Moral Hazard Under Ambiguity

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Abstract In this paper, we extend the classical Holmström and Milgrom contracting problem, by adding uncertainty on the volatility of the output for both the Agent and the Principal. We study more precisely the impact of the “Nature” playing against the Agent and the Principal, by choosing the worst possible volatility of the output. We solve the first-best and the second-best problems in this framework, and we show that optimal contracts are in a class of contracts linear with respect to the output and its quadratic variation. We also present a general modus operandi to apply our method.

Keywords Risk sharing · Moral hazard · Principal–Agent · Second-order BSDEs · Volatility uncertainty · Hamilton–Jacobi–Bellman–Isaacs PDEs

Mathematics Subject Classification 91B40 · 93E20 · C61 · C73 · D82 · J33 · M52

1 Introduction

By and large, it has now become common knowledge among the economists that almost everything in economics was to a certain degree a matter of incentives: incentives to work hard, to produce, to study, to invest, to consume reasonably, etc. At the
heart of the importance of incentives lies the fact that to quote Salanié [1] “asymmetries of information are pervasive in economic relationships, that is to say, customers know more about their tastes than firms, firms know more about their costs than the government, and all Agents take actions that are at least partly unobservable”. Starting from the 1970s, contract theory evolved from this acknowledgment. In the corresponding typical situation, a Principal (who takes the initiative of the contract) is, potentially, imperfectly informed about the actions of an Agent (who accepts or rejects the contract). The goal is to design a contract that maximises the utility of the Principal, while that of the Agent is held to a given level. Of course, the form of the optimal contracts typically depends on whether these actions are observable/contractible or not, and on whether there are characteristics of the Agent that are unknown to the Principal. There are three main types of such problems: the first-best case, or risk sharing, in which both parties have the same information; the second-best case, or moral hazard, in which the action of the Agent is hidden or not contractible; the third-best case, or adverse selection, in which the type of the Agent is hidden. We will not study this last problem and refer the interested reader to, among others, [2–4]. These problems are fundamentally linked to designing optimal incentives and are therefore present in a very large number of situations. Beyond the obvious application to the optimal remuneration of an employee, one can, for instance, think on how regulators with imperfect information and limited policy instruments can motivate banks to operate entirely in the social interest, on how a company can optimally compensate its executives, on how banks achieve optimal securitisation of mortgage loans or on how investors should pay their portfolio managers.

Concerning the so-called moral hazard problem, the first paper on continuous-time Principal–Agent problems is the seminal paper by Holmström and Milgrom [5]. They consider a Principal and an Agent with exponential utility functions and find that the optimal contract is linear. Their work was generalised notably by Schättler and Sung [6], Sung [7], using a dynamic programming and martingales approach, which is classical in stochastic control theory. The papers by Williams [8] and Cvitanić et al. [9] use the stochastic maximum principle and FBSDEs to characterise the optimal compensation for more general utility functions. A more recent seminal paper in moral hazard setting is Sannikov [10], who finds a tractable model for solving the problem with a random time of retiring the Agent and with continuous payments, rather than a lump-sum payment at the terminal time. Since then, a growing literature extending the above models has emerged, be it to include output processes with jumps [11,12], imperfect information and learning [13–16] or asset pricing [17].

Compared to the first-best problem, the moral hazard case corresponds to a Stackelberg-like game between the Principal and the Agent, in the sense that the Principal will start by trying to compute the best-reaction function of the Agent to a given contract (that is to say the optimal action chosen by the Agent given the contract), and use this action to maximise his utility over all admissible contracts. This approach does not always work, because it may be hard to solve the Agent’s stochastic control problem given an arbitrary payoff, possibly non-Markovian, and it may also be hard for the Principal to maximise over all such contracts. Furthermore, the Agent’s optimal control, if it even exists, depends on the given contract in a highly nonlinear manner, rendering the Principal’s optimisation problem even harder and obscure. For
these reasons, and as mentioned above, in its most general form the problem was also approached in the literature by the stochastic version of the Pontryagin maximal principle. Regardless of the approach used to solve the problem though, none had been able to handle the case when the Agent also controls the diffusion coefficient of the output, and not just the drift. Building upon this gap in the literature, Cvitanić et al. [18,19] have very recently developed a general approach of the problem through dynamic programming and so-called (2)BSDEs, showing that under mild conditions, the problem of the Principal could always be rewritten in an equivalent way as a standard stochastic control problem involving two state variables, namely the output itself, but also the continuation utility (or value function) of the Agent, a property which was pointed out by Sannikov in the specific setting of [10], and which was already well known by the economists, even in discrete-time models; see, for instance, Spear and Srinavastava [20]. An important finding of [18], in the context of a delegated portfolio management problem which generalises Holmström and Milgrom problem [5] to a context where the Agent can control the volatility of the (multidimensional) output process, is that in both the first-best and moral hazard problems, the optimal contracts become path-dependent, as they crucially use the quadratic variation of the output process.

Our goal in this paper is to study yet another generalisation of the Holmström and Milgrom problem [5], to a setting where the Agent only controls the drift of the output, but where the twist is that both the Principal and the Agent have some uncertainty about the future evolution of the volatility of the output. They only believe that it lies in some given random interval of $[0, +\infty]$, which is allowed to change as time passes, given the observation of the past path of the output and its realised volatility. The typical example would be of a Principal hiring an Agent to manage his firm on his behalf, with both of them trying to understand what risks they are actually taking by assuming a specific model for the volatility of the firm value. Traditional approaches, going back to [5], generally assume that this volatility is constant, or follows specific dynamics, making the whole contracting analysis subject to model uncertainty and ambiguity. Our take here is to assume that this volatility remains unknown throughout the contacting period and that both contracting parties adopt worst-case behaviours. This means that both the Principal and the Agent behave as if a third entity, and “Nature” (for lack of a better word) was playing a zero-sum game with each of them, making sure that their most pessimistic priors will be realised. The approach we follow here is directly inspired by the economics literature on Knightian uncertainty, pioneered notably by Gilboa and Schmeidler [21], and the problem is also naturally linked to the so-called situation of volatility ambiguity, which has received a lot of attention recently, both in the mathematical finance community, since the seminal paper by Denis and Martini [22], and in the economics literature; see, for instance, [23]. Our goal is therefore to first understand the complications arising from the interplay between moral hazard and ambiguity, and to give a general method to treat these problems in continuous time, with ambiguity-averse utilities. We emphasise as well that since we put no restrictions on the beliefs that the Agent and the Principal have with respect to the likely volatility scenario, in the sense that their volatility intervals may or may not have non-empty intersections, our treatment of the problem includes all possible cases of heterogeneous beliefs between the Principal and the Agent.
Before moving on to the main contributions of our paper, let us say a word about drift uncertainty, which we do not consider here. As pointed out by one of the referees, it is indeed generally much harder to estimate the drift of a diffusion than its volatility, making a situation where the drift is perfectly known, while the volatility is uncertain, slightly unrealistic. However, as will be made clear in Sect. 5, adding drift uncertainty into our model does not create any additional difficulties from the mathematical point of view, but would complicate even further our presentation, which is already quite technical. It is our point of view that our contribution stems mainly from our general treatment of volatility ambiguity, which is the main novel feature, so that we have decided to not drown our arguments in unnecessary additional technicalities and notations by adding drift uncertainty. For the interested readers, a very general model encompassing these two sources of uncertainty is currently being treated by Hernández Santibáñez and Mastrolia (see the PhD thesis [24, Chapter 4] for an early version), following the general roadmap outlined in the present paper.

The impact of volatility ambiguity on optimal contracting has not been considered before in the literature (except in a very recent paper by Sung [25], which was brought to our attention after the completion our work, and on which we expand more below). The impact of drift ambiguity has been studied by Weinschenk [26] for linear contracts in discrete time, by Szydlowski [27] and Miao and Rivera [28] who consider an extension of Sannikov’s model [10] including ambiguity about the Agent’s effort cost. Our paper also belongs to the literature on optimal contracting with learning, for which we can refer to the seminal papers of Williams [29], Prat and Jovanovic [16] and He et al. [30], or to Golosov et al. [31] for models addressing learning in the context of optimal dynamic taxation, Pavan et al. [32] for models with transferrable utility, or DeMarzo and Sannikov [14] for a setting in which both the Principal and the Agent learn about future profitability from output. What could be considered as a learning-type feature in our model is that both the Principal and the Agent have the opportunity to update their beliefs on the future realised volatility as time goes by, which gives a dynamic flavour to this aspect of their risk aversion. Notice also that the fact that both the Principal and the Agent actually learn little about the volatility as time passes is actually in accordance with the ambiguity literature (see notably Chen and Epstein [33]), which usually considers that the uncertainties are generated by ambiguous and potentially distorted signals. Epstein and Schneider [34] thus declare that “when ambiguity-averse investors process news of uncertain quality, they act as if they take a worst-case assessment of quality”. In other words, in our context, even though at some given time t the ambiguity on the past values of the volatility has been resolved, it does not allow at all to predict future values through Bayesian learning, as new uncertainties keep on arising as time passes.

Our first task in this paper is to solve the risk-sharing problem. Surprisingly, this problem is much more involved than in the classical case, since it takes a very unusual form, as a supremum of a sum of two infimum over different sets. Nonetheless, we provide a generic method (see Method 3.1) to solve it, which first focuses on a subclass of contracts similar to the ones obtained in [18, 19], and then uses calculus of variations and convex analysis to argue that the optimal contracts in the subclass of contract chosen is indeed optimal. We use it successfully in what we coin a “non-adaptative” model, where both the Principal and the Agent do not update their beliefs with regard
to volatility as time passes. Despite being restrictive, this benchmark model has the nice property that everything becomes completely explicit, and illustrates how our method can be applied in practice. We also highlight a surprising effect, which we interpret as a kind of arbitrage-like situation,\(^1\) corresponding to the situation where the volatility intervals of the Principal and the Agent are completely disjoint. In this case, the problem degenerates and the Principal can actually reach utility 0 using an appropriate sequence of contracts (we remind the reader that the exponential utility is \(-\exp(-R_p x)\), so that it is bounded from above by 0).

Next, we concentrate on the second-best problem. Our first contribution is to use the theory of second-order BSDEs developed by Soner et al. [35], and more precisely the recent well-posedness results obtained by Possamaï et al. [36], to obtain a probabilistic representation of the value function of the Agent, for any sufficiently integrable contract. In particular, this representation gives an easy access to the optimal action chosen by the Agent. Then, following the ideas of [18, 19], we concentrate our attention on a subclass of contracts, for which the Principal problem can be solved using classical dynamic programming type arguments. The main problem is then to prove that the restriction is actually without loss of generality. We emphasise that in spite of the fact that our approach is similar, in spirit, to the one used in [19], we cannot use their method of proof. Indeed, our problem is of a fundamentally different nature, because the Agent himself does not control the volatility of the output, but rather endures it. We therefore have to proceed completely differently and provide a general argument which shows, with PDE techniques, that the value function of the original and the sub-optimal problem actually solve the same PDE, which implies that they are equal by uniqueness of the solution to the latter. We believe that this general approach can actually be applied to many situations, and constitutes one of our main contributions. In addition, we obtain an extremely general result stating that if the beliefs of the Principal and the Agent are completely disjoint, then the problem always degenerates, and the Principal can reach utility 0, making the second-best and first-best problems identical. Once more, this result highlights the necessity to get rid of these arbitrage-like situations.

For simplicity and clarity, we also solve by a direct method the problem in the “non-adaptative” model mentioned above, where the identification of the two value functions can actually be obtained by simple (but tedious) algebra, constructing appropriate tight upper and lower bounds.

The rest of the paper is organised as follows. We introduce the model and the contracting problem in Sect. 2. Then Sect. 3 is devoted to the risk-sharing problem, while Sect. 4 treats the moral hazard case. We finally present some possible extensions in Sect. 5. So as to shorten the paper, we have excluded some of the most computationally intensive proofs, and we refer the reader to the online longer version of the paper [37] for the details.

After the completion of this paper, we have been made aware of a paper in preparation by Sung, where the author studied a problem similar to ours. Since a preprint version of this paper [25] has appeared during the revision of the present manuscript,

\(^1\) We would like to thank one of the referees for suggesting this interpretation.
we would like to sum up the main differences here. More detailed explanations are provided in Sect. 5 at the end of the paper. First, the ambiguity sets in [25] are assumed to have the same support for both the Agent and the Principal. In our work, we consider that the Agent and the Principal may have different beliefs on the values taken by the volatility, and prove that the Principal is always forced to work on the intersection of their ambiguity sets, when the latter is non-empty, and the problem degenerates otherwise. Second, [25] allows for drift uncertainty and more general dynamics, which can also be treated with our method; see Sect. 5. The last main difference concerns the choice of admissible contracts. Indeed, in [25] the author restricts his attention from the bounded variation part is absolutely continuous with respect to Lebesgue measure. In our paper, we claim and justify that this class of contracts is indeed the appropriate one, by taking advantage of the 2BSDE theory.

2 The Model

2.1 The Stochastic Basis

We start by giving all the necessary notations and definitions allowing us to consider the so-called weak formulation of the problem.

We will use the notation $\mathbb{R}_+:=[0, +\infty[$. Let $\Omega:=\{\omega \in C([0,T],\mathbb{R}); \omega_0=0\}$ be the canonical space equipped with the uniform norm $||\omega||_\infty:=\sup_{0 \leq t \leq T}|\omega_t|$. We then denote by $\mathbb{F}$ the canonical process, $\mathbb{P}_0$ the Wiener measure, $\mathbb{F}:=\{\mathcal{F}_t\}_{0 \leq t \leq T}$ the filtration generated by $\mathbb{B}$ and $\mathbb{F}^+:=\{\mathcal{F}_t^+, 0 \leq t \leq T\}$, the right limit of $\mathbb{F}$ where $\mathcal{F}_t^+:=\cap_{t>r}\mathcal{F}_r$. We will denote by $\mathbf{M}(\Omega)$ the set of all probability measures on $(\Omega,\mathcal{F}_T)$. For any $\mathbb{P} \in \mathbf{M}(\Omega)$, a statement holding with $\mathbb{P}$-probability 1 will be referred to as holding $\mathbb{P}$-almost surely, or $\mathbb{P}-a.s.$ for short. We also recall the so-called universal filtration $\mathbb{F}^*:=\{\mathcal{F}_t^*\}_{0 \leq t \leq T}$ defined as $\mathcal{F}_t^*:=\cap_{\mathbb{P} \in \mathbf{M}(\Omega)}\mathcal{F}_t^\mathbb{P}$, where $\mathcal{F}_t^\mathbb{P}$ is the usual augmentation under $\mathbb{P}$.

For any normed vector space $(E, \|\cdot\|_E)$ of a finite-dimensional space and any filtration $\mathbb{X}$ on $(\Omega,\mathcal{F}_T)$, we denote by $\mathcal{H}^0(E,\mathbb{X})$ the set of all $\mathbb{X}$-progressively measurable processes with values in $E$. Moreover, for all $p>0$ and for all $\mathbb{P} \in \mathbf{M}(\Omega)$, we denote by $\mathcal{H}^p(\mathbb{P},E,\mathbb{X})$ the subset of $\mathcal{H}^0(E,\mathbb{X})$ whose elements $H$ satisfy $\mathbb{E}[\int_0^T\|H_t\|_E^p\,dt]<+\infty$. The localised versions of these spaces are denoted by $\mathcal{H}^p_{\text{loc}}(\mathbb{P}, E, \mathbb{X})$.

For any subset $\mathcal{P} \subset \mathbf{M}(\Omega)$, a $\mathcal{P}$-polar set is a $\mathbb{P}$-negligible set for all $\mathbb{P} \in \mathcal{P}$, and we say that a property holds $\mathcal{P}$-quasi-surely if it holds outside some $\mathcal{P}$-polar set. We also denote by $\mathcal{H}^p(\mathbb{P}, E, \mathbb{X}) := \bigcap_{\mathbb{P} \in \mathcal{P}} \mathcal{H}^p_{\text{loc}}(\mathbb{P}, E, \mathbb{X})$. Finally, we introduce the following filtration $\mathbb{G}^\mathcal{P} := \{\mathcal{G}_t^\mathcal{P}\}_{0 \leq t \leq T}$, with $\mathcal{G}_t^\mathcal{P} := \mathcal{F}_t^* \vee \mathcal{N}^\mathcal{P}$, $t \leq T$, where $\mathcal{N}^\mathcal{P}$ is the collection of $\mathcal{P}$-polar sets, and the right-continuous limit of $\mathbb{G}^\mathcal{P}$, denoted $\mathbb{G}^\mathcal{P}^+$. For all $\alpha \in \mathcal{H}^0_{\text{loc}}(\mathbb{P}_0, [0, +\infty[, \mathbb{F})$, we define the following probability measure on $(\Omega,\mathcal{F})$

$$\mathbb{P}^\alpha := \mathbb{P}_0 \circ (X_\alpha)^{-1} \text{ where } X_t^\alpha := \int_0^t \alpha_s^{1/2} \, dB_s, \ t \in [0, T], \ \mathbb{P}_0-a.s. \quad (1)$$

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We denote by \( \mathcal{P}_S \) the collection of all such probability measures on \((\Omega, \mathcal{F}_T)\). We recall from [38] that the quadratic variation process \( \langle B \rangle \) is universally defined under any \( \mathbb{P} \in \mathcal{P}_S \), and takes values in the set of all non-decreasing continuous functions from \( \mathbb{R}_+ \) to \([0, +\infty[\). We denote for any \( p > 0 \)

\[
\hat{H}_p^\mathbb{P}(E, X) := \left\{ \gamma \in H^0(E, X) : \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^\mathbb{P}\left[ \int_0^T \| \gamma_t \|^p E \, d\langle B \rangle_t \right] < +\infty \right\}.
\]

We will denote the pathwise density of \( \langle B \rangle \) with respect to the Lebesgue measure by \( \hat{\alpha} \). Finally we recall from [39] that every \( \mathbb{P} \in \mathcal{P}_S \) satisfies the Blumenthal zero–one law and the martingale representation property. By definition, for any \( \mathbb{P} \in \mathcal{P}_S \)

\[
W^\mathbb{P}_t := \int_0^t \hat{\alpha}_s^{-1/2} \, dB_s, \quad \mathbb{P} - a.s.,
\]

is a \((\mathbb{P}, \mathcal{F})\)-Brownian motion. Notice that the probability measures in \( \mathbb{P} \in \mathcal{P}_S \) verify that the two following completed filtrations are equal

\[
\mathcal{F}^\mathbb{P} = (\mathcal{F}^{W^\mathbb{P}})^\mathbb{P},
\]

where \( \mathcal{F}^{W^\mathbb{P}} \) is the natural (raw) filtration of the process \( W^\mathbb{P} \).

