ON THE CONSISTENCY OF GRAPH-BASED BAYESIAN LEARNING
AND THE SCALABILITY OF SAMPLING ALGORITHMS

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ABSTRACT. A popular approach to semi-supervised learning proceeds by endowing the input data with a graph structure in order to extract geometric information and incorporate it into a Bayesian framework. We introduce new theory that gives appropriate scalings of graph parameters that provably lead to a well-defined limiting posterior as the size of the unlabeled data set grows. Furthermore, we show that these consistency results have profound algorithmic implications. When consistency holds, carefully designed graph-based Markov chain Monte Carlo algorithms are proved to have a uniform spectral gap, independent of the number of unlabeled inputs. Several numerical experiments corroborate both the statistical consistency and the algorithmic scalability established by the theory.

1. INTRODUCTION

The principled learning of functions is at the core of machine learning, statistics, artificial intelligence, and inverse problems. This paper is concerned with semi-supervised Bayesian learning, where unlabeled data is used to build a prior distribution on the unknown function, that is updated into a posterior distribution by the use of labeled data. Labeled data consists of pairs of function inputs and outputs, whereas unlabeled data consists only of inputs. We focus on graph-based methods, in which the geometry of the input space is learned by means of a graph, and incorporated into the prior covariance. The posterior is defined by combining the prior with a likelihood function that involves the labeled data. Our main goal is to contribute to the understanding of graph-based semi-supervised learning and its Bayesian formulation. We consider $\varepsilon$-graphs, built by connecting any two inputs whose distance is less than $\varepsilon$. Our results guarantee that, provided that the connectivity parameter $\varepsilon$ is suitably scaled with the number of inputs, the graph-based posteriors converge, as the size of the unlabeled data set grows, to a continuum posterior. Moreover we show that, under the existence of a continuum limit, carefully designed graph-based Markov chain Monte Carlo (MCMC) sampling algorithms have a uniform spectral gap, independent of the number of unlabeled data. The continuum limit theory is thus of interest in three distinct ways. First, it establishes the statistical consistency of graph-based methods in machine learning; second, it suggests suitable scalings of graph parameters of practical interest; and third, statistical consistency is shown to go hand in hand with algorithmic scalability: when graph-based learning problems have a continuum limit, algorithms that exploit this limit structure have a convergence rate that is independent of the size of the unlabeled data set. The theoretical understanding of these questions relies heavily on recently developed bounds for the asymptotic behavior of the spectra of graph-Laplacians. Our presentation aims to bring together intrinsic and extrinsic approaches to semi-supervised learning, and to highlight the similarities and differences between optimization and Bayesian formulations. We include a computational study that suggests directions for further theoretical developments, and illustrates the non-asymptotic relevance of our asymptotic results.

1.1. Problem Description. We now provide a brief intuitive problem description. For expository purposes we interpret here the unknown function of interest as heat on a manifold; a full account of our setting is given in section 2 below. We remark that our framework covers a wide class of regression and classification problems, e.g. probit and logistic, as well as a variety of Bayesian inverse problems.
We assume that we are given \( n \) inputs lying on an unknown \( m \)-dimensional manifold \( \mathcal{M} \subset \mathbb{R}^d \), \( p \) of which are labeled. The collection of input data will be denoted by \( \mathcal{M}_n = \{ x_1, \ldots, x_n \} \subset \mathcal{M} \), and we denote by \( \tilde{y} \in \mathbb{R}^p \) the vector of labels. Each label is interpreted here as a noisy measurement of the temperature at time \( t \geq 0 \) at one of the inputs. The ideal but unattainable goal would be to learn the initial temperature in the unknown manifold \( \mathcal{M} \) that is given to us by \( \tilde{y} \in \mathbb{R}^p \) the vector of labels. Each label is interpreted here as a noisy measurement of the temperature at time \( t \geq 0 \) at one of the inputs. The ideal but unattainable goal would be to learn the initial temperature in the unknown manifold \( \mathcal{M} \), described by a function \( u : \mathcal{M} \to \mathbb{R} \). Within a Bayesian setting, this would boil down to putting a prior distribution \( \pi \) on \( u \) and assuming a statistical model for the data encoded in a negative log-likelihood \( \Phi \) in order to define a posterior distribution \( \mu \) over functions on \( \mathcal{M} \) by

\[
\mu(du) \propto \exp(-\Phi(u; \tilde{y})) \pi(du).
\]

(1.1)

However, as \( \mathcal{M} \) is assumed to be unknown, we follow an intrinsic approach and aim to recover the initial temperature in the given point cloud \( \mathcal{M}_n \), described by a discrete function \( u_n : \mathcal{M}_n \to \mathbb{R} \). This will be achieved by using a graph Laplacian to define a prior measure on the temperatures and also to incorporate within the likelihood a surrogate graph-based heat equation; in this way geometric information from \( \mathcal{M} \) extracted from \( \mathcal{M}_n \) is taken into account both in the prior and the likelihood. The solution of the graph-based Bayesian approach will be a posterior distribution over discrete functions

\[
\mu_n(du_n) \propto \exp(-\Phi_n(u_n; \tilde{y})) \pi_n(du_n).
\]

(1.2)

The details on how we construct —without use of the ambient space or \( \mathcal{M} \)— the prior and likelihood surrogates \( \pi_n \) and \( \Phi_n \) are given in section 2. In the case where \( t = 0 \), observations are taken directly from the unknown function of interest, and our problem directly corresponds to semi-supervised regression or classification.

Two interpretations of our learning problem will be useful. The first is to see (1.2) as a surrogate, graph-based discretization of a Bayesian inverse problem over functions on \( \mathcal{M} \) whose posterior solution is given by equation (1.1). The second is to interpret it as a classical Bayesian linear regression problem. In the latter interpretation, \( \mathcal{M} \) may represent a low-dimensional manifold sufficient to represent features living in an extremely high dimensional ambient space (\( m \ll d \)), perhaps upon some dimensionality reduction of the given inputs; in the former, \( \mathcal{M} \) may represent the unknown physical domain of a differential equation. A direct application area of the above setting is in the recovery of an image that has been subject to noisy blurring. More broadly our framework covers —by the flexibility in the choice of misfit functional \( \Phi \)— a wide class of classification and regression learning problems that including Bayesian probit and logistic models.

Our theory on statistical consistency and algorithmic scalability concerns regimes of large number \( n \) of input data with moderate number \( p \) of labeled data. Such regime is of particular interest in semi-supervised learning applications where labeled data is often expensive to collect, but unlabeled data abounds. In a way to be made precise, consistency results guarantee that graph-based posteriors of the form (1.2) are close to a ground truth posterior of the form (1.1), while the algorithmic scalability that we establish ensures the convergence, in an \( n \)-independent number of iterations, of certain MCMC methods for graph posterior sampling. The computational cost per iteration may, however, grow with \( n \). These MCMC methods are in principle applicable in fully supervised learning, but their performance would deteriorate if both \( n \) and \( p \) are allowed to grow. Finally, we note that although our exposition is focused on semi-supervised regression, our conclusions are equally relevant for semi-supervised classification.

1.2. Literature. Here we put into perspective our framework by contrasting it with optimization and extrinsic approaches to semi-supervised learning, and by relating it to other methods for surrogate Bayesian inversion. We also give some background on MCMC algorithms.

1.2.1. Graph-Based Semi-supervised Learning. We refer to [58] for an introductory tutorial on semi-supervised learning, containing useful pointers to the literature. The question of when and how unlabeled data matters is addressed in [37]. Some key papers on graph-based methods are [57], [33], and [12]. Spectral properties of graph Laplacians, that form the core of a theoretical
justification for graph-based approaches, are studied in [8], [9], [14], [30], [47]. As already noted, a key motivation for graph-based semi-supervised learning is that high dimensional inputs can often be represented in a low-dimensional manifold, whose local geometry may be learned by imposing a graph structure on the inputs. In practice, features may be close to but not exactly on an underlying manifold. The question of how to find suitable manifold representations has led to a vast literature on dimensionality reduction techniques and manifold learning, e.g. [43], [52], [20], [7].

1.2.2. Bayesian vs. Optimization, and Intrinsic vs. Extrinsic. In this subsection we focus on the regression interpretation, with labels directly obtained from noisy observation of the unknown function. The Bayesian formulation that we consider has the advantage over traditional optimization formulations that it allows for uncertainty quantification in the recovery of the unknown function [10]. Moreover, from a computational viewpoint, we shall show that certain sampling algorithms have desirable scaling properties —these algorithms, in the form of simulated annealing, may also find application within optimization formulations [29].

The Bayesian update (1.2) is intimately related to the optimization problem

$$\min_{u_n} \langle \Delta_{\mathcal{M}_n} u_n, u_n \rangle + \Phi_n(u_n; \bar{y}). \quad (1.3)$$

Here $\Delta_{\mathcal{M}_n}$ represents the graph-Laplacian, as defined in equation (2.8) below, and the minimum is taken over square integrable functions on the point cloud $\mathcal{M}_n$. Precisely, the solution $u_n^\ast$ to (1.3) is the mode (or MAP for maximum a posteriori) of the posterior distribution $\mu_n$ in (1.2) with a Gaussian prior $\pi_n = N(0, \Delta_{\mathcal{M}_n}^{-1})$.

The Bayesian problem (1.2) and the variational problem (1.3) are intrinsic in the sense that they are constructed without reference to the ambient space (other than through its metric), working in the point cloud $\mathcal{M}_n$. In order to address the generalization problem of assigning labels to points $x \notin \mathcal{M}_n$ we use interpolation operators that turn functions defined on the point cloud into functions defined on the ambient space. We will restrict our attention to the family of $k$-NN interpolants defined by

$$[I^k_n(u_n)](x) := \frac{1}{k} \sum_{x_i \in N_k(x)} u_n(x_i), \quad x \in \mathbb{R}^d, \quad (1.4)$$

where $N_k(x)$ is the set of $k$-nearest neighbors in $\mathcal{M}_n$ to $x$; here the distance used to define nearest neighbors is that of the ambient space. Within our Bayesian setting we consider $I_n^k \mu_n$, the push-forward of $\mu_n$ by $I_n$, as the fundamental object that allows us to assign labels to inputs $x \notin \mathcal{M}_n$, and quantify the uncertainty in such inference. The need of interpolation maps also appears in the context of intrinsic variational approaches to binary classification [22] or in the context of variational problems of the form (1.3): the function $u_n^\ast$ is only defined on $\mathcal{M}_n$, and hence should be extended to the ambient space via an interpolation map $I_n$.

Intrinsic approaches contrast with extrinsic ones, such as manifold regularization [8], [9]. This method solves a variational problem of the form

$$\min_u \langle \Delta_{\mathcal{M}_n} u[M_n], u[M_n] \rangle + \Phi(u; \bar{y}) + \zeta \|u\|^2_{HK}, \quad (1.5)$$

where now the minimum is taken over functions in a reproducing kernel Hilbert space $\mathcal{H}_K$ defined over the ambient space $\mathbb{R}^d$, and $u[M_n]$ denotes the restriction of $u$ to $\mathcal{M}_n$. The kernel $K$ is defined in $\mathbb{R}^d$ and the last term in the objective functional, not present in (1.3), serves as a regularizer in the ambient space; the parameter $\zeta \geq 0$ controls the weight given to this new term. Bayesian and extrinsic formulations may be combined in future work.

In short, extrinsic variational approaches solve a problem of the form (1.5), and intrinsic ones solve (1.3) and then generalize by using an appropriate interpolation map. In the spirit of the latter, the intrinsic Bayesian approach of this paper defines an intrinsic graph-posterior by (1.2) and then this posterior is pushed-forward by an interpolation map. What are the advantages and disadvantages of each approach? Intuitively, the intrinsic approach seems more natural for label inference of inputs on or close to the underlying manifold $\mathcal{M}$. However, the extrinsic
approach is appealing for problems where no low-dimensional manifold structure is present in the input space.

1.2.3. **Surrogate Bayesian Learning.** Our learning problem can be seen as a surrogate for a ground-truth Bayesian inverse problem over functions in the underlying manifold $\mathcal{M}$ [50], [18], [19], [25]. Traditional problem formulations and sampling algorithms require repeated evaluation of the likelihood, often making naive implementations impractical. For this reason, there has been recent interest in reduced order models [14], [25], [3], [17], and in defining surrogate likelihoods in terms of Gaussian processes [42], [49], [51], or polynomial chaos expansions [54], [39]. Pseudo-marginal [5], [2] and approximate Bayesian computation methods [6] have become popular in intractable problems where evaluation of the likelihood is not possible. Bounds in Kullback-Leibler [40] and Hellinger metric [51] between surrogate and ground-truth posteriors have been established. There are two distinctive features of the graph-based models employed here. First, they involve surrogates for both the prior and the likelihood; and second, the surrogate and ground-truth posteriors live in different spaces: the former is a measure over functions on a point cloud, while the latter is a measure over functions on the continuum. The paper [23] studied the continuum limits of surrogate posteriors to the ground-truth continuum posterior. This was achieved by using a suitable topology inspired by the analysis of functionals over functions in point clouds arising in machine learning [27], [26], [48].

