Ground State Pressure and Energy Density of a Homogeneous Bose Gas in Two Dimensions

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We consider an interacting homogeneous Bose gas at zero temperature in two spatial dimensions. The properties of the system can be calculated as an expansion in powers of $g$, where $g$ is the coupling constant. We calculate the ground state pressure and the ground state energy density to second order in the quantum loop expansion. The renormalization group is used to sum up leading and subleading logarithms from all orders in perturbation theory. In the dilute limit, the renormalization group improved pressure and energy density are expansions in powers of the $T$-matrix.

I. INTRODUCTION

The remarkable achievement of Bose-Einstein condensation (BEC) of alkali atoms in harmonic traps [1–3] has created an enormous interest in the properties of dilute Bose gases (For a review, see e.g. Ref. 5 and references therein).

The homogeneous Bose gas in three dimensions has been studied in great detail over the past 50 years (For a review, see e.g. Ref. 3). At zero temperature, the quantum loop expansion is essentially an expansion in powers of $\sqrt{\rho a_0^2}$, where $a_0$ is the two-body $S$-wave scattering length and $\rho$ is the density. Lee and Yang [6] were the first to calculate the leading quantum correction to the energy density. Part of the second quantum correction to the energy density was obtained by Wu, by Hugenholz and Pines, and by Sawada [7]. Only recently a complete two-loop result has been obtained by Braaten and Nieto [8]. The result depends, in addition to the scattering length, also on the scattering amplitude for $3 \to 3$ scattering.

The homogeneous Bose gas in two dimensions has also been studied extensively. The chemical potential and ground state energy density of a two-dimensional homogeneous Bose gas were first calculated by Schick [9]. By summing up the ladder diagrams contributing to the chemical potential, he was able to show that, in the dilute limit, the relevant expansion parameter is not the coupling constant $g$, but rather the effective interaction $[\log(\rho a^2)]^{-1}$, where $\rho$ is the density and $a$ is the range of the interaction. The expansion parameter is the two-body $T$-matrix, and Schick determined the leading-order results in this expansion. Corrections to these results have been considered by several authors [10–16]. Very recently, a formal proof of the result by Schick was given by Lieb and Yngvason [17–18].

In the present paper, we reconsider the homogeneous Bose gas in two dimensions at zero temperature. We calculate the pressure and energy density of the ground state to second order in the quantum loop expansion. We also apply the renormalization group to sum up leading and subleading logarithms from all orders of perturbation theory. In the dilute limit, the renormalization group improved pressure and energy density are essentially expansions in powers of the $T$-matrix.

The outline of the paper is as follows. In Sec. II, we briefly discuss the perturbative framework developed in Ref. 8 to calculate the ground state properties of a homogeneous Bose gas. In Sec. III, we calculate the ground state pressure to two loop order. In Sec. IV, we calculate the ground state energy density through two loops. In Sec. V, we apply the renormalization group to sum up leading and subleading logarithms from all orders in perturbation theory. Finally, we summarize our results in Sec. VI. Calculational details are included in an appendix.

II. PERTURBATIVE FRAMEWORK

In this section, we discuss the perturbative framework set up in Ref. 8 to calculate the effects on the ground state from quantum fluctuations around the mean field.

The action is

$$ S = \int dt \left\{ \int d^2x \left[ i \hbar \frac{\partial \psi^*(x,t)}{\partial t} + \frac{\hbar^2}{2m} \nabla^2 + \mu \right] \psi(x,t) - \frac{1}{2} \int d^2x \int d^2x' \left[ \psi^*(x,t) \psi^*(x',t) V_0(x-x') \times \psi(x,t) \psi(x',t) \right] \right\}. \tag{1} $$

$\psi^*(x,t)$ is a complex field operator that creates a boson at the position $x$, $\mu$ is the chemical potential, and $V_0(x)$ is the two-body potential. In the following, we set $\hbar = 2m = 1$. Factors of $\hbar$ and $2m$ can be reinserted using dimensional analysis.
The action Eq. (4) has ultraviolet divergences that must be removed by adding a counterterm \( \delta \rho \) estimated by dimensional regularization, an arbitrary renormalization scale \( \mu \) can be identified with the physical one. Alternatively, one can eliminate the divergences associated with the ultraviolet divergence in the expression for the density \( \rho \). This divergence can be removed by adding a counterterm \( \delta \rho \). Alternatively, one can eliminate the divergences associated with \( \mu \) and \( \rho \) by a normal-ordering prescription of the fields in Eq. (4). The coupling constant is renormalized \( \mu \) and \( g \) are low enough, the scattering amplitude can be approximated by local interactions. If the energies \( \omega \) are high enough, the scattering amplitude can be approximated by local interactions.

