Small mass asymptotics of a charged particle in a variable magnetic field

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Abstract
We consider small mass asymptotics of the motion of a charged particle in a noisy force field combined with a variable magnetic field. The Smoluchowski-Kramers approximation does not hold in this case. We show that after a regularization of noise, a Smoluchowski-Kramers type approximation works.

1 Introduction
Consider a charged particle of mass $\mu > 0$ moving on a plane. Let the position of this particle at time $t$ be $q_t^\mu \in \mathbb{R}^2$. We may express the force field with random noise on the plane as

$$b(q_t^\mu) + \sigma(q_t^\mu) \dot{w}_t,$$

where $b: \mathbb{R}^2 \to \mathbb{R}^2$ is a vector-valued function, $\sigma: \mathbb{R}^2 \to M_2(\mathbb{R})$ is a matrix-valued function, and $\dot{w}_t \in \mathbb{R}^2$ is a two dimensional Wiener process.

Now, suppose that the motion of the particle is subject to a variable magnetic field perpendicular to the plane. The force on the particle due to this magnetic field can be expressed as

$$A(q_t^\mu) \dot{q}_t^\mu = \alpha(q_t^\mu)A_0 \dot{q}_t^\mu, \quad (1.1)$$

where $\alpha: \mathbb{R}^2 \to \mathbb{R}^+ \to \mathbb{R}$ is a positive real-valued function and

$$A_0 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The motion of this particle is governed by the Newton law, so that

$$\begin{cases} \mu \ddot{q}_t^\mu = b(q_t^\mu) + A(q_t^\mu)\dot{q}_t^\mu + \sigma(q_t^\mu) \dot{w}_t \\ q_0^\mu = q_0 \in \mathbb{R}^2, \quad \dot{q}_0^\mu = p_0 \in \mathbb{R}^2. \end{cases} \quad (1.2)$$

Notice that equation (1.2) can be rewritten as a system of first order equations

$$\begin{cases} \dot{q}_t^\mu = p_t^\mu \\ \dot{p}_t^\mu = \frac{1}{\mu} b(q_t^\mu) + \frac{1}{\mu} A(q_t^\mu) p_t^\mu + \frac{1}{\mu} \sigma(q_t^\mu) \dot{w}_t \\ q_0^\mu = q_0 \in \mathbb{R}^2, \quad p_0^\mu = p_0 \in \mathbb{R}^2. \end{cases} \quad (1.3)$$
Now, let $q_t$ be the solution of the following first order SDE with $\mu = 0$ from equation (1.2):
\[
\begin{cases}
\dot{q}_t = -A^{-1}(q_t)b(q_t) - A^{-1}(q_t)\sigma(q_t)\hat{w}_t \\
q_0 \in \mathbb{R}^2.
\end{cases}
\]

It is natural to consider the convergence of $q_t^\mu$ to $q_t$ as $\mu \downarrow 0$. The Smoluchowski-Kramers approximation tells us that in the case of the Langevin equation, that is, in the case of
\[
A(q) \equiv -cI
\]
for a constant $c > 0$, we may replace $q_t^\mu$ with $q_t$ for small $\mu$ due to the convergence
\[
\lim_{\mu \downarrow 0} E \max_{0 \leq t \leq T} |q_t^\mu - q_t| = 0
\]
([3]). However, this convergence is not satisfied for general $A(q)$. Even in the case of $A(q)$ being a constant matrix, if the real parts of the eigenvalues of $A(q)$ are nonnegative, the Smoluchowski-Kramers approximation does not hold. For example, for $A(q) \equiv A_0$, the above convergence fails due to
\[
\lim_{\mu \downarrow 0} \int_0^t \sin \left( \frac{s}{\mu} \right) dw_s \neq 0
\]
([1]). So we cannot use this approximation in our case. Nonetheless, we may regularize the problem and check a convergence similar to the Smoluchowski-Kramers approximation ([1], [3]).

Firstly, it is physically reasonable to introduce a small friction proportional to the velocity. We may write $A_\epsilon(q) = A(q) - \epsilon I$ and approximate $q_t^\mu$ with $q_t^{\mu,\epsilon}$, the solution of the following SDE
\[
\begin{cases}
\dot{q}_t^{\mu,\epsilon} = p_t^{\mu,\epsilon} \\
\dot{p}_t^{\mu,\epsilon} = \frac{1}{\mu} b(q_t^{\mu,\epsilon}) + \frac{1}{\mu} A_\epsilon(q_t^{\mu,\epsilon}) p_t^{\mu,\epsilon} + \frac{1}{\mu} \sigma(q_t^{\mu,\epsilon}) \hat{w}_t.
\end{cases}
\]

This small friction term makes the real parts of the eigenvalues of $A_\epsilon(q)$ negative and gives us an exponential decay of the term
\[
\frac{1}{\mu} \exp \left( \frac{1}{\mu} \int_0^t A_\epsilon(q_s^{\mu,\epsilon}) ds \right) = \frac{1}{\mu} \exp \left( -\frac{\epsilon}{\mu} t \right) \left( \cos \left( \frac{1}{\mu} \int_0^t \alpha(q_s^{\mu,\epsilon}) ds \right) \sin \left( \frac{1}{\mu} \int_0^t \alpha(q_s^{\mu,\epsilon}) ds \right) - \sin \left( \frac{1}{\mu} \int_0^t \alpha(q_s^{\mu,\epsilon}) ds \right) \cos \left( \frac{1}{\mu} \int_0^t \alpha(q_s^{\mu,\epsilon}) ds \right) \right)
\]
as $\mu \downarrow 0$. However, it turns out that this approximation does not support us with enough regularity for the convergence of the system. This follows from
\[
\lim_{\mu \downarrow 0} \int_0^t \frac{1}{\mu} \exp \left( -\frac{2\epsilon}{\mu} s \right) \left| \int_0^s \exp \left( \frac{\epsilon r}{\mu} \right) dw_r \right|^2 ds \neq 0.
\]

As another regularization method, we may approximate the Wiener process $w_t$ with a $\delta$-correlated smooth process $w^\delta_t$ as in [6] Example 7.3 Chapter VI. In section 2, we will prove that as $\mu \downarrow 0$ and $\delta \downarrow 0$ in the way that $\mu \epsilon^{\frac{1}{2 \epsilon}} \downarrow 0$ for each constant $C > 0$, the solution $q_t^{\mu,\delta}$ of approximated second order equation (2.2) converges to the solution $\hat{q}_t$ of first order SDE (2.5) in the sense that
\[
\lim_{\mu \downarrow 0, \delta \downarrow 0, \mu \epsilon^{\frac{1}{2 \epsilon}}} E \max_{0 \leq t \leq T} |q_t^{\mu,\delta} - \hat{q}_t| = 0.
\]
In section 3, we consider an application of this approximation, a homogenization problem.

Throughout the present paper we shall use $| \cdot |$ as the standard Euclidean norm in $\mathbb{R}^n$ and $| \cdot |_{\infty}$ as the supremum norm in spaces of functions. Moreover, symbols $C$ and $C_i$'s will indicate arbitrary large positive constants. $C$ and $C_i$'s may take different values in different places.
2 Small mass asymptotics under the regularized Wiener process

We define $w_\delta^t$ as a mollification of the Wiener process $w_t$ as in [6, Example 7.3 Chapter VI],

$$w_\delta^t = \frac{1}{\delta} \int_0^\infty w(s) \rho(\frac{s-t}{\delta}) ds,$$

(2.1)

where $\rho(t) \geq 0$ is smooth, has the support in $[0,1]$, and satisfies

$$\int_0^1 \rho(s) ds = 1.$$

Thus, $w_\delta^t$ is a smooth approximation of $w_t$ satisfying

$$\lim_{\delta \downarrow 0} \mathbb{E} \max_{0 \leq t \leq T} |w_\delta^t - w_t|^2 = 0.$$

Now, we rewrite (1.3) with $w_\delta^t$ in place of $w_t$

$$\dot{q}_\mu^t, \dot{p}_\mu^t = b(q_\mu^t) + A(q_\mu^t) p_\mu^t + \sigma(q_\mu^t) \dot{w}_\delta^t$$

(2.2)

To give enough regularity for the problem, we assume the following conditions on $b(q)$, $\sigma(q)$, and $\alpha(q)$.

