Gröbner-Shirshov basis for the finitely presented algebras defined by permutation relations of symmetric type

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Abstract
In this paper, we give a Gröbner-Shirshov basis for the finitely presented semigroup algebra \( k[S_n(Sym_n)] \) defined by permutation relations of symmetric type. As an application, by the Composition-Diamond Lemma, we obtain normal forms of elements of monoid \( S_n(Sym_n) \), which gives an answer to an open problem posted by F. Cedó, E. Jespers and J. Okniński for the symmetric group case.

AMS Mathematics Subject Classification (2000): 16S15, 16S35, 20M25.

Keywords: Gröbner-Shirshov basis, finitely presented, normal form, semigroup algebra.

1 Introduction
Let \( Sym_n \) be the symmetric group of degree \( n \) and \( H \) a subset of \( Sym_n \). Recently, F. Cedó, E. Jespers and J. Okniński introduced a new class of finitely

*Supported by the NNSF of China (11171118), the Research Fund for the Doctoral Program of Higher Education of China (20114407110007), the NSF of Guangdong Province (S2011101003374,S20121040007369), the Program on International Cooperation and Innovation, Department of Education, Guangdong Province (2012jghz0007), and the NSF of Zhanjiang Normal University (QL0902).

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presented semigroup algebra \( k[S_n(H)] \) over a field \( k \), where the monoid \( S_n(H) \) is defined by a set of generators \( x_1, x_2, \ldots, x_n \) and homogenous permutation relations, i.e.

\[
S_n(H) = \langle x_1, x_2, \ldots, x_n | x_{\sigma(1)}x_{\sigma(2)} \cdots x_{\sigma(n)} = x_1x_2 \cdots x_n, \sigma \in H \rangle.
\]

There are some results on this new algebraic structure, for example, the alternating type \([6, 8]\), the abelian type \([9]\), and the \( n \)-cyclic type \([7]\).

Let \( \varsigma \) be the cyclic permutation

\[
\varsigma = \begin{pmatrix}
1 & 2 & \cdots & n - 1 & n \\
2 & 3 & \cdots & n & 1
\end{pmatrix}
\]

By using the rewriting system method, Cédo, Jespers and Okniński \([7]\) obtained normal forms of elements of \( S_n(H) \) for the case when \( H \) is the cyclic subgroup of \( \text{Sym}_n \) generated by the cyclic permutation \( \varsigma \). They also proposed some open problems at the end of the same paper \([7]\). One of the open problems is: “For an arbitrary subgroup \( H \) of symmetric group \( \text{Sym}_n \), what does every element of \( S_n(H) \) have a unique canonical form, as is the case of the monoid defined by permutation relations of cyclic subgroup type.”

In this paper, we use the Gröbner-Shirshov bases method to study the finitely presented algebra defined by permutation relations of symmetric type \( k[S_n(\text{Sym}_n)] \). We find a Gröbner-Shirshov basis for the algebra \( k[S_n(\text{Sym}_n)] \). As an application, we get normal forms of elements of monoid \( S_n(\text{Sym}_n) \), which gives an answer to the above problem for the case when \( H \) is the symmetric group \( \text{Sym}_n \).

### 2 Composition-Diamond Lemma for associative algebra

We first cite some concepts and results from the literature \([2, 10]\) which are related to Gröbner-Shirshov bases for associative algebras.

Let \( k \) be a field, \( k\langle X \rangle \) the free associative algebra over \( k \) generated by \( X \). Denote \( X^* \) the free monoid generated by \( X \), where the empty word is the identity which is denoted by 1. For a word \( w \in X^* \), we denote the length of \( w \) by \( |w| \). Let \( X^* \) be a well ordered set. Then every nonzero polynomial \( f \in k\langle X \rangle \) has the leading word \( \bar{f} \). If the coefficient of \( \bar{f} \) in \( f \) is equal to 1, then \( f \) is called monic.

Let \( f \) and \( g \) be two monic polynomials in \( k\langle X \rangle \). Then, there are two kinds of compositions:

(i) If \( w \) is a word such that \( w = \bar{f}b = a\bar{g} \) for some \( a, b \in X^* \) with \( |\bar{f}| + |\bar{g}| > |w| \), then the polynomial \( (f, g)_w = fb - ag \) is called the intersection composition of \( f \) and \( g \) with respect to \( w \).

(ii) If \( w = \bar{f} = a\bar{g} \) for some \( a, b \in X^* \), then the polynomial \( (f, g)_w = f - agb \) is called the inclusion composition of \( f \) and \( g \) with respect to \( w \).

In (i) and (ii), the word \( w \) is called an ambiguity.
Let \( S \subseteq k\langle X \rangle \) with each \( s \in S \) monic. Then the composition \((f, g)_w \) is called trivial modulo \((S, w)\) if \((f, g)_w = \sum \alpha_i a_i s_i b_i\), where each \( \alpha_i \in k, a_i, b_i \in X^* \), \( s_i \in S \) and \( a_i s_i b_i < w \). If this is the case, then we write 
\[(f, g)_w \equiv 0 \mod (S, w) \] 
In general, for \( p, q \in k\langle X \rangle \), we write 
\[ p \equiv q \mod (S, w) \] 
which means that 
\[ p - q = \sum \alpha_i a_i s_i b_i, \] 
where each \( \alpha_i \in k, a_i, b_i \in X^* \), \( s_i \in S \) and \( a_i s_i b_i < w \).

We call the set \( S \) endowed with the well order \(<\) a Gröbner-Shirshov basis in \( k\langle X \rangle \) if any composition of polynomials in \( S \) is trivial modulo \( S \) and corresponding \( w \).

A well order \(<\) on \( X^* \) is monomial if for \( u, v \in X^* \), we have 
\[ u < v \Rightarrow w_1 u w_2 < w_1 v w_2, \] 
for all \( w_1, w_2 \in X^* \).

The following lemma was proved by Shirshov \[10\] for free Lie algebras (with deg-lex order) in 1962 (see also Bokut \[2\]). In 1976, Bokut \[3\] specialized the approach of Shirshov to associative algebras (see also Bergman \[1\]). For commutative polynomials, this lemma is known as the Buchberger’s Theorem (see \[4\] and \[5\]).

**Composition-Diamond Lemma.** Let \( k \) be a field, \( k\langle X \mid S \rangle = k\langle X \rangle / Id(S) \) and \( > \) a monomial order on \( X^* \), where \( Id(S) \) is the ideal of \( k\langle X \rangle \) generated by \( S \). Then the following statements are equivalent:

(i) \( S \) is a Gröbner-Shirshov basis in \( k\langle X \rangle \).

(ii) \( f \in Id(S) \Rightarrow \bar{f} = \bar{a} \bar{s} \bar{b} \) for some \( s \in S \) and \( a, b \in X^* \).

(iii) \( \text{Irr}(S) = \{ u \in X^* \mid u \not= \bar{a} \bar{s} \bar{b}, s \in S, a, b \in X^* \} \) is a \( k \)-linear basis of the algebra \( k\langle X \mid S \rangle \).