If the interatomic potential \( V_0(\mathbf{x}) \) is short range, it can be mimicked by local interactions. If the energies are low enough, the scattering amplitude can be approximated by \( s \)-wave scattering and the action Eq. (4) can be decomposed into three terms

\[
S = \int dt \int d^2 x \psi^* \left( \frac{i}{\partial_t} + \nabla^2 + \mu \right) \psi - \frac{1}{2} g (\psi^* \psi)^2 . \tag{4}
\]

Here, \( g \) is a coupling constant that must be tuned to reproduce some low-energy observable of the true potential \( V_0(\mathbf{x}) \).

The quantum field theory defined by the action Eq. (4) has ultraviolet divergences that must be removed by renormalization of \( \mu \) and \( g \). There is also an ultraviolet divergence in the expression for the density \( \rho \). This divergence can be removed by adding a counterterm \( \delta \rho \). Alternatively, one can eliminate the divergences associated with \( \mu \) and \( \rho \) by a normal-ordering prescription of the fields in Eq. (4). The coupling constant is renormalized \( \mu \) in the usual way by replacing the bare coupling with the physical one.

If we use a simple momentum cutoff \( M \) to cut off the ultraviolet divergences in the loop integrals, there will be terms proportional to \( M^2 \), where \( p \) is a positive integer. There are also terms that are proportional to \( \log M \). The coefficients of the power divergences depend on the regularization method and are therefore artifacts of the regulator. On the other hand, the coefficients of \( \log(M) \) are independent of the regulator and they therefore represent real physics. In this paper, we use dimensional regularization to regulate both infrared and ultraviolet divergences. In dimensional regularization, one calculates the loop integrals in \( d = 2 - 2\epsilon \) dimensions for values of \( \epsilon \) where the integrals converge. One then analytically continues back to \( d = 2 \) dimensions. With dimensional regularization, an arbitrary renormalization scale \( M \) is introduced. This scale can be identified with the simple momentum cutoff mentioned above. An advantage of dimensional regularization is that it automatically sets power divergences to zero, while logarithmic divergences show up as poles in \( \epsilon \). In two dimensions, the one-loop counterterms for the chemical potential \( \mu \) and the density \( \rho \) are quadratic ultraviolet divergences, while the one-loop counterterm for the coupling constant \( g \) is a logarithmic ultraviolet divergence. At the two-loop level, the counterterms for the chemical potential and the density are also quadratic divergences. The counterterm for the coupling constant is a double logarithmic divergence.

We next parameterize the quantum field \( \psi \) in terms of a time-independent condensate \( v \) and a quantum fluctuating field \( \tilde{\psi} \):

\[
\psi = v + \tilde{\psi} . \tag{5}
\]

The fluctuating field can be written in terms of two real fields:

\[
\tilde{\psi} = \frac{1}{\sqrt{2}} (\psi_1 + i \psi_2) . \tag{6}
\]

Substituting Eq. (4) into Eq. (5), the action can be decomposed into three terms

\[
S[\psi] = S[v] + S_{\text{free}}[\psi_1, \psi_2] + S_{\text{int}}[v, \psi_1, \psi_2] . \tag{7}
\]

\( S[v] \) is the classical action

\[
S[v] = \int dt \int d^2 x \left[ \mu v^2 - \frac{1}{2} g v^4 \right] . \tag{8}
\]

while the free part of the action is

\[
S_{\text{free}}[\psi_1, \psi_2] = \int dt \int d^2 x \left[ \frac{1}{2} \left( \dot{\psi}_1 \psi_2 - \psi_1 \dot{\psi}_2 \right) + \frac{1}{2} \psi_1 (\nabla^2 + X) \psi_1 + \frac{1}{2} \psi_2 (\nabla^2 + Y) \psi_2 \right] . \tag{9}
\]

The interaction part of the action is

\[
S_{\text{int}}[v, \psi_1, \psi_2] = \int dt \int d^2 x \left[ \sqrt{2} T \psi_1 + \frac{1}{\sqrt{2}} Z \psi_1 (\psi_1^2 + \psi_2^2) - \frac{1}{8} g (\psi_1^2 + \psi_2^2)^2 \right] . \tag{10}
\]

The sources in Eq. (10) are

\[
T = [\mu - g v^2] v , \tag{11}
\]

\[
X = [\mu - 3 g v^2] , \tag{12}
\]

\[
Y = [\mu - g v^2] , \tag{13}
\]

\[
Z = - g v . \tag{14}
\]

