Fractional Topological Phases in Generalized Hofstadter Bands with Arbitrary Chern Numbers

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We examine similarities and differences between topological flat bands with Chern numbers $C > 1$ and conventional quantum Hall multi-layers. By constructing generalized Hofstadter models that possess “color-entangled” flat bands, we provide an intuitive understanding of certain puzzling properties of $C > 1$ flat bands, which can effectively be mapped either to a single-layer or to a multi-layer model depending on the lattice configuration. We identify interacting systems in which the ground state degeneracy depends on whether the system consists of an even or odd number of unit cells along one particular direction, and discuss the relation between these observations and the previously proposed “topological nematic states.” Our study also provides a systematic way of stabilizing various fractional topological states in $C > 1$ flat bands.

Introduction – The topological origin of the integer quantum Hall (IQH) effect [1] was clarified by Thouless et. al. [2], who showed that the Hall conductance is proportional to the Chern number $C$ defined as the integral of Berry curvature over the Brillouin zone (BZ) [3]. Subsequently, Haldane constructed a two-band honeycomb lattice model in which bands with $C = ±1$ are realized in zero net magnetic field [3]; such systems are now termed “Chern insulators.” Recently, intense effort has been devoted to the study of Chern insulators with topological flat bands [5–7] and the fractional quantum Hall (FQH) states supported by them [8–26]. It has been demonstrated that the flat bands with $C = 1$ are adiabatically connected to the lowest Landau level, as also are the FQH states in these bands [22–24], which provides a simple way to understand various fractional topological states in such systems.

One difference between topological flat bands and Landau levels is that flat bands can have arbitrary Chern numbers [27–31], in contrast to Landau levels which have $C = 1$ each. It is natural to ask what are the similarities and differences between FQH states in $C > 1$ flat bands and conventional Landau levels. Using the “color-entangled Bloch basis” [32] for multi-component Landau levels (where color labels the components), the incompressible ground states at filling factor $ν = 1/(C + 1)$ [$ν = 1/(2C + 1)$] for bosons (fermions) in a $C > 1$ flat band can be interpreted as color-dependent flux inserted Halperin states [33] in some cases, but the nature of other states remains unclear. It has been argued in Ref. [34] that some bilayer FQH states realized in $C = 2$ systems have special properties: they are anisotropic; the ground state degeneracy (GSD) depends on whether the system contains an even or an odd number of unit cells along one particular direction; and lattice dislocations have projective non-Abelian braid statistics [35]. In this Letter, we provide a simple physical picture to understand these puzzling properties of $C > 1$ flat bands. In particular, we show that depending on the system size, a $C > 1$ flat band can be mapped to a single-layer or a multi-layer quantum Hall system. This naturally explains the dependence of the GSD on the parity of the number of unit cells along one direction. We confirm this understanding by explicit numerical calculations.

Color-Entangled Hofstadter Models – We construct flat bands with arbitrary Chern numbers by generalizing the Hofstadter model [36–40] following the generic scheme of Ref. [29]. One important advantage of our model is that the Berry curvature of the lowest band can be made uniform over the entire BZ. This is important because, as shown in Ref. [22], a nonuniformity in the Berry curvature usually tends to weaken or even destroy incompressible states. For the Hofstadter model defined on a square lattice, each magnetic unit cell contains $n_φ$ sites when the flux per plaquette is $2π/n_φ$ with $n_φ$ being an integer. The momentum space Hamiltonian is $H(k) = Ψ(Ψ(k))$, where $Ψ(Ψ(k)) = [a^0(0), a^1(0), \ldots, a^{n_φ−1}(0)]$ and the subscript of $a^i$ marks different sites within a magnetic unit cell. $H(k)$ is a $n_φ × n_φ$ matrix whose non-zero matrix elements are $H_{mn}(k) = 2\cos(k_y + 2mπ/n_φ)$ and $H_{mn}(k) = H_{nm}(k) = \exp(ik_x/n_φ)$ for $n = (m + 1) mod n_φ$. The $n_φ → ∞$ limit recovers the continuous limit in which Landau levels arise.

