HEAT–VISCOELASTIC PLATE INTERACTION: ANALYTICITY, SPECTRAL ANALYSIS, EXPONENTIAL DECAY

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Abstract. We consider a heat-plate interaction model where the 2-dimensional plate is subject to viscoelastic (strong) damping. Coupling occurs at the interface between the two media, where each components evolves. In this paper, we apply “low”, physically hinged boundary interface conditions, which involve the bending moment operator for the plate. We prove three main results: analyticity of the corresponding contraction semigroup on the natural energy space; sharp location of the spectrum of its generator, which does not have compact resolvent, and has the point $\lambda = -1/\rho$ in its continuous spectrum; exponential decay of the semigroup with sharp decay rate. Here analyticity cannot follow by perturbation.

1. Introduction. Fluid-structure interactions models in the physical dimensions $d = 2, 3$, have been the object of mathematical studies in the past several years, particularly since the appearance of [16] which in turn followed [29, p.121]. In this setting, the typical configuration sees a structure – modeled by a $d$-dimensional wave-type equation – surrounded by a fluid – modeled by the $d$-dimensional dynamic Stokes equation involving pressure. Thus, the overall system provides a physical illustration of hyperbolic-parabolic coupling. Moreover, coupling takes place at the common interface of the two media where the individual equations evolve and is given by differential operators. In a preliminary step, the structure was taken at first to have static interface, a case justified to be appropriate under the assumption of small, rapid oscillations of the structure [16]; see [1],[2]-[8] [9], [10], [20], [21], [23], [26], [31] for a certainly non-exhaustive list of works. These studies have provided considerable insight in the dynamical properties of the overall coupled system. Moreover, they have revealed the mathematical challenges imposed by the interface (boundary) coupling with mismatched dynamics, parabolic traces versus hyperbolic traces. This preliminary work has thus proved beneficial for subsequent

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studies, still under investigation, which have tackled the physically more attractive and mathematically more challenging case where the structure in fact moves within the fluid [17, 18, 19].

On a different but complementary direction, stimulated by biological models, it has been proposed to consider the case of a structure possessing visco-elastic damping. A comprehensive study of this case was given in [28] where the structure is a d-dimensional wave and is endowed with strong damping (same order operator), and where - as a first step in the investigation - the fluid was replaced by the simpler d-dimensional heat equation. Here abstract results of the late 80s in [12, 13, 14, 15] on damped abstract equations may only provide a motivation as well as a direct suggestion of what basic dynamical property may arise in describing the overall coupled interaction model: this is, in fact, analyticity of the s.c. semigroup describing the overall coupled dynamics. Such configuration belongs therefore now to parabolic-parabolic coupling. This and much more is established in [28], [35], [36], [37] initially in the case of a heat-viscoelastic wave interaction model. A natural generalization to the case where the heat is replaced by a fluid (with pressure) followed in [38]. This latter case introduces additional challenges. Plainly, the abstract results of the late 80s [12, 13, 14, 15] on “strongly” damped second order abstract equations (stimulated by [11]) are not directly applicable to such interactive PDE- models with strong damping of the structure, as they concerned differential operators with homogeneous boundary conditions, while fluid/heat viscoelastic wave models are driven by highly coupled boundary conditions at the interface.

The present paper is a successor of [28]. Its presentation at conferences was typically accompanied by the question: what if the structure is now a plate equation? Then, different models of the plate and its “strong” damping are possible. Here we provide a first illustration of a heat-visco-elastic physically 2-dimensional plate interaction model, with “low”, physically hinged coupled boundary conditions on the plate at the interface, thus involving only the boundary operator $B_1$. A subsequent work will consider a visco-elastic plate involving high (so called “free-type”) BC and hence will include both boundary operators $B_1$ (second order) and $B_2$ (third order), as in [27, Chapter3], [24], [25], etc.

2. The coupled PDE model, main results.

2.1. A physical 2-dimensional visco-elastic plate coupled with heat at the interface.

![Fig. 1: The Fluid-Structure Interaction](image-url)
Throughout, \( \Omega_f \subset \mathbb{R}^2 \) will denote the bounded domain on which the heat component of the coupled PDE system evolves. Its boundary will be denoted here as \( \partial \Omega_f = \Gamma_f \cup \Gamma_s \), \( \Gamma_f \cap \Gamma_s = \emptyset \), with each boundary piece being sufficiently smooth. Moreover, the geometry \( \Omega_s \), also 2-dimensional, immersed within \( \Omega_f \), will be the plate domain on which the structure component evolves with time. The coupling between the two distinct fluid (heat) and elastic dynamics occurs across the boundary interface \( \Gamma_s = \partial \Omega_s \), see Figure 1 in the previous page.

In addition, the unit normal vector \( \nu(x) \) will be directed away from \( \Omega_f \) on \( \Gamma_f \), and away from \( \Omega_s \) on \( \Gamma_s \). This choice will allow us to use the setting of [27, Chapter 3], regarding the plate on \( \Omega_s \), and quote from this reference the relevant formulas.

On this geometry in Figure 1, we thus consider the following fluid (heat) - structure PDE model in solution variables \( u = u(t, x) \) (the heat component here replacing the usual velocity field) and \( w = w(t, x) \) (the structure displacement field):

(PDE) \[
\begin{aligned}
&u_t - \Delta u = 0 \quad \text{in } (0, T) \times \Omega_f \quad (2.1a) \\
&w_{tt} + \Delta^2 w + \rho \Delta^2 w_t = 0 \quad \text{in } (0, T) \times \Omega_s \\
&w|_{\Gamma_s} = 0; \; u|_{\Gamma_s} = \frac{\partial w_t}{\partial \nu} \bigg|_{\Gamma_s} \quad \text{in } (0, T) \times \Omega_s \quad (2.1c)
\end{aligned}
\]

(BC) \[
\begin{aligned}
&\{\Delta(w + \rho w_t) + (1 - \mu)B_1(\rho w_t)|_{\Gamma_s} = \frac{\partial u}{\partial \nu} \bigg|_{\Gamma_s} \quad \text{in } (0, T) \times \Omega_s \\
&u|_{\Gamma_f} = 0 \quad \text{in } (0, T) \times \Gamma_f
\end{aligned}
\]

(IC) \[ [w(0, \cdot), w_t(0, \cdot), u(0, \cdot)] = [w_0, w_1, u_0] \in \mathbf{H}. \quad (2.1f) \]

Here, \( 0 < \rho \leq 1 \) is a constant that measures the strength of the viscoelastic damping term. Moreover, \( 0 < \mu < 1 \) is a constant (Poisson modulus, actually physically \( 0 < \mu < 1/2 \)), while \( B_1 \) is the boundary bending moment operator [27, (3.5.2a) p.208; (3C.2) p.298], [24]:

\[
B_1u = 2\nu_1\nu_2f_{xy} - \nu_1^2f_{yy} - \nu_2^2f_{xx} = -\frac{\partial^2f}{\partial \tau^2} - c(\eta)\frac{\partial f}{\partial \nu} \quad \text{on } \Gamma_s,
\]

in terms of tangent vector \( \tau = [-\nu_2, \nu_1] \) and outward normal vector \( \nu = [\nu_1, \nu_2] \) on \( \Gamma_s \). Moreover, \( c(\eta) = \text{div} \nu(\eta) \) is the mean curvature at the point \( \eta = \Gamma_s \), [27, p.298]. Accordingly, the space of well-posedness is taken to be the finite energy space

\[
\mathbf{H} \equiv \mathcal{A}_D \times L_2(\Omega_s) \times L_2(\Omega_f), \quad (2.3a)
\]

where \( \mathcal{A}_D : L_2(\Omega_s) \supset \mathcal{D}(\mathcal{A}_D) \to L_2(\Omega_s) \) is defined by

\[
\mathcal{A}_D f = \Delta f, \quad \mathcal{A}_D = H^2(\Omega_s) \cap H_0^1(\Omega_s).
\]

\( \mathbf{H} \) is a Hilbert space with the following norm-inducing inner product, where

\[
(f, g)_\Omega = \int_\Omega fg \, d\Omega:
\]

\[
\left( \begin{array}{c}
\begin{array}{c}
\nu_{1} \\
\nu_{2} \\
h
\end{array}
\end{array} \right)_\mathbf{H} = (\Delta \nu_1, \Delta \nu_1)_\Omega + (\nu_2, \nu_2)_\Omega + (h, h)_\Omega. \quad (2.4)
\]
Remark 2.1. Then physically more attractive case where $B_1 w_1$ in (2.1d) is replaced by $B_1 (w + w_1)$ is discussed in Subsection 2.5.

2.2. Abstract model of Problem (2.1): The operator $\mathcal{A}$. We rewrite problem (2.1a-f) as a first order equation:

$$
\frac{d}{dt} \begin{bmatrix} w \\ w_t \\ u \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ -\Delta^2 & -\rho \Delta^2 & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} w \\ v_1 \\ v_2 \\ \partial h \end{bmatrix} = \mathcal{A} \begin{bmatrix} w \\ v_1 \\ v_2 \\ h \end{bmatrix} \quad \text{(2.5)}
$$

where we have introduced the operator $\mathcal{A} : H \supset \mathcal{D}(\mathcal{A}) \to H$:

$$
\mathcal{A} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ -\Delta^2 & -\rho \Delta^2 & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} -\Delta^2 (v_1 + \rho v_2) \\ -\Delta^2 v_1 \\ \Delta h \end{bmatrix} \in H \quad \text{(2.6)}
$$

for $[v_1, v_2, h] \in \mathcal{D}(\mathcal{A})$. A description of $\mathcal{D}(\mathcal{A})$ is as follows:

(i) $v_1, v_2 \in \mathcal{D}(A_D) = H^2(\Omega_s) \cap H^1_0(\Omega_s)$, so that

$$
v_1|_{\Gamma_s} = 0, \quad v_2|_{\Gamma_s} = 0, \quad \frac{\partial v_2}{\partial n}|_{\Gamma_s} \in H^{1/2}(\Gamma_s) \quad \text{(2.7a)}
$$

$$
\left\{ \begin{array}{l}
\Delta^2 (v_1 + \rho v_2) = f \in L_2(\Omega_s) \\
[\Delta (v_1 + \rho v_2) + (1 - \mu) B_1 (\rho v_2)]|_{\Gamma_s} = \frac{\partial h}{\partial n}|_{\Gamma_s} \in H^{-1/2}(\Gamma_s) \\
B_1 (v_2)|_{\Gamma_s} = -c(\eta) \frac{\partial v_2}{\partial n}|_{\Gamma_s} \quad \text{first order} \end{array} \right. \quad \text{(2.7b)}
$$

by (2.2) with $v_2|_{\Gamma_s} = 0$, so that $\frac{\partial^2 v_2}{\partial n^2}|_{\Gamma_s} = 0$;

(ii) $h \in H^1(\Omega_f)$ is the solution of

$$
\Delta h \in L_2(\Omega_f), \quad h|_{\Gamma_f} \equiv 0, \quad h|_{\Gamma_s} = \frac{\partial v_2}{\partial n}|_{\Gamma_s} \in H^{1/2}(\Gamma_s) \quad \text{(2.8)}
$$

so that $\frac{\partial h}{\partial n}|_{\Gamma_s} \in H^{-1/2}(\Gamma_s)$, [28, (1.9)], as asserted in (2.7c).

(iii) Consider the elliptic problem as in [27, (3.12.9), page239], with $r \in \mathbb{R}$

$$
\left\{ \begin{array}{l}
\Delta^2 \phi = f \in L_2(\Omega_s) \\
\phi|_{\Gamma_s} = 0, \quad [\Delta (1 - \mu) B_1 \phi]|_{\Gamma_s} = g \in H^r(\Gamma_s). \end{array} \right. \quad \text{(2.9a)}
$$

Call its solution $\phi_1$ if $g \equiv 0$, and $\phi_2$ if $g \equiv 0$, so that $\phi = \phi_1 + \phi_2$.

Call $\mathcal{A}$ the operator

$$
\mathcal{A} \psi = \Delta^2 \psi, \quad \mathcal{D}(\mathcal{A}) = \{ \psi = H^4(\Omega_s) \cap H^1_0(\Omega_s) : \Delta \psi + (1 - \mu) B_1 \psi = 0 \text{ on } \Gamma_s \}. \quad \text{(2.9c)}
$$

Then $\mathcal{A}$ is a strictly positive self-adjoint operator [27, Proposition 3C.4, p.301, where the symbol $\mathcal{A}$ is used]. We then have $\phi_1 = \mathcal{A}^{-1} f \in \mathcal{D}(\mathcal{A}) \subset H^4(\Omega_s) \cap H^1_0(\Omega_s)$. We recall that $\mathcal{D}(\mathcal{A}^{1/2}) = \mathcal{D}(A_D)$, [27, p.302]: the characterization of $\|\mathcal{A}^{1/2} w\|^2$ will be given in (2.21c) below. Next, as in [27, (3.12.9), p.239], call $G$ the Green map $\phi_2 = Gg$ (with $f \equiv 0$). Then by [30, p.189, with $m = 4, m_j = 2$] for any real $r \geq -1/2$ (our case):

$$
g \in H^r(\Gamma_s) \rightarrow \phi_2 = Gg \in H^{r+\frac{\gamma}{2}}(\Omega_s) \cap H^1_0(\Omega_s). \quad \text{(2.9d)}
$$
Problem (2.7b)-(2.7c) differs from problem (2.9a)-(2.9b) by the lower order boundary term \( B_1 v_1 = -c(\eta) \frac{\partial v_1}{\partial \nu} |_{\Gamma_r} \) and thus the same gain of regularity as described for problem (2.9a)-(2.9b) applies to problem (2.7b)-(2.7c) with \( g = \frac{\partial h}{\partial \nu} |_{\Gamma_r} \in H^{-1/2}(\Gamma_s) \), i.e., \( r = -1/2 \), so that \( Gg = G \frac{\partial h}{\partial \nu} |_{\Gamma_r} \in H^2(\Omega_s) \cap H^1_0(\Omega_s) \) by (2.9d). Thus, ultimately, with respect to the description of \( \mathcal{D}(A) \) related to problem (2.7a)-(2.7c), we find, consistently with part (i), that
\[
v_1 + \rho v_2 \in H^2(\Omega_s) \cap H^1_0(\Omega_s) . \tag{2.10}
\]

**Remark 2.2.** The above description of \( \mathcal{D}(A) \) in (2.7)-(2.10) shows that the point \( \{v_1, v_2, h\} \in \mathcal{D}(A) \) enjoys a smoothing of regularity by two Sobolev units of the coordinate \( v_2 \) from \( L^2(\Omega_s) \) to \( H^2(\Omega_s) \), and a smoothing of regularity by one Sobolev unit of the coordinate \( h \) from \( L^2(\Omega_f) \) to \( H^1(\Omega_f) \), with respect to the original finite energy space \( \mathbf{H} \) in (2.3). In contrast, the first coordinate of \( v_1 \) experiences no smoothing: it is in \( \mathcal{D}(A_D) = H^2(\Omega_s) \cap H^1_0(\Omega_s) \), the first coordinate component of the space \( \mathbf{H} \). This amounts to the fact that \( A \) has non-compact resolvent \( R(\lambda, A) \) on \( \mathbf{H} \). Consistently, we shall see below in Proposition 4.1 (ii) that the point \( \lambda = -1/\rho \) belongs to the continuous spectrum of \( A: -1/\rho \in \sigma_c(A) \).

