Path Integral Formulation for Lévy Flights - Evaluation of the Propagator for Free, Linear and Harmonic Potentials in the Over- and Underdamped Limits

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Lévy flights can be described using a Fokker-Planck equation, which involves a fractional derivative operator in the position co-ordinate. Such an operator has its natural expression in the Fourier domain. Starting with this, we show that the solution of the equation can be written as a Hamiltonian path integral. Though this has been realized in the literature, the method has not found applications as the path integral appears difficult to evaluate. We show that a method in which one integrates over the position co-ordinates first, after which integration is performed over the momentum co-ordinates, can be used to evaluate several path integrals that are of interest. Using this, we evaluate the propagators for (a) free particle (b) particle subjected to a linear potential and (c) harmonic potential. In all the three cases, we have obtained results for both over damped and under damped cases.

I. INTRODUCTION

The field of anomalous transport has gained widespread interest due to its potential to explain several phenomena which fall outside the realm of simple Brownian motion [1–17]. The continuous time random walk (CTRW) is a model used to understand anomalous transport [18–21] where jump lengths and waiting time distributions are both drawn from a probability distribution function \( \psi(x, t) \). The jump length distribution is given by

\[
\lambda(x) = \int_0^\infty dt \, \psi(x, t),
\]

and the waiting time distribution is given by

\[
w(t) = \int_{-\infty}^\infty dx \, \psi(x, t).
\]

Lévy flights belong to a special class of CTRW [20, 22–25], for which, \( \psi(x, t) = \lambda(x)w(t) \). The waiting time distribution, \( w(t) \), is narrow, and this makes the process Markovian in nature [24]. The probability distribution for the jump length, \( \lambda(x) \), is Lévy stable. The easiest way to specify it is to use its Fourier transform. The most general representation of the Fourier transform of a Lévy stable distribution is given by [24],

\[
\tilde{L}_{\alpha, \beta}(p; \mu, \sigma) = \frac{e^{-\sigma |p|^\alpha}}{2\pi} \left( 1 - i\beta \frac{p}{|p|} \omega(p, \alpha) \right) + i\mu p,
\]

where,

\[
\omega(p, \alpha) = \tan \left( \frac{\pi \alpha}{2} \right), \quad \text{if} \, \alpha \neq 1
\]

\[
= -\frac{2}{\pi} \ln |p|, \quad \text{if} \, \alpha = 1
\]
and $0 < \alpha < 2$ is the Lévy index, $-1 \leq \beta \leq 1$ is the skewness parameter, $\mu \in \mathbb{R}$ is the shift parameter, and $\sigma \in \mathbb{R}^+$ determines the strength of the noise. We will write the position representation for a Lévy stable distribution as $L_{\alpha,\beta}(x; \mu, \sigma)$. This distribution has the property that it can be rewritten as

$$L_{\alpha,\beta}(x; \mu, \sigma) = \frac{1}{\sigma} L_{\alpha,\beta} \left( \frac{x-\mu}{\sigma}; 0, 1 \right).$$

(5)

The shift and strength parameters $(\mu, \sigma)$ have been absorbed into just one term in the argument. We will adopt an easy to write notation for the Lévy stable distribution, which is, $\frac{1}{\sigma} L_{\alpha,\beta} \left( \frac{x-\mu}{\sigma}; 0, 1 \right)$. If we consider the such a jump length distribution, then the Fourier transform of $\lambda(x)$ is

$$\hat{\lambda}(p) = e^{-\sigma^\alpha |p|^\alpha}.$$

(6)

The Lévy stable distribution has the following long-tailed behavior for large $x$

$$\lambda(x) \sim \sigma |x|^{-1-\alpha}, \ x \to \pm \infty.$$

(7)

The Lévy stable distribution reduces to a Gaussian for $\alpha = 2$. Unlike the Gaussian distribution, the Lévy stable distribution does not obey the usual central limit theorem. However, it obeys the Lévy-Gnedenko generalized central limit theorem and is stable under addition [26, 27].

An inverse power-law asymptotic behavior for the jump length distribution [Eq. (7)], leads to a diverging mean square displacement (MSD) at all times:

$$\langle x^2(t) \rangle = \infty.$$

(8)

This is in contrast to Brownian motion where the MSD is finite and is proportional to time. This is because, the Brownian walker takes small steps at each interval of time. On the other hand, a Lévy flier takes small steps interrupted by very long jumps (flights) in between. As a result, the variance in the step size is infinitely large [20]. Lévy flights are observed in a variety of phenomena. Search strategies of bacteria and various birds and animals, where an occasional long jump interspersed by short steps offers them significant advantages [6–11, 28], hopping on a polymer chain [12–14], diffusion in micelles [15], optical transport in a Lévy glass [16], and energy diffusion in single-molecule spectroscopy [17] are a few examples. Lévy flights have also been used to model encounters between different species (predator-prey) [29].

Lévy flights can be described by a generalization of the Fokker-Planck equation, referred to as the fractional Fokker-Planck equation (FFPE) [20, 30–35]. The FFPE for Lévy flight in the presence of an external potential $V(x)$ is given by

$$\frac{\partial P(x,t)}{\partial t} = \left\{ -D \left( -\frac{\partial^2}{\partial x^2} \right)^{\alpha/2} + \frac{\partial}{\partial x} \frac{V'(x)}{m\gamma} \right\} P(x,t).$$

(9)

In the above, $D$ is the generalized diffusion constant with dimensions $[D] = \text{meter}^{\alpha} \text{sec}^{-1}$, $V(x)$ is the potential, $\gamma$ is the friction constant, and $m$ is the mass of the particle. The solutions of the FFPE for free Lévy flight and for Lévy flight in the presence of a linear potential and a harmonic potential in the overdamped limit have been given by Jespersen et al. [36]. In their paper, they use the FFPE in the Fourier domain. For a free Lévy flight, it is

$$\frac{\partial \hat{P}_{\text{free}}(k,t)}{\partial t} = -|k|^\alpha \hat{P}_{\text{free}}(k,t),$$

(10)
which can be solved to get
\[
\tilde{P}_{\text{free}}(k,t) = e^{-D|k|^\alpha t} \tilde{P}_{\text{free}}(k,0).
\] 
(11)

Imposing the initial condition \(P_{\text{free}}(x,0) = \delta(x)\) leads to \(\tilde{P}_{\text{free}}(k,t) = e^{-D|k|^\alpha t}\) which on Fourier transformation leads to the solution
\[
P_{\text{free}}(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \ e^{-D|k|^\alpha t} e^{ikx} = \frac{1}{(Dt)^{1/\alpha}} L_{\alpha,0} \left( \frac{x}{(Dt)^{1/\alpha}} \right).
\] 
(12)

For a Lévy flight subjected to a linear potential of the form \(V(x) = -F_0 x\), the FFPE is
\[
\frac{\partial \tilde{P}_{\text{lin}}(k,t)}{\partial t} = \left( -D|k|\alpha - ik\frac{F_0}{\gamma m} \right) \tilde{P}_{\text{lin}}(k,t)
\] 
and the solution subject to the same initial condition as before is
\[
\tilde{P}_{\text{lin}}(k,t) = e^{-t \left( ik \frac{F_0}{\gamma m} + D|k|^\alpha \right)}.
\] 
(14)

This leads to
\[
P_{\text{lin}}(x,t) = P_{\text{free}} \left( x - \frac{F_0 t}{\gamma m}, t \right) = \frac{1}{(Dt)^{1/\alpha}} L_{\alpha,0} \left( \frac{x - \frac{F_0 t}{\gamma m}}{(Dt)^{1/\alpha}} \right).
\] 
(15)

Finally, for Lévy flight under a harmonic potential of the form \(V(x) = \frac{1}{2} \lambda x^2\), the FFPE is
\[
\frac{\partial \tilde{P}_{\text{har}}(k,t)}{\partial t} = -\frac{\lambda}{\gamma m} k \frac{\partial}{\partial k} \tilde{P}(k,t) - D |k|^\alpha \tilde{P}_{\text{har}}(k,t).
\] 
(16)

One now has to employ the method of characteristics [36] to obtain the solution
\[
\tilde{P}_{\text{har}}(k,t) = \exp \left( \frac{-Dm\gamma |k|^\alpha}{\alpha \lambda} \left( 1 - e^{-\alpha \lambda t/\gamma m} \right) \right).
\] 
(17)

As a result,
\[
P_{\text{har}}(x,t) = \left( \frac{\alpha \lambda}{Dm\gamma (1 - e^{-\alpha \lambda t/\gamma m})} \right)^{1/\alpha} L_{\alpha,0} \left( \frac{x}{(Dm\gamma (1 - e^{-\alpha \lambda t/\gamma m})/(\alpha \lambda))^{1/\alpha}} \right).
\] 
(18)

The path integral approach has found extensive applications in the theory of Brownian motion. As Brownian motion is a special case of Lévy flights considered here, one would expect path integrals to be very useful for Levy flights too. For example, a path integral approach for CTRW was discussed by Fredrich and Eule [37] and by Calvo et al. [34] for fractional Brownian motion. However, though the path integral for a Lévy processes has been given earlier, it has never been used for evaluating a probability distribution associated with these processes. An interesting extension of quantum mechanics was suggested by Laskin [38–43]. It consists of modifying the Schroedinger equation so that one has the operator \((-\partial^2/\partial x^2)^{\alpha/2}\) in place of the usual operator \(-\partial^2/\partial x^2\). He has used a path integral approach and has evaluated some of the associated path integrals. Our approach in the following is similar to that of Laskin, and can be used to evaluate Laskin-type path integrals. Essentially, we have the same kind of path integral, but our functionals are all real, as we are
concerned with real stochastic processes and not a generalization of quantum mechanics.

