UNKNOTTING ANNULI AND HANDLEBODY-KNOT SYMMETRY

YI-SHENG WANG

Abstract. By Thurston’s hyperbolization theorem, irreducible handlebody-knots are classified into three classes: hyperbolic, toroidal, and atoroidal cylindrical. It is known that a non-trivial handlebody-knot of genus two has a finite symmetry group if and only if it is atoroidal. The paper investigates the topology of cylindrical handlebody-knots of genus two that admit an unknotting annulus; we show that the symmetry group is trivial if the unknotting annulus is unique and of type 2.

1. Introduction

Given a subspace $X$ of an oriented manifold $M$, denoted by $(M, X)$, its mapping class group $\mathcal{MCG}(M, X)$ is defined as the group of isotopy classes of self-homeomorphisms of $M$ that preserve $X$ setwise, and its positive mapping class group $\mathcal{MCG}^+(M, X)$ is the subgroup consisting of orientation-preserving homeomorphisms. When $M = S^3$ is a 3-sphere, $\mathcal{MCG}(S^3, X)$ (resp. $\mathcal{MCG}^+(S^3, X)$) is referred to as the (resp. positive) symmetry group of $X$.

The case where $(S^3, K)$ is a knot has been studied by several authors (see Boileau-Zimmermann [4], Kodama-Sakuma [30], Kawauchi [26], for instance), and symmetry groups of a large class of knots are now determined. In particular, if the exterior of a knot is atoroidal, the symmetry group is finite, and furthermore it is either cyclic or dihedral. The present paper concerns symmetry groups of handlebody-knots of genus two, abbreviated to handlebody-knots hereafter; a knot can be viewed as a handlebody-knot of genus one. We call a handlebody-knot $(S^3, HK)$ atoroidal, acylindrical or irreducible if the exterior $E(HK) := S^3 - HK$ contains no essential torus, annulus or disks, respectively. A handlebody-knot $(S^3, HK)$ is hyperbolic if $E(HK)$ admits a complete hyperbolic structure with geodesic boundary. By Thurston’s hyperbolization theorem, equivariant torus theorem [20] and Tollefson’s fixed point theorem [39], a handlebody-knot is hyperbolic if and only if it is irreducible, atoroidal, and acylindrical.

A reducible or toroidal handlebody-knot $(S^3, HK)$ has an infinite symmetry group. In the case where $(S^3, HK)$ is trivial, Akbas [1], Cho [6] prove that the symmetry group, equivalent to the genus two Goeritz group of $S^3$, is finitely presented (see also Goeritz [12], Scharlemann [36]); finite presentation of symmetry groups of non-trivial reducible handlebody-knots has also been obtained by Koda [29].

On the other hand, by the Mostow rigidity theorem, the symmetry group of a hyperbolic handlebody-knot is always finite. The finiteness of symmetry groups of atoroidal cylindrical handlebody-knot is recently proved by Funayoshi-Koda [11]. Thus a handlebody-knot has a finite symmetry group if and only if it is non-trivial.

Date: April 12, 2021.

\footnote{The standard definition is that $(S^3, HK)$ is irreducible if there is no 2-sphere $\bar{S}^2 \subset S^3$ such that $\bar{S}^2 \cap HK$ is an essential disk of HK. In the genus two case, irreducibility of $(S^3, HK)$ is equivalent to $\partial$-irreducibility of $E(HK)$ by Tsukui [40] Theorem 1.}
and atoroidal \cite{28}. Contrary to the case of knots however, less is known about the structure of these finite symmetry groups.

The present work investigates the structure of symmetry groups of atoroidal cylindrical handlebody-knots. To state the results, we recall the classification of essential annuli in the exterior $E(HK)$ of an irreducible atoroidal handlebody-knot $(S^3, HK)$ by Koda-Ozawa \cite{28}. Such annuli are classified into four types by \cite{28} Corollary 3.18, and the classification can be described in terms of the boundary of annuli \cite{28} the proof of Theorem 3.3: Let $A$ be an essential annulus in $E(HK)$. $A$ is of type 2 if exactly one component of $\partial A$ bounds a disk in $HK$. $A$ is of type 3 if no component of $\partial A$ bounds a disk in $HK$ and there exists an essential disk in $HK$ disjoint from $\partial A$; a type 3 annulus can be further classified into two subtypes: $A$ is of type 3-2 if the components of $\partial A$ are parallel, and is of type 3-3 otherwise. $A$ is of type 4 if $\partial A$ does not bound disks in $HK$ and no essential disks in $HK$ disjoint from $\partial A$.

Recall from Koda \cite{29} an unknotting annulus $A$ of $(S^3, HK)$ is an annulus $A \subset E(HK)$ such that $(S^3, HK_A)$ is a trivial handlebody-knot, where $HK_A := HK \cup \mathcal{N}(A)$ and $\mathcal{N}(A)$ is a regular neighborhood of $A$ in $E(HK)$; in other words, $\partial HK_A$ induces a genus two Heegaard splitting of $S^3$. The existence of unknotting annuli imposes topological constraints on $(S^3, HK)$; such constraints are investigated in Section 3, and the results therein are summarized in the following.

**Theorem 1.1.** If $A$ is an unknotting annulus of an irreducible handlebody-knot $(S^3, HK)$, then $A$ is essential, and is either of type 2 or of type 3-3; furthermore, $E(HK)$ is atoroidal.