**Remark 2.3.** Throughout this paper, the constant \( \rho \) is fixed and \( 0 < \rho \leq 1 \), except for a few results (Theorem 2.2; Proposition 4.3), which include the case \( \rho = 0 \). For this reason, we have chosen not to note explicitly the dependence of \( A \) upon \( \rho \).

### 2.3. The adjoint operator \( A^* \)

We find useful to introduce the adjoint operator \( A^* \).

**Theorem 2.1.** The \( \mathbf{H} \)-adjoint operator \( A^* \) of the original operator \( A \) in (2.6)-(2.10) is the following operator
\[
A^* \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{h} \end{bmatrix} = \begin{bmatrix} 0 & -I & 0 \\ \Delta^2 & -\rho \Delta^2 & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{h} \end{bmatrix} = \Delta^2(\hat{v}_1 - \rho \hat{v}_2) \in \mathbf{H} \tag{2.11}
\]

for \( [\hat{v}_1, \hat{v}_2, \hat{h}] \in \mathcal{D}(A^*) \), which is characterized as follows:

(i) \( \hat{v}_1, \hat{v}_2 \in \mathcal{D}(A_D) = H^2(\Omega_s) \cap H^1_0(\Omega_s) \), so that
\[
\hat{v}_1|_{\Gamma_s} = 0, \quad \hat{v}_2|_{\Gamma_s} = 0, \quad \frac{\partial \hat{v}_2}{\partial \nu} |_{\Gamma_s} \in H^{1/2}(\Gamma_s); \tag{2.12a}
\]
\[
\Delta^2(\hat{v}_1 - \rho \hat{v}_2) \in L^2(\Omega_s) \tag{2.12b}
\]
\[
[\Delta(\hat{v}_1 - \rho \hat{v}_2) - (1 - \mu) B_1(\rho \hat{v}_2)]|_{\Gamma_s} = -\frac{\partial \hat{h}}{\partial \nu} |_{\Gamma_s} \in H^{-1/2}(\Gamma_s) \tag{2.12c}
\]
\[
B_1 \hat{v}_2 = -c(\eta) \frac{\partial \hat{v}_2}{\partial \nu} |_{\Gamma_s} \quad \text{(first order)} \tag{2.12d}
\]
by (2.2), since again \( \hat{v}_2|_{\Gamma_s} = 0 \) and then \( \frac{\partial^2 \hat{v}_2}{\partial \nu^2} |_{\Gamma_s} = 0 \).

(ii) \( \hat{h} \in H^1(\Omega_f) \) is the solution of
\[
\Delta \hat{h} \in L^2(\Omega_f), \quad \hat{h}|_{\Gamma_f} = 0, \quad \hat{h}|_{\Gamma_s} = \frac{\partial \hat{v}_2}{\partial \nu} |_{\Gamma_s} \in H^{1/2}(\Gamma_s) \tag{2.13}
\]
so that \( \frac{\partial \hat{h}}{\partial \nu} |_{\Gamma_s} \in H^{-1/2}(\Gamma_s) \), as asserted in (2.12c), [28, (1.9)].
The computational proof is relegated to Appendix A. On the space $H$ introduce the following bounded, symmetric operator

$$
T \equiv \begin{bmatrix}
I & 0 & 0 \\
0 & -I & 0 \\
0 & 0 & -I
\end{bmatrix} \equiv T^* \text{ on } H
$$

(2.14)

Then, one may verify the following properties, as in [28, Section 1]:

(i) \hspace{1cm} T^2 = \text{identity on } H \hspace{1cm} (2.15)

(ii) \hspace{1cm} T : D(A) \xrightarrow{onto} D(A^*); \hspace{0.5cm} T = T^{-1} : D(A^*) \xrightarrow{onto} D(A) \hspace{1cm} (2.16a)

$$
TD(A) = D(A^*); \hspace{0.5cm} TD(A^*) = D(A) \hspace{1cm} (2.16b)
$$

(iii) \hspace{1cm} TA = A^*T = A^*T^* = (TA^*)^* \text{ on } D(A) \hspace{1cm} (2.17a)

$$
A = T^{-1}A^*T \text{ on } D(A) \text{ (similarity)} \hspace{1cm} (2.17b)
$$

and $TA$ is self-adjoint with domain $D(A)$.

(iv) \hspace{1cm} (Ax, x)_H = (A^*x^*, x^*)_H, \hspace{0.5cm} \forall x \in D(A) \text{ and } x^* = Tx \in D(A^*) \hspace{1cm} (2.18a)

$$
\text{Re}(Ax, x)_H = \text{Re}(A^*x^*, x^*)_H, \hspace{0.5cm} \forall x \in D(A) \text{ and } x^* = Tx \in D(A^*). \hspace{1cm} (2.18b)
$$

(v) If $\lambda$ is an eigenvalue of $A$ with corresponding eigenvector $e$

$$
Ae = \lambda e, \hspace{0.5cm} 0 \neq e \in D(A), \hspace{1cm} (2.19a)
$$

then applying $T$ on both sides and recalling $TAe = A^*T^e$ by (2.17) yields

$$
A^*(Te) = \lambda(Te) \hspace{1cm} (2.19b)
$$

and then $\lambda$ is an eigenvalue of $A^*$ with corresponding eigenvector $(Te)$. And conversely.

In preparation for the next result, we introduce the key-shaped set $K_\rho$, $0 < \rho \leq 1$

$$
K_\rho = (-\infty, -\frac{2}{\rho\mu}) \cup \{ S_{\frac{1}{\rho\mu}}(x_0) \setminus S_{r_0}(0) \} \hspace{1cm} (2.20)
$$

where $x_0 = (-\frac{1}{\rho\mu},0)$ is a point in $\mathbb{C}$, so that $S_{\frac{1}{\rho\mu}}(x_0)$ is the open disk centered at $x_0$, of radius $\frac{1}{\rho\mu}$, bounded by the circumference: $(\alpha + \frac{1}{\rho\mu})^2 + \omega^2 = \frac{1}{\rho^2\mu^2}$. Finally $S_{r_0}(0)$ is the open disk centered at the origin of sufficiently small radius $r_0 > 0$ (see Theorem 2.2 (ii) below).
2.4. Main results.

Theorem 2.2. (Generation by $A$ and $A^*$)

(i) For any $0 \leq \rho \leq 1$, the operator $A$ in (2.6)-(2.10) as well as its adjoint $A^*$ in (2.11)-(2.12) are dissipative on the space $H$: For all $(v_1, v_2, h) \in D(A)$ and $(v_1^*, v_2^*, h^*) = T(v_1, v_2, h) \in D(A^*)$, (this exhausts all of $D(A^*)$ by (2.16)), we have complementing (2.18b) and recalling (2.19b) and recalling $0 < \mu < 1$: 

\[
\text{Re} \left( A \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right)_H = -\|\nabla h\|^2 - \rho \|A^{1/2}v_2\|^2 = -\|\nabla h\|^2 - \rho \mu \|\Delta v_2\|^2 - \rho P(v_2) 
\]

\[
= -\|\nabla h\|^2 - \rho \int_{\Omega_d} [\mu |\nabla v_2|^2 + (1 - \mu)(v_{2x}^2 + v_{2y}^2) + 2(1 - \mu)v_{2xy}^2] \, d\Omega_d \quad (2.21a) 
\]

\[
\text{Re} \left( A^* \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix}, \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} \right)_H = -\|\nabla h^*\|^2 - \rho \|A^{1/2}v^*\|^2 = -\|\nabla h^*\|^2 - \rho \mu \|\Delta v_2\|^2 - \rho P(v_2) 
\]

\[
= -\|\nabla h^*\|^2 - \rho \int_{\Omega_d} [\mu |\nabla v_2|^2 + (1 - \mu)(v_{2x}^2 + v_{2y}^2) + 2(1 - \mu)v_{2xy}^2] \, d\Omega_d \quad (2.21b) 
\]

We recall [28, Proposition 3C.4, p302, Eq. (3C.21), p302] that 

\[
\|A^{1/2}w\|_{L_2(\Omega_d)}^2 = \int_{\Omega_d} [\mu |\Delta w|^2 + (1 - \mu)(w_{xx}^2 + w_{yy}^2) + 2(1 - \mu)w_{xy}^2] \, d\Omega_d, 
\]

and we have set the non-negative term 

\[
P(v_2) \equiv \int_{\Omega_d} [(1 - \mu)(v_{2x}^2 + v_{2y}^2) + 2(1 - \mu)v_{2xy}^2] \, d\Omega_d \quad (2.21c) 
\]

(ii) For $0 \leq \rho \leq 1$, the operator $A$ is boundedly invertible on $H$: $A^{-1} \in \mathcal{L}(H)$. (The explicit expression of $A^{-1}$ is given in Remark 3.1 below.) Thus, there exists an open disk $S_{r_0}$ of the complex plane $\mathbb{C}$, centered at the origin and of suitable small radius $r_0 > 0$ such that $S_{r_0} \subset \rho(A)$, the resolvent set of $A$. Similarly for the operator $A^*$. 

\[FIG 2: \text{The set } K_{\rho}, 0 < \rho < 1 \]
(iii) Hence, $A$ is maximal dissipative on $H$ for any $0 \leq \rho \leq 1$. By Lumer-Phillips Theorem [32], $A$ generates a s.c. $C_0$-contraction semigroup $e^{At}$ on $H$:

$\begin{pmatrix} w_0 \\ w_1 \\ u_0 \end{pmatrix} \in H \rightarrow \begin{pmatrix} w(t) \\ w(t) \\ u(t) \end{pmatrix} \equiv e^{At} \begin{pmatrix} w_0 \\ w_1 \\ u_0 \end{pmatrix} \in C([0,T];H). \quad (2.22)$

(iv) The same generation result holds also for $A^\star$.

The proof is given in section 3.

Our main results are as follows:

**Theorem 2.3.**

(i) For any fixed $0 < \rho \leq 1$, the generator $A$ in (2.6) and (2.10) of the s.c. contraction semigroup $e^{At}$ asserted by Theorem 2.2 has no spectrum on the imaginary axis, and satisfies the following resolvent condition: there is a constant $C > 0$ such that

$$\| (i\omega I - A)^{-1} \|_{L(H)} = \| R(i\omega, A) \|_{L(H)} \leq \frac{C}{|\omega|}, \quad \forall |\omega| \geq \text{some } \omega_0 > 0 \text{ depending on } \rho$$

with $\omega \in \mathbb{R}$. Hence, the s.c. contraction semigroup $e^{At}$ is analytic on the finite energy space $H$, $t > 0$, [27, Thm 3E.3, p 334].

(ii) Complementing (2.23) we have that the resolvent $R(\cdot, A)$ is uniformly bounded on the imaginary axis for any fixed $0 < \rho \leq 1$: there is a constant $C > 0$ such that

$$\| R(i\omega, A) \|_{L(H)} \leq C, \quad \forall \omega \in \mathbb{R}$$

Hence, the s.c. contraction analytic semigroup $e^{At}$ is uniformly exponentially stable on $H$ for any fixed $0 < \rho \leq 1$. There exist constants $M \geq 1$, $\delta > 0$, such that [33]

$$\| e^{At} \|_{L(H)} \leq M e^{-\delta t}, \quad \forall t > 0.$$  

For an estimate of $\delta$, see $r_0$ in Theorem 2.2(ii) as well as $k$ in Proposition 4.2.

See Proposition 4.3 for $\rho = 0$. The proof is given in Section 5.

**Remark 2.4.** Once estimate (2.23) is established, then the proof in [27, p335] show that in fact, for a suitable $M > 0$, we have

$$\| R(\lambda, A) \|_{L(H)} \leq \frac{M}{|\lambda|}, \quad \lambda \neq 0, \quad \forall \lambda \in \Sigma_{\theta_1}^\circ,$$

$$\Sigma_{\theta_1}^\circ = \{ \lambda \in \mathbb{C} : 0 \leq |\text{arg}\lambda| \leq \frac{\pi}{2} + \theta_1 \} \quad (2.26a)$$

$$\Sigma_{\theta_1} = \{ \lambda \in \mathbb{C} : 0 \leq |\text{arg}\lambda| \leq \frac{\pi}{2} \}$$

Fig 3: The Triangular Sector $\Sigma_{\theta_1}$ and its Complement $\Sigma_{\theta_1}^\circ$. The Disk $S_{r_0} \subset \rho(A)$
where one may take the angle \( \theta_1, 0 < \theta_1 < \frac{\pi}{2} \), such that \( \tan(\frac{\pi}{2} - \theta_1) = \frac{C}{\kappa} \), with \( C \) the constant in (2.23), for an arbitrary fixed constant \( 0 < \kappa < 1 \). We seek the ‘largest’ possible angle \( \theta_1 < \frac{\pi}{2} \), at least after moving the vertex of the triangular sector in a nearby point. In our case, this nearby point will be \( \{-\frac{2}{\rho \mu}, 0\} \) in which case, with vertex \( x_0 = \{-\frac{2}{\rho \mu}, 0\} \) the angle \( \theta_1 \) will be arbitrarily close to \( \frac{\pi}{2} \). This is contained in the next result, which makes more precise Theorem 2.3 (i) from a spectral analysis viewpoint.