The propagator that corresponds to the free action \( S_{\text{free}}[\psi_1, \psi_2] \) in Eq. (4) is

\[
D(\omega, p) = \frac{i}{\omega^2 - \omega^2(p) + i\epsilon} \begin{pmatrix} p^2 - Y & -i\omega \\ i\omega & p^2 - X \end{pmatrix} . \tag{15}
\]
Here \( p \) is the wavevector, \( \omega \) is the frequency, and \( \epsilon(p) \) is the dispersion relation:
\[
\epsilon(p) = \sqrt{(p^2 - X)(p^2 - Y)} .
\] (16)

The value of the condensate \( v_0 \) that minimizes the classical action is given by the equation \( T = 0 \). Both the propagator Eq. (15) and the dispersion relation Eq. (16) greatly simplify at \( v_0 \), since \( Y = 0 \) here. The dispersion relation then becomes gapless, which reflects the spontaneous breakdown of the \( U(1) \)-symmetry (Goldstone's theorem). The dispersion relation is linear for small wavevectors and is quadratic for large wavevectors, which is that of a free nonrelativistic particle. The propagator is defined with an \( ice \) prescription in the usual way. The diagonal parts of the propagator are denoted by a solid and a dashed line, respectively. The off-diagonal parts are denoted by a solid-dashed and dashed-solid line, respectively.

The partition function \( Z \) can be expressed as a path integral
\[
Z = \int \mathcal{D}\psi_1 \mathcal{D}\psi_2 e^{iS[\psi_1, \psi_2]} .
\] (17)

All the thermodynamic observables can be derived from \( Z \). For instance, the free energy density \( \mathcal{F} \) is given by
\[
\mathcal{F}(\mu) = i \log \frac{Z}{VT} ,
\] (18)

where \( VT \) is the spacetime volume of the system.

The density \( \rho \) is given by the expectation value \( \langle \psi^\dagger \psi \rangle \) in the ground state. It can therefore be expressed as
\[
\rho(\mu) = -\frac{\partial \mathcal{F}(\mu)}{\partial \mu} .
\] (19)

The energy density \( \mathcal{E} \) is given by the Legendre transform of the free energy density
\[
\mathcal{E}(\rho) = \mathcal{F}(\mu) + \rho \mu .
\] (20)

At this point it is convenient to introduce the thermodynamic potential \( \Omega(\mu, v) \). The thermodynamic potential is given by all one-particle irreducible vacuum graphs and can be expanded in the number of loops
\[
\Omega(\mu, v) = \Omega_0(\mu, v) + \Omega_1(\mu, v) + \Omega_2(\mu, v) + \ldots ,
\] (21)

where the subscript \( n \) indicates the contribution from the \( n \)th order in the loop expansion. The free energy \( \mathcal{F} \) is given by all connected vacuum graphs and is independent of the condensate \( v \). If we evaluate \( \Omega \) at a value of the condensate that satisfies the condition
\[
\bar{v} = \langle \psi \rangle ,
\] (22)

it can be shown that all the one-particle reducible graphs (those that can be disconnected by cutting a single line) vanish. We then have
\[
\mathcal{F}(\mu) = \Omega_0(\mu, \bar{v}) + \Omega_1(\mu, \bar{v}) + \Omega_2(\mu, \bar{v}) + \ldots .
\] (23)

The condition Eq. (22) is equivalent to
\[
\frac{\partial \Omega}{\partial v} = 0 .
\] (24)

The loop expansion Eq. (23) does not coincide with the expansion in powers quantum corrections. To obtain the expansion in powers of quantum correction, we must expand the condensate \( \bar{v} \) about the classical minimum \( v_0 \):
\[
\bar{v} = v_0 + v_1 + v_2 + \ldots .
\] (25)

By substituting Eq. (21) into Eq. (24), we obtain the first quantum correction \( v_1 \) to the classical minimum:
\[
v_1 = -\left. \frac{\partial \Omega_1(\mu, v)}{\partial v} \right|_{v=v_0} \frac{\partial^2 \Omega_0(\mu, v)}{\partial v^2} \bigg|_{v=v_0} .
\] (26)

This first quantum correction to the free energy density is
\[
\mathcal{F}_1(\mu) = \Omega_1(\mu, v_0) ,
\] (27)

and the second quantum correction to the free energy density is
\[
\mathcal{F}_2(\mu) = \Omega_2(\mu, v_0) + v_1 \left. \frac{\partial \Omega_1(\mu, v)}{\partial v} \right|_{v=v_0} \left. + \frac{1}{2} v_1^2 \frac{\partial^2 \Omega_0(\mu, v)}{\partial v^2} \right|_{v=v_0} .
\] (28)

### III. PRESSURE TO TWO LOOPS

In this section, we calculate the pressure as a function of the chemical potential \( \mu \) and the renormalized coupling \( g \) to two loops.

