Analytic treatment of the charged black-hole-mirror bomb in the highly explosive regime

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A charged scalar field impinging upon a charged Reissner-Nordström black hole can be amplified as it scatters off the hole, a phenomenon known as superradiant scattering. This scattering process in the superradiant regime \( \omega < qQ/r_+ \) (here \( \omega, q, Q \), and \( r_\pm \) are the conserved frequency of the wave, the charge coupling constant of the field, the electric charge of the black hole, and the horizon radii of the black hole, respectively) results in the extraction of Coulomb energy and electric charge from the charged black hole. The black-hole-field system can be made unstable by placing a reflecting mirror around the black hole which prevents the amplified field from escaping to infinity. This charged black-hole-mirror system is the spherically symmetric analogue of the rotating black-hole-mirror bomb of Press and Teukolsky. In the present paper we study analytically the charged black-hole-mirror bomb in the asymptotic regime \( qQ \gg 1 \) and for mirror radii \( r_m \) in the near-horizon region \( x_m \equiv (r_m - r_+)/r_+ \ll \tau \), where \( \tau \equiv (r_+ - r_-)/r_+ \) is the dimensionless temperature of the black hole.

In particular, we derive analytic expressions for the oscillation frequencies \( \Re \omega \) and the instability growth timescales \( 1/3\omega \) of the superradiant confined fields. Remarkably, we find a simple linear scaling \( 3\omega \propto qQ/r_+ \) for the imaginary part of the resonances in the asymptotic \( qQ \gg (\tau/x_m)^2 \gg 1 \) regime, which implies that the instability timescale \( 1/3\omega \) of the system can be made arbitrarily short in the \( qQ \to \infty \) limit. The short instability timescale found in the linear regime along with the spherical symmetry of the system, make the charged bomb a convenient toy model for future numerical studies aimed to investigate the non-linear end-state of superradiant instabilities.

I. INTRODUCTION

Black holes are believed to be the most powerful source of energy in the Universe. The mass-energy \( M \) of a rotating Kerr black hole of angular-momentum \( J \) can be expressed in the form \[ M = \sqrt{M_{ir}^2 + \frac{J^2}{4M_{ir}^2}}, \] where \( M_{ir} \) is the irreducible mass of the black hole which is closely related to its surface area \( A \): \( M_{ir} = \sqrt{A/16\pi} \).

The well-known area theorem of Hawking \[ 1 \] reveals that, within the framework of classical general relativity, the black-hole surface area (and thus also the irreducible mass) cannot decrease. A physical process in which the irreducible mass of the black hole remains unchanged is known as a reversible transformation \[ 2 \]. The relation \( M = M(J, M_{ir}) \), Eq. \[ 1 \], implies that up to \( \sim 29\% \) (a fraction \( 1 - 1/\sqrt{2} \)) of the energy of a Kerr black hole is in the form of rotational energy which, in principle, can be released in a reversible process \[ 3 \] (in such a process the black-hole parameters are changed according to \( M \to M_{ir} \) and \( J \to 0 \)).

One possible mechanism to extract the rotational energy of a Kerr black hole is based on the well-known phenomenon of superradiant scattering \[ 4, 5 \]: a bosonic field of the form \( e^{im\phi}e^{-i\omega t} \) can be amplified as it scatters off a rotating Kerr black hole of angular-velocity \( \Omega_H \) if it respects the superradiance bound \[ 4 \]

\[ \omega < m\Omega_H. \] (2)

The amplification of the incident bosonic field in the superradiant regime \[ 2 \] signals a decrease in the rotational energy of the black hole \[ 1 \].

