On the Nonexpansive Operators Based on Arbitrary Metric: A Degenerate Analysis

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Abstract. We in this paper study the nonexpansive operators equipped with arbitrary metric and investigate the connections between firm nonexpansiveness, cocoerciveness and averagedness. The convergence of the associated fixed-point iterations is discussed with particular focus on the case of degenerate metric, since the degeneracy is often encountered when reformulating many existing first-order operator splitting algorithms as a metric resolvent. This work paves a way for analyzing the generalized proximal point algorithm with a non-trivial relaxation step and degenerate metric.

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1. Introduction

1.1. Nonexpansive Operators

The nonexpansive mappings were extensively studied in some early works, e.g. [3, 9, 29], and the generalizations have recently been discussed in [8, 28, 35]. Refer to [5, 6] for the comprehensive treatments.

The notion of nonexpansiveness arises primarily in connection with the study of fixed-point theory, and underlies the convergence analysis of various fixed-point iterations. Nowadays, there has been a revived interest in the design and analysis of the first-order operator splitting methods [37], of which many algorithms can be interpreted by the nonexpansive mappings, e.g. proximal forward-backward splitting algorithms [2, 19, 31], Douglas–Rachford splitting...
[21], primal-dual splitting methods [10,40]. An analysis of the operator splitting algorithms from the perspective of nonexpansive mappings is given by [32], which reinterprets a variety of algorithms by a simple Krasnosel’skii–Mann iteration built from a nonexpansive operator. More recently, the work of [17] gives a systematic overview of operator splitting algorithms based on the fixed-point theory. This demonstrates that the nonexpansive mapping still plays a central role and constantly gains the popularity and attention in the related areas.

Recently, the nonexpansive mappings have been extended to arbitrary self-adjoint and positive definite (PD) metric in various specific forms, e.g. generalized proximity operator [31], Bregman-based proximal operator [15,39], generalized resolvent [4,36], which are fundamental for analyzing proximal mapping [16], Bregman-based proximal schemes [37], variable metric Fejér sequence [18], especially under the context of operator splitting algorithms.

1.2. Motivations and Contributions

All of the existing works assume the metric to be self-adjoint and PD, e.g. [4,12,13,41]. However, in some scenarios, especially when the operator splitting schemes are reinterpreted by the generalized proximal point algorithm (with a non-trivial relaxation step\(^1\)), one has to establish the (firmly) nonexpansive results based on a non-self-adjoint operator (see [23, Lemma 3.2] and [26, Theorem 3.1, Lemma 5.4] for example). The intermediate results are essential for proving the convergence. In Sect. 2, we study the nonexpansive properties (e.g. firm nonexpansiveness, cocoerciveness and averagedness) in the context of arbitrary metric, and present the weakest possible conditions under which those properties hold. Indeed, many of the properties are shown to be valid without the self-adjoint and PD condition.

It is easy and straightforward to extend many convergence properties presented in [5,22] to the positive definite metric, by simply replacing ordinary norm \(\|\cdot\|\) by a metric-based semi-norm [11,41]. However, it would be non-trivial to extend the classic results to the case of positive semi-definite (PSD) metric, since the distance in the PSD metric cannot measure the closeness between two points in whole space due to the non-trivial null space of the metric. We refer to the positive semi-definiteness as ‘degeneracy’, which implies that the degenerate metric-based nonexpansiveness can infer the convergence in a subspace only, but not in the whole space if without additional assumptions. With particular focus on degenerate case, we in Sect. 3 analyze the properties of the metric distance for the fixed-point iterations, and further prove the convergence in the whole space under a certain mild conditions.

All of the results presented in Sect. 2 and 3 have important applications, of which a prominent example is the metric resolvent. Many operator splitting algorithms can be reinterpreted as the (degenerate) metric resolvents, and thus,

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\(^1\)The ‘non-trivial’ means that the relaxation operator is not multiple of identity operator, see [26, Eq.(2.8)], [25, Eq.(3.5)] and [24, Eq.(5.2)] for example.
the convergence properties can be easily obtained from the above results. This will be discussed in Sect. 4.

1.3. Notations

We use standard notations and concepts from convex analysis and variational analysis, which, unless otherwise specified, can all be found in the classical and recent monographs [5, 7, 34, 38].

A few more words about our notations are in order. Let $H$ be a real Hilbert space, $D$ be a nonempty subset of $H$. $P_C$ denotes a projection onto a closed subset $C \subset H$. The classes of PSD/PD linear operators are denoted by $\mathcal{M}^+/\mathcal{M}^{++}$, respectively. The classes of self-adjoint, self-adjoint and PSD/PD linear operators are denoted by $\mathcal{M}_S$, $\mathcal{M}_S^+/\mathcal{M}_S^{++}$, respectively. For our specific use, the $Q$-norm for arbitrary metric $Q$ is defined as: $\| \cdot \|_Q := \langle Q \cdot \rangle$. Here, $Q$ is not assumed to be self-adjoint and PSD, and hence, $\| \cdot \|_Q$ is not always well-defined, which is used only when $Q \in \mathcal{M}^+$. The strong and weak convergences are denoted by $\to$ and $\rightharpoonup$, respectively.

2. The Nonexpansive Properties Based on Arbitrary Metric

Throughout this paper, we assume that the nonexpansive operator $T$ is single-valued, but not necessarily injective. In fact, $T$ is generally non-injective in the case of degenerate metric (see Sect. 4 for example).

2.1. Definitions Based on Arbitrary Metric

Inspired by the work of [11], the following definitions extend the classical notions of Lipschitz continuity [5, Definition 1.47], nonexpansiveness [5, Definition 4.1], cocoerciveness [5, Definition 4.10] and averagedness [5, Definition 4.33] to arbitrary (not necessarily self-adjoint and PSD) metric $Q$.

**Definition 2.1.** Let $Q$ be arbitrary metric, then, the operator $T : D \to H$ is:

(i) $Q$-partly nonexpansive, if:

$\langle Q(x_1 - x_2) | T x_1 - T x_2 \rangle \geq \| T x_1 - T x_2 \|_Q^2$.

(ii) $Q$-nonexpansive, if:

$\| T x_1 - T x_2 \|_Q^2 \leq \| x_1 - x_2 \|_Q^2$.

(iii) $Q$-based $\xi$-Lipschitz continuous, if:

$\| T x_1 - T x_2 \|_Q^2 \leq \xi^2 \| x_1 - x_2 \|_Q^2$.

(iv) $Q$-firmly nonexpansive, if:

$\| T x_1 - T x_2 \|_Q^2 + \| (I - T)x_1 - (I - T)x_2 \|_Q^2 \leq \| x_1 - x_2 \|_Q^2$.

(v) $Q$-based $\beta$-cocoercive, if $\beta T$ is $Q$-partly nonexpansive:

$\langle Q(x_1 - x_2) | T x_1 - T x_2 \rangle \geq \beta \| T x_1 - T x_2 \|_Q^2$. 
(vi) $Q$-based $\alpha$-averaged with $\alpha \in [0,1[$, if there exists a $Q$-nonexpansive operator $K : D \mapsto H$, such that $T = (1 - \alpha)I + \alpha K$.

Remark 1. The notions of partly nonexpansive and $\beta$-cocoercive are based on arbitrary metric $Q$, not limited to self-adjoint and PSD case. Definition 2.1–(ii) and (iii) use $\| \cdot \|^2_Q$ rather than $\| \cdot \|_Q$, because $\| \cdot \|^2_\mathcal{Q}$ is always well defined for arbitrary $Q$, even if $Q$ is not PSD, as mentioned in Sect. 1.3.

The $Q$-based $\alpha$-averaged is further generalized as follows.

Definition 2.2. An operator $T : D \mapsto H$ is said to be $Q$-based $\xi$-Lipschitz $\alpha$-averaged with $\xi \in ]0, +\infty[$ and $\alpha \in ]0, 1[$, if there exists a $Q$-based $\xi$-Lipschitz continuous operator $K : D \mapsto H$, such that $T = (1 - \alpha)I + \alpha K$. In particular, if $\xi \in ]1, +\infty[$, $T$ is $Q$-weakly averaged; if $\xi \in ]0, 1]$, $T$ is $Q$-strongly averaged.

