ON THE 3-DIMENSIONAL INVARIANT FOR CYCLIC CONTACT BRANCHED COVERINGS

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Abstract. We give a formula of 3-dimensional invariant for a cyclic contact branched covering of the standard contact $S^3$.

1. Introduction

Let $\tilde{M} \to M$ be a branched covering of a 3-manifold $M$, branched along a link $K \subset M$. When $M$ has a contact structure $\xi$ and $K$ is a transverse link in the contact 3-manifold $(M, \xi)$, $\tilde{M}$ has the natural contact structure $\tilde{\xi}$. We call the contact 3-manifold $(\tilde{M}, \tilde{\xi})$ the contact branched covering of $(M, \xi)$ along the transverse link $K$.

Let $(M, \xi)$ be a $p$-fold cyclic contact branched covering of $(S^3, \xi_{std})$ (the standard contact $S^3$), branched along a transverse link $K$. In [1], Theorem 1.4, it is shown that the euler class $e(\xi)$ is zero, and the 3-dimensional invariant $d_3(\xi) \in \mathbb{Q}$ (See [2] for definition) only depends on a topological link type of $K$ and its self-linking number.

In this note, we show a direct formula of $d_3(\xi)$ in terms of its branch locus $K$.

Theorem 1.1. If a contact 3-manifold $(M, \xi)$ is a $p$-fold cyclic contact branched covering of $(S^3, \xi_{std})$, branched along a transverse link $K$, then

$$d_3(\xi) = -\frac{3}{4} \sum_{\omega \mod p = 1} \sigma_\omega(K) - \frac{p-1}{2} \text{sl}(K) - \frac{1}{2}p.$$ 

Here $\sigma_\omega(K)$ denotes the Tristram-Levine signature, the signature of $(1 - \omega)A + (1 - \bar{\omega})A^T$, where $A$ denotes the Seifert matrix for $K$, and $\text{sl}(K)$ denotes the self-linking number.

Thus, our formula tells us that $d_3(\xi)$ actually only depends on the concordance class of $K$ and the self-linking number. Also, by slice Bennequin inequality [7], it also shows that the smooth 4-genus $g_4(K)$ of $K$ gives a lower bound of $d_3(\xi)$.

Corollary 1.2. If a contact 3-manifold $(M, \xi)$ is a $p$-fold cyclic contact branched covering of $(S^3, \xi_{std})$ branched along $K$, then $d_3(\xi) \geq -\frac{3}{2}(p-1)g_4(K) - \frac{1}{2}p$.

2. Proof

Proof of Theorem 1.1. Let $(M, \xi)$ be a $p$-fold cyclic contact branched covering, branched along a transverse link $K$ in $(S, \xi_{std})$. We put the transverse link $K$ as a closed braid, the closure of an $m$-braid $\alpha$.

Let $(S, \psi)$ be the open book decomposition of $(S^3, \xi_{std})$, whose binding is the $(p, m)$-torus link. Inside $S^3$, the page $S$ is an obvious Seifert surface of the $(p, m)$-torus link which we view as the closure of the $p$-braid $(\sigma_1 \cdots \sigma_{m-1})^p$ as we illustrate in Figure 1.

Topologically, the page $S$ is the $p$-fold cyclic branched covering of the disk $D^2$, branched along $m$-points. Let $\pi : B_m = \text{MC}(D^2 \setminus \{m \text{ points}\}) \to \text{MC}(S)$ be the map induced by the branched covering map, which is explicitly written by $\pi(\sigma_i) = D_{i,1} \cdots D_{i,p-1}$ [1, Lemma 3.1]. Here $D_{i,j}$ denotes the right-handed Dehn twist along the curve $C_{i,j}$ on $S$, given in Figure 2. (Here we are assuming that $\text{MC}(S)$ acts on $S$ from left, so $D_{i,1} \cdots D_{i,p-1}$ means $D_{i,p-1}$ comes first and $D_{i,1}$ last.)

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An important observation is that in \((S^3, \xi_{std})\), the curves \(C_{i,j}\) are realized as the Legendrian unknot with \(tb = -1, \text{rot} = 0\).

