ORBIFOLD KÄHLER-EINSTEIN METRICS ON NON-\(K\)-POLYSTABLE TORIC VARIETIES

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Abstract. In this short note, we investigate the existence of orbifold Kähler-Einstein metrics on non-\(K\)-polystable toric Fano varieties. In particular, we show that every \(\mathbb{Q}\)-factorial toric variety of Picard number one and all of the 16 toric Gorenstein del Pezzo surfaces allow an orbifold Kähler-Einstein metric.

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1. Introduction

We work over the field \(\mathbb{C}\) of complex numbers. In contrast to the case of negative or zero first Chern class - where Kähler-Einstein metrics are known to always exist - due to the confirmation of the Yau-Tian-Donaldson conjecture, we know that in the case of Fano manifolds, the existence of a Kähler-Einstein metric is equivalent to the algebraic notion of \(\mathcal{K}\)-polystability [CDS15a, CDS15b, CDS15c, Tia15]. This purely smooth setting was extended in the last years to the case of klt log Fano pairs \(\mathcal{p}X,\Delta\) culminating in the analogous statement for such pairs [LXZ22, Thm. 1.6]: the existence of a singular Kähler-Einstein metric being equivalent to \(\mathcal{K}\)-polystability of the pair \((X,\Delta)\).

It was conjectured in [Don12, Conj. 1] that for non-\(\mathcal{K}\)-stable Fano manifolds, a Kähler-Einstein metric with certain cone singularities should exist. There have been found counterexamples to the original version of the conjecture [Szé13, Thm. 1]. In fact, the counterexamples given are toric Gorenstein del Pezzo surfaces, compare Theorem 3 below. On the other hand, a modified version of the conjecture [BL22, Conj. 7.4] was proven in [LXZ22, Thm. 1.8], stating that for a log Fano pair \((X,\Delta)\), there exists \(m\) and a \(\mathbb{Q}\)-divisor \(D\) in the linear system \(\frac{1}{m} - m(K_X + \Delta)\), such that \((X,\Delta + D)\) is \(\mathcal{K}\)-polystable.

However, as Donaldson remarks [Don12], the only singular metrics for which we know that ”a great deal of the standard theory can be brought to bear“ are orbifold metrics. For this, we need at least that \(X\) has quotient singularities and \(\Delta\) is snc on the smooth locus with so-called standard coefficients of the form \(1 - \frac{1}{m}\). In the case of \(\mathbb{Q}\)-factorial toric varieties, this holds for torus invariant boundaries with standard coefficients.

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1.1. Toric varieties with orbifold Kähler-Einstein metrics. As mentioned above, a toric boundary with standard coefficients indeed provides an orbifold metric. However, [LXZ22, Thm. 1.8] does not provide us with a toric boundary. We investigate the existence of such toric boundaries in the below.

We remind the reader that due to the works [DS16] in the smooth and [Zhu21] in the klt case, for $G$-varieties ($G$ a reductive group), equivariant $K$-polystability is equivalent to $K$-polystability. Moreover, for a toric weak Fano pair given by a convex polytope $P_{-(K_X+\Delta)} \subseteq M_\mathbb{Q}$, $K$-polystability is equivalent to $P_{-(K_X+\Delta)}$ having the origin as its barycenter [WZ04, BB13, Ber16].

Our first result says that as soon as we find any toric $\mathbb{Q}$-boundary, we can always produce one with standard coefficients just by scaling the polytope appropriately. This yields the following:

**Theorem 1.** Let $X$ be a projective toric variety. If $X$ allows a toric boundary $\Delta$ such that $(X, \Delta)$ is $K$-polystable, then it allows a toric boundary $\Delta'$ with standard coefficients, such that $(X, \Delta')$ is $K$-polystable as well. In particular, if $X$ is $\mathbb{Q}$-factorial, it allows an orbifold Kähler-Einstein metric.

Our first result on existence covers the maybe easiest case of toric pairs given by simplices:

**Theorem 2.** Let $X$ be a complete $\mathbb{Q}$-factorial toric variety of Picard number one. Then there is a toric boundary $\Delta$ with $(X, \Delta)$ $K$-polystable. In particular, $X$ allows an orbifold Kähler-Einstein metric.

