}^{-1} \right) g^{\rho\beta} \right) - \frac{k}{4} \tilde{G}^{\rho\sigma} R_{\rho\sigma} [g].
\]

(135)

Hence we can go to normal coordinates to simplify the comparison between \( \text{tr} \mathcal{E} \) and the Ricci scalar \( R[\tilde{G}] \). Keep in mind that the covariant derivative in Eq. (135) is with respect to the background metric \( g_{\mu\nu} \).

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Special case: \( \tilde{G} = g \) without the use of e.o.m. In that case we have
\[
G_{\mu\nu} \Gamma^\lambda \tilde{g}_{\lambda\eta} = G_{\mu\nu} G^{\rho\sigma} \Gamma^\lambda \tilde{g}_{\rho\sigma} G_{\lambda\eta} = g_{\mu\nu} \Gamma^\lambda \tilde{g}_{\rho\sigma} G^{\rho\sigma} G_{\lambda\eta} = e^\sigma \Gamma^\eta g_{\lambda\eta}.
\]
(136)

So we find for \( \text{tr} \mathcal{E} \)
\[
\text{tr} \mathcal{E} = -\frac{e^\sigma}{4} \text{tr}(\alpha^\mu \alpha^\nu G_{\mu\nu}) + \frac{k}{4} \Gamma^\lambda \Gamma^\eta g_{\lambda\eta}
= -\frac{e^\sigma}{4} k \left( g_{\mu\nu}(\nabla_\mu \theta_-1) g^{\rho\sigma} (\nabla_\nu \theta_-1) g^{\alpha\beta} - g_{\mu\nu} g^{\rho\sigma} (\nabla_\mu \theta_-1) (\nabla_\nu \theta_-1) g^{\alpha\beta}
+ g_{\mu\nu} g^{\rho\sigma} (\nabla_\mu \theta_-1) (\nabla_\rho \theta_-1) g^{\alpha\beta} \right) - \frac{k}{4} g_{\mu\nu} R_{\mu\nu} - \frac{1}{2} \Gamma^\mu \Gamma^\nu g_{\mu\nu}
+ \frac{e^\sigma}{4} k \left( \Delta_\rho \phi^i + \Gamma^\mu (\partial_\mu \phi^i) \right) \left( \Delta_\sigma \phi^j + \Gamma^\nu (\partial_\nu \phi^j) \right) \delta_{ij}
+ \frac{1}{4} k g_{\mu\nu} \Gamma^\mu \Gamma^\nu.
\]
(137)

Recalling
\[
\Delta_\rho \phi^i = \Gamma^\mu (\partial_\mu \phi^i),
\]
(140)

we see that
\[
\partial_\rho x^a \Delta_g x^b \eta_{ab} = -\Gamma^\mu \eta_{\mu\rho} + \partial_\rho \phi^i \Delta_\rho \phi^j \delta_{ij},
\]
(141)

as well as
\[
g_{\rho\mu} \Gamma^\mu = (\partial_\rho \phi^i)(\Delta_\rho \phi^j) \delta_{ij} + \Gamma^\mu (\partial_\mu \phi^i)(\partial_\rho \phi^j) \delta_{ij}
= (\partial_\rho x^a)(\Delta_g x^b) \eta_{ab} + \Gamma^\mu \eta_{\rho\mu} + \Gamma^\mu (\partial_\mu \phi^i)(\partial_\rho \phi^j) \delta_{ij}
= (\partial_\rho x^a)(\Delta_g x^b) \eta_{ab} + \Gamma^\mu g_{\rho\mu}.
\]
(142)

Therefore we have
\[
(\partial_\rho x^a)(\Delta_g x^b) \eta_{ab} = 0
\]
(143)
or

$$(\partial_\rho \phi^i)(\Delta_g \phi^j)\delta_{ij} = \Gamma^\mu \eta_{\mu\rho}. \quad (144)$$

