SUPER RICCI FLOWS FOR WEIGHTED GRAPHS

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ABSTRACT. We present a notion of super Ricci flow for time-dependent finite weighted graphs. A challenging feature is that these flows typically encounter singularities where the underlying graph structure changes. Our notion is robust enough to allow the flow to continue past these singularities. As a crucial tool for this purpose we study the heat flow on such singular time-dependent weighted graphs with changing graph structure. We then give several equivalent characterizations of super Ricci flows in terms of a discrete dynamic Bochner inequality, gradient and transport estimates for the heat flow, and dynamic convexity of the entropy along discrete optimal transport paths. The latter property can be used to show that our notion of super Ricci flow is consistent with classical super Ricci flows for manifolds (or metric measure spaces) in a discrete to continuum limit.

1. Introduction

The main purpose of the present paper is to identify a natural time evolution of weighted graphs that can be considered as a discrete analogue of (super-)Ricci flow. Its second purpose is a study of the heat equation on time-dependent weighted graphs in a general setting. The latter will serve as a tool to give robust characterizations of discrete super Ricci flows, but might also be of independent interest. Before we enter the discrete setting, let us recall the classical notion of (super-)Ricci flow for manifolds and recent developments that motivate our work.

A smooth manifold $M$ equipped with a one-parameter family $(g_t)_{t \in I}$ of Riemannian metrics evolves as a Ricci flow if $\text{Ric}_{g_t} = -\frac{1}{2} \partial_t g_t$ for all $t \in I$. It is called a super Ricci flow if instead only $\text{Ric}_{g_t} \geq -\frac{1}{2} \partial_t g_t$ is satisfied as an inequality between quadratic forms, i.e. super Ricci flows are 'super solutions' to the Ricci flow equation.

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Since the seminal work of Hamilton \[16, 17\] and Perelman \[31, 32, 33\], see also \[4, 18, 29\], Ricci flow has received a lot of attention and has become a powerful tool in many applications. A challenging feature is that the flow typically develops singularities in finite time. Currently, a lot of activity is being devoted to extend the scope of Ricci flows beyond the setting of smooth manifolds. A major challenge is to define and analyze flows that pass through singularities where dimension and/or topological type changes. Among the exciting recent contributions we mention the work of Bamler, Kleiner and Lott \[19, 1\] constructing canonical Ricci flows through singularities in dimension 3 as the limit of flows with surgery and the work of Haslhofer and Naber \[34\] characterizing Ricci flows in terms of functional inequalities on the path space, see also Cheng and Thalmeier \[5\]. Sturm \[37\] introduced a synthetic definition of super Ricci flow that applies to time-dependent metric measure spaces using optimal transport. Here, the crucial observation is that for a smooth family of Riemannian manifolds to be a super Ricci flow is equivalent to dynamic convexity of the Boltzmann entropy along geodesics in the space of probability measures equipped with the (time-dependent) $L^2$-Kantorovich distance (see Sec. 1.1 for a definition). The latter property is meaningful when the manifold is replaced with a time-dependent metric measure space and serves as a synthetic definition of super Ricci flow.

In the case of a static Riemannian metric, the super Ricci flow equation becomes $\text{Ric}_g \geq 0$ and the notion of dynamic convexity reduces to convexity of the entropy along geodesics in the Kantorovich distance, the property used as a synthetic definition of lower Ricci curvature bounds in the celebrated works of Lott, Sturm and Villani \[38, 23\].

In view of the powerful applications of Ricci flow, it seems desirable to develop a similar concept for discrete spaces, for instance as a natural way of deforming a given space to a simpler object. Unfortunately, the approach of Sturm \[37\] does not apply in this situation since the $L^2$ Kantorovich distance is degenerate if the underlying space is discrete in the sense that it does not admit geodesics. The main objective of the present article is to develop a notion of super Ricci flow that applies to discrete spaces, namely to time-dependent weighted graphs. In order to circumvent the non-existence of geodesics, we will replace the Kantorovich distance by a different distance $W$ on the space of probability measures, constructed in \[24\], that is well-adapted to the discrete setting. In the case of a static weighted graph (or Markov chain) this distance has been used successfully in \[9\] to define a notion of lower Ricci curvature bounds in the spirit of the theory of Lott, Sturm and Villani via convexity of the entropy along $W$-geodesics. Here, in the time-dependent case, super Ricci flow will be defined via dynamic convexity of the entropy.

As in the continuous case, our discrete super Ricci flows will typically produce singularities in finite time. A simple example is depicted in Figure 1. Here, several vertices collapse and the weights of the connecting edges explode at the singularity $t_1$ like $q_t = 1/(t_1 - t)$ for instance. This can be seen in analogy to the continuous example of $S^2 \times T^2$ equipped with the product $((1 - 2t)g_{S^2}) \otimes g_{T^2}$ of the scaled round and the flat metric and collapsing to $T^2$ at $t = \frac{1}{2}$. An important feature of our approach is that it allows to define discrete super Ricci flows through such singularities. In fact, we will show that discrete super Ricci flows can be characterized equivalently via a discrete dynamic Bochner inequality and via gradient and transport estimates for the heat flow. These latter characterizations hold consistently across singular times where the graph structure changes. To this end, we perform a detailed analysis of the heat flow on general time-dependent weighted graphs allowing for a variety of singular...
phenomena such as collapse and spawning of vertices, or deletion and creation of edges. In particular, we establish existence and uniqueness of the heat flow. Finally, this provides a second motivation for our investigation of the discrete setting as a ‘sandbox’ to develop methods to be used eventually also in the technically more challenging setting of continuous singular time-dependent spaces and (super) Ricci flows. For instance, the analysis of the heat flow on time-dependent metric measure space, initiated in [20], currently cannot deal with the singularities at which the base space changes.

1.1. Robust characterizations of super Ricci flows. Before we describe our main results in more detail, let us briefly recall several robust characterizations of classical super Ricci flows in terms of the heat flow and optimal transport, as they will serve as a guideline for the discrete setting.

Let \((g_t)_{t \in I}\) be a smooth family of Riemannian metrics on a (compact) manifold \(M\). We denote by \(\Delta_t\) the Laplace–Beltrami operator associated with \(g_t\). The heat flow is given by the propagator \(P_{t,s} \bar{\psi}\), defined for \(s \leq t\) as the solution to the heat equation \(\partial_t \psi = \Delta_t \psi\) with initial condition \(\psi(\cdot, s) = \bar{\psi}\). By duality, we define the heat flow on probability measures given by the propagator \(\hat{P}_{t,s} \mu\) characterized via

\[
\int \psi \, d(\hat{P}_{t,s} \mu) = \int P_{t,s} \psi \, d\mu.
\]

The \(L^2\)-Kantorovich distance on the space probability measures \(\mathcal{P}(M)\) is given by

\[
W_2(t, \mu, \nu)^2 = \inf_{\pi} \int d_t(x, y)^2 \, d\pi(x, y),
\]

where the infimum is taken over all couplings of \(\mu\) and \(\nu\) and \(d_t\) is the Riemannian distance. Finally, denote by \(\mathcal{H}_t(\mu) = \int \rho \log \rho \, d\mu_{g_t}\) for \(\mu = \rho \, d\mu_{g_t}\) the Boltzmann entropy. The connection between these objects is that the (dual) heat flow evolves as the gradient flow of the entropy w.r.t. the Kantorovich distance.

Now, the super Ricci flow equation

\[
\text{Ric}_{g_t} \geq -\frac{1}{2} \partial_t g_t \tag{1.1}
\]

is equivalent to any of the following properties:
(I) *dynamic Bochner inequality:* for all smooth functions $\psi$ on $M$ and $t \in I$:
\[
\Gamma_{2,t}(\psi) \geq \frac{1}{2} \partial_t \Gamma_t(\psi),
\]
where $\Gamma_t(\psi) := |\nabla \psi|^2_{g_t}$ and $\Gamma_{2,t}(\psi) := \frac{1}{2} \Delta_t( |\nabla \psi|^2_{g_t} ) - \langle \nabla \psi, \nabla \Delta_t \psi \rangle_{g_t}$ are the carré du champs operators associated to the Laplace–Beltrami operator $\Delta_t$.

(II) *gradient estimate:* for all smooth functions $\psi$ and $s \leq t$:
\[
\Gamma_t(P_{t,s} \psi) \leq P_{t,s} \Gamma_s(\psi),
\]

(III) *transport estimates:* for all probability measures $\mu, \nu$ and $s \leq t$:
\[
W_{2,s}(\hat{P}_{t,s} \mu, \hat{P}_{t,s} \nu) \leq W_{2,t}(\mu, \nu),
\]

(IV) *dynamic convexity of entropy:* for all $t$ and all geodesics $(\mu^a)_{a \in [0,1]}$ in $(\mathcal{P}(M), W_{2,t})$:
\[
\partial^+_a \mathcal{H}_t(\mu^1) - \partial^-_a \mathcal{H}_t(\mu^{0+}) \geq -\frac{1}{2} \partial_t W_{t-}((\mu^0, \mu^1)^2).
\]

Here and in the sequel we denote by $\partial^+_a f(a \pm)$ the upper/lower right/left derivative of $f$ at $a$, i.e. for instance
\[
\partial^+_a f(a+) := \limsup_{b \searrow a} \frac{f(b) - f(a)}{b - a}.
\]

The connection between (I) and (1.1) stems immediately from the Bochner identity $\Gamma_{2,t}(\psi) = \text{Ric}_{g_t}[\nabla \psi] + ||\text{Hess}_{g_t} \psi||^2_{\text{HS}}$. (I) and (II) are connected via a classical interpolation argument. The characterization (III) in terms of non-expansion of the transport distance under the heat flow was observed by McCann and Topping [25]. Characterization (IV) was established in [37] and should be thought of as a quantified formulation of convexity in terms of the increase of the first derivative.

In the static case it reduces to plain convexity of the entropy along geodesics characterizing non-negative Ricci curvature, see [35, 7].

As already mentioned, the advantage of the these characterizations is their robustness, i.e. that they remain meaningful in a non-smooth setting. For instance (I), (II) can be formulated for a family of time-dependent Dirichlet forms. (IV) requires only the structure of a time-dependent metric measure space $(X_t, d_t, m_t)_{t \in I}$. Sturm and the second author [20] proved that the equivalence of (I) - (IV) holds in the setting of metric measure spaces, at least under stringent regularity conditions (namely, $X_t \equiv X$ is independent of $t$, a curvature-dimension bound $\text{RCD}(K, \infty)$ holds uniformly in time, and Lipschitz controls on $d_t$ and $m_t$).

In this article, in the setting of time-dependent weighted graphs, we will obtain similar equivalent characterization (I)-(IV), where the carré du champs operators and the transport distance are replaced with suitable discrete counterparts, and where we allow for changing graph structure.

### 1.2. Main results.

Let us now discuss the content of this article in more detail.

We will consider a time-dependent family of *Markov triples* $(X_t, Q_t, \pi_t)_{t \in [0,T]}$. Here for each $t$, $X_t$ is a finite set, $\pi_t$ is a strictly positive probability measure on $X_t$, and $Q_t : X_t \times X_t \to \mathbb{R}$ is a kernel giving the transition rates of a continuous time Markov chain with the convention that $Q_t(x,y) \geq 0$ for $x \neq y$ and $Q_t(x,x) = -\sum_{y \neq x} Q_t(x,y)$. We will assume that $Q_t$ is reversible, i.e. the detailed balance condition holds:
\[
Q_t(x,y) \pi_t(x) = Q_t(y,x) \pi_t(y) \quad \forall x, y \in X_t.
\]
Equivalently, we can consider the family of weighted graphs \((\mathcal{X}_t, w_t, \pi_t)\), where \(\mathcal{X}_t\) is the set of vertices, \(\pi_t\) is the vertex weight, and the symmetric function \(w_t(x,y) := Q_t(x,y)\pi_t(x)\) is the edge-weight and the set of edges is given by \(\mathcal{Y}_t = \{\{x,y\} : w_t(x,y) > 0\}\). We will also assume that \(Q_t\) is irreducible, i.e. the associated graph is connected.

We allow the graph structure to change at a finite number of times. More precisely, we will assume that there exists a partition \(0 = t_0 < t_1 < \cdots < t_n = T\) such that \((\mathcal{X}_t, \mathcal{Y}_t) \equiv (\mathcal{X}_s, \mathcal{Y}_s)\) for all \(t \in I_i = (t_i, t_{i+1})\) and all \(i = 0, \ldots, n - 1\). During the intervals \(I_i\) we assume that \(t \mapsto \pi_t\) is Lipschitz and that \(t \mapsto Q_t\) is locally Lipschitz with limits existing in \([0, +\infty]\) as we approach the singular times, i.e. \(t \downarrow t_i\) and \(t \uparrow t_{i+1}\). If the limit of \(Q_t(x,y)\) is \(+\infty\), we assume moreover, that the accumulated transition rate explodes, i.e.

\[
\int_{t_i}^{t_{i+1}} Q_t(x,y)dt = +\infty \quad \text{resp.} \quad \int_t^{t_{i+1}} Q_t(x,y)dt = +\infty .
\] (1.2)

Moreover, the limiting weights are assumed to be compatible with the weighted graph structure at singular times \(t_i\). For a precise statement of our assumptions see Section 3.1.

The interpretation is that the graph structure can change at singular times \(t_i\) due to different phenomena:

- edges can disappear (resp. appear), corresponding to \(w_t(x,y) \rightarrow 0\) as \(t \uparrow t_i\) (resp. \(t \downarrow t_i\)),
- two vertices \(x, y\) can collapse, this happens if \(w_t(x,y) \rightarrow \infty\) as \(t \uparrow t_i\),
- a vertex can spawn new vertices (same as collapse but backwards in time).

Let us denote for \(z \in \mathcal{X}_{t_{i+1}}\) by \(C_z \subset \mathcal{X}_t\) the set of vertices that collapse onto \(z\) at \(t_{i+1}\). Similarly, let \(S_z \subset \mathcal{X}_{t+1}\) denote the set of vertices spawned by \(z\).

Our first main result (see Thm. 3.5 below) concerns the existence and uniqueness of solutions to the (dual) heat equation in this general setting. To this end, we introduce the discrete Laplacian \(\Delta_t\) and dual Laplacian \(\hat{\Delta}_t\) associated with \((\mathcal{X}_t, Q_t, \pi_t)\) acting on functions \(\psi, \sigma \in \mathbb{R}^{\mathcal{X}_t}\) via

\[
\Delta_t \psi(x) := \sum_{y \in \mathcal{X}_t} \left[\psi(y) - \psi(x)\right] Q_t(x,y),
\]

\[
\hat{\Delta}_t \sigma(x) := \sum_{y \in \mathcal{X}_t} \left[Q_t(y,x)\sigma(y) - Q_t(x,y)\sigma(x)\right].
\]

For \(0 \leq s < t \leq T\), let us define space-time during the interval \([s, t]\) by setting \(S_{s,t} := \{(r, x) : r \in [s, t], x \in \mathcal{X}_t\}\).

**Theorem 1.1.** Given \(s \in [0, T]\) and \(\bar{\psi} \in \mathbb{R}^{\mathcal{X}_s}\), there exist a unique \(\psi : S_{s,t} \rightarrow \mathbb{R}\) such that:

- \(\psi(s, \cdot) = \bar{\psi}\), the map \(t \mapsto \psi(t, \cdot)\) is differentiable on each \(I_i = (t_i, t_{i+1})\) and satisfies
  \[
  \partial_t \psi = \Delta_t \psi \quad \text{on} \quad I_i \times \mathcal{X}_i,
  \] (1.3)

- for all \(z \in \mathcal{X}_{t_i}\), \(x \in S_z\) and \(y \in C_z\) we have
  \[
  \psi(t_i, z) = \lim_{t \downarrow t_i} \psi(t, x) = \lim_{t \uparrow t_i} \psi(t, y).
  \] (1.4)

Given \(t \in [0, T]\) and \(\bar{\sigma} \in \mathbb{R}^{\mathcal{X}_t}\) there exist a unique \(\sigma : S_{0,t} \rightarrow [0, \infty)\) such that:

- \(\sigma(t, \cdot) = \bar{\sigma}\), the map \(s \mapsto \sigma(s, \cdot)\) is differentiable on each \(I_i = (t_i, t_{i+1})\) and satisfies
  \[
  \partial_s \sigma = -\hat{\Delta}_s \sigma \quad \text{on} \quad I_i \times \mathcal{X}_i,
  \] (1.5)
for all $z \in X_t$, we have
\begin{equation}
\sigma(t, z) = \sum_{x \in S_z} \lim_{s \uparrow t} \sigma(s, x) = \sum_{y \in C_z} \lim_{s \downarrow t} \sigma(s, y) .
\end{equation}

We define the heat propagator $P_{t,s} : \mathbb{R}^{X_t} \rightarrow \mathbb{R}^{X_t}$ and dual heat propagator $\hat{P}_{t,s} : \mathbb{R}^{X_t} \rightarrow \mathbb{R}^{X_t}$ by setting $P_{t,s} \tilde{\psi} = \psi(t, \cdot)$, $\hat{P}_{t,s} \bar{\sigma} = \sigma(s, \cdot)$. Note that the dual heat equation is interpreted as running backwards in time. This is natural in view of the following duality relation. Interpreting the Euclidean scalar product $\langle \psi, \sigma \rangle$ as the integral of $\psi$ against a (signed) measure $\sigma$, we have that
\begin{equation}
\langle P_{t,s} \psi, \sigma \rangle = \langle \psi, \hat{P}_{t,s} \sigma \rangle .
\end{equation}

Existence and uniqueness of solutions to (1.3) and (1.5) on the intervals $I_t$ is of course guaranteed by standard theory of ODEs. The first non-trivial aspect of the previous theorem is that the solution has a well-defined limit as we approach singular times. Here the assumption (1.2) will be crucial, which ensures that during a collapse the solution to the heat equation already equilibrates before the singular time on the group of collapsing vertices and thus leads to (1.4), similarly for the dual heat equation and spawning events. The second non-trivial aspect is that the solution can be continued from singular times. Here, the dual equation (1.5) starting from non-singular times will be used to construct the solution to (1.3) and vice-versa exploiting the duality (1.7).

In order to state our second main result on the characterization of discrete super Ricci flows, we need to introduce discrete analogues of the optimal transport distance and the carré du champs operator.

For each $t$ we consider the discrete transport distance $\mathcal{W}_t$ between probability measures $\mu^0, \mu^1 \in \mathcal{P}(X_t)$ given by
\begin{equation}
\mathcal{W}_t(\mu^0, \mu^1)^2 = \inf_{\mu, V} \left\{ \int_0^1 \frac{1}{2} \sum_{x,y \in X_t} |V_a(x, y)|^2 \frac{\Lambda(\mu^a)_t(x, y)}{a} \right\} ,
\end{equation}
where the infimum runs over all sufficiently regular curves $\mu : [0, 1] \rightarrow \mathcal{P}(X_t)$ connecting $\mu^0$ and $\mu^1$, and $V : [0, 1] \rightarrow \mathbb{R}^{X_t \times X_t}$ satisfying the discrete continuity equation
\begin{equation}
\frac{d}{da} \mu^a(x) + \frac{1}{2} \sum_{y \in X_t} \left[ V^a(x, y) - V^a(y, x) \right] = 0 ,
\end{equation}

and we write $\Lambda(\mu)_t := \Lambda(\mu(x)Q_t(x, y), \mu(y)Q_t(y, x))$, where $\Lambda(s, t) := \int_0^1 s^\alpha t^{1-\alpha} ds$ denotes the logarithmic mean of $s, t \geq 0$. This distance associated to a Markov triple has been introduced in [24] and can be thought of as a discrete analogue of the Benamou–Brenier formula for the $L^2$-Kantorovich distance.

