Modified gravity models of dark energy

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We review recent progress of modified gravity models of dark energy–based on \( f(R) \) gravity, scalar-tensor theories, braneworld gravity, Galileon gravity, and other theories. In \( f(R) \) gravity it is possible to design viable models consistent with local gravity constraints under a chameleon mechanism, while satisfying conditions for the cosmological viability. We also construct a class of scalar-tensor dark energy models based on Brans-Dicke theory in the presence of a scalar-field potential with a large coupling strength \( Q \) between the field and non-relativistic matter in the Einstein frame. We study the evolution of matter density perturbations in \( f(R) \) and Brans-Dicke theories to place observational constraints on model parameters from the power spectra of galaxy clustering and Cosmic Microwave Background (CMB).

The Dvali-Gabadazde-Porrati braneworld model can be compatible with local gravity constraints through a nonlinear field self-interaction \( \Box \phi (\partial_\mu \phi \partial^\mu \phi) \) arising from a brane-bending mode, but the self-accelerating solution contains a ghost mode in addition to the tension with the combined data analysis of Supernovae Ia (SN Ia) and Baryon Acoustic Oscillations (BAO). The extension of the field self-interaction to more general forms satisfying the Galilean symmetry \( \partial_\mu \phi \to \partial_\mu \phi + b_\mu \) in the flat space-time allows a possibility to avoid the appearance of ghosts and instabilities, while the late-time cosmic acceleration can be realized by the field kinetic energy. We study observational constraints on such Galileon models by using the data of SN Ia, BAO, and CMB shift parameters.

We also briefly review other modified gravitational models of dark energy–such as those based on Gauss-Bonnet gravity and Lorentz-violating theories.

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I. INTRODUCTION

The cosmic acceleration today has been supported by independent observational data such as the Supernovae-type Ia (SN Ia) [1, 2], the Cosmic Microwave Background (CMB) temperature anisotropies measured by WMAP [3, 4], and Baryon Acoustic Oscillations [5, 6]. The origin of dark energy responsible for this cosmic acceleration is one of the most serious problems in modern cosmology [7–20]. The cosmological constant is one of the simplest candidates for dark energy, but it is plagued by a severe energy scale problem if it originates from the vacuum energy appearing in particle physics [21].

The first step toward understanding the nature of dark energy is to clarify whether it is a simple cosmological constant or it originates from other sources that dynamically change in time. The dynamical models can be distinguished from the cosmological constant by considering the evolution of the equation of state of dark energy ($w_{DE}$). The scalar field models of dark energy such as quintessence [22–26] and k-essence [27, 28] predict a wide variety of conditions required for the viability of scalar-field models in the framework of particle physics because of a very tiny mass ($m_\phi \lesssim 10^{-33}$ eV) required for the cosmic acceleration today [29, 30].

There exists another class of dynamical dark energy models based on the large-distance modification of gravity. The models that belong to this class are $f(R)$ gravity [31, 32] (f is function of the Ricci scalar $R$), scalar-tensor theories [33, 40], braneworld models [11], Galileon gravity [42], Gauss-Bonnet gravity [43, 44], and so on. An attractive feature of these models is that the cosmic acceleration can be realized without recourse to a dark energy matter component. If we modify gravity from General Relativity (GR), however, there are tight constraints coming from local gravity tests as well as a number of observational constraints. Hence the restriction on modified gravity models is in general stringent compared to modified matter models (such as quintessence and k-essence).

For example, an $f(R)$ model of the form $f(R) = R - \mu^{2(n+1)}/R^n$ ($n > 0$) was proposed to explain the late-time cosmic acceleration [32, 33] (see also Refs. [45, 50] for early works). However this model suffers from a number of problems such as the incompatibility with local gravity constraints [51, 54], the instability of density perturbations [55, 58], and the absence of a matter-dominated epoch [54, 60]. As we will see in this review there are a number of conditions required for the viability of $f(R)$ dark energy models [53, 58, 61, 67], which stimulated to propose viable models [68, 72].

The simplest version of scalar-tensor theories is so-called Brans-Dicke theory in which a scalar field $\phi$ couples to the Ricci scalar $R$ with the Lagrangian density $\mathcal{L} = \varphi R/2 - (\omega_{BD}/2\varphi)(\nabla \varphi)^2$, where $\omega_{BD}$ is a so-called Brans-Dicke parameter [72]. GR can be recovered by taking the limit $\omega_{BD} \rightarrow \infty$. If we allow the presence of the field potential $U(\varphi)$ in Brans-Dicke theory, $f(R)$ theory in the metric formalism is equivalent to this generalized Brans-Dicke theory with the parameter $\omega_{BD} = 0$ [51, 74]. By transforming the action in generalized Brans-Dicke theory (“Jordan frame”) to an “Einstein frame” action by a conformal transformation, the theory in the Einstein frame is equivalent to a coupled quintessence scenario [72] with a constant coupling $Q$ satisfying the relation $1/(2Q^2) = 3 + 2\omega_{BD}$ [76]. For example, $f(R)$ theory in the metric formalism corresponds to the constant coupling $Q = -1/\sqrt{6}$, i.e. $\omega_{BD} = 0$. For $|Q|$ of the order of unity it is generally difficult to satisfy local gravity constraints unless some mechanism can be at work to suppress the propagation of the fifth force between the field and non-relativistic matter. It is possible for
such large-coupling models to be consistent with local gravity constraints \[ 61, 64, 68, 77, 80 \] through the so-called chameleon mechanism \[ 81, 82 \], provided that a spherically symmetric body has a thin-shell around its surface.

A braneworld model of dark energy was proposed by Dvali, Gabadadze, and Porrati (DGP) by embedding a 3-brane in the 5-dimensional Minkowski bulk spacetime \[ 41 \]. In this scenario the gravitational leakage to the extra dimension leads to a self-acceleration of the Universe on the 3-brane. Moreover a longitudinal graviton (i.e. a brane-bending mode $\phi$) gives rise to a nonlinear self-interaction of the form \( (r_c^2/m_{pl}) \Box \phi (\partial^\mu \phi \partial_\mu \phi) \) through the mixing with a transverse graviton, where $r_c$ is a cross-over scale (of the order of the Hubble radius $H_0^{-1}$ today) and $m_{pl}$ is the Planck mass \[ 83, 84 \]. In the local region where the energy density $\rho$ is much larger than $r_c^{-2} m_{pl}^2$ the nonlinear self-interaction can lead to the decoupling of the field from matter through the so-called Vainshtein mechanism \[ 85 \], which allows a possibility for the consistency with local gravity constraints. However the DGP model suffers from a ghost problem \[ 86, 88 \], in addition to the difficulty for satisfying the combined observational constraints of SN Ia and BAO \[ 89-94 \].

The equations of motion following from the self-interacting Lagrangian \( \Box \phi (\partial^\mu \phi \partial_\mu \phi) \) present in the DGP model are invariant under the Galilean shift $\partial_\mu \phi \rightarrow \partial_\mu \phi + b_\mu$ in the Minkowski background. While the DGP model is plagued by the ghost problem, the extension of the field self-interaction to more general forms satisfying the Galilean symmetry may allow us to avoid the appearance of ghosts. Nicolis et al. \[ 95 \] showed that there are only five field Lagrangians $L_i \ (i = 1, \cdots, 5)$ that respect the Galilean symmetry in the Minkowski background. In Refs. \[ 95, 96 \] these terms were extended to covariant forms in the curved space-time. In addition one can keep the equations of motion up to the second-order, while recovering the Galileon Lagrangian in the limit of the Minkowski space-time. This property is welcome to avoid the appearance of an extra degree of freedom associated with ghosts. In fact, Refs. \[ 97, 98 \] derived the viable model parameter space in which the appearance of ghosts and instabilities associated with scalar and tensor perturbations can be avoided. Moreover the late-time cosmic acceleration is realized by the existence of a stable de Sitter solution. We shall review the cosmological dynamics of Galileon gravity as well as conditions for the avoidance of ghosts and instabilities.

In order to distinguish between different models of dark energy based on modified gravitational theories, it is important to study the evolution of cosmological perturbations as well as the background expansion history of the Universe. In particular, the modified growth of matter perturbations $\delta_m$ relative to the $\Lambda$CDM model changes the matter power spectrum of large-scale structures (LSS) as well as the weak lensing spectrum \[ 99-126 \]. Moreover the modification of gravity manifests itself for the evolution of the effective gravitational potential $\psi$ related with the Integrated-Sachs-Wolfe (ISW) effect in CMB anisotropies. We shall review a number of observational signatures for the modified gravitational models of dark energy. This review is organized as follows. In Sec. II we construct viable dark energy models based on \( f(R) \) theories after discussing conditions for the cosmological viability as well as for the consistency with local gravity tests. In Sec. III we show that, in Brans-Dicke theories with large matter couplings, it is possible to design the field potential consistent with both cosmological and local gravity constraints. In Sec. IV we derive the field equations in the DGP model and confront the model with observations at the background level. In Sec. V we review the cosmological dynamics based on Galileon gravity as well as conditions for the avoidance of ghosts and Laplacian instabilities. In Sec. VI we briefly mention other modified gravity models of dark energy based on Gauss-Bonnet gravity and Lorentz-violating theories. In Sec. VII we study observational signatures of dark energy models based on \( f(R) \) gravity, Brans-Dicke theory, DGP model, and Galileon gravity, in order to confront them with the observations of LSS, CMB, and weak lensing. Sec. VIII is devoted to conclusions. Throughout the review we use the units such that $c = \hbar = k_B = 1$, where $c$ is the speed of light, $\hbar$ is reduced Planck’s constant, and $k_B$ is Boltzmann’s constant. We also adopt the metric signature $(-,+,+,+)$.  

\section{II. \( f(R) \) Gravity}

We start with the action in \( f(R) \) gravity:

\[
S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R) + \int d^4x L_M(g_{\mu\nu}, \Psi_M),
\]

where $\kappa^2 = 8\pi G$ ($G$ is a bare gravitational constant), $g$ is a determinant of the metric $g_{\mu\nu}$, $f(R)$ is an arbitrary function in terms of the Ricci scalar $R$, and $L_M$ is a matter action with matter fields $\Psi_M$. Variation of the action \( I \) with respect to $g_{\mu\nu}$ leads to the following field equation

\[
F(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu F(R) + g_{\mu\nu} \Box F(R) = \kappa^2 T_{\mu\nu},
\]
where $F(R) \equiv f_R = \partial f/\partial R$, $R_{\mu\nu}$ is a Ricci tensor, and $T_{\mu\nu} = -(2/\sqrt{-g})\delta L_M/\delta g^{\mu\nu}$ is an energy-momentum tensor of matter. The trace of Eq. (2) gives

$$3 \Box F(R) + F(R)R - 2f(R) = \kappa^2T,$$

(3)

where $T = g^{\mu\nu}T_{\mu\nu} = -\rho + 3P$. Here $\rho$ and $P$ are the energy density and the pressure of matter, respectively.

Regarding the variation of the action $\mathcal{L}$, there is another approach called the Palatini formalism $[127]$ in which $g_{\mu\nu}$ and the affine connection $\Gamma^\alpha_{\beta\gamma}$ are treated as independent variables. The resulting field equations are second-order $[132, 133]$ and the cosmological dynamics of dark energy models have been studied by a number of authors $[138, 142]$. However, $f(R)$ theory in the Palatini formalism gives rise to a large coupling between a scalar field degree of freedom and ordinary matter $[133, 137, 146, 150]$, which implies difficulty for compatibility with standard models of particle physics. This large coupling also leads to significant growth of matter density perturbations, unless the models are very close to the $\Lambda$CDM model $[151–155]$.

In the following we focus on the variational approach (so called the metric formalism) given above. The Einstein gravity without a cosmological constant corresponds to $f(R) = R$ and $F(R) = 1$, so that the term $\Box F(R)$ in Eq. (3) vanishes. Since in this case $R = -\kappa^2T = \kappa^2(\rho - 3P)$, the Ricci scalar $R$ is directly determined by matter. In $f(R)$ gravity with a non-linear term in $R$, $\Box F(R)$ does not vanish in Eq. (3). Hence there is a propagating scalar degree of freedom, $\psi \equiv F(R)$, dubbed “scalaron” in Ref. $[150]$. The trace equation (3) allows the dynamics of the scalar field $\psi$.

The de Sitter point corresponds to a vacuum solution with constant $R$. Since $\Box F(R) = 0$ at this point, we obtain

$$F(R)R - 2f(R) = 0.$$  

(4)

Since the quadratic model $f(R) = \alpha R^2$ satisfies this condition, it gives rise to an exact de Sitter solution. In the inflation model $f(R) = R + \alpha R^2$ proposed by Starobinsky $[156]$, the accelerated cosmic expansion ends when the term $\alpha R^2$ becomes smaller than the linear term $R$. It is possible to construct such $f(R)$ inflation models in the framework of supergravity $[157, 158]$.

### A. Cosmological dynamics in $f(R)$ gravity

We first study cosmological dynamics for the models based on $f(R)$ theories in the metric formalism. In order to derive conditions for the cosmological viability of $f(R)$ models we shall carry out general analysis without specifying the form of $f(R)$. We consider a flat Friedmann-Lemaître-Robertson-Walker (FLRW) background with the line element

$$ds^2 = -dt^2 + a(t)^2dx^2,$$

(5)

where $a(t)$ is a scale factor. For the matter Lagrangian $L_M$ in Eq. (1) we take into account non-relativistic matter and radiation, whose energy densities $\rho_m$ and $\rho_r$ satisfy the usual continuity equations $\dot{\rho}_m + 3H\rho_m = 0$ and $\dot{\rho}_r + 4H\rho_r = 0$ respectively. Here $H \equiv \dot{a}/a$ is the Hubble parameter and a dot represents a derivative with respect to cosmic time $t$. From Eqs. (2) and (3) we obtain

$$3FH^2 = \kappa^2(\rho_m + \rho_r) + (FR - f)/2 - 3H\dot{F},$$

(6)

$$2FH = -\kappa^2[\rho_m + (4/3)\rho_r] - \dot{F} + H\dot{F},$$

(7)

where the Ricci scalar is given by

$$R = 6(2H^2 + \dot{H}).$$

(8)

Let us introduce the following dimensionless variables:

$$x_1 = -\frac{\dot{F}}{HF}, \quad x_2 = -\frac{f}{6FH^2}, \quad x_3 = \frac{R}{6H^2}, \quad x_4 = \frac{\kappa^2\rho_r}{3FH^2},$$

(9)

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1 We also note that there is another approach for the variational principle—known as the metric-affine formalism—in which the matter Lagrangian $L_M$ depends not only on the metric $g_{\mu\nu}$ but also on the connection $\Gamma^\alpha_{\beta\gamma}$ $[128, 131]$. 

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together with the density parameters
\[ \Omega_m \equiv \frac{\kappa^2 \rho_m}{3FH^2} = 1 - x_1 - x_2 - x_3 - x_4, \quad \Omega_r \equiv x_4, \quad \Omega_{DE} \equiv x_1 + x_2 + x_3. \] (10)

It is straightforward to derive the following equations [62]:

\[
\begin{align*}
\frac{dx_1}{dN} &= -1 - x_3 - 3x_2 + x_1^2 - x_1x_3 + x_4, \quad (11) \\
\frac{dx_2}{dN} &= \frac{x_1x_3}{m} - x_2(2x_3 - 4 - x_1), \quad (12) \\
\frac{dx_3}{dN} &= -\frac{x_1x_3}{m} - 2x_3(x_3 - 2), \quad (13) \\
\frac{dx_4}{dN} &= -2x_3x_4 + x_1x_4, \quad (14)
\end{align*}
\]

where \( N = \ln a \) and

\[
\begin{align*}
m &\equiv \frac{d\ln F}{d\ln R} = \frac{Rf_{f,RR}}{f_{,R}}, \quad (15) \\
r &\equiv -\frac{d\ln f}{d\ln R} = -\frac{Rf_{f,R}}{f} = \frac{x_3}{x_2}. \quad (16)
\end{align*}
\]

From Eq. (10) one can express \( R \) as a function of \( x_3/x_2 \). Since \( m \) is a function of \( R \), it follows that \( m \) is a function of \( r \), i.e., \( m = m(r) \). The \( \Lambda \)CDM model, \( f(R) = R - 2\Lambda \), corresponds to \( m = 0 \). Then the quantity \( m \) characterizes the deviation from the \( \Lambda \)CDM model.

The effective equation of state of the system is given by

\[ w_{\text{eff}} \equiv -1 - \frac{2\dot{H}}{3H^2} = -\frac{1}{3}(2x_3 - 1). \] (17)

In the absence of radiation \( (x_4 = 0) \) the fixed points for the dynamical system (11)-(14) are

\[
\begin{align*}
P_1 : (x_1, x_2, x_3) &= (0, -1, 2), \quad \Omega_m = 0, \quad w_{\text{eff}} = -1, \quad (18) \\
P_2 : (x_1, x_2, x_3) &= (-1, 0, 0), \quad \Omega_m = 2, \quad w_{\text{eff}} = 1/3, \quad (19) \\
P_3 : (x_1, x_2, x_3) &= (1, 0, 0), \quad \Omega_m = 0, \quad w_{\text{eff}} = 1/3, \quad (20) \\
P_4 : (x_1, x_2, x_3) &= (-4, 5, 0), \quad \Omega_m = 0, \quad w_{\text{eff}} = 1/3, \quad (21) \\
P_5 : (x_1, x_2, x_3) &= \left( \frac{3m}{1 + m}, \frac{1 + 4m}{2(1 + m)^2}, \frac{1 + 4m}{2(1 + m)} \right), \quad \Omega_m = 1 - \frac{m(7 + 10m)}{2(1 + m)^2}, \quad w_{\text{eff}} = -\frac{m}{1 + m}, \quad (22) \\
P_6 : (x_1, x_2, x_3) &= \left( \frac{2(1 - m)}{1 + 2m}, \frac{1 - 4m}{m(1 + 2m)}, \frac{(1 - 4m)(1 + m)}{m(1 + 2m)} \right), \quad \Omega_m = 0, \quad w_{\text{eff}} = \frac{2 - 5m - 6m^2}{3m(1 + 2m)}. \quad (23)
\end{align*}
\]

The points \( P_5 \) and \( P_6 \) are on the line \( m(r) = -r - 1 \) in the \((r, m)\) plane.

Only the point \( P_5 \) can be responsible for the matter-dominated epoch \( (\Omega_m \simeq 1 \text{ and } w_{\text{eff}} \simeq 0) \). This is realized provided \( m \) is close to 0. In the \((r, m)\) plane the matter point \( P_5 \) exists around \((r, m) = (-1, 0)\). Either the point \( P_1 \) or \( P_6 \) can lead to the late-time cosmic acceleration. The former corresponds to a de Sitter point \( (w_{\text{eff}} = -1) \) with \( r = -2 \), in which case the condition [11] is satisfied. Depending on the values of \( m \), the point \( P_6 \) can be responsible for the cosmic acceleration [62]. In the following we shall focus on the case in which the matter point \( P_5 \) is followed by the de Sitter point \( P_1 \).

The stability of the fixed points is known by considering small perturbations \( \delta x_i \) \( (i = 1, 2, 3) \) around them [62]. For the point \( P_5 \) the eigenvalues for the 3 × 3 Jacobian matrix of perturbations are

\[
3(1 + m_5), \quad -3m_5 \pm \sqrt{m_5(256m_5^2 + 160m_5^2 - 31m_5 - 16)} \quad (4m_5(m_5 + 1)), \quad (24)
\]

where \( m_5 \equiv m(r_5) \) and \( m_5' \equiv \frac{dm}{dr}(r_5) \) with \( r_5 \approx -1 \). In the limit \(|m_5| \ll 1 \) the latter two eigenvalues reduce to \(-3/4 \pm \sqrt{-1/m_5} \). The \( f(R) \) models with \( m_5 < 0 \) show a divergence of the eigenvalues as \( m_5 \to -0 \), in which case the system cannot remain for a long time around the point \( P_5 \). For example the model \( f(R) = R - \alpha/R^n \) with \( n > 0 \)
Figure 1: Four trajectories in the \((r,m)\) plane. Each trajectory corresponds to the models: (i) ΛCDM, (ii) \(f(R) = (R^b - \Lambda)^c\), (iii) \(f(R) = R - \alpha R^n\) with \(\alpha > 0\), \(0 < n < 1\), and (iv) \(m(r) = -C(r + 1)(r^2 + ar + b)\). Here \(P_M\), \(P_A\) and \(P_B\) are the matter point \(P_5\), the de Sitter point \(P_1\), and the accelerated point \(P_6\), respectively. From Ref. [64].

and \(\alpha > 0\) falls into this category. On the other hand, if \(0 < m_5 < 0.327\), the latter two eigenvalues in Eq. (24) are complex with negative real parts. Then, provided that \(m_5' > -1\), the point \(P_5\) corresponds to a saddle point with a damped oscillation. Hence the Universe can evolve toward the point \(P_5\) from the radiation era and leave for the late-time acceleration. Then the condition for the existence of the saddle matter era is

\[m(r) \approx +0, \quad \frac{dm}{dr} > -1, \quad \text{at} \quad r = -1.\]  

(25)

The first condition implies that the \(f(R)\) models need to be close to the ΛCDM model during the matter era.

The eigenvalues for the Jacobian matrix of perturbations about the point \(P_1\)

\[-3, \quad -\frac{3}{2} \pm \sqrt{\frac{25-16/m_1}{2}},\]

(26)

where \(m_1 = m(r = -2)\). This shows that the condition for the stability of the de Sitter point \(P_1\) is \(0 < m(r = -2) \leq 1\).

(27)

The trajectories that start from the saddle matter point \(P_5\) with the condition (26) and then approach the stable de Sitter point \(P_1\) with the condition (27) are cosmologically viable.

Let us consider a couple of viable \(f(R)\) models in the \((r,m)\) plane. The ΛCDM model, \(f(R) = R - 2\Lambda\), corresponds to \(m = 0\), in which case the trajectory is a straight line from \(P_5\): \((r, m) = (-1, 0)\) to \(P_1\): \((r, m) = (-2, 0)\). The trajectory (ii) in Fig. 1 represents the model \(f(R) = (R^b - \Lambda)^c\) [64], which corresponds to the straight line \(m(r) = [(1-c)/c]r + b - 1\) in the \((r, m)\) plane. The existence of a saddle matter epoch requires the condition \(c \geq 1\) and \(bc \approx 1\). The trajectory (iii) represents the model [62, 63]

\[f(R) = R - \alpha R^n \quad (\alpha > 0, \ 0 < n < 1),\]

(28)

which corresponds to the curve \(m = n(1+r)/r\). The trajectory (iv) in Fig. 1 shows the model \(m(r) = -C(r + 1)(r^2 + ar + b)\), in which case the late-time accelerated attractor is the point \(P_6\) with \((\sqrt{3} - 1)/2 < m < 1\).

In Ref. [62] it was shown that the variable \(m\) needs to be close to 0 during the radiation-dominated epoch as well. Hence the viable \(f(R)\) models are close to the ΛCDM model, \(f(R) = R - 2\Lambda\), in the region \(R \gg R_0\) (where \(R_0\) is the present cosmological Ricci scalar). The Ricci scalar \(R\) given in Eq. (8) remains positive from the radiation era to the present epoch, as long as the it does not oscillate. As we will see in Sec. II B, we require the condition \(f_{,R} > 0\)
to avoid ghosts. Then the condition \( m > 0 \) for the presence of the matter-dominated epoch translates to \( f_{RR} > 0 \). The model \( f(R) = R - \alpha/R^n \) \((\alpha > 0, \ n > 0)\) is not viable because the condition \( f_{RR} > 0 \) is violated. We also note that the power-law models with \( f(R) = R^n \) do not give rise to a successful cosmological trajectory \([59, 60]\) (unlike the claims in Ref. [162]).

In order to derive the equation of state of dark energy to confront with SN Ia observations for the cosmologically viable models, we rewrite Eqs. (6) and (7) as follows\(^2\):

\[
3A\dot{H}^2 = k^2(\rho_m + \rho_r + \rho_{DE}) ,
\]

\[
-2A\ddot{H} = k^2[\rho_m + (4/3)\rho_r + \rho_{DE} + P_{DE}] ,
\]

where \( A \) is some constant and

\[
\begin{align*}
\kappa^2 \rho_{DE} &\equiv (1/2)(FR - f) - 3H\dot{F} + 3H^2(A - F), \\
\kappa^2 P_{DE} &\equiv \dot{F} + 2H\dot{F} - (1/2)(FR - f) - (3H^2 + 2H)(A - F) .
\end{align*}
\]

Defining \( \rho_{DE} \) and \( P_{DE} \) in this way, one can show that these satisfy the usual continuity equation

\[
\dot{\rho}_{DE} + 3H(\rho_{DE} + P_{DE}) = 0 .
\]

The dark energy equation of state related with SN Ia observations is given by \( w_{DE} \equiv P_{DE}/\rho_{DE} \). From Eqs. (29) and (30) it follows that

\[
w_{DE} = -\frac{2A\dot{H} + 3A\dot{H}^2 + \kappa^2 \rho_r/3}{3AH^2 - \kappa^2(\rho_m + \rho_r)} \simeq \frac{w_{\text{eff}}}{1 - (F/A)\Omega_m} ,
\]

where the last approximate equality in Eq. (34) is valid in the regime where the radiation density \( \rho_r \) is negligible relative to the matter density. The viable \( f(R) \) models approach the \( \Lambda \)CDM model in the past, i.e. \( F \rightarrow 1 \) as \( R \rightarrow \infty \). In order to reproduce the standard matter era for the redshifts \( z \gg 1 \), one can choose \( A = 1 \) in Eqs. (29) and (30). Another possible choice is \( A = F_0 \), where \( F_0 \) is the present value of \( F \). This choice is suitable if the deviation of \( F_0 \) from 1 is small (as in the scalar-tensor theory with a massless scalar field \([164-166]\)). In both cases the equation of state \( w_{DE} \) can be smaller than \(-1\) before reaching the de Sitter attractor \([64, 71, 167, 169]\). Thus \( f(R) \) gravity models give rise to a phantom equation of state without violating stability conditions of the system.

