Geometric stress functions, continuous and discrete

Tamás Baranyai

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Abstract

A dimension independent potential function description of stresses is derived, both for $C^1$ continuous stress distribution in a material and for the discrete case of force and moment bearing space-frames. The language of the description is multi-vector valued differential forms. The continuous description gives the Airy stress function in the planar case and the Maxwell stress function in the 3 dimensional case. The discrete stress function is derived from the continuous one, then shown to be complete and minimal.

1 Introduction

The paper continues two lines of contemporary research. One stems from the observation that stresses in a material can be treated as vector valued differential forms [1, 2]. This lead to numeric computations of elasticity problems using Discrete Exterior Calculus [3], which relies on what have been called geometric discretization of elasticity [4]. The natural question arises: What can we learn if we try to have stress functions in the language of differential forms?

The other line of research is discrete stress functions for frames, where stresses are concentrated into the axis of the rods giving a force and moment resultant and the corresponding stress function is accordingly less continuous at the rod-axes. The idea came from Maxwell [5] and his planar construction is complete, similarly to the Airy stress function [6] Maxwell took the discrete analogue of. The only three dimensional discrete stress function the author knows of suffers from incompleteness [7, 8], which suggests room for improvement.

We will derive a dimension independent differential form based continuous stress function for $C^1$ continuous stress distribution. It will give the Airy stress function for planar and the Maxwell stress function for three dimensional problems. We will use the knowledge gained through this to derive a discrete stress function that is dimension independent, complete and minimal. This derivation will be done relying on the premise that stresses are non zero only in the axes of the frames. The derivation will lead us to a conceptually new approach regarding which part of the structure the discrete stress function should correspond to. Previously these functions have been defined in cells of a polyhedral frame, while we define them corresponding to two dimensional surfaces, covering the loops in the frame. In polyhedral frames these are two-faces of the polyhedra. These two-faces coincide with cells in planar problems, hence the completeness of Maxwell’s two dimensional discrete function.

2 Notation, preliminaries

We will not make a distinction between vectors and dual vectors, as this would be mostly useful if velocities or strains would be also considered. We will only deal with stresses. Furthermore the paper is intended to be accessible to engineers as much as possible, the number of mathematical concepts used is intended to be as minimal as possible.

We will use $k$-vector valued differential forms. One way of looking at it is that one has the Grassmann algebra in $\mathbb{R}^n$ where the scalar coordinates are replaced with differential forms. (Scalar valued forms in this interpretation, and the degree of them has to be the same in all coordinates). Real valued multiplication of the coordinates is replaced with the wedge product of differential forms. We will omit the wedge-product sign if possible and denote the "directions" of the Grassmann algebra
with \( x_i, x_j, \ldots \) (for 1-vectors, 2-vectors, \ldots) while the components of the differential forms with \( d_i, d_j, \ldots \) and so on. The coordinates (functions) will be labelled by upper indices corresponding to the index-sets, for instance a 3-form coordinate of a 2-vector will have components like \( \alpha^{12,13}d_1d_2d_3 x_1x_2 \). We will mostly follow the convention to list \( x_i, d_i, x_1, d_1 \) in lexicographic order.

We will need the Hodge-duals \( \star \) of both k-vectors and k-forms (we assume the usual Euclidean metric). We denote the Hodge-star on k-vectors with \( \star_x \), mapping a k-vector to an \((n - k)\)-vector. This map may be given on the base vectors as

\[
\star_x : \bigwedge_{i \in I} x_i \mapsto (-1)^{t} \bigwedge_{j \in J} x_j 
\]

where \((1 \ldots n)\) is the disjoint union of \( I \) and \( J \) and \( t \) is the number of permutations required to bring \((I, J)\) to \((1 \ldots n)\). Similarly we have the one on scalar valued differential forms as

\[
\star_d : \bigwedge_{i \in I} d_i \mapsto (-1)^{t} \bigwedge_{j \in J} d_j 
\]

where \((1 \ldots n)\) is the disjoint union of \( I \) and \( J \) and \( t \) is the number of permutations required to bring \((I, J)\) to \((1 \ldots n)\). Under the Hodge-dual of a k-vector valued differential form we will mean the form achieved by applying composition of the two stars, i.e: \( \star = \star_x \circ \star_d = \star_d \circ \star_x \). This way \( \alpha \wedge \star \alpha \) gives an \( n \)-vector valued \( n \)-form.

