Necessary and sufficient conditions for stabilisability of
discrete-time time-varying switched systems

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Abstract
This paper is concerned with necessary and sufficient conditions for stabilisability of
time-varying discrete-time switched systems. Starting with an asymptotically stable function,
an exponentially stable function and a uniformly exponentially stable function, we succes-
sively propose necessary and sufficient conditions for asymptotic stabilisability, exponen-
tial stabilisability and uniform exponential stabilisability of time-varying switched linear
systems. Further, considering the broad applications of finite-time stability in practical sys-
tems, based on an additionally introduced concept of finite-time stable functions, we derive
a necessary and sufficient condition for finite-time stabilisability of time-varying switched
linear systems. Afterwards, three illustrative examples are given to show the applicability
of our theoretical results. In the end, we further discuss the necessary and sufficient con-
ditions for global exponential stabilisability and global uniform exponential stabilisability
of time-varying switched non-linear systems. Compared to traditional difference Lyapunov
inequalities, we release the requirement on negative definiteness of the time-difference of
Lyapunov functions.

1 | INTRODUCTION
As an important class of hybrid systems, switched systems consist of several subsystems along with a switching signal that orchestrates the switching among these subsystems (see [1, 2]). Numerous researchers focus on such systems due to their broad applications in practical multiple-mode systems such as constrained robotics, chemical processes, multi-agent systems, manufacturing systems, and traffic control (see [3–7]).

There are many interesting issues on stability analysis of switched systems, for example, the issue on proposing necessary and sufficient conditions to guarantee the stability under arbitrary switching signals (see [8, 9]), the issue on demonstrating various stability under certain constrained switching signals (see [10–13]), and the issue on analysing the existence of switching signals such that switched systems can be stable (i.e. the issue on stabilisability analysis [3, 14]). Among these issues, stabilisability analysis is an essential and significant one. However, owing to the multiple subsystems and various types of switching signals, stabilisability analysis of switched systems is complicated. For example, even when all subsystems are stable, there may exist certain switching signals such that switched systems be divergent [1]. Besides, there may also exist switching signals such that switched systems with unstable subsystems become stable. Therefore, the issue of stabilisability analysis is interesting and worthy to be paid more attention to.

Fortunately, the stabilisability problem of systems has been widely concerned and many significant contributions on this topic have been given (see [15–26]). Especially, with respect to the stabilisability of time-invariant switched systems, several valuable results are made based on Lyapunov functions method and dwell time method. For example, Fiacchini et al. [18] provided a geometrical and numerical insight on the stabilisabil-
ity criteria proposed in [16, 17]; based on multiple discontinuous Lyapunov functions and mode-dependent average dwell time, Zhao et al. [25] investigated the stabilisation problem of switched systems with unstable subsystems; several equivalent stability conditions for switched linear systems (SLs) were presented in [26]. However, the stabilisability analysis of time-
varying switched systems has been less extensively studied (see [11, 27, 28]). For instance, Chen et al. [28] researched the sufficient conditions of the asymptotic stability for positive linear

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time-varying switched systems according to time-varying copositive Lyapunov functions; Liu et al. [11] provided stabilisability criteria for switched systems with both stable and unstable subsystems based on multiple Lyapunov functions and dwell time. In particular, stabilisability of discrete-time time-varying switched systems has been even less studied.

It is known that the above-mentioned results mainly focused on the Lyapunov stability, which describes the behaviours of a system over an infinite-time interval. However, in many practical systems, it is necessary to consider their finite-time stability. Note that there are mainly two kinds of concepts of finite-time stability. One, proposed by Bhat et al., refers that the system state converges to the equilibrium point within a relatively finite time interval [29, 30]. Another, proposed by Amato et al., means that when pre-given a bound on the initial condition, states do not exceed a certain threshold during a finite-time interval [31, 32]. Many attentions have been paid on these two finite-time stability and numerous significant results emerged (see [29–43]). For example, Lu and She [42] proposed a sufficient condition of finite-time stability based on piecewise time-varying Lyapunov function; Zhao et al. [43] considered the finite-time boundedness of uncertain switched linear systems based on average dwell time method; Chen and Yang [29] investigated the sufficient conditions of the finite-time stability for time-varying continuous-time switched non-linear systems via indefinite common Lyapunov functions and indefinite multiple Lyapunov functions; for time-invariant impulsive discrete-time SLSs with and without perturbation, [36] and [33], respectively, established the corresponding sufficient conditions of the finite-time stability. We choose to investigate the finite-time stability, proposed by Amato et al., since it mainly focuses on the stability of the system during a specified interval, and the system dynamic quality can be better described when the initial state is bounded.

Inspired by the idea of using scalar functions in [24] and [41] and the time-varying Lyapunov functions in [14] and [27], we are dedicated to analysing the stabilisability of time-varying discrete-time switched systems. Firstly, resorting to asymptotically (exponentially, uniformly exponentially) stable scalar functions, we explore necessary and sufficient conditions for the asymptotic (exponentially, uniformly exponentially) stable switched non-linear systems, which can be seen in Example 1. Furthermore, considering the broad applications of finite-time stability in practical systems, we introduce a finite-time stable function and then attain a necessary and sufficient condition for the finite-time stabilisability of time-varying discrete-time SLSs, where the constraints release the traditional difference Lyapunov inequalities [45], we release the requirement on negative definiteness of the time-difference of time-varying Lyapunov functions, which can be seen in Example 1.

(2) Considering the broad applications of finite-time stability in practical systems, we introduce a finite-time stable function and then attain a necessary and sufficient condition for the finite-time stabilisability of time-varying discrete-time SLSs, where the constraints release the traditional difference Lyapunov inequalities [39] (see Example 3). Moreover, compared to [41], our finite-time stable function (see Lemma 4) is independent with the initial state.

(3) We further attain a necessary and sufficient condition for the global exponential stabilisability and global uniform exponential stabilisability of switched non-linear systems.

This paper is organised as follows. In Section 2, several relevant definitions are introduced. Specially, we additionally introduce a concept of time-varying stable functions. In Section 3, several necessary and sufficient conditions for stabilisability of discrete-time time-varying switched linear systems are proposed. Three illustrative examples are presented in Section 4. Moreover, further discussions on the stabilisability of switched non-linear systems are presented in Section 5. Finally, Section 6 concludes the paper.

2 | PRELIMINARIES

Notions: Let $\mathbb{Z}$, $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{>0}$ be the set of all integers, non-negative integers and positive integers. Denote $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ as the set of all real numbers, non-negative real numbers and positive real numbers. $\mathbb{R}^n$ denotes the space of $n$-dimensional real vectors, $\| \cdot \|$ is its Euclidean norm and for $x \in \mathbb{R}^n$, $x^T$ denotes its transpose. $\mathbb{R}^{n \times n}$ stands for the space of $n \times n$ real matrices; $I_n$ is an $n \times n$ identity matrix; and for $A \in \mathbb{R}^{n \times n}$, $A^T$ denotes its transpose and $\text{tr}(A)$ denotes its trace. Moreover, we can use $|A|$ to represent any norm of the matrix $A$ due to the equivalence between different norms, while $|A|_F$ denotes the Frobenius norm of the matrix $A$.

