INVERSE PROBLEM FOR FRACTIONAL-LAPLACIAN WITH LOWER ORDER NON-LOCAL PERTURBATIONS

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ABSTRACT. In this article, we study a model problem featuring a Lévy process in a domain with semi-transparent boundary by considering the following perturbed fractional Laplacian operator

\[ (-\Delta)^t + (-\Delta)^{s/2}_\Omega b(-\Delta)^{s/2}_\Omega + q, \quad 0 < s < t < 1 \]

on a bounded Lipschitz domain Ω ⊂ \mathbb{R}^n. While the non-locality of the fractional Laplacian \((-\Delta)^t\) depends on entire \mathbb{R}^n, in its non-local perturbation the non-locality depends on the domain Ω through the regional fractional Laplacian term \((-\Delta)^{s/2}_\Omega\) and \(b\) exhibits the semi-transparency of the process. We analyze the well-posedness of the model and certain qualitative property like unique continuation property, Runge approximation scheme considering its regional non-local perturbation. Then we move into studying the inverse problem and find that by knowing the corresponding Dirichlet to Neumann map (D-N map) of \(L_{b,c}\) on the exterior domain \(\mathbb{R}^n \setminus \Omega\), it is possible to determine the lower order perturbations \('b', 'q'\) in \(\Omega\). We also discuss the recovery of \('b', 'q'\) from a single measurement and its limitations.

1. INTRODUCTION

1.1. Model problem and motivation: In this article we consider the following non-local operator as

\[ L_{b,c} := (-\Delta)^t + (-\Delta)^{s/2}_\Omega b(-\Delta)^{s/2}_\Omega + q, \quad 0 < s < t < 1 \]

or, in the weak form

\[ \forall \varphi, \psi \in C_0^\infty(\mathbb{R}^n), \]

\[ \langle L_{b,c} \varphi, \psi \rangle := \int_{\mathbb{R}^n} (-\Delta)^{t/2} \varphi (-\Delta)^{t/2} \psi \, dx + \int_{\Omega} b(x)(-\Delta)^{s/2}_\Omega \varphi (-\Delta)^{s/2}_\Omega \psi \, dx + \int_{\Omega} q(x) \varphi \psi \, dx \]

where \(b, q\) are bounded functions defined over an open bounded regular set \(\Omega \subset \mathbb{R}^n\).

The principal part of the operator \(L_{b,c}\) is given by the fractional Laplacian operator \((-\Delta)^t\) of order 2t whose non-locality is defined over the entire \(\mathbb{R}^n\), where the sub-principal part of \(L_{b,c}\) contain another non-local operator commonly refereed as regional fractional Laplacian operator of order 2s whose non-locality is defined over \(\Omega\), along with a zero-th order local term defined in \(\Omega\).

We briefly recall the fractional Laplacian in \(\mathbb{R}^n\) which is given by

\[ (-\Delta)^t u = \mathcal{F}^{-1}\{\xi^{2t}\hat{u}(\xi)\}, \quad u \in \mathcal{S}(\mathbb{R}^n). \]

Here \(\mathcal{S}\) denotes Schwartz space in \(\mathbb{R}^n\) and \(\mathcal{F}^{-1}\) denotes the inverse Fourier transform, where \(\hat{u}(\xi) = \mathcal{F}u(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x) \, dx\). This pseudo-differential definition
(1.2) is valid for all \( t > -\frac{n}{2} \) (see [GSU16]). For \( 0 < t < 1 \), it has an equivalent integral representation (see [LPG+18]) as

\[
(-\Delta)^t u(x) = C_{n,t} \text{ p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2t}} dy, \quad x \in \mathbb{R}^n,
\]

where p.v. denotes the principal value.

Probabilistically the fractional Laplacian operator \((-\Delta)^t\) represents the infinitesimal generator of a symmetric \(2t\)-stable Lévy process in the entire space [App09]. However here we are interested to the restriction of \((-\Delta)^t\) in a bounded domain \(\Omega\). For example, as one can think of the homogeneous Dirichlet exterior value problem for the fractional Laplacian operator (e.g. \((-\Delta)^t v = g\) in \(\Omega\) and \(v = 0\) in \(\mathbb{R}^n \setminus \Omega\)) represents the infinitesimal generator of a symmetric \(2t\)-stable Lévy process for which particles are killed upon leaving the domain \(\Omega\) (see [BBC03, BV16, AVMRTM10]).

Next we recall the definition of the regional fractional Laplacian operator or censored fractional Laplacian based on the domain \(\Omega\). Formally for \(0 < s < 1\) we define \((-\Delta)^{s/2}_{\Omega}\) for \(C_c^\infty(\Omega) := \{u|\Omega : u \in C_c^\infty(\mathbb{R}^n)\}\) functions as

\[
(-\Delta)^{s/2}_{\Omega} u(x) = C_{n,s} \text{ p.v.} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+s}} dy, \quad x \in \Omega.
\]

In contrast to the fractional Laplacian, the regional fractional Laplacian \((-\Delta)^{s/2}_{\Omega}\) represents the infinitesimal generator of a censored \(s\)-stable process that is obtained from a symmetric \(s\)-stable Lévy process by restricting its measure to the domain \(\Omega\). The probabilistic meaning for such a process (and hence the operator) is that the process can only jump within the domain and is not allowed to jump outside the domain. Such process can be also exhibited through Feynman-Kac transformation. See [CZ95, MK00, GM05, GM06]. A more detailed discussion about these operators is given in Section 2.

We consider the operator \(L_{b,q}\), weighted combination of the global and the regional fractional Laplacian operator, keeping in mind the vast examples of natural domains with semi-transparent boundary. That is, after hitting the boundary of \(\Omega\), a particle can either go outside the domain \(\Omega\) or can reflect back into the domain depending on certain parameters. The coefficient \(b(x)\) denotes the transparency or permeability such a way that vanishing of \(b\) in \(\Omega\) signifies that the domain is transparent/permeable, i.e. if \(b \equiv 0\) in \(\Omega\), then the non-local part of \(L_{b,q}u(x)\) is \((-\Delta)^t u(x)\), which makes the process a \(2t\)-stable Lévy process in the entire space. In other words, it can jump anywhere in the space \(\mathbb{R}^n\) freely.

There are lots of examples of processes taking place on a domains with semi-transparent boundary viz. diffusion through a cell membrane; Photon diffusion. A good example of this kind is semipermeable cell membranes [KMS, CC72, CDMR97, Val09, Váz17, ST17]. A semipermeable membrane is a layer that only certain molecules can pass through. Semipermeable membranes can be both biological and artificial. The rate of passage depends on the pressure, concentration, and temperature of the molecules or solutes on either side, as well as the permeability of the membrane to each solute. Depending on the membrane and the solute, permeability may depend on solute size, solubility, properties, or chemistry. How the membrane is constructed to be selective in its permeability will determine the rate and the permeability. Many natural and synthetic materials thicker than a membrane are also semipermeable. One example of this is the thin film on the inside of the egg. Artificial semipermeable membranes include a variety of material
designed for the purposes of filtration, such as those used in reverse osmosis, which only allow water to pass.

1.2. Direct problem: Let us denote $\Omega_e = \mathbb{R}^n \setminus \overline{\Omega}$ and $b, q \in L^\infty(\Omega)$. We consider the following exterior boundary value problem as

$$\mathcal{L}_{b,q} u := \left( (-\Delta)^{1/2} + (-\Delta)^{s/2} b(-\Delta)^{s/2} + q \right) u = 0 \quad \text{in } \Omega,$$

$$u = f \quad \text{in } \Omega_e. \quad (1.3)$$

For suitable function class $H^1_0(\Omega_e) \ni f$ and for ‘b’ being compactly supported in $\Omega$, there exists a weak solution $u \in H^1(\mathbb{R}^n)$ of the above Dirichlet problem. The support condition on ‘b’ can be relaxed with the trade off $s \neq \frac{1}{2}$ (see (2.8) and Lemma 2.3). We refer to Section 2 for the complete details.

The associated bilinear form $B_{b,q}(\cdot, \cdot)$ on $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ of the above problem is given by

$$B_{b,q}(\varphi, \psi) = \int_{\mathbb{R}^n} (-\Delta)^{t/2} \varphi(x) (-\Delta)^{s/2} \psi(x) \, dx$$

$$+ \int_{\Omega} b(x) (-\Delta)^{s/2} \varphi(x) (-\Delta)^{s/2} \psi(x) \, dx + \int_{\Omega} q(x) \varphi(x) \psi(x) \, dx. \quad (1.4)$$

We also assume that $b, q$ are such that

$$\mathcal{L}_{b,q} \varphi = 0 \quad \text{in } \Omega, \quad \varphi = 0 \quad \text{in } \Omega_e$$

has only zero solution.

Certainly, for non-negative $b, q$ this assumption is satisfied. Under condition (1.5) there is a unique solution $u_f \in H^1(\mathbb{R}^n)$ of (1.3). Let us introduce the operator $\mathcal{N}_{b,q}(f)$, we will call it as Neumann data, through the “non-local normal derivative” (see [DROV17]) of $u_f$ as

$$\mathcal{N}_{b,q}(f) := C_{n,t} \int_{\Omega} \frac{u_f(x) - u_f(y)}{|x - y|^{n+2t}} \, dy, \quad x \in \Omega_e \quad (1.6)$$

where $u_f \in H^1(\mathbb{R}^n)$ is a unique weak solution of (1.3). The main resemblance of “non-local normal derivative” with the “local normal derivative” is explained through the following integration by-parts formula [GLX17, DROV17]

$$\int_{\Omega} v(-\Delta)^{s} w \, dx + \int_{\Omega_e} v \mathcal{N}_{b,q} w = \int_{\Omega} w(-\Delta)^{s} v \, dx + \int_{\Omega_e} w \mathcal{N}_{b,q} v$$

together with the following limiting equivalence as (see [DROV17])

$$\lim_{t \to 1} \int_{\Omega \setminus \Omega_e} v \mathcal{N}_{b,q} w \, dx = \int_{\partial \Omega} v \frac{\partial w}{\partial \nu} \, d\sigma$$

for all $v, w \in \mathcal{S}(\mathbb{R}^n)$; where $\mathcal{N}_{b,q}w$ is same as the right hand side of (1.6) with $u_f$ is replaced by any $w$ and $\nu, d\sigma$ denote the boundary normal vector and the surface measure respectively.

