Reaction-Diffusion Processes

described by

Three-State Quantum Chains and Integrability

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Abstract

The master equation of one-dimensional three-species reaction-diffusion processes is mapped onto an imaginary-time Schrödinger equation. In many cases the Hamiltonian obtained is that of an integrable quantum chain. Within this approach we search for all 3-state integrable quantum chains whose spectra are known and which are related to diffusive-reactive systems. Two integrable models are found to appear naturally in this context: the $U_q\hat{SU}(2)$-invariant model with external fields and the 3-state $U_qSU(P/M)$-invariant Perk-Schultz models with external fields. A nonlocal similarity transformation which brings the Hamiltonian governing the chemical processes to the known standard forms is described, leading in the case of periodic boundary conditions to a generalization of the Dzialoshinsky-Moriya interaction.

Key words: Reaction-diffusion models, integrable quantum chains, Bethe Ansatz

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1 Introduction

Since the pioneering work of Smoluchowski in 1917 [1], reaction-diffusion-limited processes have had a forefront position in nonequilibrium statistical Physics. They can be portrayed as bimolecular processes of the type $A+B \xrightleftharpoons{\kappa}{r} C+D$ where molecules of species $A$ and $B$ ($C$ and $D$) or two different states of the same molecule react to form $C$ and $D$ ($A$ and $B$) with a reaction rate $k(r)$. Particularly in the last decade a great amount of research has been reported (see [2]-[9] and references therein). The interest has been centered mainly on irreversible ($r=0$) and vacuum-driven chemical reactions, i.e. those for which at least one of the final products is an inert state (a precipitate or a non-reacting molecule) denoted by $\emptyset$. The commonly studied reactions are

1. Diffusion

\begin{align}
A + \emptyset & \rightleftharpoons \emptyset + A \\
B + \emptyset & \rightleftharpoons \emptyset + B
\end{align}

(1.1)

2. Interchange

\begin{align}
A + B & \rightleftharpoons B + A
\end{align}

(1.2)

3. Death

\begin{align}
A + \emptyset & \rightarrow \emptyset + \emptyset \\
B + \emptyset & \rightarrow \emptyset + \emptyset
\end{align}

(1.3)

4. One- and Two-Species Annihilation [10]

\begin{align}
A + A & \rightarrow \emptyset + \emptyset \\
B + B & \rightarrow \emptyset + \emptyset \\
A + B & \rightarrow \emptyset + \emptyset
\end{align}

(1.4)

5. Coagulation [11]

\begin{align}
A + A & \rightarrow A + \emptyset \\
B + B & \rightarrow B + \emptyset
\end{align}

(1.5)

6. Trapping [12]

\begin{align}
A + B & \rightarrow A + \emptyset \\
A + B & \rightarrow B + \emptyset
\end{align}

(1.6)

7. Mutation

\begin{align}
A + \emptyset & \rightarrow B + \emptyset \\
B + \emptyset & \rightarrow A + \emptyset
\end{align}

(1.7)
8. Transmutation

\[ A + \emptyset \rightarrow \emptyset + B \]
\[ B + \emptyset \rightarrow \emptyset + A \]  

(1.8)

9. Polymerisation \[13\]

\[ A + A \rightarrow B + \emptyset \]
\[ B + B \rightarrow A + \emptyset \]  

(1.9)

The apparent simplicity of the processes depicted above is quite deceiving. From the great variety of nonequilibrium problems, reaction-diffusion processes are one of the most difficult to tackle. An important step towards circumventing these difficulties, largely of a mathematical nature, was taken by Glauber in 1963 \[14\] and subsequently explored by many authors \[15\]. In a pursuit to understand nonequilibrium systems in terms of the more treatable equilibrium ones, he devised an ingenious way of using classical spin systems to study the problem of critical dynamics by means of a master equation approach. This opened the possibility of employing results from spin chains in the context of nonequilibrium problems. The status quo remained however practically unaltered. This was so since the developments achieved on the theory of spin chains (or more generally speaking quantum chains), which ultimately lead us into the concept of integrability \[17\], were made quite independently and the gap remained. Only recently it was realized \[18\] that a class of problems regarding shrinking domains of Ising spins could be understood in terms of the integrable six-vertex model in one of its critical manifolds \[19\]. Besides, using a master equation approach, a larger class of nonhermitian and integrable \(q\)-deformed models were shown to appear naturally as the time evolution operators of several reaction-diffusion processes \[20\].

Within this spirit, our aim in this paper is to find all three-state integrable quantum chains whose Hamiltonians are time-evolution operators of diffusion-reaction processes and whose spectra are known or can be easily calculated. Our motivation is twofold: the equivalence of the spectra guarantees us the equivalence of the phase diagram and the physical behavior of chemical systems is then in principle determined. Consider the long-time behavior of systems governed by the vacuum-driven reactions given above, as an example. The systems will relax to a final state where there are no particles left, the mean concentration for particle \(A\) (or \(B\)) decaying as

\[ c_{A,B} \sim \begin{cases} t^{-\alpha} & \text{if } \alpha < 1 \\ e^{-t/\tau} & \text{if } \alpha = 1 \\ \end{cases} \]  

(1.10)

where \(\alpha\) and \(\tau\) are characteristic of each problem. Borrowing the field-theoretic jargon we can talk of a massless and massive phase if by these we mean the behavior depicted above. In the one-dimensional annihilation and coagulation models one has \(\alpha = \frac{1}{2}\) which imply in a slower decay towards the vacuum state. The inclusion of
certain kinds of processes (or reversible reactions) leads to a local steady state, thus driving the system towards a quicker (exponential) decay-regime. We shall see in the next sections that these behaviors correspond, in the quantum chains to which these models are related, to the massless and massive regimes respectively.

Our second motivation comes from the fact that the connection with integrable systems provides us with methods of calculating physical quantities exactly. In this respect, we were particularly motivated by the work of Gwa and Spohn [21] who calculated the dynamical scaling exponent of the discrete Noisy Burgers Equation. This equation was introduced by Burgers [22] as a model for turbulent flow. The discretization leads to an asymmetric two-state diffusion problem described by the reaction

\[ A + \emptyset \xrightleftharpoons[k]{r} \emptyset + A \]

where the rates \( k \) and \( r \) are different. The dynamics of this problem is governed by the Hamiltonian of the six-vertex model.

We summarize the results of this paper in what follows. We found that a large class of chemical reactions composed of several simultaneous processes can be understood in terms of two quantum chains, namely the \( U_q\widehat{SU}(2) \)-invariant model and the 3-state \( U_qSU(P/M) \)-invariant Perk-Schultz models. Using the standard basis of matrices \((E_{kl})_{a,b} = \delta_{k,a} \delta_{l,b}\) the first chain reads [23]

\[
H' = H_0' + H_1' \\
H_0' = -\sum_{i=1}^{L-1} \left( E_{i}^{01} E_{i+1}^{10} + E_{i}^{10} E_{i+1}^{01} + E_{i}^{02} E_{i+1}^{20} + E_{i}^{20} E_{i+1}^{02} \right) + v \xi_i \xi_{i+1} \\
+w \left( \xi_i + \xi_{i+1} \right) + a \left( \xi_i - \xi_{i+1} \right) \\
H_1' = -g \sum_{i=1}^{L} \xi_i^z \\
\xi^0 = E^{11} + E^{22} \\
\xi^z = E^{11} - E^{22}
\]

where \( H_0' \) is \( U_q\widehat{SU}(2) \)-invariant and the symmetry-breaking \( H_1' \), which commutes with \( H_0' \), acts as an external field and does not spoil the integrability of the model. \( H_0' \) has the same spectrum (apart from degeneracies) as that of the spin-\( \frac{1}{2} \) Heisenberg model with an external field and a surface term

\[
H^{XXZ} = -\frac{1}{2} \sum_{i=1}^{L-1} \left\{ \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z + h \left( \sigma_i^z + \sigma_{i+1}^z \right) + a \left( \sigma_i^z - \sigma_{i+1}^z \right) + \beta \right\}
\]

The phase diagram is well known [24]: without external field, the model is massless and conformal invariant when \(-1 \leq \Delta \leq 1\), massive with a ferromagnetic ground state for \( \Delta > 1 \) and massive with an antiferromagnetic ground state for \( \Delta < -1 \). When \( h \neq 0 \) the system is massive commensurate for \( \Delta > 1 - h \) and massless incommensurate otherwise, the line \( \Delta = 1 - h \) corresponding to a Pokrovski-Talapov (PT) phase transition [25] between these two regimes. Since the spectra are equivalent, so are the phase diagrams.
To see the connection between $H'_0$ and $H_{XXZ}$ we rewrite the latter in the basis of $E_{kl}$ matrices

$$H_{XXZ} = - \sum_{i=1}^{L-1} \left( E_{i1}^0 E_{i+11}^0 + E_{i+11}^0 E_{i1}^0 \right) + v E_{i1}^{11} E_{i+11}^{11} + w \left( E_{i1}^{11} + E_{i+11}^{11} \right) + a \left( E_{i1}^{11} - E_{i+11}^{11} \right) + b$$

(1.13)

where we redefined the parameters of the Heisenberg model as follows

$$v = -2\Delta, \quad w = \Delta + h, \quad b = -\left( \frac{\Delta + \beta}{2} + h \right)$$

(1.14)

$H'_0$ is obtained from $H_{XXZ}$ by adding extra terms proportional to $E_{02}^i E_{20}^{i+1}$ and $E_{20}^i E_{02}^{i+1}$ which do not affect the spectrum and correspond, in the chemical scenario, to the exchange of the lattice configuration $B\emptyset$ to $\emptyset B$ and $\emptyset B$ to $B\emptyset$ at sites $i$ and $i + 1$ respectively. The diagonal terms are extended accordingly. The wave functions of $H'_0$ have been calculated but the effect of $H'_1$ on the phase diagram has not yet been studied [23]. We shall reinterpret the known phase-diagram in the language of chemical reactions.