The dependence of \( W^\mathbb{P} \) on the underlying probability measure is mainly due to the fact that the construction of the stochastic integral is generically done only in an almost sure sense. For want of cosmetically nicer results, we would like to be able to find a universal aggregator of this family. Using the result of [40], and, for instance, assuming that we work under the usual ZFC framework, and in addition the continuum hypothesis,\(^2\) there actually exists an aggregated version of this family, which we denote by \( W \), which is \( \mathcal{F}^* \)-adapted and a \((\mathbb{P}, \mathcal{F}^\mathbb{P})\)-Brownian motion for every \( \mathbb{P} \in \mathcal{P}_S \). Our focus in this paper will be on the following subset of \( \mathcal{P}_S \).

**Definition 2.1** \( \mathcal{P}_m \) is the subclass of \( \mathcal{P}_S \) consisting of all \( \mathbb{P} \in \mathcal{P}_S \) such that the canonical process \( B \) is a \((\mathbb{P}, \mathcal{F})\)-uniformly integrable martingale.

The actions of the Agent will be considered as \( \mathcal{F} \)-predictable processes \( a \) taking values in the compact set \([0, a_{\text{max}}]\) (for every \( \omega \)). This upper bound corresponds to a maximal effort for the Agent, and we assume that it is known by the Principal. We believe that such an assumption is reasonable, since we assume here that the Principal knows the key characteristics of the Agent, and that the latter cannot exercise arbitrarily

\(^2\) We insist on the fact that if one does not want to assume such an axiom, this is not a problem for this part of our work, and one just has to keep working with the family \((W^\mathbb{P})_{\mathbb{P} \in \mathcal{P}_S}\). However, when defining the set of admissible contracts \( \mathcal{C}^{SB} \) later in the paper, we will need it in order to define aggregated versions of stochastic integrals. If one does not want to use it, then it means that we have to restrict the control processes \( Z \) in \( \mathcal{C}^{SB} \) to ones having sufficiently regular trajectories to apply the pathwise integration theory of Karandikar [38], for instance. By standard density results, it should, however, not change the value function of the Principal. Notice also that the continuum hypothesis is one among many axioms that can be used; see [36, Footnote 4] for the weakest known ones.
large effort.\textsuperscript{3} We denote this set by $\mathcal{A}$. Next, for any subset $\mathcal{P} \subset \mathcal{P}_S$ and any $a \in \mathcal{A}$, we define

\[ \mathcal{P}^a := \left\{ Q : \frac{dQ}{dP} = E \left( \int_0^T a_s \alpha_s^{-1/2} dW_s \right), P \text{ a.s., for some } P \in \mathcal{P} \right\}. \]

We also denote $\mathcal{P}^A := \bigcup_{a \in \mathcal{A}} \mathcal{P}^a$. In particular, for every $P \in \mathcal{P}^A$ there exists a unique pair $(\alpha^P, a^P) \in \mathcal{H}_{\text{loc}}(\mathcal{P}, R_+, F) \times \mathcal{A}$ such that

\[ B_t = \int_0^t a^P_s ds + \int_0^t (\alpha^P_s)^{1/2} dW^P_s, \quad P \text{ a.s.,} \tag{3} \]

where $dW^P_s := dW_s - (\alpha^P_s)^{-1/2} a^P_s ds$, $P$ a.s. is a $(\mathcal{P}^a, \mathcal{F}^a)$-Brownian motion by Girsanov’s theorem. More precisely, for any $P \in \mathcal{P}^A$, we must have

\[ \frac{dP}{dP^a} = E \left( \int_0^T a_s \alpha_s^{-1/2} dB_s \right), \]

for some $(\alpha, a) \in \mathcal{H}_{\text{loc}}(\mathcal{P}, R_+, F) \times \mathcal{A}$ and the following equalities hold

\[ a^P(B) = a(B) \text{ and } \alpha^P(B) = \alpha(W), \text{ } dt \times P \text{ a.e.} \]

For simplicity, we will therefore sometimes denote a probability measure $P \in \mathcal{P}^A$ by $P^a$. For any subset of $\mathcal{P} \subset \mathcal{P}_m$, we also denote for any $(t, P) \in [0, T] \times \mathcal{P}$

\[ \mathcal{P}(P, t^+) := \left\{ P' \in \mathcal{P} : P' = P, \text{ on } \mathcal{F}_t^+ \right\}. \]

We also recall that for every probability measure $P$ on $\Omega$ and $\mathcal{F}$-stopping time $\tau$ taking value in $[0, T]$, there exists a family of regular conditional probability distribution (r.c.p.d. for short) $(\mathbb{P}_\omega^\tau)_{\omega \in \Omega}$ (see, for example, Stroock and Varadhan [41]), satisfying

(i) For every $\omega \in \Omega$, $\mathbb{P}_\omega^\tau$ is a probability measure on $(\Omega, \mathcal{F}_\tau)$.

(ii) For every $E \in \mathcal{F}_\tau$, the mapping $\omega \mapsto \mathbb{P}_\omega^\tau(E)$ is $\mathcal{F}_\tau$-measurable.

(iii) The family $(\mathbb{P}_\omega^\tau)_{\omega \in \Omega}$ is a version of the conditional probability measure of $P$ on $\mathcal{F}_\tau$, i.e. for every integrable $\mathcal{F}_\tau$-measurable random variable $\xi$ we have $E^\mathbb{P}[\xi | \mathcal{F}_\tau](\omega) = E^{\mathbb{P}_\omega^\tau}[\xi]$, for $P$ a.e. $\omega \in \Omega$.

(iv) For every $\omega \in \Omega$, $\mathbb{P}_\omega^T(\Omega_\tau^\omega) = 1$, where $\Omega_\tau^\omega := \{ \omega' \in \Omega, \omega'(s) = \omega(s), 0 \leq s \leq \tau(\omega) \}$.\textsuperscript{3}

\textsuperscript{3} Obviously, an extension of the present framework to model incorporating adverse selection, that is to say that the Principal does not actually know perfectly all the characteristics of the Agent, is not only interesting mathematically, but also from the point of view of applications. However, we believe that this would lead to a much more difficult problem and leave it for future research.
Furthermore, given some $P$ and a family $(Q_\omega)_{\omega \in \Omega}$ such that $\omega \mapsto Q_\omega$ is $\mathcal{F}_\tau$-measurable and $Q_\omega(\Omega_\omega^{\tau}) = 1$ for all $\omega \in \Omega$, one can then define a concatenated probability measure $P \otimes_\tau Q$ by

$$
P \otimes_\tau Q \left[ A \right] := \int_\Omega Q_\omega \left[ A \right] P(\text{d}\omega), \forall A \in \mathcal{F}_T.
$$

We conclude this introductory section by noticing that since any $a \in A$ impacts only the drift in the decomposition (3) of $B$, we directly have that for any $P_1, P_2$ subsets of $\mathcal{P}_m$

$$
\exists a \in A, P_1 \cap P_2 \neq \emptyset \iff \forall a \in A, P_1^a \cap P_2^a \neq \emptyset.
$$

### 2.2 The Contracting Problem in Finite Horizon

#### 2.2.1 The Ambiguity Sets

We consider a generalisation of the classical problem of Holmström and Milgrom [5] and fix a given time horizon $T > 0$. Here the Agent and the Principal both observe the outcome process $B$, but the Principal may not observe the action chosen by the Agent, and both of them have a “worst-case” approach to the contract, in the sense that they act as if “Nature” was playing against them by choosing the worst possible volatility of the output. More precisely, a contract will be a $\mathcal{F}_T$-measurable random variable, corresponding to the salary received by the Agent at time $T$ only. The Agent has then some beliefs about the volatility of the project, which are summed up in a family $(P_A(t, \omega))_{(t, \omega) \in [0,T] \times \Omega}$, such that for any $(t, \omega) \in [0, T] \times \Omega$, $P_A(t, \omega) \subset \mathcal{P}_m$.

The dependence in $(t, \omega)$ allows the beliefs of the Agent with regard to the volatility to change with both time and the observed randomness, that is to say with the evolution and history of the output process $B$. Similarly, we introduce a family $(\mathcal{P}_P(t, \omega))_{(t, \omega) \in [0,T] \times \Omega}$ associated with the Principal’s beliefs. Notice that since $\omega_0 = 0$ for any $\omega \in \Omega$, these sets at $t = 0$ do not depend on $\Omega$, so that we will use the simplified notations $P_A := P_A(0, \omega)$ and $P_P := P_P(0, \omega)$, for any $\omega \in \Omega$.

We emphasise that these families cannot be chosen completely arbitrarily, and have to satisfy a certain number of stability and measurability properties, which are classical in stochastic control theory. Most of these are quite technical, so that we will refrain from commenting them, and refer instead the interested reader to [36], for instance.

We will hence assume throughout the paper the following

**Assumption 2.1** For $\Psi = A, P$, we have

(i) For every $(t, \omega) \in [0, T] \times \Omega$, one has $\mathcal{P}_P(t, \omega) = \mathcal{P}_P(t, \omega, \omega^{\tau})$ and $P(\Omega^{\omega}_t) = 1$ whenever $P \in \mathcal{P}_P(t, \omega)$. The graph $[[\mathcal{P}_P]]$ of $\mathcal{P}_P$, defined by $[[\mathcal{P}_P]] := \{(t, \omega, P) : P \in \mathcal{P}_P(t, \omega)\}$, is upper semi-analytic in $[0, T] \times \Omega \times \mathcal{M}(\Omega)$.

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4 He observes it in the risk-sharing problem of Sect. 3, but not in the moral hazard case of Sect. 4.
(ii) $\mathcal{P}_\Psi$ is stable under conditioning, i.e. for every $(t, \omega) \in [0, T] \times \Omega$ and every $P \in \mathcal{P}_\Psi(t, \omega)$ together with an $\mathbb{F}$-stopping time $\tau$ taking values in $[t, T]$, there is a family of r.c.p.d. $(P_w)_{w \in \Omega}$ such that $P_w \in \mathcal{P}_\Psi(\tau(w), \omega)$ for $P$-a.e. $w \in \Omega$.

(iii) $\mathcal{P}_\Psi$ is stable under concatenation, i.e. for every $(t, \omega) \in [0, T] \times \Omega$ and $P \in \mathcal{P}_\Psi(t, \omega)$ together with a $\mathcal{F}_\tau$-stopping time $\tau$ taking values in $[t, T]$, if $(Q_w)_{w \in \Omega}$ is a family of probability measures such that $Q_w \in \mathcal{P}_\Psi(\tau(w), \omega)$ for all $w \in \Omega$ and $w \mapsto Q_w$ is $\mathcal{F}_\tau$-measurable, then the concatenated probability measure $P \otimes \tau Q \cdot \in \mathcal{P}_\Psi(t, \omega)$.

We will also need to consider the support of $\hat{\alpha}$ induced by these families.

**Definition 2.2** For $\Psi = A, P$, we denote, for any $(t, \omega) \in [0, T] \times \Omega$, by $D_\Psi(t, \omega)$ the smallest closed subset of $]0, +\infty[$ such that

$$P\left(\{w \in \Omega, \hat{\alpha}_s(w) \in D_\Psi(t + s, \omega \otimes t, w), \text{ for } a.e. s \in [0, T - t]\} \right) = 1,$$

where the concatenated path $\omega \otimes t, w \in \Omega$ is defined by

$$(\omega \otimes t, w)(s) := \omega(s) 1_{s \leq t} + (w(s) - \omega(t)) 1_{s \in (t, T]}, \ s \in [0, T].$$

At this stage, one would naturally be tempted to introduce a learning component in the problem. Indeed, the idea is that since both the Agent and the Principal observe the output $B$ and its quadratic variation $\langle B \rangle$, they should this information to update their prior beliefs, thus learning about volatility in the Bayesian sense of the term. Such problems involving filtering techniques have been considered in the contract theory literature; see, for instance, DeMarzo and Sannikov [14], Prat and Jovanovic [16], or He et al. [30], where the Principal and the Agent try to learn about the quality/ability of the Agent, or the profitability of the project. Their goal is, however, different from ours. In our setting, there is not a parameter which is not known by the Principal and/or the Agent. They are simply both risk-averse and consider that they do not have a reliable model for the volatility of the output $B$. They have their own initial beliefs about the volatility, in the form of an interval into which they consider that the volatility will remain later on, and they try to determine what is their best course of actions in the worst-case scenario for them where the most punitive path of the volatility, for their utility criterion, is realised.

This is of course a very conservative approach, but which has the merit to put clear and quantitative bounds on the risk that both the Principal and the Agent actually take when assuming that there is no model uncertainty. Such an approach is directly inspired from the so-called robust finance literature, which considers hedging and portfolio optimisation problems under volatility uncertainty; see, for instance, the recent paper [42] and the references therein for more details. Of course, the initial belief of the Principal and/or the Agent can actually prove to be wrong through time, for instance, if the volatility escapes from the bounds. Which is why we allow them to be adaptative, and change with time and realisations of the paths of $B$. In this sense, our model incorporates an aspect of learning, but since it is not related to the usual notion of filtering theory, we will instead refer to it as “adaptative”.

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Example 2.1 (Adaptative and non-adaptative models) The main example we have in mind here is the one corresponding to the so-called random $G$-expectations, introduced by Nutz [43]. The idea is to specify directly the support of $\hat{\alpha}$ and to consider, for $\Psi = A, P$, set-valued processes $D_\Psi : [0, T] \times \Omega \longrightarrow 2^{[0, +\infty]}$ which are progressively measurable in the sense of graph measurability, that is to say that for all $t \in [0, T]$, we have

$$\{(s, \omega, A) \in [0, t] \times \Omega \times [0, +\infty[. A \in D_\Psi(s, \omega)\} \in B([0, t]) \otimes F_t \otimes B([0, +\infty]),$$

where $B([0, t])$ and $B([0, +\infty[)$ are the Borel $\sigma$-algebras of $[0, t]$ and $[0, +\infty[$. In this case, the sets $P_\Psi(t, \omega)$ are defined for any $(t, \omega) \in [0, T] \times \Omega$ as being the collection of probability measures $\mathbb{P} \in \mathcal{M}(\Omega)$ such that $\tilde{\alpha}_t(w) \in D_\Psi(s + t, \omega \otimes_t w)$, for $d s \otimes d \mathbb{P} - \text{a.e.} (s, w) \in [0, T - t] \times \Omega$.

It has been shown by Nutz and van Handel [44] that the sets $P_\Psi(t, \omega)$ indeed satisfy Assumption 2.1. We can, for instance, assume that there exist processes $(\alpha^P, \alpha^A, \bar{\alpha}^P, \bar{\alpha}^A) \in (\mathcal{H}^0(\mathbb{R}_+^*, \mathcal{F}))^4$ such that for any $(t, \omega) \in [0, T] \times \Omega$

$$D_A(t, \omega) = \left[\alpha^A_t(\omega), \bar{\alpha}^A_t(\omega)\right], \quad D_P(t, \omega) = \left[\alpha^P_t(\omega), \bar{\alpha}^P_t(\omega)\right]. \quad (5)$$

This typically leads to a model in which the Principal and the Agent estimate that the volatility of the output will live in intervals, whose bounds may vary with respect to the path of the output process. An interpretation of this specification is that both the Principal and the Agent update their beliefs by observing the past realisations of the output process $B$, and therefore that there is some kind of reaction effect. To be more precise, the Principal and the Agent may fix at the beginning some updating rules for their beliefs on the volatility. For instance, if the previous path of the volatility comes closer to the bounds of the intervals, the update would consist in increasing the length of the interval by a fixed percentage. The Principal and the Agent can therefore adapt.\footnote{This is why we call this model “adaptative”. Notice nonetheless that a higher degree of generality would require to consider a learning model, without assuming that (5) holds. In this case, as mentioned before, one would have to use a filtering procedure to update the estimates of the volatility model.}

In this paper, we will explain how to solve in general the moral hazard problems associated with such a framework, by relating it to some Hamilton–Jacobi–Bellman–Isaacs partial differential equation. However, we will illustrate further our results in the only case which, to the best of our knowledge, leads to completely explicit computations. It corresponds to assuming that $\alpha^P, \alpha^A, \bar{\alpha}^P, \bar{\alpha}^A$ are constant processes. In this case, the Agent and the Principal are not learning nor adapting with time. Obviously, this can be seem as a quite unrealistic situation, and will lead in some cases to arbitrage-like results that will be commented upon. However, notice that if the bounds are sufficiently large, it is quite unlikely that the realised volatility would exit the corresponding intervals before the finite time $T$. The heterogeneity between the Principal and the Agent, due to the fact that their beliefs might be different, can

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then be understood as a simple manifestation of their potentially different aversions to model uncertainty. As such, we believe that with appropriate caution, even this case leads to interesting qualitative results, and deserves to be treated thoroughly.

