STABILITY OF RADIAL SOLUTIONS OF THE POISSON–NERNST–PLANCK SYSTEM IN ANNULAR DOMAINS

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ABSTRACT. In this paper, we consider radial solutions of the Poisson–Nernst–Planck (PNP) system with variable dielectric coefficients \( \varepsilon g(\mathbf{x}) \) in \( N \)-dimensional annular domains, \( N \geq 2 \). When the parameter \( \varepsilon \) tends to zero, the PNP system admits a boundary layer solution as a steady state, which satisfies the charge conserving Poisson–Boltzmann (CCPB) equation. For the stability of the radial boundary layer solutions to the time-dependent radial PNP system, we estimate the radial solution of the perturbed problem with global electroneutrality. We generalize the argument of the one spatial dimension case (cf. [18]) and find a new way to transform the perturbed problem. By choosing a suitable weighted norm, we then derive the associated energy law which can be used to prove that the \( H^{-1}_r \) norm of the solution of the perturbed problem decays exponentially in time with the exponent independent of \( \varepsilon \) if the coefficient of the Robin boundary condition of electrostatic potential has a suitable positive lower bound.

1. Introduction. The Poisson–Nernst–Planck (PNP) system describes the transportation of charge carriers in electrolytes, semiconductors, and so on. In recent decades, it has been used in many physical and biological applications [8, 9, 10, 16, 29, 30, 32, 35, 36, 37, 40]. There have been many results on well-posedness and long-time behavior of the solutions for the model [2, 5, 6, 13, 14, 31]. Also, various versions of the PNP system are concerned for various applications [15, 17, 20, 21, 26, 27, 38, 39]. In this paper, we consider the PNP system with variable dielectric coefficients, which takes the form

\[
\begin{align*}
n_t &= \nabla \cdot (\nabla n - n \nabla \phi) \\
p_t &= \nabla \cdot (\nabla p + p \nabla \phi) \\
\varepsilon \nabla \cdot (g \nabla \phi) &= n - p
\end{align*}
\]

for \( x \in \Omega \), and \( t > 0 \), where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^N \), \( N \geq 2 \). Here, \( n(x, t) \) and \( p(x, t) \) represent the density of anions and cations, and \( \phi \) is the electrostatic potential.

The coefficient \( \varepsilon g > 0 \) is related to the dielectric and proportional to the squared quotient of Debye length and the characteristic length of the domain \( \Omega \) [3, 11, 28, 34].

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For biological (especially physiological) applications, the coefficient $\varepsilon g$ is usually small. We regard $\varepsilon$ as a small parameter to describe the smallness and assume $g$ to be of order 1, that is, $g_0 \leq g \leq g_1$ for some positive constants $g_0$ and $g_1$. The PNP system admits boundary layer solutions with $\varepsilon > 0$ approaching zero. In order to study the Debye layers occurring in ionic solutions near the boundary when there is an applied electrostatic potential, such boundary layer solutions are employed to describe the Debye layers \[8, 19, 23, 24, 25, 31\]. The smallness of $\varepsilon$ is crucial for studying boundary layers. In contrast, there are some works concerning with the asymptotics for solutions with $\varepsilon$ large, for example, \[1\], but we will not focus on this case.

It is natural to consider the no-flux boundary condition for $n$ and $p$ to describe the insulated domain boundary. As for the electrostatic potential $\phi$, Dirichlet boundary condition is often used to describe the given surface potential. But when the capacitance effect is concerned, Robin type boundary condition for $\phi$ is used \[7, 22, 30, 41\]. Thus, we impose the boundary condition

\[
\begin{aligned}
&\left\{ \begin{array}{l}
(\nabla n - n \nabla \phi) \cdot \nu = (\nabla p + p \nabla \phi) \cdot \nu = 0 \\
\phi + \gamma_{\varepsilon} \frac{\partial \phi}{\partial \nu} = \phi_{bd}
\end{array} \right.
\end{aligned}
\] (2)

for $x \in \partial \Omega$, $t > 0$, where $\nu$ is the unit outer normal vector to $\partial \Omega$, $\phi_{bd} = \phi_{bd}(x)$ is the external electrostatic potential, and $\gamma_{\varepsilon}(x) > 0$ depends on $\varepsilon$, which can be viewed as a measure of the Stern layer thickness \[4\]. In \[24, 25\], the authors show how the height of the boundary layer depends on the ratio $\gamma_{\varepsilon}/\varepsilon$. In particular, if $\gamma_{\varepsilon}/\sqrt{\varepsilon} \to \infty$ as $\varepsilon \to 0$, then the solution $\phi$ converges to a constant uniformly and has no boundary layer.

We notice that system (1) with (2) conserves the total charges of individual ions, i.e., $\int_{\Omega} n(x, t)dx = \int_{\Omega} n(x, 0)dx = m_n$ and $\int_{\Omega} p(x, t)dx = \int_{\Omega} p(x, 0)dx = m_p$. Here, $m_n$ and $m_p$ are the total charges of anions and cations, respectively. Since most of the physical and biological systems are global electroneutral, i.e., $m_n = m_p$, we will assume that the electroneutrality holds true in the whole paper with $m_n = m_p = m_0$.

In this work, we study the stability of the PNP system in annular domains, which can be regarded as a cross section of a cylindrical core or a domain with a central obstacle. Such annular domains are also be studied in \[12, 33\]. If we assume $g(x) = g(r)$ is positive, smooth and radially symmetric in $\Omega$, $\phi_{bd} = \phi_i$ and $\gamma_{\varepsilon} = \gamma_{\varepsilon,i}$ on $\{x \in \mathbb{R}^N : |x| = R_i\}$, $i = 1, 2$, where $\phi_{i}'s$ and $\gamma_{\varepsilon,i}'s$ are constants, system (1) with boundary condition (2) has a radially symmetric steady-state solution $(n, p, \phi) = (n^0(r), p^0(r), \psi(r))$, $r = |x|$. By using the equations of $n$ and $p$ in (1), we can show that the steady-state solutions take the form

\[
\begin{aligned}
n^0 = \frac{m_0 e^{\psi}}{N \omega_N \int_{R_1}^{R_2} s^{N-1} e^{\psi(s)} ds}, \quad p^0 = \frac{m_0 e^{-\psi}}{N \omega_N \int_{R_1}^{R_2} s^{N-1} e^{-\psi(s)} ds},
\end{aligned}
\] (3)

where $\omega_N$ is the volume of the unit ball in $\mathbb{R}^N$, and $\psi$ satisfies the charge conserving Poisson–Boltzmann (CCPB) equation with boundary condition

\[
\begin{aligned}
&\left\{ \begin{array}{l}
\varepsilon r^{-(N-1)} \partial_r \left( r^{N-1} g \psi_r \right) = \frac{m_0 e^{\psi}}{N \omega_N \int_{R_1}^{R_2} s^{N-1} e^{\psi(s)} ds} - \frac{m_0 e^{-\psi}}{N \omega_N \int_{R_1}^{R_2} s^{N-1} e^{-\psi(s)} ds}, \\
\psi(R_1) - \gamma_{\varepsilon,1} \psi_r(R_1) = \phi_1 \text{ and } \psi(R_2) + \gamma_{\varepsilon,2} \psi_r(R_2) = \phi_2
\end{array} \right.
\end{aligned}
\] (4)
which is studied in [24] for one-dimensional case with constant-valued dielectric coefficients, and in [23] for general N-dimensional case with variable dielectric coefficients.

When $N = 1$, $g$ is a positive constant, the solution of (4) has boundary layers as $\varepsilon$ tends to zero [24], and the stability of the boundary layer solutions is studied in [18]. For general $N \geq 1$ and variable $g$, the solution will tend to a constant in the interior domain as $\varepsilon$ tends to zero, which is similar to the previous case. In particular, for annular domains, i.e., $\Omega = \{ x \in \mathbb{R}^N : R_1 < |x| < R_2 \}$ for some $0 < R_1 < R_2$, $N \geq 2$, the authors in [25] obtain more precise asymptotic behavior of radial solutions of (4).

From [25], if $\phi_1 = \phi_2$, solutions of (4) have no boundary layer. Without loss of generality, we assume $\phi_1 < \phi_2$. And the asymptotic behavior of the solutions of (4) is as follows:

**Lemma 1.1 (Asymptotic Behavior of Solutions of CCPB Equations [25]).** There exists a unique solution of (4) $\psi \in C^2((R_1, R_2)) \cap C^\infty((R_1, R_2))$ with $\phi_1 \leq \psi \leq \phi_2$. Moreover, there exists positive constants $\varepsilon^*$, $C$, and $M$ independent of $\varepsilon$ such that for $r \in (R_1, R_2)$, we have

$$
\left| \psi(r) - \psi \left( \frac{R_1 + R_2}{2} \right) \right| \leq C \left[ e^{-\frac{M(r-R_1)}{\sqrt{\varepsilon}}} + e^{-\frac{M(R_2-r)}{\sqrt{\varepsilon}}} \right]
$$

and

$$
|\psi_e(r)| \leq \frac{C}{\sqrt{\varepsilon}} \left( e^{-\frac{M(r-R_1)}{\sqrt{\varepsilon}}} + e^{-\frac{M(R_2-r)}{\sqrt{\varepsilon}}} \right)
$$

if $0 < \varepsilon < \varepsilon^*$. Let $\lim_{\varepsilon \to 0+} (\psi(R_1), \psi(R_2)) = (p_1, p_2)$ and $\lim_{\varepsilon \to 0+} u(r) = c$ for $r \in (R_1, R_2)$. If $0 \leq \lim_{\varepsilon \to 0+} \frac{\gamma_i}{\sqrt{\varepsilon}} = \gamma_i < \infty$ for $i = 1, 2$, then $\phi_1 \leq p_1 < c < p_2 \leq \phi_2$.

Lemma 1.1 shows that as $\varepsilon$ goes to zero, the solutions of (4) become flat in the interior and have boundary layers near the boundary if the limit $\lim_{\varepsilon \to 0+} \frac{\gamma_i}{\sqrt{\varepsilon}}$ is finite.

