FANO THREEFOLDS WITH AFFINE CANONICAL EXTENSIONS

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Abstract. Let $M$ be a smooth Fano threefold such that a canonical extension of the tangent bundle is an affine manifold. We show that $M$ is rational homogeneous.

1. Introduction

1.1. Main result. Consider a compact Kähler manifold $M$ with tangent bundle $T_M$. Any Kähler class determines a so-called canonical extension $0 \to \mathcal{O}_M \to V \to T_M \to 0$.

Then we may consider the manifold $$Z_M = \mathbb{P}(V) \setminus \mathbb{P}(T_M)$$
which is an affine bundle over $M$. In this context, we have the following conjecture [GW20], see also [HP21, Conj.1.1]:

1.1. Conjecture. Let $M$ be a projective manifold such that a canonical extension $Z_M$ is affine for some Kähler class. Then $M$ is rational homogeneous, i.e., $M = G/P$ with $G$ a semi-simple complex Lie group and $P$ a parabolic subgroup.

The aim of the paper is to prove Conjecture 1.1 for Fano threefolds:

1.2. Theorem. Let $M$ be a smooth Fano threefold. If $Z_M$ is affine for some Kähler class, then $M$ is rational homogeneous.

Fano threefolds are of course rather special manifolds, but they are a natural testing ground for Conjecture 1.1 since $Z_M$ is affine, the tangent bundle $T_M$ is big [GW20, Prop.4.2], a very restrictive property for manifolds that are not rational homogeneous [HLS22]. Thus it seems likely that potential counterexamples to Conjecture 1.1 share at least some properties of rational homogeneous spaces, e.g. we would expect them to be rationally connected or even Fano.

To prove Theorem 1.2 we first investigate the structure of extremal rays on $M$. We have shown in [HP21, Cor.1.8] that $M$ does not admit any birational Mori contraction. This will be used, via classification, to settle the case that $b_2(M) \geq 2$.

The far more difficult case is that $b_2(M) = 1$. The tangent bundle being big, we know by [HL21, Cor.1.2] that $M$ must be the del Pezzo threefold $V_5$ of degree five, unless $M$ is rational homogeneous (i.e., $P_3$ or the quadric $Q_3$). Thus we will prove - which will be surprisingly difficult -

1.3. Theorem. Let $M$ be the del Pezzo threefold of degree five. Then $Z_M$ is not Stein.

A more explicit explanation is given in Theorem 3.1.
1.B. A general strategy. When studying complements of hypersurfaces, an interesting source of examples is given by small modifications [HP21 p.2, Ott12 Ex.5.5]: given a smooth ample divisor $Y^- \subset X^-$ in a projective manifold, one tries to find a subset $C^- \subset Y^- \subset X^-$ of codimension at least two that can be “flipped”, i.e. there exists a birational map

$$X^- \dasharrow X^+$$

that is an isomorphism $X^- \setminus C^- \cong X^+ \setminus C^+$ for some subset $C^+ \subset X^+$ of codimension at least two. Let $Y^+ \subset X^+$ be the strict transform of $Y^-$, then $Y^- \subset X^-$ is never an ample divisor. Nevertheless, if $C^+ \subset X^+$, its complement

$$X^+ \setminus Y^+ \cong X^- \setminus Y^-$$

is affine. In order to decide whether a canonical extension $Z_M$ affine or not, our goal is to inverse this procedure: we aim to construct a small birational map

$$\psi : X = \mathbb{P}(V) \dasharrow \mathbb{P}(V)^- = X^-.$$

which is biholomorphic on $Z_M$ such that the strict transform $\mathbb{P}^-(T_M)$ is a big and nef divisor on $\mathbb{P}^-(V)$. There is a natural candidate for the space $X^-$: the tangent bundle $T_M$ being big, the tautological class $\zeta_V \to \mathbb{P}(V)$ is big. Therefore for some $k \gg 0$ we could consider the birational map

$$\psi : X = \mathbb{P}(V) \dasharrow X^-$$

given by the image of $\varphi_{[k\zeta_V]}$. While the indeterminacy locus of $\psi$ is contained in $\mathbb{P}(T_M) \subset \mathbb{P}(V)$ it is absolutely not clear whether such a map is an isomorphism in codimension one. Therefore we introduce the following terminology.

1.4. Definition. Let $M$ be a projective manifold, and let $\mathbb{P}(T_M) \subset \mathbb{P}(V)$ be a canonical extension associated to some Kähler class on $M$. Let

$$\psi : \mathbb{P}(V) \dasharrow \mathbb{P}(V)^-$$

be a finite sequence of antiflips (i.e., inverses of flips). Let $\mathbb{P}(T_M)^- \subset \mathbb{P}(V)^-$ be the strict transform of $\mathbb{P}(T_M)$ in $\mathbb{P}(V)^-$, and assume that $\psi|_{\mathbb{P}(T_M)}$ is again a finite sequence of antiflips. In particular, $\mathbb{P}(V)^-$ and $\mathbb{P}(T_M)^-$ are $\mathbb{Q}$-factorial with terminal singularities. We say that

$$(\mathbb{P}(V)^-, \mathbb{P}(T_M)^-)$$

is a weak Fano model of $(\mathbb{P}(V), \mathbb{P}(T_M))$, if the anticanonical divisor of $\mathbb{P}(V)^-$ is nef and big.

Verifying that $\mathbb{P}(V)^-$ is a weak Fano variety is equivalent to verifying this property for $\mathbb{P}(T_M)^-$. In fact, $\mathbb{P}(T_M) \in |\zeta_V \cdot |O_{\mathbb{P}(V)}(1)|$. Thus, if $\zeta_V^-$ denotes the induced Weil divisor class on $\mathbb{P}(V)^-$, then

$$\mathbb{P}(T_M)^- \in |\zeta_V^-|.$$

Since $-K_{\mathbb{P}(V)^-} = (\dim M + 1)\zeta_V^-$, it follows that $-K_{\mathbb{P}(T_M)^-}$ is big and nef as well. Vice versa, observe that if $(\mathbb{P}(V), \mathbb{P}(T_M))$ has a weak Fano model, then the vector bundles $V$ and $T_M$ have to be big. Constructing a weak Fano model gives a straightforward way to check the somewhat elusive property that $Z_M$ is affine.

1.5. Proposition. Let $M$ be a projective manifold, and let $\mathbb{P}(T_M) \subset \mathbb{P}(V)$ be a canonical extension associated to some Kähler class on $M$. Assume that $(\mathbb{P}(V), \mathbb{P}(T_M))$ has a weak Fano model $(\mathbb{P}(V)^-, \mathbb{P}(T_M)^-)$. Let

$$\psi : \mathbb{P}(V) \dasharrow \mathbb{P}(V)^-$$

be the corresponding sequence of antiflips and assume that the restriction $\psi|_{Z_M}$ is biholomorphic. Then we have a dichotomy:
If $\psi(Z_M) \subseteq \mathbb{P}(V)^- \setminus \mathbb{P}(T_M)^-$, then $Z_M$ is not affine.
If $\psi(Z_M) = \mathbb{P}(V)^- \setminus \mathbb{P}(T_M)^-$, then $Z_M$ is affine.

For the proof note that in the first case, the complex space $\psi(Z_M) \simeq Z_M$ is not holomorphically convex [GR04, V, §1, Thm.4], while in the second case we can apply [HP21] 3.14.

With this preparation, the strategy of the proof of Theorem 1.3 is clear: in Section 3 we will construct by hand a weak Fano model of $(\mathbb{P}(V), \mathbb{P}(T_M))$. Thanks to the explicit construction we can verify that we are in the first case of the dichotomy 1.5

1.C. Future directions. Let $M$ be a projective manifold such that a canonical extension $(\mathbb{P}(V), \mathbb{P}(T_M))$ admits a weak Fano model $\psi : \mathbb{P}(V) \to \mathbb{P}(V)^-$. Then the tangent bundle $T_M$ is big and nef in codimension one, i.e. the tautological bundle $\zeta_M \to \mathbb{P}(T_M)$ is nef in codimension one. If we allow $\psi$ to be the identity, the class of manifolds such that $T_M$ is big and nef in codimension one (resp. nef in codimension one) contains in particular the manifolds $M$ such that $T_M$ is big and nef (resp. simply nef).

1.6. Problem. What are the projective manifolds $M$ such that $T_M$ is big and nef in codimension one, but not nef?

By [HL21, Cor.1.2] there is exactly one Fano threefold $M$ with $\rho(M) = 1$ such that $T_M$ big and nef in codimension one, but not nef, namely $M = V_5$. The proof of Theorem 1.3 is quite involved and uses heavily the geometry of the del Pezzo threefold $V_5$. Thus, as a first step towards Problem 1.6 one might wonder whether there is a cohomological proof. In this context, it follows from the work of J.Zhang [Zha08] that $Z_M$ is not affine provided

$$H^i(Z_M, \Omega^j_M) \neq 0$$

some $i \geq 1, j \geq 0$ (for algebraic cohomology). These groups however seem hard to compute.

Finally note that Conjecture 1.1 has the following analytic version:

1.7. Conjecture. Let $M$ be a compact Kähler manifold such that the canonical extension is Stein for some Kähler class. Then $T_M$ is nef.

The structure of Kähler manifolds is well understood up to the case of Fano manifolds, [DPS94], in which case $M$ is conjectured to be rational homogeneous, [CP91]. Thus, Conjecture 1.7 includes

1.8. Conjecture. Let $M$ be a Fano manifold. Assume that $Z_M$ is Stein for some Kähler class. Then $Z_M$ is affine.

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2. Notation and first results

We work over the complex numbers, for general definitions we refer to [Har77, KK83]. Complex spaces and varieties will always be supposed to be irreducible and reduced.

We use the terminology of [Deb01, KM98] for birational geometry and notions from the minimal model program. We follow [Laz04] for algebraic notions of positivity. For positivity notions in the analytic setting, we refer to [Dem12].
Given a vector bundle $V$ on a manifold $B$, we denote by $\mathbb{P}(V) \to B$ the projectivisation in Grothendieck’s sense, and by $\zeta_V = \zeta_{\mathbb{P}(V)} := \mathcal{O}_{\mathbb{P}(V)}(1)$ the tautological line bundle on $\mathbb{P}(V)$. In the special case of the tangent bundle $T_M$ of a manifold $M$, we denote the tautological bundle $\zeta_{\mathbb{P}(T_M)}$ simply by $\zeta_M$.

Given any non-zero class $\alpha \in H^1(M, \Omega_M)$, usually a Kähler class, we let

$$0 \to \mathcal{O}_M \to V \to T_M \to 0$$

be the associated nonsplitting extension. We set

$$Z_M = \mathbb{P}(V) \setminus \mathbb{P}(T_M).$$

In this section we investigate the structure of Mori fibre spaces of projective manifolds $M$ such that $Z_M$ is Stein.

2.1. Theorem. Let $M$ be a projective manifold of dimension $n$, $S \subset M$ be a smooth rational curve with $C^2 < 0$ and let $\varphi : S \to S'$ be the contraction of $C$. Assume the following

- a) the restriction $N_{S/M} \otimes \mathcal{O}_C$ is trivial, or, more generally, that $\varphi_*(N_{S/M})$ is locally free,
- b) the canonical morphism $R^1\varphi_*(T_S) \to R^1\varphi_*(T_M \otimes \mathcal{O}_S)$ is injective.

