Irredundant Intervals

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Abstract. This expository note presents simplifications of a theorem due to Győri and an algorithm due to Franzblau and Kleitman: Given a family $F$ of $m$ intervals on a linearly ordered set of $n$ elements, we can construct in $O(m + n)^2$ steps an irredundant subfamily having maximum cardinality, as well as a generating family having minimum cardinality. The algorithm is of special interest because it solves a problem analogous to finding a maximum independent set, but on a class of objects that is more general than a matroid. This note is also a complete, runnable computer program, which can be used for experiments in conjunction with the public-domain software of The Stanford GraphBase.
1. Introduction. Let’s say that a family of sets is irredundant if its members can be arranged in a sequence with the following property: Each set contains a point that isn’t in any of the preceding sets.

If \( F \) is a family of sets, we write \( F^{\cup} \) for the family of all nonempty unions of elements of \( F \). When \( F \) and \( G \) are families with \( F \subseteq G^{\cup} \), we say that \( G \) generates \( F \). If \( F \) is irredundant and \( G \) generates \( F \), we obviously have \( |F| \leq |G| \), because each set in the sequence requires a new generator.

In the special case that the members of \( F \) are intervals of the real line, András Frank conjectured that the largest irredundant subfamily of \( F \) has the same cardinality as \( F \)’s smallest generating family. This conjecture was proved by Ervin Győri [4], who noted that such a result was a minimax theorem of a new type, apparently unrelated to any of the other well-known minimax theorems of graph theory and combinatorics. A constructive proof was found shortly afterwards by Franzblau and Kleitman [3], who sketched an algorithm to find a generating family and irredundant subfamily of equal cardinality. (Győri, Franzblau, and Kleitman were led to these results while studying the more general problem of finding a minimum number of subrectangles that cover a given polygon. Further information about polygon covers appears in [3] and [1].)

The purpose of this note is to describe the beautiful algorithm of Franzblau and Kleitman in full detail. Indeed, the CWEB source file that generated this document is a computer program that can be used in connection with the Stanford GraphBase [8] to find maximum irredundant subfamilies and minimum generating families of any given collection of intervals. Perhaps this new exposition will shed new light on the class of optimization problems for which an efficient algorithm exists.

According to the conventions of CWEB [9], the sections of this document are sequentially numbered 1, 2, 3, etc. In this respect we are returning to a style of exposition used by Euler and Gauss and their contemporaries. A CWEB program is also essentially a hypertext; therefore this document may also be regarded as experimental in another sense, as an attempt to find new forms of exposition appropriate to modern technology.

Note: Győri used the term “U-increasing” for an irredundant family; Franzblau and Kleitman called such intervals “independent.” Since a family of sets is a hypergraph, it seems unwise to deviate from the standard meaning of independent edges, yet “U-increasing” is not an especially appealing alternative. We will see momentarily that the term “irredundant” is quite natural in theory and practice.

2. A far-reaching generalization of Győri’s theorem was proved recently by Frank and Jordán [2], who introduced a large new family of minimax theorems related to linking systems. In particular, Frank and Jordán extended Győri’s results to intervals on a circle instead of a line. But no combinatorial algorithm is known as yet for the circular case.

Can any or all of the Franzblau/Kleitman methods be “lifted” to such more general problems? We will return to this tantalizing question after becoming familiar with Franzblau and Kleitman’s remarkable algorithm.
3. **Theory.** It is wise to study the theory underlying the Franzblau/Kleitman algorithm before getting into the program itself.

4. A family of sets is called *redundant* if it is not irredundant. Any family that contains a redundant subfamily is redundant, since any family contained in an irredundant family is irredundant.

5. If $F$ is a family of sets and $s$ is an arbitrary set, let $F|s$ denote the sets of $F$ that are contained in $s$. This operation is left-associative by convention: $F|s|t = (F|s)|t = F|(s \cap t)$.

We also write $\bigcup F$ for $\bigcup \{f \mid f \in F\}$; thus $F|\bigcup F = F$.

(An index to all the main notations and definitions that we will use appears at the end of this note.)

6. **Lemma.** A finite family $F$ is redundant if and only if there is a nonempty set $s$ such that every point of $s$ belongs to at least two members of $F|s$. (The set $s$ need not belong to $F$.)

Proof: If $F$ is irredundant there is no such $s$, because $F|s$ is irredundant and its last set in the assumed sequence contains a point that isn’t in any of the others. But if $F$ is redundant, it contains a minimal redundant subfamily $F_0$; then we have

$$f \subseteq \bigcup (F_0 \setminus \{f\}) \quad \text{for all } f \in F_0,$$

since $F_0 \setminus \{f\}$ is irredundant. It follows that every point of $s = \bigcup F_0$ is contained in at least two members of $F_0$, hence in at least two members of $F|s$ (since $F_0 \subseteq F|s$).

7. **Corollary.** A finite family $F$ of intervals on a line is redundant if and only if there is an interval $s$ such that every point of $s$ belongs to at least two intervals of $F|s$. (The set $s$ need not belong to $F$.)

Proof: Intervals are nonempty. By the proof of the preceding lemma, it suffices to consider sets $s$ that can be written $\bigcup F_0$ for some minimal redundant subfamily $F_0$. In the special case of intervals, $\bigcup F_0$ must be a single interval; otherwise $F_0$ would not be minimal.

8. Henceforth we will restrict consideration to finite families $F$ of intervals on a linearly ordered set. It suffices, in fact, to deal with integer elements; we will consider subintervals of the $n$-element set $[0 \ldots n]$. (The notation $[a \ldots b]$ stands here for the set of all integers $x$ such that $a \leq x < b$.)

If $F$ is a family of sets and $x$ is a point, we will write $N_x F$ for the number of sets that contain $x$. The corollary just proved can therefore be stated as follows: “$F$ is irredundant if and only if every interval $s \subseteq \bigcup F$ contains a point $x$ with $N_x F|s \leq 1$.” This characterization provides a polynomial-time test for irredundancy.

9. Irredundant intervals have an interesting connection to the familiar computer-science concept of *binary search trees* (see, for example, [7, §6.2.2]): A family of intervals is irredundant if and only if we can associate its intervals with a binary tree whose nodes are each labeled with an integer $x$ and an interval containing $x$.

All nodes in the left subtree of such a node correspond to intervals that are strictly less than $x$, in the sense that all elements of those intervals are $< x$; all nodes in the right subtree correspond to intervals that are strictly greater than $x$. The root of the binary tree corresponds to the interval that is last in the assumed irredundant ordering. Its distinguished integer $x$ is an element that appears in no other interval.

