The embedding structure and the shift operator
of the U(1) lattice current algebra

by

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Abstract

The structure of block-spin embeddings of the U(1) lattice current algebra is described. For an odd number of lattice sites, the inner realizations of the shift automorphism are classified. We present a particular inner shift operator which admits a factorization involving quantum dilogarithms analogous to the results of Faddeev and Volkov.

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1. Introduction

For \( N \geq 3 \), the \( U(1) \) lattice Kac Moody algebra is generated by the operators \( 1 \) and \( u_i \), \( i = 1, \ldots, N \), with relations

\[
\begin{align*}
    u_i u_j &= u_j u_i \quad \text{for} \ |i - j| \geq 2, \\
    u_i u_{i+1} &= q u_{i+1} u_i \quad \text{for} \ i = 1, \ldots, N
\end{align*}
\]

(1)

with some \( q \in \mathbb{C} \); \( u_{N+i} \equiv u_i \). We specialize to the case that \( q \) is a primitive \( p \)th root of unity, \( p > 1 \). Then \( u_i^* := u_i^{-1} \) defines a *-operation; moreover, since the elements \( u_i^p \) are all central, one can impose the further relations

\[
u_i^p = 1 \quad \text{for all } i.
\]

(2)

Relations (1) and (2) define a finite-dimensional \( C^* \)-algebra \( \mathcal{U}_N \) of dimension \( p^N \). The centre \( Z(\mathcal{U}_N) \) of \( \mathcal{U}_N \) is \( p \)-dimensional with generator

\[
c = u_1 u_2 \cdots u_N
\]

(3)

for odd \( N \), but \( p^2 \)-dimensional with generators

\[
c_1 = u_1 u_3 \cdots u_{N-1}, \quad c_2 = u_2 u_4 \cdots u_N
\]

(4)

for even \( N \).

The \( U(1) \) lattice Kac Moody algebra, as the name indicates, is designed as a discretization of the commutation relations of a chiral abelian current. Abelian and non-abelian lattice Kac Moody algebras have been studied e.g. in [1−4]. They prove to be useful in investigating the correspondence of the representation theory of quantum groups and of Kac Moody algebras [3,4]. More recently, in [2] it was shown that abelian lattice Kac Moody algebras are relevant in the analysis of integrable field theories in two dimensions.

One of the main objects in the study of the above algebra is the (periodic) shift automorphism

\[
s : \mathcal{U}_N \longrightarrow \mathcal{U}_N, \quad u_i \mapsto u_{i+1},
\]

(5)

which corresponds to the chiral time evolution operator on the lattice, see [1,2]. An important issue in those references was to find a realization of \( s \) by an inner automorphism: \( s = ad_U \) for some unitary \( U \in \mathcal{U}_N \). Since inner automorphisms leave central elements fixed, (3) and (4) make it clear that the solution \( U \) of this problem depends very much on whether \( N \) is even or odd. Indeed, for even \( N \) an inner shift operator can only exist in a representation of \( \mathcal{U}_N \) where \( c_1 = c_2 \), see [1,2]. Whereas this condition may be considered in the framework of field theory on the lattice with fixed (even) number of sites, we are mainly interested in the continuum limit \( N \to \infty \). The analysis of algebra embeddings induced by refining the lattice (see Section 2) shows that the structure of this continuum limit is in a sense much closer to that of the algebras \( \mathcal{U}_N \) with \( N \) odd than with \( N \) even. In particular, the constraint \( c_1 = c_2 \) is incompatible with the embeddings. So from this
point of view, it appears to be important to construct the shift operator for $N$ odd, which will be done in Section 3 of this letter. In Section 2, we will make the above statements on the continuum limit of the $U(1)$ lattice Kac Moody algebra more precise.

Let us first introduce some notational convention: In many equations below, factors $q^{1/2}$ will appear. For $p$ even, this cannot be avoided since the spectrum of the central element $c$ is contained in the $p$ th roots of $-1$. The choice of the sign of $q^{1/2}$ has no effect. For $p$ odd, we define $q^{1/2}$ with the help of $2^{-1} = \frac{p+1}{2}$ in the ring $\mathbb{Z}_p$ so that in this case everything could be written without any square roots of $q$. We introduce them nevertheless because this enables us to use identical formulas for $p$ even and odd in almost every equation.

2. Embeddings and the continuum limit algebra

In [4], it was shown how $\mathcal{U}_N$ can be embedded into $\mathcal{U}_{N'}$, for $N' > N$. The key idea is to associate the generators $u_i$ of $\mathcal{U}_N$ to the intervals $I_i \subset S^1$, $i = 1, \ldots, N$, of a triangulation $\mathcal{T}$ of the circle into $|\mathcal{T}| := N$ intervals. The procedure of refining $\mathcal{T}$ to $\mathcal{T}'$ with $|\mathcal{T}'| = N' > N$ by splitting the intervals, e.g. $I_i = I_{i_1} \cup \ldots \cup I_{i_{k_i}}$ with $N' = \sum_{i=1}^{N} k_i$, induces the following unital algebra homomorphism

$$\iota_{\mathcal{T}} : \mathcal{U}_{\mathcal{T}} \rightarrow \mathcal{U}_{\mathcal{T}'} , \quad u_i^{\mathcal{T}} \mapsto q^{\frac{k_i-1}{2}} u_{i_{k_i}}^{\mathcal{T}'} \cdots u_{i_1}^{\mathcal{T}'} .$$

(6)

Here we have labelled the algebras by their corresponding triangulations, and we have added a $q$-factor in comparison to [4] in order to preserve relation (2).

