Self-Consistent Large-$N$ Analytical Solutions of Inhomogeneous Condensates in Quantum $\mathbb{C}P^{N-1}$ Model

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Abstract: We give, for the first time, self-consistent large-$N$ analytical solutions of inhomogeneous condensates in the quantum $\mathbb{C}P^{N-1}$ model in the large-$N$ limit. We find a map from a set of gap equations of the $\mathbb{C}P^{N-1}$ model to those of the Gross-Neveu (GN) model (or the gap equation and the Bogoliubov-de Gennes equation), which enables us to find the self-consistent solutions. We find that the Higgs field of the $\mathbb{C}P^{N-1}$ model is given as a zero mode of solutions of the GN model, and consequently only topologically nontrivial solutions of the GN model yield nontrivial solutions of the $\mathbb{C}P^{N-1}$ model. A stable single soliton is constructed from an antikink of the GN model and has a broken (Higgs) phase inside its core, in which $\mathbb{C}P^{N-1}$ modes are localized, with a symmetric (confining) phase outside. We further find a stable periodic soliton lattice constructed from a real kink crystal in the GN model, while the Ablowitz-Kaup-Newell-Segur hierarchy yields multiple solitons at arbitrary separations.
1 Introduction

Nonlinear sigma models such as the $\mathbb{C}P^{N-1}$ model in 1+1 dimensions \cite{1,3} are known to share a number of phenomena common with 3+1 dimensional QCD, e.g. asymptotic freedom, dynamical mass generation, confinement, and instantons \cite{4-11}. The mass gap can be best shown in the large-$N$ analysis in which one solves the gap equations self-consistently, to be consistent with the Coleman-Mermin-Wagner (CMW) theorem forbidding a gapless excitations in 1+1 dimensions \cite{12,13}. The $\mathbb{C}P^{N-1}$ model, or the $\mathbb{C}P^1$ model equivalent to the $O(3)$ sigma model, appears in a wide range of physics from particle physics to condensed matter physics. The relation between the 1+1 dimensional Heisenberg antiferromagnetic spin chain and the $O(3)$ sigma model has been shown in Ref. \cite{14,15}. Recently, the quantum phase transition, so-called deconfined criticality is proposed in the antiferromagnetic system \cite{16,17}. The sigma model with topological term is known to describe the integer quantum Hall effect \cite{18}. The supersymmetric $\mathbb{C}P^{N-1}$ model was also investigated \cite{19,20} for which the all order calculation in coupling constant is possible for Gell-Mann-Low function \cite{11}, and dynamical mass gap was proved by the mirror symmetry \cite{21}. The analogy between 3+1 dimensional Yang-Mills theory and 1+1 dimensional sigma model, pointed out in Ref. \cite{4}, has been recently revealed in a rather nontrivial way; a non-Abelian vortex string in a $U(N)$ gauge theory with $N$ scalar fields in the fundamental representation carries $\mathbb{C}P^{N-1}$ moduli \cite{22,23} (see Refs. \cite{24,25} as a review), yielding a nontrivial relation between the $\mathbb{C}P^{N-1}$ model on the string worldsheet and the bulk.
gauge theory \([29, 30]\). The \(CP^{N-1}\) model defined on an interval \([31, 32]\) or on a ring \([32]\) was also studied. The \(CP^{N-1}\) model or the \(O(3)\) sigma model at finite temperature and/or density was also investigated in which Berezinskii-Kosterlitz-Thouless transition at nonzero density was examined \([34]\). One of recent developments is a resurgent structure of the \(CP^{N-1}\) model \([35, 36]\), in which a molecule of fractional instantons \([37, 38]\) called a bion, plays a crucial role. In spite of tremendous studies of the \(CP^{N-1}\) model, there was no study on inhomogeneous configurations (such as solitons) at quantum level, except for a numerical study of the \(CP^{N-1}\) model on an interval \([32]\).

