HOLOMORPHIC SECTIONAL CURVATURE, NEFNESS AND THE MIYAOKA-YAU INEQUALITY

YASHAN ZHANG

ABSTRACT. On a compact Kähler manifold, we introduce a property in terms of holomorphic sectional curvature, which is weaker than existence of a Kähler metric with semi-negative holomorphic sectional curvature. We prove that a compact Kähler manifold satisfying this property has nef canonical line bundle, does not contain any rational curve and satisfies the Miyaoka-Yau inequality. We also observe a Chern number inequality of Miyaoka-Yau type on a compact Kähler manifold $X$ with semi-ample canonical line bundle and the numerical Kodaira dimension $\nu(X) < \dim(X) - 2$.

1. Introduction

For a compact Kähler manifold, one of the most fundamental questions is: when its canonical line bundle is nef? A compact Kähler manifold with nef canonical line bundle is called a smooth minimal model, which is of great importance and interest from the viewpoint of algebraic geometry (in particular, minimal model program). In complex geometry, an important object is to give geometric criterions for positivity (e.g. nefness or ampleness) of canonical line bundle. According to conjectures of S.-T. Yau, the negativity of the holomorphic sectional curvature should closely affect the positivity of the canonical line bundle. For example, thanks to Yau’s Schwarz Lemma \cite{Yau1978}, a compact Kähler manifold $(X, \omega)$ with semi-negative holomorphic sectional curvature does not contain any rational curve and hence, if $X$ is projective, its canonical line bundle must be nef by Mori’s Cone Theorem (see e.g. \cite{Mori1988}); if in general $X$ is Kähler, then its canonical line bundle is also nef, thanks to a recent work of Tosatti-Yang \cite{TosattiYang2017} Theorem 1.1]. Moreover, a conjecture of Yau predicts that a compact Kähler manifold $(X, \omega)$ with negative holomorphic sectional curvature has ample canonical line bundle. This conjecture is solved in a breakthrough of Wu-Yau \cite{WuYau2017} provided the manifold is projective (for the surface case and threefold case, it is previously proved by B. Wong \cite{Wong2010} and Heier-Lu-B. Wong \cite{HeierLuWong2015}, respectively); Tosatti-Yang \cite{TosattiYang2017} extended Wu-Yau’s work to the Kähler case and hence proved Yau’s conjecture in full generality. Diverio-Trapani \cite{DiverioTrapani2019} and Wu-Yau \cite{WuYau2017} further proved a compact Kähler manifold $(X, \omega)$ with quasi-negative holomorphic sectional curvature has ample canonical line bundle (see P. Wong-Wu-Yau \cite{WongWuYau2019} for the special case that the manifold has Picard number one). One can also find a Kähler-Ricci flow approach in Nomura \cite{Nomura2018} for results of Wu-Yau \cite{WuYau2017} and Tosatti-Yang \cite{TosattiYang2017}.

In general, for a given positivity of canonical line bundle, one may naturally wonder what’s the “weakest/optimal” negativity condition on holomorphic sectional curvature which implies the given positivity of canonical line bundle? In contrast to the ampleness of canonical line bundle (which is equivalent to the existence of one Kähler metric of negative Ricci curvature), the nefness of canonical line bundle is defined by limiting a sequence of Kähler classes, and hence it seems not “optimal” to derive the nefness of canonical line bundle from the semi-negativity of holomorphic sectional curvature of one Kähler metric. Then it may be natural to ask:
Question 1.1. For a compact Kähler manifold, can we find some natural (negativity) properties in terms of holomorphic sectional curvature, which are weaker than the existence of a Kähler metric with semi-negative holomorphic sectional curvature but can still guarantee the nefness of canonical line bundle?

In this note, we shall propose such a property. To introduce this property, we first associate a number to a Kähler class as follows.

Definition 1.1. Let $X$ be a compact Kähler manifold and $\alpha$ a Kähler class on $X$. We define a non-negative constant $\mu_\alpha$ for $\alpha$ in the following way: $\mu_\alpha = 0$ if there exists a Kähler metric $\omega \in \alpha$ with $\sup_X H_\omega \leq 0$; otherwise we define $\mu_\alpha := \inf \{ \sup_X H_\omega | \omega \in \alpha, \omega \text{ is a Kähler metric on } X \}$.

For a compact Kähler manifold $(X, \omega)$, $\sup_X H_\omega$ means the maximum of the holomorphic sectional curvature of $\omega$ on $X$. The number $\mu_\alpha$ has some basic properties, e.g. $\mu_{c\alpha} = c^{-1} \mu_\alpha$ for any positive constant $c$, $\mu_{\alpha[Y]} \leq \mu_\alpha$ for any compact complex submanifold $Y$ of $X$, and $\mu_\alpha$ is invariant under biholomorphisms, namely, given a biholomorphism $\pi : Y \rightarrow X$ between two compact Kähler manifolds and a Kähler class $\alpha$ on $X$, we have $\mu_{\pi^* \alpha} = \mu_\alpha$.

Then we define the following property for a compact Kähler manifold.

Definition 1.2. Let $X$ be a compact Kähler manifold. We say $X$ satisfies Property (A), if there exists a sequence of Kähler classes $\alpha_i$, $i = 1, 2, \ldots$, on $X$ such that $\mu_{\alpha_i} \alpha_i \rightarrow 0$.

We mention that Property (A) is a condition for Kähler classes (not the metrics), and it may be seen as an approximate semi-negativity of holomorphic sectional curvature. Obviously, a compact Kähler manifold admitting a Kähler metric with semi-negative holomorphic sectional curvature must satisfy Property (A).

The first main result is a generalization of Tosatti-Yang [20, Theorem 1.1] to compact Kähler manifolds satisfying Property (A).

Theorem 1.1. A compact Kähler manifold satisfying Property (A) has nef canonical line bundle.

Remark 1.1. By definition, the canonical line bundle of a compact Kähler manifold is nef if $c_1(K_X)$ is a limit of a sequence of Kähler classes (in particular, nefness is a positivity in an approximate sense). From the viewpoint of the definition of nefness, our Property (A) seems a very natural corresponding property for holomorphic sectional curvature to guarantee the nefness of the canonical line bundle.

Remark 1.2. In Theorem[1.1] we generalize Tosatti-Yang [20, Theorem 1.1] in the sense of weakening the curvature condition on Kähler manifolds. In another direction, as proposed in [20, Remark 1.7], it is natural to generalize Tosatti-Yang [20, Theorem 1.1] to Hermitian manifolds; and there are also some progresses in this direction, see [28].

Remark 1.3. For an arbitrary Kähler metric $\omega$ on a compact Kähler manifold satisfying Property (A), Theorem[1.1] implies the total scalar curvature of $\omega$ is a non-positive number; if the holomorphic sectional curvature of $\omega$ does not vanish identically, then by a well-known result of Berger (see e.g. [33, page 189, Exercise 16]) we have $\inf_X H_\omega < 0$.

Our proof for Theorem[1.1] will be given in Section[2] which is based on an estimate on lower bound for existence time of the Kähler-Ricci flow in terms of the upper bound for the holomorphic sectional curvature of the initial metric. As a byproduct of our arguments,
we also observe a lower bound for the blow up rate of the maximum of holomorphic sectional curvature along the Kähler-Ricci flow with a finite time singularity, see Proposition 2.4 in Section 2.

A classical result of Yau [30] (also see Ryoden [16]) proved the non-existence of rational curves on compact Kähler manifolds admitting a Kähler metric with semi-negative holomorphic sectional curvature. Our second result extends this result to compact Kähler manifolds satisfying Property (A).

