Solution of the Dicke model for \(N = 3\)

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Abstract
The \(N = 3\) Dicke model couples three qubits to a single radiation mode via dipole interaction and constitutes the simplest quantum-optical system allowing for Greenberger–Horne–Zeilinger states. In contrast to the case \(N = 1\) (the Rabi model), it is non-integrable if the counter-rotating terms are included. The spectrum is determined analytically, employing the singularity structure of an associated differential equation. While quasi-exact eigenstates known from the Rabi model do not exist, a novel type of spectral degeneracy becomes possible which is not associated with a symmetry of the system.

(Some figures may appear in colour only in the online journal)

1. Introduction

The simplest model to describe light–matter interaction is the Rabi model, in which a two-level system (two states of a single atom in the early applications) interacts with a single mode of the radiation field, the latter being treated classically [1]. A seminal step in the analysis of the fully quantized model has been taken by Jaynes and Cummings who employed the ‘rotating-wave approximation’ (RWA) which is valid close to resonance and for coupling strengths small compared to the Rabi model, in which a two-level system (two states of a single atom in the early applications) interacts with a single mode of the radiation field, the latter being treated classically [1]. A seminal step in the analysis of the fully quantized model has been taken by Jaynes and Cummings who employed the ‘rotating-wave approximation’ (RWA) which is valid close to resonance and for coupling strengths small compared to the transition frequency [2, 3]. The ensuing model can be solved analytically in a very simple way, because the RWA introduces a strong continuous symmetry [4], rendering it superintegrable [5]. A natural generalization of the quantum Rabi model is the Dicke model, in which the radiation mode couples simultaneously to \(N\) two-level systems (qubits) [6]. It was first studied in the limit of large \(N\), because it exhibits a phase transition to a ‘super-radiant’ state for strong coupling [7–10]. Although the transition cannot be observed within a simple experimental setup [11], it can be simulated by an ensemble of laser-driven atoms coupled to a cavity mode [12] or within circuit QED [13–15]. The proposal based on cavity QED [12] has been already experimentally realized using a Bose–Einstein condensate [16].

While these developments concern the Dicke model for large \(N\), applications to quantum information technology have renewed the interest in the case of small \(N\) [17–22]. The model with three qubits allows in principle the dynamical generation of Greenberger–Horne–Zeilinger states [23] which could be of importance for future applications e.g. in quantum cryptography [22]. Possible realizations of the \(N = 3\) Dicke model within circuit QED will be able to explore the strong coupling region [24] where the RWA is not feasible and one has to consider the full model. The \(U(1)\)-symmetry induced by the RWA is so powerful that the Dicke model\(^1\) becomes integrable for arbitrary \(N\) [25], while the model including counter-rotating terms is non-integrable for all \(N \geq 2\) according to the criterion introduced in [5]. The case \(N = 1\) is the only one where Schweder’s technique [26, 27] or operator methods [28, 29] are applicable, therefore we shall employ in the following the method based on analysis of the associated differential equation in the complex domain [5].

The Dicke model for \(N = 3\) is described by the Hamiltonian (\(\hbar = 1\)),

\[
H_D = \omega a^\dagger a + \frac{\omega_0}{2} \sum_{i=1}^3 \sigma_i^z + \frac{g'}{\sqrt{3}} (a + a^\dagger) \sum_{i=1}^3 \sigma_i^x.
\]

(1)

Here \(\omega\) denotes the frequency of the radiation mode, \(a (a^\dagger)\) is its annihilation (creation) operator, \(\omega_0\) is the energy splitting of the three qubits, described by Pauli matrices \(\sigma_i^z\), which are coupled through a dipole term with strength \(g'\) to the field. Because the qubits are equivalent, \(H_D\) is rotationally invariant, leading to a splitting of the eight-dimensional spin-space into irreducible components, according to

\[
\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2}.
\]

(2)

The \(N = 3\) Dicke model is equivalent to two Rabi models and a system with spin \(S = 3/2\). We shall confine ourselves

