ON STRONG $r$-HELIX SUBMANIFOLDS AND SPECIAL CURVES

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Abstract. In this paper, we investigate special curves on a strong $r$-helix submanifold in Euclidean $n$-space $E^n$. Also, we give the important relations between strong $r$-helix submanifolds and the special curves such as line of curvature, geodesic and slant helix.

1. Introduction

In differential geometry of manifolds, an helix submanifold of $IR^n$ with respect to a fixed direction $d$ in $IR^n$ is defined by the property that tangent planes make a constant angle with the fixed direction $d$ (helix direction) in [5]. Di Scala and Ruiz-Hernández have introduced the concept of these manifolds in [5]. Besides, the concept of strong $r$-helix submanifold of $IR^n$ was introduced in [4]. Let $M \subset IR^n$ be a submanifold and let $H(M)$ be the set of helix directions of $M$. We say that $M$ is a strong $r$-helix if the set $H(M)$ is a linear subspace of $IR^n$ of dimension greater or equal to $r$ in [4].

Recently, M. Ghomi worked out the shadow problem given by H.Wente. And, He mentioned the shadow boundary in [8]. Ruiz-Hernández investigated that shadow boundaries are related to helix submanifolds in [12].

Helix hypersurfaces has been worked in nonflat ambient spaces in [6,7]. Cermelli and Di Scala have also studied helix hypersurfaces in liquid crystals in [3].

The plan of this paper is as follows. In section 2, we mention some basic facts in the general theory of strong $r$-helix, manifolds and curves. And, in section 3, we give the important relations between strong $r$-helix submanifolds and some special curves such as line of curvature, geodesic and slant helix.

2. Preliminaries

Definition 2.1. Let $M \subset IR^n$ be a submanifold of a euclidean space. A unit vector $d \in IR^n$ is called a helix direction of $M$ if the angle between $d$ and any tangent space $T_pM$ is constant. Let $H(M)$ be the set of helix directions of $M$. We say that $M$ is a strong $r$-helix if $H(M)$ is a $r$-dimensional linear subspace of $IR^n$ [4].

Definition 2.2. A submanifold $M \subset IR^n$ is a strong $r$-helix if the set $H(M)$ is a linear subspace of $IR^n$ of dimension greater or equal to $r$ [4].

Definition 2.3. A unit speed curve $\alpha : I \to E^n$ is called a slant helix if its unit principal normal $V_2$ makes a constant angle with a fixed direction $U$ [1].

Definition 2.4. Let the $(n-k)$-manifold $M$ be submanifold of the Riemannian manifold $\overline{M} = E^n$ and let $\overline{D}$ be the Riemannian connexion on $\overline{M} = E^n$. For $C^\infty$ fields $X$ and $Y$ with domain $A$ on $M$ (and tangent to $M$), define $D_XY$ and $V(X,Y)$ on $A$ by decomposing $\overline{D}_XY$ into unique tangential and normal components, respectively; thus,

$$\overline{D}_XY = D_XY + V(X,Y).$$

Then, $D$ is the Riemannian connexion on $M$ and $V$ is a symmetric vector-valued 2-covariant $C^\infty$ tensor called the second fundamental tensor. The above composition equation is called the Gauss equation [9].

Definition 2.5. Let the $(n-k)$-manifold $M$ be submanifold of the Riemannian manifold $\overline{M} = E^n$, let $\overline{D}$ be the Riemannian connexion on $\overline{M} = E^n$ and let $D$ be the Riemannian connexion on $M$. Then, the formula of Weingarten

$$\overline{D}_XN = -AN(X) + D^2N$$

for every $X$ and $Y$ tangent to $M$ and for every $N$ normal to $M$. $AN$ is the shape operator associated to $N$ also known as the Weingarten operator corresponding to $N$ and $D^2$ is the induced connexion in the

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normal bundle of $M$ ($A_N(X)$ is also the tangent component of $\overline{\nabla}_X N$ and will be denoted by $A_N(X) = \text{tang}(\overline{\nabla}_X N)$). Specially, if $M$ is a hypersurface in $E^n$, we have $(V(X, Y), N) = (A_N(X), Y)$ for all $X, Y$ tangent to $M$. So,

$$V(X, Y) = \langle V(X, Y), N \rangle N = \langle A_N(X), Y \rangle N$$

and we obtain

$$\overline{\nabla}_X Y = D_X Y + \langle A_N(X), Y \rangle N.$$  

For this definition 2.5, note that the shape operator $A_N$ is defined by the map $A_N : \kappa(M) \to \kappa(M)$, where $\kappa(M)$ is the space of tangent vector fields on $M$ and if $p \in M$, the shape operator $A_N$ is defined by the map $A_p : T_p(M) \to T_p(M)$. The eigenvalues of $A_p$ are called the principal curvatures (denoted by $\lambda_j$) and the eigenvectors of $A_p$ are called the principal vectors \cite{10,11}.