To lighten the notation, we denote a family of $Q$-based $\xi$-Lipschitz $\alpha$-averaged operators by $\mathcal{F}_{Q, \xi, \alpha}$. Obviously, Definition 2.1–(vi) is a special case of Definition 2.2 with $\xi = 1$, and thus, can be denoted by $T \in \mathcal{F}_{Q, 1, \alpha}$.

2.2. Nonexpansiveness

This section presents the nonexpansive properties in the context of arbitrary metric $Q$. First, Definition 2.1 is connected via the following results.

Lemma 2.3. $T : D \mapsto H$ is $Q$-partly nonexpansive,
   (i) if and only if $T$ is $Q$-based $1$-cocoercive;
   (ii) if $T$ is $Q$-based $\beta$-cocoercive with $Q \in \mathcal{M}^+$ and $\beta \in [1, +\infty[$;
   (iii) if and only if $T$ is $Q$-firmly nonexpansive with $Q \in \mathcal{M}_S$.

Proof. Definition 2.1 and the conditions of $Q$. □

Lemma 2.4. $T : D \mapsto H$ is $Q$-nonexpansive,
   (i) if and only if $T$ is $Q$-based $1$-Lipschitz continuous;
   (ii) if $T$ is $Q$-based $\xi$-Lipschitz continuous with $Q \in \mathcal{M}^+$ and $\xi \in ]0, 1[$;
   (iii) if $T$ is $Q$-firmly nonexpansive with $Q \in \mathcal{M}_S$.

Proof. Definition 2.1 and the conditions of $Q$. □

[5, Corollary 2.15] is also valid for arbitrary (not necessarily self-adjoint and PSD) metric $Q$, as stated below.

Lemma 2.5. The following identity holds for any $\kappa \in \mathbb{R}$ and arbitrary $Q$:
   $$\| \kappa x_1 + (1 - \kappa) x_2 \|^2_Q = \kappa \| x_1 \|^2_Q + (1 - \kappa) \| x_2 \|^2_Q - \kappa(1 - \kappa) \langle x_1 - x_2 \rangle^2_Q.$$  

Proof. Noting that $\langle (Q + Q^*) x_1 | x_2 \rangle = \| x_1 \|^2_Q + \| x_2 \|^2_Q - \| x_1 - x_2 \|^2_Q$, we have:
   $$\| \kappa x_1 + (1 - \kappa) x_2 \|^2_Q = \kappa \| x_1 \|^2_Q + (1 - \kappa)^2 \| x_2 \|^2_Q + \kappa(1 - \kappa) \langle (Q + Q^*) x_1 | x_2 \rangle$$
\[
\kappa^2 \|x_1\|_Q^2 + (1 - \kappa)^2 \|x_2\|_Q^2 + \kappa(1 - \kappa)(\|x_1\|_Q^2 + \|x_2\|_Q^2 - \|x_1 - x_2\|_Q^2)
\]
\[
= \kappa \|x_1\|_Q^2 + (1 - \kappa)^2 \|x_2\|_Q^2 - \kappa(1 - \kappa)\|x_1 - x_2\|_Q^2,
\]
which completes the proof. \[\square\]

Lemma 2.6 is an extended version of [5, Proposition 4.2] for the case of arbitrary metric \(Q\), which shows the equivalence between partly nonexpansive and firmly nonexpansive, in case of self-adjoint \(Q\).

**Lemma 2.6.** Let \(T : D \mapsto \mathcal{H}\), then, the following are equivalent:

(i) \(T\) is \(Q\)-partly nonexpansive;

(ii) \(T\) is \(Q\)-firmly nonexpansive with \(Q \in \mathcal{M}_S\);

(iii) \(I - T\) is \(Q\)-firmly nonexpansive with \(Q \in \mathcal{M}_S\);

(iv) \(2T - I\) is \(Q\)-nonexpansive with \(Q \in \mathcal{M}_S\);

(v) \(I - T\) is \(Q^*\)-partly nonexpansive.

**Proof.** (i)\(\leftrightarrow\)(ii)\(\leftrightarrow\)(iii)\(\leftrightarrow\)(iv): Definition 2.1, [5, Proposition 4.2] and Lemma 2.5.

(i)\(\leftrightarrow\)(v): By Definition 2.1–(i), we have:

\[\langle Q(I - T)x_1 - Q(I - T)x_2 | T x_1 - T x_2 \rangle \geq 0.\]

Adding \(\|((I - T)x_1 - (I - T)x_2\|_Q^2\) on both sides, we obtain:

\[\langle Q(I - T)x_1 - Q(I - T)x_2 | x_1 - x_2 \rangle \geq \|(I - T)x_1 - (I - T)x_2\|_Q^2,
\]

which leads to (v), noting that \(\langle Q(I - T)x_1 - Q(I - T)x_2 | x_1 - x_2 \rangle = \langle (I - T)x_1 - (I - T)x_2 | Q^*(x_1 - x_2) \rangle\).

**Remark 2.** As mentioned in Sect. 1.2, Lemma 2.6 is very useful for proving the convergence and asymptotic regularity of the generalized proximal point algorithm with a non-trivial relaxation step, where \(Q\) is not self-adjoint and PD. More specifically, the new metric becomes \(S = QM^{-1}\) instead of \(Q\), where \(M\) is a linear relaxation operator, such that \(S\) is self-adjoint and PD [23,41].

### 2.3. Averagedness, Cocoerciveness and Lipschitz Continuity

The following results extend [5, Proposition 4.35, Remark 4.34, Remark 4.37, Proposition 4.39, Proposition 4.40] to arbitrary metric \(Q\), which build the connections of \(Q\)-based 1–Lipschitz \(\alpha\)-averagedness (i.e. \(F_{1,\alpha}^Q\)) to other concepts.

**Lemma 2.7.** Let \(T : D \mapsto \mathcal{H}\), then, the following hold.

(i) \(T \in F_{1,\alpha}^Q\) with \(\alpha \in [0, 1]\), if and only if:

\[\|Tx_1 - Tx_2\|_Q^2 + \frac{1 - \alpha}{\alpha} \|(I - T)x_1 - (I - T)x_2\|_Q^2 \leq \|x_1 - x_2\|_Q^2.
\]

(ii) \(T\) is \(Q\)-firmly nonexpansive, if and only if \(T \in F_{1,\alpha}^Q\).

(iii) If \(T \in F_{1,\alpha}^Q\) with \(Q \in \mathcal{M}^+\) and \(\alpha \in \mathbb{[}0, \frac{1}{2}]\), then \(T\) is \(Q\)-firmly nonexpansive.
(iv) Let $\alpha \in ]0, 1[\), $\gamma \in ]0, \frac{1}{\alpha}[\), then, $T \in \mathcal{F}_{1, \gamma \alpha}^Q$, if and only if $(1-\gamma)\mathcal{I} + \gamma T \in \mathcal{F}_{1, \gamma \alpha}^Q$.

(v) $T$ is $Q$-based $\beta$-cocoercive with $Q \in \mathcal{M}_S$, if and only if $\beta T \in \mathcal{F}_{1, \frac{1}{\beta}}^Q$.

(vi) Let $T$ be $Q$-based $\beta$-cocoercive with $Q \in \mathcal{M}_S$. If $\gamma \in ]0, \frac{1}{2}[\), then, $I - \gamma T \in \mathcal{F}_{1, \frac{1}{\gamma \beta}}^Q$.

Proof. (i) [5, Proposition 4.35]–(iii) and Lemma 2.5;
(ii) [5, Remark 4.34]–(iii);
(iii) [5, Remark 4.37];
(iv) [5, Proposition 4.40].
(v) Lemma 2.7–(ii), Lemma 2.3–(iii) and Definition 2.1–(v).
(vi) Lemma 2.7–(v) and [5, Proposition 4.39].