**Remark 1.** If we consider radial solutions of the CCPB equation (4) in a ball, instead of in an annulus, it is easy to check that the (unique) solution is the constant solution $\phi_{bd}$, which has no boundary layers.

There are works using the singular perturbation theory with $\varepsilon$ as the singular perturbation parameter. Existence and uniqueness near some leading-order state are established, both for time-dependent PNP system and the steady-state equations [3, 11]. Compared with those works, our main goal is to show that the steady-state solution $(n^0, p^0, \psi)$ is stable by studying the perturbed problem (see Subsection 1.1 below). Letting $(\tilde{n}, \tilde{p}, \tilde{\phi})$ be the solution of the perturbed problem, we will prove that the $H^{-1}$-norms of $\tilde{n}$ and $\tilde{p}$ decay exponentially to zero as $t \to \infty$ provided the initial data sufficiently close to the steady state.

In [18], which considers the one-dimensional case with constant-valued dielectric coefficients, an energy law is derived by a suitable transformation so that the $H^{-1}$-norms of $\tilde{n}$ and $\tilde{p}$ can be controlled. We are motivated to generalize the idea to find some transformations and derive energy laws for the $N$-dimensional case with radial symmetry.

The main difference between one-dimensional case and higher-dimensional case is that there are coefficients $r^{N-1}$ and $r^{-(N-1)}$ in the equations. When we derive the
energy law, there will be additional terms after integration by parts, as compared to the one-dimensional case $N = 1$. In order to overcome this difficulty, we use weighted norms with the weight function $r^m$ for some $m \in \mathbb{R}$ to be chosen. By choosing $m$ suitably, we can derive the corresponding weighted estimates, which allow us to obtain $H^{-1}$ estimates of $\tilde{n}$ and $\tilde{p}$ as in one-dimensional case. In the following, we now introduce the perturbed problem and some transformation we need.

1.1. The perturbed problem. We can observe that the system (1) with (2) is invariant under rotation. If the initial data is radially symmetric, then we will have radially symmetric solutions $(n, p, \phi)$ with $n = n(r, t)$, $p = p(r, t)$, $\phi = \phi(r, t)$, $r = |x|$. Therefore, (1) can be denoted as

\[
\begin{aligned}
&\begin{cases}
  n_t = r^{-(N-1)} \partial_r \left[ r^{N-1} (n_r - n \phi_r) \right] \\
  p_t = r^{-(N-1)} \partial_r \left[ r^{N-1} (p_r + p \phi_r) \right] \\
  \varepsilon r^{-(N-1)} \partial_r \left( r^{N-1} g \phi_r \right) = n - p
\end{cases}
\end{aligned}
\]

(5)

for $r \in (R_1, R_2)$, $t > 0$, and the boundary condition (2) becomes

\[
\begin{aligned}
&\begin{cases}
  n_r - n \phi_r = p_r + p \phi_r = 0 \text{ at } r = R_1, R_2 \\
  \phi - \gamma_{r,1} \phi_r = \phi_1 \text{ at } r = R_1 \\
  \phi + \gamma_{r,2} \phi_r = \phi_2 \text{ at } r = R_2
\end{cases}
\end{aligned}
\]

(6)

In order to get the stability of the boundary layer solution $(n^0, p^0, \psi)$, we assume that the system (1) with (2) has solution

\[
(n, p, \phi) = (n^0, p^0, \psi) + (\tilde{n}, \tilde{p}, \tilde{\phi}).
\]

Then we study the perturbed problem

\[
\begin{aligned}
&\begin{cases}
  \tilde{n}_t = r^{-(N-1)} \partial_r \left[ r^{N-1} (\tilde{n}_r - n^0 \tilde{\phi}_r - \tilde{n} \psi_r - \tilde{n} \tilde{\phi}_r) \right] \\
  \tilde{p}_t = r^{-(N-1)} \partial_r \left[ r^{N-1} (\tilde{p}_r + p^0 \tilde{\phi}_r + \tilde{p} \psi_r + \tilde{p} \tilde{\phi}_r) \right] \\
  \varepsilon r^{-(N-1)} \partial_r \left( r^{N-1} g \tilde{\phi}_r \right) = \tilde{n} - \tilde{p}
\end{cases}
\end{aligned}
\]

(7)

for $r \in (R_1, R_2)$, $t > 0$, with boundary condition

\[
\begin{aligned}
&\begin{cases}
  \tilde{n}_r - n^0 \tilde{\phi}_r - \tilde{n} \psi_r - \tilde{n} \tilde{\phi}_r = \tilde{p}_r + p^0 \tilde{\phi}_r + \tilde{p} \psi_r + \tilde{p} \tilde{\phi}_r = 0 \text{ at } r = R_1, R_2 \\
  \tilde{\phi} - \gamma_{r,1} \tilde{\phi}_r = 0 \text{ at } r = R_1 \\
  \tilde{\phi} + \gamma_{r,2} \tilde{\phi}_r = 0 \text{ at } r = R_2
\end{cases}
\end{aligned}
\]

(8)

for $x \in \partial \Omega$, $t > 0$. By letting

\[
\delta = \tilde{n} - \tilde{p}, \quad \eta = \tilde{n} + \tilde{p},
\]

(9)

system (7) with boundary condition (8) becomes

\[
\begin{aligned}
&\begin{cases}
  \delta_t = r^{-(N-1)} \partial_r \left[ r^{N-1} (\delta_r - \eta^0 \delta_r - \eta \psi_r - \eta \delta_r) \right] \\
  \eta_t = r^{-(N-1)} \partial_r \left[ r^{N-1} (\eta_r - \delta^0 \eta_r - \delta \psi_r - \delta \delta_r) \right] \\
  \varepsilon r^{-(N-1)} \partial_r \left( r^{N-1} g \delta_r \right) = \delta
\end{cases}
\end{aligned}
\]

(10)
where

\[
\delta^0 = n^0 - p^0, \quad \eta^0 = n^0 + p^0.
\]

By (3), (4), and (12), \(\psi\) satisfies

\[
\varepsilon \partial_r \left( r^{N-1} g \psi_r \right) = r^{N-1} \delta^0.
\]

Moreover, (10) can be rewritten as

\[
\begin{aligned}
\begin{cases}
r^{-1} \delta_t = \partial_r \left[ r^{-1} (\delta_r - \eta^0 \phi_r - \eta \phi_r - \eta \phi_r) \right] \\
r^{-1} \eta_t = \partial_r \left[ r^{-1} (\eta_r - \delta^0 \phi_r - \delta \phi_r - \delta \phi_r) \right] \\
\varepsilon \partial_r \left( r^{-1} g \phi_r \right) = r^{-1} \delta
\end{cases}
\end{aligned}
\]

By (14) and (11), we have

\[
\frac{d}{dt} \int_{R_1}^{R_2} r^{-1} \delta dr = \frac{d}{dt} \int_{R_1}^{R_2} r^{-1} \eta dr = 0,
\]

which implies

\[
\begin{aligned}
\int_{R_1}^{R_2} r^{-N-1} \delta (r, t) dr &= \int_{R_1}^{R_2} r^{-N-1} \tilde{\delta}_0 (r) dr, \\
\int_{R_1}^{R_2} r^{-N-1} \eta (r, t) dr &= \int_{R_1}^{R_2} r^{-N-1} \tilde{\eta}_0 (r) dr
\end{aligned}
\]

for all \(t > 0\), where \(\tilde{\delta}_0 (r) = \delta (r, 0)\) and \(\tilde{\eta}_0 (r) = \eta (r, 0)\) are the initial data. Let

\[
D (r, t) = \int_{R_1}^{r} s^{-N-1} \delta (s, t) ds,
\]

\[
H (r, t) = \int_{R_1}^{r} s^{-N-1} \eta (s, t) ds
\]

for \(r \in (R_1, R_2), t > 0\). If the initial data \(\tilde{\delta}_0\) and \(\tilde{\eta}_0\) satisfy

\[
\int_{R_1}^{R_2} r^{-N-1} \delta_0 (r) dr = \int_{R_1}^{R_2} r^{-N-1} \eta_0 (r) dr = 0,
\]

then \(D\) and \(H\) satisfy

\[
D = H = 0
\]

at \(r = R_1, R_2,\) for all \(t > 0\). That is, \(D\) and \(H\) satisfy the zero Dirichlet boundary conditions.

By integrating equations of \(\delta\) and \(\eta\) in (14) from \(R_1\) to \(r\), we get

\[
D_t = r^{N-1} \left[ r^{-N-1} D_r \right] = r^{N-1} \left[ \eta^0 \phi_r - \eta \phi_r - \delta \phi_r \right],
\]

\[
H_t = r^{N-1} \left[ r^{-N-1} H_r \right] = r^{N-1} \left[ \eta^0 \phi_r - \eta \phi_r - \delta \phi_r \right]
\]
1.2. Main results.

**Theorem 1.2.** Let \((\delta, \eta, \phi)\) be a solution of (10) with boundary condition (11). Suppose that \(0 < \gamma_i, \delta_i < \gamma_{1,1} < \infty\) for \(i = 1, 2\), \(0 < g_0 \leq g \leq g_1 < \infty\), \(|g_1| \leq g_1\), and (19) hold true. There exist constants \(\varepsilon_0\) and \(K\) depending only on \(N, R_1, R_2, \delta_1, \gamma_{1,1}, \gamma_{1,2}\) such that if \(0 < \varepsilon < \varepsilon_0\) and \(\frac{\gamma_{1,1}}{\sqrt{N}} \geq K\), \(\delta_i \geq 1\), then (\(D, H\) defined by (17)--(18) satisfies

\[
\frac{1}{2} \int_{R_1}^{R_2} r^{-(N-1)} (D^2 + H^2) \, dr \\
\leq \left[ \frac{1}{2} - \frac{2}{\varepsilon^2 R_1^{2(N-1)}} g_0^2 \int_{R_1}^{R_2} r^{-(N-1)} D^2 \, dr \right] \int_{R_1}^{R_2} r^{-(N-1)} D^2 \, dr \\
- \left[ \frac{1}{8} - \frac{1}{\varepsilon^2 R_1^{2(N-1)}} g_0^2 \int_{R_1}^{R_2} r^{-(N-1)} D^2 \, dr \right] \int_{R_1}^{R_2} r^{-(N-1)} H^2 \, dr \\
- \frac{m_0}{\omega_N(R_2^N - R_1^N)\varepsilon} \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \bar{D}^2 \, dr \\
- \frac{m_0 K}{2\beta_0\beta_1\omega_N(R_2^N - R_1^N)\sqrt{\varepsilon}} \int_{R_1}^{R_2} r^{-(N-1)} g_0^{-1} \bar{D} \, dr \right] \, dr,
\]

where \(d = \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \, dr\), \(\bar{D} = D - \beta_0^{-1} d\) with \(\beta_0 = \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \, dr\).

