COVARIOGRAM OF NON-CONVEX SETS

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Abstract. The covariogram of a compact set \( A \subset \mathbb{R}^n \) is the function that to each \( x \in \mathbb{R}^n \) associates the volume of \( A \cap (A + x) \). Recently it has been proved that the covariogram determines any planar convex body, in the class of all convex bodies. We extend the class of sets in which a planar convex body is determined by its covariogram. Moreover, we prove that there is no pair of non-congruent planar polyominoes consisting of less than 9 points that have equal discrete covariogram.

1. Introduction

Let \( A \) be a compact set in \( \mathbb{R}^n \), \( n \geq 2 \), and let \( \lambda_n \) stand for the \( n \)-dimensional Lebesgue measure. The covariogram \( g_A \) of \( A \) is the function on \( \mathbb{R}^n \) defined by

\[
g_A(x) := \lambda_n(A \cap (A + x)), \quad x \in \mathbb{R}^n.
\]

This function is also called set covariance, and it coincides with the autocorrelation of the characteristic function \( 1_A \) of \( A \), i.e. \( g_A = 1_A * 1_{-A} \). The covariogram \( g_A \) is clearly unchanged by a translation or a reflection of \( A \). (The term reflection will always mean reflection in a point.) A convex body in \( \mathbb{R}^n \) is a convex compact set with non-empty interior. In 1986 Matheron [M86, p. 20] asked the following question and conjectured a positive answer for the case \( n = 2 \). (The same question was also asked independently by Adler and Pyke [AP91] in the probabilistic terms expressed by Problem 12 below.)

Covariogram problem. Does \( g_K \) determine a convex body \( K \), among all convex bodies, up to translations and reflections?

The conjecture for \( n = 2 \) has been completely settled only very recently, by Averkov and Bianchi [AB09].

Theorem 1.1 ([AB09]). Every planar convex body \( K \) is determined within all planar convex bodies by its covariogram, up to translations and reflections.

See [AB09] for further information on the covariogram problem. In general, the convexity of \( K \) is needed in Theorem 1.1 since there exist pairs of non-convex non-congruent (i.e. there exists no isometry mapping one into the other) planar polygons with equal covariogram; see Gardner, Gronchi and Zong [GGZ05] and Fig. 1. We prove some results which extend the class of bodies in which a convex body is determined by its covariogram. The main results of this type are the following two. Let \( A \) denote the class of planar regular (i.e. equal to the closure of their interior) compact sets whose interior has at most two components.

Theorem 1.2. If \( A \in A \) and \( g_A = g_K \), for some convex body \( K \) in \( \mathbb{R}^2 \), then \( A \) is convex.

Let \( B \) denote the class of planar compact sets whose boundary consists of a finite number of closed disjoint simple polygonal curves (each one with finitely many edges). The class \( B \) contains each set which is finite union of disjoint polygons, as well as sets that can be written as \( P \setminus Q \), with \( P \) and \( Q \) polygons and \( Q \subset \text{int} \ P \).
Theorem 1.3. If $A \in \mathcal{B}$ and $g_A = g_K$, for some convex body $K$ in $\mathbb{R}^2$, then $A$ is convex.

The previous theorems clearly imply that a planar convex body is determined by its covariogram both in the class $A$ and $B$. It is known that the covariogram problem is equivalent to any of the following problems (see [AB09] for a detailed explanation of each problem with references to the relevant literature):

P1 Determine a convex body $K$ by the knowledge, for each unit vector $u$ in $\mathbb{R}^n$, of the distribution of the lengths of the chords of $K$ parallel to $u$.

P2 Determine a convex body $K$ by the distribution of $X - Y$, where $X$ and $Y$ are independent random variables uniformly distributed over $K$.

P3 Determine the characteristic function $1_K$ of a convex body $K$ from the modulus of its Fourier transform $\widehat{1}_K$.

Thus the previous theorems imply a positive answer to Problems P1 and P2 both in the class $A$ and $B$, and a positive answer to Problem P3 in the class of characteristic functions of sets in $A$ or in $B$.

Propositions 3.2, 3.5, 3.7, 3.8 and Corollary 3.6, all contained in Section 3, are other results in the spirit of Theorems 1.2 and 1.3.

Some aspects of the covariogram problem are of combinatorial nature. Two finite subsets $A$ and $B$ of $\mathbb{R}^n$ are said to be homometric if $A \cap (A + x)$ and $B \cap (B + x)$ have equal cardinality for each $x \in \mathbb{R}^n$, or, equivalently, if the sets of vector differences $\{x - y : x, y \in A\}$ and $\{x - y : x, y \in B\}$ are identical counting multiplicities. One problem consists in determining all the sets which are homometric to a given set. We refer to [RS82], [LRH92] and [DGN02] for a complete algebraic solution of this problem for subsets of the real line.

A polyomino is a finite subset $A$ of $\mathbb{Z}^n$ such that the union $A + [0, 1]^n$ of lattice unit cubes has connected interior. A polyomino $A$ is convex if $A = (\text{conv } A) \cap \mathbb{Z}^n$.

We shall refer to the set $A + [0, 1]^n$ (itself called a polyomino by many authors) as the animal of the polyomino $A$. The non-convex polygons with equal covariogram presented in [CGZ03] are the animals of two homometric convex polyominoes made of 15 points. We are interested in finding a similar example with minimal number of unit squares. Since two polyominoes are homometric if and only if the associated animals have the same covariogram (see Lemma 4.1) we are interested in finding pairs of homometric polyominoes with minimal cardinality. We exhibit a pair of non-congruent homometric polyominoes made of 9 points, and we prove that this example is minimal.

Theorem 1.4. The minimum number $d$ such that there exists a pair of non-congruent homometric polyominoes in $\mathbb{Z}^2$ made of $d$ points is 9.

In terms of animals, this theorem proves that two non-congruent animals made of less than nine unit squares cannot have the same covariogram.

