Too much information: CDCL solvers need to forget and perform restarts

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Abstract

Conflict-driven clause learning (CDCL) is a remarkably successful paradigm for solving the satisfiability problem of propositional logic. Instead of a simple depth-first backtracking approach, this kind of solver learns the reason behind occurring conflicts in the form of additional clauses. However, despite the enormous success of CDCL solvers, there is still only a shallow understanding of what influences the performance of these solvers in what way.

This paper will demonstrate, quite surprisingly, that clause learning (without being able to get rid of some clauses) can not only improve the runtime but can oftentimes deteriorate it dramatically. By conducting extensive empirical analysis, we find that the runtime distributions of CDCL solvers are multimodal. This multimodality can be seen as a reason for the deterioration phenomenon described above. Simultaneously, it also gives an indication of why clause learning in combination with clause deletion and restarts is virtually the de facto standard of SAT solving in spite of this phenomenon. As a final contribution, we will show that Weibull mixture distributions can accurately describe the multimodal distributions. Thus, adding new clauses to a base instance has an inherent effect of making runtimes long-tailed. This insight provides a theoretical explanation as to why the techniques of restarts and clause deletion are useful in CDCL solvers.

1 Introduction

Since their inception in the mid-90s \cite{1,2}, CDCL solvers have proven enormously successful in solving the SAT problem. As a case in point, we refer to the annual SAT Competition\footnote{The goal of the annual SAT Competition is to promote further improvements in the field of SAT solving by hosting a competitive event where researchers can present their newest work. All submitted solvers are put up against each other to solve a pool of instances. The fastest solver wins. We refer to \url{http://www.satcompetition.org} for more information.}. CDCL solvers won several of the last competitions. In many combinatorial fields, applied problems are nowadays even solved by reducing the problem to a SAT instance and invoking a CDCL solver (see e. g. \cite{3}), despite the NP-completeness of the SAT problem\footnote{SAT solvers have also been used in the last few years to generate computerized proofs of long-standing open problems. We refer to \cite{4} for a solution of the Boolean Pythagorean Triples problem, \cite{5} for Schur number five, or \cite{6} for the resolution of Keller’s Conjecture with SAT solvers.}

The abbreviation CDCL stands for conflict-driven clause learning. The eponymous component of CDCL is clause learning, a mechanism that can enhance the simple exhaustive exploration of the search tree for possible satisfying assignments by learning...
from mistakes made and avoiding these in the future. During its execution, the solver will learn additional clauses containing this learned information (we refer to Section 2.1 for an introduction to CDCL solvers). This enables the solver to prune the search tree and avoid re-exploring similar parts. Although clause learning is heavily employed in practice, the theoretical underpinnings of the technique are not well understood.

In addition to a clause learning scheme, modern CDCL solvers also employ a technique to delete some of the new clauses from time to time when they were deemed not helpful by the solver. However, it is still largely a mystery whether this deletion process is only used to keep computation times low by having a manageable clause data set or if there is some theoretical benefit to deleting clauses.

1.1 Our contribution
To study the effect of learned clauses on CDCL solvers, we let Glucose, a leading CDCL solver, first learn the set \( L \) of all conflict clauses it encounters until a solution of a given instance \( \mathcal{I} \) is found. In a second step, we generate a multitude of different sets \( L \), where each \( L \) is a randomly sampled subset of \( L \). Finally, we call Glucose on the extended instance \( \mathcal{I} \cup L \).

Since CDCL is nowadays the leading paradigm of successful SAT solvers, one is tempted to conjecture that clause learning is always useful. However, using the described modification process, we will demonstrate in this paper that one has to be careful about this assumption. More specifically, we will show that there are a surprising number of instances where the mean runtime of the extended instances is dramatically worse than the runtime on the original instance. This also holds when using the theoretical measure of the number of conflicts that occurred towards a solution. This performance decrease is often so substantial that it cannot be explained by pure chance. This motivates the study of the runtime distribution of extended instances to shed light on the question of what influence learned clauses have on CDCL solvers.

Focusing on the runtime distribution, we obtain as our next result that the runtime distribution of Glucose is multimodal. This contrasts the recently obtained result that the runtime distribution of stochastic local search (SLS) SAT solvers can be described with one distribution (namely, a lognormal distribution) [7].

We will continue our study to determine what kind of distribution type can be used to describe this multimodal data. By conducting various statistical analyses, we will demonstrate that the runtimes of Glucose are mixed Weibull distributed. For a specific parameter range, these distributions possess the long-tailed property, which can lead to exceedingly long runtimes. This leads to a theoretical understanding of the usefulness of restart and clause deletion techniques in CDCL solvers.

1.2 Related work
This section will give a brief overview of related works in both the study of runtime distributions and the research on CDCL solvers.

**Studying runtime distributions of algorithms.** In previous works, the runtime distributions of algorithms were studied. In [8], the authors presented empirical evidence for the fact that the distribution of the effort (more precisely, the number of consistency checks) required for backtracking algorithms to solve constraint satisfaction problems randomly generated at the 50% satisfiable point can be approximated by the Weibull distribution (in the satisfiable case) and the lognormal distribution (in the unsatisfiable case). Later, these results were extended to a wider region around the 50% satisfiable
In \cite{9}, the cost profiles of combinatorial search procedures were studied. The authors showed that Pareto-Lévy type heavy tails often characterize the distributions and empirically demonstrated how rapid randomized restarts can effectively eliminate heavy-tail behavior.

In the paper \cite{7}, the hardness distribution of several SLS SAT solvers on logically equivalent modifications of a base instance was studied. The second and third author included different instance models to rule out any influence of the model. Introducing the procedure \texttt{Alfa} that we will adapt to CDCL in our work, the paper found that lognormal distributions characterize this hardness distribution perfectly. The approach of \cite{7} lends itself to the analysis of existing SLS solvers, like \texttt{GapSAT} \cite{11}. The advantage of the approach studied in \cite{7} is that the conducted work is not lost in the case of a restart: only the logically equivalent instance could be changed while keeping the current assignment. The paper \cite{12} studied the solvers \texttt{Sparrow} and \texttt{CCASAT} and found that for randomly generated instances, the lognormal distribution is a good fit for the runtime distributions. This study was performed on the domains of randomly generated and crafted instances.

Barrero et al. \cite{13} observed empirical evidence suggesting lognormally distributed runtimes in several types of population-based algorithms like evolutionary and genetic algorithms.

Studies on CDCL solvers. The reason behind the fact that CDCL algorithms also incorporate a mechanism to delete (subsets of the) learned clauses from time to time was explained by Mitchell in \cite{14}. Even when sufficient memory is available, the time required to perform unit propagation becomes impractical for extensive clause sets, thus reducing the solver’s performance. Audemard and Simons \cite{15} observed that despite this phenomenon, deleting too many learned clauses can break the learning benefit. Thus, many CDCL solvers let the maximum number of learned clauses grow exponentially. The paper \cite{15} lead to the development of the \texttt{Glucose} solver using “aggressive clause deletion” together with the “Literals Block Distance (LBD)” measure.

1.3 Organization of this paper

The rest of this paper is organized as follows. In Section 2 we will introduce the notations of the field of SAT solving that we are going to use, give a short overview of the technique of conflict-driven clause learning, and give some statistical background, especially of survival analysis. We proceed to describe the experimental setup in Section 3. Finally, Section 4 will investigate whether clause learning is useful on average. Section 5 will demonstrate that the runtime distributions exhibit a multimodal behavior. This investigation will be continued in Section 6 where it is shown that the Weibull mixture distribution is a suitable fit for the runtime distributions of CDCL solvers.