Theorem 1.2 implies the following exponential decay estimate.

**Corollary 1.** Under the same hypothesis of Theorem 1.2, if the initial data satisfy

\[
I_0 := \left[ \int_{R_1}^{R_2} r^{-(N-1)} (D^2 + H^2) \, dr \right]_{t=0} \\
\leq \min \left\{ \frac{\mu}{8} R_1^{2(N-1)} g_0^2, \frac{m_0 K R_2^{2(N-1)} g_0^2 \beta_0^2}{4\beta_0\beta_1(\beta_0^2 - \beta_1^2)\omega_N(R_2^N - R_1^N)} \right\}
\]

for some \(0 < \mu < 1\), then there exists a constant \(\lambda > 0\), independent of \(\varepsilon\), such that

\[
\int_{R_1}^{R_2} r^{-(N-1)} (D^2 + H^2) \, dr \leq I_0 e^{-\lambda t}
\]

for \(t > 0\).

Observe that \(\|D\|_{L^2((R_1, R_2))} + \|H\|_{L^2((R_1, R_2))}\) is equivalent to \(\|\delta\|_{H^{-1}(\Omega)} + \|\eta\|_{H^{-1}(\Omega)}\), and is also equivalent to \(\|\tilde{n}\|_{H^{-1}(\Omega)} + \|\tilde{\phi}\|_{H^{-1}(\Omega)}\). Therefore, Corollary 1 gives the stability with exponential decay in \(H^{-1}\)-norm, i.e., \(\|\tilde{n}\|_{H^{-1}(\Omega)} + \|\tilde{\phi}\|_{H^{-1}(\Omega)} \leq C I_0 e^{-\lambda t}\) for some positive constant \(C\). This gives the nonlinear stability of the steady-state solution \((n^0, \rho^0, \psi)\) of the \(N\)-dimensional PNP system with radial symmetry.

**Remark 2.** Our analysis is valid for Robin boundary condition with the hypothesis \(\frac{2\gamma_{1,1}}{\sqrt{N}} \geq K\), \(i = 1, 2\). It cannot approach Dirichlet boundary condition because of the lower bound \(K\). On the other hand, if we consider assume that \(\lim_{\varepsilon \to 0^+} \frac{\gamma_{1,1}}{\sqrt{N}} = \infty\),
i = 1, 2, the solutions have no boundary layer \([25]\). The analysis will be much simpler, and we can obtain analogous results as in Theorem 1.2 and Corollary 1.

2. Proof of the main results.

2.1. Energy law. In order to prove Theorem 1.2, we first derive the energy law of \((D, H)\) as follows:

**Lemma 2.1.** Let \((\tilde{\delta}, \tilde{\eta}, \tilde{\phi})\) be a solution of (10) with boundary condition (11). If (19) holds true, then \((D, H)\) satisfies

\[
\frac{1}{2} \frac{d}{dt} \int_{R_1}^{R_2} r^m (D^2 + H^2) \, dr
\]

\[
= - \int_{R_1}^{R_2} r^m \left[ D^2 + H^2 + \frac{1}{\varepsilon} g^{-1} \eta^0 D^2 \right] \, dr
\]

\[
+ \frac{1}{2} (m + N - 1)(m - 1) \int_{R_1}^{R_2} r^{m-2} (D^2 + H^2) \, dr
\]

\[
- R_1^{N-1} g(R_1) \tilde{\phi}_r(R_1,t) \int_{R_1}^{R_2} r^m g^{-1} (\eta^0 D + \delta^0 H) \, dr
\]

\[
+ (m - (N - 1)) \int_{R_1}^{R_2} r^{m-1} \psi_r DHdr - \int_{R_1}^{R_2} r^m \frac{g_r}{g} \psi_r DHdr
\]

\[
+ \frac{1}{\varepsilon} \int_{R_1}^{R_2} r^m g^{-1} \delta D DHdr
\]

\[
+ (m - (N - 1)) \int_{R_1}^{R_2} r^{m-1} \tilde{\phi}_r D DHdr - \int_{R_1}^{R_2} r^m \frac{g_r}{g} \tilde{\phi}_r D DHdr
\]

for any \(m \in \mathbb{R}\).

**Proof.** We multiply (21) by \(r^m D\), integrate it from \(R_1\) to \(R_2\), and do integration by parts. Then

\[
\frac{1}{2} \frac{d}{dt} \int_{R_1}^{R_2} r^m D^2 \, dr
\]

\[
= \int_{R_1}^{R_2} r^{m+N-1} D \left[ (r^{-(N-1)} D)_r - \eta^0 \tilde{\phi}_r - \tilde{\eta} \psi_r - \tilde{\eta} \tilde{\phi}_r \right] \, dr
\]

\[
= - \int_{R_1}^{R_2} r^{m+N-1} \left[ r^{-(N-1)} D^2_r + \eta^0 \tilde{\phi}_r D + \tilde{\eta} D \psi_r + \tilde{\eta} D \tilde{\phi}_r \right] \, dr
\]

\[
- (m + N - 1) \int_{R_1}^{R_2} r^m DD_r \, dr
\]

\[
= - \int_{R_1}^{R_2} r^{m+N-1} \left[ r^{-(N-1)} D^2_r + \eta^0 \tilde{\phi}_r D + \tilde{\eta} D \psi_r + \tilde{\eta} D \tilde{\phi}_r \right] \, dr
\]

\[
+ \frac{1}{2} (m + N - 1)(m - 1) \int_{R_1}^{R_2} r^{m-2} D^2 \, dr.
\]

Here we have used the fact that \(D = 0\) at \(r = R_1, R_2\) from (20). Similarly, we multiply (22) by \(r^m H\), integrate it from \(R_1\) to \(R_2\), and use integration by parts to
get
\[
\frac{1}{2} \frac{d}{dt} \int_{R_1}^{R_2} r^m H^2 \, dr = - \int_{R_1}^{R_2} r^{m+N-1} \left[ r^{-(N-1)} H^2 + \delta^0 \phi_r H + \tilde{\delta} H \psi_r + \tilde{\delta} H \phi_r \right] \, dr + \frac{1}{2} (m + N - 1)(m - 1) \int_{R_1}^{R_2} r^{m-2} H^2 \, dr.
\]
(26)

Here the fact that \( H = 0 \) at \( r = R_1, R_2 \) from (20) has also been used.

We recall (10)–(11) that \( \tilde{\phi} \) satisfies the Poisson’s equation
\[
\varepsilon \left( r^{N-1} g \tilde{\phi}_r \right) = r^{N-1} \tilde{\delta}
\]
(27)

for \( r \in (R_1, R_2), \, t > 0 \), with Robin boundary condition
\[
\begin{cases}
\tilde{\phi}(R_1, t) - \gamma_\epsilon, \tilde{\phi}_r(R_1, t) = 0 \\
\tilde{\phi}(R_2, t) + \gamma_\epsilon, \tilde{\phi}_r(R_2, t) = 0
\end{cases}
\]
(28)

for \( t > 0 \). By integrating (27) from \( R_1 \) to \( r \), we get
\[
\varepsilon \left( r^{N-1} g(r) \tilde{\phi}_r(r, t) - R_1^{N-1} g(R_1) \tilde{\phi}_r(R_1, t) \right) = \int_{R_1}^{r} s^{N-1} \delta(s, t) \, ds = D(r, t),
\]
which gives
\[
r^{N-1} g(r) \tilde{\phi}_r(r, t) = \frac{1}{\varepsilon} D + R_1^{N-1} g(R_1) \tilde{\phi}_r(R_1, t).
\]
(29)

Consequently,
\[
\int_{R_1}^{R_2} r^{m+N-1} \eta^0 \phi_r D \, dr = \frac{1}{\varepsilon} \int_{R_1}^{R_2} r^m g^{-1} \eta^0 D^2 \, dr + R_1^{N-1} g(R_1) \tilde{\phi}_r(R_1, t) \int_{R_1}^{R_2} r^m g^{-1} \eta^0 D \, dr,
\]
(30)

and
\[
\int_{R_1}^{R_2} r^{m+N-1} \delta^0 \phi_r H \, dr = \frac{1}{\varepsilon} \int_{R_1}^{R_2} r^m g^{-1} \delta^0 D H \, dr + R_1^{N-1} g(R_1) \tilde{\phi}_r(R_1, t) \int_{R_1}^{R_2} r^m g^{-1} \delta^0 H \, dr.
\]
(31)

Furthermore, we use (13), (17)–(20), and integration by parts to get
\[
\int_{R_1}^{R_2} r^{m+N-1} \left[ \tilde{\eta} D \psi_r + \tilde{\delta} H \psi_r \right] \, dr
= \int_{R_1}^{R_2} r^{m-(N-1)} g^{-1} \left[ r^{N-1} g \psi_r \right] [D H]_r \, dr
= -\frac{1}{\varepsilon} \int_{R_1}^{R_2} r^m g^{-1} \delta^0 D H \, dr + (N - 1 - m) \int_{R_1}^{R_2} r^{m-1} \psi_r D H \, dr
+ \int_{R_1}^{R_2} r^{m} \frac{q_r}{g} \psi_r D H \, dr.
\]
(32)
Similarly, by (27), (17)–(20), and integration by parts, we get
\[
\int_{R_1}^{R_2} r^{m+N-1} \left[ \tilde{D}\tilde{\phi}_r + \tilde{H}\tilde{\phi}_r \right] dr = -\frac{1}{\varepsilon} \int_{R_1}^{R_2} r^m g^{-1} \Delta D\phi \, dr + (N-1-m) \int_{R_1}^{R_2} r^{m-1} \tilde{\phi}_r D\phi \, dr + \int_{R_1}^{R_2} r^m g \tilde{\phi}_r D\phi \, dr.
\] (33)