Press and Teukolsky \[ 8 \] have pointed out that the mechanism of superradiant scattering can be used in order to build a powerful bomb whose energy source is the rotational energy of the black hole itself. To build this black-hole bomb one should prevent the amplified scattered field from escaping to infinity. The original suggestion of Press and Teukolsky was to surround the black hole by a reflecting mirror \[ 8, 9 \]. In this way the bosonic field [a wave packet made of frequencies in the superradiant regime \[ 2 \]] will bounce back and forth between the black hole and the mirror amplifying itself each time. As a consequence, the rotational energy extracted from the black hole by the trapped bosonic field would grow exponentially over time \[ 8 \]. Using numerical techniques, it was found in \[ 10 \] that the maximum growth rate of a scalar field (the largest imaginary part of the superradiant resonance frequency) in the Kerr black-hole-mirror system is given by

\[ 3\omega \approx 6 \times 10^{-5} M^{-1}. \] (3)

An analogous superradiant amplification of waves may take place when a charged bosonic field impinges upon a charged Reissner-Nordström (RN) black hole \[ 11 \]. In the charged case the superradiant scattering (amplification of the waves) occurs for incident waves with frequencies in the regime \[ 11 \]

\[ \omega < q\Phi_H, \] (4)
where \( q \) is the charge coupling constant of the field and
\[
\Phi_H = \frac{Q}{r_+} \tag{5}
\]
is the electric potential of the RN black hole. [Here \( Q \) and \( r_+ \) are the electric charge and horizon radius of the black hole, respectively].

The scattering of charged scalar fields off a charged RN black hole in the superradiant regime \([4]\) is governed by the scattering of Coulomb energy and electric charge from the charged black hole \([11]\). The mass-energy \( M \) of a RN black hole of charge \( Q \) can be expressed in the form \([13]\)
\[
M = M_H + \frac{Q^2}{4M_H} . \tag{6}
\]
The relation \( M = M(Q, M_H) \), Eq. \([6]\), implies that up to 50\% (!) of the energy of a charged RN black hole is in the form of a Coulomb energy which, in principle, can be released in a reversible process \([12]\) (in such a process the black-hole parameters are changed according to \( M \rightarrow M_H, Q \rightarrow 0 \)).

The physical interest in RN black holes is mainly motivated by the fact that these charged black holes share many common characteristics with the astrophysically more relevant Kerr black holes. In particular, the global spacetime structures of charged RN black holes and rotating Kerr black holes are almost identical \([13]\). It is therefore of physical interest to explore the properties of the charged black-hole-mirror bomb, which is the spherically symmetric analogue of the rotating black-hole-mirror bomb of Press and Teukolsky \([8]\).

In addition, the fact that the charged black-hole-mirror bomb has spherical symmetry (as opposed to the non-spherically symmetric Kerr black-hole spacetime) makes it a convenient toy model for future numerical studies aimed to investigate the non-linear dynamics of explosive superradiant instabilities. The present analytical study, which is restricted to the linear regime, should be regarded as a first step in this direction.

In a very interesting work, Degollado et. al. \([14]\) have recently studied this charged black-hole-mirror system (still restricted to the linear level) using numerical techniques. Remarkably, the authors of \([14]\) reported on instability growth rates of the superradiant charged confined fields which are several orders of magnitude larger than the maximal growth rate \([5] \) found for the rotating black-hole-mirror bomb.

The numerical results presented in \([14]\) indicate that \( 3\omega \) increases monotonically with increasing values of the charge coupling constant \( q \) of the field. Unfortunately, the authors of \([14]\) also stated that their numerical scheme breaks down for large values of the parameter \( q \). For this reason, the largest imaginary part of the superradiant resonance frequency reported in \([14]\) is
\[
3\omega \sim 0.07M^{-1} . \tag{7}
\]
As emphasized in \([14]\), the reported value \([7]\) is not the maximum possible value of \( 3\omega \). It is merely the maximal value of \( 3\omega \) which could be obtained numerically under the technical limitations imposed by the numerical tools used in \([14]\).

The main goal of the present paper is to explore the physical properties of the charged black-hole-mirror bomb using analytical techniques. As we shall show below, the instability growth rate of the superradiant confined charged fields (the value of \( 3\omega \)) can grow unboundedly in the \( qQ \to \infty \) limit.

II. DESCRIPTION OF THE SYSTEM

The physical system we consider consists of a charged scalar field \( \Psi \) linearly coupled to a charged RN black hole. The black-hole spacetime is described by the line element
\[
ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) , \tag{8}
\]
where
\[
f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} . \tag{9}
\]
Here \( M \) and \( Q \) are respectively the mass and electric charge of the black hole, and \( r \) is the Schwarzschild areal coordinate. The zeros of \( f(r) \),
\[
r_{\pm} = M \pm (M^2 - Q^2)^{1/2} , \tag{10}
\]
are the black-hole (event and inner) horizons.