2 Preliminaries

A literal $\ell$ over a Boolean variable $x$ is either $x$ itself or its negation $\overline{x} := \neg x$. A clause $C = (\ell_1 \lor \cdots \lor \ell_k)$ is a (possibly empty) disjunction of literals $\ell_i$. A CNF formula $\mathcal{F} = C_1 \land \cdots \land C_m$ is a conjunction of clauses. An assignment $\alpha$ for a CNF formula $\mathcal{F}$ is a function that maps some subset of the variables occurring in $\mathcal{F}$ to $\{0, 1\}$. By naturally extending $\alpha$ by the definition $\alpha(\overline{x}) := \overline{\alpha(x)}$, we can define the result of applying $\alpha$ to $C$, which we denote by $Ca$: one deletes all occurrences of literals $\ell$ from $C$, where $\alpha(\ell) = 0$; if there is a literal $\ell \in C$ with $\alpha(\ell) = 1$, then $Ca = 1$. The notation $F\alpha$ denotes the formula where all clauses containing a literal $\ell$ with $\alpha(\ell) = 1$ are deleted, and each remaining clause $C$ is replaced by $Ca$. A clause $C$ is called a logical consequence of a
formula \( \mathcal{F} \) if, for all assignments \( \alpha \) with \( \mathcal{F}_\alpha = 1 \), it also holds \( C_\alpha = 1 \). A set \( L \) of clauses is a logical consequence of \( \mathcal{F} \) if each clause \( C \in L \) is a logical consequence of \( \mathcal{F} \). We then call the formulas \( \mathcal{F} \) and \( \mathcal{F} \cup L \) logically equivalent.

### 2.1 Conflict-Driven Clause Learning

Conflict-driven clause learning SAT algorithms, or CDCL for short, are one of the most remarkable success stories in computer science. Introduced in the works [1] and [2], CDCL can yield dramatic speedups over the simple recursive depth-first backtracking approach DPLL [16, 17], that after selecting a variable \( x \) of the formula \( \mathcal{F} \) it is trying to solve, branches with calls to DPLL(\( \{x = 0\} \)) and DPLL(\( \{x = 1\} \)). While CDCL has been intensely studied, the considerable performance improvement over DPLL is still largely a mystery.

In the following, we will give a simple introduction to one of the most fundamental CDCL techniques: clause learning. Informally speaking, this can be seen as a modification of DPLL, where the algorithm adds some clauses to \( \mathcal{F} \) if it reaches a conflict, i.e., when the partial assignment constructed thus far falsifies a clause in \( \mathcal{F} \). The idea behind this is to prune the search tree and avoid having to re-explore some literal assignments that will not lead to a solution.

We will introduce clause learning mostly by example, following the exposition in [18], and refer the reader to [3, 19] for more details. As an example, consider as solver input the formula given in conjunctive normal form

\[
\begin{align*}
(\bar{x}_1 \lor x_2) \land (\bar{x}_3 \lor x_3 \lor x_4) \land (\bar{x}_3 \lor x_5 \lor x_6) \land \\
(\bar{x}_7 \lor x_8) \land (\bar{x}_8 \lor x_9) \land (x_9 \lor \bar{x}_{10}) \land (x_3 \lor x_8 \lor x_{10}).
\end{align*}
\]  

Let us suppose that the CDCL solver makes its first decision to assign \( x_1 = 1 \). The solver will always look out for clauses that only have one unassigned literal and assign this remaining literal so that the clause is satisfied. This process is called unit propagation. This process will be repeated until saturation. In our example, using unit propagation, the solver sets \( x_2 = 1 \) due to the clause \( (\bar{x}_1 \lor x_2) [x_1=1] \). It then sets \( x_5 = 0 \) because of the clause \( (\bar{x}_3 \lor x_5) [x_1=1, x_2=1] \). No further assignments can be made by unit propagation. To move things further along, the solver has to make another decision. In our example, the solver will now decide to set \( x_3 = 0 \). By unit propagation, \( x_4 = 1 \) and \( x_6 = 1 \) are assigned. Suppose, in its third decision, the solver sets \( x_7 = 1 \). Using unit propagation, the assignments \( x_8 = 1, x_9 = 0, x_{10} = 1, \) and \( x_{10} = 0 \) are made. This is a conflict since the variable \( x_{10} \) cannot be set to both 0 and 1.

During conflict analysis, the solver learns a new clause. For this process, the solver uses the implication graph that was built in stages during the execution of the algorithm (see Fig. [1]). In this graph, decision variables (in our example, \( x_1, x_3, \) and \( x_7 \)) are the source vertices. The conflict literals in our example are \( x_{10} \) and \( \bar{x}_{10} \). Furthermore, the graph includes vertices for every literal that has been assigned the value 1. A directed edge from node \( u \) to node \( v \) is included if the value of \( v \) was set by unit propagation and \( \bar{x} \) occurs in the clause that was the reason for variable \( v \) being set.

The level of an assigned variable \( x \) is defined as the number of decision variables that have been assigned before \( x \) (in our example, \( x_1, x_2, \bar{x}_7 \) have level 1; \( x_3, x_4, x_6 \) are level 2; and \( x_7, x_8, \bar{x}_7, x_{10}, \bar{x}_{10} \) are at the conflict level 3). A vertex \( v \) is called UIP (unique implication point), if all paths from the conflict level decision literal \( x_7 \) to the conflict literals run through \( v \). Here, \( x_7 \) and \( x_8 \) are UIPs. The UIP closest to the conflict literals is called first UIP (1UIP) [20]. In our example, \( x_8 \) is the 1UIP. The most popular method to learn clauses, invoked by most modern CDCL solvers, is based on the 1UIP
Fig 1. Conflict graph. The figure shows the conflict graph generated in our example run of CDCL when solving formula (1). The decision literals are marked in gray. The conflict literals are marked in red. The reason(s) for a propagation is given as label(s) of the edge(s). The first unique implication point (1UIP) is shown in blue. The 1UIP cut is the dashed blue line. The graphic is adapted from [18].

learning scheme[4] In this scheme, the implication graph will be cut such that

1. the 1UIP and all literals assigned before the conflict level are on one side,
2. while all literals assigned after the 1UIP are on the other side.

This cut, shown in Fig [1], yields the literals $x_3$ and $x_8$ as starting points of the separated edges. The clause $\neg(x_3 \land x_8) = (x_3 \lor \neg x_8) =: C$ can be shown to be a logical consequence of the original formula. This clause is called the learned clause. The solver adds this clause to the clause set.

Using so-called non-chronological backtracking, the solver would jump back to the level of the last variable in $C$ being assigned before the variable $x_8$ of the conflict level, i.e., it jumps back to level 2, where $\neg x_3$ was assigned. Then, using unit propagation on the clause $C$, the variable $x_8$ would be assigned 0.

CDCL algorithms also incorporate a mechanism to delete (subsets of the) learned clauses from time to time. As Mitchell explains in [14], this is due to the fact that even when sufficient memory is available, the time required to perform unit propagation becomes impractical for very large clause sets, thus reducing the performance of the solver.

2.2 Statistical background

In this section, we will briefly introduce the statistical tools used in this paper.

Definition 2.1 ([22]). Let $X$ be a real-valued random variable.

