Hidden vortices in a Bose-Einstein condensate in a rotating double-well potential

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(Dated: August 18, 2010)

We study vortex formation in a Bose-Einstein condensate in a rotating double-well potential. Besides the ordinary quantized vortices and elusive ghost vortices, “hidden” vortices are found distributing along the central barrier. These hidden vortices are invisible like ghost vortex but carry angular momentum. Moreover, their core size is not given by the healing length, but is strongly influenced by the external potential. We find that the Feynman’s rule can be well satisfied only after including the hidden vortices. There is no critical rotating frequency for the formation of hidden vortex while there is one for the formation of ordinary visible vortices. Hidden vortices can be revealed in the free expansion of the Bose-Einstein condensates. In addition, the hidden vortices in a Bose-Einstein condensate can appear in other external potentials, such as a rotating anisotropic toroidal trap.

PACS numbers: 03.75.Lm, 03.75.Kk, 67.85.De

I. INTRODUCTION

Double-well (DW) potential is an important model potential for its simplicity and yet richness in physics. The properties of Bose-Einstein condensates (BECs) in a DW potential have been studied extensively since the realization of BECs [1]. With the advances of technology, DW potentials for ultracold atoms can now be realized in experiments with great controllability and precision [2–4]. One particular interesting development is the possibility to rotate a DW potential via radiofrequency dressing [5]. This opens the door to study the behavior of vortices of degenerate quantum gas in a rotating DW potential. As topological defect, quantized vortices have contributed greatly to reveal the phase coherence, superfluidity and nonlinear phenomena for degenerate quantum gases, and have been the subject of extensive experimental and theoretical studies [6–22].

The DW potential also offers a unique testing ground for the well-known Feynman’s rule of vortex [23]. The Feynman’s rule is very powerful relation that links the total number of vortices with rotation angular frequency. Feynman argued that for a rotating superfluid with angular frequency Ω, the superfluid should be regarded as a classical fluid when it reaches the steady state. This leads to an important mathematical relation that the total number of vortices \( N_v \) in an area \( A \) is linearly proportional to \( Ω, \frac{2\pi \hbar N_v}{m} = 2\Omega A \). Alternatively, the Feynman’s rule can expressed as \( l_z/\hbar = N_v/2 \) with \( l_z \) being the mean angular momentum per atom at equilibrium [14, 15, 23, 24]. This rule was originally for a uniform superfluid helium, and has been intensively studied both theoretically [24, 25] and experimentally [6, 10] for a BEC trapped in single harmonic potential. It is interesting to know how the rule fares in a more complicated geometric confinement. The DW potential provides a clear opportunity to answer this question.

In this paper we have conducted a comprehensive two-dimensional (2D) numerical study of vortex formation in a BEC in a rotating DW potential. We find surprisingly that the in situ density distribution seems to violate significantly the Feynman’s rule in that the total number of vortices with visible core is significantly smaller than \( 2l_z/\hbar \). With the belief that the Feynman’s rule should hold in some form, we carefully analyze the results and find these “missing” vortices are distributed along the central barrier of the DW potential. Unlike the usual vortex, these vortices have no visible cores but have phase singularities, and their core size is not given by the healing length but is strongly influenced by the external potential. For this reason, they may be called “hidden vortices”. With the inclusion of the hidden vortices, one recovers the Feynman’s rule.

These hidden vortices remind us of ghost vortices found numerically in Ref. [24]. The ghost vortices always lie at the outskirts of the condensate where the particle density \( |\psi|^2 \) is very small. As a result, similar to hidden vortex, they show up in numerical results as phase singularities but have no visible cores. However, there are key differences: the ghost vortex carries no angular momentum while hidden vortex does; the core size of ghost vortex is determined by healing length like the usual visible vortex while that of hidden vortex by the shape of external potential. In addition, we find that with the increasing of rotation frequency \( \Omega \) the hidden vortices appear first in the DW system, followed by ghost vortices and usual visible vortices. Furthermore, the angular momentum can be put gradually into the BEC via the generation of the hidden vortices while the emergence of a visible vortex is still accompanied by a jump in the system angular momentum. Although the hidden vortices are invisible in the in situ density distribution, after free expansion...
of the BEC, they can appear in the density distribution because of their stable topological structure.