Moreover, we introduce for $\psi \in \mathbb{R}^{X_t}$ and $\mu \in \mathcal{P}(X_t)$ the integrated carré du champs operator
\begin{equation}
\Gamma_t(\mu, \psi) = \langle \nabla \psi, \nabla \cdot \Lambda(\mu)_t \rangle ,
\end{equation}
where $\nabla \psi(x, y) = \psi(y) - \psi(x)$ denotes the discrete gradient, and the multiplication with $\Lambda(\mu)_t$ is understood componentwise in $\mathbb{R}^{X_t \times X_t}$. We also introduce an integrated iterated carré du champs operator $\Gamma_{2,t}(\mu, \psi)$, see Section 2.2. These quantities should be thought of as discrete
analogues of
\[ \int \Gamma_t(\psi) d\mu, \int \Gamma_{2,t}(\psi) d\mu, \]
where \( \Gamma_t, \Gamma_{2,t} \) are the carré du champs operators associated to the Laplacian \( \Delta_t \) in the continuous setting, c.f. Section 1.1. Finally, let us denote by
\[ \mathcal{H}_t(\mu) = \sum_{x \in X_t} \log \frac{\mu(x)}{\pi_t(x)} \mu(x) \]
the relative entropy of \( \mu \in \mathcal{P}(X_t) \) w.r.t. the reference measure \( \pi_t \). Our second main result (see Theorem 4.1) is the following:

**Theorem 1.2.** Let \((X_t, Q_t, \pi_t)_{t \in [0,T]}\) be a time-dependent Markov triple satisfying (4.1) and (4.2). Then the following are equivalent:

(I) The dynamic Bochner inequality
\[ \Gamma_{2,t}(\mu, \psi) \geq \frac{1}{2} \partial_t \Gamma_t(\mu, \psi) \] (1.8)
holds for a.e. \( t \in [0,T] \) and all \( \mu \in \mathcal{P}(X_t), \psi \in \mathbb{R}^{X_t} \).

(II) The gradient estimate
\[ \Gamma_t(\mu, P_{t,s} \psi) \leq \Gamma_s(\hat{P}_{t,s} \mu, \psi) \] (1.9)
holds for all \( 0 \leq s \leq t \leq T \) and all \( \mu \in \mathcal{P}(X_t), \psi \in \mathbb{R}^{X_s} \).

(III) The transport estimate
\[ W_s(\hat{P}_{t,s} \mu, \hat{P}_{t,s} \nu) \leq W_t(\mu, \nu) \] (1.10)
holds for all \( 0 \leq s \leq t \leq T \) and all \( \mu, \nu \in \mathcal{P}(X_t) \).

(IV) The entropy is dynamically convex, i.e. for a.e. \( t \in [0,T] \) and all \( \mathcal{W}_t \)-geodesics \( (\mu^a)_{a \in [0,1]} \)
\[ \partial^+_a \mathcal{H}_t(\mu^1) - \partial^-_a \mathcal{H}_t(\mu^{0+}) \geq -\frac{1}{2} \partial_t W^2_t(\mu^0, \mu^1) \] (1.11)
A time-dependent Markov triple \((X_t, Q_t, \pi_t)_{t \in [0,T]}\) is called a discrete super Ricci flow if any of these equivalent properties hold.

The properties (I)–(IV) are natural discrete analogues of the corresponding properties characterizing classical smooth super Ricci flows discussed in Section 1.1. An essential aspect here is that the gradient estimate (II) and the transport estimate (III) are requested to hold for all \( s \leq t \), i.e. also across singular times. This is what allows us to give a consistent definition of discrete super Ricci flow through singularities where the graph structure changes.

Let us briefly comment on some ideas for the proof of Theorem 1.2. We will show that (I) implies (II) via a classical interpolation argument, considering the quantity \( \Phi(r) = \Gamma_r(\hat{P}_{t,r} \mu, P_{r,s} \psi) \) and differentiating in \( r \). The crucial observation is that \( \Phi \) is continuous at the singular times. To show that (II) implies (III) we employ a dual formulation of the discrete transport distance \( W_t \) in the spirit of the Kantorovich duality involving subsolutions to the Hamilton-Jacobi equation, that was recently established independently in [10] and [12], see Section 2.3. The reverse implication will be shown by taking \( \nu \) close to \( \mu \) on the geodesic in direction \( \nabla \psi \) and employing again the duality. The implication from (II)/(III) to (IV) constitutes the technical core of the argument. Inspired by arguments in [20], we will show
that (II) implies that the heat flow can be characterized as the gradient flow of the entropy w.r.t. $W_t$ in the sense of a dynamic evolution variational inequality. This together with (III) will imply the dynamic convexity (IV). Finally, (IV) will imply (I) after noting that $\Gamma_{2,t}(\mu,\cdot)$ coincides with the Hessian of the entropy $H_t$ at $\mu$.

Our third main result concerns the consistency of discrete super Ricci flow with classical super Ricci flows and more generally with the synthetic definition of Sturm [37] for time-dependent metric measure spaces. In Theorem 6.4 we identify a suitable notion of convergence of a sequence $(X^{(n)}, Q^{(n)}, \pi^{(n)})_{t \in I}$ of time-dependent Markov triples to a time-dependent Riemannian manifold or metric measure space $(X, d_t, m_t)_{t \in I}$ such that a limit of discrete super Ricci flows is again a super Ricci flow. To this end, we assume that maps $i_n : \mathcal{P}(X^{(n)}) \to \mathcal{P}(X)$ exist and postulate a suitable sort of $\Gamma$-convergence of the entropies and the transport distances along these maps. Under some uniform regularity assumption on the time-dependence this will suffice for the stability of super Ricci flows. In order to pass to the limit we will employ an integrated formulation of the dynamic convexity property (IV) already used in [37].

We think of the Markov triples as finer and finer discrete approximations of the spaces $(X, d_t, m_t)$. This approximation should come with a natural way of extending (and regularizing) measures from the discrete approximation to the full space, given by the maps $i_n$. The purpose of our result is to identify sufficient conditions for the stability of super Ricci flows. In practice, the $\Gamma$-convergence of entropies should be a soft requirement. Convergence of the transport distances seems harder to establish. Some results are available in the static case by Gigli and Mass [13] and Trillos [39] for a lattice resp. point cloud approximations of the torus, or by Gladbach, Maas and the second author for finite element approximation of Euclidean domains [14]. Convergence results for discrete transport distances on curved spaces remain an interesting open problem at the moment.

1.3. Connection to the literature. Let us briefly mention other related results in the literature. As already discussed, our approach is close in spirit to the synthetic approach to super Ricci flow in [37, 20] and consistent with the discrete notion of Ricci curvature bounds considered in [9]. Many other approaches to Ricci curvature for (weighted) graphs have been proposed, let us mention in particular the combinatorial notion of Forman [11], the coarse Ricci curvature by Ollivier [30] based on the $L^1$ Kantorovich distance, and approaches based on (modifications) of a discrete Bakry–Émery $\Gamma_2$ criterion, see e.g. [22, 2, 8]. A notion of discrete Ricci flow based on Forman’s combinatorial Ricci curvature has been studied recently in [41] and applied to the analysis of complex networks. Also the latter two curvature notions could be used to define a notion of (super) Ricci flow for weighted graphs. For Ollivier’s curvature this was proposed e.g. in [36] motivated by the analysis of complex cancer networks. We are not aware of any works studying these notions in more detail. A lot of activity has been devoted to discrete notions of Ricci flow in the more specific setting of triangulated surfaces. See for instance the work of Chow and Luo [6] related to circle packings. These notions have broad applications in graphics and medical imaging, for instance, see e.g. [43, 42]. A generalization of discrete Ricci flow to higher dimensional simplicial structures termed simplicial Ricci flow has recently been proposed in [28]. Also other curvature flows such as the Yamabe flow have been considered in the discrete setting [15]. An advantage of our approach is that our notion of super Ricci flow can naturally be defined through singularities. To our knowledge, this has not been considered for the other notions
discussed above. Another advantage is that our discrete super Ricci flow yields strong control on the heat flow on the evolving graph.

**Organization.** The article is organized as follows. In Section 2 we recall the notion of discrete transport distance on weighted graphs, in particular its dual formulation that will be crucial in proving the equivalence of the different characterizations of super Ricci flows. We also recall the notion of entropic Ricci bounds for weighted graphs. In Section 3 we describe in detail the setting of singular time-dependent Markov triples with changing graph structure that we consider. Then we prove existence and uniqueness of solutions to the heat equation and dual heat equation in this general setting. In Section 4 we prove equivalence of the different characterizations of super Ricci flows. Several examples of super Ricci flows are presented in Section 5. Finally, in Section 6, we discuss the consistency of our discrete notion of super Ricci flow with the notion of super Ricci flow for smooth manifolds or continuous metric measure spaces.

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### 2. Preliminaries on discrete optimal transport

Here we briefly recall the definitions of the discrete transport distance $W$ and the associated Riemannian structure introduced independently in [24, 26], and the entropic Ricci curvature bounds for finite Markov chains introduced and studied in [9]. Finally we derive a dual formulation of the transport distance.

#### 2.1. Discrete transport distance and Ricci bounds.

Let $\mathcal{X}$ be a finite set and let $Q : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ be a collection of transition rates. The operator $\Delta$ acting on functions $\psi : \mathcal{X} \to \mathbb{R}$ via

$$
\Delta \psi(x) = \sum_{y \in \mathcal{X}} Q(x, y)(\psi(y) - \psi(x))
$$

is the generator of a continuous time Markov chain on $\mathcal{X}$. We make the convention that $Q(x, x) = -\sum_{y \neq x} Q(x, y)$ for all $x \in \mathcal{X}$. We assume that $Q$ is irreducible, i.e. for all $x, y \in \mathcal{X}$ there exists a path $x_0 = x, x_1, \ldots, x_n = y$ such that $Q(x_i, x_{i+1}) > 0$. We assume moreover, that $Q$ is reversible. More precisely, we assume that there exists a strictly positive probability measure $\pi$ on $\mathcal{X}$ such that the detailed-balance condition holds:

$$
Q(x, y)\pi(x) = Q(y, x)\pi(y) \quad \forall x, y \in \mathcal{X}.
$$

A triple $(\mathcal{X}, Q, \pi)$ as above will be called a *Markov triple*.

We consider a distance $W$ on the set $\mathcal{P}(\mathcal{X})$ of probability measures on $\mathcal{X}$ defined as follows: for $\mu_0, \mu_1 \in \mathcal{P}(\mathcal{X})$ set

$$
W(\mu_0, \mu_1)^2 = \inf_{\mu, V} \left\{ \int_0^1 \mathcal{A}(\mu_t, V_t)dt : (\mu, V) \in \mathcal{CE}_1(\mu_0, \mu_1) \right\},
$$

where...

...
where \( CECT(\mu_0, \mu_1) \) denotes the collection of pairs \((\mu, V)\) satisfying the continuity equation, more precisely, the following conditions:

\[
\left\{
\begin{array}{l}
(i) \quad \mu : [0, T] \to \mathbb{R}^X \text{ is continuous;} \\
(ii) \quad \mu(0) = \mu_0, \quad \mu(T) = \mu_1; \\
(iii) \quad \mu(t) \in \mathcal{P}(X) \text{ for all } t \in [0, T]; \\
(iv) \quad V : [0, T] \to \mathbb{R}^{X \times X} \text{ is locally integrable;} \\
(v) \quad \text{For all } x \in X \text{ we have in the sense of distributions}
\end{array}
\right.
\]

\[
\frac{d\mu_t(x) + \frac{1}{2} \sum_{y \in X} (V_t(x, y) - V_t(y, x)) = 0.}
\]

\[(2.3)\]

The action \( A \) is defined via

\[
A(\mu, V) = \frac{1}{2} \sum_{x,y} \frac{V(x, y)^2}{\Lambda(\mu)(x, y)}, \quad \Lambda(\mu)(x, y) = \hat{\mu}(x, y) := \Lambda(\mu(x)Q(x, y), \mu(y)Q(y, x))
\]

where \( \Lambda \) denotes the logarithmic mean given by

\[
\theta(s, t) = \int_0^1 s^{\alpha} t^{1-\alpha} d\alpha.
\]

More precisely, we set

\[
A(\mu, V) = \frac{1}{2} \sum_{x,y} \alpha \left( V(x, y), \mu(x)Q(x, y), \mu(y)Q(y, x) \right),
\]

where the convex and lower semicontinuous function \( \alpha : \mathbb{R} \times \mathbb{R}^2_+ \to \mathbb{R} \cup \{+\infty\} \) is defined by

\[
\alpha(x, s, t) = \begin{cases} 
\frac{x^2}{\Lambda(s, t)}, & s, t \neq 0 \ , \\
0, & \Lambda(s, t) = 0 \text{ and } x = 0 \ , \\
+\infty, & \text{else}.
\end{cases}
\]

It is readily checked, that this formulation of \( W \) is equivalent to the one given in [9, Lem. 2.9], in particular, in the the definition of \( W \) one can restrict the infimum to curves \( \mu \) and \( V \) that are smooth.

It has been shown in [24] that \( W \) defines a distance on \( \mathcal{P}(X) \). It turns out that it is induced by a Riemannian structure on the interior \( \mathcal{P}_+(X) \), consisting of all strictly positive probability measures. The distance \( W \) can be seen as a discrete analogue of the Benamou–Brenier formulation [3] of the continuous \( L^2 \)-transportation cost. The role of the logarithmic mean is due to provide a discrete chain rule for the logarithm, namely \( \hat{\rho} \nabla \log \rho = \nabla \rho \), where we write \( \nabla \psi(x, y) = \psi(y) - \psi(x) \) and \( \hat{\rho}(x, y) = \Lambda(\rho(x), \rho(y)) \). The distance \( W \) is tailor-made in this way such that the discrete heat equation \( \partial_t \rho = \Delta \rho \) is the gradient flow of the relative entropy

\[
\mathcal{H}(\mu) = \sum_{x \in X} \log \frac{\mu(x)}{\pi(x)} \mu(x)
\]

w.r.t. the Riemannian structure induced by \( W \) [24, 26], making \( W \) a natural replacement of the \( L^2 \)-Kantorovich distance in the discrete setting.

Every pair of measures \( \mu_0, \mu_1 \in \mathcal{P}(X) \) can be joined by a constant speed \( W \)-geodesic \((\mu_s)_{s \in [0,1]} \). Here constant speed geodesic means that \( W(\mu_s, \mu_t) = |s - t| W(\mu_0, \mu_1) \) for all \( s, t \in [0,1] \). The geodesic is a minimizer in (2.2).
In analogy with the approach of Lott–Sturm–Villani, the following definition of a Ricci curvature lower bound has been given in [9].

**Definition 2.1.** \((\mathcal{X}, Q, \pi)\) has Ricci curvature bounded from below by \(\kappa \in \mathbb{R}\), if for any constant speed geodesic \(\{\mu_t\}_{t \in [0,1]}\) in \((\mathcal{P}(\mathcal{X}), \mathcal{W})\), we have
\[
\mathcal{H}(\mu_t) \leq (1 - t)\mathcal{H}(\mu_0) + t\mathcal{H}(\mu_1) - \frac{\kappa}{2}t(1 - t)\mathcal{W}(\mu_0, \mu_1)^2.
\]
In this case, we write \(\text{Ric}(\mathcal{X}, Q, \pi) \geq \kappa\).

### 2.2. Riemannian structure and Bochner-type inequality.

Entropic curvature bounds can be expressed equivalently via an inequality resembling Bochner’s inequality in Riemannian geometry. To this end, let us describe the Riemannian structure induced by \(\mathcal{W}\). To alleviate notation, let us denote for \(\Phi, \Psi \in \mathbb{R}^{X \times X}\) their Euclidean inner product by
\[
\langle \Phi, \Psi \rangle = \frac{1}{2} \sum_{x,y \in \mathcal{X}} \Phi(x,y)\Psi(x,y).
\]
At each \(\mu \in \mathcal{P}_s(\mathcal{X})\) the tangent space to \(\mathcal{P}_s(\mathcal{X})\) is given by \(\mathcal{T} = \{ s \in \mathbb{R}^X : \sum_x s(x) = 0 \}\). Given \(\psi \in \mathbb{R}^X\) we denote by \(\nabla \psi : \mathbb{R}^{X \times X}\) the discrete gradient of \(\psi\), i.e. the quantity \(\nabla \psi(x,y) = \psi(y) - \psi(x)\). Let \(\mathcal{G} = \{ \nabla \psi : \psi \in \mathbb{R}^X \}\) denote the set of all discrete gradient fields and note that \(\mathcal{G}\) is in bijection to the set \(\mathcal{G}' = \{ \psi \in \mathbb{R}^X : \psi(x_0) = 0 \}\). In [24, Sec. 3] it has been shown that for each \(\mu \in \mathcal{P}_s(\mathcal{X})\), the map
\[
K_{\mu} : \psi \mapsto \sum_y \nabla \psi(y,x)\Lambda(\mu)(x,y),
\]
defines a linear bijection between \(\mathcal{G}\) and the tangent space \(\mathcal{T}\). This identification can be used to define a Riemannian metric tensor on \(\mathcal{P}_s(\mathcal{X})\) by introducing the scalar product \(\langle \cdot, \cdot \rangle_\mu\) on \(\mathcal{G}\) given by
\[
\langle \nabla \psi, \nabla \varphi \rangle_\mu = \langle \nabla \psi, \nabla \varphi \cdot \Lambda(\mu) \rangle = \frac{1}{2} \sum_{x,y} \nabla \psi(x,y)\nabla \varphi(x,y)\Lambda(\mu)(x,y).
\]
Then \(W\) is the Riemannian distance associated to this Riemannian structure. Note that if we introduce the divergence of \(\Phi \in \mathbb{R}^{X \times X}\) via
\[
\nabla \cdot \Phi(x) := \frac{1}{2} \sum_{y \in \mathcal{X}} \Phi(x,y) - \Phi(y,x),
\]
we can write for short \(K_{\mu,\psi} = \nabla \cdot (\Lambda(\mu) \cdot \psi)\).

Let us introduce the following integrated carré du champs operators. For \(\mu \in \mathcal{P}(\mathcal{X})\) (resp. \(\mu \in \mathcal{P}_s(\mathcal{X})\)) and \(\psi \in \mathbb{R}^X\) set
\[
\Gamma(\mu, \psi) := \langle \nabla \psi, \nabla \psi \cdot \Lambda(\mu) \rangle = \| \nabla \psi \|_{\mu}^2,
\]
\[
\Gamma_2(\mu, \psi) := \frac{1}{2} \langle \nabla \psi, \nabla \psi \cdot \hat{\Lambda}(\mu) \rangle - \langle \nabla \psi, \nabla \Lambda(\mu) \rangle,
\]
where we have used the notation
\[
\hat{\Lambda}(\mu)(x,y) := \left[ \partial_1 \Lambda(\rho(x), \rho(y)) \Delta \rho(x) + \partial_2 \Lambda(\rho(x), \rho(y)) \Delta \rho(y) \right] Q(x,y)\pi(x),
\]
with \(\rho(x) = \mu(x)/\pi(x)\) and where the multiplication with \(\Lambda(\mu)\) and \(\hat{\Lambda}(\mu)\) is defined component wise.
Entropic Ricci bounds, i.e. convexity of the entropy along $\mathcal{W}$-geodesics, are determined by bounds on the Hessian of the entropy $\mathcal{H}$ in the Riemannian structure defined above. An explicit expression of the Hessian at $\mu \in \mathcal{P}_*(\mathcal{X})$ is given by

$$\text{Hess}\mathcal{H}(\mu)[\nabla \psi] = \Gamma_2(\mu, \psi).$$

We then have the following equivalent characterization of entropic Ricci bounds.