2. Definitions, notations and preliminaries

As usual, $S^{n-1}$ denotes the unit sphere in Euclidean $n$-space $\mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm of $x$, $x \cdot y$ denotes scalar product, while $[x, y]$ denotes the closed line segment with endpoints $x$ and $y$. For $\epsilon > 0$ the symbol $B(x, \epsilon)$ denotes the open ball centred at $x$ and with radius $\epsilon$. If $u \in S^{n-1}$, we denote by $u^\perp$ the $(n - 1)$-dimensional subspace orthogonal to $u$ and by $l_u$ the line parallel to $u$ containing the origin $o$. The symbol $\phi_{\pi/2}$ denotes counterclockwise rotation by $\pi/2$ about the origin in $\mathbb{R}^2$. We write $\lambda_k$ for $k$-dimensional Lebesgue measure in $\mathbb{R}^n$, where $k = 1, \ldots, n$, and where we identify $\lambda_k$ with $k$-dimensional Hausdorff measure.
If $A$ is a set, we denote by $|A|$, $\text{cl} A$, $\text{int} A$ and $\text{conv} A$ the cardinality, closure, interior, and convex hull of $A$, respectively. The notation for the usual orthogonal projection of $A$ on a subspace $S$ is $A|S$. A closed set $A$ is said to be regular if it coincides with the closure of its interior. If $A$ and $B$ are subsets of $\mathbb{R}^n$, their Minkowski sum is

$$A + B = \{a + b : a \in A, b \in B\}.$$ 

In particular, if $x \in \mathbb{R}^n$, then $A + x$ denotes the translate of $A$ by $x$. We also write $DA = A + (-A)$ for the difference set of $A$.

The support function of a compact set $A \subset \mathbb{R}^n$ is defined, for $x \in \mathbb{R}^n$, by

$$h_A(x) = \sup\{x \cdot y : y \in A\},$$

while the width of $A$ in direction $u \in \mathbb{S}^{n-1}$ is defined by

$$w(A,u) = h_A(u) + h_A(-u).$$

The linearity of the support function with respect to Minkowski addition implies

$$w(A,u) = (1/2)w(DA,u).$$

It is well known that, when $A$ is compact, $g_A$ is continuous and

$$\text{supp} g_A = DA,$$

where $\text{supp} f$ denotes the support of the function $f$.

Given $u \in \mathbb{S}^{n-1}$ and a compact set $A \subset \mathbb{R}^n$, the exposed face of $A$ in direction $u$ is $F(A,u) = \{x \in A : x \cdot u = h_A(u)\}$. [S93, Th. 1.7.5(c)] proves that, for a convex body $A$ and $u \in \mathbb{S}^{n-1}$,

$$F(DA,u) = F(A,u) - F(A,-u),$$

and is it not difficult to see that the previous formula is valid also for any compact set $A$.

3. Comparison between covariograms of convex and non-convex sets

Proposition 3.2 exploits the convexity of $\text{supp} g_K = DK$ when $K$ is a convex body.

**Lemma 3.1.** Given a compact set $A \subset \mathbb{R}^n$, one has $\text{conv} (DA) = D(\text{conv} A)$.

**Proof.** It suffices to prove $h_{\text{conv} (DA)} = h_{D(\text{conv} A)}$. This identity is a consequence of the linearity of the support function with respect to Minkowski addition, and of the identity $h_B = h_{\text{conv} B}$, valid for each compact set $B$. Indeed, one has

$$h_{\text{conv} (DA)} = h_{DA} = h_{A-A} = h_A + h_{-A} = h_{\text{conv} A} + h_{-\text{conv} A} = h_{D(\text{conv} A)}.$$ 

\[\square\]

**Proposition 3.2.** Let $A$ be a regular compact set of $\mathbb{R}^n$. If either $\text{supp} g_A$ is not convex, or $DA \neq D(\text{conv} A)$, then $g_A \neq g_K$, for each convex body $K \subset \mathbb{R}^n$.

**Proof.** If $A$ has the same covariogram as a convex body, then $DA$ is convex, and so $DA = \text{conv} (DA)$. By Lemma 3.1 this implies $DA = D(\text{conv} A).$ \[\square\]

It is well known (see [M75, p.86]) that, when $K$ is a convex body and $u \in \mathbb{S}^1$,

$$-\frac{\partial g_K}{\partial u}(0) = \lambda_{n-1}(K \mid u^\perp).$$

When $n = 2$, since $\lambda_1(K \mid u^\perp) = w(K,\phi_{\pi/2}u) = (1/2)w(\text{supp} g_K, \phi_{\pi/2}u)$, the formula becomes

$$-\frac{\partial g_K}{\partial u}(0) = \frac{1}{2}w(\text{supp} g_K, \phi_{\pi/2}u).$$

\[\text{(3.1)}\]

\[\text{(3.2)}\]
Propositions 3.5 and 3.7 and Theorem 1.2 exploit (3.2) to give conditions on a planar regular compact set $A$ that imply $g_A \neq g_K$ for every convex body $K$. We begin with two lemmas.

**Lemma 3.3.** Let $D$ be a bounded Lebesgue-measurable subset of $\mathbb{R}$ such that $\lambda_1(D) > 0$. Then

$$\liminf_{h \to 0} \frac{\lambda_1(D \setminus (D + h))}{h} \geq 1.$$ 

**Proof.** As $\lambda_1(D) > 0$, there exists a point $x_0$ of density for $D$, i.e. a point such that for every positive $\epsilon$ there exists $\bar{h}(\epsilon) > 0$ such that for every $h \in (0, \bar{h}(\epsilon))$ we have

$$\lambda_1((x_0 - h/2, x_0 + h/2) \cap D) \geq (1 - \epsilon)h$$

(see [C80, Cor. 6.26]). Choose $\epsilon > 0$ and let $h \in (0, \bar{h}(\epsilon))$. Let $D_0 = (x_0 - h/2, x_0 + h/2) \cap D$ and consider the sequence $(B_k)$, where