**Theorem 2.4.** Let \( 0 < \rho \leq 1 \) fixed.

(i) The resolvent operator \( R(\lambda, A) = (\lambda I - A)^{-1} \) of the generator \( A \) in (2.6)-(2.9) satisfies the following estimate

\[
\| R(\lambda, A) \|_{\mathcal{L}(H)} \leq \frac{C}{\lambda}, \quad \text{for all } \lambda \in \mathbb{C}\setminus \mathcal{K}_\rho
\]  

(2.27)

where \( \mathcal{K}_\rho \) is the (infinite) key-shaped set defined in (2.20); see Figure 2. A-fortiori, the analytic estimate (2.23) holds true.

(ii) The spectrum \( \sigma(A) \) of \( A \) is contained in the region \( \mathcal{K}_\rho \). In particular, the whole imaginary axis is in the resolvent set \( \rho(A) \) of \( A \): \( \mathrm{i}\mathbb{R} \subseteq \rho(A) \).

The proof is given in Section 6.

2.5. A physically more attractive case where the B.C. \( B_1 w_{1|\Gamma_s} \) in (2.1d) replaced by \( [B_1(w + \omega_1)]|_{\Gamma_s} \). In this subsection we briefly deal with the same model (2.1), except that the B.C. in (2.1d) is replaced with the following arguably more physical B.C.:

\[
[\Delta(w + \rho \omega_1) + (1 - \mu)B_1(w + \rho \omega_1)]|_{\Gamma_s} = \frac{\partial u}{\partial \nu}|_{\Gamma_s} \quad \text{in} \quad (0,T] \times \Gamma_s
\]  

(2.28)

Our conclusion will be that the corresponding generator is still dissipative, however in a different equivalent, yet more complicated, topology. More precisely, for \( f, g \in H^2(\Omega_s) \), let us introduce the following (well known) symmetric positive bilinear form \([27, (3C.12), p.301]\), still for \( 0 < \mu < 1 \):

\[
a(f, g) = \int_{\Omega_s} \left[ f_{xx}g_{xx} + f_{yy}g_{yy} + 2(1 - \mu)f_{xy}g_{xy} + \mu(f_{xx}g_{yy} + f_{yy}g_{xx}) \right] d\Omega_s \quad (2.29)
\]

Then our new energy space \( \mathcal{H} \) will be topologized with the inner product

\[
\begin{bmatrix}
  v_1 \\
  v_2 \\
  \bar{h}
\end{bmatrix},
\begin{bmatrix}
  \bar{v}_1 \\
  \bar{v}_2 \\
  \bar{h}
\end{bmatrix} = a(v_1, \bar{v}_1) + (v_2, \bar{v}_2)_{\Omega_s} + (h, \bar{h})_{\Omega_f}
\]  

(2.30)

instead of (2.4). On the first component space, the inner product \( (\Delta v_1, \Delta \bar{v}_1)_{\Omega_s} \) in (2.4) is topologically equivalent to the inner product defined by the form \( a(v_1, \bar{v}_1) \) for \( v_1, \bar{v}_1 \in H^2(\Omega_s) \cap H^0_0(\Omega_s) \), as called for by our model with B.C. \( w|_{\Gamma_s} = 0 \) in (2.1c). To justify our statement, in addition to the bending boundary operator \( B_1 \) in (2.2), we need to introduce the shear forces boundary operator \( B_2 \), [27, (3C.12), p.300, see also (3C.50) and (3C.54) in terms of only normal and tangential derivatives],

\[
B_2 f = \frac{\partial}{\partial \nu}[(\nu_1^2 - \nu_2^2)f_{xy} + \nu_1 \nu_2(f_{yy} - f_{xx})] \quad \text{on} \quad \Gamma_s
\]  

(2.31)
Furthermore, we need to recall the following alternative Green formula [27, Proposition C.12, p310]. For $f \in H^1(\Omega_s)$, $g \in H^2(\Omega_s)$, we have
\[(\Delta^2 f, g)_{\Omega_s} = a(f, g) + \int_{\Omega_s} \left[ \frac{\partial \Delta f}{\partial \nu} + (1 - \mu) B_2 f \right] g \, d\Gamma_s - \int_{\Gamma_s} [\Delta f + (1 - \mu) B_1 f] \frac{\partial g}{\partial \nu} \, d\Gamma_s \] (2.32)
with $a(f, g)$ defined in (2.29). We can now state and prove our claimed desired result.

**Theorem 2.5.** Let $\widetilde{A}$ be the same operator $A$ as in (2.6)-(2.8) except for incorporating the B.C.
\[\left[ \Delta(v_1 + \rho v_2) + (1 - \mu) B_1 (v_1 + \rho v_2) \right]_{\Gamma_s} = \frac{\partial h}{\partial \nu}_{\Gamma_s} \in H^{-1/2}(\Gamma_s) \] (2.33)
instead of (2.7c). Then $\widetilde{A}$ is dissipative on the space $\overline{H}$ topologized by (2.30). More precisely, we have
\[\text{Re} \left( \left\langle \widetilde{A} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right\rangle_{\overline{H}} \right) = -\rho a(v_2, v_2) - \|\nabla h\|^2 \leq 0, \quad [v_1, v_2, h] \in \mathcal{D}(\widetilde{A}) \] (2.34)
(to be compared to (2.21))

The Proof is given in Appendix B.

**Remark 2.5.** Dissipativity is of course topology-dependent. The operator $\widetilde{A}$ is not dissipative on the original space $H$ (2.3), (2.4). Notice that the new B.C. (2.33) adds, over the old B.C. (2.1d), the boundary term $B_1 v_1|_{\Gamma_s} = -c(\eta) \frac{\partial v_1}{\partial \nu}|_{\Gamma_s} \in H^{1/2}(\Gamma_s)$, $v_1 \in \mathcal{D}(A)$, see (2.7). This contributes on interior term $G(B_1 v_1|_{\Gamma_s}) \in H^{1/2+5/2}(\Omega_s) = H^3(\Omega_s)$, which is a compact perturbation on the first component space $H$. Thus, generation of a s.c. semigroup and its analyticity under the B.C. (2.33) can be readily obtained from the results of the present paper under the B.C. (2.1d). Instead, spectral properties are topology independent and do not change. We note that the inner product generated by the form $a(v_1, v_1)$ in (2.29) is much more complicated than the inner product $(\Delta v_1, \Delta v_1)$ in (2.4).

3. **Proof of Theorem 2.2.**

3.1. **Part(i): Dissipativity of $A$ as in Eq. (2.21a).** Step 1. Let $[v_1, v_2, h] \in \mathcal{D}(A)$, characterized in (2.7)-(2.10). From (2.6), recalling the topology (2.4), compute by Green’s second theorem on the plate and the Green’s first theorem on the heat. We recall that the normal vector $\nu$ is outward with respect to $\Omega_s$ on $\Gamma_s$ and outward from $\Omega_f$ on $\Gamma_f$.
\[
\begin{bmatrix} \mathcal{A} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \end{bmatrix}_{\overline{H}} = \begin{bmatrix} -\Delta^2(v_1 + \rho v_2) \\ -\Delta h \\ v_2 \end{bmatrix}_{\overline{H}} \] (3.1)
\[
= (\Delta v_2, \Delta v_1) - (\Delta^2(v_1 + \rho v_2), v_2) + (\Delta h, h) 
\]
\[
= (\Delta v_2, \Delta v_1) - (\Delta(v_1 + \rho v_2), \Delta v_2) - \left( \frac{\partial \Delta(v_1 + \rho v_2)}{\partial \nu}, v_2 \right)_{\Gamma_s} 
+ \left( \Delta(v_1 + \rho v_2), \frac{\partial v_2}{\partial \nu} \right)_{\Gamma_s} - \left( \frac{\partial h}{\partial \nu}, h \right)_{\Gamma_s} - \|\nabla h\|^2 \] (3.2)
as \(v_2|_{\Gamma_s} = 0\) by (2.7a) and \(h|_{\Gamma_s} = 0\) by (2.8). Recall now \(\Delta(v_1 + rv_2) = -(1 - 
abla \rho \nu + \frac{\partial h}{\partial \nu}|_{\Gamma_s})\) on \(\Gamma_s\) by (2.7c). We obtain

\[
\text{RHS of (3.2)} = (\Delta v_2, \Delta v_1) - (\Delta v_1, \Delta v_2) - \left\{ \rho \|\Delta v_2\|^2 + \rho(1 - \mu) \left( B_1 v_2, \frac{\partial v_2}{\partial \nu} \right)_{\Gamma_s} \right\}
\]

\[
+ \left( \frac{\partial h}{\partial \nu}, \frac{\partial v_2}{\partial \nu} \right)_{\Gamma_s} - \left( \frac{\partial h}{\partial \nu}, h \right)_{\Gamma_s} - \|\nabla h\|^2
\]

(3.3)

where the cancellation noted occurs since \(h|_{\Gamma_s} = \frac{\partial v_2}{\partial \nu}|_{\Gamma_s}\) by (2.8). Taking the real part of (3.3), and noticing that \(\Re[(\Delta v_2, \Delta v_1) - (\Delta v_1, \Delta v_2)] = \Re(z - \bar{z}) = \Re(2\text{Im}z) = 0\), we obtain

\[
\Re \left( A \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} \right) = -\Re \left( \rho \left[ \|\Delta v_2\|^2 + (1 - \mu) \left( B_1 v_2, \frac{\partial v_2}{\partial \nu} \right)_{\Gamma_s} \right] \right) - \|\nabla h\|^2
\]

(3.4)

**Step 2.** Here and through this paper, we shall invoke the following critical identities regarding the boundary operator \(B_1\), from [27, p.301-302] (with the unit normal \(\nu\) on \(\Gamma_s\) oriented outward with respect to \(\Omega_s\))

\[
\left( B_1 v_2, \frac{\partial v_2}{\partial \nu} \right)_{\Gamma_s} = \int_{\Omega_s} 2[v_{2xx}v_{yy} - v_{2xx}v_{yy}] \, d\Omega_s
\]

(3.5)

[27, Corollary 3C.3(ii), Eq. (3C.19b) p301], [27, (3C.21), (3C.22), p302], (compare with (2.21a)), so that

\[
\|\Delta v_2\|^2 + (1 - \mu) \left( B_1 v_2, \frac{\partial v_2}{\partial \nu} \right)_{\Gamma_s} = \|\Delta v_2\|^2 + 2(1 - \mu) \int_{\Omega_s} [v_{2xx}^2 - v_{2xx}v_{yy}] \, d\Omega_s
\]

(3.6)

\[
= \mu \|\Delta v_2\|^2 + \int_{\Omega_s} [(1 - \mu)(v_{2xx}^2 + v_{2yy}^2) + 2(1 - \mu)v_{2xy}^2] \, d\Omega_s
\]

(3.7)

with \(A\) the operator in (2.9e), for \(0 < \mu < 1\) as assumed, and \(P(v_2)\) defined in (2.21d). In passing from (3.6) to (3.7), we have invoked [27, Eq.(3C.22), p302]

**Step 3.** We substitute identity (3.7) on the RHS of (3.4) and obtain (2.21a), as desired. Dissipativity of \(A\) is established. Dissipativity of \(A^*\) in (2.11)-(2.13) is proved similarly, mutatis mutandis, or more simply one invokes (2.18b).

3.2. **Part (ii):** \(A^{-1} \in \mathcal{L}(H)\). By way of illustration, we first show that \(0 \notin \sigma_p(A)\), the point spectrum of \(A\). Let \(Ax = 0, \ x = [v_1, v_2, h] \in \mathcal{D}(A)\) and conclude that \(x = 0\). In fact, by (2.6), \(Ax = 0\) implies \(v_2 = 0\) in \(\Omega_s\), hence \(h|_{\Gamma_s} = \frac{\partial v_2}{\partial \nu}|_{\Gamma_s} = 0\) by (2.8), and \(\Delta^2 v_1 = 0\) in \(\Omega_s\) as well as \(\Delta h = 0\) in \(\Omega_f\). This, together with \(h|_{\Gamma_f} = 0\) in (2.8) yields \(h \equiv 0\) in \(\Omega_f\), hence \(\Delta v_1|_{\Gamma_f} = \frac{\partial h}{\partial \nu}|_{\Gamma_f} = 0\) by (2.7c). But then \(\Delta^2 v_1 = 0\) in \(\Omega_s\), \(\Delta v_1|_{\Gamma_s} = 0\) implies \(\Delta v_1 \equiv 0\) on \(\Omega_s\), which together with \(v_1|_{\Gamma_s} = 0\) (by (2.7a)) implies \(v_1 \equiv 0\) in \(\Omega_s\).
In a similar way, using now (2.11)-(2.13), one shows that 0 \not\in \sigma_p(\mathcal{A}^*)$. As a consequence, then, 0 \not\in \sigma_r(\mathcal{A})$, [34, p282], the residual spectrum of $\mathcal{A}$.

