Concavity of certain matrix trace and norm functions. II

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Abstract

We refine Epstein’s method to prove joint concavity/convexity of matrix trace functions of Lieb type $\text{Tr} f(\Phi(A^p)^{1/2}\Psi(B^q)\Phi(A^p)^{1/2})$ and symmetric (anti-) norm functions of the form $\|f(\Phi(A^p)\sigma\Psi(B^q))\|$, where $\Phi$ and $\Psi$ are positive linear maps, $\sigma$ is an operator mean, and $f(x^\gamma)$ with a certain power $\gamma$ is an operator monotone function on $(0, \infty)$. Moreover, the variational method of Carlen, Frank and Lieb is extended to general non-decreasing convex/concave functions on $(0, \infty)$ so that we prove joint concavity/convexity of more trace functions of Lieb type.

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1 Introduction

In the present paper we consider two-variable matrix functions

$$F(A, B) = f(\Phi(A^p)^{1/2}\Psi(B^q)\Phi(A^p)^{1/2}),$$

$$F(A, B) = f(\Phi(A^p)\sigma\Psi(B^q)),$$

where $A, B$ are positive definite matrices, $p, q$ are real parameters, $\Phi, \Psi$ are (strictly) positive linear maps, $\sigma$ is an operator mean, and $f$ is a real function on $(0, \infty)$. The problem of our concern is joint concavity/convexity of trace and norm functions of such $F(A, B)$ as above. The problem originated with seminal papers of Lieb [17] and Epstein [10] in 1973. In [17], motivated by a conjecture on Wigner-Yanase-Dyson skew information, Lieb established the so-called Lieb concavity/convexity for the matrix trace function $(A, B) \mapsto \text{Tr} X^*A^pX B^q$, that is a special case of (1.1) when $\Phi = X^*X$, $\Psi = \text{id}$ and $f(x) = x$. An equivalent reformulation is Ando’s matrix concavity/convexity of

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\[(A, B) \mapsto A^p \otimes B^q\] in [1]. On the other hand, in [10] Epstein developed a complex function method using theory of Pick functions, called Epstein’s method, to prove concavity of the trace function \(A \mapsto \text{Tr} \left( X^* A^p X \right)^{1/p}\).

In these years, big progress in the subject matter has been made by several authors. For instance, in [8, 9] Carlen and Lieb extensively developed concavity/convexity of the trace functions of the forms \(\text{Tr} \left( X^* A^p X \right)^s\) and \(\text{Tr} \left( A^p + B^p \right)^s\) of Minkowski type. Very recently, in [7] they with Frank made the best use of the variational formulas discovered in [9] to obtain concavity/convexity of the trace functions
\[(A, B) \mapsto \text{Tr} \left( A^{p/2} B^q A^{p/2} \right)^s,\] (1.3) a special case of the trace functions of (1.1) with \(f(x) = x^s\). In our previous papers [12, 14] we refined Epstein’s complex function method to prove joint concavity/convexity results for the trace functions of (1.1) and for the norm/trace functions of (1.2) in the case \(f(x) = x^s\). For additional relevant results see [7, 9, 14] and references therein. Moreover, it is worth noting that our problem on concavity/convexity of (1.3) also emerges from recent developments of new Rényi relative entropies relevant to quantum information theory. That is closely related to monotonicity of those relative entropies under quantum channels (i.e., completely positive and trace-preserving maps), as mentioned in the last part of [7] (see also [2] and references therein).

The present paper is a continuation of [12, 14]. In Sections 2 and 3 we further refine Epstein’s method used in [12, 14] and prove concavity/convexity theorems for the trace functions of (1.1) and for the symmetric (anti-) norm functions of (1.2) when \(f(x^\gamma)\) with a certain power \(\gamma\) is an operator monotone function on \((0, \infty)\). In Section 4 we present a general method to passage from concavity/convexity of symmetric (anti-) norm functions to that of trace functions, and apply it to obtain some general concavity/convexity result for the trace functions of (1.2). In Section 5 we extend the variational method in [9, 7] to general non-decreasing convex/concave functions on \((0, \infty)\), which enables us to obtain more concavity/convexity theorems for the trace functions of (1.1). To do this, we provide, in the appendix, some variational formulas for such functions on \((0, \infty)\), which might be of independent interest as a theory of conjugate functions (or the Legendre transform) on \((0, \infty)\).

## 2 Trace functions of Lieb type with operator monotone functions

For each \(n \in \mathbb{N}\) the \(n \times n\) complex matrix algebra is denoted by \(\mathbb{M}_n\). We write \(\mathbb{M}^+_n := \{ A \in \mathbb{M}_n : A \geq 0 \}\), the \(n \times n\) positive semidefinite matrices, and \(\mathbb{P}_n := \{ A \in \mathbb{M}_n : A > 0 \}\), the \(n \times n\) positive definite matrices. The usual trace on \(\mathbb{M}_n\) is denoted by \(\text{Tr}\). A linear map \(\Phi : \mathbb{M}_n \to \mathbb{M}_l\) is positive if \(A \in \mathbb{M}^+_n\) implies \(\Phi(A) \in \mathbb{M}^+_l\), and it
is strictly positive if \( A \in \mathbb{P}_n \) implies \( \Phi(A) \in \mathbb{P}_l \). A positive linear map \( \Phi : M_n \to M_l \) is strictly positive if and only if \( \Phi(I_n) \in \mathbb{P}_l \), where \( I_n \) (or simply \( I \)) is the identity of \( M_n \).

A real function \( h \) on \((0, \infty)\) is said to be operator monotone (resp., operator monotone decreasing) if \( A \leq B \) implies \( h(A) \leq h(B) \) (resp., \( h(A) \geq h(B) \)) for \( A, B \in \mathbb{P}_n \) of any \( n \in \mathbb{N} \). Obviously, \( h \) is operator monotone decreasing if and only if \(-h\) is operator monotone.

Let \( n, m, l \in \mathbb{N} \) and \( p, q \in \mathbb{R} \). Assume that \( (p, q) \neq (0, 0) \); otherwise our problem is trivial. Let \( f \) be a real function on \((0, \infty)\). Throughout the paper, unless otherwise stated, we assume that \( \Phi : M_n \to M_l \) and \( \Psi : M_m \to M_l \) are strictly positive linear maps. The aim of this section is to prove the next theorem concerning joint concavity/convexity of the trace function of Lieb type

\[
(A, B) \in \mathbb{P}_n \times \mathbb{P}_m \mapsto \text{Tr} f((\Phi(A^p))^{1/2} \Psi(B^q) \Phi(A^p)^{1/2}). \tag{2.1}
\]

The result was announced in the concluding remarks of [14]. Our strategy for the proof is to improve so-called Epstein’s method [10] that was also used in our previous papers [12, 14].

**Theorem 2.1.** Assume that either \( 0 \leq p, q \leq 1 \) or \(-1 \leq p, q \leq 0 \) (hence \( p + q > 0 \) or \(-p - q < 0 \) from the assumption \( (p, q) \neq (0, 0) \)). Let \( f \) be a real function on \((0, \infty)\). If \( f(x^{p+q}) \) is operator monotone (resp., operator monotone decreasing) on \((0, \infty)\), then (2.1) is jointly concave (resp., jointly convex).

When \( f(x) = x^s \) with \( s \in \mathbb{R} \), we have an important special case

\[
(A, B) \in \mathbb{P}_n \times \mathbb{P}_m \mapsto \text{Tr} \{ \Phi(A^p)^{1/2} \Psi(B^q) \Phi(A^p)^{1/2} \}^s. \tag{2.2}
\]

The most familiar case where \( \Phi = \Psi = \text{id} \) is

\[
(A, B) \in \mathbb{P}_n \times \mathbb{P}_n \mapsto \text{Tr} (A^{p/2} B^q A^{p/2})^s. \tag{2.3}
\]

Theorem 2.1 improves [13] Theorem 2.1 as follows: If either \( 0 \leq p, q \leq 1 \) and \( 0 \leq s \leq 1/(p+q) \), or \(-1 \leq p, q \leq 0 \) and \( 1/(p+q) \leq s \leq 0 \), then (2.2) is jointly concave. If either \( 0 \leq p, q \leq 1 \) and \(-1/(p+q) \leq s \leq 0 \), or \(-1 \leq p, q \leq 0 \) and \( 0 \leq s \leq -1/(p+q) \), then (2.2) is jointly convex. The concavity assertion, together with [13] Proposition 5.1 (2)], says that (2.3) is jointly concave if and only if either \( 0 \leq p, q \leq 1 \) and \( 0 \leq s \leq 1/(p+q) \), or \(-1 \leq p, q \leq 0 \) and \( 1/(p+q) \leq s \leq 0 \). This characterization result was recently established in [7] as well. On the other hand, the convexity assertion was extended to a wide variety of \((p, q, s)\) in [7] (and also in Section 5 of this paper).

A corollary of Theorem 2.1 is

**Corollary 2.2.** Assume that \( \Phi : M_n \to M_l \) and \( \Psi : M_m \to M_l \) are unital positive linear maps. Let \( 0 \leq \alpha \leq 1 \). If \( f \) is operator monotone (resp., operator monotone decreasing) on \((0, \infty)\), then

\[
(A, B) \in \mathbb{P}_n \times \mathbb{P}_m \mapsto \text{Tr} f(\exp\{\alpha \Phi(\log A) + (1 - \alpha) \Psi(\log B)\})
\]

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is jointly concave (resp., jointly convex).

Proof. We may assume that $0 < \alpha < 1$. It is easy to see that for every $A \in \mathbb{P}_n$ and $B \in \mathbb{P}_m$, 
\[
\lim_{r \searrow 0} \Phi(A^{\alpha r})^{1/r} = \exp(\alpha \Phi(\log A)), \quad \lim_{r \searrow 0} \Psi(B^{(1-\alpha)r})^{1/r} = \exp((1 - \alpha) \Psi(\log B)),
\]
from which it is also easy to verify (see the proof of [15, Lemma 3.3] for instance) that
\[
\Phi(A^{\alpha r})^{1/2} \Psi(B^{(1-\alpha)r})^{1/2} \Phi(A^{\alpha r})^{1/2} ^{1/r} \defeq \exp\{\alpha \Phi(\log A) + (1 - \alpha) \Psi(\log B)\}
\]
as $r \searrow 0$. Hence the corollary follows by taking the limit of the concavity/convexity assertions of Theorem 2.1 applied to $p = \alpha r$ and $q = (1 - \alpha)r$, since $f(x^{(p+q)/r}) = f(x)$. \hfill \Box

Proof of Theorem 2.1. First, the convexity assertion follows by applying the concavity one to $-f$. So what we need to prove is that if $h$ is an operator monotone function on $(0, \infty)$ and if either $0 \leq p, q \leq 1$ or $-1 \leq p, q \leq 0$, then
\[
(A, B) \in \mathbb{P}_n \times \mathbb{P}_m \mapsto \text{Tr} h\left\{\Phi(A^p)^{1/2} \Psi(B^q)^{1/2} \Phi(A^p)^{1/2}\right\}^{1/(p+q)}
\]
is jointly concave. Recall [11, Theorem 1.9] that an operator monotone function $h$ on $(0, \infty)$ admits an integral expression
\[
h(x) = h(1) + bx + \int_{(0,\infty)} \frac{(x - 1)(1 + \lambda)}{x + \lambda} \, d\mu(\lambda),
\]
where $b \geq 0$ and $\mu$ is a finite positive measure on $[0, \infty)$. To prove the assertion, it suffices to show that (2.4) is jointly concave when $h(x) = \text{const.}$, $h(x) = x$, and $h(x) = x/(x + \lambda)$, $\lambda \geq 0$, separately, and (2.4) is jointly convex when $h(x) = 1/(x + \lambda)$, $\lambda \geq 0$. When $h(x) = \text{const.}$, the assertion is trivial, and when $h(x) = x$ it is contained in [14, Theorem 2.1]. For the case $h(x) = x/(x + \lambda)$, it is trivial when $\lambda = 0$ so that $h(x) = 1$, and when $\lambda > 0$, by considering $h(\lambda x)$ it suffices to show the case $h(x) = x/(x + 1) = (1 + x^{-1})^{-1}$. For convexity of (2.4) for $h(x) = 1/(x + \lambda)$, when $\lambda > 0$, by considering $h(\lambda x) = \lambda^{-1}(1 - x/(x + 1))$ the assertion is reduced to concavity for $h(x) = x/(x + 1)$, and when $h(x) = 1/x$ it is in [14, Theorem 2.1]. Thus, it suffices to prove that if either $0 \leq p, q \leq 1$ or $-1 \leq p, q \leq 0$, and if $A, H \in \mathbb{M}_n$ and $B, K \in \mathbb{M}_m$ are such that $A, B > 0$ and $H, K$ are Hermitian, then
\[
\frac{d^2}{dx^2} \text{Tr} \left(I + \{\Phi((A + xH)^p)^{1/2} \Psi((B + xK)^q)\Phi((A + xH)^p)^{1/2}\}^{-1/(p+q)}\right)^{-1} \leq 0 \quad (2.5)
\]
for every sufficiently small $x > 0$.

