Gauge theory of self-similar system

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Abstract

On the basis of a dilatation invariant Lagrangian, governed equations are determined for probability density and gauge potential of the non-stationary self-similar stochastic system. It is shown that an automodel regime is observed at small time interval determined by the Tsallis' parameter $q > 1$. An exponential falling down happens at large time where the dilatation parameter and the partial scale tend to constant values.

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1 Introduction

Let us consider the one-dimensional random walker with coordinate $x$ at time $t$ – a characteristic example of the stochastic system carrying out Lévy flights [1]. In the case of self-similar system, the corresponding probability distribution $P(x, t)$ is a homogeneous function satisfying to the condition

$$P(x, t) = a^{-\alpha} \mathcal{P}(\kappa), \quad \kappa \equiv x/a$$

(1)

where $a \equiv a(t)$ is a time dependent partial scale, $\kappa$ is a dimensionless coordinate, $\alpha$ is a self-similarity index. To analyze such a system, it is convenient to use the basic conception of Jackson’s derivative $\mathcal{D}_q$, whose properties are given in Appendix. Basic advantage of this derivative for analyzing self-similar system is that the Jackson’s derivative determines
rate of the function variation with respect to dilatation \( q \), but not to the shift \( dx \to 0 \) as in usual case. This work is devoted to studying a stochastic self-similar system on the basis of such a type representation.

The paper is organized as follows. In Section 2 the governed equations for probability density and gauge potential are obtained starting from a dilatation invariant Lagrangian. Section 3 deals with the determination of the time dependencies for a characteristic scale and probability density. A steady state is shown to realize when the Jackson’s derivative of the gauge potential equals to zero. This represents a gauge condition, under which the probability distribution has the Tsallis’ form. With breaking the gauge condition, an automodel regime is observed at small time interval determined by the Tsallis’ parameter \( q > 1 \). An exponential falling down happens at large time where the dilatation parameter and the partial scale tend to a constant values. Section 4 contains a short conclusions and in Section 5 basic properties of the Jackson’s derivative are adduced.

## 2 Basic equations

In contrast to the simple case, when the scale \( a \) does not depend on the time \( t \), we study here a non-stationary self-similar system, for which the value \( a \) and the dilatation factor \( q \) are functions of \( t \). As it is known from the theory of the gauged fields [2], in such a case the system is invariant with respect to transformations \( x \to qx, \ P \to Q_qP, \ Q_q \sim q^a \) if the gradient terms \( D_qP, \ \partial P/\partial t \) are replaced by the elongated derivatives \((D_q + \epsilon)P, \ (\partial/\partial t + E)P\) with dilatational \( \epsilon \) and temporal \( E \) components of the gauge potential (hereafter the time \( t \) is measured in units of the probability relaxation time). In accordance with Eq.(33), it is easy to see that this elongated derivatives are invariant with respect to the non-stationary dilatation \( q = q(t) \) determined by the follow transformations:

\[
\begin{align*}
  x & \to qx, \quad P \to Q_qP; \\
  \epsilon & \to \epsilon - \frac{D_qQ_q}{Q_q^2} - (q - 1) \frac{D_qQ_q}{Q_q} \frac{D_qP}{P}, \quad E \to E - \dot{Q}
\end{align*}
\]  

(2)

where point denotes the time derivative for brevity.

Gauge invariant Lagrangian of the corresponding Euclidean field theory is supposed to take the form

\[
\mathcal{L} = \frac{1}{2} [(D_q + \epsilon)P]^2 + \frac{1}{2}(D_q\epsilon)^2
\]  

(3)

where the first term is caused by the gauged dilatation, the second one is the field contri-
bution. The respective dissipative function reads

\[ f = \frac{1}{2} \left[ \left( \frac{\partial}{\partial t} + E \right) P \right]^2 + \frac{\theta}{2} \left[ \left( \frac{\partial}{\partial t} + E \right) \epsilon \right]^2 \]  

(4)

where \( \theta \) is the relaxation times ratio of the gauge field and probability. As a result, the Euler equation

\[ \mathcal{D}_q \frac{\partial L}{\partial (\mathcal{D}_q z)} - \frac{\partial L}{\partial z} = -\frac{\partial f}{\partial \dot{z}}, \quad z \equiv (P, \epsilon) \]  

(5)

leads to the differential equations with partial derivatives and non-linear terms:

\[ \dot{P} + \mathcal{D}_q^2 P = -EP + \epsilon^2 P, \]  

(6)

\[ \theta \dot{\epsilon} + \mathcal{D}_q^2 \epsilon - PD_q P = -\theta EP + \epsilon P^2. \]  

(7)

The first terms in right-hand parts describe dissipative influence of external environment. Further, we will take into consideration conserved systems only, so that the time component of the gauge potential being inversely proportional to corresponding relaxation time will be put equal to zero \((E = 0)\).