**Theorem 1.2.** A compact Kähler manifold satisfying Property (A) does not contain any rational curve.

**Remark 1.4.** On the one hand, our proof for Theorem 1.1 does not depend on Theorem 1.2; on the other hand, if $X$ is a projective manifold or a compact Kähler manifold of dimension $\leq 3$, then Cone Theorem holds [11, 9] and Theorem 1.2 implies Theorem 1.1.

**Remark 1.5.** A natural question is the classification of compact Kähler manifolds satisfying Property A. For example, as a consequence of Theorem 1.2, a smooth minimal model of general type satisfying Property (A) must have ample canonical line bundle, see [10, Theorem 2] (also see [11, Lemma 2.1, 26, Lemma 5]). Moreover, by the Enriques-Kodaira classification on compact complex surfaces (see [1, Chapter VI]), Kodaira’s table of singular fibers of minimal elliptic surfaces (see [1, Chapter V]) and Theorems 1.1, 1.2, a compact Kähler surface $X$ satisfying Property (A) must be one of the followings: (1) complex torus or hyperelliptic surface, if $\text{Kod}(X) = 0$; (2) minimal elliptic surface with only singular fibers of type $mI_0$, if $\text{Kod}(X) = 1$; (3) Kähler surface with ample canonical line bundle, if $\text{Kod}(X) = 2$. For Kähler surfaces of semi-negative holomorphic sectional curvature, one can find a complete structure theorem in [8, Theorem 1.10].

Our proof for Theorem 1.2, which will be given in Section 3, is achieved by observing a positive lower bound for the maximum of holomorphic sectional curvature in terms of the existence of a rational curve. This observation is essentially due to Tosatti-Y.G. Zhang [21, Proposition 1.4, Remark 4.1], where they proved a similar result for holomorphic bisectional curvature.

Finally, we observe that the Miyaoka-Yau inequality for Chern numbers holds on compact Kähler manifolds satisfying Property (A).

**Theorem 1.3.** Assume $X$ is an $n$-dimensional ($n \geq 2$) compact Kähler manifold satisfying Property (A). Then the Miyaoka-Yau inequality holds:

$$\left(\frac{2(n+1)}{n}c_2(X) - c_1(X)^2\right) (-c_1(X))^{n-2} \geq 0.$$ (1)

It is known that the above Miyaoka-Yau inequality (1) holds in many cases; an incomplete list: surfaces of general type [12, 29], $K_X$ is ample [29], minimal manifolds of general type [23, 32, 18], minimal projective varieties of general type [5], minimal projective varieties [9] and compact Kähler manifolds whose $c_1(K_X)$ admits a smooth semi-positive representative [14].

Recall that for an $n$-dimensional compact Kähler manifold $X$ with nef canonical line bundle $K_X$, the numerical Kodaira dimension $\nu(X)$ of $X$ is defined to be

$$\nu(X) := \max\{k \in \{0, \ldots, n\} | c_1(K_X)^k \neq [0]\}.$$
Then if $\nu(X) < n - 2$, the conclusion in Theorem 1.3 holds trivially. To make a nontrivial conclusion, it is natural to replace $(-c_1(X))^{n-2}$ by $(-c_1(X))^\nu \wedge \alpha^{n-\nu} < 2$ for some Kähler/nef class $\alpha$ on $X$ (compare e.g. [3, Theorem B]). Motivated by these, we obtain the following result.

**Theorem 1.4.** Assume $X$ is an $n$-dimensional compact Kähler manifold $(n \geq 3)$ with nef $K_X$ and the numerical Kodaira dimension $\nu = \nu(X) < n - 2$.

1. If there exists a Kähler class $\alpha$ on $X$ with $\mu_\alpha = 0$, then there holds
   \[
   \left(\frac{2(n+1)}{n} c_2(X) - c_1(X)^2\right) \cdot (-c_1(X))^\nu \cdot \alpha^{n-\nu-2} \geq 0. \tag{2}
   \]

2. If $X$ satisfies Property (A) defined by Kähler classes $\alpha_i$ and, additionally, $\mu_{\alpha_i} > 0$ and $\alpha_i \to \alpha_\infty$ for some nef class $\alpha_\infty$, then there holds
   \[
   \left(\frac{2(n+1)}{n} c_2(X) - c_1(X)^2\right) \cdot (-c_1(X))^\nu \cdot \alpha_\infty^{n-\nu-2} \geq 0. \tag{3}
   \]

**Remark 1.6.** For convenience, let’s look at a special case of above results: assume $(X, \omega)$ is a compact Kähler manifold with semi-negative holomorphic sectional curvature, then Theorem 1.3 and Theorem 1.4 (1) apply, and we have
\[
\left(\frac{2(n+1)}{n} c_2(X) - c_1(X)^2\right) \cdot (-c_1(X))^\nu \cdot [\omega]^2 \geq 0,
\]
where $i = n - 2$, $j = 0$ if $\nu \geq n - 2$, and $i = \nu$, $j = n - \nu - 2$ if $\nu < n - 2$.

Our proofs for Theorems 1.3 and 1.4 will make use of Wu-Yau’s continuity equation [26, equations (3.2)] to construct certain family of Kähler metrics, see Section 4 for details.

As we mentioned before, if $K_X$ is semi-positive, i.e. the canonical class $-c_1(X)$ admits a smooth semi-positive representative, then the Miyaoka-Yau inequality [11] holds on $X$, thanks to a recent work of Nomura [13, theorem 1.1]. Similar to the above Theorem 1.3 if $\nu < n - 2$, it seems natural to extend [14, Theorem 1.1] to certain conclusions similar to the above inequalities (2) and (3). Here we observe a conclusion of such type under a stronger assumption.

**Theorem 1.5.** Assume $X$ is an $n$-dimensional compact Kähler manifold $(n \geq 3)$ with semi-ample canonical line bundle $K_X$ and the numerical Kodaira dimension $\nu = \nu(X) < n - 2$. Then for any nef class $\alpha$ on $X$ there holds
\[
\left(\frac{2(n+1)}{n} c_2(X) - c_1(X)^2\right) \cdot (-c_1(X))^\nu \cdot \alpha^{n-\nu-2} \geq 0. \tag{4}
\]

Here we have assumed semi-ampleness of $K_X$, which by definitions is stronger than semi-positivity of $K_X$, as well as nefness of $K_X$ (but the Abundance Conjecture predicts semi-ampleness of $K_X$ is equivalent to nefness of $K_X$). Also note that when $K_X$ is semi-ample, the numerical Kodaira dimension $\nu(X)$ equals to the Kodaira dimension of $X$.

We should mention that, for a minimal projective manifold $X$ with $\nu(X) < n - 2$ and an arbitrary rational Kähler class $\alpha$ on $X$, the inequality [11] is always true, see [3, Theorem B]; then our Theorem 1.5 may be regarded as a partial extension of this result to the Kähler case. It is natural to expect that Theorem 1.5 should be extended to any minimal compact Kähler manifold with $\nu(X) < n - 2$. 


Since a nef class is a limit of a sequence of Kähler classes, to prove Theorem 1.5 it suffices to check (4) for any Kähler class $\alpha$ on $X$, which can be achieved by certain arguments due to Y.G. Zhang [32] using the Kähler-Ricci flow, see Section 4 for details.

Organization of this note: Theorems 1.1 and 1.2 will be proved in Sections 2 and 3, respectively; Theorems 1.3, 1.4 and 1.5 will be proved in Section 4. In Section 5, we will discuss some basic phenomenons of Property (A).

