CAHN-HILLIARD EQUATIONS ON RANDOM WALK SPACES

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Abstract. In this paper we study a nonlocal Cahn-Hilliard model (CHE) in the framework of random walk spaces, which includes as particular cases, the CHE on locally finite weighted connected graphs, the CHE determined by finite Markov chains or the Cahn-Hilliard Equations driven by convolution integrable kernels. We consider different transitions for the phase and the chemical potential, and a large class of potentials including obstacle ones. We prove existence and uniqueness of solutions in \( L^1 \) of the Cahn-Hilliard Equation. We also show that the Cahn-Hilliard equation is the gradient flow of the Ginzburg-Landau free energy functional on an appropriate Hilbert space. We finally study the asymptotic behaviour of the solutions.

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1. Introduction

The Cahn-Hilliard model was formulated by J. W. Cahn and J. E. Hilliard ([10]) to describe the phase separation of a binary fluid or alloy. Let $\Omega \subset \mathbb{R}^N$ be the subset where the phase separation takes place. Let $u$ be the concentration of a substance, which takes the values in $[-1, 1]$. The pure phases of the material correspond to $u = 1$ and $u = -1$ while $u \in (-1, 1)$ corresponds to the transition in the interface between such phases. Denoting by $\mu$ to the chemical potential, the model, in a simplified version, is described by the following coupled system of equations

$$
\begin{align*}
\frac{\partial u}{\partial t} - \Delta \mu &= 0 \quad \text{in } (0, \infty) \times \Omega, \\
\mu &= -\varepsilon^2 \Delta u + F'(u) \quad \text{in } (0, \infty) \times \Omega,
\end{align*}
$$

(1.1)

joint with homogeneous Neumann boundary conditions and initial data, where $F(u)$ is a double-well potential, where $\varepsilon > 0$ is a small interaction parameter related to the length of the interface.

A physically relevant choice for $F$ is a singular logarithmic double-well potential

$$
F_1(r) = ((1 + r) \log(1 + r) + (1 - r) \log(1 - r)) - \frac{b}{2} r^2, \quad r \in (-1, 1), \quad b > 2,
$$

which is often approximate by regular double-well polynomial potentials, like

$$
F_2(r) = \frac{1}{4} (r^2 - 1)^2.
$$

Another physically relevant choice for $F$ is the double-well obstacle potential

$$
F_3(r) := \frac{a}{2}(1 - r^2) + I_{[-1, 1]}(r) = \begin{cases} 
\frac{a}{2}(1 - r^2) & \text{if } |r| \leq 1, \\
+\infty & \text{if } |r| > 1,
\end{cases}
$$

where $I_{[-1, 1]}$ is the indicator function of $[-1, 1]$ and $a > 0$. Here, $F_3'(r)$ must be understood as $\partial F_3(r)$, the subdifferential of $F_3$ at $r$.

Observe that, for the logarithmic potential $F_1$ and the obstacle potential $F_3$, the system provides a solution $u \in [-1, 1]$ in the admissible range, which does not happen with $F_2$. On the other hand, for the logarithmic, $u$ can not attain pure phases while this is possible in the case of the obstacle potential $F_3$.

As pointed out by Fife in [21], the Cahn-Hilliard model (1.1) is the $H^{-1}$-gradient flow of the following Ginzburg-Landau free energy

$$
\mathcal{E}(u) = \frac{\varepsilon^2}{2} \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} F(u) dx.
$$
With the term $\frac{\epsilon^2}{2} \int_\Omega |\nabla u|^2 dx$, the model is assuming a short-range interaction between particles and penalties sudden changes for the concentration measure.

From a microscopic model for lattice gas, Giacomin and Lebowitz [25] derived a nonlocal version of the Cahn-Hilliard system that takes into account long-interactions between particles. Simplifying their model, they consider the nonlocal free energy

$$E_{NL}(u) = \frac{1}{4} \int_\Omega J(x, y)|u(y) - u(x)|^2 dxdy + \int_\Omega F(u)dx,$$

where $J(x, y) = J(|y - x|)$ is a non-singular kernel, and associate to it the following local-nonlocal Cahn-Hilliard system

$$\begin{cases}
\frac{\partial u}{\partial t} + \Delta \mu = 0 & \text{in } (0, \infty) \times \Omega, \\
\mu(x) = -\int_\Omega J(x, y)(u(y) - u(x))dy + F'(u(x)) & \text{in } (0, \infty) \times \Omega,
\end{cases}$$

joint with local homogeneous Neumann boundary conditions for the first equation (non-local regional Neumann boundary conditions are implicit for the second one) and initial data.

The Cahn-Hilliard model and its variants have been widely used in different areas of science, see, e.g., [7], [22], [41] and the monograph [38]. There is also an extensive literature about nonlocal Cahn-Hilliard equations, see, e.g., [3], [4], [11], [15], [16], [17], [26], [28], [30], and the references cited therein. For an overview of early and recent references and extensions of the nonlocal Cahn-Hilliard model, we refer to [3], [37], [38].

In [27] Gal considers a double nonlocal Cahn-Hilliard equations in which also the equation of motion for mass transport is nonlocal in terms of an operator of the form

$$L(\mu) = 2P.V. \int_\Omega K(x, y)(\mu(y) - \mu(x))dy,$$

with a singular kernel $K(x, y) = K(|x - y|)$; therefore also allowing long-range interaction to occur for $\mu$ between any two points $x, y \in \Omega$. The governing system of equations takes the form

$$\begin{cases}
\frac{\partial u}{\partial t} = L(\mu) & \text{in } (0, \infty) \times \Omega, \\
\mu(x) = -\int_\Omega J(x, y)(u(y) - u(x))dy + F'(u(x)) & \text{in } (0, \infty) \times \Omega,
\end{cases}$$
which is named as a strong-to-weak interaction Cahn-Hilliard system. Recently, Gal and Shomberg have dealt with a weak-to-weak interaction Cahn-Hilliard system in [29], by considering both \( \mathcal{K}, \mathcal{J} \in L^1(\mathbb{R}^N) \).

Our aim is to study Cahn-Hilliard systems in the framework of random walk spaces, that has as a particular cases (double) nonlocal Cahn-Hilliard systems for integrable kernels, that is, the weak-to-weak iteration, and Cahn-Hilliard systems in weighted graphs. We consider different transitions for the phase and for the chemical potential.

Let \([X, \mathcal{B}, m^i, \nu_i], i = 1, 2\), be random walk spaces with \( \nu_i \) reversible, and \( m_i \)-connected (all the concepts used in this introduction can be found in the Preliminaries section). We are interested in the study of the generalized (doubly nonlocal) Cahn-Hilliard system stated on random walk spaces:

\[
\begin{aligned}
&\begin{aligned}
&u_t(t, x) = \Delta m_1 \mu(t, x), \\
&\mu(t, x) \in -\Delta m_2 u(t, x) + \partial F(u),
\end{aligned} \\
&u(0, x) = u_0(x),
\end{aligned}
\]

(1.2)

for the (nonlocal) \( m_i \)-Laplacians \((i = 1, 2)\):

\[
\Delta m_i u(t, x) = \int_\Omega (u(t, y) - u(t, x)) dm_i^x(y),
\]

and for \( F : \mathbb{R} \to ]-\infty, +\infty[ \) in the class of potentials for which its subdifferential (see (2.1)), is

\[
\partial F(r) = \gamma^{-1}(r) - cr,
\]

(1.3)

with \( \gamma \) any maximal monotone graph in \( \mathbb{R} \times \mathbb{R} \) with \( 0 \in \gamma(0) \) and \( \inf\{\text{Ran}(\gamma)\} < \sup\{\text{Ran}(\gamma)\} \), and with \( c > 0 \). This class encompasses a large set of potentials due to the generality of the conditions on \( \gamma \). Observe the three choices of potential given at the beginning belong to such class since:

\[
\partial F_1(r) = F'_1(r) = \log(1 + r) - \log(1 - r) - br,
\]

and \( \log(1 + r) - \log(1 - r) \) is the corresponding monotone function; \n
\[
\partial F_2(r) = F'_2(r) = r^3 - r,
\]

and \( r^3 \) is the corresponding monotone function; and

\[
\partial F_3(r) = \partial I_{[-1, 1]}(r) - ar,
\]

and \( \partial I_{[-1, 1]} \) is the corresponding (multivalued) monotone graph

\[
\partial I_{[-1, 1]}(r) = \begin{cases} 
(\infty, 0] & \text{if } r = -1, \\
0 & \text{if } r \in ]-1, 1[, \\
[0, +\infty) & \text{if } r = 1.
\end{cases}
\]
Observe that (1.2) can be written as follows:
\[
\begin{align*}
    u_t(t, x) &= \Delta m_1 \mu(t, x), \\
    \mu(t, x) &\in -\Delta m_2 u(t, x) + v(t, x) - cu(x, t), \\
    u(t, x) &\in \gamma(v(t, x)), \\
    u(0, x) &= u_0(x),
\end{align*}
\]
that can be reduced to the study of
\[
\begin{align*}
    u_t &= \Delta m_1 v - \Delta m_2 u + (1 - c)\Delta m_1 u + \Delta m_2 u \\
    u &\in \gamma(v) \\
    u(0) &= u_0
\end{align*}
\]
in (0, \infty) \times X, (1.4)

In this last formulation, the evolution equation
\[
    u_t = \Delta m_1 v - \Delta m_2 u + (1 - c)\Delta m_1 u + \Delta m_2 u
\]
(1.5)
corresponds to the nonlocal analogous of the Cahn-Hilliard equation
\[
    u_t - \Delta(-\Delta u + F'(u)) = 0 \quad \text{in } (0, \infty) \times \Omega.
\]
(1.6)
Observe that in (1.5) it appears the nonlocal sum of Laplacians \( \Delta m_1 \Delta m_2 - \Delta m_1 - \Delta m_2 \) playing the role of the (local) fourth-order operator \( \Delta^2 \) that appears in (1.6).

Formulation (1.4) allows us to study the (nonlocal) Cahn-Hilliard equation in a quite different way than those used for the local and nonlocal one, concretely we study it as a Lipschitz perturbation of the generalized porous medium equation,
\[
    u_t = \Delta m_1 v, \quad u \in \gamma(v),
\]
and we can do it for a large class of potentials that include the mentioned ones. At our knowledge the approach we use is new even for convolution kernels. After the preliminaries we dedicate a section to give an overview of the generalized nonlocal porous medium equation
\[
    u_t = \Delta m_1 v, \quad u \in \gamma(v),
\]
studied in [40], where, besides the nonlinearity driven by \( \gamma \), general Leray-Lions type diffusion operators are considered. We will do it for the sake of completeness since it will serve as the basis to study the existence and uniqueness of mild and strong solutions for (1.4). We also study the generalized nonlocal porous medium equation as a gradient flow in a “discrete \( H^{-1}\)-space”, that is the nonlocal version of the results by Brezis [8] for the local porous medium equation, which allows to get strong solutions in a very direct way. These results are new in this abstract setting of random walk spaces.
It is worthy to mention that we cover the study of the Cahn-Hilliard system stated on a subset $\Omega \in \mathcal{B}$ with $0 < \nu(\Omega) < \infty$ and $m_i$-connected, under (regional) homogeneous boundary conditions,

\[
\begin{aligned}
    u_t(t, x) &= \int_{\Omega} (\mu(t, y) - \mu(t, x))d(m^1)_x(y),
    & \quad (t, x) \in (0, \infty) \times \Omega, \\
    \mu(t, x) &= -\int_{\Omega} (u(t, y) - u(t, x))d(m^2)_x(y) + v(t, x) - cu(x, t),
    & \quad (t, x) \in (0, \infty) \times \Omega, \\
    u(t, x) &\in \gamma(v(t, x)),
    & \quad (t, x) \in (0, \infty) \times \Omega, \\
    u(0, x) &= u_0(x),
    & \quad x \in \Omega,
\end{aligned}
\]

where $\gamma$ is a maximal monotone graph with $0 \in \gamma(0)$ and $c > 0$, and, for which the following problem is a particular case:

\[
\begin{aligned}
    u_t(t, x) &= \int_{\Omega} J_1(x - y)(\mu(t, y) - \mu(t, x))dy,
    & \quad (t, x) \in (0, \infty) \times \Omega, \\
    \mu(t, x) &= -\int_{\Omega} J_2(x - y)(u(t, y) - u(t, x))dy + v(t, x) - cu(x, t),
    & \quad (t, x) \in (0, \infty) \times \Omega, \\
    u(t, x) &\in \gamma(v(t, x)),
    & \quad (t, x) \in (0, \infty) \times \Omega, \\
    u(0, x) &= u_0(x),
    & \quad x \in \Omega,
\end{aligned}
\]

where $\Omega$ is a bounded domain of $\mathbb{R}^N$ and $J_i : \mathbb{R}^N \rightarrow [0, +\infty[, \ i = 1, 2$, are non-singular radial kernels with mass equal to 1. Taking into account, besides the kernels, the general potentials considered here, our existence and uniqueness results can be seen as a generalization of the existence results for the weak-to-weak interaction described by Gal and Shomberg in [29].

We can put a parameter $\delta > 0$ in the model (even one can deal with $\delta = 0$),

\[
\begin{aligned}
    u_t(t, x) &= \Delta_{m^1}\mu(t, x),
    & \quad (t, x) \in (0, \infty) \times X, \\
    \mu(t, x) &\in -\delta\Delta_{m^2}u(t, x) + \partial F(u),
    & \quad (t, x) \in (0, \infty) \times X, \\
    u(0, x) &= u_0(x),
    & \quad x \in X,
\end{aligned}
\]

but for the existence results this is not relevant and we will assume $\delta = 1$. We will consider such parameter $\delta$ in the last section where we study the asymptotic behaviour of the solutions. Then we will see that if (remember that we are dealing with $F$ like in (1.3))

\[
c < \delta \text{ gap}(-\Delta_{m^2}),
\]

we have that for initial data $u_0 \in L^2$, the strong solution of (1.7) converges, as time goes to infinity, to the media of $u_0$. This shows that, for $c$ small, this problem is not...
suitable for phase separation. On the other hand, we will see that for the double-well obstacle potential, with $c \geq 2\delta$, the set of equilibria contains the functions that separate the phases.