- Its cumulative distribution function (cdf) is the function $F : \mathbb{R} \to [0, 1]$ with
  $$F_X(t) := \mathbb{P}[X \leq t].$$

- Its quantile function $Q_X : (0, 1) \to \mathbb{R}$ is given by
  $$Q_X(p) := \inf \{ t \in \mathbb{R} \mid F_X(t) \geq p \}.$$
• If there is a non-negative, integrable function \( f_X \) such that
\[
F_X(t) = \int_{-\infty}^{t} f_X(u) \, du,
\]
then we call \( f_X \) the \textit{probability density function (pdf)} of \( X \).

• The \textit{survival function} of \( X \) is given by
\[
S_X(t) := \mathbb{P} \{ X > t \} = 1 - F_X(t).
\]

We will need the fact that the quantile function is the inverse of the cdf in the next section.

2.2.1 Visual data analysis

To compare two probability distributions, we will use the explorative graphical tool of \textit{Q–Q plots}. These plots compare two distributions by plotting their quantiles against each other. If the result is a line, one can assume that the underlying distributions are the same.

\textbf{Definition 2.2.} Let \( F \) and \( G \) be two cdfs. Then the graph \((F^{-1}(p), G^{-1}(p))\) for \( 0 < p < 1 \) is called \textit{Q–Q plot} of \( F \) and \( G \).

\textbf{Remark 1} \([23]\). If \( F \) and \( G \) are identical, the Q–Q plot will be the main diagonal. If \( F(x) = G \left( \frac{x-\mu}{\sigma} \right) \), then \( F^{-1}(p) = \mu + \sigma G^{-1}(p) \). Thus, the Q–Q plot of \( F \) and \( G \) will show a linear relationship of slope \( \sigma \) and intersection \( \mu \).

In a goodness-of-fit problem, one theoretical cdf is given, and we have empirical observations drawn from the other distribution.

\textbf{Definition 2.3.} Given a sample \( y_1 \leq y_2 \leq \cdots \leq y_k \), we let \( p_i := \hat{F}_n(y_i) \) and \( x_i := Q(p_i) \), where \( Q \) is the theoretical quantile function of a theoretical distribution function \( F \). In the Q–Q plot, we plot the points \((x_i, y_i)\) for \( i = 1, \ldots, k \).

2.2.2 Survival analysis and censored data

We will use survival analysis (see [24] for an introduction to the subject) to analyze data in which the time until an event is of interest. The time until this event happens is called \textit{event time}. If all events are observed, we can estimate the cdf with the help of the observations, for which we will use the empirical cdf.

\textbf{Definition 2.4.} Let \( X_1, \ldots, X_n \) be independent, identically distributed real-valued random variables with realizations \( x_i \) of \( X_i \). Then, the \textit{empirical cumulative distribution function (ecdf)} of the sample \((x_1, \ldots, x_n)\) is defined as
\[
\hat{F}_n(t) := \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{x_i \leq t\}}, \quad t \in \mathbb{R},
\]
where \( \mathbb{1}_A \) is the indicator function of event \( A \).

Since in some of our experiments, it will turn out to be computationally infeasible to wait until all formula instances are solved, we will use a tool from non-parametric statistics to estimate the survival function of the corresponding runtime random variable. That is, we will be working with incomplete observations. To nevertheless estimate the survival function from a sample of censored survival data, we use the Kaplan–Meier product-limit estimator [25][26].
Let $T$ be a non-negative random variable (which indicates the time until an event of interest takes place, e.g., finding the solution of a formula). Let $t_1, \ldots, t_k$ be the points in time when events $1, \ldots, k$ would have happened (think of a solution for formula $F_j$ being found if the solver was not stopped) whose common distribution is that of $T$. Right-Censoring is present when we have some information about event time (e.g., the solver was still running at a certain point in time), but for some events, we do not know the exact event time (because we stopped the solver early). More precisely, to avoid excessively long runtime, we will later choose for every $j \in \{1, \ldots, k\}$ a fixed integer $c_j$ as the censoring time for event $j$ (meaning that after this time, the solving of $F_j$ will be aborted). The data available for estimating the survival function $S_T$ of the random variable $T$ then is the sequence of observations
\[ \left((\bar{t}_j, c_j)\right)_{j=1,\ldots,k} \text{ with } \bar{t}_j := \min\{t_j, c_j\}, \]
as well as censoring indicators $\text{cen}_j \in \{0, 1\}$ of the form
\[ \text{cen}_j = 0 : \iff t_j \leq c_j. \]
That is, we either know that the formula $F_j$ was solved in time (and we know the time $t_j$ needed for this), or we know that the solver was still running at the censoring time $c_j$.

**Definition 2.5** ([25,26]). The Kaplan–Meier estimator is given by
\[ \hat{S}_T(t) := \prod_{i: t_i \leq t} \left( 1 - \frac{d_i}{n_i} \right), \]
where (in our case)
- $t_i$ is a point in time when (at least one) formula was solved,
- $d_i$ is the number of experiments, where the solver finished at time $t_i$, and
- $n_i$ is the number of experiments that have not yet had an event or have not been censored up to time $t_i$.

If there are no censored observations, the Kaplan–Meier estimator reduces to one minus the empirical cumulative distribution function (see e.g. [24]), also known as the empirical survival function.

## 3 Experimental Setup

Looking at the current state of CDCL solver research, the innovation cycle boils down to improving and finding better heuristics guiding the base CDCL algorithm [27,28]. A thorough theory of clause learning\(^5\) and especially the performance of clause learning is still to be developed. For this reason, an experimental approach seems the most reasonable to investigate the effect of pre-learned clauses. Briefly summarized: we recorded all learned clauses $L$, extended the original instance with subsets $L \subseteq L$ of these pre-learned clauses, and analyzed the runtime of such extended instances compared to the original instance (see Section 3.1 for additional details on this modification process).

Our SAT solver of choice for all our experiments is Glucose 4.0 (see [15,34]). First introduced in 2009, the Glucose project, which is based on the famous Minisat solver [35], was quite successful in the past SAT Competitions. We obtained a relevant,\(^\text{\footnote{There have been slight advancements towards this in the field of proof complexity. We refer to the papers [29,33]. Still, these papers do not yet completely bridge the gap between theory and practice.}}\)

\[ \text{cen}_j = 0 : \iff t_j \leq c_j. \]
dive, and well-documented pool of instances by choosing all instances from the SAT Competition 2020 which were solved by Glucose 3.0 between 30 min and 5000 s (≈ 83 min). The upper bound comes from a time limit imposed in the SAT Competition itself, where all solvers are cut off after 5000 s. A more detailed description of all selected instances can be found in S2 Table. We also refer to the proceedings of the SAT Competition 2020 [36, 37]. A vigilant reader may notice that we have 53 instances in our pool, whereas our selection criterion applies to 61 instances of the SAT Competition 2020. We eliminated the remaining eight instances from the pool because they caused technical complications in at least one stage of our experimental setup. For example, three cases failed during clause recording as the number of learned clauses was too high and the required disk space to save all of them exceeded all reasonable limits. On the remaining five instances, Glucose ran out of RAM for some extensions. These cases could skew the runtime analysis since we don’t know how Glucose would have performed with enough memory. Therefore, we excluded them from the analysis.