We will need the scalar product of k-vector valued \( m \)-forms \( \alpha \) and \( \beta \), defined as

\[
[\alpha; \beta] := \star(\alpha \wedge \star \beta)
\]

the output of which is a real number.

Forces will correspond to 1-vectors while moments to 2-vectors. Force \( F = \sum F^ix_i \) acting at point \( x = \sum x^ix_i \) has moment \( x \wedge F \) with respect to the origin. In \( \mathbb{R}^n \) this is a 2-vector with \( \binom{n}{2} \) coordinates, as there exists a moment with respect to the ortho-complement of each plane. (A moment is a rotating effect of a force and rotations can happen in each plane of the space.) The moment introduced this way differs from the engineering moment in \( \mathbb{R}^3 \) having the sign of the second component flipped. This is captured in the relations \( a \times b = \star_x (a \wedge b), \star_x (a \times b) = a \wedge b \) (where \( a, b \in \mathbb{R}^3 \)).

Stresses in general are \((n - 1)\)-forms, that need to be integrated along \( n - 1 \) dimensional hyper-surfaces. As a consequence they will be measured in \([N/mm^{n-2}]\). As an example, we express the \( x_1 \) directional stresses in \( \mathbb{R}^3 \) as

\[
(\sigma^{1,12}d_1d_2 + \sigma^{1,13}d_1d_3 + \sigma^{1,23}d_2d_3)x_1
\]

which corresponds to components of the Cauchy stress tensor with the usual \( x, y, z \) description as

\[
\sigma^{1,12} = \sigma^{x,x} \quad \sigma^{1,13} = -\sigma^{y,x} \quad \sigma^{1,23} = \sigma^{z,x}
\]

where the minus sign comes from the lexicographic ordering \((1d_1d_3 = -1d_3d_1)\).

The exterior derivative is taken coordinate-wise on the k-vectors, we will denote it with \( d( \ ) \), omitting the parentheses if no confusion arises. The exterior derivative satisfies \( d^2 \alpha = 0 \) for all twice differentiable forms. It follows from the computational rules that given k-vector valued forms \( \alpha \) and \( \beta \) the Leibniz Rule

\[
d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg(\alpha)} \alpha \wedge d\beta
\]

holds, where \( \deg(\alpha) \) denotes the degree of \( \alpha \). We will heavily rely on the following two results: 10:

**Lemma 1** (Poincaré Lemma). If \( d\alpha = 0 \) throughout a simply connected region, then \( \exists \beta : \alpha = d\beta \).

**Theorem 1** (Fundamental Theorem of Exterior Calculus / Generalized Stokes’s Theorem). Given a compact, oriented \((p + 1)\) dimensional region \( R \), its boundary \( \partial R \) and \( p \)-form \( \alpha : \int_R d\alpha = \int_{\partial R} \alpha \).

Lower indices will have two meaning, one in each main section of the paper. In section 3 they will denote differentiation with respect to the indexed variable. In section 4 they will act as labels to describe the multiple rods or piecewise linear stress function pieces.
3 The continuous stress function

Let us cut out some volume $V$ of the material. In the absence of body forces the equilibrium of the stresses acting on its boundary surface $\partial V$ may be expressed as

$$0 = \int_{\partial V} \sigma = \int_V d\sigma$$

which must hold for all possible $V$. Thus we have

$$d\sigma = 0$$

implying the existence of $(n - 2)$-form $\psi$ such that $\sigma = d\psi$. This potential-function $\psi$ is not unique, so much so that we may prescribe some of its components to be 0. It will be apparent later that we need

$$\psi^i, \mathcal{P} = 0 \iff i \in \mathcal{P}$$

where $\mathcal{P}$ is an index-set of $n - 1$ elements. To see that this is possible, assume we have found $\alpha$ such that $\sigma = d\alpha$. We may create $(n - 3)$-form $\lambda$ as

$$\lambda^{i, \mathcal{Q}} = 0 \iff i \in \mathcal{Q}$$

$$\lambda^{i, \mathcal{Q}} = (-1)^t \int \alpha_{(i, \mathcal{Q})} dx \iff i \notin \mathcal{Q}$$

where $\mathcal{Q}$ is an index-set of $n - 2$ elements and $t$ is the number of permutations required to bring the index-set $(i, \mathcal{Q})$ to lexicographic order. We may now have $\psi = \alpha - d\lambda$, satisfying (6) and $\sigma = d\psi$ (since $d^2\lambda = d(d\lambda) = 0$). If $\sigma \in C^1$ this is always doable, we give some examples for this below.

