A one dimensional analysis of turbulence and its intermittence for the $d$-dimensional stochastic Burgers equation

A D Neate  A Truman
Department of Mathematics, University of Wales Swansea,
Singleton Park, Swansea, SA2 8PP, Wales, UK.

February 1, 2008

Abstract
The inviscid limit of the stochastic Burgers equation is discussed in terms of the level surfaces of the minimising Hamilton-Jacobi function, the classical mechanical caustic and the Maxwell set and their algebraic pre-images under the classical mechanical flow map. The problem is analysed in terms of a reduced (one dimensional) action function. We demonstrate that the geometry of the caustic, level surfaces and Maxwell set can change infinitely rapidly causing turbulent behaviour which is stochastic in nature. The intermittence of this turbulence is demonstrated in terms of the recurrence of two processes.

1 Introduction

Burgers equation has been used in studying turbulence and in modelling the large scale structure of the universe \cite{1 9 26}, as well as to obtain detailed asymptotics for stochastic Schrödinger and heat equations. \cite{10 11 28 29 30 31}. It has also played a part in Arnol’d’s work on caustics and Maslov’s works in semiclassical quantum mechanics \cite{3 4 19 20}.

We consider the stochastic viscous Burgers equation for the velocity field $v^\mu(x,t) \in \mathbb{R}^d$, where $x \in \mathbb{R}^d$ and $t > 0$,

$$\frac{\partial v^\mu}{\partial t} + (v^\mu \cdot \nabla) v^\mu = \frac{\mu^2}{2} \Delta v^\mu - \nabla V(x) - \epsilon \nabla k_t(x) \dot{W}_t,$$

with initial condition $v^\mu(x,0) = \nabla S_0(x) + O(\mu^2)$. Here $\dot{W}_t$ denotes white noise and $\mu^2$ is the coefficient of viscosity which we assume to be small.
We are interested in the advent of discontinuities in the inviscid limit of the Burgers fluid velocity \( v^0(x, t) \) where \( v^\mu(x, t) \to v^0(x, t) \) as \( \mu \to 0 \). Using the Hopf-Cole transformation \( v^\mu(x, t) = -\mu^2 \nabla \ln u^\mu(x, t) \), the Burgers equation becomes the Stratonovich heat equation,

\[
\frac{\partial u^\mu}{\partial t} = \frac{\mu^2}{2} \Delta u^\mu + \mu^{-2} V(x) u^\mu + \frac{\epsilon}{\mu^2} k_t(x) u^\mu \circ dW_t,
\]

with initial condition \( u^\mu(x, 0) = \exp \left( -\frac{S_0(x)}{\mu^2} \right) T_0(x) \), where the convergence factor \( T_0 \) is related to the initial Burgers fluid density.

Now let,

\[
A[X] := \frac{1}{2} \int_0^t X^2(s) \, ds - \int_0^t V(X(s)) \, ds - \epsilon \int_0^t k_s(X(s)) \, dW_s, \tag{1}
\]

and select a path \( X \) which minimises \( A[X] \). This requires,

\[
d\dot{X}(s) + \nabla V(X(s)) \, ds + \epsilon \nabla k_s(X(s)) \, dW_s = 0.
\]

We then define the stochastic action \( A(X(0), x, t) := \inf_{X(0)} A[X] : X(t) = x \). Setting,

\[
A(X(0), x, t) := S_0(X(0)) + A(X(0), x, t),
\]

and then minimising \( A \) over \( X(0) \), gives \( \dot{X}(0) = \nabla S_0(X(0)) \). Moreover, it follows that,

\[
S_t(x) := \inf_{X(0)} A(X(0), x, t),
\]

is the minimal solution of the Hamilton-Jacobi equation,

\[
dS_t + \left( \frac{1}{2} |\nabla S_t|^2 + V(x) \right) dt + \epsilon k_t(x) \, dW_t = 0, \quad S_{t=0}(x) = S_0(x). \tag{2}
\]

Following the work of Donsker, and Freidlin and Wentzell \cite{12}, \( -\mu^2 \ln u^\mu(x, t) \to S_t(x) \) as \( \mu \to 0 \). This gives the inviscid limit of the minimal entropy solution of Burgers equation as \( v^0(x, t) = \nabla S_t(x) \) \cite{5}.

**Definition 1.1.** The stochastic wavefront at time \( t \) is defined to be the set,

\[
W_t = \{ x : S_t(x) = 0 \}.
\]

For small \( \mu \) and fixed \( t \), \( u^\mu(x, t) \) switches continuously from being exponentially large to small as \( x \) crosses the wavefront \( W_t \). However, \( u^\mu \) and \( v^\mu \) can also switch discontinuously.
Define the classical flow map $\Phi_s : \mathbb{R}^d \to \mathbb{R}^d$ by,

$$d\dot{\Phi}_s + \nabla V(\Phi_s) d s + \epsilon \nabla k_s(\Phi_s) dW_s = 0, \quad \Phi_0 = \text{id}, \quad \dot{\Phi}_0 = \nabla S_0.$$ 

Since $X(t) = x$ it follows that $X(s) = \Phi_s(\Phi_t^{-1}(x))$, where the pre-image $x_0(x,t) = \Phi_t^{-1}(x)$ is not necessarily unique.

Given some regularity and boundedness, the global inverse function theorem gives a caustic time $T(\omega)$ such that for $0 < t < T(\omega)$, $\Phi_t$ is a random diffeomorphism; before the caustic time $v^0(x,t) = \dot{\Phi}_t(\Phi_t^{-1}(x))$ is the inviscid limit of a classical solution of the Burgers equation with probability one.

The method of characteristics suggests that discontinuities in $v^0(x,t)$ are associated with the non-uniqueness of the real pre-image $x_0(x,t)$. When this occurs, the classical flow map $\Phi_t$ focusses an infinitesimal volume of points $dx_0$ into a zero volume $dX(t)$.

**Definition 1.2.** The caustic at time $t$ is defined to be the set,

$$C_t = \left\{ x : \det \left( \frac{\partial X(t)}{\partial x_0} \right) = 0 \right\}.$$ 

Assume that $x$ has $n$ real pre-images,

$$\Phi_t^{-1}\{x\} = \{x_0(1)(x,t), x_0(2)(x,t), \ldots, x_0(n)(x,t)\},$$

where each $x_0(i)(x,t) \in \mathbb{R}^d$. Then the Feynman-Kac formula and Laplace’s method in infinite dimensions give for a non-degenerate critical point,

$$u^\mu(x,t) = \sum_{i=1}^{n} \theta_i \exp \left( -\frac{S^i_0(x,t)}{\mu^2} \right), \quad (3)$$

where $S^i_0(x,t) := S_0(x_0(i)(x,t)) + A(x_0(i)(x,t), x, t)$, and $\theta_i$ is an asymptotic series in $\mu^2$. An asymptotic series in $\mu^2$ can also be found for $v^\mu(x,t)$ [32]. Note that $S_t(x) = \min\{S^i_0(x,t) : i = 1, 2, \ldots, n\}$.

**Definition 1.3.** The Hamilton-Jacobi level surface is the set,

$$H^c_t = \left\{ x : S^i_0(x,t) = c \text{ for some } i \right\}.$$ 

The zero level surface $H^0_t$ includes the wavefront $W_t$.

As $\mu \to 0$, the dominant term in the expansion (3) comes from the minimising $x_0(i)(x,t)$ which we denote $\tilde{x}_0(x,t)$. Assuming $\tilde{x}_0(x,t)$ is unique, we obtain the inviscid limit of the Burgers fluid velocity $v^0(x,t) = \dot{\Phi}_t(\tilde{x}_0(x,t))$. 

3
If the minimising pre-image $\tilde{x}_0(x, t)$ suddenly changes value between two pre-images $x_0(i)(x, t)$ and $x_0(j)(x, t)$, a jump discontinuity will occur in $v^0(x, t)$, the inviscid limit of the Burgers fluid velocity. There are two distinct ways in which the minimiser can change; either two pre-images coalesce and disappear (become complex), or the minimiser switches between two pre-images at the same action value. The first of these occurs as $x$ crosses the caustic and when the minimiser disappears the caustic is said to be cool. The second occurs as $x$ crosses the Maxwell set and again, when the minimiser is involved the Maxwell set is said to be cool.

