A Study on the Curves of Scaling Behavior of Fractal Cities

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Abstract: The curves of scaling behavior is a significant concept in fractal dimension analysis of complex systems. However, the underlying rationale of this kind of curves for fractal cities is not yet clear. This paper is devoted to researching a set of basic problems of the scaling behavior curves in urban studies by using mathematical reasoning and empirical analysis. The main findings are as follows. First, the formula of scaling behavior curves is derived from a fractal model based on hierarchical structure of urban systems. Second, the relationships between the formula of scaling behavior curves and similarity dimension and scale-dependent fractal dimension are revealed. Third, according to the fractal dimension measurement methods, the scaling behavior curves are divided into two different types. Fourth, empirically, 1-dimensional spatial autocorrelation function of scaling behavior curves can be employed to reveal the basic property of scaling behavior curves. In comparison, scaling behavior curves are more sensitive to power law relations than the fractal distribution functions. The scaling curves can be utilized to evaluate fractal development extent of urban systems and identify scaling ranges of fractal cities. In positive studies, the curves can be used to help distinguish the boundaries between urban areas and rural areas and self-affine fractal structure behind the dynamical process of spatial correlation.

Key words: Box dimension; Cascade structure of cities; Curve of scaling behavior; Fractal cities; Radial dimension; Traffic network of cities

1. Introduction

One of basic properties of fractals is invariance understand scaling transformation. Fractals and scaling represents two different sides of the same coin. In a sense, the inherent character of fractal
systems is scaling symmetry (Mandelbrot, 1989). Therefore, where this are fractal phenomena, there is scaling behavior. Fractal studies are usually associated with scaling law (Mandelbrot, 1982; Manrubia et al, 1999; Meakin, 1998; Thomas et al, 2010). If we measure fractal dimension of cities, we always find the shadow of fractal scaling behavior curves. A curve of fractal scaling behavior is a track that reflects the change of fractal dimension with scale. The scaling behavior curves can be employed to identify fractal development in a city (Batty and Longley, 1994; Frankhauser, 1998; Frankhauser, 2008), identify urban boundaries (Tannier et al, 2011), and determine scaling ranges of fractal dimension estimation. However, a series of basic problems about fractal scaling behavior curves are still not clear for fractal and urban researchers.

This paper is devoted to exploring the mathematical essence, main types, and methods of analysis of scaling behavior curves of fractals. Using logistic reasoning, mathematical derivations, numerical experiments, and positive studies, I will try to answer the following questions. First, what is the relationships and distinctions between fractal dimension and fractal scaling behavior curves? Second, how many types can fractal scaling behavior curves be divided into? Third, how to make spatial analysis of cities and regional systems by means of fractal scaling behavior curves? The rest parts will be organized as follows. In Section 2, several basic formulae will be derived from different fractal models. The mathematical essence of the scaling behavior curves will be revealed. One dimensional spatial autocorrelation function will be employed to identify the basic properties of a scaling behavior curve. In Section 3, numerical and empirical analyses based on urban traffic networks will be made by means of theoretical models and observational data. In Sections 4 and 5, a number of related questions will be discussed, and finally, the discussion will be concluded by summarizing the main points of this study.

2. Models

2.1 Fractal dimension based on hierarchical structure

The formulae of scaling behavior curves can be derived from fractal models by means of hierarchical structure of fractal systems. Based on box-counting method, the fractal model can be expressed as an inverse power function, that is

\[ N(r) = N_0 r^{-D} \rightarrow r^{-D}, \] (1)
where \( r \) denotes linear size of fractal copies, and \( N(r) \) is the corresponding number of fractal copies, \( D \) refers to fractal dimension, and \( N_1 \) to a proportionality constant. In theory, \( N_1 = 1 \) (Liu and Liu, 1993). Thus the fractal dimension can be expressed as

\[
D = - \frac{\log N(r)}{\log r}.
\]  

(2)

By differentiation of \( \log N(r) \) and \( \log r \) with respect to \( r \), we have (Takayasu, 1990)

\[
D = - \frac{d \log N(r)}{d \log r} = - \frac{dN(r) / N(r)}{dr / r}.
\]

(3)

Equation (3) reflects a continuous fractal distribution curves. Discretizing equation (3) yields

\[
D = - \frac{\Delta \log N(r)}{\Delta \log r} = - \frac{\log N(r) - \log N(r_{i-1})}{\log r_i - \log r_{i-1}} = \frac{\Delta N(r) / N(r)}{\Delta r / r},
\]

(4)

where \( i = 1, 2, 3, \ldots, n \), is a rank variable, representing steps, levels of a self-similar hierarchy, and so on, and \( n \) denotes the number of spatial measurement for fractal dimension.