**Hypothesis 1.**

1. $b : \mathbb{R}^2 \to \mathbb{R}^2$ and $\sigma : \mathbb{R}^2 \to M_2(\mathbb{R})$ are differentiable and bounded with their derivatives.
2. $\alpha : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and bounded with its derivative. Moreover,

$$\inf_{q \in \mathbb{R}^2} \alpha(q) = \alpha_0 > 0.$$

Under Hypothesis 1, we have the relation

$$\lim_{\delta \downarrow 0} \mathbb{E} \max_{0 \leq t \leq T} |q_\mu^t, \delta - q_\mu^t|^2 = 0$$

thanks to [6, Theorem 7.2 Chapter VI] and the following property stemming from the continuous differentiability of $q_\mu^t$ in $t$:

$$\int_0^T \sigma(q_\mu^t) dw_t = \int_0^T \sigma(q_\mu^t) \circ dw_t,$$

where the integral on the right is understood in the Stratonovich sense.

What will happen if we tend $\mu \downarrow 0$ faster than $\delta \downarrow 0$? It turns out that with the smooth noise $w_\delta^t$, we have enough regularity to find the limit. This follows from the fact that for regular enough $f$,

$$\lim_{\mu \downarrow 0} \int_0^t \cos \left( \frac{1}{\mu} \int_0^s \alpha(q_\mu^r) dr \right) f(q_\mu^r) ds = 0.$$

We will discuss on this property in Lemma 2.5. Now, we are ready to state the main theorems.
Theorem 2.1. Under Hypothesis 1, there exists a constant $C > 0$ such that for any $0 < \mu \leq 1$ and $0 < \delta \leq 1$,

$$E \max_{0 \leq t \leq T} |q_{t}^{\mu, \delta} - q_{t}^{\delta}| \leq C(1 + T)\frac{\mu}{\delta^2} \exp \left( \frac{C}{\delta^2} (1 + T)^3 \right) \mu, \quad (2.3)$$

where $q_{t}^{\delta}$ is the solution of the first order differential equation

$$\begin{aligned}
q_{t}^{\delta} &= -A^{-1}(q_{t}^{\delta}) b(q_{t}^{\delta}) + A^{-1}(q_{t}^{\delta}) \sigma(q_{t}^{\delta}) \hat{w}_{t} \\
q_{0}^{\delta} &= q_{0} \in \mathbb{R}^2.
\end{aligned} \quad (2.4)$$

In particular, for any fixed $0 < \delta \leq 1$,

$$\lim_{\mu \downarrow 0} E \max_{0 \leq t \leq T} |q_{t}^{\mu, \delta} - q_{t}^{\delta}| = 0. \quad (2.6)$$

We postpone the proof of Theorem 2.1 to the end of this section. By [6, Theorem 7.2 Chapter VI], we have the following result.

Theorem 2.2. Under Hypothesis 1,

$$\lim_{\delta \downarrow 0} E \max_{0 \leq t \leq T} |q_{t}^{\delta} - \hat{q}_{t}| = 0,$$

where $\hat{q}_{t}$ is the solution of the first order stochastic differential equation

$$\begin{aligned}
\dot{q}_{t} &= -A^{-1}(\hat{q}_{t}) b(\hat{q}_{t}) - (A^{-1}(\hat{q}_{t}) \sigma(\hat{q}_{t})) \circ \hat{w}_{t} \\
\hat{q}_{0} &= q_{0} \in \mathbb{R}^2.
\end{aligned} \quad (2.5)$$

We state the combination of the above two theorems in the following corollary.

Corollary 2.3. Under Hypothesis 1, $q_{t}^{\mu, \delta}$ converges to $\hat{q}_{t}$ in probability in $C([0, T]; \mathbb{R}^2)$ as $\mu \downarrow 0$ and $\delta \downarrow 0$ so that $\mu e^{\frac{C}{\delta^2}} \downarrow 0$ for each constant $C > 0$.

For the proof of Theorem 2.1 it is necessary to find some auxiliary bounds. In the following three lemmas, we find those bounds.

First of all, in Lemma 2.4 we find a uniform bound of $|p_{t}^{\mu, \delta}|$ in $C([0, T]; \mathbb{R}^2)$ independent of $\mu$.

Lemma 2.4. Under Hypothesis 1, there exists a constant $C > 0$ such that for any $0 < \delta \leq 1$,

$$\sup_{\mu > 0} \max_{0 \leq t \leq T} |p_{t}^{\mu, \delta}| \leq C(1 + T)(1 + X_{T}) \exp \left( \frac{C}{\delta^2} (1 + T)(1 + X_{T}) \right), \quad \mathbb{P} - a.s., \quad (2.6)$$

where

$$X_{T} = \max_{0 \leq t \leq T+1} |w_{t}|. \quad (2.7)$$
Proof. Suppose \(0 \leq t \leq T\). From equation (2.2),

\[
\dot{p}_t^\mu,\delta - \frac{1}{\mu} A(q_t^\mu,\delta) p_t^\mu,\delta = \frac{1}{\mu} b(q_t^\mu,\delta) + \frac{1}{\mu} \sigma(q_t^\mu,\delta) \dot{w}_t^\delta.
\]

Multiplying both sides by

\[
\exp \left( -\frac{1}{\mu} \int_0^t A(q_s^\mu,\delta) \, ds \right),
\]

we get

\[
\left( \exp \left( -\frac{1}{\mu} \int_0^t A(q_s^\mu,\delta) \, ds \right) p_t^\mu,\delta \right) \dot{t} = \frac{1}{\mu} \exp \left( -\frac{1}{\mu} \int_0^t A(q_s^\mu,\delta) \, ds \right) b(q_t^\mu,\delta) + \frac{1}{\mu} \exp \left( -\frac{1}{\mu} \int_0^t A(q_s^\mu,\delta) \, ds \right) \sigma(q_t^\mu,\delta) \dot{w}_t^\delta.
\]

Define \(\beta_t^\mu,\delta\) as

\[
\beta_t^\mu,\delta := \int_0^t \alpha(q_s^\mu,\delta) \, ds.
\]

Considering the definition of \(A(q_t^\mu,\delta)\) in (1.1), we have

\[
\int_0^t A(q_s^\mu,\delta) \, ds = \int_0^t \alpha(q_s^\mu,\delta) \, ds A_0 = \beta_t^\mu A_0.
\]

So, we may rewrite the above equation as

\[
\left( \exp \left( -\frac{\beta_t^\mu,\delta}{\mu} A_0 \right) p_t^\mu,\delta \right) \dot{t} = \frac{1}{\mu} \exp \left( -\frac{\beta_t^\mu,\delta}{\mu} A_0 \right) b(q_t^\mu,\delta) + \frac{1}{\mu} \exp \left( -\frac{\beta_t^\mu,\delta}{\mu} A_0 \right) \sigma(q_t^\mu,\delta) \dot{w}_t^\delta.
\]

Integrating both sides with respect to \(t\) we get

\[
p_t^\mu,\delta = \exp \left( \frac{\beta_t^\mu,\delta}{\mu} A_0 \right) p_0 + \frac{1}{\mu} \exp \left( \frac{\beta_t^\mu,\delta}{\mu} A_0 \right) \int_0^t \exp \left( -\frac{\beta_s^\mu,\delta}{\mu} A_0 \right) b(q_s^\mu,\delta) \, ds
\]

\[
+ \frac{1}{\mu} \exp \left( \frac{\beta_t^\mu,\delta}{\mu} A_0 \right) \int_0^t \exp \left( -\frac{\beta_s^\mu,\delta}{\mu} A_0 \right) \sigma(q_s^\mu,\delta) \, dw_s^\delta
\]

\[
=: I_1(t) + I_2(t) + I_3(t).
\]

By the definition of \(A_0\) in (1.1), we can calculate the matrix exponentials

\[
\exp \left( \pm \frac{\beta_t^\mu,\delta}{\mu} A_0 \right) = \left( \begin{array}{cc} \cos \left( \frac{\beta_t^\mu,\delta}{\mu} \right) & \mp \sin \left( \frac{\beta_t^\mu,\delta}{\mu} \right) \\ \pm \sin \left( \frac{\beta_t^\mu,\delta}{\mu} \right) & \cos \left( \frac{\beta_t^\mu,\delta}{\mu} \right) \end{array} \right).
\]

Since (2.9) is an orthogonal matrix, for any \(v \in \mathbb{R}^2\),

\[
\left| \exp \left( \pm \frac{\beta_t^\mu,\delta}{\mu} A_0 \right) v \right| = |v|
\]