It is well known that there exists a special type of vortex called Josephson vortex (or fluxon) in a long superconducting Josephson junction [26] or between two weakly-coupled BECs [27, 28]. These Josephson vortices can be regarded as hidden vortices. However, hidden vortex is a more general notion than the Josephson vortex (fluxon) as hidden vortex can exist in non-DW potential (or non-Josephson structure). We have used rotating anisotropic toroidal trap to illustrate this point.

This paper is organized as follows. In Sec. II, we present a phenomenological model to describe the dynamics of a BEC confined in a rotating DW potential in the presence of dissipation. Hidden vortices are found in the rotating DW BEC, where the Feynman’s rule is satisfied only after including these hidden vortices. A simple and feasible scheme is proposed to observe the hidden vortices. In Sec. III, we study the vortex formation process and the critical rotating frequency in the rotating DW BEC. In Sec. IV, we discuss the hidden vortices in a BEC confined in a rotating anisotropic toroidal trap. Sec. V provides a summary and discussion.

II. HIDDEN VORTICES IN A ROTATING DW BEC AND FEYNMAN’S RULE

We consider the situation where the condensate is tightly confined in the axial direction ($\omega_z, \omega_y \ll \omega_z$) so that the system is effectively two dimensional. The DW potential is described by

$$V_{dw}(x, y) = \frac{x^2 + \lambda y^2}{4} + V_0 e^{-x^2/2\sigma^2},$$

(1)

where $V_0$ and $\sigma$ are the height and width of the potential barrier, respectively, and $\lambda = \omega_y/\omega_z$ denotes the anisotropy parameter of the harmonic trap. In the presence of dissipation, the order parameter in the frame rotating with the angular velocity $\Omega$ around the $z$ axis obeys the time-dependent Gross-Pitaevskii (GP) equation

$$(i-\gamma) \frac{\partial \psi}{\partial t} = \left[ -\left( \nabla_x^2 + \nabla_y^2 \right) + V_{dw} + c |\psi|^2 - \Omega L_z \right] \psi.$$  (2)

Here $L_z = i(\gamma \partial_x - x \partial_y)$ is the $z$ component of the angular-momentum operator, $\gamma$ characterizes the degree of dissipation, and $c$ is the 2D interatomic interaction strength. In this work, length, time, energy, angular momentum, and rotation angular frequency are in units of $d_0 = \sqrt{\hbar/2m\omega_z}$, $1/\omega_z$, $\hbar\omega_z$, $\hbar$, and $\omega_z$, respectively. The phenomenological dissipation model [2] is a variation of that in Ref. [24]. For the case of a BEC in a rotating harmonic trap, our computation results agree well with the experimental observations in Ref. [5] and the simulation results in Ref. [24].

In our calculations, we first obtain the initial ground-state order parameter in the DW potential by imaginary time propagation method [29, 31] for $\Omega = 0$. The vortex formation process is then studied by solving numerically Eq. (2) with different $\Omega$. Here we consider the BECs of $^{87}\text{Rb}$ atoms with repulsive interaction. The system parameters are chosen to be $\omega_z = \omega_y = 2\pi \times 40$ Hz, $\omega_z = 2\pi \times 800$ Hz, $V_0 = 40$, $\sigma = 0.5$, $c = 600$. In Eq. (2), the variation of $\gamma$ only influences the relaxation time scale but does not change the dynamics of vortex formation and the ultimate steady vortex structure. In our computation, we choose $\gamma = 0.03$, which corresponds to a temperature of about $0.1T_c$ [22].