The following theorem, as a main result of this part, collects the key results of $\mathcal{F}_{\xi, \alpha}^Q$.

**Theorem 2.8.** Let $T \in \mathcal{F}_{\xi, \alpha}^Q$ with $\xi \in ]0, +\infty[\) and $\alpha \in ]0, 1[\). Then, the following hold.

(i) $T$ satisfies:
$$
\|Tx_1 - Tx_2\|_Q^2 \\
\leq (1 - \alpha + \alpha \xi^{2})\|x_1 - x_2\|_Q^2 - \frac{1}{\alpha}\|(I - T)x_1 - (I - T)x_2\|_Q^2.
$$

(ii) If $Q \in \mathcal{M}^+$, $0 < \xi \leq \min\{\frac{1-\alpha}{\alpha}, 1\}$, then $T$ is $Q$-firmly nonexpansive.

(iii) If $Q \in \mathcal{M}_S^\downarrow$, $\xi \in ]0, \frac{1-\alpha}{\alpha}[\), then $T$ is $Q$-based $\beta$-cocoercive, with $\beta = \frac{1}{2}(1 + \frac{1}{1 - \alpha + \alpha \xi^{2}})$.

(iv) $I - \gamma T \in \mathcal{F}_{\frac{1}{\alpha}, \gamma (1-\alpha)}^Q$, if $\gamma \in ]0, \frac{1}{1-\alpha}[\).

(v) If $Q \in \mathcal{M}_S^\downarrow$, $\gamma \in ]0, \frac{1}{1-\alpha}[\), $\xi \leq \min\{\frac{1}{\gamma \alpha}, \frac{1-\alpha}{\alpha}, \frac{1-\alpha}{\alpha}\}$, then $I - \gamma T$ is $Q$-firmly nonexpansive.

(vi) If $Q \in \mathcal{M}_S^\downarrow$, $\gamma \in ]0, \frac{1}{1-\alpha}[\), $\xi \in ]0, \frac{1}{\gamma \alpha} - \frac{1-\alpha}{\alpha}[\), then $I - \gamma T$ is $Q$-based $\beta$-cocoercive with $\beta = \frac{1}{2}(1 + \frac{1-\alpha}{\gamma (1-\alpha) + \gamma \alpha \xi^{2}})$.

(vii) The reflected operator of $T$ follows $2T - I \in \mathcal{F}_{\xi, 2\alpha}^Q$, if $\alpha \in ]0, \frac{1}{2}[\)\

Proof. (i) By Definition 2.2, there exists a $Q$-based $\xi$-Lipschitz continuous operator $K : D \mapsto \mathcal{H}$, such that $T = (1 - \alpha)I + \alpha K$, and thus, $K = \frac{1}{\alpha}T + (1 - \frac{1}{\alpha})I$. By Lemma 2.5, we have:
$$
\|Kx_1 - Kx_2\|_Q^2 \\
= \left(1 - \frac{1}{\alpha}\right)\|x_1 - x_2\|_Q^2 + \frac{1}{\alpha}\|Tx_1 - Tx_2\|_Q^2 \\
+ \frac{1-\alpha}{\alpha^{2}}\|(I - T)x_1 - (I - T)x_2\|_Q^2.
$$
\[ \leq \xi^2 \| x_1 - x_2 \|^2_Q, \quad \text{(by Lipschitz continuity of } K) \]

which yields the desired inequality, after simple rearrangements.

(ii) If \( Q \in M^+ \), to ensure that \( T \) is \( Q \)-firmly nonexpansive, we need to let \( 1 - \alpha + \alpha \xi^2 \leq 1 \) and \( \frac{1-\alpha}{\alpha} \geq 1 \), by (1) and Definition 2.1–(iv). It yields \( \xi \in [0, 1] \) and \( \alpha \in [0, \frac{1}{2}] \).

On the other hand, rewrite (1) as:

\[
\frac{\alpha}{1-\alpha} \left\| T x_1 - T x_2 \right\|^2_Q \leq \frac{\alpha}{1-\alpha} (1 - \alpha + \alpha \xi^2) \| x_1 - x_2 \|^2_Q,
\]

(2)

The firm nonexpansiveness of \( T \) requires \( \frac{\alpha}{1-\alpha} \geq 1 \) and \( \frac{\alpha}{1-\alpha} (1 - \alpha + \alpha \xi^2) \leq 1 \), i.e. \( \xi \in [0, \frac{1-\alpha}{\alpha}] \) and \( \alpha \in [\frac{1}{2}, 1[ \). Finally, combining both conditions yields \( 0 < \xi \leq \min\{\frac{1-\alpha}{\alpha}, 1\} \).

(iii) If \( Q \in M_S \), expanding \( \| (I - T) x_1 - (I - T) x_2 \|^2_Q \), (1) is equivalent to:

\[
\frac{2(1-\alpha)}{\alpha} \langle Q(x_1 - x_2) \| T x_1 - T x_2 \rangle \geq \frac{1}{\alpha} \| T x_1 - T x_2 \|^2_Q,
\]

which yields:

\[
\langle Q(x_1 - x_2) \| T x_1 - T x_2 \rangle \geq \frac{1}{2(1-\alpha)} \| T x_1 - T x_2 \|^2_Q, \quad (3)
\]

if \( Q \in M^+ \) and \( 2 - \alpha - \frac{1}{\alpha} + \alpha \xi^2 \leq 0 \), i.e. \( \xi \leq \frac{1-\alpha}{\alpha} \).

On the other hand, if \( Q \in M_S^+ \), (2) becomes:

\[
\frac{\alpha}{1-\alpha} \left\| T x_1 - T x_2 \right\|^2_Q \\
\leq \frac{\alpha(1 - \alpha + \alpha \xi^2)}{1-\alpha} \| x_1 - x_2 \|^2_Q - \| (I - T) x_1 - (I - T) x_2 \|^2_Q \\
= \frac{\alpha(1 - \alpha + \alpha \xi^2)}{1-\alpha} \| (I - T) x_1 - (I - T) x_2 + T x_1 - T x_2 \|^2_Q \\
- \| (I - T) x_1 - (I - T) x_2 \|^2_Q \\
= \left( \frac{\alpha(1 - \alpha + \alpha \xi^2)}{1-\alpha} - 1 \right) \| (I - T) x_1 - (I - T) x_2 \|^2_Q \\
- \frac{\alpha(1 - \alpha + \alpha \xi^2)}{1-\alpha} \| T x_1 - T x_2 \|^2_Q \\
+ 2 \frac{\alpha(1 - \alpha + \alpha \xi^2)}{1-\alpha} \langle Q(x_1 - x_2) \| T x_1 - T x_2 \rangle.
\]
If $\frac{\alpha(1-\alpha+\alpha\xi^2)}{1-\alpha} \leq 1$, i.e. $\xi \leq \frac{1-\alpha}{\alpha}$, it yields:

$$
2\frac{\alpha(1-\alpha+\alpha\xi^2)}{1-\alpha} \langle Q(x_1 - x_2) | T x_1 - T x_2 \rangle \\
\geq \left( \frac{\alpha}{1-\alpha} + \frac{\alpha(1-\alpha+\alpha\xi^2)}{1-\alpha} \right) \| T x_1 - T x_2 \|^2_Q,
$$

i.e.

$$
\langle Q(x_1 - x_2) | T x_1 - T x_2 \rangle \geq \frac{1}{2} \left( 1 + \frac{1}{1-\alpha+\alpha\xi^2} \right) \| T x_1 - T x_2 \|^2_Q. \quad (4)
$$

Finally, (iii) follows by comparing (3) with (4), and noting that: $\frac{1}{1-\alpha} \leq 1 + \frac{1}{1-\alpha+\alpha\xi^2}$, if $\xi \leq \frac{1-\alpha}{\alpha}$.