Here it is worth noting that although [14, Theorem 2.1] has been referred to in the above discussion, it is in fact unnecessary in our proof of the theorem. Indeed, once (2.5) is proved, joint concavity of (2.4) for $h(x) = x$ and joint convexity of (2.4) for $h(x) = 1/x$ are obtained by taking the limits $\lambda x/(x + \lambda) \to x$ as $\lambda \to \infty$ and $\lambda^{-1}(1 - x/(x + \lambda)) \to 1/x$ as $\lambda \searrow 0$, which are the cases we referred to from [14, Theorem 2.1] in the above.

Now, assume that $0 \leq p, q \leq 1$ (and $p + q > 0$). Let $A, H, B, K$ be as in (2.5), and set $X(z) := zA + H$ and $Y(z) = zB + K$ for $z \in \mathbb{C}$. As in the proof of [14, Theorem 2.1], we see that the function

$$F(z) := \Phi(X(z)^p)^{1/2}\Psi(Y(z)^q)\Phi(X(z)^p)^{1/2}$$

is a well-defined analytic function in the upper half-plane $\mathbb{C}^+$, for which

$$\sigma(F(z)) \subset \{\zeta \in \mathbb{C} : \zeta = re^{i\theta}, \ 0 < \theta < \gamma\pi\}, \quad z \in \mathbb{C}^+,$$

where $\gamma := p + q \in (0, 2]$ and $\sigma(F(z))$ is the set of the eigenvalues of $F(z)$. Therefore, $F(z)^{-1/\gamma}$ is well-defined in $\mathbb{C}^+$ via analytic functional calculus by $\zeta^{-1/\gamma} = (r^{-1/\gamma})e^{-i\theta/\gamma}$ for $\zeta = re^{i\theta}$ ($r > 0, \ 0 < \theta < \gamma\pi$) so that $\sigma(F(z)^{-1/\gamma})$ is included in the lower half-plane $\mathbb{C}^-$ for all $z \in \mathbb{C}^+$. Hence the function $(z^{-1}I + F(z)^{-1/\gamma})^{-1}$ is a well-defined analytic function in $\mathbb{C}^+$ for which $\sigma((z^{-1}I + F(z)^{-1/\gamma})^{-1}) \subset \mathbb{C}^+$ for all $z \in \mathbb{C}^+$, so $\text{Tr}(z^{-1} + F(z)^{-1/\gamma})^{-1} \in \mathbb{C}^+$ for all $z \in \mathbb{C}^+$. Furthermore, one can choose an $R > 0$ such that $xA + H > 0$ and $xB + K > 0$ for all $x \in (R, \infty)$. Then $F(z)$ in $\mathbb{C}^+$ is continuously extended to $\mathbb{C}^+ \cup (R, \infty)$ so that

$$F(x) = \Phi((xA + H)^p)^{1/2}\Psi((xB + K)^q)\Phi((xA + H)^p)^{1/2} = x^\sigma\Phi((A + x^{-1}H)^p)^{1/2}\Psi((B + x^{-1}K)^q)\Phi((A + x^{-1}H)^p)^{1/2}, \quad x \in (R, \infty).$$

Therefore, for every $x \in (R, \infty)$ one has

$$(x^{-1}I + F(x)^{-1/\gamma})^{-1} = x(I + \{\Phi((A + x^{-1}H)^p)^{1/2}\Psi((B + x^{-1}K)^q)\Phi((A + x^{-1}H)^p)^{1/2}\}^{-1/\gamma})^{-1}.$$

Since $\text{Tr}(x^{-1}I + F(x)^{-1/\gamma})^{-1} \in \mathbb{R}$ for all $x \in (R, \infty)$, by the reflection principle we obtain a Pick function $\varphi$ on $\mathbb{C} \setminus (-\infty, R]$ such that

$$\varphi(x) = \text{Tr}(x^{-1}I + F(x)^{-1/\gamma})^{-1}, \quad x \in (R, \infty).$$

Thus, for every $x \in (0, R^{-1})$ we have

$$x\varphi(x^{-1}) = \text{Tr}\left(I + \{\Phi((A + xH)^p)^{1/2}\Psi((B + xK)^q)\Phi((A + xH)^p)^{1/2}\}^{-1/\gamma}\right)^{-1}.$$

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Now, in the same way (using Epstein’s method) as in the proof of [14, Theorem 2.1], it follows that
\[ \frac{d^2}{dx^2}(x^p(x^{-1})) \leq 0, \quad x \in (0, R^{-1}), \]
and hence (2.5) follows when \( 0 \leq p, q \leq 1 \).

Next, assume that \( -1 \leq p, q \leq 0 \) (and \( p + q < 0 \)). Set \( \hat{\Phi}(A) := \Phi(A^{-1})^{-1} \) for \( A \in \mathbb{P}_n \) and \( \hat{\Psi}(B) := \Psi(B^{-1})^{-1} \) for \( B \in \mathbb{P}_m \). Then we can write
\[
\left\{ \Phi(A^p)^{1/2}\Psi(B^q)\Phi(A^p)^{1/2} \right\}^{1/(p+q)} = \left\{ \hat{\Phi}(A^{-p})^{1/2}\hat{\Psi}(B^{-q})\hat{\Phi}(A^{-p})^{1/2} \right\}^{-1/(p+q)}.
\]

Although \( \hat{\Phi} \) and \( \hat{\Psi} \) are no longer linear, the above proof of (2.5) can work with \( \hat{\Phi} \) and \( \hat{\Psi} \) in place of \( \Phi \) and \( \Psi \) (see the proof of [14, Theorem 2.1] for more detail). Hence we have (2.5) for \( -1 \leq p, q \leq 0 \) as well.

It is obvious that if \( p, q \geq 0 \) and \( f \) can continuously extend to \( [0, \infty) \), then joint concavity/convexity in Theorem 2.1 holds true, by a simple convergence argument, for general positive (not necessarily strictly positive) linear maps \( \Phi, \Psi \) and general positive semidefinite matrices \( A, B \). This remark may be applicable in a similar situation throughout the paper.

### 3 Norm functions involving operator means

A symmetric anti-norm \( \| \cdot \|_! \) on \( \mathbb{M}_l^+ \) is a non-negative continuous functional such that \( \| \lambda A \|_! = \lambda \| A \|_! \), \( \| UAU^* \|_! = \| A \|_! \) and \( \| A + B \|_! \geq \| A \|_! + \| B \|_! \) for all \( A, B \in \mathbb{M}_l^+ \), all reals \( \lambda \geq 0 \) and all unitaries \( U \) in \( \mathbb{M}_l \). This notion is the superadditive version of usual symmetric norms (see [5] for details on anti-norms). The typical example is the Ky Fan \( k \)-anti-norm \( \| A \|_{(k)} := \sum_{j=1}^{k} \lambda_{l+1-j}(A) \) for \( 1 \leq k \leq l \), the anti-norm version of Ky Fan \( k \)-norm \( \| A \|_{(k)} := \sum_{j=1}^{k} \lambda_j(A) \), where \( \lambda_1(A) \geq \cdots \geq \lambda_l(A) \) are the eigenvalues of \( A \in \mathbb{M}_l^+ \) in decreasing order with multiplicities. For every symmetric norm \( \| \cdot \| \) on \( \mathbb{M}_l \) and every \( \alpha > 0 \) a symmetric anti-norm on \( \mathbb{M}_l^+ \) is defined as
\[
\| A \|_! := \left\{ \begin{array}{ll}
\| A^{-\alpha} \|^{1/\alpha} & \text{if } A \text{ is invertible,} \\
0 & \text{otherwise,}
\end{array} \right.
\]
that is called the derived anti-norm (see [6, Proposition 4.6]).

Throughout this section we assume that \( \sigma \) is an operator mean in the Kubo-Ando sense [16]. We consider joint concavity/convexity of the norm functions
\[
(A, B) \in \mathbb{P}_n \times \mathbb{P}_m \longrightarrow \| f(\Phi(A^p) \sigma \Psi(B^q)) \|, \tag{3.1}
\]
\[
(A, B) \in \mathbb{P}_n \times \mathbb{P}_m \longrightarrow \| f(\Phi(A^p) \sigma \Psi(B^q)) \|! \tag{3.2}
\]
for symmetric and anti-symmetric norms. Our main theorem is
Theorem 3.1. Assume that either $0 \leq p, q \leq 1$ or $-1 \leq p, q \leq 0$, and let $\gamma := \max\{p, q\}$ if $p, q \geq 0$ and $\gamma := \min\{p, q\}$ if $p, q \leq 0$. Let $f$ be a non-negative real function on $(0, \infty)$. If $f(x^\gamma)$ is operator monotone on $(0, \infty)$, then (3.2) is jointly concave for every symmetric anti-norm $\| \cdot \|_!$ on $M_+^l$. If $f(x^\gamma)$ is operator monotone decreasing on $(0, \infty)$, then (3.1) is jointly convex for every symmetric norm $\| \cdot \|$ on $M_l$.

Note that the above theorem contains [14, Theorem 3.2] as a particular case where $f(x) = x^s$. Also, the theorem gives an extension of [14, Corollary 3.6] when $\sigma$ is the arithmetic mean. The following is the special case where $B = A$, $\Psi = \Phi$ and $q = p$, which extends [14, Theorem 4.1].

Corollary 3.2. If $h$ is a non-negative and operator monotone function on $(0, \infty)$ and $0 < p \leq 1$, then the functions $A \in P_n \mapsto \| h(\Phi(A^p)^{1/p}) \|_!$ and $\| h(\Phi(A^p)^{-1/p}) \|_!$ are concave for every symmetric anti-norm $\| \cdot \|_!$, and the functions $A \in P_n \mapsto \| h(\Phi(A^{-p})^{1/p}) \|$ and $\| h(\Phi(A^{-p})^{-1/p}) \|$ are convex for every symmetric norm $\| \cdot \|$.

To prove the theorem, we first give a lemma on joint concavity of the trace function. Note that the trace-norm is a symmetric norm and an anti-symmetric norm simultaneously, so the lemma is indeed a particular case of Theorem 3.1. However, in the next section we will show that Theorem 3.1 induces joint concavity/convexity of the trace function for even more general functions $f$.

Lemma 3.3. If $0 \leq p, q \leq 1$, $\gamma := \max\{p, q\}$ and $h$ is an operator monotone function on $(0, \infty)$, then

$$(A, B) \in P_n \times P_m \mapsto \text{Tr} h\left(\{\Phi(A^p) \sigma \Psi(B^q)\}^{1/\gamma}\right)$$

is jointly concave.