For $n = 2$ both $\mathcal{P}$ and $\mathcal{Q}$ is empty and any 0-form $\psi$ is good.

For $n = 3$ only $\mathcal{Q}$ is empty. As an example the $x_2$ direction of $\alpha$ looks like

$$\alpha^2 x_2 = (\alpha^{2,1} d_1 + \alpha^{2,2} d_2 + \alpha^{2,3} d_3)x_2$$

(12)

the undesirable part being $\alpha^{2,2}$. By integrating it we get a 0-form (function) $\lambda^2 = \int \alpha^{2,2} d_2$ (here $d_2$ denotes integration with respect to variable $x_2$). The stress components pointing in the $x_2$ direction will be calculated as

$$\sigma^2 x_2 = d((\alpha^{2,1} - \lambda^2_1)d_1 + 0 d_2 + (\alpha^{2,3} - \lambda^2_3)d_3)x_2$$

$$= (-\alpha^{2,1}_2 - \lambda^1_2)d_1 d_2 + (\alpha^{2,3}_3 - \alpha^{2,1}_3 + \lambda^1_3)d_1 d_3 + (\alpha^{2,3}_2 - \lambda^1_2)d_2 d_3)x_2$$

(13)

(14)

(here lower indices denote partial differentiation). This can be compared with

$$\sigma^2 x_2 = d\alpha^2 x_2 = ((\alpha^{2,1}_2 - \alpha^{2,1}_2) d_1 d_2 + (\alpha^{2,3}_3 - \alpha^{2,1}_3) d_1 d_3 + (\alpha^{2,3}_2 - \alpha^{2,3}_2) d_2 d_3)x_2.$$  (15)

Equations (14) and (15) are the same if $\lambda^2_2 = \lambda^2_1$ and $\alpha^{2,2}_2 = \lambda^2_2$. Both are satisfied since we have $\alpha \in C^2$ due to $\sigma \in C^1$, for Equation (8) to make sense.

For $n = 4$ the $x_2$ direction of $\alpha$ looks like

$$\alpha^2 x_2 = (\alpha^{2,12} d_1 d_2 + \alpha^{2,13} d_1 d_3 + \alpha^{2,14} d_1 d_4 + \alpha^{2,23} d_2 d_3 + \alpha^{2,24} d_2 d_4 + \alpha^{2,34} d_3 d_4)x_2$$

(16)

where the undesirable parts are $\alpha^{2,12}$, $\alpha^{2,23}$ and $\alpha^{2,24}$. By integrating each with respect to $x_2$, we have

$$\lambda^2 x_2 = (\alpha^{2,1}(d_1 + 0 d_2 + \lambda^{2,3} d_3 + \lambda^{2,4} d_4)x_2 $$

$$= \lambda^{2,1} x_2 - \lambda^{2,1} x_2$$

(17)

$$\alpha^{2,12} = -\lambda^{2,1} x_2$$

(18)

with the same continuity condition. We can see (also from the definition) that the integration is always with respect to a single variable and is always possible without having to solve a system of differential equations.

The moment of the stresses acting on a small piece of hyper-surface at location $x \in \mathbb{R}^n$ is calculated as $x \wedge \sigma$ (with respect to the origin of the coordinate system). Cutting out some volume
$V$ of the material, the equilibrium of moments from the stresses acting on its boundary surface $\partial V$ may be expressed as

$$0 = \int_{\partial V} x \wedge \sigma = \int_{\partial V} d(x \wedge \sigma) = \int_{V} dx \wedge \sigma + (-1)^0 \int_{V} x \wedge d\sigma$$

(19)

for any volume $V$. Since we already know (5), we have

$$dx \wedge \sigma = 0$$

(20)

(here $dx = \sum_{i=1,\ldots,n} 1 \, dx_i$). Using this we may conclude:

$$d(dx \wedge \psi) = 0 - dx \wedge \sigma = 0$$

(21)

$$\exists \omega : d\omega = dx \wedge \psi.$$  

(22)

The question becomes: Can we reconstruct $\psi$ from $d\omega$, and if so how? For an arbitrary differential form $\alpha$ the map $\alpha \mapsto dx \wedge \alpha$ contains information loss, but for certain forms it is actually reversible. The idea can be seen in $\mathbb{R}^3$ with the cross product as:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ y \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}.$$  

(23)

