Multiplicity of closed geodesics on bumpy Finsler manifolds with elliptic closed geodesics

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January 23, 2023

Abstract

Let $M$ be a compact simply connected manifold satisfying $H^*(M;\mathbb{Q}) \cong T_{d,n+1}(x)$ for integers $d \geq 2$ and $n \geq 1$. If all prime closed geodesics on $(M, F)$ with an irreversible bumpy Finsler metric $F$ are elliptic, then either there exist exactly $\frac{dn(n+1)}{2}$ (when $d \geq 2$ is even) or $(d+1)$ (when $d \geq 3$ is odd) distinct closed geodesics, or there exist infinitely many distinct closed geodesics.

Key words: Closed geodesic, multiplicity, elliptic, bumpy Finsler metric

2000 Mathematics Subject Classification: 53C22, 58E05, 58E10.

1 Introduction and main result

Recall that a closed curve on a Finsler manifold is a closed geodesic if it is locally the shortest path connecting any two nearby points on this curve (cf. [She01]). As usual, on any Finsler manifold $(M, F)$, a closed geodesic $c : S^1 = \mathbb{R}/\mathbb{Z} \to M$ is prime if it is not a multiple covering (or, iteration) of any other closed geodesic. Here the $m$-th iteration $c^m$ of $c$ is defined by $c^m(t) = c(mt)$, $\forall m \geq 1$. The inverse curve $c^{-1}$ of $c$ is defined by $c^{-1}(t) = c(1 - t)$ for $t \in \mathbb{R}$. In the Riemannian case, two closed geodesics $c$ and $d$ are called geometrically distinct if $c(S^1) \neq d(S^1)$. However, unlike the Riemannian case, the inverse curve $c^{-1}$ of a closed geodesic $c$ on an irreversible Finsler manifold $(M, F)$ need not be a geodesic. And we call two prime closed geodesics $c$ and $d$ on $(M, F)$ distinct if there is no $\theta \in (0, 1)$ such that $c(t) = d(t + \theta)$ for all $t \in \mathbb{R}$.

For a closed geodesic $c$ on an $(n + 1)$-dimensional manifold $M$, denote by $P_c$ the linearized Poincaré map of $c$, which is a symplectic matrix, i.e., $P_c \in \text{Sp}(2n)$. We define the elliptic height

∗Partially supported by National Key R&D Program of China (Grant No. 2020YFA0713300), NNSFC (Nos. 12271268, 11671215 and 11790271) and the Fundamental Research Funds for the Central Universities. E-mail: duanhg@nankai.edu.cn.

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$e(P_c)$ of $P_c$ to be the total algebraic multiplicity of all eigenvalues of $P_c$ on the unit circle $U = \{z \in \mathbb{C} | |z| = 1\}$ in the complex plane $\mathbb{C}$. Since $P_c$ is symplectic, $e(P_c)$ is even and $0 \leq e(P_c) \leq 2n$. A closed geodesic $c$ is called elliptic if $e(P_c) = 2n$, i.e., all the eigenvalues of $P_c$ locate on $U$; irrationally elliptic if, in the homotopy component $\Omega^0(P_c)$ of $P_c$ (cf. Section 2 below for the definition), $P_c$ can be connected to the $\circ$-product of $n$ rotation matrices $R(\theta_i)$ with $\theta_i$ being irrational multiple of $\pi$ for $1 \leq i \leq n$; hyperbolic if $e(P_c) = 0$, i.e., all the eigenvalues of $P_c$ locate away from $U$; non-degenerate if $1$ is not an eigenvalue of $P_c$. A Finsler metric $F$ is called bumpy if all the closed geodesics on $(M, F)$ are non-degenerate.

The closed geodesic problem is a classical one in Riemannian geometry and dynamical system. A famous conjecture claims the existence of infinitely many distinct closed geodesics on any compact simply connected manifold $M$. This conjecture has been proved for many cases. For example, Gromoll and Meyer proved the conjecture in [GrM69] for any compact $M$ provided the Betti number sequence $\{b_p(\Lambda M)\}_{p \in \mathbb{N}}$ of the free loop space $\Lambda M$ of $M$ is unbounded. Later, for any compact simply connected manifold $M$, Vigué-Poirrier and Sullivan in [VS76] further proved that the Betti number sequence is bounded if and only if $M$ satisfies

$$H^r(M; \mathbb{Q}) \cong T_{d,n+1}(x) = \mathbb{Q}[x]/(x^{n+1} = 0)$$

(1.1)

with a generator $x$ of degree $d \geq 2$ and height $n + 1 \geq 2$, where dim $M = dn$. Note that when $d$ is odd, then $x^2 = 0$ and $n = 1$, and then $M$ is rationally homotopic to $S^d$ (cf. Remark 2.5 of [Rad89] and [Hin84]). Especially, these manifolds include the sphere $S^d$ (with $n = 1$), complex projective space $\mathbb{C}P^n$ (with $d = 2$), quaternionic projective space $\mathbb{H}P^n$ (with $d = 4$), and Cayley plane $\mathbb{C}aP^2$ (with $d = 8$ and $n = 2$) as special examples.

Among manifolds satisfying (1.1), only for Riemannian $S^2$ the above conjecture was proved by [Fra92] of Franks and [Ban93] of Bangert. When considering Finsler metrics, the situation changes dramatically. It was quite surprising that Katok in [Kat73] constructed some irreversible Finsler metrics on rank one symmetric spaces which possess finitely many $(\frac{dn(n+1)}{2})$ when $d \geq 2$ is even, or $d + 1$ when $d \geq 2$ is odd) distinct closed geodesics, and all these closed geodesics are irrationally elliptic. The geometry of Katok’s metrics was further studied by Ziller in [Zil82]. Based on Katok’s examples, Anosov in [Ano74] conjectured that the smallest number of distinct closed geodesics on any Finsler sphere $S^d$ should be $2[(d + 1)/2]$. This conjecture was proved by Bangert and Long in [BL10] for every Finsler sphere $(S^2, F)$.

It is natural to generalize this conjecture to all compact simply connected Finsler manifolds satisfying (1.1), i.e., the lower bound of the number of distinct closed geodesics on such manifolds should be $\frac{dn(n+1)}{2}$ when $d \geq 2$ is even, or $d + 1$ when $d \geq 2$ is odd, respectively. This has been proved by [DLW16a] under the bumpy metric and the nonnegative flag curvature conditions.

Recently, a great number of results about the multiplicity and stability of closed geodesics on Finsler manifolds have appeared, for example, we refer readers to some recent survey [BuK21], [DLZ20], [Lon06], [Tai10], and some research papers [Dua15], [Dua16], [DL22], [DuL07], [DuL10],
Note that Hingston in [Hin84] proved the existence of infinitely many closed geodesics on spheres if all prime closed geodesics are hyperbolic. Hofer, Wysocki and Zehnder in [HWZ98] and [HWZ03] proved that there exist either two or infinitely many prime closed geodesics on every bumpy Finsler $S^2$ provided the stable and unstable manifolds of every hyperbolic closed geodesic intersect transversally. Furthermore, most recently this result has been proved to be true only under the bumpy metric by [CHP19] of Cristofaro-Gardiner, Hutchings and Pomerleano by using the embedded contact homology method. In addition, Long and Wang in [LW08] proved that on every Finsler $S^2$, there exist either at least two irrationally elliptic closed geodesics or infinitely many closed geodesics.

Based on Katok’s examples and the above well-known results, about the number of closed geodesics on compact simply-connected manifolds with irreversible Finsler metrics, we suspect the following result holds.