Given such a tree, we obtain a suitable irredundant ordering by traversing it recursively from the leaves to the root, in postorder [6, §2.3.1]. Conversely, given an irredundant ordering, we can construct a binary tree recursively, proceeding from the root to the leaves.
10. An example might be helpful at this point. Suppose \( n = 9 \) and
\[
\begin{aligned}
f_1 &= [0 \ldots 8],
f_2 &= [0 \ldots 7],
f_3 &= [1 \ldots 6],
f_4 &= [1 \ldots 5],
f_5 &= [3 \ldots 9],
f_6 &= [2 \ldots 9].
\end{aligned}
\]

Then \( \{f_1, f_2, f_3, f_4, f_5\} \) and \( \{f_1, f_3, f_5, f_6\} \) are irredundant. (Indeed, a family of intervals is irredundant whenever its members have no repeated left endpoints or no repeated right endpoints.) These subfamilies are in fact \textit{maximally} irredundant—they become redundant when any other interval of the family is added. Therefore maximal irredundant subfamilies need not have the same cardinality; irredundant subfamilies do not form the independent sets of a matroid.

On the other hand, irredundant sets of intervals do have matroid-like properties. For example, if \( F \) is irredundant and \( F \cup \{g\} \) is redundant, there is an \( f \neq g \) such that \( F \cup \{g\} \setminus \{f\} \) is irredundant. (The proof is by induction on \( |F| \): There is \( x \in f \in F \) such that \( F = F_1 \cup \{f\} \cup F_2 \), where \( F_1 \) and \( F_2 \) correspond to the left and right subtrees of the root in the binary tree representation. If \( x \in g \), the family \( F_1 \cup \{g\} \cup F_2 \) is irredundant; if \( x < g \), there is \( f' \in F_r \) with \( F_1 \cup \{f\} \cup \{f'\} \) irredudant, by induction; if \( x > g \) there is \( f' \in F_l \) with \( (F_1 \cup \{g\} \setminus \{f'\}) \cup \{f\} \cup F_r \) irredundant.) Such near-matroid behavior makes families of intervals especially instructive.

11. Let’s say that an interval \( s \) is \textit{good} for \( F \) if \( \forall x \neq a \) \( F \upharpoonright s \leq 1 \) for some \( x \in s \); otherwise \( s \) is \textit{bad}. Franzblau and Kleitman introduced a basic reduction procedure for any family \( F \) of intervals that possesses a bad interval \( s \). Their procedure is analogous to modification along an augmenting path in other combinatorial algorithms.

Let \( [a_1 \ldots b_1], \ldots, [a_k \ldots b_k] \) be the maximal intervals in \( F \upharpoonright s \), ordered so that \( a_1 < \cdots < a_k \) and \( b_1 < \cdots < b_k \). (Notice that \( s = [a_1 \ldots b_k] \).) For example, we might have the following picture:

\[
\begin{array}{c}
\text{s} \\
\text{a}_1 \bullet \rightarrow \circ b_1 \\
\text{a}_2 \bullet \rightarrow \circ b_2 \\
\text{a}_3 \bullet \rightarrow \circ b_3 \\
\text{a}_4 \bullet \rightarrow \circ b_4 \\
\text{a}_k \bullet \rightarrow \circ b_k \\
\text{---} \rightarrow \circ \text{(non-maximal intervals make s bad)}
\end{array}
\]

If \( a_{j+1} < b_j \) for \( 1 \leq j < k \), the family \( F \) \textit{reduced} in \( s \) is defined to be
\[
F \downarrow s = F \setminus \{[a_1 \ldots b_1], \ldots, [a_k \ldots b_k]\} \cup \{[a_2 \ldots b_1], \ldots, [a_k \ldots b_k-1]\}.
\]

In the simplest case we have \( k = 1 \) and the reduced family is simply \( F \setminus \{[a_1 \ldots b_1]\} \).

If \( s \) is a \textit{minimal} bad interval for \( F \), we can prove that \( a_{j+1} < b_j \) for \( 1 \leq j < k \), hence \( F \downarrow s \) is well-defined. Indeed, point \( a_{j+1} \) must be in some interval \([c \ldots d]\) other than \([a_{j+1} \ldots b_{j+1}]\), since \( s \) is bad. We can assume that \( c < a_{j+1} \); otherwise all intervals of \( F \upharpoonright s \) would be contained in \([a_1 \ldots a_{j+1}]\) or \([a_{j+1} \ldots a_k]\) and both of these subintervals would be bad, contradicting the minimality of \( s \). But \( c < a_{j+1} \) implies that \([c \ldots d]\) is contained in some maximal \([a_i \ldots b_i]\) with \( i \leq j \). Hence \( a_{j+1} < b_i \leq b_j \).

The notation \( F \downarrow s \) is defined to be left-associative, like \( F \upharpoonright s \); that is, \( F \downarrow s \downarrow t = (F \downarrow s) \downarrow t \) and \( F \downarrow s \upharpoonright t = (F \downarrow s) \upharpoonright t \).

12. **Lemma.** If \( s \) is a \textit{minimal} bad interval for \( F \), we have \( F \subseteq (F \downarrow s)^\upharpoonright t \).

Proof: We must show that \( [a_j \ldots b_j] \in (F \downarrow s)^\upharpoonright t \) for \( 1 \leq j \leq k \). Let \( G = F \downarrow s \upharpoonright [a_j \ldots b_j] \); we will prove that \( [a_j \ldots b_j] = \bigcup G \). If \( x \in [a_j \ldots b_j] \), the badness of \( s \) implies that \( x \in t \) for some \( t \in F \upharpoonright s \) with \( t \neq [a_j \ldots b_j] \); let \( t \) be contained in the maximal interval \([a_i \ldots b_i]\). If \( i = j \), we have \( t \in G \); if \( i < j \), we have \( x \in [a_i \ldots b_i] \subseteq [a_j \ldots b_{j-1}] \in G \); and if \( i > j \), we have \( x \in [a_i \ldots b_j] \subseteq [a_{j+1} \ldots b_j] \in G \).
13. Lemma. Suppose \( s \) is a minimal bad interval for \( F \), while \( t \) is a good interval. Then \( t \) is good also for \( F \). \( s \).