**Definition 1.4.** The Maxwell set is,

$$M_t = \{ x : \exists x_0, \tilde{x}_0 \in \mathbb{R}^d \text{ s.t. } x = \Phi_t(x_0) = \Phi_t(\tilde{x}_0), x_0 \neq \tilde{x}_0 \text{ and } A(x_0, x, t) = A(\tilde{x}_0, x, t) \}.$$  

**Example 1.5 (The generic Cusp).** Let $V(x, y) = 0$, $k_t(x, y) = 0$ and $S_0(x_0, y_0) = x_0^2y_0/2$. This initial condition leads to the *generic Cusp*, a semicubical parabolic caustic shown in Figure 1. The caustic $C_t$ (long dash) is given by,

$$x_t(x_0) = t^2x_0^3, \quad y_t(x_0) = \frac{3}{2}tx_0^2 - \frac{1}{t}.$$  

The zero level surface $H^0_t$ (solid line) is,

$$x_{(t,0)}(x_0) = \frac{x_0}{2} \left( 1 \pm \sqrt{1 - t^2x_0^2} \right), \quad y_{(t,0)}(x_0) = \frac{1}{2t} \left( t^2x_0^2 - 1 \pm \sqrt{1 - t^2x_0^2} \right)$$

and the Maxwell set $M_t$ (short dash) is $x = 0$ for $y > -1/t$.

![Figure 1: Cusp and Tricorn.](image-url)
Notation: Throughout this paper \( x, x_0, x_t, x_{(t,c)} \) etc will denote vectors, where normally \( x = \Phi_t(x_0) \). Cartesian coordinates of these will be indicated using a sub/superscript where relevant; thus \( x = (x_1, x_2, \ldots, x_d) \), \( x_0 = (x_0^1, x_0^2, \ldots, x_0^d) \) etc. The only exception will be in discussions of explicit examples in two and three dimensions when we will use \((x,y)\) and \((x_0,y_0)\) etc to denote the vectors.

2 Some background

We begin by summarising some results of Davies, Truman and Zhao (DTZ) \[6, 7\]. Following equation (1), let the stochastic action be defined,

\[
A(x_0, p_0, t) = \frac{1}{2} \int_0^t \dot{X}(s)^2 \, ds - \int_0^t \left[ V(X(s)) \, ds + \epsilon k_s(X(s)) \, dW_s \right],
\]

where \( X(s) = X(s, x_0, p_0) \in \mathbb{R}^d \) and,

\[
d\dot{X}(s) = -\nabla V(X(s)) \, ds - \epsilon \nabla k_s(X(s)) \, dW_s, \quad X(0) = x_0, \quad \dot{X}(0) = p_0,
\]

for \( s \in [0, t] \) with \( x_0, p_0 \in \mathbb{R}^d \). We assume \( X(s) \) is \( \mathcal{F}_s \) measurable and unique where \( \mathcal{F}_s \) is the sigma algebra generated by \( X(u) \) up to time \( s \). It follows from Kunita \[18\]:

**Lemma 2.1.** Assume \( S_0, V \in C^2 \) and \( k_t \in C^{2,0} \), \( \nabla V, \nabla k_t \) Lipschitz with Hessians \( \nabla^2 V, \nabla^2 k_t \) and all second derivatives with respect to space variables of \( V \) and \( k_t \) bounded. Then for \( p_0 \), possibly \( x_0 \) dependent,

\[
\frac{\partial A}{\partial x_0^\alpha}(x_0, p_0, t) = \dot{X}(t) \cdot \frac{\partial X(t)}{\partial x_0^\alpha} - \dot{X}_\alpha(0), \quad \alpha = 1, 2, \ldots, d.
\]

Methods of Kolokoltov et al \[17\] guarantee that for small \( t \) the map \( p_0 \mapsto X(t, x_0, p_0) \) is onto for all \( x_0 \).

Therefore, we can define \( A(x_0, x, t) := A(x_0, p_0(x_0, x, t), t) \) where \( p_0 = p_0(x_0, x, t) \) is the random minimiser (which we assume to be unique) of \( A(x_0, p_0, t) \) when \( X(t, x_0, p_0) = x \). Thus, the stochastic action corresponding to the initial momentum \( \nabla S_0(x_0) \) is \( \mathcal{A}(x_0, x, t) := A(x_0, x, t) + S_0(x_0) \).

**Theorem 2.2.** If \( \Phi_t \) is the stochastic flow map then,

\[
\Phi_t(x_0) = x \quad \iff \quad \frac{\partial}{\partial x_0^\alpha} [A(x_0, x, t)] = 0, \quad \alpha = 1, 2, \ldots, d.
\]
The Hamilton-Jacobi level surface $H_t^c$ is found by eliminating $x_0$ between,

$$\mathcal{A}(x_0, x, t) = c, \quad \frac{\partial \mathcal{A}}{\partial x_0^\alpha}(x_0, x, t) = 0 \quad \alpha = 1, 2, \ldots, d.$$  

Alternatively, if we eliminate $x$ to give an expression in $x_0$, we have the pre-level surface $\Phi_t^{-1}H_t^c$. Similarly the caustic $C_t$ (and pre-caustic $\Phi_t^{-1}C_t$) are obtained by eliminating $x_0$ (or $x$) between,

$$\det \left( \frac{\partial^2 \mathcal{A}}{\partial x_0^\alpha \partial x_0^\beta}(x_0, x, t) \right)_{\alpha, \beta=1,2,\ldots,d} = 0, \quad \frac{\partial \mathcal{A}}{\partial x_0^\alpha}(x_0, x, t) = 0 \quad \alpha = 1, 2, \ldots, d.$$  

These pre-images are calculated algebraically and are not necessarily the topological inverse images of the surfaces $C_t$ and $H_t^c$ under $\Phi_t$.

The caustic surface can be parameterised using its pre-image by applying the stochastic flow map (a pre-parameterisation). This allows us to control the domain of the pre-images and in particular restrict them to real values. If we can locally solve the equation of the pre-caustic to give, $x_0^1 = \lambda_1$, $x_0^2 = \lambda_2$, $\ldots$, $x_0^{d-1} = \lambda_{d-1}$, $x_0^d = x_0^{d,C}(\lambda_1, \lambda_2, \ldots, \lambda_{d-1})$, then the pre-parameterisation of the caustic is $x_t(\lambda) := \Phi_t(\lambda, x_0^{d,C}(\lambda))$ where $\lambda = (\lambda_1, \ldots, \lambda_{d-1}) \in \mathbb{R}^{d-1}$.

We next outline a one dimensional analysis first described by Reynolds, Truman and Williams (RTW) \[33\].

**Definition 2.3.** The $d$-dimensional flow map $\Phi_t$ is globally reducible if for any $x = (x_1, x_2, \ldots, x_d)$ and $x_0 = (x_0^1, x_0^2, \ldots, x_0^d)$ where $x = \Phi_t(x_0)$, it is possible to write each coordinate $x_0^\alpha$ as a function of the lower coordinates. That is,

$$x = \Phi_t(x_0) \Rightarrow x_0^\alpha = \Phi_t(x_0^1, x_0^2, \ldots, x_0^{\alpha-1}, t) \quad \text{for } \alpha = d, d-1, \ldots, 2.$$  

(4)

Therefore, using Theorem 2.2 the flow map is globally reducible if we can find a chain of $C^2$ functions $x_0^d, x_0^{d-1}, \ldots, x_0^2$ such that,

$$x_0^d = \Phi_t(x_0^1, x_0^2, \ldots, x_0^{d-1}, t) \iff \frac{\partial \mathcal{A}}{\partial x_0^d}(x_0, x, t) = 0,$$

$$x_0^{d-1} = \Phi_t(x_0^1, x_0^2, \ldots, x_0^{d-2}, t) \iff \frac{\partial \mathcal{A}}{\partial x_0^{d-1}}(x_0^1, x_0^2, \ldots, x_0^{d}(\ldots), x, t) = 0,$$

$$\vdots$$

$$x_0^2 = \Phi_t(x_0^1, t) \iff$$

(4)
\[
\frac{\partial A}{\partial x_0^1}(x_0^1, x_0^2, x_0^3(x, x_0^1, x_0^2, t), \ldots, x_0^d(\ldots), x, t) = 0.
\]

This requires that no roots are repeated to ensure that none of the second derivatives of \( A \) vanish. We assume also that there is a favoured ordering of coordinates and a corresponding decomposition of \( \Phi_t \) which allows the non-uniqueness to be reduced to the level of the \( x_0^1 \) coordinate. This assumption appears to be quite restrictive. However, local reducibility at \( x \) follows from the implicit function theorem and some mild assumptions on the derivatives of \( A \) (see [22]).