3 Solution of equations

In the limits \( \epsilon \to 0, \mathcal{D}_q^2 \epsilon \to 0 \), the obtained equations (6), (7) take the form

\[ \dot{P} = -\mathcal{D}_q^2 P, \]  

(8)

\[ \theta \dot{\epsilon} = PD_q P. \]  

(9)

The first of them has the diffusion type but with inverted sign, so that self-similar system reveals running away kinetics that is inherent in hierarchical systems [4]. However, such a behaviour realizes during short time interval \( t \sim \theta \ll 1 \) only. At usual time \( t \geq 1 \), we can use the condition \( \theta \ll 1 \) of adiabatic approximation that will be used everywhere below and allow to neglect left-hand side of Eq.(9). As a result, the condition \( \mathcal{D}_q P \approx 0 \) holds true and the system passes to stationary homogeneous regime:

\[ P(x, t) = \text{const} \equiv P_{st}. \]  

(10)
In much more complicated limit $\epsilon(t) \to \text{const} \neq 0$, equation (7) is reduced to the simplest form

$$\mathcal{D}_q P = -\epsilon P$$

and we arrive at the static distribution (10) as before.

To continue analysis, let us multiply Eq.(6) by factor $\mathcal{D}_q P$ and Eq.(7) by $\mathcal{D}_q \epsilon$. Then, after addition of the obtained results we find

$$\frac{1}{2}(\mathcal{D}_q P)^2 + \frac{1}{2}(\mathcal{D}_q \epsilon)^2 = \frac{1}{2}(\epsilon P)^2 + \frac{1}{2}C^2.$$

Here we put

$$(\dot{P} - P\mathcal{D}_q \epsilon)\mathcal{D}_q P = 0$$

and fulfilled integration with constant $C^2/2$. As a result, under the assumption

$$\mathcal{D}_q \epsilon = -C, \quad C > 0$$

and condition $\mathcal{D}_q P \neq 0$, the equation (13) arrives at the exponential time dependence

$$P \propto e^{-Ct}$$

whereas the equation (12) is reduced to the form (11).

The exponential falling-down is known to be not inherent in the self-similar systems [3] and, as a consequence, we are need to put $C = 0$ in Eqs.(12), (14). We arrive then at the gauge condition

$$\mathcal{D}_q \epsilon = 0,$$

according to which the potential $\epsilon$ can be a time dependent function but does not vary with the system dilatation. Then, equation (6) takes the form $\dot{P} = 0$ meaning that the system is in steady-state, which probability distribution is obeyed to Eq.(11). Being accompanied with Eq.(31), this equation arrives at the condition $[\alpha]_q = -\epsilon$. As is ascertained in Appendix, the Jackson’s q-number $[\alpha]_q$ is reduced to the Tsallis’ q-logarithm if the steady-state probability $P_{st}$ and dilatation $q$ are connected via the follow relation:

$$P_{st}^{q-1} \equiv q^\alpha.$$
Then, the above obtained condition gives the Tsallis’ distribution [3]

$$P_{st} = [1 - (q - 1)\varepsilon]^{\frac{1}{q-1}}.$$ (18)

With breaking gauge condition (16) the self-similar system gets into non-stationary state, whose behaviour is determined by Eq.(6). Accounting definitions (34) arrives this to the algebraic form with respect to the Jackson’s derivation:

$$\dot{P} = \left(\varepsilon^2 - [\alpha]_{qq}\right)P.$$ (19)

Inserting here Eq.(1) and taking into consideration the condition $[\alpha]_q = -\varepsilon$ and relation $\dot{P} = -a^{-1}(1+\alpha)(\alpha P + \kappa P')\dot{a}$ (hereafter prime denotes the usual derivation with respect to the argument $\kappa$) we obtain

$$a^{-1}\dot{a}(\alpha P + \kappa P') = [\delta\alpha]_{qq}P$$ (20)

where the factor $[\delta\alpha]_{qq}$ stands for the term determined by Eqs.(34).