3.1 Generating extensions from learned clauses

During execution, modern CDCL solvers learn plenty of clauses. All these learned clauses are directly implied by the clauses of the initial formula \( \mathcal{F} \), which means that \( \mathcal{F} \cup L \) is logically equivalent to \( \mathcal{F} \) for all \( L \subseteq L \), with \( L \) being the set of all learned clauses. We call \( L \) an extension of the base instance \( \mathcal{F} \) and \( \mathcal{F} \cup L \) an extended instance.

| Input: Boolean formula \( \mathcal{F} \) (the base instance) |
|-------------------------------------------------------------|
| Let \( L \) be the set of all learned clauses during the execution of Glucose(\( \mathcal{F} \)) |
| \( L := \emptyset \) |
| foreach \( C \in L \) do |
| \( \text{with probability } p \text{ do } L := L \cup \{C\} \) |
| Call Glucose(\( \mathcal{F} \cup L \)) and measure the runtime and number of conflicts |

Algorithm 1. Modified version of Glucose. We used this modified version of Glucose in our experiments to model the clause learning process as a random process. Each call of this modified algorithm will use Glucose to solve an extended instance \( \mathcal{F} \cup L \). This will allow us to study the runtime distribution of Glucose.

We adapt the approach presented in [7] to CDCL solvers. We refer to Algorithm 1 which requires a pool of pre-learned clauses \( L \). These clauses were gathered by running Glucose on \( \mathcal{F} \) and logging all learned clauses to a file. The random sampling of a subset \( L \subseteq \mathbb{L} \) was implemented by independently selecting each clause in the pool with probability \( p \). This subset \( L \) will be used to study the runtime of Glucose on the extended instance \( \mathcal{F} \cup L \). For our experiments, we chose \( p = 0.01 \) and generated 5000 different extensions for each of the 53 base instances in our instance pool. That is, for each base instance \( \mathcal{F} \), we recorded the runtimes of Glucose on the extended instances \( \mathcal{F} \cup L^{(1)}, \ldots, \mathcal{F} \cup L^{(5000)} \). In this way, we can study the performance of Glucose on instances that already contain some of the learned clauses.

The scripts for generating the set \( L \) and reconstructing our sampled sets \( L \) can be found in S1 File. Overall, we produced 1.5 TB of instance data. Recording the performance, we used 265,000 calls of Algorithm 1 with sometimes surprisingly long runtimes.

3.2 The challenge of solving a myriad of hard formulas

Solving 265,000 hard Boolean formulas in a reasonable time required parallelization, for which we used Sputnik [38], and a somewhat more complex experimental setup. We
Table 1. Summary of the hardware used in our experiments.

| name            | node                  | cpu          | cores | frequency | RAM   |
|-----------------|-----------------------|--------------|-------|-----------|-------|
| Erpel           | Intel Xeon E5-2668 v3 | 32           | 2.30 GHz | 256 GB    |
| Luna            | AMD EPYC 7742         | 64           | 2.25 GHz | 256 GB    |
| BwUniCluster 2.0| Intel Xeon Gold 6230  | 40           | 2.10 GHz | 96 GB     |
|                 | Intel Xeon E5-2660 v4 | 28           | 2.00 GHz | 128 GB    |

Table notes. Erpel and Luna are standard server architectures, whereas the BwUniCluster 2.0 is an HPC cluster.

Additionally distributed the formulas over two regular servers (Luna, Erpel) and an HPC cluster (BwUniCluster 2.0). See Table 1 for more details.

After we started the experiment on just the Luna Server, we were confronted with surprisingly long runtimes on certain extended formulas. The original instances were solved after at most 95 minutes, but for some extended formulas, it took Glucose more than ten days to solve them. This led us to

(a) distribute the calls of Algorithm 1 over multiple hardware nodes, and

(b) introduce a timeout strategy.

Only 12 out of all 53 instances encountered censoring, meaning that Glucose reached the timeout limit for at least one extension. In most cases, only a few extended instances had to be aborted due to the timeout policy. More details can be found in S2 Table.

Said censoring timeouts $c_j$ were assigned to each extended formula $F_j$ according to an offset geometric distribution such that

$$c_j \sim \text{Geo}\left(\frac{1}{12 \text{h}}\right) + 5000 \text{s}.$$  

We have chosen the value of 5000 seconds since this is the cut-off in the SAT Competition. The distribution Geo($\frac{1}{12 \text{h}}$) has an expectation of 12 hours. This approach led to a total CPU time of 16 years and 1 month. All data obtained in this way can be found in S1 File.

Having used censoring in acquiring our data will make the use of survival analysis, as elaborated in Section 2.2.2, necessary.

4 Is clause learning useful on average?

Clearly, augmenting a basic DPLL solver with a clause learning mechanism is far from a modern CDCL solver. However, clause learning is arguably the most important technique used in CDCL solvers, lending its name to the paradigm. One would therefore expect that clause learning (especially when guided by state-of-the-art heuristics) is generally useful, i.e., one would expect that providing the solver with learned clauses for free does

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*Note that due to such a diverse hardware setup, runtime comparisons have to be done with caution. More details on this and other metrics can be found in Section 4.*
increase the performance of the solver when compared to the base instance, where the solver has to learn all clauses by itself.

To check this assumption, we performed a parametric test of whether the mean difference between the base instance and the 5000 (possibly censored) runtimes on the extended instances equals 0. For this, we assumed that the paired differences follow a Gaussian normal distribution. To perform this test, we used the “NADA2: Data Analysis for Censored Environmental Data” package \[39\] in R (we refer to the book \[40\] for an in-depth treatment of the statistical methods involved). For the threshold value, we used \(p = 0.05\).

In this section, we—quite surprisingly—, will demonstrate that clause learning is oftentimes useful, but there are also many instances where a dramatic negative effect can be observed. For an overview of the different effects, we refer to Table 2. Interestingly, almost all instances can be very clearly categorized in the table, in the sense that the obtained \(p\)-values are remarkably low (the exceptions are marked in the table).

### Table 2. The Effect of learned clauses (without deletion) on the runtime of Glucose.

|                          | With censoring | Without censoring |
|--------------------------|---------------|------------------|
| Positive effect          | 5             | 26               |
| No significant effect    | 0             | 1 \[†\]          |
| Negative effect          | 7             | 14 \[‡\]         |

Table notes. We say that clause learning has a **positive effect** if the mean of the runtimes required to solve the extended instances was statistically significantly smaller than the runtime required to solve the base instance. If the mean is statistically significantly greater than the runtime for the base instance, we speak of a **negative effect**. Otherwise, it has no significant effect. The base instances are grouped in the table based on whether there was at least one extended instance of this base instance where censoring occurred (with censoring) or not (without censoring). \(\triangleright\) \[†\] The observed \(p\)-value was \(p = 0.53\) for the unsatisfiable instance \text{ncc\_none\_5047\_6\_3\_3\_0\_0\_41\_p0\_01\_} (see S2 Table for the complete list of \(p\)-values). The observed effect was negative. \(\triangleright\) \[‡\] Furthermore, the non-censored instances with negative effects include an instance (namely \text{6g\_5color\_164\_100\_01}) with \(p = 0.039\). This \(p\)-value is noteworthy since all other \(p\)-values are smaller than \(2.657 \cdot 10^{-4}\) (again, see S2 Table).