A fractal is a hierarchy with cascade structure, which can be described with a power function or two exponential functions. This suggests that a power law can be decomposed into a pair of exponential laws based on self-similar hierarchical structure (Chen, 2014). The similarity ratio for linear size of fractal copies can be defined as

\[
a = \frac{r_{i-1}}{r_i}.
\]

(5)

Accordingly, the similarity ratio for the number of fractal copies can be defined as

\[
b = \frac{N_i}{N_{i-1}}.
\]

(6)

In equations (5) and (6), \( a \) and \( b \) represent two similarity ratios, respectively. By recursive relations, equations (5) and (6) can be transformed into a pair of exponential functions as follows

\[
r_i = \frac{1}{a} r_{i-1} = r_1 a^{1-i},
\]

(7)

\[
N_i = b N_{i-1} = N_1 b^{i-1}.
\]

(8)

Take the logarithm of equations (7) and (8) yields

\[
\log r_i = (1-i) \log a,
\]

(9)

\[
\log N_i = (i-1) \log b.
\]

(10)
Suppose that $i$ is a continuous variable. Derivative of equations (9) and (10) with respect to $i$ gives
\[
\frac{d \log r_i}{di} = -\log a, \quad (11)
\]
\[
\frac{d \log N_i}{di} = \log b. \quad (12)
\]
Combining equations (11) and (12) yields
\[
D = \frac{\log b}{\log a} = -\frac{d \log N_i}{d \log r_i} = \frac{\Delta N_i / N_i}{\Delta r / r_i}. \quad (13)
\]
According to equation (2), we have
\[
D = -\frac{\log N(r_i)}{\log r_i} = -\frac{d \log N(r_i)}{d \log r_i}. \quad (14)
\]
Removing the subscript $i$ in equation (14) yields the general form of fractal dimension relation.

Equation (13) is equivalent to equation (3). This suggests that fractal dimension can be evaluated by both equation (2) and equation (3). The mathematical process from equation (2) to equation (3) and then to equation (4) is simple, but the physical meaning is not clear where fractals are concerned. Equation (14) reflects the theoretical basis for defining curves of scaling behavior. The curves of fractal scaling behavior in fact reflect hierarchical structure of fractal systems.

### 2.2 Similarity dimension

The formulae of scaling behavior curves are associated with similarity dimension of fractals. The similarity dimension is based on similar ratio $a$ and $b$, and can be expressed as
\[
D = \frac{\log b}{\log a} = -\frac{\log(N(r_i) / N(r_{i-1}))}{\log(r_i / r_{i-1})} = -\frac{\log N(r_i) - \log N(r_{i-1})}{\log r_i - \log r_{i-1}}. \quad (15)
\]
On the other hand, in light of equation (14), we have
\[
D = -\frac{\log N(r_i)}{\log r_i} = -\frac{\log N(r_{i-1})}{\log r_{i-1}}. \quad (16)
\]

From equation (16) it follows
\[
D = -\frac{\log N(r_i) - \log N(r_{i-1})}{\log r_i - \log r_{i-1}} = -\frac{\log(N(r_{i-1}) / N(r_i))}{\log(r_i / r_{i-1})} = -\frac{\Delta \log(N)}{\Delta \log(r)}. \quad (17)
\]
Equation (17) is equivalent to equation (15). Suppose $\Delta \log(r) \rightarrow 0$. Then we have
\[
D = -\frac{\Delta \log(N)}{\Delta \log(r)} \rightarrow -\frac{d \log(N)}{d \log(r)}. \quad (18)
\]
Apparently, equation (14) and equation (18) reach the same goal by different routes. This suggests that the similarity dimension of fractals is the just the fractal dimension of self-similarity hierarchies. The fractal scaling behavior exponent can be derived from the similarity dimension.

### 2.3 The curves of scaling behavior based on box dimension

A formula of scaling behavior curves can be given on the basis of box-counting method. For $i=1$, we have $r_0=1, N(r_0)=1$. In this case, equation (4) can be reduced to

$$D = \frac{\log b}{\log a} = -\frac{\log N(r_i)}{\log r_i}. \quad (19)$$

By recurrence, we have

$$\frac{\log N(r_i)}{\log r_i} = \frac{\log N(r_2)}{\log r_2} = \ldots = \frac{\log N(r_i)}{\log r_i}. \quad (20)$$

If a system’s structure deviates from the power law, equation (20) will break, and we have

$$\frac{\log N(r_i)}{\log r_i} \neq \frac{\log N(r_2)}{\log r_2} \neq \ldots \neq \frac{\log N(r_i)}{\log r_i}. \quad (21)$$

This suggests non-fractals, multifractals, or scale-dependence fractals. For multifractals or scale-dependence fractals, equation (3) can be generalized to a function such as

$$D(r) = -\frac{d \log N(r)}{d \log r}, \quad (22)$$

which $D(r)$ represents the extended fractal dimension, i.e., the scale-dependent fractal dimension (Takayasu, 1990).

In fact, scale-dependent fractal dimension may suggest multifractal scaling or self-affine fractal scaling. Standard fractal structure indicates isotropic growing fractals, which self-affine scaling indicates anisotropic growing fractals. Equation (22) can be discretized as

$$D'(r) = -\frac{\Delta \log N(r)}{\Delta \log r} = -\frac{\log N(r_i) - \log N(r_{i-1})}{\log r_i - \log r_{i-1}}. \quad (23)$$

For simplicity, equation (23) can be re-expressed as formula of scaling behavior curve as follows

$$\alpha_i = -\frac{\log N_i - \log N_{i-1}}{\log r_i - \log r_{i-1}} \rightarrow D, \quad (24)$$

in which $N_i=N(r_i), N_{i-1}=N(r_{i-1}), \alpha_i=D'(r)$. Here $\alpha_i$ represents the scaling behavior exponent based on
inverse power law. This is the formula of scaling behavior curve for box dimension.

