Second Hankel Determinants and Fekete-Szegö Inequalities for Some Sub-Classes of Bi-Univalent Functions with Respect to Symmetric and Conjugate Points Related to a Shell Shaped Region

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Abstract. In this paper, we have investigated second Hankel determinants and Fekete-Szegö inequalities for some subclasses of Bi-univalent functions with respect to symmetric and Conjugate points which are subordinate to a shell shaped region in the open unit disc \(\Delta\).

Key Words: Analytic functions, univalent functions, Bi-univalent functions, second Hankel determinants, Fekete-Szegö inequalities, symmetric points, conjugate points.

AMS Subject Classifications: 30C45, 30C50, 30C80

1 Introduction

Let \(A\) be the class of all functions of the form

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k,
\]

which are analytic in the open unit disc \(\Delta = \{z : |z| < 1\}\). Let \(S\) be the class of all functions in \(A\) which are univalent in \(\Delta\).

Let \(P\) denote the family of functions \(p(z)\) which are analytic in \(\Delta\) such that \(p(0) = 1\), and \(\Re p(z) > 0\ (z \in \Delta)\) of the form \(P(z) = 1 + \sum_{n=1}^{\infty} c_n z^n\).

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For two functions \( f \) and \( g \), analytic in \( \Delta \), we say that the function \( f \) is subordinate to \( g \) in \( \Delta \) and we write it as \( f(z) \prec g(z) \) if there exists a Schwartz function \( \omega \), which is analytic in \( \Delta \) with \( \omega(0) = 0 \), \( |\omega(z)| \leq 1 \) (\( z \in \Delta \)) such that

\[
f(z) = g(\omega(z)).
\]

(1.2)

Indeed, it is known that \( f(z) \prec g(z) \Rightarrow f(0) = g(0) \) and \( f(\Delta) \subseteq g(\Delta) \).

In 1959, Sakaguchi [26] defined a subclass \( S^*_s \) of \( S \) which satisfies following condition

\[
\text{Re}\left( \frac{2zf'(z)}{f(z) - f(-z)} \right) > 0, \quad z \in \Delta.
\]

The functions in the class \( S^*_s \) are starlike with respect to symmetric points. Further Sakaguchi has shown that the functions in \( S^*_s \) are close-to-convex and hence are univalent. The concept of starlike functions with respect to symmetric points have been extended to starlike functions with respect to \( N \)-symmetric points by Ratan Chand [24] and Prithvipal Singh [21]. Ram Reddy [22] studied the class of close-to-convex functions with respect to \( N \)-symmetric points and proved that the class is closed under convolution with convex univalent functions. Das and Singh [3] introduced another class \( C_s \) namely convex functions with respect to symmetric points and satisfying the condition

\[
\text{Re}\left( \frac{2zf'(z)}{f(z) - f(-z)} \right) > 0, \quad z \in \Delta.
\]

From the definition of \( S^*_s \) and \( C_s \) it is evident that \( f \in C_s \) if and only if \( zf(z) \in S^*_s \). Ashwah and Thomas in [6] introduced another class namely the class \( S^*_c \) consisting of functions starlike with respect to conjugate points.

Let \( S^*_c \) be the subclass of \( S \) consisting of functions given by (1.1) and satisfying the condition

\[
\text{Re}\left( \frac{2zf'(z)}{f(z) + f(z-bar)} \right) > 0, \quad z \in \Delta.
\]

In terms of subordination following Ma and Minda, Ravichandran [25] defined the classes \( S^*_s(\phi) \) and \( C_s(\phi) \) as below.

A function \( f \in A \) is in the class \( S^*_s(\phi) \) if

\[
\frac{2zf'(z)}{f(z) - f(-z)} \prec \phi(z), \quad z \in \Delta.
\]

And in the class \( C_s(\phi) \) if

\[
\frac{2zf'(z)}{f(z) - f(-z)} \prec \phi(z), \quad z \in \Delta.
\]