We want to emphasize the point that the same surprising effect, namely, there are quite a few instances where adding the set \(L\) yields a deterioration of the measure studied, can be observed when using the number of conflicts occurring during a run of the solver instead of the time required to solve an instance. We refer to Table 3 for an analysis of the effect of learned clauses concerning the number of conflicts. We chose to study this additional measure due to the heterogeneous server architecture outlined in Table 1. The runtime differs very slightly between the server. However, the number of conflicts needed to solve the instance is a robust, hardware-independent, theoretical measure.

For a comparison of runtime vs. number of conflicts, we refer to Table 4. We want to point out that in each case where an opposite effect for time and number of conflicts can be observed, the effect was negative for time and positive for the number of conflicts.

As mentioned in Section 1.2, this can be explained by the observation of Mitchell [14]: the required time to perform unit propagation becomes too high for very large clause sets, which reduces the performance of the solver.

In both cases, i.e., regardless if one were to consider the runtime or the number of conflicts, the observed deterioration in performance cannot be explained by pure chance. This is all the more surprising since the set of learned clauses \(L\) is not just any random set of clauses but clauses that were learned by the same solver on its way to a
Table 3. The effect of learned clauses (without deletion) on the number of conflicts used by Glucose.

|                     | With censoring | Without censoring |
|---------------------|----------------|-------------------|
| Positive effect     | 8              | 46                |
| No significant effect| 0              | 1                 |
| Negative effect     | 2              | 4                 |

Table notes. See Table 2 for an explanation of the rows and columns. In more than 11% of the cases, a negative effect was observed. This cannot be explained by pure chance and is quite surprising. Out of the six instances with a negative effect, two were unsatisfiable, and four were satisfiable. Two of the 53 instances (both with censoring) had to be excluded from the tests due to numerical complications in the censored data paired t-test (NADA2 package).

Table 4. Comparison between the effect of adding clauses for time and number of conflicts.

|                      | With censoring | Without censoring |
|----------------------|----------------|-------------------|
| Same effect for time and number of conflicts | 7              | 30                |
| Opposite effect      | 3              | 10                |

Table notes. We excluded one instance where for at least one measure, our experiments were not able to determine if there is a positive or negative effect for that measure.

solution. Therefore, one would expect that each and any of such a clause would benefit the guidance of CDCL towards a solution.

As an explanation of this deterioration phenomenon, one should therefore consider the influence of clause deletion. Our experimental setup can be interpreted as switching off clause deletion for the set of added clauses $L$ (while keeping all other heuristics and optimizations of the solver) and learning all those clauses at once. Note that the solver will learn some additional clauses during its run and can also delete these. The set $L$, however, is fixed during the run. Seemingly, clause deletion at the right points in time is as crucial as clause learning. This statement cannot be fully explained by a blow-up of the size of the clause database, as the unit-propagation-independent measure of the number of conflicts also increased in many cases.

## 5 Multimodal behavior of runtimes

In the last section, our focus was to compare the behavior of Glucose on the unmodified base instance to the behavior on the modified instances that extended this base instance. For the following sections, we will shift our focus towards studying the runtime distribution of the modified instances.

Our precise aim in this section will be to get an understanding of the modality of the ensuing distributions. Recall that in statistics, a probability distribution with a single peak is called unimodal. Otherwise, we speak of a multimodal distribution. An easy way to inspect the modality of a distribution is to inspect the histogram of the distribution visually. This method has the additional advantage that no statistical test has to be used that can distinguish between unimodal and bimodal distributions but rely on the knowledge of the underlying distribution type (i.e., one does not need to know in advance if, e.g., the distribution can be resolved into normal distributions [41]).

Since our obtained data points are censored, we cannot immediately plot the histogram.
To overcome this obstacle, we have used the Kaplan–Meier estimator (see Definition 2.5) implemented in the Survival package [42] in R to obtain a fit of the underlying survival function. Graphically, the Kaplan–Meier survival curve is a step function with a drop each time the solver has finished an instance. The points where a drop can be observed can thus be used as an estimation basis to create the histogram. Note that in the improbable event that two instances take exactly the same time to be solved, the resulting histogram underestimates the number of instances in the corresponding bin. However, this kind of event occurs so seldom that, for all intents and purposes, we can be satisfied with the obtained estimation of the true histogram. We have printed the resulting estimated histogram of a representative instance in Fig 2. As can be clearly seen, the distribution is multimodal.

![Pdf of fitted weibull mixture model](image)

**Fig 2. Multimodal histogram of runtime distribution.** Furthermore, we used the Kaplan–Meier estimate to obtain the histogram of the runtime distribution of the instance UNSAT ME seq-sat Thoughtful p11 6 59-typed.pddl 43. We used the Expectation–maximization (EM) method to obtain the pdf of the fitted Weibull mixture model (see Definitions 6.1 and 6.2 for an introduction to this kind of distribution). The EM algorithm is an algorithm that, roughly speaking, allows cluster analysis by starting with a heuristically initialized model and by alternating between two steps. In the expectation-step ([E-step](#)), the association of the data points to the different clusters gets changed. Then, in the maximization-step ([M-step](#)), the model’s parameters get improved by using this new association of the data points. We refer to the classic paper [43] for an introduction of the algorithm (the modified algorithm for the Weibull case will be described in a forthcoming paper). The resulting fitted distribution that is seen in the plot is clearly multimodal.

To facilitate our inspection of the histograms, we also investigated the histograms for the logarithmically scaled runtimes. This method has been found to usually give a clear separation into a visible multimodal histogram if the underlying distribution is indeed multimodal (see e.g. [44,45] for the earliest applications of this technique). Using this technique, we found that a significant fraction of the instances exhibited dominant multimodal behavior.

This multimodal grouping of instances into several categories could be helpful in an investigation of the usefulness of the added clauses. We will make this thought more precise in the next section and also investigate the type of distribution underlying the model.
6 Finding the right mixture distribution type

In Section 5, we have already seen that the runtime behavior is multimodal for a substantial part of the instances. This section aims to study which types of distributions are suitable to describe this behavior. Since most well-known distributions, such as the normal distribution, are unimodal or at most bimodal, this suggests that one must resort to another type of distribution.

The presence of the many peaks points to that the different extended instances (and underlying clauses) can be divided into categories. Each category corresponds to the hardness of the extended instance, where the hardness is again not a fixed value but a random variable. If one confines oneself to a single category of extended instances, then one is (potentially) no longer confronted with multimodal behavior but can describe the remaining data by means of a unimodal distribution.

By (randomly) adding the clauses, we then end up in this category of extended instances with a certain probability. If such an analysis is conducted for each such category, we eventually obtain a description of the complete runtime behavior. Specifically, this means that for each category, the underlying runtime distribution, as well as the probability of ending up in that category, must be identified. The runtime behavior across all extended instances is then characterized by a so-called finite mixture distribution.

Definition 6.1. Let $X$ be a random variable having cdf $F_X$. Let $F_1, F_2, \ldots, F_N$ be cdfs and $p_1, p_2, \ldots, p_N$ be weights with $p_i > 0$ for all $i \in \{1, \ldots, N\}$ and $\sum_{i=1}^N p_i = 1$. If

$$F_X(x) = \sum_{i=1}^N p_i \cdot F_i(x)$$

holds for all $x \in \mathbb{R}$, then $X$ has an $N$-component (finite) mixture distribution.

In our case, $N$ can be understood as the number of categories. Furthermore, $p_i$ describes the probability of ending up in category $i$, in which case $F_i$ is the runtime distribution for category $i$. We refer to Fig 3 of a depiction of the components underlying a mixture distribution.