Proof. We may assume that $p = q$. Indeed, let $A_1, A_2 \in P_n$ and $B_1, B_2 \in P_m$, and assume that $p > q$. Since

$$\Psi\left(\left(\frac{B_1 + B_2}{2}\right)^q\right) \geq \Psi\left(\left(\frac{B_1^{q/p} + B_2^{q/p}}{2}\right)^p\right),$$

the joint concavity assertion in the case $p = q$ implies that

$$\text{Tr} h\left(\left\{\Phi\left(\left(\frac{A_1 + A_2}{2}\right)^p\right) \sigma \Psi\left(\left(\frac{B_1 + B_2}{2}\right)^q\right)\right\}^{1/p}\right) \geq \text{Tr} h\left(\left\{\Phi\left(\left(\frac{A_1^{q/p} + A_2^{q/p}}{2}\right)^p\right) \sigma \Psi\left(\left(\frac{B_1^{q/p} + B_2^{q/p}}{2}\right)^q\right)\right\}^{1/p}\right) \geq \frac{1}{2} \left[\text{Tr} h\left(\{\Phi(A_1^p)\sigma\Psi((B_1^{q/p})^p)\}^{1/p}\right) + \text{Tr} h\left(\{\Phi(A_1^p)\sigma\Psi((B_1^{q/p})^p)\}^{1/p}\right)\right]$$

is jointly concave.
\[
= \frac{1}{2} \left[ \text{Tr} (\{ \Phi(A^p) \sigma \Psi(B^q) \}^{1/p}) + \text{Tr} (\{ \Phi(A^p) \sigma \Psi(B^q) \}^{1/p}) \right].
\]

In the above we have used monotonicity of \( \sigma \) and of \( \text{Tr} h(\cdot) \). Now, let \( A, H \in \mathbb{M}_n \) and \( B, K \in \mathbb{M}_m \) be such that \( A, B > 0 \) and \( H, K \) are Hermitian. For joint concavity of the given trace function (when \( p = q \)), as in the proof of Theorem 2.1, we need to prove that
\[
\frac{d^2}{dx^2} \text{Tr} \left( I + \{ \Phi((A + xH)^p) \sigma \Psi(\{ (B + xK)^p \}^{-1/p}) \}^{-1} \right) \leq 0.
\]

(3.3)

Set \( X(z) := zA + H \) and \( Y(z) := zB + K \) for \( z \in \mathbb{C} \). As in the proof of [12, Theorem 4.3], it is seen that the function
\[
F(z) := \Phi(X(z)^p) \sigma \Psi(Y(z)^p)
\]
is an analytic functions in \( \mathbb{C}^+ \), for which
\[
\sigma(F(z)) \subset \{ \zeta \in \mathbb{C} : \zeta = re^{i\theta}, r > 0, 0 < \theta < p\pi \}, \quad z \in \mathbb{C}^+.
\]

Therefore, \( F(z)^{-1/p} \) can be defined in \( \mathbb{C}^+ \) so that \( \sigma(F(z)^{-1/p}) \subset \mathbb{C}^- \) for all \( z \in \mathbb{C}^+ \). The remaining proof of (3.3) is similar to that of Theorem 2.1. \( \square \)

**Proof of Theorem 3.1** Let \( 0 \leq p, q \leq 1 \) and \( \gamma := \max\{p, q\} \). To prove the first assertion, we need to show that if \( h \) is a non-negative and operator monotone function on \( (0, \infty) \), then the functions
\[
(A, B) \in \mathbb{P}_n \times \mathbb{P}_m \mapsto \| h(\{ \Phi(A^p) \sigma \Psi(B^q) \}^{1/\gamma}) \|_I,
\]
\[
(A, B) \in \mathbb{P}_n \times \mathbb{P}_m \mapsto \| h(\{ \Phi(A^{-p}) \sigma \Psi(B^{-q}) \}^{-1/\gamma}) \|_I
\]
are jointly concave for every symmetric anti-norm \( \| \cdot \|_I \) on \( \mathbb{M}_L^+ \).

The proof below is similar to that of [14, Theorem 3.2]. First, note that \( h \) can be extended to \( [0, \infty) \) continuously, i.e., \( h(0) := \lim_{x \to 0^+} h(x) \). For every \( A_1, A_2 \in \mathbb{P}_n, B_1, B_2 \in \mathbb{P}_m \) and for every Ky Fan \( k \)-anti-norm \( \| \cdot \|_{(k)} \), \( 1 \leq k \leq l \), there exists a rank \( k \) projection \( E \) commuting with \( \Phi((A_1 + A_2)/2)^p) \sigma \Psi((B_1 + B_2)/2)^q) \) such that
\[
\left\| h(\left\{ \Phi\left(\frac{A_1 + A_2}{2}\right)^p \right\} \sigma \Psi\left(\frac{B_1 + B_2}{2}\right)^q)^{1/\gamma} \right\|_{(k)} = \text{Tr} h \left( \left\{ E \Phi\left(\frac{A_1 + A_2}{2}\right)^p \right\} \sigma \Psi\left(\frac{B_1 + B_2}{2}\right)^q)^{1/\gamma} E \right) - h(0) \text{Tr} (I_I - E)
\]
\[
= \lim_{\varepsilon \to 0^+} \text{Tr} h \left( \left\{ (E + \varepsilon I_I) \Phi\left(\frac{A_1 + A_2}{2}\right)^p \right\} (E + \varepsilon I_I)^q \right) \sigma \Psi\left(\frac{B_1 + B_2}{2}\right)^q) (E + \varepsilon I_I)^{1/\gamma} - h(0) \text{Tr} (I_I - E).
\]
By Lemma 3.3 applied to the strictly positive linear maps $(E + \varepsilon I_l)\Phi(\cdot)(E + \varepsilon I_l)$ and $(E + \varepsilon I_l)\Psi(\cdot)(E + \varepsilon I_l)$ we obtain

$$\text{Tr } h\left( \left( (E + \varepsilon I_l)\Phi\left( \left( \frac{A_1 + A_2}{2} \right)^p \right)(E + \varepsilon I_l) \right)^{1/\gamma} \right)$$

$$\geq \frac{1}{2} \left[ \text{Tr } h\left( \{ (E + \varepsilon I_l)(\Phi(A_1^p)\sigma \Psi(B_1^q))(E + \varepsilon I_l) \}^{1/\gamma} \right) + \text{Tr } h\left( \{ (E + \varepsilon I_l)(\Phi(A_2^p)\sigma \Psi(B_2^q))(E + \varepsilon I_l) \}^{1/\gamma} \right) \right]$$

$$\rightarrow \frac{1}{2} \left[ \text{Tr } h\left( \{ E(\Phi(A_1^p)\sigma \Psi(B_1^q))E \}^{1/\gamma} \right) + \text{Tr } h\left( \{ E(\Phi(A_2^p)\sigma \Psi(B_2^q))E \}^{1/\gamma} \right) \right]$$

as $\varepsilon \searrow 0$. Since

$$\lambda_j^+(ECE) \geq \lambda_j^+(C), \quad C \in \mathbb{M}_l^+, \quad j = 1, \ldots, k,$$

where $\lambda_j^+(C)$, $1 \leq j \leq l$, denote the eigenvalues of $C$ in increasing order with multiplicities. We have

$$\text{Tr } h\left( \{ E(\Phi(A_1^p)\sigma \Psi(B_1^q))E \}^{1/\gamma} \right) - h(0)\text{Tr } (I_l - E)$$

$$= \sum_{j=1}^k h\left( \{ \lambda_j^+(E(\Phi(A_1^p)\sigma \Psi(B_1^q))E) \}^{1/\gamma} \right) \geq \sum_{j=1}^k h\left( \{ \lambda_j^+(\Phi(A_1^p)\sigma \Psi(B_1^q)) \}^{1/\gamma} \right)$$

$$= \sum_{j=1}^k \lambda_j^+(h(\{ \Phi(A_1^p)\sigma \Psi(B_1^q) \}^{1/\gamma})) = \left\| h(\{ \Phi(A_1^p)\sigma \Psi(B_1^q) \}^{1/\gamma}) \right\|_{\{k\}}$$

and similarly

$$\text{Tr } h\left( \{ E(\Phi(A_2^p)\sigma \Psi(B_2^q))E \}^{1/\gamma} \right) - h(0)\text{Tr } (I_l - E) \geq \left\| h(\{ \Phi(A_2^p)\sigma \Psi(B_2^q) \}^{1/\gamma}) \right\|_{\{k\}}.$$
\[
\begin{align*}
\geq & \frac{1}{2} \left\| \sigma \int h \left( \{ \Phi(A_1^p) \sigma \Psi(B_1^q) \}^{1/\gamma} \right)^\dagger + h \left( \{ \Phi(A_2^p) \sigma \Psi(B_2^q) \}^{1/\gamma} \right)^\dagger \right\|_1 \\
\geq & \frac{1}{2} \left[ \left\| h \left( \{ \Phi(A_1^p) \sigma \Psi(B_1^q) \}^{1/\gamma} \right) \right\|_1 + \left\| h \left( \{ \Phi(A_2^p) \sigma \Psi(B_2^q) \}^{1/\gamma} \right) \right\|_1 \right],
\end{align*}
\]
proving joint concavity of \((3.4)\).

To prove joint concavity of \((3.5)\), we note that
\[
\{ \Phi(A^{-p}) \sigma \Psi(B^{-q}) \}^{-1/\gamma} = \{ \hat{\Phi}(A^p) \sigma^* \hat{\Psi}(B^q) \}^{1/\gamma},
\]
where \(\hat{\Phi}(A) := \Phi(A^{-1})^{-1}\) for \(A \in \mathbb{P}_n\) and \(X \sigma^* Y := (X^{-1} \sigma Y^{-1})^{-1}\), the adjoint operator mean. Note that Lemma \(3.3\) holds true when \(\Phi, \Psi\) are replaced with \(\hat{\Phi}, \hat{\Psi}\), respectively (with \(\sigma^*\) in place of \(\sigma\)). Hence the above proof for \((3.4)\) shows the assertion for \((3.5)\) as well.

Next, let \(\| \cdot \|\) be a symmetric norm on \(\mathbb{M}_l\). By applying the first assertion to the operator monotone function \(h(x^{-1})^{-1}\) and the derived anti-norm \(\| A \|_1 := \| A^{-1} \|^{-1}\) for \(A \in \mathbb{P}_l\) (and \(\| A \|_1 = 0\) if \(A \in \mathbb{M}_l^+\) is singular), we see that
\[
(A, B) \in \mathbb{P}_n \times \mathbb{P}_m \mapsto \left\| h \left( \{ \Phi(A^p) \sigma \Psi(B^q) \}^{-1/\gamma} \right) \right\|^{-1},
\]
\[
(A, B) \in \mathbb{P}_n \times \mathbb{P}_m \mapsto \left\| h \left( \{ \Phi(A^{-p}) \sigma \Psi(B^{-q}) \}^{1/\gamma} \right) \right\|^{-1}
\]
are jointly concave, which implies the second assertion.

**Remark 3.4.** In the last part of the above theorem, one can take the derived anti-norm \(\| A \|_1 := \| A^{-a} \|^{-1/\alpha}\) with \(\alpha > 0\), so the second convexity assertion of Theorem \(3.1\) holds for the function \(\| \{ f(\Phi(A^p) \sigma \Psi(B^q)) \}^\alpha \|^{1/\alpha}\) for any \(\alpha > 0\) more generally than \((3.1)\). Note that if \(\| \cdot \|\) is a symmetric norm, then \(\| \cdot \|^\alpha\) is again a symmetric norm for \(\alpha \geq 1\), but this is not necessarily so for \(0 < \alpha < 1\).

### 4 Passages from norm functions to trace functions

In this section we develop an abstract method which provides passages from joint concavity/convexity of symmetric (anti-) norm functions to that of trace functions in a general form. The method is then applied to Theorem \(3.1\) (or rather [14, Theorem 3.2]) so that we have some general concavity/convexity result for trace functions involving operator means.