Therefore, by combining (25), (26), and (30)–(33), we complete the proof. \(\square\)

#### 2.2. The Choice of \(m\)

Lemma 2.1 concerns with the energy law for the weighted \(L^2\)-norms of \(D\) and \(H\) with the weight \(r^m\). On the other hand, in order to obtain the decay estimates for the weighted \(L^2\)-norms, \(m\) cannot be arbitrarily chosen. We rewrite the energy law (24) as follows:
\[
\frac{1}{2} \frac{d}{dt} \int_{R_1}^{R_2} r^m (D^2 + H^2) \, dr = -\int_{R_1}^{R_2} r^m (D^2 + H^2) \, dr + I_1 + I_2 + I_3, \tag{34}
\]
by denoting
\[
I_1 = \frac{1}{2} (m + N - 1)(m - 1) \int_{R_1}^{R_2} r^{m-2} (D^2 + H^2) \, dr,
\]
\[
I_2 = -\frac{1}{\varepsilon} \int_{R_1}^{R_2} r^m g^{-1} \phi_0 D^2 \, dr - R_1^{N-1} g(R_1) \hat{\phi}_r(R_1, t) \int_{R_1}^{R_2} r^m g^{-1} (\phi_0 D + \delta_0 H) \, dr,
\]
\[
I_3 = (m - (N - 1)) \int_{R_1}^{R_2} r^{m-1} \phi_1 D\phi \, dr - \int_{R_1}^{R_2} r^m g \hat{\phi}_r D\phi \, dr + \frac{1}{\varepsilon} \int_{R_1}^{R_2} r^m g^{-1} \Delta D\phi \, dr + (m - (N - 1)) \int_{R_1}^{R_2} r^{m-1} \hat{\phi}_r D\phi \, dr - \int_{R_1}^{R_2} r^m g \hat{\phi}_r D\phi \, dr.
\]

As a first guess, one may take \(m\) to be \(N - 1\) since \(r^{N-1}\) appears in the Jacobian of the change of variables from Cartesian to spherical coordinates, say
\[
\int_{\Omega} u(x) \, dx = N \omega_N \int_{R_1}^{R_2} r^{N-1} u(r) \, dr
\]
for radially symmetric \(u\) in \(\Omega = \{ x \in \mathbb{R}^N : R_1 < |x| < R_2 \}\). At the mean time, the integrals with coefficients \(m - (N - 1)\) in \(I_3\) will vanish to make (34) simpler. On the other hand, \(I_1 = (N - 1)(N - 2) \int_{R_1}^{R_2} r^{N-3} (D^2 + H^2) \, dr\), which in general is not bounded by \(\int_{R_1}^{R_2} r^{N-1} H^2 \, dr\). That is, \(I_1\) cannot be controlled if \(N \geq 3\). Thus, \(m\) cannot be chosen to be \(N - 1\).

In that case, we then start with those \(m\) such that \(I_1 \leq 0\). That is, we suppose that \(-(N - 1) \leq m \leq 1\). In fact, since \(I_3\) is the higher order term, it can be dominated by assuming that the initial perturbation is sufficiently small. As for \(I_2\), we
notice that it contains the boundary term $\tilde{\phi}_r(R_1, t)$. We first use the Green’s identity to express the boundary term as $R_1^{N-1}g(R_1)\tilde{\phi}_r(R_1, t) = -\frac{1}{2}\beta_\varepsilon^{-1}d$ (see Lemma 2.2), where

$$\beta_\varepsilon = \beta_0 + \frac{\gamma_{\varepsilon, 1}}{R_1^{N-1}g(R_1)} + \frac{\gamma_{\varepsilon, 2}}{R_2^{N-1}g(R_2)}$$

with $\beta_0 = \int_{R_1}^R r^{-(N-1)}g^{-1}dr$ and

$$d = \int_{R_1}^R r^{-(N-1)}g^{-1}Ddr.$$ 

And then we make use of the asymptotic behavior of the steady-state solution that $\eta^0 \sim \frac{2m_0}{\varepsilon\omega_N(R_2^N - R_1^N)}$ and $\delta^0 \sim 0$ in the interior of $\Omega$ as $\varepsilon$ tends to 0, which gives

$$I_2 \sim -\frac{2m_0}{\varepsilon\omega_N(R_2^N - R_1^N)} \left[ \int_{R_1}^{R_2} r^m g^{-1}D^2dr - \beta_\varepsilon^{-1}d \int_{R_1}^{R_2} r^m g^{-1}Ddr \right]. \quad (35)$$

By introducing the transformation $\bar{D} = D - \sigma d$ for some $\sigma \in \mathbb{R}$, we can rewrite (35) to be

$$I_2 \sim -\frac{2m_0}{\varepsilon\omega_N(R_2^N - R_1^N)} \left[ \int_{R_1}^{R_2} r^m g^{-1}\bar{D}^2dr \right. + (2\sigma - \beta_\varepsilon^{-1})d \int_{R_1}^{R_2} r^m g^{-1}\bar{D}dr + \beta_0(\sigma^2 - \beta_\varepsilon^{-1}\sigma)d^2 \right]. \quad (36)$$

The right-hand side of (36) will be non-positive, provided $\int_{R_1}^{R_2} r^m g^{-1}\bar{D}dr = 0$ and $\beta_0(\sigma^2 - \beta_\varepsilon^{-1}\sigma) > 0$. We notice that

$$\int_{R_1}^{R_2} r^m g^{-1}\bar{D}dr = \int_{R_1}^{R_2} r^m g^{-1}(D - \sigma d)dr = \int_{R_1}^{R_2} r^m g^{-1}Ddr - \sigma \int_{R_1}^{R_2} r^m g^{-1}dr \int_{R_1}^{R_2} r^{-(N-1)}g^{-1}Ddr.$$ 

Thus, in order to make $\int_{R_1}^{R_2} r^m g^{-1}\bar{D}dr = 0$, we then have to set $m = -(N-1)$ and $\sigma = \left[ \int_{R_1}^{R_2} r^{-(N-1)}g^{-1}dr \right]^{-1}$. Moreover, we have $\sigma > \beta_\varepsilon^{-1}$ if $m = -(N-1)$, and hence $\beta_0(\sigma^2 - \beta_\varepsilon^{-1}\sigma) > 0$. To summarize, if $m = -(N-1)$, we have that $I_2 \leq 0$, $I_3$ can be approximated by a sum of non-positive terms, and $I_3$ is higher order term that can be controlled by initial data.

Therefore, we will use the energy law (24) of $(D, H)$ with $m = -(N-1)$, i.e.,

$$\frac{1}{2} \frac{d}{dt} \int_{R_1}^{R_2} r^{-(N-1)}(D^2 + H^2) dr = -\int_{R_1}^{R_2} r^{-(N-1)} \left[ D_r^2 + H_r^2 + \frac{1}{\varepsilon} g^{-1}\eta^0 D^2 \right] dr - R_1^{N-1}g(R_1)\tilde{\phi}_r(R_1, t) \int_{R_1}^{R_2} r^{-(N-1)}g^{-1} \left( \eta^0 D + \delta^0 H \right) dr \quad (37)$$
Lemma 2.2. To estimate $\phi$, we need to prove the gradient estimate at boundary $r = R_1$. We can represent $\phi_r(R_1, t)$ by $D$ as follows.

\begin{align*}
\frac{2}{N - 1} \int_{R_1}^{R_2} r^{-N-1} \psi_r Ddr - \int_{R_1}^{R_2} r^{-(N-1)} g r^{-1} \psi_r Ddr
+ \frac{1}{r} \int_{R_1}^{R_2} g^{-1} \tilde{\delta} Ddr
- 2(N-1) \int_{R_1}^{R_2} r^{-N} \tilde{\phi}_r Ddr
\end{align*}

to prove Theorem 1.2.

2.3. Proof of Theorem 1.2. In order to use (37) to prove Theorem 1.2, we need to estimate $\phi_r(R_1, t)$, i.e., the gradient estimate at boundary $r = R_1$. We can represent $\phi_r(R_1, t)$ by $D$ as follows.

Lemma 2.2.

\begin{equation}
R_1^{N-1} g(R_1) \tilde{\phi}_r(R_1, t) = - \frac{1}{r} \left[ \beta_0 + \frac{\gamma_{\epsilon, 1}}{R_1^{N-1} g(R_1)} + \frac{\gamma_{\epsilon, 2}}{R_2^{N-1} g(R_2)} \right]^{-1} \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} Ddr,
\end{equation}

where

\begin{equation}
\beta_0 = \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} dr.
\end{equation}

Proof. Recall that $\tilde{\phi}$ satisfies the Poisson's equation (27) for $r \in (R_1, R_2)$, $t > 0$, with Robin boundary condition (28) for $t > 0$. By integrating (27) in $r$ over $(R_1, R_2)$, we get

\begin{equation}
\epsilon \left( R_2^{N-1} g(R_2) \tilde{\phi}_r(R_2, t) - R_1^{N-1} g(R_1) \tilde{\phi}_r(R_1, t) \right) = \epsilon \int_{R_1}^{R_2} r^{N-1} \tilde{\delta} dr = 0,
\end{equation}

which implies

\begin{equation}
R_2^{N-1} g(R_2) \tilde{\phi}_r(R_2, t) = R_1^{N-1} g(R_1) \tilde{\phi}_r(R_1, t).
\end{equation}