2.4 The curve of scaling behavior based on radial dimension

To describe scale-free phenomena of growing fractals based on core-periphery relationship, we need number-radius scaling. This method gives a local fractal parameter termed radial dimension (Frankhauser and Sadler, 1991). In theory, a radial dimension may be equal to a box dimension for a regular growing fractal (Batty and Longley, 1994). However, for a random fractal, due to method of spatial measurement or data extraction, a radial dimension is often different in value from its corresponding box dimension. Moreover, a box dimension value is independent of growing center, while a radial dimension value depends on the definition or selection of growing center. In this sense, box dimension represents a global fractal parameter, while radial dimension represents a local fractal parameter (Frankhauser, 1998). A box dimension is usually given by an inverse power law, while a radial dimension is always given by a positive power law. The model for the radial dimension is as follows

\[ N(r) = N_1 r^D \rightarrow r^D, \quad (25) \]

where \( N_1 \) denotes proportionality coefficient. Specially, we have \( N_1 = 1 \). The fractal dimension can be expressed as

\[ D = \frac{\log(N(r)/N_1)}{\log r}. \quad (26) \]

If the core-periphery relation deviates from the power law, we have

\[ D(r) = \frac{d \log(N(r)/N_1)}{d \log r} = \frac{d \log(N(r))}{d \log r} = \frac{dN(r)/N(r)}{dr/r}, \quad (27) \]

which can be discretized as

\[ D^*(r) = \frac{\Delta \log(N(r))}{\Delta \log r} = \frac{\Delta N(r)/N(r)}{\Delta r/r}. \quad (28) \]

The curve of fractal scaling behavior can be described by

\[ \alpha(r) = \frac{\log(N(r_i)) - \log(N(r_{i-1}))}{\log r_i - \log r_{i-1}} = \frac{(N(r_i) - N(r_{i-1}))/N(r_{i-1})}{(r_i - r_{i-1})/r_{i-1}}, \quad (29) \]

where \( \alpha \) represents the scale-dependent scaling exponent. According to custom, the symbol \( D^*(r) \) is replaced by \( \alpha(r) \). For simplicity, equation (29) can be expressed as (Frankhauser, 1998)
\[ \alpha_i = \frac{\log N_i - \log N_{i-1}}{\log r_i - \log r_{i-1}}. \]  

This is the formula for the scaling behavior curves based on radial dimension, which is in contrast to equation (24) based on box dimension. The parameter \( a_i \) represents the scaling exponent based on power law indicating fractal structure.

### 2.5 Autocorrelation function based on curve of scaling behavior

The key of scaling behavior lies in scaling relation of a system. One of basic properties of fractals is scaling invariance, which indicates dilation symmetry. The scaling law of fractal systems can be expressed as following scaling relation

\[ N(\zeta r) = \zeta^{+D} N(r), \]  

where \( \zeta \) denotes a ratio of scaling up or down. For scale-free scaling behavior curve, we have

\[ \frac{d \log N(\zeta r)}{d \log(\zeta r)} = \frac{d(\log(\zeta^{+D}) + \log N(\gamma))}{d(\log(\zeta) + \log(r))} = \frac{d \log N(r)}{d \log(r)}. \]  

This suggests that, for a fractal, the scaling behavior curve follows scaling law. If the scaling behavior curve departs from scaling relation, we should find the reason and effect.

For a regular fractal, a curve of scaling behavior is actually a horizontal straight line rather than a curve. The scaling exponent is equal to fractal dimension, that is, \( a_i = D \). In contrast, for a random fractal, a curve of scaling behavior is a curve that changes randomly around a horizontal straight line. The value of scaling exponent changes around the corresponding fractal dimension. Therefore, the 1-dimension spatial autocorrelation function (ACF) and the partial autocorrelation function (PACF) can be employed to evaluate the development level of fractal systems. Based on the formula for scaling behavior curve, the 1-dimensional ACF can be defined as follows

\[ \hat{\rho}(k) = \frac{\sum_{i=k+1}^{n-2} [(\alpha_i - \bar{\alpha})(\alpha_{i-k} - \bar{\alpha})]}{\sum_{i=1}^{n-2} (\alpha_i - \bar{\alpha})^2}. \]  

The PACF can be calculated by Yule-Walker’s recurrence formula (Diebold, 2007). Correspondingly, the standard error can be approximately estimated by:
Note that the number of points of scaling behavior curve is \( n-1 \) instead of \( n \). Based on the ACF and PACF, histograms and what is called “two-standard-error bands”, can be used to show whether or not there is significant difference between zero and ACF or PACF values (Diebold, 2007).

3. **Empirical analysis**

3.1 **Mathematical experiment for behavior curves**

Not all behavior curves are scaling behavior curve. If and only if a behavior curve is associated with scaling relation, it can be treated as scaling behavior curve. A simple mathematical experiment can be utilized to draw a comparison between scaling behavior curve and non-scaling behavior curve. Two mathematical models for cities can be employed to illustrate the differences between two types of behavior curves (Figures 1, 2, 3). One is Clark’s model for urban population density distribution (Clark, 1951), and the other is Smeed’s model for traffic network density distribution of cities (Smeed, 1963). Clark’s model reflects the geographical phenomena with characteristic scales and can be described with conventional mathematical methods (Takayasu, 1990), while Smeed’s model reflects the geographical phenomena without characteristic scale and cannot be described with conventional mathematical method (Batty and Longley, 1994). In short, Clark’s exponential decay process bears an effective mean representing characteristic length, while Smeed’s inverse law decay bear no determined mean and thus bear no characteristic length. The average value of Smeed’s distribution depends on the scales of measurement and must be analyzed with the ideas from scaling. Therefore, Clark’s model can be used to generate a non-scaling behavior curve, while Smeed’s model can be used to produce a scaling behavior curve. Due to the connection of the scaling exponent of Smeed’s model with fractal dimension, we can create a standard fractal scaling behavior curve by using this inverse power function (See attached File S1).