**Definition 2.4.** If \( \Phi_t \) is globally reducible then the reduced action function is the univariate function obtained by evaluating the action with equations [4],

\[
f(x, t)(x_0^1) := f(x_0^1, x, t) = A(x_0^1, x_0^2(x, x_0^1, t), x_0^d(\ldots), \ldots, x, t).
\]

**Lemma 2.5.** If \( \Phi_t \) is globally reducible, modulo the above assumptions,

\[
\left| \det \left( \frac{\partial^2 A}{(\partial x_0)^2}(x_0, x, t) \right) \right|_{x_0=(x_0^1,x_0^2(x,x_0^1,t),\ldots,x_0^d(\ldots))} = \prod_{\alpha=1}^d \left| \frac{\partial}{\partial x_0^\alpha} \right|^2 A(x_0^1, \ldots, x_0^\alpha, x_0^\alpha+1(\ldots), \ldots, x_0^d(\ldots), x, t) \bigg|_{\substack{x_0^\alpha=x_0^\alpha(\ldots) \quad \vdots \quad x_0^2=x_0^2(x,x_0^1,t) \quad \vdots \quad x_0^1=x_0^1}}
\]

where the first term is \( f''_{(x, t)}(x_0^1) \) and the last \( d - 1 \) terms are nonzero.

**Theorem 2.6.** Let the classical mechanical flow map \( \Phi_t \) be globally reducible. Then:

1. \( f'_{(x, t)}(x_0^1) = 0 \) and the equations [4] \( \iff x = \Phi_t(x_0) \),

2. \( f'_{(x, t)}(x_0^1) = f''_{(x, t)}(x_0^1) = 0 \) and the equations [4] \( \iff x = \Phi_t(x_0) \) is such that the number of real solutions \( x_0 \) changes.

**Corollary 2.7.** Let \( x_t(\lambda) \) denote the pre-parameterisation of the caustic with \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{d-1}) \in \mathbb{R}^{d-1} \). Then \( f'_{(x_t(\lambda), t)}(\lambda_1) = f''_{(x_t(\lambda), t)}(\lambda_1) = 0 \).

Consider an example where for \( x \) on one side of the caustic there are four real critical points on \( f_{(x, t)}(x_0^1) \) enumerated \( x_0^1(i)(x, t) \) for \( i = 1 \) to 4, and denote the minimising critical point by \( \bar{x}_0^1(x, t) \). Figure 2 illustrates how the minimiser jumps from (a) to (b) as \( x \) crosses the caustic if the point of
inflexion at \( x_1 = \lambda \) is the global minimiser. In this case the caustic at \( x_t(\lambda) \) is said to be cool.

We can also consider the Maxwell set in terms of the reduced action function. The Maxwell set corresponds to values of \( x \) for which \( f(x,t)(x_0^1) \) has two critical points at the same action value. If each of this pair of critical points also minimises the reduced action, then the inviscid limit of the solution to the Burgers equation will jump as shown in Figure 3 and the Maxwell set will be described as cool. Note that a Maxwell set can only exist in a region with three or more real pre-images if the reduced action is continuous.

We now use the reduced action function to find the Maxwell set. Instead of finding the Maxwell set directly it is easier to find the Maxwell-Klein set where Definition 1.4 is changed to allow \( x_0, \bar{x}_0 \in \mathbb{C}^d \).

**Theorem 2.8.** Let the reduced action function \( f(x,t)(x_0^1) \) be polynomial in all space variables. Then the set of possible discontinuities for a \( d \)-dimensional Burgers fluid velocity field in the inviscid limit is the double discriminant,

\[
D(t) := D_c \{ D_\lambda (J(x,t) - c) \} = 0,
\]

where \( D_x(p(x)) \) is the discriminant of the polynomial \( p \) with respect to \( x \).
Theorem 2.9. The double discriminant $D(t)$ factorises as,

$$D(t) = b_0^{2m-2} \cdot (C_t)^3 \cdot (B_t)^2$$

where $B_t = 0$ is the equation of the Maxwell-Klein set, $C_t = 0$ is the equation of the caustic and $b_0$ is some function of $t$ only; both $B_t$ and $C_t$ are algebraic in $x$ and $t$.

Theorem 2.9 gives the Maxwell-Klein set as an algebraic surface. It is then necessary to extract the Maxwell set from this by establishing which points have real pre-images. Alternatively, we can find the Maxwell set via a pre-parameterisation which allows us to restrict the pre-images to be real; for this we need the pre-Maxwell set which can also be found using the reduced action function.

Theorem 2.10. The pre-Maxwell set is given by the discriminant,

$$D_{x_0} \left( \frac{f_{\Phi_t(x_0),t}(x^1_0) - f_{\Phi_t(x_0),t}(\bar{x}^1_0)}{(x^1_0 - \bar{x}^1_0)^2} \right) = 0.$$  

3 Geometric results

We now summarise geometric results of DTZ and also [23]. Assume that $A(x_0, x, t)$ is $C^4$ in space variables with $\det \left( \frac{\partial^2 A}{\partial x_0^\alpha \partial x_0^\beta} \right) \neq 0$.

Definition 3.1. A curve $x = x(\gamma)$, $\gamma \in N(\gamma_0, \delta)$, is said to have a generalised cusp at $\gamma = \gamma_0$, $\gamma$ being an intrinsic variable such as arc length, if $x'(\gamma_0) = 0$.

Lemma 3.2. Let $\Phi_t$ denote the stochastic flow map and $\Phi_t^{-1} \Gamma_t$ and $\Gamma_t$ be some surfaces where if $x_0 \in \Phi_t^{-1} \Gamma_t$ then $x = \Phi_t(x_0) \in \Gamma_t$. Then, $\Phi_t$ is a differentiable map from $\Phi_t^{-1} \Gamma_t$ to $\Gamma_t$ with Frechet derivative,

$$(D\Phi_t)(x_0) = \left( -\frac{\partial^2 A}{\partial x_0^\alpha \partial x_0^\beta}(x_0, x, t) \right)^{-1} \left( \frac{\partial^2 A}{(\partial x_0^\alpha)^2}(x_0, x, t) \right).$$

Lemma 3.3 (2 dims). Let $x_0(s)$ be any two dimensional intrinsically parameterised curve, and define $x(s) = \Phi_t(x_0(s))$. Let $e_0$ denote the zero eigenvector of $\left( \frac{\partial^2 A}{(\partial x_0^\alpha)^2} \right)$ and assume that $\ker\left( \frac{\partial^2 A}{(\partial x_0^\alpha)^2} \right) = \langle e_0 \rangle$. Then, there is a generalised cusp on $x(s)$ when $s = \sigma$ if and only if either:

1. there is a generalised cusp on $x_0(s)$ when $s = \sigma$; or,

2. $x_0(\sigma) \in \Phi_t^{-1} C_t$ and the tangent $\frac{dx}{ds}(s)$ at $s = \sigma$ is parallel to $e_0$. 

9
Proposition 3.4. The normal to the pre-level surface is,
\[ n_H(x_0) = -\left( \frac{\partial^2 A}{(\partial x_0)^2} \right) \left( \frac{\partial^2 A}{\partial x_0 \partial x} \right)^{-1} \dot{X}(t, x_0, \nabla S_0(x_0)). \]

Proposition 3.5 (2 dims). Assume that at \( x_0 \in \Phi_t^{-1}H_t^c \) the normal to \( \Phi_t^{-1}H_t^c \) is non-zero, so that the pre-level surface does not have a generalised cusp at \( x_0 \). Then the level surface can only have a generalised cusp at \( \Phi_t(x_0) \) if \( \Phi_t(x_0) \in C_t \). Moreover, if \( x = \Phi_t(x_0) \in \Phi_t \{ \Phi_t^{-1}C_t \cap \Phi_t^{-1}H_t^c \} \), the level surface will have a generalised cusp.

Example 3.6 (The generic Cusp). Figure 4 shows how a point lying on three level surfaces has three distinct real pre-images each on a separate pre-level surface. A cusp only occurs on the corresponding level surface when the pre-level surface intersects the pre-caustic. Thus, provided the normal to the pre-level surface is well defined, a level surface can only have a cusp on the caustic, but it does not have to be cusped when it meets the caustic.

![Figure 4: (a) The pre-level surface (solid line) and pre-caustic (dashed), (b) the level surface (solid line) and caustic (dashed), both for the generic Cusp.](image)

Proposition 3.7. Assume that \( x \in M_t \) corresponds to exactly two pre-images on the pre-Maxwell set, \( x_0 \) and \( \tilde{x}_0 \). Then the normal to the pre-Maxwell set at \( x_0 \) is given by,
\[ n_M(x_0) = -\left( \frac{\partial^2 A}{(\partial x_0)^2}(x_0, x, t) \right) \left( \frac{\partial^2 A}{\partial x_0 \partial x}(x_0, x, t) \right)^{-1} \cdot \left( \dot{X}(t, x_0, \nabla S_0(x_0)) - \dot{X}(t, \tilde{x}_0, \nabla S_0(\tilde{x}_0)) \right). \]
**Proposition 3.8** (2 dims). Assume that at $x_0 \in \Phi_t^{-1}M_t$, $n_M(x_0) \neq 0$ so that the pre-Maxwell set does not have a generalised cusp. Then, the Maxwell set can only have a cusp at $\Phi_t(x_0)$ if $\Phi_t(x_0) \in C_t$. Moreover, if $x = \Phi_t(x_0) \in \Phi_t \left( \Phi_t^{-1}C_t \cap \Phi_t^{-1}M_t \right)$, the Maxwell set will have a generalised cusp at $x$.