2.2.2 Utilities of the Principal and the Agent

We have now all the necessary tools to specify the utility obtained by the Agent, given a contract \( \xi \), a recommended level of effort \( a \in A \) and an ambiguity set \((P_A(t, \omega))_{(t, \omega) \in [0, T] \times \Omega}\)

\[
u_0^A(\xi, a) := \inf_{P \in P_A} \mathbb{E}^P \left[ U_A \left( \xi - \int_0^T k(a_s) ds \right) \right],
\]

where \( U_A(x) := - \exp(-R_A x) \) is the utility function of the Agent, for some \( R_A > 0 \), and \( k(x) \) is his cost function, which, as usual is assumed to be increasing, strictly convex and super-linear. The value function of the Agent at time 0 is therefore

\[
U_0^A(\xi) := \sup_{a \in A} \inf_{P \in P_A} \mathbb{E}^P \left[ U_A \left( \xi - \int_0^T k(a_s) ds \right) \right].
\]

Similarly, the utility of the Principal, having an ambiguity set \((P_P(t, \omega))_{(t, \omega) \in [0, T] \times \Omega}\), when offering a contract \( \xi \) and a recommended level of effort \( a \in A \) is

\[
u_0^P(\xi, a) := \inf_{P \in P_P} \mathbb{E}^P \left[ U_P \left( B_T - \xi \right) \right],
\]

where \( U_P(x) := - \exp(-R_P x) \) is the utility function of the Principal.

Let \( R < 0 \) denote the reservation utility of the Agent. The problem of the Principal is then to offer a contract \( \xi \) as well as a recommended level of effort \( a \) so as to maximise his utility (6), subject to the constraints

\[
u_0^A(\xi, a) \geq R, \quad \nu_0^A(\xi, a) = U_0^A(\xi).
\]

The first constraint is the so-called participation constraint, while the second one is the usual incentive compatibility condition, stating that the recommended level of effort \( a \) should be the optimal response of the Agent, given the contract \( \xi \). Furthermore, we will denote by \( \mathcal{C} \) the set of admissible contracts, that is to say the set of \( \mathcal{F}_T \)-measurable random variables such that

\[
\sup_{P \in P_A^A \cup P_P^A} \mathbb{E}^P \left[ \exp(p |\xi|) \right] < +\infty, \text{ for any } p \geq 0,
\]

and we emphasise immediately that we will have to restrict a bit more the admissible contracts when solving the second-best problem, for technical reasons linked to inte-
grability assumptions. However, we postpone the exact statement to Sect. 4.3, since it requires quite an important number of preliminaries.

3 The First Best: A Problem of Calculus of Variations

In this section, we start by studying the first-best problem for the Principal, since it will serve as our main benchmark and has not been considered, as far as we know, in the preexisting literature. Moreover, we will see that the derivation is a lot more complicated than in the classical setting, so much so that, quite surprisingly compared with the classical Holmström and Milgrom [5] problem, the optimal contracts are in general not linear with respect to the final value of the output $B_T$, and are even path-dependent.

Recall that for any contract $\xi \in C$ and for any recommended effort level $a \in A$

$$u_0^P(\xi, a) = \inf_{P \in P_P^a} \mathbb{E}^P [U_P (B_T - \xi)].$$

The value function of the Principal is then

$$U_0^{P, FB} := \sup_{\xi \in C} \sup_{a \in A} \left\{ u_0^P(\xi, a) \right\},$$

(10)

where the following participation constraint has to be satisfied

$$\inf_{P \in P_A^\mu} \mathbb{E}^P \left[ U_A \left( \xi - \int_0^T k(a_s)ds \right) \right] \geq R.$$  

(11)

The value function of the Principal defined by (10) can be then rewritten

$$U_0^{P, FB} := \sup_{\xi \in C} \sup_{a \in A} \left\{ \inf_{P \in P_P^a} \mathbb{E}^P [U_P (B_T - \xi)] + \rho \inf_{P \in P_A^\mu} \mathbb{E}^P \left[ U_A \left( \xi - \int_0^T k(a_s)ds \right) \right] \right\},$$

(12)

where the Lagrange multiplier $\rho > 0$ is here to ensure that the participation constraint (11) holds.

3.1 Gâteaux Differentiability and Optimality

Once again, the main difficulty is that the sets $P_A$ and $P_P$ are too abstract to solve generally problem (12) directly, especially since we do not know whether the two infima are attained or not. In order to overcome this major difficulty, we will restrict the set of admissible contracts to the ones for which both the Principal and the Agent have indeed a worst-case measure. In order to do so, let us first introduce the following
sets of worst probabilities for $\Psi = A, P$, any contract $\xi \in C$, and any effort $a \in A$

$$\mathcal{P}^*_{\Psi, a}(\xi) := \left\{ \mathbb{P}^* \in \mathcal{P}^a_\Psi : \inf_{\mathbb{P} \in \mathcal{P}^a_\Psi} \mathbb{E}^\mathbb{P}[U_\Psi(B_T - \xi)] = \mathbb{E}^{\mathbb{P}^*}[U_\Psi(B_T - \xi)] \right\}. $$

We then define

$$\tilde{C} := \{ \xi \in C : \mathcal{P}^*_{\Psi, a}(\xi) \neq \emptyset, \ \Psi = \{ A, P \}, \ \forall a \in A \}. $$

Thus, problem (12) restricted to contracts in $\tilde{C}$ becomes

$$\tilde{U}_0^{P, FB} := \sup_{\xi \in \tilde{C}} \sup_{a \in A} \left\{ \mathbb{E}^{\mathbb{P}^*, a, \xi}_P[U_P(B_T - \xi)] + \rho \mathbb{E}^{\mathbb{P}^*, a, \xi}_A[U_A(\xi - \int_0^T k(a_s)ds)] \right\}, $$

(13)

where $\mathbb{P}^*, a, \xi_P$ and $\mathbb{P}^*, a, \xi_A$ are generic elements of $\mathcal{P}^*_{A, a}(\xi)$ and $\mathcal{P}^*_{P, a}(\xi)$, respectively.

Next, it is not extremely convenient that the two expectations above are written under different probability measures. We will therefore use their definition to bring back all the computations under $\mathbb{P}_0$. Let us start by considering the so-called Morse–Transue space on $(\Omega, \mathcal{F}_T, \mathbb{P}_0)$ (we refer the reader to the monographs [45, 46] for more details), defined by

$$M^\phi := \left\{ \xi : \Omega \rightarrow \mathbb{R} : \text{measurable, } \mathbb{E}^{\mathbb{P}_0}[\phi(a\xi)] < +\infty, \ \text{for any } a \geq 0 \right\}, $$

where $\phi$ is the Young function $\phi(x) := \exp(|x|) - 1$. Then, if $M^\phi$ is endowed with the norm

$$\|\xi\|_\phi := \sup \left\{ \mathbb{E}^{\mathbb{P}_0}[\xi g] : \mathbb{E}^{\mathbb{P}_0}[\phi(g)] \leq 1 \right\}, $$

it becomes a (non-reflexive) Banach space.

Then, for any $a \in A$ and $(\alpha^P, \alpha^A) \in H^1_{loc}(\mathbb{P}_0, \mathbb{R}^*_+, \mathcal{F}) \times H^1_{loc}(\mathbb{P}_0, \mathbb{R}^*_+, \mathcal{F})$, we consider the map $\mathcal{E}_{a}^{\alpha^P, \alpha^A} : M^\phi \rightarrow \mathbb{R}$ defined by

$$\mathcal{E}_{a}^{\alpha^P, \alpha^A}(\xi) := \mathbb{E}^{\mathbb{P}_0} \left[ e^{-R_P \left( \int_0^T a_s(X^{\alpha^P} s)ds + \int_0^T (a^P_s(B_s))^{1/2} dB_s - \xi(X^{\alpha^P}) \right)} \right. $$

$$+ \rho e^{-R_A \left( \int_0^T (\alpha^A_s(X^{\alpha^A} s))^{1/2} dB_s - \xi(X^{\alpha^A}) \right) ds} \right], $$

with

$$X^{\alpha^P}(B.) := \int_0^T a_s(B. )ds + \int_0^T (\alpha^P_s(B. ))^{1/2} dB_s, \ X^{\alpha^A}(B.) := \int_0^T a_s(B. )ds + \int_0^T (\alpha^A_s(B. ))^{1/2} dB_s.$$
Let now \((\xi, a) \in \tilde{\mathcal{C}} \times \mathcal{A}\). For \(\Psi = \{A, P\}\) and each \(\mathbb{P}^* \in \mathcal{P}^*_\Psi(\xi)\) we associate the corresponding \(\alpha^{\Psi, *}\) (recall (1)). We then have

\[
\tilde{U}_0^{P, FB} = \sup_{\xi \in \tilde{\mathcal{C}}} \sup_{a \in \mathcal{A}} \left\{ -\Xi_a^{\alpha^{P, *}, \alpha^{A, *}} \right\}.
\]

We will first interest ourselves to the maximisation with respect to \(\xi\). It can be readily checked that \(\Xi_a^{\alpha^{P, *}, \alpha^{A, *}}\) is a strictly convex mapping in \(\xi\), which is in addition proper and continuous. However, since \(M^\phi\) is not reflexive, we cannot claim that its minimum is attained. Nonetheless, we can still use the characterisation of a minimiser in terms of Gâteaux derivatives. Indeed, a random variable \(\xi \in M^\phi\) which minimises \(\Xi_a^{\alpha^{P}, \alpha^{A}}\) necessarily satisfies the following property

\[
\tilde{D}\Xi_a^{\alpha^{P, *}, \alpha^{A, *}}(\xi)[h - \xi] \geq 0, \text{ for any } h \in M^\phi, \quad (14)
\]

where \(\tilde{D}\Xi_a^{\alpha^{P, *}, \alpha^{A, *}}\) denotes the Gâteaux derivative of \(\Xi_a^{\alpha^{P, *}, \alpha^{A, *}}\) given by

\[
\tilde{D}\Xi_a^{\alpha^{P, *}, \alpha^{A, *}}(\xi)[h] = E_{\mathbb{P}}^0 \left[ R_P h(X^{a, \alpha^{P, *}}) e^{-R_P \left( \int_0^T a_t(X^{a, \alpha^{P}}) ds + \int_0^T \alpha^{P, *}_t^2 dB_t - \xi(X^{a, \alpha^{P, *}}) \right)} 
- R_A h(X^{a, \alpha^{A, *}}) e^{-R_A \left( \xi(X^{a, \alpha^{A}}) - \int_0^T k(a_t(X^{a, \alpha^{A}})) ds \right)} \right].
\]

Thus, if a contract \(\xi^* \in \tilde{\mathcal{C}}\) satisfies property (14), it is optimal for problem (13). We would like to insist on the fact that in this section, our main purpose is to propose a general method to investigate a risk-sharing problem with uncertainty on the volatility. From our understanding of the problem, we believe that it would be extremely hard, for very general sets of ambiguity \(\mathcal{P}_A\) and \(\mathcal{P}_P\), to not start by studying sub-optimal problem (13), since the latter can be put in the much more convenient form above. Nonetheless, we will show below that in the model that we coined “non-learning”, the restriction is actually without loss of generality, which gives hope to be able to treat the completely general case in future works. This, however, goes beyond the scope of the current paper.

That being said, let us describe more precisely the modus operandi that we propose for solving the risk-sharing problem.

**Method 3.1** The method is divided in two steps.

1. We restrict the study to a particular set of contracts included in \(\tilde{\mathcal{C}}\), which we justify intuitively. If this problem can be solved, we can then check whether the corresponding optimal contracts satisfy the first-order optimality conditions in (14).

2. With the solution of sub-optimal problem (13) in hand, we can then try to show that it actually coincides with the value function of problem (12).
We need of course to say something about the choice of a pertinent subset of contracts in the first step described above. As explained in [18, 19], when one deals with a problem in which the volatility of the output is controlled by the Agent, contracts which are linear (in the sense of integration) with respect to the output $B$ and its quadratic variation $\langle B \rangle_T$ play a fundamental role. We thus hope (and expect) to have optimal contracts in the following set

$$\tilde{Q} := \left\{ \xi \in \tilde{C} : \xi = \int_0^T z_t dB_t + \frac{1}{2} \gamma_t \langle B \rangle_T + \delta_t, (z, \gamma, \delta) \in H_2^P(\mathbb{R}, \mathbb{F}) \times \hat{H}_1^P(\mathbb{R}, \mathbb{F}) \times H_1(\mathbb{R}, \mathbb{F}) \right\}.$$

In this case, the contract $\xi$ appears as the terminal value of a controlled diffusion process, and we expect that the risk-sharing problem (13) may be solved using technics from stochastic control theory. Such a general resolution is again beyond the scope of this paper, but we will illustrate our method by solving completely the simplest possible case (for which the proof is already far from being trivial).

### 3.2 Application to the Non-adaptative Model

In this section, we illustrate the previous explanations within the “non-adaptative” model introduced previously in Example 2.1. We will see then that we can actually simplify even more the set $\tilde{Q}$ above and introduce the set

$$Q := \left\{ \xi \in C : \xi = zB_T + \frac{1}{2} \langle B \rangle_T + \delta, (z, \gamma, \delta) \in \mathbb{R}^3 \right\}.$$

Notice that the contracts are assumed to be in $C$ and not in $\tilde{C}$. It will actually be one of our results that $Q \subset \tilde{C}$.

From now on, noticing that any contract $\xi$ in $Q$ is uniquely defined by the corresponding triplet of processes $(z, \gamma, \delta)$. For any triplet $(z, \gamma, \delta)$, we set $\xi_{z, \gamma, \delta} := zB_T + \frac{1}{2} \langle B \rangle_T + \delta$. We thus aim at solving the sub-optimal problem

$$U_0^{P, FB} := \sup_{(z, \gamma, \delta) \in \mathcal{P}(\mathbb{F})^3} \sup_{a \in \mathcal{A}} \left\{ \inf_{P \in \mathcal{P}_P} \mathbb{E}_{P} \left[ U_P(B_T - \xi_{z, \gamma, \delta}) \right] + \rho \inf_{P \in \mathcal{P}_A} \mathbb{E}_{P} \left[ U_A \left( \xi_{z, \gamma, \delta} - \int_0^T k(a_s) ds \right) \right] \right\}.$$

We insist on the fact that such a situation is different from the original Holmström–Milgrom [5] problem, where the first-best contract was linear in $B_T$, and is thus much closer to its recent generalisation in [18] where the Agent is allowed to control the volatility of the output, where optimal contracts are shown to be linear in $B_T$ and its quadratic variation $\langle B \rangle_T$. Nonetheless, in the setting of [18], moral hazard arises from the multidimensional nature of the output process, while it comes from the worst-case attitude of both the Principal and the Agent in our framework.
3.2.1 Degeneracy for Disjoint $\mathcal{P}_P$ and $\mathcal{P}_A$

Our first result shows that if the sets of ambiguity of the Principal and the Agent are completely disjoint, then there are sequences of contracts in $\mathcal{Q}$ such that the Principal can attain the universal upper bound 0 of his utility, while ensuring that the Agent still receives his reservation utility $R_0$.

**Theorem 3.1** (i) Assume that $\alpha_P < \alpha_A$. Then, considering the sequence of contracts $(\xi^n)_{n \in \mathbb{N}^*}$ and the recommended effort $a^{\text{max}}$, with

$$
\xi^n := \frac{1}{2} n (B)_T - \frac{T}{2} n \alpha^A + \delta^*, \quad \delta^* := Tk(a^{\text{max}}) - \frac{\log(-R)}{R_A},
$$

we have $\lim_{n \to +\infty} u_0^{P,\mathcal{FB}}(\xi^n, a^{\text{max}}) = 0$ and $u_0^A(\xi^n, a^{\text{max}}) = R$, for any $n \geq 1$.

(ii) Assume that $\alpha_P > \alpha_A$. Then, considering the sequence of contracts $(\xi^n)_{n \in \mathbb{N}^*}$ and the recommended effort $a^{\text{max}}$, with

$$
\xi^n := \frac{1}{2} n (B)_T + \frac{T}{2} n \alpha^A + \delta^*, \quad \delta^* := Tk(a^{\text{max}}) - \frac{\log(-R)}{R_A},
$$

we have $\lim_{n \to +\infty} u_0^{P,\mathcal{FB}}(\xi^n, a^{\text{max}}) = 0$ and $u_0^A(\xi^n, a^{\text{max}}) = R$, for any $n \geq 1$.

Before proving this result, let us comment on it. We will see during the proof that when the sets of uncertainty for the Principal and the Agent are completely disjoint, the Principal can use the quadratic variation component in the contract in order to make appear in the exponential a term which he can make arbitrarily large, but which is not seen at all by the Agent in his utility, as it is constructed so that it disappears under the worst-case probability measure the Agent. This is therefore the combination of this difference between the worst-case measures of the Principal and the Agent, as well as the fact that their uncertainty sets are disjoints which make the problem degenerate. This is, from a mathematical point of view, quite a surprising result, which is, however, not so surprising from the economics point of view. Indeed, first of all in this case the Agent and the Principal do not somehow live in the same world, since they have totally different beliefs. Moreover, none of them learns from the observation of the realised volatility and updates his beliefs, which makes this specific case rather crude. Especially since in this case, the volatility process will undoubtedly exit one of the uncertainty intervals, without the Agent or the Principal reacting to this clear proof that their beliefs are wrong. The result above can therefore be seen as a sanity check, proving that such a modelisation does not make sense from the economic point of view, and is also degenerate mathematically. We will prove later that this phenomenon also always happens in the second-best case.

**Proof** (i) **First case:** $\alpha^A > \alpha^P$. We aim at showing that the sequence of contracts $(\xi^n)$ is a maximising sequence of contracts, allowing the Principal to reach utility 0, when recommending in addition the level of effort $a^{\text{max}}$. We have
\[ u_0^P(\xi^n, a_{\text{max}}) \]
\[ = e^{-R_P(\xi^n A - \delta^*)} \inf_{P \in \mathcal{P}_{a_{\text{max}}}} \mathbb{E}^P \left[ -e^{-R_P(\int_0^T (\xi^n A) ds) + T a_{\text{max}} - \frac{1}{2} n \int_0^T \alpha_A ds} \right] \]
\[ = -e^{-R_P(\frac{T}{2} \xi^n A - \delta^*)} \sup_{P \in \mathcal{P}_{a_{\text{max}}}} \mathbb{E}^P \left[ \mathbb{E}^P \left[ -e^{-R_P(\int_0^T (\xi^n A) ds) + R_P^2 \int_0^T \alpha_{\xi^n A} ds + \frac{R_P}{2} \int_0^T \alpha_{\xi^n A} ds} \right] \right] \]
\[ = - \exp \left( -R_P \left( a_{\text{max}} T - \delta^* + \frac{T}{2} n (\alpha^A - \alpha^P) - \frac{1}{2} R_P T \alpha^P \right) \right), \]

where we have used the fact that for any \( P \in \mathcal{P}_{a_{\text{max}}} \), we have

\[ \exp \left( \frac{1}{2} R_P^2 \int_0^T \alpha_{\xi^n A} ds + \frac{R_P}{2} \int_0^T \alpha_{\xi^n A} ds \right) \leq \exp \left( \frac{T}{2} R_P^2 \alpha^P + \frac{R_P}{2} n T \alpha^P \right), \ P - a.s., \]

and that the stochastic exponential appearing above is clearly a \( P \)-martingale for any \( P \in \mathcal{P}_{a_{\text{max}}} \), so that the value of the supremum is clear and attained for the measure \( P_{a_{\text{max}}} \).

