The Synchronization of Three Chaotic Fractional-order Lorenz Systems with Bidirectional Coupling

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Abstract. This paper mainly investigates the chaos synchronization of three identical fractional order Lorenz systems with linearly coupling. And the sufficient conditions to realize synchronization are obtained by Laplace transform theory. Numerical solutions and simulations are used for demonstration.

1. Introduction

As a mathematical topic, fractional calculus has been three centuries history[1]. However, in the last few decades many researchers pointed out that derivative and integrals of non-integer order are very suitable for description of properties of various real materials[2,3], e.g. polymers and electromagnetic waves. In comparison with classical integer order models, fractional-order derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes.

More recently, many authors began to investigate the chaotic dynamics of fractional-order dynamical systems[4-15]. According to the Poincare-Bendixson theorem, chaos can only occur in an autonomous ordinary differential equations when its dimension is at least three. But, it has been shown that the chaotic behavior can exist in a fractional order systems with order less than three, such as, the order of chaotic fractional-order Chua systems[4] is as low as 2.7. In ref. [5], chaotic behavior of fractional-order Lorenz system was studied. The fractional-order Chen chaotic system was also investigated in Refs.[7, 8]. And Ge et.al [9-14] investigated the chaotic behaviors of fractional-order modified Duffing systems, modified van der Pol system and nonlinear damped Mathieu system, respectively.

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At present, synchronization of chaotic fractional-order differential systems attracts increasing
attentions for its potential applications in secure communication and control processing[15-22]. Due to
the different definition of chaotic fractional-order differential equation(FODE) from the classical
ordinary differential equation(ODE), many methods and results of synchronization in chaotic ODE
can't directly be extended to the case of the FODE systems. Now there are some results about
synchronization between two chaotic FODE. However, for three coupled identical chaotic FODE, the
study of synchronization are much complex. Even in the classical integer order chaotic systems, there
are still a few papers investigated chaos synchronization about three coupled chaotic systems[23-26].

In this paper, we mainly study the synchronization of three identical chaotic fractional-order
Lorenz systems with bidirectional coupling, and give the sufficient conditions for achieving
synchronization by Laplace transformation theory.

There are many definitions of fractional-order derivatives. In what follows, the best-known
Riemann -Liowville definition[1] is used:

\[D^\alpha x(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{x^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau\]

Where \(m = \lceil \alpha \rceil\), i.e. \(m\) is the first integer of not less than \(\alpha\). Without loss of generality, we assume
\(0 < \alpha < 1\).

2. The synchronization of three fractional-order chaotic systems with linearly coupling
Three fractional-order chaotic systems can be coupled via many different techniques. In this paper we
only consider the unidirectional and bidirectional coupling.

A unidirectional coupling scheme among three identical chaotic FODEs with a ring connection can be given by

\[
\begin{align*}
\frac{d^\alpha X}{dt^\alpha} &= AX + G(X) + D_1(Z-X), \\
\frac{d^\alpha Y}{dt^\alpha} &= AY + G(Y) + D_2(X-Y), \\
\frac{d^\alpha Z}{dt^\alpha} &= AZ + G(Z) + D_3(Y-Z)
\end{align*}
\]

where \(X,Y,Z \in \mathbb{R}^n\) are the state vectors of the chaotic systems, respectively, \(A \in \mathbb{R}^{n \times n}\) is a constant
matrix, \(G : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is a nonlinear part of the systems (1)-(3), \(\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)^T\), \(0 < \alpha_i < 1\)
\((i = 1, 2, \cdots, n)\), and \(D_j \in \mathbb{R}^{n \times n} \ (j = 1, 2, 3)\) are the coupled parameters.

Define the error vector

\[\Xi = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} = \begin{pmatrix} X-Y \\ Y-Z \end{pmatrix}\]

which satisfies
How to realize the synchronization of systems (1)-(3) is equivalent to satisfy $\lim_{t \to \infty} \Xi(t) = 0$. According to the results of Refs. [27,28], we have

**Theorem 1.** With respect to system (5), the zero point is asymptotically stable if $|\arg(eig(M))| > \frac{\alpha \pi}{2}$. i.e. $\lim_{t \to \infty} \Xi(t) = 0$ or the systems (1)-(3) realize the synchronization.

Where $M = \begin{pmatrix} A - D_1 - D_2 & -D_1 \\ D_2 & A - D_3 \end{pmatrix}$.

The result is evident, so the proof is omitted.