There has also been an enormous amount of interest in barrier crossing by a Brownian particle since the seminal work of Kramers (see the review by Hanggi et al. [44]). More recently, the escape of particles acted upon by fractional noise was investigated in Refs. [44, 45]. Barrier crossing problem for Lévy flights has been discussed by a few authors [33, 46, 47].

In this paper we will use a Hamiltonian path integral approach for Lévy flights. This type of path integrals were introduced in the context of quantum mechanics, long ago by Garrod [48]. In these, the integration is over all paths in phase space, which are not continuous. If the action is quadratic in the momentum variable \( p \), then the momentum path integral can be done easily to get an action containing only the position and its derivative, and one gets the more common integral over paths in position space. In the case of path integrals for Lévy flights that we consider, the action expressed in phase space variables is not quadratic in \( p \), but it has the term \( |p|^\alpha \), where \( \alpha \) is a number \( (0 < \alpha < 2) \) [See Eq. (25)]. Integrals over \( p \) then lead to symmetric Lévy stable distribution, and hence one has a path integral involving a product of \( N(\rightarrow \infty) \) such distributions, which appears quite complex. It may be because of this that the path integrals for these processes has not been pursued in the literature. We show that an alternate approach in which, one first integrates over all the intermediate position variables and then integrates over the momentum variables, is quite powerful and allows us to evaluate several path integrals.

Lévy flights in the overdamped limit have been explored by other authors [20, 36, 49]. Many of the results that we find in this limit, using path integrals, have been obtained by other methods. Our method, however, allows us to evaluate the propagator for Lévy flight in a harmonic potential with a time-dependent force constant which has not been obtained by solving the FFPE. The underdamped limit for a free Lévy flight has also been studied, but only to a limited extent by earlier authors. Some of the propagators in the underdamped limit for the special case of Lévy noise with \( \alpha = 1 \) (Cauchy noise) has been obtained by West and Seshadri [50] and Garbaczewski and Olkiewicz [51]. Srokowski [49] considers a case where the process is driven by an Ornstein-Uhlenbeck equivalent for the Lévy noise and not by the Lévy noise itself. Both additive and multiplicative noise are considered in his paper. However, in the underdamped limit, the propagator for only the free Lévy flight is obtained, that too under certain conditions. Using our strategy, we are able to obtain the most general propagator in the underdamped limit for free Lévy flight and for Lévy flight in a linear and a harmonic potential. Our results reproduce the existing results under appropriate conditions. Further, the fact that one can evaluate path integral for a time-dependent harmonic potential exactly, has prompted us to study “semiclassical”-like approximations for path integrals, which can be very useful in the analysis of Kramers-like barrier crossing problem for Levy flights. We shall explore this in a forthcoming publication [52].

II. THE HAMILTONIAN PATH INTEGRAL

In this section we shall write a path integral expression for the propagator for Levy flight, subject to an arbitrary potential \( V(x) \). We start with the characteristic functional for a noise \( \eta(t) \) defined by

\[
P[p] = \langle e^{i \int_0^T dt p(t) \eta(t)} \rangle.
\]

The expectation value in Eq. (19) may be written as a path integral over the noise variable, \( \eta(t) \), as

\[
P[p] = \int D\eta \ e^{i \int_0^T dt p(t) \eta(t)} P[\eta].
\]
The probability density functional for $\eta(t)$ is

$$P[\eta] = \int_{-\infty}^{\infty} dp \, e^{-i\int_0^T dt \, p(t)\eta(t)} e^{-D \int_0^T dt |p(t)|^\alpha}. \quad (22)$$

We shall make use of the discretized version of the above integral, which may be written as

$$P(\eta_1, \eta_2, \ldots, \eta_N) = \left(\frac{1}{2\pi}\right)^N \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 \cdots \int_{-\infty}^{\infty} dp_N \exp \left(-\Delta t \sum_{n=1}^{N} \left[i p_n \eta_n + D |p_n|^\alpha \right]\right). \quad (23)$$

where we have divided the time interval $(0, T)$ into $N(\to \infty)$ equal intervals, each of length $\Delta t$ and also, $\eta_n = \int_{t_{n-1}}^{t_n} dt \, \eta(t)$. $\eta_n$ are independent Levy variables with the probability distribution for $\eta_n$ being given by $P(\eta_n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp_n \exp(-D \Delta t |p_n|^\alpha - i p_n \eta_n \Delta t)$.

The quantity that we are interested in is the probability density that a particle of mass $m$ executing a Levy flight and starting at $x_0$ at the time $t = 0$ would reach $x_f$ at the time $t = T$. The particle is assumed to obey the stochastic differential equation

$$\frac{dx}{dt} + \frac{V'(x)}{m\gamma} = \eta(t). \quad (24)$$

$\eta(t)$ is assumed to be a white Levy process, having the characteristic functional $\mathcal{P}[p] = e^{-D \int_0^T dt |p(t)|^\alpha}$, which specifies all its properties. Using the above equation, and Eq. (22), we can write the probability density functional for $x(t)$ as

$$P[x] = \int Dp \, J \, e^{-D \int_0^T dt |p(t)|^\alpha} e^{-i\int_0^T dt \, p(t) \left(\frac{x - V'(x)}{m\gamma}\right)}, \quad (25)$$

$J$ is the Jacobian involved in the transformation from $\eta(t)$ to $x(t)$. The propagator $P(x_f, T; x_0, 0)$ can now be written as

$$P(x_f, T; x_0, 0) = \int_{x(0)=x_0}^{x(T)=x_f} Dx \int Dp \, J \, e^{-D \int_0^T dt |p(t)|^\alpha} e^{-i\int_0^T dt \, p(t) \left(\frac{x - V'(x)}{m\gamma}\right)}. \quad (26)$$

At this point, it should be noted that the above equation resembles the Hamiltonian path integral of quantum mechanics, which was introduced long ago by Garrod [48]. It is better if one also introduces the discretized version of the above path integral, so that it is possible to check for the correctness of the numerical factors, if necessary. It is convenient to take the discretized version of
Eq. (24) to be given by

\[ \frac{x_n - x_{n-1}}{\Delta t} + \frac{V'(x_{n-1})}{m\gamma} = \eta_n. \]  

(27)

With this choice of discretization, the Jacobian for the transformation \( J \) becomes unity, and the discretized version reads

\[ P(x_f, T; x_0, 0) = \lim_{N \to \infty} \left( \frac{1}{2\pi} \right)^N \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_{N-1} \int_{-\infty}^{\infty} dp_1 \cdots \int_{-\infty}^{\infty} dp_N \ e^{-D\Delta t \sum_{n=1}^{N} |p_n|^\alpha} \times e^{-i \sum_{n=1}^{N} p_n \left( \frac{x_n - x_{n-1}}{\Delta t} + \frac{V'(x_{n-1})}{m\gamma} \right)} \Delta t. \]

(28)

Barkai et al. [54] have derived a fractional generalization of the Feynman-Kac equation for functionals of sub-diffusive continuous-time random walks. They have also derived a backward equation and a generalization to Lévy flights. Solutions are presented for a wide number of applications including the occupation time in half space and in an interval, the first passage time, the maximal displacement, and the hitting probability. For the particular class of processes that we are considering, it is quite easy to derive such an equation [Eq. (9) of this paper] using the above path integral. The details of this calculation is given in Appendix A. In the following, we shall write Eq. (28) as

\[ P(x_f, T; x_0, 0) = \int_{x(0)=x_0}^{x(T)=x_f} \mathcal{D}x \int \mathcal{D}p \ e^{-D \int_0^T dt |p(t)|^\alpha} e^{-i \int_0^T dt p(t) \left( \dot{x} + \frac{V'(x)}{m\gamma} \right)}. \]

(29)

Note that we have adopted the Ito discretization, which is very convenient for the problem as there is no contribution to the Jacobian from the potential, \( V(x) \), in Eq. (27). The alternate Stratanovich version of discretization is

\[ \frac{x_n - x_{n-1}}{\Delta t} + \frac{V'(x_n) + V'(x_{n-1})}{2m\gamma} = \eta_n \]

(30)

and results in a more complicated expression for the Jacobian [55].

