Non-Binary LDPC Codes with Large Alphabet Size

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Abstract—We study LDPC codes for the channel with input $x \in \mathbb{F}_q^m$ and output $y = x + z \in \mathbb{F}_q^m$. The aim of this paper is to evaluate decoding performance of $q^m$-ary non-binary LDPC codes for large $m$. We give density evolution and decoding performance evaluation for regular non-binary LDPC codes and spatially-coupled (SC) codes. We show the regular codes do not achieve the capacity of the channel while SC codes do.

I. INTRODUCTION

In 1963, Gallager invented low-density parity-check (LDPC) codes [1]. Due to sparsity of the code representation, LDPC codes are efficiently decoded by belief propagation (BP) decoders. By a powerful optimization method density evolution [2], developed by Richardson and Urbanke, messages of BP decoding can be statistically evaluated. The optimized LDPC codes can approach very close to Shannon limit [3].

In this paper, we consider non-binary LDPC codes over $\mathbb{F}_q^m$ defined by sparse parity-check matrices over $\mathrm{GL}(m, \mathbb{F}_q)$ Non-binary LDPC codes were invented by Gallager [1]. Davy and MacKay [4] found non-binary LDPC codes can outperform binary ones. Non-binary LDPC codes have captured much attention recently due to their decoding performance [5][6][7][8]. It is observed $2^m$-ary non-binary codes exhibit excellent decoding performance around at $m = 6$ over BMS channels.

Spatially-coupled (SC) codes attract much attention due to their capacity-achieving performance and a memory-efficient sliding-window decoding algorithm. Recently, SC codes are shown to prove achieve capacity of BEC [9], [10] and BMS channels [11].

In this paper, we study coding over the channel with input $x \in \mathbb{F}_q^m$ and output $y \in \mathbb{F}_q^m$. The receiver knows a subspace $V \subset \mathbb{F}_q^m$ from which $z = y - x$ is uniformly chosen. Or equivalently, the receiver receives an affine subspace $y - V := \{y - z \mid z \in V\}$ in which the input $x$ is compatible. This channel model is used in the decoding process for network coding scenario [15]. In [15], the data part of each packet is represented as $x \in \mathbb{F}_q^m$. Packets are coded by non-binary LDPC codes whose parity-check coefficients are in the general linear group $\mathrm{GL}(m, \mathbb{F}_q)$. The noise subspace $V$ is estimated by padding zero packets and using Gaussian elimination. We denote this channel by $\mathrm{CD}(m, \epsilon)$.

II. CHANNEL MODEL

In this paper, we consider channels with input $x \in \mathbb{F}_q^m$ and output $y = x + z \in \mathbb{F}_q^m$, where $z \in \mathbb{F}_q^m$ is uniformly distributed in a linear subspace $V \subset \mathbb{F}_q^m$ of dimension $\epsilon m$. It is easy to see that the channel is weakly symmetric [14]. From [14, Theorem 7.2.1], the normalized capacity is given by

$$C = \frac{1}{m} \max_{p(x)} I(X;Y) = (1 - \epsilon).$$

The channel with large $m$ was used in a decoding process of the network coding scenario [15]. In [15], the data part of each packet is represented as $x \in \mathbb{F}_q^m$. Packets are coded by non-binary LDPC codes whose parity-check coefficients are in the general linear group $\mathrm{GL}(m, \mathbb{F}_q)$. The noise subspace $V$ is estimated by padding zero packets and using Gaussian elimination. We denote this channel by $\mathrm{CD}(m, \epsilon)$.

III. CODE DEFINITION

In this section, we briefly review $(d_l, d_r)$ codes and $(d_l, d_r, L)$ codes introduced by Kudekar et al. [16]. We assume $\frac{d_l}{d_0} \in \mathbb{Z}$ and $\frac{d_l}{d_0} \geq 2$. Both $(d_l, d_r)$ codes and $(d_l, d_r, L)$ codes are defined over $\mathrm{GF}(q)$ and have parity-check matrix over $\mathrm{GF}(q)$.