In the limit $q \to \infty$, one can see with accounting asymptotics (35) that system behaves in automodel manner if conditions $aq = const$, $a^{3q-4}\dot{a} = const \equiv \tau_0^{-1}$ and equation

$$\kappa P' = (\tau_0 - \alpha)P$$ (21)

are implemented. Solution of the equation is $P \propto \kappa^{\tau_0-\alpha}$ and the time dependencies of the characteristic scale and the probability density read:

$$a^{3(\alpha-1)} = \frac{t}{\tau}, \quad P \propto x^{\tau_0-\alpha} t^{-\tau}, \quad \tau \equiv \frac{\tau_0}{3(\alpha-1)} \quad \text{at} \quad q \gg 1, \quad t < \tau.$$ (22)

Within the opposite limit $q \to 1$, a magnitude $q$ in Eq.(20) ought to put time independent and we arrive at the long-time dependencies:

$$a \propto \exp \left(\frac{t}{\tau_0}\right), \quad P \propto x^{\lambda_0-\alpha} \exp (-\lambda_0 t), \quad \lambda_0 \equiv \frac{\alpha - 1}{q - 1} \quad \text{at} \quad q \to 1, \quad t \gg \lambda_0^{-1}.$$ (23)

The coincidence condition for the time limits $\tau$ and $\lambda_0^{-1}$ in dependencies given by Eqs.(22), (23) leads to relation

$$\tau = 3(q - 1).$$ (24)
At last, we consider the case with non-zeroth second Jackson’s derivative $D^2_\epsilon = [\epsilon]_{qq}\epsilon$ determined by an index $\epsilon$. Here equation (7) gives the gauge potential

$$\epsilon = \frac{-[\alpha]_q}{1 - [\epsilon]_{qq}P^{-2}}$$  \hspace{1cm} (25)$$

that behaves in self-similar manner if the value $[\epsilon]_{qq}P^{-2}$ falls down with $q$-increase. Supposing this falling down in power form $q^{-\gamma}$ with positive index $\gamma \to 0$, we obtain the needed dependence $P(t) \propto q^\gamma(t)$, following from Eqs.(22) and condition $a(t)q(t) = \text{const}$, if $\gamma = 2\tau - 3(\varepsilon - 1) > 0$. As a result, the gauge potential index $\epsilon$ is limited by the condition

$$\varepsilon < 1 + \frac{2}{3} \tau = 1 + 2(q - 1)$$  \hspace{1cm} (26)$$

where the second equality follows from Eq.(24). Under this conditions the equation (6) accompanied with approximated result $\epsilon \approx -[\alpha]_q$, following from Eq.(25), arrives at the above obtained time dependencies (22).

The automodel regime is studied to be broken if the factor in right-hand side of Eq.(19)

$$\varepsilon^2 - [\alpha]_{qq} = [\alpha]^2_q [(1 - [\epsilon]_{qq}P^{-2})^{-2} - 1] - [\delta \alpha]_{qq} \equiv -\lambda$$  \hspace{1cm} (27)$$

becomes time independent (here the steady-state probability $P_{st}$ is determined by Eq.(18) and we take into account Eqs.(25), (34)). In such a case, the exponential decay (23) is characterized by the relaxation time $\lambda^{-1}$ instead of $\lambda_0^{-1}$. Because this regime is inherent in small values of $q$, we can use the limit $q \to 1$:

$$\lambda = \lambda_0 - \alpha^2 \left\{ \left[ 1 - \left( \varepsilon^2 + \frac{\varepsilon - 1}{q - 1} \right) e^{2\alpha} \right]^{-2} - 1 \right\}, \hspace{1cm} \lambda_0 \equiv \frac{\alpha - 1}{q - 1}. \hspace{1cm} (28)$$

Finally, under conditions $q \to 1$, $\varepsilon^2 + (\varepsilon - 1)/(q - 1) \ll e^{2\alpha}$ when $\alpha = [\alpha]_q = -\epsilon$, we have

$$\lambda = \lambda_0 \left[ 1 - 2\alpha^2 \frac{q - 1}{\alpha - 1} \left( \varepsilon^2 + \frac{\varepsilon - 1}{q - 1} \right) e^{-2\alpha} \right]. \hspace{1cm} (29)$$