The situation is rather different for an interacting fermionic theory: the Gross-Neveu (GN) \([39]\) or Nambu-Jona-Lasinon model \([40]\), exhibiting dynamical symmetry breaking of discrete or continuous chiral symmetry, thereby sharing an important property with QCD \([41, 42]\). This model is equivalent at the large-\(N\) limit or in the mean field approximation to a set of the Bogoliubov-de Gennes (BdG) equations and the gap equation, appearing in condensed matter systems such as conducting polymers \([44, 46]\), superconductors, superfluids and ultracold atomic gases \([47–49]\). Self-consistent analytical solutions such as a real kink \([41, 44]\), a twisted (complex) kink \([42]\), a real kink-anti-kink (polaron) \([41, 50]\), a real kink-anti-kink-kink \([41, 51, 52]\) and more general real solutions \([53]\) have been known. Recently, a theoretical progress has been achieved for inhomogeneous condensats in the 1+1 dimensional (chiral) GN model, e.g., the exact self-consistent and inhomogeneous condensates such as a real kink crystal \([54]\) (Larkin-Ovchinnikov(LO) state \([55]\)), a chiral spiral (Fulde-Ferrell(FF) state \([56]\)), and a twisted kink crystal \([57]\) (FF-LO state) have been found by mapping the equations to the nonlinear Schrödinger equation, and such states have been shown to be ground states in a certain region of the phase diagram for finite temperature and density \([58]\). More generally, multiple twisted kinks with arbitrary phase and positions \([59]\) can be further constructed systematically due to the integrable structure behind the model known as the Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy for the nonlinear Schrödinger equation \([60, 61]\). Recent developments include time-dependent soliton scatterings \([62]\), multi-component condensates \([63, 64]\), a ring geometry \([65]\), and an interval with a Casimir force \([66]\).

In the present work, we reveal an unexpected relation between these two completely different theories, the \(CP^{N-1}\) and GN models developed independently. By finding a map from a set of gap equations of the \(CP^{N-1}\) model to those of the GN model, we find self-consistent analytical solutions of stable inhomogeneous condensates in the quantum \(CP^{N-1}\) model, that is, a single soliton, a soliton lattice and multiple solitons at arbitrary separations.
2 Model and method

We consider the \( \mathbb{C}P^{N-1} \) model on an infinite space:

\[
S = \int dt dx \left[ (D_\mu n_i)^* (D^\mu n_i) - \lambda (n_i^* n_i - r) \right],
\]

(2.1)

where \( n^i \) \((i = 1, \cdots, N)\) are complex scalar fields, \( D_\mu = \partial_\mu - iA_\mu \), and \( \lambda(x) \) is a Lagrange multiplier. The “radius” \( r \) is known to have connection with a coupling constant \( g_{\text{YM}} \) in the Yang-Mills theory; \( r = 4\pi/g_{\text{YM}}^2 \) if we realize this model on a non-Abelian vortex in \( U(N) \) gauge theory. Here we note that the model does not have kinetic term for \( A_\mu \) and thus we focus on the case of \( A_\mu = 0 \) throughout this paper. We separate \( n_i \) fields into a classical field \( n_1 = \sigma \) (real) and \( n_i = \tau_i \) \((2, \cdots, N)\).

Integrating out the \( \tau_i \) fields, we obtain the effective action for \( \sigma \) as

\[
S_{\text{eff}} = \int dt dx \left[ (N-1) \text{Tr} \ln(-\partial_\mu \partial^\mu + \lambda) + \partial_\mu \sigma \partial^\mu \sigma - \lambda (\sigma^2 - r) \right].
\]

(2.2)

In the following we consider and the leading contribution of \( 1/N \) expansion and thus we replace \( N - 1 \) to \( N \). One can formally write down the total energy functional as

\[
E = N \sum_n \omega_n + \int dx \left[ (\partial_x \sigma)^2 + \lambda (\sigma^2 - r) \right].
\]

(2.3)

The corresponding gap equations obtained from the static condition with respect to \( \lambda \) and \( \sigma \) are

\[
\frac{N}{2} \sum_n \frac{f_n^2}{\omega_n} + \sigma^2 - r = 0,
\]

(2.4)

\[
\partial_x^2 \sigma - \lambda \sigma = 0,
\]

(2.5)

respectively, where \( f_n(x) \) and \( \omega_n \) are orthonormal eigenstates and eigenvalues of the following equation

\[
(-\partial_x^2 + \lambda) f_n(x) = \omega_n^2 f_n(x).
\]

(2.6)

We need to solve Eqs. (2.4)–(2.6) in a self-consistent manner. We here note from Eqs. (2.5) and (2.6) that \( \sigma \) is proportional to a zero mode \( f_0 \).

It is well known that assuming a uniform state in infinite system, one finds the confining (unbroken) phase with a constant \( \lambda \) to be a unique solution, to be consistent with the CMW theorem. For the case of a ring, in addition to it, there is a Higgs (broken) phase with a constant \( \sigma \) for a smaller ring \( \mathbb{R} \).