2. Nefness of canonical line bundle

We first recall a result of Royden [16, Lemma, page 552]:

Proposition 2.1. [16] Let $(X, \omega)$ be a Kähler manifold with $\sup_X H_\omega \leq A$ for a constant $A > 0$. Then for any Kähler metric $\hat{\omega}$ on $X$ we have

$$\hat{g}^{ij} \hat{g}^{\bar{m}\bar{n}} R_{ijp\bar{q}} \leq A (tr_{\hat{\omega}} \omega)^2,$$

where $\hat{\omega} = \sqrt{-1} \hat{g}^{ij} dz^i \wedge d\bar{z}^j$ and $R^\omega$ is the curvature tensor of $\omega$.

Proof. For convenience we recall here a proof due to Royden. It suffices to show that, for any given point $x \in X$ and $\xi_1, \ldots, \xi_n$ a basic of $T_x^1(X)$ which is orthonormal with respect to $\hat{\omega}$, there holds

$$\sum_{1 \leq \alpha, \beta \leq n} R^\omega (\xi_\alpha, \bar{\xi}_\alpha, \xi_\beta, \bar{\xi}_\beta) \leq A \left( \sum_{1 \leq \alpha \leq n} |\xi_\alpha|^2 \right)^2. \tag{5}$$

Recall that [16, page 552] shows, if $\sup_X H_\omega \leq A$ for any fixed $A \in \mathbb{R}$, there holds

$$\sum_{1 \leq \alpha, \beta \leq n} R^\omega (\xi_\alpha, \bar{\xi}_\alpha, \xi_\beta, \bar{\xi}_\beta) \leq \frac{1}{2} A \left( \sum_{1 \leq \alpha \leq n} |\xi_\alpha|^2 \right)^2 + \sum_{1 \leq \alpha \leq n} |\xi_\alpha|^4. \tag{6}$$

Therefore, when $A > 0$, by using an easy inequality

$$\sum_{1 \leq \alpha \leq n} |\xi_\alpha|^4 \leq \left( \sum_{1 \leq \alpha \leq n} |\xi_\alpha|^2 \right)^2,$$

we immediately conclude (5) from (6).

Proposition 2.1 is proved. \qed

Remark 2.1. In the above proof of Proposition 2.1 if in particular $\sup_X H_\omega \leq A \leq 0$, then by an obvious Cauchy inequality one has (assume $X$ is of dimension $n$)

$$\hat{g}^{ij} \hat{g}^{\bar{m}\bar{n}} R_{ijp\bar{q}} \leq A \frac{n+1}{2n} (tr_{\hat{\omega}} \omega)^2,$$

which has been very useful in solving Yau’s conjecture, see [26, 20, 4]. However, for our later applications, we will need $A > 0$ case in Proposition 2.1.

The following proposition provides an effective way to estimate the lower bound for existence time of the Kähler-Ricci flow in terms of the upper bound for the holomorphic sectional curvature of the initial metric, which will play a key role in proving Theorem 1.1.
Proposition 2.2. Let \((X, \omega)\) be an \(n\)-dimensional compact Kähler manifold and \(A = \sup_X H_\omega\). Assume \(A > 0\). Then the Kähler-Ricci flow running from \(\omega\),

\[
\begin{cases}
\partial_t \omega(t) = -\text{Ric}(\omega(t)) \\
\omega(0) = \omega,
\end{cases}
\tag{7}
\]

exists a smooth solution on \(X \times [0, \frac{1}{nA})\).

Proof. The Kähler-Ricci flow (7) is equivalent to the following parabolic Monge-Ampère equation:

\[
\begin{cases}
\partial_t \varphi(t) = \log \left( \frac{\omega - t\text{Ric}(\omega) + \sqrt{-1}\partial \bar{\partial} \varphi(t)}{\omega^n} \right) \\
\varphi(0) = 0,
\end{cases}
\tag{8}
\]

where \(\omega(t) = \omega - t\text{Ric}(\omega) + \sqrt{-1}\partial \bar{\partial} \varphi(t)\) is the solution to the Kähler-Ricci flow (7). A direct computation gives

\[
\begin{align*}
(\partial_t - \Delta_{\omega(t)})(\text{tr}_{\omega(t)}\omega) &= g(t)^{ji}g(t)^{lk}R_{ijkl} \omega - g(t)^{ji}g(t)^{lk}g^{ba}\nabla_i g^{jk} \nabla_j g \hat{g}_{ab} \\
&\leq g(t)^{ji}g(t)^{lk}R_{ijkl} \\
&\leq A(\text{tr}_{\omega(t)}\omega)^2,
\end{align*}
\tag{9}
\]

where we have used Proposition 2.1 in the last inequality.

Now we assume the Kähler-Ricci flow (7) exists a maximal solution for \(t \in [0, T)\) and assume Proposition 2.2 fails, i.e. \(T < \frac{1}{nA}\). Set \(M(t) := \sup_X \text{tr}_{\omega(t)}\omega\) for \(t \in [0, T)\), which is a smooth positive function. By applying the maximum principle in (9) we easily have

\[\partial_t M(t) \leq A(M(t))^2.\]

When \(T < \frac{1}{nA}\) and \(t \in [0, T)\), we have

\[M(t) \leq \frac{n}{1 - nAt}.\]

Set \(C_1 := \frac{n}{1 - nAt}\), which is a positive constant since by assumption \(T < \frac{1}{nA}\). Then we have proved that

\[\text{tr}_{\omega(t)}\omega \leq C_1 \tag{10}\]

on \(X \times [0, T)\). On the other hand, from (8) we have

\[(\partial_t - \Delta_{\omega(t)})\partial_t \varphi = -\text{tr}_{\omega(t)}\text{Ric}(\omega).\]

We fix a positive constant \(B\) with \(\text{Ric}(\omega) \geq -B\omega\), then by (10) we have

\[
\begin{align*}
(\partial_t - \Delta_{\omega(t)})\partial_t \varphi &= -\text{tr}_{\omega(t)}\text{Ric}(\omega) \\
&\leq B\text{tr}_{\omega(t)}\omega \\
&\leq BC_1,
\end{align*}
\]

from which one easily concludes that

\[\partial_t \varphi(t) \leq BC_1 T\]

and so

\[\omega(t)^n \leq e^{BC_1 T}\omega^n \tag{11}\]
on $X \times [0, T)$. Combining (10) and (11) gives

$$\text{tr}_{\omega(t)} \omega(t) \leq \frac{1}{(n-1)!} \left( \text{tr}_{\hat{\omega}} \omega(t) \right)^{n-1} \frac{\omega(t)^n}{\hat{\omega}^n} \leq \frac{C_1^{n-1} e^{BC_1 T}}{(n-1)!}.$$  

Therefore, we have

$$C_1^{-1} \hat{\omega} \leq \omega(t) \leq \frac{C_1^{n-1} e^{BC_1 T}}{(n-1)!} \hat{\omega}, \quad (12)$$

on $X \times [0, T)$ (note that $C_1 > n$ and hence $C_1^{-1} < \frac{C_1^{n-1} e^{BC_1 T}}{(n-1)!}$). Having (12), we can obtain uniform higher order derivatives for $\omega(t)$ on $t \in [0, T)$ and then show, as $t \to T$, $\omega(t) \to \omega(T)$ smoothly, for some Kähler metric $\omega(T)$ (see [2, 15]). Consequently, one can solve the Kähler-Ricci flow for $t \in [0, T + \epsilon)$, where $\epsilon$ is some positive number. This is a contradiction, since we have assume $T$ is the maximum existence time of (7). Therefore, there must holds $T \geq \frac{1}{nA}$.