The second class of chains which appear naturally as time-evolution operators of chemical systems are the $U_q SU(P/M)$-invariant Perk-Schultz models (PS models). The general $N$-state Hamiltonian of the PS chains can be written as

$$H\{\epsilon_1, \ldots, \epsilon_{P+M}\} = \sum_{j=1}^{L-1} U_j^{(P/M)}$$

$$= \sum_{j=1}^{L-1} \left\{ \frac{q + q^{-1}}{2} - \left[ \sum_{\alpha \neq \beta = 0}^{N-1} E_j^{\alpha \beta} E_{j+1}^{\beta \alpha} + \frac{q + q^{-1}}{2} \sum_{\alpha = 0}^{N-1} \epsilon_\alpha E_j^{\alpha \alpha} E_{j+1}^{\alpha \alpha} \right] \right\}$$

$$+ \frac{q - q^{-1}}{2} \sum_{\alpha \neq \beta = 0}^{N-1} \text{sign} (\alpha - \beta) E_j^{\alpha \alpha} E_{j+1}^{\beta \beta} \right\}$$

(1.15)

with $\epsilon_0 = \epsilon_1 = \ldots = \epsilon_{P-1} = -\epsilon_P = -\epsilon_{P+1} = \ldots = -\epsilon_{P+M-1} = 1$. Here $(P/M)$ stands for the entire partitions of $N$. These models were first introduced by Sutherland [26] for $q = 1$ and then extended by Perk and Schultz [27] to $q \neq 1$. Later [28] it was noticed that the $U_j^{(P/M)}$ are generators of the Hecke algebra $H(n)$ ($n = L - 1$) defined through

$$U_i U_i = (q + q^{-1}) U_i \quad i = 1, 2, \ldots, n$$

$$U_i U_{i+1} U_i - U_i = U_{i+1} U_i U_{i+1} - U_{i+1}$$

$$U_i U_{i+j} = U_{i+j} U_i \quad j \geq i + 2$$

(1.16)

where to each partition $(P/M)$ of $N$ corresponds an additional set of relations beyond the ones given above, defining the so-called quotients of the algebra. From a
mathematical point of view, this underlying algebraic structure has important consequences as regards the spectrum of each chain \[29\]. From the physical one, each quotient corresponds to systems with different properties. For \( P + M = 2 \), one has two chains: the \((2/0)\) chain which is the spin-\(1/2\) XXZ model and the \((1/1)\) chain which has also been extensively studied \[30\] and is believed to describe the coverage dependence on fugacity for xenon adsorption on copper \[31\]. It has a PT phase transition in its phase diagram at \( q + q^{-1} = 2 \) between a commensurate ordered phase and an incommensurate one with oscillating correlation functions. For \( P + M = 3 \) the \((2/1)\) chain has received most of the attention due to its relevance in understanding Anderson’s \( t - J \) model \[32\].

We also found a very remarkable property of the mapping of chemical systems given by combinations of rates (1.1) through (1.9) onto the quantum chains we presented. It turned out that due to the structure of the mapping not all the parameters which we started with (the different reaction rates) are important in settling the phase structure of chemical reactions. This will become clear in the next sections, were we present our results with more detail.

The third important point is that for certain chemical models it is necessary to define a nonlocal similarity transformation in order to rewrite the Hamiltonian describing them in the standard form of the PS Hamiltonians. This kind of transformation was already known to map the periodic spin-\(1/2\) XXX with an \( XY \) interaction term onto the XXZ model with general boundaries proportional to the volume of the system \[33\]. The transformation we found generalizes it to higher-state chains and also takes account of more general Dzialoshinsky-Moriya-type interactions \[34\] parametrized by different variables.

This paper is organized as follows. In the second section we introduce the formalism of the master equation on lattices and its mapping onto nearest neighbor quantum chains. The strategy we will adopt to identify chemical reactions with quantum chains is explained. Section 3 is devoted to the \( U_qSU(2) \)-invariant model, where it will serve as a workbench for the application of the ideas developed in the previous section. In section 4 we rewrite the PS models with external fields as reaction-diffusion Hamiltonians, constructing explicitly the similarity transformation which map the chemical systems onto them. In section 5 we study this transformation and generalize it in two directions: first to higher-state models parametrized by different diffusion rates and second showing that diffusion-processes on periodic lattices are mapped on chains with more general boundary conditions. Finally in section 6 we summon our results and some questions we are still faced with.

2 The Master Equation and Quantum Chains

The master equation governs the evolution of the probability distribution of Markov processes. Due to its almost universal range of applicability, it is one of the most important equations of Statistical Mechanics. Here we shall apply it in the
context of chemical processes on a chain. Consider a one-dimensional lattice with L sites and open boundaries. At each site \( j \) we define a variable \( \beta_j \) which takes \( N \) integer values (0, 1, 2, ..., \( N - 1 \)). To each possible configuration \( \{ \beta \} = \{ \beta_1, \ldots, \beta_L \} \) of the lattice realized at time \( t \) we attach a probability distribution \( P(\{ \beta \}, t) \) whose time evolution is given by the master equation

\[
\frac{\partial P(\{ \beta \}, t)}{\partial t} = \sum_{k=1}^{L-1} \left\{ -\Omega_{\beta_k,\beta_{k+1}} P(\beta_1, \ldots, \beta_L; t) \right. \\
+ \sum'_{l,m=0} \Gamma_{\beta_k,\beta_{k+1}}^{\beta_k+l,\beta_{k+1}+m} P(\beta_1, \ldots, \beta_k + l, \beta_{k+1} + m, \ldots, \beta_L; t) \right\} \quad (2.1)
\]

Here and henceforth the prime in a sum over \( l \) and \( m \) indicates exclusion of the pair \( l = m = 0 \). The \( \Gamma_{a,b}^{c,d} \) are transition rates which equal the probability that, in a unit time step and at any site \( j \), a state \((\beta_j, \beta_j+1) = (a, b)\) changes to a state \((\beta'_j, \beta'_{j+1}) = (c, d)\). We will assume throughout this paper that the transition rates depend only on links (nearest-neighbor interaction) and are homogeneous, i.e. site independent. The \( \Omega_{a,b} \) are related to the probability that a state \((a, b)\) will not change after a unit time step. From conservation of probability we see that they satisfy

\[
\Omega_{a,b} = \sum_{r,s} \Gamma_{r,s}^{a,b} \quad (2.2)
\]

With the help of these definitions and the matrices \( E^{kl} \) defined in the introduction we can rewrite the master equation (2.1) as

\[
\frac{\partial |\psi\rangle}{\partial t} = -H |\psi\rangle \quad (2.3)
\]

if we identify \( |\psi\rangle \) as the probability \( P(\{ \beta \}, t) \) and \( H \) as

\[
H = \sum_{j=1}^{L-1} H_j = \sum_{j=1}^{L-1} (U_j - T_j) \\
U_j = \sum_{a,b=0}^{N-1} \Omega_{a,b} E_{j}^{aa} E_{j+1}^{bb} \\
T_j = \sum'_{a,b,c,d=0} \Gamma_{c,d}^{a,b} E_{j}^{ca} E_{j+1}^{db} \quad (2.4)
\]

The key to the whole process of identifying chemical processes with known quantum chains lies in finding the appropriate set of rates so that we can recast \( H \) as

\[
H = H_0 + \sum_{j=1}^{L-1} (h_i + h_{i+1} + g_i - g_{i+1}) H_1 \quad (2.5)
\]
so that the first two terms on the r.h.s are equivalent to a quantum chain plus external field and surface term, and the spectrum of $H$ is independent of $H_1$. The surface term is an extra degree of freedom that we have: since they do not alter the bulk properties of the system, we can always define them. We shall look only for those chemical processes whose spectrum is equivalent to that of some quantum chain, therefore guaranteeing the equality of the phase diagram. However the wave functions are not the same. A priori one can define a similarity transformation of the form $A(\lambda)$ such that

$$A(\lambda)H A^{-1}(\lambda) = H_0 + \sum_{j=1}^{L-1} (h_i + h_{i+1} + g_i - g_{i+1}) + H_1'(\lambda)$$

(2.6)

such that for some given value $\lambda = \lambda_0$ we have $H_1'(\lambda_0) = 0$. If such a transformation were found, then the wave functions of the chemical problem could be obtained from that of the quantum chain. No solution has yet been found and the problem remains open.

### 3 The $U_qSU(2)$ model

With the formalism developed in the last section, we shall now address the problem of finding the set of chemical reactions associated to a given quantum chain. Our ultimate goal is to use the phase diagram of the chain to explain the chemistry of reaction-diffusion processes. In what follows we identify our particles with $A = 1$, $B = 2$ and inert state $= \emptyset$.

