ON THE EXISTENCE OF ASYMPTOTICALLY GOOD LINEAR CODES IN MINOR-CLOSED CLASSES

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Abstract. Let $C = (C_1, C_2, \ldots)$ be a sequence of codes such that each $C_i$ is a linear $[n_i, k_i, d_i]$-code over some fixed finite field $\mathbb{F}$, where $n_i$ is the length of the codewords, $k_i$ is the dimension, and $d_i$ is the minimum distance. We say that $C$ is asymptotically good if, for some $\varepsilon > 0$ and for all $i$, $n_i \geq i$, $k_i/n_i \geq \varepsilon$, and $d_i/n_i \geq \varepsilon$. Sequences of asymptotically good codes exist. We prove that if $C$ is a class of $\text{GF}(p^n)$-linear codes (where $p$ is prime and $n \geq 1$), closed under puncturing and shortening, and if $C$ contains an asymptotically good sequence, then $C$ must contain all $\text{GF}(p)$-linear codes. Our proof relies on a powerful new result from matroid structure theory.

1. Introduction

For a finite field $\mathbb{F}$, let $\mathbb{F}_{\text{prime}}$ denote the unique subfield of $\mathbb{F}$ of prime order. For a linear code $C$, denote the length of $C$ by $n_C$, the dimension of $C$ by $k_C$, and the minimum Hamming distance of $C$ by $d_C$. In short, $C$ is an $[n_C, k_C, d_C]$ code. A class $C$ of codes is asymptotically good if there exists $\varepsilon > 0$ such that for every $n \in \mathbb{Z}^+$ there is a code $C \in C$ of length $n_C \geq n$ satisfying $k_C \geq \varepsilon n_C$ and $d_C \geq \varepsilon n_C$.

For every finite field $\mathbb{F}$, the class of linear codes over $\mathbb{F}$ is asymptotically good, as suitable random codes have nonvanishing rate and minimum distance. Our main result can be seen as a converse to this statement. Our result involves two standard operations on linear codes. Given a code $C$, the puncturing of $C$ at $i$ is the code obtained from $C$ by removing the $i$th coordinate from each word. The shortening of $C$ at $i$ is the code obtained from $C$ by selecting only the codewords of $C$ having a 0 in position $i$, and then puncturing the resulting code at $i$.

Theorem 1.1. Let $\mathbb{F}$ be a finite field. If $C$ is an asymptotically good class of linear codes over $\mathbb{F}$, then every linear code $C'$ over $\mathbb{F}_{\text{prime}}$ can be obtained from a code $C \in C$ by a sequence of puncturings and shortenings.
In other words, the only asymptotically good classes of $\mathbb{F}$-linear codes that are closed under puncturings and shortenings are those that contain all linear codes over some field. This result was conjectured by Geelen, Gerards, and Whittle \cite{Geelen2011}. A restatement of the above theorem for prime fields is the following:

**Theorem 1.2.** Let $\mathbb{F}$ be a field of prime order. If $C$ is a proper subclass of the linear codes over $\mathbb{F}$ that is closed under puncturing and shortening, then $C$ is not asymptotically good.

When $|\mathbb{F}| = 2$, this substantially generalises results of Kashyap \cite{Kashyap2006} which show that the classes of graphic binary codes, as well as a slightly larger class of ‘almost-graphic’ codes, are not asymptotically good.

Our proof makes fundamental use of a deep theorem in structural matroid theory recently announced by Geelen, Gerards and Whittle. This theorem is one of many outcomes of the ‘matroid minors project’, the result of more than a decade of work generalising Robertson and Seymour’s graph minors structure theorem \cite{Robertson1986} to matroids representable over finite fields. While this theorem has now been stated in print \cite{Geelen2011}, its proof will stretch to hundreds of pages and has yet to be published. The reader should be aware of this contingency; for a more detailed discussion see \cite{Geelen2011}.

2. Preliminaries

Since the ideas in our proof are matroidal, we adopt the terminology of matroid theory. The correspondence between linear codes and matroids is fairly direct even when a matroid is defined in the usual way by its ground set and rank function, but for convenience we will deal with a flavour of matroid whose definition coincides exactly with that of a linear code.

**Represented matroids.** If $\mathbb{F}$ is a field, then an $\mathbb{F}$-represented matroid is a pair $M = (E, U)$, where $E$ is a finite set and $U$ is a subspace of $\mathbb{F}^E$. We often omit ‘$\mathbb{F}$-represented’ when the context is clear. We write $|M|$ for $|E|$.