**Proposition 2.2 ([9, Thm. 4.4]).** A Markov triple $(\mathcal{X}, Q, \pi)$ satisfies $\text{Ric}(\mathcal{X}, Q, \pi) \geq \kappa$ if and only if for every $\mu \in \mathcal{P}_*(\mathcal{X})$ and every $\psi \in \mathbb{R}^\mathcal{X}$ we have

$$\Gamma_2(\mu, \psi) \geq \kappa \Gamma(\mu, \psi).$$

Note that this statement is non-trivial since the Riemannian metric degenerates at the boundary of $\mathcal{P}(\mathcal{X})$. In view of (2.4), (2.5), the criterion above closely resembles (an integrated version of) the classical Bochner inequality or Bakry–Émery $\Gamma_2$-criterion. Namely, a Riemannian manifold $M$ satisfies $\text{Ric} \geq \kappa$ if and only if for every smooth function $\psi : M \to \mathbb{R}$ and probability $\mu = \rho \text{vol}$ we have:

$$\int_M \frac{1}{2} \left[ \Delta \rho |\nabla \psi|^2 - \rho (\nabla \psi, \nabla \Delta \psi) \right] \text{vol} \geq \int_M \rho |\nabla \psi|^2 \text{vol},$$

where $\nabla$ now denotes the usual gradient and $\Delta$ denotes the Laplace–Beltrami operator. In fact, the left hand side equals the Hessian of the entropy in Otto’s formal Riemannian structure on $\mathcal{P}(M)$ associated with the $L^2$-Kantorovich distance $W_2$. (2.4) and (2.5) should be seen as discrete analogues of the integrated carré du champs operators $\int \Gamma(\psi) d\mu$ and $\int \Gamma_2(\psi) d\mu$ appearing in the right resp. left hand side of Bochner’s inequality.

**2.3. Duality for discrete optimal transport.** Here, we recall a dual formulation for the discrete transport distance that has been established in [10] and which can be seen as a discrete analogue of the Kantorovich duality. A very similar result in a slightly more restrictive setting has been proven in [12] and also existence of dual optimizers has been established, see Prop. 3.10 and Thm. 5.10, 7.4 there.

**Definition 2.3 (Hamilton-Jacobi subsolution).** We say that a function $\varphi \in H^1((0,T) ; \mathbb{R}^\mathcal{X})$ is a Hamilton–Jacobi subsolution if for a.e. $t$ in $(0, T)$ we have

$$\langle \dot{\varphi}_t, \mu \rangle + \frac{1}{2} \| \nabla \varphi_t \|^2_\mu \leq 0 \quad \forall \mu \in \mathcal{P}(\mathcal{X}).$$

(2.6)

The set of all Hamilton–Jacobi subsolutions is denoted $\text{HJ}^T_\mathcal{X}$.

**Remark 2.4.** Given $\varphi \in \text{HJ}^T_\mathcal{X}$ and $\lambda > 0$, set $\varphi^\lambda_t := \lambda \varphi_{\lambda t}$. Then $\varphi^\lambda \in \text{HJ}^{\lambda T}_\mathcal{X}$.

**Theorem 2.5 (Duality formula, [10, Thm. 3.3]).** For $\mu_0, \mu_1 \in \mathcal{P}(\mathcal{X})$ we have

$$\frac{1}{2} W^2(\mu_0, \mu_1) = \sup \left\{ \langle \varphi_1, \mu_1 \rangle - \langle \varphi_0, \mu_0 \rangle : \varphi \in \text{HJ}^1_\mathcal{X} \right\}.$$  

(2.7)

This representation remains true if the supremum is restricted to functions $\varphi \in C^1([0, 1], \mathbb{R}^\mathcal{X})$ satisfying (2.6).

For the reader’s convenience, let us also recall the heuristic derivation of the duality result above. We start by introducing a Lagrange multiplier for the continuity equation constraint and write

$$\frac{1}{2} W(\mu_0, \mu_1)^2 = \inf_{\mu, \lambda} \sup_{\varphi} \left\{ \int_0^1 \frac{1}{2} \mathcal{A}(\mu_t, V_t) dt + \int_0^1 \langle \varphi_t, \dot{\mu}_t + \nabla \cdot V_t \rangle dt \right\},$$

(2.8)
where the supremum is taken over all (sufficiently smooth) functions \( \varphi : [0, 1] \rightarrow \mathbb{R}^X \) and the infimum is taken over all (sufficiently smooth) curves \( \mu : [0, 1] \rightarrow \mathbb{R}_+ \) connecting \( \mu_0 \) and \( \mu_1 \), and over all \( V : [0, 1] \rightarrow \mathbb{R}^{X \times X} \). Here we do not require that \((\mu, V)\) satisfies the continuity equation, but the inner supremum takes the value \(+\infty\) if \((\mu, V)\) does not belong to \( \mathcal{CE}_1(\mu_0, \mu_1) \). We also do not require that \( \mu \) takes values in \( \mathcal{P}(X) \), but this is automatically enforced by the continuity equation. Continuing (2.8) we obtain via integration by parts

\[
\frac{1}{2} W(\mu_0, \mu_1)^2 = \inf_{\mu, V} \sup_{\varphi} \left\{ \langle \varphi_1, \mu_1 \rangle - \langle \varphi_0, \mu_0 \rangle + \int_0^1 \frac{1}{2} A(\mu_t, V_t) - \langle \dot{\varphi}_t, \mu_t \rangle - \langle \nabla \varphi_t, V_t \rangle dt \right\} .
\]

Applying the min–max principle and calculating the infimum we obtain

\[
\frac{1}{2} W(\mu_0, \mu_1)^2 = \sup_{\varphi} \left\{ \langle \varphi_1, \mu_1 \rangle - \langle \varphi_0, \mu_0 \rangle : \varphi \in \mathcal{H} \right\} ,
\]

where \( \mathcal{H} \) is the set of \( \varphi \) such that for a.e. \( t \) and all \( \mu \) and \( V \)

\[
\frac{1}{2} A(\mu, V) - \langle \dot{\varphi}_t, \mu \rangle - \langle \nabla \varphi_t, V \rangle \geq 0 .
\]

This is due to the fact that the quantity to be minimized is positive 1-homogeneous in \((\mu, V)\), hence the infimum takes the value \(-\infty\) if \( \varphi \) does not belong to \( \mathcal{H} \). The last inequality rewrites as

\[
0 \leq \frac{1}{2} A(\mu, V) - \langle \dot{\varphi}_t, \mu \rangle - \langle \nabla \varphi_t, V \rangle = \frac{1}{4} \sum_{x,y} \left[ \frac{V(x,y)^2}{\hat{\mu}(x,y)} - 2 \nabla \varphi_t(x,y) V(x,y) \right] - \langle \dot{\varphi}_t, \mu \rangle
= \frac{1}{4} \sum_{x,y} \left( \frac{1}{\hat{\mu}(x,y)} \left[ V(x,y) - \nabla \varphi_t(x,y) \hat{\mu}(x,y) \right]^2 - |\nabla \varphi_t(x,y)|^2 \hat{\mu}(x,y) \right)
- \langle \dot{\varphi}_t, \mu \rangle .
\]

Minimizing over \( V \) we conclude that \( \varphi \in \mathcal{H} \) iff the inequality

\[
\langle \dot{\varphi}_t, \mu \rangle + \frac{1}{2} \|\nabla \varphi_t\|^2_{\hat{\mu}} \leq 0 ,
\]

holds for all \( \mu \in \mathbb{R}^X_+ \), i.e. iff \( \varphi \in \mathcal{H}^*_X \).

3. Heat equations on time-dependent Markov triples

In this section, we study the heat equation on a time-dependent Markov triple. This will be a crucial tool for the characterization of super Ricci flows in Section 4. We will first describe in Section 3.1 the setting of time-dependent Markov chains that we consider, where the state space is allowed to vary and may feature collapse or creation of vertices. We will briefly discuss in Section 3.2 the heat equation associated to a time inhomogeneous Markov chain on a fixed state space. In Section 3.3 we will give existence and uniqueness results for the heat equation and the adjoint heat equation on measures in the general singular space-time setting.
3.1. Singular discrete space-times. We consider a time dependent family of Markov triples \((X_t, Q_t, \pi_t)_{t \in [0,T]}\). Recall that this means that for each \(t \in [0,T]\), \(X_t\) is a finite set, \(Q_t\) is the matrix of transition rates \((Q_t(x,y))_{x,y \in X_t}\) with \(Q_t(x,y) \geq 0\) for \(x \neq y\) and \(Q_t(x,x) = -\sum_{y \neq x} Q_t(x,y)\), and \(\pi_t\) is a strictly positive probability measure on \(X_t\) such that \(Q_t\) is reversible w.r.t. \(\pi_t\).

**Definition 3.1.** A singular time-dependent Markov triple is a family \((X_t, Q_t, \pi_t)_{t \in [0,T]}\) of Markov triples such that there exist a partition \(0 = t_0 < t_1 < \cdots < t_n = T\), finite sets \(X_0, \ldots, X_n\) and \(X_0, \ldots, X_{n-1}\), and surjective maps \(s_i : X_i \to X_i\) and \(c_i : X_i \to X_{i+1}\) such that the following conditions hold:

1. \(X_t = \bar{X}_i\) and \(X_t = \bar{X}_i\) for \(t \in I_i := (t_i, t_{i+1})\) for all \(i = 0, \ldots, n-1\);
2. \(t \mapsto \pi_t(x)\) is Lipschitz on \(I_i\) for all \(i\) and \(x \in X_i\) and the limits
   \[
   \pi^c_t(x) := \lim_{t \uparrow t_i} \pi_t(x), \quad \pi^s_t(x) := \lim_{t \downarrow t_i} \pi_t(x)
   \]
   exist in \((0,1)\);
3. \(t \mapsto Q_t(x,y)\) is locally log-Lipschitz on \(I_i\), i.e. for each \(x \neq y\) either \(Q_t(x,y) = 0\) for all \(t \in I_i\) or \(Q_t(x,y) > 0\) for all \(t \in I_i\) and the map \(t \mapsto \log Q_t(x,y)\) is locally Lipschitz and the limits
   \[
   Q^c_t(x,y) := \lim_{t \uparrow t_i} Q_t(x,y), \quad Q^s_t(x,y) := \lim_{t \downarrow t_i} Q_t(x,y)
   \]
   exist in \([0,\infty]\). In case \(Q^s_t(x,y) = +\infty\) resp. \(Q^c_t(x,y) = +\infty\), we assume further that
   \[
   \int_{t_i}^{t_{i+1}} Q_t(x,y)dt = +\infty, \quad \text{resp.} \quad \int_{t_i}^{t_{i+1}} Q_t(x,y)dt = +\infty; \tag{3.1}
   \]
4. we have that \(c_i(x) = c_i(y) = z \in \bar{X}_{i+1}\) iff \(x \not\leftrightarrow y\) and \(s_i(x) = s_i(y) = z \in \bar{X}_i\) iff \(x \leftrightarrow y\), where we write \(x \not\leftrightarrow y\) iff there exists a path \(x = x_1, x_2, \ldots, x_n = y\) with \(Q_j^c(x_j, x_{j+1}) = +\infty\) for \(j = 0, n-1\) and similarly for \(x \leftrightarrow y\) (note that these define equivalence relations on \(X_i\) by detailed balance and \(c_i, s_i\) are the associated quotient maps);
5. we have that for \(z \in \bar{X}_i\)
   \[
   \pi^c_{t_i}(z) = \sum_{x \in c^{-1}_i(z)} \pi^s_x = \sum_{x \in c^{-1}_i(z)} \pi^c_{t_{i-1}}(x), \tag{3.2}
   \]
   and that for \(z, z' \in \bar{X}_i\)
   \[
   Q^c_{t_i}(z, z') = \frac{1}{\pi^c_{t_i}(z)} \sum_{x \in c^{-1}_i(z), x' \in c^{-1}_i(z')} Q^s_{t_i}(x, x') \pi^c_x, \tag{3.3}
   \]
   Note that (3.2), (3.3) are automatically consistent with the requirement that \(Q_{t_i}\) and \(\pi_{t_i}\) satisfy the detailed balance condition.

The interpretation of these assumption is the following. During the open intervals \(I_i = (t_i, t_{i+1})\) the graph structure does not change and we have a log-Lipschitz control on the rates. At the times \(t_i\) the topology of the graph can change and (a combination of) the following event(s) can occur:
vertices can disconnect, i.e. $Q_t(x,y) \searrow 0$ as $t \uparrow t_i$ or start to connect, i.e. $Q_t(x,y) \nearrow 0$ as $t \downarrow t_i$,

- a group of vertices can collapse to a point, here $c_{i}^{-1}(z)$ is the set of vertices of $X_i$ that collapse to $z \in X_{i+1}$, this happens iff each pair of vertices in the group is connected via a path of edges whose weights explode,

- a point can spawn a group of new vertices at later times (same as collapse but backwards in time), here $s_{i}^{-1}(z)$ is the set of vertices of $X_i$ that are spawned by $z \in \bar{X}_i$,

- collapsing happens in a controlled way, more precisely, ratios of rates inside a collapsing group have a limit and

$$
\overline{\pi}_{i}^{c-z}(x) := \pi_{i}^{c}(x) \left( \sum_{y \in c_{i}^{-1}(z)} \pi_{i}^{c}(y) \right)^{-1} \tag{3.4}
$$

can be seen as the asymptotic equilibrium measure on $c_{i}^{-1}(z)$ as we “zoom into the collapse”; similarly for spawning.

![Figure 2. A singular time-dependent Markov chain](image)

**Example 3.2.** A simple example of a singular time-dependent Markov triple satisfying these conditions is given in Figure 2. Here the transition rate $Q_t(x,y)$ is depicted with arrows from $x$ to $y$ along the edges. The three red vertices collapse at time $t_1$ to a single vertex. Afterwards, the blue vertex of $\bar{X}_1$ spawns a new vertex. Here we could set for instance $q_t = 1/(t_1 - t)$ and $r_t = 1/(t - t_1)$ so that $\int^t_1 q_t \, dt = \int^t_1 r_t \, dt = +\infty$.

In Section 5 we discuss more examples that arise as super Ricci flows and which feature a similar $1/t$ behavior of the rates at singular times.

**Remark 3.3.** From the point of view of the heat equation on the time dependent graph, one should expect that if a group of vertices collapses at $t_i$, then the solution has already equilibrated on these vertices before the collapsing time. This is the case if the average number of jumps between these vertices before $t_i$ is infinite. This is the reason why we assume (3.1). Without this condition the heat equation does not necessarily equilibrate on vertices with exploding rates before a singular time. Consider e.g. the two-point space $X_0 = \{a,b\}$ with
Then we have explicitly with \( \delta_t \) that
\[
\frac{d}{dt} \delta_t = -\delta_t q_t, \quad \delta_t = \delta_s \exp\left(-\int_s^t q_r \, dr\right).
\]

Choosing for instance \( q_t = 1/\sqrt{T_t - t} \), we see that \( \delta_t \) does not vanish as \( t \uparrow t_1 \) unless \( \delta_s = 0 \).

We will denote by \( \dot{Q}_t, \dot{\pi}_t \) the derivatives w.r.t. \( t \) of \( Q_t \) and \( \pi_t \) which exist for a.e. \( t \in [0, T] \) by the assumption of Lipschitz continuity, (2) and (3) above.

We denote by \( \Delta_t \) the Laplace operator associated to \( Q_t \) acting on function \( \psi \in \mathbb{R}^X \) via
\[
\Delta_t \psi(x) = \sum_{y \in X} \psi(y) Q_t(x, y) = \sum_{y \in X} [\psi(y) - \psi(x)] Q_t(x, y).
\]

Let us introduce the inner products on \( \mathbb{R}^{X_t} \) and \( \mathbb{R}^{X_t 	imes X_t} \) respectively given by
\[
\langle \psi, \varphi \rangle_{\pi_t} := \sum_{x \in X_t} \psi(x) \varphi(x) \pi_t(x), \quad \langle \Psi, \Phi \rangle_{\pi_t} := \frac{1}{2} \sum_{x,y \in X_t} \Psi(x,y) \Phi(x) Q_t(x,y) \pi_t(x).
\]

Then \( \Delta_t \) is symmetric, i.e. we have that \( \langle \psi, \Delta_t \varphi \rangle_{\pi_t} = \langle \Delta_t \psi, \varphi \rangle_{\pi_t} \). Moreover, we have the following integration by parts relation \( \langle \nabla \varphi, \nabla \psi \rangle_{\pi_t} = -\langle \Delta \varphi, \psi \rangle_{\pi_t} \) for all \( \varphi, \psi \in \mathbb{R}^{X_t} \).

For a function \( \sigma \in \mathbb{R}^X \) (viewed as a signed measure on \( X \)) we define the adjoint Laplace operator \( \hat{\Delta}_t \) via
\[
\hat{\Delta}_t \sigma(x) = \sum_{y \in X} Q_t(y, x) \sigma(y) = \sum_{y \neq x} Q_t(y, x) \sigma(y) - \sum_{y \neq x} Q_t(x, y) \sigma(x).
\]

Note that if \( \sigma = \rho \pi_t \) for some \( \rho \in \mathbb{R}^X \) then \( \hat{\Delta}_t \sigma = (\Delta_t \rho) \pi_t \) by the detailed balance condition. Further, denoting the Euclidean inner product \( \langle \psi, \sigma \rangle_{\pi_t} \) (viewed as the integral of \( \psi \) against \( \sigma \)) by
\[
\langle \psi, \sigma \rangle := \sum_{x \in X_t} \psi(x) \sigma(x),
\]
we have that \( \langle \Delta_t \psi, \sigma \rangle = \langle \psi, \hat{\Delta}_t \sigma \rangle \).

### 3.2. The heat equations

Let us first consider the situation of a time-dependent Markov triple \( (X_t, Q_t, \pi_t)_{t \in (0, T)} \) with a fixed space \( X \) and time-dependent rates \( Q_t \) and measure \( \pi_t \) that are locally log-Lipschitz in \( t \).

Given \( \tilde{\psi} \in \mathbb{R}^X \) and \( 0 < s < T \) we say that a function \( \psi : [s, T] \times X \to \mathbb{R} \) solves the time-dependent heat equation with initial condition \( \tilde{\psi} \) if \( t \mapsto \psi(t, x) \) is differentiable on \( (s, T) \) and continuous at \( s \) for all \( x \) and
\[
\partial_t \psi(t, x) = \Delta_t \psi(t, x) \quad \text{on} \quad (s, T) \times X,
\]
\[
\psi(s, \cdot) = \tilde{\psi}.
\]

Note that by continuity of \( t \mapsto Q_t \), there is a unique such solution. Thus we can define the heat propagator \( P_{t,s} : \mathbb{R}^X \to \mathbb{R}^X \) by setting \( P_{t,s} \tilde{\psi} = \psi(t, \cdot) \), where \( \psi \) is the above solution.

Given \( \sigma \in \mathbb{R}^X \) and \( 0 < t < T \), we say that a function \( \sigma : (0, t] \to \mathbb{R}^X \) satisfies the adjoint heat equation for measures with terminal condition \( \tilde{\sigma} \) if \( s \mapsto \sigma(s, x) \) is differentiable on \( (0, t) \) and
\[
\partial_s \sigma(s, x) = \Delta_s \sigma(s, x) \quad \text{on} \quad (0, t) \times X,
\]
\[
\sigma(t, \cdot) = \tilde{\sigma}.
\]
continuous at $t$ for all $x$ and
\[
\partial_t \sigma(s, x) = -\Delta_x \sigma(s, x) \quad \text{on } (0, t) \times \mathcal{X},
\]
\[
\sigma(t, \cdot) = \bar{\sigma}.
\]

There exist a unique such solution. We define the adjoint heat propagator $\hat{P}_{t,s} : \mathbb{R}^X \to \mathbb{R}^X$ by setting $\hat{P}_{t,s} \bar{\sigma} = \sigma(s, \cdot)$, where $\sigma$ is the above solution.

