On gravitational lensing by symmetric and asymmetric wormholes

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We discuss the peculiarities of gravitational lensing by spherically symmetric wormholes if they are not symmetric with respect to their throats. It is noticed, in particular, that wormholes always contain the so-called photon spheres, near which the photon deflection angles can be arbitrarily large, but, in general, the throat is such a sphere only for symmetric wormholes. In some cases, photons from outside can cross the throat and return back from a neighborhood of a photon sphere if the latter is located beyond the throat. Two families of generally asymmetric wormhole configurations are considered as examples: (1) anti-Fisher wormholes with a massless phantom scalar field as a source of gravity, and (2) wormholes with a zero Ricci scalar that may be interpreted as vacuum configurations in a brane world. For these metrics, the photon effective potentials and deflection angles are found and discussed.

1 Introduction

Gravitational lensing is an astrophysical phenomenon whereby the propagation of light is affected by gravitating masses. As photons travel across the Universe, their trajectories are perturbed by gravitational fields. This phenomenon has become a powerful tool for studying the distribution of mass in the Universe as well as for observing faint distant sources that would otherwise be invisible.

There is vast literature on light bending by different astrophysical bodies, beginning with Einstein’s formula verified at solar eclipses (see, e.g., [1]) and ending with the most recent papers discussing gravitational lensing by strong-field objects like black holes and wormholes, see, e.g., [2,3] for reviews. However, in most of the papers, only wormholes symmetric with respect to their throats are considered, whereas such objects are only a special subset among all wormhole space-times. In this paper we discuss some general features of gravitational lensing by asymmetric wormholes, restricting ourselves to static, spherically symmetric configurations, asymptotically flat on both sides from their throats.

We will consider two examples of such objects, one represented by the so-called anti-Fisher solution to the Einstein equations with a massless, minimally coupled phantom scalar field [4–6], the other with an anisotropic fluid source and the Ricci scalar $R = 0$ [7]. In both cases, for simplicity, we assume that both the source of the signals and the observer are located at very large distances from the deflecting body (the lens) as compared to the characteristic lengths of the lens itself. Therefore, all incident paths of photons are assumed to be initially parallel to each other. To calculate the deflection angles, we use the corresponding general formulas which can be found, e.g., in [8,9]. A more general treatment taking into account finite distances between all objects is presented in [10]. Let us also mention studies of gravitational lensing by rotating wormholes and the analysis of specific features of photon paths in the strong deflection limit, see, e.g., [11] and [3], respectively, as well as references given therein.

In Section 2 we discuss the general features of photon paths in asymmetric wormholes, Section 3 is devoted to particular examples, and Section 4 is a conclusion.

2 Photon spheres and lensing — general consideration

The general static, spherically symmetric metric can be written as

$$ds^2 = e^{2\gamma(u)}dt^2 - e^{2\alpha(u)}du^2 - e^{2\beta(u)}d\Omega^2$$

(1)

where $u$ is an arbitrary radial coordinate and $d\Omega = d\theta^2 + \sin^2\theta d\phi^2$ is the linear element on a unit sphere. Let us use the so-called quasiglobal coordinate $u = x$ defined by the condition $\alpha + \gamma = 0$,
denoting
\[ e^{2\gamma} = e^{-2\alpha} = A(x), \quad e^\beta = r(x) \] (2)
so the metric takes the form
\[ ds^2 = A(x)dt^2 - \frac{dx^2}{A(x)} - r^2(x)d\Omega^2 \] (3)
It describes a wormhole if both \( A(x) > 0 \) and \( r(x) > 0 \) for all \( x \in \mathbb{R} \), and \( r(x) \) has a minimum (called a wormhole throat) at certain \( x \), say, \( x = 0 \). To consider a wormhole as an object observable from distant weakly curved regions of space, we should require that the metric is asymptotically flat as \( x \to \infty \): the same is required for \( x \to -\infty \) if the other side of the wormhole should be equally observable. In terms of the metric \( e^{\beta} = r(x) \), it means that
\[ A(x) = A_\pm + O(1/|x|), \quad r(x) \approx c_\pm \cdot |x|, \]
\[ Ar^2 \to 1 \quad \text{as} \quad x \to \pm \infty, \] (4)
where \( A_\pm, c_\pm \) are positive constants, and the prime stands for \( d/dx \).

