We are interested in low-dimensional dynamics in an ensemble of coupled nonidentical generalized oscillators on the 3-sphere. The system of governing equations for such an ensemble is referred to as non-Abelian Kuramoto model in the literature. We establish an analogue (or an extension) of the Ott-Antonsen (OA) result for this model.

Keywords: Ott-Antonsen reduction; non-Abelian Kuramoto model; 3-sphere
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There is a growing interest of researchers in various extensions of the classical Kuramoto model to higher dimensions. It has been recognized that these extensions not only provide a paradigm for various collective phenomena, but also may have deep interpretations in Physics and applications in Cooperative Control and Artificial Intelligence.

There are several different ideas on how the Kuramoto model can be extended to higher-dimensional manifolds. One possible approach stems from the observation that the configuration space for Kuramoto oscillators is the unit circle $S^1$ that is one-dimensional Lie group. The frequencies of oscillators belong to the corresponding Lie algebra consisting of purely imaginary complex numbers. From this point of view, it seems natural to extend the model to matrix Lie groups, rather than to vector spaces or spheres. As higher-dimensional matrix Lie groups are not commutative, such models are referred to as non-Abelian Kuramoto models.

One important question is regarding the possibility to extend previous elegant results\(^5,7\) on low-dimensional dynamics in the classical (Abelian) Kuramoto model to higher-dimensional generalizations. These results provide a convenient framework for analytic study of synchronization and other collective phenomena. Besides, they unveil intriguing connections with Complex Analysis, Hyperbolic Geometry and Group Theory.

In the present paper we derive the result on Ott-Antonsen reduction to low-dimensional dynamics in non-Abelian Kuramoto model of nonidentical generalized oscillators evolving on $S^3$.

1. INTRODUCTION

Study of coupled oscillators and synchronization phenomena has a long history and can be traced back to XVII century and famous Huygens’ letter\(^1\). In 1975, Japanese physicist Yoshiki Kuramoto proposed his model that serves as a paradigmatic framework in this study. Kuramoto model\(^2,3\) describes an ensemble of $N$ coupled phase oscillators with the following governing equations

$$
\dot{\varphi}_j = \omega_j + \frac{K}{N} \sum_{i=1}^{N} \sin(\varphi_j - \varphi_i), \quad j = 1, \ldots, N.
$$

Here, $\varphi_j(t)$ and $\omega_j$ are respectively the phase and intrinsic frequency of the $j$-th oscillator, and $K$ is a global (all-to-all) coupling strength.
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In this paper we consider Kuramoto model in its most general form

\[ \dot{\phi}_j = f e^{i\phi_j} + \omega_j + \bar{f} e^{-i\phi_j}, \quad j = 1, \ldots, N. \]  

(2)

Here, \( f = f(\phi_1, \ldots, \phi_N) \) is a global coupling function.

Notice that (1) appears as a partial case of (2) for the specific choice of the coupling function

\[ f = -i \frac{K}{2N} \sum_{i=1}^{N} e^{-i\phi_i}. \]  

(3)

Furthermore, substitution \( z_j = e^{i\phi_j} \) turns (2) into the system of Riccati-like ODE’s on the unit circle in complex plane

\[ \dot{z}_j = i(f z_j^2 + \omega_j z_j + \bar{f}), \quad j = 1, \ldots, N. \]  

(4)

The notion \( \bar{f} \) stands for the complex conjugation of a complex-valued function \( f = f(z_1, \ldots, z_N) \).

One important direction in study of the Kuramoto model investigates low-dimensional dynamics and global variables in the system (2). The first theoretical result of this kind has been reported by Watanabe and Strogatz in their seminal paper\(^4\). Watanabe and Strogatz considered the simplest Kuramoto model with identical phase oscillators (\( \omega_j \equiv \omega \) for all \( j \)) and presented a change of variables that reduce the dynamical system (2) for \( N \) oscillators to the system of only three ODE’s describing the evolution of global variables.

Since the paper\(^4\), low-dimensional dynamics in the Kuramoto model has been well understood\(^5-7\). Recent studies provide a new insight into this topic by investigating hyperbolic gradient flows in the unit disc generated by the Kuramoto model, see\(^8-10\).

Most of the results on low-dimensional dynamics hold for the simplest setup with the global coupling and identical oscillators (\( \omega_j \equiv \omega \)). The exception is the paper\(^7\) of Ott and Antonsen where the system of ODE’s for global variables has been derived also for the case when oscillators are not identical, but their intrinsic frequencies \( \omega_j \) are sampled from a certain prescribed probability distribution on the real line. However, OA reduction requires some additional assumptions that will be emphasized in the present paper.

Ott and Antonsen have demonstrated that the complex order parameter satisfies a simple ODE if intrinsic frequencies are sampled, for instance, from the Cauchy (Lorentzian) distribution on the real line. This distribution is defined by the following probability density function:

\[ g(\omega) = \frac{1}{\pi} \frac{\delta}{(\omega - \omega_0)^2 + \delta^2}, \quad \omega \in (-\infty, +\infty) \]
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where \( \delta \) and \( \omega_0 \) are real parameters of the distribution.

Two additional important assumptions in the derivation of the OA result are that the number \( N \) of oscillators is large, i.e. \( N \to \infty \) and that the initial distribution of phases \( \phi_1(0), \ldots, \phi_N(0) \) is uniform on \([0,2\pi]\).②

2. NON-ABELIAN KURAMOTO MODEL ON THE 3-SPHERE

Since the seminal papers of Kuramoto, numerous variations and generalizations of his model have been introduced and studied. In particular, there is a growing interest in extensions of the Kuramoto model to higher dimensions. There are different ideas on how to extend the Kuramoto model to higher-dimensional manifolds. Some authors introduced generalized Kuramoto models on linear spaces or \( n \)-dimensional spheres, see①①-①③. In the present paper, we study extensions of a slightly different kind that are referred to as non-Abelian Kuramoto models in the literature. These models introduce generalized "oscillators" whose states are described by points on certain Lie groups with the generalized "frequencies" given by the matrices from the corresponding Lie algebras. In order to explain this, notice that the state of each oscillator in the Kuramoto model is described by a single phase \( \phi_j \in [0,2\pi] \), that is - by a point \( z_j = e^{i\phi_j} \) on the unit circle \( S^1 \). Hence, the configuration space for the classical Kuramoto model is \( N \)-torus \( T^N = S^1 \times \cdots \times S^1 \). In its turn, \( S^1 \) is the Lie group isomorphic to \( SO(2) \) and the corresponding Lie algebra \( \mathfrak{s}\mathfrak{o}(2) \) is isomorphic to the real line. Hence, frequencies \( \omega_j \) belong to \( \mathfrak{s}\mathfrak{o}(2) \) and are represented by real numbers.