As we see in Eq. (34), the presence of non-relativistic matter is important to lead to the apparent phantom behavior. We wish to stress here that for viable \( f(R) \) models constructed to satisfy all required conditions [such as the models \([43], [45], \) and \([49] \) we will discuss later] the ghosts are not present even if \( w_{DE} < -1 \). A number of authors proposed some models to realize \( w_{DE} < -1 \) without including non-relativistic matter \([170, 171]\), which means that \( w_{DE} = w_{\text{eff}} \) from Eq. (34). However, such \( f(R) \) models usually imply the presence of ghosts\(^3\), because \( w_{\text{eff}} < -1 \) corresponds to \( H > 0 \).

The observational constraints on specific \( f(R) \) models have been carried out in Refs. \([172, 175]\) from the background expansion history of the Universe (see also Refs. \([176, 180]\) for the reconstruction of \( f(R) \) models from observations). Since the deviation of \( w_{DE} \) from that in the \( \Lambda \)CDM model (\( w_{DE} = -1 \)) is not so significant \([68, 167]\), the viable models such as \([43], [45], \) and \([49] \) can be consistent with the data fairly easily. In other words we do not obtain very tight bounds on model parameters from the information of the background expansion history only. However, the models can be more strongly constrained at the level of perturbations, as we will see in Sec. VII A.

**B. Conditions for the avoidance of ghosts and tachyonic instabilities**

In this subsection we shall derive conditions for the avoidance of ghosts and tachyonic instabilities in \( f(R) \) theories. In doing so we expand the action \[(1)\] up to the second-order by considering the following perturbed metric about the FLRW background

\[
ds^2 = -(1 + 2\alpha)dt^2 - 2a(t)\partial_i \partial dt dx^i + a^2(t)(\delta_{ij} + 2\psi \delta_{ij} + 2\partial_i \partial_j \gamma) dx^i dx^j ,
\]

\(^2\) If the field equations are written in this form, we can also show that the background cosmological dynamics has a correspondence with equilibrium thermodynamics on the apparent horizon \([163]\).

\(^3\) If a late-time de Sitter solution is a stable spiral, it happens that \( w_{DE} \) oscillates around \(-1\) with a small amplitude, even for viable \( f(R) \) models. Here we are discussing the real ghosts out of this regime.
where \( \alpha, \beta, \psi, \gamma \) are scalar metric perturbations \[181\].

Introducing the perturbation \( \delta F \) for the quantity \( F = \partial f / \partial R \), one can construct the gauge-invariant curvature perturbation

\[
\mathcal{R} \equiv \psi - \frac{H}{F} \delta F.
\] (36)

Expanding the action \([11]\) without the matter source, we obtain the second-order action for the curvature perturbation \[19, 182\]

\[
\delta S^{(2)} = \int dt d^3x a^3 Q_s \left[ \frac{1}{2} \hat{\mathcal{R}}^2 - \frac{1}{2} \frac{1}{a^2} (\nabla \mathcal{R})^2 \right],
\] (37)

where

\[
Q_s = \frac{3 \dot{F}^2}{2 \kappa^2 F[H + \dot{F}/(2F)]^2}.
\] (38)

The negative sign of \( Q_s \) corresponds to a ghost field because of the negative kinetic energy. Hence the condition for the avoidance of ghosts is given by

\[
F > 0.
\] (39)

For the matter sector the ghost does not appear for \( \rho_M (1 + w_M)/w_M > 0 \) (where \( w_M \) is the equation of state for the matter fluid) \[183\], which is satisfied for radiation \( (w_M = 1/3) \) and non-relativistic matter \( (w_M \approx +0) \).

If \( Q_s \) is positive, the action (37) can be written in the following form by introducing the new variables \( u = z_s \mathcal{R} \) and \( z_s = a \sqrt{Q_s} \):

\[
\delta S^{(2)} = \int d\tau d^3x \left[ \frac{1}{2} u''^2 - \frac{1}{2} (\nabla u)^2 + \frac{1}{2} z_s'' z_s^2 \right],
\] (40)

where a prime represents a derivative with respect to the conformal time \( \tau = \int a^{-1} dt \). Equation (40) shows that the scalar degree of freedom has the effective mass

\[
M_s^2 \equiv - \frac{1}{a^2} z_s'' = \frac{\dot{Q}_s^2}{4Q_s^2} - \frac{\ddot{Q}_s}{2Q_s} - \frac{3H \dot{Q}_s}{2Q_s}
\]

\[
= - \frac{72F^2 H^4}{(2FH + f,RR R)^2} + \frac{1}{3} F \left( \frac{288H^3 - 12HR}{2FH + f,RR R} + \frac{1}{f,RR} \right) + \frac{f,RR R^2}{4F^2} - 24H^2 + \frac{7}{6} R,
\] (41)

where we have eliminated the term \( \dot{H} \) by using the background equations.

In Fourier space the perturbation \( u \) satisfies the equation of motion

\[
u'' + (k^2 + M_s^2 a^2) u = 0.
\] (42)

For \( k^2/a^2 \gg M_s^2 \), the propagation speed \( c_s \) of the field \( u \) is equivalent to the speed of light \( c \). Hence, in \( f(R) \) gravity, the gradient instability associated with negative \( c_s^2 \) is absent. For small \( k \) satisfying \( k^2/a^2 \ll M_s^2 \), we require that \( M_s^2 > 0 \) to avoid the tachyonic instability of perturbations. The viable dark energy models based on \( f(R) \) theories need to satisfy the condition \( R f,RR \ll F \) (i.e. \( m = R f,RR / f,RR \ll 1 \)) at early cosmological epochs in order to have successful cosmological evolution from radiation domination till matter domination. At these epochs the mass squared is approximately given by

\[
M_s^2 \approx \frac{F}{3f,RR}.
\] (43)

Under the no-ghost condition \[39\] the tachyonic instability is absent for

\[
f,RR > 0.
\] (44)

The viable \( f(R) \) dark energy models have been constructed to satisfy the conditions \[39\] and \[44\] in the regime \( R \geq R_1 \), where \( R_1 \) is the Ricci scalar at the late-time de Sitter point. Moreover we require that the models are
consistent with the conditions (25) and (27). The model (28) can be consistent with all these conditions, but the local gravity constraints demand that the variable \( m \) is very much smaller than 1 in the regions of high density (i.e. \( R \gg R_0 \), where \( R_0 \) is the cosmological Ricci scalar today). In the model (28) one has \( m \simeq n(-r - 1) \) around \( r \simeq -1 \). For the consistency with local gravity constraints we require that \( n \lesssim 10^{-10} \) \[73\], but in this case the deviation from the ΛCDM model around the present epoch \( (R \approx R_0) \) is very small.

If the variable \( m \) behaves as \( m = C(-r - 1)^p \) with \( p > 1 \) in the region \( R \gg R_0 \), then it is possible to satisfy local gravity constraints (i.e. \( m \ll 0.01 \sim 0.1 \) for \( R \gg R_0 \)) while at the same time showing deviations from the ΛCDM \( (m \gtrsim 0.01 \sim 0.1 \text{ for } R \approx R_0) \). The models constructed in this vein are

\[
(A) \quad f(R) = R - \mu R_c \frac{(R/R_c)^{2n}}{(R/R_c)^{2n} + 1} \quad \text{with } \mu > 0, \ n > 0 \text{ and } R_c > 0, \\
(B) \quad f(R) = R - \mu R_c \left[1 - (1 + R^2/R_c^2)^{-n}\right] \quad \text{with } \mu > 0, \ n > 0 \text{ and } R_c > 0, \\
\]

which were proposed by Hu and Sawicki \[68\] and Starobinsky \[69\], respectively. \( R_c \) is roughly of the order of the present cosmological Ricci scalar \( R_0 \) for \( \mu \) and \( n \) of the order of unity. The models (A) and (B) asymptotically behave as

\[
f(R) \simeq R - \mu R_c [1 - (R^2/R_c^2)^{-n}] \quad \text{for } R \gg R_c, \tag{47}
\]

which gives \( m(r) = C(-r - 1)^{2n+1} \).

Another viable model that leads to the even rapid decrease of \( m \) toward the past is \[71\]

\[
(C) \quad f(R) = R - \mu R_c \tanh(R/R_c) \quad \text{with } \mu > 0, \ R_c > 0. \tag{48}
\]

Other similar models were proposed by Appleby and Battye \[70\] and Linder \[72\].

In what follows we shall discuss local gravity constraints on the above models.

### C. Local gravity constraints on \( f(R) \) gravity models

Let us proceed to discuss local gravity constraints on \( f(R) \) gravity models. In the region of high density like Earth or Sun, the Ricci scalar \( R \) is much larger than the background cosmological value \( R_0 \). In this case the linear expansion of \( R = R_0 + \delta R \) cannot be justified. In such a non-linear regime the chameleon mechanism \[81, 82\] plays an important role for the \( f(R) \) models to satisfy local gravity constraints \[61, 64, 68, 77–80\] (see also Refs. \[184–190\]).

To discuss the chameleon mechanism in \( f(R) \) gravity, it is convenient to transform the action (11) to the so-called Einstein frame action via the conformal transformation \[191\]:

\[
g_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \Omega^2 = F. \tag{49}
\]

The action in the Einstein frame includes a linear term in \( \tilde{R} \), where the tilde represents quantities in the Einstein frame. Introducing a new scalar field \( \phi = \sqrt{3/2k^2} \ln F \), we obtain the action in the Einstein frame, as \[19, 191\]

\[
S_E = \int d^4x \sqrt{-g} \left\{ \frac{1}{2k^2} \tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right\} + \int d^4x \mathcal{L}_M(F^{-1} \tilde{g}^{\mu\nu}, \Psi_M), \tag{50}
\]

where

\[
V(\phi) = \frac{RF - f}{2k^3 F^2}. \tag{51}
\]

In the Einstein frame the scalar field \( \phi \) directly couples with non-relativistic matter. The strength of this coupling depends on the conformal factor \( \Omega = \sqrt{F} \). We define the coupling \( Q \) as

\[
Q \equiv -\frac{\Omega_\phi}{\Omega} = -\frac{F_\phi}{2F} = -\frac{1}{\sqrt{6}}, \tag{52}
\]

which is of the order of unity in \( f(R) \) gravity. If the field potential \( V(\phi) \) is absent, the field propagates freely with a large coupling \( Q \). Since a potential \[71\] with a gravitational origin is present in \( f(R) \) gravity, it is possible for \( f(R) \) dark energy models to satisfy local gravity constraints through the chameleon mechanism \[61, 68, 78, 80\].
In a spherically symmetric space-time under a weak gravitational background (i.e. neglecting the backreaction of gravitational potentials), variation of the action \( \mathcal{S} \) with respect to the scalar field \( \phi \) leads to

\[
\frac{d^2 \phi}{d\tilde{r}^2} + \frac{2}{\tilde{r}} \frac{d\phi}{d\tilde{r}} = \frac{dV_{\text{eff}}}{d\phi},
\]

where \( \tilde{r} \) is a distance from the center of symmetry, and \( V_{\text{eff}}(\phi) \) is an effective potential defined by

\[
V_{\text{eff}}(\phi) = V(\phi) + e^{Q\phi} \rho^*.
\]

Here \( \rho^* \) is a conserved quantity in the Einstein frame, which is related to the density \( \tilde{\rho} \) in the Jordan frame via the relation \( \rho^* = e^{3Q\phi} \tilde{\rho} \). By the end of this section we use the unit \( \kappa = 1 \).

We assume that a spherically symmetric body has a constant density \( \rho^* = \rho_A \) inside the body (\( \tilde{r} < \tilde{r}_c \)) and that the density outside the body (\( \tilde{r} > \tilde{r}_c \)) is \( \rho^* = \rho_B \). The mass \( M_c \) of the body and the gravitational potential \( \Phi_\circ \) at the radius \( \tilde{r}_c \) are given by \( M_c = (4\pi/3)\tilde{r}_c^3 \rho_A \) and \( \Phi_\circ = M_c / 8\pi \tilde{r}_c \), respectively. The effective potential \( V_{\text{eff}}(\phi) \) has two minima at the field values \( \phi_A \) and \( \phi_B \) satisfying \( V'_\text{eff}(\phi_A) = 0 \) and \( V'_\text{eff}(\phi_B) = 0 \), respectively (here a prime represents a derivative with respect to \( \phi \)). The former corresponds to the region with a high density that gives rise to a large mass squared \( m_A^2 = V''_{\text{eff}}(\phi_A) \), whereas the latter to the lower density region with a smaller mass squared \( m_B^2 = V''_{\text{eff}}(\phi_B) \). When the “dynamics” of the field \( \phi \) with the field equation \( \Box \phi = 0 \) is studied, we need to consider the effective potential \( (-V_{\text{eff}}) \) so that it has two maxima at \( \phi = \phi_A \) and \( \phi = \phi_B \).

We impose the two boundary conditions \( (d\phi/d\tilde{r})(\tilde{r} = 0) = 0 \) and \( \phi(\tilde{r} \to \infty) = \phi_B \). The field \( \phi \) is at rest at \( \tilde{r} = 0 \) and begins to roll down the potential when the matter-coupling term \( Q_P A e^{Q\phi} \) becomes important at a radius \( \tilde{r}_1 \) in Eq. \( (53) \). As long as \( \tilde{r}_1 \) is close to \( \tilde{r}_c \) such that \( \Delta \tilde{r}_c = \tilde{r}_c - \tilde{r}_1 \ll \tilde{r}_c \), the body has a thin-shell inside the body. The field acquires a sufficient kinetic energy in the thin-shell regime (\( \tilde{r}_1 < \tilde{r} < \tilde{r}_c \)) and hence the field climbs up the potential hill outside the body (\( \tilde{r} > \tilde{r}_c \)).

The field profile can be obtained by matching the solutions of Eq. \( (53) \) at the radius \( \tilde{r} = \tilde{r}_1 \) and \( \tilde{r} = \tilde{r}_c \). Neglecting the mass term \( m_B \), the thin-shell field profile outside the body is given by \( (192) \)

\[
\phi(\tilde{r}) = \phi_B - \frac{2Q_{\text{eff}} GM_c}{\tilde{r}}.
\]

where

\[
Q_{\text{eff}} \simeq 3Q\epsilon_{\text{th}}, \quad \epsilon_{\text{th}} = \frac{\phi_B - \phi_A}{6Q\Phi_\circ}.
\]

Here \( \epsilon_{\text{th}} \) is called a thin-shell parameter. Under the conditions \( \Delta \tilde{r}_c/\tilde{r}_c \ll 1 \) and \( 1/(m_A \tilde{r}_c) \ll 1 \), the thin-shell parameter is approximately given by \( (192) \)

\[
\epsilon_{\text{th}} \simeq \frac{\Delta \tilde{r}_c}{\tilde{r}_c} + \frac{1}{m_A \tilde{r}_c}.
\]

Provided that \( \epsilon_{\text{th}} \ll 1 \), the amplitude of the effective coupling \( Q_{\text{eff}} \) can be much smaller than 1. It is then possible for the \( f(R) \) models (\( |Q| = 1/\sqrt{\tilde{g}} \)) to be consistent with local gravity experiments. Originally the thin-shell solution was derived by assuming that the field is frozen in the region \( 0 < \tilde{r} < \tilde{r}_1 \) \( (81, 82) \). In this case the thin-shell parameter is given by \( \epsilon_{\text{th}} \simeq \Delta \tilde{r}_c/\tilde{r}_c \), which is different from Eq. \( (57) \). However, this difference is not important because the condition \( \Delta \tilde{r}_c/\tilde{r}_c \gg 1/(m_A \tilde{r}_c) \) is satisfied for most of viable models \( (192) \).

Consider the bound on the thin-shell parameter from the possible violation of equivalence principle (EP). The tightest bound comes from the solar system tests of weak EP using the free-fall acceleration of Moon (\( a_{\text{Moon}} \)) and Earth (\( a_\oplus \)) toward Sun \( (82) \). The experimental bound on the difference of two accelerations is given by \( (193) \)

\[
\frac{|a_{\text{Moon}} - a_\oplus|}{(a_{\text{Moon}} + a_\oplus)/2} < 10^{-13}.
\]

Provided that Earth, Sun, and Moon have thin-shells, the field profiles outside the bodies are given by Eq. \( (55) \), with the replacement of corresponding quantities. The acceleration induced by a fifth force with the field profile \( \phi(F) \) and the effective coupling \( Q_{\text{eff}} \) is \( a_{\text{fifth}} = |Q_{\text{eff}} \nabla \phi(F)| \). Using the thin-shell parameter \( \epsilon_{\text{th}} \) for Earth, the accelerations \( a_\oplus \) and \( a_{\text{Moon}} \) toward Sun (mass \( M_\odot \)) are

\[
a_\oplus \approx \frac{GM_\odot}{\tilde{r}^2} \left[ 1 + 18Q^2 \epsilon_{\text{th}, \odot}^2 \frac{\Phi_\odot}{\Phi_\odot} \right], \quad a_{\text{Moon}} \approx \frac{GM_\odot}{\tilde{r}_c^2} \left[ 1 + 18Q^2 \epsilon_{\text{th}, \odot}^2 \frac{\Phi_\odot^2}{\Phi_\odot \Phi_{\text{Moon}}} \right],
\]

\( (59) \).
where $\Phi_\odot \simeq 2.1 \times 10^{-6}$, $\Phi_\oplus \simeq 7.0 \times 10^{-10}$, and $\Phi_{\text{Moon}} \simeq 3.1 \times 10^{-11}$ are the gravitational potentials of Sun, Earth and Moon, respectively. Then the condition (58) translates to

$$\epsilon_{\text{th,}\odot} < 8.8 \times 10^{-7}/|Q|.$$  \hspace{1cm} (60)

Since the condition $|\phi_B| \gg |\phi_A|$ is satisfied for viable $f(R)$ models (as we will see below), we have $\epsilon_{\text{th,}\odot} \simeq \phi_B/(6Q\Phi_\odot)$ from Eq. (58). Hence the condition (60) corresponds to

$$|\phi_B| < 3.7 \times 10^{-15}.$$  \hspace{1cm} (61)

Let us consider local gravity constraints on the $f(R)$ models given in Eqs. (45) and (46). In the region of high density where local gravity experiments are carried out, it is sufficient to use the asymptotic form given in Eq. (47). In order for these models to be responsible for the present cosmic acceleration, $R_\epsilon$ is roughly the same order as the cosmological Ricci scalar $R_0$ today for $\mu$ and $n$ of the order of unity. For the functional form (47) we have the following relations

$$F = e^{2\phi/\sqrt{\sigma}} = 1 - 2n\mu(R/R_\epsilon)^{-2n+1},$$  \hspace{1cm} (62)

$$V_{\text{eff}}(\phi) \simeq \frac{1}{2} \mu R_\epsilon e^{-4\phi/\sqrt{\sigma}} \left[1 - (2n + 1) \left(-\frac{\phi}{\sqrt{6n}\mu}\right)^{2n/(2n+1)}\right] + \rho^* e^{-\phi/\sqrt{\sigma}}.$$  \hspace{1cm} (63)

Inside and outside the body the effective potential (63) has minima at

$$\phi_A \simeq -\sqrt{6n}\mu(R_\epsilon/\rho_A)^{2n+1}, \quad \phi_B \simeq -\sqrt{6n}\mu(R_\epsilon/\rho_B)^{2n+1}.$$  \hspace{1cm} (64)

If $\rho_A \gg \rho_B$, then one has $|\phi_B| \gg |\phi_A|$.

The bound (61) translates into

$$\frac{n \mu}{x_1^{2n+1}} \left(\frac{R_1}{\rho_B}\right)^{2n+1} < 1.5 \times 10^{-15}.$$  \hspace{1cm} (65)

Here $x_1$ is defined by $x_1 \equiv R_1/R_\epsilon$, where $R_1$ is the Ricci scalar at the late-time de Sitter fixed point $P_1$ given in Eq. (18). Let us consider the model described by the Lagrangian density (17) for $R \geq R_1$. If we use the models (45) and (46), then there are some modifications for the estimation of $R_1$. However this change is not significant when we place constraints on model parameters.

The de Sitter solution for the model (17) satisfies $\mu = x_1^{2n+1}/[2(x_1^{2n} - n - 1)]$. Substituting this relation into Eq. (65), it follows that

$$\frac{n}{2(x_1^{2n} - n - 1)} \left(\frac{R_1}{\rho_B}\right)^{2n+1} < 1.5 \times 10^{-15}.$$  \hspace{1cm} (66)

For the stability of the de Sitter point we require that $m(R_1) < 1$, which translates into the condition $x_1^{2n} > 2n^2 + 3n + 1$. Hence the term $n/[2(x_1^{2n} - n - 1)]$ in Eq. (66) is smaller than 0.25 for $n > 0$.

Let us use the simple approximation that $R_1$ and $\rho_B$ are of the orders of the present cosmological density $10^{-29}$ g/cm$^3$ and the baryonic/dark matter density $10^{-24}$ g/cm$^3$ in our galaxy, respectively. From Eq. (66) we obtain the constraint

$$n > 0.9.$$  \hspace{1cm} (67)

Thus $n$ is not required to be much larger than unity. Under the condition (67), as $R$ decreases to the order of $R_\epsilon$, one can cosmologically see an appreciable deviation from the $\Lambda$CDM model. The deviation from the $\Lambda$CDM model appears when $R$ decreases to the order of $R_\epsilon$. The model (18) also shows similar behavior. If we consider the model (28), it was shown in Ref. 78 that the bound (61) gives the constraint $n < 3 \times 10^{-10}$. Hence the deviation from the $\Lambda$CDM model is very small. The models (45) and (46) are carefully constructed to satisfy local gravity constraints, while at the same time the deviation from the $\Lambda$CDM model appears even for $n = O(1)$. Note that the model (48) can easily satisfy local gravity constraints because of the rapid approach to the $\Lambda$CDM in the regime $R \gg R_\epsilon$.

In the strong gravitational background (such as neutron stars), Kobayashi and Maeda (194, 195) pointed out that for the model (46) it is difficult to obtain thin-shell solutions inside a spherically symmetric body with constant density. For chameleon models with general couplings $Q$, a thin-shell field profile was analytically derived in Ref. 196 by employing a linear expansion in terms of the gravitational potential $\Phi_\epsilon$ at the surface of a compact object with
constant density. Using the boundary condition set by analytic solutions, Ref. 196 also numerically confirmed the existence of thin-shell solutions for $\Phi_r \lesssim 0.3$ in the case of inverse power-law potentials $V(\phi) = M^{3+n}\phi^{-n}$. Ref. 197 also showed that static relativistic stars with constant density exists for the model (46). The effect of the relativistic pressure is important around the center of the body, so that the field tends to roll down the potential quickly unless the boundary condition is carefully chosen. Realistic stars have densities $\rho_A (r)$ that globally decrease as a function of $r$. The numerical simulation of Refs. 198, 199 showed that thin-shell solutions are present for the $f(R)$ model (46) by considering a polytropic equation of state even in the strong gravitational background (see also Ref. 200).

III. SCALAR-TENSOR GRAVITY

There is another class of modified gravity called scalar-tensor theories in which the Ricci scalar $R$ is coupled to a scalar field $\phi$. One of the simplest examples is the so-called Brans-Dicke theory with the action

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \phi R - \frac{\omega_{BD}}{2\varphi} \left( \nabla \varphi \right)^2 - U(\varphi) \right] + \int d^4x L_M (g_{\mu\nu}, \Psi_M),$$

(68)

where $\omega_{BD}$ is a constant (called the Brans-Dicke parameter), $U(\varphi)$ is a field potential, and $L_M$ is a matter Lagrangian that depends on the metric $g_{\mu\nu}$ and matter fields $\Psi_m$. The original Brans-Dicke theory (73) does not have the field potential. As we will see below, metric $f(R)$ gravity discussed in Sec. II is equivalent to the Brans-Dicke theory with $\omega_{BD} = 0$.

A. Scalar-tensor theories and the matter coupling in the Einstein frame

The general action for scalar-tensor theories can be written as

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} f(\varphi, R) - \frac{1}{2} \zeta(\varphi)(\nabla \varphi)^2 \right] + \int d^4x L_M (g_{\mu\nu}, \Psi_M),$$

(69)

where $f$ depends on the scalar field $\varphi$ and the Ricci scalar $R$, $\zeta$ is a function of $\varphi$. We choose the unit $\kappa^2 = 8\pi G = 1$. The action (69) covers a wide variety of theories such as $f(R)$ gravity ($f(\varphi, R) = f(R)$, $\zeta = 0$), Brans-Dicke theory ($f = \varphi R$ and $\zeta = \omega_{BD}/\varphi$), and dilaton gravity ($f = e^{-\varphi} R$ and $\zeta = -e^{-\varphi}$).

Let us consider theories of the type

$$f(\varphi, R) = F(\varphi) R - 2U(\varphi).$$

(70)

In order to avoid the appearance of ghosts we require that $F(\varphi) > 0$. Under the conformal transformation (49) with the conformal factor $\Omega = \sqrt{F}$, the action (69) can be transformed to that in the Einstein frame:

$$S_E = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{1}{2} \tilde{R} - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right] + \int d^4x L_M (\tilde{g}_{\mu\nu} F^{-1}(\phi), \Psi_M),$$

(71)

where

$$V = U/F^2.$$  

(72)

We have introduced a new scalar field $\phi$ in order to make the kinetic term canonical:

$$\phi \equiv \int d\varphi \sqrt{\frac{3}{2} \left( \frac{F_{,\varphi}}{F} \right)^2 + \frac{\zeta}{F}},$$

(73)

We define the coupling between dark energy and non-relativistic matter in the Einstein frame:

$$Q \equiv \frac{F_{,\phi}}{2F} = - \frac{F_{,\varphi}}{F} \left[ \frac{3}{2} \left( \frac{F_{,\varphi}}{F} \right)^2 + \frac{\zeta}{F} \right]^{-1/2}.$$  

(74)

Recall that in metric $f(R)$ gravity we have that $Q = -1/\sqrt{6}$. If $Q$ is a constant, the following relations hold from Eqs. (73) and (74):

$$F = e^{-2Q\phi}, \quad \zeta = (1 - 6Q^2) F \left( \frac{d\phi}{d\varphi} \right)^2.$$

(75)
Then the action \( (69) \) in the Jordan frame can be written as \( [76] \)

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} F(\phi)R - \frac{1}{2(1 - 6Q^2)} F(\phi)(\nabla \phi)^2 - U(\phi) \right] + \int d^4x \mathcal{L}_M(g_{\mu\nu}, \Psi_M). \tag{76}
\]

In the limit that \( Q \to 0 \), the action \( [76] \) reduces to the one for a minimally coupled scalar field \( \phi \) with the potential \( U(\phi) \). The transformation of the Jordan frame action \( [76] \) via a conformal transformation \( \tilde{g}_{\mu\nu} = F(\phi)g_{\mu\nu} \) gives rise to the Einstein frame action \( [71] \) with a constant coupling \( Q \). The action \( [71] \) is equivalent to the action \( [50] \) with \( \tilde{g}_{\mu\nu} = e^{-2Q\phi}g_{\mu\nu} \).

