EFFECTIVE $d = 2$ SUPERSYMMETRIC LAGRANGIANS
FROM $d = 1$ SUPERMATRIX MODELS

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ABSTRACT

We discuss $d = 1, \mathcal{N} = 2$ supersymmetric matrix models and exhibit the associated $d = 2$ collective field theory in the limit of dense eigenvalues. From this theory we construct, by the addition of several new fields, a $d = 2$ supersymmetric effective field theory, which reduces to the collective field theory when the new fields are replaced with their vacuum expectation values. This effective theory is Poincare invariant and contains perturbative and non-perturbative information about the associated superstrings. We exhibit instanton solutions corresponding to the motion of single eigenvalues and discuss their possible role in supersymmetry breaking.

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1. Introduction

Non-perturbative interactions in string theory are believed to determine a number of important quantities, such as the strength of supersymmetry breaking [1]. Matrix models [2] offer a unique opportunity to learn about non-perturbative aspects of string theory. The $d = 1$ matrix models have related to them string theories with a low number of degrees of freedom, propagating in 1+1 space-time dimensions [3]. These matrix models have the power to describe non-perturbative phenomena in the associated string theories. Moreover, there are indications that some non-perturbative features are common to all string theories [4]. By studying the generic features of non-perturbative behavior in 1+1 dimensional string theories, one may therefore learn about more realistic string theories, such as those in four dimensions.

To use $d = 1$ matrix models for the purpose of understanding non-perturbative effects in string theory, it is essential to first construct the complete two-dimensional effective Lagrangian for the associated $d = 2$ string theory. Once that is achieved, one can look for non-perturbative phenomena, such as instantons. This entire program has already been carried out [5], with positive results, in the case of $d = 1$ bosonic matrix models. In this case, the nonperturbative effective Lagrangian of the strings was constructed and its fundamental symmetry, a non-compact shift symmetry, $\zeta \rightarrow \zeta + c$, in one of its bosonic fields $\zeta$, was shown to be broken by instantons in a single eigenvalue of the matrix model. This was done using the methods of collective field theory [6]. A notable property of the collective field theory is the presence of a space-dependent coupling parameter. This has been consistently interpreted as deriving from a field dependent coupling in an effective theory which reduces to the collective field theory when the field in question attains a space-dependent vacuum expectation value. In this paper we extend this previous result to the supersymmetric case. We choose the simplest interesting construction, which involves a $d = 1, \mathcal{N} = \infty$ supersymmetric matrix model [7]. The simpler $d = 1, \mathcal{N} = \infty$ matrix model is not of interest since it is a non-interacting theory.

To arrive at a collective field description of a matrix model, it is necessary to first isolate the sub-theory of the matrix eigenvalues. The process of extracting the eigenvalue theory
from the supermatrix model and then correctly identifying a canonical collective field description is rather involved. This has previously been attempted by several groups \[8, 9, 10, 11, 12\] with only partial success. The first half of this paper is devoted to a careful analysis of this problem. A feature of the collective field theories obtained in this manner is that they are not Poincare invariant and not supersymmetric. A related feature is the existence of a coupling parameter in the collective field theory which is space-dependent. The second half of this paper is devoted to describing, reconciling, and interpreting these facts. We interpret the non-Poincare invariant, non-supersymmetric collective field theory as deriving from a particular Poincare invariant, supersymmetric effective theory when certain “heavy” fields in the effective theory are frozen in their vacuum expectation values (VEV’s).

An important question is which \( d = 2 \) supersymmetry the effective theory should have. We demonstrate that, of all \( d = 2, (p,q) \) supersymmetries, it is \((1,1)\) supersymmetry which appropriately relates to the \( d = 1, \mathcal{N} = 2 \) supersymmetric matrix model. The construction of a \( d = 2 \) supersymmetric effective superstring Lagrangian using matrix models can be accomplished in several related ways. An important consistency check relevant to the work described in this paper results from the demonstration that each of these ways yield the same results. We illustrate the various possibilities in figure 1.

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Figure 1. Different ways to obtain a supersymmetric effective Lagrangian using matrix models

The boxes in this figure represent intermediate steps in the construction of the supersymmetric effective theory. The upper line represents the transformation from a bosonic matrix model through various associated bosonic theories. The steps labeled 1-3 thus respectively represent the extraction of the bosonic eigenvalue theory through a suitable restriction of the Hilbert space, the construction of the bosonic collective field theory from the eigenvalue theory, and finally the identification of a bosonic effective theory which is Poincare invariant and which reduces to the bosonic collective field theory when a “heavy” field is frozen in its VEV. The bottom line represents an analogous derivation starting from a supersymmetric matrix model. Thus, the steps labeled \( I – III \) respectively represent the extraction of a supersymmetric eigenvalue theory by a suitable restriction of the Hilbert space, the con-
struction of the associated collective field theory, and finally the identification of an effective field theory which is both Poincare invariant and supersymmetric and which reduces to the collective field theory when “heavy” fields are replaced with their VEV’s. The steps labeled $A - D$ represent direct supersymmetrizations of the various bosonic theories. Thus, a supersymmetric effective theory can be obtained by supersymmetrizing the bosonic effective theory. Alternatively, any intermediate bosonic theory could be supersymmetrized and then the remaining steps in the bottom line could be followed until a supersymmetric effective theory is obtained. In this paper we emphasize the route, labeled in figure 1 by $A - I - II - III$, from a bosonic matrix model to a supersymmetric effective theory. However, at every step we check that our result is, in fact, the appropriate supersymmetrization of the corresponding bosonic theory. We thus show that the diagram indicated in figure 1 commutes completely.

The paper is structured as follows.

In section 2 we discuss the bosonic matrix model. We define the theory, quantize it, and explain how a quantum mechanics of matrix eigenvalues can be extracted from the theory upon suitable restriction of the Hilbert space.

In section 3 we discuss the supersymmetric analog of the bosonic matrix model. We introduce a $d = 1, N = 2$ supersymmetric matrix model, quantize the theory and show how a supersymmetric quantum mechanics can be extracted from the theory upon suitable restriction of the Hilbert space. Not suprisingly, this is more subtle than the bosonic case.

In section 4 we represent the supersymmetric quantum mechanics, extracted from the supersymmetric matrix model in section 3, in terms of collective fields. This is done by introducing a new spatial parameter $x$, which is a continuous extension of the discrete eigenvalue index. The collective fields aggregate the distinct matrix eigenvalues into fields defined over $x$ and $t$. We show that the large $N$ limit can be taken in two distinct ways and that these two ways can be taken independently over any regions of $x$. In the first of these, the high density case, the eigenvalues “pack” densely over $x$. The high density collective fields defined in this way are ordinary two dimensional fields. In the second case, only a finite number of eigenvalues populate the associated region of $x$. In this section we present the mathematical details of the derivation of the high density collective field theory related to
the $d = 1, \mathcal{N} = 2$ supersymmetric matrix model. We discuss subtleties of regularization and canonicalization of the theory. We show that the supersymmetric collective field theory possesses a single coupling parameter which blows up at finite points in space. We end this section by exhibiting the high density collective field theory, with canonically normalized fields. As mentioned above, the theory is neither Poincare invariant nor supersymmetric.

In section 5 we derive a (1,1) supersymmetric effective theory which reduces to the collective field theory derived in section 4 when certain fields are frozen in their VEV’s. We discuss two dimensional $(p, q)$ supersymmetry and demonstrate that only (1,1) supersymmetry is compatible with the collective field theory. The Poincare invariant, (1,1) supersymmetric effective field Lagrangian derived in this section is the essential result of this paper.

In section 6 we solve the Euclidean equations of motion and find solutions corresponding to the motion of individual eigenvalues in the low density regions. We exhibit, explicitly, the eigenvalue instantons alluded to above. We then briefly describe how we expect these instantons to break the supersymmetry of the effective theory.

2. The Bosonic Matrix Model

In this section we briefly review the bosonic matrix model. We begin with a description of the classical theory and its symmetries. We then quantize the theory and show that an effective quantum theory involving only the matrix eigenvalues can be constructed provided the Hilbert space is suitably restricted. The results of this section are known. We discuss them here in order to motivate the extension to the supersymmetric case and also to set our notation.

The fundamental variable in the bosonic matrix model is a time-dependent $N \times N$ Hermitian matrix, $M(t)$. Its dynamics are described by the Lagrangian,

$$L(\dot{M}, M) = \frac{1}{2} \text{Tr} \dot{M}^2 - V(M).$$

The potential is taken to be polynomial,

$$V(M) = \sum_n a_n \text{Tr} M^n,$$

where the $a_n$ are real coupling parameters. The mass dimension of $M$ is $-\frac{1}{2}$ so that the $a_n$ have positive mass dimension $(n + 2)/2$. The momentum conjugate to $M$ is the $N \times N$
Hermitian matrix \( \Pi_M(t) = \dot{M} \). It follows that the associated Hamiltonian is given by

\[
H(\Pi_M, M) = \frac{1}{2} Tr \Pi_M^2 + V(M).
\] (2.3)

The matrix \( M \) remains Hermitian under the transformation \( M \to U^\dagger M U \) where \( U \) is an arbitrary \( N \times N \) complex matrix. The Lagrangian is invariant under such a transformation provided it is global and that \( U \in U(N) \). Thus the classical theory possesses a global \( U(N) \) symmetry. We proceed to quantize this theory. As stated above, it is of great interest to extract an effective quantum theory of matrix eigenvalues. This procedure is complicated by the fact that it is necessary to suitably restrict the Hilbert space of states in order to diagonalize the matrix momentum operator \( \hat{\Pi}_M \). This poses a difficulty when attempting to express the quantum theory using path integral language, which is the natural language for a discussion of the nonperturbative issues which are our main concern. This complication is subtle and the extraction of the effective eigenvalue theory using path integrals from the outset, although possible, is relatively complicated. Such procedures are explained at various places in the literature, e.g. [13, 5]. An equivalent procedure is to first extract the relevant eigenvalue theory using canonical operator quantum mechanics. The passage to a path integral description is then straightforward. We proceed with a description of this method. A detailed discussion of the following calculation is given in Appendix A.

In the \( M \) basis, the operator \( \hat{\Pi}_M \), constructed to satisfy \([\hat{\Pi}_{Mij}, \hat{M}_{kl}] = -i \delta_{ik} \delta_{jl}\), is given by

\[
\hat{\Pi}_{Mij} = -i \frac{\partial}{\partial M_{ij}}.
\] (2.4)

Thus, the quantum operator Hamiltonian is

\[
\hat{H} = -\frac{1}{2} \sum_{ij} \frac{\partial}{\partial M_{ij}} \frac{\partial}{\partial M_{ji}} + V(M).
\] (2.5)

Since \( M \) is Hermitian, there exists, at every time \( t \), a unitary matrix, \( U(t) \), such that \( M = U^\dagger \lambda U \), where \( \lambda \) is a time dependent diagonal matrix consisting of the eigenvalues of \( M \). This is a useful parameterization of \( M \). The operator \( \partial / \partial M \) can then be decomposed into a sum of operators involving \( \partial / \partial \lambda \) and \( \partial / \partial U \). Then, by restricting attention to only those states \( |s > \) which are annihilated by \( \partial / \partial U \), the \( U(N) \) “singlet” sector of the Hilbert
space, it can be shown that $\hat{H}|s> = \hat{H}_s|s>$, where
\begin{equation}
\hat{H}_s = \sum_i \left\{ -\frac{1}{2} \frac{\partial^2}{\partial \lambda_i^2} - \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \frac{\partial}{\partial \lambda_i} + V(\lambda_i) \right\}.
\end{equation}

(2.6)

Since $\hat{\Pi}_{\lambda_i} = i\partial/\partial \lambda_i$ is the momentum operator conjugate to $\hat{\lambda}_i$ this effective Hamiltonian can be expressed as
\begin{equation}
\hat{H}_s = \sum_i \left\{ \frac{1}{2} \hat{\Pi}_{\lambda_i}^2 - i \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \hat{\Pi}_{\lambda_i} + V(\lambda_i) \right\}.
\end{equation}

(2.7)

It is then straightforward, using well known techniques, to show that the quantum mechanics of the singlet sector is governed by the following partition function,
\begin{equation}
Z_N(a_n) = \int [d\lambda] \exp i \int dt L_s(\dot{\lambda}, \lambda),
\end{equation}

(2.8)

where
\begin{equation}
L_s(\dot{\lambda}, \lambda) = \sum_i \left\{ \frac{1}{2} \dot{\lambda}_i^2 - V_{eff}(\lambda_i) \right\}
\end{equation}

(2.9)

and
\begin{equation}
V_{eff}(\lambda_i) = V(\lambda_i) + \frac{1}{2} (\sum_{j \neq i} \frac{1}{\lambda_j - \lambda_i})^2.
\end{equation}

(2.10)

Once again, a detailed derivation of this result is given in Appendix A. The Lagrangian $L_s$ given in (2.9) is the appropriate Lagrangian for studying the dynamics of the $U(N)$ singlet sector of the bosonic quantum mechanical matrix model.

3. A Supersymmetric Matrix Model

In this section we present a supersymmetric matrix model. After introducing the classical theory and its symmetries, we then quantize the theory and show that the effective quantum theory of the matrix eigenvalues reduces to a supersymmetric quantum mechanics provided the Hilbert space is suitably restricted. This particular model was originally presented by Marinari and Parisi[7]. We present a brief discussion of it here as it is essential to the main results in this paper. The reduction to supersymmetric quantum mechanics has also been presented elsewhere[8], but we present an alternative method which we find illuminating.

There are many $d = 1$ supersymmetries enumerated by the number of supersymmetric charges, $N$. The simplest nontrivial supersymmetric extension of the bosonic theory
presented in the last section involves a $d = 1, \mathcal{N} = 2$ supersymmetry. This is because $d = 1, \mathcal{N} = 1$ supersymmetry does not admit interactions. We extend the bosonic theory by letting the fundamental variable be a time-dependent $N \times N$ matrix whose elements are $d = 1, \mathcal{N} = 2$ complex superfields. We further restrict this matrix to be Hermitian. The reason for choosing Hermitian matrices rather than real symmetric matrices is the following. Matrix models involving real symmetric matrices generate triangulations of string worldsheets which are both orientable and non-orientable whereas Hermitian matrix models describe only orientable worldsheets. Since we want our matrix model to describe a two-dimensional supersymmetric string theory, we must assume the existence of supersymmetry on the associated string worldsheet. The worldsheet is thus a spin manifold and a spin manifold is necessarily orientable. This motivates the choice of Hermitian matrices. We wish to point out that the complex $d = 1, \mathcal{N} = \in$ supermultiplets are reducible under supersymmetry. Regardless of this fact, the supersymmetric quantum mechanics of the matrix eigenvalues, which we will extract from the matrix model, will involve only real irreducible $d = 1, \mathcal{N} = \in$ multiplets, since the diagonal elements of a Hermitian matrix are real. We now present the details of the classical theory.

As just described, the fundamental variable of the supermatrix model is a time-dependent, $N \times N, d = 1, \mathcal{N} = \in$ Hermitian matrix superfield,

$$\Phi_{ij} = M_{ij}(t) + i\theta_1 \Psi_{1ij}(t) + i\theta_2 \Psi_{2ij} + i\theta_1 \theta_2 F_{ij}(t), \quad (3.1)$$

where $\theta_1$ and $\theta_2$ are real anticommuting parameters, $M_{ij}$ and $F_{ij}$ are $N \times N$ bosonic Hermitian matrices and $\Psi_{1ij}$ and $\Psi_{2ij}$ are $N \times N$ fermionic Hermitian matrices. We note that $(\theta \Psi_{ij})^\dagger = \Psi_{ij}^\dagger \theta = -\theta (\Psi_{ji}^\dagger)$. Thus, $\Phi_{ij} = \Phi_{ij}^\dagger$. The Lagrangian is

$$L = \int d\theta_1 d\theta_2 \left\{ \frac{1}{2} Tr D_1 \Phi D_2 \Phi + iW(\Phi) \right\}, \quad (3.2)$$

where the superpotential, $W$, is a polynomial in $\Phi$,

$$W(\Phi) = \sum_n b_n Tr \Phi^n, \quad (3.3)$$

$b_n$ are real coupling parameters, and $D_I$ are superspace derivatives,

$$D_I = \frac{\partial}{\partial \theta_I} + i\theta_I \frac{\partial}{\partial t} \quad (3.4)$$
for $I = 1, 2$. In component fields, the Lagrangian reads

$$L = \sum_{ij} \left\{ \frac{1}{2} (\dot{M}_{ij} \dot{M}_{ji} + F_{ij} F_{ji}) + \frac{\partial W(M)}{\partial M_{ij}} F_{ij} \right\} - \frac{i}{2} \sum_{ij} (\dot{\Psi}_{1ij} \dot{\Psi}_{1ji} + \dot{\Psi}_{2ij} \dot{\Psi}_{2ji}) - i \sum_{ijkl} \Psi_{1ij} \frac{\partial^2 W(M)}{\partial M_{ij} \partial M_{kl}} \Psi_{2kl}. \quad (3.5)$$

The $d = 1, N = \in$ supersymmetry transformation law on the component matrices is given by

$$\begin{align*}
\delta M_{ij} &= i\eta^1 \Psi_{1ij} + i\eta^2 \Psi_{2ij}, \\
\delta \Psi_{1ij} &= \eta^1 \dot{M}_{ij} + \eta^2 F_{ij}, \\
\delta \Psi_{2ij} &= \eta^2 \dot{M}_{ij} - \eta^1 F_{ij}, \\
\delta F_{ij} &= i\eta^2 \dot{\Psi}_{1ij} - i\eta^1 \dot{\Psi}_{2ij},
\end{align*} \quad (3.6)$$

where $\eta^1$ and $\eta^2$ are anticommuting constants. It is straightforward to check that this is a symmetry of the Lagrangian (3.3). The momenta conjugate to the matrices $M, \Psi_1$ and $\Psi_2$ are

$$\begin{align*}
\Pi_{M_{ij}} &= \dot{M}_{ij}, \\
\Pi_{\Psi_{1ij}} &= -\frac{i}{2} \Psi_{1ij}, \\
\Pi_{\Psi_{2ij}} &= -\frac{i}{2} \Psi_{2ij}.
\end{align*} \quad (3.7)$$

Thus, the Hamiltonian is given by

$$H = \sum_{ij} \left\{ \frac{1}{2} \Pi_{M_{ij}} \Pi_{M_{ji}} - F_{ij} F_{ji} - \frac{\partial W(M)}{\partial M_{ij}} F_{ij} \right\} + i \sum_{ijkl} \Psi_{1ij} \frac{\partial^2 W(M)}{\partial M_{ij} \partial M_{kl}} \Psi_{2kl}. \quad (3.8)$$

Note that $H$ does not depend on $\Pi_{\Psi_{1ij}}$ for $I = 1, 2$. Now, $\Phi_{ij}$ remains a Hermitian matrix of superfields under the transformation $\Phi \rightarrow U^\dagger \Phi U$ where $U$ is an arbitrary $N \times N$ matrix of complex numbers. The Lagrangian is invariant under such a transformation provided that $U \in U(N)$. Thus the classical theory possesses a global $U(N)$ symmetry.