The dynamics of the charged scalar field \( \Psi \) in the charged RN spacetime is governed by the Klein-Gordon wave equation \([15, 17]\)
\[
[(\nabla^2 - iqA^\nu)(\nabla_\nu - iqA_\nu) - \mu^2]\Psi = 0 , \tag{11}
\]
where \( A_\nu = -\delta_\nu^0 Q/r \) is the electromagnetic potential of the black hole. Here \( q \) and \( \mu \) are respectively the charge and mass of the field \([15]\). One may decompose the field \( \Psi \) in the form
\[
\Psi_{lm}(t, r, \theta, \phi) = e^{i\omega t} S_{lm}(\theta) R_{lm}(r)e^{-i\nu t} , \tag{12}
\]
where \( \omega \) is the conserved frequency of the mode and \( \{ l, m \} \) are respectively the spherical harmonic index and the azimuthal harmonic index of the mode (we shall henceforth omit the indices \( l \) and \( m \) for brevity). The sign of \( 3\omega \) determines whether the solution is stable (decaying in time with \( 3\omega < 0 \)) or unstable (growing in time with \( 3\omega > 0 \)). Stationary modes are characterized by \( 3\omega = 0 \).

Substituting the decomposition \([12]\) into the Klein-Gordon wave equation \([11]\), one finds \([15, 17]\) that \( R(r) \) and \( S(\theta) \) obey radial and angular equations both of confluent Heun type \([19, 20]\) coupled by a separation constant \( K_l = l(l + 1) \), where \( l \geq |m| \) is an integer. The radial wave equation is given by \([15, 17]\)
\[
\Delta \frac{d}{dr} \left( \Delta \frac{dR}{dr} \right) + UR = 0 , \tag{13}
\]
where
\[ \Delta \equiv r^2 - 2Mr + Q^2, \] (14)
and
\[ U \equiv (\omega r^2 - qQr)^2 - \Delta[\mu^2 r^2 + l(l + 1)]. \] (15)

We are interested in solutions of the radial equation with the physical boundary conditions of purely ingoing waves at the black-hole horizon and a vanishing field at the location \( r_m \) of the mirror \([10, 14]\). That is,
\[ R \sim e^{-i(\omega r^2 + qQr + \tau)} \quad \text{as} \quad r \to r_+ \quad (y \to -\infty), \] (16)
and
\[ R(r = r_m) = 0. \] (17)

Here the “tortoise” radial coordinate \( y \) is defined by \( dy = (r^2/\Delta)dr \). The boundary condition \([10]\) describes an outgoing flux of energy and charge from the charged black hole for scattered fields in the superradiant regime \([10, 11, 14]\).

The boundary conditions \([10, 17]\) single out a discrete set of complex resonances known as Boxed Quasi-Normal (BQN) \([11, 14]\). The main goal of the present paper is to determine these characteristic resonances analytically in the superradiant regime \([1]\). To that end, it is convenient to define new dimensionless variables

\[ x \equiv \frac{r - r_+}{r_+}; \quad \tau \equiv \frac{r_+ - r}{r_+}; \quad \omega r_+ \equiv qQ + \epsilon, \] (18)

in terms of which the radial wave equation \([13]\) becomes
\[ x(x + \tau) \frac{d^2 R}{dx^2} + (2x + \tau) \frac{dR}{dx} + VR = 0, \] (19)

where
\[ V \equiv K^2/x(x + \tau) - [\mu^2 r_+^2(x + 1)^2 + l(l + 1)] \] (20)
with
\[ K \equiv (qQ + \epsilon)x^2 + (qQ + 2\epsilon)x + \epsilon. \] (21)

As we shall show below, the radial equation \([19]\) is amenable to an analytic treatment in the double limit
\[ qQ \gg 1 \quad \text{with} \quad x \ll \tau. \] (22)

In this asymptotic regime the radial equation \([19]\) can be approximated by
\[ x \frac{d^2 R}{dx^2} + x \frac{dR}{dx} + V_{\text{near}} R = 0, \] (23)
where
\[ V_{\text{near}} = \tau^{-2}[(qQ + 2\epsilon)x + \epsilon]^2. \] (24)

III. THE STATIONARY RESONANCES OF THE BLACK-HOLE-MIRROR SYSTEM

We shall first analyze the stationary resonances of the charged black-hole-mirror system. In particular, in the present section we shall derive an analytical formula for the discrete radii of the mirror, \( \{r_{\text{stat}}(\tau, qQ; n)\} \), which satisfy the stationary resonance condition \( \Im \omega = 0 \).