Proof: Let \( s = [a \ldots b], t = [c \ldots d], l = \min(a, c), r = \max(b, d) \). Suppose \( N_x F|[l \ldots d] \geq 2 \) and \( N_x F|[c \ldots r] \geq 2 \) for all \( x \in t \). Then \( l = a < c < d < b = r \), because \( t \) is good for \( F \). By the minimality of \( s \), there is some \( x \in [a \ldots d] \) with \( N_x F|[a \ldots d] \cdot x \leq 1 \). Since \( N_x F|[a \ldots d] \geq 2 \) we have \( x < c \). Furthermore, \( x \) is in some interval \( (F|s) \setminus F|[a \ldots d] \), because \( s \) is bad; so \( x \) is in some maximal \( [a_j \ldots b_j] \) with \( a \leq a_j \leq x < c < d < b_j \leq b \). It follows that none of the intervals \( [a_1 \ldots b_1], \ldots, [a_k \ldots b_k], [a_2 \ldots b_1], \ldots, [a_k \ldots b_k] \) are contained in \( t \), hence \( F|s \upharpoonright t = F|t \).

On the other hand, suppose \( N_x F|[l \ldots d] \leq 1 \) for some \( x \in t \). We will show that \( N_x F|s \upharpoonright t \leq 1 \). This assertion can fail only if \( x \) lies in some interval \( [a_{j+1} \ldots b_j] \subseteq t \) newly added to \( F|s \). Then \( x \in [a_j \ldots b_j] \subseteq [l \ldots d] \), and \( x \in [a_{j+1} \ldots b_{j+1}] \notin [l \ldots d] \), hence \( b_j \leq d < b_{j+1} \); it follows that \( j \) is uniquely determined, and the only interval containing \( x \) in \( F|s \upharpoonright t \) is \( [a_{j+1} \ldots b_j] \). A similar argument applies if \( N_x F|[c \ldots r] \leq 1 \).

14. Corollary. If \( s \) is a minimal bad interval for \( F \), we have \( N_x F|s = N_x F - N_x s \).

Proof: The proof of the preceding lemma shows in particular that none of the intervals \( [a_{j+1} \ldots b_j] \) are already present in \( F \) before the reduction. And if \( x \in s \), suppose \( x \) lies in \( [a_i \ldots b_i], [a_{i+1} \ldots b_{i+1}], \ldots, [a_j \ldots b_j] \); then \( x \) lies in \( [a_{i+1} \ldots b_i], \ldots, [a_j \ldots b_{j-1}] \) after reduction, a net change of \(-1\).

15. The Franzblau/Kleitman algorithm has a very simple outline: We let \( G_0 = F \) and repeatedly set \( G_{k+1} = G_k \upharpoonright s_k \), where \( s_k \) is the leftmost minimal bad interval for \( G_k \), until we finally reach a family \( G_r \) in which no bad intervals remain. This must happen sooner or later, because \( |G_k| = |F| - k \). The final irredundant family \( G = G_r \) generates \( F \), because \( F \subseteq G_r^c \) for all \( k \) by the lemma of §12. Franzblau and Kleitman proved the nontrivial fact that \( |G| \) is the size of the maximum irredundant subfamily of \( F \); hence \( G \) is a minimum generating family.

16. It is tempting to try to prove the optimality of \( G \) by a simpler, inductive approach in which we “grow” \( F \) one interval at a time, updating its maximum irredundant set and minimum generating set appropriately. But experiments show that the maximum irredundant set can change drastically when \( F \) receives a single new interval, so this direct primal-dual approach seems doomed to failure. The indirect approach is more difficult to prove, but no more difficult to program. So we will proceed to develop further properties of Franzblau and Kleitman’s reduction procedure [3]. The key fact is a remarkable theorem that we turn to next.

17. Theorem. The same final family \( G = G_r \) is obtained when \( s_k \) is chosen to be an arbitrary (not necessarily leftmost) minimal bad interval of \( G_k \) in the reduction algorithm. Moreover, the same multiset \( \{s_0, \ldots, s_{r-1}\} \) of minimal bad intervals arises, in some order, regardless of the choices made at each step.

Proof: We use induction on \( r \), the maximum number of steps to convergence among all reduction procedures that begin with a family \( F \). If \( r = 0 \), the result is trivial, and if \( F \) has only one minimal bad interval the result is immediate by induction. Suppose therefore that \( s \) and \( t \) are distinct minimal bad intervals of \( F \). We will prove later that \( t \) is a minimal bad interval for \( F|s \), and that \( F|s \upharpoonright t = F|t \upharpoonright s \). Let \( r' \) be the maximum distance to convergence from \( F|s \), and \( r'' \) the maximum from \( F|t \); then \( r' \) and \( r'' \) are less than \( r \), and induction proves that the final result from \( F|s \) is the final result from \( F|s \upharpoonright t = F|t \upharpoonright s \), which is the final result from \( F|t \). (Readers familiar with other reduction algorithms, like that of [5], will recognize this as a familiar “diamond lemma” argument. We construct a diamond-shaped diagram with four vertices: \( F, F|s, F|t, \) and a common outcome of \( F|s \) and \( F|t \).) This completes the proof, except for two lemmas that will be demonstrated below; their proofs have been deferred so that we could motivate them first.
18. This theorem and the lemma of §13 have an important corollary: Let $S = \{s_0, \ldots, s_{r-1}\}$ be the multiset of minimal bad intervals determined by the algorithm from $F$, and let $t$ be any interval. Then $S \{t\}$ is the multiset of minimal bad intervals determined by the algorithm from $F\{t\}$. This holds because an interval $s \subseteq t$ is bad for $F$ if and only if it is bad for $F\{t\}$. Minimal bad intervals within $t$ never appear again once they are removed, and we can remove them first.

Reducing a minimal bad interval $s$ when $s$ is contained in a bad interval $t$ may make $t$ good, or leave it bad, or make it minimally bad. If $s$ is minimally bad for $F$, it might also be minimally bad for $F\{t\}$.

19. Now we are ready for the coup de grâce and the pièce de résistance: After the reduction algorithm has computed the irredundant generating family $G = G_r$ and the multiset $S$ of minimal bad intervals, we can construct an irredundant subfamily $F'$ of $F$ with $|F'| = |G|$ by constructing a binary search tree as described in §9. The procedure is recursive, starting with an initial interval $t = [0..n]$ that contains $F$: The tree defined for $F\{t\}$ is empty if $F\{t\}$ is empty. Otherwise it has a root node labeled with $x$ and with any interval of $F\{t\}$ containing $x$, where $x$ is an integer such that $N_x G^{(t)} = 1$; here $G^{(t)}$ is the final generating set that is obtained when the reduction procedure is applied to $F\{t\}$. A suitable interval containing $x$ exists, because every element of $G^{(t)}$ is an intersection of intervals in $F\{t\}$. The left subtree of the root node is the binary search tree for $F\{t\} (t \cap [0..x))$; the right subtree is the binary search tree for $F\{t\} (t \cap [x+1..n])$.