Since the set of all triangulations of $S^1$ is directed (i.e. for any two $\mathcal{T}$ and $\mathcal{T}'$ there is a common refining $\mathcal{T}''$), the collection of all $\mathcal{U}_{\mathcal{T}}$ and all $\iota_{\mathcal{T}}$, forms a directed system and one can define the inductive limit

$$\mathcal{U}^\infty = \lim_{\rightarrow \mathcal{T}} \mathcal{U}_{\mathcal{T}} .$$

(7)

$\mathcal{U}^\infty$ may be understood as $(\bigcup_{\mathcal{T}} \mathcal{U}_{\mathcal{T}}) / \sim$, where $x_{\mathcal{T}} \in \mathcal{U}_{\mathcal{T}}$ and $y_{\mathcal{T}'} \in \mathcal{U}_{\mathcal{T}'}$ are equivalent, $x_{\mathcal{T}} \sim y_{\mathcal{T}'}$, iff there exists a common refining $\mathcal{T}''$ of $\mathcal{T}$ and $\mathcal{T}'$ such that $\iota_{\mathcal{T}''} (x_{\mathcal{T}}) = \iota_{\mathcal{T}''} (y_{\mathcal{T}'})$ in $\mathcal{U}_{\mathcal{T}''}$. As was explained in [4], this algebraic version of the continuum limit also carries a representation of the chiral conformal group $\text{Diff}_+(S^1)$, which is installed more or less “by brute force” since the collection $\bigcup_{\mathcal{T}} \mathcal{U}_{\mathcal{T}}$ contains one algebra $\mathcal{U}_{\mathcal{T}}$ per triangulation, in particular both $\mathcal{U}_{\mathcal{T}}$ and $\mathcal{U}_{g\mathcal{T}}$ for any conformal transformation $g \in \text{Diff}_+(S^1)$.

At the same time, this implies that $\mathcal{U}^\infty$ is rather difficult to handle, simply because of its size: For a fixed length $N$ of triangulations, there is already a continuum of associated algebras $\mathcal{U}_{\mathcal{T}}$ in the collection, which by definition are all isomorphic to $\mathcal{U}_N$. Any two of these – $\mathcal{U}_{\mathcal{T}_1}$ and $\mathcal{U}_{\mathcal{T}_2}$, say, with $|\mathcal{T}_1| = |\mathcal{T}_2| = N$ – are embedded into some $\mathcal{U}_{\mathcal{T}'}$ – one can take $|\mathcal{T}'| = 2N$ –, and standard facts on finite-dimensional algebras tell that the images $\iota_{\mathcal{T}_1}^{\mathcal{T}'} (\mathcal{U}_{\mathcal{T}_1})$ and $\iota_{\mathcal{T}_2}^{\mathcal{T}'} (\mathcal{U}_{\mathcal{T}_2})$ are unitarily equivalent in $\mathcal{U}_{\mathcal{T}'}$ (see e.g. [5]), but they need not coincide. Therefore we expect that also the inductive limit $\mathcal{U}^\infty$ is a huge object if the equivalence relation $\sim$ from above is used.
In this paper, we will adopt a different notion of continuum limit, which gives a smaller limiting algebra \( U_\infty \) and is defined in closer analogy to what is done “on the lattice” when the number of sites is sent to infinity. We start from a countable system \( (T_N)_{N=3}^\infty \) of triangulations where \( N = |T_N| \) and \( T_{N+1} \) is a refining of \( T_N \); again we associate an algebra \( U_N \) to \( T_N \). This means that we have replaced all the unitarily equivalent algebras \( U_T \) with \( |T| = N \) from (7) by just one “reference copy” \( U_N \). Together with the embeddings \( \iota_N, \iota_{N'} \) as in (6) – with an obvious change of notation from \( T \) to \( N \) – we get a directed system \((U_N, \iota_{N'})\) so that the inductive limit

\[
U_\infty = \lim_{N \to \infty} U_N \tag{8}
\]

is well defined. \( U_\infty \) in principle depends on the specific sequence of triangulations; we have suppressed this fact in the above notation because at least the isomorphism class of \( U_\infty \) does not, as the following propositions will show.

We would like to mention that \( U_\infty \) still allows for the appealing physical interpretation of the refining-embedding procedure (6) as an inverse block-spin transformation [4]; the limiting algebras \( U_\infty, U_\infty \) by definition display block-spin transformation invariance, which is a lattice remnant of full conformal invariance.