Our next result concerns the symmetry group of a handlebody-knots $(S^3, HK)$ admitting a unique unknotting annulus $A$; that is, any other unknotting annulus of the same type is isotopic to $A$ in $E(HK)$. Examples of handlebody knots admitting a unique unknotting annulus include infinite families of handlebody-knots in Motto \cite{33}, Lee-Lee \cite{31}, and Koda \cite{29} Example 3.

**Theorem 1.2.** An irreducible handlebody-knot admitting a unique unknotting annulus of type 2 has a trivial symmetry group.

**Corollary 1.3.** An irreducible handlebody-knot admitting a unique unknotting annulus of type 2 is chiral.

Detecting chirality of handlebody knots is in general a challenging task; several methods are employed (e.g. Motte \cite{33}, Lee-Iwakiri \cite{21}, Ishii-Iwakiri-Jang-Oshiro \cite{22}, Ishii-Kishimoto-Ozawa \cite{24}) to determine the chirality of handlebody knots in the Ishii-Kishimoto-Moriuchi-Suzuki knot table \cite{23}. Theorem 1.1 can be obtained by standard 3-manifold techniques, and classification of knot tunnels of a trefoil knot, whereas Theorem 1.2 relies on results on knot-tunnels-preserving homeomorphisms by Cho-McCullough \cite{7}, and finiteness theorems on symmetry group of spatial graphs and handlebody-knots in Cho-Koda \cite{8} and Funayoshi-Koda \cite{11}. We also make use of mapping class groups of surfaces (see Farb-Margalit \cite{10}, Özgür-Şahin \cite{34}, and homotopy types of embedding spaces of subpolyhedra in a surface as discussed in Yagasaki \cite{41}, \cite{42}.

## 2. Preliminaries

In this section we fix the convention, and recall some results in \cite{8}, \cite{7}, \cite{29}, \cite{11}, \cite{42} needed in subsequent sections.

---

\footnote{In the genus two case, non-triviality and atoroidality imply irreducibility.}
Throughout the paper, we work in the piecewise linear category, given subpolyhedra $X_1, \ldots, X_n$ of a manifold $M$, the space of self-homeomorphisms of $M$ preserving $X_i$, $i = 1, \ldots, n$, setwise (resp. pointwise) is denoted by
\[ \text{Homeo}(M, X_1, \ldots, X_n) \quad (\text{resp. Homeo}(M, \text{rel } X_1, \ldots, X_n)) , \]
and the mapping class group of $(M, X_1, \ldots, X_n)$ is defined as
\[ \mathcal{MCG}(M, X_1, \ldots, X_n) := \pi_0(\text{Homeo}(M, X_1, \ldots, X_n)) \]
(\text{resp. } \mathcal{MCG}(M, \text{rel } X_1, \ldots, X_n) := \pi_0(\text{Homeo}(M, \text{rel } X_1, \ldots, X_n)) ).

The “+” subscript is added when considering the subspace or the subgroup consisting of orientation-preserving homeomorphisms
\[ \text{Homeo}_+(M, X_1, \ldots, X_n) \quad (\text{resp. Homeo}_+(M, \text{rel } X_1, \ldots, X_n)) \]
\[ \mathcal{MCG}_+(M, X_1, \ldots, X_n) \quad (\text{resp. } \mathcal{MCG}_+(M, \text{rel } X_1, \ldots, X_n)) . \]

If $f$ is a self-homeomorphism of $(M, X_1, \ldots, X_n)$, we denote by $[f]$ the mapping class it represents.

Given a subpolyhedron $X$ of $M$, $\hat{X}$ denotes the interior of $X$, and $\mathcal{R}(X; M)$ a regular neighborhood of $X$ in $M$, or simply $\mathcal{R}(X)$ when $M$ is clear from the context. The exterior $E(X)$ of $X$ in $M$ is the the complement $\mathcal{R}(X; M)$ if $X$ has codimension greater than zero, and is the closure of $M - X$ otherwise. Submanifolds of a manifold $M$ are understood to be proper except in some obvious case where submanifolds are in $\partial M$, and intersection of two submanifolds are assumed to be transverse. A surface in a three-manifold is essential if it is non-boundary parallel, incompressible, and $\partial$-incompressible.

**Lemma 2.1** ([2], Corollary 11.3, Theorem 16.2]). Let $\Gamma$ be the union of a tunnel number one knot $K$ or link $L$ and a tunnel $\tau$, and $f \in \text{Homeo}(S^3, \Gamma)$. Then
- If $f \in \text{Homeo}_+(S^3, \Gamma)$ swaps two arcs of $K$, then $K$ is trivial or $\tau$ is the upper or lower tunnel of a two-bridge knot $K$.
- If $f \notin \text{Homeo}_+(S^3, \Gamma)$, then $\tau$ is the tunnel of either a trivial knot $K$, or a trivial link or Hopf link $L$.

Assume $X$ is a subpolyhedron of a surface $\Sigma$ of genus two. We denote by $\text{Emb}(X, \Sigma)_0$ (resp. $\text{Homeo}(X)_0$) the component of the space of embeddings of $X$ in $M$ (resp. self-homeomorphisms of $X$) containing the inclusion (resp. identity). Let
\[ r : \text{Homeo}_+(\Sigma, X) \to \text{Homeo}_+(X) \quad \text{and} \quad r : \text{Homeo}_+(\Sigma) \to \text{Emb}(X, \Sigma) \]
be the restriction maps, which are Serre fibrations [13], [14].