1.3. Inverse problem: We are interested in studying the inverse problem of recovering the unknown coefficients $b, q$ in $\Omega$ from the non-local Cauchy data $(f, \mathcal{N}_{b,q}(f))$ in some open subset of $\Omega_e \times \Omega_e$ (possibly different open subsets of $\Omega_e$ for two components of the data). Assuming $b$ and $q$ are compactly supported in $\Omega$ we can recover the coefficients $b, q$ in $\Omega$ from the non-local Cauchy data $(f|_W, \mathcal{N}_{b,q}(f)|_{\overline{W}})$, $\forall f \in C^\infty_c(W)$ for some open sets $W, \overline{W} \subset \Omega_e$ (see Theorem 1.1).
This kind of inverse problem is often addressed as the Calderón problem. In the standard Calderón problem [Cal80] the objective is to determine the electrical conductivity of a medium from voltage and current measurements on its boundary. Study of the inverse boundary value problems have a long history, in particular, in the context of electrical impedance tomography; in seismic and medical imaging; as well as in inverse scattering problems. We refer to a thorough discussion in [Uh14] and the reference therein for a rigorous understanding of this topic.

The study of Calderón type inverse problem for non-local operators began with the recent article [GSU16], where the authors address the inverse problem of determining the potential ‘q’ in fractional Schrödinger operator \((-\Delta)^t + q(x))\), 0 < t < 1 in \(\Omega\) from the corresponding Dirichlet Neumann map in the exterior domain \(\Omega_e\). In [AS17] the authors study the stability estimates in the way of recovering the potential ‘q’. Later it has been shown that with a single measurement \((f, \mathcal{N}_q(f))\) it is possible to recover and reconstruct the potential ‘q’ in \(\Omega\) (see [GASU18]). Subsequent problem of recovering ‘q’ for the anisotropic fractional elliptic operator \((-\text{div} A(x)\nabla)^t + q(x))\) (0 < t < 1), which is more delicate, has been successfully considered in [GLX17].

Here, in a further generalization of this kind inverse problems, we are interested in determining two unknown potentials \(b, q\) in the perturbed non-local operator \(\mathcal{L}_{b,q}\), where apart from the zeroth order perturbation additionally we have a 2s-order non-local perturbation to the principal 2t-order fractional Laplacian operator.

**Theorem 1.1 (All measurements).** Let \(\Omega \subset \mathbb{R}^n\), \(n \geq 1\) be an open bounded set with Lipschitz boundary and \(\mathcal{L}_{b_1,q_1}, \mathcal{L}_{b_2,q_2}\) be such that the assumption (1.5) is satisfied. We assume \(b_1, b_2, q_1, q_2 \in L^\infty(\Omega)\) compactly supported in \(\Omega\). Let \(\mathcal{N}_{b_1,q_1}(f) = \mathcal{N}_{b_2,q_2}(f)\) on \(\tilde{W}\), for all \(f \in \mathcal{C}_c^\infty(\tilde{W})\), where \(\tilde{W}, \tilde{W} \subset \Omega_e\) be some nonempty open sets. Then \(q_1 = q_2\) and \(b_1 = b_2\) in \(\Omega\).

Next we state a qualitative results for the regional fractional Laplacian, which helps us to solve the inverse problem discussed above.

**Theorem 1.2.** Let \(\Omega\) be a bounded Lipschitz domain in \(\mathbb{R}^n\), and \(\mathcal{O} \subset \Omega\) be any open subset compactly contained in \(\Omega\). Let \(u \in H^a(\Omega)\), 0 < a < 1.

1. (Strong Unique Continuation) If \(u = 0 = (-\Delta)^a_{\Omega} u\) on \(\mathcal{O}\) then \(u \equiv 0\) in \(\Omega\).
2. (Runge approximation) The following space

\[X_\mathcal{O} := \{ v|_\mathcal{O} : v \in H^2(\Omega), (-\Delta)^a_{\Omega} v = 0 \text{ in } \mathcal{O} \}\]

is dense in \(L^2(\mathcal{O})\).

**Remark 1.3.** We assume the support condition on \(b_j, q_j\) as stated in the Theorem 1.4 and 1.1 in order to use the above result, which gives density of the solutions of the operator \((-\Delta)^a_{\Omega} \) on an open subset compactly contained in \(\Omega\).

1.4. Single Measurement: We also investigate the possibility of recovering \(b\) and \(q\) on suitable subsets of \(\Omega\) subject to only a single measurement of the non-local Cauchy data \((f|_W, \mathcal{N}_q(f)|_{\partial W})\). Let \(u_f\) be the unique solution of the problem \(\mathcal{L}_{b,q} u_f = 0\) in \(\Omega\) and \(u_f|_{\partial W} = f\). If \(u_f = 0\) in some non-empty open subset \(E \subset \Omega\), and \(b \neq 0\) in \(\Omega\) then we can’t really conclude \(u_f = 0\) everywhere in \(\mathbb{R}^n\), although in the absence of \(b\) can, see [GSU16]; i.e. a single measurement is enough to determine \(q\) in the absence of \(b\), which has been shown in [GASU18]. So in this
case ($b \neq 0$ in $\Omega$) for any $\varphi \in C_c(E)$
\begin{equation}
\mathcal{L}_{b,q}u_f = 0 \quad \text{in } \Omega \implies \mathcal{L}_{b,(q+\varphi)}u_f = 0 \quad \text{in } \Omega,
\end{equation}
with the same Dirichlet and Neumann data. Therefore, it is impossible to recover $q$ on $E$ from the single measurement of $f$ and $N_{b,q}f$. Similarly, if $(-\Delta)^{s/2}_{b,q}u_f = 0$ in some non-empty open subset $F \subset \Omega$, then it is impossible to recover $b$ on $F$ from the single measurement $(f, N_{b,q}(f))$. Therefore, an optimal claim is to recover $b$ and $q$ on the support of $(-\Delta)^{s/2}_{b,q}u_f$ and $u_f$ respectively. In Theorem 1.4 we prove the above optimal claim, i.e. we recover $b$ and $q$ on the support of $(-\Delta)^{s/2}_{b,q}u_f$ and $u_f$ in $\Omega$ respectively.

The surprising part to observe here that even one measurement of DN data can determine two unknowns. If the supports of $b$ and $q$ are contained in the support of $(-\Delta)^{s/2}_{b,q}u_f$ and $u_f$ in $\Omega$ respectively, then the complete recovery of $b$ and $q$ follows from that single measurement. If not, definitely all the measurements (i.e. varying $f$ in the Cauchy data) provides the recovery of the unknown coefficients $b$, $q$. In addition, two suitable measurements $(f_1, N_{b,q}(f_1))$, $l = 1, 2$ can determine the coefficients completely in $\Omega$ provided the support of the solutions $u_{f_l}$ and $(-\Delta)^{s/2}_{b,q}u_{f_l}$ have disjoint zero sets corresponding to the measurements $f_1$ and $f_2$.

We now state our main result.

**Theorem 1.4 (Single measurement).** Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$ be an open bounded set with Lipschitz boundary and $\mathcal{L}_{b_1,q_1}$, $\mathcal{L}_{b_2,q_2}$ be such that the assumption (1.5) is satisfied. We assume $b_1, b_2, q_1, q_2 \in C_c(\Omega)$. Let $f \in H^1_0(\Omega_\varepsilon)$ be a fixed non-zero function and for $j = 1, 2$, $(u_f)_j \in H^1(\mathbb{R}^n)$ solves $\mathcal{L}_{b_j,q_j}(u_f)_j = 0$ in $\Omega$ with $(u_f)_j = f$ in $\Omega_\varepsilon$. If $N_{b_1,q_1}(f) = N_{b_2,q_2}(f)$ on $\tilde{W}$, where $\tilde{W} \subset \Omega_\varepsilon$ be some nonempty open set, then $(u_f)_1 = (u_f)_2$ in $\Omega$, $q_1 = q_2$ on the support of $(u_f)_1$ in $\Omega$ and $b_1 = b_2$ on the support of $(-\Delta)^{s/2}_{b,q}u_f$ in $\Omega$.

**Remark 1.5.** Let us write $u_f = (u_f)_1 = (u_f)_2$.

- It is evident from the Theorem 1.4 that if the support of $(-\Delta)^{s/2}_{b,q}u_f$ contains the support of $b_1$, $b_2$ and the support of $u_f$ contains the support of $q_1$, $q_2$ then that single measurement determines $b_1 = b_2$ and $q_1 = q_2$ in $\Omega$.
- On the other hand, we only recover $b$ and $q$ on the support of $(-\Delta)^{s/2}_{b,q}u_f$ and $u_f$ in $\Omega$ respectively. In particular it can be seen that $N_{b,q}(f) = N_{\tilde{b},\tilde{q}}(f)$, where $\tilde{b} = b + \varphi_1$ and $\tilde{q} = q + \varphi_2$, for some $\varphi_1, \varphi_2 \in L^\infty(\Omega)$ supported outside the support of $(-\Delta)^{s/2}_{b,q}u_f$ and $u_f$ in $\Omega$ respectively, (see Equation (1.7) and (1.4)).
- We would like to emphasize the fact that even two measurements; viz. $(f, N_{b_j,q_j}(f))$ and $(g, N_{b_j,q_j}(g))$, $j = 1, 2$ are sufficient for uniqueness $b_1 = b_2$ and $q_1 = q_2$ in $\Omega$, subject to the condition that zero sets of $u_f$, $u_g$ and $(-\Delta)^{s/2}_{b,q}u_f$, $(-\Delta)^{s/2}_{b,q}u_g$ are disjoint in $\Omega$ respectively.

1.5. **Outline of the proof.** While in the local case, in principal, solving the inverse problem of determining unknown coefficients of an elliptic (or hyperbolic) operator relies on constructing special family of solution; For example, the complex geometric optics solutions for Schrödinger [SU87] or magnetic-Schrödinger operators [NSU95] etc. or geometric optics solution for elastic wave operators [SU91, Uhl04], and reducing the problem to a question of inverting geodesic ray transform for
functions and tensors. Here in the non-local analogue of the inverse problem our method is based on the strong unique continuation property and subsequent Runge approximation scheme for the fractional Laplacian operators, developed in (cf. [GSU16, GLX17]), and for the regional fractional Laplacian which we develop here (cf. Theorem 1.2).

The paper is organized as follows. In Section 2 we present an extensive discussion about the non-local operators and the Dirichlet problem (1.3). In Section 3 we address the inverse problem through single measurement and derive an identity (3.2) as an initial step to recover the unknown coefficients. Section 4 is devoted to the unique continuation and Runge approximation scheme for regional fractional Laplacian, essentially the proof of Theorem 1.2. It also discusses on relevant forward problems namely Dirichlet exterior value problem (see Subsection 4.1 and 4.2) for non-local regional fractional Laplacian operator. Finally in the Section 5 we show the recovery of the lower order perturbations 'b' and 'q' of $L_{b,q}$ and complete the proof of the Theorem 1.4 and Theorem 1.1.