Note that if $\bar{\sigma} = \rho t$, then we have $\hat{P}_{t,s} \bar{\sigma} = \rho_s \bar{\sigma}$, where $\rho_s$ solves the adjoint heat equation
\[
\partial_s \rho(s, x) = -\Delta_x \rho(s, x) - \hat{p}_s(x) \rho(s, x),
\]
where $p_s = \log \pi_s$. Note that $P_{t,s}$ and $\hat{P}_{t,s}$ are linear operators. Moreover, they are adjoint in the following sense: for all $0 < s < t < T$ and $\bar{\sigma} \in \mathbb{R}^X$ we have
\[
\langle P_{t,s} \bar{\psi}, \bar{\sigma} \rangle = \langle \bar{\psi}, \hat{P}_{t,s} \bar{\sigma} \rangle. \tag{3.5}
\]

Indeed, setting $\psi_r = P_{r,s} \bar{\psi}$ and $\sigma_r = \hat{P}_{t,r} \bar{\sigma}$ for $s \leq r \leq t$ we have
\[
\frac{d}{dr}(\psi_r, \sigma_r) = \langle \Delta_r \psi_r, \sigma_r \rangle - \langle \psi_r, \Delta_r \sigma_r \rangle = 0.
\]

We note the following maximum and positivity principles for the (adjoint) heat equation.

**Lemma 3.4.** For all $0 < s < t < T$ and $\bar{\psi}, \bar{\sigma} \in \mathbb{R}^X$ and $x \in \mathcal{X}$ we have that
\[
\min_{y \in \mathcal{X}} \bar{\psi}(y) \leq P_{t,s} \bar{\psi}(x) \leq \max_{y \in \mathcal{X}} \bar{\psi}(y), \tag{3.6}
\]
\[
\min_{y \in \mathcal{X}} \bar{\sigma}(y) \leq \hat{P}_{t,s} \bar{\sigma}(x) \leq \max_{y \in \mathcal{X}} \bar{\sigma}(y). \tag{3.7}
\]
Moreover, $P_{t,s} \bar{\psi}$ and $\hat{P}_{t,s} \bar{\sigma}$ are strictly positive provided that $\bar{\psi}$ and $\bar{\sigma}$ are non-negative and not identically 0.

**Proof.** Let us first show that $P_{t,s} \bar{\psi} \geq 0$ whenever $\bar{\psi} \geq 0$. For this define $\psi^-_t := \max\{-P_{t,s} \bar{\psi}, 0\}$ and $\psi_t = P_{t,s} \bar{\psi}$. For $s < t < T$, $r \mapsto \log \pi_r$ is Lipschitz on $[s, t]$ with some constant $L$. Thus, we obtain
\[
0 \leq \frac{1}{2} \sum_{x,y \in \mathcal{X}} (\psi^-_t(x) - \psi^-_t(y))^2 Q_t(x, y) \pi_t(x)
\leq \frac{1}{2} \sum_{x,y \in \mathcal{X}} (\psi^-_t(x) - \psi^-_t(y))((\psi_t(x) - \psi_t(y))Q_t(x, y) \pi_t(x)
= \sum_{x \in \mathcal{X}} \psi^-_t(x) \Delta_t \psi_t(x) \pi_t(x) = \sum_{x \in \mathcal{X}} \psi^-_t(x) \partial_t \psi_t(x) \pi_t(x)
= \frac{1}{2} \sum_{x \in \mathcal{X}} \partial_t (\psi^-_t(x))^2 \pi_t(x) \leq \frac{1}{2} e^{Lt} \partial_t \sum_{x \in \mathcal{X}} e^{-Lt} (\psi^-_t(x))^2 \pi_t(x),
\]
and in particular
\[
0 = \sum_{x \in \mathcal{X}} e^{-Lt} (\psi^-_s(x))^2 \pi_s(x) \geq \sum_{x \in \mathcal{X}} e^{-Lt} (\psi^-_t(x))^2 \pi_t(x),
\]
which implies that $\psi_t \geq 0$.

Now, let $m = \min_{y \in \mathcal{X}} \bar{\psi}(y)$ and $M = \max_{y \in \mathcal{X}} \bar{\psi}(y)$. Then (3.6) follows similarly by choosing $\psi^-_t := \max\{-P_{t,s} \bar{\psi} - m, 0\}$ and $\psi^+_t := \max\{P_{t,s} \bar{\psi} - M, 0\}$ respectively.
To show (3.7) it suffices to note that

$$\dot{\bar{\psi}}(x) = \langle \dot{\psi}, \bar{P}_{t,s} \bar{\sigma} \rangle = \langle P_{t,s} \dot{\psi}, \bar{\sigma} \rangle$$

and to apply (3.6).

The last statement follows from the fact that due to the Lipschitz assumption, the transition rates can be controlled on each compact subinterval of \((0, T)\) and then applying standard results for time-homogeneous Markov chains and the duality (3.5).

In particular, we see that the heat equation preserves constants, i.e. \(P_{t,s} \bar{\psi} \equiv c\) provided \(\bar{\psi} \equiv c\). On the other hand, the adjoint heat equation preserves mass, i.e.

$$\sum_{x \in \mathcal{X}} \bar{P}_{t,s} \bar{\sigma}(x) = \sum_{x \in \mathcal{X}} \bar{\sigma}(x).$$

(this follows form (3.5) choosing \(\psi \equiv 1\)). Combining with the maximum principle, we see that the adjoint heat equation preserves probability measures, i.e. \(\bar{P}_{t,s} \mu \in \mathcal{P}(\mathcal{X})\) provided \(\mu \in \mathcal{P}(\mathcal{X})\).

Using the propagator, the (adjoint) heat equation reads

$$\partial_t P_{t,s} \bar{\psi} = \Delta_t P_{t,s} \bar{\psi}, \quad \partial_s P_{t,s} \bar{\sigma} = -\Delta_s \bar{P}_{t,s} \bar{\sigma}.$$ 

We can also take the derivative in the other time parameter, obtaining

$$\partial_s P_{t,s} \bar{\psi} = -P_{t,s} \Delta_s \bar{\psi}, \quad \partial_s \bar{P}_{t,s} \bar{\sigma} = \bar{P}_{t,s} \Delta_t \bar{\sigma}.$$ (3.9)

This follows by noting that for \(h > 0\) we have

$$P_{t,s+h} \bar{\psi} - P_{t,s} \bar{\psi} = P_{t,s} \left[ \bar{\psi} - P_{s+h, s} \bar{\psi} \right] = -P_{t,s} \int_s^{s+h} \Delta_r P_{r,s} \bar{\psi} dr,$$

and then dividing by \(h\) and letting \(h \downarrow 0\). Similarly, one argues for the left derivative and for the adjoint equation.

### 3.3. The heat equations on singular space-times.

Now, let us consider to the general setting of Section 3.1 and consider a singular time-dependent Markov triple \((\mathcal{X}_t, Q_t, \pi_t)_{t \in [0,T]}\) according to Definition 3.1. We will show existence and uniqueness of solutions to the heat equations on functions and measures across singular times. To this end for \(0 \leq s < t \leq T\), let us define space-time during the interval \([s, t]\) by setting

$$\mathcal{S}_{s,t} := \{(r, x) : r \in [s, t], x \in \mathcal{X}_r\}.$$ (3.10)

**Theorem 3.5.** Given \(s \in [0, T]\) and \(\bar{\psi} \in \mathbb{R}^{\mathcal{X}_s}\) there exist a unique function \(\psi : \mathcal{S}_{s,T} \rightarrow \mathbb{R}\) with the following properties:

1. \(\psi(s, \cdot) = \bar{\psi}\),
2. \(t \mapsto \psi(t, \cdot)\) is differentiable on \(I_i = (t_i, t_{i+1})\) and satisfies \(\partial_t \psi(t, x) = \Delta_t \psi(t, x)\) on \(I_i \times \mathcal{X}_i\),
3. for all \(z \in \mathcal{X}_i, x \in s_i^{-1}(z)\) and \(y \in c_{i-1}^{-1}(z)\) we have

$$\psi(t, z) = \lim_{t \downarrow t_i} \psi(t, x) = \lim_{t \uparrow t_i} \psi(t, y).$$ (3.11)

Given \(t \in [0, T]\) and \(\bar{\sigma} \in \mathbb{R}^{\mathcal{X}_t}\) there exist a unique function \(\sigma : \mathcal{S}_{0,t} \rightarrow [0, \infty)\) with the following properties:

1. \(\sigma(t, \cdot) = \bar{\sigma}\),
(ii) \( s \mapsto \sigma(s, \cdot) \) is differentiable on \( I_i = (t_i, t_{i+1}) \) and satisfies \( \partial_t \sigma(s, x) = -\Delta \sigma(s, x) \) on \( I_i \times \mathcal{X}_i \),

(iii) for all \( z \in \mathcal{X}_i \) we have

\[
\sigma(t_i, z) = \sum_{x \in s_i^{-1}(z)} \lim_{y \to z} \sigma(s, x) = \sum_{y \in c_i^{-1}(z)} \lim_{s \to t_i} \sigma(s, y). 
\tag{3.12}
\]

We define the heat propagator \( P_{t,s} : \mathbb{R}^{\mathcal{X}_i} \to \mathbb{R}^{\mathcal{X}_i} \) and adjoint heat propagator \( \hat{P}_{t,s} : \mathbb{R}^{\mathcal{X}_i} \to \mathbb{R}^{\mathcal{X}_i} \) by setting

\[
P_{t,s} \tilde{\psi} = \psi(t, \cdot), \quad \hat{P}_{t,s} \tilde{\sigma} = \sigma(s, \cdot),
\]

where \( \tilde{\psi} \) and \( \tilde{\sigma} \) are the solutions given by the previous theorem with initial/terminal condition \( \tilde{\psi} \) and \( \tilde{\sigma} \) respectively. We have the following properties of the propagators.

**Proposition 3.6.** For any \( 0 \leq s \leq r \leq t \leq T \) we have

\[
P_{t,s} = P_{t,r} \circ P_{r,s}, \quad \hat{P}_{t,s} = \hat{P}_{r,s} \circ \hat{P}_{t,r}.
\]

Moreover, for \( \tilde{\psi} \in \mathbb{R}^{\mathcal{X}_i}, \tilde{\sigma} \in \mathbb{R}^{\mathcal{X}_i} \) we have

\[
\langle P_{t,s} \tilde{\psi}, \tilde{\sigma} \rangle = \langle \tilde{\psi}, \tilde{\sigma} \rangle.
\tag{3.13}
\]

We have the maximum principle, i.e. we have

\[
\min_{y \in \mathcal{X}_s} \tilde{\psi}(y) \leq P_{t,s} \tilde{\psi}(x) \leq \max_{y \in \mathcal{X}_s} \tilde{\psi}(x) \quad \forall x \in \mathcal{X}_i,
\]

\[
\min_{y \in \mathcal{X}_s} \tilde{\sigma}(y) \leq \hat{P}_{t,s} \tilde{\sigma}(x) \leq \max_{y \in \mathcal{X}_s} \tilde{\sigma}(x) \quad \forall x \in \mathcal{X}_i.
\]

Moreover, \( P_{t,s} \tilde{\psi} \) and \( \hat{P}_{t,s} \tilde{\sigma} \) are strictly positive provided \( \tilde{\psi} \) and \( \tilde{\sigma} \) are non-negative and not identically 0. Finally, we have that

\[
\sum_{x \in \mathcal{X}_s} \hat{P}_{t,s} \tilde{\sigma}(x) = \sum_{x \in \mathcal{X}_s} \tilde{\sigma}(x).
\]

In particular, for \( \hat{\mu} \in \mathcal{P}(\mathcal{X}_i) \) we have \( \hat{P}_{t,s} \hat{\mu} \in \mathcal{P}(\mathcal{X}_s) \).

**Proof.** These properties follow immediately from the corresponding properties during each interval \( I_i \) established Section 3.2, in particular Lemma 3.4, together with the boundary conditions (3.11), (3.12).

The asymptotics of solutions at singular times can be described in more detail.

**Proposition 3.7.** We have that

\[
P_{t_{i+1},s} \tilde{\psi}(z) = \sum_{x \in c_i^{-1}(z)} \tilde{\psi}(x) \hat{p}^z_i(x) + O(|t_{i+1} - s|),
\tag{3.14}
\]

\[
P_{t,t_i} \tilde{\psi}(x) = \tilde{\psi}(z) + O(|t - t_i|), \quad x \in s_i^{-1}(z).
\tag{3.15}
\]

Similarly, for the adjoint equation, we have

\[
\hat{P}_{t_{i+1},s} \tilde{\sigma}(z) = \tilde{\sigma}(z) \hat{p}^z_i(x) + O(|t_{i+1} - s|), \quad x \in c_i^{-1}(z),
\tag{3.16}
\]

\[
\hat{P}_{t,t_i} \tilde{\sigma}(x) = \sum_{x \in s_i^{-1}(z)} \tilde{\sigma}(x) + O(|t - t_i|).
\tag{3.17}
\]
Moreover, we have for all $x,y \in c^{-1}_i(z)$ and $t \in (t_i, t_{i+1})$

$$|P_{t,s}\psi(x) - P_{t,s}\psi(y)| \leq C \exp\left(- \int_{s}^{t} Q^*_z(r) \, dr \right), \quad \text{(3.18)}$$

for a suitable constant $C$ depending on $\psi$ and $\pi^*_i$, where $Q^*_z(r) = \min\{Q_r(x,y) : x,y \in c^{-1}_i(z), Q^*_r(x,y) = \infty\}$ and $\pi^*_i = \inf\{\pi_t(x) : x \in c^{-1}_i(z), r \in (s \wedge t_i, t_{i+1})\} > 0$. An analogous estimate holds for the density $(\hat{P}_{t,s}\sigma)/\pi_s$ as $s \downarrow t_i$.

Finally, we have

$$\lim_{s \downarrow t_i} \hat{P}_{t,s}\hat{\sigma}(x) = \hat{P}_{t,t_i}\hat{\sigma}(z)\hat{\pi}_i^{s,z}(x), \quad x \in s_i^{-1}(z). \quad \text{(3.19)}$$

The proof of Proposition 3.7 will follow alongside the one of Theorem 3.5.

**Proof of Theorem 3.5.** It suffices to consider the case of a single interval $0 = t_0 < t_1 = T$. The general case with multiple intervals and singular times then follows immediately by concatenating solutions on different intervals. For simplicity, we write $s = s_0, c = c_0$.

**Step 1:** Recall that for $t_0 < s < t_1$ and $\psi \in \mathbb{R}^{\mathcal{X}_0}$ there exists a unique solution $\psi = P_{s,t}\psi$ to the heat equation on $[s, t_1) \times \mathcal{X}_0$ with $\psi(s, \cdot) = \bar{\psi}$. We will show that for all $z \in \bar{\mathcal{X}}_1$ and $x \in c^{-1}(z)$ the limit $\psi^{c,z}$ of $\psi(t,x)$ as $t \uparrow t_1$ exists and is independent of $x$. This will allow to define the propagator $P_{s,t_1} : \mathbb{R}^{\mathcal{X}_0} \to \mathbb{R}^{\mathcal{X}_1}$ by setting $P_{s,t_1}\bar{\psi}(z) = \psi^{c,z}$. Obviously, this way, $P_{s,t_1}$ will still be linear and satisfy the propagator identity and the maximum principle.

By the maximum principle, Lemma 3.4, $\psi$ is uniformly bounded on $[s, t_1) \times \mathcal{X}_0$. Assume first that $c^{-1}(z) = \{x\}$ is a singleton. Then $Q_t(x, y)$ is uniformly bounded on $[s, t_1)$ and has a limit as $t \uparrow t_1$ for all $y \in \mathcal{X}_0$. From the heat equation

$$\partial_t \psi(t,x) = \sum_y [\psi(t,y) - \psi(t,x)] Q_t(x, y)$$

we thus infer that $t \mapsto \psi(t, x)$ is Lipschitz on $[s, t_1)$ and thus has a limit as $t \uparrow t_1$.

Assume now that $c^{-1}(z)$ is not a singleton and put

$$m(t) := \sum_{x \in c^{-1}(z)} \psi(t,x) \pi_t(x), \quad v(t) := \sum_{x \in c^{-1}(z)} |\psi(t,x) - m(t)|^2 \pi_t(x).$$

We calculate

$$\frac{d}{dt} m(t) = \sum_{x \in c^{-1}(z)} \Delta_t \psi(t,x) \pi_t(x) + \psi(t,x) \pi_t(x)$$

$$= \sum_{x,y \in c^{-1}(z)} [\psi(t,y) - \psi(t,x)] Q_t(x, y) \pi_t(x)$$

$$+ \sum_{x \in c^{-1}(z), y \notin c^{-1}(z)} [\psi(t,y) - \psi(t,x)] Q_t(x, y) \pi_t(x)$$

$$+ \sum_{x \in c^{-1}(z)} \psi(t,x) \pi_t(x).$$

The first sum vanishes by the detailed balance condition. In the second sum $Q_t(x, y)$ remains bounded as $t \uparrow t_1$. Together with the maximum principle and the assumption that $\pi_t$ is Lipschitz we infer that $t \mapsto m(t)$ is Lipschitz and the limit $m(t_1) := \lim_{t \uparrow t_1} m(t)$ exists.
Similarly, using the detailed balance condition we calculate
\[
\frac{d}{dt} \psi(t) = \sum_{x \in c^{-1}(z)} 2[\psi(t,x) - m(t)][\Delta \psi_t(x) - \dot{m}(t)] \pi_t(x) + |\psi(t,x) - m(t)|^2 \pi_t(x)
\]
\[
= \sum_{x,y \in c^{-1}(z)} -[\psi(t,y) - \psi(t,x)]^2 Q_t(x,y) \pi_t(x)
+ 2 \sum_{x \in c^{-1}(z), y \notin c^{-1}(z)} [\psi(t,x) - m(t)][\psi(t,y) - \psi(t,x)] Q_t(x,y) \pi_t(x)
- 2 \sum_{x \in c^{-1}(z)} [\psi(t,x) - m(t)] \dot{m}(t) \pi_t(x) + |\psi(t,x) - m(t)|^2 \pi_t(x).
\]
As before the terms in the last two lines are uniformly bounded by some constant \(C\) as \(t \uparrow t_1\).

On the other hand, one readily checks by expanding the square that
\[
\int_{c^{-1}(z)} \psi_t(x) \pi_t(x) \geq 2Q_\ast(t) v(t),
\]
where \(Q_\ast(t)\) is maximal such that \(Q_t(x,y) \geq Q_\ast(t)\) for all \(x, y \in c^{-1}(z)\) with \(Q_0^0(x,y) = \infty\).

We conclude that \(\psi(t,x)\) converges to \(m(t_1)\) for all \(x \in c^{-1}(z)\) as \(t \uparrow t_1\). In particular, using that \(|\psi(t,y) - \psi(t,x)|^2 \leq 4v(t)/\pi^2\), we have established (3.18).

Finally, let us show in addition that for \(z = \delta_x\) for \(x \in c^{-1}(z)\) for some \(z \in \bar{X}\) we have
\[
P_{t_1,t} \tilde{\psi} = \pi^{c,z}(x) \delta_z + O(|t_1 - t|)\]  
and thus also (3.14) by linearity.

Indeed, for this \(\tilde{\psi}\) we have \(m(t) = \pi_t(x)\). Since \(m\) and \(\pi\) are Lipschitz we have
\[
P_{t_1,t} \tilde{\psi}(z) = \lim_{r \uparrow t_1} \sum_{x \in c^{-1}(z)} \pi^{c,z}(x) \psi(r,x) = \lim_{r \uparrow t_1} \pi_t(c^{-1}(z))^{-1} m(r)
= \pi_t(c^{-1}(z))^{-1} m(t) + O(|t_1 - t|) = \pi^{c,z}(x) + O(|t_1 - t|).
\]

Arguing similarly, we show that for \(z' \neq z\) we have \(P_{t_1,t} \tilde{\psi}(z') = 0 + O(|t_1 - t|)\).