**Lemma 2.2** ([2], [11]). If $\pi_1(X)$ is non-cyclic, then $\text{Emb}(X, \Sigma)_0$ is contractible. If $X$ is a finite union of circles, then the natural homomorphism
\[ \pi_0(r^{-1}(\text{Homeo}_+(X)_0)) \to \pi_0(r^{-1}(\text{Emb}(X, \Sigma)_0)) \]
is an isomorphism.

**Proof.** The first statement is a special case in [22, Theorem 1.2]. To see the second statement, we note that by [22, the proof of Theorem 1.2] and the Serre fibration [11]
\[ \text{Emb}(S^1, \Sigma)_0 \to \text{Emb}(X, \Sigma)_0 \to \text{Emb}(X', \Sigma)_0 \]
where $X = S^1 \cup X'$, the natural map
\[ \text{Homeo}_+(X)_0 \to \text{Emb}(X, \Sigma)_0 \]
is a homotopy equivalence. The assertion then follows from the map of Serre fibrations.
A meridian system $\mathcal{D}$ of a handlebody $H$ is a set of disjoint, non-parallel, meridian disks $\{D_1, \ldots, D_n\}$ in $H$ such that the exterior of $\bigcup_{i=1}^n D_i$ consists of only 3-balls. Every meridian system determines a trivalent spine of $H$. In particular, given a handlebody-knot $(S^3, HK)$ and a meridian system $\{D_1, D_2, D_3\}$ of HK, the induced spine is either a spatial $\theta$-curve or handcuff graph. Given a spatial graph $(S^3, \Gamma)$, TSG$(S^3, \Gamma)$ denotes the topological symmetry group [38], which is the image of $MCG(S^3, \Gamma)$ in $MCG(\Gamma)$; note that if $\Gamma$ is a handcuff graph, $MCG(\Gamma)$ is the dihedral group $D_4$.

The next two lemmas follow from the Alexander trick and [13, 15 Section 2], [16, Theorem 1] (see also [8, Section 2], [29, Section 2]).

**Lemma 2.3.** Given a handlebody-knot $(S^3, HK)$, let $\{D_1, D_2, D_3\}$ be a meridian system of HK, and $\Gamma$ the induced spatial graph. Then

1. the natural homomorphism $MCG(S^3, HK, D_1 \cup D_2 \cup D_3) \rightarrow MCG(S^3, HK)$ is injective;
2. the natural homomorphism given by the Alexander trick $MCG(S^3, HK, D_1 \cup D_2 \cup D_3) \xrightarrow{\sim} MCG(S^3, \Gamma)$ is an isomorphism.

**Lemma 2.4.** Given a handlebody-knot $(S^3, HK)$ and an essential annulus $A$ in $E(HK)$, the natural homomorphism $MCG(S^3, HK, A) \rightarrow MCG(S^3, HK)$ is injective.

The next lemma is a direct consequence of [29, Theorems 2.5 and 3.2].

**Lemma 2.5.** If $\Gamma$ is a spine of an irreducible atoroidal handlebody-knot HK, then $MCG(S^3, \Gamma) \simeq$ TSG$(S^3, \Gamma) < D_4$.

The next theorem, strengthening the finiteness result in [29, Theorem 4.4] follows from [11].

**Theorem 2.6 (11).** The symmetry group of an atoroidal cylindrical handlebody-knot is finite.

**Remark 2.1.** [11] the proof of Theorem 4.3] shows that every element in the symmetry group of an atoroidal cylindrical handlebody-knot is of finite order. Theorem 2.6 then follows from the injections [8], [15] $MCG(S^3, HK) \rightarrow MCG(HK) \rightarrow MCG(\partial HK)$ and $MCG(\partial HK)$ being virtually torsion free [2].
We now review some properties of the mapping class group of a four-times-punctured sphere \( S_{0,4} \). Up to change of basis, the mapping class group \( \text{MCG}(S_{0,4}) \) (resp. \( \text{MCG}^+(S_{0,4}) \)) is canonically isomorphic to
\[
PGL(2, \mathbb{Z}) \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2) \quad \text{(resp. PSL(2, \mathbb{Z}) \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2))} \quad (2.1)
\]
via the homomorphisms
\[
PGL(2, \mathbb{Z}) \to \text{MCG}(S_{0,4}) \quad \text{(resp. PSL(2, \mathbb{Z}) \to \text{MCG}^+(S_{0,4}))}
\]
given by linear homomorphisms [10, Section 2.2.5], where the factor \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) in (2.1) corresponds to the four hyperelliptic involutions.

On the other hand, it is known [9], [34] that every element of order 2 in \( PGL(2, \mathbb{Z}) \) is conjugate to one of the matrices:
\[
\begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}, 
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}, 
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} \quad (2.2)
\]
The first two matrices correspond to reflection across the plane containing all punctures and two punctures, respectively, while the third matrix corresponds to a rotation by \( \pi \) with two punctures on the axis of rotation. Similarly, we have every element of order 2 in \( PSL(2, \mathbb{Z}) \) is conjugate to
\[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\]
(see [10, Chapter 7]).

Denote by \( \sigma \) the composition
\[
\text{MCG}(S_{0,4}) \to S_4 \to \mathbb{Z}_2
\]
induced by even and odd permutations on punctures of \( S_{0,4} \), where \( S_4 \) is the permutation group on four punctures. An element \( x \in \text{MCG}(S_{0,4}) \) is even if \( \sigma(x) = 0 \), and is odd if \( \sigma(x) = 1 \). For instance, only the first matrix in (2.2) is even. We call a partition \( \mathcal{P} \) of punctures of \( S_{0,4} \) a grouping if each part of \( \mathcal{P} \) contains two punctures. There are three possible groupings. A grouping \( \mathcal{P} \) is realized by a loop \( l \subset S_{0,4} \) if \( l \) separates one part of \( \mathcal{P} \) from the other.

