Quantum Groups and Von Neumann Theorem

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Abstract

We discuss the $q$ deformation of Weyl-Heisenberg algebra in connection with the von Neumann theorem in Quantum Mechanics. We show that the $q$-deformation parameter labels the Weyl systems in Quantum Mechanics and the unitarily inequivalent representations of the canonical commutation relations in Quantum Field Theory.

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1 Introduction

The von Neumann theorem in Quantum Mechanics (QM) states that for systems with finite number of degrees of freedom the representations of the canonical commutation relations (ccr), also called the Weyl representations, are each other unitarily equivalent. In the limit of infinite number of degrees of freedom, namely in Quantum Field Theory (QFT), the space of physical states splits into unitarily inequivalent representations[1]. Thus in QFT, contrarily to QM, one has to single out the appropriate representation to properly describe the system under study: different, i.e. unitarily inequivalent, representations describe different physical situations. Typical examples are the ones related with spontaneous breakdown of symmetry where different phases are described by different representations, e.g. the superconducting phase and the normal phase are described by states belonging to orthogonal (unitarily inequivalent) Hilbert spaces: the Bogoliubov transformations in superconductivity are (non-unitary) canonical transformations which relate such different representations[2] (of course, one works at finite volume where the transformations are formally well defined and at the end of the computations one goes to the infinite volume limit).

In QFT the representations of the ccr may be generally labelled by some physically significant parameter, e.g. in spontaneously broken symmetry theories different values of the order parameter are associated to different representations; in thermal field theories temperature labels inequivalent representations[2]; in the case of dissipative systems the label is the time variable which thus play the role of a parameter[3].

In this paper our purpose is to show that $q$-deformations of the Weyl-Heisenberg algebra ($q$-WH)[4] do indeed introduce a labelling of the Weyl representations in QM and of the inequivalent representations in QFT, the label being the $q$-deformation parameter.

The fact that $q$-WH algebra is in this way associated with the foliation of the space of the states in QFT is remarkable since some light may thus be shed on the physical meaning of the deformation parameter in view of the physical content of the representations; moreover, $q$-parametrization may signal the presence of some finite spacing in the theory since $q$-WH algebra has been shown[5] to be related with systems characterized by some finite (space or time) scale. The relation of inequivalent representations with $q$-WH algebra is also interesting from the mathematical point of view since the rich mathematical structure underlying $q$-algebras is thus recognized to underly QFT, too.

The properties of $q$-algebras will not be discussed in this paper; we simply remind that they are deformations of the enveloping algebras of Lie algebras and have Hopf algebra structure[4]. In particular, the $q$-WH algebra has the properties of Hopf superalgebra[4]. The interest in $q$-algebras arose in non-linear dynamical systems as well as in statistical mechanics, in conformal theories, in solid state physics, etc.. $q$-WH algebra has been studied in the framework of the entire analytic functions theory and has been related with theta functions and with Bloch functions[5]. In ref. [7] $q$-WH algebra has been related with dissipative systems in QFT and with thermal field theories. In order to show that the $q$-deformation parameter labels the inequivalent representa-
tions in QFT, we need first to study the Weyl representations and the von Neumann theorem in QM; this is done in the following section 2. The realization of \( q \)-WH algebra in terms of finite difference operators\[5\] and its relation with the generator of Bogoliubov transformations are presented in section 3. The parametrization of the Weyl representations by the \( q \)-parameter and its extension to the infinite volume limit in QFT is finally established in section 4.

## 2 The von Neumann uniqueness theorem

In this section we present the general formalism to introduce von Neumann theorem in QM and we show how to parametrize the representations of the ccr.