**Step 2:** Now, we fix \(t_0 < t < t_1\) and \(\tilde{\sigma} \in \mathcal{P}(X_0)\). Recall that there exist a unique solution \(\sigma = \tilde{P}_{t,t_0} \tilde{\sigma}\) to the adjoint heat equation on \((t_0,t) \times X_0\) with \(\sigma(t,\cdot) = \tilde{\sigma}\). We will show that for all \(z \in \bar{X}\) and \(x \in s^{-1}(z)\) the limit \(\sigma^x(t) := \lim_{s \downarrow t_0} \sigma(s,x)\) exists in \((0,1)\) and that we have
\[
\sigma^x(t) = \tilde{\sigma}^{s,z}(x) \sigma^{s,z}, \quad \sigma^{s,z} := \sum_{x \in s^{-1}(z)} \sigma^x(x)\]  
This will allow to define the propagator \(\tilde{P}_{t,t_0} : X_0 \to \mathbb{R}^0\) by setting \(\tilde{P}_{t,t_0} \tilde{\sigma}(z) = \sigma^{s,z}\).

Obviously, this way, \(\tilde{P}_{t,t_0}\) will still be linear and satisfy the propagator identity and the maximum principle. Moreover, we obtain (3.19).
If $s^{-1}(z) = \{ x \}$ is a singleton, we infer similarly as in the first step, that $s \mapsto \sigma(s, x)$ is Lipschitz on $(t_0, t]$ and thus the limit $\sigma^s(x)$ exists.

Assume that $s^{-1}(z)$ is not a singleton. Note that the density $\rho(s, x) := \sigma(s, x)/\pi_s(x)$ satisfies the adjoint heat equation

$$\partial_s \rho(s, x) = -\Delta_s \rho(s, x) - \dot{\rho}_s(x) \rho(s, x).$$

By the assumptions the second term remains bounded as $s \downarrow t_0$. Reversing time we can thus argue as in the first step to see that $\rho(s, x)$ converges to a constant $\bar{\rho}$ as $s \downarrow t_0$ independent of $x$. Since $\pi_s$ has a limit $\pi^s(x)$ we infer that $\lim_{s \downarrow t_0} \sigma(s, x) = \bar{\rho} \pi^s(x)$, which immediately implies (3.21).

Finally, let us show in addition that for $\bar{\sigma} = \delta_x$ for some $x \in s^{-1}(z)$, $z \in \mathcal{X}_0$ we have

$$\hat{P}_{t,t_0} \delta_x = \delta_z + O(|t - t_0|),$$

and thus (3.17) by linearity.

Let us put

$$m(s) := \sum_{x \in s^{-1}(z)} \sigma(s, x).$$

We have $m(t) = 1$ and $\lim_{s \downarrow t_0} m(s) = \hat{P}_{t,t_0} \bar{\sigma}(z)$. We calculate

$$\frac{d}{ds} m(s) = \sum_{x \in s^{-1}(z)} -\dot{\Delta}_s \sigma(s, x) = \sum_{x \in s^{-1}(z), y \in \mathcal{X}_0, y \neq x} -\dot{\sigma}(s, y) Q_s(y, x) + \sigma(s, x) Q_s(x, y)

= \sum_{x, y \in s^{-1}(z), y \neq x} -\dot{\sigma}(s, y) Q_s(y, x) + \sigma(s, x) Q_s(x, y) + \sum_{x \in s^{-1}(z), y \neq s^{-1}(z)} -\dot{\sigma}(s, y) Q_s(y, x) + \sigma(s, x) Q_s(x, y).$$

The first sum in the right hand side vanishes by symmetry. In the second sum $Q_s(x, y)$ remains bounded as $s \downarrow t_0$. Thus $s \mapsto m(s)$ is Lipschitz which yields (3.22).

**Step 3:** We show that given $\tilde{\psi} \in \mathcal{X}_0$ there exist a unique solution $\hat{\psi}$ on $(t_0, t_1) \times \mathcal{X}_0$ such that

$$\hat{\psi}(z) = \lim_{t \downarrow t_0} \psi(t, x) \quad \forall z \in \mathcal{X}_0, \quad x \in s^{-1}(z).$$

This will allow to define the propagator $P_{t,t_0}$ for all $t \in [t_0, t_1]$. To show uniqueness let $\psi$ be any such solution. Then for any $t_0 < s < t < t_1$ and $\tilde{\sigma} \in \mathbb{R}^{\mathcal{X}_0}$ we have

$$\langle \psi(t, \cdot), \tilde{\sigma} \rangle = \langle \psi(s, \cdot), \hat{P}_{t,s} \tilde{\sigma} \rangle \xrightarrow{\text{step 2}} \sum_{z \in \mathcal{X}_0} \tilde{\psi}(z) \hat{P}_{t,t_0} \tilde{\sigma},$$

using the assumption on $\psi$ and the convergence of the solution to the adjoint equation from step 2. Thus the solution $\hat{\psi}$ is uniquely determined. To show existence, we define $\psi(t, \cdot)$ via $\langle \psi(t, \cdot), \tilde{\sigma} \rangle = \langle \tilde{\psi}, \hat{P}_{t,t_0} \tilde{\sigma} \rangle$ for $\tilde{\sigma} \in \mathbb{R}^{\mathcal{X}_0}$. Using (3.22) we see that $\psi$ has the correct limit as $t \downarrow t_0$. It remains to verify that it is a solution. To this end it suffices to show that extending (3.9) for $t_0 < t < t_1$ we have

$$\partial_t \hat{P}_{t,t_0} \tilde{\sigma} = \hat{P}_{t,t_0} \hat{\Delta} \tilde{\sigma}.$$  

(3.23)
Indeed, from this we obtain immediately
\[ \frac{d}{dt} \langle \psi(t, \cdot), \bar{\sigma} \rangle = \langle \bar{\psi}, \hat{P}_{t,t_0} \hat{\Delta} \bar{\sigma} \rangle = \langle \psi(t, \cdot), \hat{\Delta} \bar{\sigma} \rangle = \langle \Delta_t \psi(t, \cdot), \bar{\sigma} \rangle. \]

Let us show (3.23). For \( t_0 < s < t \) we obtain integrating (3.9)
\[ \hat{P}_{t+h,s} \bar{\sigma} - \hat{P}_{t,s} \bar{\sigma} = \int_t^{t+h} \hat{P}_{r,s} \hat{\Delta} \bar{\sigma} \, dr. \]

Noting that the rates \( Q_r \) are bounded for \( r \in [t, t+h] \) and thanks to the maximum principle we can thus first pass to the limit \( s \downarrow t_0 \) by dominated convergence. Again thanks to the maximum principle, linearity, and the continuity assumption on the rates, the map \( r \mapsto \hat{P}_{r,t_0} \hat{\Delta} \bar{\sigma}(x) \) is continuous. Thus we can divide by \( h \) and let \( h \downarrow 0 \) to obtain the claim (arguing similarly for the left derivative).

**Step 4:** Similarly, we show that given \( \bar{\sigma} \in \bar{X}_1 \) there exist a unique solution \( \sigma \) on \( (t_0, t_1) \times X_0 \) such that
\[ \bar{\sigma}(z) = \sum_{x \in c^{-1}(z)} \lim_{s \uparrow t_1} \sigma(s, x) \quad \forall z \in \bar{X}_1. \]

This will allow to define the propagator \( \hat{P}_{t,s} \) for all \( s \in [t_0, t_1] \).

To show uniqueness let \( \sigma \) be any such solution. Then for any \( t_0 < s < t < t_1 \) and \( \tilde{\psi} \in \mathbb{R}^{X_0} \) we have
\[ \langle \tilde{\psi}, \sigma(s, \cdot) \rangle = \langle P_{t,s} \tilde{\psi}, \sigma(t, \cdot) \rangle \lim_{t \uparrow t_1} \sum_{z \in \bar{X}_1} P_{t_1,t} \tilde{\psi}(z) \sigma(z), \]

using the assumption on \( \sigma \) and the convergence of the solution to the heat equation from step 1. Thus the solution \( \mu \) is uniquely determined. To show existence we define \( \sigma(s, \cdot) \) via
\[ \langle \tilde{\psi}(s, \cdot), \sigma(s, \cdot) \rangle = \langle P_{t,s} \tilde{\psi}, \hat{\mu} \rangle \text{ for } \tilde{\psi} \in \mathbb{R}^{X_0}. \]

Using (3.20) shows that this solution has the correct limit as \( s \uparrow t_1 \). Similarly as before one can show that this is a solution to the heat equation by showing that \( \partial_s P_{t,s} \tilde{\psi} = -P_{t,s} \Delta \tilde{\psi} \) extending (3.9).

Note that the propagators \( P_{t_0, t} \) and \( \hat{P}_{t_1,t} \) constructed in steps 3 and 4 by construction satisfy the adjointness relation (3.13).

Finally, note that (3.15) and (3.16) follow from (3.17) and (3.14) by the adjointness (3.13). \( \square \)

4. **Characterizations of super Ricci flows**

In this section we will give several equivalent characterizations of discrete super Ricci flows. These will be formulated in terms of a time-dependent Bochner inequality, gradient estimates for the heat propagator, transport estimates for the dual heat propagator, and dynamic convexity of the entropy.

Throughout this section \( (X_i, Q_i, \pi_i)_{i \in [0,T]} \) will be a singular time-dependent Markov triple according to Definition 3.1. We additionally make the following assumption on the growth of the transition rates that go to infinity at singular times: For each \( z \in \bar{X}_{i+1} \) we assume that
\[ Q_{i,\max}^{z,c}(t) \exp \left( -2 \int_0^t Q_{i,\min}^{z,c}(r) \, dr \right) \rightarrow 0 \quad \text{as } t \nearrow t_{i+1}, \]

where we set \( Q_{i,\max}^{z,c}(t) = \max \{ Q_i(x,y) : x,y \in c_i^{-1}(z), Q_i(x,y) = \infty \} \) and \( Q_{i,\min}^{z,c}(t) = \min \{ Q_i(x,y) : x,y \in c_i^{-1}(z), Q_i(x,y) = \infty \} \) are the maximal resp. minimal diverging rates in a collapsing region. Note that \( Q_{i,\min}^{z,c}(t) \rightarrow \infty \) as \( t \nearrow t_{i+1}. \)
Moreover for all \( z \in \tilde{X}_t \) we assume that

\[
(t - t_i)^2 Q_{i, \max}^z(t) \to 0 \quad \text{as} \quad t \searrow t_i,
\]

where \( Q_{i, \max}^z(t) = \max\{Q_i(x, y) : x, y \in s_{i}^{-1}(z), Q_i^t(x, y) = \infty\} \).

To state the defining properties of super Ricci flows, let us introduce or recall the following central objects. We will denote by \( \Gamma_t \) and \( \Gamma_{2,t} \) the integrated carré du champs operator associated to the Markov triple \((X_t, Q_t, \pi)\), c.f. (2.4), (2.5), i.e.

\[
\Gamma_t(\mu, \psi) := \langle \nabla \psi, \nabla \psi \cdot \Lambda_t(\mu) \rangle, \\
\Gamma_{2,t}(\mu, \psi) := \frac{1}{2} \langle \nabla \psi, \nabla \psi \cdot \hat{\Delta} \Lambda_t(\mu) \rangle - \langle \nabla \psi, \nabla \Delta_t \psi \cdot \Lambda_t(\mu) \rangle,
\]

where we write \( \Lambda_t(\mu)(x, y) = \Lambda(\mu(x)Q_t(x, y), \mu(y)Q_t(y, x)) \) and \( \hat{\Delta} \Lambda_t(\mu) \) is defined as in Section 2.2. Moreover, we introduce the time-derivative of the \( \Gamma \)-operator given by

\[
\partial_t \Gamma_t(\mu, \psi) := \langle \nabla \psi, \nabla \psi \cdot \partial_t \Lambda_t(\mu) \rangle,
\]

where we set

\[
\partial_t \Lambda_t(\mu)(x, y) = \partial_t \Lambda(\mu(x)Q_t(x, y), \mu(y)Q_t(y, x)) \mu(x)Q_t(x, y) \\
+ \partial_2 \Lambda(\mu(x)Q_t(x, y), \mu(y)Q_t(y, x)) \mu(y)Q_t(y, x).
\]

Note that by the Lipschitz assumption on the transition rates, \( \partial_t \Gamma_t \) is well defined for a.e. \( t \in (0, T) \). Further let us denote by \( \mathcal{W}_t \) the discrete transport distance associated to \((X_t, Q_t, \pi_t)\). Finally, we denote by \( \mathcal{H}_t \) the relative entropy w.r.t. \( \pi_t \).

With this we have the following result.

**Theorem 4.1.** Let \((X_t, Q_t, \pi_t)_{t \in [0, T]}\) be a singular time-dependent Markov triple satisfying (4.1) and (4.2). Then the following are equivalent

(I) The Bochner inequality

\[
\Gamma_{2,t}(\mu, \psi) \geq \frac{1}{2} \partial_t \Gamma_t(\mu, \psi)
\]

holds for a.e. \( t \in [0, T] \) and all \( \mu \in \mathcal{P}(X_t), \psi \in \mathbb{R}^{X_t} \).

(II) The gradient estimate

\[
\Gamma_t(\mu, P_{t,s}\psi) \leq \Gamma_s(\hat{P}_{t,s} \mu, \psi)
\]

holds for all \( 0 \leq s \leq t \leq T \) and all \( \mu \in \mathcal{P}(X_t), \psi \in \mathbb{R}^{X_t} \).

(III) The transport estimate

\[
\mathcal{W}_t(\hat{P}_{t,s} \mu, \hat{P}_{t,s} \nu) \leq \mathcal{W}_s(\mu, \nu)
\]

holds for all \( 0 \leq s \leq t \leq T \) and all \( \mu, \nu \in \mathcal{P}(X_t) \).

(IV) The entropy is dynamically convex, i.e. for a.e. \( t \in [0, T] \) and all \( W_t \)-geodesics \((\mu^a)_{a \in [0,1]} \)

\[
\partial_a^+ \mathcal{H}_t(\mu^1) - \partial_a^- \mathcal{H}_t(\mu^0) \geq -\frac{1}{2} \partial_t \mathcal{W}_t^2(\mu^0, \mu^1).
\]

**Definition 4.2.** A time-dependent Markov triple \((X_t, Q_t, \pi_t)_{t} \) is called a super Ricci flow if any of the equivalent properties of the previous theorem holds.
The proof of Theorem 4.1 will be given in the following subsections. We will show the following implications: $(I) \iff (II)$, $(II) \iff (III)$, $(IV) \Rightarrow (I)$, $(II)$ implies the dynamic EVI property of the heat flow, which together with $(III)$ implies $(IV)$.

### 4.1. Bochner formula and gradient estimates

In this section we prove the implication $(I) \iff (II)$.

**Proof of $(I) \Rightarrow (II)$.

**Step 1:** We will first show that (4.5) holds for $t_i < s < t < t_{i+1}$. To this end, fix $\mu \in \mathcal{P}(\mathcal{X}_t)$ and $\psi \in \mathbb{R}^{\mathcal{X}_t}$ and for $s \leq r \leq t$ set $\mu_r = \bar{P}_{t,r} \mu$ and $\psi_r = P_{r,s} \psi$. Then we have

\[
\frac{d}{dr} \Gamma_r(\mu_r, \psi_r) = \sum_{x,y} \nabla \psi_r \nabla \Delta_r \psi_r \Lambda_r(\mu_r)(x,y) + \frac{1}{2} \sum_{x,y} |\nabla \psi_r|^2 \left[ - \partial_1 \Lambda_r(\mu_r) \hat{\Delta}_r \mu_r(x,y) - \partial_2 \Lambda_r(\mu_r) \hat{\Delta}_r \psi_r(x,y) \right] + \frac{1}{2} \sum_{x,y} |\nabla \psi_r|^2 \left[ \partial_1 \Lambda_r(\mu_r) \mu_r(x,y) \hat{Q}_r(x,y) + \partial_2 \Lambda_r(\mu_r) \mu_r(y) \hat{Q}_r(x,y) \right],
\]

where we have put for brevity $\partial_1 \Lambda_r(\mu)(x,y) = \partial_1 \Lambda(\mu) Q_r(x,y,\mu) Q_r(y,x)$ and similarly for $\partial_2 \Lambda_r(\mu)$. Inserting the definition of $\Gamma_2$ and $\partial_1 \Gamma_r$ we obtain

\[
\frac{d}{dr} \Gamma_r(\mu_r, \psi_r) = -2 \Gamma_2(r, \mu_r, \psi_r) + \partial_1 \Gamma_r(\mu_r, \psi_r) \leq 0,
\]

where the last inequality follows from (4.4). Integrating over $r \in (s, t)$ then yields the gradient estimate (4.5).

