The End of Optimism?
An Asymptotic Analysis of Finite-Armed Linear Bandits

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Abstract

Stochastic linear bandits are a natural and simple generalisation of finite-armed bandits with numerous practical applications. Current approaches focus on generalising existing techniques for finite-armed bandits, notably the optimism principle and Thompson sampling. While prior work has mostly been in the worst-case setting, we analyse the asymptotic instance-dependent regret and show matching upper and lower bounds on what is achievable. Surprisingly, our results show that no algorithm based on optimism or Thompson sampling will ever achieve the optimal rate, and indeed, can be arbitrarily far from optimal, even in very simple cases. This is a disturbing result because these techniques are standard tools that are widely used for sequential optimisation. For example, for generalised linear bandits and reinforcement learning.

1 INTRODUCTION

The linear bandit is the simplest generalisation of the finite-armed bandit. Let \( A \subseteq \mathbb{R}^d \) be a finite set that spans \( \mathbb{R}^d \) with \( |A| = k \) and \( \|x\|_2 \leq 1 \) for all \( x \in A \). A learner interacts with the bandit over \( n \) rounds. In each round \( t \) the learner chooses an action (arm) \( A_t \in A \) and observes a payoff \( Y_t = \langle A_t, \theta \rangle + \eta_t \) where \( \eta_t \sim \mathcal{N}(0,1) \) is Gaussian noise and \( \theta \in \mathbb{R}^d \) is an unknown parameter. The optimal action is \( x^* = \arg\max_{x \in A} \langle x, \theta \rangle \), which is not known since it depends on \( \theta \). The assumption that \( A \) spans \( \mathbb{R}^d \) is non-restrictive, since if \( \text{span}(A) \) has rank \( r < d \), then one can simply use a different basis for which all but \( r \) coordinates are always zero and then drop them from the analysis. The Gaussian assumption can be relaxed to 1-subgaussian for our upper bound, but is needed for the lower bound. Our performance measure is the expected pseudo-regret (from now on just the regret), which is given by

\[
R_n^2(n) = \mathbb{E}\left[ \sum_{t=1}^{n} \langle x^* - A_t, \theta \rangle \right],
\]

where the expectation is taken with respect to the actions of the strategy and the noise. There are a number of algorithms designed for minimising the regret, all of which use one of two algorithmic designs. The first is the principle of optimism in the face of uncertainty, which was originally applied to finite-armed bandits by Agrawal [1995], Katehakis and Robbins [1995], Auer et al. [2002] and many others, and more recently to linear bandits [Auer, 2002, Dani et al., 2008, Abbasi-Yadkori et al., 2011, 2012]. The second algorithm design is Thompson sampling, which is an old algorithm [Thompson, 1933] that has experienced resurgence in popularity because of its impressive practical performance and theoretical guarantees for finite-armed bandits [Kaufmann et al., 2012, Korda et al., 2013]. Thompson sampling has also recently been applied to linear bandits with good empirical performance [Chapelle and Li, 2011] and near-minimax theoretical guarantees [Agrawal and Goyal, 2013].

While both approaches lead to practical algorithms (especially Thompson sampling), we will show they are fundamentally flawed in that algorithms based on these ideas cannot be close to asymptotically optimal. Along the way we characterise the optimal achievable asymptotic regret and design a strategy achieving it. This is an important message because optimism and Thompson sampling are widely used beyond the finite-armed case. Examples include generalised linear bandits [Filippi et al., 2010], spectral bandits [Valko et al., 2014], and even learning in Markov decision processes [Auer et al., 2010, Gopalan and Mannor, 2015].

The disadvantages of these approaches is obscured in the worst-case regime, where both are quite close to optimal. One might question whether or not the asymptotic analysis is relevant in practice. The gold standard would be instance-dependent finite-time guarantees.
like what is available for finite-armed bandits, but historically the asymptotic analysis has served as a useful guide towards understanding the trade-offs in finite-time. Besides hiding the structure of specific problems, pushing for optimality in the worst-case regime can also lead to sub-optimal instance-dependent guarantees. For example, the MOSS algorithm for finite-armed bandits is minimax optimal, but far from finite-time optimal [Audibert and Bubeck, 2009]. For these reasons we believe that understanding the asymptoics of a problem is a useful first step towards optimal finite-time instance-dependent guarantees that are most desirable.

It is worth mentioning that partial monitoring (a more complicated online learning setting) is a well known example of the failure of optimism [Bartók et al., 2014]. Although related, the partial monitoring framework is more general than the bandit setting because the learner may not observe the reward even for the action they take, which means that additional exploration is usually necessary in order to gain information. Basic results in partial monitoring are concerned with characterizing which instance is easier or harder than bandit instances. More recently, the question of asymptotic instance optimality was studied in finite stochastic partial monitoring [Komiya et al., 2015], and the special setting of learning with side information [Wu et al., 2015]. While the algorithms derived in these works served as inspiration, the analysis and the algorithms do not generalise in a simple direct fashion to the linear setting, which requires a careful study of how information is transferred between actions in a linear setting.

2 NOTATION

For positive semidefinite $G$ (written as $G \succeq 0$) and vector $x$ we write $\|x\|_G = x^T G x$. The Euclidean norm of a vector $x \in \mathbb{R}^d$ is $\|x\|$ and the spectral norm of a matrix $A$ is $\|A\|$. The pseudo-inverse of a matrix $A$ is denoted by $A^+$. The mean of arm $x \in A$ is $\mu_x = \langle x, \theta \rangle$ and the optimal mean is $\mu^* = \max_{x \in A} \mu_x$. Let $x^* \in A$ be any optimal action such that $\mu_{x^*} = \mu^*$. The sub-optimality gap of arm $x$ is $\Delta_x = \mu^* - \mu_x$ and $\Delta_{\min} = \min \{ \Delta_x : x \in A \}$ and $\Delta_{\max} = \max \{ \Delta_x : x \in A \}$. The number of times arm $x$ has been chosen after round $t$ is denoted by $T^x(t) = \sum_{s=1}^t 1 \{ A_s = x \}$ and $T_x(t) = \sum_{s=1}^t 1 \{ \mu_{A_s} = \mu^* \}$. A policy $\pi$ is consistent if for all $\theta$ and $p > 0$ it holds that $R^\pi_n(n) = o(n^p)$. Note that this is equivalent to $R^\pi_n(n) = O(n^p)$ and also to $\limsup_{n \to \infty} \log(R^\pi_n(n))/\log(n) \leq 0$. When more precise, we will use the more precise Landau notation $o(n) \in O(b_n)$ (also with $\Omega$, $o$ and $\omega$). Vectors in $\mathbb{R}^k$ will often be indexed by the action set, which we assume has an arbitrary fixed order. For example, we might write $\alpha \in \mathbb{R}^k$ and refer to $\alpha_x \in \mathbb{R}$ for some $x \in A$.

3 LOWER BOUND

We note first that the finite-armed UCB algorithm of Agrawal [1995], Katehakis and Robbins [1995] can be used on this problem by disregarding the structure on the arms to achieve an asymptotic regret of

$$\limsup_{n \to \infty} \frac{R_{\text{UCB}}^\pi(n)}{\log(n)} = \sum_{x \in A: \Delta_x > 0} \frac{2}{\Delta_x}.$$ 

This quantity depends linearly on the number of suboptimal arms, which may be very large (much larger than the dimension) and is very undesirable. Nevertheless we immediately observe that the asymptotic regret should be logarithmic. The following theorem and its corollary characterises the optimal asymptotic regret.

**Theorem 1.** Fix $\theta \in \mathbb{R}^d$ such that there is a unique optimal arm. Let $\pi$ be a consistent policy and let $G_n = \mathbb{E} \left[ \sum_{t=1}^n A_t A_t^T \right]$, which we assume is invertible for sufficiently large $n$. Then for all suboptimal $x \in A$ it holds that

$$\limsup_{n \to \infty} \log(n) \|x - x^*\|_{G_n^{-1}}^2 \leq \frac{\Delta_x^2}{2}.$$ 

The astute reader may recognize $\|x - x^*\|_{G_n^{-1}}$ as the leading factor in the width of the confidence interval for estimating the gap $\Delta_x$ using a linear least squares estimator. The result says that this width has to shrink at least logarithmically with a specific constant. Before the proof of Theorem 1 we present a trivial corollary and some consequences. The assumption that $G_n$ is eventually invertible can be relaxed. In fact, if $G_n$ is not eventually invertible, then the algorithm must suffer linear regret on some problem. This is quite natural because a singular $G_n$ implies the algorithm has not explored at all in some direction. The proof of this fact may be found in Appendix C.