(2.10)
so that

\[ |I_1(t)| \leq |p_0|. \quad (2.11) \]

As \( A_0 \) and \( A_0^{-1} \) commute, we have

\[
I_2(t) = \exp \left( \frac{\beta_t^{\mu,\delta}}{\mu} A_0 \right) \int_0^t \left( -\frac{\alpha(q_s^{\mu,\delta})}{\mu} A_0 \exp \left( -\frac{\beta_s^{\mu,\delta}}{\mu} A_0 \right) \right) \left( -\frac{1}{\alpha(q_s^{\mu,\delta})} A_0^{-1} b(q_s^{\mu,\delta}) \right) ds
\]

\[
= \exp \left( \frac{\beta_t^{\mu,\delta}}{\mu} A_0 \right) \left[ \exp \left( -\frac{\beta_s^{\mu,\delta}}{\mu} A_0 \right) \left( -\frac{1}{\alpha(q_s^{\mu,\delta})} A_0^{-1} b(q_s^{\mu,\delta}) \right) \right]_0^t
\]

\[
- \int_0^t \exp \left( \frac{\beta_s^{\mu,\delta}}{\mu} A_0 \right) \left( \nabla \alpha(q_s^{\mu,\delta}) \cdot p_s^{\mu,\delta} A_0^{-1} b(q_s^{\mu,\delta}) - \frac{1}{\alpha(q_s^{\mu,\delta})} A_0^{-1} D_b(q_s^{\mu,\delta}) p_s^{\mu,\delta} \right) ds
\]

\[
= \exp \left( \frac{\beta_t^{\mu,\delta}}{\mu} A_0 \right) \left( -\frac{1}{\alpha(q_t^{\mu,\delta})} A_0^{-1} b(q_t^{\mu,\delta}) - \frac{1}{\alpha(q_0^{\mu,\delta})} A_0^{-1} b(q_0) \right)
\]

\[
- \int_0^t \exp \left( \frac{\beta_t^{\mu,\delta} - \beta_s^{\mu,\delta}}{\mu} A_0 \right) \left( \nabla \alpha(q_s^{\mu,\delta}) \cdot p_s^{\mu,\delta} A_0^{-1} b(q_s^{\mu,\delta}) - \frac{1}{\alpha(q_s^{\mu,\delta})} A_0^{-1} D_b(q_s^{\mu,\delta}) p_s^{\mu,\delta} \right) ds.
\]

\[
(2.12)
\]

The same method can be used for \( I_3(t) \) and we get

\[
I_3(t) = \frac{1}{\mu} \exp \left( \frac{\beta_t^{\mu,\delta}}{\mu} A_0 \right) \int_0^t \exp \left( -\frac{\beta_s^{\mu,\delta}}{\mu} A_0 \right) \sigma(q_s^{\mu,\delta}) \omega_s^{\delta,\delta} ds
\]

\[
= -\frac{1}{\alpha(q_t^{\mu,\delta})} A_0^{-1} \sigma(q_t^{\mu,\delta}) \omega_t^{\delta} + \frac{1}{\alpha(q_0^{\mu,\delta})} A_0^{-1} \sigma(q_0) \omega_0^{\delta}
\]

\[
- \int_0^t \exp \left( \frac{\beta_t^{\mu,\delta} - \beta_s^{\mu,\delta}}{\mu} A_0 \right) \left( \nabla \alpha(q_s^{\mu,\delta}) \cdot p_s^{\mu,\delta} A_0^{-1} \sigma(q_s^{\mu,\delta}) \omega_s^{\delta} - \frac{1}{\alpha(q_s^{\mu,\delta})} A_0^{-1} D_b(q_s^{\mu,\delta}) p_s^{\mu,\delta} \omega_s^{\delta} \right) ds.
\]

\[
(2.13)
\]

To find bounds for \( I_2(t) \) and \( I_3(t) \), we need bounds for \( \omega_t^{\delta} \) and \( \omega_0^{\delta} \). In view of equation \((2.11)\), we note that \( (w_t^{\delta})^{(n)} \), the \( n \)th derivative of \( w_t^{\delta} \) with respect to \( t \), satisfies

\[
(w_t^{\delta})^{(n)} = \frac{(-1)^n}{\delta^n} \int_0^\infty w(s) \rho^{(n)}(s - t) ds = \frac{(-1)^n}{\delta^n} \int_0^1 w(t + \delta r) \rho^{(n)}(r) dr.
\]

Hence, for any \( 0 \leq t \leq T \),

\[
|(w_t^{\delta})^{(n)}| \leq \frac{1}{\delta^n} \int_0^1 |w(t + \delta s)||\rho^{(n)}(s)| ds
\]

\[
\leq \frac{1}{\delta^n} \max_{0 \leq s \leq T + \delta} |w(t)| \int_0^1 |\rho^{(n)}(s)| ds = \frac{C(n)}{\delta^n} \frac{1}{\max_{0 \leq s \leq T + 1} |w(t)|}.
\]
where $C(n)$ is a constant depending on $n$.

Letting

$$X_T := \max_{0 \leq t \leq T+1} |w_t|,$$

we have

$$\max_{0 \leq t \leq T} |(w^\delta_t)^{(n)}| \leq \frac{C(n)}{\delta^n} X_T.$$

In particular, we can find a constant $C > 0$ such that

$$\max_{0 \leq t \leq T} |\dot{w}^\delta_t| \leq \frac{C}{\delta} X_T$$

and

$$\max_{0 \leq t \leq T} |\ddot{w}^\delta_t| \leq \frac{C}{\delta^2} X_T.$$

Now, we are ready to find bounds for $I_2(t)$ and $I_3(t)$. Applying Hypothesis [1] (2.10), and (2.14) to (2.12) and (2.13), we get

$$|I_2(t)| \leq \left| \frac{1}{\alpha(q) A_0} A_0^{-1} b(q^\mu) \right| + \left| \frac{1}{\alpha(q_0) A_0} A_0^{-1} \exp \left( \frac{\beta^\mu}{\mu} A_0 \right) b(q_0) \right|$$

$$+ \int_0^t \exp \left( \frac{\beta^\mu}{\mu} A_0 \right) \left( \nabla \alpha(q^\mu \cdot p^\mu A_0^{-1} b(q^\mu) - \frac{1}{\alpha(q_0)} A_0^{-1} Db(q^\mu) p^\mu \right) ds$$

$$\leq C_1 + C_2 + C_3 \int_0^t |p^{\mu,\delta}_s| ds + C_4 \int_0^t |p^{\mu,\delta}_s| ds = C_5 + C_6 \int_0^t |p^{\mu,\delta}_s| ds$$

and

$$|I_3(t)| \leq \left| \frac{1}{\alpha(q_0) A_0} A_0^{-1} \sigma(q^\mu \delta) w^\delta \right| + \left| \frac{1}{\alpha(q) A_0} A_0^{-1} \exp \left( \frac{\beta^\mu}{\mu} A_0 \right) \sigma(q) w^\delta \right|$$

$$+ \int_0^t \exp \left( \frac{\beta^\mu}{\mu} A_0 \right) \left( \nabla \alpha(q^\mu \cdot p^\mu A_0^{-1} \sigma(q^\mu) w^\delta - \frac{1}{\alpha(q_0)} A_0^{-1} D\sigma(q^\mu) p^\mu \bar{w}^\delta \right)$$

$$+ \left| \frac{1}{\alpha(q_0) A_0} A_0^{-1} \sigma(q^\mu \delta) w^\delta \right| ds$$

$$\leq \frac{C_7}{\delta} X_T + \frac{C_8}{\delta} X_T + \frac{C_9}{\delta} X_T \int_0^t |p^{\mu,\delta}_s| ds + \frac{C_{10}}{\delta} X_T \int_0^t |p^{\mu,\delta}_s| ds + \frac{C_{11}}{\delta^2} X_T \int_0^t |p^{\mu,\delta}_s| ds$$

$$\leq \frac{C_{12}}{\delta^2} (1 + t) X_T + \frac{C_{13}}{\delta} X_T \int_0^t |p^{\mu,\delta}_s| ds.$$ (2.15)

Applying three inequalities (2.11), (2.15), and (2.16) to (2.8), we get a bound for $p^{\mu,\delta}_t$.

$$|p^{\mu,\delta}_t| \leq |I_1(t)| + |I_2(t)| + |I_3(t)|$$

$$\leq |p| + C_5 + C_6 \int_0^t |p^{\mu,\delta}_s| ds + \frac{C_{12}}{\delta^2} (1 + t) X_T + \frac{C_{13}}{\delta} X_T \int_0^t |p^{\mu,\delta}_s| ds$$

$$\leq \frac{C_{14}}{\delta^2} (1 + t)(1 + X_T) + \frac{C_{15}}{\delta}(1 + X_T) \int_0^t |p^{\mu,\delta}_s| ds.$$ (2.16)
By Gronwall’s lemma,
\[
|p_t| \leq C_{14}(1 + t)(1 + X_T) \exp \left( \frac{C_{15}}{\delta} (1 + X_T)t \right)
\]
\[
\leq C(1 + t)(1 + X_T) \exp \left( \frac{C}{\delta} (1 + t)(1 + X_T) \right)
\]
for sufficiently large \( C > 0 \). The last inequality came from the fact that the term \( \frac{1}{\delta^2} \) can be absorbed in the term \( e^{\frac{C}{\delta}} \) for large \( C \).