1.2.4. **Markov Chain Monte Carlo.** MCMC is a popular class of algorithms for posterior sampling. Here we consider certain Metropolis–Hastings MCMC methods that construct a Markov chain that has the posterior as invariant distribution by sampling from a user-chosen proposal and accepting/rejecting the samples using a general recipe. Posterior expectations are then approximated by averages with respect to the chain’s empirical measure. The generality of Metropolis–Hastings algorithms is a double-edged sword: the choice of proposal may have a dramatic impact on the convergence of the chain. Even for a given form of proposal, parameter tuning is often problematic. These issues are exacerbated in learning problems over functions, as traditional algorithms often break-down.

The pCN algorithm, introduced in [11], allows for robust sampling of infinite dimensional functions provided that the target is suitably defined as a change of measure. Indeed, the key idea of the method is to exploit this change of measure structure, that arises naturally in Bayesian nonparametrics but also in the sampling of conditioned diffusions and elsewhere. Robustness is understood in the sense that, when pCN is used to sample functions projected onto a finite $D$-dimensional space, the rate of convergence of the chain is independent of $D$. This was already observed in [11] and [10], and was further understood in [32] by showing that projected pCN methods have a uniform spectral gap, while traditional random walk does not.

In this paper we substantiate the use of graph-based pCN MCMC algorithms [10] in semi-supervised learning. The key idea is that our continuum limit results provide the necessary change of measure structure for the robustness of pCN. This allows us to establish their uniform spectral gap in the regime where the continuum limit holds. Namely, we show that if the number $p$ of labeled data is fixed, then the rate of convergence of graph pCN methods for sampling graph posterior distributions is independent of $n$. We remark that pCN addresses some of the challenges arising from sampling functions, but fails to address challenges arising from tall data. Some techniques to address this complementary difficulty are reviewed in [4].

1.3. **Paper Organization and Main Contributions.** A thorough description of our setting is given in section 2. Algorithms for posterior sampling are presented in section 3. Section 4 contains our main theorems on continuum limits and uniform spectral gaps. Finally, a computational study is conducted in section 5. All proofs and technical material are collected in an appendix.

The two main theoretical contributions of this paper are Theorem 4.2—establishing statistical consistency of intrinsic graph methods generalized by means of interpolants—and Theorem 4.6—establishing the uniform spectral gap for graph-based pCN methods under the conditions
required for the existence of a continuum limit. Both results require appropriate scalings of the graph connectivity with the number of inputs. An important contribution of this paper is the analysis of truncated graph-priors that retain only the portion of the spectra that provably approximates that of the ground-truth continuum. From a numerical viewpoint, our experiments illustrate parameter choices that lead to successful graph-based inversion, highlight the need for a theoretical understanding of the spectrum of graph Laplacians and of regularity of functions on graphs, and show that the asymptotic consistency and scalability analysis set forth in this paper is of practical use outside the asymptotic regime.

2. Setting

Throughout, $\mathcal{M}$ will denote an $m$-dimensional, compact, smooth manifold embedded in $\mathbb{R}^d$. We let $\mathcal{M}_n := \{x_1, \ldots, x_n\}$ be a collection of i.i.d. samples from the uniform distribution on $\mathcal{M}$. We are interested in learning functions defined on $\mathcal{M}_n$ by using the inputs $x_i$ and some output values, obtained by noisy evaluation at $p \leq n$ inputs of a transformation of the unknown function. The learning problem in the discrete space $\mathcal{M}_n$ is defined by means of a surrogate, graph-based discretization of a continuum learning problem defined over functions on $\mathcal{M}$. We view the continuum problem as a ground-truth case where full geometric information of the input space is available. We describe the continuum learning setting in subsection 2.1, followed by the discrete learning setting in subsection 2.2. We will denote by $L^2(\gamma)$ the space of functions on the underlying manifold that are square integrable with respect to the uniform measure $\gamma$. We use extensively that functions in $L^2(\gamma)$ can be written in terms of the (normalized) eigenfunctions $\{\psi_i\}_{i=1}^{\infty}$ of the Laplace Beltrami operator $\Delta_{\mathcal{M}}$. We denote by $\{\lambda_i\}_{i=1}^{\infty}$ the associated eigenvalues of $-\Delta_{\mathcal{M}}$, assumed to be in non-decreasing order and repeated according to multiplicity. Analogous notations will be used in the graph-based setting, with scripts $n$.

2.1. Continuum Learning Setting. Our ground-truth continuum learning problem consists of the recovery of a function $u \in L^2(\gamma)$ from data $\tilde{y} \in \mathbb{R}^p$. The data $\tilde{y}$ is assumed to be a noisy observation of a vector $\tilde{v} \in \mathbb{R}^p$ obtained indirectly from the function of interest $u$ as follows:

$$u \in L^2(\gamma) \mapsto \tilde{v} := O \circ \mathcal{F}(u) \mapsto \tilde{y}.$$  

Here $\mathcal{F} : L^2(\gamma) \to L^2(\gamma)$ is interpreted as a forward map representing, for instance, a mapping from inputs to outputs of a differential equation. As a particular case of interest, $\mathcal{F}$ may be the identity map in $L^2(\gamma)$. The map $O : L^2(\gamma) \to \mathbb{R}^p$ is interpreted as an observation map, and is assumed to be linear and continuous. The Bayesian approach that we will now describe proceeds by specifying a prior on the unknown function $u$, and a noise model for the generation of data $\tilde{y}$ given the vector $\tilde{v} = O \circ \mathcal{F}(u)$. The solution is a posterior measure $\mu$ over functions on $\mathcal{M}$, supported on $L^2(\gamma)$.

2.1.1. Continuum Prior. We assume a Gaussian prior distribution $\pi$ on the unknown initial condition $u \in L^2(\gamma)$:

$$\pi = N(0, C_u), \quad C_u = (\alpha I - \Delta_{\mathcal{M}})^{-s/2}, \quad (2.1)$$

where $\alpha \geq 0$, and $s > m$ so that $u \sim \pi$ belongs to $L^2(\gamma)$ almost surely, and $\Delta_{\mathcal{M}}$ denotes the Laplace Beltrami operator. The parameter $s$ characterizes, in a well-understood manner, the almost sure regularity of draws from $\pi$ [50].

Draws $u \sim \pi$ can be obtained, for instance, via the Karhunen-Loève expansion

$$u(x) = \sum_{i=1}^{\infty} (\alpha + \lambda_i)^{-s/4} \xi_i \psi_i(x), \quad \xi_i \text{i.i.d. } N(0, 1). \quad (2.2)$$

Equation (2.1) corresponds to the covariance operator description of the measure $\pi$. The covariance function representation of this operator may be advantageous in the derivation of regression formulae — see the appendix.
2.1.2. Continuum Forward and Observation Maps. In what follows we take, for concreteness, the forward map $\mathcal{F} = \mathcal{F}^t$ to be the solution of the heat equation on $\mathcal{M}$ up to a given time $t \geq 0$. That is, we set
\[
\mathcal{F} u \equiv \mathcal{F}^t u := e^{t \Delta_{\mathcal{M}}} u.
\]
Note that $\mathcal{M}$ plays two roles in definition of $\mathcal{F}^t$: it determines both the physical domain of the heat equation and the Laplace Beltrami operator $\Delta_{\mathcal{M}}$. If $t = 0$, the forward map is the identity operator in $L^2(\gamma)$.

We now describe our choice and interpretation of observation maps. Let $x_1, \ldots, x_p \in \mathcal{M}$, and let $\delta > 0$ be small. For $w \in L^2(\gamma)$ we define the $j$-th coordinate of the vector $\mathcal{O}w$ by
\[
[\mathcal{O}w](j) := \frac{1}{\gamma(B_\delta(x_j) \cap \mathcal{M})} \int_{B_\delta(x_j) \cap \mathcal{M}} w(x)\gamma(dx), \quad 1 \leq j \leq p,
\]
where $B_\delta(x_j)$ denotes the Euclidean ball of radius $\delta$ centered at $x_j$. At an intuitive level, and in our numerical investigations, we see $\mathcal{O}$ as the point-wise evaluation map at the inputs $x_j$:
\[
\mathcal{O}w = [w(x_1), \ldots, w(x_p)]^t \in \mathbb{R}^p.
\]
The definition in (2.4) allows us, however, to perform rigorous analysis in an $L^2$ sense. Henceforth we denote $\mathcal{G} := \mathcal{O} \circ \mathcal{F}$.

2.1.3. Data and Noise Models. Having specified the forward and observation maps $\mathcal{F}$ and $\mathcal{O}$, we assume that the label vector $\vec{y} \in \mathbb{R}^p$ arises from noisy measurement of $\mathcal{O} \circ \mathcal{F}(u) \in \mathbb{R}^p$. A noise-model will be specified via a function $\phi^\theta : \mathbb{R}^p \rightarrow \mathbb{R}^p$. We postpone the precise statement of assumptions on $\phi^\theta$ to section 4. Two guiding examples, covered by the theory, are given by
\[
\phi^\theta(\vec{w}) := \frac{1}{2\sigma^2}|\vec{y} - \vec{w}|^2, \quad \text{or} \quad \phi^\theta(\vec{w}) := -\sum_{i=1}^p \log\left(\Psi(\vec{y}(i)\vec{w}(i); \sigma)\right),
\]
where $\Psi$ denotes the CDF of a centered univariate Gaussian with variance $\sigma^2$. The former noise model corresponds to Gaussian i.i.d. noise in the observation of each of the $p$ coordinates of $\mathcal{G}u$. The latter corresponds to probit classification, and a noise model of the form $\vec{y}(i) = S(v(i) + \eta_i)$ with $\eta_i$ i.i.d. $N(0, \sigma^2)$, and $S$ the sign function. For label inference in Bayesian classification the posterior obtained below needs to be pushed-forward via the sign function, see [10].

2.1.4. Continuum Posterior. The Bayesian solution to the ground-truth continuum learning problem is a continuum posterior measure
\[
\mu(du) \propto \exp(-\phi^\theta(\mathcal{G}u))\pi(du)
\]
\[
=: \exp(-\Phi(u; \vec{y}))\pi(du),
\]
that represents the conditional distribution of $u$ given the data $\vec{y}$. Equation (2.6) defines the negative log-likelihood function $\Phi$, that represents the conditional distribution of $\vec{y}$ given $u$. The posterior $\mu$ contains —in a precise sense [55]— all the information on the unknown input $u$ available in the prior and the data.

2.2. Discrete Learning Setting. We consider the learning of functions defined on a point cloud $\mathcal{M}_n := \{x_1, \ldots, x_n\} \subset \mathcal{M}$. The underlying manifold $\mathcal{M}$ is assumed to be unknown. We suppose to have access to the same data $\vec{y}$ as in the continuous setting, and that the inputs $x_1, \ldots, x_p$ in the definition of $\mathcal{O}$ correspond to the first $p$ points in $\mathcal{M}_n$. Thus, the data may be interpreted as noisy measurements of the true temperature at the first $p$ points in the cloud at time $t \geq 0$. The aim is to construct —without knowledge of $\mathcal{M}$— a surrogate “posterior” measure $\mu_n$ over functions in $\mathcal{M}_n$ representing the initial temperatures at each point in the cloud.

Analogous to the continuous setting, we will denote by $L^2(\gamma_n)$ the space of functions on the cloud that are square integrable with respect to the uniform measure $\gamma_n$ on $\mathcal{M}_n$. It will be convenient to view, formally, functions $u_n \in L^2(\gamma_n)$ as vectors in $\mathbb{R}^n$. We then write $u_n \equiv [u_n(1), \ldots, u_n(n)]^t$, and think of $u_n(i)$ as evaluation of the function $u_n$ at $x_i$. 

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[55]: Reference to a specific document or page number.
The surrogate posteriors are built by introducing a surrogate prior, and surrogate forward and observation maps \( \mathcal{F}_n : L^2(\gamma_n) \to L^2(\gamma_n) \) and \( \mathcal{O}_n : L^2(\gamma_n) \to \mathbb{R}^p \) for \( \mathcal{F} \) and \( \mathcal{O} \). The same noise-model and data as in the continuum case will be used. We start by introducing a graph structure in the point cloud. Surrogate priors and forward maps are defined via a graph-Laplacian that summarizes the geometric information available in the point cloud \( \mathcal{M}_n \).