For the case \( q^p = 1 \), the continuum limit (8) is very handy, since all the \( U_N \) are multi matrix algebras; in other words, \( U_\infty \) is a so-called AF-algebra after taking the \( C^* \)-closure, see e.g. [5] – “AF” short for “approximately finite-dimensional”. Therefore \( U_\infty \) is determined up to isomorphism by the Bratteli diagram (see e.g. [5, 6]) of the inclusions \( \iota_N^N: U_N \to U_{N'} \), which in turn is given as soon as we know the decomposition of \( U_N \) into simple matrix factors, and how these are embedded into those of \( U_{N'} \) under \( \iota_N^N \) [7].

**Proposition 1**

\[
U_N = \bigoplus_{i=0}^{p-1} U_N^i \quad \text{with} \quad U_N^i \cong M_{\frac{N-1}{2}}(\mathbb{C}) \quad \text{for all } i \text{ if } N \text{ is odd},
\]

\[
U_N = \bigoplus_{k,l=0}^{p-1} U_N^{k,l} \quad \text{with} \quad U_N^{k,l} \cong M_{\frac{N-2}{2}}(\mathbb{C}) \quad \text{for all } k, l \text{ if } N \text{ is even}.
\]

**Proof:** The number of simple factors in \( U_N \) is just given by the dimension of the centre. Let \( e_i^{(N)} \) (\( i = 0, \ldots, p-1; \ N \text{ odd} \)) resp. \( e_{kl}^{(N)} \) (\( k, l = 0, \ldots, p-1; \ N \text{ even} \)) denote the minimal central projections of \( U_N \), i.e. \( U_N^i = e_i^{(N)} U_N \) and \( U_N^{k,l} = e_{kl}^{(N)} U_N \). It is an easy exercise in root of unity calculations to show that

\[
e_i^{(N)} = \frac{1}{p} \sum_{n=0}^{p-1} q^{in} q^{-\frac{N-2}{2n}} a^n, \tag{9}
\]

\[
e_{kl}^{(N)} = \frac{1}{p^2} \sum_{n,m=0}^{p-1} q^{kn+lm} c_1^n c_2^m; \tag{10}
\]
of course, the numeration of the minimal central projections is arbitrary, the one we have
chosen above (involving the \( N - 2 \) in the second \( q \)-factor in (9)) will lead to simple formulas
for the embeddings \( \iota_N \), later on. In order to show that all the \( e^{(N)}_i \), resp. all the \( e^{(N)}_{kl} \), have
the same rank, we just use the automorphisms \( R_1 : u_i \mapsto q^{\delta_{i1}} u_i \) and \( R_2 : u_i \mapsto q^{\delta_{i2}} u_i \) of
\( U_N \) to permute the minimal central projections:

\[
\begin{align*}
R_1(e^{(N)}_i) &= e^{(N)}_{i+1}, \\
R_1(e^{(N)}_{kl}) &= e^{(N)}_{k+1,l}, \\
R_2(e^{(N)}_{kl}) &= e^{(N)}_{k,l+1};
\end{align*}
\]

the subscripts are understood modulo \( p \).

As for the embeddings, it is sufficient to study “elementary” ones \( \iota^N := \iota^{N+1}_N \) associated
with cutting just one interval of \( T_N \) into two pieces.

**Proposition 2**

\[
\begin{align*}
\iota^N(U^{k,l}_N) &\subset U^{k,l+1}_{N+1} \text{ (mod } p), \quad N \text{ even, } \\
\iota^N(U^i_N) &= \bigoplus_{k+l=i \text{(mod } p)} U^{k,l}_{N+1}, \quad N \text{ odd.}
\end{align*}
\]

**Proof:** Let \( \delta^{(p)}_{n,m} \) denote the Kronecker symbol defined as \( \delta^{(p)}_{n,m} = 1 \) if \( n \equiv m \) (mod \( p \)) and
zero otherwise. For the special elementary embedding

\[
\begin{align*}
\iota^N(u^{(N)}_i) &= u^{(N+1)}_i \quad \text{for } i = 1, \ldots, N - 1, \\
\iota^N(u^{(N)}_N) &= q^{1/2} u^{(N+1)}_{N+1} u^{(N+1)}_N,
\end{align*}
\]

one can verify by direct computation that

\[
\begin{align*}
\iota^N(e^{(N)}_{kl}) \cdot e^{(N+1)}_i &= \delta^{(p)}_{k+l,i} \iota^N(e^{(N)}_{kl}), \quad N \text{ even, } \\
\iota^N(e^{(N)}_i) &= \sum_{k+l=i} e^{(N+1)}_{kl}, \quad N \text{ odd.} \quad (11)
\end{align*}
\]

Any other elementary embedding (cutting an interval other than \( I_N \) ) is obtained from this
\( \iota^N \) by applying shift automorphisms \( s^{(N)} \), \( s^{(N+1)} \) as in (5) several times. But

\[
\begin{align*}
s^{(N)}(e^{(N)}_i) &= e^{(N)}_i \quad N \text{ odd, } \\
s^{(N)}(e^{(N)}_{kl}) &= e^{(N)}_{lk} \quad N \text{ even,} \quad (12)
\end{align*}
\]

so that the formulas (11) are valid for arbitrary elementary embeddings. By definition of
the minimal central projections, the equations (11) imply the claim.  

