Tight Sum-of-squares Lower Bounds for Binary Polynomial Optimization Problems

ADAM KURPISZ, Business School, Bern University of Applied Sciences and ETH Zürich, Department of Mathematics
SAMULI LEPPÄNEN, Axpo Solutions AG
MONALDO MASTROLILLI, SUPSI-IDSIA - Dalle Molle Institute for Artificial Intelligence

For binary polynomial optimization problems of degree $2d$ with $n$ variables Sakaue, Takeda, Kim, and Ito [33] proved that the $\left\lceil \frac{n+2d-1}{2} \right\rceil$th semidefinite (SDP) relaxation in the SoS/Lasserre hierarchy of SDP relaxations provides the exact optimal value. When $n$ is an odd number, we show that their analysis is tight, i.e., we prove that $\frac{n+2d-1}{2}$ levels of the SoS/Lasserre hierarchy are also necessary.

Laurent [24] showed that the Sherali-Adams hierarchy requires $n$ levels to detect the empty integer hull of a linear representation of a set with no integral points. She conjectured that the SoS/Lasserre rank for the same problem is $n - 1$. In this article, we disprove this conjecture and derive lower and upper bounds for the rank.

CCS Concepts: • Theory of computation → Discrete optimization; Design and analysis of algorithms; Computational complexity and cryptography;

Additional Key Words and Phrases: Sum-of-squares

ACM Reference format:
Adam Kurpisz, Samuli Leppänen, and Monaldo Mastrolilli. 2024. Tight Sum-of-squares Lower Bounds for Binary Polynomial Optimization Problems. ACM Trans. Comput. Theory 16, 1, Article 3 (March 2024), 16 pages. https://doi.org/10.1145/3626106

1 INTRODUCTION

We study binary polynomial optimization problems (BPOP). BPOP consists in minimizing an $n$-variate polynomial $f(x)$ over a subset of vertices of the Boolean hypercube. In an unconstrained
version the problem takes the following form:
\[
\min_{x \in \{0,1\}^n} f(x).
\]

The unconstrained BPOP captures many fundamental optimization problems. Prominent examples include the MAXCUT problem and the more general Boolean MAX k-csp, which can be modeled using a polynomial \( f(x) \) of degree 2 and \( k \), respectively. The generality of the BPOP comes at the cost of computational complexity. Even for the degree of \( f \) equal to 2 solving the unconstrained BPOP exactly is NP-hard. This motivates the search for methods to solve BPOP efficiently with a possibly small deviation from the optimal value. Candidate methods are the class of lift-and-project techniques, also known as hierarchies of relaxations.

The Sum-of-Squares (SoS)/Lasserre hierarchy of semidefinite programming (SDP) relaxations [23, 31] is one of the most-studied solution methods for general polynomial optimization problems, the class of problems that includes BPOP. The strength of the SoS method emerged for a wide variety of problems in combinatorial optimization [1, 12, 28], robust estimation [17], dictionary learning [4, 34], tensor completion and decomposition [5, 16, 32], and problems arising from statistical physics [10, 18]. The hierarchy is parameterized by a parameter \( t \) called the level. The larger the level \( t \), the tighter the relaxation. At level \( t \), the relaxation is solvable by semidefinite programming involving matrices of size \( n^{O(t)} \). For arbitrary constrained BPOP, the SoS hierarchy is guaranteed to find the exact optimal value at level at most \( n \).

Studying the performance of the SoS method for BPOP attracted a lot of attention. For the case of degree of \( f \) equal to 2, called a quadratic BPOP, Laurent [25] conjectured that at level \( \lceil \frac{n}{2} \rceil \) the relaxation provides the exact optimal value. In Reference [25] she also provided a matching lower bound showing that \( \lceil \frac{n}{2} \rceil \) levels are not enough for finding the integer cut polytope of the complete graph with \( n \) nodes when \( n \) is odd. A similar lower bound was proved before by Grigoriev [13] in the context of the Knapsack problem. The Laurent conjecture was resolved by Fawzi, Saunderson, and Parrilo [9]. They showed that \( \lceil \frac{n}{2} \rceil \) levels are enough to exactly solve any unconstrained quadratic BPOP. Subsequently, building upon the result in Reference [9], Sakaue, Takeda, Kim, and Ito [33] extended the result of [9] and showed that the SoS hierarchy requires at most \( \lceil (n + d − 1)/2 \rceil \) levels to find the exact optimal value of any unconstrained BPOP of degree \( d \) with \( n \) variables. Note that, for \( d = 2 \), the two upper bounds \([9, 33]\) coincide when \( n \) is odd. Moreover, [33] studied an interesting special case, when \( n \) is even and \( f \) consists of only even degree monomials, which is the case for e.g., the MAXCUT problem. In this case the bound in Reference [33] improves to \( \lceil (n + d − 2)/2 \rceil \), again matching the bound of Reference[9]. Finally, Sakaue et al. [33] numerically confirmed that for some degrees, their bound is tight for certain instances of unconstrained BPOP with 8 variables. Very recently, the performance of the SoS method for the BPOP problem was studied from the approximation perspective. In a beautiful result, Slot and Laurent [35] leveraged Fourier analysis of Boolean functions to prove that for the normalized polynomial \( f \) of degree \( d \), the worst-case difference between the optimal solution of BPOP and the solution returned by the level \( t \) SoS hierarchy is of the order \( C_d(1/2 − \sqrt{r(1−r)}) \), for a constant \( C_d \) depending only on \( d \), and \( r \) \( \in \) \([0,1/2]\) such that \( t \approx r \cdot n \). Note that for \( t/n \to 1/2 \) the difference tends to 0, thus it asymptotically recovers the result from Reference[9]. Moreover, in a breakthrough result, Lee, Raghavendra, and Steurer [26] built on the work of Grigoriev/Laurent [13, 25] to show that, for odd \( n \), any sum of squares of degree \( \lfloor n/2 \rfloor \) polynomials has \( \ell_1 \)-error at least \( 2^{n−2}/\sqrt{n} \) in approximating the following quadratic function:

\[
 f(x) = (\|x\|_1 − \lfloor n/2 \rfloor)(\|x\|_1 − \lfloor n/2 \rfloor − 1).
\]

As a result, they proved that for the class of MAX-CSPs the SoS relaxation yields the “optimal” SDP approximation, meaning that SDPs of polynomial-size are equivalent in power to those
arising from $O(1)$ rounds of the SoS relaxations. Finally, the performance of the SoS method for the family of quadratic polynomials generalizing the polynomial $f$ in Equation (1) was studied in Reference [22].

Our Results. In this article, we give two results concerning the power of the SoS hierarchy for BPOP. Our first result for unconstrained BPOP shows that the bound given by Sakaue et al. [33] is tight for polynomials with even degree and an odd number of variables. More precisely, we consider BPOPs of the form $\min_{x \in [0,1]^n} f_d(x)$ where $f_d(x)$ is a degree $2d$ (for $d \geq 1$) polynomial defined as follows:

$$f_d(x) = (||x||_1 - \lfloor n/2 \rfloor + d - 1)^{2d}$$

where $k^\ell = k(k - 1) \cdots (k - r + 1)$ denotes the falling factorial. For $d = 1$, we have $f_1(x) = f(x)$, where $f(x)$ is the polynomial defined in Equation (1) and considered in References [26, 27]. We show that for odd $n = 2m + 1$, while $f_d(x)$ is non-negative over $\{-1, 1\}^n$, the SoS relaxation allows negative values for $f_d(x)$, even at level $\frac{n + 2d - 1}{2} - 1 + m + d - 1$, which leads to the following theorem:

**Theorem 3.1** For odd $n$, the SoS relaxation of minimizing $f_d$ requires at least $\frac{n + 2d - 1}{2} + 1$ levels to find the exact optimum.