**Corollary 3.9** (2 dims). Assuming that $n_H(x_0) \neq 0$ and $n_M(x_0) \neq 0$, then when the pre-Maxwell set intersects the pre-caustic, it touches a pre-level surface. Moreover, there is a cusp on the pre-Maxwell set which also intersects the same pre-level surface.

**Example 3.10** (The polynomial swallowtail). Let $V = 0$, $k_t = 0$ and $S_0(x_0,y_0) = x_0^5 + x_0^2y_0$. From Proposition 3.8, the cusps on the Maxwell set correspond to the intersections of the pre-curves (points 3 and 6 on Figure 5). But from Corollary 3.9, the cusps on the Maxwell set also correspond to the cusps on the pre-Maxwell set (points 2 and 5). Each cusp on the pre-Maxwell set lies on the same level surface as a point of intersection between the pre-caustic and pre-Maxwell set (i.e. 3 and 5, 2 and 6).

![Figure 5](image_url)

Figure 5: (a) The pre-Maxwell set (solid line) and pre-caustic (dashed), (b) the Maxwell set (solid) and caustic (dashed), for the polynomial swallowtail.

The Maxwell set terminates when it reaches the cusps on the caustic (points 1 and 4). These points satisfy the condition for a generalised cusp but, instead of appearing cusped, the curve stops and the parameterisation begins again in the sense that it maps back exactly onto itself. This follows because every point on the Maxwell set has at least two real pre-images, and so by pre-parameterising the Maxwell set, we effectively sweep it out twice. All of the pre-surfaces touch at the cusps on the caustic.

These results can be extended to three dimensions.
Theorem 3.11 (3 dims). Let

\[ x \in \text{Cusp}(H^c_t) = \{ x \in \Phi_t (\Phi_t^{-1} C_t \cap \Phi_t^{-1} H^c_t), x = \Phi_t(x_0), n_H(x_0) \neq 0 \} . \]

Then in three dimensions, with probability one, \( T_H(x) \) the tangent space to the level surface at \( x \) is at most one dimensional.

Theorem 3.12 (3 dims). Let,

\[ x \in \text{Cusp}(M_t) = \{ x \in \Phi_t (\Phi_t^{-1} C_t \cap \Phi_t^{-1} M_t), x = \Phi_t(x_0), n_M(x_0) \neq 0 \} . \]

Then in three dimensions, with probability one, \( T_M(x) \), the tangent space to the Maxwell set at \( x \) is at most one dimensional.

4 Swallowtail perestroikas

The geometry of a caustic or wavefront can suddenly change with singularities appearing and disappearing \[2\]. We consider the formation or collapse of a swallowtail using some earlier works of Cayley and Klein. Here we provide a summary of results from \[22\].

In Cayley’s work on plane algebraic curves, he describes the possible triple points of a curve \[25\] by considering the collapse of systems of double points which would lead to the existence of three tangents at a point. The four possibilities are shown in Figure 6. The systems will collapse to form a triple point with respectively, three real distinct tangents, three real tangents with two coincident, three real tangents all of which are coincident, or one real tangent and two complex tangents. We are interested in the interchange between the last two cases which Felix Klein investigated \[14, 16\].

![Figure 6: Cayley’s triple points.](image)

As indicated in Section 2, we often parameterise the caustic and level surfaces using a pre-parameterisation in which we restrict the parameter to be real to only consider points with real pre-images. This does not allow there to be any isolated double points on these curves. We now let the pre-parameter vary throughout the complex plane and consider when this maps to real...
points. We begin with a family of curves of the form \( x_t(\lambda) = (x_t^1(\lambda), x_t^2(\lambda)) \) where each \( x_t^\alpha(\lambda) \) is real analytic in \( \lambda \). If \( \text{Im}\{x_t(a+i\eta)\} = 0 \), it follows that \( x_t(a+i\eta) = x_t(a-i\eta) \), so this is a “complex double point” of the curve \( x_t(\lambda) \).

**Proposition 4.1.** If a swallowtail on the curve \( x_t(\lambda) \) collapses to a point where \( \lambda = \tilde{\lambda} \) when \( t = \tilde{t} \) then \( \frac{dx_t}{d\lambda}(\tilde{\lambda}) = \frac{d^2x_t}{d\lambda^2}(\tilde{\lambda}) = 0 \).

**Proposition 4.2.** Assume that there exists a neighbourhood of \( \tilde{\lambda} \in \mathbb{R} \) such that \( \frac{dx_t}{d\lambda}(\lambda) \neq 0 \) for \( t \in (\tilde{t} - \delta, \tilde{t}) \) where \( \delta > 0 \). If a complex double point joins the curve \( x_t(\lambda) \) at \( \lambda = \tilde{\lambda} \) when \( t = \tilde{t} \) then \( \frac{dx_t}{d\lambda}(\tilde{\lambda}) = \frac{d^2x_t}{d\lambda^2}(\tilde{\lambda}) = 0 \).

These give a necessary condition for the formation or destruction of a swallowtail, and for complex double points to join or leave the main curve.

**Definition 4.3.** A family of parameterised curves \( x_t(\lambda) \), (where \( \lambda \) is some intrinsic parameter) for which \( \frac{dx_t}{d\lambda}(\tilde{\lambda}) = \frac{d^2x_t}{d\lambda^2}(\tilde{\lambda}) = 0 \) is said to have a point of swallowtail perestroika when \( \lambda = \tilde{\lambda} \) and \( t = \tilde{t} \).

As with generalised cusps, we have not ruled out further degeneracy at these points. Moreover, as Cayley highlighted, these points are not cusped and are barely distinguishable from an ordinary point of the curve [23].

We now apply these ideas to the caustic where \( x_t(\lambda) \) will denote the pre-parameterisation. The “complex caustic” is found by allowing the parameter \( \lambda \) to vary over the complex plane. We are interested in the complex double points if they join the main caustic at some finite critical time \( \tilde{t} \) where \( \eta_t \to 0 \) as \( t \uparrow \tilde{t} \); at such a point a swallowtail can develop.

**Example 4.4.** Let \( V(x, y) = 0, k_t(x, y) \equiv 0 \) and \( S_0(x_0, y_0) = x_0^5 + x_0^6y_0 \). The caustic has no cusps for times \( t < \tilde{t} \) and two cusps for times \( t > \tilde{t} \) where, \( \tilde{t} = \frac{1}{2}\sqrt{2} \left( \frac{27}{4} \right)^{3/4} = 2.5854 \ldots \)

At the critical time \( \tilde{t} \) the caustic has a point of swallowtail perestroika as shown in Figure 7. There are five complex double points before the critical time and four afterwards. The remaining complex double points do not join the main caustic and so do not influence its behaviour for real times.

Figure 7: Caustic plotted at corresponding times.
Unsurprisingly, these phenomena are not restricted to caustics. There is an interplay between the level surfaces and the caustics, characterised by their pre-images.

**Proposition 4.5.** Assume that in two dimensions at $x_0 \in \Phi_t^{-1}H_c^t \cap \Phi_t^{-1}C_t$ the normal to the pre-level surface $n_H(x_0) \neq 0$ and the normal to the pre-caustic $n_C(x_0) \neq 0$ so that the pre-caustic is not cusped at $x_0$. Then $n_C(x_0)$ is parallel to $n_H(x_0)$ if and only if there is a generalised cusp on the caustic.

**Corollary 4.6.** Assume that in two dimensions at $x_0 \in \Phi_t^{-1}H_c^t \cap \Phi_t^{-1}C_t$ the normal to the pre-level surface $n_H(x_0) \neq 0$. Then at $\Phi_t(x_0)$ there is a point of swallowtail perestroika on the level surface $H_c^t$ if and only if there is a generalised cusp on the caustic $C_t$ at $\Phi_t(x_0)$.

**Example 4.7.** Let $V(x, y) = 0$, $k_t(x, y) = 0$, and $S_0(x_0, y_0) = x_0^5 + x_0^6y_0$. Consider the behaviour of the level surfaces through a point inside the caustic swallowtail at a fixed time as the point is moved through a cusp on the caustic. This is illustrated in Figure 8. Part (a) shows all five of the level surfaces through the point demonstrating how three swallowtail level surfaces collapse together at the cusp to form a single level surface with a point of swallowtail perestroika. Parts (b) and (c) show how one of these swallowtails collapses on its own and how its pre-image behaves.