**Step 2:** Now, we establish (4.5) across a singular time, i.e. for $t_{i-1} < s < t_i < t < t_{i+1}$. This then readily implies (4.5) for all $s, t$. From the previous step we obtain for $\varepsilon > 0$ sufficiently small

\[
\Gamma_i(\mu, P_{t,s}) \leq \Gamma_{t_i + \varepsilon}(\hat{P}_{t_{i-1} + \varepsilon} \mu, P_{t_{i-1} + \varepsilon, s} \psi) ,
\]

\[
\Gamma_s(\hat{P}_{t,i} \mu, \psi) \geq \Gamma_{t_i - \varepsilon}(\hat{P}_{t_{i-1} - \varepsilon} \mu, P_{t_{i-1} - \varepsilon, s} \psi).
\]

Thus, it will be sufficient to show

\[
\lim_{\varepsilon \to 0} \Gamma_{t_i - \varepsilon}(\hat{P}_{t_{i-1} - \varepsilon} \mu, P_{t_{i-1}, s} \psi) = \Gamma_{t_i}(\hat{P}_{t_{i-1}} \mu, P_{t_{i-1}, s} \psi) = \lim_{\varepsilon \to 0} \Gamma_{t_i + \varepsilon}(\hat{P}_{t_{i+1} + \varepsilon} \mu, P_{t_{i+1}, s} \psi).
\] (4.8)

Let us first show that the first identity in (4.8). For this let $z \in \mathcal{X}_t$ and write $\mu_z = \hat{P}_{t_{i-1} + \varepsilon} \mu$, $\psi_z = P_{t_{i-1}, s} \psi$, $\Lambda_z = \Lambda_{t_i - \varepsilon}$, and $c = c_{i-1}$. Then, using (3.18), we estimate for $x, y \in c^{-1}(z)$:

\[
|\nabla \psi_z|^2(x,y) \Lambda_z(\mu_z)(x,y) \leq C \exp \left( - \int_s^{t_i - \varepsilon} Q_{t_i, \min}(r) dr \right) Q_{t_i, \max}(t_i - \varepsilon).
\]

Hence we find with the assumption (4.1) that for all $z \in \mathcal{X}_t$

\[
\lim_{\varepsilon \to 0} \sum_{x, y \in c^{-1}(z)} \frac{1}{2} |\nabla \psi_z|^2(x,y) \Lambda_z(\mu_z)(x,y) = 0.
\]
Moreover, for $z \neq z' \in \mathcal{X}_i$ with Theorem 3.5 and (3.3) we find
\[
\lim_{\varepsilon \to 0} \sum_{x \in e^{-1}(z), y \in e^{-1}(z')} |\nabla \psi_\varepsilon|^2(x, y) \Lambda_e(\mu_\varepsilon)(x, y)
= \sum_{x \in e^{-1}(z), y \in e^{-1}(z')} |\nabla \psi|^2(z, z') \Lambda \left( \mu_0(z) \frac{\pi_i^r(x)}{\pi_i(z)} Q^r_i(x, y), \mu_0(z') \frac{\pi_i^r(y)}{\pi_i(z')} Q^r_i(y, x) \right).
\]

From the positive 1-homogeneity of $\Lambda$ we have that $\Lambda(r, s) + \Lambda(r', s') = \Lambda(r + r', s + s')$ whenever $r = \lambda r'$ and $s = \lambda s'$ for some $\lambda \geq 0$. Since we have $Q^r_i(x, y) \pi_i^r(x) = Q^r_i(y, x) \pi_i^r(y)$, we deduce that
\[
\lim_{\varepsilon \to 0} \sum_{x \in e^{-1}(z), y \in e^{-1}(z')} |\nabla \psi_\varepsilon|^2(x, y) \Lambda_e(\mu_\varepsilon)(x, y)
= |\nabla \psi|^2(z, z') \Lambda \left( \mu_0(z) \frac{\pi_i^r(x)}{\pi_i(z)} Q^r_i(x, y), \mu_0(z') \frac{\pi_i^r(y)}{\pi_i(z')} Q^r_i(y, x) \right)
= |\nabla \psi|^2(z, z') \Lambda \left( \mu_0(z) Q_{t_i}(z, z'), \mu_0(z') Q_{t_i}(z', z) \right),
\]
where we used again (3.3). Summing over all $z \neq z' \in \mathcal{X}_i$ yields the first identity in (4.8). Let us now show the second identity. We write $s_i = s$ and $\mu_\varepsilon = \tilde{P}_{t_i, t_i+\varepsilon} \mu$, $\psi_\varepsilon = P_{t_i+\varepsilon, s} \psi$, and $\Lambda_e = \Lambda_{t_i+\varepsilon}$. Then, by (3.15) we obtain for $z \in \mathcal{X}_i$ and $x, y \in s^{-1}(z)$:
\[
|\nabla \psi_\varepsilon|^2(x, y) \Lambda_e(\mu_\varepsilon)(x, y) \leq 4C \varepsilon^2 Q^r_{t_i, \text{max}}(t_i + \varepsilon)
\]
for some constant $C$. Hence we deduce from (4.2) that
\[
\lim_{\varepsilon \to 0} \sum_{x, y \in s^{-1}(z)} |\nabla \psi_\varepsilon|^2(x, y) \Lambda_{t_i+\varepsilon}(\mu_\varepsilon)(x, y) = 0.
\]
For $z \neq z' \in \mathcal{X}_i$ with Theorem 3.5 and (3.19) we find similarly as above, that
\[
\lim_{\varepsilon \to 0} \sum_{x \in s^{-1}(z), y \in s^{-1}(z')} |\nabla \psi_\varepsilon|^2(x, y) \Lambda_{t_i+\varepsilon}(\mu_\varepsilon)(x, y)
= |\nabla \psi|^2(z, z') \sum_{x \in s^{-1}(z), y \in s^{-1}(z')} \Lambda \left( \mu_0(z) \frac{\pi_i^{s,z}}{\pi_i(z)} Q^s_i(x, y), \mu_0(z') \frac{\pi_i^{s,z}(y)}{\pi_i(z')} Q^s_i(y, x) \right)
= |\nabla \psi|^2(z, z') \Lambda \left( \mu_0(z) Q_{t_i}(z, z'), \mu_0(z') Q_{t_i}(z', z) \right).
\]
Summing over all $z, z' \in \mathcal{X}_i$ yields the second identity in (4.8). This finishes the proof. \hfill \Box

Proof of (11). Consider $t_i < s < t < t_{i+1}$ for some $i$ and set again $\mu_r = \tilde{P}_{t_r} \mu$ and $\psi_r = P_{r, s} \psi$ for $\mu \in \mathcal{P}(\mathcal{X}_i)$ and $\psi \in \mathbb{R}^d$. Arguing similarly as before, we find
\[
0 \geq \Gamma_t(\mu, \tilde{P}_{t, s} \psi) - \Gamma_s(\tilde{P}_{t, s} \mu, \psi) = \int_s^t \frac{d}{dr} \Gamma_r(\mu_r, \psi_r) \, dr = \int_s^t -2 \Gamma_{2,r}(\mu_r, \psi_r) + \partial_r \Gamma_r(\mu_r, \psi_r) \, dr.
\]
Dividing by $t - s$ and letting $s \to t$ the Lebesgue differentiation theorem implies that for a.e. $t \in (t_i, t_{i+1})$ we have
\[
\Gamma_{2,t}(\mu, \psi) \geq \frac{1}{2} \partial_t \Gamma_t(\mu, \psi),
\]
4.2. Transport estimates. In this section, we will prove the implication (II) ⇔ (III).

To this end, we will use the dual characterization of the discrete transport distance given by Theorem 2.5. We denote by $\text{HJ}_{X}$ the set of Hamilton–Jacobi subsolutions on the interval $[0,1]$ for the triple $(X_t, Q_t, \pi_t)$. Further, we need an observation on the that metric tensor can be expressed as a limit of distances. For this, recall from Section 2.2 that on a Markov triple $(X(Q,\pi))$ the metric $\mathcal{W}$ is induced by a Riemannian metric tensor on $\mathcal{P}_s(X)$ which is given for $\mu \in \mathcal{P}_s(X)$ by $g_{\mu}(s) = \langle \nabla \psi, \nabla \psi \rangle_{\mu} = \langle \nabla \psi, \nabla \psi \cdot \Lambda(\mu) \rangle$, where we have identified the tangent space $\mathcal{T}$ with the space of discrete gradients $\mathcal{G}$ (resp. the set $\mathcal{G}'$ of functions modulo constants) via the map

$$ s = K_\mu \psi = -\nabla \cdot \left( \nabla \psi \cdot \Lambda(\mu) \right). $$

In other words, we have $g_{\mu}(s) = \langle s, K_\mu^{-1}s \rangle =: G(\mu, s)$. Note that for any $\omega \in \mathcal{T}$, $\psi \in \mathcal{G}'$, and $\mu \in \mathcal{P}_s(X)$ we have

$$ \Gamma(\mu, \psi) = \langle \nabla \psi, \nabla \psi \cdot \Lambda(\mu) \rangle = \langle \psi, K_\mu \psi \rangle \geq 2\langle \omega, \psi \rangle - \langle \omega, K_\mu^{-1}\omega \rangle = 2\langle \omega, \psi \rangle - G(\mu, \omega), \quad (4.9) $$

and we have equality if $\omega = K_\mu \psi$. The following is a direct consequence of the observation that $\mathcal{W}$ is the Riemannian distance associated to $g_{\mu}$.

**Lemma 4.3.** For any $C^1$-curve $(\mu^a)_{a \in [0,1]}$ in $\mathcal{P}_s(X)$ we have

$$ \lim_{s \to 0} \frac{1}{2s^2} \mathcal{W}(\mu^0, \mu^a)^2 = G(\mu^0, \mu^0). \quad (4.10) $$

**Proof of (II) ⇒ (III).**

Fix $0 \leq s \leq t \leq T$ and let $(\varphi^a)_{a \in [0,1]}$ be a Hamilton–Jacobi subsolution in $\text{HJ}_{X}^1$. The gradient estimate $(4.5)$ implies that $(P_{t,s} \varphi^a)_a$ is again a Hamilton–Jacobi subsolution in $\text{HJ}_{X}^1$. Indeed for any $\mu \in \mathcal{P}(X_t)$ we have

$$ \langle \partial_a P_{t,s} \varphi^a, \mu \rangle = \langle \varphi^a, \hat{P}_{t,s} \mu \rangle \leq -\frac{1}{2} \Gamma_s(\hat{P}_{t,s} \mu, \varphi^a) \leq -\frac{1}{2} \Gamma_t(\mu, P_{t,s} \varphi^a). $$

Thus, by the duality result Theorem 2.5 we have

$$ \langle \varphi^1, \hat{P}_{t,s} \mu \rangle - \langle \varphi^0, \hat{P}_{t,s} \nu \rangle = \langle P_{t,s} \varphi^1, \mu \rangle - \langle P_{t,s} \varphi^0, \nu \rangle \leq \frac{1}{2} \mathcal{W}(\mu, \nu)^2. $$

Taking the supremum over $\varphi$ and using again Theorem 2.5 yields the claim. \qed

**Proof of (III) ⇒ (II).**

It suffices to show $(4.5)$ for strictly positive measures, i.e. $\mu \in \mathcal{P}_s(X_t)$. The statement for general $\mu \in \mathcal{P}(X_t)$ follows by approximation. Fix $\mu \in \mathcal{P}_s(X_t)$ and $\psi \in \mathbb{R}^{X_t}$ and put $\psi_r = P_{r,s} \psi$ for $r \in [s,t]$. Let $(\mu^{a})_{a \in [0,1]}$ be a curve such that

$$ \dot{\mu}^0 + \nabla \cdot \left( \Lambda_t(\mu^0) \cdot \nabla \psi_t \right) = 0. $$

For instance, one could take $\mu^a = \mu - a \varepsilon \nabla \cdot (\Lambda_t(\mu) \nabla \psi_t)$ for $\varepsilon$ sufficiently small. Finally, put $\mu^a_r = \hat{P}_{t,r} \mu^a$ and $w_r = \mu^a_r = \hat{P}_{t,r} \mu^0$. Note that $r \mapsto \langle w_r, \psi_r \rangle$ is constant. Now, we deduce from
the transport estimate using (4.10) and (4.9) that
\[
\Gamma_s(\mu_s, \psi_s) \geq \langle w_s, \psi_s \rangle - G_s(\mu_s, w_s) = \langle w_s, \psi_s \rangle - \lim_{a \downarrow 0} \frac{1}{a^2} W_a(\mu_s^0, \mu_s^a)^2 \\
\geq \langle w_t, \psi_t \rangle - \lim_{a \downarrow 0} \frac{1}{a^2} W_a(\mu_t^0, \mu_t^a)^2 = \langle w_t, \psi_t \rangle - G_t(\mu_t, w_t) \\
= \Gamma_t(\mu_t, \psi_t).
\]
This proofs the claim. \(\square\)

4.3. Entropy and convexity. In this section we prove the implication (IV) \(\Rightarrow (I)\).

Let us first observe the following. Let \((\mu^a)_{a \in (-\varepsilon, \varepsilon)}\) be a \(\mathcal{W}_1\)-geodesic in \(\mathcal{P}_s(\mathcal{X})\). Note that in this case we have
\[
\mathcal{W}_t(\mu^b, \mu^c)^2 = \frac{1}{(c - b)^2} \int_b^c \|\nabla \psi^a\|^2_{\mu^a, t} da,
\]
\[
\dot{\mu}^a + K_{\mu^a, t} \nabla \psi^a = 0
\]
and where \(\langle \nabla \psi^2 \rangle_{\mu, t}\) denotes the inner product on the tangent space at \(\mu\) associated to \((\mathcal{X}, Q_t, \pi_t)\). For \(s \neq t\) we have with the same choice of \(\psi\)
\[
\mathcal{W}_s(\mu^b, \mu^c)^2 \leq \frac{1}{(c - b)^2} \int_b^c \|\nabla \psi^a\|^2_{\mu^a, s} da.
\]
This implies that
\[
\partial_t \mathcal{W}_t(\mu^b, \mu^c)^2 \leq \frac{1}{(c - b)^2} \int_b^c -\partial_t ^{-} \|\nabla \psi^a\|^2_{\mu^a, t} da,
\]
where the minus is due to the fact that \(K_{\mu, t}\) is the inverse of the metric tensor.

**Proof of (IV) \(\Rightarrow (I)\).**

Let \(\mu \in \mathcal{P}_s(\mathcal{X})\) and \(\psi \in R^X\). Let \((\mu^a)_{a \in (-\varepsilon, \varepsilon)}\) be the geodesic starting in \(\mu\) with initial velocity \(\nabla \psi\) and let \(\psi^a\) as above. Then dynamic convexity together with (4.11) implies that
\[
\langle \text{Hess}_t \mathcal{H}_t(\mu) \nabla \psi, \nabla \psi \rangle_{\mu, t} = \frac{d^2}{da^2} \mathcal{H}(\mu^a)|_{a=0} \geq \frac{1}{2} \partial_t ^{-} \|\nabla \psi\|^2_{\mu^a, t},
\]
where we used Proposition 16.2 in [40]. To finish the proof, it suffices to recall that from Section 2.2 and the definition of \(\partial_t \Gamma_t\) that for every \(t\) where \(t \mapsto Q_t\) is differentiable we have
\[
\langle \text{Hess}_t \mathcal{H}_t(\mu) \nabla \psi, \nabla \psi \rangle_{\mu, t} = \Gamma_{2, t}(\mu, \psi),
\]
\[
\partial_t ^{-} \|\nabla \psi\|^2_{\mu^a, t} = \partial_t \Gamma_t(\mu, \psi).
\]
\(\square\)

4.4. Dynamic EVI. In this section we prove the implication (II) \(\Rightarrow (VI)\). More precisely, we will show that the gradient estimate (II) implies the dynamic EVI\(^{-}\) property for the heat flow, which together with the transport estimate (III) implies dynamic convexity (I).

To this end we introduce in the spirit of [20] an analogue of the transport distance across different time slices. We fix an interval \(I_i = (t_i, t_{i+1})\) between two singular times for some \(i\) and given \(s, t \in (t_i, t_{i+1})\) for some \(i\) and \(\overrightarrow{\mu}^0, \mu^1 \in \mathcal{P}(\mathcal{X})\) we define
\[
\frac{1}{2} \mathcal{W}_s(t)(\overrightarrow{\mu}^0, \mu^1)^2 = \sup \{ \langle \varphi^1, \mu^1 \rangle - \langle \varphi^0, \mu^0 \rangle : \varphi \in \mathcal{HJ}_{s, t} \},
\]
(4.12)
where $HJ_{s,t}$ denotes the set of all $C^1$ functions $\varphi : [0, 1] \to \mathbb{R}^X$ satisfying

$$
\langle \dot{\varphi}^a, \mu \rangle + \frac{1}{2} \| \nabla \varphi^a \|^2_{\mu, \theta(a)} \leq 0 \quad \forall \mu \in \mathcal{P}(X), \ a \in (0, 1), \tag{4.13}
$$

where $\theta(a) := s + a(t - s)$. Note that this is not a distance in the usual sense since $W_{s,t}(\mu^0, \mu^1) \neq W_{s,t}(\mu^1, \mu^0)$. In the rest of this section we drop the index $i$ and write $X$ instead of $X_i$ and $I$ for $I_i$.

Due to the local Lipschitz continuity in time of $Q$ and $\pi$ we have the following control.

**Lemma 4.4.** For each compact subinterval $J \subset I$ there exists a constant $L > 0$ such that for all $\mu^0, \mu^1 \in \mathcal{P}(X)$ and $s, t \in J$:

$$
e^{-L[s-t]} W_t(\mu^0, \mu^1)^2 \leq W_s(\mu^0, \mu^1)^2 \leq e^{L[s-t]} W_t(\mu^0, \mu^1)^2, \tag{4.14}
$$

$$
e^{-L[s-t]} W_s(\mu^0, \mu^1)^2 \leq W_t(\mu^0, \mu^1)^2 \leq e^{L[s-t]} W_s(\mu^0, \mu^1)^2. \tag{4.15}
$$

**Proof.** Recall that by assumption the maps $r \mapsto Q_r$ and $r \mapsto \pi_r$ are log-Lipschitz on $J$ for some constant constant $L$, i.e. for all $x, y \in X$ and $s, t \in I$ we have

$$
e^{-L[s-t]} Q_s(x, y) \leq Q_t(x, y) \leq e^{L[s-t]} Q_s(x, y), \ e^{-L[s-t]} \pi_s(x) \leq \pi_t(x) \leq e^{L[s-t]} \pi_s(x). \tag{4.16}
$$

The estimate (4.14) follows immediately from this and the definition of $W_t$.

To show (4.15) let $\varphi \in HJ^1_X$ be a Hamilton–Jacobi subsolution with respect to $(X, Q_s, \pi_s)$. Then applying (4.16) yields

$$
\langle \dot{\varphi}^a, \mu \rangle \leq -\frac{1}{2} \| \nabla \varphi^a \|^2_{\mu, s} \leq -\frac{1}{2} e^{-L[s-t]} \| \nabla \varphi^a \|^2_{\mu, \theta(a)} \quad \forall \mu \in \mathcal{P}(X), \ a \in [0, 1]. \tag{4.17}
$$

Set $\bar{\varphi}^a := e^{-L[s-t]} \varphi^a$. Then $\bar{\varphi}$ solves

$$
\langle \dot{\bar{\varphi}}^a, \mu \rangle \leq -\frac{1}{2} \| \nabla \bar{\varphi}^a \|^2_{\mu, \theta(a)} \quad \forall \mu \in \mathcal{P}(X), \ a \in [0, 1]
$$

and

$$
e^{-L[s-t]} \left( \langle \varphi^1, \mu^1 \rangle - \langle \varphi^0, \mu^0 \rangle \right) = \langle \bar{\varphi}^1, \mu^1 \rangle - \langle \bar{\varphi}^0, \mu^0 \rangle. \tag{4.18}
$$

Hence

$$
e^{-L[s-t]} \left( \langle \varphi^1, \mu^1 \rangle - \langle \varphi^0, \mu^0 \rangle \right) \leq \frac{1}{2} W_{s,t}(\mu^0, \mu^1)^2.
$$

Taking the supremum among all such $\varphi$ yields by Theorem 2.5

$$
W_s(\mu^0, \mu^1)^2 \leq e^{-L[s-t]} W_{s,t}(\mu^0, \mu^1)^2,
$$

which proves the left bound in (4.15). The other bound follows analogously. \hfill \square

**Definition 4.5.** We say that a curve $(\mu_t)_{t \geq 0}$ is a dynamic (upward) EVI-dyn flow for the entropy if it is locally absolutely continuous on $(0, \infty)$ and continuous at 0 and for all $t \in (0, T)$ and all $\sigma \in \mathcal{P}(X)$ we have

$$
\frac{1}{2} \partial_s W_{s,t}(\mu_s, \sigma)^2 \geq H_t(\mu_t) - H_t(\sigma).
$$

**Proposition 4.6 (Gradient estimate implies EVI-dyn).** Assume that the gradient estimate (4.5) holds for on the interval $I$. For $\mu \in \mathcal{P}(X)$ and $\tau \in I$ let $\dot{\mu}_t = \dot{P}_{\tau\tau} \mu$ denote the dual heat flow starting from $\mu$. Then for all $s, t \in I$ with $s \leq t \leq \tau$ and $\sigma \in \mathcal{P}(X)$ we have:

$$
H_s(\mu_s) - H_t(\sigma) \leq \frac{1}{2(t-s)} \left[ W_t(\mu_t, \sigma)^2 - W_{s,t}(\mu_s, \sigma)^2 \right] - (t-s) \int_0^1 \langle \dot{\mu}_{\theta(a)}, \mu^a \rangle da. \tag{4.18}
$$
Next, we calculate

\[ \langle \varphi^1, \mu_1^1 \rangle - \langle \varphi^0, \mu_0^0 \rangle - \int_0^1 \frac{1}{2} A_r(\mu^a, V^a) \, da \]
\[ \leq (t - s) \left[ H_t(\mu_1^1) - H_s(\mu_0^0) \right] - (t - s)^2 \int_0^1 \langle \dot{p}_{\theta(a)}, \mu_0^0 \rangle \, da . \]  

(4.19)

Recall the shorthand notation \( \| \nabla \psi \|_{\mu, r}^2 = \Gamma_r(\mu, \psi) \).