We now establish directly that, in fact, 0 \not\in \rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$. Let $[v_1^*, v_2^*, h^*] \in \mathcal{H}$. We seek to solve

$$(\star) \quad \mathcal{A}[v_1, v_2, h] = [v_1^*, v_2^*, h^*]$$

uniquely and explicitly for $[v_1, v_2, h] \in \mathcal{D}(\mathcal{A})$, with inverse $\mathcal{A}^{-1}$ being bounded on $\mathcal{L}(\mathcal{H})$. By (2.6)-(2.11), we first obtain

$$v_2 = v_1^* \in \mathcal{D}(\mathcal{A}_D) \Rightarrow \begin{cases} \Delta h = h^* \\ h|_{\Gamma_f} = 0, \quad h|_{\Gamma_s} = \frac{\partial v_2}{\partial \nu}|_{\Gamma_s} = \frac{\partial v_1^*}{\partial \nu}|_{\Gamma_s} \in H^{1/2}(\Gamma_s). \end{cases}$$

(3.8a)

Thus, with $D_s$ the Dirichlet map on $\Omega_s$, $(D_s g = \psi \leftrightarrow \{ \Delta \psi = 0 \in \Omega_s, \psi = g \text{ on } \Gamma_s \})$, with $D_s$: $H^{1/2}(\Gamma_s) \rightarrow H^1(\Omega_s)$ [30], and $A_D$ defined in (2.3b), we solve for $h$ uniquely

$$h = A_D^{-1} h^* + D_s \frac{\partial v_1^*}{\partial \nu}|_{\Gamma_s} \in H^1(\Omega_f), \quad \text{hence} \quad \frac{\partial h}{\partial \nu}|_{\Gamma_s} \in H^{-1/2}(\Gamma_s),$$

(3.9)

as below (2.8), a next implication of $(\star)$ is (recall (2.6), (2.7c-2.7d))

$$\begin{cases} -\Delta^2 (v_1 + \rho v_1^*) = v_2^* \in L_2(\Omega_s) \\ \Delta(v_1 + \rho v_1^*)|_{\Gamma_s} = -(1 - \mu) B_1(\rho v_1^*)|_{\Gamma_s} + \frac{\partial h}{\partial \nu}|_{\Gamma_s} \\ = -\rho (1 - \mu) c(\eta) \frac{\partial v_1^*}{\partial \nu}|_{\Gamma_s} + \frac{\partial h}{\partial \nu}|_{\Gamma_s} \equiv g \in H^{-1/2}(\Gamma_s) \end{cases}$$

(3.10a)

(3.10b)

(3.10c)

With $v_1^* \in \mathcal{D}(\mathcal{A}_D) \equiv H^2(\Omega_s) \cap H_0^1(\Omega_s)$, we have $\frac{\partial v_1^*}{\partial \nu}|_{\Gamma_s} \in H^{1/2}(\Gamma_s)$, hence $g \in H^{-1/2}(\Gamma_s)$ by (3.9) and (3.10b). The solution $\psi = v_1 + \rho v_1^*$ of problem (3.10a-3.10c) is given by $\psi = \psi_1 + \psi_2$, with $\psi_1 = -A_D^{-2} v_2^* \in \mathcal{D}(A_D^2)$ and $\psi_2 = \tilde{G}_2 g = -A_D^{-1} D_s g$, [27, (3.6.3),(3.6.6), p212], so that $g \in H^{-1/2}(\Gamma_s)$ yields $D_s g \in L_2(\Omega_s)$, hence $\psi \in \mathcal{D}(\mathcal{A}_D)$. Ultimately, $\psi \equiv v_1 + \rho v_1^* \in \mathcal{D}(\mathcal{A}_D)$, explicitly via (3.10b) and (3.9)

$$v_1 + \rho v_1^* = -A_D^{-2} v_2^* - A_D^{-1} D_s \left\{ \rho (1 - \mu) c(\eta) \frac{\partial v_1^*}{\partial \nu}|_{\Gamma_s} + \frac{\partial}{\partial \nu} \left[ A_D^{-1} h^* + D_s \frac{\partial v_1^*}{\partial \nu}|_{\Gamma_s} \right] \right\} \in \mathcal{D}(\mathcal{A}_D) \equiv H^2(\Omega_s) \cap H_0^1(\Omega_s)$$

(3.11)

from which we obtain $v_1 \in \mathcal{D}(\mathcal{A}_D)$ explicitly. Expression (3.11) holds true also for $\rho = 0$. Thus the solution $[v_1^*, v_2^*, h^*] \in \mathcal{D}(\mathcal{A})$ has been obtained uniquely and explicitly in terms of $[v_1^*, v_2^*, h^*] \in \mathcal{H}$, with $\mathcal{A}^{-1} \in \mathcal{L}(\mathcal{H})$. Theorem 2.2 is established.

**Remark 3.1.** In effect, putting together $v_2 = v_1^*$, (3.9) and (3.11), we have obtained the following explicit expression for $\mathcal{A}^{-1}$
\[\begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = A^{-1} \begin{bmatrix} v_1^* \\ v_2^* \\ h^* \end{bmatrix} = \begin{bmatrix} -\rho v_1^* + Tv_1^* - A_D^{-2} v_2^* - A_D^{-1} D_s \frac{\partial v_1^*}{\partial \nu} (A_D^{-1} h^*)_{\Gamma_s} \\ D_s \frac{\partial v_1^*}{\partial \nu} + A_D^{-1} h^* \end{bmatrix} \] (3.12)

\[Tv_1^* = -A_D^{-1} D_s \left\{ \rho (1 - \mu) c(\eta) \frac{\partial v_1^*}{\partial \nu} \bigg|_{\Gamma_s} + \frac{\partial}{\partial \nu} \left( D_s \left( \frac{\partial v_1^*}{\partial \nu} \right) \right) \right\} \] (3.13)

Notice that we can solve problem (3.10a)-(3.10c) in a different way: First solve
\[
\left\{ \begin{aligned}
\Delta(\Delta(v_1 + \rho v_1^*)) &= -v_2^* \in L_2(\Omega_s) \\
[\Delta(v_1 + \rho v_1^*)]_{\Gamma_s} &= \left[ \rho (1 - \mu) c(\eta) \frac{\partial v_1^*}{\partial \nu} + \frac{\partial h}{\partial \nu} \right]_{\Gamma_s}
\end{aligned} \right.
\] (3.14a)

whose solution is (recall that \(\frac{\partial h}{\partial \nu} \big|_{\Gamma_s}\) is explicit from (3.9))
\[
\Delta(v_1 + \rho v_1^*) = -A_D^{-1} v_2^* + D_s \left[ \rho (1 - \mu) c(\eta) \frac{\partial v_1^*}{\partial \nu} + \frac{\partial h}{\partial \nu} \right]_{\Gamma_s}
\] (3.15)

This coupled with the Boundary Condition (3.10c), in turn, yields
\[v_1 + \rho v_1^* = A_D^{-1} \left\{ -A_D^{-1} v_2^* + D_s \left[ (1 - \mu) c(\eta) \frac{\partial v_1^*}{\partial \nu} + \frac{\partial h}{\partial \nu} \right] \right\} \] (3.16)

so that (3.16) gives another explicit representation for \(v_1\) and hence for \(A^{-1}\).

4. Spectral (Eigenvalues) analysis. We next present an eigenvalues analysis of the operator \(A\). This will be much extended in Section 6, in proving Theorem 2.4, which, in particular, provides information on the entire spectrum of \(A\). The proof here is more direct and much simpler than in Section 6. Specializing the treatment of Section 6, one obtains a different but parallel proof of Proposition 4.1 below, See Remark 6.1.

Proposition 4.1. Assume \(0 < \rho \leq 1\).

(i) Let \(\lambda = \alpha + \omega \), \(\alpha \leq 0, \omega \neq 0\), be a point outside of the open disk \(S_r = 1/\rho\), \(x_0 = (-1/\rho \mu, 0)\), defined below (2.20); i.e. satisfying
\[
(\alpha + \frac{1}{\rho \mu})^2 + \omega^2 - (\frac{1}{\rho \mu})^2 > 0, \quad \text{or} \quad \rho \mu (\alpha^2 + \omega^2) + 2\alpha > 0 \quad (4.1)
\]

Then
\[
\lambda = \alpha + i\omega \notin \sigma_p(A), \quad \lambda = \alpha + i\omega \notin \sigma_p(A^*), \quad \text{hence} \quad (\lambda + i\omega) \notin \sigma_r(A), \quad (4.2)
\]
where \(\sigma_p(\cdot)\) = point spectrum, \(\sigma_r(\cdot)\) = residual spectrum. In particular, (4.1) applies to all points \(\lambda = i\omega\) of the imaginary axis. (The point \(\lambda = 0\) was shown to be in \(\rho(A)\) in Theorem 2.2 (ii)). By contrast, the case \(\rho = 0\) is given in Proposition 4.2.

(ii) The point \(\lambda = -\frac{1}{\rho} (\alpha = -\frac{1}{\rho}, \omega = 0)\) belongs to the continuous spectrum of \(A\) as well as to the continuous spectrum of \(A^*: -\frac{1}{\rho} \in \sigma_c(A), \text{ and } -\frac{1}{\rho} \in \sigma_c(A^*)\).
Proof. We provide a proof for $\mathcal{A}$. The proof for $\mathcal{A}^*$ is similar, mutatis mutandis, or we simply invoke property (v) of $\mathcal{T}$ is Section 2.3. Regarding the last assertion in (4.2) for $\sigma_\mathcal{F}(\mathcal{A})$, recall via [34, p282] that if $\lambda = \alpha + i\omega$ were in $\sigma_\mathcal{F}(\mathcal{A}^*)$, then $\lambda = \alpha + i\omega$ would be in $\sigma_\mathcal{F}(\mathcal{A}^*)$, which is excluded under the LHS of (4.2).

Proof that $\lambda = \alpha + i\omega \notin \sigma_\mathcal{F}(\mathcal{A})$, where $\alpha + i\omega \in \mathcal{S}_{\frac{1}{r}}(x_0)$, the closed complement of the open disk $\mathcal{S}_{\frac{1}{r}}(x_0)$.

**Step 1.** With $\alpha \leq 0$, $\omega \in \mathbb{R}$, we consider the eigenvalue problem

$$\mathcal{A} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_2 \\ -\Delta^2(v_1 + \rho v_2) \\ \Delta h \end{bmatrix} = (\alpha + i\omega) \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \quad [v_1, v_2, h] \in \mathcal{D}(\mathcal{A}) \tag{4.3}$$

$$v_2 = (\alpha + i\omega)v_1; \quad -\Delta^2(v_1 + \rho v_2) = (\alpha + i\omega)v_2; \quad \Delta h = (\alpha + i\omega)h. \tag{4.4}$$

From the third $h$-relation in (4.4) via Green’s First Theorem, recalling that $h|_{\Gamma_s} = 0$ by (2.8) and that $\nu$ is inward to $\Omega_f$ over $\Gamma_s$, we obtain

$$\left(\Delta h, h\right) = -\left(\frac{\partial h}{\partial \nu}, h\right)_{\Gamma_s} - \|\nabla h\|^2 = (\alpha + i\omega)\|h\|^2. \tag{4.5}$$

Next, inner product the second relation in (4.4) with $v_2$, apply Green’s second theorem, recall $v_2|_{\Gamma_s} = 0$ from (2.7a), and obtain

$$- (\Delta(v_1 + \rho v_2), \Delta v_2) = \left(\frac{\partial \Delta(v_1 + \rho v_2)}{\partial \nu}, v_2\right)_{\Gamma_s} + \left(\Delta(v_1 + \rho v_2), \frac{\partial v_2}{\partial \nu}\right)_{\Gamma_s} = (\alpha + i\omega)\|v_2\|^2. \tag{4.6}$$

Next, we recall $v_2 = (\alpha + i\omega)v_1$ from (4.4), $\Delta(v_1 + \rho v_2) = -\rho(1 - \mu)B_1v_2 + \frac{\partial h}{\partial \nu}$ on $\Gamma_s$ from (2.7c), and $h = \frac{\partial v_2}{\partial \nu}$ on $\Gamma_s$ from (2.8), to rewrite (4.6) as

$$(-\alpha + i\omega)\|\Delta v_1\|^2 - \rho \left\{\|\Delta v_2\|^2 + (1 - \mu) \left(B_1v_2, \frac{\partial v_2}{\partial \nu}\right)_{\Gamma_s}\right\} + \left(\frac{\partial h}{\partial \nu}, h\right)_{\Gamma_s} = (\alpha + i\omega)\|v_2\|^2. \tag{4.7}$$

We now sum up (4.5) and (4.7) and obtain after a cancellation of the boundary term $\left(\frac{\partial h}{\partial \nu}, h\right)_{\Gamma_s}$:

$$\|\nabla h\|^2 + \alpha \left\{\|\Delta v_1\|^2 + \|v_2\|^2 + \|h\|^2\right\} + \rho \left\{\|\Delta v_2\|^2 + (1 - \mu) \left(B_1v_2, \frac{\partial v_2}{\partial \nu}\right)_{\Gamma_s}\right\}$$

$$+ i\omega \left[\|\Delta v_2\|^2 + \|v_2\|^2 - \|\Delta v_1\|^2\right] = 0; \tag{4.8}$$

We next critically invoke [27, Eq.(3C.19b), p301], the identity already recalled in (3.6) and (3.7) of Section 3, to rewrite the term in brackets $\{\}$ in (4.8) and obtain

$$\|\nabla h\|^2 + \alpha \left\{\|\Delta v_1\|^2 + \|v_2\|^2 + \|h\|^2\right\}$$

$$+ \rho \left\{\mu\|\Delta v_2\|^2 + \int_{\Omega_s} [(1 - \mu)(v_{2,sx}^2 + v_{2,sy}^2) + 2(1 - \mu)v_{2,xy}^2] d\Omega_s\right\}$$

$$+ i\omega \left[\|\Delta v_2\|^2 + \|v_2\|^2 - \|\Delta v_1\|^2\right] = 0; \tag{4.9}$$
Thus, in conclusion, the point 

\[ \lambda \]

was closed disk

Second Case.

\[ v \equiv \lambda \]

\[ i \equiv (2.21c) \text{ or } (3.7) \]

\[ ||\nabla h||^2 + \alpha \left( ||\Delta v_1||^2 + ||v_2||^2 + ||\lambda||^2 \right) + \rho ||A^{1/2}v_2||^2 + i\omega \left( ||h||^2 + ||v_2||^2 - ||\Delta v_1||^2 \right) = 0 \]

(4.10)

**Step 2.** As a preliminary analysis, let \( \alpha = 0 \); i.e. consider the point \( i \omega \) on the imaginary axis. Then, the real part of (4.10) implies \( \nabla h = 0 \) i.e. \( h \equiv 0 \) on \( \Omega_f \), since \( h|_{\Gamma_s} = 0 \) by (2.8); as well as \( v_2 \equiv 0 \) in \( \Omega_s \), since \( A \) is strictly positive [27, Proposition 3C.4, p.301] and \( \rho > 0 \). Then the first equation in (4.4) yields \( v_1 \equiv 0 \) on \( \Omega_s \) for \( \omega \neq 0 \). Moreover, also for \( \omega = 0 \), \( v_2 \equiv 0 \) implies that \( \Delta^2 v_1 \equiv 0 \) in \( \Omega_s \) by the second equation in (4.4), as well as \( \Delta v_1 \equiv 0 \) on \( \Gamma_s \) by (2.7c). Hence we obtain \( \Delta v_1 \equiv 0 \) on \( \Omega_s \) which along with \( v_1 = 0 \) on \( \Gamma_s \) by (2.7a) implies \( v_1 \equiv 0 \). Thus \( i\mathbb{R} \notin \sigma_p(A) \), and \( i\mathbb{R} \notin \sigma_r(A) \) by Property (v) of \( T \) in Section 2.2. Finally, \( i\mathbb{R} \notin \sigma_r(A) \) [34, p.282].