If \( \alpha = 2 \), then the integrals over \( p_n \)s in Eq. (28) are simple Gaussian integrals and can be done easily to get the usual path integrals for Brownian motion. For other values of \( \alpha \), each integral over \( p_n \) will result in a stable distribution of the form

\[ \int_{-\infty}^{\infty} dp_n \ e^{-D\Delta t |p_n|^\alpha} e^{-ip_n \left( \frac{x_n - x_{n-1}}{m\gamma} + \frac{V'(x_{n-1})}{m\gamma} \Delta t \right)} = \frac{2\pi}{(D\Delta t)^{1/\alpha}} \mathcal{L}_{\alpha,0} \left( \frac{x_n - x_{n-1} + \frac{V'(x_{n-1})}{m\gamma} \Delta t}{(D\Delta t)^{1/\alpha}} \right). \]

(31)

Therefore, Eq. (28) will become a product of \( N \) such stable distributions, where \( N \to \infty \) and this product has to be integrated over \( x_n \), which seems formidable. This is probably the reason why a path integral approach for Lévy flights has been used rarely in the literature.

Our approach to the path integral in Eq. (29) is simple. If \( V(x) \) is at the most a quadratic function of \( x \), then it is possible to first integrate over the position co-ordinates exactly leaving us with a product of Dirac delta functions involving the \( p_n \)s. The integrals over \( p_n \) can then be performed easily to get the final answer. Using the procedure delineated here, we will obtain \( P(x_f, T; x_0, 0) \) for the free Lévy flight and Lévy flight in the presence of linear and harmonic potentials and thereby, reproduce the results of Jespersen et al. [36]. For the harmonic potential, we will present results for both time-independent as well as time-dependent force constants. The
result for time-independent force constant is given in the paper by Jespersen et al. [36]. However, the propagator for the Lévy flight in a harmonic potential with a time-dependent force constant is a new result. Also, using the same approach, we will obtain the most general propagator in the underdamped limit for free Lévy flight and for Lévy flight in a linear potential and a harmonic potential with a time-independent force constant, which are also new results. The procedure can also be applied to find “semiclassical” approximations for more complicated \( V(x) \) and can be used to analyze the Kramers problem for a Levy particle [52]. We adopt the following notations: In the overdamped limit, the propagator will have the superscript ‘od’ (\( P^{\text{od}} \)), and in the underdamped limit, it has the superscript ‘ud’ (\( P^{\text{ud}} \)). We also write \( p(T) = p_T \) and \( p(0) = p_0 \).

III. APPLICATION TO VARIOUS POTENTIALS IN THE OVERDAMPED LIMIT

A. Free Lévy flight

For a free particle, \( V(x) = 0 \). For this case, Eq. (28) becomes

\[
P^{\text{od}}_{\text{free}}(x_f, T; x_0, 0) = \lim_{N \to \infty} \left( \frac{1}{2\pi} \right)^N \int_{-\infty}^{\infty} dp_N \prod_{n=1}^{N-1} \int_{-\infty}^{\infty} dp_n e^{-D\Delta t \sum_{n=1}^{N} |p_n|^\alpha} e^{-ip_N x_f} e^{ip_1 x_0} \times \int_{-\infty}^{\infty} \prod_{n=1}^{N-1} dx_n e^{i(p_{n+1}-p_n)x_n}. \tag{32}
\]

After performing integrals over all \( x_i \)s, the resulting expression is

\[
P^{\text{od}}_{\text{free}}(x_f, T; x_0, 0) = \lim_{N \to \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} dp_N e^{-D|p_N|^\alpha \Delta t} e^{-ip_N x_N} \left( \prod_{n=1}^{N-1} \int_{-\infty}^{\infty} dp_n e^{-D|p_n|^\alpha \Delta t} \delta(p_{n+1}-p_n) \right) e^{ip_1 x_0}. \tag{33}
\]

Integrating over \( p_n \) with \( n = 1 \) to \( N-1 \) gives

\[
P^{\text{od}}_{\text{free}}(x_f, T; x_0, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp_N e^{-DT|p_N|^\alpha} e^{-ip_N(x_f-x_0)}, \tag{34}
\]

which leads to [36]

\[
P^{\text{od}}_{\text{free}}(x_f, T; x_0, 0) = \frac{1}{(DT)^{1/\alpha}} L_{0,0} \left( \frac{x_f-x_0}{(DT)^{1/\alpha}} \right). \tag{35}
\]

In the following, one can proceed with similar discretized integrals to evaluate the propagator for other problems. The discretized version of the path integral is tedious to perform, and it is far easier if one adopted a continuum version of the same integral. To illustrate the approach, we will do the above integral using the continuum approach. The approach has the difficulty that the propagator is determined only to within a multiplicative factor. However, this is not a problem as one can determine the factor using the normalization condition on the propagator. In all the further calculations, we will use the continuum version only. The propagator for free Lévy flight in the continuum version is

\[
P^{\text{od}}_{\text{free}}(x_f, T; x_0, 0) = \int_{x(0)=x_0}^{x(T)=x_f} Dx \int Dp \ e^{-D\int_0^T dt |p(t)|^\alpha} e^{-i \int_0^T \dot{p}(t) \dot{x}(t)}. \tag{36}
\]
On performing the integral in the exponent, \( \int_0^T dt \ p(t)\dot{x}(t) \), by parts, we get

\[
P_{free}^{od}(x_f, T; x_0, 0) = \int Dp \ e^{-D \int_0^T dt \ |p(t)|^\alpha \ e^{-i(pT x_f - p_0 x_0)}} \int_{x(0) = x_0}^{x(T) = x_f} Dx \ e^{i \int_0^T dt \ \dot{p}(t)x(t)}
\]

\[
= \int Dp \ e^{-D \int_0^T dt \ |p(t)|^\alpha \ e^{-i(pT x_f - p_0 x_0)}} \delta[\dot{p}(t)].
\]  

(37)

In the above equation, \( \delta[\dot{p}(t)] \) stands for Dirac delta functional, which results from the path integral over \( x \). Dirac delta functional implies that

\[
\dot{p}(t) = 0 \Rightarrow p(t) = p_0, \text{ a constant}.
\]  

(38)

On performing the integral over \( p(t) \), taking into account the delta functional and determining the multiplicative factor from the normalization condition, we obtain

\[
P_{free}^{od}(x_f, T; x_0, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp_0 \ e^{-D|p_0|^\alpha T e^{-ip_0(x_f - x_0)}} = \frac{1}{(DT)_{1/\alpha}} L_{\alpha,0}\left( \frac{x_f - x_0}{m\gamma T} \right).
\]  

(39)

This is the same result that we obtained using the discretized version of the path integral in Eq. (35).

**B. Linear potential**

We now analyze Lévy flight under the influence of a linear potential of the form \( V(x) = -F_0 x \).

The propagator is

\[
P_{lin}^{od}(x_f, T; x_0, 0) = \int_{x(0) = x_0}^{x(T) = x_f} Dx \int Dp \ e^{-D \int_0^T dt \ |p(t)|^\alpha \ e^{-i(pT x_f - p_0 x_0)}} \ e^{i \int_0^T dt \ p(t)\dot{x}(t)} \left( \dot{x}(t) - \frac{F_0}{m\gamma} \right).
\]  

(40)

On performing the integral in the exponent, \( \int_0^T dt \ p(t) \left( \dot{x}(t) - \frac{F_0}{m\gamma} \right) \), by parts,

\[
P_{lin}^{od}(x_f, T; x_0, 0) = \int Dp \ e^{-D \int_0^T dt \ |p(t)|^\alpha \ e^{-i(pT x_f - p_0 x_0)}} \ e^{i \int_0^T dt \ p(t)\dot{x}(t)} \int_{x(0) = x_0}^{x(T) = x_f} Dx \ e^{i \int_0^T dt \ \dot{p}(t)x(t)}
\]

(41)

Integrating over all the paths in position space with fixed end points, we get

\[
P_{lin}^{od}(x_f, T; x_0, 0) = \int Dp \ e^{-D \int_0^T dt \ |p(t)|^\alpha \ e^{-i(pT x_f - p_0 x_0)}} \ e^{i \int_0^T dt \ p(t)\dot{x}(t)} \delta[\dot{p}(t)].
\]  

(42)

The Dirac delta functional \( \delta[\dot{p}(t)] \), implies that

\[
\dot{p}(t) = 0 \Rightarrow p(t) = p_0, \text{ a constant}.
\]  

(43)

Performing the path integration over \( p(t) \) accounting for the delta functional and determining the multiplicative factor such that \( P_{lin}^{od} \) is normalized, gives

\[
P_{lin}^{od}(x_f, T; x_0, 0) = P_{free}^{od}(x_f - F_0 T/m\gamma, T; x_0, 0).
\]  

(44)

If \( x_0 = 0 \), this is just the result of Jespersen et al. presented in Eq. [15].
It may be noted that for $\mu \neq 0$ and $\beta = 0$ in Eq. (21), the equations will turn out to be exactly the same as Lévy flight in a linear potential. For the case where $\beta \neq 0$, this procedure, where we first integrate over position coordinates first, is still valid. As an example, let us consider free Lévy flight where $\beta \neq 0$ and $\mu \neq 0$. After performing the integration over position variables, we will be left with the integral

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dp_0 e^{-D|p_0|^\alpha T(1-\beta \frac{m_0}{\alpha} \omega(p_0, \alpha))} e^{-ip_0(x_f-x_0-\mu)} = \frac{1}{(DT)^{1/\alpha}} L_{\alpha, \beta} \left( \frac{x_f-x_0-\mu}{(DT)^{1/\alpha}} \right).$$  \hspace{1cm} (45)

The final distribution is Lévy stable, just that it is not symmetric, as expected.