We consider a system in which particles $A$ and $B$ diffuse to the right and to the left symmetrically, with rates equal to unity

$$\begin{align*}
\begin{cases}
A + \emptyset &\rightarrow \emptyset + A & \text{rate } \Gamma_{1,0}^{1,0} = \Gamma_{1,0}^{0,1} = 1 \\
B + \emptyset &\rightarrow \emptyset + B & \text{rate } \Gamma_{2,0}^{1,0} = \Gamma_{2,0}^{0,2} = 1
\end{cases}
\end{align*}$$

(3.1)

In addition to these processes, we also allow the particles to react according to the following 24 vacuum-driven rates:

- annihilation

$$\begin{align*}
\begin{cases}
A + A &\rightarrow \emptyset + \emptyset & \Gamma_{0,0}^{1,1} \\
B + B &\rightarrow \emptyset + \emptyset & \Gamma_{0,0}^{2,2} \\
A + B &\rightarrow \emptyset + \emptyset & \Gamma_{0,0}^{1,2} \\
B + A &\rightarrow \emptyset + \emptyset & \Gamma_{0,0}^{2,1}
\end{cases}
\end{align*}$$

(3.2)
• coagulation

\[
\begin{align*}
A + A &\rightarrow A + \emptyset & \Gamma_{1,0}^{1,1} \\
A + A &\rightarrow \emptyset + A & \Gamma_{0,1}^{1,1} \\
B + B &\rightarrow B + \emptyset & \Gamma_{2,0}^{2,2} \\
B + B &\rightarrow \emptyset + B & \Gamma_{0,2}^{2,2}
\end{align*}
\] (3.3)

• death

\[
\begin{align*}
A + \emptyset &\rightarrow \emptyset + \emptyset & \Gamma_{0,0}^{1,0} \\
\emptyset + A &\rightarrow \emptyset + \emptyset & \Gamma_{0,0}^{0,1} \\
B + \emptyset &\rightarrow \emptyset + \emptyset & \Gamma_{0,0}^{2,0} \\
\emptyset + B &\rightarrow \emptyset + \emptyset & \Gamma_{0,0}^{0,2}
\end{align*}
\] (3.4)

• polymerisation

\[
\begin{align*}
A + A &\rightarrow B + \emptyset & \Gamma_{2,0}^{1,1} \\
A + A &\rightarrow \emptyset + B & \Gamma_{0,2}^{1,1} \\
B + B &\rightarrow A + \emptyset & \Gamma_{1,0}^{2,2} \\
B + B &\rightarrow \emptyset + B & \Gamma_{0,2}^{2,2}
\end{align*}
\] (3.5)

• trapping

\[
\begin{align*}
A + B &\rightarrow A + \emptyset & \Gamma_{1,0}^{1,2} \\
B + A &\rightarrow \emptyset + A & \Gamma_{0,1}^{2,1} \\
A + B &\rightarrow B + \emptyset & \Gamma_{2,0}^{1,2} \\
B + A &\rightarrow \emptyset + B & \Gamma_{0,2}^{2,1} \\
A + B &\rightarrow \emptyset + A & \Gamma_{0,1}^{1,2} \\
B + A &\rightarrow A + \emptyset & \Gamma_{2,1}^{2,1} \\
A + B &\rightarrow \emptyset + B & \Gamma_{0,2}^{1,2} \\
B + A &\rightarrow B + \emptyset & \Gamma_{2,0}^{2,1}
\end{align*}
\] (3.6)

With these processes and the technique developed in section 2 we obtain the following Hamiltonian

\[
H = H_0 + H_1
\]

\[
H_0 = \sum_{i=0}^{L-1} \left\{ - \left( E_{i+1}^{01} E_i^{10} + E_i^{10} E_{i+1}^{01} + E_i^{02} E_{i+1}^{20} + E_i^{20} E_{i+1}^{02} \right) \right. \\
+ \left. \left( \Gamma_{0,0,0}^{0,1} + 1 \right) E_{i+1}^{00} E_i^{11} + \left( \Gamma_{0,0,0}^{0,2} + 1 \right) E_i^{00} E_{i+1}^{22} \right. \\
+ \left. \left( \Gamma_{0,0,0}^{1,0} + 1 \right) E_i^{11} E_{i+1}^{00} + \left( \Gamma_{0,0,0}^{2,0} + 1 \right) E_i^{22} E_{i+1}^{00} \right\}
\]

8
which can be written as

\[ H_{1} = \sum_{i=1}^{L-1} \left\{ \Gamma_{0,0}^{0,1} E_{i,i+1}^{0,0} + \Gamma_{0,0}^{0,1} E_{i,i+1}^{0,1} + \Gamma_{0,0}^{0,1} E_{i,i+1}^{0,2} + \Gamma_{0,0}^{0,1} E_{i,i+1}^{0,3} + \Gamma_{0,0}^{0,1} E_{i,i+1}^{0,4} + \Gamma_{0,0}^{0,1} E_{i,i+1}^{0,5} + \Gamma_{0,0}^{0,1} E_{i,i+1}^{0,6} + \Gamma_{0,0}^{0,1} E_{i,i+1}^{0,7} \right\} \]  

(3.7)

The spectrum of \( H \) coincides with that of \( H_{0} \). We found this solution numerically, by successive trials: we started out with a minimum set of reactions, namely those corresponding to diffusion and we added vacuum-driven rates one by one until we had a 2-body Hamiltonian structure resembling the structure of the quantum chain we were interested in. On top of these, we added more processes until we reached the point were the spectrum obtained was different. This type of property is characteristic of nonhermitian phenomena and has an important physical implication: since the spectrum does not depend on each parameter of \( H_{1} \) independently but only on their combinations which appear in \( H_{0} \), our system has 8 effective parameters, which are the 8 sums inside parentheses of equation (3.7) \(^2\). Yet, the problem of identifying this effective chemical Hamiltonian with the chain given by \( H' \) in (1.11) requires the reduction to an even smaller set of parameters since \( H' \) depends only on four. The solution is to find the proper way of recombining the eight effective parameters of the chemical model into a final set of four. We will solve the problem in steps.

First, to avoid carrying too heavy a notation throughout the text we rename groups of rates regarding the same category of processes as follows \((i = 1, 2)\)

\[ A_{i} = \Gamma_{0,0}^{0,i} \text{ annihilation} \]

\[ A_{1,2} = \Gamma_{0,0}^{1,2} \pm \Gamma_{0,0}^{2,1} \text{ annihilation} \]

\(^2\)Due to conservation of probability each diagonal element of a Hamiltonian must equal the sum of the nondiagonal entries in that same column. We can see this if we look at the definitions of rates in section 2 and realize that the conservation of the probabilities implies in a relation among rates which can be written as \( \sum_{k \neq l} H_{kl} = H_{ll} \).
\[ D_i^+ = \Gamma_{0,0}^0 + \Gamma_{0,0}^i \text{ death} \]
\[ C_i^+ = \Gamma_{0,0}^i + \Gamma_{0,0}^i \text{ coagulation} \]
\[ P_1^+ = \Gamma_{0,0}^{1,1} + \Gamma_{0,0}^{1,1} \text{ polymerisation} \]
\[ P_2^+ = \Gamma_{0,0}^{2,1} + \Gamma_{0,0}^{2,1} \text{ polymerisation} \]
\[ T^\pm = [\Gamma_{0,0}^{1,2} + \Gamma_{0,0}^{1,2}] + [\Gamma_{0,0}^{1,2} + \Gamma_{0,0}^{1,2}] \text{ trapping} \] (3.8)

The problem is solved through the rearrangement of the rates into the following variables

\[
\begin{align*}
v &= A_1 + C_1^+ + P_1^+ - D_1^+ - 2 \\
w &= \frac{D_1^+ + D_2^+}{4} + 1 \\
g &= \frac{D_2^+ - D_1^+}{4} \\
a &= -\frac{D_1^- + D_2^-}{4} \\
l &= \frac{D_2^- - D_1^-}{4} \quad (3.9)
\end{align*}
\]

together with 3 conditions on the rates

\[
\begin{align*}
2w + 2g + v &= A_2 + C_2^+ + P_2^+ \\
4w + 2v &= A_{12}^+ + T^+ \\
4l &= A_{12}^- + T^- \quad (3.10)
\end{align*}
\]

After some simple algebraic manipulation, we can rewrite \( H_0 \) as a new \( \tilde{H}_0 \) which reads

\[
\tilde{H}_0 = \sum_{i=1}^{L-1} -E_i^0 E_{i+1}^{10} + E_i^{10} E_{i+1}^{01} + E_i^{02} E_{i+1}^{20} + E_i^{20} E_{i+1}^{02} + w(\varepsilon_i^0 + \varepsilon_{i+1}^0) + g(\varepsilon_i^z + \varepsilon_{i+1}^z) + a(\varepsilon_i^0 - \varepsilon_{i+1}^0) + l(\varepsilon_i^z - \varepsilon_{i+1}^z) + \nu \varepsilon_i^0 \varepsilon_{i+1}^0 \\
\varepsilon_i^0 &= E_i^{11} + E_i^{22} \quad \varepsilon^z = E_i^{11} - E_i^{22} \quad (3.11)
\]

Comparing this expression with \( H_0' \) of eq. (1.11) we conclude that they have the same phase diagram since the term in \( \tilde{H}_0 \) having the parameter \( l \) as coefficient is a surface contribution. To interpret the phase diagram of the chemical model in terms of the phase diagram of the \( XXZ \) model we still have to recover the \( U_q SU(2) \)-symmetric spectrum, which can be done by requiring that \( g = l = 0 \). In terms of chemical rates this condition means

\[
\begin{align*}
D_1^+ &= D_2^+ \\
D_1^- &= D_2^- \quad (3.12)
\end{align*}
\]
This amounts to saying that $A$ and $B$ are indistinguishable as we can see by examining equations (3.9), (3.10) and (3.12). It suffices now to identify the parameters of the Heisenberg chain with the rates of our chemical model. We obtain

$$h = \frac{A_1 + C_1^+ + P_1^+}{2}$$
$$\Delta = 1 + \frac{D_1^+ - \left(A_1 + C_1^+ + P_1^+\right)}{2}$$
$$a = -\frac{D_1^-}{2}$$
$$\beta = -1 - \frac{A_1 + C_1^+ + P_1^+ + D_1^+}{2}$$ (3.13)

The analysis is straightforward. Making $h = 0$ implies that no rates but death survive. In this situation

$$\Delta = 1 + \frac{D_1^+}{2}$$
$$\beta = -\Delta$$ (3.14)

By varying the death rate we go from a massive regime to a massless one, that is the time evolution for the concentration of particles has an exponential or algebraic fall-off respectively. This can be understood on physical grounds: being a ‘one-particle’ process, death happens irrespectively of any other processes occurring in the system, i.e. it is not diffusion-limited since any particle can die alone. It therefore outruns the characteristic time scale set by diffusion and brings about a quicker decay. With an external field we have

$$\Delta + h - 1 = \frac{D_1^+}{2}$$ (3.15)

If we take death with a non-zero probability, then from the equation above one sees that our system is massive ferromagnetic. In Chemistry this means having a ground state with no particles (we identify spin up with $\emptyset$) By varying the rate of death we approach the line on the phase diagram given by $\Delta + h = 1$. The system undergoes a PT transition \[25\] when the energy of the state with just one particle equals that of the state with no particles and it becomes the ground state. We have a level-crossing: since death is absent the system can evolve to a final steady state where only one particle is left.