Define

\begin{equation}
G(r) = - \int_{r}^{R_2} s^{-(N-1)} g(s)^{-1} ds,
\end{equation}

then $G(R_1) = -\beta_0$, $G(R_2) = 0$, and $G''(r) = r^{-(N-1)} g^{-1}$. By using (27)–(28) and integration by parts, we have

\begin{align*}
\int_{R_1}^{R_2} r^{-(N-1)} g^{-1} Ddr
= \int_{R_1}^{R_2} G r Ddr = - \int_{R_1}^{R_2} G r^{N-1} \tilde{\delta} dr
- \varepsilon \int_{R_1}^{R_2} G \left[ r^{N-1} g \tilde{\phi}_r \right] dr
= -\varepsilon \beta_0 R_1^{N-1} g(R_1) \tilde{\phi}_r(R_1, t) + \varepsilon \int_{R_1}^{R_2} \tilde{\phi}_r dr
= -\varepsilon \beta_0 R_1^{N-1} g(R_1) \tilde{\phi}_r(R_1, t) + \varepsilon \left( \tilde{\phi}(R_2, t) - \tilde{\phi}(R_1, t) \right)
= -\varepsilon \beta_0 R_1^{N-1} g(R_1) \tilde{\phi}_r(R_1, t) - \varepsilon \left( \gamma_{\epsilon, 0} \tilde{\phi}_r(R_2, t) + \gamma_{\epsilon, 1} \tilde{\phi}_r(R_1, t) \right).
\end{align*}

(41)
(40) and (41) imply
\[ \tilde{\phi}_r(R_1, t) = -\frac{1}{\varepsilon} \left[ \beta_0 R_1^{N-1} g(R_1) + \gamma_{\varepsilon, 1} + \gamma_{\varepsilon, 2} \frac{R_1^{N-1} g(R_1)}{R_2^{N-1} g(R_2)} \right]^{-1} \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} D dr. \]

This complete the proof. \(\square\)

Now, we let
\[ \beta_\varepsilon = \beta_0 + \frac{\gamma_{\varepsilon, 1}}{R_1^{N-1} g(R_1)} + \frac{\gamma_{\varepsilon, 2}}{R_2^{N-1} g(R_2)} \tag{42} \]
and
\[ d = \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} D dr. \tag{43} \]

Then, by (38), (37) becomes
\[ \frac{1}{2} \frac{d}{dt} \int_{R_1}^{R_2} r^{-(N-1)} (D^2 + H^2) dr \]
\[ = -\int_{R_1}^{R_2} r^{-(N-1)} \left[ D_r^2 + H_r^2 + \frac{1}{\varepsilon} g^{-1} \eta^0 D^2 \right] dr \tag{44} \]
\[ + \frac{1}{\varepsilon} \beta_\varepsilon^{-1} d \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} (\eta^0 D + \delta^0 H) dr \]
\[ - 2(N-1) \int_{R_1}^{R_2} r^{-N} \psi_r D H dr - \int_{R_1}^{R_2} r^{-(N-1)} \frac{g}{g} \psi_r D H dr \]
\[ + \frac{1}{\varepsilon} \int_{R_1}^{R_2} r^{-(N-1)} \tilde{\delta} D H dr \]
\[ - 2(N-1) \int_{R_1}^{R_2} r^{-N} \tilde{\phi}_r D H dr - \int_{R_1}^{R_2} r^{-(N-1)} \frac{g}{g} \tilde{\phi}_r D H dr. \]

The right-hand side of (44) can be divided into three part:

\[ I = -\int_{R_1}^{R_2} r^{-(N-1)} \left[ D_r^2 + H_r^2 + \frac{1}{\varepsilon} g^{-1} \eta^0 D^2 \right] dr, \]
\[ II = \frac{1}{\varepsilon} \beta_\varepsilon^{-1} d \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} (\eta^0 D + \delta^0 H) dr \]
\[ - 2(N-1) \int_{R_1}^{R_2} r^{-N} \psi_r D H dr - \int_{R_1}^{R_2} r^{-(N-1)} \frac{g}{g} \psi_r D H dr, \]
\[ III = \frac{1}{\varepsilon} \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \tilde{\delta} D H dr \]
\[ - 2(N-1) \int_{R_1}^{R_2} r^{-N} \tilde{\phi}_r D H dr - \int_{R_1}^{R_2} r^{-(N-1)} \frac{g}{g} \tilde{\phi}_r D H dr, \]

where \( I \leq 0 \) has a good sign, and \( III \) is the higher order terms. In order to estimate \( d \) appearing in \( II \), we introduce
\[ \bar{D} = D - \beta_0^{-1} d. \] (45)
Notice that the definition (45) gives
\[ \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \, Ddr = 0. \tag{46} \]
Then, for the integral \( \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \eta^0 \, Ddr \) in \( II \), we have
\[
- \frac{1}{\varepsilon} \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \eta^0 \, D^2 dr + \frac{1}{\varepsilon \beta_0^{-1}} d \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \eta^0 \, Ddr
\]
\[
= - \frac{1}{\varepsilon} \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \eta^0 \, D^2 dr - \frac{1}{\varepsilon \beta_0^{-1}(\beta_0^{-1} - \beta_0^{-1})} d^2 \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \eta^0 \, dr \tag{47}
\]
\[- \frac{1}{\varepsilon} (2\beta_0^{-1} - \beta_0^{-1}) d \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \eta^0 \, Ddr.
\]
Now, we need some properties of the steady state \( \eta^0 \) to estimate the right-hand side of (47).

**Lemma 2.3.** If \( 0 < \varepsilon < \varepsilon^* \), then \( \eta^0 \) satisfies
\[
\eta^0 \geq \frac{2m_0}{\omega_N(R_2^N - R_1^N)} \left[ 1 - \frac{2CNR_{N-1}g^{\phi_2 - \phi_1}}{M(R_2^N - R_1^N)} \sqrt{\varepsilon} \right] \tag{48}
\]
and
\[
\left| \eta^0 - \frac{2m_0}{\omega_N(R_2^N - R_1^N)} \right| \leq \frac{8m_0CNR_{N-1}g^{\phi_2 - \phi_1}}{\omega_NM(R_2^N - R_1^N)^2} \sqrt{\varepsilon}, \tag{49}
\]
where constants \( \varepsilon^* \), \( C \), and \( M \) are the same as in Lemma 1.1.

**Proof.** We can use Lemma 1.1 to get the lower bound of \( \eta^0 \) as follows. From the definitions of \( \eta^0 \), \( p^0 \), and \( \eta^0 \), i.e., (3) and (12), we have \( \eta^0 = \frac{m_0 e^{-\psi}}{N \omega_N \int_{R_1}^{R_2} s^{N-1} e^{-\psi(s)} \, ds} \). Recall Lemma 1.1 that \( \phi_1 \leq \psi \leq \phi_2 \). Then, by the mean value theorem, we have
\[
\left| e^{\psi(r)} - e^{c_\varepsilon} \right| \leq e^{\phi_2} |\psi(r) - c_\varepsilon|
\]
and
\[
\left| e^{-\psi(r)} - e^{-c_\varepsilon} \right| \leq e^{-\phi_1} |\psi(r) - c_\varepsilon|
\]
for all \( r \in [R_1, R_2] \), where \( c_\varepsilon = \psi \left( \frac{R_1 + R_2}{2} \right) \). Therefore, we use Lemma 1.1 to get
\[
\eta^0 \geq \min_{r \in [R_1, R_2]} \left\{ \frac{m_0 e^{\psi(r)}}{N \omega_N \int_{R_1}^{R_2} s^{N-1} e^{\psi(s)} \, ds} + \frac{m_0 e^{-\psi(r)}}{N \omega_N \int_{R_1}^{R_2} s^{N-1} e^{-\psi(s)} \, ds} \right\}
\]
\[
\geq \min_{r \in [R_1, R_2]} \left\{ \frac{m_0 e^{\psi(r)}}{N \omega_N \int_{R_1}^{R_2} s^{N-1} e^{\psi(s)} \, ds} \right\}
\]
\[
+ \frac{m_0 e^{-\psi(r)}}{N \omega_N \int_{R_1}^{R_2} s^{N-1} e^{-\psi(s)} \, ds}
\]
\[
\geq \min_{r \in [R_1, R_2]} \left\{ \frac{m_0 e^{\psi(r)}}{N \omega_N \int_{R_1}^{R_2} s^{N-1} e^{\psi(s)} \, ds} \right\}
\]
\[
+ \frac{m_0 e^{-\psi(r)}}{N \omega_N \int_{R_1}^{R_2} s^{N-1} e^{-\psi(s)} \, ds}
\]
Then we have proved (49).

And for (49), if we divide \([R_1, R_2]\) into two parts

\[
A = \left\{ r \in [R_1, R_2] : \eta^0 \geq \frac{2m_0}{\omega_N(R_2^N - R_1^N)} \right\}
\]

and

\[
B = \left\{ r \in [R_1, R_2] : \eta^0 < \frac{2m_0}{\omega_N(R_2^N - R_1^N)} \right\},
\]

then we get

\[
\left| \eta^0 - \frac{2m_0}{\omega_N(R_2^N - R_1^N)} \right| = 0 \text{ on } A
\]

and

\[
\left| \eta^0 - \frac{2m_0}{\omega_N(R_2^N - R_1^N)} \right| = 2 \left( \frac{2m_0}{\omega_N(R_2^N - R_1^N)} - \eta^0 \right) \text{ on } B.
\]

By (48), on \(B\), we have

\[
\frac{2m_0}{\omega_N(R_2^N - R_1^N)} - \eta^0 \\
\leq \frac{2m_0}{\omega_N(R_2^N - R_1^N)} - \frac{2m_0}{\omega_N(R_2^N - R_1^N)} \left[ 1 - \frac{2CN R_2^{N-1} e^{\phi_2 - \phi_1}}{M(R_2^N - R_1^N) \sqrt{\varepsilon}} \right] \\
\leq \frac{4m_0 CN R_2^{N-1} e^{\phi_2 - \phi_1}}{\omega_N M(R_2^N - R_1^N)^2 \sqrt{\varepsilon}}.
\]

Then we have proved (49). \(\square\)

Lemma 2.3 immediately implies

\[
\int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \eta^0 dr \\
\geq \frac{m_0}{\omega_N(R_2^N - R_1^N)} \int_{R_1}^{R_2} r^{-(N-1)} dr = \frac{\beta_0 m_0}{\omega_N(R_2^N - R_1^N)}
\]