$$B_k = (D_0 \cap (D + h) \cap ... \cap (D + (k - 1)h)) \setminus (D + kh).$$

We have $D_0 = \bigcup_{i=1}^{\infty} B_i$. In fact, the inclusion $D_0 \supseteq \bigcup_{i=1}^{\infty} B_i$ is trivial; while if $x \in D_0$, then there exists $k$ such that $x \notin D + kh$, since $D$ is bounded. Let $k$ be the smallest integer such that $x \notin D + kh$. Then $x \in B_k \subset \bigcup_{i=1}^{\infty} B_i$. If $i \neq j$, then $B_i \cap B_j = \emptyset$. In fact, when $i < j$ we have $B_i \cap (D + ih) = \emptyset$ and $B_j \subset (D + ih)$. It follows that

$$\sum_{i=1}^{\infty} \lambda_1(B_i) = \lambda_1(D_0) \geq (1 - \epsilon)h.$$ 

If $i \neq j$, then $(B_i - (i - 1)h) \cap (B_j - (j - 1)h) = \emptyset$. In fact, $B_i \subset (x_0 - h/2, x_0 + h/2)$ implies $B_i - (i - 1)h \subset I_i := (x_0 - (i - 1)h - h/2, x_0 - (i - 1)h + h/2)$ and $I_i \cap I_j = \emptyset$. Let us also remark that $B_i \subset (D + (i - 1)h) \setminus (D + ih)$ and so $B_i - (i - 1)h \subset D \setminus (D + h)$. Thus,

$$\sum_{i=1}^{\infty} \lambda_1(B_i) = \sum_{i=1}^{\infty} \lambda_1(B_i - (i - 1)h) \leq \lambda_1(D \setminus (D + h)).$$

The inequalities 3.3 and 3.4 imply the statement. \hfill $\square$

**Lemma 3.4.** Let $D$ be a bounded Lebesgue-measurable subset of $\mathbb{R}$ such that there exist $2r$ points $a_1, b_1, \ldots, a_r, b_r \in \mathbb{R} \setminus D$, with $a_1 < b_1 < a_2 < b_2 < \ldots < a_r < b_r$, for which $\lambda_1(D \cap [a_i, b_i]) > 0$, $i = 1, \ldots, r$, and $\lambda_1(D \cap [b_i, a_{i+1}]) = 0$, $i = 1, \ldots, r - 1$. Then

$$\liminf_{h \to 0} \frac{\lambda_1(D \setminus (D + h))}{h} \geq r.$$ 

**Proof.** Let $D_i = D \cap [a_i, b_i]$ and let $h$ satisfy $|h| < \min_{i \neq j} |b_i - a_j|$. Observe that if $i \neq j$, then $D_i \cap (D_j + h) = \emptyset$. Therefore

$$g_D(h) = \lambda_1(D \cap (D + h)) = \lambda_1 \left( \bigcup_{i=1}^{r} (D_i \cap (D_i + h)) \right) = \sum_{i=1}^{r} \lambda_1((D_i \cap (D_i + h)).$$

Thus,

$$\lambda_1(D \setminus (D + h)) = \lambda_1(D) - g_D(h) = \lambda_1(D) - \sum_{i=1}^{r} \lambda_1((D_i \cap (D_i + h)) =$$

$$= \lambda_1(D) - \sum_{i=1}^{r} (\lambda_1(D_i) - \lambda_1(D_i \setminus (D_i + h))) = \sum_{i=1}^{r} \lambda_1(D_i \setminus (D_i + h)).$$

The statement follows by applying Lemma 3.3 to each set $D_i$. \hfill $\square$
Let \( A \) be a regular compact set of \( \mathbb{R}^2 \), \( u \in \mathbb{S}^1 \) and \( y \in u^\perp \). Let us set
\[
f_{A,u}(y) = \liminf_{h \to 0} \frac{\lambda_1((A \setminus (A + hu)) \cap (y + l_u))}{h}.
\]

**Proposition 3.5.** Let \( A \subset \mathbb{R}^2 \) be a regular compact set for which there exists a direction \( u \in \mathbb{S}^1 \) such that

(i) \( \lambda_1(A \cap (y + l_u)) > 0 \) for \( \lambda_1 \)-a.e. \( y \in \text{conv}(A|u^\perp) \) and

(ii) \( \lambda_1(\{y \in u^\perp : f_{A,u}(y) \geq 2\}) > 0 \).

Then \( g_A \neq g_K \), for every convex body \( K \) in \( \mathbb{R}^2 \).

**Proof.** If \( (\partial g_A/\partial u)(0) \) does not exist, the statement follows by \( \text{(8.2)} \). Otherwise
\[
-\frac{\partial g_A}{\partial u}(0) = \lim_{h \to 0} \frac{\lambda_1((A \setminus (A + hu)) \cap (y + l_u))}{h} = \lim_{h \to 0} \int_{\text{conv}(A|u^\perp)} \frac{\lambda_1((A \setminus (A + hu)) \cap (y + l_u))}{h} d\lambda_1(y),
\]
since \( A|u^\perp = \text{conv}(A|u^\perp) \), by Assumption \( \text{(i)} \) and the fact that \( A|u^\perp \) is closed.