Then $Z_M$ is not Stein for any Kähler class.

2.2. Remark. The technical conditions in Theorem 2.1 can be easily checked in a number of situations:

- If $C^2 = -1$, then $R^1\varphi_*(T_S) = 0$, so the second condition is trivial. Thus if $N_{S/M} \otimes \mathcal{O}_C$ is trivial, Theorem 2.1 applies.
- If $\dim M = 3$ and $S \cdot C \leq 0, C^2 = -1$, Theorem 2.1 also applies: by what precedes this is clear if $S \cdot C = 0$. If $S \cdot C < 0$, then $N_{S/M} \otimes \mathcal{O}_C \simeq N_C^{\otimes k}$ with $k \in \mathbb{N}$. Therefore $N_{S/M} \otimes \mathcal{O}_U \simeq \mathcal{O}_U(kC)$ in a neighborhood $U$ of $C$, thus $\varphi_*(N_{S/M})$ is locally free.
- If $\dim M = 3$ and $S \cdot C < 0, C^2 \leq -2$, the same arguments applies provided $N_{S/M} \otimes \mathcal{O}_C = N_C^{\otimes k}$ with $k \in \mathbb{N}$
- The injectivity of the canonical morphism $R^1\varphi_*(T_S) \to R^1\varphi_*(T_M \otimes \mathcal{O}_S)$ is satisfied if the tangent bundle sequence

$$0 \to T_S \to T_M \otimes \mathcal{O}_S \to N_{S/M} \to 0$$

splits.

2.3. Problem. If $\dim M = 3$, generalize Theorem 2.1 to the case that $C$ is a smooth rational curve with $C^2 \leq -2$ and $S \cdot C \leq 0$.

Proof of Theorem 2.1. The canonical extension defining $Z_M = \mathbb{P}(V_M) \setminus \mathbb{P}(T_M)$ restricts to an exact sequence

(1)\hspace{1cm}0 \to \mathcal{O}_S \to V_M \otimes \mathcal{O}_S \to T_M \otimes \mathcal{O}_S \to 0

Applying $\varphi_*$ and using $R^1\varphi_*(\mathcal{O}_S) = 0$ yields an exact sequence

(2)\hspace{1cm}0 \to \mathcal{O}_{S'} \to \varphi_*(V_M \otimes \mathcal{O}_S) = V' \to \varphi_*(T_M \otimes \mathcal{O}_S) =: T' \to 0.

We set

$$\tilde{Z}_S := \mathbb{P}(V_M \otimes \mathcal{O}_S) \setminus \mathbb{P}(T_M \otimes \mathcal{O}_S)$$

and

$$\tilde{Z}_{S'} := \mathbb{P}(V') \setminus \mathbb{P}(T').$$
Claim A. Sequence \(2\) satisfies the extension property of [HP21, Lemma 4.9] near the singular point \(s\) of \(S'\). This will in turn be a consequence of [HP21, Thm.4.5], Part 3.

Given this, we conclude as follows. Denote by \(\pi : \tilde{Z}_S \rightarrow S\) and \(\tau : \tilde{Z}_{S'} \rightarrow S'\) the projections. Assume that \(Z_M\) is Stein. Then the closed subspace \(\tilde{Z}_S \subset Z_M\) is Stein, hence \(\tilde{Z}_S \setminus \pi^{-1}(C)\) is Stein as well, since \(\pi^{-1}(C)\) is a Cartier divisor [GR04, V, §1, Thm.1 d]). On the other hand, \(\tilde{Z}_S \setminus \pi^{-1}(C) \simeq \tilde{Z}_{S'} \setminus \tau^{-1}(s)\), and \(\tilde{Z}_{S'} \setminus \tau^{-1}(s)\) is not Stein by [HP21, Lemma 4.9], a contradiction.

Proof of Claim A. The extension class \(\zeta_{M|S} \in H^1(S, \Omega^1_M \otimes O_S)\) defining our given extension projects onto a class \(\zeta_S \in H^1(S, \Omega^1_S)\) defining a canonical extension

\[
0 \rightarrow O_S \rightarrow V_S \rightarrow T_S \rightarrow 0
\]

such that the following diagram commutes

(3)

\[
\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
\rightarrow O_S & \rightarrow V_S & \rightarrow T_S & \rightarrow 0 & & = & & = & & =
\end{array}
\]

By assumption, the canonical morphism

\[
R^1\varphi_*(T_S) \rightarrow R^1\varphi_*(T_M \otimes O_S)
\]

is injective. Thus pushing forward the whole diagram to \(S'\) we obtain a commutative diagram of exact sequences

(4)

\[
\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & 0 & & 0 \\
\rightarrow O_{S'} & \rightarrow \varphi_*V_S & \rightarrow \varphi_*T_S & \rightarrow 0 & & = & & = & & =
\end{array}
\]

\[
\begin{array}{ccccccccc}
\varphi_*N_{S/M} & \rightarrow \varphi_*N_{S'/M}
\end{array}
\]

5
Taking biduals this defines a commutative diagram of reflexive sheaves

\[\cdots \rightarrow 0 \rightarrow \mathcal{O}_{S'} \rightarrow (\varphi_* \mathcal{V}_S)^{**} \rightarrow (\varphi_* \mathcal{T}_S)^{**} \rightarrow 0 \]

\[\cdots \rightarrow 0 \rightarrow \mathcal{O}_{S'} \rightarrow (\varphi_* (\mathcal{V}_M \otimes \mathcal{O}_S))^{**} \rightarrow (\varphi_* (\mathcal{T}_M \otimes \mathcal{O}_S))^{**} \rightarrow 0 \]

\[\varphi_* \mathcal{N}_{S/M} = (\varphi_* \mathcal{N}_{S/M})^{**} \rightarrow (\varphi_* \mathcal{N}_{S/M})^{**} = \varphi_* (\mathcal{N}_{S/M}) \]

Claim B. The sequences in this diagram are also exact.

Proof of Claim B. We start by observing that the first row is exact: the class \(\zeta_S\) determines canonically a class \(\zeta'_S\) on \(S'\), see [HP21, 4.7], thereby defining an extension of \(\mathcal{T}_S\) by \(\mathcal{O}_{S'}\). Since \((\varphi_* \mathcal{T}_S)^{**} \simeq \mathcal{T}_{S'}\), the first line coincides in codimension one with this extension, so they coincide.

Note also that the statement is clear in the complement of the point \(s\), so we can replace \(S'\) by a Stein neighbourhood \(U'\) of the point \(s\). Recall that \(\varphi_* \mathcal{N}_{S/M}\) is locally free. Since the columns of (4) are exact, they split on the Stein neighbourhood \(U'\), i.e.,

\[\varphi_* (\mathcal{V}_M \otimes \mathcal{O}_S) \otimes \mathcal{O}_{U'} \simeq (\varphi_* \mathcal{V}_S \otimes \mathcal{O}_{U'}) \oplus \mathcal{O}_{U'}^{\dim \mathcal{M} - 2}\]

and analogously for \(\varphi_* (\mathcal{T}_M \otimes \mathcal{O}_S) \otimes \mathcal{O}_{U'}\). In particular taking biduals does not change the exactness of the columns. Thus we are left to show the exactness of the second row, but this follows from a diagram chase. This shows Claim B.

The first row of Diagram (5) reads

\[0 \rightarrow \mathcal{O}_{S'} \rightarrow \mathcal{V}_{S'} \rightarrow \mathcal{T}_{S'} \rightarrow 0,\]

hence has the extension property by [HP21, Thm.4.5], proof of Part 3. Thus the middle row of Diagram (5) has the extension property as well. But restricted to \(S' \setminus \{s\}\), this sequence is just Sequence (2). Hence Sequence (2) has the extension property, too, which settles Claim A and finishes the proof.

2.4. Corollary. Let \(M\) be a projective manifold such that \(Z_M\) is Stein for some Kähler class. Let \(f : M \rightarrow N\) be a fibre space and \(\dim N = \dim M - 2\). Then any smooth fibre \(S\) is a minimal surface, i.e., does not contain any \((-1)\)-curve.

In particular, if \(f\) is a Mori fibre space, then any smooth fibre of \(f\) is either \(\mathbb{P}^2\) or \(\mathbb{P}^1 \times \mathbb{P}^1\).

Proof. Since \(S\) is a smooth fibre of the fibration \(f\), the normal bundle \(\mathcal{N}_{S/M}\) is trivial. Thus Theorem 2.1 applies to any \((-1)\)-curve in \(S\), cf. Remark 2.2.

2.5. Corollary. Let \(M\) be a smooth projective threefold such that \(Z_M\) is Stein for some Kähler class. Let \(f : M \rightarrow N\) be an elementary Mori contraction to a surface \(N\). Then \(f\) is a \(\mathbb{P}^1\)-bundle.
Proof. By [Mor82], the fibration $f$ is either a $\mathbb{P}^1$-bundle or a conic bundle with singular fibres, with singular fibres in codimension one being a line pair. Assume the latter and let $H \subset N$ be a general hyperplane section. Then $S = \varphi^{-1}(H)$ is a smooth surface, and $f|_S : S \to H$ contains line pairs as singular fibres. Choose a singular fibre $f^{-1}(x_0)$ and let $C \subset f^{-1}(x_0)$ be an irreducible component. Then $C \subset S$ is a $(-1)$-curve and $S \cdot C = 0$. But then by Theorem 2.1, $Z_M$ cannot be Stein. Hence $f$ must be a $\mathbb{P}^1$-bundle. $\square$

3. The del Pezzo threefold of degree five

We fix the following setup for the whole section: we denote by $M := \text{Gr}(2, 5) \cap \mathbb{P}^6 \subset \mathbb{P}^9$ the del Pezzo threefold of degree five, often quoted as $V_5$. We let

$$Y = \mathbb{P}(T_M)$$

with projection $\pi : \mathbb{P}(T_M) \to M$ and tautological line bundle $\zeta_M = \mathcal{O}_{\mathbb{P}(T_M)}(1)$. We denote by $V$ the canonical extension and set

$$X = \mathbb{P}(V),$$

so that $Z_M = X \setminus Y$. Recall that $\zeta_V$ denotes the tautological line bundle on $X$ and that $Y \in |\zeta_V|$. Our goal is to prove

3.1. Theorem. Let $M$ be the Fano threefold $V_5$. Then $Z_M$ is not Stein. More precisely, there exists an antiflip $\psi : \mathbb{P}(V) \to \mathbb{P}(V)^-$ with the following properties

a) $\psi$ induces a biholomorphic map

$$Z_M \to \mathbb{P}(V)^- \setminus (\mathbb{P}(T_M)^- \cup A)$$

where $\mathbb{P}(T_M)^-$ is the strict transform of $\mathbb{P}(Z_M)$ and $A$ is (non-empty) analytic set of codimension two, which is not contained in $\mathbb{P}(T_M)^-$;

b) $\mathbb{P}(T_M)^-$ is an antiflip of $\mathbb{P}(T_M)$;

c) $\mathbb{P}(V)^-$ and $\mathbb{P}(T_M)^-$ are weak $\mathbb{Q}$-Fano;

d) $\mathbb{P}(V)^- \setminus \mathbb{P}(T_M)^-$ is affine.