\(^1\) Properly called ‘Tavis–Cummings model’ if the RWA is used.
in the following to the $S = 3/2$ model with four-dimensional spin-space. The Hamiltonian reads with $\omega = 1$, $\Delta = \omega_0/2$ and $g = g'/\sqrt{3}$,

$$H_D = a^\dagger a + 2\Delta \hat{J}_z + 2g(a + a^\dagger)\hat{J}_z,$$

(3)

and $\hat{J}_x$ and $\hat{J}_y$ are generators of SU(2) in the spin-3/2 representation. $H_D$ possesses a $\mathbb{Z}_2$-symmetry (parity), $\hat{P} = e^{i\pi a^\dagger a} \otimes \hat{R}$, with the involution $\hat{R}$ acting in spin-space as $\hat{R}\hat{J}_x\hat{R} = \hat{J}_y$, $\hat{R}\hat{J}_y\hat{R} = -\hat{J}_x$. We have $PH_D\hat{P} = H_D$. The total Hilbert space $L^2(\mathbb{R}) \otimes \mathbb{C}^4$ splits into two invariant subspaces (parity chains) labelled by the eigenvalues $\pm 1$ of $\hat{P}$ [24]. This discrete symmetry is familiar from the Rabi model and renders it integrable because $\hat{P}$ has as many irreducible representations as the dimension of the state space of the (single) qubit [5]. The same symmetry is present in the $S = 3/2$ Dicke model. Nevertheless, the symmetry leads to a considerable simplification of the analytical solution.

2. The spectrum

The operator $\hat{R}$ becomes a simple reflection after transformation of $H_D$ into ‘spin-boson’ form, using a $SO(4)$-transformation $OH_D\hat{O} = H_{\text{ab}}$, with

$$H_{\text{ab}} = a^\dagger a + 2\Delta \hat{J}_z + 2g(a + a^\dagger)\hat{J}_z.$$  

(4)

As in the Rabi model, the parity invariance can be used to partially diagonalize $H_{\text{ab}}$. Define $\hat{T} = e^{i\pi a^\dagger a}$ and

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & \hat{T} & 0 & -\hat{T} \\ 0 & -\hat{T} & 0 & \hat{T} \end{pmatrix}.$$  

(5)

Then $U^\dagger H_{\text{ab}}U = H_+ + H_-$, where $H_\pm$ acts in $\mathcal{H}_\pm$; $\mathcal{H}_+$ and $\mathcal{H}_-$ are the two mutually orthogonal subspaces with fixed parity. We have

$$H_\pm = a^\dagger a + \Delta \begin{pmatrix} 0 & \sqrt{3} \\ \sqrt{3} & \pm 2\hat{T} \end{pmatrix} - g(a + a^\dagger) \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}. $$  

(6)

We shall now represent the continuous degree of freedom in the Bargmann space $\mathcal{B}$, spanned by analytic functions $f(z)$ [30]; $\mathcal{H}_\pm$ is isomorphic to $\mathcal{B} \cong \mathbb{C}^2$. The operator $\hat{T}$ acts on elements of $\mathcal{B}$ as $(\hat{T} f)(z) = f(-z)$. The eigenvalue equation $H_\pm \psi = E\psi$ takes with $\psi = (\phi_1(z), \phi_2(z))^T$ the form of a non-local system of linear ordinary differential equations in the complex domain,

$$z\frac{d}{dz} \phi_1(z) + \sqrt{3}\Delta \phi_2(x) - 3gz\phi_1(z) - 3g\frac{d}{dz} \phi_1(z) = E\phi_1(z),$$

(7)

$$z\frac{d}{dz} \phi_2(z) + \sqrt{3}\Delta \phi_1(z) - gz\phi_2(z) - g\frac{d}{dz} \phi_2(z) + 2\Delta \phi_2(-z) = E\phi_2(z).$$

(8)

With the definitions $\tilde{\phi}_j(z) = \phi_j(-z)$, $j = 1, 2$ and denoting the derivative with a prime, we obtain the following local system of the first order,

$$(z - 3g)\phi'_1 = (E + 3gz)\phi_1 - \sqrt{3}\Delta \phi_2,$$

(9)

$$(z - g)\phi'_2 = (E + gz)\phi_2 - \sqrt{3}\Delta \phi_1 - 2\Delta \phi_2,$$

(10)

$$(z + 3g)\tilde{\phi}'_1 = (E - 3gz)\tilde{\phi}_1 - \sqrt{3}\Delta \tilde{\phi}_2,$$

(11)

$$(z + g)\tilde{\phi}'_2 = (E - gz)\tilde{\phi}_2 - \sqrt{3}\Delta \tilde{\phi}_1 - 2\Delta \tilde{\phi}_2.$$  

