Mean-field transport in stratified and/or rotating turbulence

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ABSTRACT

\textbf{Context.} The large-scale magnetic fields of stars and galaxies are often described in the framework of mean-field dynamo theory. At moderate magnetic Reynolds numbers, the transport coefficients defining the mean electromotive force can be determined from simulations. This applies analogously also to the mean-field theory of passive scalar transport.

\textbf{Aims.} The mean electromotive force is investigated in the kinematic framework, that is, ignoring the back-reaction of the magnetic field on the fluid velocity, under the assumption of axisymmetric turbulence determined by the presence of either rotation, density stratification, or both (providing the corresponding directions are aligned to each other). As for the mean passive scalar flux, an analogous approach is used. As an alternative to convection, forced turbulence in an isothermal layer is considered. When using standard ansatzes, the mean magnetic transport is then determined by nine, and the mean passive scalar transport by four coefficients. We give results for all these transport coefficients.

\textbf{Methods.} The test-field method and the test-scalar method are used, where transport coefficients are determined by solving sets of equations with properly chosen mean magnetic fields or mean scalars. These methods are adapted to mean fields which may depend on all three space coordinates.

\textbf{Results.} Anisotropy of turbulent diffusion is found to be moderate in spite of rapid rotation or strong density stratification. Contributions to the mean electromotive force determined by the symmetric part of the gradient tensor of the mean magnetic field, which were ignored in several earlier investigations, turn out to be important. In stratified rotating turbulence, the \( \alpha \)-effect is strongly anisotropic, suppressed along the rotation axis on large length scales, but strongly enhanced at intermediate length scales. Also the \( \Omega \times \mathbf{J} \) effect is enhanced at intermediate length scales. The turbulent passive scalar diffusion typically almost twice as large as the magnetic turbulent diffusivity. Both magnetic and passive scalar diffusion are slightly enhanced along the rotation axis, but decreased if there is gravity.

\textbf{Conclusions.} The test-field and test-scalar methods provide powerful tools for analyzing transport properties of axisymmetric turbulence. Future applications are proposed ranging from anisotropic turbulence due to the presence of a uniform magnetic field to inhomogeneous turbulence where the specific entropy is nonuniform, for example. Some of the contributions to the mean electromotive force which have been ignored in several earlier investigations, in particular those given by the symmetric part of the gradient tensor of the mean magnetic field, turn out to be of significant magnitude.

\textbf{Key words.} magnetohydrodynamics (MHD) – hydrodynamics – turbulence – Sun: dynamo

1. Introduction

Stellar mixing length theory is a rudimentary description of turbulent convective energy transport. The mixing length theory of turbulent transport goes back to Prandtl (1925) and, in the stellar context, to Vitense (1953). The simplest form of turbulent transport is turbulent diffusion, which quantifies the mean flux of a given quantity, e.g., momentum, concentration of chemicals, specific entropy or magnetic fields, down the gradient of its mean value. In all these cases essentially a Fickian diffusion law is established, where the turbulent diffusion coefficient is proportional to the rms velocity of the turbulent eddies and the effective mean free path of the eddies or their correlation length.

Mean-field theories, which have been elaborated, e.g., for the behavior of magnetic fields or of passive scalars in turbulent media, go beyond this concept. In the case of magnetic fields, the effects of turbulence occur in a mean electromotive force, which is related to the mean magnetic field and its derivatives in a tensorial fashion. Examples for effects described by the mean magnetic field alone, without spatial derivatives, are the \( \alpha \)-effect (Steenbeck et al., 1966) and the pumping of mean magnetic flux (Rädler, 1966, 1968; Roberts & Soward, 1975); for more information on these topics see, e.g., Krause & Rädler (1980) or Brandenburg & Subramanian (2005). Likewise the mean passive scalar flux contains a pumping effect (Elperin et al., 1996). In both the magnetic and the passive scalar cases turbulent diffusion occurs, which is in general anisotropic. The coupling between the mean electromotive force and the magnetic field and its derivatives, or mean passive scalar flux and the mean scalar and its derivatives, is given by turbulent transport coefficients.

On the analytic level of the theory the determination of these transport coefficients is only possible with some approximations. The most often used one is the second-order correlation approximation (SOCA), which has delivered so far many important results. Its applicability is however restricted to certain ranges of parameters like the magnetic...
Reynolds number or the Péclet number. In spite of this restriction, SOCA is an invaluable tool, because it allows a rigorous treatment within the limits of its applicability. It is in particular important for testing numerical methods that apply in a wider range.

In recent years it has become possible to compute the full set of turbulent transport coefficients numerically from simulations of turbulent flows. The most accurate method for that is the test-field method (Schrinner et al., 2005, 2007). In addition to the equations describing laminar and turbulent flows, one solves a set of evolution equations for the small-scale magnetic or scalar fields which result from given mean fields, the test fields. By selecting a sufficient number of independent test fields, one obtains a corresponding number of mean electromotive forces or mean scalar fluxes and can then compute in a unique way all the associated transport coefficients.

Most of the applications of the test-field method are based on spatial averages that are taken over two coordinates. In the magnetic case this approach has been applied to a range of different flows including isotropic homogeneous turbulence (Sur et al., 2008; Brandenburg et al., 2008a), homogeneous shear flow turbulence (Brandenburg et al., 2008b) without and with helicity (Mitra et al., 2009), and turbulent convection (Käpylä et al., 2009). One of the main results is that in the isotropic case, for magnetic Reynolds numbers $R_m$ larger than unity, the turbulent diffusivity is given by $\frac{1}{2} \tau u_{rms}^2$, where the correlation time $\tau$ is, to a good approximation, given by $\tau = (u_{rms} k_f)^{-1}$. Here, $u_{rms}$ is the rms velocity of the turbulent small-scale flow and $k_f$ is the wavenumber of the energy-carrying eddies. For smaller $R_m$, the turbulent diffusivity grows linearly with $R_m$. Furthermore, if the turbulence is driven isotropically by polarized waves, the flow becomes helical and there is an $\alpha$ effect. In the kinematic regime (for weak magnetic fields), the $\alpha$ coefficient is proportional to $\nabla \cdot \mathbf{u}$, where $\omega = \nabla \times \mathbf{u}$ is the vorticity of the small-scale flow, $\mathbf{u}$. In the passive scalar case, test scalars are used to determine the transport coefficients. Results have been obtained for anisotropic flows in the presence of rotation or strong magnetic fields (Brandenburg et al., 2009), linear shear (Madarassy & Brandenburg, 2010), and for irrotational flows (Rädler et al., 2011).

The present paper deals with the magnetic and the passive scalar case in the above sense. Its goal is to compute the transport coefficients for axisymmetric turbulence, that is, turbulence with one preferred direction, given by the presence of either rotation or density stratification or, if the relevant directions coincide, of both. (Axisymmetric turbulence can be defined by requiring that any averaged quantity depending on the turbulent velocity field is invariant under any rotation of this field about the preferred axis.) Note that a dynamo-generated magnetic field will in general violate the assumption of axisymmetric turbulence. To avoid this problem while still being able to investigate the general effects arising from only one preferred direction, we assume such fields to be weak so as not to affect the assumption of axisymmetry of the turbulence. An imposed uniform magnetic field in the preferred direction would still be allowed, but this case will not be investigated in this paper; see Brandenburg et al. (2009) for numerical investigations of passive scalar transport with a uniform field.

Except for a few comparison cases, we always consider flows in a slab between stress-free boundaries. This is the simplest example of flows that are nonvanishing on the boundary and compatible with axisymmetric turbulence. To facilitate comparison with earlier work on forced turbulence, we consider an isothermal layer even in the density-stratified case, i.e., there is no convection, and the flow is driven by a prescribed random forcing, just as we did in the case of homogeneous turbulence. This is similar to earlier work on forced homogeneous turbulence, but now we will be able to address questions regarding vertical pumping as well as helicity production and $\alpha$ effect in the presence of rotation. This setup allows us to isolate effects of density stratification from those originating from the nonuniformities of turbulence intensity and local correlation length. In addition to isothermal stratification, we assume an isothermal equation of state and thus do not consider an equation for the specific entropy. Hence, no Brunt-Väisälä oscillations can occur. This assumption would need to relaxed for studying turbulent convection, which will be the subject of a future investigation.