Leaving the realm of simplices, things instantly get more complicated: for instance, in Lemma 3.1, we show that a simplex has its barycenter at the origin if and only if its dual has its barycenter at the origin, which is no longer true for polytopes with more vertices. This substantially complicates the calculations.

We are not able to show the existence of toric $\mathbb{Q}$-boundaries with $(X, \Delta)$ $K$-polystable in general. The very reason is that as soon as $P$ is not a simplex (i.e. $X$ has a class group of rank greater than 1), the barycenter is not a linear function of the vertices (or defining half-planes). The existence of $\mathbb{Q}$-boundaries thus amounts to solving a system of polynomial diophantine equations for each $X$. This is a serious issue even in the case of the 16 well-known toric Gorenstein del Pezzo surfaces. However, we obtain a positive result for these surfaces:

**Theorem 3.** Let $X$ be one of the 16 toric Gorenstein del Pezzo surfaces. Then $X$ allows a toric orbifold boundary $\Delta$ with $(X, \Delta)$ $K$-polystable.

In the following pictures of the corresponding dual Fano polygons $P_\mathbb{X} \subseteq N$, we attach (coprime) integer stretching factors $m_\rho$ to the vertices, which yield the orbifold boundary $\Delta = \sum_{\rho \in \Sigma(1)} (1 - 1/m_\rho) D_\rho$:
Obviously, the polytopes marked only with 1’s correspond to the surfaces that are already $K$-polystable with no need for a boundary. For the polygons with more than three vertices, we do not know if other stretching factors - which are not multiples by a common factor of the given ones - exist.

1.2. $K$-stability in terms of the (log) Cox ring. Since all toric varieties have a polynomial Cox ring, the grading by the class group alone must encode the $K$-polystability of a toric variety. In [BM21], the authors introduced the notion of the log Cox ring of a pair $(X, \Delta)$, which is the right object to study in this context, since it takes into account the boundary $\Delta$. For a toric orbifold boundary $\Delta$, the log class group $Cl(X, \Delta)$ is the quotient of orbifold Weil divisors (Q-divisors that become integral on orbifold charts) by linear equivalence. The log Cox ring is the associated divisorial algebra. It’s spectrum $\hat{X}_\Delta$ - the log characteristic space - allows a good quotient $\hat{X}_\Delta \to X$ by the diagonalizable group $H_{(X, \Delta)} := \text{Spec } \mathbb{C}[Cl(X, \Delta)]$ which ramifies over $\Delta$, with order $m_i$. In this setting, we have the following characterization of $K$-polystability:

**Theorem 4.** Let $X$ be a Q-factorial toric variety of Picard number one and dimension $n$ and $\Delta = \sum_{\rho \in \Sigma(1)} (1 - 1/m_{\rho}) D_{\rho}$ a toric orbifold boundary. Then the following are equivalent:

1. $(X, \Delta)$ is $K$-polystable.
2. The barycenter of $P_{-\langle K_X + \Delta \rangle} = \text{conv}(m_{\rho} u_{\rho}, \rho \in \Sigma(1)) \subseteq NQ$ is 0.
3. The orbifold universal cover of $(X_\text{reg}, \Delta)$ is $(\mathbb{P}^n, \emptyset)$.
4. There is a subgroup $\mathbb{Z} \leq Cl(X, \Delta)$ such that $\text{Spec } \mathbb{C}[\mathbb{Z}] \cong \mathbb{C}^*$ acts with weights $(1, \ldots, 1)$ on $\hat{X}_\Delta$.

Again, this characterization breaks down for higher Picard numbers due to the more complicated formulae for the barycenter in these cases.

1.3. $\mathbb{R}$-boundaries on toric varieties. When we allow $\mathbb{R}$-boundaries $\Delta_\mathbb{R}$, the situation becomes more tractable. However, it is questionable how meaningful the existence of such an $\mathbb{R}$-boundary is.

**Theorem 5.** Let $X$ be a projective toric variety. Then $X$ admits a toric $\mathbb{R}$-boundary $\Delta_\mathbb{R}$, such that $-(K_X + \Delta_\mathbb{R})$ is ample and the barycenter of $P_{-\langle K_X + \Delta_\mathbb{R} \rangle}$ is the origin, if one of the following holds:

1. $X$ is a surface.
2. $X$ is Q-factorial Gorenstein Fano (of arbitrary dimension).
The proof in the surface case is an extension of the proof of projectivity of toric surfaces, so to say. The intersection points of the rays of the fan of a toric surface with a circle are the vertices of a polytope, which provides us with an ample divisor. Now if we move around the circle, we can easily produce a polytope with it’s barycenter sitting at the origin. Even taking a rational approximation of the circle does not give us a \( \mathbb{Q} \)-boundary in general, since it’s center may be irrational.