For the metric \( e^{\beta} = r(x) \), the null geodesic equations (assuming photon motion in the equatorial plane \( \theta = \pi/2 \) without losing generality) has two well-known integrals
\[ i = E/A(x), \quad |\dot{\phi}| = L/r^2(x), \] (5)
where dots denote \( d/d\sigma \), \( \sigma \) being an affine parameter along the geodesics; \( E \) is the conserved energy parameter, and \( L \) the conserved angular momentum. With these integrals, the condition \( u^\mu u_\mu = 0 \) for the photon 4-velocity \( u^\mu \) (comprising an integral of the radial component of the null geodesic equations) reads
\[ \dot{x}^2 + L^2/A(x)/r^2(x) = E^2. \] (6)
This is the energy conservation law at photon motion: the first term plays the role of kinetic energy while the second one that of a potential, \( V(x) = L^2/A/r^2 \). The photon motion is only possible in the range of \( x \) where \( E \geq V(x) \). In particular, circular photon orbits (\( \dot{x} = 0 \)) can take place on such spheres \( x = x_{ph} \), where \( V' = 0 \), that is, according to \( i \),
\[ A/r^2 = E^2/L^2, \quad rA' - 2Ar' = 0. \] (7)
Spheres \( i \) (the so-called photon spheres) play an important role in gravitational lensing since the deflection angle of a photon approaching such a sphere along its tangent tends to infinity. It is therefore of interest to compare the location of photon spheres in different space-times.

Thus, in a Schwarzschild black hole space-time \((r = x, \quad A = 1 - 2m/r)\) with mass \( m \), the photon sphere is located at \( r = 3m \) (3/2 of the horizon radius).

For wormhole space-times, the Schwarzschild mass, describing the metric at large \(|x|\) (assuming that it is asymptotically flat on both sides of the throat), is in general not the only parameter determining their shape: there is at least one more characteristic length, the throat radius \( r_{th} = \min r(x) \). Furthermore, the wormhole may be symmetric or asymmetric with respect to the throat, which, without loss of generality, can be placed at \( x = 0 \). If the wormhole is symmetric, both \( r(x) \) and \( A(x) \) are even functions, hence \( r' = A' = 0 \) at the throat, and by \( i \) the throat is inevitably a photon sphere. (This is also evident by symmetry: a photon launched from a point on the throat in one of the angular directions, that is, with \( \dot{x} = 0 \), has no reason to leave the sphere \( x = 0 \) to either positive or negative \( x \).) A symmetric wormhole may have, but not necessarily has, other photon spheres, depending on the behavior of the metric functions \( A \) and \( r \), as was recently discussed in \([12]\). It is necessary to note that the claim of \([12]\) that a throat is necessarily a photon sphere is true for symmetric wormholes only.

If the wormhole is asymmetric, such that \( A'(0) \neq 0 \), then by \( i \) the throat \( x = 0 \) is not a photon sphere. On the other hand, since in a twice asymptotically flat wormhole \( V(x) \to 0 \) at both infinities, it is clear that there should be at least one photon sphere as a maximum of \( V(x) \), but the actual number and allocation of photon spheres depends on the particular metric. One should also note that photon paths along such a sphere are stable if \( V(x) \) has there a minimum and are unstable otherwise.

In what follows we will find the positions of photon spheres and the corresponding features of gravitational lensing for some examples of wormhole metrics.

To describe gravitational lensing, one can use the general formulas for asymptotically flat static, spherically symmetric space-times, see, e.g., \([8,9]\). We will assume that both the source and the ob-
server are located at very large distances from the deflecting body (the lens) as compared to the characteristic lengths of the lens itself. Then all incident paths of photons are initially parallel to each other.

In this case, in our notations, the deflection angle $\alpha$, found from null geodesics in the metric \([13]\), is given by

$$\alpha = \alpha(x_0) = I(x_0) - \pi,$$

$$I(x_0) = \frac{1}{2} \int_{x_0}^{\infty} \frac{dx}{r(x)\sqrt{r^2(x) - A^2(x)}}, \quad \text{(8)}$$

where $b = L/E$ is the impact parameter characterizing a particular null geodesic with the conserved energy parameter $E$ and angular momentum $L$ according to \([5]\): $x_0$ is the coordinate value corresponding to the nearest approach of the photon to the strong field region and can be found from the condition $dr/d\sigma = 0$ which leads to

$$A(x_0)b^2 = r^2(x_0), \quad \text{(9)}$$

and $I(x_0)$ in Eq. \((8)\) turns into

$$I(x_0) = \frac{1}{2} \int_{x_0}^{\infty} \frac{r_0dx}{r(x)\sqrt{r^2(x)A(x_0) - r_0^2A(x)}}, \quad \text{(10)}$$

where $r_0 = r(x_0)$. Equations \((8)\) and \((10)\) will be further used for finding specific wormhole lensing characteristics.

