Instantons in Large Order of the Perturbative Series

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ABSTRACT

Behavior of the Euclidean path integral at large orders of the perturbation series is studied. When the model allows tunneling, the path-integral functional in the zero instanton sector is known to be dominated by bounce-like configurations at large order of the perturbative series, which causes non-convergence of the series. We find that in addition to this bounce the perturbative functional has a subleading peak at the instanton and anti-instanton pair, and its sum reproduces the non-perturbative valley.

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1. Introduction

In quantum theories that allow tunneling phenomena the perturbative series is known to diverge.\cite{1-4} Existence of instanton configuration for real coupling leads to the result that perturbative coefficient behaves as $c_n \sim n^{n/2}$. This leads to non-summable (not even Borel-summable) perturbative series. Of course, this does not mean that such a theory is ill-defined. In fact, one has to take into account the “non-perturbative” effects. However, these “non-perturbative” effects have some overlapping with the perturbative series at large order. Therefore, it is of fundamental importance to seek a method to do converging calculation, incorporating the perturbative as well as non-perturbative contributions. This kind of analysis becomes of practical interest when one is faced with the problem of tunneling at higher energies, such as the baryon and lepton number violation process of the standard model at TeV energies.\cite{5-7}

In previous papers,\cite{8,9} the authors carried out analytical studies of the perturbative series at large orders. We used saddle-point approximation to the path-integral functional at large orders to identify the dominating configurations. We found that bounce-like configurations dominate and that configurations was responsible for the diverging (non-Borel-summable) nature of the perturbative series. This was further confirmed by various numerical calculations.

In this letter, we apply the above formalism to the study of the contribution of the non-perturbative configuration to the perturbative functional.
2. Analysis

For completeness we first go through the saddle point approximation for the perturbative functional. The actions of the models we deal with are second order polynomials of the coupling constant, which we denote by $g$.

\[ S = c_0 - gc_1 + g^2c_3 \]  

(1)

A one-dimensional quantum-mechanical model that fits in this category is,

\[ S[\phi, g] = \int d\tau \left[ \frac{1}{2} \left( \frac{\partial \phi}{\partial \tau} \right)^2 + \frac{1}{2} \phi^2 (1 - g\phi)^2 \right]. \]

(2)

The instanton solution corresponding to the tunneling from $\phi = 0$ to $\phi = 1/g$ through the double-well potential is known to have action $S^{(1)} = 1/6g^2$. This action leads to the WKB tunneling amplitude $e^{-1/6g^2}$.

In evaluating perturbative expansion of the partition function,

\[ Z = \int \mathcal{D}\phi \ e^{-S[\phi, g]}, \]

(3)

in the zero-instanton sector, we first expand integrand in powers of $g$;

\[ e^{-S[\phi, g]} = \sum_{n=0}^{\infty} g^n F_n[\phi]. \]

(4)

The perturbative functional $F_n[\phi]$ can be expressed as a contour integral in the complex $g$-plane,

\[ F_n[\phi] = \frac{1}{2\pi i} \oint \frac{dg}{g} e^{-(S[\phi, g] + n \log g)}. \]

(5)

This can be evaluated using the saddle point approximation for the exponent $S + n \log g$ (which we denote by $\tilde{S}$ hereafter) in the $n \to \infty$ limit. There are always two
saddle points. Their positions depend on the behavior of the function $\phi(\tau)$: In case $D \equiv 1 - (8nc_2/c_1^2) > 0$, there are two saddle points on the real axis of the $g$-plane, while otherwise there is a complex conjugate pair of saddle points. In the first case, we have found that we should choose the contour to go through the saddle-point of smaller absolute value. In the latter case ($D < 0$), the contour should be chosen to go through both saddle points, which yields the following expression:

$$F_n[\phi] = \frac{2}{\sqrt{2\pi}} \text{Im} \left[ \frac{1}{\sqrt{\tilde{S}'' g^2}} \frac{e^{-S[\phi,g_-]}}{g^n} \right],$$

(6)

where we denoted the saddle point in the lower half-plane by $g_-$,

$$g_- = g_0 e^{-i\theta}, \quad g_0 \equiv \sqrt{\frac{n}{2c_2}}, \quad \cos \theta \equiv \frac{c_1}{\sqrt{8nc_2}}.$$

(7)

In this case, since the contribution of the pair of saddle points made (6) different from the simple $e^{-S}$ form, the equation that determines the dominating configuration is different from ordinary equation of motion. This allowed the dominating configuration to be a bounce solution, which did not exist in the original theory.

3. Instantons in perturbative functional

In order to look at the behavior of the perturbative functional, we first numerically evaluate $F_n[\phi]$ in a two-dimensional subspace of the functional space of $\phi(\tau)$, which contain both the bounce-like configuration and the instanton pair. We choose this space to be the space of two parameters $d$ and $\phi_0$, which determine
\( \phi(\tau) \) as in the following;

\[
\phi^{(II)}(\tau) = \frac{\phi_0}{(1 + e^{-\tau-d/2}) (1 + e^{\tau+d/2})}.
\] (8)