#### A. Mean-field free energy

The thermodynamic potential in the mean-field approximation is
\[
\Omega_0(\mu, v) = -\mu v^2 + \frac{1}{2} g v^4 .
\] (29)

The mean-field free energy is given by the classical thermodynamic potential evaluated at the classical minimum \( v_0 \):
\[
\mathcal{F}_0(\mu) = \frac{\mu^2}{2g} .
\] (30)
B. One-loop free energy

The one-loop contribution to the free energy is

$$\mathcal{F}_1(\mu) = \mathcal{F}_{1a}(\mu) + \Delta_1 \mathcal{F},$$  \hspace{1cm} (31)$$

where

$$\mathcal{F}_{1a}(\mu) = \frac{i}{2} \int \frac{d\omega}{2\pi} \int \frac{d^d p}{(2\pi)^d} \log \det D^{-1},$$ \hspace{1cm} (32)

and $\Delta_1 \mathcal{F}$ is the one-loop counterterm

$$\Delta_1 \mathcal{F} = -\frac{\mu^2}{g} \Delta_1 \mu + \frac{\mu^2}{2g^2} \Delta_1 g.$$ \hspace{1cm} (33)

The propagator $D(\omega, p)$ is evaluated at the classical minimum, where $Y = 0$. By integrating over $\omega$, Eq. (32) becomes

$$\mathcal{F}_{1a}(\mu) = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \epsilon(p) \left[ 1 - L - \frac{1}{2} + O(\epsilon) \right],$$ \hspace{1cm} (34)

where $I_{m,n}$ is defined in the appendix. Using Eq. (34) in the appendix, we obtain

$$\mathcal{F}_{1a}(\mu) = -\frac{\mu^2}{16\pi} \left[ 1 - L - \frac{1}{2} + O(\epsilon) \right],$$ \hspace{1cm} (35)

where $L = \log (\mu/2M^2)$. The counterterms at one-loop are

$$\Delta_1 \mu = 0,$$ \hspace{1cm} (36)

$$\Delta_1 g = \frac{g^2}{8\pi \epsilon}.$$ \hspace{1cm} (37)

Adding Eqs. (35) and (37), we obtain the one-loop contribution to the free energy

$$\mathcal{F}_1(\mu) = -\frac{\mu^2}{16\pi} \left[ L + \frac{1}{2} \right].$$ \hspace{1cm} (38)

Note that $\mathcal{F}_1(\mu)$ is independent of the coupling constant $g$. Adding Eqs. (30) and (39), we obtain the one-loop approximation to the free energy:

$$\mathcal{F}_{0+1}(\mu) = -\frac{\mu^2}{g} + \frac{\mu^2}{16\pi} \left[ L + \frac{1}{2} \right],$$ \hspace{1cm} (39)

where the coupling constant is evaluated at the scale $M$, $g = g(M)$.

C. Two-loop free energy

The two-loop contribution to the free energy $\mathcal{F}$ is given by the one-particle irreducible potential $\Omega_2(\mu, v)$ is given by the one-loop counterterm $\Delta \mathcal{F}$ is

$$\Delta \mathcal{F} = -\frac{\mu^2}{g} \Delta_1 \mu + \frac{\mu^2}{2g^2} \Delta_1 g.$$ \hspace{1cm} (33)

The two-loop contribution to the thermodynamic potential $\Omega_2(\mu, v)$ is given by

$$\Omega_2(\mu, v) = \frac{g^2}{8} J + \frac{g}{16} \left[ 3I_{1,1}^2 + 2I_{-1,-1}I_{1,1} + 3I_{-1,-1}^2 \right],$$ \hspace{1cm} (40)

where

$$J = 6J_{0,0,1} - J_{-1,-1,1} - 3J_{1,1,1} - 2J_{-1,0,0},$$ \hspace{1cm} (41)

and integrals $I_{l,m,n}$ are defined in the appendix. The first correction $v_1$ is given by Eq. (20):

$$v_1 = -\frac{g\mu}{8\pi} \left[ 3I_{1,1} + I_{-1,-1} \right].$$ \hspace{1cm} (42)

The two-loop contribution to the free energy is then

$$\mathcal{F}_2(\mu) = \frac{g^2}{8} J + \frac{g}{16} \left[ I_{1,1}^2 - 2I_{-1,-1}I_{1,1} + 3I_{1,1}^2 \right] + \Delta_2 \mathcal{F}.$$ \hspace{1cm} (43)