Recalling how the cross product is the combination of the wedge product and the Hodge dual, the two maps become

$$\psi \mapsto \bar{\psi} = *(dx \wedge \psi)$$

(24)

$$\bar{\psi} \mapsto * (dx \wedge \bar{\psi}).$$  

(25)

For differential forms $\psi$ satisfying the orthogonality condition $[\psi; dx] = 0$ map (24) interchanges the indices as $\psi^{i,j} = (-1)^n \psi^{j,i}$ taking the vector valued $(n-2)$-form to a $(n-2)$-vector valued 1-form; furthermore map (25) is the inverse of map (24). The condition (6) is sufficient (but not necessary) to satisfy $[\psi; dx] = 0$, but getting rid of redundant parameters is useful in general, so let us parametrize the $(n-2)$-form $\omega$ to satisfy (22) as follows: If $\alpha = d\omega$ then

$$\alpha^{ij} = 0 \iff (i \in P \text{ and } j \in P)$$

(26)

and equivalently

$$\omega^{ij} = 0 \iff (i \in Q \text{ or } j \in Q)$$

(27)

must hold. (For $n = 2$ $\omega$ is an arbitrary $0$-form.) In other words for any $x, x_j$ moment component there is a single non-zero component $\omega^{ij}$, exactly the one where $(1\ldots n)$ is the disjoint union of $(i,j)$ and $Q$.

To sum up the usage of what has been derived: Pick any $(n-2)$-form $\omega$ satisfying (27) and the boundary conditions corresponding to the problem. The stresses are determined as

$$\sigma = d(*(dx \wedge d\omega)).$$  

(28)

### 3.1 Mechanical interpretation

We may note, that

$$d(x \wedge \psi) = dx \wedge \psi + x \wedge \sigma$$

(29)

$$x \wedge \psi = \omega + \psi^M + \delta$$

(30)

where $d\delta = 0$ and $d\psi^M = x \wedge \sigma$. The form $\psi^M$ is a suitable (non unique) potential function used to get the moment of the stresses with respect to the origin, as follows: Cut the structure in half along a hyper-surface and denote the surface of the cut by $S$! Then the moments of the stresses along $S$ are given as

$$\int_S x \wedge \sigma = \int_{\partial S} \psi^M.$$  

(31)

Thus integrating the stress function $\int_{\partial S} \omega$ provides a "correction term" to get correct moment values from the expression $\int_{\partial S} (x \wedge \psi)$. In case of $n = 2$ the stress function is actually a moment-function (a 0-form, having moment values over the plane). If the two endpoints of the cut are $u, v \in \mathbb{R}^2$, then $\int_{\partial S} \omega = \omega(v) - \omega(u)$. (See for instance Phillips [11].)
3.2 Relation with earlier continuous stress functions

Using the notation of Sadd [12]: For \( n = 2 \) the stress function is the Airy stress function, \( \omega = \phi \). Here the Hodge dual of bi-vector \( x_1x_2 \) is a 0-vector, that acts like a scalar. The Hodge dual of a bi-vector valued 1-form is a scalar valued 1-form.

For \( n = 3 \) we have given a 1-form whose components correspond to the Maxwell stress function as \( \omega^{12,3} = -\Phi^{32}, \omega^{13,2} = \Phi^{22} \) and \( \omega^{23,1} = -\Phi^{11} \).

One could also embed the components of the Morera stress function into \( \omega \) as \( \omega^{13,1} = -\omega^{23,2} = \Phi^{12}, -\omega^{12,1} = \omega^{23,3} = \Phi^{13} \) and \( -\omega^{13,2} = \omega^{13,3} = \Phi^{23} \) (the symmetry would be visible in cyclic and not lexicographic labelling). The symmetry of \( \omega \) would mean \( [\psi] \Delta x = 0 \) would still hold and Equation (30) would still work. However, as \( n \) grows the number of these parameters would grow faster than is strictly necessary, hence we did not include this approach in our generalization.

4 The discrete stress function

In his work on stress functions, Maxwell [5] noted that given a structure concentrated to lines (a planar truss or a frame for instance), the internal force distribution may be described by a piecewise linear, \( C^0 \) continuous version of the Airy stress function implying \( d\omega = \sigma = 0 \) where there is no structure (between the rods) and \( d\omega \) not being defined where there is structure. Later it was proposed that one can consider moment-bearing frames, where the stress function need not be even \( C^0 \) continuous, see [14]. The planar case is somewhat degenerate as although \( d\psi = 0 \) holds \( \psi \) cannot be the derivative of anything since it is a 0-form. Regardless it is not hard to see that \( d\psi = \sigma = 0 \) also implies the piecewise linearity of \( \omega \). We build up the chain of implications for \( n \geq 3 \) below, show the \( n = 3 \) case in detail, then generalize.