\(^1\)We note that the large \( N \) limit is considered to obtain the self-consistent equations and the rest does not rely on the large \( N \). Furthermore, the mean field approximation (for finite \( N \)) also yields the same self-consistent equations. Thus the results in the following are expected to be qualitatively correct even in the case of finite \( N \).
One of the main results of this paper is a map from those equations to the gap equation and eigenvalue equation for the GN model. In order to reduce the number of equations, we introduce the new field \( \Delta \) such as

\[
\Delta^2 + \partial_x \Delta = \lambda(x).
\]

By using this function, we find a solution to Eq. (2.5):

\[
\sigma = A \exp \left[ \int^x dy \Delta(y) \right],
\]

where \( A \) is the integral constant. The energy in Eq. (2.3) can be rewritten as

\[
E_{\text{tot}} = N \sum_n \omega_n - r \int_{-\infty}^{\infty} dx (\Delta^2 + \partial_x \Delta) + \sigma \partial_x \sigma|_{-\infty}^{\infty}.
\]

The rather nontrivial step is to rewrite Eq. (2.6) as [See Appendix A]

\[
\begin{pmatrix}
0 & \partial_x + \Delta \\
-\partial_x + \Delta & 0
\end{pmatrix}
\begin{pmatrix}
f_n \\
g_n
\end{pmatrix}
= \omega_n \begin{pmatrix}
f_n \\
g_n
\end{pmatrix},
\]

where \( g_n \)'s are auxiliary fields and the elimination of \( g_n \) yields Eq. (2.6). We note that Eq. (2.10) together with Eq. (2.7) describes a supersymmetric quantum mechanics, in which the potential \( \lambda \) is given by the superpotential \( \Delta \) [68]. Eq. (2.10) is the positive energy part of the BdG or Andreev equation which corresponds to the Hartree-Fock equation of the GN model with \( N \) flavors [See Appendix B]

\[
L_{\text{GN}} = \bar{\psi} i \not{\partial} \psi + \frac{g^2}{2} (\bar{\psi} \psi)^2.
\]

The corresponding Hartree-Fock equation becomes \( H \psi = E \psi \), with \( H = -i \gamma^5 \partial_x + \gamma^0 \Delta \), where \( \gamma^5 = -\sigma_2 \) and \( \gamma^0 = \sigma_1 \) with the Pauli matrices \( \sigma_i \). Here \( \Delta \) (real) satisfies \( \langle \bar{\psi} \psi \rangle = -\Delta / g \), which is called a gap equation. It is known that the \( \mathbb{Z}_2 \) symmetry is spontaneously broken in the GN model, yielding two discrete vacua.

With a help of \( g_n = (-\partial_x + \Delta) f_n / \omega_n \), one can show that \( g_n \) automatically gives an orthonormal set if \( f_n \) gives an orthonormal set. Eq. (2.10) has the particle-hole symmetry which enables us to obtain the set \( \{-\omega_n, \tilde{f}_n, \tilde{g}_n\} \) from the set \( \{\omega_n, f_n, g_n\} \) by \( \tilde{f}_n = f_n \) and \( \tilde{g}_n = -g_n \). By taking the derivative of Eq. (2.4) with respect to \( x \) and by substituting Eqs. (2.8) and \( \omega_n g_n = (-\partial_x + \Delta) f_n \) into that, we obtain

\[
\Delta = N \frac{2r}{2r} \sum_n \tilde{f}_n \tilde{g}_n = N \frac{2r}{2r} \sum_n \tilde{f}_n \tilde{g}_n,
\]

which has the same form with the gap equation for the GN model. Here we note that corresponding fermionic coupling \( N g^2 = N/2r \) is proportional to the 't Hooft coupling in an underlying \( U(N) \) gauge theory \( N g^2_{YM} \). Since we solve the differentiated
one instead of Eq. (2.4) itself, we need to fix the integration constant $A$ for $\sigma$ by substituting Eq. (2.8) into Eq. (2.4). For the BdG equation (2.10) and gap equation (2.12), various exact self-consistent solutions are already known. From Eq. (2.10) one can immediately find the zero mode solution

$$
f_0(x) \propto \exp \left[ \int_{x_0}^{x} dy \Delta(y) \right],
$$

where the corresponding auxiliary field is $g_0(x) = 0$. The zero mode solutions $f_0$ in the $\mathbb{C}P^{N-1}$ and GN models are identical. As denoted below Eq. (2.4), the Higgs field $\sigma(x)$ in the $\mathbb{C}P^{N-1}$ model is proportional to the zero mode, thereby exists only when corresponding $\Delta$ in the GN model is topologically nontrivial with allowing a normalizable zero mode \[.]

