Birational geometry of algebraic varieties with a pencil of Fano complete intersections

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We prove birational superrigidity of generic Fano fiber spaces $V/\mathbb{P}^1$, the fibers of which are Fano complete intersections of index 1 and dimension $M$ in $\mathbb{P}^{M+k}$, provided that $M \geq 2k+1$. The proof combines the traditional quadratic techniques of the method of maximal singularities with the linear techniques based on the connectedness principle of Shokurov and Kollár. Certain related results are also considered.

Bibliography: 23 titles.

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Introduction

0.1. Fano complete intersections. Fix integers $k \geq 2$, $M \geq 2k+1$ and a $k$-uple of integers $(d_1, \ldots, d_k) \in \mathbb{Z}^k_+$, satisfying the conditions

$$d_k \geq \ldots \geq d_1 \geq 2 \quad \text{and} \quad d_1 + \ldots + d_k = M + k.$$ 

The symbol $\mathbb{P}$ stands for the complex projective space $\mathbb{P}^{M+k}$. Take homogeneous polynomials $f_i \in H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d_i)) \setminus \{0\}$, $i = 1, \ldots, k$, on $\mathbb{P}$. By the symbol

$$F(f_1, \ldots, f_k) = F(f_*)$$

we denote the closed algebraic set

$$\{f_1 = \ldots = f_k = 0\} \subset \mathbb{P}.$$ 

Let

$$\mathcal{F} \subset \prod_{i=1}^{k} \mathbb{P}(H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d_i)))$$

be the space of \textit{Fano complete intersections} of type $(d_1, \ldots, d_k)$, that is, $\mathcal{F}$ is the set of \textit{irreducible reduced} complete intersections of codimension $k$ in $\mathbb{P}$. Every variety $F \in \mathcal{F}$ is a Fano variety of index one and dimension $M$. The anticanonical degree of the variety $F$ is $d = d_1 \ldots d_k$. Let $\mathcal{F}_{\text{sm}} \subset \mathcal{F}$ be the space of \textit{smooth} Fano complete intersections.
Let $F = F(f_1, \ldots, f_k) \in \mathcal{F}$ be a Fano complete intersection, $x \in F$ a point. Let $\mathbb{C}^{M+k} \subset \mathbb{P}$ be a standard affine chart, $(z_1, \ldots, z_{M+k})$ a system of linear coordinates on $\mathbb{C}^{M+k}$ with the origin at the point $x$. For each $i = 1, \ldots, k$ we have the presentation

$$f_i = q_{i,1} + \ldots + q_{i,d_i},$$

where $q_{i,j}(z_*)$ is a homogeneous polynomial of degree $j$. Recall

**Definition 0.1** [1]. A smooth point $x \in F$ is regular, if the set of polynomials

$$\{q_{i,j} \mid 1 \leq i \leq k, 1 \leq j \leq d_i, (i, j) \neq (k, d_k)\},$$

(1)

consisting of all homogeneous polynomials $q_{i,j}$, except for the very last one $q_{k,d_k}$, makes a regular sequence in $O_{x, F}$, that is, the system of equations

$$\{q_{i,j} = 0 \mid (i, j) \neq (k, d_k)\}$$

(2)

defines a closed set of dimension one in $\mathbb{C}^{M+k}$, that is, a finite set of lines passing through the origin.

Projectivizing $\mathbb{C}^{M+k}$, one can formulate the regularity condition in the following way: the system (2) defines a zero-dimensional set in $\mathbb{P}^{M+k-1}$. If $x \in F$ is a regular point, then there are at most finitely many lines on $F$ passing through $x$. (The converse, generally speaking, is not true.)

Note that regularity of the sequence (1) implies smoothness of the point $x \in F$. In Definition 1 we intentionally assumed that the point is smooth, in order to be able to extend the regularity condition to singular points.

**Definition 0.2.** A singular point $x \in F$ is regular, if

(i) it is a non-degenerate quadratic singularity;

(ii) the system of equations (2) defines a closed set of dimension two in $\mathbb{C}^{M+k}$ (respectively, a curve in $\mathbb{P}^{M+k-1}$), the linear span of each irreducible component of which is

$$T = \{q_{1,1} = q_{2,1} = \ldots = q_{k,1} = 0\}.$$

If $x \in F$ is a singularity, then the linear forms $q_{i,1}, i = 1, \ldots, k$, are linear dependent. The regularity of the point $x \in F$ means that, deleting from the set (1) exactly one linear form, say $q_{1,e}$, we obtain a regular sequence, that is, the system of equations

$$\{q_{i,j} = 0 \mid (i, j) \notin \{(1, e), (k, d_k)\}\}$$

(3)

defines a two-dimensional set in $\mathbb{C}^{M+k}$ (respectively, a curve in $\mathbb{P}^{M+k-1}$). In particular, codim $T = k - 1$ and the tangent cone $T_x F \subset T$ is a non-degenerate quadric. Moreover, it follows from the regularity condition that, replacing in the set (1) the linear form $q_{1,e}$ by an arbitrary linear form $l(z_1, \ldots, z_{M+k-1})$, such that $l \mid T \neq 0$, we obtain a regular sequence, since neither component of the closed set (3) is contained in the hyperplane $l = 0$.

The following fact was proved in [1].