![Figure 8](image-url)

Figure 8: (a) All level surfaces (solid line) through a point as it crosses the caustic (dashed line) at a cusp, (b) one of these level surfaces with its complex double point, and (c) its real pre-image.
5 Real turbulence

The geometric results of Section 3 showed that cusps (or in three dimensions curves of cusps) on the level surfaces occur where the pre-level surface intersects the pre-caustic. As time passes, the cusps or curves of cusps will appear and disappear on the level surfaces as the pre-curves move.

**Definition 5.1.** Real turbulent times are defined to be times $t$ at which there exist points where the pre-level surface $\Phi_t^{-1}H^c_t$ and pre-caustic $\Phi_t^{-1}C_t$ touch.

Real turbulent times correspond to times at which there is a change in the number of cusps or cusped curves on the level surface $H^c_t$.

In $d$-dimensions, assuming $\Phi_t$ is globally reducible, let $f(x,t)(x_0^c)$ denote the reduced action function and $x_t(\lambda)$ denote the pre-parameterisation of the caustic.

**Theorem 5.2.** The real turbulent times $t$ are given by the zeros of the zeta process $\zeta^c_t$ where,

$$\zeta^c_t := f(x_t(\lambda),t)(\lambda_1) - c,$$

$\lambda$ satisfies,

$$\frac{\partial}{\partial \lambda_\alpha}f(x_t(\lambda),t)(\lambda_1) = 0 \quad \text{for } \alpha = 1, 2, \ldots, d,$$  \hspace{1cm} (5)

and $x_t(\lambda)$ is on the cool part of the caustic.

**Proof.** At real turbulent times there is a change in the cardinality,

$$\# \{ \lambda_d = \lambda_d(\lambda_1, \ldots, \lambda_{d-1}) : f(x_t(\lambda_1, \ldots, \lambda_{d-1}, \lambda_d), t)(\lambda_1) = c \}.$$  \hspace{1cm} $\square$

5.1 White noise in $d$-orthogonal directions

We now consider the Burgers fluid under the potential $V(x) = 0$ and the noise $\sum_{\alpha=1}^d \nabla k_\alpha(x) W_\alpha(t)$ where $W_\alpha$ are $d$-independent Wiener processes and $k_\alpha(x) = x_\alpha$ with $x = (x_1, x_2, \ldots, x_d)$. The Burgers equation is then,

$$\frac{\partial v^\mu}{\partial t} + (v^\mu \cdot \nabla) v^\mu = \frac{\mu^2}{2} \Delta v^\mu - \epsilon \dot{W}(t),$$  \hspace{1cm} (6)

where $W(t) = (W_1(t), W_2(t), \ldots, W_d(t))$.

**Proposition 5.3.** The stochastic action corresponding to the Burgers equation (6) is,

$$A(x_0, x, t) = \frac{|x - x_0|^2}{2t} + \epsilon \frac{\xi}{t}(x - x_0) \cdot \int_0^t W(s) \, ds - \epsilon x \cdot W(t)$$

$$- \frac{\epsilon^2}{2} \int_0^t |W(s)|^2 \, ds + \frac{\epsilon^2}{2t} \left| \int_0^t W(s) \, dw \right|^2 + S_0(x_0).$$
Proof. The action is derived as in Section 2 using integration by parts.

**Proposition 5.4.** If \( x_t^\epsilon(\lambda) \) denotes the pre-parameterisation of the random caustic for the stochastic Burgers equation (6) and \( x_0^\epsilon(\lambda) \) denotes the pre-parameterisation of the deterministic caustic (the \( \epsilon = 0 \) case) then,

\[
x_t^\epsilon(\lambda) = x_t^0(\lambda) - \epsilon \int_0^t W(u) \, du.
\]

Proof. Follows from Definition 1.2 and Theorem 2.2.

Using Propositions 5.3 and 5.4, we can find the zeta process explicitly.

**Theorem 5.5.** In \( d \)-dimensions, the zeta process for the stochastic Burgers equation (6) is,

\[
\zeta_t^c = f(x_0^\epsilon(\lambda), t, \lambda_1) - \epsilon x_t^0(\lambda) \cdot W(t) + \epsilon^2 W(t) \cdot \int_0^t W(s) \, ds - \frac{\epsilon^2}{2} \int_0^t |W(s)|^2 \, ds - c,
\]

where \( f(x_0^\epsilon(\lambda_1), t) \) is the deterministic reduced action function, \( x_t^0(\lambda) \) is the deterministic caustic and \( \lambda \) must satisfy the stochastic equation,

\[
\nabla_\lambda \left( f(x_0^\epsilon(\lambda_1), t) - \epsilon x_t^0(\lambda) \cdot W(t) \right) = 0. \tag{7}
\]

Proof. Follows from Theorem 5.2 having derived by induction the reduced action function from Theorem 5.3.

Equation (7) shows that the value of \( \lambda \) used in the zeta process may be either deterministic or random. In the two dimensional case this gives,

\[
0 = \left( \nabla_x f(x_0^\epsilon(\lambda_1), t) - \epsilon W(t) \right) \cdot \frac{dx_t^0}{d\lambda}(\lambda), \tag{8}
\]

which has a deterministic solution for \( \lambda \) corresponding to a cusp on the deterministic caustic. This point will be returned to in Section 5.3.

### 5.2 Recurrence, Strassen and Spitzer

One of the key properties associated with turbulence is the intermittent recurrence of short intervals during which the fluid velocity varies infinitely rapidly.

Using the law of the iterated logarithm, it is a simple matter to show formally that if there is a time \( \tau \) such that \( \zeta_\tau = 0 \), then there will be infinitely many zeros of \( \zeta_t^c = 0 \) in some neighbourhood of \( \tau \). This will make the set of zeros...
of ζ_t a perfect set and will result in a short period during which the fluid velocity will vary infinitely rapidly. However, this formal argument is not rigorous as it will not hold on some set of times t of measure zero [24].

The intermittent recurrence of turbulence will be demonstrated if we can show that there is an unbounded increasing sequence of times at which the zeta process is zero.

We begin by indicating the derivation of Strassen’s form of the law of the iterated logarithm from the theory of large deviations [27, 35]. Consider a complete separable metric space X with a family of probability measures P_ε defined on the Borel sigma algebra of X.

**Definition 5.6.** The family of probability measures P_ε obeys the large deviation principle with a rate function I if there exists a lower semicontinuous function I: X → [0, ∞) where:

1. for each l ∈ ℝ the set \{x : I(x) ≤ l\} is compact in X,
2. for each closed set C ⊂ X, \(\limsup_{\epsilon \to 0} \epsilon \ln P_\epsilon(C) \leq -\inf_{x \in C} I(x)\),
3. for each open set G ⊂ X, \(\liminf_{\epsilon \to 0} \epsilon \ln P_\epsilon(G) \geq -\inf_{x \in G} I(x)\).

Let X = C_0[0, 1] where C_0[0, 1] is the space of continuous functions f : [0, 1] → ℝ^d with f(0) = 0. Let W(t) be a d-dimensional Wiener process and P_ε be the distribution of \(\sqrt{\epsilon}W(t)\) so that P_1 is the Wiener measure.

**Theorem 5.7.** For the measure P_ε the large deviation principle holds with a rate function,

\[
I(f) = \begin{cases} 
\frac{1}{2} \int_0^1 \dot{f}(t)^2 \, dt & : f(t) \text{ absolutely continuous and } f(0) = 0, \\
\infty & : \text{otherwise}.
\end{cases}
\]

**Definition 5.8.** The set of Strassen functions is defined by,

\[K = \{f \in C_0[0,1] : 2I(f) \leq 1\}.\]

**Theorem 5.9** (Strassen’s Law of the Iterated Logarithm). Let \(Z_n(t) = (2n \ln \ln n)^{-\frac{1}{2}}W(nt)\) for \(n \geq 2\) and \(0 \leq t \leq 1\) where W(t) is a d-dimensional Wiener process. For almost all paths ω the subset \(\{Z_n(t) : n = 2, 3, \ldots\}\) is relatively compact with limit set K.

Following the ideas of RTW, this theorem can be applied to the zeta process to demonstrate its recurrence.
Corollary 5.10. There exists an unbounded increasing sequence of times \( t_n \) for which \( Y_{t_n} = 0 \), almost surely, where,

\[
Y_t = W(t) \cdot \int_0^t W(s) \, ds - \frac{1}{2} \int_0^t |W(s)|^2 \, ds,
\]

and \( W(t) \) is a \( d \)-dimensional Wiener process.