Here \( n = 1, 2, 3, ... \) is the resonance parameter. It is worth noting that the minimum radius of the mirror, \( r_{\text{min}} \equiv r_{\text{stat}}(\tau, qQ, n = 1) \), would mark the boundary between stable and unstable black-hole-mirror configurations: configurations with \( r_m < r_{\text{min}} \) would be stable \((\Im \omega < 0)\) whereas configurations with \( r_m > r_{\text{min}} \) would be unstable \((\Im \omega > 0)\).

The stationary resonances of the system (modes with \( \Im \omega = 0 \)) are described by the field \([12]\) with the critical frequency for superradiance \([\text{see Eq. } (4)]\):
\[ \omega_c = \frac{qQ}{r_+}. \] (25)

The critical frequency \([26]\) corresponds to the value \( \epsilon = 0 \) \([\text{see Eq. } (13)]\). Taking cognizance of Eqs. \([23, 24]\) with \( \epsilon = 0 \), one finds that the ‘stationary’ radial field \( R_{\text{stat}}(x) \) \([23]\) is described by the Bessel function of the first kind \([\text{see Eq. } 9.1.1 \text{ of } [19]]\):
\[ R_{\text{stat}}(x) = J_0 \left( \frac{qQ}{x} \right). \] (26)

Taking cognizance of the boundary condition \( \dot{R}(x = x_m) = 0 \), which is dictated by the presence of the reflecting mirror \([\text{see Eq. } (17)]\), one finds that the stationary resonances of the charged field correspond to the discrete radii
\[ x_m^{\text{stat}}(\tau, qQ; n) = \frac{\tau}{qQ} \times j_{0,n} \quad ; \quad n = 1, 2, 3, ... \] (27)

of the mirror. Here \( j_{0,n} \) is the \( n \)-th zero of the Bessel function \( J_0(x) \). The real zeros of the Bessel functions were studied by many authors \([19, 24]\). For completeness, we state here the first three zeros of \( J_0(x) \) \([24]\): \( j_{0,1} = 2.4048, j_{0,2} = 5.5201, \) and \( j_{0,3} = 8.6537 \).

It is worth emphasizing that the smallest ‘stationary’ radius of the mirror, \( x_{\text{min}} \equiv x_m^{\text{stat}}(\tau, qQ; n = 1) \), corresponds to the innermost location of the mirror (for given values of the parameters \( qQ \) and \( \tau \)) which allows the extraction of the Coulomb energy from the charged black hole. In other words, the dimensionless radius \( x_{\text{min}} \) marks the onset of instability in the black-hole-mirror system: black-hole-mirror configurations with \( x_m < x_{\text{min}} \) are stable \((\Im \omega < 0)\) whereas black-hole-mirror configurations with \( x_m > x_{\text{min}} \) are unstable \((\Im \omega > 0)\).

Note that the solution \([27]\) with \( qQ \gg 1 \) is consistent with the near-horizon condition \( x_m \ll \tau \) that we assumed above \([\text{see Eq. } (22)]\). In particular, one finds the interesting property
\[ x_{\text{min}} \to 0 \quad \text{as} \quad qQ \to \infty. \] (28)
That is, the reflecting mirror can be placed arbitrarily close to the black-hole horizon \((r_{\text{min}} \to r_+)^\) in the \(q Q \to \infty\) asymptotic limit.

IV. RAPIDLY GROWING SUPERRADIANT INSTABILITIES

The solution of the radial equation \((23)\) obeying the ingoing boundary condition \((10)\) at the black-hole horizon is given by \([10, 19]\)

\[
R = x^{-i\epsilon/\tau} _2F_1(1/2, 1/2 - 2i Q - 4i\epsilon; 1 - 2i \epsilon; -x/\tau),
\]

where \(_2F_1(a, b; c; z)\) is the hypergeometric function.