4 The Perk Schultz chains

4.1 The $U_qSU(P/M)$-invariant chain

It is a well established fact that the Hamiltonian of the 6-vertex model is the time-evolution operator for the two-state asymmetric diffusion process \[18\]. On the same
ground we expect that the PS chains will play the role of time-evolution operators of higher-state asymmetric diffusion processes since they are the Hamiltonians of higher-state ice models. This picture is however far from complete and can be extended to encompass more general chemical systems if we reinterpret the additional reactions on chemical Hamiltonians as external fields in the PS chains they are mapped onto. For the sake of completeness, we present first the results without external fields and then we proceed with the more general models.

Following \[16\], we first consider a system in which particles \(A\) and \(B\) diffuse and interchange positions on the lattice according to

\[
\begin{align*}
A + \emptyset & \rightarrow \emptyset + A & \text{rate} & \Gamma_R \\
\emptyset + A & \rightarrow A + \emptyset & & \Gamma_L \\
B + \emptyset & \rightarrow \emptyset + B & & \Gamma_R \\
\emptyset + B & \rightarrow B + \emptyset & & \Gamma_L \\
B + A & \rightarrow A + B & & \Gamma_R \\
A + B & \rightarrow B + A & & \Gamma_L
\end{align*}
\]

(4.1)

With these processes and the rates defined above we get a Hamiltonian which reads

\[
H = \sum_{j=1}^{L-1} \left\{ \Gamma_L (E_{00}^{11} E_{j+1}^{11} + E_{00}^{22} E_{j+1}^{22} + E_{11}^{22} E_{j+1}^{22}) + \Gamma_R (E_{11}^{00} E_{j+1}^{00} + E_{22}^{00} E_{j+1}^{22}) - \Gamma_L (E_{10}^{01} E_{j+1}^{02} + E_{20}^{01} E_{j+1}^{22}) - \Gamma_R (E_{01}^{01} E_{j+1}^{10} + E_{02}^{02} E_{j+1}^{21}) \right\}
\]

(4.2)

Defining \(\sqrt{\Gamma_L \Gamma_R} = q\), which measures the asymmetry of the diffusion, and the diffusion constant \(\sqrt{\Gamma_L \Gamma_R} = D\), which sets the time scale of the problem, this can be rewritten as

\[
\frac{H_{(\epsilon_\alpha)}}{D} = \sum_{j=1}^{L-1} \left\{ \frac{q + q^{-1}}{2} - \left[ q^{-1} \sum_{\alpha > \beta}^2 E_{j+1}^{\alpha \beta} E_{j+1}^{\beta \alpha} + q \sum_{\alpha < \beta}^2 E_{j+1}^{\alpha \beta} E_{j+1}^{\beta \alpha} \right] \right. \\
+ \left. \frac{q + q^{-1}}{2} \sum_{\alpha=0}^2 \epsilon_\alpha E_{j+1}^{\alpha \alpha} E_{j+1}^{\alpha \alpha} + \frac{q - q^{-1}}{2} \sum_{\alpha \neq \beta}^2 \text{sign}(\alpha - \beta) E_{j+1}^{\alpha \alpha} E_{j+1}^{\beta \beta} \right\}
\]

(4.3)

where \((\epsilon_0, \epsilon_1, \epsilon_2) = (1, 1, 1)\). Comparing with eq. (1.15) we see that this is the \(U_q SU(3/0)\) chain in a non-standard form. To bring it to the standard form we have to consider a nonlocal similarity transformation \(S\) which is given by

\[
S = \sum_{\alpha_1, \alpha_2, \ldots, \alpha_L = 0}^2 q^{\frac{1}{2} \sum_{j>i=1}^{L} \text{sign}(\alpha_j - \alpha_i)} E_{\alpha_1}^{\alpha_1} \otimes E_{\alpha_2}^{\alpha_2} \otimes \cdots \otimes E_{\alpha_L}^{\alpha_L}
\]

(4.4)

The implications of this transformation coming from boundary terms are discussed in detail in the next section. If we apply it to the Hamiltonian (4.3) we obtain the
standard $U_qSU(3/0)$ chain which reads

$$S \frac{H_{\{1,1,1\}}}{D} S^{-1} = \frac{H^{(3/0)}}{D} = \sum_{j=1}^{L-1} \left\{ \frac{q + q^{-1}}{2} - \sum_{\alpha \neq \beta}^2 E_j^{\alpha \beta} E_{j+1}^{\beta \alpha} \right\} + \frac{q + q^{-1}}{2} \sum_{\alpha = 0}^2 E_j^{\alpha \alpha} E_{j+1}^{\alpha \alpha} + \frac{q - q^{-1}}{2} \sum_{\alpha \neq \beta}^2 \text{sign}(\alpha - \beta) E_j^{\alpha \alpha} E_{j+1}^{\beta \beta} \right\} \right\} \right\}$$

(4.5)

We use two notations to differentiate between what we call the non-standard (non-hermitian) and standard (hermitian) forms of the chain.

To reproduce the $U_qSU(2/1)$-invariant PS model, two new processes have to be added, namely those corresponding to coagulation of $B$

$$\left\{ \begin{array}{l} B + B \to B + \emptyset \quad \text{rate} \quad \Gamma_{2,0}^{2,2} \\ B + B \to \emptyset + B \quad \Gamma_{0,2}^{2,2} \end{array} \right\} \quad \text{(4.6)}$$

such that $\Gamma_{2,0}^{2,2} + \Gamma_{0,2}^{2,2} = D(q + q^{-1})$. The main difference now in comparison to the first case we studied is that we will have an extra diagonal contribution equal to $\Gamma_{2,0}^{2,2} + \Gamma_{0,2}^{2,2}$ and two new nondiagonal pieces proportional to each of these two rates. The Hamiltonian that we find in this case reads

$$\frac{H}{D} = \frac{H_{\{1,1,-1\}}}{D} - \sum_{j=1}^{L-1} \left\{ \frac{\Gamma_{2,0}^{2,2}}{D} E_j^{22} E_{j+1}^{02} - \frac{\Gamma_{0,2}^{2,2}}{D} E_j^{22} E_{j+1}^{20} \right\} \right\} \right\} \right\}$$

(4.7)

where $H_{\{1,1,-1\}}$ is obtained from (4.3) with $(\epsilon_0, \epsilon_1, \epsilon_2) = (1, 1, -1)$. The spectrum of $H$ is equal to the spectrum of $H_{\{1,1,-1\}}$ and again we can study only the properties of the latter as long as we are interested only in the phase diagram. $H_{\{1,1,-1\}}$ is the non-standard representation of the (2/1) PS chain. With the similarity transformation of eq. $(4.4)$ we obtain the standard $U_qSU(2/1)$-invariant PS model

$$S H_{\{1,1,-1\}} S^{-1} = H^{(2/1)} \quad \text{(4.8)}$$

where $H^{(2/1)}$ is given by (1.13) with $(\epsilon_0, \epsilon_1, \epsilon_2) = (1, 1, -1)$.

Finally, the last PS chain can be reproduced by the inclusion of the rates corresponding to coagulation of $A$

$$\left\{ \begin{array}{l} A + A \to A + \emptyset \quad \text{rate} \quad \Gamma_{1,0}^{1,1} \\ A + A \to \emptyset + A \quad \Gamma_{0,1}^{1,1} \end{array} \right\} \quad \text{(4.9)}$$

with $\Gamma_{1,0}^{1,1} + \Gamma_{0,1}^{1,1} = D(q + q^{-1})$. The Hamiltonian in this case reads

$$\frac{H}{D} = \frac{H_{\{1,-1,-1\}}}{D} - \sum_{j=1}^{L-1} \left\{ \frac{\Gamma_{1,0}^{1,1}}{D} E_j^{11} E_{j+1}^{01} - \frac{\Gamma_{0,1}^{1,1}}{D} E_j^{11} E_{j+1}^{10} \right\} \right\} \right\} \right\}$$

(4.10)
The same argument applies: the spectrum of $H$ is equal to the spectrum of $H_{\{1,-1,-1\}}$, which is the non-standard $U_qSU(1/2)$-invariant PS model. This is in turn equivalent to the standard form through the transformation $S$

$$S H_{\{1,-1,-1\}} S^{-1} = H^{(1/2)} \quad (4.11)$$

We would like to point out one particular feature of the two last chains. Once we find the spectrum of the $(2/1)$ chain, the spectrum of the $(1/2)$ can be obtained in a straightforward manner: we reverse the sign of the whole spectrum of the $(2/1)$ model and an overall constant equal to $(L-1)(q + q^{-1})$, which is the highest energy of the $(2/1)$ chain (this value is indeed the highest energy for all $L$-site $(P/M)$ chains with $q$ real and non-zero $P$ and $M$. An outline proof is given in ref. [29]). In the chemical scenario however, they correspond to different physical pictures and the reason is the positivity of the spectrum.