Let us shortly describe the contents of the paper. In Section 2 we recall all the notions about random walk spaces required in this paper. Section 3 deals with the generalized porous medium equation in random walk spaces, that is one of the main tools used in the next section. We recall the results obtained in [40], and prove some results by means of the Hilbertian theory that are interesting by itself. In section 4 we obtain the main results about existence and uniqueness of solutions of the Cahn-Hilliard Equation. First we study the Cauchy problem in $L^1$. As a consequence, we also obtain the existence and uniqueness of the regional Neumann problem. Moreover we prove that the Cahn-Hilliard equation is the gradient flow of the Ginzburg-Landau free energy functional in an adequate Hilbert space when the random walks have the same invariant measure. Finally, in Section 5 we obtain some properties of the solutions and of their asymptotic behaviour, that particularly apply to finite weighted discrete graphs and non-smooth potentials.

2. Preliminaries

2.1. Convex functions and subdifferentials. Let $H$ be a real Hilbert space with scalar product $\langle \cdot , \cdot \rangle_H$ and norm $\| u \|_H = \sqrt{\langle u , u \rangle_H}$. Given a function $F : H \to [-\infty , \infty ]$, we call the set $D(F) := \{ u \in H : F(u) < +\infty \}$ the effective domain of $F$, and $F$ is said to be proper if $D(F)$ is non-empty. Further, we say that $D(F)$ is lower semi-continuous if for every $c \in \mathbb{R}$, the sublevel set

$$ E_c := \{ u \in D(F) : F(u) \leq c \} $$

is closed in $H$.

Given a convex proper function $F : H \to [-\infty , \infty ]$, its subdifferential is defined by

$$ \partial_H F := \{ (u , h) \in H \times H : F(u + v) - F(u) \geq \langle h , v \rangle_H \ \forall \ v \in D(F) \} . $$

This concept can be extended to any proper function $F : H \to [-\infty , \infty ]$ by defining the Gateaux subdifferential of $F$ as

$$ \partial_H F := \left\{ (u , h) \in H \times H : \liminf_{t \to 0^+} \frac{F(u + tv) - F(u)}{t} \geq \langle h , v \rangle_H \ \forall \ v \in D(F) \right\} , \quad (2.1) $$

which, if $F$ is convex, is reduced to the usual subdifferential.

We say that $\alpha : [a , b] \to H$ is an absolutely continuous curve, and we write $\alpha \in AC([a , b]; H)$, if there exists $g \in L^1([a , b])$ such that

$$ \| \alpha(y) - \alpha(x) \|_H \leq \int_x^y g(t) dt \ \ \forall \ a \leq x \leq y \leq b. $$
We define

\[ \text{AC}_{\text{loc}}(0, \infty); H) := \{ \alpha : [0, \infty[ \to H : \alpha \in \text{AC}([a, b]; H) \text{ for all } a < b \}. \]

The following result is given in [1, Proposition 11.4].

**Proposition 2.1.** Let \( I \) be an open interval of \( \mathbb{R} \). Then any \( u \in \text{AC}(I; H) \) is differentiable \( \mathcal{L}^1 \)-a.e. \( t \in I \), \( \| u' \| \in \mathcal{L}^1 (I) \) and the fundamental theorem of calculus

\[ u(t) - u(s) = \int_s^t u'(r) \, dr \quad \forall s, t \in I \]

holds.

The following definitions are given in [1]

**Definition 2.2.** Given a proper function \( F : H \to [\infty, \infty] \), we say that \( u : (0, \infty) \to \text{D}(F) \) is a gradient flow of \( F \) if \( u \in \text{AC}_{\text{loc}}(0, \infty); H) \) and

\[ u'(t) + \partial_H F(u(t)) \ni 0 \quad \text{for } \mathcal{L}^1 \text{-a.e. } t \in (0, \infty). \]

We say that \( u \) starts from \( u_0 \in H \) if \( \lim_{t \to 0} u(t) = u_0 \).

We are now in position to state the celebrated *Brezis-Komura Theorem* (see [9], [32], or [1]).

**Theorem 2.3.** Let \( F : H \to [\infty, \infty] \) be a proper convex and lower semi-continuous functional. Given \( f \in \mathcal{L}^2(0, T; H) \) and \( u_0 \in \overline{\text{D}(F)} \) there exists a unique strong solution of the abstract Cauchy problem

\[ u'(t) + \partial_H F(u(t)) \ni f(t) \quad \text{for } \mathcal{L}^1 \text{-a.e. } t \in (0, \infty) \] (2.2)

such that \( u(0) = u_0 \), that is \( u \in C([0, +\infty[; H) \cap \text{AC}_{\text{loc}}((0, \infty); H) \) and satisfies (2.2).

In the case \( f = 0 \), if we denote \( S(t)u_0 := u(t) \), the unique strong solution of the abstract Cauchy problem (2.2), then \( S(t) : \overline{\text{D}(F)} \to H \) is a continuous semigroup satisfying the \( T \)-contraction property

\[ \| (S(t)u_0 - S(t)v_0) \| \leq \| (u_0 - v_0) \| \quad \forall t > 0, \ u_0, v_0 \in \overline{\text{D}(F)}. \]

2.2. **Random walk spaces.** We recall some concepts and results about random walk spaces given in [34], [35] and [36].

Let \( (X, \mathcal{B}) \) be a measurable space such that the \( \sigma \)-field \( \mathcal{B} \) is countably generated. A random walk \( m \) on \( (X, \mathcal{B}) \) is a family of probability measures \( (m_x)_{x \in X} \) on \( \mathcal{B} \) such that \( x \mapsto m_x(B) \) is a measurable function on \( X \) for each fixed \( B \in \mathcal{B} \).

The notation and terminology chosen in this definition comes from Ollivier’s paper [39]. As noted in that paper, geometers may think of \( m_x \) as a replacement for the notion of balls
around $x$, while in probabilistic terms we can rather think of these probability measures as defining a Markov chain whose transition probability from $x$ to $y$ in $n$ steps is

$$dm^*_x(y) := \int_{z \in X} dm_z(y)dm^*_{x-1}(z), \quad n \geq 1$$

and $m^*_x = \delta_x$, the dirac measure at $x$.

**Definition 2.4.** If $m$ is a random walk on $(X, \mathcal{B})$ and $\mu$ is a $\sigma$-finite measure on $X$. The convolution of $\mu$ with $m$ on $X$ is the measure defined as follows:

$$\mu * m(A) := \int_X m_x(A)\mu(x) \quad \forall A \in \mathcal{B},$$

which is the image of $\mu$ by the random walk $m$.

**Definition 2.5.** If $m$ is a random walk on $(X, \mathcal{B})$, a $\sigma$-finite measure $\nu$ on $X$ is invariant with respect to the random walk $m$ if

$$\nu * m = \nu.$$

The measure $\nu$ is said to be reversible if moreover, the detailed balance condition

$$dm_x(y)d\nu(x) = dm_y(x)d\nu(y)$$

holds true.

**Definition 2.6.** Let $(X, \mathcal{B})$ be a measurable space where the $\sigma$-field $\mathcal{B}$ is countably generated. Let $m$ be a random walk on $(X, \mathcal{B})$ and $\nu$ an invariant measure with respect to $m$. The measurable space together with $m$ and $\nu$ is then called a random walk space and is denoted by $[X, \mathcal{B}, m, \nu]$.

**Definition 2.7.** Let $[X, \mathcal{B}, m, \nu]$ be a random walk space. If $(X, d)$ is a Polish metric space (separable completely metrizable topological space), $\mathcal{B}$ is its Borel $\sigma$-algebra and $\nu$ is a Radon measure (i.e. $\nu$ is inner regular and locally finite) and we denote it by $[X, d, m, \nu]$.

**Definition 2.8.** Let $[X, \mathcal{B}, m, \nu]$ be a random walk space. We say that $[X, \mathcal{B}, m, \nu]$ is $m$-connected if, for every $D \in \mathcal{B}$ with $\nu(D) > 0$ and $\nu$-a.e. $x \in X$,

$$\sum_{n=1}^{\infty} m^*_x(D) > 0.$$

**Definition 2.9.** Let $[X, \mathcal{B}, m, \nu]$ be a random walk space and let $A, B \in \mathcal{B}$. We define the $m$-interaction between $A$ and $B$ as

$$L_m(A, B) := \int_A \int_B dm_x(y)d\nu(x) = \int_A m_x(B)d\nu(x).$$
The following result gives a characterization of $m$-connectedness in terms of the $m$-interaction between sets.

**Proposition 2.10.** ([34, Proposition 2.11], [36, Proposition 1.34]) Let $[X, \mathcal{B}, m, \nu]$ be a random walk space. The following statements are equivalent:

(i) $[X, \mathcal{B}, m, \nu]$ is $m$-connected.

(ii) If $A, B \in \mathcal{B}$ satisfy $A \cup B = X$ and $L_m(A, B) = 0$, then either $\nu(A) = 0$ or $\nu(B) = 0$.

(iii) If $A \in \mathcal{B}$ is a $\nu$-invariant set then either $\nu(A) = 0$ or $\nu(X \setminus A) = 0$.

Let us see now some examples of random walk spaces.

**Example 2.11.** Consider the metric measure space $(\mathbb{R}^N, d, \mathcal{L}^N)$, where $d$ is the Euclidean distance and $\mathcal{L}^N$ the Lebesgue measure on $\mathbb{R}^N$ (which we will also denote by $|.|$). For simplicity, we will write $dx$ instead of $d\mathcal{L}^N(x)$. Let $J : \mathbb{R}^N \to [0, +\infty[$ be a measurable, nonnegative and radially symmetric function verifying $\int_{\mathbb{R}^N} J(x)dx = 1$. Let $m_J$ be the following random walk on $(\mathbb{R}^N, d)$:

$$m^J_x(A) := \int_A J(x - y)dy \quad \text{for every } x \in \mathbb{R}^N \text{ and every Borel set } A \subset \mathbb{R}^N.$$  

Then, applying Fubini’s Theorem it is easy to see that the Lebesgue measure $\mathcal{L}^N$ is reversible with respect to $m^J$. Therefore, $[\mathbb{R}^N, d, m^J, \mathcal{L}^N]$ is a reversible metric random walk space.

**Example 2.12.** [Weighted discrete graphs] Consider a locally finite weighted discrete graph $G = (V(G), E(G))$, where $V(G)$ is the vertex set, $E(G)$ is the edge set and each edge $(x, y) \in E(G)$ (we will write $x \sim y$ if $(x, y) \in E(G)$) has a positive weight $w_{xy} = w_{yx}$ assigned. Suppose further that $w_{xy} = 0$ if $(x, y) \notin E(G)$. Note that there may be loops in the graph, that is, we may have $(x, x) \in E(G)$ for some $x \in V(G)$ and, therefore, $w_{xx} > 0$. Recall that a graph is locally finite if every vertex is only contained in a finite number of edges.

A finite sequence $\{x_k\}_{k=0}^n$ of vertices of the graph is called a path if $x_k \sim x_{k+1}$ for all $k = 0, 1, ..., n - 1$. The length of a path $\{x_k\}_{k=0}^n$ is defined as the number $n$ of edges in the path. With this terminology, $G = (V(G), E(G))$ is said to be connected if, for any two vertices $x, y \in V$, there is a path connecting $x$ and $y$, that is, a path $\{x_k\}_{k=0}^n$ such that $x_0 = x$ and $x_n = y$. Finally, if $G = (V(G), E(G))$ is connected, the graph distance $d_G(x, y)$ between any two distinct vertices $x, y$ is defined as the minimum of the lengths of the paths connecting $x$ and $y$. Note that this metric is independent of the weights.
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For \( x \in V(G) \) we define the weight at \( x \) as

\[
d_x := \sum_{y \sim x} w_{xy} = \sum_{y \in V(G)} w_{xy},
\]

and the neighbourhood of \( x \) as \( N_G(x) := \{ y \in V(G) : x \sim y \} \). Note that, by definition of locally finite graph, the sets \( N_G(x) \) are finite. When all the weights are 1, \( d_x \) coincides with the degree of the vertex \( x \) in a graph, that is, the number of edges containing \( x \).

For each \( x \in V(G) \) we define the following probability measure

\[
m^G_x := \frac{1}{d_x} \sum_{y \sim x} w_{xy} \delta_y.
\]

It is not difficult to see that the measure \( \nu_G \) defined as

\[
\nu_G(A) := \sum_{x \in A} d_x, \quad A \subset V(G),
\]

is a reversible measure with respect to this random walk. Therefore, \([V(G), \mathcal{B}, m^G, \nu_G] \) is a reversible random walk space (\( \mathcal{B} \) is the \( \sigma \)-algebra of all subsets of \( V(G) \)) and \([V(G), d_G, m^G, \nu_G] \) is a reversible metric random walk space.

In Machine Learning Theory ([23], [24]), an example of a weighted discrete graph is a point cloud in \( \mathbb{R}^N \), \( V = \{ x_1, \ldots, x_n \} \), with edge weights \( w_{x_i,x_j} \) given by

\[
w_{x_i,x_j} := \eta(|x_i - x_j|), \quad 1 \leq i, j \leq n,
\]

where the kernel \( \eta : [0, \infty) \to [0, \infty) \) is a radial profile satisfying

(i) \( \eta(0) > 0 \), and \( \eta \) is continuous at 0,
(ii) \( \eta \) is non-decreasing,
(iii) and the integral \( \int_0^\infty \eta(r)r^N dr \) is finite.

Example 2.13. [Markov chains] Let \( K : X \times X \to \mathbb{R} \) be a Markov kernel on a countable space \( X \), i.e.,

\[
K(x,y) \geq 0 \quad \forall x, y \in X, \quad \sum_{y \in X} K(x,y) = 1 \quad \forall x \in X.
\]

Then, if

\[
m^K_x(A) := \sum_{y \in A} K(x,y), \quad x \in X, \ A \subset X
\]

and \( \mathcal{B} \) is the \( \sigma \)-algebra of all subsets of \( X \), \( m^K \) is a random walk on \( (X, \mathcal{B}) \).
Recall that, in discrete Markov chain theory terminology, a measure $\pi$ on $X$ satisfying
\[ \sum_{x \in X} \pi(x) = 1 \quad \text{and} \quad \pi(y) = \sum_{x \in X} \pi(x) K(x, y) \quad \forall y \in X, \]
is called a stationary probability measure (or steady state) on $X$. Of course, $\pi$ is a stationary probability measure if, and only if, $\pi$ is an invariant probability measure with respect to $m^K$. Consequently, if $\pi$ is a stationary probability measure on $X$, then $[X, B, m^K, \pi]$ is a random walk space.