4 Discussion

The above offered formalism is based on the dilatation invariant Lagrangian (3) and dissipative function (4) to describe the conserved non-stationary self-similar stochastic system.
Behaviour of such a system is determined by the probability density distribution (1) and
the gauge potential $\epsilon$ (the latter is reduced to the ratio of the microstate energy to tem-
perature in usual case of thermodynamic systems). The non-linear differential equations
with partial derivatives, Eqs.(6), (7) are obtained to analyze the system kinetics. It is
occurred that under gauge condition (16), when the microstate energy is independent on
dilatation, the system is in the steady-state characterized by the Tsallis’ distribution (18).
With gauge breaking, the automodel regime (22) realizes for a time less than bounded
magnitude (24) determined by the difference $q - 1$. For more values of time, when the
dilatation becomes constant, the system passes to the usual exponential regime (23).

Finally, we comment on limitations of our approach. The main of these is that the system
under consideration is conserved, so that influence of external environment have been
put equal to zero (the value $E = 0$ in Eqs.(6), (7)). Accounting such type terms for
hierarchical systems shows that the time component of the gauge potential is reduced to
the linear differential operator $E \equiv -(\partial/\partial x)F(x) + (\partial^2/\partial^2 x)D(x)$ to be anti-dissipative
in the physical meaning (here $F(x)$ is a drift force and $D(x)$ is a diffusion coefficient) [5,
6]. Due to these terms, the above found exponential regime will be suppressed and the
self-similar system will behave in automodel manner during whole time interval.

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5 Appendix. Basic properties of the Jackson’s derivative

The Jackson’s derivative is defined by equation

$$D_q f(x) \equiv \frac{f(qx) - f(x)}{q - 1}, \quad q \neq 1 \quad (30)$$

that is reduced to the usual derivative in the limit $q \to 1$. Apparently, for a homogeneous
function with a self-similarity index $\alpha$ the Jackson’s derivative is reduced to Jackson’s
q-number $[\alpha]_q$:

$$D_q f(x) = [\alpha]_q f(x), \quad [\alpha]_q \equiv \frac{q^\alpha - 1}{q - 1}. \quad (31)$$
It is easily to see that the value $[\alpha]_q \to \alpha$ in the limit $q \to 1$ and increases as $q^{\alpha-1}$ at $q \to \infty$ (we propose $\alpha > 1$). On the other hand, the Tsallis’ q-logarithm function $\ln_q x \equiv (x^{q-1} - 1)/(q - 1)$ can be represented in the form of the Jackson’s q-number with index $\alpha = (q - 1) \ln x/\ln q$. Accompanied Eq.(31) this relation and apparent equality [3]

$$\ln_q(xy) = \ln_q x + \ln_q y + (q - 1)(\ln_q x)(\ln_q y)$$

lead to important rule for the Jackson’s derivative:

$$\mathcal{D}_q [f(x)g(x)] = [\mathcal{D}_q f(x)] g(x) + f(x) [\mathcal{D}_q g(x)] + (q - 1) [\mathcal{D}_q f(x)][\mathcal{D}_q g(x)].$$

The Jackson’s derivative of the second order is determined as follows:

$$\mathcal{D}_p \mathcal{D}_q f(x) = \mathcal{D}_p \{[\alpha]_q f(x)\} = [\alpha]_{pq} f(x),$$

$$[\alpha]_{pq} = [\alpha]_p [\alpha]_q + [\delta \alpha]_{pq}, \quad [\alpha]_p = \frac{p^\alpha - 1}{p - 1}, \quad [\delta \alpha]_{pq} = \frac{p^\alpha ((pq)^\alpha - pq)}{(p - 1)(pq - 1)}.$$

In proposition $\alpha > 1$, the value $[\delta \alpha]_{pq}$ has the follow asymptotics:

$$[\delta \alpha]_{pq} \to \frac{\alpha - 1}{p - 1} \quad \text{at} \quad p, q \to 1,$$

$$[\delta \alpha]_{pq} \to p^{2(\alpha - 1)} q^{(\alpha - 1)} \quad \text{at} \quad p, q \to \infty.$$

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