If $A$ is an $\mathbb{F}$-matrix with column set $E$ such that rowspace($A$) = $U$, then we write $M = M(A)$; we call $A$ a generator matrix for $M$ and say that $A$ generates $M$. For $X \subseteq E$ we write $u|X$ for the restriction of the vector $u$ to those coordinates indexed by $X$, we write $U|X$ for the space \{u|X : u \in U\}, and we write $r_M(X)$ for the dimension of $U|X$. We denote $r_M(E)$ simply by $r(M)$. We call $r_M$ the rank function of $M$. 
If $C$ is a $q$-ary $[n, k, d]$ code, then $M = (\{1, \ldots, n\}, C)$ is a matroid with $|M| = n$ and $r(M) = k$. The rank function simply gives the dimension of this code and of all its puncturings.

If $A_2$ is an $F$-matrix obtained from an $F$-matrix $A_1$ by nonzero column scalings, then $M(A_1)$ and $M(A_2)$ are formally distinct but share the same rank function. We give a name to this equivalence: two $F$-represented matroids $(E, U_1)$ and $(E, U_2)$ are projectively equivalent if $U_2 = \{uD : u \in U_1\}$ for some nonsingular diagonal matrix $D$.

Matroid terminology. Matroid duality coincides with linear code duality. The dual matroid $M^*$ of an $F$-represented matroid $(E, U)$ is defined to be $(E, U^\perp)$, where $U^\perp$ denotes the orthogonal complement of $U$. For $X \subseteq E$ we write $M\setminus X$ for $(E \setminus X, U|_{(E \setminus X)})$ and $M/X$ for $(M^*\setminus X)^*$; these are the matroids obtained from $M$ by deletion and contraction of $X$ respectively; these operations correspond to puncturing and shortening of codes. If $N$ is an $F$-represented matroid that is projectively equivalent to $M/C \setminus D$ for some disjoint subsets $C$ and $D$ of $E$, then we say $N$ is a minor of $M$. A class $\mathcal{M}$ of matroids is minor-closed if $\mathcal{M}$ is closed under taking minors and isomorphism.

A set $X \subseteq E$ is spanning in $M$ if $r_M(X) = r(M)$ and independent in $M$ if $r_M(X) = |X|$. If $X$ is not independent then $X$ is dependent. A circuit of $M$ is a minimal dependent set and a cocircuit of $M$ is a minimal dependent set of $M^*$, or equivalently a minimal set $C$ satisfying $r(M \setminus C) < r(M)$. For a matroid $M$, we write $g(M)$ for the size of a smallest circuit of $M$, also called the girth of $M$.

If $C$ is a $q$-ary $[n, k, d]$ code, then $M = (\{1, \ldots, n\}, C)$ is a matroid with $g(M^*) = d$. For that reason we sometimes write $d(M)$ for $g(M^*)$.

Connectivity. A notion fundamental in matroid theory that arises in our proof is that of connectivity. Informally, a matroid has low connectivity if it can be obtained by ‘gluing’ two smaller matroids together on a low-dimensional subspace. There are many notions of matroid connectivity but we just need one; for $t \in \mathbb{Z}^+$, a matroid $M = (E, U)$ is vertically $t$-connected if, for every partition $(A, B)$ of $E$ satisfying $r_M(A) + r_M(B) < r(M) + t - 1$, either $A$ or $B$ is spanning in $M$. For instance, $M$ is vertically 2-connected if and only if $M$ cannot be written as the direct sum of two positive-rank matroids. Note that other authors often use a more restrictive definition of “vertically $t$-connected,” which implies ours.

Frame matroids. An $F$-frame matrix is an $F$-matrix in which every column has at most two nonzero entries, and an $F$-represented frame matroid is a matroid having an $F$-frame matrix as a generator matrix.
For a group $\Gamma$, a $\Gamma$-labelled digraph is a pair $(G, \Sigma)$, where $G = (V, E)$ is a directed graph (allowing loops and multiple edges) and $\Sigma : E \to \Gamma$ is an assignment of a label in $\Gamma$ to every arc of $G$. There is a well-known and natural correspondence between $F$-frame matroids and $F^\times$-labelled digraphs; a full treatment is given in [4] and a reader familiar with these concepts can skip this subsection, where we just give the minimum definitions and observations we will need.