Our goal in the present section is to determine analytically the resonance frequency \(\omega = \omega(\tau, q Q, x_m)\) which satisfies the mirror-like boundary condition \(R(x = x_m) = 0\) for a given radius \(x_m\) of the mirror. It proofs useful to use the ansatz \((25)\)

\[
\epsilon = -q Q x_m (1 - \delta)
\]

with \(|\delta| \ll 1\) [see Eq. \((41)\) below], where the unknown quantity \(\delta = \delta(\tau, q Q, x_m)\) is to be determined below. This quantity contains within it the information about the instability timescale (the value of \(3\omega\)) which characterizes the composed black-hole-mirror system in the superradiant regime.

In the asymptotic regime \((26)\)

\[
q Q \gg \frac{\tau}{x_m} \gg 1
\]

one may use the large-\(|b|\) asymptotic expansion \([27, 28]\)

\[
_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(c - a)} (-bz)^{-a} [1 + O(|b|^{-1})]
\]

of the hypergeometric function in order to express the boundary condition \(R(x = x_m) = 0\) in the form

\[
\frac{\Gamma(1/2)}{\Gamma(1/2 - 2i\epsilon/\tau)} (1 + x_m/\tau)^{-2i Q - 4i\epsilon - 2i \epsilon} \times [-(1/2 - 2i Q - 4i\epsilon)x_m/\tau]^{-2i \epsilon} = 1.
\]

Substituting \((32)\) into \((33)\) one can write the resonance condition in the form

\[
i^2(-1/2 + x_m/\tau)^{-2i Q - 4i\epsilon - 2i \epsilon} \times [(1/2 - 2i Q - 4i\epsilon)x_m/2i\epsilon]^{-2i \epsilon} = 1.
\]

Taking the logarithm of both sides of Eq. \((35)\), one finds

\[
\epsilon \left[ \ln \left( -\frac{q Q x_m}{\epsilon} \right) + 1 + \frac{8 \epsilon + i}{4 q Q} + x_m (2 - \frac{1}{\tau}) \right] + q Q x_m (1 - x_m/2\tau) - \frac{1}{4} \tau (\pi + i \ln 2) = 0
\]

for the resonance condition.

Substituting the ansatz \((30)\) into Eq. \((36)\), one finds after some tedious algebra that the resonance condition \((36)\) can be expressed as a quadratic equation for the dimensionless quantity \(\delta = \delta(\tau, q Q, x_m)\)

\[
i q Q x_m \delta^2 - 4 i q Q x_m^2 \delta + i q Q x_m^2 + \frac{x_m}{2
\]

\[
+ q Q x_m (1 - x_m/2\tau) - \frac{1}{4} \tau (\pi + i \ln 2) = 0.
\]

In the asymptotic regime

\[
q Q \gg \left(\frac{x_m}{\tau}\right)^2 \gg 1,
\]

one finds from \((38)\) the simple solution \([32]\)

\[
\delta \simeq i \sqrt{\frac{x_m}{\tau}}
\]

Taking cognizance of Eqs. \((18), (30),\) and \((40)\), we finally find

\[
\Im \omega = \frac{q Q}{\tau} \sqrt{\frac{x_m^3}{\tau}}
\]

for the imaginary part of the resonance frequency \((33)\).

Note that \(\Im \omega > 0\) [see Eq. \((41)\)], which implies an instability of the charged black-hole-mirror system. Moreover, the simple linear scaling \(\Im \omega \sim q Q\) found for the imaginary part of the resonance frequency implies an instability timescale \(1/3 \omega\) which can be made arbitrarily short in the \(q Q \to \infty\) limit.

V. SUMMARY AND DISCUSSION

Motivated by the well-known phenomenon of superradiant instability of a rotating Kerr black hole enclosed in a cavity, we have explored here the analogous phenomenon of superradiant instability of a charged
Reissner-Nordström black hole enclosed in a reflecting cavity. Imposing a mirror-like boundary condition on a charged scalar field in the vicinity of the black-hole horizon, it was shown that the confined field grows exponentially over time in the superradiant regime [3]. In particular, we derived analytic expressions for the oscillation frequency \( \omega \) [see Eqs. (18) and (30)] and the instability growth timescale \( 1/3\omega \) [see Eq. (41)] of the confined charged field in the asymptotic regime \( qQ \gg 1 \).