In principle, one can choose arbitrary cdfs $F_1$ to $F_N$. However, it is common practice to choose the cdfs from the same family of (parametric) distributions, where only the
parameters of the distribution differ. For example, a popular model are Gaussian mixture distributions in which the cdfs describe normal distributions varying with respect to their expected value and variance. Therefore, the question arises of which parametric distribution type should be chosen for the mixture distribution. We shall argue that a distribution based on the so-called Weibull distribution is an appropriate type of distribution.

**Definition 6.2** ([46]). A random variable $X$ with pdf

$$f_X(x) = \begin{cases} \frac{k}{a} \cdot \left(\frac{x-\ell}{a}\right)^{k-1} \cdot e^{-\left(\frac{x-\ell}{a}\right)^k}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

is 3-parameter **Weibull distributed** with parameters $k \in \mathbb{R}^+$ (shape), $a \in \mathbb{R}^+$ (scale), and $\ell \in \mathbb{R}$ (location). The cdf of $X$ is given by

$$F_X(x) = \begin{cases} 1 - e^{-\left(\frac{x-\ell}{a}\right)^k}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

If the location parameter $\ell$ is zero, we call the distribution 2-parameter **Weibull distributed** or just **Weibull distributed**.

We first start by analyzing instances that can be described with only one component (i.e., a 1-component mixture distribution). The idea behind this is that one can derive information about the instances that require more than one component. A suitable graphical tool for this analysis is provided by Q–Q plots, where the observed quantiles are plotted against the theoretical quantiles of a given distribution (recall Section 2.2.1). In the following, we consider the required CPU time until the respective instance is solved, i.e., either a satisfying assignment is constructed, or a proof of unsatisfiability is established.

![Q–Q plot](image)

**Fig 4. Q–Q plot.** The Q–Q plot for instance crafted.n11_d6.c4.num19 was obtained by the quantiles of a fitted 3-parameter Weibull distribution and the data quantiles. The plot appears as a straight line. The correlation coefficient calculates to 0.9997979. For reference, the identity is given in gray.

As an example, we consider the data from one instance in Fig 4. We use a fitted 3-parameter Weibull distribution as the theoretical distribution. As can be seen here, the
Q–Q plot yields a straight line, indicating that the theoretical distribution can describe the empirical data very well. So we can conclude that a 3-parameter Weibull distribution is a suitable description for this instance. This is not only the case in this example. For example, one can use the correlation coefficient to measure how linear a certain relationship is. A correlation coefficient of 0.999 describes an extremely strong linear relationship. We used this value and examined all Q–Q plots. In total, 12 instances reach a correlation coefficient of 0.999 if a fitted 3-parameter Weibull distribution is used as theoretical distribution. This suggests that a substantial number of instances can be described by a single Weibull distribution. It should also be emphasized that other typical distribution types, such as the normal or lognormal distribution, do not provide good fits. While Weibull distributions describe a considerable fraction of the instances, this begs the question of what to do with the remaining instances.

First, it should be emphasized that especially the instances where multimodality is strong cannot be described by a single Weibull distribution. Then, however, it is natural to assume that the individual components of a mixture distribution follow Weibull distributions. We will pursue this line of thought in more detail in the following.

Graphical analyses are well suited to argue that Weibull distributions are appropriate. First, we examine how the Weibull distribution behaves at the left tail, i.e., the behavior if \( x \) approaches 0. It is well known that Weibull distributions have a linear appearance on a log–log plot of the cdf at the left tail. To see this, we investigate the logarithm of the cdf \( F \) with location parameter \( \ell = 0 \) for \( x \geq 0 \):

\[
\log F(x) = \log \left( 1 - e^{-\left(\frac{x}{a}\right)^k} \right).
\]

Plugging the Taylor expansion of \( \exp(-x) \) into this equation yields:

\[
\log F(x) = \log \left( 1 - \left[ 1 - \left(\frac{x}{a}\right)^k + \frac{(\frac{x}{a})^{2k}}{2!} - \cdots \right] \right)
= \log \left( \left(\frac{x}{a}\right)^k - \frac{(\frac{x}{a})^{2k}}{2!} + \cdots \right).
\]

Considering the behavior as \( x \) approaches 0, we notice that the trailing terms approach 0 much faster than \( (x/a)^k \) and thus can be neglected. Hence, we obtain:

\[
\log F(x) \approx \log \left( \left(\frac{x}{a}\right)^k \right) = k \cdot \log x - k \cdot \log a.
\]

By substituting \( z = \log x \), one finds that the cdf \( F \) indeed appears linearly in the neighborhood of zero on a log–log plot.

For mixture distributions, this method is useful for making statements about the smallest component. Suppose that \( F(x) = \sum_{i=1}^{N} p_i F_i(x) \) is the cdf of a mixture distribution. Here, \( F_1 \) is the cdf of a Weibull distribution, and for small \( x \), we have \( F_1(x) \gg 0 \) and \( F_2(x) \approx F_3(x) \approx \cdots \approx F_N(x) \approx 0 \). Thus, \( F(x) \approx p_1 F_1(x) \) is also valid; moreover, due to the reasoning above, the cdf \( F \) appears linearly on a log–log plot in the neighborhood of zero. Conversely, one can argue that log–log plots of the cdf are suitable for evaluating whether the smallest cdf can be characterized by a Weibull distribution.

Another popular method of analyzing Weibull distributions is to examine the survival function \( S(x) = 1 - F(x) \). In particular, the survival function transformed as follows is used:

\[
\log \left( -\log S(x) \right) = \log \left( -\log e^{-\left(\frac{x}{a}\right)^k} \right)
= \log \left( \left(\frac{x}{a}\right)^k \right) = k \cdot \log x - k \cdot \log a.
\]
In other words, a Weibull distribution appears linear if the survival function is double logarithmized in this manner and the $x$-axis is singly logarithmized. We can apply this graphical tool to determine whether the largest component in a mixture distribution can be described by a Weibull distribution.

Again, suppose that $F(x) = \sum_{i=1}^{N} p_i F_i(x)$ is the cdf of a mixture distribution. Here, $F_N$ is the cdf of a Weibull distribution, and for large $x$, we have $F_N(x) \ll 1$ and $F_1(x) \approx F_2(x) \approx \cdots \approx F_{N-1}(x) \approx 1$. Thus, we have

$$S(x) = 1 - F(x) = 1 - \sum_{i=1}^{N} p_i F_i(x) \approx 1 - \sum_{i=1}^{N-1} p_i F_N(x) = p_N - p_N F_N(x) = p_N \left(1 - F_N(x) \right).$$

By the above argument, the doubly logarithmized survival function $S$ and singly logarithmized $x$-axis will appear approximately linear for large $x$. Conversely, such a plot can also be used to deduce whether the largest component can be described by a Weibull distribution.

These two plot types are therefore suitable for finding out whether the extreme values, i.e., particularly short and particularly long runs, are described by Weibull distributions, respectively. Thus, as before, we examine the CPU times and investigate them with the help of these two plot types.

![Cumulative distribution function](image1)

![Survival function](image2)

**Fig 5. Inspection of the smallest and largest component of the Weibull mixture model.** Based on the Kaplan–Meier estimator, the estimations of the cdf and survival function of the instance bivium-40-200 are shown. Both the left and the right tails appear as straight lines (depicted in red). The plot of the cdf is a log–log plot, while the plot of the survival function is a log log–log plot. The gray area marks the confidence interval. This suggests that the smallest and largest component of the underlying mixture model are Weibull distributions. (a) Estimation of the cdf. (b) Estimation of the survival function.