If $A$ is an $F$-frame matrix with row set $V$ and column set $E$, then a graph representation of $A$ is an $F^\times$-labelled digraph $(G, \Sigma)$, where $G = (V, E)$ and $(G, \Sigma)$ satisfies the following conditions:

- If $A[e]$ has two nonzero entries in rows $x$ and $y$, then $e$ is an arc of $G$ from $x$ to $y$ with label $-A[e, x]A[e, y]^{-1}$ or an arc of $G$ from $y$ to $x$ with label $-A[e, y]A[e, x]^{-1}$, and
- If $A[e]$ has exactly one nonzero entry, then $e$ is a loop of $G$ at $x$ with arbitrary label in $F^\times - \{1\}$.
- If $A[e] = 0$, then $e$ is a loop of $G$ with label 1.

It is clear that one frame matrix may have many graph representations, and that graph representations always exist unless $|F| = 2$, where a frame matrix having a column with exactly one nonzero entry has no graph representations since $F^\times - \{1\}$ is empty. However, appending a ‘parity’ row to a binary frame matrix yields another row-equivalent frame matrix where every column has even support. Since we can remove redundant rows from an arbitrary frame matrix to still have a frame matrix and append a single row in this way in the binary case, we have the following statement, which we apply freely.

**Proposition 2.1.** If $M$ is an $F$-represented frame matroid, then there is a generator matrix $A$ of $M$ having a graph representation and at most $r(M) + 1$ rows.

A cycle or path of a digraph $G$ will denote any cycle or path of the underlying undirected graph of $G$. Let $C$ be a cycle of $G$, and $v_1, e_1, v_2, e_2, \ldots, e_{k-1}, v_k = v_1$ be a corresponding alternating sequence of vertices and arcs of $G$ (there are two choices for this sequence). Let $C^+$ be the set of $e_i$ such that $e_i$ is directed from $v_i$ to $v_{i+1}$ in $G$, and $C^-$ be the set of all other $e_i$. The sign of $C$ (relative to this sequence) is $\prod_{e \in C^+} \Sigma(e) \prod_{e \in C^-} \Sigma(e)^{-1}$. We say that $C$ is a balanced cycle of $(G, \Sigma)$ if $\text{sign}(C) = 1$. Note that a cycle being balanced does not depend on the choice of sequence of vertices and edges. There is a well-known characterisation of the set of circuits of $M(A)$ in terms of the balanced and unbalanced cycles of $(G, \Sigma)$, but we will just use the following weaker statement, which is fairly straightforward to check by considering linear dependencies in the columns of $A[C]$. 
Proposition 2.2. If \((G, \Sigma)\) is a graph representation of an \(\mathbb{F}\)-frame matrix \(A\) and \(C\) is a balanced cycle of \((G, \Sigma)\) or a connected subgraph of \(G\) of minimum degree 2 which is not a cycle of \(G\), then \(C\) is dependent in \(M(A)\).

We say \((G, \Sigma')\) was obtained from \((G, \Sigma)\) by resigning if, for some \(\gamma \in \Gamma\) and for some partition \((U, W)\) of the vertices of \(G\), we have

\[
\Sigma'(e) = \begin{cases} 
\gamma \Sigma(e) & \text{if } e \text{ runs from } U \text{ to } W; \\
\gamma^{-1} \Sigma(e) & \text{if } e \text{ runs from } W \text{ to } U; \\
\Sigma(e) & \text{otherwise.}
\end{cases}
\]

It is easily checked that \((G, \Sigma)\) and \((G, \Sigma')\) have the same collection of balanced cycles. In the representation this corresponds to scaling the rows indexed by \(W\) by a factor \(\gamma\).

In what follows, we need to assume that the graph representation of a frame matroid is connected, which means that there is a path between every pair of vertices. It is easy to show that a vertically 3-connected matroid, with no elements \(e\) such that \(r_M(e) = 0\), has the property that every pair of elements is in a circuit. A consequence of this is:

Lemma 2.3. Let \((G, \Sigma)\) be a graph representation of an \(\mathbb{F}\)-represented frame matroid \(M\). If \(M\) is vertically 3-connected and has no loops, then \(G\) is connected.

Corollary 2.4. Let \(M\) be an \(\mathbb{F}\)-represented frame matroid. If \(M\) is vertically 3-connected, then \(M\) has a graph representation \((G, \Sigma)\) with \(G\) connected.