In (8), if we choose \( \phi_0 = 1/g \) and \( d \) large enough, we obtain instanton and anti-instanton pair. On the other hand, if we choose \( d \sim 0 \), (8) is similar to bounce of height \( \phi_0 \). [Since \( \phi^{(II)}(\tau) \) is invariant under \( (d, \phi_0) \rightarrow (-d, \phi_0 e^d) \), only \( d > 0 \) region shall be considered.] For analysis, we have substituted (8) in (2) and obtained the expression \( S(d, \phi_0, g) \). The exact integrand \( e^{-S(d,\phi_0,g)} \) is plotted in Fig.1. The ridge at \( \phi_0 = 1/g \) is the “valley” (of action) in the valley methods.[10–12] We have carried out the \( g \)-expansion (4) analytically (using Mathematica). In Fig.2, we plot the resulting finite-order sum,

\[
F_N(d, \phi_0) = \sum_{n=0}^{N} g^n F_n(d, \phi_0) .
\] (9)

As the order \( N \) increases, the peak structure at \( d \sim 0 \) rises. This corresponds to the fact that the perturbative functional \( F_n[\phi] \) is dominated by bounce, as was shown in Re. 8 and 9. Furthermore, it is seen that the ridge structure (valley) is reproduced at higher orders of the perturbation expansion. More specifically, as the order increases, the ridge structure extends to larger separation \( d \).

Let us now carry out analytical study of what has been observed above. We take a pair of instanton and anti-instanton separated by a large distance \( d \) and approximate the value of (9) as a function of \( d \). We first note that the configuration has

\[
c_1 = \frac{1}{g^3}(d - 3 + ...), \quad c_2 = \frac{1}{2g^4}(d + \frac{11}{3} + ...).
\] (10)

[The omitted parts are non-leading and are irrelevant.] We find that for \( d < 8ng^2 \),
$D < 0$. In estimating value of $F_n[\phi]$ for that case, the most important factor is the exponent $\text{Re}(S)$. Using (10) and (7), we find that

$$\text{Re}(S) = S[\phi^{(II)}, g] + \delta S(g, d), \quad \delta S(g, d) = \frac{d}{2g^2} - \frac{n}{2}. \quad (11)$$

The contribution of the $g^n$ factor in (6) should also be looked at, since potentially it might have large contribution. The major part comes from the absolute value $g_0$, which is now

$$g_0 = g \sqrt{\frac{g^2 n}{d}}, \quad (12)$$

which adds logarithmic contributions to $\delta S$ in (11). Combining (11) and (12), we find that

$$\text{Re} \left[ \frac{e^{-S[\phi,g]}}{g^n} \right] = \frac{e^{-S[\phi^{(I)},g]}}{g^n} e^{-\Delta},$$

$$\Delta = \frac{n}{2} \left( \frac{d}{ng^2} - 1 + \log \frac{ng^2}{d} \right). \quad (13)$$

Therefore we find that in the $n$-th order perturbative polynomial $F_n[\phi]$ has a peak at the instanton pair with distance

$$d = ng^2 + \text{(non-leading terms)}. \quad (14)$$

Further, since this pair has the “right” weight, $e^{-1/3g^2} / g^n$, as $g^n F_n[\phi]$ is summed up, the valley is reproduced to larger and larger $d$ with the right weight $e^{-1/3g^2}$. This agrees with what can be observed in Fig.2.
4. Discussions

In this letter we have seen that the perturbative functional integrand $F_n[\phi]$ have peaks at instanton and anti-instanton pair at distance $d \sim ng^2$. This implies the following: Since the perturbative coefficient behaves as $g^n n^{n/2}$ for $n \gg 1$, it starts to diverge at around $n \sim 1/g^2$. Therefore if there exists a way to separate the “perturbative” contribution from the “non-perturbative” one (i.e. instanton pairs), so that both give convergent results, the effective cut-off of the perturbative series should be at $n \sim 1/g^2$. On the other hand, according to (14), if the perturbation series is cut-off at $n \sim 1/g^2$, it does not contain instanton pairs of separation larger than $O(1)$. Since the width of instantons are of order one, for $d > 1$ the instanton pair is well-separated. Thus the cutoff at $n \sim 1/g^2$ serves to cutoff the well-separated (and therefore definitely “non-perturbative”) instanton and anti-instanton pairs. [This is the advantage not shared by the method of fundamental region discussed in Ref.9]. Thus our result means that the cut-off at $n \sim 1/g^2$ not only makes the perturbative series well-behaved, but also separates the well-defined non-perturbative valley.

The extension of the analysis to the field theories including the gauge theories is straightforward. The corresponding results will be published in near future, together with the details of the analysis presented here.\textsuperscript{[13]}
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FIGURE CAPTIONS

1) The plot of $e^{-S(d,\phi_0,0.4)}$ in the $(d, \phi_0)$ space. The axis $\phi = 0$ is the classical minimum, where the functional is equal to 1. The ridge at $\phi_0 = 1/g$ for large $d$ corresponds to the pair of instanton and anti-instanton with distance $d$. The time-integration is actually done only for $\tau > 0$ region. Therefore the height of the ridge is determined by one instanton action $1/6g^2$.

2) Three dimensional plots of (9), i.e., the integrand $e^{-S}$ cut-off at $N$-th order. $N$ is chosen to be 2, 4, 6, 8 for Figs. (a), (b), (c), (d), consequently. The axis, scales, and the value of $g$ are the same as in Fig.1. The plots are cut off from above at height 1.
This figure "fig1-1.png" is available in "png" format from:

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