*Any smooth variety $F \in \mathcal{F}$, regular at every point $x \in F$, is birationally super-rigid. In particular, $F$ has no non-trivial structures of a rationally connected fiber
space (and, moreover, non-trivial structures of a fiber space into varieties of negative Kodaira dimension), $F$ is non-rational and the groups of birational and biregular automorphisms of the variety $F$ coincide: $\text{Bir } F = \text{Aut } F$.

Since the singular Fano complete intersections form a divisor $\mathcal{F}_{\text{sing}} = \mathcal{F} \setminus \mathcal{F}_{\text{sm}}$, for any Fano fiber space $\pi: V \to \mathbb{P}^1$, each fiber of which $F_t = \pi^{-1}(t)$, $t \in \mathbb{P}^1$, is a Fano complete intersection, $F_t \in \mathcal{F}$, there are singular fibers $F_t \in \mathcal{F}_{\text{sing}}$ (unless $V = F \times \mathbb{P}^1$ for some $F \in \mathcal{F}_{\text{sm}}$, but we do not consider these fiber spaces here).

Let $\mathcal{F}_{\text{reg}} \subset \mathcal{F}$ be the set of complete intersections, satisfying the regularity condition at every point (smooth or singular).

**Proposition 0.1.** The following estimate holds: $\text{codim}_F (\mathcal{F} \setminus \mathcal{F}_{\text{reg}}) \geq 2$.

**Proof.** The computations of [1] show that the complete intersections $F_t \in \mathcal{F}$, non-regular at at least one smooth point $x \in F$, form a subset of codimension $> 2$ in $\mathcal{F}$ (see [1, p. 76]). On the other hand, it is obvious that the varieties $F_t$ that have at least one non-regular singular point, form a proper closed subset in $\mathcal{F}_{\text{sing}}$.

Q.E.D. for the proposition.

0.2. **Fano fiber spaces.** Let $\pi: V \to \mathbb{P}^1$ be a Fano fiber space, the fibers of which are complete intersections of type $(d_1, \ldots, d_k)$ in $\mathbb{P}$, that is, $F_t = \pi^{-1}(t) \in \mathcal{F}$ for $t \in \mathbb{P}^1$. For the variety $V$ we assume the following:

- $V$ is smooth,
- $A^1 V = \text{Pic } V = \mathbb{Z}K_V \oplus \mathbb{Z}F$, $A^2 V = \mathbb{Z}K_V^2 \oplus \mathbb{Z}H_F$,

where $H_F = (-K_V \cdot F)$ is the anticanonical section of the fiber. By the symbols $A^1_1 V \subset A^1_2 V$ and $A^2_1 V \subset A^2_2 V$ we denote the closed cones of the pseudoeffective cycles of codimension one (that is, divisors) and two, respectively. Set also $A^1_{\text{mov}} V \subset A^1_2 V$ to be the closed cone generated by the classes of movable divisors in $A^1_2 V = A^1 V \otimes \mathbb{R}$.

Now let us formulate the main result of this paper.

**Theorem 1.** Assume that the Fano fiber space $V/\mathbb{P}^1$ satisfies the following conditions:

(i) (the regularity condition) $F_t \in \mathcal{F}_{\text{reg}}$ for every point $t \in \mathbb{P}^1$,

(ii) (the $K^2$-condition of depth 2) $K_V^2 + 2H_F \notin \text{Int } A^2_2 V$.

Then for any movable linear system $\Sigma \subset |-nK_V + lF|$ with $l \in \mathbb{Z}_+$ its virtual and actual thresholds of canonical adjunction coincide, $c_{\text{virt}}(\Sigma) = c(\Sigma) = n$. If, moreover, the fiber space $V/\mathbb{P}^1$ satisfies the condition

(iii) (K-condition) $-K_V \notin \text{Int } A^1_{\text{mov}} V$,

then the variety $V$ is birationally superrigid.

**Corollary 1.** Assume that the Fano fiber space $V/\mathbb{P}^1$ satisfies the conditions (i)-(iii) of the theorem above. Then the projection $\pi: V \to \mathbb{P}^1$ is the only non-trivial structure of a fibration into varieties of negative Kodaira dimension on $V$. The variety $V$ is non-rational, its groups of birational and biregular automorphisms coincide: $\text{Bir } V = \text{Aut } V$; for a generic $V$ this group is trivial.
Proof of the corollary. These claims follow from birational superrigidity in the standard way, see [2-5].

0.3. Explicit constructions. Let $a_* = \{0 = a_0 \leq a_1 \leq \ldots \leq a_{M+k}\}$ be a non-decreasing sequence of non-negative integers, $E = \bigoplus_{i=0}^{M+k} \mathcal{O}_{\mathbb{P}^1}(a_i)$ a locally free sheaf on $\mathbb{P}^1$, $X = \mathbb{P}(E)$ the corresponding projective bundle in the sense of Grothendieck. Obviously, we have

$$\text{Pic } X = \mathbb{Z} L_X \oplus \mathbb{Z} R, \quad K_X = -(M + k + 1)L_X + (a_X - 2)R,$$

where $L_X$ is the class of the tautological sheaf, $R$ is the class of a fiber of the morphism $\pi: X \to \mathbb{P}^1$, $a_X = a_1 + \ldots + a_{M+k}$. Furthermore, we get $L_X^{M+k+1} = a_X$.