Asymptotically good matroids. Finally, we redefine asymptotic goodness, this time for matroids. For \(\alpha, \beta \in \mathbb{R}\) we say a sequence \(M_0, M_1, \ldots\) of matroids is \((\alpha, \beta)\)-good if \(|M_i| \geq i\), \(r(M_i) \geq \alpha |M_i|\) and \(g(M_i^\ast) \geq \beta |M_i|\) for each \(i \in \mathbb{Z}_+\). A class \(\mathcal{M}\) of matroids is asymptotically good if \(\mathcal{M}\) contains an \((\alpha, \beta)\)-good sequence for some \(\alpha, \beta \in \mathbb{R}^+\). Note that \(\alpha, \beta \leq 1\) for any such \(\alpha, \beta\). For any finite field \(\mathbb{F}\), the class of all \(\mathbb{F}\)-represented matroids is such a class.

3. Connectivity

Our goal in this section is to show that, to prove our main result, it suffices to focus on highly connected matroids.

Lemma 3.1. If \(t \in \mathbb{Z}\) and \(\mathcal{M}\) is an asymptotically good minor-closed class of matroids, then the class of vertically \(t\)-connected matroids in \(\mathcal{M}\) is asymptotically good.
Proof. For each $\beta \in \mathbb{R}^+$ let $A_\beta$ denote the set of all $\alpha \in \mathbb{R}^+$ such that $M$ contains an $(\alpha, \beta)$-good sequence. Let $B = \{\beta \in \mathbb{R}^+ : A_\beta \neq \emptyset\}$. By assumption, $B$ is a nonempty interval with $\inf(B) = 0$ and $\sup(B) \leq 1$; let $\beta_{\max} = \sup(B)$. Each nonempty $A_x$ is also such an interval; for each $x \in B$, let $\alpha_{\max}(x) = \sup(A_x)$, noting that $\alpha_{\max}(x) \leq 1$.

Let $\beta \in \left(\frac{2}{3}\beta_{\max}, \beta_{\max}\right)$ and let $\delta = \frac{\beta}{1 - \beta}$. Set $\alpha_{\max} = \alpha_{\max}(\beta)$ and let $\alpha \in ((1 + \frac{1}{2}\delta)^{-1}\alpha_{\max}, \alpha_{\max})$. We have $\beta \in B$ and $\alpha \in A_\beta$; let $M_0, M_1, \ldots$ be an $(\alpha, \beta)$-good sequence of matroids in $\mathcal{M}$. Since $2\beta > \beta_{\max}$ we have $2\beta \notin B$ so $A_{2\beta} = \emptyset$ and in particular $\frac{1}{2}\alpha \notin A_{2\beta}$. Moreover, we have $(1 + \delta)\alpha > \alpha_{\max}$ so $(1 + \delta)\alpha \notin A_\beta$. There are therefore integers $m_1$ and $m_2$ such that and every matroid $M \in \mathcal{M}$ with $|M| \geq m_1$ satisfies $g(M^*) < 2|\beta|M$ or $r(M) < \frac{1}{2}\alpha|M|$, and every matroid $M \in \mathcal{M}$ with $|M| \geq m_2$ satisfies $g(M^*) < |\beta|M$ or $r(M) < \alpha(1 + \delta)|M|$. Let $m = \max(m_1, m_2, 8\alpha^{-1})$.

3.1.1. For each $n \in \mathbb{Z}$ with $n \geq 2\beta^{-1}m$, the matroid $M_n$ is vertically $t$-connected.

Proof. Suppose for a contradiction that $n \geq 2\beta^{-1}m$ and $M = M_n$ is not vertically $t$-connected. Let $(X_1, X_2)$ be a partition of $E(M)$ with $r_M(X_1) + r_M(X_2) = r(M) + t' - 1$ with $t' < t$ and $r_M(X_1), r_M(X_2) \geq t'$. Let $N_1 = M/X_2$ and $N_2 = M/X_1$. Note that $N_1, N_2 \in \mathcal{M}$ and that $r(N_i) = r(M) - r_M(X_{3-i}) > r(M) - t + 1$, so $r(N_1) + r(N_2) > r(M) - 2t$. Since every cocircuit of $N_1$ or $N_2$ is a cocircuit of $M$, we have $g(M^*) \leq \min(g(N_1^*), g(N_2^*)).$ We have

$$\beta|M| \leq g(M^*) \leq \min(g(N_1^*), g(N_2^*)) \leq \frac{1}{2}(g(N_1^*) + g(N_2^*))$$

and, since $|M| = |N_1| + |N_2|$, we have either $g(N_1^*) \geq 2\beta|N_1|$ or $g(N_2^*) \geq 2\beta|N_2|$. We may assume that the first case holds. Since each of $X_1$ and $X_2$ contains a cocircuit of $M$, we have $|N_i| = |X_i| \geq g(M^*) \geq \beta|M| \geq m$ for each $i \in \{1, 2\}$. Since $|N_1| + |N_2| = |M|$, this implies that $|N_1| \geq \frac{\beta}{1 - \beta}|N_2|$. Moreover, $m \geq m_1$ gives $r(N_1) < \frac{1}{2}\alpha|N_1|$. We have