Non-Abelian Kuramoto (NAK) models are first introduced in papers①④ and①⑤ on some Lie groups, notably on special orthogonal groups \( SO(n) \) or unitary groups \( SU(n) \). This framework includes the classical Kuramoto model as a particular case, obtained for the group \( SO(2) \). Hence the term non-Abelian Kuramoto model that precisely describes the most important difference, as groups \( SO(n) \) and \( SU(n) \) are not Abelian for \( n > 2 \) and \( n > 1 \) respectively. From this point of view, one might refer to the Kuramoto model on \( S^1 \) as Abelian Kuramoto model.

NAK models provide a new paradigm with many potential applications and exciting physical interpretations that are still to be investigated. One direction of research deals with the low-dimensional behavior in NAK models①⑦⑧. However, it seems that universal results of this kind that would be valid for all possible underlying manifolds (i.e. Lie groups) are not possible, due to their different topological properties. In the present paper we restrict our attention to the NAK model on the special unitary group \( SU(2) \) with the group manifold \( S^3 \). This particular model
Ott-Antonsen reduction for the non-Abelian Kuramoto model on the 3-sphere allows for an elegant mathematical analysis and, at the same time, might have many intriguing interpretations in Physics and Biology.

In order to introduce coordinates on $S^3$, we will use the algebra of quaternions. Points on $S^3$ will be represented by unit quaternions $q_j$. The corresponding Lie algebra is isomorphic to the set of pure quaternions. Hence, intrinsic frequencies of generalized oscillators are given by pure quaternions. Then the model in general form is written as the system of quaternion-valued ODE’s (QODE) (see 12)

$$
\dot{q}_j = q_j f(q_j) + w_j q_j - \bar{f}, \quad j = 1, \ldots, N.
$$

Here, $f = f(q_1, \ldots, q_N)$ is a quaternion-valued function that defines the coupling between generalized oscillators. The notion $\bar{f}$ stands for the quaternionic conjugate of the quaternion $f$, i.e. if $f = f_1 + i \cdot f_2 + j \cdot f_3 + k \cdot f_4$, then $\bar{f} = f_1 - i \cdot f_2 - j \cdot f_3 - k \cdot f_4$. For each oscillator $j$, $w_j$ denotes a pure quaternion that represents its (left) intrinsic frequency.

It is easy to verify (see, for instance, 15) that the system (5) preserves $S^3$. In other words, $q_j(0) \in S^3$ for $j = 1, \ldots, N$ implies that $q_j(t) \in S^3$ for all $t > 0$ and $j = 1, \ldots, N$.

Underline that (5) is an extension of the Abelian Kuramoto model (2). Indeed, assume that $M$ is a unit circle on $S^3$ and pure quaternions $w_j$ are of the form $w_j = i \cdot \omega_j + j \cdot 0 + k \cdot 0$. Then, $q_j(0) \in M$ for all $j = 1, \ldots, N$ implies that $q_j(t) \in M$ for all $t > 0$. This follows from the fact that $M$ is a subgroup of $S^3$ with the Lie algebra consisting of pure complex numbers. Hence, restriction of the dynamics (5) from $S^3$ to the invariant submanifold $M$ yields precisely (2).

**Remark 1** As multiplication of quaternions is not commutative, generalized oscillators in NAK models in general can have left and right frequencies. Thus, in the most general setup the system (5) would contain an extra term $q_j u_j$ on the right hand side. Here, $u_j$ is another pure quaternion, representing the right intrinsic frequency of the oscillator $j$. However, we omit this term for the sake of simpler notations throughout the paper. Notice that the presence of the term $q_j u_j$ would require just a slightly more complicated notations in the paper without significant qualitative difference.

Substitution of the coupling function of following form

$$
f = -\frac{K}{2N} \sum_{i=1}^{N} \bar{q}_i
$$

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turns (5) into the "standard" NAK model on $S^3$

$$\dot{q}_j = -\frac{K}{2N} \sum_{i=1}^{N} q_j \bar{q}_i q_j + w_j q_j + \frac{K}{2N} \sum_{i=1}^{N} q_i, \quad j = 1, \ldots, N.$$ (7)

In this context, $K$ is a real number interpreted as a coupling strength.

Low-dimensional dynamics, global variables and constants of motion for the model (5) with identical frequencies have been explained in17. The result therein can be regarded as an analogue (or an extension) of the result of Marvel et al.5 for the Abelian Kuramoto model.

In the present paper we establish an analogue of the OA result for the model on $S^3$ with non-identical frequencies. Analogously to the paper7, we assume that intrinsic frequencies are sampled from a certain prescribed probability distribution on $\mathbb{R}^3$. Along the way, we will introduce several additional restrictive assumptions that are necessary prerequisites in derivation of the OA reduction both for the models on $S^1$ and $S^3$.

In the next Section we derive the OA reduction for the model (5). Notice that our method of derivation differs to some extent from the one in7. Our reasoning is based on results from Quaternionic Analysis. Underline, however, that completely analogous reasoning (with referring to the results from Complex Analysis) can be conducted to obtain OA result for the Abelian Kuramoto model.

In Section4 we address particular case (7) and obtain ODE for the real order parameter.