2. Mean-field concept in turbulent transport

2.1. Mean electromotive force

The evolution of the magnetic field $\mathbf{B}$ in an electrically conducting fluid is assumed to obey the induction equation, $\partial \mathbf{B} / \partial t = \nabla \times (\mathbf{U} \times \mathbf{B} - \eta \mathbf{J})$, (1)

where $\mathbf{U}$ is the velocity and $\eta$ the microscopic magnetic diffusivity of the fluid, and $\mathbf{J}$ is defined by $\mathbf{J} = \nabla \times \mathbf{B}$ (so that $\mathbf{J}/\mu_0$ with $\mu_0$ being the magnetic permeability is the electric current density). We define mean fields as averages, assume that the averaging satisfies (exactly or approximately) the Reynolds rules, and denote averaged quantities by overbars. The mean magnetic field $\overline{\mathbf{B}}$ is then governed by $\partial \overline{\mathbf{B}} / \partial t = \nabla \times (\overline{\mathbf{U}} \times \overline{\mathbf{B}} + \overline{\mathbf{E}} - \eta \overline{\mathbf{J}})$, (2)

where $\overline{\mathbf{E}} = \overline{\mathbf{u} \times \mathbf{b}}$ is the mean electromotive force resulting from the correlation of velocity and magnetic field fluctuations, $\mathbf{u} = \mathbf{U} - \overline{\mathbf{U}}$ and $\mathbf{b} = \overline{\mathbf{B}} - \overline{\mathbf{B}}$.

We focus attention on the mean electromotive force $\overline{\mathbf{E}}$ in cases in which the velocity fluctuations $\mathbf{u}$ constitute axisymmetric turbulence, that is, turbulence with one preferred direction, which we describe by the unit vector $\hat{e}$. Until further notice we accept the traditional assumption according to which $\overline{\mathbf{E}}$ in a given point in space and time is a linear homogeneous function of $\overline{\mathbf{B}}$ and its first spatial derivatives in this point. Then, $\overline{\mathbf{E}}$ can be represented in the form

$$\overline{\mathbf{E}} = -\alpha_\perp \overline{\mathbf{B}} - (\alpha_0 - \alpha_\perp) (\hat{e} \cdot \overline{\mathbf{B}}) \hat{e} - \gamma \hat{e} \times \overline{\mathbf{B}} - \beta_\perp \overline{\mathbf{J}} - (\beta_\parallel - \beta_\perp) (\hat{e} \cdot \overline{\mathbf{J}}) \hat{e} - \delta \hat{e} \times \overline{\mathbf{J}} - \kappa_\perp \overline{\mathbf{K}} - (\kappa_0 - \kappa_\perp) (\hat{e} \cdot \overline{\mathbf{K}}) \hat{e} - \mu \hat{e} \times \overline{\mathbf{K}}$$

The Reynolds rules imply that $\overline{F + G} = \overline{F} + \overline{G}$, $\overline{F} = \overline{F}$, $\overline{F \cdot G} = \overline{F} \overline{G}$, $\partial \overline{F}/\partial x = \overline{\partial F/\partial x}$ and $\partial \overline{F}/\partial t = \overline{\partial F/\partial t}$ for any fluctuating quantities $\overline{F}$ and $\overline{G}$.
with nine coefficients $\alpha_1, \alpha_2, \ldots, \mu$.\footnote{Note that the signs in front of some individual terms on the right-hand side of (3), in particular of those with $\alpha_1$ and $\alpha_2$ (perpendicular and parallel to $\alpha$ effect) as well as $\gamma$ (pumping in the $z$ direction), may differ from the signs used in other representations.} Like $\mathbf{J} = \nabla \times \mathbf{B}$, also $\mathbf{K}$ is determined by the gradient tensor $\mathbf{\nabla} \mathbf{B}$. While $\mathbf{J}$ is given by its antisymmetric part, $\mathbf{K}$ is a vector defined by $\mathbf{K} = \hat{\epsilon} \cdot (\nabla \mathbf{B})^\parallel$ with $\mathbf{B}$ being the symmetric part of $\mathbf{\nabla} \mathbf{B}$. A more detailed explanation of (3) is given in Appendix A. If $\hat{\epsilon}$ is understood as polar vector (for example $\nabla \mathbf{H} / |\nabla \mathbf{H}|$, where $\mathbf{H}$ is the mean mass density), then $\mathbf{K}$ is axial and $\gamma, \beta_1, \beta_2$ and $\mu$ are true scalars, but $\alpha_1, \alpha_2, \delta, \kappa_1,$ and $\kappa_2$ pseudoscalars. ( Scalars are invariant but pseudoscalars change sign if the turbulent velocity field is reflected at a point or at a plane containing the preferred axis.) Sometimes it is useful to interpret $\hat{\epsilon}$ as an axial vector (for example $\mathbf{\Omega}/|\mathbf{\Omega}|$ with $\mathbf{\Omega}$ being an angular velocity). Then, $\mathbf{K}$ is a polar vector, $\beta_1, \beta_2, \delta, \kappa_1,$ and $\mu$ are true scalars but $\alpha_1, \alpha_2,$ and $\gamma$ pseudoscalars.

We may split $\mathbf{E}$ and $\mathbf{B}$ into parts $\mathbf{E}_\parallel$ and $\mathbf{B}_\parallel$ perpendicular to $\hat{\epsilon}$ and parts $\mathbf{E}_\perp$ and $\mathbf{B}_\perp$ parallel to it. Then (3) can be written in the form

$$
\begin{align*}
\mathbf{E}_\perp & = -\alpha_2 \mathbf{B}_\perp - \gamma \hat{\epsilon} \times \mathbf{B}_\perp - \beta_1 \mathbf{J}_\perp - \delta \hat{\epsilon} \times \mathbf{J}_\perp - \kappa_1 \mathbf{K}_\perp - \mu \hat{\epsilon} \times \mathbf{K}_\perp, \\
\mathbf{E}_\parallel & = -\alpha_1 \mathbf{B}_\parallel - \beta_1 \mathbf{J}_\parallel - \kappa_1 \mathbf{K}_\parallel.
\end{align*}
$$

(4)

Let us return to (3). In the simple case of homogeneous isotropic turbulence we have $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$, and all remaining coefficients vanish. Then, (3) takes the form $\mathbf{E} = \alpha \mathbf{B} - \eta \mathbf{J}$ with properly defined $\alpha$ and $\eta$. These two coefficients have been determined by test-field calculations (Sur et al., 2008; Brandenburg et al., 2008a). In several previous studies of $\mathbf{E}$, more general kinds of turbulence (that is, not only axisymmetric turbulence) have been considered, but with a less general definition of mean fields, which were just horizontal averages. More precisely, Cartesian coordinates $(x, y, z)$ were adopted and the averages were taken over all $x$ and $y$ so that they depend on $z$ and $t$ only (Brandenburg et al., 2008a,b). This definition implies remarkable simplifications. Of course, we then have $\mathbf{J} = 0$. Further, there are no zero-components of $\nabla \mathbf{B}$ other than $B_{\parallel z}$ and $B_{\perp z}$, for $\nabla \cdot \mathbf{B} = 0$ requires $B_{\parallel z} = 0$, and these components can be expressed as components of $\mathbf{J}$, viz. $\mathbf{B}_{\parallel z} = \mathbf{J}_z$ and $\mathbf{B}_{\perp z} = -\mathbf{J}_z$. (Here and in what follows, commas denote partial derivatives.) This again implies $\mathbf{K} = -\frac{1}{2} \hat{\epsilon} \mathbf{J}$. As a consequence, this definition of mean fields reduces (3) to

$$
\begin{align*}
\mathbf{E} = -\alpha_1 \mathbf{B} - (\alpha_2 - \alpha_1) (\hat{\epsilon} \cdot \mathbf{B}) \hat{\epsilon} - \gamma \hat{\epsilon} \times \mathbf{B} - \beta_1 \mathbf{J} - \delta \hat{\epsilon} \times \mathbf{J} - \kappa_1 \mathbf{K},
\end{align*}
$$