$$\alpha|M| \leq r(M)$$

$$< r(N_1) + r(N_2) + 2t$$

$$< \frac{1}{2}\alpha|N_1| + r(N_2) + 2t$$

$$\leq \frac{2}{3}\alpha|N_1| + r(N_2),$$

Where the last line uses $\frac{1}{4}\alpha|N_1| \geq \frac{1}{4}\alpha m \geq 2t$. From this and $|M| = |N_1| + |N_2|$ we get

$$r(N_2) \geq \frac{1}{4}\alpha|N_1| + \alpha|N_2|$$

$$= \alpha(1 + \delta)|N_2|$$
Now $|N_2| \geq m \geq m_2$ so $g(N_2^*) < \beta|N_2|$ by choice of $m_2$. However $g(M^*) \leq g(N_2^*)$ and $\beta|N_2| < \beta|M|$, so $g(M^*) < \beta|M|$, contradicting the definition of $M$. \hfill \Box

By the claim, all but finitely many terms of the sequence $M_0, M_1, \ldots$ are vertically $t$-connected, so the class of vertically $t$-connected matroids in $\mathcal{M}$ is asymptotically good, as required. \hfill \Box

4. The Structure Theorem

For each field $\mathbb{F}$ of prime characteristic $p$, we write $\mathbb{F}_{\text{prime}}$ for the unique subfield of $\mathbb{F}$ with $p$ elements. In this section we state the deep structural result on which our proof is based. Essentially the theorem states that for any minor-closed class $\mathcal{M}$ of $\mathbb{F}$-represented matroids not containing all $\mathbb{F}_{\text{prime}}$-represented matroids, the highly connected members of $\mathcal{M}$ are 'close' to being an $\mathbb{F}$-represented frame matroid or its dual. We need to define our notion of distance.

Our distance metric is based on ‘lifts’ and ‘projections’. If $M_1 = (E, U_1)$ and $M_2 = (E, U_2)$ are $\mathbb{F}$-represented matroids and there is an $\mathbb{F}$-represented matroid $M$ with ground set $E \cup \{e\}$ satisfying $M \setminus e = M_1$ and $M/e = M_2$, then we say that $M_2$ is an elementary projection of $M_1$ and $M_1$ is an elementary lift of $M_2$. For arbitrary $\mathbb{F}$-represented matroids $M_1$ and $M_2$ on a common ground set $E$, we write dist($M_1, M_2$) for the minimum number of elementary lifts/projections required to transform $M_1$ into $M_2$. (It is clear that any matroid on $E$ can be transformed into the rank-0 matroid on $E$ by a finite sequence of projections, so this distance is always finite.) It is easy to see that, if dist($M_1, M_2$) $\leq k$, then there is a matroid $M$ with ground set $E \cup C \cup D$ satisfying $M \setminus D/C = M_1$ and $M/D \setminus C = M_2$, where $|C| + |D| \leq k$. Each elementary lift and projection can change the rank of a subset by at most one. From this we can deduce the following lemma.

**Lemma 4.1.** Let $M, N$ be $\mathbb{F}$-represented matroids with dist($M, N$) $\leq k$. If $M$ is vertically $t$-connected, then $N$ is vertically $(t - 2k)$-connected.

The structure theorem, a weakened combination of Lemma 2.1 and Theorems 3.2 and 3.3 in [3], can now be stated.

**Theorem 4.2.** Let $\mathbb{F}$ be a finite field and let $\mathcal{M}$ be a minor-closed class of $\mathbb{F}$-represented matroids not containing all projective geometries over $\mathbb{F}_{\text{prime}}$. There exists $k \in \mathbb{Z}^+$ such that every vertically $k$-connected matroid $M$ in $\mathcal{M}$ satisfies dist($M, N$) $\leq k$ or dist($M^*, N$) $\leq k$ for some $\mathbb{F}$-represented frame matroid $N$. 

5. Small Circuits

In this section we show that if $M$ is a rank-$r$ matroid with has significantly more than $r$ elements and $M$ or $M^*$ is close to a frame matroid or its dual, then the girth of $M$ is at most logarithmic in $r$. The primal case is slightly more difficult and will result from the following corollary of a result of Alon, Hoory and Linial \([1]\) observed by Kashyap \([2]\):

**Lemma 5.1.** If $G$ is a graph with girth $g$ and average degree $\delta > 2$, then $g \leq 4 + \frac{\log(|V(G)|)}{\log(\delta - 1)}$.