These two propositions show that the Bratteli diagram of \( (U_N, \iota^N_N) \) in (8) is independent
of the chosen sequence of triangulations, and that it splits into \( p \) disconnected, identical
subdiagrams, each of which looks as follows: On the \( N \)th floor, \( N \) odd, there is just one
dot marked $p^{N-1}$ (representing one of the simple factors of $U_N$, say $U_N^i$); the $N + 1$st floor consists of $p$ dots, each marked $p^{N+1-2}$ (which represent the factors $U_{N+1}^{k,l}$ with $k + l \equiv i \pmod{p}$). From the single dot on the $N$th floor, one line goes to every dot on the $N + 1$st floor; from every such dot there is one line to the single dot (marked $p^{N+1-2}$) on the $N + 2$nd floor.

After “telescoping” the Bratteli diagram, see [5], such that only the odd floors with embeddings $(\iota^{N+1} \circ \iota^N) : U_N \to U_{N+2}$ are left, the subdiagrams of $U_\infty$ become even simpler: There is one dot per floor and $p$ lines between consecutive floors. One can see this directly if one first shows that $\iota^{N+1} (\iota^N (e_i^{(N)})) = e_i^{(N+2)}$ and then compares the rank of the projections $e_i^{(N+2)}$ and $e_i^{(N)}$: The ratio equals the number of lines since $\iota^{N+1} \circ \iota^N$ is unital.

AF-algebras that possess an approximation by a tower of full matrix algebras, as is the case for the subdiagrams of $U_\infty$, are called UHF-algebras (“uniformly hyperfinite”), see [5, 8]. They are completely determined (up to isomorphism) by giving the infinite “product” of numbers of lines in their Bratteli diagram; in our particularly simple case we can write

$$U_\infty \cong \bigoplus_{\alpha=1}^p M_{p^\alpha} .$$

Note that one can regard $M_{p^\alpha}$ as an infinite tensor product $M_{p^\alpha} = M_p(\mathbb{C})^\otimes\infty$.

All in all, we have shown that it is indeed the algebras $U_N$ with $N$ odd that determine the essential features of the continuum limit as defined in (8); in particular, the centre of $U_\infty$ is spanned by the images (in $U_\infty$) of the $e_i^{(N)}$ and is also $p$-dimensional.

3. Construction of the shift operator

In [1, 2], an expression for an inner shift operator $U \in U_N$ with $s = ad_U$ was given for the case $N$ even under the additional restriction $c_1 = c_2$:

$$U_{even} = r(u_2)r(u_3)\cdots r(u_N)$$

The main feature of this solution is the appearance of the function $r(w)$ which is determined up to a constant factor by the functional equation $(\gamma \in \mathbb{C})$

$$wr(w) = \gamma r(q^{-1}w) ;$$

here $q$ need not be a root of unity. The function $r(w)$ enjoys the property $r(w) = r(w^{-1})$ and for $\gamma = q^{1/2}$ can be factorized as

$$r(w) = S(w)S(w^{-1}) \quad \text{with} \quad \frac{S(q^{1/2}w)}{S(q^{-1/2}w)} = \frac{1}{1+w} .$$

If $q = e^z, |q| < 1$, one can show that [9]

$$S(w) = \exp\{\frac{1}{e} \text{Li}_2(-w) + \mathcal{O}(e)\}$$
where
\[ \text{Li}_2(w) = \sum_{k=1}^{\infty} \frac{w^k}{k^2} \]
is the Euler dilogarithm function; that is why \( S(w) \) is called “quantum dilogarithm”.

In the following, two constructions for a unitary shift operator of \( \mathcal{U}_N \), \( N \) odd, will be given, the first one starting from formula (14) and producing a “correction factor”, the second one giving a formula for the most general unitary shift operator. We would like to point out in advance that, although abstractly \( U \) is unique up to invertible central factors, the concrete expression for it is highly non-unique. This makes the existence of the beautiful formula (14) for even \( N \) only the more remarkable. The ones for \( N \) odd turn out to be somewhat more complicated.

Besides the function \( r(w) \) from (15) we will need \( \tilde{r}(w) \) satisfying the similar relation
\[ w\tilde{r}(w) = \tilde{\gamma}\tilde{r}(q^{-2}w) \tag{17} \]
for some \( \tilde{\gamma} \in \mathbb{C} \). Both functions may exist for \( q^p \neq 1 \) as well (see [1, 2] for explicit solutions), and accordingly some of the formulas in Proposition 3 below are valid whenever invertible solutions to (15) and (17) exist. But we are mainly interested in the case when both \( q^p = 1 \) and the argument \( w \) is such that \( w^p = 1 \). Then the constants \( \gamma \) and \( \tilde{\gamma} \) cannot be chosen arbitrarily.