By using \(D_{i,j}\), \(\psi\) is written by
\[
\psi = \pi(\sigma_{m-1} \cdots \sigma_2 \sigma_1) = (D_{m-1,1} \cdots D_{m-1,p-1}) \cdots (D_{2,1} \cdots D_{2,p-1})(D_{1,1} \cdots D_{1,p-1}).
\]

Also, \((S, \phi = \pi(\alpha))\) gives the open book decomposition of \((M, \xi)\).

First we draw the surgery diagram of \((M, \xi)\) from its open book decomposition \((S, \phi)\), following the discussion in [5, Section 3]. We take a factorization of the braid \((\sigma_{m-1} \cdots \sigma_2 \sigma_1)\alpha\)

\[
(\sigma_{m-1} \cdots \sigma_2 \sigma_1)\alpha = \sigma_{i_1}^\varepsilon_1 \cdots \sigma_{i_n}^\varepsilon_n \quad (\varepsilon_j \in \{\pm 1\}, \ i_j \in \{1, \ldots, m-1\})
\]

by the standard generators \(\{\sigma_{i}^\pm, \ldots, \sigma_{m-1}^\pm\}\) of \(B_m\). By replacing each \(\sigma_{i}^\pm\) in (2.1) with the sequence of Dehn twists \((D_{i,1} \cdots D_{i,p-1})^{\pm 1}\), we have the factorization of \(\psi^{-1}\phi\) by Dehn twists \(D_{i,j}^{\pm 1}\),

\[
(2.2) \quad \psi^{-1}\phi = \prod_{j=1}^{n}(D_{i,j,1} \cdots D_{i,j,p-1})^{\varepsilon_j}.
\]

For each Dehn twist \(D_{i,j}^{\pm 1}\) in the factorization (2.2), we put a curve \(C_{i,j}\) on distinct pages on the open book \((S, \psi)\), so that it is a Legendrian unknot with \(tb = -1, \text{rot} = 0\) in \((S^3, \xi_{std})\). Then \((M, \xi)\) is obtained by the contact surgery along the resulting Legendrian link. Here the surgery coefficient of a component is \((-1)\) (resp. \((+1)\)) if it comes from a positive (resp. negative) Dehn twist.

The factor \(\sigma_{i}^\pm\) in the factorization (2.1) gives a sequence of Dehn twists \((D_{i,1} \cdots D_{i,p-1})\) in the factorization (2.2). The Legendrian curves \(C_{1,1}, \ldots C_{1,p-1}\), put in different pages (so that \(C_{1,p-1}\) comes first and \(C_{1,1}\) last), produce the \((p-1)\) component Legendrian link as we draw in Figure 2 (a). Similarly, \(\sigma_{i}^{-1}\) in the factorization (2.1) gives a sequence of Dehn twists \((D_{i,p-1} \cdots D_{i,1})^{\pm 1}\) in the factorization (2.2), which produce the \((p-1)\) component Legendrian unlink as we draw in Figure 2 (b).

These local contributions of surgery links interact each other, whose linking patterns can be chased by looking the page \(S\) in Figure 1, as we summarize as follows (cf. [5, Fig 11, Remark 3.3]):

![Fig 1. Page S of the open book (S, ψ) inside S^3.](image)

![Fig 2. The contribution of σ_i^±1 in the resulting contact surgery diagram](image)
Observation 1. Let $L_{i_k}^{t_k} = C_{i_k,1} ∪ \cdots ∪ C_{i_k,p-1}$ and $L_{i_l}^{t_l} = C_{i_l,1} ∪ \cdots ∪ C_{i_l,p-1}$ be the sub Legendrian links in the contact surgery diagram of $M$, that comes from the $k$-th factor $\sigma_{i_k}^{t_k}$ and $l$-th factor $\sigma_{i_l}^{t_l}$ in the factorization (2.1), with $k < l$.