The next three results extend the above lemma to matroids that are close to frame matroids. As before, a cycle of a graph refers strictly to a set of its edges.

**Lemma 5.2.** Let $t \in \mathbb{Z}^+$ and $\beta \in \mathbb{R}^+$. If $G$ is a graph with $|V(G)| = n \geq \max \left( \frac{4t}{\beta \log(1+\beta)} \right)^2, (1+\beta)e^4$ and $|E(G)| \geq (1+\beta)n$, then $G$ has a collection of $t$ pairwise edge-disjoint cycles, each of size at most $2 \log \frac{n}{\log(1+\beta)}$.

**Proof.** Let $m = |E(G)|$ and $\alpha = 2(\log(1+\beta))^{-1}$. By choice of $n$ we have $\frac{\alpha \log n}{n} < \frac{\alpha}{\sqrt{n}} < \frac{1}{2} \beta$. Let $C$ be a maximal collection of pairwise disjoint cycles of $G$ such that $|E(C)| \leq \alpha \log n$ for each $C \in C$. Assume for a contradiction that $|C| < t$; let $G' = G \cup C$. We have $\frac{|E(G')|}{n} \geq \frac{m - \alpha \log(n)}{n} \geq 1 + \beta - \frac{\alpha \log(n)}{n} > 1 + \frac{1}{2} \beta$, so the average degree $\delta'$ of $G'$ is at least $2 + \beta$. By maximality of $C$, the graph $G'$ has girth $g' > \alpha \log(n)$, so Lemma 5.1 gives

$$\alpha \log(n) < 4 + \frac{\log(n)}{\log(\delta' - 1)} \leq 4 + \frac{\log(n)}{\log(1 + \beta)}.$$  

Rearranging gives $\log(n) < 4 \log(1+\beta)$, contradicting $n \geq (1+\beta)e^4$. □

**Corollary 5.3.** Let $\mathbb{F}$ be a finite field of order $q$, let $\beta \in \mathbb{R}^+$ and $t \in \mathbb{Z}^+$. Let $M$ be an $\mathbb{F}$-represented frame matroid with graph representation $(G, \Sigma)$, such that $G$ is connected. If $M$ satisfies

$$r(M) \geq \max \left( |\mathbb{F}|, (2q - 3)\beta^{-1} + 1 - q, \frac{1}{q-1} \left( \frac{4t}{\beta \log(1+\beta)} \right)^2, \frac{1+\beta}{q-1}e^4 \right)$$

and $|M| \geq (1 + q \beta)r(M)$, then there is a set $X \subseteq E(M)$ such that $r_M(X) \leq |X| - t$ and $|X| \leq \frac{4t \log r(M)}{\log(1+\beta)}$.

**Proof.** Let $A$ be an $\mathbb{F}$-frame matrix generating $M$ and let $(G, \Sigma)$ be a graph representation of $A$. Pick a spanning tree $T$ of $G$. By repeatedly
resigning, we may assume that the edges of $T$ have sign $1$. Note that $|E(T)| \geq r(M) - 1$.

Let $G^+$ denote the undirected graph with vertex set $V(G) \times \mathbb{F}^\times$ and edge set

$$\{(e^- (e), (e^+(e), e)) : e \in T, e \in \mathbb{F}\}$$

$$\cup \{(e^- (e), 1), (e^+(e), \text{sign}(e)) : e \in E(G) \setminus T\};$$

that is, we take $q - 1$ copies of $T$, and each directed edge $(u, v)$ not in $T$ with sign $\gamma$ connects the copy of $T$ corresponding to $1$ with the copy of $T$ corresponding to $\gamma$. It is easy to check that each cycle of $G^+$, by projection onto the first coordinate, corresponds to either a balanced cycle of $(G, \Sigma)$, or a subgraph of $G$ of minimum degree 2 that is not a cycle. It follows from Proposition 2.2 that every cycle of $G^+$ is dependent in $M$. Now,