Before quantizing the theory, we eliminate the auxiliary matrix $F_{ij}$. Its equation of motion reads

$$F_{ij} = -\frac{\partial W(M)}{\partial M_{ji}}. \quad (3.9)$$
Eliminating $F_{ij}$ with this equation, the Lagrangian then becomes

$$L = \sum_{ij} \left\{ \frac{1}{2} \dot{M}_{ij} \dot{M}_{ji} - \frac{1}{2} \frac{\partial W(M) \partial W(M)}{\partial M_{ij}} \right\} - \frac{i}{2} \sum_{ij} \left( \Psi_{1ij} \dot{\Psi}_{1ji} + \Psi_{2ij} \dot{\Psi}_{2ji} \right) - \frac{i}{2} \sum_{ijkl} \Psi_{1ij} \partial_{M_{ij}} \partial_{M_{kl}} W_{ijkl} \Psi_{2kl}. \quad (3.10)$$

This is symmetric with respect to the nonlinear $d = 1, N = \epsilon$ supersymmetry transformation,

$$\delta M_{ij} = i \eta^1 \Psi_{1ij} + i \eta^2 \Psi_{2ij},$$
$$\delta \Psi_{1ij} = \eta^1 \dot{M}_{ij} - \eta^2 \frac{\partial W(M)}{\partial M_{ij}},$$
$$\delta \Psi_{2ij} = \eta^2 \dot{M}_{ij} + \eta^1 \frac{\partial W(M)}{\partial M_{ji}}. \quad (3.11)$$

and also with respect to the $U(N)$ transformation

$$M \rightarrow U^\dagger M U,$$
$$\Psi_1 \rightarrow U^\dagger \Psi_1 U,$$
$$\Psi_2 \rightarrow U^\dagger \Psi_2 U. \quad (3.12)$$

With $F_{ij}$ eliminated, the classical Hamiltonian becomes

$$H = \sum_{ij} \left\{ \frac{1}{2} \Pi_{M_{ij}} \Pi_{M_{ji}} + \frac{1}{2} \frac{\partial W(M) \partial W(M)}{\partial M_{ij}} \right\} + \frac{i}{2} \sum_{ijkl} \Psi_{1ij} \Psi_{2kl} \frac{\partial^2 W(M)}{\partial M_{ij} \partial M_{kl}}. \quad (3.13)$$

We proceed to quantize the theory.

Canonical quantization is achieved by promoting the matrices $M, \Pi_M$ and $\Psi_I$ to operators and by imposing the following relations,

$$[\hat{\Pi}_{M_{ij}}, \hat{M}_{kl}] = -i \delta_{ik} \delta_{jl},$$
$$\{ \hat{\Psi}_{Iij}, \hat{\Psi}_{Jkl} \} = \delta_{IJ} \delta_{ik} \delta_{jl}. \quad (3.14)$$

For the fermions, it is useful to define complex operators,

$$\hat{\Psi} = \frac{1}{\sqrt{2}}(\hat{\Psi}_1 + i \hat{\Psi}_2),$$
$$\hat{\bar{\Psi}} = \frac{1}{\sqrt{2}}(\hat{\Psi}_1 - i \hat{\Psi}_2). \quad (3.15)$$
It then follows that
\[ \{ \hat{\Psi}_{ij}, \hat{\Psi}_{kl} \} = \delta_{ik}\delta_{jl}. \]  
(3.16)

We can thus choose \( \hat{\Psi} \) and \( \hat{\bar{\Psi}} \), respectively, to be annihilation and creation operators for fermions. The quantum operator Hamiltonian can now be written
\[ \hat{H} = \sum_{ij} \left\{ \frac{1}{2} \hat{\Pi}_{Mij} \hat{\Pi}_{Mji} + \frac{1}{2} \frac{\partial W(\hat{M})}{\partial \hat{M}_{ij}} \frac{\partial W(\hat{M})}{\partial \hat{M}_{ij}} \right\} + \frac{1}{2} \sum_{ijkl} [\hat{\bar{\Psi}}_{ij}, \hat{\Psi}_{kl}] \frac{\partial^2 W(\hat{M})}{\partial \hat{M}_{ij} \partial \hat{M}_{kl}}. \]  
(3.17)

Upon appropriate restriction to a subspace of the full Hilbert space, this theory reduces to a supersymmetric quantum mechanics. We proceed to show this. A detailed derivation of the following calculation is given in Appendix B.

We work in the \( M \) basis, so that \( \hat{\Pi}_{M} = -i\partial/\partial M \). We then parameterize \( M \) in terms of its eigenvalues and angular variables, as discussed in section 2. Thus, \( M = U^\dagger \lambda U \), where \( \lambda \) is a diagonal matrix of time-dependant eigenvalues and \( U(t) \) is a unitary matrix. The operator \( \partial/\partial M \) is then decomposed into a sum of operators involving \( \partial/\partial \lambda \) and \( \partial/\partial U \). We define a “rotated” fermion matrix \( \chi = U\Psi U^\dagger \). Note that \( U \) diagonalizes \( M \) but that \( \chi \) is not diagonal. It is possible to show, on states \( |S> \) which are annihilated by both \( \partial/\partial U \) and by \( \hat{\chi}_{ij} \), where \( i \neq j \), the \( U(N) \) “singlet” sector of the Hilbert space, that \( \hat{H}|S> = \hat{H}_S|S> \), where
\[ \hat{H}_S = \sum_i \left\{ \frac{1}{2} \hat{\Pi}_{\lambda i}^2 + i \frac{\partial w}{\partial \lambda_i} \hat{\Pi}_{\lambda i} + \frac{1}{2} \left( \frac{\partial W}{\partial \lambda_i} \right)^2 + \frac{1}{2} \frac{\partial w}{\partial \lambda_i} \frac{\partial W}{\partial \lambda_i} \right\} + \frac{1}{2} \sum_{ij} [\hat{\chi}_{ii}, \hat{\chi}_{jj}] \frac{\partial^2 w}{\partial \lambda_i \partial \lambda_j}. \]  
(3.18)

In (3.18), and henceforth, we abbreviate \( \chi_{ii} \) by writing \( \chi_i, \hat{\Pi}_{\lambda i} = -i\partial/\partial \lambda_i \), and
\[ w = -\sum_i \sum_{j \neq i} \ln |\lambda_i - \lambda_j|. \]  
(3.19)

We note that \( w \) has the following properties,
\[ \frac{\partial w}{\partial \lambda_i} = -\sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}, \]  
(3.20)

and
\[ \frac{\partial^2 w}{\partial \lambda_m \partial \lambda_n} = \begin{cases} \frac{\sum_{k \neq i} 1/(\lambda_i - \lambda_k)^2}{m = n} \\ -1/(\lambda_m - \lambda_n)^2 \quad ; m \neq n \end{cases} \]  
(3.21)
It is then straightforward, using well known techniques, to show that the quantum mechanics of the singlet sector is governed by the following partition function,

\[ Z_N(b_n) = \int [d\lambda][d\bar{\chi}][d\chi] \exp\left(i \int dt L_S\right) \]  \hspace{1cm} (3.22)

where

\[ L_S = \sum_i \left\{ \frac{1}{2} \dot{\lambda}_i^2 - \frac{1}{2} \left( \frac{\partial W_{\text{eff}}}{\partial \lambda_i} \right)^2 - i \frac{1}{2} (\bar{\chi}_i \dot{\chi}_i - \dot{\bar{\chi}}_i \chi_i) \right\} - \sum_{ij} \bar{\chi}_i \chi_j \frac{\partial^2 W_{\text{eff}}}{\partial \lambda_i \partial \lambda_j}, \]  \hspace{1cm} (3.23)

and

\[ W_{\text{eff}}(\lambda_i) = W(\lambda_i) + w(\lambda_i) \]  \hspace{1cm} (3.24)

For convenience we rewrite this Lagrangian as follows,

\[ L = \sum_i \left\{ \frac{1}{2} \dot{\lambda}_i^2 - \frac{1}{2} \left( \frac{\partial W}{\partial \lambda_i} \right)^2 - \frac{\partial w}{\partial \lambda_i} \frac{\partial W}{\partial \lambda_i} - \frac{1}{2} \left( \frac{\partial w}{\partial \lambda_i} \right)^2 - i \frac{1}{2} (\bar{\chi}_i \dot{\chi}_i - \dot{\bar{\chi}}_i \chi_i) \right\} \]

\[ - \sum_{ij} \left\{ \frac{\partial^2 W}{\partial \lambda_i \partial \lambda_j} \bar{\chi}_i \chi_j + \frac{\partial^2 w}{\partial \lambda_i \partial \lambda_j} \bar{\chi}_i \chi_j \right\} \]  \hspace{1cm} (3.25)

In passing from (3.23) to (3.25) we have dropped the subscript $S$. It is henceforth assumed that we are describing only the singlet sector of the matrix model. Once again, a detailed derivation of this result is given in Appendix B.
4. Supersymmetric Collective Field Theory

In this section we introduce the notion of collective fields. This is a powerful construction which will allow us, in subsequent sections, to investigate the physics of the supersymmetric matrix model. The formalism is particularly suited to studying this model in the large $N$ limit.

We begin by introducing a continuous real parameter, $x$, constrained to lie in the interval $-L/2 < x < L/2$. On this line segment define “collective fields”,

\[
\begin{align*}
\varphi(x,t) &= \sum_i \Theta(x - \lambda_i(t)) \\
\psi(x,t) &= -\sum_i \delta(x - \lambda_i(t))\chi_i(t) \\
\bar{\psi}(x,t) &= -\sum_i \delta(x - \lambda_i(t))\bar{\chi}_i(t).
\end{align*}
\]  

The parameter $x$ is a continuous extension of the discrete eigenvalue index. For finite $N$ the two dimensional fields $\varphi$, $\psi$, and $\bar{\psi}$ have a finite number of independent modes. They are thus not ordinary unconstrained fields. Eventually, we take the limit $N \to \infty$, $L \to \infty$. We can take this limit in one of two ways; we may let $N/L \to \text{finite}$ or we may let $N/L \to \infty$. In the first case the average density of eigenvalues over $x$ remains finite. In this case, the collective fields remain unwieldy as mathematical tools. In the second case, however, the density of eigenvalues becomes infinite. In this case, it can be shown, by representing the theta and delta functions by Gaussian integrals and then taking the desired limit, that, modulo a subtlety which we will discuss below, the collective fields shed their constraints and become ordinary two dimensional fields. We proceed to represent the eigenvalue Lagrangian, \eqref{eq:3.24}, in terms of the collective fields defined in \eqref{eq:4.1}. We begin by keeping both $N$ and $L$ finite. The collective field representation of the eigenvalue Lagrangian is then nothing more than a reparameterization. We then take the $N \to \infty$, $L \to \infty$ limit. We will see that a careful reevaluation of the significance of the collective field Lagrangian is then warranted.

4.1 Finite $N$ Collective Field Theory

Using the definitions \eqref{eq:4.1} it is easily seen that

\[
\int dx \frac{\varphi'^2}{2\varphi} = \sum_i \frac{1}{2} \int dx \{ \sum_j \delta(x - \lambda_j) \} \delta(x - \lambda_i) \lambda_i^2
\]
where a dot represents a time derivative and a prime represents a derivative with respect to $x$. This offers an alternative representation of the first term in the eigenvalue Lagrangian (3.25). We can apply similar techniques to the terms,

\begin{equation}
\frac{1}{2} \sum_{i} \dot{\lambda}_i^2 = \int dx \frac{\dot{\phi}^2}{2\varphi'} \tag{4.3}
\end{equation}

\begin{equation}
-\frac{1}{2} \sum_{i} \left( \frac{\partial W}{\partial \lambda_i} \right)^2 = -\frac{1}{2} \int dx \varphi' W'(x)^2 \tag{4.4}
\end{equation}

\begin{equation}
-\frac{i}{2} \sum_{i} \bar{x}_i \dot{x}_i = \int dx \left\{ -\frac{1}{2} \bar{\psi} \dot{\psi} + \frac{i}{2} \frac{\dot{\varphi}}{\varphi^2} \bar{\psi} \dot{\psi}' \right\} \tag{4.5}
\end{equation}

\begin{equation}
\sum_{ij} \frac{\partial^2 W}{\partial \lambda_i \partial \lambda_j} \bar{x}_i x_j = \int dx \frac{W''(x)}{\varphi'} \bar{\psi} \psi, \tag{4.6}
\end{equation}

where, on the right hand side,

\begin{equation}
W(x) = \sum_n b_n x^n. \tag{4.7}
\end{equation}

The other terms in the eigenvalue Lagrangian contain factors of $w$. Since $w = -\sum_i \sum_{j \neq i} \ln |\lambda_i - \lambda_j|$, we must properly regulate the collective field expression. We will now discuss this is some detail.

Generically, we encounter terms of the following sort,

\begin{equation}
\sum_{i} \sum_{j \neq i} \frac{f(\lambda_i, \lambda_j)}{\lambda_i - \lambda_j} = \sum_{i} \sum_{j \neq i} \int dx dy \delta(x - \lambda_i) \delta(y - \lambda_j) \frac{f(x,y)}{x-y} = \sum_{ij} \int dx dy \delta(x - \lambda_i) \delta(y - \lambda_j) \frac{f(x,y)}{x-y} = \int dx dy \varphi'(x) \varphi'(y) \frac{f(x,y)}{x-y}. \tag{4.8}
\end{equation}

The symbol $\int$ designates “principal part” of the integral, which is defined as follows. Given an integral over a real function with a simple pole,

\begin{equation}
\int_{-L/2}^{+L/2} dx \frac{\varphi(x)}{x-y}, \tag{4.9}
\end{equation}

...
where $\phi(x)$ is analytic and $y$ is a constant, we note that, for $|y| < L/2$, the integral is not a-priori well defined. The “principle part” of the integral is defined by the following limiting procedure,

$$
\int_{-L/2}^{+L/2} \frac{\phi(x)}{x - y} \, dx = \lim_{\epsilon \to 0} \left( \int_{-L/2}^{y-\epsilon} dx + \int_{y+\epsilon}^{+L/2} dx \right) \frac{\phi(x)}{x - y}.
$$

(4.10)

This removes the point $x = y$ from the range of integration. Thus, in passing from the first line in (4.8) to the second we have shifted the regulator $j \neq i$ onto the continuous coordinate by implicitly invoking $x \neq y$. Note that when $N$ is finite, which in this subsection it is, equation (4.8) is merely a series of identities. The more complicated case when $N \to \infty$ will be discussed in detail later. We can now compute the remaining terms in the collective field Lagrangian.

Using the techniques discussed above it is straightforward to prove that

$$
\sum_i \left( \frac{\partial w}{\partial \lambda_i} \right)^2 = \frac{1}{3} \int dx dy dz \frac{\varphi'(x)\varphi'(y)\varphi'(z)}{(x - y)(x - z)},
$$

(4.11)

$$
- \sum_i \frac{\partial w}{\partial \lambda_i} \frac{\partial W}{\partial \lambda_i} = \int dx dy \frac{\varphi'(x)\varphi'(y)}{(x - y)} W'(x),
$$

(4.12)

$$
\sum_{ij} \frac{\partial^2 w}{\partial \lambda_i \partial \lambda_j} \bar{\psi}_i \psi_j = -\int dx dy \frac{1}{(x - y)^2} \left\{ \bar{\psi}(x)\psi(y) - \frac{\varphi''(y)}{\varphi'(x)} \bar{\psi}(x)\psi(x) \right\}
$$

$$
= \int \frac{1}{(x - y)} \left\{ \bar{\psi}(x)\psi'(y) - \frac{\varphi''(y)}{\varphi'(x)} \bar{\psi}(x)\psi(x) \right\}.
$$

(4.13)

To obtain the last line of (4.13) we have integrated by parts. We may now assemble the full collective field Lagrangian. Inserting the expressions (4.3)-(4.6) and (4.11)-(4.13), into the Lagrangian (3.25), we obtain

$$
L = \int dx \left\{ \frac{\dot{\psi}^2}{2\varphi'} - \frac{1}{2} \varphi' W'(x)^2 + \frac{\varphi''(x)}{\varphi'} \bar{\psi}\psi \right.
$$

$$
- \frac{1}{2\varphi'} (\bar{\psi}\dot{\psi} + \dot{\bar{\psi}}\psi) + \frac{i}{2} \frac{\dot{\phi}}{\varphi^2} (\bar{\psi}\psi' - \bar{\psi}'\psi) \right\}
$$

$$
+ \frac{1}{3} \int dx dy dz \frac{\varphi'(x)\varphi'(y)\varphi'(z)}{(x - y)(x - z)},
$$

$$
+ \int dx dy \frac{\varphi'(x)\varphi'(y)}{(x - y)} W'(x)
$$

$$
+ \int \frac{1}{(x - y)} \left\{ \bar{\psi}(x)\psi'(y) - \frac{\varphi''(y)}{\varphi'(x)} \bar{\psi}(x)\psi(x) \right\}.
$$

(4.14)
4.2 The $N$-dependence Of The Superpotential.