(iv) Expanding $\| (I - \gamma T) x_1 - (I - \gamma T) x_2 \|^2_Q$, (1) is equivalent to:

$$
\langle (Q + Q^*) (x_1 - x_2) | T x_1 - T x_2 \rangle \geq \frac{1}{1-\alpha} \| T x_1 - T x_2 \|^2_Q
$$

Then, we have:

$$
\| (I - \gamma T) x_1 - (I - \gamma T) x_2 \|^2_Q \\
= \| x_1 - x_2 \|^2_Q - \gamma \langle (Q + Q^*) (x_1 - x_2) | T x_1 - T x_2 \rangle + \| \gamma T x_1 - \gamma T x_2 \|^2_Q \\
\leq \| x_1 - x_2 \|^2_Q - \gamma \frac{1}{1-\alpha} \| T x_1 - T x_2 \|^2_Q + \gamma \frac{\alpha^2\xi^2 - (1-\alpha)^2}{1-\alpha} \\
\| x_1 - x_2 \|^2_Q + \| \gamma T x_1 - \gamma T x_2 \|^2_Q \\
= \left( 1 + \gamma \frac{\alpha^2\xi^2 - (1-\alpha)^2}{1-\alpha} \right) \| x_1 - x_2 \|^2_Q - \left( \frac{1}{\gamma(1-\alpha)} - 1 \right) \| \gamma T x_1 - \gamma T x_2 \|^2_Q.
$$

Let $I - \gamma T \in F_{\xi',\alpha'}^Q$, then, by (1), we have $\frac{1-\alpha}{\alpha'} = \frac{1-\alpha}{\gamma(1-\alpha)} - 1$ and $1-\alpha' + \alpha'\xi^2 = 1 + \frac{\alpha^2\xi^2 - (1-\alpha)^2}{1-\alpha}$, which yields $\alpha' = \gamma(1-\alpha)$ and $\xi' = \frac{\alpha\xi}{1-\alpha}$.

(v) Theorem 2.8–(ii) and (iv).

(vi) Theorem 2.8–(iii) and (iv).

(vii) We deduce that:

$$
\| (2T - I) x_1 - (2T - I) x_2 \|^2_Q \\
= 2\| T x_1 - T x_2 \|^2_Q - \| x_1 - x_2 \|^2_Q + 2 \| (I - T) x_1 \\
- (I - T) x_2 \|^2_Q \quad \text{(by Lemma 2.5)} \\
\leq 2(1-\alpha + \alpha\xi^2) \| x_1 - x_2 \|^2_Q - \frac{2(1-\alpha)}{\alpha} \| (I - T) x_1 - (I - T) x_2 \|^2_Q \\
- \| x_1 - x_2 \|^2_Q + 2 \| (I - T) x_1 - (I - T) x_2 \|^2_Q \quad \text{(by (2.1) )} \\
= (1 - 2\alpha + 2\alpha\xi^2) \| x_1 - x_2 \|^2_Q - \frac{1 - 2\alpha}{2\alpha} \| (2T - 2T) x_1 - (2T - 2T) x_2 \|^2_Q.
$$
Let $2\mathcal{T} - \mathcal{I} \in \mathcal{F}_{\xi, \alpha}^Q$. Thus, we have $\frac{1-\alpha'}{\alpha} = \frac{1-2\alpha}{2\alpha}$ and $1 - \alpha' + \alpha'\xi'^2 = 1 - 2\alpha + 2\alpha\xi^2$, i.e. $\alpha' = 2\alpha$, and $\xi' = \xi$. \hfill \Box

Two corollaries follow from Theorem 2.8.

Corollary 2.9 [Further results of Theorem 2.8–(iii)]. Let $\mathcal{T} \in \mathcal{F}_{\xi, \alpha}^Q$ with $\xi \in [0, +\infty[$ and $\alpha \in ]0, 1[$, then, the following hold.

(i) If $\xi \leq \min\{\frac{1-\alpha}{\alpha}, \frac{1}{\gamma} - \frac{1-\alpha}{\alpha}\}$, then, $\mathcal{T}$ is $Q$-based $\beta$-cocoercive with $\beta \in [1, +\infty[$, strongly $\alpha$-averaged, and $Q$-firmly nonexpansive.

(ii) If $\alpha \in ]0, \frac{1}{2}[\), $\xi \in ]1, \frac{1-\alpha}{\alpha}\]$], then, $\mathcal{T}$ is $Q$-based $\beta$-cocoercive with $\beta \in [0, 1[$, and weakly $\alpha$-averaged.

Proof. By Theorem 2.8-(iii), $\mathcal{T}$ is $\beta$-cocoercive, with $\beta = \frac{1}{2} \left(1 + \frac{1}{1-\alpha + \alpha\xi^2}\right)$. The proof is completed by comparing $\beta$ in with $1$. \hfill \Box

Corollary 2.10 [Further results of Theorem 2.8–(vi)]. Let $\mathcal{T} \in \mathcal{F}_{\xi, \alpha}^Q$ with $\xi \in [0, +\infty[$ and $\alpha \in ]0, 1[$. If $\gamma \in ]0, \frac{1}{\gamma} [\), then, the following hold.

(i) If $\xi \leq \min\{\frac{1-\alpha}{\alpha}, \frac{1}{\gamma} - \frac{1-\alpha}{\alpha}\}$, then, $\mathcal{T} - \gamma\mathcal{T}$ is $Q$-based $\beta$-cocoercive with $\beta \in [1, +\infty[$, strongly $\alpha$-averaged, and $Q$-firmly nonexpansive.

(ii) If $\gamma \in ]0, \frac{1}{2(1-\alpha)}[\), $\xi \in ]1, \frac{1-\alpha}{\alpha}\]$], then, $\mathcal{T} - \gamma\mathcal{T}$ is $Q$-based $\beta$-cocoercive with $\beta \in [0, 1[$, and weakly $\alpha$-averaged.

Proof. The proof is completed by comparing $\beta$ in Theorem 2.8–(vi) with $1$. \hfill \Box

Lemma 2.11. Let the operator $\mathcal{T} : D \mapsto \mathcal{H}$ be $Q$-based $\beta$-cocoercive with $Q \in \mathcal{M}_S$ and $\beta \in ]\frac{1}{2}, +\infty[$. Then, the following hold.

(i) If $\gamma \in ]0, 2\beta[$, then, $\mathcal{T} \in \mathcal{F}_{\frac{1}{2\beta - 1}, 1-\frac{1}{2\beta}} Q$, $\mathcal{T} - \gamma\mathcal{T} \in \mathcal{F}_{\frac{1}{2\beta}, \frac{1}{2\beta}} Q$.

(ii) If $Q \in \mathcal{M}_S^+$, $\gamma \in ]0, \beta[$, $\mathcal{T} - \gamma\mathcal{T}$ is $Q$-based $1$-cocoercive (i.e. $Q$-partly nonexpansive).

Proof. (i) If $Q \in \mathcal{M}_S$, we have:

\[
\left\| (\mathcal{I} - \mathcal{T})x_1 - (\mathcal{I} - \mathcal{T})x_2 \right\|^2_Q \\
= \left\| x_1 - x_2 \right\|^2_Q - 2 \langle Q(x_1 - x_2), (\mathcal{T}x_1 - \mathcal{T}x_2) \rangle + \left\| \mathcal{T}x_1 - \mathcal{T}x_2 \right\|^2_Q \\
\leq \left\| x_1 - x_2 \right\|^2_Q - 2\beta \left\| \mathcal{T}x_1 - \mathcal{T}x_2 \right\|^2_Q + \left\| \mathcal{T}x_1 - \mathcal{T}x_2 \right\|^2_Q \\
= \left\| x_1 - x_2 \right\|^2_Q - (2\beta - 1) \left\| \mathcal{T}x_1 - \mathcal{T}x_2 \right\|^2_Q,
\]

which yields:

\[
\left\| \mathcal{T}x_1 - \mathcal{T}x_2 \right\|^2_Q \leq \frac{1}{2\beta - 1} \left\| x_1 - x_2 \right\|^2_Q - \frac{1}{2\beta - 1} \left\| (\mathcal{I} - \mathcal{T})x_1 - (\mathcal{I} - \mathcal{T})x_2 \right\|^2_Q.
\]

Thus, if $\mathcal{T} \in \mathcal{F}_{\xi, \alpha}^Q$, by (1), we have: $\frac{1-\alpha'}{\alpha} = \frac{1}{2\beta - 1}$ and $1 - \alpha' + \alpha'\xi'^2 = \frac{1}{2\beta - 1}$, i.e. $\alpha' = 1 - \frac{1}{2\beta - 1}$ and $\xi' = \frac{1}{2\beta - 1}$.

