CONGRUENCES FOR DIMENSIONS OF SPACES OF SIEGEL CUSP FORMS AND 4-CORE PARTITIONS

CHIRANJIT RAY, MANAMI ROY, AND SHAOYUN YI

Abstract. Using the relationship between Siegel cusp forms of degree 2 and cuspidal automorphic representations of GSp(4, \(\mathbb{A}_\mathbb{Q}\)), we derive some congruences involving dimensions of spaces of Siegel cusp forms of degree 2 and the class number of \(\mathbb{Q}(\sqrt{-p})\). We also obtain some congruences between the 4-core partition function \(c_4(n)\) and dimensions of spaces of Siegel cusp forms of degree 2.

1. Introduction

There is a well known connection between Siegel modular forms of degree 2 and automorphic representations of the adelic group GSp(4, \(\mathbb{A}_\mathbb{Q}\)); for more details see [2, 12]. Let \(S_k(\Gamma_p)\) be the space of Siegel cusp forms of degree 2, weight \(k\), and level \(p\) with respect to a congruence subgroup \(\Gamma_p\) of \(\text{Sp}(4, \mathbb{Q})\). Here we consider the following congruence subgroups: the full modular group \(\text{Sp}(4, \mathbb{Z})\); the paramodular group \(K(N)\) of level \(N\), the Klingen congruence subgroup \(\Gamma_0'(N)\) of level \(N\), and the Siegel congruence subgroup \(\Gamma_0(N)\) of level \(N\) defined as follows, respectively.

\[
K(N) = \left[ \begin{array}{cccc}
\mathbb{Z} & \mathbb{N} & \mathbb{Z} & \mathbb{Z} \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z}
\end{array} \right] \cap \text{Sp}(4, \mathbb{Q}),
\]

\[
\Gamma_0'(N) = \left[ \begin{array}{cccc}
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z}
\end{array} \right] \cap \text{Sp}(4, \mathbb{Z}),
\]

\[
\Gamma_0(N) = \left[ \begin{array}{cccc}
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\
\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z}
\end{array} \right] \cap \text{Sp}(4, \mathbb{Z}).
\]

Definition 1.1. Let \(k \geq 3\) be a positive integer, and let \(p\) be a prime. Let \(S_k(p, \Omega)\) be the set of cuspidal automorphic representations \(\pi \cong \otimes_{v \leq \infty} \pi_v\) of GSp(4, \(\mathbb{A}_\mathbb{Q}\)) with trivial central character satisfying the following properties:

(i) \(\pi_\infty\) is the lowest weight module with minimal \(K\)-type \((k, k)\); it is a holomorphic discrete series representation.

(ii) \(\pi_v\) is unramified for each \(v \neq p, \infty\).

(iii) \(\pi_p\) is an Iwahori-spherical representation of GSp(4, \(\mathbb{Q}_p\)) of type \(\Omega\).

We denote \(s_k(p, \Omega) := \#S_k(p, \Omega)\).

Here the representation type \(\Omega\) is one of the types listed in [11, Table 3]: I, IIa, IIb, IIIa, . . . , VId. The dimensions of spaces \(S_k(\Gamma_p)\) of Siegel cusp forms of degree 2 and level \(p\) with respect to the congruence subgroups in (1) are connected to the numbers \(s_k(p, \Omega)\); see [10]. We discuss this relationship briefly in Section 2. In this article, we obtain some congruences
between dimensions of spaces \( S_k(\Gamma_p) \) of Siegel cusp forms and class numbers of imaginary quadratic fields \( \mathbb{Q}(\sqrt{-p}) \) utilizing the explicit formulas between dimensions of \( S_k(\Gamma_p) \) and the quantities \( s_k(p, \Omega) \).

The connection between class numbers of imaginary quadratic fields and dimensions of spaces of cusp forms may seem surprising, but it has been studied before. In [14], Wakatsuki proved some congruences between class number of imaginary quadratic fields and dimensions of spaces of elliptic cusp forms. The congruence for elliptic cusp forms case follows from Yamauchi’s formula [15] for traces of Atkin–Lehner involutions involving the class number of \( \mathbb{Q}(\sqrt{-p}) \). In the same paper, Wakatsuki also obtained some congruences between class number of imaginary quadratic fields \( \mathbb{Q}(\sqrt{-p}) \) and dimensions of spaces of vector-valued Siegel cusp forms. Here, we prove some congruences for scalar-valued Siegel cusp forms in Section 3; to be more precise, see Theorem 3.1 and Theorem 3.2.

The class number of imaginary quadratic fields appears in the study of certain partition functions, like \( t \)-core partitions. For positive integers \( t \), we let \( c_t(n) \) denote the number of \( t \)-core partitions of \( n \). The arithmetic properties of \( c_t(n) \) has been of interest in combinatorial number theory and representation theory; for example see [4, 5]. The 4-core partitions arise naturally in the modular representation theory of finite general linear groups. In [9], Ono and Sze studied 4-cores partitions using Gauss’ theory of class numbers. In Section 4, we discuss some new congruences on 4-core partitions. By the formula that connects class numbers of imaginary quadratic fields and 4-core partitions, we obtain some congruences between 4-core partitions and dimensions of spaces of Siegel cusp forms modulo 4 in Corollary 4.1. As a consequence, we prove that for positive integer \( n \) such that \( 8n + 5 \) is a prime number and for any positive integer \( k \),

\[
(2) \quad c_4(n) \equiv \dim \mathbb{C} S_{4k}^{\text{new}}(\Gamma_0^{(1)}(8n + 5)) \quad (\text{mod } 2).
\]

Here \( S_{4k}^{\text{new}}(\Gamma_0^{(1)}(8n + 5)) \) denotes the space of elliptic newforms of weight \( k \) and level \( 8n + 5 \) with respect to the congruence subgroup \( \Gamma_0^{(1)}(N) = \left[ \begin{smallmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{smallmatrix} \right] \cap \text{SL}(2, \mathbb{Z}) \). One can readily obtain from (2) that \( c_4(n) \) is odd when \( 8n + 5 \) is a prime number; this result is also proven in [9, Remark 1] using Gauss’s genus theory.

