Gonek, Graham, and Lee have shown recently that the Riemann Hypothesis (RH) can be reformulated in terms of certain asymptotic estimates for twisted sums with von Mangoldt function $\Lambda$. Building on their ideas, for each $k \in \mathbb{N}$, we study twisted sums with the generalized von Mangoldt function $\Lambda_k(n) := \sum_{d|n} \mu(d) \left( \log \frac{n}{d} \right)^k$

and establish similar connections with RH. For example, for $k = 2$ we show that RH is equivalent to the assertion that, for any fixed $\varepsilon > 0$, the estimate

$$\sum_{n \leq x} \Lambda_2(n) n^{-iy} = \frac{2x^{1-iy}(\log x - C_0)}{(1 - iy)} - \frac{2x^{1-iy}}{(1 - iy)^2} + O\left(x^{1/2}(x + |y|)^\varepsilon\right)$$

holds uniformly for all $x, y \in \mathbb{R}, x \geq 2$; hence, the validity of RH is governed by the distribution of almost-primes in the integers. We obtain similar results for the function

$$\Lambda^k := \Lambda \star \cdots \star \Lambda,$$

the $k$-fold convolution of the von Mangoldt function.

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1. Introduction

1.1. Background. Let $\zeta(s)$ be the Riemann zeta function. In terms of the complex variable $s = \sigma + it$, the Riemann Hypothesis (RH) is the assertion that all of the nontrivial zeros of $\zeta(s)$ lie on the line $\sigma = \frac{1}{2}$; in other words,

$$\sigma > \frac{1}{2} \implies \zeta(\sigma + it) \neq 0.$$ 

In a recent paper, Gonek, Graham, and Lee [1] have shown that a necessary and sufficient condition for the truth of the Riemann Hypothesis is that for any fixed constants $\varepsilon, B > 0$, one has the uniform estimate

$$\sum_{n \leq x} \Lambda(n) n^{-iy} = \frac{x^{1-iy}}{1-iy} + O(x^{1/2}|y|^\varepsilon) \quad (2 \leq x \leq |y|^B), \quad (1.1)$$

where $\Lambda$ is the von Mangoldt function; see [1, Thm. 1] and its proof. In the present paper, we study two distinct families of sums that are similar to the sum $\sum_{n \leq x} \Lambda(n) n^{-iy}$ above, and we show that the appropriate analogue of (1.1) also yields a necessary and sufficient condition for the truth of RH. Our results suggest that, in addition to influencing the distribution of primes, the zeros of the zeta function also exert a strong influence on the distribution of almost-primes and other related classes of integers.

1.2. Statement of results. In this paper, we study twisted sums of the form

$$\psi^k(x, y) := \sum_{n \leq x} \Lambda^k(n) n^{-iy} \quad (k \in \mathbb{N}, \ x, y \in \mathbb{R}), \quad (1.2)$$

where $\Lambda^k$ denotes the $k$-fold convolution of the von Mangoldt function $\Lambda$:

$$\Lambda^k := \underbrace{\Lambda \star \cdots \star \Lambda}_{k \text{ copies}}.$$

We have

$$\sum_{n=1}^{\infty} \frac{\Lambda^k(n)}{n^s} = (-1)^k \left\{ \frac{\zeta'}{\zeta} \right\}^k \quad (\sigma > 1). \quad (1.3)$$

Note that the function $\Lambda^k$ is supported on the set of natural numbers that have at most $k$ distinct prime divisors, i.e., positive integers for which $\omega(n) \leq k$.

**Theorem 1.1.** Fix $k \in \mathbb{N}$. If the Riemann Hypothesis is true, then

$$\psi^k(x, y) = (-1)^k \text{Res}_{w=1-iy} \left( \left\{ \frac{\zeta'}{\zeta}(w + iy) \right\}^k \frac{x^w}{w} \right) + O(x^{1/2} \{\log(x + |y|)\}^{2k+1})$$

holds uniformly for all $x, y \in \mathbb{R}, x \geq 2$, where the implied constant depends only on $k$. The residual term can be omitted if $|y| > \sqrt{x}$, and the exponent $2k+1$ can be replaced by $2$ in the case that $k = 1$.

The proof is a straightforward application of Perron’s formula coupled with bounds on $\zeta'(s)/\zeta(s)$. When $k = 1$, we follow closely the proof of [1, Thm. 1] (see also [2, Thm. 13.1]). For larger values of $k$, a slightly different approach is needed since we do not know that the zeros of the zeta function are all simple (this is not a problem when $k = 1$). At the cost of an extra factor $\log(x + |y|)$ in
the error term, we avoid this issue by shifting our line of integration close to but not beyond the vertical line $\sigma = \frac{1}{2}$.

Theorem 1.1 (with $k := 1$) yields the uniform estimate

$$\sum_{n \leq x} \Lambda(n)n^{-iy} = \frac{x^{1-iy}}{1-iy} + O(x^{1/2}\{\log(x + |y|)\}^2) \quad (x, y \in \mathbb{R}, \, x \geq 2)$$

under RH. This result strengthens the estimate (1.1) obtained in [1]. Moreover, taking $y := 0$, we recover the well known result of von Koch [5] which asserts that, under RH, one has

$$\psi(x) = x + O(x^{1/2}(\log x)^2) \quad (x \geq 2),$$

where $\psi$ is the Chebyshev function. As another example, Theorem 1.1 (with $k := 2$) provides the conditional estimate

$$\sum_{n \leq x} (\Lambda \ast \Lambda)(n)n^{-iy} = \frac{x^{1-iy}(\log x - 2C_0)}{1-iy} - \frac{x^{1-iy}}{(1-iy)^2} + O(x^{1/2}\{\log(x + |y|)\}^5)$$

holds, where $C_0$ is the Euler-Mascheroni constant. In particular, under RH we have

$$\sum_{n \leq x} (\Lambda \ast \Lambda)(n) = x(\log x - 2C_0 - 1) + O(x^{1/2}(\log x)^5).$$

Our next result is the following strong converse of Theorem 1.1.

**Theorem 1.2.** Fix $k \in \mathbb{N}$, and suppose that for any $\varepsilon > 0$ the estimate

$$\psi^k(x, y) = (-1)^k \mathop{\text{Res}}_{w=1-iy} \left( \left\{ \frac{\zeta'}{\zeta}(w + iy) \right\}^k \frac{x^w}{w} \right) + O(x^{1/2}(x + |y|)^\varepsilon)$$

holds uniformly for all $x, y \in \mathbb{R}$, $x \geq 2$, where the implied constant depends only on $k$ and $\varepsilon$. Then the Riemann Hypothesis is true.