One can compare \( [76] \) with the action \( [63] \) in Brans-Dicke theory. Setting \( \phi = F = e^{-2Q\phi} \), one finds that two actions are equivalent if the parameter \( \omega_{BD} \) is related to \( Q \) via the relation \( [76, 82] \)

\[
3 + 2\omega_{BD} = \frac{1}{2Q^2}. \tag{77}
\]

Using this relation, we find that the General Relativistic limit \( (\omega_{BD} \to \infty) \) corresponds to the vanishing coupling \( (Q \to 0) \). Since \( Q = -1/\sqrt{6} \) in metric \( f(R) \) gravity, this corresponds to the Brans-Dicke parameter \( \omega_{BD} = 0 \) \([51, 74]\). The experimental bound on \( \omega_{BD} \) for a massless scalar field is given by \( \omega_{BD} > 40000 \) \([193, 201]\), which translates into the condition

\[
|Q| < 2.5 \times 10^{-3} \quad (\text{for a massless field}). \tag{78}
\]

In such cases it is difficult to find a large difference relative to the uncoupled quintessence model. In the presence of the field potential, however, it is possible for large coupling models \((|Q| \sim 1)\) to satisfy local gravity constraints via the chameleon mechanism \([76]\).

The above Brans-Dicke theory is one of the examples in scalar-tensor theories. In general the coupling \( Q \) is field-dependent apart from Brans-Dicke theory. If we consider a nonminimally coupled scalar field with \( F(\phi) = 1 - \xi \phi^2 \) and \( \zeta(\phi) = 1 \), then it follows that \( Q(\phi) = \xi \phi/[1 - \xi \phi^2(1 - 6\xi)]^{1/2} \). The cosmological dynamics in such a theory have been studied by a number of authors \([33, 40, 202, 203, 207, 211]\). If the field is nearly massless during most of the cosmological epochs, the coupling \( Q \) needs to be suppressed to avoid the propagation of the fifth force.

In the following we shall study the cosmological dynamics and local gravity constraints on the constant coupling models based on the action \( [76] \) with \( F(\phi) = e^{-2Q\phi} \).

**B. Cosmological dynamics in Brans-Dicke theory**

We study the cosmological dynamics for the Jordan frame action \( [76] \) in the presence of a non-relativistic fluid with energy density \( \rho_m \) and a radiation fluid with energy density \( \rho_r \). We regard the Jordan frame as a physical frame due to the usual conservation of non-relativistic matter \( (\rho_m \propto a^{-3}) \). In the flat FLRW background variation of the action \( [76] \) with respect to \( g_{\mu\nu} \) and \( \phi \) gives the following equations of motion

\[
3FH^2 = (1 - 6Q^2)F\phi^2/2 + U - 3H\dot{F} + \rho_m + \rho_r, \tag{79}
\]

\[
2F\dot{H} = -(1 - 6Q^2)F\phi^2 - \dot{F} + H\dot{F} - \rho_m - (4/3)\rho_r, \tag{80}
\]

\[
(1 - 6Q^2)F[\ddot{\phi} + 3H\dot{\phi} + \dot{F}/(2F)\dot{\phi}] + U_{,\phi} + QFR = 0. \tag{81}
\]

Let us introduce the following variables

\[
x_1 = \frac{\phi}{\sqrt{6}H}, \quad x_2 = \frac{1}{H} \sqrt{\frac{U}{3F}}, \quad x_3 = \frac{1}{H} \sqrt{\frac{\rho_r}{3F}}, \tag{82}
\]

and

\[
\Omega_m = \frac{\rho_m}{3FH^2}, \quad \Omega_{rad} = x_1^2, \quad \Omega_{DE} = (1 - 6Q^2)x_1^2 + x_2^2 + 2\sqrt{6}Qx_1. \tag{83}
\]

These satisfy the relation \( \Omega_m + \Omega_{rad} + \Omega_{DE} = 1 \) from Eq. \( [79] \). Using Eqs. \( [79]-[81] \), we obtain the differential equations for \( x_1, x_2 \) and \( x_3 \):

\[
\frac{dx_1}{dN} = \frac{\sqrt{6}}{2} (\lambda x_2^2 - \sqrt{6}x_1) + \frac{\sqrt{6}Q}{2} \left[ (5 - 6Q^2)x_1^2 + 2\sqrt{6}Qx_1 - 3x_2^2 + x_3^2 - 1 \right] - x_1 \frac{\dot{H}}{H^2}, \tag{84}
\]

\[
\frac{dx_2}{dN} = \frac{\sqrt{6}}{2} (2Q - \lambda)x_1x_2 - x_2 \frac{\dot{H}}{H^2}, \tag{85}
\]

\[
\frac{dx_3}{dN} = \sqrt{6}Qx_1x_3 - 2x_3 - x_3 \frac{\dot{H}}{H^2}, \tag{86}
\]
where $N = \ln a$, $\lambda = -U_{,\phi}/U$, and

$$\frac{\dot{H}}{H^2} = -\frac{1 - 6Q^2}{2} \left(3 + 3x_1^2 - 3x_2^2 + x_2^3 - 6Q^2x_1^2 + 2\sqrt{6}Qx_1\right) + 3Q(\lambda x_2^2 - 4Q).$$

(87)

The effective equation of state of the system is given by $w_{\text{eff}} = -1 - 2\dot{H}/(3H^2)$.

If $\lambda$ is a constant, one can derive the fixed points of the system [84]-[86] in the absence of radiation ($x_3 = 0$) [76]:

- (a)

$$(x_1, x_2) = \left(\frac{\sqrt{6}Q}{3(2Q^2 - 1)}, 0\right), \quad \Omega_m = \frac{3 - 2Q^2}{3(1 - 2Q^2)^2}, \quad w_{\text{eff}} = \frac{4Q^2}{3(1 - 2Q^2)}.$$  

(88)

- (b)

$$(x_1, x_2) = \left(\frac{1}{\sqrt{6}Q \pm 1}, 0\right), \quad \Omega_m = 0, \quad w_{\text{eff}} = \frac{3 \mp \sqrt{6}Q}{3(1 \pm \sqrt{6}Q)}.$$  

(89)

- (c)

$$(x_1, x_2) = \left(\frac{\sqrt{6}}{6(4Q^2 - Q\lambda - 1)} \left[6 - \lambda^2 + 8Q\lambda - 16Q^2\right]^{1/2} \right), \quad \Omega_m = 0, \quad w_{\text{eff}} = \frac{-20Q^2 - 9Q\lambda - 3 + \lambda^2}{3(4Q^2 - Q\lambda - 1)}.$$  

(90)

- (d)

$$(x_1, x_2) = \left(\frac{\sqrt{6}}{2\lambda \sqrt{3 + 2Q\lambda - 6Q^2}} \right), \quad \Omega_m = 1 - \frac{3 - 12Q^2 + 7Q\lambda}{\lambda^2}, \quad w_{\text{eff}} = -\frac{2Q}{\lambda}.$$  

(91)

- (e)

$$(x_1, x_2) = (0, 1), \quad \Omega_m = 0, \quad w_{\text{eff}} = -1.$$  

(92)

The point (e) corresponds to the de Sitter point, which exists only for $\lambda = 4Q$ [this can be confirmed by setting $\dot{\phi} = 0$ in Eqs. [76]-[81]].

We first study the case of non-zero values of $Q$ with constant $\lambda$, i.e. for the exponential potential $U(\phi) = U_0 e^{-\lambda \phi}$. We do not consider the special case of $\lambda = 4Q$. The matter-dominated era can be realized either by the point (a) or by the point (d). If the point (a) is responsible for the matter era, the condition $Q^2 \ll 1$ is required. We then have $\Omega_m \approx 1 + 10Q^2/3 > 1$ and $w_{\text{eff}} \approx 4Q^2/3$. When $Q^2 \ll 1$ the scalar-field dominated point (c) yields an accelerated expansion of the Universe provided that $-\sqrt{2} + 4Q < \lambda < \sqrt{2} + 4Q$. Under these conditions the point (a) is followed by the late-time cosmic acceleration. The scaling solution (d) can give rise to the equation of state, $w_{\text{eff}} \approx 0$ for $|Q| \ll |\lambda|$. In this case, however, the condition $w_{\text{eff}} < -1/3$ for the point (c) gives $\lambda^2 < 2$. Then the energy fraction of the pressureless matter for the point (d) does not satisfy the condition $\Omega_m \approx 1$. From the above discussion the viable cosmological trajectory for constant $\lambda$ corresponds to the sequence from the point (a) to the scalar-field dominated point (c) under the conditions $Q^2 \ll 1$ and $-\sqrt{2} + 4Q < \lambda < \sqrt{2} + 4Q$.

We shall proceed to the case where $\lambda$ varies with time. The fixed points derived above for constant $\lambda$ can be regarded as the “instantaneous” fixed points, provided that the time scale of the variation of $\lambda$ is smaller than that of the cosmic expansion. The matter era can be realized by the point (d) with $|Q| \ll |\lambda|$. The solutions finally approach either the de Sitter point (e) with $\lambda = 4Q$ or the accelerated point (c).

In the following we focus on the case in which the matter era with the point (d) is followed by the accelerated epoch with the de Sitter solution (e). To study the stability of the point (e) we define a variable $x_4 \equiv F$, satisfying the following equation

$$\frac{dx_4}{dN} = -2\sqrt{6}Qx_1x_4.$$  

(93)
Considering the $3 \times 3$ matrix for perturbations $\delta x_1$, $\delta x_2$ and $\delta x_4$ around the point (e), we obtain the eigenvalues

$$-3, \quad -\frac{3}{2} \left[ 1 \pm \sqrt{1 - \frac{8}{3} F_1 Q \frac{d\lambda}{dF}(F_1)} \right],$$

where $F_1 \equiv F(\phi_1)$ is the value of $F$ at the de Sitter point with the field value $\phi_1$. Since $F_1 > 0$, we find that the de Sitter point is stable under the condition

$$Q \frac{d\lambda}{dF}(F_1) \geq 0, \quad \text{i.e.,} \quad \frac{d\lambda}{d\phi}(\phi_1) \leq 0. \tag{95}$$

Let us consider the $f(R)$ model (47) in which the models (45) and (46) are recovered in the regime $R \gg R_c$. Since $e^{2\phi/\sqrt{6}} = 1 - 2n\mu(R/R_c)^{-(2n+1)}$, the potential $U = (FR - f)/2$ is given by

$$U(\phi) = \frac{nR_c}{2} \left[ 1 - \frac{2n + 1}{(2n\mu)^{2n/(2n+1)}} \left( 1 - e^{2\phi/\sqrt{6}} \right)^{2n/(2n+1)} \right], \tag{96}$$

In this case the slope of the potential, $\lambda = -U_{,\phi}/U$, is

$$\lambda = -\frac{4ne^{2\phi/\sqrt{6}}}{\sqrt{6}(2n\mu)^{2n/(2n+1)}} \left[ 1 - \frac{2n + 1}{(2n\mu)^{2n/(2n+1)}} \left( 1 - e^{2\phi/\sqrt{6}} \right)^{-2n/(2n+1)} \left( 1 - e^{2\phi/\sqrt{6}} \right)^{-1/(2n+1)} \right]. \tag{97}$$

In the deep matter-dominated epoch during which the condition $R/R_c \gg 1$ is satisfied, the field $\phi$ is very close to zero. For $n$ and $\mu$ of the order of unity, we have $|\lambda| \gg 1$ at this stage. Hence the matter era can be realized by the instantaneous fixed point (d). As $R/R_c$ gets smaller, $|\lambda|$ decreases to the order of unity. If the solutions reach the point $\lambda = 4Q = -4/\sqrt{6}$ and satisfy the stability condition $d\lambda/dF \leq 0$, then the final attractor corresponds to the de Sitter fixed point (e).

For the theories with general couplings $Q$, it is possible to construct a scalar-field potential that is the generalization of (51). One example is (70)

$$U(\phi) = U_0 \left[ 1 - C(1 - e^{-2Q\phi})^p \right] \quad (U_0 > 0, \ C > 0, \ 0 < p < 1). \tag{98}$$

The $f(R)$ model (47) corresponds to $Q = -1/\sqrt{6}$ and $p = 2n/(2n + 1)$. The slope of the potential is given by

$$\lambda = \frac{2CpQe^{-2Q\phi}(1 - e^{-2Q\phi})^{p-1}}{1 - C(1 - e^{-2Q\phi})^p}. \tag{99}$$

We have $U(\phi) \rightarrow U_0$ for $\phi \rightarrow 0$ and $U(\phi) \rightarrow U_0(1 - C)$ in the limits $\phi \rightarrow \infty$ (for $Q > 0$) and $\phi \rightarrow -\infty$ (for $Q < 0$).

The field is nearly frozen around the value $\phi = 0$ during the deep radiation and matter epochs. In these epochs we have $R \approx \rho_m/F$ from Eqs. (79)-(81) by noting that $U_0$ is negligibly small compared to $\rho_m$ or $\rho_r$. Using Eq. (51), it follows that $U_{,\phi} + Q \rho_m \approx 0$. Hence, in the high-curvature region, the field $\phi$ evolves along the instantaneous minima given by

$$\phi_m \approx \frac{1}{2Q} \left( \frac{2U_0 pC}{\rho_m} \right)^{1/(1-p)}. \tag{100}$$

The field value $|\phi_m|$ increases for decreasing $\rho_m$. As long as the condition $\rho_m \gg 2U_0 pC$ is satisfied, we have $|\phi_m| \ll 1$ from Eq. (100).

For field values around $\phi = 0$ one has $|\lambda| \gg 1$ from Eq. (99). Hence the instantaneous fixed point (d) can be responsible for the matter-dominated epoch provided that $|Q| \ll |\lambda|$. The variable $F = e^{-2Q\phi}$ decreases in time irrespective of the sign of the coupling $Q$ and hence $0 < F < 1$. The de Sitter solution corresponds to $\lambda = 4Q$, that is

$$C = \frac{2}{(1 - F_1)^{p-1} [2 + (p - 2)F_1]}, \tag{101}$$

This solution is present as long as the solution of this equation exists in the region $0 < F_1 < 1$.

From Eq. (99) the derivative of $\lambda$ with respect to $\phi$ is

$$\frac{d\lambda}{d\phi} = -\frac{4CpQ^2 F(1 - F)^p - 2[1 - pF - C(1 - F)^p]}{[1 - C(1 - F)^p]^2}. \tag{102}$$
Figure 2: The evolution of $\Omega_{\text{DE}}, \Omega_m, \Omega_{\text{rad}}$ and $w_{\text{eff}}$ in Brans-Dicke theory with the potential \((98)\). The model parameters are $Q = 0.01$, $p = 0.2$ and $C = 0.7$ with the initial conditions $x_1 = 0$, $x_2 = 2.27 \times 10^{-7}$, $x_3 = 0.7$, and $x_4 = -5.0 \times 10^{-13}$. From Ref. \[76\].

The de Sitter point is stable under the condition $1 - pF_1 > C(1 - F_1)^p$. Using Eq. \(101\) this translates into

$$F_1 > 1/(2 - p).$$

When $0 < C < 1$ one can show that $d\lambda/d\phi < 0$ is always satisfied. Hence the solutions approach the de Sitter attractor after the end of the matter era. When $C > 1$, the de Sitter point is stable under the condition \(102\). If this condition is violated, the solutions choose another stable fixed point [such as the point (c)] as an attractor.

The above discussion shows that, if $0 < C < 1$, the matter point (d) can be followed by the stable de Sitter solution (e). In Fig. 2 we plot the evolution of $\Omega_{\text{DE}}, \Omega_m, \Omega_{\text{rad}},$ and $w_{\text{eff}}$ for $Q = 0.01$, $p = 0.2$ and $C = 0.7$. This shows that the viable cosmological trajectory can be realized for the potential \((98)\). In order to confront with SN Ia observations, it is possible to rewrite Eqs. \((79)\) and \((80)\) in the forms of Eqs. \((29)\) and \((30)\) by defining the dark energy density $\rho_{\text{DE}}$ and the pressure $P_{\text{DE}}$ in the similar way. It was shown in Ref. \[76\] that the phantom equation of state as well as the cosmological constant boundary crossing can be realized for the field potentials $U(\phi)$ satisfying local gravity constraints.

C. Local gravity constraints on Brans-Dicke theory

We study local gravity constraints on Brans-Dicke theory described by the action \[76\]. In the absence of the potential $U(\phi)$ we already mentioned that the Brans-Dicke parameter $\omega_{\text{BD}}$ is constrained to be $\omega_{\text{BD}} > 4.0 \times 10^4$ from solar-system experiments. This gives the upper bound \(78\) on the coupling $Q$ between the field $\phi$ and non-relativistic matter in the Einstein frame. This bound also applies to the case of a nearly massless field with the potential $U(\phi)$ in which the Yukawa correction $e^{-Mr}$ is close to unity (where $M$ is the scalar field mass and $r$ is an interaction length).

In the presence of the field-potential it is possible for large coupling models ($|Q| \sim 1$) to satisfy local gravity constraints provided that the mass $M$ of the field $\phi$ is sufficiently large in the region of high density. In fact, the potential \[98\] is designed to have a large mass in the high-density region, so that it can be compatible with experimental tests of gravity through the chameleon mechanism. In the following we study the model \[98\] and derive the conditions under which local gravity constraints can be satisfied. If we make a conformal transformation for the action \[98\], the action in the Einstein frame is given by \[76\] with $F(\phi) = e^{-2Q\phi}$. We can use the results obtained in Sec. \[11C\] because thin-shell solutions have been derived for the general coupling $Q$. 

As in the case of $f(R)$ gravity, we consider a configuration in which a spherically symmetric body has a constant density $\rho_A$ inside the body and that the density outside the body is given by $\rho = \rho_B$ ($\ll \rho_A$). Under the condition $|Q\phi| \ll 1$, we obtain $V_{,\phi} \approx -2U_0 Q p C (2Q\phi)^{p-1}$ for the potential $V = U/F^2$ in the Einstein frame. Then the field values at the potential minima inside and outside the body are

$$
\phi_A \approx \frac{1}{2Q} \left( \frac{2U_0 p C}{\rho_A} \right)^{1/(1-p)}, \quad \phi_B \approx \frac{1}{2Q} \left( \frac{2U_0 p C}{\rho_B} \right)^{1/(1-p)}.
$$

In order to realize the accelerated expansion today, the energy scale $U_0$ is required to be the same order as the square of the present Hubble parameter $H_0$, i.e., $U_0 \sim H_0^2 \sim \rho_0$, where $\rho_0 \approx 10^{-29}$ g/cm$^3$ is the cosmological density today. The baryonic/dark matter density in our galaxy corresponds to $M/\phi \sim 10^{12}$ g/cm$^3$. When $\phi < \phi_0$ the chameleon mechanism to work, because the condition $1/(m_A^2 c^2) \ll 1$ is satisfied unless $C \ll 1$. The field mass squared $m_A^2 \equiv V_{,\phi\phi}$ at $\phi = \phi_A$ is approximately given by

$$
m_A^2 \approx \frac{1-p}{(2p p C)^{1/(1-p)}} Q^2 \left( \frac{\rho_A}{U_0} \right)^{(2-p)/(1-p)} U_0,
$$

which means that $m_A$ can be much larger than $H_0$ because of the condition $\rho_A \gg U_0$. This large mass allows the chameleon mechanism to work, because the condition $1/(m_A^2 c^2) \ll 1$ is satisfied.

The bound $|Q| \lesssim 10^{-4}$ coming from the violation of equivalence principle in the solar system translates into

$$
(2U_0 p C/\rho_B)^{1/(1-p)} < 7.4 \times 10^{-15} |Q|.
$$

Let us consider the case in which the solutions finally approach the de Sitter solution (e). At the point (e), one has $3F_1 H_0^2 = U_0[1 - C(1-F_1)^p]$ with $C$ given in Eq. (101). Hence we get the following relation

$$
U_0 = 3H_0^2 \frac{2 + (p-2)F_1}{p}.
$$

Plugging this into Eq. (106), it follows that

$$
(R_1/\rho_B)^{1/(1-p)} (1 - F_1) < 7.4 \times 10^{-15} |Q|,
$$

where $R_1 = 12H_0^2$ is the Ricci scalar at the de Sitter point. Since the term $(1 - F_1)$ is smaller than 1/2 from the condition $\rho_B = 10^{-24}$ g/cm$^3$, we obtain the following bound

$$
p > 1 - \frac{5}{13.8 - \log_{10} |Q|}.
$$

When $|Q| = 10^{-1}$ and $|Q| = 1$ we have $p > 0.66$ and $p > 0.64$, respectively. Thus the model can be compatible with local gravity experiments even for $|Q| = \mathcal{O}(1)$.

In Ref. [29] it was shown that in order to satisfy both local gravity and cosmological constraints the chameleon potentials in the Einstein frame need to be of the form $V(\phi) = M^4 [1 + f(\phi)]$, where the function $f(\phi)$ is smaller than 1 today and $M$ is a mass that corresponds to the dark energy scale ($M \sim 10^{-12}$ GeV). The potential $V(\phi) = M^4 \exp[\mu(M/\phi)^n]$ is of those viable candidates, but the allowed model parameter space is severely constrained by the 2006 Eöt-Wash experiment [213]. Unless the parameter $\mu$ is unnaturally small ($\mu \lesssim 10^{-5}$), this potential is incompatible with local gravity constraints for $\{n, Q\} = \mathcal{O}(1)$.

On the other hand, the chameleon potential $V(\phi) = V_0[1 - \mu(1 - e^{-Q\phi})^n]$ can satisfy both local gravity and cosmological constraints. In Ref. [79] this potential is consistent with the constraint coming from 2006 Eöt-Wash experiments as well as the WMAP bound on the variation of the field-dependent mass [214] for natural model parameters.

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4 In the Einstein frame the potential takes the form $V(\phi) = U/F^2 = U_0 e^{4Q\phi} [1 - C(1-e^{-2Q\phi})^p]$, so in the region $|Q\phi| \ll 1$ this potential is similar to $V(\phi) = V_0[1 - \mu(1 - e^{-Q\phi})^n]$. 
IV. DGP MODEL

In this section we review braneworld models of dark energy motivated by string theory. In braneworlds standard model particles are confined on a 3-dimensional (3D) brane embedded in 5-dimensional bulk with large extra dimensions [215, 216]. Dvali, Gabadadze, and Forrati (DGP) [41] proposed a braneworld model in which the 3-brane is embedded in a Minkowski bulk with infinitely large extra dimensions. One can recover Newton’s law by adding a 4D Einstein-Hilbert action sourced by the brane curvature to the 5D action [217]. The presence of such a 4D term may be induced by quantum corrections coming from the bulk gravity and its coupling with matter on the brane. In the DGP model the standard 4D gravity is recovered at small distances, whereas the effect from the 5D gravity manifests itself for large distances. Interestingly one can realize the self cosmic acceleration without introducing a dark energy component [218, 219] (see also Ref. 220).

A. Self-accelerating solution

The action of the DGP model is given by

$$S = \frac{1}{2\kappa_{(5)}^2} \int d^5X \sqrt{-\tilde{g}} \tilde{R} + \frac{1}{2\kappa_{(4)}^2} \int d^4X \sqrt{-g} R - \int d^5X \sqrt{-\tilde{g}} \mathcal{L}_M,$$

where $\tilde{g}_{AB}$ is the metric in the 5D bulk and $g_{\mu\nu} = \partial_\nu X^A \partial_\mu X^B g_{AB}$ is the induced metric on the brane with $X^A(x^c)$ being the coordinates of an event on the brane labelled by $x^c$. The 5D and 4D gravitational constants, $\kappa_{(5)}^2$ and $\kappa_{(4)}^2$, are related with the 5D and 4D Planck masses, $M_{(5)}$ and $M_{(4)}$, via

$$\kappa_{(5)}^2 = 1/M_{(5)}^3, \quad \kappa_{(4)}^2 = 1/M_{(4)}^2.$$

The first and second terms in Eq. (110) correspond to Einstein-Hilbert actions in the 5D bulk and on the brane, respectively. There is no contribution to the Lagrangian $\mathcal{L}_M$ from the bulk because we are considering a Minkowski bulk. Then the matter action consists of a brane-localized matter whose action is given by $\int d^4\sqrt{-\tilde{g}} (\sigma + \mathcal{L}_M^{brane})$, where $\sigma$ is the 3-brane tension and $\mathcal{L}_M^{brane}$ is the Lagrangian density on the brane. Since the tension is unrelated to the Ricci scalar $R$, it can be adjusted to be zero (as we do in the following).