2. The forward problem

2.1. Fractional Laplacian and fractional Sobolev space. Let $0 < a < 1$ and consider the fractional Laplacian in $\mathbb{R}^n$ for Schwartz class functions

$$\forall x \in \mathbb{R}^n, \quad (-\Delta)^a u(x) = \mathcal{F}^{-1}\{|\xi|^{2a} \hat{u}(\xi)|, \quad u \in \mathcal{S}(\mathbb{R}^n).$$

We note that, $(-\Delta)^a u$ is not a Schwartz class function due to its lack of decay near infinity, in particular, $(-\Delta)^a u$ decays at infinity as $|x|^{-n-2a}$, see [Lan72].

It enjoys the following integration by parts formula in $\mathbb{R}^n$ in $L^2$ sense (i.e. $(-\Delta)^a u \in L^2(\mathbb{R}^n)$, $u \in \mathcal{S}(\mathbb{R}^n)$ for $0 < a < 1$) as

$$\int_{\mathbb{R}^n} (-\Delta)^a u v dx = \int_{\mathbb{R}^n} (-\Delta)^{a/2} u (-\Delta)^{a/2} v dx, \quad \forall u, v \in \mathcal{S}(\mathbb{R}^n)$$

consequently,

$$\int_{\mathbb{R}^n} (-\Delta)^a u v dx = \int_{\mathbb{R}^n} (-\Delta)^a v u dx, \quad \forall u, v \in \mathcal{S}(\mathbb{R}^n).$$

There are many equivalent definitions of the fractional Laplacian, see [Kwa17]. For instance, it is given by the principal value integral as $(0 < a < 1)$

$$(-\Delta)^a u(x) = C_{n,a} \text{p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2a}} dy$$

$$= C_{n,a} \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B(x, \epsilon)} \frac{u(x) - u(y)}{|x - y|^{n+2a}} dy,$$

where $C_{n,a}$ is some constant given by $\frac{4^a \Gamma(\frac{n+a}{2})}{\pi^{n/2} \Gamma(-a)}$ (see [DNPV12]), and $B(x, \epsilon)$ is a ball in $\mathbb{R}^n$ centered at $x$ with radius $\epsilon > 0$. Due to the singularity of the kernel, the difference $u(x) - u(y)$ in the numerator, which vanishes at the singularity, provides a regularization, together with averaging of positive and negative parts allows the principal value to exist at least for smooth $u$ with sufficient decay. However, when $a \in (0, \frac{1}{2})$, the integral is not really singular near $x$. Indeed, for $u \in \mathcal{S}(\mathbb{R}^n)$ and
Next we extend (with the well-known Aronszajn-Slobodeckij inner products) operator \( \langle \cdot, \cdot \rangle \) where \( - (2.1) \) follows
\[
\|\nabla u\|_{L^\infty} \int_{\mathcal{B}(x,1)} \frac{dy}{|x - y|^{n + 2a - 1}} + \|u\|_{L^\infty} \int_{\mathcal{R}^n \setminus \mathcal{B}(x,1)} \frac{dy}{|x - y|^{n + 2a}}.
\]
and both of the integrals in the right hand side are finite. Note that we used only \( C^1 \) regularity and its boundedness. Moreover, by using the \( C^2 \) regularity and its boundedness, in general for \( a \in (0, 1) \) we can write the fractional Laplacian with a non principal value integral as (see [BV16])
\[
\forall x \in \mathcal{R}^n,
(-\Delta)^a u(x) = - \frac{C_{n,a}}{2} \int_{\mathcal{R}^n} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{n + 2a}} \, dy, \quad u \in \mathcal{S}(\mathcal{R}^n).
\]
Next we extend \( (-\Delta)^a \) on larger spaces, in particular on \( H^r(\mathcal{R}^n) \) Sobolev space for any \( r \in \mathcal{R} \).
Let us start with recalling the definition of the fractional order Sobolev space \( H^r(\mathcal{R}^n) \). One way to define the space \( H^r(\mathcal{R}^n) \) is to use the Fourier transform and define
\[
H^r(\mathcal{R}^n) : = \{ u \in L^2(\mathcal{R}^n) : \langle \xi \rangle^r \hat{u}(\xi) \in L^2(\mathcal{R}^n) \}, \quad \forall r \in \mathcal{R},
\]
where \( \langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}} \). The fractional Laplacian extends as a bounded map (see [GSU16])
\[
(-\Delta)^a : H^r(\mathcal{R}^n) \rightarrow H^{r - 2a}(\mathcal{R}^n)
\]
whenever \( r \in \mathcal{R} \) and \( a \in (0, 1) \).
One can equivalently characterize the space \( H^r(\mathcal{R}^n) \) for \( 0 < r < 1 \) as
\[
H^r(\mathcal{R}^n) = \{ u \in L^2(\mathcal{R}^n) : \frac{u(x) - u(y)}{|x - y|^{r + 2a}} \in L^2(\mathcal{R}^n \times \mathcal{R}^n) \}
\]
with the well-known Aronszajn-Slobodeckij inner products [AF03] and what it follows
\[
\frac{C_{n,r}}{2} \int_{\mathcal{R}^n} \int_{\mathcal{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{r + 2a}} \, dy \, dx = \langle (-\Delta)^{r/2} u, (-\Delta)^{r/2} v \rangle_{L^2(\mathcal{R}^n)},
\]
for all \( u, v \in H^r(\mathcal{R}^n) \). Following that, we assign the graph norm on \( H^r(\mathcal{R}^n) \) for \( 0 < r < 1 \) as
\[
\|u\|^2_{H^r(\mathcal{R}^n)} = \|u\|^2_{L^2(\mathcal{R}^n)} + \|(-\Delta)^{r/2} u\|^2_{L^2(\mathcal{R}^n)}.
\]
2.2. Regional fractional Laplacian. Let us consider any open set \( \mathcal{O} \subset \mathcal{R}^n \) with Lipschitz boundary. Let \( 0 < a < 1 \), now we introduce the following non-local operator \( (-\Delta)^a_\mathcal{O} \) defined in the domain \( \mathcal{O} \) over the class of \( C^\infty(\mathcal{O}) \) functions as
\[
\forall x \in \mathcal{O}, \quad (-\Delta)^a_\mathcal{O} u(x) = C_{n,a} \lim_{\epsilon \rightarrow 0} \int_{\mathcal{O} \setminus \mathcal{B}(x, \epsilon)} \frac{u(x) - u(y)}{|x - y|^{n + 2a}} \, dy, \quad u \in C^\infty(\mathcal{O}).
\]
The following results ensures that the above limit always exists pointwise for each \( x \in \mathcal{O} \) at least for smooth functions \( u \in C^\infty(\mathcal{O}) \). Let us define the function space
\[
L^1 \left( \mathcal{O}, \frac{1}{(1 + |x|)^{n + 2a}} \right) := \left\{ u \in L^1(\mathcal{O}) : \int_{\mathcal{O}} \frac{|u(x)|}{(1 + |x|)^{n + 2a}} \, dx < \infty \right\}
\]
Proposition 2.1. [GM05, GM06] Let $O \subset \mathbb{R}^n$ be an open set and $u \in L^1 \left( O, \frac{1}{(1+|x|)^{n+\alpha}} \right)$ for some $0 < \alpha < 1$.

1. If $u$ is a Hölder continuous function at $x_0 \in O$ with the exponent $\beta > a$, then $(-\Delta)^\beta_O u(x_0)$ exists for $a \in (0, \frac{1}{2})$.
2. If $u$ is $C^1$ regular and all the first derivatives of $u$ are Hölder continuous at $x_0 \in O$ with the exponent $\beta > a - \frac{1}{2}$, then $(-\Delta)^\beta_O u(x_0)$ exists for $a \in (\frac{1}{2}, 1)$.

Clearly (2.2) is well defined ($(-\Delta)^\beta_O(x) \neq \infty, x \in O$) for $u \in C^2(O) \cap L^\infty(O)$, or in particular for $C^\infty(\overline{O})$ functions which automatically take care of $L^1(1+|x|^{n+\alpha})$-ness of the function. In general $(-\Delta)^a_O$ $(0 < a < 1)$ maps smooth functions into smooth functions, in particular

$$(-\Delta)^a_O : C^{k+2}(O) \rightarrow C^k(O) \quad \forall k \geq 1.$$ 

This result has been established in [MY15] for all dimensions.

Next we mention the following result in a bounded domain $O$ which gives the integrability aspect of $(-\Delta)^a_O$.

Proposition 2.2. [GM05, GM06] Let $O$ be a bounded domain.

1. If $u \in C^\beta(O)$ for some $\beta > a$, then $(-\Delta)^a_O u$ is continuous on $O$ and $(-\Delta)^a_O u \in L^\infty(O)$ whenever $a \in (0, \frac{1}{2})$.
2. If $u \in C^{1+\beta}(O)$ for some $\beta > a - \frac{1}{2}$, then $(-\Delta)^a_O u$ is continuous on $O$ and $(-\Delta)^a_O u \in L^1(O)$ for $a \in (\frac{1}{2}, 1)$.

Let $O$ be any open set with Lipschitz boundary and $C_c^\infty(\overline{O})$ denotes the restriction of all $C_c^\infty(\mathbb{R}^n)$ (compactly supported in $\mathbb{R}^n$) functions on $\overline{O}$. Then we have the following integration by parts formula (see [Gua06])

$$\int_O (-\Delta)^a_O u v \, dx = \frac{C_{n,r}}{2} \int_O \int_O \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2a}} \, dy \, dx, \quad \forall u,v \in C_c^\infty(\overline{O})$$

Consequently, we have

$$\int_O (-\Delta)^a_O u v \, dx = \int_O (-\Delta)^a_O u v \, dx, \quad \forall u,v \in C_c^\infty(\overline{O}).$$

In particular, when $O$ is also bounded, then the above integration by parts identity holds for all $u, v \in C^\infty(\overline{O})$.

Clearly when $O = \mathbb{R}^n$ the regional fractional Laplacian coincides with the definition of the usual fractional Laplacian $(-\Delta)^a$, $(0 < a < 1)$. Moreover, what it follows for $u \in C_c^\infty(O)$ the regional fractional Laplacian can be identified with the fractional Schrödinger operator $((-\Delta)^a - \varphi_a)$ in $O$ $(0 < a < 1)$ as

$$\forall x \in O, \quad (-\Delta)^a_O u(x) = (-\Delta)^a u(x) - \varphi_a(x) u(x), \quad \forall u \in C_c^\infty(O)$$

where the potential

$$\varphi_a(x) = C_{n,a} \int_{\mathbb{R}^n \setminus O} \frac{1}{|x - y|^{n+2a}} \, dy$$

admitting (see [Che17])

$$\varphi_a \in C^{\alpha,1}_{loc}(O), \text{ i.e. a locally Lipschitz function}$$

$$\varphi_a \in C^{\alpha,1}_{loc}(O), \text{ i.e. a locally Lipschitz function}$$
which follows from the relation (2.1).