### 3 Self-consistent analytical solutions

In the GN model, a constant gap $\Delta = m$ is a solution which can be called the Bardeen-Cooper-Schrieffer (BCS) phase, whereas that for $m = 0$ is called a normal phase. We show that the BCS and normal phases in the GN model correspond to the confining and Higgs phases in the $\mathbb{C}P^{N-1}$ model, respectively. For the constant solution, $\omega_n = \sqrt{(\pi n/L)^2 + m^2}$ and the degenerated eigenfunctions are $f_n^{(1)} = \sqrt{2} \sin \pi n x / L$, $f_n^{(2)} = \sqrt{2} \cos \pi n x / L$. For both the cases, $g_n^{(i)}(x) = (-\partial_x + m) f_n / \omega_n$ ($i = 1, 2$). Here we consider the periodic boundary condition in domain $[-L/2, L/2]$. The infinite system can be obtained by taking the proper limit of $L \to \infty$. The substitution $\Delta = m$ and corresponding eigenstates into Eq. (2.12) yields

$$
m = \frac{N}{r} \sum_n \frac{m}{\omega_n},
$$

while Eq. (2.4) becomes

$$
\sigma^2 = r - \frac{N}{\sum_n \omega_n}.
$$

We find that the condition (3.1) for $m \neq 0$ and (3.2) for $\sigma = 0$ are equivalent

$$
1 = \frac{N}{r} \sum_n \frac{1}{\omega_n},
$$

which gives the well known renormalization condition of the coupling constant $g^2 = 4\pi / r$. This results in two possibilities $\{ \lambda = m^2, \sigma = 0 \}$ (confining phase) and $\{ \lambda = 0, \sigma = \text{const} \}$ (Higgs phase), but only the former satisfies the gap equation (2.4) and the latter is not allowed in the infinite system \[11, 33\].

The solution $\Delta = -m \tanh mx$ is known as a topological kink solution interpolating two discrete vacua of the GN model, which has a zero mode localized...
near the kink. In the case of kink solution, the eigenvalue is the same with the constant solution \( \omega_n = \sqrt{(\pi n/L)^2 + m^2} \) while the degenerated eigenfunctions are

\[ f_n^{(i)} = (\partial_x - m \tanh mx)g_n^{(i)}/\omega_n \]

with \( g_n^{(1)} = \sqrt{2} \sin \pi nx/L, \quad g_n^{(2)} = \sqrt{2} \cos \pi nx/L \). We also have a normalizable zero mode \( f_0(x) \propto 1/\cosh mx, \quad g_0(x) = 0 \). Thus Eq. (2.13) yields

\[ -m \tanh mx = \frac{N}{r} \sum_n \frac{-m \tanh mx}{\omega_n}, \tag{3.4} \]

which indeed gives the same condition with Eq. (3.1). On the other hand, Eq. (2.4) implies

\[ \sigma^2 = r - N \sum_n \frac{1}{\omega_n} + \frac{m^2}{\cosh^2 mx} N \sum_n \frac{1}{\omega_n^3}. \tag{3.5} \]

In the case of \( m \neq 0 \), Eq. (3.4) yields Eq. (3.3) and we reach at

\[ \sigma = \frac{m}{\cosh mx} \sqrt{N \sum_n \frac{1}{\omega_n^3}}, \tag{3.6} \]

which has a bright solitonic profile. Again, it is indeed proportional to the zero mode solution. In this case, the mass gap function becomes

\[ \lambda(x) = m^2(1 - 2 \cosh^{-2} mx), \tag{3.7} \]

which has a gray soliton configuration and is called the Pöschl-Teller potential \[28\]. Since all the eigenenergies of this solution are non-negative, the solution is stable. In Fig. 4 we plot the configuration of \( \sigma(x) \) and the mass gap function \( \lambda(x) \). The energy of the soliton can be calculated by the energy \( E_s \) for the soliton configuration in Eqs. (3.6) and (3.7) subtracted by \( E_0 \) for the confining phase (\( \sigma_0 = 0 \) and \( \lambda_0 = m^2 \)), for both of which the third term in Eq. (2.9) vanishes from the equation of motion (2.5) and \( \omega_n \)'s are the same. We thus obtain

\[ E_s - E_0 = \int_{-\infty}^{\infty} dx r(\lambda_0 - \lambda_s) = 4rm. \tag{3.8} \]