Lemma 2.7. Let \( [f] \in \text{MCG}(S_{0,4}) \) be even and of order 2.

1. If \( [f] \in \text{MCG}^+(S_{0,4}) \), then \( f \) can be isotoped such that, for any grouping \( \mathcal{P} \), \( f \) preserves a loop realizing it.
2. If \( [f] \notin \text{MCG}^+(S_{0,4}) \), then \( f \) can be isotoped such that only one grouping cannot be realized by loops preserved by \( f \).

Proof. If \( [f] \in \text{MCG}^+(S_{0,4}) \), then \( [f] \) is conjugate to
\[
(id, x) \in \text{PSL}(2, \mathbb{Z}) \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2),
\]
namely, a hyperelliptic involution, which preserves loops realizing all three groupings. If \( [f] \notin \text{MCG}^+(S_{0,4}) \), then \( [f] \) is conjugate to
\[
\left( \begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}, x \right) \in \text{PGL}(2, \mathbb{Z}) \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2),
\]
that is, the composition of a hyperelliptic involution and a reflection across the plane containing all punctures, which preserves loops realizing all but one grouping. \( \square \)

3. UNKNOTTING ANNULI

From now on we let \((S^4, \text{HK})\) be an irreducible handlebody-knot, and \( A \) an annulus in \( E(\text{HK}) \). Recall that \( \text{HK}_A = \text{HK} \cup \mathcal{N}(A; E(\text{HK})) \).

Lemma 3.1 (Essentiality). If \( \text{HK}_A \) is a handlebody, then \( A \) is essential in \( E(\text{HK}) \).
Proof. Observe first that $A$ cannot be boundary-parallel, for otherwise, the boundary $\partial HK_A$ would be non-connected. For the same reason, at least one component of $\partial A$ is non-separating and hence essential in $\partial HK$. Particularly, if $A$ is compressible, then $E(HK)$ is $\partial$-reducible, contradicting the irreducibility of $(S^3, HK)$.

Suppose $A$ is $\partial$-compressible, and $D$ is a compressing disk. Then the boundary of a regular neighborhood $N(A \cup D; E(HK))$ of $A \cup D$ consists of an annulus parallel to $A$ and a disk $D'$ in $E(HK)$. Since $(S^3, HK)$ is irreducible, $\partial D'$ bounds a disk in $\partial HK$. This implies $A$ is either boundary parallel or compressible, but neither can happen. □

Lemma 3.2 (Types of Annuli). If $HK_A$ is a handlebody, then $A$ is either of type 2 or of type 3-3.

Proof. Note first that, since $HK_A$ is a handlebody, at most one component of $\partial A$ bounds a disk in $HK$; also by Lemma 3.1, $A$ is essential.

Let $C \subset A$ be an essential circle in $A$, and denote by $A^\perp$ the annulus $C \times I \subset A \times I \simeq N(A; E(HK))$, where $I$ is an interval. $A^\perp$ is a non-separating annulus in the handlebody $HK_A$, so it is compressible, or $\partial$-compressible, or both in $HK_A$ by [5, Lemma 9], [17, Lemma 2.4].

If $A^\perp$ is compressible in $HK_A$, exactly one component of $\partial A$ bounds a meridian disk in $HK$ since $A$ is essential. This implies $A$ is of type 2. If $A^\perp$ is incompressible, then it is necessarily $\partial$-compressible. Compressing $A^\perp$ with a compressing disk $D'$, we obtain a non-separating disk $D$ in $HK_A$ disjoint from $A$. In particular, $A^\perp$ can be obtained by the boundary of a regular neighborhood of the union of $D$ and an arc $\alpha \subset \partial HK_A$ transverse to $D'$ and connecting to the same side of $D$.

Since $A^\perp$ is incompressible in $HK_A$, it is also incompressible in the solid torus $V = HK_A - \mathfrak{N}(D; HK_A)$, and therefore $A^\perp$ separates $V$ into two solid tori $V_+, V_-$ with two disk components $D_\pm = \partial \mathfrak{N}(D) - \partial HK$ in $V_\pm$ (Fig. 3.1) and the core of $A^\perp$ some multiples of longitudes of $V_+$ and $V_-$, respectively. This implies $A$ is of type 3-3. □

Figure 3.1. $D_\pm, \alpha$ and $A^\perp$.

Theorem 3.3. If $A$ is unknotting, then $(S^3, HK)$ is atoroidal.

Proof. We prove by contradiction. Suppose $T$ is an incompressible torus in $E(HK)$ such that $\# A \cap T$ is minimized. By Lemma 3.1, $A$ is essential, so every component of $A \cap T$ is essential in $A$ and $T$. Since $HK_A$ is atoroidal and $T \cap HK = \emptyset$, $\# T \cap A$ is a positive even number. And by Proposition 3.2, $A$ is either of type 2 or of type 3-3.