The number of nodes in this tree is $|G|$. For $x$ is the integer in the label of the root, $G$ has one interval containing $x$, and its other intervals are $G([0..x])$ and $G([x+1..n])$. The family $G^{(t)}$ is not the same as $G\{t\}$; but we do have $|G^{(t)}| = |G\{t\}|$ when $t$ has the special form $[0..x] \cup [x+1..n]$, because in such cases $F\downarrow s \{t\}$ has the same cardinality as $F\{t\}$ when $s$ is a minimal bad interval and $s \not\subseteq t$. For example, if $t = [0..x)$ and $b_j \leq x < b_{j+1}$, we have $F\downarrow s \{t\} = F\{t\} \setminus \{[a_1..b_1], \ldots, [a_j..b_j]\} \cup \{[a_2..b_1], \ldots, [a_{j+1}..b_j]\}$.

20. It is not necessary to compute each $G^{(t)}$ from scratch by starting with $F\{t\}$ and applying the reduction algorithm until it converges, because the binary tree construction algorithm requires only a knowledge of the incidence function $N_x G^{(t)}$. This function is easy to compute, because $N_x F\downarrow s = N_x F - N_x s$ by §14; therefore

$$N_x G^{(t)} = N_x F\{t\} - N_x S\{t\}.$$  

21. All the basic ideas of Franzblau and Kleitman’s algorithm have now been explained. But we must still carry out a careful analysis of some fine points of reduction that were claimed in the proof of the main theorem. If $s$ and $t$ are distinct minimal bad intervals, the lemma of §13 implies that no bad subintervals of $t$ appear in $F\downarrow s$; we also need to verify that $t$ itself remains bad.

**Lemma.** If $s$ is a minimal bad interval for $F$ and $t$ is a bad interval such that $s \not\subseteq t$, then $t$ is bad for $F\downarrow s$.

**Proof:** Let $s = [a..b)$ and $t = [c..d]$. We can assume by left-right symmetry that $a < c$. Then $b < d$, by minimality of $s$. Assume that $t$ isn’t bad for $F\downarrow s$. The subfamily $F\{t\}$ must contain at least one of the maximal intervals $[a_j..b_j]$ of $F\{s\}$ that are deleted during the reduction; hence $c \leq a_j < b_{j-1} \leq b$.

Let $j$ be minimal with $a_j \geq c$. Then

$$F\downarrow s \{t\} = F\{t\} \setminus \{[a_j..b_j], \ldots, [a_k..b_k]\} \cup \{[a_j..b_{j-1}], \ldots, [a_k..b_{k-1}]\};$$

so the elements of $t$ that are covered once less often are the elements of $[b_{j-1}..b_k)$. Suppose $y \in [b_{j-1}..b_l)$ for some $l \geq j$. Then $y \in [a_i..b_l)$ for $i \geq j$, and $y \in [a_i..b_l) \subseteq s \cap t$, and $y$ is in some other interval $f \subseteq s$. The maximal interval containing $f$ must be $[a_i..b_l)$ for some $l \geq i$, hence $f \subseteq s \cap t$. Thus $N_x F\{s \cap t\} \geq 2$ for all $y \in [b_{j-1}..b_l)$. But $s \cap t$ is good, so there must be a point $x \in [c..b_{j-1}]$ with $N_x F\{s \cap t\} \leq 1$. We also have $N_x F\{t\} \geq 2$, since $t$ is bad, so there’s an interval in $F\{t\} \setminus F\{s \cap t\}$ that contains $x$. This interval contains $[x..b_k)$. Consequently $N_y F\{t\} \geq 3$ for all $y \in [b_{j-1}..b_l)$. 


22. **Lemma.** If \( s \) and \( t \) are minimal bad intervals for \( F \) and \( s \neq t \), we have \( F \downarrow s \downarrow t = F \downarrow t \downarrow s \).

Proof: Let the maximal intervals in \( F|s \) and \( F|t \) be \([a_1 \ldots b_1], \ldots, [a_k \ldots b_k]\) and \([c_1 \ldots d_1], \ldots, [c_l \ldots d_l]\), respectively, where \( a_1 < c_1 \). The lemma is obvious unless \( F|(s \cap t) \) is nonempty, so we assume that \( c_1 < b_k < d_1 \). Let \( x \in s \cap t \) have \( N_x F|(s \cap t) = 1 \), and let \( f \) be the interval of \( F|(s \cap t) \) that contains \( x \). Let \( p \) be maximal with \( a_p < c_1 \), and let \( q \) be minimal with \( d_q > b_k \). Since \( N_x F|s > 1 \), there is an interval \([a_j \ldots b_j]\) containing \( x \) with \( j \leq p \); thus \( x \in [a_p \ldots b_p] \). Similarly \( x \in [c_q \ldots d_q] \). Furthermore, if \( p < k \) we have \([a_{p+1} \ldots b_{p+1}] \subseteq s \cap t \); hence either \([a_{p+1} \ldots b_{p+1}] = f \) or \( a_{p+1} > x \). If \( q > 1 \) we have either \([c_{q-1} \ldots d_{q-1}] = f \) or \( d_{q-1} \leq x \).

If \( p = k \) or \( f \neq [a_{p+1} \ldots b_{p+1}] \), any newly added intervals \([a_{j+1} \ldots b_j]\) for \( p < j < k \) in \( F \downarrow s \) are properly contained in \([c_q \ldots d_q]\), so they remain in \( F \downarrow s \downarrow t \). Thus we can easily describe the compound operation \( F \downarrow s \downarrow t \) in detail:

Delete \([a_1 \ldots b_1], \ldots, [a_k \ldots b_k]\); insert \([a_2 \ldots b_1], \ldots, [a_k \ldots b_{k-1}]\); delete \([c_1 \ldots d_1], \ldots, [c_l \ldots d_l]\); insert \([c_2 \ldots d_1], \ldots, [c_l \ldots d_{l-1}]\).