Proof. If \( h(n) = (2n \ln \ln n)^{-\frac{1}{2}} \) and \( x(t) \in K \) then there exists an increasing sequence \( n_i \) such that, \( Z_{n_i}(t) = h(n_i)W(n_i) \to x(t) \), as \( i \to \infty \).

Consider the behaviour of each term in \( h(n_i)^2n_i^{-1}W_{t_i} \). Firstly, by applying Lebesgue’s dominated convergence theorem,

\[
h(n_i)^2n_i^{-1}W(n_i) \cdot \int_0^{n_i} W(s) \, ds \to x(1) \cdot \int_0^1 x(r) \, dr,
\]

and,

\[
h(n_i)^2n_i^{-1} \int_0^{n_i} |W(s)|^2 \, ds \to \int_0^1 |x(r)|^2 \, dr,
\]

as \( i \to \infty \).

Now let \( x(t) = (x_1(t), x_2(t), \ldots, x_d(t)) \) where \( x_\alpha(t) = d^{-\frac{1}{2}}t \) for each \( \alpha = 1, 2, \ldots, d \). Therefore, from equations (9) and (10), there is an increasing sequence of times \( t_i \) such that,

\[
h(t_i)^2t_i^{-1}Y_{t_i} \to \frac{1}{2} - \frac{1}{6} = \frac{1}{3},
\]

as \( i \to \infty \).

Alternatively, let

\[
x_\alpha(t) = \begin{cases} (d)^{-\frac{1}{2}}t : & 0 \leq t \leq \frac{1}{3}, \\ (d)^{-\frac{1}{2}}(\frac{2}{3} - t) : & \frac{1}{3} \leq t \leq 1, \end{cases}
\]

for \( \alpha = 1, 2, \ldots d \). Therefore, using equations (9) and (10) there is an increasing sequence of times \( \tau_i \) such that,

\[
h(\tau_i)^2\tau_i^{-1}Y_{\tau_i} \to -\frac{1}{54} - \frac{1}{27} = -\frac{1}{18}.
\]

Thus, the sequence \( t_i \) is an unbounded increasing infinite sequence of times at which \( Y_t > 0 \), and the sequence \( \tau_i \) is an unbounded increasing infinite sequence of times at which \( Y_t < 0 \).
Corollary 5.11. If \( h(t)^2 t^{-1} f^0_{x_0^2(\lambda),t}(\lambda_1) \to 0 \) and \( h(t)t^{-1} \sum_{i=0}^d x_i^0(\lambda) \to 0 \), then the zeta process \( \zeta^c_t \) is recurrent.

A stronger condition on recurrence can be found in the two dimensional case if we work with small \( \epsilon \) and neglect terms of order \( \epsilon^2 \) so that,

\[
\zeta^c_t = f^0_{x_0^2(\lambda),t}(\lambda_1) - \epsilon x_1^0(\lambda) \cdot W(t) - c.
\] (11)

For this we use Spitzer’s theorem, a proof of which can be found in Durrett [8].

Theorem 5.12 (Spitzer’s Theorem). Let \( D(t) = D_1(t) + iD_2(t) \) be a complex Brownian motion where \( D_1 \) and \( D_2 \) are independent, \( D_1(0) = 1 \) and \( D_2(0) = 0 \). Define the process \( \theta_t \) as the continuous process where \( \theta_0 = 0 \) and

\[
\cos(\theta_t - \phi_t) = \frac{1}{\pi} \int_{-\infty}^{y} \frac{dx}{1 + x^2}.
\]

The process \( \theta_t \) gives the angle swept out by \( D(t) \) in time \( t \), counting anti-clockwise loops as \(-2\pi\) and clockwise loops as \(2\pi\).

Let \( A : \mathbb{R}^+ \to \mathbb{R}^2 \) and consider the behaviour of the process, \( Y_t = A(t) \cdot W(t) \). Assuming that \( A(t) \neq 0 \), let \( \phi_t \) and \( \theta_t \) measure the windings around the origin of \( A(t) \) and \( W_t \) respectively. Then, \( Y_t = \epsilon |A(t)||W(t)| \cos(\phi_t - \theta_t) \).

Therefore, for \( Y_t = 0 \) we require \( \cos(\phi_t - \theta_t) = 0 \), so that the two vectors \( A(t) \) and \( W(t) \) are perpendicular to each other. (Alternatively, this would be satisfied trivially if \( A(t) \) were periodically zero with \( t \).)

Corollary 5.13. The small noise zeta process (11) is recurrent if there exists a bounded function \( h(t) \) where \( h : \mathbb{R}^+ \to \mathbb{R}^+ \) such that,

\[
h(t) \left( f^0_{x_0^2(\lambda),t}(\lambda_1) - c \right) \to 0,
\]

as \( t \to \infty \) and there exists a function \( n_t \) such that \( n_t \to \infty \) with,

\[
\frac{4\pi^2n_t^2 - \phi_t^2}{(\ln t)^2} < \frac{1}{4} \quad \text{and} \quad \frac{n_t \ln t}{16\pi^2n_t^2 - 4\phi_t^2 + (\ln t)^2} \to 0,
\]

as \( t \to \infty \) where \( A(t) = \epsilon h(t)x_1^0(\lambda) \).
5.3 Examples in two and three dimensions

We now consider an explicit example in two dimensions. Since the parameter $\lambda \in \mathbb{R}$, equation (11) reduces to,

$$0 = \frac{d}{d\lambda} f(x_t(\lambda), t)(\lambda) = \nabla x f(x_t(\lambda), t)(\lambda) \cdot \frac{dx_t}{d\lambda}(\lambda). \quad (12)$$

This gives three different forms of turbulence:

1. ‘zero speed turbulence’ where $\nabla f(x_t(x_0^1), t) = \dot{X}(t) = 0$. so the Burgers fluid has zero velocity.

2. ‘orthogonal turbulence’ where $\nabla f(x_t(\lambda), t)(\lambda)$ is orthogonal to $\frac{dx_t}{d\lambda}(\lambda)$ so that the caustic tangent is orthogonal to the Burgers fluid velocity.

3. ‘cusped turbulence’ where $\frac{dx_t}{d\lambda}(\lambda) = 0$, so there is a generalised cusp on the caustic at $x_t(\lambda)$.

As discussed previously, cusped turbulence will occur at deterministic values of $\lambda$ and will also correspond to points of swallowtail perestroika on the level surfaces. As such, it is not only the simplest form to analyse, but also the most important. The categorisation of turbulence leads to a factorisation of equation (12).

**Example 5.14 (The generic Cusp).** For the generic Cusp, the zeta process reduces to,

$$\zeta_t^c = -\frac{3 \lambda^4 t}{8} + \frac{\lambda^6 t^3}{2} - \epsilon \left( \lambda^3 t^2 W_1(t) - \frac{W_2(t)}{t} + \frac{3}{2} \lambda^2 t W_2(t) \right)$$

$$+ \epsilon^2 \left( W(t) \cdot \int_0^t W(s) \, ds - \frac{1}{2} \int_0^t |W(s)|^2 \, ds \right) - c,$$

where $\lambda$ must be a root of,

$$0 = \frac{3}{2} \lambda t \left( 2\lambda^4 t^2 - \lambda^2 - 2 \epsilon \{ \lambda t W_1(t) + W_2(t) \} \right).$$

The factor $\lambda = 0$ corresponds to the cusp on the caustic while the roots of the second factor correspond to orthogonal and zero speed turbulence. Firstly, if $\lambda = 0$ then from Corollary 5.11, the zeta process is recurrent. Therefore, the turbulence occurring at the cusp on the generic Cusp caustic is recurrent.

Alternatively, for large times it can be shown formally that the four roots which give rise to orthogonal and zero speed turbulence all tend towards zero. Thus all four roots tend towards the cusp and consequently, the zeta processes associated with each root will be recurrent.

Moreover, it can be shown that the whole caustic is cool and so all of these points of turbulence will be genuine.
Next consider the three dimensional case. Thus \( \lambda \in \mathbb{R}^2 \) and equation (13) becomes the pair,

\[
0 = \nabla_x f(x_t(\lambda), t) (\lambda_1) \cdot \frac{dx_t}{d\lambda_1}(\lambda), \quad 0 = \nabla_x f(x_t(\lambda), t) (\lambda_1) \cdot \frac{dx_t}{d\lambda_2}(\lambda). \tag{13}
\]

In direct correlation to the two dimensional case, we can categorise three dimensional turbulence depending on how we solve equations (13):

1. ‘zero speed turbulence’ where again \( \nabla_x f(x_t(\lambda), t) (\lambda_1) = 0 \),

2. ‘orthogonal turbulence’ where all three vectors \( \nabla_x f(x_t(\lambda), t) (\lambda_1), \frac{dx_t}{d\lambda_1}(\lambda) \) and \( \frac{dx_t}{d\lambda_2}(\lambda) \) are mutually orthogonal.