**Step 3.** For \( \omega \neq 0 \), the imaginary part of (4.10) yields

\[ ||h||^2 + ||v_2||^2 \equiv ||\Delta v_1||^2 \]

which substituted into the real part of (4.9) leads to

\[ ||\nabla h||^2 + 2\alpha ||\Delta v_1||^2 \]

\[ + \rho \left\{ \mu ||\Delta v_2||^2 + \int_{\Omega_s} \left[ (1 - \mu) (v_{2,x}^2 + v_{2,y}^2) + 2(1 - \mu) v_{2,x}^2 \right] d\Omega_s \right\} = 0 \]

(4.12)

We now critically use that the integral \( \int_{\Omega_s} \) over \( \Omega_s \); i.e. the term \( P(v_2) \) in (2.21d) is non-negative for \( 0 < \mu < 1 \) (as assumed). Since \( ||\Delta v_1||^2 = (\alpha^2 + \omega^2)||\Delta v_1||^2 \) by the first equation of (4.4), we finally arrive at the identity:

\[ ||\nabla h||^2 + [\rho \mu (\alpha^2 + \omega^2) + 2\alpha] ||\Delta v_1||^2 + \rho P(v_2) = 0, \quad \omega \neq 0. \]

(4.13)

**First Case.** Assume first that

\[ \rho \mu (\alpha^2 + \omega^2) + 2\alpha > 0, \quad \text{or} \quad \left( \alpha + \frac{1}{\rho\mu} \right)^2 + \omega^2 > \frac{1}{\rho^2 \mu^2}, \quad \omega \neq 0. \]

(4.14)

i.e. that the point \( \lambda = \alpha + i\omega, \omega \neq 0 \), lies outside of the closed disk \( S_{r = \frac{1}{\rho\mu}}(x_0) \) with center \( x_0 = \left\{ -\frac{1}{\rho\mu}, 0 \right\} \) and radius \( r = \frac{1}{\rho\mu} \). Then identity (4.13) implies via (4.14):

\[ \nabla h \equiv 0 \text{ in } \Omega_f \quad \Rightarrow \quad h \equiv 0 \text{ in } \Omega_f \text{ by } h|_{\Gamma_f} = 0 \text{ in } (2.8); \]

\[ \Delta v_1 \equiv 0 \text{ in } \Omega_s \quad \Rightarrow \quad v_1 \equiv 0 \text{ in } \Omega_s \text{ by } v_1|_{\Gamma_s} = 0 \text{ in } (2.7a); \]

(4.15)

finally \( v_2 \equiv 0 \) by (4.4).

Or else recall (4.11) whereby \( \Delta v_1 \equiv 0 \text{ in } \Omega_s \) implies \( h \equiv 0 \) in \( \Omega_f \) and \( v_2 \equiv 0 \) in \( \Omega_s \). Thus, in conclusion, the point \( \lambda = \alpha + i\omega, \alpha \leq 0, \omega \neq 0 \), which lies outside the closed disk \( S_{r = \frac{1}{\rho\mu}}(x_0) \) is not an eigenvalue of \( A \), as claimed.

**Second Case.** Assume now that \( \rho \mu (\alpha^2 + \omega^2) + 2\alpha = 0, \quad \text{or} \quad \left( \alpha + \frac{1}{\rho\mu} \right)^2 + \omega^2 = \left( \frac{1}{\rho\mu} \right)^2 \); i.e. that the point \( \lambda = \alpha + i\omega, \omega \neq 0 \), lies on the circumference of the disk \( S_{r = \frac{1}{\rho\mu}}(x_0) \). Then by (4.13), \( h = 0 \text{ in } \Omega_f \), as before, as well as \( P(v_2) \equiv 0 \). By (2.21d) (where \( 0 < \mu < 1 \)) we deduce that \( \Delta v_2 \equiv 0 \text{ in } \Omega_s \). Thus along with \( v_2|_{\Gamma_s} = 0 \) by (2.7a) implies \( v_2 \equiv 0 \text{ in } \Omega_s \), hence \( v_1 \equiv 0 \text{ in } \Omega_s \), as \( \omega \neq 0 \).
Thus, part (i) of Proposition 4.1 for $\mathcal{A}$ has been established. A similar proof for $\mathcal{A}^*$ applies mutatis mutandis, or simply invoke Property (v) of $\mathcal{T}$ in Section 2.2. It then follows [34, p. 282] that such $\lambda = \alpha + i\omega$ outside of the open disk $\mathcal{S}_{\rho,\pi}(x_0)$ is not in $\sigma_r(\mathcal{A})$.

(ii) For sake of illustration, let us first show that $-\frac{1}{\rho} \notin \sigma_p(\mathcal{A})$. Thus, let $\mathcal{A}x = -\frac{1}{\rho}x$, $x = [v_1, v_2, h] \in \mathcal{D}(\mathcal{A})$ and show that $x = 0$. First, by the first relation in (4.4) with $\alpha = -\frac{1}{\rho}$, $\omega = 0$ we have $v_1 + \rho v_2 \equiv 0$ in $\Omega_s$. Hence, $\Delta^2(0) = 0 = -\frac{1}{\rho}v_2$ still by (4.4); thus $v_2 = 0$ and then $v_1 = 0$. Moreover, $h|_{\Gamma_s} = \frac{\partial v_2}{\partial \nu}|_{\Gamma_s} = 0$ by (2.8).

Furthermore, $\frac{\partial h}{\partial \nu} = \Delta(0) + (1 - \mu)B_1(0) = 0$ on $\Gamma_s$ by (2.7c). This together with $h|_{\Gamma_s} = 0$ by (2.8) yields an over-determined boundary problem for $\Delta h = -h$ on $\Omega_f$, and hence implies $h \equiv 0$ in $\Omega_f$, by uniqueness as desired.

The proof that $-\frac{1}{\rho} \notin \sigma_p(\mathcal{A}^*)$ is similar, invoking $\mathcal{A}^*$ on (2.11)-(2.13), or simply invoke Property (v) of $\mathcal{T}$ in Section 2.2. It follows then [34, p282] that $-\frac{1}{\rho} \notin \sigma_r(\mathcal{A})$.

We now show that $-\frac{1}{\rho} \notin \rho(\mathcal{A})$, the resolvent set of $\mathcal{A}$, and therefore it must be $-\frac{1}{\rho} \notin \sigma_c(\mathcal{A})$. Let $[v_1^*, v_2^*, h^*] \in \mathbf{H}$ be arbitrary. We seek to solve $(-\frac{1}{\rho}I - \mathcal{A})[v_1, v_2, h] = [v_1^*, v_2^*, h^*]$ for $[v_1, v_2, h] \in \mathcal{D}(\mathcal{A})$, as a bounded map with respect to $[v_1^*, v_2^*, h^*]$. This would imply by (2.6) that $v_1 + \rho v_2 = -\rho v_1^* \in \mathcal{D}(\mathcal{A}_D)$, so that $\Delta^2(v_1 + \rho v_2) = -\rho \Delta^2 v_1^* \notin L_2(\Omega_s)$, contrary to requirement (2.7b). Proposition 4.1 is proved.

**Proposition 4.2.** Let $0 < \rho \leq 1$. The following result, complementary to Proposition 4.1, holds true: the corresponding operator $\mathcal{A}$ has no eigenvalue in the vertical strip

\[ -k < \alpha \leq 0, \quad \omega \in \mathbb{R}\setminus\{0\}, \quad k > 0 \quad (4.16) \]

placed in the left of the imaginary axis (the segment $-k < \alpha < -r_0$ ($r_0$ in Theorem 2.2 (ii)) is not covered). The positive constant $k$ is quantified below in the proof. A similar result holds for $\mathcal{A}^*$.

**Proof.** We return to identity (4.12), rewritten now as

\[ \|\nabla h\|^2 + 2\alpha(\|h\|^2 + \|v_2\|^2) + \rho|\mathcal{A}^{1/2}v_2|^2 = 0, \quad \omega \neq 0 \quad (4.17) \]

recalling (4.11) and (2.21c). Next, to the variable $h$ we apply Poincare inequality, as $h|_{\Gamma_s} = 0$ by (2.8). Moreover, we recall that the operator $\mathcal{A}$ in (2.9c) is strictly positive self-adjoint on $L_2(\Omega_s)$. Thus we can write

\[ c_h\|h\|^2 \leq \|\nabla h\|^2 \quad \text{and} \quad c_s\|v_2\|^2 \leq \|\mathcal{A}^{1/2}v_2\|^2 \]

(4.18)

$c_s = \mu_1 = \text{smallest (positive) eigenvalue of } \mathcal{A}$. Using (4.18) in (4.17) yields

\[ [c_h + 2\alpha]\|h\|^2 + [\rho c_s + 2\alpha]\|v_2\|^2 \leq 0, \quad \omega \neq 0 \quad (4.19) \]

Thus, if

\[ c_h + 2\alpha > 0 \quad \text{and} \quad \rho c_s + 2\alpha > 0; \quad \text{i.e.} \, 0 \geq \alpha > -\frac{1}{2}\min\{c_h, \rho c_s\} = -k, \quad (4.20) \]
then \( h \equiv 0 \) in \( \Omega_f \), \( v_2 \equiv 0 \) in \( \Omega_s \), hence \( v_1 \equiv 0 \) in \( \Omega_s \), as desired.

**Proposition 4.3.** Let \( \rho = 0 \) and \( \alpha = 0 \). Then, the corresponding operator \( A \) (which is boundedly invertible by Theorem 2.2 (ii), see its inverse in Remark 3.1) has infinitely many eigenvalues \( \pm i \sqrt{\mu_n} \) on the imaginary axis, where \( \{\mu_n\}_{n=1}^{\infty} \) are the positive eigenvalues, \( \mu_n \to +\infty \) of the operator \( \Delta^2 \) with “hinged” B.C.:

\[
\begin{align*}
\Delta^2 \phi_n &= \mu_n \phi_n \quad \text{in} \quad \Omega_s \\
\Delta \phi_n |_{\Gamma_s} &= 0, \quad \phi_n |_{\Gamma_s} = 0 \quad \text{in} \quad \Omega_s
\end{align*}
\]

**Proof.** With \( \alpha = 0 \) and \( \rho = 0 \), we return to identity (4.12) and obtain \( \nabla h \equiv 0 \) hence \( h \equiv 0 \) in \( \Omega_f \) by (2.8). Thus, the eigenvalue problem (4.3) specializes to \( v_2 = i \omega v_1 \),

\[
v_2 = i \omega v_1; \quad \begin{cases} 
\Delta^2 v_1 = -i \omega v_2 = \omega^2 v_1 & \text{in} \quad \Omega_s \\
\Delta v_1 = \frac{\partial h}{\partial \nu} \equiv 0 & \text{in} \quad \Gamma_s \\
v_1 = 0 & \text{on} \quad \Gamma_s
\end{cases}
\]

Thus there are infinitely many eigenvalues \( \omega_n^2 = \mu_n \) with corresponding eigenfunctions \( v_{1n}, n = 1, 2, \ldots \), as desired.

5. **Proof of Theorem 2.3.** (i) We first prove (2.23), that is analyticity. The constant \( 0 < \rho \leq 1 \) is fixed. We already know from Theorem 2.2 (ii) that \( i \omega \in \rho(A), |\omega| < \omega_0 \) for some positive \( \omega_0 \), and actually \( i \omega \in \rho(A) \) by Proposition 4.1 (ii).

**Step 1.** Given \( \{v^*_1, v^*_2, h^*\} \in H \) and \( \omega \in \mathbb{R} \setminus \{0\} \), we first seek to solve the equation

\[
(i \omega I - A) \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} i \omega v_1 - v_2 \\ i \omega v_2 + \Delta^2 (v_1 + \rho v_2) \\ i \omega h - \Delta h \end{bmatrix} = \begin{bmatrix} v^*_1 \\ v^*_2 \\ h^* \end{bmatrix}.
\]

in terms of \( \{v_1, v_2, h\} \in \mathcal{D}(A) \) uniquely. For \( \lambda = i \omega, \omega \neq 0 \), we have, recalling (2.6)

\[
AR(i \omega I, A) = \begin{bmatrix} v^*_1 \\ v^*_2 \\ h^* \end{bmatrix} = \begin{bmatrix} 0 & I & 0 \\ -\Delta^2 & -\rho \Delta^2 & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix} = \begin{bmatrix} v_2 \\ -\Delta^2 (v_1 + \rho v_2) \\ \Delta h \end{bmatrix}.
\]

The analyticity condition (2.23): there is a constant \( C > 0 \) such that

\[
\|R(i \omega, A)\|_{L(H)} \leq \frac{C}{|\omega|}, \quad \forall |\omega| > \text{some} \ \omega_0 > 0.
\]

is equivalent (since \( AR(\lambda, A) = -I + \lambda R(\lambda, A) \)) to showing: there is a constant \( C > 0 \) such that

\[
\|AR(i \omega I, A)\|_{L(H)} \leq C, \quad \forall |\omega| > \text{some} \ \omega_0 > 0.
\]

These constants will depend on \( 0 < \rho \leq 1 \) fixed. In view of (5.2), condition (5.4) is in turn equivalent to showing the following estimate, recalling the topology of \( H \)
in (2.4) (all norms are $L_2$-norm in their respective domains): there is a constant $C > 0$ such that
\[
\|\Delta v_2\|^2 + \|\Delta^2 (v_1 + \rho v_2)\|^2 + \|\Delta h\|^2 \leq C \left[ \|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2 \right],
\]
\[
\forall |\omega| > \text{some } \omega_0 > 0. \quad (5.5)
\]

Below, we shall in fact establish a more precise inequality than (5.5): see the final estimate (5.20).