Furthermore, a stationary probability measure $\pi$ is said to be reversible for $K$ if the following detailed balance equation holds:
\[ K(x, y) \pi(x) = K(y, x) \pi(y) \quad \forall x, y \in X. \]
This balance condition is equivalent to
\[ dm^K_x(y) d\pi(x) = dm^K_y(x) d\pi(y) \quad \forall x, y \in X. \]

Note that, given a locally finite weighted discrete graph $G = (V(G), E(G))$ as in Example 2.12, there is a natural definition of a Markov chain on the vertices. Indeed, define the Markov kernel $K_G : V(G) \times V(G) \to \mathbb{R}$ as
\[ K_G(x, y) := \frac{1}{d_x} w_{xy}. \]
Then, $m^G$ and $m^{K_G}$ define the same random walk. If $\nu_G(V(G))$ is finite, the unique reversible probability measure with respect to $m^G$ is given by
\[ \pi_G(x) := \frac{1}{\nu_G(V(G))} \sum_{z \in V(G)} w_{xz}. \]

Example 2.14. Given a random walk space $[X, B, m, \nu]$ and $\Omega \in B$ with $\nu(\Omega) > 0$, let
\[ m^\Omega_x(A) := \int_A dm_x(y) + \left( \int_{X \setminus \Omega} dm_x(y) \right) \delta_x(A) \quad \text{for every } A \in B_\Omega \text{ and } x \in \Omega. \]
Then, $m^\Omega$ is a random walk on $(\Omega, B_\Omega)$ and it is easy to see that $\nu \ll \Omega$ is invariant with respect to $m^\Omega$. Therefore, $[\Omega, B_\Omega, m^\Omega, \nu \ll \Omega]$ is a random walk space. Moreover, if $\nu$ is reversible with respect to $m$ then $\nu \ll \Omega$ is reversible with respect to $m^\Omega$. Of course, if $\nu$ is a probability measure we may normalize $\nu \ll \Omega$ to obtain the random walk space
\[ \left[ \Omega, B_\Omega, m^\Omega, \frac{1}{\nu(\Omega)} \nu \ll \Omega \right]. \]
Note that, if $[X, d, m, \nu]$ is a metric random walk space and $\Omega$ is closed, then $[\Omega, d, m^\Omega, \nu \ll \Omega]$ is also a metric random walk space, where we abuse notation and denote by $d$ the restriction of $d$ to $\Omega$. 
In particular, in the context of Example 2.11, if \( \Omega \) is a closed and bounded subset of \( \mathbb{R}^N \), we obtain the metric random walk space \([\Omega, d, m^J, \mathcal{L}^N \mathbf{L}, \Omega]\) where \( m^J := (m^J)^\Omega; \)
that is,
\[
m^J_x(\Omega) := \int_A J(x - y)dy + \left( \int_{\mathbb{R}^n \setminus \Omega} J(x - z)dz \right) d\delta_x
\]
for every Borel set \( A \subset \Omega \) and \( x \in \Omega \).

2.3. The nonlocal gradient, divergence and Laplace operators. Let us introduce the nonlocal counterparts of some classical concepts.

**Definition 2.15.** Let \([X, \mathcal{B}, m, \nu]\) be a random walk space. Given a function \( u : X \to \mathbb{R} \) we define its *nonlocal gradient* \( \nabla u : X \times X \to \mathbb{R} \) as
\[
\nabla u(x, y) := u(y) - u(x) \quad \forall x, y \in X.
\]
Moreover, given \( z : X \times X \to \mathbb{R} \), its *\( m \)-divergence* \( \text{div}_m z : X \to \mathbb{R} \) is defined as
\[
(\text{div}_m z)(x) := \frac{1}{2} \int_X (z(x, y) - z(y, x))dm_x(y).
\]

**Definition 2.16.** If \( \nu \) is an invariant measure with respect to \( m \), we define the linear operator \( M_m \) on \( L^1(X, \nu) \) into itself as follows
\[
M_m f(x) := \int_X f(y)dm_x(y), \quad f \in L^1(X, \nu).
\]
\( M_m \) is called the *averaging operator* on \([X, \mathcal{B}, m]\)

Note that, if \( f \in L^1(X, \nu) \) then, using the invariance of \( \nu \) with respect to \( m \),
\[
\int_X \int_X |f(y)|dm_x(y)d\nu(x) = \int_X |f(x)|d\nu(x) < \infty,
\]
so \( f \in L^1(X, m_x) \) for \( \nu \)-a.e. \( x \in X \), thus \( M_m \) is well defined from \( L^1(X, \nu) \) into itself.

**Remark 2.17.** Let \( \nu \) be an invariant measure with respect to \( m \). It follows that
\[
\|M_m f\|_{L^1(X, \nu)} \leq \|f\|_{L^1(X, \nu)} \quad \forall f \in L^1(X, \nu),
\]
so that \( M_m \) is a contraction on \( L^1(X, \nu) \). In fact, since \( M_m f \geq 0 \) if \( f \geq 0 \), we have that \( M_m \) is a positive contraction on \( L^1(X, \nu) \).

Moreover, by Jensen’s inequality, we have that,
\[
\|M_m f\|_{L^2(X, \nu)}^2 \leq \int_X f^2(x)d\nu(x) = \|f\|_{L^2(X, \nu)}^2.
\]
Therefore, \( M_m \) is a linear operator in \( L^2(X, \nu) \) with domain
\[
D(M_m) = L^1(X, \nu) \cap L^2(X, \nu).
\]
Consequently, if $\nu(X) < +\infty$, $M_m$ is a bounded linear operator from $L^2(X, \nu)$ into itself satisfying $\|M_m\| = \|M_m\|_{B(L^2(X, \nu), L^2(X, \nu))} \leq 1$. ■

We define the (nonlocal) Laplace operator as follows.

**Definition 2.18.** Let $[X, \mathcal{B}, m, \nu]$ be a random walk space, we define the $m$-Laplace operator (or $m$-Laplacian) from $L^1(X, \nu)$ into itself as $\Delta_m := M_m - I$, i.e.,

$$\Delta_m u(x) = \int_X u(y)dm_x(y) - u(x) = \int_X (u(y) - u(x))dm_x(y), \ u \in L^1(X, \nu).$$

Note that $\Delta_m f(x) = \text{div}_m(\nabla f)(x)$.

**Remark 2.19.** We have that $\|\Delta_m f\|_1 \leq 2\|f\|_1$ and

$$\int_X \Delta_m f(x)d\nu(x) = 0 \ \forall f \in L^1(X, \nu).$$

As in Remark 2.17 we obtain that $\Delta_m$ is a linear operator in $L^2(X, \nu)$ with domain $D(\Delta_m) = L^1(X, \nu) \cap L^2(X, \nu)$.

Moreover, if $\nu(X) < +\infty$, $\Delta_m$ is a bounded linear operator in $L^2(X, \nu)$ satisfying $\|\Delta_m\| \leq 2$. ■

We define the energy functional $\mathcal{H}_m : L^2(X, \nu) \to [0, +\infty]$ defined as

$$\mathcal{H}_m(f) = \begin{cases} \frac{1}{4} \int_{X \times X} (f(x) - f(y))^2dm_x(y)d\nu(x) & \text{if } f \in L^2(X, \nu) \cap L^1(X, \nu), \\ +\infty, & \text{else.} \end{cases}$$

We denote

$$D(\mathcal{H}_m) = L^2(X, \nu) \cap L^1(X, \nu).$$

In [34] it is proved that

$$-\Delta_m = \partial_{L^2(X, \nu)} \mathcal{H}_m.$$
Proposition 2.20. (Integration by parts formula) Let \([X, \mathcal{B}, m, \nu]\) be a reversible random walk space. Then,

\[
\int_X f(x) \Delta_m g(x) d\nu(x) = -\frac{1}{2} \int_{X \times X} \nabla f(x, y) \nabla g(x, y) d(\nu \otimes m_x)(x, y)
\]

(2.3)

for \(f, g \in L^1(X, \nu) \cap L^2(X, \nu)\). In particular for \(f \in D(\mathcal{H}_m)\), we have

\[
\mathcal{H}_m(f) = -\frac{1}{2} \int_X f(x) \Delta_m f(x) d\nu(x).
\]

Definition 2.21. We say that \([X, \mathcal{B}, m, \nu]\) satisfies a Poincaré inequality if there exists \(\lambda > 0\) such that

\[
\lambda \|f\|^2_{L^2(X, \nu)} \leq \mathcal{H}_m(f) \quad \text{for all} \quad f \in L^2(X, \nu) \quad \text{with} \quad \int_X f d\nu = 0.
\]

In [34] (see also [36]) it is shown that under quite general assumptions such an inequality holds true for the examples of random walk spaces given in Subsection 2.2.

The spectral gap of \(-\Delta_m\) is defined as

\[
\text{gap}(-\Delta_m) := \inf \left\{ \frac{2\mathcal{H}_m(f)}{\|f\|^2} : f \in D(\mathcal{H}_m), \|f\| \neq 0, \int_X f d\nu = 0 \right\}.
\]

We have that \(\frac{1}{2} \text{gap}(-\Delta_m)\) is the best constant in the Poincaré inequality. Moreover, it is well-known that \(\text{gap}(-\Delta_m) \leq 2\).

In [35] (see also [36]) we introduce and study the following concept that will be used later on.

Definition 2.22. Let \([X, \mathcal{B}, m, \nu]\) be a random walk space and let \(E \subset X\) be \(\nu\)-measurable. For a point \(x \in X\) we define its \(m\)-mean curvature as

\[
\mathcal{H}_{\partial E}^m(x) := \int_X (X_{E \setminus E}(y) - \chi_E(y)) dm_x(y) = 1 - 2m_x(E),
\]

which takes values in \([-1, 1]\).

3. THE CAUCHY PROBLEM FOR THE GENERALIZED POROUS MEDIUM EQUATION

Assume that \([X, \mathcal{B}, m, \nu]\) is a \(m\)-connected random walk space with \(\nu\) reversible, and \(\nu(X) < +\infty\). In this section we will study the Cauchy problem for the generalized porous medium equation under two points of view.

Let \(\gamma\) be a maximal monotone graph in \(\mathbb{R} \times \mathbb{R}\) such that \(0 \in \gamma(0)\), and set

\[
\gamma^- := \inf\{\text{Ran}(\gamma)\}, \quad \gamma^+ := \sup\{\text{Ran}(\gamma)\}.
\]
We assume $\gamma^{-} < \gamma^{+}$. Let us also define
\[
j_{\gamma}^{*}(r) := \int_{0}^{r} (\gamma^{-1})^{0}(s) ds,
\]
for $r \in D(\gamma^{-1})$ (the effective domain of $\gamma^{-1}$), where $(\gamma^{-1})^{0}(r)$ is the element of $\gamma^{-1}(r)$ of least norm. We have that
\[
\partial j_{\gamma}^{*}(r) = \gamma^{-1}(r).
\]

Consider the problem
\[
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta m v &= f & \text{in } (0, T) \times X, \\
u &\in \gamma(v) & \text{in } (0, T) \times X, \\
u(0, x) &= u_{0}(x) & x \in X.
\end{aligned}
\] (3.1)

Problem (3.1), for different choices of $\gamma$, gives rise to important examples. For instance, if $\gamma^{-1}(r) = |r|^{m-1}r$, and $m > 1$, it corresponds to the (nonlocal) porous medium equation, if $m = 1$ is the (nonlocal) heat equation, and if $0 < m < 1$, it is the fast diffusion equation. For
\[
\gamma^{-1}(r) = \begin{cases} 
  r & \text{if } r < 0, \\
  [0, 1] & \text{if } 0 \leq r \leq 1, \\
  r - 1 & \text{if } r \geq 1,
\end{cases}
\]
it deals with a (nonlocal) Stephan problem. While (nonlocal) Hele-Shaw type problems correspond to a choice like
\[
\gamma(r) = \begin{cases} 
  0 & \text{if } r < 0, \\
  [0, 1] & \text{if } r = 0, \\
  1 & \text{if } r > 0.
\end{cases}
\]

As we will see another important examples of $\gamma$ are the related with the graphs that appear in the Ginzburg-Landau free energy for the Cahn-Hilliard system.

3.1. The $L^1$-Theory. Let $T > 0$ and $f \in L^{1}(0, T; L^{1}(X, \nu))$. The study of existence and uniqueness of solution of Problem (3.1) was done in [40] by means of the Nonlinear Semigroup Theory ([12], [13], [14], [2]). Problem (3.1) is written as an abstract Cauchy problem in $L^{1}(X, \nu)$ associated with a $T$-accretive operator.

**Definition 3.1.** Define in $L^{1}(X, \nu)$ the operator $B_{\gamma}^{m}$ as $(u, \dot{u}) \in B_{\gamma}^{m}$ if:
\[
u, \dot{u} \in L^{1}(X, \nu) \text{ and there exists } v \in L^{2}(X, \nu) \text{ with}
\]
\[
u \in \gamma(v) \text{ } \nu \text{-a.e.}
\]
such that

$$-\Delta_m v = \hat{u}.$$  

With this operator at hand, Problem (3.1) can be rewritten as the following abstract Cauchy problem:

$$\begin{cases}
    u'(t) + B^m_\gamma(u(t)) \ni f(t), & t > 0, \\
    u(0) = u_0.
\end{cases}$$  

(3.2)

The following facts are proved in [40, Theorems 3.2, 3.3, 3.4] (for, using the notation of such reference, $\gamma = \beta$, $a_p(x, y, r) = r$ and $\Omega_1 \cup \Omega_2 = X$):

**Proposition 3.2 ([40]).** Assume $[X, B, m, \nu]$ satisfies a Poincaré inequality. Then:

1. The domain of the operator $B^m_\gamma$ satisfies:

$$D(B^m_\gamma)^{L^1(\Omega)} = \{ u \in L^1(X, \nu) : \gamma^- \leq u \leq \gamma^+ \}.$$  

2. $B^m_\gamma$ is $T$-accretive in $L^1(X, \nu)$ and satisfies the range condition:

$$\left\{ u \in L^2(X, \nu) : \nu(X) \gamma^- < \int_X u d\nu < \nu(X) \gamma^+ \right\} \subset R(I + \lambda B^m_\gamma) \quad \forall \lambda > 0.$$  

3. For any $T > 0$, and for any $u_0 \in D(B^m_\gamma)^{L^1(\Omega)}$ and $f \in L^1(0, T; L^1(\Omega, \nu))$ satisfying

$$\nu(X) \gamma^- < \int_X u_0 d\nu + \int_0^T \int_X f d\nu dt < \nu(X) \gamma^+,$$  

$$\forall 0 \leq t \leq T,$$  

(3.3)

there exists a unique mild-solution $u \in C([0, T] : L^1(\Omega, \nu))$ of Problem (3.2) (hence of Problem (3.1)).