Let us consider a system of \( M \) degrees of freedom. By setting \( \hbar = 1 \), the position and the momentum operators are introduced as usual in the Schrödinger representations as \( \hat{x}_j \rightarrow x_j \) and \( \hat{p}_j \rightarrow -i \frac{\partial}{\partial x_j} \), \( j = 1, 2, \ldots, M \), respectively, with commutation relations

\[
[\hat{x}_j, \hat{p}_k] = i\delta_{jk}, \quad [\hat{x}_j, \hat{x}_k] = 0, \quad [\hat{p}_j, \hat{p}_k] = 0, \quad \forall j, k \tag{1}
\]

The operators of creation and annihilation are

\[
c_j \equiv \frac{1}{\sqrt{2}}(\hat{x}_j + i \hat{p}_j), \quad c_j^\dagger \equiv \frac{1}{\sqrt{2}}(\hat{x}_j - i \hat{p}_j), \tag{2}
\]

with commutators

\[
[c_j, c_k^\dagger] = \delta_{jk}, \quad [c_j, c_k] = 0, \quad [c_j^\dagger, c_k^\dagger] = 0, \quad \forall j, k \tag{3}
\]

As well known, since \( \hat{x} \) and \( \hat{p} \) are unbounded operators, it is necessary to introduce the Weyl system of unitary operators

\[
U(\alpha) \equiv \exp \left( i \sum_{j=1}^{M} \alpha_j \hat{p}_j \right), \quad V(\beta) \equiv \exp \left( i \sum_{j=1}^{M} \beta_j \hat{x}_j \right) \tag{4}
\]

with \( \alpha, \beta \in \mathbb{R}^M, \alpha \equiv (\alpha_1, ..., \alpha_M), \beta \equiv (\beta_1, ..., \beta_M) \). The ccr (1) (and (3)) are then represented by

\[
U(\alpha)U(\alpha') = U(\alpha + \alpha'), \quad V(\beta)V(\beta') = U(\beta + \beta'), \quad U(\alpha)V(\beta) = e^{i\alpha \beta}V(\beta)U(\alpha) \tag{5}
\]

In terms of the Weyl operators \( W(z) \),

\[
W(z) \equiv W(\alpha + i\beta) = e^{i\alpha \beta}V(\sqrt{2}\alpha)U(\sqrt{2}\beta) \tag{6}
\]

with the complex variable \( z \equiv \alpha + i\beta \in \mathbb{C}^M \), eqs. (5) lead to

\[
W(z_1)W(z_2) = \exp (-iIm(z_1^* z_2))W(z_1 + z_2) \tag{7}
\]
In the transition from QM to QFT, i.e. from finite to infinite number of degrees of freedom, one must operate on the complex linear space $E\mathbb{C} = E + iE$, instead of working on $CM$. Here $E$ denotes a real linear space of square-integrable functions $f$; we will denote by $F = f + ig$, $f, g \in E$, the elements of $E\mathbb{C}$. The scalar product $\langle F_1, F_2 \rangle$ in $E\mathbb{C}$ is defined through the scalar product $(f, g)$ in $E$:

$$\langle F_1, F_2 \rangle = \langle f_1 + ig_1, f_2 + ig_2 \rangle = (f_1, f_2) + (g_1, g_2) + i[(f_1, g_2) - (f_2, g_1)]$$

(8)

In QFT the Weyl operators and their algebra (6) and (7) become

$\quad \quad \quad \quad W(F) \equiv W(f + ig) = e^{i(f \cdot g)}V(\sqrt{2}f)U(\sqrt{2}g)$

$$W(F_1)W(F_2) = \exp(-iIm(F_1, F_2))W(F_1 + F_2)$$

(9)

We stress that the use of complex linear space $E\mathbb{C}$ in QFT is required to smear out spatial integrations of field operators by means of test functions $f$.

The von Neumann uniqueness theorem in QM can be now stated as follows [1,8]: Each Weyl system with finite number $M$ of degrees of freedom is unitarily equivalent to the Schrödinger representation. Each reducible Weyl system with finite number $M$ of degrees of freedom is the direct sum of irreducible representations; hence it is a multiple of the Schrödinger representation.

We observe that unitarily equivalence among the Weyl systems is lost in the limit of infinite number of degrees of freedom, $M \to \infty$: thus in QFT the Weyl systems provide unitarily inequivalent representations of the ccr.