Before that, we observe

\[ H^r(\Omega) = \{ u \in L^2(\Omega) : \frac{u(x) - u(y)}{|x - y|^{\frac{n+2r}{2}}} \in L^2(\Omega \times \Omega) \} \]

Let \( H^r_0(\Omega) \) denote the closure of \( C^\infty_c(\Omega) \) functions in \( H^r(\Omega) \). Then the dual of \( H^r_0(\Omega) \) is \( H^{-r}(\Omega) = \{ u|_\Omega : u \in H^{-r}(\mathbb{R}^n) \} \). Moreover, whenever \( 0 < r < \frac{1}{2} \), we have \( H^r_0(\Omega) = H^r(\Omega) \). We also denote \( H^r(\Omega) \) as the closure of \( C^\infty_c(\Omega) \) functions in \( H^r(\mathbb{R}^n) \) and for \( \Omega \) being Lipschitz we identify the space as \( H^r(\Omega) = H^r_0(\Omega) \) for \( 0 < r < 1 \).

Next we extend the definition of \( (-\Delta)^a_{\Omega} \) weakly in \( (H^a(\Omega))^* \) \( (0 < a < 1) \). Since \( C^\infty_c(\Omega) \) is dense in \( H^a(\Omega) \), so for \( u, v \in H^a(\Omega) \), one defines \( (-\Delta)^a_{\Omega} u \in (H^a(\Omega))^* \) through the integration by parts formula (2.3), which becomes the duality bracket as

\[
\int_{\Omega} (-\Delta)^a_{\Omega} u \, v \, dx = \frac{C_{a, n}}{2} \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2a}} \, dy \, dx
\]

admitting

\[
\left| \int_{\Omega} (-\Delta)^a_{\Omega} u \, v \, dx \right| \leq ||(u(x) - u(y))||_{L^2(\Omega \times \Omega)}^2 \left\| (v(x) - v(y)) \right\|^2_{L^2(\Omega \times \Omega)}
\]

Thus

\[
(-\Delta)^a_{\Omega} : H^a(\Omega) \mapsto (H^a(\Omega))^* \text{ is continuous.}
\]

Next we examine for being \( u \in H^a_0(\Omega) \) \( (0 < a < 1) \), whether \( (-\Delta)^{a/2}_{\Omega} u \in L^2(\Omega) \). Before that, we observe

\[
\forall \Omega' \subseteq \Omega, \ (-\Delta)^{a/2}_{\Omega} u|_{\Omega'} \in L^2(\Omega'), \ u \in H^a_0(\Omega),
\]

which follows from the relation (2.4) and the fact (2.5). As by extending the function \( u \in H^a_0(\Omega) \) by 0 in \( \mathbb{R}^n \) and write \( u \in H^a(\mathbb{R}^n) \). So \( (-\Delta)^{a/2} u \in L^2(\mathbb{R}^n) \) (see (2.1)), in particular \( (-\Delta)^{a/2} u \in L^2(\Omega) \). Now in \( \Omega \) from (2.4) we have

\[
(-\Delta)^{a/2}_{\Omega} u + \varphi_{a/2}(x) u = (-\Delta)^{a/2} u \in L^2(\Omega).
\]

Now \( \forall \Omega' \subseteq \Omega \), since \( \varphi_{a/2}(x) u|_{\Omega'} \in L^2(\Omega') \) (c.f. (2.5)), ensures (2.8).
Lemma 2.3. Let $\mathcal{O}$ be a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary. Let $u \in H^0_\alpha(\mathcal{O})$ and $a \in (0,1) \setminus \{\frac{1}{2}\}$, then $(-\Delta)^{a/2}_\mathcal{O} u \in L^2(\mathcal{O})$ and
\begin{equation}
\|u\|_{L^2(\mathcal{O})} + \|(-\Delta)^{a/2}_\mathcal{O} u\|_{L^2(\mathcal{O})} \leq \|u\|_{H^\alpha(\mathcal{O})}.
\end{equation}

Remark 2.4. For the case of $a = \frac{1}{2}$, $(-\Delta)^{1/4}_\mathcal{O} u$ might not be in $L^2(\mathcal{O})$, for $u \in H^{1/2}_0(\mathcal{O})$, see the counter example in [Dyd04, Section 2].

Proof of Lemma 2.3. From (2.9) it is reduced to investigate whether $\varphi_{a/2}(x)u \in L^2(\mathcal{O})$. Note that from (2.6) we have $\varphi_{a/2}(x) \sim (\text{dist } (x, \mathcal{O}))^{-a}$ for $x \in \mathcal{O}$ and $0 < a < 1$. We recall the following fractional Hardy inequality from [Dyd04] as follows:
\begin{equation}
\forall u \in H^\alpha_0(\mathcal{O}), \quad \int_{\mathcal{O}} \frac{|u(x)|^2}{(\text{dist } (x, \mathcal{O}))^{2a}} \, dx \leq C \left( \int_{\mathcal{O}} |u|^2 \, dx + \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{(u(x) - u(y))^2}{|x-y|^{n+2a}} \, dy \, dx \right),
\end{equation}
and whenever $0 < a < \frac{1}{2}$
\begin{equation}
\forall u \in H^\alpha_0(\mathcal{O}), \quad \int_{\mathcal{O}} \frac{|u(x)|^2}{(\text{dist } (x, \mathcal{O}))^{2a}} \, dx \leq C \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{(u(x) - u(y))^2}{|x-y|^{n+2a}} \, dy \, dx
\end{equation}
where $C = C(\mathcal{O}, n, a)$.

Thus for $u \in H^\alpha_0(\mathcal{O})$ in both cases (i.e. (2.11),(2.12) holds) we can say $(-\Delta)^{a/2}_\mathcal{O} u \in L^2(\mathcal{O})$ Moreover from (2.9) we have
\begin{equation}
\|(\Delta)^{a/2}_\mathcal{O} u\|_{L^2(\mathcal{O})} \leq \|\varphi_{a/2} u\|_{L^2(\mathcal{O})} + \|(\Delta)^{a/2}_\mathcal{O} u\|_{L^2(\mathcal{O})}
\end{equation}

and
\begin{align*}
&\|(\Delta)^{a/2}_\mathcal{O} u\|_{L^2(\mathcal{O})}^2 \\
&= C_{n,a} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x-y|^{n+2a}} \, dy \, dx \\
&= C_{n,a} \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{(u(x) - u(y))^2}{|x-y|^{n+2a}} \, dy \, dx \quad \text{(since } u = 0 \text{ in } \mathbb{R}^n \setminus \mathcal{O}) \\
&+ 2C_{n,a} \int_{\mathcal{O}} \int_{\mathcal{O}} \left( \frac{(u(x))^2}{\int_{\mathbb{R}^n \setminus \mathcal{O}} \frac{1}{|x-y|^{n+2a}} \, dy} \right) \, dx \\
&\leq C_{n,a} \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{(u(x) - u(y))^2}{|x-y|^{n+2a}} \, dy \, dx + C \int_{\mathcal{O}} (\text{dist } (x, \mathcal{O}))^{2a} \, dx \\
&\leq \|u\|^2_{H^\alpha_0(\mathcal{O})}.
\end{align*}

So,
\begin{equation*}
\|u\|_{L^2(\mathcal{O})} + \|(\Delta)^{a/2}_\mathcal{O} u\|_{L^2(\mathcal{O})} \leq C\|u\|_{H^\alpha_0(\mathcal{O})}.
\end{equation*}

This completes the proof of the lemma. \qed
2.3. **Existence, uniqueness and stability estimate of the solution of (1.3):**

Here we address the question of existence and uniqueness of the solution of the problem (1.3). Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ with Lipschitz boundary. We consider the following inhomogeneous problem

\[
\mathcal{L}_{b,q} u := \left( (-\Delta)^{t/2} + (-\Delta)^{s/2} b(-\Delta)^{s/2} \right) u = F \quad \text{in } \Omega, \\
u = f \quad \text{in } \Omega_c,
\]

where $F \in H^{-t}(\Omega)$ and $f \in H^t_0(\Omega_c)$ and $b \in L^\infty(\Omega)$ with compact support in $\Omega$ and $q \in L^\infty(\Omega)$. From the fact (2.8) and Lemma 2.3 it is evident that either we need to take $b$ to be compactly supported, bounded function in $\Omega$ or we can relax the support condition of $b$ by not allowing $s = 1/2$.

Let us define the bilinear form

\[
\mathcal{B}_{b,q} : H^t(\mathbb{R}^n) \times H^t(\mathbb{R}^n) \rightarrow \mathbb{R}
\]

as

\[
\mathcal{B}_{b,q}(\varphi, \psi) = \int_{\mathbb{R}^n} (-\Delta)^{t/2} \varphi(x) (-\Delta)^{t/2} \psi(x) \, dx \\
+ \int_{\Omega} b(x) (-\Delta)^{s/2} \varphi(x) (-\Delta)^{s/2} \psi(x) \, dx + \int_{\Omega} q(x) \varphi(x) \psi(x) \, dx.
\]

We will say $u \in H^t(\mathbb{R}^n)$ to be a weak solution of (2.14) if for every $\varphi \in C_c^\infty(\Omega)$ we get

\[
\mathcal{B}_{b,q}(u, \varphi) = \langle F, \varphi \rangle, \quad \text{with } u = f \text{ in } \Omega_c.
\]

Extending $f \in H^t_0(\Omega_c)$ by 0 in $\Omega$ we have $f \in H^t(\mathbb{R}^n)$ and observe that it is equivalent to consider the problem for $v = (u - f) \in \tilde{H}^t(\Omega)$ given as

\[
\mathcal{L}_{b,q} v = F - (-\Delta)^t f \quad \text{in } \Omega, \\
v = 0 \quad \text{in } \Omega_c.
\]

Let us call $\tilde{F} := F - (-\Delta)^t f$ and it is see that $\tilde{F} \in H^{-t}(\Omega)$. Equivalently, in terms of the bilinear form the above problem implies

\[
\mathcal{B}_{b,q}(v, \varphi) = \langle \tilde{F}, \varphi \rangle, \quad \text{with } v \in \tilde{H}^t(\Omega) \quad \text{and } \forall \varphi \in \tilde{H}^t(\Omega).
\]

In order to prove the existence and uniqueness of the solution of (2.14), now, we will prove the existence and uniqueness of a solution $v \in \tilde{H}^t(\Omega)$ solving (2.16).