This strategy clearly does not work anymore in higher dimensions. Here, the \( \mathbb{Q} \)-factorial Gorenstein condition guarantees that at least for one maximal cone, we have no restrictions on how far we can move away the vertices (lying on the rays of the cone).

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2. Preliminaries

2.1. Log pairs and their singularities. Let \( X \) be a normal variety and \( \Delta \) be an effective \( \mathbb{Q} \)-divisor. We call \((X, \Delta)\) a log pair if \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier. In case \( 0 \leq \Delta \leq 1 \), we call \( \Delta \) a boundary. Then for a log resolution \( f: Y \to X \), we define the discrepancies of \( K_X + \Delta \) to be the coefficients at exceptional prime divisors of the divisor \( K_Y - f^*(K_X + \Delta) \). We say that \((X, \Delta)\) is a klt pair, if \( \Delta < 1 \) and the discrepancies are greater than \(-1\). We call \((X, \Delta)\) log Fano, if it is klt and \(-(K_X + \Delta)\) is ample. Moreover, we say that \( X \) is of klt type (Fano type), if there exists a boundary \( \Delta \) with \((X, \Delta)\) klt (log Fano).

2.2. Toric Geometry. We follow [CLS11]. Let \( X \) be a toric variety with acting torus \( T \). As usual, by \( M \) and \( N \) we denote the dual lattices of characters and one-parameter subgroups of \( T \), respectively. Then \( X = X_\Sigma \) for some polyhedral fan \( \Sigma \) in \( N_\mathbb{Q} \). Every ray \( \rho \) of \( \Sigma \) is associated with a \( T \)-invariant prime divisor \( D_\rho \), and these generate the group of \( T \)-invariant Weil divisors. We denote the primitive ray generators by \( u_\rho \). Elements \( m \in M_\mathbb{Q} \) define \( T \)-invariant \( \mathbb{Q} \)-principal divisors \( D_m \) in the following way:

\[
D_m = \sum_{\rho \in \Sigma} -\langle m, u_\rho \rangle D_\rho.
\]

Since the Picard group of affine toric varieties \( X_\Sigma \) is trivial, consequently Cartier divisors on \( X_\Sigma \) are just given by collections \((m_\sigma)_{\sigma \in \Sigma}\) such that \( \langle m_\sigma, u_\rho \rangle = \langle m_\tau, u_\rho \rangle \) whenever \( \rho \) is a common ray of \( \sigma \) and \( \tau \) [CLS11, Thm. 4.2.8]. Obviously, it suffices to specify \( m_\sigma \) for the maximal cones of \( \Sigma \). The situation gets even simpler if we consider ample divisors. For those, the \( m_\sigma \) are pairwise distinct and form the vertices of a convex polytope \( P_D \subseteq M_\mathbb{Q} \) [CLS11, Cor. 6.1.16]. Moreover, the normal fan of \( P_D \) is \( \Sigma \) and the vertices of the dual polytope \( P_D^\vee \) are supported on the rays of \( \Sigma \). If we denote such a vertex supported on \( \rho \) by \( v_\rho \), then the value \( a_\rho \) of \( D \) at \( D_\rho \) is the rational number satisfying \( a_\rho v_\rho = u_\rho \). In particular, \( P_D^\vee \) is a lattice polytope if and only if the \( a_\rho \) are of the form \( 1/k \) with \( k \in \mathbb{Z} \).

A \( T \)-invariant canonical divisor on a toric variety is given by \( K_{X_\Sigma} = -\sum_{\rho \in \Sigma} a_\rho D_\rho \). So for a boundary \( \Delta = \sum_{\rho \in \Sigma} a_\rho D_\rho \), the vertices of the polytope \( P_{-(K_X + \Delta)} \) are given by \( v_\rho = \frac{1}{1-a_\rho} u_\rho \).