$$|E(G^+)| = (q - 1)|E(T)| + |M| - |E(T)| \geq (q - 2)(r(M) - 1) + |M|$$

$$\geq (q - 1)(1 + \beta)r(M) + \beta r(M) - (q - 2)$$

$$\geq (1 + \beta)|V(G^+)| - (q - 1)(1 + \beta) + \beta r(M) - q + 2$$

$$\geq (1 + \beta)|V(G^+)|,$$

where we use $|E(T)| \geq r(M) - 1$ first, $|M| \geq (1 + q\beta)r(M)$ second, $|V(G^+)| \leq (q - 1)(r(M) + 1)$ third, and $r(M) \geq (2q - 3)\beta^{-1} + 1 - q$ last. By our lower bounds for $n = |V(G^+)| \geq (q - 1)r(M)$ and Lemma 5.2 the graph $G^+$ contains $t$ pairwise disjoint cycles, each of size at most

$$\frac{2 \log n}{\log(1 + \beta)} \leq \frac{4 \log r(M)}{\log(1 + \beta)}.$$ Each of these cycles of $G$ is dependent in $M$, and it is thus easy to see that their union satisfies the result. \qed

**Lemma 5.4.** Let $k \in \mathbb{Z}^+$, $\beta \in \mathbb{R}^+$, and $\mathbb{F}$ be a finite field. There exists $c \in \mathbb{Z}$ such that if $M$ is a rank-$r$ $\mathbb{F}$-represented matroid with $r > 2$ and $|M| \geq (1 + \beta)r$ and there is an $\mathbb{F}$-represented, connected frame matroid $N$ satisfying $\text{dist}(M, N) \leq k$, then $g(M) \leq c \log r$.

**Proof.** Let $\beta' = \frac{1}{2} \beta$ and $c \geq \max\left(\frac{4(k + 1)\beta^{-1} + 1 - q}{|\mathbb{F}|}, \left(\frac{4(k + 1)}{\beta' \log(1 + \beta')}\right)^2\right)$

$(1 + \beta')c^4 + k$ be an integer so that $4(k + 1) \log(x + k) \leq c \log(1 + \beta') \log x$ for all $x \geq c$. Since $\beta' > 0$ we know that $M$ has a circuit, so $g(M) \leq r + 1$. If $r \leq c - 1$ then $g(M) \leq r + 1 \leq c \leq c \log r$, so the result holds; we may thus assume that $r \geq c$.

Let $M^+$ be a matroid and $C, D \subseteq E(M^+)$ be sets of size at most $k$ so that $M^+ / C \setminus D = M$ and $M^+ / D \setminus C = N$. Since each $Y \subseteq E(M)$ satisfies $r_{M^+}(Y) \geq r_M(Y) \geq r_{M^+}(Y) - r_{M^+}(C) \geq r_{M^+}(Y) - k$ and a similar statement holds for $N$, we have $|r_M(Y) - r_N(Y)| \leq k$ for
each \( Y \subseteq E(M) \). In particular, \( r + k \geq r(N) \geq r - k \geq c - k \). By choice of \( c \), Corollary 5.3 implies there is a set \( X \subseteq E(N) \) with \( r_N(X) \leq |X| - (k+1) \) and \( |X| \leq \frac{4(k+1)\log(r(N))}{\log(1+\beta)} \leq \frac{4(k+1)\log(r+k)}{\log(1+\beta)} \leq c \log r. \)

Now \( r_M(X) \leq r_N(X) + k \leq |X| \), so \( X \) contains a circuit of \( M \) and therefore \( g(M) \leq c \log r \), as required. \( \square \)

We now deal with the case when \( M \) is close to the dual of a frame matroid. Here we show that the girth is bounded above by a constant.

**Lemma 5.5.** Let \( k \in \mathbb{Z}^+ \), \( \beta \in \mathbb{R}^+ \), and \( \mathbb{F} \) be a finite field. There exists \( c \in \mathbb{Z} \) so that, if \( M \) is a nonempty \( \mathbb{F} \)-represented matroid such that \( |M| \geq (1 + \beta)r(M) \) and there is an \( \mathbb{F} \)-represented frame matroid \( N \) with \( \text{dist}(M^*, N) \leq k \) and connected graph, then \( g(M) \leq c \).

**Proof.** We may assume that \( \beta \leq 1 \). Let \( c = \lceil 12\beta^{-1}(3k+1) \rceil \). If \( |M| \leq c \) then the result clearly holds, so we will assume otherwise. Let \( M^+ \) be a matroid and \( C, D \subseteq E(M^+) \) be sets of size at most \( k \) so that \( M^+ / C \setminus D = M^* \) and \( M^+/D \setminus C = N \). As before, we have \( |r_M(Y) - r_N(Y)| \leq k \) for each \( Y \subseteq E(M^+) \). Let \( A \) be an \( \mathbb{F} \)-frame matrix generating \( N \) with \( r(N) \) rows (or \( r(N) + 1 \) rows if \( |\mathbb{F}| = 2 \)), and \( (G, \Sigma) \) be an \( \mathbb{F}^\ast \)-labelled graph associated with \( N \). We have \( |M^*| = |M| \geq (1 + \beta)r(M) = (1 + \beta)(|M^*| - r(M^*)) \), giving \( |M^*| \leq (1 + \beta^{-1})r(M^*) \leq 2\beta^{-1}r(M^*) \).