Before we consider the case $N \to \infty, L \to \infty$ we should first discuss a relevant issue concerning the $N$ dependance of the superpotential. Recall that the superpotential was expressed as a polynomial,

$$W(x) = \sum_n b_n x^n.$$  \hspace{1cm} (4.15)

It turns out, if the coefficients $b_n$ depend on $N$ in a specific manner, that, when the limit $N \to \infty$ is taken, the matrix partition function actually describes an ensemble of two dimensional super-Riemann surfaces. This is what allows us to interpret the matrix models as describing string theory. Since our interest in matrix models is to help us better understand string theory, we should accordingly impose that the coefficients $b_n$ have the appropriate $N$ dependence. The correct dependence is that $b_n$ should scale as $N^{1-n/2}$. If we write $b_n = \frac{1}{n!} N^{1-n/2} \tilde{c}_n$, where the $\tilde{c}_n$ do not depend on $N$, the superpotential becomes

$$W(x) = \sum_n \frac{1}{n!} N^{1-n/2} \tilde{c}_n x^n.$$  \hspace{1cm} (4.16)

Since $N$ is finite, we can also make the following shift,

$$x \to x + \sqrt{N} \beta,$$  \hspace{1cm} (4.17)

where $\beta$ is an arbitrary real constant. This induces a shift in the superpotential,

$$W(x) \to \sum_n \frac{1}{n!} N^{1-n/2} c_n x^n,$$  \hspace{1cm} (4.18)

where

$$c_n = \sum_m \frac{1}{m!} \beta^m \tilde{c}_{m+n}. $$  \hspace{1cm} (4.19)

By choosing $\beta$ appropriately we can consistently drop one of the coupling parameters $c_n$. A natural choice is to take $c_2 = 0$, which requires

$$\sum_m \frac{1}{m!} \beta^m \tilde{c}_{m+2} = 0.$$  \hspace{1cm} (4.20)

We will henceforth assume that $\beta$ satisfies (4.20). It is useful to exhibit explicitly the three $x$-dependent functions which appear in the collective field Lagrangian. They are,

$$W'(x) = \sqrt{N} c_1 + \frac{1}{2} \frac{c_3}{\sqrt{N}} x^2 + \frac{1}{6} \frac{c_4}{N} x^3 + \cdots ,$$
\[
W'(x)^2 = Nc_1^2 + c_1c_3x^2 + \frac{1}{3}\frac{c_1c_4}{\sqrt{N}}x^3 + \cdots,
\]
\[
W''(x) = \frac{c_3}{\sqrt{N}}x + \cdots. \quad (4.21)
\]

For any finite \(x\), all terms involving \(c_n\) for \(n \geq 4\) vanish as \(N \to \infty\). Since this is the limit of interest in the remaining part of this paper, we can therefore, without loss of generality, neglect all \(c_n\) for \(n \geq 4\). Since \(c_2\) has independently been set to zero by shifting \(x\), the most general superpotential for our purposes is of the form
\[
W(x) = Nc_0 + \sqrt{N}c_1x + \frac{1}{6}\frac{c_3}{\sqrt{N}}x^3. \quad (4.22)
\]

An important qualitative aspect of this superpotential depends on the sign of the product \(c_1c_3\). Specifically, in the large \(N\) limit, the potential, \(\frac{1}{2}W'(x)^2\), will be a parabola which is concave up if \(c_1c_3 < 0\) or concave down if \(c_1c_3 > 0\). The interesting physics, which we will discuss below, depends crucially on the existence of a local maximum in this potential. We will therefore take \(c_1c_3 < 0\).

4.3 \(N \to \infty\) Collective Field Theory

We now take the limit \(N \to \infty, L \to \infty\). As noted above, this limit can be taken in one of two ways. We will discuss each of these possibilities in detail. It is important to note that we may take the limit in either manner, independently, within any given region of \(x\).

a) “Low Density” case: The first possibility is that \(N \to \infty, L \to \infty\) but, over the range \(x_1 < x < x_1 + l_1\), \(N/l_1\) remains finite. In this case the density of eigenvalues remains sparse. Under this circumstance, within this region, the collective fields (4.1) contain only a finite number of independent modes. The collective fields must then satisfy constraints which are simply the definitions (4.1). The collective field Lagrangian (4.14) applies to the physics in this region, but it must be understood that \(\varphi\) and \(\psi\) are constrained and this fact must be duly accounted for. Because of the definitions (4.1) and the fact that (4.14) is merely a rewriting of (3.25), the natural way to avoid this complication is to simply use (3.25) to describe the physics of the individual eigenvalues. Any eigenvalue behavior which is deduced using (3.25) can then be cast in collective field language by invoking (4.1).

b) “High Density” case: The other possibility is that \(N \to \infty, L \to \infty\) such that, within a region \(x_2 < x < x_2 + l_2\), \(N/l_2 \to \infty\). In this case the eigenvalues become dense. The
collective fields become, modulo a subtlety to be discussed below, unconstrained, ordinary two dimensional fields. In this limit the eigenvalue Lagrangian (3.25) becomes less useful. This is because it is difficult to interpret the sums over an infinite number of unspecified, dense eigenvalues. The collective field Lagrangian (4.14) offers a more useful description of the system. However, some care must now be taken in evaluating the last three terms of (4.14). We would like to use equation (4.10) to perform the integrations in these terms. In the case of finitely separated eigenvalues there is not a problem, as discussed above. However, for densely packed eigenvalues, these integrals diverge and have to be regulated. We propose a regulation procedure which is implemented using properties of complex integration. In this way sensible finite results can be obtained, but there are important subtle ambiguities. We proceed with an analysis of this issue.

Consider an analytic function, \( \phi(z) \), where \( z = x + iy \), and assume that \( \phi(z) \rightarrow 0 \) as \( z \rightarrow \infty \). Over a contour which traverses the real axis, \( x = (-\infty, +\infty) \) and then closes back in either the upper or lower half plane we have,

\[
\oint dz \frac{\phi(z)}{z-a} = \int_{-\infty}^{+\infty} dx \frac{\phi(x)}{x-a}. \tag{4.23}
\]

This is because the contribution from the contour at infinity vanishes. Deform the contour around the pole using using a semicircle of radius \( \epsilon \).

Figure 2. Contour of integration \( C_+ \)

We consider two possibilities. In the first, we choose a contour, which we denote \( C_+ \), which follows the real axis from \(-\infty\) to the point \( x = a - \epsilon \), then follows a semicircle, \( \gamma_+ \), around the pole in the upper half plane to the point \( x = a + \epsilon \), follows the \( x \) axis to \(+\infty\), and then closes back in the upper half plane. In the second case we consider the mirror image contour, \( C_- \) in the lower half plane. The small semicircle is then denoted \( \gamma_- \). The contour \( C_+ \) is depicted in Figure 2. We can now apply the Cauchy-Riemann theorem. Since the contribution at infinity vanishes, we see that

\[
\oint_{C_\pm} dz \frac{\phi(z)}{z-a} = \int_{-\infty}^{+\infty} dx \frac{\phi(x)}{x-a} + \lim_{\epsilon \rightarrow 0} \int_{\gamma_\pm} dz \frac{\phi(z)}{z-a}. \tag{4.24}
\]
It is easy to show, using polar coordinates, that
\[
\lim_{\epsilon \to 0} \int_{\gamma_{\pm}} \frac{\phi(z)}{z-a} = \mp i \pi \phi(a). \tag{4.25}
\]
The left hand side of (4.24) vanishes since the full contour does not encompass any poles. Thus,
\[
\int_{-\infty}^{+\infty} dx \frac{\phi(x)}{x-a} = \pm i \pi \phi(a). \tag{4.26}
\]
Note the sign ambiguity. This is due to the ambiguity concerning which of the two contours \(C_{\pm}\) we can choose when performing the integration. Note also that the right hand side of (4.26) is imaginary. This may appear peculiar but it is the only mathematically consistent way to make finite sense out of this irregular integral. The sign ambiguity has physical significance to the collective field theory as we demonstrate shortly.

We proceed to discuss the last three terms of (4.14) sequentially. The first of these is evaluated as follows,
\[
\frac{1}{3} \int dx dy dz \frac{\varphi'(x) \varphi'(y) \varphi'(z)}{(x-y)(x-z)} = \frac{1}{3} \int dx \varphi'(x) \left( \int dy \frac{\varphi'(y)}{x-y} \right) \left( \int dz \frac{\varphi'(z)}{x-z} \right) = \frac{1}{3} \int dx \varphi'(x) (\pm i \pi \varphi'(x)) (\pm i \pi \varphi'(x)) = \pm \frac{\pi^2}{3} \int dx \varphi'(x)^3. \tag{4.27}
\]
In the second line of (4.27) the two ambiguous signs are independent so that the final result has an ambiguous sign. This sign determines the signature of the two-dimensional spacetime metric. We next consider the term
\[
\int dx dy \frac{\varphi'(x) \varphi'(y)}{x-y} W'(x), \tag{4.28}
\]
which we denote by \(\mathcal{O}\). Using (4.26), we see that when \(N \to \infty\) this expression becomes antihermitian. The collective field Lagrangian must be Hermitian. Recall, however, that the original definition of this term is given by the left hand side of (4.13), which is real. This term may then be decomposed as follows,
\[
\sum_i \frac{\partial w}{\partial \lambda_i} \frac{\partial W}{\partial \lambda_i} = a \left\{ \sum_i \frac{\partial w}{\partial \lambda_i} \frac{\partial W}{\partial \lambda_i} \right\} + (1-a) \left\{ \sum_i \frac{\partial w}{\partial \lambda_i} \frac{\partial W}{\partial \lambda_i} \right\}^* = a \mathcal{O} + (1-a) \mathcal{O}^*. \tag{4.29}
\]
where $a$ is an arbitrary real parameter. Therefore, in the limit $N \to \infty$,

$$
\sum_i \frac{\partial w}{\partial \lambda_i} \frac{\partial W}{\partial \lambda_i} \to \pm (2a - 1)i\pi \int dx \varphi'(x)^2 W'(x).
$$

(4.30)

We can then choose $a = 1/2$ and this term vanishes. That is, in the limit $N \to \infty$ we can take the next to last term in (4.14) to be zero. This is the unique consistent prescription which yields a Hermitian collective field Lagrangian in this limit. We now turn to the remaining term in the collective field Lagrangian (4.14). It is useful to express the fermion fields in terms of the real and imaginary parts,

$$
\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)
$$

$$
\bar{\psi} = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2).
$$

(4.31)

The remaining term in the Lagrangian (4.14) then reads

$$
-\frac{1}{2} \int dx dy \left\{ \frac{1}{x-y} \bar{\psi}(x) \psi'(y) - \frac{\varphi''(y)}{\varphi'(x)} \bar{\psi}(x) \psi(x) \right\}
$$

$$
= -\frac{1}{2} \int dx dy \left\{ \psi_1(x) \psi_1'(y) + \psi_2(x) \psi_2'(y) + i\psi_1(x) \psi_2'(y) - i\psi_2(x) \psi_1'(y) - \frac{\varphi''(y)}{\varphi'(x)} \bar{\psi}(x) \psi(x) \right\}.
$$

(4.32)

Using (4.26) we see that the last three terms in (4.32) become antihermitian as $N \to \infty$.

Since the Lagrangian must be Hermitian, we treat this problem in exact analogy with the previous discussion. That is, we reexpress (4.32) in terms of the lefthand side of (4.13). This real expression is then decomposed exactly as in (4.29), and $a$ is chosen to be 1/2 to cancel the antihermitian terms. However, unlike the previous case the final result is nonvanishing.

We find that, in the $N \to \infty$ limit, we can consistently take the last term in (4.14) to be

$$
\int dx \left\{ \pm \frac{i\pi}{2} \psi_1(x) \psi_1'(x) \pm \frac{i\pi}{2} \psi_2(x) \psi_2'(x) \right\}.
$$

(4.33)

The two ambiguous signs in this expression are independant. As will be seen, the choice of these signs determines the “chiralities” of the two dimensional fermions.
We have now consistently interpreted the principle part terms in the collective field Lagrangian in the “dense” $N \to \infty$, $L \to \infty$ limit. Before exhibiting the full resultant Lagrangian we note the following facts. First, from (4.21), we see that, as $N \to \infty$,

$$W''(x) \to 0$$

$$W'(x)^2 \to \Lambda - \omega^2 x^2,$$  \hspace{0.5cm} (4.34)

where $\Lambda = \frac{1}{2} N c_1^2$ and $\omega^2 = -c_1 c_3$ are positive constants. It follows from the first of these expressions that we can neglect the third term in (4.14). Second, recall that the definition (4.1) implies that

$$\int \varphi'(x) dx = N. \hspace{0.5cm} (4.35)$$

This constraint continues to hold in the large $N$ limit, despite the fact that $\varphi$ then has an infinite number of independent modes. We cannot then arbitrarily vary the Lagrangian to obtain the field equations. The easiest way to handle this is to introduce constraint (4.35) into the Lagrangian by means of a Lagrange multiplier, in which case field $\varphi$ becomes completely unconstrained. The correct procedure for doing this is described in detail in Appendix C. The end result of this complicated procedure, however, is simply a modification of the constant $\Lambda$. It turns out that all subsequent results are correct if we simply take $\Lambda = 0$ and treat $\varphi$ as an unconstrained field. This is the subtlety concerning the continuous field $\varphi$ alluded to several times above. The interested reader is referred to Appendix C for a proof of this.

Combining (4.14) with all of the facts just discussed, the effective $N \to \infty$, $L \to \infty$ high density collective field Lagrangian reads

$$L = \int dx \left\{ \frac{\dot{\varphi}^2}{2 \varphi'} \pm \frac{\pi^2}{6} \varphi'^3 + \frac{1}{2} \omega^2 x^2 \varphi' \right.$$  
$$- \frac{i}{2 \varphi'} (\psi_1 \dot{\psi}_1 + \psi_2 \dot{\psi}_2) \pm \frac{i \pi}{2} \psi_1 \dot{\psi}_1' \pm \frac{i \pi}{2} \psi_2 \dot{\psi}_2'$$  
$$+ \frac{i}{2 \varphi^2} (\psi_1 \psi_1' + \psi_2 \psi_2'). \right\} \hspace{0.5cm} (4.36)$$

There are three notable features of this Lagrangian. The first is that it is neither translationally invariant, nor Lorentz invariant, nor supersymmetric. The second is that the kinetic energy terms are not in canonical form, and the third is that there are ambiguous signs.
The first issue, that the theory does not have the appropriate symmetry for a realistic two-dimensional field theory will be resolved in the next section. For now we defer a discussion of this point. The second issue is resolved in the next subsection when we canonicalize the theory by redefining our fields and by redefining our spatial coordinate. We will now address the issue of the ambiguous signs. There is physics in the choice of each of these ambiguous signs. The first of them dictates the signature of the two-dimensional spacetime metric, which we desire to be Minkowskian. The appropriate choice for the first ambiguous sign then turns out to be the minus sign. The remaining two ambiguous signs dictate chiralities for the respective fermions. For reasons of supersymmetry to be discussed, we require that the two fermion fields have opposite chirality. We then choose the second ambiguous sign to be a minus and the third to be a plus. To be clear, we rewrite the collective field Lagrangian with these sign choices,

\[ L = \int dx \left\{ \frac{\dot{\phi}^2}{2\phi'} - \frac{\pi^2}{6} \phi'^3 + \frac{1}{2} \omega^2 x^2 \phi' \right. \\
- \frac{i}{2\phi'}(\psi_1 \dot{\psi}_1 + \psi_2 \dot{\psi}_2) - \frac{i\pi}{2} \psi_1 \psi_1' + \frac{i\pi}{2} \psi_2 \psi_2' \\
\left. + \frac{i}{2\phi'^2}(\psi_1 \psi_1' + \psi_2 \psi_2') \right\}. \tag{4.37} \]

This expression is the $N \to \infty, L \to \infty$ high density collective field Lagrangian which is compatible with Minkowski spacetime and two dimensional supersymmetry. Note that the bosonic part of this Lagrangian is identical to the bosonic collective field theory Lagrangian derived in [5].
4.4 Canonical High Density Collective Field Theory

In order to identify the canonical fields of the theory we have to shift the field $\varphi$ around a solution to its equation of motion. We then have to perform a coordinate transformation. In this subsection we will perform these operations and arrive at a canonical Lagrangian for the high density collective field theory. We begin by listing the equations of motion derived from (4.37). They are

\[
\begin{align*}
\partial_t (\dot{\varphi} \varphi') - \frac{1}{2} \partial_x (\dot{\varphi}^2 + \pi^2 \varphi'^2 - \omega^2 x^2) &+ \partial_x \left\{ \frac{i}{2\varphi} (\psi_1 \dot{\psi}_1 + \psi_2 \dot{\psi}_2) - \frac{i \dot{\varphi}}{\varphi^3} (\psi_1 \psi_1' + \psi_2 \psi_2') \right\} \\
&+ \partial_t \left\{ \frac{i}{2\varphi^2} (\psi_1 \psi_1' + \psi_2 \psi_2') \right\} = 0,
\end{align*}
\]

\[
\begin{align*}
\partial_t (\dot{\psi}_1) - \partial_x (\dot{\varphi} \varphi_1 - \pi \psi_1) = 0,
\end{align*}
\]

\[
\begin{align*}
\partial_t (\dot{\psi}_2) - \partial_x (\dot{\varphi} \varphi_2 + \pi \psi_2) = 0,
\end{align*}
\]

for the $\varphi$ field and for the $\psi_1, \psi_2$ fields, respectively. We focus on solutions $(\varphi, \psi_1, \psi_2) = (\tilde{\varphi}_0, \tilde{\psi}_{10}, \tilde{\psi}_{20})$ which have the following property, $\dot{\varphi}_0 = \dot{\psi}_{10} = \dot{\psi}_{20} = 0$. (We denote classical solutions with both a tilde and a subscript 0 for reasons to become clear below). That is, we are interested in static, purely bosonic solutions. The second two equations in (4.38) are then solved automatically and the first becomes

\[
\partial_x (\pi^2 \tilde{\varphi}_0^2 - \omega_0^2 x^2) = 0.
\]

This implies that

\[
\tilde{\varphi}_0' = \frac{1}{\pi} \sqrt{\omega_0^2 x^2 - 1/g},
\]

where $g$ is an arbitrary integration constant. However, we are interested exclusively in the case $g > 0$, since this case yields the interesting physics, as we will discuss. It then follows that $\tilde{\varphi}_0$ is only defined for $|x| \geq 1/(\omega \sqrt{g})$. This is a very significant fact. It turns out that we cannot canonically define the dense collective field theory in the region $|x| \leq 1/(\omega \sqrt{g})$. As we will see, there are other problems with this region as well. Notably, the theory, properly expressed in terms of canonical fields, possesses a space-dependent coupling parameter which
actually blows up at the boundaries of this region. We will discuss this issue at length below, but before proceeding we will say a few words about our interpretation of this. The high density collective field theory is only valid in the region $|x| \geq 1/(\omega \sqrt{g})$. The infinite number of eigenvalues, defined over $x$, densely populate only the “exterior” region. Within the region $|x| \leq 1/(\omega \sqrt{g})$, exist only a finite number of eigenvalues. Their behavior is described, not by the high density collective field theory, but, more properly, by the eigenvalue Lagrangian (3.23). The actual mechanics of how the physics in the different regions is patched together will be described later. For the moment, we will continue to focus on the high density theory which is defined only in the regions $|x| \geq 1/(\omega \sqrt{g})$.