**Corollary 2.** Let $\pi$ be a consistent policy, $\theta \in \mathbb{R}^d$ such that there is a unique optimal arm in $A$. Then

$$\limsup_{n \to \infty} \log(n) \|x\|_{G_n^{-1}}^2 \leq \frac{\Delta_x^2}{2}$$

and also $\limsup_{n \to \infty} \frac{R^\pi_n(n)}{\log(n)} \geq c(A, \theta),$

where $c(A, \theta)$ is defined as the solution to the following

$$\sum_{x \in A: \Delta_x > 0} \frac{2}{\Delta_x}.$$ 


optimisation problem:

\[
\inf_{\alpha \in (0, \infty)^A} \sum_{x \in A} \alpha(x) \Delta x, \quad \text{subject to} \\
\|x\|_{H^{-1}(\alpha)}^2 \leq \frac{\Delta^2}{2}, \quad \forall x \in A^-,
\]

where \( H(\alpha) = \sum_{x \in A} \alpha(x)xx^T \).

As with the previous result, in (1) the reader may recognize the leading term of the confidence width for estimating the mean reward of \( x \). Unsurprisingly, the width of this confidence interval has to shrink at least as fast as the width of the confidence interval for estimating the gap \( \Delta_x \). The intuition underlying the optimisation problem (2) is that no consistent strategy can escape allocating samples so that the gaps of all suboptimal actions are identified with high confidence, while a good strategy will also minimise the regret subject to the identifiability condition. The proof of Corollary 2 is given in Appendix B.

**Example 3** (Finite armed bandits). Suppose \( k = d \) and \( \mathcal{A} = \{e_1, \ldots, e_k\} \) be the standard basis vectors. Then

\[
c(\mathcal{A}, \theta) = \sum_{x \in \mathcal{A}, \Delta x > 0} \frac{2}{\Delta x},
\]

which recovers the lower bound by Lai and Robbins [1985].

**Example 4.** Let \( \alpha > 1 \) and \( d = 2 \) and \( \mathcal{A} = \{x_1, x_2, x_3\} \) with \( x_1 = (1, 0) \) and \( x_2 = (0, 1) \) and \( x_3 = (1 - \varepsilon, \alpha \varepsilon) \) and \( \theta = (1, 0) \). Then \( c(\mathcal{A}, \theta) = 2\alpha^2 \) for all sufficiently small \( \varepsilon \). The example serves to illustrate the interesting fact that \( c(\mathcal{A} - \{x_2\}, \theta) = 2\varepsilon^{-1} \gg c(\mathcal{A}, \theta) \), which means that the problem becomes significantly harder if \( x_2 \) is removed from the action-set. The reason is that \( x_1 \) and \( x_3 \) are pointing in nearly the same direction, so learning the difference is very challenging. But determining which of \( x_1 \) and \( x_3 \) is optimal is easy by playing \( x_2 \). So we see that in linear bandits there is a complicated trade-off between information and regret that makes the structure of the optimal strategy more interesting than in the finite setting.

The closest prior work to our lower bound is by Komiyama et al. [2015] and Agrawal et al. [1989]. The latter consider stochastic partial monitoring when the reward is part of the observation. In this setting in each round, the learner selects one of finitely many actions and receives an observation from a distribution that depends on the action chosen and an unknown parameter, but is otherwise known. While this model could cover our setting, the results in the paper are developed only for the case when the unknown parameter belongs to a finite set, an assumption that all the results of the paper heavily depend on. Komiyama et al. [2015] on the other hand restricts partial monitoring to the case when the observations belong to a finite set, while the parameter belongs to the unit simplex. While this problem also has a linear structure, their results do not generalize beyond the discrete observation setting.

### 4 PROOF OF THEOREM 1

We make use of two standard results from information theory. The first is a high probability version of Pinsker’s inequality.

**Lemma 5.** Let \( \mathbb{P} \) and \( \mathbb{P}' \) be measures on the same measurable space \((\Omega, \mathcal{F})\). Then for any event \( A \in \mathcal{F} \),

\[
\mathbb{P}(A) + \mathbb{P}'(A^c) \geq \frac{1}{2} \exp \left( -\text{KL}(\mathbb{P}, \mathbb{P}') \right),
\]

where \( A^c \) is the complementer event of \( A \) \((\Omega \setminus A) \) and \( \text{KL}(\mathbb{P}, \mathbb{P}') \) is the relative entropy between \( \mathbb{P} \) and \( \mathbb{P}' \), which is defined as \(+\infty\) if \( \mathbb{P} \) is not absolutely continuous with respect to \( \mathbb{P}' \), and is \( \int_{\Omega} d\mathbb{P}(\omega) \log \frac{d\mathbb{P}}{d\mathbb{P}'}(\omega) \) otherwise.

This result follows easily from Lemma 2.6 of Tsybakov [2008].

The second lemma is sometimes called the information processing lemma and shows that the relative entropy between measures on sequences of outcomes for the same algorithm interacting with different bandits can be decomposed in terms of the expected number of times each arm is chosen and the relative entropies of the distributions of the arms. There are many versions of this result (e.g., Agrawal et al. [1999] and Gerchinovitz and Lattimore [2016]). To state the result, assume without the loss of generality that the measure space underlying the action-reward sequence \((A_1, Y_1, \ldots, A_n, Y_n) \in \Omega_n \approx (\mathcal{A} \times \mathbb{R})^n \) and \( A_t \) and \( Y_t \) are the respective coordinate projections: \( A_t(a_1, y_1, \ldots, a_n, y_n) = a_t \) and \( Y_t(a_1, y_1, \ldots, a_n, y_n) = y_t, 1 \leq t \leq n \).

**Lemma 6.** Let \( \mathbb{P} \) and \( \mathbb{P}' \) be the probability measures on the sequence \((A_1, Y_1, \ldots, A_n, Y_n) \in \Omega_n \) for a fixed bandit policy \( \pi \) interacting with a linear bandit with standard Gaussian noise and parameters \( \theta \) and \( \theta' \) respectively. Under these conditions the KL divergence of \( \mathbb{P} \) and \( \mathbb{P}' \) can be computed exactly and is given by

\[
\text{KL}(\mathbb{P}, \mathbb{P}') = \frac{1}{2} \sum_{x \in \mathcal{A}} \mathbb{E}[T_x(n)] (x, \theta - \theta')^2,
\]

where \( \mathbb{E} \) is the expectation operator induced by \( \mathbb{P} \).

**Proof of Theorem 1.** Recall that \( x^* \) is the optimal arm, which we assumed to be unique. Let \( x \in \mathcal{A} \) be
a suboptimal arm (so $\Delta_x > 0$) and $A \subset \Omega_n$ be an event to be chosen later. Rearranging (3) gives

$$\text{KL}(\mathbb{P}, \mathbb{P}') \geq \log \left( \frac{1}{2P(A) + 2P'(A')} \right)$$

and recalling that $\tilde{G}_n = \mathbb{E} \left[ \sum_{t=1}^n A_t A_t^\top \right]$, together with Lemma 6 we get that

$$\frac{1}{2} \| \theta - \theta' \|^2_{\tilde{G}_n} = \text{KL}(\mathbb{P}, \mathbb{P}') \geq \log \left( \frac{1}{2P(A) + 2P'(A')} \right).$$  \hspace{1cm} (5)

Now we choose $\theta'$ “close” to $\theta$, but in such a way that $\langle x - x^*, \theta' \rangle > 0$, meaning in the bandit determined by $\theta'$ the optimal action is not $x^*$. Selecting $A = \{ T_x(n) \leq n/2 \}$ ensures that $\mathbb{P}(A) + \mathbb{P}'(A')$ is small, because $\pi$ is consistent. Intuitively, this holds because if $\mathbb{P}(A)$ is large then $x^*$ is not used much in $\theta$, hence $R_n \doteq R_n^\pi(n)$ must be large. If $\mathbb{P}'(A')$ is large, then $x^*$ is used often in $\theta'$, hence $R_n' \doteq R_n^\pi(n)$ must be large. But from the consistency of $\pi$ we know that both $R_n$ and $R_n'$ are sub-polynomial. Let $\varepsilon > 0$ and $H \geq 0$ ($H \in \mathbb{R}^{d \times d}$) to be chosen later and define $\theta'$ by

$$\theta' = \theta + \frac{H(x - x^*)}{\| x - x^* \|^2_H} (\Delta_x + \varepsilon),$$  \hspace{1cm} (6)

where we also restrict $H$ so that $\| x - x^* \|^2_H > 0$. Then,

$$\langle x - x^*, \theta' \rangle = \langle x - x^*, \theta \rangle + \Delta_x + \varepsilon = \varepsilon > 0.$$  \hspace{1cm} (7)

Hence the mean reward of $x$ is higher than that of $x^*$ in $\theta'$.