Our proof is an adaptation of the second half of the proof of [1, Thm. 1]

We also consider twisted sums with the *generalized von Mangoldt function* (see, for example, [3, §1.4]). More specifically, we study sums of the form

$$\psi_k(x, y) := \sum_{n \leq x} \Lambda_k(n)n^{-iy} \quad (k \in \mathbb{N}, \, x, y \in \mathbb{R}),$$

where $\Lambda_k := \mu \ast L^k$ with $\mu$ the Möbius function and $L$ the natural logarithm; in other words,

$$\Lambda_k(n) := \sum_{d|n} \mu(d) \left( \log \frac{n}{d} \right)^k. \quad (1.4)$$

Note that

$$\sum_{n=1}^{\infty} \frac{\Lambda_k(n)}{n^s} = (-1)^k \frac{\zeta^{(k)}(s)}{\zeta(s)} \quad (\sigma > 1). \quad (1.5)$$

The function $\Lambda_k$ is supported on the set of natural numbers that have no more than $k$ distinct prime divisors.
Theorem 1.3. Fix \( k \in \mathbb{N} \). If the Riemann Hypothesis is true, then the estimate

\[
\psi_k(x, y) = (-1)^k \left\{ \frac{\zeta^{(k)}(w + iy)}{1 + \mu} \right\} + O\left(x^{1/2}\{\log(x + |y|)\}^{2k+1}\right)
\]

holds uniformly for all \( x, y \in \mathbb{R}, \ x \geq 2 \), where the implied constant depends only on \( k \). The residual term can be omitted if \( |y| > \sqrt{x} \), and the exponent \( 2k+1 \) can be replaced by \( 2 \) in the case that \( k = 1 \).

For example, Theorem 1.3 (with \( k := 2 \)) asserts that the conditional estimate

\[
\sum_{n \leq x} \Lambda_2(n) n^{-iy} = \frac{2x^{1-iy}(\log x - C_0)}{1 - iy} - \frac{2x^{1-iy}}{(1 - iy)^2} + O\left(x^{1/2}\{\log(x + |y|)\}^5\right)
\]

holds uniformly for all \( x, y \in \mathbb{R}, \ x \geq 2 \). In particular, under RH we have

\[
\sum_{n \leq x} \Lambda_2(n) = 2x(\log x - C_0 - 1) + O\left(x^{1/2}(\log x)^5\right).
\]

Theorem 1.4. Fix \( k \in \mathbb{N} \), and suppose that for any \( \varepsilon > 0 \) the estimate

\[
\psi_k(x, y) = (-1)^k \left\{ \frac{\zeta^{(k)}(w + iy)}{1 + \mu} \right\} + O\left(x^{1/2}(x + |y|)^\varepsilon\right)
\]

holds uniformly for all \( x, y \in \mathbb{R}, \ x \geq 2 \), where the implied constant depends only on \( k \) and \( \varepsilon \). Then the Riemann Hypothesis is true.

We expect that similar results can be achieved for a much wider class of arithmetic functions.

2. SOME AUXILIARY RESULTS

In this section, we present a series of technical lemmas that are used below in the proofs of our theorems.

Lemma 2.1. For all \( k, n \in \mathbb{N} \) we have

\[
0 \leq \Lambda^k(n) \leq \Lambda_k(n) \leq (\log n)^k.
\]

Proof. Let \( L \) be the natural logarithm function, and define \( \Lambda^0 \) and \( \Lambda_0 \) to be the indicator function of the number one. Since \( \Lambda \) takes only nonnegative values, it follows that \( \Lambda^k \geq 0 \), and the upper bound \( \Lambda_k \leq L^k \) is well known; see, for example, [3, (1.45)]. It remains to show that \( \Lambda^k \leq \Lambda_k \). Using the identity \( \Lambda_{m+1} = L \Lambda_m + \Lambda \ast \Lambda_m \) (see [3, (1.44)]) we have

\[
\Lambda^{\ell - 1} \ast \Lambda_{k-\ell+1} - \Lambda^{\ell} \ast \Lambda_{k-\ell} = \Lambda^{\ell - 1} \ast (L \Lambda_{k-\ell}) \geq 0.
\]

Summing this bound with \( \ell = 1, \ldots, k \), we see that \( \Lambda_k \geq \Lambda^k \). \( \square \)

Lemma 2.2. For all \( m \in \mathbb{N}, \ y \in \mathbb{R}, \) and \( s \in \mathbb{C} \) with \( \sigma > 2 \), we have

\[
\int_1^\infty \frac{1}{(w - 1)^m} \left( \frac{w^{s-iy}}{w - iy} \right) x^{-s} \ dx = \frac{1}{s - 1} \left( \frac{1}{(s - 2 - iy)^m} - \frac{1}{(iy - 1)^m} \right).
\]
Proof. We compute
\[
\text{Res}_{w=1} \left( \frac{1}{(w-1)^m} \frac{x^{w-iy}}{w - iy} \right) = \frac{1}{(m-1)!} \lim_{w \to 1} d_{m-1} \left( \frac{x^{w-iy}}{w - iy} \right)
\]
\[= \frac{1}{(m-1)!} \lim_{w \to 1} \sum_{j=0}^{m-1} \binom{m-1}{j} x^{w-iy} (\log x)^{m-j-1} \frac{(-1)^j j!}{(w - iy)^{j+1}}
\]
\[= - \sum_{j=0}^{m-1} \frac{x^{1-iy} (\log x)^{m-1-j}}{(m-1-j)! (iy - 1)^{j+1}}.
\] (2.1)

Using the identity
\[
\int_1^\infty x^{1-iy} (\log x)^{m-1-j} x^{-s} \, dx = \frac{(m-j-1)!}{(s-2+iy)^{m-j}},
\]
we get that
\[
\int_1^\infty \text{Res}_{w=1} \left( \frac{1}{(w-1)^m} \frac{x^{w-iy}}{w - iy} \right) x^{-s} \, dx = - \sum_{j=0}^{m-1} \frac{1}{(iy - 1)^{j+1} (s-2+iy)^{m-j}}
\]
\[= \frac{1}{s-1} \sum_{j=0}^{m-1} \frac{1}{(iy - 1)^j (s-2+iy)^{m-j}} - \frac{1}{(iy - 1)^{j+1} (s-2+iy)^{m-j-1}}.
\]

Evaluating the telescoping sum, the lemma follows. □

Lemma 2.3. For all \( k \in \mathbb{N} \) and \( x, y \in \mathbb{R} \) with \( x \geq 1 \), we have
\[
\text{Res}_{w=1-iy} \left( \left\{ \zeta'(w) \right\}^k \frac{x^w}{w} \right) \ll \frac{x(\log x)^{k-1}}{|y| + 1}
\]
and
\[
\text{Res}_{w=1-iy} \left( \left\{ \zeta^{(k)}(w) \right\} \frac{x^w}{w} \right) \ll \frac{x(\log x)^{k-1}}{|y| + 1},
\]
where the implied constants depend only on \( k \).