If \( n \geq 3 \), the non-existence of stresses outside the rod axes imply the existence of \( n - 3 \) form \( \Phi \) such that \( d\Phi = \psi \). This also gives us

\[
d(dx \wedge \Psi) = -d\omega \Rightarrow \omega = \gamma - dx \wedge \Psi : d\gamma = 0
\]  

(32)

implying the existence of \( n - 3 \)-form \( \Omega \) such that \( \gamma = d\Omega \). Returning to the potential function in Equation (31), we may express the moment of the stresses as

\[
\int_{\partial S} \psi M = \int_{\partial S} x \wedge \psi - \omega = \int_{\partial S} x \wedge d\Psi - (\gamma - dx \wedge \Psi) = \int_{\partial S} x \wedge \Psi - \Omega
\]  

(33)

where we used \( d(x \wedge \Psi) = dx \wedge \Psi + x \wedge d\Psi \). This shape would only make sense if \( \psi \) and \( \omega \) would be differentiable in the whole of \( \partial S \). This will not be the case in general, we will have parts in which they are continuous, but the actual equilibrium will depend on what happens at the \( C^0 \) discontinuities. The use of this equation is that it tells us the shape of the stress function pieces on the continuous parts.

4.1 The discrete stress function in 3 dimensions

Consider a tetrahedral piece of material, vertices of the tetrahedron denoted by \( p_1, p_2, p_3, p_4 \), point \( p \) being inside the tetrahedron (Figure [1]). We want to replace this with 4 pieces of rods running from \( p_i \) to \( p \) connected in a force and moment-bearing way. Choosing \( q_1, q_2, q_3 \) to be points close to \( p_i \) on the edges of the tetrahedron, the force resultant of the stresses (in the continuous case) acting on the area enclosed by lines \( q_i, q_j \) may be calculated via line-integrals of 1-forms \( \psi_{ij} \) on the respective lines as

\[
F_i = \int_{q_1}^{q_2} \psi_{12} + \int_{q_2}^{q_3} \psi_{23} + \int_{q_3}^{q_1} \psi_{31}.
\]  

(34)

Here the three 1-forms are the same, we labelled them according to the curve-segments to introduce the logic of building up the resultant from parts.

Distorting the geometry of the tetrahedron into the discontinuous case, we naturally get triangles \( p, p_i, p_j \) spanning planes where the continuity of the line-integral may break. This is enough to treat the tetrahedron locally and polyhedral frames show an example where global treatment can be easily done. For now we will label these surfaces \( D_i \), and move on, a good global definition will be given later relying on observations we still have to make. The reader may think of polyhedral frames until then.

5
The discrete version of Equation (34) will be

\[ F_1 = \lim_{q_i \to p_1} \left( \Psi_{12}(q_2) - \Psi_{12}(q_1) + \Psi_{23}(q_3) - \Psi_{23}(q_2) + \Psi_{31}(q_3) - \Psi_{31}(q_1) + \sum_{i=1}^{3} \pm \Psi_i(q_i) \right) \]  

(35)

where \( \Psi_{ij} \) are the potential functions as introduced above, while \( \pm \Psi_i \) are the discrete, orientation sensitive jumps corresponding to the line-integral passing through the surfaces \( D_i \) spanned by the rods. (Here every unique label may denote a unique form.) Observing

\[ \Psi_{ij}(q_i) - \Psi_{ij}(q_j) \to \Psi_{ij}(p_1) - \Psi_{ij}(p_1) = 0 \]  

(36)

shows we do need the jumps, since they will carry all the information we need. We will treat \( \Psi_i \) and \( \Omega_i \) as functions continuous in \( \mathbb{R}^3 \), having an effect in our expressions when the line-integrals pass the respective surfaces \( D_i \). This way each of their partial derivatives exist, which would not be the case if we defined them having \( D_i \) as domains. We will support this with a mechanical interpretation on the whole of \( \mathbb{R}^3 \).