Equation (4.40) can be integrated. Doing this we find the most general purely bosonic, static solution to the equations of motion derived from the high density collective field theory. The result is

$$\tilde{\varphi}_0(x) = \begin{cases} a_- + \frac{\pi}{2\sqrt{\pi}} \omega^2 x^2 - 1/g + \frac{\pi}{2\sqrt{\pi}} \ln(-\sqrt{\omega x} + \sqrt{\omega x^2 - 1/g}) & ; x \leq -\frac{1}{\sqrt{g}} \\ a_+ + \frac{\pi}{2\sqrt{\pi}} \omega^2 x^2 - 1/g - \frac{\pi}{2\sqrt{\pi}} \ln(+\sqrt{\omega x} + \sqrt{\omega x^2 - 1/g}) & ; x \geq \frac{1}{\sqrt{g}} \end{cases} \quad (4.41)$$

The parameters $a_+$ and $a_-$ are independent arbitrary integration constants. We now take the solution $\tilde{\varphi}_0$ as a background and define a new field, $\zeta$, as the fluctuation around this background,

$$\varphi = \tilde{\varphi}_0(x) + \frac{1}{\sqrt{\pi}} \zeta. \quad (4.42)$$

Expressed in terms of the shifted field, $\zeta$, the Lagrangian (4.37) reads

$$L = \int dx \left\{ \frac{1}{2} \frac{\dot{\zeta}^2}{\tilde{\varphi}_0'(x)} - \frac{\sqrt{\pi}}{6} \zeta^3 - \frac{\pi}{2} \tilde{\varphi}_0'(x) \zeta'^2 - \frac{i}{2\sqrt{\pi}} \left( \psi_1 \psi_1' + \psi_2 \psi_2' \right) - \frac{i\pi}{2} \psi_1 \psi_1' + \frac{i\pi}{2} \psi_2 \psi_2' + \frac{i}{2} \left( \frac{1}{\sqrt{\pi}} \dot{\zeta} \right)^2 \left( \psi_1 \psi_1' + \psi_2 \psi_2' \right) \right\} + \frac{\pi^2}{3} \int dx \tilde{\varphi}_0'(x)^3 \quad (4.43)$$

It is now possible to perform a coordinate transformation in order to render both the bosonic and fermionic kinetic energies canonical. The appropriate choice is to define a spatial coordinate $\tau$ by

$$\tau'(x) = \frac{1}{\pi} (\tilde{\varphi}_0'(x))^{-1}$$
\[ \int \frac{1}{\sqrt{\omega^2 x^2 - 1/g}}. \] (4.44)

Integrating this, we find

\[ \tau(x) = \begin{cases} 
(\tau_0 - \frac{\sigma}{2}) - \frac{1}{\omega} \ln(\sqrt{g\omega^2 x^2} + \sqrt{g\omega^2 x^2 - 1}) & ; x \leq \frac{-1}{\omega \sqrt{g}} \\
(\tau_0 + \frac{\sigma}{2}) + \frac{1}{\omega} \ln(\sqrt{g\omega^2 x^2} + \sqrt{g\omega^2 x^2 - 1}) & ; x \geq \frac{+1}{\omega \sqrt{g}}.
\end{cases} \] (4.45)

where \( \tau_0 \) and \( \sigma \) are independent integration constants. Although the dense collective field theory is not defined in the region \( |x| < \frac{1}{\omega \sqrt{g}} \), we can, and will, continue the definition of \( \tau \) into this region. We require that \( \tau(x) \) and \( \tau'(x) \) match at the boundary. The following is then a suitable choice,

\[ \tau(x) = \begin{cases} 
(\tau_0 - \frac{\sigma}{2}) - \frac{1}{\omega} \ln(\sqrt{g\omega^2 x^2} + \sqrt{g\omega^2 x^2 - 1}) & ; x \leq \frac{-1}{\omega \sqrt{g}} \\
\tau_0 + \frac{\sigma}{2} \sin^{-1}(x \omega \sqrt{g}) & ; \frac{-1}{\omega \sqrt{g}} < x < \frac{+1}{\omega \sqrt{g}} \\
(\tau_0 + \frac{\sigma}{2}) + \frac{1}{\omega} \ln(\sqrt{g\omega^2 x^2} + \sqrt{g\omega^2 x^2 - 1}) & ; x \geq \frac{+1}{\omega \sqrt{g}}.
\end{cases} \] (4.46)

The inverse of this transformation is given by

\[ x(\tau) = \begin{cases} 
\frac{-1}{\omega \sqrt{g}} \cosh\{\omega(\tau - \tau_0 + \sigma/2)\} & ; \tau \leq (\tau_0 - \frac{\sigma}{2}) \\
\frac{1}{\omega \sqrt{g}} \sin\{\frac{\pi}{\sigma}(\tau - \tau_0)\} & ; (\tau_0 - \frac{\sigma}{2}) < \tau < (\tau_0 + \frac{\sigma}{2}) \\
\frac{+1}{\omega \sqrt{g}} \cosh\{\omega(\tau - \tau_0 - \sigma/2)\} & ; \tau \geq (\tau_0 + \frac{\sigma}{2})
\end{cases} \] (4.47)

This transformation is depicted in Figure 3.

Figure 3. The \( x - \tau \) transformation.

It is easily seen that \( \tau_0 \) is the position, in \( \tau \) space, of the center of the low density region and that \( \sigma \) is the width of this region. We may now express the background solution in terms of \( \tau \). To avoid confusion, we define \( \varphi_0(\tau) = \tilde{\varphi}_0(x(\tau)) \). This explains the use of the tilde. Using (4.41) and (4.46), this is

\[ \varphi_0(\tau) = \begin{cases} 
a_- + \frac{1}{2\pi g}(\tau - \tau_0 + \frac{\sigma}{2}) + \frac{1}{4\pi \omega g} \sinh\{2\omega(\tau - \tau_0 + \frac{\sigma}{2})\} & ; \tau \leq (\tau_0 - \frac{\sigma}{2}) \\
a_+ + \frac{1}{2\pi g}(\tau - \tau_0 - \frac{\sigma}{2}) + \frac{1}{4\pi \omega g} \sinh\{2\omega(\tau - \tau_0 - \frac{\sigma}{2})\} & ; \tau \geq (\tau_0 + \frac{\sigma}{2}).
\end{cases} \] (4.48)
In the region $|x| \geq 1/(\omega \sqrt{g})$, it is useful to define a function

$$
\tilde{g}(x) = \pi^{-3/2}(z'_0(x))^{-2} = \frac{\sqrt{\pi}}{\omega^2 x^2 - \frac{1}{g}}.
$$

(4.49)

In terms of $\tau$, we then have $g(\tau) = \tilde{g}(x(\tau))$, which reads

$$
g(\tau) = \begin{cases} 
g_- (\tau) & ; \tau \leq (\tau_0 - \sigma/2) \\
g_+ (\tau) & ; \tau \geq (\tau_0 + \sigma/2) 
\end{cases},
$$

(4.50)

where

$$
g_{\pm}(\tau) = 4\sqrt{\pi g} \frac{\frac{1}{\kappa} e^{\mp 2\omega(\tau - \tau_0)}}{(1 - \frac{1}{\kappa} e^{\mp 2\omega(\tau - \tau_0)})^2},
$$

(4.51)

and $\kappa$ is a dimensionless constant,

$$
\kappa = \exp (\omega \sigma),
$$

(4.52)

which relates the width, $\sigma$, of the low density region in $\tau$ space to the natural length scale in the matrix model, $1/\omega$. As we will see momentarily, function $g_{\pm}$ is the coupling parameter in the high density collective field theory, expressed, canonically, in $\tau$ space. It is a space-dependent coupling, and is plotted in figure 4.

Figure 4. Space dependent coupling parameter $g_{\pm}$.

Before exhibiting the collective field Lagrangian in $\tau$ space, we will first discuss one small issue. That is, in $\tau$ space, the fermionic kinetic energy has the correct normalization only if we trivially scale the fields $\psi_1$ and $\psi_2$. Toward this end, we define

$$
\psi_+ = \frac{2^{1/4}}{\sqrt{\pi}} \psi_1, \\
\psi_- = \frac{2^{1/4}}{\sqrt{\pi}} \psi_2.
$$

(4.53)

The “$\pm$” is a useful notation in two dimensions. As we will see, this designation relates to the Lorentz structure of these fields. Now, using the coordinate transformation (4.46), and the definitions (4.51), (4.53), we can write the high density collective field Lagrangian as follows,

$$
L = \int d\tau \left\{ \frac{1}{2}(\dot{\zeta}^2 - \zeta'^2) - \frac{i}{\sqrt{2}} (\psi_+ \dot{\psi}_+ - \psi_+ \psi'_+) - \frac{i}{\sqrt{2}} (\psi_- \dot{\psi}_- + \psi_- \psi'_-) \right\}
$$

25
\[-\frac{1}{2} \frac{g(\tau)\dot{\zeta}^2\zeta'}{1 + g(\tau)\zeta'} - \frac{1}{6} g(\tau)\dot{\zeta}^3 \]
\[+ \frac{i}{\sqrt{2}} \frac{g(\tau)\zeta'}{1 + g(\tau)\zeta'} \left( \psi_+\dot{\psi}_+ + \psi_-\dot{\psi}_- \right) \]
\[+ \frac{i}{\sqrt{2}} \frac{g(\tau)\dot{\zeta}}{(1 + g(\tau)\zeta')^2} \left( \psi_+\psi'_+ + \psi_-\psi'_- \right) \}
\[+ \frac{1}{3} \int d\tau \frac{1}{g(\tau)^2}, \tag{4.54} \]

where now the prime means \( \partial/\partial \tau \). This is the \( N \to \infty, L \to \infty \) high density collective field Lagrangian with canonically normalized kinetic energy terms. The bosonic terms of this Lagrangian are identical to the canonical bosonic collective field theory. We reiterate that (4.54) is only valid in the high density regions \( \tau \leq (\tau_0 - \sigma/2) \) and \( \tau \geq (\tau_0 + \sigma/2) \). In the region \( (\tau_0 - \sigma/2) < \tau < (\tau_0 + \sigma/2) \) there are only a finite number of eigenvalues, whose dynamics is best described by Lagrangian (3.23).

5. The Supersymmetric Effective Theory

The Lagrangian (4.54) has kinetic energy terms for both the bosonic field, \( \zeta \) and for the fermionic fields, \( \psi_\pm \), which are canonically normalized for a flat two-dimensional spacetime. The interaction terms, however, involve an explicit spatially-dependent coupling, \( g(\tau) \), which violates Poincaré invariance. We interpret \( g(\tau) \) to be the vacuum expectation value (VEV) of a function of an additional “heavy” field, which we denote by \( \alpha \). Furthermore, we infer the existence of an effective theory involving \( \alpha \), as well as \( \zeta, \psi_+ \) and \( \psi_- \), which reproduces (4.54) when \( \alpha \) is replaced by its \( \tau \)-dependent VEV. Additionally, we postulate that the effective theory possesses a two-dimensional supersymmetry. It follows that, in addition to \( \alpha \), we must introduce its fermionic superpartners, \( \chi_+ \) and \( \chi_- \) which, of course, have vanishing VEV’s. The field \( \alpha \) and its superpartners \( \chi_+ \) and \( \chi_- \) are assumed to be heavy. We do not consider their fluctuations but rather treat them as frozen in their VEV’s. Although we have a ready interpretation of the light fields \( \zeta, \psi_+ \), and \( \psi_- \) as being comprised of modes related to the singlet sector of the underlying matrix model, we do not attempt a similar interpretation of the heavy fields. A precise explanation for treating these fields as suggested is beyond the scope of this paper. We can only offer at this point a motivation from two dimensional string theory. Two dimensional string theory contains, in its associated effective low-energy Lagrangian, a massive dilaton multiplet which is frozen at its VEV as a consequence of
gauge symmetry. Furthermore, although the underlying matrix theory is supersymmetric, it is not obvious that our effective two dimensional theory must also be supersymmetric. Nevertheless, we proceed in this section to demonstrate that, indeed, there exists an effective two-dimensional theory which is Poincare invariant and supersymmetric, which involves both heavy fields and light fields and which, when the heavy fields are replaced by their VEV’s, reproduces the collective field Lagrangian (4.54).

5.1 Two Dimensional \((p,q)\) Supersymmetry

It is well known that, because the two-dimensional Lorentz group is abelian, the possible \(d = 2\) supersymmetries have a rich structure \([14, 15]\). Specifically, there exist supersymmetries with any number of left-chiral fermionic generators, \(Q_{A-}, A = 1, \ldots, p\), and any independent number, \(q\), of right-chiral fermionic generators, \(Q_{I+}, I = 1, \ldots, q\). There are thus an infinite number of two-dimensional supersymmetries enumerated by the respective numbers of left- and right-chiral fermionic generators, \(p\) and \(q\). These are called \((p,q)\) supersymmetries. The generators must satisfy the following algebra,

\[
\{Q_{A-}, Q_{B-}\} = -2i\delta_{AB}\partial_- \\
\{Q_{I+}, Q_{J+}\} = -2i\delta_{IJ}\partial_+ \\
\{Q_{A-}, Q_{I+}\} = 0,
\]

(5.1)

where indices \(A, B, \ldots\) run from 1 to \(p\) and the indices \(I, J, \ldots\) run from 1 to \(q\). Which of these, if any, is the appropriate supersymmetry of our theory?

We may quickly narrow the range of possibilities by the following observations. The collective field Lagrangian, (4.54), describes all light fields in the effective theory. Since the collective field Lagrangian has only one bosonic, \(\zeta\), and two fermionic, \(\psi_{\pm}\), degrees of freedom, and since the light fields and heavy fields cannot belong to the same supersymmetric multiplet, it follows that the number of fermions in the fundamental matter multiplet of the relevant supersymmetry is at most two. Since the number of fermions in the fundamental matter multiplet is the same as the number of supersymmetry generators, it follows that \(p + q \leq 2\). There are, therefore, only five possibilities, \((1,0), (0,1), (2,0), (0,2),\) or \((1,1)\) supersymmetry. We will now examine each of these possibilities in turn.
a) (1,0) and (0,1) supersymmetry: There is only one matter multiplet for either (1,0) or (0,1) supersymmetry. It contains one boson and one fermion. The light sector of the effective theory would need two such multiplets to accommodate the required number of fermions. The theory would then have two bosons as well as two fermions. Since we require that there be only one light boson, it is impossible to properly describe the necessary degrees of freedom using (1,0) or (0,1) supersymmetry. This case is thus ruled out.

b) (2,0) and (0,2) supersymmetry: A (0,2) superspace has coordinates $z^{\mu} = (x^{\pm}, \theta^{\pm}, \bar{\theta}^{\pm})$ where $\theta^{\pm}$ and $\bar{\theta}^{\pm}$ are complex conjugates. The covariant superspace derivatives are given by

$$
D_+ = \frac{\partial}{\partial \theta^+} + i \bar{\theta}^+ \partial_+
$$

$$
\bar{D}_+ = \frac{\partial}{\partial \bar{\theta}^+} + i \theta^+ \partial_+.
$$

(5.2)

Note that $(\partial/\partial \theta)^* = -(\partial/\partial \bar{\theta})$, and $\bar{D} = -D^*$. An irreducible “chiral” superfield, $\Phi$ is obtained by imposing the differential constraint $\bar{D}_+ \Phi = 0$. In component fields such a superfield reads

$$
\Phi = \phi + i\sqrt{2} \theta^+ \psi_+ + i \theta^+ \bar{\theta}^+ \partial_+ \phi,
$$

(5.3)

where $\phi$ and $\psi$ are complex. The complex conjugate, $\bar{\Phi}$, satisfies $D_+ \bar{\Phi} = 0$ and is called “antichiral”. It is given by

$$
\bar{\Phi} = \phi^* + i\sqrt{2} \bar{\theta}^+ \bar{\psi}_+ - i \theta^+ \bar{\theta}^+ \partial_+ \phi^*.
$$

(5.4)

(We recall that $(\theta \psi)^* = \bar{\psi} \bar{\theta} = -\bar{\theta} \bar{\psi}$.) The (0,2) transformation law, on the component fields, is

$$
\delta \phi = i \eta^+ \psi_+ \\
\delta \psi_+ = \bar{\eta}^+ \partial_+ \phi \\
\delta \phi^* = i \bar{\eta}^+ \bar{\psi}_+ \\
\delta \bar{\psi}_+ = \eta^+ \partial_+ \phi^*.
$$

(5.5)

It can be shown that these are the only irreducible representations of (0,2) supersymmetry. We exhibit the above details in order to allow the following possibility. First of all, we note
that the irreducible representations of (0,2) supersymmetry each have two bosonic and two fermionic degrees of freedom. By our reasoning above, we should rule out this supersymmetry from consideration immediately since the effective theory is desired to have only one light boson. However, it is tempting to try to accommodate the new boson $\alpha$ along with the light fields $\zeta$, $\psi_+$ and $\psi_-$ in a single supersymmetric multiplet. We would then have to find some alternative explanation for neglecting the fluctuations of the extra boson, but this would avoid the otherwise necessary addition of extra heavy fermions. The (0,2) multiplet is particularly suited to this idea because its fundamental representation has exactly two bosons and two fermions. It turns out, however, after an extensive analysis of this possibility, that this supersymmetry can be ruled out even if we allow this last idea. This is because it is impossible to construct an interaction Lagrangian using the superfields (5.3) and (5.4) which has the appropriate derivative structure indicated in (4.54). This problem is related to the existence of the derivative in the highest components of $\Phi$ and $\bar{\Phi}$. The same reasoning applies to (2,0) supersymmetry. We can thus rule out (2,0) and (0,2) as candidate supersymmetries of the effective theory.

c) (1,1) supersymmetry: The sole remaining possibility is (1,1) supersymmetry. As we will show, it is indeed possible to construct an effective theory with this supersymmetry and with the desired relationship to the collective field theory derived in the last section. We proceed to describe (1,1) supersymmetry in some detail. We will then derive the effective theory, discuss its equations of motion, solve these equations of motion, and show that when the heavy fields are replaced by their VEV’s that the high density collective field Lagrangian (4.54) is recovered.