A time-varying switched non-linear system which contains $N$ subsystems can be formed as

$$x(t + 1) = f_{\sigma(t)}(t, x(t)), x(h_0) = x_0, t \geq h_0, \quad (1)$$

where $t, t_0 \in \mathbb{Z}_{\geq 0}$, $x(\cdot) \in \mathbb{R}^n$, $\sigma(\cdot): \mathbb{Z}_{\geq 0} \rightarrow \Gamma = \{1, 2, \ldots, N\}$ is a piecewise constant map defined as the switching signal, and
$N$ with $N \geq 1$ represents the number of subsystems of the switched system (1). $\sigma(t) = i$ means that the $i$th subsystem is active at time $t$. Indeed, Equation (1) models a system that can switch between $N$ subsystems of the form

$$x(t + 1) = f_i(t, x(t)), \quad x(h_i) = x_0, \quad t \geq h_i, \quad i \in \Gamma,$$

where $f_i : \mathbb{Z}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies $f_i(t, 0) = 0$ for each $i \in \Gamma$ and all $t \in \mathbb{Z}_{\geq 0}$. Moreover, a time-varying switched linear system can be described by

$$x(t + 1) = A_{\sigma(t)}(t)x(t), \quad x(h_i) = x_0, \quad t \geq h_i, \quad (2)$$

we assume that $A_i$ is non-singular for all $t \geq h_i$, all $h_i \in \mathbb{Z}_{\geq 0}$ and all $i \in \Gamma$. Note that any entry in matrix $A_{\sigma(t)}$ changes with $t$ and can be a non-linear function with respect to $t$.

The aim of this paper is to investigate the stabilisability of discrete-time switched systems. To this purpose, we first need the following definitions selected from [1, 39].

**Definition 1.** A function $x(t) : \mathbb{Z}_{\geq 0} \to \mathbb{R}^n$ is called the trajectory (or solution) of system (1) if there is a switching signal $\sigma(t)$ such that $x(t + 1) = f_{\sigma(t)}(t, x(t))$ for all $t \geq h_i$, where $x(h_i) = x_0$.

Under a given switching signal $\sigma(t)$, we usually use $x_{\sigma(t)}(t, h_i, x_0)$ to represent the trajectory of the system (1) starting from the initial state $x_0$ at initial instant $h_0 \in \mathbb{Z}_{\geq 0}$. For simplicity, we can use $x_{\sigma(t)}(t, h_0, x_0)$ to denote the trajectory $x_{\sigma(t)}(t, h_0, x_0)$. Without confusion, $x_{\sigma(t)}(t, h_0, x_0)$ can be expressed as $x(t, h_0, x_0)$ or even $x(t)$.

**Definition 2.** System (1) is called

(i) stabilisable with respect to the origin if there is a switching signal $\sigma$ satisfying that for any $h_i \in \mathbb{Z}_{\geq 0}$ and any $\epsilon > 0$, there exists a constant $\delta(\sigma, h_i, \epsilon) > 0$ such that $\|x_{\sigma(t, h_0, x_0)}\| < \epsilon$ for all $t \geq h_i$ when $\|x_0\| < \delta(\sigma, h_i, \epsilon)$;

(ii) globally asymptotically stabilisable with respect to the origin if there is a switching signal $\sigma$ such that system (1) is both stable and $\lim_{t \to +\infty} \|x_{\sigma(t, h_0, x_0)}\| = 0$ for any $h_i \in \mathbb{Z}_{\geq 0}$ and all $x_0 \in \mathbb{R}^n$;

(iii) globally exponentially stabilisable with respect to the origin if for any $h_i \in \mathbb{Z}_{\geq 0}$ there is a switching signal $\sigma$, constants $\beta(h_i) \geq 1$ and $\alpha > 0$ such that $\|x_{\sigma(t, h_0, x_0)}\| \leq \beta(h_i) e^{-\alpha(t - h_i)}\|x_0\|$ for all $t \geq h_i$ and all $x_0 \in \mathbb{R}^n$;

(iv) globally uniformly exponentially stabilisable with respect to the origin if $\beta(h_i)$ in Item (iii) is independent of $h_i$.

**Definition 3.** Given an initial time $h_0$, a positive constant $F$, a positive definite matrix $\Lambda$ and a positive definite matrix-valued sequence $Y(t) : F = \{h_0, \ldots, h_0 + F\} \to \mathbb{R}^{\times n}$ with $Y(h_0) < \Lambda$, system (1) is said to be finite-time stabilisable with respect to $[h_0, F, \Lambda, Y(t)]$ if there exists a finite-time switching signal $\sigma(t) : F \to \Gamma$ such that system (1) is finite-time stable with respect to $[h_0, F, \Lambda, Y(t)]$, that is,

$$x_0^T \Lambda x_0 \leq 1 \Rightarrow x_{\sigma(t, h_0, x_0)}^T (t) Y(t) x_{\sigma(t, h_0, x_0)} < 1, \quad \forall t \in F.$$

Moreover, for system (2) under a given switching signal $\sigma(t)$, we know that $x_{\sigma(t)}(t, h_0, x_0) = \Phi_{\sigma(t)}(t, h_0) x_0$ for all $t \geq h_0$, where $\Phi_{\sigma(t)}(t, h_0) = A_{\sigma(t-1)}(t-1) A_{\sigma(t-2)}(t-2) \cdots A_{\sigma(h_0+1)}(t_0 + 1) A_{\sigma(h_0)}(h_0)$. Thus, we can alternatively use the state transition matrix $\Phi_{\sigma(t)}(t, h_0)$ of system (2) to obtain its asymptotic stabilisability, exponential stabilisability, uniform exponential stabilisability and finite-time stabilisability as follows according to the works in [39, 44, 45].

**Lemma 1.** System (2) is

(i) stabilisable with respect to the origin if and only if there is a switching signal $\sigma$ such that for any $h_i \in \mathbb{Z}_{\geq 0}$, there exists a scalar $c(h_i) > 0$ satisfying $|\Phi_{\sigma(t)}(t, h_0)| \leq c(h_i)$ for any $t \geq h_i$;

(ii) globally asymptotically stabilisable with respect to the origin if and only if there is a switching signal $\sigma$ such that system (2) becomes stable and $\lim_{t \to +\infty} |\Phi_{\sigma(t)}(t, h_0)| = 0$ for any $h_i \in \mathbb{Z}_{\geq 0}$;

(iii) globally exponentially stabilisable with respect to the origin if and only if there is a switching signal $\sigma$ such that for any $h_i \in \mathbb{Z}_{\geq 0}$, there exist a scalar $\beta(h_i) \geq 1$ and a constant $\alpha > 0$ such that $|\Phi_{\sigma(t)}(t, h_0)| \leq \beta(h_i) e^{-\alpha(t - h_i)}$ for any $t \geq h_i$;

(iv) globally uniformly exponentially stabilisable with respect to the origin if and only if $\beta(h_i)$ in Item (iii) is independent of $h_i$.