In other words, $(\mathbb{P}(V), \mathbb{P}(T_M))$ has the weak Fano model $(\mathbb{P}(V)^-, \mathbb{P}(T_M)^-)$. 

3.A. Geometry of $\mathbb{P}(T_M)$. We denote by $\mathcal{O}_M(1)$ the restriction of the tautological bundle to $M$. It is well-known [Sha99 Prop.3.2.4] that $\mathcal{O}_M(1)$ is the ample generator of $\text{Pic}(M)$.

The threefold $M$ has been studied by many authors, e.g., by [MUS83, FN89, San14]; here we mainly follow the exposition in [CS16 Ch.7]. The manifold $M$ is almost homogeneous with automorphism group $\text{Aut}(M) \simeq \text{PSL}_2 \mathbb{C}$ [CS16 Prop.7.1.10] and has three orbits [CS16 Thm.7.1.4, Thm.7.1.9]:

- The unique closed orbit is a smooth rational curve $B$ which is a rational normal curve of degree six in $M \subset \mathbb{P}^6$. We have [CS16 Lemma 7.2.8]

$$N_{B/M} \simeq \mathcal{O}_B(5)^{\oplus 2}. \tag{6}$$
There is a two-dimensional orbit and we denote its closure by $\bar{S}$: it is obtained as the locus of lines $l \subset M$ that are tangent lines of $B$. By [FN89, Cor.1.2] these are exactly the lines $l \subset M$ such that

$$T_M \otimes \mathcal{O}_l \simeq \mathcal{O}_l(2) \oplus \mathcal{O}_l(1) \oplus \mathcal{O}_l(-1).$$

The divisor $\bar{S}$ is an element of the anticanonical system $|\mathcal{O}_M(2)|$.

The three-dimensional orbit is $M \setminus \bar{S}$.

The threefold $M \subset \mathbb{P}^6$ is covered by lines, and its Fano surface of lines is isomorphic to $\mathbb{P}^2$ [FN89, Thm.I]. [San14, Rem.2.28]. We denote by

$$q : \mathcal{U} \rightarrow \mathbb{P}^2$$

the universal family of lines on $M$ and by $e : \mathcal{U} \rightarrow M$ the evaluation morphism. By [CS16, Thm.7.1.9], [FN89, Lemma 2.3] the evaluation morphism has degree three and is étale over $M \setminus \bar{S}$.

The lines of splitting type (7) are parametrised by a smooth conic $C \subset \mathbb{P}^2$ [CS16, Thm.7.1.9] and we denote by

$$q_{|\mathcal{U}_C} : \mathcal{U}_C \rightarrow C$$

the restriction of the universal family. By [CS16, Thm.7.1.9(iii)] the lines contained in $\bar{S}$ are disjoint, so the evaluation map $e_C : \mathcal{U}_C \rightarrow \bar{S}$ is injective. Thus we can identify $e_C$ with the normalisation of $\bar{S}$. The singular locus of $\bar{S}$ is the curve $B$ [CS16, Lemma 7.2.2(i)], so we see that $e_C$ is an isomorphism on $\mathcal{U}_C \setminus \tilde{B}$ where by $\tilde{B} \subset \mathcal{U}_C$ we denote the set-theoretical pre-image of the curve $B \subset \bar{S}$. The surface $\mathcal{U}_C$ is isomorphic to a quadric surface $\mathbb{P}^1 \times \mathbb{P}^1$ [CS16, Lemma 7.2.2(vi)]. In order to keep the notation transparent we will write

$$\mathcal{U}_C \simeq C \times \mathbb{P}^2,$$

and identify the morphism $q_{|\mathcal{U}_C}$ with the projection $p_C : C \times \mathbb{P}^2 \rightarrow C$. We also know [CS16, Lemma 7.2.3(iii)] that

$$\tilde{B} \in |\mathcal{O}_{C \times \mathbb{P}^2}(1,1)|$$

Since $K_{\bar{S}} \simeq \mathcal{O}_{\bar{S}}$ by adjunction and $K_{C \times \mathbb{P}^2} \simeq \mathcal{O}_{C \times \mathbb{P}^2}(-2,-2)$, the map $e_C$ ramifies with multiplicity two along $B$. Finally note that by [CS16, Lemma 7.2.3] we have

$$e_C^* \mathcal{O}_M(1) \simeq \mathcal{O}_{C \times \mathbb{P}^2}(5,1).$$

Observe that the line bundle $\zeta_M \otimes \pi^* \mathcal{O}_M(1)$ is nef and globally generated [FL22, Prop.6.1]: indeed $\Omega_{2\mathbb{P}^2}(3)$ is globally generated, and we have a surjection

$$\Omega_{2\mathbb{P}^2}(3) \otimes \mathcal{O}_M \twoheadrightarrow \Omega^2_M(3) \simeq T_M(1).$$

Denote by

$$\varphi_M : \mathbb{P}(T_M) \rightarrow \mathbb{P}^N$$

the morphism defined by the global sections of $\zeta_M \otimes \pi^* \mathcal{O}_M(1)$.

Let $l \subset M$ be a line parametrised by $C \subset \mathbb{P}^2$, i.e., $l$ has splitting type (7). Then we denote by

$$\tilde{l} \subset \mathbb{P}(T_M)$$

the lifting given by the quotient line bundle $T_M \otimes \mathcal{O}_l \rightarrow \mathcal{O}_l(-1)$. Note that by construction

$$c_1(\zeta_M \otimes \pi^* \mathcal{O}_M(1)) \cdot \tilde{l} = 0,$$

so the curve $\tilde{l}$ is contracted by $\varphi_M$ onto a point. Let

$$S \subset \mathbb{P}(T_M)$$
be the union of the curves $\tilde{l}$.

We can get a better description of $S$ by looking at the vector bundle $e_C^* T_M \otimes O_{U_C}$: the surface $U_C$ is the ramification divisor of $e$, so the tangent map $T_e$ has rank two in the generic point of $U_C$ and the image of $T_{\tilde{e}|U_C}$ is given by the image of

$$T_{U_C} \to e^* T_M \otimes O_{U_C}. $$

Since $e_C$ is not immersive along the curve $\tilde{B} \subset U_C$ and since $\tilde{B}$ is embedded into $M$ via $e$, the map $T_{U_C} \to e^* T_M$ has rank one along $\tilde{B}$. Thus the image of the injective map $T_{U_C} \to e^* T_M$ is not saturated, and we denote by

$$K \subset e^* T_M \otimes O_{U_C}$$

the saturation. Since $T_{U_C} \simeq T_{C \times \tilde{l}} \simeq T_{U_C/C} \oplus p_C^* T_C$ and the evaluation map $e$ is immersive on the fibres of the universal family (the images are lines), the map

$$T_{U_C/C} \to e^* T_M \otimes O_{U_C}$$

has rank one in every point. Therefore we have an extension

$$0 \to T_{U_C/C} \to K \to L \to 0$$

with $L$ a reflexive sheaf of rank one and a non-zero morphism $p_C^* T_C \to L$. Since $p_C^* T_C \to L$ vanishes along $B$ and $O_{U_C}(B) \simeq O_{C \times \tilde{l}}(1,1)$ we have $L \simeq O_{C \times \tilde{l}}(2,0) \otimes O_{C \times \tilde{l}}(m,m)$ with $m$ the vanishing order. We claim that $m = 1$: restricting to a line $l \subset U_C$ we have

$$T_{U_C} \otimes O_l \simeq O_l(2) \oplus O_l \to e^* T_M \otimes O_l \simeq O_l(2) \oplus O_l(1) \oplus O_l(-1).$$

Since $K$ is the saturation of $T_{U_C}$ we obtain that $K \otimes O_l \simeq O_l(2) \oplus O_l(1)$. Thus we have $m = 1$.

The following statement summarises our construction:

**3.2. Lemma.** The evaluation map $e_C : U_C \to \tilde{S} \subset M$ factors through a morphism $\tilde{e}_C : U_C \to S \subset \mathbb{P}(T_M)$ such that for a line $l \subset \tilde{S}$, the set-theoretical preimage $l \subset S$ is given by the quotient line bundle

$$T_M \otimes O_l \to O_l(-1).$$

Moreover we have $\tilde{e}_C^* \zeta_M \simeq O_{C \times \tilde{l}}(7,-1)$.

**Remark.** At this point we do not know whether $U_C \to S$ is an isomorphism, this will be shown in Lemma 3.4.

**Proof.** By what precedes we have an extension

$$0 \to K \to e_C^* T_M \otimes O_{U_C} \to Q \to 0$$

where $K \subset e_C^* T_M \otimes O_{U_C}$ is a rank two subbundle given by an extension

$$0 \to O_{C \times \tilde{l}}(0,2) \to K \to O_{C \times \tilde{l}}(3,1) \to 0.$$ 

Since by (12) one has $e_C^* \omega_M^* = e_C^* O_M(2H) = O_{C \times \tilde{l}}(10,2)$, we obtain that $Q \simeq O_{C \times \tilde{l}}(7,-1)$. Let $\tilde{e}_C : U_C \to S \subset \mathbb{P}(T_M)$ be the map determined by the quotient $e_C^* T_M \to Q$. Then $\tilde{e}_C$ factors $e_C$ and by the universal property of the tautological bundle one has $\tilde{e}_C^* \zeta_M \simeq O_{C \times \tilde{l}}(7,-1)$. Note also that the restriction of $Q$ to a line $l \subset U_C$ is $O_l(-1)$. Since $l$ identifies to its image in $\tilde{S}$, we see that $\tilde{e}_C(l)$ corresponds to the lifting $T_M \otimes O_l \to O_l(-1)$.

**3.3. Lemma.** (Jie Liu) The morphism $\varphi_M$ contracts exactly the curves $\tilde{l}$ onto points.
This statement was communicated to us by Jie Liu, his proof uses the blow-up construction from [CS16, Sect.7.2]. For our purpose it is more convenient to use the description of $e^*T_M \otimes \mathcal{O}_{U_C}$.

**Proof.** Let $\hat{D} \subset \mathbb{P}(T_M)$ be a curve contracted by $\varphi_M$ and set $D := \pi(\hat{D})$.

Assume first that $D \not\subset \hat{S}$. Since $T_M$ is globally generated in the complement of $\hat{S}$, we have $c_1(\xi_M) \cdot D \geq 0$ and hence $c_1(\xi_M \otimes \pi^*\mathcal{O}_M(1)) \cdot \hat{D} > 0$, a contradiction.