A. $(d_l, d_r)$-Codes

Let $H(d_l, d_r)$ be an $Md_l \times Md_r$ sparse binary matrix of column weight $d_l$ and row weight $d_r$. The Tanner graph of $(d_l, d_r, L)$ code is obtained by making $M$ copies of protographs of $H(d_l, d_r, L)$ and connecting edges among the same edge types. $H(d_l, d_r, m)$ is given by replacing 1 with a randomly chosen non-zero elements in $\mathrm{GL}(m, \mathbb{F}_q)$ and replacing 0 with 0 $\in \mathrm{GL}(m, \mathbb{F}_q)$, where $\mathrm{GL}(m, \mathbb{F}_q)$ is the set of all non-singular $\mathbb{F}_q$-valued matrix of size $m \times m$. The resultant matrix $H(d_l, d_r, m)$ can be viewed as a $\mathrm{GL}(m, \mathbb{F}_q)$-valued matrix of size $Md_l \times Md_r$.

B. $(d_l, d_r, L)$-Codes

The $(d_l, d_r, L)$ codes are defined by the following protograph codes [17]. The adjacency matrix of the protograph is referred to as a base matrix. The base matrix of $(d_l, d_r, L)$ code is given as follow. Let $H(d_l, d_r, L)$ be an $(L + d_l - 1) \times \frac{d_l}{d_0} L$...
band binary matrix of band size \(d_r \times d_t\) and column weight \(d_l\), where the band size is height \(\times\) width of the band. We refer to \(L\) as coupling number. For example

\[
H(d_l = 4, d_r = 8, L = 9) = \begin{bmatrix}
\end{bmatrix}.
\]

The Tanner graph of \((d_l, d_r, L)\) code is obtained by making \(M\) copies of protographs of \(H(d_l, d_r, L)\) and connecting edges among the same edge types. The parameter \(M\) is referred to as lifting number. The matrix \(H(d_l, d_r, L, M)\) is given by replacing each 1 in \(H(d_l, d_r, L)\) with an \(M \times M\) random permutation and each 0 with an \(M \times M\) zero matrix. \(H(d_l, d_r, L, M, m)\) is given by replacing 1 with a randomly chosen non-zero elements in \(\text{GL}(m, \mathbb{F}_q)\) and replacing 0 with \(0 \in \text{GL}(m, \mathbb{F}_q)\), where \(\text{GL}(m, \mathbb{F}_q)\) is the set of all non-singular \(\mathbb{F}_q\)-valued matrix of size \(m \times m\). The resultant matrix \(H(d_l, d_r, L, M, m)\) can be viewed as a \(\text{GL}(m, \mathbb{F}_q)\)-valued matrix of size \((L + d_l - 1) M \times \frac{d_r}{d_l} LM\).

### IV. Decoding Algorithm

Let \(\mathcal{H}\) be a \(\text{GL}(m, \mathbb{F}_q)\)-valued matrix given by the construction above. Denote row and column size of \(\mathcal{H}\) by \(\mathcal{M}\) and \(\mathcal{N}\), respectively. Denote the \((i, j)\)-th entry of \(\mathcal{H}\) by \(h_{i,j} \in \text{GL}(m, \mathbb{F}_q)\). Then a codeword \((x_1, \ldots, x_N) \in (\mathbb{F}_q^m)^N\) satisfies parity-check equations

\[
\sum_{j \in \partial i} h_{i,j} x_j = 0, \quad \text{for } i = 1, \ldots, M
\]

for \(i = 1, \ldots, M\) where \(\partial i := \{ j \in \{1, \ldots, N\} \mid h_{i,j} \neq 0\}\).

Sum-product algorithm (SPA) [18] is employed to decode. Without loss of generality, we can assume all-zero codeword was sent to make analysis easier [19]. The SPA tries to marginalize the following function with respect to each \(x_j\) \((j = 1, \ldots, N)\).

\[
\prod_{j=1}^{N} \Pr(Y_j = y_j \mid X_j = x_j) \prod_{i=1}^{M} \mathbb{I}\left(\sum_{j \in \partial i} h_{i,j} x_j = 0\right),
\]

where \(\mathbb{I}[\cdot]\) is the indicator function. The SPA message forms a uniform probability vector over a subset of \(\mathbb{F}_q^m\). The support of each sum-product message forms a linear subspace of \(\mathbb{F}_q^m\) [19].