For \( \gamma = q^{1/2} \) and \( w^p = 1 \), \( w^* = w^{-1} \),
\[ r(w) = \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} q^{k^2} w^k \tag{18} \]
is a unitary solution of (15): \( r(w)^*r(w) = 1 \). The proof is straightforward using the equation \( \sum_{n=0}^{p-1} q^{kn} = p \delta^{(p)}_{k,0} \) for a primitive root of unity.

The (up to constant factors) most general solutions of (17) with \( q^p = 1 \) and \( w^p = 1 \) are
\[ \tilde{r}_l(w) = \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} q^{k(k+1)} q^{-kl} w^k \quad , \quad \tilde{\gamma}_l = q^l \tag{19} \]
for \( l \equiv 0, \ldots, p - 1 \pmod{p} \), which again are essentially theta functions. With \( w^* = w^{-1} \), these \( \tilde{r}_l(w) \) are unitary if \( p \) is odd, whereas we have
\[ \tilde{r}_l(w)^*\tilde{r}_l(w) = 1 + (-1)^{p/2+1-l} w^{p/2} \tag{20} \]
for even \( p \); note that then \( q^{p/2} = -1 \). (20) shows that, for \( p \) even, a unitary solution of (17) can be found only for special arguments. For instance, assume that
\[ w^{p/2} = \sum_{i=0}^{p-1} (-1)^{i+1} e_i^{(N)} \quad \in Z(\mathcal{U}_N) \tag{21} \]
where \( e_i^{(N)} \) are the minimal central projections (9) of \( \mathcal{U}_N \). Then

\[
\tilde{r}(w) := \frac{1}{\sqrt{2}} \sum_{i=0}^{p-1} \tilde{r}_{i+p/2}(e_i^{(N)} w)
\]

(22)

provides a unitary solution of (17). Here, in the different simple factors \( \mathcal{U}_N^i \) of \( \mathcal{U}_N \), different \( \tilde{r}_l(w) \) functions (and \( \tilde{\gamma}_l \)) have been used. Now we are ready to state our first result on the shift operator:

**Proposition 3** Denote by \( W \) the product

\[
W = r(u_2) r(u_3) \cdots r(u_N)
\]

(23)

and by \( u_1 \) the operator

\[
u_1 = \gamma u_N u_{N-2} \cdots u_3 u_1^{-1} u_2^{-1} u_4^{-1} \cdots u_{N-1}^{-1}
\]

(24)

with \( \gamma = q^{1/2} \) as before. A unitary inner shift operator \( U \in \mathcal{U}_N \) for \( N \) odd, \( q^p = 1 \), can be written in the form

\[
U = V \cdot W
\]

(25)

for \( p \) odd, \( V \) is given by

\[
V = \tilde{r}(\tilde{\gamma}q^{-2}u_1)
\]

(26)

where \( \tilde{r}(w) \) can be any \( \tilde{r}_l(w) \) of the solutions (19) of the relation (17) with the associated factors \( \tilde{\gamma}_l \) from the functional equation;

for \( p \) even, \( V \) is given by

\[
V = \left( \sum_{i=0}^{p-1} e_i^{(N)} x^{i+p/2} \right) \tilde{r}(q^{-2}u_1)
\]

(27)

with the function \( \tilde{r}(w) \) defined in (22) and

\[
x = (u_3 u_5 \cdots u_N)^{-1}
\]

If \( q^p \neq 1 \) and invertible solutions of (15) and (17) exist, the formula of the \( p \) odd case can be taken to define an invertible inner shift operator of the \( U(1) \) lattice Kac Moody algebra.

Proof: From (1) and the functional equation (15) one deduces that

\[
\begin{align*}
   r(u_i)u_{i-1}r(u_i)^{-1} &= \gamma u_i u_{i-1} \\
   r(u_i)u_{i+1}r(u_i)^{-1} &= \gamma u_i^{-1} u_{i+1} \\
   r(u_i)u_jr(u_i)^{-1} &= u_j \quad \text{for } |i-j| \geq 2
\end{align*}
\]

(28)
This in turn implies that the operator $W \in U_N$ is almost the desired shift, namely
\[
W u_{i+1} = u_{i+1} \quad \text{for } i = 2, \ldots, N - 1,
\]
\[
W u_N W^{-1} = u_1 u_1,
\]
\[
W u_1 W^{-1} = u_1^{-1} u_2,
\]
with $u_1$ as in (24). $u_1$ commutes with $u_i$ for $i = 3, \ldots, N$, but
\[
W u_1 = u_1 + 1
\]
\[
W u_2 W^{-1} = u_1 u_2,
\]
\[
W u_3 W^{-1} = u_1^{-1} u_2,
\]

with $u_1$ as in (24). $u_1$ commutes with $u_i$ for $i = 3, \ldots, N$, but
\[
W u_1 = W u_1 W^{-1} = u_1 + 1
\]
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W u_3 W^{-1} = u_1^{-1} u_2,
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\]
\[
W u_3 W^{-1} = u_1^{-1} u_2,
\]