2. Preliminaries

In this section we review some basic results about modular forms, Siegel modular forms and the relationship of Siegel modular forms with automorphic representations of \( \text{GSp}(4, \mathbb{A}_\mathbb{Q}) \). The algebraic group \( \text{GSp}(4) \) is defined by

\[
(3) \quad \text{GSp}(4) := \{ g \in \text{GL}(4): ^t g J g = \lambda(g) J, \; \lambda(g) \in \text{GL}(1) \}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.
\]

The function \( \lambda \) is called the multiplier homomorphism. The kernel of this function is the symplectic group \( \text{Sp}(4) \). Let \( Z \) be the center of \( \text{GSp}(4) \) and \( \text{PGSp}(4) = \text{GSp}(4)/Z \).

Let \( \mathcal{H}_2 \) be the Siegel upper half space of degree 2, i.e., \( \mathcal{H}_2 \) consists of all symmetric complex \( 2 \times 2 \) matrices whose imaginary part is positive definite.

**Definition 2.1.** A Siegel modular form \( f : \mathcal{H}_2 \rightarrow \mathbb{C} \) of degree 2 and weight \( k \) with respect to a congruence subgroup \( \Gamma_N \) of \( \text{Sp}(4, \mathbb{Q}) \) is a holomorphic function satisfying the following transformation property

\[
(f|_k g)(Z) = \det(CZ + D)^{-k} f((AZ + B)(CZ + D)^{-1}) = f(Z) \quad \text{for } g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_N.
\]
Let $\Gamma_N$ be one of the congruence subgroups given in (1). We call a Siegel modular form $f$ a cusp form if
\[
\lim_{\lambda \to \infty} (f|_k g)([1^\lambda \tau]) = 0 \quad \text{for all } g \in \text{Sp}(4, \mathbb{Q}), \tau \in \mathcal{H},
\]
where $\mathcal{H}$ is the complex upper half plane. Now, let $f \in S_k(\Gamma_p)$ be an eigenform. We say that $f$ gives rise to an irreducible automorphic representation if its adelization generates an irreducible cuspidal automorphic representation $\pi \cong \otimes \pi_v$ of $\text{GSp}(4, \mathbb{A}_\mathbb{Q})$ (in the sense of [12, Section 3]). The automorphic representation $\pi$ associated to any such $f$ has trivial central character and hence may be viewed as an automorphic representation of $\text{PGSp}(4, \mathbb{A}_\mathbb{Q})$. Moreover, it follows from [10, Section 2.1] that such $\pi \cong \otimes \pi_v$ is an element of $S_k(p, \Omega)$, where $\pi_p$ is an Iwahori-spherical representation of type $\Omega$. In fact, every eigenform $f \in S_k(\Gamma_p)$ arises from a vector in $\pi_p^{C_p}$. Here $C_p$ is one of the compact open subgroups in $\text{GSp}(4, \mathbb{Q}_p)$ that corresponds to the congruence subgroup $\Gamma_p$ of $\text{Sp}(4, \mathbb{Q})$ given in (1). So, we have the following relationship between dimension of $S_k(\Gamma_p)$ and the quantities $s_k(p, \Omega)$

\[
\dim_{\mathbb{C}} S_k(\Gamma_p) = \sum_{\Omega} \sum_{\pi \in S_k(p, \Omega)} \dim \pi_p^{C_p} = \sum_{\Omega} s_k(p, \Omega) d_{C_p, \Omega}.
\]

Here $d_{C_p, \Omega}$ is the common dimension for all Iwahori-spherical representations of type $\Omega$. These quantities are given explicitly in [11, Table 3]. For the purpose of this article, we will use (4) to get some congruences for $\dim_{\mathbb{C}} S_k(\Gamma_p)$. In order to compute $s_k(p, \Omega)$ explicitly one needs to look at the representations in $S_k(p, \Omega)$ inside different Arthur packets. In particular, the Arthur packets types $(G)$, $(Y)$ and $(P)$ are known as the general type, the Yoshida type and the Saito-Kurokawa type, respectively. See [1, 13] for more details about the Arthur packets for $\text{GSp}(4)$. Let $S_k^{(s)}(p, \Omega)$ be the set of those $\pi \in S_k(p, \Omega)$ that lie in an Arthur packet of type $(s)$ and $s_k^{(s)}(p, \Omega) = \# S_k^{(s)}(p, \Omega)$. Then we get

\[
\dim_{\mathbb{C}} S_k(\Gamma_p) = d_{\mathbb{C}}(G) + d_{\mathbb{C}}(Y) + d_{\mathbb{C}}(P).
\]

A summary of the quantities $s_k^{(s)}(p, \Omega)$ that are considered in (4) and their explicit formulas is given in [10, Section 2.2].

