Original Papers

**Influence of an Embedded Condition on Stress Distribution on an Infinite Plate under Tension Containing Elastic Elliptic Plate**

Mixed Boundary-Value Problem in Consideration of Partial Contact Region

by

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This paper is concerned with the mixed boundary-value problem of an infinite plate with an elliptic hole, into which a smooth elastic elliptic plate is inserted. Since the infinite plate is subjected to a uniaxial tension at infinity, separations are produced along the boundary between the inserted plate and the hole. The contact pressure between the inserted plate and the hole is expressed in a convergent series whose differential form is also convergent, so that the stress and displacement generated along the boundary can be numerically analyzed by the point-matching method. Using the numerical results for various elliptic shapes, the influence of fit tolerance condition and elastic moduli of plate and inserted elliptic plate is shown with the stress distribution around an elliptic hole. The effects of the magnitude of fit tolerance, load and the shape of an embedded elliptic plate on the condition of stress concentration are analytically clarified in this study. The main results are as follows: [ I ] The contact region, stress, and the displacement depend only on the one parameter containing the magnitude of fit tolerance and load. [ II ] When the fit tolerance is zero the contact region and stress distribution of elastic plate do not vary with the magnitude of load. [ III ] When the fit tolerance is not zero, the contact region and stress distribution of the elastic plate vary with the magnitude of tensile load.

**Key words:**
Elasticity, Mixed Boundary Value Problem, Stress Concentration, Partial Contact Problem, Elliptic Hole, Infinite Elastic Plate, Elastic Elliptic Insertion, Insert Condition

1 Introduction

The presence of a hole or an insertion in a plate gives rise to stress concentrations that may cause either yielding or cracking. Therefore, stress analysis of an elastic plate under load with an elliptic insertion is of importance in connection with the strength of materials and strength design of machine elements. For this reason, many reports dealing with these elliptic insertion problems have been presented.

Elliptic insertion or inclusion problems are roughly categorized into two groups, i.e., the perfect bonding condition and the simply inserted one. Most of these reports have discussed the perfectly bonded elliptic insertion in elastic plate, while some reports are concerned with a simply inserted problem. If an elliptic insertion is embedded in an infinite plate and these surfaces are not bonded together, separations are sometimes produced along either boundary because of load. This problem cannot be easily analyzed, in this case, a mixed boundary-value problem must be solved for the unknown contact pressure over an unknown contact region.

Hitherto, some researchers have analyzed for the simply insertion problem with an elliptic insertion. Wilson formulated the problem for a rigid elliptic insertion when the initial hole and insertion were of the same shape, and Goree solved this problem numerically. On the other hand, one of the authors has clarified the effects of fit tolerance and load on contact region and contact pressure distribution of an infinite plate with an elliptic hole which was filled with a smooth, elliptic and rigid insertion, by analyzing both stress and displacement under uniaxial tensile load taking into account their fit tolerance. In our analysis, fit tolerance between the elliptic insertion and hole is considered. In this case, the contact region has a constant value independent of tensile load when the magnitude of fit tolerance is zero, but when the magnitude of fit tolerance is not zero, the contact region changes with tensile load, so that the stress distribution varies with tensile load.

This report expands the method of analysis used in the former report, so as to be applicable to an elastic elliptic insertion problem, and clarifies the effects of fit tolerance, tensile load, elliptic shape and material constant of elastic plate on contact region and distribution of contact pressures, for the case when separation between an elliptic insertion and an elliptic hole partially occurs under tensile load.

2 Method of Analysis

When an elastic elliptic plate is inserted into an elliptic hole of an infinite plate, and the infinite plate is tensioned...
uniformly in direction of \( y \) axis, as shown in Fig. 1, if the elliptic hole and the inserted plate are not bonded together they may separate depending on material constants, fit tolerance, and magnitude of tensile load. To analyze the mixed boundary value problem, we derived a solution for the infinite plate, whose pressure distribution at the hole was unknown under a tensile load in direction of \( y \) axis from infinite distance, and a solution of the elastic elliptic plate whose pressure distribution at the outer surface was unknown, and then equalizing both displacements at the contact region, determined the pressure distribution at the contact region.