Our second result, for constrained BPOP, concerns comparing the SoS hierarchy to other lift-and-project methods. A commonly used benchmark for comparing hierarchies is to find the smallest level at which they find the convex hull of a given set of integral points $P$, called the rank of $P$. Benchmark problems are usually given as an intersection of the set $[0,1]^n$ and a polytope. Examples of such results include References [11, 13–15, 25, 29, 36]. One of the well-known benchmark problems is the Empty Integral Hull problem that consists in detecting that the following set is empty:

$$K = \{0, 1\}^n \cap \left\{ x \in [0,1]^n \mid \sum_{r \in R} x_r + \sum_{r \in N \setminus R} (1 - x_r) \geq \frac{1}{2} \text{ for all } R \subseteq N \right\}. \quad (3)$$

In Reference [24], Laurent shows that the Sherali-Adams hierarchy solves the Empty Integral Hull problem only after $n$ levels. She then conjectures the SoS rank of $K$ is $\rho(K) = n - 1$. The polytope $K$ has been used earlier to show that $n$ iterations are needed also for the following procedures: the Lovász-Schrijver $N_\circ$ operator (with positive semidefiniteness) [11], the Lovász-Schrijver $N_\circ$ operator combined with taking Chvátal cuts [6], and the $N_\circ$ operator combined with taking Gomory mixed integer cuts (equivalent to disjunctive cuts) [7]. In this article, we disprove Laurent’s conjecture and show that indeed the SoS rank of $K$ is bounded between $\Omega(\sqrt{n})$ and $n - \Omega(n^{1/3})$.

**Theorem 4.2** The SoS rank of the Empty Integral Hull problem is bounded by $\Omega(\sqrt{n}) \leq \rho(K) \leq n - \Omega(n^{1/3})$.

Interestingly, Au [2] and the authors of this article [20] independently considered the rank of a variation of the set $K$ where on the right-hand side of the inequalities there is an exponentially small constant instead of $\frac{1}{2}$. Both works show that the rank of the modified $K$ is exactly $n$. Finally, in a very recent paper [19] (see also a followup [22]) the result was further improved, showing that for the Empty Integral Hull parametrized by right-hand side constant of $1/B$ in Equation (3) the SoS rank is greater than or equal to the minimum $d$, which satisfies: $B^{-1} \geq \frac{e^{2n}}{\sum_{k=\infty} (\frac{1}{x})}$. For $B = 2$ this gives a lower bound of $\lfloor n/2 \rfloor$ and an upper bound of $\lfloor \frac{n}{2} + \sqrt{n \log 2n} \rfloor$.

In our proofs, we demonstrate the use of a recent theorem of the authors [21] that simplifies the positive semidefiniteness (PSD) condition of the SoS hierarchy when the problem formulation
is highly symmetric. A similar result was proved by [27], as noted in Reference [27]. Our first result is obtained by showing that a certain conical combination of solutions with non-integral relaxation value to the SoS relaxation for the function (1) gives a negative SoS relaxation value for the polynomials (2) of degree 2d. Then, for the first and the second result, we apply the theorem in Reference [21] to reduce the PSDness condition into showing that a particular inequality is satisfied for every polynomial with a certain form. Showing that the inequality is satisfied (lower bounds) or cannot be satisfied (upper bounds) then boils down to evaluating or approximating a certain combinatorial sum. Our results also answer the question in Reference [27] regarding the applications of the theorem of References [21, 27].

2 THE SUM-OF-SQUARES HIERARCHY

In this article, we consider the SoS hierarchy when applied to (i) unconstrained 0/1 polynomial optimization problems and (ii) approximating the convex hull of the set

\[ P = \{ x \in \{0,1\}^n \mid g_\ell(x) \geq 0, \forall \ell \in [p] \}, \]  

where \( g_\ell(x) \) are linear constraints and \( p \) a positive integer. The form of the SoS hierarchy we use in this article is equivalent to the one used in literature (see, e.g., References [3, 23, 24]) and follows from applying a change of basis to the dual certificate of the refutation of the proof system (see Reference [21] for the details on the change of basis and Reference [30] for discussion on the connection to the proof system). We use this change of basis to obtain a useful decomposition of the moment matrices as a sum of rank one matrices of special kind.

For any \( I \subseteq N = \{1, \ldots, n\} \), let \( x_I \) denote the 0/1 solution obtained by setting \( x_i = 1 \) for \( i \in I \), and \( x_i = 0 \) for \( i \in N \setminus I \). For a function \( f : \{0,1\}^n \to \mathbb{R} \), we denote by \( f(x_I) \) the value of the function evaluated at \( x_I \). In the SoS hierarchy defined below there is a variable \( y_I^N \) that can be interpreted as the “relaxed” indicator variable for the solution \( x_I \), which can also be seen as the pseudo-probability measure on \( \{0,1\}^n \) that allows negative probabilities. We point out that in this formulation of the hierarchy the number of variables \( \{ y_I^N : I \subseteq N \} \) is exponential in \( n \), but this is not a problem in our context, since we are interested in proving lower and upper bounds rather than solving an optimization problem.

Let \( P_t(N) \) be the collection of subsets of \( N \) of size at most \( t \in \mathbb{N} \). For every \( I \subseteq N \), the \( q \)-zeta vector \( Z_I \in \mathbb{R}^{P_q(N)} \) is a 0/1 vector with \( J \)-th entry (\( |J| \leq q \)) equal to 1 if and only if \( J \subseteq I \).\(^1\)

Note that \( Z_I Z_I^\top \) is a rank one matrix and the matrices considered in Definitions 2.1 and 2.2 are linear combinations of these rank one matrices.

To simplify the presentation, we define the SoS hierarchy separately for polynomial optimization problems and for the integer hull approximation.

**Definition 2.1.** The \( t \)-th round SoS hierarchy relaxation of \( \min_{x \in \{0,1\}^n} f(x) \), denoted by \( \text{SoS}_t(f) \), is the optimization problem with variables \( \{ y_I^N \in \mathbb{R} : \forall I \subseteq N \} \) of the form

\[
\min_{y_I^N \in \mathbb{R}^n} \sum_{I \subseteq N} y_I^N f(x_I) \tag{5}
\]

s.t.

\[
\sum_{I \subseteq N} y_I^N = 1, \tag{6}
\]

\[
\sum_{I \subseteq N} y_I^N Z_I Z_I^\top \geq 0, \text{ where } Z_I \in \mathbb{R}^{P_t(N)}. \tag{7}
\]

\(^1\)To keep the notation simple, we do not emphasize the parameter \( q \), as the dimension of the vectors should be clear from the context.
Definition 2.2. The th round SoS hierarchy relaxation for the set \( P \) as given in Equation (4), denoted by \( \text{SoS}_t(P) \), is the set of variables \( \{y_I^N \in \mathbb{R} : \forall I \subseteq N \} \) that satisfy

\[
\sum_{I \subseteq N} y_I^N = 1, \tag{8}
\]

\[
\sum_{I \subseteq N} y_I^N Z_I Z_I^\top \geq 0, \text{ where } Z_I \in \mathbb{R}^{P_{\ell+1}(N)} \tag{9}
\]

\[
\sum_{I \subseteq N} g_{\ell}(x_I) y_I^N Z_I Z_I^\top \geq 0, \forall \ell \in [p], \text{ where } Z_I \in \mathbb{R}^{P_{\ell}(N)}. \tag{10}
\]

It is straightforward to see that the SoS hierarchy formulation given in Definition 2.2 is a relaxation of the integral polytope. Indeed, consider any feasible integral solution \( x_I \in P \) and set \( y_I^N = 1 \) and the other variables to zero. This solution clearly satisfies (8) and (9), because the rank one matrix \( Z_I Z_I^\top \) is \textit{positive semidefinite (PSD)}, and condition (10), since \( x_I \in P \).

For a set \( Q \subseteq [0, 1]^n \), we define the projection from \( \text{SoS}_t(Q) \) to \( \mathbb{R}^n \) as \( x_I = \sum_{i \in I \subseteq N} y_I^N \) for each \( i \in \{1, \ldots, n\} \). The \textit{SoS rank} of \( Q, \rho(Q) \), is the smallest \( t \) such that \( \text{SoS}_t(Q) \) projects exactly to the convex hull of \( Q \cap \{0, 1\}^n \).