For some $k$-uple $(b_1, \ldots, b_k) \in \mathbb{Z}_+^k$ let

$$G_i \in |d_i L_X + b_i R|$$

be irreducible divisors such that the complete intersection

$$V = G_1 \cap \ldots \cap G_k \subset X$$

is a smooth subvariety. The projection $\pi|_V: V \to \mathbb{P}^1$ is denoted by the same symbol $\pi$, the fiber $\pi^{-1}(t) \subset V$ by the symbol $F_t$, the restriction $L_X|_V$ by $L$. Obviously,

$$\text{Pic } V = \mathbb{Z} L \oplus \mathbb{Z} F, \quad K_V = -L + (a_X + b_X - 2)F,$$

where $b_X = b_1 + \ldots + b_k$. It is easy to check the formulae

$$(L^M \cdot F) = (H_F \cdot L^{M-1}) = d, \quad L^{M+1} = d(a_X + \sum_{i=1}^{k} \frac{b_i}{d_i}),$$

where $d = d_1 \ldots d_k$ is the degree of the fiber. From this we get:

$$(-K_V \cdot L^M) = d(2 - \sum_{i=1}^{k} \frac{d_i - 1}{d_i} b_i)$$

and

$$(K_V^2 \cdot L^{M-1}) = d(4 - a_X - \sum_{i=1}^{k} \frac{2d_i - 1}{d_i} b_i).$$

Since the linear system $|L|$ is free, these formulae immediately imply

**Proposition 0.2.** (i) If $a_X + \sum_{i=1}^{k} \frac{2d_i - 1}{d_i} b_i \geq 2$ then the $K^2$-condition of depth 2 holds: $K^2_V - 2H_F \notin \text{Int } A^2_+ V$.

(ii) If $\sum_{i=1}^{k} \frac{d_i - 1}{d_i} b_i \geq 2$, then $-K_V \notin \text{Int } A^1_+ V$ and the more so, $-K_V \notin \text{Int } A^1_{\text{mov}} V$.

(iii) If the inequality just above is strict, then $-K_V \notin A^1_+ V$. 

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1 Proof of birational superrigidity

In this section we prove Theorem 1. The proof consists of two parts: firstly, we formulate a sufficient condition of birational superrigidity (Theorem 2), secondly, we check this condition for varieties with a pencil of Fano complete intersections.

1.1. The method of maximal singularities. In this subsection we consider Fano fiber spaces $V/\mathbb{P}^1$, not assuming that the fibers $F_t$, $t \in \mathbb{P}^1$, are taken from some particular family of Fano varieties. We assume only that $V$ is a smooth variety, that the conditions $A^1 V = \text{Pic} V = \mathbb{Z} K_V \oplus \mathbb{Z} F$, $A^2 V = \mathbb{Z} K^2_V \oplus \mathbb{Z} H_F$ hold, where $H_F = (-K_V \cdot F)$ is the ample anticanonical section of the fiber, and that the fibers $F_t$, $t \in \mathbb{P}^1$, have at most isolated factorial singularities, and moreover $\text{Pic} F_t = A^1 F_t = \mathbb{Z} K_{F_t}$ and $A^2 F_t = \mathbb{Z} K^2_{F_t}$ for every $t \in \mathbb{P}^1$. The symbols $A^1 V$, $A^2 V$ and $\text{Alb} V$ mean the same as above. The general idea of the method of maximal singularities is to reduce the problem of birational rigidity of a Fano fiber space $V/\mathbb{P}^1$ to certain problems of numerical geometry of its fibers and to numerical characteristics of “twistedness” of the fiber space $V/\mathbb{P}^1$ over the base. In this subsection we formulate a sufficient condition of birational superrigidity, realizing one of the versions of such reduction. For another versions, see [3,4].

By the degree of an irreducible subvariety $Y \subset V$, contained in a fiber, $Y \subset F_t$ (such subvarieties are said to be vertical), we mean the integer

$$\deg Y = (Y \cdot (-K_V)^{\dim Y}).$$

By the degree of an irreducible subvariety $Y \subset V$, covering the base $\mathbb{P}^1$, $\pi(Y) = \mathbb{P}^1$ (such subvarieties are said to be horizontal), we mean the integer

$$\deg Y = (Y \cdot F \cdot (-K_V)^{\dim Y - 1}).$$

Definition 1.1 [4]. The fiber space $V/\mathbb{P}^1$ satisfies

- the condition $(v)$, if for any irreducible vertical subvariety $Y$ of codimension two (that is, $Y \subset F_t$ is a prime divisor, $t = \pi(Y)$) and any smooth point $o \in F_t$ the inequality

$$\frac{\text{mult}_o Y}{\deg Y} \leq \frac{2}{\deg V}$$

holds;
• the condition (f), if for any irreducible vertical subvariety \(Y\) of codimension three (that is, \(\text{codim}_F Y = 2, F = F_t \supset Y\)) and any smooth point of the fiber \(o \in F\) the following inequality holds:

\[
\frac{\text{mult}_o Y}{\deg Y} \leq \frac{4}{\deg V}.
\]

(4)