**Step 2.** We take the inner product of the third identity in (5.1) with $\Delta h$ and obtain:
\[
i\omega (h, \Delta h) - \|\Delta h\|^2 = (h^*, \Delta h), \text{ or via Green’s first theorem, recalling } h|_{\Gamma_s} = 0 \text{ by (2.8) and the orientation of } \nu \text{ on } \Gamma_s
\]
\[
- i\omega \left( h, \frac{\partial h}{\partial \nu} \right)_{\Gamma_s} - i\omega \|\nabla h\|^2 - \|\Delta h\|^2 = (h^*, \Delta h). \quad (5.6)
\]

Next we take the inner product of the second identity in (5.1) with $\Delta (v_1 + \rho v_2)$ and obtain:
\[
i\omega (v_2, \Delta^2 (v_1 + \rho v_2)) + \|\Delta^2 (v_1 + \rho v_2)\|^2 = (v_2^*, \Delta^2 (v_1 + \rho v_2)); \text{ or via Green’s second theorem, recalling } v_2|_{\Gamma_s} = 0 \text{ by (2.7a)}:
\]
\[
i\omega (\Delta v_2, \Delta (v_1 + \rho v_2)) + i\omega \left( v_2, \frac{\partial \Delta (v_1 + \rho v_2)^*}{\partial \nu} \right)_{\Gamma_s}
- i\omega \left( \frac{\partial v_2}{\partial \nu}, \Delta (v_1 + \rho v_2) \right)_{\Gamma_s} + \|\Delta^2 (v_1 + \rho v_2)\|^2 = (v_2^*, \Delta^2 (v_1 + \rho v_2)). \quad (5.7)
\]

Recalling now $\Delta (v_1 + \rho v_2) = -\rho (1 - \mu) B_1 v_2 + \frac{\partial h}{\partial \nu}$ on $\Gamma_s$ by (2.7c) as well as
\[
\frac{\partial v_2}{\partial \nu} \big|_{\Gamma_s} = h|_{\Gamma_s} \text{ by (2.8)}, \text{ we rewrite (5.7) as}
\]
\[
i\omega (\Delta v_2, \Delta v_1) + i\rho \omega \|\Delta v_2\|^2 + \|\Delta^2 (v_1 + \rho v_2)\|^2
+ i\rho \omega (1 - \mu) \left( \frac{\partial v_2}{\partial \nu}, B_1 v_2 \right)_{\Gamma_s} - i\omega \left( h, \frac{\partial h}{\partial \nu} \right)_{\Gamma_s} = (v_2^*, \Delta^2 (v_1 + \rho v_2)). \quad (5.8)
\]

**Step 3.** Next we subtract (5.6) from (5.8), and obtain a cancellation of the boundary term $(h, \frac{\partial h}{\partial \nu})_{\Gamma_s}$. In the resulting expression, we next use the first equation of (5.1) to obtain the following expression for the first term on LHS of (5.8)
\[
i\omega (\Delta v_2, \Delta v_1) = - (\Delta v_2, \Delta (i\omega v_1)) = - (\Delta v_2, \Delta (v_2^* + v_1^*)) = - \|\Delta v_2\|^2 - (\Delta v_2, \Delta v_1^*). \quad (5.9)
\]

Thus, we finally arrive at the following identity:
\[
\|\Delta h\|^2 + \|\Delta^2 (v_1 + \rho v_2)\|^2 - \|\Delta v_2\|^2 + i\omega \|\nabla h\|^2
+ i\rho \omega \left\{ \|\Delta v_2\|^2 + \left( \frac{\partial v_2}{\partial \nu}, (1 - \mu) B_1 v_2 \right)_{\Gamma_s} \right\}
= (v_2^*, \Delta^2 (v_1 + \rho v_2)) + (\Delta v_2, \Delta v_1^*) - (h^*, \Delta h). \quad (5.10)
\]
We now invoke the critical identity (3.7) (of real terms) of Section 3 for the term \{ \} in brackets in (5.10). We then obtain
\begin{align*}
\| \Delta h \|^2 + \| \Delta^2 (v_1 + \rho v_2) \|^2 - \| \Delta v_2 \|^2 \\
+ i \omega \left\{ \| \nabla h \|^2 + \rho \mu \| \Delta v_2 \|^2 + \rho \int_{\Omega_s} [(1 - \mu)(v_{2xx}^2 + v_{2yy}^2) + 2(1 - \mu)v_{2xy}^2] \, d\Omega_s \right\} \\
= (v_2^2, \Delta^2 (v_1 + \rho v_2)) + (\Delta v_2, \Delta v_1^*) - \langle h^*, \Delta h \rangle. \quad (5.11)
\end{align*}

**Step 4.** Taking the real part of (5.11) yields
\begin{align*}
\| \Delta h \|^2 + \| \Delta^2 (v_1 + \rho v_2) \|^2 \\
= \| \Delta v_2 \|^2 + \text{Re} \langle v_2^2, \Delta^2 (v_1 + \rho v_2) \rangle + \text{Re} (\Delta v_2, \Delta v_1^*) - \lambda \langle h^*, \Delta h \rangle, \quad (5.12)
\end{align*}
from which we then obtain for \( \varepsilon > 0 \) small enough:
\begin{align*}
(1 - \varepsilon) \left[ \| \Delta h \|^2 + \| \Delta^2 (v_1 + \rho v_2) \|^2 \right] \leq (1 + \varepsilon) \| \Delta v_2 \|^2 + C \varepsilon \left[ \| \Delta v_1^* \|^2 + \| v_2^2 \|^2 + \| h^* \|^2 \right]. \quad (5.13)
\end{align*}

**Step 5.** We now take the imaginary part of identity (5.11). We obtain
\begin{align*}
\omega \left\{ \| \nabla h \|^2 + \rho \mu \| \Delta v_2 \|^2 + \rho \int_{\Omega_s} [(1 - \mu)(v_{2xx}^2 + v_{2yy}^2) + 2(1 - \mu)v_{2xy}^2] \, d\Omega_s \right\} \\
= \text{Im} \langle v_2^2, \Delta^2 (v_1 + \rho v_2) \rangle + \text{Im} (\Delta v_2, \Delta v_1^*) - \text{Im} \langle h^*, \Delta h \rangle. \quad (5.14)
\end{align*}
The integral term \( \int_{\Omega_s} \) over \( \Omega_s \), i.e. \( P(v_2) \) as in (2.21d), is positive for \( 0 < \mu < 1 \), as assumed. From (5.14), we ready obtain for \( \varepsilon > 0 \) small enough, as \( \rho \mu < 1 \):
\begin{align*}
(|w| \rho \mu - \varepsilon) \left[ \| \Delta v_2 \|^2 + \| \nabla h \|^2 + P(v_2) \right] \leq \varepsilon \left[ \| \Delta h \|^2 + \| \Delta^2 (v_1 + \rho v_2) \|^2 \right] + \widetilde{C} \varepsilon \left[ \| \Delta v_1^* \|^2 + \| v_2^2 \|^2 + \| h^* \|^2 \right]. \quad (5.15)
\end{align*}
where we have recalled the non-negative Interior Term \( P(v_2) = \int_{\Omega_s} [(1 - \mu)(v_{2xx}^2 + v_{2yy}^2) + 2(1 - \mu)v_{2xy}^2] \, d\Omega_s > 0 \) from (2.21d) for convenience. Taking \( (|w| \rho \mu - \varepsilon) > 0 \), some \( \omega_1 > 0 \), (naturally, the proof fails for \( \rho = 0 \)), we obtain
\begin{align*}
\left[ \| \Delta v_2 \|^2 + \| \nabla h \|^2 + P(v_2) \right] \\
\leq \frac{\varepsilon}{\omega_1} \left[ \| \Delta h \|^2 + \| \Delta^2 (v_1 + \rho v_2) \|^2 \right] + \frac{\widetilde{C} \varepsilon}{\omega_1} \left[ \| \Delta v_1^* \|^2 + \| v_2^2 \|^2 + \| h^* \|^2 \right]. \quad (5.16)
\end{align*}

**Step 6.** On the LHS of estimate (5.16), we drop the positive term \( \| \nabla h \|^2 + P(v_2) \) and substitute the term \( \| \Delta v_2 \|^2 \) in the RHS of (real part) estimate (5.13). We obtain
\begin{align*}
\| \Delta h \|^2 + \| \Delta^2 (v_1 + \rho v_2) \|^2 \leq \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right) \frac{\varepsilon}{\omega_1} \left[ \| \Delta h \|^2 + \| \Delta^2 (v_1 + \rho v_2) \|^2 \right] \\
+ \left( \frac{C \varepsilon}{1 - \varepsilon} + \frac{\widetilde{C} \varepsilon}{\omega_1} \right) \left[ \| \Delta v_1^* \|^2 + \| v_2^2 \|^2 + \| h^* \|^2 \right], \quad \forall (|w| \rho \mu - \varepsilon) > \omega_1 > 0; \quad (5.17)
\end{align*}
By Theorem 2.2 (ii), inequality (5.5). Theorem 2.3 (i) is proved.

Proof of Theorem 2.4: Estimate (2.27). Step 1. Given 

\[
\begin{bmatrix}
v_1 \\
v_2 \\
h
\end{bmatrix} = R((\alpha + i\omega), A) \begin{bmatrix}
v_1^* \\
v_2^* \\
h^*
\end{bmatrix}
\]

in terms of \( \{v_1, v_2, h\} \in D(A) \) uniquely with bounded inverse. We have again via (2.6)

\[
\begin{bmatrix}
v_1 \\
v_2 \\
h
\end{bmatrix} = R((\alpha + i\omega), A) \begin{bmatrix}
v_1^* \\
v_2^* \\
h^*
\end{bmatrix}
\]

Step 7. To complete the proof of the final estimate (5.5), we substitute inequality (5.18) in the RHS of the (imaginary part) estimate (5.16). We thus obtain recalling (2.21d)

\[
\|\Delta v_2\|^2 + \|\nabla h\|^2 + \int_{\Omega_s} [(1 - \mu)(v_{xx}^2 + v_{yy}^2) + 2(1 - \mu)v_{xy}^2] \, d\Omega_s
\]

Step 8. Combining estimates (5.18) and (5.19), we finally arrive at

\[
\|\Delta^2(v_1 + v_2)\|^2 + \|\Delta h\|^2 + \|\nabla h\|^2 + \int_{\Omega_s} [(1 - \mu)(v_{xx}^2 + v_{yy}^2) + 2(1 - \mu)v_{xy}^2] \, d\Omega_s
\]

\[
\leq \text{const}_{\varepsilon, \omega} \left[\|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2\right], \quad \forall (|\omega| \rho \mu - \varepsilon) > \omega_1 > 0.
\]
\[ A R ((\alpha + i\omega), \mathcal{A}) \begin{pmatrix} v_1^* \\ v_2^* \\ h^* \end{pmatrix} = \begin{bmatrix} 0 & I & 0 \\ -\Delta^2 & -\rho\Delta^2 & 0 \\ 0 & 0 & \Delta \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ h \end{pmatrix} = \begin{bmatrix} -\Delta^2(v_1 + \rho v_2) \\ \Delta h \end{pmatrix}. \] (6.3)

The sought-after (analyticity) condition (2.27) is therefore explicitly rewritten as follows via (6.3): there exists a constant \( C > 0 \) such that, recalling the topology (2.4)

\[
\left\{ \|\Delta v_2\|^2 + \|\Delta^2(v_1 + \rho v_2)\|^2 + \|\Delta h\|^2 \leq C \left( \|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2 \right) \right\}
\]

for all \( \lambda = \alpha + i\omega \) in \( \rho(A) \backslash \mathcal{K}_\rho \); that is, outside the set \( \mathcal{K}_\rho \) defined in (2.20) (Figure 2) (6.4)

This estimate (6.4) – that gives analyticity and a much more precise description of the spectrum location for \( A \) over (2.26a), which is a consequence of (2.23) – is proved in the present section. The proof is a more delicate extension of the proof of Theorem 2.3 of Section 5.

Orientation (with reference to Figure 4) In the proof of statement (6.4) given below, we shall proceed as follows, after obtaining the basic identities

(i) We start with an arbitrarily small positive number \( r_1 > 0 \), which we hold fixed throughout.

(ii) Henceforth, the points \( \{\alpha, \omega\} \), \( \alpha < 0 \), will be further constrained to satisfy the preliminary constraint: \( |\alpha| > r_1 \) arbitrarily small; that is, to lie in the half plane \( \alpha < -r_1 < 0 \). See below (6.14). This allows one to select a number \( \epsilon_1 > 0 \) such that \( 1 + \epsilon_1 < \frac{|\alpha|}{r_1} \).

(iii) Next, we preassign an arbitrarily small number \( \epsilon_2 \), and select a number \( \epsilon > 0 \) such that \( \frac{\epsilon}{r_1} < \epsilon_2 \). This is, in effect, the number \( \epsilon > 0 \) that works throughout the proof. Such choice implies that \( k_0\epsilon = \rho \mu \epsilon + \frac{\epsilon}{r_1} \) is arbitrarily small, as needed below (6.25).

(iv) Finally, the points \( \{\alpha, \omega\} \) will be constrained to satisfy two further requirements

\[
(\rho \mu - \epsilon)(\alpha^2 + \omega^2) + 2\alpha > 0; \text{ and } \epsilon|\alpha| \leq |\omega|
\]

The first condition says that \( \{\alpha, \omega\} \) must lie outside the disk centered at \( \left\{ \frac{-1}{\rho \mu - \epsilon}, 0 \right\} \), of radius \( \frac{1}{\rho \mu - \epsilon} \), that is with circumference: \( \left( \alpha + \frac{1}{\rho \mu - \epsilon} \right)^2 + \omega^2 = \left( \frac{1}{\rho \mu - \epsilon} \right)^2 \). The second condition says that \( \{\alpha, \omega\} \) must lie outside the (open) triangular sector (in the half-plane \( \alpha < -r_1 < 0 \)) delimited by the half-line \( \omega = -\epsilon\alpha \) for \( \omega > 0 \), and the half-line \( \omega = \epsilon\alpha \) for \( \omega < 0 \); the first passing through the origin and the point \( \{-1, \epsilon\} \); the second passing through the origin and the point \( \{1, -\epsilon\} \).