We remark that often in toric geometry, people work with the polytope \( P_D \subseteq M_\mathbb{Q} \). We will work with \( P_D^\vee \subseteq N_\mathbb{Q} \) instead. Both approaches are of course equivalent, but while all \( P_D \) for a fixed toric variety \( X \) satisfy the condition that the facet normals do not depend on \( D \), for \( P_D^\vee \) this means that the vertices always stay on the same ray of \( \Sigma \), which for us is more convenient. However, in order to achieve \( K \)-polystability, we need the barycenter of \( P_D \subseteq M_\mathbb{Q} \) to lie at the origin.
3. Proofs of the main statements

Proof of Theorem 1. Let $q$ be the lowest common multiple of the numerators of the coefficients $a_\rho$ of $-(K_X + \Delta)$. Stretching the polytope $P'_{(K_X + \Delta)}$ by $q$ yields an ample divisor $L$ if $-(K_X + \Delta)$ was ample. It also doesn’t change the barycenter. Moreover, the coefficients $l_\rho = a_\rho/q$ of $L$ are of the form $1/b_\rho$, where $b_\rho \in \mathbb{N}$. Thus we can write

$$L = \sum_{\rho \in \Sigma} \frac{1}{b_\rho} D_\rho = -(K_X + \Delta'),$$

where the coefficients of $\Delta'$ are $(1 - 1/b_\rho)$. So the claim is proven. 

The following is an easy and useful observation in the case of simplices, that we haven’t found in the literature.

Lemma 3.1. Let $P \subseteq \mathbb{Q}^n$ be a simplex. Then $b_p = 0$ if and only if $b_{P^\vee} = 0$.

Proof. Since $(P^\vee)^\vee = P$, we only have to prove that $b_{P^\vee} = 0$ if $b_p = 0$. So, assuming $b_p = 0$ and applying a change of basis, we are in the situation that the vertices of $P$ are $a_i = e_i$ for $1 \leq i \leq n$ (where $(e_i)_i$ is the standard basis), and $a_{n+1} = -\sum e_i$.

The facets of $P^\vee$ are given by the $n + 1$ hyperplanes $\{x_i = -1\}$ $(1 \leq i \leq n)$ and $\{\sum x_i = 1\}$. Thus in the dual basis $(e^i)_i$, the vertices of $P^\vee$ are given by

$$b_k := ne^k + \sum_{k \neq i=1}^n -e^i \quad \text{for} \quad 1 \leq k \leq n, \quad \text{and} \quad b_{k+1} := \sum_{i=1}^n -e^i.$$

Thus $(n + 1)b_{P^\vee} = \sum_{k=1}^{n+1} b_k = 0$ and the claim is proven. 

Proof of Theorem 2. Let $X = X_\Sigma$ be a complete $\mathbb{Q}$-factorial toric variety of Picard number one. The fan $\Sigma$ has $n + 1$ ray generators $u_1, \ldots, u_{n+1} \subseteq N$, such that any $n$-tuple of them is linearly independent over $\mathbb{Q}$ and $u_1, \ldots, u_{n+1}$ generate $N_{\mathbb{Q}}$ as a cone. For any tuple $b := (b_i)_{1 \leq i \leq n+1}$ of strictly positive rational numbers, the simplex $P_b = \text{conv}(b_1u_1, \ldots, b_{n+1}u_{n+1})$ has the $b_iu_i$ as its vertices. By Lemma 3.1, if the barycenter of $P_b$ lies at the origin, then the barycenter of its dual does so. The barycenter of $P_b$ is given by

$$\frac{1}{n+1} \sum_{i=1}^{n+1} b_iu_i.$$

Since the $u_i$ generate $N_{\mathbb{Q}}$ as a cone, in particular $-u_{n+1}$ lies in the interior of cone$(u_1, \ldots, u_n)$. Thus we can find $b_1, \ldots, b_n \in \mathbb{Q}_{>0}$ and $b_{n+1} = 1$, such that $\sum_{i=1}^n b_iu_i = -b_{n+1}u_{n+1}$. Stretching the polytope $P_b$ such that all $b_i \geq 1$, it yields a KE-boundary $0 \leq \Delta < 1$ via $P_b' = P'_{(K_X + \Delta)}$.

Then, the existence of a $\Delta'$ with standard coefficients follows from Theorem 1. 