No two of these intervals are identical, so \( F \downarrow t \downarrow s \) gives the same result. (If \( f = [c_{q-1} \ldots d_{q-1}] \), the family \( F \downarrow t \) has \( f \) replaced by \([c_q \ldots d_{q-1}] \subseteq f \subseteq [a_i \ldots b_i] \) for some \( i \leq p \), so \([c_q \ldots d_{q-1}] \) is not maximal in \( F \downarrow t \downarrow s \).)

The remaining case \( f = [a_{p+1} \ldots b_{p+1}] = [c_{q-1} \ldots d_{q-1}] \) needs to be considered specially, since we can’t delete this interval twice. The following picture might help clarify the situation:

Suppose \( g \) is an interval of \( F|t \) that is contained in \([c_q \ldots d_q]\) if and only if \( j = q - 1 \). Then \( g \) contains a point < \( c_q \). If \( x \in g \) we have \( g = f \), since \( g \subseteq s \cap t \). Otherwise \( g \not\subseteq [c_{q-1} \ldots x) \subseteq [c_{q-1} \ldots b_p] = [a_{p+1} \ldots b_p] \). It follows that the maximal intervals of \( F \downarrow s|t \) are:

\([c_1 \ldots d_1], \ldots, [c_{q-2} \ldots d_{q-2}], [c_{q-1} \ldots b_p], [c_q \ldots d_q], \ldots, [c_l \ldots d_l] \).

These intervals are replaced in \( F \downarrow s \downarrow t \) by:

\([c_2 \ldots d_1], \ldots, [c_{q-1} \ldots d_{q-2}], [c_q \ldots b_p], [c_{q+1} \ldots d_q], \ldots, [c_l \ldots d_{l-1}] \).

Thus \( F \downarrow s \downarrow t \) is formed almost as in the previous case, but with \([a_{p+1} \ldots b_p]\) and \([c_q \ldots d_{q-1}] \) replaced by \([c_q \ldots b_p]\). And we get precisely the same intervals in \( F \downarrow t \downarrow s \).

(Is there a simpler proof?)
23. Practice. The computer program in the remainder of this note operates on a family of intervals defined by a graph on \( n + 1 \) vertices \( \{0, 1, \ldots, n\} \). We regard an edge between \( u \) and \( v \) as the half-open interval \([u, v)\), when \( u < v \).

Graphs are represented as in the algorithms of the Stanford GraphBase [8], and the reader of this program is supposed to be familiar with the elementary conventions of that system.

The program reads two command-line parameters, \( m \) and \( n \), and an optional third parameter representing a random-number seed. (The seed value is zero by default.) The Franzblau/Kleitman algorithm is then applied to the graph \( \text{random\_graph}(n + 1, m, 0, 0, 0, 0, 0, 0, 0, \text{seed}) \), a random graph with vertices \( \{0, 1, \ldots, n\} \) and \( m \) edges. Alternatively, the user can specify an arbitrary graph as input by typing the single command-line parameter \( -g \langle \text{filename} \rangle \); in this case the named file should describe the graph in \( \text{save\_graph} \) format (as in the \( \text{MILES\_SPAN} \) program of [8]).

When the computation is finished, a minimal generating family and a maximal irredundant subfamily will be printed on the standard output file.

If a negative value is given for \( n \), the random graph is reversed from left to right; each interval \([a, b)\) is essentially replaced by \([-b, -a)\) (but minus signs are suppressed in the output). This feature lends credibility to the correctness of our highly asymmetric algorithm and program, because we can verify the fact that the minimum generating family of the mirror image of \( F \) is indeed the mirror image of \( F \)'s minimum generating family.

In practice, the algorithm tends to be interesting only when \( m \) and \( n \) are roughly equal. If \( n \) is large compared to \( m \), we can remove any vertices of degree zero; such vertices aren’t the endpoint of any interval. If \( m \) is large compared to \( n \), we can almost always find \( n \) irredundant intervals by inspection. The running time in general is readily seen to be \( O(mn + n^2) \).

```c
#define panic(k)
{
    fprintf(stderr,"Oops, we’re out of memory! (Case %d)\n",k);
    return k;
}
#include "gb_graph.h" /* the GraphBase data structures */
#include "gb_rand.h" /* the random_graph generator */
#include "gb_save.h" /* the restore_graph generator */

⟨Preprocessor definitions⟩

Graph *F; /* the graph that defines intervals */
Graph *G; /* a graph of intervals that generate F */

⟨Subroutines 33⟩

main(argc, argv)
int argc; /* the number of command-line arguments, plus 1 */
char *argv[]; /* the command-line arguments */
{
    register Vertex *t, *u, *v, *w, *x; /* current vertices of interest */
    register Arc *a, *b, *c; /* current arcs of interest */
    ⟨Scan the command-line options and generate F 24⟩;
    ⟨Compute G and S by the Franzblau/Kleitman algorithm 26⟩;
    if (gb_trouble_code) panic(1);
    ⟨Construct an irredundant subfamily of F with the cardinality of G 32⟩;
    ⟨Print the results 36⟩;
    return 0; /* this is the normal exit */
}
24. (Scan the command-line options and generate $F$ 24) $\equiv$

```c
{ int m = 0, n = 0, seed = 0;
if (argc $\geq$ 3 $\land$ sscanf(argv[1], "%d", &m) $\equiv$ 1 $\land$ sscanf(argv[2], "%d", &n) $\equiv$ 1) {
if (argc $>$ 3) sscanf(argv[3], "%d", &seed);
if (m < 0) {
  m = −m; /* we assume the user meant to negate $n$ instead of $m$ */
  n = −n;
}
if (n $\geq$ 0) $F$ = random_graph($n$ + 1, $m$, 0, 0, 0, 0, 0, 0, seed);
else {
  $G$ = random_graph($−n$ + 1, $m$, 0, 0, 0, 0, 0, 0, seed);
  $\langle$ Set $F$ to the mirror image of $G$ 25 $\rangle$
  gb_recycle($G$);
}
}
else if (argc $\equiv$ 2 $\land$ strncmp(argv[1], "−g", 2) $\equiv$ 0) $F$ = restore_graph(argv[1] + 2);
else {
  fprintf(stderr,"Usage: %s $m$ $n$ \[seed\] | %s −gfoo.gb\n", argv[0], argv[0]);
  return 2;
}
if (¬$F$) {
  fprintf(stderr,"Sorry, can’t create the graph! (error code %ld)\n", panic_code);
  return 3;
}
print("Applying Franzblau/Kleitman to "$F$'s: ", $F$-id);
}

This code is used in section 23.