**Conjecture A.** (cf. [DLW16a]) Let $M$ be a compact simply connected manifold satisfying (1.1), and $F$ be any irreversible Finsler metric $F$ on $M$. Denote by $\mathcal{N}(M,F)$ the number of the distinct closed geodesics on $(M,F)$. Then there holds

- $\mathcal{N}(M,F) = \frac{dn(n+1)}{2}$ or $+\infty$, if $d$ is even;
- $\mathcal{N}(M,F) = d + 1$ or $+\infty$, if $d$ is odd.

Furthermore, if $\mathcal{N}(M,F) < +\infty$, then all these closed geodesics are irrationally elliptic.

Motivated by Katok’s examples and the above Conjecture, we will establish the following result, which gives a partial answer to Conjecture A in the bumpy case.

**Theorem 1.1.** Let $M$ be a compact simply connected manifold satisfying (1.1). If all prime closed geodesics on $(M,F)$ with an irreversible bumpy Finsler metric $F$ are elliptic, then either $\mathcal{N}(M,F) \in \{\frac{dn(n+1)}{2}, +\infty\}$ when $d$ is even, or $\mathcal{N}(M,F) \in \{d + 1, +\infty\}$ when $d$ is odd.

Note that when the manifold $(M,F)$ is the sphere $(S^d,F)$, Theorem 1.1 has been proved in [DL16] under more stronger conditions. On the other hand, the methods in [DL16] depend heavily on the monotonicity of iterated indices of closed geodesics, which has been guaranteed since the Morse index $i(c) \geq \dim M - 1$ there for every closed geodesic $c$.

In this paper, note that $\dim M = dn$, but the Morse index can only be estimated to be greater than and equal to $(d - 1)$ (see Claim 2 in Section 3), which will lead to the loss of monotonicity of the iterated indices of closed geodesics. In order to overcome this difficulty, we will make use of some methods and arguments developed most recently by the first author and his co-workers, especially the enhanced common index jump methods established in [DLW16a] and [DLW16b].

This paper is organized as follows. In Section 2, we review the critical point theory and index iteration theory for closed geodesics. In Section 3, we give the proof of Theorem 1.1. Next let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural integers, non-negative integers, integers, rational
numbers, real numbers, and complex numbers respectively. We use only singular homology modules with $\mathbb{Q}$-coefficients. The following notations will be used in this paper.

$$
\begin{aligned}
\gamma &\in (Finsler manifold) \\
(F) &\;\text{natural structure of Hilbert manifold on which the group} \ F \\
\text{Especially,} \; \varphi(a) &\;\text{even.}
\end{aligned}
$$

It is $C_1$ with numbers, real numbers, and complex numbers respectively. We use only singular homology modules with $\mathbb{Q}$-coefficients. The following notations will be used in this paper.

$$
\begin{aligned}
[a] &\;\max\{k \in \mathbb{Z} \mid k \leq a\}, \quad E(a) = \min\{k \in \mathbb{Z} \mid k \geq a\}, \\
\varphi(a) &\;E(a) - [a], \\
\{a\} &\;a - [a].
\end{aligned}
$$

Especially, $\varphi(a) = 0$ if $a \in \mathbb{Z}$, and $\varphi(a) = 1$ if $a \notin \mathbb{Z}$.

## 2 Morse theory and index iteration theory of closed geodesics

### 2.1 Morse theory of closed geodesics

Let $(M, F)$ be a compact Finsler manifold, the space $\Lambda = \Lambda M$ of $H^1$-maps $\gamma : S^1 \to M$ has a natural structure of Hilbert manifold on which the group $S^1 = \mathbb{R}/\mathbb{Z}$ acts continuously by isometries (cf. [Kli78]). This action is defined by $(s \cdot \gamma)(t) = \gamma(t + s)$ for all $\gamma \in \Lambda$ and $s, t \in S^1$. For any $\gamma \in \Lambda$, the energy functional is defined by

$$
E(\gamma) = \frac{1}{2} \int_{S^1} F(\gamma(t), \dot{\gamma}(t))^2 dt.
$$

It is $C^{1,1}$ and invariant under the $S^1$-action. The critical points of $E$ of positive energies are precisely the closed geodesics $\gamma : S^1 \to M$. The index form of the functional $E$ is well defined along any closed geodesic $c$ on $M$, which we denote by $E''(c)$. As usual, we denote by $i(c)$ and $\nu(c)$ the Morse index and nullity of $E$ at $c$. In the following, we denote by

$$
\Lambda^\kappa = \{d \in \Lambda \mid E(d) \leq \kappa\}, \quad \Lambda^{\kappa-} = \{d \in \Lambda \mid E(d) < \kappa\}, \quad \forall \kappa \geq 0.
$$

For a closed geodesic $c$ we set $\Lambda(c) = \{\gamma \in \Lambda \mid E(\gamma) < E(c)\}$.

For $m \in \mathbb{N}$ we denote the $m$-fold iteration map $\phi_m : \Lambda \to \Lambda$ by $\phi_m(\gamma)(t) = \gamma(mt)$, for all $\gamma \in \Lambda, t \in S^1$, as well as $\gamma^m = \phi_m(\gamma)$. If $\gamma \in \Lambda$ is not constant then the multiplicity $m(\gamma)$ of $\gamma$ is the order of the isotropy group $\{s \in S^1 \mid s \cdot \gamma = \gamma\}$. For a closed geodesic $c$, the mean index $\hat{i}(c)$ is defined as usual by $\hat{i}(c) = \lim_{m \to \infty} i(e^m)/m$. Using singular homology with rational coefficients we consider the following critical $\mathbb{Q}$-module of a closed geodesic $c \in \Lambda$:

$$
\overline{\mathcal{C}}_*(E, c) = H_*\left((\Lambda(c) \cup S^1 \cdot c)/S^1, \Lambda(c)/S^1\right).
$$

**Proposition 2.1.** (cf. Satz 6.11 of [Rad92]) Let $c$ be a prime closed geodesic on a bumpy Finsler manifold $(M, F)$. Then there holds

$$
\overline{\mathcal{C}}_*(E, c^m) = \begin{cases} 
\mathbb{Q}, & \text{if } i(c^m) - i(c) \in 2\mathbb{Z} \quad \text{and} \quad q = i(c^m), \\
0, & \text{otherwise}.
\end{cases}
$$

**Definition 2.2.** (cf. Definition 1.6 of [Rad89]) For a closed geodesic $c$, let $\gamma_c \in \{\pm \frac{1}{2}, \pm 1\}$ be the invariant defined by $\gamma_c > 0$ if and only if $i(c)$ is even, and $|\gamma_c| = 1$ if and only if $i(c^2) - i(c)$ is even.
Theorem 2.3. (cf. Theorem 3.1 of [Rad89] and Satz 7.9 of [Rad92]) Let \((M, F)\) be a compact simply connected bumpy Finsler manifold of \(\dim M = dn\) satisfying \(H^*(M, \mathbb{Q}) = T_{d,n+1}(x)\). Denote by \(\{c_k\}_{1 \leq k \leq q}\) the prime closed geodesics on \((M, F)\) with positive mean indices. Then
\[
\sum_{j=1}^{q} \frac{\gamma_{c_j}}{i(c_j)} = B(d, n) = \begin{cases} 
-\frac{n(n+1)d}{2d(n+1)-4}, & \text{d is even}, \\
\frac{d+1}{2d-2}, & \text{d is odd}.
\end{cases}
\] (2.3)

Let \((X, Y)\) be a space pair with \(S^1\)-action such that the Betti numbers \(b_i = \dim H_i(X/S^1, Y/S^1; \mathbb{Q})\) are finite for all \(i \in \mathbb{Z}\). The following result gives the precise information about the Betti number sequence of \(\Lambda M\).