Proof. Using the Laurent series development of \( \left\{ \zeta'(s)/\zeta(s) \right\}^k \) at \( s = 1 \), we write
\[
\left\{ \frac{\zeta'}{\zeta}(s) \right\}^k = f(s) + g(s), \quad f(s) := \sum_{m=1}^{k} \frac{a_m}{(s-1)^m},
\]
where \( g \) is analytic in a neighborhood of one. Therefore,
\[
\text{Res}_{w=1-iy} \left( \left\{ \frac{\zeta'}{\zeta}(w + iy) \right\}^k \frac{x^w}{w} \right) = \text{Res}_{w=1} \left( \left\{ \frac{\zeta'}{\zeta}(w) \right\}^k \frac{x^{w-iy}}{w - iy} \right)
\]
\[= \sum_{m=1}^{k} a_m \text{Res}_{w=1} \left( \frac{1}{(w-1)^m} \frac{x^{w-iy}}{w - iy} \right).
\]

For each \( m = 1, \ldots, k \), we have by (2.1):
\[
\text{Res}_{w=1} \left( \frac{1}{(w-1)^m} \frac{x^{w-iy}}{w - iy} \right) \ll \frac{x(\log x)^{k-1}}{|y| + 1},
\]
and so we obtain the first bound of the lemma. The proof of the second bound is similar. □

3. Bounds on $\zeta^{(k)}/\zeta$ and $(\zeta'/\zeta)^k$

In this section, any implied constants in the symbols $O$ and $\ll$ may depend (where obvious) on the parameters $k$, $\ell$, and $\delta$, but are absolute otherwise. For any complex number $s = \sigma + it$, we denote $\tau := |t| + 2$.

**Lemma 3.1.** For any real number $T_1 \geq 2$, there exists $T \in [T_1, T_1 + 1]$ such that

$$\frac{\zeta'}{\zeta}(\sigma + iT) \ll (\log T)^2 \quad (-1 \leq \sigma \leq 2). \quad (3.1)$$

We also have

$$\frac{\zeta'}{\zeta}(-1 + it) \ll \log \tau \quad (t \in \mathbb{R}). \quad (3.2)$$

**Proof.** See [2, Lemma 12.2] and [2, Lemma 12.4]. □

In addition to Lemma 3.1 we use a conditional bound on the logarithmic derivative of $\zeta(s)$; see Proposition 3.4 below. We achieve this via the following technical lemmas.

**Lemma 3.2.** Assume RH. For any $x \geq 2$, the bounds

$$\sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right) \ll \log(x\tau) \log \tau \quad (3.3)$$

and

$$\sum_{\rho} \frac{1}{(s - \rho)^k} \ll (\log x)^k \log \tau \quad (k \geq 2) \quad (3.4)$$

hold uniformly throughout the region

$$\mathcal{R}_x := \{s \in \mathbb{C} : \frac{1}{2} + \frac{1}{\log x} \leq \sigma \leq 2, \ |s - 1| > \frac{1}{100}\},$$

where the sums in (3.3) and (3.4) run over the zeros of $\zeta(s)$ in the critical strip.

**Proof.** For every integer $n$, let $\mathcal{Z}_n$ be the multiset consisting of zeros $\rho = \frac{1}{2} + i\gamma$ of the zeta function that satisfy the equivalent conditions

$$n \leq t - \gamma \leq n + 1 \quad \iff \quad t - n - 1 \leq \gamma \leq t - n,$$

where each zero $\rho$ in $\mathcal{Z}_n$ appears a number of times that is equal to its multiplicity. Note that

$$|s - \rho| = |\sigma - \frac{1}{2} + i(t - \gamma)| \geq \max\left\{ \frac{1}{\log x}, |n| - 1 \right\} \quad (s \in \mathcal{R}_x, \ \rho \in \mathcal{Z}_n). \quad (3.5)$$

Since $\zeta(s)$ has no zeros in the critical strip with $\gamma \in [-14, 14]$, it follows that $\mathcal{Z}_n = \emptyset$ unless $|t - n| > 10$ (say). Thus, denoting

$$\tau_n := |t - n| + 2,$$

we have

$$|\rho| = \left| \frac{1}{2} + i\gamma \right| \asymp |\gamma| \asymp \tau_n \quad (\rho \in \mathcal{Z}_n). \quad (3.6)$$
and (cf. [2, Thm. 10.13])
\[ |\mathcal{Z}_n| \ll \log \tau_n. \]  
(3.7)

Let \( s \in \mathcal{R}_x \) be fixed in what follows, and define
\[ S_{k,n} := \sum_{\rho \in \mathbb{Z}_n} f_k(\rho) \quad (k \in \mathbb{N}, \ n \in \mathbb{Z}), \]
where
\[ f_1(\rho) := \left| \frac{1}{s - \rho} + \frac{1}{\rho} \right| \quad \text{and} \quad f_k(\rho) := \left| \frac{1}{(s - \rho)^k} \right| \quad (k \geq 2). \]
and observe that
\[ \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right) \ll \sum_{n \in \mathbb{Z}} S_{1,n} \quad \text{and} \quad \sum_{\rho} \frac{1}{(s - \rho)^k} \ll \sum_{n \in \mathbb{Z}} S_{k,n} \quad (k \geq 2). \]

If \(|n| \leq 10\), then by (3.5) and (3.6):
\[ f_1(\rho) \ll \frac{1}{|s - \rho|} + \frac{1}{|\rho|} \ll \log x \quad (\rho \in \mathcal{Z}_n), \]
and
\[ f_k(\rho) = \frac{1}{|s - \rho|^k} \ll (\log x)^k \quad (k \geq 2, \ \rho \in \mathcal{Z}_n), \]
Using (3.7) we get that
\[ \sum_{|n| \leq 10} S_{k,n} \ll (\log x)^k \sum_{|n| \leq 10} \log \tau_n \ll (\log x)^k \log \tau \quad (k \in \mathbb{N}). \]
(3.8)