(5)

where $\beta_1 = \beta_2 + \frac{1}{2} \mu$ and $\delta_1 = \delta - \frac{1}{2} \kappa_1$. Of course, $\alpha_1, \alpha_2,$ $\gamma, \beta_1$ and $\delta_1$ are independent of $x$ or $y$. Clearly, $\beta_1$ and $\mu$ as well as $\delta$ and $\kappa_1$ have no longer independent meanings. From (2) we may conclude that $\partial \mathbf{B}_z / \partial t = 0$. If we restrict ourselves to applications in which $\mathbf{B}_z$ vanishes initially, it does so at all times and the term with $\alpha_1 - \alpha_2$ in (5) disappears. Then, only the four coefficients $\alpha_1, \gamma, \beta_1$ and $\delta_1$ are of interest. They can be determined by test-field calculations using two test fields independent of $x$ and $y$ (Brandenburg et al., 2008a,b).

In this paper we go in several respect beyond the assumptions mentioned so far. Firstly, we relax the assumption that $\tilde{\mathbf{E}}$ in a given point in space is a homogeneous function of $\mathbf{B}$ and its first spatial derivatives in this point. Instead, we admit a non-local connection between $\tilde{\mathbf{E}}$ and $\mathbf{B}$. For simplicity, however, we further on assume that $\tilde{\mathbf{E}}$ at a given time depends only on $\mathbf{B}$ at the same time, that is, we remain with an instantaneous connection between $\tilde{\mathbf{E}}$ and $\mathbf{B}$. This approximation requires that the mean field varies slowly on a time scale much longer than the turnover time of the turbulence; see Hubbard & Brandenburg (2009) for a more general treatment of rapidly changing fields. Secondly, we consider mean fields no longer as averages over all $x$ and $y$. We define $\tilde{\mathbf{E}}$ at a point $(x, y)$ in a plane $z = 0$ by averaging over some surroundings of this point in this plane so that it still depends on $x$ and $y$. In that sense we generalize (3) so that

$$
\tilde{\mathbf{E}}(x) = -\int (\alpha_1(x, z) \tilde{\mathbf{B}}(x - z) - \alpha_2(x, z) \mathbf{B}(x - z)) (\hat{\epsilon} \cdot \mathbf{B}(x - z)) \hat{\epsilon} - \beta_1(x, z) \mathbf{J}(x - z) - \delta(x, z) \hat{\epsilon} \times \mathbf{J}(x - z) - \kappa_1(x, z) \mathbf{K}(x - z) - \mu(x, z) \hat{\epsilon} \times \mathbf{K}(x - z) d^3 \xi.
$$

As a consequence of the axisymmetry of the turbulence, the coefficients $\alpha_1, \alpha_2, \ldots, \mu$ depend only via $\xi_2^2 + \xi_3^2$ on $\xi_2$ and $\xi_3$. We consider them also as symmetric in $\xi_2$. The integration is over all $\xi$ space. Of course, $\tilde{\mathbf{E}}, \mathbf{B}, \mathbf{J},$ and $\mathbf{K}$ may depend on $t$. For simplicity, however, the argument $t$ has been dropped.

Let us subject (6) to a Fourier transformation with respect to $\xi$. We define it by

$$
F(\xi) = (2\pi)^{-3} \int F(k) \exp(ik \cdot \xi) d^3 k.
$$

(7)

Remembering the convolution theorem we obtain

$$
\begin{align*}
\tilde{\mathbf{E}}(x) = -& (2\pi)^{-3} \int (\tilde{\alpha}_1(x, k) \tilde{\mathbf{B}}(k) + (\tilde{\alpha}_2(x, k) - \tilde{\alpha}_1(x, k)) (\hat{\epsilon} \cdot \tilde{\mathbf{B}}(k)) \hat{\epsilon} - \tilde{\beta}_1(x, k) \mathbf{J}(k) + (\tilde{\beta}_2(x, k) - \tilde{\beta}_1(x, k)) (\hat{\epsilon} \cdot \tilde{\mathbf{J}}(k)) \hat{\epsilon} - \tilde{\delta}(x, k) \hat{\epsilon} \times \mathbf{J}(k) + \tilde{\kappa}_1(x, k) \mathbf{K}(k) + (\tilde{\kappa}_2(x, k) - \tilde{\kappa}_1(x, k)) (\hat{\epsilon} \cdot \tilde{\mathbf{K}}(k)) \hat{\epsilon} + \tilde{\mu}(x, k) \hat{\epsilon} \times \mathbf{K}(k)) \exp(ik \cdot x) d^3 k;
\end{align*}
$$

(8)

see Chatterjee et al. (2011) for a corresponding relation in the case of horizontally averaged magnetic fields that depend only on $z$. Like $\alpha_1, \alpha_2, \ldots, \mu$, the $\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\mu}$ are real quantities. They depend only via $k_\perp = (k_2^2 + k_3^2)^{1/2}$
There are interesting analogies between turbulent transport and the concentration of an admixture in a fluid, e.g., the concentration of an admixture in a fluid, is given by

\[
\alpha = \int_0^\infty \alpha(x, \xi) \cos k_x \xi_x \cos k_y \xi_y \cos k_z \xi_z \, dk^3 \xi
\]

and analogous relations for \( \alpha, \ldots, \mu \). We note that \( \alpha, \ldots, \mu \), taken at \( k = 0 \), agree with \( \alpha, \ldots, \mu \) in Equation (3).

2.2. Mean passive scalar flux

There are interesting analogies between turbulent transport and the concentration of an admixture in a fluid, e.g., the concentration of an admixture in a fluid, is given by

\[
\frac{\partial C}{\partial t} = -\nabla \cdot (UC - D\nabla C),
\]

where \( D \) is the microscopic (molecular) diffusivity. Then the mean scalar \( \bar{C} \) has to satisfy

\[
\frac{\partial \bar{C}}{\partial t} = -\nabla \cdot (\bar{U} \bar{C} + F - D\nabla \bar{C}),
\]

where \( \bar{F} = \bar{w} \) is the mean passive scalar flux, \( \bar{u} \) stands again for the fluctuations of the velocity and \( c = C - \bar{C} \) for the fluctuations of \( C \). Consider again axisymmetric turbulence with a preferred direction given by the unit vector \( \hat{e} \). Assume that \( \bar{F} \) in a given point in space and time is determined by \( C \) and its gradient \( \bar{G} = \nabla \bar{C} \) in this point. Then we have

\[
\bar{F} = -\gamma \bar{C} \hat{e} - \beta^\perp \bar{G} - (\beta^\parallel - \beta^\perp) (\hat{e} \cdot \bar{G}) \hat{e} - \delta^\perp \hat{e} \times \bar{G},
\]

with coefficients \( \gamma, \beta^C, \beta^\parallel, \beta^\perp \) and \( \delta^\perp \). If \( \hat{e} \) is a polar vector, \( \gamma \) is a scalar but \( \delta^\perp \) a pseudoscalar, and if \( \hat{e} \) is an axial vector, \( \gamma \) is a pseudoscalar but \( \delta^\perp \) a scalar, while \( \beta^\parallel \) and \( \beta^\perp \) are always scalars. We note that \( \nabla \cdot (\delta^\perp \hat{e} \times \bar{G}) \) is only unequal zero if \( \delta^\perp \) is not constant but varies in the direction of \( \hat{e} \times \bar{G} \).