Let \(n, m, l\) be fixed and a function \(F : \mathbb{P}_n \times \mathbb{P}_m \to \mathbb{P}_l\) be given, for which we consider the following conditions:

(a) \((A, B) \in \mathbb{P}_n \times \mathbb{P}_m \mapsto \| F(A, B) \|_1\) is jointly concave for every symmetric anti-norm \(\| \cdot \|_1\).

(a)' \((A, B) \in \mathbb{P}_n \times \mathbb{P}_m \mapsto \| F(A, B) \|_{\{k\}}\) is jointly concave for the Ky Fan \(k\)-anti-norms \(\| \cdot \|_{\{k\}}, 1 \leq k \leq l\).
(b) \((A, B) \in \mathbb{P}_n \times \mathbb{P}_m \mapsto \| F(A, B)^{-1} \|\) is jointly convex for every symmetric norm \(\| \cdot \|\).

(b)' \((A, B) \in \mathbb{P}_n \times \mathbb{P}_m \mapsto \| F(A, B)^{-1}\|_{(k)}\) is jointly convex for the Ky Fan \(k\)-norms \(\| \cdot \|_{(k)}, 1 \leq k \leq l\).

(c) \((A, B) \in \mathbb{P}_n \times \mathbb{P}_m \mapsto \text{Tr} f(F(A, B))\) is jointly concave for every non-decreasing concave function \(f\) on \((0, \infty)\).

(d) \((A, B) \in \mathbb{P}_n \times \mathbb{P}_m \mapsto \text{Tr} f(F(A, B)^{-1})\) is jointly convex for every non-decreasing convex function \(f\) on \((0, \infty)\).

**Theorem 4.1.** Concerning conditions stated above we have

\[(a) \iff (a)' \implies (b) \implies (b)' \implies (d),\]

\[(a) \implies (c) \implies (d).\]

**Proof.** \((a) \implies (a)\)' and \((b) \implies (b)\)' are trivial. \((a)' \implies (a)\) follows from [5, Lemma 4.2] as in the proof of Theorem 3.1 and \((b)\)' \implies \((b)\) is similar (see [13, Proposition 4.4.13]).

\((a) \implies (b)\) follows from [6, Propositions 4.6], as used in the last part of the proof of Theorem 3.1.

\((a) \implies (c)\). Let \(A_1, A_2 \in \mathbb{P}_n\) and \(B_1, B_2 \in \mathbb{P}_m\). Let \(\alpha_1 \geq \cdots \geq \alpha_l, \alpha'_1 \geq \cdots \geq \alpha'_l\) and \(\alpha''_1 \geq \cdots \geq \alpha''_l\) be the eigenvalues of \(F((A_1 + A_2)/2, (B_1 + B_2)/2), F(A_1, B_1)\) and \(F(A_2, B_2)\), respectively, in decreasing order with multiplicities. Joint concavity in \((a)\) for the Ky Fan anti-norms \(\| \cdot \|_{(k)}\) means that

\[
\sum_{i=1}^{k} \alpha_{l+1-i} \geq \sum_{i=1}^{k} \frac{\alpha'_{l+1-i} + \alpha''_{l+1-i}}{2}, \quad 1 \leq k \leq l,
\]

that is, we have the weak majorization

\[
(-\alpha_{l+1-i})_{i=1}^{l} \prec_w \left( -\frac{\alpha'_{l+1-i} + \alpha''_{l+1-i}}{2} \right)_{i=1}^{l}.
\]

Now, assume that \(f\) is an non-decreasing concave function on \((0, \infty)\). Since \(-f(-x)\) is non-decreasing and convex on \((-\infty, 0)\), we obtain

\[
- \sum_{i=1}^{l} f(\alpha_{l+1-i}) \leq - \sum_{i=1}^{l} f\left( \frac{\alpha'_{l+1-i} + \alpha''_{l+1-i}}{2} \right)
\]

and hence

\[
\sum_{i=1}^{l} f(\alpha_i) \geq \sum_{i=1}^{l} f\left( \frac{\alpha'_i + \alpha''_i}{2} \right) \geq \sum_{i=1}^{l} \frac{f(\alpha'_i) + f(\alpha''_i)}{2}
\]

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thanks to concavity of \( f \). This means that
\[
\text{Tr} f \left( F \left( \frac{A_1 + A_2}{2}, \frac{B_1 + B_2}{2} \right) \right) \geq \frac{\text{Tr} f(F(A_1, B_1)) + \text{Tr} f(F(A_2, B_2))}{2}.
\]

(b) \( \Rightarrow \) (d). Let \( \alpha_i, \alpha'_i \) and \( \alpha''_i \) be defined as above corresponding to \( F(A, B)^{-1} \) instead of \( F(A, B) \). Joint convexity in (b) for the Ky Fan norms \( \| \cdot \|_X \) means the weak majorization
\[
(\alpha_i)_{i=1}^l \prec_w \left( \frac{\alpha'_i + \alpha''_i}{2} \right)_{i=1}^l.
\]
If \( f \) is non-decreasing and convex on \((0, \infty)\), then
\[
\sum_{i=1}^l f(\alpha_i) \leq \sum_{i=1}^l f \left( \frac{\alpha'_i + \alpha''_i}{2} \right) \leq \frac{\sum_{i=1}^l f(\alpha'_i) + f(\alpha''_i)}{2}
\]
so that
\[
\text{Tr} f \left( F \left( \frac{A_1 + A_2}{2}, \frac{B_1 + B_2}{2} \right)^{-1} \right) \leq \frac{\text{Tr} f(F(A_1, B_1)^{-1}) + \text{Tr} f(F(A_2, B_2)^{-1})}{2}.
\]

(c) \( \Rightarrow \) (d) immediately follows from the fact that if \( f \) is non-decreasing and convex on \((0, \infty)\), then \(-f(x^{-1})\) is non-decreasing and concave on \((0, \infty)\). \( \square \)

**Corollary 4.2.** Let \( \sigma \) be an operator mean and \( f \) be a real function on \((0, \infty)\). Assume that either \( 0 \leq p, q \leq 1 \) or \(-1 \leq p, q \leq 0\), and let \( \gamma := \max\{p, q\} \) if \( p, q \geq 0 \) and \( \gamma := \min\{p, q\} \) if \( p, q \leq 0\). If \( f(x^\gamma) \) is non-decreasing and concave on \((0, \infty)\), then
\[
(A, B) \in \mathbb{P}_n \times \mathbb{P}_m \mapsto \text{Tr} f(\Phi(A^p) \sigma \Psi(B^q))
\]
is jointly concave. If \( f(x^{-\gamma}) \) is non-decreasing and convex on \((0, \infty)\), then (4.1) is jointly convex.

Indeed, Theorem 3.1 (also [14, Theorem 3.2]) implies that the function \( \Phi(A, B) := \{\Phi(A^p) \sigma \Psi(B^q)\}^{1/\gamma} \) for \((A, B) \in \mathbb{P}_n \times \mathbb{P}_m\) satisfies condition (a) above, so by Theorem 4.1, we have the assertions by rewriting conditions (c) and (d).

**Remark 4.3.** When \( f \) is non-decreasing and concave on \((0, \infty)\), it is straightforward to see that the function \( \text{Tr} f(\Phi(A^p) \sigma \Psi(B^q)) \) is jointly concave in \((A, B)\) when \( 0 \leq p, q \leq 1\). Indeed, one has
\[
\Phi \left( \left( \frac{A_1 + A_2}{2} \right)^p \right) \sigma \Psi \left( \left( \frac{B_1 + B_2}{2} \right)^p \right) \geq \frac{\Phi(A_1^p) + \Phi(A_2^p)}{2} \sigma \left( \Psi(B_1^q) + \Psi(B_2^q) \right) \geq \frac{\Phi(A_1^p) \sigma \Psi(B_1^q) + \Phi(A_2^p) \sigma \Psi(B_2^q)}{2}
\]
thanks to joint concavity of \( \sigma \). Since \( \text{Tr} f(\cdot) \) is monotone and concave on \( \mathbb{P}_t \), we have the conclusion. The real merit of Corollary 4.2 is that it holds under the weaker assumption of \( f(x^\gamma) \) being concave.
Remark 4.4. The assumptions on $p, q$ and $f$ for the joint concavity assertion in Corollary 4.2 are considered optimal from the following facts:

- Let $p, s \neq 0$. If $A \in \mathbb{P}_2 \mapsto \text{Tr} (X^* A^p X)^s$ is concave for any invertible $X \in \mathbb{M}_2$, then either $0 < p \leq 1$ and $0 < s \leq 1/p$, or $-1 \leq p \leq 0$ and $1/p \leq s < 0$ (see [14, Proposition 5.1 (1)]).

- For the case where $p = q = 1$ and $\sigma$ is the geometric mean, the numerical function $f(x^{1/2} y^{1/2})$ must be jointly concave in $x, y > 0$, which implies that $f$ is non-decreasing and concave.

- Let $p, q \geq 0$ and $\gamma := \max\{p, q\}$. For the case where $\sigma$ is the arithmetic mean, the numerical function $f(x^p + y^q)$ must be jointly concave in $x, y > 0$, which implies that $f(x^\gamma)$ is concave.

The next corollary gives concavity/convexity of one-variable trace functions of Epstein type. The first assertion (1) will repeatedly be used in the next section.

Corollary 4.5. Let $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_l$ be a strictly positive linear map.

(1) If $0 < p \leq 1$ and $f$ is a non-decreasing concave function on $(0, \infty)$, then

$$A \in \mathbb{P}_n \mapsto \text{Tr} f(\Phi(A^p)^{1/p}) \quad \text{and} \quad \text{Tr} f(\Phi(A^{-p})^{-1/p})$$

are concave.

(2) Assume that $\Phi$ is CP (i.e., completely positive). If $1 \leq p \leq 2$ and $f$ is a non-decreasing convex function on $(0, \infty)$, then

$$A \in \mathbb{P}_n \mapsto \text{Tr} f(\Phi(A^p)^{1/p})$$

is convex.

Indeed, (1) is specialization of Corollary 4.2 to the case where $B = A$, $\Psi = \Phi$ and $q = p$. Moreover, it is obvious that Theorem 4.1 holds for a one-variable function $F : \mathbb{P}_n \rightarrow \mathbb{P}_l$ as well. Applying this to [14, Theorem 4.2] gives (2).

In particular, Corollary 4.5 covers the result in [8, Theorem 1.1] that for every $X \in \mathbb{M}_n$, the function $A \in \mathbb{M}_n^+ \mapsto \text{Tr} (X^* A^p X)^{q/p}$ is concave if $0 < p \leq 1$ and $0 \leq q \leq 1$, and is convex if $1 \leq p \leq 2$ and $q \geq 1$.

Remark 4.6. Compared the above (2) with (1) it might be expected that, under the same assumption of (3), the function $A \in \mathbb{P}_n \rightarrow \text{Tr} f(\Phi(A^{-p})^{-1/p})$ is convex for $1 \leq p \leq 2$. In particular, when $\Phi = K \cdot K^* : \mathbb{M}_n \rightarrow \mathbb{M}_n$ with an invertible $K \in \mathbb{M}_n$, this is certainly true since $\Phi(A^{-p})^{-1/p} = (K A^{-p} K^*)^{-1/p} = (K^{s-1} A^p K^{-1})^{1/p}$. However, it is not true when $\Phi : \mathbb{M}_n \rightarrow \mathbb{M}_l$ is a general CP map. For instance, let $E$ be an
orthogonal projection in $\mathbb{M}_n$, and let $\Phi : \mathbb{M}_n \to \mathbb{E}\mathbb{M}_n\mathbb{E} (\cong \mathbb{M}_l$ where $l := \dim E)$ be defined by $\Phi(X) = EXE$ for $X \in \mathbb{M}_n$. Then the assertion applied to $f(x) = x^s$ for $s \geq 1$ would imply that $A \in \mathbb{P}_n \mapsto \text{Tr} \Phi(A^{-p})^{-s/p}$ is convex for $1 \leq p \leq 2$. For example, let $n = 2$, $E = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$, $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & t \end{bmatrix}$ and $A_2 = \begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix}$ for $t > 0$. We then compute

$$\text{Tr} \Phi \left( \left( \frac{A_1 + A_2}{2} \right)^{-p} \right)^{-s/p} = \left( \frac{1 + t}{2} \right)^s,$$

$$\text{Tr} \Phi(A_1^{-p})^{-s/p} = \text{Tr} \Phi(A_2^{-p})^{-s/p} = \left( \frac{1 + t^{-p}}{2} \right)^{-s/p}.$$

For any $p, s > 0$, since $(1 + t)/2 > ((1 + t^{-p})/2)^{-1/p}$ for $t \neq 1$, we see that $A \in \mathbb{P}_2 \mapsto \text{Tr} \Phi(A^{-p})^{-s/p}$ is not convex.