We next stack two identical Hofstadter lattices together. In Fig. 1(a), the same orbitals in different layers are aligned together, which results in a Hofstadter bilayer that is essentially the same as a bilayer quantum Hall system. In Fig. 1(b), the $m$-th orbital in one layer is aligned with the $(m + n_φ/2)$-th orbital in the other layer. As pointed out in Ref. [29], this stacking pattern reduces the size of the magnetic unit cell by half. Instead of having two degenerate $C = 1$ lowest bands, the model shown in Fig. 1(b) possess a single lowest band with $C = 2$. We emphasize that these two systems are equivalent insofar as the behavior of the bulk is concerned, since they correspond to two different gauge choices. However, as shown below, they can behave differently when periodic boundary conditions (PBCs) are imposed. This difference has
FIG. 1. (color online) In both panels, two Hofstadter layers are displayed on the left where the indices of orbitals in a magnetic unit cell are shown in parentheses and the numbers on the bonds indicate the phases associated with the complex hopping matrix element along the $y$ direction in units of $\pi$. In panel (a), a Hofstadter bilayer is obtained by stacking the two layers together. In panel (b), the two Hofstadter layers are shifted relative to each other and then stacked together. The latter gives a color-entangled Hofstadter model with $C = 2$ in which the size of the magnetic unit cell is reduced by a factor of two. In both cases, there are two orbitals on each lattice site (colored in red and blue) in the resulting models on the right and their indices are given in parentheses.

a direct impact on the topology and symmetry of the system, which is a key to understanding the physics of $C > 1$ bands.

In general, we can construct a band with an arbitrary Chern number $C$ by stacking $C$ layers of Hofstadter lattices and aligning the orbitals labeled by $m$, $m + n_{\phi}/C$, $\cdots$, and $m + (C - 1)n_{\phi}/C$ in different layers (where $m \in [0, 1, \cdots, n_{\phi}/C - 1]$). The momentum space Hamiltonian is very similar to the single layer Hofstadter model discussed above, except that the off-diagonal term $H_{mn}(k)$ is replaced by $\exp(i k_x C / n_{\phi})$. In Appendix A, we show the similarity between these models and the color-entangled Bloch basis [32]. This suggests that our models, as well as those constructed in Ref. [29], can be referred to as “color-entangled” topological flat band models.

**Boundary Conditions, Topology and Symmetry** – We use the example shown Fig. 1 to explain the difference between the simple bilayer model and the color-entangled one. We impose PBCs and denote the number of magnetic unit cells along the $x$ and $y$ directions as $N_x$ and $N_y$, respectively. The $C = 2$ model has two fundamentally distinct topologies determined by the parity of $N_x$: for odd $N_x$ [Fig. 2 (a)], it can be unfolded to produce a single Hofstadter layer with $C = 1$ by tracking the black lines which represent hoppings along the $x$ direction; for even $N_x$ [Fig. 2 (b)], it contains two decoupled Hofstadter layers each having $C = 1$. The underlying reason for the parity dependence is that the hopping from one magnetic unit cell to the next also alters the color index.

To study interacting many particle states, we need to understand the behavior of 2-body interaction terms when the system is unfolded. The six terms shown in Fig. 2 (c) provide such information. For both even and odd $N_x$, (1) is still an onsite term and both (3) and (6) turn out to be intra-layer nearest neighbor (NN) terms. If $N_x$ is even, (2), (4) and (5) become, respectively, an inter-layer onsite term, an inter-layer NN term, and an inter-layer NN term. On the other hand, when $N_x$ is odd, (2), (4) and (5) all result in interactions, in the single unfolded layer, that extend over a range comparable to the system size. In Appendix B, we discuss the nature of the gapped ground states observed in previous works [27,31] in light of these observations.

The fact that the unfolding of the model depends only on the parity of $N_x$ but not of $N_y$ signifies a reduction of rotational symmetry. To gain insight into this issue, we note that the simple Hofstadter model on a square lattice has four-fold rotational symmetry $C_4$ (up to gauge transformations) even though the magnetic unit cell usually has only two-fold rotational symmetry $C_2$. This conclusion is valid when the system contains an integral number of magnetic unit cells, which is automatically satisfied for a Hofstadter multi-layer with PBCs. In contrast, since the unit cell of the color-entangled $C = 2$ Hofstadter model is half as large as the original magnetic unit cell, the $C_4$ symmetry of the parent Hofstadter model is inherited only when $N_x$ is even, but is reduced to $C_2$ symmetry when $N_x$ is odd.
Since the system can have two distinct structures depending on the parity of $N_x$, one may expect that the GSD of some fractional topological states would also depend on $N_x$. These states can thus be considered a realization of the “topological nematic states” proposed in Ref. 34. However, it should be emphasized that in our systems the $C_4$ symmetry is not broken spontaneously, as in previously studied nematic states 41–42, but results from the model Hamiltonian itself through boundary conditions. For bosons, we seek realization of the $C$-layer wave function