1. **Continuity of the bilinear form $\mathcal{B}_{b,q} (\cdot, \cdot)$:** Let $\Omega' \subseteq \Omega$ so that $b$ is supported inside $\Omega'$. Let $\varphi, \psi \in \tilde{H}^t(\Omega)$, then by using $\|(-\Delta)^{s/2} \varphi\|_{L^2(\Omega')} \leq C \|\varphi\|_{H^t(\Omega)} \leq \|\varphi\|_{H^t(\Omega)}$ (c.f. (2.8) and $0 < s < t < 1$), we have

\[
|\mathcal{B}_{b,q}(\varphi, \psi)| \leq \|(-\Delta)^{t/2} \varphi\|_{L^2(\mathbb{R}^n)} \|(-\Delta)^{t/2} \psi\|_{L^2(\mathbb{R}^n)} \\
+ \|b\|_{L^\infty(\Omega)} \|(-\Delta)^{s/2} \varphi\|_{L^2(\Omega')} \|(-\Delta)^{s/2} \psi\|_{L^2(\Omega')} \\
+ \|q\|_{L^\infty(\Omega)} \|\varphi\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)}, \\
\leq C \|\varphi\|_{H^t(\mathbb{R}^n)} \|\psi\|_{H^t(\mathbb{R}^n)}.
\]

2. **Coercivity of the bilinear form $\mathcal{B}_{b,q}$ on $\tilde{H}^t(\Omega)$:** Let $0 < s < t < 1$ and $\varphi \in \tilde{H}^t(\Omega)$. Then using the Hardy-Littlewood-Sobolev inequality (see [Ste70]) on a bounded domain

\[
\|\varphi\|_{L^2(\Omega)} \leq C \|\varphi\|_{L^{\frac{2n}{n-2t}}(\Omega)} \leq C \|(-\Delta)^{t/2} \varphi\|_{L^2(\mathbb{R}^n)}
\]
we obtain
\[ B_{b,q}(\varphi, \varphi) \geq ||(-\Delta)^{t/2} \varphi||^2_{L^2(\mathbb{R}^n)} \]
\[ - ||b||_{L^\infty(\Omega)} ||(-\Delta)^{t/2} \varphi||^2_{L^2(\Omega)} - ||q||_{L^\infty(\Omega)} ||\varphi||^2_{L^2(\Omega)} \]
\[ \geq ||\varphi||^2_{H^s(\Omega)} - \lambda ||\varphi||^2_{H^t(\Omega)}, \]  
(2.18)  
where \( \lambda = \max\{2||b||_{L^\infty(\Omega)}, 2||q||_{L^\infty(\Omega)}\}. 

Having the compact inclusions
\[ \tilde{H}^s(\Omega) \hookrightarrow \tilde{H}^t(\Omega) \hookrightarrow L^2(\Omega), \quad \text{for} \ 0 < s < t < 1, \]
one gets
\[ ||\varphi||^2_{H^t(\Omega)} \leq \frac{1}{2\lambda} ||\varphi||^2_{H^t(\Omega)} + C_\lambda ||\varphi||^2_{L^2(\Omega)}, \quad \text{for} \ \varphi \in \tilde{H}^t(\Omega). \]
Therefore, combining the above estimates we get
(2.19)  
\[ B_{b,q}(\varphi, \varphi) + C_\lambda ||\varphi||^2_{L^2(\Omega)} \geq \frac{1}{2} ||\varphi||^2_{\tilde{H}^t(\Omega)}, \quad \text{for} \ \varphi \in \tilde{H}^t(\Omega). \]

By Riesz-representation theorem there exists a unique
\[ v = G(\vec{F}, \mu) \in \tilde{H}^t(\Omega) \]
such that
\[ B_{b,q}(v, \varphi) + \mu \langle v, \varphi \rangle = \langle \vec{F}, \varphi \rangle, \quad \forall \varphi \in \tilde{H}^t(\Omega), \quad \mu \geq C_\lambda, \]
and it follows
\[ B_{b,q}(v, \varphi) - \theta \langle v, \varphi \rangle = \langle \vec{F}, \varphi \rangle, \quad \text{for} \ v = G\left((\mu + \theta)v + \vec{F}, \mu\right), \quad \forall \theta \in \mathbb{R}. \]

Observe that \( G(\cdot, \mu) \) is a bounded operator from \( \left(\tilde{H}^t(\Omega)\right)^* \to \tilde{H}^t(\Omega) \) and by using compact Sobolev embedding result we conclude
(2.20)  
\[ G(\cdot, \mu) : L^2(\Omega) \to L^2(\Omega) \]
is a compact, self adjoint, positive definite operator. Therefore, from the standard spectral analysis \( G(\cdot, \mu) \) has a discreet spectrum \( \Sigma \subset \mathbb{R} \). That is if \( \theta \not\in \Sigma \) then there is a unique solution \( v \in \tilde{H}^t(\Omega) \) solving
\[ \mathcal{L}_{b,q} v - \theta v = \vec{F} \quad \text{in} \ \Omega, \quad v \in \tilde{H}^t(\Omega), \quad \theta \in \mathbb{R} \setminus \Sigma, \]
and subsequently a unique \( u \in H^t(\mathbb{R}^n) \) solving
\[ \mathcal{L}_{b,q} u - \theta u = F \quad \text{in} \ \Omega, \quad u \in \mathbb{R} \setminus \Sigma, \]
\[ u = f \quad \text{in} \ \Omega_{\epsilon}. \]

If we assume that \( \text{(cf. (1.5))} \) that 0 is not an eigenvalue of \( \mathcal{L}_{b,q} \), i.e. \( 0 \not\in \Sigma \), then we can choose, in particular, \( \theta = 0 \) above to get existence and uniqueness of the solution of (2.14).

3. Stability estimate:  
Now we will show that if \( u \) is the unique solution of (2.14) then the following stability estimate is true:
\[ ||u||_{H^t(\mathbb{R}^n)} \leq C \left(||F||_{H^{-t}(\Omega)} + ||f||_{H^t_0(\Omega_\epsilon)}\right). \]

In order to show that, observe that from (2.19) we get
\[ ||v||_{\tilde{H}^t(\Omega)}^2 \leq C ||v||_{L^2(\Omega)}^2 + B_{b,q}(v, v). \]
Now as $v$ solves the (2.16) we get $\mathcal{B}_{b,q}(v,v) = \langle \tilde{F}, v \rangle$. Hence,

$$\|v\|^2_{\tilde{H}^t(\Omega)} \leq C \|v\|^2_{L^2(\Omega)} + |\langle \tilde{F}, v \rangle|$$

$$\leq C \left( \|v\|_{L^2(\Omega)} + \|\tilde{F}\|_{\tilde{H}^t(\Omega)^*} \right) \|v\|_{\tilde{H}^t(\Omega)}$$

or,

$$\|v\|_{\tilde{H}^t(\Omega)} \leq C \left( \|v\|_{L^2(\Omega)} + \|\tilde{F}\|_{H^{-t}(\Omega)} \right)$$

Now by putting $u = v + f$ and $\tilde{F} = F - (-\Delta)^t f$ with $\|\tilde{F}\|_{H^{-t}(\Omega)} \leq \|F\|_{H^{-t}(\Omega)} + \|f\|_{H^t(\Omega)}$ one gets

$$\|u\|_{H^t(\mathbb{R}^n)} \leq C \left( \|u\|_{L^2(\Omega)} + \|F\|_{H^{-t}(\Omega)} + \|f\|_{H^t(\Omega)} \right).$$

Moreover, by using the compactness of the inverse operator $G$ introduced in (2.20), one finally shows

$$\|u\|_{H^t(\mathbb{R}^n)} \leq C \left( \|F\|_{H^{-t}(\Omega)} + \|f\|_{H^t(\Omega)} \right).$$

Hence, the stability estimate follows.

2.4. The Dirichlet to Neumann Map: Let $\Omega$ be a bounded Lipschitz domain. Let us define the Dirichlet to Neumann map $\Lambda_{b,q} : \tilde{H}^t(\Omega) \mapsto \left( \tilde{H}^t(\Omega) \right)^*$ $(0 < t < 1)$ as

$$\langle \Lambda_{b,q} f, \psi \rangle := \mathcal{B}_{b,q}(u_f, \psi) \quad \text{for } \psi \in \tilde{H}^t(\Omega)$$

where $u_f$ is the unique solution of $\mathcal{L}_{b,q} u_f = 0$ in $\Omega$ and $u_f = f$ on $\Omega_e$. In particular,

$$\langle \Lambda_{b,q} f, \psi \rangle = \int_{\mathbb{R}^n} \left( (-\Delta)^{t/2} u_f \right) \left( (-\Delta)^{t/2} \psi \right) + \int_{\mathbb{R}^n} b \left( (-\Delta)^{s/2} u_f \right) \left( (-\Delta)^{s/2} \psi \right)$$

$$= \int_{\Omega_e} \left( (-\Delta)^{t} u_f \right) \psi.$$

So,

$$\Lambda_{b,q} f := (-\Delta)^{t} u_f, \quad \text{in } \Omega_e.$$  \hspace{1cm} (2.21)

Let $u$ be a bounded $C^2$ functions in $\mathbb{R}^n$. Then, we introduce the “non-local normal derivative” of $u$ in $\Omega_e$ as

$$\mathcal{N}_{b,c} u(x) = C_{n,t} \int_{\Omega} \frac{u(x) - u(y)}{|x - y|^{n+2t}} dy, \quad x \in \Omega_e. \hspace{1cm} (2.22)$$

Next we recall the relation between $\Lambda_{b,q}(f)$ and the non-local normal derivative $\mathcal{N}_{b,c}(f)$ in the following proposition:

**Proposition 2.5.** [GSU16] One has

$$\Lambda_{b,q} f = \mathcal{N}_{b,q} u_f - mf + \left( -\Delta \right)^t (E_0 f)|_{\Omega_e}, \quad f \in H^t_0(\Omega_e)$$

where, for $\gamma > -1/2$, $\mathcal{N}_{b,q}$ is the map

$$\mathcal{N}_{b,q} : H^t(\mathbb{R}^n) \rightarrow H^t_{loc}(\Omega_e), \quad \mathcal{N}_{b,q} u = mu|_{\Omega_e} + \left( -\Delta \right)^t (\chi_{\Omega} u)|_{\Omega_e}$$

where $m \in C^\infty(\Omega_e)$ is given by $m(x) = c_{n,t} \int_{\Omega} \frac{1}{|x - y|^{n+2t}} dy$ and $\chi_{\Omega}$ is the characteristic function of $\Omega$. Also, $E_0$ is extension by zero. If $u \in L^2(\mathbb{R}^n)$, then $\mathcal{N}_{b,q} u \in L^2_{loc}(\Omega_e)$ is given a.e. by the formula (2.22).
The result shows that knowing $\Lambda_{b,q}f|_W$ is equivalent to knowing $\mathcal{N}_{b,q}u_f|_W$ for some $f \in C_c^\infty(W)$ for $W \subset \Omega_e$, since they differ by the quantities which are independent of $b$ and $q$.