First of all, let’s examine the behavior curve based on negative exponential decay. Clark’s model is suitable for describing the density decay regularity of urban population distribution (Feng, 2002; Wang and Zhou, 1999). The standard form of Clark’s model is

\[
p(r) = \rho_0 e^{-r/t_0} ,
\]

(35)
where $r$ is the distance to city center, $\rho(r)$ is the average population density corresponding the radius $r$, the coefficient $\rho_0$ represents the population density of city center, and $r_0$ refers to the characteristic radius of population distribution. For a density distribution, integrating $\rho(r)$ over $r$ based on 2-dimension space yields a cumulative distribution, that is

$$P(r) = 2\pi \int_0^r x\rho(x)dx,$$  \hspace{1cm} (36)

where $P(r)$ denotes the total population within the scope of radius $r$. Where Clark’s model is concerned, the cumulative distribution function is

$$P(r) = 2\pi \rho_0 \int_0^r x e^{-x/r_0} dx = 2\pi \rho_0^2 [1 - (1 + \frac{r}{r_0})e^{-r/r_0}],$$ \hspace{1cm} (37)

Suppose that $\rho_0=30000$, $r_0=5$. We have

$$P(r) = 1500000\pi [1 - (1 + \frac{r}{5})e^{-r/5}].$$ \hspace{1cm} (38)

The scaling behavior can be expressed as

$$\alpha_i = \frac{\log P_i - \log P_{i-1}}{\log r_i - \log r_{i-1}}.$$ \hspace{1cm} (39)

In theory, equation (39) can be approximated by the following formula

$$\alpha_i = \frac{(P_i - P_{i-1})/P_{i-1}}{(r_i - r_{i-1})/r_{i-1}}.$$ \hspace{1cm} (40)

For equation (40), the precondition is that $r$ is a continuous variable.

![Figure 1 The standard behavior curves of the urban density based on Clark’s model and Smeed’s model](image_url)
(Note: The first behavior curve, Figure 1(a), is generated by using equations (38) and (39), the second behavior curve, Figure 1(b), is created by using equations (30) and (43). The former is not a scaling behavior curve, while the latter is a regular fractal scaling behavior curve.)

Figure 1 The random behavior curves of the urban density based on Clark’s model and Smeed’s model

(Figure 2) The random behavior curves of the urban density based on Clark’s model and Smeed’s model

(Note: The first behavior curve, Figure 2(a), is generated by introducing normal white noises into the curve in Figure 1(a), the second behavior curve, Figure 2(b), is created by introducing white noises into the curve in Figure 1(b). The former is not a scaling behavior curve, while the latter is a random fractal scaling behavior curve.)

Then, let’s investigate the behavior curve based on inverse power law decay. Smeed’s model is suitable for describing the density decay regularity of urban traffic (Chen and Long, 2021). The standard form of Smeed model is

\[ \rho(r) = \rho_1 r^{D-d}, \]  \hspace{1cm} (41)

where \( \rho_1 \) represents the population density near city center, \( D \) is the fractal dimension of traffic network, and \( d=2 \) refers to Euclidean dimension of embedding space (Batty and Longley, 1994; Chen et al., 2019). The other symbols are the same as those in equation (35). In terms of equation (36), the cumulative distribution of traffic network is

\[ N(r) = 2\pi\rho_1 \int_0^r x^{D-d+1} \, dx = \frac{2\pi\rho_1}{D} r^D, \]  \hspace{1cm} (42)

where \( N(r) \) denotes the total length of streets and roads or total number of traffic nodes within the scope of radius \( r \). Suppose that \( \rho_1=10000 \), \( D=1.7 \). We have density distribution \( \rho(r)=10000r^{0.3} \). So, the cumulative distribution is
\[ N(r) = \frac{20000\pi}{1.7} r^{1.7}. \]  

Thus, the scaling behavior curve can be expressed as equation (30), which can be approximated by

\[ \alpha_i = \frac{(N_i - N_{i-1})}{(r_i - r_{i-1})} \]. \]  

For equation (44), the precondition is also that \( r \) is a continuous variable.

If behavior curves are generated by mathematical models, the results will be in sharp contrast. The differences help us distinguish fractal scaling behavior curve from the noncaling behavior curves in the real world. Using equations (38) and (39), we can generate a standard non-scaling behavior curve based on Clark’s model (Figure 1(a)). Introducing a white noise sequence following normal distribution into the curve yields a random non-scaling behavior curve (Figure 2(a)). Based on the random non-scaling behavior curve, we can calculate the ACF values (Figure 3(a)). In contrast, using equations (30) and (43), we can create a standard scaling behavior curve based on Smeed’s model for fractal cities (Figure 1(b)). Introducing a normal white noise sequence into this curve yields a random scaling behavior curve (Figure 2(b)). Based on the random non-scaling behavior curve, we can calculate the ACF values (Figure 3(b)). The difference between the two types of behavior curve can be tabulated as follows (Table 1).

| Type            | Non-scaling behavior curve based on Clark’s model | Scaling behavior curve based on Smeed’s model |
|-----------------|--------------------------------------------------|---------------------------------------------|
| Standard curve  | A significant curve                              | Horizontal straight line                    |
| Actual curve    | A significant curve attached white noise          | A horizontal straight line attached white noise |
| ACF             | Significantly more than twice the standard error | Less than twice the standard error          |

**Note:** The scaling behavior curve of a real-world fractal is often not a horizontal straight line, but the local trend line take on a horizontal line. Generally speaking, the middle section shows a horizontal straight trend, and the straight segment is regarded as *scaling range*. 