$\mathcal{T} - \gamma\mathcal{T}$ follows from Theorem 2.8-(iv).

(ii) Theorem 2.8-(iii) and Lemma 2.11-(i). \hfill \Box
Part of the results in Theorem 2.8, Corollary 2.9 and Corollary 2.10 is summarized in Fig.1, where FNE stands for ‘Q–firmly nonexpansive’. We can see that $T \in F_{Q,\xi,\alpha}$ could be Q–firmly nonexpansive for $\alpha > 1/2$, at the expense of stricter condition on the Lipschitz constant $\xi \leq \frac{1-\alpha}{\alpha} < 1$.

**Remark 3.** Many results in Sect. 2.3 are useful for analyzing the relaxed version of fixed point iterations (see Sect. 3.3), and proving linear convergence under stronger conditions (e.g. the case of $\xi \in ]0,1[$ in Proposition 3.2 and Corollary 3.4).

### 3. The Associated Fixed-Point Iterations

#### 3.1. Assumptions

The convergence of the fixed-point iterations associated with the non-degenerate metric-based nonexpansive operator $T$ has been well understood in literature, see [4,5,18,41] for some typical results. In this sequel, we focus on the degenerate case only. More specifically, we make the following assumption on $Q$:

**Assumption 1.** $Q \in \mathcal{M}_S^+$, such that $\ker Q \setminus \{0\} \neq \emptyset$.

**Remark 4.** Assumption 1 implies that $Q$ has a non-trivial null space in the degenerate case, which is the focus of our discussion.

We also make several assumptions on $T$:

**Assumption 2.** (i) $T \in F_{Q,\xi,\alpha}$ with $\alpha \in ]0,1[$ and $\xi \in ]0,1]$, i.e. $T$ is $Q$–strongly averaged (see Definition 2.2).

(ii) The set $\text{Fix}T := \{x \in D | x = Tx\}$ is non-empty.
(iii) \( T : D \mapsto H \) is demiclosed, where \( D \) is a nonempty weakly sequentially closed subset of \( H \).

(iv) \( T \) satisfies \( \| T x_1 - T x_2 \| \leq L \| x_1 - x_2 \|_Q \) for some constant \( L \).

Remark 5. Assumption 2-(i) is a conventional condition, commonly used in classical results, to guarantee the basic nonexpansiveness. In (ii), the existence of fixed point set of \( T \) is a subtle assumption, which, however, is reasonable in many applications, as shown in Sect. 4.

(iii) implies that \( \text{gra} T \) is sequentially closed in \( H_{\text{weak}} \times H_{\text{strong}} \), which is useful to prove the convergence. Generally speaking, (iv) is a rather restrictive condition, which indicates that all of the useful information of \( T x \) lies in \( \text{ran} Q \), instead of the whole space \( H \). We will see that (iii) and (iv) are essential to prove the boundedness of \( \{ x_k \}_{k \in \mathbb{N}} \) and the strong convergence of \( x_k - T x_k \to 0 \) in \( H \), as \( k \to \infty \). In addition, many degenerate metric resolvents (e.g. discussed in Sect. 4) satisfy this rigid requirement.

We define a fixed point as \( x^* \in \text{Fix} T \). The \( Q \)-based solution distance and sequential error of the \( k \)-th iterate are defined by \( \| x^k - x^* \|_Q \) and \( \| x^{k+1} - x^k \|_Q \), respectively. The \( Q \)-based sequential error is closely related to \( Q \)-asymptotic regularity, which is an extended version of asymptotically regular [27,30].

Definition 3.1. A mapping \( T : D \mapsto H \) is \( Q \)-asymptotically regular, if \( \| T^k x - T^{k+1} x \|_Q \to 0 \), as \( k \to \infty \), \( \forall x \in D \). Here, \( T^k \) is defined as: \( T^k := T \circ \cdots \circ T \) \( k \) times.

Clearly, if \( T \) is \( Q \)-asymptotically regular, the \( Q \)-based sequential error vanishes, as \( k \to \infty \). However, it does not necessarily yield the strong convergence of \( x^k - x^{k+1} \to 0 \), due to the degeneracy of \( Q \). Nonetheless, if Assumption 2-(iv) is taken into account, we can obtain the following important observation:

Fact 1. Under Assumption 2-(iv), if \( T \) is \( Q \)-asymptotically regular, then it is also asymptotically regular.

Proof. By Assumption 2-(iv), we have \( \| T^{k+1} x - T^{k+2} x \| \leq L \| T^k x - T^{k+1} x \|_Q \), which yields the desired result by taking \( k \to \infty \).

3.2. Banach-Picard Iteration

Considering the scheme

\[
x^{k+1} := T x^k,
\]

the properties of metric-based distances of (5) are given as follows.

Proposition 3.2 [Convergence in \( \text{ran} Q \)]. Let \( x^0 \in D \), \( \{ x^k \}_{k \in \mathbb{N}} \) be a sequence generated by (5). Denote \( \nu := 1 - \alpha + \alpha \xi^2 \). Under Assumptions 1 and 2-(i-ii), the following hold.

(i) \( T \) is \( Q \)-asymptotically regular.
(ii) [Sequential error] \( \|x^{k+1} - x^k\|_q \) has the pointwise sublinear convergence rate of \( O(1/\sqrt{k}) \):
\[
\|x^{k+1} - x^k\|_q \leq \frac{1}{\sqrt{k} + 1} \sqrt{\frac{\alpha}{1 - \alpha}} \|x^0 - x^*\|_q, \quad \forall k \in \mathbb{N}.
\]

(iii) [\( q \)-linear convergence] If \( \xi \in ]0, 1[ \), both \( \|x^k - x^*\|_q \) and \( \|x^k - x^{k+1}\|_q \) are \( q \)-linearly convergent with the rate of \( \sqrt{\nu} \).

(iv) [\( r \)-linear convergence] If \( \alpha \in ]1 - \frac{1}{\sqrt{2}}, 1[ \), \( \xi \in ]0, \sqrt{1 - \frac{2 - \sqrt{2}}{2\alpha}}[ \), \( \|x^k - x^{k+1}\|_q \) is globally \( r \)-linearly convergent w.r.t. \( \|x^0 - x^*\|_q \):
\[
\|x^k - x^{k+1}\|_q \leq \sqrt{\frac{2\alpha(1 - \nu)}{(1 - \alpha)\nu}} \cdot \nu^{\frac{k+1}{2}} \|x^0 - x^*\|_q.
\]

(v) [Weak/strong convergence in \( \text{ran} Q \)] If \( \xi = 1 \) or \( \xi \in ]0, 1[ \), there exists \( x^* \in \text{Fix} T \), such that \( \sqrt{Q}x^k \to \alpha \) or \( \to \sqrt{Q}x^* \) respectively, as \( k \to \infty \).

Proof. (i) Taking \( x_1 = x^k \) and \( x_2 = x^* \in \text{Fix} T \) in (1), we obtain:
\[
\|x^{k+1} - x^*\|_q^2 \leq \nu \|x^k - x^*\|_q^2 - \frac{1 - \alpha}{\alpha} \|x^k - x^{k+1}\|_q^2.
\]
\[
(6)
\]
Noting \( \nu \in ]1 - \alpha, 1[ \), and summing up (6) from \( k = 0 \) to \( K \) yields:
\[
\sum_{k=0}^{K} \|x^k - x^{k+1}\|_q^2 \leq \frac{\alpha}{1 - \alpha} \|x^0 - x^*\|_q^2.
\]
\[
(7)
\]
Taking \( K \to \infty \), we have: \( \sum_{k=0}^{\infty} \|x^k - x^{k+1}\|_q^2 = \frac{\alpha}{1 - \alpha} \|x^0 - x^*\|_q^2 < +\infty \), which implies that \( \lim_{k\to\infty} \|x^k - x^{k+1}\|_q = 0 \).