(12)

The system (9)–(12) has four regular singular points at $z = \pm g, \pm 3g$, and an irregular singular point at infinity [31]. The latter has rank 1 as in the Rabi model [32], therefore a solution $\psi_0$ of (7), (8) is an eigenvector of $H_\pm$ with eigenvalue $E$ if and only if $\phi_1(z)$ and $\phi_2(z)$ are analytic in the whole complex plane [30]. The indicial analysis of (9)–(12) shows that the exponents at the points $z = \pm 3g$ are 0 and $E + 9g^2$, whereas at $z = \pm g$ they are $E + g^2$. The exponent zero is three-fold degenerate at all regular singularities. This is the major difference to the Rabi model, which has only two regular singular points at $\pm g$ and the exponent zero is non-degenerate at each of these points. Formal solutions of (9)–(12) in terms of power series in $z - z_j$ are possible in regions $D_j$, with $z_j = jq$, $j = \pm 1, \pm 3$, see figure 1. The expansion around $z_1 = g$ reads for $j = 1, 2$,

$$\phi_j(z) = \sum_{n=0}^{\infty} a_{j,n}(z - g)^n, \quad \tilde{\phi}_j(z) = \sum_{n=0}^{\infty} \tilde{a}_{j,n}(z - g)^n.$$  

(13)

The expansion (13) is absolutely convergent in $D_1$, with radius of convergence $2g$ and $\phi_j(z), \tilde{\phi}_j(z)$ are analytic at $z_j$. Likewise, there is an expansion of $\psi_j(z), \tilde{\psi}_j(z)$ around $z_j$, which we denote as $\psi_j(z), \tilde{\psi}_j(z)$,

$$\psi_j(z) = \sum_{n=0}^{\infty} a_{j,n}(z - 3g)^n, \quad \tilde{\psi}_j(z) = \sum_{n=0}^{\infty} \tilde{a}_{j,n}(z - 3g)^n.$$  

(14)

The series (14) converges in $D_3$ and $\psi_j(z), \tilde{\psi}_j(z)$ are analytic at $z_j$. Using the identification $\phi_j(z) = \phi_j(-z)$ and $\psi_j(z) = \psi_j(-z)$, (13) and (14) furnish series expansions of $\phi_j(z)$ in regions $D_{-1}$ and $D_{-3}$, respectively. These series lead for arbitrary $E$ to functions which are analytic at their expansion.
points but develop branch-cuts at the other singular points. The discrete set of eigenvalues \(E_n\), \(n = 0, 1, 2, \ldots\) is determined by the condition that all four expansions describe the same function \(\phi_j(z)\), i.e.
that they are analytic continuations of each other [32].

Inserting the ansatz (13) into (9)–(12) yields the following coupled recurrence relations for \(n \geq 0\) and with \(x = E + g^2\),
\[
2g(n + 1)(x - (n + 1))a_{2,n+1} = (3\Delta^2 - n^2 + 2n - 2g^2 - T^2 - 2g^2)a_{2,n} + g(1 - 4x - 2g^2 + 4n - 4)\bar{a}_{2,n-1} - 3g^2\bar{a}_{2,n-2} + 4\Delta g(n + 1)\bar{a}_{2,n-1} + 2\Delta(x + 2g^2 - n)\bar{a}_{2,n} + 6\Delta g\bar{a}_{2,n-1},
\]
and
\[
8g^2(n + 2)(n + 1)\bar{a}_{2,n+2} = -2g(n + 1)(3n + 2 - 3x + 8g^2)\bar{a}_{2,n+1} + (3\Delta^2 - n^2 + 2n - 16g^2) - 4g^2 - (x - 2g^2)(x - 4g^2)\bar{a}_{2,n} + (4x - 10g^2 - 3 - 4n)\bar{a}_{2,n-1} - 3g^2\bar{a}_{2,n-2} - 8\Delta g(n + 1)\bar{a}_{2,n+1} + 2\Delta(x + 4g^2 - n)\bar{a}_{2,n} - 6\Delta g\bar{a}_{2,n-1}.
\]