Since \( \sigma \) has a localized profile function, soliton core is in the Higgs (broken) phase where the \( \mathbb{C}P^{N-1} \) modes are localized, while the bulk is in the confining (symmetric) phase, in contrast to a uniform system allowing only the confining phase in infinite system to be consistent with the CMW theorem. It is known that the correlation function behaves at large distance as \( x^{-1/N} \) in 1+1 dimension \[30\], which inhibits the long-range order for finite \( N \). Here we have obtained the Higgs phase localized with length \( \sim 1/m \), thus the robustness of our solitonic solution is expected if \( N \) is sufficiently large as \( \ln(1/m) \ll N \).

\[ ^2 \] It is also the case of the \( \mathbb{C}P^{N-1} \) model on a ring: The Higgs phase is allowed for a smaller ring \[33\].
Figure 1. The configuration of $\sigma$ (solid line) and $\lambda$ (dashed line) for $\Delta = -m \tanh mx$ (dotted line). Here we normalize as $\sigma(0) = 1$ and $m = 1$.

Figure 2. The bright soliton lattice configuration of $\sigma$ (solid line) and $\lambda$ (dashed line) for $\nu = 10^{-2}$ (left figure) and $\nu = 1 - 10^{-2}$ (right figure). The auxiliary field $\Delta$ (dotted line) are also plotted. Here we set, $m = 1$ and normalize the peak of $\sigma$ to be 1.

The above solutions can be obtained from a soliton lattice obtained from a real kink crystal in the GN model:

\[ \Delta(x) = m \text{sn}(mx, \nu), \]  

(3.9)

where sn, cn, and dn (appearing later) are the Jacobi functions and $\nu$ is elliptic parameter. Here the periodicity of the above solution is given by $\ell = 4K(\nu)/m$, where $K(\nu)$ is a complete elliptic integral of the first kind. This solution together with Eq. (2.8) gives a soliton lattice:

\[ \sigma = A \left[ -\sqrt{\nu} cn(x, \nu) + dn(x, \nu) \right]^{\pm \sqrt{\nu}}. \]  

(3.10)

In Fig. 2, we plot the mass gap function $\lambda$ and $\sigma$ for $\nu = 10^{-2}$ and $\nu = 1 - 10^{-2}$. The auxiliary field $\Delta$ are also plotted. The Higgs field $\sigma$ in this solution has a bright soliton lattice profile. By taking $\nu = 1$ limit for $\Delta = m \text{sn}(mx + K(\nu), \nu)$, $\lambda$ becomes constant and $\sigma = 0$ in the whole system. This limit corresponds to the constant solution discussed above. On the other hand, $\Delta = m \text{sn}(mx + 2K(\nu), \nu)$
reduces to \( \lambda = m[1 - 2/cosh^2 mx] \) and \( \sigma(x) \propto m/cosh mx \). This corresponds to the kink solution [See Appendix A]. Our periodic soliton solutions can be put on a ring, while the previous studies on the \( \mathbb{C}P^{N-1} \) model on a ring dealt with only constant configurations [3].

4 Higher order self-consistent analytical solutions

In the GN model, the integrable structure enables us to systematically construct all possible exact self-consistent solutions [61, 62]. The above solutions belong to the lowest order \((n = 1)\) of the AKNS hierarchy (denoted by AKNS\(_n\) for \(n = 1, 2, \cdots\)) for the nonlinear Schrödinger equation [21, 62] [See Appendix B]. The configuration of a kink-anti-kink (polaron) in the GN model [50] (in AKNS\(_2\)) does not yield a nontrivial solution in the \( \mathbb{C}P^{N-1} \) model, while the three kink solution (in AKNS\(_3\)) [43, 51, 52]

\[
\Delta = k \tanh[kx - k\delta + R] - \frac{\omega_b e^R[\sinh(m_+ x - k\delta + 2R) + \sinh(m_- x + k\delta)]}{\cosh(m_+ x - k\delta + 2R) + e^{2R} \cosh(m_- x + k\delta)},
\]

does. Here \( \omega_b = \sqrt{m^2 - k^2}, R = (1/2) \ln(m_+/m_-) \), and \( m_\pm = m \pm k \). In Fig. 3 we plot the configurations of \( \sigma, \lambda, \) and \( \Delta \) for various parameter choices. The symmetric case \( \delta = 0 \) (a) looks like a double copy of a single soliton in Fig. 1. For larger \( \delta \) the middle kink is closer to the right anti-kink than the left anti-kink in \( \Delta \) as (b), and then the amplitude of the Higgs field \( \sigma \) localized in the right soliton of \( \lambda \) decreases with increasing \( \delta \). On the other hand, the parameter \( k \) controls the soliton-soliton distance [(a), (c), and (d)]. The two solitons merge for larger \( k \) and eventually becomes one soliton in \( k \to 1 \). This is possible because the three kink solution belongs to the same topological sector with the single kink solution in the GN model. In general, AKNS\(_{2k+1}\) \((k = 1, 2, \cdots)\) yields solutions of \( k \) solitons with arbitrary positions exhibiting the similar behaviors.