For convenience of notations the ratio of the multiplicity to the degree is written down in the sequel by one symbol

\[
\frac{\text{mult}_o Y}{\deg Y} = \frac{\text{mult}_o Y}{\deg V}.
\]

Let \(\Sigma \subset | - nK_V + lF|\) be a movable linear system, \(l \in \mathbb{Z}_+\). Recall [1-6]

**Definition 1.2.** An exceptional divisor \(E\) of a birational morphism \(\varphi: \tilde{V} \to V\), where \(\tilde{V}\) is a smooth projective variety (we can restrict ourselves by the morphisms \(\varphi\) of the type \(\varphi_N \circ \ldots \circ \varphi_1\), where \(\varphi_i\) is a blow up with an irreducible center), is a maximal singularity of the linear system \(\Sigma\), if the Noether-Fano inequality

\[
\nu_E(\Sigma) > na(E)
\]

(5)

holds, where \(\nu_E(\Sigma) = \text{ord}_E \varphi^* \Sigma\) is the multiplicity of the pull back of a general divisor of the system \(\Sigma\) along \(E\), \(a(E)\) is the discrepancy, \(n \in \mathbb{Z}_+\) was defined above.

For \(n \geq 1\) the inequality \(\nu_E(\Sigma) > na(E)\) means that the pair \((V, \frac{1}{n}\Sigma)\) is non-canonical and \(E \subset \tilde{V}\) is its non-canonical singularity. The irreducible subvariety

\[B = \varphi(E) = \text{centre } (E, V)\]

is called the center of the non-canonical (maximal) singularity \(E\).

**Theorem 2.** Assume that the Fano fiber space \(V/\mathbb{P}^1\) satisfies the generalized \(K^2\)-condition of depth 2, that is,

\[K^2_V + 2H_F \not\in \text{Int } A^2_+ V;\]

and the conditions \((v)\) and \((f)\), formulated above.

(i) If the center of every maximal singularity of a movable linear system \(\Sigma \subset | - nK_V + lF|\) with \(l \in \mathbb{Z}_+\) is not a singular point of a fiber, then the virtual and actual thresholds of canonical adjunction of the system \(\Sigma\) coincide: \(c_{\text{virt}}(\Sigma) = c(\Sigma)\).

(ii) If the assumption of (i) holds for any movable linear system on \(V\) and the variety \(V\) satisfies the \(K\)-condition, that is,

\[-K_V \not\in \text{Int } A^1_+ \text{mov} V,\]

then the variety \(V\) is birationally superrigid.

For the proof of the theorem, see [4]. Now we just note that the claim (ii) follows from (i) in an obvious way. Theorem 2 reduces proving birational superrigidity to checking the \(K^2\)-condition, \(K\)-condition, the conditions \((v)\) and \((f)\) and, finally, to excluding the maximal singularities, the center of which is a singular point of a fiber. Note that

\[\frac{\text{mult}_o Y}{\deg Y} = \frac{\text{mult}_o Y}{\deg V}.
\]
• the $K^2$- and $K$-conditions are checked in a routine way, usually it is an easy thing to do, and, in a sense, a “majority” of Fano fiber spaces satisfies these conditions, see Proposition 0.2 above;

• usually the conditions $(v)$ and $(f)$ are known, given that every fiber is birationally superrigid, since it is via checking the inequality of the condition $(f)$ that birational superrigidity of a Fano variety is usually being proved, whereas the condition $(v)$ follows from $(f)$ in an easy way (see below);

• it is excluding of a maximal singularity lying over a singular point of a fiber that has ever been the hardest part of the proof, its heart [3,7], however, employing the connectedness principle of Shokurov and Kollár [8,9] makes it possible to considerably simplify this part, in the way in which it is done below, even slightly relaxing the conditions of general position compared to [3,7].

1.2. Proof of Theorem 1. The condition $(f)$ was shown in [1]. Let us prove $(v)$. Let $Y \subset F = F_t$ be a prime divisor, $o \in Y$ a point. Take a general hyperplane $H \subset \mathbb{P}$, tangent to $F$ at the point $o$, that is, $H \supset T_o F$. Set $T = H \cap F$. By generality, $Y \neq T$, so that $Y_T = (Y \circ T)$ is a well defined effective cycle of codimension two on $F$, and moreover,

$$\frac{\text{mult}_o Y_T}{\deg Y_T} \geq 2 \frac{\text{mult}_o Y}{\deg Y}.$$ 

Now the condition $(f)$ implies $(v)$.

It remains to check that the center of a maximal singularity of the pair $(V, \frac{1}{n} \Sigma)$ cannot be a singular point of a fiber. Assume the converse: in the notations of Sec. 1.1 centre$(E, V) = o \in F$ is a singularity of the fiber. Let $\lambda: F^+ \to F$ be the blow up of the point $o$, $\lambda^{-1}(o) = E^+ \subset F^+$ the exceptional divisor. The blow up $\lambda$ can be looked at as the restriction of the blow up $\lambda_{\mathbb{P}}: \mathbb{P}^+ \to \mathbb{P}$ of the point $o$ on $\mathbb{P}$, so that $E^+ \subset E$ is a non-singular quadric of dimension $M - 1$, where $E = \lambda_{\mathbb{P}}^{-1}(o) \cong \mathbb{P}^{M+k-1}$ is the exceptional divisor.