Together with (17) this leads to relations similar to (28):
\[
\tilde{r}(u_1) u_1 \tilde{r}(u_1)^{-1} = u_1 (\tilde{\gamma}^{-1} q^2 u_1)^{-1},
\]
\[
\tilde{r}(u_1) u_2 \tilde{r}(u_1)^{-1} = (\tilde{\gamma}^{-1} q^2 u_1) u_2,
\]

so that $U = \tilde{r}(\tilde{\gamma} q^{-2} u_1) W$ indeed satisfies the shift relation $U u_i = u_{i+1} U$ for all $u_i$. Since $u_1$ is unitary and $u_1^p = 1$, the first factor, i.e. $V$ from (26), is well defined and unitary for $p$ odd.

If $p$ is even, however, we have to take into account the extra $w^{p/2}$ term in (20): Fortunately, $u_1^{p/2}$ is just given by the rhs of (21) and therefore a unitary $\tilde{r}(u_1)$ and also $\tilde{r}(q^{-2} u_1)$ can be defined as in (22). Then the relations (29) have to be read in each simple factor of $U_N$ separately, with $\tilde{\gamma}_{i+p/2} = q^{i+p/2}$ in the factor $U_N^i$. This leaves us with an operator $U' := \tilde{r}(q^{-2} u_1) W$ shifting $u_2, \ldots, u_{N-1}$ in the correct way but producing extra ($i$-dependent) $\tilde{\gamma}$ factors when acting on $u_N, u_1$:
\[
U' u_N U'^{-1} \cdot e_i^{(N)} = \tilde{\gamma}_{i+p/2} u_1 \cdot e_i^{(N)},
\]
\[
U' u_1 U'^{-1} \cdot e_i^{(N)} = \tilde{\gamma}_{i+p/2} u_2 \cdot e_i^{(N)},
\]

One finds that these $\tilde{\gamma}$ factors cannot be absorbed by changing the argument of $\tilde{r}$ as we did for $p$ odd, since this would spoil the property (21) and would render $\tilde{r}(\tilde{\gamma} q^{-2} u_1)$ non-unitary. Instead, we seek a unitary operator $x \in U_N$ with the properties
\[
x u_i x^{-1} = u_i \quad \text{for } i = 3, \ldots, N,
\]
\[
x u_1 x^{-1} = q^{-1} u_1,
\]
\[
x u_2 x^{-1} = q u_2,
\]

Since $u_1$ as in the Proposition. Note that $y := u_2 u_4 \cdots u_{N-1} u_1$ satisfies the same commutation relations, and that $u_1 \sim (yx)^{-1}, c \sim yx^{-1}$. After splitting into the simple factors, the unitary operator
\[
\sum_{i=0}^{p-1} e_i^{(N)} x_{i+p/2}
\]
then acts by cancelling the superfluous $\tilde{\gamma}$s. Altogether, we arrive at formula (27) for $V$ in the case when $p$ is even.

For further remarks on the object $u_1$, we refer to the concluding section. But let us mention here that the splitting of $U$ into separate shift operators for the simple factors, which was necessary for $p$ even, is not so surprising if one identifies the simple factors with inequivalent superselection sectors of the model. Of course, every sector has its own chiral time evolution operator so that from this point of view the surprise is rather that the splitting is not necessary if $p$ is odd.

Since the centre of $U_N$ is $p$-dimensional for $N$ odd, there is a $p$-torus of unitary shift operators, and Proposition 3 provides just one example. In the following, we will give a formula for arbitrary shift operators by determining a “basis” of unitary shift operators in the simple factors $U_i^N$; these shifts are unique up to a phase.

We introduce some notation first: Let $d = \gcd(N, p)$ be the greatest common divisor of $N$ and $p$, and $N = N’d$, $p = p’d$. The natural $\mathbb{Z}_N$-action on $\mathbb{Z}_p$ induces a partition of \{0, \ldots, p−1\} into $d$ cycles $C_j = \{j, j+d, \ldots, j+(p’−1)d\}$, $j = 0, \ldots, d−1$; put differently, the $C_j$ are the cosets $\mathbb{Z}_p/C_0$.

**Proposition 4** For $k = 0, \ldots, p−1$, let $U^{(k)}$ be the operator

$$U^{(k)} = \frac{1}{\sqrt{d}} \frac{1}{p^{\frac{N-1}{2}}} \sum_{r_1, \ldots, r_N=0}^{p-1} [k] q^{e(r_1,\ldots,r_N)} u_1^{r_1} \cdots u_N^{r_N} \in U_N$$

where

$$e(r_1,\ldots,r_N) = -r_2(r_1 + r_N) - r_3(r_2 + r_1 + r_N) - \ldots - r_{N−1}(r_{N−2} + \ldots + r_1 + r_N)$$

and $\sum[k]$ indicates that the summation variables are subject to $\sum_i r_i \equiv k \pmod{p}$.