**Lemma 2.** Given an initial time $h_0 \in \mathbb{Z}_{\geq 0}$, a positive constant $F$, a finite-time switching signal $\sigma(t) : F = \{h_0, \ldots, h_0 + F\} \to \Gamma$, a positive definite matrix $\Lambda$ and a positive definite matrix-valued sequence $Y(t) : F \to \mathbb{R}^{\times n}$ with $Y(h_0) < \Lambda$, system (2) is said to be finite-time stabilisable with respect to $[h_0, F, \Lambda, Y(t)]$ if and only if there exists a finite-time switching signal $\sigma(t) : F \to \Gamma$ such that $\Phi_{\sigma(t)}(t, h_0) Y(t) \Phi_{\sigma(t)}(t, h_0) < \Lambda$ for any $t \in F \setminus \{h_0\}$.

Afterwards, we introduce a scalar time-varying linear system of form

$$y(t + 1) = \mu(t)y(t), \quad y(h_0) = y_0, \quad (3)$$

where $\mu(t) : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $y(t) : \mathbb{R} \to \mathbb{R}$. Denote $\phi(t, h_0)$ as the state transition matrix of system (3). Clearly, $\phi(t, h_0) = \prod_{k=h_0}^{t-1} \mu(k)$ for any $t \geq h_0$ and any $h_0 \in \mathbb{Z}_{\geq 0}$.

For system (3), we firstly introduce the following definition [23].

**Definition 4.** For system (3), the function $\mu(t) : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called

(i) a globally asymptotically stable function if system (3) is globally asymptotically stable (GAS);

(ii) a globally exponentially stable function if system (3) is globally exponentially stable (GES);
(iii) a globally uniformly exponentially stable function if system (3) is globally uniformly exponentially stable (GUES).

Then, inspired by the above stable functions, we additionally introduce the concept of finite-time stable functions as follows.

**Definition 5.** For system (3) with the pre-given initial time $t_0$ and a positive constant $F$, the scalar function $\mu(t) : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$ is called a finite-time stable function if for arbitrary two positive constants $\Lambda_0$ and $Y_0$ with $Y_0 < \Lambda_0$, system (3) is finite-time stable with respect to $[t_0, F, \Lambda_0, Y_0]$.

**Remark 1.** Note that in Definition 5, when positive constants $\Lambda_0$ and $Y_0$ are pre-given, the scalar function $\mu(t) : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$ can be called a finite-time stable function with respect to $[t_0, F, \Lambda_0, Y_0]$ if system (3) is finite-time stable with respect to $[t_0, F, \Lambda_0, Y_0]$.

Based on Definition 4, we can easily obtain the following lemma, which has been studied in [24] and will be used in Sections 3 and 5.

**Lemma 3.** For system (3), the function $\mu(t) : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$ is

(i) a globally asymptotically stable function if and only if
\[
\lim_{t \to \infty} \prod_{k=t}^{t-1} \mu(k) = 0 \text{ for all } t_0 \in \mathbb{Z}_{\geq 0};
\]

(ii) a globally exponentially stable function if and only if for any $t_0 \in \mathbb{Z}_{\geq 0}$, there exist a $\hat{\beta}(t_0) \geq 1$ and a constant $\alpha > 0$ such that
\[
\prod_{k=t}^{t-1} \mu(k) \leq \hat{\beta}(t_0) e^{\alpha(t-t_0)} \text{ for all } t \geq t_0;
\]

(iii) a globally uniformly exponentially stable function if and only if there exist a $\hat{\beta} \geq 1$ and a constant $\alpha > 0$ such that
\[
\prod_{k=t}^{t-1} \mu(k) \leq \hat{\beta} e^{\alpha(t-t_0)} \text{ for all } t \geq t_0 \text{ and all } t_0 \in \mathbb{Z}_{\geq 0}.
\]

Further, based on our additionally introduced Definition 5, we can derive the following lemma, which will also be used in Section 3.

**Lemma 4.** For system (3) with the pre-given initial time $t_0$ and a positive constant $F$, the scalar function $\mu(t) : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$ is a finite-time stable function if and only if
\[
\prod_{k=t}^{t-1} \mu^2(k) \leq 1 \text{ for any } t \in \mathcal{P} \setminus \{t_0\}.
\]

**Proof.** (1) Sufficiency:

For system (3) with the pre-given initial time $t_0$ and a positive constant $F$, if $\prod_{k=t}^{t-1} \mu^2(k) \leq 1 \text{ for any } t \in \mathcal{P} \setminus \{t_0\}$, we can directly obtain that $Y_0 \prod_{k=t}^{t-1} \mu^2(k) \leq \Lambda_0 Y_0 \text{ for arbitrary two constants } \Lambda_0$ and $Y_0$ with $Y_0 < \Lambda_0$. This implies that for arbitrary constants $\Lambda_0$ and $Y_0$ with $Y_0 < \Lambda_0$ and all $t \in \mathcal{P} \setminus \{t_0\}$,
\[
Y_0 \prod_{k=t}^{t-1} \mu^2(k) < 1 \text{ when } \Lambda_0 Y_0 \leq 1. \text{ Therefore, for arbitrary positive constants } \Lambda_0 \text{ and } Y_0 \text{ with } Y_0 < \Lambda_0, \text{ system (3) is finite-time stable with respect to } [t_0, F, \Lambda_0, Y_0].
\]

**3.1 STABILIZABILITY ANALYSIS OF TIME-VARYING SWITCHED LINEAR SYSTEMS**

Inspired by the idea of using scalar functions in [24] and [41] and the idea of using time-varying Lyapunov functions in [14] and [27], we in this section are dedicated to utilise an asymptotically stable function, an exponentially stable function, an uniformly exponentially stable function and a finite-time stable function to establish necessary and sufficient conditions for the global asymptotic stabilisability, global exponential stabilisability, global uniform exponential stabilisability and finite-time stabilisability of time-varying discrete-time SLSs, respectively. Note that Zhou and Zhao [24] considered the stability of the time-varying linear dynamical systems, which can be regarded as a special case of time-varying switched linear systems with a single subsystem.