Let \( A^{(2)} = \{y \in u^\perp : f_{A,u}(y) \geq 2\} \). By virtue of Fatou’s lemma we have
\[
-\frac{\partial g_A}{\partial u}(0) \geq \int_{\text{conv}(A|u^\perp) \setminus A^{(2)}} \liminf_{h \to 0} \frac{\lambda_1((A \setminus (A + hu)) \cap (y + l_u))}{h} d\lambda_1(y) + \int_{A^{(2)}} \liminf_{h \to 0} \frac{\lambda_1((A \setminus (A + hu)) \cap (y + l_u))}{h} d\lambda_1(y).
\]

Thus, by Lemmas \( \text{(8.3)} \) and \( \text{(8.4)} \) and Assumptions \( \text{(i)} \) and \( \text{(ii)} \), we have
\[
-\frac{\partial g_A}{\partial u}(0) \geq \int_{\text{conv}(A|u^\perp) \setminus A^{(2)}} d\lambda_1(y) + \int_{A^{(2)}} 2 d\lambda_1(y) > \lambda_1(\text{conv}(A|u^\perp)).
\]

Since \( \lambda_1(\text{conv}(A|u^\perp)) = w(A, \phi_{\pi/2}u) \), and \( w(A, \phi_{\pi/2}u) = (1/2)w(\text{supp } g_A, \phi_{\pi/2}u) \) (because \( \text{supp } g_A = D_A \)), we have
\[
-\frac{\partial g_A}{\partial u}(0) > \frac{1}{2} w(\text{supp } g_A, \phi_{\pi/2}u).
\]

This inequality and \( \text{(8.2)} \) imply \( g_A \neq g_K \), for every convex body \( K \) in \( \mathbb{R}^2 \). \( \square \)

**Corollary 3.6.** Let \( A \subset \mathbb{R}^2 \) be a regular compact set such that \( \text{int } A \) has finitely many components. Assume that there exist \( u \in \mathbb{S}^1 \) and \( a_1, a_2 \in \text{int } A \) such that \( A|u^\perp \) is a segment and \( [a_1, a_2] \) is parallel to \( u \) and meets \( \mathbb{R}^2 \setminus A \). Then \( g_A \neq g_K \), for every convex body \( K \) in \( \mathbb{R}^2 \).

**Proof.** It suffices to prove that the assumptions of Proposition \( \text{(8.4)} \) are satisfied. The assumptions of the corollary imply that \( (\text{int } A)|u^\perp \) consists of finitely many intervals and \( A|u^\perp \) is a regular closed set. Thus \( \lambda_1(A|u^\perp \setminus (\text{int } A)|u^\perp) = 0 \). Since, \( \lambda_1(A \cap (y + l_u)) \) is positive when \( y \in (\text{int } A)|u^\perp \), Assumption \( \text{(i)} \) of Proposition \( \text{(8.4)} \) is satisfied.

Let \( b \in [a_1, a_2]|\setminus A \), and let \( \epsilon > 0 \) be such that \( B(a_i, \epsilon) \subset A, i = 1, 2, \) and \( B(b, \epsilon) \subset \mathbb{R}^2 \setminus A \). If \( y \in u^\perp \cap B(a_i, |u^\perp|, \epsilon) \), then \( A \cap (y + l_u) \) contains two closed non-degenerate intervals separated by a non-degenerate interval contained in \( \mathbb{R}^2 \setminus A \). Thus, by Lemma \( \text{(8.3)} \) \( u^\perp \cap B(a_i, |u^\perp|, \epsilon) \subset \{y \in u^\perp : f_{A,u}(y) \geq 2\} \) and Assumption \( \text{(ii)} \) of Proposition \( \text{(8.5)} \) is satisfied. \( \square \)

**Proof of Theorem \( \text{(1.2)} \)** We argue by contradiction. Assume that \( \text{int } A \) has two components \( A_1 \) and \( A_2 \), and let \( a_1 \) and \( a_2 \) belong respectively to \( A_1 \) and \( A_2 \), and be such that \( [a_1, a_2] \) meets \( \mathbb{R}^2 \setminus A \). Let \( u \) be the direction of the segment \( [a_1, a_2] \). The set \( (\text{int } A)|u^\perp \) is an interval, because \( A_1|u^\perp \) and \( A_2|u^\perp \) are intervals and, by the
integral is finite, then we may apply the Lebesgue dominated convergence Theorem

Corollary 3.6 gives a contradiction.

Lemma. In this case (3.2) implies

For brevity, let

Lemma 3.4 implies that, for every integer

Since each summand in the last sum of the previous formula is less than or equal to

Lemma 3.4 implies that, for every integer

(3.7)

Proof. Let us first prove that, for each

(3.8)

Since each summand in the last sum of the previous formula is less than or equal to

to

Lemma 3.4 implies that, for every integer

(3.8)

Formulas (3.7) and (3.8) imply

If

If

the Lebesgue dominated convergence Theorem to the last integral in (3.5), and we have



\[ \frac{\partial g_A}{\partial u}(0) = \int_{\text{conv}(A[u^+])} N(y) \, d\lambda_A(y). \]
Since
\[
\int_{\text{conv}(A|u^+)} N(y) \, d\lambda_1(y) = \sum_{i=0}^{\infty} \int_{y \in \text{conv}(A|u^+): N(y) = i} i \, d\lambda_1(y)
\]
\[
= \sum_{i=0}^{\infty} i \lambda_1 \{ y \in \text{conv}(A|u^+): N(y) = i \},
\]
and \((1/2)w(\text{supp} \, g_A, \phi_\pi/2u) = w(A, \phi_\pi/2u) = \sum_{i=0}^{\infty} \lambda_1 \{ y \in \text{conv}(A|u^+): N(y) = i \}, \tag{3.9}
\]
implies
\[
\frac{\partial g_A}{\partial u}(0) \neq \frac{1}{2} w(\text{supp} \, g_A, \phi_\pi/2u).
\]
Again, \(\text{3.10}\) implies \(g_A \neq g_K\) for every convex body \(K\).

The next result is valid for sets of any dimension. The covariogram \(g_A\) provides both \(\lambda_n(A) = g_A(0)\) and \(\lambda_n(DA) = \lambda_n(\text{supp} \, g_A)\). Since when \(A\) is convex \(\lambda_n(A)\) and \(\lambda_n(DA)\) are related by the Rogers-Shephard and the Brunn-Minkowski inequalities, we obtain some conditions on \(g_A\) which are necessary for \(A\) to be convex.