25. (Set $F$ to the mirror image of $G$ 25) $\equiv$

```c
$F$ = gb_new_graph($G$-$n$);
if (¬$F$) panic(4);
make_compound_id($F$,"reflect","G","n");
for ($v = G$-vertices; $u = F$-vertices + $F$-$n$ − 1; $v < G$-vertices + $G$-$n$; $u$, $v$++) {
  $v$-clone = $u$;
  $w$-name = gb_save_string($v$-name);
}
for ($v = G$-vertices; $v < G$-vertices + $G$-$n$; $v$++)
  for ($a = v$-arcs; $a = a$-next) gb_new_arc($v$-clone, $a$-tip-clone, 1);
```

This code is used in section 24.
26. **The main algorithm.** We follow the outline of §15.

\[
\begin{align*}
\langle \text{Compute } G \text{ and } S \text{ by the Franzblau/Kleitman algorithm} \rangle & \equiv \\
\langle \text{Make } G \text{ a copy of } F, \text{ retaining only leftward arcs} \rangle & \equiv \\
\text{for } (v = G\text{-vertices} + 2; \ v < G\text{-vertices} + G\text{-n}; \ v++) & \\
\langle \text{Reduce all minimal bad intervals with right endpoint } v \text{ and record them in } S \rangle & \\
\end{align*}
\]

This code is used in section 23.

27. The algorithm doesn’t need pointers from the left endpoint of an interval to the right endpoint; leftward pointers are sufficient. (This observation makes the reduction procedure faster.)

While copying \( F \), we remove its rightward arcs, and we assign length 1 to all its leftward arcs. Later on, we will represent intervals of \( S \) by recording them in \( F \) as leftward arcs of length \(-1\).

We also clear two utility fields of \( F \)’s vertices, since they will be used by the algorithm later. (They might be nonzero, if \( F \) was supplied with the \(-g\) option.)

```c
#define clone u.V /* the vertex in \( F \) that matches a vertex in \( G \), or vice versa */
```

\[
\begin{align*}
\langle \text{Make } G \text{ a copy of } F, \text{ retaining only leftward arcs} \rangle & \equiv \\
\text{switch_to_graph}(\Lambda); /* prepare to return to graph } F \text{ later */} \\
G = gb\_new\_graph(F\text{-n}); /* a graph with nameless vertices and no arcs */ \\
\text{if } (\neg G) \text{ panic}(5); \\
\text{for } (u = F\text{-vertices}, \ v = G\text{-vertices}; \ u < F\text{-vertices} + F\text{-n}; \ u++, \ v++) & \\
\text{w-name} = gb\_save\_string(u\text{-name}); \\
\text{for } (a = u\text{-arcs}, \ b = \Lambda; \ a = a\text{-next}) & \\
\text{if } (a\text{-tip} \geq u) & \{ /* we will remove the non-leftward arc } a */ \\
\text{if } (b) \text{ b-next} = a\text{-next}; \\
\text{else } u\text{-arcs} = a\text{-next}; \\
\} \text{ else } & \{ /* we will copy the leftward arc } a */ \\
\text{gb\_new\_arc}(v, a\text{-tip}\_clone, 0); /* the length in } G \text{ is } 0 */ \\
a\_len = 1; /* but in } F \text{ the length is } 1 */ \\
b = a; /* } b \text{ points to the last non-removed arc */} \\
\}
\]

```c
\text{if } (gb\_trouble\_code) \text{ panic}(6); \\
\text{switch_to_graph}(F); /* now we can add arcs to } F \text{ again */}
```

This code is used in section 26.
28. Here's the most interesting part of the program, algorithmwise. Given a vertex \( v \), we want to find the largest \( u \), if any, such that \([u..v)\) is bad for the intervals in \( G \). So we sweep through the intervals \([u..v)\) from right to left, decreasing \( u \) until it reaches a limiting value \( t \). Here \( t \) is the least upper bound on a left endpoint that could guarantee double coverage of all points in \([u..v)\).

The utility field \( x\cdot count \) records the number of intervals with left endpoint \( x \) and right endpoint \( w \) in the range \( u < w \leq v \). Another utility field \( x\cdot link \) is used to link vertices with nonzero counts together so that we can clear the counts to zero again afterwards.

It's easy to see that each iteration of the \( \text{while} \) loop in this section takes at most \( O(m+n) \) steps. The actual computation time is, however, usually much faster.

This program is designed to work correctly when \( G \) contains more than one arc from \( u \) to \( v \). Duplicate arcs are discarded as a special case of the reduction procedure.

```c
#define count z.I /* coverage decreases by this much when we pass to the left */
#define link y.V /* pointer to a vertex whose count field needs to be zeroed later */

(Reduce all minimal bad intervals with right endpoint \( v \) and record them in \( S \))

\[ \text{while} (1) \{ \\
\text{int coverage} = 0; /* the number of intervals} \subseteq [t..v)\) that contain \( u \) */
\text{int potential} = 0; /* sum of } x\cdot count \) for \( x < t \) */
\text{Vertex } *\text{cleanup} = \Lambda; /* head of the list of vertices with nonzero count */
\text{for} (u = v, t = v - 1; u > t; u--) \{ \\
\text{coverage} -= u\cdot count; /* we prepare to decrease } u \)
\}
\text{if} (\text{coverage} + \text{potential} < 2) \{ /* there's no bad interval ending at } v \)
\text{goto } done\_with\_v;
\}
\text{while} (\text{coverage} < 2) \{ \\
\text{t}--; \\
\text{coverage} += t\cdot count;
\text{potential} -= t\cdot count;
\}
\text{gap}\_\text{new}\_\text{arc}(v\cdot clone, u\cdot clone, -1); /* } [u..v)\) is minimally bad; we record it in \( S = -F \) */
\text{(Replace } G \text{ by } G\downarrow [u..v)\) 31);} \\
\text{(Clean up all } count \text{ fields 30);} /* now we'll try again */
\}
\text{done\_with\_v: ;}
```

This code is used in section 26.
29. \(\langle\text{Update the counts for all intervals ending at } u\rangle\) \(\equiv\)
\[
\text{for } (a = u\text{-arcs}; a; a = a\text{-next}) \{ \\
w = a\text{-tip}; \\
\text{if } (w\text{-count} \equiv 0) \{ \\
w\text{-link} = \text{cleanup}; \\
\text{cleanup} = w; \\
\} \\
w\text{-count}++; \\
\text{if } (w \geq t) \text{ coverage}++; \\
\text{else } \text{potential}++; \\
\}
\]
This code is used in section 28.