$$\Psi(\{z_1\}, \ldots, \{z^C\}) = \prod_{\alpha=1}^{C} \Phi_{\alpha}^\dagger(\{z^\alpha\})$$  \hspace{1cm} (1)$$

with $p = 1$ or $2$, where $z = x + iy$ is the complex coordinate and its superscript indicates the layer it describes, $\Phi_{1/2}(\{z^\alpha\})$ is the Laughlin $1/2$ state and $\Phi_{2/3}(\{z^\alpha\})$ is the Jain $2/3$ state 45 (they are the bosonic analogs of the Laughlin $1/3$ state 40 and the Jain $2/5$ state 47 for fermions). For fermions, we consider the Halperin $331$ state 33

$$\Psi(\{z_1, z^2\}) = \prod_{i<j} (z_i^1 - z_j^1) \overline{\Phi}_{\alpha} \prod_{i,j} (z_i^1 - z_j^2)$$  \hspace{1cm} (2)$$

In conventional quantum Hall multi-layers, these states can be realized when intra-layer interaction is stronger than inter-layer interaction, which motivates our Hamiltonians given below. One expects the following GSD of the bosonic states: when $N_x$ is not a multiple of $3$, a single-layer $p/(p+1)$ state is realized in $\phi$ and it has GSD $p+1$; if the system is mapped to $C$ layers, we have $C$ decoupled $p/(p+1)$ state which gives GSD $(p+1)^C$. The fermionic case is more complicated, but it has been shown in Ref. 34 that the GSD of the Halperin $331$ state in a $C = 2$ band is $8$ when $N_x$ is even and $4$ if $N_x$ is odd.

**Exact Diagonalization –** The consequence of lattice topology can be seen explicitly in exact diagonalization. We denote the number of particles by $N$, and the total number of plaquettes (to be distinguished with the numbers of unit cells) in the $x$ and $y$ directions by $L_x$ and $L_y$. For simplicity, we choose the magnetic unit cell such that it contains only one plaquette in the $y$ direction, so $L_y$ is always equal to $N_y$. For bosons, we consider on-site interactions $H_B = \sum_{\sigma} \sum_{\tau} U_{\sigma \tau} : n_i(\sigma) n_i(\tau) :$, where $\cdot \cdot \cdot$ denotes normal ordering and $n_i(\sigma)$ is the number operator for particle of color $\sigma$ on site $i$. Here we choose $U_{\sigma \tau} = 1$ and $U_{\sigma \tau} = 0.03$ for $\sigma \neq \tau$. For fermions, we consider intra-color NN and inter-color onsite interactions through the Hamiltonian: $H_F = \sum_{\langle ij \rangle} \sum_{\sigma} U_{\sigma} : n_i(\sigma) n_j(\sigma) : + \sum_{\langle i \rangle} \sum_{\tau \neq \sigma} V_{\sigma \tau} : n_i(\sigma) n_i(\tau) :, \langle \cdot \rangle$ denotes nearest neighbors with $U_L = 1$ and $V_{\sigma \tau} = 0.5$. The many-body Hamiltonian is projected into the partially occupied lowest band 10, which is analogous to the lowest Landau level projection commonly used in theoretical studies of FQH states. Because of the translational symmetry, the many-body eigenstates can be labeled by total momenta $K_x$ and $K_y$ along the $x$ and $y$ directions.

The number of plaquettes in the $x$ direction for given $N_x$ and $N_y$ values is chosen to ensure that this system is close to isotropic ($i.e.$ has aspect ratio close to 1). As the size of the unit cell increases, the wave function of a particle spreads over a larger area and the interaction between two particles becomes weaker. To compare systems with different $N_\phi$, we normalize the energy scale using the total energy of two particles in a system with $N_x = 1$ and $N_y = 1$, and $L_x = 12$; (c) $N = 12$, $N_x = 1$, $N_y = 18$, and $L_x = 12$; (d) $N = 12$, $N_x = 2$, $N_y = 9$, and $L_x = 12$. The numbers above some energy levels indicate degeneracies that may not be resolved by inspection.

