IKEHARA-TYPE THEOREM INVOLVING BOUNDEDNESS

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ABSTRACT. Let $\sum a_n/n^z$ be a Dirichlet series with nonnegative coefficients that converges to a sum function $f(z) = f(x + iy)$ for $x > 1$. Setting $s_N = \sum_{n \leq N} a_n$, the paper gives a necessary and sufficient condition for boundedness of $s_N/N$. As $x \searrow 1$, the quotient $f(x + iy)/(x + iy)$ must converge to a pseudomeasure $q(1 + iy)$, the distributional Fourier transform of a bounded function. The paper also gives an optimal estimate for $s_N/N$ under the ‘real condition’ $f(x) = O\{1/(1 - x)\}$.

1. INTRODUCTION

We recall the famous Tauberian theorem of Ikehara:

Theorem 1.1. Suppose that the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^z} \quad \text{with} \quad a_n \geq 0 \quad \text{and} \quad z = x + iy$$

converges throughout the half-plane $\{x > 1\}$, so that the sum function $f(z)$ is analytic there. Suppose furthermore that there is a constant $A$ such that the difference

$$g(z) = f(z) - \frac{A}{z - 1}$$

has an analytic or continuous extension to the closed half-plane $\{x \geq 1\}$. Then the partial sums $s_N = \sum_{n \leq N} a_n$ satisfy the limit relation

$$s_N/N \to A \quad \text{as} \quad N \to \infty.$$ 

The theorem is often called ‘Wiener–Ikehara theorem’ because Ikehara studied with Wiener, and applied the new Tauberian method that Wiener was developing in the years 1926–1931; see [4], [10], [11] and cf. [5], [6]. Ikehara’s theorem led to a greatly simplified proof of the prime number theorem.
In [7] the author obtained a two-way form of the theorem. Given $f(z)$ of the form (1.1), the following condition is necessary and sufficient for (1.3):
The difference $g(z) = g(x + iy)$ must have a distributional limit $g(1 + iy)$ for $x \searrow 1$ which is locally equal to a pseudofunction. That is, on every finite interval $(-B, B)$, the distribution $g(1 + iy)$ must be equal to a pseudofunction which may depend on $B$. A pseudofunction is the distributional Fourier transform of a bounded function that tends to 0 at infinity. It can also be characterized as a tempered distribution which is locally given by Fourier series with coefficients that tend to 0. A pseudofunction may have nonintegrable singularities, but not as strong as first-order poles.

In connection with Ikehara’s theorem one may ask (cf. Mhaskar [8]) what condition on $f(z)$ would suffice for the conclusion that
\[(1.4) \quad s_N/N = O(1) \quad \text{as} \quad N \to \infty.\]

Unlike the situation in the case of power series $\sum a_n z^n$, it is not enough when $f(\cdot)$ satisfies the ‘real condition’
\[(1.5) \quad f(x) = O\{1/(x - 1)\} \quad \text{as} \quad x \searrow 1.\]

**Proposition 1.2.** For Dirichlet series (1.1) with sum $f(z)$, condition (1.5) implies the estimate
\[(1.6) \quad s_N/N = O(\log N),\]
and this order-estimate is best possible.

See Section 2. In Section 3 we will prove

**Theorem 1.3.** Let the series $\sum a_n/n^z$ with coefficients $a_n \geq 0$ converge to $f(z) = f(x + iy)$ for $x > 1$. Setting $s_N = \sum_{n \leq N} a_n$ as before, the sequence $\{s_N/N\}$ will remain bounded if and only if the quotient
\[(1.7) \quad q(x + iy) = \frac{f(x + iy)}{x + iy} \quad (x > 1)\]
converges in the sense of tempered distributions to a pseudomeasure $q(1 + iy)$ as $x \searrow 1$.

A pseudomeasure is the distributional Fourier transform of a bounded measurable function. It has local representations by Fourier series with uniformly bounded coefficients. A simple example is given by the delta distribution or Dirac measure. The following pseudomeasure is the boundary
distribution of an analytic function:
\[
\frac{1}{+0+iy} \overset{\text{def}}{=} \lim_{x \searrow 0} \frac{1}{x+iy} = \lim_{x \searrow 0} \int_0^\infty e^{-xt} e^{-iyt} dt.
\]
It is the Fourier transform of the Heaviside function \(1_+(t)\), which equals 1 for \(t \geq 0\) and 0 for \(t < 0\). Pseudomeasures can have no singularities worse than first-order poles; cf. (3.2) below.