Proposition 2.2 is proved. □

Remark 2.2. The $A = 0$ case was proved by Nomura in [13, Proof of Theorem 1.1], which gives a Kähler-Ricci flow proof for [20, Theorem 1.1]. The above Proposition 2.2 extends Nomura’s work [13] to the general case with arbitrary positive upper bound $A$.

A consequence of Proposition 2.2 is the following

**Proposition 2.3.** Let $X$ be an $n$-dimensional compact Kähler manifold and $\alpha$ Kähler class on $X$. Denote $\lambda_\alpha := \sup \{ t > 0 | \alpha + 2\pi t c_1 (K_X) > 0 \}$. Then

$$\lambda_\alpha \geq \frac{1}{n\mu_\alpha}. \quad (13)$$

**Proof.** For any small $\epsilon > 0$, we choose a Kähler metric $\omega_\epsilon \in \alpha$ with $\sup_X H_{\omega_\epsilon} \leq \mu_\alpha + \epsilon$. Then we consider the Kähler-Ricci flow $\omega_\epsilon(t)$ running from $\omega_\epsilon$ on $X$,

$$\begin{cases} \partial_t \omega_\epsilon(t) = -\text{Ric}(\omega_\epsilon(t)) \\ \omega_\epsilon(0) = \omega_\epsilon, \end{cases} \quad (14)$$

which by Proposition 2.2 has a smooth solution on $[0, \frac{1}{n(\mu_\alpha + \epsilon)})$. On the other hand, recall the Kähler class along the Kähler-Ricci flow (14) is given by

$$[\omega_\epsilon(t)] = \alpha + 2\pi t c_1 (K_X).$$

So we have

$$\lambda_\alpha \geq \frac{1}{n(\mu_\alpha + \epsilon)}. \quad (15)$$

Since $\epsilon$ is an arbitrary positive constant, we conclude from (15) that

$$\lambda_\alpha \geq \frac{1}{n\mu_\alpha}.$$  

Proposition 2.3 is proved. □

Now we can give a
Proof of Theorem 1.1. If there exists some $i_0$ with $\mu_{\alpha_{i_0}} = 0$, then by Proposition 2.3 we see $\lambda_{\alpha_{i_0}} = \infty$ and hence $K_X$ is nef. So in the following we may assume $\mu_{\alpha_i} > 0$ for all $i$. Again, by Proposition 2.3 we have

$$\lambda_{\alpha_i} \geq \frac{1}{n\mu_{\alpha_i}}.$$ 

In particular,

$$[\alpha_i] + \frac{\pi}{n\mu_{\alpha_i}} c_1(K_X)$$

is a Kähler class, or equivalently,

$$\frac{n\mu_{\alpha_i}}{\pi} [\alpha_i] + c_1(K_X)$$

is a Kähler class. On the other hand, by the Definition 1.2 of Property (A), we easily have that

$$c_1(K_X) = \lim_{i \to \infty} \left( \frac{n\mu_{\alpha_i}}{\pi} [\alpha_i] + c_1(K_X) \right),$$

which by definition means $K_X$ is nef.

Theorem 1.1 is proved. □

Remark 2.3. A consequence of Theorem 1.1 is that, for every Kähler class $\alpha$ on an $n$-dimensional compact Kähler manifold $X$ with $K_X$ not nef, $\mu_{\alpha} > 0$. In fact we have an effective lower bound for $\mu_{\alpha}$ in certain cases as follows. Recall a result of Berger (see e.g. [33, page 189, Exercise 16]): for every Kähler metric $\omega \in \alpha$, its scalar curvature $S_\omega$ satisfies

$$\int_{\mathbb{CP}^{n-1}} H_\omega(x) \omega_{FS}^{n-1} = \frac{2}{n(n+1)} S_\omega(x)$$

for every $x \in X$, here $H_\omega(x)$ has been regarded as a function on $\mathbb{CP}^{n-1}$. Note that

$$\int_X S_\omega \omega^n = \int_X nRic(\omega) \wedge \omega^{n-1} = 2\pi nc_1(X) \cdot \alpha^{n-1},$$

which implies

$$\sup_X S_\omega \geq \frac{2\pi nc_1(X) \cdot \alpha^{n-1}}{\alpha^n}.$$ 

Assume $\sup_X S_\omega = S_\omega(x_0)$, then using (16) gives

$$\sup_X H_\omega \geq \sup_X H_\omega(x_0) \geq \frac{2}{n(n+1)} \int_{\mathbb{CP}^{n-1}} \omega_{FS}^{n-1} S_\omega(x_0) \geq C_n \frac{2\pi c_1(X) \cdot \alpha^{n-1}}{\alpha^n},$$

where $C_n := \frac{2}{(n+1) \int_{\mathbb{CP}^{n-1}} \omega_{FS}^{n-1}} = \frac{2(n-1)!}{(n+1)! \pi^n}$, and hence

$$\mu_{\alpha} \geq C_n \frac{2\pi c_1(X) \cdot \alpha^{n-1}}{\alpha^n}. \quad (17)$$

Therefore, if in particular $X$ is a Fano manifold and $\alpha$ ia a Kähler class on $X$, then (17) provides a positive lower bound for $\mu_{\alpha}$; for example, setting $\alpha = 2\pi c_1(X)$ in (17) reads

$$\mu_{2\pi c_1(X)} \geq C_n.$$
We finish this section by discussing a byproduct of Proposition 2.2. The arguments for Proposition 2.2 in fact proved the following:

**Proposition 2.4.** Let $X$ be an $n$-dimensional compact Kähler manifold and $\omega(t)$ a solution to the Kähler-Ricci flow (7) on the maximal time interval $[0, T)$. Assume $T < \infty$. Then for every $t \in [0, T)$ we have

$$(T - t) \sup_X H_{\omega(t)} \geq \frac{1}{n}.$$ 

**Proof.** Note that by the maximal existence time theorem for the Kähler-Ricci flow [2, 22, 19], the assumption $T < \infty$ implies $K_X$ is not nef, and hence by Theorem 1.1 (or [20, Theorem 1.1]) $\sup_X H_{\omega(t)} > 0$ for every $t \in [0, T)$. Next, for any fixed $t_0 \in [0, T)$, we have a Kähler-Ricci flow $\omega(t + t_0)$ with the maximal time interval $[0, T - t_0)$ (here we have used again the maximal existence time theorem for the Kähler-Ricci flow [2, 22, 19]). Then applying Proposition 2.2 gives

$$T - t_0 \geq \frac{1}{n \cdot \sup_X H_{\omega(t_0)}},$$

or equivalently,

$$(T - t_0) \sup_X H_{\omega(t_0)} \geq \frac{1}{n}.$$ 

Proposition 2.4 is proved. □

**Remark 2.4.** In the setting of Proposition 2.4, applying a general fact of the Ricci flow [3, Lemma 8.6] gives

$$(T - t) \sup_X |\text{Rm}(\omega(t))| \geq \frac{1}{4},$$

which is proved by using the evolution equation of $|\text{Rm}|$ and the maximum principle (see [3, Lemma 8.6] for details). On the one hand, our Proposition 2.4 implies $T - t \sup_X |\text{Rm}(\omega(t))| \geq \frac{1}{n}$ and hence provides an alternative arguments for the above (18) in the Kähler-Ricci flow category (but the lower bound $\frac{1}{n}$ is less optimal when $n \geq 5$); on the other hand, Proposition 2.4 can be regarded as an improvement of (18) in the Kähler-Ricci flow category.