30. \(\langle\text{Clean up all count fields}\rangle\) \(\equiv\)
\[
\text{for } (w = \text{cleanup}; w; w = w\text{-link}) \ w\text{-count} = 0;
\]
This code is used in section 28.

31. The reduction process is kind of cute too.
\(\langle\text{Replace } G \text{ by } G\downarrow[u..v]\rangle\) \(\equiv\)
\[
\text{for } (a = v\text{-arcs}, c = \Lambda, w = v; a; c = a, a = a\text{-next}) \\
\text{if } (a\text{-tip} \geq u \land a\text{-tip} < w) \ w = a\text{-tip}, b = c; \\
\text{/* now } [w..v) \text{ is the longest interval from } v \text{ inside } [u..v); \text{ we’ll remove it */} \\
\text{if } (b) \ b\text{-next} = b\text{-next}\text{-next}; \\
\text{else } v\text{-arcs} = v\text{-arcs}\text{-next}; \\
\text{/* the remaining job is to shorten the other maximal arcs in } [u..v); */ \\
\text{for } (t = v - 1; w > u; t--) \{ \\
\text{for } (a = t\text{-arcs}, x = w; a; a = a\text{-next}) \\
\text{if } (a\text{-tip} \geq u \land a\text{-tip} < x) \ x = a\text{-tip}, b = a; \\
\text{if } (x < w) \ b\text{-tip} = w, w = x; \text{/* } [x..t) \text{ is the longest interval from } t */ \\
\}
\]
This code is used in section 28.
32. The dénouement. Now we build a binary tree in the original graph $F$, by filling in some of the utility fields of $F$'s vertices. If a node in the tree is labeled with $x$ and with the interval $[u, v)$, we represent it by $x\text{left} = u$ and $x\text{right} = v$; the subtrees of this node are $x\text{llink}$ and $x\text{rlink}$. The root of the whole tree is $F\text{root}$.

The $rlink$ field happens to be the same as the $count$ field, but this is no problem because the $rlink$ is never changed or examined until after the $count$ has been reset to zero for the last time.

```c
#define left x.V /* left endpoint of interval labeling this node */
#define right w.V /* right endpoint of interval labeling this node */
#define llink v.V /* left subtree of this node */
#define rlink z.V /* right subtree of this node */
#define root uu.V /* root node of the binary tree for this graph */
```

$\langle$ Construct an irredundant subfamily of $F$ with the cardinality of $G \ 32 \rangle \equiv$

$$F\text{root} = \text{make_tree}(F\text{vertices}, F\text{vertices} + F^n - 1);$$

This code is used in section 23.

33. With a little care we could maintain a stack within $F$ itself, but it’s easier to use recursion in C. Let’s just hope the system programmers have given us a large enough runtime stack to work with.

This subroutine is based on the trick explained in §20.

$\langle$ Subroutines 33 $\rangle \equiv$

```c
Vertex *make_tree(t, w)
{
    register Vertex *u, *v, *x;
    register Arc *a;
    ⟨Find a vertex $x$ with $N_x F|[t \ldots w) - N_x S|[t \ldots w) = 1 \ 34⟩;)
    if (!x) return Λ; /* $F|[t \ldots w)$ is empty */
    ⟨Find an interval $[u \ldots v$) such that $x \in [u \ldots v) \subseteq [t \ldots w) \ 35⟩;)
    x\text{left} = u;
    x\text{right} = v;
    x\text{llink} = \text{make_tree}(t, x);
    x\text{rlink} = \text{make_tree}(x + 1, w);
    return x;
}
```

See also section 37.

This code is used in section 23.
34. A subtle bug is avoided here when we realize that a vertex might already be in the cleanup list when its count is zero.

\[\text{Find a vertex } x \text{ with } N_x F[t..w] - N_x S[t..w] = 1\]

\[
\begin{align*}
\text{register int } & \text{coverage} = 0; \quad \text{/* coverage in } F \text{ minus } S */ \\
\text{Vertex } & \text{cleanup} = w + 1; \quad \text{/* } w + 1 \text{ is a sentinel value */} \\
\text{for } (v = w, x = \Lambda; v > t; v--) \{ \\
& \text{coverage} = -v\text{-count}; \quad \text{/* now coverage refers to } N_{v-1} */ \\
& \text{for } (a = v\text{-arcs}; a; a = a\text{-next}) \{ \\
& \quad u = a\text{-tip}; \\
& \quad \text{if } (u \geq t) \{ \\
& \quad \quad \text{if } (u\text{-link }= \Lambda) \quad u\text{-link} = \text{cleanup}, \text{cleanup} = u; \\
& \quad \quad u\text{-count} += a\text{-len}; \quad \text{/* the length is } +1 \text{ for } F, -1 \text{ for } S */ \\
& \quad \quad \text{coverage} += a\text{-len}; \\
& \quad \} \\
& \} \\
& \text{if } (\text{coverage }= 1) \{ \\
& \quad x = v - 1; \text{ break;} \\
& \} \\
& \text{if } (\neg x \land \text{cleanup } \leq w) \quad \text{fprintf(stderr, "This can't happen!\n")}; \\
\text{while } (\text{cleanup } \leq w) \{ \\
& \quad v = \text{cleanup}\text{-link}; \\
& \quad \text{cleanup}\text{-count} = 0; \\
& \quad \text{cleanup}\text{-link} = \Lambda; \\
& \quad \text{cleanup} = v; \\
& \}
\end{align*}
\]

This code is used in section 33.

35. \(\text{Find an interval } [u..v) \text{ such that } x \in [u..v) \subseteq \{t..w\}\)\]

\[
\begin{align*}
\text{for } (v = w; v > x; v--) \{ \\
& \text{for } (a = v\text{-arcs}; a; a = a\text{-next}) \\
& \quad \text{if } (a\text{-len } > 0) \{ \\
& \quad \quad u = a\text{-tip}; \\
& \quad \quad \text{if } (u \leq x \land u \geq t) \quad \text{goto done;} \\
& \quad \} \\
& \}
\end{align*}
\]

\text{done:}

This code is used in section 33.