Next, suppose that \(|n| > 10\). For \( k \geq 2 \), using (3.5) and (3.7) we have
\[ f_k(\rho) = \frac{1}{|s - \rho|^k} \ll \frac{1}{|n|^k} \quad \Rightarrow \quad S_{k,n} \ll \sum_{\rho \in \mathbb{Z}_n} \frac{1}{|n|^k} \ll \frac{\log \tau_n}{|n|^k} \]
\[ \Rightarrow \quad \sum_{|n| > 10} S_{k,n} \ll \log \tau. \]

Combined with (3.8), this completes the proof of (3.4). When \( k = 1 \), we have by (3.5), (3.6), and (3.7):
\[ f_1(\rho) = \left| \frac{s}{(s - \rho)\rho} \right| \ll \frac{\tau}{|n|\tau_n} \quad \Rightarrow \quad S_{1,n} \ll \sum_{\rho \in \mathbb{Z}_n} \frac{\tau}{|n|\tau_n} \ll \frac{\tau \log \tau_n}{|n|\tau_n}. \]

Recalling the definition of \( \tau_n \), we see that
\[ \tau_n \asymp \begin{cases} \tau & \text{if } 10 < |n| \leq \frac{1}{2}|t|, \\ |t - n| + 2 & \text{if } \frac{1}{2}|t| < |n| \leq 2|t|, \\ |n| & \text{if } |n| > 2|t|. \end{cases} \]

In addition to this, \( \tau_n \ll \tau \) in the second range above, and \(|n| \gg \tau \) in the last two ranges. Putting everything together, we have
\[ \sum_{|n| > 10} S_{1,n} \ll \sum_{10 < |n| \leq \frac{1}{2}|t|} \frac{\log \tau}{|n|} + \sum_{\frac{1}{2}|t| < |n| \leq 2|t|} \frac{\log \tau}{|t - n| + 2} + \sum_{|n| > 2|t|} \frac{\tau \log |n|}{|n|^2} \ll (\log \tau)^2. \]

Combined with (3.8), this completes the proof of (3.3). \( \square \)
Let $f$ be a twice-differentiable function on $(0, \infty)$ such that

(i) $f(u) \to 0$ and $f'(u) \to 0$ as $u \to \infty$;
(ii) $f''$ is integrable on $(1, \infty)$.

The relation

$$\sum_{n=1}^{\infty} f'(n) = -f(1) + f'(1) + \int_{1}^{\infty} f''(u)\{u\} \, du,$$

which is immediate using Riemann-Stieltjes integration, plays an important role in the proof of the following lemma.

**Lemma 3.3.** Let $\delta > 0$ be fixed. The estimates

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O(\tau^{-1})$$

and

$$\frac{d^\ell}{ds^\ell} \frac{\Gamma'(s)}{\Gamma(s)} = (-1)^{\ell-1}(\ell - 1)! \, s^{-\ell} + O(\tau^{-\ell-1}) \quad (\ell \geq 1)$$

hold uniformly throughout the half-plane $\{\sigma \geq \delta\}$.

**Proof.** The gamma function $\Gamma$ can be defined as the reciprocal of the Weierstrass product (see, e.g., [4, Chap. XII]):

$$\Gamma(s) := s^{-1} e^{-\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right)^{-1} e^{s/n},$$

which implies that

$$\frac{\Gamma'(s)}{\Gamma(s)} = -\frac{1}{s} - \gamma + s \sum_{n=1}^{\infty} \frac{1}{n(n + s)}.$$  

Using (3.9) with

$$f(u) := \log \left(\frac{u}{u + s}\right),$$

we have

$$s \sum_{n=1}^{\infty} \frac{1}{n(n + s)} = \log(s + 1) + 1 - \frac{1}{s + 1} - \int_{1}^{\infty} \{u\} \left(\frac{1}{u^2} - \frac{1}{(u + s)^2}\right) \, du$$

$$= \log s + \gamma + O(\tau^{-1}).$$

Combining this with (3.12), the first statement of the lemma is proved.

For any positive integer $\ell$, from (3.12) we deduce that

$$\frac{d^\ell}{ds^\ell} \frac{\Gamma'(s)}{\Gamma(s)} = (-1)^{\ell-1} \left\{ \frac{1}{s^{\ell+1}} + \sum_{n=1}^{\infty} \frac{1}{(n + s)^{\ell+1}} \right\}.$$  

Applying (3.9) with

$$f(u) := -\frac{1}{\ell(u + s)\ell},$$
we have
\[
\sum_{n=1}^{\infty} \frac{1}{(n+s)^{\ell+1}} = \frac{1}{\ell(s+1)^{\ell}} + \frac{1}{(s+1)^{\ell+1}} - (\ell + 1) \int_{1}^{\infty} \frac{\{u\}}{(u+s)^{\ell+2}} du
\]
\[= \frac{1}{\ell(s+1)^{\ell}} + O(\tau^{-\ell-1}).\]
Combining this with (3.13), we obtain the second statement of the lemma. \(\square\)

PROPOSITION 3.4. Assume RH. For any \(k \in \mathbb{N}\) and \(x \geq 2\), the bounds
\[
\left\{ \frac{\zeta'(s)}{\zeta(s)} \right\}^k \ll \left( \log(x\tau) \log \tau \right)^k
\]
and
\[
\frac{\zeta^{(k)}(s)}{\zeta(s)} \ll \left( \log(x\tau) \log \tau \right)^k
\]
hold uniformly throughout the region
\[
\mathcal{R}_x := \left\{ s \in \mathbb{C} : \frac{1}{2} + \frac{1}{\log x} \leq \sigma \leq 2, |s - 1| > \frac{1}{100} \right\}.
\]

Proof. It suffices to prove (3.15), since (3.14) follows from case \(k = 1\) of (3.15).

Let \(z_j := \zeta^{(j)}/\zeta\) for each \(j \in \mathbb{N}\). Our goal is to show that
\[
z_j \ll \left( \log(x\tau) \log \tau \right)^j \quad (s \in \mathcal{R}_x)
\]
holds for all \(j\), where the implied constant depends only on \(j\).

For \(j = 1\) we have (cf. [2, (10.29)])
\[
z_1(s) = B + \frac{1}{2} \log \pi - \frac{1}{s-1} - \frac{1}{2} \Gamma'(s/2 + 1) + \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right),
\]
where \(B\) is a constant, and the sum runs over the zeros of \(\zeta(s)\) in the critical strip. The case \(j = 1\) of (3.16) follows from (3.17) and Lemmas 3.2 and 3.3.