### 4.2 The $U_qSU(P/M)$-invariant chain with external fields

We now generalize the results above to include the most general set of vacuum-driven processes which can be written as a pure nonhermitian Hamiltonian which does not contribute to the spectrum of the PS chains. We analysed this problem on the computer and found that beyond the 24 rates of section 3 we can include one more set of reactions, namely those corresponding to mutation and transmutation of $A$. They are defined through

$$
\begin{align*}
& A + \emptyset \to B + \emptyset \quad \text{rate} \quad \Gamma_{1,0}^{1,0} \\
& \emptyset + A \to \emptyset + B \quad \Gamma_{0,1}^{0,1} \\
& A + \emptyset \to \emptyset + B \quad \Gamma_{1,0}^{1,0} \\
& \emptyset + A \to B + \emptyset \quad \Gamma_{0,1}^{0,1}
\end{align*}
\quad (4.12)
\]

For these reactions we define the following combination of rates

$$
\begin{align*}
M_1^\pm &= \Gamma_{0,1}^{0,1} \pm \Gamma_{2,0}^{1,0} \\
X_1^\pm &= \Gamma_{2,0}^{0,1} \pm \Gamma_{0,2}^{1,0}
\end{align*}
\quad (4.13)
\]

The problem is the same of section 3: the original chemical process we start with has more parameters than the PS chains. The question is whether it is possible to reduce these rates to the appropriate set of parameters in a way that these can be rewritten as external fields and surface terms. We found a positive answer in all three cases studied. Using the master equation approach we get a Hamiltonian which can be written as two separate pieces as follows

$$
\frac{H}{D} = \frac{H_{\{\epsilon_0\}}}{D} + \sum_{i=1}^{L-1} (h_i + h_{i+1} + g_i - g_{i+1}) + H^{(1)} \quad (4.14)
$$

14
together with the conditions

\[
\begin{align*}
M_{i+}^+ + X_{i+}^+ + D_{i+}^+ &= C_{i+}^+ + P_{i+}^+ + A_1 \\
D_{i+}^+ &= C_{i+}^+ + P_{i+}^+ + A_2 \\
A_{i2}^+ + T^+ &= M_{i+}^+ + X_{i+}^+ + D_{i+}^+ + D_{i-}^+ \\
A_{i2}^- + T^- &= D_{i-}^+ - (M_{i+}^+ + X_{i+}^- + D_{i-}^-)
\end{align*}
\]
In this case we get the following relations between field, surface contributions and reaction rates

\[
\begin{align*}
\frac{M_1^+ + X_1^+ + D_1^+}{2D} & = h_1 \\
\frac{M_1^- + X_1^- + D_1^-}{2D} & = -g_1 \\
\frac{D_2^+}{2D} & = -h_2 \\
\frac{D_2^-}{2D} & = -g_2
\end{align*}
\] (4.18)

For these identifications to hold we have to impose the extra set of relations among rates

\[
\begin{align*}
M_1^+ + T_1^+ + D_1^+ & = C_1^+ + P_1^+ + A_1 \\
D_2^+ & = C_2^+ + P_2^+ + A_2 - D(q + q^{-1}) \\
A_{12}^+ + T^+ & = M_1^+ + X_1^+ + D_1^+ + D_2^+ \\
A_{12}^- + T^- & = D_2^- - (M_1^- + X_1^- + D_1^-)
\end{align*}
\] (4.19)

iii) The $U_qSU(1/2)$ chain

For the last model we have

\[
\begin{align*}
\frac{M_1^+ + X_1^+ + D_1^+}{2D} & = -h_1 \\
\frac{M_1^- + X_1^- + D_1^-}{2D} & = -g_1 \\
\frac{D_2^+}{2D} & = -h_2 \\
\frac{D_2^-}{2D} & = -g_2
\end{align*}
\] (4.20)

The consistency conditions for these equations are

\[
\begin{align*}
M_1^+ + T_1^+ + D_1^+ & = C_1^+ + P_1^+ + A_1 - D(q + q^{-1}) \\
D_2^+ & = C_2^+ + P_2^+ + A_2 - D(q + q^{-1}) \\
A_{12}^+ + T^+ & = M_1^+ + X_1^+ + D_1^+ + D_2^+ \\
A_{12}^- + T^- & = D_2^- - (M_1^- + X_1^- + D_1^-)
\end{align*}
\] (4.21)

We observed that if in place of mutation and transmutation of $A$ we considered the
corresponding processes for the particle $B$

$$\begin{align*}
\emptyset + B &\rightarrow \emptyset + A & \text{rate} & \Gamma_{0,1}^{0,2} \\
B + \emptyset &\rightarrow \emptyset + A & & \Gamma_{0,1}^{2,0} \\
\emptyset + B &\rightarrow A + \emptyset & & \Gamma_{1,0}^{0,2} \\
B + \emptyset &\rightarrow A + \emptyset & & \Gamma_{1,0}^{2,0}
\end{align*}$$

(4.22)

from which we define the relation of rates

$$M_2^\pm = \Gamma_{0,1}^{0,2} \pm \Gamma_{1,0}^{2,0}$$

$$X_2^\pm = \Gamma_{1,0}^{0,2} \pm \Gamma_{0,1}^{2,0}$$

(4.23)

the spectrum also remained invariant, but not if we took mutations and transmutations for both particles at the same time. The reason is that the inclusion of both processes in the system yields a local steady regime given by the reversible process $A + \emptyset \rightleftharpoons B + \emptyset$. In the presence of reversible reactions we expect a totally different physical picture which translates itself, among other things, into a different spectrum. The phase diagram of the chemical processes can now be in principle explained in terms of the Physics of the PS models. Unfortunately only the phase diagram in the absence of fields is known: this is however not so interesting because it implies then that many processes do not survive - and the whole mapping only makes sense if one is able to study non-trivial cases. We will return to this point in a future publication.

5 Similarity Transformation and Boundary Conditions

We shall now consider more thoroughly the transformation given by the matrix $S$ which we used to rewrite our 3-state Hamiltonians describing diffusion processes as the standard PS chains. We shall address here two points: the first regards the extention of $S$ not only to higher-state models but also to multi-parameter diffusion processes, i.e. those characterized by a larger set $\{q\} = \{q_1, q_2, \cdots\}$ of parameters in substitution to the one-parameter diffusion we dealt with so far. Second, we want to look at the effect of $S$ on periodic chains and the physical features it induces.

5.1 Multi-parameter and Higher-State Diffusion Processes

As before we consider a system in which $(N-1)$ particles diffuse with $\emptyset$ and interchange places among themselves. We define the following rates

$$A_x + A_y \rightleftharpoons A_y + A_x \quad \Gamma_{A_y,A_x}^{A_x,A_y}, \Gamma_{A_x,A_y}^{A_y,A_x} \quad x > y = 0, 1, \cdots, N - 1$$

(5.1)
from which we also define \( \frac{N(N-1)}{2} \) parameters \( q_{xy} \)

\[
\sqrt{\frac{A_y A_x}{A_x A_y}} = q_{xy}
\]

with \( q_{xy} = q_{yx}^{-1} \). These are the generalization of the parameter \( q \) defined in the previous sections. We impose a homogeneous time scale

\[
\sqrt{\Gamma_{A_1,A_2}} = \cdots = \sqrt{\Gamma_{A_{N-2},A_{N-1}} A_{N-1} A_{N-2} \Gamma_{0}} = \mathcal{D}
\]

and obtain a nonhermitian Hamiltonian which reads

\[
\frac{H}{\mathcal{D}} = \sum_{j=1}^{L-1} \left\{ \sum_{x \neq y = 0}^{N-1} q_{xy} \left( E_{kk}^{xx} E_{kk+1}^{yy} - E_{kk}^{xy} E_{kk+1}^{yx} \right) \right\}
\]

We want to find a similarity transformation which makes \( H \) hermitian. The (nonlocal!) similarity transformation which accomplishes this reads

\[
S = \sum_{\alpha_1,\alpha_2,\cdots,\alpha_L = 0}^{N-1} \left( \prod_{x>y=0}^{N-1} (q_{xy}) f_{xy}(\alpha_1,\alpha_2,\cdots,\alpha_L) \right) E^{\alpha_1 \alpha_1} \otimes E^{\alpha_2 \alpha_2} \otimes \cdots \otimes E^{\alpha_L \alpha_L}
\]

with the functions \( f_{xy} \) given by

\[
f_{xy}(\alpha_1,\alpha_2,\cdots,\alpha_L) = \frac{1}{2} \sum_{n>m=1}^{L} (\delta_{\alpha_n,x} \delta_{\alpha_m,y} - \delta_{\alpha_n,y} \delta_{\alpha_m,x})
\]

We point out in the expression for the function that the order of its arguments is very important since they index the sites on the chain, that is \( f(\cdots,\alpha_j,\alpha_{j+1},\cdots) \neq f(\cdots,\alpha_{j+1},\alpha_j,\cdots) \).

**Proof**

For clarity, we will consider the diagonal and nondiagonal pieces of the Hamiltonian separately. Since \( S(\sum_j H_{jj+1}) S^{-1} = \sum_j S H_{jj+1} S^{-1} \), it suffices to consider the action of \( S \) on the 2-site operator only.