2. Proof of Proposition 1.2

The proof consists of two parts.
(i) Let \(f(z) = \sum a_n/n^z\) with \(a_n \geq 0\) as in (1.1) satisfy the real condition (1.5). Setting \(x = x_N = 1 + 1/\log N\), one finds that
\[
\sigma_N \overset{\text{def}}{=} \sum_{n \leq N} \frac{a_n}{n} \leq e \sum_{n \leq N} \frac{a_n}{n} e^{-(\log n)/\log N} \leq ef(x_N) = \mathcal{O}(\log N).
\]
A crude estimate now gives the result of (1.6):
\[
s_N = \sum_{1}^{N} n \frac{a_n}{n} \leq N\sigma_N = \mathcal{O}(N \log N).
\]
(ii) For the second part we use an example.

Lemma 2.1. Let
\[
a_n = \begin{cases} 
2^{2^k+k} & \text{for } n = 2^{2^k}, \ k = 1, 2, \ldots, \\
0 & \text{for all other } n.
\end{cases}
\]
Then
\[
f(x) = \sum \frac{a_n}{n^x} = \mathcal{O}\{1/(x - 1)\} \quad \text{as } x \searrow 1,
\]
but
\[
\text{for } N = 2^{2^k}, \quad \text{one has } s_N \geq a_N = (1/\log 2)N \log N.
\]
Proof. Take \(x = 1 + \delta\) with \(0 < \delta < 1\). Then
\[
f(x) = \sum \frac{2^{2^k+k}}{2^{2^k_x}} = \sum \frac{2^k}{2^{2^k\delta}}.
\]
Observe that the graph of
\[
h(t) = \frac{2^t}{2^{2^k\delta}} \quad (0 < t < \infty)
\]
is rising to a maximum at some point \( t = t_0(\delta) \) and then falling. Thus the sum for \( f(x) \) is majorized by the integral of \( h(t) \) over \((0, \infty)\) plus the value \( h(t_0) \). Both have the form \( \text{const}/\delta \), hence (2.2).

Now take \( N \) of the form \( 2^{2^k} \), so that \( \log N = 2^k \log 2 \). Then
\[
a_N = 2^{2^k + k} = N2^k = N(\log N)/\log 2.
\]

\[\square\]

3. Proof of Theorem 1.3

Note that the distributional convergence in the theorem is convergence in the Schwartz space \( \mathcal{S}' \). In other words,
\[(3.1) \quad \langle q(x + iy), \phi(y) \rangle \to \langle q(1 + iy), \phi(y) \rangle \quad \text{as } x \downarrow 1\]
for all testing functions \( \phi(y) \in \mathcal{S} \), that is, all rapidly decreasing \( C^\infty \) functions; see Schwartz [9] or Hörmander [3].

**Proof of Theorem 1.3.** Let \( f(z) \) be the sum of the Dirichlet series in the theorem. Now define \( s(v) = \sum_{n \leq v} a_n \), so that \( s(v) = 0 \) for \( v < 0 \) and \( s_N = s(N) = \mathcal{O}(N^{1+\varepsilon}) \) for every \( \varepsilon > 0 \). Integrating by parts, one obtains a representation for \( q(z) = f(z)/z \) as a Mellin transform:
\[
q(z) = (1/z) \int_{1-}^{\infty} v^{-z} ds(v) = \int_{1}^{\infty} s(v)v^{-z-1}dv \quad (x > 1).
\]
The substitution \( v = e^t \) gives \( q(z) \) as a shifted Laplace transform of \( S(t) = e^{-t}s(e^t) \):
\[
q(z) = \int_{0}^{\infty} s(e^t)e^{-zt}dt = \int_{0}^{\infty} S(t)e^{-(z-1)t}dt \quad (x > 1).
\]