So, we have
\[
\max_{0 \leq t \leq T} |p_t| \leq C(1 + T)(1 + X_T) \exp \left( \frac{C}{\delta} (1 + T)(1 + X_T) \right).
\]

\[\Box\]

**Remark 1.** Note that by Lemma 2.4, \( q_{t}^{\mu, \delta} \) is Lipschitz continuous with its Lipschitz constant independent of \( \mu \) on the interval \([0, T]\). That is, for \( 0 \leq t_1 \leq t_2 \leq T \),
\[
|q_{t_2}^{\mu, \delta} - q_{t_1}^{\mu, \delta}| \leq C(T, \delta, X_T)|t_2 - t_1| \quad \mathbb{P} - a.s. \tag{2.17}
\]

Next, we find a bound of the integral of a highly oscillating function. The result is similar to that of the Riemann-Lebesgue lemma.

**Lemma 2.5.** Under Hypothesis 1, there exists a constant \( C > 0 \) such that for any \( \mu > 0 \) and \( 0 < \delta \leq 1 \), and for any bounded Lipschitz continuous function \( f : \mathbb{R}^2 \to \mathbb{R} \) with the Lipschitz constant \( K_f \),
\[
\left| \int_0^t \cos \left( \frac{\beta_s^{\mu, \delta}}{\mu} \right) f(q_s^{\mu, \delta}) ds \right| + \left| \int_0^t \sin \left( \frac{\beta_s^{\mu, \delta}}{\mu} \right) f(q_s^{\mu, \delta}) ds \right| \leq C_1(t, \delta, X_T, f) \mu \tag{2.18}
\]
and
\[
\left| \int_0^t \cos \left( \frac{\beta_s^{\mu, \delta}}{\mu} \right) f(q_s^{\mu, \delta}) \dot{w}^\delta_s ds \right| + \left| \int_0^t \sin \left( \frac{\beta_s^{\mu, \delta}}{\mu} \right) f(q_s^{\mu, \delta}) \dot{w}^\delta_s ds \right| \leq C_2(t, \delta, X_T, f) \mu \tag{2.19}
\]
\[\mathbb{P} - a.s., \text{ where} \]
\[
C_1(t, \delta, X_T, f) = C(1 + t)^2(|f|_\infty + K_f)(1 + X_T) \exp \left( \frac{C}{\delta} (1 + t)(1 + X_T) \right)
\]
and
\[
C_2(t, \delta, X_T, f) = C(1 + t)^2(|f|_\infty + K_f)^2(1 + X_T)^2 \exp \left( \frac{C}{\delta} (1 + t)(1 + X_T) \right)
\]
\[\text{Proof.} \] Since \( \alpha(q_s^{\mu, \delta}) \) is strictly positive, \( \beta_s^{\mu, \delta} \) is strictly increasing, so that
\[
u = \frac{\beta_s^{\mu, \delta}}{\mu}
\]
provides a good change of variables. 

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Then, as
\[ du = \frac{\alpha(q^\mu_\delta)}{\mu} ds, \] (2.20)
we have
\[ \int_0^t \cos \left( \frac{\beta^\mu_\delta}{\mu} \right) f(q^\mu_\delta) ds = \mu \int_0^{\beta^\mu_\delta} \cos(u) \frac{f(q^\mu_\delta)}{\alpha(q^\mu_\delta)} du. \]

If we define
\[ g^\mu_\delta(u) := \frac{f(q^\mu_\delta(u))}{\alpha(q^\mu_\delta)}, \] (2.21)
we get
\[ \left| \int_0^t \cos \left( \frac{\beta^\mu_\delta}{\mu} \right) f(q^\mu_\delta) ds \right| = \left| \mu \int_0^{\beta^\mu_\delta} \cos(u) g^\mu_\delta(u) du \right| \]
\[ = \left| \mu \sum_{k=0}^{\left[ \frac{\beta^\mu_\delta}{2\pi\mu} \right] - 1} \int_{2\pi k}^{2\pi(k+1)} \cos(u) g^\mu_\delta(u) du + \mu \int_{2\pi \left[ \frac{\beta^\mu_\delta}{2\pi\mu} \right]}^{\beta^\mu_\delta} \cos(u) g^\mu_\delta(u) du \right| \]
\[ \leq \mu \sum_{k=0}^{\left[ \frac{\beta^\mu_\delta}{2\pi\mu} \right] - 1} \int_{2\pi k}^{2\pi(k+1)} \cos(u) \left( g^\mu_\delta(u) - g^\mu_\delta(2\pi k) \right) du \]
\[ + \mu \int_{2\pi \left[ \frac{\beta^\mu_\delta}{2\pi\mu} \right]}^{\beta^\mu_\delta} \cos(u) g^\mu_\delta(u) du \]
\[ = \left| I_1(t) \right| + \left| I_2(t) \right|. \] (2.22)

We first find a bound of \( |I_1(t)| \). Note that from Hypothesis 1
\[ 0 < \alpha_0 \leq \alpha(q) \leq |\alpha|_{\infty}. \]
where $K$ is the Lipschitz constant for $\alpha(q)$ and $K_f$ is the Lipschitz constant for $f(q)$.

From Lemma 2.4, we have

$$\left| \frac{d}{du} y_{\mu,\delta}(u) \right| = \left| \frac{\mu}{\alpha(q_{s(u)})} \right| \leq C(t, \delta, X_t) \frac{\mu}{\alpha_0},$$

where

$$C(t, \delta, X_t) := C(1 + t)(1 + X_t) \exp \left( \frac{C}{\delta}(1 + t)(1 + X_t) \right).$$

So, from (2.23),

$$|g^{\mu,\delta}(u) - g^{\mu,\delta}(2\pi k)| \leq C_2 C(t, \delta, X_t)(|f|_\infty + K_f)\mu|u - 2\pi k|.$$  

This implies

$$\left| \frac{\beta^{\mu,\delta}}{2\pi\mu} \right| -1 \leq \mu \sum_{k=0}^{2\pi(k+1)} \int_{2\pi k}^{2\pi(k+1)} |\cos(u)||g^{\mu,\delta}(u) - g^{\mu,\delta}(2\pi k)|du$$

$$\leq \mu \sum_{k=0}^{2\pi(k+1)} \int_{2\pi k}^{2\pi(k+1)} C_2 C(t, \delta, X_t)(|f|_\infty + K_f)\mu|u - 2\pi k|du$$

$$= \mu \sum_{k=0}^{2\pi(k+1)} C_2 C(t, \delta, X_t)(|f|_\infty + K_f)\mu|u - 2\pi k|$$

$$= C_2 C(t, \delta, X_t)(|f|_\infty + K_f)\mu^2 2\pi^2 \left| \frac{\beta^{\mu,\delta}}{2\pi\mu} \right|.$$
Since 
\[ \beta^\mu,\delta_t \leq |\alpha|_\infty t, \]
we get 
\[ |I_1(t)| \leq C_2 C(t, \delta, X_t)(|f|_\infty + K_f) \mu \pi |\alpha|_\infty t \]
\[ = C_3 C(t, \delta, X_t)(|f|_\infty + K_f)t\mu. \] (2.25)