The instability timescale \( T_{\text{ins}} \) which characterizes the composed black-hole-mirror system in the asymptotic regime \( qQ \gg (\tau/x_m)^2 \gg 1 \) is given by

\[
T_{\text{ins}} = 1/3\omega = \frac{r_+ \sqrt{\tau/x_m^2}}{qQ}.
\]

(42)

Remarkably, the simple scaling \( T_{\text{ins}} \sim r_+ / qQ \) found in the asymptotic regime [39] implies that the instability growth timescale of the confined superradiant modes can be made arbitrarily short in the \( qQ \to \infty \) limit (in particular, \( T_{\text{ins}} \) can be made much shorter than the dynamical timescale set by the mass \( M \) of the black hole).

It should be emphasized that the current analytic study is restricted to the linear regime. As we have shown, the instability (exponential growth) of the confined superradiant modes can be revealed at this linear level. However, a fully non-linear numerical simulation of the charged scalar field dynamics [33] is required in order to explore the end-state of this superradiant instability. One possible stationary end-state of the system may be described by the stationary resonances discussed in Sec. III. It has also been suggested that the end-point of the instability is attained after a violent boson explosion [36].

It is well known that RN black holes undergo a Schwinger discharge on very short timescales [37]. Thus, the charged black hole bomb probably has a limited astrophysical relevance. This charged black-hole-mirror system should instead be regarded as a simple toy-model for the astrophysically more relevant rotating black-hole-bomb.

In this respect, the charged black-hole-mirror model has two important advantages over the astrophysically more realistic rotating black-hole bomb [14]:

1. Unlike the rotating Kerr black-hole spacetime which is not spherically symmetric, the charged black-hole bomb can be ignited by spherical modes. This spherical symmetry of the charged model is expected to facilitate future non-linear numerical studies of the superradiant instabilities.

2. The unstable modes of the rotating black-hole bomb are characterized by very small growth rates [see Eq. (3)]. One is therefore forced to use very long numerical integration times in order to observe these weak superradiant instabilities (the numerical integration time which is required in order to observe the characteristic instabilities of the rotating black-hole-mirror system should be of the order of \( t_{\text{num}} \sim 10^5 M \), see Eq. (3). These extremely long integrations times may introduce numerical errors into the system. On the other hand, we have seen that the charged black-hole-mirror bomb is characterized by an instability timescale (42) which can be made much shorter than the instability timescale (3) of the rotating black-hole-mirror bomb. Thus, moderate integration times would probably be sufficient in order to explore the non-linear end-state of the charged black-hole bomb.

These two important advantages of the charged black-hole-mirror bomb make this system a convenient toy model for future numerical studies aimed to investigate the non-linear dynamics of the explosive superradiant instabilities.

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[22] In obtaining Eq. (24) we have assumed that $(|qQ + iε|) x^2 \ll (|qQ + 2ix| x + iε)$ in Eq. (23). In the $ε = 0$ case that we shall consider in Sec. III below, this inequality is satisfied in the regime $x_m < 1$ [see Eq. (22)]. For the $ε = -qQ x_m (1 - δ)$ case that we shall consider in Sec. IV below, this inequality is satisfied in the regime $x_m < |δ|$. In addition, we have assumed that $R^2 (x (x + τ) \gg μ^2 τ^2 (x + 1)^2 + 1) + 1$ in Eq. (20). In the $ε = 0$ case that we shall consider in Sec. III below, this inequality is satisfied in the regime $qQ \gg τ x_m$ [see Eq. (21)] below.

[23] That is, the radial field $R(x)$ which corresponds to the stationary condition $3ω = 0$.

[24] K. T. Tang, Mathematical methods for engineers and scientists3: Fourier analysis, partial differential equations and variational models (Springer, New York, 2006).

[25] Note that we take $Re < 0$ in the ansatz (30), which corresponds to the superradiant regime $\delta$ [see Eq. (18)].

[26] Note that the regime (31) corresponds to $x_m \ll x_m < τ$ [see Eq. (27)].