If a subset \( S \) of \( k\langle X \rangle \) is not a Gröbner-Shirshov basis, then we can add to \( S \) all nontrivial compositions of polynomials of \( S \), and by continuing this process (may be infinitely) many times, we eventually obtain a Gröbner-Shirshov basis \( S^{\text{comp}} \). Such a process is called the Shirshov algorithm.

Let \( M = \langle X \mid S \rangle \) be a monoid presentation. Then \( S \) is a subset of \( k\langle X \rangle \) and hence one can find a Gröbner-Shirshov basis \( S^{\text{comp}} \). We also call \( S^{\text{comp}} \) a Gröbner-Shirshov basis of monoid \( M \). The set \( \text{Irr}(S^{\text{comp}}) = \{ u \in X^* \mid u \not= \bar{a} \bar{s} \bar{b}, a, b \in X^*, s \in S^{\text{comp}} \} \) is a \( k \)-linear basis of \( k\langle X \mid S \rangle \) which is also normal forms of elements of monoid \( M \).
3 A Gröbner-Shirshov basis for $k[S_n(Sym_n)]$

Let $S_n(Sym_n)$ be the finitely presented monoid defined by permutation relations of symmetric type, i.e.

$$S_n(Sym_n) = \langle x_1, x_2, \ldots, x_n | x_\sigma(1)x_\sigma(2) \cdots x_\sigma(n) = x_1 x_2 \cdots x_n, \sigma \in Sym_n \rangle,$$

where $Sym_n$ is the symmetric group of degree $n$.

We give some notations which will be used in this section. Let $\varepsilon \in Sym_n$ be the identity map of $Sym_n$ and $Sym_n^0 = Sym_n \setminus \{\varepsilon\}$. Let $\mathbb{N}$ be the set of positive integers. Denote $n = \{1, 2, \ldots, n\}$, and $[n_1, n_2] = \{n_1, n_1 + 1, \ldots, n_2\}$ for any $n_1, n_2 \in n$ and $n_1 \leq n_2$. For any $\sigma \in Sym_n$, denote

$$X_\sigma := x_\sigma(1)x_\sigma(2) \cdots x_\sigma(n),$$

in particular,

$$X_\varepsilon := x_1 x_2 \cdots x_n.$$

For any $x_{i_1}, x_{i_2}, \ldots, x_{i_m} \in X$, $m \geq 2$, define

$$x_{i_1} x_{i_2} \cdots x_{i_m} := x_{j_1} x_{j_2} \cdots x_{j_m},$$

where $j_1, j_2, \ldots, j_m$ is the permutation of $i_1, i_2, \ldots, i_m$ such that $j_1 \leq j_2 \leq \cdots \leq j_m$. For example, $x_2 x_3 x_1 x_2 x_3 x_4 x_5 = x_2 x_2 x_3 x_3 x_4 x_5$.

Let $X = \{x_1, x_2, \ldots, x_n\}$, $x_1 < x_2 < \cdots < x_n$ and “$<$” the degree-lexicographic order on $X^*$. Denote

$$S = \{X_\sigma - X_\varepsilon | \sigma \in Sym_n^0\}$$

and $\overline{S}$ the subset of $k\langle X \rangle$ consisting of the following polynomials:

1. $X_\sigma - X_\varepsilon$,
2. $x_i X_\varepsilon - X_\varepsilon x_i$,
3. $x_i x_1^m X_\varepsilon - x_1^m X_\varepsilon x_i$,
4. $X_\varepsilon x_{i_1} x_{i_2} \cdots x_{i_{m+1}} - X_\varepsilon x_{i_1} x_{i_2} \cdots x_{i_{m+1}}$, $x_{i_1} x_{i_2} \cdots x_{i_{m+1}} > x_{i_1} x_{i_2} \cdots x_{i_{m+1}}$, $x_{i_1} x_{i_2} \cdots x_{i_{m+1}}$,
5. $x_i x_{i_1} x_{i_2} \cdots x_{i_m} x_1 - x_1 x_{i_1} x_{i_2} \cdots x_{i_m}$,

where $\sigma \in Sym_n^0$, $m \geq 1$, $2 \leq i, i_1, i_2, \ldots, i_{m+1} \leq n$.

**Lemma 3.1** $k[S_n(Sym_n)] = k\langle X \rangle | S = k\langle X \rangle | \overline{S}$.

**Proof.** For any $s_1, s_2 \in k\langle X \rangle$, we write $s_1 \equiv_I s_2$ if $s_1 - s_2 \in Id(S)$. Since $S \subseteq \overline{S}$, we just have to prove that $\overline{S} \subseteq Id(S)$. It suffices to prove that $s \equiv_I 0$ for any $s \in \overline{S}$.

For $2 \leq i \leq n$, there exist $\sigma_1, \sigma_2 \in Sym_n^0$ such that $X_{\sigma_1} x_i = x_i X_{\sigma_2}$. Therefore

$$x_i X_\varepsilon - X_\varepsilon x_i = (x_{\sigma_1} - x_\varepsilon) x_i - x_i (x_{\sigma_2} - x_\varepsilon) \equiv_I 0.$$
Now we use induction on $m$ to prove that all the polynomials of type $3, 4, 5$ are in $Id(S)$.

(a) For $m = 1$ and $2 \leq i \leq n$, there exist $\sigma_1, \sigma_2 \in Sym^0_i$ such that $x_{\sigma_1} x_i x_1 x_i = x_{i} x_1 x_{\sigma_2}$. Therefore
\[
x_i x_1 x_1 x_i - x_1 x_1 x_i \\
= (x_{\sigma_1} - x_1) x_1 x_i - x_i x_1 (x_{\sigma_2} - x_1) + x_1 (x_{\sigma_1} - x_1) x_i \\
\equiv_I 0,
\]
where $\xi$ is the cyclic permutation defined by (1).

(b) For $m = 1$ and $2 \leq i_2 < i_1 \leq n$, there exist $\sigma_1, \sigma_2 \in Sym^0_n$ such that $x_{\sigma_1} x_{i_2} x_{i_1} = x_{i_1} x_{i_2} x_{\sigma_2}$. Therefore,
\[
x_{i_2} x_{i_1} x_{i_2} x_{i_1} - x_{i_1} x_{i_2} x_{i_1} x_{i_2} \\
= x_{i_2} x_{i_1} x_{i_2} x_{i_1} - x_{i_1} x_{i_2} x_{i_1} x_{i_2} (by\ \text{type}\ 2) \\
\equiv_I (x_{\sigma_1} - x_1) x_{i_2} x_{i_1} - x_{i_1} x_{i_2} (x_{\sigma_2} - x_1) \\
\equiv_I 0.
\]

(c) For $m = 1$ and $2 \leq i_1 \leq n$, since $x_{\sigma_1} x_1 x_1 = x_{i_1} x_1 x_{\sigma_1}$, we have
\[
x_{\sigma_1} x_1 x_1 - x_1 x_1 x_1 \\
= (x_{\sigma_1} x_1 - x_1 x_{\sigma_1}) x_1 + x_1 x_1 x_1 - x_1 x_1 x_1 \\
\equiv_I (x_{\sigma_1} x_1 - x_1 x_{\sigma_1}) x_1 + (x_1 x_1 x_1 - x_1 x_1 x_1) \\
\equiv_I 0\ (by\ (a)\ and\ \text{type}\ 2).
\]

Now we assume that all the polynomials of type $3, 4, 5$ are in $Id(S)$ for $m' = 1 \leq m' < m$.