**Numerical Results –** The state represented by Eq. 1 is realized in $C = 2$ and $3$ topological flat bands at filling factor $\nu = p/(p+1)$. The filling factor for lattice models is defined as $\nu = N/(N_x N_y)$, to be contrasted with the conventional multi-layer systems where it is defined as the ratio of the total number of particles and the number of orbitals in a single layer.] Energy spectra of bosons on the $C = 2$ and $C = 3$ models are presented in Figs. 3 and 4. For $C = 2$, the GSD at $1/2$ is 2 if $N_x$ is odd and $2^2 = 4$ if $N_x$ is even; the GSD at $2/3$ is 3 if $N_x$ is odd and $3^2 = 9$ if $N_x$ is even. For $C = 3$, the GSD at $1/2$ is $2^3 = 8$ if $N_x$ is a multiple of 3 and 2 otherwise; the GSD at $2/3$ is $3^3 = 27$ if $N_x$ is a multiple of 3 and 3 otherwise. These results clearly support our analysis above in terms of mapping into a single- or multi-layer system. We have further confirmed that, when the $C = 2$ and $3$ models are mapped to a single layer, their energy spectra and particle entangle-

FIG. 3. Energy spectra of bosons on the $C = 2$ model at filling factors $1/2$ [(a) and (b)] and $2/3$ [(c) and (d)]. The parameters are as follows: (a) $N = 10$, $N_x = 1$, $N_y = 20$, and $L_x = 15$; (b) $N = 10$, $N_x = 2$, $N_y = 10$, and $L_x = 12$; (c) $N = 12$, $N_x = 1$, $N_y = 18$, and $L_x = 12$; (d) $N = 12$, $N_x = 2$, $N_y = 9$, and $L_x = 12$. The numbers above some energy levels indicate degeneracies that may not be resolved by inspection.
The Halperin 331 state corresponds to filling factor 1/4 in a $C = 2$ system. We present in Fig. 5 energy spectra of fermions on the $C = 2$ model at 1/4. The GSD is 4 for $N_x = 1$ and 8 for $N_x = 2$, which is consistent with the expectation for the 331 state. If $N_x$ is further increased, however, the gap in the energy spectrum becomes less clear and we cannot identify 8 quasi-degenerate ground states when $N = 8$, $N_x = 4$, $N_y = 8$, and $L_x = 16$. This is due to the fact that, when the model is mapped to a bilayer system for even $N_x$, the intra-color NN terms across boundaries between unit cells turn into inter-layer NN terms, which would weaken or destroy the 331 state. One can choose less natural Hamiltonians that contain no inter-layer NN terms after mapping into a bilayer system to produce 331 state with a clear gap [23].

We finally note that the simple Hofstadter model has recently been realized for $^{87}$Rb [13, 44]. It is possible that a practical method to realize the color-entangled Hofstadter models with $C > 1$ can be designed along the same line. The tunability of interaction in ultracold atomic systems [51, 52] is likely to prove essential for stabilizing the various states discussed above.

In conclusion, we have constructed color-entangled Hofstadter models with arbitrary Chern numbers, which clarify many aspects of topological flat bands with $C > 1$ in a physically intuitive manner. We have also demonstrated the existence of exotic fractional topological phases in such systems by extensive exact diagonalization studies.

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Appendix A. Hofstadter Model and Bloch Basis

The color-entangled Bloch basis was introduced in Ref. [32] for a C-component Landau level to understand lattice models with Chern number C. We now demonstrate that this basis is closely related to the Hofstadter models considered in the main text. The moment space single-particle Hamiltonian of the four band C = 2 model shown in Fig. 1 is

\[
H(k) = \Psi(k) \begin{pmatrix}
2 \cos(k_y) & e^{ik_z/2} \\
e^{-ik_z/2} & 2 \cos(k_y + \pi/2) \\
0 & e^{-ik_z/2} \\
0 & 2 \cos(k_y + \pi)
\end{pmatrix} \Psi(k)
\]

(S1)

for bosons and the Hamiltonian

\[
\tilde{H}_F = \sum_{\langle ij \rangle} \sum_{\sigma : n_i(\sigma)n_j(\sigma)} + \sum_i \sum_{\sigma \neq \tau} : n_i(\sigma)n_i(\tau) :
\]

(S4)

for fermions. The energy spectra as well as particle entanglement spectra match the results obtained using color-entangled Bloch basis.