In Section5 as an another example, we analyze an extension of the model of conformists and contrarians proposed by Hong and Strogatz in16. By writing out ODE’s for global variables in this model we illustrate that the OA reduction technique established on $S^3$ degenerates to the classical one for the Abelian Kuramoto model when restricted onto a low dimensional torus.

Finally, Section6 contains some concluding remarks and the discussion on future directions of research.

3. OA REDUCTION FOR THE MODEL ON THE 3-SPHERE

In this Section, we derive OA result for the general NAK model (5) on $S^3$.

Denote by $O$ the set of all oscillators and by $O_w$ the set of all oscillators whose intrinsic left frequencies are equal to $w$. Furthermore, denote by $Q_0$ the set of all pure quaternions. Notice that $Q_0$ is isomorphic to the set $\mathfrak{su}(2)$ of all skew-Hermitian zero-trace matrices and also to the vector
Ott-Antonsen reduction for the non-Abelian Kuramoto model on the 3-sphere space $\mathbb{R}^3$. Then

$$O = \bigcup_{w \in Q_0} O_w.$$ 

Consider the group $SL(2, \mathbb{H})$ of all quaternionic Möbius (linear fractional) transformations. Denote by $G_{\mathbb{H}}$ the subgroup of $SL(2, \mathbb{H})$ that consists of those transformations that preserve $S^3$. For the definition of quaternionic Möbius transformations and results of Quaternionic Analysis, we refer to $^{19}$.

**Proposition 1** For each pure quaternion $w$, all oscillators from $O_w$ evolve by the action of the same one-parametric family $m_t^w$ from $G_{\mathbb{H}}$. More precisely, for each $t > 0$ there exists a Möbius transformation $m_t^w$, such that

$$q_j(t) = m_t^w(q_j(0)), \quad \text{for all oscillators } j \text{ that belong to } O_w.$$  

Moreover, the mapping $t \to m_t^w$ is continuous in the topology of $G_{\mathbb{H}}$.

**Proof** Oscillators that belong to the same class $O_w$ have equal intrinsic frequencies $w$ and, hence, satisfy the same ODE (5). Let $q_j(t)$ be a solution of (5) with initial condition $q_j(0) = q_0$. Then, oscillators $q_j$ evolve by the action of one-parametric family of Möbius transformations from $G_{\mathbb{H}}$. More precisely, there exists a one-parametric family of transformations $m_t$, such that $q_j(t) = m_t(q_0)$ for each $t > 0$. This is the known fact, it can be proven by a direct substitution and derivation with respect to $t$, see, for instance $^{17}$. The relation between Riccati ODE’s on homogeneous manifolds and linear fractional transformations are exposed in detail in the book $^{20}$. Further, we use the fact that $G_{\mathbb{H}}$ is a Lie group. From the general Lie group theory, it follows that at each $t > 0$ the solution of (5) is given by a certain Möbius transformation of initial point $q_0$, as stated in (8). □

General Möbius transformation from $G_{\mathbb{H}}$ can be written in the following form

$$m_t^w(q) = (\bar{p}q\bar{r} + a)(1 + \bar{a}pqr)^{-1},$$  

where $p$ and $r$ are unit quaternions and $a = a_1 + ia_2 + ja_3 + ka_4$ is a quaternion, such that $|a| = a_1^2 + a_2^2 + a_3^2 + a_4^2 \leq 1$. Hence, $a$ can be identified with a point in the unit ball $B^4$ (interior of $S^3$).

Underline that parameters of Möbius transformation (9) are quaternions $p, r$ and $a$ that depend on $t > 0$ and $w \in Q_0$. Moreover, $p, r$ and $a$ satisfy a certain system of ODE’s with respect to $t$, see $^{17}$. In this Section, we will derive ODE for $a \equiv a(t; w)$ (see Proposition 4 below), as it is one
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of the crucial ingredients in the OA reduction. On the other hand, as we will see in what follows, ODE’s for $p \equiv p(t;w)$ and $r \equiv r(t;w)$ will be irrelevant in the present context, as we will work under the following important assumptions:

(A1) We consider the system in thermodynamic limit, when $N \to \infty$. The distribution of oscillators at each moment $t \geq 0$ is described by a density function $p(q,w,t) : S^3 \times Q_0 \times \mathbb{R}_+ \to \mathbb{R}$. The function $p(q,w,t)$ is continuous with respect to all variables.

(A2) Initial distribution of oscillators is uniform on $S^3$, i.e. $p(q,w,0) = const$. Moreover, for each $w \in Q_0$ initial distribution of oscillators from $O_w$ is uniform on $S^3$, i.e. $p(q,w,0) = const$ for all $w \in Q_0$.

For each $w \in Q_0$, denote by $\zeta^w(t)$ the centroid of all oscillators from $O_w$. If the number of oscillators in $O_w$ is finite, the centroid is simply defined as $\zeta^w(t) = \frac{1}{N_w} \sum_{j \in O_w} q_j(t)$, where $N_w$ is the number of oscillators in $O_w$. However, as we consider the system in thermodynamic limit, $N_w$ is infinite and the notion of centroids requires a bit more careful elaboration. To that aim, we represent a unit quaternion through angular (Hopf) coordinates in the following way

$$q = e^{i\phi} \cos \theta + e^{i\psi} \sin \theta \cdot j,$$

where $\phi, \psi \in [0, 2\pi]$ and $\theta \in (0, \frac{\pi}{2})$.

Introduce the following integrals over $S^3$

$$I_1^w = \int_0^{2\pi} \int_0^{2\pi} \int_0^{\pi/2} \left( e^{i\phi} \cos \theta + e^{i\psi} \sin \theta \cdot j \right) \cdot p(\phi, \psi, \theta, w, t) d\phi d\psi d\theta$$

and

$$I_2^w = \int_0^{2\pi} \int_0^{2\pi} \int_0^{\pi/2} p(\phi, \psi, \theta, w, t) d\phi d\psi d\theta.$$

Centroid of all oscillators that belong to $O_w$ is defined as $\zeta^w(t) = I_1^w / I_2^w$.