We may split \( \bar{F} \) and \( \bar{G} \) into parts \( \bar{F}^\perp \) and \( \bar{G}^\perp \) perpendicular to \( \hat{e} \), and parts \( \bar{F}^\parallel \) and \( \bar{G}^\parallel \) parallel to it, and give (12) the form

\[
\bar{F}^\perp = -\beta^\perp \bar{G}^\perp - \delta^\perp \hat{e} \times \bar{G}^\perp,
\]

\[
\bar{F}^\parallel = -\gamma \hat{e} \bar{C}^\parallel - \beta^\parallel \bar{G}^\parallel.
\]

Let us now relax the assumption that \( \bar{F} \) in a given point in space and time is determined by \( \bar{C} \) and \( \bar{G} \) in this point. Analogously to the magnetic case we consider a non-local but instantaneous connection between \( \bar{F} \) and \( \bar{C} \). Then we have

\[
\bar{F}(x) = -\int_0^\infty \left( \gamma(x, \xi) \hat{e} \bar{C}(x - \xi)
+ \beta^\perp(x, \xi) \bar{G}(x - \xi)
+ (\beta^\parallel(x, \xi) - \beta^\perp(x, \xi)) (\hat{e} \cdot \bar{G}(x - \xi)) \hat{e}
+ \delta^\perp(x, \xi) \hat{e} \times \bar{G}(x - \xi) \right) \, dk^3 \xi.
\]

As \( \alpha, \ldots, \mu \) in the magnetic case, \( \gamma, \beta^C, \beta^\parallel, \beta^\perp \) and \( \delta^\perp \) depend only via \( \xi_x^2 + \xi_y^2 \) on \( \xi_x \) and \( \xi_y \), and we consider them also as symmetric in \( \xi_z \). The integration is again over all \( \xi \) space. Note that \( \bar{F}, \bar{C}, \) and \( \bar{G} \) may, even if it is not explicitly indicated, depend on \( t \). Applying the Fourier transformation defined by (7) on (14), we arrive at

\[
\bar{F}(x) = -\int_0^\infty \left( \gamma(x, \xi) \hat{e} \bar{C}(k)
+ \beta^\perp(x, k) \bar{G}(k)
+ (\beta^\parallel(x, k) - \beta^\perp(x, k)) (\hat{e} \cdot \bar{G}(k)) \hat{e}
+ \delta^\perp(x, k) \hat{e} \times \bar{G}(k) \right) \exp(ik \cdot x) \, dk^3 k,
\]

where \( \gamma, \beta^C, \beta^\parallel, \beta^\perp \) and \( \delta^\perp \) are real quantities. They depend only via \( k_x^2 + k_y^2 \) on \( k_x \) and \( k_y \), and only via \( k_z \) on \( k_z \), and they satisfy relations analogous to (9). We note that \( \gamma^C, \beta^\perp, \beta^\parallel, \delta^\perp \) at \( k = 0 \) agree with \( \gamma, \beta^C, \beta^\parallel, \delta^\perp \) in (12).

3. Simulating the turbulence

We assume that the fluid is compressible and its flow is governed by the equations

\[
\frac{DU}{Dt} = f + g - \nabla h - 2\Omega \times U + \rho^{-1} \nabla \cdot (2\nu S),
\]

\[
\frac{Dh}{Dt} = -c_s^2 \nabla \cdot U.
\]

Here, \( f \) means a random force which primarily drives isotropic turbulence (e.g., Haugen et al., 2004), \( g \) the gravitational force, and \( h \) the specific enthalpy. An isothermal equation of state, \( p = \rho c_s^2 \), has been adopted with a constant isothermal sound speed \( c_s \). In general a fluid flow in a rotating system is considered, \( \Omega \) is the angular velocity which defines the Coriolis force. As usual \( \rho \) means the mass density, \( \nu \) the kinematic viscosity and \( S \) the trace-free rate of strain tensor, \( S_{ij} = \frac{1}{2} (\partial U_{i,j} + \partial U_{j,i}) - \frac{1}{3} \delta_{ij} \nabla \cdot U \). The influence of the magnetic field on the fluid motion, that is the Lorentz force, is ignored throughout the paper.

The numerical simulation is carried out in a cubic domain of size \( L^3 \), so the smallest wavenumber is \( k_1 = 2\pi/L \). In most of the cases a density stratification is included with \( \rho = (0, 0, -g) \), so the density scale height is \( H = c_s^2/g \). The number of scale heights across the domain is equal to \( \Delta \ln \rho \), where \( \Delta \) denotes the difference of values at the two edges of the domain. Since \( L/H \rho = 2\pi \) we have a density contrast of \( \exp 2\pi \approx 535 \). The forcing is assumed to work with an average wavenumber \( k_1 \). The scale separation ratio is then given by \( k_1/k_1 \), for which we usually adopt the value 5. This means that we have about 5 eddies in each of the three coordinate directions.

The flow inside the considered domain depends on the boundary conditions. Unless indicated otherwise we take the top and bottom surfaces \( z = z_1 \) and \( z = z_2 \) with \( z_2 = -z_1 = L/2 \) as stress-free and adopt periodic boundary conditions for the outer surfaces.

4. Computing the transport coefficients

4.1. Test-field method

In the magnetic case the coefficients \( \alpha, \alpha, \ldots, \mu \) are determined by the test-field method (Schrinner et al., 2005,
We recall that test-fields need not to be solenoidal (see Schrinner et al., 2005, 2007; Brandenburg et al., 2008a). This method works with a set of test fields \( \overline{B} \), called \( \overline{B}^T \), and the corresponding mean electromotive forces \( \overline{E} \), called \( \overline{E}^T \). For the latter we have \( \overline{E}^T = \mathbf{u} \times \overline{B}^T \), where the \( \overline{B}^T \) obey

\[
\frac{\partial \mathbf{a}^T}{\partial t} = \mathbf{U} \times \overline{B}^T + (\mathbf{u} \times \overline{B}^T)' + \eta \nabla^2 \mathbf{a}^T, \quad (17)
\]

with \( \mathbf{U} \) and \( \mathbf{u} \) taken from the solutions of (16). For the boundaries \( z = \text{const} \) we choose conditions which correspond to an adjacent perfect conductor, for the \( x \) and \( y \) directions periodic boundary conditions.

We define four test fields by

\[
\begin{align*}
\overline{B}^{1s} &= (B_0, s x y s z, 0, 0), \\
\overline{B}^{ic} &= (B_0, s x y c z, 0, 0) \\
\overline{B}^{2s} &= (0, 0, B_0, s x y s z), \\
\overline{B}^{2c} &= (0, 0, B_0, s x y c z)
\end{align*}
\]

with a constant \( B_0 \). Here and in what follows we use the abbreviations

\[
\begin{align*}
s x &= \sin k_x x, & c x &= \cos k_x x \\
s y &= \sin k_y y, & c y &= \cos k_y y \\
s z &= \sin k_z z, & c z &= \cos k_z z
\end{align*}
\]

We recall that test-fields need not to be solenoidal (see Schrinner et al., 2005, 2007).

We define the mean electromotive forces which correspond to the test fields (18) by \( \overline{E}^{1s}, \overline{E}^{ic}, \overline{E}^{2s}, \) and \( \overline{E}^{2c} \). With the presentation (6) and relations (9) we find

\[
\begin{align*}
\overline{E}^{1s}_x &= -B_0 (\tilde{\alpha}_1 s x s y s z - (\tilde{\delta} - \frac{1}{2} \tilde{\kappa}_1) k_x s x s y s z) \\
\overline{E}^{1s}_y &= -B_0 (\tilde{\gamma} s y s z y s + (\tilde{\beta}_1 + \frac{1}{2} \tilde{\mu}) k_x s z y s c y s z) \\
\overline{E}^{1s}_z &= B_0 \tilde{\beta}_1 k_y s x c y s z
\end{align*}
\]

and corresponding relations for \( \overline{E}^{ic}, \overline{E}^{2s}, \) and \( \overline{E}^{2c} \), with \( s x, s y, s z, c x, c y, c y s z, \) and \( c x s y s z \) and averaging over all \( x, y, \) and \( z \), we obtain a system of equations, which can be solved for \( \tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\mu} \). The result reads

\[
\begin{align*}
\tilde{\alpha}_1 &= -\langle B^{ss} (s x E_{1s} - s y E_{1c}) \rangle \\
\tilde{\alpha}_2 &= -\langle B^{ss} (s z E_{1s} + c z E_{1c}) \rangle \\
\tilde{\gamma} &= -\langle B^{ss} (s y E_{1s} + c y E_{1c}) \rangle \\
\tilde{\beta}_1 &= -\frac{1}{2} \langle B^{ss} (c z E_{1s} - s z E_{1c}) + B^{sc} (s z E_{2s} + c z E_{2c}) \rangle \\
\tilde{\beta}_2 &= -\frac{1}{2} \langle B^{ss} (c y E_{1s} - s y E_{1c}) - B^{sc} (s y E_{2s} + c y E_{2c}) \rangle \\
\tilde{\gamma} &= \langle B^{sc} (s y E_{1s} + c y E_{1c}) \rangle \\
\tilde{\delta} &= \frac{1}{2} \langle B^{ss} (c z E_{1s} - s z E_{1c}) - B^{sc} (s z E_{2s} + c z E_{2c}) \rangle
\end{align*}
\]