Lemma 2.4. (cf. Theorem 2.4 of [Rad89] and Lemma 2.5 and Lemma 2.6 of [DuL 10]) Let \(M\) be a compact simply connected manifold satisfying \(H^*(M; \mathbb{Q}) \cong T_{d,n+1}(x)\) for integers \(d \geq 2\) and \(n \geq 1\).

(i) When \(d\) is odd (which implies that \(n = 1\)), i.e. \(M\) is rationally homotopic to the sphere \(S^d\), then the Betti numbers of the free loop space of \(S^d\) are given by
\[
b_i(\Lambda S^d) = \text{rank} H_i(\Lambda S^d/S^1, \Lambda^0 S^d/S^1; \mathbb{Q}) = \begin{cases} 
2, & \text{if } i \in K \equiv \{k(d-1) \mid 2 \leq k \in \mathbb{N}\}, \\
1, & \text{if } i \in \{d-1 + 2k \mid k \in \mathbb{N}_0\} \setminus K, \\
0, & \text{otherwise}.
\end{cases}
\] (2.4)

For any integer \(k \geq d - 1\), there holds
\[
\sum_{i=0}^{k} b_i(\Lambda S^d) = \left[ \frac{k}{d-1} \right] + \left[ \frac{k}{2} \right] - \frac{d-1}{2}.
\] (2.5)

(ii) When \(d\) is even, let \(D = d(n+1) - 2\) and
\[
\Omega(d, n) = \{k \in 2\mathbb{N} - 1 \mid k_1 D \leq k - (d - 1) = k_1 D + k_2 d \leq k_1 D + (n - 1)d \}
\]
for some \(k_1 \in \mathbb{N}\) and \(k_2 \in [1, n - 1]\).

Then the Betti numbers of the free loop space of \(M\) are given by
\[
b_i(\Lambda M) = \text{rank} H_i(\Lambda M/S^1, \Lambda^0 M/S^1; \mathbb{Q}) = \begin{cases} 
0, & \text{if } i \text{ is even or } i \leq d - 2, \\
\left[ \frac{i-(d-1)}{d} \right] + 1, & \text{if } i \in 2\mathbb{N} - 1 \text{ and } d - 1 \leq i < d - 1 + (n - 1)d, \\
n + 1, & \text{if } i \in \Omega(d, n), \\
n, & \text{otherwise}.
\end{cases}
\] (2.6)

For any integer \(k \geq dn - 1\), we have
\[
\sum_{i=0}^{k} b_i(\Lambda M) = \frac{n(n+1)d}{2D}(k - (d - 1)) - \frac{n(n-1)d}{4} + 1 + \Theta_{d,n}(k).
\] (2.7)
where
\[
\Theta_{d,n}(k) = \left\{ \frac{D}{dn} \left\{ \frac{k - (d - 1)}{D} \right\} \right\} - \left( \frac{2}{d} + \frac{d - 2}{dn} \right) \left\{ \frac{k - (d - 1)}{D} \right\} - \left( \frac{D}{2} \left\{ \frac{k - (d - 1)}{D} \right\} \right\}.
\]  

(2.8)

2.2 Index iteration theory of closed geodesics

In [Lon99], Y. Long established the basic normal form decomposition of symplectic matrices. Based on this result the precise iteration formulae of indices of symplectic paths has been established in [Lon00]. Note that this index iteration formulae works for Morse indices of iterated closed geodesics (cf. [LL02], [Liu05] and Chapter 12 of [Lon02]). Since every closed geodesic on a compact simply-connected manifold \( M \) is orientable, then by Theorem 1.1 of [Liu05], the Morse index of a closed geodesic \( c \) on such manifold coincides with the index of a corresponding symplectic path.

As in [Lon02], denote by

\[
N_1(\lambda, a) = \begin{pmatrix} \lambda & a \\ 0 & \lambda \end{pmatrix}, \quad \text{with } \lambda = \pm 1, \ a \in \mathbb{R},
\]

(2.9)

\[
H(b) = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}, \quad \text{with } b \in \mathbb{R} \setminus \{0, \pm 1\},
\]

(2.10)

\[
R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \text{with } \theta \in (0, \pi) \cup (\pi, 2\pi),
\]

(2.11)

\[
N_2(e^{\theta \sqrt{-1}}, B) = \begin{pmatrix} R(\theta) & B \\ 0 & R(\theta) \end{pmatrix}, \quad \text{with } \theta \in (0, \pi) \cup (\pi, 2\pi) \text{ and } B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \text{ with } b_j \in \mathbb{R}, \ \text{and } b_2 \neq b_3.
\]

(2.12)

Here \( N_2(e^{\theta \sqrt{-1}}, B) \) is non-trivial if \((b_2 - b_3) \sin \theta < 0\), and trivial if \((b_2 - b_3) \sin \theta > 0\) as defined in [Lon02] and Definition 1.8.11 of [Lon02].

As in [Lon02], the \( \circ \)-sum of any two real matrices is defined by

\[
\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \circ \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.
\]

Let \( J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \) be the standard symplectic matrix, where \( I_n \) is the identity matrix. As usual, the symplectic group is defined by \( \text{Sp}(2n) = \{ M \in GL(2n, \mathbb{R}) \mid M^TJM = J \} \). For every
Let \( P \in \text{Sp}(2n) \), the homotopy set \( \Omega(P) \) of \( P \) in \( \text{Sp}(2n) \) is defined by

\[
\Omega(P) = \{ Q \in \text{Sp}(2n) \mid \sigma(Q) \cap U = \sigma(P) \cap U \equiv \Gamma \text{ and } \nuP(\omega) = \nuP(\omega) \ \forall \ \omega \in \Gamma \},
\]

where \( \sigma(\omega) \) denotes the spectrum of \( P \), \( \nuP(\omega) = \dim \ker C(P - \omega I) \) for \( \omega \in U \). The homotopy component \( \OmegaP(\omega) \) of \( P \) in \( \text{Sp}(2n) \) is defined by the path connected component of \( \Omega(P) \) containing \( P \). Then the following decomposition theorem is proved in [Lon99] and [Lon00].

**Theorem 2.5.** (cf. Theorem 1.8.10, Lemma 2.3.5 and Theorem 8.3.1 of [Lon02]) For every \( P \in \text{Sp}(2n) \), there exists a continuous path \( f : [0, 1] \to \OmegaP(\nu) \) such that \( f(0) = P \) and

\[
f(1) = N_1(1, 1)\sigmaP \circ I_{2p_0} \circ N_1(1, -1)\sigmaP \circ (-I_{2q_0}) \circ N_1(-1, -1)\sigmaP \\
\circ N_2(e^{q_\alpha \sqrt{-1}}, A_1) \circ \cdots \circ N_2(e^{q_\alpha \sqrt{-1}}, A_r) \circ N_2(e^{r_\beta \sqrt{-1}}, B_1) \circ \cdots \circ N_2(e^{r_\beta \sqrt{-1}}, B_r) \\
\circ \rho(\theta_0) \circ \cdots \circ \rho(\theta_r) \circ \rho(\theta_{r+1}) \circ \cdots \circ \rho(\theta_0) \circ H(b)^{\sigmaP},
\]

where \( \theta_j \in (0, 1) \setminus Q \) for \( 1 \leq j \leq r \) and \( \theta_j \in ((0, 1) \setminus \{1/2\}) \cap Q \) for \( r + 1 \leq j \leq \tilde{r} \); \( N_2(e^{q_\alpha \sqrt{-1}}, A_j) \)'s are nontrivial and \( N_2(e^{r_\beta \sqrt{-1}}, B_j) \)'s are trivial, and non-negative integers \( p_\pm, p_0, q_\pm, q_0, r, r_0, h \) satisfy

\[
p_\pm + p_0 + p_+ + q_\pm + q_0 + q_+ + \tilde{r} + 2r_0 + 2r_0 + h = n.
\]