Let us consider now the problem of labelling each Weyl system in a way to preserve the Weyl algebra. To this aim we consider the following transformations

$$\alpha_i \mapsto \rho \alpha_i, \quad \beta_i \mapsto \frac{1}{\rho} \beta_i, \quad i = 1, 2, \rho \neq 0$$

(10)

These are canonical transformations since $Im(z_1^*z_2)$ is left invariant under them and the Weyl algebra (6) is therefore preserved:

$$W^{(\rho)}(z_1)W^{(\rho)}(z_2) = \exp(-iIm(z_1^*z_2))W^{(\rho)}(z_1 + z_2), \quad \rho \neq 0$$

(11)

where $W^{(\rho)}(z) \equiv W(\rho \alpha + i\frac{1}{\rho} \beta)$. We thus conclude that the transformation parameter $\rho$ labels the Weyl systems.

In QFT, i.e. for $M \to \infty$, we use the canonical transformations

$$g_j \to \rho \ g_j, \quad f_j \to \frac{1}{\rho} \ f_j, \quad \rho \neq 0$$

$$Im(F_1, F_2) \to Im(F_1, F_2)$$

(12)

and, in such a limit, the representations $W^{(\rho)}(F)$, $W^{(\rho')}(F)$, $\rho \neq \rho'$, are each other unitarily inequivalent[1,8,9].

3
In this section we establish the relation between the \(q\)-WH algebra and the Bogoliubov transformations to be used in section 4.

From now on, for simplicity we limit ourself to one degree of freedom \((M = 1)\); extension to many (finite) degrees of freedom is straightforward. Let us consider the Weyl-Heisenberg (WH) algebra realized in terms of the set of operators \(\{a, a^\dagger, N\}\), with relations:

\[
[a, a^\dagger] = I, \quad [N, a] = -a, \quad [N, a^\dagger] = a^\dagger
\]

The \(q\)-deformation of the WH (\(q\)-WH) algebra realized in terms of the set of operators \(\{a_q, \hat{a}_q, N_q \equiv N; q = e^\epsilon \in \mathbb{C}\}\), with \(a_q \to a, \quad \hat{a}_q \to a^\dagger\) as \(q \to 1\), is:

\[
[a_q, \hat{a}_q] = q^N, \quad [N, a_q] = -a_q, \quad [N, \hat{a}_q] = \hat{a}_q
\]

Since in our study of the \(q\)-deformation we want to preserve the analytic properties of the Lie algebra structure, it is useful to work with the space \(\mathcal{F}\) of the (entire) analytical functions. We therefore will adopt the Fock-Bargmann representation (FBR) where the operators

\[
\begin{align*}
\hat{a}_q &\to \zeta , \\
a_q &\to \frac{d}{d\zeta} , \\
N &\to \zeta \frac{d}{d\zeta} , \\
\zeta &\in \mathcal{C}
\end{align*}
\]

provide a realization of the WH algebra (13). The \(q\)-WH algebra (14) is realized by the operators

\[
\hat{a}_q \to \zeta , \quad a_q \to D_q , \quad N \to \zeta \frac{d}{d\zeta}
\]

with the finite difference operator \(D_q\) (also called the \(q\)-derivative operator) defined by

\[
D_q \ f(\zeta) = \frac{f(q \zeta) - f(\zeta)}{(q - 1) \zeta} , \quad q \neq 1 , \quad f \in \mathcal{F}
\]

Clearly, eq. (17) gives the usual derivative in the limit of the deformation parameter \(\epsilon\) going to zero (or \(q \to 1\)), and in the same limit the FBR (15) is obtained from (16). One readily obtains

\[
[a_q, \hat{a}_q] \ f(\zeta) = q^N \ f(\zeta) = f(q \zeta) , \quad f \in \mathcal{F}
\]

The \(q\)-deformation of the WH algebra is therefore strictly related with the finite difference operator \(D_q\) \((q \neq 1)\). This suggests to us that the \(q\)-deformation of the operator algebra should arise in the presence of lattice or discrete structures.