Firstly, for system (2), we obtain the following necessary and sufficient condition of its global asymptotic stabilisability.
Theorem 1. System (2) is globally asymptotically stabilizable if and only if there exist a GAS function \( \mu(t) : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0} \) and a non-singular matrix \( P(t) : \mathbb{Z}_{\geq 0} \to \mathbb{R}^{n \times n} \) with \( P^T(t) = P(t) \) such that

(i) for all \( t \in \mathbb{Z}_{\geq 0} \), \( I_n \leq P(t) \), and

(ii) for all \( t \in \mathbb{Z}_{\geq 0} \), an \( i_t \in \Gamma \) can be found to satisfy that \( A_i(t)P(t + 1)A_i(t) \leq \mu^2(t)P(t) \).

Moreover, system (2) is globally exponentially stabilizable if and only if there exist a GES function \( \mu(t) : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0} \) and a non-singular matrix \( P(t) : \mathbb{Z}_{\geq 0} \to \mathbb{R}^{n \times n} \) with \( P^T(t) = P(t) \) such that conditions (i) and (ii) are satisfied.

Proof. (1) Sufficiency:

For proving the global asymptotic (exponential) stabilizability of system (2), we aim to construct a switching signal \( \sigma(t) \) such that system (2) is GAS under it. For this, we define the switching signal \( \sigma(t) \) as \( \sigma(t) = i_t \) for all \( t \in \mathbb{Z}_{\geq 0} \). Then, define time-varying Lyapunov function \( \phi(t,x(t)) = x^T(t)P(t)x(t) \), where \( x(t) \) denotes the trajectory \( x_{\sigma(t)}(t,t_0,x_0) \) of system (2) under the above well-defined switching signal.

Clearly, due to condition (i),

\[
V(t,x(t)) = x^T(t)P(t)x(t) \geq \|x(t)\|^2 \tag{4}
\]

for all \( t \geq t_0 \), \( t_0 \in \mathbb{Z}_{\geq 0} \) and any initial state \( x_0 \). Further, for any initial state \( x_0 \), all \( t \geq t_0 \) and \( t_0 \in \mathbb{Z}_{\geq 0} \), we can derive from condition (ii) that

\[
V(t+1,x(t+1)) = x^T(t+1)P(t+1)x(t+1) = x^T(t)A_i^T(\sigma(t))P(t+1)A_i(\sigma(t))x(t) \leq \mu^2(t)x^T(t)P(t)x(t) \tag{5}
\]

\[
\ldots \leq \phi^2(t+1,t_0)x_0^TP(t_0)x_0 \leq \phi^2(t+1,t_0)\|P(t_0)\|\|x_0\|^2,
\]

where \( |P(t_0)| \) represents the spectral norm of matrix \( P(t_0) \). Therefore, it follows from inequalities (4) and (5) that

\[
\|x(t)\|^2 \leq V(x(t)) \leq \phi^2(t,t_0)\|P(t_0)\|\|x_0\|^2, \tag{6}
\]

which implies that

\[
\|x(t)\|^2 \leq \phi^2(t,t_0)\|P(t_0)\|\|x_0\|^2. \tag{7}
\]

That is, \( |\Phi_{\sigma}(t,t_0)| \|x_0\| \leq \phi(t_0)\sqrt{|P(t_0)|}\|x_0\| \), \( \forall t \geq t_0, \forall t_0 \in \mathbb{Z}_{\geq 0}, \forall x_0 \). Due to the arbitrariness of \( x_0 \) and the definition of the norm of an operator defined by a matrix, we have

\[
|\Phi_{\sigma}(t,t_0)| \leq \phi(t_0)\sqrt{|P(t_0)|}. \tag{8}
\]

Since \( \mu(t) \) is a GAS function, from inequality (6) and Lemma 3, we can conclude that the system (2) is GAS under the above constructed switching signal \( \sigma \), that is, the sufficiency of the global asymptotic stabilizability is proved due to Lemma 1.

Moreover, if \( \mu(t) \) is a GES function, we can derive from inequality (6) and Lemma 3 that there exist a \( \beta(t_0) \geq 0 \) and a constant \( \alpha > 0 \) such that

\[
|\Phi_{\sigma}(t,t_0)| \leq \alpha e^{-\beta(t_0)}, \forall t, \forall t_0 \in \mathbb{Z}_{\geq 0}.
\]

Thus, condition (i) is satisfied.

Next, due to the above well-defined switching signal \( \sigma \) and \( \Phi_{\sigma}(t_0,t+1) = A_i^{-1}(\sigma(t_0))A_i^{-1}(\sigma(t))A_i^{-1}(\sigma(t)) \ldots A_i^{-1}(\sigma(t_0)) \), we have that, for all \( t \geq t_0 \) and \( t_0 \in \mathbb{Z}_{\geq 0} \), an index \( i_t \in \Gamma \) can be found
such that
\[ A_{i_{k}}^{T}(t)P(t+1)A_{i_{k}}(t) \]
\[ = nA_{i_{k}}^{T}(t)\Phi^{2}(t+1,h_{i})\Phi^{T}_{\sigma}(h_{i},t+1)\Phi_{\sigma}(h_{i},t+1)A_{i_{k}}(t) \]
\[ = n\mu^{2}(t)\Phi^{2}(t,h_{i})A_{i_{k}}^{T}(t)\Phi^{T}_{\sigma}(h_{i},t+1)\Phi_{\sigma}(h_{i},t+1)A_{i_{k}}(t) \]
\[ = n\mu^{2}(t)\Phi^{2}(t,h_{i})\Phi^{2}_{\sigma}(t,h_{i})\Phi_{\sigma}(t,h_{i}) \]
\[ = \mu^{2}(t)P(t), \]
which indicates that condition (ii) is satisfied. Thus, the necessity of the global asymptotic stabilisability is proved.

We in the following prove the necessity of the global exponential stabilisability. Similar with the above proof process, for all \( t \in \mathbb{Z}_{\geq 0} \), an index \( i_{k} \in \Gamma \) can be found such that \( \sigma(t) = i_{k} \). Then, we define \( \mu(t) = \frac{\|\Phi_{\sigma}(t+1,h_{i})\|_{F}}{\|\Phi_{\sigma}(h_{i})\|_{F}} \) and \( \phi(h_{i}) = \prod_{k=0}^{t-1} \mu(k) \). It is clear that \( \phi(t,h_{i}) = ||\Phi_{\sigma}(h_{i},0)||_{F}^{-1}||\Phi_{\sigma}(h_{i},t)||_{F} = \frac{n^{-\frac{1}{2}}||\Phi_{\sigma}(t,h_{i})||_{F}}{n^{-\frac{1}{2}}||\Phi_{\sigma}(h_{i},0)||_{F}} \). Moreover, according to Lemma 1, there exist a scalar \( \beta(h_{i}) \) \( \geq 1 \) and a positive constant \( \alpha \) such that for all \( t \geq h_{i} \) and all \( t \in \mathbb{Z}_{\geq 0} \), \( \beta(t,h_{i}) \leq n^{-\frac{1}{2}}\beta(h_{i}) e^{-\alpha(t-h_{i})} \). Obviously, due to Lemma 3, \( \mu(t) \) is a GUES function. From the rest proof of the necessity of the global asymptotic stabilisability, the necessity of the global exponential stabilisability can be obtained.