3. Single Measurement and the Inverse Problem

Here we exploit the strong unique continuation property of the fractional Laplacian $(-\Delta)^t$, $0 < t < 1$ and apply it to approach the non-local inverse problem through a single measurement.

Let us begin with recalling the strong unique continuation property for fractional Laplacian operator (see [GSU16, Theorem 1.2]), which goes as follows:

**Proposition 3.1 (SUCP).** Let $u \in H^{-r}(\mathbb{R}^n)$, $r > 0$. If $u = (-\Delta)^t u = 0$ in some non-empty open set $\mathcal{O} \subset \mathbb{R}^n$, then $u \equiv 0$ in $\mathbb{R}^n$.

As a direct application of the above SUCP we have the following Runge approximation result (cf. [non-empty open set]). Let us consider the set

$$X_{\mathcal{O}, W} := \{ v|_{\mathcal{O}} : (-\Delta)^t v = 0, \text{ in } \mathcal{O}, \ v|_{\partial \Omega_e} = f, \forall f \in C_c^\infty(W) \}$$

where $W$ be some open bounded subset of $\Omega_e := \mathbb{R}^n \setminus \overline{\mathcal{O}}$.

**Proposition 3.2 (Runge approximation).** The set $X_{\mathcal{O}, W}$ is dense in $L^2(\mathcal{O})$.

3.1. Inverse problem for $\mathcal{L}_{b,q}$. Following the above SUCP result, we get the following. Let $u_k \in H^t(\mathbb{R}^n)$, $k = 1, 2$ solve

$$\mathcal{L}_{b,q_k} u_k = \left( (-\Delta)^t + (-\Delta)^{s/2} b_k(-\Delta)^{s/2} + q_k \right) u_k = 0 \quad \text{in } \Omega,$$

with the same exterior data, that is

$$u_1 = u_2 = f \quad \text{in } \Omega_e.$$

By our assumption in Theorem 1.4, there exists a fixed function $f \in H^t_0(\Omega_e)$ such that

$$\mathcal{N}_{b_1,q_1}(f)|_{\overline{W}} = \mathcal{N}_{b_2,q_2}(f)|_{\overline{W}},$$

where $\overline{W} \subset \Omega_e$ be some open set. Hence, by using the Proposition 2.5 and (2.21) we get

$$(-\Delta)^t u_1|_{\overline{W}} = (-\Delta)^t u_2|_{\overline{W}}.$$

Hence we have

$$(-\Delta)^t(u_1 - u_2)|_{\overline{W}} = 0 = (u_1 - u_2)|_{\overline{W}}$$

and consequently from the above Proposition 3.1 we have

$$u_1 = u_2 \text{ on } \mathbb{R}^n.$$

Let us now denote $u = u_1 = u_2$ in $\mathbb{R}^n$. Therefore we have

$$\left( (-\Delta)^t + (-\Delta)^{s/2} b_1(-\Delta)^{s/2} + q_1 \right) u = 0, \quad \text{in } \Omega$$

and

$$\left( (-\Delta)^t + (-\Delta)^{s/2} b_2(-\Delta)^{s/2} + q_2 \right) u = 0, \quad \text{in } \Omega,$$

which reduces to

$$(-\Delta)^{s/2}(b_1 - b_2)(-\Delta)^{s/2} u + (q_1 - q_2)u = 0, \quad \text{in } \Omega;$$

or one can equivalently write the above equation as

$$\int_{\Omega}(b_1 - b_2)(-\Delta)^{s/2} u(-\Delta)^{s/2} \varphi + \int_{\Omega}(q_1 - q_2)u \varphi = 0, \quad \forall \varphi \in C_c^\infty(\Omega).$$
Our goal is now to recover $b_j$, $q_j$, as stated in the Theorem 1.4 and 1.1, which we do in Section 5, following the Section 4 where we develop the required machineries.

4. SUCP and Runge approximation scheme for $(-\Delta)_{\Omega}^a$ operator

We want to prove the following strong unique continuation property for the regional fractional Laplacian $(-\Delta)_{\Omega}^a$, $0 < a < 1$.

Lemma 4.1 (SUCP). Let $\Omega \subset \mathbb{R}^n$, be a bounded Lipschitz domain. Let $v \in H^a(\Omega)$, $0 < a < 1$. If $v = (-\Delta)_{\Omega}^a v = 0$ on an open subset $O \Subset \Omega$, then $v = 0$ in $\Omega$.

Proof. Let us take $v \in H^a(\Omega)$ and extend it by zero in $\Omega \setminus O$ to write $v \in \tilde{H}^a(\Omega)$. From (2.4) with using the fact $v = (-\Delta)_{\Omega}^a v = 0$ in $\Omega$ we simply obtain $v = (-\Delta)_{\tilde{O}}^a v = 0$ in $O$. Consequently, from Proposition 3.1 we obtain $v \equiv 0$, or $v = 0$ in $\Omega$.

Next we prove the following Runge approximation result for the regional fractional Laplacian.

Lemma 4.2 (Runge approximation). Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, and $O \Subset \Omega$ be an open subset compactly contained in $\Omega$. Then we will show $X_O := \{v|_O : v \in H^a(\Omega), (-\Delta)_{\Omega}^a v = 0 \text{ in } \Omega\}$ is dense in $L^2(O)$.

Before going to prove the above lemma we move into the following discussion on Dirichlet problems for the regional fractional Laplacian. We also refer [Che17, Gua06] where certain Dirichlet problems for regional fractional Laplacian has been studied.

4.1. Dirichlet boundary value problem for $(-\Delta)_{\Omega}^a$, $0 < a < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. In this section we will study the existence, uniqueness and stability results of the weak solution of the equation

\begin{align}
(-\Delta)_{\Omega}^a v &= f, \quad \text{in } \Omega \\
v &= g, \quad \text{on } \partial \Omega.
\end{align}

We will specify $f, g$ later.

Homogeneous Case: Let us begin with the homogeneous boundary value problem. Let $f \in H^{-a}(\Omega)$. Then we say $v_f \in H_0^a(\Omega)$, $0 < a < 1$, be the weak solution of

\begin{align}
(-\Delta)_{\Omega}^a v &= f \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial \Omega\end{align}

if for all $w \in H_0^a(\Omega)$

\begin{align}
\int_{\Omega} \int_{\Omega} \frac{(v_f(x) - v_f(y))(w(x) - w(y))}{|x-y|^{n+2a}} \, dy \, dx &= (f, w)_{H^{-a}(\Omega), H_0^a(\Omega)}.
\end{align}
In order to show the existence, uniqueness and stability we define the corresponding bilinear form $B_{\Omega} : H^a_0(\Omega) \times H^a_0(\Omega) \to \mathbb{R}$ as 

$$B_{\Omega}(\varphi, \psi) = \int_{\Omega} \int_{\Omega} \frac{(\varphi(x) - \varphi(y))(\psi(x) - \psi(y))}{|x - y|^{n+2a}} \, dy \, dx.$$ 

Then it is easy to see the bilinear form is continuous on $H^a_0(\Omega) \times H^a_0(\Omega)$ since 

$$(4.4) \quad |B_{\Omega}(\varphi, \psi)| \leq \|\varphi\|_{H^a_0(\Omega)} \|\psi\|_{H^a_0(\Omega)}$$

and coercive over $H^a_0(\Omega)$ space due to the Poincaré inequality [Pon16] as 

$$(4.5) \quad B_{\Omega}(\varphi, \varphi) \geq C\|\varphi\|_{H^a_0(\Omega)}^2.$$ 

Therefore for a given $f \in H^{-a}(\Omega)$, we have a unique weak solution of (4.2) in $H^a_0(\Omega)$ with the stability estimate 

$$\|v_f\|_{H^a_0(\Omega)} \leq \|f\|_{H^{-a}(\Omega)}.$$ 

**Inhomogeneous Case:** Let $G \in H^a(\Omega)$ then from (2.7) we know $(-\Delta)^a_{\Omega}G \in (H^a(\Omega))^*$, $0 < a < 1$. Now we are interested in the following inhomogeneous problem 

$$(4.6) \quad (-\Delta)^a_{\Omega}v = f \quad \text{in } \Omega, \quad (v - G) \in H^a_0(\Omega).$$

Clearly by considering $w = (v - G) \in H^a_0(\Omega)$, it solves 

$$(-\Delta)^a_{\Omega}w = f - (-\Delta)^a_{\Omega}G \in H^{-a}(\Omega),$$

and by the previous discussion we have a unique weak solution in $H^a_0(\Omega)$.

Now when $1 > a > \frac{1}{2}$, then we can define the trace of $G$ as $g = G|_{\partial \Omega} \in H^{a-\frac{1}{2}}(\partial \Omega)$ and as it stands $u \in H^a(\Omega)$ weakly solves the inhomogeneous problem 

$$(-\Delta)^a_{\Omega}v = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial \Omega.$$ 

**Proposition 4.3.** Let $\Omega \subset \mathbb{R}^n$, bounded Lipschitz domain. Let $f \in H^{-a}(\Omega)$ and $G \in H^a(\Omega)$, $0 < a < 1$. Then there exists a unique weak solution $v \in H^a(\Omega)$ solving (4.6) with the following stability estimate as 

$$\|v\|_{H^a(\Omega)} \leq C \left(\|f\|_{H^{-a}(\Omega)} + \|G\|_{H^a(\Omega)}\right).$$ 

**Corollary 4.4.** The operator 

$$((-\Delta)^a_{\Omega})^{-1} : H^{-a}(\Omega) \to H^a_0(\Omega)$$

is one-one, onto and bounded.

4.2. **Dirichlet exterior value problem for $(-\Delta)^a_{\Omega}$, $0 < a < 1$**. Let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^n$, and $\mathcal{O} \subset \subset \Omega$ be an open subset compactly contained in $\Omega$. We consider the following non-local problem 

$$(4.7) \quad (-\Delta)^a_{\Omega} = f \quad \text{in } \mathcal{O}$$

$$u = g \quad \text{in } \Omega \setminus \mathcal{O}.$$ 

We will specify $f, g$ later. 