The counterterm $\Delta_2 \mathcal{F}$ is

$$\Delta_2 \mathcal{F} = \frac{1}{2} I_{1,1} \Delta_1 \mu - \frac{\mu}{g} \Delta_2 \mu + \frac{\mu^2}{2g^2} \Delta_2 g - \frac{1}{2g} (\Delta_1 \mu)^2 + \frac{\mu}{g^2} \Delta_1 \mu \Delta_1 g - \frac{\mu^2}{2g^2} \Delta_1 g^2.$$ \hspace{1cm} (44)

At the two-loop level, the counterterms are given by

$$\Delta_2 \mu = 0,$$ \hspace{1cm} (45)

$$\Delta_2 g = \frac{g^3}{64\pi^2 \epsilon^2}.$$ \hspace{1cm} (46)

The integrals $I_{l,m,n}$ are ultraviolet divergent. After subtracting the divergent part, the remainder must be calculated numerically. We evaluate the expression in the appendix. The result is

$$J = -2I_{0,1} \left[ I_{1,1} + I_{-1,-1} \right] + J_{\text{num}},$$ \hspace{1cm} (47)
where \( J_{\text{num}} = -3.52 \times 10^{-5} \mu \). The final result for the two-loop contribution to the free energy is

\[
\mathcal{F}_2(\mu) = -\frac{g\mu^2}{64\pi^2} [1 + C].
\]  

(48)

Here \( C = 2.78 \times 10^{-3} \). Adding Eqs. (30), (39), and (48), we obtain our final result for the free energy to second order in the quantum loop expansion

\[
\mathcal{F}_{0+1+2}(\mu) = -\frac{\mu^2}{2g} + \frac{\mu^2}{16\pi} \left[ L + \frac{1}{2} \right] - \frac{g\mu^2}{64\pi^2} [1 + C].
\]  

(49)

### D. Pressure to two loops

The pressure \( \mathcal{P} \) is given by \(-\mathcal{F}\). The pressure through two loops is given by minus the sum of Eqs. (30), (39) and (48):

\[
\mathcal{P}_{0+1+2}(\mu) = \frac{\mu^2}{2g} - \frac{\mu^2}{16\pi} \left[ L + \frac{1}{2} \right] + \frac{g\mu^2}{64\pi^2} [1 + C].
\]  

(50)

The coupling constant \( M \frac{dM}{dM} g = \beta(g)\),

\[
\text{where the } \beta\text{-function is a polynomial in } g. \text{ Normally, the } \beta\text{-functions are known only up to a certain order in the quantum loop expansion. In the present case, the one-loop result for the } \beta\text{-function is exact and } \beta(g) = \frac{g^2}{4\pi} [13]. \text{ From Eq. (51), one can easily check that our result Eq. (50) for the two-loop pressure is independent of the scale } M \text{ up to correction of order } g^2.
\]

### IV. ENERGY DENSITY TO TWO LOOPS

In this section, we derive the energy density \( E \) as a function of the density \( \rho \) and the renormalized coupling \( g \) to two loops.

#### A. Mean field energy density

Using Eqs. (13) and (14), the density in the mean-field approximation is

\[
\rho_0(\mu) = \frac{\mu}{g}.
\]  

(52)

The chemical potential is obtained by inverting Eq. (52):

\[
\mu_0(\rho) = g\rho.
\]  

(53)

Using Eqs. (14) and (52), the energy density in the mean-field approximation is given by

\[
E_0(\rho) = \frac{1}{2} g\rho^2.
\]  

(54)

#### B. One-loop energy density

Using Eqs. (15) and (16), we obtain the density in the one-loop approximation

\[
\rho_{0+1}(\mu) = \frac{\mu}{g} - \frac{\mu}{8\pi} \left[ L + \frac{1}{2} \right].
\]  

(55)

Inverting Eq. (55) to obtain \( \mu \) as a function of \( \rho \), one finds

\[
\mu_{0+1}(\rho) = g\rho + \frac{g^2\rho^2}{8\pi} \left[ L + \frac{1}{2} \right].
\]  

(56)

where \( L = \log(g/2M^2) \) and \( g = g(M) \). Using Eqs. (19), (58) and (59), the energy density in the one-loop approximation becomes

\[
E_{0+1}(\rho) = \frac{1}{2} g\rho^2 + \frac{g^2\rho^2}{16\pi} \left[ L + \frac{1}{2} \right].
\]  

(57)

This agrees with the result obtained by Lozano [20], and by Haugset and Ravndal [21].