Case 1: $A$ is of type 2. Let $B \subset T$ be an annulus with $B \cap A = \partial B$, and $B' \subset A$ be the annulus cut off by $\partial B$ (Fig. 3.2a). Push $B \cup B'$ slightly away from HK, we obtain a torus $T_B \subset E(HK_A)$. Let $V$ be the closure of the component of $S^3 - T_B$ with $V \cap HK = \emptyset$. Then $HK_A$ being atoroidal implies that $V$ is a solid torus. Since
A is of type 2, \( \partial B' \) bounds a disk in \( S^3 - \hat{V} \). Therefore the core of \( B' \) is a longitude of \( V \), and hence

\[ \pi_1(B') \to \pi_1(V) \]

is an isomorphism. In particular, \( B' \) is parallel to \( B \) through \( V \), so one can isotope \( A \) to decrease \( \#A \cap T \), a contradiction.

**Case 2:** \( A \) is of type 3-3. Let \( U \supset HK \) be the solid torus bounded by \( T \).

**Subcase 2i:** \( T \cap A \subset U \) is meridional. There exists an annulus \( B \subset A \) with \( B \cap T = \partial B \) and \( B \not\subset U \). Let \( B' \subset T \) be an annulus cut off by \( \partial B \) (Fig. 3.2b).

Let \( T_B \) be the union \( B \cup B' \), and \( V \) be the component of \( S^3 - T_B \) not containing \( HK \). Since \( T_B \) has less intersection with \( A \) than \( T \) does, \( V \) is necessarily a solid torus. On the other hand, because \( T \cap A \) are meridional in \( \partial U \), the core of \( B' \) is a longitude of \( V \), and hence \( B', B \) are parallel through \( V \). Isotoping \( T \) through \( V \) gives a contradiction to the minimality of \( \#A \cap T \).

**Subcase 2ii:** \( T \cap A \subset U \) is non-meridional. We first prove that \( \#T \cap A \) is at most 2. If \( \#T \cap A > 2 \), then there is an annulus \( B \subset A \) with \( B \cap T = \partial B \) and \( B \not\subset U \). Let \( T_B \) be the union \( B \cup B' \), and \( V \) be the component of \( S^3 - T_B \) not containing \( HK \). Since \( T_B \) has less intersection with \( A \) than \( T \) does, \( V \) is necessarily a solid torus. On the other hand, because \( T \cap A \) are meridional in \( \partial U \), the core of \( B' \) is a longitude of \( V \), and hence \( B', B \) are parallel through \( V \). Isotoping \( T \) through \( V \) gives a contradiction to the minimality of \( \#A \cap T \).
induces an isomorphism on \( \pi_1 \), the core of \( U \), and hence the core of \( A_m \), is a torus knot by the classification of Seifert fiber structure of \( S^3 \). Therefore \( \text{HK}_A \) can be identified with \( \mathcal{N}(A_m; E(U)) \cup \mathcal{N}(\gamma; U) \), where \( \gamma \) is an arc in \( U \) with \( \partial \gamma \subset \partial A_m \). Since \( \text{HK}_A \) is trivial, \( \gamma \) can be viewed as a tunnel of a torus knot (the core of \( A_m \)).

By the classification of tunnels of a torus knot \( [3], [32] \) (see also \( [7] \)), there is an isomorphism \( f \in \text{Homeo}(\mathcal{N}(A_m; E(U))) \) sending \( f(\gamma) \) to an arc \( t \) isotopic to an essential arc in \( A_1 \) (or \( A_2 \)) in \( \text{Homeo}(U, \text{rel} \ U) \). One can further isotope \( f \) such that it preserves \( A_1 \) with \( f \) still sending \( \gamma \) to \( t \). Thus \( (S^3, \text{HK}) \) can be identified with a regular neighborhood of the union of \( \partial A_m = \partial A_i \) and an essential arc of \( A_i, i = 1 \) or 2, in \( S^3 \), which is reducible, however.

\[ \square \]

4. Symmetry groups

**Definition 4.1.** \( (S^3, \text{HK}) \) admits a unique unknotted annulus \( A \) if given another unknotted annulus \( A' \) of the same type, there exists a path \( F_t \in \text{Homeo}(S^3, \text{HK}) \) such that \( F_i = \text{id} \) and \( F_1(A') = A \).

If \( (S^3, \text{HK}) \) admits a unique unknotted annulus \( A \), there are isomorphisms

\[ \mathcal{MCG}(S^3, \text{HK}, \mathcal{N}(A)) \simeq \mathcal{MCG}(S^3, \text{HK}, A) \simeq \mathcal{MCG}(S^3, \text{HK}) \quad (4.1) \]

by Lemma 2.4. Before restating the main result, we recall that essential annuli of type 2 can be divided into two subtypes \([11]\).

**Definition 4.2.** Let \( \Gamma \) be a type 2 unknotted annulus of \( (S^3, \text{HK}) \), and \( D \subset \text{HK} \) be a meridian disk bounded by a component of \( \partial A \). Then \( \Gamma \) is of type 2-1 if \( D \) is separating, and is of type 2-2 otherwise.

**Theorem 4.1.** If \( (S^3, \text{HK}) \) admits a unique unknotted annulus of type 2, then \( \mathcal{MCG}(S^3, \text{HK}) = 1 \).