Consider first the action of \( S \) on the diagonal piece of the Hamiltonian. We have

\[
S H_{jj+1}^{diag} S^{-1} = \sum_{\{\alpha\}}^{N-1} \prod_{x>y}^{N-1} (q_{xy}) f_{xy}(\alpha_1,\alpha_2,\cdots,\alpha_L) E^{\alpha_1 \alpha_1} \otimes E^{\alpha_2 \alpha_2} \otimes \cdots \otimes E^{\alpha_L \alpha_L}
\]

\[
\times \left( \sum_{a \neq b = 0}^{N-1} q_{ab} E_a^{aa} E_{j+1}^{bb} \right)
\]

\[
\times \sum_{\{\beta\}}^{N-1} \prod_{w>z}^{N-1} (q_{wz}) f_{wz}(\beta_1,\beta_2,\cdots,\beta_L) E^{\beta_1 \beta_1} \otimes E^{\beta_2 \beta_2} \otimes \cdots \otimes E^{\beta_L \beta_L}
\]

18
We rearrange the terms in the expression above as follows

\[
\frac{SH_{j,j+1}^{(\text{diag})}}{D} S^{-1} = \sum_{\{\alpha\}} \sum_{\{\beta\}} \prod_{x>y=0}^{N-1} \prod_{w>z=0}^{N-1} \left( q_{xy}^{(f_{xy}(\alpha_1,\alpha_2,\ldots,\alpha_L))} q_{wz}^{-f_{wz}(\beta_1,\beta_2,\ldots,\beta_L)} \right) \\
\times E^{\alpha_1\alpha_1} E^{\beta_1\beta_1} \otimes E^{\alpha_2\alpha_2} E^{\beta_2\beta_2} \otimes \cdots \otimes 1^{(j)} \otimes 1^{(j+1)} \otimes \cdots \otimes E^{\alpha_L\alpha_L} E^{\beta_L\beta_L} \\
\times \sum_{\alpha_j,\beta_j=1}^{N-1} \sum_{\alpha_{j+1},\beta_{j+1}}^{N-1} \left( q_{ab} E_{j+1}^{\alpha_1 \alpha_j} E_{j}^{\alpha_j \beta_j} \otimes E_{j+1}^{\alpha_{j+1} \alpha_j} E_{j}^{\alpha_j \beta_{j+1}} \right) \quad (5.8)
\]

Since \( x, y, w \) and \( z \) serve only to index the same set \( \{q\} \) of diffusion parameters, we can take \( x = w, y = z \). Also, the multiplication properties of the matrices \( E^{\alpha\beta} \) gives us

\[
E^{pq} E^{rs} = \delta_{q,r} E^{ps}
\]

We have therefore

\[
\frac{SH_{j,j+1}^{(\text{diag})}}{D} S^{-1} = \sum_{\{\alpha\}} \sum_{\{\beta\}} \prod_{x>y=0}^{N-1} \prod_{a\neq b=0}^{N-1} \left( q_{xy}^{(f_{xy}(\alpha_1,\alpha_2,\ldots,\alpha_L))} q_{ab} E_{j+1}^{\alpha_1 \alpha_j} E_{j}^{\alpha_j \beta_j} \right) \\
\times E^{\alpha_1\alpha_1} E^{\alpha_2\alpha_2} \otimes \cdots \otimes 1^{(j)} \otimes 1^{(j+1)} \otimes \cdots \otimes E^{\alpha_L\alpha_L} E^{\beta_L\beta_L} \sum_{\alpha_{j+1},\beta_{j+1}=1}^{N-1} \sum_{a\neq b=0}^{N-1} \left( q_{ab} E_{j}^{\alpha_1 \alpha_j} E_{j+1}^{\alpha_j \beta_j} \delta_{\alpha_{j+1},\beta_{j+1}} \right) \quad (5.10)
\]

Taking into account the Kronecker’s deltas we have

\[
\frac{SH_{j,j+1}^{(\text{diag})}}{D} S^{-1} = \sum_{\{\alpha\}} \prod_{x>y=0}^{N-1} q_{xy}^{(f_{xy}(\alpha_1,\alpha_2,\ldots,\alpha_L))} \sum_{a\neq b=0}^{N-1} q_{ab} E_{j+1}^{\alpha_1 \alpha_j} E_{j}^{\alpha_j \beta_j} \left( \sum_{\alpha_{j+1},\beta_{j+1}=1}^{N-1} \sum_{a\neq b=0}^{N-1} \left( q_{ab} E_{j}^{\alpha_1 \alpha_j} E_{j+1}^{\alpha_j \beta_j} \delta_{\alpha_{j+1},\beta_{j+1}} \right) \right) \\
\times E^{\alpha_1\alpha_1} E^{\alpha_2\alpha_2} \otimes \cdots \otimes 1^{(j)} \otimes 1^{(j+1)} \otimes \cdots \otimes E^{\alpha_L\alpha_L} E^{\beta_L\beta_L} \quad (5.11)
\]

The exponent of \( q_{xy} \) is clearly zero. Summing over \( \{\alpha\} \) we have finally

\[
\frac{SH_{j,j+1}^{(\text{diag})}}{D} S^{-1} = \sum_{a\neq b=0}^{N-1} q_{ab} 1 \otimes 1 \otimes \cdots \otimes E_{(j)}^{\alpha a} \otimes E_{(j+1)}^{\beta b} \otimes \cdots \otimes 1 \\
= \frac{H_{j,j+1}^{(\text{diag})}}{D} \quad (5.12)
\]

This concludes the proof that the diagonal elements of the Hamiltonian are not changed by the transformation generated by \( S \).

To see the effect of \( S \) on the nondiagonal piece we proceed as before. We consider only the 2-body operator acting on sites \((j, j+1)\) and we specialize to one given value of the pair \((a, b)\). We have

\[
S \frac{H_{j,j+1}^{(\text{nond})}}{D} S^{-1} = S \left( q_{ab} E_{j+1}^{\alpha a} E_{(j+1)}^{\alpha b} + q_{ab}^{-1} E_{j}^{\beta a} E_{(j)}^{\beta b} \right) a > b \quad S^{-1} \quad (5.13)
\]
As in the diagonal case after multiplying the $E^{\alpha \beta}$ matrices on each site we obtain

$$S\frac{H_{j,j+1}^{(nond)}}{D} S^{-1} = \sum_{\{\alpha\}} \sum_{\{\beta\}} \prod_{x>y=0} \left( q_{xy}^{E_{j,j+1},\alpha_1,\alpha_2,\ldots,\alpha_L} - f_{xy}(\alpha_1,\alpha_2,\ldots,\alpha_L) \right) \times E^{\alpha_1} \delta_{\alpha_1,\beta_1} \times \cdots \times E^{\alpha_L} \delta_{\alpha_L,\beta_L}$$

$$\times \left( q_{ab} E_{j,j+1}^{ab} E_{j,j+1}^{ab} \delta_{\alpha_j,\beta_j,a} + q_{ab}^{-1} E_{j,j+1}^{ba} E_{j,j+1}^{ba} \delta_{\alpha_j,\beta_j,b} \right)_{a>b}$$

$$\times \delta_{\alpha_j+1,\beta_j} \delta_{\alpha_j,\beta_j+1}$$

(5.14)

This reduces to

$$S\frac{H_{j,j+1}^{(nond)}}{D} S^{-1} = \sum_{\{\alpha\}} \sum_{\{\beta\}} \prod_{x>y=0} q_{xy}^{E_{j,j+1},\alpha_1,\alpha_2,\ldots,\alpha_L} - f_{xy}(\alpha_1,\alpha_2,\ldots,\alpha_L)$$

$$\times E^{\alpha_1} \times \cdots \times E^{\alpha_L} \times \left( q_{ab} E_{j,j+1}^{ab} E_{j,j+1}^{ab} \delta_{\alpha_j,\beta_j,a} + q_{ab}^{-1} E_{j,j+1}^{ba} E_{j,j+1}^{ba} \delta_{\alpha_j,\beta_j,b} \right)_{a>b}$$

(5.15)

We have to analyse the exponent of $q_{xy}$ now. We once again point out that the order of the arguments of the functions $f$ is important. With this in mind we have

$$f_{xy}(\alpha_1,\ldots,\alpha_j,\alpha_{j+1},\ldots,\alpha_L) = f_{xy}(\alpha_1,\ldots,\alpha_{j+1},\alpha_j,\ldots,\alpha_L)$$

$$= \frac{1}{2} (\delta_{\alpha_{j+1},x} \delta_{\alpha_j,y} - \delta_{\alpha_{j+1},y} \delta_{\alpha_j,x}) = \frac{1}{2} (\delta_{\alpha_{j+1},x} \delta_{\alpha_j,y} - \delta_{\alpha_{j+1},y} \delta_{\alpha_j,x})$$

(5.16)

Substituting this expression on (5.15) we get

$$S\frac{H_{j,j+1}^{(nond)}}{D} S^{-1} = \sum_{\{\alpha\}} \sum_{\{\beta\}} \prod_{x>y=0} \delta_{\alpha_{j+1},x} \delta_{\alpha_j,y} - \delta_{\alpha_{j+1},y} \delta_{\alpha_j,x}$$

$$\times \left( q_{ab} E_{j,j+1}^{ab} E_{j,j+1}^{ab} \delta_{\alpha_j,\beta_j,a} + q_{ab}^{-1} E_{j,j+1}^{ba} E_{j,j+1}^{ba} \delta_{\alpha_j,\beta_j,b} \right)_{a>b}$$

$$\times E^{\alpha_1} \times \cdots \times E^{\alpha_L}$$

(5.17)

Summing over $\{\alpha\}$ gives us an identity matrix on each site. The deltas also ‘kill’ the sum over $\alpha_j, \alpha_{j+1}$ such that we will be left with

$$S\frac{H_{j,j+1}^{(nond)}}{D} S^{-1} = \prod_{x>y=0} \left( q_{xy}^{E_{j,j+1}^{ab} E_{j,j+1}^{ab} q_{ab} E_{j,j+1}^{ba} E_{j,j+1}^{ba}} + q_{xy}^{E_{j,j+1}^{ba} E_{j,j+1}^{ba} q_{ab}^{-1} E_{j,j+1}^{ba} E_{j,j+1}^{ba}} \right)_{a>b}$$

$$= \left( q_{ab}^{-1} q_{ab} E_{j,j+1}^{ba} E_{j,j+1}^{ba} + q_{ab} q_{ab}^{-1} E_{j,j+1}^{ba} E_{j,j+1}^{ba} \right)_{a>b}$$

$$= \left( E_{j,j+1}^{ab} E_{j,j+1}^{ab} + E_{j,j+1}^{ba} E_{j,j+1}^{ba} \right)_{a>b}$$

(5.18)
This concludes our proof.