According to the Refs. [24,25], the three identical chaotic FODEs via bidirectional coupling can be written as

$$\frac{d^\alpha X}{dt^\alpha} = AX + G(X) + D_1(Y + Z - 2X),$$

(6)

$$\frac{d^\alpha Y}{dt^\alpha} = AY + G(Y) + D_2(Z + X - 2Y),$$

(7)

and

$$\frac{d^\alpha Z}{dt^\alpha} = AZ + G(Z) + D_3(X + Y - 2Z).$$

(8)

The corresponding error dynamics can be given as

$$\frac{d^\alpha \Xi}{dt^\alpha} = \begin{pmatrix} \frac{d^\alpha E_1}{dt^\alpha} \\ \frac{d^\alpha E_2}{dt^\alpha} \end{pmatrix} = \begin{pmatrix} A - 2D_1 - D_2 & D_2 - D_1 \\ D_2 - D_3 & A - D_2 - 2D_3 \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} + \begin{pmatrix} G(X) - G(Y) \\ G(Y) - G(Z) \end{pmatrix},$$

(9)

**Theorem 2.** With respect to system (9), the zero point is asymptotically stable if $|\arg(eig(N))| > \frac{\alpha \pi}{2}$. i.e. $\lim_{t \to \infty} \Xi(t) = 0$ or the systems (6)-(8) realize the synchronization. Where

$$N = \begin{pmatrix} A - 2D_1 - D_2 & D_2 - D_1 \\ D_2 - D_3 & A - D_2 - 2D_3 \end{pmatrix}.$$
\[
\begin{align*}
\frac{d^{\alpha_1}x}{dt^{\alpha_1}} &= a(y - x) \\
\frac{d^{\alpha_2}y}{dt^{\alpha_2}} &= cx - y - xz, \\
\frac{d^{\alpha_3}z}{dt^{\alpha_3}} &= -bz + xy
\end{align*}
\] (10)

where the parameters \( a, b \) and \( c > 0 \), \( 0 < \alpha_i < 1, (i = 1, 2, 3) \). Especially, when \( \alpha_1 = 0.985 \), \( \alpha_2, 3 = 0.99 \) and \((a, b, c) = (10, \frac{8}{3}, 28)\), the fractional-order Lorenz system (10) has a chaotic attractor as shown in fig.1.

The three identical chaotic fractional order Lorenz systems with bidirectional coupling can be written as
\[
\begin{align*}
\frac{d^{\alpha_1}x_1}{dt^{\alpha_1}} &= a(y_1 - x_1) + d_{11}(x_1 + x_2 - 2x_1) \\
\frac{d^{\alpha_2}y_1}{dt^{\alpha_2}} &= cx_1 - y_1 - x_1z_1 + d_{12}(y_1 + y_2 - 2y_1), \\
\frac{d^{\alpha_3}z_1}{dt^{\alpha_3}} &= -bz_1 + x_1y_1 + d_{13}(z_1 + z_2 - 2z_1)
\end{align*}
\]

\[
\begin{align*}
\frac{d^{\alpha_1}x_2}{dt^{\alpha_1}} &= a(y_2 - x_2) + d_{21}(x_1 + x_3 - 2x_2) \\
\frac{d^{\alpha_2}y_2}{dt^{\alpha_2}} &= cx_2 - y_2 - x_2z_2 + d_{22}(y_1 + y_3 - 2y_2) \\
\frac{d^{\alpha_3}z_2}{dt^{\alpha_3}} &= -bz_2 + x_2y_2 + d_{23}(z_1 + z_3 - 2z_2)
\end{align*}
\] (12)

and
\[
\begin{align*}
\frac{d^{\alpha_1}x_3}{dt^{\alpha_1}} &= a(y_3 - x_3) + d_{31}(x_1 + x_2 - 2x_3) \\
\frac{d^{\alpha_2}y_3}{dt^{\alpha_2}} &= cx_3 - y_3 - x_3z_3 + d_{32}(y_1 + y_2 - 2y_3) \\
\frac{d^{\alpha_3}z_3}{dt^{\alpha_3}} &= -bz_3 + x_3y_3 + d_{33}(z_1 + z_2 - 2z_3)
\end{align*}
\] (13)

Define the error vector \( \Xi = (e_{11}, e_{12}, e_{13}, e_{21}, e_{22}, e_{23})^T = (x_1 - x, y_1 - y, z_1 - z, x_2 - x, y_2 - y, z_2 - z)^T \), then the fractional order dynamics system can be obtained:
\[
\begin{align*}
\frac{d^\alpha e_{11}}{dt^\alpha} &= -(a + 2d_{11} + d_{21})e_{11} + ae_{12} - (d_{11} - d_{21})e_{21} \\
\frac{d^\alpha e_{12}}{dt^\alpha} &= -ce_{11} - (1 + 2d_{12} + d_{22})e_{12} - (d_{12} - d_{22})e_{22} - x_1e_{13} - z_1e_{11} \\
\frac{d^\alpha e_{13}}{dt^\alpha} &= -(b + 2d_{13} + d_{23})e_{13} - (d_{13} - d_{23})e_{23} + x_1e_{12} + y_1e_{11} \\
\frac{d^\alpha e_{21}}{dt^\alpha} &= -(a + d_{21} + 2d_{31})e_{21} + ae_{22} - (d_{31} - d_{21})e_{11} \\
\frac{d^\alpha e_{22}}{dt^\alpha} &= -ce_{21} - (1 + d_{22} + 2d_{32})e_{22} - (d_{32} - d_{22})e_{12} - x_2e_{23} - z_2e_{21} \\
\frac{d^\alpha e_{23}}{dt^\alpha} &= -(b + d_{23} + 2d_{33})e_{23} - (d_{33} - d_{23})e_{13} + x_2e_{22} + y_2e_{21}
\end{align*}
\] (14)