If we take
\[
z = w(\zeta) = R \left( \zeta + \frac{\lambda}{\zeta} \right), \quad R > 0, \quad -1 < \lambda < 1 \tag{1}
\]
\[
z = x + iy, \quad \zeta = \rho e^{i\vartheta}, \quad i = \sqrt{-1}
\]
the polar coordinates in the \( \zeta \) plane become elliptic coordinates in the \( z \)-plane. We put
\[
a = R(1 + \lambda), \quad b = R(1 - \lambda) \tag{2}
\]
That is
\[
R = a + b, \quad \lambda = (a - b)/(a + b)
\]
then from Eq. (1),
\[
x^2/a^2 + y^2/b^2 = 1 \tag{3}
\]
The unit circle in the \( \zeta \) plane maps into an ellipse [length of principal axes: \( 2a \) (on the \( x \)-axis), \( 2b \) (on the \( y \)-axis)] in the \( z \)-plane.

If we use two complex stress functions, \( \varphi(\zeta) \) and \( \psi(\zeta) \), then the equation of equilibrium and the condition of compatibility are satisfied and the stress components \( \sigma_{\rho\rho}, \sigma_{\rho\vartheta}, \tau_{\rho\vartheta} \) and displacement components \( u_{\rho}, v_{\vartheta} \) in elliptic coordinates, can be expressed\(^{12} \) as
\[
\begin{align*}
\sigma_{\rho\rho} + \sigma_{\vartheta\vartheta} &= 2 \left[ \varphi(\zeta) + \overline{\psi}(\zeta) \right], \\
\sigma_{\rho\vartheta} - i2\sigma_{\vartheta\rho} &= \frac{\alpha_{0}^2}{\rho^2 w(\zeta)} \left[ w(\zeta) \phi(\zeta) + \psi(\zeta) \right], \\
2G(u_{\rho} + iv_{\vartheta}) &= \frac{\zeta w(\zeta)}{\rho^2 w(\zeta)} \left[ k \varphi(\zeta) - w(\zeta) \phi(\zeta) + \psi(\zeta) \right] - \frac{w(\zeta)}{\rho^2 w(\zeta)} \left[ k \psi(\zeta) - w(\zeta) \psi(\zeta) - \phi(\zeta) \right].
\end{align*}
\]
\[
\Phi(\zeta) = \varphi(\zeta)/w(\zeta),
\]
for plane stress \( \kappa = (3 - \nu)/(1 + \nu) \)
for plane strain \( \kappa = (3 - \nu)/(3 - \nu) \)
where \( G \) and \( \nu \) denote shear modulus and Poisson ratio, \( \phi'(\zeta) \) is the first derivation of \( \Phi(\zeta) \) and \( \overline{\psi'(\zeta)} \) is the conjugate complex quantity of \( \phi'(\zeta) \), \( \psi'(\zeta) \) is the first derivation of \( \psi'(\zeta) \).

To obtain the solution for an infinite plate with a vacant elliptic hole under uniform tension \( \sigma_{0} \) in the \( y \)-direction at infinity, we express complex stress functions in the following manner.
\[
\begin{align*}
\varphi(\zeta) &= \frac{\sigma_{0} R}{4\zeta} \left( \zeta^2 - 2 - \lambda \right), \\
\psi(\zeta) &= \frac{\sigma_{0} R}{2\zeta} \left( \zeta^4 - (1 + \lambda)^2 \zeta^2 - 1 \right),
\end{align*}
\]
\[
\begin{align*}
\frac{\sigma_{0}}{2G} \frac{\partial u_{\rho}}{\partial \vartheta} &= \frac{1}{\sqrt{1 + \lambda^2 - 2\lambda \cos 2\vartheta}}, \\
\frac{\sigma_{0}}{\sigma_{0}} \frac{\partial \varphi}{\partial \vartheta} &= \frac{(\kappa + 1)(1 + \lambda)(1 + \lambda - 2 \cos 2\vartheta)}{4\sqrt{1 + \lambda^2 - 2\lambda \cos 2\vartheta}}.
\end{align*}
\]
where infinitely far stresses are \( \sigma_{\nu} = \sigma_{0} \) and \( \sigma_{\vartheta} = \tau_{\vartheta\vartheta} = 0 \).