**Proof.** Case 1: \( \text{A is of type 2-1} \). In this case, \( A \) determines a meridian system \( \{D_1, D_2, D_3\} \), where \( D_3 \) is a separating disk bounding a component \( \alpha_3 \) of \( \partial A \), and \( D_1, D_2 \) are meridian disks of the two tori \( \text{HK} - \mathcal{N}(D_3; \text{HK}) \), respectively, one of which, say \( D_1 \), intersecting with the other component \( \alpha_1 \) of \( \partial A \) at a single point (Fig. 1.1a). Since \( A \) is unique, \( D_1, D_2, D_3 \) are unique in \( \text{HK} \) up to isotopy, and therefore the injection

\[ \mathcal{MCG}(S^3, \text{HK}, D_1 \cup D_2 \cup D_3) \rightarrow \mathcal{MCG}(S^3, \text{HK}) \quad (4.2) \]

is an isomorphism. On the other hand, by Lemma 2.3 if \( \Gamma \) is a spatial graph associated to the meridian system \( \{D_1, D_2, D_3\} \) of \( \text{HK} \), then there are isomorphisms

\[ \text{TSG}(S^3, \Gamma) \simeq \mathcal{MCG}(S^3, \Gamma) \simeq \mathcal{MCG}(S^3, \text{HK}, D_1 \cup D_2 \cup D_3) \quad (4.3) \]

4.2 and 4.3 together imply \( \mathcal{MCG}(S^3, \text{HK}) \prec \mathcal{D}_4 \).

**Claim:** \( \mathcal{MCG}_+(S^3, \text{HK}) \) is trivial.

Given \( f \in \text{Homeo}_+(S^3, \text{HK}) \), it may be assumed that \( f \in \text{Homeo}_+(S^3, \text{HK}, D_1 \cup D_2 \cup D_3) \) by 4.2, and \( f(A) = A \) by the uniqueness of \( A \). \( f \) does not permute \( \{D_1, D_2, D_3\} \), and cannot reverse two sides of \( D_1 \) and \( D_3 \) since \( f \) is orientation-preserving. We may hence assume that \( f \) restricts to the identity on \( D_1 \) and \( \alpha_1 \).

On the other hand, \( f \) might reverse two sides of \( D_2 \). By 4.2, if \( f \) fixes two sides of \( D_2 \), then \( f \) is the identity in \( \mathcal{MCG}_+(S^3, \text{HK}) \), and if \( f \) reverses two sides of \( D_2 \), \( f \) is an element of order 2 in \( \mathcal{MCG}(S^3, \text{HK}) \).

Now suppose \( [f] \) is non-trivial in \( \mathcal{MCG}_+(S^3, \text{HK}) \). By Theorem 3.3 and \([8\), Lemma 2.3], \( \mathcal{MCG}_+(E(\text{HK}), \partial E(\text{HK})) \) is trivial, and therefore the composition

\[ \mathcal{MCG}_+(S^3, \text{HK}) \rightarrow \mathcal{MCG}_+(\text{HK}) \rightarrow \mathcal{MCG}_+(\partial \text{HK}) \quad (4.4) \]
is injective. In particular, \([f|_{\partial HK}]\) is an element of order 2 in \(\text{MCG}_+((\partial HK))\). On the other hand, there is a Serre fibration

\[
\text{Homeo}_+(\partial HK, \text{rel } \alpha_1 \cup \partial D_1) \to r^{-1}(\text{Emb}(\alpha_1 \cup \partial D_1, \partial HK)_0)
\]

derived from \([4, \text{Theorem 1.1}]\), where \(r : \text{Homeo}(\partial HK) \to \text{Emb}(\alpha_1 \cup \partial D_1, \partial HK)\) is the restriction map. Since \(E\) is injective. In particular, \([f|_{\partial HK}]\) is injective, and hence \([f|_{\partial HK}]\) is an element of order 2 in \(\text{MCG}_+((\partial HK), \text{rel } \alpha_1 \cup \partial D_1)\),

which however, is isomorphic to the torsion free group

\[
\text{MCG}_+((\partial HK, \text{rel } \alpha_1 \cup \partial D_1) \to \text{MCG}_+((\partial HK))
\]

is injective, and hence \([f|_{\partial HK}]\) is an element of order 2 in \(\text{MCG}_+((\partial HK, \text{rel } \alpha_1 \cup \partial D_1)\),

This gives a contradiction, and therefore \(\text{MCG}_+(S^3, HK) = 1\).

**Claim:** \(\text{MCG}(S^3, HK) = 1\).

Suppose \(\text{MCG}(S^3, HK)\) is non-trivial, and \(f \in \text{Homeo}(S^3, HK)\) is an orientation-reversing homeomorphism. It may be assumed that \(f(\mathfrak{N}(A)) = \mathfrak{N}(A)\) by the uniqueness of \(A\). Denote by \(D_{\pm}^3\) two non-parallel meridian disks in \(HK_A\) that are parallel to \(D_3\) in \(HK\). Then \(\{D_2, D_{+}^3, D_{-}^3\}\) is a meridian disk system of \(HK_A\); let \(\Gamma_A\) be the associated handcuff graph.

Since \(f\) preserves \(D_3\), one can isotope \(f\) such that it preserves \(D_{\pm}^3\), and hence \(f\) represents an element in

\[
\text{MCG}(S^3, HK_A, D_2, D_{+}^3, D_{-}^3) \simeq \text{MCG}(S^3, \Gamma_A).
\]

Notice that \(f\) does not permute \(\{D_2, D_{+}^3, D_{-}^3\}\) since it keeps the two sides of \(D_3\).

Let \(L\) be constituent link of \(\Gamma_A\), and \(\gamma\) the connecting arc. Then \(\gamma\) is an unknotting tunnel of \(L\), and \(f\) induces an orientation-reversing homeomorphism preserving \(L, \gamma\). By Lemma 2.1, \(\gamma\) is the tunnel of either a trivial link or a Hopf link (Fig. 4.2a). In the former, \((S^3, HK)\) is trivial, contradicting the assumption, whereas in the latter, \((S^3, HK)\) is the handlebody-knot 41 in \([23, \text{Table 1}]\) (Fig. 4.2b), which has two non-isotopic unknotting annuli of type 2. Therefore \(\text{MCG}(S^3, HK) = 1\).