4. Let $u_0, \tilde{u}_0 \in D(B^m_\gamma)^{L^1(\Omega)}$ and $f, \tilde{f} \in L^1(0, T; L^1(\Omega, \nu))$, satisfying the corresponding assumption (3.3), and $u, \tilde{u}$ the respective mild solutions of Problem (3.2), then

$$\int_X (u(t, x) - \tilde{u}(t, x))^+ d\nu(x) \leq \int_X (u_0(x) - \tilde{u}_0(x))^+ d\nu(x) +$$

$$+ \int_0^t \int_X (f(s, x) - \tilde{f}(s, x))^+ \nu(x) ds, \quad \forall 0 \leq t \leq T.$$  

5. If in addition $u_0 \in L^2(\Omega, \nu)$ and $\int\int_X j^*_\gamma(u_0) d\nu < +\infty$, and $f \in L^2(0, T; L^2(X, \nu))$, the mild solution belongs to $W^{1,1}(0, T; L^2(\Omega))$, therefore it is a strong solution.

In the next section we see that strong solutions can be also obtained via a gradient flow in an adequate Hilbert space. This will be used later on to get strong solutions for the Cahn-Hilliard problem.
3.2. The Hilbertian theory. Now we study the nonlocal version of the results by Brezis [8] for the local porous medium equation. In order to do this let us introduce a Hilbertian structure in random walk spaces. The particular case of finite weighted graphs and $\gamma^{-1}$ an increasing function was considered in [20] where the discrete porous medium arise as gradient flow of certain entropy functionals with respect to suitable non-local transportation metrics.

Let $[X, \mathcal{B}, m, \nu]$ be a $m$-connected random walk space with $\nu$ reversible, $\nu(X) < +\infty$, and assume that $[X, \mathcal{B}, m, \nu]$ satisfies a Poincaré inequality.

We have that $\Delta_m$ is a linear bounded operator in $L^2(X, \nu)$, and by the ergodicity of $\nu$, we have

$$\text{Ker}(\Delta_m) = \text{Lin}\{\chi_X\}.$$ 

Since $\Delta_m$ is selfadjoint in $L^2(X, \nu)$ and $[X, \mathcal{B}, m, \nu]$ satisfies a Poincaré inequality, we have that

$$\text{Ran}(\Delta_m) = L^2_0(X, \nu) := \{u \in L^2(X, \nu) : \int_X u(x) d\nu(x) = 0\}$$

and moreover

$$\Delta_m : L^2_0(X, \nu) \rightarrow L^2_0(X, \nu)$$

is bijective. For $v \in L^2_0(X, \nu)$, $\Delta_m^{-1}v$ denotes the preimage of $v$ via this bijection.

From now on we will denote

$$H_m^{-1}(X, \nu) := \text{Ran}(\Delta_m) = L^2_0(X, \nu)$$

endowed the inner product

$$\langle v_1, v_2 \rangle_{H_m^{-1}} := -\int_X \Delta_m^{-1}v_1 v_2 d\nu.$$ 

By the integration by parts formula (2.3), we have

$$\langle v_1, v_2 \rangle_{H_m^{-1}} = \frac{1}{2} \langle \nabla \phi_1, \nabla \phi_2 \rangle_{L^2(X \times X, d(\nu \otimes m_x))},$$

being $\phi_1, \phi_2 \in H_m^{-1}$ the unique functions such that $\Delta_m \phi_i = v_i$, $i = 1, 2$, that is

$$\langle v_1, v_2 \rangle_{H_m^{-1}} = \frac{1}{2} \langle \nabla \Delta_m^{-1}v_1, \nabla \Delta_m^{-1}v_2 \rangle_{L^2(X \times X, d(\nu \otimes m_x))}.$$ 

Let $\|\cdot\|_{H_m^{-1}}$ be the induced norm by such inner product.

**Proposition 3.3.** Assume that the random walk space $[X, \mathcal{B}, m, \nu]$ satisfies a Poincaré inequality (with constant $\lambda$). Then, the Hilbert space $H_m^{-1}(X, \nu)$ is isomorphic to the Hilbert space $(L^2_0(\Omega, \nu), \|\cdot\|_{L^2(\Omega, \nu)})$ with

$$\|v\|_{L^2(X, \nu)} \leq \sqrt{\frac{2}{\lambda}} \|v\|_{H_m^{-1}} \leq \frac{1}{\lambda} \|v\|_{L^2(X, \nu)}.$$
Proof. Given $v \in H^{-1}_m(X, \nu)$, applying Poincaré’s inequality, we have
\[
\|(\Delta^{-1}_m v)\|_{L^2(X,\nu)}^2 \leq \frac{1}{\lambda} H_m(\Delta^{-1}_m v)
\]
\[
= \frac{1}{4\lambda} \int_{X \times X} \nabla \Delta^{-1}_m v \cdot \nabla \Delta^{-1}_m v d(\nu \otimes m_x) = \frac{1}{2\lambda} \|v\|_{H^{-1}_m}^2.
\] (3.4)
Then
\[
\|v\|_{L^2(X,\nu)}^2 = \|\Delta_m(\Delta^{-1}_m v)\|_{L^2(X,\nu)}^2 \leq 4 \|\Delta^{-1}_m v\|_{L^2(X,\nu)}^2 \leq \frac{2}{\lambda} \|v\|_{H^{-1}_m}^2.
\]
Hence
\[
\|v\|_{L^2(X,\nu)} \leq \sqrt{\frac{2}{\lambda}} \|v\|_{H^{-1}_m}.
\]
On the other hand, applying Cauchy-Scharz and (3.4), we have
\[
\|v\|_{H^{-1}_m}^2 = -\int_X v \Delta^{-1}_m v d\nu \leq \|v\|_{L^2(X,\nu)} \|\Delta^{-1}_m v\|_{L^2(X,\nu)} \leq \|v\|_{L^2(X,\nu)} \|v\|_{H^{-1}_m} \leq \frac{1}{\sqrt{2\lambda}} \|v\|_{H^{-1}_m}.
\]
Hence
\[
\|v\|_{H^{-1}_m} \leq \frac{1}{\sqrt{2\lambda}} \|v\|_{L^2(X,\nu)}.
\]

Our aim is to study Problem (3.1) as a gradient flow in the Hilbert space $H^{-1}_m(X, \nu)$. For this, we consider the energy functional $\Psi_\gamma : H^{-1}_m(X, \nu) \to [0, +\infty]$ defined by
\[
\Psi_\gamma(u) := \begin{cases} \int_X j_\gamma^*(u(x)) d\nu(x) & \text{if } j_\gamma^*(u) \in L^1(X, \nu), \\ +\infty & \text{otherwise.} \end{cases}
\]

**Theorem 3.4.** We have that $\Psi_\gamma$ is convex and lower semi-continuous on $H^{-1}_m(X, \nu)$. Moreover,
\[
\partial_{H^{-1}_m(X,\nu)} \Psi_\gamma = \left\{ (u, w) \in H^{-1}_m(X, \nu) \times H^{-1}_m(X, \nu) : w = -\Delta_m v, v \in L^2(X,\nu), u(x) \in \gamma(v(x)) \nu \text{ a.e. } x \in X \right\}.
\]

**Proof.** Obviously $\Psi_\gamma$ is convex. On the other hand, we have that $\Psi_\gamma$ is lower semi-continuous in $L^1(X,\nu)$ (see for example [2, Proposition 2.7]), then by Proposition 3.3, we get that $\Psi_\gamma$ is lower semi-continuous on $H^{-1}(X,\nu)$. Therefore, $\partial_{H^{-1}_m(X,\nu)} \Psi_\gamma$ is a maximal monotone operator.
We consider now the operator
\[ A_\gamma = \{ (u, w) \in H^{-1}_m(X, \nu) \times H^{-1}_m(X, \nu) : w = -\Delta_m v, \] 
\[ \nu \in L^2(X, \nu), \ u(x) \in \gamma(v(x)) \nu - \text{a.e. } x \in X \}. \]

Let us see now that (this implies monotonicity of \( A_\gamma \)). For that we will see first that \( A_\gamma \subset \partial_{H^{-1}_m(X, \nu)} \Psi_\gamma \) (this implies monotonicity of \( A_\gamma \)) and next that \( A_\gamma \) is in fact maximal monotone.

Given \((u, w) \in A_\gamma\), we have \( w = -\Delta_m v, \) with \( v \in L^2(X, \nu), \ u(x) \in \gamma(v(x)) \nu - \text{a.e. } x \in X \). Then, \( v(x) \in \gamma^{-1}(u(x)) = \partial j_\gamma^*(u(x)) \nu - \text{a.e. } x \in X \), and consequently, if \( \tilde{u} \in H^{-1}(X, \nu), \)
\[ j_\gamma^*(\tilde{u}(x)) - j_\gamma^*(u(x)) \geq v(x)(\tilde{u}(x) - u(x)) \quad \nu - \text{a.e. } x \in X, \]
and hence
\[ \Psi_\gamma(\tilde{u}) - \Psi_\gamma(u) \geq \int_X v(x)(\tilde{u}(x) - u(x))d\nu(x) = \int_X (-\Delta_m^{-1} w)(\tilde{u} - u)d\nu = \langle w, \tilde{u} - u \rangle_{H^{-1}_m}. \]

Therefore \((u, w) \in \partial_{H^{-1}_m(X, \nu)} \Psi_\gamma\), and, consequently, \( A_\gamma \subset \partial_{H^{-1}_m(X, \nu)} \Psi_\gamma \).

By Minty’s Theorem, to see that \( A_\gamma \) is maximal monotone, we need to show that verifies the range condition
\[ R(I + A_\gamma) = H^{-1}_m(X, \nu). \]

Then, we must show that given \( z \in H^{-1}(X, \nu)\), there exists \( u \in H^{-1}_m(X, \nu) \) such that \((u, z - u) \in A_\gamma\). Now, this is equivalent to show that there exists \( u \in L^2_0(X, \nu) \) such that \((u, z - u) \in B_\gamma^m \cap L^2_0(X, \nu) \times L^2_0(X, \nu)\) which is true, thanks to Proposition 3.2.2. Indeed, there exists \( u \in L^1(X, \nu) \) such that \((u, z - u) \in B_\gamma^m\), but it is easy that, in fact, \( u \in L^2_0(X, \nu)\).

By the above theorem and the Brezis-Komura Theorem (Theorem 2.3), we have:

**Theorem 3.5.** For every initial data \( u_0 \in L^2(X, \nu) \) with
\[ \int_X j_\gamma^*(u_0)d\nu < +\infty, \]
\( T > 0, \) and any \( f \in L^2(0, T; H^{-1}_m(X, \nu)) \), there exists a unique strong solution of the abstract Cauchy problem
\[
\begin{align*}
\begin{cases}
    u'(t) + \partial \Psi_\gamma(u(t)) \ni f(t), & 0 \leq t \leq T, \\
    u(0) = u_0.
\end{cases}
\end{align*}
\]

(3.5)
By the characterization of the operator $\partial \partial_{H^{-1}(X,\nu)} \Psi_{\gamma}$ obtained in Theorem 3.4, we have the unique strong solution of Problem (3.5) is the only function $u \in C(0,T;H^{-1}(X,\nu))$ that satisfies

$$\begin{cases}
\frac{\partial u}{\partial t} - \Delta_m v(t) \ni f(t) & \text{a.e } t \in (0,T), \\
u(t) \in \gamma(v(t)) & \text{a.e } t \in (0,T), \\
u(0) = u_0.
\end{cases}$$

Therefore, the mild solutions given in Proposition 3.2 are in fact strong solutions under the conditions of the above result. Under this point of view we get moreover the following contraction principle: for $u_0, \tilde{u}_0 \in L^2(X,\nu)$ with $\int_X j_\gamma^*(u_0) d\nu, \int_X j_\gamma^*(\tilde{u}_0) d\nu < +\infty$, and $f, \tilde{f} \in L^1(0,T;H^{-1}_m(\Omega,\nu))$, and $u, \tilde{u}$ the respective strong solutions of Problem (3.6), then

$$||u(t) - \tilde{u}(t)||_{H^{-1}_m} \leq ||u_0 - \tilde{u}_0||_{H^{-1}_m} + \int_0^t ||f(s) - \tilde{f}(s)||_{H^{-1}_m} ds, \quad \forall 0 \leq t \leq T.$$  

**Remark 3.6.** Observe, that we can obtain the same result for data $u_0 \in L^2(X,\nu)$ with $\int_X j_\gamma^*(u_0) d\nu < +\infty$, by using a translation argument. Set

$$b = \frac{1}{\nu(X)} \int_X u_0 d\nu.$$

Since $j_\gamma^*$ is convex, we have

$$j_\gamma^*(b) = j_\gamma^* \left( \frac{1}{\nu(X)} \int_X u_0 d\nu \right) \leq \frac{1}{\nu(X)} \int_X j_\gamma^*(u_0) d\nu < +\infty.$$

Set $\tilde{\gamma}$ defined via

$$\tilde{\gamma}^{-1}(s) := \gamma^{-1}(s + b) - (\gamma^{-1})^0(b).$$

Then, for $\tilde{u}_0 := u_0 - b$,

$$\int_X j_\gamma^*(\tilde{u}_0) d\nu = \int_X j_\gamma^*(u_0) d\nu - \nu(X)j_\gamma^*(b) < +\infty.$$ 

Therefore, since $\tilde{u}_0 \in H^{-1}(X,\nu)$, for $T > 0$ and $f \in L^2(0,T;H^{-1}_m(X,\nu))$, there exists a unique strong solution $\tilde{u}(t)$ of problem

$$\begin{cases}
\tilde{u}'(t) + \partial \Psi_{\tilde{\gamma}}(\tilde{u}(t)) \ni f(t), & 0 \leq t \leq T, \\
\tilde{u}(0) = \tilde{u}_0.
\end{cases}$$
Then, there exists \( \tilde{v}(t) \in \tilde{\gamma}^{-1}(\tilde{u}(t)) \) such that
\[
\frac{\partial \tilde{u}}{\partial t} - \Delta_m \tilde{v}(t) \ni f(t) \quad \text{a.e } t \in (0, T).
\]

Now, if we define \( u(t) := \tilde{u}(t) + b \) and \( v(t) := \tilde{v}(t) + (\gamma^{-1})^0(b) \), we have
\[
v(t) = \tilde{v}(t) + (\gamma^{-1})^0(b) \in \tilde{\gamma}^{-1}(\tilde{u}(t)) + (\gamma^{-1})^0(b) = \gamma^{-1}(u(t)).
\]

Consequently, \( u(t) \) is a strong solution of problem (3.6).