## 5 More general trace functions of Lieb type

In this section we are concerned with joint concavity/convexity of the functions

$$(A, B) \in \mathbb{P}_n \times \mathbb{P}_m \mapsto \text{Tr} f(\Phi(A^p)^{1/2} \Psi(B^q) \Phi(A^p)^{1/2}), \quad (5.1)$$

$$(A, B) \in \mathbb{P}_n \times \mathbb{P}_m \mapsto \text{Tr} f(\Phi(A^{-p})^{1/2} \Psi(B^{-q}) \Phi(A^{-p})^{1/2})^{-1}). \quad (5.2)$$

The form (5.2) is the rewriting of (5.1) by replacing $p, q, f$ with $-p, -q, f(x^{-1})$. The form (5.1) of trace functions was already treated in Section 2 but we here consider its joint concavity/convexity problem for more varieties of functions $f$ on $(0, \infty)$ and of real parameters $p, q$. Our strategy here is to extend the method adopted in [7, Section 4]. To do this, we have to prepare some technical results on variational formulas of trace functions, which we will summarize in Appendix A.

We first give a lemma which will be useful in the proofs of the theorems below.

**Lemma 5.1.** Assume that $-1 \leq q \leq 0$. Then:

(a) The function $B \in \mathbb{P}_m \mapsto \Psi(B^q)^{-1}$ is operator concave, and $B \in \mathbb{P}_m \mapsto \Psi(B^{-q})^{-1}$ is operator convex. Hence, if $f$ is an non-decreasing and concave (resp., convex) function on $(0, \infty)$, then $\text{Tr} f(\Psi(B^q)^{-1})$ (resp., $\text{Tr} f(\Psi(B^{-q})^{-1})$) is concave (resp., convex) in $B \in \mathbb{P}_m$.

(b) The functions

$$(X, B) \in \mathbb{M}_l \times \mathbb{P}_m \mapsto X^* \Psi(B^q)X \quad \text{and} \quad X^* \Psi(B^{-q})^{-1}X$$

are jointly operator convex. Hence, if $f$ is a non-decreasing and convex function on $(0, \infty)$, then $\text{Tr} f(X^* \Psi(B^q)X)$ and $\text{Tr} X^* \Psi(B^{-q})^{-1}X$ are jointly convex in $(X, B) \in \mathbb{M}_l \times \mathbb{M}_m$. 

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Proof. (a) Operator concavity of $B \in \mathbb{P}_m \mapsto \Psi(B^q)^{-1}$ is [14, Lemma 3.4], and operator convexity of $\Psi(B^{-q})^{-1}$ is similar, so we omit the proof. The latter assertion is immediately seen from monotonicity and concavity/convexity of $\text{Tr} f(\cdot)$ on $\mathbb{P}_l$. (Note that the concavity assertion for $\text{Tr} f(\Psi(B^q))$ is also an immediate consequence of Corollary 4.3(1).)

(b) First, recall a well-known fact [18, Theorem 1] that the function $(X, Y) \in \mathbb{M}_l \times \mathbb{P}_l \mapsto X^*Y^{-1}X$ is jointly operator convex. Let $X_1, X_2 \in \mathbb{M}_n$ and $B_1, B_2 \in \mathbb{P}_m$. Since $B \in \mathbb{P}_m \mapsto \Psi(B^q)^{-1}$ is operator concave by (a), we have

$$
\Psi\left(\left(\frac{B_1 + B_2}{2}\right)^q\right) \leq \left(\frac{\Psi(B_1^q)^{-1} + \Psi(B_2^q)^{-1}}{2}\right)^{-1}
$$

and hence

$$
\left(\frac{X_1 + X_2}{2}\right)^* \Psi\left(\left(\frac{B_1 + B_2}{2}\right)^q\right) \left(\frac{X_1 + X_2}{2}\right)
\leq \left(\frac{X_1 + X_2}{2}\right)^* \left(\frac{\Psi(B_1^q)^{-1} + \Psi(B_2^q)^{-1}}{2}\right)^{-1} \left(\frac{X_1 + X_2}{2}\right)
\leq \frac{X_1^* \Psi(B_1^q)X_1 + X_2^* \Psi(B_2^q)X_2}{2}
$$

thanks to joint operator convexity mentioned above. For the latter function, since $((B_1 + B_2)/2)^{-q} \geq (B_1^{-q} + B_2^{-q})/2$, we have

$$
\Psi\left(\left(\frac{B_1 + B_2}{2}\right)^{-q}\right)^{-1} \leq \left(\frac{\Psi(B_1^{-q}) + \Psi(B_2^{-q})}{2}\right)^{-1}
$$

and thus the assertion follows as above. The latter assertion is immediate as in (a). \hfill \square

The next theorem gives a sufficient condition for (5.1) and (5.2) to be jointly concave.

Theorem 5.2. Let $f$ be a non-decreasing (resp., non-increasing) function on $(0, \infty)$ and $0 \leq p, q \leq 1$. If either $f(x^{1+p})$ or $f(x^{1+q})$ is concave (resp. convex) on $(0, \infty)$, then the functions (5.1) and (5.2) are jointly concave (resp., jointly convex).

Proof. The concavity assertion follows by applying the concavity one to $-f$. So we may confine the proof to the concavity assertion. When $p = 0$ and $0 \leq q \leq 1$, the assertion reduces to concavity of $B \in \mathbb{P}_m \mapsto \text{Tr} f(\Phi(I)^{1/2} \Psi(B^q) \Phi(I)^{1/2})$ and $\text{Tr} f\left(\Psi(I)^{1/2} \Psi(B^{-q}) \Phi(I)^{1/2}\right)^{-1}$. This immediately follows from operator concavity of $x^q$ (for the former) and from Lemma 5.1(a) (for the latter). The situation is similar when $0 \leq p \leq 1$ and $q = 0$. So we assume that $0 < p, q \leq 1$ and $f(x^{1+p})$ is concave on $(0, \infty)$. For every $A \in \mathbb{P}_n$ and $B \in \mathbb{P}_m$, by (b) and (c) of Lemma A.2 with $r = p$ we have

$$
\text{Tr} f(\Phi(A^p)^{1/2} \Psi(B^q) \Phi(A^p)^{1/2}) = \inf_{Y \in \mathbb{P}_l} \left\{ \text{Tr} Y \Phi(A^p)^{1/2} \Psi(B^q) \Phi(A^p)^{1/2} - \text{Tr} \tilde{f}(Y) \right\}
$$
wider class of functions containing (2.2)). However, Theorem 5.2 gains an advantage that it is applicable to a

range of \((p, q, s)\) for joint convexity of (2.2) covered by Theorem 5.2 is: \(0 \leq p, q \leq 1\) and \(s \leq 0\), or \(-1 \leq p, q \leq 0\) and \(-\max\{1/(1 - p), 1/(1 + q)\} \leq s \leq 0\), which is smaller than the best possible range covered by Theorem 2.1 (see the paragraph containing (2.2)). However, Theorem 5.2 gains an advantage that it is applicable to a wider class of functions \(f\), as demonstrated in Example A.4. On the other hand, the range of \((p, q, s)\) for joint convexity of (2.2) covered by Theorem 5.2 is: \(0 \leq p, q \leq 1\) and \(s \leq 0\), or \(-1 \leq p, q \leq 0\) and \(s \geq 0\), which includes the range by Theorem 2.1.

The rest of the section is devoted to more results on joint convexity of (5.1) and (5.2).
Theorem 5.3. Let $f$ be a non-decreasing function on $(0, \infty)$ and $-1 < p \leq 0$. Assume that $f(x^{1+p})$ is convex on $(0, \infty)$. Then:

1. For every $q \in [-1,0] \cup [1,2]$ the function (5.1) is jointly convex.
2. For every $q \in [-1,0]$ the function (5.2) is jointly convex.
3. If $\Psi = \text{id}$ with $\mathbb{M}_m = \mathbb{M}_t$, then (5.2) is jointly convex for every $q \in [1,2]$.

Proof. Let $-1 < p \leq 0$ and $f$ be a non-constant and non-decreasing function on $(0, \infty)$. Assume that $f(x^{1+p})$ is convex on $(0, \infty)$, hence so is $f$.

1. When $p = 0$, (5.1) reduces to $B \mapsto \text{Tr} f(\Phi(I)^{1/2}B^q\Phi(I)^{1/2})$, whose concavity is immediately seen. So assume that $-1 < p < 0$. For every $A \in \mathbb{P}_n$ and $B \in \mathbb{P}_m$, by (b) and (c) of Lemma A.1 with $r = -p$ we have, as in the proof of Theorem 5.2

$$\text{Tr} f(\Phi(A)^{1/2}B^q\Phi(A)^{1/2}) = \sup_{X \in \mathbb{P}_n} \{ \text{Tr} \Phi(A)^{1/2}Y \Phi(A)^{1/2}B^q - \text{Tr} \hat{f}(Y) \} = \sup_{X \in \mathbb{P}_n} \{ \text{Tr} X \Psi(B^q)X - \hat{f}(X \Phi(A)^{-1}X) \}. \tag{5.4}$$

For any fixed $X \in \mathbb{P}_n$, since $\hat{f}(x^{-p})$ is non-decreasing and concave on $(0, \infty)$ by Lemma A.1(c) with $r = -p$, it follows from Corollary 4.5(1) that

$$A \in \mathbb{P}_n \mapsto \text{Tr} \hat{f}(X \Phi(A)^{-1}X) = \text{Tr} \hat{f}(((X^{-1}\Phi(A^{-1})X^{-1})^{1/p})^{-p})$$

is concave. Moreover, when $q \in [-1,0] \cup [1,2]$, the function $B \in \mathbb{P}_m \mapsto \text{Tr} X \Psi(B^q)X$ is convex so that joint convexity of (5.1) follows.

2. When $p = 0$ or $q = 0$, the assertion is immediate from Lemma 5.1(a). When $-1 < p < 0$ and $-1 \leq q < 0$, we may replace (5.4) with

$$\text{Tr} \hat{f}(\Phi(A^{-p})^{1/2}B^{-q}\Phi(A^{-p})^{1/2}) = \text{Tr} \hat{f}(\Phi(A^{-p})^{1/2}B^{-q})^{-1}\Phi(A^{-p})^{-1/2}) = \sup_{X \in \mathbb{P}_n} \{ \text{Tr} X \Psi(B^{-q})^{-1}X - \hat{f}(X \Phi(A^{-p})X) \}.$$ 

For any fixed $X \in \mathbb{P}_i$, it follows from Corollary 4.5(1) that

$$A \in \mathbb{P}_n \mapsto \text{Tr} \hat{f}(X \Phi(A^{-p})X) = \text{Tr} \hat{f}(((X \Phi(A^{-p})X)^{-1})^{1/p})^{-p})$$

and

$$B \in \mathbb{P}_m \mapsto -\text{Tr} X \Psi(B^{-q})^{-1}X = -\text{Tr} ((X^{-1}\Psi(B^{-q})X^{-1})^{-1/q}q$$

are concave. Hence joint convexity of (5.2) follows.