2.2.1. Geometric Graph and Graph-Laplacian. We endow the point cloud with a graph structure. We focus on \( \varepsilon \)-neighborhood graphs: an input is connected to every input within a distance of \( \varepsilon \). A popular alternative are \( k \)-nearest neighbor graphs, where an input is connected to its \( k \) nearest neighbors. The influence of the choice of graphs in unsupervised learning is studied in [38].

First, consider the kernel function \( K : [0, \infty) \to [0, \infty) \) defined by
\[
K(r) := \begin{cases} 
1 & \text{if } r \leq 1, \\
0 & \text{otherwise}.
\end{cases}
\] (2.7)

For \( \varepsilon > 0 \) we let \( K_\varepsilon : [0, \infty) \to [0, \infty) \) be the rescaled version of \( K \) given by
\[
K_\varepsilon(r) := \frac{m + 2}{n^2 \alpha_m \varepsilon^{m+2}} K\left(\frac{r}{\varepsilon}\right),
\]
where \( \alpha_m \) denotes the volume of the \( m \)-dimensional unit ball. We then define the weight \( W_n(x_i, x_j) \) between \( x_i, x_j \in \mathcal{M}_n \) by
\[
W_n(x_i, x_j) := K_\varepsilon(|x_i - x_j|),
\]
for a given choice of parameter \( \varepsilon = \varepsilon_n \), where we have made the dependence of the connectivity rate \( \varepsilon \) on \( n \) explicit. In order for the graph-based learning problems to be consistent in the large \( n \) limit, \( \varepsilon \) should be scaled appropriately with \( n \) — see subsection [11]. Figure 1 shows three geometric graphs \((\mathcal{M}_n, W_n)\) with fixed \( n \) and different choices of connectivity \( \varepsilon \).

\[
\begin{align*}
\text{Figure 1. Geometric graphs with } & n = 500, \text{ and } \varepsilon = n^{-1/4}, 2n^{-1/4}, \text{ and } 3n^{-1/4} \\
& \text{from left to right.}
\end{align*}
\]

We now define the graph Laplacian of the geometric graph \((\mathcal{M}_n, W_n)\) by
\[
\Delta_{\mathcal{M}_n} := D_n - W_n,
\] (2.8)
where \( D \) is the degree matrix of the weighted graph, i.e., the diagonal matrix with diagonal entries \( D_{ii} = \sum_{j=1}^{n} W_n(x_i, x_j) \). We remark that several definitions of graph Laplacian co-exist in the literature; the one above is some times referred to as the unnormalized graph Laplacian [53]. As will be made precise, the performance of the learning problems considered here is largely determined by the behavior of the spectrum of the graph Laplacian. Throughout we denote its eigenpairs by \( \{\lambda_i^n, \psi_i^n\}_{i=1}^n \), and assume that the eigenvalues are in non-decreasing order.
2.2.2. Graph Prior. A straight-forward discrete analogue to (2.1) suggests to endow the unknown function \( u_n \) with a prior
\[
\tilde{\pi}_n = N(0, C_{u_n}), \quad C_{u_n} := (\alpha I_n + \Delta_{M_n})^{-s/2}, \quad (2.9)
\]
defined in terms of truncation of the priors \( \tilde{\pi}_n \), retaining only the portion of the spectra of \( \Delta_{M_n} \) that provably approximates that of \( -\Delta_{M} \).

Precisely, we define the graph priors \( \pi_n \) as the law of \( u_n \) given by
\[
u_n = \sum_{i=1}^{k_n} (\alpha + \lambda_i^n)^{-s/4} \xi_i^n, \quad \xi_i^{i.i.d} \sim N(0, 1), \quad (2.10)
\]
where \( k_n \leq n \) may be chosen freely with the restrictions that \( k_n \to \infty \) and \( \lim_{n \to \infty} k_n \varepsilon_n^m = 0 \). Such choice is possible as long as the connectivity \( \varepsilon_n \) decays with \( n \).

2.2.3. Graph Forward and Observation Maps. We define a surrogate forward map \( F : L^2(\gamma_n) \to L^2(\gamma_n) \) by
\[
F_n u_n \equiv F_n^t u_n := e^{-t \Delta_{M_n}} u_n, \quad (2.11)
\]
where \( t \geq 0 \) is given as in the continuum setting. Likewise, for \( \delta > 0 \) as in (2.4) we define a surrogate observation map \( O_n : L^2(\gamma_n) \to \mathbb{R}^p \) by
\[
[O_n w](j) := \frac{1}{n \gamma_n(B_\delta(x_j))} \sum_{k : x_k \in B_\delta(x_j) \cap M_n} w(k), \quad 1 \leq j \leq p.
\]
As in the continuum setting, \( O_n \) should be thought of as point-wise evaluation at the inputs \( \{x_i\}_{i=1}^p \) and we denote \( G_n := O_n \circ F_n \).

2.2.4. Data and Likelihood. For the construction of graph posteriors we use the same data \( \tilde{y} \) and noise model \( \phi : \mathbb{R}^p \to \mathbb{R}^p \) as in the continuum case —see subsection 2.1.3.

2.2.5. Graph Posterior. We define the graph-posterior measure \( \mu_n \) by
\[
\mu_n(du) \propto \exp(-\phi(\tilde{y}, G_n u_n)) \pi_n(du_n) =: \exp(-\Phi_n(u_n; \tilde{y})) \pi_n(du_n), \quad (2.12)
\]
where \( \pi_n \) is the (truncated) graph prior defined as the law of (2.10), and the above expression defines the function \( \Phi_n \), interpreted as a surrogate negative log-likelihood.

In subsection 4.1 we will contrast the above “truncated” graph-posteriors to the “untruncated” graph-posteriors
\[
\bar{\mu}_n(du) \propto \exp(-\phi(\tilde{y}, G_n u_n)) \bar{\pi}_n(du_n) =: \exp(-\Phi_n(u_n; \tilde{y})) \bar{\pi}_n(du_n), \quad (2.13)
\]
obtained by using the prior \( \bar{\pi}_n \) in equation 2.1.

3. Posterior Sampling: pCN and Graph-pCN

The continuum limit theory developed in [23] and stated in subsection 4.1 suggests viewing graph posteriors \( \mu_n \) as discretizations of a posterior measure over functions on the underlying manifold. Again, these discretizations are robust for fixed \( p \) and growing number of total inputs \( n \). This observation substantiates the idea introduced in [10] of using certain version of the pCN MCMC method [11] for robust sampling of graph posteriors. We review the continuum pCN method in subsection 3.1 and the graph pCN counterpart in 3.2.
3.1. Continuum pCN. In practice, sampling of functions on the continuum always requires a discretization of the infinite dimensional function, usually defined in terms of a mesh and possibly a series truncation. A fundamental idea is that algorithmic robustness with respect to discretization refinement can be guaranteed by ensuring that the algorithm is well defined in function space, prior to discretization [50]. This insight led to the formulation of the pCN method for sampling of conditioned diffusions [11], and of measures arising in Bayesian non-parametrics in [15]. The pCN method for sampling the continuum posterior measure (2.6) is given in Algorithm 1.

Algorithm 1 Continuum pCN

Set \( j = 0 \) and pick any \( u^{(0)} \in L^2(\gamma) \).
Propose \( \tilde{u}^{(j)} = (1 - \beta^2)^{1/2}u^{(j)} + \beta \zeta^{(j)} \), where \( \zeta^{(j)} \sim N(0, C_u) \).
Set \( u^{(j+1)} = \tilde{u}^{(j)} \) with probability
\[
a(u^{(j)}, \tilde{u}^{(j)}) := \min\left\{1, \exp\left(\Phi(u^{(j)}) - \Phi(\tilde{u}^{(j)})\right)\right\},
\]
Set \( u^{(j+1)} = u^{(j)} \) otherwise.
\( j \to j + 1 \).

Posterior expectations of suitable test functions \( f \) can then be approximated by empirical averages
\[
\mu(f) \approx \frac{1}{J} \sum_{j=1}^{J} f(u^{(j)}) = S^N(f).
\] (3.1)

The user-chosen parameter \( \beta \in [0, 1] \) in Algorithm 2 monitors the step-size of the chain jumps: larger \( \beta \) leads to larger jumps, and hence to more state space exploration, more rejections and slower probing of high probability regions. Several robust discretization properties of Algorithm 1—that contrast with the deterioration of traditional random walk approaches—have been proved in [32]. Note that the acceptance probability is determined by the potential \( \Phi \) (here interpreted as the negative log-likelihood) that defines the density of posterior with respect to prior. In the extreme case where \( \Phi \) is constant, moves are always accepted. However, if the continuum posterior is far from the continuum prior, the density will be far from constant. This situation may arise, for instance, in cases where \( p \) is large or the observation noise \( \sigma \) is small. A way to make posterior informed proposals that may lead to improved performance in these scenarios has been proposed in [44].

3.2. Graph pCN. The graph pCN method is described in Algorithm 2 and is defined in complete analogy to the continuum pCN, Algorithm 1. When considering a sequence of problems with fixed \( p \) and increasing \( n \), the continuum theory intuitively supports the robustness of the method. Moreover, as indicated in [10] the parameter \( \beta \) may be chosen independently of the value of \( n \).

Algorithm 2 Graph pCN

Set \( j = 0 \) and pick any \( u_n^{(0)} \in L^2(\gamma_n) \).
Propose \( \tilde{u}_n^{(j)} = (1 - \beta^2)^{1/2}u_n^{(j)} + \beta \zeta_n^{(j)} \), where \( \zeta_n^{(j)} \sim N(0, C_{u_n}) \).
Set \( u_n^{(j+1)} = \tilde{u}_n^{(j)} \) with probability
\[
a_n(u_n^{(j)}, \tilde{u}_n^{(j)}) := \min\left\{1, \exp\left(\Phi_n(u_n^{(j)}) - \Phi_n(\tilde{u}_n^{(j)})\right)\right\},
\]
Set \( u_n^{(j+1)} = u_n^{(j)} \) otherwise.
\( j \to j + 1 \).
Our experiments in section 5 confirm this robustness, and also investigate the deterioration of the acceptance rate when both \( n \) and \( p \) are large.

4. Main Results

4.1. Continuum Limits. The paper [23] established large \( n \) asymptotic convergence of the untruncated graph-posteriors \( \tilde{\mu}_n \) in (2.13) to the continuum posterior \( \mu \) in (2.6). The convergence was established in a topology that combines Wasserstein distance and an \( L^2 \)-type term in order to compare measures over functions in the continuum with measures over functions in graphs.

**Proposition 4.1** (Theorem 4.4 in [23]). Suppose that \( s > 2m \) and that

\[
\frac{(\log(n))^{p_m}}{n^{1/m}} \ll \varepsilon_n \ll \frac{1}{n^{1/s}}, \quad \text{as } n \to \infty,
\]

where \( p_m = 3/4 \) for \( m = 2 \) and \( p_m = 1/m \) for \( m \geq 3 \). Then, the untruncated graph-posteriors \( \tilde{\mu}_n \) converge towards the posterior \( \mu \) in the \( \mathcal{P}(L^2) \) sense.

We refer to [27] for the construction of the space \( TL^2 \) that allows to compare functions on \( M_n \) with functions on \( M \), and to [23] for the construction of \( P(TL^2) \), space which allows us to compare probabilities over functions on \( M_n \) with probabilities over functions on \( M \). It is important to note that here convergence refers to the limit of fixed labeled data set of size \( p \), and growing size of unlabeled data. In order for the continuum limit to hold, the connectivity of the graph \( \varepsilon_n \) needs to be carefully scaled with \( n \) as in (4.1).

At an intuitive level, the lower bound on \( \varepsilon_n \) guarantees that there is enough averaging in the limit to recover a meaningful deterministic quantity. The upper bound ensures that the graph priors converge appropriately towards the continuum prior. At a deeper level, the lower bound is an order one asymptotic estimate for the \( \infty \)-optimal transport distance between the uniform and uniform empirical measure on the manifold [26], that hinges on the points \( x_1, \ldots, x_n \) lying on the manifold \( M \): if the inputs were sampled from a distribution whose support is close to \( M \), but whose intrinsic dimension is \( d \) and not \( m \), then the lower bound would be written in terms of \( d \) instead of \( m \). The upper bound, on the other hand, relies on the approximation bounds (5.1) of the continuum spectrum of the Laplace-Beltrami by the graph Laplacian.

We now present a new result on the stability of intrinsically constructed posteriors, generalized to \( M \) by interpolation via the map \( I_n := I_{1,n} \) —see (1.4); this is the most basic interpolant that can be constructed exclusively from the point cloud \( M_n \) and the metric on the ambient space. Other than extending the theory to cover the important question of generalization, there is another layer of novelty in Theorem 4.2: graph-posteriors are constructed with truncated priors, and the upper-bound in the connectivity \( \varepsilon_n \) in (4.1) is removed.