$$R_n = \sum_x \Delta_x \mathbb{E} \left[ T_x(n) \right] \geq \Delta_{\min} \mathbb{E} \left[ \mathbf{1} \{ T_x(n) \leq n/2 \} \right]$$

$$\geq \Delta_{\min} \mathbb{E} \left[ \mathbf{1} \{ T_x(n) \leq n/2 \} \right] \frac{n}{2}$$

$$= \frac{\Delta_{\min} n}{2} \mathbb{P} \left( T_x(n) \leq n/2 \right).$$

On the other hand, introducing $\Delta'_n = \max_x \{ \langle x, \theta' \rangle - \langle y, \theta' \rangle \}$ and $\mathbb{E}'$ to denote the expectation operator induced by $\mathbb{P}'$ and using that by (7), $x^*$ is suboptimal in $\theta'$, we also have

$$R'_n = \sum_x \Delta'_n \mathbb{E}' \left[ T_x(n) \right] \geq \Delta'_n \mathbb{E}' \left[ T_x(n) \right]$$

$$\geq \varepsilon \mathbb{E}' \left[ \mathbf{1} \{ T_x(n) > n/2 \} \right]$$

$$\geq \varepsilon \frac{n}{2} \mathbb{P}' \left( T_x(n) > n/2 \right).$$

Adding up the two inequalities and lower bounding $\varepsilon + \Delta_{\min}$ by $2\varepsilon$, which holds when $\varepsilon \leq \Delta_{\min}$ (which we assume from now on), we get

$$\frac{R_n + R'_n}{\varepsilon n} \geq \mathbb{P} \left( T_x(n) \leq n/2 \right) + \mathbb{P}' \left( T_x(n) > n/2 \right),$$  \hspace{1cm} (8)

which completes the proof that $\mathbb{P} \left( T_x(n) \leq n/2 \right) + \mathbb{P}' \left( T_x(n) > n/2 \right)$ is indeed small. Now we calculate the term on the left-hand side of (5). Using the definition of $\theta'$, we get

$$\frac{1}{2} \| \theta - \theta' \|^2_{\tilde{G}_n} = \frac{(\Delta_x + \varepsilon)^2}{2} \| x - x^* \|^2_H \frac{n}{n/2}$$

$$= \frac{(\Delta_x + \varepsilon)^2}{2 \| s \|^2_{\tilde{G}_n^{-1}}} \rho_n(H)$$

where in the last line we introduced

$$\rho_n(H) = \frac{\| s \|^2_{\tilde{G}_n^{-1}} \| s \|^2_H}{\| s \|^2_H}.$$  \hspace{1cm} (9)

Combining this with (8), (5) and some algebra gives

$$\frac{(\Delta_x + \varepsilon)^2}{2 \log(n)} \| s \|^2_{\tilde{G}_n^{-1}} \geq 1 - \frac{\log(n)}{\log(n)}.$$  \hspace{1cm} (10)

Now take a subsequence $\{ \tilde{G}_{n_k} \}_{k=1}^\infty$ such that

$$c \doteq \lim_{n \to \infty} \log(n) \| s \|^2_{\tilde{G}_{n_k}} = \lim_{k \to \infty} \log(n_k) \| s \|^2_{\tilde{G}_{n_k}}.$$  \hspace{1cm} (11)

Hence,

$$\liminf_{n \to \infty} \frac{\rho_n(H)}{\log(n) \| s \|^2_{\tilde{G}_{n_k}}} \leq \liminf_{k \to \infty} \frac{\rho_n(H)}{\log(n_k) \| s \|^2_{\tilde{G}_{n_k}}}$$

$$= \liminf_{k \to \infty} \frac{\rho_n(H)}{\log(n_k) \| s \|^2_{\tilde{G}_{n_k}}}$$

$$\leq \frac{c}{c}.$$  \hspace{1cm} (11)

Let $\tilde{H}_n = \tilde{G}_{n_k}^{-1} \| s \|^2_{\tilde{G}_{n_k}}$. A simple calculation gives $\rho_n(H) = \| s \|^2_{\tilde{H}_n} \| s \|^2_{\tilde{H}_n^{-1}} \| s \|^2_H$ and hence if $H$ is any cluster point of $\{ \tilde{H}_{n_k} \}_k$, say, the subsequence $\{ \tilde{H}_{n_k} \}_k$ of the subsequence $\{ \tilde{H}_{n_k} \}_k$ converges to $H$, and $\| s \|^2_H > 0$ then

$$\liminf_{k \to \infty} \| s \|^2_{\tilde{H}_{n_k}} \| s \|^2_{\tilde{H}_{n_k}^{-1}} \| s \|^2_H$$

$$\leq \lim_{k \to \infty} \| s \|^2_{\tilde{H}_{n_k}} \| s \|^2_{\tilde{H}_{n_k}^{-1}} \| s \|^2_H$$

$$= \| s \|^2_H \| s \|^2_H = 1,$$  \hspace{1cm} (11)
showing that
\[
1 \leq \lim inf_{n \to \infty} \frac{(\Delta_x + \varepsilon)^2 \rho_n(H)}{2 \log(n)} \leq \frac{(\Delta_x + \varepsilon)^2}{2c}.
\]
Since \(\varepsilon > 0\) was arbitrary small, the result will follow once we establish that \(\|s\|_H > 0\). To show this, assume on the contrary that \(\|s\|_H = 0\). This implies that \(HS = 0\) and through \(\ker(H) = \ker(H^{-1})\) it also implies that \(H^{-1}s = 0\). Let \(H_\gamma = H + \gamma I\), where \(I\) is the \(d \times d\) identity matrix. Then, \(H_\gamma s = \gamma s\), so \(\|s\|_H^2 = \gamma \|s\| > 0\) and thus
\[
\lim inf_{k \to \infty} \rho_n(H_\gamma) \leq \lim_{k \to \infty} \|s\|_{H_\gamma}^2 \leq \|s\|_{H_\gamma}^2 = \|s\|_{H_\gamma}^2 = \|s\|_{H^{-1}}^2 = 0.
\]
Chaining (10), (11) and the last display gives \(1 \leq 0\), a contradiction. Thus, \(\|s\|_H > 0\) must hold, finishing the proof.

**Remark 7.** The uniqueness assumption of the theorem can be lifted at the price of more work and by slightly changing the theorem statement. In particular, the theorem statement must be restricted to those suboptimal actions \(x \in \mathcal{A}^*\) that can be made optimal by changing \(\theta\) to \(\theta'\), while none of the optimal actions \(\mathcal{A}^*(\theta) = \{x \in \mathcal{A} : \langle x, \theta \rangle = \max_{y \in \mathcal{A}} \langle y, \theta \rangle\}\) are optimal. That is, the statement only concerns \(x \in \mathcal{A}\) such that \(x \notin \mathcal{A}^*(\theta)\) but there exists \(\theta' \in \mathbb{R}^d\) such that \(\mathcal{A}^*(\theta') \cap \mathcal{A}^*(\theta) = \emptyset\) and \(x \in \mathcal{A}^*(\theta')\). The choice of \(\theta'\) would still be as before, except that \(x^*\) is selected as the optimal action under \(\theta\) that maximizes \(c(H, \theta) = \inf_{x' \in \mathcal{A}^*(\theta)} \langle x - x', x - x^* \rangle_H\). Then, in the proof, \(T_x(n)\) has to be redefined to be \(\sum_{x \in \mathcal{A}^*(\theta)} T_x(n)\) (the total number of times an optimal action is chosen), and at the end one also needs to show that the chosen \(H\) satisfies \(c(H, \theta) > 0\).

## 5 CONCENTRATION

Before introducing the new algorithm we analyse the concentration properties of the least squares estimator. Our results refine the existing guarantees by Abbasi-Yadkori et al. [2011], and are necessary in order to obtain asymptotic optimality. Let \(G_t\) be the Gram matrix after round \(t\) defined by \(G_t = \sum_{s \leq t} A_s A_s^T\) and \(\hat{\theta}(t) = G_t^{-1} \sum_{s=1}^t A_s Y_s\) be the empirical (least squares) estimate, where \(A_s\) is selected based on \(A_1, Y_1, \ldots, A_s-1, Y_{s-1}\) and \(Y_s = \langle A_s, \theta \rangle + \eta_s, \eta_s \sim N(0,1)\). We will only use \(\hat{\theta}(t)\) for rounds \(t\) when \(G_t\) is invertible. The empirical estimate of the suboptimal gaps is \(\hat{\Delta}_x(t) = \max_{y \in \mathcal{A}} \hat{\mu}_y(t) - \hat{\mu}_x(t), \) where \(\hat{\mu}_x(t) = \langle x, \hat{\theta}(t) \rangle\). We will also use the notation \(\hat{\mu}(t)\) and \(\hat{\Delta}(t) \in \mathbb{R}^k\) for vectors of empirical means and sub-optimality gaps (indexed by the arms).