A bound for \( |I_2(t)| \) can be found relatively easily. From (2.21),
\[ |I_2(t)| \leq \mu \int_{2\pi}^{\beta^\mu,\delta \frac{t}{2\pi \mu}} \left| \frac{f(q^\mu,\delta_{s(u)})}{\alpha(q^\mu,\delta_{s(u)})} \right| du \]
\[ \leq \mu \int_{2\pi}^{\beta^\mu,\delta \frac{t}{2\pi \mu}} \frac{|f|_\infty}{\alpha_0} du \]
\[ \leq \mu 2\pi \frac{|f|_\infty}{\alpha_0} = C_4 |f|_\infty \mu. \] (2.26)

So, from (2.22), (2.25), (2.26), and (2.21),
\[ \left| \int_0^t \cos \left( \frac{\beta^\mu,\delta_s}{\mu} \right) f(q^\mu,\delta_s) \delta_s ds \right| \leq C_3 C(t, \delta, X_t)(|f|_\infty + K_f)t\mu + C_4 |f|_\infty \mu \]
\[ \leq C_5 (1 + t)^2(|f|_\infty + K_f)(1 + X_t) \exp \left( \frac{C_6}{\delta} (1 + t)(1 + X_t) \right) \mu. \]

This proves inequality (2.18) for the cosine part. The sine part can be treated analogously. Now consider inequality (2.19). As in (2.22),
\[ \left| \int_0^t \cos \left( \frac{\beta^\mu,\delta_s}{\mu} \right) f(q^\mu,\delta_s) \delta_s ds \right| \leq \mu \sum_{k=0}^{\beta^\mu,\delta \frac{t}{2\pi \mu} - 1} \int_{2\pi k}^{2\pi(k+1)} \cos(u) \left( g^\mu,\delta_1(u) - g^\mu,\delta_1(2\pi k) \right) du \]
\[ + \mu \int_{2\pi}^{\beta^\mu,\delta \frac{t}{2\pi \mu}} g^\mu,\delta_1(u) du \]
\[ = |I_1(t)| + |I_2(t)|, \] (2.27)
where
\[ g^\mu,\delta_1(u) := \frac{f(q^\mu,\delta_{s(u)})}{\alpha(q^\mu,\delta_{s(u)})} \beta^\mu,\delta_s(u). \]
By a similar argument as in (2.23), we obtain

\[ |g_1^{\mu,\delta}(u) - g_1^{\mu,\delta}(2\pi k)| = \left| \frac{f(q_{s(u)}^{\mu,\delta})}{\alpha(q_{s(u)}^{\mu,\delta})} \dot{u}_s^{\mu,\delta} - \frac{f(q_{s(2\pi k)}^{\mu,\delta})}{\alpha(q_{s(2\pi k)}^{\mu,\delta})} \dot{u}_s^{\mu,\delta} \right| \leq \frac{1}{\alpha_0^2} \left( |f|_\infty \max_{0 \leq s \leq t} \{|u_s^{\mu,\delta}|\} K \left| q_{s(u)}^{\mu,\delta} - q_{s(2\pi k)}^{\mu,\delta} \right| + |\alpha|_\infty \max_{0 \leq s \leq t} \{|\dot{u}_s^{\mu,\delta}|\} |q_{s(u)}^{\mu,\delta} - q_{s(2\pi k)}^{\mu,\delta}| \right) + |\alpha|_\infty |f|_\infty \max_{0 \leq s \leq t} \{|\dot{u}_s^{\mu,\delta}|\} |s(u) - s(2\pi k)| \right) .

Considering inequalities in (2.14) and Remark 1,

\[ |g_1^{\mu,\delta}(u) - g_1^{\mu,\delta}(2\pi k)| \leq \frac{C_7}{\delta} (|f|_\infty + K_f) X_t |q_{s(u)}^{\mu,\delta} - q_{s(2\pi k)}^{\mu,\delta}| + \frac{C_8}{\delta^2} |f|_\infty X_t |s(u) - s(2\pi k)| \leq \frac{C_9}{\delta} (|f|_\infty + K_f) C(t, \delta, X_t) |s(u) - s(2\pi k)| + \frac{C_8}{\delta^2} |f|_\infty X_t |s(u) - s(2\pi k)| \leq \frac{C_9}{\delta^2} (|f|_\infty + K_f) C(t, \delta, X_t) + |f|_\infty X_t |s(u) - s(2\pi k)| .

Note that from (2.20)

\[ |\frac{d}{du} s(u)| \leq \frac{\mu}{\alpha_0} .

Therefore, if we assume that \(0 < \delta \leq 1\), we have

\[ |g_1^{\mu,\delta}(u) - g_1^{\mu,\delta}(2\pi k)| \leq \frac{C_9}{\delta^2} (|f|_\infty + K_f) C(t, \delta, X_t) |s(u) - s(2\pi k)| + \frac{C_8}{\delta^2} |f|_\infty X_t |s(u) - s(2\pi k)| \leq \frac{C_10}{\delta^2} (|f|_\infty + K_f) C(t, \delta, X_t) X_t |s(u) - s(2\pi k)| .

By the same procedures as in (2.25) and (2.26),

\[ |I_1(t)| \leq \frac{C_10}{\delta^2} (|f|_\infty + K_f) C(t, \delta, X_t) X_t |s(u)| \]

and

\[ |I_2(t)| \leq \frac{C_11}{\delta} |f|_\infty X_t |s(u)| .

Now from (2.27), we get

\[ \int_0^t \cos \frac{\beta_s^{\mu,\delta}}{\mu} f(q_{s(u)}^{\mu,\delta}) \dot{u}_s^{\mu,\delta} ds \leq \frac{C_10}{\delta^2} (|f|_\infty + K_f) C(t, \delta, X_t) X_t t \mu + \frac{C_11}{\delta} |f|_\infty X_t \mu \]

\[ \leq C_12 (1 + t) \frac{|f|_\infty + K_f}{1 + X_t} \left( \frac{C_13}{\delta} (1 + t)(1 + X_t) \right) \mu .

The last inequality was from (2.24) and the fact that \(\frac{1}{\delta}\) or \(\frac{1}{\delta^2}\) can be absorbed in \(e^{\frac{t}{\delta}}\) in \(C(t, \delta, X_t)\).

In the next lemma, we show that the expectation of the exponential of the uniform norm of the two dimensional Wiener process in \(C([0, t + 1]; \mathbb{R}^2)\) is finite. This property will be used at the end of the proof of the main theorem.
Lemma 2.6. For any integer \( n \geq 0 \) and \( a \in \mathbb{R}^+ \), there exists a constant \( C(n) > 0 \) such that

\[
E \left( (1 + X_t)^n e^{(1+X_t)a} \right) \leq C(n)(1 + t)^{\frac{2n}{2}} e^{2(1+t)a^2 + a},
\]

where

\[
X_t = \max_{0 \leq s \leq t+1} |w_s|,
\]

Proof. We have

\[
E \left( (1 + X_t)^n e^{(1+X_t)a} \right) \leq C_1(n) e^{a} \sum_{k=0}^{n} E(X_t^k e^{aX_t}) \leq C_1(n) e^{a} \sum_{k=0}^{n} \left( E(X_t^2)^{\frac{k}{2}} \right) \left( E(e^{2aX_t}) \right)^{\frac{k}{2}},
\]

where \( C_1(n) \) is a constant depending on \( n \).

Since \( w_s = (w_s^1, w_s^2) \), where \( w_s^1 \) and \( w_s^2 \) are independent one dimensional Wiener processes, defining

\[
X_{i,t} := \max_{0 \leq s \leq t+1} |w_s^i|
\]

for \( i = 1, 2 \), we have

\[
X_t \leq X_{1,t} + X_{2,t}.
\]

From

\[
E(X_t^{2k}) \leq E((X_{1,t} + X_{2,t})^{2k}) \leq C_2(n) E(X_{1,t}^{2k} + X_{2,t}^{2k})
\]

\[
= 2C_2(n) E(X_{1,t}^{2k})
\]

for \( k = 1, 2, ..., n \) and

\[
E(e^{2aX_t}) \leq E(e^{2a(X_{1,t} + X_{2,t})}) = E(e^{2aX_{1,t}})^2,
\]

we get

\[
E \left( (1 + X_t)^n e^{(1+X_t)a} \right) \leq C_3(n) e^{a} \sum_{k=0}^{n} \left( E(X_{1,t}^{2k}) \right)^{\frac{k}{2}} E(e^{2aX_{1,t}}).
\]

To find bounds for \( E(X_{1,t}^{2k}) \) or \( E(e^{2aX_{1,t}}) \), we need to know a bound for the distribution of \( X_{1,t} \). We use the symmetry of the Wiener process and the reflection principle to find this bound.