**Proposition 1.1.** There exists a hyperplane section $B$ of the quadric $E^+ \subset E$, satisfying the inequality

$$\text{mult}_B(\lambda^*\Sigma_F) > 2n.$$ 

**Proof:** it follows from the connectedness principle of Shokurov and Kollár [8,9], for the details see [5 , Sec. 3].

Let $D \in \Sigma_F = \Sigma|_F$ be an effective divisor on $F$, $D \in |nH_F|$. By Proposition 1.1, the inequality

$$\text{mult}_o D + 2 \text{mult}_B D^+ > 4n$$

holds, where $D^+ \subset F^+$ is its strict transform on $F^+$. Let $H \subset \mathbb{P}$ be a general hyperplane, containing the point $o$ and cutting out $B$, that is,

$$H^+ \cap E^+ = (H^+ \cap E) \cap E^+ = B,$$
$H^+ \subset \mathbb{P}^+$ is the strict transform. Set $T = H \cap F$. The variety $T$ is a complete intersection of type $(d_1, \ldots, d_k)$ in $H = \mathbb{P}^{M+k-1}$ with an isolated quadratic singularity at the point $o$. The effective divisor $D_T = (D \circ T)$ on $T$ satisfies the inequality
$$\text{mult}_o D_T > 4n.$$ (6)

Obviously, $D_T \in |nH_T|$, where $H_T$ is the hyperplane section of $T \subset \mathbb{P}^{M+k-1}$. By linearity, one may assume the divisor $D_T$ to be prime, that is, an irreducible subvariety of codimension one.

Now, repeating the arguments of [1, Sec. 2] word for word, we obtain a contradiction.

It is possible to repeat the arguments word for word due to the stronger regularity condition at the point $o \in F$: the hyperplane section $T \subset \mathbb{P}^{M+k-1}$ satisfies the ordinary regularity condition at this point.

This scheme of arguments was suggested and first used in [10, Sec. 3] for the pencils of Fano hypersurfaces.

Q.E.D. for Theorem 1.

**Remark 1.1.** Starting with the pioneer paper [6] and up to [4], the technique of excluding infinitely near maximal singularities almost always was quadratic, that is, making use of the operation of taking the self-intersection of the movable linear system $\Sigma$. As the proof above shows, for certain types of maximal singularities the linear technique, based on the connectedness principle of Shokurov and Kollár (which, in its turn, is based on the Kawamata-Viehweg vanishing theorem [11-13]), is more effective. Combining the quadratic and linear methods makes it possible to simplify the proof and in some cases relax the conditions of general position (compare [3,4,14]).

## 2 Related results

### 2.1. Divisorially canonical complete intersections

**Definition 2.1** [5]. We say that a primitive Fano variety $F$ is **divisorially canonical**, or satisfies the condition $(C)$ (respectively, is **divisorially log canonical**, or satisfies the condition $(L)$), if for any effective divisor $D \in |-nK_F|$, $n \geq 1$, the pair

$$(F, \frac{1}{n}D)$$ (7)

has canonical (respectively, log canonical) singularities. If the pair (7) has canonical singularities for a general divisor $D \in \Sigma \subset |-nK_F|$ of any movable linear system $\Sigma$, then we say that $F$ satisfies the condition of movable canonicity, or the condition $(M)$.

The following fact was proved in [5].

Assume that primitive Fano varieties $F_1, \ldots, F_K$, $K \geq 2$, satisfy the conditions $(L)$ and $(M)$. Then their direct product

$$V = F_1 \times \ldots \times F_K$$

is divisorially canonical.
is birationally superrigid.

To prove the condition \((C)\), one needs much stronger regularity conditions than to prove the condition \((M)\), which already implies birational superrigidity. In the notations of Sec. 0.1 let us give

**Definition 2.2.** A smooth point \(x \in F\) satisfies the **stronger regularity condition** \((R^+)\), if for every linear form \(l(z_i)\), that does not vanish identically on the tangent space

\[
T_xF = \{q_{1,1} = \ldots = q_{k,1} = 0\},
\]

the following set of polynomials:

\[\{l\} \cup \{q_{i,j}|1 \leq i \leq k, 1 \leq j \leq d_i, (i,j) \notin \{(k,d_k),(k-1,d_{k-1})\}\}\]

provided that \(d_{k-1} = d_k\), and

\[\{l\} \cup \{q_{i,j}|1 \leq i \leq k, 1 \leq j \leq d_i, (i,j) \notin \{(k,d_k),(k,d_k-1)\}\}\]

provided that \(d_{k-1} \leq d_k - 1\), makes a regular sequence in \(O_{x,F}\).

**Proposition 2.1.** When \(k \geq 2\), \(M \geq 4k + 1\), there exists a non-empty Zariski open subset \(F_{\text{reg}}^+ \subset F_{\text{reg}}\) of smooth Fano complete intersections, satisfying the condition \((R^+)\) at every point.