**Case 2:** \(A\) is of type 2-2. As with the previous case, we first prove

**Claim:** \(\text{MCG}_+(S^3, HK) = 1\).

Given an element \(f \in \text{Homeo}_+(S^3, HK)\), it may be assumed \(f(A) = A\). Denote by \(\alpha_1, \alpha_2\) the components of \(\partial A\) and fix an orientation of \(\alpha_1, \alpha_2\). \(f\) cannot permute \(\{\alpha_1, \alpha_2\}\), but it might reverse their orientation (simultaneously) (Fig. 4.1b).
Theorem 2.6 [f] is a torsion element in $\mathcal{MCG}_+ (\mathbb{S}^3, HK)$, so $[f|_{\partial HK}]$ is also a torsion element in $\mathcal{MCG}_+ (\partial HK)$ via the injection (4.4).

Subcase 1: $f$ preserves the orientation of $\alpha_1, \alpha_2$. Since the restriction of $f$

$f|_{\partial HK} \in r^{-1}(\mathcal{E}mb (\alpha_1 \cup \alpha_2, \partial HK)_0) \subset \text{Homeo}_+ (\partial HK)$

induces a torsion element in

$\pi_0 (r^{-1}(\mathcal{E}mb (\alpha_1 \cup \alpha_2, \partial HK)_0))$

$[f|_{\partial HK}]$ is a torsion element in $\pi_0 (\text{Homeo}_+ (\partial HK, \alpha_1, \alpha_2))$ by Lemma 2.2.

Now, consider the cutting homomorphism cut [10] Proposition 3.20:

$0 \to \langle t_1, t_2 \rangle \to \mathcal{MCG}_+ (\partial HK, \alpha_1, \alpha_2) \cong \mathcal{MCG}_+ (\partial HK - \alpha_1 \cup \alpha_2)$ (4.5)

where the Dehn twists $t_i$ about $\alpha_i$, $i = 1, 2$ generate its kernel.

Since $[f|_{\partial HK - \alpha_1 \cup \alpha_2}]$ is a torsion element in $\mathcal{MCG}_+ (\partial HK - \alpha_1 \cup \alpha_2)$, and every non-trivial torsion element in $\mathcal{MCG}_+ (\partial HK - \alpha_1 \cup \alpha_2)$, the positive mapping class group of a four-times-punctured sphere, permutes some punctures, $[f|_{\partial HK - \alpha_1 \cup \alpha_2}]$ is the trivial element in $\mathcal{MCG}_+ (\partial HK - \alpha_1 \cup \alpha_2)$. Thus by (4.5), $[f|_{\partial HK}]$ is trivial in $\mathcal{MCG}_+ (\partial HK, \alpha_1, \alpha_2)$ as well. Thus $f$ can be isotoped in $\text{Homeo}_+ (\mathbb{S}^3, HK, \alpha_1, \alpha_2)$ such that $f|_{\partial HK} = \text{id}$. The fact [15], [8, Lemma 2.3]

$\mathcal{MCG}_+ (HK, \text{rel} HK) = 1 = \mathcal{MCG}_+ (E(HK), \text{rel} E(HK))$

implies that $f$ can be further isotoped to id in $\text{Homeo}_+ (\mathbb{S}^3, \text{rel} \partial HK)$, so $[f] \in \mathcal{MCG}_+ (\mathbb{S}^3, HK)$ is trivial.

Subcase 2: $f$ reverses the orientation of $\alpha_1, \alpha_2$. The preceding argument implies that $[f]$ is an element of order 2 in $\mathcal{MCG}_+ (\mathbb{S}^3, HK, \alpha_1, \alpha_2)$, and hence

$[f|_{\partial HK - \alpha_1 \cup \alpha_2}] \in \mathcal{MCG}_+ (\partial HK - \alpha_1 \cup \alpha_2)$

is an even element of order two. By Lemma 2.7 one can isotope $f|_{\partial HK - \alpha_1 \cup \alpha_2}$ such that it preserves a loop $\alpha_3$ separating punctures induced by $\alpha_1$ and punctures induced by $\alpha_2$. Following from the isotopy extension theorem [19] Theorem 2), it may be assumed that $f|_{\partial HK}$ preserves $\alpha_3 \subset \partial HK$, which separates $\alpha_1$ and $\alpha_2$ and therefore bounds a separating meridian disk $D_3$ as $\alpha_2$ is the boundary of a meridian disk $D_3$ (Fig. 4.1b).

Let $D_1$ be a meridian disk of HK disjoint from $D_3$ with $D_1 \cap \alpha_1$ a point. Then $f$ can be isotoped in $\text{Homeo}_+ (\mathbb{S}^3, HK, \alpha_1, \alpha_2)$ such that $f$ preserves $D_1$, $i = 1, 2, 3$. Let $D_1^\pm$ be non-parallel meridian disks of $HK_A$ that are parallel to $D_2$ in HK. Then since it may be assumed that $f(\mathcal{N}(A)) = \mathcal{N}(A)$ by the uniqueness of $A$, one can view $f$ as an element in $\text{Homeo}_+ (\mathbb{S}^3, HK_A, D_1^+, D_2^\pm \cup D_3^+ \cup D_3^-)$). Furthermore, $f(D_2^+) = D_2^-$ because $f$ reverses the orientation of $\alpha_2$.