3. ‘subcaustic turbulence’ where the vectors \( \frac{dx_t}{d\lambda_1}(\lambda) \) and \( \frac{dx_t}{d\lambda_2}(\lambda) \) are linearly dependent. The “subcaustic” is the region of the caustic where the tangent space drops one or more dimensions. In three dimensions it corresponds to folds in the caustic.

As in the two dimensional case, it follows from Proposition 5.4 that the values of \( \lambda \) that determine the subcaustic are deterministic. However, unlike the two dimensional case, subcaustic turbulence only occurs at points where the Burgers fluid velocity is orthogonal to the subcaustic. Hence, we are selecting random points on a deterministic curve, and so subcaustic turbulence involves random values of \( \lambda \). Again the categorisation of turbulence leads to a factorisation in equations (13).

**Example 5.15 (The butterfly).** Let \( S_0(x_0, y_0, z_0) = x_0^2 y_0 + x_0^2 z_0 \), this gives a butterfly caustic – the three dimensional analogue of the generic Cusp. The zeta process is,

\[
\zeta_t^c = \lambda_1^3 \lambda_2 - \frac{3}{2} \lambda_1^4 t - 4 \lambda_6 t^2 + \frac{9}{2} \lambda_1 \lambda_2 t - 12 \lambda_1^5 \lambda_2 t^2 - 27 \lambda_1^7 \lambda_2 t^2 + 8 \lambda_1^6 t^3 + 36 \lambda_1^8 t^3 + \frac{81}{2} \lambda_1^{10} t^3 + \epsilon \left( (3 \lambda_1^2 \lambda_2 t - 4 \lambda_1^3 t^2 - 9 \lambda_1^5 t^2)W_1(t) \right. \\
- (\lambda_2 + \lambda_1^2 t)W_2(t) + (3 \lambda_1 \lambda_2 + \frac{1}{2} - 3 \lambda_1^2 t - \frac{9}{2} \lambda_1^4 t)W_3(t) \\
\left. + \epsilon^2 \left( W(t) \cdot \int_0^t W(s) \, ds \right) - \frac{1}{2} \int_0^t |W(s)|^2 \, ds \right) - c,
\]

where \( \lambda = (\lambda_1, \lambda_2) \) must satisfy,

\[
0 = 135t^3 \lambda_1^6 + 96t^3 \lambda_1^5 - 63 \lambda_1^2 t^2 \lambda_6 + (16t^3 - 8t) \lambda_1^5 - (20 \lambda_2 + 15 \epsilon W_1(t)) t^2 \lambda_1^4 + (6 \lambda_2^2 - 2 - 6 \epsilon W_3(t)) t \lambda_1^3 + (\lambda_2 + 4 \epsilon W_1(t) - \epsilon W_2(t)) \lambda_1^2 \\
+ 2 \left( \lambda_2 W_1(t) - W_3(t) \right) t \lambda_1 + \epsilon \lambda_2 W_3(t),
\]

\[
0 = -27t^2 \lambda_1^7 + 12t^2 \lambda_5 + 9t \lambda_1^4 \lambda_2 + \lambda_3^5 + 3 \epsilon W_1(t) \lambda_1^2 + 3 \epsilon W_3(t) \lambda_1 - \epsilon W_2(t).
\]
Eliminating $\lambda_2$ gives the factorisation,

$$
0 = (54t^2\lambda_1^7 + 6t^2\lambda_5^5 + \lambda_1^3 + 3\epsilon W_1(t)\lambda_1^2 + 3\epsilon W_3(t)\lambda_1 - \epsilon W_2(t) ) \\
\times (\lambda_1^3 - 3\epsilon W_3(t)\lambda_1 + 2\epsilon W_2(t)) ,
$$

(14)

where the first factor gives zero speed and orthogonal turbulence while the second factor gives subcaustic turbulence.

For large times, it can be shown formally that of the seven roots corresponding to zero and orthogonal turbulence, five should tend to $\lambda = (0, 0)$ and so should give a recurrent zeta process. None of the remaining roots give recurrence [21].

### 5.4 The harmonic oscillator potential

It is not always necessary to resort to Strassen's law to demonstrate the recurrence of turbulence; some systems have an inherent periodicity which produces such behaviour. The following two dimensional example is taken from RTW [33] in which a single Wiener process acts in the $x$ direction.

**Example 5.16.** Let $k_1(x, y) = x$, $V(x, y) = \frac{1}{2}(x^2 \omega_1^2 + y^2 \omega_2^2)$ and $S_0(x_0, y_0) = f(x_0) + g(x_0)y_0$ where $f, f', f'', g, g', g''$ are zero when $x_0 = \alpha$ and $g''(\alpha) \neq 0$. Then the zeta process for turbulence at $\alpha$ is given by,

$$
\zeta_t = -\frac{\omega_2}{4g''(\alpha)} \sin(2\omega_2 t) \csc^2(\omega_1 t) \{\sin(\omega_1 t) f''(\alpha) + \omega_1 \cos(\omega_1 t)\}^2 \\
+ \epsilon \csc(\omega_1 t) R_t - \frac{1}{4} \alpha^2 \omega_1 \sin(2\omega_1 t) - c,
$$

where $R_t$ is a stochastic process which is well defined for all $t$.

Therefore, $\zeta_t \to \pm \infty$ as $t \to \frac{k\pi}{\omega_1}$ because $\csc^2(k\pi) = \infty$ where the sign depends upon the sign of $-\frac{\sin(2\omega_2 t)}{g''(\alpha)}$. Thus, it is possible to construct an unbounded increasing sequence of times at which $\zeta_t$ switches between $\pm \infty$ and so by continuity and the intermediate value theorem there will almost surely exist an increasing unbounded sequence $\{t_k\}$ at which $\zeta_{t_k} = 0$.

### 6 Complex turbulence

We now consider a completely different approach to turbulence based on the work of Section 4. Let $(\lambda, x^2_{0,C}(\lambda))$ denote the parameterisation of the pre-caustic so that $x_t(\lambda) = \Phi_t\left(\lambda, x^2_{0,C}(\lambda)\right)$ is the pre-parameterisation of the caustic. When,

$$Z_t = \text{Im} \left\{ \Phi_t(a + i\eta, x^2_{0,C}(a + i\eta)) \right\},$$

(22)
is random, the values of $\eta(t)$ for which $Z_t = 0$ will form a stochastic process. The zeros of this new process will correspond to points at which the real pre-caustic touches the complex pre-caustic.

**Definition 6.1.** The complex turbulent times $t$ are defined to be times $t$ when the real and complex pre-caustics touch.

The points at which these surfaces touch correspond to swallowtail perestroikas on the caustic.

**Theorem 6.2.** Let $x_t(\lambda)$ denote the pre-parameterisation of the caustic where $\lambda \in \mathbb{R}$ and $x_t(\lambda)$ is a real analytic function. If at time $\tilde{t}$ a swallowtail perestroika occurs on the caustic when $\lambda = \tilde{\lambda}$ then,

$$f'_{x_t(\tilde{\lambda}), \tilde{t}}(\tilde{\lambda}) = f''_{x_t(\tilde{\lambda}), \tilde{t}}(\tilde{\lambda}) = f'''_{x_t(\tilde{\lambda}), \tilde{t}}(\tilde{\lambda}) = 0.$$ 

Assuming that $f_{x_t}(x_0^1)$ is a polynomial in $x_0^1$ we can use the resultant to state explicit conditions for which this holds.

**Lemma 6.3.** Let $g$ and $h$ be polynomials of degrees $m$ and $n$ respectively with no common roots or zeros. Let $f = gh$ be the product polynomial. Then the resultant,

$$R(f, f') = (-1)^{mn} \left( \frac{m!n!}{N!} \right) \frac{f^{(N)}(0)}{g^{(m)}(0)h^{(n)}(0)} R(g, g') R(h, h') R(g, h)^2,$$

where $N = m + n$ and $R(g, h) \neq 0$.