C. Harmonic potential with a time-dependent Force Constant

We will now consider Lévy flight in a harmonic potential with a time-dependent force constant, for which the results have not been reported in the literature. It is interesting to note that our method is very easy when applied to this problem, while in comparison solving the corresponding fractional differential equation would be much more involved. We write the potential as $V(x) = \frac{1}{2} \lambda(t)x^2$ and find an analytical expression for the propagator using the Hamiltonian path integral approach. The path integral for the propagator is

$$P_{\text{har}}^{\text{od}}(x_f, T; x_0, 0) = \int_{x(0)=x_0}^{x(T)=x_f} D\phi \int_{x(0)=x_0}^{x(T)=x_f} Dp e^{-\frac{1}{2} \int_0^T dt \{ |p(t)|^2 + 2m_\gamma \phi(t) / \gamma \}} e^{i \int_0^T dt p(t) \left( \dot{x}(t) + \frac{\lambda(t)}{m_\gamma} x(t) \right)}.$$  \hspace{1cm} (46)

On performing the integral in the exponent, $\int_0^T dt p(t) \dot{x}(t)$, by parts,

$$P_{\text{har}}^{\text{od}}(x_f, T; x_0) = \int Dp e^{-\frac{1}{2} \int_0^T dt \{ |p(t)|^2 + 2m_\gamma \phi(t) / \gamma \}} \left[ \int_{x(0)=x_0}^{x(T)=x_f} D\phi \right] e^{i \int_0^T dt \left( \phi(t) - \frac{\lambda(t)}{m_\gamma} p(t) \right) x(t)}.$$  \hspace{1cm} (47)

The delta functional leads to the relation

$$\phi(t) - \frac{\lambda(t)}{m_\gamma} p(t) = 0 \Rightarrow p(t) = p_0 e^{\int_0^t dt' \lambda(t')/m_\gamma}.$$  \hspace{1cm} (48)

After performing the integration over the delta functional and making a change of integration variable $\bar{p}_0 = p_0 e^{\int_0^T dt' \lambda(t')/m_\gamma}$, the propagator, $P_{\text{har}}^{\text{od}}(x_f, T; x_0, 0)$ along with the multiplicative factor is

$$P_{\text{har}}^{\text{od}}(x_f, T; x_0) = \frac{1}{\left( D \int_0^T dt e^{-\alpha \int_0^T dt' \lambda(t')/m_\gamma} \right)^{1/\alpha}} L_{\alpha, 0} \left( \frac{x_f-x_0 e^{-\int_0^T dt' \lambda(t')/m_\gamma}}{\left( D \int_0^T dt e^{-\alpha \int_0^T dt' \lambda(t')/m_\gamma} \right)^{1/\alpha}} \right).$$  \hspace{1cm} (49)

For time-independent force constant, the result in Eq. (49) reduces to the following

$$P_{\text{har}}^{\text{od}}(x_f, T; x_0) = \frac{1}{\left( Dm_\gamma (1-e^{-\frac{\alpha \lambda T}{m_\gamma}}) / (\alpha \lambda) \right)^{1/\alpha}} L_{\alpha, 0} \left( \frac{x_f-x_0 e^{-\frac{\alpha \lambda T}{m_\gamma}}}{\left( Dm_\gamma (1-e^{-\frac{\alpha \lambda T}{m_\gamma}}) / (\alpha \lambda) \right)^{1/\alpha}} \right).$$  \hspace{1cm} (50)
This result is identical to that of Jespersen et al. in Eq. (18) for the initial condition \( x_0 = 0 \). Clearly as \( T \to \infty \) the above function approaches \( \frac{1}{(Dm\gamma)^{1/\alpha}} L_{0,0} \left( \frac{x_f}{(Dm\gamma)^{1/\alpha}} \right) \), showing that the probability distribution attains a steady value at infinite time. The average energy of the particle is given by the expression \( \frac{1}{2} \lambda \langle x_f^2 \rangle \). Since the mean square displacement of a Lévy distribution always diverges [Eq. (18)], the average energy of the particle also diverges.

**IV. Lévy Flight in the Underdamped Limit**

The use of our procedure, enables us to obtain the most general propagator in the underdamped limit, which is a new result. In the underdamped limit, the inertial term in the Langevin equation cannot be ignored and the motion is governed by the equation

\[
\frac{\ddot{x}(t)}{\gamma} + \dot{x}(t) + \frac{V'(x(t))}{m\gamma} = \eta(t).
\]  

(51)

Here also, we will use the Hamiltonian path integral formulation in its continuum version. We will present results for the free Lévy flight and also in the presence of a linear potential and a harmonic potential with a time-independent force constant.

**A. Free Lévy Flight**

The propagator for free Lévy flight in the underdamped regime is

\[
P_{\text{free}}^{ud}(x_f, t; x_0, 0) = \int_{x(0)=x_0}^{x(T)=x_f} Dx \int Dp \ e^{-D \int_0^T dt \ |p(t)|^\alpha} e^{-\int_0^T dt \ p(t) \left( \frac{\ddot{x}(t)}{\gamma} + \dot{x}(t) \right)}.
\]  

(52)

Integrating \( \int_0^T dt \ p(t) \left( \frac{\ddot{x}(t)}{\gamma} + \dot{x}(t) \right) \) in the exponent by parts we get

\[
\int_0^T dt \ p(t) \left( \frac{\ddot{x}(t)}{\gamma} + \dot{x}(t) \right) = \frac{p_T v_f - p_0 v_0 - \dot{p}_T x_f + \dot{p}_0 x_0}{\gamma} + p_T x_f - p_0 x_0 + \int_0^T dt \left( \frac{\ddot{p}(t)}{\gamma} - \dot{p}(t) \right) x(t),
\]  

(53)

where, \( v_0 \) and \( v_f \) are the initial and final velocities, respectively. Since the differential equation governing the motion is second order [Eq. (51)], the propagator will depend on the initial and final positions as well as the velocities. The propagator is

\[
P_{\text{free}}^{ud}(x_f, v_f, t; x_0, v_0, 0) = \int Dp \ e^{-D \int_0^T dt \ |p(t)|^\alpha} e^{-i \left( \frac{p_T v_f - p_0 v_0 - \dot{p}_T x_f + \dot{p}_0 x_0}{\gamma} + p_T x_f - p_0 x_0 + \int_0^T dt \left( \frac{\ddot{p}(t)}{\gamma} - \dot{p}(t) \right) x(t) \right)}
\]

\[
= \int Dp \ e^{-D \int_0^T dt \ |p(t)|^\alpha} e^{-i \left( \frac{p_T v_f - p_0 v_0 - \dot{p}_T x_f + \dot{p}_0 x_0}{\gamma} + p_T x_f - p_0 x_0 \right) \delta (\ddot{p}(t) - \gamma \dot{p}(t))}.
\]  

Dirac delta functional leads to the condition

\[
\ddot{p}(t) - \gamma \dot{p}(t) = 0,
\]  