For the stress and displacement components of an elliptic hole boundary, we put \( \overline{\psi'(\zeta)} \) above each symbol.
Consider the case in which a elliptic plate and an infinite plate make contact and the friction at the contact region \((-\alpha \leq \theta \leq \alpha\) ), as shown in Fig. 1, is assumed to be negligible. Since the contact pressure is symmetric with respect to both the $x$ and $y$ axis, it can then be expressed as follows:

$$\sigma_p = \frac{k}{k+1} p_k f_k^m(\theta),$$

$$f_k^m(\theta) = \begin{cases} 0 & (-\pi/2 \leq \theta < \alpha) \\ \frac{1}{k+1} (-1)^k \cos \frac{k\pi}{\alpha} & (\alpha \leq \theta \leq \alpha) \\ \frac{1}{k+1} & (\alpha < \theta \leq \pi/2) \end{cases} \quad (7)$$

where $p_k$ is an unknown constant with the units of stress. Figure 2 shows the example of distribution function $f_k^m(\theta)$. Expanding $f_k^m(\theta)$ into a Fourier's series, we have

$$f_k^m(\theta) = kA_0 + \sum_{n=1,2,\ldots} kA_n \cos 2n\theta \quad (8)$$

where

$$kA_0 = \frac{\alpha}{\pi},$$

$$kA_n = -\frac{1}{n^2 \alpha^2} \sin 2n\alpha \left( \frac{1}{n^2 \alpha^2} \right) \quad (n = 1, 2, \ldots) \quad (9)$$

To obtain the solution for an infinite plate for the case when pressure $p_k f_k^m(\theta)$ acts on the surface of an elliptic hole, we express complex stress functions as follows:

$$\psi'(\zeta) = p_k R \left[ \sum_{n=1,2,\ldots} kA_n \left( \frac{1}{2n+1} - \frac{\zeta^2}{2n-1} \right)^{2n} \right]$$

$$\psi(\zeta) = p_k R \left[ \sum_{n=1,2,\ldots} kA_n \left\{ \frac{1}{2n+1} - \frac{\zeta^2}{2n-1} \right\} \right] \quad (10)$$

Then, all stress components of plate become zero at infinity, and the stress and the displacement at the elliptic hole boundary is as follows:

$$\sigma_p = kA_0 + \sum_{n=1,2,\ldots} kA_n \cos 2n\theta$$

$$\frac{kA_0}{p_k} = \frac{\alpha}{\pi}$$

$$\frac{kA_n}{p_k} = kA_n \left( \frac{(1+2\zeta^2) \sin 2n\alpha}{c^2 - \lambda} \right) = \sum_{n=1,2,\ldots} kA_n \left( \frac{1}{2n+1} - \frac{\zeta^2}{2n-1} \right) \quad (11)$$

The stress-free conditions in the region of separation

$$\sigma_p = kA_0 + \sum_{n=1,2,\ldots} kA_n \cos 2n\theta$$

$$\sigma_p = 0 \quad (12)$$

For solution of the case when pressure $p_k f_k^m(\theta)$ acts on the surface of elliptic plate, expressing complex stress functions as follows:

$$\psi'(\zeta) = \frac{1}{2} p_k R \left[ \sum_{n=1,2,\ldots} kA_n \left( \frac{1}{2n+1} - \frac{\zeta^2}{2n-1} \right)^{2n} \right]$$

$$\psi(\zeta) = p_k R \left[ \sum_{n=1,2,\ldots} kA_n \left\{ \frac{1}{2n+1} - \frac{\zeta^2}{2n-1} \right\} \right] \quad (13)$$