**Remark 2** In the proof of Proposition 7 we have used the Lie-group property of $G_\mathbb{H}$. In order to determine the dimension of this group, consider the representation (9) involving parameters $p, r$ and $a$. As $p$ and $r$ are unit quaternions, each of them is determined by three real parameters (for instance, by three angles $\phi, \psi$ and $\theta$, as above). On the other hand, quaternion $a$ can be identified with a point in $B^4$ and includes four real parameters. Hence, general Möbius transformation from $G_\mathbb{H}$ is determined by 10 real parameters and we conclude that the dimension of $G_\mathbb{H}$ is equal to 10.
Remark 3 Denote by $\mathcal{M}(S^3)$ the space of all density functions on $S^3$. Fix an arbitrary $w \in Q_0$ and initial density function. Then, density function $p(q,w,t)$ of the population evolves on a certain finite-dimensional invariant submanifold $\mathcal{M}_0 \subseteq \mathcal{M}(S^3)$ that is determined by the initial distribution $p(q,w,t)$. Due to Proposition 1, $\mathcal{M}_0$ lies in the orbit of Lie group $G_{\mathbb{H}}$. As explained in the previous Remark, the dimension of this Lie group equals 10. Hence, $\mathcal{M}_0$ is a submanifold of a real dimension 10.

Remark 4 In the rest of this Section we will derive some important results under assumptions (A1) and (A2). To this end, we emphasize some special properties of the particular case when the initial distribution of oscillators is uniform on $S^3$. Due to rotational symmetries the dimension of an invariant submanifold $\mathcal{M}_0$ is reduced. Indeed, unit quaternions $p$ and $r$ in (9) are responsible for rotations on $S^3$, and they are irrelevant in the rotational-invariant case. Hence, points on $\mathcal{M}_0$ are determined exclusively by a quaternion $a$ that contains 4 real parameters. We conclude that under assumption (A2), the dimension of invariant submanifold $\mathcal{M}_0$ equals four.

In Remark 4 we have explained that under assumptions (A1) and (A2), the distribution of oscillators evolves on a certain 4-dimensional invariant submanifold $\mathcal{M}_0$. It turns out that this specific submanifold is well known and can be explicitly characterized.

Proposition 2 Under (A1) and (A2) the distribution of oscillators from $O_w$ for any $w \in Q_0$ at each moment $t > 0$ is given by the following density function on $S^3$:

$$p(y,a(t;w)) = \frac{1}{I_2^2} \frac{1}{2\pi^2} \left(1 - |a(t;w)|^2\right)^3 \frac{1}{|y - a(t;w)|^2}, \quad y \in S^3. \quad (10)$$

Here, $a(t;w) \in B^4$ is a parameter of the distribution (10).

The function $p(y,a(t;w))$ is called Poisson kernel on $S^3$.

Proof From Proposition 1 it follows that the distribution of oscillators evolves by the action of $G_{\mathbb{H}}$. This means that at each moment $t$ the distribution of oscillators is obtained as a certain Möbius transformation of an initial distribution. Due to (A2), the initial distribution of oscillators is uniform on $S^3$. However, it is known (see, for instance, 21) that Möbius transformations of the uniform distribution yield Poisson kernels, i.e. distributions described by density functions of the form (10).
Proposition 3  Under assumptions (A1) and (A2), for each $w \in Q_0$ the centroid $\zeta^w(t)$ of the group $O_w$ evolve by the action of $m^w_t$ on the point $0 \in B^4$, that is

$$\zeta^w(t) = m^w_t(0).$$  

Moreover, $\zeta^w(t) = a(t;w)$.

Proof  Due to Proposition 2, distribution of oscillators at each moment $t$ has a density function of the form (10). It is well known (and easy to check) that the centroid of this distribution is $a(t;w)$. We have also established that the submanifold of Poisson kernels is invariant under the action of $G_{\mathbb{H}}$. On the other hand, it is known that the centroid of a Poisson kernel evolves by the same Möbius transformation. More precisely, let $p_2(y,b(t;w)) = m(p_1(y,a(t;w)))$, where $p_1$ is a function of the form (10) and $m \in G_{\mathbb{H}}$. Then we can claim that $p_2$ is also a function of the form (10) and $b(t;w) = m(a(t;w))$. Therefore, $p_2(y,b(t;w)) = p_1(y,m(a(t;w)))$. This reasoning justifies equality (11).

Furthermore, from (A2) we have that $\zeta^w(0) = 0$. Substitution $q = 0$ into (9) yields $m^w_t(0) = a \equiv a(t;w)$. Combining the last observation with (11), we obtain $\zeta^w(t) = a \equiv a(t;w)$.  

In conclusion, parameter $a(t;w)$ turns out to be the quaternionic order parameter for the set $O_w$.

Remark 5  In the proof of Proposition 3, we have used special properties of the class of functions on $S^3$ that are named Poisson kernels. These special properties are related to the notion of conformal barycenter of a probability measure on $S^3$ (and, more generally, on spheres $S^n$). As we do not have enough space to explain this notion here, we refer to the paper of Douady and Earle for the definition of conformal barycenter and this elegant mathematical construction. Poisson kernels are the only functions on $S^3$ for which centroids coincide with conformal barycenters.

In order to shed some light on this, denote by $\mu$ a probability measure on $S^3$ and by $m$ a certain transformation from $G_{\mathbb{H}}$. Let $C(\mu)$ and $B(\mu)$ be respectively centroid and conformal barycenter of the measure $\mu$. Both $C(\mu)$ and $B(\mu)$ are points in $B^4$ (interior of $S^3$). In general, $C(m(\mu)) \neq m(C(\mu))$. However, it is always true that $B(m(\mu)) = m(B(\mu))$. In other words, when a probability measure on $S^3$ is transformed by a Möbius transformation $m$, its conformal barycenter is transformed by the same transformation $m$, while the centroid is not.

However, if the measure $\mu$ has a density function of the form (10), then $C(\mu) = B(\mu)$ and, hence, $C(m(\mu)) = m(C(\mu))$. 