Let \( \gamma \in \mathcal{P}_r(2n) = \{ \gamma \in C([0, \tau], \text{Sp}(2n)) \mid \gamma(0) = I \} \), we extend \( \gamma(t) \) to \( t \in [0, m\tau] \) for every \( m \in \mathbb{N} \) by

\[
\gamma^m(t) = \gamma(t - j\tau)\gamma(t)^j \quad \forall \ j\tau \leq t \leq (j + 1)\tau \text{ and } j = 0, 1, \ldots, m - 1.
\]

Denote the basic normal form decomposition of \( P = \gamma(\tau) \) by (2.13). Then we have

\[
i(\gamma^m) = m(i(\gamma) + p_+ + p_0 - \tilde{r}) + 2 \sum_{j=1}^r E \left( \frac{mj_0}{2\pi} \right) - \tilde{r} \\
-p_\pm - p_0 - 1 + (-1)^m \left( \frac{q_\pm}{2} + q_0 + q_+ \right) + 2 \sum_{j=1}^r \Phi \left( \frac{m\alpha_j}{2\pi} \right) - 2r_0.
\]

Let

\[
\mathcal{M} \equiv \{ N_1(1, 1); \ N_1(-1, a_2), a_2 = \pm 1; \ R(\theta), \ \theta \in [0, 2\pi); \ H(-2) \}.
\]

By Theorems 8.1.4-8.1.7 and 8.2.1-8.2.4 of [Lon02], we have

**Proposition 2.6.** Every path \( \gamma \in \mathcal{P}_r(2) \) with end matrix homotopic to some matrix in \( \mathcal{M} \) has odd index \( i(\gamma) \). Every path \( \xi \in \mathcal{P}_r(2) \) with end matrix homotopic to \( N_1(1, -1) \) or \( H(2) \), and every path \( \eta \in \mathcal{P}_r(4) \) with end matrix homotopic to \( N_2(\omega, B) \) have even indices \( i(\xi) \) and \( i(\eta) \).

The common index jump theorem (cf. Theorem 4.3 of [LZ02]) for symplectic paths has become one of the main tools in studying the multiplicity and stability of periodic orbits in Hamiltonian
and symplectic dynamics. Recently, the following enhanced common index jump theorem has been obtained by Duan, Long and Wang in ([DLW16a]).

**Theorem 2.7.** (cf. Theorem 3.5 of [DLW16a]) Let \( \gamma_k \in P_{\tau_k}(2n) \) for \( k = 1, \cdots, q \) be a finite collection of symplectic paths. Let \( M_k = \gamma_k(\tau_k) \). We extend \( \gamma_k \) to \([0, +\infty)\) by (2.15) inductively. Suppose

$$
\dot{i}(\gamma_k, 1) > 0, \quad \forall k = 1, \cdots, q.
$$

Then for any fixed integer \( \bar{m} \in \mathbb{N} \), there exist infinitely many \((q+1)\)-tuples \((N, m_1, \cdots, m_q) \in \mathbb{N}^{q+1}\) such that for all \( 1 \leq k \leq q \) and \( 1 \leq m \leq \bar{m} \), there holds

$$
\nu(\gamma_k, 2m_k - m) = \nu(\gamma_k, 2m_k + m) = \nu(\gamma_k, m), \quad (2.19)
$$

$$
i(\gamma_k, 2m_k + m) = 2N + i(\gamma_k, m), \quad (2.20)
$$

$$
i(\gamma_k, 2m_k - m) = 2N - i(\gamma_k, m) - 2(S_{M_k}^+(1) + Q_k(m)), \quad (2.21)
$$

$$
i(\gamma_k, 2m_k) = 2N - (S_{M_k}^+(1) + C(M_k) - 2\Delta_k), \quad (2.22)
$$

where \( S_{M_k}^\pm(\omega) \) is the splitting number of \( M_k \) at \( \omega \) (cf. Definition 9.1.4 of [Lon02]) and

$$
C(M_k) = \sum_{0<\theta<2\pi} S_{M_k}^-(e^{\sqrt{-1}\theta}), \quad \Delta_k = \sum_{0\{m_k\theta/\pi\} < \delta} S_{M_k}^-(e^{\sqrt{-1}\theta}), \quad Q_k(m) = \sum_{\sigma(\frac{m_k\theta}{\pi}) = \sigma(\frac{m_0\theta}{\pi}) = 0} S_{M_k}^-(e^{\sqrt{-1}\theta}). \quad (2.23)
$$

More precisely, by (4.10), (4.40) and (4.41) in [LZ02], we have

$$
m_k = \left[ \frac{N}{Mi(\gamma_k, 1)} \right] + \chi_k \bar{M}, \quad 1 \leq k \leq q, \quad (2.24)
$$

where \( \chi_k = 0 \) or \( 1 \) for \( 1 \leq k \leq q \) and \( \frac{M_0}{\pi} \in \mathbb{Z} \) whenever \( e^{\sqrt{-1}\theta} \in \sigma(M_k) \) and \( \frac{\theta}{\pi} \in \mathbb{Q} \) for some \( 1 \leq k \leq q \). Furthermore, for any fixed \( M_0 \in \mathbb{N} \), we may further require \( M_0 | N \), and for any \( \epsilon > 0 \), we can choose \( N \) and \( \{\chi_k\}_{1 \leq k \leq q} \) such that

$$
\left| \left\{ \frac{N}{Mi(\gamma_k, 1)} \right\} - \chi_k \right| < \epsilon, \quad 1 \leq k \leq q. \quad (2.25)
$$

### 3 Proof of Theorem 1.1

Let \((M, F)\) be a compact simply-connected manifold with a bumpy, irreversible Finsler metric \( F \) and satisfy \( H^*(M; \mathbb{Q}) \cong T_{d,n+1}(x) \) of \( \dim M = dn \), where integers \( d \geq 2 \) and \( n \geq 1 \). In order to prove Theorem 1.1, We make the following assumption