**Theorem 8.** For any \(\delta \in [1/n, 1]\), \(n\) sufficiently large and \(t_0 \in \mathbb{N}\) such that \(G_{t_0}\) is almost surely non-singular,
\[
\mathbb{P}\left(\exists t \geq t_0, x : |\hat{\mu}_x(t) - \mu_x| \geq \sqrt{\|x\|_{G_{t_0}^{-1}}^2 f_n, \delta} \right) \leq \delta,
\]
where for some \(c > 0\) universal constant
\[
f_n, \delta = 2 \left(1 + \frac{1}{\log(n)}\right) \log(1/\delta) + cd \log(d \log(n))
\]
The result improves on the elegant concentration guarantee of Abbasi-Yadkori et al. [2011] because asymptotically we have \(f_{n,1/n} \sim 2 \log(n)\), while there it was \(2d \log(n)\). Note that the restriction on \(\delta\) may be relaxed with a small additional argument. The proof of Theorem 8 relies on a peeling argument and is given in Appendix A. For the remainder we abbreviate \(f_n = f_{n,1/n}\) and \(g_n = f_{n,1/log(n)}\), which are chosen so that
\[
\mathbb{P}\left(\exists t \geq t_0, x : |\hat{\mu}_x(t) - \mu_x| \geq \sqrt{\|x\|_{G_{t_0}^{-1}}^2 f_n} \right) \leq \frac{1}{n},
\]
\[
\mathbb{P}\left(\exists t \geq t_0, x : |\hat{\mu}_x(t) - \mu_x| \geq \sqrt{\|x\|_{G_{t_0}^{-1}}^2 g_n} \right) \leq \frac{1}{\log(n)}.
\]

## 6 OPTIMAL STRATEGY

A barycentric spanner of the action space is a set \(B = \{x_1, \ldots, x_d\} \subseteq \mathcal{A}\) such that for any \(x \in \mathcal{A}\) there exists an \(\alpha \in [-1, 1]^d\) with \(x = \sum_{i=1}^d \alpha_i x_i\). The existence of a barycentric spanner is guaranteed because \(\mathcal{A}\) is finite and spans \(\mathbb{R}^d\) [Awerbuch and Kleinberg, 2004]. We propose a simple strategy that operates in three phases called the **warm-up** phase, the **success** phase and the **recovery** phase. In the warm-up the algorithm deterministically chooses its actions from a barycentric spanner to obtain a rough estimate of the sub-optimality gaps. The algorithm then uses the estimated gaps as a substitute for the true gaps to determine the optimal pull counts for each action, and starts implementing this strategy. Finally, if an anomaly is detected that indicates the inaccuracy of the estimated gaps then the algorithm switches to the recovery phase where it simply plays UCB.

**Definition 9.** For any \(\Delta \in [0, \infty]^k\) define \(T_x(\Delta) \in [0, \infty]^k\) to be a solution to the optimisation problem
\[
\min_{T \in [0, \infty]^k} \sum_{x \in \mathcal{A}} T_x \Delta_x \text{ subject to } \|x\|_{H_T}^2 \leq \frac{\Delta_x^2}{f_n} \text{ for all } x \in \mathcal{A}, \text{ where } H_T = \sum_{x \in \mathcal{A}} T_x x x^T.
\]
Algorithm 1 Optimal Algorithm

1: Input: $\mathcal{A}$ and $n$
2: // Warmup phase
3: Find a barycentric spanner: $B = \{x_1, \ldots, x_d\}$
4: Choose each arm in $B$ exactly $\lceil \log^{1/2}(n) \rceil$ times
5: // Success phase
6: $\varepsilon_n \leftarrow \max_{x \in \mathcal{A}} \|x\|_{G_t^{-1}} g_n^{1/2}$, $t \leftarrow n + 1$
7: $\hat{\Delta} \leftarrow \hat{\Delta}(t-1)$ and $\hat{T} \leftarrow T_n(\hat{\Delta})$ and $\hat{\mu} \leftarrow \hat{\mu}(t-1)$
8: while $t \leq n$ and $\|\hat{\mu} - \hat{\mu}(t-1)\|_{\infty} \leq 2\varepsilon_n$ do
9: Play actions $x$ with $T_x(t) \leq T_x$, $t \leftarrow t + 1$
10: end while
11: // Recovery phase
12: Discard all data and play UCB until $t = n$.

Theorem 10. Assuming that $x^*$ is unique, the strategy given in Algorithm 1 satisfies

$$
\limsup_{n \to \infty} \frac{R_n^*(n)}{\log(n)} \leq c(\mathcal{A}, \theta) \text{ for all } \theta \in \mathbb{R}^d .
$$

7 PROOF OF THEOREM 10

We analyse the regret in each of the three phases. The warm-up phase has length $d\lceil \log^{1/2}(n) \rceil$, so its contribution to the asymptotic regret is negligible. There are two challenges. The first is to show that the recovery phase happens with probability at most $1/\log(n)$. Then, since the regret in the recovery phase is logarithmic by known results for UCB, this ensures that the expected regret incurred in the recovery phase is also negligible. The second challenge is to show that the expected regret incurred during the success phase is asymptotically matching the lower bound in Theorem 1.

The set of rounds when the algorithm is in the warm-up/success/recovery phases are denoted by $T_{\text{warm}}$, $T_{\text{succ}}$, and $T_{\text{rec}}$, respectively. We introduce two failure events that occur when the errors in the empirical estimates of the arms are excessively large. Let $F_n$ be the event that there exists an arm $x$ and round $t \geq d$ such that

$$
\hat{\mu}_x(t) - \mu_x \geq \sqrt{\|x\|^2_{G_t^{-1}} g_n}.
$$

Similarly, let $F'_n$ be the event that there exists an arm $x$ and round $t \geq d$ such that

$$
\hat{\mu}_x(t) - \mu_x \geq \sqrt{\|x\|^2_{G_t^{-1}} f_n}.
$$

Theorem 8 with $t_0 = d$ and (12) imply that $\mathbb{P}(F_n) \leq 1/\log(n)$ and $\mathbb{P}(F'_n) \leq 1/n$. The failure events determine the quality of the estimates throughout time. The following two lemmas show that if $F_n$ does not occur then the regret is asymptotically optimal, while if $F'_n$ occurs then the regret is logarithmic with some constant factor that depends only on the problem (determined by the action set $\mathcal{A}$ and the parameter $\theta$). Since $F'_n$ occurs with probability at most $1/\log(n)$, the contribution of the latter component is negligible asymptotically.

Lemma 11. If $F_n$ does not occur then Algorithm 1 never enters the recovery phase. Furthermore,

$$
\limsup_{n \to \infty} \mathbb{E}\left[ \frac{\mathbb{1}\{\text{not } F_n\} \sum_{i \in T_{\text{rec}}} \Delta_{Ai}}{\log(n)} \right] \leq c(\mathcal{A}, \theta) .
$$

Before proving Lemma 11 we need a naive bound on the solution to the optimisation problem, the proof of which is given in Appendix D.

Lemma 12. Let $T = T_n(\Delta)$ for any $n$. Then

$$
\sum_{x : \Delta_x > 0} T_x \leq \frac{2d^3 f_n \Delta_{\text{max}}}{\Delta_{\text{min}}} .
$$

Proof of Lemma 11. First, if $t = d\lceil \log^{1/2}(n) \rceil$ is the round at the end of the warm-up period then by the definition of the algorithm there is a barycentric spanner $B = \{x_1, \ldots, x_d\}$ and $T_x(t) = \lceil \log^{1/2}(n) \rceil$ for $1 \leq i \leq d$. Let $x \in \mathcal{A}$ be arbitrary. Then, by the definition of the barycentric spanner, we can write $x = \sum_{i=1}^d \alpha_i x_i$ where $\alpha_i \in [-1, 1]$ for all $i$. Therefore,

$$
\|x\|_{G_t^{-1}} \leq \sum_{i=1}^d \|x_i\|_{G_t^{-1}} \leq \frac{d}{\log^{1/4}(n)} .
$$

Recalling the definition of $\varepsilon_n$ in the algorithm we have

$$
\varepsilon_n = \max_{x \in \mathcal{A}} \|x\|_{G_t^{-1}} \sqrt{g_n} = O\left( \frac{d \log^{1/2}(\log(n))}{\log^{1/4}(n)} \right) .
$$

Consider the case when $F_n$ does not hold. Then, for all arms $x$ and rounds $t$ after the warm-up period we have

$$
|\hat{\mu}_x(t) - \mu_x| \leq \|x\|_{G_t^{-1}} \sqrt{g_n} \leq \varepsilon_n ,
$$