Hence, we obtain \( u_0^P(\xi^n, a_{\text{max}}) \rightarrow 0 \) when \( n \rightarrow +\infty \). Since \( U_{0,P}^{FB} \leq 0 \), we deduce that the sequence \( (\xi^n) \) approaches the best utility for the Principal when \( n \) goes to \( +\infty \). It remains to prove that for any \( n \in \mathbb{N}^* \), \( \xi^n \) is admissible, and satisfies the participation constraint. Indeed,

\[ \inf_{P \in \mathcal{P}_{a_{\text{max}}}} \mathbb{E}^P \left[ U_A(\xi^n - Tk(a_{\text{max}})) \right] \]
\[ = \inf_{P \in \mathcal{P}_{a_{\text{max}}}} \mathbb{E}^P \left[ - \exp \left( -R_A(\xi^n - Tk(a_{\text{max}})) \right) \right] \]
\[ = \inf_{P \in \mathcal{P}_{a_{\text{max}}}} \mathbb{E}^P \left[ - \exp \left( -R_A \left( \frac{1}{2} n \int_0^T \alpha_{\xi^n A} ds - \frac{T}{2} n \alpha^A + \delta^* - Tk(a_{\text{max}}) \right) \right) \right] \]
\[ = - \exp \left( -R_A \left( \delta^* - Tk(a_{\text{max}}) \right) - \frac{T}{2} n \alpha^A \right) \sup_{P \in \mathcal{P}_{a_{\text{max}}}} \mathbb{E}^P \left[ \exp \left( -R_A n \int_0^T \alpha_{\xi^n A} ds \right) \right] \]
\[ = -e^{-R_A(\delta^* - Tk(a_{\text{max}}))} = R. \]

(ii) *Second case:* \( \alpha^A < \alpha^P \). The proof is similar so that we omit it. \qed
3.2.2 Optimal Contracts with Intersecting Uncertainty Sets

We now study the non-degenerating case by applying Method 3.1. The proof of our main result below are based on fastidious but easy computations. We therefore omit all of them to simplify the reading of the paper and we refer the interested reader to the online version [37] for all the details. We introduce the following partition of the set $Q$

$$Q^\gamma := \{ \xi = (z, \gamma, \delta) \in Q : \gamma < -RP(1-z)^2 \},$$
$$Q^d := \{ \xi = (z, \gamma, \delta) \in Q : -RP(1-z)^2 = \gamma \},$$
$$Q^u := \{ \xi = (z, \gamma, \delta) \in Q : \gamma = RAz^2 \},$$
$$Q^\overline{\gamma} := \{ \xi = (z, \gamma, \delta) \in Q : \gamma > RAz^2 \},$$
$$Q^{\|\gamma\|} := \{ \xi = (z, \gamma, \delta) \in Q : -RP(1-z)^2 < \gamma < RAz^2 \}.$$

We can now give our main result stating that the optimal contract in the first-best problem belongs to $Q$.

**Theorem 3.2** We have

(i) Assume that $\alpha^A = \alpha^P$. Then, the set

$$\overline{Q^\gamma} := \{ \xi^* = (z^*, \gamma^*, \delta^*) \in Q : z^* = \frac{RP}{RA + RP}, \quad \gamma^* \geq RA|z^*|^2, \quad \delta^* = Tk(a^*) - \frac{RP}{RA + RP} T a^* + \frac{\alpha^P T}{2} \left( \frac{RA^2}{(RA + RP)^2} - \gamma^* \right) - \frac{1}{RA} \log(-R) \},$$

is the subset of optimal contracts in $Q$ for the first-best problem (10) with the optimal recommended level effort $a^* := \text{argmax} (k(a) - a)$.

(ii) Assume that $\alpha^A < \alpha^P < \alpha^A$. Then, an optimal contract is given by

$$\xi^* := z^* B_T + \frac{\gamma^*}{2} \langle B \rangle_T + \delta^*, \quad \gamma^* = RA(z^*)^2, \quad \text{and} \quad z^* := \frac{RP}{RA + RP}, \quad \delta^* := Tk(a^*) - \frac{RP}{RA + RP} T a^* - \frac{1}{RA} \log(-R).$$

(iii) Assume that $\alpha^A = \alpha^P$. Then, the set

$$\overline{Q^{\|\gamma\|}} := \{ \xi^* = (z^*, \gamma^*, \delta^*) \in Q : z^* = \frac{RP}{RA + RP}, \quad \gamma^* \in [-RP(1-z^*)^2, RA|z^*|^2], \quad \delta^* = Tk(a^*) - \frac{RP}{RA + RP} T a^* + \frac{\alpha^P T}{2} \left( \frac{RA^2}{(RA + RP)^2} - \gamma^* \right) - \frac{1}{RA} \log(-R) \}.$$
is the subset of optimal contracts in $Q$ for the first-best problem (10) with the optimal recommended level effort $a^\star := \arg\max (k(a) - a)$.

(iv) Assume that $\alpha^P = \overline{\alpha}^A$. Then, the set

$$
Q^\star := \left\{ \xi^\star = (z^\star, \gamma^\star, \delta^\star) \in Q : 
\begin{align*}
\frac{R_P}{R_A + R_P} T a^\star + \frac{\alpha^P T}{2} & \left( \frac{R_A R_p^2}{(R_A + R_P)^2} - \gamma^\star \right) \\
- \frac{1}{R_A} \log(-R) \end{align*}
\right\},
$$

is the subset of optimal contracts in $Q$ for the first-best problem (10) with the optimal recommended level effort $a^\star := \arg\max (k(a) - a)$.

(v) Assume that $\overline{\alpha}^P < \overline{\alpha}^A < \overline{\alpha}^P$. Then, an optimal contract is given by

$$
\xi^\star := z^\star B_T + \gamma^\star \langle B \rangle_T + \delta^\star,
$$

where $\gamma^\star = -R_P|1 - z^\star|^2$, and

$$
\begin{align*}
z^\star & := \frac{R_P}{R_A + R_P}, \quad \delta^\star := Tk(a^\star) - \frac{R_P}{R_A + R_P} T a^\star + \frac{\overline{\alpha}^A T}{2} \frac{R_A R_P}{R_A + R_P} \\
& \quad - \frac{1}{R_A} \log(-R).
\end{align*}
$$

### 3.2.3 Comments and Comparison with the Case without Ambiguity

Using Theorem 3.2, we recover the classical result that when $\alpha^P = \overline{\alpha}^P = \alpha_A = \overline{\alpha}^A =: \alpha$ (that is to say when there is no ambiguity), the optimal first-best contract is given by

$$
z^\star B_T + \frac{R_A R_p^2 \alpha}{2(R_A + R_P)^2} T + Tk(a^\star) - \frac{R_P}{R_A + R_P} T a^\star - \frac{1}{R_A} \log(-R), \quad (16)
$$

which provides the Principal with utility

$$
-(-R_0) \frac{R_P}{R_A} \exp \left( R_P T \left( k(a^\star) - a^\star + \alpha \frac{R_A R_P}{2 (R_A + R_P)} \right) \right). \quad (17)
$$

Therefore, as mentioned above, the first main difference with the ambiguity case is that in our framework, one has in general to rely on path-dependent contracts using the quadratic variation of the output. There is nonetheless an exception. Indeed, in the case where $\overline{\alpha}^A = \overline{\alpha}^P$, the choice $\gamma^\star = 0$ is allowed, so that there is a linear optimal contract in this case (which coincides with (16)), and in this case only. Furthermore, in the three cases $\overline{\alpha}^A = \overline{\alpha}^P$, $\overline{\alpha}^A = \alpha^P$, $\overline{\alpha}^P = \alpha^A$, we have identified uncountably many optimal
contracts in the class $Q$. This is really different from the case without ambiguity, where the optimal contract is essentially unique. This phenomenon may require to add a Planner (exogenous or inside the Principal/Agent system) who provides criterion to select among the set of optimal contracts.

Finally, let us compare the utility of the Principal can get out of the problem (since the Agent always receives his reservation utility, there is nothing to compare for him). Again by Theorem 3.2, whenever we have $\alpha^A \leq \bar{\alpha}^P \leq \alpha^A$, the Principal receives

$$-(R)^{-\frac{R_P}{\bar{\alpha}_A}} \exp \left( R_P T \left( k(a^*) - a^* + \frac{\bar{\alpha}^P}{2} \frac{R_A R_P}{R_A + R_P} \right) \right),$$

which is always less than (17), for any $\alpha \in [\bar{\alpha}^P, \alpha^P]$, which means that, as intuition would dictate, the Principal is worse off compared to the case where he would not have any aversion to ambiguity.

Then, when we have $\alpha^P \leq \bar{\alpha}^A \leq \alpha^P$, the Principal gets

$$-(R)^{-\frac{R_P}{\bar{\alpha}_A}} \exp \left( R_P T \left( k(a^*) - a^* + \frac{\bar{\alpha}^A}{2} \frac{R_A R_P}{R_A + R_P} \right) \right),$$

which is actually larger than (17) if $\alpha \geq \bar{\alpha}^A$. In other words, compared to a situation where the Principal would have no ambiguity, but were more pessimistic than the Agent and believed in a level of volatility higher than $\bar{\alpha}^A$, the ambiguity-averse Principal actually obtains a larger utility.

The situation is the same, though even more extreme, when $\bar{\alpha}^P < \alpha^A$ or $\alpha^A < \alpha^P$, since the Principal can reach utility 0 and is therefore always better off compared to the case without ambiguity. We nonetheless insist once more on the fact that these results are obviously in part due to the “non-learning” assumption we made on both the Principal and the Agent, and a deeper understanding of the problem would obviously be achieved from studying more realistic situations. We emphasise that in the second-best problem treated below, we will give general results allowing to solve completely the problem for general ambiguity sets, with solutions which are amenable to numerical computations, thus opening the door to such an exploration.

4 Moral Hazard and the Second-Best Problem

We now study the so-called second-best problem, corresponding to a Stackelberg-like equilibrium between the Principal and the Agent. Now, the Principal has no control (or cannot observe) the effort level chosen by the Agent. Hence, his strategy is to first compute the best-reaction function of the Agent to a given contract, and to determine his corresponding optimal effort (if it exists) and then use this in his own utility function to maximise over all the contracts. Obviously, the above approach can only work if the Principal can actually find the optimal effort of the Agent. Therefore, the set of admissible contracts in the second-best setting must at least be reduced to the contracts $\xi$ such that there exists (possibly several) $a^* \in A$ with
As we will see below, this set of contracts is actually equal to \( \mathcal{C} \), so that the above restriction is without loss of generality.

**Remark 4.1** Before turning to the solution to the moral hazard problem, notice that solving the problem of the Agent only involves looking at the contract \( \xi \) on the support of his beliefs set \( \mathcal{P}_A \). Therefore, the only information about \( \xi \) that we can obtain from solving the Agent’s problem will be in a \( \mathcal{P}_A \)-quasi sure sense. Therefore, the Principal will always have a degree of freedom when choosing the contracts on the support of \( \mathcal{P}_P \setminus \mathcal{P}_A \). This will be important later on. Indeed, the intuition then dictates that since the Principal can penalise infinitely the Agent outside of the support of the measures associated with his beliefs, the only thing that should matter is what happens on the intersection of \( \mathcal{P}_A \) and \( \mathcal{P}_P \). We will prove this fact rigorously, and that when this intersection is empty, the problem degenerates again as in the first best, so that this meaningless case, from the economic point of view, is immediately ruled out by our approach to the problem.

The main steps of the resolution procedure we will follow are

- First, we solve the problem of the Agent by solving a utility maximisation problem with ambiguity on the volatility, by reducing it to the solution of a 2BSDE. At this step, given a very general contract \( \xi \) without any restriction on it, we can find the best-reaction effort of the Agent in terms of the solution \( (Y, Z, K) \) of this 2BSDE.
- Secondly, in view of the best-reaction effort of the Agent found previously, the problem of the Principal seems to be hard to solve, due to the fact that the process \( K \) obtained in the 2BSDE is singular in general, and satisfies a complicated minimality condition. This suggests to introduce a restrictive class of contracts for which the controlled process \( K \) has a smooth decomposition.
- Finally, we show that there is no restriction to consider contracts in the restrictive set of smooth \( K \).

### 4.1 The Agent’s Problem

The aim of this section is to prove that despite the generality of our setting, the utility of the Agent, as well as his optimal effort, can always be characterised completely for any admissible contract \( \xi \in \mathcal{C} \). Our result relies essentially on the recent theory of second-order BSDEs, introduced by Soner, Touzi and Zhang [35], and revisited in a framework suitable for our purpose by Possamaî, Tan and Zhou [36].

Before starting, we will need to introduce the following spaces

- \( \mathcal{D}^{\exp}_{\mathcal{P}_A} \) is the set of processes \( Y, \mathcal{G}^{\mathcal{P}_A}_t \)-progressively measurable, \( \mathcal{P}_A \)-q.s. càdlàg, and such that

\[
\sup_{\mathcal{P} \in \mathcal{P}_A} \mathbb{E}^\mathcal{P} \left[ \exp \left( p \sup_{0 \leq t \leq T} |Y_t| \right) \right] < +\infty, \; \forall p \geq 0.
\]
\( \mathbb{P}_A \) is the set of processes \( K, G^P_A \)-predictable, \( P_A \) \( q.s. \) càdlàg, non-decreasing, null at 0 with
\[
\sup_{P \in P_A} \mathbb{E}^P[K_T] < +\infty, \ \forall p \geq 0.
\]
Let us next introduce the following 2BSDE, holding for any \( t \in [0, T] \), and \( P_A \) \( q.s. \).
\[
Y_t = \xi - \int_t^T \left( \frac{RA}{2} |Z_s|^2 \hat{\alpha}_s + \inf_{a \in [0, a_{\max}]} \{k(a) - aZ_s\} \right) ds - \int_t^T Z_s \hat{\alpha}_s^{1/2} dW_s - \int_t^T dK_s. \tag{18}
\]
We say that the triplet \( (Y, Z, K) \in \mathbb{D}_{P_A}^{\text{exp}} \times \cup_{p \geq 0} \mathbb{H}^P_A (\mathbb{R}, G^P_A) \times \mathbb{I}_{P_A} \) is the maximal solution to (18) if it indeed satisfies (18) \( P_A \) \( q.s. \), if \( K \) satisfies the following minimality condition for any \( P \in P_A \)
\[
K_t = \operatorname{essinf}_{P' \in P_A(t^+)} \mathbb{E}^{P'} \left[ K_T | \mathcal{F}_t \right], \ \ P - a.s., \tag{19}
\]
and if for any other solution \( (Y', Z', K') \in \mathbb{D}_{P_A}^{\text{exp}} \times \cup_{p \geq 0} \mathbb{H}^P_A (\mathbb{R}, G^P_A) \times \mathbb{I}_{P_A} \), we have \( Y_t \geq Y'_t, \ t \in [0, T], P_A \) \( q.s. \).

Our main result for this section is then the following representation, whose proof is deferred to Appendix.

**Proposition 4.1** For any \( \xi \in \mathcal{C} \), the value function of the Agent verifies
\[
U_A^0(\xi) = -\exp(-RA Y_0), \quad \text{and the optimal effort of the Agent is given by the unique} \quad (a^*(Z_s))_{s \in [0, T]} \quad \text{which satisfies}
\[
\inf_{a \in [0, a_{\max}]} \{k(a) - aZ_s\} = k(a^*(Z_s)) - a^*(Z_s)Z_s, \ s \in [0, T],
\]
where \( (Y, Z) \) is the maximal solution to (18). Furthermore, \( \xi \in \mathcal{C} \) if and only if
\[
Y_0 \geq -\frac{\log(-R)}{RA} =: R_0. \tag{20}
\]

**4.2 Admissible Contracts**

We have thus solved the problem of the Agent for any \( \xi \in \mathcal{C} \), in the sense that we have systematically found his optimal action for a given \( \xi \in \mathcal{C} \). Along the way, we proved that any \( \xi \in \mathcal{C} \) had the following decomposition
\[
\xi = Y_0 + \int_0^T g(Z_s, \hat{\alpha}_s) ds + \int_0^T Z_s \hat{\alpha}_s^{1/2} dW_s + \int_0^T dK_s, \ P_A \ q.s., \tag{21}
\]
for some \( Z \in \cup_{p \geq 0} \mathbb{H}_p^P(A, \mathbb{G}^P_A) \), \( Y_0 \geq R_0 \) and some \( K \in \mathbb{P}_A \) satisfying the minimality condition (19), where we have defined for simplicity for any \((z, a) \in \mathbb{R}^2\)

\[
g(z, a) := \frac{R_A}{2} |z|^2 a + \inf_{a' \in [0,a_{\max}]} \{ k(a') - a'z \}.
\]

Notice that, as already mentioned in Remark 4.1, we only retrieved information about \( \xi \) in a \( \mathcal{P}_A-q.s. \) sense.

