Hamiltonian mappings and circle packing phase spaces: numerical investigations

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In a previous paper we introduced examples of Hamiltonian mappings with phase space structures resembling circle packings. We now concentrate on one particular mapping and present numerical evidence which supports the conjecture that the set of circular resonance islands is dense in phase space.

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I. INTRODUCTION

Piecewise isometries are iterated mappings which preserve distances on each of a number of disjoint regions of space. Such maps and their variants are currently enjoying considerable attention in the literature [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12], and arise naturally in the context of digital filters [13, 14, 15, 16, 17, 18, 19, 20, 21] and dual billiards [22, 23, 24]. In a recent article [25] three new piecewise isometries were introduced, each corresponding to a particular phase space geometry: planar, hyperbolic or spherical. As examples of kicked Hamiltonian systems, all of these maps invite physical interpretations. For instance, in the presence of damping, the planar map describes oscillations in a water-filled U-tube under the periodic kicking of bubbles [26].

In the present paper we restrict our attention to the mapping defined on a sphere. It has a dynamical behaviour which is dominated by an infinite number of stable periodic orbits of arbitrarily long period. The resonance islands associated with these orbits are circular discs which fill in the phase space in a manner resembling circle packings [27, 28, 29, 30].

A circle packing is a set with empty interior and whose complement is the union of disjoint open circular discs. Circle packings with zero Lebesgue measure are called complete. We have previously conjectured [25] that the phase space structure produced by this map is indeed a circle packing, but unlikely to be complete. We now present numerical evidence which supports this belief. Our choice to concentrate on the spherical map was due to the compactness of its phase space. This proves to be of a great advantage in the numerical computations. However all three of the maps described in [25] are conjectured to produce circle packings.

Numerical studies have led Ashwin et al [1, 2] to put forward similar conjectures for the sawtooth standard map (see also Goetz [7] for proofs). This mapping might be considered an analogue of those described above for the torus. However the fundamental question about the ‘packed’ nature of the phase space remained untested. To this endeavour the notion of riddling was developed [11, 31, 32]. Also of interest in the literature is the concept of a fat

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Fat fractals are loosely defined to be sets with a fractal structure but non-zero measure. The circle packings described above might be considered examples of such sets.

To test whether the set of circular resonances is dense in phase space, we will employ a symbolic labeling of the periodic orbits and use special symmetries in the mapping, to efficiently calculate a large class of the discs. Our numerical studies indicate that this class may in fact itself pack the phase space, and therefore, is the entire set.

Our paper is organized as follows. In the next section we will give an overview of the mapping to be analyzed, stating three main conjectures concerning its behaviour. In section III we present numerical evidence which supports these conjectures. Finally, in section IV we will discuss our results.

II. THE MAP

The Hamiltonian under consideration is that of a kicked linear top

\[ H(J, t) = \omega J_3 + \mu |J_1| \sum_{n=-\infty}^{\infty} \delta(t - n), \]

where \( \mu, \omega \in [0, 2\pi] \) are parameters and \( (J)_i = J_i = \epsilon_{ijk}x_jp_k \) \((i = 1, 2, 3)\) are the three components of angular momentum for a particle confined to a sphere, normalized such that \( J \cdot J = 1 \). The evolution of \( J \) is governed by the equations

\[ \dot{J}_i = \{J_i, H\}, \quad \{J_i, J_j\} = \epsilon_{ijk}J_k \]

where \( \{\cdot, \cdot\} \) are the Poisson brackets. Their solution can be written as the mapping

\[
\begin{bmatrix}
J_1^{n+1} \\
J_2^{n+1} \\
J_3^{n+1}
\end{bmatrix} =
\begin{bmatrix}
\cos \omega & -\sin \omega & 0 \\
\sin \omega & \cos \omega & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos \mu s^n - \sin \mu s^n & \cos \mu s^n \\
0 & \sin \mu s^n & \cos \mu s^n
\end{bmatrix}
\begin{bmatrix}
J_1^n \\
J_2^n \\
J_3^n
\end{bmatrix}
\equiv F(s^n)J^n \equiv FJ^n
\]

where \( s^n \equiv \text{sgn} J_1^n \), which takes \( J \) from just before a kick to one period later. Here \( \text{sgn} \) is the signum function with the convention \( \text{sgn} 0 = 0 \).