**Homogeneous Case:** Let us begin with the homogeneous boundary value problem. Let $f \in H^{-a}(\mathcal{O})$. Then we say $v_f \in H^a_0(\Omega) := \{v \in H^a(\Omega) : \text{Sppt } v \subseteq \overline{\mathcal{O}}\}$, $0 < a < 1$, be the weak solution of 

$$(4.8) \quad (-\Delta)^a_{\Omega}v = f \quad \text{in } \Omega, \quad v = 0 \quad \text{in } \Omega \setminus \mathcal{O}.$$
if for all \( w \in C_c^\infty(\mathcal{O}) \)
\[
(4.9) \quad \mathcal{B}_\Omega(v_f, w) = \langle f, w \rangle_{H^{-a}(\mathcal{O}), H^a_0(\mathcal{O})}
\]
holds, where the bilinear form
\[
\mathcal{B}_\Omega : H^a_0(\Omega) \times H^a_0(\Omega) \to \mathbb{R}
\]
as introduced in (4.4). The bilinear is continuous over \( H^a_0(\Omega) \times H^a_0(\Omega) \) and coercive over \( H^a_0(\Omega) \) space, as mentioned before (cf. Subsection 4.1). So it is also coercive over \( H^a_0(\mathcal{O}) \) space for \( \mathcal{O} \subseteq \Omega \) i.e.
\[
(4.10) \quad \int_{\Omega} \int_{\Omega} \frac{(\varphi(x) - \varphi(y))^2}{|x - y|^{n+2a}} \, dy \, dx \geq C\|\varphi\|_{H^a_0(\mathcal{O})}^2, \quad \forall \varphi \in C_c^\infty(\mathcal{O}).
\]
Therefore for a given \( f \in H^{-a}(\mathcal{O}) \), we have a unique weak solution of (4.8) in \( H^a_0(\mathcal{O}) \) with the stability estimate
\[
\|v_f\|_{H^{-a}(\mathcal{O})} \leq \|f\|_{H^{-a}(\mathcal{O})}.
\]

**Inhomogeneous Case:** Let \( G \in H^a(\Omega) \) then from (2.7) we know \((-\Delta)^a_\Omega G \in (H^a(\Omega))^*\), \( 0 < a < 1 \). Now we are interested in the following inhomogeneous problem
\[
(-\Delta)^a_\Omega v = f \quad \text{in} \quad \mathcal{O}, \quad (v - G) \in H^a_0(\mathcal{O}).
\]
Clearly by considering \( w = (v - G) \in H^a_0(\mathcal{O}) \), it solves
\[
(-\Delta)^a_\Omega w = f - (-\Delta)^a_\Omega G \in H^{-a}(\mathcal{O}),
\]
and by the previous discussion we have a unique weak solution in \( H^a_0(\mathcal{O}) \).

**Proposition 4.5.** Let \( \Omega \subset \mathbb{R}^n \), bounded Lipschitz domain, and \( \mathcal{O} \subseteq \Omega \) be an open subset compactly contained in \( \Omega \). Let \( f \in H^{-a}(\mathcal{O}) \) and \( G \in H^a(\Omega) \) with \( G = g \) in \( \Omega \setminus \mathcal{O}, \) \( 0 < a < 1 \). Then there exists a unique weak solution \( v \in H^a(\Omega) \) solving (4.7) with the following stability estimate as
\[
\|v\|_{H^a(\Omega)} \leq C \left( \|f\|_{H^{-a}(\mathcal{O})} + \|G\|_{H^a(\Omega)} \right).
\]

**Corollary 4.6.** The operator
\[
((-\Delta)^a_\Omega)^{-1} : H^{-a}(\mathcal{O}) \to H^a_0(\Omega)
\]
is one-one, onto and bounded.

Let \( \mathcal{O}_1, \mathcal{O}_2 \subseteq \Omega \), and let us consider the following problem
\[
(-\Delta)^a_{\mathcal{O}_1} u = f \quad \text{in} \quad \mathcal{O}_1
\]
\[
(-\Delta)^a_{\mathcal{O}_2} u = 0 \quad \text{in} \quad \mathcal{O}_2
\]
\[
u = 0 \quad \text{in} \quad \Omega \setminus (\mathcal{O}_1 \cup \mathcal{O}_2).
\]
We denote \( \mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2 \), and \( f \in H^{-a}_{\mathcal{O}_1}(\mathcal{O}) \). Then we want to see whether the operator
\[
((-\Delta)^a_{\mathcal{O}_1})^{-1} : H^{-a}_{\mathcal{O}_1}(\mathcal{O}) \to H^a_{\mathcal{O}_1}(\Omega)
\]
is onto over the subspace \( H^a(\mathcal{O}_1) \subset H^a_{\mathcal{O}_1}(\Omega) \) or not. Of course, it is one-one and bounded.
Lemma 4.7. The operator

\[ (\mathcal{L} : H^a_{\text{div}}(\Omega))^{-1} \rightarrow H^a_{\text{div}}(\Omega) \]

is onto over the subspace \( H^a(\Omega_1) \subset H^a_{\text{div}}(\Omega) \).

Proof. Let \( u \in H^a(\Omega_1) \). Then we consider a new function \( v \in H^a_{\text{div}}(\Omega) \) as

\[
\begin{align*}
(\mathcal{L}) v & = 0 \quad \text{in } \Omega_2 \\
v & = u \quad \text{in } \Omega_1 \\
v & = 0 \quad \text{in } \Omega \setminus (\Omega_1 \cup \Omega_2).
\end{align*}
\]

(4.12)

Note that \((\mathcal{L}) v \in H^a_{\text{div}}(\Omega_1)\). Now by considering \( f = ((\mathcal{L}) v) \mid_{\Omega_1} \) in \( \Omega_1 \) we prove our claim. □

4.3. Runge approximation: Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^n \), and \( \Omega \subset \Omega \) be an open subset compactly contained in \( \Omega \). Then we will show

\[ X_{\Omega} := \{ v \mid_{\Omega} : v \in H^a(\Omega), (\mathcal{L}) v = 0 \text{ in } \Omega \} \]

is dense in \( L^2(\Omega) \).

Proof of Lemma 4.2. Thanks to Hahn-Banach theorem applied to \( L^2(\Omega) \) subspace topology, it is enough to show that

\[
\langle w, v \rangle_{L^2(\Omega)} = 0 \quad \forall v \in X_{\Omega}, \quad \text{then } w = 0 \text{ in } \Omega.
\]

(4.13)

Let us assume that there is \( 0 \neq w \in L^2(\Omega) \), such that \( \langle w, v \rangle_{L^2(\Omega)} = 0 \) for all \( v \in X_{\Omega} \). Then we consider the function \( \varphi \in H^a_{\text{div}}(\Omega) \) as a unique weak solution of

\[
(\mathcal{L}) \varphi = w \quad \text{in } \Omega \\
\varphi = 0 \quad \text{in } \Omega \setminus \Omega.
\]

(4.14)

Now from (4.13) and (4.14) we have

\[
0 = \langle w, v \rangle_{L^2(\Omega)} = \langle (\mathcal{L}) \varphi, v \rangle_{L^2(\Omega)} = \langle (\mathcal{L}) \varphi, v \rangle_{L^2(\Omega)} - \langle (\mathcal{L}) \varphi, v \rangle_{L^2(\Omega)}
\]

or,

\[
\langle (\mathcal{L}) \varphi, v \rangle_{L^2(\Omega \setminus \Omega)} = \langle (\mathcal{L}) \varphi, v \rangle_{L^2(\Omega)} = \langle \varphi, (\mathcal{L}) v \rangle_{L^2(\Omega)} = 0
\]

(4.15)

for all \( v \in X_{\Omega} \).

Since \( v \mid_{\Omega \setminus \Omega} \in H^a(\Omega \setminus \Omega) \) is arbitrary, this implies that

\[
(\mathcal{L}) \varphi = 0 \quad \text{in } \Omega \setminus \Omega.
\]

(4.16)

Consequently we end up with having

\[ \varphi = (\mathcal{L}) \varphi = 0 \text{ in } \Omega \setminus \Omega, \]

which implies that \( \varphi \equiv 0 \) in \( \Omega \) thanks due to strong unique continuation principal (cf. Lemma 4.1) and \( w = 0 \) in \( \Omega \). This proves (4.13) and essentially the Lemma 4.2. □
Let us assume \( \mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2 \) be a non empty open subset in \( \Omega \). Let us consider the function \( v \in H^a(\Omega) \) defined as a solution of

\[
\begin{align*}
(-\Delta)^a_{\Omega_1} v &= 0 & \text{in } \mathcal{O}_1 \\
(-\Delta)^a_{\Omega_2} v &= f & \text{in } \mathcal{O}_2 \\
v &= 0 & \text{in } \Omega \setminus \mathcal{O}.
\end{align*}
\]  

(4.17)

We are interested to know by varying \( f \) whether one can get the denseness of \( v \) in \( L^2(\mathcal{O}_1) \). The answer is “yes”. We prove in the following lemma. Let us define the space

\[X_{\mathcal{O}_1, \mathcal{O}_2} = \{v|_{\mathcal{O}_1} : v \in H^a(\Omega), v \text{ solves (4.17) for } f \in H^{-a}_{\mathcal{O}_2}(\mathcal{O})\}.
\]

**Lemma 4.8.** \( X_{\mathcal{O}_1, \mathcal{O}_2} \) is dense in \( L^2(\mathcal{O}_1) \).

**Proof.** We exactly follow the proof of Lemma 4.2. We show that

\[
\begin{align*}
\langle w, v \rangle_{L^2(\mathcal{O}_1)} &= 0 \quad \forall v \in X_{\mathcal{O}_1, \mathcal{O}_2}, \quad \text{then } w = 0 \text{ in } \mathcal{O}_1.
\end{align*}
\]  

(4.18)

Let us assume that there is \( 0 \neq w \in L^2(\mathcal{O}_1) \), such that \( \langle w, v \rangle_{L^2(\mathcal{O}_1)} = 0 \) for all \( v \in X_{\mathcal{O}_1, \mathcal{O}_2} \). Then we consider the function \( \varphi \in H^a(\Omega) \) as a unique weak solution of

\[
\begin{align*}
(-\Delta)^a_{\Omega_1} \varphi &= w & \text{in } \mathcal{O}_1 \\
\varphi &= 0 & \text{in } \Omega \setminus \mathcal{O}_1.
\end{align*}
\]  

(4.19)

Now from (4.17), (4.19) and (4.18), and we have (similar to (4.15))

\[
\begin{align*}
\langle (-\Delta)^a_{\Omega_1} \varphi, v \rangle_{L^2(\mathcal{O}_2)} &= \langle (-\Delta)^a_{\Omega_1} \varphi, v \rangle_{L^2(\Omega \setminus \mathcal{O}_1)} \\
&= \langle (-\Delta)^a_{\Omega_1} \varphi, v \rangle_{L^2(\Omega)} - \langle (-\Delta)^a_{\Omega_1} \varphi, v \rangle_{L^2(\mathcal{O}_1)} \\
&= \langle \varphi, (-\Delta)^a_{\Omega_1} v \rangle_{L^2(\Omega)} - \langle w, v \rangle_{L^2(\mathcal{O}_1)} \\
&= 0
\end{align*}
\]

for all \( v \in X_{\mathcal{O}_1, \mathcal{O}_2} \).