Denote by $\Gamma_A$ the dual $\theta$-graph induced by $(HK_A, D_1^+ \cup D_2^+ \cup D_3)$, and by $K$ the constituent knot dual to $D_2^+$, and by $\gamma$ the arc dual to $D_3$. Then $\gamma$ is a tunnel.
of $K$. Via the isomorphism

$$\text{MCG}(S^3, \Gamma_A) \simeq \text{MCG}(S^3, HK_A, D_2^+ \cup D_2^- \cup D_3^+),$$

$f$ can be isotoped such that it preserves $K, \gamma$ and swaps two arcs of $K$. Applying Lemma 2.1, we see $K$ is a 2-bridge knot and $\gamma$ is the lower or upper tunnel, and by the classification of tunnels of a 2-bridge knot, $HK$ is a regular neighborhood of the union of $K$ and one of the four other tunnels [27], contradicting $(S^3, HK)$ is non-trivial.

Claim: $\text{MCG}(S^3, HK) = 1$.

Let $f \in \text{Homeo}(S^3, HK)$ be orientation-reversing. It may be assumed that $f(\pi(A)) = \pi(A)$, and $f^2$ is isotopic to id in $\text{Homeo}(S^3, HK, \pi(A))$ by Lemma 2.4 and (1.1).

Denote by $A^\pm$ the two annular components of $\pi(A) \cap \partial HK_A$. Then $[f|_{\partial HK_A}]$ and $[f|_{\partial HK_A - A^+ \cup A^-}]$ are of order two in $\text{MCG}(\partial HK_A, A^+ \cup A^-)$ and $\text{MCG}(\partial HK_A - A^+ \cup A^-)$, respectively, the latter being the mapping class group of a 4-times-punctured sphere. Since $f$ either swaps $A^+, A^-$ or preserves them, $[f|_{\partial HK_A - A^+ \cup A^-}]$ is even.

By Lemma 2.7, $f|_{\partial HK_A - A^+ \cup A^-}$ can be isotoped such that it preserves a loop $\alpha_0$ that separates one boundary component of $A^+$ (resp. of $A^-$) from the other. By the isotopy extension theorem [10] Theorem 2], $f|_{\partial HK_A}$ can be isotoped in $\text{Homeo}(\partial HK_A, A^+ \cup A^-)$, with $f|_{A^+ \cup A^-}$ fixed, such that $f(\alpha_0) = \alpha_0$.

Let $D_2^+ \subset HK_A$ be non-separating disks bounded by the core of $A^\pm$, and $D_0$ a non-separating disk bounded by $\alpha_0$. $\{D_2^+, D_2^-, D_0\}$ gives a meridian disk system of $HK_A$, and $f$ can be further isotoped in $\text{Homeo}(S^3, HK_A)$ such that it preserves $D_2^+ \cup D_2^-$ and $D_0$. Denote by $\Gamma_A$ the spatial graph associated to $(HK_A, D_2^+, D_2^-, D_0)$, and by $K$ the constituent knot dual to $D_2^+$, and by $\gamma$ the arc dual to $D_0$, which is a tunnel of $K$.

By Lemma 2.3, we have the isomorphism

$$\text{MCG}(S^3, \Gamma_A) \simeq \text{MCG}(S^3, HK_A, D_2^+ \cup D_2^- \cup D_3^+),$$

and hence $f$ induces an orientation-reversing homeomorphism preserving $K, \gamma$. However, this implies $\gamma$ is the tunnel of a trivial knot $K$ by Lemma 2.1. In particular, $D_2^+$ are primitive disks of the Heegaard splitting induced by $\partial HK_A \subset S^3$; therefore $A$ is $\partial$-compressible—or, in fact, $(S^3, HK)$ is trivial. Thus $\text{MCG}(S^3, HK) = 1$. □

REFERENCES

[1] E. Akbuz, A presentation for the automorphisms of the 3-sphere that preserve a genus two Heegaard splitting, Pacific J. Math. 236 (2008), 201–222.
[2] H. Bass, A. Lubotzky, Automorphisms of groups and of schemes of finite type, Israel J. Math. 44 (1983), 1–22.
[3] M. Boileau, M. Rost, H. Zieschang, On Heegaard decompositions of torus knot exteriors and related Seifert fibre spaces, Math. Ann. 279 (1988), 553–581.
[4] M. Boileau, B. Zimmermann, Symmetries of nonelliptic Montesinos links, Math. Ann. 277 (1987), 563–584.
[5] F. Bonahon, J.-P. Otal, Scindements de Heegaard des espaces lenticulaires, Ann. Sci. Éc. Norm. Supér. 16 (1983), 451–466.
[6] S. Cho, Homeomorphisms of the 3-sphere that preserve a heegaard splitting of genus two, Proc. Am. Math. Soc. 136 (2008), 1113–1123.
[7] S. Cho, D. McCullough, Cabling sequences of tunnels of torus knots, Algebr. Geom. Topol. 9 (2009), 1–20.
[8] S. Cho, Y. Koda, Topological symmetry groups and mapping class groups for spatial graphs, Michigan Math. J. 62 (2013), 131–142.
[9] G. Dresden, P. Panthi, A. Shrestha, J. Zhang, Finite subgroups of the extended modular group, Rocky Mountain J. Math. 4 (2019), 1123–1127.
[10] B. Farb, D. Margalit, A Primer on Mapping Class Groups, Princeton University Press, (2011).