(21)

The angle brackets indicate averaging over \( x, y, \) and \( z \). Although the relations (21) and (22) contain \( k_x, k_y \) and \( k_z \) as independent variables, the \( \tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\mu} \) should vary only via \( k_\parallel = (k_x^2 + k_y^2)^{1/2} \) with \( k_x \) and \( k_y \), and only via \( k_\perp \) with \( k_z \).

4.2. Test-scalar method

In the passive-scalar case the coefficients \( \gamma^C, \beta^C, \) and \( \delta^C \) are determined by the test-scalar method with test scalars \( \overline{C}^T \) and the corresponding fluxes \( \overline{F}^T \). For the latter, we have \( \overline{F}^T = \mathbf{u} \overline{C}^T \), where \( \mathbf{u} \) obeys

\[
\frac{\partial \mathbf{c}}{\partial t} = -\nabla \cdot (\mathbf{U} \mathbf{c}) + (\mathbf{u} \mathbf{c})' - D \nabla \mathbf{c}, \quad (23)
\]

Again \( \mathbf{U} \) and \( \mathbf{u} \) are taken from the solutions of (16).

We define two test-scalars \( \overline{C}^{Ts} \) and \( \overline{C}^{Tc} \) by

\[
\overline{C}^{Ts} = C_0 s x s y s z, \quad \overline{C}^{Tc} = C_0 s x y c z,
\]

where \( C_0 \) is a constant and the abbreviations (19) are used. From (14) we then have

\[
\begin{align*}
\overline{F}^{Ts}_x &= -C_0 (\tilde{\beta}_1 k_y s x s y s z - \tilde{\delta}_1 k_x s x s y s z) \\
\overline{F}^{Ts}_y &= -C_0 (\tilde{\beta}_2 k_y s x c y s z - \tilde{\delta}_2 k_x s y c z) \\
\overline{F}^{Ts}_z &= -C_0 (\tilde{\alpha}_1 s x s y s z + \tilde{\kappa}_1 k_y s x c y s z)
\end{align*}
\]

(25)

and analogous relations for \( \overline{F}^{Tc} \), with \( s x, s y, s z, c x, c y, c y s z, \) and \( c x s y s z \) replaced by \( c z \) and \( -s z, \) respectively.

Analogous to the magnetic case, we assume that \( \gamma^C, \beta^C, \) \( \delta^C \), and \( \delta^C \) are independent of \( x \) and \( y \) but may depend on \( z \). Analogous to (21) we find here

\[
\begin{align*}
\tilde{\gamma}^C &= -\langle c^{ss} (s z F_{1s} + c z F_{1c}) \rangle \\
\tilde{\beta}_1^C &= -\langle c^{sc} (s z F_{2s} + c z F_{2c}) \rangle = -\langle C^{sc} (s z F_{2s} + c z F_{2c}) \rangle \\
\tilde{\beta}_2^C &= -\langle c^{ss} (s z F_{2s} + c z F_{2c}) \rangle \\
\tilde{\delta}^C &= \langle c^{sc} (s z F_{2s} + c z F_{2c}) \rangle = \langle C^{sc} (s z F_{2s} + c z F_{2c}) \rangle
\end{align*}
\]

(26)

where \( c^{ss}, c^{sc}, C^{sc}, \) and \( C^{sc} \) are defined like \( b^{ss}, B^{ss}, B^{sc}, \) and \( B^{sc} \), with \( C_0 \) at the place of \( B_0 \). The angle brackets indicate again averaging over \( x \) and \( y \). Note that \( \tilde{\gamma}^C, \tilde{\beta}_1^C, \tilde{\beta}_2^C, \) and \( \tilde{\delta}^C \) should depend only via \( k_\parallel = (k_x^2 + k_y^2)^{1/2} \) on \( k_x \) and \( k_y \), and only via \( k_\perp \) on \( k_z \).
4.3. Validation using the Roberts flow

For a validation of our test-field procedure for the determination of the coefficients occurring in (3) we rely on the Roberts flow. We define it here by

\[ u = u_0 ( - \cos k_0 x \sin k_0 y, \sin k_0 x \cos k_0 y, 2 f \cos k_0 x \cos k_0 y ), \]  

with some wavenumber \( k_0 \) and a factor \( f \) which characterizes the ratio of the magnitude of \( u_z \) to that of \( u_x \) and \( u_y \). We further define mean fields as averages over \( x \) and \( y \) with an averaging scale which is much larger than the period length \( 2 \pi / k_0 \) of the flow pattern. When calculating the mean electromotive force \( \overline{\mathbf{E}} \) for this flow, we assume that it is a linear homogeneous function of \( \overline{\mathbf{B}} \) and its first spatial derivatives and adopt the second-order correlation approximation. Although the Roberts flow is far from being axisymmetric, the result for \( \overline{\mathbf{E}} \) can be written in the form (3), and we have

\[ \alpha_\perp = \frac{u_0^2 f}{2 \eta k_0}, \quad \alpha_\parallel = \gamma = 0 \]

\[ \beta_\perp = \frac{u_0^2 (1 + 4 f^2)}{16 \eta k_0^2}, \quad \beta_\parallel = \frac{u_0^2}{8 \eta k_0^2}, \quad \delta = 0 \]  

(28)

\[ \kappa_\perp = \kappa_\parallel = 0, \quad \mu = -\frac{u_0^2 (1 - 4 f^2)}{8 \eta k_0^2} = 2 (\beta_\perp - \beta_\parallel). \]

It agrees with and can be deduced from results reported in Rädler et al. (2002a,b). As for the passive scalar case, an analogous analytical calculation of the mean scalar flow \( \overline{\mathcal{F}} \) leads to (12) with

\[ \gamma^C = 0, \quad \beta^C_\perp = \frac{u_0^2}{8 D k_0^2}, \quad \beta^C_\parallel = \frac{u_0^2 f^2}{2 D k_0^2}, \quad \delta^C = 0. \]  

(29)

We may proceed from the local connection of \( \overline{\mathbf{E}} \) with \( \overline{\mathbf{B}} \) and its derivatives considered in (3) to the non-local ones given by (6) or (8). As a consequence of the deviation of the flow from axisymmetry, we can then no longer justify that coefficients like \( \alpha_\perp (\xi) \) depend only via \( \xi_x^2 + \xi_y^2 \) on \( \xi_x \) and \( \xi_y \), and coefficients like \( \tilde{a}_\perp (k) \) only via \( k_x \) on \( k_x \) and \( k_y \). This applies analogously to the connection of \( \overline{\mathcal{F}} \) with \( \overline{\mathcal{C}} \) and its derivatives and to coefficients like \( \beta_\perp (\xi) \) and \( \tilde{\beta}_\perp (k) \). A test-field calculation of the coefficients \( \tilde{a}_\perp, \tilde{a}_\parallel, \ldots, \tilde{\mu}, \) as well as \( \tilde{\gamma}^C, \ldots, \tilde{\delta}^C \), has been carried out under the conditions of the second-order correlation approximation with \( \mathbf{u} \) given by (27) and \( f = 1/\sqrt{2} \). Figure 1 shows the results obtained for \( \tilde{a}_\perp, \tilde{\beta}_\perp, \tilde{\beta}_\parallel, \) and \( \tilde{\mu}, \) as well as \( \tilde{\gamma}^C, \ldots, \tilde{\delta}^C \), as functions of \( k_x/k_f \), with \( k_f = \sqrt{2} k_0 \), for two fixed ratios \( k_f/k_\perp \). In the limit \( k_f/k_\perp \ll 1 \) these coefficients take just the values of \( a_\perp, \beta_\perp, \beta_\parallel, \mu, \beta^C_\parallel, \) and \( \beta^C_\perp \) given in (28) and (29). For larger values of \( k_f/k_\perp \), as to be expected, the \( \tilde{a}_\perp, \tilde{\beta}_\perp, \tilde{\beta}_\parallel, \tilde{\mu}, \beta^C_\parallel, \) and \( \beta^C_\perp \) depend also on the ratio of \( k_x \) and \( k_y \).