**ECG** Suppose that all prime closed geodesics on \((M, F)\) are elliptic, and the total number of distinct closed geodesics is finite, denoted by \( \{c_k\}_{k=1}^q \).
When the Finsler metric $F$ is bumpy, there holds $p_- + p_0 + p_+ + q_- + q_0 + q_+ + \bar{r} - r = 0$ and there does not exist $\alpha_j$ or $\beta_j$ which is the rational multiple of $\pi$ in (2.13). Furthermore, if the closed geodesic is elliptic, then there also holds $h = 0$. So by Theorem 2.5, under the assumption (ECG), for every prime closed geodesic $c_k$, $\forall 1 \leq k \leq q$, the basic normal form decomposition of the linearized Poincaré map $P_{c_k}$ possesses the following form

$$f_{c_k}(1) = R(\theta_1^k) \circ \cdots \circ R(\theta_{r_k}^k) \circ N_2(e^{\alpha_k^k \sqrt{-1}}, A_{r_k}^k) \circ \cdots \circ N_2(e^{\alpha_{r_k+1}^k \sqrt{-1}}, A_{r_{r_{k+1}}}^k) \circ N_2(e^{\beta_1^k \sqrt{-1}}, B_1^k) \circ \cdots \circ N_2(e^{\beta_{r_{k_0}}^k \sqrt{-1}}, B_{r_{k_0}}^k),$$

where $\frac{\theta_j^k}{2\pi} \in (0, 1) \setminus \mathbb{Q}$ for $1 \leq j \leq r_k$, $\frac{\alpha_j^k}{2\pi} \in (0, 1) \setminus \mathbb{Q}$ for $1 \leq j \leq r_k^*$, $\frac{\beta_j^k}{2\pi} \in (0, 1) \setminus \mathbb{Q}$ for $1 \leq j \leq r_{k_0}$, and

$$r_k + 2r_{k^*} + 2r_{k_0} = dn - 1. \quad (3.1)$$

Therefore, by (2.16) we obtain the index iteration formula of $c_k^m$ for $1 \leq k \leq q$

$$i(c_k^m) = m(i(c_k) - r_k) + 2 \sum_{j=1}^{r_k} \left[ \frac{m\theta_j^k}{2\pi} \right] + r_k, \quad \nu(c_k^m) = 0, \quad \forall m \geq 1. \quad (3.2)$$

**Claim 1.** Under (ECG), for every $1 \leq k \leq q$, there holds $i(c_k^m) = dn - 1 \pmod{2}$, $\forall m \geq 1$.

In fact, by Proposition 2.6, (3.1) and the homotopy invariant and the symplectic additivity of indices (cf. Theorem 6.2.7 of [Lon02]), we obtain

$$i(c_k) = dn - 1 \pmod{2}, \quad 1 \leq k \leq q. \quad (3.3)$$

By (3.1), (3.2) and (3.3), it yields

$$i(c_k^{m+1}) - i(c_k^m) = (i(c_k) - r_k) + 2 \sum_{j=1}^{r_k} \left[ \frac{(m+1)\theta_j^k}{2\pi} \right] - 2 \sum_{j=1}^{r_k} \left[ \frac{m\theta_j^k}{2\pi} \right]$$

$$= i(c_k) + 2r_{k^*} + 2r_{k_0} - (dn - 1) \pmod{2}$$

$$= i(c_k) - (dn - 1) \pmod{2}$$

$$= 0 \pmod{2}, \quad \forall 1 \leq k \leq q, \quad m \geq 1. \quad (3.4)$$

Now (3.3) and (3.4) finished the proof of Claim 1.

Define the Morse-type numbers

$$M_p = \sum_{k=1}^{q} M_p(k) \equiv \sum_{k=1}^{q} \# \{ m \geq 1 \mid i(c_k^m) = p, \overline{C}_p(E, c_k^m) \neq 0 \}, \quad p \in \mathbb{Z}, \quad (3.5)$$

which, through the following Morse inequalities (cf. Theorem 1.4.3 of [Cha93]), relates the Betti numbers $b_p \equiv b_p(\Lambda M)$ defined in Lemma 2.4

$$M_p \geq b_p, \quad (3.6)$$

$$M_p - M_{p-1} + \cdots + (-1)^p M_0 \geq b_p - b_{p-1} + \cdots + (-1)^p b_0, \quad \forall p \in \mathbb{N}_0. \quad (3.7)$$
Since Claim 1 implies $i(c^m_k) - i(c_k) \in 2\mathbb{Z}$, $\forall m \geq 1$ and $1 \leq k \leq q$. By Proposition 2.1 we obtain

$$M_p(k) = \#\{m \geq 1 \mid i(c^m_k) = p\}, \quad \forall p \in \mathbb{Z}, \quad \forall 1 \leq k \leq q,$$

(3.8)

which, together with the fact $i(c^m_k) = dn - 1 \pmod{2}$, $\forall m \geq 1$ from Claim 1, implies

$$M_p = 0, \quad \forall p = dn \pmod{2}, \quad p \in \mathbb{N}_0.$$

(3.9)

On the other hand, it follows from Lemma 2.4 that

$$b_p \equiv b_p(\Lambda M) = 0, \quad \forall p = dn \pmod{2}, \quad p \in \mathbb{N}_0.$$

(3.10)

Then by the Morse inequalities (3.6) and (3.7), we obtain

$$M_p = b_p, \quad \forall p = dn - 1 \pmod{2}, \quad p \in \mathbb{N}_0.$$

(3.11)

In summary, under the assumption (ECG), the Morse inequalities become the identities

$$M_p = b_p, \quad \sum_{i=0}^{p} (-1)^i M_i = \sum_{i=0}^{p} (-1)^i b_i, \quad \forall p \in \mathbb{N}_0.$$

(3.12)

**Claim 2.** Under (ECG), there holds $i(c_k) \geq d - 1$ and the mean index $\hat{i}(c_k) > 0$, $\forall 1 \leq k \leq q$. Moreover, $\#\{1 \leq k \leq q \mid i(c_k) = d - 1\} = 1$.

In fact, if there exists at least one closed geodesic $c_{k_0}$ such that $0 \leq i(c_{k_0}) < d - 1$, then it follows from (3.5) and (3.8) that $M_{i(c_{k_0})} = \sum_{k=1}^{q} M_{i(c_{k_0})}(k) \geq 1$. Note that $b_i = 0$, $\forall 0 \leq i < d - 1$ by Lemma 2.4. So by (3.12) we get the following contradiction

$$1 \leq M_{i(c_{k_0})} = b_{i(c_{k_0})} = 0.$$

(3.13)

So there holds $i(c_k) \geq d - 1$, $\forall 1 \leq k \leq q$.

Assume $\hat{i}(c_k) = 0$ holds for some $1 \leq k \leq q$, then it yields $i(c^m_k) = 0$ for any $m \geq 1$ by Corollary 4.2 of [LL02]. This contradicts the fact $i(c_k) \geq d - 1 \geq 1$. Thus the mean index $\hat{i}(c_k)$ of every closed geodesic $c_k$ must be positive.

In addition, by Lemma 2.4 and (3.12), it yields $M_{d-1} = b_{d-1} = 1$. So by (3.8), we have

$$M_{d-1} = \sum_{k=1}^{q} M_{d-1}(k) = \sum_{k=1}^{q} \#\{m \geq 1 \mid i(c^m_k) = d - 1\} = 1.$$

(3.14)

So there exist only one pair $(k_0, m_0)$ with some $1 \leq k_0 \leq q$ and $m_0 \in \mathbb{N}$ satisfying $i(c^{m_0}_{k_0}) = d - 1$. Note that there always holds $i(c^m_k) \geq i(c_k), \forall m \geq 1$ by the Bott-type formulae (cf. Theorem A of [Bot56] and Theorem 9.2.1 of [Lon02]). Since $i(c_{k_0}) \geq d - 1$, so the unique possibility is $m_0 = 1$, i.e., $\#\{1 \leq k \leq q \mid i(c_k) = d - 1\} = 1$.