### V. Density Evolution Analysis of \((d_l, d_r)\)-Codes

Denote the message subspace sent along a randomly picked edge connecting symbol nodes to check nodes at the \(t\)-th iteration by \(V(t)\). Similarly, denote the message subspace sent along a randomly picked edge connecting symbol nodes to symbol nodes at the \(t\)-th iteration by \(U(t)\). The initial message subspace \(V(0)\) is given by a uniformly random subspace of dimension \(m\). Density evolution [19] gives update equations of \(V(t)\) and \(U(t)\) as follows.

\[
U(t) = \sum_{i=1}^{d_r-1} V_i^{(t)},
\]

\[
V(t) = V(0) \cap \bigcap_{i=1}^{d_t-1} U_i^{(t)}.
\]

where \(U_i^{(t)}\) and \(V_i^{(t)}\) are iid copies of \(U(t)\) and \(V(t)\), respectively and \(V_1 + V_2 := \{ v_1 + v_2 \mid v_1, v_2 \in V_1, V_2 \}\). If \(V(t)\) becomes \(\{0\}\), decoding is successfully completed.

It is not easy to track \(V(t)\). Instead, we track the dimension of \(V(t)\). We define \(\xi(t)\) in order to predict the \(\dim V(t)\).

**Definition 1**: Define

\[
\xi^{t+1} = (\xi(t)) \boxdot (d_r - 1),
\]

\[
\xi(t) = \epsilon \oplus (\xi^{(t)}) \boxdot (d_t - 1),
\]

\[
\xi(0) = \epsilon
\]

where for \(\xi_1, \xi_2 \in \{0, 1\}\)

\[
\xi_1 \boxdot \xi_2 := \max(\xi_1 + \xi_2 - 1, 0),
\]

\[
\xi_1 \oplus \xi_2 := \min(\xi_1 + \xi_2, 1).
\]

Next Lemma shows \(\frac{1}{m} \dim V(t)\) converges to \(\xi(t)\) in probability.

**Lemma 1**: For any \(\delta > 0\) and \(\epsilon > 0\), there exists \(m’\) such that for \(m > m’\)

\[
\Pr\{\left| \dim V(t) - \xi(t) m\right| < \delta m\} > 1 - \epsilon.
\]

**Proof**: Let \(V_1\) be a uniformly random subspace of dimension \(d_1\) in \(\mathbb{F}_q^m\), and \(V_2\) a uniformly random subspace of dimension \(d_2\). Then from [12, Proposition 4.4], it holds that for any \(k \geq 0\) and \(m \geq 0\),

\[
\Pr\{d_1 \boxdot d_2 \leq \dim(V_1 \cap V_2) < d_1 \oplus d_2 + k\}
\]

\[
\geq 1 - q^{-k - \max(0, m - d_1 - d_2)},
\]

\[
\Pr\{d_1 \oplus d_2 - k \leq \dim(V_1 + V_2) < d_1 \boxdot d_2\}
\]

\[
\geq 1 - q^{-k - \max(0, m - d_1 - d_2)},
\]

where, with abuse of notation, we define \(\boxdot\) and \(\oplus\) for \(d_1, d_2 \in \mathbb{N}\) as follows

\[
d_1 \boxdot d_2 := \max(d_1 + d_2 - m, 0),
\]

\[
d_1 \oplus d_2 := \min(d_1 + d_2, m).
\]

For \(\xi_1 := d_1 / m\) and \(\xi_2 := d_2 / m\) it follows that

\[
\Pr\left\{\left| \frac{\dim(V_1 \cap V_2)}{m} - \xi_1 \boxdot \xi_2 \right| < \frac{k}{m}\right\}
\]

\[
\geq \Pr\{d_1 \boxdot d_2 \leq \dim(V_1 \cap V_2) < d_1 \oplus d_2 + k\}
\]

\[
\geq 1 - q^{-k - m \max(0,1 - \xi_1 - \xi_2)}.
\]

From this, for sufficiently large \(m\) such that \(\frac{1}{m} < \delta\) and \(q^{-k - m \max(0,1 - \xi_1 - \xi_2)} < \epsilon\), it holds that

\[
\Pr\left\{\left| \frac{\dim(V_1 \cap V_2)}{m} - \xi_1 \boxdot \xi_2 \right| < \delta\right\} \geq 1 - \epsilon.
\]
Hence, we obtain that for all
Next, we claim that
use induction. Under the assumption that
Proof
Similarly, we have
The union bound of the two probabilities gives
Using the triangle inequality and the fact that \( \square \) is a continuous function, we have
The same argument is valid for any combinations of \( \square \) and \( \boxdot \) of \( V_i^{(0)} \) (\( i = 0, 1, \ldots \)). \( V^{(t)} \) is an instance of the combinations. Hence the thesis holds.
\[
\Pr \{ \left| \frac{\text{dim}(V_1 \cap V_2 \cap V_3)}{m} - \frac{\text{dim}(V_1 \cap V_2 \cap V_3)}{m} \right| < \delta \} \geq 1 - 2\epsilon.
\]
Discussion 1: From Lemma 1, it follows that even a single parity-check code is enough to achieve the capacity when \( m \) is infinite. However the aim of this paper is not to design codes for \( \text{CD}(m, \epsilon) \), but evaluate the performance of non-binary codes for large \( m \).