The Saito-Kurokawa type $(P)$ and the Yoshida type $(Y)$ are two kinds of liftings from elliptic cuspidal automorphic representations. As a consequence, both $s_k^{(P)}(p, \Omega)$ and $s_k^{(Y)}(p, \Omega)$ are related to dimensions of spaces of elliptic modular forms. The following lemma is useful for finding the quantities $s_k(p, \Omega)$ for the representations of Saito-Kurokawa type and Yoshida type. This result can be obtained from the work of [15], and it is explicitly given in [7, Theorem 2.2]. Here, $S_k^{\text{new}}(\Gamma_0^{(1)}(p))$ is the new subspace of weight $k$ elliptic cusp forms on the congruence subgroup $\Gamma_0^{(1)}(p)$ of $\text{SL}(2, \mathbb{Z})$. The plus and minus spaces $S_k^{\pm, \text{new}}(\Gamma_0^{(1)}(p))$ are the space spanned by the eigenforms in $S_k^{\text{new}}(\Gamma_0^{(1)}(p))$ which have the sign $\pm 1$ in the functional equation of their $L$-functions.

**Lemma 2.2.** For $p > 3$ and even $k \geq 2$,

\[
\dim_{\mathbb{C}} S_k^{\pm, \text{new}}(\Gamma_0^{(1)}(p)) = \left\lfloor \frac{1}{2} \right\rfloor \dim_{\mathbb{C}} S_k^{\text{new}}(\Gamma_0^{(1)}(p)) \pm \frac{1}{2} \left( \frac{1}{2} h(\Delta_p) b - \delta_{k, 2} \right),
\]
where \( h(\Delta_p) \) is the class number of \( \mathbb{Q}(\sqrt{-p}) \) and

\[
\begin{align*}
  (6) \quad & b = \begin{cases}
    1 & \text{if } p \equiv 1 \pmod{4}, \\
    2 & \text{if } p \equiv 7 \pmod{8}, \\
    4 & \text{if } p \equiv 3 \pmod{8}.
  \end{cases} \quad \text{and} \quad \\
  \delta_{k,2} = \begin{cases}
    1 & \text{if } k = 2, \\
    0 & \text{if } k \neq 2.
  \end{cases}
\end{align*}
\]

For \( k > 2 \),

\[
\begin{align*}
  & \dim_{\mathbb{C}} S_k^{\pm, \text{new}}(\Gamma_0^{(1)}(2)) = \frac{1}{2} \dim_{\mathbb{C}} S_k^{\text{new}}(\Gamma_0^{(1)}(2)) \pm \begin{cases}
    \frac{1}{2} & \text{if } k \equiv 0, 2 \pmod{8}, \\
    0 & \text{else.}
  \end{cases} \\
  & \dim_{\mathbb{C}} S_k^{\pm, \text{new}}(\Gamma_0^{(1)}(3)) = \frac{1}{2} \dim_{\mathbb{C}} S_k^{\text{new}}(\Gamma_0^{(1)}(3)) \pm \begin{cases}
    \frac{1}{2} & \text{if } k \equiv 0, 2, 6, 8 \pmod{12}, \\
    0 & \text{else.}
  \end{cases}
\end{align*}
\]

We note that, when \( \Omega \) is one of the types \( \text{IIb}, \text{Vb}, \text{VIc} \), there is a global representation in \( S_k(p, \Omega) \). When \( \Omega \) is of type \( \text{VIb} \), there are global representations in \( S_k^{(*)}(p, \Omega) \) for all three types \( (G), (Y), \) and \( (P) \).

From now on we assume \( k \geq 3 \). Then, by Lemma 2.2 and [10, (3.10)], for \( p \geq 5 \) we get the following identities

\[
\begin{align*}
  s_k(p, \text{Vb}) &= \begin{cases}
    0 & \text{if } k \text{ is odd}, \\
    \frac{1}{2} \dim_{\mathbb{C}} S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(p)) - \frac{1}{4} h(\Delta_p)b & \text{if } k \text{ is even.}
  \end{cases} \\
  s_k^{(P)}(p, \text{VIb}) &= \begin{cases}
    0 & \text{if } k \text{ is odd}, \\
    \frac{1}{2} \dim_{\mathbb{C}} S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(p)) + \frac{1}{4} h(\Delta_p)b & \text{if } k \text{ is even.}
  \end{cases} \\
  s_k(p, \text{VIc}) &= \begin{cases}
    0 & \text{if } k \text{ is odd}, \\
    \frac{1}{2} \dim_{\mathbb{C}} S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(p)) - \frac{1}{4} h(\Delta_p)b & \text{if } k \text{ is even.}
  \end{cases}
\end{align*}
\]

Similarly, by Lemma 2.2 and [10, (3.11)], we have \( s_k^{(Y)}(p, \text{VIb}) = 0 \) for \( p = 2, 3 \), and for \( p \geq 5 \) we have

\[
(8) \quad s_k^{(Y)}(p, \text{VIb}) = \frac{1}{2} \dim_{\mathbb{C}} S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(p)) \dim_{\mathbb{C}} S_2^{\text{new}}(\Gamma_0^{(1)}(p)) + \frac{(-1)^k}{8} h(\Delta_p)^2 b^2 - \frac{(-1)^k}{4} h(\Delta_p)b.
\]

3. Congruences for dimensions of spaces of Siegel cusp forms

In this section, we derive some congruences modulo 16 and modulo 4 involving dimensions of spaces of Siegel cusp forms of degree 2, the class number of \( \mathbb{Q}(\sqrt{-p}) \), and dimensions of spaces of elliptic modular newforms.