This completes the proof of Claim 2.
By (3.2) and Claim 2, it yields
\[
\hat{i}(c_k) = i(c_k) - r_k + \sum_{j=1}^{r_k} \frac{\theta_j^k}{\pi} > 0, \tag{3.15}
\]
which, together with (3.2), implies that for any \(m, l \in \mathbb{N}\), there holds
\[
i(c_k^{m+l}) - i(c_k^l) = m(i(c_k) - r_k) + 2 \sum_{j=1}^{r_k} \left[ \frac{(m + l)\theta_j^k}{2\pi} \right] - 2 \sum_{j=1}^{r_k} \left[ \frac{\theta_j^k}{2\pi} \right] \\
\geq m(i(c_k) - r_k) + \sum_{j=1}^{r_k} \frac{m\theta_j^k}{\pi} - 2r_k \\
= m\hat{i}(c_k) - 2r_k, \quad \forall \, 1 \leq k \leq q. \tag{3.16}
\]

It follows from (3.15) and (3.16) that there exists sufficiently large \(m \in \mathbb{N}\) such that \(i(c_k^{m+l}) - i(c_k^l) \geq 0\), \(\forall \, l \geq 1\). So the positive integer \(\bar{m}\) defined by
\[
\bar{m} = \max_{1 \leq k \leq q} \left\{ \min \{\hat{m} \in \mathbb{N} \mid i(c_k^{m+l}) \geq i(c_k^l), \, \forall \, l \geq 1, \, m \geq \hat{m} \} \right\} \tag{3.17}
\]
is well-defined and finite.

For the integer \(\bar{m}\) defined in (3.17), it follows from Theorem 2.7 that there exist infinitely many \((q + 1)\)-tuples \((N, m_1, \ldots, m_q) \in \mathbb{N}^{q+1}\) such that for any \(1 \leq k \leq q\), there holds
\[
\bar{m} + 2 \leq \min_{1 \leq k \leq q} \{2m_k\}, \tag{3.18}
\]
\[
i(c_k^{2m_k-m}) = 2N - i(c_k^m), \quad 1 \leq m \leq \bar{m}, \tag{3.19}
\]
\[
i(c_k^{2m_k}) = 2N - C(M_k) + 2\Delta_k, \tag{3.20}
\]
\[
i(c_k^{2m_k+m}) = 2N + i(c_k^m), \quad 1 \leq m \leq \bar{m}. \tag{3.21}
\]
where \(M_k = P_{c_k} \in \text{Sp}(2(2n - 1))\) is the linearized Poincaré map of \(c_k\). Note that in the bumpy case, \(S_{\hat{M}_k}(1) = 0\) and \(Q_k(m) = 0\) holds for all \(m \in \mathbb{N}\).

Next we continue the proof of Theorem 1.1 in two cases according to the parity of \(d\).

**Case 1.** \(H^*(M; \mathbb{Q}) \cong T_{d,n+1}(x)\) with even \(d \geq 2\).

We use three steps to carry out the proof of Theorem 1.1 in this case.

**Step 1.** *The existence of \(\frac{dn(n+1)}{4}\) prime closed geodesics.*

On one hand, since \(i(c_k^m) \geq i(c_k)\) for any \(m \geq 1\), so by (3.19), (3.21) and the fact \(i(c_k) \geq d-1 \geq 1\) for \(1 \leq k \leq q\) from Claim 2, it yields
\[
i(c_k^{2m_k-m}) = 2N - i(c_k^m) \leq 2N - i(c_k) \leq 2N - 1, \quad 1 \leq m \leq \bar{m}, \tag{3.22}
\]
\[
i(c_k^{2m_k+m}) = 2N + i(c_k^m) \geq 2N + i(c_k) \geq 2N + 1, \quad 1 \leq m \leq \bar{m}. \tag{3.23}
\]
On the other hand, for any $\bar{m} + 1 \leq m < 2m_k$, by the definition (3.17) of $\bar{m}$ (note that $m - 1 \geq \bar{m}$), (3.19), (3.21) and $i(c_k) \geq 1$, we have

\begin{align*}
i(c_k^{2m_k - m}) &\leq i(c_k^{2m_k - m + (m - 1)}) = i(c_k^{2m_k - 1}) = 2N - i(c_k) \leq 2N - 1, \quad (3.24) \\
i(c_k^{2m_k + m}) &\geq i(c_k^{2m_k + 1}) + (m - 1) \geq i(c_k^{2m_k + 1}) = 2N + i(c_k) \geq 2N + 1, \quad (3.25)
\end{align*}

where notice that the second inequality actually holds for any $m \geq \bar{m} + 1$.

Therefore, for every $1 \leq k \leq q$, by (3.22)-(3.25), we obtained the following estimates

\begin{align*}
i(c_k^{2m_k - m}) &\leq 2N - i(c_k) \leq 2N - 1, \quad \forall 1 \leq m < 2m_k, \quad (3.26) \\
i(c_k^{2m_k}) &= 2N - C(M_k) + 2\Delta_k, \quad (3.27) \\
i(c_k^{2m_k + m}) &\geq 2N + i(c_k) \geq 2N + 1, \quad \forall m \geq 1. \quad (3.28)
\end{align*}

**Claim 3.** For $N \in \mathbb{N}$ in Theorem 2.7 satisfying (3.26)-(3.28) and $2NB(d, n) \in 2\mathbb{N}$, we have

$$\sum_{1 \leq k \leq q} 2m_k \gamma_{c_k} = 2NB(d, n). \quad (3.29)$$

In fact, choose $\epsilon < \frac{1}{1 + 2M} \frac{1}{\sum_{1 \leq k \leq q} |\gamma_{c_k}|}$, by Theorem 2.3, (2.24) and (2.25) of Theorem 2.7, it yields

$$\left| 2NB(d, n) - \sum_{k=1}^{q} 2m_k \gamma_{c_k} \right| = \left| \sum_{k=1}^{q} \frac{2N \gamma_{c_k}}{i(c_k)} - \sum_{k=1}^{q} 2\gamma_{c_k} \left( \left\lfloor \frac{N}{M i(c_k)} \right\rfloor + \chi_k \right) \bar{M} \right|$$

$$\leq 2\bar{M} \sum_{k=1}^{q} |\gamma_{c_k}| \left\lfloor \frac{N}{M i(c_k)} \right\rfloor - \chi_k$$

$$< 2\bar{M} \epsilon \sum_{k=1}^{q} |\gamma_{c_k}| < 1. \quad (3.30)$$

Since every $2m_k \gamma_{c_k}$ is an integer, Claim 3 is proved.

Now by Proposition 2.1, Definition 2.2 and Claim 1, it yields

\begin{align*}
\sum_{m=1}^{2m_k} (-1)^{i(c_k^m)} \dim \overline{C}_{i(c_k^m)}(E, c_k^m) &= \sum_{i=0}^{m_k-1} \sum_{m=2i+1}^{2i+2} (-1)^{i(c_k^m)} \dim \overline{C}_{i(c_k^m)}(E, c_k^m) \\
&= \sum_{i=0}^{m_k-1} \sum_{m=1}^{2} (-1)^{i(c_k^m)} \dim \overline{C}_{i(c_k^m)}(E, c_k^m) \\
&= m_k \sum_{m=1}^{2} (-1)^{i(c_k^m)} \dim \overline{C}_{i(c_k^m)}(E, c_k^m) \\
&= 2m_k \gamma_{c_k}, \quad \forall 1 \leq k \leq q, \quad (3.31)
\end{align*}

where the second equality follows from Proposition 2.1 and the fact $i(c_k^{m+2}) - i(c_k^m) \in 2\mathbb{Z}$ for all $m \geq 1$ from Claim 1, and the last equality follows from Proposition 2.1 and Definition 2.2.
By (3.28) and Proposition 2.1, it follows that all $c_k^{2m_k + m}$'s with $m \geq 1$ and $1 \leq k \leq q$ have no contribution to the alternating sum $\sum_{p=0}^{2N}(-1)^p M_p$. Similarly again by Proposition 2.1 and (3.26), all $c_k^{2m_k - m}$'s with $1 \leq m < 2m_k$ and $1 \leq k \leq q$ only have contribution to $\sum_{p=0}^{2N}(-1)^p M_p$.