(i) Suppose that the sequence \( \{s_N/N\} \) is bounded. Then \( S(t) \) is bounded, \( |S(t)| \leq M \), say. Hence
\[(3.2) \quad |q(z)| \leq \frac{M}{x - 1} \quad \text{for } x > 1.
\]

Thus the boundary singularities of \( q(z) \) on the line \( \{x = 1\} \) can be no worse than first-order poles. We will verify that in the sense of distributions,
\[
q(x + iy) \to q(1 + iy) \overset{\text{def}}{=} \hat{S}(y),
\]
where \( \hat{S}(y) \) denotes the distributional Fourier transform of \( S(t) \). Indeed, for fixed \( x > 1 \), the function \( q(x + iy) \) is the Fourier transform of \( S_x(t) = S(t)e^{-(x-1)t} \). Since \( |S(t)| \leq M \) and \( S(t) = 0 \) for \( t < 0 \), the functions
$S_x(t)$ converge to $S(t)$ boundedly as $x \searrow 1$, hence in the sense of tempered distributions. Since distributional Fourier transformation is continuous on the Schwartz space $\mathcal{S}'$, it follows that $q(x+iy)$ converges to the distributional Fourier transform of $S(t)$ – in this case a pseudomeasure.

(ii) Conversely, suppose that $q(x+iy) = \hat{S}_x(y)$ converges to a pseudomeasure as $x \searrow 1$, symbolically written as $q(1+iy)$. Then $q(1+iy)$ is the Fourier transform $\hat{H}(y)$ of a bounded function $H(t)$. By the continuity of inverse Fourier transformation, this implies that $H(t)$ is the distributional limit of $S_x(t) = S(t)e^{-(x-1)t}$ as $x \searrow 1$. But the latter limit is equal to $S(t)$:

$$<S_x(t), \phi_0(t)> = \int_0^\infty S_x(t)\phi_0(t)dt \to <S(t), \phi_0(t)>$$

for all $C^\infty$ functions $\phi_0(t)$ of compact support. It follows that $S(t) = H(t)$ on $\mathbb{R}$, hence bounded. □

4. Final remarks

Let $\pi_2(N)$ denote the number of prime twins $(p, p+2)$ with $p \leq N$. The famous twin-prime conjecture (TPC) of Hardy and Littlewood [2] asserts that for $N \to \infty$,

$$\pi_2(N) \sim 2C_2 \text{li}_2(N) = 2C_2 \int_2^N \frac{dt}{\log^2 t} \sim 2C_2 \frac{N}{\log^2 N}.$$  

Here $C_2$ is the ‘twin-prime constant’,

$$C_2 = \prod_{p \text{ prime}, p>2} \left\{ 1 - \frac{1}{(p-1)^2} \right\} \approx 0.6601618.$$  

For the discussion of the TPC it is convenient to introduce the modified counting function

$$\psi_2(N) \overset{\text{def}}{=} \sum_{n \leq N} \Lambda(n)\Lambda(n+2),$$

where $\Lambda(k)$ denotes von Mangoldt’s function. Since $\Lambda(k) = \log p$ if $k = p^a$ for some prime number $p$ and $\Lambda(k) = 0$ otherwise, the TPC turns out to be equivalent to the asymptotic relation

$$\psi_2(N) \sim 2C_2N \quad \text{as} \quad N \to \infty.$$
It is natural then to introduce the Dirichlet series

\[
(4.4) \quad D_2(z) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{\Lambda(n)\Lambda(n + 2)}{n^z} \quad (z = x + iy, \ x > 1).
\]

By a sieving argument, cf. Halberstam and Richert [1], one has \(\pi_2(N) = O(N / \log^2 N)\), or equivalently, \(\psi_2(N) = O(N)\). By Theorem 1.3 another equivalent statement is that the quotient \(D_2(x + iy)/(x + iy)\) converges distributionally to a pseudomeasure as \(x \searrow 1\). And finally, by the two-way Ikehara–Wiener theorem referred to in Section 1, the TPC is equivalent to the conjecture that the difference \(D_2(z) - 2C_2/(z - 1)\) has local pseudofunction boundary behavior as \(x \searrow 1\).

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