Theorem 3.1. Let \( h(\Delta_p) \) be the class number of \( \mathbb{Q}(\sqrt{-p}) \). For \( k \geq 3 \) and \( p \geq 5 \), we have the following congruence relations.

(i) For \( p \equiv 1 \pmod{4} \)

\[
(-1)^{k-1} h(\Delta_p)^2 - \begin{cases}
    4 h(\Delta_p) & \text{if } k \text{ is odd,} \\
    0 & \text{if } k \text{ is even.}
  \end{cases}
\]

\[
\equiv 4 \left( \dim_{\mathbb{C}} S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(p)) \dim_{\mathbb{C}} S_2^{\text{new}}(\Gamma_0^{(1)}(p)) - (-1)^{k-1} \dim_{\mathbb{C}} S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(p)) \right) \\
+ 8 \left( \dim_{\mathbb{C}} S_k(K(p)) - \dim_{\mathbb{C}} S_k(\Gamma_0(p)) \right) \pmod{16}.
\]
(ii) For $p \equiv 7 \pmod{8}$

\[
(-1)^{k-1} h(\Delta_p)^2 - \begin{cases} 
2h(\Delta_p) & \text{if } k \text{ is odd} \\
0 & \text{if } k \text{ is even}
\end{cases} 
\]

\[
\equiv \dim_{C} S_{2k-2}(\Gamma_0^{(1)}(p)) \dim_{C} S_{2k-2}(\Gamma_0^{(1)}(p)) - (-1)^{k-1} \dim_{C} S_{2k-2}(\Gamma_0^{(1)}(p)) 
+ 2 \left( \dim_{C} S_k(K(p)) - \dim_{C} S_k(\Gamma_0(p)) \right) \pmod{4}.
\]

(iii) For $p \equiv 3 \pmod{8}$

\[
2 \left( \dim_{C} S_k(K(p)) - \dim_{C} S_k(\Gamma_0(p)) \right) 
\equiv (-1)^{k-1} \dim_{C} S_{2k-2}(\Gamma_0^{(1)}(p)) - \dim_{C} S_{2k-2}(\Gamma_0^{(1)}(p)) \dim_{C} S_{2k-2}(\Gamma_0^{(1)}(p)) \pmod{4}.
\]

**Proof.** From [10, (2.5) and (2.6)], we have

\[
s_k^{(G)}(p, \text{VIA}/b) = s_k^{(G)}(p, \text{VIA}) = s_k^{(G)}(p, \text{VIb}),
\]

\[
s_k(p, \text{IIIA} + \text{VIA}/b) := s_k^{(G)}(p, \text{IIIA}) + s_k^{(G)}(p, \text{VIA}/b).
\]

Using (9) and replacing $s_k(p, \text{VIIb})$ by $s_k^{(G)}(p, \text{VIIb}) + s_k^{(Y)}(p, \text{VIIb}) + s_k^{(P)}(p, \text{VIIb})$ in [10, (3.12)], we obtain the following identity

\[
s_k(p, \text{IIIA} + \text{VIA}/b) = \frac{1}{2} \dim_{C} S_k(\Gamma_0(p)) - \frac{1}{2} \dim_{C} S_k(K(p)) - s_k(p, \text{I}) - s_k(p, \text{IIb})
- \frac{1}{2} s_k(p, \text{VIb}) - \frac{1}{2} s_k(p, \text{VIIb}) + \frac{1}{2} s_k(p, \text{VIIc}).
\]

Using (7) and (8) we obtain

\[
2(s_k(p, \text{IIIA} + \text{VIA}/b) + s_k(p, \text{I}) + s_k(p, \text{IIb})) + \dim_{C} S_k(K(p)) - \dim_{C} S_k(\Gamma_0(p)) 
= (-1)^{k-1} \frac{1}{2} \dim_{C} S_{2k-2}(\Gamma_0^{(1)}(p)) - \frac{1}{2} \dim_{C} S_{2k-2}(\Gamma_0^{(1)}(p)) \dim_{C} S_{2k-2}(\Gamma_0^{(1)}(p)) 
+ (-1)^{k-1} \frac{1}{8} h(\Delta_p)^2 b^2 - \begin{cases} 
\frac{1}{7} h(\Delta_p) b & \text{if } k \text{ is odd,} \\
0 & \text{if } k \text{ is even.}
\end{cases}
\]

(i) If $p \equiv 1 \pmod{4}$, we have $b = 1$. Then we get

\[
16(s_k(p, \text{IIIA} + \text{VIA}/b) + s_k(p, \text{I}) + s_k(p, \text{IIb})) + 8 \dim_{C} S_k(K(p)) - 8 \dim_{C} S_k(\Gamma_0(p)) 
= 4(-1)^{k-1} \dim_{C} S_{2k-2}(\Gamma_0^{(1)}(p)) - 4 \dim_{C} S_{2k-2}(\Gamma_0^{(1)}(p)) \dim_{C} S_{2k-2}(\Gamma_0^{(1)}(p)) 
+ (-1)^{k-1} h(\Delta_p)^2 - \begin{cases} 
4h(\Delta_p) & \text{if } k \text{ is odd,} \\
0 & \text{if } k \text{ is even.}
\end{cases}
\]