Based on our analysis in the main text and the previous section, the C-color Hofstadter model or Bloch basis can be mapped to a single layer if N_x is not a multiple of C, but the nature of the gapped states here is unclear. As shown in Fig. 2, some local interaction terms in Eq. (S2) and Eq. (S4) induce special long-range correlations when the system is unfolded to a single layer, but their exact forms in the continuum are not known without analytical calculations. To test this interpretation more explicitly, we have tested many different Hamiltonians for particles in a one-component Landau level on torus and found that some choices of system-dependent unnatural long-range interactions (in addition to short-range ones) indeed produce gapped ground states at filling factor 1/3 (1/5) for bosons (fermions).

Appendix B. Ground States at \( \nu = 1/(C + 1) \) and \( \nu = 1/(2C + 1) \)

For certain color-independent Hamiltonians and the C-color Bloch basis, zero energy ground state occur at filling factor \( \nu = 1/(C + 1) \) \( \nu = 1/(2C + 1) \) for bosons (fermions) [32]. These states correspond to color-dependent flux inserted version of the Halperin states [32,33] when N_x is a multiple of C (i.e. in the cases where the Bloch basis can be mapped to a multi-layer system). We have confirmed that these FQH states also appear in our color-entangled Hofstadter models with Chern number C by using the Hamiltonian

\[
\tilde{H}_B = \sum_i \sum_{\sigma \tau} : n_i(\sigma)n_i(\tau) :
\]

(S3)

for bosons on the \( \nu = 1 \) model and the \( \nu = 1/2 \) model. For instance, we can also consider the \( \nu = 1 \) model with the Hamiltonian

\[
\tilde{H}_B = \sum_i \sum_{\sigma \tau} : n_i(\sigma)n_i(\tau) :
\]

(S3)

for bosons on the \( \nu = 1 \) model and the \( \nu = 1/2 \) model. For instance, we can also consider the \( \nu = 1 \) model with the Hamiltonian
ness of the eigenvalues, we scale them by a factor of 10 in Fig. S1). The GSD in both Fig. S1(a) and (c) are seen to be 3; this is consistent with the GSD of bosonic Pfaffian state in a $C = 1$ system, as expected from the mapping into a single layer. For Figs. S1(b) and (d), the GSD is $3^C$, indicating that the system consists of $C$ decoupled Pfaffian states.

Here we generalize our considerations to the square lattice $C = 2$ model [29], which can be obtained by stacking two checkerboard lattices together and shift them relative to each other along the $a_x$ direction defined in Fig. S2. The checkerboard lattice model [6] contains two orbitals per unit cell and there are NN, next NN and second next NN hopping terms. The NN hopping terms connect the two types of orbitals and this brings out certain additional subtleties. For the usual choice of the primitive translation vectors $a_x$ and $a_y$ (Fig. S2), the hoppings along both these directions are associated with changes of color index. A system is mapped to a single layer when one of $N_x$ and $N_y$ is odd, and two decoupled layers when both are even. Insight into the physics of this model is given by choosing instead $a_x$ and $\tilde{a}_y$ to define the unit cell. In this case, the color index of a particle does not change during hopping along the $\tilde{a}_y$ direction, and a system may be mapped to a single layer or two layers depending only on the parity of $N_x$. The momentum space single-particle Hamiltonian in this case is

$$H_S = 2t_3 \left[ \cos(2k_x) + \cos(2k_y - 2k_x) \right] \mathbb{I} + \sqrt{2}t_1 \left[ \cos(k_x) + \cos(k_y - k_x) \right] \sigma_x - \sqrt{2}t_1 \left[ \cos(k_x) - \cos(k_y - k_x) \right] \sigma_y - 4t_2 \sin(k_x) \sin(k_y - k_x)\sigma_z$$

where $t_1 = 1, t_2 = 1/(2 + \sqrt{2})$ and $t_3 = 1/(2\sqrt{2} + 2)$. $\mathbb{I}$ is the identity matrix and $\sigma_x,\sigma_y,\sigma_z$ are the Pauli matrices. Note that this is different from that given in Ref. [29], because we are using different lattice translation vectors. We use 2-body onsite interaction given by

$$H_S = \sum_i :n_i(A)n_i(A) + n_i(B)n_i(B) + 0.06n_i(A)n_i(B):$$

(the small interaction between $A$ and $B$ is used to split some degeneracies and increase the speed of exact diagonalization). The energy spectra of bosons on this model are shown in Fig. S3. We see that the GSD is 2 when $N_x = 5$ and 4 when $N_x = 4$, which can be understood along the same lines as for the models discussed in the main text.