Thus for the Morse-type numbers $M_p$'s defined by (3.5), by (3.31) we have

$$
\sum_{p=0}^{2N}(-1)^p M_p = \sum_{k=1}^{q} \sum_{1 \leq m \leq 2m_k \atop i(c_k^m) \leq 2N} (-1)^i(c_k^m) \dim C_{i(c_k^m)}(E, c_k^m)
$$

$$
= \sum_{k=1}^{q} \sum_{m=1}^{2m_k} (-1)^i(c_k^m) \dim C_{i(c_k^m)}(E, c_k^m)

- \sum_{1 \leq k \leq q \atop i(c_k^{2m_k}) \geq 2N+1} (-1)^i(c_k^{2m_k}) \dim C_{i(c_k^{2m_k})}(E, c_k^{2m_k})
$$

$$
= \sum_{k=1}^{q} 2m_k \gamma c_k - \sum_{1 \leq k \leq q \atop i(c_k^{2m_k}) \geq 2N+1} (-1)^i(c_k^{2m_k}) \dim C_{i(c_k^{2m_k})}(E, c_k^{2m_k}). \quad (3.32)
$$

In order to precisely know the contribution of the iterate $c_k^{2m_k}$ of $c_k$ to the alternating sum $\sum_{p=0}^{2N}(-1)^p M_p(k)$ for $1 \leq k \leq q$, we define

$$
\mathfrak{N}_+ = \#\{1 \leq k \leq q \mid i(c_k^{2m_k}) \geq 2N + 1\}, \quad (3.33)
$$

$$
\mathfrak{N}_- = \#\{1 \leq k \leq q \mid i(c_k^{2m_k}) \leq 2N - 1\}. \quad (3.34)
$$

Note that by Claim 1, there holds $i(c_k^{2m_k}) - i(c_k) \in 2\mathbb{N}_0$, $i(c_k) \in 2\mathbb{N} - 1$, $\forall 1 \leq k \leq q$. Thus by Theorem 2.3, Claim 3, (3.12), (3.32), the definition of $\mathfrak{N}_+$ and Lemma 2.4, we have

$$
-\frac{Ndn(n+1)}{D} + \mathfrak{N}_+ = 2NB(d, n) + \mathfrak{N}_+
$$

$$
= \sum_{k=1}^{q} 2m_k \gamma c_k + \mathfrak{N}_+
$$

$$
= \sum_{p=0}^{2N} (-1)^p M_p = \sum_{p=0}^{2N} (-1)^p b_p
$$

$$
= - \sum_{2k-1=1}^{2N-1} b_{2k-1}(\Lambda M)
$$

$$
= - \frac{dn(n+1)}{2D} (2N - d) + \frac{dn(n-1)}{4} - 1 - \Theta_{d,n}(2N - 1), \quad (3.35)
$$

which implies

$$
\mathfrak{N}_+ = \frac{d^2n(n+1)}{2D} + \frac{dn(n-1)}{4} - 1 - \Theta_{d,n}(2N - 1). \quad (3.36)
$$
Note that we can assume that \( N \) is a multiple of \( D = d(n + 1) - 2 \) by Theorem 2.7. So there holds \( \{\frac{2N-1-(d-1)}{D}\} = 1 - \frac{d}{D} = \frac{dn-2}{D} \). Then by (2.8) we have

\[
\Theta_{d,n}(2N-1) = \frac{dn-2}{dn} - \frac{2n + d - 2}{dn} \left( 1 - \frac{d}{D} \right) - n \left\{ \frac{dn-2}{2} \right\} - \left\{ \frac{dn-2}{d} \right\} = \frac{(d-2)D + 2d - d^2}{Dd} - \left\{ \frac{2}{d} \right\} = 1 - \frac{2}{d} - \left\{ \frac{2}{d} \right\} - \frac{d-2}{D}.
\]

which, together with (3.36), yields

\[
\mathcal{R}_+ = \frac{d^2(n+1)}{2D} + \frac{dn(n-1)}{4} - 1 + \frac{d-2}{D} = \frac{dn(n+1)}{4}.
\]  

(3.37)

**Step 2.** The existence of other \( \frac{dn(n+1)}{4} \) prime closed geodesics.

Now we can apply Theorem 2.7 again and find another \((q+1)\)-tuple \((N', m'_1, \ldots, m'_q) \in \mathbb{N}^{q+1}\) such that similarly to (3.26)-(3.28) for every \(1 \leq k \leq q\), there holds

\[
i(c_k^{2m'_k}) \leq 2N' - i(c_k) \leq 2N' - 1, \quad \forall 1 \leq m < 2m_k, \tag{3.38}
\]

\[
i(c_k^{2m'_k}) = 2N' - C(M_k) + 2\Delta'_k, \tag{3.39}
\]

\[
i(c_k^{2m'_k}) \geq 2N' + i(c_k) \geq 2N' + 1, \quad \forall m \geq 1, \tag{3.40}
\]

where

\[
\Delta'_k = \sum_{0 < \phi < \pi, |m| < \phi K} S_{M_k}(e^{\sqrt{2}q}),
\]

and \( \Delta_k \) and \( \Delta'_k \) satisfy the following relationship (also cf. (42) in Theorem 2.8 of [HW22])

\[
\Delta_k + \Delta'_k = C(M_k), \quad 1 \leq k \leq q. \tag{3.41}
\]

Similarly, note that \( i(c_k^{2m'_k}) - i(c_k) \in 2\mathbb{N}_0, i(c_k) \in 2\mathbb{N} - 1, \quad \forall 1 \leq k \leq q, \) we define

\[
\mathcal{R}'_+ = \#\{1 \leq k \leq q \mid i(c_k^{2m'_k}) \geq 2N' + 1\}, \tag{3.42}
\]

\[
\mathcal{R}'_- = \#\{1 \leq k \leq q \mid i(c_k^{2m'_k}) \leq 2N' - 1\}. \tag{3.43}
\]

So by (3.39) and (3.41) it yields

\[
i(c_k^{2m'_k}) = 2N' - C(M_k) + 2(C(M_k) - \Delta_k) = 2N' + C(M_k) - 2\Delta_k. \tag{3.44}
\]

Note that when \( i(c_k^{2m'_k}) \geq 2N' + 1 \), it yields \( C(M_k) - 2\Delta_k \geq 1 \) by (3.44), then it follows from (3.27) that \( i(c_k^{2m_k}) = 2N - C(M_k) + 2\Delta_k \leq 2N - 1 \), and vice versa. So, by (3.27), (3.39), (3.44) and the definitions of \( \mathcal{R}_\pm \) and \( \mathcal{R}'_\pm \), it yields

\[
\mathcal{R}_\pm = \mathcal{R}'_\mp. \tag{3.45}
\]
By Proposition 2.1 and (3.40), it follows that all \(c^2m_k+m\)'s with \(m \geq 1\) and \(1 \leq k \leq q\) have no contribution to the alternating sum \(\sum_{p=0}^{2N'}(-1)^p M_p\). Similarly also by Proposition 2.1 and (3.38), all \(c^2m_k-m\)'s with \(m \geq 1\) and \(1 \leq k \leq q\) only have contribution to \(\sum_{p=0}^{2N'}(-1)^p M_p\).