(ii) Taking \( x_1 = x^k \) and \( x_2 = x^{k+1} \) in (1), we have:
\[
\|x^{k+1} - x^{k+2}\|_q^2 \leq \nu \|x^k - x^{k+1}\|_q^2.
\]
\[
(8)
\]
\( \nu \in ]1 - \alpha, 1[ \) implies that \( \|x^k - x^{k+1}\|_q \) is non–increasing. Then, (ii) follows from (7).

(iii) If \( \xi \in ]0, 1[ \), (6) yields that \( \|x^{k+1} - x^*\|_q^2 \leq \nu \|x^k - x^*\|_q^2 \), where \( \nu \in ]1 - \alpha, 1[ \). The \( Q \)-based sequential error follows from (8).

(iv) If \( \xi \in ]0, 1[ \), combining (8) with (7) yields:
\[
\left( \nu^{-k} + \nu^{-(k-1)} + \ldots + 1 \right) \|x^k - x^{k+1}\|_q^2 \leq \frac{\alpha}{1 - \alpha} \|x^0 - x^*\|_q^2,
\]
which leads to:
\[
\|x^k - x^{k+1}\|_q^2 \leq \frac{\alpha(1 - \nu)}{(1 - \alpha)\nu} \cdot \frac{1}{\nu^{-(k+1)} - 1} \|x^0 - x^*\|_q^2.
\]
Clearly, if $\nu^{-(k+1)} - 1 \geq \frac{1}{2} \nu^{-(k+1)}$, (i.e. $k \geq \frac{\ln 2}{\ln(1/\nu)} - 1$), $\|x^k - x^{k+1}\|^2_\mathcal{Q}$ is $r$–linearly convergent w.r.t. $\|x^0 - x^*\|^2_\mathcal{Q}$:  

$$\|x^k - x^{k+1}\|^2_\mathcal{Q} \leq \frac{2\alpha(1-\nu)}{(1-\alpha)\nu} \cdot \nu^{k+1} \|x^0 - x^*\|^2_\mathcal{Q}.$$  

Furthermore, if $\frac{\ln 2}{\ln(1/\nu)} - 1 \leq 1$, the $r$–linear convergence is globally valid for $\forall k \in \mathbb{N}$. This condition can be simplified as $\xi^2 \leq 1 - \frac{2-\sqrt{2}}{2\alpha}$.

(v) If $\xi = 1$, the weak convergence of $\{\sqrt{\mathcal{Q}}x^k\}_{k \in \mathbb{N}}$ is clear, by basic non-expansive properties [5, Theorem 5.14-(i), Example 5.18] of Fejér monotonicity [5, Proposition 5.4, Theorem 5.5].

In the case of $\xi \in ]0,1[$, the linear convergence of $\{\sqrt{\mathcal{Q}}x^k\}_{k \in \mathbb{N}}$ immediately follows by [5, Theorem 5.12].

**Remark 6.** As emphasized above, one cannot conclude from Proposition 3.2 the convergence of $\{x_k\}_{k \in \mathbb{N}}$ in the whole space, since the $\mathcal{Q}$–metric distance does not infer anything about the projection of $x_k$ onto $\ker \mathcal{Q}$, which, however, has to be taken into account for the convergence in the whole space.

The following theorem is a main result of this paper, which shows the convergence of $x^k$ in $\mathcal{H}$ under additional Assumption 2-(iii–iv). The proof adopts some techniques in [41, Theorem 2.1].

**Theorem 3.3** [Weak convergence in $\mathcal{H}$]. Let $x^0 \in D$, $\{x^k\}_{k \in \mathbb{N}}$ be a sequence generated by (5). Under Assumptions 1 and 2, if $\xi = 1$, then there exists $x^* \in \text{Fix} T$, such that $x^k \to x^*$, as $k \to \infty$.

**Proof.** Following the reasoning of the well-known Opial’s lemma [33]2, the proof is divided into 4 steps3:

(i) for every $x^* \in \text{Fix} T$, $\lim_{k \to \infty} \|x^k - x^*\|_\mathcal{Q}$ exists;

(ii) the sequence $\{x^k\}_{k \in \mathbb{N}}$ is bounded;

(iii) if $x^{k_i} \to x^*$ weakly in $\mathcal{H}$ for a subsequence $k_i \to \infty$, then $x^* \in \text{Fix} T$;

(iv) $\{x^k\}_{k \in \mathbb{N}}$ possesses at most one weak sequential cluster point in $\text{Fix} T$.

(i) (6) shows that $\{\|x^k - x^*\|_\mathcal{Q}\}_{k \in \mathbb{N}}$ is non-increasing, and bounded from below (always being non-negative), and thus, convergent, i.e. $\lim_{k \to \infty} \|x^k - x^*\|_\mathcal{Q}$ exists.

(ii) By Assumption 2-(iv), we have:

$$\|x^{k+1} - x^*\| = \|T x^k - T x^*\| \leq L \|x^k - x^*\|_\mathcal{Q} \leq L \|x^0 - x^*\|_\mathcal{Q}, \quad \forall k \in \mathbb{N},$$

where the last inequality comes from (i). It implies that $\{x^k\}_{k \in \mathbb{N}}$ is bounded.

(iii) Since $\{x^k\}_{k \in \mathbb{N}}$ is bounded, by [5, Lemma 2.45], $\{x^k\}_{k \in \mathbb{N}}$ has at least one weak sequential cluster point, i.e. $\{x^{k_i}\}_{i \in \mathbb{N}}$ has a subsequence $\{x^{k_i}\}_{i \in \mathbb{N}}$ that weakly converges to a point $x^*$, denoted by $x^{k_i} \rightharpoonup x^*$, as $k_i \to \infty$. Our aim

---

2Refer to [5, Lemma 2.47] or [1, Lemma 2.1] for the Opial’s argument.
3This line of reasoning is very similar to Fejér monotonicity, see [5, Proposition 5.4, Theorem 5.5] for example.
is to show that $x^* \in \text{Fix} T$, and more generally, every weak sequential cluster point of $\{x^k\}_{k \in \mathbb{N}}$ belongs to $\text{Fix} T$. To this end, combining Assumption 2-(iv) with the claim (i), the weakly convergent subsequence $\{x^{k_i}\}_{k_i \in \mathbb{N}}$ satisfies:

$$\|x^{k_{i+1}} - x^{k_i + 1}\| = \|T x^{k_i} - T x^{k_i + 1}\| \leq L \|x^{k_i} - x^{k_i + 1}\| \to 0,$$  

as $k_i \to \infty$, which shows that $x^{k_i} \to x^*$ as $k_i \to \infty$, as $k_i \to \infty$ (this is also Fact 1). Since $x^{k_i} \to x^*$, it follows that $x^* - T x^* = 0$, due to the demiclosedness of $T$ (i.e. Assumption 2-(iii) that implies that gra($T - T$) is sequentially closed in $\mathcal{H}_{\text{weak}} \times \mathcal{H}_{\text{strong}}$). Thus, for every weak sequential cluster point $x^*$ of $\{x^k\}_{k \in \mathbb{N}}$, $x^* \in \text{Fix} T$.