Thus we may assume that $D \subset \hat{S}$. By (14) we have an extension

$$0 \to K \otimes \epsilon_C^*\mathcal{O}_M(1) \to \epsilon_C^*(T_M \otimes \mathcal{O}_M(1)) \to Q \otimes \epsilon_C^*\mathcal{O}_M(1) \to 0.$$  

Since $\epsilon_C$ is finite and $K$ is nef by (15), the vector bundle $K \otimes \epsilon_C^*\mathcal{O}_M(1)$ is ample. Thus the non-ample locus of $\xi_M \otimes \pi^*\mathcal{O}_M(1)$ is the image of $\mathbb{P}(Q \otimes \epsilon_C^*\mathcal{O}_M(1)) \subset \mathbb{P}(\epsilon_C^*(T_M \otimes \mathcal{O}_M(1)))$. By construction this is exactly the surface $S$, so we have $D \subset S$. By Lemma 3.3 we have

$$\mathbb{P}(\epsilon_C^*(T_M \otimes \mathcal{O}_M(1))) \simeq O_{C \times \{7, -1\}} \otimes O_{C \times \{5, 1\}} \simeq O_{C \times \{12, 0\}},$$

so $\xi_M \otimes \pi^*\mathcal{O}_M(1)$ restricted to $S$ is nef, not big and has degree zero exactly on the curves $\tilde{l}$.

**3.4. Lemma.** We have

$$\text{Exc}(\varphi_M) = S \simeq \mathbb{P}^1 \times \mathbb{P}^1,$$

i.e. the surface $S$ is smooth and $\epsilon_C : U_C \to S$ is an isomorphism.

**Proof.** By Lemma 3.3 the first statement is clear. Since $\varphi_M$ contracts no other curves, the image $\varphi_M(S)$ has dimension one, so $S \to \varphi_M(S)$ is a fibration with set-theoretical fibres the curves $\tilde{l}$. Since $U_C \to \hat{S}$ is bijective, the normalisation of $S$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, so

$$\mathbb{P}^1 \times \mathbb{P}^1 \to S \to \varphi_M(S)$$

identifies to one of the projections. Thus $\varphi_M(S)$ is a smooth rational curve: in fact, $\text{PSL}_2 \mathbb{C}$ acts on the image of $\varphi_M$ and $\varphi_M(S)$ is an orbit as image of the 1-dimensional orbit $B$. We also know that $S$ is smooth in the complement of the preimage of $B \subset \hat{S}$ and this preimage maps surjectively onto $\varphi_M(S)$, so every fibre of $S \to \varphi_M(S)$ meets the smooth locus of $S$.

We now argue by contradiction: if $p \in S$ is a singular point, the fibre $\varphi_M^*\varphi_M(p)$ is a Cartier divisor meeting a singular point, hence is singular. Yet $\varphi_M^*\varphi_M(p)$ is isomorphic to $\tilde{l}$ in its general point. Thus the Cartier divisor $\varphi_M^*\varphi_M(p)$ is reduced and coincides with the set-theoretical fibre $\tilde{l}$. Thus $\varphi_M^*\varphi_M(p)$ is smooth, a contradiction. \hfill \Box

**Remark.** In order to keep the notation transparent, we will write

$$S \simeq C \times \tilde{l},$$

and identify $\varphi_M|_S$ with the projection $p_C : C \times \tilde{l} \to C$.

The base locus $B|_{\xi_M}$ of $|\xi_M|$ is easy to compute:

**3.5. Lemma.** $B|_{\xi_M} = S \cup \mathbb{P}(N_{B/M}).$

**Proof.** From the description of the orbits of $\text{Aut}(M)$ at the start of this subsection it is clear that $B|_{\xi_M}$ is contained in $\pi^{-1}(\mathcal{S})$. Clearly $S \subset B|_{\xi_M}$, since the restriction of $\xi_M$ to $S$ is not pseudoeffective. Let $x \in \mathcal{S} \setminus B$. Then

$$\text{Ker}(H^0(M, T_M) \to T_{M,x})$$

...
has dimension one, and so has
\[ \text{Ker} \left( H^0(\mathbb{P}(T_M), \zeta_M) \to H^0(\pi^{-1}(x), \zeta_M) \right). \]
Thus \((BS|\zeta_M|) \cap \pi^{-1}(x)\) is a single point, i.e., an irreducible component \(S' \neq S\) of \(BS|\zeta_M|\) is mapped into \(B\).

Finally, since the image of the restriction map
\[ H^0(M, T_M) \to H^0(B, T_M \otimes \mathcal{O}_B) \]
is equal to \(H^0(B, T_B)\), it follows that \(\mathbb{P}(N_{B/M}) \subset BS|\zeta_M|\) and that \(\pi^{-1}(B) \not\subset BS|\zeta_M|\). □

Recall that \(\mathbb{P}(T_M) = Y\) has a contact structure \([\text{LeB95}]\): there exists a corank one subbundle \(\mathcal{F} \subset T_Y\) that fits into an exact sequence
\[
0 \to \mathcal{F} \to T_Y \to \zeta_M \to 0.
\]
Moreover the Lie-bracket induces a morphism
\[
\mathcal{F} \otimes \mathcal{F} \to \zeta_M
\]
that is every non-degenerate, so we have an isomorphism
\[
\mathcal{F} \simeq \mathcal{F}^* \otimes \zeta_M. \tag{17}
\]
We also have an exact sequence
\[
0 \to T_{Y/M} \to \mathcal{F} \to \Omega_{Y/M} \otimes \zeta_M \to 0. \tag{18}
\]
We will now determine the normal bundle of a curve \(\tilde{l} \subset Y = \mathbb{P}(T_M)\):

### 3.6. Lemma

We have
\[ T_Y \otimes \mathcal{O}_{\tilde{l}} \simeq \mathcal{O}_{\tilde{l}}(2) \oplus \mathcal{O}_{\tilde{l}} \oplus \mathcal{O}_{\tilde{l}}(-1) \oplus \mathcal{O}_{\tilde{l}}(-2)^{\oplus 2} \]

**Proof.** The curve \(\tilde{l}\) is the section of \(\mathbb{P}(T_M \otimes \mathcal{O}_l) \to l\) corresponding to the negative quotient \(\mathcal{O}_l(2) \oplus \mathcal{O}_l(1) \oplus \mathcal{O}_l(-1) \to \mathcal{O}_l(-1)\). Thus we have
\[ T_{Y/M} \otimes \mathcal{O}_{\tilde{l}} \simeq T_{\mathbb{P}(T_M \otimes \mathcal{O}_l)/l} \otimes \mathcal{O}_{\tilde{l}} \simeq N_{i/\mathbb{P}(T_M \otimes \mathcal{O}_l)} \simeq \mathcal{O}_{\tilde{l}}(-3) \oplus \mathcal{O}_{\tilde{l}}(-2). \]
Since \(\zeta_M \cdot \tilde{l} = -1\) this implies that
\[ \Omega_{Y/M} \otimes \zeta_M \otimes \mathcal{O}_{\tilde{l}} \simeq \mathcal{O}_{\tilde{l}}(2) \oplus \mathcal{O}_{\tilde{l}}(1). \]
Sequence \((18)\) therefore reads
\[ 0 \to \mathcal{O}_{\tilde{l}}(-3) \oplus \mathcal{O}_{\tilde{l}}(-2) \to \mathcal{F} \otimes \mathcal{O}_{\tilde{l}} \to \mathcal{O}_{\tilde{l}}(2) \oplus \mathcal{O}_{\tilde{l}}(1) \to 0 \]
Thus if we write
\[ \mathcal{F} \otimes \mathcal{O}_{\tilde{l}} \simeq \oplus_{i=1}^{a_1} \mathcal{O}_{\tilde{l}}(a_i) \]
with \(a_1 \geq a_2 \geq a_3 \geq a_4\), then \(a_1 \leq 2\).

Consider now the natural inclusion \(T_{\tilde{l}} \hookrightarrow T_Y \otimes \mathcal{O}_{\tilde{l}}\): since \(\zeta_M \cdot \tilde{l} = -1\) we deduce from \((16)\) that \(T_{\tilde{l}} \hookrightarrow \mathcal{F} \otimes \mathcal{O}_{\tilde{l}}\). Hence we have \(a_1 = 2\). Since by \((17)\) we have \(\mathcal{F} \otimes \mathcal{O}_{\tilde{l}} \simeq \mathcal{F}^* \otimes \mathcal{O}_{\tilde{l}}(-1)\) this also implies that \(a_4 = -3\). Since \(c_1(\mathcal{F}) \cdot \tilde{l} = -2\), we are left with two possibilities:
\[ \mathcal{F} \otimes \mathcal{O}_{\tilde{l}} \simeq \mathcal{O}_{\tilde{l}}(2) \oplus \mathcal{O}_{\tilde{l}} \oplus \mathcal{O}_{\tilde{l}}(-1) \oplus \mathcal{O}_{\tilde{l}}(-3) \]
or
\[ \mathcal{F} \otimes \mathcal{O}_{\tilde{l}} \simeq \mathcal{O}_{\tilde{l}}(2) \oplus \mathcal{O}_{\tilde{l}}(1) \oplus \mathcal{O}_{\tilde{l}}(-2) \oplus \mathcal{O}_{\tilde{l}}(-3). \]
Let us exclude the latter case: since \(S \subset Y\) is a smooth surface, we have an injective morphism of vector bundles
\[ \mathcal{O}_{\tilde{l}}(2) \oplus \mathcal{O}_{\tilde{l}} \simeq T_S \otimes \mathcal{O}_{\tilde{l}} \hookrightarrow T_Y \otimes \mathcal{O}_{\tilde{l}}. \]
Using again that $\zeta_M \cdot \tilde{l} = -1$ and (16), the inclusion lifts to an injective morphism of vector bundles
\[
\mathcal{O}_l(2) \oplus \mathcal{O}_l \hookrightarrow \mathcal{F} \otimes \mathcal{O}_l.
\]
Yet if $\mathcal{F} \otimes \mathcal{O}_l \simeq \mathcal{O}_l(2) \oplus \mathcal{O}_l(1) \oplus \mathcal{O}_l(-2) \oplus \mathcal{O}_l(-3)$, such an injective morphism does not exist.