It is worthwhile mentioning that the relation Eq. (143) for the general case $\tilde{G} \neq g$ turns out to be an e.o.m [3] and thus this equation usually holds only for on-shell geometries.

$$(\Delta_g \phi^i + \Gamma^\mu(\partial_\mu \phi^i))(\Delta_g \phi^j + \Gamma^\nu(\partial_\nu \phi^i))\delta_{ij} = \left((\Delta_g \phi^i)(\Delta_g \phi^j) + 2\Gamma^\mu(\partial_\mu \phi^i)(\Delta_g \phi^j)\right) + \Gamma^\mu \Gamma^\nu(\partial_\mu \phi^i)(\partial_\nu \phi^i)\delta_{ij} = (\Delta_g \phi^i)(\Delta_g \phi^j)\delta_{ij} + 2\Gamma^\nu \eta_{\mu\nu} + \Gamma^\mu \Gamma^\nu g_{\mu\nu} \quad (145)$$

$$\begin{align*}
\text{tr} & \mathcal{E} = -\frac{e^{-\sigma}}{4}\text{tr}(a^\mu a^\nu G_{\mu\nu}) + \frac{k}{4}\Gamma^\lambda \Gamma^\eta g_{\lambda\eta} \\
& = -\frac{e^{-\sigma}}{4} k \left(g^{\mu\nu}(\nabla_\mu \theta^{-1}_{\nu\alpha})(\nabla^\rho \theta^{-1}_{\sigma\beta})g^{\alpha\beta} - g^{\mu\nu}g^{\rho\sigma}(\nabla_\mu \theta^{-1}_{\rho\alpha})(\nabla_\nu \theta^{-1}_{\sigma\beta})g^{\alpha\beta}\right) + g^{\mu\nu}g^{\rho\sigma}(\nabla_\mu \theta^{-1}_{\rho\alpha})(\nabla_\nu \theta^{-1}_{\sigma\beta})g^{\alpha\beta} - \frac{k}{4} R[g] - \frac{1}{2}\Gamma^\nu \Gamma^\mu g_{\mu\nu} \\
& + \frac{k}{4}(\Delta_g x^a)(\Delta_b x^b)\eta_{ab} + \frac{k}{4}\Gamma^\nu \Gamma^\mu g_{\mu\nu} + \frac{k}{4}\Gamma^\mu \Gamma^\nu g_{\mu\nu} \\
& = -\frac{e^{-\sigma}}{4} k \left(g^{\mu\nu}(\nabla_\mu \theta^{-1}_{\nu\alpha})(\nabla^\rho \theta^{-1}_{\sigma\beta})g^{\alpha\beta} - g^{\mu\nu}g^{\rho\sigma}(\nabla_\mu \theta^{-1}_{\rho\alpha})(\nabla_\nu \theta^{-1}_{\sigma\beta})g^{\alpha\beta}\right) + g^{\mu\nu}g^{\rho\sigma}(\nabla_\mu \theta^{-1}_{\rho\alpha})(\nabla_\nu \theta^{-1}_{\sigma\beta})g^{\alpha\beta} - \frac{k}{4} R[g] + \frac{k}{4}(\Delta_g x^a)(\Delta_b x^b)\eta_{ab}. \quad (146)
\end{align*}$$

Due to the antisymmetry of $\theta^{\mu\nu}$ and since $\theta^{\mu\nu}$ also fulfills the Jacobi identity we have

$$\nabla_\rho \theta^{-1}_{\mu\nu} + \nabla_\nu \theta^{-1}_{\nu\rho} + \nabla_\mu \theta^{-1}_{\rho\nu} = 0 \quad (147)$$