If $i \in C_j$, then $U^{(j/2)} e_i^{(N)}$ is the (up to a phase unique) unitary shift operator of the simple factor $U_i^N$ of $U_N$ for $i = 0, \ldots, p−1$; we use $2^{-1} = \frac{d+1}{2}$ in $\mathbb{Z}_d$ to define $j/2$ if $j$ is odd.

If $U$ is an arbitrary unitary shift operator of $U_N$, there exist complex numbers $\lambda_i$ with $|\lambda_i|^2 = 1$ for $i = 0, \ldots, p−1$ such that

$$U = \sum_{j=0}^{d-1} \left( \sum_{i \in C_j} \lambda_i e_i^{(N)} \right) U^{(j/2)}.$$  \hspace{1cm} (32)

Remark: For definiteness, we have labelled the cycles $C_j$ by the particular representative $j \in C_j$, but of course any other would do as well. Accordingly, in the Proposition one can replace $U^{(j/2)}$ by any $U^{(j’/2)}$ with $j’ \in C_j$; note, however, that the coefficients $\lambda_i$ in (32) depend on the basis of shift operators in the simple factors.

Proof: We first have to show that the operators $U^{(k)}$ from (30,31) satisfy the relations

$$U^{(k)} u_i = u_{i+1} U^{(k)}$$

\hspace{1cm} (33)
with all the generators $u_i$ of $U_N$: Suppose e.g. that $i = 2, \ldots, N - 1$; then
\[ u_i^{r_1} \cdots u_i^{r_{i+1}} \cdots u_N^{r_N} \cdot u_i = q^{r_i'-r_{i+1}'-1} u_{i+1}^{r_1} \cdots u_i^{r_i'} u_{i+1}^{r_{i+1}'} \cdots u_N^{r_N} \]
with $r_i' := r_i + 1$, $r_{i+1}' := r_{i+1} - 1$; this substitution does not interfere with the summation restriction in $\sum [k]$, and furthermore
\[ e(r_1, \ldots, r_i, r_{i+1}, \ldots, r_N) + r_i - r_{i+1} - 1 = e(r_1, \ldots, r_i', r_{i+1}', \ldots, r_N) . \]
The cases $i = 1, N$ are worked out similarly. Note that one way to arrive at the operators $U^{(k)}$ from the Proposition is just to make an Ansatz for it with $e(r_1, \ldots, r_N)$ given by some undetermined quadratic form, and then to solve for (33) – leading to (31) as a particularly simple solution.

Thus the operators $U^{(k)}$ have the right shift commutation relations but they are not unitary unless $d = 1$: (33) implies that $U^{(k)} U^{(k)} \in Z(U_N)$ so that we have to determine the real numbers $\mu_i^{(k)}$ in the decomposition
\[ U^{(k)} U^{(k)} = \sum_{i=0}^{p-1} \mu_i^{(k)} e_i^{(N)} \]
in order to see which $U^{(k)}$ projects onto a unitary shift operator in which simple factor $U_N'$. The computation of $\mu_i^{(k)}$ is a little tedious, so we will only list the various steps: One first inserts (30,31) into the lhs of (34), uses the summation restrictions to eliminate two of the $2N$ summation variables, and brings the product of $u_i$’s into the standard form as in (30) again; here additional $q$-factors arise. Next a substitution of summation variables is applied such that only half of the new ones appear in the exponents of $U_N$-generators. It turns out that the other $N - 1$ summations can be performed with the help of $\sum_{k=0}^{p-1} q^k n = p \delta_n^{(p)}$, which on the whole yields a factor $p^{N-1}$ as well as conditions on the remaining summation variables: Namely, they all have to be equal (modulo $p$) to each other, and the last free variable, $r$ say, is subject to $NR \equiv 0 \pmod{p}$; this means that the sum is over $r = 0, p', \ldots, (d-1)p'$ only; we arrive at
\[ U^{(k)} U^{(k)} = \frac{1}{d} \sum_{r=0}^{p-1} q^{e(r,\ldots,r)+2kr} u_1^r \cdots u_N^r . \]

With our definition (9) of the minimal central projections $e_i^{(N)}$, we have
\[ u_1^r \cdots u_N^r = \sum_{i=0}^{p-1} q^{\frac{N-2}{2} r^2 - ir} e_i^{(N)} \]
so that
\[ \mu_i^{(k)} = \frac{1}{d} \sum_{r=0}^{p-1} q^{(2k-i)r} \]
here we have used \( Nr \equiv 0 \pmod{p} \) to simplify the \( q \)-exponent. This condition, by the way, also shows that \( \mu_{i+N}^{(k)} \equiv \mu_i^{(k)} \pmod{p} \), which means that \( \mu_i^{(k)} \) depends only on the cycle \( C_i \) and not on the representative \( i \in C_i \). Finally, note that the sum in (35) ranges over all \( d \)th roots of unity, raised to the power \( 2k - i \), therefore
\[
\mu_i^{(k)} = \delta_{2k,i}^{(d)} ;
\]
in particular, \( \mu_i^{(k)} = 1 \) for \( k = i/2 \) as claimed.