Let \( \hat{\mathcal{C}} \) be the set of random variables \( \xi \) such that there exist processes \((Z, K) \in \cap_{p \geq 0} \mathbb{H}_p^P(A, \mathbb{G}^P_A) \times \mathbb{P}_A \) and some \( \hat{\xi} \in \mathcal{C} \) such that

\[
\xi = \begin{cases} 
Y_0, Z, K := Y_0 + \int_0^T g(Z_s, \hat{\alpha}_s) ds + \int_0^T Z_s \hat{\alpha}_s^{1/2} dW_s + K_T, \mathcal{P}_A-q.s., \\
\hat{\xi}, \mathcal{P}_P \setminus \mathcal{P}_A-q.s. 
\end{cases}
\] (22)

The above reasoning proves that \( \hat{\mathcal{C}} \cap \mathcal{C} = \mathcal{C} \), and that an arbitrary element of \( \hat{\mathcal{C}} \) will belong to \( \mathcal{C} \) if \((Z, K)\) satisfy appropriate integrability conditions ensuring that \( \xi \) satisfies (9). We let \( \mathcal{K} \) be the corresponding set of processes \( Z \) and \( K \), which, for technical reasons, verify in addition that

\[
\mathcal{E}(-R_P \int_0^T \frac{1}{2} \hat{\alpha}_s^2 (1 - Z_s) dW_s^{q^*(Z)}) \text{ is a } \mathbb{P}-\text{martingale, } \forall \mathbb{P} \in \mathcal{P}_P^{q^*(Z)}.
\]

Finally, we will take as admissible contracts the following class

\[
\mathcal{C}^{SB} := \left\{ \xi \in \mathcal{C} : \exists (Y_0, Z, K) \in [R_0, +\infty] \times \mathcal{K}, \xi = Y_0 + \int_0^T g(Z_s, \hat{\alpha}_s) ds + Z_s \hat{\alpha}_s^{1/2} dW_s + K_T, \mathcal{P}_A-q.s. \right\}
\]

Remark 4.2 2BSDEs are actually fundamental to our approach. Whatever one does, the problem of the Agent is always non-Markovian with respect to the output variable \( B \), as it would be in general way to restrictive to consider that compensation of the Agent should only be based on the terminal value of the output \( B_T \). Any non-trivial model, except for the classical Holmström and Milgrom model [5], actually leads to optimal contracts which are not Markovian in the output (this is the case already in Sannikov’s [10] celebrated model). However, it generally turns out that despite this fact, the problem of the Principal becomes Markovian in both the output and the continuation utility of the Agent (which itself is a non-Markovian function of the output). This is why the continuous-time literature on moral hazard can use PDE techniques to treat the problem of the Principal. Now, in order to prove rigorously that the value function of the Principal indeed only depends on these two state variables, one has to understand thoroughly the non-Markovian problem of the Agent, which is where the 2BSDEs theory becomes necessary in our approach.
4.3 The Principal’s Problem

4.3.1 General Formulation of the Problem and Degeneracies

Define for simplicity of notations

\[ P_a \star (Z \cdot P_a \cap P) : \quad P_a \star (Z \cdot P_a \setminus P) \]

From Remark 4.1 and the previous section, the Principal’s problem can always be written as

\[
U_0^P := \sup_{(Y_0, Z, K) \in [0, \infty) \times K \times C} \min \left\{ U_0^P (\xi Y_0, Z, K), \inf_{P \in P_a \star (Z \cdot P)} \mathbb{E}_P [U_P (B_T - \xi)] \right\}.
\]

(23)

with

\[
U_0^P (\xi Y_0, Z, K) := \inf_{P \in P_a \star (Z \cdot P)} \mathbb{E}_P [U_P (B_T - \xi Y_0, Z, K)]
\]

To simplify our study, we set the following assumption, which merely imposes a boundedness property on the volatilities in which the Principal believes.

**Assumption 4.1** There exists some positive constant \( M \) such that for any \((t, \omega, \mathbb{P}) \in [0, T] \times \Omega \times \mathbb{P}(t, \omega),\)

\[
0 \leq \hat{\alpha}_s \leq M, \quad \mathbb{P} - a.s.
\]

We then have the following result, which states that as soon as the beliefs of the Principal and the Agent are disjoint, the problem degenerates, and the Principal can obtain his maximal possible utility, that is to say 0. In other words, we are in a situation that is reminiscent of an arbitrage opportunity, and should be ruled out as meaningless in our setting.

**Proposition 4.2** Let Assumption 4.1 hold. If \( \mathcal{P}_{P} \cap \mathcal{P}_{A} = \emptyset \), then \( U_0^P = 0 \).

**Proof** First notice that from (4), we automatically have

\[
U_0^P = \sup_{\xi \in \mathcal{C}} \inf_{P \in P_a \star (Z \cdot P)} \mathbb{E}_P [U_P (B_T - \xi)].
\]

Let \( c \in \mathbb{R} \) be such that \(-e^{-RA(c-k(0)T)} = R\). We define the following contract in \( \mathcal{C} \).

\[
\xi^n = \begin{cases} 
  c, & \mathcal{P}_A - q.s., \\
  -n, & \mathcal{P}_P \setminus \mathcal{P}_A - q.s.
\end{cases}
\]

(24)
The utility that the Agent receives is then

\[ U^A_0(c) = \sup_{a \in A} E^P \left[ -e^{-R_A (c - \int_0^T k(a_s) ds)} \right] = -e^{-R_A (c - k_0 T)} = R, \]

using the definition of \( c \). We now turn to the Principal’s utility. We directly have for any \( a \in A \)

\[ U^P_0 \geq \inf_{P \in \mathcal{P}_p \setminus \mathcal{P}_A} E^P \left[ \mathcal{U}_P \left( B_T + n \right) \right] \]

\[ = e^{-R_P n} \inf_{P \in \mathcal{P}_p \setminus \mathcal{P}_A} E^P \left[ -e^{-R_P B_T} \right] \]

\[ = e^{-R_P n} \inf_{P \in \mathcal{P}_p \setminus \mathcal{P}_A} E^P \left[ -E \left( -R_P \int_0^T \hat{\alpha}^a_1 dW^a_s \right) e^{-R_P \int_0^T a_s ds} \right] \]

\[ \geq -e^{-R_P n + R_P^2 T / 2}. \]

Thus, \( U^P_0 \geq \lim_{n \to +\infty} \inf_{P \in \mathcal{P}_p \setminus \mathcal{P}_A} E^P \left[ \mathcal{U}_P \left( B_T + n \right) \right] = 0. \) Since the reverse inequality is trivial, this ends the proof.

The case \( \mathcal{P}_p \cap \mathcal{P}_A = \emptyset \) can be seen as a non-realistic approach to the problem, since the Agent and the Principal have completely disjoint estimates on the volatility. We now turn to a more realistic case, where we assume that \( \mathcal{P}_p \cap \mathcal{P}_A \neq \emptyset \). We then have the following proposition which simplifies greatly the problem of the Principal (23). Again, let us insist on the fact that though intuitively expected, this results needed to be proved rigorously.

**Proposition 4.3** Let Assumption 4.1 hold. If \( \mathcal{P}_p \cap \mathcal{P}_A \neq \emptyset \),

\[ U^P_0 = \sup_{(Y_0, Z, K) \in [0, \infty] \times \mathbb{R}} \inf_{P \in \mathcal{P}_p \setminus \mathcal{P}_A} E^P \left[ \mathcal{U}_P \left( B_T - \hat{\xi} Y_0, Z, K \right) \right]. \]  

**Proof** We have \( U^P_0 = \sup_{\xi \in C^{SB}} \tilde{U}^P_0 (\xi) \), where for any \( \xi \in C^{SB} \)

\[ \tilde{U}^P_0 (\xi) := \sup_{\xi \in C^{SB}} \min \left\{ U^P_0 (\xi), \inf_{P \in \mathcal{P}_p \setminus \mathcal{P}_A} E^P \left[ \mathcal{U}_P \left( B_T - \hat{\xi} \right) \right] \right\}. \]

Notice that the first term in the minimum does not depend on \( \hat{\xi} \). Furthermore, we know by Proposition 4.2 that for any \( \xi \in C^{SB} \)

\[ 0 = \sup_{\xi \in C^{SB}} \inf_{P \in \mathcal{P}_p \setminus \mathcal{P}_A} E^P \left[ \mathcal{U}_P \left( B_T - \hat{\xi} \right) \right] \geq U^P_0 (\xi). \]

Therefore, \( \tilde{U}^P_0 (\xi) = U^P_0 (\xi). \)
From now on, we will always assume that \( \mathcal{P}_P \cap \mathcal{P}_A \neq \emptyset \) and that Assumption 4.1 holds, since we have already solved the other case in Proposition 4.2.

### 4.3.2 A Sub-Optimal Version of (25)

From (22) and Proposition 4.3, we know that we can restrict our attention to contracts of the form \( \xi_{Y_0, Z, K} \). However, in order to solve this problem, we actually need to have more information on the non-decreasing process \( K \). Using similar intuitions as the ones given in [18, 19], we expect that when the contract \( \xi \) is sufficiently “smooth”, we can find a \( \mathbb{G}^{\mathcal{P}A} \)-predictable process \( \Gamma \) such that for every \( \mathbb{P} \in \mathcal{P}_{A}^{\star}(Z, K) \)

\[
K_t = \int_0^t \left( \frac{1}{2} \alpha_s \Gamma_s - \inf_{a \in \mathbb{R}} \left\{ \frac{1}{2} a \Gamma_s + 1_{a \in \mathcal{D}(s, B)} \right\} \right) ds,
\]

where the indicator function \( 1_{a \in \mathcal{D}(t, \omega)} \) is the one from convex analysis, and is equal to 0 when \( a \) indeed belongs to \( \mathcal{D}(t, \omega) \), and \( +\infty \) otherwise. However, in general, such a decomposition for \( K \) is not true for every \( \xi \in \mathcal{C}^{SB} \). We will therefore start by solving the Principal problem for a particular subclass of contracts in \( \mathcal{C}^{SB} \) such that the process \( \Gamma \) exists, and then show, under appropriate assumptions, that the Principal’s value function is not actually affected by this restriction. For simplicity, we denote by \( \mathcal{K} \) the set of processes \( Z \) and \( \Gamma \) such that \( \Gamma \) is \( \mathbb{G}^{\mathcal{P}A} \)-predictable and \( (Z, K_{\Gamma}) \in \mathcal{K} \), where

\[
K_{\Gamma}^t := \int_0^t \left( \frac{1}{2} \alpha_s \Gamma_s - \inf_{a \in \mathbb{R}} \left\{ \frac{1}{2} a \Gamma_s + 1_{a \in \mathcal{D}(s, B)} \right\} \right) ds.
\]

Building upon (21) and (26), we consider the class \( \mathcal{C}^{SB} \subset \mathcal{C}^{SB} \) of contracts \( \xi \) admitting the decomposition \( \xi = Y_{t, Z, \Gamma} \) for some \( Y_0 \geq R_0 \), and some \( (Z, \Gamma) \in \mathcal{K} \), where for any \( t \in [0, T] \), and \( \mathcal{P}_A - q.s. \)

\[
Y_{y, Z, \Gamma} := Y_0 + \int_0^t \left( \frac{1}{2} \alpha_s \Gamma_s - \inf_{a \in \mathbb{R}} \left\{ \frac{1}{2} a \Gamma_s + 1_{a \in \mathcal{D}(s, B)} \right\} + g(Z_s, \alpha_s) \right) ds
+ \int_0^t Z_s \alpha_s^{1/2} dW_s.
\]

**Remark 4.3** Let us say a word about implementability of the contracts in the class \( \mathcal{C}^{SB} \). They can actually be rewritten, \( \mathcal{P}_A - q.s. \), as follows

\[
Y_{T, Z, \Gamma} = Y_0 + \frac{1}{2} \int_0^T \Gamma_s d(B)_s - \int_0^T \left( \inf_{a \in \mathbb{R}} \left\{ \frac{1}{2} a \Gamma_s + 1_{a \in \mathcal{D}(s, B)} \right\} + g(Z_s, \alpha_s) \right) ds
+ \int_0^T Z_s dB_s.
\]

Therefore, the Principal should be able to reward the Agent using the path of \( B \), which could easily be done in practice using stocks on the value \( B \) of the firm, or forward
contracts, for instance. (One would then of course have to approximate the stochastic integral by a finite Riemann sum.) The term involving the quadratic variation is more complex, however. Since it is deeply linked to the volatility of the output, one could try to replicate it using stock options. This is in line with the classical managerial compensation using stock options, for instance, as the latter are clearly correlated with realised volatility; see, for instance, Coles et al. [47] for associated empirical studies. It can also be interpreted as a sort of volatility swap arrangement between the Principal and the Agent, transferring volatility uncertainties from one party to the other.

Another important point to realise is that, in general, the Principal can substitute in the contract the quadratic variation by
\[ d(B^2_t) \] instead, without losing too much utility. This point has been raised recently by Aïd, Possamaï and Touzi [48] in a Principal–Agent problem for electricity pricing where the Agent (the client) controls the variability of his consumption. Their numerical results show that the loss of utility for the Principal when doing this substitution is not that large in many situations. We believe that this result should still hold in our framework, which would provide a more practical way to implement the optimal contract.

We define the following sub-optimal problem for the Principal

\[
 U^P_0 := \sup_{Y_0 \geq R_0} U^P_0(Y_0), \tag{28}
\]
where for any \( Y_0 \geq R_0 \)

\[
 U^P_0(Y_0) := \sup_{(Z, \Gamma) \in \mathcal{K}} \inf_{P \in \mathcal{P} \cap \mathcal{P}_A} \mathbb{E}_P \left[ U_P \left( B_T - Y_0^T - Z_t^T - \Gamma_t^T \right) \right].
\]

Notice first that by linearity of \( Y_0^T, Z_t^T, \Gamma_t^T \) in \( Y_0 \) and the fact that \( U_P \) is non-decreasing, we deduce immediately that \( U^P_0(R) = U^P_0(R_0) \). Now, all the interest of concentrating on \( U^P_0(R) \) is that it is simply the value function of zero-sum stochastic differential game, under weak formulation, with the two controlled state variables \( B_t \) and \( Y_0^T, Z_t^T, \Gamma_t^T \), with controls \( (Z, \Gamma) \in \mathcal{K} \) for the Principal and \( \alpha \) for the “Nature”, and dynamics

\[
 B_t = \int_0^t a^*(Z_s) \, ds + \int_0^t \tilde{\alpha}_s^{1/2} \, dW^a_s, \quad t \in [0, T], \ \mathcal{P}_A \cap \mathcal{P}_P - q.s.,
\]
\[
 Y_t^{R_0, Z_t^T, \Gamma_t^T} = R_0 + \int_0^t \left( \frac{1}{2} \tilde{\alpha}_s \Gamma_s^T - \inf_{a \in \mathbb{R}} \left\{ \frac{1}{2} a \Gamma_s + 1_{a \in D(s, B)} \right\} \right) \, ds
\]
\[
 + \frac{R_A}{2} \tilde{\alpha}_s [Z_s]^2 + k(a^*(Z_s))) \, ds
\]
\[
 + \int_0^t Z_s \tilde{\alpha}_s^{1/2} \, dW^a_s, \quad t \in [0, T], \ \mathcal{P}_A \cap \mathcal{P}_P - q.s.
\]

Under this form, the sub-optimal problem of the Principal becomes amenable to the dynamic programming approach to differential games, and in particular, in a Markovian setting, \( U^P_0 \) can be linked to the associated Hamilton–Jacobi–Bellman–Isaacs

\[ \square \] Springer
(HJBI for short) PDE. The aim of the next section is to take advantage of this representation of the value function to prove that the restriction to contracts in $C_{SB}$ is, under natural assumptions, without loss of generality.

### 4.3.3 The Hamilton–Jacobi–Bellman–Isaacs Approach

As mentioned above, we now restrict our attention to the Markovian setting, which requires the following assumption.

**Assumption 4.2 (M)** For $\Psi \in \{A, P\}$ and any $(s, \omega) \in [0, T] \times \Omega$, we have, abusing notations slightly

$$D_\Psi (s, \omega) = D_\Psi (s, B_s (\omega)).$$

We next define for any $0 \leq t \leq s \leq T$, $(Z, K) \in \mathcal{K}$ and $y \in \mathbb{R}$

$$Y^{t, y, Z, K}_s := y + \int_t^s \left( \frac{RA}{2} |Z_r|^2 \bar{\alpha}_r + \inf_{a \in [0, a_{\max}]} \{k(a) - aZ_r\} \right) dr$$

$$+ \int_t^s Z_r \tilde{\alpha}_r^{1/2} dW_r + \int_t^T dK_r.$$

Since our setting is now Markovian, we can simplify the notations for the ambiguity sets of the Principal and the Agent to $\mathcal{P}_P (t, x)$ and $\mathcal{P}_A (t, x)$, for any $(t, x) \in [0, T] \times \mathbb{R}$. Consider now, with obvious notations, the dynamic version of the value function of the Principal

$$u(t, x, y) := \sup_{(Z, K) \in \mathcal{K}} \inf_{\mathbb{P} \in \mathcal{P}_A^\star (Z, x)} \mathbb{E}_\mathbb{P} \left[ \mathcal{U}_P \left( B_T - Y^{t, y, Z, K}_T \right) \right].$$

Since it is clear from the definition of $Y^{t, y, Z, K} = y + Y^{t, 0, Z, K}$, we deduce immediately that

$$u(t, x, y) = -e^{R_T y} u(t, x),$$

where

$$u(t, x, y) := \inf_{(Z, K) \in \mathcal{K}} \sup_{\mathbb{P} \in \mathcal{P}_A^\star (Z, x)} \mathbb{E}_\mathbb{P} \left[ e^{-R_T (B_T - Y^{t, 0, Z, K}_T)} \right].$$ (29)

We are now going to make a series of assumptions concerning the function $u$ defined above. These assumptions will be related to standard properties of stochastic control.

---

6 Actually, our approach would also work in non-Markovian case, provided that one uses the recently developed theory of viscosity solutions for path-dependent PDEs, in a series of papers by Ekren, Keller, Ren, Touzi and Zhang [49–51]. We preferred to present our arguments in the Markovian case to avoid additional technicalities. See, however, Sect. 5 for a specific non-Markovian case.
or stochastic differential games, and although expected and standard, they may be quite hard to prove in an extremely general setting. We will comment on this further after the statement of the assumptions themselves.