To properly treat the sign sensitivity, assign an orientation to each surface \( D_i \), which can be captured by the normal vector \( n_i \) (at the point of intersection with the path of integration). Let \( t_i \) denote the tangent vector of the path of integration, at the point of intersection with \( D_i \). The force in the bar (acting on the rod-star that the tetrahedron becomes, expressed in the global frame) will be

\[ F_1 = \sum_{i=1}^{3} \text{sign}(\langle n_i, t_i \rangle) \Psi_i(p_1) \]  

(37)

where \( \text{sign}(\ ) \) denotes signum function and \( \langle n_i, t_i \rangle \) is the usual scalar product. The requirement of sign consistency is not the distribution of \( n_i \) but the fact that one chooses one and sticks to it through all the calculations.

With this established, we may observe that in the discontinuous case we actually want \( \Psi_i \) to be constants, since we want the line-integral around the rod-axis to give the same force resultant even if the exact path is perturbed in the stress-free region.

We may similarly write up the moment of the force system with respect to the origin as

\[ M_1 = \lim_{q_i \to p_1} \left( \sum_{i=1}^{3} q_i \wedge \Psi_i(q_i) - \Omega_i(q_i) \right) = p_1 \wedge F_1 - \sum_{i=1}^{3} \Omega_i(p_1). \]  

(38)

The moment of the force system at \( p_1 \) with respect to \( p_1 \) is \( M_1 - p_1 \wedge F_1 = -\sum_{i=1}^{3} \Omega_i(p_1) \).
Thus to each surface $D_i$ (polyhedron face in the polyhedral case) corresponds a force system, and the stress function $\Omega_i$ is $-1$ times the moment of the force system with respect to point $q \in D_i$. As a consequence

$$d\Omega_i = dx \wedge \Psi_i,$$

(39)

holds (which is the discrete analogue of Equation (22)) and the force components stored in $\Psi_i$ may be restored from the derivative, for instance in the way introduced in the continuous case. The sign sensitive sum can then be computed.

Another way of looking at this is that any force system may be expressed as the vector pair $(F_i, M_i)$ (the moment is again taken with respect to the origin.) The stress function is $\Omega_i(x) = x \wedge F_i - M_i$ which makes sense on the whole of $\mathbb{R}^3$. The internal forces in the rod admit this decomposition as well. Thus and Equations (37) and (38) can be expressed together as

$$(F_1, M_1) = \sum_{i=1}^{3} \text{sign}(\langle n_i, t_i \rangle)(F_i, M_i)$$

(40)

which is just a 6-dimensional vectorial sum.

4.1.1 Usage

The original idea of Maxwell turned loads and support reactions into internal forces by adding fictitious bars and at least one node representing the "world" outside the structure, where the loads and support reactions meet. This way the equilibrium of the entire structure is expressed by the equilibrium of the added node(s). This imposes boundary conditions on the stress-function, not unlike the case of traditional stress functions.

To see an example what these boundary conditions mean, consider a structure in the shape of a triangular prism, with moment-bearing bars on the edges (Figure 2). Let us label the positions of top nodes $p_1$ to $p_3$ and the bottom nodes $p_4$ to $p_6$. Let us load the structure with loads $L_1$ to $L_3$ acting at the respective nodes and support it with supports $S_4$ to $S_6$ again at the respective nodes. To incorporate the loads and supports into the structure add node at point $p_7$ inside the prism, representing the world. (The exact location does not matter, as the fictitious bars have no material and can not fail. One could move it even outside the prism, but this way the connectivity relations are easy to visualize.) The fictitious bars will be $p_i p_7$. Denoting the stress functions with the $3$ indices of the points their corresponding surfaces contain, the boundary conditions imposed by the loads may be expressed as

$$L_1 = \Omega_{127} + \Omega_{137} + \Omega_{167}$$

(41)

$$L_2 = -\Omega_{127} + \Omega_{237} + \Omega_{247}$$

(42)

$$L_3 = -\Omega_{137} - \Omega_{237} + \Omega_{357}$$

(43)

where $L_i$ are the opposite of the moment-fields determined by each load. The sign changes in the three expressions are due to the opposite direction of traverse of surfaces $D_{ijk}$. If one has boundary conditions on the supports they appear here, for instance if force $S_4$ had to pass through point $p_4$ one would have

$$S_4(p_4) = 0 = -\Omega_{247}(p_4) + \Omega_{457}(p_4) + \Omega_{467}(p_4).$$

(44)

These equations are just linear and are easy to solve, the non-determined stress-function parameters may be chosen freely.