**Proof** is obtained by a routine dimension count by the methods of [1,15]. The scheme of arguments is as follows. Without loss of generality assume that \(q_{i,1} \equiv z_{M+i}\). It is necessary to estimate the codimension of the set of non-regular sequences

\[\{l\} \cup \{\tilde{q}_{i,j}|1 \leq i \leq k, 2 \leq j \leq d_i, (i,j) \notin \{(k,d_k),(k-1,d_{k-1})\}\}\]

(respectively,

\[\{l\} \cup \{\tilde{q}_{i,j}|1 \leq i \leq k, 2 \leq j \leq d_i, (i,j) \notin \{(k,d_k),(k,d_k-1)\}\}\]

for the case \(d_{k-1} \leq d_k - 1\), where \(\tilde{l}, \tilde{q}_{ij}\) are homogeneous polynomials in the variables \(z_1, \ldots, z_M\). We need this codimension to be at least \(2M + 1\). For \(M \geq 4k + 1\) a straightforward combination of the methods of [1] and [15] gives the required inequality which proves the proposition.

**Remark 2.1.** More refined computations make it possible to prove that the set \(F_{\text{reg}}^+\) is non-empty (by the same methods) under somewhat weaker assumptions for \(M,k\).

**Theorem 3.** The complete intersection \(F \in F_{\text{reg}}^+\) satisfies the condition \((C)\) for \(M \geq 4k + 1, d_k \geq 8\).

**Proof.** Assume the converse: for some divisor \(D \in |nH|\) the pair \((F, \frac{1}{m}D)\) is not canonical, that is, it has a maximal singularity \(E \subset \tilde{V}\),

\[
\nu_E(D) > na(E),
\]
where $\varphi: \tilde{V} \to V$ is a sequence of blow ups, $E$ is an exceptional divisor. We may assume $D$ to be irreducible. Since for any irreducible subvariety $B \subset F$ of dimension $\geq k$ we have $\text{mult}\, B D \leq n$ (see below Sec. 2.3), we obtain the inequality

$$\text{mult}_x D + \text{mult}_Y D^+ > 2n$$

for some point $x \in F$ and a hyperplane $Y \subset E^+$ in the exceptional divisor $E^+ \subset F^+$ of the blow up of the point $x$, $\lambda: F^+ \to F$, $E^+ = \lambda^{-1}(x)$, see [5,10] for the details. Let $T \subset F$ be a general hyperplane section, containing the point $x$ and satisfying the condition $T^+ \cap E^+ = Y$, where $T^+ \subset F^+$ is the strict transform of the divisor $T$. By generality, we get $\text{Supp} D \not\subset T$, so that $D_T = (D \circ T)$ is an effective divisor on $T$, $D_T \in |nH_T|$, satisfying the inequality $\text{mult}_x D_T > 2n$.

Let us show that this is impossible. In order to do that, we apply to the effective divisor $D_T$ on the complete intersection $T$ the technique of hypertangent divisors in exactly the same way as was done for a subvariety of codimension two in [1, Sec. 2]. As a result, we obtain the estimate

$$\frac{\text{mult}_x D_T}{\deg F} \leq \frac{2}{\deg F} \cdot \max \{ 1, \frac{3}{4} \cdot \frac{d_k}{d_k - 1} \cdot \frac{d_k}{d_k - 1} \} ,$$

where $d_+ = d_k$, if $d_{k-1} = d_k$, and $d_+ = d_k - 1$, otherwise. If $M \geq 4k + 1$ and $d_k \geq 8$, this implies the inequality $\text{mult}_x D_T \leq 2n$, which is what we need.

Q.E.D. for the theorem.

2.2. Structures of relative Kodaira dimension zero. By the relative Kodaira dimension of the fiber space $\beta: W \to S$ we mean the Kodaira dimension of a fiber of general position $\beta^{-1}(s)$, $s \in S$. Notation: $\kappa(W/S)$.

**Proposition 2.2.** Let $F$ be an arbitrary primitive Fano variety, $\chi: F \dasharrow W$ a structure of a fiber space, that is, a birational map. Assume that the inequality

$$\dim S + \kappa(W/S) < \dim W$$

holds (that is, the fiber of the fiber space $W/S$ is not a variety of general type). Then for any movable linear system $\Sigma_S$ on $S$ the pair $(F, \frac{1}{n}(\chi^{-1})^*\beta^*\Sigma_S)$ is not terminal, where $\Sigma = (\chi^{-1})^*\beta^*\Sigma_S \subset |-nK_F|$ is the strict transform of the system $\beta^*\Sigma_S$ on $F$.

**Proof** is almost word for word the same as the proof of existence of a maximal singularity in the case of Kodaira dimension $-\infty$. Assume the converse: the pair $(F, \frac{1}{n}\Sigma)$ is terminal. Let $\varphi: \tilde{F} \to F$ be a resolution of singularities of the map $\chi$, $\psi = \chi \circ \varphi$ the composite map, a birational morphism. By the assumption, for each exceptional divisor $E \subset \tilde{F}$ of the morphism $\varphi$ we have $\nu_E(\Sigma) < na(E)$. Let $D \in \Sigma$ be a general divisor. Since there are finitely many exceptional divisors, for some $n_+ \in \mathbb{Q}_+$, $n_+ < n$, we get in $A^1_{\tilde{F}}$:

$$\tilde{D} + n_+ \tilde{K} = (n - n_+) \varphi^*(-K_F) + D^\sharp,$$

where $\tilde{D}$ is the strict transform of $D$, $\tilde{K}$ is the canonical class of $\tilde{F}$, $D^\sharp$ is an effective divisor. Therefore, for $N \gg 0$ the linear system