**Proof.** Let us put \( g_0^a := \log \rho_0^a \), where \( \mu_0^a = \rho_0^a \pi_{\theta(a)} \). We first calculate

\[
\frac{d}{da} \langle \varphi^a, \mu_0^a \rangle = \langle \dot{\varphi}^a, \mu_0^a \rangle + (t - s) \langle \varphi^a, -\Delta_{\theta(a)} \mu_0^a \rangle + \langle P_{\tau, \theta(a)} \varphi^a, \dot{\mu}_0^a \rangle
\]
\[ \leq -\frac{1}{2} | \nabla \varphi^a |_{\mu_0^a, \theta(a)}^2 + (t - s) \langle \nabla \varphi^a, \nabla g_0^a \rangle_{\mu_0^a, \theta(a)} + \langle P_{\tau, \theta(a)} \varphi^a, \dot{\mu}_0^a \rangle \]
\[ =: I_1 . \]

Here, we have used that \( \varphi \) is a HJ-subsolution and the fact that for any \( \varphi \in \mathbb{R}^X \) and \( \mu = \rho \pi_r \in \mathcal{P}(X) \) we have that \( \langle \varphi, \Delta_r \mu \rangle = -\langle \nabla \varphi, \nabla \log \rho \rangle_{\mu_r} \).

Next, we calculate

\[
\frac{d}{da} H_{\theta(a)}(\mu_0^a) = \frac{d}{da} \sum_x \log \frac{\mu_0^a(x)}{\pi_{\theta(a)}(x)} \mu_0^a(x)
\]
\[ = \langle \log \mu_0^a - \log \pi_{\theta(a)}, \frac{d}{da} \mu_0^a \rangle - \langle \frac{d}{da} \log \pi_{\theta(a)}, \mu_0^a \rangle
\]
\[ = (t - s) \langle g_0^a, -\Delta_{\theta(a)} \mu_0^a \rangle + \langle P_{\tau, \theta(a)} g_0^a, \dot{\mu}_0^a \rangle - (t - s) \langle \dot{p}_{\theta(a)}, \mu_0^a \rangle
\]
\[ = (t - s) | \nabla g_0^a |_{\mu_0^a, \theta(a)}^2 + \langle P_{\tau, \theta(a)} g_0^a, \dot{\mu}_0^a \rangle - (t - s) \langle \dot{p}_{\theta(a)}, \mu_0^a \rangle
\]
\[ =: I_2 . \]  

(4.21)
If we set \( f^a := \varphi^a + (t-s)g^0 \) we can estimate further

\[
I_1 + (t-s) \cdot I_2 = \frac{1}{2} |\nabla \varphi^a|_{\mu^a_t, \theta(a)}^2 + (t-s) \langle \nabla g^0_t, \nabla f^a \rangle_{\mu^a_t, \theta(a)} \\
+ \langle P_{r, \theta(a)}, f^a, \dot{\mu}^a \rangle - (t-s)^2 \langle \dot{\phi}_{\theta(a)}, \mu^a \rangle \\
\leq \frac{1}{2} \mathcal{A}_r(\mu^a, V^a) - (t-s)^2 \langle \dot{\phi}_{\theta(a)}, \mu^a \rangle \\
+ \frac{1}{2} |\nabla P_{r, \theta(a)} f^a|_{\mu^a_t, \theta(a)}^2 - \frac{1}{2} |\nabla \varphi^a|_{\mu^a_t, \theta(a)}^2 \\
+ (t-s) \langle \nabla g^0_t, \nabla f^a \rangle_{\mu^a_t, \theta(a)} \\
\leq \frac{1}{2} \mathcal{A}_r(\mu^a, V^a) - (t-s)^2 \langle \dot{\phi}_{\theta(a)}, \mu^a \rangle \\
+ \frac{1}{2} |\nabla f^a|_{\mu^a_t, \theta(a)}^2 - \frac{1}{2} |\nabla \varphi^a|_{\mu^a_t, \theta(a)}^2 \\
+ (t-s) \langle \nabla g^0_t, \nabla f^a \rangle_{\mu^a_t, \theta(a)} \\
\leq \frac{1}{2} \mathcal{A}_r(\mu^a, V^a) - (t-s)^2 \langle \dot{\phi}_{\theta(a)}, \mu^a \rangle .
\]

Here, we have used the gradient estimate in the second inequality and in the first inequality the fact that for every \( r \) and \( \psi \in \mathbb{R}^\mathcal{X} \) we have

\[
\langle \psi, \dot{\mu}^a \rangle \leq \mathcal{A}_r(\mu^a, V^a) + \frac{1}{2} \|\nabla \psi\|_{\mu^a_t, \theta(a)}^2 ,
\]

where \( V \) is such that \( (\mu, V) \in \mathcal{C}E_1 \).

Now, the claim follows immediately by integrating the last estimate in \( a \) from 0 to 1. □

**Proof of Proposition 4.6.** By [9, Lemma 2.9] we can find a sequence of \( C^1 \) curves \( (\mu_n, V_n) \in \mathcal{C}E_1(\mu_t, \sigma) \) such that \( \lim_n \int_0^1 \mathcal{A}_r(\mu^a_n, V^a_n) da = W_t(\mu_t, \sigma)^2 \). Now we can apply Proposition 4.7 with \( \tau = t \) to the curves \( (\mu^a_n)_{a \in [0,1]} \) and take the limit in \( n \) and the supremum over HJ-subolutions \( (\varphi^a)_a \) using (4.12). □

**Proof of (II) ⇒ (IV).** Fix \( t \in I \) and let \( (\mu^a)_{a \in [0,1]} \) be a \( W_t \)-geodesic. From the estimate (4.18) applied to \( \tau = t \) and \( \mu = \mu^a, \sigma = \mu^0 \) we obtain for \( s < t \), setting \( \mu^a_s = \bar{P}_{t,s} \mu^a \):

\[
\mathcal{H}_t(\mu^0) - \mathcal{H}_s(\mu^a_s) \geq \frac{1}{2(t-s)} \left[ W_s(t, \mu^a_s, \mu^0) - W_t(\mu^a, \mu^0) \right] - (t-s)L \\
\geq \frac{1}{2(t-s)} \left[ W_s(\mu^a_s, \mu^0) - W_t(\mu^a, \mu^0) \right] - \frac{1}{2} a^2 L - (t-s)L .
\]

Here, we have used that \( |\bar{p}| \leq L \) and the control (4.15). Similarly, choosing \( \mu = \mu^{1-a}, \sigma = \mu^1 \) we obtain:

\[
\mathcal{H}_t(\mu^1) - \mathcal{H}_s(\mu^{1-a}_s) \geq \frac{1}{2(t-s)} \left[ W_s(\mu^{1-a}_s, \mu^1)^2 - W_t(\mu^{1-a}, \mu^1)^2 \right] - \frac{1}{2} a^2 L - (t-s)L .
\]

Moreover, the contraction estimate yields

\[
W_s(\mu^a_s, \mu^{1-a}_s)^2 \leq W_t(\mu^a, \mu^{1-a})^2 .
\]
Adding (4.22) and (4.23) multiplied by 1/a and (4.24) multiplied by 1/(1 - 2a) we obtain
\[
\frac{1}{a} \left[ H_t(\mu^0) - H_s(\mu_s^0) + H_t(\mu^1) - H_s(\mu_s^{1-a}) \right] \\
\geq \frac{1}{2(t-s)} \left[ \frac{1}{a} W_s(\mu^0, \mu_s^0)^2 + \frac{1}{1 - 2a} W_s(\mu_s^a, \mu_s^{1-a})^2 + \frac{1}{a} W_s(\mu_s^{1-a}, \mu^1)^2 \\
- \frac{1}{a} W_t(\mu^0, \mu_s^a)^2 - \frac{1}{1 - 2a} W_t(\mu_s^a, \mu_s^{1-a})^2 - \frac{1}{a} W_t(\mu_s^{1-a}, \mu^1)^2 \right] \\
- aL - (t-s)\frac{2L}{a}
\]
Now, taking first the lim sup as s ↗ t and then the lim sup as a ↘ 0 yields (6.5). □

4.5. Reverse Poincaré inequality for super Ricci flows. We finish this section by showing that a reverse Poincaré inequality holds on discrete super Ricci flows. A similar result is expected for super Ricci flows of metric measure spaces and is currently investigation [21]. In fact it is expected that local Poincaré inequalities and other Harnack type inequalities can be expected for super Ricci flows of metric measure spaces and is currently investigation [21].

Theorem 4.8 (Reverse Poincaré inequality). Let \((X_t, Q_t, \pi_t)_{t \in [0, T]}\) be a super-Ricci flow. Then the one-sided local Poincaré inequality holds, i.e. for all s ≤ t and all \(\mu \in \mathcal{P}(X_t)\), \(\psi \in \mathbb{R}^{X_t}\) we have
\[
\langle P_{t,s}(\psi^2), \mu \rangle - \langle (P_{t,s}\psi)^2, \mu \rangle \geq 2(t-s)\Gamma_t(\mu, P_{t,s}\psi).
\]

Proof. Define for \(s \leq r \leq t\) the function \(h(r) = \langle (P_{r,s}\psi)^2, \hat{P}_{t,r}\mu \rangle\). Then for a.e. \(r \in (s, t)\)
\[
h'(r) = -\langle \Delta_r(P_{r,s}\psi)^2, \hat{P}_{t,r}\mu \rangle + 2\langle P_{r,s}\psi \Delta_r P_{r,s}\psi, \hat{P}_{t,r}\mu \rangle.
\]
Note that for the Laplacian satisfies for all \(\psi \in \mathbb{R}^{X_t}\)
\[
\Delta_r \psi^2(x) = 2\psi(x) \Delta_r \psi(x) + \sum_{y \in X_r} |\nabla \psi|^2(x, y) Q_r(x, y).
\]
Consequently we have
\[
h'(r) = -\sum_{x, y \in X_r} |\nabla P_{r,s}\psi|^2(x, y) Q_r(x, y) \hat{P}_{t,r}\mu(x)
\]
\[
= -\sum_{x, y \in X_r} |\nabla P_{r,s}\psi|^2(x, y) \frac{Q_r(x, y) \hat{P}_{t,r}\mu(x) + Q_r(x, y) \hat{P}_{t,r}\mu(y)}{2}
\]
\[
\leq -2\Gamma_r(\hat{P}_{t,r}\mu, P_{r,s}\psi)
\]
where we used the reversibility of the chain and that the logarithmic mean is dominated by the arithmetic mean, i.e. \(\Lambda(s, t) \leq (s + t)/2\). The gradient estimate readily implies
\[
h'(r) \leq -2\Gamma_t(\mu, P_{t,s}\psi).
\]
Noting that \(r \mapsto h(r)\) is continuous on \([r, s]\) we can integrate the last estimate to prove the claim. □
5. Examples

A first elementary example of super Ricci flows are static Markov triples with non-negative Ricci curvature.

Example 5.1. Let $(X, Q, \pi)$ be a Markov triple with $\text{Ric}(X, Q, \pi) \geq 0$ and let $Q_t = Q$ and $\pi_t = \pi$ for all $t$. Then $(X, Q_t, \pi_t)$ is a super Ricci flow. Indeed, by Proposition 2.2 we have that $\Gamma_{2,t}(\mu, \psi) \geq 0$ for all $\mu \in \mathcal{P}_*(X)$, $\psi \in \mathbb{R}^X$ and obviously we have $\partial_t \Gamma_t(\mu, \psi) = 0$.

More generally, any homogeneous Markov triple with a positive (negative) lower Ricci bound gives rise to a shrinking (expanding) soliton-like super Ricci flow.

Example 5.2. Let $(X, Q, \pi)$ be a Markov triple with $\text{Ric}(X, Q, \pi) \geq \kappa$ for some $\kappa \in \mathbb{R}$. Define

$$L_t = \frac{1}{1 - 2\kappa R}$$

and put $Q_t = L_t Q$ and $\pi_t = \pi$ for $t \in I$ with $I = [0, 1/2\kappa R]$ if $\kappa > 0$ or $I = [0, \infty)$ if $\kappa \leq 0$. Then $(X, Q_t, \pi_t)_{t \in I}$ is a super Ricci flow. Indeed, for all $\mu \in \mathcal{P}_*(X)$, $\psi \in \mathbb{R}^X$ we have that:

$$\Gamma_{2,t}(\mu, \psi) = L_t^2 \cdot \Gamma_{0,t}(\mu, \psi) \geq \Gamma_{0,t}(\mu, \psi) = L_t \kappa \cdot \Gamma_t(\mu, \psi).$$

Moreover, we have

$$\partial_t \Gamma_t(\mu, \psi) = \hat{L}_t \cdot \Gamma_0(\mu, \psi) = \frac{\hat{L}_t}{L_t} \cdot \Gamma_t(\mu, \psi).$$

Since $L_t$ satisfies the ODE $\hat{L}_t = 2\kappa L_t \hat{L}_t$, we have $\Gamma_{2,t}(\mu, \psi) \geq \frac{1}{2} \partial_t \Gamma_t(\mu, \psi)$ as required.

We can interpret growing transition rates as a shrinking of the corresponding graph and decreasing rates as an expansion. Thus in the case $\kappa > 0$ the super Ricci flow collapses to a point at time $t_1 = 1/2\kappa R$.

We can combine these effects to produce examples of flows evolving across singular times featuring collapse and expansion of vertices. To this end we recall the notion of product of two Markov triples. Given Markov triples $(X^1, Q^1, \pi^1)$, $(X^2, Q^2, \pi^2)$ we denote by $(X^1 \times X^2, Q^1 \otimes Q^2, \pi^1 \otimes \pi^2)$ the Markov chain evolving on the product space taking independent jumps in each factor, i.e. for distinct pairs $(x^1, x^2), (y^1, y^2) \in X^1 \times X^2$ we set

$$Q^1 \otimes Q^2((x^1, x^2), (y^1, y^2)) := \begin{cases} Q^1(x^1, y^1), & x^2 = y^2, \\ Q^2(x^2, y^2), & x^1 = y^1, \\ 0, & \text{else}. \end{cases}$$

This chain is reversible w.r.t. the product measure $\pi^1 \otimes \pi^2$.

Example 5.3. Let $(Y, Q^Y, \pi^Y), (Z, Q^Z, \pi^Z)$ be Markov triples with $\text{Ric}(Y) \geq 0$, $\text{Ric}(Z) \geq \kappa > 0$. Then the time-dependent triple

$$(X_t, Q_t, \pi_t) := \begin{cases} (Y, Q^Y, \pi^Y) \otimes (Z, L_t Q^Z, \pi^Z), & 0 \leq t < t_1 := 1/2\kappa, \\ (Y, Q^Y, \pi^Y), & t \geq t_1, \end{cases}$$

with $L_t = 1/(1 - 2\kappa t)$ is a super Ricci flow.

Indeed, it is readily checked that this choice of rates satisfies the condition in Sections 3.1 and 4. In view of Theorem 4.1 we only need to check that Bochner’s inequality is satisfied for
a.e. \( t \). This follows from the same argument as in the previous examples together with the fact that for \( t < t_1 \) we have (see the e.g. the proof of tensorization principle [9, Thm. 6.2]):

\[
\Gamma_{2,t}(\mu, \psi) \geq \sum_{z \in \mathcal{Z}} \Gamma_{Y}^2(\mu(\cdot, z), \psi(\cdot, z)) + L^2 \sum_{y \in \mathcal{Y}} \Gamma_{Z}^2(\mu(y, \cdot), \psi(y, \cdot)) ,
\]

where \( \Gamma_{Y}^2, \Gamma_{Z}^2 \) denote the integrated carré du champs operators calculated for the triples on \( \mathcal{Y} \) and \( \mathcal{Z} \) respectively.

In the previous example, at the singular time \( t_1 \) a positively curved factor collapses to a point. Similarly, we can consider an evolution where at a singular time each vertex explodes into a chain with negative Ricci bound.

**Example 5.4.** Let \((\mathcal{Y}, Q^Y, \pi^Y),(\mathcal{Z}, Q^Z, \pi^Z)\) be Markov triples with \( \text{Ric}(\mathcal{Y}) \geq 0, \text{Ric}(\mathcal{Z}) \geq \kappa \) for \( \kappa < 0 \). Then the time-dependent triple

\[
(X_t, Q_t, \pi_t) := \begin{cases} (\mathcal{Y}, Q^Y, \pi^Y), & 0 \leq t \leq t_1, \\ (\mathcal{Y}, Q^Y, \pi^Y) \otimes (\mathcal{Z}, L_t Q^Z, \pi^Z), & t \geq t_1, \end{cases}
\]

with \( L_t = -1/2\kappa(t - t_1) \) is a super Ricci flow.

**Example 5.5.** Consider the time-dependent two-point space \( \{(a, b), Q_t, \pi\}_{t \in (0,T)} \) and assume \( Q_t(a, b) = Q_t(b, a) = p_t \) with \( p_t = \frac{1}{1-4p_0} p_0 \). This is a super Ricci flow up to collapsing time \( T = \frac{1}{4p_0} \) as we have seen in Example 5.2 and it is optimal since

\[
\Gamma_{2,t}(\mu, \psi) \geq 2p_t \Gamma_t(\mu, \psi) = \frac{1}{2} \partial_t \Gamma_t(\mu, \psi),
\]

and \( 2p_t \) is the optimal lower Ricci bound for each \( \{(a, b), Q_t, \pi\} \), i.e. the optimal constant in the first inequality, see [24, Prop.2.12].

**6. Stability of super Ricci flows**

In this section we show that our notion of discrete super Ricci flow is consistent with classical super Ricci flows on manifolds, and more generally the synthetic notion considered in [37], in a discrete to continuum limit. More precisely, we show that if a sequence of discrete super Ricci flows (with some uniform control on the distances) converges to a time-dependent continuous metric measure space in a suitable weak sense then the latter is a super Ricci flow in the sense of [37].

Let us first recall the definitions. A time-dependent metric measure space is a family \((X, d_t, m_t)_{t \in I}\) for an (left open) interval \( I \subset \mathbb{R} \), where \( X \) is a compact Polish space and for each \( t \), \( m_t \) is a Borel probability measure on \( X \) and \( d_t \) is a geodesic metric on \( X \) generating the given topology. One also assumes that all measures \( m_t \) are absolutely continuous w.r.t. each other, more precisely there exists a bounded measurable function \( f : I \times X \to \mathbb{R} \) and a probability measure \( m \) such that \( m_t = e^{-f_t} m \) for all \( t \in I \). We denote by \( \mathcal{H}_t(\mu) := \text{Ent}(\mu|m_t) \) the Boltzmann entropy of \( \mu \in \mathcal{P}(X) \) relative to \( m_t \) given by \( \mathcal{H}_t(\mu) = \int \rho \log \rho d m_t \), provided \( \mu = \rho m_t \) and \(+ \infty \) else. Note that \( \mathcal{H}_t(\mu) = \text{Ent}(\mu|m) + \int f_t d \mu \), thus in particular the condition \( \mathcal{H}_t(\mu) < \infty \) is independent of \( t \). We denote by \( W_{2,t} \) the \( L^2 \)-Kantorovich distance associated to \( d_t \), i.e. for \( \mu, \nu \in \mathcal{P}(X) \)

\[
W_{2,t}^2(\mu, \nu) = \inf_q \int d_t(x, y)^2 dq(x, y),
\]

where the infimum runs over all couplings of \( \mu \) and \( \nu \).
Definition 6.1 ([37, Def. 2.4]). A time-dependent mm-space \((X, d_t, m_t)_{t \in I}\) is a super Ricci flow if the Boltzmann entropy is dynamically convex, i.e.: for a.e. \(t \in I\) and every \(\mu^0, \mu^1 \in \mathcal{P}(X)\) there exists an \(W_{2,t}\)-geodesic (\(\mu^a\))\(a \in [0,1]\) connecting \(\mu^0\) to \(\mu^1\) such that \(a \mapsto \text{Ent}(\mu^a|m_t)\) is absolutely continuous on \([0,1]\) and
\[
\partial_+ \mathcal{H}_t(\mu^1) - \partial_- \mathcal{H}_t(\mu^0) \geq -\frac{1}{2} \partial_t W_{2,t}(\mu^0, \mu^1) .
\] (6.1)

We now introduce a suitable notion of convergence of a sequence of time-dependent Markov triples to a time-dependent continuous mm-space.