Using again the extension (16) we deduce
\[
T_Y \otimes \mathcal{O}_l \simeq \mathcal{O}_l(2) \oplus \mathcal{O}_l \oplus \mathcal{O}_l(-1)^{\oplus 2} \oplus \mathcal{O}_l(-3)
\]
or
\[
T_Y \otimes \mathcal{O}_l \simeq \mathcal{O}_l(2) \oplus \mathcal{O}_l \oplus \mathcal{O}_l(-1) \oplus \mathcal{O}_l(-2)^{\oplus 2}.
\]
Thus, it remains to exclude the first case which is exactly the case when the restricted contact sequence
\[
0 \to \mathcal{F} \otimes \mathcal{O}_l \to T_Y \otimes \mathcal{O}_l \to \zeta_M \otimes \mathcal{O}_l \to 0
\]
splits. By [LeB95, Prop.2.5], the extension class $a$ of this sequence is the image of
\[
\frac{2\pi i}{2} [-K_Y]_l \in H^1(\tilde{l}, \Omega_Y \otimes \mathcal{O}_l)
\]
under a canonical map
\[
\mu : H^1(\tilde{l}, \Omega_Y \otimes \mathcal{O}_l) \to H^1(\tilde{l}, \mathcal{F} \otimes \zeta_M \otimes \mathcal{O}_l).
\]
Moreover, the kernel of $\mu$ is contained in $H^1(\tilde{l}, \zeta_M^* \otimes \mathcal{O}_l) = 0$, so $\mu$ is injective. Further, $-K_Y \cdot \tilde{l} < 0$, hence $a \neq 0$. □

Note that the proof above shows that the inclusion $T_S \hookrightarrow T_Y \otimes \mathcal{O}_S$ factors through an inclusion
\[
T_S \hookrightarrow \mathcal{F} \otimes \mathcal{O}_S.
\]
inducing an exact sequence
\[
0 \to T_S \to \mathcal{F} \otimes \mathcal{O}_S \to \Omega_S \otimes \zeta_M \to 0.
\]
Sequences (16) and (19) now yield an exact sequence
\[
0 \to \Omega_S \otimes \zeta_M \to N_{S/Y} \to \zeta_M \otimes \mathcal{O}_S \to 0.
\]

3.7. Proposition. We have an exact sequence
\[
0 \to \mathcal{O}_{C \times \tilde{l}}(-7, 2)^{\oplus 2} \to N_{S/Y}^* \to \mathcal{O}_{C \times \tilde{l}}(-5, 1) \to 0.
\]

Proof. By Lemma 3.2 we have
\[
\zeta_M \otimes \mathcal{O}_S \simeq \mathcal{O}_{C \times \tilde{l}}(7, -1).
\]
Since $T_S \simeq \mathcal{O}_{C \times \tilde{l}}(2, 0) \oplus \mathcal{O}_{C \times \tilde{l}}(0, 2)$, we deduce from (20) an exact sequence
\[
0 \to \mathcal{O}_{C \times \tilde{l}}(-7, 1) \to N_{S/Y}^{\ast} \to \mathcal{O}_{C \times \tilde{l}}(-5, 1) \oplus \mathcal{O}_{C \times \tilde{l}}(-7, 3) \to 0.
\]
By Lemma 3.6 we have $N_{S/Y}^* \otimes \mathcal{O}_l \simeq \mathcal{O}_l(1) \oplus \mathcal{O}_l(2)^{\oplus 2}$. Thus if $G$ denotes the kernel of $N_{S/Y}^* \to \mathcal{O}_{C \times \tilde{l}}(-5, 1)$, it sits in a non-split extension
\[
0 \to \mathcal{O}_{C \times \tilde{l}}(-7, 1) \to G \to \mathcal{O}_{C \times \tilde{l}}(-7, 3) \to 0.
\]
Denoting by $p_C$ the projection onto $C$, the bundle $(p_C)_* (G \otimes \mathcal{O}_{C \times \tilde{l}}(0, -2))$ has rank two, and the evaluation morphism
\[
p_C^*(p_C)_* (G \otimes \mathcal{O}_{C \times \tilde{l}}(0, -2)) \to G \otimes \mathcal{O}_{C \times \tilde{l}}(0, -2)
\]
is surjective. It is straightforward to see that
\[
(p_C)_* (G \otimes \mathcal{O}_{C \times \tilde{l}}(-7, 1)) \simeq (p_C)_* \mathcal{O}_{C \times \tilde{l}}(-7, 1) \simeq \mathcal{O}_C(-7)^{\oplus 2}.
\]
Thus we have $G \simeq \mathcal{O}_{C \times \tilde{l}}(-7, 2)^{\oplus 2}$ which proves our claim. □
3.B. Geometry of $\mathbb{P}(V)$ and construction of an antiflip. Let us recall that the vector bundle $V$ is given by an extension
$$0 \to \mathcal{O}_M \to V \to T_M \to 0$$
associated to the Kähler class $c_1(\mathcal{O}_M(1)) \in H^1(M, \Omega_M)$. Let $\tilde{\pi} : X = \mathbb{P}(V) \to M$ denote the projectivisation; thus $Y = \mathbb{P}(T_M) \subset X$, and we have an exact sequence
$$0 \to T_Y \to T_X \otimes \mathcal{O}_Y \to N_{Y/X} \simeq \mathcal{O}_{\mathbb{P}(V)}(1) \otimes \mathcal{O}_Y \simeq \zeta_M \to 0.$$ 
Let
$$\varphi_V : \mathbb{P}(V) \to \mathbb{P}_M$$
be the morphism associated with the base point free linear system $[\zeta_V \otimes \tilde{\pi}^* \mathcal{O}_M(H)]$.

3.8. Proposition. We have the following properties:

a) $\varphi_V|_Y = \varphi_M$;

b) the exceptional locus of $\varphi_V$ is $S$, and $\varphi_V$ contracts exactly the curves $\tilde{I}$;

c) $T_X \otimes \mathcal{O}_I = \mathcal{O}_I(2) \oplus \mathcal{O} \oplus \mathcal{O}_I(-1)^{\oplus 2} \oplus \mathcal{O}_I(-2)^{\oplus 2}$.

d) $N_{S/X} \otimes \mathcal{O}_I \simeq \mathcal{O}_I(-1)^{\oplus 2} \oplus \mathcal{O}_I(-2)^{\oplus 2}$.

Proof. The exact sequence
$$0 \to \mathcal{O}_M(1) \to V \otimes \mathcal{O}_M(1) \to T_M \otimes \mathcal{O}_M(1) \to 0$$
and Kodaira vanishing shows that $H^0(M, V \otimes \mathcal{O}_M(1)) \to H^0(M, T_M \otimes \mathcal{O}_M(1))$ is surjective, and the first statement follows. The second statement is then immediate from Lemma 3.3.

The third and fourth statement follow from Lemma 3.6 and the exact sequence (21). □

3.9. Proposition. We have an exact sequence
$$0 \to \mathcal{O}_{C \times I}(-7, 2)^{\oplus 2} \to N_{S/X}^* \to T \to 0$$
where $T$ is a rank two vector bundle given by an extension
$$0 \to \mathcal{O}_{C \times I}(-7, 1) \to T \to \mathcal{O}_{C \times I}(-5, 1) \to 0.$$

Proof. By Proposition 3.7 there is an exact sequence
$$0 \to \mathcal{O}_{C \times I}(-7, 2)^{\oplus 2} \to N_{S/Y}^* \to \mathcal{O}_{C \times I}(-5, 1) \to 0.$$
Consider the exact sequence
$$0 \to N_{Y/X}^* \otimes \mathcal{O}_S \simeq \zeta_M^* \otimes \mathcal{O}_S \simeq \mathcal{O}_{C \times I}(-7, 1) \to N_{S/X}^* \to N_{S/Y}^* \to 0.$$

By Proposition 3.3(d) the direct image sheaf $(p_C)_* (N_{S/X}^* \otimes \mathcal{O}_{C \times I}(0, -2))$ has rank two. Moreover we can repeat the argument from the proof of Proposition 3.7 to establish a subbundle $\mathcal{O}_{C \times I}(-7, 2)^{\oplus 2} \to N_{S/X}^*$. The description of $T$ is now straightforward. □

We will now start the construction of the antiflip $X \dashrightarrow X^{-}$.

Step 1. Let
$$\mu_1 : X_1 \to X$$
be the blow-up of $X_1$ along $S$, and denote by $Y_1$ the strict transform of $Y$. Denote by
$$E_1 \simeq \mathbb{P}(N_{S/X}^*) \to S$$
the exceptional divisor, and by
$$F_1 \simeq \mathbb{P}(N_{S/Y}^*)$$
the exceptional locus of the induced blow-up $Y_1 \to Y$. By Proposition 3.9 and Proposition 3.7 there exists a surface

$$S_1 = \mathbb{P}(\mathcal{O}_{C \times I}(-5, 1)) \subset F_1 \subset E_1$$

(22) corresponding to a quotient $N_{S_1/X}^* \to \mathcal{O}_{C \times I}(-5, 1)$.

3.10. Lemma. We have

$$N_{S_1/F_1}^* \simeq \mathcal{O}_{C \times I}(-2, 1)^{\oplus 2}$$

and

$$N_{S_1/Y_1}^* \simeq p_C^* W_C \otimes \mathcal{O}_{C \times I}(0, 1)$$

(23) where $W_C$ is a rank three vector bundle on $C$ given by an extension

$$0 \to \mathcal{O}_C(-5) \to W_C \to \mathcal{O}_C(-2)^{\oplus 2} \to 0.$$  

Proof. The expression for the conormal bundle $N_{S_1/F_1}^* = N_{\mathbb{P}(\mathcal{O}_{C \times I}(-5, 1))}^{\mathbb{P}(N_{S_i/Y_i})}$ is immediate from Proposition 3.7. For the computation of $N_{S_1/Y_1}^*$ consider the exact sequence

$$0 \to N_{F_1/Y_1}^* \otimes \mathcal{O}_{S_1} \simeq \mathcal{O}_{C \times I}(-5, 1) \to N_{S_1/Y_1}^* \to N_{S_1/F_1}^* \simeq \mathcal{O}_{C \times I}(-2, 1)^{\oplus 2} \to 0$$

and observe that $N_{S_1/F_1}^* \otimes \mathcal{O}_I \simeq \mathcal{O}_I(1)^{\oplus 3}$. Note that $S_1 \simeq S = C \times \tilde{I}$, and let $p_C : S_1 \to C$ denote the projection. Twisting the exact sequence with $\mathcal{O}_{C \times I}(0, -1)$ and pushing forward via $p_C$ we obtain an exact sequence

$$0 \to \mathcal{O}_C(-5) \to W_C := (p_C)_*(N_{S_1/F_1}^* \otimes \mathcal{O}_{C \times I}(0, -1)) \to \mathcal{O}_C(-2)^{\oplus 2} \to 0.$$  

Since the relative evaluation map

$$p_C^* \left( (p_C)_*(N_{S_1/F_1}^* \otimes \mathcal{O}_{C \times I}(0, -1)) \right) \to N_{S_1/F_1}^* \otimes \mathcal{O}_{C \times I}(0, -1)$$

is surjective, this shows the statement. \hfill \square

Using the notation of Proposition 3.9 we set

$$T_1 := \mathbb{P}(T) \subset \mathbb{P}(N_{S_1/X}) = E_1,$$

so that $T_1 \cap Y_1 = S_1$. In order to simplify the notation, we denote by $\mu_1 : T_1 \to S_1$ the restriction of $\mu_1$ to $T_1$. Since $T \otimes \mathcal{O}_I = \mathcal{O}_I(1) \oplus \mathcal{O}_I(1)$, we may write

$$T \simeq p_C^*(\mathcal{O}_C) \otimes \mathcal{O}_{C \times I}(0, 1)$$

(24) with a rank 2-vector bundle $U_C \to C$ so that $T_1 = \mathbb{P}(T) \simeq \mathbb{P}(p_C^*(\mathcal{O}_C))$ and we have a commutative diagram

$$T_1 \simeq \mathbb{P}(T) \simeq S_1 \times_C \mathbb{P}(U_C)$$

$$\begin{array}{ccc}
T_1 & \simeq & \mathbb{P}(T) \\
\mu_1 & & \phi \\
S_1 & \simeq & C \times \tilde{I} \\
\overline{\mu} & & \pi_C \\
C & & \mathbb{P}(U_C) \\
\end{array}$$

Let

$$\mu_2 : X_2 \to X_1$$

be the blow-up of $X_1$ along $T_1$, and denote by $E_2 \subset X_2$ the exceptional divisor. For simplicity’s sake denote by $\mu_2 : E_2 \to T_1$ the restriction of $\mu_2$ to $E_2 \simeq \mathbb{P}(N_{T_1/X_1}^*)$.  