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ACF-based on Clark’s model

(b) ACF based on Smeed’s model

Figure 3 Histograms of 1-dimensional spatial ACF of the non-scaling behavior curve based on Clark’s model and scaling behavior curves based on Smeed’s model

(Note: The ACF values are computed by using equation (33). The two lines in the histograms is called “two-standard-error bands”, according to which we can know whether or not there is significant difference between zero and ACF or PACF values. See File S1 for the PACF.)

3.2 Case analysis of Changchun city

In the real world, scaling behavior is always associated with complex systems. If a system bear a single scaling process, the scaling behavior curve takes on a horizontal straight line. However, if a system possesses multiple scaling processes, which are mixed together, the scaling behavior curve will be more complicated than the one based on standard model and it is hard to differentiate it from a non-scaling behavior curve. By means of ACF, we can reveal some fractal property of complex systems through fractal scaling behavior curves. One of complex geographical systems is traffic or transportation network, which proved to bear fractal nature (Chen et al, 2019; Frankhauser, 1990;
Kim et al., 2003; Lu and Tang, 2004; Sun, 2007). A number of studies show that urban road and street networks can be described with fractal geometry (Bai and Cai, 2008; Lu et al., 2016; Prada et al., 2019; Rodin and Rodina, 2000; Sahitya and Prasad, 2020; Sun et al., 2012; Wang et al., 2017). Moreover, railway networks can also be treated as fractal systems (Benguigui and Daoud, 1991; Valério et al., 2016). In this case, the fractal scaling curve can be used to study the fractal characteristics of traffic network. An example is to find the fractal growing center for the traffic network of Changchun city of China by using scaling behavior curves (See the attached File S2). Two measurements can be utilized to describe traffic fractals: one is traffic lines, i.e., streets and roads, and the other is points, i.e., nodes of traffic lines. There are two possible fractal growing centers for Changchun city’s traffic networks: one the railway station, and the other, central business district (CBD) (Chen et al., 2019).

![Figure 4](image-url) The double logarithmic relationships between the distance to city centers and total

\[
N(r) = 2.0382r^{1.7823} \\
R^2 = 0.9978
\]

\[
N(r) = 2.6088r^{1.7213} \\
R^2 = 0.9962
\]

\[
L(r) = 5.2513r^{1.8219} \\
R^2 = 0.9993
\]

\[
L(r) = 5.3619r^{1.8265} \\
R^2 = 0.9984
\]
The traffic network of Changchun city takes on statistically self-similar properties. Both the intra-urban traffic lines and traffic nodes follow fractal scaling law. Smeed’s model can be well fitted the observational data of urban density distribution based on road lengths and node numbers. In comparison, traffic lines seem to exhibit more significant fractal properties than traffic nodes. On the other, the fractal growth centered on railway station seems to be more significant than that centered on CBD (Figure 4). The fractal scaling behavior curve based on traffic lines seem to be more stable than ones based on traffic nodes (Figure 5).

**Figure 5** The fractal scaling behavior curves of the traffic network of Changchun city, China
The scaling behavior curves are given by using equation (30), but for traffic lines, the number \( N(r) \) was replaced by total length \( L(r) \).

For Changchun city, railway station represents the starting point of city development, and CBD reflects the current possible center of the city. ACF provides important information for evaluating the traffic fractals. The results are as below. The ACF values of the scaling behavior curves of traffic nodes and lines centered on railway station shows no significant autcorrelation. This is close to standard random scaling behavior curve and suggests a believable fractal structure (Figures 6(a) and...
The ACF values of the scaling behavior curve of traffic lines and nodes centered on CBD displays weak autocorrelation (Figure 6(b) and (d)). The self-similar fractal structure is not very satisfying. A conclusion can be reached that, as far as fractal growing is concerned, the city center of Changchun is railway station rather than CBD. This inference is helpful for us to know the city development from the prospective of urban evolution.

3.3 Case analysis of Beijing and Guangzhou cities

As a comparison, the next study is Beijing traffic network and Guangzhou traffic network. The property and source of original data of the two cities for traffic systems differ from that of Changchun city (Chen and Long, 2021). In light of log-log plot reflecting the scaling relationships between urban radius and traffic road length, Beijing’s traffic network departs significantly from the scaling trend line, while Guangzhou’s traffic network takes on scaling character to a degree (Table 7). The scaling behavior curves deviates from the horizontal trend line to some extent. For Guangzhou’s traffic network is concerned, the departure is not very serious. However, for Beijing’s traffic network, the scaling behavior curve looks like the scaling behavior curve based on Clark’s model instead of that based on Smeed’s model (Figure 8). Actually, the traffic network density can be modeled by a gamma function rather than an inverse power function. The ACF values of the scaling behavior curve of Beijing’s traffic network show strong autocorrelation. Relatively, autocorrelation of the scaling behavior curve of Guantzhou’s traffic network is not so strong (Figure 9).

![Double logarithmic relationships between the distance to city centers and total length of road lines of Beijing and Guangzhou cities, China (2016)](image)
(Note: The original data were cited from the complementary material of Chen and Long (2021).)