The coefficients \(a_{1,n}, \bar{a}_{1,n}\) read in terms of the \(a_{2,n}, \bar{a}_{2,n}\) as
\[
a_{1,n} = -\frac{1}{\sqrt{3\Delta}}[(n - x)\bar{a}_{2,n} - g\bar{a}_{2,n-1} + 2\Delta\bar{a}_{2,n}],
\]
\[
\bar{a}_{1,n} = -\frac{1}{\sqrt{3\Delta}}[(n - x + 2g^2)\bar{a}_{2,n} + g\bar{a}_{2,n-1} + 2\Delta\bar{a}_{2,n}].
\]

The system (15), (16) cannot be reduced to a linear three-term recurrence relation, therefore continued-fraction techniques are not applicable to the present model [33]. Moreover, the relations (15)–(18) are not sufficient to determine the eigenfunctions uniquely because there are three linear independent solutions of (15), (16) which are analytic at \(z_1\). The initial conditions for (15), (16) are given by the set \(\{a_{2,0}, \bar{a}_{2,0}, \bar{a}_{2,1}\}\) and the three solutions \(\{\phi_j^{(k)}(z), \bar{\phi}_j^{(k)}(z)\}\) of (9)–(12) analytic at \(z = g\) are obtained by setting one element of the set to 1 and the others to 0.

The general solution of this type is therefore given as
\[
\phi_j(z; \{\gamma_k\}) = \sum_{k=1}^{3}\gamma_k\phi_j^{(k)}(z),
\]
\[
\bar{\phi}_j(z; \{\gamma_k\}) = \sum_{k=1}^{3}\gamma_k\bar{\phi}_j^{(k)}(z),
\]
with \(\gamma_k\) to be determined. In the same way, we obtain for the \(\psi_j(z), \bar{\psi}_j(z)\) the recurrences,
\[
2g(n + 1)(n - x - 8g^2)a_{2,n+1} = (3\Delta^2 - n^2 + 2(2x + 16g^2) - (x + 8g^2)(x + 2g^2))a_{2,n} + g(4n - 3 - 4x - 14g^2)a_{2,n-1} - 3g^2a_{2,n-2} + 2\Delta(x + 8g^2 - n)\bar{a}_{2,n} + 6\Delta g\bar{a}_{2,n-1}.
\]
and
\[
2g^2(n + 2)(n + 1)\bar{a}_{2,n+2} = 2g(n + 1)(5x - 5n - 3 - 32g^2)\bar{a}_{2,n+1} + (3\Delta^2 - n^2 + 2n(2x - 32g^2)
\]
\[
- 6g^2 - (x - 10g^2)(x - 4g^2)\bar{a}_{2,n} + g(4x - 4n + 3 - 22g^2)\bar{a}_{2,n-1} - 3g^2\bar{a}_{2,n-2} - 12\Delta g(n + 1)\bar{a}_{2,n+1} + 2\Delta(x - 10g^2 - n)\bar{a}_{2,n} - 6\Delta g\bar{a}_{2,n-1}.
\]

For the \(a_{1,n}, \bar{a}_{1,n}\) we have,
\[
a_{1,n} = -\frac{1}{\sqrt{3\Delta}}[(n - x - 2g^2)a_{2,n} - g\bar{a}_{2,n-1} + 2g(n + 1)a_{2,n+1} + 2\Delta\bar{a}_{2,n}],
\]
\[
\bar{a}_{1,n} = -\frac{1}{\sqrt{3\Delta}}[(n - x + 4g^2)\bar{a}_{2,n} + g\bar{a}_{2,n-1} + 4g(n + 1)\bar{a}_{2,n+1} + 2\Delta\bar{a}_{2,n}].
\]

The solution space of (9)–(12) analytic at \(z = g\) is likewise three-dimensional, determined by the initial set \(\{a_{2,0}, \bar{a}_{2,0}, \bar{a}_{2,1}\}\) and the general solution reads
\[
\psi_j(z; \{c_k\}) = \sum_{k=1}^{3}c_k\psi_j^{(k)}(z),
\]
\[
\bar{\psi}_j(z; \{c_k\}) = \sum_{k=1}^{3}c_k\bar{\psi}_j^{(k)}(z),
\]
with three unknown constants \(c_k\). Equations (19) and (24) are the most general solutions analytic at \(g\) and \(3g\), respectively, if the spectral parameter \(x\) is not a positive integer (see (15)) or satisfies \(x + 8g^2 = n, n = 0, 1, 2, \ldots\) (20). These special values for \(x\) determine the two types of baselines in the model where the exceptional spectrum is located (the Rabi model has only one type of baseline). We shall now determine the condition under which all four sets of series expansions for \(\phi_j(z), \bar{\phi}_j(z)\) describe the same functions, which are therefore analytic in the whole complex plane and correspond to an eigenvector \(\{(\phi_j, \bar{\phi}_j)^T\}\) of \(H_z\).