(i) For $2 \leq i \leq n$, since $x_1 x_{\xi} = x_{\xi} x_1$, we have
\[
x_i x_1 x_1 x_i - x_1 x_1 x_i \\
= x_i x_1 x_1 x_i - x_1 x_1 x_i \\
\equiv_I x_i x_1 x_1 x_i - x_1 x_1 x_i (by\ \text{induction}) \\
\equiv_I x_{i} x_1 x_1 x_i - x_1 x_1 x_i \\
\equiv_I 0.
\]

This shows that all polynomials of type $3$ are in $Id(S)$.

(ii) For $2 \leq i_1, i_2, \cdots, i_{m+1} \leq n$, let
\[
x_{i_1} = \max\{x_{i_1}, x_{i_2}, \cdots, x_{i_{m+1}}\}.
\]

There are two cases to consider.
Case 1: If \( x_{it} = x_{im+1} \), then
\[
X_x x_{i_1} x_{i_2} \cdots x_{i_m} x_{im+1} = X_x x_{i_1} x_{i_2} \cdots x_{im+1} \quad (\text{by induction})
\]
\[
\equiv_I x_{i_1} x_{i_2} \cdots x_{im+1} x_{im+1} - x_{i_1} x_{i_2} \cdots x_{im+1} \quad (\text{by induction})
\]
\[
\equiv_I 0.
\]

Case 2: If \( x_{it} \neq x_{im+1} \), then by induction, we have
\[
x_x x_{i_1} \cdots x_{i_t} \cdots x_{i_m} x_{im+1} \equiv_I x_{i_1} \cdots x_{i_{t-1}} x_x x_{i_t} \cdots x_{i_m} x_{im+1} - x_x x_{i_1} \cdots x_{i_t} \cdots x_{i_m} x_{im+1} \quad (\text{by type 2})
\]
\[
\equiv_I x_{i_1} \cdots x_{i_{t-1}} x_x x_{i_t} \cdots x_{i_m+1} x_{i_t} - x_x x_{i_1} \cdots x_{i_t} \cdots x_{i_m} x_{im+1} \quad (\text{by induction})
\]
\[
\equiv_I x_x x_{i_1} \cdots x_{i_{t-1}} x_{i_{t+1}} \cdots x_{i_m+1} x_{i_t} - x_x x_{i_1} \cdots x_{i_t} \cdots x_{i_m} x_{im+1}
\]
\[
\equiv_I 0.
\]

This shows that all polynomials of type 4 are in \( Id(S) \).

(iii) For \( 2 \leq i_1, i_2, \ldots, i_m \leq n \), we have,
\[
x_x x_{i_1} x_{i_2} \cdots x_{im} x_{im+1} - x_{i_1} x_x x_{i_1} x_{i_2} \cdots x_{im} \quad (\text{by type 2})
\]
\[
\equiv_I x_{i_1} x_x x_{i_1} x_{i_2} \cdots x_{im+1} x_{i_1} - x_x x_{i_1} x_{i_2} \cdots x_{im} \quad (\text{by induction})
\]
\[
\equiv_I x_{i_1} x_x x_{i_1} x_{i_2} \cdots x_{im} x_{i_1} - x_{i_1} x_x x_{i_1} x_{i_2} \cdots x_{im} \quad (\text{by type 3})
\]
\[
\equiv_I 0.
\]

This shows that all polynomials of type 5 are in \( Id(S) \).

The proof is complete.

The following theorem is the main result in this paper.

**Theorem 3.2** With the degree-lexicographic order on \( X^* \), \( \tilde{S} \) is a Gröbner-Shirshov basis in \( k(X) \).

**Proof.** Let \( f_i \) or \( f_i' \) be the polynomial of type \( i \) in \( \tilde{S}, \ i = 1, 2, \ldots, 5 \) and \( \sigma, \sigma' \in \text{Sym}_0 \).

Denote \( i \wedge j \) the composition of the polynomials of type \( i \) and type \( j \).

All possible compositions of the polynomials in \( S \) are only as below:

\[
1 \wedge 1, f_1 = x_{i_1} \quad 1 \wedge 2, f_1' = x_{i_1}, f_1'' = x_{i_1}, w = x_{i_1} x_{i_2} \cdots x_{i_r} x_{i_{r+1}} x_{i_{r+2}} \cdots x_{i_{n-r}}, x_{i_1} x_{i_2} \cdots x_{i_r} \Delta, x_{i_1} = x_{i_1} x_{i_2} \cdots x_{i_r}, x_{i_1} = x_{i_1} x_{i_2} \cdots x_{i_r}, x_{i_1} \in \text{Sym}_r, 1 \leq r < n.
\]
1 \wedge 2, f_1 = x_\sigma - x_\tau, f_2 = x_i x_\tau - x_i x_i, w = x_{i_1} \cdots x_i x_{i_1}, x_\sigma = x_{i_1} \cdots x_i x_{i_1}, x_{i_1} x_{i_2} \cdots x_{i_{n-1}} x_{i_1} x_{i_2} + 1, \{i_1, i_2, \ldots, i_t\} = [n-t, n] \setminus \{i\}, 0 \leq n - t - 1 < i, 2 \leq i \leq n.

1 \wedge 3, there are two cases. Let f_1 = x_\sigma - x_\tau and f_3 = x_i x_1^m x_\tau - x_i x_1^m x_i.

w_1 = x_\sigma x_1^m x_\tau, x_\sigma = x_{i_1} x_{i_2} \cdots x_{i_{n-2}} x_{i_1} x_{i_2}, \{i_1, i_2, \ldots, i_{n-2}\} = n \setminus \{i\}, m \geq 1, 2 \leq i \leq n.

w_2 = x_\sigma x_1^m x_\tau, x_\sigma = x_i x_{i_2} \cdots x_{i_{n-1}} x_i, \{i_1, i_2, \ldots, i_{n-1}\} = n \setminus \{i\}, m \geq 1, 2 \leq i \leq n.

1 \wedge 4, f_1 = x_\sigma - x_\tau, f_4 = x_i x_{i_1} x_{i_2} \cdots x_{i_{m+1}} - x_i x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, w = x_{i_1} x_{i_2} \cdots x_i x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, x_\sigma = x_{i_1} x_{i_2} \cdots x_i x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, \{j_1, j_2, \ldots, j_t\} = \{n - t + 1, n\}, 2 \leq i_1, i_2, \ldots, i_{m+1} \leq n, m \geq 1, 1 \leq t \leq n - 1, x_{i_1} x_{i_2} \cdots x_{i_{m+1}} > x_{i_1} x_{i_2} \cdots x_{i_{m+1}}.

2 \wedge 1, f_2 = x_i x_\tau - x_{i_1} x_i, f_1 = x_\sigma - x_\tau, w = x_i x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, x_\sigma = x_{i_1} x_{i_2} \cdots x_i x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, \{j_1, j_2, \ldots, j_t\} = \{n - t + 1, n\}, 2 \leq i_1, i_2, \ldots, i_{m+1} \leq n, m \geq 1, 1 \leq t \leq n - 1.