To finish we would like to indicate how in the one-parameter \( N \)-state diffusion process this expression simplifies. The rates are defined according to

\[
A_x + A_y \iff A_y + A_x \quad \Gamma_R, \Gamma_L \quad x > y = 0, 1, \ldots, N - 1
\]  

(5.19)

for which the \( L \)-site Hamiltonian reads

\[
\frac{H}{D} = \sum_{k=1}^{L-1} \left\{ \frac{q + q^{-1}}{2} - \left[ q^{-1} \sum_{\alpha > \beta = 0}^{N-1} E_k^{\alpha \beta} E_k^{\beta \alpha} + q \sum_{\alpha < \beta = 0}^{N-1} E_k^{\alpha \beta} E_k^{\beta \alpha} \right] + \frac{q + q^{-1}}{2} \sum_{\alpha = 0}^{N-1} E_k^{\alpha \alpha} E_{k+1}^{\alpha \alpha} \right\} (5.20)
\]

This is the non-standard \( U_qSU(N/0) \)-invariant PS model. To derive the standard form we consider the matrix \( S \) from eq. (5.5) with \( q_{xy} = q \) for all \( x, y \). By observing that in this case we have

\[
f_1(\alpha_1, \alpha_2, \ldots, \alpha_L) + \cdots + f_{N-1}(\alpha_1, \alpha_2, \ldots, \alpha_L) = \frac{1}{2} \sum_{n>m=1}^{L} \text{sign}(\alpha_n - \alpha_m) \]  

(5.21)

we get for \( S \) the following expression

\[
S = \sum_{\alpha_1, \alpha_2, \ldots, \alpha_L = 0}^{N-1} q^{\frac{1}{2}} \sum_{n>m=1}^{L} \text{sign}(\alpha_n - \alpha_m) E^{\alpha_1 \alpha_1} \otimes E^{\alpha_2 \alpha_2} \otimes \cdots \otimes E^{\alpha_L \alpha_L} \]  

(5.22)

It is a simple algebraic exercise to bring the chain (5.20) to its standard form under the action of \( S \). Actually the result holds for any of the \( U_qSU(P/M) \)-invariant PS chains with \( P + M = N \).

### 5.2 Periodic Boundary Conditions

Due to its nonlocality, the effect of \( S \) on free or periodic chains is different. In the first case it brings the original non-hermitian chemical model to a hermitian quantum chain. In the second case, it brings the non-hermitian chemical model to a hermitian quantum chain but with general boundary terms which imply in a generalized Dzialoshinsky-Moriya-type of interaction in the bulk [34]. This transformation was known for many years for 2-state systems [33].

To understand the effect of \( S \) on periodic chains, it is appropriate to start with the simplest system, namely a particle \( A \) diffusing to the left and right with rates

\[
A + \emptyset \iff \emptyset + A \quad \Gamma_R, \Gamma_L
\]  

(5.23)

where \( \mathcal{D} \) and \( q \) are defined in the usual way (see section 4). The Hamiltonian is given by

\[
\frac{H}{\mathcal{D}} = \sum_{k=1}^{L} \left\{ \frac{q + q^{-1}}{2} - \frac{q + q^{-1}}{2} \sum_{\alpha = 0}^{1} E_k^{\alpha \alpha} E_{k+1}^{\alpha \alpha} - \frac{q - q^{-1}}{2} \sum_{\alpha \neq \beta = 0}^{1} \text{sign}(\alpha - \beta) E_k^{\alpha \alpha} E_{k+1}^{\beta \beta} \right\} - q^{-1} E_k^{10} E_{k+1}^{10} - q E_k^{01} E_{k+1}^{10} \]  

(5.24)
It is convenient to change to the more familiar basis of Pauli matrices, by making the identification

\[ E_{00} = \frac{1 + \sigma^z}{2} \]
\[ E_{11} = \frac{1 - \sigma^z}{2} \]
\[ E_{01} = \sigma^+ \]
\[ E_{10} = \sigma^- \]  
(5.25)

In this basis the Hamiltonian can be written as

\[ \frac{H}{\mathcal{D}} = -\frac{1}{2} \sum_{k=1}^{L} \left\{ 2q^{-1} \sigma_k^+ \sigma_{k+1}^- + 2q \sigma_k^- \sigma_{k+1}^+ + \frac{q + q^{-1}}{2} \sigma_k^z \sigma_{k+1}^z - \frac{q - q^{-1}}{4} \right\} \]  
(5.26)

The constant term is irrelevant in our discussion and we will drop it for the time being. Applying eq. (5.22) to the remaining terms in the chain we get

\[ S \frac{H}{\mathcal{D}} S^{-1} = \frac{\bar{H}}{\mathcal{D}} = -\frac{1}{2} \sum_{k=1}^{L-1} \left\{ 2 \sigma_k^+ \sigma_{k+1}^- + 2 \sigma_k^- \sigma_{k+1}^+ + \frac{q + q^{-1}}{2} \sigma_k^z \sigma_{k+1}^z \right\} \]
\[ -\frac{1}{2} \left\{ 2q^L \sigma_L^- \sigma_1^+ + 2q^{-L} \sigma_L^+ \sigma_1^- + \frac{q + q^{-1}}{2} \sigma_L^z \sigma_1^z \right\} \]  
(5.27)

We can bring this under a common summation sign if we define the boundaries as

\[ \sigma_{L+1}^z = \sigma_1^z \]
\[ \sigma_{L+1}^+ = q^L \sigma_1^+ \]
\[ \sigma_{L+1}^- = q^{-L} \sigma_1^- \]  
(5.28)

With this definition we have

\[ \frac{\bar{H}}{\mathcal{D}} = -\frac{1}{2} \sum_{k=1}^{L} \left\{ 2 \sigma_k^+ \sigma_{k+1}^- + 2 \sigma_k^- \sigma_{k+1}^+ + \frac{q + q^{-1}}{2} \sigma_k^z \sigma_{k+1}^z \right\} \]  
(5.29)

Since \( q \) is real, we have a boundary term which is proportional to the volume of the system, changing the whole structure of the problem. In other words, the dynamics of diffusion processes in a chain with open boundaries are different from that of a periodic chain [35].

For higher-state diffusion models on periodic lattices we will get, after applying \( S \), chains which cannot be put in terms of simple generalized boundaries. Rather we get a bulk interaction. To see this we first note that the whole effect of \( S \) is to change the off-diagonal terms of the 2-body chemical Hamiltonian at the boundary, namely \( E_{1b}^a E_{1a}^b \) and \( E_{Lb}^a E_{Lb}^a \). In the 2-state case there are only two possibilities for the pair \((a, b)\): \((1, 0)\) and \((0, 1)\). Since opposite pairs are related by an inversion of the power of \( q \), we have effectively one result which is \( q^L \). In higher-state cases, the
power depends on the particular \((a, b)\) chosen. We present the results for the 3-state model and the for the general \(N\)-state case in what follows.

i) 3-state diffusion process

Here we have \((a, b) = (0, 1); (0, 2); (1, 2)\) and the reversed pairs. The action of \(S\) yields

\[
S\{q^{-1}E_1^{01}E_L^{10}\}S^{-1} = q^{-2}E_1^{01} \otimes M_1 \otimes M_1 \otimes \cdots \otimes M_1 \otimes E_L^{10}
\]

\[
M_1 = \begin{pmatrix} q \qquad q \\ 1 \end{pmatrix}
\] \hspace{1cm} (5.30)

for \((a, b) = (0, 1)\). Exchanging \(a, b\) we get the same structure but with \(q\) replaced by \(q^{-1}\) as expected. For the pair \((a, b) = (1, 2)\) we obtain

\[
S\{q^{-1}E_1^{12}E_L^{21}\}S^{-1} = q^{-2}E_1^{12} \otimes M_2 \otimes M_2 \otimes \cdots \otimes M_2 \otimes E_L^{21}
\]

\[
M_2 = \begin{pmatrix} 1 \qquad q \\ q \end{pmatrix}
\] \hspace{1cm} (5.31)

Finally for \((a, b) = (0, 2)\) we have

\[
S\{q^{-1}E_1^{02}E_L^{20}\}S^{-1} = q^{-2}E_1^{02} \otimes M_1M_2 \otimes M_1M_2 \otimes \cdots \otimes M_1M_2 \otimes E_L^{21}
\] \hspace{1cm} (5.32)

We see now that in the case under study, the effect of the transformation cannot be put in the boundary terms in contradistinction to the 2-state problem - the interaction is ‘spread’ over the bulk. The structure is analogous in higher-state models as it is shown next.

ii) \(N\)-state diffusion process

Assume that \((a, b) = (i, j)\) such that \(i < j = 0, 1, \ldots, N-1\). We obtain in this case

\[
S\{q^{-1}E_1^{ij}E_L^{ji}\}S^{-1} = q^{-2}E_1^{ij} \otimes M \otimes M \otimes \cdots \otimes M \otimes E_L^{ji}
\] \hspace{1cm} (5.33)
where $M$ is an $N \times N$ matrix which reads

$$M = \begin{pmatrix} 1 & & & & & \cdots & & \cdots & & \cdots \\ & \ddots & & & & & \ddots & & & \\ & & 1 & & & & & \ddots & & \cdots \\ & & & q & & & \ddots & & \cdots & & \cdots \\ & & & & q^2 & & \ddots & & \cdots & & \cdots \\ & & & & & \ddots & & \ddots & & \cdots & & \cdots \\ & & & & & & \ddots & & \ddots & & \cdots & & \cdots \\ & & & & & & & 1 & & \cdots & \cdots \\ & & & & & & & & 1 & \cdots & \cdots \\ \end{pmatrix}$$

(5.34)

where $q$ appears at the $i$-th and $j$-th positions in the diagonal. All other $M_{kk}$ are equal to 1 for $k < i$ and $k > j$ and equal to $q^2$ in the case $i < k < j$. The interaction is spread along the bulk as we can see from eq. (5.33).