We expressed modulus of rigidity as $G'$, Poisson's ratio as $\nu'$, displacement components as $u'_\rho$, for inserted elliptic plate respectively. For stress and displacement components of elliptic hole boundary and outer surface of inserted elliptic plate, we put $\nu' = \nu$ above each symbol. In the case when the contact pressure is expressed as $\sum_{k=1,2,\ldots} p_k f_k^m(\theta)$, the displacements of an elliptic hole boundary and surface of an elliptic plate can be obtained as the summation of $k$ in Eq. (11) or (13), respectively.

Consider the case in which an elliptic plate is inserted in a forced-fit condition, which means that the elliptic plate is slightly larger than the elliptic hole of the plate. When $\sigma_0$ attains and exceeds a certain value, then the separation starts and develops with an increase in tensile load. However, in our analysis, contrariwise assuming a contact region $\alpha$ we obtained the magnitude of fit tolerance and tensile load.

In the case when an infinite plate with a friction less elliptic plate is tensioned uniformly in the $y$-direction at infinity and when the elliptic plate and hole contact at the region $\alpha$ as shown in Fig. 1, by expressing the pressure distribution at the elliptic hole boundary as $\sum_{k=1,2,\ldots} p_k f_k^m(\theta)$, the stress-free conditions in the region of separation

$$\sigma_p = kA_0 + \sum_{n=1,2,\ldots} kA_n \cos 2n\theta = 0 \quad (14)$$

are already satisfied in Eq.(7). Hence, we can determine both the load $\sigma_0$ and the unknown constants $p_k$ ($k = 1, 2, 3, \ldots, K$), so as to satisfy the continuity condition of displacement in contact region

$$u'_\rho - u'_{\rho} = I_0 \quad (0 \leq \theta \leq \alpha) \quad (15)$$

where $I_0$ is magnitude of fit tolerance. For this purpose, by selecting ($K+1$) points in the contact region and satisfying Eq.(15) at these points, we can get a $(K+1)$ dimension equation of the first degree as follows:
can be determined with contact region. Hence, if the fit $\bar{\theta} = 1/2$, and stress and
tensile load $\sigma = 0.5$, the displacement of elliptic hole boundary
expression of the relation $0 \leq \theta \leq \alpha$, that is, when there is no
contact pressure in the contact region ($0 \leq \theta \leq \alpha$). Figure 4 shows
the distributions of difference in displacement $(\bar{u}_{\rho} - \bar{u}'_{\rho})$. The displacement of elliptic hole boundary $\bar{u}_{\rho}$
and that of outer surface of plate $\bar{u}'_{\rho}$ are measured from the
initial situation before inserting. Hence the value $(\bar{u}_{\rho} - \bar{u}'_{\rho})$ at
($0 \leq \theta \leq \alpha$) is the fit tolerance $I_0$. Therefore, in the case of
$\alpha = 0.2$ the tightly-fitting plate produces separation in the
region $0.2 \leq \theta \leq 1.0$ under load. The contact region under the
tensile load becomes larger with an increase in fit tolerance.

3 Numerical Calculations and Results

We performed calculations on the cases of ratio of rigidities $\Gamma = 2, 1, 0.5$ for elliptic shape $\lambda = 3/5, 1/3, 0, -1/3$
and $-3/5$, that is, $b/a = 0.25, 0.5, 1.0, 2.0$ and 4.0, assuming the
condition of plane stress, and Poisson ratio $\nu = \nu' = 0.3$.
In all cases, we selected 51 points at equal intervals in the
contact region, including both ends, and determined $\xi_k$ at
these points using Eq. (16). We then calculated both stress and
displacement using Eq. (18). Rewriting Eq. (15) for the
condition of displacement in the contact region in the
non-dimensional form, we obtained

$$\frac{2G}{\sigma_0 R} \left[ (\bar{u}_{\rho} - \bar{u}'_{\rho}) - I_0 \right] = 0, \ (0 \leq \theta \leq \alpha) \quad (19)$$

This equation was exactly satisfied at the selected points.
We could evaluate the preciseness of the calculated results by
the value of the left side of Eq. (19) at the midway between
two selected points. In this calculation, we used terms of an
infinite series equation up to $n = 600$, and validated the
In this calculation, we used terms of an expression of the value of the left side of Eq. (19) at the midway between the selected points. We could evaluate the preciseness of the calculated results by comparing the calculated values with the exact values. The results show that the present approach is accurate.