Therefore for all $s$, $t$ after the warm-up period we have $|\hat{\mu}_x(t) - \hat{\mu}_x(s)| \leq 2\varepsilon_n$, which means the success phase never ends and so the first part of the lemma is proven. It remains to bound the regret. Since we are only concerned with the asymptotics we may take $n$ to be large enough so that $2\varepsilon_n \leq \Delta_{\text{min}}/2$, which implies that $\hat{\Delta}_x = 0$. For $T_n(\Delta)$, the solution to the optimisation problem in Definition 9 with the true gaps, it holds that

$$
\limsup_{n \to \infty} \sum_{x \neq x^*} T_{n,x}(\Delta_x) \Delta_x = c(\mathcal{A}, \theta) .
$$

Letting $T^* = T_n(\Delta)$ and $1 + \delta_n = \max_{x : \Delta_x > 0} \frac{\Delta^2_x}{\hat{\Delta}^2_x}$, we have

$$
\|x\|^2_{H_{[1 + \delta_n] T^*}} = \frac{\|x\|^2_{H_{\Delta^2_x}}} {1 + \delta_n} \leq \frac{\Delta^2_x}{(1 + \delta_n) f_n} \leq \frac{\hat{\Delta}^2_x}{f_n} .
$$
Therefore, \( \sum_{x \neq x^*} T_x \Delta_x \leq (1 + \delta_n) \sum_{x \neq x^*} T_x \Delta_x \), where \( T = (T_x)_x \equiv T_\delta(n) \). Also,
\[
1 + \delta_n = \max_{x : \Delta_x > 0} \frac{\Delta_x^2}{\Delta_x} \leq \max_{x : \Delta_x > 0} \frac{\Delta_x^2}{(\Delta_x - 2 \varepsilon_n)^2} \\
= \max_{x : \Delta_x > 0} \left( 1 + \frac{4(\Delta_x - \varepsilon_n)\varepsilon_n}{(\Delta_x - 2 \varepsilon_n)^2} \right) \leq 1 + \frac{16 \varepsilon_n}{\Delta_{\min}}, \quad (14)
\]
where in the last inequality we used the fact that \( 0 \leq 2 \varepsilon_n \leq \Delta_{\min}/2 \). Then the regret in the success phase is
\[
\sum_{t \in \text{rec.}} \Delta_{A_t} \leq \sum_{x \neq x^*} T_x \Delta_x \\
= \sum_{x \neq x^*} T_x \Delta_x + \sum_{x \neq x^*} T_x (\Delta_x - \Delta_x) \\
\leq (1 + \delta_n) \sum_{x \neq x^*} T_x \Delta_x + 2 \varepsilon_n \sum_{x \neq x^*} T_x \\
\leq (1 + \delta_n) \sum_{x \neq x^*} T_x \Delta_x + 2 \varepsilon_n \sum_{x \neq x^*} ((1 + \delta_n) T_x^* + T_x).
\]
The result follows by taking the limit as \( n \) tends to infinity and from Lemma 12 and (13) and (14), together with the reverse Fatou lemma.

Our second lemma shows that provided \( F'_n \) fails, the regret in the success phase is at most logarithmic:

**Lemma 13.** It holds that:
\[
\limsup_{n \to \infty} \frac{\mathbb{E} [ \mathbb{I} \{ F_n \text{ and not } F'_n \} \sum_{t \in \text{rec.}} \Delta_{A_t} ]}{\log(n)} = 0.
\]
The proof follows by showing the existence of a constant \( m \) that depends on \( \mathcal{A} \) and \( \theta \), but not \( n \) such that the regret suffered in the success phase whenever \( F'_n \) does not hold is almost surely at most \( m \log(n) \). The result follows from this because \( \mathbb{P}(F_n) \leq 1/\log(n) \). See Appendix E for details.

**Proof of Theorem 10.** We decompose the regret into the regret suffered in each of the phases:
\[
R_\theta^*_c(n) = \mathbb{E} \left[ \sum_{t \in \text{rec.}} \Delta_{A_t} + \sum_{t \in \text{rec.}} \Delta_{A_t} + \sum_{t \in \text{rec.}} \Delta_{A_t} \right].
\]
The warm-up phase has length \( d \lfloor \log^{1/2}(n) \rfloor \), which contributes asymptotically negligibly to the regret:
\[
\limsup_{n \to \infty} \frac{\mathbb{E} [ \sum_{t \in \text{rec.}} \Delta_{A_t} ]}{\log(n)} = 0. \quad (16)
\]
By Lemma 11, the recovery phase only occurs if \( F_n \) occurs and \( \mathbb{P}(F_n) \leq 1/\log(n) \). Therefore by well-known guarantees for UCB [Bubeck and Cesa-Bianchi, 2012] there exists a universal constant \( c > 0 \) such that
\[
\mathbb{E} \left[ \sum_{t \in \text{rec.}} \Delta_{A_t} \right] = \mathbb{E} \left[ \sum_{t \in \text{rec.}} \Delta_{A_t} | T_{\text{rec.}} \neq \emptyset \right] \mathbb{P}(T_{\text{rec.}} \neq \emptyset) \\
\leq \frac{ck \log(n)}{\Delta_{\min}} \mathbb{P}(T_{\text{rec.}} \neq \emptyset) \leq \frac{ck}{\Delta_{\min}}.
\]
Therefore
\[
\limsup_{n \to \infty} \frac{\mathbb{E} [ \sum_{t \in \text{rec.}} \Delta_{A_t} ]}{\log(n)} = 0. \quad (17)
\]
Finally we use the previous lemmas to analyse the regret in the success phase:
\[
\mathbb{E} \left[ \sum_{t \in \text{rec.}} \Delta_{A_t} \right] = \mathbb{E} \left[ \mathbb{I} \{ \text{not } F_n \} \sum_{t \in \text{rec.}} \Delta_{A_t} \right] \\
+ \mathbb{E} \left[ \mathbb{I} \{ F_n \text{ and not } F'_n \} \sum_{t \in \text{rec.}} \Delta_{A_t} \right] \\
+ \mathbb{E} \left[ \mathbb{I} \{ F'_n \} \sum_{t \in \text{rec.}} \Delta_{A_t} \right]. \quad (18)
\]
By (12), the last term satisfies
\[
\limsup_{n \to \infty} \frac{\mathbb{E} [ \mathbb{I} \{ F'_n \} \sum_{t \in \text{rec.}} \Delta_{A_t} ]}{\log(n)} \leq \limsup_{n \to \infty} \frac{n \Delta_{\max} \mathbb{P}(F'_n)}{\log(n)} = 0.
\]
The first two terms in (18) are bounded using Lemmas 11 and 13, leading to
\[
\limsup_{n \to \infty} \frac{\mathbb{E} [ \sum_{t \in \text{rec.}} \Delta_{A_t} ]}{\log(n)} \leq c(\mathcal{A}, \theta).
\]
Substituting the above display together with (16) and (17) into (15) completes the result.

### 8 SUB-OPTIMALITY OF OPTIMISM AND THOMPSON SAMPLING

We now argue that algorithms based on optimism or Thompson sampling cannot be close to asymptotically optimal. In each round \( t \) an optimistic algorithm constructs a confidence set \( \mathcal{C}_t \subseteq \mathbb{R}^d \) and chooses \( A_t \) according to \( A_t = \arg \max_{x \in \mathcal{A}} \max_{\theta \in \mathcal{C}_t} \mathbb{E}(x, \theta) \). In order to proceed we need to make some assumptions on \( \mathcal{C}_t \), otherwise one can define a “confidence set” to ensure any behaviour at all. First of all, we will assume that \( \mathbb{P}(\exists t \leq n : \theta \notin \mathcal{C}_t) = O(1/n) \). That is, that the probability that the true parameter is ever outside the confidence set is not too large. Second, we assume that
\( C_t \subseteq E_t \) where \( E_t \) is the ellipsoid about the least squares estimator given by

\[
E_t = \left\{ \hat{\theta} : \| \hat{\theta} - \tilde{\theta} \|_{C_t}^2 \leq \alpha \log(n) \right\},
\]

where \( \alpha \) is some constant and \( \hat{\theta}(t) \) is the empirical estimate of \( \theta \) based on the observations so far. Existing algorithms based on confidence all use such confidence sets. Standard wisdom when designing optimistic algorithms is to use the smallest confidence set possible, so an alternative algorithm that used a different form of confidence set would normally be advised to use the intersection \( C_t \cap E_t \), which remains valid with high probability by a union bound. If the optimistic algorithm is not consistent, then its regret is not logarithmic on some problem and so diverges relative to the optimal strategy. Suppose now that the algorithm is consistent. Then we design a bandit on which its asymptotic regret is worse than optimal by an arbitrarily large constant factor.