### 2.1 Using Symmetry to Simplify the PSDness Conditions

In this section, we present a theorem given in Reference [21] that can be used to simplify the PSDness conditions (7), (9), and (10) when the problem formulation is very symmetric. More precisely, the theorem can be applied whenever the solutions and constraints are symmetric in the sense that \( y_I^N = w_J^N \) whenever \( |I| = |J| \) where \( w_I^N \) is understood to denote either \( y_I^N \) or \( g_{\ell}(x_I) y_I^N \). In what follows, we denote by \( \mathbb{R}[x] \) the ring of polynomials with real coefficients and by \( \mathbb{R}[x]_d \) the polynomials in \( \mathbb{R}[x] \) with degree less or equal to \( d \).

**Theorem 2.1 ([21]).** For any \( t \in \{1, \ldots, n\} \), let \( S_t \) be the set of univariate polynomials \( G_h(k) \in \mathbb{R}[k], \text{ for } h \in \{0, \ldots, t\}, \) that satisfy the following conditions:

\[
G_h(k) \in \mathbb{R}[k]_{2t},
\]

\[
G_h(k) = 0 \quad \text{for } k \in \{0, \ldots, h-1\} \cup \{n-h+1, \ldots, n\}, \text{ when } h \geq 1,
\]

\[
G_h(k) \geq 0 \quad \text{for } k \in \{h-1, n-h+1\}.
\]

For any fixed set of values \( \{w_k^N \in \mathbb{R} : k = 0, \ldots, n\} \), if the following holds:

\[
\sum_{k=h}^{n-h} \binom{n}{k} w_k^N G_h(k) \geq 0 \quad \forall G_h(k) \in S_t \tag{14}
\]

then

\[
\sum_{k=0}^{n} w_k^N \sum_{I \subseteq N, \ |I|=k} Z_I Z_I^\top \geq 0,
\]

where \( Z_I \in \mathbb{R}^{P_{\ell}(N)} \).

Note that the polynomial \( G_h(k) \) in condition (13) is nonnegative in a \textit{real interval}, and in condition (12) it is zero over a set of integers. Moreover, the constraints (14) are trivially satisfied for \( h > \lfloor n/2 \rfloor \).
3 TIGHTNESS OF THE SOS UPPER BOUNDS FOR UNCONSTRAINED BPOPS

In Reference [33] it is shown that the SoS hierarchy exactly solves any unconstrained BPOP of degree r with n variables after ⌈n−r−1/2⌉ levels. We show that this bound is tight for certain values of n and r, by giving a polynomial of degree r = 2d for d ≥ 1 that is non-negative over the hypercube, and show that when n = 2m + 1, m ≥ d, the SoS relaxation of the corresponding BPOP attains a negative value at level t = n+2d−1/2 − 1 = m + d − 1.

More precisely, we consider the degree 2d polynomial
\[ f_d(x) = (||x||_1 + d - m - 1)^{2d}, \]
where k! = k(k − 1) ⋅ ⋅ ⋅ (k − r + 1) denotes the falling factorial and ||x||_1 = \sum_i x_i. For the sake of convenience, we denote by f_d(k) the univariate polynomial evaluated at any point x with \sum_i x_i = k. We obtain the following result:

**Theorem 3.1.** For odd n, the SoS relaxation of minimizing f_d requires at least \( \frac{n+2d-1}{2} \) levels to find the exact optimum.

3.1 Proof of Theorem 3.1

The case d = 1. The polynomial f_1(x) is connected to the MAXCUT problem in the complete graph of n = 2m + 1 vertices in the following way: Let x ∈ \{0, 1\}^n denote any partition of the vertices into two sets in the natural way. Then, the maximal cut is achieved whenever \( \sum_i x_i \) is either m or m + 1, and m(m + 1) − f_1(x) counts the edges in the cut. Therefore, the SoS hierarchy is not able to exactly solve the MAXCUT problem if it allows for solutions with negative values of the objective function (5).

It is shown in Reference [21] that
\[ y_I^N[\alpha] = (n + 1)^{2d} \frac{\alpha}{n + 1} \frac{(-1)^{n-|I|}}{\alpha - |I|} \quad \forall I \subseteq N \]
is a feasible solution to the SoS hierarchy (as given in Definition 2.1) at level \( [\alpha] \) for any non-integer 0 < \( \alpha < \frac{n}{2} \). Since the value of the solution only depends on the size of the set I, we denote by \( y_k^N[\alpha] \) any \( y_I^N[\alpha] \) with |I| = k. As a consequence of the proof in Reference [21] it follows that for any non-integer 0 < \( \alpha \leq n \), \( \sum_{I=0}^n (n)_k y_k^N[\alpha] = 1 \). Furthermore, it is shown that the objective function attains the value \( \sum_{k=0}^n (n)_k y_k^N[\alpha] f_1(k) = f_1(\alpha) \) and that in particular for \( \alpha = \frac{n}{2} \), \( f_1(\alpha) = -\frac{1}{4} \) at level \( t = [\alpha] = m \). Next, we generalize this approach to \( f_d(x) \).

**Polynomials of degree 2d.** Consider the following assignment:
\[ z_k^N = (2d - 2)! (n + 1) \frac{\alpha}{n + 1} \frac{(-1)^{n-k}}{(\alpha + d - 1 - k)^{2d-1}} \quad \forall k \in \{0, \ldots, n\}. \]

We show that for this assignment, the SoS hierarchy objective (5) attains a negative value (see Lemma 3.7) and (7) is satisfied. For convenience, we do not actually show that (6) is satisfied and in fact it is not. We show, however, that \sum_k (n)_k z_k^N > 0, which implies that with proper normalization also (6) can be satisfied (see Lemma 3.4).

---

\[ \text{(2) The same solution was earlier considered in different basis by References [14, 25] for the Knapsack and MaxCut problems, respectively, to show that the SoS hierarchy does not exactly solve the aforementioned problems at level } [\frac{n}{2}] \text{. In the basis considered in References [14, 25] the so-called pseudo-expectation for a set } I \in [n] \text{ is expressed as } \hat{E}[\prod_{i \in I} x_i] = (\frac{a}{|I|})^{|I|}, \text{ which has a nice combinatorial interpretation.} \]

\[ \text{(3) For } d > 1, \text{ the assignment, in contrast to the case } d = 1, \text{ does not have an easy and intuitive pseudo-expectation form.} \]
First, we prove that the assignment \( z_k^N \) can be written as a conical combination of the solutions \( y_k^N[\cdot] \) in (16). We begin with the following lemma about partial fraction decompositions:

**Lemma 3.2.** For any \( b \in \mathbb{N}_+ \) and \( a \in \mathbb{R} \) the following identity holds:

\[
\frac{1}{(x-a)^b} = \frac{(-1)^{b-1-i}}{i!(b-1-i)!} \frac{1}{(x-a-i)}.
\]

**Proof.** It is known that given two polynomials \( P(x) \) and \( Q(x) = (x-a_1)(x-a_2) \cdots (x-a_n) \), where the \( a_i \) are distinct constants and \( \deg P < n \), the rational polynomial \( \frac{P(x)}{Q(x)} \) can be decomposed into (see Reference [37])

\[
\frac{P(x)}{Q(x)} = \sum_{i=1}^n \frac{P(a_i)}{Q'(a_i)} \frac{1}{(x-a_i)},
\]

where \( Q'(x) \) is the derivative of \( Q(x) \). In our case, since \( P(x) = 1 \) and \( Q(x) = \prod_{i=0}^{b-1} (x-a) \), we get

\[
\frac{1}{(x-a)^b} = \sum_{i=0}^{b-1} \frac{1}{\prod_{j \neq i} (a+i-(a+j))} \frac{1}{(x-a-i)} = \sum_{i=0}^{b-1} \frac{(-1)^{b-1-i}}{i!(b-1-i)!} \frac{1}{(x-a-i)}. \quad \square
\]

Now, we can express the assignment \( z_k^N \) as a conical combination of the solutions \( y_k^N[\cdot] \).