(ii) If $p \equiv 7 \pmod{8}$, we have $b = 2$. Then we get

\[
4(s_k(p, \text{IIIA} + \text{VIA}/b) + s_k(p, \text{I}) + s_k(p, \text{IIb})) + 2 \dim_{C} S_k(K(p)) - 2 \dim_{C} S_k(\Gamma_0(p)) 
= (-1)^{k-1} \dim_{C} S_{2k-2}(\Gamma_0^{(1)}(p)) - \dim_{C} S_{2k-2}(\Gamma_0^{(1)}(p)) \dim_{C} S_{2k-2}(\Gamma_0^{(1)}(p)) 
+ (-1)^{k-1} h(\Delta_p)^2 - \begin{cases} 
2h(\Delta_p) & \text{if } k \text{ is odd,} \\
0 & \text{if } k \text{ is even.}
\end{cases}
\]
(iii) If $p \equiv 3 \pmod{8}$, we have $b = 4$. Then we get
\[ 4(s_k(p, \Pi a + V I a/b) + s_k(p, I) + s_k(p, II b)) + 2 \dim S_k(K(p)) - 2 \dim S_k(\Gamma_0(p)) \]
\[ = (-1)^{k-1} \dim S_{2k-2}^\new(\Gamma_0^{(1)}(p)) - \dim S_{2k-2}^\new(\Gamma_0^{(1)}(p)) \dim S_2^\new(\Gamma_0^{(1)}(p)) \]
\[ + (-1)^{k-1} \cdot 4h(\Delta_p)^2 = \begin{cases} 
4h(\Delta_p) & \text{if } k \text{ is odd}, \\
0 & \text{if } k \text{ is even}.
\end{cases} \]

The desired congruences now follow from above discussions. \hfill \square

**Theorem 3.2.** Let $h(\Delta_p)$ be the class number of $\mathbb{Q}(\sqrt{-p})$. For $k \geq 3$ and $p \geq 5$, we have the following congruence relations.

(i) For $p \equiv 1 \pmod{4}$

\[ (-1)^k h(\Delta_p)^2 \equiv -4 \left( \dim S_{2k-2}^\new(\Gamma_0^{(1)}(p)) \dim S_2^\new(\Gamma_0^{(1)}(p)) + \dim S_{2k-2}^\new(\Gamma_0^{(1)}(p)) \right) \]
\[ + 8 \left( \dim S_k(K(p)) + \dim S_k(\Gamma_0(p)) \right) \pmod{16}. \]

(ii) For $p \equiv 7 \pmod{8}$

\[ (-1)^k h(\Delta_p)^2 \equiv - \dim S_{2k-2}^\new(\Gamma_0^{(1)}(p)) \dim S_2^\new(\Gamma_0^{(1)}(p)) - \dim S_{2k-2}^\new(\Gamma_0^{(1)}(p)) \]
\[ + 2 \left( \dim S_k(K(p)) + \dim S_k(\Gamma_0(p)) \right) \pmod{4}. \]

(iii) For $p \equiv 3 \pmod{8}$

\[ 2 \left( \dim S_k(K(p)) + \dim S_k(\Gamma_0(p)) \right) \]
\[ \equiv \dim S_{2k-2}^\new(\Gamma_0^{(1)}(p)) + \dim S_{2k-2}^\new(\Gamma_0^{(1)}(p)) \dim S_2^\new(\Gamma_0^{(1)}(p)) \pmod{4}. \]

**Proof.** We have the following formula for $s_k(p, Va)$ from the proof of [10, Theorems 3.5–3.8]

\[ s_k(p, Va) = \dim S_k(\Gamma_0'(p)) - \frac{1}{2} \dim S_k(\Gamma_0(p)) - \frac{3}{2} \dim S_k(K(p)) + s_k(p, I) \]
\[ + s_k(p, II b) + s_k(p, V b) + \frac{1}{2}s_k(p, V I b) + \frac{1}{2}s_k(p, V I c). \]

Using (7) and (8) we obtain

\[ 2(s_k(p, V a) - s_k(p, I) - s_k(p, II b) - s_k(p, V b)) \]
\[ - 2 \dim S_k(\Gamma_0'(p)) + \dim S_k(\Gamma_0(p)) + 3 \dim S_k(K(p)) \]
\[ = \frac{1}{2} \dim S_{2k-2}^\new(\Gamma_0^{(1)}(p)) + \frac{1}{2} \dim S_{2k-2}^\new(\Gamma_0^{(1)}(p)) \dim S_2^\new(\Gamma_0^{(1)}(p)) + (-1)^k \frac{1}{8} h(\Delta_p)^2 b^2. \]

Then, by the same arguments as in Theorem 3.1, we get the desired congruences. \hfill \square

Similarly, if we use the following identity from the proof of [10, Theorems 3.5–3.8]

\[ s_k(p, \Pi a) = \dim S_k(K(p)) - 2s_k(p, I) - s_k(p, II b) - s_k(p, V b) - s_k(p, V I c), \]

we obtain a congruence that relates dimension of the space $S_{2k-2}^\new(\Gamma_0^{(1)}(p))$ of elliptic newforms and the class number of $\mathbb{Q}(\sqrt{-p})$. This result also follows from [14, Theorem 3.3].
Theorem 3.3. Let $h(\Delta_p)$ be the class number of $\mathbb{Q}(\sqrt{-p})$. For $k \geq 3$ and $p \geq 5$, we have the following congruences

\[
\dim_C S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(p)) \equiv \frac{1}{2} h(\Delta_p) b = \begin{cases} 
\frac{1}{2} h(\Delta_p) & \text{if } p \equiv 1 \pmod{4}, \\
 0 & \text{if } p \equiv 0 \pmod{4}, \\
 h(\Delta_p) & \text{if } p \equiv 7 \pmod{8}, \\
 0 & \text{if } p \equiv 3 \pmod{8}.
\end{cases}
\]

Next we consider the results for $p = 2, 3$. We have to treat $p = 2, 3$ separately because the formulas in Lemma 2.2 are different for these two primes.