In Fig[5] we exemplarily consider one instance. Note that both the left and the right tails appear as straight lines. Using the reasoning presented above, we can therefore infer that for both cases, a Weibull distribution is appropriate to characterize the left and the right tail, respectively. On the one hand, Weibull distributions describe both the left and the right tail and, for some cases, the entire support. On the other hand, it is
common practice to use only one type of distribution for mixed distributions. Therefore, we argue that the runtime distributions can be described by Weibull distributions.

One can derive some highly intriguing insights into the operation of CDCL solvers from the knowledge that Weibull distributions describe the runtime behavior of such solvers. First, if the shape parameter $k$ of the Weibull distribution is less than 1, then the distribution has the so-called long-tail property [47].

Fig 6. Long-/Heavy-Tails. This figure shows various plots of the instance 6g5color_164_100.01. This is an example of an instance with a long-tailed runtime distribution. (a) The plot shows the histogram of runtimes (in gray) and the fitted pdf (in red). Both are shifted to the left by the minimal time $T_0$ required to solve any extended instance. The obtained shape parameter of the fit is $k = 0.88 < 1$. Thus, the distribution is long-tailed. (b) We have plotted the logarithm of the tail of the distribution, i.e., $\log S(x)$. By visual inspection, one can see that it decays sub-linearly. In this case, $\lim \inf_{x \to \infty} -\log P[X > x]/x = 0$. This property characterizes the class of so-called heavy-tailed distributions (a superset of the class of long-tailed distributions) [47]. Intuitively, this means that the algorithm has a non-vanishing probability of requiring very long runtimes. For comparison, we have plotted the logarithmic survival function of an exponentially distributed random variable with the same expectation in blue. The logarithm of the tail of such an exponential distribution decays linearly. (c) Zoomed in version of (b). This clearly shows the sublinear decay by focusing on the curvature.

Roughly speaking, this property indicates that the algorithm either finishes (relatively)}
quickly or takes exceedingly long. For an illustration of a long-tailed runtime distribution and a further elaboration on the subject of the long-tailed property, refer to Fig. What is remarkable about this is that restarts have been proven to be useful for long-tailed distributions. This means that the algorithm can be accelerated by reinitializing it from scratch. In our context, a restart consists of discarding all added clauses \( L \) that were added to the original instance \( \mathcal{F} \). Instead, a new set of clauses \( L' \) is sampled from the base set \( L \) that will then be added to the original instance \( \mathcal{F} \). Of course, the search tree is also reset to the top level at the same time. In the context of CDCL solvers, this procedure is referred to as “clause deletion” and “restarts.”

Therefore, the observation that Weibull distributions describe the runtime behavior implies that clause deletions and restarts are useful in the context of CDCL solvers, i.e., they improve the runtime. However, what is remarkable about this is not the mere observation that these two techniques improve the runtime because this fact has already been shown empirically (see e.g. [2,15,50,51]). It is more interesting that we are reaching a conclusion as to why these techniques have a positive effect on the performance. Adding new clauses to the base instance thus has the inherent effect of making runtimes long-tailed. While the added clauses usually improve the performance, there is a non-negligible chance that the performance will deteriorate (sometimes drastically). The easiest way to circumvent this problem is to delete the learned clauses and reset the search tree periodically.

It is also worth repeating the observations on Table 3. Here, the effect of adding clauses is measured by the number of conflicts. This table tells us that, contrary to common belief, the degraded performance is not only due to the increase in the size of the base instance \( \mathcal{F} \) and thus due to a more considerable overhead for each propagation. Instead, it implies that some clauses lead the CDCL algorithm itself astray, i.e., to a path in the search tree that does not yield a solution.

Since clause deletions and restarts are useful because Weibull distributions describe the runtimes, this shifts the question to why adding clauses causes Weibull distributions. A possible starting point is provided by the Fisher–Tippett–Gnedenko theorem. Roughly speaking, this theorem states that the minima and maxima of independent and identically distributed random variables converge to one of three distribution types (under the condition that they converge at all). The Weibull distribution is one of these three distribution types. This suggests that the reason for the observed runtimes may be a minimum or maximum process. For example, the runtimes could be significantly influenced by the quality of the “best” or “worst” clause, where by the quality of the clause, we mean the extent to which the clause guides the heuristics of the CDCL algorithm towards a solution. However, this is only a hypothesis that should be further investigated in other future research.

7 Conclusion

We have modeled the technique of clause learning in CDCL solvers by solving new logically equivalent formulas of a base instance. This allowed us to analyze the resulting runtime distribution.

We have provided compelling evidence that this distribution is a Weibull mixture model, completing the runtime distribution study for both paradigms of SAT solvers. In addition, the Weibull fit was suitable for both multimodal and unimodal instances. Because the underlying distribution is Weibull, adding new clauses thus has an inherent effect of making runtimes long-tailed in both SLS and CDCL solvers. The long-tailed

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8Long-tails are not equivalent to heavy-tails or powerlaws. Following [48] and [47], we say that a positive, real-valued random variable \( X \) is long-tailed, if and only if for all \( x \in \mathbb{R}^+ \) it holds \( \mathbb{P} [X > x] > 0 \), and for all \( y \in \mathbb{R}^+ \) it holds \( \lim_{x \to \infty} \mathbb{P} [X > x + y] / \mathbb{P} [X > x] = 1 \).
runtime distribution in CDCL solvers yields additional motivation to improve on existing clause deletion schemes. These are not only needed to speed up unit propagation steps but are indispensable tools to avoid getting stuck in the tail of the distribution and ultimately avoid excessively long solving times. Additionally, the long-tailed property theoretically explains why restarts are useful for CDCL algorithms.

We furthermore provided a hypothesis for the suitability of the Weibull distribution by invoking the Fisher–Tippett–Gnedenko theorem. It seems reasonable that runtimes are heavily influenced by the quality of the “best” and “worst” clauses. An analysis of these clause qualities, especially in the context of LBD [15], seems like a fruitful pursuit for further research.

Supporting information

S1 File. Generated data and evaluations. We have provided all data of this paper in the repository [56] (doi:10.5281/zenodo.5902373). This collection contains the scripts for obtaining the sets \( L \) and reconstructing our sampled sets \( L \). Furthermore, all data obtained by calling \texttt{Glucose}(\( F \cup L \)) can be found. Additionally, we included visual and statistical evaluations used in this paper.