4. THE CAHN-HILLIARD EQUATIONS ON RANDOM WALK SPACES

4.1. The Cauchy problem in \( L^1 \). Let \([X, \mathcal{B}, m^1, \nu^1]\) be a \( m^1 \)-connected random walk space such that \( \nu^1 \) is reversible and \( 0 < \nu^1(X) < +\infty \). Let \([X, \mathcal{B}, m^2, \nu^2]\) be a random walk space such that \( \nu^2 \) is invariant and \( 0 < \nu^2(X) < +\infty \). Assume moreover that
\[
\nu^1 \ll \nu^2
\]
and
\[
R := \frac{d\nu^1}{d\nu^2} \in L^\infty(X, \nu^2).
\]
The above hypothesis implies that \( L^1(X, \nu^2) \subset L^1(X, \nu^1) \) and
\[
m ||f||_{L^1(X, \nu^1)} \leq ||f||_{L^1(X, \nu^2)},
\]
with \( m = \frac{1}{||R||_{L^\infty(X, \nu^2)}} \). We moreover assume that \( L^1(X, \nu^1) \) is continuously embedded in \( L^1(X, \nu^2) \) with
\[
||f||_{L^1(X, \nu^2)} \leq M ||f||_{L^1(X, \nu^1)}.
\]

We consider the doubly nonlocal system
\[
\begin{aligned}
&u_t(t, x) = \Delta_{m^1}\mu(t, x), & (t, x) \in (0, \infty) \times X, \\
&\mu(t, x) \in -\Delta_{m^2}u(t, x) + \partial F(u(t, x)), & (t, x) \in (0, \infty) \times X, \\
&u(0, x) = u_0(x), & x \in X,
\end{aligned}
\]
where
\[
\partial F(r) = \gamma^{-1}(r) - cr,
\]
being \( \gamma \) a maximal monotone graph in \( \mathbb{R} \) with \( 0 \in \gamma(0) \) and \( \gamma^- < \gamma^+ \), and \( c > 0 \).

**Definition 4.1.** Given the random walks \( m^1, m^2 \), we define its convolution \( m^1 \ast m^2 \) as the random walk defined by
\[
\int_X \phi(y)dm^1 \ast m^2_x(y) := \int_X \left( \int_X \phi(y)dm^2_x(y) \right) dm^1_z(z).
\]
Lemma 4.2. Let \( u \in L^1(X, \nu_1) \). We have
\[
\Delta_{m_1}(\Delta_{m_2}u)(x) = \Delta_{m_1}\Delta_{m_2}u(x) - \Delta_{m_1}u(x) - \Delta_{m_2}u(x) \quad \text{for all } x \in X.
\]

Proof. We have
\[
\Delta_{m_1}(\Delta_{m_2})u(x) = M_{m_1}(\Delta_{m_2}u)(x) - \Delta_{m_2}u(x) = M_{m_1}(M_{m_2}u - u)(x) - \Delta_{m_2}u(x).
\]
Now
\[
M_{m_1}(M_{m_2}u)(x) = \int_X M_{m_2}u(z) dm_x^1(z) = \int_X \left( \int_X u(y) dm_x^2(y) \right) dm_x^1(z)
\]
\[
= \int_X u(y) (m^1 * m^2)_x(y) = M_{m_1 \ast m_2}u(x).
\]
Hence
\[
\Delta_{m_1}(\Delta_{m_2})u(x) = M_{m_1 \ast m_2}u(x) - M_{m_1}u(x) - \Delta_{m_2}u(x),
\]
that is
\[
\Delta_{m_1}(\Delta_{m_2})u(x) = \Delta_{m_1 \ast m_2}u(x)(x) - \Delta_{m_1}u(x) - \Delta_{m_2}u(x). \quad \square
\]

As consequence of the above lemma, we can rewrite (4.1) as
\[
\begin{align*}
&\begin{cases}
  u_t = \Delta_{m_1}v - \Delta_{m_1 \ast m_2}u + (1 - c)\Delta_{m_1}u + \Delta_{m_2}u & \text{in } (0, \infty) \times X, \\
  u \in \gamma(v) & \text{in } (0, \infty) \times X, \\
  u(0) = u_0 & \text{in } X.
\end{cases}
  
& \text{(4.3)}
\end{align*}
\]

We define the operator \( G : L^1(X, \nu_1) \to L^1(X, \nu_1) \) as
\[
G(u) = \Delta_{m_1 \ast m_2}u + (c - 1)\Delta_{m_1}u - \Delta_{m_2}u.
\]

As consequence of Remark 2.19, we have the following result.

Lemma 4.3. The operator \( G \) is Lipschitz continuous in \( L^1(X, \nu_1) \) and in \( L^2(X, \nu_1) \).

Proof. By Remark 2.19 we have:
1. \( \Delta_{m_1} \) is 2-Lipschitz continuous in \( L^1(X, \nu_1) \) and in \( L^2(X, \nu_1) \).
2. \( \Delta_{m_2} \) is 2-Lipschitz continuous in \( L^1(X, \nu_2) \) and in \( L^2(X, \nu_2) \). Hence, for \( f \in L^1(X, \nu_1) \),
\[
||\Delta_{m_2}f||_{L^1(X, \nu_1)} \leq \frac{1}{m}||\Delta_{m_2}f||_{L^1(X, \nu_2)} \leq \frac{2}{m}||f||_{L^1(X, \nu_2)} \leq \frac{M}{m}||f||_{L^1(X, \nu_1)}.
\]

And, for \( f \in L^2(X, \nu_1) \), with a similar argument,
\[
||\Delta_{m_2}f||_{L^2(X, \nu_1)}^2 \leq \frac{M}{m}||f||_{L^2(X, \nu_1)}^2.
\]
3. For \( f \in L^1(X, \nu_1) \),

\[
\|\Delta_{m^1} (\Delta_{m^2} f)\|_{L^1(X, \nu_1)} \leq 2\|\Delta_{m^2} f\|_{L^1(X, \nu_1)} \leq \frac{M}{m}\|f\|_{L^1(X, \nu_1)}.
\]

And, for \( f \in L^2(X, \nu_1) \),

\[
\|\Delta_{m^1} (\Delta_{m^2} f)\|_{L^2(X, \nu_1)} \leq 4\|\Delta_{m^2} f\|_{L^2(X, \nu_1)} \leq \frac{4^2 M}{m}\|f\|_{L^2(X, \nu_1)}.
\]

Now, by means of the operator

\[ B_\gamma := B_{\gamma}^{m^1}, \]

as given in Definition 3.1, but for the random walk \([X, B, m^1, \nu_1]\), we can rewrite (4.3) as the abstract Cauchy problem

\[
\begin{cases}
  u'(t) + (B_\gamma + G)(u(t)) \ni 0, & t > 0, \\
  u(0) = u_0.
\end{cases}
\]  

(4.4)

By Proposition 3.2 and Lemma 4.3, we have that \( B_\gamma + G + L_G I \) is an accretive operator in \( L^1(X, \nu_1) \), being \( L_G \) the Lipschitz constant of \( G \), with \( \frac{D(B_\gamma + G + L_G I)}{L^1(X, \nu_1)} = \{u \in L^1(X, \nu_1) : \gamma^- \leq u \leq \gamma^+\} \). We are going to see that Problem (4.3), via its abstract formulation given by (4.4), has mild solutions for a large class of general initial data:

**Theorem 4.4.** Assume that the random walk space \([X, B, m^1, \nu_1]\) satisfies a Poincaré inequality. For \( u_0 \in L^1(X, \nu_1) \), \( \gamma^- \leq u_0 \leq \gamma^+ \), such that

\[
\nu(X) \gamma^- < \int_X u_0 d\nu_1 < \nu(X) \gamma^+.
\]  

(4.5)

Problem (4.3) has a unique mild solution.

**Proof.** Let \( u_0 \in \{u \in L^1(X, \nu_1) : \gamma^- \leq u \leq \gamma^+\} \) satisfying condition (4.5), and \( 0 < \tilde{T} < 1/L_G \), where \( L_G \) is the Lipschitz constant of \( G \) in \( L^1(X, \nu_1) \). And set, for a fixed \( u_0 \) as in the hypothesis, the functional

\[ \mathfrak{F} : C([0, \tilde{T}] : L^1(X, \nu_1))) \rightarrow C([0, \tilde{T}] : L^1(X, \nu_1)) \]

given by \( \mathfrak{F}(z) = u_z \) the mild solution of Problem (3.2) (or Problem (3.1)) with initial datum \( u_0 \) and \( f = G(z) \) (observe that \( G(z) \in L^1(0, \tilde{T}; L^1(X, \nu_1)) \)). This functional is well defined thanks to Proposition 3.2, and, by item 4. in such proposition and Lemma 4.3, it is contractive with constant \( \tilde{T} L_G \), that we are assuming less than 1. Then it has a unique fix point \( u \) which is the unique mild solution of Problem (4.3). Indeed, for \( \epsilon > 0 \), there exists \( u_\epsilon \) a solution of an \( \frac{\epsilon}{2(1+T L_G)} \)-discretization in \([0, \tilde{T}]\) of \( u' + B_\gamma u \ni f \), for \( f = -G(u) \), with \( u_\epsilon(0) = u_0 \). But since \( G \) is \( L_G \)-Lipschitz continuous, this \( u_\epsilon \) is a solution of an \( \epsilon \)-discretization in \([0, \tilde{T}]\) of \( u' + (B_\gamma + G)u \ni g \), for \( g = 0 \), with \( u_\epsilon(0) = u_0 \).
and we have existence and uniqueness of mild solution since $B_\gamma + G + L_G I$ is an accretive operator in $L^1(X, \nu_1)$ and the initial datum is in $D((B_\gamma + G + L_G I)^{1/2}(X, \nu_1))$. Finally we can extend the solutions up to any $T > 0$ since continuity extension in time holds true for mild solutions.

**Remark 4.5.** We have that mild solutions are strong solutions when $\gamma^-$ and $\gamma^+$ are finite and

$$\int_X j^*_\gamma(u_0) d\nu_1 < +\infty.$$  

(4.6)

Indeed, since the mild solution $u$ is bounded between $\gamma^-$ and $\gamma^+$, then we have that $G(u) \in L^2(0, T; L^2(X, \nu_1))$. Therefore, we can apply 5. in Proposition 3.2.

In Subsection 4.3, for $\nu_1 = \nu_2 = \nu$, we see that we have strong solutions for any initial data in $L^2(X, \nu)$ satisfying (4.6). □

**Remark 4.6.** Observe that there is mass preservation:

$$\int_\Omega u(t) d\nu_1 = \int_\Omega u_0 d\nu_1 \quad \forall t \in [0, T].$$

In fact, this is clear for strong solutions, but also for mild solutions, since this property is inherited from the stationary schemes. □

**Example 4.7.** 1. Consider the potential $F_3(u) = A(1 - u^2) + I_{[-1,1]}(u)$ given in the Introduction. Let $\gamma$ be the inverse graph of $\partial I_{[-1,1]}$,

$$\gamma(r) = \begin{cases} 
-1 (= \gamma^-) & \text{if } r < 0, \\
[-1, 1] & \text{if } r = 0, \\
1 (= \gamma^+) & \text{if } r > 0.
\end{cases}$$

Any initial datum $u_0 \in L^1(X, \nu_1), -1 \leq u_0 \leq 1$, is in $L^2(X, \nu_1)$ and satisfies

$$\int_X j^*_\gamma(u_0) d\nu_1 < +\infty,$$

since such condition, in this case, is equivalent to have $u_0(x) \in [-1, 1], x \in X$. Therefore we will always have strong solutions for any initial datum $u_0 \in L^1(X, \nu_1), -1 \leq u_0 \leq 1$, satisfying

$$-\nu_1(X) < \int_X u_0 d\nu_1 < \nu_1(X).$$

But, if $u_0$ satisfies

$$-\nu_1(X) = \int_X u_0 d\nu_1,$$
which is equivalent to say that the initial datum is the pure phase

\[ u_0 = -1, \]

then the pure phase \( u(t) = -1 \) is the solution of Problem (4.3). Similarly, for the other pure phase \( u_0 = 1 \), \( u(t) = 1 \) is the solution of the problem.

2. For the case with the potential \( F_1(r) = ((1 + r) \log(1 + r) + (1 - r) \log(1 - r)) - \frac{c}{2} r^2 \), and datum \( u_0 \in L^1(X, \nu_1) \), \( -1 \leq u_0 \leq 1 \),

\[-\nu_1(X) < \int_X u_0 d\nu_1 < \nu_1(X),\]

and

\[ \int_X j^*_\gamma(u_0) d\nu_1 < +\infty, \]

where here \( \gamma^{-1}(r) = \log(1 + r) - \log(1 - r) \), we will have strong solutions.