3. When $\Psi = \text{id}$ and $1 \leq q \leq 2$, the assertion follows similarly to the above proof of (2) since $\text{Tr} X \Psi(B^{-q})^{-1}X = \text{Tr} XB^qX$ is convex in $B$.

\[\square\]

Theorem 5.4. Let $f$ be a non-decreasing function on $(0, \infty)$ and $1 < p \leq 2$. Assume that $f(x^{p-1})$ is convex on $(0, \infty)$.
(1) If $\Phi$ is CP, then (5.1) is jointly convex for every $q \in [-1, 0]$.

(2) If $\Phi = \text{id}$ with $\mathbb{M}_n = \mathbb{M}_1$, then (5.2) is jointly convex for every $q \in [-1, 0]$.

Proof. Let $1 < p < 2$, $-1 \leq q \leq 0$ and $f$ be a non-constant and non-decreasing function on $(0, \infty)$. Assume that $f(x^{\frac{p}{2}})$ is convex on $(0, \infty)$, hence so is $f$.

(1) Assume that $\Phi$ is CP. We take the Stinespring representation

$$\Phi(Z) = K \pi(Z) K^*, \quad Z \in \mathbb{M}_n,$$

where $\pi : \mathbb{M}_n \rightarrow \mathbb{M}_{nk}$ is a representation and $K : \mathbb{C}^n \rightarrow \mathbb{C}^l$ is a linear map (see, e.g., [1, Theorem 3.1.2]). We have

$$\mathbb{E}(\Phi(B^q)\Phi(A^p)^{1/2}) = \mathbb{E}(\Phi(B^q)^{1/2}K\pi(A^p)K^*\Phi(B^q)^{1/2})$$

$$= \mathbb{E}(\pi(A)^{p/2}K\pi(B^q)K\pi(A)^{p/2}) - \alpha_0,$$

where $\alpha_0 := f(0)\text{Tr}(I_{nk} - P_0)$ with $P_0$ the orthogonal projection onto the range of $K^*K$. Assume that $1 < p < 2$. Letting $\tilde{\Psi}(\cdot) := K^*\Psi(\cdot)K : \mathbb{M}_m \rightarrow \mathbb{M}_{nk}$, by (b) and (c) of Lemma A.1 with $r = 2 - p$ we further have

$$\mathbb{E}(\Phi(A^p)^{1/2}\Phi(B^q)\Phi(A^p)^{1/2})$$

$$= \sup_{Y \in \mathbb{P}_{nk}} \left\{ \mathbb{E}(\pi(A)^{q/2}Y\pi(A)^{p/2} - \mathbb{E}\tilde{\Psi}(Y)) \right\} - \alpha_0$$

$$= \sup_{Y \in \mathbb{P}_{nk}} \left\{ \mathbb{E}(\pi(A)^{q/2}Y\pi(A)^{p/2} - \mathbb{E}\tilde{\Psi}(Y)) \right\} - \alpha_0$$

$$= \sup_{X \in \mathbb{P}_{nk}} \left\{ \mathbb{E}(\pi(A)^{q/2}Y\pi(A)^{p/2} - \mathbb{E}\tilde{\Psi}(Y)) \right\} - \alpha_0.$$

For any fixed $X \in \mathbb{P}_{nk}$, since $\hat{f}(x^{2-p})$ is non-decreasing and concave on $(0, \infty)$ by Lemma A.1(c) with $r = 2 - p$, Corollary 3.3(1) implies that

$$A \in \mathbb{P}_n \mapsto \mathbb{E}(\pi(A)^{q/2}Y\pi(A)^{p/2} - \mathbb{E}\tilde{\Psi}(Y))$$

is concave. Moreover, $(A, B) \in \mathbb{P}_n \times \mathbb{P}_m \mapsto \mathbb{E}(\pi(A)^{q/2}Y\pi(A)^{p/2} - \mathbb{E}\tilde{\Psi}(Y))X$ is convex due to Lemma 5.1(b). Here, although $\tilde{\Psi}$ is not necessarily strictly positive, we can simply take the convergence from strictly positive maps. Joint convexity of (5.1) thus follows. The case $p = 2$ also follows by using Lemma 5.1(b) to (5.5) directly.

(2) Assume that $\Phi = \text{id}$ with $\mathbb{M}_n = \mathbb{M}_1$. The function (5.2) in this case is $\mathbb{E}(\Phi(A^p)^{1/2}\Phi(B^q)\Phi(A^p)^{1/2})$. As in the above proof for (5.1) we have

$$\mathbb{E}(\Phi(A^p)^{1/2}\Phi(B^q)\Phi(A^p)^{1/2}) = \sup_{X \in \mathbb{P}_l} \left\{ \mathbb{E}(\pi(A)^{q/2}Y\pi(A)^{p/2} - \mathbb{E}\tilde{\Psi}(Y))X \right\}.$$

Hence the assertion follows since $(A, B) \in \mathbb{P}_n \times \mathbb{P}_m \mapsto \mathbb{E}(\pi(A)^{q/2}Y\pi(A)^{p/2} - \mathbb{E}\tilde{\Psi}(Y))X$ is jointly convex by Lemma 5.1(b). \qed
Remark 5.5. For $1 \leq q \leq 2$ (resp., $-1 \leq q \leq 0$) joint convexity of (3) of Theorem 5.3 (resp., (2) of Theorem 5.4) in a slightly more general case where $\Psi$ (resp., $\Phi$) is invertible $K : M_n \rightarrow M_l$ with an invertible $K \in M_l$. However, this is not true when $\Psi : M_m \rightarrow M_l$ (resp., $\Phi : M_n \rightarrow M_l$) is a general CP map. For instance, let $-1 < p \leq 0$, $f(x) = x^s$ with $s \geq 1/(1+p)$, and $E$ be an orthogonal projection in $M_n$. Let $\Psi : M_n \rightarrow EM_nE$ (resp., $\Phi : M_l \rightarrow M_l$) be as defined in Remark 4.6. Then joint convexity of (5.2) would imply in particular that $B_\Psi (\Phi)$ of (3) of Theorem 5.3 (resp., (2) of Theorem 5.4) in a slightly more general case where $\Psi$ (resp., $\Phi$) is CP, we may write $\tilde{K} = \bar{K} \bar{\pi}$ with a representation $\bar{\pi} : M_m \rightarrow M_{mk}$ and a linear map $K : \mathbb{C}^{nk} \rightarrow \mathbb{C}^l$.

Theorem 5.6. Let $-1 \leq p \leq 0$ and $f$ be a non-decreasing concave function on $(0, \infty)$ such that $\lim_{x \rightarrow \infty} f(x)/x = 0$. Assume that $f(x^{2+p})$ is convex on $(0, \infty)$. If $q = 2$ and $\Phi, \Psi$ are CP, then (5.1) is jointly convex.

Proof. Assume that $q = 2$ and $\Phi, \Psi$ are CP. Write $\Psi(Z) = K \pi(Z) K^*$ with a representation $\pi : M_m \rightarrow M_{mk}$ and a linear map $K : \mathbb{C}^{nk} \rightarrow \mathbb{C}^l$, and let $\Phi(\cdot) := K^* \Phi(\cdot) K : M_n \rightarrow M_{nk}$. Then (5.1) with $q = 2$ is written as $\inf_{Y \in \mathbb{P}_{mk}} \{ \Tr \pi(B) Y^{1-p} \pi(B) \Phi(A^p) - \Tr \tilde{f}(Y^{1-p}) \} - \alpha_0$. Since $\tilde{f}(x^{1-p})$ is concave on $(0, \infty)$ by Lemma A.2(d) with $r = 1+p$, it follows that $Y \in \mathbb{P}_{mk} \rightarrow \Tr \tilde{f}(Y^{1-p})$ is concave. Hence, by [9] Lemma 2.3 it suffices to show that $(A, B, Y) \in \mathbb{P}_n \times \mathbb{P}_m \times \mathbb{P}_{mk} \rightarrow \Tr \pi(B) Y^{1-p} \pi(B) \Phi(A^p)$ is jointly convex. When $p = 0$, this holds by [13] Theorem 1. So assume that $-1 < p < 0$. Since $\Phi$ is CP, we may write $\tilde{K} = \bar{K} \bar{\pi}$ with a representation $\bar{\pi} : M_m \rightarrow M_{nk}$ and a linear map $\bar{K} : \mathbb{C}^{nk} \rightarrow \mathbb{C}^{nk}$. Then

$$\Tr \pi(B) Y^{-1-p} \pi(B) \Phi(A^p) = \Tr \bar{K}^* \pi(B) Y^{-1-p} \pi(B) \bar{\pi}(A)^p,$$

which is jointly convex in $(A, B, Y)$ by [17] Corollary 2.1.

The theorems proved above of course holds also when the roles of $p, \Phi$ and $q, \Psi$ are interchanged. In the case of power functions $f(x) = x^s$ we have a variety of ranges of $(p,q,s)$ for joint convexity of (2.2) from the above theorems, which are listed in the following as well as their counterparts where $p, \Phi$ and $q, \Psi$ are interchanged:
\(0 \leq p, q \leq 1, s \leq 0, \) or \(-1 \leq p, q \leq 0, s \geq 0,\) by Theorem 5.2.

(ii) \(-1 \leq p \leq 0, 1 \leq q \leq 2, s \geq \min\{1/(p + 1), 1/(q - 1)\}\) (with convention \(1/(-1 + 1) = 1/(1 - 1) = \infty\), and \(\Psi\) is CP, by Theorems 5.3(1) and 5.4(1).

(iii) \(0 \leq p \leq 1, -2 \leq q \leq -1, s \leq \max\{1/(p - 1), 1/(q + 1)\}\) (with convention \(1/(1 - 1) = 1/(1 + 1) = \infty\)), and \(\Psi = \text{id}\), by Theorems 5.3(3) and 5.4(2).

(iv) \(-1 \leq p \leq 0, q = 2, s \geq 1/(2 + p),\) and \(\Phi, \Psi\) are CP, by Theorems 5.6 and 5.4(1).

For the function (2.3) (when \(\Phi = \Psi = \text{id}\)), the convexity results in the cases (ii), (iii) and (iv) are contained in [7], as seen from \(\text{Tr} \left(A^{p/2}B^{q}A^{p/2}ight)^{s} = \text{Tr} \left(A^{-p/2}B^{-q}A^{-p/2}ight)^{-s}\).

Compared with the necessary conditions in [14, Proposition 5.4(2)], the missing region for joint convexity in this situation is only

\[-1 < p < 0, \quad 1 < q < 2, \quad \frac{1}{p + q} \leq s (\neq 1) < \min\left\{\frac{1}{p + 1}, \frac{1}{q - 1}\right\},\]

and its counterparts where \((p, q, s)\) are replaced with \((-p, -q, -s)\) and/or \(p, q\) are interchanged. Here, note that joint convexity is known when \(-1 \leq p \leq 0, 1 \leq q \leq 2, s = 1 \geq 1/(p + q)\), due to Ando [7]. In connection with the above missing region, it might be expected that Theorem 5.6 and its proof are also valid in the case where \(-1 \leq p \leq 0, 1 \leq q \leq 2\) and \(p + q \geq 1\). But this does not seem possible due to [7, Theorem 3.2].

A Variational formulas of trace functions

In this appendix we provide some variational formulas, which have played an essential role in Section 5, but which may also be of independent interest. For the convenience in exposition let us introduce the following classes of functions on \((0, \infty)\):

\[\mathcal{F}_{\text{convex}}^{\rho}(0, \infty)\] is the set of non-decreasing convex real functions \(f\) on \((0, \infty)\) such that \(\lim_{x \to \infty} f(x)/x = +\infty\).

\[\mathcal{F}_{\text{concave}}^{\rho}(0, \infty)\] is the set of non-decreasing concave real functions \(f\) on \((0, \infty)\) such that \(\lim_{x \to \infty} f(x)/x = 0\).

Note that affine functions \(ax + b (a \geq 0)\) are excluded from \(\mathcal{F}_{\text{convex}}^{\rho}(0, \infty)\), and so are \(ax + b (a > 0)\) from \(\mathcal{F}_{\text{concave}}^{\rho}(0, \infty)\).