For \( x \geq 0 \),

\[
P(\max_{0 \leq s \leq T} |w_s^1| > x) = P(\max_{0 \leq s \leq T} \{w_s^1\} > x) \cup \{ \min_{0 \leq s \leq T} \{w_s^1\} < -x \}
\]

\[
\leq P(\max_{0 \leq s \leq T} \{w_s^1\} > x) + P(\min_{0 \leq s \leq T} \{w_s^1\} < -x)
\]

\[
= 2P(\max_{0 \leq s \leq T} \{w_s^1\} > x).
\]

By the reflection principle,

\[
P(\max_{0 \leq s \leq T} |w_s^1| > x) \leq 4P(w_T^1 > x) = 4 \int_{x}^{\infty} \frac{1}{\sqrt{2\pi T}} e^{-\frac{y^2}{2T}} dy.
\]

So, for \( T = t + 1 \),

\[
P(X_t > x) \leq 4 \int_{x}^{\infty} \frac{1}{\sqrt{2\pi (t+1)}} e^{-\frac{y^2}{2(t+1)}} dy.
\]
Using inequality (2.30),
\[
E(e^{2aX_{1,t}}) = \int_0^\infty P(e^{2aX_{1,t}} > x)dx = \int_0^\infty P(X_{1,t} > \frac{1}{2a} \ln x)dx \\
\leq \int_0^\infty 4P(w_{t+1}^1 > \frac{1}{2a} \ln x)dx = 4 \int_0^\infty \int_{\frac{1}{2a} \ln x}^\infty \frac{1}{\sqrt{2\pi(t+1)}} e^{-\frac{y^2}{2(t+1)}} dy dx \\
\leq 4e^{2(t+1)a^2}
\]
and
\[
E(X_{1,t}^{2k}) = \int_0^\infty P(X_{1,t}^{2k} > x)dx = \int_0^\infty P(X_{1,t} > x^{\frac{1}{2k}})dx \\
\leq \int_0^\infty 4P(w_{t+1}^1 > x^{\frac{1}{2k}})dx = 4 \int_0^\infty \int_{x^{\frac{1}{2k}}}^\infty \frac{1}{\sqrt{2\pi(t+1)}} e^{-\frac{y^2}{2(t+1)}} dy dx \\
\leq C_4(n)(t+1)^k
\]
for \(k = 1, 2, ..., n\).
Applying these bounds to (2.29),
\[
E \left( (1 + X_t)^n e^{(1+X_t)a} \right) \leq C_5(n)e^a \sum_{k=0}^n (t+1)^k e^{2(t+1)a^2} \\
\leq C_6(n)(t+1)^{\frac{n}{2}} e^{2(t+1)a^2+a}.
\]
The last inequality was from Young’s inequality:
\[
\sum_{k=0}^n (t+1)^\frac{k}{2} \leq C_7(n)(1 + (t+1)^{\frac{n}{2}}) \leq 2C_7(n)(t+1)^{\frac{n}{2}}.
\]
Finally, we are ready to prove the main theorem, Theorem 2.1.

Proof of Theorem 2.1 Consider \(0 \leq t \leq T\). First, we find representations of \(q_{t}^\mu,\delta\) and \(q_{t}^\delta\). Integrating equations (2.8) and (2.4),
\[
q_{t}^\mu,\delta = q_0 + \int_0^t \exp \left( \frac{\beta_{\mu,\delta}}{\mu} A_0 \right) p_0 ds + \int_0^t \exp \left( \frac{\beta_{\mu,\delta}}{\mu} A_0 \right) \int_0^s \exp \left( -\frac{\beta_{\mu,\delta}}{\mu} A_0 \right) b(q_{r}^\mu,\delta) dr ds \\
+ \int_0^t \exp \left( \frac{\beta_{\mu,\delta}}{\mu} A_0 \right) \int_0^s \exp \left( -\frac{\beta_{\mu,\delta}}{\mu} A_0 \right) \sigma(q_{r}^\mu,\delta) dw_r^\delta ds
\]
and
\[
q_{t}^\delta = q_0 - \int_0^t A^{-1}(q_{s}^\delta) b(q_{s}^\delta) ds - \int_0^t A^{-1}(q_{s}^\delta) \sigma(q_{s}^\delta) dw_s^\delta \\
= q_0 - \int_0^t \frac{1}{\alpha(q_{s}^\delta)} A_0^{-1} b(q_{s}^\delta) ds - \int_0^t \frac{1}{\alpha(q_{s}^\delta)} A_0^{-1} \sigma(q_{s}^\delta) dw_s^\delta.
\]
Subtracting $q^\delta_t$ from $q^{\mu,\delta}_t$,

\[
q^{\mu,\delta}_t - q^\delta_t = \int_0^t \exp \left( \frac{\beta^{\mu,\delta}_s}{\mu} A_0 \right) p_0 \, ds \\
+ \left( \frac{1}{\mu} \right) \int_0^t \exp \left( \frac{\beta^{\mu,\delta}_s}{\mu} A_0 \right) \int_0^t \exp \left( - \frac{\beta^{\mu,\delta}_r}{\mu} A_0 \right) b(q^{\mu,\delta}_r) \, dr \, ds + \int_0^t \frac{1}{\alpha(q^{\mu,\delta}_s)} A_0^{-1} b(q^{\mu,\delta}_r) \, ds \\
+ \left( \frac{1}{\mu} \right) \int_0^t \exp \left( \frac{\beta^{\mu,\delta}_s}{\mu} A_0 \right) \int_0^t \exp \left( - \frac{\beta^{\mu,\delta}_r}{\mu} A_0 \right) \sigma(q^{\mu,\delta}_r) \, dw^{\delta}_r \, ds + \int_0^t \frac{1}{\alpha(q^{\mu,\delta}_s)} A_0^{-1} \sigma(q^{\mu,\delta}_r) \, dw^{\delta}_r \\
- \left( \int_0^t \frac{1}{\alpha(q^{\mu,\delta}_s)} A_0^{-1} b(q^{\mu,\delta}_s) \, ds - \int_0^t \frac{1}{\alpha(q^{\delta}_s)} A_0^{-1} b(q^{\delta}_s) \, ds \right) \\
- \left( \int_0^t \frac{1}{\alpha(q^{\mu,\delta}_s)} A_0^{-1} \sigma(q^{\mu,\delta}_s) \, dw^{\delta}_s \ - \ \int_0^t \frac{1}{\alpha(q^{\delta}_s)} A_0^{-1} \sigma(q^{\delta}_s) \, dw^{\delta}_s \right) \\
= I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t).
\]

To get a bound for $q^{\mu,\delta}_t - q^\delta_t$, we will find bounds for the terms from $I_1(t)$ to $I_5(t)$. First, consider $I_1(t)$.

From (2.9), expressing $p_0 = \left( \begin{array}{c} p_0^1 \\ p_0^2 \end{array} \right)$, we have

\[
|I_1(t)| = \left| \int_0^t \cos \left( \frac{\beta^{\mu,\delta}_s}{\mu} \right) ds p_0^1 - \int_0^t \sin \left( \frac{\beta^{\mu,\delta}_s}{\mu} \right) ds p_0^2 \right| \\
= |p_0| \left( \left| \int_0^t \cos \left( \frac{\beta^{\mu,\delta}_s}{\mu} \right) ds \right| + \left| \int_0^t \sin \left( \frac{\beta^{\mu,\delta}_s}{\mu} \right) ds \right| \right) \\
\leq C |p_0| (1 + t)^2 (1 + X t) e^{C(1+t)(1+X r)} \mu.
\]

In the last inequality, we used Lemma 2.5.