#### C. Two-loop energy density

Using Eqs. (19) and (20), we obtain the density in the two-loop approximation:

\[
\rho_{0+1+2}(\mu) = \frac{\mu}{g} - \frac{\mu}{32\pi^2} \left[ L + \frac{1}{2} \right].
\]  

(58)

Inverting Eq. (58), we obtain the two-loop expression for the chemical potential:

\[
\mu_{0+1+2}(\rho) = g\rho + \frac{g^2\rho^2}{32\pi^2} \left[ L + \frac{1}{2} \right]
\]  

\[
+ \frac{g^3\rho^3}{64\pi^2} \left[ L^2 + 3L - 2C \right].
\]  

(59)

The final result for the two-loop energy density then becomes

\[
E_{0+1+2}(\rho) = \frac{1}{2} g\rho^2 + \frac{g^2\rho^2}{16\pi} \left[ L + \frac{1}{2} \right]
\]  

\[
+ \frac{g^3\rho^3}{128\pi^2} \left[ L^2 + 2L - 1 - 2C \right].
\]  

(60)

Using the renormalization group equation (51) for the running coupling constant, we see that the two-loop results for the density, chemical potential, and energy density are independent of the renormalization scale \( M \) up to corrections of order \( g^4 \).
V. RENORMALIZATION GROUP

In Secs. III and IV, we showed that the results for the pressure, density, chemical potential, and energy density are independent of the arbitrary renormalization scale \( M \) that is introduced with dimensional regularization.

The two-loop results for the pressure and density include a logarithm \( L \). The renormalization group can be used to absorb this logarithm. The solution to the renormalization group equation (51) is

\[
g(M) = \frac{g(M)}{1 - \frac{g(M)}{\alpha \pi} \log \left( \frac{M_\infty^2}{M^2} \right)},
\]

where \( M_\infty \equiv g(M_0) \). In the same way, we can sum leading (of the form \( g^{n+1} \bar{L}^n \), where \( n = 1, 2, 3 \ldots \)) and subleading logs (of the form \( g^{n+m} \bar{L}^m \), where \( n = 1, 2, 3 \ldots \) and \( m = 2, 3 \ldots \)) from all orders of perturbation theory in the chemical potential and energy density by using a running coupling constant \( g(M_0) \) and choosing the renormalization scale \( M_0 = \sqrt{\mu/2} \). These terms are generated by expanding out Eq. (61) in powers of \( L \). The result is

\[
\rho(\mu) = \frac{\mu}{g_0} - \frac{\mu}{32 \pi} + \frac{g_0 \mu}{64 \pi^2} [1 + C] \quad \text{and} \quad \mu(\rho) = \frac{\mu}{2g_0} - \frac{\mu}{32 \pi} + \frac{g_0 \mu}{64 \pi^2} [1 + C],
\]

where \( g_0 \equiv g(M_0) \). Essentially, these two-body \( T \)-matrix and so this is the relevant expansion parameter. By substituting these expressions for the coupling constant into to the expressions for the pressure, density, chemical potential, and energy density, we obtain the renormalization group improved results

\[
\mathcal{P}(\mu) = -\frac{\mu^2 L}{16 \pi} \left[ 1 + \frac{1}{2} L^{-1} + 2(1 + C) L^{-2} \right],
\]

\[
\rho(\mu) = -\frac{\mu L}{8 \pi} \left[ 1 + L^{-1} + 2(1 + C) L^{-2} \right],
\]

\[
\mu(\rho) = -\frac{8 \pi \rho}{L} \left[ 1 + (\log[-\bar{L}/8 \pi] - 1) \bar{L}^{-1} + \log^2(\log[-\bar{L}/8 \pi]) \bar{L}^{-2} - 3 \log[-\bar{L}/8 \pi]) \bar{L}^{-2} - 2 C \bar{L}^{-2} \right],
\]

\[
\mathcal{E}(\rho) = -\frac{4 \pi \rho^2}{L} \left[ 1 + \frac{1}{2} (2 \log[-\bar{L}/8 \pi] - 1) \bar{L}^{-1} + \log^2(\log[-\bar{L}/8 \pi]) \bar{L}^{-2} - 2 \log[-\bar{L}/8 \pi]) \bar{L}^{-2} - (1 + 2 C) \bar{L}^{-2} \right],
\]

where where \( L = [\log(\mu a^2/2)] \) and \( \bar{L} = [\log(\rho a^2/2)] \). Eqs. (61)-(65) are the main result of the present paper. The leading order results were first obtained by Schick [11, 12], while the leading corrections have been calculated in Refs. [13, 14]. The last terms are new result.