It would be desirable to rewrite the formulas for the shift operators given above into a product of \( r \)-functions similar to Proposition 3, if possible with more symmetric arguments. This would allow us to make contact with the quantum dilogarithm and in addition would perhaps lead to expressions valid for \( q^p \neq 1 \) as well. Unfortunately, we have not been able to find a nice factorization of a shift operator for general \( N \) yet. The difficulties start with choosing an appropriate unitary shift operator from the \( p \)-torus of solutions given above. Then, it is advisable to approach the factorization problem from a general point of view – since by simply “guessing” arguments of \( r \)-functions which are to be split off an expression like (30) one typically arrives at non-factorizable “remainders”. Generally, factorization of an operator like the \( U^{(k)} \) in Proposition 4 is achieved if we can find new summation variables which diagonalize the quadratic form in the \( q \)-exponent. However, the latter is then not just given by \( e(r_1, \ldots, r_N) \) from (31), but there are additional \( q \)-factors arising from re-grouping the generators \( u_i \) into more complicated expressions, which are in turn dictated by the substitution of variables. All in all, this “\( q \)-diagonalization problem” leads to equations considerably more involved than in the commutative case.

4. Further comments

The main part of this paper was addressed to the construction of the shift operator of \( U_N \) for \( N \) odd – not merely in order to complete the picture begun in [1, 2], but rather because the considerations of Section 2 suggest that \( N \) odd is the more important case. Unfortunately, the results we have obtained do not look as nice as formula (14) for the \( N \) even case – where, however, some complications are hidden in the necessary condition \( c_1 = c_2 \). Indeed, the only difference between formulas (14) and (25) is the additional factor involving the operator \( u_1 \) in the latter; but observe that \( u_1 \sim t^{N-1}(c_2c_1^{-1}) \) where \( t^{N-1} \) is an elementary embedding \( t^{N-1} : U_{N-1} \longrightarrow U_N \) induced by cutting the interval \( I_1 \) of \( T_{N-1} \) into two pieces, compare (6). Thus, \( u_1 \) appears simply because \( c_1 \neq c_2 \) in general. Stated differently, if we were to restrict \( U \) as in (25) to the subalgebra \( t^{N-1}(U_{N-1}) \subset U_N \) and to impose \( c_1 = c_2 \) in \( U_{N-1} \), the “correction factor” \( V \) from Proposition 3 would disappear. This fact is also evidence for the consistency of the shift operator construction with the embedding structure.

If one is interested in the continuum limit of the model and of the shift operator – a continuum limit defined by the embedding procedure of [4] –, then one is forced to consider the shift operator for \( N \) odd since the condition \( c_1^{(N-1)} = c_2^{(N-1)} \) is not mapped to \( c_1^{(N+1)} = c_2^{(N+1)} \).
One may expect that the $N$ odd expression (25) will approach a simpler limit when $N \to \infty$, because we would like to think of $u_1$ as some kind of “lattice artefact”: to $u_1$ a string of $u_{i-1}^{-1}$ is attached which hopefully “averages out” in the limit of large $N$.

We think that also the formulas given in Proposition 4 might prove useful in this context, e.g. if one wants to check rigorously whether the shift operators of $U_N$ have some limit in $U_\infty$ (or affiliated to this algebra).

Investigating this continuum limit further is in itself an interesting project. Among the obvious problems are to clarify the relation between $U^\infty$ of [4] and the smaller algebra $U_\infty$ used here, and to decide whether the full conformal symmetry is restored in $U_\infty$. For field theoretic applications it is very important to find representations with proper ground states.

One might also be able to construct a direct link between the (limiting) shift operator and the $L_0$-component of the energy momentum tensor. Since the shift operator can be expressed through the quantum dilogarithm (16), such a connection could e.g. be useful to find generalizations of the existing (classical) dilogarithm identities for the central charge and the conformal dimensions of some conformal field theories, see e.g. references [10]. On the other hand, there are quantum dilogarithm formulas for the scattering matrices of some 1+1-dimensional integrable quantum field theories [1, 2, 11]. Because of all these connections, it appears to be important to explore this function further.

Let us finally remark that the system of algebras $U_N$ may also serve as a “toy model” for applying the algebraic approach to quantum field theory, see e.g. [12] and references therein; in particular, one can define “local algebras” associated to proper intervals of $S^1$. We expect that, as soon as questions concerning the continuum limit and its relevant representations have been clarified, the U(1) lattice Kac Moody algebra could become much more than a toy model and serve as a basis for free field constructions to obtain other (conformal and maybe even massive) field theories – see [4] for further details. In this context, the fact that our basic model leads to an equally basic operator algebra is rather encouraging, since in the theory of operator algebras there is a number of constructions leading from UHF-algebras to more involved and more interesting algebras.

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