5.2 (1,1) Supersymmetry

A (1,1) superspace has coordinates $z^M = (x^\pm, \theta^+, \theta^-)$. The supersymmetry generators are given by

$$ Q_+ = \frac{\partial}{\partial \theta^+} - i\theta^+ \partial_+ $$
$$ Q_- = \frac{\partial}{\partial \theta^-} - i\theta^- \partial_. $$

(5.6)
and satisfy the algebra,
\[
\{ Q_\pm, Q_\mp \} = -2i \partial_\pm
\]
\[
\{ Q_+, Q_- \} = 0.
\]
(5.7)

The covariant superspace derivatives are
\[
D_+ = \frac{\partial}{\partial \theta^+} + i \theta^+ \partial_+
\]
\[
D_- = \frac{\partial}{\partial \theta^-} + i \theta^- \partial_-
\]
(5.8)

The fundamental irreducible representation is a real superfield, \( \Phi_1 = \Phi \), which, in component fields, is given by
\[
\Phi_1 = \zeta + i \theta^+ \psi_+ + i \theta^- \psi_- + i \theta^+ \theta^- Z,
\]
(5.9)

where \( \zeta \) and \( Z \) are real and commuting and \( \psi_+ \) and \( \psi_- \) are real anticommuting spinors. The (1,1) transformation law for the component fields is
\[
\delta \zeta = i \eta^+ \psi_+ + i \eta^- \psi_-
\]
\[
\delta \psi_+ = \eta^+ \partial_+ \zeta + \eta^- Z
\]
\[
\delta \psi_- = \eta^- \partial_- \zeta - \eta^+ Z
\]
\[
\delta Z = i \eta^- \partial_- \psi_+ - i \eta^+ \partial_+ \psi_-.
\]
(5.10)

Depending on the dynamics of the theory, the highest component of this multiplet can be either a nonphysical auxiliary degree of freedom or a physical propagating field. The light sector is required to have one physical boson and two physical fermions. This can be accommodated by the superfield \( \Phi_1 \) provided field \( Z \) is auxiliary. We therefore choose the Lagrangian so that this is the case. We also require the existence of a massive sector which includes the bosonic field, \( \alpha \). We must then introduce a second, “heavy” superfield, \( \Phi_2 \), given, in components, as follows,
\[
\Phi_2 = \alpha + i \theta^+ \chi_+ + i \theta^- \chi_- + i \theta^+ \theta^- A.
\]
(5.11)

As in \( \Phi_1 \), \( \alpha \) is a real, physical boson, \( \chi_+ \) and \( \chi_- \) are real, physical fermions, and \( A \) is an additional boson whose status as auxiliary or physical depends on the form of the effective
Lagrangian. The (1,1) transformation law on the component fields of $\Phi_2$ reads

$$
\begin{align*}
\delta \alpha &= i\eta^+ \chi_+ + i\eta^- \chi_- \\
\delta \chi_+ &= \eta^+ \partial_+ \alpha + \eta^- A \\
\delta \chi_- &= \eta^- \partial_- \alpha - \eta^+ A \\
\delta A &= i\eta^- \partial_- \chi_+ - i\eta^+ \partial_+ \chi_-.
\end{align*}
$$

We would like to use the two superfields, $\Phi_1$ and $\Phi_2$, to construct a (1,1) supersymmetric Lagrangian that reproduces the high density collective field Lagrangian, \((4.54)\), when the heavy fields are replaced by their VEV’s. The discussion so far in this section has demonstrated conclusively that this is the minimal prescription which could conceivably satisfy this criterion.

5.3 The (1,1) Supersymmetric Effective Theory

We proceed to construct the effective theory using the two (1,1) superfields introduced above. For convenience, we list these superfields again,

$$
\begin{align*}
\Phi_1 &= \zeta + i\theta^+ \psi_+ + i\theta^- \psi_- + i\theta^+ \theta^- Z \\
\Phi_2 &= \alpha + i\theta^+ \chi_+ + i\theta^- \chi_- + i\theta^+ \theta^- A.
\end{align*}
$$

Using these two superfields and the differential operators, $D_+, D_-, \partial_+$, and $\partial_-$, we build the effective theory piecemeal, order by order in the coupling, $g(\tau)$. We begin with the free part of the collective field Lagrangian, \((4.54)\), and find the relevant supersymmetric expression involving $\Phi_1$ and $\Phi_2$ which reproduces it when the equations of motion are used. We then include the part of \((4.54)\) which is linear in $g(\tau)$ and modify our construction appropriately. Proceeding in this manner, we eventually discover the entire supersymmetric effective theory. We end this section by exhibiting the complete effective theory, listing its equations of motion, and showing that when the equations of motion are used, the collective field Lagrangian is recovered.

a) $0^{th}$ order:
The free (noninteracting) part of the collective field Lagrangian, (4.54), involving the fields \( \zeta, \psi_+, \) and \( \psi_- \) is given by

\[
L_{01} = \frac{1}{2}(\dot{\zeta}^2 - \zeta'^2) - i\psi_+ \partial_- \psi_+ - i\psi_- \partial_+ \psi_-.
\]  

This can be written manifestly supersymmetrically using \( \Phi_1 \). The appropriate super-Lagrangian is

\[
L_{01}^{(\text{eff})} = \int d\theta^+ d\theta^- D_+ \Phi_1 D_- \Phi_1
= \frac{1}{2}(\dot{\zeta}^2 - \zeta'^2) - i\psi_+ \partial_- \psi_+ - i\psi_- \partial_+ \psi_- + Z^2.
\]  

Field \( Z \) has no dynamics. It is therefore auxiliary and can be eliminated using its equation of motion, which reads \( Z = 0 \). Eliminating \( Z \), (5.15) becomes

\[
L_{01}^{(\text{eff})} = \frac{1}{2}(\dot{\zeta}^2 - \zeta'^2) - i\psi_+ \partial_- \psi_+ - i\psi_- \partial_+ \psi_-,
\]  

which is precisely the free collective field Lagrangian for \( \zeta, \psi_+, \) and \( \psi_- \) given in (5.14).

We also need to introduce kinetic energy for the heavy multiplet, \( \Phi_2 \). Additionally, we need to introduce a mass term or some suitable alternative interaction for \( \Phi_2 \). Since the last term in (4.54) involves only \( g(\tau) \), we presume that it is the vestige of the pure \( \Phi_2 \) kinetic energy Lagrangian which must become a function of \( \tau \) only when the equations of motion are used. Using (4.51), the last term in (4.54), \((3g(\tau)^2)^{-1}\), can be written as follows

\[
L_{02} = \frac{\kappa^2}{48\pi g^2}(e^{\omega|\tau-\tau_0|} - \frac{1}{\kappa}e^{-\omega|\tau-\tau_0|})^4.
\]  

We pose the important hypotheses that the equations of motion admit the following solution,

\[
<\alpha> = e^{-\omega|\tau-\tau_0|}
<\chi^{\pm}> = 0
<A> = 0.
\]  

This amounts to the requirement that

\[
<\mathcal{L}_{\alpha}^{(\text{eff})}> = \frac{\kappa^2}{48\pi g^2}(<\alpha^{-1}> - \frac{1}{\kappa} <\alpha>)^4.
\]  

32
It is expedient to first construct a relevant Lagrangian involving only $\alpha$, which we call $L^{(\text{eff})}_\alpha$, and then to supersymmetrize the result to include the full multiplet, $\Phi_2$. We must first decide what sort of structure we expect $L^{(\text{eff})}_\alpha$ to have. An obvious guess would be

$$F_1(\alpha)\partial_+\alpha\partial_-\alpha - \omega^2 F_2(\alpha),$$

(5.20)

where $F_1(\alpha)$ and $F_2(\alpha)$ are polynomials in $\alpha$. It turns out that there do exist $F_1$ and $F_2$ such that (5.20) is both compatible with the exponential solution (5.18) and with the desired property that $<L^{(\text{eff})}_\alpha>=L_{02}$. There is a convoluted impediment to the supersymmetrization of this choice, however. This is related to the fact that the supersymmetrization of $\alpha^n$, which is $\int d\theta^+d\theta^-\Phi^m_\alpha$, generates interactions of the sort $\alpha^{m-1}A$. The equation of motion for $A$ then involves powers of $\alpha$, that is $<A>\neq 0$. Under this circumstance it is impossible to construct a Lagrangian which reproduces the necessary higher-order interactions between $\Phi_1$ and $\Phi_2$. An interesting resolution to this problem is the following. We take

$$L^{(\text{eff})}_\alpha = F_1(\alpha)\partial_+\alpha\partial_-\alpha - \frac{1}{\omega^2} F_2(\alpha)(\partial_+\alpha\partial_-\alpha)^2$$

(5.21)

That is, we consider higher derivative interactions. As we will see, this generalizes supersymmetrically in such a way that $<A>=0$. How then do we determine the functions $F_1(\alpha)$ and $F_2(\alpha)$? There are two criteria for this which, together, uniquely specify these functions. First, the equation of motion derived from (5.21) need allow (5.18) as a solution, and, secondly, using this solution, we must have $<L^{(\text{eff})}_\alpha>=L_{02}$. To resolve the first issue, we compute the equation of motion using (5.21). This reads,

$$2F_1(\alpha)\partial_+\partial_-\alpha + F'_1(\alpha)\partial_+\alpha\partial_-\alpha + \frac{2}{\omega^2} F_2(\alpha)\partial_+^2\partial_-^2\alpha + \frac{1}{\omega^2} F'_2(\alpha)\left\{2\partial_+\alpha\partial_+\partial_-\alpha + (\partial_+\partial_-\alpha)^2 + 2\partial_+(\partial_-\alpha\partial_+\partial_-\alpha)\right\} + \frac{2}{\omega^2} F''_2(\alpha)\partial_+\alpha\partial_-\alpha\partial_+\partial_-\alpha = 0.$$  

(5.22)

If $\alpha = \exp(-\omega|\tau - \tau_0|)$, then this equation becomes

$$4\alpha F_1(\alpha) + 2\alpha^2 F'_1(\alpha) - 2\alpha F_2(\alpha) - 7\alpha^2 F'_2(\alpha) - 2\alpha^3 F''_2(\alpha) = 0.$$  

(5.23)
If we express $F_1$ and $F_2$ as follows

$$
F_1(\alpha) = \sum_n a_n \alpha^n \\
F_2(\alpha) = \sum_n b_n \alpha^n,
$$

then (5.23) requires that

$$
b_n = \frac{2}{1+2n}a_m. \tag{5.25}
$$

To resolve the second issue, we can use (5.25) to rewrite equation (5.21) as

$$
L_{\alpha}^{(\text{eff})} = \sum_n a_n \alpha^n \left\{ \partial_+ \alpha \partial_- \alpha - \frac{2}{1+2n} \frac{1}{\omega^2} (\partial_+ \partial_- \alpha)^2 \right\}. \tag{5.26}
$$

Substituting (5.18), we find that

$$
< L_{\alpha}^{(\text{eff})} > = -\omega^2 \sum_n a_n \frac{1+n}{1+2n} < \alpha >^n \\
= -\omega^2 \sum_n c_n < \alpha >^n. \tag{5.27}
$$

Combining (5.23) and (5.27), we find

$$
a_n = \frac{1+2n}{1+n} c_{n+2} \\
b_n = \frac{2}{1+n} c_{n+2}. \tag{5.28}
$$

The coefficients $c_n$ are easily determined from (5.13). The coefficients $a_n$ and $b_n$ are then computed with (5.28). This then determines both $F_1(\alpha)$ and $F_2(\alpha)$. The result is that

$$
F_1(\alpha) = -\frac{\kappa^2}{48\pi g^2 \omega^2} \left( \frac{11}{5} \alpha^{-6} - \frac{28}{3} \frac{1}{\kappa} \alpha^{-4} + 18 \frac{1}{\kappa^2} \alpha^{-2} - 4 \frac{1}{\kappa^3} + \frac{5}{3} \frac{1}{\kappa^4} \alpha^2 \right) \\
F_2(\alpha) = -\frac{\kappa^2}{48\pi g^2 \omega^2} \left( -\frac{2}{5} \alpha^{-6} + \frac{8}{3} \frac{1}{\kappa} \alpha^{-4} - \frac{12}{\kappa^2} \alpha^{-2} - \frac{8}{\kappa^3} + \frac{2}{3} \frac{1}{\kappa^4} \alpha^2 \right). \tag{5.29}
$$

These functions look rather peculiar. This is a consequence of the fact that equation (5.21) is not a unique prescription for determining the pure $\alpha$ Lagrangian. We could, for instance, have included interactions of the form $\omega^{-P}(\partial_+ \partial_- \alpha)^P$ where $P$ is a completely arbitrary exponent. The corresponding $\alpha$ Lagrangian would then be different. Our construction is,
however, the simplest example which has the appropriate relationship to the collective field theory. The supersymmetrization of (5.21) is given by the following super-Lagrangian,

\[ \mathcal{L}_{02}^{(\text{eff})} = \int d\theta^+ d\theta^- \left\{ F_1(\Phi_2)D_+\Phi_2D_-\Phi_2 - \frac{1}{\omega^2} F_2(\Phi_2)\partial_-D_+\Phi_2\partial_+D_-\Phi_2 \right\}. \]  

(5.30)

In components this reads

\[ \mathcal{L}_\alpha^{(\text{eff})} = F_1(\alpha)\partial_+\alpha\partial_-\alpha - \frac{1}{\omega^2} F_2(\alpha)(\partial_+\partial_-\alpha)^2 \]

\[ + F_1(\alpha)A^2 - \frac{1}{\omega^2} F_2(\alpha)\partial_+A\partial_-A \]

\[ - iF_1(\alpha)\chi_+\partial_-\chi_+ - \frac{i}{\omega^2} F_2(\alpha)\partial_-\chi_+\partial_+\partial_-\chi_+ \]

\[ - iF_1(\alpha)\chi_-\partial_+\chi_- - \frac{i}{\omega^2} F_2(\alpha)\partial_+\chi_-\partial_-\partial_+\chi_- \]  

(5.31)

Clearly, the equations of motion admit solution (5.18). Also, \( \langle \mathcal{L}_\alpha^{(\text{eff})} \rangle = \mathcal{L}_{02} \) by construction.

b) 1st order:

The collective field Lagrangian (4.54) can be expanded in powers of \( g(\tau) \),

\[ \mathcal{L} = (\mathcal{L}_{01} + \mathcal{L}_{02}) + \mathcal{L}_1 + \mathcal{L}_2 + \cdots. \]

(5.32)

We have supersymmetrized \( \mathcal{L}_{01} \) and \( \mathcal{L}_{02} \) above. The term linear in \( g(\tau) \) reads

\[ \mathcal{L}_1 = g(\tau) \left\{ -\frac{1}{6} (\dot{\zeta}^3 + 3\dot{\zeta}\ddot{\zeta}) \right. \]

\[ + \frac{i}{\sqrt{2}} \dot{\zeta}'(\psi_+\psi_+ + \psi_-\psi_-) \]

\[ + \frac{i}{\sqrt{2}} \dot{\zeta}(\psi_+\psi_+ + \psi_-\psi_-) \} \].

(5.33)

This may be extended supersymmetrically as follows. First of all we note that

\[ \partial_+(\Phi_1\partial_+\Phi_2) = \dot{\alpha}' - \alpha'\zeta' \]

\[ \rightarrow \omega < \alpha > \zeta' \]  

(5.34)
\[
\partial_{\pm} \Phi_1 \partial_{\mp} \Phi_2 = \dot{\alpha} \zeta' - \alpha' \dot{\zeta} \\
\rightarrow \omega < \alpha > \dot{\zeta},
\]
(5.35)

where \( A_+ B_- = A_+ B_- + A_- B_+ \), \( A_+ B_- = A_+ B_- - A_- B_+ \), the arrow implies that \( \alpha \rightarrow < \alpha > : \exp (-\omega |\tau - \tau_0|) \), \( \chi_+ \rightarrow 0 \) and \( \big| \) indicates the lowest component of the indicated superfield expression. Expressions (5.34) and (5.33) are useful when used in conjunction with the following facts. If \( \alpha \rightarrow < \alpha >, \chi_\pm \rightarrow 0 \), and \( A \rightarrow 0 \), then

\[
\int d\theta^+ d\theta^- F(\dot{\Phi}_1, \Phi'_1; \Phi_2, \dot{\Phi}_2, \Phi'_2) D_{(+} \Phi_1 D_{-} \Phi_2 \\
\rightarrow \omega < \alpha > \left\{ -F |\dot{\zeta} + \frac{i}{\sqrt{2}} \delta F}{\delta \dot{\Phi}_1} |(\psi_+ \dot{\psi}_+ + \psi_- \dot{\psi}_-) \\
+ \frac{i}{\sqrt{2}} \delta F}{\delta \Phi'_1} |(\psi_+ \dot{\psi}_+ + \psi_- \dot{\psi}_-) \right\}
\]
(5.36)

and

\[
\int d\theta^+ d\theta^- G(\dot{\Phi}_1, \Phi'_1; \Phi_2, \dot{\Phi}_2, \Phi'_2) D_+ \Phi_2 D_- \Phi_2 \\
\rightarrow -\frac{1}{2} \omega^2 < \alpha >^2 G |.
\]
(5.37)

Note that (5.37) only involves \( < \alpha (\tau) >, \dot{\zeta} \) and \( \zeta' \) when \( \Phi_2 \) is replaced by its VEV. Notice also that the fermionic part of the right hand side of (5.36) has the same structure as the fermionic part of (5.33). We can thus use (5.36) and (5.37) to find the correct functions \( F \) and \( G \) to reproduce the fermionic part of the order \( g(\tau) \) collective field Lagrangian. Implementing this procedure, we see immediately, from comparing (5.33) and (5.36), that we require

\[
\frac{\delta F}{\delta \Phi_1} \bigg| \rightarrow f(<\alpha>) \zeta' \\
\frac{\delta F}{\delta \Phi'_1} \bigg| \rightarrow f(<\alpha>) \dot{\zeta},
\]
(5.38)

where \( f(<\alpha>) = f(\exp (-\omega |\tau - \tau_0|)) = g(\tau) \). By comparison with (5.11), we find that

\[
f(\alpha) = 4 \sqrt{\pi} g \frac{1}{1 - \frac{1}{2} \alpha^2},
\]
(5.39)
where $\kappa$ is defined in (4.52). Using (5.34) and (5.35), an appropriate function, $F$, is immediately seen to be

$$F = \frac{f(\Phi_2)}{\omega^3 \Phi_2^3} \partial_{(+} \Phi_1 \partial_{-}) \Phi_2 \partial_{(+} \Phi_1 \partial_{-}) \Phi_2. \quad (5.40)$$