(iv) We need to show that $\{x^k\}_{k \in \mathbb{N}}$ cannot have two distinct weak sequential cluster point in $\text{Fix} T$. To this end, let $x_1^*, x_2^* \in \text{Fix} T$ be two cluster points of $\{x^k\}_{k \in \mathbb{N}}$. Set $\eta_1 = \lim_{k \to \infty} \|x^k - x_1^*\|_Q$, and $\eta_2 = \lim_{k \to \infty} \|x^k - x_2^*\|_Q$. Take a subsequence $\{x^{k_i}\}$ weakly converging to $x_1^*$, as $k_i \to \infty$. From the identity

$$\|x^k - x_1^*\|^2_Q - \|x^k - x_2^*\|^2_Q = \|x^* - x_2^*\|^2_Q + 2 \langle (Q x_1^* - x_2^*) - (Q x^* - x_2^*), x_2^* - x^k \rangle,$$

we deduce that $\eta_1 - \eta_2 = -\|x_1^* - x_2^*\|^2_Q$ by taking $k \to \infty$ on both sides. Similarly, take a subsequence $\{x^{k_i}\}$ weakly converging to $x_2^*$, as $i \to \infty$, which yields that $\eta_1 - \eta_2 = \|x_1^* - x_2^*\|^2_Q$. Consequently, $\|x^k - x_2^*\|_Q = 0$, i.e. $x^k \to x_2^* \in \ker Q$. Furthermore, Assumption 2-(iv) yields $\|T x_1^* - T x_2^*\|_Q \leq L \|x_1^* - x_2^*\|_Q = 0$, which results in $T x_1^* = T x_2^*$, and thus, $x_1^* = x_2^*$, since $x_1^*, x_2^* \in \text{Fix} T$. This shows the uniqueness of the weak sequential cluster point, denoted by $x^*$.

Finally, to summarize, $\{x^k\}_{k \in \mathbb{N}}$ is bounded and possesses a unique weak sequential cluster point $x^* \in \text{Fix} T$. Then, the weak convergence is established by [5, Lemma 2.46].

It is much easier to prove the strong convergence of (5) in the case of $\xi \in ]0, 1[$.

**Corollary 3.4 [Strong convergence in $\mathcal{H}$].** Let $x^0 \in D$, $\{x^k\}_{k \in \mathbb{N}}$ be a sequence generated by (5). Under Assumptions 1 and 2, if $\xi \in ]0, 1[$, then there exists $x^* \in \text{Fix} T$, such that $x^k \to x^*$, as $k \to \infty$.

**Proof.** If $\xi \in ]0, 1[$, then $\nu < 1$. For $x^* \in \text{Fix} T$, combining Assumption 2-(iv) with (6), it yields:

$$\|x^{k+1} - x^*\|^2 = \|T x^k - T x^*\|^2 \leq L \|x^k - x^*\|^2_\mathcal{Q} \leq L \nu^k \|x^0 - x^*\|^2_\mathcal{Q},$$

which concludes the strong convergence of $x^k \to x^*$, as $k \to \infty$. 

The following results build the connection of the convergence properties with the cocoerciveness of $T$.

**Proposition 3.5 [Convergence of (5)].** Let $x^0 \in D$, $\{x^k\}_{k \in \mathbb{N}}$ be a sequence generated by (5), with $T$ being $Q$-based $\beta$-cocoercive with $\beta \in [1, +\infty[$. Under Assumptions 1 and 2-(i-ii), the following hold.
Proof. By Lemma 2.11–(i), we have:

\[ \|x^{k+1} - x^k\|_Q \leq \frac{1}{\sqrt{k+1}} \cdot \sqrt{2} - 1 \|x^0 - x^*\|_Q, \quad \forall k \in \mathbb{N}. \]

(iii) [q–linear convergence] If \( \beta \in ]1, +\infty[ \), both \( \|x^k - x^*\|_Q \) and \( \|x^k - x^{k+1}\|_Q \) are q–linearly convergent with the rate of \( \frac{1}{\sqrt{2} - 1} \).

(iv) [r–linear convergence] If \( \beta \in [\sqrt{1 + \frac{1}{2}}, +\infty[ \), \( \|x^k - x^{k+1}\|_Q \) is globally r–linearly convergent w.r.t. \( \|x^0 - x^*\|_Q \):

\[ \|x^k - x^{k+1}\|_Q \leq 2 \sqrt{2} - 1 \cdot (2 \beta - 1)^{-\frac{k}{2}} \|x^0 - x^*\|_Q. \]

(v) [Weak/strong convergence in ran\( Q \)] If \( \beta = 1 \) or \( \beta \in ]1, +\infty[ \), there exists \( x^* \in \text{Fix}T \), such that \( \sqrt{Q} x^k \rightarrow x^* \) respectively, as \( k \rightarrow \infty \).

(vi) [Weak/strong convergence in \( H \) Under Assumptions 1 and 2, if \( \beta = 1 \) or \( \beta \in ]1, +\infty[ \), then there exists \( x^* \in \text{Fix}T \), such that \( x^k \rightarrow x^* \) respectively, as \( k \rightarrow \infty \).

The rest of proof is similar to Proposition 3.2, Theorem 3.3 and Corollary 3.4.

Remark 7. Theorem 3.3 and Corollary 3.4 are closely linked to Proposition 3.5, if \( T \in \mathcal{F}^Q_{\xi, \alpha} \) is also \( \beta \)-cocoercive. This connection can be immediately obtained by Theorem 2.8–(iii). Indeed, if \( \xi \leq \min\{\frac{1}{\alpha}, 1\} \), \( T \in \mathcal{F}^Q_{\xi, \alpha} \) is \( Q \)-firmly nonexpansive (by Theorem 2.8–(ii)), and also \( \beta \)-cocoercive with \( \beta = \frac{1}{2}(1 + \frac{1}{1 - \alpha + \alpha \xi}) \geq 1 \) (by Theorem 2.8–(iii)). According to Theorem 3.3 and Corollary 3.4, \( \xi \in [0, 1] \) is sufficient to guarantee the convergence, while \( T \) is not necessarily \( Q \)-firmly nonexpansive. This implies that the \( Q \)-firm nonexpansiveness of \( T \) is an over–sufficient condition for the convergence of (5).

If \( T \in \mathcal{F}^Q_{\xi, \alpha} \) is \( \beta \)-cocoercive, \( \xi \leq \min\{\frac{1}{\alpha}, 1\} \) guarantees the convergence (by Proposition 3.5), while \( T \) is also \( Q \)-firmly nonexpansive (by Theorem 2.8–(ii)). In this sense, Proposition 3.5 is somewhat a special case of Theorem 3.3 and Corollary 3.4. Note that in Theorem 3.3 and Corollary 3.4, the convergence condition \( \xi \in [0, 1] \) cannot guarantee the cocoerciveness of \( T \). For instance, when \( \alpha \in ]\frac{1}{2}, +\infty[ \) and \( \xi \in ]\frac{1}{\alpha}, 1\] , (5) is convergent, but \( T \) is not cocoercive.
3.3. Krasnosel’skiĭ–Mann Algorithm

Consider the iteration:

\[ x^{k+1} := x^k + \gamma (T x^k - x^k) := T_\gamma x^k, \quad (9) \]

where \( T_\gamma = I - \gamma (I - T) \) and \( T \in \mathcal{F}_{\xi, \alpha}^Q \).

**Corollary 3.6** [Convergence of (9)]. Let \( x^0 \in D \), \( \{x^k\}_{k \in \mathbb{N}} \) be a sequence generated by (9). Denote \( \nu := 1 - \gamma_\alpha + \gamma_\alpha \xi^2 \). Under Assumption 1 and 2-(i-ii), if \( \gamma \in [0, 1/\alpha] \), the following hold.

(i) \( T \) is \( Q \)-asymptotically regular.

(ii) [Sequential error] \( \|x^{k+1} - x^k\|_Q \) has the pointwise sublinear convergence rate of \( O(1/\sqrt{k}) \):

\[
\|x^{k+1} - x^k\|_Q \leq \frac{1}{\sqrt{k+1}} \sqrt{\frac{\gamma_\alpha}{1 - \gamma_\alpha}} \|x^0 - x^*\|_Q, \quad \forall k \in \mathbb{N}.
\]

(iii) [\( q \)-linear convergence] If \( \xi \in [0, 1] \), both \( \|x^k - x^*\|_Q \) and \( \|x^k - x^{k+1}\|_Q \) are \( q \)-linearly convergent with the rate of \( \sqrt{\nu} \).