Fig. 1(a) shows the steady density distribution $|\psi|^2$ at $t = 250$ for the DW potential rotating with $\Omega = 0.9$. From this in situ density distribution, we see a pair of ordinary vortex lattices with triangular structure as expected from the rotating DW configuration. However, by simply counting, we find something surprising. In Fig. 1(a), the total number of vortices is $N_v = 18$. Numerical results show that the mean angular momentum per atom $l_z = \int \int \psi^* L_z \psi dxdy/\int \int |\psi|^2 dxdy$ is about $l_z \approx 16 \gg N_v/2$. It seems that the Feynman’s rule [14, 15, 23, 24] is no longer satisfied. For other rotation frequencies, this seemingly significant violation of Feynman’s rule is also found. With the belief that the Feynman’s rule should always hold, this violation indicates that some angular momentum is missing and is not manifested in the form of ordinary vortices.

To find the missing angular momentum, we look into the phase distribution of $\psi(x, y, t = 250)$, which is plotted in Fig. 1(b). We find that besides the phase singularities corresponding to the above mentioned vortices, there are other phase defects, distributed along the central barrier and the outskirts of the cloud. The initial reaction is that these phase defects, which are invisible in the in situ density distribution, are ghost vortices discussed in Ref. [24]. Since ghost vortices are known not to carry angular momentum, it seems that these invisible phase defects can not account for the missing angular momentum. However, a more careful examination shows otherwise.

The phase singularities along the central barrier are not ghost vortices and they carry angular momentum. To see this, we assume $N_b$ is the total number of phase singularities along the central barrier, and $N_l = (N_v + N_b)$ is the sum of $N_v$ and $N_b$. From Fig. 1(b), we have $N_b = 16$. If we include them, the Feynman’s rule $l_z \simeq N_l/2 = (N_v + N_b)/2$ is well satisfied, indicating that these phase singularities do carry angular momentum like the usual visible vortices. We have checked other rotating frequencies and that the Feynman’s rule can always be well satisfied by including the phase defects along the central barrier. In Fig. 1(c), we have plotted the dependence of $l_z$ on $N_l/2$ for different rotation angular frequencies. The solid line is the Feynman’s rule $l_z = N_l/2$ while the solid circles are the numerical results of $l_z$ at equilibrium. The excellent agreement between them strongly support that phase singularities along the central barrier carry angular momentum, and thus are not ghost vor-
moments. With the local density approximation, our numerical calculations do show that the hidden vortices carry significant angular momentum while the angular momentum due to a ghost vortex can be negligible.

Even though the hidden vortices are invisible in the in situ density distribution as shown in Fig. 1(a), we find numerically that they show up in the cloud after free expansion (see the following discussion). This makes it possible to observe and test the existence of these hidden vortices experimentally.

We use the state shown in Figs. 1(a)-1(b) as an example. After a short expansion time, the state begins to look very differently. In Fig. 2, the density distribution and phase distribution at the expansion time $\tau = 4$ is plotted. We see clearly in Figs. 2(a)-2(b) that, besides eighteen vortices already shown in the in situ density distribution [see Fig. 1(a)], a series of new ordinary vortices appear along the symmetric axis of two condensates. These new visible vortices originate physically from the hidden vortices. It is not difficult to understand the revelation of the hidden vortices during the free expansion. The core of a vortex (hidden, ghost, or visible) is also a velocity singularity, where the velocity approaches infinity. Because the kinetic energy should be finite, during the free expansion where the angular momentum is conserved, no atoms will be allowed into the core area. As a result, the core is stable and will not be destroyed. At the same time, as the two BECs begin to overlap, atoms begin to move into the central region and fill the space between hidden vortex cores, rendering them visible. No hidden vortices can be revealed during free expansion. As shown in Fig. 2(c), there still exist several remnant hidden vortices along the boundary between the two flows [see Fig. 2(c)], which is due to the strong repulsion and pushing of the newly formed visible vortices during the expansion. Note that in our simulation, we did not find the generation of the vortex-antivortex pairs predicted in the numerical study of the interference of sliced BECs without rotation [33].

Because the ghost vortices lie on the outskirts, the locations of their phase defects move outward during the free expansion. Consequently, the density in the regime of the ghost vortices is always negligible. This is the reason that the ghost vortices will not become visible vortices during the expansion.