Because the functions \(\{(\phi_j(z), \bar{\phi}_j(z), \bar{\phi}_j(z), \bar{\phi}_j(z))\}\) satisfy the same differential equation of the first order as \(\{(\psi_j(z), \bar{\psi}_j(z), \bar{\psi}_j(z), \bar{\psi}_j(z))\}\), both sets will coincide in \(D_2\) if they coincide at one or more points \(z_0 \in D_2\). This yields four equations for the functions in (19) and (24). Furthermore, \(\{(\phi_j(z), \bar{\phi}_j(z), \bar{\phi}_j(z), \bar{\phi}_j(z))\}\) and \(\{(\bar{\phi}_j(z), \bar{\phi}_j(z), \bar{\phi}_j(z), \bar{\phi}_j(z))\}\) satisfy the same differential equation and coincide in all of \(D_0 = D_1 \cap D_2\) if they do so at a point \(z_0 \in D_0\). Obviously, only two of the four equations are independent if \(z_0 = 0\). \((\phi_j(z), \bar{\phi}_j(z))\) \((\phi_j(z), \bar{\phi}_j(z))\) is then the analytic continuation of \((\phi_j(z), \bar{\phi}_j(z))\) into the disc \(D_{-3}\). But because \(\phi_j(z) = \bar{\psi}_j(z)\) for \(z \in D_2\), it follows that \(\bar{\psi}_j(z)\) is the analytic continuation of \(\phi_j(z)\) (and therefore of \(\phi_j(z)\)) into the disc \(D_{-3}\). The six equations
\[
\phi_j(z_0) = \psi_j(z_0), \quad \bar{\phi}_j(z_0) = \bar{\psi}_j(z_0), \quad \phi_j(0) = \bar{\phi}_j(0),
\]
for \(j = 1, 2\) and \(z_0 \in D_2\) are equivalent to the analyticity of \(\phi_j(z)\) in \(C\). A non-trivial solution of (25) can be found if the parameters \(\{\gamma_k, c_k\}\) are not all zero. The functions \(\phi_j(z), \bar{\phi}_j(z), \ldots\) depend parametrically on the energy
Define then the matrix $M_+ (x, z_0)$ as

$$
M_+ \begin{pmatrix}
\psi_1^{(1)}(z_0) & \psi_2^{(1)}(z_0) & \psi_1^{(2)}(z_0) & -\phi_1^{(1)}(z_0) & -\phi_2^{(1)}(z_0) & -\phi_1^{(2)}(z_0) & -\phi_2^{(2)}(z_0) \\
\psi_1^{(1)}(z_0) & \psi_2^{(1)}(z_0) & \psi_1^{(2)}(z_0) & -\phi_1^{(1)}(z_0) & -\phi_2^{(1)}(z_0) & -\phi_1^{(2)}(z_0) & -\phi_2^{(2)}(z_0) \\
\psi_1^{(1)}(z_0) & \psi_2^{(1)}(z_0) & \psi_1^{(2)}(z_0) & -\phi_1^{(1)}(z_0) & -\phi_2^{(1)}(z_0) & -\phi_1^{(2)}(z_0) & -\phi_2^{(2)}(z_0) \\
0 & 0 & 0 & \delta \phi_1^{(1)}(0) & \delta \phi_2^{(1)}(0) & \delta \phi_1^{(2)}(0) & \delta \phi_2^{(2)}(0) \\
0 & 0 & 0 & \delta \phi_1^{(1)}(0) & \delta \phi_2^{(1)}(0) & \delta \phi_1^{(2)}(0) & \delta \phi_2^{(2)}(0) \\
\end{pmatrix}
$$