(iv) [\( r \)-linear convergence] If \( \gamma \alpha \in [1 - \frac{1}{\sqrt{2}}, 1] \), \( \xi \in [0, 1] \), \( \gamma_\alpha \in [0, 1] \), \( \|x^k - x^{k+1}\|_Q \) is globally \( r \)-linearly convergent w.r.t. \( \|x^0 - x^*\|_Q \):

\[
\|x^k - x^{k+1}\|_Q \leq \sqrt{\frac{2\gamma_\alpha (1 - \nu)}{(1 - \gamma_\alpha)\nu}} \nu^{\frac{k+1}{2}} \|x^0 - x^*\|_Q.
\]

(v) [Weak/strong convergence in \( \text{ran} Q \)] If \( \xi = 1 \) or \( \xi \in [0, 1] \), there exists \( x^* \in \text{Fix} T_\gamma \), such that \( \sqrt{Q} x^k \rightharpoonup \text{or} \rightarrow \sqrt{Q} x^* \) respectively, as \( k \to \infty \).

(vi) [Weak/strong convergence in \( \mathcal{H} \)] Under Assumptions 1 and 2, if \( \xi = 1 \) or \( \xi \in [0, 1] \), then there exists \( x^* \in \text{Fix} T_\gamma \), such that \( x^k \rightharpoonup \text{or} \rightarrow x^* \) respectively, as \( k \to \infty \).

**Proof.** First, we claim that \( \text{Fix} T_\gamma = \text{Fix} T \). Indeed, \( x^* \in \text{Fix} T_\gamma \iff x^* = x^* - \gamma (x^* - T x^*) \iff x^* = T x^* \iff x^* \in \text{Fix} T \).

If \( \gamma < \frac{1}{\alpha} \), we deduce by Theorem 2.8–(iv) that:

\[ T \in \mathcal{F}_{\xi, \alpha}^Q \implies \mathcal{R} = \mathcal{I} - T \in \mathcal{F}_{\xi, 1-\alpha, 1}^Q \implies T_\gamma = \mathcal{I} - \gamma \mathcal{R} \in \mathcal{F}_{\xi, \gamma \alpha}^Q. \]

The rest of the proof is similar to Proposition 3.2, Theorem 3.3 and Corollary 3.4, just replacing \( \alpha \) by \( \gamma \alpha \), provided that \( \gamma < \frac{1}{\alpha} \). \( \square \)

4. Application to Metric Resolvent

4.1. Basic Properties

Consider the metric resolvent\(^4\):

\[
T := (A + Q)^{-1} Q, \quad (10)
\]

\(^4\)It is also called \( F \)-resolvent in [4] or warped resolvent [13].
where $A: \mathcal{H} \mapsto 2^\mathcal{H}$ is a set-valued maximally monotone operator, $Q \in \mathcal{M}_2^\perp$. It is easy to show that $T \in \mathcal{F}^Q_{1,\frac{1}{2}}$, $I - T \in \mathcal{F}^Q_{1,\frac{1}{2}}$ [11, 41]. Furthermore, if $A$ is $\mu$-strongly monotone, $T \in \mathcal{F}^Q_{\frac{\|Q\|}{2\mu+\|Q\|}, \frac{\|Q\|}{2\mu+2\|Q\|}}$, $I - T \in \mathcal{F}^Q_{\frac{\|Q\|}{2(\|Q\|+\mu)}}$. Then, the convergence properties of the Banach-Picard iteration:

$$x^{k+1} := (A + Q)^{-1} Q x^k \quad (11)$$

immediately follow from Proposition 3.2, Theorem 3.3 and Corollary 3.4, by substituting $\xi$ and $\alpha$ with proper quantities, if the corresponding assumptions are satisfied.

Considering the Krasnosel’skii-Mann iteration:

$$x^{k+1} := x^k + \gamma ((A + Q)^{-1} Q x^k - x^k) \quad (12)$$

it is easy to show that $T_\gamma \in \mathcal{F}^Q_{1,\frac{1}{2}}$ from the proof of Corollary 3.6. Furthermore, if $A$ is $\mu$-strongly monotone, $T_\gamma \in \mathcal{F}^Q_{\frac{\|Q\|}{2\mu+\|Q\|}, \frac{\|Q\|}{2\mu+2\|Q\|}}$. Then, the convergence properties of (12) follow from Corollary 3.6, if the corresponding assumptions are fulfilled.

4.2. Reinterpretation of Primal-Dual Hybrid Gradient Algorithm

The primal-dual hybrid gradient (PDHG) algorithm, for solving $\min_u f(u) + g(Au)$, is given as [14, 20]:

$$\begin{aligned}
  s^{k+1} &:= \text{prox}_{\sigma g^*} (s^k + \sigma A u^k), \\
  u^{k+1} &:= \text{prox}_{\tau f} (u^k - \tau A^* (2s^{k+1} - s^k)).
\end{aligned} \quad (13)$$

It exactly fits into the form of metric resolvent (11):

$$\begin{bmatrix}
  s^{k+1} \\
  u^{k+1}
\end{bmatrix} = \begin{bmatrix}
  \partial g^* & -A \\
  A^* & \partial f
\end{bmatrix}\begin{bmatrix}
  \frac{1}{\sigma} I \\
  \frac{1}{\tau} I
\end{bmatrix}^{-1} \begin{bmatrix}
  \frac{1}{\sigma} I \\
  \frac{1}{\tau} I
\end{bmatrix} \begin{bmatrix}
  A \\
  Q
\end{bmatrix} \begin{bmatrix}
  s^k \\
  u^k
\end{bmatrix}. \quad (14)$$

For this specific case of (14), we have the following basic observations:

- $A$ is maximally monotone;
- $Q$ is self-adjoint and PSD, if $\tau \sigma \leq \frac{1}{\|A^* A\|};$
- $T \in \mathcal{F}^Q_{1,\frac{1}{2}}$;
- $\text{Fix} T = \text{zer} A$.

Here, Assumption 2-(ii) is satisfied, as long as $\text{zer} A \neq \emptyset$, i.e. there exists a point $(u^*, s^*)$ satisfying the Karush-Kuhn-Tucker conditions. This is a reasonable assumption under this context.

Based on the above results, it is needless to discuss the non-degenerate case when $\tau \sigma < \frac{1}{\|A^* A\|}$. We are mainly concerned with the degenerate metric when $\tau \sigma = \frac{1}{\|A^* A\|}$. We now claim that (14) satisfies Assumption 2-(iv). Indeed,

[^5]: Here, the functions $f$ and $g$ are assumed to be proper, lower semi-continuous and convex.
\[ \|T x_1 - T x_2\| \leq \|(A + Q)^{-1} Q x_1 - (A + Q)^{-1} Q x_2\| \]
\[ \leq \|(A + Q)^{-1}\| \cdot \|\sqrt{Q}\| \cdot \|x_1 - x_2\|_Q, \]

where
\[(A + Q)^{-1} : (s, u) \mapsto (\text{prox}_{\sigma g^*(\sigma s)}, \text{prox}_{\tau f^*}(-2\tau A^* \text{prox}_{\sigma g^*(\sigma s)} + \text{prox}_{\tau f}(\tau u)).\]

This is a composition of Lipschitz functions, and the Lipschitz constant \(L\) is not relevant in this context. Finally, we have verified that (14) with degenerate metric \(Q\) satisfies all of Assumptions 1 and 2, and thus, the results in Sect. 2 and 3 can be applied.

5. Concluding Remarks

We investigated in details the nonexpansive mappings in the context of arbitrary metric, and particularly discussed the convergence of the associated fixed-point iterations under the setting of degenerate metric. There are more prospective applications of our results. Besides from PDHG, more splitting algorithms can be reformulated as the metric resolvent, many of them correspond to degenerate metric. In addition, our results can be extended to analyze more related concepts, e.g. generalized proximity operator, Bregman proximal map, variable metric Fejér sequence, especially equipped with degenerate metric.

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Declarations

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