Then the components $C_{i_k,s}$ and $C_{i_l,t}$ link forms a (topological) positive Hopf link, if and only if $i_k \in \{i_l, i_l + 1\}$. Otherwise, two components $C_{i_k,s}$ and $C_{i_l,t}$ are disjoint. (See Figure 3.)

Figure 3. How the local contribution of $\sigma_i^{\pm 1}$ in the contact surgery diagram links each other. Here we illustrate contributions for $\sigma_i$ (depicted by black line) and $\sigma_{i_l}^{-1}$ (depicted by gray line) with $k < l$ in the factorization (2.1), for the case $i_k \in \{i_l, i_l + 1\}$.

The contact surgery diagram provides a 4-manifold $X$ that bounds $M$. By [2] Corollary 3.6,

$$d_3(\xi) = \frac{1}{4} (-3\sigma(X) - 2\chi(X)) + q,$$

where $q$ is the number of $(+1)$-contact surgeries, and $\chi(X)$ is the euler characteristic of $X$. Note that the term $\xi^2$ in the formula [2] Corollary 3.6 disappears since each component of the surgery link has zero rotation number. Let $e_+$ and $e_-$ be the number of positive and negative Dehn twist in the factorization (2.1). Since each factor $\sigma_i^{\pm 1}$ produces $(p - 1)$ $(\pm 1)$ contact surgeries along unknots,

$$d_3(\xi) = \frac{3}{4} \sigma(X) - \frac{1}{2} ((p - 1)e_+ + (p - 1)e_- + 1) + (p - 1)e_-$$

$$= \frac{3}{4} \sigma(X) - \frac{p - 1}{2} (e_+ - e_-) - \frac{1}{2}.$$

By Bennequin’s formula $sl(K) = e_+ - e_- + 1$, hence

(2.3) $$d_3(\xi) = -\frac{3}{4} \sigma(X) - \frac{p - 1}{2} sl(K) - \frac{p}{2}.$$

It remains to compute $\sigma(X)$. Take a factorization of the braid $\alpha$ given by

(2.4) $$\alpha = \sigma_{m-1} \cdots \sigma_{i_1} \sigma_{i_1}^{\pm 1} \cdots \sigma_{i_m}^{\pm 1} \quad (\varepsilon_j \in \{\pm 1\}, \quad i_j \in \{1, \ldots, m - 1\}).$$

Let $\Sigma \subset S^3 = \partial B^4$ be the canonical Seifert surface of $K$ that comes from the factorization (2.4). Namely, $\Sigma$ is made of $m$ disks $\{D_1, \ldots, D_m\}$, with twisted bands connecting $i$-th and $(i + 1)$-st disk for each $\sigma_i^{\pm 1}$ in the factorization (2.1).

The following is the most crucial observation in our computation.

Claim 2.1. Let $W$ be the $p$-fold cyclic branched covering of $B^4$ branched along $\Sigma$ (pushed into the interiors of $B^4$). Then $X$ is diffeomorphic to $W$.

Proof of Claim. We draw a Kirby diagram of $W$, following [1] Section 2 (see also [1] Section 6.3]).

Take a handle decomposition of $\Sigma$ so that the 0-handle is $D_1 \cup h_1 \cup D_2 \cup \cdots \cup h_{m-1} \cup D_m$, where $h_i$ is the twisted band coming from the $(m - i)$-th factor $\sigma_i$ of the factorization (2.4), and that the 1-handles are the rest of twisted bands. We put $\Sigma$ in the 3-space so that the 0-handle is the unit disc in the $x$-$y$ plane, and that 1-handles are contained in the upper half-space. Then the
Kirby diagram of $W$ is obtained by “symmetrizing” the cores of 1-handles of $\Sigma$ whose framings are determined by the framings of the core of 1-handles. Except the simplest case $p = 2$, which we will explicitly illustrate later in Example 5.2, the diagram is complicated and it is not easy to draw the whole diagram – however, the contribution of 1-handle in the resulting Kirby diagram, and how they interact each other is simple. See Figure 4 below.

![Figure 4](image)

**Figure 4.** Branched covering of Seifert surface: 1-handle contribution, and how these contributions interact each other (when they are nested).