$$\Sigma_+ = |N!(\tilde{D} + n_+ \tilde{K})|$$
defines a birational map $\gamma: \tilde{F} \dashrightarrow F^+ \subset \mathbb{P}^{\dim \Sigma_+}$. Let $W(s) = \beta^{-1}(s)$ be a fiber of general position, $Y(s) \subset \tilde{F}$ its strict transform on $\tilde{F}$. Obviously, the linear system $\Sigma_{+} | Y(s)$ also defines a birational map, so that $\psi_{*}\Sigma_{+} | W(s)$ defines a birational map, either. However,

$$\psi_{*}\Sigma_{+} | W(s) = |N! n_{+} K_{W(s)}|$$

is a subsystem of the pluricanonical system of the fiber $W(s)$, whereas by assumption $\kappa(W(s)) < \dim W(s)$. A contradiction. Q.E.D. for the proposition.

**Proposition 2.3.** Let $\pi: V \to \mathbb{P}^1$ be a fibration into primitive Fano varieties, as described in Sec. 0.2. Assume in addition that

$$-K_{V} \notin A_{\text{mov}} V.$$

Then for any structure of a fiber space $\chi: V \dashrightarrow W$ with $\kappa(W/S) < \dim(W/S) = \dim W - \dim S$ the pair $(V, \frac{1}{n}(\chi^{-1})_{*}\beta_{*}\Sigma_{S})$ is not terminal, where $\Sigma = (\chi^{-1})_{*}\beta_{*}\Sigma_{S} \subset |-nK_{V} + \ell F|$. When $\kappa(W/S) = 0$, Proposition 2.3 can be refined. Let $\varphi: \tilde{V} \to V$ be a resolution of singularities of $\chi$, $\psi = \chi \circ \varphi$ the composite map, a birational morphism. If the pair $(V, \frac{1}{n}(\chi^{-1})_{*}\beta_{*}\Sigma_{S})$ is non-canonical, let $\mathcal{M} = \{E_{1}, \ldots, E_{k}\}$ be the set of all maximal singularities of the system $\Sigma$, that is, the exceptional divisors $E \subset \tilde{V}$, satisfying the strict (that is, the usual) Noether-Fano inequality $\nu_{E}(\Sigma) > na(E)$. If the pair $(V, \frac{1}{n}(\chi^{-1})_{*}\beta_{*}\Sigma_{S})$ is canonical, set $\mathcal{M} = \emptyset$. Now set

$$\mathcal{M}_{t} = \{E \in \mathcal{M} | \text{centre}(E) = \varphi(E) \subset F_{t}\}$$

to be the set of maximal singularities, the centers of which are contained in the fiber $F_{t}$ over some point $t \in \mathbb{P}^1$. Set also

$$\mathcal{M}^{h} = \mathcal{M} \setminus \left( \bigcup_{t \in \mathbb{P}^1} \mathcal{M}_{t} \right)$$

to be the set of horizontal maximal singularities, the centers of which cover the base $\mathbb{P}^1$.

**Proposition 2.4.** In the notations above assume that $\kappa(W/S) = 0$ and the structure $\chi$ is not fiber-wise with respect to $\pi$, that is, the strict transform $(\chi^{-1})_{*}\beta_{*}(s)$ of a general fiber of the fibration $\beta: W \to S$ on $V$ covers the base $\mathbb{P}^1$. Then $\mathcal{M} \neq \emptyset$. If, in addition, $\mathcal{M}^{h} = \emptyset$, then the following inequality holds:

$$\max_{t \in \mathbb{P}^1} \max_{\{E \in \mathcal{M}_{t}\}} \frac{\nu_{E}(\Sigma) - na(E)}{\nu_{E}(F_{t})} \geq l \quad (8)$$
Proof of the fact that $\mathcal{M} \neq \emptyset$ is word for word the same as the proof of Proposition 2.3: assume the converse, take $n_+ = n$ and use the fact that we obtain a linear system with a non-empty movable part, since the system $|F|$ is movable. Taking into account that $\chi$ is not fiber-wise with respect to $\pi$, this contradicts the condition $\kappa(W/S) = 0$ and proves the existence of a maximal singularity. Proof of the second claim is word for word the same as the proof of Proposition 1.3 in [4], with the only difference: by what has been just said, the linear system

$$|lF - \Sigma_{E \in \mathcal{M}}(\nu_E(\Sigma) - na(E))|$$

cannot have a non-empty movable part. In [4] the case under consideration was $\kappa(W/S) = -\infty$, so the system (9) had to be empty. For this reason the inequality (5) turns out to be non-strict. Q.E.D. for the proposition.

Remark 2.2. As one can see from the arguments of this section, description of the structures of a fibration of relative Kodaira dimension zero (in fact, of any non-maximal Kodaira dimension for primitive Fano varieties) is completely similar to the case of negative Kodaira dimension. If it is possible to study the structures of negative Kodaira dimension for a certain class of Fano varieties or Fano fiber spaces, then the very same arguments (with minimal modifications) work successfully for the structures with $\kappa = 0$, either. This observation belongs to Cheltsov: [16,17,18,19] reproduced the arguments of [15,20 and 21, 22 and 23, 2], respectively, which gave a description of $K$-trivial structures on the corresponding varieties.