Proposition 3.4. Suppose $k \geq 3$. Then, modulo 4 we get

\[
2 \dim_C S_k(K(2)) - 2 \dim_C S_k(\Gamma_0(2)) \equiv \begin{cases} 
- \dim_C S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(2)) & \text{if } k \equiv 0 \pmod{4}, \\
\dim_C S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(2)) + 1 & \text{if } k \equiv 1 \pmod{4}, \\
- \dim_C S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(2)) - 1 & \text{if } k \equiv 2 \pmod{4}, \\
\dim_C S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(2)) & \text{if } k \equiv 3 \pmod{4},
\end{cases}
\]

and

\[
2 \dim_C S_k(K(3)) - 2 \dim_C S_k(\Gamma_0(3)) \equiv \begin{cases} 
- \dim_C S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(3)) & \text{if } k \equiv 0 \pmod{6}, \\
\dim_C S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(3)) + 1 & \text{if } k \equiv 1, 5 \pmod{6}, \\
- \dim_C S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(3)) - 1 & \text{if } k \equiv 2, 4 \pmod{6}, \\
\dim_C S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(3)) & \text{if } k \equiv 3 \pmod{6}.
\end{cases}
\]

Proof. Since $s_k^{(Y)}(p, VIb) = 0$ for $p = 2, 3$, from (10) we get

\[
2(s_k(p, IIIa + VIa/b) + s_k(p, I) + s_k(p, IIb) + \dim_C S_k(K(p)) - \dim_C S_k(\Gamma_0(p))) = s_k(p, VIc) - s_k^{(P)}(p, VIlb).
\]

(11)

If $k$ is odd, it follows from Lemma 2.2 and [10, (3.10)] that

\[
s_k(2, VIc) = \frac{1}{2} \dim_C S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(2)) + \begin{cases} 
\frac{1}{2} & \text{if } 2k - 2 \equiv 0 \pmod{8}, \\
0 & \text{else.}
\end{cases}
\]

\[
= \frac{1}{2} \dim_C S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(2)) + \begin{cases} 
\frac{1}{2} & \text{if } k \equiv 1 \pmod{4}, \\
0 & \text{if } k \equiv 3 \pmod{4}.
\end{cases}
\]

\[
s_k(3, VIc) = \frac{1}{2} \dim_C S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(3)) + \begin{cases} 
\frac{1}{2} & \text{if } 2k - 2 \equiv 0, 8 \pmod{12}, \\
0 & \text{else.}
\end{cases}
\]

\[
= \frac{1}{2} \dim_C S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(3)) + \begin{cases} 
\frac{1}{2} & \text{if } k \equiv 1, 5 \pmod{6}, \\
0 & \text{if } k \equiv 3 \pmod{6}.
\end{cases}
\]

If $k$ is even, by Lemma 2.2 and [10, (3.10)] we obtain

\[
s_k^{(P)}(2, VIb) = \frac{1}{2} \dim_C S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(2)) + \begin{cases} 
\frac{1}{2} & \text{if } 2k - 2 \equiv 2 \pmod{8}, \\
0 & \text{else.}
\end{cases}
\]
\[ = \frac{1}{2} \dim_{\mathbb{C}} S_{2k-2}^{\text{new}}(\Gamma_0(1)(2)) + \begin{cases} \frac{1}{2} & \text{if } k \equiv 2 \pmod{4}, \\ 0 & \text{if } k \equiv 0 \pmod{4}. \end{cases} \]

\[ s_k^p(3, \text{VI}) = \frac{1}{2} \dim_{\mathbb{C}} S_{2k-2}^{\text{new}}(\Gamma_0(1)(3)) + \begin{cases} \frac{1}{2} & \text{if } 2k - 2 \equiv 2, 6 \pmod{12}, \\ 0 & \text{else}. \end{cases} \]

Then we get the following

\[ 4(s_k(2, \text{IIIa} + \text{VIa}/b) + s_k(2, I) + s_k(2, \text{IIb})) = 2 \dim_{\mathbb{C}} S_k(K(2)) - 2 \dim_{\mathbb{C}} S_k(\Gamma_0(2)) \]

\[ = \begin{cases} - \dim_{\mathbb{C}} S_{2k-2}^{\text{new}}(\Gamma_0(1)(2)) & \text{if } k \equiv 0 \pmod{4}, \\ \dim_{\mathbb{C}} S_{2k-2}^{\text{new}}(\Gamma_0(1)(2)) + 1 & \text{if } k \equiv 1 \pmod{4}, \\ - \dim_{\mathbb{C}} S_{2k-2}^{\text{new}}(\Gamma_0(1)(2)) - 1 & \text{if } k \equiv 2 \pmod{4}, \\ \dim_{\mathbb{C}} S_{2k-2}^{\text{new}}(\Gamma_0(1)(2)) & \text{if } k \equiv 3 \pmod{4}. \end{cases} \]

and

\[ 4(s_k(3, \text{IIIa} + \text{VIa}/b) + s_k(3, I) + s_k(3, \text{IIb})) = 2 \dim_{\mathbb{C}} S_k(K(3)) - 2 \dim_{\mathbb{C}} S_k(\Gamma_0(3)) \]