5 Summary

We have found the map from the GN model to the \( \mathbb{C}P^{N-1} \) model, which enables us to construct, for the first time, the exact self-consistent stable inhomogeneous solutions of the \( \mathbb{C}P^{N-1} \) model; a single soliton, a soliton lattice and multiple solitons with arbitrary separations. The Higgs (broken) phase appears inside the soliton cores where the Higgs field \( \sigma \) has bright solitonic profiles and the \( \mathbb{C}P^{N-1} \) moduli are confined.

It is an open question whether there is a map to the chiral GN model with continuous chiral symmetry, which allows a variety of complex solutions. In the (chiral) GN model, the inhomogeneous phase is stabilized at the low temperature and high density [58], or in the presence of a chiral chemical potential, equivalent to
Figure 3. The two bright soliton configuration of $\sigma$ (solid line) and $\lambda$ (dashed line). The auxiliary field $\Delta$ (dotted line) are also plotted. Here we set, $m = 1$. In Fig. (a) and (b) we plot the case of $\delta = 0$ and $\delta = 2$, respectively, with $k = 1 - 10^{-3}$. In Fig. (c) and (d) we plot the case of $k = 1 - 10^{-2}$ and $k = 1 - 10^{-1}$, respectively, with $\delta = 0$. In the figure, we normalize $\sigma$ such that the height of the highest peak is 1.

the constant Zeeman magnetic field on the superconductivity [47]. Such analogies in the $\mathbb{C}P^{N-1}$ model may imply a possibility of a crystalline phase. While our periodic soliton lattice can be put on a ring, an extension to an interval [31, 32] is also possible to calculate a Casimir force [71], since the exact solutions in the GN model on an interval have been found recently [67]. Another relation between the the $\mathbb{C}P^{N-1}$ model and the GN model in $2 + 1$ dimensions has recently been found in Ref. [72] in which the large-$N$ free energy densities for the both theories are found to be remarkably similar. Though it would be important to see whether the similar structure also appears in the $1 + 1$ dimensions, we leave it as a future problem. The connection between our formalism and the bosonization scheme in $1 + 1$ dimensions should be also important. The former gives the coincidence of the self-consistent equations in $\mathbb{C}P^{N-1}$ model and the GN model, whereas the latter yields the sine-Gordon model as the bosonized model of the GN model [70]. We also leave it as a future problem. Physical consequences of our solitons on a non-Abelian vortex in supersymmetric gauge theories [25–28] or dense QCD [68] will be an important problem to be explored.
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A Alternative mapping

In this Appendix, we show an alternative map from the Gross-Neveu model to the $\mathbb{C}P^{N-1}$ model. In our formalism, $f_n$’s are chosen as upper components of BdG equation $(u_n = f_n, v_n = g_n)$ in

\[
\begin{pmatrix}
0 & \partial_x + \Delta \\
-\partial_x + \Delta & 0
\end{pmatrix}
\begin{pmatrix}
u_n \\
v_n
\end{pmatrix} = \omega_n
\begin{pmatrix}u_n \\
v_n
\end{pmatrix}, \quad (A.1)
\]

with

\[
\lambda = \Delta^2 + \partial_x \Delta, \quad \sigma \propto \exp \left( \int^x dy \Delta \right). \quad (A.2)
\]

For the same $\Delta$, one can also define

\[
\tilde{\lambda} = \Delta^2 - \partial_x \Delta, \quad \tilde{\sigma} \propto \exp \left( - \int^x dy \Delta \right). \quad (A.3)
\]

These functions satisfy

\[
\begin{align*}
\partial_x^2 \tilde{\sigma} - \tilde{\lambda} \sigma &= 0, \quad (A.4) \\
(-\partial_x^2 + \tilde{\lambda})v_n &= \omega_n^2 v_n, \quad (A.5) \\
(N/2r) \sum_n u_n v_n &= \Delta. \quad (A.6)
\end{align*}
\]

This implies that the lower component can also be mapped to the $\mathbb{C}P^{N-1}$ model $(v_n = f_n, u_n = g_n)$ with the Higgs field $\tilde{\sigma}$ and the mass gap function $\tilde{\lambda}$. Thus the single $\Delta$ corresponds to two solutions in the $\mathbb{C}P^{N-1}$ model (for $\Delta = m$, those are identical).