In order to study the cosmological dynamics on the brane (located at $y = 0$), we take a metric of the form:

$$ds^2 = -n^2(\tau,y)d\tau^2 + a^2(\tau,y) \gamma_{ij} dx^i dx^j + dy^2,$$

where $\gamma_{ij}$ represents a maximally symmetric space-time with a constant curvature $K$. The 5D Einstein equations are

$$\tilde{G}_{AB} = \tilde{R}_{AB} - \frac{1}{2} \tilde{R} \tilde{g}_{AB} = \kappa_{(5)}^2 \tilde{T}_{AB},$$

where $\tilde{R}_{AB}$ is the 5D Ricci tensor, $\tilde{T}_{AB}$ is the sum of the energy momentum tensor $T_{AB}^{(brane)}$ on the brane and the contribution $\tilde{U}_{AB}$ coming from the scalar curvature of the brane:

$$\tilde{T}_{AB} = T_{AB}^{(brane)} + \tilde{U}_{AB}.$$

Since we are considering a homogeneous and isotropic Universe on the brane, one can write $T_{AB}^{(brane)}$ in the form

$$T_{AB}^{(brane)} = \delta(y) \text{diag}(-\rho_M, P_M, P_M, P_M, 0),$$

where $\rho_M$ and $P_M$ are functions of $\tau$ only. The non-vanishing components coming from the Ricci scalar $R$ of the brane are

$$\tilde{U}_{00} = \frac{3}{\kappa_{(4)}^2} \left(\frac{\dot{a}^2}{a^2} + K \frac{n^2}{a^2}\right) \delta(y),$$

$$\tilde{U}_{ij} = -\frac{1}{\kappa_{(4)}^2} \left(\frac{\dot{a}^2 + \dot{a} / a}{a^2} + 2 \frac{\dot{a}}{a} - 2 \frac{\ddot{a}}{a} - K\right) \gamma_{ij} \delta(y),$$

where $n$ is the conformal time and $a(t)$ the scale factor. The Greek indxes $i,j$ stand for the 3D coordinates $x^1, x^2, x^3$. The 4D space-time is expanding ($\dot{a}^2 > 0$).
where a dot represents a derivative with respect to \( \tau \). The non-vanishing components of the 5D Einstein tensor \( \tilde{G}_{AB} \) are \([218, 221, 222]\)

\[
\tilde{G}_{00} = 3 \left[ \frac{\dot{a}^2}{a^2} - n^2 \left( \frac{a''}{a} + \frac{a'^2}{a^2} \right) + K \frac{n^2}{a^2} \right], \\
\tilde{G}_{ij} = a^2 \left( \frac{2a''}{a} + \frac{n^2}{n} + \frac{a'^2}{a^2} + 2 \frac{a'^n}{an} \right) + a^2 \left( \frac{a''}{a} - \frac{a'^2}{a^2} - \frac{\dot{a}n}{an} \right) - K \gamma_{ij}, \\
\tilde{G}_{05} = 3 \left( \frac{\dot{a}n}{an} - \frac{\dot{a}'}{a} \right), \\
\tilde{G}_{55} = 3 \left( \frac{a'^2}{a^2} + \frac{a'n'}{an} \right) - \frac{3}{n^2} \left( \frac{\dot{a}'}{a} + \frac{\dot{a}}{a^2} - \frac{\dot{a}n}{an} \right) - 3 \frac{K}{a^2},
\]

where a prime represents a derivative with respect to \( y \).

Assuming no flow of matter along the 5-th dimension, we have \( \tilde{T}_{05} = 0 \) and hence \( \tilde{G}_{05} = 0 \). Then Eqs. \((118)\) and \((121)\) can be written as

\[
\tilde{G}_{00} = -\frac{3n^2}{2a^2a'} I', \quad \tilde{G}_{55} = -\frac{3}{2a^2a} \hat{I},
\]

where

\[
I \equiv (a'a)^2 - \frac{(\dot{a}a)^2}{n^2} - Ka^2.
\]

Since we are considering the Minkowski bulk, we have \( \tilde{G}_{00} = 0 \) and \( \tilde{G}_{55} = 0 \) locally in the bulk. This gives \( I' = 0 \) and \( \hat{I} = 0 \). Integrations of these equations lead to

\[
(a'a)^2 - \frac{(\dot{a}a)^2}{n^2} - Ka^2 + C = 0,
\]

where \( C \) is a constant independent of \( \tau \) and \( y \).

We shall find solutions of the Einstein equations \((115)\) in the vicinity of \( y = 0 \). The metric needs to be continuous across the brane in order to have a well-defined geometry. However, its derivatives with respect to \( y \) can be discontinuous at \( y = 0 \). The Einstein tensor is made of the metric up to the second derivatives with respect to \( y \), so the Einstein equations with a distributional source are written in the form \([218, 221, 222]\)

\[
g'' = T \delta(y),
\]

where \( \delta(y) \) is a Dirac’s delta function. Integrating this equation across the brane gives

\[
[g'] = T, \quad \text{where} \quad [g'] \equiv g'(0^+) - g'(0^-).
\]

The jump of the first derivative of the metric is equivalent to the energy-momentum tensor on the brane.

Equations \((118)\) and \((119)\) include the derivatives \( a'' \) and \( n'' \) of the metric. Integrating the Einstein equations \( \tilde{G}_{00} = \kappa_{(5)}^2 \tilde{T}_{00} \) and \( \tilde{G}_{ij} = \kappa_{(5)}^2 \tilde{T}_{ij} \) across the brane, we obtain

\[
\frac{a'}{a_b} = -\frac{\kappa_{(5)}^2}{3} \rho_M + \frac{\kappa_{(5)}^2}{\kappa_{(4)}^2 n_b^2} \left( \frac{\dot{a}b}{a_b} + \frac{K n_b^2}{a_b^2} \right),
\]

\[
\frac{n'}{n_b} = \frac{\kappa_{(5)}^2}{3} (3P_M + 2\rho_M) - \frac{\kappa_{(5)}^2}{\kappa_{(4)}^2 n_b^2} \left( \frac{\dot{a}b}{a_b} + \frac{\dot{a}b}{a_b} n_b - \frac{\dot{a}b}{a_b} + K n_b^2 \right),
\]

where the subscript “\( b \)” represents the quantities on the brane.

We assume the symmetry \( y \leftrightarrow -y \), in which case \( [a'] = 2a''(0^+) \) and \( [n'] = 2n''(0^+) \). Substituting Eq. \((127)\) into Eq. \((124)\), we obtain the modified Friedmann equation on the brane:

\[
\epsilon \sqrt{H^2 + \frac{K}{a_b^2} - \frac{C}{a_b^2}} = \frac{\kappa_{(5)}^2}{2\kappa_{(4)}^2} \left( H^2 + \frac{K}{a_b^2} \right) - \frac{\kappa_{(5)}^2}{6} \rho_M.
\]
where \( H \equiv \dot{a}_b/(a_b n_b) \) is the Hubble parameter and \( \epsilon = \pm 1 \) is the sign of \([a']\). The constant \( C \) can be interpreted as the term coming from the 5D bulk Weyl tensor \([218, 219, 223]\). Since the Weyl tensor vanishes for the Minkowski bulk, we set \( C = 0 \) in the following discussion. We introduce a length scale
\[
r_c \equiv \frac{\kappa^2 (5)}{2 \kappa^2 (4)} = \frac{M^2 (4)}{2 M (5)}.
\]
Then Eq. (129) can be written as
\[
\epsilon \frac{r_c}{\kappa^2} \sqrt{H^2 + \frac{K}{a^2}} = H^2 + \frac{K}{a^2} - \frac{2}{3} \rho_M ,
\]
where we have omitted the subscript “\(b\)” for the quantities at \(y = 0\).

Plugging the junction conditions (127) and (128) into the (05) component of the Einstein equations, \( \tilde{G}_{05} = 0 \), the following matter conservation equation holds on the brane:
\[
d \rho_M / dt + 3 H (\rho_M + P_M) = 0 ,
\]
where \( t \) is the cosmic time related to the time \( \tau \) via the relation \( dt = n_b d\tau \). If the equation of state, \( w_M = P_M/\rho_M \), is specified, the cosmological evolution is known by solving Eqs. (131) and (132).

For a flat geometry \( (K = 0) \), Eq. (131) reduces to
\[
H^2 - \frac{\epsilon}{r_c} H = \frac{\kappa^2 (4)}{3} \rho_M .
\]
If the crossover scale \( r_c \) is much larger than the Hubble radius \( H^{-1} \), the first term in Eq. (133) dominates over the second one. In this case the standard Friedmann equation, \( H^2 = \kappa^2 (4) \rho_M /3 \), is recovered. On the other hand, in the regime \( r_c < H^{-1} \), the presence of the second term in Eq. (133) leads to a modification to the standard Friedmann equation. In the Universe dominated by non-relativistic matter \( (\rho_M \propto a^{-3}) \), the Universe approaches a de Sitter solution for the branch \( \epsilon = +1 \):
\[
H \rightarrow H_{dS} = 1/r_c .
\]
We can realize the cosmic acceleration today provided that \( r_c \) is of the order of the present Hubble radius \( H_0^{-1} \).

**B. Observational constraints on the DGP model and other aspects of the model**

Equation (131) can be written as
\[
H^2 + \frac{K}{a^2} = \left( \sqrt{\frac{\kappa^2 (4)}{3} \rho_M + \frac{1}{4r_c^2}} + \frac{1}{2r_c} \right)^2 .
\]
For the matter on the brane, we consider non-relativistic matter with the energy density \( \rho_m \) and the equation of state \( w_m = 0 \). We then have \( \rho_m = \rho_m^{(0)} (1 + z)^3 \) from Eq. (132). Let us introduce the following density parameters
\[
\Omega_K^{(0)} = - \frac{K}{a_0^2 H_0^2} , \quad \Omega_r^{(0)} = \frac{1}{4r_c^2 H_0^2} , \quad \Omega_m^{(0)} = \frac{\kappa^2 (4) \rho_m^{(0)}}{3 H_0^2} .
\]
Then Eq. (135) reads
\[
H^2(z) = H_0^2 \left[ \Omega_K^{(0)} (1 + z)^2 + \left\{ \sqrt{\Omega_m^{(0)} (1 + z)^3 + \Omega_r^{(0)}} + \sqrt{\Omega_r^{(0)}} \right\}^2 \right] .
\]
The normalization condition at \( z = 0 \) is given by
\[
\Omega_m^{(0)} + \Omega_K^{(0)} + 2 \sqrt{1 - \Omega_K^{(0)} \Omega_r^{(0)}} = 1 .
\]
Figure 3: Observational constraints on the DGP model from the SNLS data [225] (solid thin), the BAO [5] (dotted), and the CMB shift parameter from the WMAP 3-year data [226] (dot-dashed). The thick line represents the curve (139) for the flat model ($\Omega_K(0) = 0$). The figure labels $\Omega_m$ and $\Omega_{rc}$ correspond to $\Omega_m(0)$ and $\Omega_{rc}(0)$, respectively. From Ref. [91].

For the flat universe ($K = 0$) this relation corresponds to

$$\Omega_{rc}(0) = (1 - \Omega_m(0))^2/4.$$  \hspace{1cm} (139)

The parametrization (137) of the Hubble parameter together with the normalization (138) can be used to place observational constraints on the DGP model at the background level [89–93]. In Ref. [89] the authors found a significantly worse fit to Supernova Ia (SN Ia) data and the distance to the last-scattering surface [3] relative to the $\Lambda$CDM model. In Refs. [90] and [92] the authors showed that the flat DGP model is disfavored from the combined data analysis of SN Ia [224, 225] and BAO [5]. In Fig. 3 we show the joint observational constraints [91] from the data of SNLS [225], BAO [5], and the CMB shift parameter [226]. While the flat DGP model can be consistent with the SN Ia data, it is under strong observational pressure by adding the data of BAO and the CMB shift parameter. The open DGP model gives a slightly better fit relative to the flat model [91, 93]. The joint analysis using the data of SN Ia, BAO, CMB, gamma ray bursts, and the linear growth factor of matter perturbations show that the flat DGP model is incompatible with current observations [94].

In the DGP model a brane-bending mode $\phi$ (i.e. longitudinal graviton) gives rise to a field self-interaction of the form $\Box \phi (\partial^\mu \phi \partial_\mu \phi)$ through a mixing with the transverse graviton [83–84]. This can lead to the decoupling of the field $\phi$ from gravitational dynamics in the local region by the so-called Vainshtein mechanism [85]. The General Relativistic behavior can be recovered within a radius $r_s = (r_g r)^{1/3}$, where $r_g$ is the Schwarzschild radius of a source. Since $r_s$ is larger than the solar-system scales, the DGP model can evade local gravity constraints [83]. However, the DGP model is plagued by a strong coupling problem for typical distances smaller than 1000 km [228]. Some regularization methods have been proposed to avoid the strong coupling problem, such as smoothing out the delta profile on the brane [230, 231] or re-using the delta function profile but in a higher-dimensional brane [232, 233].

As we will see in VII C, the analysis of 5D cosmological perturbations on the scales larger than $r_s$ shows that the DGP model contains a ghost mode in the scalar sector of the gravitational field [86–88]. There are several ways of the generalization of the DGP model to avoid the appearance of ghosts. One way is to consider the 6D braneworld set-up as in the Cascading gravity [235]. Another is to generalize the field self-interaction term $\Box \phi (\partial^\mu \phi \partial_\mu \phi)$ to more general forms in the 4D gravity [42]. In Sec. V we shall discuss the latter approach (“Galileon gravity”) in detail.
V. GALILEON GRAVITY

In the DGP model the field derivative self-coupling $\Box \phi (\partial^\mu \phi \partial^\mu \phi)$, arising from a brane-bending mode, allows the decoupling of the field from matter within a Vainshtein radius. In the local regions where solar-system experiments are carried out, the field is nearly frozen through the non-linear self-interaction. This is different from the chameleon mechanism in which the presence of the field potential with a matter coupling gives rise to a minimum with a large mass in the regions of high density.

Under the Galilean shift $\partial_\mu \phi \to \partial_\mu \phi + b_\mu$, the field equation following from the Lagrangian $\Box \phi (\partial^\mu \phi \partial^\mu \phi)$ is unchanged in the Minkowski space-time. The generalization of the nonlinear field Lagrangian to more general cases may be useful, e.g., to overcome the ghost problem associated with the DGP model. In fact Nicolis et al. [42] derived five Lagrangians that lead to the field equations invariant under the Galilean shift $\partial_\mu \phi \to \partial_\mu \phi + b_\mu$ in the Minkowski space-time. The scalar field respecting the Galilean symmetry is dubbed “Galileon”. Each of the five terms only leads to second-order differential equations, keeping the theory free from unstable spin-2 ghost degrees of freedom.

If we extend the analysis in Ref. [42] to that in the curved space-time, the Lagrangians should be promoted to covariant forms. Deffayet et al. [93, 94] derived covariant Lagrangians $L_i$ ($i = 1, \cdots, 5$) that keep the field equations up to second-order, while recovering the five Lagrangians derived by Nicolis et al. in the Minkowski space-time. This can be achieved by introducing field-derivative couplings with the Ricci scalar $R$ and the Einstein tensor $G_{\mu \nu}$ in the expression of $L_{1,5}$. Since the existence of those terms affects the effective gravitational coupling, the Galileon gravity based on the covariant Lagrangians $L_i$ ($i = 1, \cdots, 5$) can be classified as one of modified gravitational theories.

The cosmological dynamics including the terms up to $L_4$ and $L_5$ have been studied by a number of authors [97, 236]. In particular Refs. [97, 98] have shown that, for the covariant Galileon theory having de Sitter attractors, cosmological solutions with different initial conditions converge to a common trajectory— a tracker solution. Moreover there is a viable parameter space in which the conditions for the avoidance of ghosts and Laplacian instabilities of scalar and tensor perturbations are satisfied.

The generalization of Galileon gravity, which mostly corresponds to the modification of the term $L_3 = \Box \phi (\partial^\mu \phi \partial^\mu \phi)$, has been also extensively studied recently [237–253]. One application is to introduce the non-linear field self-interaction of the form $\xi (\phi) \Delta (\partial^\mu \phi \partial^\mu \phi)$ in the action of (generalized) Brans-Dicke theories [183, 238, 240, 242], where $\xi$ is a function of $\phi$. For suitable choices of the function $\xi (\phi)$, there exist de Sitter (dS) solutions responsible for dark energy even in the absence of the field potential. The cosmology based on a further general term $G (\phi, X) \Box \phi$ has been discussed in the context of either dark energy and inflation [246, 247, 252, 253, 256] and perturbations are satisfied.

In the following we review the cosmological dynamics in the Galileon dark energy model based on the covariant Lagrangians $L_i$ ($i = 1, \cdots, 5$) and study observational constraints on the model. We will also discuss the modified version of Galileon gravity in which the term $L_3 = \Box \phi (\partial^\mu \phi \partial^\mu \phi)$ is generalized.

A. Cosmology of a covariant Galileon field

We start with the covariant Galileon gravity described by the action [93, 96]

$$S = \int d^4 x \sqrt{-g} \left[ \frac{M^2_{\text{pl}}}{2} R + \frac{1}{2} \sum_{i=1}^5 c_i L_i \right] + \int d^4 x L_M, \quad (140)$$

where $M_{\text{pl}} = (8\pi G)^{-1/2}$ is the reduced Planck mass, and $c_i$’s are constants. The five Lagrangians $L_i$ ($i = 1, \cdots, 5$) satisfying the Galilean symmetry in the limit of the Minkowski space-time are given by

$$L_1 = M^2 \phi, \quad L_2 = (\nabla \phi)^2, \quad L_3 = (\Box \phi)(\nabla \phi)^2 / M^3, \quad L_4 = (\nabla \phi)^2 \left[ 2(\Box \phi)^2 - 2 \phi_{\mu \nu} \phi^{\mu \nu} - R(\nabla \phi)^2 / 2 \right] / M^6,$$

$$L_5 = (\nabla \phi)^2 [(\Box \phi)^3 - 3(\Box \phi) \phi_{\nu \mu \rho} \phi^{\nu \mu \rho} + 2 \phi_{\mu \nu} \phi_{\nu \rho} \phi_{\rho \mu} - 6 \phi^{\mu \nu \rho} \phi^{\rho \mu} / G_{\mu \nu}] / M^9, \quad (141)$$

where $M$ is a constant having a dimension of mass, and $G_{\mu \nu}$ is the Einstein tensor. For the matter Lagrangian $L_M$ we take into account perfect fluids of non-relativistic matter (energy density $\rho_m$, equation of state $w_m = 0$) and radiation (energy density $\rho_r$, equation of state $w_r = 1/3$).

Let us consider the FLRW metric with the cosmic curvature $K$:

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right]. \quad (142)$$

Variation of the action (140) with respect to $g_{\mu \nu}$ leads to the following equations of motion

$$3M^2_{\text{pl}} H^2 = \rho_{\text{DE}} + \rho_m + \rho_r + \rho_K, \quad (143)$$

$$3M^2_{\text{pl}} H^2 + 2M^2_{\text{pl}} H = -P_{\text{DE}} - \rho_r / 3 + \rho_K / 3, \quad (144)$$
where $\rho_K \equiv -3K M_{pl}^2/a^2$, and

$$
\rho_{DE} \equiv -c_1 M^3 \dot{\phi}/2 - c_2 \dot{\phi}^2/2 + 3c_3 H \dot{\phi}^3/M - 45c_4 H^2 \dot{\phi}^4/(2M^6) + 21c_5 H^3 \dot{\phi}^5/M^9,
$$

$$
P_{DE} \equiv c_1 M^3 \dot{\phi}/2 - c_2 \dot{\phi}^2/2 - c_3 \dot{\phi}^2 \dot{\phi}/M^2 + 3c_3 \dot{\phi}^3 (8H \dot{\phi} + (3H^2 + 2H) \ddot{\phi})/(2M^6)
-3c_5 H \dot{\phi}^4 [5H \dot{\phi} + 2(H^2 + \dot{H}) \dot{\phi}]/M^9.
$$

The matter fluids satisfy the continuity equations $\dot{\rho}_m + 3H \rho_m = 0$ and $\dot{\rho}_r + 4H \rho_r = 0$. We define the dark energy equation of state $w_{DE}$ and the effective equation of state $w_{eff}$, as

$$
w_{DE} \equiv \frac{P_{DE}}{\rho_{DE}}, \quad w_{eff} \equiv -1 - \frac{2\ddot{H}}{3H^2}.
$$

Using the continuity equation $\dot{\rho}_{DE} + 3H (\rho_{DE} + P_{DE}) = 0$, it follows that $w_{DE} = w_{eff} = \Omega_{DE}/(3H \Omega_{DE})$.

Since we are interested in the case where the late-time cosmic acceleration is realized by the field kinetic energy, we set $c_1 = 0$ in the following discussion$^5$. Then the de Sitter solution ($H = H_{ds} = \text{constant}$) can be present for $\dot{\phi} = \dot{\phi}_{ds} = \text{constant}$. We normalize the mass $M$ to be $M^3 = M_{pl} H_{ds}^2$, which gives $M \approx 10^{-40} M_{pl}$ for $H_{ds} \approx 10^{-60} M_{pl}$.

Defining $x_{ds} \equiv \dot{\phi}_{ds}/(H_{ds} M_{pl})$, Eqs. (143) and (144) lead to the following relations at the de Sitter solution:

$$c_2 x_{ds}^2 = 6 + 9\alpha - 12\beta, \quad c_3 x_{ds}^3 = 2 + 9\alpha - 9\beta,
$$

where

$$\alpha \equiv c_4 x_{ds}^4, \quad \beta \equiv c_5 x_{ds}^5.
$$

It is convenient to use the variables $\alpha$ and $\beta$, because the coefficients of physical quantities and dynamical equations can be expressed by $\alpha$ and $\beta$. The relations (148) do not change under the rescaling $x_{ds} \rightarrow \gamma x_{ds}$ and $c_i \rightarrow c_i/\gamma^4$, where $\gamma$ is a real constant. Then the rescaled choice of $c_i$ can provide the same physics.

In order to study the cosmological dynamics, we introduce the following dimensionless variables:

$$r_1 \equiv \frac{\dot{\phi}_{ds} H_{ds}}{\phi H}, \quad r_2 \equiv \frac{1}{r_1} \left( \frac{\dot{\phi}}{\phi_{ds}} \right)^4.
$$

At the de Sitter solution $r_1 = 1$ and $r_2 = 1$. We define the dark energy density parameter

$$\Omega_{DE} \equiv \frac{\rho_{DE}}{3M_{pl}^2 H^2} = -(2 + 3\alpha - 4\beta) r_2^3/2 + (2 + 9\alpha - 9\beta) r_2^4 - 15\alpha r_1 r_2/2 + 7\beta r_2.
$$

Then Eq. (143) can be written as $\Omega_{DE} + \Omega_m + \Omega_r + \Omega_K = 1$, where $\Omega_m \equiv \rho_m/(3M_{pl}^2 H^2)$, $\Omega_r \equiv \rho_r/(3M_{pl}^2 H^2)$, and $\Omega_K \equiv \rho_K/(3M_{pl}^2 H^2) = -K/(aH)^2$.

The autonomous equations for the variables $r_1$, $r_2$, $\Omega_r$, and $\Omega_K$ are given by $[98, 259]$

$$r_1' = \frac{1}{\Delta} \left[ (r_1 - 1) r_1 \left[ r_1 (r_1 (3\alpha + 4\beta - 2) + 6\alpha - 5\beta) - 5\beta \right] \right. \times \left[ 2(\Omega_r - \Omega_K + 9) + 3r_2 \left( (r_1 (3\alpha + 4\beta - 2) + 2r_1^2 (9\alpha - 9\beta + 2) - 15\alpha r_1 + 14\beta) \right) \right],
$$

$$r_2' = \frac{1}{\Delta} \left[ r_2 (6r_1^2 (45\alpha^2 - 4(9a + 2\beta) + 36\beta^2) - (9\alpha - 9\beta - 33) + (3\alpha - 4\beta - 2) - 3r_2 (-2(20\alpha + 89)\beta + 15\alpha (9a + 2) + 356\beta^2)) - 3r_1 \alpha (-28\Omega_r + 28\Omega_K + 123r_2 \beta + 36) + 10\beta (-11\Omega_r + 11\Omega_K + 21r_2 \beta - 3 + 3r_1 r_2 (9\alpha - 9\beta + 1) + 2 (2 - 9\beta^2) + 3r_1^2 r_2 (3\alpha - 4\beta + 2)^2
+ 3r_1 r_2 (9\alpha - 9\beta + 2)(3\alpha - 4\beta + 2)) \right],
$$

$$\Omega_r' = \Omega_r (-4 - 2H'/H),
$$

$$\Omega_K' = \Omega_K (-2 - 2H'/H),
$$

$^5$ In this case the only solution in the Minkowski background ($H = 0$) corresponds to $\dot{\phi} = 0$ for $c_2 \neq 0$. 
where a prime represents a derivative with respect to $N = \ln a$, and
\[
\Delta \equiv 2r_1^2 r_2 (72\alpha^2 + 30\alpha (1 - 5\beta) + (2 - 9\beta)^2) + 4r_1^4 [9r_2 (5\alpha^2 + 9\alpha\beta + (2 - 9\beta)\beta) + 2(9\alpha - 9\beta + 2)] \\
+ 4r_1^2 [-3r_2 (-2(15\alpha + 1)\beta + 3\alpha (9\alpha + 2) + 4\beta^2) - 3\alpha + 4\beta - 2] - 24r_1\alpha (16r_2\beta + 3) + 10\beta (21r_2\beta + 8) .
\] (156)

The Hubble parameter follows from the equation $H'/H = -5r_1'/(4r_1) - r_2'/(4r_2)$. The solutions to Eqs. (154) and (155) are given by $\Omega_r(N) = \Omega_r(0) e^{-4N H_0^2}/H^2(N)$ and $\Omega_K(N) = \Omega_K(0) e^{-2N H_0^2}/H^2(N)$ respectively, where the subscript "(0)" represents the values today ($N = 0$).

From Eqs. (152) and (153) we find that there are three distinct fixed points: (A) $(r_1, r_2) = (0, 0)$, (B) $(r_1, r_2) = (1, 0)$, and (C) $(r_1, r_2) = (1, 1)$. As we have already mentioned, the point (C) corresponds to the de Sitter solution. By considering homogeneous perturbations about this point, we can show that the de Sitter solution (C) is always classically stable [98]. The point (B) is a tracker solution found in Ref. [97], along which the field velocity evolves as $\dot{\phi} \propto 1/H$. During the radiation and matter eras the variable $r_2$ is much smaller than 1. The fixed point (B) is followed by the stable de Sitter point (C) once $r_2$ grows to the order of 1. If the initial conditions of both $r_1$ and $r_2$ in the radiation era are much smaller than 1, then the solutions are close to the point (A) at the initial stage. At late times the solutions approach the tracker at $r_1 = 1$. Depending on the initial values of $r_1$, the epoch at which the solutions reach the tracker is different. In the following we consider the background evolution in two regimes: (i) $r_1 = 1$ and (ii) $r_1 \ll 1$ in more detail.