**Theorem 4.2.** Suppose that \( s > 2m \) and that

\[
\frac{(\log(n))^{p_m}}{n^{1/m}} \ll \varepsilon_n \ll 1, \quad \text{as } n \to \infty,
\]

where \( p_m \) is as in Proposition 4.1. Then, with probability one,

\[
\mathcal{I}_n\mu_n \to_{\mathcal{P}(L^2)} \mu, \quad \text{as } n \to \infty.
\]

The proof is presented in Appendix B. Similar results hold for more general interpolants as long as they are uniformly bounded and consistent when evaluated at the eigenfunctions of graph Laplacians (see Remark B.1.)

4.2. Uniform Spectral Gaps for Graph-pCN Algorithms. The aim of this subsection is to establish that, in a precise and rigorous sense, the graph-pCN method in Algorithm 2 is insensitive to the increase of the number \( n \) of input data provided that the number \( p \) of labeled data is fixed and a continuum limit exists. This behavior contrasts dramatically with other sampling methodologies such as the random walk sampler. One could characterize the robustness of MCMC algorithms in terms of uniform spectral gaps.
We start by defining the spectral gap for a single Markov chain with state space an arbitrary separable Hilbert space $\mathcal{H}$. We consider two notions of spectral gap, one with respect to a Wasserstein distance with respect to some distance like function $\tilde{d}$ and one in terms of $L^2$.

**Definition 4.3** (Wasserstein spectral gaps). Let $P$ be the transition kernel for a discrete time Markov chain with state space $\mathcal{H}$. Let $\tilde{d}: \mathcal{H} \times \mathcal{H} \to [0,1]$ be a distance like function (i.e. a symmetric, lower-semicontinuous function satisfying $\tilde{d}(u,v) = 0$ if and only if $u = v$). Without the loss of generality we also denote by $\tilde{d}$ the Wasserstein distance (1-OT distance) on $\mathcal{P}(\mathcal{H})$ induced by $\tilde{d}$ (see (C.1)). Then $P$ has spectral gap if there exists positive constants $C, \lambda$ such that

$$\tilde{d}(P^j \mu, P^j \nu) \leq C \exp(-\lambda j) d(\mu, \nu), \quad \forall \mu, \nu \in \mathcal{P}(\mathcal{H}), \quad \forall j \in \mathbb{N}.$$ 

In the above $\mathcal{P}(\mathcal{H})$ stands for the set of Borel probability measures on $\mathcal{H}$.

**Definition 4.4** ($L^2$-spectral gaps). Let $P$ be the transition kernel for a discrete time Markov chain with state space $\mathcal{H}$ and suppose that $\mu$ is invariant under $P$. $P$ is said to have $L^2_\mu$-spectral gap $1 - \exp(-\lambda)$ (for $\lambda > 0$) if for every $f \in L^2(\mathcal{H}; \mu)$ we have

$$\frac{\| Pf - \mu(f) \|^2_{L^2(\mathcal{H}; \mu)}}{\| f - \mu(f) \|^2_{L^2(\mathcal{H}; \mu)}} \leq \exp(-\lambda).$$

In the above, $\mu(f) := \int_{\mathcal{H}} f(u) d\mu(u)$ and $Pf(u) := \int_{\mathcal{H}} f(v) P(u, dv)$.

Having defined the notion of spectral gap for a single Markov chain, the notion of uniform spectral gap for a family of Markov chains is defined in an obvious way. Namely, if $\{P_n\}_{n \in \mathbb{N}}$ is a family of Markov chains, with perhaps different state spaces $\{\mathcal{H}_n\}_{n \in \mathbb{N}}$, we say that the family of Markov chains has uniform Wasserstein spectral gap with respect to a family of distance like functions $\{d_n\}$, if the Markov chains have spectral gaps with constants $C, \lambda$ which can be uniformly controlled, respectively, from above and away from zero. Likewise the chains are said to have uniform $L^2$-gaps (with respect to respective invariant measures) if the constant $\lambda$ can be uniformly bounded away from zero. We remark that Wasserstein spectral gaps imply uniqueness of invariant measures.

In what follows we use the following assumption:

**Assumption 4.5.** Let $\beta \in (0,1]$. For a certain fixed $\bar{y} \in \mathbb{R}^p$ we assume the following conditions on $\phi^\beta: \mathbb{R}^p \to \mathbb{R}$.

1. For every $K > 0$ there exists $c \in \mathbb{R}$ such that if $\bar{v}, \bar{w} \in \mathbb{R}^p$ satisfy

$$|\bar{w} - \sqrt{1 - \beta^2 \bar{v}}| \leq K$$

then,

$$\phi^\beta(\bar{v}) - \phi^\beta(\bar{w}) \geq c.$$

2. (Linear growth of local Lipschitz constant) There exists a constant $L$ such that

$$|\phi^\beta(\bar{v}_1) - \phi^\beta(\bar{v}_2)| \leq L \max\{ |\bar{v}_1|, |\bar{v}_2|, 1 \} |\bar{v}_1 - \bar{v}_2|, \quad \forall \bar{v}_1, \bar{v}_2 \in \mathbb{R}^p.$$

In Appendix [D] we show that the Gaussian model and the probit model satisfy these assumptions.

In what follows it is convenient to use $\mathcal{H}$ as a placeholder for one of the spaces $L^2(\gamma_n)$ $n \in \mathbb{N}$ or the space $L^2(\gamma)$. Likewise $P$ is a placeholder for the transition kernel associated to the pCN scheme from section [2] defined on $\mathcal{H}$ for each choice of $\mathcal{H}$. We are ready to state our second main theorem:

**Theorem 4.6** (Uniform Wasserstein spectral gap). Let $\theta > 0, \eta > 0$. For each choice of $\mathcal{H}$ let $d: \mathcal{H} \times \mathcal{H} \to [0,1]$,

$$d(u,v) := \min\{ 1, \frac{\bar{d}(u,v)}{\theta} \}, \quad u, v \in \mathcal{H}$$

where $\bar{d}(u,v)$ is a distance function satisfying $\bar{d}(u,v) = 0$ if and only if $u = v$.
be a rescaled and truncated version of the distance
\[ d(u, v) := \inf_{T, \psi \in A(T, u, v)} \int_0^T \exp(\eta \|\psi\|) dt, \]
\[ A(T, u, v) := \{ \psi \in C^1([0, T]; \mathcal{H}) : \psi(0) = u, \ \psi(T) = v, \ \|\dot{\psi}\| = 1 \}. \]
Finally, let \( \tilde{d} \) be the distance-like function
\[ \tilde{d}(x, y) := \sqrt{d(x, y)(1 + V(x) + V(y))}, \quad u, v \in \mathcal{H} \]
where
\[ V(u) := \|u\|^2, \quad u \in \mathcal{H}. \]
Then, under the assumptions of Theorem 4.2 and the assumptions 4.5, \( \theta > 0 \) and \( \eta > 0 \) can be chosen independently of the specific choice of \( \mathcal{H} \) in such a way that
\[ \tilde{d}(P^j \nu_1, P^j \nu_2) \leq C \exp(-\lambda j) \tilde{d}(\nu_1, \nu_2), \quad \forall \nu_1, \nu_2 \in \mathcal{P}(\mathcal{H}), \quad \forall j \in \mathbb{N}, \]
for constants \( C, \lambda \) that are independent of the choice of \( \mathcal{H} \).

A few remarks help clarify our results.

**Remark 4.7.** Notice that the distance \( d \) is a Riemannian distance whose metric tensor changes in space and in particular gets larger for points that are far away from the origin (notice that the choice \( \eta = 0 \) returns the canonical distance on \( \mathcal{H} \)). In particular, points that are far away from the origin have to be very close in the canonical distance in order to be close in the \( d \) distance. This distance was considered in [32]. We would also like to point out that the exponential form of the metric tensor can be changed to one with polynomial growth given the choice of \( V \).

**Remark 4.8.** Theorem 4.6 is closely related to Theorem 2.14 in [32]. There, uniform spectral gaps are obtained for the family of pCN kernels indexed by the different truncation levels of the Karhunen Loève expansion of the continuum prior. For that type of discretization, all distributions are part of the same space; this contrasts with our set-up where the discretizations of the continuum prior are the graph priors.

Due to the reversibility of the kernels associated to the pCN algorithms, Theorem 4.6 implies uniform \( L^2 \)-spectral gaps. Notice that the Wasserstein gaps imply uniqueness of invariant measures (which are precisely the graph and continuum posteriors for each setting) and hence there is no ambiguity when talking about \( L^2 \)-spectral gaps.

**Corollary 4.9.** Under the assumptions of Theorem 4.2 and the assumptions 4.5 the kernel associated to the pCN algorithm has an \( L^2 \)-spectral gap independent of the choice of \( \mathcal{H} \).

The proof of Theorem 4.6 and its corollary are presented in Appendix C.

**Remark 4.10.** Uniform spectral gaps may be used to find uniform bounds on the asymptotic variance of empirical averages [36].

5. **Numerical Study**

In the numerical experiments that follow we take \( \mathcal{M} = \mathcal{S} \) to be the two-dimensional sphere in \( \mathbb{R}^3 \). Our main motivation for this choice of manifold is that it allows us to expediently make use of well-known closed formulse [41] for the spectrum of the spherical Laplacian \( \Delta_{\mathcal{M}} = \Delta_{\mathcal{S}} \) in the continuum setting that serves as our ground truth model. We recall that \(-\Delta_{\mathcal{S}}\) admits eigenvalues \( l(l+1) \), \( l \geq 0 \), with corresponding eigenspaces of dimension \( 2l + 1 \). These eigenspaces are spanned by spherical harmonics [41]. In subsections 5.1, 5.2 and 5.3 we study, respectively, the spectrum of graph Laplacians, continuum limits, and the scalability of pCN methods.

### 5.1. Spectrum of Graph Laplacians.

The asymptotic behavior of the spectra of graph-Laplacians is crucial in the theoretical study of consistency of graph-based methods. In subsection 5.1.1 we review approximation bounds that motivate our truncation of graph-priors, and in subsection 5.1.2 we comment on the theory of regularity of functions on graphs.
5.1.1. **Approximation Bounds.** Quantitative error bounds for the difference of the spectrum of the graph Laplacian and the spectrum of the Laplace-Beltrami operator are given in \[13\] and \[21\]. Those results imply that, with very high probability,

\[
\left| 1 - \frac{\lambda_i^n}{\lambda_i} \right| \leq C \left( \frac{\delta_n}{\epsilon_n} + \epsilon_n \sqrt{\lambda_i} \right), \quad \forall i, \tag{5.1}
\]

where \(\delta_n\) denotes the \(\infty\)-optimal transport distance \[26\] between the uniform and the uniform empirical measure on the underlying manifold. The important observation here is that the above estimates are only relevant for the first portion of the spectra (in particular for those indices \(i\) for which \(\epsilon_n \sqrt{\lambda_i}\) is small). The truncation point at which the estimates stop being meaningful can then be estimated combining (5.1) and Weyl’s formula for the growth of eigenvalues of the Laplace-Beltrami operator on a compact Riemannian manifold of dimension \(m\) \[23\]. Namely, from \(\lambda_i \sim i^{2/m}\) we see that \(\epsilon_n \sqrt{\lambda_i} \ll 1\) as long as \(i = 1, \ldots, k_n\) and

\[1 \ll k_n \ll \frac{1}{\epsilon_n^m}.
\]

This motivates our truncation point for graph priors in equation (2.10).

Figure 2 illustrates the approximation bounds (5.1). The figure shows the eigenvalues of the graph Laplacian for three different choices of connectivity length scale \(\epsilon\) and three different choices of number \(n\) of inputs in the graph; superimposed is the spectra of the spherical Laplacian. We notice the flattening of the spectra of the graph Laplacian and, in particular, how the eigenvalues of the graph Laplacian start deviating substantially from those of the Laplace-Beltrami operator after some point in the \(x\)-axis. As discussed in \[21\], the estimates (5.1) are not necessarily sharp, and may be conservative in suggesting where the deviations start to happen.

![Figure 2](image)

**Figure 2.** Spectra of spherical and graph Laplacians in red and blue, respectively. Charts are arranged such that \(\epsilon\) varies as \([1, 2, 3] \times n^{-1/4}\) horizontally and \(n\) varies as \([1000, 500, 100]\)' vertically.

5.1.2. **Regularity of Discrete Functions.** We numerically investigate the role of the parameter \(s\) in the discrete regularity of functions \(u_n \in L^2(\gamma_n)\) sampled from \(\pi_n\). We focus on studying the oscillations of a function within balls of radius \(\epsilon_n\). More precisely, we consider

\[
[\text{osc}_{\epsilon_n}(u_n)](x_i) := \max_{x, z \in B_{\epsilon_n}(x_i) \cap M_n} |u_n(x) - u_n(z)|, \quad i = 1, \ldots, n.
\]
For given $s = 2, 3, \ldots, 8$ we take 100 samples $u_n \sim \pi_n$, and we normalize so that

$$\langle \Delta_n^s u_n, u_n \rangle_{L^2(\gamma_n)} = 1.$$  

We then compute the maximum value of $\text{osc}_{\varepsilon_n}(u_n)(x_i)$ over all $i = 1, \ldots, n$ and over all samples $u_n$ and plot the outcome against $s$. The results are shown in Figure 3.