Lemma A.1. (a) For each \(f \in \mathcal{F}_{\text{convex}}^{\rho}(0, \infty)\) define

\[\hat{f}(t) := \sup_{x > 0}\{xt - f(x)\}, \quad t \in (0, \infty)\].

Then \(\hat{f} \in \mathcal{F}_{\text{convex}}^{\rho}(0, \infty)\) and \(f \mapsto \hat{f}\) is an involutive bijection on \(\mathcal{F}_{\text{convex}}^{\rho}(0, \infty)\), i.e., \(\hat{\hat{f}} = f\) for all \(f \in \mathcal{F}_{\text{convex}}^{\rho}(0, \infty)\).
(b) For every $f \in \mathcal{F}_{\text{convex}}(0, \infty)$ and $B \in M_n^+$,

$$\text{Tr } f(B) = \sup_{A \in \mathbb{P}_n} \{ \text{Tr } AB - \text{Tr } \hat{f}(A) \},$$

where $f$ is continuously extended to $[0, \infty)$.

(c) Let $f$ be a non-constant and non-decreasing function on $(0, \infty)$ and $0 < r < 1$. Then $f(x^{1-r})$ is convex on $(0, \infty)$ if and only if $f \in \mathcal{F}_{\text{convex}}(0, \infty)$ and $\hat{f}(x^r)$ is concave on $(0, \infty)$.

**Proof.** (a) Let $f \in \mathcal{F}_{\text{convex}}(0, \infty)$ and $t \in (0, \infty)$. Since $f(0+) := \lim_{x \to 0+} f(x)$ exists in $\mathbb{R}$ and $xt - f(x) = x(t - f(x)/x) \to -\infty$ as $x \to \infty$, it follows that $\hat{f}(t)$ is defined as a finite value. By definition it is clear that $\hat{f}$ is convex and non-decreasing. For any $x > 0$ fixed, since $\hat{f}(t)/t \geq x - f(x)/t \to x$ as $t \to \infty$, we have $\lim_{t \to \infty} \hat{f}(t)/t = +\infty$, so $\hat{f} \in \mathcal{F}_{\text{convex}}(0, \infty)$. To show that $f \mapsto \hat{f}$ is an involutive bijection, we appeal to the duality of conjugate functions (or the Legendre transform) on $\mathbb{R}$. For each $f \in \mathcal{F}_{\text{convex}}(0, \infty)$ we extend $f$ to a continuous convex function $\tilde{f}$ on the whole $\mathbb{R}$ by $\tilde{f}(x) := f(0+)$ for $x \leq 0$. Then it is plain to see that the conjugate function $\hat{\tilde{f}}(t) := \sup_{x \in \mathbb{R}} \{ xt - \tilde{f}(x) \}$ is

$$\hat{\tilde{f}}(t) = \begin{cases} +\infty & \text{if } t < 0, \\ -f(0+) = \hat{f}(0+) & \text{if } t = 0, \\ \hat{f}(t) & \text{if } t > 0. \end{cases}$$

Due to the duality for conjugate functions, we have for $x > 0$,

$$f(x) = \sup_{t \in \mathbb{R}} \{ xt - \hat{f}(t) \} = \sup_{t > 0} \{ xt - \tilde{f}(t) \} = \hat{\tilde{f}}(x).$$

(b) To prove the assertion, we may assume that $B \in M_n^+$ is diagonal so that $B = \text{diag}(b_1, \ldots, b_n)$ with $b_1 \geq \cdots \geq b_n$. Since $f(x) = \sup_{t > 0} \{ tx - \hat{f}(t) \}$ for $x \geq 0$, we have

$$\text{Tr } f(B) = \sum_{i=1}^n f(b_i) = \sup_{a_1, \ldots, a_n > 0} \sum_{i=1}^n \{ a_i b_i - \hat{f}(a_i) \}$$

$$= \sup_{A = \text{diag}(a_1, \ldots, a_n) \in \mathbb{P}_n} \{ \text{Tr } AB - \text{Tr } \hat{f}(A) \}$$

$$\leq \sup_{A \in \mathbb{P}_n} \{ \text{Tr } AB - \text{Tr } \hat{f}(A) \}.$$

On the other hand, for every $A \in \mathbb{P}_n$ with eigenvalues $a_1 \geq \cdots \geq a_n$, since $\text{Tr } AB \leq \sum_{i=1}^n a_i b_i$ by majorization (see, e.g., [3 (III.19)], [13 Corollary 4.3.5]), we have

$$\text{Tr } AB - \text{Tr } \hat{f}(A) \leq \sum_{i=1}^n \{ a_i b_i - \hat{f}(a_i) \} \leq \sum_{i=1}^n f(b_i) = \text{Tr } f(B),$$

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and so
\[
\sup_{A \in \mathbb{P}_n} \{ \text{Tr} AB - \text{Tr} \hat{f}(A) \} \leq \text{Tr} f(B).
\]

(c) Let \( f \) be a non-constant and non-decreasing function on \((0, \infty)\) and \(0 < r < 1\). Assume that \( \hat{f}(x) := f(x^{1-r}) \) is convex on \((0, \infty)\). Then it immediately follows that \( f \) is convex on \((0, \infty)\). Since \( f(x^{1-r})/x^{1-r} = (\hat{f}(x)/x)x^{r} \to +\infty \) as \( x \to \infty \), we have \( f \in \mathcal{F}_{\text{convex}}(0, \infty) \). To show concavity of \( \hat{f}(x^{r}) \), we can assume that \( f \) is \( C^2 \) (even \( C^\infty \)) on \((0, \infty)\). Indeed, let \( \phi \) be a \( C^\infty \) function on \( \mathbb{R} \) supported on \([-1, 1]\) such that \( \phi(x) \geq 0 \) and \( \int_{-1}^{1} \phi(x) \, dx = 1 \). For each \( \varepsilon > 0 \) define a function \( f_\varepsilon \) on \((0, \infty)\) by
\[
f_\varepsilon(x) := \int_{-1}^{1} \phi(t) f(x^{-\varepsilon} t) \, dt, \quad x \in (0, \infty).
\]
Note that this product type regularization \( f_\varepsilon \) is \( h_\varepsilon(\log x), x > 0 \), where \( h_\varepsilon \) is the usual (additive type) regularization \( h_\varepsilon(s) := \int_{-1}^{1} \phi(t) h(s - \varepsilon t) \, dt \) of \( h(s) := f(e^s), s \in \mathbb{R} \) (see, e.g., [3] pp. 146–147, [13] Appendix A.2). Then, \( f_\varepsilon \) is \( C^\infty \) on \((0, \infty)\) and \( f_\varepsilon \to f \) as \( \varepsilon \downarrow 0 \) uniformly on any bounded closed interval of \((0, \infty)\). It is clear that \( f_\varepsilon \) satisfies the same assumption as \( f \). Moreover, we see that \( \hat{f}_\varepsilon \to \hat{f} \) as \( \varepsilon \downarrow 0 \) uniformly on any bounded closed interval of \((0, \infty)\), whose proof is given in Lemma A.3 below for completeness. So we may prove the conclusion for \( f_\varepsilon \) in place of \( f \).

By taking the limit as \( x \to 0^+ \) of the equation
\[
\frac{d}{dx} f(x^{1-r}) = (1-r)x^{-r}f'(x^{1-r}),
\]
we see that \( f'(0+) = 0 \). We can approximate \( f \) by \( g_\varepsilon(x) := f(x) + \varepsilon x^{1/(1-r)} \) for \( \varepsilon > 0 \) so that \( g_\varepsilon \) satisfies the same assumption as \( f \) and \( \hat{g}_\varepsilon(x) \to \hat{f}(x) \) as \( \varepsilon \downarrow 0 \). Hence we furthermore assume that \( f''(x) > 0 \) for all \( x > 0 \) and so \( f'(x) \) is strictly increasing on \((0, \infty)\). Now, compute the second derivative of \( f(x^{1-r}) \) as
\[
\frac{d^2}{dx^2} f(x^{1-r}) = (1-r)x^{-r-1}\{(1-r)x^{1-r}f''(x^{1-r}) - rf'(x^{1-r})\}, \tag{A.2}
\]
and therefore
\[
(1-r)x^{1-r}f''(x^{1-r}) - rf'(x^{1-r}) \geq 0, \quad x > 0. \tag{A.3}
\]
For every \( t > 0 \), since \( f'(0+) = 0 \) and \( \lim_{x \to \infty} f'(x) = \lim_{x \to \infty} f(x)/x = +\infty \), there is a unique \( x_0 > 0 \) such that \( f'(x_0) = t \) and thus \( xt - f(x) \) on \( x > 0 \) takes the maximum at \( x = x_0 = (f')^{-1}(t) \). Hence
\[
\hat{f}(t) = t(f')^{-1}(t) - f((f')^{-1}(t)).
\]
We further compute
\[
\frac{d}{dt} \hat{f}'(t) = rt^{r-1}(f')^{-1}(t) + t^{r-1} \left( \frac{rt^{r-1}}{f''((f')^{-1}(t))} - f'(((f')^{-1}(t))) \right) \frac{rt^{r-1}}{f''((f')^{-1}(t))}
\]

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Lemma A.2. \( f \) is \( \text{concave} \) on \((0, \infty)\) if and only if \( \hat{f} \) is \( \text{convex} \) on \((0, \infty)\).

Let \( (f')^{-1}(t') \) so that \( t' = f'(x) \), we have
\[
\frac{d^2}{dt^2} \hat{f}(t) = \frac{rt'^{-2}}{f''((f')^{-1}(t'))} \{(r-1)(f')^{-1}(t')f''((f')^{-1}(t')) + rt'\}.
\]

which is \( \leq 0 \) thanks to (A.3) and \( f''(x) > 0 \). Hence \( \hat{f}(x) \) is \( \text{convex} \) on \((0, \infty)\).

To prove the converse, assume that \( \hat{f} \) and hence \( \hat{f} \) are in \( \mathcal{F}_\text{convex}(0, \infty) \). By interchanging \( f \) with \( \hat{f} \) and \( r \) with \( 1 - r \) it suffices to prove that if \( \hat{f}(x^{1-r}) \) is \( \text{convex} \) on \((0, \infty)\), then \( \hat{f}(x^r) \) is \( \text{convex} \) on \((0, \infty)\). By Lemma \( \text{A.3} \) we can assume as in the first part of the proof that \( f \) is \( C^2 \) on \((0, \infty)\). By approximating \( f \) by \( f(x) + \varepsilon x^{1/(1-r)} \) as \( \varepsilon \searrow 0 \), we can furthermore assume that \( f''(x) > 0 \) for all \( x > 0 \). From (A.2) we have
\[
(1-r)x f''(x) - r f'(x) \leq 0, \quad x > 0.
\]

Let
\[
\alpha := f'(0+) = \lim_{x \to 0+} f'(x) \in [0, +\infty).
\]

It is clear that \( \hat{f}(t) = -f(0+) \) for all \( t \in (0, \alpha] \) (if \( \alpha > 0 \)). So it remains to prove that
\[
\frac{d^2}{dt^2} \hat{f}(t) \geq 0, \quad t > \alpha^{1/r}.
\]

When \( t > \alpha^{1/r} \), i.e., \( t^r > \alpha \), we can define \( x := (f')^{-1}(t') \) and compute (A.4) in the same way as above. Hence (A.6) follows from (A.5). \( \square \)

Concerning the assertion (c) above we need in Section 5 its “only if” part only while we give it as “if and only if” for completeness.

**Lemma A.2.** (a) For each \( f \in \mathcal{F}^\prime_\text{concave}(0, \infty) \) define
\[
\hat{f}(t) := \inf_{x > 0} \{xt - f(x)\}, \quad t \in (0, \infty).
\]

Then \( \hat{f} \in \mathcal{F}^\prime_\text{concave}(0, \infty) \) and \( f \mapsto \hat{f} \) is an involutive bijection on \( \mathcal{F}^\prime_\text{concave}(0, \infty) \), i.e., \( \hat{\hat{f}} = f \) for all \( f \in \mathcal{F}^\prime_\text{concave}(0, \infty) \).