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3.11. Lemma. The divisor $E_2$ is a $\mathbb{P}^1$-bundle $\mu_3 : E_2 \to W$ over a fourfold $W$ that is a $\mathbb{P}^2$-bundle $p_F : W \simeq \mathbb{P}(U_F) \to \mathbb{P}(U_C)$.
Moreover we have a rank two quotient bundle
\[ U_F \to (\pi_C^*\mathcal{O}_C(-7)^{\oplus 2}) \otimes \zeta_{\mathbb{P}(U_C)}^* \]
such that
\[ \mathbb{P}(N^*_{T_1/E_1}) \simeq T_1 \times_{\mathbb{P}(U_C)} \mathbb{P}((\pi_C^*\mathcal{O}_C(-7)^{\oplus 2}) \otimes \zeta_{\mathbb{P}(U_C)}^*) \]
and the restriction of $\mu_3$ to $\mathbb{P}(N^*_{T_1/E_1}) \subset E_2$ corresponds to the projection on the second factor.

Proof. Since $T_1 = \mathbb{P}(T) \subset \mathbb{P}(N^*_{S/X}) = E_1$ we have $N^*_{T_1/E_1} \simeq (\mu_1^*\mathcal{O}_{C \times Y}(-7, 2)^{\oplus 2}) \otimes \zeta_{\mathbb{P}(T)}^*$ by Proposition 3.9. Consider now the exact sequence
\[ 0 \to N^*_{E_1/X_1} \otimes \mathcal{O}_T(0, 1) \to N^*_{T_1/X_1} \to (\mu_1^*\mathcal{O}_{C \times Y}(-7, 1)^{\oplus 2}) \otimes \zeta_{\mathbb{P}(T)}^* \to 0. \]
Twisting by $\mu_1^*\mathcal{O}_{C \times Y}(0, -1)$ and pushing forward via $q_F$ we argue as in the proof of Lemma 3.10 to obtain
\[ (\pi_C^*\mathcal{O}_C(-7)^{\oplus 2}) \otimes \zeta_{\mathbb{P}(U_C)}^* \]
where $\zeta_{\mathbb{P}(U_C)}$ is the tautological bundle on $\mathbb{P}(U_C)$. Thus the exact sequence above simplifies to
\[ 0 \to q_F^*\zeta_{\mathbb{P}(U_C)} \otimes \mu_1^*\mathcal{O}_{C \times Y}(0, 1) \to N^*_{T_1/X_1} \to (\mu_1^*\mathcal{O}_{C \times Y}(-7, 1)^{\oplus 2}) \otimes q_F^*\zeta_{\mathbb{P}(U_C)}^* \to 0. \]
Now the conclusion for $E_2$ immediate, since
\[ E_2 \simeq \mathbb{P}(N^*_{T_1/E_1}) \simeq \mathbb{P}(q_F^*U_F) \simeq T_1 \times_{\mathbb{P}(U_C)} \mathbb{P}(U_F). \]

For the statement about $\mathbb{P}(N^*_{T_1/E_1})$ just observe that the isomorphism $N^*_{T_1/X_1} \simeq q_F^*U_F \otimes \mu_1^*\mathcal{O}_{C \times Y}(0, 1)$ maps $N^*_{T_1/E_1}$ onto $q_F^*((\pi_C^*\mathcal{O}_C(-7)^{\oplus 2}) \otimes \zeta_{\mathbb{P}(U_C)}^*) \otimes \mu_1^*\mathcal{O}_{C \times Y}(0, 1)$. □

We summarise the situation in a commutative diagram
\[ \begin{array}{ccc}
E_2 & \simeq & \mathbb{P}(N^*_{T_1/X_1}) \simeq T_1 \times_{\mathbb{P}(U_C)} \mathbb{P}(U_F) \\
& & \\
& & \mu_3 \\
& & \\
& & \mu_2 \\
& & \\
& & \mu_1 \\
& & \\
S_1 & \simeq & C \times \tilde{l} \\
& & \\
& & \pi_C \\
C & \simeq & \mathbb{P}(U_C) \\
& & \\
& & \text{id} \\
& & W \simeq \mathbb{P}(U_F) \\
& & \\
\end{array} \]

Let $\tilde{l} \simeq \mathbb{P}^1$ be a fibre of the fibration $q_F$. Then it is not difficult to see that
\[ N^*_{I/X_1} \simeq \mathcal{O}_{\tilde{l}} \oplus \mathcal{O}_{\tilde{l}}(1)^{\oplus 3}, \]
the trivial part corresponding to the pull-back of $T_{\mathbb{P}(U_C)}$. It is well-known that a rational curve $\mathbb{P}^1 \subset Q$ in a smooth fourfold $Q$ with $N^*_{I/Q} \simeq \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3}$ can be
flipped, i.e. there exists a smooth fourfold $Q^-$ containing a $\mathbb{P}^2$ with $\mathcal{N}^*_P/Q^- \cong \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$ and $Q \setminus \mathbb{P}^1 \cong Q^- \setminus \mathbb{P}^1$. Indeed this is the inverse of the smooth fourfold flip described by Kawamata [Kaw89, Thm.1.1]. The next statement shows that in our situation we can construct a relative version of Kawamata’s antiflip:

3.12. Proposition. The fibration $\mu_3 : E_2 \to W$ extends to a bimeromorphic map $\mu_3 : X_2 \to X_3$ of compact complex manifolds such that $X_2$ is the blow-up of $X_3$ along $W$.

Remark. At this point we do not claim that $X_3$ is a projective manifold, although this will follow from Section 5.

Proof. Fujiki’s criterion [Fuj73, Thm.1] guarantees the existence of $\mu_3$ if we check that the conormal bundle $\mathcal{N}^*_{E_2/X_2}$ is ample on the fibres of the $\mathbb{P}^1$-bundle $\mu_3$. We will prove that $\mathcal{N}^*_{E_2/X_2}$ has degree one on the fibres, so the bimeromorphic map is a blow-up of the manifold $X_3$ along $W$.

Recall first that by (26) we have

$$\mathcal{N}^*_{T_1/X_1} \cong q^*_F U_{\mathbb{P}} \otimes \mu_1^* \mathcal{O}_{C \times \mathbb{P}}(0,1).$$

Since $\mathcal{N}^*_{E_2/X_2} \cong \zeta_{P(\mathcal{N}^*_{T_1/X_1})}$ we obtain

$$\mathcal{N}^*_{E_2/X_2} \cong \mu_3^* \zeta_{U_{\mathbb{P}}} \otimes \mu_2^* \mu_1^* \mathcal{O}_{C \times \mathbb{P}}(0,1).$$

Since $E_2 \cong T_1 \times_{\mathbb{P}(U_{\mathbb{P}})} W$, the $\mu_3$-fibres map isomorphically onto the $q_F$-fibres. Since $T_1 \cong S_1 \times_{\mathbb{C}} \mathbb{P}(U_{\mathbb{C}})$ the $q_F$-fibres map isomorphically onto the $p_C$-fibres. Yet $\mu_1^* \mathcal{O}_{C \times \mathbb{P}}(0,1)$ has degree one on the $p_C$-fibres, so the statement follows.

3.13. Proposition. Denote by $Y_3 \subset X_3$ the strict transform of $Y_1 \subset X_1$ under the “Kawamata anti-flip” $\mu_3 \circ \mu_2^{-1}$. Then $Y_3$ does not contain $W$.

Proof. Let $Y_2 \subset X_2$ be the strict transform of $Y_1$. The statement is equivalent to showing that $Y_2$ is disjoint from a general exceptional curve of $\mu_3$. Since $T_1$ is not contained in $Y_1$ we have $Y_2 \cong \mu_2^* Y_1$. Since the $\mu_3$-fibres map onto $q_F$-fibres it is sufficient to show that $Y_1 \cdot \mathcal{I} = 0$ where $\mathcal{I} \subset T_1$ is a $q_F$-fibre. By definition, cf. (23), the surface $S_1$ is given by the quotient $\mathcal{N}^*_S/X \to \mathcal{N}^*_S/Y \to \mathcal{O}_{C \times \mathbb{P}}(-5,1)$. By Proposition 5.10 the quotient map $\mathcal{N}^*_S/X \to \mathcal{O}_{C \times \mathbb{P}}(-5,1)$ factors into

$$\mathcal{N}^*_S/X \to T \to \mathcal{O}_{C \times \mathbb{P}}(-5,1),$$

so we have $Y_1 \cap T_1 = S_1$ and we are left to show that $S_1$ is disjoint from the general $q_F$-fibre. By (23) we have

$$T \cong p_C^* U_C \otimes \mathcal{O}_{C \times \mathbb{P}}(0,1)$$

and $U_C \to C$ is given by an extension

$$0 \to \mathcal{O}_C(-7) \to U_C \to \mathcal{O}_C(-5) \to 0.$$  

Hence we obtain that the isomorphism $T_1 = \mathbb{P}(T) \to \mathbb{P}(p_C^* U_C)$ maps the surface $S_1 = \mathbb{P}(\mathcal{O}_{C \times \mathbb{P}}(-5,1))$ onto

$$\mathbb{P}(p_C^* \mathcal{O}_C(-5)) \cong q_F^* \mathbb{P}(\mathcal{O}_C(-5))$$

where in the last step we consider $\mathbb{P}(\mathcal{O}_C(-5))$ as a divisor in $\mathbb{P}(U_C)$. In particular its intersection number with a general $q_F$-fibre is zero.

3.14. Proposition. The strict transform $E_1 \subset X_3$ of $E_1$ has a $\mathbb{P}^3$-bundle structure $\mu_4 : E_1 \to C \times D$, where $D \cong \mathbb{P}^1$. Moreover the fourfold $W$ is not contained $E_1$. 
Proof. We start considering the strict transform $E'_1 \subset Y_2$, i.e., the blowup of $E_1$ along $S_1$. Since $E_1 \cong \mathbb{P}(N^*_{S/X})$ and since $T_1 \subset E_1$ corresponds to the quotient

$$0 \to K := \mathcal{O}_{C \times \tilde{l}}(-7, 2)^{\oplus 2} \to N^*_{S/X} \to T \to 0,$$

the blowup $E'_1$ has a $\mathbb{P}^2$-bundle structure $\psi : E'_1 \to \mathbb{P}(K)$. This $\mathbb{P}^2$-bundle structure corresponds to a rank three vector bundle $Q$ given by an extension

$$(28) \quad 0 \to Q \to p^* T \otimes \zeta^{*}_{\mathbb{P}(K)} \to 0,$$

where $p : \mathbb{P}(K) \to S$ is the natural map and $\zeta_{\mathbb{P}(K)}$ the tautological divisor. Denote by $N \cong \mathbb{P}(N^*_{T_1/E_1})$ the exceptional divisor of $E'_1 \to E_1$. With the notation of (28) the exceptional divisor corresponds to the quotient $Q \to p^* T \otimes \zeta^{*}_{\mathbb{P}(K)}$.