Now, let’s consider $I_2(t)$. Note that the commutativity of $A_0$ and $A_0^{-1}$ justifies the commuta-
tivity of matrix exponentials. Applying integration by parts,

\[
\frac{1}{\mu} \int_0^t \exp \left( \frac{\beta_{s,\delta}}{\mu} A_0 \right) \int_0^s \exp \left( -\frac{\beta_{r,\delta}}{\mu} A_0 \right) b(q_r^{s,\delta}) \, dr \, ds \\
= \int_0^t \frac{\alpha(q_{s,\delta}^{s,\delta})}{\mu} A_0 \exp \left( \frac{\beta_{s,\delta}}{\mu} A_0 \right) \frac{1}{\alpha(q_{s,\delta}^{s,\delta})} A_0^{-1} \int_0^s \exp \left( -\frac{\beta_{r,\delta}}{\mu} A_0 \right) b(q_r^{s,\delta}) \, dr \, ds \\
- \int_0^t \frac{1}{\alpha(q_{s,\delta}^{s,\delta})} A_0^{-1} \int_0^t \exp \left( -\frac{\beta_{r,\delta}}{\mu} A_0 \right) b(q_r^{s,\delta}) \, ds \\
= \exp \left( \frac{\beta_{s,\delta}}{\mu} A_0 \right) \frac{1}{\alpha(q_{s,\delta}^{s,\delta})} A_0^{-1} \int_0^t \exp \left( -\frac{\beta_{r,\delta}}{\mu} A_0 \right) b(q_r^{s,\delta}) \, ds \\
- \int_0^t \exp \left( \frac{\beta_{s,\delta}}{\mu} A_0 \right) \nabla \alpha(q_{s,\delta}^{s,\delta}) \cdot \frac{p_{s,\delta}}{\alpha(q_{s,\delta}^{s,\delta})^2} A_0^{-1} \int_0^s \exp \left( -\frac{\beta_{r,\delta}}{\mu} A_0 \right) b(q_r^{s,\delta}) \, dr \, ds \\
- \int_0^t \frac{1}{\alpha(q_{s,\delta}^{s,\delta})} A_0^{-1} b(q_s^{s,\delta}) \, ds.
\]

This yields

\[
|I_2(t)| = \left| \exp \left( \frac{\beta_{t,\delta}}{\mu} A_0 \right) \frac{1}{\alpha(q_{t,\delta}^{s,\delta})} A_0^{-1} \int_0^t \exp \left( -\frac{\beta_{s,\delta}}{\mu} A_0 \right) b(q_r^{s,\delta}) \, ds \\
- \int_0^t \exp \left( \frac{\beta_{s,\delta}}{\mu} A_0 \right) \nabla \alpha(q_{s,\delta}^{s,\delta}) \cdot \frac{p_{s,\delta}}{\alpha(q_{s,\delta}^{s,\delta})^2} A_0^{-1} \int_0^s \exp \left( -\frac{\beta_{r,\delta}}{\mu} A_0 \right) b(q_r^{s,\delta}) \, dr \, ds \right| \\
\leq \left| \exp \left( \frac{\beta_{t,\delta}}{\mu} A_0 \right) \frac{1}{\alpha(q_{t,\delta}^{s,\delta})} A_0^{-1} \int_0^t \exp \left( -\frac{\beta_{s,\delta}}{\mu} A_0 \right) b(q_r^{s,\delta}) \, ds \right| \\
+ \left| \int_0^t \exp \left( \frac{\beta_{s,\delta}}{\mu} A_0 \right) \nabla \alpha(q_{s,\delta}^{s,\delta}) \cdot \frac{p_{s,\delta}}{\alpha(q_{s,\delta}^{s,\delta})^2} A_0^{-1} \int_0^s \exp \left( -\frac{\beta_{r,\delta}}{\mu} A_0 \right) b(q_r^{s,\delta}) \, dr \, ds \right|.
\]

Considering (2.10) and Hypothesis II

\[
|I_2(t)| \leq \frac{1}{\alpha_0} \int_0^t \exp \left( -\frac{\beta_{s,\delta}}{\mu} A_0 \right) b(q_r^{s,\delta}) \, ds + \int_0^t \left| \nabla \alpha \right|_{\infty} \left| p_{s,\delta} \right| |p_{s,\delta}| \int_0^s \exp \left( -\frac{\beta_{r,\delta}}{\mu} A_0 \right) b(q_r^{s,\delta}) \, dr \, ds.
\]

Applying Lemma 2.3,

\[
|I_2(t)| \leq C_1 (1 + t)^2 (1 + X_t) \exp \left( \frac{C_2}{\delta} (1 + t)(1 + X_t) \right) \mu \\
+ \int_0^t \left| p_{s,\delta} \right| C_3 (1 + s)^2 (1 + X_s) \exp \left( \frac{C_4}{\delta} (1 + s)(1 + X_s) \right) \mu \, ds \\
\leq C_1 (1 + t)^2 (1 + X_t) \exp \left( \frac{C_2}{\delta} (1 + t)(1 + X_t) \right) \mu \\
+ C_3 (1 + t)^2 (1 + X_t) \exp \left( \frac{C_4}{\delta} (1 + t)(1 + X_t) \right) \mu \int_0^t \left| p_{s,\delta} \right| \, ds.
\]
Note that by Lemma 2.4

\[
\int_0^t |p_s^\mu| ds \leq C_5(1 + t)(1 + X_t) \exp \left( \frac{C_6}{\delta}(1 + t)(1 + X_t) \right) t
\]

and so,

\[
|I_2(t)| \leq C_1(1 + t)^2(1 + X_t) \exp \left( \frac{C_2}{\delta}(1 + t)(1 + X_t) \right) \mu
\]

\[
+ C_6(1 + t)^4(1 + X_t)^2 \exp \left( \frac{C_7}{\delta}(1 + t)(1 + X_t) \right) \mu
\]

\[
\leq C(1 + t)^4(1 + X_t)^2 \exp \left( \frac{C}{\delta}(1 + t)(1 + X_t) \right) \mu.
\]

We can apply a similar procedure as in getting the bound for \(I_2(t)\) in the case of \(I_3(t)\) and get the bound

\[
|I_3(t)| \leq C(1 + t)^4(1 + X_t)^3 \exp \left( \frac{C}{\delta}(1 + t)(1 + X_t) \right) \mu.
\]

Now, we find a bound for \(I_4(t)\). From the expression of \(I_4(t)\) in (2.31),

\[
|I_4(t)| = \left| \int_0^t A_0^{-1} \left( \frac{1}{\alpha(q_s^\mu)}b(q_s^\mu) - \frac{1}{\alpha(q_s^\mu)}b(q_s^\delta) \right) ds \right|
\]

\[
= \left| \int_0^t A_0^{-1} \left( \frac{b(q_s^\mu)\alpha(q_s^\delta) - b(q_s^\delta)\alpha(q_s^\mu)}{\alpha(q_s^\mu)\alpha(q_s^\mu)} \right) ds \right|
\]

\[
\leq \int_0^t \left| \frac{b(q_s^\mu)\alpha(q_s^\delta) - b(q_s^\delta)\alpha(q_s^\mu)}{\alpha(q_s^\mu)\alpha(q_s^\mu)} \right| ds
\]

\[
\leq \frac{1}{\alpha_0^2} \int_0^t |b(q_s^\mu)\alpha(q_s^\delta) - b(q_s^\delta)\alpha(q_s^\mu)| ds + |b(q_s^\mu)\alpha(q_s^\delta) - b(q_s^\delta)\alpha(q_s^\mu)| ds
\]

\[
\leq \frac{1}{\alpha_0^2} \int_0^t |\alpha| |b(q_s^\mu) - b(q_s^\delta)| + |b(q_s^\mu) - b(q_s^\delta)| \alpha(q_s^\mu)\alpha(q_s^\mu) | ds
\]

\[
\leq \frac{1}{\alpha_0^2} \int_0^t |\alpha| |b(q_s^\mu) - b(q_s^\delta)| + |\alpha(q_s^\mu) - \alpha(q_s^\delta)| | ds
\]

\[
\leq \frac{1}{\alpha_0^2} \int_0^t |\alpha| |K| |q_s^\mu - q_s^\delta| + |\alpha(q_s^\mu) - \alpha(q_s^\delta)| ds
\]

\[
\leq C \int_0^t |q_s^\mu - q_s^\delta| ds,
\]

where \(K\) is the Lipschitz constant for both \(b(q)\) and \(\alpha(q)\).