(54)
which has the solution
\[ p(t) = c_1 + c_2 e^{\gamma t}, \tag{55} \]
where \( c_1 = \frac{p_T - p_0 e^{\gamma T}}{1 - e^{-\gamma T}} \) and \( c_2 = \frac{p_0 - p_T}{1 - e^{-\gamma T}} \). On performing the integral over the delta functional, the path integral over all \( p_T \) and \( p_0 \) reduces to just two integrals over \( p_T \) and \( p_0 \). Therefore, the unnormalized propagator is
\[
P^{ud}_{\text{free}}(x_f, v_f; T; x_0, v_0, 0) = \int_{-\infty}^{\infty} dp_0 \int_{-\infty}^{\infty} dp_T e^{-D f_T^0 dt} \left| p_T \frac{1 - e^{-\gamma T}}{1 - e^{-\gamma T}} + p_0 e^{\gamma T} - p_T (x_f - x_0) \right|^\alpha \exp \left[ -i \left\{ p_T \frac{v_f}{\gamma} - p_0 \frac{v_0}{\gamma} + \frac{p_0 e^{\gamma T} - p_T}{1 - e^{-\gamma T}} (x_f - x_0) \right\} \right].
\tag{56} \]
After normalizing, the propagator becomes
\[
P^{ud}_{\text{free}}(x_f, v_f; T; x_0, v_0, 0) = \frac{1}{4\pi^2 \gamma (1 - e^{-\gamma T})} \int_{-\infty}^{\infty} dp_0 \int_{-\infty}^{\infty} dp_T e^{-D f_T^0 dt} \left| p_T \frac{1 - e^{-\gamma T}}{1 - e^{-\gamma T}} + p_0 e^{\gamma T} - p_T (x_f - x_0) \right|^\alpha \exp \left[ -i \left\{ p_T \frac{v_f}{\gamma} - p_0 \frac{v_0}{\gamma} + \frac{p_0 e^{\gamma T} - p_T}{1 - e^{-\gamma T}} (x_f - x_0) \right\} \right].
\tag{57} \]
For \( \alpha = 2 \), the above expression corresponds to Brownian motion, the integrals can be performed exactly and, the propagator we obtain matches with the result given in the seminal paper by Chandrasekhar [56]. For \( \alpha \neq 2 \), it seems difficult to evaluate the double integral in Eq. (57). In order to simplify the evaluation of the propagator, we make a change of variables from \( \{p_0, p_T\} \) to \( \{q_1, q_2\} \), such that \( p_T = q_1 \) and \( p_0 = q_1 q_2 \). The details of this transformation are given in Appendix B. On making the change of variables, we get
\[
P^{ud}_{\text{free}}(x_f, v_f; T; x_0, v_0, 0) = \frac{1}{2\pi^2 \gamma (1 - e^{-\gamma T})} \int_{-\infty}^{\infty} dq_1 e^{-\Theta_1 |q_1|^\alpha} \cos (\Lambda_1 q_1) \tag{58} \]
where,
\[
\Theta_1 = D \int_0^T dt \left| e^{-\gamma t} - e^{-\gamma T} \right| + q_1 \frac{1 - e^{-\gamma t}}{1 - e^{-\gamma T}} \alpha,
\Lambda_1 = \left( \frac{v_f}{\gamma} - \frac{e^{-\gamma T} (x_f - x_0)}{1 - e^{-\gamma T}} \right) - q_2 \left( \frac{v_0}{\gamma} - \frac{(x_f - x_0)}{1 - e^{-\gamma T}} \right). \tag{59} \]
The integral over \( q_1 \) can be performed exactly for \( \alpha = 1 \) and written as
\[
P^{ud}_{\text{free}}(x_f, v_f; T; x_0, v_0, 0) = \frac{1}{2\pi^2 \gamma (1 - e^{-\gamma T})} \int_{-\infty}^{\infty} dq_2 \frac{\Theta_1^2 - \Lambda_1^2}{(\Theta_1^2 + \Lambda_1^2)^2}. \tag{60} \]
For other values of \( \alpha \), the integral over \( q_1 \) cannot be performed exactly. However, we expand the cosine function in Eq. (58) and then perform the integral over \( q_1 \), which results in
\[
P^{ud}_{\text{free}}(x_f, v_f; T; x_0, v_0) = \frac{1}{2\pi^2 \gamma (1 - e^{-\gamma T})} \int_{-\infty}^{\infty} dq_2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \Gamma \left( \frac{2(n + 1)}{\alpha} \right) \frac{\Lambda_1^{2n}}{\Theta_1^{2(n+1)/\alpha}}. \tag{61} \]
The sum does not converge for \( \alpha < 1 \). As \( T \to \infty \), \( P^{ud}_{\text{free}}(x_f, v_f; T; x_0, v_0) \to 0 \) because \( \Theta_1 \to \infty \) in this limit for any non-zero value of \( q_2 \).
More specific propagators can be derived from $P_{\text{free}}^{ud}(x_f, v_f, T; x_0, v_0, 0)$. For the case of Brownian motion, the propagators $P_{\text{free}}^{ud}(x_f, T; x_0, v_0, 0)$ and $P_{\text{free}}^{ud}(v_f, T; v_0, 0)$ were given in Chandrasekhar’s paper. Similar exact results can be obtained for Lévy flights using our formalism. The details of the calculation are provided in Appendix C. The propagator

$$P_{\text{free}}^{ud}(x_f, T; x_0, v_0, 0) = \int_{-\infty}^{\infty} dv_f P_{\text{free}}^{ud}(x_f, v_f, T; x_0, v_0, 0)$$

$$= \frac{1}{(1 - e^{-\gamma T})\Theta_2^{1/\alpha}} L_{\alpha,0} \left( \frac{\Lambda_2}{\Theta_2^{1/\alpha}} \right), \quad (62)$$

with

$$\Theta_2 = D \int_0^T dt \left( \frac{1 - e^{-\gamma t}}{1 - e^{-\gamma T}} \right)^\alpha = \frac{D B \left( 1 - e^{-\gamma T}; 1 + \alpha, 0 \right)}{\gamma (1 - e^{-\gamma T})^\alpha}, \quad (63)$$

and

$$\Lambda_2 = \frac{v_0}{\gamma} - \frac{x_f - x_0}{1 - e^{-\gamma T}}. \quad (64)$$

In the limit $T \to \infty$, $\Theta_2 \to \infty$ and therefore, in this limit, $P_{\text{free}}^{ud}(x_f, T; x_0, v_0, 0) \to 0$.

On integrating over all possible values of $x_f$ of $P_{\text{free}}^{ud}(v_f, x_f, T; v_0, 0)$, we get

$$P_{\text{free}}^{ud}(v_f, T; v_0, 0) = \int_{-\infty}^{\infty} dx_f P_{\text{free}}^{ud}(x_f, v_f, T; x_0, v_0, 0)$$

$$= \frac{1}{\Theta_3^{1/\alpha}} L_{\alpha,0} \left( \frac{\Lambda_3}{\Theta_3^{1/\alpha}} \right), \quad (65)$$

where

$$\Theta_3 = \frac{D}{\alpha \gamma^{1-\alpha}} \left( 1 - e^{-\alpha \gamma T} \right), \quad (66)$$

and

$$\Lambda_3 = v_f - v_0 e^{-\gamma T}. \quad (67)$$

The stationary distribution for $P_{\text{free}}^{ud}(v_f, T; x_0, v_0, 0)$ is obtained by letting $T \to \infty$, which gives

$$P_{\text{free, st}}^{ud}(v_f) = \frac{1}{(D/(\alpha \gamma^{1-\alpha}))^{1/\alpha}} L_{\alpha,0} \left( \frac{v_f}{(D/(\alpha \gamma^{1-\alpha}))^{1/\alpha}} \right). \quad (68)$$

This reduces to the Maxwell Boltzmann velocity distribution for $\alpha = 2$. $P_{\text{free}}^{ud}(x_f, T; x_0, v_0, 0)$ and $P_{\text{free}}^{ud}(v_f, T; v_0, 0)$ for the special case of Cauchy noise (i.e. $\alpha = 1$) have been obtained earlier \[50, 51\]. For $\alpha = 2$, we reproduce the expressions given in Chandrasekhar’s paper \[56\].
B. Linear Potential

The propagator for the Lévy flight under a linear potential of the form $V(x) = -F_0 x$ in the underdamped regime is

$$P_{\text{lin}}^{\text{ud}}(x_f, v_f, T; x_0, v_0) = \int_{x(0)=x_0}^{x(T)=x_f} Dx \int Dp \ e^{-D \int_0^T dt \ |p(t)|^\alpha} e^{-i \int_0^T dt \ p(t) \left( \frac{\dot{x}(t)}{\gamma} + \dot{x}(t) - \frac{F_0}{m \gamma} \right)}.$$

(69)

Since, $V'(x)$ is independent of the position in this case, the part of the above expression dependent on position is exactly similar to the case of free Lévy flight. On following the same procedure as we did for free Lévy flight, we obtain the normalized propagator to be

$$P_{\text{lin}}^{\text{nd}}(x_f, v_f, T; x_0, v_0) = \frac{1}{4\pi^2 \gamma (1 - e^{-T \gamma})} \int_{-\infty}^{\infty} dp_0 \int_{-\infty}^{\infty} dp_T \ e^{-D \int_0^T dt \ \frac{p_0 (1 - e^{-T \gamma} + p_T (e^{-T \gamma} + e^{-T \gamma})}{1 - e^{-T \gamma}}} \times$$

$$\exp \left[ -i \left\{ \frac{v_f - F_0}{m \gamma} - p_0 \frac{v_0 - F_0}{m \gamma} + p_0 - p_T e^{-T \gamma} \left( x_f - x_0 - \frac{F_0 T}{m \gamma} \right) \right\} \right].$$

(70)

From the above expression it is easy to see that the case of Lévy flight under a linear potential is equivalent to free Lévy flight with $v_f \to \left( v_f - \frac{F_0}{m \gamma} \right)$, $v_0 \to \left( v_0 - \frac{F_0}{m \gamma} \right)$ and $x_f \to \left( x_f - \frac{F_0 T}{m \gamma} \right)$. On making these substitutions in the propagators for free Lévy flight we can obtain the corresponding propagators for Lévy flight in a linear potential.