Note also that since $K \cong \mathcal{O}_{C \times \tilde{l}}(-7, 2)^{\oplus 2}$, we have

$$\mathbb{P}(K) \cong C \times \tilde{l} \times D,$$

with $D \cong \mathbb{P}^1$. Hence we have fibration

$$\tau : E'_1 \to \mathbb{P}(K) \to C \times D$$

such that the fibres are $\mathbb{P}^2$-bundles over $\tilde{l}$. Restricting (28) to $\tilde{l}$ we see that

$$Q \otimes \mathcal{O}_l \cong \mathcal{O}_l \oplus \mathcal{O}_l(-1)^{\oplus 2},$$

so the fibres of $\tau$ are isomorphic to the blow-up of $\mathbb{P}^3$ along a line, and the exceptional divisor of $\mathbb{P}(Q \otimes \mathcal{O}_l) \to \mathbb{P}^3$ is given by the second projection of

$$N \cap \psi^{-1}(\tilde{l}) \cong \mathbb{P}(p^* T \otimes \zeta^{*}_{\mathbb{P}(K)} \otimes \mathcal{O}_l) \cong \mathbb{P}(\mathcal{O}_l(-1)^{\oplus 2}) \cong \tilde{l} \times \mathbb{P}^1.$$

Yet these are exactly the fibres of the fibration

$$N \to \mathbb{P}((\pi^{*}_{C} \mathcal{O}_C(-7)^{\oplus 2}) \otimes \zeta^{*}_{\mathbb{P}(U_C)})$$

given by Lemma 3.11 i.e. the restriction of $\mu_3$ to $N$. Since the divisor $N \subset E'_1$ has degree minus one on these curves, we can again apply Fujiki’s criterion [Fuj75, Thm.1] to obtain a birational morphism

$$E'_1 \to Z \to C \times d$$

such that $\mu_4 : Z \to C \times d$ is a $\mathbb{P}^3$-bundle. Since the fibres of $E'_1 \to Z$ coincide with the fibres of $E'_1 \to E_1$, we have $Z \cong E_1$.

Finally let us show that $W$ is not contained in $E_1$. Since $\mu_3$ is an isomorphism in the complement of $E_2$, this comes down to show that

$$\mu_3(E'_1 \cap E_2) = \mu_3(N) = \mu_3(\mathbb{P}(N^*_{T_1/E_1}))$$

is a proper subset of $W$. Yet by Lemma 3.11 the birational map $\mu_3$ maps the fourfold $N$ onto the threefold $\mathbb{P}((\pi^{*}_{C} \mathcal{O}_C(-7)^{\oplus 2}) \otimes \zeta^{*}_{\mathbb{P}(U_C)})$. Thus it is a proper subset of the fourfold $W$. \hfill \Box

Finally we construct $\mu_4 : X_3 \to X_4$.

3.15. Proposition. The restriction of the normal bundle $N_{E_1/X_3}$ to the fibres of the $\mathbb{P}^3$-bundle $\mu_4 : \tilde{E}_1 \to C \times D$ is isomorphic to $\mathcal{O}_{\mathbb{P}^3}(-2)$.

In particular, by Fujiki’s criterion [Fuj75, Thm.1], there exists a birational morphism $\mu_4 : X_3 \to X_4$ to a normal compact complex space $X_4$ contracting $\tilde{E}_1$ onto $C \times D$. 

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Proof. We use the notation from the proof of Proposition 3.14. Since $W$, the center of the blow-up $\mu_3$, is not contained in $\tilde{E}_1$, we have

$$E'_1 = \mu_3^*E_1.$$

Let $C \subset E'_1$ be a line in a fibre of the $\mathbb{P}^2$-bundle $E'_1 \to \mathbb{P}(K)$. Then $\mu_3$ maps $C$ isomorphically to a line in a fibre of $\mu_4$. Hence it suffices to show that $E'_1 \cdot C = -2$. Now

$$E'_1 = \mu_3^*E_1 - E_2,$$

and $E'_1 \cap E_2 = N$ is a $\mathbb{P}^1$-subbundle of the $\mathbb{P}^2$-bundle $E'_1 \to \mathbb{P}(K)$. Hence $N \cdot C = 1$, and we are left to show that

$$\mu_3^*E_1 \cdot C = E_1 \cdot \mu_2(C) = -1.$$

The curve $\mu_2(C)$ is a line in a fibre of the $\mathbb{P}^2$-bundle $E_1 \to S_1$, and $E_1$ is the exceptional locus of the blow-up of $X$ along $S_1$. Hence $E_1 \cdot \mu_2(C) = -1$. □

3.16. Lemma. Denote by $X'$ the Stein factorisation of the image of the morphism $\varphi_V : X = \mathbb{P}(V) \to \mathbb{P}_M$. There exists a bimeromorphic morphism

$$\varphi^- : X_4 \to X'$$

such that the exceptional locus coincides with $\mu_4(W) \cup (C \times D)$. More precisely the restriction of $\varphi^-$ to $C \times D$ corresponds to the projection onto $C$, and the restriction to $\mu_4(W)$ corresponds to the fibration $\pi_C \circ \pi_F$ (cf. Diagram (27)).

Proof. Consider the birational morphism

$$\tau : X_2 \xrightarrow{\mu_3 \circ \mu_2} X \xrightarrow{\varphi_V} X',$$

and recall that $\varphi_V$ simply contracts $S \cong C \times \tilde{l}$ onto $C$. By the rigidity lemma this morphism factors through $\mu_4 \circ \mu_3 : X_2 \to X_4$ if we show that the $(\mu_4 \circ \mu_3)$-fibres are contained in $\tau$-fibres. The exceptional locus of $\mu_4 \circ \mu_3$ is equal to $E'_1 \cup E_2$ and the restriction of $\mu_4 \circ \mu_3$ to the exceptional divisors is given by

$$E'_1 \mu_3 \xrightarrow{\mu_4} \tilde{E}_1 \rightarrow C \times D$$

and

$$\mu_3 : E_2 \to W.$$

By (27) and (23) the $\mu_3$-fibres map onto $pc$-fibres, so for $E_2$ the property is clear. For $E'_1$ recall from the proof of Proposition 3.13 that we have a $\mathbb{P}^2$-bundle structure $E'_1 \to C \times \tilde{l} \times D$ so that the fibres of $E'_1 \to C \times D$ are $\mathbb{P}^2$-bundles over $\tilde{l}$. The restriction of $\tau$ to $E'_1$ coincides with

$$E'_1 \to C \times \tilde{l} \times D \to C,$$

so we obtain the desired property for $E'_1$. □

4. Proofs of the main results

Proof of Theorem 1.3. We use the notation introduced in Subsection 3.13. Although the birational map

$$\mu_3 \circ (\mu_1 \circ \mu_2)^{-1} : X \dashrightarrow X_3$$

is not an isomorphism in codimension one, we can show that $Z_M$ is not Stein. In fact, we have

$$Z_M \cong X \setminus Y \cong X_2 \setminus (Y_2 \cup E'_1 \cup E_2) \cong X_3 \setminus (Y_3 \cup \tilde{E}_1 \cup W).$$

By Proposition 3.13 and 3.14 the codimension two subset $B$ is not contained in the divisors $Y_3 \cup \tilde{E}_1$. Thus by [GR04, V, §1, Thm.4] the complex space $Z_M$ is not Stein. □
codimension one we have left to show nefness. Since and 
\[ X^− := \mathbb{P}(V)^− := X_4, \quad \psi := (\mu_4 \circ \mu_3) \circ (\mu_1 \circ \mu_2)^−1 : X \dashrightarrow X^− \]
and denote by \( Y^− := \mathbb{P}(T_M)^− := Y_4 \) the strict transform of \( Y = \mathbb{P}(T_M) \) in \( X_4 \). By construction of \( \mu_1 \) and \( \mu_2 \), the map
\[ X_2 \setminus (E'_1 \cup E_2) \to X \setminus S \]
is an isomorphism. By construction of \( \mu_3 \) and \( \mu_4 \), the map
\[ X_2 \setminus (E'_1 \cup E_2) \to X^− \setminus (\mu_4(W) \cup (C \times D)) \]
is an isomorphism. Thus it is clear that \( \psi \) is an isomorphism in codimension one.

Set \( A = \mu_4(W) \cup (C \times D) \). Since \( W \) is not contained in \( Y_3 \) by Proposition 3.13 and since \( \mu_4 \) is an isomorphism at the generic point of \( W \) by Proposition 3.14, the fourfold \( \mu_4(W) \) is not contained in \( Y^− \). This shows statement \( a \).

By what precedes the bimeromorphic map \( \psi \) induces an isomorphism
\[ Y \setminus S \to Y^− \setminus (A \cap Y^−). \]
Since \( \mu_4(B) \) is not contained in \( Y^− \), the intersection \( A \cap Y^− \) has codimension at least two in \( Y^− \). Thus \( Y \dashrightarrow Y^− \) is an isomorphism in codimension one. To show that \( X \dashrightarrow X^− \) is antiflip as well as \( Y \dashrightarrow Y^− \), it suffices that \( -K_{Y^−} \) is \( \varphi^∗|_{Y^−} \)-ample. This is shown in Proposition 4.1 below. Thus statement \( b \) is established.

For the proof of \( c \), note that by construction it is clear that both spaces \( X^− \) and \( Y^− \) are \( \mathbb{Q} \)-factorial with terminal singularities. Since \( -K_{X^−} \) is \( \varphi^∗\)-ample by \( b \), the bimeromorphic morphism \( \varphi^− : X^− \to X^′ \) is projective. Since \( X^′ \) is a projective variety, the complex space \( X^− \) is thus projective. Thus it suffices to show that the anticanonical divisors are big and nef. Since \( \psi \) and \( \varphi^∗|_{Y^−} \) are isomorphisms in codimension one we have
\[ -K_{X^−} = \psi^∗(-K_X) = \psi^∗(4c_1(\zeta_V)) \]
and
\[ -K_{Y^−} = (\varphi^∗|_{Y^−} )^∗(-K_Y) = (\varphi^∗|_{Y^−} )^∗(3c_1(\zeta_M)). \]

Since \( T_M \) is big, the bigness of the anticanonical divisor is thus clear and we are left to show nefness. Since
\[ Y^− \in \frac{1}{4} (-K_{X^−}), \]
it suffices to show that \( -K_{Y^−} \) is nef. This is settled in Proposition 4.1.

Finally, statement \( d \) follows from [HP21, Thm.1.2]. \( \square \)

4.1. Proposition. The anticanonical divisor of \( Y^− \) is nef. Furthermore, the anticanonical divisor of \( Y^− \) is relatively ample with respect to the morphism \( \varphi^− \).

Proof. We set \( \nu_j := \mu_1|_{Y_j} \), and
\[ F'_1 := E'_1 \cap Y_2, \quad \tilde{F}_1 := E_1 \cap Y_3. \]
It is not hard to check that all the intersections are transversal, in particular we have \( \nu_2^∗F_1 = F'_1 + F_2 \).