### 3. Non-existence of rational curve

In this section, we prove Theorem 1.2. To this end, we need the following observation. This observation is essentially due to Tosatti-Y.G. Zhang [21, Proposition 1.4, Remark 4.1], where they proved a similar result for the maximum of holomorphic bisectional curvature.

**Proposition 3.1.** If $(X, \omega)$ is a compact Kähler manifold containing a rational curve $C$, then

$$\sup_X H_\omega \geq \frac{\pi}{32 \int_C \omega}.$$ 

**Proof.** This follows from an almost identical argument given in Tosatti-Y.G. Zhang [21, page 2939], the only modification is replacing Yau’s Schwarz lemma [30] by the one of Royden given in Proposition 2.1. For convenience, we present some details by following [21, page 2939].

Set $A = \sup_X H_\omega$. Firstly, since there exists a rational curve, by Yau’s Schwarz Lemma [30] we know $A > 0$. The existence of a rational curve $C$ in $X$ also implies a non-constant
holomorphic map \( f : \mathbb{C} \to X \). Let \( \omega_C \) be the Euclidean metric on \( \mathbb{C} \). By applying a direct computation and Proposition 2.1 one has
\[
\Delta_{\omega_C} (\operatorname{tr}_{\omega_C} f^* \omega) \geq -A (\operatorname{tr}_{\omega_C} f^* \omega)^2.
\] (19)
on \( \mathbb{C} \). Having (19), one can apply the identical arguments in [21, page 2939] to complete the proof.
Proposition 3.1 is proved. □

We are ready to give a Proof of Theorem 1.2. We prove Theorem 1.2 by contradiction. Assume Theorem 1.2 fails, i.e. there exists a rational curve \( C \) in \( X \). Then we apply Proposition 3.1 with any \( \omega_i \in \alpha_i \) to see that
\[
\frac{\pi}{32} \int_C \alpha_i = \frac{\pi}{32} \int_C \omega_i \leq \sup_X H_{\omega_i},
\]
and hence
\[
\frac{\pi}{32} \int_C \alpha_i \leq \mu_{\alpha_i},
\]
i.e.
\[
\int_C \mu_{\alpha_i} \alpha_i \geq \frac{\pi}{32}.
\] (20)
However, by the Definition 1.2 of Property (A), we easily have that
\[
\lim_{i \to \infty} \left( \int_C \mu_{\alpha_i} \alpha_i \right) = 0.
\] (21)
By combining (20) and (21), we obtain a contradiction. Theorem 1.2 is proved. □

4. MIYAOKA-YAU INEQUALITY

In this section, we first prove Theorem 1.3.

Proof of Theorem 1.3. Note that, in view of Remark 1.5 and Yau’s result [29], we only need to prove the case that \( X \) satisfies \( \int_X c_1(K_X)^n = 0 \), i.e. \( K_X \) is not big.

By the curvature tensor decomposition and Chern-Weil theory (see [32, page 2752-2753]; also see [18, page 96] or [14, Proposition 2.1]), we have, for any Kähler metric \( \omega \) on \( X \),
\[
\left( \frac{2(n + 1)}{n} c_2(X) - c_1(X)^2 \right) \cdot [\omega]^{n-2} \geq \frac{n + 2}{4\pi^2 n^2 (n - 1)} \int_X |\operatorname{Ric}(\omega) + \omega|_n^2 \omega^n.
\] (22)
Therefore, to prove Theorem 1.3 it suffices to construct a family of Kähler metrics \( \tilde{\omega}_i \) on \( X \) satisfying both of the followings as \( i \to \infty \):
(1) \( \tilde{\omega}_i \to 2\pi c_1(K_X) \);
(2) \( \int_X |\operatorname{Ric}(\tilde{\omega}_i) + \tilde{\omega}_i|_{\tilde{\omega}_i}^2 \tilde{\omega}_i^n \to 0 \).

We now use Wu-Yau’s continuity equation: for any given Kähler metric \( \omega_i \in \alpha_i \), consider
\[
\operatorname{Ric}(\omega_i(t)) = -\omega_i(t) + t\omega_i.
\] (23)
By using Yau’s theorem [31] and Proposition 2.3 one easily sees that (23) has a smooth solution \( \omega_i(t) \) for \( t \in (n\mu_{\alpha_i}, \infty) \) (in fact, by Theorem 1.1 the solution exists for \( t \in (0, \infty) \)).
Moreover, by Proposition 2.1 and arguments in [20, Section 2] we know, for any $\epsilon > 0$, if we choose $\omega_i \in \alpha_i$ with $\sup_X H_{\omega_i} \leq \mu_{\alpha_i} + \epsilon$, then the solution $\omega_i(t)$ to (23) satisfies

$$tr_{\omega_i(n\mu_{\alpha_i} + 2n\epsilon)} \omega_i \leq \frac{1}{\epsilon},$$

which in particular implies

$$|\omega_i|_{\omega_i(n\mu_{\alpha_i} + 2n\epsilon)}^2 \leq \frac{n}{\epsilon^2}.$$

We separate discussions into two cases:

**Case (1):** there exists some $i_0$ with $\mu_{\alpha_{i_0}} = 0$. In this case, we choose a family of Kähler metrics $\omega_{i_0,\epsilon} \in \alpha_0$ with $\sup_X H_{\omega_{i_0,\epsilon}} \leq \epsilon$ and set $\tilde{\omega}_\epsilon := \omega_{i_0,\epsilon}(2n\epsilon)$, where $\omega_{i_0,\epsilon}(t)$ is the solution to

$$Ric(\omega_{i_0,\epsilon}(t)) = -\omega_{i_0,\epsilon}(t) + t\omega_{i_0,\epsilon}.$$  

Obviously, $[\tilde{\omega}_\epsilon] = 2\pi c_1(K_X) + 2n\epsilon[\alpha_{i_0}] \to 2\pi c_1(K_X)$ as $\epsilon \to 0$. Moreover, since $Ric(\tilde{\omega}_\epsilon) + \tilde{\omega}_\epsilon = 2n\epsilon\omega_{i_0,\epsilon}$ and (25) means $|\omega_{i_0,\epsilon}|_{\tilde{\omega}_\epsilon}^2 \leq \frac{n}{\epsilon^2}$, we have

$$\int_X |Ric(\tilde{\omega}_\epsilon) + \tilde{\omega}_\epsilon|_{\tilde{\omega}_\epsilon}^2 \tilde{\omega}_\epsilon^n \leq 4n^3 \int_X \tilde{\omega}_\epsilon^n \to 4n^3(2\pi)^n \int_X c_1(K_X)^n = 0.$$  

**Case (2):** $\mu_{\alpha_i} > 0$ for every $i$. In this case, we choose $\omega_i \in \alpha_i$ with $\sup_X H_{\omega_i} \leq 2\mu_{\alpha_i}$ and set $\tilde{\omega}_i := \omega_i(3n\mu_{\alpha_i})$. Then by the Definition 1.2 of Property (A) $[\tilde{\omega}_i] = 2\pi c_1(K_X) + 3n\mu_{\alpha_i}\omega_i \to 2\pi c_1(K_X)$ as $i \to \infty$. Moreover, (25) means $|\omega_i|_{\tilde{\omega}_i}^2 \leq \frac{n}{\mu_{\alpha_i}}^2$, and hence, as $i \to \infty$,

$$\int_X |Ric(\tilde{\omega}_i) + \tilde{\omega}_i|_{\tilde{\omega}_i}^2 \tilde{\omega}_i^n = \int_X |3n\mu_{\alpha_i}\omega_i|_{\tilde{\omega}_i}^2 \tilde{\omega}_i^n \leq 9n^3 \int_X \tilde{\omega}_i^n \to 9n^3(2\pi)^n \int_X c_1(K_X)^n = 0.$$  

Combining Cases (1) and (2), we have proved Theorem 1.3.