1. Tracker solution ($r_1 = 1$)

Along the tracker ($r_1 = 1$) the dark energy density parameter (151) is given by
\[
\Omega_{DE} = r_2 ,
\] (157)
which is much smaller than 1 during the radiation and matter eras. From Eqs. (154) and (155) we obtain $r_2'/r_2 = 8 + 2\Omega_r'/\Omega_r$. This is integrated to give
\[
r_2 = d_1 a^8 \Omega_r^2 ,
\] (158)
where $d_1$ is a constant. From Eqs. (153) and (154) we have $\Omega_K'/\Omega_K - \Omega_r'/\Omega_r = 2$, which is integrated to give
\[
\frac{\Omega_K}{\Omega_r} = d_2 a^2 , \quad \text{with} \quad d_2 = \frac{\Omega_K(0)}{\Omega_r(0)} .
\] (159)

Substituting Eqs. (158) and (159) into Eq. (154), we obtain the cosmologically viable solution to $\Omega_r$, as
\[
\Omega_r = \frac{-1 + d_3 a - d_2 a^2 + \sqrt{4d_1 a^8 + (-1 + d_3 a - d_2 a^2)^2}}{2d_1 a^8} ,
\] (160)
where $d_3$ is another constant. Since the density parameter (160) evolves as $\Omega_r \simeq 1 + d_3 a$ in the early time ($a \ll 1$), this demands the condition $d_3 < 0$ (provided $\Omega_{DE} > 0$). Using the density parameters today, the constants $d_1$ and $d_3$ can be expressed as $d_1 = [1 - \Omega_m(0) - \Omega_r(0) - \Omega_K(0)]/\Omega_r(0)^2$ and $d_3 = -\Omega_m(0)/\Omega_r(0)$. Since $\Omega_r \propto \rho_r/H^2 \propto 1/(a^4 H^2)$, we have $H^2/H_0^2 = (\Omega_r(0)/\Omega_r)(1/a^4)$. Using Eq. (160), the Hubble parameter can be expressed in terms of the redshift $z = 1/a - 1$:
\[
\left( \frac{H(z)}{H_0} \right)^2 = \frac{1}{2} \Omega_K(0) (1 + z)^2 + \frac{1}{2} \Omega_m(0) (1 + z)^3 + \frac{1}{2} \Omega_r(0) (1 + z)^4 \\
+ \sqrt{1 - \Omega_m(0) - \Omega_r(0) - \Omega_K(0) + \frac{(1 + z)^2}{4} \left[ \Omega_K(0) + \Omega_m(0) (1 + z) + \Omega_r(0) (1 + z)^2 \right]^2} ,
\] (161)
which is useful to test the viability of the tracker solution from observations.

On the tracker, the equations of state defined in Eq. (147) are given by
\[
w_{DE} = -\frac{\Omega_r - \Omega_K + 6}{3(r_2 + 1)} , \quad w_{eff} = \frac{\Omega_r - \Omega_K - 6r_2}{3(r_2 + 1)} .
\] (162)
significantly, it can change the diameter distance as well as the luminosity distance relative to the flat Universe.

During the cosmological sequence of radiation ($\Omega_r \simeq 1$, $|\Omega_K| \ll 1$, $r_2 \ll 1$), matter ($\Omega_r \ll 1$, $|\Omega_K| \ll 1$, $r_2 \ll 1$), and de Sitter ($\Omega_r \ll 1$, $|\Omega_K| \ll 1$, $r_2 = 1$) eras, the dark energy equation of state evolves as $w_{DE} = -7/3 \to -2 \to -1$, whereas the effective equation of state evolves as $w_{\text{eff}} = 1/3 \to 0 \to -1$. This peculiar evolution of $w_{DE}$ for the tracker corresponds to the case (e) in Fig. 4. Although the effect of the cosmic curvature does not affect the dynamics of $w_{DE}$ significantly, it can change the diameter distance as well as the luminosity distance relative to the flat Universe.

The epoch at which the solutions reach the tracking regime $r_1 \simeq 1$ depends on model parameters and initial conditions. The approach to this regime occurs later for smaller initial values of $r_1$, see Fig. 4. In Ref. [98] it was shown that the tracker is stable in the direction of $r_1$ by considering a homogeneous perturbation $\delta r_1$. This means that once the solutions reach the tracker the variable $r_1$ does not repel away from 1. If $r_1 \lesssim 2$ initially, numerical simulations show that the solutions approach the tracker with the late-time cosmic acceleration. Meanwhile, for the initial conditions with $r_1 \gtrsim 2$, the dominant contribution to $\Omega_{DE}$ comes from the Lagrangian $\mathcal{L}_2$, so that the field energy density decreases rapidly as in the standard massless scalar field.

2. Solutions in the regime $r_1 \ll 1$

There is another case in which the solutions start to evolve from the regime $r_1 \ll 1$ (where the term $\mathcal{L}_5$ gives the dominant contribution to the field dynamics). In this regime, the variables $r_1$ and $r_2$ satisfy the following approximate equations

$$ r_1' \simeq \frac{9 + \Omega_r - \Omega_K + 21 \beta r_2}{8 + 21 \beta r_2} r_1, \quad r_2' \simeq \frac{3 + 11 \Omega_r - 11 \Omega_K - 21 \beta r_2}{8 + 21 \beta r_2} r_2. $$

(163)

As long as $\{\beta r_2, |\Omega_K|\} \ll 1$, the evolution of $r_1$ and $r_2$ during the radiation (matter) era is given by $r_1 \propto a^{5/4}$ and $r_2 \propto a^{7/4}$ ($r_1 \propto a^{9/8}$ and $r_2 \propto a^{3/8}$). Then the field velocity grows as $\dot{\phi} \propto t^{3/8}$ during the radiation era and $\dot{\phi} \propto t^{1/4}$ during the matter era. The evolution of $\dot{\phi}$ is slower than that for the tracker (i.e. $\dot{\phi} \propto t$).

In the regime $r_1 \ll 1$ the equations of state are

$$ w_{DE} \simeq -\frac{1 + \Omega_r - \Omega_K}{8 + 21 \beta r_2}, \quad w_{\text{eff}} \simeq \frac{8 \Omega_r - 8 \Omega_K - 21 \beta r_2}{3(8 + 21 \beta r_2)}. $$

(164)
Provided that \( \{ \beta r_2, |\Omega_K| \} \ll 1 \), one has \( w_{\text{DE}} \simeq -1/4 \), \( w_{\text{eff}} \simeq 1/3 \) during the radiation era and \( w_{\text{DE}} \simeq -1/8 \), \( w_{\text{eff}} \simeq 0 \) during the matter era. This evolution of \( w_{\text{DE}} \) is quite different from that for the tracker solution.

In Fig. 4 the variation of \( w_{\text{DE}} \) is plotted for a number of different initial conditions with \( r_1 \ll 1 \) [which correspond to the cases (a)-(d)]. As expected, the solutions start to evolve from the value \( w_{\text{DE}} \simeq -1/4 \) in the radiation era. For larger initial values of \( r_1 \), they approach the tracker earlier. This tracking behavior also occurs in the presence of the cosmic curvature \( K \). [258]

3. Conditions for the avoidance of ghosts and Laplacian instabilities

Let us find a model parameter space in which the appearance of ghosts and instabilities can be avoided in covariant Galileon gravity. In doing so, we need to study a linear perturbation theory on the FLRW background. For simplicity we focus on the flat Universe with \( K = 0 \). Let us consider the perturbed metric

\[
\text{d} s^2 = -[1 + 2\Psi(t, x)]\text{d}t^2 + \partial_i\chi(t, x)\text{d}t\text{d}x^i + a^2(t)[1 + 2\Phi(t, x)]\text{d}x^2,
\]

where \( \Psi, \Phi, \) and \( \chi \) are scalar metric perturbations. We have chosen the gauge \( \delta \phi = 0 \) without a non-diagonal scalar perturbation in the spatial part of the metric, i.e. \( \partial_i\gamma = 0 \) [181]. Taking into account two perfect fluids with the equations of state \( w_i = P_i/\rho_i \) (\( i = 1, 2 \)), there are three propagating scalar degrees of freedom. The velocity potentials \( v_i \) (\( i = 1, 2 \)) of perfect fluids are related with the energy-momentum tensor \( T^{(i)}_{\mu\nu} \), as \( T^{(i)}_{\mu\nu} = - (\rho_i + P_i) \partial_\mu v_i + \partial_\nu v_i \) (\( i = 1, 2 \)).

Introducing the vector \( \vec{Q} = (v_1, v_2, \Phi) \) and expanding the action [140] until the second-order, we obtain the second-order action for scalar perturbations [98] (see also Refs. [183, 260, 261]):

\[
\delta S^{(2)}_S = \frac{1}{2} \int d^4 x\, a^3 \left[ \hat{Q}^i A \hat{Q}^i - \frac{1}{a^2} \nabla \hat{Q}^i C \nabla \hat{Q}^i - \hat{Q}^i B \hat{Q}^i - \hat{Q}^i D \hat{Q}^i \right],
\]

where the fields \( \Psi \) and \( \chi \) are integrated out. \( A, C \) and \( D \) are 3 \times 3 symmetric matrices and \( B \) is an antisymmetric matrix (for which we do not write explicit forms).

In order to avoid the appearance of ghosts we require that the matrix \( A \) is positive definite. This corresponds to the conditions \((1 + w_1)\rho_1/w_1 > 0\), \((1 + w_2)\rho_2/w_2 > 0\), and

\[
\frac{Q_S}{M^4_{\text{pl}}} = -\frac{6(1 + \mu_1)(\mu_2 + \mu_1 \mu_2 - 2\mu_3 - \mu_3^2)}{(1 + \mu_3)^2} > 0,
\]

where

\[
\begin{align*}
\mu_1 & \equiv 3\alpha r_1 r_2/2 - 3\beta r_2, \\
\mu_2 & \equiv (3\alpha - 4\beta + 2)r_1^2 r_2/2 - 2(9\alpha - 9\beta + 2)r_1^2 r_2 + 45\alpha r_1 r_2^2/2 - 28\beta r_2, \\
\mu_3 & \equiv -(9\alpha - 9\beta + 2)r_1^2 r_2/2 + 15\alpha r_1 r_2^2/2 - 21\beta r_2/2.
\end{align*}
\]

The propagation speeds \( c_S \) of three scalar degrees of freedom are known by solving the equation

\[
\det(c^2_S A - C) = 0.
\]

For the two perfect fluids we have \( c_S^2 = w_1 \) and \( c_S^2 = w_2 \), which are are positive for both radiation and non-relativistic matter. The third stability condition associated with another scalar degree of freedom is given by

\[
c_S^2 = \frac{(1 + \mu_1)^2(2\mu_3^2 - (1 + \mu_3)(5 + 3w_{\text{eff}}) + 4\Omega_r + 3\Omega_m) - 4\mu_1(1 + \mu_1)(1 + \mu_2) + 2(1 + \mu_3)^2(1 + \mu_4)}{6(1 + \mu_1)(\mu_1 + \mu_2 + \mu_1 \mu_2 - 2\mu_3 - \mu_3^2)} > 0,
\]

where

\[
\mu_4 = -\alpha r_1 r_2/2 - 3\beta r_2(r_1^2/r_1 + r_1^2/r_2)/4.
\]

Let us consider tensor perturbations with \( \delta g_{ij} = a^2 h_{ij} \), where \( h_{ij} \) is traceless (\( h^i_i = 0 \)) and divergence-free (\( h_{ij}^\alpha_j = 0 \)). We expand the action [140] at second-order in terms of the two polarization modes, \( h_{ij} = h_\parallel \epsilon_{ij}^\parallel + h_\perp \epsilon_{ij}^\perp \), where \( \epsilon_{ij}^\parallel \) and \( \epsilon_{ij}^\perp \) are the polarization tensors. For the polarization mode \( h_\parallel \), the second-order action is given by

\[
\delta S_T^{(2)} = \frac{1}{2} \int d^4 x a^3 Q_T \left[ h_\parallel^2 - \frac{c^2_T}{a^2} (\nabla h_\parallel)^2 \right].
\]
The conditions for the avoidance of ghosts and Laplacian instabilities of tensor perturbations correspond, respectively, to

\[ \frac{Q_T}{M_{pl}^2} = \frac{1}{2} + \frac{3}{4} \alpha r_1 r_2 - \frac{3}{2} \beta r_2 > 0, \]  
\[ c_T^2 = \frac{2r_1 (2 - 3 \alpha r_1 r_2 - 3 \beta r_2 r_1')}{2r_1 (2 + 3 \alpha r_1 r_2 - 6 \beta r_2)} > 0. \]  

The same conditions also follow from \( h_\phi \).

In the regime \( r_1 < 1 \) and \( r_2 < 1 \) one has \( Q_S/M_{pl}^2 \simeq 60 \beta r_2 \) and \( Q_T \simeq 1/2 \). For the initial conditions with \( r_2 > 0 \) we require that \( \beta > 0 \) to avoid the scalar ghost. Since \( c_T^2 \approx (1 + \Omega_r)/40 \) and \( c_T^2 \approx 1 + 3 \beta r_2 (5 - 3 \Omega_r)/8 \approx 1 \), there are no Laplacian instabilities of scalar and tensor perturbations in this regime.

In the tracking regime characterized by \( r_1 = 1 \) (either \( r_2 < 1 \) or \( r_2 = 1 \)), the conditions (177), (172), (175), and (176) give the bounds on the parameters \( \alpha \) and \( \beta \). In the regime \( r_2 < 1 \) these conditions translate to

\[ Q_S \simeq 3(2 - 3 \alpha + 6 \beta) r_2 > 0, \]  
\[ c_S^2 \simeq \frac{8 + 10 \alpha - 9 \beta + \Omega_r (2 + 3 \alpha - 3 \beta)}{3(2 - 3 \alpha + 6 \beta)} > 0. \]  

For the branch \( r_2 > 0 \) the first condition reduces to \( 2 - 3 \alpha + 6 \beta > 0 \). Since \( c_T^2 \approx 1 - r_2 (4 \alpha + 3 \beta + 3 \beta \Omega_r)/2 \approx 1 \) and \( Q_T/M_{pl}^2 = 1/2 + 3(\alpha - 2 \beta) r_2/4 > 0 \), the tensor modes do not provide additional constraints. At the de Sitter point \( (r_2 = 1) \) we require that

\[ \frac{Q_S}{M_{pl}^2} = \frac{4 - 9(\alpha - 2 \beta)^2}{3(\alpha - 2 \beta)^2} > 0, \]  
\[ \frac{Q_T}{M_{pl}^2} = \frac{1}{4} (2 + 3 \alpha - 6 \beta) > 0, \]  
\[ c_T^2 = \frac{(\alpha - 2 \beta)(4 + 15 \alpha^2 - 48 \alpha \beta + 36 \beta^2)}{2[4 - 9(\alpha - 2 \beta)^2]} > 0, \]  
\[ c_S^2 = \frac{2 - \alpha}{2 + 3 \alpha - 6 \beta} > 0. \]  

If \( \beta > 0 \), it can happen that \( c_T^2 \) has a minimum during the transition from the regime \( r_2 < 1 \) to \( r_2 \approx 1 \) [97, 98]. This value tends to decrease as \( \beta \) approaches 1. Imposing that \( c_T^2 > 0 \) at the minimum, we obtain the bound

\[ \alpha < 12 \sqrt{\beta - 9 \beta - 2}. \]  

In Fig. 5 we plot the parameter space in the \((\alpha, \beta)\) plane constrained by the conditions (177)-(181). Clearly there are viable model parameters satisfying all the theoretical constraints.
4. Observational constraints on Galileon cosmology from the background cosmic expansion history

Since the evolution of the dark energy equation of state in covariant Galileon gravity is rather peculiar, the observational data related with the background cosmic expansion history may place tight constraints on the model. Especially the analytic formula (161) for the tracker is useful for such a purpose. In Ref. [259], the authors confronted the Galileon model by using the observational data of SN Ia (Constitution [262] and Union2 sets [263]), the CMB (WMAP7) shift parameters [4], and BAO (SDSS7) [6].

If either of the SN Ia data (Constitution or Union2) is used in the data analysis, the $\chi^2$ for the tracker is similar to that in the $\Lambda$CDM model. In the presence of the cosmic curvature $K$, the tracker solution is compatible with the individual observational bound constrained from either CMB or BAO. However, the combined data analysis of Constitution+BAO+CMB shows that the difference of $\chi^2$ between the tracker and the $\Lambda$CDM is $\delta \chi^2 \sim 22$ (or $\sim 4.3\sigma$). This means that the tracker is severely disfavored with respect to the $\Lambda$CDM. A similar conclusion was reached from the combined data analysis of Union2+BAO+CMB. The reason for this incompatibility is that the SN Ia data favor the large values of $\Omega_m^{(0)}$ ($\gtrsim 0.32$), whereas the CMB and BAO data constrain smaller values of $\Omega_m^{(0)} (\lesssim 0.27)$.

The general solutions starting from the regime $r_1 \ll 1$ finally approach the tracker as $r_1$ grows to 1. In Ref. [259] the authors carried out the likelihood analysis for such general solutions and found that the solutions approaching the tracker at late times (such as the case (a) in Fig. 4) are favored from the combined data analysis. In the flat FLRW background the best-fit model parameters are $\alpha = 1.411 \pm 0.056$, $\beta = 0.422 \pm 0.022$ (Constitution+CMB+BAO, 68% CL), and $\alpha = 1.404 \pm 0.057$, $\beta = 0.419 \pm 0.023$ (Union2+CMB+BAO, 68% CL).

For several fixed values of $\Omega_K^{(0)}$ it was shown that the late-time tracking solutions can be consistent with the data, apart from the models with largely negative $\Omega_K^{(0)}$ such as $\Omega_K^{(0)} \lesssim -0.01$. For example, the general solutions with $\Omega_K^{(0)} = 0.01$ and the model parameters $(\alpha, \beta) = (1.862, 0.607)$ give the similar value of $\chi^2$ to that in the the $\Lambda$CDM. In this case the Akaike-Information-Criteiron (AIC) statistics [264] also have the same support for the two models (see also Ref. [259]).

The Bayesian-Information-Criterion (BIC) statistics [266] show that the general solutions, with all 4 parameters ($\alpha, \beta, \Omega_m^{(0)}, \Omega_K^{(0)}$) are varied, are not particularly favored over the $\Lambda$CDM model. This mainly comes from the statistical property that the numbers of model parameters are larger than those in the flat $\Lambda$CDM. In fact the late-time tracking solutions with a non-zero cosmic curvature can be well consistent with the combined data analysis at the background level.

B. Generalized Galileon gravity

In Sec. [31] we showed that in Brans-Dicke theory with the coupling $Q$ of the order of unity the presence of the field potential allows a possibility for the consistency with local gravity constraints through the chameleon mechanism. Another way to recover the General Relativistic behavior in the regions of high density is to introduce the Galileon-like field self-interaction. Silva and Koyama [238] studied Brans-Dicke theory in the presence of the term $\xi(\phi)\Box \phi (\partial_\mu \phi \partial^\mu \phi)$ [which is the generalization of the term $\mathcal{L}_3 = \Box \phi (\partial_\mu \phi \partial^\mu \phi)$]. The action of this theory is given by

$$S = \int d^4 x \sqrt{-g} \left[ \frac{1}{2} \phi R - \frac{\omega_{\text{BD}}}{2\phi} (\nabla \phi)^2 + \xi(\phi) \Box \phi (\partial_\mu \phi \partial^\mu \phi) \right] + \int d^4 x \mathcal{L}_M. \quad (182)$$

If $\xi(\phi) \propto \phi^{-2}$, there exists a de Sitter solution that can be responsible for the late-time acceleration. As in the Galileon model discussed in Sec. [4A] the field is nearly frozen during the radiation and matter eras through the cosmological Vainshtein mechanism, but it finally approaches the de Sitter solution characterized by $\phi = \text{constant}$. Moreover, as in the DGP model, the Vainshtein radius can be much larger than the solar system scale, so that the General Relativistic behavior can be recovered in the local region [238].

We may consider more general theories described by the action [242]

$$S = \int d^4 x \sqrt{-g} \left[ \frac{1}{2} F(\phi) R + B(\phi) X + \xi(\phi) \Box \phi (\partial_\mu \phi \partial^\mu \phi) \right] + \int d^4 x \mathcal{L}_M, \quad (183)$$

where $X = -g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi / 2$, and $F(\phi)$, $B(\phi)$, $\xi(\phi)$ are functions of $\phi$. From the requirement of having de Sitter solutions responsible for dark energy, it is possible to restrict the functional forms of $F(\phi)$, $B(\phi)$, and $\xi(\phi)$. In the presence of non-relativistic matter (energy density $\rho_m$) and radiation (energy density $\rho_r$), the field equations are given
Since $x$ in which the field is nearly frozen during the radiation and matter eras. The field starts to evolve at the late cosmological epoch. Brans-Dicke theory described by the action (182) corresponds to $\phi$ $\lambda$ where $\mu = (\lambda/\mu^3)(\phi/M_{pl})^{-n}$, we can solve Eqs. (184) and (185) for $x$ and $H$ at the de Sitter point. These conditions are satisfied for the following functions

$$F(\phi) = M_{pl}^2(\phi/M_{pl})^{3-n}, \quad B(\phi) = \omega(\phi/M_{pl})^{1-n}, \quad \xi(\phi) = (\lambda/\mu^3)(\phi/M_{pl})^{-n},$$

(186)

where $M_{pl} \approx 10^{18}$ GeV is the reduced Planck mass, $\mu$ ($>0$) is a constant having a dimension of mass, and $\omega$ and $\lambda$ are dimensionless constants. One can show that the coupling $\lambda$ must be positive for the consistency of theories [242]. The Brans-Dicke theory described by the action (182) corresponds to $n = 2$ with the Brans-Dicke parameter $\omega_{BD} = \omega$. Since $F(\phi)$ is constant for $n = 3$, the theory with $n = 3$ corresponds to k-essence minimally coupled gravity.

From Eqs. (184) and (185) we obtain the following algebraic equations at the de Sitter fixed point:

$$\omega = -\frac{n(n - 3)^2x_{dS}^3 + (n - 3)(n - 12)x_{dS}^3 - 6(n - 5)x_{dS} + 18}{x_{dS}^4 + 3},$$

(187)

$$\lambda = \frac{\mu^3}{M_{pl}H_{dS}^2} \frac{[(n - 3)x_{dS} - 2][(n - 3)x_{dS} - 3]}{2x_{dS}^4 + 3},$$

(188)

where $x_{dS}$ and $H_{dS}$ are the values of $x$ and $H$ at the de Sitter point, respectively. We fix the mass scale $\mu$ to be $\mu = (M_{pl}H_{dS}^2)^{1/3}$, where we have used $H_{dS} \approx 10^{-68} M_{pl}$. For given $\omega$ and $n$, the quantity $x_{dS}$ is determined by solving Eq. (187). Then the dimensionless constant $\lambda$ is known from Eq. (188).

In order to recover the General Relativistic behavior in the early cosmological epoch we require that the field initial value $\phi_i$ is close to $M_{pl}$ from Eq. (189). The quantity $x$ is much smaller than 1 in the early cosmological epoch, so that the field is nearly frozen during the radiation and matter eras. The field starts to evolve at the late cosmological epoch in which $x$ grows to the order of unity. Introducing the dimensionless quantities $y = \mu x^2 H^2/H_{dS}^2$ and $\Omega_r = \rho_r/(3F H^2)$, one can show that the fixed point corresponding to the matter era corresponds to $(x, y, \Omega_r) = (0, (3 - n)/6, 0)$ [242]. Since $y$ is positive definite, it follows that $n \leq 3$.

The conditions for the avoidance of ghosts and Laplacian instabilities are known by employing the method presented in Sec. V A 3. Provided $F(\phi) > 0$, we require that $x > 0$ to avoid ghosts during the cosmological evolution from the radiation era to the epoch of cosmic acceleration [242]. The stability of the de Sitter point is automatically ensured for $n \leq 3$ and $x_{dS} > 0$. From Eq. (187) the parameter $\omega$ is restricted in the range

$$\omega < -n(n - 3)^2,$$

(189)

The field propagation speed squared during the radiation and matter eras is given by $c_s^2 \approx 6/5$ and $c_s^2 \approx 2/3$, respectively in which case no instabilities of linear perturbations are present. Meanwhile, at the de Sitter solution, we have [242]

$$c_s^2 = \frac{(n - 2)[(n - 3)(n - 4)x_{dS}^2 - 8(n - 3)x_{dS} + 6]x_{dS}}{(n - 3)(3n^2 - 10n + 12)x_{dS}^4 + 12(n - 5)x_{dS}^2 + 18(n - 8)x_{dS} - 108},$$

(190)

which is positive for $n \geq 2$. Hence the parameter $n$ is restricted in the range

$$2 \leq n \leq 3,$$

(191)

which includes Brans-Dicke theory with the action (182) as a specific case ($n = 2$).

In Ref. [239], the authors studied the evolution of matter density perturbations and showed that, for the model with $n = 2$, there is an anti-correlation between the cross-correlation of large scale structure and the integrated Sachs-Wolfe effect in CMB anisotropies. We shall discuss the main reason of this anti-correlation in Sec. VII D. This property will be useful to distinguish the above model from the $\Lambda$CDM in future observations.
VI. OTHER MODIFIED GRAVITY MODELS OF DARK ENERGY

In this section we briefly discuss other classes of modified gravity models of dark energy. These include (i) Gauss-Bonnet gravity with a scalar coupling \( f(\phi)\mathcal{G} \), (ii) \( R/2 + f(\mathcal{G}) \) gravity, and (iii) Lorentz-violating models.