VI. SUMMARY

In the present paper, we have studied a two-dimensional interacting homogeneous Bose gas at zero temperature. We have calculated the ground state pressure and energy density to second order in the quantum loop expansion. The results are independent of the arbitrary renormalization scale \( M \).

We have applied the renormalization group to sum up leading and subleading logarithms from all orders in perturbation theory. The renormalization group improved pressure and density are expansions in \( [\log(\mu a^2/2)]^{-1} \), while the chemical potential and energy density are expansions in \( [\log(\rho a^2/2)]^{-1} \). These are essentially expansions in the two-body \( T \)-matrix, and we have obtained the ground state pressure, density, chemical potential, and energy density to next-to-next-to-leading order.

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APPENDIX A: FORMULAS

The loop integrals that appear in our calculations involve integrations over the energy $\omega$ and the spatial momentum $p$. The energy integrals are evaluated using contour integration.

The specific one-loop integral needed is
\[
\int \frac{d\omega}{2\pi} \frac{1}{|\omega^2 - \epsilon^2(p) + i\epsilon|} = -\frac{i}{2\epsilon(p)}. \tag{A.1}
\]

The specific two-loop integrals needed are
\[
\int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} \frac{1}{|\omega_1^2 - \epsilon^2(p) + i\epsilon|} \times \\
\frac{\omega_2^2 - \epsilon^2(q) + i\epsilon}{[(\omega_1 + \omega_2)^2 - \epsilon^2(r) + i\epsilon]} = \\
\frac{4\epsilon(p)\epsilon(q)\epsilon(r)(\epsilon(p) + \epsilon(q) + \epsilon(r))}{4\epsilon(r)[\epsilon(p) + \epsilon(q) + \epsilon(r)]}, \tag{A.2}
\]
\[
\int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} \frac{\omega_1\omega_2}{|\omega_1^2 - \epsilon^2(p) + i\epsilon|} \times \\
\frac{\omega_2^2 - \epsilon^2(q) + i\epsilon}{[(\omega_1 + \omega_2)^2 - \epsilon^2(r) + i\epsilon]} = \\
\frac{1}{4\epsilon(r)\epsilon(p) + \epsilon(q) + \epsilon(r)}. \tag{A.3}
\]

Here $r = |p + q|$.

Some of the one-loop momentum integrals are infrared divergent or ultraviolet divergent or both. They can be written in terms of the integral $I_{m,n}$, which is defined by
\[
I_{m,n} = \left(\frac{\epsilon^7 M^2}{4\pi}\right)^e \int \frac{d^dp}{(2\pi)^d} \frac{p^{2m}}{p^n (p^2 + \Lambda^2)^{n/2}}. \tag{A.4}
\]

Here, $M$ is a renormalization scale that ensures that $I_{m,n}$ has the canonical dimension also for $d \neq 2$. $\gamma \approx 0.5772$ is the Euler-Mascheroni constant. With dimensional regularization, $I_{m,n}$ is given by the formula
\[
I_{m,n} = \frac{\Omega_d}{(2\pi)^d} \left(\frac{\epsilon^7 M^2}{4\pi}\right)^e \frac{\Lambda^{d+2m-2n}}{\Gamma\left(d/2\right)} \Gamma\left(d/2 + m\right)\Gamma\left(n - m - \frac{d}{2}\right), \tag{A.5}
\]

where $\Omega_d = 2\pi^{d/2}/\Gamma[d/2]$ is the area of the $d$-dimensional sphere.

The integrals $I_{m,n}$ satisfy the relations
\[
\frac{d}{d\Lambda^2} I_{m,n} = -\frac{n}{2} I_{m+1,n+2}, \tag{A.6}
\]
\[
(d + 2m - n) I_{m,n} = n I_{m+2,n+2}, \tag{A.7}
\]
\[
\Lambda^2 I_{m,n} = I_{m-1,n-2} - I_{m+1,n}. \tag{A.8}
\]

The first relation follows directly from the definition of $I_{m,n}$. The second relation follows from integration by parts, while the last is simply an algebraic relation.