Now \( k \leq r(M^*) - k \leq r(N) \leq |V(G)| + 1 \leq 2|V(G)| \), so

\[
|E(G)| = |N| = |M^*| \leq 2\beta^{-1}(|V(G)| + k) \leq 6\beta^{-1}|V(G)|.
\]

Therefore \( G \) has average degree at most \( 12\beta^{-1} \). If \( |V(G)| < 2(k + 1) \), then \( 2(k + 1) - 1 \geq r(N) \geq r(M^*) - k \) so \( r(M^*) \leq 3k + 1 \). This gives \( |M^*| \leq 2\beta^{-1}(3k+1) < c \), a contradiction. Therefore \( |V(G)| \geq 2(k+1) \), so there is a collection of \( 2(k+1) \) vertices of \( G \) whose degrees sum to at most \( 12\beta^{-1}(k + 1) \), so there is a set \( F \subseteq E(G) \) such that \( |F| \leq 12\beta^{-1}(k + 1) \leq c \) and \( r(N \setminus F) \leq r(N) + 1 - 2(k + 1) \) (since \( A \) has at most \( r(N) + 1 \) rows and \( A[E(N) - F] \) has at least \( 2(k + 1) \) zero rows). Now

\[
r(M^* \setminus F) \leq r(N \setminus F) + k \leq r(N) - 2k - 1 + k \leq r(M^*) - 1.
\]

Thus \( F \) contains a cocircuit of \( M^* \), which is a circuit of \( M \), giving \( g(M) \leq c \). \( \square \)

6. The Main Result

The following theorem implies Theorem 1.1.
Theorem 6.1. Let $\mathcal{F}$ be a finite field. If $\mathcal{M}$ is a minor-closed class of $\mathcal{F}$-represented matroids, then $\mathcal{M}$ is asymptotically good if and only if $\mathcal{M}$ contains all projective geometries over $\mathcal{F}_{\text{prime}}$.

Proof. If $\mathcal{M}$ contains all projective geometries over $\mathcal{F}_{\text{prime}}$, then $\mathcal{M}$ contains all $\mathcal{F}_{\text{prime}}$-represented matroids so is clearly asymptotically good. Suppose that $\mathcal{M}$ does not contain all projective geometries over $\mathcal{F}_{\text{prime}}$ but is asymptotically good. By Theorem 4.2, there is an integer $k$ so that every vertically $k$-connected matroid $M \in \mathcal{M}$ satisfies $\text{dist}(M, N) \leq k$ or $\text{dist}(M^*, N) \leq k$ for some $\mathcal{F}$-represented frame matroid $N$; moreover, $N$ is vertically 3-connected by Lemma 4.1, so by Corollary 2.4 we may assume the associated graph is connected. Let $\mathcal{M}_k$ denote the class of vertically $(2k+3)$-connected matroids in $\mathcal{M}$. Note that every such matroid is also vertically $k$-connected.

By Lemma 3.1, the class $\mathcal{M}_k$ is asymptotically good, so $\mathcal{M}_k$ contains an $(\alpha, \alpha)$-good sequence for some $\alpha \in (0, 1)$. Let $\beta = (1 - \alpha)^{-1} - 1$ and let $c$ be the maximum of the two integers given by Lemmas 5.4 and 5.5 for $\mathcal{F}$, $\beta$ and $k$. Let $n_0$ be an integer so that $c \log n < \alpha n$ for all $n \geq n_0$.

There is a matroid $M \in \mathcal{M}_k$ so that $|M| \geq n_0$, $d(M) = g(M^*) \geq \alpha |M|$ and $r(M) \geq \alpha |M|$. The last inequality gives $|M^*| \geq (1 + \beta)r(M^*)$, so by Lemma 5.4 or 5.5 we have $d(M) \leq c \log r(M^*) \leq c \log |M^*| < \alpha |M^*$, a contradiction.

□

Acknowledgements

We thank Navin Kashyap for his careful reading and very useful advice on the manuscript.

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