S2 Table. Instance pool. This supporting table describes the instances used for our experiments. The instance \texttt{bivium-40-200-0s0-0x92fc13b11169afbb2ef11a684d9fe9a19e743cd6aa5ce23fb5-19} was abbreviated by \texttt{bivium-40-200} in the table. The column \( cen \) denotes the number of censored data points of each instance, i.e., \( \sum_{j=1}^{5000} cen_j \), where \( cen_j \) is the censoring indicator introduced in Equation (2). The columns \( Z_{time} \) and \( p_{time} \) report the results of the \( t \)-test for CPU time as described in Section 4. Similarly, the columns \( Z_{conf} \) and \( p_{conf} \) report the results of the \( t \)-test for the number of conflicts as described in Section 4. In both cases, a negative value of the test statistic \( Z \) signifies a positive effect for the mean of the extended instances. All values were rounded to two places. The values “—” in the table denote the two instances where complications in the \( t \)-test for the censored number of conflict data occurred. The table itself can be found on the following pages in landscape mode.
| instance                               | cen | $Z_{\text{time}}$ | $p_{\text{time}}$ | $Z_{\text{confl}}$ | $p_{\text{confl}}$ |
|----------------------------------------|-----|-------------------|-------------------|-------------------|-------------------|
| 3bitadd.32.cnf.gz.CP3-cnfmiter         | 8   | -118.79           | 0                 | -271.14           | 0                 |
| 59-129706                              | 1   | -253.62           | 0                 | -321.23           | 0                 |
| 6g_5color_164_100_01                   | 0   | 2.06              | 0.04              | -50.04            | 0                 |
| abw-K-dwt_234 mtx-w55                  | 0   | -6167.81          | 0                 | -4888.98          | 0                 |
| bivium-40-200                          | 0   | -10026.97         | 0                 | -178525.36        | 0                 |
| crafted_n11_d6_c4_num19                | 0   | -1247.17          | 0                 | -2295.24          | 0                 |
| cz-alt-3-7                            | 0   | -3480.32          | 0                 | -5057.18          | 0                 |
| DLTM_twitter249_74_10                  | 0   | -3526.23          | 0                 | -3923.37          | 0                 |
| DLTM_twitter799_70_13                  | 0   | 43.73             | 0                 | 17.60             | 2.55 · 10^{-69}   |
| Kakuro-easy-117-ext.xml hg.5           | 0   | -93.85            | 0                 | -164.09           | 0                 |
| Kakuro-easy-125-ext.xml hg.4           | 0   | -395.68           | 0                 | -407.96           | 0                 |
| Kakuro-easy-132-ext.xml hg.9           | 17  | 54.51             | 0                 | 23.08             | 6.57 · 10^{-118}  |
| Kakuro-easy-149-ext.xml hg.4           | 0   | -190.97           | 0                 | -383.81           | 0                 |
| Kakuro-easy-154-ext.xml hg.4           | 0   | -313.40           | 0                 | -758.95           | 0                 |
| LABS_n038_goal002                     | 0   | -8917.91          | 0                 | -18594.88         | 0                 |
| LABS_n071_goal001-sc2013               | 0   | -915.23           | 0                 | -2133.75          | 0                 |
| logistics-unsat-logistics-rotate-11t5.sat05-1141.reshuffled-07 | 19  | 201.15            | 0                 | -51597.71         | 0                 |
| ls16-normalized.cnf.gz.CP3-cnfmiter    | 134 | -11.76            | 5.96 · 10^{-32}   | -123.27           | 0                 |
| mm-2x3-8-sb.1.sat05-475.reshuffled-07  | 0   | -6888.62          | 0                 | -14486.58         | 0                 |
| ncc_none_12477_5_3_3_0_0_435991723     | 2   | 220.05            | 0                 | -503.92           | 0                 |
| ncc_none_12477_5_3_3_1_0_435991723     | 0   | 70.03             | 0                 | -12.33            | 6.16 · 10^{-35}   |
| ncc_none_2_18_9_3_0_0_435991723        | 0   | -94.07            | 0                 | -303.30           | 0                 |
| ncc_none_3047_7_3_3_1_0_1              | 0   | 79.94             | 0                 | 20.15             | 2.59 · 10^{-90}   |
| ncc_none_5047_6_3_3_0_0_41             | 0   | 0.63              | 0.53              | -38.21            | 0                 |
| ncc_none_5047_6_3_3_3_0_435991723      | 0   | 123.00            | 0                 | -19.46            | 2.54 · 10^{-84}   |
| newpol34-4                             | 8   | -132.70           | 0                 | -229.29           | 0                 |
| preimage_80r_491m_160h_seed_407         | 0   | 8.11              | 4.88 · 10^{-16}   | -104.52           | 0                 |
| preimage_80r_492m_160h_seed_136         | 0   | -1513.85          | 0                 | -1562.72          | 0                 |
| preimage_80r_493m_160h_seed_249         | 0   | -39.30            | 0                 | -172.73           | 0                 |
| instance                                      | cen | $Z_{time}$ | $P_{time}$ | $Z_{confl}$ | $P_{confl}$ |
|-----------------------------------------------|-----|------------|------------|-------------|-------------|
| problem_23.smt2                               | 276 | 225.17     | 0          | 112.68      | 0           |
| QG7a-gensys-ukn009.sat05-3849.reshuffled-07   | 0   | -1200.66   | 0          | -4072.55    | 0           |
| SAT_ME_opt_snake_p15.pddl_25                   | 0   | -107.38    | 0          | -73.32      | 0           |
| SAT_P_opt_snake_p02.pddl_32                    | 0   | 56.78      | 0          | 3.76        | 1.69 · 10^{-4} |
| size_5_5_5_i019_r12                           | 2403| 36.96      | 4.70 · 10^{-299} | —           | —           |
| sqrt_inq_3.c                                  | 53  | 65.71      | 0          | —           | —           |
| sted1_0x0_u438-636                            | 0   | -2212.69   | 0          | -2600.06    | 0           |
| sted2_0x0_u219-342                            | 0   | -402.64    | 0          | -493.07     | 0           |
| sted3_0x1c3-147                               | 0   | 35.87      | 7.93 · 10^{-282} | -83.77      | 0           |
| sted5_0x0-157                                 | 0   | 20.92      | 3.19 · 10^{-97} | -58.98      | 0           |
| UNSAT_ME_seq-opt_Tidybot_p17.pddl_29           | 0   | -56.36     | 0          | -87.24      | 0           |
| UNSAT_ME_seq-opt_Tidybot_p19.pddl_29           | 0   | -191.08    | 0          | -165.07     | 0           |
| UNSAT_ME_seq-sat_Thoughtful_p11_6_53-typed.pddl_49 | 0   | -66.30    | 0          | -166.79     | 0           |
| UNSAT_ME_seq-sat_Thoughtful_p11_6_59-typed.pddl_43 | 0   | 13.13     | 2.28 · 10^{-39} | -77.32      | 0           |
| UNSAT_ME_seq-sat_Thoughtful_p11_6_62-typed.pddl_47 | 0   | -5.15     | 2.66 · 10^{-7} | -108.34     | 0           |
| UNSAT_MS_opt_snake_p06.pddl_30                 | 0   | 48.99      | 0          | 8.95        | 3.60 · 10^{-19} |
| UNSAT_MS_opt_termes_p04.pddl_79                | 0   | 133.17     | 0          | -156.46     | 0           |
| UNSAT_MS_opt_termes_p11.pddl_65                | 0   | -50.30     | 0          | -104.72     | 0           |
| UNSAT_P_opt_snake_p02.pddl_31                  | 0   | 18.20      | 4.93 · 10^{-74} | -149.79     | 0           |
| UNSAT_P_sat_snake_p05.pddl_30                  | 0   | 55.44      | 0          | -78.71      | 0           |
| UNSAT_P_seq-opt_Barman_p435.1.pddl_32          | 0   | 65.31      | 0          | -99.10      | 0           |
| UNSAT_P_seq-opt_Barman_p435.2.pddl_32          | 0   | -31.90     | 2.56 · 10^{-223} | -208.84     | 0           |
| w15                                           | 0   | -18343.15  | 0          | -10427.54   | 0           |
| w19-5.1                                       | 0   | -9522.45   | 0          | -8114.50    | 0           |
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