4.4. Dimensionless parameters and related issues

Within the framework of this paper, the coefficients \( a_\perp, a_\parallel, \mu \) as well as \( \tilde{a}_\perp, \tilde{a}_\parallel, \ldots, \tilde{\mu}, \) and likewise \( \gamma^C, \beta^C_\parallel, \ldots, \beta^C_\perp, \delta^C \), and \( \tilde{\gamma}^C, \tilde{\beta}^C_\parallel, \ldots, \tilde{\delta}^C \), have to be considered as functions of several dimensionless parameters. In the magnetic case these are the magnetic Reynolds number \( R_m = u_{\text{rms}} / \eta k_f \).
and the magnetic Prandtl number $P_m = \nu/\eta$, in the passive scalar case the Péclet number $Pe = u_{rms}/Dk_1$ and the Schmidt number $Sc = \nu/D$, further the Mach number $Ma = u_{rms}/c$, the gravity parameter $Gr = g/c^2k_1$, the Coriolis number $Co = 2\Omega/u_{rms}k_1$, as well as the scale separation ratio $k_1/k_3$.

Throughout the rest of the paper we give the coefficients $\alpha_\perp, \alpha_\parallel, \gamma$, and $\gamma^C$ as well as $\tilde{\alpha}_\perp, \alpha_\parallel^C, \gamma$, and $\gamma^C$ in units of $u_{rms}/3$, the remaining coefficients $\beta_\perp, \ldots, \delta^C$ and $\tilde{\beta}_\perp, \ldots$, $\delta^C$ in units of $u_{rms}/3k_1$. The numerical calculations deliver these coefficients as functions of $z$ and $t$. To avoid boundary effects, we average these results over $-2 \leq k_3z \leq 1$ (see Figure 3 below). The resulting time series are averaged over a range where the results are statistically stationary, i.e., there is no trend in the time series. Error bars are defined by comparing the maximum departure of an average over any one third of the time series with the full time average.

In the case of isotropic turbulence it has been observed that many of the transport coefficients enter an asymptotic regime as soon as $R_m$ exceeds unity (Sur et al., 2008). While this should be checked in every new case again (see below), it is important to realize that, according to several earlier results (see also Brandenburg et al., 2009), only values of $R_m$ below unity are characteristic of the diffusively dominated regime, while for $R_m$ exceeding unity the transport coefficients turn out to be nearly independent of the value of $R_m$.

We are often interested in the limit $k_1, k_\parallel \to 0$, in which the $\tilde{\alpha}_\perp, \tilde{\alpha}_\parallel, \ldots, \delta^C$ turn into the $\alpha_\perp, \alpha_\parallel \ldots \delta^C$. In this limit, however, the test fields and test scalars defined by (18) and (24) vanish. Unless specified otherwise, we approach this limit by choosing the smallest possible non-zero $|k_2|, |k_3|$, and $|k_4|$, that is, by putting $k_3 = k_2 = k_1$.

In the figures of the next section results for $\tilde{\alpha}_\perp, \tilde{\alpha}_\parallel, \ldots$ $\delta^C$ are represented. In all cases in which they are considered as results for the limit $k_1, k_\parallel \to 0$ they are simply denoted as $\alpha_\perp, \alpha_\parallel \ldots \delta^C$ in the text.

5. Results

5.1. Homogeneous rotating turbulence

Let us first consider homogeneous turbulence in a rotating system, that is, under the influence of the Coriolis force. The angular velocity $\Omega$ responsible for this force defines the preferred direction of the turbulence, $\mathbf{e} = \Omega/|\Omega|$. In this case we expect only contributions to the mean electromagnetic force $\mathbf{F}$ from a spatially varying mean magnetic field $\mathbf{B}$, and contributions to the passive scalar flux $\mathbf{F}$ from a spatially varying mean passive scalar $C$. That is, in (3) we have only the terms with $\beta_\perp, \beta_\parallel, \delta, \alpha_\perp, \alpha_\parallel, \gamma$, and $\delta$ and in (12) only those with $\beta_\perp^C, \beta_\parallel^C$, and $\delta^C$. The terms with $\beta_\perp$ and $\beta_\parallel$, as well as those with $\beta_\perp^C$ and $\beta_\parallel^C$, characterize anisotropic mean-field diffusivities, and that with $\delta$ corresponds to the “$\Omega \times \mathbf{F}$ effect” (Rädler, 1969a,b, 1976; Krause & Rädler, 1971, 1980), while the $\delta^C$ term vanishes underneath the divergence and is therefore without interest.

Figure 2 shows the dependence of the aforementioned coefficients on $Co$ for $R_m \approx Pe \approx 9$ and $k_3/k_1 = 5$. The values of $\beta_\perp, \beta_\parallel, \beta_\perp^C$, and $\beta_\parallel^C$, which remain finite for $Co \to 0$, are always close together. The other four coefficients vary linearly with $Co$ as long as $Co$ is small. Specifically, we find $\delta \approx -0.1Co, \delta^C \approx -Co$, as well as $\tilde{\kappa}_\perp \approx -0.3Co$ and $\tilde{\kappa}_\parallel \approx -Co$. These coefficients reach maxima at $Co \approx 1$. For rapid rotation, $|Co| \gg 1$, all coefficients approach zero like $1/Co$. In particular, we have $\beta_\perp \approx 1/Co$ and the same for $\beta_\parallel, \beta_\perp^C$, and $\beta_\parallel^C$, further $\tilde{\kappa}_\perp \approx -0.5Co, \kappa_\parallel \approx -1.2Co$, $\delta \approx -0.3/Co$, and $\delta^C \approx -0.6/Co$. Furthermore, we find that, within error bars, $\alpha_\perp, \alpha_\parallel, \gamma$, and $\gamma^C$ are indeed zero.

5.2. Stratified turbulence

Owing to the presence of boundary conditions at the top and bottom of our domain and the lack of scale separation for our default choice of $k_1/k_3 = 5$, the turbulence is in all cases anisotropic, even if gravity is negligible. The ratio of the vertical and horizontal velocity components, $2u_2^2/u_1^2$, is no longer, as in the isotropic case, equal to unity. For moderate stratification ($g/c^2k_1 \approx 1$), not too large $|z|$, and $k_3/k_1 = 5$, it takes a value of about 0.9. It decreases when the ratio $k_3/k_1$ is decreased; see Table 1. Figure 3 shows the $z$ dependence of $2u_2^2/u_1^2$. For strong stratification and a high degree of scale separation, e.g. $k_3/k_1 = 30$, the mentioned ratio comes close to unity. Note, however, that smaller values of $2u_2^2/u_1^2$ can be achieved in the non-isothermal case when the effects of buoyancy become important.
Likewise, in the representation (12) of $E$ only the four coefficients $\gamma$, $\beta_\parallel$, $\beta_\perp$, and $\mu$ can be non-zero. Likewise, in the representation (12) of $F$ only the three coefficients $\gamma^C$, $\beta^C_\parallel$, and $\beta^C_\perp$ can be non-zero. Figure 4 shows their dependence on Gr. It appears that $\gamma$ is always close to zero, while $\gamma^C$ shows a linear increase for not too strong gravity. At the same time, $\beta_\parallel$, $\beta_\perp$, $\beta^C_\parallel$, and $\beta^C_\perp$ remain approximately constant. We find that $\mu$ is negative and its modulus is mildly increasing with increasing stratification, but the error bars are large.