Let \( d = 2 \) and \( e_1 = (1, 0) \) and \( e_2 = (0, 1) \) be the standard basis vectors. The counter-example (illustrated in Figure 1) is very simple with \( A = \{e_1, e_2, x\} \) where \( x = (1 - \varepsilon, 8\alpha \varepsilon) \). The true parameter is given by \( \theta = e_1 \), which means that \( x^* = e_1 \) and \( \Delta_x = \varepsilon \). Suppose a consistent optimistic algorithm has chosen \( T_{e_2}(t - 1) \geq 4\alpha \log(n) \) and that \( \theta \in C_t \). Then,

\[
\max_{\theta \in C_t} \langle e_2, \hat{\theta} \rangle \leq \langle e_2, \hat{\theta}(t - 1) \rangle + \sqrt{\|e_2\|^2 \log(n)} \leq 2 \sqrt{\|e_2\|^2 \log(n)} \leq 1.
\]

But because \( \theta \in C_t \), the optimistic value of the optimal action is at least \( \langle e_1, \hat{\theta} \rangle = 1 \), which means that \( A_t \neq e_2 \). We conclude that if \( \theta \in C_t \) for all rounds, then the optimistic algorithm satisfies \( T_{e_2}(t - 1) \leq 1 + 4\alpha \log(n) \).

By the assumption that \( \theta \in C_t \) with probability at least \( 1 - 1/n \) we bound \( \mathbb{E}[T_{e_2}(n)] \leq 2 + 4\alpha \log(n) \). By consistency of the optimistic algorithm and our lower bound (Theorem 1) we have

\[
\limsup_{n \to \infty} \log(n) \| x - e_1 \|_{G_n^{-1}}^2 \leq \frac{\varepsilon^2}{2},
\]

Therefore by choosing \( \varepsilon \) sufficiently small we conclude that \( \limsup_{n \to \infty} \mathbb{E}[T_x(n)] / \log(n) = \Omega(1/\varepsilon^2) \) and so the asymptotic regret of the optimistic algorithm is at least

\[
\limsup_{n \to \infty} \frac{\mathbb{E}[ \text{Reg}(n) ]}{\log(n)} = \Omega \left( \frac{1}{\varepsilon} \right).
\]

However, for small \( \varepsilon \) the optimal regret for this problem is \( c(A, \theta) = 128\alpha^2 \) and so by choosing \( \varepsilon \ll \alpha \) we can see that the optimistic approach is sub-optimal by an arbitrarily large constant factor. The intuition is that the optimistic algorithms very quickly learn that \( e_2 \) is a sub-optimal arm and stop playing it. But as it turns out, the information gained by choosing \( e_2 \) is sufficiently valuable that an optimal algorithm should use it for exploration.

Thompson sampling has also been proposed for the linear bandit problem [Agrawal and Goyal, 2013]. The standard approach uses a nearly flat Gaussian prior (and so posterior), which means that essentially the algorithm operates by sampling \( \theta_t \) from \( \mathcal{N}(\hat{\mu}(t), \alpha G_t^{-1}) \) and choosing the arm \( A_t = \arg \max_{x \in A} \langle x, \theta_t \rangle \). Why does this approach fail? By the assumption of consistency we expect that the optimal arm will be played all but logarithmically often, which means that the posterior will concentrate quickly about the value of the optimal action so that \( \langle x^*, \theta_t \rangle \approx \mu^* \). Then using the same counter-example as for the optimistic algorithm we see that the likelihood that \( \langle e_2 - e_1, \theta_t \rangle \geq 0 \) is vanishingly small once \( T_{e_2}(t - 1) = \Omega(\alpha \log(n)) \) and so Thompson sampling will also fail to sample action \( e_2 \) sufficiently often.

9 SUMMARY

We characterised the optimal asymptotic regret for linear bandits with Gaussian noise and finitely many actions in the sense of Lai and Robbins [1985]. The results highlight a surprising fact that all reasonable algorithms based on optimism can be arbitrarily worse than optimal. While this behaviour has been observed before in more complicated settings (notably, partial monitoring), our results are the first to illustrate this issue in a setting only barely more complicated than finite-armed bandits. Besides this we improve the self-normalised concentration guarantees by Abbasi-Yadkori et al. [2011] by a factor of \( d \) asymptotically.

As usual, we open more questions than we answer. While the proposed strategy is asymptotically optimal, it is also extraordinarily naive and the analysis is far from showing finite-time optimality. For this reason we think the most pressing task is to develop efficient and practical algorithms that exploit the available information in a way that Thompson sampling and optimism do not. There are two natural research directions towards this goal. The first is to push the optimisation approach used here and also by Wu et al. [2015], but applied more “smoothly” without discarding data or long phases. The second is to generalise information-theoretic ideas used (for instance) by Russo and Van Roy [2014] or Reddy et al. [2016].
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A PROOF OF THEOREM 8

Recall that $A_t$ is the action chosen in round $t$ and that $\eta_t = X_t - \langle A_t, \theta \rangle$ is the noise term, which we assumed to be a standard Gaussian. Let $S_t = \sum_{s=1}^t A_s \eta_s$. By assumption, $\|A_t\| \leq 1$ for all $t \geq 1$.

**Lemma 14.** Let $n \in \mathbb{N}$ and $\varepsilon > 0$ and $\sigma^2 > 0$. Let $X_1, X_2, \ldots, X_n$ be a sequence of Gaussian random variables adapted to filtration $F_1, F_2, \ldots$ such that $\mathbb{E}[X_t | F_{t-1}] = 0$. Define $\sigma_t^2 = \text{Var}[X_t | F_{t-1}]$ and assume that $\sigma_t^2 \leq \sigma^2$ almost surely. Then

$$\mathbb{P}\left( \exists t \leq n : \sum_{s=1}^t X_s \geq \sqrt{2 \gamma_n V_t \log \left( \frac{N}{\delta} \right)} \right) \leq \delta,$$

where $V_t = \max \left\{ \varepsilon, \sum_{s=1}^t \sigma_t^2 \right\}$ and

$$\gamma_n = 1 + \frac{1}{\log(n)} \quad \text{and} \quad N = 1 + \left\lceil \frac{\log(n \sigma^2 / \varepsilon)}{\log(\gamma_n)} \right\rceil.$$

**Proof.** For $\psi \in \mathbb{R}$ define

$$M_{t, \psi} = \exp\left( \sum_{s=1}^t \psi X_s - \frac{\psi^2 \sigma_t^2}{2} \right).$$

If $\tau \leq n$ is a stopping time with respect to $F$, then as in the proof [Abbasi-Yadkori et al., 2011, Lemma 8] we have $\mathbb{E}[M_{\tau, \psi}] \leq 1$. Therefore, by Markov’s inequality we have

$$\mathbb{P}\left( M_{\tau, \psi} \geq 1/\delta \right) \leq \delta. \tag{19}$$

For $k \in \{1, 2, \ldots, N\}$ define

$$\psi_k = \sqrt{\frac{2}{\varepsilon \gamma_{k-1}} \log \left( \frac{N}{\delta} \right)}.$$

Then rearranging (19) leads to

$$\mathbb{P}\left( \exists k \in [N] : \sum_{t=1}^{\tau} X_t \geq \frac{1}{\psi_k} \log \left( \frac{N}{\delta} \right) + \frac{\psi_k V_\tau}{2} \right) \leq \delta.$$

Therefore letting

$$k^* = \min \left\{ k \in [N] : \psi_k \geq \sqrt{2 \log(N/\delta)/V_\tau} \right\}$$

leads to

$$\delta \geq \mathbb{P}\left( \sum_{t=1}^{\tau} X_t \geq \frac{1}{\psi_{k^*}} \log \left( \frac{N}{\delta} \right) + \frac{\psi_{k^*} V_\tau}{2} \right) \geq \mathbb{P}\left( \sum_{t=1}^{\tau} X_t \geq \sqrt{2 \gamma_{n} V_\tau \log \left( \frac{N}{\delta} \right)} \right).$$

The result is completed by choosing stopping time $\tau$ by $\tau = \min(n, \tau_n)$, where

$$\tau_n = \min \left\{ t \leq n : \sum_{s=1}^{t} X_s \geq \sqrt{2 \gamma_{n} V_\tau \log \left( \frac{N}{\delta} \right)} \right\}.$$

**Lemma 15.** Let $\delta \in [1/n, 1)$ and $\lambda \in \mathbb{R}^d$ with $\|\lambda\| \leq 1$. Then

$$\mathbb{P}\left( \exists t \leq n : \langle \lambda, S_t \rangle \geq \sqrt{\frac{1}{n^2} \|\lambda\|^2 \log(n) \sqrt{h_{n, \delta}}} \right) \leq \delta,$$

where

$$h_{n, \delta} = 2 \left( 1 + \frac{1}{\log(n)} \right) \log \left( \frac{c \log(n)}{\delta} \right)$$

with some universal constant $c \geq 1$.