Next, we give a proof for Theorem 1.4.

**Proof of Theorem 1.4.** Let’s first look at the item (1), so we have a Kähler class $\alpha$ with $\mu_{\alpha} = 0$. Then by the same arguments in above proof for Theorem 1.3, we can construct a sequence of Kähler metric $\tilde{\omega}_\epsilon$ for $\epsilon > 0$ satisfying

1. $[\tilde{\omega}_\epsilon] = 2\pi c_1(K_X) + 2n\epsilon\alpha$;
2. $\int_X |Ric(\tilde{\omega}_\epsilon) + \tilde{\omega}_\epsilon|_{\tilde{\omega}_\epsilon}^2 \tilde{\omega}_\epsilon^n \leq 4n^3 \int_X \tilde{\omega}_\epsilon^n$
Observe that, when $\nu < n - 2$,
\[
\frac{1}{\epsilon^{n-\nu-2}} \left( \frac{2(n+1)}{n} c_2(X) - c_1(X)^2 \right) \cdot [\tilde{\omega}_i]^{n-2}
= \frac{1}{\epsilon^{n-\nu-2}} \left( \frac{2(n+1)}{n} c_2(X) - c_1(X)^2 \right) \cdot (-2\pi c_1(X) + 2n\epsilon\alpha)^{n-2}
= \frac{1}{\epsilon^{n-\nu-2}} \left( \frac{2(n+1)}{n} c_2(X) - c_1(X)^2 \right) \cdot \left( \sum_{k=0}^{\nu} \binom{n-2}{k} (2\pi k (2n\epsilon) n^{-k-2} (-c_1(X))^k \cdot \alpha^{n-k-2}) \right)
\to \left( \frac{n-2}{\nu} \right) (2\pi)^{\nu}(2n)^{n-\nu-2} \left( \frac{2(n+1)}{n} c_2(X) - c_1(X)^2 \right) \cdot (-c_1(X))^k \cdot \alpha^{n-k-2},
\]
as $\epsilon \to 0$. On the other hand,
\[
\frac{1}{\epsilon^{n-\nu-2}} \int_X \left|\text{Ric}\left(\tilde{\omega}_i\right) + \tilde{\omega}_i \cdot \tilde{\omega}_i^n\right|
\leq \frac{4n^3}{\epsilon^{n-\nu-2}} \int_X \tilde{\omega}_i^n
= \frac{4n^3}{\epsilon^{n-\nu-2}} \int_X \sum_{k=0}^{\nu} \binom{n}{k} (2\pi k (2n\epsilon) n^{-k-2} (-c_1(X))^k \cdot \alpha^{n-k}) \cdot (\tilde{\omega}_i)^k \cdot \alpha^n
\leq C\epsilon^2 \to 0,
\]
as $\epsilon \to 0$. Plugging (29) and (30) into (22) gives the desired result (2).

Next we look at the item (2). Obviously, we only need to discuss the case $\alpha_\infty \neq [0]$. When $\alpha_\infty \neq [0]$, we must have $\mu_{\alpha_i} \to 0$ as $i \to \infty$. By the same arguments in above proof for Theorem 1.3, we can construct a sequence of Kähler metric $\tilde{\omega}_i$ for $\epsilon > 0$ satisfying

(1) $[\tilde{\omega}_i] = 2\pi c_1(K_X) + 3n\mu_{\alpha_i} \alpha_i$;
(2) $\int_X |\text{Ric}(\tilde{\omega}_i) + \tilde{\omega}_i \cdot \tilde{\omega}_i^n| \leq 9n^3 \int_X \tilde{\omega}_i^n$

Similar to the item (1), we can show that, as $i \to \infty$,
\[
\frac{1}{\mu_{\alpha_i}^{n-\nu-2}} \left( \frac{2(n+1)}{n} c_2(X) - c_1(X)^2 \right) \cdot [\tilde{\omega}_i]^{n-2}
\to \left( \frac{n-2}{\nu} \right) (2\pi)^{\nu}(3n)^{n-\nu-2} \left( \frac{2(n+1)}{n} c_2(X) - c_1(X)^2 \right) \cdot (-c_1(X))^k \cdot \alpha_\infty^{n-k-2},
\]
and
\[
\frac{1}{\mu_{\alpha_i}^{n-\nu-2}} \int_X |\text{Ric}(\tilde{\omega}_i) + \tilde{\omega}_i \cdot \tilde{\omega}_i^n| \leq C\mu_{\alpha_i}^2 \to 0.
\]

Plugging (31) and (32) into (22) gives the desired result (3).

Theorem 1.4 is proved. \hfill \Box

Finally, we give a

Proof of Theorem 1.5. We may assume without loss of any generality that $\alpha$ is a Kähler class on $X$. The proof we discuss here is a simple modification of arguments in Y.G.
Zhang [32] (also see [11]). For an arbitrary Kähler class \( \alpha \) on \( X \) and a Kähler metric \( \omega_0 \in \alpha \), we consider the Kähler-Ricci flow \( \omega = \omega(t) \in [0, \infty) \) running from \( \omega_0 \):

\[
\left\{ \begin{aligned}
\partial_t \omega(t) &= -\text{Ric}(\omega(t)) - \omega(t) \\
\omega(0) &= \omega_0,
\end{aligned} \right.
\]

along which the Kähler class satisfies \( [\omega(t)] = -2\pi (1 - e^{-t})c_1(X) + e^{-t} \alpha \). Note that, as in the above proof for Theorem 1.4 we easily have, as \( t \to \infty \),

\[
e^{(n-\nu-2)t} \left( \frac{2(n+1)}{n} c_2(X) - c_1(X)^2 \right) \cdot [\omega(t)]^{n-2}
\]

\[
e^{(n-\nu-2)t} \left( \frac{2(n+1)}{n} c_2(X) - c_1(X)^2 \right) \cdot (-1 - e^{-t})2\pi c_1(X) + e^{-t} \alpha)^{n-2}
\]

\[
\to \left( \frac{n-2}{\nu} \right) (2\pi)^\nu \left( \frac{2(n+1)}{n} c_2(X) - c_1(X)^2 \right) \cdot (-c_1(X))^\nu \cdot \alpha^{n-\nu-2}, \tag{33}
\]

Then, to complete the proof we make the following

**Claim:** as \( t \to \infty \) there holds

\[
e^{(n-\nu-2)t} \int_X |\text{Ric}(\omega(t)) + \omega(t)|^2_{\omega(t)} \omega(t)^n \to 0. \tag{34}
\]

To see this, we need the followings:

\[
\partial_t \omega(t)^n = -(S(t) + n)\omega(t)^n,
\]

\[
\partial_t S(t) = \Delta_{\omega(t)} S(t) + |\text{Ric}(\omega(t)) + \omega(t)|^2_{\omega(t)} - (S(t) + n).
\]