Now suppose (3.16) holds for all positive integers \(j < k\), where \(k \geq 2\). On the one hand, differentiating both sides of (3.17) precisely \(k - 1\) times, we have
\[
z_1^{(k-1)}(s) = \frac{(-1)^k (k-1)!}{(s-1)^k} - \frac{1}{2k} \frac{d^{k-1}}{ds^{k-1}} \Gamma'(s/2 + 1) + \sum_{\rho} \frac{(-1)^{k-1} (k-1)!}{(s - \rho)^k}.
\]
Using Lemmas 3.2 and 3.3 again, this implies that the bound
\[
z_1^{(k-1)}(s) \ll (\log x)^k \log \tau
\]
holds uniformly in \(\mathcal{R}_x\). On the other hand, by the quotient rule, one has the simple relation
\[
z_j' = z_{j+1} - z_j z_m \quad (m \in \mathbb{N}).
\]
Using an inductive argument, one can show that
\[
z_m^{(n)} = z_{m+n} + P_n(z_1, \ldots, z_{m+n-1}) \quad (m, n \in \mathbb{N}),
\]
where each \(P_n\) is a polynomial in \(\mathbb{Z}[X_1, \ldots, X_{m+n-1}]\) whose monomials all have the form \(\prod_j X_j^{n_j}\) for some integers \(n_j \geq 0\) that satisfy \(\sum_j jn_j = m + n\). In particular, we have
\[
z_1^{(k-1)} - z_k = P_{k-1}(z_1, \ldots, z_{k-1}),
\]
and each monomial term occurring in the polynomial on the right side satisfies the uniform bound
\[
\prod_{j=1}^{k-1} z_j(s)^n_j \ll \prod_{j=1}^{k-1} \left( \log(x\tau) \log \tau \right)^{n_j} = \left( \log(x\tau) \log \tau \right)^k \quad (s \in \mathcal{R}_x).
\]
In other words,
\[
z_1^{(k-1)}(s) - z_k(s) \ll \left( \log(x\tau) \log \tau \right)^k \quad (s \in \mathcal{R}_x).
\]
Combining this bound with (3.18), it follows that (3.16) holds when \( j = k \). This completes the induction, and the proposition is proved. \( \square \)

4. Proof of Theorems 1.1 and 1.3

Both theorems are proved in parallel.

Let the integer \( k \in \mathbb{N} \) be fixed throughout. Below, any implied constants in the symbols \( O \) and \( \ll \) may depend (where obvious) on \( k \) but are independent of other parameters.

For the proof of Theorem 1.1, we set
\[
a_n(y) := \Lambda^k(n)n^{-iy} \quad \text{and} \quad \alpha(y, s) := \sum_{n=1}^{\infty} \frac{a_n(y)}{n^s} = (-1)^k \left\{ \frac{\zeta'}{\zeta}(s + iy) \right\}^k.
\]
Note that
\[
\sum_{n \leq x} a_n(y) = \psi^k(x, y).
\]
On the other hand, for the proof of Theorem 1.3, we put
\[
a_n(y) := \Lambda_k(n)n^{-iy} \quad \text{and} \quad \alpha(y, s) := \sum_{n=1}^{\infty} \frac{a_n(y)}{n^s} = (-1)^k \frac{\zeta^{(k)}}{\zeta}(s + iy),
\]
and we have
\[
\sum_{n \leq x} a_n(y) = \psi_k(x, y).
\]
In either case, our aim is to show that
\[
\sum_{n \leq x} a_n(y) = \text{Res}_{w=1-iy} \left( \alpha(y, w) \frac{x^w}{w} \right) + O\left( x^{1/2} \{ \log(x + |y|) \}^{2k+1} \right),
\]
where the exponent \( 2k+1 \) can be replaced by \( 2 \) in the case that \( k = 1 \). Note that Lemma 2.3 guarantees that the residual term can be omitted when \( |y| > \sqrt{x} \).

Let \( x, y \in \mathbb{R} \) with \( x \geq 2 \). Let
\[
\sigma_0 := 1 + 1/\log x \quad \text{and} \quad T \in [\sqrt{x} + 10, \sqrt{x} + 11],
\]
where \( T \) is chosen so that (3.2) holds. Adjusting \( T \) slightly, we can assume that \( T \) is not the ordinate of any zero of the zeta function. By Perron’s formula (see [2, Thm. 5.2 and Cor. 5.3]) we have
\[
\sum_{n \leq x} a_n(y) = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \alpha(y, s) \frac{x^s}{s} ds + O(E_1 + E_2 + E_3), \quad (4.1)
\]
where the error terms are given by
\[ E_1 := \sum_{x/2 < n < 2x} a_n(0) \min \left\{ 1, \frac{x}{T|n|} \right\}, \quad E_2 := \frac{x}{T} \sum_{n=1}^{\infty} a_n(0), \]
and
\[ E_3 := \begin{cases} a_n(0) & \text{if } x \in \mathbb{N}, \\ 0 & \text{otherwise}. \end{cases} \]
Since \( a_n(0) = \Lambda^k(n) \) or \( \Lambda_k(n) \), Lemma 2.1 shows that
\[ E_1 \ll \frac{x(\log x)^{k+1}}{T} \ll x^{1/2}(\log x)^{k+1} \quad \text{and} \quad E_3 \ll (\log x)^{k}. \quad (4.2) \]
Moreover, as
\[ \left| \sum_{n=1}^{\infty} \frac{a_n(0)}{n^{\sigma_0}} \right| = |\alpha(0, \sigma_0)| = \left| \frac{\zeta'(\sigma_0)}{\zeta(\sigma_0)} \right|^k \quad \text{or} \quad \left| \frac{\zeta^{(k)}(\sigma_0)}{\zeta(\sigma_0)} \right|, \]
using the Laurent series development of \((\zeta'(s)/\zeta(s))^k\) or \((\zeta^{(k)}(s)/\zeta(s))\) at \( s = 1 \), we derive the bound
\[ E_2 \ll \frac{x(\log x)^{k}}{T} \ll x^{1/2}(\log x)^{k}. \quad (4.3) \]
Combining (4.2) and (4.3) with (4.1), we derive the estimate
\[ \sum_{n \leq x} a_n(y) = \frac{1}{2\pi i} \int_{\sigma_0-iT}^{\sigma_0+iT} \alpha(y, s) \frac{x^s}{s} ds + O\left( x^{1/2}(\log x)^{k+1} \right). \quad (4.4) \]

Next, we shift the line of integration of the integral in (4.4). For this, we study the cases \( k = 1 \) and \( k \geq 2 \), separately.