**Proof.** We prepare to use the previous lemma. First note that

$$\langle \lambda, S_t \rangle = \sum_{s=1}^{t} \eta_s \langle \lambda, A_s \rangle.$$

Since $\eta_s$ is a standard Gaussian, the predictable variance of the term inside the sum is $\sigma_t^2 = \langle \lambda, A_t \rangle^2 \leq \|\lambda\|^2 \|A_t\|^2 \leq 1$. Therefore

$$\sum_{s=1}^{t} \sigma_s^2 = \lambda^\top \sum_{s=1}^{t} A_s A_s^\top \lambda = \|\lambda\|^2 g_s.$$

Therefore the result follows by the previous lemma with $X_t = \eta_t \langle \lambda, A_t \rangle$ and $\varepsilon = 1/(n^2 \log(n)^3)$ and $\sigma^2 = 1$. $\square$
The following lemma can be extracted from the proof of Theorem 1 in Abbasi-Yadkori et al. [2011].

**Lemma 16.** Assume that \( \{A_x\} \) is such that for some \( t_0 > 0 \), \( G_{t_0} \) is non-singular almost surely. Then, for some \( c > 0 \) universal constant,

\[
P \left( \exists t \geq t_0 : \|S_t\|_{G_t}^2 \geq cd \log(n/\delta) \right) \leq \delta.
\]

**Proof of Theorem 8.** Let \( \varepsilon > 0 \) be some small real number to be tuned subsequently and choose \( C \subseteq \mathbb{R}^d \) to be a finite covering set such that for all \( x \in A \) and \( t \) with \( G_t \) non-singular there exists a \( \lambda \in C \) such that \( \lambda = (I + \varepsilon)G_t^{-1} x \), where \( \varepsilon \) is some diagonal matrix (possibly depending on \( x \) and \( G_{t_0}^{-1} \)) with entries bound in \( [0, \varepsilon] \). Of course \( G_t \) is a random variable, so we insist the existence of \( \lambda \) is almost sure (that is, no matter how the actions are taken). We defer calculating the necessary size \( N = |C| \) until later. Let \( \delta_1 = \delta/(N + 1) \) and \( F_\lambda \) be the event that

\[
F_\lambda = \{ \exists t \geq t_0 : \|S_t\|_{G_t}^2 \geq cd \log(n/\delta_1) \}.
\]

Then a union bound and Lemma 15 leads to

\[
P (\cup_{\lambda \in C} F_\lambda) \leq N \delta_1.
\]  

By Lemma 16, for \( G = \{ \exists t \geq t_0 : \|S_t\|_{G_t}^2 \geq cd \log(n/\delta_1) \} \), we have

\[
P (G) \leq \delta_1.
\]  

Another union bound shows that the \( P (\cup_{\lambda \in C} F_\lambda \cup G) \leq (N + 1) \delta_1 = \delta \). From now on we assume that neither \( F = \cup_{\lambda \in C} F_\lambda \), nor \( G \) occurs and let \( x \in A \) be arbitrary and for \( t \geq t_0 \) let \( \lambda \in C \) be such that \( \lambda = (I + \varepsilon)G_t^{-1} x \) where \( \varepsilon \) is diagonal with entries in \( [0, \varepsilon] \). Then

\[
\hat{u}_x(t) - u_x = \langle G_t^{-1} x, S_t \rangle = \langle G_t^{-1} x - \lambda, S_t \rangle + \langle \lambda, S_t \rangle
\]

\[
\leq \|G_t^{-1} x - \lambda\|_{G_t} \|S_t\|_{G_t} + \|S_t\|_{G_t}^2 \delta \|G_t^{-1}\|_{h_n, \delta_1}.
\]

We bound each term separately using matrix algebra and the assumption that the failure events \( F \) and \( G \) do not occur:

\[
\|G_t^{-1} x - \lambda\|_{G_t} = \|G_t^{-1} x - \varepsilon G_t^{-1} x\|_{G_t} = \|G_t^{1/2} G_t^{-1/2} G_t^{-1/2} x\|_F \leq \|G_t^{1/2} G_t^{-1/2} x\|_F \leq \sqrt{d} \|\nu\|_\infty \leq \varepsilon \sqrt{d}.
\]

Therefore if \( \varepsilon = 1/(d^{3/2} \log(n)) \), then the first term in (23) is bounded by

\[
\|G_t^{-1} x - \lambda\|_{G_t} \|S_t\|_{G_t} = O(1) \cdot \|x\|_{G_t}.
\]

For the second term we proceed similarly:

\[
\| \nu \|_{G_t}^2 = \| G_t^{-1} x + \varepsilon G_t^{-1} x \|_{G_t}^2 \leq \|x\|_{G_t}^2 \left( 1 + \varepsilon \sqrt{d} \right)^2 = (1 + o(1)) \|x\|_{G_t}^2.
\]

Therefore, assuming \( n \) is large enough so that \( 1/n^2 \leq \|x\|/n \leq \|x\|_{G_t}^2 \) (in the unique case that \( \|x\| = 0 \) we simply note that the following equality holds trivially), we have

\[
\|G_t^{-1} x - \lambda\|_{G_t} \|S_t\|_{G_t} = O(1) \sqrt{\|x\|_{G_t}^2 \delta_1}.
\]

Substituting the above expression along with (24) into (23) leads to

\[
\hat{u}_x(t) - u_x = (1 + o(1)) \sqrt{\|x\|_{G_t}^2 \delta_1},
\]

Finally we note that \( C \) can be chosen in such a way that for suitably large universal constant \( c > 0 \) its size is \( \log N = O(d \log d \log(n)) \). This follows by treating each arm \( x \in A \) separately and noting that \( \|x\|/n \leq \|G_t^{-1} x\| \leq \|x\| \). Then letting \( J = \log(n)/\log(1 + \varepsilon) \) we have

\[
C_x = \left\{ \frac{\|x\| (1 + \varepsilon)^j}{n} : 0 \leq j \leq J \right\}.
\]

The theorem is completed by using the definition of \( h_n, \delta_1 \) in Lemma 15.

\[\square\]

**B PROOF OF COROLLARY 2**

Let \( A^* = A \setminus \{x^*\} \) be the set of suboptimal actions. To see (1), it suffices to show that for every consistent policy \( \pi \) and vector \( y \in \mathbb{R}^d \),

\[
\lim_{n \to \infty} \log(n) y^T G_n^{-1} x^* = 0.
\]
The proof hinges on the fact that $\mathbb{E}[T_x(n)] \in \Omega(n)$ and for $x \in A^∗$, $\mathbb{E}[T_x(n)] \in \cap_{p>0} O(n^p)$. Indeed, these follow from the assumption that $\pi$ is consistent and such as for any $p > 0$, $O(n^p) \ni R_x^p(n) = \sum_{x \in A} \Delta_x \mathbb{E}[T_x(n)]$, so $\mathbb{E}[T_x(n)] \in \cap_{p>0} O(n^p)$ indeed, and thus also $\mathbb{E}[T_x(n)] \in \Omega(n)$.

Let us return to proving (25). Clearly, it is enough to see this in the two cases: when $y = x^*$ and when $y$ and $x^*$ are perpendicular. Consider first when $y = x^*$. Then, from $G_n \geq \mathbb{E}[T_x(n)]x^*(x^*)^\top$ it follows that $G^{-1}_n \leq (\mathbb{E}[T_x(n)])^{-1}x^*(x^*)^\top$ and hence $\log(n)(x^*)^\top G^{-1}_n x^* \leq \frac{\log(n)}{(\mathbb{E}[T_x(n)])^2} ||x^*||^2 \to 0$ as $n \to \infty$.

Now consider the case when $y$ and $x^*$ are perpendicular. Let $v = \hat{G}^{-1}y$. Then, it must hold that $\hat{G}v = y$. Using the definition of $\hat{G}$, $y = \mathbb{E}[T_x(n)]x^*(x^*)^\top v + \sum_{x \in A^*} \mathbb{E}[T_x(n)]xx^\top v$. Since by assumption, $y$ and $x^*$ are perpendicular, $0 = (x^*)^\top y = \mathbb{E}[T_x(n)]||x^*||^2 (x^*)^\top v + \sum_{x \in A^*} \mathbb{E}[T_x(n)](x^*)^\top xx^\top v$. Hence, $\log(n)(x^*)^\top v = -\log(n) \sum_{x \in A^*} \mathbb{E}[T_x(n)](x^*)^\top xx^\top v \leq \frac{\log(n)}{||x^*||^2}$ converges to zero as $n \to \infty$. This finishes the proof of (25) and thus of (1).