Note that it is difficult to directly solve the necessary and sufficient condition of Theorem 1. We can combine Theorem 1 and the average dwell time method to construct computable sufficient conditions for the asymptotic and exponential stability analysis, respectively. Then, based on the S-procedure, semi-definite program and sum of squares decomposition, which is well used in [14, 46], the computation of \( P(t) \) can be similarly solved if replacing \( t \in \mathbb{Z}_{\geq 0} \) with \( t \in \mathbb{R}_{\geq 0} \).

Next, combining the above result with a special form of \( P(t) \), a necessary and sufficient condition for the global uniform exponential stabilisability of system (2) can be proposed via a GUES function as follows.

**Theorem 2.** System (2) is globally uniformly exponentially stabilisable if and only if there exist a GUES function \( \mu(t) : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0} \), a positive constant \( \varepsilon \geq 1 \) and a non-singular matrix \( P(t) : \mathbb{Z}_{\geq 0} \to \mathbb{R}^{n \times n} \) with \( P^{T}(t) = P(t) \) such that for all \( t \in \mathbb{Z}_{\geq 0} \), \( I_{n} \leq P(t) \leq \varepsilon I_{n} \); and

(i) for all \( t \in \mathbb{Z}_{\geq 0} \), \( I_{n} \leq P(t) \leq \varepsilon I_{n} \), and

(ii) for all \( t \in \mathbb{Z}_{\geq 0} \), an \( i_{k} \in \Gamma \) can be found to satisfy that
\[ A_{i_{k}}^{T}(t)P(t+1)A_{i_{k}}(t) \leq \mu^{2}(t)P(t). \]

Proof. (1) Sufficiency:

Similar to the proof process of Theorem 1, since \( \mu(t) \) is a GUES function, we can derive from inequality (6), Lemma 3 and condition (i) that there exist constants \( \varepsilon \geq 1, \beta \geq 1 \) and \( \alpha > 0 \) such that for all \( t \geq h_{i} \) and all \( h_{i} \in \mathbb{Z}_{\geq 0} \), \( ||\Phi_{\sigma}(h_{i},t)||_{F} \leq \sqrt{\mu(t)} \beta e^{-\alpha(t-h_{i})} \leq \sqrt{\beta} e^{-\alpha(t-h_{i})} \). Obviously, system (2) is GUES under the above constructed switching signal \( \sigma \). Then, due to Lemma 1, the sufficiency is proved.

(2) Necessity:

We can derive from the global uniform exponential stabilisability of system (2) that there exists a switching signal \( \sigma \) such that system (2) is GUES under it. Thus, for all \( t \in \mathbb{Z}_{\geq 0} \), there exists an index \( i_{k} \in \Gamma \) such that \( \sigma(t) = i_{k} \). Then, according to Lemma 1, there exist positive constants \( \beta \geq 1 \) and \( \alpha > 0 \) such that for all \( t \geq h_{i} \) and all \( h_{i} \in \mathbb{Z}_{\geq 0} \),
\[ ||\Phi_{\sigma}(h_{i},t)||_{F} \leq \beta e^{-\alpha(t-h_{i})}. \]

Next, for all \( t \in \mathbb{Z}_{\geq 0} \), define \( P(t) \) as
\[ P(t) = \sum_{k=t}^{\infty} e^{\varepsilon(k-t-\frac{1}{2})} \Phi^{T}_{\sigma}(k,t) \Phi_{\sigma}(k,t), \]
where \( 0 < \varepsilon < 2\alpha \). Clearly, \( P^{T}(t) = P(t) \) for all \( t \in \mathbb{Z}_{\geq 0} \). Then, owing to inequalities (11) and (12), we have
\[ |P(t)| \leq \sum_{k=t}^{\infty} \beta^{2} e^{(2\varepsilon-2\alpha)(k-t)} = \frac{\beta^{2}}{1-e^{2\varepsilon-2\alpha}} \]
for all \( t \in \mathbb{Z}_{\geq 0} \). Moreover, we can derive from \( P(t) = I_{n} + \sum_{k=t+1}^{\infty} e^{\varepsilon(k-t-\frac{1}{2})} \Phi^{T}_{\sigma}(k,t) \Phi_{\sigma}(k,t) \), that \( P(t) \geq I_{n} \) for all \( t \in \mathbb{Z}_{\geq 0} \).

Thus, \( I_{n} \leq P(t) \leq \varepsilon I_{n} \), where \( \epsilon = \frac{\beta^{2}}{1-e^{2\varepsilon-2\alpha}} \geq 1 \), that is, condition (i) is satisfied. Next, due to the above well-defined switching signal \( \sigma \), we obtain that for all \( t \in \mathbb{Z}_{\geq 0} \), an index \( i_{k} \) can be found such that
\[ A_{i_{k}}^{T}(t)P(t+1)A_{i_{k}}(t) \]
\[ = \sum_{k=t+1}^{\infty} A_{i_{k}}^{T}(t) e^{\varepsilon(k-t-\frac{1}{2})} \Phi^{T}_{\sigma}(k,t+1) \Phi_{\sigma}(k,t+1)A_{i_{k}}(t) \]
\[ = e^{-\varepsilon} \sum_{k=t+1}^{\infty} e^{\varepsilon(k-t-\frac{1}{2})} \Phi^{T}_{\sigma}(k,t) \Phi_{\sigma}(k,t) \]
\[ = e^{-\varepsilon} P(t) - e^{-\varepsilon} I_{n}, \]
which indicates that \( A_{i_{k}}^{T}(t)P(t+1)A_{i_{k}}(t) \leq e^{-\varepsilon} P(t) \), that is, condition (ii) is satisfied if letting \( \mu(t) = e^{-\frac{\varepsilon}{2}} \) for all \( t \in \mathbb{Z}_{\geq 0} \). Clearly, \( \mu(t) = e^{-\frac{\varepsilon}{2}} \) is a GUES function due to Lemma 3. Thus, the necessity is proved.

In Theorem 2, we have proposed a necessary and sufficient condition for the global uniform exponential stabilisability of system (2). Clearly, we can similarly utilise the average dwell time method to construct a computable sufficient condition for the uniform exponential stability of system (2). Then, the computation issue can be solved based on the S-procedure, semi-definite program and sum of squares decomposition [14, 46].
[47, 48] have proposed sufficient conditions for the uniform exponential stability of periodic time-varying systems based on a time-varying periodic Lyapunov matrix $P(t)$. Motivated by this, we can combine Theorem 2, the periodic Lyapunov matrix $P(t)$ and dwell time method to construct another tractable sufficient condition formed by time-invariant matrix inequalities. Then, this tractable sufficient condition can be solved by LMI Matlab toolbox.