5.2.1. Stratified nonrotating turbulence
For axisymmetric turbulence in a nonrotating system showing any kind of stratification in the representation (3) of $E$ only the four coefficients $\gamma$, $\beta_\parallel$, $\beta_\perp$, and $\mu$ can be non-zero. Likewise, in the representation (12) of $F$ only the three coefficients $\gamma^C$, $\beta^C_\parallel$, and $\beta^C_\perp$ can be non-zero. Figure 4 shows their dependence on Gr. It appears that $\gamma$ is always close to zero, while $\gamma^C$ shows a linear increase for not too strong gravity. At the same time, $\beta_\parallel$, $\beta_\perp$, $\beta^C_\parallel$, and $\beta^C_\perp$ remain approximately constant. We find that $\mu$ is negative and its modulus is mildly increasing with increasing stratification, but the error bars are large.

5.2.2. Stratified rotating turbulence
For turbulence under the influence of gravity and rotation, all nine coefficients $\alpha_\parallel$, $\alpha_\perp$, $\mu$, $\beta_\parallel$, $\beta_\perp$, $\beta^C_\parallel$, $\beta^C_\perp$, $\gamma$, $\delta$, $\omega$, $\epsilon$, $\kappa_{\parallel}$, and $\kappa_{\perp}$ may well be unequal to zero. If $n + m\leq 2$, all nine coefficients may indeed be non-zero.

Results for stratified rotating turbulence are shown in Figure 5. The error bars are now bigger than either with just rotation or just stratification. For $\alpha \to 0$, the coefficients $\beta_\parallel$, $\beta_\perp$, $\mu$, $\beta^C_\parallel$, $\beta^C_\perp$, and $\delta^C$ remain finite. As $\alpha$ is increased, their moduli show some decline. On the other hand the moduli of $\alpha_\parallel$, $\alpha_\perp$, $\gamma$, $\delta$, $\kappa_{\parallel}$, and $\kappa_{\perp}$ increase with $\alpha$ as long as it is smaller than some value below unity but decrease again for larger $\alpha$. Both $\alpha_\parallel$ and $\alpha_\perp$ are negative, which is expected for $g$ and $\Omega$ being antiparallel to each other. Interestingly, $\mu$ is finite for small values of $\alpha$, in agreement with the result when there is only stratification (Figure 4), but with a modest amount of rotation, $\mu$ is suppressed and grows only when $\alpha$ has reached values around unity.

5.3. Wavenumber dependence
So far we have considered the coefficients $\tilde{\alpha}_{\parallel}$, $\tilde{\alpha}_{\perp}$, $\ldots$, $\tilde{\gamma}^C$ in the limit $k = |k| \to 0$, that is, $k_{\parallel}, k_{\perp} \to 0$. However, their behavior for larger $k$, in particular for $k$ up to several $k_{\parallel}$, is of interest, too. Most of them decrease like $k^{-2}$ as $k$ grows and can be fitted to a Lorentzian profile, as has been found in earlier calculation using the test-field method; see Brandenburg et al. (2008a), where in fact the dependence on $k_{\parallel}$ was considered. Even earlier work that was not based on the test-field method showed a declining trend (Miesch et al., 2000; Brandenburg & Sokoloff, 2002). Nevertheless, as is shown in Figure 6, there are also some coefficients that first increase with $k_{\parallel}$, have a maximum near $k_{\parallel} = k_{\parallel}$ and only then decrease with growing $k_{\parallel}$. Examples for such a

![Fig. 3. Anisotropy $2u_1^2/u_1^2$ of nonrotating turbulence for different stratifications, $g/c_k^2 k_1$, and different degrees of scale separation, $k_1/k_1$.](image)

![Fig. 4. Gr dependence of the transport coefficients in a model with density stratification but zero rotation, $P_m = Sc = 1$, $R_m \approx 22$, $Co = 0$, $k_f/k_1 = 5$.](image)

| $g/c_k^2 k_1$ | $\rho_{rot}/\rho_{nop}$ | $2u_1^2/u_1^2$ |
|-------------|----------------|----------------|
|             | 1.5k_1 | 5k_1 | 30k_1 |
| 0           | 0      | 0.84 | 0.99 | 1.00 |
| 0.5         | 23     | 0.84 | 0.97 | 1.00 |
| 1           | 540    | 0.66 | 0.90 | 0.99 |
behavior are $\tilde{\alpha}_\parallel$, $\tilde{\delta}$, and $\tilde{\kappa}_\perp$, while $\tilde{\kappa}_\parallel$ peaks slightly below $k_\parallel/k_f = 0.5$.

The dependence of the coefficients under discussion on $k_\perp$ is shown in Figure 7. Note that our test fields vanish for $k_\perp = 0$, so no values are shown for this case. Note also that $-\tilde{\alpha}_\parallel$, $-\tilde{\delta}$, $-\tilde{\kappa}_\parallel$, and $-\tilde{\beta}_\perp$, which all have maxima for $k_\parallel/k_f \approx 1$, show a clear monotonic decline with $k_\perp$. Only $-\tilde{\kappa}_\perp$ has a maximum for intermediate values of both $k_\parallel/k_f$ and $k_\perp/k_f$.

Most of the results presented in Figure 7 have been calculated with $k_x = k_y$, a few single ones for $\tilde{\alpha}_\perp$, $\tilde{\beta}_\perp$, $\tilde{\kappa}_\perp$, and $\tilde{\beta}_\perp^C$ also with $k_x/k_y = 0.75$ and $k_x/k_y = 0.2$. While the results for $\tilde{\beta}_\perp$ and $\tilde{\beta}_\perp^C$ agree well for all these values of $k_x/k_y$, there are significant discrepancies with $\tilde{\alpha}_\perp$ and $\tilde{\kappa}_\perp$.

5.4. Dependencies on $R_m$ and $Pe$

Let us finally consider the dependence of all transport coefficients on $R_m$ or $Pe$ for a case where they are all expected to be finite. Therefore we choose again the case with $Co = 1$ and $Gr = 0.16$, which was also considered in Figures 5–7, and keep $P_m = Sc = 1$.

The results are shown in Figure 8. As expected, all quantities increase approximately linearly with $R_m$ if $R_m < 1$, or with $P_m$ if $P_m < 1$, and seem to level off to constant values for larger values of $R_m$, or $P_m$, although the uncertainty tends to increase significantly.

6. Conclusions

In this paper we have dealt with the mean electromotive force and the mean passive scalar flux in axisymmetric turbulence and have calculated the transport coefficients that define these quantities. Unlike most of the earlier work, we have no longer assumed that mean fields are defined as planar averages but admit a dependence on all three space coordinates. The number of test fields and test scalars is the same (4 and 2, respectively) as in earlier work using planar averages, so the computational cost is unchanged.

We may conclude from general symmetry considerations that the mean electromotive force $\mathbf{E}$ has altogether nine contributions: three defined by the mean magnetic field $\mathbf{B}$, three by the mean current density $\mathbf{J}$, and three by the vector $\mathbf{K}$, which is the projection of the symmetric part of the gradient tensor $\nabla \mathbf{B}$ of the magnetic field on the preferred direction. In many representations of $\mathbf{E}$ the last three contri-
Fig. 7. Same as Figure 6, but $k_\perp$ dependence, $k_\parallel = k_1$. The filled and open circles denote results for $\alpha_\perp$, $\beta_\perp$, $\kappa_\perp$, and $\beta_\perp C$ obtained with $k_x/k_y = 0.75$ [$k_\perp = (3, 4, 0)k_1$] and $k_x/k_y = 0.2$ [$k_\perp = (1, 5, 0)k_1$], respectively.