### 3 Examples

#### 3.1 Lensing by anti-Fisher wormholes

One of the simplest known wormhole solutions in general relativity is the so-called anti-Fisher solution to the Einstein-minimally coupled massless scalar field system in the case where the scalar field $\phi$ is phantom, and its Lagrangian is $L_\phi = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ \([14]\). The wormhole branch of the anti-Fisher solution may be written in the form

$$ds^2 = e^{-2m u} dt^2 - \frac{k^2 e^{2m u}}{\sin^2 ku} \left[ \frac{k^2 du^2}{\sin^2 ku} + d\Omega^2 \right],$$

$$\phi = Cu, \quad k^2 + m^2 = C^2/2, \quad \text{(11)}$$

where $k > 0$, $m$ and $C$ are integration constants, and $C$ has the meaning of a scalar charge. The radial coordinate $u$ is harmonic: it satisfies the “gauge” condition $\alpha = 2\beta + \gamma$ in terms of the metric \([11]\), which implies $\Box u = 0$. Without loss of generality we assume $0 < u < \pi/k$, and the metric \((11)\) is asymptotically flat at both ends of this range, where the spherical radius $r = e^\beta$ is infinite. The constant $m$ is the Schwarzschild mass at spatial infinity $u = 0$, as verified by comparison with the Schwarzschild metric at small $u$. At the other spatial infinity, $u = \pi/k$, the Schwarzschild mass is $m_- = -me^{-m\pi/k}$. Thus, if the mass is positive at one end, $m > 0$, it is negative on the other end and larger than $m$ in absolute value\(^2\).

The solution \((11)\) is transformed to the quasiglobal coordinate $x$ (which is approximately equal to the conventional radial coordinate at large radii) by the substitution

$$ku = \arccot(x/k), \quad x = k\cot ku, \quad \text{(12)}$$

whence (preserving the notation \((12)\) for $u$)

$$ds^2 = e^{-2mu}dt^2 - e^{2mu}[d\Omega^2 + (k^2 + x^2)d\Omega^2]. \quad \text{(13)}$$

The two flat spatial infinities correspond to $x \to \pm \infty$. The wormhole throat, where $r$ has its minimum, is located at $x = x_\text{th} = m$ and has the size

$$r_\text{th} = \sqrt{m^2 + k^2} \exp \left( \frac{m}{k} \arccot \frac{m}{k} \right). \quad \text{(14)}$$

The photon sphere parameters found using \((7)\) are

$$x_\text{ph} = 2m, \quad r_\text{ph} = \sqrt{4m^2 + k^2} \exp \left( \frac{m}{k} \arccot \frac{2m}{k} \right). \quad \text{(15)}$$

The photon sphere coincides with the throat only in the massless case $m = 0$, often called the Ellis wormhole, although in fact H. Ellis in in \([5]\) considered the general anti-Fisher solution.

For numerical estimates of light deflection angles by an anti-Fisher wormhole, we put $k = 1$, thus specifying an arbitrary length scale, and the parameter $m$ also becomes dimensionless. Our length unit is then equal to the throat radius $r_\text{th}$ for $m = 0$ but rather strongly differs from it for $m \neq 0$, see Fig. 1. The same figure also shows the photon sphere radius $r_\text{ph}$: its difference from $r_\text{th}$ looks rather small, apparently because $r(x)$ is a slowly varying function close to its minimum.

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\(^1\)The term “anti-Fisher” has been suggested in order to recall that the corresponding solution for a canonical scalar field was first obtained by I.Z. Fisher in 1948 \([13]\).

\(^2\)This inequality was discussed by H. Ellis in his recent paper \([15]\) as a possible source of dark energy due to net negative mass of such wormholes.
Deflection angles

This expression can be used to calculate the deflection angles \( \alpha(x_0) = I(x_0) - \pi \) for \( x_0 > 2m \), and the results are presented in Fig. 2. The calculation confirms that the deflection angles diverge near the photon spheres \( x_{\text{ph}} = 2m \). In the massless case (Ellis wormhole) the deflections are especially small except for the immediate neighborhood of the throat. As could be expected, negative masses create negative deflections (out from the lens), but close to the photon sphere they anyway become positive. To approach this sphere, located beyond the throat, photons have also to travel partly beyond the throat and only then return to "our" side of the wormhole.

With \( m > 0 \), the nature of deflections is more or less conventional. What can be marked as a general feature of wormholes is the photon sphere location close to \( r = m \), whereas for a Schwarzschild black hole its radius is \( r_{\text{ph}} = 3m/2 \). It is an evident potentially observable distinction.

### 3.2 Lensing by some wormholes with \( R = 0 \)

Symmetric and asymmetric wormholes with anisotropic fluid sources were obtained in [7] among other solutions to the Einstein equations under the assumptions

\[
R = 0, \quad r(x) = \sqrt{x^2 + b^2}, \quad b = \text{const} > 0, \quad (17)
\]

where \( R \) is the Ricci scalar, and the metric is assumed in the form (3). The condition \( R = 0 \) allows for treating such solutions as those describing either the gravitational field of an anisotropic fluid source or as vacuum configurations in an RS2-type brane world, like those considered in [20], since \( R = 0 \) follows from the brane-world modified 4D Einstein equations [21] in vacuum.