To put $\Sigma$ in such a convenient position, first we flip the 1st disc $D_1$, by untwisting the band $h_1$ (see Figure 5(a) –(d)). This simplifies the 0-handle of $\Sigma$, and iterating this flipping procedure, eventually we put $\Sigma$ in such a convenient position (see Figure 6). In this process, all 1-handles gains negative half twist, so in the final position, the framing of 1-handle is $(-1)$ if it comes from positive generator, and is 0 if it comes from negative generator.

![Figure 5](image)

**Figure 5.** Putting the Seifert surface $\Sigma$ into a nice position, by flipping $D_1$ along $h_1$. Here we draw 1-handle by a line with $\pm$-sign coming from corresponding generator $\sigma^\pm_i$. The gray box inside $\pm$ represents positive and negative half twist.

From this procedure, we observe:

**Observation 2.** The 1-handles $h_k$ and $h_l$ of $\Sigma$, coming from the $k$-th and $l$-th factor $\sigma^\pm_i$ in the factorization (2.1) ($k < l$), nest each other in Figure 6 if and only if $i_k \in \{i_l, i_l + 1\}$ (see Figure 7).

Recall that each component of the contact surgery diagram of $M$ has $tb = -1$, so $(-1)$ and $(+1)$ contact surgery corresponds to $-2$ and $0$ topological surgery, respectively. Hence each factor
Figure 6. The canonical Seifert surface Σ, put in a convenient position for drawing Kirby diagram. A box represents the positive half twist, and each 1-handle depicted by line has either 0 or $-1$ framing.

Figure 7. How 1-handles of Σ nest each other, when we put Σ in a convenient position as in Figure 6.

$$i_k \in \{i_l, i_l + 1\} \quad i_k \not\in \{i_l, i_l + 1\}$$

σ$^{±1}$ in (2.1) contributes the same $(p - 1)$ components framed link in the surgery diagram of $X$ and $W$ (compare Figure 2 and Figure 4). Moreover, from Observation 1 and Observation 2 these local contributions link other part of the diagrams, in exactly the same way (compare Figure 3 and Figure 4). Thus, comparisons of the construction of the surgery diagrams for $X$ and $W$ proves that they are completely the same diagram.

Claim 2.1, together with a well-known fact on Tristram-Levine signature (see [6, Theorem 12.6], for example) shows

$$(2.5) \quad \sigma(X) = \sigma(W) = \sum_{\omega: p=1} \sigma_\omega(K).$$

The equalities (2.3) and (2.5) completes the proof.

Example 2.2 (The case double branched covering). In the case $p = 2$, the contact double branched covering, it is much easier to treat and draw the surgery diagram of $X$ and $W$. Here we give more explicit illustrations of surgery diagrams.

Let $(M, \xi)$ be a contact double branched covering branched along the closure of an $m$-braid $\alpha$. We begin with the open book $(S, \psi)$ whose binding is $(m, 2)$-torus link. To visualize its symmetry, we view the the page $S$ as the $(m - 1)$-times plumbing of an annulus $A_i$, that is the boundary of the positive Hopf link, as illustrated in Figure 8. As an element of the mapping class group of $S$, the standard generator $\sigma_i$ lifts to the right-handed Dehn twist along the core of an annulus $A_i$.

By taking a factorization of the braid $\alpha$, following the discussion in the proof of Theorem 1.1 we get a contact surgery diagram of $(M, \xi)$, as we draw in Figure 9. On the other hand, the Kirby diagram of $W$ is obtained by “doubling” the core of 1-handles of the canonical Seifert surface Σ, as we show in Figure 10.

Now one immediately see that these two diagrams are the same.

References

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Figure 8. Page $S$ of an open book of $(S^3, \xi_{std})$ whose binding is the $(m, 2)$-torus link.

Figure 9. A contact surgery diagram of contact double branched covering. A box represents the positive half twist.

Figure 10. Kirby Diagram for double branched covering along the canonical Seifert surface $\Sigma$.

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