**Lemma 3.3.** The assignment (17) can be decomposed as a conical combination of \( y_k^N[\cdot] \):

\[
z_k^N = \sum_{j=0}^{2d-2} a_j y_k^N [n/2 + d - 1 - j] \quad \forall k \in \{0, \ldots, n\}
\]

for positive

\[
a_j = \left( \frac{2d-2}{2} \right) \frac{\left( \frac{n}{2} + d - 1 \right)_{n-j}}{\left( \frac{n}{2} - d + 1 + j \right)_{n-j}}.
\]

**Proof.** By Lemma 3.2, we get that

\[
\frac{1}{(\frac{n}{2} + d - 1 - k)_{2d-1}} = \sum_{j=0}^{2d-2} \frac{(-1)^{2d-2-j}}{j!(2d-2-j)!} \frac{1}{(\frac{n}{2} + d - 1 - k - j)}
\]

and by writing

\[
\left( \frac{n}{2} - d + 1 \right)_{n+1} = \left( \frac{n}{2} - d + 2 - j \right)_{2d-2-j} \left( \frac{n}{2} - d + 1 - j \right)_{2d-2-j}
\]

and using raising factorial notation, \( (-b)^m = (-1)^m b^m \), we get that

\[
z_k^N = \sum_{j=0}^{2d-2} \frac{(2d-2)!}{j!(2d-2-j)!} \frac{(-\frac{n}{2} + d - 2 - j)_{2d-2-j}}{(\frac{n}{2} + d - 1 - j)_{2d-2-j}} \left( \frac{n}{2} + d - 1 - j \right)_{n+1} \frac{1}{(\frac{n}{2} + d - 1 - k - j)}
\]

\[
= \sum_{j=0}^{2d-2} \frac{(2d-2)!}{j!} \frac{(\frac{n}{2} + d - 2 + j)_{2d-2-j}}{(\frac{n}{2} + d - 1 - j)_{2d-2-j}} y_k^{N}[n/2 + d - 1 - j] \frac{1}{(\frac{n}{2} + d - 1 - k - j)}
\]

\[
= \sum_{j=0}^{2d-2} \frac{(2d-2)!}{j!} \frac{(\frac{n}{2} + d - 1)_{n-j}}{(\frac{n}{2} - d + 1 + j)_j} y_k^{N}[n/2 + d - 1 - j] \quad \square
\]

**Lemma 3.4.** We have \( \sum_{k=0}^{n} (\binom{n}{k}) z_k^N > 0 \) for every odd \( n, n = 2m + 1, \) and \( d \in [m] \).
Proof. The proof follows by recalling that for every \( \alpha \in [0, n] \setminus \mathbb{Z} \), \( \sum_{i=0}^{n} \binom{n}{k} g_{k}[\alpha] = 1 \) and by the fact that all the coefficients in the decomposition in Lemma 3.3 are positive. \( \square \)

Now, we show that the assignment (17) is a feasible solution for the SoS hierarchy at level \( t = m + d - 1 \). The assignment (17) is symmetric, and so by Theorem 2.1 (see (14)) is enough to prove that for \( t = m + d - 1 \),

\[
\sum_{k=0}^{n} \binom{n}{k} z_{k}^{N} G_{h}(k) \geq 0 \quad \forall G_{h}(k) \in S_{t}.
\]  

(18)

We first note that the assignment (17) attains positive values for every integer \( k \in \{m - d + 1, \ldots, m + d\} \). Indeed, let us start considering the term \( \binom{\frac{n}{2} - d + 1}{n+1} \frac{m-d+2}{2} \frac{m+d}{2} \frac{(-1)^{m+d}}{(-1)^{p}} \). (19)

Similarly, for \( k = m - d + 1 + p \) with \( p \in \{0, \ldots, 2d - 1\} \), the following identity holds:

\[
\binom{n}{2} \frac{2d-1}{d-1-k} = \binom{2d-3}{2} \frac{2d-1-p}{2} \frac{1}{2} \frac{(-1)^{p}}. \]  

(20)

By using (19) and (20) in (17), we have (for \( k = m - d + 1 + p \)):

\[
z_{k}^{N} = (2d-2)! (n+1) \frac{\binom{n}{2} \frac{m-d+2}{2} \frac{m+d}{2} \frac{(-1)^{m+d}}{(-1)^{p}}}{(n+1)! (2d-3-p) \frac{2d-1-p}{2} \frac{1}{2} \frac{(-1)^{p}}}. \]  

(21)

Note that

\[
\frac{(-1)^{m+d}}{(-1)^{p}} = (-1)^{2(m+d-p)} = 1.
\]

It follows that (17) is always positive. Thus, (18) is always satisfied whenever \( h \geq m - d + 1 \) by the definition of the polynomials \( G_{h} \in S_{t} \).

It follows that it is enough to prove that (18) is satisfied for \( h \leq m - d \), which is implied if the following is true:

\[
\sum_{k=0}^{n} \binom{n}{k} z_{k}^{N} P(k) \geq 0
\]

for every polynomial \( P(x) \in \mathbb{R}[x]_{2d} \) that is nonnegative in the interval \([m - d + 1, m + d]\).

Lemma 3.5. For any polynomial \( P(x) \in \mathbb{R}[x]_{2d(m+d-1)} \), we have

\[
\sum_{k=0}^{n} \binom{n}{k} z_{k}^{N} P(k) = \sum_{j=0}^{2d-2} a_{j} P \left( \frac{n}{2} + d - 1 - j \right).
\]

Proof. Let \( g(k) = \binom{\frac{n}{2} + d - 1 - k}{2d-1} \) be the polynomial of degree \( 2d - 1 \) that corresponds to the denominator in \( z_{k}^{N} \) (see (17)). By the polynomial remainder theorem, there are unique polynomials \( Q(k) \) and \( R(k) \) such that \( P(k) = g(k)Q(k) + R(k) \), where \( \deg(Q(x)) \leq \deg(P) - \deg(g) \leq n - 2 \) and \( \deg(R(k)) < 2d - 1 \), and for the remainder it holds that \( R(r) = P(r) \) for all the roots \( r \) of polynomial \( g(k) \). Then

\[
\sum_{k=0}^{n} \binom{n}{k} z_{k}^{N} P(k) = \sum_{k=0}^{n} \binom{n}{k} z_{k}^{N} g(k)Q(k) + \sum_{k=0}^{n} \binom{n}{k} z_{k}^{N} R(k).
\]
Here, \( \sum_{k=0}^{n} \binom{n}{k} z_k^N g(k) Q(k) = 0 \), as \( \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^c = 0 \) for every \( c \leq n - 1 \). The latter holds for the following reasons: By the binomial theorem, we have that

\[
(x - 1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k (-1)^{n-k}.
\]

With respect to \( x \), differentiate \( c \) times, with \( c \leq n - 1 \):

\[
n^c (x - 1)^{n-c} = \sum_{k=0}^{n} \binom{n}{k} k^c x^k (-1)^{n-k}.
\]

Set \( x = 1 \) and obtain

\[
\sum_{k=0}^{n} \binom{n}{k} k^c (-1)^{n-k} = 0.
\]

Note that \((-1)^{n-k} = (-1)^n (-1)^{-k} = (-1)^n (-1)^k\) and the claim follows.

We remark here that if the level \( t \) is greater than \( m + d - 1 \), then the polynomial \( Q \) can be of degree \( n \) or more and this reasoning fails.