(50)
if $0 < \varepsilon < \varepsilon_1 := \left[ \frac{M(R_2^N - R_1^N)}{4CN^2 r_0^2 \epsilon^{2-\sigma_1}} \right]^2$, and moreover,

$$\int_{R_1}^{R_2} r^{N-1} g^{-1} \left| \eta^0 - \frac{2m_0}{\omega_N(R_2^N - R_1^N)} \right| dr \leq g_0^{-1} \int_{R_1}^{R_2} r^{N-1} \left| \eta^0 - \frac{2m_0}{\omega_N(R_2^N - R_1^N)} \right| dr \leq g_0^{-1} \int_{R_1}^{R_2} r^{N-1} \left( \eta^0 - \frac{2m_0}{\omega_N(R_2^N - R_1^N)} \right) dr + \frac{8m_0CN_2R_2N_2^{-1}e^{\phi_2 - \phi_1}}{g_0\omega_N M(R_2^N - R_1^N)^2} \sqrt{\varepsilon} \int_{R_1}^{R_2} r^{N-1} dr \leq \frac{8m_0CR_2N_2^{-1}e^{\phi_2 - \phi_1}}{g_0\omega_N M(R_2^N - R_1^N)^2} \sqrt{\varepsilon}.$$  

(51)

Here we have used (3) and (12), which gives

$$\int_{R_1}^{R_2} r^{N-1} \left( \eta^0 - \frac{2m_0}{\omega_N(R_2^N - R_1^N)} \right) dr = 0.$$

By (50), we have

$$- \frac{1}{\varepsilon \beta_0^{-1}}(\beta_0^{-1} - \beta_\varepsilon^{-1})d^2 \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \eta^0 dr \leq - \frac{m_0(\beta_0^{-1} - \beta_\varepsilon^{-1})}{\omega_N(R_2^N - R_1^N)\varepsilon} d^2 \leq - \frac{m_0}{\beta_0\beta_1\omega_N(R_2^N - R_1^N)\varepsilon} \left( \frac{\gamma_{\varepsilon,1}}{R_1^N g(R_1)} + \frac{\gamma_{\varepsilon,2}}{R_2^N g(R_2)} \right) d^2,$$  

(52)

where

$$\beta_1 := \beta_0 + \frac{\gamma_{1,1}}{R_1^N g(R_1)} + \frac{\gamma_{1,2}}{R_2^N g(R_2)}.$$  

And by (46) and (51), we obtain

$$\left| \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \left| \eta^0 - \frac{2m_0}{\omega_N(R_2^N - R_1^N)} \right| D^2 dr \right| \leq \frac{1}{\varepsilon} \left( 2\beta_0^{-1} - \beta_\varepsilon^{-1} \right) \left( \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \left| \eta^0 - \frac{2m_0}{\omega_N(R_2^N - R_1^N)} \right| dr \right)^\frac{1}{2} \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \left| \eta^0 - \frac{2m_0}{\omega_N(R_2^N - R_1^N)} \right| D^2 dr \leq \frac{1}{\varepsilon} \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \left| \eta^0 - \frac{2m_0}{\omega_N(R_2^N - R_1^N)} \right| D^2 dr + \frac{2(2\beta_0^{-1} - \beta_\varepsilon^{-1})^2 m_0 C R_2N_2^{-1} e^{\phi_2 - \phi_1}}{g_0\omega_N M(R_2^N - R_1^N)^2} \sqrt{\varepsilon} d^2.$$  

(53)
Combining (47), (52), and (53), and using (49), we get

\[ - \frac{1}{\varepsilon} \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \eta^0 D^2 dr + \frac{1}{\varepsilon} \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \eta^0 D dr \]

\[ \leq - \frac{1}{\varepsilon} \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \bar{D}^2 dr \]

\[ + \frac{1}{\varepsilon} \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \eta^0 - \frac{2m_0}{\omega_N (R_2^N - R_1^N)} \bar{D}^2 dr \]

\[ \leq \left( \frac{2m_0}{\omega_N (R_2^N - R_1^N) \varepsilon} - \frac{8m_0 CN R_2^{N-1} e^{\phi_2 - \phi_1}}{\omega_N (R_2^N - R_1^N)^2 \varepsilon} \right) \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \bar{D}^2 dr \]

\[ \leq - \left( \frac{2m_0}{\omega_N (R_2^N - R_1^N) \varepsilon} \right) \left[ \frac{1}{\beta_0 \beta_1 \varepsilon} \left( \frac{\gamma_{\epsilon, 1}}{R_1^{N-1} g(R_1)} + \frac{\gamma_{\epsilon, 2}}{R_2^{N-1} g(R_2)} \right) \right. 

\[ \left. \quad - \frac{2(2\beta_0^{-1} - \beta_1^{-1})^2 m_0 CR_2^{N-1} e^{\phi_2 - \phi_1}}{g_0 M} \right] d^2. \]

Next, for the other three integrals, which contain $\sigma^0$ and $\psi_r$, in $II$, we will use the following gradient estimate for $\psi$.

\[ \int_{R_1}^{R_2} \psi_r^2 dr \leq \frac{2C^2}{\varepsilon} \int_{R_1}^{R_2} \left( e^{-\frac{2M(r-R_1)}{\varepsilon \sigma^0}} + e^{-\frac{2M(r-R_1)}{\varepsilon \sigma^0}} \right) \]

\[ \leq \frac{2C^2}{M \sqrt{\varepsilon}} \left( 1 - e^{-\frac{M(R_2-R_1)}{\varepsilon \sigma^0}} \right) \]

\[ \leq \frac{2C^2}{M \sqrt{\varepsilon}}. \]

This estimate (55) directly comes from Lemma 1.1. Then, by (13) and integration by parts, we have

\[ \frac{1}{\varepsilon} \beta_1^{-1} d \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \sigma^0 H dr \]

\[ = \beta_1^{-1} d \int_{R_1}^{R_2} r^{-2(N-1)} g^{-1} \left( r^{N-1} g \psi_r \right) H dr \]

\[ = 2(N-1) \beta_1^{-1} d \int_{R_1}^{R_2} r^{-N} \psi_r H dr + \beta_1^{-1} d \int_{R_1}^{R_2} r^{-(N-1)} g \psi_r H dr \]

\[ - \beta_1^{-1} d \int_{R_1}^{R_2} r^{-(N-1)} \psi_r H dr. \]

We recall (45) that $D = \bar{D} + \beta_0^{-1} d$, which implies

\[ \frac{1}{\varepsilon} \beta_1^{-1} d \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \sigma^0 H dr \]
\[
- 2(N - 1) \int_{R_1}^{R_2} r^{-N} \psi_r \, DH \, dr - \int_{R_1}^{R_2} r^{-(N-1)} \frac{g}{g} \psi_r \, DH \, dr \\
= - 2(N - 1) \int_{R_1}^{R_2} r^{-N} \psi_r \, DH \, dr - \int_{R_1}^{R_2} r^{-(N-1)} \frac{g}{g} \psi_r \, DH \, dr \\
- \beta_{e}^{-1} d \int_{R_1}^{R_2} r^{-(N-1)} \psi_r \, H_r \, dr + 2(N - 1)(\beta_{e}^{-1} - \beta_{0}^{-1})d \int_{R_1}^{R_2} r^{-N} \psi_r \, H_r \, dr \\
+ (\beta_{e}^{-1} - \beta_{0}^{-1})d \int_{R_1}^{R_2} r^{-(N-1)} \frac{g}{g} \psi_r \, H_r \, dr.
\]

We shall estimate (56) term by term. By using Poincaré’s inequalities

\[
\|u\|_{L^p}^2((R_1, R_2)) \leq C_{\infty} \int_{R_1}^{R_2} r^{-(N-1)} u_r^2 \, dr, \quad (57)
\]

\[
\|u\|_{L^2}^2((R_1, R_2)) \leq C_2 \int_{R_1}^{R_2} r^{-(N-1)} u_r^2 \, dr \quad (58)
\]

for any \(u \in H_0^1((R_1, R_2))\), where \(C_{\infty}, C_2\) are constants depending only on \(R_1\) and \(R_2\), we have

\[
\left| 2(N - 1) \int_{R_1}^{R_2} r^{-N} \psi_r \, DH \, dr \right| \\
\leq 2(N - 1) \|H\|_{L^\infty} \left[ \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \, D^2 \, dr \right]^{\frac{1}{2}} \left[ \int_{R_1}^{R_2} r^{-N-1} g \psi_r^2 \, dr \right]^{\frac{1}{2}} \\
\leq \frac{1}{8} \int_{R_1}^{R_2} r^{-(N-1)} H_r^2 \, dr \\
+ 8(N - 1)^2 C_{\infty} \left[ \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \, D^2 \, dr \right] \left[ \int_{R_1}^{R_2} r^{-N-1} g \psi_r^2 \, dr \right] \\
\leq \frac{1}{8} \int_{R_1}^{R_2} r^{-(N-1)} H_r^2 \, dr + \frac{16(N - 1)^2 C_{\infty} g_1}{M \rho_1^{N+1} \sqrt{\varepsilon}} \int_{R_1}^{R_2} r^{-(N-1)} g \psi_r^2 \, dr,
\]

and

\[
\left| \int_{R_1}^{R_2} r^{-(N-1)} \frac{g}{g} \psi_r \, DH \, dr \right| \\
\leq \|H\|_{L^\infty} \left[ \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \, D^2 \, dr \right]^{\frac{1}{2}} \left[ \int_{R_1}^{R_2} r^{-N-1} g \psi_r^2 \, dr \right]^{\frac{1}{2}} \\
\leq \frac{1}{8} \int_{R_1}^{R_2} r^{-(N-1)} H_r^2 \, dr \\
+ 2C_{\infty} \left[ \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \, D^2 \, dr \right] \left[ \int_{R_1}^{R_2} r^{-N-1} g \psi_r^2 \, dr \right] \\
\leq \frac{1}{8} \int_{R_1}^{R_2} r^{-(N-1)} H_r^2 \, dr + \frac{4C_{\infty} g_1^2}{M \rho_1^{N+1} g_0 \sqrt{\varepsilon}} \int_{R_1}^{R_2} r^{-(N-1)} g \psi_r^2 \, dr.
\]
Next,

\[
\begin{align*}
& \left| \beta_\varepsilon^{-1} d \int_{R_1}^{R_2} r^{-(N-1)} \psi_r H \, dr \right| \\
& \leq \frac{1}{8} \int_{R_1}^{R_2} r^{-(N-1)} H^2 \, dr + 2 \beta_\varepsilon^{-2} d^2 \int_{R_1}^{R_2} r^{-(N-1)} \psi_r^2 \, dr \\
& \leq \frac{1}{8} \int_{R_1}^{R_2} r^{-(N-1)} H^2 \, dr + \frac{4C^2}{\beta_0^2 M R_1^{N-1} \varepsilon} d^2.
\end{align*}
\]