Thus, through carrying out arguments similar to (3.35)-(3.37), by Claim 3, the definition of \(\mathfrak{N}'\) and Lemma 2.4, together with (3.45), we obtain

\[
\mathfrak{N}_+ = \mathfrak{N}'_+ = \frac{dn(n+1)}{4}. \tag{3.46}
\]

Since \(i(c^2m_k) \neq 2N\), \forall \, 1 \leq k \leq q\) by Claim 1, it yields \(q = \mathfrak{N}_+ + \mathfrak{N}_-\) by the definitions (3.33) and (3.34) of \(\mathfrak{N}_\pm\). Then by (3.37) and (3.46), we get

\[
q = \mathfrak{N}_+ + \mathfrak{N}_- = \frac{dn(n+1)}{4} + \frac{dn(n+1)}{4} = \frac{dn(n+1)}{2}. \tag{3.47}
\]

This completes the proof of Theorem 1.1 in this case.

**Case 2.** \(H^*(M; \mathbb{Q}) \cong T_{d,n+1}(x)\) with odd \(d \geq 3\).

Note that in this case there holds \(n = 1\) and \(d \geq 3\), which implies that \(M\) is rationally homotopic to the sphere \(S^d\).

Now we use two steps to carry out the proof of Theorem 1.1 in this case.

**Step 3.** The existence of \((d-1)\) prime closed geodesics.

Firstly, note that there holds \(i(c_k) \geq d-1 \geq 2\), \forall \, 1 \leq k \leq q\) by Claim 2. Therefore, by Theorem 2.7 and some similar arguments as those above (3.26)-(3.28), for \(1 \leq k \leq q\), we obtain

\[
i(c_k^{2m-k-m}) \leq 2N - i(c_k) \leq 2N - 2, \quad \forall \, 1 \leq m < 2m_k, \tag{3.48}
\]

\[
i(c_k^{2m_k}) = 2N - C(M_k) + 2\Delta_k, \tag{3.49}
\]

\[
i(c_k^{2m_k+m}) \geq 2N + i(c_k) \geq 2N + 2, \quad \forall \, m \geq 1. \tag{3.50}
\]

Therefore, similarly to the equation (3.32), we have

\[
\sum_{p=0}^{2N+1}(-1)^p M_p = \sum_{k=1}^{q} 2m_k \gamma_{c_k} - \sum_{1 \leq k \leq q} \sum_{i(c_k^{2m_k}) \geq 2N+2} (-1)^{i(c_k^{2m_k})} \dim \overline{C}_{i(c_k^{2m_k})}(E, c_k^{2m_k}). \tag{3.51}
\]

Define

\[
\hat{\mathfrak{N}}_+ = \#\{1 \leq k \leq q \mid i(c_k^{2m_k}) \geq 2N + 2\}, \tag{3.52}
\]

\[
\hat{\mathfrak{N}}_- = \#\{1 \leq k \leq q \mid i(c_k^{2m_k}) \leq 2N - 2\}. \tag{3.53}
\]
Note that by Claim 1, there holds $i(c_k^{2m_k}) - i(c_k) \in 2\mathbb{N}_0$, $i(c_k) \in 2\mathbb{N}$, $\forall 1 \leq k \leq q$. Then, by Theorem 2.3, (3.12), Claim 3, (3.51), the above definition of $\hat{N}_+$, and Lemma 2.4, we have

$$\frac{N(d + 1)}{d - 1} - \hat{N}_+ = 2NB(d, 1) - \hat{N}_+ = \sum_{k=1}^{q} 2m_k\gamma c_k - \hat{N}_+$$

$$= \sum_{p=0}^{2N+1} (-1)^p M_p = \sum_{p=0}^{2N+1} (-1)^p b_p$$

$$= \sum_{2k=0}^{2N} b_{2k}(\Lambda M)$$

$$= \frac{N(d + 1)}{d - 1} - \frac{d - 1}{2},$$

which yields

$$\hat{N}_+ = \frac{d - 1}{2}. \quad (3.55)$$

By Theorem 2.7 again, there exists another $(q + 1)$-tuple $(N', m'_1, \ldots, m'_q) \in \mathbb{N}^{q+1}$ such that (3.48)-(3.50) hold with $N$ and $m_k$s being replaced by $N'$ and $m'_k$s respectively. Then, denote by $\hat{N}'_+$ and $\hat{N}'_-$ the numbers similarly defined by (3.52)-(3.53) instead of using $N'$ and $m'_k$s. According to their definitions, obviously these numbers satisfy the relationship

$$\hat{N}_\pm = \hat{N}'_\pm. \quad (3.56)$$

Similarly to the same arguments from (3.46) and using the relationship (3.56), we obtain

$$\hat{N}_- = \hat{N}'_+ = \frac{d - 1}{2}. \quad (3.57)$$

Therefore from (3.55) and (3.57) it follows

$$\hat{N}_+ + \hat{N}_- = \frac{d - 1}{2} + \frac{d - 1}{2} = d - 1. \quad (3.58)$$

**Step 4. The existence of other two prime closed geodesics.**

Denote by $\{c_k\}_{k=1}^{d-1}$ the $(d - 1)$ prime closed geodesics found in Step 3. For these closed geodesics, there holds $i(c_k^{2m_k}) \neq 2N$ by the definitions of $\hat{N}_+$ and $\hat{N}_-$, which, together with (3.48) and (3.50), yields

$$i(c_k^{m}) \neq 2N, \quad \forall m \geq 1, 1 \leq k \leq d - 1. \quad (3.59)$$

Then by Proposition 2.1 it yields

$$M_{2N}(k) = \# \{m \geq 1 \mid i(c_k^{m}) = 2N, \overline{C}_{2N}(E, c_k^{m}) \neq 0\} = 0, \quad \forall 1 \leq k \leq d - 1. \quad (3.60)$$
Since $N$ can be chosen to be the multiple of $d - 1$ by Theorem 2.7, by Lemma 2.4 and (3.12), it yields $M_{2N} = b_{2N}(\Lambda M) = 2$. Therefore, by Proposition 2.1, (3.48), (3.50) and (3.60) it yields

$$2 = M_{2N} = \sum_{k=1}^{q} M_{2N}(k)$$

$$= \sum_{k=d}^{q} \#\{m \geq 1 \mid i(c_k^m) = 2N, \overline{C}_{2N}(E, c_k^m) \neq 0\}$$

$$= \sum_{k=d}^{q} \#\{m = 2m_k \mid i(c_k^m) = 2N, \overline{C}_{2N}(E, c_k^m) \neq 0\},$$

(3.61)

where the final equality follows from (3.48) and (3.50).

Now it follows from (3.61) and Proposition 2.1 that there exist exactly two prime closed geodesic $c_d$ and $c_{d+1}$ such that

$$i(c_k^{2m_k}) = 2N, \quad k = d, d + 1,$$

(3.62)

which implies that $c_d$ and $c_{d+1}$ are distinct from $\{c_k\}_{k=1}^{d-1}$ by (3.59). This completes the proof of Step 4.

Now notice that $dn - 1 = d - 1$ is even in this case. By Claim 1, it yields $i(c_k^{2m_k}) \notin \{2N - 1, 2N + 1\}, \forall 1 \leq k \leq q$. Therefore it yields $q = \hat{N}_+ + \hat{N}_- + 2$ by (3.62), the definitions (3.52) and (3.53) of $\hat{N}_\pm$. Then by (3.58) and (3.62), we get

$$q = \hat{N}_+ + \hat{N}_- + 2 = (d - 1) + 2 = d + 1.$$ 

(3.63)

This completes the proof of Theorem 1.1 in this case.

**Acknowledgement.** The authors sincerely thank the referee for her/his careful reading and valuable comments.

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