(26)

with $\delta \phi_j^{(k)}(0) = \tilde{\phi}_j^{(k)}(0) - \phi_j^{(k)}(0)$. It follows that the $G$-function of the Dicke model for positive parity,

$$
G^D_+(x, z_0) = \det M_+(x, z_0),
$$

(27)

is zero for all $z_0 \in D_2$, if and only if $x = E + g^2$ corresponds to an element of the spectrum of $H_+$. The discrete set of zeros $x_n$ with $G^D_+(x_n, z_0) = 0$ for $n = 0, 1, 2, \ldots$ determines the regular spectrum $\sigma_+(H_+) = \{x_n - g^2\}_{n \in \mathbb{N}_0}$ of $H_+$ [5]. The regular spectrum of $H_-$ is given in an analogous manner, starting from recurrences (15), (16), (20), (21) by reversing the sign of the coupling terms between $\alpha_n$ and $\bar{\alpha}_n$, resp. $\alpha_n$ and $\bar{\alpha}_n$ (see (6)), constructing functions $\phi_j^{(k)}(x, z)$, $\tilde{\phi}_j^{(k)}(x, z)$, $\bar{\phi}_j^{(k)}(x, z)$, $\phi_{\pm}(x, z)$, and the matrix $M_-(x, z_0)$. Figure 2 shows $G^D_+(x, 2g)$ for $g = 0.25$ and $\Delta = 0.7$.

The functions $\phi_j^{(k)}(x, z)$, $\tilde{\phi}_j^{(k)}(x, z)$, $\bar{\phi}_j^{(k)}(x, z)$ entering $M_-(x, z_0)$ are not related in a simple manner to their counterparts $\phi_j^{(k)}(x, z)$, $\tilde{\phi}_j^{(k)}(x, z)$, $\bar{\phi}_j^{(k)}(x, z)$ for positive parity, in contrast to the case of the Rabi model. There one has only one pair of functions $\{\phi_{\pm}(x, z), \phi_{\pm}(x, z)\}$ and the $G$-function of the Rabi model reads simply

$$
G^R_+(x) = \tilde{\phi}_+(x, 0) - \phi_+(x, 0).
$$

(28)

Moreover, $\phi_{\pm}(x, z) = \phi_{\pm}(x, z)$. It follows at once that the regular spectrum of the Rabi model is not degenerate between states of different parity, as $G^R_+(x) = G^R_-(x) = 0$ implies $\tilde{\phi}_+(x, z) \equiv 0$. On the other hand, regular states with the same parity cannot be degenerate because the formal solution $\phi_{\pm}(x, z)$, analytic at $z = g$, is unique for $x \notin \mathbb{N}$. The only possibility for degenerate eigenvalues in the Rabi model occurs therefore in the exceptional spectrum, where $x \in \mathbb{N}$. The two degenerate states at these points have different parity because there are only two linear independent formal solutions of the eigenvalue equation and the pole of $G^R_+(x)$ at integer $x$ is lifted in both $G^D_+(x)$ and $G^D_- (x)$, which corresponds to a condition satisfied by the model parameters $g$ and $\Delta$ [34, 5].

As mentioned above, $G^D_+(x, z_0)$ has two different types of baselines, located at $x = n, n \in \mathbb{N}$ (first kind) and $x = -8g^2, n \in \mathbb{N}_0$ (second kind). Although $G^D_+(x, z_0)$ has pole singularities in $x$ at these values for general $g, \Delta$, there may be (exceptional) eigenvalues $E_2^1 = n - g^2$, resp. $E_2^- = n - 9g^2$, if the singularity is lifted in $G^D_+(x, z_0)$ or $G^D_-(x, z_0)$. This exceptional solution, however, is usually non-degenerate, because there is no single lifting condition (as in the Rabi model), valid for both parities.

Therefore, a quasi-exact spectrum in the sense of the Rabi model is not present in the Dicke model. If one defines the quasi-exact spectrum differently, by demanding that the eigenfunctions are polynomial in $z$ (apart from a common factor), this possibility is not ruled out in principle, although the set $[n, g, \Delta]$ would then have to satisfy three consistency equations [34], making a solution with integer $n$ unlikely. In fact, as these solutions lie necessarily on baselines and are parity degenerate, the consistency equations given by Kuik and Lewenstein must comprise the two independent lifting conditions for $G^D_+(x, z_0)$. On the other hand, $G^D_+(x, z_0) = G^D_-(x, z_0)$ is no longer tantamount to vanishing of the wave-functions $\phi_{\pm}(x, z)$ themselves, therefore regular eigenvalues of the Dicke model may well be parity degenerate. Figures 3 and 4 show the