6 Conclusions

We must start this summary with one question: what is the purpose of this whole technique? Does it bring any new information to what is already known regarding reaction-diffusion processes?

Reaction-diffusion models are per se a very rich and fascinating field, from a physical and mathematical point of view. The range of physical phenomena which they encompass is extensive, from dispersive transport in amorphous silicon [36] to polymerisation processes like thermal soliton-antisoliton interaction of trans-polyacetylene [13], to name a few. Experimental studies have also been conducted on one-dimensional coagulation models [37]. This alone justifies their study and any new method which might shed some light into the structure of these problems is highly welcome.

We showed in this paper that a large class of chemical processes can be understood in terms of the $U_qSU(P/M)$-invariant Perk-Schultz models and the $U_qSU(2)$-invariant model with external fields. Since both are integrable, this opens a new perspective, namely of employing analytical methods to calculate physical quantities of interest exactly. We showed how the phase diagram in the latter case completely settles the phase diagram in the chemical model, the corresponding massless and massive regimes yielding two types of time-dependence for the decay of the particles’ concentration. We showed also what processes correspond to a field in the language of quantum chains and that not each and every processes governs the Physics but combinations of them.

In the Perk-Schultz models little is known about the phase diagram with external fields. Since the effect of a field on the spectrum is trivial, the problem is a feasible one. However the limitation comes mostly from hindrances in numerical techniques.
To calculate the phase diagram one needs data from large lattice sizes in order to make good extrapolations. Here the integrability of the model comes in handy. The Bethe Ansatz Equations which allow for good spectral evaluations are not known in the case of open chains. For periodic chains we must take care. The chemical Hamiltonians associated to the Perk-Schultz models are usually not in a standard form. We found a similarity transformation which brings them to the standard form, and whose properties were not yet discussed in the literature except for the 2-state processes. This transformation induces terms of physical relevance in the bulk which are generalizations of the Dzialoshinski-Moriya interaction, thus confirming the fact that the Physics of nonequilibrium problems depends on the boundary conditions\cite{35}. The three-state Perk-Schultz models with these new interactions are soluble through the Bethe Ansatz technique, which in this case yields the following set of coupled nonlinear equations \cite{38}

$$
\begin{align*}
\varepsilon_0^{\gamma(N_0+N_1)} e^{M_1 \varepsilon_1} & \left( \frac{\sinh(\lambda_k^{(0)} + \epsilon_0 \gamma/2)}{\sinh(\lambda_k^{(0)} - \epsilon_0 \gamma/2)} \right)^N = \\
& \prod_{\alpha=1}^{M_0} \frac{\sinh(\lambda_k^{(0)} - \lambda_\alpha^{(0)} + \epsilon_1 \gamma)}{\sinh(\lambda_k^{(0)} - \lambda_\alpha^{(0)} - \epsilon_0 \gamma)} \\
& \times \prod_{\alpha=1}^{M_1} \frac{\sinh(\lambda_\alpha^{(0)} - \lambda_\alpha^{(1)} + \epsilon_1 \gamma/2)}{\sinh(\lambda_\alpha^{(0)} - \lambda_\alpha^{(1)} - \epsilon_1 \gamma/2)} \\
& \times \prod_{\alpha=1}^{M_1} \frac{\sinh(\lambda_\alpha^{(1)} - \lambda_\alpha^{(1)} - \epsilon_2 \gamma)}{\sinh(\lambda_\alpha^{(1)} - \lambda_\alpha^{(1)} + \epsilon_1 \gamma)}
\end{align*}
$$

where \{\epsilon_\alpha\} are the parameters of the Perk-Schultz chains, and \gamma is related to \(q = \exp(\gamma)\). The \(N_i\)’s correspond to the particular number of particles of type \(i\) in each charge sector and the \(M_i\)’s equal the number of roots of the set of coupled equations. They are obtained from the \(N_i\)’s through

$$
M_i = L - (N_0 + N_1 + \cdots + N_i)
$$

where \(L\) is the lattice size. We shall analyse this problem in a future publication.

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References

[1] M. V. Smoluchowski, Z. Phys. Chem. 92 (1917), 215
[2] M. Bramson and D. Griffeath, Ann. Prob. 8 (1980), 183
[3] D. C. Torney and H. M. McConnel, J. Phys. Chem. 87 (1983), 1941
[4] K. Kang, P. Meakin, J. H. Oh and S. Redner, J. Phys. A 17 (1984), L60
[5] Z. Racz, Phys. Rev. Lett. 55 (1985), 1107
[6] A. A. Lushnikov, Phys. Lett. 120A (1987), 135
[7] M. Bramson and J. L. Lebowitz, Phys. Rev. Lett. 61 (1988), 2397
[8] V. Kusovkov and E. Kotomin, Rep. Prog. Phys. 51 (1988), 2397
[9] D. ben-Avraham, M. A. Burschka and C. R. Doering, J. Stat. Phys. 60 (1990), 695
[10] D. Toussaint and F. Wilczek, J. Chem. Phys. 78 (1983), 2642; K. Kang and S. Redner, Phys. Rev. Letters 52 (1984), 955; V. Privman, J. Stat. Phys. 72 3,4 (1993), 845
[11] K. Kang and S. Redner, Phys. Rev. A30 (1984), 2833
[12] D. Chopard, M. Droz, T. Karapiperis and Z. Rácz, Phys. Rev. E47,1 (1993), R40; M. A. Rodriguez, G. Abramson, H. Wio and A. Bru, Phys. Rev. E48,2 (1993), 829
[13] J. L. Spouge, Phys. Rev. Lett. 60 (1988), 871
[14] R. J. Glauber, J. Math. Phys. 4 (1963), 294
[15] S. P. Heims, J. Chem. Phys. 45 (1966), 370; K. Kawasaki, Phys. Rev. 145 (1966), 224; L. P. Kadanoff and J. Swift, Phys. Rev. 165 (1968), 310
[16] F. C. Alcaraz, M. Droz, M. Henkel and V. Rittenberg, Ann. Phys. 230 (1994), 250
[17] H. J. De Vega, Int. J. Mod. Phys. A 4,10 (1989), 2371
[18] D. Kandel, E. Domany and B. Nienhuis, J. Phys. A: Math Gen. 23 (1990), L755 and references therein.
[19] R. J. Baxter in ‘Exactly Solved Models in Statistical Mechanics’, Academic Press, London, 1982
[20] F. C. Alcaraz and V. Rittenberg, Phys. Lett. B 314 (1993), 377
[21] L.-H. Gwa ans H. Spohn, Phys. Rev. A 46,2 (1992), 844
[22] J. M. Burgers in ‘The Nonlinear Diffusion Equation’, Riedel, Boston, 1974
[23] F. C. Alcaraz, D. Arnaudon, V. Rittenberg and M. Scheunert, ‘Hubbard-like Models in the Infinite Repulsion Limit and Finite-Dimensional Representations of the Affine Algebra’, CERN Preprint CERN-TH.6935/93, to appear in ‘Int. J. Mod. Phys. B’

[24] J. D. Johnson and B. M. Mckoy, Phys. Rev A46 (1972), 1613; M. Takahashi and M. Suzuki, Prog. Theor. Phys. 48 (1972), 2187

[25] V. L. Pokrovskii and A. L. Talapov, Sov. Phys. JETP 51 (1980), 134

[26] B. Sutherland, Phys. Rev. B12 (1975), 3795

[27] J. H. H. Perk and C. L. Schultz in ‘Non-Linear Integrable Systems, Classical Theory and Quantum Theory’, ed. M. Jimbo and T. Miwa, World Scientific, Singapore, 1981; C. L. Schultz, Phys. Rev. Letters 46 (1981), 629, Physica 122 (1983), 71

[28] T. Deguchi, J. Phys. Soc. Japan 58 (1989), 3441; T. Deguchi and Y. Akutsu, J. Phys. A. 23 (1990), 1861

[29] P. P. Martin and V. Rittenberg, Int. J. Mod. Phys. A7, Suppl.1B (1992), 707

[30] H. Saleur in ‘Trieste Conference on Recent Developments in Conformal Field Theories’, World Scientific, 1989

[31] M. Jaubert, A. Glachand, M. Bienfat and G. Boato, Phys. Rev. Letters 46 (1981), 1676

[32] P. W. Anderson, Science 235 (1987), 1196; P. A. Bares and G. Blatter, Phys. Rev. Letters 64 (1990), 2567

[33] J. H. H. Perk, private communication to V. Rittenberg.

[34] I. Dzialoshinsky, J. Phys. Chem. Solids 4 (1958), 241; T. Moriya, Phys. Rev. 117 (1960), 635 and in ‘Magnetism’, eds. G. T. Rado and H. Suhl, Academic Press, London, 1963

[35] M. Henkel and G. Schütz, ‘Boundary-Induced Phase Transitions in Equilibrium and Non-Equilibrium Problems’, Univ. Genève preprint UGVA-DPT 07-826 (1993); F. C. Alcaraz, ‘Exact Steady States of Asymmetric Diffusion and Two-Species Annihilation with Back Reaction from the Ground State of Quantum Spin Models’, Universidade Federal de São Carlos - Brazil preprint (1994)

[36] M. Hvam and M. H. Brodsky, Phys. Rev. Lett. 46 (1981), 371; J. Orenstein and M. Kastner, Phys. Rev. Lett. 46 (1981), 1421 and references therein

[37] R. Kroon, H. Fleurent and R. Sprik, Phys. Rev. E 47,4 (1993), 2462

[38] S. R. Dahmen, in preparation.