4.1.2 Completeness and minimality

We will argue using the above shape where loads and supports are turned into internal forces. This is in contrast to the completeness investigation of continuous stress functions [5, 14], where the question of completeness can not be investigated without considering what the boundary of the solid is like [15, 16]. Since the supports and loads are turned into parts of the structure, we don’t prescribe boundary conditions as they would only exclude certain loads and we are interested in parametrizing the general case.

**Theorem 2.** The proposed discrete 3D stress function is complete and minimal.
Figure 2: An example to handle loads and supports as fictitious internal bars. The world outside the structure is represented by node $p_7$, the equilibrium of the loads and supports correspond to the equilibrium of the fictitious node.

Proof. We have to show that whatever internal force distribution, that is in equilibrium is given in the structure, there is a unique corresponding stress function built from the appropriate components. Recall how the force components can be calculated through component-wise independent summation (Equation (40)), where the topology of the structure determines the summations. We will calculate the stress function coordinates component-wise, row by row. Let $g_i$ denote the $x_1$ directional force component of each stress-function, where $i$ runs on all the surfaces of the structure. Let $f_k$ denote the $x_1$ directional component of the bar-force in bar $k$ ($k$ runs on all the bars). We will manipulate a diagram that will act as a topological aid to write up the correct equations. The starting shape of this diagram will be the actual structure. For each $i$ in ascending order we may do the following:

Find the loop (shortest path along bars) in the topological diagram enclosing (only) the surface $D_i$ (Figure 3, left). Choose any bar $k$ in the loop and express $f_k$ as

$$f_k = \pm g_i + \sum_{j \in \mathcal{J}_i} \pm g_j \quad (45)$$

where $\mathcal{J}_i$ is some index set. After this equation is written up contract the loop in the topological diagram, unifying the involved nodes to a single one. By contracting the loop we make sure not to use component $g_i$ again. This way the index set $\mathcal{J}_i$ will satisfy: $j \in \mathcal{J}_i \implies j > i$.

After we do this for all $i$ we get a system of linear equations $A g = f$, whose coefficient matrix $A$ is square, upper triangular and each element on the main diagonal is $\pm 1$. The determinant of this matrix is $\pm 1$ (the product of the elements on the main diagonal), thus it is invertible and we may solve for $g$. We may also repeat the whole procedure for the other 5 components of $(F_i, M_i)$.

As such we may find a suitable stress function distribution to any force system in the structure, that is in equilibrium. We may also note that the number of loops equals six-times the degree of static indeterminacy, implying that in the general case of a frame we can not get away with less stress function parameters.

We still have to see, that choosing different bars in each loop does not give a different stress-distribution, or in other words $f$ contains force components that parametrize the self-stresses of the structure. This can be seen by doing the loop-contraction procedure backwards. At each backwards-step $f_k$ may be used as a parameter and the rest of the unknowns in the step may be calculated from the static equilibrium equations (see Figure 4). At each backwards-step the number of added nodes equals the number of unknown components and there is an independent equilibrium equation corresponding to each node. The mechanical interpretation of the equilibrium equation of a fictitious node (one that contains a loop contracted into it) is the sum of all the equilibrium equations of the nodes that were present in the loop contracted into it. As the procedure restores the internal force distribution of the structure, the proof is concluded. \qed

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Figure 3: Contracting loop $i$ corresponding to equation $i$ in determining the stress-function.

Figure 4: Doing the contraction procedure backwards to determine non-parameter rod forces.
4.1.3 Automatic equilibrium

**Theorem 3.** The proposed discrete 3D stress function gives internal force systems that are in static equilibrium.

**Proof.** To see how $\sum F$ holds in case of the tetrahedral element one can note how during the summation of forces each surface $D_i$ is passed two times (see Figure 5), thus after some rearrangement we have

$$\sum_j F_j = \sum_k (\Psi_k - \Psi_k) = 0$$

where $j \in \{1\ldots4\}$ and $k \in \{1\ldots6\}$. Similarly, we can write up the sum of the moments with respect to $p$, which after some rearrangement will take the shape of

$$\sum_j M_j(p) = \sum_k (\Psi_k(p) - \Psi_k(p)) = 0$$

with the same indices. These arguments naturally generalize to connections where not 4 rods meet, a straight rod segment can be considered the meet of 2 rods.