Definition 2.6. If $\alpha$ is a (unit speed) curve in $M$ with $C^\infty$ unit tangent $T$, then $V(T, T)$ is called normal curvature vector field of $\alpha$ and $k_T = \|V(T, T)\|$ is called the normal curvature of $\alpha$ \cite{9}.

3. MAIN THEOREMS

Theorem 3.1. Let $M$ be a strong r-helix hypersurface and $H(M) \subset E^n$ be the set of helix directions of $M$. If $\alpha : I \subset R \to M$ is a (unit speed) line of curvature (not a line) on $M$, then $d_j \notin Sp\{N, T\}$ along the curve $\alpha$ for all $d_j \in H(M)$, where $T$ is the tangent vector field of $\alpha$ and $N$ is a unit normal vector field of $M$.

Proof. We assume that $d_j \in Sp\{N, T\}$ along the curve $\alpha$ for any $d_j \in H(M)$. Then, along the curve $\alpha$, since $M$ is a strong r-helix hypersurface, we can decompose $d_j$ in tangent and normal components:

$$d_j = \cos(\theta_j)N + \sin(\theta_j)T \quad (3.1)$$

where $\theta_j$ is constant. From (3.1), by taking derivatives on both sides along the curve $\alpha$, we get:

$$0 = \cos(\theta_j)N' + \sin(\theta_j)T' \quad (3.2)$$

Moreover, since $\alpha$ is a line of curvature on $M$,

$$N' = \lambda_n^\alpha \quad (3.3)$$

along the curve $\alpha$. By using the equations (3.2) and (3.3), we deduce that the system $\{\alpha', T\}$ is linear dependent. But, the system $\{\alpha', T\}$ is never linear dependent. This is a contradiction. This completes the proof. \hfill \Box

Theorem 3.2. Let $M$ be a submanifold with $(n-k)$ dimension in $E^n$. Let $\overline{\nabla}$ be Riemannian connexion (standard covariant derivative) on $E^n$ and $D$ be Riemannian connexion on $M$. Let us assume that $M \subset E^n$ be a strong r-helix submanifold and $H(M) \subset E^n$ be the space of the helix directions of $M$. If $\alpha : I \subset R \to M$ is a (unit speed) geodesic curve on $M$ and if $\langle V_2, \xi_j \rangle$ is a constant function along the curve $\alpha$, then $\alpha$ is a slant helix in $E^n$, where $V_2$ is the unit principal normal of $\alpha$ and $\xi_j$ is the normal component of a direction $d_j \in H(M)$.

Proof. Let $T$ be the unit tangent vector field of $\alpha$. Then, from the formula Gauss in Definition (2.4),

$$\overline{\nabla}_T T = D_T T + V(T, T) \quad (3.4)$$

According to the Theorem, since $\alpha$ is a geodesic curve on $M$,

$$D_T T = 0 \quad (3.5)$$

So, by using (3.4),(3.5) and Frenet formulas, we have:

$$\overline{\nabla}_T T = k_1 V_2 = V(T, T)$$

That is, the vector field $V_2 \in \vartheta(M)$ along the curve $\alpha$, where $\vartheta(M)$ is the normal space of $M$. On the other hand, since $M$ is a strong r-helix submanifold, we can decompose any $d_j \in H(M)$ in its tangent and normal components:

$$d_j = \cos(\theta_j)\xi_j + \sin(\theta_j)T_j \quad (3.6)$$

where $\theta_j$ is constant. Moreover, according to the Theorem, $\langle V_2, \xi_j \rangle$ is a constant function along the curve $\alpha$ for the normal component $\xi_j$ of a direction $d_j \in H(M)$. Hence, doing the scalar product with $V_2$ in each part of the equation (3.6), we obtain:

$$\langle d_j, V_2 \rangle = \cos(\theta_j) \langle V_2, \xi_j \rangle + \sin(\theta_j) \langle V_2, T_j \rangle \quad (3.7)$$
Since \( \cos(\theta_j) \langle V_2, \xi_j \rangle \) = constant and \( \langle V_2, T_j \rangle = 0 \) (\( V_2 \in \vartheta(M) \)) along the curve \( \alpha \), from (3.7) we have:
\[
\langle d_j, V_2 \rangle = \text{constant}.
\]
along the curve \( \alpha \). Consequently, \( \alpha \) is a slant helix in \( E^n \). 

**Theorem 3.3.** Let \( M \) be a submanifold with \((n-k)\) dimension in \( E^n \). Let \( \overline{D} \) be Riemannian connexion (standard covariant derivative) on \( E^n \) and \( D \) be Riemannian connexion on \( M \). Let us assume that \( M \subset E^n \) be a strong \( r \)-helix submanifold and \( H(M) \subset E^n \) be the space of the helix directions of \( M \). If \( \alpha : I \subset IR \to M \) is a (unit speed) curve on \( M \) with the normal curvature function \( k_T = 0 \) and if \( \langle V_2, T_j \rangle \) is a constant function along the curve \( \alpha \), then \( \alpha \) is a slant helix in \( E^n \), where \( V_2 \) is the unit principal normal of \( \alpha \) and \( T_j \) is the tangent component of a direction \( d_j \in H(M) \).

**Proof.** Let \( T \) be the unit tangent vector field of \( \alpha \). Then, from the formula Gauss in Definition (2.4),
\[
\overline{D}_TT = D_T T + V(T, T)
\]
(3.8)

According to the Theorem, since the normal curvature \( k_T = 0 \),
\[
V(T, T) = 0
\]
(3.9)

So, by using (3.8),(3.9) and Frenet formulas, we have:
\[
\overline{D}_TT = k_1 V_2 = D_T T.
\]
That is, the vector field \( V_2 \in T_{\alpha(t)}M \), where \( T_{\alpha(t)}M \) is the tangent space of \( M \). On the other hand, since \( M \) is a strong \( r \)-helix submanifold, we can decompose any \( d_j \in H(M) \) in its tangent and normal components:
\[
d_j = \cos(\theta_j) \xi_j + \sin(\theta_j) T_j
\]
(3.10)

where \( \theta_j \) is constant. Moreover, according to the Theorem, \( \langle V_2, T_j \rangle \) is a constant function along the curve \( \alpha \) for the tangent component \( T_j \) of a direction \( d_j \in H(M) \). Hence, doing the scalar product with \( V_2 \) in each part of the equation (3.10), we obtain:
\[
\langle d_j, V_2 \rangle = \cos(\theta_j) \langle V_2, \xi_j \rangle + \sin(\theta_j) \langle V_2, T_j \rangle
\]
(3.11)

Since \( \sin(\theta_j) \langle V_2, \xi_j \rangle \) = constant and \( \langle V_2, \xi_j \rangle = 0 \) (\( V_2 \in T_{\alpha(t)}M \)) along the curve \( \alpha \), from (3.11) we have:
\[
\langle d_j, V_2 \rangle = \text{constant}.
\]
along the curve \( \alpha \). Consequently, \( \alpha \) is a slant helix in \( E^n \). 

**Definition 3.1.** Given an Euclidean submanifold of arbitrary codimension \( M \subset IR^n \). A curve \( \alpha \) in \( M \) is called a line of curvature if its tangent \( T \) is a principal vector at each of its points. In other words, when \( T \) (the tangent of \( \alpha \)) is a principal vector at each of its points, for an arbitrary normal vector field \( N \in \vartheta(M) \), the shape operator \( A_N \) associated to \( N \) says \( A_N(T) = \text{tang}(-\overline{D}_TN) = \lambda_j T \) along the curve \( \alpha \), where \( \lambda_j \) is a principal curvature and \( \overline{D} \) be the Riemannian connexion (standard covariant derivative) on \( IR^n \) [2].