By Lemma 3.3, we can write the sum with the remainder polynomial \( R(k) \) as

\[
\sum_{j=0}^{2d-2} a_j \sum_{k=0}^{n} \binom{n}{k} y_k^N \left[ \frac{n}{2} + d - 1 - j \right] R(k)
\]

and, again by the polynomial reminder theorem applied to polynomial \( R(k) \), for every \( j \in \{0, \ldots, 2d-2\} \),

\[
R(k) = \left( \frac{n}{2} + d - 1 - j \right) S_j(k) + R\left( \frac{n}{2} + d - 1 - j \right),
\]

for some polynomial \( S_j(k) \), and as before, since the degree of \( R \) is less or equal to \( 2d - 2 \), we have \( \sum_{j=0}^{2d-2} (-1)^k \binom{n}{k} S_j(k) = 0 \). Thus, since \( R(r) = P(r) \) for all the roots \( r = \frac{n}{2} + d - 1 - j \), for \( j \in \{0, \ldots, 2d-2\} \), of the polynomial \( g(k) \), the above reduces to

\[
\sum_{j=0}^{2d-2} a_j \left( \frac{n}{2} + d - 1 - j \right) \sum_{k=0}^{n} \binom{n}{k} y_k^N \left[ \frac{n}{2} + d - 1 - j \right] = 1
\]

\[
= \sum_{j=0}^{2d-2} a_j P \left( \frac{n}{2} + d - 1 - j \right).
\]

By Lemma 3.5, we immediately obtain the following corollary:

**Corollary 3.6.** For any polynomial \( P(x) \in \mathbb{R}[x]_{2(m+d-1)} \) such that \( P(x) \geq 0 \) for \( x \in [m - d + 1, m + d] \), we have

\[
\sum_{k=0}^{n} \binom{n}{k} z_k^N P(k) \geq 0.
\]

**Proof.** By Lemma 3.5, we have that

\[
\sum_{k=0}^{n} \binom{n}{k} z_k^N P(k) = \sum_{j=0}^{2d-2} a_j P \left( \frac{n}{2} + d - 1 - j \right),
\]

which is positive, since it is a conical combination of points at which polynomial \( P \) is positive. \( \square \)
It remains to show that the objective value of the SoS hierarchy (5) attains a negative value.

**Lemma 3.7.** The sum $\sum_{k=0}^{n} \binom{n}{k} z_k^N f_d(k)$ is negative for every odd $n = 2m + 1$, for any positive integer $m$ and $d \in \{1, \ldots, m\}$.

**Proof.** By Lemma 3.5, the assignment $z_k^N$ is such that

$$
\sum_{k=0}^{n} \binom{n}{k} z_k^N f_d(k) = \sum_{j=0}^{2d-2} a_j f_d \left( \frac{n}{2} + d - 1 - j \right).
$$

Then, the claim is proved by showing that the following function $g(d, n)$ is negative for every odd $n = 2m + 1$, for any positive integer $m$ and $d \in [m]$. Formally, that

$$
g(d, n) = \left(\frac{2d-2}{2}\right) \left(\frac{n}{2} + d - 1\right)^{j} \left(\frac{n}{2} - d + 1 + j\right)^{j} (2d - 3/2 - j)^{2d} < 0.
$$

More precisely, we show that the following identity holds (where $!!$ denotes the double factorial):

$$
g(d, n) = (2d - 3/2)^{2d} \cdot \frac{4^{d-1}(2d - 2)!!(2d - 1)!!}{(d - 1)!(4d - 3)!!} \cdot \frac{(2m - 2d + 3)!!(m - 1)!}{(m - d)!(2m + 1)!!}.
$$

By simple inspection it is easy to see that (22) is negative and the claim follows.

We start by rewriting $g(d, n)$ by using the following (easy to check) identities:

$$
\left(\frac{2d-2}{2}\right) \left(\frac{n}{2} + d - 1\right)^{j} = \frac{(2 - 2d)^{j} \left(1 - d - \frac{n}{2}\right)^{j}}{j!}
$$

$$
\left(\frac{n}{2} - d + 1 + j\right)^{j} = \frac{(2 - d + \frac{n}{2})^{j}}{j!}
$$

$$
(2d - 3/2 - j)^{2d} = \frac{(3/2)^{j}}{(3/2 - 2d)^{j}}
$$

By the above identities, we have that

$$
g(d, n) = \sum_{j=0}^{2d-2} \left(\frac{2d-2}{2}\right) \left(\frac{n}{2} + d - 1\right)^{j} \left(\frac{n}{2} - d + 1 + j\right)^{j} (2d - 3/2 - j)^{2d}
$$

$$
= (2d - 3/2)^{2d} \sum_{j=0}^{2d-2} \frac{(2 - 2d)^{j} \left(1 - d - \frac{n}{2}\right)^{j}}{(2 - d + \frac{n}{2})^{j} (3/2 - 2d)^{j}} \cdot \frac{1}{j!}
$$

$$
= (2d - 3/2)^{2d} \sum_{j=0}^{\infty} \frac{(2 - 2d)^{j} \left(1 - d - \frac{n}{2}\right)^{j}}{(2 - d + \frac{n}{2})^{j} (3/2 - 2d)^{j}} \cdot \frac{1}{j!}
$$

$$
= (2d - 3/2)^{2d} \cdot 3F_2 \left[ \begin{array}{c} a \ b \ c \\ 1 + a - b \ 1 + a - c \end{array} \left| 1 \right] \right],
$$

where $3F_2[a, b, c; 1/a - b, 1/a - c; 1] = \sum_{j=0}^{\infty} \frac{(a)^{j}(b)^{j}(c)^{j}}{(1+a-b)^{j}(1+a-c)^{j}} \cdot \frac{1}{j!}$ is the generalized hypergeometric series with $a = 2 - 2d$, $b = 1 - d - n/2$ and $c = 3/2$. 

ACM Transactions on Computation Theory, Vol. 16, No. 1, Article 3. Publication date: March 2024.
Note that \(1 + a/2 - b - c = \frac{n-1}{2} > 0\) and by using Dixon’s identity [8] for the generalized hypergeometric series \(3F_2[1+a-b,1+a-c;1]\) (when \(\Re(1 + a/2 - b - c) > 0\)), we have
\[
3F_2\left[\begin{array}{cc} a & b \\ 1+a-b & 1+a-c \end{array};1\right] = \frac{\Gamma(1 + \frac{1}{2}a)\Gamma(1 + a - b)\Gamma(1 + a - c)\Gamma(1 + \frac{1}{2}a - b - c)}{\Gamma(1 + a)\Gamma(1 + \frac{1}{2}a - b)\Gamma(1 + \frac{1}{2}a - c)\Gamma(1 + a - b - c)} = \frac{\Gamma(2-d)\Gamma(5/2-d+n)\Gamma(3/2-2d)\Gamma(n)}{\Gamma(3-2d)\Gamma(3/2-m)\Gamma(1/2-d)\Gamma(1-d+m)}.
\]

Note that \(\Gamma(2-d) = (1-d)\Gamma(1-d)\) and \((3-2d) = 2(1-d)(1-2d)\Gamma(1-2d)\). By using the Euler’s reflection formula, we have that \(\frac{\sin(\pi d)}{\pi} = \frac{1}{\Gamma(1-d)\Gamma(d)}\) and \(\frac{\sin(\pi d)}{\pi} = \frac{1}{\Gamma(1-2d)\Gamma(2d)}\), and by the integrality of \(d\), we have that
\[
\frac{\Gamma(1-d)}{\Gamma(1-2d)} = \frac{\sin(\pi 2d)(2d-1)!}{\sin(\pi d)(d-1)!} = \frac{2\cos(\pi d)(2d-1)!}{(d-1)!} = \frac{2(-1)^d(2d-1)!}{(d-1)!}.
\]
Therefore,
\[
\frac{\Gamma(2-d)}{\Gamma(3-2d)} = \frac{(-1)^{d+1}(2d-2)!}{(d-1)!}.
\]
Recall that for nonnegative integer values of \(x\), we have \(\Gamma(\frac{1}{2} - x) = \frac{(-2)^x}{(2n-1)!} \sqrt{2\pi}\) and \(\Gamma(\frac{1}{2} + x) = \frac{(2x-1)!!}{2^{x-1}} \sqrt{2\pi}\), and the following holds:
\[
3F_2\left[\begin{array}{cc} a & b \\ 1+a-b & 1+a-c \end{array};1\right] = (-1)^{d+1}(2d-2)! \frac{(m-1)!}{(d-1)!} \frac{\Gamma(\frac{5}{2} - d + m)\Gamma(\frac{3}{2} - 2d)}{\Gamma(\frac{3}{2} + m)\Gamma(\frac{3}{2} - d)} = \frac{4^{d-1}(2d-2)!(2d-1)! (2m-2d+3)!(m-1)!}{(d-1)!(4d-3)! (m-d)!(2m+1)!}.
\]
By simple inspection, we see that \(3F_2[1+a-b,1+a-c;1]\) is always positive and \(g(d, n)\) (see (23)) is negative as claimed. \(\Box\)