Set \( g = \nu_1 \circ \nu_2 : Y_2 \to Y \) and \( h = \nu_4 \circ \nu_3 : Y_2 \to Y^− = Y_4 \). Since \( S_1 \) is contained in \( F_1 \), we have
\[ -K_{Y_2} = g^∗(-K_Y) - 2F'_1 - 4F_2. \]
Since $\nu_1$ is a smooth blowup and along a center that is not contained in $\tilde{F}_1$, (Proposition 3.14) and $\nu_1$ contracts the $\mathbb{P}^2$-bundle $\tilde{F}_1 \to C \times D$ onto $C \times D$ with $N_{\tilde{F}_1/Y_3} \otimes \mathcal{O}_{Y_2} \simeq \mathcal{O}_{Y_2}(-2)$, we have

$$-K_{Y_2} = h^*(-K_{Y_1}) - \frac{3}{2} F_1' - F_2.$$

Thus we obtain

$$h^*(-K_{Y_1}) = g^*(-K_Y) - \frac{3}{2} F_1' - 3F_2 = g^*(-K_Y) - 3\nu_2 F_1 + \frac{3}{2} F_1'.$$

Recall first that $-K_Y = 3\zeta_M$ and $\text{Bs}([\zeta_M]) = S \cup Z$ where $Z := \mathbb{P}(N_{B/M})$. Thus we have

$$\text{Bs}((\psi_Y)_* [\zeta_M]) \subset h(F_1' \cup F_2 \cup Z_2)$$

where $Z_2 \subset Y_2$ is the strict transform of $Z$.

**Step 1.** $h^*(-K_{Y_1})$ is nef on $Z_2$. Since $N_{B/M} = \mathcal{O}_B(5) \oplus \mathcal{O}_B(5)$ by (6), we have $Z \simeq B \times \mathbb{P}^1$. Moreover since $N^*_B(M) \simeq N_{B/M} \otimes \mathcal{O}_B(-10)$, the blowup of $M$ along $B$ embeds into $Y = \mathbb{P}(T_M)$. Thus we can apply [CS10] Lemma 7.2.3(iii)] to see that the intersection $Z \cap S$ is not transversal and $[Z \cap S] = 2\Delta$ where $\Delta \in [\mathcal{O}_{B\times \mathbb{P}^1}(1, 1)]$. Since $\nu_1$ is the blowup of $Y$ along $S$, the intersection of the strict transform $Z_1 \subset Y_1$ with the exceptional divisor $F_1$ has still support along $\Delta$, yet now the intersection is transversal. Thus we obtain that

$$\mathcal{O}_{Z_2}(F_1) \simeq \mathcal{O}_{B \times \mathbb{P}^1}(1, 1).$$

We claim that

$$F_1 \cap Z_1 = S_1 \cap Z_1.$$

Indeed by (22) the surface $S_1 \subset F_1$ corresponds to the quotient $N^*_{S/Y} \to \mathcal{O}_{C\times \mathbb{I}}(-5, 1)$, which is the quotient determined by the direct factor $\mathcal{O}_{C\times \mathbb{I}}(2, 0)$, i.e. the relative tangent bundle of the projection $S \simeq C \times \mathbb{I} \to \mathbb{I}$. Yet these curves are tangent to $Z$ along $\Delta$, so $\mathbb{P}(\mathcal{O}_{C\times \mathbb{I}}(-5, 1) \otimes \mathcal{O}_\Delta)$ is contained in $Z_1$. This shows the claim.

Since $\nu_2$ is the blow-up of $Y_1$ along $S_1$ and $S_1 \cap Z_2 = F_1 \cap Z_1 \subset Z_1$ is a Cartier divisor, the strict transform $Z_2 \subset Y_2$ is isomorphic to $Z_1 \simeq Z \simeq B \times \mathbb{P}^1$. Now we can finally compute the restriction: since $N_{B/M} \simeq \mathcal{O}_B(5) \oplus \mathcal{O}_B(5)$, we have

$$\mathcal{O}_{Z}(K_Y) \simeq \zeta^3_M \otimes \mathcal{O}_{Z} \simeq \mathbb{O}_B(3) \otimes \mathcal{O}_Z \simeq \mathcal{O}_{B \times \mathbb{P}^1}(15, 3).$$

Hence $\mathcal{O}_{Z_2}(g^*(-K_Y)) \simeq \mathcal{O}_{B \times \mathbb{P}^1}(15, 3)$. By (30) we have $\mathcal{O}_{Z_2}(\nu_2^* F_1) \simeq \mathcal{O}_{B \times \mathbb{P}^1}(1, 1)$, hence

$$\mathcal{O}_{Z_2}(g^*(-K_Y) - 3\nu_2^* F_1 + \frac{3}{2} F_1') \simeq \mathcal{O}_{B \times \mathbb{P}^1}(12, 0) \otimes \mathcal{O}_{Z_2}(\frac{3}{2} F_1'),$$

is nef, since any effective divisor on $Z_2 \simeq B \times \mathbb{P}^1$ is nef. By (29) this finishes the proof of Step 1.

**Step 2.** $h^*(-K_{Y_1})$ is nef on $F_2$. Since $\nu_2$ is the blowup of $S_1 \subset Y_1$ and $S_1 \simeq S$ we obtain from Lemma 5.2 that

$$\mathcal{O}_{F_2}(g^*(-K_{Y_1})) \simeq g^* \mathcal{O}_{C\times \mathbb{I}}(21, -3).$$

Furthermore $F_2 \simeq \mathbb{P}(N^*_{S_1/Y_1})$ and $F_1' \cap F_2 \simeq \mathbb{P}(N^*_{S_1/F_1})$, so by (22)

$$\mathcal{O}_{F_2}(F_1') \simeq \mathcal{O}_{\mathbb{P}(N^*_{S_1/Y_1})}(1) \otimes \nu_2^* \mathcal{O}_{C\times \mathbb{I}}(5, -1).$$

Since $\mathcal{O}_{F_2}(F_2) \simeq \mathcal{O}_{\mathbb{P}(N^*_{S_1/Y_1})}(-1)$ we obtain from (22) that

$$\mathcal{O}_{F_2}(2h^*(-K_{Y_1})) = (\mathcal{O}_{\mathbb{P}(N^*_{S_1/Y_1})}(1) \otimes \nu_2^* \mathcal{O}_{C\times \mathbb{I}}(9, -1))^\otimes 3.$$
Yet by (29) we have $N^*_i \times Y \simeq p^*_C W_C \otimes O_{C \times i}(0,1)$, so
$$N^*_i \otimes O_{C \times i}(9,-1) \simeq p^*_C W_C \otimes O_{C \times i}(9,0)$$
is nef by the extension defining $W_C$ (cf. Lemma 5.10).

Note that this also shows that $\mu_3$ maps $F_2$ onto $P(W_C)$ and $P(W_C) \subset Y_4$ is one of the two irreducible components of the exceptional locus of $\varphi^{-1}_Y$. Since $-K_Y$ is a positive multiple of the tautological class along the fibres of $P(W_C) \to C$, we obtain the first half of the second statement.

**Step 3.** $h^*(-K_{Y_4})$ is nef on $F'_1$. We proceed analogously to the proof of Proposition 5.14 since $F_1 \simeq P(N^*_i/Y)$ and since $S_1 \subset F_1$ corresponds to the quotient
$$0 \to K := O(-7,2) \otimes N^*_i \to \mathbb{P}(N^*_i/Y) \to O(5,1) \to 0,$$the blowup $F'_1$ has a $\mathbb{P}^1$-bundle structure $\psi : F'_1 \to \mathbb{P}(K)$. This $\mathbb{P}^2$-bundle structure corresponds to a rank two vector bundle $Q_Y$ given by an extension
$$0 \to O \to Q_Y \to p^*O(5,1) \otimes \zeta^*_p(K) \to 0,$$
where $p : \mathbb{P}(K) \to S$ is the natural map and $\zeta_p(K)$ the tautological divisor. The exceptional divisor $N$ of the blow-up $F'_1 \to F_1$, i.e. the intersection $F_2 \cap F_1$, is given by the quotient $Q \to p^*O(5,1) \otimes \zeta^*_p(K)$. Thus we have
$$F_2 \cap F_1 = N.$$Note also that $v^*_2 \zeta_p(N^*_i/Y)(1) \simeq p^*\zeta_p(K) \otimes O_{F'_1}(N)$, so
$$O_{F'_1}(v^*_2 F_1) \simeq v^*_2 \zeta_p(N^*_i/Y)(-1) \simeq p^*\zeta_p(K) \otimes O_{F'_1}(-N).$$

Thus by (29) we have
$$O_{F'_1}(2h^*(-K_{Y_4})) \simeq O_{F'_1}(2g^*(-K_Y) - 3v^*_2 F_1 - 3F_2) \simeq O_{F'_1}(2g^*(-K_Y)) \otimes p^*\zeta^*_p(K)^3.$$Since $K \simeq O(-7,2) \otimes N$ and $O_S(-K_Y) = O_{C \times i}(21,-3)$, we see that
$$O_{F'_1}(2g^*(-K_Y)) \otimes p^*\zeta^*_p(K)^3 \simeq O_{P(O_{C \times i}(7,0)^{\otimes 2})}(3)$$is nef. This equation also shows that the restriction of $-K_{Y_4}$ to $C \times D$ is isomorphic to $O_{C \times D}(21,3)$. Since $C \times D$ is one of the two irreducible components of the exceptional locus of $\varphi^{-1}_Y$, we obtain the second half of the second statement.

**Step 4.** Conclusion. By Step 2 and 3, the morphism $\varphi^{-1}$ is projective, in fact $-K_Y$ is relatively ample. Since the image of $\varphi^{-1}$ is a projective variety, the complex space $Y_4 = Y$ is thus projective. In particular we can verify the nefness of the anticanonical divisor by computing the intersection with curves in $Y_4$.

Since $-K_{Y_4} \simeq 3(\varphi_Y)_* \zeta_M$, it is sufficient to show that $-K_{Y_4}$ is nef on $h(F'_1 \cup F_2 \cup Z_2)$. Yet by the Steps 1-3 the divisor $h^*(-K_{Y_4})$ is nef on $F'_1 \cup F_2 \cup Z_2$, this finishes the proof.

**Proof of Theorem 1.2.** Suppose first that $\rho(M) \geq 2$. By [HP21, Cor.1.8], any contraction $f : M \to N$ of an extremal ray is a fibre space. Further, by Corollaries 2.4 and 2.5, the fibration $f$ is either a $\mathbb{P}^1$-bundle over the surface $N$, a $\mathbb{P}^2$-bundle or a quadric bundle over $\mathbb{P}^1$. Going through the classification [MM82, MM83, MM93], only the following cases remain:

a) $M = \mathbb{P}(T\mathbb{P}^2)$;
b) $M = \mathbb{P}^2 \times \mathbb{P}^1$;
c) $M = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. 

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We are therefore reduced to the case $\rho(M) = 1$. Since $Z_M$ is affine, the tangent bundle $T_M$ is big [GW20, Prop.4.2]. Hence we know by [HL21, Cor.1.2] that $M \simeq \mathbb{P}^3, M \simeq \mathbb{Q}^3$ or $M = V_5$, the del Pezzo threefold of degree 5. This last case is excluded by Theorem 3.1.

\[ \Box \]

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