Under the above constraints (i) through (iv) for the points \( \{\alpha, \omega\} \) – illustrated in Fig. 4 – we obtain estimate (6.27) in Step 10. Finally, since \( \epsilon > 0 \) is arbitrarily small (since so was the preassigned \( \epsilon_2 > 0 \)), and \( r_1 > 0 \) is arbitrarily small, the estimate (6.27) holding true for each such \( \epsilon > 0 \) and \( r_1 > 0 \) implies the desired final estimate (6.4).
Step 2. We take the inner product of the third identity (6.1c) with $\Delta h$ and obtain 
\[(\alpha + i\omega)(h, \Delta h) - \|\Delta h\|^2 = (h^*, \Delta h); \] or via Green’s first theorem recalling also $h|_{\Gamma_s} = 0$ by (2.8) as well as the inward orientation of $\nu$ on $\Omega_f$,
\[- (\alpha + i\omega)(h, \partial h/\partial \nu)_{\Gamma_s} - (\alpha + i\omega)\|\nabla h\|^2 - \|\Delta h\|^2 = (h^*, \Delta h). \quad (6.5)\]

Next, we take the inner product of the second identity (6.1b) with $\Delta^2(v_1 + \rho v_2)$ and obtain: 
\[(\alpha + i\omega)(v_2, \Delta^2(v_1 + \rho v_2)) + \|\Delta^2(v_1 + \rho v_2)\|^2 = (v_2^*, \Delta^2(v_1 + \rho v_2)); \] or via Green’s second theorem, recalling $v_2|_{\Gamma_s} = 0$ in (2.7a) as well as $\nu$ being outward to $\Omega_s$ on $\Gamma_s$:
\[(\alpha + i\omega)(\Delta v_2, \Delta(v_1 + \rho v_2)) + (\alpha + i\omega)(v_2, \frac{\partial \Delta(v_1 + \rho v_2)}{\partial \nu})_{\Gamma_s}
\- (\alpha + i\omega)(v_2, \Delta(v_1 + \rho v_2))_{\Gamma_s} - \|\Delta^2(v_1 + \rho v_2)\|^2 = (v_2^*, \Delta^2(v_1 + \rho v_2)). \quad (6.6)\]

Recalling now $\Delta(v_1 + \rho v_2) = -\rho(1 - \mu)B_1v_2 + \frac{\partial h}{\partial \nu}$ on $\Gamma_s$ by (2.7c) as well as 
$\frac{\partial v_2}{\partial \nu} |_{\Gamma_s} = h|_{\Gamma_s}$ by (2.8), we re-write (6.6) as
\[(\alpha + i\omega)(\Delta v_2, \Delta v_1) + \rho(\alpha + i\omega)\|\Delta v_2\|^2 + \|\Delta^2(v_1 + \rho v_2)\|^2
\+ \rho(\alpha + i\omega)(1 - \mu)(\frac{\partial v_2}{\partial \nu}, B_1v_2_{\Gamma_s})_{\Gamma_s} - (\alpha + i\omega)(h, \frac{\partial h}{\partial \nu})_{\Gamma_s} = (v_2^*, \Delta^2(v_1 + \rho v_2)). \quad (6.7)\]

As to the first term on the LHS of (6.7) since by (6.1a),
\[v_1 = (v_2 + v_1^*)/(\alpha + i\omega) = (\alpha - i\omega)(v_2 + v_1^*)/(\alpha^2 + \omega^2),\]
we rewrite it as
\[(\alpha + i\omega)(\Delta v_2, \Delta v_1) = \frac{(\alpha + i\omega)^2}{\alpha^2 + \omega^2} \left[ \|\Delta v_2\|^2 + (\Delta v_2, \Delta v_1^*) \right]\]
\[= \frac{(\alpha^2 - \omega^2) + i2\alpha\omega}{\alpha^2 + \omega^2} \|\Delta v_2\|^2 + \frac{(\alpha + i\omega)^2}{\alpha^2 + \omega^2} (\Delta v_2, \Delta v_1^*). \quad (6.8)\]

So that (6.7) is rewritten as
\[\|\Delta^2(v_1 + \rho v_2)\|^2 + \rho(\alpha + i\omega) \left\{ \|\Delta v_2\|^2 + (1 - \mu) \left( \frac{\partial v_2}{\partial \nu}, B_1 v_2 \right)_{\Gamma_s} \right\} \]
\[+ \frac{(\alpha^2 - \omega^2) + i2\alpha\omega}{\alpha^2 + \omega^2} \|\Delta v_2\|^2 - \rho(\alpha + i\omega) \left( h, \frac{\partial h}{\partial \nu} \right)_{\Gamma_s} \]
\[= (v_2^*, \Delta^2(v_1 + \rho v_2)) - \frac{\alpha + i\omega)^2}{\alpha^2 + \omega^2} (\Delta v_2, \Delta v_1^*). \quad (6.9)\]

We next invoke the fundamental identities (3.6), (3.7) for the term within brackets \{\} in (6.9). In doing so, we recall the positive term \(P(v_2)\) in (2.21d), so that \{\} = \(\mu\|\Delta v_2\|^2 + P(v_2)\). This way, we re-write (6.9) in its final form as
\[\|\Delta^2(v_1 + \rho v_2)\|^2 + \rho(\alpha + i\omega) \left\{ \mu\|\Delta v_2\|^2 + P(v_2) \right\} + \frac{(\alpha^2 - \omega^2) + i2\alpha\omega}{\alpha^2 + \omega^2} \|\Delta v_2\|^2 \]
\[- \rho(\alpha + i\omega) \left( h, \frac{\partial h}{\partial \nu} \right)_{\Gamma_s} = (v_2^*, \Delta^2(v_1 + \rho v_2)) - \frac{(\alpha + i\omega)^2}{\alpha^2 + \omega^2} (\Delta v_2, \Delta v_1^*). \quad (6.10)\]

**Step 3.** Next, subtract (6.5) from (6.10), thus canceling the boundary term \(\left( h, \frac{\partial h}{\partial \nu} \right)_{\Gamma_s}\). We then arrive at the final identity of interest
\[\|\Delta^2(v_1 + \rho v_2)\|^2 + \|\Delta h\|^2 + \frac{\alpha^2 - \omega^2}{\alpha^2 + \omega^2} \|\Delta v_2\|^2 + \alpha \|\nabla h\|^2 + \rho \alpha \left[ \mu\|\Delta v_2\|^2 + P(v_2) \right] \]
\[+ i\omega \left\{ \rho \mu\|\Delta v_2\|^2 + P(v_2) \right\} + \frac{2\alpha}{\alpha^2 + \omega^2} \|\Delta v_2\|^2 + \|\nabla h\|^2 \]
\[= (v_2^*, \Delta^2(v_1 + \rho v_2)) - (h^*, \Delta h) - \frac{(\alpha + i\omega)^2}{\alpha^2 + \omega^2} (\Delta v_2, \Delta v_1^*), \quad (6.11)\]
where the LHS has been split into real and imaginary parts.

**Step 4.** We now take the real part of the (6.11). We obtain (recall \(\alpha < 0\))
\[\|\Delta^2(v_1 + \rho v_2)\|^2 + \|\Delta h\|^2 = \left[ |\alpha| \rho \mu + \frac{\omega^2 - \alpha^2}{\alpha^2 + \omega^2} \right] \|\Delta v_2\|^2 + |\alpha| \|\nabla h\|^2 + \rho P(v_2) \]
\[+ \Re \left\{ (v_2^*, \Delta^2(v_1 + \rho v_2)) - (h^*, \Delta h) - \frac{(\alpha + i\omega)^2}{\alpha^2 + \omega^2} (\Delta v_2, \Delta v_1^*) \right\}. \quad (6.12)\]

**Step 5.** Next we take the imaginary part of (6.11) which we re-write again, but opting now for a different distribution of terms. We obtain
\[ \omega \left\{ \rho \mu + \frac{2\alpha}{\alpha^2 + \omega^2} \right\} \| \Delta v_2 \|^2 + \rho P(v_2) + \| \nabla h \|^2 \]

\[ = \text{Im} \left\{ (v_2^*, \Delta^2(v_1 + \rho v_2)) - (h^*, \Delta h) - \frac{(\alpha + i\omega)^2}{\alpha^2 + \omega^2} (\Delta v_2, \Delta v_1^*) \right\}. \quad (6.13) \]

**Step 6.** We return to the real part identity (6.12). Here and below we shall use freely that

\[ |\omega^2 - \alpha^2| \leq 1, \quad \frac{|(\alpha + i\omega)|}{\alpha^2 + \omega^2} \equiv 1. \]

On the right-hand side of (6.12), we penalize the terms \( \Delta^2(v_1 + \rho v_2) \) and \( \Delta h \) by \( \varepsilon > 0 \) and the term \( \Delta v_2 \) by \( \varepsilon_1 > 0 \). We obtain

\[ (1 - \varepsilon) \left[ \| \Delta^2(v_1 + \rho v_2) \|^2 + \| \Delta h \|^2 \right] \leq \left[ \alpha |\rho\mu + 1 + \varepsilon_1 | \right] \| \Delta v_2 \|^2 + |\alpha| \| \nabla h \|^2 + |\alpha| \rho P(v_2) + C_{\varepsilon, \varepsilon_1} \left\{ \| \Delta v_1^* \|^2 + \| v_2^* \|^2 + \| h^* \|^2 \right\}. \quad (6.14) \]

We next take \( |\alpha| > r_1 > 0 \), with \( r_1 \) fixed but arbitrarily small (Selection (i) in the Orientation), \( 1 + \varepsilon_1 < \frac{1}{r_1} |\alpha| \), and setting \( k_0 = \rho \mu + \frac{1}{r_1} \), we obtain

\[ (1 - \varepsilon) \left[ \| \Delta^2(v_1 + \rho v_2) \|^2 + \| \Delta h \|^2 \right] \leq k_0 |\alpha| \| \Delta v_2 \|^2 + |\alpha| \| \nabla h \|^2 + |\alpha| \rho P(v_2) + C_{\varepsilon, \varepsilon_1} \left\{ \| \Delta v_1^* \|^2 + \| v_2^* \|^2 + \| h^* \|^2 \right\}. \quad (6.15) \]

Next, with \( \varepsilon > 0 \) chosen above, we take \( \varepsilon |\alpha| \leq |\omega| \) (Selection (iv) in the Orientation), thereby rewriting (6.15) in its definitive form

\[ (1 - \varepsilon) \left[ \| \Delta^2(v_1 + \rho v_2) \|^2 + \| \Delta h \|^2 \right] \leq \frac{1}{\varepsilon} \left[ k_0 |\alpha| \| \Delta v_2 \|^2 + \varepsilon |\alpha| \| \nabla h \|^2 + \varepsilon |\alpha| \rho P(v_2) \right] + C_{\varepsilon, \varepsilon_1} \left\{ \| \Delta v_1^* \|^2 + \| v_2^* \|^2 + \| h^* \|^2 \right\} \quad (6.16) \]

\[ \leq \frac{1}{\varepsilon} \left[ k_0 |\omega| \| \Delta v_2 \|^2 + |\omega| \| \nabla h \|^2 + |\omega| \rho P(v_2) \right] + C_{\varepsilon, \varepsilon_1} \left\{ \| \Delta v_1^* \|^2 + \| v_2^* \|^2 + \| h^* \|^2 \right\}. \quad (6.17) \]

The passage from (6.16) to (6.17) is critical using \( \varepsilon |\alpha| \leq |\omega| \).

**Step 7.** We return to the imaginary part identity (6.13). With \( \varepsilon > 0 \) arbitrarily fixed above, impose further on \( \{\alpha, \omega\} \) that (Selection(iv) in the Orientation)

\[ 0 < \varepsilon < \left[ \rho \mu + \frac{2\alpha}{\alpha^2 + \omega^2} \right], \quad \text{that is,} \quad 0 < (\rho \mu - \varepsilon)(\alpha^2 + \omega^2) + 2\alpha \] or

\[ \left( \alpha + \frac{1}{\rho \mu - \varepsilon} \right)^2 + \omega^2 - \left( \frac{1}{\rho \mu - \varepsilon} \right)^2 > 0 \]

(6.18)

that is, that the point \( (\alpha, \omega) \) lies outside the disk

\[ \left( \alpha + \frac{1}{\rho \mu - \varepsilon} \right)^2 + \omega^2 = \left( \frac{1}{\rho \mu - \varepsilon} \right)^2 \]
Thus using (6.18) on the LHS of (6.13) we obtain

\[ \varepsilon |\omega| \|\Delta v_2\|^2 + |\omega| \rho P(v_2) + |\omega| \|\nabla h\|^2 \]

\[ \leq \varepsilon^3 \left[ \|\Delta^2(v_1 + \rho v_2)\|^2 + \|\Delta h\|^2 \right] + \frac{\varepsilon^2}{2} \|\Delta v_2\|^2 + C_\varepsilon \left\{ \|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2 \right\}. \]

(6.19)

or

\[ \left( \varepsilon |\omega| - \frac{\varepsilon^2}{2} \right) \|\Delta v_2\|^2 + |\omega| \rho P(v_2) + |\omega| \|\nabla h\|^2 \]

\[ \leq \varepsilon^3 \left[ \|\Delta^2(v_1 + \rho v_2)\|^2 + \|\Delta h\|^2 \right] + C_\varepsilon \left\{ \|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2 \right\}. \]

(6.20)

Impose now the condition on \( \varepsilon \) so that

\[ \frac{\varepsilon}{2} |\omega| < |\omega| - \frac{\varepsilon^2}{2}, \quad \text{equivalently} \quad \frac{\varepsilon^2}{2} < \frac{\varepsilon}{2} |\omega|, \quad \text{or} \quad 0 < \varepsilon < |\omega|. \]