In terms of the operators \(\tilde{c}\) and \(\tilde{c}^\dagger\) defined by

\[
\tilde{c} \equiv \frac{1}{\sqrt{2}}(\hat{\zeta} + i \hat{p}_\zeta) , \quad \tilde{c}^\dagger \equiv \frac{1}{\sqrt{2}}(\hat{\zeta} - i \hat{p}_\zeta)
\]
with $\hat{p}_\zeta = -i \frac{d}{d\zeta}$ and usual ccr, we have

$$\zeta \frac{d}{d\zeta} f(\zeta) = \left[ \frac{1}{2} (\zeta^2 - \zeta^{t_2}) - \frac{1}{2} \right] f(\zeta), \quad f \in \mathcal{F}$$  \hfill (20)

and, in $\mathcal{F}$,

$$[a_q, \hat{a}_q] = q^N = \frac{1}{\sqrt{q}} \exp \left( \frac{\epsilon}{2} (\zeta^2 - \zeta^{t_2}) \right) \equiv \frac{1}{\sqrt{q}} S(\epsilon), \quad q \neq 0$$ \hfill (21)

Eq. (21) shows that, for $\epsilon \in \mathcal{R}$, $\sqrt{q} [a_q, \hat{a}_q] \equiv S(\epsilon)$ is the generator of the Bogoliubov transformation

$$\zeta \rightarrow \zeta(\epsilon) = S^{-1}(\epsilon) \zeta S(\epsilon) = \zeta \cosh \epsilon - \zeta^{t_1} \sinh \epsilon$$ \hfill (22)

$$\zeta^{t_1} \rightarrow \zeta^{t_1}(\epsilon) = S^{-1}(\epsilon) \zeta^{t_1} S(\epsilon) = \zeta^{t_1} \cosh \epsilon - \zeta^{t_1} \sinh \epsilon$$

We close this section by observing that in ref. [5] it has been shown that $\sqrt{q} [a_q, \hat{a}_q]$ also acts as the generator of the squeezed coherent states[10].

4 $q$-WH algebra and the von Neumann theorem

By using the result (21) and (22) it is easy now to show that the parameter $q$ labels the Weyl systems in QM and the inequivalent representations in QFT.

We consider the Weyl operator (6) (for $M = 1$ for simplicity) and parameterize the representations by implementing (10). We observe that due to the definitions (4) the transformations (10) can be equivalently thought as applied to $\hat{x}$ and $\hat{p}$ instead than to $\alpha$ and $\beta$ (or to $f$ and $g$):

$$\hat{x} \mapsto \hat{x}(\rho) \equiv \rho \hat{x}, \quad \hat{p} \mapsto \hat{p}(\rho) \equiv \frac{1}{\rho} \hat{p}, \quad \rho \neq 0$$ \hfill (23)

The action variable $J = \int p \, dx$ is invariant under the transformations (22); this clarifies the physical meaning of the invariance of the area $Im(z_1^* z_2)$ under (10). Of course, (22) is a canonical transformation: $[\hat{x}, \hat{p}] = i \Rightarrow [\hat{x}(\rho), \hat{p}(\rho)] = i$. In terms of $c$ and $c^{t_1}$ in the Schrödinger representation (see eq. (2)) eqs. (22) read

$$\hat{x} \mapsto \hat{x}(\rho) \equiv \rho \hat{x} = \frac{1}{\sqrt{2}} \left( \rho c + \rho c^{t_1} \right)$$ \hfill (24)

$$\hat{p} \mapsto \hat{p}(\rho) \equiv \frac{1}{\rho} \hat{p} = \frac{-i}{\sqrt{2}} \left( \frac{1}{\rho} c - \frac{1}{\rho} c^{t_1} \right), \quad \rho \neq 0$$
We may thus introduce the operators $c(\rho)$ and $c^\dagger(\rho)$ as
\[
c \mapsto c(\rho) \equiv \frac{1}{\sqrt{2}}(\hat{x}(\rho) + i\hat{p}(\rho)) = \frac{1}{2}\left[\left(\rho + \frac{1}{\rho}\right) c + \left(\rho - \frac{1}{\rho}\right) c^\dagger\right]
\]
(25)
\[
c^\dagger \mapsto c^\dagger(\rho) \equiv \frac{1}{\sqrt{2}}(\hat{x}(\rho) - i\hat{p}(\rho)) = \frac{1}{2}\left[\left(\rho + \frac{1}{\rho}\right) c^\dagger + \left(\rho - \frac{1}{\rho}\right) c\right]
\]
By assuming $\rho$ real and defining
\[
u \equiv \frac{1}{2}\left(\rho + \frac{1}{\rho}\right), \quad \psi \equiv \frac{1}{2}\left(\rho - \frac{1}{\rho}\right)
\]
(26)
so that $u^2 - \nu^2 = 1$, eqs. (25) are then recognized to be nothing but Bogoliubov transformations. Let $\rho \equiv q^{-1} = e^{-\epsilon}, \epsilon \neq \infty$ and real. Eqs. (25) are then put in the more familiar form
\[
c(\epsilon) = c \cosh \epsilon - c^\dagger \sinh \epsilon
\]
\[
c^\dagger(\epsilon) = c^\dagger \cosh \epsilon - c \sinh \epsilon
\]
(27)
where we used the notation $c(\epsilon) \equiv c(\rho)$. We thus reach the (well known[1,8]) conclusion that the representations $W(z)$ and $W(q) \equiv W(\rho/z)$ are connected by Bogoliubov transformations. The $\rho$-parametrization is called the Bogoliubov parameterization.