**Figure 3.** The figure shows the maximum (and its logarithm) amplitude of oscillations for different values of the regularity parameter $s$.

This experiment illustrates the regularity of functions with bounded $H^s_n$ semi-norm

$$\|u_n\|_{H^s_n}^2 := \sum_{i=1}^{k_n} (\lambda^n_i)^s \langle u_n, \psi^n_i \rangle_{L^2(\gamma_n)}^2.$$  

As expected, higher values of $s$ enforce more regularity on the functions. Notice that here we only consider functions $u_n$ in the support of $\pi_n$ and hence we remove the effect of high eigenfunctions of $\Delta_n$ (which may be irregular). In particular, the regularity of the functions $u_n$ must come from the regularity of the first eigenvectors of $\Delta_n$ together with the growth of $(\lambda^n_i)^s$. To the best of our knowledge nothing is known about regularity of eigenfunctions of graph Laplacians. Studying such regularity properties is an important direction to explore in the future as we believe it would allow us to go beyond the $L^2$ set-up that we consider for the theoretical results in this paper. In that respect we would like to emphasize that the observation maps considered for the theory of this work are defined in terms of averages and not in terms of pointwise evaluations, but that for our numerical experiments we have used the latter.

A closely related setting in which discrete regularity has been mathematically studied is in the context of graph $p$-Laplacian semi-norm (here $p$ denotes an arbitrary number greater than one, and is not to be confused with the number $p$ of labeled data points). Lemma 4.1 in [48] states that, under the assumptions on $\varepsilon_n$ from Theorem 4.2, for all large enough $n$ and for every discrete function $u_n$ satisfying

$$\mathcal{E}_n^{(p)}(u_n) := \frac{1}{n^2 \varepsilon_n^p} \sum_{i,j} K \left( \frac{|x_i - x_j|}{\varepsilon_n} \right) \|u_n(x_i) - u_n(x_j)\|^p = 1,$$

it holds

$$\text{osc}_{\varepsilon_n}(u_n)(x_i) \leq C^{1/p} n^{1/p} \varepsilon_n, \quad \forall i = 1, \ldots, n.$$  

This estimate allows to establish uniform convergence (and not simply convergence in $TL^2$) of discrete functions towards functions defined at the continuum level. More precisely, suppose
that $p > m$ and that $\varepsilon_n \ll \frac{1}{n^{1/\theta}}$. Let $\{u_n\}_{n \in \mathbb{N}}$ be a sequence with $u_n \in L^2(\gamma_n)$ converging to a function $u \in L^2(\gamma)$ in the $TL^2$ sense and for which

$$\sup_{n \in \mathbb{N}} \mathcal{E}_n^{(p)}(u_n) < \infty.$$ 

Then, $u$ must be continuous (in fact Hölder continuous with Hölder constant obtained from the Sobolev embedding theorem) and moreover

$$\max_{i=1,\ldots,n} |u_n(x_i) - u(x_i)| \to 0, \quad \text{as } n \to \infty.$$ 

This is the content of Lemma 4.5 in [48]. This type of result rigorously justifies pointwise evaluation of discrete functions with bounded graph $p$-Laplacian seminorm and the stability of this operation as $n \to \infty$.

5.2. Continuum Limits.

5.2.1. Set-up. For the remainder of section 5 we work under the assumption of Gaussian observation noise, so that

$$\Phi(u; y) = \frac{1}{2\sigma^2} |y - G(u)|^2, \quad \Phi_n(u_n, y) = \frac{1}{2\sigma^2} |y - G_n(u_n)|^2.$$ (5.2)

The synthetic data $y$ in our numerical experiments is generated by drawing a sample $\eta \sim N(0, \sigma^2 I_{p \times p})$, and setting

$$y = G(u^\dagger) + \eta,$$

where $u^\dagger$ is the function in the left panel of Figure 4. We consider several choices of $t \geq 0$, number $p$ of labeled data points, and size of observation noise $\sigma > 0$. The parameters $s$ and $\alpha$ in the prior measures are fixed to $s = 5$, $\alpha = 1$ throughout.

The use of Gaussian observation noise, combined with the linearity of our forward and observation maps, allows us to derive closed formulae for the graph and continuum posteriors. We do so in the appendix.

![Figure 4](image.png)

**Figure 4.** From left to right: Initial condition $u^\dagger$ used as ground truth to generate our synthetic data; heat at $t = 0.5$ with initial condition $u^\dagger$; and draw from the continuous prior.
5.2.2. Numerical Results. Here we complement the theory by studying the effect that various model parameters have in the accurate approximation of continuum posteriors by graph posteriors. We emphasize that the continuum posteriors serve as a gold standard for our learning problem: graph posteriors built with appropriate choices of connectivity $\varepsilon$ result in good approximations to continuum posteriors; however, reconstruction of the unknown function $u^\dagger$ is not accurate if the data is not informative enough. In such case, MAPs constructed with graph or continuum posteriors may be far from $u^\dagger$.

In all the figures involving graph-posterior means, these are represented by using a $k$-NN interpolant, as defined in equation (1.4), with $k = 4$. Figure 5 shows a graph-prior draw represented in the point cloud (left), and the associated 4-NN interpolant (right). All the plots have been obtained using the (graph) pCN algorithm. The pCN algorithm was run for $10^5$ iterations, and the last $10^4$ samples were used to compute quantities of interest (e.g. means and variances).

Figure 6 shows graph and continuum posteriors with $t = 0$, $t = 0.1$, and $t = 0.3$. For these plots, a suitable choice of graph connectivity $\varepsilon$ was taken. In all three cases we see remarkable similarity between the graph and continuum posterior means. However, recovery of the initial condition with $t = 0.3$ is unsuccessful: the data does not contain enough information to accurately reconstruct $u^\dagger$. Figure 7 shows graph-posterior means computed in the regime of the first row of Figure 6 using the three graphs in Figure 1. Note that the spectra of the associated graph-Laplacians is represented in Figure 2. It is clear that inappropriate choice of $\varepsilon$ leads to poor approximation of the continuum posterior, and here also to poor recovery of the initial condition $u^\dagger$. This is unsurprising in view of the dramatic effect of the choice of $\varepsilon$ in the approximation properties of the spectrum of the spherical Laplacian, as shown in Figure 2. Note that while the numerical results are outside the asymptotic regime ($n = 1000$ throughout), they illustrate the role of $\varepsilon$. Theorem 4.2 establishes appropriate scalings for successful graph-learning in the large $n$ asymptotic.

![Figure 5](image5.png)

**Figure 5.** Draw from the discrete graph prior on the left, and the corresponding representation visualized using a 4-nearest-neighbors interpolation on the right. Parameters are $\varepsilon = 2n^{-1/4}$, $n = 1000$.  
Figure 6. Means of the discrete and continuum posterior distributions are compared; the plots $P_{i,j}$ are arranged such that $P_{i,1}$ are graph-posterior means, $P_{i,2}$ are continuum posterior means, and $P_{i,3}$ are the differences row-wise. $P_{1,j}$, $P_{2,j}$, $P_{3,j}$ differ in the choice of the time parameter. They are, from the top, $t = [0, 0.1, 0.3]$.

Figure 7. Graph-posterior means computed with the graph-pCN algorithm. All parameters of the learning problem are fixed to $t = 0$, $\sigma = 0.1$, $n = 1000$, and $p = 200$. The three plots show three choices of graph connectivities $\varepsilon = [1, 2, 3] \times n^{-1/4}$ as in Figure 1.
Figure 8. Effect of the parameter $\sigma$ on graph-pCN algorithm. When $\sigma$ is prohibitively small, here $\sigma = 0.001$, the chain fails to mix poorly. With more noise, here $\sigma = 0.1$, the chain mixes rapidly.

Figure 9. Shown here is the graph-pCN’s chain mixing and converging for different values of the parameter $t$. Other parameter values for both chains are the same; note that the variation from $t = 0.1$ to $t = 10$ does not significantly affect the characteristics of the chain.
5.3. Algorithmic Scalability. It is important to stress that the large \( n \) robust performance of pCN methods established in this paper hinges on the existence of a continuum limit for the measures \( \mu_n \). Indeed, the fact that the limit posterior \( \mu \) over infinite dimensional functions can be written as a change of measure from the limit prior \( \pi \) has been rigorously shown to be equivalent to the limit learning problem having \textit{finite} intrinsic dimension [1]. In such a case, a key principle for the robust large \( n \) sampling of the measures \( \mu_n \) is to exploit the existence of a limit density, and use some variant of the dominating measure to obtain proposal samples. It has been established —and we do so here in the context of graph-based methods— that careful implementation of this principle leads to robust MCMC and importance sampling methodologies [32], [1].

A further point to note is that —even though from a theoretical and applied viewpoint it is clearly desirable that the data is informative— computational challenges in Bayesian settings often arise when the data is highly informative. This is so in the context of importance sampling and particle filters [1], [46], where certain notion of distance between prior and proposal characterizes the algorithmic complexity. In the context of the pCN MCMC algorithms, if \( \Phi \) is constant then the algorithm has acceptance probability 1. On the other hand, large Lipschitz constant of \( \Phi \) (which translates to a posterior that is far from the prior) leads to small spectral gap. Indeed, tracking the spectral gap of pCN in terms of model parameters via the understanding of Lipschitz constants is in principle possible, and will be the subject of further work. In particular, small observation noise \( \sigma \) leads to deterioration of the pCN performance, see Figure [8]. This issue may be alleviated by the use of the generalized version of pCN introduced in [44]. Figures [9] and [10] investigate the role of the parameters \( t \) and \( p \). All these figures show the posterior mean at one of the inputs, and the true graph posterior means have been computed with the formulae in the appendix.

Table [11] shows the large \( n \) robustness of pCN methods, while also exhibits its deterioration in the fully supervised case \( n = p \). The table shows the average acceptance probability with model parameters \( \beta = 0.01, p = 200, \varepsilon_n = 2n^{-1/4} \) for the semi-supervised setting, and same parameters but with \( p = n \) for the fully supervised case. The corresponding graph-posterior means are shown in Figure [11].
Table 1. Average acceptance probability for the graph pCN in the semi-supervised setting with constant data-set of size $p = 200$ and increasing number of unlabeled data.

| $n$   | 300 | 600 | 900 | 1200 | 1500 | 2000 |
|-------|-----|-----|-----|------|------|------|
| Acceptance Probability | 0.230 | 0.245 | 0.237 | 0.249 | 0.236 | 0.239 |

Table 2. Deterioration of the average acceptance probability in a fully-supervised setting with $n = p$. The parameter $\beta$ was held constant at $\beta = 0.01$. Additionally, $\varepsilon = 2n^{-1/4}$ and $t = 0$.

| $n = p$ | 300 | 600 | 900 | 1200 | 1500 | 2000 |
|---------|-----|-----|-----|------|------|------|
| Acceptance Probability | 0.4536 | 0.3144 | 0.2360 | 0.1924 | 0.1644 | 0.1100 |

Figure 11. Graph pCN’s robustness with respect to a changing value of $n$. In all plots $P_{i,j}$ above, $p = 200$, $t = 0.1$, $\sigma = 0.1$, and $\varepsilon = 2n^{-1/4}$. The plots are arranged such that $n = [300, 600, 900]$ for $P_{1,j}$ and $n = [1200, 1500, 2000]$ for $P_{2,j}$. The average acceptance probability remains constant with fixed $\beta$, as shown in Table 1.

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Here we exploit the linearity of the forward and observation maps to compute, under a Gaussian observation noise assumption \(5.2\), the mean and covariance of the Gaussian graph and continuum posteriors. These formulae could be useful to understand the approximation of continuum posteriors by graph posteriors, and to provide benchmark for posteriors computed with MCMC methods. For the derivations we use the covariance function representation of Gaussian measures and the theory of Gaussian process regression in \(42\). Throughout we assume that \(s\) is large enough so that the formulae below are well-defined.