2 \wedge 2, f_2 = x_i x_\tau - x_{i_1} x_{i_1}, f_2 = x_n x_\tau - x_{i_1} x_{i_1}, w = x_i x_\tau, 2 \leq i \leq n.

2 \wedge 3, f_2 = x_i x_\tau - x_{i_1} x_{i_1}, f_3 = x_n x_1^m x_\tau - x_1^m x_\tau, w = x_i x_\tau, 2 \leq i \leq n, m \geq 1.

2 \wedge 4, f_4 = x_i x_{i_1} x_{i_2} \cdots x_{i_{m+1}} - x_i x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, w = x_{i_1} x_{i_2} \cdots x_i x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, m \geq 1, 2 \leq i, i_1, i_2, \ldots, i_{m+1} \leq n, x_{i_1} x_{i_2} \cdots x_{i_{m+1}} > x_{i_1} x_{i_2} \cdots x_{i_{m+1}}.

2 \wedge 5, f_2 = x_i x_\tau - x_{i_1} x_{i_1}, f_5 = x_i x_{i_1} x_{i_2} \cdots x_{i_{m+1}} - x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, w = x_{i_1} x_{i_2} \cdots x_i x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, m \geq 1, 2 \leq i, i_1, i_2, \ldots, i_{m+1} \leq n.

3 \wedge 1, f_3 = x_i x_1^m x_\tau - x_1^m x_\tau, f_1 = x_\sigma - x_\tau, w = x_i x_1^m x_\tau, x_\sigma = x_{i_1} x_{i_2} \cdots x_i x_{i_1} \cdots x_{i_{m+1}}, \{i_1, i_2, \ldots, i_t\} = [1, t], 1 \leq t \leq n - 1, 2 \leq i \leq n, m \geq 1.

3 \wedge 2, f_3 = x_i x_1^m x_\tau - x_1^m x_\tau, f_2 = x_n x_\tau - x_{i_1} x_{i_1}, x_\tau = x_i x_1^m x_\tau, 2 \leq i \leq n, m \geq 1.

3 \wedge 3, f_3 = x_i x_1^m x_\tau - x_1^m x_\tau, f_3 = x_n x_1^m x_\tau - x_1^m x_\tau, w = x_i x_1^m x_\tau, 2 \leq i \leq n, m, m_1 \geq 1.

3 \wedge 4, f_3 = x_i x_1^m x_\tau - x_1^m x_\tau, f_4 = x_i x_{i_1} x_{i_2} \cdots x_{i_{m+1}} - x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, w = x_i x_1^m x_\tau, 2 \leq i, i_1, i_2, \ldots, i_{m+1} \leq n, m, m_1 \geq 1, x_{i_1} x_{i_2} \cdots x_{i_{m+1}} >
$x_{i_1} \cdots x_{i_{m+1}}$

$3 \wedge 5, f_3 = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}} x_1 - x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, w = x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, 2 \leq i_1, i_2, \ldots, i_{m+1} \leq n, m, m_1 \geq 1.$

$4 \wedge 1, \text{there are two cases. Let } f_4 = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}} - x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, f_1 = x_x - x_x, m \geq 1, 2 \leq i_1, i_2, \ldots, i_{m+1} \leq n, x_{i_1} x_{i_2} \cdots x_{i_{m+1}} > x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}},$

$w = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, 2 \leq i_1, i_2, \ldots, i_{m+1} \leq n, \{j_1, j_2, \ldots, j_{t-1}\} = n \backslash \{t+1, n\} \cup \{i_1, i_2, \ldots, i_{m+1}\}, t - m - 1 \geq 1.$

$4 \wedge 2, f_4 = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}} - x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, f_2 = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, w = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, m \geq 1, 2 \leq i_1, i_2, \ldots, i_{m+1} \leq n, x_{i_1} x_{i_2} \cdots x_{i_{m+1}} > x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}.$

$4 \wedge 3, f_4 = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}} - x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, f_3 = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, w = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, 2 \leq i_1, i_2, \ldots, i_{m+1} \leq n, m, m_1 \geq 1,$

$x_{i_1} x_{i_2} \cdots x_{i_{m+1}} > x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}.$

$5 \wedge 1, \text{there are two cases. Let } f_5 = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}} x_1 - x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, f_1 = x_x - x_x, 2 \leq i_1, i_2, \ldots, i_{m+1} \leq n, m \geq 1.$

$w = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, 2 \leq i_1, i_2, \ldots, i_{m+1} \leq n, x_{i_1} x_{i_2} \cdots x_{i_{m+1}} > x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}.$

$5 \wedge 2, f_5 = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}} x_1 - x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, f_2 = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, w = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, 2 \leq i_1, i_2, \ldots, i_{m+1} \leq n, m \geq 1.$

$5 \wedge 3, f_5 = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}} x_1 - x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, f_3 = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, w = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, 2 \leq i_1, i_2, \ldots, i_{m+1} \leq n, m, m_1 \geq 1.$

$5 \wedge 4, f_5 = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}} x_1 - x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, f_4 = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, w = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, 2 \leq i_1, i_2, \ldots, i_{m+1}, j_1, j_2, \ldots, j_{m+1} \leq n, m, m_1 \geq 1.$

$5 \wedge 5, f_5 = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}} x_1 - x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, f_5' = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, w = x_x x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, 2 \leq i_1, i_2, \ldots, i_{m+1}, j_1, j_2, \ldots, j_{m+1} \leq n, m, m_1 \leq n.$

We prove that all the above compositions are trivial. Here, we just check $1 \wedge 1, 1 \wedge 4, 2 \wedge 5$. Others are similarly proved.
For $1 \wedge 1$, there are two cases to consider.

Case 1: If $1 \notin \{i_1, i_2, \ldots, i_t\}$, then

\[
1 \wedge 1 = f_1 x_{t_{s(1)}} x_{t_{s(2)}} \cdots x_{t_{s(r)}} - x_{i_1} x_{i_2} \cdots x_{i_t} f'_1
\]
\[
= -x_r x_{t_{s(1)}} x_{t_{s(2)}} \cdots x_{t_{s(r)}} + x_1 x_{i_1} x_{i_2} \cdots x_{i_t} x_r
\]
\[
= -x_r x_{t_{s(1)}} x_{t_{s(2)}} \cdots x_{t_{s(r)}} + x_r x_{i_1} x_{i_2} \cdots x_{i_t} \quad \text{(by type 2)}
\]
\[
= -x_r x_{t_{s(1)}} x_{t_{s(2)}} \cdots x_{t_{s(r)}} + x_r x_{i_1} x_{i_2} \cdots x_{i_t} \quad \text{(by type 4)}
\]
\[
= 0 \mod \langle \tilde{S}, w \rangle,
\]