**CASE 1 (\( k = 1 \)).** In this case, for both Theorems 1.1 and 1.3 we have
\[ \alpha(y, s) = -\frac{\zeta'}{\zeta}(s + iy). \]
Let \( \mathcal{C} \) be the rectangular contour in \( \mathbb{C} \) that connects
\[ \sigma_0 - iT \rightarrow \sigma_0 + iT \rightarrow -1 + iT \rightarrow -1 - iT \rightarrow \sigma_0 - iT. \]
Along the horizontal segment \( \sigma = \sigma + iT \) with \( \sigma \in (-1, \sigma_0) \), by (3.1) we see that
\[ \alpha(y, s) \ll \mathcal{L}^2, \quad \text{where} \quad \mathcal{L} := \log(x + |y|); \]
consequently,
\[ \int_{-1+iT}^{\sigma_0+iT} \alpha(y, s) \frac{x^s}{s} ds \ll \frac{\mathcal{L}^2}{T} \int_{-1}^{\sigma_0} x^\sigma d\sigma \ll \frac{x^{\sigma_0} \mathcal{L}^2}{T \log x} \ll x^{1/2} \mathcal{L}^2. \quad (4.5) \]
Similarly,
\[ \int_{-1-iT}^{\sigma_0-iT} \alpha(y, s) \frac{x^s}{s} ds \ll x^{1/2} \mathcal{L}^2. \quad (4.6) \]
On the other hand, using (3.2) we have \( \alpha(y, s) \ll \mathcal{L} \) along the vertical segment \( s = -1 + it \) with \( |t| < T \); thus,
\[ \int_{-1-iT}^{-1+iT} \alpha(y, s) \frac{x^s}{s} ds \ll x^{-1} \mathcal{L} \int_{-T}^{T} \frac{dt}{|t| + 1} \ll x^{-1} \mathcal{L}^2. \quad (4.7) \]
Combining (4.5), (4.6), and (4.7) with our previous estimate (4.4), we have
\[
\sum_{n \leq x} a_n(y) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \alpha(y, s) \frac{x^s}{s} ds + O(x^{1/2}L^2). \tag{4.8}
\]
Since \(k = 1\), the poles of the integrand inside the contour are all simple. If \(|y| < T\), there is a pole at \(s = 1 - iy\) with residue
\[
- \text{Res}_{w=1-iy} \left( \frac{\zeta'(w + iy)}{\zeta(w + iy)} \right) = \frac{x^{1-iy}}{1-iy}.
\]
The pole at \(s = 0\) contributes has residue \(-\zeta'(iy)/\zeta(iy) \ll L\), which can be thrown into the error term (for the bound, see [2, Lemma 12.1]). The remaining poles occur at points \(s = \rho - iy\) for which \(\rho = \beta + i\gamma\) is a zero of the zeta function with \(|\gamma - y| < T\). Arguing as in the proof of Lemma 3.2, for each integer \(n \in [0, T]\) there are at most \(O(L)\) zeros \(\rho = \frac{1}{2} + i\gamma\) of \(\zeta(s)\) such that \(n \leq |\gamma - y| \leq n + 1\); hence the sum of the residues from such zeros is
\[
- \sum_{\rho=\beta+i\gamma \atop |\gamma-y|<T} \frac{x^{\rho-iy}}{\rho-iy} \ll x^{1/2} \sum_{0 \leq n \leq T} \sum_{n \leq |\gamma-y| \leq n+1} \frac{1}{n+1} \ll x^{1/2} L \sum_{0 \leq n \leq T} \frac{1}{n+1} \ll x^{1/2} L^2.
\]
Putting everything together, both theorems are proved for \(k = 1\).

**Case 2** (\(k \geq 2\)). In this case, let \(\mathcal{C}\) be the contour
\[
\sigma_0 - iT \longrightarrow \sigma_0 + iT \longrightarrow \frac{1}{2} + \frac{1}{\log x} + iT \longrightarrow \frac{1}{2} + \frac{1}{\log x} - iT \longrightarrow \sigma_0 - iT.
\]
Since every number \(s + iy = \sigma + i(T + y)\) with \(\sigma \in (\frac{1}{2} + \frac{1}{\log x}, \sigma_0)\) is contained in the region \(\mathcal{R}_x\) defined in Proposition 3.4, and for such \(s + iy\) one has
\[
\tau = |T + y| + 2 \ll x + |y|,
\]
by the proposition we get that
\[
\alpha(y, s) \ll (\log(x\tau) \log \tau)^k \ll L^{2k}, \quad L := \log(x + |y|). \tag{4.9}
\]
Consequently,
\[
\int_{\frac{1}{2} + \frac{1}{\log x} + iT}^{\sigma_0 + iT} \alpha(y, s) \frac{x^s}{s} ds \ll \frac{L^{2k}}{T} \int_{\frac{1}{2} + \frac{1}{\log x}}^{\sigma_0} x^\sigma d\sigma \ll \frac{x^{\sigma_0} L^{2k}}{T \log x} \ll x^{1/2} L^{2k}. \tag{4.10}
\]
Similarly,
\[
\int_{\frac{1}{2} + \frac{1}{\log x} - iT}^{\sigma_0 - iT} \alpha(y, s) \frac{x^s}{s} ds \ll x^{1/2} L^{2k}. \tag{4.11}
\]
On the vertical segment, \(s = \frac{1}{2} + \frac{1}{\log x} + it\) with \(|t| < T\), the number \(s + iy\) again lies in \(\mathcal{R}_x\). For such \(s + iy\) one has
\[
\tau = |t + y| + 2 \ll x + |y|,
\]
thus Proposition 3.4 again yields the uniform bound (4.9) along the vertical segment. Consequently,
\[
\int_{\frac{1}{2} + \frac{1}{\log x} + iT}^{\frac{1}{2} + \frac{1}{\log x} + iT} \alpha(y, s) \frac{x^s}{s} ds \ll x^{\frac{1}{2} + \frac{1}{\log x}} L^{2k} \int_{-T}^{T} \frac{dt}{|t| + 1} \ll x^{1/2} L^{2k+1}. \tag{4.12}
\]
Combining (4.10), (4.11), and (4.12) with our previous estimate (4.4), we have

\[ \sum_{n \leq x} a_n(y) = \frac{1}{2\pi i} \oint_{\mathcal{C}} \alpha(y, s) \frac{x^s}{s} ds + O(x^{1/2} \mathcal{L}^{2k+1}). \]  

(4.13)

The result follows from Cauchy’s theorem, since the only pole of the integrand with \( \sigma > \frac{1}{2} \) occurs at the point \( s = 1 - iy \). Note that this pole does not lie inside \( \mathcal{C} \) when \( |y| > T \), and in that case the integral vanishes from (4.13).