Thus

$$\int d\theta^+ d\theta^- F D_{(+} \Phi_1 D_{-)} \Phi_2$$

$$\rightarrow -f(\alpha) \dot{\zeta}^2 \zeta' + \frac{i}{\sqrt{2}} \zeta' (\dot{\psi}_+ \psi_+ + \dot{\psi}_- \psi_-)$$

$$+ \frac{i}{\sqrt{2}} \dot{\zeta} (\dot{\psi}_+ \psi'_+ + \dot{\psi}_- \psi'_-). \quad (5.41)$$

By construction, the fermionic part of this expression reproduces the fermionic part of the collective field Lagrangian at first order in $g(\tau)$. In order that we also match the bosonic part of the order $g(\tau)$ collective field Lagrangian, we must add to this result another supersymmetric expression to supply the difference, which is

$$f(<\alpha>) \{ -\frac{1}{6} \zeta'^3 + \frac{1}{2} \zeta^2 \zeta' \}. \quad (5.42)$$

Accordingly, we use (5.37) to determine the function $G$, which is found to be

$$G = \frac{f(\Phi_2)}{\omega^5 \Phi_2^5} \left\{ \frac{1}{3} (\partial_{(+} \Phi_1 \partial_{-)} \Phi_2)^3 - (\partial_{(+} \Phi_1 \partial_{-)} \Phi_2)^2 \partial_{(+} \Phi_1 \partial_{-)} \Phi_2 \right\}. \quad (5.43)$$

The complete supersymmetrization of the first order interactions in the collective field Lagrangian is then given by combining (5.36) and (5.37), where functions $F$ and $G$ are specified in (5.40) and (5.43) respectively. The result is

$$L^{(eff)}_1 = \int d\theta^+ d\theta^- \left\{ \frac{f(\Phi_2)}{\omega^3 \Phi_2^3} \partial_{(+} \Phi_1 \partial_{-)} \Phi_2 \partial_{(+} \Phi_1 \partial_{-)} \Phi_2 D_{(+} \Phi_1 D_{-)} \Phi_2$$

$$+ \frac{1}{3} \frac{f(\Phi_2)}{\omega^5 \Phi_2^5} (\partial_{(+} \Phi_1 \partial_{-)} \Phi_2)^3 D_+ \Phi_2 D_- \Phi_2$$

$$- \frac{f(\Phi_2)}{\omega^5 \Phi_2^5} (\partial_{(+} \Phi_1 \partial_{-)} \Phi_2)^2 \partial_{(+} \Phi_1 \partial_{-)} \Phi_2 D_+ \Phi_2 D_- \Phi_2 \right\} \quad (5.44)$$

where function $f$ is defined in (5.39). Adding this result to (5.15) and (5.30), the supersymmetric effective theory, up to first order in $g(\tau)$, is

$$L^{(eff)} = \int d\theta^+ d\theta^- \left\{ D_+ \Phi_1 D_- \Phi_1$$

$$+ \frac{1}{3} \frac{f(\Phi_2)}{\omega^5 \Phi_2^5} (\partial_{(+} \Phi_1 \partial_{-)} \Phi_2)^3 D_+ \Phi_2 D_- \Phi_2$$

$$- \frac{f(\Phi_2)}{\omega^5 \Phi_2^5} (\partial_{(+} \Phi_1 \partial_{-)} \Phi_2)^2 \partial_{(+} \Phi_1 \partial_{-)} \Phi_2 D_+ \Phi_2 D_- \Phi_2 \right\} \quad (5.45)$$

where function $f$ is defined in (5.39). Adding this result to (5.15) and (5.30), the supersymmetric effective theory, up to first order in $g(\tau)$, is
\[ +F_1(\Phi_2)D_+\Phi_2D_-\Phi_2 - \frac{1}{\omega^2}F_2(\Phi_2)\partial_-D_+\Phi_2\partial_+D_+\Phi_2 \]
\[ +\frac{f(\Phi_2)}{\omega^3\Phi_2^3}(\partial_+\Phi_1\partial_-\Phi_2\partial_+\Phi_1\partial_-\Phi_2D_+(\Phi_1D_-)\Phi_2 \]
\[ +\frac{1}{3}\frac{f(\Phi_2)}{\omega^5\Phi_2^5}(\partial_+\Phi_1\partial_-\Phi_2)^2D_+\Phi_2D_-\Phi_2 \]
\[ -\frac{f(\Phi_2)}{\omega^3\Phi_2^3}(\partial_+\Phi_1\partial_-\Phi_2)^2\partial_+(\Phi_1D_-)\Phi_2D_+\Phi_2D_-\Phi_2 \} . \quad (5.45) \]

In terms of component fields, this becomes

\[ \mathcal{L}^{(\text{eff})} = +\partial_+\zeta\partial_-\zeta + Z^2 - i\psi_+\partial_-\psi_+ - i\psi_-\partial_+\psi_- \]
\[ +F_1(\alpha)\partial_+\alpha\partial_-\alpha - \frac{1}{\omega^2}(\partial_+\partial_+\alpha)^2 \]
\[ +F_1(\alpha)A^2 - \frac{1}{\omega^2}F_2(\alpha)\partial_+A\partial_-A \]
\[ -iF_1(\alpha)\chi_+\partial_-\chi_+ - \frac{i}{\omega^2}F_2(\alpha)\partial_-\chi_+\partial_+\partial_-\chi_+ \]
\[ -iF_1(\alpha)\chi_-\partial_+\chi_- - \frac{i}{\omega^2}F_2(\alpha)\partial_+\chi_-\partial_-\partial_+\chi_- \]
\[ +\sum_n \mathcal{O}(\alpha^n\chi_+\chi_- + \alpha^{n-1}A\chi_+\chi_-) . \]
\[ -\frac{1}{2}f(\alpha)\zeta'^2\zeta' - \frac{1}{6}f(\alpha)\zeta'^3 \]
\[ +\frac{i}{\sqrt{2}}f(\alpha)\zeta'(\psi_+\dot{\psi_+} + \psi_-\dot{\psi_-}) + \frac{i}{\sqrt{2}}f(\alpha)\zeta(\psi_+\psi'_+ - \psi_-\psi'_-) \]
\[ +\mathcal{O}\left\{ \partial\zeta(\psi_+ + \chi_+ + Z\psi_+ + Z\chi_+ + A\psi_+ + A\psi_+\chi) + \psi\psi_+ + \psi\chi_+ \right\} \quad (5.46) \]

In the above analysis we have implicitly assumed that \( <\alpha>=\exp(-\omega|\tau - \tau_0|) \), \( <\chi_\pm>=<A>=0 \) and \( <Z>=0 \) remain solutions of the equations of motion at the order \( g(\tau) \) level. We now show that this assumption is indeed correct. The \( \alpha \) equation of motion is given by (5.22) modified by terms of order \( A^2 \), order \( \chi^2 \), and of order \( \partial\zeta \). By construction, the functions \( F_1 \) and \( F_2 \) ensure that (5.22) is satisfied by the exponential solution. If \( <A>=<\chi_\pm>=0 \) and if \( <\zeta>=\text{constant} \), then the \( \alpha \) equation is still satisfied by the same exponential. The fields \( \chi_\pm, A \) and \( Z \) all occur at least bilinearly or coupled to \( \partial\zeta \) in the component Lagrangian (5.46). Also, the field \( \zeta \) always appears with derivatives and at least quadratically. It is clear then that the following is a solution to the set of component field
equations derived from (5.46),

\[
< \alpha > = \exp \left( -\omega |\tau - \tau_0| \right)
\]

\[
< \zeta > = \text{constant}
\]

\[
< \chi_\pm > = 0
\]

\[
< \psi_\pm > = 0
\]

\[
< A > = 0
\]

\[
< Z > = 0.
\]

(5.47)

To expose the relation of the component Lagrangian (5.46) to the collective field Lagrangian, we replace the fields \( \alpha, \chi_\pm, \) and \( A \) with the VEV’s listed above. We replace the fields \( \zeta \) and \( \psi_\pm \) with fields shifted around their VEV’s. But since \( < \psi_\pm > = 0 \) and \( < \partial \zeta > = 0 \), the shifted fields appear coupled precisely as do the unshifted fields. For this reason we do not distinguish, notationally, the fields \( \zeta \) and \( \psi_\pm \) in the effective Lagrangian (5.46) from the “shifted” fields appearing in the collective field Lagrangian (4.54). The field \( Z \) is auxiliary. Since it always appears either bilinearly or coupled to \( \chi_\pm \) it follows, since we do not exhibit the fluctuation of \( \chi_\pm \) around its vanishing VEV, that the auxiliary field \( Z \) is of no consequence to the shifted Lagrangian. Implementing the process of replacing fields with VEV’s and shifting \( \zeta \) and \( \psi_\pm \) around their VEV’s, we recover, from the supersymmetric Lagrangian (5.46), the collective field Lagrangian (4.54) to first order in \( g(\tau) \).

c) All orders:

The procedure outlined above may be carried out to all orders in the coupling \( g(\tau) \). It is straightforward to do this, so we will simply quote the result. This is, however, a non-trivial statement. The Lagrangian is highly non-linear and we find it remarkable that it in fact exists. The all orders supersymmetric effective theory is

\[
\mathcal{L}^{(\text{eff})} = \int d\theta_+ d\theta_- \left\{ D_+ \Phi_1 D_- \Phi_1 
\right.
\]

\[
+ F_1(\Phi_2) D_+ \Phi_2 D_- \Phi_2 - \frac{1}{\omega_2^2} F_2(\Phi_2) \partial_- D_+ \Phi_2 \partial_+ D_- \Phi_2
\]

\[
+ \frac{f(\Phi_2)}{\omega_2^2} \frac{\partial_+ (\Phi_1 \partial_- \Phi_2) \partial_+ \Phi_1 \partial_- \Phi_2}{\partial_+ \Phi_1 \partial_- \Phi_2} D_+ D_- \Phi_2
\]

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In terms of component fields, this reads

$$L^{(\text{eff})} = +\partial_+ \zeta \partial_- \zeta + Z^2 - i\psi_+ \partial_+ \psi_+ - i\psi_- \partial_- \psi_-$$

$$+ F_1(\alpha) \partial_+ \alpha \partial_- \alpha - \frac{1}{\omega^2} F_2(\alpha) (\partial_+ \partial_- \alpha)^2$$

$$+ F_1(\alpha) A^2 - \frac{1}{\omega^2} F_2(\alpha) \partial_+ A \partial_- A$$

$$- i F_1(\alpha) \chi_+ \partial_- \chi_+ - \frac{i}{\omega^2} F_2(\alpha) \partial_- \chi_+ \partial_+ \chi_+$$

$$- i F_1(\alpha) \chi_- \partial_+ \chi_- - \frac{i}{\omega^2} F_2(\alpha) \partial_+ \chi_- \partial_- \chi_-$$

$$+ \sum_n O(\alpha^n \chi_+ \chi_- + \alpha^{n-1} A \chi_+ \chi_-).$$

$$- \frac{1}{2} \frac{f(\alpha)}{1 + f(\alpha)} \zeta' \zeta' - \frac{1}{6} f(\alpha) \zeta'^3$$

$$+ \frac{i}{\sqrt{2}} \frac{f(\alpha)}{1 + f(\alpha)} \zeta' (\psi_+ \dot{\psi}_+ + \psi_- \dot{\psi}_-) + \frac{i}{\sqrt{2}} \frac{f(\alpha)}{1 + f(\alpha)} \zeta' (\psi_+ \dot{\psi}_+ + \psi_- \dot{\psi}_-)$$

$$+ O\left\{ \partial_\tau (\psi \chi + \chi \psi + Z \psi \chi + Z \chi \psi + A \psi \psi + A \psi \chi + \psi \chi + \psi \chi \right\} \quad (5.49)$$

The component field equations of motion derived from (5.48) are of the same type as those derived from the first order Lagrangian (5.46) and the solution (5.47) satisfies these equations as well. By replacing all fields by their VEV’s and then shifting \(\zeta\) and \(\psi\) around their VEV’s, we recover, from the supersymmetric Lagrangian (5.48), the \(\tau\)-dependent collective field Lagrangian (4.54). This, then, is the supersymmetric effective theory which we had set out to construct. This is the essential result of this paper.

Since the Lagrangian (5.48) is Poincare invariant and supersymmetric, there exists a more general class of solutions to the field equations than those given in (5.47). The more general solution is given by

$$< \alpha > = \exp \left\{ \omega \left[ |t - t_0| \sinh \theta_0 - |\tau - \tau_0| \cosh \theta_0 \right] \right\}$$

$$< \zeta > = \text{constant}$$

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\(< \chi_\pm > = \eta_0^\pm < \alpha > \\
< \psi_\pm > = 0 \\
< A > = 0 \\
< Z > = 0, \quad (5.50)\)

where \(\theta_0\) is a Lorentz zero mode, \(t_0\) and \(\tau_0\) are translational zero modes, and \(\eta_0^\pm\) are supersymmetric zero modes. The solution (5.47) corresponds to the choice \(\theta_0 = \eta_0^\pm = 0\).

The results derived in this section are sufficiently complicated that, for clarity, we will now recapitulate them. The main result is the (1,1) supersymmetric effective Lagrangian,

\[
\mathcal{L}^{(e_{ff})} = \int d\theta^+d\theta^- \left\{ D_+ \Phi_1 D_+ \Phi_1 + F_1(\Phi_2)D_+ \Phi_2 D_- \Phi_2 - \frac{1}{\omega^2} F_2(\Phi_2) \partial_- D_+ \Phi_2 \partial_+ D_- \Phi_2 - \frac{f(\Phi_2)}{\omega^3 \Phi_2^3} \frac{1 + \frac{f(\Phi_2)}{\omega \Phi_2} \partial_+ \Phi_1 \partial_- \Phi_2}{\partial_+ \Phi_1 \partial_- \Phi_2} D_+ \Phi_2 D_- \Phi_2 + \frac{1}{3} \frac{f(\Phi_2)}{\omega^5 \Phi_2^5} (\partial_+ \Phi_1 \partial_- \Phi_2)^3 D_+ \Phi_2 D_- \Phi_2 - \frac{f(\Phi_2)}{\omega^3 \Phi_2^3} \frac{1 + \frac{f(\Phi_2)}{\omega \Phi_2} \partial_+ \Phi_1 \partial_- \Phi_2}{\partial_+ \Phi_1 \partial_- \Phi_2} D_+ \Phi_2 D_- \Phi_2 \right\}, \quad (5.51)\]

where

\[
F_1(\Phi_2) = -\frac{1}{48\pi\kappa\omega} g^2 \left( \frac{11}{5} \kappa^3 - \frac{28}{3} \kappa^2 + 18 - \frac{5}{3} \kappa \right) \\
F_2(\Phi_2) = -\frac{1}{48\pi\kappa\omega} g^2 \left( -\frac{2}{5} \kappa^3 + \frac{8}{3} \kappa^2 - 12 \kappa + \frac{2}{3} \kappa \right) \\
f(\Phi_2) = 4\sqrt{\pi} g \left( \frac{1}{\kappa} \right) \frac{\Phi_2^2}{(1 - \frac{1}{\kappa} \Phi_2^2)^2}. \quad (5.52)\]

Note that the effective Lagrangian has three independent parameters, \(\omega, g\) and \(\kappa\). The parameter \(\omega\) is a mass, \(g\) is an inverse mass, and \(\kappa\) is a dimensionless number. The equations of motion derived from (5.51) are satisfied by the solution (5.50). This solution is labeled by the translational zero modes \(t_0\) and \(\tau_0\), a Lorentz zero mode \(\theta_0\) and by supersymmetric zero modes \(\eta_0^\pm\). In a preferred frame of reference, \(\theta_0 = \eta_0^+ = \eta_0^- = 0\), and the solution (5.50) becomes equivalent to (5.47). If we substitute the solution (5.47) into (5.51), thus freezing
the “heavy” fields $\alpha, \chi_+$ and $\chi_-$ at their VEV’s, and shift the light fields $\zeta, \psi_-$ and $\psi_+$ around their VEV’s, we recover the collective field Lagrangian (4.54) derived from the $d = 1, \mathcal{N} = \mathfrak{e}$ supersymmetric matrix model. In this expression the coupling parameter $g(\tau) = f(<\alpha>)$ is given by (4.50) and (4.51), and is plotted in figure 4. Notice that this coupling parameter blows up at the boundaries of a region centered at $\tau = \tau_0$ with width $\sigma = \frac{1}{\omega} \ln \kappa$. Outside of this region, the high density collective field theory (4.54) is valid. Within the region $|\tau - \tau_0| < \sigma/2$ or, equivalently, in the region $|x| < 1/(\omega \sqrt{g})$ however, the collective field theory must describe a finite number of eigenvalues. In this case, the appropriate form for the collective field theory is given in (4.14). This Lagrangian is completely equivalent to the original eigenvalue Lagrangian (3.23) or (3.25), which is, in most cases, easier to use. We now turn to a discussion of instanton-like solutions to the Euclidean equations of motion of this low density eigenvalue theory.