C. Harmonic Potential

In the underdamped limit of friction, we analyze Lévy flight in a harmonic potential of the form $V(x) = \frac{1}{2} \lambda x^2$, with a time-independent force constant. The propagator, $P_{\text{ud}}^{\text{har}}(x_f, v_f, T; x_0, v_0)$, is

$$P_{\text{ud}}^{\text{har}}(x_f, v_f, T; x_0, v_0) = \int Dx \int Dp \ e^{-D \int_0^T dt \ |p(t)|^\alpha} e^{-i \int_0^T dt \ p(t) \left( \frac{\dot{x}(t)}{\gamma} + \dot{x}(t) + \frac{\lambda x(t)}{m \gamma} \right)}.$$

(71)

Integrating $\int_0^T dt \ p(t) \left( \frac{\dot{x}(t)}{\gamma} + \dot{x}(t) + \frac{\lambda x(t)}{m \gamma} \right)$ in the exponent by parts gives

$$\int_0^T dt \ p(t) \left( \frac{\dot{x}(t)}{\gamma} + \dot{x}(t) + \frac{\lambda x(t)}{m \gamma} \right) = \frac{p_T v_f - p_0 v_0 - \dot{x}_f \dot{x}_0 + \ddot{x}_0 x_0}{\gamma} + p_T \dot{x}_f - p_0 x_0 +$$

$$\int_0^T dt \ \left( \frac{\ddot{x}(t)}{\gamma} - \ddot{x}(t) + \lambda \frac{\dot{x}(t)}{m \gamma} \right) x(t).$$

(72)

Using this result in the propagator we obtain,

$$P_{\text{ud}}^{\text{har}}(x_f, v_f, T; x_0, v_0) = \int Dp \ e^{-D \int_0^T dt \ |p(t)|^\alpha} e^{-i \left( \frac{p_T v_f - p_0 v_0 - \dot{x}_f \dot{x}_0 + \ddot{x}_0 x_0}{\gamma} + p_T \dot{x}_f - p_0 x_0 \right)} \times$$

$$\int Dx \ e^{-i \int_0^T dt \ \left( \frac{\ddot{x}(t)}{\gamma} - \ddot{x}(t) + \lambda \frac{\dot{x}(t)}{m \gamma} \right) x(t)}$$

(73)

$$= \int Dp \ e^{-D \int_0^T dt \ |p(t)|^\alpha} e^{-i \left( \frac{p_T v_f - p_0 v_0 - \dot{x}_f \dot{x}_0 + \ddot{x}_0 x_0}{\gamma} + p_T \dot{x}_f - p_0 x_0 \right)}$$

$$\delta \left[ \ddot{x}(t) - \gamma \ddot{x}(t) + \frac{\lambda}{m} \dot{x}(t) \right].$$

(74)
Delta functional implies,
\[ \dot{p}(t) - \gamma \dot{p}(t) + \frac{\lambda}{m} p(t) = 0 \]  
\[ : p(t) = e^{\frac{2\gamma t}{T}} \left( c_1 \cosh \left( \frac{\beta}{2} t \right) + c_2 \sinh \left( \frac{\beta}{2} t \right) \right). \]

where \( \beta = \sqrt{\gamma^2 - \frac{4\lambda}{m}} \), \( c_1 = p_0 \) and \( c_2 = \frac{p_T e^{-\frac{\gamma T}{2}} - p_0 \cosh \left( \frac{\beta T}{2} \right)}{\sinh \left( \frac{\beta T}{2} \right)} \). Though we have shown the working only for the case where \( \gamma^2 > \frac{4\lambda}{m} \), our analysis is equally valid in general. Integral over the delta functional will result in the in a double integral over \( p_T \) and \( p_0 \). The resultant unnormalized propagator is

\[
P_{\text{har}}^{ud}(x_f, v_f, T; x_0, v_0) = \int_{-\infty}^{\infty} dp_0 \int_{-\infty}^{\infty} dp_T e^{-D \int_0^T dt \left| \frac{p_T e^{-\gamma T} \sinh \left( \frac{\beta T}{T} \right) + p_0 \sinh \left( \frac{\beta (T-t)}{T} \right)}{\sinh \left( \frac{\beta T}{2} \right)} \right|^2} \times \\
e^{-i \left\{ p_T v_f - p_0 v_0 + p_T \left( \gamma - \beta \coth \left( \frac{\beta T}{2} \right) \right) + \gamma e \left( \frac{\gamma - \beta \coth \left( \frac{\beta T}{2} \right)}{2} \right) x_f - p_0 \left( \gamma + \beta \coth \left( \frac{\beta T}{2} \right) \right) \right\}_0}.
\]

Upon determining the normalization constant, the propagator is

\[
P_{\text{har}}^{ud}(x_f, v_f, T; x_0, v_0) = \frac{e^{\frac{2\gamma T}{T}} \beta}{8\pi^2 \gamma^2 \sinh \left( \frac{\beta T}{2} \right)} \int_{-\infty}^{\infty} dp_0 \int_{-\infty}^{\infty} dp_T e^{-D \int_0^T dt \left| \frac{p_T e^{-\gamma T} \sinh \left( \frac{\beta T}{T} \right) + p_0 \sinh \left( \frac{\beta (T-t)}{T} \right)}{\sinh \left( \frac{\beta T}{2} \right)} \right|^2} \times \\
e^{-i \left\{ \frac{p_T}{\gamma} \left( v_f + \frac{\gamma - \beta \coth \left( \frac{\beta T}{2} \right)}{2} x_f + e^{-\frac{\gamma T}{2} \beta \coth \left( \frac{\beta T}{2} \right) x_0} \right) + p_0 \left( -v_0 - \frac{\gamma + \beta \coth \left( \frac{\beta T}{2} \right)}{2} x_0 + e^{\frac{\gamma T}{2} \beta \coth \left( \frac{\beta T}{2} \right) x_0} \right) \right\}_0}.
\]

In order to obtain a more elegant expression for the propagator, we perform a simple change of variables where \( \tilde{p}_T = \frac{p_T}{\gamma} \), \( \tilde{p}_0 = -\frac{p_0 e^{-\gamma T}}{\gamma \sinh \left( \frac{\beta T}{2} \right)} \) and obtain

\[
P_{\text{har}}^{ud}(x_f, v_f, T; x_0, v_0) = \frac{\beta}{8\pi^2} \int_{-\infty}^{\infty} d\tilde{p}_0 \int_{-\infty}^{\infty} d\tilde{p}_T e^{-D \int_0^T dt \left| \frac{e^{-\gamma T} \left( \tilde{p}_0 \sinh \left( \frac{\beta T}{T} \right) + \tilde{p}_T \sinh \left( \frac{\beta (T-t)}{T} \right) \right)}{\sinh \left( \frac{\beta T}{2} \right)} \right|^2} \times \\
e^{-i \left\{ \tilde{p}_T \left( v_f + \frac{\gamma - \beta \coth \left( \frac{\beta T}{2} \right)}{2} x_f + e^{-\frac{\gamma T}{2} \beta \coth \left( \frac{\beta T}{2} \right) x_0} \right) + \tilde{p}_0 \left( -v_0 - \frac{\gamma + \beta \coth \left( \frac{\beta T}{2} \right)}{2} x_0 + e^{\frac{\gamma T}{2} \beta \coth \left( \frac{\beta T}{2} \right) x_0} \right) \right\}_0}.
\]

It is possible to get the stationary distribution by allowing \( T \to \infty \). Noting that we are considering the case \( \gamma > \beta \), and letting \( T \to \infty \) leads to

\[
P_{\text{har, st}}^{ud}(x_f, v_f) = \frac{\beta}{8\pi^2} \int_{-\infty}^{\infty} d\tilde{p}_0 \int_{-\infty}^{\infty} d\tilde{p}_T e^{-D \int_0^\infty dt \left| \frac{e^{-\gamma T} \left( \tilde{p}_0 \sinh \left( \frac{\beta T}{T} \right) + \tilde{p}_T e^{-\beta t/2} \right)}{\sinh \left( \frac{\beta T}{2} \right)} \right|^2} \times \\
e^{-i \left\{ \tilde{p}_T \left( v_f + \frac{\gamma - \beta \coth \left( \frac{\beta T}{2} \right)}{2} x_f \right) + \tilde{p}_0 \frac{\beta x_f}{2} \right\}_0}.
\]