3. Consider now the regular polynomial potential \( F_2(u) = \frac{1}{4} (u^2 - 1)^2 \). In this case, \( \gamma^{-1}(r) = r^3 \), and \( \gamma^- = -\infty \) and \( \gamma^+ = +\infty \). Therefore we have existence of mild solutions for any initial datum in \( L^1(X, \nu_1) \). For \( \nu_1 = \nu_2 = \nu \) and for data in \( L^2(X, \nu) \) satisfying \( \int_X j^*_\gamma(u_0) d\nu < +\infty \), which in this case is equivalent to ask for data \( u_0 \in L^4(X, \nu) \), we also have strong solutions on account of the results given in Subsection 4.3. ■

4.2. The regional Neumann problem. Let \([X, \mathcal{B}, m^1, \nu_1]\) be a random walk space such that \( \nu_1 \) is reversible and \( \nu_1(X) < +\infty \). Let \([X, \mathcal{B}, m^2, \nu_2]\) be a random walk space such that \( \nu_1 \) is invariant and \( \nu_2(X) < +\infty \).

Let \( \Omega \in \mathcal{B} \) with \( 0 < \nu_1(\Omega) < +\infty, i = 1, 2 \). Consider the random walk spaces

\[ [\Omega, \mathcal{B}_\Omega, (m^i)^\Omega, \nu_i\mathcal{L}\Omega], \quad i = 1, 2, \]

given in Example 2.14; assume \([\Omega, \mathcal{B}_\Omega, (m^1)^\Omega, \nu_1\mathcal{L}\Omega]\) is \((m^1)^\Omega\)-connected. Assume moreover that

\[ \nu_1\mathcal{L}\Omega \ll \nu_2\mathcal{L}\Omega, \]

and

\[ \frac{d\nu_1\mathcal{L}\Omega}{d\nu_2\mathcal{L}\Omega} \in L^\infty(\Omega, \nu_2). \]

Assume also that that \( L^1(\Omega, \nu_1) \) is continuously embedded in \( L^1(\Omega, \nu_2) \).
Then, if we apply the results of Subsection 4.1 to these random walk spaces, we have that Problem (4.1) corresponds to the following Cahn-Hilliard problem

$$\begin{cases}
u(t, x) 
= \int_{\Omega} (u(t, y) - u(t, x)) d(m^2)(y), & (t, x) \in (0, \infty) \times \Omega, \\
\mu(t, x) 
= -\int_{\Omega} (u(t, y) - u(t, x)) d(m^1)(y) + \partial F(u(t, x)), & (t, x) \in (0, \infty) \times \Omega, \\
u(0, x) 
= u_0(x), & x \in \Omega.
\end{cases} \quad (4.7)$$

As consequence of Theorem 4.4 and Theorem 4.13 below, we have the following result about existence and uniqueness of solutions to Problem (4.7).

**Theorem 4.8.** Assume, joint to the previous assumptions, that the random walk space $[\Omega, B_{\Omega}, (m^1)^{\Omega}, \nu_1 \llcorner \Omega]$ satisfies a Poincaré inequality. For $u_0 \in L^1(\Omega, \nu_1)$, $\gamma^- \leq u_0 \leq \gamma^+$, such that

$$\nu(\Omega)\gamma^- < \int_{\Omega} u_0 dv_1 < \nu(\Omega)\gamma^+,$$

Problem (4.7) has a unique mild solution.

Moreover, if $\nu_1 \llcorner \Omega = \nu_2 \llcorner \Omega$ and it is also reversible with respect to $m^2$, for $u_0 \in L^2(\Omega, \nu_1)$ such that

$$\int_{\Omega} j^*_\gamma(u_0) d\nu_1 < +\infty,$$

Problem (4.7) has a unique strong solution.

4.3. **The gradient flow.** Fife showed in [21] that the local system (1.1) is the gradient flow of the Ginzburg-Landau free energy

$$\mathcal{E}_l(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} F(u) dx,$$

respect the the scalar product

$$\langle v_1, v_2 \rangle_{H^{-1}} := \langle \nabla \phi_1, \nabla \phi_2 \rangle_{L^2},$$

where $v_1, v_2 \in H^{-1} := \left\{ v \in L^2(\Omega) : \int_{\Omega} v = 0 \right\}$ and $\phi_1, \phi_1 \in H^{-1}$ are the unique solutions of the Neumann problem

$$\begin{cases}
\Delta \phi_i = v_i & \text{in} \ \Omega, \\
\frac{\partial \phi_i}{\partial \eta} = 0 & \text{on} \ \partial \Omega.
\end{cases}$$

This approach is frequently used in the literature to deal with the different versions (local-nonlocal, nonlocal-nonlocal) of the Cahn-Hilliard system. We will obtain a similar development in the ambient space of random walk spaces.
Let \([X, \mathcal{B}, m^1, \nu]\) be a \(m^1\)-connected random walk space such that \(\nu\) is reversible with respect to \(m^1\) and \(\nu(X) < +\infty\). Let \([X, \mathcal{B}, m^2, \nu]\) be a random walk space such that \(\nu\) is reversible with respect to \(m^2\). Observe we are imposing the restrictive assumption that both spaces have the same invariant and reversible measure. This assumption is always satisfied if we are dealing with nonlocal kernels \(m^1\) and \(m^2\) in \(\mathbb{R}^N\) like in Example 2.11 since the Lebesgue measures is an invariant and reversible measure for both, they are also satisfied for certain graphs with different, but related, weights defining \(m^1\) and \(m^2\).

Since we are dealing with potentials \(F\) such that \(\partial F(r) = \gamma^{-1}(r) - cr\), with \(\gamma^{-1}\) possibly multivalued, the Ginzburg-Landau free energy functional is defined as

\[
\mathcal{E} : H_{m^1}^{-1}(X, \nu) \to [\infty, +\infty],
\]

with

\[
\mathcal{E}(u) := \mathcal{H}(u) + \Psi(u) - \frac{c}{2} \int_X u^2 d\nu,
\]

where \(\mathcal{H} : H_{m^1}^{-1}(X, \nu) \to \mathbb{R}\) is

\[
\mathcal{H}(u) := \frac{1}{4} \int_{X \times X} (u(x) - u(y))^2 d\nu(x, y),
\]

\(\Psi : H_{m^1}^{-1}(X, \nu) \to ]\infty, +\infty]\) is

\[
\Psi(u) := \begin{cases} 
\int_X j_{\gamma}^\ast(u(x)) d\nu(x) & \text{if } j_{\gamma}^\ast(u) \in L^1(X, \nu), \\
+\infty & \text{otherwise}.
\end{cases}
\]

Set

\[
\mathcal{K}(u) := \frac{c}{2} \int_X u^2 d\nu.
\]

Applying Theorem 3.4, we have the following result.

**Lemma 4.9.** The functional \(\Psi\) is convex and lower semi-continuous in \(H_{m^1}^{-1}(X, \nu)\) and

\[
\partial_{H_{m^1}^{-1}(X, \nu)} \Psi = \left\{ (u, w) \in H_{m^1}^{-1}(X, \nu) \times H_{m^1}^{-1}(X, \nu) : w = -\Delta_{m^1} v, \right. \\
\left. v \in L^2(X, \nu), \ u(x) \in \gamma(v(x)) \nu\text{-a.e. } x \in X \right\}.
\]

**Lemma 4.10.** The operator \(\mathcal{H}\) is proper, convex and continuous in \(H_{m^1}^{-1}(X, \nu)\).

**Proof.** Obviously \(\mathcal{H}\) is proper and convex, and the continuity follows directly from Proposition 3.3. In fact, let \(\{u_n\}_n \subset H_{m^1}^{-1}(X, \nu)\) and \(u \in H_{m^1}^{-1}(X, \nu)\) be such that

\[
\| \cdot \|_{H_{m^1}^{-1}(X, \nu)} - \lim_n u_n = u.
\]
Then, from Proposition 3.3,

\[ u_n \to u \quad \text{in} \quad L^2(X, \nu). \]

Therefore, on account of this convergence, we get

\[ \mathcal{H}(u) = \lim_{n \to \infty} \mathcal{H}(u_n). \]

**Theorem 4.11.** We have that \( \mathcal{H} + \Psi \) is convex and lower semi-continuous in \( H^{-1}_{m^1}(X, \nu) \). Moreover, \( w \in \partial_{H^{-1}_{m^1}(X, \nu)}(\mathcal{H} + \Psi)(u) \) if and only if

\[ \exists \nu \in L^2(X, \nu), \text{ with } u(x) \in \gamma(\nu(x)) \text{ } \nu \text{-a.e., such that } w = \Delta_{m^1}(\Delta_{m^2} u) - \Delta_{m^1} \nu. \]

**Proof.** By Lemma 4.9 and Lemma 4.10, we have \( \mathcal{H} + \Psi \) is convex and lower semi-continuous in \( H^{-1}_{m^1}(X, \nu) \). Now, since \( \text{int}(D(\mathcal{H})) \cap D(\Psi) = L^2_0(X, \nu) \cap L^1(X, \nu) \neq \emptyset \), applying [9, Corollary 2.11], we have

\[ \partial_{H^{-1}_{m^1}(X, \nu)}(\mathcal{H} + \Psi)(u) = \partial_{H^{-1}_{m^1}(X, \nu)} \mathcal{H} + \partial_{H^{-1}_{m^1}(X, \nu)} \Psi. \]

Hence, \( w \in \partial_{H^{-1}_{m^1}(X, \nu)}(\mathcal{H} + \Psi)(u) \) if and only if

\[ \exists w_1 \in \partial_{H^{-1}_{m^1}(X, \nu)} \mathcal{H}(u) \text{ and } \exists w_2 \in \partial_{H^{-1}_{m^1}(X, \nu)} \Psi(u) \text{ such that } w = w_1 + w_2. \]

Now, by Lemma 4.9, there exists \( \nu \in L^2(X, \nu), \text{ } u(x) \in \gamma(\nu(x)) \text{ } \nu \text{-a.e. } x \in X \), such that \( w_2 = -\Delta_{m^1} \nu. \)

On the other hand, since \( \mathcal{H} \) is convex, we have \( w_1 \in \partial_{H^{-1}_{m^1}(X, \nu)} \mathcal{H}(u) \) in and only if

\[ \liminf_{t \to 0^+} \frac{1}{t} (\mathcal{H}(u + th) - \mathcal{H}(u)) \geq \langle w_1, h \rangle_{H^{-1}_{m^1}(X, \nu)} \quad \forall h \in H^{-1}_{m^1}(X, \nu). \]

Now, for \( h \in H^{-1}_{m^1}(X, \nu) \), we have

\[ \liminf_{t \to 0^+} \frac{1}{t} (\mathcal{H}(u + th) - \mathcal{H}(u)) = \frac{1}{2} \int_{X \times X} \nabla u(x, y) \nabla h(x, y) d(\nu \otimes (m^2))(x, y) \]

\[ = -\int_X \Delta_{m^2} u d\nu = -\int_X \Delta_{m^1}^{-1}(\Delta_{m^3}(\Delta_{m^2} u)) h d\nu = \langle \Delta_{m^1}(\Delta_{m^2} u), h \rangle_{H^{-1}_{m^1}(X, \nu)}. \]

Therefore, \( w_1 = \Delta_{m^3}(\Delta_{m^2} u). \)

As consequence of Theorem 4.11, we can rewrite Problem (4.1) as

\[ \begin{cases} 
  u'(t) + \partial_{H^{-1}_{m^1}(X, \nu)}(\mathcal{H} + \Psi)(u(t)) - B(u(t)) \ni 0, & t > 0, \\
  u(0) = u_0. 
\end{cases} \]  

(4.9)
being
\[ \mathcal{B}(u) := -c \Delta_{m_1} u. \]

**Proposition 4.12.** The operator \( \mathcal{B} \) is \( H^{-1}_{m_1} \)-Lipschitz continuous.

**Proof.** Since \( \mathcal{B} \) is \( L^2(X, \nu) \)-Lipschitz continuous, the result is consequence of Proposition 3.3. \( \square \)

**Theorem 4.13.** Assume that the random walk space \([X, \mathcal{B}, m_1, \nu]\) satisfies a Poincaré inequality. Then, for any \( u_0 \in L^2(X, \nu) \) with
\[ \int_X j_{\gamma}^*(u_0) d\nu < +\infty, \quad (4.10) \]
there exists a unique strong solution of Problem (4.1). Moreover, the mild solution given in Theorem 4.4 is such strong solution under the above conditions.

**Proof.** Having in mind Theorem 4.11 and Proposition 4.12, applying [9, Proposition 3.12], we have the result for initial data in \( L^2_0(X, \nu) \). Now, using the translation argument given in Remark 3.6, we also have strong solutions for data \( u_0 \in L^2(X, \nu) \) since we have that \( -c \Delta_{m_1} u \in L^2(0, T; H^{-1}_{m_1}(X, \nu)) \). \( \square \)

**Remark 4.14.** Problem (4.1), or its equivalent expression given in Problem (4.9), is the gradient flow in \( H^{-1}_{m_1}(X, \nu) \) of the Ginzburg-Landau free energy functional \( \mathcal{E} \) given in (4.8):
\[ \begin{cases} 
  u'(t) + \partial_{H^{-1}_{m_1}(X, \nu)} \mathcal{E}(u(t)) \ni 0, & t > 0, \\
  u(0) = u_0.
\end{cases} \quad (4.11) \]
Indeed, by definition (see (2.1)), \( h \in \partial_{H^{-1}_{m_1}(X, \nu)} \mathcal{E}(u) \) if and only if
\[ \liminf_{t \to 0^+} \frac{\mathcal{E}(u + tv) - \mathcal{E}(u)}{t} \geq \langle h, v \rangle_{H^{-1}_{m_1}(X, \nu)} \quad \forall v \in H^{-1}_{m_1}(X, \nu). \]
Now, by convexity, \( \widetilde{h} \in \partial_{H^{-1}_{m_1}(X, \nu)} (\mathcal{H} + \Psi)(u) \) if and only if
\[ \liminf_{t \to 0^+} \frac{(\mathcal{H} + \Psi)(u + tv) - (\mathcal{H} + \Psi)(u)}{t} \geq \langle \widetilde{h}, v \rangle_{H^{-1}_{m_1}(X, \nu)} \quad \forall v \in H^{-1}_{m_1}(X, \nu). \quad (4.12) \]
And, on the other hand, for \( v \in H^{-1}_{m_1}(X, \nu) \),
\[ \exists \lim_{t \to 0^+} \frac{-\frac{c}{2} \int_X (u + tv)^2 d\nu + \frac{c}{2} \int_X u^2 d\nu}{t} = -\langle cu, v \rangle_{L^2(X, \nu)} \]
\[ = -\langle c \Delta_{m_1} u, v \rangle_{L^2(X, \nu)} = \langle c \Delta_{m_1} u, v \rangle_{H^{-1}_{m_1}(X, \nu)}. \quad (4.13) \]
Therefore adding up (4.12) and (4.13), (4.9) can be written as (4.11).