4 RANK BOUNDS FOR DETECTING A PARTICULAR EMPTY INTEGRAL HULL

In Reference [24] Laurent considers the representation of the empty set as (3) and shows that the Sherali-Adams procedure requires \(n\) levels to detect that \(K = \emptyset\). She conjectures that the SoS rank of \(K = n - 1\). In this section, we disprove this conjecture and derive a lower and upper bound for the SoS rank of \(K\).

We start with the following observation that will be used to prove the main claim:

**Lemma 4.1.** For any \(t \geq 1\), if \(\text{SoS}_t(K) \neq \emptyset\), then the solution \(y_t^N = \frac{1}{2\pi}\) for each \(I \subseteq N\) is a feasible solution for \(\text{SoS}_t(K)\).

**Proof.** Let \(d\) be either \(t\) or \(t + 1\). First, we observe that there is a one-to-one correspondence between vectors \(v \in \mathbb{R}^{d\ell(N)}\) and multilinear polynomials \(p(x)\) of degree \(d\) over \(\{0, 1\}^n\). Using the basis where \(w_I\) is the coefficient of the monomial \(\prod_{i \in I} x_i\) in \(p(x)\), it can be easily seen that \(v^T z_I = p(x_I)\), since \(x_I^2 = x_i\) on \(\{0, 1\}^n\).

Assume that one of the matrices associated with the SoS hierarchy is PSD (where \(z_I^N\) is either \(y_I^N\) or \(g_e(x_I)y_I^N\)), in other words
\[
v^T \left( \sum_{I \subseteq N} z_I^N z_I^T z_I^T \right) v = \sum_{I \subseteq N} z_I^N (z_I^T v)^2 = \sum_{I \subseteq N} z_I^N p^2(x_I) \geq 0
\]
for every vector \(v \in \mathbb{R}^d\) and thus for every multilinear polynomial \(p\) of degree \(d\). Then, consider the polynomial \(q_S(x)\) “rotated” by the set \(S \subseteq N\) defined as \(q_S(x_I) = p(x_I, S)\) for every point \(x_I\).
In words, the polynomial $q_S$ evaluates the polynomial $p$ such that it replaces $x_i$ by $1 - x_i$ if $i \in S$. Thus, $q_S$ and $p$ have the same degree. Then,
\[
\sum_{i \le N} z_i^N q^2(x_i) = \sum_{i \le N} z_i^N p^2(x_i) = \sum_{i \le N} z_i^N (Z_{i \Delta S}^T v)^2,
\]
showing that the matrix $\sum_{i \le N} z_i^N Z_{i \Delta S} Z_{i \Delta S}^T$ is PSD if the matrix in (24) is PSD.

For the constraints of the polytope $K$ as in (3), we have $g_R(x_i) = |N \setminus (R \cap I)| - \frac{1}{2}$. By this observation for any $S \subseteq N$ it holds that $g_{R \Delta S}(x_{i \Delta S}) = g_R(x_i)$.

Now, assume that there exists a solution $y^N$ to $\text{SoS}_I(K)$. Then, for any $S \subseteq N$ also the solution $u^N$ such that $u^N_{i} = y^N_{i \Delta S}$ must be feasible. This is because for the solution $u^N$, we can write the PSDness condition of the constraint $R$ as
\[
\sum_{i \le N} u^N_i g_R(x_i) Z_i Z_i^T = \sum_{i \le N} y^N_i g_{R \Delta S}(x_i) Z_{i \Delta S} Z_{i \Delta S}^T
\]
which must be PSD by assumption and the above discussion. Averaging over all $S$ yields a symmetric solution to $\text{SoS}_I(K)$. \hfill \Box

The following result derives a lower and upper bound for the SoS rank of $K$:

**Theorem 4.2.** The SoS rank of the Empty Integral Hull problem is bounded by $\Omega(\sqrt{n}) \leq \rho(K) \leq n - \Omega(n^{\frac{1}{3}})$.

**Proof.**

The upper bound. By Lemma 4.1, the solution $y^N_{I} = \frac{1}{2^n}$ for each $I \subseteq N$ is feasible to $\text{SoS}_I(K)$ unless $\text{SoS}_I(K) = \emptyset$. Let us assume that such a solution is feasible and consider the constraint of $K$ corresponding to $R = N$. Then, $g_R(x_i)$ is negative only when $I = \emptyset$.

To analyze the PSDness, we apply Theorem 2.1. Notice that in this case to satisfy (14) it is enough to satisfy the inequality for the polynomial $G_0(k)$. Indeed, note that for each inequality in (14), the only potentially negative term is the constraint $g_R$ for $R = N$, since $y_I = 1/2^n \geq 0$ for all $I \subseteq N$ and $G_k(k)$ is nonnegative in the interval $[h - 1, n - h - 1]$. However, $g_N(k)$ is nonnegative for $k \in [1, n]$ and thus all the terms in all summands for $h \in \{1, \ldots, n\}$ are nonnegative and the inequalities in (14) holds trivially for all $h \neq 0$.

Therefore, the PSDness condition (10), by Theorem 2.1 for $h = 0$, reduces to (14) of the form
\[
\sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^n} \left(k - \frac{1}{2}\right) p^2(k) \geq 0
\]
for every polynomial $P$ of degree $t$. Importantly, what is not mentioned in the statement of Theorem 2.1, in this case the PSDness condition actually becomes an if and only if condition (see Theorem 7 in Reference [21]). Therefore, showing that (25) is not satisfied implies that the PSDness condition (10) is not satisfied.

We now fix the polynomial as $P(k) = \prod_{i=1}^{t} (n - k - i + 1)$, i.e., such that $P$ has the roots at $n, n - 1, \ldots, n - t + 1$ and argue that such a polynomial can never satisfy (25) when $t$ is large. Indeed, rewriting the condition using this polynomial, removing the redundant factor $\frac{1}{2^n}$ and moving the negative term to the right-hand side, we have the necessary requirement for the positive semidefiniteness that
\[
\sum_{k=1}^{n-t} \binom{n-t}{k} \left(k - \frac{1}{2}\right) \prod_{i=1}^{t} (n - k - i + 1)^2 \geq \frac{1}{2} \prod_{i=1}^{t} (n - i + 1)^2.
\]
Notice that now the sum goes up to \( n - t \) only, since all the terms \( k > n - t \) are 0 by our choice of the polynomial. By dividing both sides by the positive term \( \prod_{i=1}^{t} (n - i + 1)^2 \) and observing that 
\[
\frac{\prod_{i=1}^{t} (n - k - i + 1)}{\prod_{i=1}^{t} (n - i + 1)} = \frac{(n-t)^k}{n^k},
\]
the condition further simplifies to
\[
\sum_{k=1}^{n-t} \binom{n}{k} \left( k - \frac{1}{2} \right) \left( \frac{(n-t)^k}{n^k} \right)^2 \geq \frac{1}{2}.
\]  
(26)