**Assumption 4.3 (PPD)** The map \( v \) is continuously differentiable in time on \([0, T]\), and twice continuously differentiable with respect to \( x \) on \( \mathbb{R} \). Besides, for any family \( \{ \theta^{Z,K,P} : (Z, K, P) \in \mathcal{K} \times \mathcal{P}_{PA}^{\mathbb{R}^4(Z)} \} \) of stopping times independent of \( \mathcal{F}_t \), we have

\[
v(t, x) = \inf_{(Z,K) \in \mathcal{K}} \sup_{P \in \mathcal{P}_{PA}^{\mathbb{R}^4(Z)}(t,x)} \mathbb{E}^P \left[ v(\theta^{Z,K,P}, B_{\theta^{Z,K,P}}) e^{R_P Y_{t,0,Z,K,P}^{0,Z,K,P}} \right]. \tag{30}
\]

Furthermore, there exists \((Z^*, K^*) \in \mathcal{K}\) such that

\[
v(t, x) = \sup_{P \in \mathcal{P}_{PA}^{\mathbb{R}^4(Z^*)}(t,x)} \mathbb{E}^P \left[ v(\theta^{Z^*,K^*,P}, B_{\theta^{Z^*,K^*,P}}) e^{R_P Y_{t,0,Z^*,K^*,P}^{0,Z^*,K^*,P}} \right]. \tag{31}
\]

First of all, the smoothness assumption on \( v \) is for simplicity and because we do not want to add an extra layer of technicalities by having to rely on the notion of viscosity solutions. However, the same line of reasoning would still go through in that case. Next, roughly speaking, (30) means that we are assuming that \( v \) satisfies the dynamic programming principle. For standard stochastic control problems, this is a result known to hold in extremely general settings (see, for instance, [52] for a recent account). However, it has been known that this is a much more complex problem for differential games, since the seminal paper on the subject by Fleming and Souganidis [53]. However, such results have already been proved in the literature, for instance, by Buckdahn and Li [54], or Bouchard, Moreau and Nutz [55]. The fact that we take it as an assumption here is once more mainly for simplicity, and because we want to give a general idea on the strategy of proof that we think should be used. The verification itself of whether the dynamic programming principle holds should be done on a case by case basis. We also want to insist on the fact that we only require this principle to hold in order to be able to relate the value function of the game to the associated HJBI PDE. Therefore, any other approach not requiring this principle and still allowing to prove such a relationship would work for us. This would be the case for the so-called stochastic viscosity solutions introduced by Bayraktar and Sîrbu; for instance, see [56–58]. Finally, Relation (31) simply stipulates that there is an optimal control in the maximisation part of the problem of the Principal. This could be in principle relaxed to the existence of \( \varepsilon \)-optimal controls.

Define next the map \( G : [0, T] \times \mathbb{R}^6 \times \mathbb{R}_+ \longrightarrow \mathbb{R} \) for any \((t, x, v, p, q, z, \gamma, \alpha) \in [0, T] \times \mathbb{R}^5 \times \mathbb{R}_+ \) by

\[
G(t, x, v, p, q, z, \gamma, \alpha) = a^*(z)p + \left( \frac{R_A}{2} \alpha |z|^2 + k(a^*(z)) \right)
+ \frac{1}{2} \alpha \gamma - \inf_{\tilde{a} \in \mathbb{R}_+} \left\{ \frac{1}{2} \tilde{a} \gamma + 1_{\tilde{a} \in \mathcal{D}_A(t, x)} \right\} R_P v
+ \frac{1}{2} \alpha q + \frac{1}{2} |z|^2 R_P^2 v + \alpha z R_P v + 1_{\alpha \in \mathcal{D}_P(t, x)}.
\]
We can then introduce the following HJBI equation, which should be related to our problem.

\[- \partial_t \psi(t, x) - \sup_{\alpha \in \mathbb{R}^+} \inf_{(z, \gamma) \in \mathbb{R}^2} G(t, x, \psi, \partial_x \psi, \partial_{xx} \psi, z, \gamma, \alpha) = 0, \quad (t, x) \in [0, T] \times \mathbb{R}, \]

\[\psi(T, x) = e^{-R P x}, \quad x \in \mathbb{R}. \quad (32)\]

Our strategy now is to prove that both the value function of the original and sub-optimal problems of the Principal solve this PDE. Then, by a uniqueness argument (which will require a further assumption), we will be able to affirm that they are equal. First, we introduce the notion of (classical) super-solution to PDE (32).

**Definition 4.1** A map \(v\) from \([0, T] \times \mathbb{R}\) into \(\mathbb{R}\) is a smooth super-solution to PDE (32) if \(v\) is once continuously differentiable in time and twice continuously differentiable with respect to \(x\) and satisfies

\[- \partial_t v(t, x) - \sup_{\alpha \in \mathbb{R}^+} \inf_{(z, \gamma) \in \mathbb{R}^2} G(t, x, v(t, x), \partial_x v(t, x), \partial_{xx} v(t, x), z, \gamma, \alpha) \geq 0, \quad (t, x) \in [0, T] \times \mathbb{R}, \]

\[v(T, x) = e^{-R P x}, \quad x \in \mathbb{R}. \quad (33)\]

We now assume that a comparison theorem for the HJBI equation (32) holds, and that the latter admits a unique classical solution, which by a standard verification argument would then be equal to the dynamic version of the value function of the sub-optimal problem of the Principal. This is why (34) holds in this case.

**Assumption 4.4 (C)** There exists a unique smooth solution \(\psi\) to PDE (32) such that \(\psi\) is once continuously differentiable with respect to time and twice continuously differentiable with respect to \(x\) such that

\[- e^{R P R_0} \psi(0, 0) = U_0^P. \quad (34)\]

Moreover, assume that \(v\) is a smooth super-solution to PDE (32) then for all \((t, x) \in [0, T] \times \mathbb{R}\) we have \(\psi(t, x) \leq v(t, x)\).

We finally consider an extra technical assumption, ensuring that the maximum in the Hamiltonian of the HJBI equation is attained and that the corresponding maximiser is sufficiently “smooth”.

**Assumption 4.5 (A)** We assume that for any \((t, x, v, p, q) \in [0, T] \times \mathbb{R}^4\), there exists a maximiser \(\tilde{\alpha}(t, x, v, p, q)\) for the map

\[\alpha \mapsto \inf_{(z, \gamma) \in \mathbb{R}^2} G(t, x, v, p, q, z, \gamma, \alpha),\]

such that the following SDE has a unique strong solution

\[X_t = \int_0^t \tilde{\alpha}(s, X_s, v(s, X_s), \partial_x v(s, X_s), \partial_{xx} v(s, X_s)) dB_s, \quad \mathbb{P}_0 - a.s.\]

The following lemma will be useful in the sequel.
Lemma 4.1 Under Assumptions (M), (PPD) and (A), we can define for any \((t, x) \in [0, T] \times \mathbb{R}\) a probability measure \(\widetilde{P} \in \mathcal{P}_{PA}^{a*(Z^*)}(t, x)\) such that

\[
\tilde{\alpha}_t = \tilde{\alpha}(t, B_t, v(t, B_t), \partial_x v(t, B_t), \partial_{xx} v(t, B_t)), \quad dt \otimes \widetilde{P} - \text{a.e.} \tag{35}
\]

Moreover, let \(\varphi\) be a map from \([0, T] \times \mathbb{R}\) into \(\mathbb{R}^+ \setminus \{0\}\). Then, there is a sequence \((\Gamma^{*, n})_{n \in \mathbb{N}}\) of \(\mathcal{G}^{PA}\)-predictable processes such that

\[
\mathbb{E}_{\widetilde{P}} \left[ \int_0^T \varphi(r, B_r) dK^*_r - \int_0^T \varphi(r, B_r) k^{*, n}_r dr \right] \longrightarrow 0, \tag{36}
\]

where

\[
k^{*, n}_r := \frac{1}{2} \alpha^*_r \Gamma^{*, n}_r - \inf_{a \in \mathbb{R}} \left\{ \frac{1}{2} a \Gamma^{*, n}_r + \mathbf{1}_{a \in \mathcal{D}(r, B_r)} \right\}, \quad r \in [0, T].
\]

Proof Assumption (A) provides directly the existence of the probability \(\widetilde{P} \in \mathcal{P}_{PA}^{a*(Z^*)}(t, x)\). Next, we set

\[
H_T := \int_0^T \varphi(r, B_r) dK^*_r \geq 0, \quad \widetilde{P} - \text{a.s.}
\]

Then, since \(\varphi\) is a positive function, there exists by standard density arguments a sequence of nonnegative predictable processes \((k^{*, n})_{n \in \mathbb{N}}\) such that

\[
\mathbb{E}_{\widetilde{P}} \left[ H_T - \int_0^T \varphi(r, B_r) k^{*, n}_r dr \right] \longrightarrow 0, \quad n \to +\infty.
\]

Fix some \(\omega\) in the support of the probability measure \(\widetilde{P}\). We will now prove that the map

\[
\gamma \mapsto \frac{1}{2} \tilde{\alpha}(t, B_t(\omega), v(t, B_t(\omega)), \partial_x v(t, B_t(\omega)), \partial_{xx} v(t, B_t(\omega))) \gamma
\]

\[
- \inf_{a \in \mathbb{R}} \left\{ \frac{1}{2} a \gamma + \mathbf{1}_{a \in \mathcal{D}(t, B_t(\omega))} \right\},
\]

is surjective from \(\mathbb{R}\) into \(\mathbb{R}^+_\). First notice that since \(\tilde{\alpha}(t, B_t(\omega), v(t, B_t(\omega)), \partial_x v(t, B_t(\omega)), \partial_{xx} v(t, B_t(\omega)) \in \mathcal{D}(t, B_t(\omega))\), we have \(s_t(\omega, \gamma) \geq 0\). Indeed, \(s_t(\omega, 0) = 0\), and if \(\gamma > 0\), we have

\[
s_t(\omega, \gamma) = \frac{1}{2} \gamma \left( \tilde{\alpha}(t, B_t(\omega), v(t, B_t(\omega)), \partial_x v(t, B_t(\omega)), \partial_{xx} v(t, B_t(\omega)))
\]

\[
- a_-(t, B_t(\omega)) \right) \geq 0,
\]

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and if \( \gamma < 0 \),
\[
 s_t(\omega, \gamma) = \frac{1}{2} \gamma \left( \bar{a}(t, B_t(\omega), v(t, B_t(\omega)), \partial_x v(t, B_t(\omega)), \partial_{xx} v(t, B_t(\omega))) - a_+(t, B_t(\omega)) \right) \geq 0,
\]
where we defined
\[
a_-(t, B_t(\omega)) := \inf_{a \in \mathbb{R}} \left\{ a + 1_{a \in \mathcal{D}(t, B_t(\omega))} \right\}, \quad a_+(t, B_t(\omega)) := \sup_{a \in \mathbb{R}} \left\{ a - 1_{a \in \mathcal{D}(t, B_t(\omega))} \right\}.
\]
Next, it is clear that if
\[
a_-(t, B_t(\omega)) < \bar{a}(t, B_t(\omega), v(t, B_t(\omega)), \partial_x v(t, B_t(\omega)), \partial_{xx} v(t, B_t(\omega))) \]
\[
< a_+(t, B_t(\omega)),
\]
then \( s_t(\omega, \gamma) \) goes to \(+\infty\) as \( \gamma \) goes to \(\pm \infty\). Besides, if
\[
\tilde{a}(t, B_t(\omega), v(t, B_t(\omega)), \partial_x v(t, B_t(\omega)), \partial_{xx} v(t, B_t(\omega))) = a_-(t, B_t(\omega)) \quad \text{(resp.} a_+(t, B_t(\omega))),
\]
then by letting \( \gamma \to -\infty \) (resp. \( \gamma \to +\infty \)) we still have \( s_t(\omega, \gamma) \to +\infty \). Hence, the surjectivity of \( s_t(\omega, \cdot) \) from \( \mathbb{R} \) into \( \mathbb{R}^+ \) is clear, since this map is also continuous. Thus, using a classical measurable selection argument we deduce that for any \( n \in \mathbb{N} \), there exists a \( \mathcal{F}_{\mathcal{P}^A}^A \)-predictable process \( \Gamma^{n,*} \in \mathbb{I}_{\mathcal{P}^A} \) such that
\[
k_t^{n,*}(\cdot) = s_t(\cdot, \Gamma_t^{n,*}(\cdot)).
\]
Thus, approximation (36) holds. \( \square \)

**Proposition 4.4** Let Assumptions (M), (PPD), (A) and (C) hold. Then \( v \) is a supersolution of HJB equation (32) in the sense of Definition 4.1.

**Proof** We proceed by contradiction. We assume that there exists \( (t_0, x_0) \in [0, T[ \times \mathbb{R} \) and \( \delta > 0 \)
\[
-\partial_t v(t_0, x_0) - \sup_{a \in \mathbb{R}^+} \inf_{(z, \gamma) \in \mathbb{R}^2} G(t_0, x_0, v(t_0, x_0), \partial_x v(t_0, x_0), \partial_{xx} v(t_0, x_0), z, \gamma, \alpha) \leq -3\delta < 0.
\]
Using the continuity of the function \( v \) and Assumption (A), we know that for any \( (t, x) \in [0, T[ \times \mathbb{R} \) there exists \( \tilde{a}(t, x, v(t, x), \partial_x v(t, x), \partial_{xx} v(t, x)) \in \mathbb{R}^+ \) and there exists \( \varepsilon > 0 \) such that for any \( (t, x) \) in \( \mathcal{V}(t_0, x_0) := [t_0, t_0 + \varepsilon] \times B(x_0, \varepsilon) \), where \( B(x_0, \varepsilon) \) denotes the ball centred at \( x_0 \) with radius \( \varepsilon \), we have for any \( (z, \gamma) \in \mathbb{R}^2 \)
\[
\partial_t v(t, x) + G(t, x, v(t, x), \partial_x v(t, x), \partial_{xx} v(t, x), z, \gamma, \alpha) \quad \tilde{a}(t, x, v(t, x), \partial_x v(t, x), \partial_{xx} v(t, x)) \geq 2\delta.
\]
For any $v := (Z, \Gamma) \in \mathcal{K}$ and $\mathbb{P} \in \mathcal{P}^a(Z)(t_0, x_0) \cap \mathcal{P}^a^*(Z)(t, x)$, let $\theta_{v, \mathbb{P}}$ be the first exit time of $(t, B_t, Y_{t,0,v})$ of $\mathcal{V}(t_0, x_0) \times \mathcal{B}(y_0, \varepsilon)$. Using Assumption (PPD) we have

$$0 = \inf_{\nu \in \mathcal{K}} \sup_{\mathbb{P} \in \mathcal{P}^a(Z)(t_0, x_0)} \mathbb{E}^\mathbb{P} \left[ v(\theta_{v, \mathbb{P}}, B_{\theta_{v, \mathbb{P}}})e^{R_P Y_{t,0,v}^0} - v(t_0, x_0) \right]$$

$$= \sup_{\mathbb{P} \in \mathcal{P}^a^*(Z)(t_0, x_0)} \mathbb{E}^\mathbb{P} \left[ v(\theta_{v^*, \mathbb{P}}, B_{\theta_{v^*, \mathbb{P}}})e^{R_P Y_{t,0,v^*}^0} - v(t_0, x_0) \right],$$

with $v^* := (Z^*, K^*)$. Since Assumption (A) holds, we have a probability $\mathbb{F} \in \mathcal{P}^a(Z)(t_0, x_0) \cap \mathcal{P}^a^*(Z)(t_0, x_0)$ such that (35) holds. Denote for simplicity

$$\tilde{\alpha}_r := \tilde{\alpha}(r, B_r, v(r, B_r), \partial_x v(r, B_r), \partial_{xx} v(r, B_r)).$$

We thus obtain by applying Itô's formula

$$0 \geq \mathbb{E}^{\mathbb{F}} \left[ v(\theta_{v^*, \mathbb{F}}, B_{\theta_{v^*, \mathbb{F}}})e^{R_P Y_{t,0,v^*}^0} - v(t_0, x_0) \right]$$

$$= \mathbb{E}^\mathbb{F} \left[ \int_{t_0}^{\theta_{v^*, \mathbb{F}}} e^{R_P Y_{t,0,v^*}^0} (\partial_t v(r, B_r)$$

$$+ G(r, B_r, v(r, B_r), \partial_x v(r, B_r), \partial_{xx} v(r, B_r), Z_r^*, \Gamma_r^{n,*}, \tilde{\alpha}_r)$$

$$+ \int_{t_0}^{\theta_{v^*, \mathbb{F}}} e^{R_P Y_{t,0,v^*}^0} v(r, B_r) (dK_r^* - k_r^{*n} dr) \right],$$

using the same notation that those in Lemma 4.1 with the choice $\varphi = v$ (we recall that by definition $v > 0$). From (36), we deduce that there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$,

$$\mathbb{E}^{\mathbb{F}} \left[ \int_{t_0}^{\theta_{v^*, \mathbb{F}}} e^{R_P Y_{t,0,v^*}^0} v(r, B_r) (dK_r^* - k_r^{*n} dr) \right] \geq -\delta.$$ 

Using (37), we finally get for $n \geq n_0$

$$0 \geq \mathbb{E}^{\mathbb{F}} \left[ v(\theta_{v^*, \mathbb{F}}, B_{\theta_{v^*, \mathbb{F}}})e^{R_P Y_{t,0,v^*}^0} - v(t_0, x_0) \right] \geq \delta > 0,$$

which provides the desired contradiction. Thus, $f$ is a super-solution of (32). \qed

**Corollary 4.1** Let Assumptions 4.1, (M), (PPD), (A), (C) hold. Then,

$$U_0^P = U_{-0}^P$$

(38)
Proof From Proposition 4.4 together with Assumption (C), we deduce that

\[ U_P^0 = -e^{R_P R_0} v(0, 0) \leq -e^{R_P R_0} \psi(0, 0) = U_P^0. \]

The other inequality being clear by definition, this concludes the proof. \(\Box\)

Remark 4.4 It is worth noting that the approach we have described is by no means the only way to prove the desired result. Among all our assumptions, the most complicated ones to check in practice are (PPD) and (C). We have already explained how assumption (PPD) could be completely avoided if one uses the theory of stochastic viscosity solutions. This is actually the approach used in the work in preparation [24].