Definition 6.2. A sequence \((\mathcal{X}^{(n)}, Q^{(n)}, \pi^{(n)})_{t \in I}\) of time-dependent Markov triples converges to a time-dependent mm-space \((X, d_t, m_t)_{t \in I}\) if there exist maps \(i_n : \mathcal{P}(\mathcal{X}^{(n)}) \to \mathcal{P}(X)\) such that:

(i) for each \(J = (r, s) \subset I\) and for each family of sequences \(\mu_i^{n,0}, \mu_i^{n,1} \in \mathcal{P}(\mathcal{X}^{(n)})\) for \(t \in J\) such that \(i_n(\mu_i^{n,j})dt \to \mu_i^j dt\) weakly as measures on \(X \times [r, s]\) for \(j = 0, 1\) and some families \(\mu_i^0, \mu_i^1 \in \mathcal{P}(X)\):
\[
\int_J \mathcal{H}_t(\mu_i^j) dt \leq \liminf_{n \to \infty} \int_J \mathcal{H}_t^{(n)}(\mu_i^{n,j}) dt ,
\]
\[
\int_J W_{2,t}(\mu_i^{n,0}, \mu_i^{n,1})^2 dt \leq \liminf_{n \to \infty} \int_J W_t^{(n)}(\mu_i^{n,0}, \mu_i^{n,1})^2 dt ,
\]

(ii) for each \(J = (r, s) \subset I\) and for each \(\mu^0, \mu^1 \in \mathcal{P}(X)\) there exist sequences \(\mu_i^{n,j} \in \mathcal{P}(\mathcal{X}^{(n)})\), such that for \(j = 0, 1\) we have \(i_n(\mu_i^{n,j}) \to \mu^j\) weakly and:
\[
\int_J \mathcal{H}_t(\mu^j) dt = \lim_{n \to \infty} \int_J \mathcal{H}_t^{(n)}(\mu_i^{n,j}) dt ,
\]
\[
W_{2,t}(\mu^0, \mu^1) = \lim_{n \to \infty} W_t^{(n)}(\mu_i^{n,0}, \mu_i^{n,1})\text{ for a.e. } t \in J .
\]

Remark 6.3. The intuition behind is that we think of \(\mathcal{X}^{(n)}\) as finer and finer discretizations of \(X\) and the Markov generators \(\Delta^{(n)}\) associated to \(Q^{(n)}\) as discretizations of the canonical Laplacian on \((X, d, m)\). For instance, \(\mathcal{X}^{(n)}\) could be the set of vertices of a mesh in \(X\) and the map \(i_n\) a suitable extension of a measure on \(\mathcal{X}^{(n)}\) to a measure on \(X\) via interpolation or convolution with a mollifying kernel.

In this scenario, we might expect additional properties of the maps \(i_n\) that allow to verify the assumptions of Definition 6.2. Motivated by the results on Gromov–Hausdorff convergence of discrete transport distances in [13, 39, 14], we could expect that \(i_n\) are approximate isometries, i.e.
\[
|W_{2,t}(i_n(\mu^0), i_n(\mu^1)) - W_t^{(n)}(\mu^0, \mu^1)| \leq \varepsilon_n
\]
for all \(\mu^0, \mu^1 \in \mathcal{P}(\mathcal{X}^{(n)})\) and a.e. \(t \in I\) and for all \(\mu \in \mathcal{P}(X)\) there exist \(\mu_n \in \mathcal{P}(\mathcal{X}^{(n)})\) such that
\[
W_{2,t}(i_n(\mu_n), \mu) \leq \varepsilon_n ,
\]
for a.e. \(t \in I\), where \(\varepsilon_n \to 0\) as \(n \to \infty\).

Note that by our assumptions in Section 3.1, if \((\mathcal{X}, Q_t, \pi_t)_{t \in I}\) is a super Ricci flow with constant base space, then \(t \mapsto \pi_t(x)\) is in particular continuous and bounded away from 0 and \(\infty\). Thus there exists a bounded continuous \(f : I \times X \to \mathbb{R}\) with \(\pi_t = e^{-f_t} \pi_{t_0}\) for some \(t_0 \in I\).
We need the following additional notion of control on the time regularity of the flows: We say that a time-dependent Markov triple \((X, Q_t, \pi_t)\) is moderate if there exists a function \(t \mapsto \lambda_t\) in \(L^1_{\text{loc}}(I)\) such that
\[
Q_t(x, y) \geq L_{t,s}Q_s(x, y), \quad \forall s \leq t, \ x, y \in \mathcal{X},
\]
where
\[
L_{t,s} := \exp \left( -\int_s^t \lambda_r \right).
\]
We call \(\lambda\) the control function in this case.

Note that the control \((6.2)\) on the rates immediately implies the control \(\mathcal{W}_t(\mu, \nu)^2 \geq L_{t,s}\mathcal{W}_s(\mu, \nu)^2\) on the transportation costs for all \(s \leq t\) and \(\mu, \nu \in \mathcal{P}(\mathcal{X})\) and in turn
\[
\partial_t \mathcal{W}_{t-}(\mu, \nu)^2 \geq -\lambda_t \mathcal{W}_t(\mu, \nu)^2, \quad \text{a.e. } t, \ \mu, \nu \in \mathcal{P}(\mathcal{X}). \tag{6.3}
\]

We have the following stability result for discrete super Ricci flows.

**Theorem 6.4.** Let \((\mathcal{X}^{(n)}, Q_t^{(n)}, \pi_t^{(n)})_{t \in I}\) be a sequence of moderate super Ricci flows with control function \(\lambda\) and such that \(\text{diam}(\mathcal{P}(\mathcal{X}^{(n)}), \mathcal{W}_t^{(n)}) \leq L\) for all \(n\) and \(t \in I\), which converges to a time-dependent mm-space \((X, d_t, m_t)_{t \in I}\). Then \((X, d_t, m_t)_{t \in I}\) is a super Ricci flow.

The proof of stability follows from the fact that under the control \((6.2)\) the dynamic convexity property can be reformulated in an integrated way, following the reasoning in [37, Thm. 3.3] for stability of super Ricci flows of mm-spaces, see in particular [37, Thm. 1.15, Prop. 2.21]. For the reader’s convenience we recapitulate this in the present setting in the first and the last step of the proof.

**Proof.** **Step 1:** By assumption (c.f. Theorem 4.1) we have dynamic convexity of the entropy \(\mathcal{H}_t^{(n)} = \text{Ent}(\cdot | \pi_t^{(n)})\) on \((\mathcal{X}^{(n)}, Q_t^{(n)}, \pi_t^{(n)})\). I.e. for a.e. \(t \in I\), every \(n\), and every \(\mathcal{W}_t^{(n)}\)-geodesic \(\mu^{n,a}\) such that
\[
\partial_a^+ \mathcal{H}_t^{(n)}(\mu^{n,1}) - \partial_a^- \mathcal{H}_t^{(n)}(\mu^{n,0}) \geq -\frac{1}{2} \partial_t^- \mathcal{W}_{t-}(\mu^{n,0}, \mu^{n,1})^2. \tag{6.4}
\]

We will first pass to an integrated version of \((6.4)\) in space and time. More precisely, we claim: for every \(n\), every \(J = (r, s) \subset I\) and every measurable family of \(\mathcal{W}_t^{(n)}\)-geodesics \((\mu_t^{n,a})_{a \in [0,1]}\) connecting \(\mu^{n,0}, \mu^{n,1}\) for \(t \in J\) and every \(\tau \in (0, \frac{1}{2})\) we have that
\[
\mathcal{H}_t^{(n)}(\mu^{n,1}) - \mathcal{H}_t^{(n)}(\mu^{n,1-\tau}) + \mathcal{H}_t^{(n)}(\mu^{n,0}) - \mathcal{H}_t^{(n)}(\mu^{n,\tau})
\geq -\frac{\tau}{2(s-r)} \left[ \mathcal{W}_s^{(n)}(\mu^{n,0}, \mu^{n,1})^2 - \mathcal{W}_r^{(n)}(\mu^{n,0}, \mu^{n,1})^2 \right] - \tau^2 \mathcal{W}_{s,t}^{(n)}(\mu^{n,0}, \mu^{n,1})^2, \tag{6.5}
\]
where we have put
\[
\mathcal{H}_t^{(n)}(\mu^{n,a}) := \frac{1}{s-r} \int_r^s \mathcal{H}_t^{(n)}(\mu_t^{n,a})dt, \quad \mathcal{W}_{s,t}^{(n)}(\mu^{n,0}, \mu^{n,1})^2 := \frac{1}{s-r} \int_r^s \lambda_t \mathcal{W}_t^{(n)}(\mu^{n,0}, \mu^{n,1})^2 dt.
\]
To show this, first note that for all $t$ we have a curvature bound $\text{Ric}(\mathcal{X}^{(n)}, Q_t^{(n)}, \pi_t^{(n)}) \geq \kappa_t$ for some $\kappa_t \in \mathbb{R}$ by [27, Theorem 4.1] implying that $\mathcal{H}_t^{(n)}$ is semiconvex along $\mathcal{W}_t^{(n)}$-geodesics. Thus for the geodesics $a \mapsto \mu_{t,a} := \mu_{t,a(1-2\sigma)}$ we can write

$$H_t^{(n)}(\mu_{t,\tau}) - H_t^{(n)}(\mu_{t,0}) = \int_0^\tau \partial_\sigma H_t^{(n)}(\mu_{t,\sigma}) d\sigma = \frac{1}{1-2\sigma} \int_0^\tau \partial_\sigma H_t^{(n)}(\mu_{t,\sigma}) |_{a=0} d\sigma ,$$

and $H_t^{(n)}(\mu_{t,1}) - H_t^{(n)}(\mu_{t,1-\tau}) = \int_0^\tau \frac{1}{1-2\sigma} \partial_\sigma H_t^{(n)}(\mu_{t,\sigma}) |_{a=1} d\sigma$. Adding these identities and using (6.4) for the geodesics $\mu_{t,a}$ yields

$$H_t^{(n)}(\mu_{t,1}) - H_t^{(n)}(\mu_{t,1-\tau}) + H_t^{(n)}(\mu_{t,0}) - H_t^{(n)}(\mu_{t,\tau}) \geq \frac{1}{2} \int_0^\tau \frac{1}{1-2\sigma} \partial_\sigma H_t^{(n)}(\mu_{t,\sigma}, \mu_{t,1-\sigma})^2 d\sigma .$$

Noting that for the $\mathcal{W}_t^{(n)}$-geodesics $\mu_{t,a}$, we have

$$\partial_\tau W_t^{(n)}(\mu_{t,0}, \mu_{t,1})^2 \geq \frac{1}{1-2\sigma} \partial_\tau W_t^{(n)}(\mu_{t,\sigma}, \mu_{t,1-\sigma})^2 + \frac{1}{\sigma} \partial_\tau W_t^{(n)}(\mu_{t,0}, \mu_{t,\sigma})^2$$

$$+ \frac{1}{\sigma} \partial_\tau W_t^{(n)}(\mu_{t,1-\sigma}, \mu_{t,1})^2 ,$$

and the bound (6.3) and integrating in $\sigma$ then yields

$$H_t^{(n)}(\mu_{t,1}) - H_t^{(n)}(\mu_{t,1-\tau}) + H_t^{(n)}(\mu_{t,0}) - H_t^{(n)}(\mu_{t,\tau}) \geq -\frac{\tau}{2} \partial_\tau W_t^{(n)}(\mu_{t,0}, \mu_{t,1})^2 - \tau^2 \lambda_t W_t^{(n)}(\mu_{t,0}, \mu_{t,1})^2 . \tag{6.6}$$

To obtain (6.5) it suffices to integrate (6.6) in $t$ on $(r, s)$ using that $W_{s,r}^{(n)}(\mu, \nu)^2 - W_{r}^{(n)}(\mu, \nu)^2 \geq \int_r^s \partial_\tau W_t^{(n)}(\mu, \nu)^2 dt$. This last estimate can be deduced from the fact that $W_{s,s}^{(n)}(\mu, \nu)^2 - W_{r}^{(n)}(\mu, \nu)^2 \geq \int_r^s \eta dt$ for some $\eta \in L^1_{\text{loc}}$, more precisely by the lower log-Lipschitz bound we can take $\eta = \exp(\int_s^t \lambda_u du) \lambda_u L^2$, where $L$ is a uniform bound on $\mathcal{W}_t^{(n)}$.

**Step 2:** Now, we pass to the limit in (6.5) as $n \to \infty$. We claim that for every $J = (r, s) \subset I$ and every $\mu^0, \mu^1 \in \mathcal{P}(X)$ there exists a measurable family of $W_{2,\mathcal{C}}$-geodesics $(\mu_t^a)_{a}$ connecting $\mu^0, \mu^1$ for $t \in J$ such that for every $\tau \in (0, \frac{1}{2})$ we have that

$$\mathcal{H}_t(\mu^1) - \mathcal{H}_t(\mu^{1-\tau}) + \mathcal{H}_t(\mu^0) - \mathcal{H}_t(\mu^0) \geq -\frac{\tau}{2} \mathcal{W}_{2,\mathcal{C}}(\mu^0, \mu^1)^2 - \tau^2 \lambda_t \mathcal{W}_{2,\mathcal{C}}(\mu^0, \mu^1)^2 \tag{6.7}$$

where we have put again

$$\mathcal{H}_t(\mu_t^a) := \frac{1}{s-r} \int_r^s \mathcal{H}_t(\mu_t^a) dt , \quad \mathcal{W}_{2,\mathcal{C}}(\mu^0, \mu^1)^2 := \frac{1}{s-r} \int_r^s \lambda_t \mathcal{W}_{2,\mathcal{C}}(\mu^0, \mu^1)^2 dt .$$

Indeed, by Definition 6.2 we can find sequences $\mu_{n,0}, \mu_{n,1} \in \mathcal{P}(\mathcal{X}^{(n)})$ such that $\iota_n(\mu_{n,j}) \to \mu^j$ weakly and $\mathcal{H}_{t}^{(n)}(\mu_{n,j}) \to \mathcal{H}_t(\mu^j)$ for $j = 0, 1$ as $n \to \infty$ as well as $\mathcal{W}_{2,\mathcal{C}}(\mu_{n,0}, \mu_{n,1}) \to \mathcal{W}_{2,\mathcal{C}}(\mu^0, \mu^1)$ for a.e. $t \in J$. By the uniform bound on $\mathcal{W}_{t}^{(n)}$ this implies also that $\mathcal{W}_{2,\mathcal{C}}(\mu_{n,0}, \mu_{n,1}) \to \mathcal{W}_{2,\mathcal{C}}(\mu^0, \mu^1)$. By the previous step there exist a family of $\mathcal{W}_t^{(n)}$-geodesics $(\mu_{t,a}^{(n)})$ for $t \in J$ connecting $\mu_t^{a(0)}$ and $\mu_{t,a(1-2\sigma)}$ for which (6.5) holds. Let $\tilde{\mu}_{t,a}^{(n)}$ be the image of $\mu_{t,a}^{(n)}$ under $\iota_n$ and put $\tilde{\mu}_{t,a}(dx, dt) = \tilde{\mu}_{t,a}^{(n)}(dx) dt$. By compactness of $X \times J$, we can find measures $\mu^a(dx, dt)$ for $a \in [0, 1] \cap \mathbb{Q}$ such that up to extracting a subsequence we have that $\tilde{\mu}_{t,a}(dx, dt) \to \mu^a(dx, dt)$ weakly. It is readily checked that the limiting measures take the
form $\mu^a(dx, dt) = \mu^b_t(dx)dt$ for a family of measures $\mu^a_t(dx) \in \mathcal{P}(X)$. Again by Definition 6.2 we have for all rational $a, b$:

$$W_{\mathcal{J}}(\mu^a_j, \mu^b_j)^2 \leq \liminf_n W_{\mathcal{J}^{(n)}}(\mu^a_n, \mu^b_n)^2 = \liminf_{n}(b-a)^2 W_{\mathcal{J}^{(n)}}(\mu^0_n, \mu^1_n)^2 = (b-a)^2 W_{\mathcal{J}}(\mu^0, \mu^1)^2$$

where we have set $W_{\mathcal{J}}(\mu^a_j, \mu^b_j)^2 = \frac{1}{|\mathcal{J}|} \int_{\mathcal{J}} W_{2,t}(\mu^a_t, \mu^b_t)^2 dt$, analogously for $W_{\mathcal{J}}$. In particular, we obtain

$$\frac{1}{a} W_{\mathcal{J}}(\mu^0, \mu^1)^2 + \frac{1}{1-a} W_{\mathcal{J}}(\mu^a_j, \mu^1)^2 \leq W_{\mathcal{J}}^2(\mu^0, \mu^1)^2 .$$

Note that this implies that for a.e. $t \in J$ and all rational $a \in [0,1]$

$$\frac{1}{a} W_{2,t}(\mu^0, \mu^a_t)^2 + \frac{1}{1-a} W_{2,t}(\mu^a_t, \mu^1)^2 = W_{2,t}^2(\mu^0, \mu^1)^2 , \quad (6.8)$$

since the “$\geq$” in (6.8) holds by the triangle inequality. Further we entail from Definition 6.2 that for all rational $\tau \in [0,1]$.

$$\mathcal{H}_{\mathcal{J}}(\mu^a_{J, \tau}) \leq \frac{1}{s-r} \int_r^s \liminf_n \mathcal{H}_{\mathcal{J}^{(n)}}(\mu^a_{n, \tau}) dt \leq \liminf_n \mathcal{H}_{\mathcal{J}^{(n)}}(\mu^a_{J, \tau}),$$

and similarly $\mathcal{H}_{\mathcal{J}}(\mu^{1-\tau}_{J, \tau}) \leq \liminf_n \mathcal{H}_{\mathcal{J}_{\tau}^{(n)}}(\mu^{1-\tau}_{J, \tau})$. Thus, we can pass to the limit (inferior) in (6.5) to obtain (6.7) at all rational $\tau$.

To conclude, we note that (6.8) implies that $W_{2,t}(\mu^0_t, \mu^1_t) = |b-a| W_{2,t}(\mu^0, \mu^1)$ for almost every $t \in J$ and all rational $a, b$. Thus, by completeness of $(\mathcal{P}(X), W_{2,t})$, we can extend for a.e. $t$ the family $(\mu^0_t, \mu^1_t)_{a \in [0,1]}$ to a $W_{2,t}$-geodesic $(\mu^a_t)_{a \in [0,1]}$. By lower semicontinuity of the relative entropy and Fatou’s Lemma we extend the estimate (6.7) to all $\tau \to (0, \frac{1}{2})$.

**Step 3:** Finally, we deduce from (6.7) that $(X, d, m_t)$ is a super Ricci flow in the sense of Definition 6.1. To do so, note that we can choose a common family of geodesics for all rational $r \leq s$. Then we let $r, s \to t$ using Lebesgue’s density theorem to obtain for a.e. $t$ that

$$\mathcal{H}_t(\mu^0) - \mathcal{H}_t(\mu^{1-\tau}_t) + \mathcal{H}_t(\mu^0) - \mathcal{H}_t(\mu^1_t) \geq -\frac{\tau}{2} \partial_t W_{2,t}(\mu^0, \mu^1^2 - \tau^2 \lambda_t W_{2,t}(\mu^0, \mu^1)^2 . \quad (6.9)$$

Then it suffices to divide by $\tau$ and let $\tau \to 0$ in (6.9) to obtain (6.1).

\[\square\]

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