III. VORTEX FORMATION PROCESS AND CRITICAL ROTATING FREQUENCY FOR A ROTATING DW BEC

The vortex formation process with the rotating DW potential is drastically different from the one with a single harmonic potential. There is a critical angular frequency for a rotating single-well potential to create vortices: when the angular frequency $\Omega$ is below the critical angular frequency, only ghost vortices are formed at the outskirts of the BEC cloud and the cloud does
FIG. 2: (color online) (a) Density distribution, (b) density contour, and (c) phase distribution after the cloud free expands for \( \tau = 4 \). Before the free expansion, the system has rotated with \( \Omega = 0.9 \) for \( t = 250 \). The darker color area or contour indicate the lower density or phase.

not carry any significant angular momentum; when \( \Omega \) is larger than the critical frequency, visible vortices begin to appear along with a jump in the angular momentum \([24]\). For the rotating DW system, the vortex formation starts with a pair of hidden vortices. As seen in Figs. 3(a)-3(e), the hidden vortex pair begin their formation at the ends of the potential barrier, then move towards the center. This is followed by a sequence of other hidden vortex pairs. Ghost vortices begin to appear only after several pairs of hidden vortices are already formed. Eventually at the critical rotating frequency \( \Omega_c = 0.59 \), a pair of ordinary visible vortices are formed [see Fig. 3(f)], along with a jump in the system angular momentum [see Fig. 3(g)].

The dependence of the angular momentum per atom \( l_z \) on the rotating frequency \( \Omega \) is shown in Fig. 3(g). It is clear from the figure that the angular momentum \( l_z \) increases gradually and continuously with \( \Omega \) until a jump occurs at \( \Omega_c = 0.59 \). Along with Figs. 3(a)-3(e), this means that the hidden vortices can gradually increase the system angular momentum as they move towards the center. The ghost vortex has no capacity to carry angular momentum. As demonstrated clearly in Figs. 3(d)-3(e), even as a pair of ghost vortices move towards the center and eventually become a pair of ordinary visible vortices, the change in the angular momentum is very sudden as witnessed by the jump in Fig. 3(g).

In virtue of data fit, another interesting feature in Fig. 3(g) is that, for \( \Omega < \Omega_c \), \( l_z \) increases linearly; in contrast, for \( \Omega \geq \Omega_c \), \( l_z \) grows exponentially. These two different regimes marked by well linear growth and perfectly exponential growth are likely associated with the fact that the hidden vortices only form along the central barrier but the ordinary visible vortices can emerge in the whole regions of the cloud.

IV. HIDDEN VORTEXES IN A BEC CONFINED IN A ROTATING TOROIDAL TRAP

Hidden vortex can exist in non-DW potentials. To illustrate this point, we consider a BEC confined in a rotating toroidal trap. The toroidal trap is given by

\[
V_{\text{tt}}(x, y) = \frac{x^2 + y^2}{4} + V_0 e^{-(\alpha x^2 + y^2/\alpha)/2\sigma^2}, \quad (3)
\]

where \( \alpha \) characterizes the anisotropy of the 2D central barrier in the toroidal trap. \( \alpha = 1 \) corresponds to a circular toroidal trap, which has recently been studied by Aftalion et al. [34]. We focus on the anisotropic (deformed) toroidal trap \( \alpha \neq 1 \), where the lack of the rotation symmetry excludes the formation possibility of multi-quantized vortex (giant vortex) in the center of the trap for sufficiently large rotation frequency \( \Omega \) and sufficiently narrow barrier. The numerical procedure for this toroidal trap is identical to the DW potential. In Fig. 4 we display the steady density distributions \( |\psi|^2 \) (left) and the corresponding phase distributions of \( \psi \) (right) at \( t = 250 \) for the anisotropic toroidal trap rotating with \( \Omega = 0.6 \) (top) and \( \Omega = 0.9 \) (bottom), respectively. The parameters are \( V_0 = 40, \alpha = 0.8, \sigma = 1, \gamma = 0.03, \) and \( c = 600 \).