Case 2: If $1 \in \{i_1, i_2, \ldots, i_r\}$, say, $x_1 = x_{i_1} = x_{i_{s(t)}}, 1 \leq s, t \leq r$, then by type 5 and 4, we have

\[
1 \wedge 1 = f_1 x_{t_{s(1)}} x_{t_{s(2)}} \cdots x_{t_{s(r)}} - x_{i_1} x_{i_2} \cdots x_{i_t} f'_1
\]
\[
= -x_r x_{t_{s(1)}} x_{t_{s(2)}} \cdots x_{t_{s(r)}} + x_1 x_{i_1} x_{i_2} \cdots x_{i_t} x_r
\]
\[
= -x_r x_{t_{s(1)}} x_{t_{s(2)}} \cdots x_{t_{s(r)}} + x_r x_{i_1} x_{i_2} \cdots x_{i_t} \quad \text{(by type 4)}
\]
\[
= -x_r x_{t_{s(1)}} x_{t_{s(2)}} \cdots x_{t_{s(r)}} + x_r x_{i_1} x_{i_2} \cdots x_{i_t} \quad \text{(by type 4)}
\]
\[
= 0 \mod \langle \tilde{S}, w \rangle.
\]

\[
1 \wedge 4 = f_1 x_{n-t+1} x_{n-t+2} \cdots x_n x_{i_1} x_{i_2} \cdots x_{i_{m+1}} - x_{j_1} x_{j_2} \cdots x_{j_{t} f'_1}
\]
\[
= -x_r x_{n-t+1} x_{n-t+2} \cdots x_n x_{i_1} x_{i_2} \cdots x_{i_{m+1}} + x_{j_1} x_{j_2} \cdots x_{j_t} x_r
\]
\[
= -x_r x_{n-t+1} x_{n-t+2} \cdots x_n x_{i_1} x_{i_2} \cdots x_{i_{m+1}} + x_r x_{j_1} x_{j_2} \cdots x_{j_t} x_r
\]
\[
= -x_r x_{n-t+1} x_{n-t+2} \cdots x_n x_{i_1} x_{i_2} \cdots x_{i_{m+1}} + x_r x_{j_1} x_{j_2} \cdots x_{j_t} x_r
\]
\[
= 0 \mod \langle \tilde{S}, w \rangle.
\]

\[
2 \wedge 5 = f_2 x_{x_1} x_{x_2} \cdots x_{x_m} x_1 - x_i f_5
\]
\[
= x_r x_{x_1} x_{x_2} \cdots x_{x_m} x_1 - x_i x_1 x_r x_{x_1} x_{x_2} \cdots x_{x_m}
\]
\[
= x_r x_{x_1} x_{x_2} \cdots x_{x_m} x_1 - x_1 x_r x_{x_1} x_{x_2} \cdots x_{x_m}
\]
\[
= x_1 x_r x_{x_1} x_{x_2} \cdots x_{x_m} - x_1 x_r x_{x_1} x_{x_2} \cdots x_{x_m}
\]
\[
= \mod \langle \tilde{S}, w \rangle.
\]

The proof is complete.

By the Composition-Diamond Lemma and Theorem 3.2, we have the following corollary.

**Corollary 3.3** The set

\[
Irr(\tilde{S}) = (X^* \setminus \bigcup_{\sigma \in Sym_n} X^* \{x_\sigma\} X^*) \bigcup \{x_1^{m_1} x_r x_2^{m_2} \cdots x_n^{m_n} | m_i \geq 0, i = 1, 2, \ldots, n\}
\]

is a $k$-linear basis of algebra $k[Sym_n]$. Moreover, $Irr(\tilde{S})$ is normal forms of elements of monoid $S_n(Sym_n)$. 

9
4 Appendix

In this section, we will check that all the compositions are trivial.

1 \land 1, \ f_1 = x_\sigma - x_\varepsilon, f'_1 = x_\sigma' - x_\varepsilon, \ w = x_{i_1} x_{i_2} \cdots x_{i_t}, \Delta x_{i_1} x_{i_2} \cdots x_{i_{n(r)}}, x_{\sigma'} = x_{i_1} x_{i_2} \cdots x_{i_r} \Delta, x_{\sigma'} = \Delta x_{i_1} x_{i_2} \cdots x_{i_{n(r)}}, \pi \in Sym_r, 1 \leq r < n.

For 1 \land 1, there are two cases to consider.

Case 1: If 1 \notin \{i_1, i_2, \ldots, i_r\}, then

1 \land 1 = \ f_1 x_{i_1} x_{i_2} \cdots x_{i_{n(r)}} - x_{i_1} x_{i_2} \cdots x_{i_r} f'_1
= -\Delta x_{i_1} x_{i_2} \cdots x_{i_{n(r)}} + x_{i_1} x_{i_2} \cdots x_{i_r} (by type 2)
= 0 \mod(\mathcal{S}, w).

Case 2: If 1 \in \{i_1, i_2, \ldots, i_r\}, say, x_1 = x_{i_1} = x_{i_{n(s)}}, 1 \leq s, t \leq r, then by type 5 and 4, we have

1 \land 1 = \ f_1 x_{i_1} x_{i_2} \cdots x_{i_{n(r)}} - x_{i_1} x_{i_2} \cdots x_{i_r} f'_1
= -\Delta x_{i_1} x_{i_2} \cdots x_{i_{n(r)}} + x_{i_1} x_{i_2} \cdots x_{i_r} (by type 4)
= 0 \mod(\mathcal{S}, w).

1 \land 2 = \ f_1 x_{n-t} \cdots x_n - x_{i_1} x_{i_2} \cdots x_{i_t} f_2
= x_{n-t} x_{n-t} \cdots x_n - x_{i_1} x_{i_2} \cdots x_{i_t} x_{i_1} x_{i_2} x_{i_1}
= x_{i_1} x_{i_2} \cdots x_{i_n-t} - x_{i_1} x_{i_2} \cdots x_{i_t} x_{i_1} x_{i_2}
= 0 \mod(\mathcal{S}, w)

1 \land 3, \ w_1 = x_{\sigma'} x_{n}^{m-1} x_\varepsilon, x_{\sigma} = x_{i_1} x_{i_2} \cdots x_{i_{n-2}} x_{i_1} x_{1}, \{i_1, i_2, \ldots, i_{n-2}\} = n \setminus \{i, 1\}, m \geq 1, 2 \leq i \leq n.

w_2 = x_{\sigma} x_{1}^{m} x_\varepsilon, x_{\sigma} = x_{i_1} x_{i_2} \cdots x_{i_{n-2}} x_{i_1}, \{i_1, i_2, \ldots, i_{n-1}\} = n \setminus \{i\}, m \geq 1, 2 \leq i \leq n.