Figure 8 The fractal scaling behavior curves of the road networks of Beijing and Guangzhou cities, China (2016)
(Note: The scaling behavior curves are created by using equation (30), but the number \( N(r) \) was replaced by total length \( L(r) \).)

If a scaling behavior curve takes on a horizontal trend line, it suggests fractal property. On the other hand, if the scaling behavior curve depart horizontal trend line, the thing cannot be judged simply. For a self-similar fractal city of isotropic growth, the scaling behavior curve shows a horizontal trend line. However, for a self-affine fractal city of anisotropic growth, the density distribution curve looks similar to negative exponential decay, indicated by Clark’s model. In this case, the fractal scaling behavior curve may be an exponential-like curve rather than a horizontal straight line. If the geographical phenomenon is a multifractal system, the scaling behavior may be very complicated. A fractal city in the real world is always the result of the superposition of multiple fractal structures.

4. Discussion

The formulae of fractal scaling behavior curves can be derived by three different methods. The first is to take the derivative of fractal dimension formulae, the second is to derive it from the model of self-similar hierarchies, and the third is to derive it from similarity dimension. Based on global fractal parameters and box-counting method, the fractal scaling behavior exponent is negative, while
based on local fractal parameter and radius-area scaling method, the fractal scaling behavior exponent is positive. All these formula based on power law and inverse power law can be unified into the same logic framework. The basic property of a fractal scaling behavior curve can be reflected by ACF analysis. According to the fractal development state or fractal structure characteristics, the scaling behavior curves can be divided into three categories. Among these categories, the second category can be divided into two subclasses, and the third category can be divided into four subclasses (Table 2). If a city system is not fractal, the behavior curve bear no scaling nature and does not satisfy the equation (32). In this case, the behavior curve takes on a concave or convex curve. However, if the city system is of multifractal scaling, or self-affine scaling, the behavior curve will also be a concave or convex curve. Self-affine fractal curve may be caused by two reasons. There may be two reasons for the self-affine fractal curve: one is anisotropic growth, and the other is the wrong positioning of the growth center of isotropic growth.

Figure 9 Histograms of 1-dimensional spatial ACF of the scaling behavior curves of Beijing’s and Guangzhou’s traffic networks

The curve of scaling behavior is a useful tool in the studies of fractal cities. First, the curves of scaling behavior can be used to evaluate the development level and property of fractal structure of cities. If a city bear fractal structure, both ACF and PACF come between the two standard error bands. In contrast, if the fractal structure of a city is not developed, partial values of ACF or PACF will break through the two standard error bands. On the other hand, if the fractal dimension depends on measurement scales, ACF or PACF will also go beyond the two standard error bands. In this case,
the scaling behavior curve may reflect non-fractal phenomena, self-affine fractals, or multifractals. Second, the curves of scaling behavior can be utilized to identify the scaling range for fractal dimension estimation. In many cases, the power law relationships for fractal development only appear within certain scale limits. Reflected in log-log plots for fractal dimension estimation, a straight line segment appears in the scattered points. The straight line part is so-called scaling range. In empirical analyses, it is difficult to identify the lower limit and upper limit of the scaling range. Due to the sensitivity of scaling behavior curve to measurement scale, it can be employed to determine the scaling range of fractal development of cities. Third, the curves of scaling behavior can be employed to identify urban boundaries and growing center of fractal cities. For a fractal city, the lower limit of the scaling range indicates an equivalent circle of an urban envelope. An urban envelope is the closed urban boundary line (Batty and Longley, 1994; Longley et al., 1991). In this sense, the scaling behavior curve can be used to identify urban boundary (Tannier et al., 2011). In urban studies, it is significant to determine when and where a city fractal appears (Benguigui et al., 2000). For radial fractal dimension, we have to identify properly the growing center of the city. In this respect, ACF analysis of fractal scaling behavior curve shows useful information for reveal urban growth center.

### Table 2 Classification of scaling behavior curves and their formula expression

| Type                 | Theoretical formula | Empirical formula | Scaling process     |
|----------------------|---------------------|-------------------|---------------------|
| Global scaling       | $\alpha_i = D$      | $\hat{\alpha}_i = D + \epsilon_i$ | Monofractals        |
| Local scaling        | $\alpha_i = \begin{cases} d_E = 2, & r < r_i \\ D, & r_i \leq r \leq r_u \\ d_f = 0, & r > r_u \end{cases}$ | $\hat{\alpha}_i = \begin{cases} d_E + \epsilon_i = 2 + \epsilon_i, & r < r_i \\ D + \epsilon_i, & r_i \leq r \leq r_u \\ d_f + \epsilon_i = \epsilon_i, & r > r_u \end{cases}$ | Monofractals, Local developed fractal |
| Abnormal scaling     | $\alpha_i = f(r_i)$ | $\hat{\alpha}_i = f(r_i) + \epsilon_i$ | Scale-dependent fractal, Self-affine scaling, Multifractals, Non-fractals |

The fractal scaling behavior exponent can be generalized from a constant to a function. For the
local fractal model based on area-radius scaling, the derivative of fractal unit number $N(r)$ with respect to radius $r$ is

$$\alpha(r) = \frac{dN(r)}{dr} = DN_r^{D-1} = \frac{D}{r} N(r).$$  \hspace{1cm} (45)

Discretizing equation (45) yields

$$\alpha(r) = \frac{dN(r)}{dr} \rightarrow \alpha^*(r) = \frac{\Delta N(r)}{\Delta r} = \frac{N(r_i) - N(r_{i-1})}{r_i - r_{i-1}}.$$  \hspace{1cm} (46)

It is easy to demonstrate that equation (46) follows scaling law, that is

$$\alpha(\zeta r) = \frac{D}{\zeta r} N(\zeta r) = \zeta^{D-1} \alpha(r) = \zeta^b \alpha(r),$$  \hspace{1cm} (47)

in which $\zeta$ denotes scale dilation factor, and the scaling exponent is

$$b = D - 1,$$  \hspace{1cm} (48)

where $D$ refers to the local fractal dimension based on area-radius scaling.