A. Gauss-Bonnet gravity with a scalar coupling

In addition to the Ricci scalar \( R \), we can construct other scalar quantities coming from the Ricci tensor \( R_{\mu\nu} \) and the Riemann tensor \( R_{\mu\nu\alpha\beta} \), i.e. \( P \equiv R_{\mu\nu}R^{\mu\nu} \) and \( Q \equiv R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \) \cite{267}. It is possible to avoid the appearance of spurious spin-2 ghosts by taking a Gauss-Bonnet (GB) combination \cite{268, 269, 271}, defined by

\[
\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}.
\]

(192)

A simple model that can be responsible for the cosmic acceleration today is \cite{43}

\[
S = \int \! \mathrm{d}^4x \sqrt{-\mathcal{g}} \left[ \frac{1}{2} R - \frac{1}{2} (\nabla \phi)^2 - V(\phi) - f(\phi)\mathcal{G} \right] + \int \! \mathrm{d}^4x \, \mathcal{L}_M,
\]

(193)

where \( V(\phi) \) and \( f(\phi) \) are functions of a scalar field \( \phi \). The coupling of the field with the GB term appears in low-energy effective string theory \cite{274}. For the exponential potential \( V(\phi) = V_0 e^{-\lambda \phi} \) and the coupling \( f(\phi) = (f_0/\mu) e^{\mu \phi} \), the cosmological dynamics were studied in Refs. \cite{30, 272} \cite{273, 274}. In this model there exists a de Sitter solution due to the presence of the GB term. In Refs. \cite{272, 273} it was also found that the late-time de-Sitter solution is preceded by a scaling matter era.

Koivisto and Mota \cite{272} placed observational constraints on the model \cite{103} with the exponential potential \( V(\phi) = V_0 e^{-\lambda \phi} \), by using the Gold data set of SN Ia together with the CMB shift parameter data of WMAP. The parameter \( \lambda \) is constrained to be \( 3.5 < \lambda < 4.5 \) at the 95% confidence level. In Ref. \cite{274}, they also included the constraints coming from the BAO, LSS, big bang nucleosynthesis, and solar system data. This joint analysis showed that the model is strongly disfavored by the data. In Ref. \cite{273} it was also found that tensor perturbations are subject to negative instabilities when the GB term dominates the dynamics (see also Refs. \cite{278, 279} for related works).

Amendola et al. \cite{290} studied local gravity constraints on the above model and showed that the density parameter \( \Omega_{\text{GB}} \) coming from the GB term is required to be strongly suppressed for the compatibility with solar-system experiments (which is typically of the order of \( \Omega_{\text{GB}} \sim 10^{-30} \)). The above discussion indicates that the GB term with the scalar-field coupling \( f(\phi)\mathcal{G} \) can hardly be the source for dark energy.

B. \( R/2 + f(\mathcal{G}) \) gravity

The general Lagrangian including the scalar quantities constructed from the Ricci scalar \( R \), the Ricci tensor \( R_{\mu\nu} \) and the Riemann tensor \( R_{\mu\nu\alpha\beta} \) is given by \( \mathcal{L} = f(R, P, Q) \). The dark energy models based on these theories have been studied in Refs. \cite{272, 273, 274, 275}. In order to avoid spurious spin-2 ghosts we need to choose the GB combination \cite{192}, i.e. \( \mathcal{L} = f(R, Q - 4P) \) \cite{280, 281}.

The cosmological dynamics based on the action,

\[
S = \int \! \mathrm{d}^4x \sqrt{-\mathcal{g}} \left[ \frac{1}{2} R + f(\mathcal{G}) \right] + \int \! \mathrm{d}^4x \, \mathcal{L}_M,
\]

(194)

have been studied by a number of authors \cite{14, 291, 296}. In order to ensure the stability of radiation/matter solutions, we need to satisfy the condition \( f_{,\mathcal{G}} > 0 \) for all \( \mathcal{G} \). We also require the regularities of the functions \( f \), \( f_{,\mathcal{G}} \), and \( f_{,\mathcal{G}^2} \) \cite{293}. There exists a de Sitter solution responsible for dark energy, whose stability requires the condition \( 0 < H_{\text{dS}} f_{,\mathcal{G}}(H_{\text{dS}}) < 1/384 \). Moreover \( f_{,\mathcal{G}^2} \) must approach +0 in the limit \( |\mathcal{G}| \to \infty \).

In Ref. \cite{293} the authors proposed a number of \( f(\mathcal{G}) \) models satisfying these conditions (see also Ref. \cite{294}). One of such models is given by

\[
f(\mathcal{G}) = \lambda \frac{\mathcal{G}}{\sqrt{\mathcal{G}_*}} \arctan \left( \frac{\mathcal{G}}{\mathcal{G}_*} \right) - \alpha \lambda \sqrt{\mathcal{G}_*},
\]

(195)

where \( \alpha, \lambda \) and \( \mathcal{G}_* \) are constants. The numerical simulation of Ref. \cite{293} shows that the model \cite{195} is cosmologically viable at least at the background level. Moreover it can be consistent with solar-system constraints for a wide range of model parameters \cite{297}.
In order to study the viability of the theories described by the action \((194)\), further, let us consider the evolution of matter density perturbations in the presence of a perfect fluid with the barotropic equation of state \(w_M = P_M/\rho_M\). In Ref. 298, it was shown that, for small scales (i.e. for large momenta \(k\)), there are two different scalar propagation speeds. One of them corresponds to the mode for the fluid, i.e. \(c_2^2 = w_M\), whereas another is given by

\[
c^2_2 = 1 + \frac{2\dot{H}}{H^2} + \frac{1 + w_M \kappa^2 \rho_M}{1 + 4\mu} \frac{3\dot{H}^2}{3H^2},
\]

where

\[
\mu = H\dot{f}_{,GG}. \tag{197}
\]

The parameter \(\mu\) characterizes the deviation from the \(\Lambda\)CDM model (note that the linear term \(f = cG\) does not give rise to any contribution to the field equation). For viable \(f(G)\) models we have \(|\mu| < 1\) at high redshifts 292. Since the background evolution during the radiation/matter domination is given by \(3H^2 \simeq \kappa^2 \rho_M\) and \(\dot{H}/H^2 \simeq -(3/2)(1+w_M)\), it follows that

\[
c^2_2 \simeq -1 - 2w_M. \tag{198}
\]

The Laplacian instability at small scales is absent only for \(w_M < -1/2\). Since \(w_M = 1/3\) and \(w_M = 0\) during the radiation and matter eras, respectively, the perturbations with large momentum modes are unstable. This leads to violent growth of matter density perturbations incompatible with the observations of large-scale structure 292,298.

By considering the full perturbation equations, one can show that the onset of the negative instability corresponds to

\[
\mu \approx (aH/k)^2. \tag{199}
\]

Even when \(\mu\) is much smaller than 1, we can always find a wave number \(k\) (\(\gg aH\)) satisfying the condition \((199)\). For the scales smaller than that determined by the wave number in Eq. \((199)\), the linear perturbation theory breaks down. Hence the background solutions cannot be trusted for those scales, which makes the theory unpredictable. The Laplacian instability can be avoided only for \(\mu = 0\), which corresponds to the \(\Lambda\)CDM model. The above property persists irrespective of the forms of \(f(G)\).

For more general theories described by the Lagrangian density \(f(R,G)\) it is possible to avoid such Laplacian instabilities 295, depending on the models 300 (see also Refs. 301,304). It may be of interest to construct some viable dark energy models in such theories.

C. Lorentz violating models

The modified gravity models such as \(f(R)\) gravity and Galileon gravity can give rise to the phantom dark energy equation of state \(w_{DE} < -1\) without violating the conditions for the appearance of ghosts and instabilities. In the models with a broken Lorentz invariance it is also possible to realize \(w_{DE} < -1\) without pathological behavior in the Ultra-Violet (UV) region 305,307 (see also Ref. 308 for a review). In order to construct Lorentz violating models without pathological behavior of phantoms, one may start with a field theory consistent at energy scales from zero to the UV cutoff scale \(\mathcal{M}\) and then deform the theory in the Infra-Red (IR) in such a way that its behavior at high energies remains healthy. Although the weak energy condition is violated in the homogenous background, pathological states are present below a certain low scale \(\epsilon\) only. Provided that \(\epsilon\) is close to the Hubble scale, a theory of this sort should be acceptable.

Let us consider a Lorentz violating model with two-derivative kinetic terms with healthy behavior below the scale \(\mathcal{M} \geq 309\) plus one-derivative term suppressed by the small parameter \(\epsilon \ll 1\). The model has a vector field \(B_{\mu}\) and a scalar field \(\Phi\) with a potential \(V(B,\Phi)\). The Lagrangian is given by

\[
\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(1)} + \mathcal{L}^{(0)} + \mathcal{L}_M, \tag{200}
\]

where \(\mathcal{L}^{(2)} = -\frac{1}{2} \alpha(\Xi) g^\nu\lambda D_{\nu}B_{\lambda}D^\nu B_{\lambda} + \frac{1}{2} \beta(\Xi) D_{\nu}B_{\lambda}D^\nu D_{\mu}B_{\lambda} \frac{B^{\nu}B^{\lambda}}{\mathcal{M}^2} + \frac{1}{2} \partial_{\mu}\Phi \partial^\mu\Phi\), \(\mathcal{L}^{(1)} = \epsilon \partial_{\mu}\Phi B^\mu\), and \(\mathcal{L}^{(0)} = -V(B,\Phi)\), \(\mathcal{L}_M\) (\(\mathcal{M}\) is the UV cut-off scale). The dimensionless parameters \(\alpha\) and \(\beta\) are the functions of \(\Xi\), and \(\epsilon\) is a free positive parameter that characterizes an IR scale. The Lorentz invariance is broken for \(\Xi \neq 0\).
In spatially homogeneous background we have

\[ B_0 = X, \quad B_1 = 0, \quad \Phi = \phi, \]  

(203)

where \( B_0 \) and \( B_1 \) are time and space components of \( B_\mu \). In the flat FLRW background with non-relativistic matter and radiation for the matter Lagrangian \( \mathcal{L}_M \), we obtain the following equations of motion

\[
H^2 = \frac{8 \pi G}{3} \left[ \frac{1}{2} \gamma \dot{X}^2 - \frac{3\alpha}{2} H^2 X^2 + \frac{1}{2} \phi^2 + W(\phi) + U(X) + \rho_m + \rho_r \right],
\]

(204)

\[
\gamma \left( \dot{X} + 3H \dot{X} \right) + \frac{1}{2} \gamma X \ddot{X} + \frac{3}{2} \alpha_X H^2 X^2 + 3\alpha H^2 X - \epsilon \dot{\phi} + V_\phi = 0,
\]

(205)

\[
\ddot{\phi} + 3H \dot{\phi} + \epsilon (\dot{X} + 3HX) + V_{\phi} = 0,
\]

(206)

where \( \gamma(X) = X^2 \beta(X)/M^2 - \alpha(X) \).

For the separable potential \( V = m^2 \dot{\phi}^2/2 - M^2 X^2/2 \) with constant \( X \) the cosmological dynamics of the above system have been studied in detail in Refs. [306, 307]. In what follows we assume that both \( \alpha \) and \( \gamma \) are constants. Provided \( \epsilon/m > \sqrt{2\alpha/3} \) one can show that there is a de Sitter solution with \( H = M/\sqrt{3\alpha} \), at which \( \phi \) and \( X \) are frozen. In the early cosmological epoch the field \( X \) is close to 0, whereas the field \( \phi \) slowly rolls down its potential (with the energy density dominating over that of \( X \)).

Prior to the epoch of de Sitter cosmic acceleration there is a transient phantom regime \((w_{DE} < -1)\) characterized by \( H < M/\sqrt{3\alpha} \) in which the field \( \phi \) rolls up its potential. The Hubble parameter slowly increases toward the de Sitter value \( H_{dS} = M/\sqrt{3\alpha} \). In Fig. 6 we plot the evolution of the dark energy equation of state \( w_{DE} = P_{DE}/\rho_{DE} \) and the effective equation of state \( w_{\text{eff}} = 1 - 2\dot{H}/(3H^2) \) together with the density parameters of dark energy, non-relativistic matter, and radiation\(^6\). Clearly the solution undergoes the period with \( w_{DE} < -1 \) before reaching the de Sitter attractor.

\(^6\) Here the definition of \( \rho_{DE} \) and \( P_{DE} \) is \( \rho_{DE} = \gamma \dot{X}^2/2 + \dot{\phi}^2/2 + V - 3\alpha H^2 X^2/2 \) and \( P_{DE} = \gamma \dot{X}^2/2 + \dot{\phi}^2/2 - V + \epsilon \dot{\phi} X + \alpha \dot{H} X^2 + 2\alpha H X X + 3\alpha H^2 X^2/2 \).
The phantom equation of state can be realized without having ghosts, tachyons or superluminal modes in the UV region. In the IR region characterized by \( p \lesssim \epsilon \), either tachyons or ghosts appear for the spatial momenta \( p \) smaller than \( \sqrt{(\epsilon^2 - M^2)/\alpha - m^2} \) \([306, 307]\). The presence of tachyons at IR scales leads to the amplification of large-scale field perturbations whose wavelengths are roughly comparable to the present Hubble radius. There are two tachyonic regions of spatial momenta in this model: (a) one is sub-horizon and its momenta are characterized by \( M^2/\gamma < p^2 < (\epsilon^2 - M^2)/\alpha - m^2 \); (b) another is super-horizon with the momenta \( 0 < p^2 < m^2 M^2/\epsilon^2 \). In the region (a) there is a parameter space in which the perturbations always remain smaller than the homogenous fields. In the region (b) the growth of the perturbations is suppressed by the factor \( m^2/\epsilon^2 \).

There are other classes of Lorentz violating models such as ghost condensate \([305]\) and Horava-Lifshitz gravity \([311]\) (see also Refs. \([312]\)-\([317]\)). The application of such scenarios to dark energy has been studied by a number of authors, see e.g., \([318, 320]\). It remains to see whether such Lorentz-violating models can be observationally distinguished from other dark energy models.

### VII. OBSERVATIONAL SIGNATURES OF MODIFIED GRAVITY

In order to confront modified gravity models with the observations of large-scale structure and CMB, we discuss the evolution of density perturbations in four modified gravity models: (i) \( f(R) \) gravity, (ii) scalar-tensor gravity, (iii) DGP braneworld model, and (iv) Galileon gravity. We also discuss observables to confront with weak lensing observations.

#### A. \( f(R) \) gravity

Let us first consider metric \( f(R) \) gravity in the presence of non-relativistic matter. We take the following perturbed metric in a longitudinal gauge about the flat FLRW background with scalar metric perturbations \( \Phi \) and \( \Psi \) \([181]\):

\[
\text{d}x^2 = -(1 + 2\Psi)\text{d}t^2 + a^2(1 + 2\Phi)\delta_{ij}\text{d}x^i\text{d}x^j.
\]  

The energy momentum tensors of a non-relativistic perfect fluid are decomposed into background and perturbed parts, as \( T_0^0 = -\rho_m - \delta\rho_m \) and \( T_0^a = -\rho_m v_m,\alpha \) (\( v_m \) is a velocity potential).

The equations for matter perturbations, in the Fourier space, are given by \([321, 323]\):

\[
\delta\dot{\rho}_m + 3H\delta\rho_m = -\rho_m \left[ 3\dot{\Phi} + (k^2/a^2)v_m \right],
\]

\[
\dot{v}_m + Hv_m = \Psi/a,
\]

where \( k \) is a comoving wave number. We define the gauge-invariant matter density perturbation \( \delta_m \), as

\[
\delta_m = \delta\rho_m/\rho_m + 3Hv, \quad \text{where} \quad v = av_m.
\]

Then Eqs. (208) and (209) yield

\[
\delta\dot{m} = -(k^2/a^2)v - 3(\dot{\Phi} - Hv),
\]

\[
\dot{v} = \Psi,
\]

from which we obtain

\[
\delta\dot{m} + 2H\delta m + (k^2/a^2)\Psi = 3\dot{B} + 6H\dot{B},
\]

where \( B \equiv -\dot{\Phi} + Hv \).

In \( f(R) \) gravity the quantity \( F(R) = \partial f/\partial R \) has a perturbation \( \delta F \). In the following we use the unit \( \kappa^2 = 8\pi G = 1 \), but we restore gravitational constant \( G \) when it is required. For the action given in Eq. (2), we obtain the linearized perturbation equations in Fourier space \([324, 326]\):

\[
-k^2/a^2\Phi + 3H(H\Psi - \dot{\Phi}) = \frac{1}{2F} \left[ 3H\delta F - \left( 3\dot{H} + 3H^2 - \frac{k^2}{a^2} \right) \delta F - 3H\dot{F}\Psi - 3\dot{F}(H\Psi - \dot{\Phi}) - \delta\rho_m \right],
\]

\[
\delta\dot{F} + 3H\delta F + \left( \frac{k^2}{a^2} + M^2 \right) \delta F = \frac{1}{3}\delta\rho_m + \dot{F}(3H\Psi + \dot{\Psi} - 3\dot{\Phi}) + (2\dot{\Phi} + 3H\dot{\Phi})\Psi,
\]

\[
\Psi + \Phi = -\delta F/F.
\]
In Eq. (215) we have introduced the mass term

$$M^2 \equiv \frac{1}{3} \left( \frac{F}{f_{,RR}} - R \right).$$ (217)

For viable dark energy models the condition $F/f_{,RR} \gg R$ is satisfied during most of the cosmological epoch, so that $M^2 \approx F/(3f_{,RR})$. This is equivalent to the mass squared $M^2$ introduced in Eq. (43), which is required to be positive to avoid the tachyonic instability.

For the observations of large-scale structure and weak lensing we are interested in the modes deep inside the Hubble radius ($k \gg aH$). In the following we employ the quasi-static approximation under which the dominant terms in Eqs. (213) - (216) correspond to those including $k^2/a^2$, $\delta\rho_m$ (or $\rho_m$) and $M^2$. We then obtain the following approximate relations from Eqs. (213) - (216):

$$\ddot{\delta}_m + 2H\dot{\delta}_m + (k^2/a^2)\Psi = 0,$$ (218)

$$\Phi = \frac{1}{2F} \left( \frac{a^2}{k^2} \delta\rho_m - \delta F \right), \quad \Psi = -\frac{1}{2F} \left( \frac{a^2}{k^2} \delta\rho_m + \delta F \right),$$ (219)

$$\ddot{\delta}F + 3H\dot{\delta}F + (k^2/a^2 + M^2) \delta F = \delta\rho_m/3.$$ (220)

The evolution of perturbations is different depending on whether $M^2$ is larger or smaller than $k^2/a^2$. We shall discuss two cases: (A) $M^2 \gg k^2/a^2$ and (B) $M^2 \ll k^2/a^2$, separately. For viable $f(R)$ models the mass squared $M^2$ is large in the past and it gradually decreases with time. Hence the transition from the regime (A) to the regime (B) can occur in the past, depending on the wave numbers $k$.

1. Evolution of perturbations in the regime: $M^2 \gg k^2/a^2$

The solutions to Eq. (220) are given by the sum of the oscillating solution $\delta F_{osc}$ obtained by setting $\delta\rho_m = 0$ and the special solution $\delta F_{ind}$ of Eq. (220) induced by the presence of matter perturbations $\delta\rho_m$. The oscillating part $\delta F_{osc}$ satisfies the equation $(a^{3/2}\delta F_{osc})' + M^2(a^{3/2}\delta F_{osc}) \approx 0$. Using the WKB approximation, we obtain the following solution:

$$\delta F_{osc} \propto a^{-3/2} f_{,RR}^{1/4} \cos \left( \int \frac{1}{\sqrt{f_{,RR}}} \, dt \right),$$ (221)

where we have used the approximation $F \approx 1$.

For the analytic estimation of the oscillating mode we take the model (47), which corresponds to the asymptotic form of the models (15) and (16) in the region $R \gg R_c$. During the matter era in which the background Ricci scalar evolves as $R^{(0)} = 4/(3L^2)$, the quantity $f_{,RR}$ has a dependence $f_{,RR} \propto R^{-2(n+1)} \propto t^{4(n+1)}$ for the model (47). Then the evolution of the perturbation, $\delta R_{osc} = \delta F_{osc}/f_{,RR}$, is given by

$$\delta R_{osc} \approx c t^{-(3n+4)} \cos(c_0 t^{-2(n+1)})$$ (222)

where $c$ and $c_0$ are constants. Unless the coefficient $c$ is chosen to be very small, as we go back to the past, the perturbation $\delta R_{osc}$ dominates over the background value $R^{(0)}(\propto t^{-2})$. Since the Ricci scalar can be negative, this leads to the violation of the stability conditions ($f_{,RR} > 0$ and $F > 0$).

The special solution $\delta F_{ind}$ to Eq. (220) can be derived by neglecting the first and second terms relative to others, giving

$$\delta F_{ind} \approx \delta\rho_m/(3M^2), \quad \delta R_{ind} \approx \delta\rho_m.$$ (223)

Under the condition $|\delta F_{osc}| \ll |\delta F_{ind}|$, one has $\delta F \approx \delta\rho_m/(3M^2)$ and hence $\Psi = -\Phi = -(a^2/k^2)\delta\rho_m/(2F)$. Then the matter perturbation equation (218) reduces to

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\rho_m \delta_m/F = 0,$$ (224)

where we have reproduced the gravitational constant $G$. In Refs. 330, 331 the perturbation equations have been derived without neglecting the oscillating mode.
During the matter-dominated epoch \((\Omega_m = \rho_m/(3FH^2) \simeq 1)\), Eq. (221) has the growing-mode solution
\[
\delta_m \propto t^{2/3}.
\] (225)

This is the same evolution as that in standard General Relativity. From Eq. (223), the matter-induced mode evolves as \(\delta F_{\text{ind}} \propto t^{2(n+2)/3}\) and \(\delta R_{\text{ind}} \propto t^{-4/3}\). Compared to the oscillating mode \(\delta F = \delta R_{\text{osc}} + \delta R_{\text{ind}}\), \(\delta R_{\text{ind}}\) decreases more slowly and hence it dominates over \(\delta R_{\text{osc}}\) at late times. The evolution of the perturbation \(\delta R = \delta R_{\text{osc}} + \delta R_{\text{ind}}\) relative to the background value \(R(0)\) is given by
\[
\delta R/R(0) \simeq c_1 t^{-(3n+2)} \cos(c_0 t^{-\phi}) + c_2 t^{2/3},
\] (226)
where \(c_1\) and \(c_2\) are constants. In order to avoid the dominance of the oscillating mode at the early cosmological epoch, we require that the coefficient \(c_1\) is suppressed relative to \(c_2\). 

This fine-tuning of initial conditions is related with the singularity problem raised by Frolov [333]. The field \(\phi = \sqrt{3/2k^2}\ln F\) in the Einstein frame has a weak singularity at \(\phi = 0\) (at which the curvature \(R\) and the mass \(M\) go to infinity with a finite potential \(V\)). Unless the oscillating mode of the field perturbation \(\delta \phi\) is strongly suppressed relative to the background field \(\phi(0)\), the system can access the curvature singularity. This past singularity can be cured by taking into account the \(R^2\) term [334]. Note that the \(f(R)\) models proposed in Ref. [333] [e.g., \(f(R) = R - \alpha R_0 \ln(1 + R/R_c)\)] to cure the singularity problem satisfy neither the local gravity constraints [336] nor observational constraints of large-scale structure [337]. There are some works for the construction of unified models of inflation and dark energy based on \(f(R)\) theories [338, 339, 340], but the smooth transition between two accelerated epochs without crossing the point \(f_{,RR} = 0\) is not easy unless the forms of \(f(R)\) are carefully chosen [334].

2. Evolution of perturbations in the regime: \(M^2 \ll k^2/a^2\)

Since the mass \(M\) decreases as \(M \propto t^{-2(n+1)}\), the modes initially in the region \(M^2 \gg k^2/a^2\) can enter the regime \(M^2 \ll k^2/a^2\) during the matter-dominated epoch. It is sufficient to consider the matter-induced mode because the oscillating mode is already suppressed during the evolution in the regime \(M^2 \gg k^2/a^2\). The matter-induced special solution of Eq. (220) in the regime \(M^2 \ll k^2/a^2\) is approximately given by
\[
\delta F_{\text{ind}} \simeq \frac{a^2}{3k^2} \delta \rho_m.
\] (227)

From Eq. (219) the gravitational potentials satisfy
\[
\Psi = \frac{4}{3} \frac{1}{2F} \frac{a^2}{k^2} \delta \rho_m, \quad \Phi = \frac{2}{3} \frac{1}{2F} \frac{a^2}{k^2} \delta \rho_m.
\] (228)

Plugging Eq. (228) into Eq. (218), the matter perturbation obeys the following equation
\[
\ddot{\delta}_m + 2H \dot{\delta}_m - \frac{4}{3} \cdot 4\pi G \rho_m \delta_m/F = 0.
\] (229)

During the matter-dominated epoch \((\Omega_m \simeq 1\) and \(a \propto t^{2/3}\)), we obtain the following evolution
\[
\delta_m \propto t^{(\sqrt{3} - 1)/6}.
\] (230)

The growth rate of \(\delta_m\) gets larger compared to that in the regime \(M^2 \gg k^2/a^2\).

3. Matter power spectra and the ISW effect

If the transition from the regime \(M^2 \gg k^2/a^2\) to the regime \(M^2 \ll k^2/a^2\) occurs during the matter era, the evolution of matter perturbations changes from \(\delta_m \propto t^{2/3}\) to \(\delta_m \propto t^{(\sqrt{3} - 1)/6}\). We use the subscript “\(k\)” for the quantities at which \(k\) is equal to \(aM\), whereas the subscript “\(N\)” is used at which the accelerated expansion starts \((\ddot{a} = 0)\). While the redshift \(z_A\) is independent of \(k\), \(z_k\) depend on \(k\) and also on the mass \(M\).

For the model (47) the variable \(m = R_{f,RR}/f_R\) can grow fast from the regime \(m \ll (aH/k)^2\) (i.e., \(M^2 \gg k^2/a^2\)) to the regime \(m \gg (aH/k)^2\) (i.e., \(M^2 \ll k^2/a^2\)). In fact, \(m\) can grow to the order of 0.1 even if \(m\) is much smaller than \(10^{-6}\) in the deep matter era. For the sub-horizon modes relevant to the galaxy power spectrum, the transition at
Figure 7: The matter power spectra $P_{\delta_m}(k)$ in Brans-Dicke theory with the potential (47) with the correspondence $Q = -1/\sqrt{6}$ and $p = 2n/(2n + 1)$. Each case corresponds to (a) $Q = 0.7$, $p = 0.6$, $C = 0.9$, (b) $Q = -1/\sqrt{6}$, $p = 0.6$, $C = 0.9$, (c) the $\Lambda$CDM model, and (d) the $\Lambda$CDM model with a nonlinear halo-fitting ($\sigma_s = 0.78$ and shape parameter $\Gamma = 0.2$). The model parameters are $\Omega_{m0}^{[0]} = 0.28$, $H_0 = 3.34 \times 10^{-4}$ $h$ Mpc$^{-1}$, $n_s = 1$ and $\delta_H^2 = 3.2 \times 10^{-10}$. From Ref. [341].