(b) For every \( f \in \mathcal{F}^\prime_\text{concave}(0, \infty) \) and \( B \in \mathbb{P}_n \),
\[
\text{Tr} f(B) = \inf_{A \in \mathbb{P}_n} \{\text{Tr} AB - \text{Tr} \hat{f}(A)\}.
\]
(c) Let $f$ be a non-decreasing function on $(0, \infty)$ and $r > 0$. If $f(x^{1+r})$ is concave on $(0, \infty)$, then $f \in \mathcal{F}_\text{concave}(0, \infty)$ and $\tilde{f}(x^{-r})$ is convex on $(0, \infty)$.

(d) Let $f \in \mathcal{F}_\text{concave}(0, \infty)$ and $r > 0$. If $f(x^{1+r})$ is convex on $(0, \infty)$, then $\tilde{f}(x^{-r})$ is concave on $(0, \infty)$.

**Proof.** (a) Let $f \in \mathcal{F}_\text{concave}(0, \infty)$ and $t \in (0, \infty)$. Since $xt - f(x)$ is convex in $x \in (0, \infty)$ and $xt - f(x) = x(t - f(x)/x) \to +\infty$ as $x \to \infty$, it follows that $\tilde{f}(t)$ is defined as a finite value. By definition, $\tilde{f}$ is concave and non-decreasing. For any $x > 0$ fixed, since $\tilde{f}(t)/t \leq f(x)/x$ for $t \to \infty$, we have $\lim_{t \to \infty} \tilde{f}(t)/t = 0$, so $\tilde{f} \in \mathcal{F}_\text{concave}(0, \infty)$. To show that $f \mapsto \tilde{f}$ is an involutive bijection, we extend $f$ to $\tilde{f}$ on the whole $\mathbb{R}$ by $\tilde{f}(0) := \lim_{x \to 0^+} f(x)$ (possibly $-\infty$) and $\tilde{f}(x) = -\infty$ for $x < 0$. Then $-\tilde{f}$ is a lower semicontinuous convex function on $\mathbb{R}$, and the conjugate function $(-\tilde{f})^*(t) := \sup_{x \in \mathbb{R}} \{xt + \tilde{f}(x)\}$ is given as

$$(-\tilde{f})^*(t) = \begin{cases} -\tilde{f}(-t) & \text{if } t < 0, \\ f(\infty) := \lim_{x \to \infty} f(x) & \text{if } t = 0, \\ +\infty & \text{if } t > 0. \end{cases}$$

Due to the duality of conjugate functions, we have for $x > 0$,

$$-f(x) = -\tilde{f}(x) = \sup_{t \in \mathbb{R}} \{xt - (-\tilde{f})^*(t)\} = \sup_{t < 0} \{xt + \tilde{f}(-t)\}$$

by taking account of $(-\tilde{f})^*(0) = f(\infty) = -\lim_{t \to 0^+} \tilde{f}(t)$. Therefore,

$$f(x) = \inf_{t < 0} \{x(-t) - \tilde{f}(-t)\} = \inf_{t > 0} \{xt - \tilde{f}(t)\} = \tilde{f}(x).$$

(b) The proof is similar to that of Lemma A.1(b). We may use the majorization $\text{Tr} AB \geq \sum_{i=1}^n a_i b_{n+1-i}$ for $A, B \in \mathbb{P}_n$ with the respective eigenvalues $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$.

(c) Let $f$ be a non-decreasing function on $(0, \infty)$ and $r > 0$. Assume that $\tilde{f}(x) := f(x^{1+r})$ is concave on $(0, \infty)$. Then it immediately follows that $f$ is concave on $(0, \infty)$. Since $f(x^{1+r})/x^{1+r} = (\tilde{f}(x)/x)/x^r \to 0$ as $x \to \infty$, we have $f \in \mathcal{F}_\text{concave}(0, \infty)$. To show convexity of $\tilde{f}(x^{-r})$, the regularization (A.1) and Lemma A.3 below can be employed so that we may assume that $f$ is $C^2$ on $(0, \infty)$. By approximating $f$ by $f(x) + \varepsilon x^{1/(1+r)}$ as $\varepsilon \searrow 0$, we may assume that $\lim_{x \to 0^+} f'(x) = +\infty$ and $f''(x) < 0$ for all $x > 0$ and so $f'(x)$ is strictly decreasing on $(0, \infty)$. Since

$$\frac{d^2}{dx^2} f(x^{1+r}) = (1 + r)x^{-1}\{(1 + r)x^{1+r}f''(x^{1+r}) + rf'(x^{1+r})\},$$

we have

$$(1 + r)x f''(x) + rf'(x) \leq 0, \quad x > 0.$$
For every $t > 0$, since $\lim_{x \to 0^+} f'(x) = +\infty$ and $\lim_{x \to \infty} f'(x) = \lim_{x \to \infty} f(x)/x = 0$, there is a unique $x_0 > 0$ such that $f'(x_0) = t$ and thus $xt - f(x)$ on $x > 0$ takes the minimum at $x = x_0 = (f')^{-1}(t)$. We hence have $\hat{f}(t) = t(f')^{-1}(t) - f((f')^{-1}(t))$ and, as in the proof of Lemma A.1(c),

$$\frac{d^2}{dt^2} \hat{f}(t-r) = \frac{rt-r^2}{f''((f')^{-1}(t-r))} \{(1+r)(f')^{-1}(t-r)f''((f')^{-1}(t-r)) + rt-r\}.$$

Let $x := (f')^{-1}(t-r)$ so that $t-r = f'(x)$ we have

$$\frac{d^2}{dt^2} \hat{f}(t-r) = \frac{rt-r^2}{f''(x)} \{(1+r)xf''(x) + rf'(x)\}, \quad (A.9)$$

which is $\geq 0$ thanks to (A.8). Hence $\hat{f}(x-r)$ is convex on $(0, \infty)$.

(d) Assume that $f \in \mathcal{F}'_{\text{convex}}(0, \infty)$ and $f(x+r)$ is convex on $(0, \infty)$. We may assume that $f$ is $C^2$ as before. Approximating $f$ by $f(x) + \varepsilon x^{1/(1+r)}$ as $\varepsilon \searrow 0$ we may further assume that $f'(x) < 0$ for all $x > 0$. From (A.7) we have $(1+r)xf''(x) + rf'(x) \geq 0$ for all $x > 0$. Let $\alpha := \lim_{x \to 0^+} f'(x) \in (0, +\infty]$. If $\alpha < +\infty$, then $f(0+) := \lim_{x \to 0^+} f(x)$ exists in $\mathbb{R}$ and $\hat{f}(t) = -f(0+)$ for all $t \geq \alpha$. So it suffices to prove that

$$\frac{d^2}{dt^2} \hat{f}(t-r) \leq 0, \quad t-r < \alpha, \quad \text{i.e.,} \quad t > \alpha^{-1/r},$$

which indeed holds since we have (A.9) with $x := (f')^{-1}(t-r)$ when $t-r < \alpha$. \qed

**Lemma A.3.** Let $f \in \mathcal{F}'_{\text{convex}}(0, \infty)$ (resp., $f \in \mathcal{F}'_{\text{concave}}(0, \infty)$) and $f_\varepsilon$ be defined by (A.1) for each $\varepsilon > 0$. Then $f_\varepsilon \in \mathcal{F}'_{\text{convex}}(0, \infty)$ (resp., $f_\varepsilon \in \mathcal{F}'_{\text{concave}}(0, \infty)$) and $f_\varepsilon \to \hat{f}$ as $\varepsilon \searrow 0$ uniformly on any bounded closed interval of $(0, \infty)$.

**Proof.** Assume that $f \in \mathcal{F}'_{\text{convex}}(0, \infty)$. By definition (A.1) it is obvious that $f_\varepsilon$ is non-decreasing and convex on $(0, \infty)$. It is also obvious that $\lim_{x \to \infty} f_\varepsilon(x)/x = +\infty$ follows from the same property of $f$. Hence $f_\varepsilon \in \mathcal{F}'_{\text{convex}}(0, \infty)$ for any $\varepsilon > 0$. To prove the latter assertion, it suffices to show that $f_\varepsilon(t) \to f(t)$ for every $t > 0$, for it is plain to see that a pointwise convergent sequence of convex functions on $(0, \infty)$ is equicontinuous on any bounded closed interval of $(0, \infty)$. Let $t > 0$ be arbitrary. Choose a $\xi \geq 0$ such that $\hat{f}(t) = \xi t - f(\xi)$ (where $f(0) = f(0+)$). For every $\delta > 0$ we have $|f_\varepsilon(\xi) - f(\xi)| < \delta$ and so $\hat{f}_\varepsilon(t) \geq \xi t - f_\varepsilon(\xi) \geq \hat{f}(t) - \delta$ for any sufficiently small $\varepsilon > 0$. Hence $\liminf_{\varepsilon \searrow 0} \hat{f}_\varepsilon(t) \geq \hat{f}(t)$. Now, suppose by contradiction that $\hat{f}_\varepsilon(t) \not\to \hat{f}(t)$ as $\varepsilon \searrow 0$; then there are a $\delta_0 > 0$ and a sequence $0 < \varepsilon_n \searrow 0$ such that $\hat{f}_\varepsilon(t) \geq \hat{f}(t) - \delta_0$ for all $n$. Choose a sequence $x_n \geq 0$ such that $\hat{f}_\varepsilon(t) = x_n t - f_\varepsilon(x_n)$. By taking a subsequence we may assume that $x_n \to x_0 \in [0, \infty]$. If $x_0 = 0$, then

$$\hat{f}(t) + \delta_0 \leq \hat{f}_\varepsilon(t) = x_n t - f_\varepsilon(x_n) \to f(0) \leq \hat{f}(t) \quad \text{as} \quad n \to \infty,$$
Let $0 < r < 1$. Besides $f(x) = x^s$ with $s \geq 1/(1-r)$ the following are examples of non-decreasing convex functions $f$ such that $f(x^{1-r})$ is convex on $(0, \infty)$:

- For any $s \geq 1/(1-r)$ and $\alpha > 0$, $f(x) = (x - \alpha)^s_+$ or $f(x) = (x^s - \alpha^s)_+$.

- For $s_1, s_2 \geq 1/(1-r)$ and $\alpha > 0$,
  
  $$
  f(x) = \begin{cases} 
  x^{s_1} & \text{if } 0 < x \leq \alpha, \\
  \beta(x^{s_2} - \alpha^{s_2}) + \alpha^{s_1} & \text{if } x \geq \alpha,
  \end{cases}
  $$

  where $\beta \geq (s_1/s_2)^{s_1-s_2}$.

(2) Let $r > 0$. Besides $f(x) = x^s$ with $0 < s \leq 1/(1+r)$ and $f(x) = \log x$ the following are examples of non-decreasing concave functions $f$ such that $f(x^{1+r})$ is concave on $(0, \infty)$:

- For any $0 < s \leq 1/(1+r)$ and $\alpha > 0$,
  
  $$
  f(x) = \begin{cases} 
  x^s - \alpha x & \text{if } 0 < x \leq (s/\alpha)^{1/(1-s)}, \\
  (1-s)(s/\alpha)^s/(1-s) & \text{if } x \geq (s/\alpha)^{1/(1-s)}.
  \end{cases}
  $$

- For $0 < s_1, s_2 \leq 1/(1+r)$ and $\alpha > 0$,
  
  $$
  f(x) = \begin{cases} 
  x^{s_1} & \text{if } 0 < x \leq \alpha, \\
  \beta(x^{s_2} - \alpha^{s_2}) + \alpha^{s_1} & \text{if } x \geq \alpha,
  \end{cases}
  $$

  where $0 < \beta \leq (s_1/s_2)^{s_1-s_2}$. 

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