At \( \Omega = 0.6 \), two visible vortices appear in the \textit{in situ} density distribution as shown in Fig. 4(a). Moreover, there is an ellipsoid density hole in the trap center which looks like a giant vortex. In the phase distribution displayed in Fig. 4(b), we see that besides two phase singularities corresponding to the two visible vortices, there

FIG. 3: (color online) Phase distributions of \( \psi \) at \( t = 250 \) for different rotating frequencies: (a) \( \Omega = 0.1 \); (b) \( \Omega = 0.12 \); (c) \( \Omega = 0.5 \); (d) \( \Omega = 0.58 \); (e) \( \Omega = 0.59 \), respectively. (f) The density distribution \( |\psi|^2 \) at \( t = 250 \) for \( \Omega = 0.59 \). (g) Angular momentum per atom \( l_z \) versus \( \Omega \). The darker color area indicates the lower phase or density.
are other phase defects, which are distributed along the long axis of the central barrier and at the outskirts of the cloud. The four singly quantized phase defects along the long axis of the central barrier show that the ellipsoid density hole is not a giant vortex. Our numerical simulation further indicates that the four single-quantized phase defects carry angular momentum and satisfy the Feynman’s rule together with the two visible vortices. Therefore, they are four singly quantized hidden vortices. On the other hand, the phase defects at the outskirts of the cloud are ghost vortices because they contribute no angular momentum to the system.

With the increase of rotation frequency, more vortices nucleate and a triangular vortex lattice forms eventually [see Fig. 4(c)]. At the same time, as shown in Fig. 4(d), more hidden vortices also show up in the central barrier region, e.g., there are six hidden vortices for $\Omega = 0.9$. However, these hidden vortices do not form triangular lattice, and they are still distributed along the long axis of the ellipsoid central barrier. This shows from another perspective that the hidden vortex is different from the usual visible vortex. The local enlargements of Figs. 4(b) and 4(d) are given in Fig. 5, where the circles denote the positions of hidden vortices.

FIG. 4: (color online) Density distributions $|\psi|^2$ (left) and phase distributions of $\psi$ (right) at $t = 250$ for the toroidal trap rotating with $\Omega = 0.6$ (top) and $\Omega = 0.9$ (bottom), respectively. The corresponding parameters are $V_0 = 40$, $\alpha = 0.8$, $\sigma = 1$, $\gamma = 0.03$, and $c = 600$. The value of the phase varies continuously from 0 to $2\pi$. The darker color area indicates the lower density or phase.

FIG. 5: (color online) (a) Local enlargement of Fig. 4(b) and (b) that of Fig. 4(d), where the circles mark the positions of hidden vortices.

V. DISCUSSION AND SUMMARY

In summary, we have investigated numerically the formation of vortices in a rotating double-well potential. We found that, other than usual visible vortex and ghost vortex, there exists another type of vortex, which we call hidden vortex. Unlike the usual visible vortex, these hidden vortices are invisible in the in situ density distribution. They differ also from the ghost vortex by being able to carry angular momentum. In addition, the core size of hidden vortex is not given by the healing length, and is strongly influenced by the shape of external potential. Only after including the hidden vortices can the Feynman’s rule be satisfied.

Hidden vortex has appeared in literature in other names. For example, the magnetic fluxons in a superconducting long Josephson junction in a parallel magnetic field [26], Josephson vortices between two long parallel coupled atomic BECs [27], and rotational fluxons of BECs in rotating coplanar double-ring traps [28]. The giant vortices (or, sometimes, called “phantom vortices”) in a cylindrical hard-walled bucket or a quadratic plus quartic trap [33] or a circular toroidal trap [34] can also be regarded as a form of hidden vortex. However, as we have illustrated in the last section with an anisotropic toroidal trap, hidden vortex can occur in many settings other than previously mentioned structure or potentials. Therefore, hidden vortex is a more general notion that encompass all the essential features of Josephson vortex and giant vortex. At the same time, these names, such as Josephson vortex and giant vortex, each coined for a special potential, show that it is necessary to distinguish hidden vortex from the usual visible vortex and the ghost vortex.

Acknowledgments

We thank B. L. Lv, K. J. Jiang, Y. P. Zhang and L. Mao for helpful discussions. This work was supported by the NSFC (10825417, 10875165, 10847143), the NSF of Shandong Province (Q2007A01), and Ph.D. Foundation
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