Further, the fractal scaling behavior curves analysis can be generalized to other fields involving scaling. Scaling is one of basic character of complex systems. Owing to scaling properties, a system bear no characteristic scale and thus cannot be described with conventional mathematical tools; owing to a system cannot be analyzed by conventional mathematical and quantitative methods, it is treated as one of complex systems. Besides fractals, famous scaling phenomena include allometric growth, complex network, $1/f$ noise, and Zipf’s law (Figure 10). Among these scaling phenomena, Zipf’s law and $1/f$ noise are the signatures of self-organized criticality and complexity (Bak, 1996). Fractal geometry, allometry, and complex network theory compose the foundations for new science of cities (Batty, 2008; Batty, 2013). By introducing white noise into a mathematical model, we can simulate various scaling behavior curves for above-mentioned scale-free phenomena (See attached File S1). All these scale curves show a horizontal trend and no significant autocorrelation.

The novelty of this work rests with two aspects. First, a systematic logic framework of scaling behavior curves is presented. The formulae of scaling behavior curve are derived from different fractal models. Second, the autocorrelation and partial autocorrelation function are utilized to analyze fractal scaling behavior. Histograms and “two-standard-error bands” are employed to illustrate the statistical property of scaling behavior curve. The main shortcomings of this study is as below: First, the scaling behavior curve is based on monofractals rather multifractals. In the real world, the fractal systems are chiefly of multifractal structure instead of monofractal structure.
Second, the four types of scaling behavior curves cannot be effectively distinguished, that is, non-fractals, self-affine fractals, multifractals, and scale-dependence fractals.

Figure 10 The scaling behavior curves of the general scaling phenomena: Zipf’s law, allometric growth, 1/f noise, and complex network
(Note: The scaling behavior curves were generated by means of Zipf’s rank-size relation, allometric scaling law, 1/f frequency spectrum relation, and link-node relation in scale-free network of cities. See File S4 for the dataset.)

5. Conclusions

A fractal scaling behavior curve proved to be a curve of changing fractal dimension with scales. Based on the theoretical derivation, numerical analysis, and empirical analysis, the main conclusions
can be reached as follows. **First, a curve of fractal scaling behavior is a derivative or companion of a fractal model.** It can be derived from some kind of mathematical model of fractals. A fractal model is always based on cumulative distribution, while a fractal scaling behavior curve of based on density distribution. A density distribution is more sensitive to random disturbance or local differences than cumulative distribution. Therefore, the curves of fractal scaling behavior can be employed to reveal local feature or variety in details in fractal pattern or dynamic process. **Second, the curves of fractal scaling behavior can be divided into four types.** The first is the global scaling type. For regular fractals, this type of curve appears as a horizontal straight line; for random fractals, this type of curve fluctuates randomly and slightly around a horizontal straight line. The second is the local scaling type. The middle section of this curve is shown as a horizontal straight segment or an approximate horizontal straight segment. This type of curves reflect prefractals with local development. The third is the abnormal scaling curve. No horizontal straight line or straight segment can be found or revealed. The scaling behavior of a possible fractal system shows a scale-dependence curve. This type of curves suggests self-affine fractals, multifractals, scale-dependence fractals, or non-fractals. **Third, 1-dimensional spatial ACF is an effective tools for revealing the statistic properties of scaling behavior curves of fractals.** In practice, ACF can be combined with PACF. For fractal structure, the ACF and PACF come within the “two-standard-error bands”. If ACF and PACF go beyond the two-standard-error bands, the fractal structure is less developed, or it contain multifractal scaling process or self-affine scaling process or scale-dependence fractal process. Especially, ACF and PACF can be used to identify scaling range of fractal measurement. The upper limit of scaling range can be employed to find urban boundary line. For a growing city fractal, ACF and PACF can be used to identify the growth center of the city.

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**References**

Bai CG, Cai XH (2008). Fractal characteristics of transportation network of Nanjing city. *Geographical Research*, 27(6): 1419-1426 [In Chinese]
Bak P (1996). *How Nature Works: the Science of Self-organized Criticality*. New York: Springer-Verlag

Batty M (2008). The size, scale, and shape of cities. *Science*, 319: 769-771

Batty M (2013). *The New Science of Cities*. Cambridge, MA: MIT Press

Batty M, Longley PA (1994). *Fractal Cities: A Geometry of Form and Function*. London: Academic Press

Benguigui L, Czamanski D, Marinov M, Portugali J (2000). When and where is a city fractal? *Environment and Planning B: Planning and Design*, 27(4): 507–519

Benguigui L, Daoud M (1991). Is the suburban railway system a fractal? *Geographical Analysis*, 23: 362-368