Next, we upper bound the sum on the left-hand side of (26) by considering a generic element for any \( 1 \leq k \leq n - t \). Any element can be bounded by
\[
\binom{n}{k} \left( k - \frac{1}{2} \right) \left( \frac{(n-t)^k}{n^k} \right)^2 \leq \frac{n^k e^k}{k^k} \frac{(n-t)^{2k}}{(n-k)^{2k}} \leq e^k \frac{k^{k-1}}{(n-k)^2} \frac{n(n-t)^2}{t^2}.
\]

Next, we use \( \frac{e^k}{k^{k-1}} < 3 \) for any \( k \) and \( \frac{1}{n-k} \leq \frac{1}{t} \) for \( k \leq n - t \). Then, for \( t \geq n - o(\sqrt{n}) \), it holds \( \frac{n(n-t)^2}{t^2} < 1 \), so we can approximate
\[
e^k \frac{k^{k-1}}{(n-k)^2} \frac{n(n-t)^2}{t^2} \leq 3 \frac{n(n-t)^2}{t^2}.
\]

Now, the sum on the left-hand side of (26) is upper bounded by \( 3(n-t) \frac{n(n-t)^2}{t^2} \) and thus the solution is never feasible to SoS\(_t\) (K) if
\[
3 \frac{n(n-t)^2}{t^2} < \frac{1}{2}.
\]

Setting \( t = n - Cn^{\frac{3}{2}} \) satisfies the inequality asymptotically for an appropriate constant \( C \).

The lower bound. We show that the symmetric solution \( y^N_1 = \frac{1}{2^n} \) is feasible for SoS\(_t\) (K) when \( t \) is \( \Omega(\sqrt{n}) \). Again, by symmetry, it is enough to show that the moment matrix of one constraint is PSD, and again we consider the constraint corresponding to \( R = N \). Therefore, we need to show that (25) is satisfied for any choice of the polynomial \( P \) with degree less or equal to \( t \). Writing the polynomial \( P \) in root form with roots \( r_i, i = 1, \ldots, t \), we get similarly as in (26) the condition
\[
\sum_{k=1}^{n} \binom{n}{k} \left( k - \frac{1}{2} \right) \prod_{i=1}^{t} \left( \frac{k - r_i}{r_i} \right)^2 \geq \frac{1}{2}.
\]  
(27)

Now, we seek for a lower bound for the sum on the left-hand side and find the condition on \( t \) such that the lower bound still exceeds \( \frac{1}{2} \).

By Lemma 4.3, which we state and prove below, the roots \( r_i \) can be assumed to be real and to be located in the interval \( [0, n] \). Furthermore, we can assume that the polynomial has degree of exactly \( t \). Then, we look for the worst-case assignment for the roots.

No matter how the roots are located, there exist at least one non-zero point \( k \in N \) such that \( |k - r_i| \geq \frac{n}{2(t+1)} \) for every root \( r_i \) and \( k \geq \left\lfloor \frac{n}{2(t+1)} \right\rfloor \). In the worst case, the smallest of such points is \( \left\lfloor \frac{n}{2(t+1)} \right\rfloor \). Let \( u = \left\lfloor \frac{n}{2(t+1)} \right\rfloor \) be this point. Then, (27) is satisfied if we can show that
\[
\binom{n}{u} \left( u - \frac{1}{2} \right) \left( \frac{u^{2t}}{\prod_{i=1}^{t} r_i^{2}} \right) \geq \frac{1}{2}.
\]

Next, the worst case of the location for the roots in this formulation is \( r_i = n \) for every \( i = 1, \ldots, t \), since all the roots can be assumed to be less or equal to \( n \). We can also get rid of the term \( u - \frac{1}{2} \),
since it is always greater than 1. We then obtain that (27) holds if
\[
\left(\frac{n}{u}\right) \left(\frac{n}{2(n+1)}\right)^{2t} \geq \frac{1}{2}.
\]
Next, we use the inequality \(\left(\frac{n}{u}\right) > \frac{n}{u^n}\) and the fact that \(\frac{1}{2(n+1)} \geq \frac{1}{4n}\) to get that if \(\frac{n}{u^n}(4t)^{-2t} \geq \frac{1}{2}\) holds, then the solution is feasible for SoS\(_t\)(\(K\)). We have that \(n \geq tu\), so the above is satisfied if
\[
\frac{(tu)^u}{u^n}(4t)^{-2t} \geq \frac{1}{2} \iff t^n(4t)^{-2t} \geq \frac{1}{2}.
\]
We have that \(u \geq \frac{n}{4t}\), so it is enough to satisfy \(t^n(4t)^{-2t} \geq \frac{1}{2}\), which is equivalent to \(4^{-2t} t^n \geq \frac{1}{2}\).

If here \(t = \frac{\sqrt{n}}{4}\), we need to then satisfy \(4^{-\sqrt{n}} \sqrt{n^\sqrt{n}/2} \geq \frac{1}{2}\), which holds asymptotically in \(n\). Thus, (25) is satisfied for any choice of the polynomial \(P\) with degree less or equal to \(t\) for \(t = \Omega(\sqrt{n})\). \(\square\)

The following lemma is used within the proof of 4.2. For the polynomial in (27), it shows that the roots \(r_i\) can be assumed to be real and to be located in the interval \([0, n]\). Furthermore, we can assume that the polynomial has degree of exactly \(t\).

**Lemma 4.3.** For the polynomial in (27), we have

(a) all the roots \(r_1, \ldots, r_t\) are real numbers,

(b) all the roots \(r_1, \ldots, r_t\) are in the range, \(1 \leq r_j \leq n\) for all \(j = 1, \ldots, t\),

(c) the degree of the polynomial is exactly \(t\).

**Proof.** The proofs follow by inspecting (27).

(a) Assume that some of the roots are complex and recall that complex roots of polynomials with real coefficients appear in conjugate pairs, i.e., \(r_{2j-1} = a_j + bj, r_{2j} = a_j - bj\) for \(j = 1, \ldots, q\). Let \(P'(k)\) be the polynomial with all real roots such that \(r_{2j-1}' = r_{2j}' = \sqrt{a_j^2 + b_j^2}\) for \(j = 1, \ldots, q\) and \(r_j' = r_j, j > 2q\).

For any \(k \in N\) and \(j \in [t]\), a simple calculation shows that
\[
\left(\frac{r_{2j-1} - k}{r_{2j-1}}\right)^2 \left(\frac{r_{2j} - k}{r_{2j}}\right)^2 \geq \left(\frac{r_{2j-1}' - k}{r_{2j-1}'}\right)^2 \left(\frac{r_{2j}' - k}{r_{2j}'}\right)^2.
\]

Hence,
\[
\sum_{j=1}^{t} \sum_{k=1}^{n} \left(\frac{n}{k}\right) \left(\frac{k - 1}{2}\right) \left(\frac{r_j - k}{r_j}\right)^2 \geq \sum_{j=1}^{t} \sum_{k=1}^{n} \left(\frac{n}{k}\right) \left(\frac{k - 1}{2}\right) \left(\frac{r_j' - k}{r_j'}\right)^2.
\]

(b) Assume exactly one of the roots is negative, i.e., \(r_1 = -a\), for \(a > 0\). Let \(P'(k)\) be the univariate polynomial with all positive roots such that \(r_1' = a\) and \(r_j' = r_j, j > 1\). We have then for any \(k \in N\) that
\[
\left(\frac{-a - k}{-a}\right)^2 \geq \left(\frac{a - k}{a}\right)^2.
\]

Similarly, let \(P(k)\) be the univariate polynomial with \(r_1 \in (0, 1)\) and \(r_j \geq 1\), for \(j > 1\). Again, let \(P'(k)\) be the univariate polynomial with \(r_1 = 1\) and \(r_j' = r_j, j > 1\). Again for any \(k \in N\)
\[
\left(\frac{r_1 - k}{r_1}\right)^2 \geq \left(\frac{1 - k}{1}\right)^2.
\]
Finally, let $P(k)$ be the univariate polynomial with $r_t = an$ for $a > 1$ and $r_j \in [1, n]$, for $j \neq t$. Let $P'(k)$ be the univariate polynomial with $r_t = n$ and $r'_j = r_j, j \neq t$. As in the above cases, for any $k \in \mathbb{N}$, we have

\[
\frac{(an - k)^2}{an} \geq \frac{(n - k)^2}{n}.
\]