According to (17), in such solutions the wormhole throat is located at \( x = 0 \), and its radius is \( r_{\text{th}} = b \). Assuming \( b = 1 \) for numerical calculations, we choose the throat radius as the length unit.

As examples, we take solutions to the equation \( R(x) = 0 \), having the form

\[
A'' + \frac{4x}{1+x^2}A' + \frac{2(2+x^2)}{(1+x^2)^2}A = \frac{2}{1+x^2}, \quad (18)
\]

under the initial conditions \( A(0) = 0.5 \) and \( A'(0) = 0, \pm 0.5 \). The solutions are obtained numerically. With \( A'(0) = 0 \) we obtain an even function \( A(x) \), corresponding to a symmetric wormhole, while those obtained with \( A'(0) = \pm 0.5 \) describe significantly asymmetric wormholes, see Fig. 3. Fig. 4 shows the behavior of the effective potential for photon motion. The positions of its maximum values show the location of photon spheres: \( x_{\text{ph}} = 0 \) and \( r_{\text{ph}} = 1 \) for the symmetric solution \( A'(0) = 0 \), \( x_{\text{ph}} \approx 0.3031 \) and \( r_{\text{ph}} \approx 1.0445 \) for the asymmetric solutions \( A'(0) = \pm 0.5 \).

Figure 5 shows the deflection angles \( \alpha(x_0) \) for these three examples of wormhole configurations. The plots confirm that the deflection blows up if a
Figure 3: The metric function $A(x)$ as a solution to Eq. (18) with $A(0) = 0.5$ and $A'(0) = -0.5$, 0, 0.5 (curves 1, 2, 3, respectively).

Figure 4: The quantity $V(x) = A(x)/r^2(x)$ for the solutions shown in Fig. 3; the maxima of $V(x)$ show the positions of photon spheres.

Figure 5: Deflection angles $\alpha$ as functions of the closest approach coordinate $x_0$ for wormholes with $R = 0$ and $A(x)$ shown in Fig. 3 (the curves are numbered accordingly).

3. If a photon sphere is located beyond the throat, some photons can also travel beyond the throat and return back into the initial region.

4. The deflection angle dependence on the closest approach coordinate $x_0$ is not always monotonic and depends on the particular wormhole geometry.

5. We confirm the already known observation that the location of photon spheres is not directly related to the Schwarzschild mass of a wormhole; the latter can even be massless. However, in known examples the size of a photon sphere is smaller than that of a Schwarzschild black hole of the same mass, which may be an observational distinction between black holes and wormholes.

As particular examples, we have considered anti-Fisher wormholes and those with a zero Ricci scalar that may be interpreted as vacuum configurations in a brane world. One can recall that anti-Fisher wormholes are unstable under radial perturbations [16,18] and therefore cannot be regarded viable. However, we used them here as convenient examples of generally asymmetric wormholes (symmetric only in the massless case $m = 0$), convenient for studying the asymmetry effects in gravitational lensing. One can also recall that the same metric can be obtained with another source of gravity having the same stress-energy tensor in the static case but quite different dynamic properties, including stability, as exemplified in [19] for Ellis wormholes.

In future studies it makes sense, among other things, to clarify the photon behavior close to the photon spheres in generic asymmetric wormholes, to take into account finite distances between the wormhole, the source and the observer, affecting photon approaches a photon sphere, but the behavior of $\alpha(x_0)$ contains some peculiar features. Thus, a “zigzag” of curve 1 at small positive $x$ is evidently related to a quick rise of $A(x)$ while approaching the throat ($x = 0$) from the positive side. One can also observe that the radii of the photon spheres are rather close to the throat radius, at least if the wormhole asymmetry is not too large.

4 Concluding remarks

We have discussed some features of gravitational lensing by asymmetric wormholes and can make the following observations.

1. All asymptotically flat static, spherically symmetric wormholes possess photon spheres near which there can be arbitrarily large deflection angles of incident photons.

2. A photon sphere is necessarily located on the throat if the wormhole is symmetric with respect to it, while in generic asymmetric wormholes the throat is, in general, not a photon sphere.
the observable picture in the sky, and to describe the gravitational lensing picture in the case of multiple photon spheres (see, e.g., [12]). It would also be of interest to consider the effect of long wormhole throats [22] on gravitational lensing and to extend the study to other kinds of wormholes such as those known in scalar-tensor and $f(R)$ theories of gravity (see, e.g. [6, 23, 24]) and those with “trapped ghosts” where a scalar field is phantom only close to the throat [24,25].

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