Moreover, Theorems 1 and 2 deal with the behaviour of a system within a sufficiently long time interval, while, in practical systems, the behaviour of a system on a finite time interval is also worthy to be concentrated on. Therefore, we in the following are dedicated to proposing a necessary and sufficient condition for the finite-time stabilisability of system (2).

**Theorem 3.** Given an initial time $t_0$, a positive constant $F$, a positive definite matrix $\Lambda$ and a positive definite matrix-valued sequence $Y(t): F = \{t_0, ..., t_0 + F\} \rightarrow \mathbb{R}^{n\times n}$ with $Y(t_0) < \Lambda$, system (2) is finite-time stable with respect to $t \in F$, for all $t$.

(i) for all $t \in F \setminus \{t_0\}$, $P(t) > Y(t)$; and

(ii) for all $t \in F \setminus \{t_0 + F\}$, an $i_t \in \Gamma$ can be found to satisfy that $A_{i_t}^T(t)P(t+1)A_{i_t}(t) \leq \mu^2(t)P(t)$.

**Proof.** (1) Sufficiently:

We aim to prove that system (2) is finite-time stabilisable via constructing a switching signal $\sigma(t)$ such that system (2) is finite-time stable under it. For this, define a finite-time switching signal $\sigma(t)$ as $\sigma(t) = i_t$ for all $t \in F$ and define time-varying Lyapunov function $V(t, x(t)) = x^T(t)P(t)x(t)$. Then, according to the well-defined switching signal, condition (ii) and $P(t_0) < \Lambda$, we have

$$V(t + 1, x(t + 1)) = x^T(t)A_{i_t}^T(t)P(t + 1)A_{i_t}(t)x(t)$$

$$\leq \mu^2(t)x^T(t)P(t)x(t)$$

$$\leq \sum_{k=t_0}^{t} \mu^2(k)x^T(k)P(k)x(k)$$

(14)

for all $t \in F \setminus \{t_0 + F\}$. Thus, we can further derive from inequality (14), condition (i) and Lemma 4 that, when

$$x^T(t_0)\Lambda x(t_0) < 1$$

for all $t \in F \setminus \{t_0 + F\}$, $x^T(t)Y(t)x(t) < x^T(t)P(t)x(t)$

$$< \sum_{k=t_0}^{t} \mu^2(k)x^T(k)\Lambda x(k)$$

$$< 1.$$
\[ -c \Phi_{\sigma(t)}^{\top}(t, t_0) \Lambda \Phi_{\sigma(t)}(t, t_0) = 0, \]

which indicates that
\[ A_1^{\top}(t)P(t) + P(t)A_2(t) = 0. \]

Thus, we can further attain that \( A_1^{\top}(t)P(t) + P(t)A_2(t) < 0 \) for all \( t \in T \setminus \{t_0 + F \} \). Moreover, due to the property of positive definite matrix, there definitely exists a constant \( M_0 \) with \( 0 < M_0 < 1 \) such that \( A_1^{\top}(t)P(t) + P(t)A_2(t) \leq M_0P(t) \). Then, define
\[ \mu(t) = \sqrt{M_0} \quad \text{for all} \quad t \in F, \]

it is obvious that \( \prod_{k=0}^{t-1} \mu^2(k) \leq 1 \) for all \( t \in T \setminus \{t_0 \} \), which indicates that \( \mu(t) \) is a finite-time stable function and condition (ii) is satisfied. Thus, the necessity is proved.

In Theorem 3, resorting to the additionally proposed finite-time stable function, we propose a necessary and sufficient condition for the finite-time stabilisability of discrete-time time-varying SLSs, which releases the requirement of the non-increasing of Lyapunov functions. Note that [38] obtained necessary and sufficient conditions for single continuous-time time-varying linear systems based on a Lyapunov function non-increasing on a series of subregions. Besides, from Lemma 4, the constraint of our proposed finite-time stable function \( \mu(t) \) is independent with initial state, \( \Lambda \) and \( P(t_0) \) in Theorem 3, which indicates that \( \mu(t) \) is a GAS function and \( P(t) \) is obtained tractable sufficient conditions in our future work inspired by [14, 46-48].

### 4 Illustrative Examples

In this section, three illustrative examples will be presented to manifest the applicability and validity of our theoretical results obtained above.

**Example 1.** Consider a switched system described by
\[
\dot{x}(t + 1) = A_{\sigma(t)}(t)x(t), \quad x(t_0) = x_0,
\]

where \( A_1(t) = \begin{bmatrix} 0 & 1 \\ \frac{3}{5} & 1 \end{bmatrix} \), \( A_2(t) = \begin{bmatrix} 0 & 1 \\ \frac{3}{5} & 1 \end{bmatrix} \), and \( \sigma(t) \in \Gamma = \{1, 2\}. \)

For verifying the global asymptotic stabilisability of system (18), a GAS function \( \mu(t) : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) and a \( P(t) : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}^{2 \times 2} \) are required to satisfy the conditions in Theorem 1.

**Figure 1** The value of \( x_1(t, t_0, x_0) \) and \( x_2(t, t_0, x_0) \) with \( t_0 = 0 \) and \( x_0 = [-10, 5]^T \).

Firstly, define
\[
\mu(t) = \begin{cases} 
\frac{2\sqrt{2}}{\sqrt{(1-t)^2 + 1}}, & t \in [t_k, t_k + 3) \\
\frac{2\sqrt{2}}{1 + t}, & t \in [t_k + 3, t_k + 9) 
\end{cases}
\]

with \( t_{k+1} = t_k + 9, t_0 = 0 \) and \( k \in \mathbb{Z}_{\geq 0} \).

Then, we can easily obtain that \( \lim_{t \to \infty} \prod_{k=0}^{t-1} \mu(k) = 0 \) for all \( t_0 \in \mathbb{Z}_{\geq 0} \) since \( \mu(t) < \frac{1}{2} \) for all \( t \geq 7 \). Hence, \( \mu(t) \) is a GAS function due to Lemma 3. Moreover, let
\[
P(t) = \begin{bmatrix} 8t + 6 & 3 \\ 3 & 8t + 6 \end{bmatrix}
\]

for all \( t \in \mathbb{Z}_{\geq 0} \). Obviously, \( P(t) = P^T(t) \) and \( L_0 \leq P(t) \) for all \( t \in \mathbb{Z}_{\geq 0} \), that is, condition (i) is verified. Further, it is easy to verify that \( A_1^{\top}(t)P(t) + P(t)A_2(t) \leq \mu^2(t)P(t) \) for all \( t \in [t_k, t_k + 3] \) with \( t_0 = 0, t_{k+1} = t_k + 9 \) and all \( k \in \mathbb{Z}_{\geq 0} \); and \( A_1^{\top}(t)P(t) + P(t)A_2(t) \leq \mu^2(t)P(t) \) for all \( t \in [t_k + 9, t_{k+1} + 9] \) with \( t_0 = 0, t_{k+1} = t_k + 9 \) and all \( k \in \mathbb{Z}_{\geq 0} \). Thus, condition (ii) of Theorem 1 is satisfied for all \( t \in \mathbb{Z}_{\geq 0} \).