We start with the continuum case. Set \(v := Fu\). The prior \(2.1\) on \(u\) induces a prior on \(v \sim GP(0, c_v(x, \tilde{x}))\), where

\[
c_v(x, \tilde{x}) = \sum_{i=1}^{\infty} e^{-2\lambda_i t} (\alpha + \lambda_i)^{-s/2} \psi_i(x) \psi_i(\tilde{x}). \tag{A.1}
\]
Then, we have a regression problem for \( v \) given data \( \vec{y} = [y_1, \ldots, y_p]' \)
\[
y_i = v(x_i) + \eta_i, \quad \eta_i \sim N(0, \gamma^2)
\]
in the form of \([12]\). The posterior distribution of \( v|\vec{y} \) is thus given by a Gaussian process \( GP(m_{v|\vec{y}}(x), c_{v|\vec{y}}(x, \tilde{x})) \), with
\[
m_{v|\vec{y}}(x) = c_v(x, X)'(c_v(X, X) + \gamma^2 I)^{-1}\vec{y},
\]
\[
c_{v|\vec{y}}(x, \tilde{x}) = c_v(x, \tilde{x}) - c_v(x, X)'(c_v(X, X) + \gamma^2 I)^{-1}c_v(\tilde{x}, X),
\]
where we use the following notations:
\[
c_v(x, X) := [c_v(x, x_1), \ldots, c_v(x, x_p)]' \in \mathbb{R}^p,
\]
\[
c_v(X, X) := (c_v(x_i, x_j))_{1 \leq i, j \leq p} \in \mathbb{R}^{p \times p}.
\]
Now the posterior of interest \( \mu \) on \( u \) given \( \vec{y} \) can be recovered by running the heat equation backwards. Namely, we have that \( \mu = GP(m_{u|\vec{y}}(x), c_{u|\vec{y}}(x, \tilde{x})) \) with
\[
m_{u|\vec{y}}(x) = c_w(x, X)'(c_v(X, X) + \gamma^2 I)^{-1}y,
\]
\[
c_{u|\vec{y}}(x, \tilde{x}) = c_u(x, \tilde{x}) - c_u(x, X)'(c_v(X, X) + \gamma^2 I)^{-1}c_u(\tilde{x}, X),
\]
where \( c_v(x, X) \) is a vector made of evaluations of the covariance function of \( w := F^{1/2}u \) at the test and training points. Precisely, its \( j \)-th entry is given by
\[
c_u(x, X)_j = \sum_{i=1}^{\infty} e^{-\lambda_i t}(\alpha + \lambda_i)^{-s/2} \psi_i(x) \psi_i(x_j). \tag{A.3}
\]

There are several points to note about equation (A.2). First, the predictive mean is a linear function of the data \( y \), hence a linear predictor. It is indeed the best linear predictor in a mean-squared error sense \([49]\). Second, since \( c_v(X, X) + \gamma^2 I \) is positive definite, \( c_{u|\vec{y}}(x, \tilde{x}) \leq c_u(x, \tilde{x}) \); thus, conditioning reduces the uncertainty. Moreover, in the limit of noiseless observations \( (\gamma = 0) \) and \( t = 0 \) we recover that \( c_{u|\vec{y}}(x_i, x_j) = 0 \) in the training points. However, even with noiseless observations this is not true if \( t > 0 \). Finally, note the well-known fact that the the posterior covariance \( c_{u|\vec{y}} \) does not depend on the observed data \( \vec{y} \).

Formulae in the discrete setting can be obtained in a similar way, and we omit the details. Plugging in the data \( \vec{y} \) from the continuum setting, we deduce that \( \mu_n = N(m_{u_n|\vec{y}}(x_k), c_{u_n|\vec{y}}(x_k, x_l)) \), with
\[
m_{u_n|\vec{y}}(x_k) = c_{w_n}(x, X)'(c_{v_n}(X, X) + \gamma^2 I)^{-1}\vec{y},
\]
\[
c_{u_n|\vec{y}}(x_k, x_l) = c_{w_n}(x_k, x_l) - c_{w_n}(x_k, X)'(c_{v_n}(X, X) + \gamma^2 I)^{-1}c_{w_n}(x_l, X). \tag{A.4}
\]

In the above equations, all objects indexed by \( n \) constitute straightforward analogues of objects in the continuum, constructed using the graph spectrum rather than the continuum one.

**Appendix B. Proof of Theorem 4.2**

We want to show that
\[
\mathcal{I}_{n\uparrow} \mu_n \rightarrow_{\mathcal{P}(L^2(\gamma))} \mu, \quad \text{as } n \rightarrow \infty. \tag{B.1}
\]

**Step 0:** The proof of Theorem 4.1. in \([23]\) shows that
\[
\pi_n \rightarrow_{\mathcal{P}(TL^2)} \pi, \quad \text{as } n \rightarrow \infty,
\]
under the assumptions of Theorem 4.2 (in particular removing the upper bound assumption on \( \varepsilon_n \) from Theorems 4.1 and 4.4 in \([23]\)). Likewise the proof of Theorem 4.4 in \([23]\) establishes the \( \Gamma \)-convergence of the energies
\[
J_n(\nu_n) := D_{KL}(\nu_n||\pi_n) + \int_{L^2(\gamma_n)} \phi_n(u_n; y)d\nu_n(u_n), \quad \mu_n \in \mathcal{P}(L^2(\gamma_n)),
\]

towards the energy
\[
J(\nu) = D_{\text{KL}}(\nu\|\pi) + \int_{L^2(\gamma)} \phi(u; y)d\nu(u), \quad \nu \in \mathcal{P}(L^2(\gamma))
\]
in the $\mathcal{P}(TL^2)$-sense, under the assumptions of Theorem 4.2. In particular,
\[
\mu_n \to_{\mathcal{P}(TL^2)} \mu, \quad n \to \infty,
\]

because $\mu_n$ is the minimizer of $J_n$ and $\mu$ is the minimizer of $J$ (see the variational characterization of posterior distributions in [24]).

**Step 1:** We claim that $\{I_n(\mu_n)\}_{n \in \mathbb{N}}$ is pre-compact with respect to the weak convergence of probability measures on $L^2(\gamma)$. By Lemma 5.1 in [23] it is enough to show that

(i) $\sup_{n \in \mathbb{N}} D_{\text{KL}}(I_n(\mu_n)\|I_n(\pi_n)) < +\infty$; and

(ii) $I_n(\pi_n) \to_{\mathcal{P}(L^2(\gamma))} \pi$.

Let us start with (i). Step 0 implies that
\[
\lim_{n \to \infty} \min_{\nu_n} J_n(\nu_n) = \min_{\nu} J(\nu) < +\infty.
\]
Given that $\mu_n$ is the minimizer of $J_n$ and $\mu$ is the minimizer of $J$, it follows that
\[
\lim_{n \to \infty} J_n(\mu_n) = J(\mu) < +\infty.
\]

Combining the previous fact with the chain of inequalities
\[
D_{\text{KL}}(I_n(\mu_n)\|I_n(\pi_n)) \leq D_{\text{KL}}(\mu_n\|\pi_n) \leq J_n(\mu_n)
\]
gives (i).

We now show (ii). Consider an orthonormal basis of eigenvectors $\{\psi^n_i, \ldots, \psi^n_k\}$ of $\Delta_{M_n}$ and an orthonormal basis $\{\psi_1, \ldots, \psi_n, \ldots\}$ of eigenfunctions of $\Delta_M$. By the results in [28] we can assume without the loss of generality that, for all $j \in \mathbb{N}$,
\[
\psi^n_j \to_{TL^2} \psi_j, \quad n \to \infty.
\]

Let $(\hat{\Omega}, \hat{F}, \hat{\pi})$ be a probability space supporting i.i.d. random variables $\{\xi_i\}_{i \in \mathbb{N}}$ with $\xi_i \sim N(0,1)$ and consider
\[
X_n = \sum_{i=1}^{k_n}(\alpha + \lambda^n_i)^{-s/4}\xi_i\psi^n_i, \quad X = \sum_{i=1}^{\infty}(\alpha + \lambda_i)^{-s/4}\xi_i\psi_i,
\]
where, recall, $k_n$ is the truncation level of the prior $\pi_n$. Notice that $X_n \sim \pi_n$, $X \sim \pi$ and $I_n(X_n)$ is distributed according to $I_n(\pi_n)$. For any fixed $i = 1, \ldots, k_n$ it follows from the first part of the proof of Theorem 1.10 in [21] that
\[
\|I_n(\psi^n_i)\|_{L^2(\gamma)} \leq \|I_n(\psi^n_i) - \psi_i\|_{L^2(\gamma)} + \|\psi_i\|_{L^2(\gamma)} \leq C, \quad \text{(B.2)}
\]
where $C$ is a constant independent of $i = 1, \ldots, k_n$ and $n$. It then follows that for every $l \in \mathbb{N}$,
\[
\|I_n(X_n) - X\|_{L^2(\gamma)} \leq \left\|\sum_{i=1}^{l}(\alpha + \lambda^n_i)^{-s/4}\xi_iI_n(\psi^n_i) - \sum_{i=1}^{l}(\alpha + \lambda_i)^{-s/4}\xi_i\psi_i\right\|_{L^2(\gamma)}
\]
\[
+ \sum_{i=l}^{k_n}(\alpha + \lambda^n_i)^{-s/4}|\xi_iI_n(\psi^n_i)\|_{L^2(\gamma)} + \sum_{i=l}^{\infty}(\alpha + \lambda_i)^{-s/4}|\xi_i||\psi_i\|_{L^2(\gamma)}
\]
\[
\leq \left\|\sum_{i=1}^{l}(\alpha + \lambda^n_i)^{-s/4}\xi_iI_n(\psi^n_i) - \sum_{i=1}^{l}(\alpha + \lambda_i)^{-s/4}\xi_i\psi_i\right\|_{L^2(\gamma)} + C\sum_{i=l}^{\infty}(\alpha + \lambda_i)^{-s/4}|\xi_i|,
\]
where $C$ is a constant that does not depend on $n$; we have used the bounds (B.2) on $\|I_n(\psi^n_i)\|_{L^2(\gamma)}$ and the bounds [5,1] for $\lambda^n_i$ in terms of $\lambda_i$ for $i = 1, \ldots, k_n$. We can then take expectations and
\[ \limsup_{n \to \infty} \mathbb{E} \left( \| J_n(X_n) - X \|_{L^2(\gamma)} \right) \leq C \sum_{i=1}^{\infty} (\alpha + \lambda_i)^{-s/4}. \]

Since the above is true for every \( l \) and the series is convergent, (ii) follows.

An application of Lemma 5.1 in [23] allows us to deduce that \( \{ I_{n;\mu} \}_{n \in \mathbb{N}} \subseteq \mathcal{P}(L^2(\gamma)) \) is pre-compact and, moreover, that each of its cluster points is a measure that is absolutely continuous with respect to \( \pi \). We can then assume without the loss of generality that, for some \( \tilde{\mu} \in \mathcal{P}(L^2(\gamma)), \]

\[ I_{n;\mu} \to \mathcal{P}(L^2(\gamma)) \tilde{\mu}, \quad \text{as } n \to \infty. \]

**Step 2:** To show (B.1) it is then enough to prove that the finite dimensional projections of \( \tilde{\mu} \) coincide with those of \( \mu \). More precisely, we identify \( u \in L^2(\gamma) \) with the infinite vector \((u_1, u_2, \ldots)\) denoting the coefficients of \( u \) in the basis \( \{ \psi_1, \psi_2, \ldots \} \) and define \( \text{Proj}_j(u) := \sum_{i=1}^{j} u_i \psi_i \); we need to show that for arbitrary \( j \in \mathbb{N} \) we have

\[ \text{Proj}_j \tilde{\mu} = \text{Proj}_j \mu. \]

From Step 0 and Skorohod’s theorem, we know there exists a probability space \((\tilde{\Omega}, \tilde{F}, \tilde{P})\) supporting random variables \( \{ X^\gamma_n \}_{n \in \mathbb{N}} \) and \( X^\gamma \) with \( X^\gamma_n \sim \mu_n \) and \( X^\gamma \sim \mu \) and for which \( X^\gamma_n \to_{TL^2} X^\gamma \) for \( \tilde{P} \)-a.e. \( \tilde{\omega} \in \tilde{\Omega} \). We can then write

\[ X^\gamma_n = \sum_{i=1}^{k_n} a^i_n \psi^i_n, \quad X^\gamma = \sum_{i=1}^{\infty} a_i \psi_i, \]

for some random variables \( a^i_n \) and \( a_i \). Notice that the continuity of inner products with respect to \( TL^2 \)-convergence (see Proposition 2.6 in [25]) implies that

\[ \lim_{n \to \infty} a^i_n = a_i, \quad \tilde{P} \text{-a.e.} \]

Now, for every fixed \( l \geq j \) we can write

\[ \text{Proj}_j(I_n(X^\gamma_n)) = \sum_{i=1}^{l} a^i_n \text{Proj}_j(I_n(\psi^i_n)) + \sum_{i=l+1}^{k_n} a^i_n \text{Proj}_j(I_n(\psi^i_n)). \]  

(B.3)

The left hand side of the above expression is seen to converge weakly towards \( \text{Proj}_{j+1} \tilde{\mu} \) because \( I_n(X^\gamma_n) \sim I_{n;\mu}, \ I_{n;\mu} \to_{\mathcal{P}(L^2(\gamma))} \tilde{\mu}, \) and because \( \text{Proj}_j \) is continuous. On the other hand, the first term on the right hand side is seen to converge \( \tilde{P} \)-a.e. towards \( \sum_{i=1}^{j} a_i \text{Proj}_j(\psi_i) = \sum_{i=1}^{j} a_i \psi_i \) because

\[ I_n(\psi^i_n) \to_{L^2(\gamma)} \psi_i, \quad \text{as } n \to \infty, \]

which follows from Theorem 1.10 in [21] (it is at this stage that we need the extra technical condition on \( \varepsilon_n \)); in particular this term converges weakly towards \( \text{Proj}_{j+1} \mu \). To show \( \text{Proj}_{j+1} \tilde{\mu} = \text{Proj}_{j+1} \mu \) it is then enough, by Slutsky’s theorem, to prove that \( \| \sum_{i=l+1}^{k_n} a^i_n \text{Proj}_j(I_n(\psi^i_n)) \|_{L^2(\gamma)} \) converges in probability towards zero.