By calculations in [32], we have

\[
\int_X |\text{Ric}(\omega(t)) + \omega(t)|^2_{\omega(t)} \omega(t)^n
\]

\[
= \int_X (\partial_t S(t)) \omega(t)^n + \int_X (S(t) + n) \omega(t)^n
\]

\[
= \partial_t \left( \int_X S(t) \omega(t)^n \right) + \int_X (S(t) + 1)(S(t) + n) \omega(t)^n,
\]

and so

\[
e^{(n-\nu-2)t} \int_X |\text{Ric}(\omega(t)) + \omega(t)|^2_{\omega(t)} \omega(t)^n
\]

\[
= e^{(n-\nu-2)t} \partial_t \left( \int_X S(t) \omega(t)^n \right) + e^{(n-\nu-2)t} \int_X (S(t) + 1)(S(t) + n) \omega(t)^n
\]

\[
= \partial_t \left( e^{(n-\nu-2)t} \int_X S(t) \omega(t)^n \right) + e^{(n-\nu-2)t} \int_X (S(t) + (\nu + 3)S(t) + n) \omega(t)^n. \tag{35}
\]

Set \( L(t) := e^{(n-\nu-2)t} \int_X S(t) \omega(t)^n \). By direct computation we have

\[
L(t) = e^{(n-\nu-2)t} \int_X S(t) \omega(t)^n
\]

\[
= ne^{(n-\nu-2)t}(2\pi c_1) \cdot (-2\pi (1 - e^{-t})c_1(X) + e^{-t} \alpha)^{n-1}
\]

\[
= \sum_{k=0}^{\nu-1} A_k (1 - e^{-t})^k e^{(k-\nu-1)t},
\]

where

\[
A_k = \frac{(2\pi c_1)^{n-1}}{k!(n-2k)!} (-2\pi (1 - e^{-t})c_1(X) + e^{-t} \alpha)^{n-1}.
\]
where $A_k$’s are some constant only depending on $c_1(X), \alpha, n$ and $k$, and hence we can find a positive constant $C$ such that

$$L(t) \leq Ce^{-2t}. \quad (36)$$

and

$$\partial_t L(t) \leq Ce^{-2t}. \quad (37)$$

Using (36) and the easy fact $[\omega(t)]^n \leq Ce^{-(n-\nu)t}$, we see that the second term in (35) satisfies

$$e^{(n-\nu-2)t} \int_X (S(t)^2 + (\nu + 3)S(t) + n)\omega(t)^n$$

$$= e^{(n-\nu-2)t} \int_X S(t)^2\omega(t)^n + (\nu + 3)L(t) + ne^{(n-\nu-2)}[\omega(t)]^n$$

$$\leq e^{(n-\nu-2)t} \int_X S(t)^2\omega(t)^n + 2Ce^{-2t},$$

where $C$ is some uniform positive constant.

Now recall a result of Song-Tian [17] that the scalar curvature $S(t)$ is uniformly bounded on $X \times [0, \infty)$ when $K_X$ is semi-ample (this is the only place using the semi-ampleness of $K_X$), and so

$$e^{(n-\nu-2)t} \int_X (S(t)^2 + (\nu + 3)S(t) + n)\omega(t)^n$$

$$\leq Ce^{(n-\nu-2)t} \int_X \omega(t)^n + Ce^{-2t}$$

$$\leq Ce^{-2t}. \quad (38)$$

Plugging (37) and (38) into (35) gives

$$e^{(n-\nu-2)t} \int_X |\text{Ric}(\omega(t)) + \omega(t)|^2_{\omega(t)} \omega(t)^n \leq Ce^{-2t},$$

which completes the proof of Claim.

Now the (4) follows by plugging (33) and (34) into (22).

Theorem 1.5 is proved. □

5. Remarks on Property (A)

In this section, we shall make some more remarks on Property (A). Let’s begin with the following one, which directly follows from the well-known decreasing property of holomorphic sectional curvature on submanifolds.

**Proposition 5.1.** Let $X$ be a compact Kähler manifold and $Y$ a compact complex submanifold of $X$. If $X$ satisfies Property (A), then $Y$ also satisfies Property (A).

Having Proposition 5.1, all the conclusions in Theorems 1.1, 1.2 and 1.3 hold for every compact complex submanifold of $X$, provided $X$ satisfies Property (A). For example, as in [26, Corollary] and [20, Remark 1.5], we can conclude that if $X$ is a compact Kähler manifold satisfying Property (A), then every compact complex submanifold of $X$ has nef canonical line bundle and does not contain any rational curve.

Next we observe that Property (A) is preserved under the product of Kähler manifolds.
Proposition 5.2. Let $X, Y$ be two compact Kähler manifolds and $Z := X \times Y$ the product manifold with the product complex structure. If both $X$ and $Y$ satisfy Property (A), then $Z$ also satisfies Property (A).

Proof. To complete the proof, we need the following

Claim: Given a Kähler metric $\omega$ on $X$ and a Kähler metric $\eta$ on $Y$, if we choose two positive constants $A_1$ and $A_2$ satisfying $\sup_X H_\omega \leq A_1$ and $\sup_Y H_\eta \leq A_2$, then the product Kähler metric $\chi := \omega + \eta$ on $Z$ satisfies $\sup_Z H_\chi \leq A_1 + A_2$.

This Claim can be easily checked by using the definition of holomorphic sectional curvature and the product structure; so we omit the details here.

By assumption, we fix a sequence of Kähler classes $\alpha_i$ on $X$ and $\beta_i$ on $Y$ such that $\mu_\alpha, \alpha_i \to 0$ and $\mu_\beta, \beta_i \to 0$. We separate discussions into three cases as follows:

Case 1: there exist a Kähler class $\alpha$ on $X$ with $\mu_\alpha = 0$ and a Kähler class $\beta$ on $Y$ with $\mu_\beta = 0$. In this case we have $\mu_{\alpha + \beta} = 0$. Indeed, for any $\epsilon > 0$ we choose Kähler metrics $\omega_\epsilon \in \alpha$ and $\eta_\epsilon \in \beta$ such that $\sup_X H_{\omega_\epsilon} \leq \epsilon$ and $\sup_Y H_{\eta_\epsilon} \leq \epsilon$, and define $\chi_\epsilon := \omega_\epsilon + \eta_\epsilon$ be the product Kähler metric on $Z$. By the above Claim we have $\sup_Z H_{\chi_\epsilon} \leq 2\epsilon$, and so $\mu_{\alpha + \beta} = 0$.

Case 2: there exists a Kähler class $\alpha$ on $X$ with $\mu_\alpha = 0$ and $\mu_\beta_i > 0$ for every $i$. For a fixed sequence of positive numbers $\epsilon_i$ with $\epsilon_i \to 0$, we can choose Kähler metrics $\omega_i \in \alpha$ with $\sup_X H_{\omega_i} \leq \epsilon_i$. Next we set $\tilde{\beta}_i := \frac{\mu_\beta_i}{\epsilon_i} \beta_i$, which is a sequence of Kähler metrics on $Y$ with $\mu_{\tilde{\beta}_i} = \epsilon_i$. Notice that $\epsilon_i \tilde{\beta}_i = \mu_\beta_i \beta_i \to 0$. Now we choose for every $i$ a Kähler metric $\eta_i \in \tilde{\beta}_i$ with $\sup_Y H_{\eta_i} \leq 2\epsilon_i$ and define Kähler metrics $\chi_i := \omega_i + \eta_i$ on $Z$. By the above Claim we know $\sup_Z H_{\chi_i} \leq 3\epsilon_i$, and hence

$$\mu_{\alpha + \tilde{\beta}_i}(\alpha + \tilde{\beta}_i) \leq 3\epsilon_i(\alpha + \tilde{\beta}_i) \to 0.$$

Case 3: $\mu_{\alpha_i} > 0$ and $\mu_{\beta_i} > 0$ for every $i$. Similar to Case 2, we may define a sequence of Kähler classes on $Y$ by $\tilde{\beta}_i := \frac{\mu_{\beta_i}}{\mu_{\alpha_i}} \beta_i$, which satisfies $\mu_{\tilde{\beta}_i} = \mu_{\alpha_i}$ and $\mu_{\alpha_i} \tilde{\beta}_i = \mu_{\tilde{\beta}_i} \beta_i = \mu_{\beta_i} \beta_i \to 0$. Also we can choose Kähler metrics $\omega_i \in \alpha_i$ and $\eta_i \in \tilde{\beta}_i$ with $\sup_X H_{\omega_i} \leq 2\mu_{\alpha_i}$ and $\sup_Y H_{\eta_i} \leq 2\mu_{\tilde{\beta}_i} = 2\mu_{\alpha_i}$. Therefore,

$$\mu_{\alpha_i + \tilde{\beta}_i}(\alpha_i + \tilde{\beta}_i) \leq 4\mu_{\alpha_i}(\alpha_i + \tilde{\beta}_i) \to 0.$$

Proposition 5.2 is proved. \qed

Finally, we make a remark on projective case.