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**Remark 6** For the classical Kuramoto model with identical oscillators, the above facts are exposed in the paper of Marvel et al.\(^5\). Our reasoning up to this point was mainly inspired by this paper.

Notice that exposition therein is more transparent, as Marvel et al. consider the model (2) with identical frequencies \(\omega_j \equiv \omega\). In addition, an explicit formula, relating conformal barycenter and centroid of distributions on \(S^1\) is derived therein, see\(^5\) (56).

**Proposition 4** For each \(w \in Q_0\) the centroid \(a(t; w)\) satisfies the Riccati-like QODE of the form (5), that is
\[
\frac{d}{dt} a(t; w) = a f a + w a - \bar{f}.
\] (12)

**Proof** Due to Proposition \(^3\) \(a(t; w)\) evolves by the action \(m_t^w\). Then, by the same argument as in the proof of Proposition \(^1\) \(a(t; w)\) satisfies QODE (12). \(\square\)

Now, assume that intrinsic frequencies of oscillators are random variables on \(Q_0 \equiv \mathbb{R}^3\) with a certain probability density function \(g(w)\). As \(Q_0\) is isomorphic to \(\mathbb{R}^3\), we think of \(g(w)\) as a p.d.f. of a probability distribution on \(\mathbb{R}^3\).

We introduce quaternionic order parameter of the whole population \(O\). To this aim, write a pure quaternion \(w\) as \(w = w_2 \cdot i + w_3 \cdot j + w_4 \cdot k\), where \(w_2, w_3, w_4 \in \mathbb{R}\). Then the centroid of the whole population \(O\) is an integral of \(a(t; w)\) over all frequencies \(w \in Q_0\)
\[
\zeta(t) = \int_{Q_0} g(w) \cdot a(t; w) \cdot dv(w) = \int \int \int g(w_2, w_3, w_4) a(t; w_2, w_3, w_4) dw_2 dw_3 dw_4. \tag{13}
\]

In the above formula \(v(w)\) is the volume element on \(Q_0 \equiv \mathbb{R}^3\).

**Remark 7** In order to clarify an analogy with the OA result for the classical Kuramoto model, emphasize that QODE (12) is analogous to the complex-valued ODE (6) from\(^2\). Underline, however, that (12) is derived in more general context, since we allow general coupling function \(f\). Hence, for the full analogy one should substitute particular coupling function (6) into (12). This particular case will be written out in the next Section. In addition, integral equation (13) is analogous to the equation (7) in\(^2\).

Now, we have ODE (12) for \(a(t; w)\) and (13) for the quaternionic order parameter \(\zeta(t)\). In general, one cannot derive ODE for \(\zeta\) from this. However, it can be done for some specific probability distributions \(g(w)\). For the most notable example, we introduce one more assumption.
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(A3) The distribution of intrinsic frequencies is the harmonic measure on $Q_0 \equiv su(2)$ with the p.d.f.

$$g(w) = \frac{1}{\pi^2 \left[ (\tilde{q}_2 + \tilde{q}_3 + \tilde{q}_4 - w_2 - w_3 - w_4)^2 + \tilde{q}_1^2 \right]}. \quad (14)$$

Here, $\tilde{q} = \tilde{q}_1 + i \cdot \tilde{q}_2 + j \cdot \tilde{q}_3 + k \cdot \tilde{q}_4$ with $\Re \tilde{q} = \tilde{q}_1 < 0$, and $w$ is a pure quaternion: $w = i \cdot w_2 + j \cdot w_3 + k \cdot w_4$.

The probability distribution with p.d.f. (14) is the 3-dimensional Cauchy (Lorentzian) distribution.

Finally, introduce one more assumption on $a(t; w)$ as a function of $w \in Q_0$.

(A4) For each $t$ the function $a(t; w)$ can be analytically extended from the space $Q_0$ of pure quaternions to the half-space of all quaternions $\tilde{w} = \tilde{w}_1 + \tilde{w}_2 \cdot i + \tilde{w}_3 \cdot j + \tilde{w}_4 \cdot k$, such that $\tilde{w}_1 < 0$.

Assume (A3) and (A4) and substitute (14) into (13). Then Poisson integral formula from Quaternionic Analysis yields

$$\zeta(t) = a(t; \tilde{q}). \quad (15)$$

The last formula concludes the derivation of OA result for the general NAK model on $S^3$. The main results of our reasoning are QODE (12) and the formula (15) for the global quaternionic order parameter $\zeta$.

Now we can analyze different partial cases by plugging specific coupling functions $f$ into (12). This yields different QODE’s for quaternionic order parameter and real-valued ODE’s for real order parameter.

We will illustrate the most common particular case in the next Section.

4. **OA REDUCTION FOR THE STANDARD MODEL**

In this Section we address the particular model (7). In order to do this, it suffices to plug coupling function (6) into QODE (12) and use the equality (15).

Assume that the distribution of frequencies has p.d.f. (14) with the parameter $\tilde{q} = -1 + i \cdot 0 + j \cdot 0 + k \cdot 0$.

Substitution of $\tilde{q}$ and (15) into (12) yields QODE for $\zeta$

$$\frac{d}{dt} \zeta = -\frac{K}{2} \zeta \bar{\zeta} \zeta - \zeta + \frac{K}{2} \zeta. \quad (16)$$
Ott-Antonsen reduction for the non-Abelian Kuramoto model on the 3-sphere

Introduce real order parameter \( \rho = |\zeta|, \ 0 \leq \rho \leq 1 \). Then, QODE (16) can be written as

\[
\frac{d}{dt} \zeta = -\frac{K}{2} \zeta \rho^2 - \left(1 - \frac{K}{2}\right) \zeta.
\]  

(17)

In order to derive ODE for the real order parameter \( \rho \), we represent quaternion \( \zeta \) in the polar form

\[
\zeta = \rho (\cos \theta + \nu \sin \theta) = \rho e^{\theta \nu},
\]

where \( \theta \in (0, 2\pi) \) and \( \nu \) is a pure quaternion.