Via the following computation

$$G^{\mu\nu} G^{\rho\sigma}(\nabla_\mu \theta^{-1}_{\rho\alpha})(\nabla_\nu \theta^{-1}_{\sigma\beta})g^{\alpha\beta} = G^{\mu\nu} G^{\rho\sigma} \left(\nabla_\alpha \theta^{-1}_{\mu\rho} + \nabla_\rho \theta^{-1}_{\alpha\mu}\right) \left(\nabla_\beta \theta^{-1}_{\nu\sigma} + \nabla_\sigma \theta^{-1}_{\beta\nu}\right) g^{\alpha\beta} = G^{\mu\nu} G^{\rho\sigma} \left((\nabla_\alpha \theta^{-1}_{\mu\rho})(\nabla_\beta \theta^{-1}_{\nu\sigma}) + 2(\nabla_\rho \theta^{-1}_{\alpha\mu})(\nabla_\sigma \theta^{-1}_{\beta\nu}) \right) g^{\alpha\beta} + (\nabla_\rho \theta^{-1}_{\mu\rho})(\nabla_\sigma \theta^{-1}_{\nu\sigma}) \right) g^{\alpha\beta} \quad (148)$$
we see that

\[ G^{\mu\nu}G^{\rho\sigma}\left( (\nabla_\alpha\theta_\mu^{-1})(\nabla_\beta\theta_\nu^{-1}) + 2(\nabla_\mu\theta_\alpha^{-1})(\nabla_\nu\theta_\alpha^{-1}) \right)g^{\alpha\beta} = 0 \]  

(149)

In the case of \( \tilde{G}_{\mu\nu} = g_{\mu\nu} \) we hence have

\[ g^{\mu\nu}g^{\rho\sigma}(\nabla_\mu\theta_\alpha^{-1})(\nabla_\nu\theta_\alpha^{-1}) = 2g^{\mu\nu}g^{\rho\sigma}(\nabla_\mu\theta_\alpha^{-1})(\nabla_\nu\theta_\alpha^{-1}). \]  

(150)

Also, in that case the following relation holds

\[ g^{\mu\nu}\nabla_\mu\theta_\nu^{-1} = 0. \]  

(151)

To see this consider the covariant derivative acting on the Poisson structure

\[ g^{\mu\nu}\nabla_\mu\theta_\nu^{-1} = g^{\mu\nu}\partial_\mu\theta_\nu^{-1} - g^{\mu\nu}\Gamma^\lambda_{\mu\nu}\theta_\lambda^{-1} - g^{\mu\nu}\Gamma^\lambda_{\mu\rho}\theta_\nu^{-1}. \]  

(152)

Using

\[ g^{\mu\nu}\Gamma^\lambda_{\mu\nu}\theta_\lambda^{-1} = \left( \frac{1}{2}g^{\mu\nu}g^{\lambda\eta}(\partial_\mu g_{\eta\nu}) - \frac{1}{2}g^{\mu\nu}g^{\lambda\eta}(\partial_\eta g_{\mu\nu}) - \frac{1}{2}(\partial_\rho g^{\lambda\nu}) \right)\theta_\nu^{-1} \]

\[ = g^{\mu\nu}g^{\lambda\eta}(\partial_\mu g_{\eta\nu})\theta_\nu^{-1} \]

\[ = -e^{-\sigma}\theta^{\mu\eta}(\partial_\mu g_{\eta\nu}) \]  

we can see that

\[ g^{\mu\nu}\nabla_\mu\theta_\nu^{-1} = g^{\mu\nu}(\partial_\mu\theta_\nu^{-1}) + e^{-\sigma}\theta^{\mu\eta}(\partial_\mu g_{\eta\nu}) - \Gamma^\lambda_{\mu\nu}\theta_\lambda^{-1} \]

\[ = g^{\mu\nu}(\partial_\mu\theta_\nu^{-1}) + e^{-\sigma}\theta^{\mu\eta}(\partial_\mu g_{\eta\nu}) \]

\[ + e^{-\sigma}(\partial_\eta\theta^{\lambda\eta})\theta^{\alpha\beta}g_{\alpha\beta}\theta_\lambda^{-1} + e^{-\sigma}\theta^{\lambda\alpha}\theta^{\eta\beta}(\partial_\rho g_{\eta\beta})\theta_\nu^{-1} \]  

(153)

\[ = 0 \]