Remark 5.1. Recall that if $X$ is a projective manifold, then $K_X$ is nef by definition means $\int_X c_1(K_X) \geq 0$ for any irreducible 1-dimensional subvariety $C$ on $X$ (this is equivalent to that $c_1(K_X)$ is a limit of Kähler classes when $X$ is projective). Correspondingly, it seems natural to introduce the following definition: we say a projective manifold $X$ satisfies Property $(A^*)$ if for any irreducible 1-dimensional subvariety $C$ on $X$ there exists a sequence of Kähler classes $\alpha_i^C$, $i = 1, 2, \ldots$, on $X$ such that $\int_X \mu_{\alpha_i^C} c_1^C \to 0$. Then our arguments for Theorems 11 and 12 can be easily applied to show that a projective manifold satisfying Property $(A^*)$ has nef canonical line bundle and does not contain any rational curve.
The author is grateful to Professors Huai-Dong Cao and Gang Tian for their interest in this work, and constant encouragement and support. The author is also grateful to Professor Gang Tian for useful conversations on Chern number inequality, Professor Valentino Tosatti for many valuable discussions on papers [4, 20, 26] in Spring 2017 and Professor Chengjie Yu for useful comments on a previous version.

REFERENCES

[1] Barth, W., Hulek, K., Peters, C. and Van de Ven, A., Compact complex surfaces, 2nd edition, Springer, Berlin (2004)
[2] Cao, H.-D., Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds, Invent. Math. 81 (2), 359-372 (1985)
[3] Chow, B., Lu, P. and Ni, L., Hamilton’s Ricci flow, Graduate Studies in Mathematics, 77. American Mathematical Society, Providence, RI; Science Press, New York, 2006
[4] Diverio, S. and Trapani, S., Quasi-negative holomorphic sectional curvature and positivity of the canonical bundle, to appear in J. Differential Geom., [arXiv:1606.01381]
[5] Greb, D., Kebekus, S., Peternell, T. and Taji, B., The Miyaoka-Yau inequality and uniformisation of minimal models, [arXiv:1511.08822]
[6] Guenancia, H. and Taji, B., Orbifold stability and Miyaoka-Yau inequality for minimal pairs, [arXiv:1611.05981]
[7] Heier, G., Lu, S. and Wong, B., On the canonical line bundle and negative holomorphic sectional curvature, Math. Res. Lett., 17 (6), 1101-1110, 2010
[8] Heier, G., Lu, S. and Wong, B., Kähler manifolds of semi-negative holomorphic sectional curvature, J. Differential Geom. 104 (2016), no. 3, 419-441
[9] Höring, A. and Peternell, T., Minimal model for Kähler threefolds, Invent. Math. 203 (2016), no.1, 217-264
[10] Kawamata, Y., On the length of an extremal rational curve, Invent. Math. 105 (1991) 609-611
[11] Matsuki, K., Introduction to the Mori program, Universitext, Springer-Verlag, New York, 2002
[12] Miyaoka, Y., On the Chern numbers of surfaces of general type, Invent. Math. 42 (1977), 225-237
[13] Nomura, R., Kähler manifolds with negative holomorphic sectional curvature, Kähler-Ricci flow approach, Int. Math. Res. Not., https://doi.org/10.1093/imrn/rnx075
[14] Nomura, R., Miyaoka-Yau inequality on compact Kähler manifolds with semi-positive canonical bundle, [arXiv:1802.05425]
[15] Phong, D.H., ˇSeˇsum, N. and Sturm, J., Multiplier ideal sheaves and the Kähler-Ricci flow, Comm. Anal. Geom. 15 (2007), no. 3, 613-632
[16] Royden, H.L., Ahlfors-Schwarz lemma in several complex variables, Comment. Math. Helv. 55 (1980), no. 4, 547-558
[17] Song, J. and Tian, G., Bounding scalar curvature for global solutions of the Kähler-Ricci flow, Amer. J. Math. 138 (2016) no. 3, 683-695
[18] Song, J. and Wang, X., The greatest Ricci lower bound, conical Einstein metrics and the Chern number inequality, Geom. Topol. 20 (2016) 49-102
[19] Tian, G. and Zhang, Z., On the Kähler-Ricci flow on projective manifolds of general type, Chinese Ann. Math. Ser. B 27 (2006), no. 2, 179-192
[20] Tosatti, V. and Yang, X., An extension of a theorem of Wu-Yau, J. Differential Geom. 107 (2017), no. 3, 573-579
[21] Tosatti, V. and Zhang, Y.G., Infinite-time singularities of the Kähler-Ricci flow, Geom. Topol. 19 (2015), no. 5, 2925-2948
[22] Tsuji, H., Existence and degeneration of Kähler-Einstein metrics on minimal algebraic varieties of general type, Math. Ann. 281, 123-133 (1988)
[23] Tsuji, H., Stability of tangent bundles of minimal algebraic varieties, Topology, 27, no. 4 (1988), 429-442
[24] Wong, B., The uniformization of compact Kähler surfaces of negative curvature, J. Differential Geom. 16 (1981), no. 3, 407-420
[25] Wong, P.-M., Wu, D. and Yau, S.-T., Picard number, holomorphic sectional curvature and ampleness, Proc. Amer. Math. Soc. 140 (2), 621-626, 2012
[26] Wu, D. and Yau, S.-T., Negative holomorphic curvature and positive canonical bundle, Invent. Math. 204, no. 2 (2016), 595-604
[27] Wu, D. and Yau, S.-T., A remark on our paper ”Negative holomorphic curvature and positive canonical bundle”, Comm. Anal. Geom. 24 (4), 901-912 (2016)
[28] Yang, X. and Zheng, F., On real bisectional curvature for Hermitian manifolds, to appear in Trans. AMS, arXiv:1610.07165
[29] Yau, S.-T., Calabi’s conjecture and some new results in algebraic geometry, Proc. Nat. Acad. Sci. U.S.A. 74 (1977), no. 5, 1798-1799
[30] Yau, S.-T., A general Schwarz lemma for Kähler manifolds, Amer. J. Math. 100, no. 1 (1978), 197-203.
[31] Yau, S.-T., On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I, Comm. Pure Appl. Math. 31 (1978) 339-411
[32] Zhang, Y.G., Miyaoka-Yau inequality for minimal projective manifolds of general type, Proc. Amer. Math. Soc. 137 (2009), no. 8, 2749-2754
[33] Zheng, F., Complex Differential Geometry, In: AMS/IP Studies in Advanced Mathematics, vol. 18, American Mathematical Society, Providence (2000)

Beijing International Center for Mathematical Research, Peking University, Beijing 100781, China
E-mail address: yashanzh@pku.edu.cn