For instance, in the case of the kink solution, we obtain

\[
\begin{align*}
\Delta &= m \tanh mx, \quad (A.7) \\
\lambda &= m^2, \quad \sigma = 0, \quad (A.8) \\
\tilde{\lambda} &= m^2 (1 - 2 \text{sech}^2 mx), \quad \tilde{\sigma} = A \text{sech} mx, \quad (A.9)
\end{align*}
\]
whereas in the case of the anti-kink solution, we obtain
\[ \Delta = -m \tanh mx, \]  
\[ \lambda = m^2 (1 - 2 \text{sech}^2 mx), \quad \sigma = A \text{sech} mx, \]  
\[ \tilde{\lambda} = m^2, \quad \tilde{\sigma} = 0. \]  
(A.10) (A.11) (A.12)

Thus both the solutions correspond to the same solution in the $\mathbb{C}P^{N-1}$ model.

## B Chiral Gross-Neveu model, Bogoliubov-de Gennes equation, and AKNS hierarchy

In this Appendix, we briefly summarize the self-consistent treatment of Gross-Neveu model studied in Refs. [61, 62]. The Lagrangian of the chiral Gross-Neveu model with $N$ flavor is given by
\[ L = \bar{\psi}i\partial_\tau \psi + \frac{g^2}{2} \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma^5 \psi)^2 \right], \]  
(B.1)

where $g > 0$. By introducing the auxiliary fields $\Delta_1 = -g^2 \langle \bar{\psi}\psi \rangle$ and $\Delta_2 = -g^2 \langle \bar{\psi}i\gamma^5 \psi \rangle$, and by taking the large $N$ approximation (or mean field approximation) one can obtain the following effective Lagrangian
\[ L_{\text{eff}} = \bar{\psi}i\partial_\tau \psi + (\Delta_1 \bar{\psi}\psi + \Delta_2 \bar{\psi}i\gamma^5 \psi) - \frac{1}{2g^2} (\Delta_1^2 + \Delta_2^2). \]  
(B.2)

Thus we obtain the following total energy
\[ E_{\text{tot}} = \int dx \bar{\psi}(H\psi + \frac{1}{2g^2} (\Delta_1^2 + \Delta_2^2)), \]  
(B.3)

with the Bogoliubov-de Gennes (BdG) Hamiltonian
\[ H = -i\gamma^0 \gamma^1 \frac{d}{dx} - \gamma^0 \left( \Delta_1 + i\gamma^5 \Delta_2 \right). \]  
(B.4)

The consistency condition of the auxiliary field $\Delta_1$ and $\Delta_2$ are called the gap equations
\[ \langle \bar{\psi}\psi \rangle = -\frac{1}{g^2} \Delta_1, \quad \langle \bar{\psi}i\gamma^5 \psi \rangle = -\frac{1}{g^2} \Delta_2, \]  
(B.5)

which must be solved in a consistent manner with the BdG equation $H\psi = E\psi$. Here the left hand sides of the gap equations can be, respectively, rewritten as $N \langle \bar{\psi}_1 \psi_1 \rangle$ and $N \langle \bar{\psi}_1 i\gamma^5 \psi_1 \rangle$, since the $N$ flavors gives the same contributions, e.g., $\langle \bar{\psi}_1 \psi_1 \rangle = \langle \bar{\psi}_2 \psi_2 \rangle = \cdots = \langle \bar{\psi}_N \psi_N \rangle$. Thus we can rewrite the gap equations as
\[ \Delta_1 = -g^2 N \langle \bar{\psi}_1 \psi_1 \rangle, \quad \Delta_2 = -g^2 N \langle \bar{\psi}_1 i\gamma^5 \psi_1 \rangle. \]  
(B.6)
In the following, we use the chiral representation $\gamma_0 = \sigma_1$, $\gamma_1 = -i\sigma_2$, and $\gamma_5 = \sigma_3$.

For the BdG Hamiltonian, the Gor’kov resolvent $R(x; E) = 1/\langle x|(H - E)|x\rangle$ satisfies the Dikii-Eilenberger equation
\begin{equation}
\partial_t R(x; E) \sigma_3 = [Q(E, \Delta), R(x; E) \sigma_3],
\end{equation}
with the constraint $R_\sigma$. The AKNS system, one can systematically expand the resolvent between BdG system to the AKNS system, by using the machinery of the integrable Dikii-Eilenberger equation
\begin{equation}
\partial_t Q - \partial_x R_{\sigma_3} + [Q, R_{\sigma_3}] = 0, \quad \partial_x \psi = Q \psi, \quad \partial_t \psi = R_{\sigma_3} \psi,
\end{equation}
where $\Delta = \Delta_1 - i\Delta_2$. We note that the BdG equation can be written as $\partial_x \psi = Q \psi$. The Gor’kov resolvent must satisfies the conditions $\det R = -\frac{1}{4}$, $\text{Tr} Q R_{\sigma_3} = 0$, and $R^t = R$.