**Proposition 3.8.** Let \(A \subset \mathbb{R}^n\) be a regular compact set. If \(A\) is convex, then
\[
\left(\frac{2n}{n}\right)^{-1} \lambda_n(\text{supp} \, g_A) \leq g_A(0) \leq 2^{-n} \lambda_n(\text{supp} \, g_A) \tag{3.9}
\]
and, for each \(u \in \mathbb{S}^{n-1}\),
\[
\left(\frac{2n-2}{n-1}\right)^{-1} \lambda_{n-1}(\text{supp} \, g_A \mid u^+) \leq \frac{\partial g_A}{\partial u}(0) \leq 2^{1-n} \lambda_{n-1}(\text{supp} \, g_A \mid u^+) \tag{3.10}
\]

\(\text{Proof.}\) The Rogers-Shephard and the Brunn-Minkowski inequalities (see \(\text{S93 Th. 7.3.1}\)) state that, when \(A \subset \mathbb{R}^n\) is convex, we have
\[
\left(\frac{2n}{n}\right)^{-1} \lambda_n(DA) \leq \lambda_n(A) \leq 2^{-n} \lambda_n(DA).
\]
Thus \(\text{3.9}\) is an immediate consequence of the previous inequalities and of the identities \(g_A(0) = \lambda_n(A)\) and \(DA = \text{supp} \, g_A\). The same inequalities, applied to the \((n-1)\)-dimensional convex body \(A \mid u^+\), give
\[
\left(\frac{2n-2}{n-1}\right)^{-1} \lambda_{n-1}(D(A \mid u^+)) \leq \lambda_{n-1}(A \mid u^+) \leq 2^{1-n} \lambda_{n-1}(D(A \mid u^+)) \tag{3.10}
\]
The identity \(D(A \mid u^+) = (DA) \mid u^+ = \text{supp} \, g_A \mid u^+\) and \(\text{3.1}\) imply \(\text{3.10}\). \(\square\)

In order to critically discuss the previous results, let us present some examples (see Figures 1 and 2). Let \(Q = [0, 1]^2\). The set \(B\) is obtained by placing four squares of edge 1/4 inside and in the corner of \(Q\), so that \(\text{conv} \, B = Q\). To prove that \(g_B\) differs from the covariogram of any convex body one cannot use Proposition \(\text{3.7}\) because \(\text{3.6}\) is false, but one can use Proposition \(\text{3.2}\) since \(DB \neq DQ\) is not convex. The set \(C\) is constructed as follows. Divide \(Q\) in \(d^2\) equal squares. We obtain a grid of \((d+1)^2\) points. The body \(C\) is the subset of \(Q\) which is the union of the four squares of edge 1/d touching the four vertex of \(Q\) and of \((d+1)^2 - 16\) little squares of edge \(\epsilon = (1 - 4/d)/((d+1)^2 - 16)\) contained in \(Q\) and containing the points of the grid outside the four squares already considered. It results that \(C\) does not satisfy condition \(\text{3.6}\) in Proposition \(\text{3.7}\) and, moreover, \(DC = DQ\) is convex. In this case, what proves that \(g_C\) differs from the covariogram of a convex body when \(d\) is large is Proposition \(\text{3.8}\) since \(\text{3.9}\) is not satisfied by \(C\) (because \(1/6 \lambda_1(DQ) > g_C(0)\) when \(d\) is large). Choose \(\epsilon\) so that \(0 < \epsilon < (1 - 4/d)/((d+1)^2 - 16)\). The set
$E$ (see Fig. 2) is constructed by adding another square in the centre of $C$ of edge $1 - 4/d - ((d+1)^2 - 16)/d$ (actually, a little bit longer than this, to compensate for the little squares included in this central square which disappear so that (3.6) does not hold). The set $E$ does not satisfy (3.6), we have $DE$ convex and, when $d$ is large and $\epsilon$ is small even (3.9) and (3.10) are satisfied. The fact that a set like $E$ does not have the covariogram equal to that of a convex body is a consequence of Theorem 1.3.

In order to prove Theorem 1.3 we need a lemma computing a second order distributional derivative of $g_A$. These computations are made in [B09, Lemma 4.2] when $A$ is a convex polytope in $\mathbb{R}^n$, and can be repeated, almost without any change, also when $A \in \mathcal{B}$. Let $C^\infty_0(\mathbb{R}^2)$ denote the class of infinitely differentiable functions on $\mathbb{R}^2$ with compact support. We recall that $|A|$ denotes the cardinality of $A$.

**Lemma 3.9.** Let $A \in \mathcal{B}$ and $F_1, \ldots, F_m$ be the edges of the polygons which constitute $A$. Let $\nu_i$, $i = 1, \ldots, m$, be the unit outer normal vector of $A$ at $F_i$, $w \in \mathbb{S}^1$, $I_p = \{(i, j) : F_i$ is parallel to $F_j\}$ and $I_{np} = \{(i, j) : F_i$ is not parallel to $F_j\}$. Then, for $f \in C^\infty_0(\mathbb{R}^2)$, we have

\[
\frac{\partial^2 g_A}{\partial w^2}(f) = \sum_{(i, j) \in I_{np}} \frac{w \cdot \nu_i \cdot w \cdot \nu_j}{\sqrt{1 - (w \cdot \nu_i)^2}} \int_{\mathbb{R}^2} |F_i \cap (F_j + z)| f(z) \, d\lambda_2(z) + \sum_{(i, j) \in I_p} w \cdot \nu_i \cdot w \cdot \nu_j \int_{F_i - F_j} \lambda_1(F_i \cap (F_j + z)) f(z) \, d\lambda_1(z).
\]
Both sums in the right hand side of (3.11) are uniquely determined by $g_A$.