2.3. The structures of zero Kodaira dimension on varieties with a pencil of complete intersections. In [1] a proof of the following fact was sketched.

Proposition 2.5. Let $F \in \mathcal{F}_{\text{reg}}$ be a smooth regular Fano complete intersection. Then any structure of a fiber space of relative Kodaira dimension zero $\chi: F \to W$, $\kappa(W/S) = 0$, is a pencil: $\dim S = 1$.

Recall the main steps of the proof. Let $W/S$ and $\chi$ be as above. Then the pair $(F, (\chi^{-1})_*\beta^*\Sigma_S \subset | - nK_F|)$, is non-terminal (Proposition 2.2). However, it is canonical [1]. In the notations of the proof of Proposition 2.2, let $E \subset \tilde{F}$ be a non-terminal singularity of the system $\Sigma$, that is, $\nu_E(\Sigma) = na(E)$, $B = \varphi(E) \subset F$ its center. If $\text{codim} B \geq 4$ or $\text{codim} B = 3$, but the inequality $\text{mult}_B \Sigma < 2n$ holds, then the technique of counting multiplicities [15] immediately gives the inequality

$$\text{mult}_B Z > 4n^2;$$

where $Z = (D_1 \circ D_2)$ is the self-intersection of the linear system $\Sigma$, $D_i \in \Sigma$ are general divisors. However, it was proved in [1, Sec. 2] that (10) is impossible. Therefore, either $\text{codim} B = 3$ and $\text{mult}_B \Sigma = 2n$, or $\text{codim} B = 2$ and $\text{mult}_B \Sigma = n$.

In any case the blow up of the subvariety $B$ realizes a non-terminal singularity of the system $\Sigma$. If $\text{codim} B = 2$, then $Z = n^2 B$, that is, the self-intersection of the system $\Sigma$ has no movable part. Thus the system $\Sigma$ is composed from a pencil, $\dim S = 1$, as we claimed.

To complete the proof, it remains to exclude the first case when $\text{codim} B = 3$. It can be done, using the technique of [1, Sec. 2]. For instance, if $d_k = \max\{d_i\} \geq 5$,
then the equality $\text{mult}_B Z = 4n^2$ implies that for every point $x \in B$ each component of the effective cycle $Z$ is of the form $T_1 \cap T_2$, where $T_1 \neq T_2$ are sections of $F$ by hyperplanes, tangent to $F$ at $x$. This is, of course, impossible (a section of $F$ by any plane $P \subset \mathbb{P}$ of codimension two has at most a curve of singular points). Cheltsov noted that it is easier to exclude the case $\text{codim} B = 3$, using the cone technique [15]. Namely, the following claim holds.

**Lemma 2.1.** For any irreducible subvariety $Y \subset F$ of dimension $k$ the inequality $\text{mult}_Y \Sigma \leq n$ holds.

**Proof.** Let $x \in \mathbb{P} \setminus F$ be a point of general position, $C(x) \subset \mathbb{P}$ the cone with the vertex at $x$ and the base $Y$. It is easy to see that $C(x) \cap F = Y \cup R(x)$, where $R(x) \subset F$ is the residual curve. For a sufficiently general point $x$ the curve $R(x)$ is irreducible. The family of residual curves $R(x)$ sweeps out $F$. At the points of intersection $y \in R(x) \cap Y$ the line $L_{x,y}$, connecting the points $x$ and $y$, is tangent to $F$. Thus for a general point $x \in \mathbb{P}$

$$R(x) \cap \text{Sing} Y = \emptyset$$

(by a trivial dimension count). Furthermore (see [15]),

$$(R(x) \cdot Y)_{C(x)} = \deg R(x).$$

Now if $\text{mult}_Y \Sigma > n$, we immediately get a contradiction in exactly the same way as in [15] (computing the intersection index

$$\Sigma_{y \in R(x) \cap Y} (R(x) \cdot D)_{T_F(y)}$$

for a general divisor $D \in \Sigma$). Q.E.D. for the lemma.

Proof of Proposition 2.5 is complete.

The arguments above extend immediately to the relative case.

**Theorem 4.** Assume that the Fano fiber space $V/\mathbb{P}^1$ satisfies the following conditions:

(i) $F_t \in \mathcal{F}_{\text{reg}}$ for any point $t \in \mathbb{P}^1$,
(ii) $K_V^2 + 2H_F \notin \text{Int} A^2_+ V$,
(iii) $-K_V \notin A^1_{\text{mov}} V$.

Then for every structure $\chi: V \dashrightarrow W$ of a fiber space of relative Kodaira dimension zero, $\kappa(W/S) = 0$, we get $\dim S = 2$, and moreover, the structure $\chi$ is compatible with $\pi$, that is, there is a rational dominant map $\lambda: S \dashrightarrow \mathbb{P}^1$ such that $\lambda \circ \beta \circ \chi = \pi$.

**Proof.** If the structure $\chi$ is not compatible with $\pi$, we apply Proposition 2.4 and obtain a contradiction word for word as in the proof of Theorem 1. Therefore, the structure $\chi$ is fiber-wise. Thus the problem is reduced to describing the structures of relative Kodaira dimension zero on a fiber of general position, that is, to Proposition 2.5?.

Q.E.D. for the theorem.
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