5. Proof of Theorems 1.2 and 1.4

As in the previous section, we prove both theorems in parallel.

Let the integer \( k \geq 2 \) be fixed throughout, and fix \( \varepsilon > 0 \). In what follows, any implied constants in the symbols \( O \) and \( \ll \) may depend (where obvious) on \( k \) and \( \varepsilon \) but are independent of other parameters.

We continue to use notation introduced in the previous section. In particular, for the proof of Theorem 1.2 we have \( \alpha(0, s) = (-1)^k \{ \zeta'(s)/\zeta(s) \}^k \), and for the proof of Theorem 1.4, \( \alpha(0, s) = (-1)^k \zeta(k)(s)/\zeta(s) \). In either case, the Laurent series development of \( \alpha(0, s) \) at \( s = 1 \) has the form

\[ \alpha(0, s) = f(s) + g(s), \quad f(s) := \sum_{m=1}^{k} \frac{a_m}{(s-1)^m}, \quad g(s) := \sum_{n=0}^{\infty} b_n (s-1)^n. \]

Both \( f \) and \( g \) continue meromorphically to the entire complex plane. Following the method of [1], let us denote

\[ \Psi(x, y) := \sum_{n \leq x} a_n(y) \]

(hence \( \Psi = \psi^k \) or \( \psi_k \)) and

\[ R(x, y) := \Psi(x, y) - \operatorname{Res}_{w=1-iy} \left( \alpha(y, w) \frac{x^w}{w} \right). \]

Our hypothesis (in both theorems) is that

\[ R(x, y) \ll x^{1/2} (x + |y|)^{\varepsilon}. \]  

(5.1)

Note that

\[ R(x, y) = \Psi(x, y) - \operatorname{Res}_{w=1} \left( f(w) \frac{x^{w-iy}}{w-iy} \right) \]

since \( g \) is analytic in a neighborhood of one.

Let \( H \) be the meromorphic function defined in half-plane \( \sigma > 2 \) by

\[ H(s) := \int_{1}^{\infty} R(x, y) x^{-s} dx = \int_{1}^{\infty} \Psi(x, y) x^{-s} dx - \int_{1}^{\infty} \operatorname{Res}_{w=1} \left( f(w) \frac{x^{w-iy}}{w-iy} \right) x^{-s} dx. \]

In the same half-plane, we have

\[ \int_{1}^{\infty} \Psi(x, y) x^{-s} dx = \int_{1}^{\infty} \left( \sum_{n \leq x} a_n(y) \right) x^{-s} dx = \sum_{n=1}^{\infty} a_n(y) \int_{n}^{\infty} x^{-s} dx \]

\[ = \sum_{n=1}^{\infty} a_n(y) \cdot \frac{n^{1-s}}{s-1} = \frac{1}{s-1} \alpha(y, s-1). \]
Moreover, using Lemma 2.2 we have
\[
\int_1^\infty \text{Res}_{w=1} \left( f(w) \frac{x^{w-iy}}{w-iy} \right) x^{-s} \, dx = \sum_{m=1}^{k} a_m \int_1^\infty \text{Res}_{w=1} \left( \frac{1}{(w-1)^m w-iy} \right) x^{-s} \, dx
\]
\[
= \frac{1}{s-1} \sum_{m=1}^{k} a_m \left( \frac{1}{(s-2+iy)^m} - \frac{1}{(iy-1)^m} \right)
\]
\[
= \frac{1}{s-1} \left( f(s-1+iy) + C \right),
\]
where \( C \) is a constant that depends only on \( k \) and \( y \). Therefore,
\[
H(s) = \frac{1}{s-1} \left( \alpha(y, s-1) - f(s-1+iy) - C \right).
\]
For the sake of clarity, we adopt the following notation:
\[
\hat{s} := \sigma - 1, \quad \hat{t} := y + t, \quad \hat{s} := \hat{s} + i\hat{t} = s - 1 + iy, \quad K(\hat{s}) := H(\hat{s} + 1 - iy) = H(s).
\]
Then, throughout the half-plane \( \hat{s} > 1 \) we have
\[
K(\hat{s}) = \frac{1}{\hat{s} - iy} \left( \alpha(0, \hat{s}) - f(\hat{s}) - C \right) = \frac{1}{\hat{s} - iy} (g(\hat{s}) - C),
\]
and so it is clear that \( K(\hat{s}) \) has meromorphic continuation to the entire complex plane. Taking into account that \( g(\hat{s}) \) is analytic at \( \hat{s} = 1 \), and that the only pole of \( f(\hat{s}) \) occurs at \( \hat{s} = 1 \), we see that the only poles of \( K(\hat{s}) \) in the half-plane \( \hat{s} > 0 \) occur at zeros of the zeta function. For any such zero \( \rho \), let
\[
k_*(\rho) := \text{the order of the pole of } K(\hat{s}) \text{ at } \hat{s} = \rho.
\]
If \( \alpha(0, \hat{s}) = (-1)^k \{ \zeta'(\hat{s})/\zeta(\hat{s}) \}^k \) (Theorem 1.2), then \( k_*(\rho) = k \) for every zero \( \rho \) of the zeta function. When \( \alpha(0, s) = (-1)^k \zeta^{(k)}(s)/\zeta(s) \) (Theorem 1.4), each zero \( \rho \) gives rise to a pole of \( K(\hat{s}) \) of order \( k_*(\rho) \leq k \). These statements are easily verified by examining the Taylor series development of \( \zeta(\hat{s}) \) at the point \( \hat{s} = \rho \).