$M^2 = k^2/a^2$ typically occurs at the redshift $z_k$ larger than 1 (provided that $n = O(1)$). For the mode $k/(a_0H_0) = 300$ one has $z_k = 4.83$ for $n = 1$ and $z_k = 2.49$ for $n = 2$. As $n$ gets larger, the period of non-standard evolution of $\delta_m$ becomes shorter because $z_k$ tends to be smaller. Since the scalaron mass evolves as $M \propto t^{-2(n+1)}$ for the model (47), the time $t_k$ has a scale-dependence $t_k \propto k^{-3/(6n+4)}$. This means that the smaller-scale modes cross the transition point earlier. The matter power spectrum $P_{\delta_m} = |\delta_m|^2$ at the time $t_A$ shows a difference compared to the case of the $\Lambda$CDM model:

$$\frac{P_{\delta_m}(t_A)}{P_{\delta_{m, \Lambda CD M}}(t_A)} = \left(\frac{t_A}{t_k}\right)^2 \left(\frac{\sqrt{33} - 5}{6n + 4}\right) \propto k^{-3/(6n+4)}. \quad (231)$$

From Fig. 7, we find that the matter power spectrum in the $f(R)$ model (47) with $n = 3/4$ is in fact larger than that in the $\Lambda$CDM model on smaller scales.

The galaxy matter power spectrum is modified by this effect. Meanwhile the CMB spectrum is hardly affected except for very large scales (for the multipoles $\ell = O(1)$) at which the Integrated Sachs-Wolfe (ISW) effect becomes important. Hence there is a difference for the spectral indices of two power spectra, i.e.

$$\Delta n(t_A) = \frac{\sqrt{33} - 5}{6n + 4}. \quad (232)$$

For larger $n$ the redshift $z_k$ can be as close as $z_A$, which means that the estimation (232) is not necessarily valid in such cases. Moreover the estimation (232) does not take into account the evolution of $\delta_m$ after $z = z_A$ to the present epoch ($z = 0$). It was found in Ref. [71] that the estimation (232) agrees well with the numerically obtained $\Delta n(t_A)$ for $n \leq 2$.

In order to discuss the growth rate of matter perturbations, it is customary to introduce the growth index [341]

$$f_\delta \equiv \frac{\dot{\delta}_m}{H\delta_m} = (\Omega_m)^\gamma, \quad (233)$$

where $\Omega_m = \kappa^2\rho_m/(3H^2)$. In the $\Lambda$CDM model $\gamma$ is nearly constant for the redshifts $0 < z < 1$, i.e. $\gamma \simeq 0.55$ [342, 343]. In $f(R)$ gravity, if the perturbations are in the GR regime ($M^2 \gg k^2/a^2$) today, $\gamma$ is close to 0.55. On the other hand, if the transition to the scalar-tensor regime occurs at the redshift $z_k$ larger than 1, the growth index tends to be smaller than 0.55 [344, 345]. Since $0 < \Omega_m < 1$, the smaller $\gamma$ implies a larger growth rate.
For the wave numbers relevant to the linear regime of the matter power spectra \((0.01 \text{ h Mpc}^{-1} \lesssim k \lesssim 0.2 \text{ h Mpc}^{-1})\), where \(h \approx 0.7\) the viable \(f(R)\) models (15)-(18) can give rise to the growth index \(\gamma_0 \approx 0.40\) today. Depending on the wave numbers \(k\), \(\gamma_0\) can be dispersed in the regime \(0.40 \lesssim \gamma_0 \lesssim 0.55\), or \(\gamma_0\) can show the convergence in the regime \(0.40 \lesssim \gamma_0 \lesssim 0.43\). Moreover, when \(\gamma_0\) is small, the growth index exhibits a large variation even for low redshifts \((0 < z < 1)\). Although the present observational constraints on \(\gamma\) are quite weak, the unusual evolution of \(\gamma\) can be useful to distinguish the \(f(R)\) models from the \(\Lambda\)CDM in future observations.

After the system enters the epoch of cosmic acceleration, the wave number \(k\) can again become smaller than \(aH\). Hence the \(k\)-dependence is not necessarily negligible even for \(z < z_A\). However numerical simulations show that \(\Delta n(t_0)\) is not much different from \(\Delta n(t_\Lambda)\) derived by Eq. (232). Thus the analytic estimation (232) is certainly reliable to place constraints on model parameters except for \(n \gg 1\). Observationally we do not find any strong difference for the slopes of the spectra of LSS and CMB. If we take the mild bound \(\Delta n(t_\Lambda) < 0.05\), we obtain the constraint \(n \geq 20\). In this case the local gravity constraint (67) is also satisfied.

For the wave numbers \(k \gtrsim 0.2 \text{ h Mpc}^{-1}\) we need to take into account the non-linear effect of density perturbations. In Refs. [346-349] the authors carried out \(N\)-body simulations for the \(f(R)\) model (40) (see also Refs. [350, 351]). Hu and Sawicki (HS) [352] proposed a fitting formula to describe the non-linear power spectrum based on the halo model. Koyama et al. [353] studied the validity of the HS fitting formula by comparing it with the results of \(N\)-body simulations and showed that the HS fitting formula can reproduce the power spectrum in \(N\)-body simulations for the scales \(k < 0.5\text{ h Mpc}^{-1}\). In the quasi non-linear regime a normalized skewness, \(S_3 = (\delta_m^3)/(\delta_m^2)^{3/2}\), has been evaluated in \(f(R)\) gravity and in Brans-Dicke theories [354]. The skewness in \(f(R)\) dark energy models differs only by a few percent relative to the value \(S_3 = 34/7\) in the \(\Lambda\)CDM model.

The modified growth of matter perturbations also affects the evolution of the gravitational potentials \(\Psi\) and \(\Phi\). The effective potential \(\psi = \Phi - \Psi\) is important to discuss the ISW effect on the CMB as well as the weak lensing observations [355]. For the modes deep inside the Hubble radius, Eq. (219) gives

\[
\psi = 3 \Omega_m \delta_m \left(\frac{aH}{k}\right)^2.
\]

Note that for the large-scale modes relevant to the ISW effect in CMB we need to solve the full perturbation equations without using the quasi-static approximation on sub-horizon scales. For viable \(f(R)\) models, however, numerical simulations show that the result (254) can be trusted even for the wave numbers close to the Hubble radius today (i.e. \(k \gtrsim a_0 H_0\)) [153]. In the \(\Lambda\)CDM model the potential \(\psi\) remains constant during the standard matter era, but it decays after the system enters the accelerated epoch, producing the ISW contribution for low multipoles on the CMB power spectrum. In \(f(R)\) gravity the additional growth of matter perturbations in the region \(z < z_k\) changes the evolution of \(\psi\).

From the observation of the CMB angular power spectrum, the constraint on the deviation parameter \(B \equiv (\dot{R}H/R\dot{H})/m\) from the \(\Lambda\)CDM is weak. The value of \(B\) today is constrained to be \(B_0 < 4.3\) [356]. There is another observational constraint coming from the angular correlation between the CMB temperature field and the galaxy number density field induced by the ISW effect. The avoidance of a large anti-correlation between the observational data of CMB and LSS places an upper bound of \(B_0 \lesssim 1\). Since this roughly corresponds to \(m(z = 0) \lesssim 1\), the CMB observations do not provide tight constraints on \(f(R)\) models relative to the matter power spectrum of LSS. In weak lensing observations, the modified evolution of the lensing potential \(\psi\) directly leads to the change even for the small-scale shear power spectrum [340, 355, 357]. Hence this can be a powerful tool to constrain \(f(R)\) gravity models from future observations.

### B. Brans-Dicke theory

Let us proceed to discuss the evolution of matter perturbations in Brans-Dicke theory described by the action (76) with the potential \(U(\phi)\) and the coupling \(F(\phi) = e^{-2Q\phi}\). We are mainly interested in large coupling models with \(|Q|\) of the order of unity [76, 81, 82], as this gives rise to significant deviation from the \(\Lambda\)CDM. We define the field mass squared to be \(M^2 = U_{,\phi\phi}\). If the scalar field is light such that the condition \(M < H_0\) is always satisfied irrespective of high or low density regions, the coupling \(Q\) is constrained to be \(|Q| \lesssim 10^{-3}\) from local gravity constraints. Meanwhile, if the mass \(M\) in the region of high density is much larger than that on cosmological scales, the model can satisfy local gravity constraints under the chameleon mechanism even if \(|Q|\) is of the order of unity. Cosmologically the mass \(M\) can decrease from the past to the present, which can allow the transition from the “GR regime” to the “scalar-tensor regime” as it happens in \(f(R)\) gravity. An example of the field potential showing this behavior is given by Eq. (18).

As in \(f(R)\) gravity, the matter perturbation \(\delta_m\) obeys Eq. (213). The difference appears in the expression of the
gravitational potential \( \Psi \). In Fourier space the scalar metric perturbations obey the following equations \( [76, 325, 326] \):

\[
\frac{-k^2}{a^2} \Phi + 3H(H \Psi - \Phi) = -\frac{1}{2F} \left[ \omega \delta \phi + \frac{1}{2}(\omega_{,\phi} \phi^2 - F,_{\phi} R + 2V,_{\phi}) \delta \phi \right]
\]

\[
+ \left( 3H + 3H^2 - \frac{k^2}{a^2} \right) \delta F - 3H \delta \dot{F} + (3H \dot{F} - \omega \ddot{\phi}) \Psi + 3F(H \Psi - \Phi) + \delta \rho_m ,
\]

\[
\delta \ddot{\phi} + \left( 3 \dot{H} + \frac{\omega_{,\phi}}{\omega} \right) \delta \phi + \left[ \frac{k^2}{a^2} + \left( \frac{\omega_{,\phi}}{\omega} \right) \phi^2 + \left( \frac{2U_{,\phi} - F_{,\phi} R}{2\omega} \right) \phi \right] \delta \phi
\]

\[
= \dot{\phi} \Psi + \left( 2\dot{\phi} + 3H \phi + \frac{\omega_{,\phi}}{\omega} \phi^2 \right) \Psi + 3\phi(\dot{H} \Psi - \Phi) + \frac{1}{2\omega} F,_{\phi} \delta R ,
\]

\[
\Psi + \Phi = -\frac{\delta F}{F} = \frac{-F,_{\phi}}{F} \delta \phi ,
\]

where \( \delta \phi \) is the perturbation of the field \( \phi \), \( \omega = (1 - 6Q^2) F \), and

\[
\delta R = 2 \left[ 3(\dot{\Phi} - H \Psi) - 12H(\dot{H} \Psi - \Phi) + \left( \frac{k^2}{a^2} - 3\dot{H} \right) \Psi + 2(k^2/a^2)\Phi \right] .
\]

Provided that the field is sufficiently heavy to satisfy the conditions \( M^2 \gg R \), one can employ the approximation \( [(2U_{,\phi} - F_{,\phi} R)/2\omega],_{\phi} \simeq M^2/\omega \) in Eq. (235). The solution to Eq. (235) consists of the sum of the matter-induced mode \( \delta \phi_{\text{ind}} \) sourced by the matter perturbation and the oscillating mode \( \delta \phi_{\text{osc}} \), i.e. \( \delta \phi = \delta \phi_{\text{ind}} + \delta \phi_{\text{osc}} \) (as in the case of \( f(R) \) gravity).

In order to know the evolution of the matter-induced mode we employ the quasi-static approximation on sub-horizon scales. Under this approximation, we have \( \delta R_{\text{ind}} \simeq 2(k^2/a^2)[\dot{\Phi} - (F,_{\phi}/F) \delta \phi_{\text{ind}}] \) from Eqs. (235) and (238), where the subscript “ind” represents the matter-induced mode. Then from Eq. (235) we find

\[
\delta \phi_{\text{ind}} \simeq -\frac{2QF}{(k^2/a^2)(1 - 2Q^2)F + M^2} \frac{k^2}{a^2} \Phi .
\]

Using Eqs. (235) and (237) we obtain

\[
\frac{k^2}{a^2} \Psi = -\frac{\delta \rho_m}{2F} \left( \frac{k^2}{a^2}(1 + 2Q^2)F + M^2 \right) , \quad \frac{k^2}{a^2} \Phi = \frac{\delta \rho_m}{2F} \left( \frac{2Q^2}{k^2/a^2}F + M^2 \right).
\]

In the massive limit \( M^2/F \gg k^2/a^2 \), we recover the standard result of General Relativity. In the massless limit \( M^2/F \ll k^2/a^2 \), it follows that \( (k^2/a^2) \Psi \simeq -(\delta \rho_m/2F)(1 + 2Q^2) \) and \( (k^2/a^2) \Phi \simeq (\delta \rho_m/2F)(1 - 2Q^2) \).

Plugging Eq. (240) into Eq. (241), we obtain the equation for matter perturbations \( [76] \):

\[
\ddot{\delta}_m + 2H \dot{\delta}_m - 4\pi G_{\text{eff}} \delta \rho_m = 0 ,
\]

where the effective gravitational coupling is

\[
G_{\text{eff}} = \frac{G}{F} \left( \frac{k^2}{a^2}(1 + 2Q^2)F + M^2 \right) .
\]

We have recovered the bare gravitational constant \( G \). In the massless limit \( M^2 \ll k^2/a^2 \) this reduces to

\[
G_{\text{eff}} \simeq \frac{G}{F} (1 + 2Q^2) = \frac{G}{F} \left( 4 + 2\omega_{\text{BD}} \right) ,
\]

where in the last line we have used the relation \( [77] \) between the coupling \( Q \) and the Brans-Dicke parameter \( \omega_{\text{BD}} \). In \( f(R) \) gravity we have \( \omega_{\text{BD}} = 0 \) and hence \( G_{\text{eff}} = 4G/(3F) \).

Let us consider the evolution of the oscillating mode of perturbations. Using Eqs. (235) and (236) for sub-horizon modes \( (k^2/a^2) \gg H^2 \), the gravitational potentials can be expressed by \( \delta \phi_{\text{osc}} \) (note that \( \delta \rho_m = 0 \) for the oscillating mode). From Eq. (238) the perturbation of \( R \) corresponding to the oscillating mode is given by

\[
\delta R_{\text{osc}} \simeq 6Q \left[ \delta \phi_{\text{osc}} + 3H \delta \phi_{\text{osc}} + (k^2/a^2) \delta \phi_{\text{osc}} \right] .
\]
Substituting Eq. (244) into Eq. (236), it follows that
\[ \delta \phi_{\text{osc}} + 3H \delta \phi_{\text{osc}} + \left( k^2/a^2 + M^2/F \right) \delta \phi_{\text{osc}} \approx 0, \] (245)
which is valid in the regime \( M^2 \gg R \).

When \(|Q| = O(1)\) the field potential \( U(\phi) \) is required to be heavy in the region of high density for the consistency with local gravity constraints. We take the potential \(|Q|\) as an example of a viable model. During the matter era the field \( \phi \) settles down at the instantaneous minima characterized by the condition \(|Q|\). Then we have that \( \phi \propto \rho_m^{1/3} \) and \( M^2 \propto \rho_m^{2/3} \) during the matter-dominated epoch. The field \( \phi \) is initially heavy to satisfy the condition \( M^2/F \gg k^2/a^2 \) for the modes relevant to the galaxy power spectrum. Depending upon the model parameters and the mode \( k \), the mass squared \( M^2 \) can be smaller than \( k^2/a^2 \) during the matter era \(|Q|\).

In the regime \( M^2/F \gg k^2/a^2 \) the matter perturbation equation \(241\) reduces to the standard one in Einstein gravity, which gives the evolution \( \delta \propto t^{2/3} \). For the model \(|Q|\) the matter-induced mode of the field perturbation evolves as \( \delta \propto \rho_m/M^2 \propto t^{2/3} \). Meanwhile, the WKB solution to Eq. \(245\) is given by \( \delta \propto t^{2/3} \cos \left( c t^{2/3} \right) \), where \( c \) is a constant. Since the background field \( \phi \) during the matter era evolves as \( \phi \propto t^{1/3} \), we find
\[ \delta \phi = \left( \delta \phi_{\text{ind}} + \delta \phi_{\text{osc}} \right) / \phi \simeq c_1 t^{2/3} + c_2 t^{1/2} \cos \left( c t^{2/3} \right). \] (246)

As long as the oscillating mode is initially suppressed relative to the matter-induced mode, the latter remains the dominant contribution in the subsequent cosmic expansion history.

In the regime \( M^2/F \ll k^2/a^2 \) the effective gravitational coupling is given by Eq. \(243\). Solving Eq. \(241\) in this case, we obtain the solution
\[ \delta \propto t^{2-1/\sqrt{Q}}. \] (247)
Setting \( Q = -1/\sqrt{6} \), this recovers the solution \( \delta \propto t^{\sqrt{55}-1/6} \) in metric \( f(R) \) gravity.

The potential \(|Q|\) has a heavy mass \( M \) much larger than \( H \) in the deep matter-dominated epoch, but it gradually decreases with time. Depending on the modes \( k \), the system crosses the point \( M^2/F = k^2/a^2 \) at \( t = t_k \) during the matter-dominated epoch. Since the field mass evolves as \( M \propto t^{-2/3} \) during the matter era, the time \( t_k \) has a scale-dependence given by \( t_k \propto k^{-3/4} \). Since \( 0 < p < 1 \) the smaller scale modes (i.e. larger \( k \)) cross the transition point earlier. During the matter era the mass squared is approximately given by
\[ M^2 \simeq \frac{1 - p}{(2 \rho_p C)^{1/(1-p)}} Q^2 \left( \frac{\rho_m}{V_0} \right) ^{2/3} U_0. \] (248)
Using the relation \( \rho_m = 3F_0 \Omega_m^{(0)} H_0^2 (1 + z)^3 \), the critical redshift \( z_k \) at time \( t_k \) can be estimated as
\[ z_k \simeq \left[ \left( \frac{k}{a_0 H_0} \right) \frac{1}{Q} \right] ^{(2(1-p))/2} \left( \frac{2 \rho_p C}{(1-p)^{1-p}} \left( \frac{3F_0 \Omega_m^{(0)}}{2-p} \right)^{2-p} H_0^2 \right) ^{1/2} - 1, \] (249)
where \( a_0 \) is the scale factor today. The critical redshift increases for larger \( k/(a_0 H_0) \) and for smaller \( p \). If \( k/(a_0 H_0) = 600 \) and \( p = 0.7 \), Eq. \(249\) gives \( z_k = 3.9 \).

Defining the growth rate of matter perturbations as in Eq. \(233\), it follows that the asymptotic values of \( f_\delta \) in the regions \( t < t_k \) and \( t \gg t_k \) are given by \( f_\delta = 1 \) and \( f_\delta = (\sqrt{25 + 48Q^2} - 1)/4 \), respectively. Numerical simulations show that the growth rate reaches a maximum value around the end of the matter era and then it starts to decrease during the epoch of cosmic acceleration \(|Q|\). The observational constraint on \( f_\delta \) reported by McDonald et al. \(|Q|\) is \( f_\delta = 1.46 \pm 0.49 \) around the redshift \( z = 3 \), whereas the data reported by Viel and Haehnelt \(|Q|\) in the redshift range \( 2 < z < 4 \) show that even the value \( f_\delta = 2 \) can be allowed in some of the observations. If we use the criterion \( f_\delta < 2 \) with the analytic estimation \( f_\delta = (\sqrt{25 + 48Q^2} - 1)/4 \), we obtain the bound \( Q < 1.08 \). Note that the growth index today can be smaller than 0.4 for \(|Q| \) larger than 0.4, so this will be also useful to place tight bounds on \( Q \) in future observations.

The relative difference of the matter power spectrum \( P_m \) at time \( t = t_\Lambda \) (at which \( \ddot{a} = 0 \)) from that in the LCDM is given by
\[ \frac{P_m(t_\Lambda)}{P_m^{\Lambda\text{CDM}}(t_\Lambda)} = \left( \frac{t_\Lambda}{t_k} \right) ^{2(1/p)} \left( \frac{25 + 48Q^2 - 19}{1 - 4p} \right) \propto k^{(1-p)(\sqrt{25 + 48Q^2} - 19)}. \] (250)
In Fig. [7] we plot the matter power spectrum for $Q = 0.7$ and $p = 0.6$, which deviates from that in the $\Lambda$CDM model on small scales. The estimation (250) shows fairly good agreement with numerical results [340].

From Eqs. (250) we find that the effective gravitational potential $\psi = \Phi - \Psi$ satisfies the same equation as (253). Since the ISW effect induced by the modified evolution of $\psi$ is limited on large-scale CMB perturbations irrelevant to the galaxy power spectrum, there is a difference between the spectral indices of the matter power spectrum and of the CMB spectrum on the scales, $k > 0.01 h \text{Mpc}^{-1}$:

$$\Delta n(t_\Lambda) = \frac{(1 - p)(\sqrt{25 + 48Q^2} - 5)}{4 - p}. \quad (251)$$

This reproduces the result (232) in $f(R)$ gravity by setting $Q = -1/\sqrt{6}$ and $p = 2n + 1$. If we use the criterion $\Delta n(t_\Lambda) < 0.05$, as in the case of the $f(R)$ gravity, we obtain the bounds $p > 0.957$ for $Q = 1$ and $p > 0.855$ for $Q = 0.5$. As long as $p$ is close to 1, it is possible to satisfy both cosmological and local gravity constraints for $|Q| \lesssim 1$.

C. DGP model

In this section we study the evolution of linear matter perturbations in the DGP braneworld model. The discussion below is valid for the wavelengths larger than the Vainshtein radius $r_*$. For the radius $r$ smaller than $r_*$ the non-linear effect coming from the brane-bending mode becomes crucially important. The perturbed metric in the 5-dimensional longitudinal gauge with four scalar metric perturbations $\Psi, \Phi, B, E$ is given by [34, 361]

$$\text{ds}^2 = -(1 + 2\Psi)n(t, y)^2 \text{d}t^2 + (1 + 2\Phi)A(t, y)^2 \delta_{ij} \text{d}x^i \text{d}x^j + 2r_c B_i \text{d}x^i \text{d}y + (1 + 2E)\text{d}y^2, \quad (252)$$

where the brane is located at $y = 0$ in the 5-th dimension characterized by the coordinate $y$ (we are considering a flat FLRW spacetime on the brane). Note that $B$ can be identified as a brane bending mode describing a perturbation of the brane location and that $r_c$ is the crossover scale defined in Eq. (130). The background solution describing the self-accelerating Universe is [218]

$$n(t, y) = 1 + H(1 + \ddot{H}/H^2)y, \quad A(t, y) = a(t)(1 + Hy). \quad (253)$$

The Hubble parameter $H = \dot{a}/a$ satisfies Eq. (133) with $\epsilon = +1$.

In what follows we neglect the terms suppressed by the factor $aH/k \ll 1$ because we are considering sub-horizon perturbations. We also ignore the terms such as $(A'/A)\Phi'$, where a prime represents a derivative with respect to $y$. This comes from the fact that $\Phi'$ is of the order of $(k/a)\Phi$, as we will show later. The time-derivative terms can be also dropped under the quasi-static approximation on sub-horizon scales. Then the perturbed 5-dimensional Einstein tensors $\delta G^\alpha_\beta$ obey the following equations locally in the bulk [361]:

$$\delta G^0_0 = 3\Phi'' + \frac{2}{A^2} \nabla^2 \Phi + \frac{2}{A^2} (E - r_c B') - 2 \frac{r_c}{A^2} \left( \frac{A'}{A} \right) \nabla^2 B = 0, \quad (254)$$

$$\delta G^i_j = - \frac{1}{A^2} (\nabla^i \nabla_j - \delta^i_j \nabla^2)(\Phi + \Psi + E - r_c B') + \frac{r_c}{A^2} (\nabla^i \nabla_j - \delta^i_j \nabla^2) \left( \frac{A'}{A} + \frac{n'}{n} \right) B = 0, \quad (255)$$

$$\delta G^5_5 = - (\Psi' + 2\Phi'), i = 0, \quad (256)$$

$$\delta G^5_5 = \frac{1}{A^2} \nabla^2 (\Psi + 2\Phi) - \frac{r_c}{A^2} \left( 2 \frac{A'}{A} + \frac{n'}{n} \right) \nabla^2 B = 0. \quad (257)$$

Taking the divergence of the traceless part of Eq. (255), we obtain

$$\frac{\nabla^2}{A^2} (\Phi + \Psi + E - r_c B') - \frac{r_c}{A^2} \left( \frac{A'}{A} + \frac{n'}{n} \right) \nabla^2 B = 0. \quad (258)$$

The consistency between Eqs. (256) and (257) requires that

$$B' = 0, \quad \Psi' + 2\Phi' = 0. \quad (259)$$

From Eqs. (257) and (258) we find

$$\frac{\nabla^2}{A^2} (E - r_c B') = - \frac{1}{2} \frac{\nabla^2}{A^2} \Psi + \frac{r_c n'}{2A^2 n} \nabla^2 B. \quad (260)$$
Substituting Eqs (257) and (260) into Eq. (254) together with the use of Eq. (259), we obtain
\[
\Psi'' + \frac{\nabla^2}{A^2} \Psi - \frac{n'}{n} \frac{r_c}{A^2} \nabla^2 B = 0.
\] (261)

Under the sub-horizon approximation \((k/aH \gg 1)\) the solution to Eq. (261), upon the Fourier transformation, is given by
\[
\Psi - \frac{n'}{n} r_c B = \left[ c_1 (1 + H y)^{-k/aH} + c_2 (1 + H y)^{k/aH} \right],
\] (262)
where \(c_1\) and \(c_2\) are integration constants. In order to avoid the divergence of the perturbation in the limit \(y \to \infty\) we choose \(c_2 = 0\).