For the second part we start with

$$\frac{R_x^p(n)}{\log(n)} = \sum_{x \in A^*} \frac{\mathbb{E}[T_x(n)]}{\log(n)} \Delta_x.$$ 

Then $\alpha_n(x) = \frac{\mathbb{E}[T_x(n)]}{\log(n)}$ is asymptotically feasible for $n$ large. Indeed, $\hat{G}_n = \log(n)H(\alpha_n)$, hence $\hat{G}^{-1}_n = H^{-1}(\alpha_n)/\log(n)$ and so

$$\Delta^2_2 \geq \limsup_{n \to \infty} \log(n) ||x||^2_{\hat{G}^{-1}_n} = \limsup_{n \to \infty} ||x||^2_{H^{-1}(\alpha_n)}.$$ 

Thus for any $\varepsilon > 0$ and $n$ large enough, $||x||^2_{H^{-1}(\alpha_n)} \leq \Delta^2_2/2 + \varepsilon$ and also

$$\frac{R_x^p(n)}{\log(n)} = \sum_{x \in A^*} \frac{\mathbb{E}[T_x(n)]}{\log(n)} \Delta_x \geq c_\varepsilon(A, \theta),$$

where $c_\varepsilon(A, \theta)$ is the solution to the optimisation problem (2) where $\Delta^2_2/2$ is replaced by $\Delta^2_2/2 + \varepsilon$. Hence, $\liminf_{n \to \infty} \frac{R_x^p(n)}{\log(n)} \geq c_\varepsilon(A, \theta)$. Since $\varepsilon > 0$ was arbitrary and $\inf_{x \in A^*} c_\varepsilon(A, \theta) = c(A, \theta)$, we get the desired result. $\square$

C PROOF THAT THE GRAM MATRIX IS EVENTUALLY NON-SINGULAR

Let $\pi$ be a consistent strategy and $A$ and $\theta$ be the action-set and parameter for a linear bandit. Define $A' = \{x : \sum_{t=1}^n 1\{A_t = x\} > 0\}$ to be the set of arms that are played at least once with non-zero probability. We proceed by contradiction. Suppose that $G_n$ is singular for all $n$. Then there exists an $x \in A$ such that $x \notin \text{span } A'$. Decompose $x = y + z$ where $y \in \text{span } A'$ and $z \in \text{span } A^\perp$ is non-zero and in the orthogonal complement of the subspace spanned by $A'$. Therefore $(w, z) = 0$ for all $w \in A'$. Define an alternative bandit with the same action-set and parameter $\theta' = \theta + 2\Delta_{\max} z$. Then $(w, \theta - \theta') = 0$ for all $w \in A'$. Therefore the bandits determined by $\theta$ and $\theta'$ appear identical to the algorithm, and in particular, $\mathbb{E}[(\sum_{t=1}^n 1\{A_t \notin A'\})] = 0$, and yet by construction we have

$$R_{\theta'}(n) \geq \Delta_{\max}(\theta') \left[ \sum_{t=1}^n 1\{A_t \in A'\} \right] = n\Delta_{\max}.$$ 

Therefore the regret is linear for $\theta'$, which implies that $\pi$ is not consistent. Therefore for sufficiently large $n$ we have $G_n$ is non-singular.

D PROOF OF LEMMA 12

Let $B \subseteq A$ be a barycentric spanner and let $S \subseteq [0, \infty)^k$ be an alternative to $T$ given by

$$S_x = \begin{cases} \infty, & \text{if } x = x^*; \\ \frac{2d^2}{\Delta_{\min}^2}, & \text{if } x \in B; \\ 0, & \text{otherwise}. \end{cases}$$

Then $\|x\|_{H^1}^2 = 0$ and for $x^* \neq y \in A$ we have

$$\|y\|_{H^1}^2 \leq \left( \sum_{x \in B} \frac{\|x\|^2_{H^1}}{\Delta_{\min}^2} \right)^2 \leq \left( \frac{\Delta_{\min}}{\sqrt{2d}} \right)^2 \leq \frac{\Delta_2^2}{2f_n}. $$

Therefore

$$\sum_{x: \Delta_x > 0} T_x \leq \frac{1}{\Delta_{\min}} \sum_{x: \Delta_x > 0} T_x \Delta_x \leq \frac{2d^3 f_n \Delta_{\min} \Delta_{\max}}{\Delta_{\min}^3}. \quad \square$$

E PROOF OF LEMMA 13

The proof of Lemma 13 requires one more technical result.

Lemma 17. Let $\varepsilon > 0$ and recall the definition of $T_n(\hat{\Delta})$ given in Definition 9. For $m \in \mathbb{N}$ define

$$S_{n,m}(\hat{\Delta}) = \min \left\{ mf_n, T_n(\hat{\Delta}) \right\}.$$
Then there exists an $m$ such that for all $n \in \mathbb{N}$ and $x \in \mathcal{A}$
\[
\|x\|^2_{\mathcal{H}_{n,m}(\Delta)} \leq \max \left\{ \frac{\Delta^2}{f_n}, \frac{\Delta^2_{\min}}{f_n}, \frac{\Delta^4_{\min}}{16 f_n} \right\}.
\]

**Proof of Lemma 13.** Assume that $F_n'$ does not hold. We consider three cases.

**Case 1.** $\hat{\Delta}_{x^*} > 0$.
**Case 2.** $\hat{\Delta}_{x^*} = 0$ and $\hat{\Delta}_{\min} > \Delta_{\min}/4$.
**Case 3.** $\hat{\Delta}_{x^*} = 0$ and $\hat{\Delta}_{\min} \leq \Delta_{\min}/4$.

The idea is to show that in each case the regret is at most logarithmic, with a leading constant that depends on $\theta$ and $\mathcal{A}$, but not on the observed samples. Treating each case separately.

**Case 1** Recall that $\hat{\Delta} \in \mathbb{R}^k$ (indexed by the actions) is the empirical estimate of the sub-optimality gaps after the warm-up phase. Let $x$ be the sub-optimal arm for which $\hat{\Delta}_x = 0$. By the definition of the optimisation problem this arm will be played in every while loop. Let $t$ be the first round when for all $x$ it holds that
\[
\|x\|^2_{\mathcal{H}_t} \leq \max \left\{ \frac{\Delta^2}{f_n}, \frac{\Delta^2_{\min}}{f_n}, \frac{\Delta^4_{\min}}{16 f_n} \right\}.
\]
By Lemma 17 there exists a constant $m_1$ depending only on $\mathcal{A}$ and $\theta$ such that
\[
t \leq m_1 f_n.
\]
By the assumption that $F_n'$ does not hold (and its definition) we have
\[
\hat{\mu}_x(t) \geq \mu_x \geq \mu_x - \Delta_{\min}/4
\]
where $t_0 = d[\log^{1/2}(n)]$ is the round at the end of the warm-up phase. Therefore if $n$ is sufficiently large that $\Delta_{\min}/2 \geq \epsilon_n$, then
\[
\hat{\mu}_x(t) \geq \mu_x \Delta_x - \Delta_{\min}/4
\]
which by the fact that $\max \{a, b\} \geq (a + b)/2$ for all $a, b \in \mathbb{R}$ implies that the success phase of the algorithm ends. Therefore if $n$ is sufficiently large, then in case 1 the regret in the success phase is at most
\[
\sum_{t \in T_{\text{succ.}}} \Delta_{A_t} \leq \Delta_{\max} m_1 f_n.
\]

**Case 2** Recall that $\hat{T}$ is the strategy used in the success phase based on samples collected in the warm-up phase. Since $\Delta_{x^*} = 0$ and $\Delta_{\min} \geq \Delta_{\min}/4$, by Lemma 12 it holds that
\[
\sum_{x \neq x^*} \hat{f}_x \leq \frac{2 \cdot 4^3 d^3 f_n \Delta_{\max}}{\Delta_{\min}^3}.
\]
And again we have that for sufficiently large $n$ that the regret in the success phase is at most
\[
\sum_{t \in T_{\text{succ.}}} \Delta_{A_t} \leq \frac{2 \cdot 4^3 d^3 f_n \Delta_{\max}^2}{\Delta_{\min}^3}.
\]

**Case 3** For the final case we assume that $\Delta_{x^*} = 0$ and there exists an $x$ for which $\Delta_x \leq \Delta_{\min}/4$. Let $t$ be the first time-step when for all $x \in \mathcal{A}$ it holds that
\[
\|x\|^2_{\mathcal{H}_t} \leq \max \left\{ \frac{\Delta^2_{\min}}{f_n}, \frac{\Delta^4_{\min}}{64 f_n} \right\}.
\]

Then by Lemma 17 there exists a constant $m_2$ that is independent of $\Delta$ and $n$ such that $t \leq m_2 f_n$. Then since $F_n'$ does not hold we have
\[
\hat{\mu}_x(t) - \mu_{x^*}(t_0) \geq \hat{\mu}_x(t) - \mu_{x^*}(t_0)
\]
which by the fact that $\max \{a, b\} \geq (a + b)/2$ for all $a, b \in \mathbb{R}$ implies that the success phase of the algorithm ends. Therefore if $n$ is sufficiently large, then in case 1 the regret in the success phase is at most
\[
\sum_{t \in T_{\text{succ.}}} \Delta_{A_t} \leq \Delta_{\max} m_1 f_n.
\]