Proof. The definition of derivative in the sense of distributions implies $(\partial A/\partial w)(f) = -\int_A \partial f(x)/\partial w \, dx$. Thus, by the Divergence Theorem, we have
\[
\frac{\partial A}{\partial w}(f) = -\sum_{i=1}^m w \cdot \nu_i \delta_{F_i}(f),
\]
where $\delta_{F_i}(f) = \int_{F_i} f(x) d\lambda_1(x)$. Since $g_A = 1_A * 1_{-A}$, we can write
\[
\frac{\partial^2 g_A}{\partial w^2}(f) = \left( \frac{\partial A}{\partial w} + \frac{\partial A}{\partial w} \right)(f) = -\sum_{i,j=1}^m w \cdot \nu_i \cdot w \cdot \nu_j (\delta_{F_i} * \delta_{F_j})(f).
\]

A direct computation (see [B09, Lemma 4.2] for the details) proves
\[
(\delta_{F_i} * \delta_{F_j})(f) = \int_{F_i \cap (F_j + z)} \lambda_1(F_i \cap (F_j + z)) f(z) \, d\lambda_1(z)
\]
when $F_i$ and $F_j$ are parallel, and
\[
(\delta_{F_i} * \delta_{F_j})(f) = (1 - (\nu_i \cdot \nu_j)^2)^{-1/2} \int_{\mathbb{R}^2} |F_i \cap (F_j + z)| f(z) \, d\lambda_2(z)
\]
when $F_i$ and $F_j$ are not parallel. These formulas give (3.11).

Both sums in the right-hand side of (3.11) are determined because, roughly speaking, the first sum corresponds to the absolutely continuous part of the derivative and the second sum to its singular part (see [B09] for the details).

Proof of Theorem 1.3. Let $F_i$, $\nu_i$ and $I_p$ be as in the statement of Lemma 3.9. Consider the distribution defined by the second sum in (3.11). This distribution determines its support, which we denote by $S(A, w)$, and determines
\[
d(x) := \sum_{(i,j) \in I_p} w \cdot \nu_i \cdot w \cdot \nu_j \cdot \lambda_1(F_i \cap (F_j + x)),
\]
for $\lambda_1$-a.e. $x \in S(A, w)$. Note that $S(A, w) \subset \cup_{(i,j) \in I_p} A_{\nu_i \nu_j} \cap (F_i \cap F_j)$. Choose any $i \in \{1, \ldots, m\}$ and let $I_{\nu_i} = \{j \in 1, \ldots, m : \nu_j = \pm \nu_i\}$. We recall that $\phi_{\pi/2} \nu_i$ is a rotation of $\nu_i$ by $\pi/2$. Then, for any $h > 0$ sufficiently small, we have
\[
d(h \phi_{\pi/2} \nu_i) = \sum_{j \in I_{\nu_i}} (w \cdot \nu_j)^2 \lambda_1(F_j \cap (F_j + h \phi_{\pi/2} \nu_i)) = (w \cdot \nu_i)^2 \sum_{j \in I_{\nu_i}} (\lambda_1(F_j) - h).
\]
Choose $w$ so that $w \cdot \nu_i \neq 0$. Since the previous function is different from 0, $S(A, w)$ contains a segment containing $o$ and parallel to $F_i$. Moreover, we have
\[
\frac{\partial d}{\partial \phi_{\pi/2} \nu_i}(0) = -(w \cdot \nu_i)^2 |I_{\nu_i}|,
\]
and this formula provides the number of edges of $A$ parallel to $F_i$.

The set $K$ is a convex polygon, because $DK$ coincides with $\text{supp} g_A$, which is a polygon. Since $g_A = g_K$ the distribution considered above has the same features as the corresponding one for a convex polygon. This implies the following consequences.

C1 The number of edges of $A$ parallel to $F_i$ is at most two.

C2 We have
\[
S(A, w) \subset (\partial \text{supp} g_A) \cup \left( \bigcup_{i \in I_p, \nu_i \neq 0} \nu_i \right),
\]
and each segment in $S(A, w)$ is parallel to an edge of $\text{supp} g_A$. 
We need to prove only (2) since (1) is obvious. Assume \( A \) convex polygon. If \( F_i \) and \( F_j \) are parallel and \( i \neq j \) then \( \nu_i = -\nu_j \) and, by (2.1), \( F_i - F_j \) is an edge of \( DA = \text{supp } DA \). Moreover, when \( i = j \) \( F_i - F_j \) is a segment contained in \( \nu_i^{\perp} \). Since \( DA \) has an edge orthogonal to \( \nu_i \), for each \( i \), by (2.1) with \( u = \nu_i \), the property is proved.

To conclude the proof of the theorem we argue by contradiction and assume \( A \) non-convex. We have \( DA = D(\text{conv } A) \), because otherwise \( g_A \neq g_K \), by Proposition 3.2. Consider the edges of \( A \) not contained in \( \partial(\text{conv } A) \). We may assume that they are \( F_1, \ldots, F_d \), for some \( d < m \), Let us distinguish the following three cases.

1) There exists an edge \( F_k, k \in \{1, \ldots, d\} \), which is not parallel to any edge of \( \text{conv } A \). In this case, if we choose \( w \) so that \( w \cdot \nu_k \neq 0 \), \( S(A, w) \) contains a segment parallel to \( F_k \) which is not parallel to any edge of \( D(\text{conv } A) = DA = \text{supp } g_A \). This contradicts (2).

2) There exists an edge \( F_k, k \in \{1, \ldots, d\} \), parallel to exactly one edge \( M \) of \( \text{conv } A \). Let us show that \( M \) is an edge of \( A \). We can write \( M = [m_1, m_2] \), with \( m_1, m_2 \in A \). Let \( u \) be the unit outer normal vector to \( \text{conv } A \) at \( M \), i.e. \( M = F(\text{conv } A, u) \). The hypothesis defining this case implies that \( F(\text{conv } A, -u) \) is not an edge and is a single point \( m \). Thus, \( F(D(\text{conv } A), u) = [m_1, m_2] = m \), by (2.1). As \( DA = D(\text{conv } A) \), (2.1) implies \( M = F(A, u) \).