\[2\] Non-degenerate exceptional solutions appear in the Rabi model as well, but then the pole is not lifted in $\phi_{\pm}(x, 0)$ but only in the difference, either $\phi_{\pm}(x, 0) - \phi_{\pm}(x, 0)$ or $\phi_{\pm}(x, 0) - \phi_{\pm}(x, 0)$. These solutions are exceptional but not quasi-exact.
Dicke and Rabi spectra for fixed $\Delta$ and varying $g$. It is apparent that the coupling between exceptional eigenvalues and degeneracies renders the Rabi spectrum much more ‘regular’ than the Dicke spectrum, apart from the complication coming from two kinds of baselines in the latter.

All regular eigenvalues of the Rabi model with fixed parity correspond to unique eigenfunctions because the exponent 0 of the indicial equation at each singular point is non-degenerate [31, 34] and there exists at most one solution analytic at $z = \pm g$. In contrast, we have three solutions analytic at each of the $z_j$ in the case of the $N = 3$ model. Although generally there is only one solution analytic at all four singular points, the possibility is not excluded that the kernel of $M(x, z_0; g, \Delta)$ has dimension $\geq 1$ at some value $x$, which would correspond to a degeneracy within a given parity chain $H_{\alpha g}$. Indeed, for this it is necessary that the linear term in the characteristic polynomial of $M(x, z_0; g, \Delta)$ vanishes, providing a condition to be satisfied by the model parameters $g$ and $\Delta$, in analogy to the equation determining the quasi-exact spectrum of the Rabi model [34]. On has to note here that this equation is not independent from $\det M(x, z_0; g, \Delta) = 0$, because the value of $x$ is not restricted to integers. Both equations form a coupled system to determine the triple $(x_{\text{deg}}, R_{\text{deg}}, \Delta_{\text{deg}})$ where two states with equal parity are degenerate. If such a triple exists, a level crossing at $E = x_{\text{deg}}^2 - R_{\text{deg}}$ will appear within the corresponding parity chain $H_{\alpha g}$. These degeneracies would not originate from a global symmetry of the model and thus are not accidental, because the degenerate states do not belong to dynamically decoupled subspaces. The numerical investigations done so far have shown no hint to this novel type of degeneracy yet. However, in a recent work on the $N = 2$ model with inequivalent qubits, Chilingaryan and Rodriguez-Lara have discovered level crossings within spectra with fixed parity [35]. This is a strong indication that the phenomenon predicted here for the $N = 3$ model is not forbidden by some special feature of the matrix $M(x, z_0; g, \Delta)$.

3. Conclusions

We have computed the spectrum of the Dicke model for three qubits analytically using the technique based on formal solutions in the Bargmann space, where the spectral condition corresponds to analyticity in the whole complex plane [30, 26, 5]. In contrast to the $N = 1$ model, the argument used by Schweber to derive a continued-fraction representation of the spectral condition is not applicable, because the formal solutions are not given in terms of linear three-term recurrence relations [33]. If one defines the model on a truncated Hilbert space, matrix-valued continued fractions could be employed in principle [36], but this approach suffers from ambiguities and can be justified only in the case $N = 1$, by its formal equivalence to Schweber's method [37].

The Dicke model shows much more spectral ‘irregularities’ than the Rabi model, whose spectral graph is restricted by the position of the quasi-exact eigenvalues, confining degeneracies to the baselines. The $N = 3$ model possesses two types of baselines, but they govern only the asymptotics for strong coupling (figure 3); the quasi-exact spectrum does not exist and levels corresponding to different parity intersect within the regular spectrum, while the exceptional spectrum is non-degenerate. On the other hand, the structure of the $G$-function for $N = 3$ predicts degeneracies within the parity chains, which are forbidden for $N = 1$. These degeneracies are not related to a symmetry of the model and do not imply integrability. They cannot be termed ‘accidental’ either, as the degenerate states do not belong to different invariant subspaces. However, if they appear in sufficient number, which seems to be possible for large $N$, it could lead to a level statistics resembling Poissonian behaviour, which has been found numerically for $N \geq 20$ in the coupling region below the quantum phase transition [38]. The implications of this novel type of degeneracy for the notion of (non-)integrability in systems with less than two continuous degrees of freedom [5] have yet to be explored.

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