Noticing that

\[
|\beta_\varepsilon^{-1} - \beta_0^{-1}| = \frac{1}{\beta_0 \beta_\varepsilon} \left[ \frac{\gamma_{\varepsilon,1}}{R_1^{N-1} g(R_1)} + \frac{\gamma_{\varepsilon,2}}{R_2^{N-1} g(R_2)} \right]
\]

\[
\leq \frac{1}{\beta_0^2} \left[ \frac{\gamma_{1,1}}{R_1^{N-1} g(R_1)} + \frac{\gamma_{1,2}}{R_2^{N-1} g(R_2)} \right],
\]

we have

\[
\begin{align*}
& \left| 2(N-1)(\beta_\varepsilon^{-1} - \beta_0^{-1}) d \int_{R_1}^{R_2} r^{-N} \psi_r H \, dr \right| \\
& \leq \frac{2(N-1)}{\beta_0^3 R_1^N} \left[ \frac{\gamma_{1,1}}{R_1^{N-1} g(R_1)} + \frac{\gamma_{1,2}}{R_2^{N-1} g(R_2)} \right] |d||\psi_r|_{L_2^2((R_1, R_2))}|H|_{L_2^2((R_1, R_2))} \\
& \leq \frac{1}{8C_2} |H|^2_{L_2^2((R_1, R_2))} + \frac{8C_2(N-1)^2}{\beta_0^4 R_1^{2N}} \left[ \frac{\gamma_{1,1}}{R_1^{N-1} g(R_1)} + \frac{\gamma_{1,2}}{R_2^{N-1} g(R_2)} \right] d^2 |\psi_r|_{L_2^2((R_1, R_2))}^2 \\
& \leq \frac{1}{8} \int_{R_1}^{R_2} r^{-(N-1)} H^2 \, dr + \frac{16C^2 C_2(N-1)^2}{\beta_0^4 R_1^{2N} M \varepsilon} \left[ \frac{\gamma_{1,1}}{R_1^{N-1} g(R_1)} + \frac{\gamma_{1,2}}{R_2^{N-1} g(R_2)} \right]^2 d^2.
\end{align*}
\]

Similarly,

\[
\begin{align*}
& \left| (\beta_\varepsilon^{-1} - \beta_0^{-1}) d \int_{R_1}^{R_2} r^{-(N-1)} \frac{g_r}{g} \psi_r H \, dr \right| \\
& \leq \frac{1}{8} \int_{R_1}^{R_2} r^{-(N-1)} H^2 \, dr \\
& + \frac{4C^2 C_2 g_1^2}{\beta_0^4 R_1^{2(N-1)} g_0^2 M \varepsilon} \left[ \frac{\gamma_{1,1}}{R_1^{N-1} g(R_1)} + \frac{\gamma_{1,2}}{R_2^{N-1} g(R_2)} \right]^2 d^2.
\end{align*}
\]

Therefore, (56) implies

\[
\begin{align*}
& \frac{1}{\varepsilon} \beta_\varepsilon^{-1} d \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} g^0 H \, dr \\
& - 2(N-1) \int_{R_1}^{R_2} r^{-N} \psi_r H \, dr - \int_{R_1}^{R_2} r^{-(N-1)} \frac{g_r}{g} \psi_r DH \, dr
\end{align*}
\]
\[
\leq \frac{5}{8} \int_{R_1}^{R_2} r^{-(N-1)} H^2 dr \\
+ \left[ \frac{16(N-1)C^2 C_\infty g_1}{MR_1^{N+1} \sqrt{\varepsilon}} + \frac{4C^2 C_\infty g_1^2}{MR_1^{N-1} g_0 \sqrt{\varepsilon}} \right] \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} D^2 dr \\
+ \left\{ \frac{4C^2}{\beta_0^2 M R_1^{N-1} \sqrt{\varepsilon}} \\
+ \frac{16C^2 C_2(N-1)^2 g_0^2 + 4C^2 C_2 g_0^2}{\beta_0^4 R_1^{2N} g_0^2 M \sqrt{\varepsilon}} \left[ \gamma_{1,1} R_1^{N-1} g(R_1) + \gamma_{1,2} R_2^{N-1} g(R_2) \right]^2 \right\} d^2.
\]

Putting (54) and (59) into (44), then we obtain

\[
\frac{1}{2} \frac{d}{dt} \int_{R_1}^{R_2} r^{-(N-1)} (D^2 + H^2) dr \\
\leq - \int_{R_1}^{R_2} \left[ r^{-(N-1)} D_r^2 + \frac{3}{8} r^{-(N-1)} H_r^2 \right] dr \\
- \left[ \frac{2m_0}{\omega_N (R_2^N - R_1^N) \varepsilon} - \frac{8m_0 C N R_2^{N-1} e^{\phi_2 - \phi_1}}{\omega_N (R_2^N - R_1^N) \varepsilon} \right] \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} D^2 dr \\
- \left\{ \frac{m_0}{\beta_0 \beta_1 \omega_N (R_2^N - R_1^N) \varepsilon} \left( \frac{\gamma_{1,1}}{R_1^{N-1} g(R_1)} + \frac{\gamma_{1,2}}{R_2^{N-1} g(R_2)} \right) \\
- \frac{2(2\beta_0^{-1} - \beta_1^{-1})^2 m_0 C R_2^{N-1} e^{\phi_2 - \phi_1}}{\omega_N (R_2^N - R_1^N) g_0 M} - \frac{4C^2}{\beta_0^2 M R_1^{N-1}} \\
- \frac{16C^2 C_2(N-1)^2 g_0^2 + 4C^2 C_2 g_0^2}{\beta_0^4 R_1^{2N} g_0^2 M} \left[ \frac{\gamma_{1,1}}{R_1^{N-1} g(R_1)} + \frac{\gamma_{1,2}}{R_2^{N-1} g(R_2)} \right]^2 \right\} \sqrt{\varepsilon} \\
+ \frac{1}{\varepsilon} \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \delta D H dr \\
- 2(N-1) \int_{R_1}^{R_2} r^{-N} \phi_r D H dr - \int_{R_1}^{R_2} r^{-(N-1)} g_r \phi_r D H dr.
\]

Therefore, if \( \varepsilon \leq \min\{\varepsilon_2, \varepsilon_3, \varepsilon_4\} \), where

\[
\varepsilon_2 := \left[ \frac{M (R_2^N - R_1^N)}{24CN R_2^{N-1} e^{\phi_2 - \phi_1}} \right]^2, \\
\varepsilon_3 := \left[ \frac{m_0 M R_1^{N+1}}{48(N-1)^2 \omega_N (R_2^N - R_1^N) C^2 C_\infty g_1} \right]^2, \\
\varepsilon_4 := \left[ \frac{m_0 M R_1^{N-1} g_0}{12 \omega_N (R_2^N - R_1^N) C^2 C_\infty g_1^2} \right]^2.
\]
and \( \frac{1}{\sqrt{\varepsilon}} \left( \frac{\gamma_1}{g(R_1)} + \frac{\gamma_2}{g(R_2)} \right) \geq K := \max \{ K_1, K_2, K_3 \} \), where
\[
K_1 := \frac{12\beta_0^2\beta_1 (2\beta_0^{-1} - \beta_1^{-1})^2 C R_2^{N-1} e^{\phi_2 - \phi_1}}{g_0 M},
\]
\[
K_2 := \frac{24\beta_1 C^2 \omega_N (R_2^N - R_1^N)}{\beta_0 m_0 MR_1^{N-1}},
\]
\[
K_3 := \frac{24 C^2 \beta_2 \omega_N (R_2^N - R_1^N) (4(N-1)^2 g_0^2 + R_1^2 g_1^2)}{m_0 \beta_0^3 R_1^{2N} g_0^3 M}
\]
\[
\left[ \frac{\gamma_{1,1}}{R_1^{N-1} g(R_1)} + \frac{\gamma_{1,2}}{R_2^{N-1} g(R_2)} \right]^2,
\]
then (60) becomes
\[
\frac{1}{2} \frac{d}{dt} \int_{R_1}^{R_2} r^{-(N-1)} (D^2 + H^2) \, dr
\leq - \int_{R_1}^{R_2} \left[ r^{-(N-1)} D_\tau^2 + \frac{3}{8} r^{-(N-1)} H_\tau^2 \right] \, dr
\]
\[
- \frac{m_0}{\omega_N (R_2^N - R_1^N) \varepsilon} \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \delta D^2 \, dr
- \frac{m_0 K}{2 \beta_0 \beta_1 \omega_N (R_2^N - R_1^N) \sqrt{\varepsilon}} \, d^2
+ \frac{1}{\varepsilon} \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \tilde{\delta} D H \, dr
- 2(N-1) \int_{R_1}^{R_2} r^{-N} \tilde{\phi}_r D H \, dr
- \int_{R_1}^{R_2} r^{-(N-1)} \frac{g_r}{g} \tilde{\phi}_r D H \, dr.
\]