\[ = \begin{cases} - \dim_{\mathbb{C}} S_{2k-2}^{\text{new}}(\Gamma_0(1)(3)) & \text{if } k \equiv 0 \pmod{6}, \\ \dim_{\mathbb{C}} S_{2k-2}^{\text{new}}(\Gamma_0(1)(3)) + 1 & \text{if } k \equiv 1, 5 \pmod{6}, \\ - \dim_{\mathbb{C}} S_{2k-2}^{\text{new}}(\Gamma_0(1)(3)) - 1 & \text{if } k \equiv 2, 4 \pmod{6}, \\ \dim_{\mathbb{C}} S_{2k-2}^{\text{new}}(\Gamma_0(1)(3)) & \text{if } k \equiv 3 \pmod{6}. \end{cases} \]

Hence the stated congruences follow from above equations. \(\square\)

4. Congruences for 4-core partition functions

A partition of a positive integer \(n\) is any non-increasing sequence of positive integers whose sum is \(n\). If \(\pi = \pi_1 + \pi_2 + \cdots + \pi_m\) be a partition of \(n\), where \(\pi_1 \geq \pi_2 \geq \cdots \geq \pi_m\), then the Ferrers–Young diagram of \(\pi\) is an array of nodes with \(\pi_p\) nodes in the \(p\)-th row.

\[
\begin{array}{c}
\bullet \quad \cdots \quad \bullet \\
\bullet \quad \cdots \\
\vdots \\
\bullet \quad \cdots \\
\end{array}
\]

\(\pi_1\) nodes
\(\pi_2\) nodes
\(\vdots\)
\(\pi_m\) nodes

The \((p, q)\)-hook is the set of nodes directly below and directly to the right of the \((p, q)\)-node, as well as the \((p, q)\)-node. The hook number, \(H(p, q)\), is the total number of nodes on the \((p, q)\)-hook. For a positive integer \(t\), a \(t\)-core partition of \(n\) is a partition of \(n\) in which none of the hook numbers are divisible by \(t\). Suppose \(c_t(n)\) denote the number of \(t\)-core partitions
of $n$. Now we illustrate the Ferrers–Young diagram of the partition $5 + 4 + 2$ of 11 with the corresponding hook numbers shown in the graph.

\[
\begin{array}{ccccccc}
& 7 & 6 & 4 & 3 & 1 \\
5 & 4 & 2 & 1 \\
2 & 1 \\
\end{array}
\]

Therefore it is clear that if $t > 7$ then $5 + 4 + 2$ is a $t$-core partition of 11. The generating function for $c_t(n)$ is

\[
\sum_{n=0}^{\infty} c_t(n)q^n := \prod_{n=1}^{\infty} \frac{(1-q^n)^t}{(1-q^n)}.
\]

In this article we focus on 4-core partitions. Hirschhorn and Sellers [6], proved that

\[
c_4(9n + 2) \equiv 0 \pmod{2},
\]

\[
c_4(9n + 8) \equiv 0 \pmod{2}.
\]

Analyzing the action of the Hecke operators on the space of integer weight cusp forms Boylan [3] proved that

\[
c_4(Bn - 5) \equiv 0 \pmod{2},
\]

where $B$ is product of any five distinct odd primes. Regarding the positivity of 4-core partition, Ono [8] proved that $c_4(n) > 0$ for all $n \geq 0$.

It is well known that the number of 4-core partitions $c_4(n)$ of $n$ is equal to the number of representations of $8n+5$ in the form $x^2+2y^2+2z^2$ with $x, y, z$ odd positive integers. Let $H(D)$ denote the Hurwitz class numbers of binary quadratic forms of discriminant $D < 0$. There is relationship between the Hurwitz class numbers and the class number of imaginary quadratic fields. Let $h(\Delta_n)$ be the class number for imaginary quadratic fields $\mathbb{Q}(\sqrt{-n})$, where $\Delta_n < 0$ is the discriminant of $\mathbb{Q}(\sqrt{-n})$. Here $\Delta_n$ is a fundamental discriminant given by

\[
\Delta_n = \begin{cases} 
-4n & \text{if } n \equiv 1, 2 \pmod{4}, \\
-n & \text{if } n \equiv 3 \pmod{4}.
\end{cases}
\]

Suppose $D = \Delta_n f^2$ where $\Delta_n < 0$ is a fundamental discriminant for a squarefree integer $n$. Then we have

\[
H(D) = \frac{2h(\Delta_n)}{w(\Delta_n)} \sum_{d \mid f} \mu(d) \left( \frac{\Delta_n}{d} \right) \sigma_1(f/d),
\]

where $w(\Delta_n)$ is the number of units in the ring of integers of $\mathbb{Q}(\sqrt{-n})$, $\mu(n)$ is the Möbius function and $\sigma_1(n)$ is the sum of the divisors of $n$. In particular, when $D < 0$ is a fundamental discriminant and $D \neq 3, 4$ then we have $H(D) = h(D)$. There is very interesting connection between 4-core partition functions and the Hurwitz class numbers.