Conclusions have been ignored. Our results underline that this simplification is in general not justified. The corresponding coefficients $\kappa_\perp$, $\kappa_\parallel$ and $\mu$ are in general not small compared to $\beta_\perp$, $\beta_\parallel$ and $\delta$.

It has been known since long that a stratification of the turbulence intensity, that is, a gradient of $u^2$, causes a pumping of magnetic flux (Rädler, 1969a). It remained however uncertain whether the same effect occurs if a preferred direction is given by a gradient of the mean mass density $\bar{\rho}$ while the turbulence intensity is spatially constant. In our calculations, which correspond to this assumption, the value of $\gamma$ is not clearly different from zero. This suggests that a gradient of the mass density alone is not sufficient for pumping, what is also in agreement with results of Brandenburg et al. (2011). This is even more remarkable as the corresponding coefficient $\gamma C$ which describes the transport of a mean passive scalar is noticeably different from zero. Pumping down the density gradient is indeed expected (Elperin et al., 1995). An explanation of these results would be very desirable.

In homogeneous rotating turbulence, apart from an anisotropy of the mean-field conductivity, the $\Omega \times \mathbf{J}$ effect occurs (Rädler, 1969b). In the passive scalar case again an anisotropy of the mean diffusivity is possible. Even if the flux proportional to $\Omega \times \nabla C$ is non-zero, it cannot influence $C$.

Let us turn to the induction effects described by $\mathbf{K}$. If the preferred direction is given by a polar vector, the corresponding contribution to the mean electromotive force can only be proportional to $\hat{e} \times \mathbf{K}$. We found such a contribution in the case of the Roberts flow and also, for turbulence subject the Coriolis force, in the results presented in Figure 2 and Figures 4–7.

Contributions to the mean electromotive force as described here by $\mathbf{K}$ occur also in earlier calculations, e.g. Kitchatinov et al. (1994) or Rüdiger & Brandenburg (1995). As a consequence of other notations, however, this is not always obvious. For example, Rüdiger & Brandenburg (1995) consider a mean electromotive force of the form

$$\mathbf{E} = -\eta_\parallel \mathbf{J} + (\eta_\parallel - \eta_T)(\mathbf{z} \times \mathbf{J}) = -\eta_T \mathbf{z} \times \nabla \mathbf{B}$$

with two coefficients $\eta_\parallel$ and $\eta_T$ (equation (18) of their paper with $\mu_0 \mathbf{J}$ in the sense of the definition introduced here, replaced by $\mathbf{J}$; $\mathbf{z}$ is our $\hat{e}$). It is equivalent to our representations (3) or (4) of $\mathbf{E}$ if we put there $\beta_\perp = \frac{1}{2}(\eta_\parallel + \eta_T)$, $\beta_\parallel = \eta_T$, $\mu = \eta_\parallel - \eta_T$ and all other coefficients equal to zero. This implies $\beta_\perp - \beta_\parallel = \mu/2$, which is in agreement
with the relation for $\mu$ in equation (28) for the Roberts flow. The latter equality is also approximately obeyed for turbulence in the presence of rotation, stratification, and both; see Figures 2 and 4 and 5, respectively.

If both rotation and density stratification are present, there is also an $\alpha$ effect, which is necessarily anisotropic. For strong rotation $|\alpha|_1$ is only half as big as $|\alpha|_L$. For $\alpha \approx 1$, we have $\beta_L > |\beta|_1$, which means that the magnetic diffusivity along the rotation axis is enhanced. The same is true for the passive scalar diffusivity, where enhanced diffusivity along the rotation axis means $\beta^S_L > |\beta^S|_1$. Moreover, stratification has the effect of decreasing the magnetic and passive scalar diffusivity in the vertical direction. In the presence of rotation and density stratification all three contributions to the mean electromotive force described by $K$ are in general non-zero. Here, $|\kappa_L|$ is smaller than $|\kappa|_1$.

The present work is applicable to investigations of stellar convection either with or without rotation, and it would provide a more comprehensive description of turbulent transport properties than what has been available so far (Käpylä et al., 2009). The methods utilized in this paper can be extended to a large class of phenomena in which turbulence with just one preferred direction plays an important role. Examples for that include turbulence under the influence of a strong magnetic field and/or an externally applied electric field leading to a current permeating the system. Turbulence generated by the Bell (2004) instability is an example. Other examples include all types of inhomogeneous turbulence with only one preferred direction. In addition to density stratification, there can be a systematic variation of the turbulence intensity in one direction. A further example is entropy inhomogeneity combined with gravity giving rise to Brunt-Väisälä oscillations. Pumping effects also exist in homogeneous flows if the turbulence is the center for Parallel Computers at the Royal Institute of Technology in Stockholm and the National Supercomputer Centers in Linköping.

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Appendix A: Derivation of relation (3)

We start from the aforementioned assumption according to which $\overline{E}$ is linear and homogeneous in $\overline{B}$ and its first spatial derivatives, $\overline{E}_i = a_{ij} B_j + b_{ijk}(\nabla \overline{B})_{jk}$. (A.1)

Here $a_{ij}$ and $b_{ijk}$ are tensors determined by the fluid flow. The gradient tensor $(\nabla \overline{B})_{jk}$ can be split into a antisymmetric part, which can be expressed by $\mathbf{J}$, and a symmetric part $(\nabla \overline{B})^S_{jk}$. Therefore we may also write $\overline{E}_i = a_{ij} B_j - b_{ij} J_j - c_{ijk} (\nabla \overline{B})^S_{jk}$ (A.2)

with new tensors $b_{ij}$ and $c_{ijk}$, the latter being symmetric in $j$ and $k$. From the further assumption that the flow constitutes an axisymmetric turbulence we may conclude that $a_{ij}$, $b_{ij}$ and $c_{ijk}$ are axisymmetric tensors. Defining the preferred direction by the unit vector $\mathbf{e}$ we then have

$$a_{ij} = a_{i} \delta_{j} + a_{2} \epsilon_{ij} \epsilon_{k} + a_{3} \epsilon_{ij} \epsilon_{k},$$

$$b_{ij} = b_{i} \delta_{j} + b_{2} \epsilon_{ij} \epsilon_{k} + b_{3} \epsilon_{ij} \epsilon_{k},$$

$$c_{ijk} = c_{1} \delta_{k} \epsilon_{i} + c_{2} (\delta_{i} \epsilon_{k} + \delta_{k} \epsilon_{i}) + c_{3} \epsilon_{i} \epsilon_{j} \epsilon_{k}.$$ (A.3)

with coefficients $a_1, a_2, \ldots, a_4$ determined by the fluid flow. Taking (A.2) and (A.3) together and considering that

$$(\delta_{i} \epsilon_{j} + \delta_{k} \epsilon_{j})(\nabla \overline{B})^S_{jk} = 2 K_{i},$$

$$\left(\epsilon_{ij} \epsilon_{k} + \epsilon_{ik} \epsilon_{j}\right)(\nabla \overline{B})^S_{jk} = -2(\mathbf{e} \times K)_{i},$$

we find

$$\overline{E} = a_{1} \mathbf{B} - a_{2} \mathbf{e} \times \mathbf{B} - a_{3}(\mathbf{e} \cdot \mathbf{B}) \mathbf{e}$$

$$+ b_{1} \mathbf{J} - b_{2} \mathbf{e} \times J - b_{3}(\mathbf{e} \cdot J) \mathbf{e}$$

$$+ c_{1} \mathbf{K} - c_{2} \mathbf{e} \times \mathbf{K} + c_{3}(\mathbf{e} \cdot \mathbf{K}) \mathbf{e}.$$ (A.5)

Since $(\nabla \overline{B})_{ii} = 0$ there is no contribution with $c_1$. With a proper renaming of the coefficients (A.5) turns into (3).

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