As for (PPD), one should not expect in practice to have smooth solutions to PDE (32), and viscosity solutions should be used instead. That being said, the comparison theorem could still prove to be complicated to prove in general situations, but it only one of the possible ways to prove the required inequality. If more direct arguments can be used, we feel that it would definitely prove faster and more convenient to use them.

4.3.4 Application to the Non-adaptative Model

In this section, we concentrate on the “non-adaptative” model. It is easy to check in this setting that Assumptions 4.1, (M) and (A) hold. We will now identify a smooth solution to the corresponding HJBI PDE. By the above calculations, we immediately have that

\[ U_P^0 = \sup_{(Z, \Gamma) \in \mathcal{U}} \inf_{P \in \mathcal{P}} \mathbb{E}[ - \mathcal{E} - R_P \int_0^T \tilde{\alpha}_s^{1/2} (1 - Z_s) dW_s^{\alpha^*_s(Z_s)} ] \]

where

\[ H(\alpha, z, \gamma) := a^*(z) - k(a^*(z)) - \frac{\alpha}{2} \left( R_A z^2 + R_P (1 - z)^2 \right) - \frac{1}{2} \alpha \gamma \]

In order to pursue the computations, we need to specify a form for the cost function \(k\). Namely, we will assume in what follows that

**Assumption 4.6**

The cost function of the Agent is quadratic, defined, for some \(k > 0\), by

\[ k(a) := ka^2 / 2, \ a \geq 0. \]

We deduce from Proposition 4.1 that the Agent chooses the control \(a^*(z) = \frac{z}{k}\). Hence, Equality (39) can be rewritten
\[ H(\alpha, z, \gamma) = \frac{2z - z^2}{2k} - \frac{\alpha}{2} \left( RAz^2 + RP(1 - z)^2 \right) - \frac{1}{2} \left( \alpha \gamma - \inf_{a \in [\alpha_A, \bar{\alpha}_A]} \{ a \gamma \} \right) \]
\[ =: H^z(\alpha, z) + H^\gamma(\alpha, \gamma), \] (40)

where
\[ H^z(\alpha, z) := \frac{2z - z^2}{2k} - \frac{\alpha}{2} \left( RAz^2 + RP(1 - z)^2 \right), \]
\[ H^\gamma(\alpha, \gamma) := -\frac{1}{2} \alpha \gamma + \inf_{\alpha \in [\alpha_A, \bar{\alpha}_A]} \left\{ \frac{1}{2} \alpha \gamma \right\}. \]

Notice that for any \( \alpha \geq 0 \)
\[ H(\alpha_A, z, 0) = H(\alpha, z, -RAz^2 - RP(1 - z)^2). \] (41)

The following lemma computes the maximum of the map \((z, \gamma) \mapsto H(\alpha, z, \gamma)\), depending on the value of \( \alpha \in \mathbb{R}_+ \).

**Lemma 4.2** We distinguish three cases.

(i) If \( \alpha_A \leq \alpha \leq \bar{\alpha}_A \), then \((z, \gamma) \mapsto H(\alpha, z, \gamma)\) admits a (global) maximum at
\[ z^*(\alpha) := \frac{1 + k \alpha R_P}{1 + \alpha k (R_A + R_P)}, \gamma^*: = 0. \] (42)

(ii) If \( \alpha < \alpha_A \), \( \gamma \mapsto H(\alpha, z, \gamma) \) is increasing and attains its maximum at \( \gamma^* = +\infty \), with \( H(\alpha, z, \gamma^*) = +\infty \).

(iii) If \( \bar{\alpha}_A < \alpha \), \( \gamma \mapsto H(\alpha, z, \gamma) \) is decreasing and attains its maximum at \( \gamma^* = -\infty \), with \( H(\alpha, z, \gamma^*) = +\infty \).

**Proof** We have
\[ \frac{\partial H}{\partial z}(\alpha, z, \gamma) = \frac{\partial H^z}{\partial z}(\alpha, z) = \frac{1}{k} - \frac{z}{k} - \alpha (RAz - RP(1 - z)), \]
so that
\[ \frac{\partial H}{\partial z}(\alpha, z, \gamma) = 0 \iff z = z^*(\alpha) := \frac{1 + k \alpha R_P}{1 + \alpha k (R_A + R_P)}. \]

Since \( z \mapsto H^z(\alpha, z) \) is concave for any \( \alpha \geq 0 \), we deduce that the maximum of \( H^z \) is attained at \( z^*(\alpha) \). Furthermore, for any \( \gamma \neq 0 \)
\[ \frac{\partial H}{\partial \gamma}(\alpha, z, \gamma) = \frac{\partial H^\gamma}{\partial \gamma}(\alpha, \gamma) = \frac{1}{2} (\alpha - \alpha_A) 1_{\gamma > 0} + \frac{1}{2} (\bar{\alpha}_A - \alpha) 1_{\gamma < 0}. \]

If \( \alpha_A \leq \alpha \leq \bar{\alpha}_A \), then \((z, \gamma) \mapsto H(\alpha, z, \gamma)\) admits a global maximum at \((z^*(\alpha), 0)\) which proves (i). Then, (ii) and (iii) are clear. \( \square \)
It is then a tedious but easy task to check that the dynamic version of the Agent’s utility $U^P_0$ is a smooth solution to the PDE (32). It turns out in this case that we do not need Assumption (PPD) nor (C) to conclude the desired inequality. As per our discussion in Remark 4.4, we will actually use a direct approach to prove that $U^P_0 \geq U^P_0$. We can now state the main result of this section, which gives the optimal contracts for the second-best problem, when contracts are restricted to the class $\mathcal{C}^{SB}$.

**Theorem 4.1** Let Assumption 4.6 hold. Define for any $\alpha \geq 0$

$$z^*(\alpha) := \frac{1 + k\alpha R_P}{1 + \alpha k(R_A + R_P)}.$$ 

(i) If $\alpha^A \leq \alpha^P \leq \alpha^A$, then an admissible optimal contract is given by $\xi R_0, z^*(\alpha^P), 0$.

In this case,

$$U^P_0 = -\exp \left(-R_P (TH(a^P, z^*(\alpha^P), 0) - R_0)\right).$$

(ii) If $\alpha^P \leq \alpha^A \leq \alpha^P$, then an admissible optimal contract is given by $\xi R_0, z^*(\alpha^A), \gamma^*$, where $\gamma^* := -R_A (z^*(\alpha^A))^2 - R_P (1 - z^*(\alpha^A))^2$. In this case,

$$U^P_0 = -\exp \left(-R_P (TH(a^A, z^*(\alpha^A), \gamma^*) - R_0)\right).$$

(iii) Assume that $\alpha^P < \alpha^A$. Then $U^P_0 = 0$.

(iv) Assume that $\alpha^A < \alpha^P$. Then $U^P_0 = 0$.

We now prove the following result.

**Theorem 4.2** Let Assumption 4.6 hold. Then

$$U^P_0 = U^P_0.$$

**Proof** First of all, notice that when $\alpha^P < \alpha^A$ or $\alpha^A < \alpha^P$, we have $U^P_0 = 0$, so that $U^P_0 = 0$ as well. Let us now assume that $\alpha^A \leq \alpha^P \leq \alpha^A$. Since $K_T \geq 0$, we easily have

$$U^P_0 \leq \sup_{\xi \in \mathcal{C}^{SB}} E_{\mathcal{F}^T} \left[ z^*(\alpha^P) \exp \left(-R_P \int_0^T (a^P) \frac{1}{2} (1 - Z_s^\xi) dW_s \right) e^{R_P \int_0^T f(z_s^\xi, \alpha^P) ds} \right].$$

Then, we easily have that the map $z \mapsto f(z, a^P)$ attains its maximum at $z^*(\alpha^P)$, where it is actually equal to $H(a^P, z^*(\alpha^P), 0)$. Thus, by Theorem 4.1(i)

$$U^P_0 \leq e^{R_P R_0} e^{-R_P TH(a^P, z^*(\alpha^P), 0)} = U^P_0.$$

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Assume now that $\alpha_P^P \leq \overline{\alpha_A} \leq \alpha_P$. Then, with the same arguments, we have by Theorem 4.1(ii)

$$U_0^P \leq \sup_{\xi \in C_{SB}} \mathbb{E}^{a^*(\xi)} \left[ -\mathcal{E} \left( -R_P \int_0^T (\overline{\alpha_A})^{1/2} (1 - Z_{\xi s}) d W_a^s \right) \right] \leq e^{R_P R_0} e^{-R_P T H(\overline{\alpha_A}, z^*, (\overline{\alpha_A}), 0)} = U_0^P.$$ 

\[4.4\text{ Comments}\]

The comparison with the case without ambiguity is actually very similar to the first-best problem. First, notice that when $\alpha_P^P \in [\alpha_A, \alpha_A]$, an optimal contract can be chosen to be linear in the terminal value of the output, and it is actually the exact same contract as the optimal one for a Principal who would only believe in a constant volatility process equal to $\alpha_P^P$. Since the utility of the Principal is then a decreasing function of the volatility, this means that the Principal always gets less utility than in a context without ambiguity.

However, as soon as $\alpha_A \in [\alpha_P^P, \alpha_P^P]$, the second-best optimal contract makes use of the quadratic variation of the output and is therefore path-dependent. Besides, as in the first-best case, the Principal may get an higher utility level than in the case without ambiguity.

Finally, in the degenerated cases (iii) and (iv) of Theorem 4.1, we have seen that the optimal effort for the Agent is equal to 0 since $z^* = 0$ and $a^*(z^*) = 0$, on the contrary to the first-best problem where, in the same case, the optimal level of effort for the Agent, chosen by the Principal to obtained his best utility 0, was $a_{\max}$. Hence, to solve the second-best problem, the Agent does not provide any effort and attains his reservation utility. It can be explained by the fact that in the second-best problem, an optimal contract is a Stackelberg equilibrium, where the Principal has to anticipate the reaction of the Agent given an admissible contract, unlike the first-best problem for which the Principal chooses the level of effort for the Agent.

\[5\text{ Possible Extensions and Comparison with the Literature}\]

In this section, we examine several potential generalisations of the problem at hand, and we try to explain how to tackle it in each case.

\[5.1\text{ More General Dynamics}\]

The first possible extension would be to consider an output with more general dynamics. Typically, one could have a general non-Markovian model where
\[ B_t = \int_0^t b_s \left( B_s, a_s^p, \alpha_s^p \right) \, ds + \int_0^t \sigma_s \left( B_s, \alpha_s^p \right) \, dW_s^p, \ P - a.s., \]

that is to say that the impact of the effort choice of the Agent on the drift of the output is now nonlinear, and the value of this drift may also depend on the past values of the output itself, which could model some synergy effects. Furthermore, the cost function \( k \) could also take the form \( k_s \left( B_s, a_s, \alpha_s \right) \).

In the first-best problem, if the map \( b \) actually only depends on \( a \), and not on \( B \) and \( \alpha \), and if \( k \) does not depend on \( B \), then it is not difficult to see that our approach will still work, albeit with more complicated computations. Notably, the optimal effort of the Agent will either be \( a_{\text{max}} \) or any (deterministic) minimiser of \( a \mapsto -k_t(x,a,\alpha) - b_t(x,a,\alpha)z \) (which exist since \( k \) is super-linear). It is, however, not clear to us how to handle the general dynamics.

In the second-best problem, the representation of the value function of the Agent in terms of 2BSDEs will always work, provided that one can indeed check that it is well posed (which requires obviously some assumptions on \( k \) and \( b \)). Then, in the Markovian case, our approach depicted above using the Hamilton–Jacobi–Bellman–Isaacs (HJBI) equation

\[
- \partial_t u(t, x, y) - \sup_{a \in \mathbb{R}^+} \inf_{(x, y) \in \mathbb{R}^2} \left\{ b_t(x, a_t(x, z, \alpha), \alpha) \partial_x u(t, x, y) + \left( k_t(x, a_t(x, z, \alpha), \alpha) - \inf_{a \in \mathbb{R}^+} \left\{ \frac{1}{2} \sigma_t^2(x, a) \gamma + 1_{a_t^2(x, a) \in D_{A(t,x)}} \right\} \right) \partial_y u(t, x, y) + \frac{1}{2} \sigma_t^2(x, \alpha) \left( \partial_{xx} u + (\gamma + R_A z^2) \partial_y u \right)(t, x, y) + \frac{1}{2} \sigma_t^2(x, \alpha) (z^2 \partial_{yy} u + z \partial_{xy} u)(t, x, y) + 1_{a_t^2(x, \alpha) \in D_{P(t,x)}} \left\{ \right. \right. \right. \\
\left. \left. \left. (t, x, y) \in [0, T] \times \mathbb{R}^2, \quad v(T, x, y) = U_P(x - y), \quad (x, y) \in \mathbb{R}^2, \right. \right. \right. \}
\]

where \( a_t^*(x, z, \alpha) \) is the unique, for simplicity,\(^{7}\) minimiser of the map \( a \mapsto k_t(x,a,\alpha) - b_t(x,a,\alpha)z \).

A particular non-Markovian case could prove very interesting in this framework. It would correspond to the case where for \( \Psi = A, P \)

\[ D\Psi(t, \omega) = D\Psi(t, B_t(\omega), \langle B \rangle_t(\omega)), \quad (t, \omega) \in [0, T] \times \Omega, \]

which means that the Principal and the Agent update their beliefs according to both the current value of the output and of its quadratic variation. In this case, we could simply consider \( \langle B \rangle \) as an additional state variable, with dynamics

\(^{7}\) If the minimiser is not unique, then we assume as usual that the Principal has sufficient bargaining power to make the Agent choose the best minimiser for him. This means that one has also to take the supremum over all minimisers in the Hamiltonian above.
\[ \langle B \rangle_t = \int_0^t \sigma_s^2 \left( B_s, \alpha_s^P \right) \, ds, \ \mathbb{P} - a.s., \]

and have then an HJBI PDE with 3 state variables, thus avoiding to have to rely on the theory of path-dependent PDEs (recall Footnote 6).

5.2 More General Utility Functions

Another possible generalisation would be to go beyond the case of exponential utility functions for the Principal and the Agent. As usual, if the utility of the Agent is separable (that is to say if the cost comes out of the utility), then the 2BSDE characterisation of his value function would still hold. This means that in general, the criterion of the Agent could take the form

\[ U_A^0(\xi) := \sup_{a \in A} \inf_{P \in \mathcal{P}_a} \mathbb{E}_P \left[ K_{0,T}^a(\xi) - \int_0^T K_{0,s}^a k_s(B_s, a_s, \alpha_s^P) \, ds \right], \]

where the utility function \( U_A \) can now be arbitrary, and where we have added a discount factor defined for any controls \( a \) and \( \alpha^P \)

\[ K_{s,t}^a := \exp \left( - \int_s^t r_u(B_u, a_u, \alpha_u^P) \, du \right), \ 0 \leq s \leq t \leq T. \]

Once again, one could write down an HJBI equation similar to the one above to study the problem of Principal. This is actually the main purpose of the work in preparation [24] where the authors use the method outlined in the current paper to treat the general case above. This is a further justification of the generality of our proposed method.

5.3 Detailed Comparison with [25]

As mentioned in Introduction, independently of our work, Sung [25] has studied a similar model of Principal–Agent with ambiguity. For the sake of understanding the specificities of these two approaches, we will now list the main differences.

(i) The modelisation considered in [25] is roughly the same as the one we described in Sect. 5.1 in terms of the dynamics of the output. As explained, our approach would work similarly in such a context.

(ii) There is a first important difference in terms of the ambiguity sets. Indeed, [25] considers that the Principal and the Agent share the exact same ambiguity set \( D \), which is defined through a map \( \pi \) satisfying the KKT conditions or the Slater constraint qualification conditions (see [25, page 12] for more details). In our work, we do not require any of these conditions, and our modelisation allows for general, and different ambiguity sets for the Principal and the Agent. The only assumptions we impose on these sets are the ones necessary so that the dynamic programming principle holds, which are necessary to ensure that the problem
of the Agent is time-consistent, and allows as a consequence to characterise it through 2BSDEs. It is then a mathematically proved result that the analysis can be reduced without loss of generality to the intersection of the ambiguity sets of the Principal and the Agent.

(iii) Another important difference lies in the choice of admissible contracts. In [25], the author takes right from the start as class of admissible contracts the terminal values at time $T$ of some semi-martingales with a triplet of characteristics which is absolutely continuous with respect to the Lebesgue measure, of a form similar to our $\xi Y_0, Z, \Gamma$; see Equation (11) and Theorem 1 in [25]. In [25], the author justifies this choice through informal arguments, with which we obviously agree, and claims (see [25, Footnote 14]) that the restriction is without loss of generality, but without any proof. We do not understand this point, since, as we proved it in Proposition 4.1, the value function of the Agent is always represented through a 2BSDE, in which there is the non-decreasing process $K$. However, it is known that such a process is not always absolutely continuous, for any choice of $\xi$. Indeed, Peng, Song and Zhang [59] have characterised completely the set of random variables for which this was the case in the context of our non-adaptative model, and proved that integrability was not a sufficient condition. Therefore, it is a result by itself, and actually one of the important contributions of our work, to justify that the restriction is without loss of generality, which we explain how to do for general models, and prove under natural assumptions.