In 1997, Ono and Sze [9] showed that 4-core partitions naturally arise in algebraic number theory. In particular they proved that, if $8n + 5$ is square-free integer, then

\[
(12) \quad c_4(n) = \frac{1}{2} h(-32n - 20).
\]

Note that by Dirichlet’s Theorem on primes in arithmetic progression, there are infinitely many primes of the form $8n + 5$. Next, we prove a congruence for $c_4(n)$ using the results from the previous section.
Corollary 4.1. Suppose $8n + 5$ is a prime number and $k \geq 5$. Then, modulo 4, we have

$$(-1)^{k-1}c_4(n)^2 - \begin{cases} 2c_4(n) & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases} - 2\left( \dim_{\mathbb{C}} S_k(K(8n + 5)) - \dim_{\mathbb{C}} S_k(\Gamma_0(8n + 5)) \right)$$

$$\equiv \left( \dim_{\mathbb{C}} S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(8n + 5)) \right) \dim_{\mathbb{C}} S_{2}^{\text{new}}(\Gamma_0^{(1)}(8n + 5)) - (-1)^{k-1} \dim_{\mathbb{C}} S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(8n + 5)) \right)$$

and

$$(-1)^{k}c_4(n)^2 - 2\left( \dim_{\mathbb{C}} S_k(K(8n + 5)) - \dim_{\mathbb{C}} S_k(\Gamma_0(8n + 5)) \right)$$

$$\equiv - \left( \dim_{\mathbb{C}} S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(8n + 5)) \dim_{\mathbb{C}} S_{2}^{\text{new}}(\Gamma_0^{(1)}(8n + 5)) + \dim_{\mathbb{C}} S_{2k-2}^{\text{new}}(\Gamma_0^{(1)}(8n + 5)) \right).$$

Proof. Let $p$ be a prime of the form $8n + 5$. In particular, $p \equiv 1 \pmod{4}$. It follows from (12) that the class number of $Q(\sqrt{-p})$ is given by $h(\Delta_p) = h(-32n - 20) = 2c_4(n)$. Then the first congruence follows from part (i) of Theorem 3.1 and the second congruence follows from part (i) of Theorem 3.2. □

Finally, by adding two different congruences in Corollary 4.1 when $k$ is odd, we get the following result.

Corollary 4.2. Suppose $8n + 5$ is a prime number and $k$ is a positive integer. Then

$$c_4(n) \equiv \dim_{\mathbb{C}} S_{4k}^{\text{new}}(\Gamma_0^{(1)}(8n + 5)) \pmod{2}.$$

Acknowledgements. We would like to thank Ken Ono for helpful advice. The first author has carried out this work at Harish-Chandra Research Institute, affiliated with Homi Bhabha National Institute (Department of Atomic Energy, India), as a Postdoctoral Fellow. We would also like to thank the referee for the detailed comments and suggestions.

References

[1] James Arthur. Automorphic representations of $GSp(4)$. In Contributions to automorphic forms, geometry, and number theory, pages 65–81. Johns Hopkins Univ. Press, Baltimore, MD, 2004.

[2] Mahdi Asgari and Ralf Schmidt. Siegel modular forms and representations. *Manuscripta Math.*, 104(2):173–200, 2001.

[3] Matthew Boylan. Congruences for $2^t$-core partition functions. *J. Number Theory*, 92(1):131–138, 2002.

[4] Frank Garvan, Dongsu Kim, and Dennis Stanton. Cranks and $t$-cores. *Invent. Math.*, 101(1):1–17, 1990.

[5] Michael D. Hirschhorn and James A. Sellers. Some amazing facts about 4-cores. *J. Number Theory*, 60(1):51–69, 1996.

[6] Michael D. Hirschhorn and James A. Sellers. Two congruences involving 4-cores. *Electron. J. Combin.*, 3(2):Research Paper 10, approx. 8, 1996. The Foata Festschrift.

[7] Kimball Martin. Refined dimensions of cusp forms, and equidistribution and bias of signs. *J. Number Theory*, 188:1–17, 2018.

[8] Ken Ono. A note on the number of $t$-core partitions. *Rocky Mountain J. Math.*, 25(3):1165–1169, 1995.

[9] Ken Ono and Lawrence Sze. 4-core partitions and class numbers. *Acta Arith.*, 80(3):249–272, 1997.
[10] Manami Roy, Ralf Schmidt, and Shaoyun Yi. On counting cuspidal automorphic representations for $\text{GSp}(4)$. *Forum Math.*, 33(3):821–843, 2021.

[11] Ralf Schmidt. Iwahori-spherical representations of $\text{GSp}(4)$ and Siegel modular forms of degree 2 with square-free level. *J. Math. Soc. Japan*, 57(1):259–293, 2005.

[12] Ralf Schmidt. Archimedean aspects of Siegel modular forms of degree 2. *Rocky Mountain J. Math.*, 47(7):2381–2422, 2017.

[13] Ralf Schmidt. Packet structure and paramodular forms. *Trans. Amer. Math. Soc.*, 370(5):3085–3112, 2018.

[14] Satoshi Wakatsuki. Congruences modulo 2 for dimensions of spaces of cusp forms. *J. Number Theory*, 140:169–180, 2014.

[15] Masatoshi Yamauchi. On the traces of Hecke operators for a normalizer of $\Gamma_0(N)$. *J. Math. Kyoto Univ.*, 13:403–411, 1973.

Department of Mathematics, Harish-Chandra Research Institute, U.P. 211019, INDIA.  
*Email address:* chiranjitray.m@gmail.com

Department of Mathematics, Fordham University, Bronx, New York 10458, USA.  
*Email address:* manami.roy.90@gmail.com

Department of Mathematics, University of South Carolina, Columbia, SC 29208, USA.  
*Email address:* yishaoyun926@gmail.com