Substitution of the above polar form into (17) yields

\[
\dot{\rho} e^{\theta \nu} + \rho \dot{\theta} \nu e^{\theta \nu} + \rho \dot{\nu} e^{\theta \nu} = -\frac{K}{2} \rho^3 e^{\theta \nu} - \left(1 - \frac{K}{2}\right) \rho e^{\theta \nu}.
\]

Multiplying the last equation by \( e^{-\theta \nu} \) from the right we get

\[
\dot{\rho} + \rho \dot{\theta} \nu + \rho \dot{\nu} \theta = -\frac{K}{2} \rho^3 - \left(1 - \frac{K}{2}\right) \rho.
\]

(18)

Somewhat unexpectedly, ODE for the real order parameter turns out to be exactly the same as in the classical case, c.f. (11).

We can also consider the model (7) where the p.d.f. (14) is centered at arbitrary quaternion \( \tilde{q} \), with \( \Re \tilde{q} < 0 \). Represent \( \tilde{q} \) in the polar form \( \tilde{q} = r e^{\beta \mu} \). Then QODE for \( \zeta \) is given by

\[
\frac{d}{dt} \zeta = -\frac{K}{2} \zeta \tilde{q} \zeta - \tilde{q} \zeta + \frac{K}{2} \zeta = -\frac{K}{2} \zeta \rho^2 - (re^{\beta \mu} - \frac{K}{2}) \zeta.
\]  

(19)

Using the polar representation of \( \zeta = \rho e^{\theta \nu} \), we obtain

\[
\rho e^{\theta \nu} + \rho \dot{\nu} \theta e^{\theta \nu} = -\frac{K}{2} \rho^3 e^{\theta \nu} - re^{\beta \mu} \rho e^{\theta \nu} + \frac{K}{2} \rho e^{\theta \nu}.
\]

Multiplying by \( e^{-\theta \nu} \) from the right

\[
\dot{\rho} + \rho \dot{\nu} \theta + \rho \dot{\theta} \nu = -\frac{K}{2} \rho^3 - re^{\beta \mu} \rho + \frac{K}{2} \rho,
\]

and we obtain ODE for real order parameter

\[
\rho = \left(\frac{K}{2} - r \cos \beta\right) \rho - \frac{K}{2} \rho^3.
\]  

(20)
Comparing the last ODE with (18) we conclude that changing the center and width of the distribution (14) results in replacing 1 in (18) by a real number $r \cos \beta$. Hence, change of the distribution of intrinsic frequencies (14) does not bring any new dynamical effects, but changes equilibrium values of $\rho$.

5. OA REDUCTION FOR THE CONFORMISTS AND CONTRARIANS MODEL ON THE 3-SPHERE

For the second illustrative example, we extend the model of conformists and contrarians proposed in [16].

Following Hong and Strogatz, consider the population of $N$ oscillators divided into two groups: $N_1$ oscillators are attractively coupled to the mean field (“conformists”), while $N_2$ oscillators are repulsed by the mean field (“contrarians”). Here, $N_1 + N_2 = N$. Denote by $p_1 = N_1/N$, $p_2 = N_2/N$ the portions of conformists and contrarians in the population, $p_1 + p_2 = 1$.

We introduce the upper index in order to specify the group: $q_i^{(1)}$, $i = 1, \ldots, N_1$ stand for the states of conformists, and $q_i^{(2)}$, $i = 1, \ldots, N_2$ are states of the contrarians. Then, each oscillator satisfies QODE (5) with the global coupling function

$$f = \frac{K^{(l)} N_1}{2N} \sum_{i=1}^{N_1} q_i^{(1)} + \frac{K^{(l)} N_2}{2N} \sum_{i=1}^{N_2} q_i^{(2)}, \quad l = 1, 2.$$ 

The difference between the two groups is that $K^{(1)}>0$ for conformists and $K^{(2)}<0$ for contrarians.

Introduce centroids (quaternionic order parameters) of the two groups:

$$\zeta_1 = \frac{1}{N_1} \sum_{i=1}^{N_1} q_i^{(1)}, \quad \zeta_2 = \frac{1}{N_2} \sum_{i=1}^{N_2} q_i^{(2)}.$$

Then, the coupling function is rewritten as

$$f = -\frac{K^{(l)}}{2} (p_1 \bar{\zeta}_1 + p_2 \bar{\zeta}_2), \quad l = 1, 2.$$ 

In order to apply the theoretical result, consider the system in thermodynamical limit and suppose that assumptions (A1)-(A4) are satisfied. Due to (A3), intrinsic frequencies in both groups have the 3-dimensional Cauchy distribution. In general, the distributions within groups can have different parameters. Denote by $\tilde{q}^{(1)}$ and $\tilde{q}^{(2)}$ quaternionic parameters of the probability distributions (14) for conformists and contrarians, respectively. Then, substitution of (15) into (12) and (11) yields the result

$$f = -\frac{K^{(l)}}{2} (p_1 \bar{\tilde{q}}^{(1)} + p_2 \bar{\tilde{q}}^{(2)}), \quad l = 1, 2.$$ 

The difference between the two groups is that $K^{(1)}>0$ for conformists and $K^{(2)}<0$ for contrarians.
taking \( K^{(1)} = -K^{(2)} = K \) yield QODE’s for quaternionic order parameters

\[
\dot{\zeta}_1 = -\frac{p_1K}{2}\zeta_1\zeta_1 - \frac{p_2K}{2}\zeta_1\tilde{\zeta}_2 + \tilde{\phi}^{(1)}\zeta_1 + \frac{p_1K}{2}\zeta_1 + \frac{p_2K}{2}\zeta_2 \\
\dot{\zeta}_2 = -\frac{p_2K}{2}\zeta_2\zeta_2 - \frac{p_1K}{2}\zeta_2\tilde{\zeta}_1 + \tilde{\phi}^{(2)}\zeta_2 + \frac{p_2K}{2}\zeta_2 + \frac{p_1K}{2}\zeta_1.
\]

(21)

In order to be more specific, suppose that the parameters of probability distributions of frequencies for the two populations are given by \( \tilde{\phi}^{(1)} = (\tilde{\phi}_1^{(1)}, \tilde{\phi}_2^{(1)}, \tilde{\phi}_3^{(1)}, \tilde{\phi}_4^{(1)}) = (-\gamma_1, 0, 0, 0) \) and \( \tilde{\phi}^{(2)} = (\tilde{\phi}_1^{(2)}, \tilde{\phi}_2^{(2)}, \tilde{\phi}_3^{(2)}, \tilde{\phi}_4^{(2)}) = (-\gamma_2, 0, 0, 0) \) respectively. This essentially means that mean frequencies in both populations are zero. Real numbers \( \gamma_1 > 0 \) and \( \gamma_2 > 0 \) stand for "widths" of the distributions of frequencies for the two groups.