The junction condition at the brane can be expressed in terms of an extrinsic curvature \(K_{\mu \nu}\) and an energy-momentum tensor on the brane [362]:
\[
K_{\mu \nu} - Kg_{\mu \nu} = -\kappa^2 (5) T_{\mu \nu}/2 + r_c G_{\mu \nu},
\] (263)
where \(K \equiv K_{\mu}{}^{\mu}\). The extrinsic curvature is defined as
\[
K_{\mu \nu} = h_\lambda{}^{\mu} \nabla_\lambda n_\nu,
\]
where \(n_\nu\) is the unit vector normal to the brane and \(h_{\mu \nu} = g_{\mu \nu} - n_\mu n_\nu\) is the induced metric on the brane. The \((0,0)\) and spatial components of the junction condition (263) give
\[
2 a^2 \nabla^2 \Phi = -\kappa^2 (4) \delta \rho_m + \frac{1}{a^2} \nabla^2 B - \frac{3}{r_c} \Phi',
\] (264)
\[
\Phi + \Psi = B,
\] (265)
\[
\Psi' + 2 \Phi' = 0,
\] (266)
where \(\delta \rho_m\) is the matter perturbation on the brane. Equation (266) is consistent with the latter of Eq. (259).

From Eq. (262) it follows that \(\Phi' \sim (k/a) \Phi\) in Fourier space. For the perturbations whose wavelengths are much smaller than the cross-over scale \(r_c\), i.e., \(r_c k/a \gg 1\), the term \((3/r_c) \Phi'\) in Eq. (264) is much smaller than \((k^2/a^2) \Phi\). In Fourier space Eq. (264) is approximately given by
\[
\frac{2 k^2}{a^2} \Phi = \frac{\kappa^2 (4)}{2} \delta \rho_m + \frac{k^2}{a^2} B.
\] (267)

Using the projection of Eq. (257) as well as Eqs. (265) and (267), we find that metric perturbations \(\Psi\) and \(\Phi\) obey the following equations
\[
\frac{k^2}{a^2} \Psi = -\frac{\kappa^2 (4)}{2} \left( 1 + \frac{1}{3 \beta} \right) \delta \rho_m, \quad \frac{k^2}{a^2} \Phi = \frac{\kappa^2 (4)}{2} \left( 1 - \frac{1}{3 \beta} \right) \delta \rho_m,
\] (268)
where
\[
\beta(t) \equiv 1 - \frac{2 r_c}{3} \left( \frac{2 A' + n'}{n} \right) = 1 - 2 H r_c \left( 1 + \frac{H}{3 H^2} \right).
\] (269)

The matter perturbation \(\delta_m\) satisfies the same form of equation as given in (218) for the modes deep inside the horizon [360]. Substituting the former of Eq. (268) into Eq. (218), we find that the matter perturbation obeys the following equation [367, 360]
\[
\ddot{\delta}_m + 2 H \dot{\delta}_m - 4 \pi G_{\text{eff}} \rho_m \delta_m = 0,
\] (270)
where
\[
G_{\text{eff}} = \left( 1 + \frac{1}{3 \beta} \right) G.
\] (271)

From Eq. (268) the effective gravitational potential \(\psi = \Phi - \Psi\) obeys the same equation as (234).

In the deep matter era one has \(H r_c \gg 1\) and hence \(\beta \simeq -H r_c\), so that \(\beta\) is largely negative \((|\beta| \gg 1)\). In this regime the evolution of the matter perturbation is similar to that in General Relativity \((\delta_m \propto t^{2/3})\). The solutions finally approach the de Sitter attractor characterized by \(H_{\text{dS}} = 1/r_c\). At the de Sitter solution one has \(\beta \simeq 1 - 2 H r_c \simeq -1\).
Since $1 + 1/(3\beta) \approx 2/3$, the growth rate in this regime is smaller than that in GR. The growth index is approximately given by $\gamma \approx 0.68$, which is different from the value $\gamma \approx 0.55$ in the $\Lambda$CDM model. If the future imaging survey of galaxies can constrain $\gamma$ within 20%, it may be possible to distinguish the DGP model from the $\Lambda$CDM model [363].

Comparing Eq. (271) with the effective gravitational coupling (243) in Brans-Dicke theory with a massless limit (or the absence of the field potential), we find that the Brans-Dicke parameter $\omega_{BD}$ has the following relation with $\beta$:

$$
\omega_{BD} = \frac{3}{2}(\beta - 1).
$$

(272)

Since $\beta < 0$ for the self-accelerating DGP solution, this implies that $\omega_{BD} < -3/2$. Since in this case the kinetic energy of a scalar field degree of freedom is negative in the Einstein frame, the DGP model contains a ghost mode. The solution in another branch of the DGP model is not plagued by this problem, because the minus sign of Eq. (269) is replaced by the plus sign. The self accelerating solution in the original DGP model can be realized at the expense of an appearance of the ghost state.

D. Galileon gravity

1. Covariant Galileon gravity

In covariant Galileon gravity described by the action (140) the evolution of matter density perturbations was studied in Ref. [364]. In spite of the complexities of full perturbation equations, they are simplified under the quasi-static approximation on sub-horizon scales. Under this approximation the matter perturbation obeys the same equation as in Ref. [364]. In spite of the complexities of full perturbation equations, they are simplified under the quasi-static approximation on sub-horizon scales. Under this approximation the matter perturbation obeys the same equation as (271).

We define the effective gravitational potential $\psi = \Phi - \Psi$ as well as the anisotropic parameter $\eta = -\Phi/\Psi$. Under the quasi-static approximation on sub-horizon scales we obtain [364]

$$
\psi \approx 3 \frac{G_{eff}}{G} \left[ 1 + \frac{\eta}{2} \Omega_m \delta_m \left( \frac{aH}{k} \right)^2 \right],
$$

(273)

where $\Omega_m$ and $\delta_m$ are the density parameter and the perturbation of non-relativistic matter, respectively. In $f(R)$ gravity, Brans-Dicke theory, and the DGP model the effective gravitational potential obeys Eq. (273) for the modes deep inside the Hubble radius. In Galileon gravity the combination $(G_{eff}/G)(1 + \eta)/2$ is different from 1. This means that the effective gravitational potential may acquire some additional growth compared to other models. In Galileon cosmology there are three different regimes characterized by (i) $r_1 \ll 1$, $r_2 \ll 1$, (ii) $r_1 = 1$, $r_2 \ll 1$, and (iii) $r_1 = 1$, $r_2 = 1$, where $r_1$ and $r_2$ are defined in Eq. (150). In these regimes we can estimate $G_{eff}$ and $\eta$ as follows [364]. We stress that these analytic results are valid for the modes deep inside the Hubble radius.

- (i) $r_1 \ll 1$, $r_2 \ll 1$

  Expanding $G_{eff}$ and $\eta$ about $r_1 = 0$, $r_2 = 0$, it follows that

$$
\frac{G_{eff}}{G} = 1 + \left( \frac{255}{8} \beta + \frac{211}{16} \alpha r_1 \right) r_2 + \mathcal{O}(r_2^2),
\eta = 1 + \left( \frac{129}{8} \beta + \frac{589}{16} \alpha r_1 \right) r_2 + \mathcal{O}(r_2^2),
$$

(274)

where $\alpha$ and $\beta$ are defined by Eq. (149). Since $\beta > 0$ to avoid ghosts (for the branch $r_2 > 0$), we have $G_{eff} > G$ and $\eta > 1$ in this regime. This means that the growth rates of $\delta_m$ and $\psi$ are larger than those in the $\Lambda$CDM model.

- (ii) $r_1 = 1$, $r_2 \ll 1$

  Expansion of $G_{eff}$ and $\eta$ about $r_2 = 0$ gives

$$
\frac{G_{eff}}{G} = 1 + \frac{291\alpha^2 + 702\beta^2 - 933\alpha \beta + 20\alpha - 84\beta + 4}{2(10\alpha - 9\beta + 8)} r_2 + \mathcal{O}(r_2^2),
\eta = 1 - \frac{3(126\alpha^2 + 306\beta^2 - 406\alpha \beta + 4\alpha - 30\beta)}{2(10\alpha - 9\beta + 8)} r_2 + \mathcal{O}(r_2^2).
$$

(275)

The evolution of $G_{eff}$ and $\eta$ depends on both $\alpha$ and $\beta$. If $\alpha = 1.4$ and $\beta = 0.4$, for example, we have $G_{eff}/G \approx 1 + 4.31r_2$ and $\eta \approx 1 - 5.11r_2$, respectively. In this case $G_{eff} > G$, but $\eta$ is smaller than 1.
\[ \beta = 0 \]

for the modes \( k_a \delta \) panel of Fig. 8 we find that, on larger scales, the growth of \( \psi \) Eq. (274). Then the growth rates of \( z \) for the wave numbers (a) \( k = 300aH_0 \), (b) \( k = 10aH_0 \), and (c) \( k = 5aH_0 \). Note that \( \delta_m/a \) and \( \psi \) are divided by their initial amplitudes \( \delta_m(t_i)/a(t_i) \) and \( \psi(t_i) \), respectively, so that their initial values are normalized to be 1. The bold dotted lines show the results obtained under the quasi-static approximation on sub-horizon scales. From Ref. [364].

- (iii) \( r_1 = 1, r_2 = 1 \)

At the dS point we have

\[
G_{\text{eff}} = \frac{1}{3(\alpha - 2 \beta)}; \quad \eta = 1, \quad (277)
\]

which means that there is no anisotropic stress.

Recall that the late-time tracking solutions are favored from observational constraints at the background level. In this case the solutions start from the regime (i) and finally approach the de Sitter fixed point with a short period of the regime (ii). In Fig. 8 we plot \( \delta_m/a \) and \( \psi \) versus the redshift \( z \) for \( \alpha = 1.37 \) and \( \beta = 0.44 \) with the background initial conditions \( r_1 = 0.03 \) and \( r_2 = 0.003 \). In this case the solutions approach the tracker at late times. The initial conditions of perturbations are chosen to recover the GR behavior in the asymptotic past. Note that these results are obtained by numerical integration of the full perturbation equations without quasi-static approximations. For the mode \( k = 300aH_0 \) the numerical result shows excellent agreement with that obtained under the quasi-static approximation on sub-horizon scales. The difference starts to appear for the modes \( k/(a_0 H_0) < O(10) \). From the left panel of Fig. 8 we find that, on larger scales, the growth of \( \delta_m \) tends to be less significant. For the modes \( k \gg a_0 H_0 \) the matter perturbation evolves faster than \( a \) during the matter era.

From the right panel of Fig. 8 we find that, unlike the ΛCDM model, \( \psi \) changes in time even during the matter era for the modes \( k \gg aH \). Before reaching the tracker we have \( G_{\text{eff}}/G \approx 1 + 255 \beta r_2/8 \approx 1 + 129 \beta r_2/8 \) from Eq. (274). Then the growth rates of \( \psi \) and \( \delta_m \) get larger than those in GR. In particular the term \( (G_{\text{eff}}/G)(1 + \eta)/2 \) in Eq. (273) is larger than 1, which leads to the additional growth of \( \psi \) to that coming from \( \delta_m \). In Galileon gravity the unusual behavior of the anisotropic parameter \( \eta \) leads to the non-trivial evolution of perturbations. For the model parameters \( \alpha = 1.37 \) and \( \beta = 0.44 \), Eq. (274) gives \( G_{\text{eff}} \approx 0.68G \) at the de Sitter fixed point. Since in this case \( G_{\text{eff}} \) is smaller than \( G \), \( \psi \) begins to decrease at some point after the matter era.

For the large-scale modes relevant to the ISW effect in CMB anisotropies \( k/(a_0 H_0) \lesssim 10 \), \( \psi \) is nearly constant in the early matter-dominated epoch. However, as we see in Fig. 8, \( \psi \) exhibits temporal growth during the transition from the matter era to the epoch of cosmic acceleration. The characteristic variation of \( \psi \) in the Galileon model may leave interesting observational signatures on the large-scale CMB anisotropies.

For the model parameters constrained by SN Ia (Union2)+CMB+BAO data sets, i.e. \( \alpha = 1.404 \pm 0.057 \) and \( \beta = 0.419 \pm 0.023 \) [250], the effective gravitational coupling at the de Sitter solution is restricted in the range
Provided exhibits more or less the similar property to that shown in Fig. 8. If the model parameters are close to the upper limit $\alpha = 2 \beta + 2/3$ of the allowed parameter space at the background level (i.e. $G_{\text{eff}}$ is close to 0.5G at the de Sitter point), the parameter $\eta$ tends to show a divergence during the transition from the matter era to the epoch of cosmic acceleration. If $G_{\text{eff}}$ is larger than 0.66G, we find that such divergent behavior is typically avoided. For the viable model parameters the evolution of $\delta_m$ and $\psi$ exhibits more or less the similar property to that shown in Fig. 8.

2. Modified Galileon gravity

Finally we study modified Galileon theories in which the term $\Box \phi (\partial_{\mu} \phi \partial^{\mu} \phi)$ is generalized to $\xi (\phi) \Box \phi (\partial_{\mu} \phi \partial^{\mu} \phi)$. Let us consider general theories described by the action

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} f(R, \phi, X) + \xi (\phi) \Box \phi (\partial_{\mu} \phi \partial^{\mu} \phi) \right] + \int d^4x L_M,$$

(278)

where

$$f(R, \phi, X) = f_1(R, \phi) + f_2(\phi, X).$$

(279)

We introduce two mass scales associates with the field $\phi$ and the scalar gravitational degree of freedom, respectively, as

$$M^2_{\phi} = - f_{,\phi \phi} / 2, \quad M^2_R = F / (3F, R),$$

(280)

where $F \equiv \partial f / \partial R$.

The full perturbation equations for the perturbed metric are given in Ref. [183]. Under the quasi-static approximation on sub-horizon scales the matter perturbation satisfies Eq. (270) with the effective gravitational coupling [183]

$$G_{\text{eff}} = \frac{1}{8\pi F} \frac{1 + 4s_1}{1 + 3s_1} \left( 1 + \frac{[F_{,\phi} + 2(1 + 4s_1) \xi \dot{\phi}^2]^2}{(1 + 4s_1) \mu F} \right),$$

(281)

where

$$\mu \equiv (1 + 3s_1)(f_{,X} + 2s_2) + 3F_{,\phi}^2 / F + 2F_{,\phi} \phi^2 / F - 2\xi \dot{\phi}^4(1 + 4s_1) / F,$$

(282)

$$s_1 \equiv k^2 / (3a^2 M_R^2), \quad s_2 \equiv a^2 M_R^2 / k^2.$$

(283)

The anisotropic parameter $\eta = - \Phi / \Psi$ is given by

$$\eta = \frac{(1 + 2s_1)(f_{,X} + 2s_2) + 2F_{,\phi}^2 / F + 4\xi [2(1 + 2s_1)(\dot{\phi} + 2H \dot{\phi}) - 2F_{,\phi} \phi^2 / F - 2\xi \dot{\phi}^4(1 + 4s_1) / F]}{(1 + 4s_1)(f_{,X} + 2s_2) + 4F_{,\phi}^2 / F + 8\xi (1 + 4s_1)(\dot{\phi} + 2H \dot{\phi})}.$$

(284)

The effective gravitational potential $\psi = \Phi - \Psi$ obeys the equation of the form (278), whose explicit form is

$$\psi = 3\Omega_m \delta_m \left( \frac{aH}{k} \right)^2 \frac{(1 + 3s_1)[f_{,X} + 2s_2 + 8\xi(\dot{\phi} + 2H \phi)] + 3F_{,\phi}^2 / F - 2\xi F_{,\phi} \phi^2 / F}{(1 + 3s_1)[f_{,X} + 2s_2 + 8\xi(\dot{\phi} + 2H \phi)] + 3F_{,\phi}^2 / F - 4\xi [F_{,\phi} \phi^2 / F + \xi \dot{\phi}^4(1 + 4s_1) / F]},$$

(285)

where $\Omega_m = \rho_m / (3FH^2)$.

In the massive limits $M_{\phi}^2 \to \infty$ and $M_R^2 \to \infty$, i.e. $s_1 \to 0$ and $s_2 \to \infty$ we recover the standard results in GR: $G_{\text{eff}} \approx 1 / (8\pi F)$, $\eta \approx 1$, and $\psi \approx 3\Omega_m \delta_m (aH/k)^2$. The difference appears in the regimes characterized by the conditions $s_1 \gtrsim 1$ and $s_2 \lesssim 1$, as it happens for the late-time evolution of perturbations in $f(R)$ gravity and in Brans-Dicke theory.

For the theories with $\xi = 0$, Eq. (285) gives the standard relation $\psi = 3\Omega_m \delta_m (aH/k)^2$. If $\xi \neq 0$, then this relation no longer holds. In this case $\psi$ is not directly related with $\delta_m$ due to the additional contribution from the $\xi$-dependent term. This can be regarded as the main reason for the anti-correlation between the ISW effect in CMB and the large-scale structure found for the model [182].

In Ref. [183] the authors derived conditions for avoiding the appearance of ghosts and Laplacian instabilities. Provided $F > 0$ the tensor ghosts do not appear. Since $c_T^2 = 1$ for the theories (278), tensor perturbations have
no Laplacian instabilities. For the theories with \( f_{,RR} \neq 0 \) the conditions for the avoidance of ghosts and Laplacian instabilities of scalar perturbations are given, respectively, by

\[
24\xi H \dot{\phi} - 8\xi,\phi \dot{\phi}^2 + f,_{XX} \dot{\phi}^2 - F^2_{,X} \dot{\phi}^2 / F_R > 0, \tag{286}
\]
\[
f,_{X} + 8(\ddot{\phi} + 2H \dot{\phi})\xi - 16\xi^2 \dot{\phi}^2 / (3F) > 0. \tag{287}
\]

Similar conditions have been also derived for the theories with \( f_{,RR} \neq 0 \). The dark energy models based on the action \[278\] need to be constructed to satisfy these conditions.

### E. Observables in weak lensing

We have shown that modified gravity models of dark energy generally lead to changes for the growth rate of matter perturbations compared to the ΛCDM model. Since there are two free functions that determine the first-order metrics \( \Psi \) and \( \Phi \), dark energy models can be classified according to how the gravitational potentials are linked to \( \delta_m \). In order to quantify this, we introduce two quantities \( q(k, t) \) and \( \zeta(k, t) \) defined by

\[
(k^2/a^2)\Phi = 4\pi G q \delta_m \rho_m, \tag{288}
\]
\[
(\Phi + \Psi) / \Phi = \zeta, \tag{289}
\]

where \( G \) is the 4-dimensional bare gravitational constant. The ΛCDM model corresponds to \( q = 1 \) and \( \zeta = 0 \) (note that the cosmological constant does not cluster). Modified gravity models give rise to different values of \( q \) and \( \zeta \) relative to those in the ΛCDM model. Therefore the functions \( q \) and \( \zeta \) characterize gravitational theories for first-order scalar perturbations on small scales.

In Brans-Dicke theory discussed in Sec. \[286\] the gravitational potentials are given by Eq. \[240\] on sub-horizon scales. In this case we have

\[
q = 1 - 1/(3\beta) , \quad \zeta = 2/(1 - 3\beta). \tag{291}
\]

In the deep matter era one has \( |\beta| \gg 1 \), so that \( q \approx 1 \) and \( \zeta \approx 0 \). The deviation from \( (q, \zeta) = (1, 0) \) appears when \( |\beta| \) decreases to the order of unity, i.e., when the Universe enters the epoch of cosmic acceleration.

In order to confront dark energy models with the observations of weak lensing, it may be convenient to introduce the following quantity \[355\]

\[
\Sigma \equiv q(1 - \zeta / 2). \tag{292}
\]

From Eqs. \[288\] and \[289\] we find that the effective gravitational potential \( \psi = \Phi - \Psi \) associated with weak lensing observations can be expressed as

\[
\psi = 8\pi G (a^2/k^2) \rho_m \delta_m \Sigma. \tag{293}
\]

In the DGP model and in Brans-Dicke theory we have \( \Sigma = 1 \) and \( \Sigma = 1/F \), respectively. In (modified) Galileon theories the term \( \Sigma \) is of more complicated forms, see e.g., Eq. \[255\].

The effect of modified gravity theories manifests itself in weak lensing observations in at least two ways. One is the multiplication of the term \( \Sigma \) on the r.h.s. of Eq. \[283\]. Another is the modification of the evolution of \( \delta_m \). The latter depends on two parameters \( q \) and \( \zeta \), or equivalently, \( \Sigma \) and \( \zeta \). Thus two parameters \( (\Sigma, \zeta) \) will be useful to detect signatures of modified gravity theories from future surveys of weak lensing.
VIII. CONCLUSIONS

We have reviewed modified gravitational models of dark energy responsible for the cosmic acceleration today. In addition to cosmological constraints such as the presence of a matter era followed by a stable de Sitter solution, we require that the models satisfy local gravity constraints. There are two mechanisms for the recovery of GR behavior in the regions of high density. The first one is the chameleon mechanism in which the mass of a scalar-field degree of freedom depends on the matter density in the surrounding environment. The chameleon mechanism can be at work in \( f(R) \) gravity and Brans-Dicke theory, as long as the field potential is designed to have a large mass in the regions of high density. The second one is the Vainshtein mechanism in which the nonlinear effect of scalar-field self interactions leads to the recovery of GR at small distances. This can be applied to the DGP braneworld model and Galileon gravity.

The modified gravity models can give rise to the phantom equation of state of dark energy \((w_{DE} < -1)\) without having ghosts, tachyons, and Laplacian instabilities. The deviation of \( w_{DE} \) from that in the \( \Lambda \)CDM model \((w_{DE} = -1)\) is not so significant in the models based on \( f(R) \) gravity and Brans-Dicke theory, so these models can be compatible with the observational constraints at the background cosmology fairly easily. On the other hand, the tracker solution in covariant Galileon gravity, which has \( w_{DE} = -2 \) during the matter era, is disfavored from the joint data analysis of SN Ia, BAO, and CMB shift parameters. However the late-time tracking without a significant deviation from \( w_{DE} = -1 \) is allowed observationally.

In order to confront the modified gravity models with the observations of large-scale structure, CMB, and weak lensing, we have also discussed the evolution of matter density perturbations. In the models based on \( f(R) \) and Brans-Dicke theory there is a “General Relativistic” regime in which the field is heavy such that \( M^2 \gg k^2/a^2 \). At late times this is followed by a “scalar-tensor” regime \((M^2 \ll k^2/a^2)\) in which the gravitational law is modified from that in General Relativity. In Brans-Dicke theory the evolution of matter perturbations during the matter era changes from \( \delta_m \propto t^{2/3} \) to \( \delta_m \propto t^{2(\sqrt{25+48Q^2}-1)/6} \) where \( Q \) is related to the Brans-Dicke parameter via the relation \( 3 + 2\omega_{BD} = 1/(2Q^2) \). The effective gravitational couplings \( G_{\text{eff}} \) in the DGP and Galileon models are independent of the wave numbers \( k \). This reflects the fact that the field is massless in those models. In the DGP model the growth rate of \( \delta_m \) is smaller than that in the \( \Lambda \)CDM, which is associated with the appearance of ghosts. In (modified) Galileon gravity, a non-trivial relation between the effective gravitational potential \( \psi \) and the matter perturbation \( \delta_m \) leads to the extra growth of \( \psi \).

We summarize the current status of each modified gravity model of dark energy.

- (i) In \( f(R) \) gravity there are some viable dark energy models such as \( 45, 46, \) and \( 48 \), which can be consistent with both cosmological and local gravity constraints. For the models \( 45, 46 \) the local gravity constraints can be satisfied for \( n > 0.9 \) under the chameleon mechanism. If we use the criterion that the difference between the spectral indices between the matter power spectrum and the CMB spectrum is smaller than 0.05, then we obtain the bound \( n \geq 2 \). In these models the initial conditions of perturbations need to be chosen such that the oscillating mode does not dominate over the matter-induced mode in the early Universe. This is associated with a weak singularity problem about the divergence of the mass squared \( M^2 \sim F/(3f_{RR}) \) for \( R \rightarrow \infty \), but it can be circumvented by including higher-curvature terms such as \( R^2 \) to the Lagrangian.

- (ii) In Brans-Dicke theory, even if the coupling \( Q \) is of the order of unity, it is possible to design field potentials consistent with both cosmological and local gravity constraints. One of the examples is the potential \( 98 \), which is motivated by the \( f(R) \) models \( 45 \) and \( 46 \). Depending on the couplings \( Q \) the growth of matter perturbations is different \((Q = -1/\sqrt{6} \) in metric \( f(R) \) gravity). From the observational constraints on the growth rate of \( \delta_m, |Q| \) is required to be smaller than the order of 1.

- (iii) In the DGP model the self-acceleration is realized through the gravitational leakage to the extra dimension, but the joint analysis using the data of SN Ia, BAO, and CMB shift parameters shows that the model is in tension with observations. Moreover the linear perturbation theory beyond the Vainshtein radius shows that the model contains a ghost mode with the effective Brans-Dicke parameter \( \omega_{BD} \) smaller than \(-3/2\). However, the modification of the DGP model like Cascading gravity can alleviate this problem.

- (iv) In covariant Galileon gravity there is a tracker that attracts solutions with different initial conditions to a common trajectory. The joint observational constraints at the background cosmology shows that the late-time tracking solutions are favored from the data. Cosmological perturbations in (modified) Galileon theories exhibit peculiar features because of non-trivial relations between the effective gravitational potential \( \psi \) and the matter perturbation \( \delta_m \).

- (v) The dark energy models described by a Gauss-Bonnet term with a scalar coupling \( F(\phi)G \) do not satisfy both cosmological and local gravity constraints. The generalized Gauss-Bonnet model in which the Lagrangian
density is given by $\frac{R}{2} + f(G)$ is plagued by a serious problem of the Laplacian instability in the presence of matter fluids. There are some viable Lorentz-violating models of dark energy in which the phantom equation of state can be realized without having ghosts, tachyons, and Laplacian instabilities. We hope to find some signatures for the modification of gravity in future high-precision observations. This will shed new light on the nature of dark energy.

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