The mapping is deceptively simple. It first rotates the eastern hemisphere \((J_1 > 0)\) through an angle \( \mu \) and the western hemisphere \((J_1 < 0)\) through an angle \(-\mu \). The great circle \( J_1 = 0 \) remains fixed. Then the entire sphere is rotated about the \( J_3 \) axis through an angle \( \omega \). Thus the map rotates every point on the sphere in a piecewise linear fashion except on \( J_1 = 0 \) where its Jacobian is singular. Surprisingly, this leads to a phase space of great structural complexity. An example is plotted in figure 1. The eastern hemisphere is shown in black, the western in gray. To simplify our study we will restrict ourselves to the case \( \mu = \omega = \pi(\sqrt{5} - 1) \), as shown in the figure.

Note that if an orbit of our map does not have a point on \( J_1 = 0 \), then its Lyapunov exponents are zero, making it stable. If however, it does have a point on \( J_1 = 0 \), then its Lyapunov exponents are undefined. We overcome this problem by simply defining an unstable orbit to be one with a point on \( J_1 = 0 \), and all others stable. The unstable set is defined to be the closure of the set of all images and preimages of the great circle \( J_1 = 0 \). In figure 1 the unstable set is in black and gray. The circular holes are resonances consisting
FIG. 1: The phase space when $\mu = \omega = \pi(\sqrt{5} - 1)$.

entirely of stable orbits. At their center lies a stable periodic orbit. All other points inside the resonances rotate about the central periodic orbit in a linear fashion. The period of rotation is an irrational multiple of $\pi$ for almost all values of the parameters $\mu$ and $\omega$, hence their circular shape. If the period of rotation happens to chance upon a rational multiple of $\pi$, then the resonance forms a polygon (see [25, 40]).

We have previously shown [25] that it is possible to label every point of a periodic orbit of least period $n$ with a unique sequence $\{s^k = 0, \pm 1\}_{k=0..n-1}$. The position of the point is given by the solution of

$$J = F(s^{n-1})F(s^{n-2}) \ldots F(s^0)J \equiv RJ$$

(4)

lying on the unit sphere with $\text{sgn} J_1 = s^0$. The other $n - 1$ points of the periodic orbit are found by substituting the $n - 1$ cycles of the original sequence into (4). Hence, if $\{J^k\}_{k=0..n-1}$ is a periodic orbit then the sequence corresponding to its first point is $\{\text{sgn} J^k\}_{k=0..n-1}$. The orbit is unstable if and only if this sequence contains 0. Note that the (unnormalized) solution of (4) is simply the axis of rotation of $R$, namely

$$J = (R_{23} - R_{32}, R_{31} - R_{13}, R_{12} - R_{21}).$$

(5)
Although every periodic point is uniquely represented by a sequence, not every sequence represents a periodic point. Hence we still need to find which sequences are legitimate. This may be done by checking each of the $2^n$ different possible sequences for a period-$n$ orbit. However, this is computationally expensive for large periods. An alternative approach is to exploit symmetries of our mapping.

If we decompose our map into involutions \[ F = I_2 \circ I_1 = \tilde{I}_2 \circ \tilde{I}_1 \] (6)

where

\[ I_1 J = (-J_1, J_2 \cos(\mu \text{sgn} J_1) - J_3 \sin(\mu \text{sgn} J_1), J_2 \sin(\mu \text{sgn} J_1) + J_3 \cos(\mu \text{sgn} J_1)) \] (7)

\[ I_2 J = (-J_1 \cos \omega - J_2 \sin \omega, -J_1 \sin \omega + J_2 \cos \omega, J_3) \] (8)

\[ \tilde{I}_1 J = (J_1, -J_2 \cos(\mu \text{sgn} J_1) + J_3 \sin(\mu \text{sgn} J_1), J_2 \sin(\mu \text{sgn} J_1) + J_3 \cos(\mu \text{sgn} J_1)) \] (9)

\[ \tilde{I}_2 J = (J_1 \cos \omega + J_2 \sin \omega, J_1 \sin \omega - J_2 \cos \omega, J_3) \] (10)

and

\[ I_1^2 = I_2^2 = \tilde{I}_1^2 = \tilde{I}_2^2 = 1 \] (11)

we obtain the four symmetry lines

\[ S_1 : J_1 = 0 \] (12)

\[ S_2 : J_1 = -J_2 \tan(\omega/2) \] (13)

\[ \tilde{S}_1 : J_2 = J_3 \tan(\mu/2) \text{sgn} J_1 \] (14)

\[ \tilde{S}_2 : J_2 = J_1 \tan(\omega/2) \] (15)

as the fixed lines (Fix(I) $\equiv \{x | Ix = x\}$) of our involutions. Given that $IFI = F^{-1}$ for all the above involutions, it is not difficult to prove the following theorem \[24].