**Proof.** See [22].

Since $f'_{x_t(\lambda), t}(x_0^1)$ is a polynomial in $x_0$ with real coefficients, its zeros are real or occur in complex conjugate pairs. Of the real roots, $x_0 = \lambda$ is repeated. So,

$$f'_{x_t(\lambda), t}(x_0^1) = (x_0^1 - \lambda)^2 Q_{\lambda, t}(x_0^1) H_{\lambda, t}(x_0^1),$$

where $Q$ is the product of quadratic factors,

$$Q_{\lambda, t}(x_0^1) = \prod_{i=1}^{q} \left\{ (x_0^1 - a_i^1)^2 + (\eta_i^1)^2 \right\},$$

and $H_{\lambda, t}(x_0^1)$ the product of real factors corresponding to real zeros. This gives,

$$f'''_{x_t(\lambda), t}(x_0^1) \big|_{x_0^1 = \lambda} = 2 \prod_{i=1}^{q} \left\{ (\lambda - a_i^1)^2 + (\eta_i^1)^2 \right\} H_{\lambda, t}(\lambda).$$
We now assume that the real roots of $H$ are distinct as are the complex roots of $Q$. Denoting $f'''_{(x_t(\lambda),t)}\big|_{x_0^t=\lambda}$ by $f'''_t(\lambda)$ etc, a simple calculation gives

$$|R_\lambda(f'''_t(\lambda), f^{(4)}_t(\lambda))| =$$

$$K_t \prod_{k=1}^{q} (\eta_k)\prod_{j \neq k} \left\{ (a_k^t - a_j^t)^4 + 2((\eta_k^t)^2 + (\eta_j^t)^2)(a_k^t - a_j^t)^2 + ((\eta_k^t)^2 - (\eta_j^t)^2)^2 \right\}$$

$$\times |R_\lambda(H, H')| |R_\lambda(Q, H)|^2,$$

$K_t$ being a positive constant. Thus, the condition for a swallowtail perestroika to occur is that

$$\rho_\eta(t) := \left| R_\lambda(f'''_t(\lambda), f^{(4)}_t(\lambda)) \right| = 0,$$

where we call $\rho_\eta(t)$ the resultant eta process.

When the zeros of $\rho_\eta(t)$ form a perfect set, swallowtails will spontaneously appear and disappear on the caustic infinitely rapidly. As they do so, the geometry of the caustic will rapidly change. Moreover, Maxwell sets will be created and destroyed with each swallowtail that forms and vanishes as when a swallowtail forms it contains a region with two more pre-images than the surrounding space. This will add to the turbulent nature of the solution in these regions. We call this ‘complex turbulence’ occurring at the turbulent times which are the zeros of the resultant eta process.

Complex turbulence can be seen as a special case of real turbulence which occurs at specific generalised cusps of the caustic. Recall that when a swallowtail perestroika occurs on a curve, it also satisfies the conditions for having a generalised cusp. Thus, the zeros of the resultant eta process must coincide with some of the zeros of the zeta process for certain forms of cusped turbulence. At points where the complex and real pre-caustic touch, the real pre-caustic and pre-level surface touch in a particular manner (a double touch) since at such a point two swallowtail perestroikas on the level surface have coalesced.

Thus, our separation of complex turbulence from real turbulence can be seen as an alternative form of categorisation to that outlined in Section 5.3 which could be extended to include other perestroikas.

Acknowledgement

One of us (AT) would like to record his indebtedness to John T Lewis as his teacher, mentor and friend. This paper could not have been written without
John’s inspirational work on large deviations which underlies our work.

References

[1] Arnol’d V I, Shandarin S F and Zeldovich Y B 1982 The large scale structure of the universe 1 Geophys. Astrophys. Fluid Dyn. 20 111–30

[2] Arnol’d V I 1986 Catastrophe Theory (Berlin: Springer-Verlag)

[3] Arnol’d V I 1989 Mathematical Methods of Classical Mechanics (New York: Springer-Verlag)

[4] Arnol’d V I 1990 Singularities of Caustics and Wave Fronts. Mathematics and its Applications (Soviet Series) 62 (Dordrecht: Kluwer Academic Publishers Group)

[5] Dafermos C 2000 Hyperbolic Conservation Laws in Continuum Physics. Grundlehren der Mathematischen Wissenschaften 325 (Berlin: Springer-Verlag)

[6] Davies I M, Truman A and Zhao H 2002 Stochastic heat and Burgers equations and their singularities I - geometric properties J. Math. Phys. 43 3293-328

[7] Davies I M, Truman A and Zhao H 2005 Stochastic heat and Burgers equations and their singularities II. Analytical properties and limiting distributions J. Math. Phys. 46 043515

[8] Durrett R 1984 Brownian motion and martingales in analysis (Belmont: Wadsworth)

[9] E Weinan, Khanin K, Mazel A and Sinai Y 2000 Invariant measures for Burgers equations with stochastic forcing Ann. Math. 151 877-960

[10] Elworthy K D, Truman A and Zhao H Stochastic elementary formulae on caustics 1: One dimensional linear heat equations UWS MRRS Preprint

[11] Elworthy K D, Truman A and Zhao H 2005 Generalised Ito formulae and space-time Lebesgue-Stieltjes integrals of local times To appear in Seminaire de Probabilites Strasbourg Vol. 40

[12] Freidlin M I and Wentzell A D 1998 Random Perturbations of Dynamical Systems (New York: Springer-Verlag)
[13] Gilmore R 1981 *Catastrophe Theory for Scientists and Engineers* (New York: John Wiley)

[14] Hwa R C and Teplitz V L 1966 *Homology and Feynman integrals* (New York: W A Benjamin)

[15] Kac M 1959 *Probability and Related Topics in Physical Science* (New York: Interscience Publishers)

[16] Klein F 1922 Über den Verlauf der Abelschen Integrale bei den Kurven vierten Grades in Gesammelte Mathematische Abhandlungen II ed Fricke R and Vermeil H (Berlin: Springer)

[17] Kolokoltsov V N, Schilling R L and Tyukov A E 2004 Estimates for multiple stochastic integrals and stochastic Hamilton-Jacobi equations *Rev. Mat. Iberoamericana* 20 333-80

[18] Kunita H 1984 *Stochastic differential equations and stochastic flows of homeomorphisms*, in *Stochastic Analysis and Applications, Advances in Probability and Related Topics. Vol. 7* ed Pinsky M A (New York: Marcel Dekker)

[19] Maslov V P 1972 *Perturbation Theory and Asymptotic Methods* (Paris: Dunod)

[20] Maslov V P and Fedoriuk M V 1981 *Semi-Classical Approximation in Quantum Mechanics. Mathematical Physics and Applied Mathematics Vol. 7* (Dordrecht: Riedel Publishing Company)

[21] Neate A D 2005 *A one dimensional analysis of the singularities of the d-dimensional stochastic Burgers equation* PhD thesis UWS

[22] Neate A D and Truman A 2005 A one dimensional analysis of real and complex turbulence and the Maxwell set for the stochastic Burgers equation *J. Phys. A: Math. Gen.* 38 7093–127

[23] Neate A D and Truman A 2005 The Maxwell set for the stochastic Burgers equation *In preparation.*

[24] Reynolds C 2002 *On the polynomial swallowtail and cusp singularities of stochastic Burgers equations* PhD thesis UWS

[25] Salmon G 1934 *A Treatise on the Higher Plane Curves* (New York: G E Stechert Co)
[26] Shandarin S F and Zeldovich Y B 1989 The large scale structure of the universe 2: turbulence, intermittency, structures in a self gravitating medium *Rev. Mod. Phys.* 6 185-220

[27] Stroock D W 1984 *An introduction to the theory of large deviations* (New York: Springer-Verlag)

[28] Truman A and Zhao H 1995 The stochastic Hamilton-Jacobi equations and related topics: a survey *LMS Lecture Note Ser. 216* (Cambridge: Cambridge University Press) p 287

[29] Truman A and Zhao H 1996 The stochastic Hamilton-Jacobi equations, stochastic heat equations and Schrödinger equations *Stochastic Analysis and Applications. Proc. of the 5th Gregynog Symp. held in Powys July 9–14 1995* ed. Davies I M et al (River Edge NJ: World Scientific) p 441–64

[30] Truman A and Zhao H 1996 On stochastic diffusion equations and stochastic Burgers equations *J. Math. Phys.* 37 283–307

[31] Truman A and Zhao H 1996 Quantum mechanics of charged particles in random electromagnetic fields *J. Math. Phys.* 37 3180–97

[32] Truman A and Zhao H 1998 Stochastic Burgers equations and their semi classical expansions *Comm. Math. Phys.* 194 231-48

[33] Truman A, Reynolds C N and Williams D 2003 Stochastic Burgers equations in d-dimensions – a one dimensional analysis: Hot and cool caustics and intermittence of stochastic turbulence *Probabilistic Methods in Fluids* ed Davies I M et al (Singapore: World Scientific) pp 239–62

[34] Van Der Waerden 1949 *Modern Algebra* Vols. 1 and 2 (New York: Frederick Ungar Publishing)

[35] Varadhan S R S 1984 *Large deviations and applications* (Philadelphia: SIAM)