Combining the above discussions, Theorem 1 is satisfied, which indicates that system (18) is globally asymptotically stabilisable. Actually, switched system (18) is GAS (see Figure 1) under the switching signal
\[
\sigma(t) = \begin{cases} 
1, & t \in [t_k, t_k + 3) \\
2, & t \in [t_k + 3, t_{k+1}] 
\end{cases}
\]
with \( t_{k+1} = t_k + 9 \), \( t_0 = 0 \) and \( k \in \mathbb{Z}_{\geq 0} \). Note that Figure 2 illustrates the releasing of the requirement on negative definiteness of the time-difference of Lyapunov functions compared to the traditional difference Lyapunov inequalities in [45].

Example 2. Consider a switched system described by

\[
\begin{align*}
x(t + 1) &= A_2(t)x(t), \quad x(t_0) = x_0,
\end{align*}
\]

where \( A_1(t) = \begin{bmatrix} 1/8 & -1/8 & 0 \\ 0 & -1/10 & 0 \\ 0 & -1/8 & 1/10 \end{bmatrix} \), \( A_2(t) = \begin{bmatrix} 0 & 2^{-t-3} & 2^{-t-3} \\ 0 & 2^{-t-3} & 0 \\ 0 & 0 & 0 \end{bmatrix} \) and \( \sigma(t) \in \Gamma = \{1, 2\} \).

For verifying the global exponential stabilisability of system (19), a GES function \( \mu(t) : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) and a \( P(t) : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}^{2 \times 2} \) are required to satisfy the conditions in Theorem 1.

Firstly, define \( \mu(t) = 2^{-t+4}(-1)^{\frac{t}{2}} \), we can obtain from [24] that \( \mu(t) \) is a GES function. Moreover, set

\[
P(t) = \begin{bmatrix}
1 & 2^{-t+2} & 0 \\
0 & 1 & 2^{-t+2} \\
0 & 0 & 1
\end{bmatrix}
\]

for all \( t \in \mathbb{Z}_{\geq 0} \). Obviously, \( P(t) = P^T(t) \) and \( I_2 \leq P(t) \leq 8I_2 \) for all \( t \in \mathbb{Z}_{\geq 0} \), that is, condition (i) of Theorem 1 is verified. Further, it is easy to verify that \( \mu^T(t)P(t+1)A_1(t) \leq \mu^2(t)P(t) \) for all \( t \in [t_0, t_0 + 1] \) with \( t_0 = 0 \), \( t_{k+1} = t_k + 4 \) and all \( k \in \mathbb{Z}_{\geq 0} \); and \( \mu^T(t)P(t+1)A_2(t) \leq \mu^2(t)P(t) \) for all \( t \in [t_k + 1, t_k + 4] \) with \( t_0 + 4 \), \( t_{k+1} = t_k + 4 \) and all \( k \in \mathbb{Z}_{\geq 0} \). Thus, condition (ii) of Theorem 1 is satisfied for all \( t \in \mathbb{Z}_{\geq 0} \).

Combining the above discussions, Theorem 1 is satisfied, which indicates that system (19) is globally exponentially stabilisable. Actually, switched system (19) is GES (see Figure 3) under the switching signal

\[
\sigma(t) = \begin{cases}
1, & t \in [t_k, t_k + 1) \\
2, & t \in [t_k + 1, t_k + 4]
\end{cases}
\]

where \( t_{k+1} = t_k + 4 \), \( t_0 = 0 \) and \( k \in \mathbb{Z}_{\geq 0} \). Note that Figure 4 shows the value of Lyapunov function \( V(t(\sigma(t))) \) for system (19) with \( t_0 = 0 \) and \( x_0 = [-1, 1]^T \).

Example 3. Consider a switched system described by

\[
\begin{align*}
x(t + 1) &= A_{2\sigma(t)}(t)x(t), \quad x(t_0) = x_0
\end{align*}
\]

where \( A_1(t) = \begin{bmatrix} t^2+1 & 0 \\ 0 & t^2+1 \end{bmatrix} \), \( A_2(t) = \begin{bmatrix} 1 & t + 1 \\ 0 & t + 1 \end{bmatrix} \) and \( \sigma(t) \in \Gamma = \{1, 2\} \).

Given a finite time interval \( \mathcal{P} = \{0, ..., 32\} \), we attempt to analyse the finite-time stabilisability of system (20) with respect to \( [0, 32, A, Y(t)] \), where \( A = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \) and \( Y(t) = \begin{bmatrix} t + 1 & 0 \\ 0 & t + 1 \end{bmatrix} \) with \( Y(0) < A \). For this, a function \( \mu(t) : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{>0} \) and a matrix \( P(t) : \mathcal{P} \rightarrow \mathbb{R}^{2 \times 2} \) are required to satisfy the conditions in Theorem 3.

Firstly, define

\[
\mu(t) = \begin{cases}
\sqrt{40 - t}, & t \in [t_k, t_k + 2) \\
\sqrt{12t + 1}, & t \in [t_k + 2, t_k + 4)
\end{cases}
\]

with \( t_{k+1} = t_k + 4 \), \( t_0 = 0 \) and \( 0 \leq k \leq 7 \). Then, we can easily obtain that \( \prod_{k=0}^{31} \mu^2(k) \leq 1 \), which implies that \( \mu(t) \) is a finite-
DISCUSSIONS ON SWITCHED NON-LINEAR SYSTEMS

Non-linear systems have been adequately discussed. Naturally, the question arises whether it is possible to derive similar necessary and sufficient conditions for the time-varying switched non-linear systems. Since there has been no common approach to construct proper Lyapunov functions for the necessary condition of the asymptotic stabilisability and finite-time stabilisability, we here only consider necessary and sufficient conditions for the global exponential stabilisability and global uniform exponential stabilisability of switched non-linear system (1).