Proof of Theorem 3. For the 16 given polygons, it can be easily verified that stretching the vertices with the respective factors, the barycenter of the dual polygon moves to the origin. In most cases, we found the stretching factors by combining Maple’s solve function with brute force searching for integer solutions. However, depending mainly on the Picard number, we used different strategies, which we explain in the below.

- The case of Picard number one.

This is of course a special case of Theorem 2. In particular, for a Fano simplex $P \subseteq \mathbb{Q}^2$ with vertices $v_1, v_2, v_3$, we only have to solve the system of linear diophantine equations

$$x_1v_{11} + x_2v_{21} + x_3v_{31} = 0, \quad x_1v_{12} + x_2v_{22} + x_3v_{32} = 0,$$

in order to get a toric boundary with standard coefficients.
• The case of Picard number two.

For this class of del Pezzo surfaces, we exemplarily investigate the weighted blowup $X$ of $\mathbb{P}_2$ with one $A_1$-singularity. The fan of $X$ has ray generators

$$u_1 := (0, 1), \quad u_2 := (1, 0), \quad u_3 := (-1, -1), \quad u_4 := (-1, 1).$$

Consider the polytope $P := P_{a_1, a_2, a_3, a_4}$ with vertices $a_i u_i$ for $a_i \in \mathbb{Q}$. The dual polytope $P^\vee$ is the intersection of the half planes:

$$H_1 = \left\{ x \geq -\frac{1}{a_1} \right\}, \quad H_2 = \left\{ y \geq -\frac{1}{a_2} \right\}, \quad H_3 = \left\{ y \leq -x + \frac{1}{a_3} \right\}, \quad H_4 = \left\{ y \geq x - \frac{1}{a_4} \right\}.$$

From this, we compute the vertices of $P^\vee$ to be:

$$v_1 = \left( -\frac{1}{a_2 + 1/a_3} \right), \quad v_2 = \left( \frac{2a_3}{a_3} + \frac{1}{a_4} \right), \quad v_3 = \left( \frac{1}{a_1} + \frac{1}{a_4} \right), \quad v_4 = \left( -\frac{1}{a_2} \right).$$

We compute the barycenter $b_{P^\vee}$ of $P^\vee$ as the weighted difference of the barycenters of the triangles $\Delta_1 := \text{conv}(v_1, v_4, (1/a_1 + 1/a_3, -1/a_1)^t)$ and $\Delta_2 := \text{conv}(v_2, v_3, (1/a_1 + 1/a_3, -1/a_1)^t)$, which yields:

$$6\text{vol}(P^\vee) b_{P^\vee} = \left( \frac{1}{a_4} + \frac{1}{a_1} \right) \left( \frac{1}{a_1} + \frac{1}{a_3} + \frac{1}{a_2} \right) \left( \frac{1}{a_2} + \frac{1}{a_3} - \frac{2}{a_4} \right)$$

$$- \left( \frac{1}{2a_3} - \frac{1}{2a_4} + \frac{1}{a_1} \right) \left( \frac{1}{a_1} + \frac{1}{a_3} + \frac{1}{a_4} \right) \left( -\frac{2}{a_1} + \frac{3}{2a_3} - \frac{1}{2a_4} \right).$$

Setting the right hand side equal to zero yields a system of two homogeneous polynomial equations of degree four in the variables $a_1, \ldots, a_4$, which we aim to solve over $\mathbb{Q}_{>0}$ (yielding a solution over $\mathbb{N}$ by scaling). Furthermore, since the $v_i$ are required to be the vertices of their convex hull, we have the constraints

$$2a_3 a_4 + a_1 a_4 > a_1 a_3, \quad a_1 a_4 + a_1 a_2 > a_2 a_4.$$

The strategy here and in the other cases of Picard number two was to let $a_1$ and $a_2$ run through the natural numbers, solve for $a_3$ and $a_4$ and check if they are rational and if the above constraints are fulfilled. E.g., in the present case, for $a_1 = 3$ and $a_2 = 5$, we got $a_3 = \frac{77}{4}$ and $a_4 = \frac{41}{4}$, and all checks were positive. Scaling by $23 \cdot 31$ yields the boundary with standard coefficients from Theorem 3. In other cases, we had to go to much higher values for $a_1$ and $a_2$.

• The case of Picard number three.