Lemma 2:
\[
\sup \{ \epsilon \in [0, 1] \mid \lim_{t \to \infty} \xi^{(t)} = 0 \} = \frac{1}{d_r - 1}.
\]
Proof: It is easy to see that
\[
\xi^{(d_r - 1)} = \min((d_r - 1)\xi, 1),
\]
\[
\epsilon \square \xi^{(d_r - 1)} = \max((d_l - 1)\xi + \epsilon - (d_l - 1), 0).
\]
First, we claim that \( \xi^{(t)} \geq \frac{1}{d_r - 1} \) for \( t \geq 1 \) if \( \epsilon \geq \frac{1}{d_r - 1} \). We use induction. Under the assumption that \( \xi^{(t)} \geq \frac{1}{d_r - 1} \), we can see that
\[
\xi^{(t+1)} = \min((d_r - 1)\xi^{(t)} + 1,
\]
\[
\xi^{(t+1)} = \max((d_l - 1)\xi^{(t+1)} + \epsilon - (d_l - 1), 0) = \epsilon.
\]
Hence, we obtain that for all \( t \geq 0 \),
\[
\xi^{(t)} = \xi^{(0)} \geq \frac{1}{d_r - 1}.
\]
Next, we claim that \( \lim_{t \to \infty} \xi^{(t)} = 0 \) if \( 0 \leq \epsilon < \frac{1}{d_r - 1} \).
It follows that \( 0 \leq \xi^{(t)} < 1 \), (1) and (2) can be rewritten respectively by
\[
\xi^{(t+1)} = \max((d_l - 1)(d_r - 1)\xi^{(t)} + \epsilon - (d_l - 1), 0).
\]
This can be solved as
\[
\xi^{(t)} = \max\left(\frac{(d_l - 1)(d_r - 1)\xi^{(0)} + \epsilon - (d_l - 1)}{1 - (d_l - 1)(d_r - 1)}, 0\right).
\]
From this, it can be seen that if \( \epsilon < \frac{1}{d_r - 1} \), \( \xi^{(t)} \) is monotonically decreasing down to 0.

We define the threshold which shows how good the \( (d_l, d_r) \) code is. If \( \epsilon < \epsilon(d_l, d_r) \), \( (d_l, d_r) \) codes achieve vanishing decoding error probability.

Definition 2: We define the threshold of \( (d_l, d_r) \) codes as follows.
\[
\epsilon(d_l, d_r) = \sup \{ \epsilon \in [0, 1] \mid \lim_{t \to \infty} \lim_{m \to \infty} \text{dim} V^{(t)} = 0 \}.
\]
We say that the \( (d_l, d_r) \) codes achieve capacity of \( \text{CD}(m, \epsilon) \) when \( \epsilon(d_l, d_r) = \frac{1}{d_r - 1} \).

From Lemma 1, Lemma 2 we have the following theorem.

Theorem 1: For \( d_l \geq 2 \), \( \epsilon(d_l, d_r) = \frac{1}{d_r - 1} \).

VI. DENSITY EVOLUTION ANALYSIS OF \( (d_l, d_r, L) \)-CODES

Denote the message subspace sent along a randomly picked edge connecting symbol nodes to check nodes at the \( t \)-th iteration from section \( i \) to section \( j \) by \( V^{(t)}_{i,j} \). Similarly, denote the message subspace sent along a randomly picked edge connecting check nodes to symbol nodes at the \( t \)-th iteration from section \( i \) to section \( j \) by \( U^{(t)}_{i,j} \).