The covariant e.o.m. that was derived in [3]

\[ \tilde{G}^{\mu\nu}\nabla_\mu(e^{\sigma}\theta_\nu^{-1}) = e^{-\sigma}\tilde{G}_{\mu\nu}\partial_\mu(G^{\kappa\lambda}g_{\kappa\lambda}) \]  

(155)

reduces for \( \tilde{G}_{\mu\nu} = g_{\mu\nu} \) to

\[ g^{\mu\nu}\nabla_\mu\theta_\nu^{-1} = 0, \]  

(156)

which has the form of a homogeneous Maxwell equation. So for \( \tilde{G} = g \) this relation is actually an identity.

Our final result for \( \text{tr}\mathcal{E} \) in case of \( \tilde{G}_{\mu\nu} = g_{\mu\nu} \) is

\[ \text{tr}\mathcal{E} = \frac{e^{\sigma}}{4}k g^{\mu\nu}g^{\rho\sigma}(\nabla_\mu\theta_\rho^{-1})(\nabla_\nu\theta_\nu^{-1})g^{\alpha\beta} - \frac{k}{4}R[g] + \frac{k}{4}(\Delta_g x^a)(\Delta_g x^b)\eta_{ab}. \]  

(157)

Thus we have shown that in the case of \( \tilde{G}_{\mu\nu} = g_{\mu\nu} \), for a covariance proof it is not necessary to use e.o.m.
Special case $\tilde{G} = g$ using e.o.m \( \text{tr} \mathcal{E} \) is now very simple,
\[
\text{tr} \mathcal{E} = \frac{e^\sigma}{4} k g^{\mu \nu} g^{\rho \sigma} (\nabla_\mu \theta_\rho^{-1})(\nabla_\sigma \theta_\nu^{-1}) g^{\alpha \beta} - \frac{k}{4} R[g]. \tag{158}
\]

10 Appendix C: Expressing \( R \) in normal coordinates

We evaluate the Ricci scalar in normal coordinates. First of all note that
\[
\tilde{G}^{\rho \sigma} \partial_\mu \tilde{G}_{\rho \sigma} = g^{\rho \sigma} \partial_\mu g_{\rho \sigma} \equiv 0. \tag{159}
\]

Using also
\[
\tilde{G}^{\mu \nu} \tilde{G}^{\rho \sigma} (\partial_\mu \partial_\rho \tilde{G}_{\nu \sigma}) = -\tilde{G}^{\mu \nu} (\partial_\mu \tilde{G}^{\rho \sigma}) (\partial_\rho \tilde{G}_{\nu \sigma}) + \tilde{G}^{\mu \nu} (\partial_\rho \tilde{G}^{\rho \sigma}) (\partial_\sigma \tilde{G}_{\nu \sigma}) - \partial_\mu \partial_\nu \tilde{G}^{\mu \nu} \tag{160}
\]
we can simplify the Ricci scalar Eq. (74) and we obtain
\[
R[\tilde{G}] = e^{-\sigma} \left\{ - \frac{3}{2} G^{\mu \nu} (\partial_\mu \sigma)(\partial_\nu \sigma) - 3 G^{\mu \nu} (\partial_\mu \partial_\nu \sigma) + (\partial_\mu G^{\mu \nu})(\partial_\nu \sigma) - \frac{1}{2} G^{\mu \nu} (\partial_\mu \sigma)(G^{\rho \sigma} \partial_\nu G_{\rho \sigma}) \right. \\
\left. - \frac{1}{2} G^{\mu \nu} (\partial_\mu G^{\rho \sigma})(\partial_\nu G_{\rho \sigma}) - \partial_\mu \partial_\nu G^{\mu \nu} - G^{\mu \nu} (G^{\rho \sigma} \partial_\mu \partial_\nu G_{\rho \sigma}) - \frac{3}{4} G^{\mu \nu} (\partial_\mu G^{\rho \sigma})(\partial_\nu G_{\rho \sigma}) \right\}. 
\]