Since \( v|_{\mathcal{O}_2} \in H^a(\mathcal{O}_2) \) arbitrary, thanks due to the Lemma 4.7. This implies that \( (-\Delta)^a_{\Omega_1} \varphi = 0 \) in \( \mathcal{O}_2 \). Consequently we end up with having \( \varphi = (-\Delta)^a_{\Omega_1} \varphi = 0 \) in \( \mathcal{O}_2 \) implies that \( \varphi \equiv 0 \) in \( \Omega \) (cf. Lemma 4.1) and \( w = 0 \) in \( \mathcal{O}_1 \). This completes the proof. \( \square \)

4.4. **Runge approximation result for** \( \mathcal{L}_{b,q} \). We end this section by proving one more Runge approximation result. The following result provides the direct recovery of the coefficients \( b \) and \( q \) from all measurements.

Let us recall the operator \( \mathcal{L}_{b,q} \) introduced in (1.1) on \( \Omega \). Let \( \mathcal{O} \subseteq \Omega \) be an open set compactly contained in \( \Omega \) and consider the sets

\[X := \{(-\Delta)^{a/2}_{\mathcal{O}} v|_{\mathcal{O}} : \mathcal{L}_{b,q} v = 0, \text{ in } \Omega, v|_{\partial \mathcal{O}} = f, \forall f \in C^\infty_c(W)\}
\]

and

\[Y := \{v|_{\Omega} : \mathcal{L}_{b,q} v = 0, \text{ in } \Omega, v|_{\partial \mathcal{O}} = f, \forall f \in C^\infty_c(W)\}
\]

where \( W \) be some open bounded subset of \( \Omega_e \).

**Proposition 4.9.** Let \( \mathcal{L}_{b,q}, X, Y \) be defined as above. Then

1. For any \( F \in L^2(\mathcal{O}) \) and any \( \epsilon > 0 \) then there exists a \( u \in X \) such that

\[\|F - u\|_{L^2(\mathcal{O})} < \epsilon.\]
(2) The set $Y$ is dense in $L^2(\Omega)$.

Proof. (1) We observe that, it is enough to prove the result for $F \in H^s_0(\Omega)$ for some $0 < s < 1$, since $H^s_0(\Omega)$ is dense in $L^2(\Omega)$. Let $F \in H^s_0(\Omega)$ such that $\langle F, \bar{v} \rangle_{L^2(\Omega)} = 0$, for all $\bar{v} \in X$, then we show $F = 0$ in $\Omega$. Since $\bar{v} = (-\Delta)^{s/2} v|_{\Omega} \in L^2(\Omega)$ for some $v \in H^1(\mathbb{R}^n)$ solving $\mathcal{L}_{b,q} v = 0$ in $\Omega$. Consequently, we have

$$\langle F, (-\Delta)^{s/2} v \rangle_{L^2(\Omega)} = 0.$$

Now extending $F$ by 0 outside $\Omega$ we have $F \in H^s_0(\Omega)$ and we claim $(-\Delta)^{s/2} F \in L^2(\Omega)$. It follows from the fact $(-\Delta)^{s/2} F + \varphi_{s/2}(x) F = (-\Delta)^{s/2} F \in L^2(\Omega)$ and from (2.5) along with $F = 0$ in $\Omega \setminus \mathcal{O}$ implies $\varphi_{s/2}(x) F|_{\Omega} \in L^2(\Omega)$, which establishes our claim. Next we write

$$0 = \langle F, (-\Delta)^{s/2} v \rangle_{L^2(\Omega)} = \left\langle \left( (-\Delta)^{s/2} F \right), v \right\rangle_{L^2(\Omega)}.$$

Since $(-\Delta)^{s/2} F \in L^2(\Omega)$, there is $w \in H^t(\mathbb{R}^n)$, $0 < t < 1$ such that

$$\mathcal{L}_{b,q} w = (-\Delta)^{s/2} F \quad \text{in } \Omega, \quad w = 0 \text{ in } \Omega_c.$$

Therefore we get,

$$0 = \left( \mathcal{L}_{b,q} w, v \right)_{L^2(\Omega)} = \left( w, \mathcal{L}_{b,q} v \right)_{L^2(\Omega)} - \left( (-\Delta)^t w, v \right)_{L^2(\Omega_c)}.$$

Since $\mathcal{L}_{b,q} v = 0$ in $\Omega$, thus

$$\left( (-\Delta)^t w, f \right)_{\Omega_c} = \left( (-\Delta)^t w, v \right)_{\Omega_c} = 0, \quad \forall f \in C^\infty_c(W).$$

Hence, $(-\Delta)^t w = 0 = w \text{ in } W \subset \Omega_c$. Consequently, by SUCP we have $w \equiv 0$, that $(-\Delta)^{s/2} F \equiv 0 \text{ in } \Omega$. Using SUCP for the Regional fractional Laplacian operator as $(-\Delta)^{s/2} F = 0 = F \text{ in } \Omega \setminus \mathcal{O}$, we get $F = 0$ in $\Omega$ (c.f. Lemma 4.1). Hence the part (1) follows.

(2) The proof of part (2) follows exactly similar way like part (1). Let $G \in L^2(\Omega)$ and $(G, v)_{L^2(\Omega)} = 0$, for all $v \in Y$. Then we will show that $G = 0$ in $\Omega$ to prove our claim.

Let $w \in H^t(\mathbb{R}^n)$ solves $\mathcal{L}_{b,q} w = G$ in $\Omega$ and $w = 0$ in $\Omega_c$. Then we have 0 =

$$0 = \left( \mathcal{L}_{b,q} w, v \right)_{L^2(\Omega)} = \left( w, \mathcal{L}_{b,q} v \right)_{L^2(\Omega)} - \left( (-\Delta)^t w, v \right)_{L^2(\Omega_c)}.$$

Since $\mathcal{L}_{b,q} v = 0$ in $\Omega$, thus

$$\left( (-\Delta)^t w, f \right)_{\Omega_c} = \left( (-\Delta)^t w, v \right)_{\Omega_c} = 0 \text{ for all } f \in C^\infty_c(W).$$

Hence, $(-\Delta)^t w = 0$ in $W \subset \Omega_c$. Consequently, by SUCP we have $w \equiv 0$ and $G = 0$ in $\Omega$. Hence the proof follows.

5. Recovery of the lower order perturbations of $\mathcal{L}_{b,q}$

In this section we will apply our machinery been developed in the previous sections to recover the lower order perturbations. Let us recall the integral identity (3.2) from Section 3, which we have obtained by the using the single measurement $f$ satisfying the assumption mentioned in Theorem 1.4. We have

$$\int_{\Omega} \left( (b_1 - b_2) \left( (-\Delta)^{s/2} u \right) \left( (-\Delta)^{s/2} \varphi \right) + \int_{\Omega} (q_1 - q_2) u \varphi \right) = 0, \quad \forall \varphi \in C^\infty_c(\Omega).$$

Let $\mathcal{O} \subset \Omega$ contains the compact supports of $b_1, b_2, c_1, c_2$ in $\Omega$. Since

$$X_\mathcal{O} = \{ \varphi|_{\mathcal{O}} : \varphi \in H^{s/2}(\Omega) : (-\Delta)^{s/2} v = 0 \text{ in } \mathcal{O}, 0 < s < 1 \}$$
is dense in $L^2(\Omega)$ (c.f. Lemma 4.2) So from the $L^2$-density of $C^\infty_0(\Omega)$ in $H^{s/2}(\Omega) \cap X_\Omega$ for $0 < s < 1$, we can conclude

$$\int_{\Omega} (q_1 - q_2) u \varphi = 0, \quad \forall \varphi \in X_\Omega$$

and this implies

(5.2) $$(q_1 - q_2) u = 0 \quad \text{in } \Omega.$$  

Plugging this information in the integral identity (5.1) now we have

(5.3) $$\int_{\Omega} (b_1 - b_2) \left( (\Delta)^{s/2}_\Omega u \right) \left( (\Delta)^{s/2}_\Omega \varphi \right) = 0, \quad \forall \varphi \in H^{s/2}(\Omega).$$

Next we show $(b_1 - b_2) \left( (\Delta)^{s/2}_\Omega u \right) = 0$ in $\Omega$. Let us choose $\varphi \in H^{s/2}(\Omega)$ be a weak solution of

$$\left( (\Delta)^{s/2}_\Omega u \right) \varphi = (b_1 - b_2) \left( (\Delta)^{s/2}_\Omega u \right) \in L^2(\Omega), \quad \text{in } \Omega,$$

$$\varphi = 0, \quad \text{on } \partial\Omega.$$

Then plugging this in (5.3) we clearly obtain

(5.4) $$(b_1 - b_2) \left( (\Delta)^{s/2}_\Omega u \right) = 0 \in \Omega.$$  

Observe the relations obtained in (5.2) and (5.4), that is

(5.5) $$(b_1 - b_2) \left( (\Delta)^{s/2}_\Omega u \right) = 0 = (q_1 - q_2) u \quad \text{in } \Omega.$$  

If $(b_1 - b_2)$ and $(q_1 - q_2)$ are continuous, then

(5.6) $$B = \{ x \in \Omega : (b_1 - b_2)(x) \neq 0 \} \quad \text{and} \quad C = \{ x \in \Omega : (q_1 - q_2)(x) \neq 0 \}$$

are open subsets in $\Omega$. If $B, C$ are non-empty, then from (5.5), we get $(-\Delta)^{s/2}_\Omega u$ and $u$ are zero on the open sets $B$ and $C$ respectively.

From Lemma 4.1 it is evident that $B$ and $C$ are disjoint open sets, as they have any intersection it would lead to $u \equiv 0$ in $\Omega$.

We also observe that $B, C$ can’t be compliment of each other in $\Omega$, i.e. $B \cup C = \Omega$.

Since the exterior value problem

$$(-\Delta)^{s/2}_\Omega u = 0 \quad \text{in } B, \quad u = 0 \quad \text{in } C, \quad \text{and } B \cup C = \Omega$$

has only $u = 0$ solution in $\Omega$ (cf. Proposition 4.5). This completes the proof of Theorem 1.4. \hfill \Box

**Proof of Theorem 1.1.** We begin with the identity (5.5) what we have obtained for each single $f \in C^\infty_c(W)$ where $W \subset \Omega$, as

$$(b_1 - b_2)(-\Delta)^{s/2}_\Omega u_f = 0 = (q_1 - q_2) u_f, \quad \text{in } \Omega \quad \text{and } u_f = f \in C^\infty_c(W).$$

Now let us recall the density result concerning $(-\Delta)^{s/2}_\Omega u_f$ and $u_f$, proved in the Proposition 4.9, by varying $f \in C^\infty_c(W)$ one obtains $b_1 = b_2$ and $q_1 = q_2$ respectively in $\Omega$. \hfill \Box
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