5. Some properties of the solution

In this section we obtain some properties of the solutions and of their asymptotic behaviour.

In the next result we see that, for \( L^\infty(X, \nu) \)-bounded initial data, strong solutions stay bounded in \( L^\infty(0, T; L^\infty(X, \nu)) \), therefore the solutions obtained in Theorem 4.8 coincide with the solutions obtained, with a different method, by Gal and Shomberg in [29] for the case considered in such paper. Concretely, they impose that the potential \( F \in C^2(\mathbb{R}) \) satisfies

\[
F'(0) = 0 \quad \text{and} \quad F''(r) + a_K(x) \geq c_0 \quad \text{for all} \quad r \in \mathbb{R} \quad \text{and a.e.}\; x \in \Omega,
\]

for some constant \( c_0 > 0 \), where \( a_K(x) := \int_{\Omega} K(x - y) dy \) (being \( K \) the interaction kernel playing the role of \( m^2 \)). This case includes, for example, the double-well potential \( F_2 \) given in the Introduction but not \( F_1 \) or \( F_3 \). Therefore the above existence and uniqueness result generalize the results by Gal and Shomberg. At our knowledge, Theorem 4.4 and Theorem 4.10 give new existence and uniqueness results for the Cahn-Hilliard problem by the generality of the potentials and the transitions that can be considered on the system, and the large class of initial data.

**Proposition 5.1.** Under the conditions in Theorem 4.13, assume moreover that \( u_0 \in L^p(X, \nu) \), \( 2 \leq p \leq +\infty \). Then, for \( u \) being the unique strong solution of Problem (4.1), there exists a constant \( C > 0 \) (which depends only on \( c \) given in (4.2)) such that

\[
||u(t)||_{L^p(X, \nu)} \leq ||u_0||_{L^p(X, \nu)} e^{CT} \quad \text{for all} \; 0 < t < T.
\]

**Proof.** We know that \( u \in W^{1,1}_{\text{loc}}((0, T); L^2(X, \nu) \cap C([0, T] : L^2(X, \nu)), u(0) = u_0 \), and there exists \( v(t) \in L^2(X, \nu), v(t) \in \gamma(u(t)) \nu_1\text{-a.e, such that, for almost every} \; t \in (0, T),

\[
u_t = \Delta m_1 v - \Delta m_{1, m_2} u + (1 - c)\Delta m_1 u + \Delta m_2 u \quad \text{in} \; (0, T) \times X.
\]

Let \( p \geq 2 \) and \( j_{p, k}(r) \) be the primitive of the nondecreasing function \( r|T_k(r)|^{p-2} \). Multiplying the above equation by \( u|T_k(u)|^{p-2} \), integrating over \( X \), and misleading nonnegative
terms, we get
\[
\frac{d}{dt} \int_X j_{k,p}(u(t))d\nu \leq \int_X -\Delta_{m^1,m^2} u(t,x) \left( |u| T_k(u) |^{p-2} \right) (t,x)d\nu(x)
\]
\[\quad + (c - 1)^+ \int_X -\Delta_{m^1} u(t) \left( u(t) |T_k(u(t)) |^{p-2} \right) d\nu
\]
\[= \int_X \int_X -(u(t,y) - u(t,x)) \left( u(t,x) |T_k(u(t,x)) |^{p-2} \right) d(m^1 * m^2)_x(y)d\nu(x)
\]
\[\quad + (c - 1)^+ \int_X \int_X -(u(t,y) - u(t,x))u(t,x) \left( u(t,x) |T_k(u(t,x)) |^{p-2} \right) dm^1_xd\nu(x)
\]
Now,
\[-(a - b)|T_k(b)|^{p-2} \leq (|a| + |b|)|T_k(b)|^{p-2} \leq 2a^2 |T_k(a)|^{p-2} + 2b^2 |T_k(b)|^{p-2}, \quad (5.2)
\] and, a direct calculation shows that
\[r^2 |T_k(r)|^{p-2} \leq pj_{k,p}(r). \quad (5.3)
\]
Then, by (5.2) and (5.3), (5.1) yields
\[
\frac{d}{dt} \int_X j_{k,p}(u(t))d\nu \leq 2p \int_X \int_X (j_{k,p}(u(t,x)) + j_{k,p}(u(t,y))) d(m^1 * m^2)_x(y)d\nu(x)
\]
\[\quad + 2p(c - 1)^+ \int_X \int_X (j_{k,p}(u(t,x)) + j_{k,p}(u(t,y))) dm^1_xd\nu(x)
\]
Now, it is easy to see that \(\nu\) is invariant with respect to \(m^1 * m^2\). Therefore, using the invariance of \(\nu\) with respect to \(m^1\) and \(m^1 * m^2\), (5.4) gives
\[
\frac{d}{dt} \int_X j_{k,p}(u(t))d\nu \leq Cp \int_X j_{k,p}(u(t))d\nu,
\] with \(C = 4 \max\{c,1\}\). Then, from Grönwall’s lemma,
\[
\int_X j_{k,p}(u(t))d\nu \leq e^{CpT} \int_X j_{k,p}(u_0)d\nu \quad \forall 0 \leq t \leq T.
\] Hence
\[
\left( \int_X pj_{k,p}(u(t))d\nu \right)^{1/p} \leq e^{CT} \left( \int_X pj_{k,p}(u_0)d\nu \right)^{1/p} \quad \forall 0 \leq t \leq T,
\] and, taking limits as \(k\) goes to \(+\infty\) in the above expression we get
\[
\left( \int_X |u(t)|^p d\nu \right)^{1/p} \leq e^{CT} \left( \int_X |u_0|^p d\nu \right)^{1/p} \quad \forall 0 \leq t \leq T. \quad (5.5)
\]
This gives the proof if $p$ is finite. For $p = +\infty$, we have just to take limits as $p$ goes to $+\infty$ in (5.5) to get

$$||u(t)||_{L^\infty(X,\nu)} \leq ||u_0||_{L^\infty(X,\nu)}e^{CT} \quad \forall 0 < t < T.$$

\[\square\]

**Remark 5.2.** A similar result can be obtained for strong solutions of Problem (4.1) under the general conditions of Subsection 4.1. ■

Let us see that the solution to Problem (4.1) satisfies an energy identity. This will allow to get an uniform estimate in time for the $L^2$-norm of the solution under natural extra conditions (see Corollary 5.4).

**Proposition 5.3.** Under the conditions in Theorem 4.13, for $u$ the strong solution to Problem (4.1), set $\tilde{E}(t) := E(u(t))$. Then

$$\frac{d}{dt} \tilde{E}(t) = -\frac{1}{2} \int_{X \times X} |\nabla \mu(t)|^2 d(\nu \otimes (m^1)_x) \quad \text{for a.e. } t > 0,$$

where $\mu = -\Delta m^2 u + v - cu$, $v \in \gamma^{-1}(u)$, as corresponding to the definition of strong solution.

**Proof.** We have that

$$\begin{cases} u'(t) + \partial_{H_{m^1}^{-1}(X,\nu)}(H + \Psi))u(t)) - c\Delta m^1 u \geq 0, & t > 0, \\ u(0) = u_0. \end{cases}$$

Now, by [9, Lemme 3.3], for almost every $t > 0$,

$$\frac{d}{dt} \left((H + \Psi)(u(t))\right) = \langle h, u'(t)\rangle_{H_{m^1}^{-1}(X,\nu)} \quad \forall h \in \partial_{H_{m^1}^{-1}(X,\nu)}(H + \Psi)(u(t)).$$

Then, since

$$c\Delta m^1 u - u'(t) \in \partial_{H_{m^1}^{-1}(X,\nu)}(H + \Psi)(u(t))$$

and

$$u'(t) = \Delta m^1 \mu(t),$$

we have

$$\frac{d}{dt} \left((H + \Psi)(u(t))\right) = \langle c\Delta m^1 u - u'(t), u'(t)\rangle_{H_{m^1}^{-1}(X,\nu)}$$

$$= \langle c\Delta m^1 u, u'(t)\rangle_{H_{m^1}^{-1}(X,\nu)} - \langle \Delta m^1 \mu(t), \Delta m^1 \mu(t)\rangle_{H_{m^1}^{-1}(X,\nu)}$$

$$= \frac{d}{dt} \left(K(u(t))\right) - \langle \Delta m^1 \mu(t), \Delta m^1 \mu(t)\rangle_{H_{m^1}^{-1}(X,\nu)},$$
and consequently,
\[
\frac{d}{dt} \left( (\mathcal{H} + \Psi - \mathcal{K})(u(t)) \right) = -\langle \Delta m^1 \mu(t), \Delta m^1 \mu(t) \rangle_{H^{-1}_m(X, \nu)},
\]
\[
= -\frac{1}{2} \int_{X \times X} |\nabla \mu(t)|^2 d(\nu \otimes (m^1)_x).
\]

\[\square\]

**Corollary 5.4.** Assume the conditions in Theorem 4.13 and suppose \([X, B, m^2, \nu]\) satisfies a Poincaré inequality and the potential \(j^*_r(r) - \frac{C}{2} r^2\) is bounded from below. For \(u\) the strong solution to Problem (4.1), we have that 
\[
\{u(t) : t > 0\} \text{ is bounded in } L^2(X, \nu).
\]

**Proof.** As a consequence of Proposition 5.3 we have that \(\tilde{E}(t)\) is absolutely continuous and nonincreasing, therefore
\[
E(u(t)) \leq E(u_0) \quad \forall t > 0.
\]
Hence, since the potential \(F\) is bounded from below, we have
\[
\mathcal{H}(u(t)) \leq C, \quad \forall t \geq 0.
\]
Finally, using the Poincaré inequality for \([X, B, m^2, \nu]\), from (5.6), we get the thesis. \(\square\)

5.1. **Asymptotic behaviour.** In [29] a very interesting analysis is done on the asymptotic behaviour of the solutions for the case of smooth convolution kernels and smooth potential. Their assumptions guaranties a regularization and a Lojasiewicz-Simon inequality. With the generality of the nonlocal interactions and potentials studied here it is not clear if one can get such tools. Nevertheless we can prove some facts.

From now on we will assume that we are under the conditions in Theorem 4.13. Then, we can define a semigroup \(T(t) : L^2(X, \nu) \to L^2(X, \nu)\), such that, for every \(u_0 \in L^2(X, \nu)\), \(T(t)u_0 := u(t)\) is the unique strong solution of problem (4.1). For \(u_0 \in L^2(X, \nu)\), we define its omega limit set
\[
\omega(u_0) := \{w \in L^2(X, \nu) : \exists t_n \to +\infty \text{ such that } T(t_n)u_0 \to w \text{ in } L^2(X, \nu)\}.
\]

We also consider the set of *equilibria solutions* of the problem,
\[
\mathbb{E} = \mathbb{E}(T(t)) := \{u \in L^2(X, \nu) : T(t)u = u \quad \forall t \geq 0\}
\]
that can be characterized as
\[
\mathbb{E} = \{u \in L^2(X, \nu) : \exists \mu \text{ constant with } \mu + \Delta m^2 u + cu \in \gamma^{-1}(u)\}.
\]
Theorem 5.5. Assume that the random walk space \([X, \mathcal{B}, m^2, \nu]\) also satisfies a Poincaré inequality. Let \(u_0 \in L^2(X, \nu)\) be and assume that
\[
\gamma^- < \frac{1}{\nu(X)} \int_X u_0 d\nu < \gamma^+, \tag{5.7}
\]
and also that the omega limit set \(\omega(u_0)\) is not empty. Then,
\[
\omega(u_0) \subset E. \tag{5.8}
\]

Proof. Let \(u(t) := T(t)u_0\) be, and \(\mu = -\Delta_m^2 u + v - cu, v \in \gamma^-(u)\), as corresponding to the definition of strong solution.

Given \(u_\infty \in \omega(u_0)\), there exists a sequence \(t_n \to +\infty\) such that
\[
u(t_n) \to u_\infty \text{ in } L^2(X, \nu). \tag{5.11}
\]

From Proposition 5.3 and the Poincaré inequality we have that
\[
\int_0^\infty \int_X |\mu(s, x) - \overline{\mu}(s)|^2 d\nu(x) ds \leq C, \tag{5.9}
\]
where \(\overline{\mu}(s) = \frac{1}{\nu(X)} \int_X \mu(s, x) d\nu(x)\), and
\[
\alpha_n := \int_{t_n}^\infty \int_X |\mu(s, x) - \overline{\mu}(s)|^2 d\nu(x) ds \leq 2\lambda m^2 (\tilde{\mathcal{E}}(t_n) - \tilde{\mathcal{E}}_\infty) \to 0,
\]
where \(\tilde{\mathcal{E}}_\infty := \lim_{t \to +\infty} \tilde{\mathcal{E}}(t)\).

Let us see that there exists \(\tilde{t}_n \in [t_n, t_n + C/\sqrt{\alpha_n}]\) such that
\[
\int_X |\mu(\tilde{t}_n, x) - \overline{\mu}(\tilde{t}_n)|^2 d\nu(x) \leq \sqrt{\alpha_n}. \tag{5.10}
\]

Indeed, arguing by contradiction, if for almost all \(s \in [t_n, t_n + C/\sqrt{\alpha_n}]\) we have
\[
\int_X |\mu(s, x) - \overline{\mu}(s)|^2 d\nu(x) > \sqrt{\alpha_n},
\]
then,
\[
\int_{t_n}^{t_n + C/\sqrt{\alpha_n}} \int_X |\mu(s, x) - \overline{\mu}(s)|^2 d\nu(x) > C,
\]
which gives a contradiction with (5.9).