6. Eigenvalue Instantons

In this section we will construct solutions to the Euclidean field equations derived from the low density eigenvalue Lagrangian (3.23). The Euclideanized version of Lagrangian (3.23), is given by

$$L_E = \sum_i \left\{ \frac{1}{2} \dot{\lambda}_i^2 + \frac{1}{2} (\partial W_{\text{eff}}/\partial \lambda_i)^2 + \frac{i}{2} \chi_1 \dot{\chi}_1 - \frac{i}{2} \chi_2 \dot{\chi}_2 \right\} + i \sum_{ij} \chi_i \chi_j \frac{\partial^2 W_{\text{eff}}}{\partial \lambda_i \partial \lambda_j},$$

(6.1)

where the dot means differentiation with respect to Euclidean time $\theta$. In this expression,

$$W_{\text{eff}}(\lambda) = W(\lambda) + w(\lambda),$$

(6.2)

where $W(\lambda)$ is the superpotential, and $w(\lambda)$ is the modification, given in (3.19), which results from the restriction of the underlying matrix model to its singlet sector. In the static ground states discussed above no eigenvalues populate the low density region. In this section we will describe additional solutions to the Euclidean field equations in which only one eigenvalue populates the low density region. The modification to the superpotential, $w(\lambda)$, induces only a local inter-eigenvalue force. Therefore, if only a single eigenvalue exists in the low density region, we can neglect $w(\lambda)$ in the Lagrangian. The dynamics of this single eigenvalue and
its fermionic superpartners is then described by the following Euclideanized Lagrangian,

\[ L_E = \frac{1}{2} \dot{\lambda}^2 + \frac{1}{2} (\frac{\partial W}{\partial \lambda})^2 - \frac{i}{2} \chi_1 \dot{\chi}_1 + \frac{i}{2} \chi_2 \dot{\chi}_2 + i \chi_1 \chi_2 \frac{\partial^2 W}{\partial \lambda^2}. \]  \( 6.3 \)

This Lagrangian is symmetric under the Euclidean \( d = 1, \mathcal{N} = 2 \) supersymmetry transformation,

\[
\delta \lambda = i \eta^1 \chi_1 + i \eta^2 \chi_2 \\
\delta \chi_1 = + \eta^1 \dot{\lambda} - \eta^2 W' \\
\delta \chi_2 = - \eta^2 \dot{\lambda} + \eta^1 W' \]  \( 6.4 \)

The Euclidean field equations derived from (6.3) are

\[ \ddot{\lambda} - W' W'' = 0 \]
\[ \dot{\chi}_1 - W'' \chi_2 = 0 \]
\[ \dot{\chi}_2 - W'' \chi_1 = 0, \]  \( 6.5 \)

where \( W' = \partial W/\partial \lambda \) and \( W'' = \partial^2 W/\partial \lambda^2 \). Now, recall that the superpotential depends on \( N \), the total number of eigenvalues in the matrix model. Specifically, from (4.22), we have

\[ W(\lambda) = N c_0 + \sqrt{N} c_1 \lambda + \frac{1}{6} \sqrt{\frac{c_3}{N}} \lambda^3, \]  \( 6.6 \)

where \( c_0, c_1 \) and \( c_3 \) are arbitrary finite constants. It follows that (6.5) become

\[ \ddot{\lambda} + \omega^2 = \frac{1}{2} a^2 \lambda^3 \]
\[ \dot{\chi}_1 = a \lambda \chi_2 \]
\[ \dot{\chi}_2 = a \lambda \chi_1, \]  \( 6.7 \)

where \( \omega^2 = - c_1 c_3 > 0 \) and \( a = c_3 / \sqrt{N} \). Since \( N \) is very large, \( a \) is very small. Thus, for finite \( N \), the field equations (6.7) are only slight perturbations from the following system,

\[ \ddot{\lambda} + \omega^2 \lambda = 0 \]
\[ \dot{\chi}_1 = 0 \]
\[ \dot{\chi}_2 = 0. \]  \( 6.8 \)
In the limit $N \to \infty$, these equations become exact. The general solution to the single-eigenvalue Euclidean field equations in the large $N$ limit is then

$$
\lambda^* = A \sin \{\omega (\theta - \theta_0)\} + B \cos \{\omega (\theta - \theta_0)\}
$$

$$
\chi_1^* = -\eta_{10}
$$

$$
\chi_2^* = -\eta_{20},
$$

(6.9)

where $A, B$ and $\theta_0$ are real commuting constants and $\eta_{10}$ and $\eta_{20}$ are real anticommuting constants. We want to consider, for reasons to be discussed below, solutions, which we denote by $\lambda^{(+)}$ and $\chi^{(+)}_{1,2}$, satisfying the following boundary condition,

$$
\lambda^{(+)}|_{\theta = \theta_0 \mp \pi/(2\omega)} = \mp \frac{1}{\omega \sqrt{g}}
$$

$$
\dot{\lambda}^{(+)}|_{\theta = \theta_0 \mp \pi/(2\omega)} = 0.
$$

(6.10)

It follows from (6.3) that

$$
\lambda^{(+)} = \frac{-1}{\omega \sqrt{g}} \sin \{\omega (\theta - \theta_0^{(+)})\}
$$

$$
\chi^{(+)}_{1} = -\eta_{10}^{(+)}
$$

$$
\chi^{(+)}_{2} = -\eta_{20}^{(+)}.
$$

(6.11)

Similarly, we consider solutions $\lambda^{(-)}$ and $\chi^{(-)}_{1,2}$ which satisfy the boundary condition,

$$
\lambda^{(-)}|_{\theta = \theta_0 \mp \pi/(2\omega)} = \pm \frac{1}{\omega \sqrt{g}}
$$

$$
\dot{\lambda}^{(-)}|_{\theta = \theta_0 \mp \pi/(2\omega)} = 0.
$$

(6.12)

Thus,

$$
\lambda^{(-)} = \frac{1}{\omega \sqrt{g}} \sin \{\omega (\theta - \theta_0^{(-)})\}
$$

$$
\chi^{(-)}_{1} = -\eta_{10}^{(-)}
$$

$$
\chi^{(-)}_{2} = -\eta_{20}^{(-)}.
$$

(6.13)

The parameters $\theta_0^{(\pm)}$, $\eta_{10}^{(\pm)}$ and $\eta_{20}^{(\pm)}$ are zero-modes associated with these solutions. It is very enlightening to reexpress these results in the language of collective field theory. To do this,
we use the definitions (4.1), which for the single eigenvalue case in Euclidean time are

\[
\varphi(x, \theta) = \Theta(x - \lambda(\theta))
\]

\[
\psi_1(x, \theta) = \delta(x - \lambda(\theta)) \chi_1(\theta)
\]

\[
\psi_2(x, \theta) = \delta(x - \lambda(\theta)) \chi_2(\theta).
\]

(6.14)

Recall that in the high density region we let \( \varphi = \tilde{\varphi}_0(x) + \frac{1}{\sqrt{\pi}} \zeta \), where \( \tilde{\varphi}_0(x) \) was the vacuum solution given in (4.41). Here, in the low density region, we will also express \( \varphi \) as \( \varphi = \tilde{\varphi}_0(x) + \frac{1}{\sqrt{\pi}} \zeta \). Now, however, \( \tilde{\varphi}_0(x) = 0 \) and, hence, \( \varphi = \frac{1}{\sqrt{\pi}} \zeta \). Substituting solutions (6.11) and (6.13) into (6.14) yields the collective field theory vacuum configurations

\[
\zeta^{(\pm)}(x, \theta) = \sqrt{\pi} \Theta \left( x \mp \frac{1}{\omega \sqrt{g}} \sin[\omega(\theta - \theta_0^{(\pm)})] \right)
\]

\[
\psi_1^{(\pm)}(x, \theta) = \delta \left( x \mp \frac{1}{\omega \sqrt{g}} \sin[\omega(\theta - \theta_0^{(\pm)})] \right) \eta_{10}^{(\pm)}
\]

\[
\psi_2^{(\pm)}(x, \theta) = \delta \left( x \mp \frac{1}{\omega \sqrt{g}} \sin[\omega(\theta - \theta_0^{(\pm)})] \right) \eta_{20}^{(\pm)}.
\]

(6.15)

These expressions are valid over the region \(-1/(\omega \sqrt{g}) < x < +1/(\omega \sqrt{g})\). Outside of this region, for \(|x| \geq 1/(\omega \sqrt{g})\) (or, equivalently, for \(|\tau| \geq \sigma/2\), the high density collective field theory is valid and the vacuum solution is given by (5.47). Thus, for \(|x| \geq 1/(\omega \sqrt{g})\), we take \( \zeta^{(\pm)} = \text{constant} \) and \( \psi_1^{(\pm)} = \psi_2^{(\pm)} = 0 \). We can then match these vacuum solutions at both \( x = -1/(\omega \sqrt{g}) \) and \( x = +1/(\omega \sqrt{g}) \), and therefore extend the configurations (6.13) over all of space. This requires that, for \( x < -1/(\omega \sqrt{g}) \), we choose \( \zeta^{(\pm)} = \psi_1^{(\pm)} = \psi_2^{(\pm)} = 0 \), and, for \( x > +1/(\omega \sqrt{g}) \), we choose \( \zeta^{(\pm)} = \sqrt{\pi} \) and \( \psi_1^{(\pm)} = \psi_2^{(\pm)} = 0 \). The \((+)\) configurations are depicted in figure 5. The configurations \( \zeta^{(\pm)} \) are kinks which move across the low density region in Euclidean time. Since they are kinks, these configurations are topologically stable. The \( \zeta^{(\pm)} \) describe an eigenvalue moving in Euclidean time from \( x = \mp 1/\sqrt{\omega^2 g} \) at \( \theta = \theta_0 - \pi/(2\omega) \) to \( x = \pm 1/\sqrt{\omega^2 g} \) at \( \theta = \theta_0 + \pi/(2\omega) \). This represents a quantum mechanical tunneling of an eigenvalue across the low density region. For the cases where \( \eta_{10}^{(\pm)} \neq 0 \) or \( \eta_{20}^{(\pm)} \neq 0 \) the eigenvalues are accompanied by fermions which also tunnel across the low density region at the same time. Notice that configurations with a superscript \((+)\) represent tunneling from left to right and that configurations with a superscript \((-)\) represent tunneling from right to left.
Now recall from (4.47) that in the low density region $|x| < 1/(\omega \sqrt{g})$, we have $x = \frac{1}{\omega \sqrt{g}} \sin\{\frac{\pi}{\sigma} (\tau - \tau_0)\}$. Using this transformation, we can represent the above vacuum in $\tau$ space. Thus,

$$\zeta^{(\pm)}(\tau, \theta) = \sqrt{\pi} \Theta \left( \frac{1}{\omega \sqrt{g}} [\sin \frac{\pi}{\sigma} (\tau - \tau_0) \mp \sin(\theta - \theta_0^{(\pm)})] \right)$$

$$\psi_1^{(\pm)}(\tau, \theta) = \delta \left( \frac{1}{\omega \sqrt{g}} [\sin \frac{\pi}{\sigma} (\tau - \tau_0) \mp \sin(\theta - \theta_0^{(\pm)})] \right) \eta_{10}^{(\pm)}$$

$$\psi_2^{(\pm)}(\tau, \theta) = \delta \left( \frac{1}{\omega \sqrt{g}} [\sin \frac{\pi}{\sigma} (\tau - \tau_0) \mp \sin(\theta - \theta_0^{(\pm)})] \right) \eta_{20}^{(\pm)}.$$ (6.16)

These expressions are valid for $(\tau_0 - \sigma/2) < \tau < (\tau_0 + \sigma/2)$. As discussed above, outside this region, for $\tau < (\tau_0 - \sigma/2)$, we take $\zeta^{(\pm)} = \psi_1^{(\pm)} = \psi_2^{(\pm)} = 0$, and for $\tau > (\tau_0 + \sigma/2)$, we take $\zeta^{(\pm)} = \sqrt{\pi}$ and $\psi_1^{(\pm)} = \psi_2^{(\pm)} = 0$. The instanton background is thus defined over all of $(\theta, \tau)$ space. One eigenvalue instantons were also discussed in a different context in [16].

It is tempting to conjecture that the instantons described above actually break the two-dimensional (1,1) supersymmetry of the effective action. The reasons for this speculation are the following. The quantum mechanical instantons are expected to break the $d = 1, \mathcal{N} = \in$ supersymmetry. This is a known phenomenon in supersymmetric quantum mechanics[17, 18]. The $d = 2, (1,1)$ supersymmetry is apparently a consequence of the $d = 1, \mathcal{N} = \in$ supersymmetry. The exact connection between the two supersymmetries is not well understood, however. It is thus reasonable to assume that, if the underlying $d = 1, \mathcal{N} = \in$ is broken, so will be the $d = 2, (1,1)$ supersymmetry of the effective theory.

The question of whether supersymmetry breaking indeed occurs in the effective theory and whether it is due to single eigenvalue instantons is currently under active investigation[19].

7. Conclusions

We have derived a two-dimensional (1,1) supersymmetric effective Lagrangian which reduces to the collective field Lagrangian describing the most general $d = 1, \mathcal{N} = \in$ supersymmetric matrix model in the large $N$ limit, when certain “heavy” fields are frozen at their
VEV’s. Additionally, we have shown that the dynamics of the light fields in the effective theory include a space-dependent coupling parameter which blows up at finite points and delineates a special zone in which quantum mechanical, not field theoretical, considerations need to be incorporated into the physical picture of the system. The quantum mechanical aspect of the effective field theory relates to individual eigenvalue dynamics of the underlying matrix model. It is a “stringy” aspect of the effective theory. We have also indicated how these eigenvalue instantons might induce supersymmetry breaking in the effective field theory.

Appendix A: Derivation of the Effective Singlet Sector Lagrangian for the Quantum Bosonic Matrix Model

In this Appendix we provide a discussion of some results cited in section 2. Specifically, what follows is a detailed calculation of the effective $U(N)$ singlet sector Lagrangian relevant to the quantum mechanical bosonic matrix model.

The Hamiltonian for a bosonic matrix model is

$$H = \frac{1}{2} Tr \Pi_M^2 + V(M), \quad (A.1)$$

where

$$V(M) = \sum_n a_n Tr M^n \quad (A.2)$$

and $M$ is a time dependent Hermitian $N \times N$ matrix. Canonically quantizing, we replace $M_{ij}$ with the operator $\hat{M}_{ij}$ and $\Pi_{M,ij}$ with $\hat{\Pi}_{M,ij}$. We work in the $M$ basis, where $\hat{\Pi}_{M,ij} = -i \partial / \partial M_{ij}$. We would like to express the quantum operator Hamiltonian, $\hat{H}$, in terms of the matrix eigenvalues, $\lambda_i$ and their conjugate momenta, $\hat{\Pi}_{\lambda_i} = -i \partial / \partial \lambda_i$. Toward this goal we embark on the following discussion. Given a unitary matrix, $U_{ij}$,

$$\sum_k U_{ik}^\dagger U_{kj} = \delta_{ij}, \quad (A.3)$$

we have the relation

$$\frac{\partial U_{ij}^\dagger}{\partial U_{kl}} = -U_{ik}^\dagger U_{lj}^\dagger. \quad (A.4)$$
If $M_{ij}$ is a Hermitian matrix, then there exists a $U_{ij}$ such that the unitary transformation,

$$M_{ij} = \sum_k U_{ik}^\dagger \lambda_k U_{kj},$$

(A.5)

relates $M_{ij}$ to the the diagonal matrix $\lambda_{ij} = \lambda_i \delta_{ij}$ which consists of the eigenvalues of $M_{ij}$.

Given (A.4) and (A.3), it it readily found that

$$\{ \frac{\partial M_{ij}}{\partial \lambda_l} \}^{-1} = U_{jl}^\dagger U_{li}$$

(A.6)

and

$$\{ \frac{\partial M_{ij}}{\partial U_{ls}} \}^{-1} = \sum_{k \neq l} \frac{U_{jk}^\dagger U_{ks} U_{li}}{\lambda_l - \lambda_k}.$$  

(A.7)

Now, using the chain rule,

$$\frac{\partial}{\partial M_{ij}} = \sum_l \{ \frac{\partial M_{ij}}{\partial \lambda_l} \}^{-1} \frac{\partial}{\partial \lambda_l} + \sum_{ls} \{ \frac{\partial M_{ij}}{\partial U_{ls}} \}^{-1} \frac{\partial}{\partial U_{ls}}$$

(A.8)

$$= \sum_l U_{jl}^\dagger U_{li} \frac{\partial}{\partial \lambda_l} + \sum_{ls} \sum_{k \neq l} \frac{U_{jk}^\dagger U_{ks} U_{li}}{\lambda_l - \lambda_k} \frac{\partial}{\partial U_{ls}}.$$  

(A.9)

With (A.9), it is straightforward, if tedious, using (A.3) and (A.4), to show that

$$\sum_{ij} \sum_j \frac{\partial}{\partial M_{ij}} \frac{\partial}{\partial M_{ji}} = \sum_l \left\{ \frac{\partial^2}{\partial \lambda_l^2} + 2 \sum_{j \neq i} \frac{1}{\lambda_j - \lambda_i} \frac{\partial}{\partial \lambda_j} \right\}$$

$$- \sum_i \sum_{j \neq i} \frac{1}{(\lambda_i - \lambda_j)^2} \left\{ \sum_s U_{is} \frac{\partial}{\partial U_{is}} + \sum_s \sum_t U_{is} U_{jt} \frac{\partial}{\partial U_{is}} \frac{\partial}{\partial U_{jt}} \right\}.$$  

(A.10)

This relation also holds when $M, \lambda$ and $U$ are replaced with quantum operators $\hat{M}, \hat{\lambda}$ and $\hat{U}$. Our interest is in the quantum theory. Henceforth, we restrict our attention to quantum states which have no $U_{ij}$ dependance. This subspace of the Hilbert space consists of $U(N)$ singlets. The last two terms in (A.10) annihilate this subspace and can therefore be ignored.

We have the relations, $\hat{\Pi}_{M_{ij}} = -i \frac{\partial}{\partial M_{ij}}$ and $\hat{\Pi}_{\lambda_i} = -i \frac{\partial}{\partial \lambda_i}$, so that (A.10) can now be rewritten

$$\sum_{ij} \frac{1}{2} \hat{\Pi}_{M_{ij}} \hat{\Pi}_{M_{ji}} = \sum_l \left\{ \frac{\hat{\Pi}_\lambda^2}{2} - i \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \hat{\Pi}_\lambda \right\}.$$  

(A.11)

Also, somewhat trivially,

$$V(\hat{M}) = \sum_n a_n Tr M^n$$  

(A.12)
\[ n \sum a_n \sum \lambda_i^n = \sum_i V(\lambda_i). \quad (A.13) \]

Thus, given \((A.1)\), \((A.11)\), and \((A.14)\), the quantum operator Hamiltonian relevant to the \(U(N)\) singlet sector of the matrix model is

\[ \hat{H}_S = \sum_i \left\{ \frac{1}{2} \hat{\Pi}_{\lambda_i}^2 - i \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} \hat{\Pi}_{\lambda_i} + V(\lambda_i) \right\} \quad (A.15) \]

The partition function is given by

\[ Z_N(a_n) = \int [d\Pi_\lambda] [d\lambda] \exp i \int dt \sum_i \left\{ \Pi_{\lambda_i} \dot{\lambda}_i - H \right\} \quad (A.16) \]

\[ = \int [d\Pi_\lambda] [d\lambda] \exp i \int dt \sum_i \left\{ -\frac{1}{2} \Pi_{\lambda_i}^2 + (\dot{\lambda}_i + i \sum_{j \neq i} \frac{1}{\lambda_j - \lambda_i}) \Pi_{\lambda_i} - V(\lambda_i) \right\} \quad (A.17) \]

The \([d\Pi_\lambda]\) integration is gaussian and is easily performed. The result, ignoring an irrelevant constant prefactor, is

\[ Z_N(a_n) = \int [d\lambda] \exp i \int dt \sum_i \left\{ \frac{1}{2} (\dot{\lambda}_i + i \sum_{j \neq i} \frac{1}{\lambda_j - \lambda_i})^2 - V(\lambda_i) \right\} \quad (A.18) \]

\[ = \int [d\lambda] \exp i \int dt \sum_i \left\{ \frac{1}{2} \dot{\lambda}_i^2 - \frac{1}{2} \left( \sum_{j \neq i} \frac{1}{\lambda_j - \lambda_i} \right)^2 - V(\lambda_i) \right\} \quad (A.19) \]

\[ \equiv \int [d\lambda] \exp i \int dt L_S(\dot{\lambda}, \lambda). \quad (A.20) \]

In passing from \((A.18)\) to \((A.19)\), we drop the cross term because

\[ \int dt \sum_i \sum_{j \neq i} \frac{1}{\lambda_j - \lambda_i} \dot{\lambda}_i = - \int dt \sum_{j \neq i} \sum_{i \neq j} \frac{\dot{\lambda}_i - \dot{\lambda}_j}{\lambda_i - \lambda_j} \]

\[ = - \sum_{j \neq i} \sum_i \int dt \partial_t \ln(\lambda_i - \lambda_j) \]

\[ = 0. \quad (A.23) \]

Thus, the \(U(N)\) singlet sector of a quantum mechanical bosonic matrix model is governed by an effective lagrangian

\[ L_S(\dot{\lambda}, \lambda) = \sum_i \left\{ \frac{1}{2} \dot{\lambda}_i^2 - V(\lambda_i) - \frac{1}{2} \left( \sum_{j \neq i} \frac{1}{\lambda_j - \lambda_i} \right)^2 \right\}. \quad (A.24) \]

This is the result cited in section 2.
Appendix B: Derivation of the Supersymmetric Quantum Mechanics as a Subsector of the Supersymmetric Quantum Matrix Model

In this Appendix we provide a discussion of results cited in section 3. Specifically, what follows is a detailed extraction of the supersymmetric quantum mechanics as a subspace of the full supersymmetric quantum matrix model.