The connection with the $q$-WH algebra (14) is immediately seen by realizing that from the holomorphy condition on $f(\zeta), f \in F$, with $\zeta = x + iy$, we have $\frac{d}{d\zeta}f(\zeta) = \frac{d}{dx}f(x)$, so that $\hat{p}_\zeta = \hat{p}$ and eqs.(22) give
\[
S^{-1}(\epsilon) \hat{c} S(\epsilon) \rightarrow c(\epsilon) \quad \text{and} \quad S^{-1}(\epsilon) \hat{c}^\dagger S(\epsilon) \rightarrow c^\dagger(\epsilon)
\]
(28)
as $y \rightarrow 0$.

We therefore conclude that, in the $y \rightarrow 0$ limit, the $q$-WH algebra commutator $[a_q, \hat{a}_q]$ acts (up to the $\sqrt{q}$ factor) as the generator of the Bogoliubov transformations (27) relating $W(z)$ with $W(q)(z)$, the $q$-deformation parameter labelling the Weyl representations.

We also observe that same conclusion is reached by working since the beginning in the complex $\zeta$-plane, $\zeta = x + iy$, with functions $f(\zeta) \in F$ (in the extension to the $\zeta$-plane the Weyl operator (6) acquires the phase factor $e^{-\sqrt{2}a_{\zeta}}$ (cf. eq.(6) for $M = 1$)).

In the limit of infinite degrees of freedom the representations $W(q)(F), W(q')(F), q \neq q'$, are unitarily inequivalent[1,8,9]. Different values of the $q$-deformation parameter thus label unitarily inequivalent representations in QFT.

5 Conclusions

We have shown that the $q$-deformation parameter of the $q$-WH algebra labels the Weyl systems in QM and the unitarily inequivalent representations of the ccr in QFT. We
found that the generator of the Bogoliubov transformations which relate Weyl systems or representations labelled by different values of $q$ is (up to a c-number factor) the $q$-WH algebra commutator $[a_q, \hat{a}_q]$.

The von Neumann theorem in QM has been discussed in relation with the parameterization of the Weyl representations through Bogoliubov transformations. The quantum commutation rules (1) specify the set of canonically conjugate operators $(\hat{x}_i, \hat{p}_j)$ only up to the transformations (22) (which are in fact canonical transformations) even if irreducibility is assumed\([9]\), so that the set of canonical operators is fully specified only when the value of $\epsilon = \log q$ is also given, or, in other words, when the representation $W^{(q)}(z)$ is assigned. This is a trivial problem in QM where Weyl systems with different $q$ labels are each other unitarily (and therefore physically) equivalent (the von Neumann theorem). Not so in QFT where representations with different $q$ labels are each other unitarily inequivalent and thus the assignment of the representation for a specific value of $q$ is physically relevant, e.g. in spontaneously broken symmetry theories where different physical phases of the system are described by different (inequivalent) representations. The procedure of tuning the $q$-parameter in the $q$-WH algebra may be thus understood as "tunneling" through inequivalent representations in QFT.

The $q$-parametrization of the representations of the ccr provides remarkable physical interpretation of the $q$-deformation parameter also because $q$-WH algebra has been related with coherent states, squeezed states, discretized (periodic) systems\([5]\), thermal field theories and quantum dissipation\([7]\).

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