(iv) The last difference lies in the methods used to solve the problem itself. First of all, the specification of the model is similar in both papers, which had to be expected, since this is the natural way to give the weak formulation for stochastic control problems or differential games. As for the problem of the Agent, since [25] concentrates on a class of contracts similar to our $C^{SB}$, the problem becomes a simple verification result, and does not have to rely on the 2BSDE theory. The main difference resides in the approach to the Principal’s problem. In [25, Theorems 3 and 4], the author characterises the value function of the Principal through some predictable process $Z^P$, for which existence is obtained, but no explicit construction is given. The only exemple where the author manages to compute this $Z^P$ roughly corresponds to our “non-adaptative” model; see [25, Proposition 1]. In this regard, our method based on HJBI PDEs (or path-dependent PDEs in the non-Markovian case, see Pham and Zhang [60]) provides a clear way to compute, at least numerically, both the value function of the Principal and the associated optimal contract, even in very general models. We believe that this is of the utmost importance for the practical application of this theory.

(v) Finally, we would like to point out that we believe that [25] may be more finance oriented, and as a result provides arguments and explanations which may prove more accessible for a less technical audience. We therefore believe that the two

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8 One could argue that it suffices to use the same arguments as in [19] to obtain this result; however, their argument does not go through in this case, as we already explained earlier.

9 Numerical schemes in the Markovian case are by now extremely well-known schemes for PPDEs have recently been considered by Zhang and Zhuo [61], and Ren and Tan [62].
papers complement each other very well, and [25] is an excellent companion to our work, especially in terms of studying the managerial implications of the results that we have both obtained, which are described at length in [25].

6 Conclusions

In this paper, we developed a consistent framework to deal with both first- and second-best contracting situations involving some uncertainty on the parameters of the model, from the point of view of both the Principal and the Agent. As already mentioned, the main difficulties are related to setting up rigorously the weak formulation of the problem, especially when the uncertainty impacts the volatility of the output process, and on proving that a reasonably natural class of contracts can be considered without loss of generality. In order to not complicate even more our presentation, we have chosen to work under assumption which are clearly not the most general ones, and we refer the reader to the PhD thesis [24] for a first tentative to improve the results of this paper. Another avenue of generalisation would be to consider a setting where the Agent can control both the drift and the volatility of the output, while there is also some uncertainty impacting them both. In this case, our approach would fail, as the value function of the Agent would no longer be represented by a 2BSDE.

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Appendix

Proof of Proposition 4.1 Our fist step is to look at the dynamic version of the value function of the Agent. Fix some \( a \in \mathcal{A} \). We refer to the papers [44,63] for the proofs that, for any \( \mathcal{F}_T \)-measurable contract \( \xi \in \mathcal{C} \), one can define a process, which we denote by \( u^A_t(\xi, a) \) (denoted by \( Y_t \) in [63]), which is càdlàg, \( \mathcal{G}^{P_A} \)-adapted (recall that for any \( a \in \mathcal{A}, \mathcal{G}^{P_A} = \mathcal{G}^{P_A} \), since the polar sets of \( P_A \) are the same as the polar sets of \( P_A \)) and such that

\[
\begin{align*}
&u^A_t(\xi, a) = \text{essinf}_{\mathbb{P}} \mathbb{E}^{\mathbb{P}} \left[ U_A \left( \xi - \int_t^T k(a_s)ds \right) \right]_{\mathcal{F}_t}, \mathbb{P} - a.s., \text{ for all } \mathbb{P} \in \mathcal{P}^a_A. \\
&\text{(43)}
\end{align*}
\]

Notice that since \( \xi \in \mathcal{C} \), it has exponential moments of any order, so that since in addition the effort process \( a \) is bounded, we have that \( u^A(\xi, a) \) has moments of any order, in the sense that

\[
\sup_{a \in \mathcal{A}} \sup_{\mathbb{P} \in \mathcal{P}^a_A} \mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq t \leq T} |u^A_t(\xi, a)|^p \right] < +\infty, \text{ for all } p \geq 0, \quad (44)
\]
where we have used the generalised Doob inequality for sub-linear expectations given in Proposition A.1 in [64].

Moreover, by [63, step 2 in the proof of Theorem 2.3], 
\[ e^{R_A \int_0^T k(a_s)ds} u^A_t(\xi, a) \]

is a \((P, \mathcal{G}^{P_A+})\)-sub-martingale for every \(P \in \mathcal{P}_A^a\), and by [63, step 3 in the proof of Theorem 2.3], there is a \(\mathcal{G}^{P_A}\)-predictable process \(\tilde{Z}\), and a family of non-decreasing and \(\mathcal{F}_P\)-predictable processes \((\tilde{K}_P)_{P \in \mathcal{P}_A^a}\), such that, for all \(P \in \mathcal{P}_A^a\)

\[
e^{R_A \int_0^T k(a_s)ds} u^A_t(\xi, a) = e^{R_A \int_0^T k(a_s)ds} U_A(\xi) - \int_t^T \tilde{Z}_s \alpha_s^{1/2} dW_s - \tilde{K}^P_T + \tilde{K}^P_t, \quad P - a.s.
\]

Notice also that since every probability measure in \(\mathcal{P}_A\) is equivalent, by definition, to a probability measure in \(\mathcal{P}_A^a\) (and conversely), the above also holds \(P - a.s., \) for any \(P \in \mathcal{P}_A\), with the convention that we will still denote by \(\tilde{K}\) the non-decreasing process associated with \(P \in \mathcal{P}_A^a\) or \(\mathcal{P}_A\). Moreover, using the aggregation result of [40], we can actually aggregate the family \(\tilde{K}\) into a universal process, which is \(\mathcal{G}^{P_A}\)-predictable, and which we denote by \(\tilde{K}\).

Define

\[
Y^a_t := -\ln \left( -u^A_t(\xi, a) \right), \quad Z^a_t := -\frac{e^{-R_A \int_0^T k(a_s)ds}}{R_A u^A_t(\xi, a)} \tilde{Z}_t, \quad K^a_t := -\int_0^t \frac{e^{-R_A \int_0^T k(a_s)dr}}{R_A u^A_t(\xi, a)} d\tilde{K}_r.
\]

We have, after some computations, for all \(P \in \mathcal{P}_A\)

\[
Y^a_t = \xi - \int_t^T \left( \frac{R_A}{2} \left| Z^a_t \right|^2 \alpha_s + k(a_s) - a_s Z^a_t \right) ds - \int_t^T Z^a_t \alpha_s^{1/2} dW_s
\]

\[
- \int_t^T dK^a_s, \quad P - a.s.
\]

Now notice that by (44), we immediately have

\[
\sup_{a \in A} \sup_{P \in \mathcal{P}_A} \mathbb{E}^P \left[ \exp \left( p \sup_{0 \leq t \leq T} |Y^a_t| \right) \right] < +\infty, \text{ for every } p \geq 0.
\]

Moreover, remember that by (43), we also have for every \(P \in \mathcal{P}_A^a\), by the exact same arguments as above applied under any fixed measure \(P \in \mathcal{P}_A\), that

\[
Y^a_t = \text{essinf}_{P' \in \mathcal{P}_A^a(P, t^+)} Y^{P'}_{t^+} \quad P - a.s., \quad (45)
\]

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where for any $\mathbb{P} \in \mathcal{P}^a_A$, $(Y^\mathbb{P}, Z^\mathbb{P})$ is the unique solution to the following BSDE defined under $\mathbb{P}$

$$\mathcal{Y}^\mathbb{P} = \xi - \int_t^T \left( \frac{R_A}{2} |Z^\mathbb{P}_s| \hat{\alpha}_s + k(a_s) - a_s Z^\mathbb{P}_s \right) \, ds$$

$$- \int_t^T Z^\mathbb{P}_s \hat{\alpha}_s^{1/2} \, dW_s, \quad \mathbb{P} - \text{a.s.}$$

Then, using (44), we can follow the proof of Lemma 3.1 in [65] to obtain that $Z^a$ actually belongs to the BMO space defined in [65] (see Section 2.3.2). Then, we can follow exactly the proof of Theorem 6.1 in [65] to obtain with (45) that for any $\mathbb{P} \in \mathcal{P}^a_A$

$$K^a_t = \operatorname{essinf}_{\mathbb{P}' \in \mathcal{P}^a_A(\mathbb{P}, t^+)} \mathbb{P}'[K^a_T | \mathcal{F}_t], \quad \mathbb{P} - \text{a.s.}$$

Therefore, $(Y^a_t, Z^a_t)$ is the unique solution to the (quadratic linear) 2BSDE with terminal condition $\xi$ and generator $R_A/2 \hat{\alpha}_s + k(a_s) - a_s Z^a_s$ (see, for instance, Definition 2.3 of [65]).

The final step of the proof is now to relate the family $(Y^a)_{a \in \Lambda}$ to the solution of the 2BSDE (18). Before proceeding, let us explain why the 2BSDE (18) does indeed admit a maximal solution. First of all, the corresponding quadratic BSDEs admit a maximal solution, because, since the infimum in the generator is over a compact set, the generator of the BSDE is bounded from above by a function with linear growth in $z$. The existence of a maximal solution is then direct from Proposition 4 of [66]. Furthermore, since this maximal solution is obtained as a monotone approximation of Lipschitz BSDEs, it satisfies a comparison theorem. Hence, we can apply first Proposition 2.1 of [36] to obtain the existence of a maximal solution of the 2BSDE, in the sense of Definition 4.1 of [36], and then use Remark 4.1 of [36] to aggregate the family of non-decreasing processes into $K$. (We remind the reader that all the measures in $\mathcal{P}^a_A$ satisfy the predictable martingale representation property.)

In particular, we have the following representation for any $\mathbb{P} \in \mathcal{P}_A$,

$$Y_t = \operatorname{essinf}_{\mathbb{P}' \in \mathcal{P}^a_A(\mathbb{P}, t^+)} \mathcal{Y}^\mathbb{P}'_t, \quad \mathbb{P} - \text{a.s.}, \quad (46)$$

where for any $\mathbb{P} \in \mathcal{P}^a_A$, $(\mathcal{Y}^\mathbb{P}, \mathcal{Z}^\mathbb{P})$ is the maximal solution of the quadratic BSDE

$$\mathcal{Y}^\mathbb{P} = \xi - \int_t^T \left( \frac{R_A}{2} |\mathcal{Z}^\mathbb{P}_s| \hat{\alpha}_s + \inf_{a \in [0, a_{\max}]} \left\{ k(a) - a \mathcal{Z}^\mathbb{P}_s \right\} \right) \, ds$$

$$- \int_t^T \mathcal{Z}^\mathbb{P}_s \hat{\alpha}_s^{1/2} \, dW_s, \quad \mathbb{P} - \text{a.s.}$$

10 Well-posedness is clear here, since we have easily that $\mathcal{Y}^\mathbb{P}_{-a} = -\frac{1}{R_A} \log \left( -\mathbb{E}^\mathbb{P}[\mathcal{U}/A] \left( \xi - \int_t^T k(a_s) \, ds \right) | \mathcal{F}_t \right), \quad \mathbb{P} - \text{a.s.}$

11 In this result, $\xi$ and $Y^a$ are assumed to be bounded, but the proof generalises easily to our setting where $Y^a$ satisfies (44).
Now it is a classical result dating back to [67–69] (see also [70] for a similar result using 2BSDEs) that, using the comparison theorem satisfied by the maximal solution of the 2BSDEs (which is automatically inherited from the one satisfied by the BSDEs)

\[ Y_0 = \sup_{a \in A} Y_0^a = \sup_{a \in A} \inf_{P \in \mathcal{P}_A^a} Y_0^P, \]

so that \( U_A^0(\xi) = -\exp(-R_A Y_0) \). Furthermore, it is then clear, since the function \( k \) is strictly convex that there is some \( a^*(Z_s) \in A \) such that

\[
\inf_{a \in [0, a_{\text{max}}]} \{ k(a) - a Z_s \} = k(a^*(Z_s)) - a^*(Z_s) Z_s, \ s \in [0, T].
\]

This implies that \( Y_0 = \inf_{P \in \mathcal{P}_A^{a^*(Z_s)}} Y_0^P, a^*(Z_s) \).

**Proof of Theorem 4.1** We recall Definition (42) for any \( \alpha \geq 0 \)

\[
z^*(\alpha) := \frac{1 + k\alpha R_P}{1 + \alpha k(R_A + R_P)}.
\]

We begin with the proof of (i). Assume that \( \underline{\alpha} \leq \alpha^A \leq \overline{\alpha} \), then

\[
U_0^P \geq \inf_{P \in \mathcal{P}_P^{z^*(\overline{\alpha})}} \mathbb{E}^P \left[ -(R_P \int_0^T \frac{1}{\alpha_s^2} (1 - z^*(\overline{\alpha}_s)) dW_s) e^{R_P \int_0^T H(\alpha, z^*(\overline{\alpha}_s), 0) ds} \right].
\]

Then we have for any \( \alpha \geq 0 \)

\[
H(\alpha, z^*(\alpha), 0) = -\frac{\alpha}{2} R_P + \frac{(1 + \alpha k R_P)^2}{2k(1 + \alpha k(R_A + R_P))}.
\]

Hence,

\[
\frac{\partial H}{\partial \alpha}(\alpha, z^*(\alpha), 0) = \frac{-R_A (1 + 2k\alpha R_P + k^2 \alpha^2 R_P (R_A + R_P))}{2(1 + \alpha k(R_A + R_P))^2} \leq 0, \ \forall \alpha \in [\underline{\alpha}, \overline{\alpha}].
\]

Therefore, \( U_0^P \geq -e^{R_P R_0 - R_P \int_0^T H(\alpha, z^*(\overline{\alpha}), 0) ds} \). Indeed, \( z^*(\overline{\alpha}_s) \) is bounded so that the stochastic exponential is trivially a true martingale. We now turn to the converse inequality; we have

\[ \square \]
\[
\begin{align*}
\frac{U_0^P}{\alpha} \leq \sup_{(Z, \Gamma) \in \mathcal{R}} \mathbb{E}^{\mathbb{P}^{\alpha}(Z)} \left[ -\mathcal{E} \left( -R_P \int_0^T (a^* P)^{1/2}(1 - Z_s) dW_{a^*}(Z) \right) \right]
\end{align*}
\]

According to Lemma 4.2(i), we obtain \(\frac{U_0^P}{\alpha} \leq \mathcal{E} \left( R_P R_0 e^{-R_P T H(\alpha P, z^*(\alpha P), 0)} \right)\). Hence, if \(\alpha^A \leq \alpha P \leq \alpha^P\), then \(\frac{U_0^P}{\alpha} = \mathcal{E} \left( R_P R_0 e^{-R_P T H(\alpha P, z^*(\alpha P), 0)} \right)\). We now prove that the contract \(\xi R_0, z^*(\alpha P), 0 \in \mathcal{C}^S_B\) is indeed optimal. We have

\[
\inf_{\mathcal{P} \in \mathcal{P}^{\alpha}(\alpha P)} \mathbb{E}[\mathcal{P}] \left[ -\mathcal{E} \left( -R_P \int_0^T \alpha \hat{\mathcal{A}}^\alpha \left( 1 - z^*(\alpha P) dW_{a^*}(\alpha P) \right) e^{R_P \left( R_0 - \int_0^T H(\alpha P, z^*(\alpha P), 0) ds \right)} \right] = \frac{U_0^P}{\alpha},
\]

since by definition (40) of \(H, \alpha \mapsto H(\alpha, z^*(\alpha P), 0)\) is decreasing, so that the above infimum is attained for the measure \(\mathcal{P}^{\alpha}(\alpha P)\).

**We now turn to the proof of (ii).** Assume that \(\alpha P \leq \alpha^A \leq \alpha P\). On the one hand,

\[
\inf_{\mathcal{P} \in \mathcal{P}^{\alpha}(\alpha A)} \mathbb{E}[\mathcal{P}] \left[ -\mathcal{E} \left( -R_P \int_0^T \alpha \hat{\mathcal{A}}^\alpha \left( 1 - z^*(\alpha A) dW_{a^*}(\alpha A) \right) e^{R_P \left( R_0 - \int_0^T H(\alpha A, z^*(\alpha A), 0) ds \right)} \right] \leq \frac{U_0^P}{\alpha},
\]

where \(\gamma^* := -R_A \left( z^*(\alpha A) \right)^2 - R_P \left( 1 - z^*(\alpha A) \right)^2\). Thus, using Relation (41), we have

\[
\frac{U_0^P}{\alpha} \geq \mathcal{E} \left( R_P R_0 e^{-R_P T H(\alpha A, z^*(\alpha A), 0)} \right).
\]

On the other hand, since \(\alpha^A \in [\alpha P, \alpha P]\)

\[
\frac{U_0^P}{\alpha} \leq \mathcal{E} \left( R_P R_0 \sup_{(Z, \Gamma) \in \mathcal{R}} \mathbb{E}^{\mathbb{P}^{\alpha}(Z)} \left[ -\mathcal{E} \left( -R_P \int_0^T \alpha \hat{\mathcal{A}}^\alpha \left( 1 - Z_s) dW_{a^*}(Z) \right) e^{R_P \left( R_0 - \int_0^T H(\alpha A, \alpha, \alpha A) ds \right)} \right] \right).
\]

By using Lemma 4.2(i), we obtain \(\frac{U_0^P}{\alpha} \leq \mathcal{E} \left( R_P R_0 e^{-R_P T H(\alpha A, z^*(\alpha A), 0)} \right)\).

We consider now a contract \(\xi R_0, z^*(\alpha A), \gamma^*\) and we show that \(\frac{U_0^P}{\alpha} (\xi R_0, z^*(\alpha A), \gamma^*) = \frac{U_0^P}{\alpha}\). We have
\[
\inf_{P \in P_\mathbf{a}_\star \left( z \star \left( \alpha A \right) \right)} \mathbb{E}^P \left[ -\mathcal{E} \left( -R_P \int_0^T \alpha_s^{1/2} \left( 1 - z^\star (\alpha_s) \right) dW_s \alpha_s^\star (z^\star (\alpha_s)) \right) \right] = -e^{R_P R_0} e^{-R_P T H(\alpha^\star, z^\star (\alpha^\star), 0)} = U_0^P,
\]

since \( H(\alpha, z^\star (\alpha^\star), \gamma^\star) \) is actually independent of \( \alpha \). The last two results are immediate consequences of Proposition 4.2.

\[\square\]

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