Chen YG (2012). The mathematical relationship between Zipf's law and the hierarchical scaling law. *Physica A: Statistical Mechanics and its Applications*, 391(11): 3285-3299

Chen YG (2014). The spatial meaning of Pareto’s scaling exponent of city-size distributions. *Fractals*, 22(1-2):1450001

Chen YG, Long YQ (2021). Spatial signal analysis based on wave-spectral fractal scaling: A case of urban street networks. *Applied Sciences*, 11(1): 87

Chen YG, Wang YH, Li XJ (2019). Fractal dimensions derived from spatial allometric scaling of urban form. *Chaos, Solitons & Fractals*, 2019, 126: 122-134

Clark C (1951). Urban population densities. *Journal of Royal Statistical Society*, 114(4): 490-496

Diebold FX (2007). *Elements of Forecasting* (4th ed.). Mason, Ohio: Thomson/South-Western

Feng J (2002). Modeling the spatial distribution of urban population density and its evolution in Hangzhou. *Geographical Research*, 21(5): 635-646 [In Chinese]

Frankhauser P (1990). Aspects fractals des structures urbaines. *Espace Géographique*, 19(1): 45–69

Frankhauser P (1998). The fractal approach: A new tool for the spatial analysis of urban agglomerations. *Population: An English Selection*, 10(1): 205-240

Frankhauser P (2008). Fractal geometry for measuring and modeling urban patterns. In: S. Albeverio, D. Andrey, P. Giordano, and A. Vancheri (eds). *The Dynamics of Complex Urban Systems: An Interdisciplinary Approach*. Heidelberg: Physica-Verlag, pp213-243

Frankhauser P, Sadler R (1991). Fractal analysis of agglomerations. In: M. Hilliges (ed.). *Natural Structures: Principles, Strategies, and Models in Architecture and Nature*. Stuttgart: University of Stuttgart, pp 57-65
Kim KS, Benguigui L, Marinovc M (2003). The fractal structure of Seoul’s public transportation system. *Cities*, 20(1): 31-39

Liu SD, Liu SK (1993). *An Introduction to Fractals and Fractal Dimension*. Beijing: China Meteorological Press [In Chinese]

Longley PA, Batty M, Shepherd J (1991). The size, shape and dimension of urban settlements. *Transactions of the Institute of British Geographers (New Series)*, 16(1): 75-94

Lu YM, Tang JM (2004). Fractal dimension of a transportation network and its relationship with urban growth: a study of the Dallas-Fort Worth area. *Environment and Planning B: Planning and Design*, 31: 895-911

Lu ZM, Zhang H, Southworth F, Crittenden J (2016). Fractal dimensions of metropolitan area road networks and the impacts on the urban built environment. *Ecological Indicators*, 70: 285-296

Mandelbrot BB (1982). *The Fractal Geometry of Nature*. New York: W. H. Freeman and Company

Mandelbrot BB (1989). Fractal geometry: what is it, and what does it do? *Proceedings of the Royal Society of London A: Mathematical and Physical Sciences*, 423 (1864): 3-16

Manrubia SC, Zanette DH, Solé RV (1999). Transient dynamics and scaling phenomena in urban growth. *Fractals*, 7(1): 1-8

Meakin P (1998). *Fractal, Scaling and Growth far from Equilibrium*. Cambridge: Cambridge University Press

Prada D, Montoya S, Sanabria M, Torres F, Serrano D, Acevedo A (2019). Fractal analysis of the influence of the distribution of road networks on the traffic. *Journal of Physics: Conference Series*, 1329: 012003

Rodin V, Rodina E (2000). The fractal dimension of Tokyo’s streets. *Fractals*, 8(4): 413-418

Sahitya KS, Prasad CSRK (2020). Modelling structural interdependent parameters of an urban road network using GIS. *Spatial Information Research*, 28(3): 327–334

Smeed RJ (1963). Road development in urban area. *Journal of the Institution of Highway Engineers*, 10(1): 5-30

Sun Z, Zheng JF, Hu HT (2012). Fractal pattern in spatial structure of urban road networks. *International Journal of Modern Physics B*, 26(30): 1250172

Sun ZZ (2007). The study of fractal approach to measure urban rail transit network morphology. *Journal of Transportation Systems Engineering and Information Technology*, 7(1): 29-38 [in Chinese]
Takayasu H (1990). *Fractals in the Physical Sciences*. Manchester: Manchester University Press

Tannier C, Thomas I, Vuidel G, Frankhauser P (2011). A fractal approach to identifying urban boundaries. *Geographical Analysis*, 43(2): 211-227

Thomas I, Frankhauser P, Frenay B, Verleysen M (2010). Clustering patterns of urban built-up areas with curves of fractal scaling behavior. *Environment and Planning B*, 37(5): 942-954

Thomas I, Frankhauser P, Frenay B, Verleysen M (2010). Clustering patterns of urban built-up areas with curves of fractal scaling behavior. *Environment and Planning B*, 37(5): 942-954

Valério D, Lopes AM, Machado JAT (2016). Entropy analysis of a railway network complexity. *Entropy*, 18(11): 388

Wang FH, Zhou YX (1999). Modeling urban population densities in Beijing 1982-90: suburbanisation and its causes. *Urban Studies*, 36: 271-287

Wang H, Luo S, Luo TM (2017). Fractal characteristics of urban surface transit and road networks: Case study of Strasbourg, France. *Advances in Mechanical Engineering*, 9(2): 1–12