The above equation implies that \( P_{\text{har, st}}^{ud}(x_f, v_f) \) is the two-dimensional Fourier transform of a well behaved function. This guarantees that the steady state distribution is well-behaved for any \( 2 \geq \alpha > 0 \).
In Eq. (79), after performing the coordinate transformation from \( \{ \tilde{p}_0, \tilde{p}_T \} \) to \( \{ q_1, q_2 \} \) as we did for the free particle, we get

\[
P_{\text{har}}^{ud}(x_f, v_f, T; x_0, v_0) = \frac{\beta}{4\pi^2\alpha} \int_{-\infty}^{\infty} dq_2 \int_{-\infty}^{\infty} dq_1 q_1 e^{-\Omega_1 |q_1|^\alpha} \cos(\chi_1 q_1). \tag{81}
\]

where,

\[
\Omega_1 = D \int_0^T dt \left| \gamma e^{-\gamma t/2} \left( \frac{\sinh \left( \frac{\beta(T-t)}{2} \right)}{\sinh \left( \frac{\beta T}{2} \right)} + q_2 \sinh \left( \frac{\beta T}{2} \right) \right) \right|^\alpha,
\]

\[
\chi_1 = \left( v_f + \frac{\gamma - \beta \coth \left( \frac{\beta T}{2} \right)}{2} x_f + e^{-\frac{\gamma T}{2}} \beta \csch \left( \frac{\beta T}{2} \right) \frac{x_0}{2} \right) + q_2 \left( -v_0 e^{-\frac{\gamma T}{2}} \sinh \left( \frac{\beta T}{2} \right) + \frac{\beta x_f}{2} - \gamma \sinh \left( \frac{\beta T}{2} \right) + \beta \cosh \left( \frac{\beta T}{2} \right) e^{-\frac{\gamma T}{2}} x_0 \right). \tag{82}
\]

For \( \alpha = 1 \), the integral over \( q_1 \) can be performed exactly resulting in the expression

\[
P_{\text{har}}^{ud}(x_f, v_f, T; x_0, v_0) = \frac{\beta}{4\pi^2} \int_{-\infty}^{\infty} dq_2 \frac{\Omega_1^2 - \chi_1^2}{(\Omega_1^2 + \chi_1^2)^2}. \tag{83}
\]

This enables us to calculate the time evolution of the probability distribution easily (see Fig. 1). For other values of \( \alpha \), for which the integral over \( q_1 \) cannot be performed exactly, we expand the cosine function in Eq. (81) as a sum and perform the integral over \( q_1 \). This results in

\[
P_{\text{har}}^{ud}(x_f, v_f, T; x_0, v_0) = \frac{\beta}{4\pi^2\alpha} \int_{-\infty}^{\infty} dq_2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \Gamma \left( \frac{2n+1}{\alpha} \right) \frac{\chi_1^{2n}}{\Omega_1^{2(n+1)/\alpha}}. \tag{84}
\]

This sum does not converge for \( \alpha < 1 \).

Specific forms of the propagator \( P_{\text{har}}^{ud}(x_f, T; x_0, v_0, 0) \) and \( P_{\text{har}}^{ud}(v_f, T; x_0, v_0, 0) \) can be obtained from \( P_{\text{har}}^{ud}(x_f, v_f, T; x_0, v_0, 0) \) by integrating over \( v_f \) and \( x_f \) respectively. \( P_{\text{har}}^{ud}(x_f, T; x_0, v_0, 0) \) is

\[
P_{\text{har}}^{ud}(x_f, T; x_0, v_0, 0) = \int_{-\infty}^{\infty} dv_f P_{\text{har}}^{ud}(x_f, v_f, T; x_0, v_0, 0) = \frac{1}{\Omega_2^{1/\alpha}} L_{\alpha,0} \left( \frac{\chi_2}{\Omega_2^{1/\alpha}} \right) \tag{85}
\]

where,

\[
\Omega_2 = D \int_0^T dt \left( \frac{2\gamma}{\beta} e^{-\gamma t/2} \sinh \left( \frac{\beta T}{2} \right) \right)^\alpha = D \frac{\gamma^\alpha}{\beta^{\alpha+1}} B \left( 1 - e^{-\beta T}; \alpha + 1, \frac{\alpha}{2} \left( \frac{\gamma}{\beta} - 1 \right) \right)
\]

\[
\chi_2 = x_f - \left( \cosh \left( \frac{\beta T}{2} \right) + \frac{\gamma}{\beta} \sinh \left( \frac{\beta T}{2} \right) \right) e^{-\frac{\gamma T}{2}} x_0 - \frac{v_0}{\beta} e^{-\frac{\gamma T}{2}} \sinh \left( \frac{\beta T}{2} \right). \tag{86}
\]

The stationary distribution for the propagator \( P_{\text{har}}^{ud}(x_f, T; x_0, v_0, 0) \) is

\[
P_{\text{har, st}}^{ud}(x_f) = \int_{-\infty}^{\infty} dv_f P_{\text{har}}^{ud}(x_f, v_f, T; x_0, v_0, 0) = \frac{1}{\Omega_{2,\alpha}^{1/\alpha}} L_{\alpha,0} \left( \frac{\chi_{2,\alpha}}{\Omega_{2,\alpha}^{1/\alpha}} \right) \tag{87}
\]
where,
\[ \Omega_{2,\text{st}} = D \int_0^\infty dt \left( \frac{2\gamma}{\beta} e^{-\gamma t/2} \sinh \left( \frac{\beta t}{2} \right) \right)^\alpha = D \frac{\gamma^\alpha}{\beta^{\alpha+1}} B \left( \frac{\alpha}{2}, \frac{\gamma}{\beta} - 1, \alpha + 1 \right), \]
\[ \chi_{2,\text{st}} = x_f. \]

Also,
\[ P_{\text{dir}}(v_f,T;x_0,v_0,0) = \int_{-\infty}^\infty dx_f \ P_{\text{dir}}(x_f,v_f,T;x_0,v_0,0) = \frac{\beta}{2\Omega_3^{1/\alpha}} L_{\alpha,0} \left( \frac{\chi_3}{\Omega_3^{1/\alpha}} \right) \]
where,
\[ \Omega_3 = D \int_0^T dt \left| \gamma e^{-\gamma t/2} \left( \cosh \left( \frac{\beta t}{2} \right) - \frac{\gamma}{\beta} \sinh \left( \frac{\beta t}{2} \right) \right) \right|^\alpha, \]
\[ \chi_3 = v_f + \left( \frac{\gamma}{\beta} \sinh \left( \frac{\beta T}{2} \right) - \cosh \left( \frac{\beta T}{2} \right) \right) e^{-\gamma T/2} v_0 + \left( \frac{(\gamma^2 - \beta^2) \sinh \left( \frac{\beta T}{2} \right)}{2\beta} \right) e^{-\gamma T/2} x_0. \]

The stationary distribution for the propagator \( P_{\text{dir}}(x_f,T;x_0,v_0,0) \) is
\[ P_{\text{dir},\text{st}}(v_f) = \int_{-\infty}^\infty dx_f \ P_{\text{dir}}(x_f,v_f,T;x_0,v_0,0) = \frac{\beta}{2\Omega_{3,\text{st}}^{1/\alpha}} L_{\alpha,0} \left( \frac{\chi_{3,\text{st}}}{\Omega_{3,\text{st}}^{1/\alpha}} \right) \]
where,
\[ \Omega_{3,\text{st}} = D \int_0^\infty dt \left| \gamma e^{-\gamma t/2} \left( \cosh \left( \frac{\beta t}{2} \right) - \frac{\gamma}{\beta} \sinh \left( \frac{\beta t}{2} \right) \right) \right|^\alpha, \]
\[ \chi_{3,\text{st}} = v_f. \]

For \( \alpha = 2 \), we obtain the results of Chandrasekhar \[56\].

The propagators in the case where \( \gamma^2 < \frac{4\lambda}{m} \) can be obtained from the expressions derived for \( \gamma^2 > \frac{4\lambda}{m} \) by replacing \( \cosh \left( \frac{\beta T}{2} \right) \) with \( \cos \left( \frac{\beta T}{2} \right) \) and \( \frac{1}{\beta} \sinh \left( \frac{\beta T}{2} \right) \) with \( \frac{1}{\beta} \sin \left( \frac{\beta T}{2} \right) \) where, \( \beta = \sqrt{\frac{4\lambda}{m} - \gamma^2} \).

For \( \gamma^2 = \frac{4\lambda}{m} \), the appropriate propagators can be obtained by replacing \( \cosh \left( \frac{\beta T}{2} \right) \) with \( \cos \left( \frac{\beta T}{2} \right) \) and \( \frac{1}{\beta} \sinh \left( \frac{\beta T}{2} \right) \) with \( \frac{1}{\beta} \sin \left( \frac{\beta T}{2} \right) \) in the expressions obtained for \( \gamma^2 > \frac{4\lambda}{m} \). The expectation value of the energy of a particle executing Lévy flight in a harmonic potential in the underdamped limit will be given by \( \frac{1}{2} m (v_f^2) + \frac{1}{2} \lambda (x_f^2) \). Since both the position and the velocity distributions are Lévy stable, as given in Eq. \[83\] and \[84\] respectively, both \( \langle x_f^2 \rangle \) and \( \langle v_f^2 \rangle \) diverge.