Let us now see that
\[
u(\tilde{t}_n) \to u_\infty \text{ in } L^2(X, \nu). \tag{5.11}
\]
In fact,

\[
|u(\tilde{t}_n) - u(t_n)|_{L^2(X, \nu)} = \left\| \int_{t_n}^{\tilde{t}_n} \frac{d}{ds} u(s) ds \right\|_{L^2(X, \nu)}
\]

\[
= \left\| \int_{t_n}^{\tilde{t}_n} \Delta_m \mu(s) ds \right\|_{L^2(X, \nu)} = \left\| \int_{t_n}^{\tilde{t}_n} \Delta_m (\mu(s) - \bar{\mu}(s)) ds \right\|_{L^2(X, \nu)}
\]

\[
\leq (t_n - \tilde{t}_n)^{1/2} \left( \int_{t_n}^{\tilde{t}_n} \|\Delta_m (\mu(s) - \bar{\mu}(s))\|_{L^2(X, \nu)}^2 ds \right)^{1/2}
\]

\[
\leq 2(t_n - \tilde{t}_n)^{1/2} \left( \int_{t_n}^{t_n+\infty} \|\mu(s) - \bar{\mu}(s)\|_{L^2(X, \nu)}^2 ds \right)^{1/2}
\]

\[
= 2 (t_n - \tilde{t}_n)^{1/2} (\alpha_n)^{1/2} \leq 2C^{1/2} (\alpha_n)^{1/4} \to 0.
\]

Now we see that \(\{\bar{\mu}(\tilde{t}_n)\}_n\) is bounded. In fact, if

\[
\lim \bar{\mu}(\tilde{t}_n) = +\infty,
\]

then, from (5.10), we have that

\[
\mu(\tilde{t}_n, \cdot) \to +\infty \quad \nu\text{-a.e.}
\]

But, since

\[
\mu(\tilde{t}_n) = -\Delta_m u(\tilde{t}_n) + v(\tilde{t}_n) - cu(\tilde{t}_n), \quad v(\tilde{t}_n) \in \gamma^{-1}(u(\tilde{t}_n)),
\]

and we have (5.11), we arrive at

\[
u(\tilde{t}_n) \to \gamma^+ \quad \text{in } L^2(X, \nu),
\]

which is impossible since the mass is preserved and we are taking \(\gamma^- < \frac{1}{\nu(X)} \int_X u_0 d\nu < \gamma^+\). Similarly we also arrive to a contradiction if we suppose that \(\lim \bar{\mu}(\tilde{t}_n) = -\infty\).

Since \(\{\bar{\mu}(\tilde{t}_n)\}_n\) is bounded, then we have that there exist a subsequence, that we denote equal, and a constant \(\mu_\infty\) such that

\[
\lim_n \bar{\mu}(\tilde{t}_n) = \mu_\infty.
\]

Therefore, from (5.10), we also have

\[
\mu(\tilde{t}_n, \cdot) \to \mu_\infty \quad \text{in } L^2(X, \nu).
\]

And, from the convergences obtained, we easily arrive to

\[
\mu_\infty = -\Delta_m u_\infty + v_\infty - cu_\infty, \quad v_\infty \in \gamma^{-1}(u_\infty),
\]

that is, \(u_\infty\) us a stationary solution of our Cahn-Hilliard problem. Hence we have proved (5.8). \(\Box\)
Remark 5.6. (1) Let us point out that, for $F_3(r) = \frac{c}{2}(1 - r^2) + I_{[-1,1]}(r)$, the assumption (5.7) is natural, otherwise the solution is trivial, see Remark 4.7.

(2) Observe that

$$E(w) = \tilde{E}_\infty, \quad \text{for all } w \in \omega(u_0).$$

In fact, given $w \in \omega(u_0)$, there exists $t_n \to +\infty$ such that $u(t_n) \to w$ in $L^2(X, \nu)$. Then, since $E$ is lower semi-continuous, we have

$$E(w) \leq \liminf_{n \to \infty} E(u(t_n)) = \tilde{E}_\infty.$$

On the other hand, given $t > 0$, if $t \leq t_n$, since $E$ is non-increasing, we have

$$\tilde{E}(t) \geq \tilde{E}(t_n) \geq E(w).$$

Thus $\tilde{E}(t)$ is bounded from below on the orbit $\{u(t) : t \geq 0\}$, and by using again the monotonicity of $\tilde{E}(t)$, we have there exist $\lim_{t \to \infty} \tilde{E}(t) \geq E(w)$. Therefore, $E(w) = \tilde{E}_\infty$. $\blacksquare$

For the obstacle potential $F_3$, it is interesting to know when, for some $\nu$-measurable $D \subset X$, we have

$$\chi_D - \chi_{X \setminus D} \in \mathbb{E},$$

that is, when there exists equilibria solutions that divide the space in two pure phases without interface between them. In the next result we will be that this happen under some geometrical condition on $D$. Consider here that we are dealing with

$$\begin{cases}
  u_t(t, x) = \Delta_m \mu(t, x), & (t, x) \in (0, \infty) \times X, \\
  \mu(t, x) \in -\delta \Delta_m u(t, x) + \partial F(u(t, x)), & (t, x) \in (0, \infty) \times X, \\
  u(0, x) = u_0(x), & x \in X,
\end{cases} \quad (5.12)$$

with $\delta > 0$ (for which the same existence and uniqueness result holds true).

**Proposition 5.7.** Suppose we have Problem (5.12) with the obstacle potential $F_3$. Let $D \subset X$ be $\nu$-measurable such that $\int_X (\chi_D - \chi_{X \setminus D}) d\nu = \int_X u_0$. If

$$1 + \frac{1}{2} \left( \sup_{x \in D} \mathcal{H}_{\partial D}^m(x) + \sup_{x \in X \setminus D} \mathcal{H}_{\partial(X \setminus D)}^m(x) \right) \leq \frac{c}{\delta}, \quad (5.13)$$

then

$$\chi_D - \chi_{X \setminus D} \in \mathbb{E}. \quad (5.14)$$
Proof. To prove (5.14), we need the existence of a constant $\mu$ such
\[
\mu + \delta \Delta m^2(x_D - x_{X \setminus D})(x) + c(x_D(x) - x_{X \setminus D}(x)) \in \gamma^{-1}(x_D(x) - x_{X \setminus D}(x)).
\]
Therefore, since $\Delta m^2(x_D - x_{X \setminus D})(x) = 2m^2_x(D) - 1 - (x_D(x) - x_{X \setminus D}(x))$, we need the existence of a constant $\mu$ such that
\[
\mu + \delta(2m^2_x(D) - 1) + (c - \delta)(x_D(x) - x_{X \setminus D}(x)) \in \gamma^{-1}(x_D(x) - x_{X \setminus D}(x));
\]
which is equivalent to ask for
\[
\mu + \delta(2m^2_x(D) - 1) + (c - \delta)(x_D(x) - x_{X \setminus D}(x)) \geq 0 \text{ if } x \in D,
\]
\[
\mu + \delta(2m^2_x(D) - 1) - (c - \delta) \leq 0 \text{ if } x \in X \setminus D.
\]
that is,
\[
\frac{\mu}{\delta} \geq 1 - 2m^2_x(D) - \left(\frac{c}{\delta} - 1\right) \text{ if } x \in D,
\]
\[
\frac{\mu}{\delta} \leq 1 - 2m^2_x(D) + \left(\frac{c}{\delta} - 1\right) \text{ if } x \in X \setminus D.
\]
Then, we can find a constant $\mu$ satisfying the above inequalities if
\[
1 + \sup_{x \in X \setminus D} m^2_x(D) - \inf_{x \in D} m^2_x(D) \leq \frac{c}{\delta}.
\]
Using that $1 - 2m^2_x(D) = \mathcal{H}^m_{\partial D}(x) = -\mathcal{H}^m_{\partial(X \setminus D)}(x)$, the above inequality is equivalent to (5.13), and the proof concludes. $\square$

Remark 5.8. Observe that if $c \geq 2\delta$ then (5.13) holds, and therefore
\[
(5.14) \text{ also holds for any } c \geq 2\delta.
\]
Then, an small $\delta$, or a large $c$ in the potential, ensures the existence of equilibria with only pure states regions. $\blacksquare$

In the next result we are dealing with strong solutions of Problem (5.12).

**Proposition 5.9.** Assume we are under the assumptions of Theorem 5.5. If
\[
c < \delta \text{ gap}(-\Delta m^2),
\]
then
\[
\omega(u_0) = \{\overline{u_0}\},
\]
where $\overline{u_0} = \frac{1}{\nu(X)} \int_X u_0 d\nu$, which is equivalent to have
\[
\lim_{t \to +\infty} u(t) = \overline{u_0}.
\]
Proof. Let $u_\infty, \tilde{u}_\infty \in \omega(u_0)$. As in the proof of Theorem 5.5, we can find constants $\mu$ and $\tilde{\mu}_\infty$ such that

$$
\mu_\infty = -\delta \Delta_{m^2} u_\infty + v_\infty - cu_\infty, \quad v_\infty \in \gamma^{-1}(u_\infty),
$$

and

$$
\tilde{\mu}_\infty = -\delta \Delta_{m^2} \tilde{u}_\infty + \tilde{v}_\infty - c\tilde{u}_\infty, \quad \tilde{v}_\infty \in \gamma^{-1}(\tilde{u}_\infty).
$$

Then,

$$
\begin{align*}
 j_\gamma^*(\tilde{u}_\infty) - j_\gamma^*(u_\infty) & \geq \int_X (\mu_\infty + \delta \Delta_{m^2} u_\infty + cu_\infty)(\tilde{u}_\infty - u_\infty) d\nu \\
 j_\gamma^*(u_\infty) - j_\gamma^*(\tilde{u}_\infty) & \geq \int_X (\tilde{\mu}_\infty + \delta \Delta_{m^2} \tilde{u}_\infty + c\tilde{u}_\infty)(u_\infty - \tilde{u}_\infty) d\nu.
\end{align*}
$$

Hence, since we have preservation of mass,

$$
\begin{align*}
 j_\gamma^*(\tilde{u}_\infty) - j_\gamma^*(u_\infty) & \geq \int_X (\mu_\infty + \delta \Delta_{m^2} u_\infty + cu_\infty)(\tilde{u}_\infty - u_\infty) d\nu \\
 \delta j_\gamma^*(u_\infty) - j_\gamma^*(\tilde{u}_\infty) & \geq \int_X (\delta \Delta_{m^2} u_\infty + c\tilde{u}_\infty)(u_\infty - \tilde{u}_\infty) d\nu.
\end{align*}
$$

Then adding both last expressions

$$
0 \geq -\delta \int_X \Delta_m (\tilde{u}_\infty - u_\infty) dm^2_x(y)(\tilde{u}_\infty - u_\infty) d\nu(x) - c \int_X (\tilde{u}_\infty - u_\infty)^2 d\nu,
$$

and integrating by parts we get,

$$
0 \geq \frac{\delta}{2} \int_{X \times X} |\nabla (\tilde{u}_\infty - u_\infty)|^2 d(\nu \otimes m^2_x) - c \int_X (\tilde{u}_\infty - u_\infty)^2 d\nu.
$$

Hence, since $\int_X (\tilde{u}_\infty - u_\infty) d\nu = 0$, from the Poincaré inequality, we obtain

$$
\delta \text{gap}(-\Delta_{m^2}) \int_X |\tilde{u}_\infty - u_\infty|^2 d\nu \leq c \int_X |\tilde{u}_\infty - u_\infty|^2 d\nu.
$$

Therefore, since we are assuming (5.15) we have that $\tilde{u}_\infty = u_\infty$, that is, the omega limit set is a singleton,

$$
\omega(u_0) = \{u_\infty\}.
$$

Now, since $\bar{u}_0 \in ]\gamma^-, \gamma^+]$, there exists (a constant) $\beta \in \gamma^{-1}(\bar{u}_0)$. Therefore we can take $[u_\infty, v_\infty, \mu_\infty] = [\bar{u}_0, \beta, \beta - c\bar{u}_0]$ in the above computations, and we get (5.16). \qed

Let us point out that by the above result we have that if (5.15) holds, then this model is not suitable for phase separation.
5.2. **The case of finite weighted discrete graphs.** Let \( G = (V(G), E(G)) \) be a finite graph and suppose that we have two sets of weights \( w^1_{xy}, w^2_{xy} \) such that

\[
d_x := \sum_{y \sim x} w^1_{xy} = \sum_{y \sim x} w^2_{xy}, \quad \text{for all } x \in V(G). \tag{5.17}
\]

So,

\[
\nu_G(A) := \sum_{x \in A} d_x, \quad A \subset V(G),
\]

is a invariant measure for \((m^i_x), i = 1, 2\), being

\[
m^i_x := \frac{1}{d_x} \sum_{y \sim x} w^i_{xy} \delta_y, \quad i = 1, 2.
\]

Then, we can consider the random walk spaces \([V(G), m^i, \nu_G], i = 1, 2\) (see Example 2.12), that we assume for both to be \(m^i\)-connected.

We have that each random walk space \([V(G), m^i, \nu_G], i = 1, 2\), satisfies a Poincaré inequality, and also (4.10) holds for any \(u_0 \in L^2(V(G), \nu_G)\). Therefore, by Theorem 4.13, there exists a unique solution strong solution \(u(t)\) of Problem (4.1) for any initial data \(u_0 \in L^2(V(G), \nu_G)\). Since \(L^2(V(G), \nu_G)\) is finite dimensional, if we assume that the potential \(j^\gamma(r) - \frac{c}{2} r^2\) is bounded from bellow, then, by Corollary 5.4, we have

\[
\{u(t) : t > 0\} \text{ is bounded in } L^2(V(G), \nu_G),
\]

from where we get that

\[
\{u(t) : t > 0\} \text{ is relatively compact in } L^2(V(G), \nu_G).
\]

And, therefore, the \(\omega\)-limit set \(\omega(u_0)\) is not empty. Hence, as a consequence of Theorem 5.5, we have the following result.

**Theorem 5.10.** Let \( G = (V(G), E(G)) \) be a finite graph satisfying (5.17). Assume that the potential \(j^\gamma(r) - \frac{c}{2} r^2\) is bounded from bellow. Let \(u_0 \in L^2(V, \nu_G)\) be and assume that

\[
\gamma^- < \frac{1}{\nu_G(V)} \int_V u_0 d\nu_G < \gamma^+.
\]

Then,

\[
\omega(u_0) \subset \mathbb{E}.
\]

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