The condition of plane stress, and Poisson ratio \( \nu \) that appear in the problem can be determined.

We solved Eq. (16) to determine unknown constants. The expressions of the stress components were obtained in a non-dimensional form, we obtained

\[
\sigma_{\theta\theta} = \left( \frac{2G}{\sigma_0 R} \right)^2 \frac{K_{10}}{K_{22}} \text{cos}(\gamma) + \frac{K_{12}}{K_{22}} \text{cos}(\gamma) \quad \text{and} \quad \sigma_{\theta\phi} = \left( \frac{2G}{\sigma_0 R} \right)^2 \frac{K_{10}}{K_{22}} \text{cos}(\gamma) - \frac{K_{12}}{K_{22}} \text{cos}(\gamma)
\]

Finally, solving Eq. (16), we can determine the relation

\[
\sigma_{\theta\theta} = \left( \frac{2G}{\sigma_0 R} \right)^2 \frac{K_{10}}{K_{22}} \text{cos}(\gamma) + \frac{K_{12}}{K_{22}} \text{cos}(\gamma) \quad \text{and} \quad \sigma_{\phi\phi} = \left( \frac{2G}{\sigma_0 R} \right)^2 \frac{K_{10}}{K_{22}} \text{cos}(\gamma) - \frac{K_{12}}{K_{22}} \text{cos}(\gamma)
\]

where \( K_{10} = \sum_{k} \frac{1}{k^2} \) and \( K_{12} = \sum_{k} \frac{1}{k^2} \) are the constants. We selected 51 points at equal intervals in the contact region, including both ends, and determined the contact region, that is, when there is no contact in the region \( 0 < \alpha < 1 \), the elliptic hole boundary is deformed by the tightly-fitting plate producing separation in the region \( 1 < \alpha \). The displacement of elliptic hole boundary, \( \alpha = 1 \) can be determined with the tightly-fitting plate.

Hence, if the fit tolerance \( \alpha \) is small, the elliptic hole boundary deforms freely, and that of outer surface of plate \( 0 < \alpha < 1 \) shows the distributions of difference in displacement.

\[
\sigma_{\theta\theta} = \left( \frac{2G}{\sigma_0 R} \right)^2 \frac{K_{10}}{K_{22}} \text{cos}(\gamma) + \frac{K_{12}}{K_{22}} \text{cos}(\gamma) \quad \text{and} \quad \sigma_{\phi\phi} = \left( \frac{2G}{\sigma_0 R} \right)^2 \frac{K_{10}}{K_{22}} \text{cos}(\gamma) - \frac{K_{12}}{K_{22}} \text{cos}(\gamma)
\]

where \( K_{10} = \sum_{k} \frac{1}{k^2} \) and \( K_{12} = \sum_{k} \frac{1}{k^2} \) are the constants. We selected 51 points at equal intervals in the contact region, including both ends, and determined the contact region, that is, when there is no contact in the region \( 0 < \alpha < 1 \), the elliptic hole boundary is deformed by the tightly-fitting plate producing separation in the region \( 1 < \alpha \).

The displacement of elliptic hole boundary, \( \alpha = 1 \) can be determined with the tightly-fitting plate. The displacements of elliptic hole boundary, \( \alpha = 1 \) and contact region, \( 0 < \alpha < 1 \) show the distributions of difference in displacement.

Fig. 5 Contact region for fit tolerance and load.