Theorem 4. System (1) is globally exponentially stabilisable if and only if there exist a GES function \( \mu(t) : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \), an integer \( p \geq 1 \), a positive definite function \( c(t) \) and a function \( V(t, x) : \mathbb{Z}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \) such that

(i) for all \( t \in \mathbb{Z}_{\geq 0} \) and all \( x \in \mathbb{R}^n \), \( \|x\|^{2p} \leq V(t, x) \leq c(t)\|x\|^{2p} \); and
Moreover, system (1) is globally uniformly exponentially stabilisable if and only if there exist a GUES function \( \mu(t) : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{>0} \), an integer \( p \geq 1 \), a positive constant \( \epsilon \) and a function \( V(t, x) : \mathbb{Z}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0} \) such that conditions (i) with \( c(t) = \epsilon \) and (ii) are satisfied.

**Proof.** (1) Sufficiency:

For proving the global exponential stabilisability of system (1), we aim to construct a switching signal \( \sigma(t) \) such that system (1) is GES under it. For this, define the switching signal \( \sigma(t) = i_t \) for all \( t \in \mathbb{Z}_{\geq 0} \). Then, for any initial state \( x_0 \), all \( t \geq t_0 \) and all \( t_0 \in \mathbb{Z}_{\geq 0} \), we can derive from condition (ii) that

\[
V(t+1, x(t+1)) = V(t+1, f_{i_t}(t, x(t))) 
\leq \mu^{(p)}(t)V(t, x(t)) 
\leq \phi^{(p)}(t+1, h_t) V(h_t, x(h_t)).
\]

Further, it follows from inequality (21) and condition (i) that \( \|x(t)\|^{2p} \leq V(t, x(t)) \leq \phi^{(p)}(h_0) V(h_0, x_0) \|x_0\|^{2p} \), which implies that

\[
\|x(t)\| \leq \phi^{(p)}(h_0) V(h_0, x_0) \|x_0\|^{2p}.
\]

Then, since \( \mu(t) \) is a GES function, we can derive from inequality (22) and Lemma 3 that system (1) is GES under the above constructed switching signal \( \sigma \), that is, the sufficiency of the global exponential stabilisability of system (1) is proved due to Definition 2.

Obviously, if \( \mu(t) \) is a GUES function, we can derive from inequality (22) and Lemma 3 that system (1) is GUES under the above constructed switching signal \( \sigma \), that is, the sufficiency of the global uniform exponential stabilisability of system (1) is proved.

(2) Necessity:

We can derive from the global exponential stabilisability of system (1) that there exists a switching signal \( \sigma \), constants \( \beta(h_0) \geq 1 \) and \( \alpha > 0 \) such that \( \|x_0(t, t_0, x_0)\| \leq \beta(h_0) e^{-\alpha(t-t_0)} \|x_0\| \) for all \( t \geq t_0 \) and all \( x_0 \in \mathbb{R}^n \). Thus, for all \( t \in \mathbb{Z}_{\geq 0} \), there exists an index \( i_t \in \Gamma \) such that \( \sigma(t) = i_t \). Then, define

\[
V(t, x) = \sum_{k=t}^{\infty} e^{(k-t)}(x^T(k, t, x)x(k, t, x))^p,
\]

where \( 0 < \epsilon < 2\alpha \). Clearly,

\[
V(t, x) \geq \|x\|^{2p}.
\]

Moreover,

\[
V(t, x) \leq \sum_{k=t}^{\infty} e^{(k-t)}(\beta(t))^p e^{-2\alpha(k-t)} \|x\|^{2p} = \sum_{k=t}^{\infty} e^{(2\alpha(k-t))} \|x\|^{2p} = \frac{1}{1 - e^{2\alpha \|x\|^{2p}}}.\]

Thus, condition (i) is satisfied via defining \( c(t) = \frac{1}{1 - e^{2\alpha \|x\|^{2p}}} \). Besides,

\[
V(t+1, f_{i_t}(t, x)) = \sum_{k=t+1}^{\infty} e^{(k-t)}(x^T(k, t+1, f_{i_t}(t, x))x(k, t+1, f_{i_t}(t, x)))^p
\leq e^{-\epsilon} V(t, x) + e^{-\epsilon \|x\|^{2p}}
\leq e^{-\epsilon} V(t, x).
\]

Letting \( \mu(t) = e^{-\frac{\epsilon}{\|x\|^{2p}}} \), it follows that \( \mu(k) = e^{-\frac{\epsilon}{\|x(k)\|^{2p}}} \), which implies that \( \mu(t) \) is a GES function. Combining inequalities (24)–(26), we can conclude that the necessity of the global exponential stabilisability of system (1) is proved.

Moreover, when system (1) is global uniform exponential stabilisability, we define the required \( V(t, x) \) as Equation (23). Then, we can also obtain inequalities (24), (25) with \( \beta(t) = \beta \) and (26). Besides, \( \mu(t) = e^{-\frac{\epsilon}{\|x\|^{2p}}} \) is a GUES function. Thus, the necessity of the global uniform exponential stabilisability of system (1) can be directly proved.

Motivated by [14, 46–48], we can similarly combine Theorem 4, the average dwell time method and even periodic function \( V(t, x) \) to construct computable sufficient conditions for the global exponential stability and global uniform exponential stability of system (1), respectively. Then, the computation issue can be solved.

**Remark 2.** Note that the function formed by (23) cannot qualify as a Lyapunov function for proving the necessity of the global asymptotic stabilisability since the convergence of series \( \sum_{k=t}^{\infty} e^{(k-t)}(x^T(k, t, x)x(k, t, x))^p \) cannot be guaranteed. Likewise, there is still no proper Lyapunov function that can be chosen to derive the necessity of the finite-time stabilisability for general time-varying single non-linear systems not even to say switched non-linear systems [38]. Thus, more relaxed necessary and sufficient conditions for the asymptotic stabilisability and finite-time stabilisability of system (1) are still required to explore.
6 | CONCLUSION

This paper has provided several necessary and sufficient conditions for the stabilisability of time-varying discrete-time switched systems. Specifically, necessary and sufficient conditions for the asymptotic (exponential, uniform exponential) stabilisability of time-varying SLSs are proposed by resorting to asymptotically (exponentially, uniformly exponentially) stable functions. Moreover, a necessary and sufficient condition for finite-time stabilisability of time-varying discrete-time SLSs is further obtained via an additionally introduced finite-time stable function. Specially, we further attain the necessary and sufficient conditions for global (uniform) exponential stabilisability of time-varying switched non-linear systems. Based on these stable functions, we had released the requirement on negative definiteness of the time-difference of Lyapunov functions, compared to traditional difference Lyapunov inequalities.

Our possible future work is to investigate the computation mechanism of the required $P(t)$ (or $V(t,x)$) and stable function $\mu(t)$ based on [14, 46–48]. Besides, motivated by the work in [11, 19, 21, 46, 49, 50], we would like to further research on constructing switching signals to stabilising a time-varying switched system via utilising the idea of multiple Lyapunov functions and average dwell time method. Moreover, based on [51–54], it merits further investigation on the computation of basin of attraction via stable functions for locally asymptotically stable time-varying switched systems.

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