Next give a list of terms that appear in the Ricci scalar in normal coordinates. Also here we exploit the e.o.m. Eq. [15],
\[
G^{\mu \nu} (\partial_\mu \sigma)(\partial_\nu \sigma) = (\partial_\mu \theta^{\mu \alpha})(\partial_\nu \theta^{\nu \beta}) g_{\alpha \beta} \\
G^{\mu \nu} (\partial_\mu \partial_\nu \sigma) = \frac{1}{2} G^{\mu \nu} (\theta^{\rho \sigma} \partial_\rho \partial_\sigma g_{\mu \nu}) + \frac{1}{2} G^{\mu \nu} (\partial_\mu \theta^{\rho \sigma}) (\partial_\nu \theta^{\rho \sigma}) \\
\left. + \frac{1}{2} G^{\mu \nu} (g^{\rho \sigma} \partial_\rho \partial_\sigma g_{\mu \nu}) \right. \tag{162}
\]
\[
(\partial_\mu G^{\mu \nu})(\partial_\nu \sigma) \equiv e^{\rho \sigma} (\partial_\mu \theta^{\rho \alpha})(\partial_\nu \theta^{\nu \beta}) g_{\alpha \beta} \\
(G^{\rho \sigma} \partial_\mu G_{\rho \sigma}) = -4 \partial_\mu \sigma \\
G^{\mu \nu} (\partial_\mu \partial_\nu g_{\rho \sigma}) = -4 \partial_\mu \sigma \tag{163}
\]
\[ G^{\mu \nu}(\partial_\mu G^{\rho \sigma})(\partial_\nu G_{\rho \sigma}) = - (\partial_\mu \theta^{\mu \alpha})(\partial_\nu \theta^{\nu \beta})g_{\alpha \beta} - 2G^{\mu \nu}(\partial_\mu \theta^{\mu \alpha})(\partial_\nu \theta^{-1}_{\alpha}) - G^{\mu \nu}G^{\rho \sigma}(\partial_\mu \theta^{-1}_{\rho \alpha})(\partial_\sigma \theta^{-1}_{\nu \beta}) \]
\[ \partial_\mu \partial_\nu G^{\mu \nu} \big|_{\text{com}} = \theta^{\mu \alpha}(\partial_\mu \partial_\nu \theta^{\nu \beta})g_{\alpha \beta} + (\partial_\mu \theta^{\mu \alpha})(\partial_\nu \theta^{\nu \beta})g_{\alpha \beta} \]
\[ = \frac{1}{2}G^{\mu \nu}(\theta^{\mu \alpha}(\partial_\nu \theta^{\nu \beta}) + (\partial_\mu \theta^{\mu \alpha})(\partial_\nu \theta^{\nu \beta}) + (\partial_\mu \theta^{\mu \alpha})(\partial_\nu \theta^{\nu \beta})g_{\alpha \beta} \]

Using these we get our final result
\[ R[\tilde{G}] = e^{-\sigma} \left\{ \frac{1}{2} (\partial_\mu \theta^{\mu \alpha})(\partial_\nu \theta^{\nu \beta}) \eta_{\alpha \beta} + \frac{1}{2} (\partial_\mu \theta^{\mu \alpha})(\partial_\nu \theta^{\nu \beta}) \eta_{\alpha \beta} \right. \]
\[ + \frac{1}{2} G^{\mu \nu} G^{\rho \sigma}(\partial_\mu \theta^{-1}_{\rho \alpha})(\partial_\nu \theta^{-1}_{\sigma \beta}) \eta_{\alpha \beta} - \frac{1}{2} G^{\mu \nu} G^{\rho \sigma}(\partial_\mu \theta^{-1}_{\rho \alpha})(\partial_\nu \theta^{-1}_{\sigma \beta}) \eta_{\alpha \beta} \]
\[ - \frac{1}{2} G^{\mu \nu}(g^{\rho \sigma}(\partial_\mu \partial_\nu g_{\rho \sigma})) \right\}. \]

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