Take Laplace transformation in both sides of Eq.(14), and let \( E_i(s) = L(e_i) \) \((i=1,2,3; j=1,2,3)\), one has

\[
\begin{align*}
\mathcal{L}^{-1}E_{11}(s) - s^{\alpha-1}E_{11}(0) &= -(a + 2d_{11} + d_{21})E_{11}(s) + aE_{12}(s) - (d_{11} - d_{21})E_{21}(s) \\
\mathcal{L}^{-1}E_{12}(s) - s^{\alpha-1}E_{12}(0) &= -cE_{11}(s) - (1 + 2d_{12} + d_{22})E_{12}(s) - (d_{12} - d_{22})E_{22}(s) - L(x_1e_{13}) - L(z_1e_{11}) \\
\mathcal{L}^{-1}E_{13}(s) - s^{\alpha-1}E_{13}(0) &= -(b + 2d_{13} + d_{23})E_{13}(s) - (d_{13} - d_{23})E_{23}(s) + L(x_1e_{12}) + L(y_1e_{11}) \\
\mathcal{L}^{-1}E_{21}(s) - s^{\alpha-1}E_{21}(0) &= -(a + d_{21} + 2d_{31})E_{21}(s) + aE_{22}(s) - (d_{31} - d_{21})E_{11}(s) \\
\mathcal{L}^{-1}E_{22}(s) - s^{\alpha-1}E_{22}(0) &= -cE_{21}(s) - (1 + d_{22} + 2d_{32})E_{22}(s) - (d_{32} - d_{22})E_{12}(s) - L(x_2e_{23}) - L(z_2e_{21}) \\
\mathcal{L}^{-1}E_{23}(s) - s^{\alpha-1}E_{23}(0) &= -(b + d_{23} + 2d_{33})E_{23}(s) - (d_{33} - d_{23})E_{13}(s) + L(x_2e_{22}) + L(y_2e_{21})
\end{align*}
\] (15)

Fig. 1. The fractional-order Lorenz chaotic attractor

From Eq.(15), it can follow that
\[ E_{11}(s) = \frac{1}{s^{\alpha_1} + a + 2d_{11} + d_{21}} (s^{\alpha_{11}}e_{11}(0) + aE_{12}(s) - (d_{11} - d_{21})E_{21}(s)) \]
\[ E_{13}(s) = \frac{1}{s^{\alpha_3} + b + 2d_{13} + d_{23}} (s^{\alpha_{13}}e_{13}(0) + (d_{23} - d_{13})E_{23}(s) + L(x_1e_{12}) + L(y_2e_{11})) \]
\[ E_{21}(s) = \frac{1}{s^{\alpha_1} + a + d_{21} + 2d_{31}} (s^{\alpha_{21}}e_{21}(0) + aE_{22}(s) + (d_{21} - d_{31})E_{11}(s)) \]
\[ E_{23}(s) = \frac{1}{s^{\alpha_3} + b + d_{23} + 2d_{33}} (s^{\alpha_{23}}e_{23}(0) + (d_{23} - d_{33})E_{13}(s) + L(x_2e_{22}) + L(y_3e_{21})) \]  

According to the Final-value theorem of the Laplace transformation[29], we have

\[
\dot{e}_{11}(t) = \lim_{s \to 0} se_{11}(s) = \frac{a}{a + 2d_{11} + d_{21}} \lim_{s \to 0} e_{12}(t) + \frac{d_{21} - d_{11}}{a + 2d_{11} + d_{21}} \lim_{s \to 0} e_{21}(t) \\
\dot{e}_{21}(t) = \lim_{s \to 0} se_{21}(s) = \frac{a}{a + d_{21} + 2d_{31}} \lim_{s \to 0} e_{22}(t) + \frac{d_{21} - d_{31}}{a + d_{21} + 2d_{31}} \lim_{s \to 0} e_{11}(t) 
\]