Consider now a Cartesian coordinate system so that \((0, 1) = u \). Clearly \( M \) and \( F_k \) are parallel to the \( x \)-axis. Among the edges \( F_1, \ldots, F_d \) parallel to the \( x \)-axis consider those with the smallest \( y \)-coordinate. Among these edges consider the edge \( F_m \) with largest abscissa (see Fig. 3). Let \( x_0 \) be the translation which maps the point with smallest abscissa of \( M \) to the point with largest abscissa of \( F_m \), and let \( x_1 = x_0 - (h, 0) \) with \( h > 0 \) sufficiently small. Note that \([x_1, x_0] \subset F_m - M \). We claim that, if \( w \cdot (0, 1) \neq 0 \), then \([x_1, x_0] \subset S(A, w) \). Indeed, let \( x \in [x_1, x_0] \) and consider the pairs of edges \( F_i \) and \( F_j \) of \( A \) such that

\[
\lambda_1(F_i \cap (F_j + x)) > 0. 
\tag{3.13}
\]

If \( F_i \) and \( F_j \) are parallel to the \( x \)-axis, then we have necessarily \( F_i = F_m \) and \( F_j = M \), by the choice of \( x_0 \). If \( F_i \) and \( F_j \) are not parallel to the \( x \)-axis, then (3.13) is false except possibly for finitely many \( x \in [x_1, x_0] \). Therefore, \( \lambda_1 \)-a.e. in \([x_1, x_0] \) we have \( d(x) = \pm w \cdot (0, 1) \lambda_1(F_m \cap (M + x)) \neq 0 \). This proves the claim.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Figure3}
\caption{The intersection of \( A \) and \( A + x \) for \( x \in [x_1, x_0] \)}
\end{figure}
The segment \([x_1, x_0]\) is not contained in a line through \(a\), as \(F_m\) is not aligned with \(M\). Let us prove \([x_1, x_0] \not\subset \partial \text{supp} \, g_A\). Let \(l_1\) and \(l_2\) be the lines parallel to the \(x\)-axis supporting \(A\), with \(M \subset l_2\). The edges of \(\text{supp} \, g_A\) parallel to the \(x\)-axis are contained in \(\pm (l_1 - l_2)\). On the other hand, we have \([x_1, x_0] \subset F_m - M \not\subset l_1 - l_2\), because \(F_1 \not\subset l_1\) \((l_1 \cap A \subset l_1 \cap \text{conv} \, A = F(\text{conv} \, A, 0, -1))\) and \(F(\text{conv} \, A, 0, -1))\) is not an edge, by the hypothesis defining this case. This proves \([x_1, x_0] \not\subset \partial \text{supp} \, g_A\).

These properties of \([x_1, x_0]\) contradict \((\exists)\). 

3) There exists an edge \(F_k\), \(k \in \{1, \ldots, d\}\) parallel to a pair of antipodal parallel edges \(M\) and \(N\) of \(\text{conv} \, A\). Let us show that at least one of the inequalities

\[
\lambda_1(M \cap \partial A) > 0 \quad \text{and} \quad \lambda_1(N \cap \partial A) > 0
\]

holds. Let \(u \in \mathbb{S}^1\) be such that \(M = F(\text{conv} \, A, u)\) and \(N = F(\text{conv} \, A, -u)\) and assume that both inequalities are false. The geometric structure of \(A\) implies that both \(F(A, u)\) and \(F(A, -u)\) consist of a finite number of points. Consequently, \(F(DA, u)\) consists of a finite number of points, by \((2.1)\), contradicting \(DA = D(\text{conv} \, A)\).

If exactly one of the previous inequalities holds, then the proof is concluded as in the previous case. If both inequalities hold, then \(A\) has at least three edges orthogonal to \(u\). This contradicts \((\bar{H})\).

The above three cases complete all the possibilities. \(\square\)

4. Non-convex sets with equal covariogram

Gardner, Gronchi and Zong \([\text{GGZ05}]\) presents a pair of non-congruent non-convex polygons \(P\) and \(Q\) with equal covariogram. The polygons \(P\) and \(Q\) are

![Two non-congruent non-convex polygons with equal covariogram](from [GGZ05])

the animals associated to two homometric convex polyominoes consisting of fifteen points. We are interested in finding similar examples with minimal cardinality. Let us first prove that two animals have the same covariogram if and only if the corresponding polyominoes are homometric. The “if” part is proved, in a more general setting, in \([\text{GGZ05}]\).

Lemma 4.1. Let \(A\) and \(B\) be finite subsets of \(\mathbb{Z}^n\) and let \(\bar{A} = A + [0, 1]^n\) and \(\bar{B} = B + [0, 1]^n\). Then \(g_{\bar{A}} = g_{\bar{B}}\) if and only if \(A\) and \(B\) are homometric.

Proof. Let \(Q = [0, 1]^n\). \([\text{GGZ05}]\) proves the following formulas, valid for any \(x \in \mathbb{R}^n\),

\[
(4.1) \quad g_A(x) = \sum_{z \in \mathbb{Z}^n} |A \cap (A + z)|g_Q(z + x), \quad g_B(x) = \sum_{z \in \mathbb{Z}^n} |B \cap (B + z)|g_Q(z + x).
\]

If \(A\) and \(B\) are homometric these formulas imply \(g_{\bar{A}} = g_{\bar{B}}\). Assume now \(g_{\bar{A}} = g_{\bar{B}}\) and choose \(w \in \mathbb{Z}^n\). The support of \(g_Q(\cdot - w)\) is \(DQ + w = [-1, 1]^n + w\). Since
$Z^n \cap \text{int}([-1, 1]^n + w) = \{w\}$, we have $g_Q(z - w) = 0$ for each $z \in Z^n$, $z \neq w$. Thus $g_A(-w) = g_B(-w)$ and (4.1) imply $|A \cap (A + w)| = |B \cap (B + w)|$.

Since $|A \cap (A + w)| = |B \cap (B + w)| = 0$ when $w \notin Z^n$, the previous identity implies $A$ and $B$ homometric.