1 \land 3 = \ f_1 x_{i_1}^{m-1} x_\varepsilon - x_{i_1} x_{i_2} \cdots x_{i_{n-2}} f_3
\[ x_1^{m-1}x_2 - x_1x_2 \cdots x_{i+n-2}x_1^m x_1 \]

\[ x_1^{m-1}x_2 - x_1x_2 \cdots x_{i+n-2}x_1 \]

\[ x_1^{m-1}x_2 \cdots x_n - x_1^{m+1}x_2 \cdots x_{n-2}x_1 \]

\[ x_1^{m+1}x_2 \cdots x_n - x_1^{m+1}x_2 \cdots x_n \]

\[ 0 \text{ mod}(S, w_1) \]

\[ 1 \land 3 = f_1x_1^m x_2 - x_1x_{i_2} \cdots x_{i_{n-1}}^f \]

\[ x_1^{m}x_2 - x_1x_2 \cdots x_{i_{n-1}}^x \]

\[ x_1^{m}x_2 - x_1x_2 \cdots x_{i_{n-1}}^x \]

\[ x_1^{m+1}x_2 \cdots x_n - x_1^{m+1}x_2 \cdots x_{n-2}x_1 \]

\[ x_1^{m+1}x_2 \cdots x_n - x_1^{m+1}x_2 \cdots x_n \]

\[ 0 \text{ mod}(S, w_2) \]

where \( \{i_1', i_2', \ldots, i_{n-2}'\} = [2, n] \backslash \{i\} \).

\[ 1 \land 4, f_1 = x_\sigma - x_e, f_4 = x_e x_1x_2 \cdots x_{i_{m+1}} - x_1^e x_1x_2 \cdots x_{i_{m+1}}, w = x_j x_j' \cdots x_{j_m} x_{i_2} \cdots x_{i_{m+1}}, x_\sigma = x_j x_j' \cdots x_{j_{m+1}} x_{i_2} \cdots x_{i_{m+1}}, n \in [n-t+1, n], 2 \leq i_1, i_2, \ldots, i_{m+1} \leq n, m \geq 1, 1 \leq t \leq n-1, x_{i_1}x_{i_2} \cdots x_{i_{m+1}} > x_j x_j' \cdots x_{j_{m+1}}. \]

\[ 1 \land 4 = f_1 x_{n-t+1} \cdots x_n x_{i_1} x_{i_2} \cdots x_{i_{m+1}} - x_j x_j' \cdots x_{j_{m+1}} \]

\[ 1 \land 4 = f_1 x_{n-t+1} \cdots x_n x_{i_1} x_{i_2} \cdots x_{i_{m+1}} - x_j x_j' \cdots x_{j_{m+1}} \]

\[ 1 \land 5, f_1 = x_\sigma - x_e, f_5 = x_e x_1x_2 \cdots x_{i_{m+1}} - x_1^e x_1x_2 \cdots x_{i_{m+1}}, w = x_j x_j' \cdots x_{j_m} x_{i_2} \cdots x_{i_{m+1}}, x_\sigma = x_j x_j' \cdots x_{j_{m+1}} x_{i_2} \cdots x_{i_{m+1}}, n \in [n-t+1, n], 2 \leq i_1, i_2, \ldots, i_{m+1} \leq n, m \geq 1, 1 \leq t \leq n-1. \]

\[ 1 \land 5 = f_1 x_{n-t+1} \cdots x_n x_{i_1} x_{i_2} \cdots x_{i_{m+1}} - x_j x_j' \cdots x_{j_{m+1}} \]

\[ 2 \land 1, f_2 = x_i x_e - x_{e_i} x_i, f_1 = x_\sigma - x_e, w = x_j x_j' \cdots x_{j_{m}} x_{i_2} \cdots x_{j_{m+1}}, x_\sigma = x_{j_{t+1}} x_{j_1} x_{j_2} \cdots x_{j_{m}}, n \in [t+1, n], 2 \leq i \leq n, 1 \leq t \leq n-1. \]
\[ f_2 = x_i x_e - x_e x_i, \] 
\[ f'_2 = x_n x_e - x_e x_n, \] 
\[ w = x_i x_e x_i, \] 
\[ 2 \leq i \leq n. \]

\[ 2 \land 2, \] 
\[ f_2 = x_i x_e - x_e x_i, \] 
\[ f'_2 = x_n x_e - x_e x_n, \] 
\[ w = x_i x_e x_i, \] 
\[ 2 \leq i \leq n. \]

\[ 2 \land 3, \] 
\[ f_2 = x_i x_e - x_e x_i, \] 
\[ f'_3 = x_n x_e x_1 - x_1 x_n, \] 
\[ w = x_i x_e x_i, \] 
\[ 2 \leq i \leq n, \] 
\[ m \geq 1. \]

\[ 2 \land 4, \] 
\[ f_2 = x_i x_e - x_e x_i, \] 
\[ f'_4 = x_e x_i x_1 x_2 \cdots x_{i+1} - x_1 x_2 \cdots x_{i+1}, \] 
\[ w = x_i x_e x_i x_1 x_2 \cdots x_{i+1}, \] 
\[ m \geq 1, \] 
\[ 2 \leq i, i_1, i_2, \ldots, i_{m+1} \leq n, \] 
\[ x_i x_1 x_2 \cdots x_{i+1}. \]}
\[ 3 \land 1, \ f_3 = x_i x_1^m x_e - x_1^m x_i x_i, \ f_1 = x_\sigma - x_e, \ w = x_i x_1^m x_e x_i \cdots x_i, \ x_\sigma = x_{t+1} x_{t+2} \cdots x_n x_i \cdots x_i, \{ i_1, i_2, \ldots, i_l \} = [1, t], 1 \leq t \leq n-1, 2 \leq i \leq n, m \geq 1. \]

\[
3 \land 1 \quad = \quad f_3 x_{i_1} \cdots x_{i_i} - x_i x_1^m x_1 \cdots x_i f_1 \\
\equiv x_1^m x_e x_{i_1} x_{i_1} \cdots x_{i_i} - x_i x_1^m x_1 \cdots x_i x_e \\
\equiv x_1^{m+1} x_e x_{i_1} x_{i_1} \cdots x_{i_i} - x_1^m x_e x_{i_1+2} \cdots x_i \\
\equiv 0 \ mod(\mathcal{S}, w).
\]

where \( \{ i_2', i_3', \ldots, i_l' \} = [2, t] \).

\[ 3 \land 2, \ f_3 = x_i x_1^m x_e - x_1^m x_i x_i, \ f_2 = x_n x_e - x_e x_n, \ w = x_i x_1^m x_e x_\sigma, 2 \leq i \leq n, \ m \geq 1. \]

\[
3 \land 2 \quad = \quad f_3 x_e - x_i x_1^m x_1 \cdots x_{n-1} f_2 \\
\equiv x_1^m x_e x_i x_e - x_i x_1^m x_1 \cdots x_{n-1} x_e x_n \\
\equiv x_1^{m+1} x_e x_i x_2 \cdots x_n - x_1^m x_e x_2 \cdots x_n \\
\equiv 0 \ mod(\mathcal{S}, w).
\]

\[ 3 \land 3, \ f_3 = x_i x_1^m x_e - x_1^m x_i x_i, \ f_3' = x_n x_1^m x_e - x_1^m x_i x_n, \ w = x_i x_1^m x_e x_1^m x_e, \ 2 \leq i \leq n, \ m, m_1 \geq 1. \]

\[
3 \land 3 \quad = \quad f_3 x_1^m x_e - x_i x_1^m x_1 \cdots x_{n-1} f_3' \\
\equiv x_1^m x_e x_i x_1^m x_e - x_i x_1^m x_1 \cdots x_{n-1} x_1^m x_e x_n \\
\equiv x_1^{m+m+1} x_e x_i x_2 \cdots x_n - x_1^m x_1^{m+1} x_e x_i x_2 \cdots x_n \\
\equiv 0 \ mod(\mathcal{S}, w).
\]