To see this, first notice that

\[ \left\| \sum_{i=l+1}^{k_n} a^i_n \text{Proj}_j(I_n(\psi^i_n)) \right\|_{L^2(\gamma)} \leq C \sum_{i=l+1}^{k_n} |a^i_n|. \]

Fix \( t > 0 \). Observe that the expression

\[ \limsup_{n \to \infty} \tilde{P} \left( \left\| \sum_{i=l+1}^{k_n} a^i_n \text{Proj}_j(I_n(\psi^i_n)) \right\|_{L^2(\gamma)} > t \right) \]
is independent of $l$. Then,

$$q_j(t) := \limsup_{n \to \infty} \hat{P}\left( \left\| \sum_{i=l+1}^{k_n} a^n_i \text{Proj}_j(\mathcal{I}_n(\psi^n_i)) \right\|_{L^2(\gamma)} > t \right)$$

$$\leq \limsup_{n \to \infty} \hat{P}\left( \sum_{i=l+1}^{k_n} |a^n_i| > \frac{t}{C} \right).$$

On the other hand, identifying the elements in the support of $\pi_n$ with $\mathbb{R}^{k_n}$ (i.e. writing $u_n \in \text{supp}(\pi_n)$ in the basis $\{\psi^n_1, \ldots, \psi^n_{k_n}\}$) and letting $A_{n,l,t}$ be the set

$$A_{n,l,t} := \left\{ x \in \mathbb{R}^{k_n} : \sum_{i=l+1}^{k_n} |x_i| > \frac{t}{C} \right\},$$

we see that

$$\hat{P}\left( \sum_{i=l+1}^{k_n} |a^n_i| > \frac{t}{C} \right) = \mu_n(A_{n,l,t}) = \frac{1}{Z_n} \int_{A_{n,l,t}} \exp(-\Phi_n(x))d\pi_n(x) \leq \frac{1}{Z_n} \pi_n(A_{n,l,t}),$$

and hence

$$\limsup_{n \to \infty} \hat{P}\left( \sum_{i=l+1}^{k_n} |a^n_i| > \frac{t}{C} \right) \leq \frac{1}{Z} \pi \left( \{ u \in L^2(\gamma) : \sum_{i=l+1}^{\infty} |u_i| > \frac{t}{C} \} \right).$$

In the above $Z$ and $Z_n$ are the normalization constants from (1.1) and (1.2) respectively. Therefore,

$$q_j(t) \leq \frac{1}{Z} \pi \left( \{ u \in L^2(\gamma) : \sum_{i=l+1}^{\infty} |u_i| > \frac{t}{C} \} \right).$$

Taking now the limit as $l \to \infty$ of the right hand side of the above expression we deduce that $q_j(t) = 0$. Since this is true for arbitrary $t > 0$ we deduce that indeed $||\sum_{i=l+1}^{k_n} a^n_i \text{Proj}_j(\mathcal{I}_n(\psi^n_i))||_{L^2(\gamma)}$ converges in probability towards zero and the proof is now complete.

**Remark B.1.** In the above proof we have used results from [21] on Voronoi extensions, but it is clear that analogue results can be deduced for more general interpolants $\{\mathcal{I}_n\}_{n \in \mathbb{N}}$ as long as one can show the following:

1. (Uniform $L^2$-boundedness) There is a constant $C > 0$ such that $||\mathcal{I}_n(\psi^n_i)||_{L^2(\gamma)} \leq C$ for every $i = 1, \ldots, k_n$ and for every $n$.

2. (Consistency) For every $i \in \mathbb{N}$ we have $\mathcal{I}_n(\psi^n_i) \to L^2(\gamma) \psi_i$.

**Appendix C. Proof of Theorem 4.6**

The proof of Theorem 4.6 is based on the proof of Theorem in [32] which in turn makes use of the following weak form of Harris theorem from [31]. We let $\mathcal{H}$ be a separable Hilbert space and for a distance like function $\tilde{d} : \mathcal{H} \times \mathcal{H} \to [0, \infty)$ define the associated Wasserstein distance ($1$-OT distance) on $\mathcal{P}(\mathcal{H})$

$$\tilde{d}(\mu, \nu) := \inf_{\theta \in \Gamma(\mu, \nu)} \int_{\mathcal{H} \times \mathcal{H}} \tilde{d}(u, w)d\theta(u, w), \quad \mu, \nu \in \mathcal{P}(\mathcal{H}),$$

where $\Gamma(\mu, \nu)$ denotes the set of couplings between $\mu$ and $\nu$.

**Theorem C.1** (Weak Harris Theorem; Theorem 4.7 in [31]). Let $\mathcal{H}$ be a separable Hilbert space and let $P$ be a transition kernel for a discrete time Markov chain with state space $\mathcal{H}$ for which the following conditions are satisfied:
(1) (Lyapunov functional) There exists a lower semi-continuous function $V : \mathcal{H} \to [0, \infty)$ such that
\[
P V(u) := \int_{\mathcal{H}} V(w) P(u, dw) \leq l V(u) + K, \quad \forall u \in \mathcal{H},
\]
where $K > 0$ and $0 < l < 1$ are some constants.

(2) (d-contraction) There exists a distance like function $d : \mathcal{H} \times \mathcal{H} \to [0, 1]$ and a constant $\varrho \in (0, 1)$ such that, for all $u, w \in \mathcal{H}$ with $d(u, w) < 1$,
\[
d(u, w) \leq \varrho.
\]

(3) (d-smallness of level sets of $V$) For the distance like function $d$ above, the functional $V$ and the constant $K$ in (C.2), there exists $\vartheta \in (0, 1)$ such that, for all $u, w$ with $V(u), V(w) \leq 4K$,
\[
d(u, w) \leq \vartheta.
\]

Then, the Markov chain $P$ has a $\tilde{d}$-Wasserstein spectral gap where $\tilde{d}$ is the distance like function:
\[
\tilde{d}(P^j \mu, P^j \nu) \leq C \exp(-\lambda j) \tilde{d}(\mu, \nu), \quad \forall \mu, \nu \in \mathcal{P}(\mathcal{H}), \quad \forall j \in \mathbb{N}.
\]

Remark C.2. As remarked in [31], we highlight that the second hypothesis is an assumption that holds for points $u, w$ with $d(u, w) < 1$ and that nothing is being stated about points for which $d(u, w) = 1$. The observation here is that even if one can not deduce a Wasserstein spectral gap for the distance like function $d$, one can still obtain a Wasserstein spectral gap for the distance like function $\tilde{d}$.

It is possible to quantify the constants $\lambda$ and $C$ in the conclusion of Theorem C.1 in terms of the parameters $l, K, \varrho, \vartheta$. Here, however, we are simply interested in pointing out how changing the parameters in the assumptions affect the constants in the conclusions. In particular, as can be seen from the proof in [31], the larger each of the parameters $l, K, \varrho, \vartheta$ is, the larger the constant $C$ and the smaller the constant $\lambda$ gets. In other words, enlarging any of the parameters $l, K, \varrho, \vartheta$ results on a worse spectral gap. This observation is relevant in order to obtain uniform spectral gaps for a sequence of Markov chains. Namely, suppose that we have Markov kernels $\{P_n\}_{n \in \mathbb{N}}$ (with perhaps different state spaces) for which we can find distance like functions $\{d_n\}_{n \in \mathbb{N}}$ and Lyapunov functionals $\{V_n\}_{n \in \mathbb{N}}$ satisfying the conditions in theorem C.1 with constants $l, K, \varrho, \vartheta$ (independent of $n$). We can then deduce that the constants $\lambda > 0$ and $C > 0$ in the conclusion of the weak Harris theorem can be chosen independently of $n$. It is precisely this observation that is exploited in [32].

It is then important to highlight the main differences between our set-up and the one in [32]. First, the Markov kernels that we consider in this paper are not defined on the same state space and in particular the log-likelihoods $\Phi_n, \Phi$, although related, are different. Secondly, our discretization of the continuum prior $\pi$ is the surrogate $\pi_n$ supported on $L^2(\gamma_n)$ and not the discretization constructed by truncating the Karhunen Loève expansion of the continuum prior. These differences in the set-ups, however, do not prevent us from using the same proof in [32] thanks to the following three observations.

(1) (Uniform control on local Lipschitz constants of log-likelihoods)

**Lemma C.3.** There exists a constant $L > 0$ such that for every $r > 0$ and $n \in \mathbb{N}$
\[
\sup_{u_n, v_n \in B^n_r} \frac{|\Phi_n(u_n) - \Phi_n(v_n)|}{\|u_n - v_n\|} \leq L r, \quad \sup_{u, v \in B_r} \frac{|\Phi(u) - \Phi(v)|}{\|u - v\|} \leq L r,
\]
where in the above $B^n_r$ (or $B_r$) denotes the ball in $L^2(\gamma_n)$ ($L^2(\gamma)$) centered at the origin and with radius $r$. 

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Proof. Recall that
\[ \Phi_n(u_n) = \phi^\gamma(G_n(u_n)), \quad u_n \in L^2(\gamma_n), \]
and so, thanks to Assumptions 4.5 on \( \phi^\gamma \), we get
\[ |\Phi_n(u_n) - \Phi_n(v_n)| \leq |\phi^\gamma(G_n(u_n)) - \phi^\gamma(G_n(v_n))| \]
\[ \leq C_1 \max\{|G_n(u_n)|, |G_n(v_n)|, 1\} |G_n(u_n) - G_n(v_n)|. \]
Now, recall that the vector \( G_n(u_n) - G_n(v_n) \in \mathbb{R}^p \) has coordinates
\[ [G_n(u_n) - G_n(v_n)]_i = \frac{1}{\gamma_n(B_\delta(x_i))} \|F_n(u_n) - F_n(v_n)\|_{L^2(\gamma_n)}, \quad i = 1, \ldots, p. \]
From the Cauchy-Schwartz inequality it follows that
\[ ||G_n(u_n) - G_n(v_n)||_i \leq \frac{1}{(\gamma_n(B_\delta(x_i)))^{1/2}} \|F_n(u_n) - F_n(v_n)\|_{L^2(\gamma_n)} \]
\[ \leq \frac{1}{(\gamma_n(B_\delta(x_i)))^{1/2}} \|u_n - v_n\|_{L^2(\gamma_n)}, \]
where in the last line we have used the fact that \( F_n \) is a linear map as well as the fact that it is a contraction. Since
\[ \gamma_n(B_\delta(x_i)) \to \gamma(B_\delta(x_i)), \quad \text{as } n \to \infty, \]
(C.3)
it follows that
\[ |G_n(u_n) - G_n(v_n)| \leq C_2 \|u_n - v_n\|_{L^2(\gamma)}, \]
where \( C_2 \) is independent of \( u_n, v_n \in L^2(\gamma_n) \) or \( n \in \mathbb{N} \). Therefore, there exists a constant \( C_3 \) (independent of \( u_n, v_n \in L^2(\gamma_n) \) or \( n \in \mathbb{N} \)) such that
\[ |\Phi_n(u_n) - \Phi_n(v_n)| \leq C_3 \max\{\|u_n\|_{L^2(\gamma_n)}, \|v_n\|_{L^2(\gamma_n)}, 1\} \|u_n - v_n\|_{L^2(\gamma_n)}. \]
Naturally the same analysis holds for \( \Phi \) and this finishes the proof. \( \square \)

Remark C.4. We remark that the same conclusions in the previous lemma hold for non-linear forward maps \( F_n, F \) that are (uniformly in \( n \)) Lipschitz and have (uniformly in \( n \)) linear growh.