36. \(\text{Print the results}\)\]

\[
\begin{align*}
\text{printf("Minimum generating family:");} \\
\text{for } (v = G\text{-vertices } + 1; v < G\text{-vertices } + G\text{-n}; v++) \\
\quad \text{for } (a = v\text{-arcs}; a; a = a\text{-next}) \quad \text{printf("\%s..\%s", a\text{-tip-name}, v\text{-name});} \\
\quad \text{printf("\nMaximum irredundant family:");} \\
\text{postorder_print}(F\text{-root}); \\
\text{printf("\n");}
\end{align*}
\]

This code is used in section 23.
37. There's just one subroutine to go. This is textbook stuff.

\[ \langle \text{Subroutines 33} \rangle + \equiv \]

\begin{verbatim}
\textbf{void postorder\_print}(x)
    \textbf{Vertex }*x;
    
    \{ if \ (x) \ 
       \text{postorder\_print}(x\_llink);
       \text{postorder\_print}(x\_rlink);
       \text{printf}("\%s[\%s..\%s]", x\_name, x\_left\_name, x\_right\_name);
    \}
\}
\end{verbatim}
38. Comments and extensions. The program just presented incorporates several refinements to the implementation sketched by Franzblau and Kleitman in [3], and the author hopes that readers will enjoy finding them in the code. The Stanford GraphBase provides convenient data structures, by means of which it was possible to make the program short and sweet. However, most of the key ideas (except for the make_tree procedure) can be found in [3].

39. Lubiw [10] discovered that the algorithm of Franzblau and Kleitman can be generalized so that it finds optimum irredundant subfamilies and generating families in an appropriate sense when the points of the underlying line have been given arbitrary nonnegative weights. It should be interesting and instructive to extend the program above so that it handles this more general problem.

40. The introductory remarks in §2 mention the recent breakthrough by Frank and Jordán [2], who showed (among many other things) that Győri’s theorem can be extended to intervals on a circle as well as a line. Such a generalization was surprising because the size of a minimum generating set might be strictly larger than the size of a maximum irredundant subfamily of cyclic intervals. For example, the $n$ intervals $[k \ldots k+2)$ for $0 \leq k < n$ on the ring of integers mod $n$ are obviously redundant; if we leave out any one of them, the remaining $n - 1$ intervals will cover all $n$ points. However, these cyclic intervals cannot be generated by fewer than $n$ subintervals: No $n - 1$ subintervals of length 1 will do the job, and if $[k \ldots k+2)$ is one of the generators the remaining $n - 1$ intervals require $n - 1$ further generators because they are irredundant.

Győri’s minimax principle is restored, however, if we change the definition of irredundant families. We can say that $F$ is irredundant if each $f \in F$ has a distinguished element $f_* \in f$, such that whenever $f$ and $f'$ are distinct sets of $F$ we have either $f_* \notin f'$ or $f_*' \notin f$. If $F$ is irredundant in this sense, and if $G$ generates $F$, it is not difficult to prove that $|F| \leq |G|$: There is a $g_f \in G$ for each $f \in F$, with the property that $f_* \in g_f \subseteq f$; our new definition guarantees that $g_f \neq g_{f'}$, when $f \neq f'$.

According to this new definition, the intervals $[k \ldots k+2)$ for $0 \leq k < n$, modulo $n$, are irredundant whenever $n > 2(d - 1)$, because we can let $[k \ldots k+2) = k$. Frank and Jordán showed that if $F$ is any family of intervals modulo $n$ with the property that each intersection $f \cap f'$ of two of its members is either empty or a single interval, then the size of $F$’s smallest generating family is the size of its largest irredundant subfamily under the new definition.

41. For intervals on a line, Győri [4] had already observed that both definitions of irredundancy are equivalent. Suppose a system of representatives $f_\ast \in f$ is given for all $f$ in some family $F$ of intervals on a line, such that $f \neq f'$ implies $f_\ast \notin f'$ or $f'_\ast \notin f$. If we cannot arrange those intervals in a sequence $f^{(1)}$, $f^{(2)}$, $\ldots$, $f^{(n)}$ such that $f_s^{(j)} \notin f^{(j-1)} \cup \cdots \cup f^{(1)}$ for $1 < j \leq n$, there must be some cycle of intervals such that $f_s^{(j)} \in f^{(2)}$, $f_s^{(2)} \in f^{(3)}$, $\ldots$, $f_s^{(m)} \in f^{(1)}$, where $m > 2$. Consider the shortest such cycle, and suppose $f_s^{(1)} = \min_{j=1}^m f_s^{(j)}$. We cannot have $f^{(k-1)} < f_s^{(k)}$ for $1 < k \leq m$, because $f_s^{(m)} \in f^{(1)}$; let $k > 1$ be minimum such that $f^{(k-1)}$ is not strictly less than $f_s^{(k)}$. Then $f^{(k-1)}$ must be strictly greater than $f_s^{(k)}$, and we have $f_s^{(1)} < f_s^{(k)} < f^{(k-1)}$. There is some $j$ with $1 < j < k$ and $f_s^{(j-1)} < f_s^{(k)} < f^{(1)}$; since $f_s^{(j-1)} \in f^{(j)}$ we have $f_s^{(k)} \notin f^{(j)}$, a shorter cycle. This contradiction shows that no cycles exist.

42. Frank and Jordán gave another criterion for irredundancy that works also for general families of intervals on a circle when large intervals might wrap around so that their intersection $f \cap f'$ consists of two disjoint intervals. In such cases they allow $\{f_\ast, f'_\ast\} \subseteq f \cap f'$, but only if $f_\ast$ and $f'_\ast$ lie in different components of $f \cap f'$. For example, the intervals $[k \ldots k+2)$ for $0 \leq k < n$, modulo $n$, are irredundant by this definition for all $n > d$. Once again the minimax theorem for generating families and irredundant subfamilies remains valid, in this extended sense.

43. The algorithms presented by Frank and Jordán [2] for such problems require linear programming as a subroutine. Therefore it would be extremely interesting to find a purely combinatorial procedure, analogous to the algorithm of Franzblau and Kleitman, either for the wrap-restricted situation of §40 or for the more general setup of §42.
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⟨Construct an irredundant subfamily of $F$ with the cardinality of $G$⟩  Used in section 23.
⟨Find a vertex $x$ with $N_x F[\{t \ldots w\}] - N_x S[\{t \ldots w\}] = 1⟩  Used in section 33.
⟨Find an interval $[u \ldots v]$ such that $x \in [u \ldots v] \subseteq [t \ldots w]⟩  Used in section 33.
⟨Make $G$ a copy of $F$, retaining only leftward arcs⟩  Used in section 26.
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