\[ 3 \land 4, \ f_3 = x_i x_1^m x_e - x_1^m x_i x_i, \ f_4 = x_e x_{i_1} \cdots x_{i_{m+1}}, \ x_e x_{i_1} \cdots x_{i_{m+1}}, \ w = x_i x_1^m x_e x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, \ 2 \leq i, i_1, i_2, \ldots, i_{m+1} \leq n, \ m, m_1 \geq 1, x_{i_1} \cdots x_{i_{m+1}} > x_{i_1} \cdots x_{i_{m+1}}. \]

\[
3 \land 4 \quad = \quad f_3 x_{i_1} \cdots x_{i_{m+1}} - x_i x_1^m f_4 \\
\equiv x_1^m x_e x_{i_1} x_{i_1} \cdots x_{i_{m+1}} - x_i x_1^m x_e x_{i_1} \cdots x_{i_{m+1}} \\
\equiv x_1^m x_e x_{i_1} x_{i_1} \cdots x_{i_{m+1}} - x_1^m x_e x_{i_1} x_{i_1} \cdots x_{i_{m+1}} \\
\equiv 0 \ mod(\mathcal{S}, w).
\]

\[ 3 \land 5, \ f_3 = x_i x_1^m x_e - x_1^m x_i x_i, \ f_5 = x_e x_{i_1} x_{i_2} \cdots x_{i_{m+1}} - x_1 x_i x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, \ w = x_i x_1^m x_e x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, \ 2 \leq i, i_1, i_2, \ldots, i_{m+1} \leq n, \ m, m_1 \geq 1. \]

\[
3 \land 5 \quad = \quad f_3 x_{i_1} x_{i_2} \cdots x_{i_{m+1}} x_1 - x_i x_1^m f_5
\]

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\[ f_4 = x_{i_1} x_{i_2} \cdots x_{i_{m+1}} - x_{i_1} x_{i_2} \cdots x_{i_{m+1}}. \]

4 \land 1, there are two cases. Let

\[ f_4 = x_{i_1} x_{i_2} \cdots x_{i_{m+1}} - x_{i_1} x_{i_2} \cdots x_{i_{m+1}}, \]
\[ f_1 = x_{i_1} - x_{i_2}, \quad m \geq 1, \quad 2 \leq i_1, i_2, \ldots, i_{m+1} \leq n, \quad x_{i_1} x_{i_2} \cdots x_{i_{m+1}} > x_{i_1} x_{i_2} \cdots x_{i_{m+1}}. \]

\[ w_1 = x_{i_1} x_{i_2} \cdots x_{i_{m+1}} + x_{i_1} x_{i_2} \cdots x_{i_{m+1}} = x_{i_1} x_{i_2} \cdots x_{i_{m+1}}. \]
\[ \{j_1, j_2, \ldots, j_t\} = n \\{\{i_{m+2-n+1}, \ldots, i_{m+1}\}. \]

\[ w_2 = x_{i_1} x_{i_2} \cdots x_{i_{m+1}} + x_{i_1} x_{i_2} \cdots x_{i_{m+1}} = x_{i_1} x_{i_2} \cdots x_{i_{m+1}}. \]

\[ 2 \leq i_1, i_2, \ldots, i_{m+1} \leq n, \quad \{j_1, j_2, \ldots, j_t\} = n \\{\{t+1, n\} \cup \{i_1, i_2, \ldots, i_{m+1}\}. \]

\[ t - m - 1 \geq 1. \]

\[ 4 \land 1 = \left\{ f_4 x_{j_1} x_{j_2} \cdots x_{j_t} - x_{i_1} x_{i_2} \cdots x_{i_{m+1-n}+1} \right\}. \]

\[ = x_{i_1} x_{i_2} \cdots x_{i_{m+1-n}+1} x_{i_1} - x_{i_1} x_{i_2} \cdots x_{i_{m+1-n}+1} x_{i_1} \]

\[ = 0 \mod (S, w_1). \]

where \( \{j_1, j_2, \ldots, j_t\} = n \\{\{i_{m+2-n+1}, \ldots, i_{m+1}\}. \]

\[ 4 \land 1 = \left\{ f_4 x_{j_1} x_{j_2} \cdots x_{j_t} - x_{i_1} x_{i_2} \cdots x_{i_{m+1-n}+1} \right\}. \]

\[ = x_{i_1} x_{i_2} \cdots x_{i_{m+1-n}+1} x_{i_1} - x_{i_1} x_{i_2} \cdots x_{i_{m+1-n}+1} x_{i_1} \]

\[ = 0 \mod (S, w_1). \]

where \( \{j_1, j_2, \ldots, j_t\} = \{2, 4\} \cup \{i_1, i_2, \ldots, i_{m+1}\}. \]

\[ 4 \land 2 = \left\{ f_4 x_{j_1} x_{j_2} \cdots x_{j_t} - x_{i_1} x_{i_2} \cdots x_{i_{m+1}} \right\}. \]

\[ = x_{i_1} x_{i_2} \cdots x_{i_{m+1}} x_{i_1} - x_{i_1} x_{i_2} \cdots x_{i_{m+1}} x_{i_1} \]

\[ = 0 \mod (S, w). \]

\[ 4 \land 3, \quad f_4 = x_{i_1} x_{i_2} \cdots x_{i_{m+1-n}} - x_{i_1} x_{i_2} \cdots x_{i_{m+1-n}}. \]

\[ \frac{f_3}{x_{i_1} x_{i_2} \cdots x_{i_{m+1-n}} + x_{i_1} x_{i_2} \cdots x_{i_{m+1-n}} x_{i_1} x_{i_2} \cdots x_{i_{m+1-n}}} = \frac{0}{x_{i_1} x_{i_2} \cdots x_{i_{m+1-n}}}. \]

\[ 4 \land 3 = \frac{f_4 x_{i_1} x_{i_2} \cdots x_{i_{m+1-n}} - x_{i_1} x_{i_2} \cdots x_{i_{m+1-n}}}{x_{i_1} x_{i_2} \cdots x_{i_{m+1-n}}}. \]

\[ = 0 \mod (S, w). \]

\[ 4 \land 3 = \frac{f_4 x_{i_1} x_{i_2} \cdots x_{i_{m+1-n}} - x_{i_1} x_{i_2} \cdots x_{i_{m+1-n}}}{x_{i_1} x_{i_2} \cdots x_{i_{m+1-n}}}. \]
\[ f_5 = x_\varepsilon x_{i_1} x_{i_2} \cdots x_{i_m} x_1 - x_1 x_{i_1} x_{i_2} \cdots x_{i_m}, \]
\[ f_1 = x_\sigma - x_\varepsilon, \quad 2 \leq i_1, i_2, \cdots, i_m \leq n, \ m \geq 1. \]

\[ w_1 = x_\varepsilon x_{i_1} \cdots x_{i_m} x_1 x_{j_1} \cdots x_{j_{n+1-m-2}}, \ x_\sigma = x_{i_1} x_{i_2} \cdots x_{i_m} x_1 x_{j_1} \cdots x_{j_{n+1-m-2}}, \]
\[ 2 \leq i_1, i_2, \cdots, i_m \leq n, \ \{j_1, j_2, \ldots, j_{n+1-m-2}\} = n\{\{i_1, i_1+1, \ldots, i_m\} \cup \{1\}\}. \]