In two dimensions, these integrals have logarithmic and power ultraviolet divergences. The power divergences are set to zero in dimensional regularization, while the logarithmic divergences appear as poles in $\epsilon$. The specific integrals are
\[
I_{0,-1} = -\frac{\Lambda^4}{32\pi} \frac{1}{\epsilon} - L - \frac{1}{2} + \frac{1}{2} \left[ L^2 + L - \frac{5}{2} + \frac{\pi^2}{2} \right] + \mathcal{O}\left(\epsilon^2\right), \tag{A.9}
\]
\[
I_{-1,-1} = \frac{\Lambda^2}{8\pi} \frac{1}{\epsilon} - L + 1 + \frac{1}{2} \left[ (L - 1)^2 + 1 + \frac{\pi^2}{2} \right] + \mathcal{O}\left(\epsilon^2\right), \tag{A.10}
\]
\[
I_{1,1} = -\frac{\Lambda^2}{8\pi} \frac{1}{\epsilon} - L - 1 + \frac{1}{2} \left[ (L + 1)^2 - 4 + \frac{\pi^2}{2} \right] + \mathcal{O}\left(\epsilon^2\right), \tag{A.11}
\]
\[
I_{0,1} = \frac{1}{4\pi} \left[ \frac{1}{\epsilon} - L + \frac{1}{2} \right] \left[ L^2 + \frac{\pi^2}{2} \right] + \mathcal{O}\left(\epsilon^2\right), \tag{A.12}
\]
where $L = \log(\Lambda^2/4M^2)$.

The two-loop integrals needed can be expressed in terms of $J_{l,m,n}$
\[
J_{l,m,n} = \left(\frac{\epsilon^7 M^2}{4\pi}\right)^e \int \frac{d^dp}{(2\pi)^d} \frac{d^dq}{(2\pi)^d} \times \\
\frac{p^2 + \Lambda^2}{p^2 + \Lambda^2 + \sqrt{p^2 + \Lambda^2}^m} \left[ q^2 + \sqrt{q^2 + \Lambda^2}^m \right] \frac{r^2 + \sqrt{r^2 + \Lambda^2}^m}{r^2 + \sqrt{r^2 + \Lambda^2}^m}. \tag{A.13}
\]

In two dimensions, these integrals have quadratic and double logarithmic divergences that cancel in the particular combination in Eq. (43), leaving us with a logarithmically divergent integral. We write the integral as
\[
J = J_{\text{div}} + J_{\text{num}}, \tag{A.14}
\]

where the ultraviolet divergence of the integral $J$ has been isolated:
\[
J_{\text{div}} = 2 \left(\frac{\epsilon^7 M^2}{4\pi}\right)^e \int \frac{d^dp}{(2\pi)^d} \left[ \frac{2 - \frac{p^2 + \Lambda^2}{\sqrt{p^2 + \Lambda^2}}}{\sqrt{p^2 + \Lambda^2}} \right] \frac{d^dq}{(2\pi)^d} \frac{1}{q^2 + \sqrt{q^2 + \Lambda^2}^m}. \tag{A.15}
\]

The first term inside the square brackets in Eq. (A.15) has only a power divergence, and so it is set to zero in dimensional regularization. In terms of $I_{m,n}$, the remaining terms can be written as
\[
J_{\text{div}} = -2I_{0,1} \left[ I_{1,1} + I_{-1,-1} \right]. \quad (A.16)
\]

The remaining finite part of the integral \( J \) can be evaluated directly in two dimensions and reads

\[
J_{\text{num}} = \frac{d^2 p}{(2\pi)^2} \int \frac{d^2 q}{(2\pi)^2} \left\{ \frac{6p}{\sqrt{p^2 + \Lambda^2}} - \frac{2}{p} \sqrt{p^2 + \Lambda^2} \right. \\
- \frac{3pq}{\sqrt{p^2 + \Lambda^2} \sqrt{q^2 + \Lambda^2} \sqrt{q^2 + \Lambda^2}} \\
- \frac{r \sqrt{p^2 + \Lambda^2} \sqrt{q^2 + \Lambda^2}}{pq \sqrt{r^2 + \Lambda^2}} \right. \\
\left. \times \frac{1}{p \sqrt{p^2 + \Lambda^2} + q \sqrt{q^2 + \Lambda^2} + r \sqrt{r^2 + \Lambda^2}} \\
- \frac{2}{q \sqrt{q^2 + \Lambda^2}} \left[ 2 - \frac{p}{\sqrt{p^2 + \Lambda^2}} \right] \\
- \frac{\sqrt{\Lambda^2 + \Lambda^2}}{p} \right\}. \quad (A.17)
\]

Since the only scale in the integrand in Eq. (A.17) is \( \Lambda \), it follows from dimensional analysis that \( J_{\text{num}} \) is proportional to \( \Lambda^2 \). The numerical value is

\[
J_{\text{num}} = -1.76 \times 10^{-5} \Lambda^2. \quad (A.18)
\]

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