**Theorem.** Let $\{J^k\}_{k=0,n-1}$ be a periodic orbit of $F$ with least period $n$. Suppose that the point $J^0$ lies on a symmetry line of $F$ ($S_1$, $S_2$, $\tilde{S}_1$ or $\tilde{S}_2$). Then no other points of the orbit lie on the symmetry line if $n$ is odd, or exactly one other, $J^{n/2}$, if $n$ is even.

This property reduces the task of finding periodic orbits to a one-dimensional search. However, if we wish to find all periodic orbits of a given period, then we must first establish that every periodic orbit must have a point on at least one of the four given symmetry lines. By definition, every unstable orbit has a point on $S_1$, and thus, we have the following fact.

**Fact.** Every unstable periodic orbit has a point on $S_1$.

In \[25\] we made the following conjecture based on preliminary numerical work.

**Conjecture 1.** For almost all $\omega$ and $\mu$, every stable periodic orbit has a point on at least one of $S_2$, $\tilde{S}_1$ or $\tilde{S}_2$.

The conjecture is not true for all $\omega$ and $\mu$ since, as remarked above, the rotation of a resonance may become a rational multiple of $\pi$ when the parameters take on particular values. However this occurs on a set of measure zero.
The above theorem, and the corresponding symmetric periodic orbits for which it refers, have become folklore in the theory of reversible dynamical systems [42]. Proponents of symmetry methods include Birkhoff [43], De Vogelaere [44] and Greene et al. [45]. Unfortunately, one generally finds that many other periodic orbits will be present which do not have points on a symmetry line. Thus, the mapping we are considering is conjectured to be of a very special class for which all periodic orbits are of the symmetric type.

We now turn our attention to the unstable set. The complicated structure of circular resonances with decreasing radii displayed in figure I suggests the following conjecture.

Conjecture 2. The unstable set has empty interior.

That is, the unstable set is a circle packing of the unit sphere. We also put forward a third conjecture.

Conjecture 3. The unstable set has positive Lebesgue measure.

Or equivalently, the circle packing is not complete. Numerical evidence which supports the above three conjectures will now be presented.

III. NUMERICAL EVIDENCE

Suppose we ignore the fact that the stable symmetric periodic orbits may only form a subset of the set of all stable periodic orbits. If we find that the set of circular resonances associated with these orbits pack the phase space (i.e. the set of resonance discs is dense on the surface of the unit sphere), then we are left with only one conclusion: both Conjecture 1 and 2 must be correct. Furthermore, we are now in a position to test Conjecture 3. This is precisely the approach we will take. Consider the following trivial property.

Suppose $J^0$ lies on a symmetry line corresponding to the involution $I$ ($IJ^0 = J^0$), and the iterate $J^m$ also lies on this same symmetry line ($IF^mJ^0 = F^mJ^0$). Then,

$$F^{2m}J^0 = F^mF^mJ^0 = F^mIF^mJ^0 = IF^{-m}I^2F^mJ^0 = IF^0 = J^0$$

and hence, $J^0$ must be a periodic point with period $2m$. To find the orbit’s least period we construct the unique sequence $\{\text{sgn } J^k\}_{k=0..2m-1}$ and locate the shortest word which generates the orbit. If $J^m$ was the first iterate to hit the symmetry line, then either the period of the orbit is odd with period $m$, or even with period $2m$.

Thus we only need to find where each symmetry line intersects with its image; an effortless task to solve numerically by virtue of the simplicity of our mapping. Every image of a symmetry line is just a collection of arc segments on the sphere.

Now suppose that we have just found a particular periodic orbit. If we are to gain some insight into the above three conjectures we will need to find the size of its circular resonances. Note that the resonance of at least one point of the orbit must touch the great circle $J_1 = 0$ ($S_1$). This is because the boundary of each resonance forms part of the unstable set, and hence, iterates arbitrarily close to $S_1$. Consequently

$$\sigma = \min_{k=0..n-1} |J^k_1|$$

where $n$ is the period of the orbit, and $\sigma$ is perpendicular distance between the boundary of the circular resonance, and the line connecting its center (the periodic point) with the origin.
All resonances of the same periodic orbit are of equal size. The sections of the symmetry line which intersect the orbit's resonances contain no other periodic orbits and may be ignored for the remainder of our search.

For the symmetry lines $S_2$, $\tilde{S}_1$ and $\tilde{S}_2$, the above procedure was run for $m = 5 \times 10^5$ iterations of the mapping. Hence all symmetric periodic orbits with odd periods $\leq 5 \times 10^5$ and even periods $\leq 1 \times 10^6$ were found, producing over 22 billion circles.