(6.21)

which is a-fortiori implied by the above choices \( \varepsilon |\alpha| \leq |\omega| \) below (6.15) and (6.18) for \( r_1 \leq |\alpha| \). We finally obtain from (6.21) used in (6.20):

\[ \frac{\varepsilon}{2} |\omega| \|\Delta v_2\|^2 + |\omega| \|\nabla h\|^2 + |\omega| \rho P(v_2) \]

\[ \leq \varepsilon^3 \left[ \|\Delta^2(v_1 + \rho v_2)\|^2 + \|\Delta h\|^2 \right] + C_\varepsilon \left\{ \|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2 \right\}. \]

(6.22)

Then (6.22) implies the following two inequalities, the second of which in (6.23b)

\[ |\omega| \|\nabla h\|^2 + |\omega| \rho P(v_2) \leq \varepsilon^3 \left[ \|\Delta^2(v_1 + \rho v_2)\|^2 + \|\Delta h\|^2 \right] + C_\varepsilon \left\{ \|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2 \right\} \]

(6.23a)

and

\[ |\omega| \|\Delta v_2\|^2 \leq 2\varepsilon^2 \left[ \|\Delta^2(v_1 + \rho v_2)\|^2 + \|\Delta h\|^2 \right] + C_\varepsilon \left\{ \|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2 \right\}. \]

(6.23b)

**Step 8.** We return to estimate (6.17), and on its RHS we invoke (6.23a-6.23b). We obtain

(1 - \( \varepsilon \)) \left[ \|\Delta^2(v_1 + \rho v_2)\|^2 + \|\Delta h\|^2 \right]

\[ \leq \frac{1}{\varepsilon} \left\{ 2k_0 \varepsilon^2 \left[ \|\Delta^2(v_1 + \rho v_2)\|^2 + \|\Delta h\|^2 \right] \right\} + \frac{1}{\varepsilon} \left\{ \varepsilon^3 \left[ \|\Delta^2(v_1 + \rho v_2)\|^2 + \|\Delta h\|^2 \right] \right\}

+ C_{\varepsilon, \varepsilon_1} \left\{ \|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2 \right\}. \]

(6.24)

Finally, (6.24) implies, as desired,

\[ \|\Delta^2(v_1 + \rho v_2)\|^2 + \|\Delta h\|^2 \leq C_{\varepsilon, \varepsilon_1, \varepsilon_2} \left\{ \|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2 \right\}, \]

(6.25)

since \[ 1 - \varepsilon - 2k_0 \varepsilon - \varepsilon^2 \] > 0 by restricting further \( \varepsilon \) to have \( \frac{2\varepsilon}{r_1} < \varepsilon_2 \) with \( \varepsilon_2 \) arbitrarily small, so that recalling \( k_0 = \rho \mu + \frac{1}{r_1} \), we have \( 2k_0 \varepsilon = 2\rho \mu \varepsilon + \frac{2\varepsilon}{r_1} \) arbitrarily small.
Estimate (6.25) is the first explicit contribution towards proving the sought-after estimate (6.4).

**Step 9.** We return to estimate (6.22). On its LHS, we drop the positive terms $|\omega||\nabla h|^2 + |\omega|\rho P(v_2)$, while for its first term we use: \( \frac{\varepsilon^2}{2} r_1 \leq |\alpha| \leq \frac{\varepsilon}{2} |\omega| \) according to prior selections \( \varepsilon |\alpha| \leq |\omega| \) and \( |\alpha| > r_1 \). On the RHS of (6.22), we invoke (6.25). We thus obtain, as desired
\[
\|\Delta v_2\|^2 \leq C_{\varepsilon, x_1, x_2} \left\{ \|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2 \right\}
\] (6.26)

**Step 10.** Summing up (6.25) and (6.26), we finally obtain
\[
\begin{cases}
\|\Delta v_2\|^2 + \|\Delta^2(v_1 + \rho v_2)\|^2 + \|\Delta h\|^2 \leq C \left\{ \|\Delta v_1^*\|^2 + \|v_2^*\|^2 + \|h^*\|^2 \right\}
\end{cases}
\]
for all \( \lambda = \alpha + i\omega \), such that the point \((\alpha, \omega)\) lies outside the disk
\[
\left(\alpha + \frac{1}{\rho \mu - \varepsilon}\right)^2 + \omega^2 = \left(\frac{1}{\rho \mu - \varepsilon}\right)^2,
\]
and outside the triangular sector \( \omega = \pm \varepsilon \alpha \), in the half-plane \( \lambda < -r_1 < 0 \).

(6.27)

Since \( \varepsilon > 0 \) arbitrarily small, and \( r_1 > 0 \) arbitrarily small, then (6.27) proves (6.4). Theorem 2.4 is established.

**Remark 6.1.** Here we provide another proof of Proposition 4.1 (i), Eqs. (4.1), (4.2), that there is no eigenvalue of \( \mathcal{A} \) (or \( \mathcal{A}^* \)) outside the open disk \( S_{\rho \mu}(x_0), x_0 = \end{equation}
\{ - \frac{1}{\rho \mu}, 0 \} \) with \( \omega \neq 0 \). Consider the eigenvalue problem (4.3), or equivalently (6.1) with \( \{v_1^*, v_2^*, h^*\} = 0 \). Our starting point is now identity (6.11) (with \( \{v_1^*, v_2^*, h^*\} = 0 \) – which was obtained by use of higher order multipliers \( \Delta^2(v_1 + \rho v_2), \Delta h \), rather than identity (4.9) – which was obtained in Section 4 by use of lower order multipliers \( v_2, h \).

**Step 1.** We return to identity (6.11) (with \( \{v_1^*, v_2^*, h^*\} = 0 \)). In the present proof – unlike the one in Section 4 – we only need to use its imaginary part, which we now re-write as follows (see (6.13))
\[
\omega \left\{ \rho \mu + \frac{2\alpha}{\alpha^2 + \omega^2} \right\} \|\Delta v_2\|^2 + \rho P(v_2) + \|\nabla h\|^2 \right\} = 0
\] (6.28)

**Case 1.** Let \( \omega \neq 0 \), and \( \rho \mu + \frac{2\alpha}{\alpha^2 + \omega^2} > 0 \), equivalently \( \rho \mu (\alpha^2 + \omega^2) + 2\alpha > 0 \), as assumed in (4.14) first case. Then (6.28) implies
(i) \( \Delta v_2 \equiv 0 \) in \( \Omega_s \), which along with \( v_2|_{\Gamma_s} = 0 \) in (2.7a) \( \rightarrow v_2 \equiv 0 \) in \( \Omega_s \); hence \( \rightarrow v_1 \equiv 0 \) in \( \Omega_s \);
(ii) \( \nabla h \equiv 0 \) in \( \Omega_f \), which along with \( h|_{\Gamma_f} = 0 \) in (2.8) \( \rightarrow h \equiv 0 \) in \( \Omega_f \).

Thus, \( \{v_1, v_2, h\} = 0 \) and Case 1 is re-proved.

**Case 2.** Let now \( w \neq 0 \) and \( \rho \mu + \frac{2\alpha}{\alpha^2 + \omega^2} = 0 \), as assumed in the second case of Section 4. Then (6.28) implies (i) \( \nabla h \equiv 0 \) and \( h \equiv 0 \) in \( \Omega_f \) as before; (ii) \( P(v_2) \equiv 0 \) in \( \Omega_s \). By the definition of \( P \) in (2.21d), \( P(v_2) \equiv 0 \) in \( \Omega_s \rightarrow v_{2x_s}^2 + v_{2y_s}^2 \equiv 0 \) or \( v_{2x_s} \equiv v_{2y_s} \equiv 0 \), or \( \Delta v_2 \equiv 0 \) in \( \Omega_s \) (for \( 0 < \mu < 1 \) as assumed). This, then yield \( v_2 \equiv 0 \) as before, as \( v_2|_{\Gamma_s} = 0 \). Finally \( v_1 \equiv 0 \) in \( \Omega_s \), with \( \omega \neq 0 \). Case 2 is proved. So Proposition 4.1 (i) has been re-proved.
Appendix A: The adjoint operator $A^*$. Here we establish Theorem 2.1 by identifying the adjoint operator $A^*$. Let $[v_1, v_2, h] \in D(A)$ described in (2.7)-(2.10), and let $[\hat{v}_1, \hat{v}_2, \hat{h}] \in D(A^*)$ described in (2.12) and (2.13). Recalling $A$ in (2.6) and the topology of $H$ in (2.4), we compute recalling the orientation of $\nu$

$$
\begin{align*}
\left( A \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{h} \end{bmatrix} \right)_H &= \left( \begin{bmatrix} 0 & -\Delta^2 v_2 \\ -\Delta^2 & -\rho \Delta^2 v_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ h \end{bmatrix}, \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \hat{h} \end{bmatrix} \right)_H \\
&= \left( -\Delta^2 (v_1 + \rho v_2), \hat{v}_2 \right)_H = (\Delta v_2, \Delta \hat{v}_1) - (\Delta (v_1 + \rho v_2), \Delta \hat{v}_2) - \left( \frac{\partial \Delta (v_1 + \rho v_2)}{\partial \nu}, \hat{v}_2 \right)_{\Gamma_s} \\
&\quad + \left( \Delta (v_1 + \rho v_2), \frac{\partial \hat{v}_2}{\partial \nu} \right)_{\Gamma_s} + (h, \Delta \hat{h}) - \left( \frac{\partial h}{\partial \nu}, \hat{h} \right)_{\Gamma_s} + \left( h, \frac{\partial \hat{h}}{\partial \nu} \right)_{\Gamma_s}.
\end{align*}
$$

(A.1)

In going from (A.2) to (A.3), we have used Green’s second theorem for the second and third terms in (A.2), along with $\hat{v}_2|_{\Gamma_s} = 0$ by (2.12a) and $h|_{\Gamma_s} = 0$ by (2.8) and $\hat{h}|_{\Gamma_s} = 0$ by (2.13). We next recall $\Delta (v_1 + \rho v_2) = -\rho (1 - \mu) B_1 v_2 + \frac{\partial h}{\partial \nu} = \rho (1 - \mu) c(\eta) \frac{\partial v_2}{\partial \nu} + \frac{\partial h}{\partial \nu}$ on $\Gamma_s$ by (2.7c) along with $B_1 v_2 = -c(\eta) \frac{\partial v_2}{\partial \nu}$ on $\Gamma_s$ by (2.7d). We re-write (A.3) as

\begin{align*}
\text{RHS of (A.3)} &= (\Delta v_2, \Delta (\hat{v}_1 - \rho \hat{v}_2)) - (\Delta v_1, \Delta \hat{v}_2) + \rho (1 - \mu) \left( c(\eta) \frac{\partial v_2}{\partial \nu}, \frac{\partial \hat{v}_2}{\partial \nu} \right)_{\Gamma_s} \\
&\quad + \left( \frac{\partial h}{\partial \nu}, \frac{\partial \hat{v}_2}{\partial \nu} \right)_{\Gamma_s} + (h, \Delta \hat{h}) - \left( \frac{\partial h}{\partial \nu}, \hat{h} \right)_{\Gamma_s} + \left( h, \frac{\partial \hat{h}}{\partial \nu} \right)_{\Gamma_s}.
\end{align*}

(A.4)

In (A.4) we have recalled $\hat{h} = \frac{\partial \hat{v}_2}{\partial \nu}$ on $\Gamma_s$ by (2.13) leading to the noted cancellation. Next, we apply again Green’s second theorem on the first term in (A.4) and obtain since $v_2|_{\Gamma_s} = 0$ by (2.7a),

\begin{align*}
\text{RHS of (A.4)} &= (v_2, \Delta^2 (\hat{v}_1 - \rho \hat{v}_2)) + \left( \frac{\partial \hat{v}_2}{\partial \nu}, \Delta (\hat{v}_1 - \rho \hat{v}_2) \right)_{\Gamma_s} - \left( v_2, \frac{\partial \Delta (\hat{v}_1 - \rho \hat{v}_2)}{\partial \nu} \right)_{\Gamma_s} \\
&\quad + \left( \frac{\partial v_2}{\partial \nu}, \rho (1 - \mu) c(\eta) \frac{\partial \hat{v}_2}{\partial \nu} \right)_{\Gamma_s} - (\Delta v_1, \Delta \hat{v}_2) + (h, \Delta \hat{h}) + \left( h, \frac{\partial \hat{h}}{\partial \nu} \right)_{\Gamma_s}.
\end{align*}

(A.5)

$$
\begin{align*}
&= \langle v_2, \Delta^2 (\hat{v}_1 - \rho \hat{v}_2) \rangle + \langle \frac{\partial v_2}{\partial \nu}, \Delta (\hat{v}_1 - \rho \hat{v}_2) + \rho (1 - \mu) c(\eta) \frac{\partial \hat{v}_2}{\partial \nu} \rangle_{\Gamma_s}
\end{align*}
$$


where the cancellation noted in (A.6) uses (2.12c) and (2.12d) as well as $h = \frac{\partial v_2}{\partial v}$ on $\Gamma_s$ by (2.8). Combining (A.1) - (A.6) we finally obtain

$$
(A.7)
$$

for $[v_1, v_2, h] \in D(A)$ in (2.7)-(2.10) and $[\tilde{v}_1, \tilde{v}_2, \tilde{h}] \in D(A^*)$ in (2.12) and (2.13). Theorem 2.1 is proved.

Appendix B: Proof of Theorem 2.5, on the model with the B.C. in (2.1d) replacing $B_1w_1|_{\Gamma_s}$ with $[B_1(w + w_1)]|_{\Gamma_s}$. (Compare with Section 3.1) For $[v_1, v_2, h] \in D(A)$, we compute via (3.1), (2.30) and (2.32)

$$
(B.1)
$$

$$
(B.2)
$$

$$
(B.3)
$$

since $v_2|_{\Gamma_s} = 0$ and $h|_{\Gamma_s} = 0$ in $\tilde{A}$, see (2.7a), (2.8). Next we use

$$
(B.4)
$$

Moreover, in (B.3), we use the B.C. (2.33) as well as $h|_{\Gamma_s} = \frac{\partial v_2}{\partial v}|_{\Gamma_s}$ from (2.8). We then obtain from (B.3), (B.4)

$$
(B.5)
$$

and Theorem 2.5 is established.
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