By a similar method, a bound for \(I_5(t)\) can be found also. We have

\[
|I_5(t)| \leq \frac{C}{\delta} X_t \int_0^t |q_s^\mu - q_s^\delta| ds.
\]
Combining these results and applying the bounds of $I_1(t)$ to $I_5(t)$ to (2.31), we obtain
\[ |q^\mu,\delta_t - q^\delta_t| \leq C|p_0|(1 + t)^2(1 + X_t) \exp \left( \frac{C}{\delta}(1 + t)(1 + X_t) \right) \mu \]
\[ + C(1 + t)^4(1 + X_t)^2 \exp \left( \frac{C}{\delta}(1 + t)(1 + X_t) \right) \mu \]
\[ + C(1 + t)^4(1 + X_t)^3 \exp \left( \frac{C}{\delta}(1 + t)(1 + X_t) \right) \mu \]
\[ + C \int_0^t |q^\mu,\delta_s - q^\delta_s| ds + \frac{C}{\delta} X_t \int_0^t |q^\mu,\delta_s - q^\delta_s| ds \]
\[ \leq C(1 + t)^4(1 + X_t)^3 \exp \left( \frac{C}{\delta}(1 + t)(1 + X_t) \right) \mu + \frac{C}{\delta}(1 + X_t) \int_0^t |q^\mu,\delta_s - q^\delta_s| ds. \]

Then, from the Gronwall’s lemma, we can conclude
\[ |q^\mu,\delta_t - q^\delta_t| \leq C(1 + t)^4(1 + X_t)^3 \exp \left( \frac{C}{\delta}(1 + t)(1 + X_t) \right) \mu \exp \left( \frac{C}{\delta}(1 + X_t)t \right) \]
\[ \leq C(1 + t)^4(1 + X_t)^3 \exp \left( \frac{C}{\delta}(1 + t)(1 + X_t) \right) \mu. \]

This gives
\[ \max_{0 \leq t \leq T} |q^\mu,\delta_t - q^\delta_t| \leq C(1 + T)^4(1 + X_T)^3 \exp \left( \frac{C}{\delta}(1 + T)(1 + X_T) \right) \mu, \]

So that, by taking expectation and applying Lemma 2.6,
\[ E \max_{0 \leq t \leq T} |q^\mu,\delta_t - q^\delta_t| \leq E[C(1 + T)^4(1 + X_T)^3 \exp \left( \frac{C}{\delta}(1 + T)(1 + X_T) \right) \mu] \]
\[ \leq C(1 + T)^4 \mu E[(1 + X_T)^3 \exp \left( \frac{C}{\delta}(1 + T)(1 + X_T) \right)] \]
\[ \leq C(1 + T)^4 \mu(1 + T)^3 \exp \left( 2(T + 1) \frac{C^2}{\delta^2}(1 + T)^2 + \frac{C}{\delta}(1 + T) \right) \]
\[ \leq C(1 + T)^{\frac{3}{2}} \exp \left( \frac{C}{\delta^2}(1 + T)^3 \right) \mu. \]

\[ \square \]

3 Homogenization

In this section, we consider the case of a fast oscillating periodic magnetic field. Consider the solution $q^\mu,\delta,\epsilon_t$ of
\[ \begin{cases} 
\mu q^\mu,\delta,\epsilon_t = b(q^\mu,\delta,\epsilon_t) + \alpha \left( \frac{q^\mu,\delta,\epsilon_t}{\epsilon} \right) A_0 q^\mu,\delta,\epsilon_t + w^\delta_t \\
q^\mu,\delta,\epsilon_0 = q_0 \in \mathbb{R}^2, \quad q_0^\mu,\delta,\epsilon = p_0 \in \mathbb{R}^2,
\end{cases} \]

where $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a 1-periodic function and $\epsilon > 0$ is a constant. By periodicity of $\alpha$, we can consider the domain of $\alpha$ as $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$, the two dimensional unit torus. In this case, a unique
weak limit of the process \( q_{t}^{\mu,\delta,\epsilon} \) as \( \mu \downarrow 0 \), \( \delta \downarrow 0 \), and \( \epsilon \downarrow 0 \) in order exists and we find this limit by applying homogenization results in the literature \([3, 2, 4, 3, 8, 7]\) to our system. Note that we solve for \( \sigma(q) \equiv I \) for computational convenience. In general, if \( \sigma(q) \sigma(q)^{\ast} \) is positive definite for all \( q \in \mathbb{R}^{2} \), we can find a weak limit. For the proof of homogenization results, we need more restrictive assumptions than Hypothesis 1.

**Hypothesis 2.**

1. \( b : \mathbb{R}^{2} \rightarrow \mathbb{R}^{2} \) is twice continuously differentiable and bounded with its derivatives.
2. \( \alpha : \mathbb{R}^{2} \rightarrow \mathbb{R} \) is twice continuously differentiable and bounded with its derivatives. Moreover, \( \inf_{q \in \mathbb{R}^{2}} \alpha(q) = \alpha_{0} > 0 \).

**Proposition 3.1.** Under Hypothesis 2, \( q_{t}^{\mu,\delta,\epsilon} \) converges to \( \hat{q}_{t} \) weakly as \( \mu \downarrow 0 \), \( \delta \downarrow 0 \), and \( \epsilon \downarrow 0 \) in order, where \( \hat{q}_{t} \) solves

\[
\begin{aligned}
\dot{\hat{q}}_{t} &= \hat{b}(\hat{q}_{t}) + \hat{\sigma} \dot{w}_{t} \\
\hat{q}_{0} &= q_{0} \in \mathbb{R}^{2}.
\end{aligned}
\]

Here,

\[
\hat{b}(q) = \left( \frac{1}{\int_{\mathbb{T}^{2}} \alpha(q) dq} \int_{\mathbb{T}^{2}} (I - D\chi(q)) dq A_{0} \right) b(q)
\]

and

\[
\hat{\sigma} \hat{\sigma}^{\ast} = \frac{1}{\int_{\mathbb{T}^{2}} \alpha(q) dq} \int_{\mathbb{T}^{2}} \frac{1}{\alpha(q)} (I - D\chi(q)) (I - D\chi(q))^{\ast} dq
\]

with \( \chi(q) = (\chi^{1}(q), \chi^{2}(q)) \) solving

\[
L \chi^{i}(q) = -\frac{1}{2\alpha^{3}(q)} \frac{\partial \alpha}{\partial q^{i}(q)},
\]

where \( L \) is the operator

\[
L = \frac{1}{2} \frac{1}{\alpha^{2}(q)} \Delta q - \frac{1}{2} \frac{\nabla \alpha(q)}{\alpha^{3}(q)} \cdot \nabla q.
\]

**Proof.** By Corollary 2.3 as \( \mu \downarrow 0 \) first and \( \delta \downarrow 0 \), \( q_{t}^{\mu,\delta,\epsilon} \rightarrow \hat{q}_{t}^{\epsilon} \) in probability in \( C([0,T];\mathbb{R}^{2}) \), where \( \hat{q}_{t}^{\epsilon} \) solves

\[
\begin{aligned}
\dot{\hat{q}}_{t}^{\epsilon} &= -\frac{1}{\alpha(\frac{\hat{q}_{t}^{\epsilon}}{\epsilon})} A_{0}^{-1} b(\hat{q}_{t}^{\epsilon}) - \frac{1}{\alpha(\frac{\hat{q}_{t}^{\epsilon}}{\epsilon})} A_{0}^{-1} \circ \dot{\hat{w}}_{t} \\
\hat{q}_{0}^{\epsilon} &= q_{0} \in \mathbb{R}^{2}.
\end{aligned}
\]

(3.1)

Considering

\[
A_{0}^{-1} = -A_{0}
\]

from the definition of \( A_{0} \) in (1.1), writing (3.1) in Itô integral, we get

\[
\dot{\hat{q}}_{t}^{\epsilon} = \frac{1}{\alpha(\frac{\hat{q}_{t}^{\epsilon}}{\epsilon})} \hat{b}(\hat{q}_{t}^{\epsilon}) - \frac{1}{2\epsilon} \frac{\nabla \alpha(\frac{\hat{q}_{t}^{\epsilon}}{\epsilon})}{\alpha^{3}(\frac{\hat{q}_{t}^{\epsilon}}{\epsilon})} + \frac{1}{\alpha(\frac{\hat{q}_{t}^{\epsilon}}{\epsilon})} \hat{\dot{w}}_{t},
\]

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where
\[ \tilde{b}(q) := A_0 b(q) \]
and
\[ \tilde{w}_t := A_0 w_t. \]

Note that \( \tilde{w}_t \) is also a Wiener process in \( \mathbb{R}^2 \).

Under Hypothesis 2, we can apply [8, Theorem 6.1, Chapter 3] to \( \tilde{q}_t^\varepsilon \).

The normalized solution \( m(q) \) of the adjoint equation \( L^* m(q) = 0 \) can be easily found as
\[ m(q) = \frac{1}{\int_{\mathbb{R}^2} \alpha(q) dq} \alpha(q) \]
and the statement of the proposition follows.

\[ \square \]

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