To investigate properties of the unstable set we will define the radius of a circle on the unit sphere to be

$$ r \equiv 2 \sin(\arcsin(\sigma)/2) $$

(18)

which allows us to write the spherical area covered by the resonance as $\pi r^2$. Other definitions for the radius such as $\sigma$, or the angular radius $\arcsin(\sigma)$, are also suitable since all are equivalent in the limit $\sigma \to 0$.

The proliferation of circles with smaller and smaller radii is displayed in figure 2(a). Here the radius of the resonance versus the period of the orbit is plotted on a logarithmic scale. Note that we can only be sure that we have all circles of a particular radius for radii $\lesssim 10^{-5.5}$ (the upper dashed line in figure 2(a)).
Next we define the quantity $N(r)$ as the number of circles of radius larger than $r$. It is found that $N(r)$ follows the scaling law

$$N(r) = Ar^{-d}$$  \hspace{1cm} (19)$$
(see figure 2(b)). Linear regression on a data set in the range $10^{-4} \leq r \leq 10^{-5.5}$ reveals the exponent to be $d \approx 1.8$. If the unstable set is indeed a circle packing, then the above scaling law implies that $d$ will be its circle packing exponent

$$d = \sup \left\{ x : \sum_{j=1}^{\infty} r_j^x = \infty \right\} = \inf \left\{ y : \sum_{j=1}^{\infty} r_j^y < \infty \right\}$$  \hspace{1cm} (20)$$
where the radii $r_j$ are labeled to form a non-increasing sequence: $r_1 \geq r_2 \geq \ldots$. The circle packing exponent is known as the Besicovitch-Taylor index in one dimension [46], and is equivalent to the exterior dimension $d_x$ and uncertainty exponent $\alpha$ of Grebogi et al [35]: $d = d_x = 2 - \alpha$ [37].

To investigate the Lebesgue measure of the unstable set we define the residual area

$$R(r) \equiv 4\pi - \pi \sum_{j=1}^{j(r)} r_j^2$$  \hspace{1cm} (21)$$
where $j(r)$ is the largest $j$ such that $r_j > r$. This quantity follows the scaling law

$$R(r) = R_0 + Br^{2-d}.$$  \hspace{1cm} (22)$$
Using linear regression on our data set (see figure 2(c)) we find that $R_0/4\pi \approx 0.1169$, and hence, the set of resonances (for the symmetric periodic orbits) measure less than 89% of the sphere. Note that the above scaling law also implies that $2 - d = \beta$, where $\beta$ is the fatness exponent of Farmer [33].

Finally we ask the most critical question: Does the set of resonances for the symmetric periodic orbits pack the surface of the sphere? A numerical scheme for answering this question might be to consider the set $C_r$ of all circular resonances with radius larger than $r$. If we define the quantity $s(r)$ to be the radius of the largest circle which will ‘fit’ into the complement of this set, then the unstable set has empty interior if and only if $s(r) \to 0$ as $r \to 0$. In figure 2(d) we plot $s(r)$ on a logarithmic scale and find that the scaling law

$$s(r) = Cr^\rho$$  \hspace{1cm} (23)$$
is approximately obeyed. The exponent might be thought of as describing the ‘completeness’ of the circle packing, and invites the general definition

$$\rho \equiv \lim_{r \to 0} \frac{\log s(r)}{\log r}.$$  \hspace{1cm} (24)$$
When constructing Apollonian packings, $s(r)$ is the radius of the next circle to be packed, and hence $\rho = 1$. In our case $\rho \approx 0.6$. This estimate, however, might prove to be a poor one. Note that $s(r)$ has been calculated only for $r \lesssim 10^{-4}$. Ideally we would like to use our entire set of circles and increase the graph to $r = 10^{-5.5}$. However the current plot took over a year to produce. Our procedure for finding all symmetric circles with radius larger than $r$ is extremely efficient. However we are unaware of an efficient algorithm for the calculation of $s(r)$. 
IV. CONCLUSION

In this paper we have outlined a procedure to efficiently calculate all symmetric periodic orbits up to a given period for the spherical map of [25]. When $\mu = \omega = \pi(\sqrt{5} - 1)$, the circular resonance islands corresponding to the stable orbits are found to initially follow a scaling law which indicates that they ‘pack’ the surface of the unit sphere. This would imply that all periodic orbits are of the symmetric type. Under this assumption, the unstable set has empty interior, but appears not to be of zero Lebesgue measure. The latter result agrees with Ashwin et al. [1, 2] for the sawtooth standard map. As a final remark, it might prove fruitful to study the index $\rho$ [24] as an indicator for nowhere dense sets. Measures of connectedness have already been defined [17].

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