The Hamiltonian for an $N = 2$ supersymmetric matrix model is

$$H = \sum_{ij} \left\{ \frac{1}{2} \Pi_{Mij} \Pi_{Mij} + \frac{1}{2} \frac{\partial W(M)}{\partial M_{ij}} \frac{\partial W(M)}{\partial M_{ij}} \right\} + \frac{i}{2} \sum_{ijkl} [\hat{\Psi}_{ij}, \Psi_{kl}] \frac{\partial^2 W(M)}{\partial M_{ij} \partial M_{kl}}, \quad (B.1)$$

where

$$W(M) = \sum_n b_n Tr M^n, \quad (B.2)$$

$M$ is a time-dependent commuting $N \times N$ Hermitian matrix, $\Psi$ is an anticommuting $N \times N$ Hermitian matrix, and $\hat{\Psi}$ is the Hermitian conjugate of $\Psi$. Canonically quantizing, we replace $M_{ij}$ with the operator $\hat{M}_{ij}$, $\Pi_{Mij}$ with $\hat{\Pi}_{Mij}$, $\Psi_{ij}$ with $\hat{\Psi}_{ij}$ and $\bar{\Psi}_{ij}$ with $\hat{\bar{\Psi}}$. We also impose the following relations

$$[\hat{\Pi}_{Mij}, \hat{M}_{kl}] = -i \delta_{ik} \delta_{jl}$$

$$\{\hat{\Psi}_{ij}, \hat{\Psi}_{kl}\} = \delta_{ik} \delta_{jl}. \quad (B.3)$$

We henceforth work in the $M$ basis, where $\hat{\Pi}_{Mij} = -i \partial / \partial M_{ij}$. The operators $\Psi$ and $\bar{\Psi}$ are annihilation and creation operators for fermions. We parameterize $M_{ij}$ as follows,

$$M_{ij} = \sum_k U_{ik}^\dagger \lambda_k U_{kj}, \quad (B.4)$$

where $\lambda_k$ are the eigenvalues of $M_{ij}$ and $U_{ij}$ is a unitary matrix. This is always possible since $M$ is Hermitian. We use the same matrix $U$ to define a “rotated” fermion matrix, $\chi_{ij}$,

$$\Psi_{ij} = \sum_{kl} U_{kl}^\dagger \chi_{kl} U_{ij}. \quad (B.5)$$

Using the chain rule, as discussed in Appendix A, it follows that

$$\frac{\partial}{\partial M_{ij}} = \sum_l U_{jl}^\dagger U_{li} \frac{\partial}{\partial \lambda_l} + \sum_{ls} \sum_{k \neq l} \frac{U_{jk}^\dagger U_{ks} U_{li}}{\lambda_l - \lambda_k} \frac{\partial}{\partial U_{ls}}. \quad (B.6)$$
It is then straightforward to demonstrate that
\[
\frac{\partial}{\partial M_{ij}} \frac{\partial}{\partial M_{lk}} = \sum_{ab} (U_{ai} U_{aj}^\dagger)(U_{bl} U_{kb}^\dagger) \frac{\partial}{\partial \lambda_a} \frac{\partial}{\partial \lambda_b} \\
+ 2 \sum_a \sum_{b \neq a} (U_{ai} U_{aj}^\dagger)(U_{bl} U_{kb}^\dagger) \frac{1}{\lambda_a - \lambda_b} \frac{\partial}{\partial \lambda_a} \\
+ \mathcal{O}(\frac{\partial}{\partial U}).
\]  
(B.7)

We work with a restricted Hilbert space consisting only of states which are annihilated by \(\partial/\partial U\). We therefore disregard the last term in (B.7). Note that (A.10) is recovered when (B.7) is acted on with \(\sum_l \sum_k \delta_{il} \delta_{jk}\), as expected. It is useful to define a function
\[
w = -\sum_i \sum_{j \neq i} \ln |\lambda_i - \lambda_j|
\]  
(B.8)

which has the following properties,
\[
\frac{\partial w}{\partial \lambda_i} = -\sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}
\]  
(B.9)

and
\[
\frac{\partial^2 w}{\partial \lambda_m \partial \lambda_n} = \begin{cases} 
\sum_{k \neq i} 1/(\lambda_i - \lambda_k)^2 & \text{; } m = n \\
-1/(\lambda_m - \lambda_n)^2 & \text{; } m \neq n
\end{cases}
\]  
(B.10)

The first two terms in (B.1) are identical to the Hamiltonian treated in Appendix A. To connect with the notation used in Appendix A, we define
\[
V(M) = \frac{1}{2} Tr\left(\frac{\partial W(M)}{\partial M}\right)^2.
\]  
(B.11)

The result, (A.15), is directly applicable. Using (B.10) and (B.11), and noting that \(\hat{\Pi}_{\lambda_i} = -i\partial/\partial \lambda_i\), (A.13) and, hence, the first two terms of (B.1) can be written as follows,
\[
\sum_{ij} \left\{ \frac{1}{2} \hat{\Pi}_{M_{ij}} \hat{\Pi}_{M_{ji}} + \frac{1}{2} \frac{\partial W(M)}{\partial M_{ij}} \frac{\partial W(M)}{\partial M_{ji}} \right\} \\
= \sum_i \left\{ \frac{1}{2} \hat{\Pi}_{\lambda_i}^2 + i \frac{\partial w}{\partial \lambda_i} \hat{\Pi}_{\lambda_i} + \frac{1}{2} \left( \frac{\partial W(\lambda_i)}{\partial \lambda_i} \right)^2 \right\}.
\]  
(B.12)
We now concentrate on the last term in (B.1). Using the relation (B.3) it is easily seen that
\[
\sum_{ijkl} \bar{\Psi}_{ij} \Psi_{kl} \frac{\partial^2 W(M)}{\partial M_{ij} \partial M_{kl}} = -\frac{1}{2} \sum_{ij} \frac{\partial^2 W(M)}{\partial M_{ij} \partial M_{ji}} + \sum_{ijkl} \bar{\Psi}_{ij} \Psi_{kl} \frac{\partial^2 W(M)}{\partial M_{ij} \partial M_{kl}}.
\] (B.13)

Now, using (B.7) and (B.5) it is straightforward to show the following,
\[
\sum_{ij} \frac{\partial^2 W(M)}{\partial M_{ij} \partial M_{ji}} = \sum_i \left\{ \frac{\partial^2 W(\lambda)}{\partial \lambda_i^2} - 2 \frac{\partial w(\lambda)}{\partial \lambda_i} \frac{\partial W(\lambda)}{\partial \lambda_i} \right\} + 2 \sum_i \sum_{j \neq i} \bar{\chi}_{ij} \chi_{ji} \frac{1}{\lambda_i - \lambda_j} \frac{\partial W(\lambda)}{\partial \lambda_i}.
\] (B.14)

We now further restrict the Hilbert space to include only those states \(|S\rangle\) which are annihilated by “off-diagonal” fermionic creation operators, \(\chi_{ij}\), where \(i \neq j\). The last term in (B.14) then annihilates this subspace of states and can be neglected. We abbreviate the diagonal fermions, \(\chi_{ii}\), by denoting them \(\chi_i\). Using the quantization condition, \(\{ \bar{\chi}_i, \chi_j \} = \delta_{ij}\), it is now straightforward to show that
\[
\frac{1}{2} \sum_{ijkl} \bar{\Psi}_{ij} \Psi_{kl} \frac{\partial^2 W(M)}{\partial M_{ij} \partial M_{kl}} = \sum_i \frac{\partial w(\lambda)}{\partial \lambda_i} \frac{\partial W(\lambda)}{\partial \lambda_i} + \frac{1}{2} \sum_{i} [\bar{\chi}_i, \chi_j] \frac{\partial^2 W(\lambda)}{\partial \lambda_i \partial \lambda_j}.
\] (B.15)

Immediately, by combining (B.1) and (B.12), we find that
\[
\hat{H}_S = \frac{1}{2} \hat{\Pi}_i^2 + \frac{\partial w}{\partial \lambda_i} \hat{\Pi}_i + \frac{1}{2} \left( \frac{\partial W}{\partial \lambda_i} \right)^2 + \frac{\partial w}{\partial \lambda_i} \frac{\partial W}{\partial \lambda_i} + \frac{1}{2} \sum_{ij} [\bar{\chi}_i, \chi_j] \frac{\partial^2 W(\lambda)}{\partial \lambda_i \partial \lambda_j},
\] (B.16)

where the subscript \(S\) indicates the restriction to the singlet, fermion diagonal subspace of states. Over the singlet sector, the partition function is
\[
Z_N(b_n) = \int [d\Pi_\lambda][d\lambda][d\chi][d\bar{\chi}] \exp i \int dt \sum_{ij} \bar{\chi}_i \chi_j \frac{\partial^2 w}{\partial \lambda_i \partial \lambda_j} \times \exp i \int dt \sum_i \{ \Pi_\lambda \dot{\lambda}_i - i \bar{\chi}_i \dot{\chi}_i - H_S \}
\] (B.17)
In this expression, the first exponential factor is a Jacobian associated with the parameterization of the fermion fields. It is necessary because the measure on the Hilbert space becomes nontrivial when we restrict to the “diagonal” states, \( \chi_i \), which is essentially a choice of curvilinear coordinates in functional space. Inserting (B.16) and rearranging, this partition function can be expressed as

\[
Z_N(b_n) = \int [d\Pi_\lambda][d\lambda][d\chi][d\bar{\chi}]
\times \exp i \int dt \left\{ \frac{1}{2} \Pi_{\lambda_i}^2 + (\dot{\lambda}_i + i\frac{\partial w}{\partial \lambda_i})\Pi_{\lambda_i} \right\}
\times \exp i \int dt \left\{ \sum_i \left[ -\frac{1}{2} \left( \frac{\partial W}{\partial \lambda_i} \right)^2 - \frac{\partial w}{\partial \lambda_i} \frac{\partial W}{\partial \lambda_i} - i\bar{\chi}_i \chi_i \right] + \sum_{ij} \bar{\chi}_i \chi_j \frac{\partial^2 (W + w)}{\partial \lambda_i \partial \lambda_j} \right\}.
\]  

The Gaussian \([d\Pi_\lambda]\) integration is straightforward and the details are the same as those described in Appendix A. Performing the \([d\Pi_\lambda]\) integration, it is readily found that

\[
Z_N(b_n) = \int [d\lambda][d\chi][d\bar{\chi}] \exp i \int dt L_S,
\]  

where

\[
L_S = \sum_i \left\{ \frac{1}{2} \dot{\lambda}_i^2 - \frac{1}{2} \left( \frac{\partial^2 (W + w)}{\partial \lambda_i^2} \right)^2 - i(\bar{\chi}_i \dot{\chi}_i - \bar{\chi}_i \dot{\chi}_i) \right\} - \sum_{ij} \bar{\chi}_i \chi_j \frac{\partial^2 (W + w)}{\partial \lambda_i \partial \lambda_j}.
\]  

This is the result cited in section 3.

**Appendix C: Proper Implementation of the Collective Field Constraint Condition**

In this Appendix we discuss a technical issue associated with the proper implementation of constraints when constructing the high density collective field theory. This issue is relevant to subsections 4.3 and 4.4. The field equations shown in (4.38) were derived from the Lagrangian (4.37) with \( \Lambda = 0 \) by the usual variation method. At the end of subsection 4.3 we made the assertion that this procedure gives rise to the correct canonical theory and could be used with impunity. We proceed to prove this.
It follows from the definitions (4.1), that the field $\varphi$ in equation (4.37) must satisfy the following constraint equation,

$$\int \varphi'(x) dx = N, \quad (C.1)$$

even when the large $N$ limit is taken. It can be shown that this is the only constraint which the high density fields are required to satisfy. When varying the Lagrangian (4.37) to derive field equations, this constraint must be accounted for. A powerful way to implement the constraint is to amend the Lagrangian by the addition of a Lagrange multiplier term. In this case, the purely bosonic part of the collective field Lagrangian becomes

$$L_B = \int dx\left\{\frac{\dot{\varphi}^2}{2\varphi'} - \frac{\pi^2}{6}\varphi'^3 + \frac{1}{2}(\omega^2 x^2 - \Lambda)\varphi' + \mu(\varphi' - \frac{N}{L})\right\}, \quad (C.2)$$

where $\mu$ is the Lagrange multiplier. The fermionic parts of the Lagrangian and the fermionic field equations are unaffected by this concern and we omit them from this discussion. We then vary the Lagrangian (C.2) with respect to both $\varphi$, which is now unconstrained, and to $\mu$. This gives us a coupled system of equations which determine both the stable configuration, $\tilde{\varphi}_0(x)$, which will depend on $\mu$, as well as a relation between $\mu$, $N$ and $L$. Doing this, we find the $\varphi$ equation and the $\mu$ equation, respectively, to be

$$\partial_t(\dot{\varphi} - \frac{1}{2}\partial_x(\frac{\dot{\varphi}^2}{\varphi^2} + \pi^2\varphi'^2 - \omega^2x^2 + \Lambda - 2\mu)) = 0$$

$$\int dx\varphi'(x) = N. \quad (C.3)$$

The first of these equations is solved, for the static case $\dot{\varphi} = 0$, by the following expression,

$$\tilde{\varphi}_0' = \frac{1}{\pi}\sqrt{\omega^2x^2 - \frac{1}{g}}, \quad (C.4)$$

where $\frac{1}{g} = 2\mu - \Lambda + C$ and $C$ is an arbitrary integration constant. There are two possibilities. Either $g > 0$ or $g < 0$. We will consider each of these cases independently.

a) $g > 0$: In this case, $\tilde{\varphi}_0'$ is only defined for $(\omega\sqrt{g})^{-1} \leq |x| \leq L/2$. The second equation in (C.3) then requires that

$$\frac{2}{\pi}\int_{(\omega\sqrt{g})^{-1}}^{L/2} dx\sqrt{\omega^2x^2 - \frac{1}{g}} = N. \quad (C.5)$$
Integrating and performing some algebra, this equation becomes

\[ N = \frac{1}{2\pi\omega} \left\{ \frac{1}{2} L^2 \omega^2 - \frac{1}{g} \ln\left(\frac{L^2 \omega^2}{g}\right) - \frac{1}{g} \right\} + \mathcal{O}\left(\frac{1}{L^2}\right). \]  

(C.6)

b) \( g < 0 \): In this case, \( \varphi' \) is defined for all \( |x| \leq L/2 \). The second equation of (C.3) then requires that

\[ \frac{2}{\pi} \int_0^{L/2} dx \sqrt{\omega^2 x^2 - \frac{1}{g}} = N. \]  

(C.7)

Integrating and performing some algebra, this equation becomes

\[ N = \frac{1}{2\pi\omega} \left\{ \frac{1}{2} L^2 \omega^2 - \frac{1}{g} \ln\left(\frac{L^2 \omega^2}{|g|}\right) - \frac{1}{g} \right\} + \mathcal{O}\left(\frac{1}{L^2}\right). \]  

(C.8)

Combining the above results we see that, regardless of the sign of \( g \), the constraint equation (C.1) is embodied completely in the following relation,

\[ N = \frac{1}{2\pi\omega} \left\{ \frac{1}{2} L^2 \omega^2 - \frac{1}{g} \ln\left(\frac{L^2 \omega^2}{|g|}\right) - \frac{1}{g} \right\} + \mathcal{O}\left(\frac{1}{L^2}\right). \]  

(C.9)

Now, recall that it was necessary to specify the \( N \) dependence of the coefficients of the superpotential for large \( N \). One result of this is that \( \Lambda = \frac{1}{2} N c_1^2 \). In much the same manner, we must now specify the large \( N \) behavior of the new coefficients \( \mu \) and \( C \). The appropriate choice is

\[ 2\mu + C = \frac{1}{2} N c_1^2. \]  

(C.10)

It follows that, for large \( N \), \( \frac{1}{g} \) is a finite constant. With this in mind, let us analyze (C.9) in the limit of large \( N \) and large \( L \). Since \( \frac{1}{g} \) is a constant in this limit, it is clear that this equation simplifies to

\[ N = \frac{\omega L^2}{4\pi}. \]  

(C.11)

That is, not only do we take the \( N \to \infty, L \to \infty \) limit but we must do so in such a way that (C.11) is satisfied. Note that, since \( N/L \propto L \to \infty \), this condition is compatible with the high density eigenvalue condition. The result of all this is, in fact, very simple. It implies that (C.11), with an arbitrary constant \( \frac{1}{g} \), is the solution of the \( \varphi \) equation of motion in the appropriate \( N \to \infty, L \to \infty \) limit. Note that this is exactly the result that would have been obtained from the high density Lagrangian if we simply took \( \Lambda = 0 \), treated \( \varphi \) as an
unconstrained field and ignored the question of the constraint (1.33). In this case $\frac{1}{g}$ would arise as an arbitrary integration constant. This justifies the statements made in subsection 4.3.

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