There are three surfaces of this type, and all of them are mirror symmetric. This suggests to try to reduce the number of variables from five to three by attaching the same stretching factors $a_i$ to mirror pairs of vertices. Then we proceed similarly as in the case of Picard number two. This strategy was successful in all cases, which can be seen from the symmetry of the stretching factors in Theorem 3.

• The case of Picard number four.

There is only one surface of this type, which is already $K$-polystable.

Proof of Theorem 4. The equivalence of (1) and (2) follows from [BB13, Thm. 1.2] and Lemma 3.1.

Now assume that (2) holds, i.e. the barycenter of $P := \text{conv}(m_\rho u_\rho)$ is zero, where $\Delta = \sum(1 - 1/m_\rho)D_\rho$ and $u_\rho$ are the primitive lattice generators of $\Sigma_X$. Choose some numbering $\rho_1, \ldots, \rho_n$ of the columns of $\Sigma_X$. Then the matrix with columns $m_\rho u_{\rho_1}, \ldots, m_\rho u_{\rho_n}$ yields a lattice homomorphism (and a vector space isomorphism) which - since $b_p = 0$ - maps the cones of the fan $\Sigma_{2^n}$ to the cones of $\Sigma_X$ and thus by [CLS11,
Thm. 3.3.4] yields a toric morphism $\mathbb{P}^n \to X$. This morphism ramifies over $D_\rho$ exactly with order $m_\rho$ and thus due to (the log version of) [CLS11, Thm. 12.1.10] corresponds to the orbifold universal cover of $(X, \Delta)$. So (3) follows from (2).

Again by [CLS11, Thm. 12.1.10], since $\pi_1^{\text{orb}}(X_{\text{reg}}, \Delta) = N/N(\Sigma, \Delta)$, this group is a quotient of $\text{Cl}(X, \Delta)$ by some subgroup $H$, such that $\tilde{X}_\Delta/H \times X_\Delta$ is the orbifold universal cover of $(X, \Delta)$. Now this group is isomorphic to $\mathbb{Z}$ and acts with weights $(1, \ldots, 1)$ if and only if $\tilde{X}_\Delta = (\mathbb{P}^n, \emptyset)$. So (3) and (4) are equivalent.

Finally assume that (3) holds. Then the covering $\mathbb{P} \to X$ is toric and again by [CLS11, Thm. 3.3.4] yields a lattice homomorphism (and a vector space isomorphism) mapping the cones of $\Sigma_{\text{reg}}$ to the cones of $\Sigma_X$. This homomorphism maps the barycenter of $P^\rho_{\text{reg}}$ - which is the origin - to the barycenter of $P^\rho_{-(K_X+\Delta)}$ - which therefore is the origin. Thus (2) follows from (3) and the claim is proven.

Proof of Theorem 5. We first treat the surface case. So let $X$ be a complete toric surface with corresponding fan $\Sigma$. Since $\dim X = 2$, the fan is uniquely determined by the rays $\rho$. Let $C_x \subseteq N_\mathbb{R}$ be a disc with diameter 1 and center $x$. We denote by $\partial C_x$ the corresponding circle.

Denote by $u_{\rho, x}$ the intersection point of the ray $\rho$ with the circle $C_x$. As long as the origin is inside $C_x$, the $u_{\rho, x}$ are the (distinct) vertices of a convex polytope $P_\rho$. However, they are not rational in general. We denote by $b_x$ the barycenter of $P^\rho_{\text{reg}}$. We can choose a small disc $C'$ centered at the origin, such that $\{b_x \mid x \in \partial C'\}$ is a (continuous) curve around the origin.

Due to continuity of the barycenter, the set $\{b_x \mid x \in C'\}$ contains the origin, say $0 = b_y$. If we enlarge $P_y$ such that $r_\rho u_{\rho, y} = u_y$ holds with $r_\rho \geq 1$ (where by $u_{\rho, y}$ we still denote the vertex of the enlarged polytope lying on $\rho$), then it gives us an $\mathbb{R}$-boundary $0 \leq \Delta_\mathbb{R} < 1$ with the desired properties.

We come to the $\mathbb{Q}$-factorial Gorenstein case in arbitrary dimension. The claim here follows from a continuity argument as well. We start with the polytope $P := P^\rho_{-(K_X+\Delta)} = \text{conv}(u_{\rho, \rho})$, where as usual the $u_\rho$ are the ray generators. Since $P^\rho = P_{-K_X}$ is a convex polytope, it’s barycenter $c_\rho$ lies in it’s interior.