and

\[
\dot{e}_{13}(t) = \lim_{s \to 0} se_{13}(s) = \frac{d_{23} - d_{13}}{b + 2d_{13} + d_{23}} \lim_{s \to 0} e_{23}(t) + \frac{1}{b + 2d_{13} + d_{23}} \lim_{s \to 0} L(x_1e_{12}(t) + L(y_2e_{11}(t))) \\
\dot{e}_{23}(t) = \lim_{s \to 0} se_{23}(s) = \frac{d_{23} - d_{33}}{b + d_{23} + 2d_{33}} \lim_{s \to 0} e_{13}(t) + \frac{1}{b + d_{23} + 2d_{33}} \lim_{s \to 0} L(x_2e_{22}(t) + L(y_3e_{21}(t))) 
\]

\[ \text{Theorem 3.} \text{ The zero point of error dynamics system (14) is globally asymptotically stable when the any two of } E_{ij}(s) (i, j = 1, 2) \text{ are bounded, and, } \frac{d_{23} - d_{13}}{a + 2d_{11} + d_{21}} \cdot \frac{d_{21} - d_{31}}{a + d_{21} + 2d_{31}} \neq 1, \frac{d_{23} - d_{13}}{b + 2d_{13} + d_{23}} \cdot \frac{d_{23} - d_{33}}{b + d_{23} + 2d_{33}} \neq 1. \text{ i.e. the synchronization of the coupled systems (11)-(13) is realized.} \]

\[ \text{Proof.} \text{ Without loss of generality, we assume that } E_{12}(s) \text{ and } E_{22}(s) \text{ are bounded, it follows that} \]
\[ \lim_{t \to \infty} e_{12}(t) = \lim_{t \to \infty} e_{22}(t) = 0. \text{ From (17), one has} \]
\[ \lim_{t \to \infty} e_{11}(t) = \frac{d_{21} - d_{11}}{a + 2d_{11} + d_{21}} \lim_{t \to \infty} e_{21}(t) \]
\[ \lim_{t \to \infty} e_{21}(t) = \frac{d_{21} - d_{31}}{a + d_{21} + 2d_{31}} \lim_{t \to \infty} e_{11}(t) \]

Again, according to \[ \frac{d_{21} - d_{11}}{a + 2d_{11} + d_{21}} \cdot \frac{d_{21} - d_{31}}{a + d_{21} + 2d_{31}} \neq 1, \text{ one gets} \lim_{t \to \infty} e_{11}(t) = \lim_{t \to \infty} e_{21}(t) = 0. \]

Furthermore, due to the attractiveness of the chaotic fractional order Lorenz system (10), it must be bounded. \text{i.e. there exists a constant } G \text{ such that } |x_i| \leq G, |y_i| \leq G \text{ and } |z_i| \leq G (i = 1, 2, 3). \text{ Then}
\[
\lim_{s \to 0} s(L(x,e_{12}(t)) + L(y,e_{11}(t))) = \lim_{t \to \infty} (e_{12}(t) + e_{11}(t)) = 0,
\]
and
\[
\lim_{s \to 0} s(L(x,e_{22}(t)) + L(y,e_{21}(t))) = \lim_{t \to \infty} (e_{22}(t) + e_{21}(t)) = 0.
\]

Fig. 2. the errors graph of three coupled fractional-order Lorenz chaotic attractors

So Eq. (18) can be simplify as
\[
\lim_{t \to \infty} e_{13}(t) = \frac{d_{23} - d_{13}}{b + 2d_{23} + d_{33}} \lim_{t \to \infty} e_{23}(t), \quad \lim_{t \to \infty} e_{23}(t) = \frac{d_{23} - d_{33}}{b + d_{23} + 2d_{33}} \lim_{t \to \infty} e_{13}(t).
\]

According to \(\frac{d_{23} - d_{13}}{b + 2d_{23} + d_{33}} \cdot \frac{d_{23} - d_{33}}{b + d_{23} + 2d_{33}} \neq 1\), it follows that \(\lim_{t \to \infty} e_{13}(t) = \lim_{t \to \infty} e_{23}(t) = 0\). Hence, \(\lim_{t \to \infty} e_{ij}(t) = 0\) \((i = 1, 2; j = 1, 2, 3)\). The proof is complete.

According to the Adams predictor-corrector scheme in Refs. [8,15-21], the simulation of the results can be seen in fig. 2, where the initial values of three systems (11)-(13) are \(x_1(0) = 10, y_1(0) = 5, z_1(0) = 7\); \(x_2(0) = 6, y_2(0) = 9, z_2(0) = 6\) and \(x_3(0) = 9, y_3(0) = -1, z_3(0) = 4\), respectively.

3. Conclusion
In this paper we have mainly studied the synchronization of three fractional-order Lorenz systems with linearly coupling. Based on the Laplace transformation theory, the sufficient conditions of synchronization are obtained. The corresponding numerical simulations are given to verify the feasibility of the results.

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