The initial message subspace \( V^{(0)}_i \) is given by a uniformly random subspace of dimension \( m \) for \( i \in \{0, \ldots, L - 1\} \) and \( V^{(0)}_i = \{0\} \) for \( i \notin \{0, \ldots, L - 1\} \). Density evolution gives update equations of \( V^{(t)} \) and \( U^{(t)} \) as follows.
\[
V^{(t)}_{i,j} = V^{(0)}_{i,j} \cap \left( \bigcap_{k=0, k \neq j}^{d_l - 1} U^{(t)}_{i+k,k} \right),
\]
\[
V^{(t)}_i = V^{(0)}_i \cap \left( \bigcap_{k=0}^{d_l - 1} U^{(t)}_{i+k,k} \right).
\]

Definition 3: For \( i \in \{0, \ldots, L - 1\} \), we set define
\[
\xi^{(0)}_i = \xi^{(0)}_{i,j} = 0
\]
For \( i \in \{0, \ldots, L - 1\} \), define
\[
\xi^{(t+1)}_{i,j} = \max\left(\frac{(d_l - 1)(d_r - 1)\xi^{(t)}_{i,j} + \epsilon - (d_l - 1)}{1 - (d_l - 1)(d_r - 1)}, 0\right),
\]
\[
\xi^{(t)}_{i,j} = \max\left(\frac{(d_l - 1)(d_r - 1)\xi^{(0)}_{i,j} + \epsilon - (d_l - 1)}{1 - (d_l - 1)(d_r - 1)}, 0\right).
\]
Lemma 3: For any $\delta > 0$ and $\epsilon > 0$, there exists $m'$ such that for $m > m'$
\[
\Pr\{\{\dim V_{i,j}^{(t)} - \xi_{i,j}^{(t)} m\} < \delta m\} > 1 - \epsilon,
\]
\[
\Pr\{\{\dim V_{i}^{(t)} - \xi_i^{(t)} m\} < \delta m\} > 1 - \epsilon.
\]
Proof: The proof is similar to that of Lemma 1 and hence omitted.

Lemma 4:
\[
\sup_\epsilon \{\epsilon \in [0, 1] | \lim_{t \to \infty} \xi_i^{(t)} = 0, \ i = 0, \ldots, L - 1\} = \frac{d_i}{d_r}.
\]
Proof: It is sufficient to show that if $\epsilon = \frac{d_i}{d_r}$, $\xi_i = 0$. This is due to the fact that $\frac{d_i}{d_r}$ is the Shannon threshold. First let us check messages from check nodes at section 0 to variable nodes at section 0.
\[
\zeta_{0,0}^{(1)} = \epsilon \square \cdots \square \epsilon = \frac{d_i}{d_r} \left(\frac{d_r}{d_i} - 1\right) = 1 - \frac{d_i}{d_r}.
\]
We employ peeling decoder [19, p. 30] instead of SPA at section 0. The threshold should be the same [19].
\[
\zeta_0^{(1)} = \zeta_{0,0}^{(1)} + \zeta_{0,1}^{(1)} + \cdots + \zeta_{0,d_l-1}^{(1)} - \epsilon - d_l
\]
\[
\leq \zeta_{0,0}^{(1)} + \zeta_{0,1}^{(1)} + \cdots + 1 + \epsilon - d_l
\]
\[
= \zeta_0^{(1)} + \epsilon - 1 = 0.
\]
This implies all symbols at section 0 can be successfully decoded. This reduces $(d_l, d_r, L)$-code to $(d_l, d_r, L-1)$-code. Repeat the decoding step $L$ times then all symbols will be decoded.

Definition 4: We define BP threshold of $(d_l, d_r, L)$ codes as follows.
\[
\epsilon(d_l, d_r, L) = \sup_\epsilon \{\epsilon \in [0, 1] | \lim_{t \to \infty} \lim_{m \to \infty} \dim V_i^{(t)} = 0\},
\]
where $V_i^{(t)}$ is defined in (3).
From Lemma 3, Lemma 4 and the fact that the $(d_l, d_r, L)$ codes have rate $1 - \frac{d_l}{d_l} - \frac{d_r}{L}$, we have the following theorem.

Theorem 2: In the limit of large $m$, the $(d_l, d_r, L)$ codes have threshold $1 - \frac{d_i}{d_i}$. In the limit of large coupling number $L$, the $(d_l, d_r, L)$ codes achieve the capacity of CD$(m, \epsilon)$.
\[
\lim_{L \to \infty} \epsilon(d_l, d_r, L) = \frac{d_i}{d_r},
\]
\[
\lim_{L \to \infty} \lim_{m \to \infty} R(d_l, d_r, L) = 1 - \frac{d_i}{d_r}.
\]

VII. CONCLUSION

We investigated decoding performance of $Q^m$-ary non-binary LDPC codes for large $m$ over CD$(m, \epsilon)$. We gave density evolution and decoding performance evaluation for regular non-binary LDPC codes and SC codes. We show the regular codes do not achieve the capacity of the channel while SC codes do.

VIII. CONCLUSION

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