It follows that in each case it holds

\[
\sum_{k=1}^{n} \left( \binom{n}{k} \left( k - \frac{1}{2} \right) \prod_{j=1}^{t} \left( \frac{r_j - k}{r_j} \right)^2 \right) \geq \sum_{k=1}^{n} \left( \binom{n}{k} \left( k - \frac{1}{2} \right) \prod_{j=1}^{t} \left( \frac{r'_j - k}{r'_j} \right)^2 \right).
\]

(c) Let $P(k)$ be the univariate polynomial with degree $s < t$ with all real roots $r_j$. Let $P'(k)$ be the polynomial of degree $t$ with all real roots such that $r'_j = r_j, j \leq s$ and $r'_j = n$ for $s < j \leq t$. For any $k \in \mathbb{N}$, we have

\[
1 \geq \frac{(n - k)^2}{n}.
\]

Hence,

\[
\frac{(r_1 - k)^2}{r_1} \cdots \frac{(r_s - k)^2}{r_s} \geq \frac{(r_1 - k)^2}{r_1} \cdots \frac{(r_s - k)^2}{r_s} \frac{(n - k)^{2(t-s)}}{n}
\]

and finally

\[
\sum_{k=1}^{n} \left( \binom{n}{k} \left( k - \frac{1}{2} \right) \prod_{j=1}^{s} \frac{(r_j - k)}{r_j} \right)^2 \geq \sum_{k=1}^{n} \left( \binom{n}{k} \left( k - \frac{1}{2} \right) \prod_{j=1}^{t} \frac{(r'_j - k)}{r'_j} \right)^2.
\]

\[\square\]

ACKNOWLEDGMENTS

The authors would like to express their gratitude to Alessio Benavoli for helpful discussions.

REFERENCES

[1] S. Arora, S. Rao, and U. V. Vazirani. 2009. Expander flows, geometric embeddings and graph partitioning. J. ACM 56, 2 (2009), 51:1–53:7.
[2] Yu Hin Au. 2014. A Comprehensive Analysis of Lift-and-project Methods for Combinatorial Optimization. PhD thesis. University of Waterloo.
[3] Boaz Barak, Fernando G. S. L. Brandão, Aram Wettroth Harrow, Jonathan A. Kelner, David Steurer, and Yuan Zhou. 2012. Hypercontractivity, sum-of-squares proofs, and their applications. In STOC ’12. 307–326.
[4] B. Barak, J. A. Kelner, and D. Steurer. 2015. Dictionary learning and tensor decomposition via the sum-of-squares method. In STOC’15. 143–151.
[5] B. Barak and A. Moitra. 2016. Noisy tensor completion via the sum-of-squares hierarchy. In COLT’16. 417–445.
[6] Dima Grigoriev. 2001. Complexity of positivstellensatz proofs for the knapsack. J. Assoc. Comput. Mach. 42, 6 (1995), 1115–1145.
[7] Michel X. Goemans and Levent Tunçel. 2001. When does the positive semidefiniteness constraint help in lifting procedures? Math. Oper. Res. 26, 4 (2001), 796–815.
[8] M. X. Goemans and D. P. Williamson. 1995. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. J. Assoc. Comput. Mach. 42, 6 (1995), 1115–1145.
[9] Dima Grigoriev. 2001. Complexity of positivstellensatz proofs for the knapsack. Comput. Complex. 10, 2 (2001), 139–154.
[14] Dima Grigoriev. 2001. Linear lower bound on degrees of positivstellensatz calculus proofs for the parity. Theoret. Comput. Sci. 259, 1-2 (2001), 613–622.
[15] Dima Grigoriev and Nicolai Vorobjov. 2001. Complexity of null-and positivstellensatz proofs. Ann. Pure Appl. Logic 113, 1-3 (2001), 153–160.
[16] S. B. Hopkins, T. Schramm, J. Shi, and D. Steurer. 2016. Fast spectral algorithms from sum-of-squares proofs: Tensor decomposition and planted sparse vectors. In STOC’16. 178–191.
[17] P. Kothari, J. Steinhardt, and D. Steurer. 2018. Robust moment estimation and improved clustering via sum of squares. In STOC’18.
[18] Dmitriy Kunisky and Afonso S. Bandeira. 2019. A tight degree 4 sum-of-squares lower bound for the Sherrington-Kirkpatrick Hamiltonian. CoRR abs/1907.11686 (2019).
[19] Adam Kurpisz. 2019. Sum-of-squares bounds via Boolean function analysis. In ICALP’19. 79:1–79:15.
[20] Adam Kurpisz, Samuli Leppänen, and Monaldo Mastrolilli. 2017. On the hardest problem formulations for the 0/1 Lasserre hierarchy. Math. Oper. Res. 42, 1 (2017), 135–143.
[21] Adam Kurpisz, Samuli Leppänen, and Monaldo Mastrolilli. 2020. Sum-of-squares hierarchy lower bounds for symmetric formulations. Math. Program. 182, 1 (2020), 369–397.
[22] Adam Kurpisz, Aaron Potechin, and Elias Samuel Wirth. 2021. SoS certification for symmetric quadratic functions and its connection to constrained Boolean hypercube optimization. In ICALP’21. 90:1–90:20.
[23] Jean B. Lasserre. 2001. Global optimization with polynomials and the problem of moments. SIAM J. Optim. 11, 3 (2001), 796–817.
[24] Monique Laurent. 2003. A comparison of the Sherali-Adams, Lovász-Schrijver, and Lasserre relaxations for 0-1 programming. Math. Oper. Res. 28, 3 (2003), 470–496.
[25] Monique Laurent. 2003. Lower bound for the number of iterations in semidefinite hierarchies for the cut polytope. Math. Oper. Res. 28, 4 (2003), 871–883.
[26] James R. Lee, Prasad Raghavendra, and David Steurer. 2015. Lower bounds on the size of semidefinite programming relaxations. In STOC’15. 567–576.
[27] Troy Lee, Anupam Prakash, Ronald de Wolf, and Henry Yuen. 2016. On the sum-of-squares degree of symmetric quadratic functions. In CCC’16. 17:1–17:31.
[28] László Lovász. 1979. On the Shannon capacity of a graph. IEEE Trans. Inf. Theor. 25 (1979), 1–7.
[29] Claire Mathieu and Alistair Sinclair. 2009. Sherali-Adams relaxations of the matching polytope. In STOC’09. 293–302.
[30] Raghu Meka, Aaron Potechin, and Avi Wigderson. 2015. Sum-of-squares lower bounds for planted clique. In STOC’15. 87–96.
[31] Pablo Parrilo. 2000. Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization. PhD thesis. California Institute of Technology.
[32] A. Potechin and D. Steurer. 2017. Exact tensor completion with sum-of-squares. In COLT’17. 1619–1673.
[33] Shinsaku Sakaue, Akiko Takeda, Sunyoung Kim, and Naoki Ito. 2017. Exact semidefinite programming relaxations with truncated moment matrix for binary polynomial optimization problems. SIAM J. Optim. 27, 1 (2017), 565–582.
[34] T. Schramm and D. Steurer. 2017. Fast and robust tensor decomposition with applications to dictionary learning. In COLT’17. 1760–1793.
[35] Lucas Slot and Monique Laurent. 2021. Sum-of-squares hierarchies for binary polynomial optimization. In IPCO’21. 43–57.
[36] Tamon Stephen and Levent Tunçel. 1999. On a representation of the matching polytope via semidefinite liftings. Math. Oper. Res. 24, 1 (1999), 1–7.
[37] Wikipedia. 2023. Partial fraction decomposition. Retrieved from https://en.wikipedia.org/wiki/Partial_fraction_decomposition#Procedure

Received 15 October 2021; accepted 26 September 2023