The last step is to estimate higher-order terms in (61). Recalling the Poisson’s equation for \( \tilde{\phi} \), i.e., (27), we do integration by parts to get
\[
\frac{1}{\varepsilon} \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \tilde{\delta} D H \, dr
= \int_{R_1}^{R_2} r^{-2(N-1)} g^{-1} \left( r^{-N} \tilde{g}_r \tilde{\phi}_r \right) D H \, dr
= 2(N-1) \int_{R_1}^{R_2} r^{-N} \tilde{\phi}_r D H \, dr + \int_{R_1}^{R_2} r^{-(N-1)} \frac{g_r}{g} \tilde{\phi}_r D H \, dr
- \int_{R_1}^{R_2} r^{-(N-1)} \tilde{\phi}_r (D_r H + D H_r) \, dr,
\]
which gives
\[
\frac{1}{\varepsilon} \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \tilde{\delta} D H \, dr
- 2(N-1) \int_{R_1}^{R_2} r^{-N} \tilde{\phi}_r D H \, dr
- \int_{R_1}^{R_2} r^{-(N-1)} \frac{g_r}{g} \tilde{\phi}_r D H \, dr
= \int_{R_1}^{R_2} r^{-(N-1)} \tilde{\phi}_r (D_r H + D H_r) \, dr,
\]
and we can rewrite (61) as

\[
\frac{1}{2} \frac{d}{dt} \int_{R_1}^{R_2} r^{-(N-1)} (D^2 + H^2) \, dr \\
\leq - \int_{R_1}^{R_2} \left[ r^{-(N-1)} D_r^2 + \frac{1}{4} r^{-(N-1)} H_r^2 \right] \, dr \\
- \frac{m_0}{\omega_N(R_2^N - R_1^N) \varepsilon} \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \tilde{D}^2 \, dr - \frac{m_0 K}{2\beta_0 \beta_1 \omega_N(R_2^N - R_1^N) \varepsilon^2} \, d^2 \\
+ \int_{R_1}^{R_2} r^{-(N-1)} \tilde{\phi}_r (D_r H + DH_r) \, dr.
\]

By integrating (27) from \( R_1 \) to \( r \) and using Lemma 2.2, we have

\[
r^{N-1} g \tilde{\phi}_r = \frac{1}{\varepsilon} D + R_1^{N-1} g(R_1) \tilde{\phi}_r(R_1) \\
= \frac{1}{\varepsilon} (D - \beta^{-1}_\varepsilon d) \\
= \frac{1}{\varepsilon} \tilde{D} + \frac{1}{\varepsilon} (\beta^{-1}_0 - \beta^{-1}_\varepsilon) d.
\]

Here we have used the definition of \( \tilde{D} \) (45). This gives

\[
\tilde{\phi}_r = \frac{1}{\varepsilon} r^{-(N-1)} g^{-1} \tilde{D} + \frac{1}{\varepsilon} r^{-(N-1)} g^{-1} (\beta^{-1}_0 - \beta^{-1}_\varepsilon) d.
\]

By (63),

\[
\int_{R_1}^{R_2} r^{-(N-1)} \tilde{\phi}_r (D_r H + DH_r) \, dr \\
= \frac{1}{\varepsilon} \int_{R_1}^{R_2} r^{-2(N-1)} g^{-1} \tilde{D}(D_r H + DH_r) \\
+ (\beta^{-1}_0 - \beta^{-1}_\varepsilon) d(D_r H + DH_r) \, dr \\
= \frac{1}{\varepsilon} \int_{R_1}^{R_2} r^{-2(N-1)} g^{-1} \left[ \tilde{D}D_r H + \tilde{D}DH_r \\
+ (\beta^{-1}_0 - \beta^{-1}_\varepsilon) dD_r H + (\beta^{-1}_0 - \beta^{-1}_\varepsilon) dDH_r \right] \, dr.
\]

We can use Hölder’s inequality, Young’s inequality, and Poincaré’s inequality to get

\[
\left| \frac{1}{\varepsilon} \int_{R_1}^{R_2} r^{2(N-1)} g^{-1} \tilde{D}D_r H \, dr \right| \\
\leq \frac{1}{\varepsilon} R_1^{-(N-1)} \| H \|_{L_\infty} \left[ \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \tilde{D}^2 \, dr \right]^{\frac{1}{2}} \left[ \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} D_r^2 \, dr \right]^{\frac{1}{2}} \\
\leq \frac{1}{8C_\infty} \| H \|_{L_\infty}^2 \\
+ \frac{2}{\varepsilon^2} R_1^{-(N-1)} \left[ \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} \tilde{D}^2 \, dr \right] \left[ \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} D_r^2 \, dr \right].
\]

≤ \frac{1}{8} \int_{R_1}^{R_2} r^{-(N-1)} H_r^2 dr \
+ \frac{2}{\varepsilon^2} R_1^{-2(N-1)} g_0^{-2} \left[ \int_{R_1}^{R_2} r^{-(N-1)} D^2 dr \right] \left[ \int_{R_1}^{R_2} r^{-(N-1)} D_r^2 dr \right]. \tag{65}

Here we have used the fact that
\int_{R_2}^{R_1} r^{-(N-1)} g^{-1} D^2 dr = \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} D^2 dr + \beta_0^{-1} d^2
≥ \int_{R_1}^{R_2} r^{-(N-1)} g^{-1} D^2 dr.

Similarly,
\left| \frac{1}{\varepsilon} \int_{R_1}^{R_2} r^{-2(N-1)} g^{-1} \bar{D} D H_r dr \right|
≤ \frac{1}{4} \int_{R_1}^{R_2} r^{-(N-1)} D_r^2 dr \tag{66}
+ \frac{1}{\varepsilon^2} R_1^{-2(N-1)} g_0^{-2} \left[ \int_{R_1}^{R_2} r^{-(N-1)} D^2 dr \right] \left[ \int_{R_1}^{R_2} r^{-(N-1)} H_r^2 dr \right],

\left| \frac{1}{\varepsilon} (\beta_0^{-1} - \beta_1^{-1}) d \int_{R_1}^{R_2} r^{-2(N-1)} g^{-1} D_r H dr \right|
≤ \frac{1}{4} \int_{R_1}^{R_2} r^{-(N-1)} D_r^2 dr \tag{67}
+ \frac{1}{\varepsilon^2} (\beta_0^{-1} - \beta_1^{-1})^2 R_1^{-2(N-1)} g_0^{-2} d^2 \int_{R_1}^{R_2} r^{-(N-1)} H^2 dr,

and
\left| \frac{1}{\varepsilon} (\beta_0^{-1} - \beta_1^{-1}) d \int_{R_1}^{R_2} r^{-2(N-1)} g^{-1} D_r H_r dr \right|
≤ \frac{1}{8} \int_{R_1}^{R_2} r^{-(N-1)} H_r^2 dr \tag{68}
+ \frac{2}{\varepsilon^2} (\beta_0^{-1} - \beta_1^{-1})^2 R_1^{-2(N-1)} g_0^{-2} d^2 \int_{R_1}^{R_2} r^{-(N-1)} D_r^2 dr.

Putting (64)–(68) into (61), we obtain
\begin{align*}
\frac{1}{2} \frac{d}{dt} & \int_{R_1}^{R_2} r^{-(N-1)} (D^2 + H^2) dr \\
\leq & - \left[ \frac{1}{2} - \frac{2}{\varepsilon^2} R_1^{-2(N-1)} g_0^{-2} \int_{R_1}^{R_2} r^{-(N-1)} D^2 dr \right] \int_{R_1}^{R_2} r^{-(N-1)} D_r^2 dr \tag{69} \\
& - \left[ \frac{1}{8} - \frac{1}{\varepsilon^2} R_1^{-2(N-1)} g_0^{-2} \int_{R_1}^{R_2} r^{-(N-1)} D^2 dr \right] \int_{R_1}^{R_2} r^{-(N-1)} H_r^2 dr.
\end{align*}
Then Theorem 1.2 is proved by letting $\varepsilon_2 = \frac{m_0}{2\beta_0\beta_1\omega_N(R_2^N - R_1^N)\sqrt{\varepsilon}}$.

2.4. Proof of Corollary 1. By Theorem 1.2, i.e., (23),

$$\frac{d}{dt} \int_{R_1}^{R_2} r^{-(N-1)}(D^2 + H^2) \, dr \leq 0,$$

provided that

$$\int_{R_1}^{R_2} r^{-(N-1)} D^2 \, dr \leq \frac{1}{8} R_1^{2(N-1)} g_0^2 \varepsilon^2$$

and

$$\int_{R_1}^{R_2} r^{-(N-1)} (2D^2 + H^2) \, dr \leq \frac{m_0 K R_1^{2(N-1)} g_0^2 \varepsilon^{3/2}}{2\beta_0\beta_1(\beta_0^{-1} - \beta_1^{-1})^2 \omega_N(R_2^N - R_1^N)}.$$

If the initial data satisfy

$$I_0 \leq \min \left\{ \frac{\mu}{8} R_1^{2(N-1)} g_0^2 \varepsilon^2, \frac{m_0 K R_1^{2(N-1)} g_0^2 \varepsilon^{3/2}}{4\beta_0\beta_1(\beta_0^{-1} - \beta_1^{-1})^2 \omega_N(R_2^N - R_1^N)} \right\}$$

for some $0 < \mu < 1$, then (70) implies

$$I(t) := \int_{R_1}^{R_2} r^{-(N-1)} (D^2 + H^2) \, dr \leq I_0 \leq \min \left\{ \frac{\mu}{8} R_1^{2(N-1)} g_0^2 \varepsilon^2, \frac{m_0 K R_1^{2(N-1)} g_0^2 \varepsilon^{3/2}}{8\beta_0\beta_1(\beta_0^{-1} - \beta_1^{-1})^2 \omega_N(R_2^N - R_1^N)} \right\}$$

for $t > 0$. This gives

$$\frac{d}{dt} \int_{R_1}^{R_2} r^{-(N-1)} (D^2 + H^2) \, dr \leq -\frac{1}{2}(1 - \mu) \int_{R_1}^{R_2} r^{-(N-1)} D^2 \, dr - \frac{1}{8}(1 - \mu) \int_{R_1}^{R_2} r^{-(N-1)} H^2 \, dr \leq -\frac{1}{8}(1 - \mu) C_2^{-1} \int_{R_1}^{R_2} r^{-(N-1)} (D^2 + H^2) \, dr$$

for $t > 0$, where $C_2$ is the constant for Poincaré’s inequality (58). Therefore, we complete the proof of Corollary 1 with $\lambda = \frac{1 - \mu}{8C_2}$.
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