Fig. 6 Contact stress distributions on the elliptic hole boundary. (\( b/\alpha = 0.5 \), \( \Gamma = 1 \))

Fig. 7 Maximum contact pressure for fit tolerance and load.

Fig. 8 Circumferential stress distributions on the elliptic hole boundary. (\( b/\alpha = 0.5 \), \( \Gamma = 1 \))
Figure 5 shows the relation of fit tolerance, tensile load and contact region. For example in the case of \( b/\alpha = 0.5 \) and \( \Gamma = 1 \), if the relation of \( 2G I_0 / (\sigma_0 R) \leq -1.026 \) is satisfied, the contact region is not produced under tensile load, and if the relation of \( 2G I_0 / (\sigma_0 R) \geq 1.572 \) is satisfied, no separation will be formed under tensile load. Table 1 shows values of \( 2G I_0 / (\sigma_0 R) \) for \( \alpha = 0 \) or \( \alpha = 1.0 \). The results of \( \Gamma = 2 \), 1 and 0.5 show the same tendency. Results for the case of \( b/\alpha = 1.0 \) and \( \Gamma = 1 \) were given by Yamamoto and Tumura. The results perfectly agree with Fig.5.

Figure 6 shows the stress \( \sigma_p \) distribution at the elliptic hole boundary in the case of \( b/\alpha = 0.5 \) and \( \Gamma = 1 \). Furthermore, it shows that the contact pressure increases with enlargement of the contact region. Figure 7 shows the relation of maximum contact pressure and magnitude of fit tolerance, and that the maximum contact pressure increases with an increase in magnitude of fit tolerance and \( \Gamma \). Moreover, it shows that the contact pressure increases with a decrease in amount of \( b/\alpha \). (in the case when elliptic shape is slender in the load direction).

Figure 8 shows a distribution of stress \( \sigma_p \) at the elliptic hole boundary in the case of \( \Gamma = 1 \) and \( b/\alpha = 0.5 \). Figure 9 shows the relation of the value of \( \sigma_p \) at \( \theta = 0 \), \( \alpha \) and 1, and the magnitude of fit tolerance. It also indicates that the maximum value occurs at \( \theta = \alpha \) in the cases of \( b/\alpha = 0.25 \), 0.5 and 1.0 when the magnitude of fit tolerance is small, and at \( \theta = 0 \) when the magnitude of fit
tolerance is large. However, in the case of $b/\alpha =4$, the maximum value occurs at $t = \sqrt{2}$ when the magnitude of fit tolerance is small when the magnitude of fit tolerance is large, the maximum value occurs at $t = 1$ and the minimum value occurs at $t = 0$. The results of $\Delta = 2$, $1$ and $0.5$ show the same tendency.

Conditions of contact region, stress and displacement are determined with the non-dimensional parameter $2G I_i / (\sigma R)$ and elliptic shape $b/\alpha$. Since the parameter $2G I_i / (\sigma R)$ is always zero when fit tolerance is zero, that is, the elliptic plate is slightly larger than the elliptic hole, this parameter approaches zero from a positive value and the contact region decreases with load. If fit tolerance is negative, that is, the elliptic plate is smaller than the elliptic hole, then this parameter approaches zero from a negative value and the contact region increases with tensile load.

4 Conclusion

We analyzed the mixed boundary-value problems of an infinite plate with an inserted elastic elliptic plate, under tensile load, taking into account the rigidity of elliptic plate and fit tolerance. In this analysis, we expressed the distribution of contact stress by summing up a convergent series and applying a point-matching method to satisfy the condition of contact boundary. We clarified the effects of rigidity of elliptic plate, fit tolerance and elliptic shape on the stress distribution and contact region.

The main results are as follows:

[1] The contact region, stress, and displacement depend only on the one parameter containing the magnitude of fit tolerance and load.

[II] When the fit tolerance is zero, the contact region and stress distribution of elastic plate does not vary with the magnitude of load.

[III] When the fit tolerance is not zero, the contact region and stress distribution of elastic plate varies with the magnitude of load.

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