The Dikii-Eilenberger equation and the BdG equation can be rewritten as $\partial_t Q - \partial_x R_{\sigma_3} + [Q, R_{\sigma_3}] = 0$, $\partial_x \psi = Q \psi$, $\partial_t \psi = R_{\sigma_3} \psi$, with the constraint $\partial_t Q = 0$. The first equation is the integrable condition (zero curvature condition) of this system; $\partial_x \partial_t \psi = \partial_t \partial_x \psi$. Since we find the connection between BdG system to the AKNS system, by using the machinery of the integrable system, one can systematically expand the resolvent $R_{\sigma_3}$ which yields AKNS system as
\begin{equation}
R_{\sigma_3} = \sum_{j=1}^{n+2} c_j V^{(j)}, \quad V^{(n)} = \sum_{j=0}^{n-1} (2E)^{n-k} M^{(j)},
\end{equation}
where $c_j$’s are positive constants. Here $M_{i,j}^{(i)}$ components of the matrices $M^{(i)}$ satisfy $M_{11}^{(i)} = -M_{22}^{(i)}$, $M_{12}^{(i)} = (M_{21}^{(i)})^*$, and first few components are given by
\begin{align}
M_{11}^{(0)} &= -\frac{i}{2}, & M_{12}^{(0)} &= 0, \\
M_{11}^{(1)} &= 0, & M_{12}^{(1)} &= i\Delta, \\
M_{11}^{(2)} &= -i|\Delta|^2, & M_{12}^{(2)} &= \partial_x \Delta, \\
M_{11}^{(3)} &= -2i\Im(\Delta^* \partial_x \Delta), & M_{12}^{(3)} &= \partial_x^2 \Delta - 2|\Delta|^2, \\
M_{11}^{(4)} &= 2i\Re(\Delta^* \partial_x^2 \Delta) - 2i|\partial_x \Delta|^2 - 3|\Delta|^4, \\
M_{12}^{(4)} &= -\partial_x^2 \Delta + 6|\Delta|^2 \partial_x \Delta.
\end{align}
The higher components are calculable with a help of the following formula
\begin{equation}
\frac{i}{2} [\sigma_3, M^{(n+1)}] = \partial_x M^{(n)} + [M^{(1)}, M^{(n)}].
\end{equation}
We can also obtain the nonlinear Schrödinger equations for this system as $\sum_{j=1}^{n+1} c_j M_{12}^{(j)} = 0$. The AKNS$_0$, AKNS$_1$, AKNS$_2$ for instance, yield
\begin{align}
-\frac{i}{2} \partial_x \Delta + c_1 \Delta &= 0, \\
-\frac{1}{4} (\partial_x^2 \Delta - 2|\Delta|^2 \Delta) - c_1 \frac{1}{2} \partial_x \Delta + c_2 \Delta &= 0, \\
\frac{i}{8} (\partial_x^3 \Delta - 8|\Delta|^2 \partial_x^2 \Delta) - c_1 \frac{1}{4} (\partial_x^2 \Delta - 2|\Delta|^2 \Delta) - c_2 \frac{1}{2} \partial_x \Delta + c_3 \Delta &= 0.
\end{align}
The fermionic solutions are also calculable as

$$\psi_1^2 = CV_{12} \sqrt{\frac{iV_{11} - \omega}{iV_{11} + \omega}} \exp \left[ i\omega \int_0^x dx \left( \frac{U_{12}}{V_{12}} + \frac{U_{21}}{V_{21}} \right) \right], \quad (B.20)$$

$$\psi_1^2 = -CV_{21} \sqrt{\frac{iV_{11} + \omega}{iV_{11} - \omega}} \exp \left[ i\omega \int_0^x dx \left( \frac{U_{12}}{V_{12}} + \frac{U_{21}}{V_{21}} \right) \right], \quad (B.21)$$

where $\psi = (\psi_1, \psi_2)^T$ and $C$ is the normalization constant. The square-root of those function must be taken such as $v/u = iV_{21}/(iV_{11} - \omega)$.

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