The following proposition is known in the literature on homometric sets (see Rosenblatt and Seymour [RS82]). It provides a method to construct pairs of homometric sets in any dimension. In some cases the obtained sets are polyominoes.

**Proposition 4.2.** Let $A$ and $B$ be subsets of $\mathbb{Z}^n$. Assume that each point of $A + B$ (and of $A - B$) can be written in an unique way as sum of a point of $A$ and of a point of $B$ (of $-B$, respectively). Then $A + B$ and $A - B$ are homometric sets.

The example provided in [GGZ05] can be obtained using this construction. The pair of homometric polyominoes in Fig. 4 can be written as $A + B$ and $A - B$, where $A$ and $B$ are the finite sets in Fig. 5. Consider now the two sets of three points, $L$ and $2L$, in Fig. 6 and the two sets $2L + L$ and $2L - L$. These two sets are homometric polyominoes made of nine points. The corresponding animals are non-congruent.

**Figure 5** The polyominoes in Fig. 4 are equal to $A + B$ and $A - B$.

**Figure 6** Two non-congruent homometric polyominoes made of 9 points, and the associated animals.

Another pair of animals made of nine squares which are not translations or reflections (with respect to a point) of each other is presented in [DGN05, Fig. 1]. The corresponding polyominoes are convex and one animal is the reflection of the other with respect to a line.

**Proof of Theorem 1.1** Let us consider two polyominoes $A, B \subset \mathbb{Z}^2$ and the co-varioograms of $\bar{A} = A + [0, 1]^2$ and $\bar{B} = B + [0, 1]^2$. Obviously, $g_A = g_B$ implies $\bar{D}A = \bar{D}B$ as $\bar{D}A = \text{supp} g_A$ and $\bar{D}B = \text{supp} g_B$. Thus, the widths of $A$ and $\bar{B}$
in the coordinate directions are equal. This implies that the minimum rectangle with edges parallel to the coordinate axes containing \( A \) has to be equal to the one containing \( B \).

Let us denote by \( dP(h \times b) \) the class of \( d \)-polyominoes (polyominoes consisting of \( d \) points) \( A \) such that the minimal rectangular container of \( A + [0,1]^2 \) has height \( h \) and basis \( b \). Let us remark that commonly polyominoes are classified up to all the symmetries with respect to the coordinates axes. Here, however, we will classify polyominoes up to translations and reflections in a point, i.e. we identify two polyominoes in \( dP(h \times b) \) if they are reflections or translations of each other.

We consider now the \( d \)-polyominoes, for each \( d = 1, \ldots, 8 \). It suffices to consider in the proof only polyominoes in \( dP(h \times b) \), with \( h \leq b \). Indeed, the polyominoes in \( dP(h \times h) \) are obtained from those in \( dP(h \times b) \) by a rotation of \( \pi/2 \), and, moreover, a polyomino in \( dP(h \times b) \) cannot have the same covariogram of one in \( dP(h \times h) \), unless \( h = b \), for the reason explained above.

The case \( d = 1 \) and \( d = 2 \) are trivial because there exist only one 1-polyomino and only one 2-polyomino that belongs to \( 2P(1 \times 2) \).

The class \( 3P(1 \times 3) \) contains one element, while \( 3P(2 \times 2) \) contains two elements. The two polyominoes in \( 3P(2 \times 2) \) cannot have the same covariogram as their difference bodies are not equal.

For \( d = 4 \) the only class \( 4P(h \times b) \) with more than one element is \( 4P(2 \times 3) \). The five sets in \( 4P(2 \times 3) \) have different difference bodies.

For \( d = 5 \) there are six elements in \( 5P(2 \times 4) \), three elements in \( 5P(2 \times 3) \) and twelve elements in \( 5P(3 \times 3) \). None of these sets has difference body equal to that of another set in the same class.

The elements in \( 6P(h \times b) \), in \( 7P(h \times b) \) and in \( 8P(h \times b) \) have been analysed using the simple algorithm described in the appendix. In the case of \( 6 \)-polyominoes, \( 7 \)-polyominoes and \( 8 \)-polyominoes the algorithm stops without finding a pair of homometric polyominoes.

\[ \square \]

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5. APPENDIX

The diagram in Fig. 7 describes the algorithm used in the proof of Theorem 1.4. We briefly explain what each subprogram does.

**Generate sets:** this function generates all possible sets of eight (respectively seven, six) points of a grid of $\mathbb{Z}^2$ with at most four (respectively four, three) rows and eight (respectively seven, six) columns.

**Check if the interior is connected:** this function chooses a point $x_1$ of the selected set and constructs the component containing the point. Successively it establishes if this component coincides with the whole set. It works with two lists of points. At the beginning the first list $L_1$ contains only $x_1$, whereas the second list $L_2$ contains all the other points of the set. Among the points in $L_2$, the program transfers in $L_1$ those whose distance from $x_1$ is unitary. Successively, the program considers the second point in $L_1$ and repeats the process. The algorithm stops when it has considered the last point in $L_1$. The set is connected if at the end $L_2$ is empty.

**Check translations or reflections:** this function computes the vector differences of each point of the first set $P_i$ with the corresponding (in the lexicographic order) point of the other set, $P_j$. If all these differences are equal then the two sets are translations of each other. If some of these differences are not equal, then the function computes the vector differences of each point of the first set with the corresponding (in the lexicographic order) point of the second set, previously reflected and ordered. If all these differences are equal, the two sets are reflections of each other. Otherwise $P_i$ and $P_j$ are not one translations or reflections of each other.

**Create and compare the two sets of vector differences:** this function generates for the pair $(P_i, P_j)$ the vector differences sets $DP_i$ and $DP_j$. Successively, it orders $DP_i$ and $DP_j$ according to the lexicographic order and compute the vector differences of each point of $DP_i$ with the corresponding point of $DP_j$. If all these vectors are equal to the null vector, then $P_i$ and $P_j$ are homometric. Otherwise they are not homometric.
Figure 7 The algorithm used in the proof of Theorem 1.4.