Phase space distributions for Lévy flights under a harmonic potential have been explored previously (see references \[57, 58\]). In the case usual Brownian motion, \( x_f \) and \( v_f \) are uncorrelated and one obtains elliptical contours for the phase space distribution. However, for Lévy flights this is no longer true - an inhomogeneous stationary phase space distribution was observed \[57, 58\]. Since, our method gives exact expressions for propagators at any time, we can study the time evolution of the phase space distribution, while the previous papers were concerned only with the steady state distribution function. In Fig. (1(a),(b),(c),(d)), we show the time evolution of the phase space distribution for \( \alpha = 1 \) with the initial conditions, \( v_f = 0 \) and \( x_f = 0 \). In Fig. (1(e)), we also show the stationary distribution calculated simply as the product of \( P_{\text{dir}}(x_f,T;v_0,x_0,0) \) and \( P_{\text{dir}}(v_f,T;v_0,x_0,0) \) in the large \( T \) limit. We observe that there is correlation between \( x_f \) and \( v_f \).
which does not die off as $T \to \infty$. Our findings are consistent with the inhomogeneity in phase space distribution reported in the literature \cite{57, 58}.

V. CONCLUSIONS

We have shown that Hamiltonian path integrals offer an appealing way to study Lévy flights in both the overdamped and underdamped limits of friction. Though some results are already available, the method adopted to obtain them is interesting. In addition, they enable us to obtain expressions for several propagators for Lévy flights, which seem difficult to obtain using other methods.

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VII. APPENDIX

A. Fractional Fokker-Planck Equation from the path integral

Here we derive the FFPE starting from the discretized of the path integral as given in Eq. \cite{28}. Let $P(x, t + \Delta t; x_0, 0)$ be the probability of finding the particle at $x$ at the time $t + \Delta t$ given that it started at $x_0$ at the time $t = 0$. Then the path integral prescription of Eq. \cite{28} enables us to calculate this probability density in terms of $P(x', t; x_0, 0)$, the probability distribution at the time $t$, as

$$P(x, t + \Delta t; x_0, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dx' e^{-D|p|^\alpha \Delta t} e^{ip\{(x-x') + V'(x')/\gamma m\}\Delta t} P(x', t; x_0, 0). \quad (93)$$

Since $\Delta t$ is infinitesimally small, we can expand the exponential $e^{-D|p|^\alpha + ipV'(x')/\gamma m}\Delta t$ up to linear order in $\Delta t$ and obtain,

$$P(x, t + \Delta t; x_0, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dx' \left(1 - D|p|^\alpha \Delta t + i\Delta t V'(x')/\gamma m\right) e^{ip(x-x')} P(x', t; x_0, 0). \quad (94)$$

Defining the operator $\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2}$ by

$$D \left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2} P(x, t; x_0, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dx' D|p|^\alpha e^{ip(x-x')} P(x', t; x_0, 0), \quad (95)$$

we can write Eq. \cite{94} as

$$P(x, t + \Delta t; x_0, 0) = P(x, t; x_0, 0) + \Delta t \left(-D \left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha/2} + \frac{\partial}{\partial x} \frac{V'(x)}{\gamma m}\right) P(x, t; x_0, 0). \quad (96)$$
From the above equation, we obtain the FFPE given in Eq. (9) which is

$$\frac{\partial P(x,t; x_0,0)}{\partial t} = \left( -D \left( -\frac{\partial^2}{\partial x^2} \right)^{\alpha/2} + \frac{\partial}{\partial x} V'(x) \right) P(x,t; x_0,0).$$

(97)

B. Transformation of coordinates from \{p_0, p_T\} to \{q_1, q_2\}

The propagators \(P^{ud}_{free}(x_f, v_f; T; x_0, v_0,0)\) and \(P^{ud}_{har}(x_f, v_f; T; x_0, v_0,0)\) can be written in a generalized form as

$$P_{gen} = \int_{-\infty}^{\infty} dp_T \int_{-\infty}^{\infty} dp_0 e^{-\int_0^T dt |a(t)p_T + b(t)p_0|^\alpha} e^{-i(gp_T + hp_0)}$$

(98)

which shows the formal dependence of \(p_0\) and \(p_T\). We will not work out the steps explicitly for \(P^{ud}_{free}(x_f, v_f; T; x_0, v_0,0)\) and \(P^{ud}_{har}(x_f, v_f; T; x_0, v_0,0)\) but only for \(P_{gen}\). The propagators of interest can be obtained from \(P_{gen}\). Using the transformation \(\{p_0, p_T\} \to \{q_1, q_2\}\) with \(p_T = q_1\) and \(p_0 = q_1 q_2\), we get

$$P_{gen} = 2 \int_{-\infty}^{\infty} dq_2 \int_{-\infty}^{\infty} dq_1 e^{-|q_1|^\alpha} \int_0^T dt |a(t) + b(t)q_2|^\alpha \cos((g + hq_2)q_1).$$

(99)

Upon expanding the cosine series and performing the integral over \(q_1\), we get

$$P_{gen} = \frac{2}{\alpha} \int_{-\infty}^{\infty} dq_2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \Gamma \left( \frac{2(n + 1)}{\alpha} \right) \frac{(g + hq_2)^{2n}}{\left( \int_0^T dt |a(t) + b(t)q_2|^\alpha \right)^{2(n+1)/\alpha}}.$$  

(100)

Note that the above sum can be evaluated exactly for \(\alpha = 1\) or 2 and lead to Cauchy or exponential functions. Further, the sum does not converge if \(\alpha < 1\).

C. Calculation of the propagator \(P^{ud}_{free}(x_f, T; x_0, v_0,0)\)

Our procedure enables us to calculate the most general propagator in the underdamped regime for both free Lévy flight and that under a linear and a harmonic potential. From these, we showed results for more specific propagators such as \(P^{ud}_{free}(x_f, T; x_0, v_0,0)\), \(P^{ud}_{free}(v_f, T; v_0,0)\), \(P^{ud}_{har}(x_f, T; x_0, v_0,0)\), and \(P^{ud}_{har}(v_f, T; x_0, v_0,0)\). Here, in the appendix, we will show the steps involved in calculating \(P^{ud}_{free}(x_f, T; x_0, v_0,0)\). The rest of the propagators can be obtained in a similar
fashion.

\[
P^{\text{ud}}_{\text{free}}(x_f, T; x_0, v_0, 0) = \int_{-\infty}^{\infty} dv_f \int_{-\infty}^{\infty} dp_T \int_{-\infty}^{\infty} dp_0 \left[ p_0 \left( 1-e^{-\gamma T} \right) - p_T \left( e^{-\gamma T} - e^{-\gamma t} \right) \right] e^{-i \left( -p_0 \frac{v_0}{\gamma} + \frac{p_0 - p_T}{1-e^{-\gamma T}} (x_f - x_0) \right)} d\tau \left( p_T \right)
\]

Performing the integration over the delta function we obtain

\[
P^{\text{ud}}_{\text{free}}(x_f, T; x_0, v_0, 0) = \frac{1}{2\pi \left( 1-e^{-\gamma T} \right)} \int_{-\infty}^{\infty} dp_0 \int_{-\infty}^{\infty} dp_T \left[ p_0 \left( 1-e^{-\gamma T} \right) - p_T \left( e^{-\gamma T} - e^{-\gamma t} \right) \right] e^{-i \left( -p_0 \frac{v_0}{\gamma} + \frac{p_0 - p_T}{1-e^{-\gamma T}} (x_f - x_0) \right)} d\tau \left( p_T \right)
\]

\[= \frac{1}{\left( D \int_0^T dt \left( 1-e^{-\gamma t} \right)^{\frac{1}{\alpha}} \right)^{1/\alpha} L^{\alpha,0} \left( \frac{v_0 \left( 1-e^{-\gamma T} \right)}{\gamma} - (x_f - x_0) \right) \left( D \int_0^T dt \left( 1-e^{-\gamma t} \right)^{\frac{1}{\alpha}} \right)^{1/\alpha}}
\]
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FIG. 1: The figures (a)-(d) show the time evolution of $P_{\text{har}}^{ud}(v_f, x_f, T; v_0, x_0, 0)$ starting with $v_0 = 0$ and $x_0 = 0$. Figure (e) shows the long time behavior of the product of $P_{\text{har}}^{ud}(x_f, T; v_0, x_0, 0)$ and $P_{\text{har}}^{ud}(v_f, T; v_0, x_0, 0)$. (a) $P_{\text{har}}^{ud}(v_f, x_f, T; 0, 0, 0)$ at $T=1$; (b) $P_{\text{har}}^{ud}(v_f, x_f, T; 0, 0, 0)$ at $T=3$; (c) $P_{\text{har}}^{ud}(v_f, x_f, T; 0, 0, 0)$ at $T=5$; (d) $P_{\text{har}}^{ud}(v_f, x_f, T; 0, 0, 0)$ at long time; (e) Product of the stationary distributions $P_{\text{har}}^{ud}(x_f)$ and $P_{\text{har}}^{ud}(v_f)$.