Our aim is to show that (5.1) implies RH. Suppose, on the contrary, that (5.1) holds, and \( \rho_0 = \beta_0 + i\gamma_0 \) is a zero of the zeta function with \( \beta_0 > \frac{1}{2} \). Let \( m \) be the multiplicity of \( \rho_0 \), and define
\[
\kappa(\hat{s}) := \frac{(\hat{s} - 1)^k \zeta(\hat{s})^k}{(\hat{s} - \rho_0)^{mk-k_*(\rho)+1} (\hat{s} + 2)^k}, \quad h(s) := \kappa(s-1+iy) = \kappa(\hat{s}).
\]
The function \( \kappa(\hat{s}) \) is meromorphic for \( \hat{s} \in \mathbb{C} \) and has been crafted so that (among other things) the product \( \kappa(\hat{s})K(\hat{s}) \) has no pole in the half-plane \( \hat{s} > 0 \) other than a simple pole at \( \hat{s} = \rho_0 \).

We begin by observing that
\[
\frac{1}{2\pi i} \int_{3-i\infty}^{3+i\infty} h(s)H(s)e^{s \log x} \, ds = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \kappa(\hat{s})K(\hat{s})e^{(\hat{s}+1-iy) \log x} \, d\hat{s}.
\]
In the latter integral, we shift the line of integration left to the line \( \hat{s} = \frac{1}{2} \), passing only the pole at \( \hat{s} = \rho_0 \); the residue at that point is \( c x^{\rho_0 - iy} + 1 \), where
\[
c := \frac{(-1)^k (\rho_0 - 1)^k \zeta^{(m)}(\rho_0))^k}{((m-1)!)^k (\rho_0 - iy)(\rho_0 + 2)^k} \quad \text{(Theorem 1.2)}.
\]
or
\[
c := (-1)^k \binom{k}{k} (m(m-1) \ldots (m-k_s(\rho_0) + 1) \zeta^{(m+k-k_s(\rho_0))}(\rho_0)(\zeta^{(m)}(\rho_0))^{k-1}}{(m!)^{k-1}(m+k-k_s(\rho_0))!(\rho_0 - iy)(\rho_0 + 2)^{4k}} \tag{Theorem 1.4}
\]

Applying the bounds\(^1\)
\[
\zeta^{(j)}(\frac{1}{4} + iv) \ll 1 + |v| \quad \text{and} \quad f(\frac{1}{4} + iv) \ll 1 \quad (v \in \mathbb{R}, \ j = 0, 1, \ldots, k),
\]
we derive that
\[
\left| \int_{1/4 - i\infty}^{1/4 + i\infty} \kappa(\hat{s})K(\hat{s})e^{(\hat{s} + 1 - iy)\log x}d\hat{s} \right| \ll x^{5/4} \int_{-\infty}^{\infty} \left| \kappa(\frac{1}{4} + iv)K(\frac{1}{4} + iv) \right| dv \ll x^{5/4}.
\]

Consequently,
\[
\frac{1}{2\pi i} \int_{3 - i\infty}^{3 + i\infty} h(s)H(s)e^{s\log x}ds = c_0 \rho_0 - iy + 1 + O(x^{5/4}). \tag{5.2}
\]

Next, we evaluate the left side of (5.2) in a different way. Put
\[
w(u) := \frac{1}{2\pi i} \int_{3 - i\infty}^{3 + i\infty} h(s)e^{usu} ds. \tag{5.3}
\]

We have
\[
h(s)e^{usu} = \frac{(s + iy - 2)^k \zeta(s - 1 + iy)^k e^{usu}}{(s - 1 + iy - \rho_0)^{mk - k_s(\rho_0) + 1}(s + iy + 1)^{4k}}.
\]

When \(u < 0\), we use the uniform bound
\[
h(s)e^{usu} \ll \frac{e^{\sigma u}}{(1 + |y + t - \gamma_0|)^{mk - k_s(\rho_0) + 1}(1 + |y + t|)^{3k}} \quad (\sigma \geq 3).
\]

Shifting the line of integration in the integral (5.3) right to +\(\infty\), we conclude that \(w(u) = 0\). On other hand, when \(u \geq 0\) we shift the line of integration left to the line \(\sigma = -\frac{5}{4}\). We pass a pole of order 4\(k\) at the point \(s = -1 - iy\), which contributes a residue of size \(O(1)\). Using the bound (cf. [2, Cor. 10.5])
\[
|\zeta(s - 1 + iy)| \ll (1 + |y + t|)^{11/4} \quad (\sigma = -\frac{5}{4}),
\]

it follows that
\[
h(s)e^{usu} \ll \frac{e^{-5u/4}}{(1 + |y + t - \gamma_0|)^{mk - k_s(\rho_0) + 1}(1 + |y + t|)^{k/4}}
\]
on the line \(\sigma = -\frac{5}{4}\), so the integral on the new line converges absolutely. To summarize, we have shown that
\[
w(u) = \begin{cases} 
0 & \text{if } u < 0, \\
O(1) & \text{if } u \geq 0.
\end{cases} \tag{5.4}
\]

\(^1\)More precisely, one has \(\zeta^{(j)}(\frac{1}{4} + iv) \ll (1 + |v|)^{1/2}(\log(|v| + 2))^{j}\) for fixed \(j = 0, 1, \ldots, k\).
Now,
\[
\frac{1}{2\pi i} \int_{3-i\infty}^{3+i\infty} h(s) H(s) e^{s \log x} \, ds = \frac{1}{2\pi i} \int_{3-i\infty}^{3+i\infty} h(s) \left( \int_{1}^{\infty} R(z, y) z^{-s} \, dz \right) e^{s \log x} \, ds
\]
\[
= \int_{1}^{\infty} R(z, y) \left( \frac{1}{2\pi i} \int_{3-i\infty}^{3+i\infty} h(s) e^{s \log x - \log z} \, ds \right) \, dz
\]
\[
= \int_{1}^{\infty} R(z, y) w(\log x - \log z) \, dz \ll \int_{1}^{x} R(z, y) \, dz,
\]
where we used (5.4) in the last step.

Combining the previous bound with (5.2) we see that
\[
x^{\beta_0 + 1} \ll \left| c \, x^{\rho_0 - iy + 1} \right| \ll \int_{1}^{x} R(z, y) \, dz + x^{5/4},
\]
where \( \beta_0 \) is the real part of \( \rho_0 \). Recalling (5.1) we have
\[
\int_{1}^{x} R(z, y) \, dz \ll \int_{1}^{x} z^{1/2}(z + |y|)^{\varepsilon} \, dz \ll x^{3/2}(x + |y|)^{\varepsilon},
\]
and therefore
\[
x^{\beta_0 + 1} \ll x^{3/2}(x + |y|)^{\varepsilon}
\]
for every \( \varepsilon > 0 \). Since \( \beta_0 > \frac{1}{2} \), this leads to the desired contradiction, concluding the proof of the theorems.

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