\[ w_2 = x_\varepsilon x_{i_1} \cdots x_{i_m} x_1 x_{j_1} x_{j_2} \cdots x_{i_m} x_1 x_{j_1} \cdots x_{j_{n+1-m}}, \]
\[ 2 \leq i_1, i_2, \cdots, i_m \leq n, \ \{j_1, j_2, \ldots, j_{n+1-m}\} = n\{\{i_1, i_2, \ldots, i_m\} \cup [t+1, n] \cup \{1\}\}, \]
\[ 1 \leq m \leq n-2, \ t-m \geq 0. \]

\[ 5 \land 1 \]
\[ f_5 x_{j_1} \cdots x_{j_{n+1-m-2}} - x_\varepsilon x_{i_1} x_{i_2} \cdots x_{i_{n-1}} f_1 \]
\[ = x_1 x_\varepsilon x_{i_1} x_{i_2} \cdots x_{i_m} x_{j_1} \cdots x_{j_{n+1-m-2}} - x_\varepsilon x_{i_1} \cdots x_{i_{n-1}} x_\varepsilon \]
\[ = x_1 x_\varepsilon x_{i_1} x_{i_2} \cdots x_{i_m} x_{j_1} \cdots x_{j_{n+1-m-2}} - x_1 x_\varepsilon x_{i_1} \cdots x_{i_{n-1}} x_\varepsilon \]
\[ = x_1 x_\varepsilon x_{i_1} x_{i_2} \cdots x_{i_m} x_{j_1} \cdots x_{j_{n+1-m-2}} - x_1 x_\varepsilon x_{i_1} \cdots x_{i_{n-1}} x_\varepsilon \]
\[ = 0 \text{ mod}(\tilde{S}, w_1). \]

\[ 5 \land 1 \]
\[ f_5 x_{j_1} \cdots x_{j_{n+1-m}} - x_1 x_2 \cdots x_t f_1 \]
\[ = x_1 x_\varepsilon x_{i_1} x_{i_2} \cdots x_{i_m} x_{j_1} \cdots x_{j_{n+1-m}} - x_1 x_2 \cdots x_t x_\varepsilon \]
\[ = x_1 x_\varepsilon x_{i_1} x_{i_2} \cdots x_{i_m} x_{j_1} \cdots x_{j_{n+1-m}} - x_1 x_\varepsilon x_{i_1} \cdots x_{i_{n-1}} x_t \]
\[ = 0 \text{ mod}(\tilde{S}, w_2). \]

\[ 5 \land 2 \]
\[ f_5 = x_\varepsilon x_{i_1} x_{i_2} \cdots x_{i_m} x_1 - x_1 x_\varepsilon x_{i_1} x_{i_2} \cdots x_{i_m}, \ f_2 = x_{i_m} x_\varepsilon - x_\varepsilon x_{i_m} \]
\[ w = x_\varepsilon x_{i_1} x_{i_2} \cdots x_{i_m} x_\sigma, \ 2 \leq i_1, i_2, \cdots, i_m \leq n, \ m \geq 1. \]

\[ 5 \land 2 \]
\[ f_5 x_2 \cdots x_n - x_\varepsilon x_{i_1} \cdots x_{i_{m-1}} f_2 \]
\[ = x_1 x_\varepsilon x_{i_1} x_{i_2} \cdots x_{i_m} x_2 \cdots x_n - x_\varepsilon x_{i_1} \cdots x_{i_{m-1}} x_\varepsilon x_{i_m} \]
\[ = x_1 x_\varepsilon x_{i_1} x_{i_2} \cdots x_{i_m} x_2 \cdots x_n - x_1 x_\varepsilon x_{i_1} \cdots x_{i_{m-1}} x_2 \cdots x_{i_m} \]
\[ = x_1 x_\varepsilon x_{i_1} x_{i_2} \cdots x_{i_m} x_2 \cdots x_n - x_1 x_\varepsilon x_{i_1} \cdots x_{i_{m-1}} x_2 \cdots x_{i_m} \]
\[ = 0 \text{ mod}(\tilde{S}, w). \]

\[ 5 \land 3 \]
\[ f_5 = x_\varepsilon x_{i_1} x_{i_2} \cdots x_{i_m} x_1 - x_1 x_\varepsilon x_{i_1} x_{i_2} \cdots x_{i_m}, \ f_4 = x_{i_m}^{m_1} x_\varepsilon - x_1^{m_1} x_\varepsilon x_{i_m} \]
\[ w = x_\varepsilon x_{i_1} x_{i_2} \cdots x_{i_m} x_\varepsilon, \ 2 \leq i_1, i_2, \cdots, i_m \leq n, \ m, m_1 \geq 1. \]
\[5 \land 3 = f_5 x_1^{m_1-1} x_e - x_e x_{i_1} \cdots x_{i_{m-1}} f_3\]
\[= x_1 x_e x_{i_1} x_{i_2} \cdots x_{i_{m-1}} x_e - x_e x_{i_1} \cdots x_{i_{m-1}} x_1^{m_1} x_e\]
\[= x_1^{m_1+1} x_e x_{i_1} \cdots x_{i_m} x_2 \cdots x_n - x_1^{m_1+1} x_e x_{i_1} \cdots x_{i_m} x_2 \cdots x_n\]
\[= 0 \mod (S, w).\]

\[5 \land 4, f_5 = x_e x_{i_1} x_{i_2} \cdots x_{i_m} x_1 - x_1 x_e x_{i_1} x_{i_2} \cdots x_{i_m}, f_4 = x_e x_{i_1} x_{i_2} \cdots x_{i_{m+1}}\]
\[= x_1 x_e x_{i_1} \cdots x_{i_m} x_2 \cdots x_{i_{m+1}} - x_e x_{i_1} x_{i_2} \cdots x_{i_{m+1}}.\]

\[5 \land 5, f_5 = x_e x_{i_1} x_{i_2} \cdots x_{i_{m+1}} x_1 - x_1 x_e x_{i_1} x_{i_2} \cdots x_{i_{m+1}} f'_5 = x_e x_{i_1} x_{i_2} \cdots x_{i_{m+1}} x_1\]
\[= x_1 x_e x_{i_1} \cdots x_{i_{m+1}} x_2 \cdots x_{i_{m+1}} x_1 - x_e x_{i_1} x_{i_2} \cdots x_{i_{m+1}} x_1.\]

\[5 \land 5, f_5 = x_e x_{i_1} x_{i_2} \cdots x_{i_{m+1}} x_1 - x_1 x_e x_{i_1} x_{i_2} \cdots x_{i_{m+1}} f'_5 = x_e x_{i_1} x_{i_2} \cdots x_{i_{m+1}} x_1\]
\[= x_1^2 x_e x_{i_1} \cdots x_{i_{m+1}} x_2 \cdots x_{i_{m+1}} x_1 - x_1^2 x_e x_{i_1} x_{i_2} \cdots x_{i_{m+1}} x_1\]
\[= 0 \mod (S, w).\]

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