[27] See http://link.springer.com/content/pdf/bbm3A978-3-540-40914-42F1.pdf; F. W. J. Olver, Unsolved problems in the asymptotic estimation of special functions, in Theory and Application of Special Functions (Edited by R. Askey, Academic Press, New York, 1975, pp. 99-142).

[28] The asymptotic expansion of $qF_i (a, b; c; z)$, Eq. (62), is valid in the regime $|b| \gg \max (|a|, |c|, |z|)$ with $|b| > 1$. The first inequality requires $qQ > |ε|/τ$, a condition which is satisfied in the regime (31) [see Eq. (30)]. The second inequality requires $qQ x_m/τ > 1$, a condition which is also satisfied in the regime (31).

[29] Here we have used the relation $i = e^{iπ(2n + 1)/2}$, which implies $ln i = iπ(2n + 1/2)$, where here $n = 0, 1, 2, \ldots$ is the resonance parameter. We shall henceforth consider the fundamental mode with $n = 0$. In order to obtain the higher-order resonances one should replace $π → π(1 + 4n)$ in the equations below. The imaginary part $3ω$ of the resonance frequency is found to be a decreasing function of the resonance parameter $n$ [see Eq. (54) below with $π → π(1 + 4n)$].

[30] We have used the fact that $x_m/τ \ll 1$ [see Eq. (31)], which implies $ln (1 + x_m/τ) \approx x_m/τ - (x_m/τ)^2/2$. In addition, we have used the facts that $1/qQ \ll 1$ and $|ε|/qQ = O(x_m) \ll 1$ [see Eq. (30)], which imply $ln (1/2 - 2qQ - 4iτ) \approx ln(-2qQ + (8e + i)qQ/4).

[31] Here we have used the ansatz (30) with $|δ| < 1$, which implies $ln(-qQ x_m/τ) = ln(1 - δ) \approx δ + δ^2/2$. In addition, we neglected sub-leading terms which are factors $O(x_m) \ll 1$ and $O(δ) \ll 1$ smaller than the dominant terms which appear in Eq. (67).

[32] Note that the solution (18) for $δ$ with $x_m/τ \ll 1$ [see Eq. (31)] is consistent with the condition $|δ| < 1$ that we assumed above. In addition, the requirement $|δ| \gg x_m$ (22) is satisfied thanks to the inequalities $x_m < τ < |τ|/|x|.

[33] Taking cognizance of Eqs. (18) and (30), one finds $3ω/τ = O(\sqrt{x_m/τ}) \ll x_m \ll 1$.

[34] In the intermediate asymptotic regime $1 \ll τ/x_m \ll qQ < \sqrt{n/x_m^2}$ one finds from (18) the solution $δ \approx \sqrt{(π + t ln 2)/τ \sqrt{2qQ x_m}}$, which can be approximated as $δ \approx [1 + iτ(2τ)]^{-1} \ln 2 \sqrt{π τ \sqrt{2qQ x_m}}$. Note that this solution with $τ/qQ x_m \ll 1$ is consistent with the condition $|δ| \ll 1$ that we assumed above. In addition, the requirement $|δ| \gg x_m$ (22) is satisfied thanks to the inequalities $τ/qQ x_m \gg x_m/τ \geq x_m \gg x_m^2$. Taking cognizance of Eqs. (18) and (30), one finds $3ω/τ \approx \ln 2 \sqrt{7qQ x_m}/8\pi$, which implies $3ω/τ = O(\sqrt{x_m} \sqrt{qQ}) \ll x_m \ll 1$ in this intermediate asymptotic regime. In this regime one also finds $T_{ins} = 1/3ω = O(τ_{ins}/τ \sqrt{qQ x_m})$ for the instascale.

[35] S. Hod and T. Piran, Phys. Rev. D 55, Rapid Communication R440 (1997) arXiv:gr-qc/9606057; S. Hod and T. Piran, Phys. Rev. D 55, 3485 (1997) arXiv:gr-qc/9606093.

[36] H. Yoshino and H. Kodama, Prog. Theor. Phys. 128, 153 (2012).

[37] J. Schwinger, Phys. Rev. 82, 664 (1951); T. Damour and R. Ruffini, Phys. Rev. Lett. 35, 463 (1975).