(2) (Dominating limiting measure) We make use of a “limiting measure” that dominates the measures \( \pi_n \) in the sense described below. Notice that we can not use the continuum prior \( \pi \), but a slight modification of it will suffice.

Lemma C.5. There exists a large enough \( \rho > 0 \), such that the Gaussian measure
\[ \pi^\rho := N\left(0, (1 + \rho)^2(\alpha I - \Delta_M)^{-s}\right), \]
satisfies
\[ \int_{L^2(\gamma_n)} g(\|u_n\|_{L^2(\gamma_n)}) d\pi_n(u_n) \leq \int_{L^2(\gamma)} g(\|u\|_{L^2(\gamma)}) d\pi^\rho(u), \]
for every \( n \in \mathbb{N} \) and every increasing function \( g : [0, \infty) \to \mathbb{R} \). In particular, for every \( r > 0 \) and every \( n \in \mathbb{N} \)
\[ \pi_n \left( L^2(\gamma_n) \setminus B_r^\rho \right) \leq \pi^\rho \left( L^2(\gamma) \setminus B_r \right). \]
Proof. Thanks to inequality (4.6), we can find \( \rho > 0 \) such that for every \( n \in \mathbb{N} \)
\[ \frac{1}{(\alpha + \lambda_i^2)^s} \leq \frac{1 + \rho}{(\alpha + \lambda_i)^s}, \quad \forall i = 1, \ldots, k_n. \]
Using the Karhunen Loève expansion to represent random variables of \( \pi_n \) and \( \pi^\rho \) we can easily deduce the inequality for the measures of complements of balls (last inequality). The inequality for a general increasing function \( g \) follows from a standard approximation with increasing step functions. \( \square \)
(3) (Uniform lower bound for acceptance probability) The next lemma provides uniform control on the acceptance probability of the pCN algorithm when a proposal lies within a fixed distance of a contracted version of the current state of the chain. More precisely:

**Lemma C.6.** Let \( a(u, v) \) be the acceptance probability in Algorithm 1 for continuum pCN and \( a_n(u_n, v_n) \) the acceptance probability in Algorithm 2 for graph pCN. Fix an arbitrary \( r > 0 \). Then, there exists \( c \in \mathbb{R} \) such that

\[
\inf_{v_n \in B^r_n(\sqrt{1-\beta^2} v_n)} a_n(u_n, v_n) \geq \exp(c) > 0, \quad \inf_{w \in B_r(\sqrt{1-\beta^2} v)} a(v, w) \geq \exp(c) > 0
\]

for arbitrary \( v_n \in L^2(\gamma_n) \), \( v \in L^2(\gamma) \) and \( n \in \mathbb{N} \).

**Proof.** First of all notice that

\[
\|G_n\| \leq \|O_n\| \cdot \|F_n\| \leq \|O_n\|
\]

where in the last inequality we have used the fact that \( F_n \) is a contraction. Thanks to (C.3) it follows that

\[
\|O_n\| \to \|O\|, \quad \text{as } n \to \infty,
\]

and in particular we can find a constant \( \tilde{K} \) (independent of \( n \)) such that

\[
\|G_n\| \leq \tilde{K}.
\]

Let \( v_n, w_n \in L^2(\gamma_n) \) be such that \( w_n \in B^r_n(\sqrt{1-\beta^2} v_n) \). Then,

\[
|G_n(w_n) - \sqrt{1-\beta^2} G_n(v_n)| = |G_n(w_n) - \sqrt{1-\beta^2} v_n| \leq \|G_n\| \|w_n - \sqrt{1-\beta^2} v_n\|_{L^2(\gamma_n)} \leq \tilde{K} r =: K.
\]

From Assumptions 4.5 we deduce that

\[
\Phi_n(v_n) - \Phi_n(w_n) = \phi^\beta(G_n(v_n)) - \phi^\beta(G_n(w_n)) \geq c,
\]

for a \( c \) that is independent of \( n \). Hence,

\[
\inf_{v_n \in B^r_n(\sqrt{1-\beta^2} v_n)} a(v_n, w_n) \geq \exp(c) > 0.
\]

Naturally the same analysis holds for \( \Phi \) and this finishes the proof. □

**Remark C.7.** We remark that the same conclusions in the previous lemma hold for non-linear forward maps \( F_n, F \) that are (uniformly in \( n \)) Lipschitz, have (uniformly in \( n \)) linear growth and are positively homogeneous of degree one.

**Proof of Theorem 4.6.** Lemmas C.3, C.5 and C.6 allow us to use the same proof of Theorem in [32] (where in our case the analysis is done with \( \pi^\delta \) from Lemma C.5) and check that the conditions of the weak Harris theorem (with distance like functional \( d \) and Lyapunov functional \( V \) as in the statement of our theorem) are satisfied with constants \( l, K, \varrho, \vartheta \) that are independent of the discretization.

□

**Proof of Corollary 4.7.** By Proposition 2.8 and Lemma 2.9 in [32], and the reversibility of the Markov kernel of the pCN algorithm, it is enough to check that the space

\[
\text{Lip}(\tilde{d}) \cap L^\infty(\mathcal{H}; \mu),
\]

is dense in \( L^2(\mathcal{H}; \mu) \). Here \( \tilde{d} \) denotes the distance-like function from Theorem 4.6 and \( \mu \) stands for the invariant measure of the Markov chain (in this case the posterior distribution). In the finite dimensional case (i.e. \( \mathcal{H} = L^2(\gamma_n) \)) this is a simple consequence of a standard mollification argument. More precisely, it follows from the following observations:

1. For every \( R > 0 \), \( \|\cdot\|\)-Lipschitz functions on \( \mathcal{B}^n_R \) are also \( \tilde{d} \)-Lipschitz on \( \mathcal{B}^n_R \).
2. \( \|\cdot\|\)-Lipschitz functions on \( \mathcal{B}^n_R \) are dense in \( L^2(\mathcal{B}^n_R; \mu) \) (by mollification).
(3) $f \in L^2(\mathcal{H}; \mu)$ can be approximated with $\{f_k\}_{k \in \mathbb{N}}$, where

$$f_k(u) := \eta_k(||u||) \min\{\max\{f(u), -k\}, k\}, \quad u \in \mathcal{H},$$

where $\eta_k : [0, \infty) \to [0, 1]$ is a smooth cut-off function which satisfies $\eta_k(r) = 1$ if $r < k$ and $\eta_k(r) = 0$ if $r > 2k$.

For the infinite dimensional case it is enough to reduce the problem to the finite dimensional case. This reduction is achieved as follows. Without the loss of generality an arbitrary element $u \in \mathcal{H}$ can be written as $u = (u_1, u_2, \ldots)$ and for every $k \in \mathbb{N}$ we may consider the projection:

$$\Pi^c_k : u \in \mathcal{H} \mapsto (u_{k+1}, u_{k+2}, \ldots),$$

and the measure $\mu^c_k := \Pi^c_k \mu$. For an arbitrary $f \in L^2(\mathcal{H}; \mu)$, we can then define the sequence $\{f_k\}_{k \in \mathbb{N}} \subseteq L^2(\mathcal{H}; \mu)$ defined by

$$f_k(u) := \int f(u_1, \ldots, u_k, v_{k+1}, v_{k+2}, \ldots) d\mu^c_k(v_{k+1}, v_{k+2}, \ldots), \quad u \in \mathcal{H}$$

Notice that for each $k$ the function $f_k$ depends only on the first $k$ coordinates of $u$ and so we can apply the result for the finite dimensional case to approximate $f_k$ with functions in $Lip(\bar{d}) \cap L^\infty(\mathcal{H}; \mu)$. From the straightforward fact that $f_k \to L^2(\mathcal{H}; \mu)$, the approximation of functions in $L^2(\mathcal{H}; \mu)$ with functions in $Lip(\bar{d}) \cap L^\infty(\mathcal{H}; \mu)$ now follows. 

\[\square\]

### Appendix D. Verification of Hypotheses for Gaussian and Probit Noise Models

#### D.1. Gaussian

Let us show that the Gaussian model satisfies the assumptions [4.5]

(1) Let $K > 0$ and let $\tau > 0$ be such that $(1 - \tau) > (1 + \tau)(1 - \beta^2)$. For such $\tau$ choose $R = R_\tau > 0$ large enough so that if $\bar{u} \in \mathbb{R}^p$ satisfies $|\bar{u}| \geq R$ then

$$(1 - \tau)|\bar{u}|^2 \leq |\bar{u} - \bar{y}|^2 \leq (1 + \tau)|\bar{u}|^2.$$  

Let $\bar{v}, \bar{w} \in \mathbb{R}^p$ be such that $|\bar{w} - \sqrt{1 - \beta^2} \bar{v}| \leq K$. If $|\bar{w}| \leq R + K$, then

$$|\bar{v} - \bar{y}|^2 - |\bar{v} - \bar{y}|^2 \geq 0 - 2|\bar{y}|^2 - 2(R + K)^2.$$ 

On the other hand, if $|\bar{w}| \leq R + K$, we see that

$$R + K \leq |\bar{w}| \leq \sqrt{1 - \beta^2} |\bar{v}| + K,$$

and it follows that

$$|\bar{v} - \bar{y}|^2 - |\bar{v} - \bar{y}|^2 \geq (1 - \tau)|\bar{v}|^2 - (1 + \tau)|\bar{w}|^2 \geq ((1 - \tau) - (1 + \tau)(1 - \beta^2))|\bar{v}|^2 - 2(1 + \tau)\sqrt{1 - \beta^2} K |\bar{v}| - (1 + \tau) K^2 \geq C_1,$$

for some real number $C_1$.

From the above analysis we deduce that for every $\bar{v}, \bar{w}$ with $|\bar{w} - \sqrt{1 - \beta^2} \bar{v}| \leq K$,

$$\phi^\bar{y}(\bar{v}) - \phi^\bar{y}(\bar{w}) \geq c.$$

for some $c \in \mathbb{R}$.

(2) The second assumption is easily seen to be satisfied by the Gaussian model
D.2. **Probit.** Let us show that the probit model satisfies the assumptions 4.5.

(1) Let $K > 0$ and consider $\vec{v}, \vec{w} \in \mathbb{R}^p$ such that $|\vec{w} - \sqrt{1 - \beta^2} \vec{v}| \leq K$. Then,

$$|y_i w_i - \sqrt{1 - \beta^2} y_i v_i| = |w_i - \sqrt{1 - \beta^2} v_i| \leq |\vec{v} - \sqrt{1 - \beta^2} \vec{w}| \leq K, \quad i = 1, \ldots, p,$$

where the first equality follows from the fact that $y_i \in \{-1, 1\}$. In particular,

$$|y_i w_i| \leq K + \sqrt{1 - \beta^2} |y_i v_i|, \quad i = 1, \ldots, p$$

(D.1)

Notice that the function $t \in \mathbb{R} \mapsto -\log(\Psi(t))$ is decreasing. Hence, if $y_i w_i > -(1/(1 - \sqrt{1 - \beta^2}) + 1)K$ we see that

$$-\log(\Psi(y_i v_i)) - (-\log(\Psi(y_i v_i))) \geq 0 + \log(\Psi(-(1/(1 - \sqrt{1 - \beta^2}) + 1)K)).$$

On the other hand, if $y_i w_i < -(1/(1 - \sqrt{1 - \beta^2}) + 1)K$ we deduce from (D.1) that $y_i v_i < -K/(1 - \sqrt{1 - \beta^2}) < 0$ and from (D.2) we deduce

$$y_i v_i \leq \sqrt{1 - \beta^2} y_i v_i - K \leq y_i w_i,$$

from where it follows that

$$-\log(\Psi(y_i v_i)) - (-\log(\Psi(y_i v_i))) \geq 0.$$

From the above analysis we deduce that for every $\vec{v}, \vec{w}$ with $|\vec{w} - \sqrt{1 - \beta^2} \vec{v}| \leq K$,

$$\phi^{\vec{y}}(\vec{v}) - \phi^{\vec{y}}(\vec{w}) \geq c := p \log(\Psi(-(1/(1 - \sqrt{1 - \beta^2}) + 1)K)).$$

(2) Let us now check that the probit model satisfies the second assumption on $\phi^{\vec{y}}$. Since the function

$$g : t \in \mathbb{R} \mapsto -\log(\Psi(t))$$

is decreasing, convex, and converges to zero as $t \to \infty$, the first assumption on $\phi^{\vec{y}}$ will hold if we can show that

$$\limsup_{t \to -\infty} \frac{|g'(t)|}{|t|} < \infty.$$

This however follows from the fact that

$$g'(t) = \frac{-e^{-t^2/2}}{\int_{-\infty}^{t} e^{-r^2/2} dr},$$

and the well known fact that

$$\frac{e^{-t^2/2}}{2|t|} \leq \int_{-\infty}^{t} e^{-r^2/2} dr,$$

for all negative enough $t$.

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