Substituting these parameters into (21) we get

\[
\dot{\zeta}_1 = -\frac{p_1K}{2}\rho_1^2\zeta_1 - \frac{p_2K}{2}\rho_1^2\zeta_1\zeta_1 - \gamma_1\zeta_1 + \frac{p_1K}{2}\zeta_1 + \frac{p_2K}{2}\zeta_2 \\
\dot{\zeta}_2 = -\frac{p_2K}{2}\rho_2^2\zeta_2 - \frac{p_1K}{2}\rho_2^2\zeta_2\zeta_2 - \gamma_2\zeta_2 + \frac{p_2K}{2}\zeta_2 + \frac{p_1K}{2}\zeta_1.
\]

(22)

Set \( \zeta_1 = \rho_1 u_1, \zeta_2 = \rho_2 u_2 \), where \( u_1 \) and \( u_2 \) are unit quaternions and \( 0 \leq \rho_1, \rho_2 \leq 1 \) are real order parameters for conformists and contrarians respectively. In order to introduce the new coordinates, we represent unit quaternions by the three angles:

\[
\zeta_1 = \rho_1(e^{i\varphi_1} \cos \theta_1 + e^{i\psi_1} \sin \theta_1 \cdot j), \quad \zeta_2 = \rho_2(e^{i\varphi_2} \cos \theta_2 + e^{i\psi_2} \sin \theta_2 \cdot j)
\]

where \( \varphi_1, \varphi_2, \psi_1, \psi_2 \in \left[0, 2\pi\right] \) and \( \theta_1, \theta_2 \in \left(0, \frac{\pi}{2}\right) \). Then, the system of QODE’s (22) in variables \( (\rho_1, \varphi_1, \psi_1, \theta_1, \theta_2) \) reads

\[
\begin{aligned}
\dot{\rho}_1 &= -\frac{p_1K}{2}\rho_1^3 - \gamma_1\rho_1 + \frac{p_1K}{2}\rho_1 - \frac{p_2K}{2}\rho_2 \cos(\varphi_1 - \varphi_2)(\rho_1^2 - 1) \cos \theta_1 \cos \theta_2 - \frac{p_2K}{2}\rho_2 \sin(\psi_1 - \psi_2)(\rho_1^2 - 1) \sin \theta_1 \sin \theta_2; \\
\dot{\varphi}_1 &= -\frac{p_2K}{2}\rho_2 \rho_1 \sin(\varphi_1 - \varphi_2) \cos \theta_1 \cos \theta_2 + \tan \theta_1 \sin \theta_1 \sin \theta_2 + \frac{p_2K}{2}\rho_2 \cos \theta_1 \sin(\varphi_2 - \varphi_1); \\
\dot{\psi}_1 &= -\frac{p_2K}{2}\rho_2 \rho_1 \sin(\psi_1 - \psi_2) \sin \theta_1 \sin \theta_2 + \cot \theta_1 \cos \theta_1 \sin \theta_2 + \frac{p_2K}{2}\rho_2 \sin \theta_1 \sin(\psi_2 - \psi_1); \\
\dot{\theta}_1 &= \frac{p_2K}{2}\rho_1 \rho_2 \cos(\psi_2 - \psi_1) \cos \theta_1 \sin \theta_2 + \frac{p_2K}{2}\rho_1 \rho_2 \sin(\varphi_2 - \varphi_1) \sin \theta_1 \cos \theta_2 - \frac{p_2K}{2}\rho_1 \cos(\varphi_2 - \varphi_1) \cos \theta_1 \cos \theta_2 - \frac{p_2K}{2}\rho_1 \sin(\psi_2 - \psi_1) \sin \theta_1 \cos \theta_2.
\end{aligned}
\]
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for conformists, and

\[
\begin{align*}
\dot{\rho}_2 &= \frac{\gamma_2}{2} \rho_2^3 - \frac{\gamma_2}{2} \rho_2 - \frac{\rho_2}{2} \rho_2 + \frac{\rho_1}{2} \rho_1 \cos(\varphi_2 - \varphi_1)(\rho_2^2 - 1) \cos \theta_1 \cos \theta_2 + \\
&\quad + \frac{\rho_1}{2} \rho_1 \cos(\psi_2 - \psi_1)(\rho_2^2 - 1) \sin \theta_1 \sin \theta_2; \\
\dot{\phi}_2 &= -\frac{\rho_1}{2} \rho_1 \rho_2 \sin(\varphi_1 - \varphi_2)(\cos \theta_1 \cos \theta_2 + \tan \theta_1 \sin \theta_2) + \\
&\quad + \frac{\rho_1}{2} \rho_1 \cos \theta_1 \sin(\varphi_2 - \varphi_1); \\
\dot{\psi}_2 &= -\frac{\rho_1}{2} \rho_1 \rho_2 \sin(\psi_1 - \psi_2)(\sin \theta_1 \sin \theta_2 + \cot \theta_2 \cos \theta_2 \sin \theta_1) + \\
&\quad + \frac{\rho_1}{2} \rho_1 \sin \theta_1 \sin(\psi_2 - \psi_1); \\
\dot{\theta}_2 &= -\frac{\rho_1}{2} \rho_1 \rho_2 \cos(\varphi_2 - \varphi_1) \cos \theta_2 \sin \theta_1 + \frac{\rho_1}{2} \rho_2 \cos(\varphi_2 - \varphi_1) \sin \theta_2 \cos \theta_1 + \\
&\quad + \frac{\rho_1}{2} \rho_2 \cos(\varphi_2 - \varphi_1) \cos \theta_1 \sin \theta_2 - \cos(\psi_2 - \psi_1) \sin \theta_1 \cos \theta_2)
\end{align*}
\]

for contrarians.