4.1.4 Global definition for $D_i$

In the continuous case typically there is material in all points in the domain and there is static indeterminacy "everywhere". We define the continuous stress functions correspondingly "everywhere". For space frames the static indeterminacy is tied to the loops in the structure; one can not solve the static problem because the loop has no free end to start determining internal forces from. Thus to properly define surfaces $D_i$ globally for more complicated structures, we should define one inside each "independent" loop, each loop introducing $n + \binom{n}{2}$ static indeterminacies. These loops may be rigorously given by the generators of the fundamental group of the graph of the structure. See for instance section 1.2 in the algebraic topology book of Hatcher [17].

4.2 Dimension independent generalization

In Figure 1 we integrated along what could be considered a boundary of a triangular cross-section of a rod. Generalizing this we integrate along what could be the $n - 2$ dimensional boundary of an $n - 1$ dimensional cross-section. Surfaces $D_i$ puncture the boundary at points $q_i$, since $D_i$ runs along the axis of the rods, orthogonal of the cross-section. As such in the generalization of Equation (35) the single indexed parts will be always present. The terms with the multi-indices turn into $n - 3$ integrals and vanish in the limit. The force system in the bar is determined in general by the single

\[ \sum_j F_j = \sum_k (\Psi_k - \Psi_k) = 0 \]
Figure 6: Orientation on the boundary loop of the discontinuity surfaces

indexed part, similarly to Equation (40), but we do have to generalize the sign convention. We have seen in the proof of Theorem 2 that we have to deal with orientation of loops that are the boundaries of surfaces $D_i$. In general this may be done by choosing an orientation on each surface $D_i$: if we pick point $c_i$ in it and place $b_{i,1}, b_{i,2}$ orthonormal base vectors there. The positive direction of rotation in the plane is represented by $b_{i,1} \wedge b_{i,2}$ (see Figure 6). If we cut out an element of the rod, it will have two outwards pointing axial vectors $a$ and $-a$. When determining the internal force system at the endpoints, the stress function will appear with positive sign if the respective axial vector $\pm a$ causes a positive directional rotation around $c_i$. If we wish to express the internal forces of the structure at point $r$ with outward normal $a$ we may formally do this as

$$ (F, M) = \sum_{i=1}^{m} \text{sign}((r - c_i) \wedge a; b_{i,1} \wedge b_{i,2}))(F_i, M_i) $$

where the bar is involved in $m$ surfaces.

After this we observe that the proof of Theorem 2 is actually dimension independent in the sense that the row-by-row nature of the proof allows vectors of arbitrary size. Thus we conclude that

**Corollary 1.** The dimension independent discrete stress function is complete and minimal.

To see the self-equilibrated nature of the general case we may look at Figure 5 and note that locally each surface $D_i$ is spanned by the centre of the rod-star (point $p$) and two endpoints of the rods ($p_i$ and $p_j$), regardless of the dimension of the space the problem is in. When determining the total sum of forces each surface $D_i$ and the corresponding stress function is taken into account with one positive and one negative orientation. Similar logic applies to the moment equilibrium, thus we have

**Corollary 2.** The dimension independent discrete stress function gives internal force systems that are in static equilibrium.

The method given here can incorporate planar problems if they are embedded in at least $\mathbb{R}^3$, with potential functions $\Omega$ and $\Psi$ having some constant 0 components.

## 5 Conclusion

We provided a differential form based dimension independent potential function description of stresses, that gave the Airy stress function in the planar case and the Maxwell stress function in the 3 dimensional case. The existence of the function in simply connected manifolds is equivalent with static equilibrium (inside the material, we did not look into boundary conditions).

We derived a dimension independent piecewise linear discrete stress function for frames from the continuous one. We given stress function pieces corresponding to each two dimensional surface covering each loop in the structure, which are two-faces for polyhedral frames. Each loop prevents the structure from being simply connected, compare with the continuous case.
above. Thus the number of function pieces in the description reflects the topology of the structure. In comparison, the previous discrete three-dimensional stress function \cite{7, 8} defines stress function pieces in polyhedral (three-) cells and is incomplete. Our approach is complete and minimal.

It is not expected that people will solve elasticity problems for $n > 3$, the point of the work is not this. Having a dimension independent formulation tends to capture the phenomena that are the root of the corresponding problem and these formulations are the most grateful to build on. Having derived the discrete stress function from the continuous one is one such example (along with solving a previously open problem). We arrived at an observation about how the discrete stress function is tied to the mechanical behaviour of structures (loops cause static indeterminacy and we shall define our discrete stress function components corresponding to them) literally by taking the definition "stresses are 0 everywhere except the rod axes" at face value and plugging it in the differential form shaped continuous stress-function.

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