**Theorem 3.4.** Let \( M \) be a submanifold with \((n-k)\) dimension in \( E^n \) and let \( \overline{D} \) be Riemannian connexion (standard covariant derivative) on \( E^n \). Let us assume that \( M \subset E^n \) be a strong \( r \)-helix submanifold and \( H(M) \subset E^n \) be the space of the helix directions of \( M \). If \( \alpha : I \to M \) is a line of curvature with respect to the normal component \( N_j \in \vartheta(M) \) of a direction \( d_j \in H(M) \) and if \( N_j \in \vartheta(M) \) along the curve \( \alpha \), then \( d_j \in SP \{ T \}^\perp \) along the curve \( \alpha \), where \( T \) is the unit tangent vector field of \( \alpha \).

**Proof.** We assume that \( \alpha : I \to M \) is a line of curvature with respect to the normal component \( N_j \in \vartheta(M) \) of a direction \( d_j \in H(M) \). Since \( M \) is a strong \( r \)-helix submanifold, we can decompose \( d_j \in H(M) \) in its tangent and normal components:
\[
d_j = \cos(\theta_j) N_j + \sin(\theta_j) T_j
\]
where \( \theta_j \) is constant. So, \( \langle N_j, d_j \rangle \) = constant and by taking derivatives on both sides along the curve \( \alpha \), we get \( \langle N_j, d_j \rangle = 0 \). On the other hand, since \( \alpha : I \to M \) is a line of curvature with respect to the \( N_j \in \vartheta(M) \),
\[
A_{N_j}(T) = \text{tang}(-\overline{D}_TN_j) = \text{tang}(-N_j) = \lambda_j T
\]
along the curve \( \alpha \). According to this Theorem, \( N_j \in \vartheta(M) \) along the curve \( \alpha \). Hence,
\[
\text{tang}(N_j) = -N_j = \lambda_j T
\]
(3.12)
Therefore, by using the equalities \( \langle N_j', d_j \rangle = 0 \) and (3.12), we obtain:
\[
(T, d_j) = 0
\]
along the curve \( \alpha \). This completes the proof. \( \square \)

**Theorem 3.5.** Let \( M \) be a submanifold with \( (n-k) \) dimension in \( E^n \) and let \( \overline{\nabla} \) be Riemannian connexion (standart covariant derivative) on \( E^n \). Let us assume that \( M \subset E^n \) be a strong \( r \)-helix submanifold and \( H(M) \subset E^n \) be the space of the helix directions of \( M \). If \( \alpha : I \rightarrow M \) is a curve in \( M \) and if the system \( \{T_j', T\} \) is linear dependent along the curve \( \alpha \), where \( T_j' \) is the derivative of the tangent component \( T_j \) of a direction \( d_j \in H(M) \) and \( T \) the tangent to the curve \( \alpha \), then \( \alpha \) is a line of curvature in \( M \).

**Proof.** Since \( M \) is a strong \( r \)-helix submanifold, we can decompose \( d_j \in H(M) \) in its tangent and normal components:
\[
d_j = \cos(\theta_j)N_j + \sin(\theta_j)T_j
\]
where \( \theta_j \) is constant. If we take derivative in each part of the equation (3.13) along the curve \( \alpha \), we obtain:
\[
0 = \cos(\theta_j)N_j' + \sin(\theta_j)T_j'
\]
From (3.14), we can write
\[
N_j' = -\tan(\theta_j)T_j'
\]
So, for the tangent component of \(-N_j'\), from (3.15) we can write:
\[
A_{N_j}(T) = \tan(-\overline{\nabla}_T N_j) = \tan(-N_j') = \tan(\tan(\theta_j)T_j)
\]
along the curve \( \alpha \). According to the hypothesis, the system \( \{T_j', T\} \) is linear dependent along the curve \( \alpha \). Hence, we get \( T_j' = \lambda_j T \). And, by using the equation (3.16), we have:
\[
A_{N_j}(T) = \tan(\tan(\theta_j)T_j)
\]
and
\[
A_{N_j}(T) = \tan(\tan(\theta_j)\lambda_j T)
\]
Moreover, since \( T \in \pi(M) \), \( \tan(\tan(\theta_j)\lambda T) = (\tan(\theta_j)\lambda_j)T = k_j T \). Therefore, from (3.17), we have:
\[
A_{N_j}(T) = k_j T
\]
It follows that \( \alpha \) is a line of curvature in \( M \) for \( N_j \subset \partial(M) \). This completes the proof. \( \square \)

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