For a tuple $b := (b_\rho)_{\rho \in \Sigma}$ of rational numbers, we denote $P_b := \text{conv}(b_\rho u_\rho)$. Since $X$ is $\mathbb{Q}$-factorial, for every $\sigma \in \Sigma$ and every $v \in \sigma$, there is a unique representation $v = \sum_{\rho \in \sigma} a_{v, \rho} u_\rho$. Setting

$$b(\sigma, v) := (b_\rho)_{\rho \in \Sigma} \text{ with } b_\rho = \begin{cases} 1 + a_{v, \rho} & \text{if } \rho \in \sigma \\ 1 & \text{else,} \end{cases}$$

we see that since $X$ was Gorenstein, $P_{b(\sigma, v)}$ has vertices $b_\rho u_\rho$. The mapping $v \mapsto c_{P^\rho_{b(\sigma, v)}}$ from $\sigma$ to $M_\mathbb{R}$ is continuous and it’s image is a ‘deformed cone’ with vertex $c_{P^\rho}$. We can define a map on the whole of $N_\mathbb{R}$ by

$$f: N_\mathbb{R} \to M_\mathbb{R}; \sigma \ni v \mapsto c_{P^\rho_{b(\sigma, v)}},$$

which is surjective. Thus there exists $\sigma \in \Sigma$ and $v \in \sigma$, such that $c_{P^\rho_{b(\sigma, v)}} = 0$. By construction, $P_{b(\sigma, v)}$ has vertices $b_\rho u_\rho$ and thus defines an $\mathbb{R}$-boundary $\Delta_\mathbb{R}$ via $P^\rho_{-(K_X+\Delta)} = P_{b(\sigma, v)}$. □

References

[BB13] Robert J. Berman and Bo Berndtsson. Real Monge-Ampère equations and Kähler-Ricci solitons on toric log Fano varieties. Ann. Fac. Sci. Toulouse Math. (6), 22(4):649–711, 2013.

[Ber16] Robert J. Berman. K-polystability of Q-Fano varieties admitting Kähler-Einstein metrics. Invent. Math., 203(3):973–1025, 2016.

[BL22] Harold Blum and Yuchen Liu. Openness of uniform K-stability in families of Q-Fano varieties. Ann. Sci. Éc. Norm. Supér. (4), 55(1):1–41, 2022.

[BM21] Lukas Braun and Joaquín Moraga. Iteration of cox rings of klt singularities, 2021.
[CDS15a] Xiuxiong Chen, Simon Donaldson, and Song Sun. Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities. *J. Amer. Math. Soc.*, 28(1):183–197, 2015.

[CDS15b] Xiuxiong Chen, Simon Donaldson, and Song Sun. Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than $2\pi$. *J. Amer. Math. Soc.*, 28(1):199–234, 2015.

[CDS15c] Xiuxiong Chen, Simon Donaldson, and Song Sun. Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches $2\pi$ and completion of the main proof. *J. Amer. Math. Soc.*, 28(1):235–278, 2015.

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[Don12] S. K. Donaldson. Kähler metrics with cone singularities along a divisor. In *Essays in mathematics and its applications*, pages 49–79. Springer, Heidelberg, 2012.

[DS16] Ved Datar and Gábor Székelyhidi. Kähler-Einstein metrics along the smooth continuity method. *Geom. Funct. Anal.*, 26(4):975–1010, 2016.

[LXZ22] Yuchen Liu, Chenyang Xu, and Ziquan Zhuang. Finite generation for valuations computing stability thresholds and applications to K-stability. *Ann. of Math. (2)*, 196(2):507–566, 2022.

[Szé13] Gábor Székelyhidi. A remark on conical Kähler-Einstein metrics. *Math. Res. Lett.*, 20(3):581–590, 2013.

[Tia15] Gang Tian. K-stability and Kähler-Einstein metrics. *Comm. Pure Appl. Math.*, 68(7):1085–1156, 2015.

[WZ04] Xu-Jia Wang and Xiaohua Zhu. Kähler-Ricci solitons on toric manifolds with positive first Chern class. *Adv. Math.*, 188(1):87–103, 2004.

[Zhu21] Ziquan Zhuang. Optimal destabilizing centers and equivariant K-stability. *Invent. Math.*, 226(1):195–223, 2021.