Detailed bifurcation analysis of this system seems very demanding, if possible at all. However, some partial conclusions can be obtained analytically. For instance, one can reduce the dynamics to the torus \(T\) by setting \(\theta_1(t) = \theta_2(t) = \theta_0\). Denote by \(\Psi = \psi_1 - \psi_2\) and \(\Phi = \varphi_1 - \varphi_2\). Noticing that \(\dot{\theta}_1(t) = \dot{\theta}_2(t) = 0\), we have \(\cos \Phi = \cos \Psi\). Then, the above dynamics can be reduced to

\[
\begin{align*}
\dot{\rho}_1 &= -\gamma_1 \rho_1 - \frac{\rho_1}{2} \rho_1(\rho_1^2 - 1) - \frac{\rho_2}{2} \rho_2 \cos \Phi(\rho_2^2 - 1) \\
\dot{\rho}_2 &= -\gamma_2 \rho_2 + \frac{\rho_2}{2} \rho_2(\rho_2^2 - 1) + \frac{\rho_1}{2} \rho_1 \cos \Phi(\rho_2^2 - 1)
\end{align*}
\]

with the angular difference \(\Phi\) governed by

\[
\Phi = K \left[ \frac{(p_1 - p_2)}{2} \rho_1 \rho_2(\cos^2 \theta_0 - \tan \theta_0 \sin^2 \theta_0) + \left( \frac{p_1 \rho_1}{2} \rho_2 - \frac{p_2 \rho_2}{2} \rho_1 \right) \right] \sin \Phi.
\]

At the steady state \(\Phi = 0\), system \((23)\) is precisely the system obtained by Hong and Strogatz in\textsuperscript{16}. This reflects the fact that our result for the model on \(S^3\) is an extension of the previous result of Ott and Antonsen.

Hence, restriction of the dynamics to the tori on the 3-sphere yields the system of ODE’s from\textsuperscript{16}. Underline, however, that we do not claim stability of these tori, as we were not able to prove it rigorously.

6. CONCLUSION AND OUTLOOK

We have derived an analogue of OA result for the NAK model on the 3-sphere \(S^3\). This makes a new step in the understanding of low-dimensional dynamics for the model on \(S^3\), since the case of identical oscillators has been exposed in\textsuperscript{17}.
Previously, analogous results for the Abelian Kuramoto model have been reported in \cite{2} for non-identical oscillators and in \cite{5} for identical oscillators.

As we have pointed out in Section 2, the Kuramoto model on $S^3$ is of a special interest, due to the enormous role of the group $SU(2)$ in Physics.

On our way we had to introduce several important assumptions (A1)-(A4). Assumptions (A1) and (A2) are very restrictive, reflecting the fact that OA result is valid only in thermodynamic limit and only if the initial distribution of oscillators is uniform. This makes OA reduction an excellent analytic tool in study of various effects of heterogeneity in large population on synchronization process. However, it requires the caution when trying to verify these theoretical results experimentally, as such setup can be achieved only approximately in simulations.

We emphasize that absolutely analogous restrictive assumptions are also required for the classical OA result on $S^1$. Assumptions (A1) and (A2) have been frequently overseen in the literature dealing with (or exploiting) OA reduction.

Our derivation is based on some theorems from Quaternionic Analysis. Classical OA result can be derived using analogous reasoning and analogous facts from Complex Analysis. In particular, one of key tools in the paper \cite{2} of Ott and Antonsen is the Poisson integral formula from Complex Analysis. In the same manner, we have referred to the Poisson integral formula from Quaternionic Analysis in order to deal with heterogeneous population on $S^3$.

Note that Ott and Antonsen in their paper have presented a somewhat different way of reasoning. Their derivation starts from the continuity equation (the first order PDE) for the density function on the circle. The second step is to plug Poisson kernel (decomposed into Fourier series) into this PDE as an ansatz. We have avoided this way of reasoning, as it would require introducing a continuity equation on $S^3$ and representing Poisson kernels on $S^3$ as Fourier series of spherical harmonics with quaternionic coefficients.

We have conducted our analysis in Section 3 for the general model \cite{5}, with an arbitrary coupling function $f$. After that, in Section 4, we have obtained ODE’s for order parameters for the particular model \cite{7}.

This approach paves the way to consider various coupling functions instead of \cite{6}. Ott and Antonsen have derived their result for the standard coupling function at the first place, but adapted it afterwards in order to analyze the Kuramoto-Sakaguchi model and the system with external forcing. Our approach allows to do something similar easily, however, non-Abelian Kuramoto-Sakaguchi model and the model with external forcing on $S^3$ are still to be precisely introduced.
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As we have mentioned in the introduction, the NAK model on $S^3$ is an extension of the classical Kuramoto model. Hence the reduction established in this paper should degenerate to the previous OA result when considering the classical Kuramoto model. This is shown in Section 5 through the example of the conformists and contrarians model. We find that the order parameter for model on $S^1$ also evolves on a low dimensional torus of the phase space where the quaternion-valued order parameter belongs to. However, there is still a long way to give a clear bifurcation scenario for the conformists and contrarians model on $S^3$, because attractiveness of the torus has not been proven, which might require a subtle global analysis from ODE theory.

At several crucial points of our derivation we have used the group property of $S^3$. It seems that $S^1$ and $S^3$ are the only spheres where such results on low-dimensional behavior can be established ($S^7$ might be the third possible exception). This implies the more general question: For which manifolds similar results about low-dimensional behavior in non-Abelian Kuramoto model can be obtained? This remains an open (and important, in our point of view) question. We presume that the answer might be related to Riccati ODE’s on manifolds and Siegel domains. In any case, this seems like a mathematically demanding question.

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