Torsion Theories and Coverings of $V$-Groups

Aline Michel

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Abstract
For a commutative, unital and integral quantale $V$, we generalize to $V$-groups the results developed by Gran and Michel for preordered groups. We first of all show that, in the category $\mathcal{V}$-$\text{Grp}$ of $V$-groups, there exists a torsion theory whose torsion and torsion-free subcategories are given by those of indiscrete and separated $V$-groups, respectively. It turns out that this torsion theory induces a monotone-light factorization system that we characterize, and it is then possible to describe the coverings in $\mathcal{V}$-$\text{Grp}$. We next classify these coverings as internal actions of a Galois groupoid. Finally, we observe that the subcategory of separated $V$-groups is also a torsion-free subcategory for a pretorsion theory whose torsion subcategory is the one of symmetric $V$-groups. As recently proved by Clementino and Montoli, this latter category is actually not only coreflective, as it is the case for any torsion subcategory, but also reflective.

Keywords $V$-group · Torsion theory · Categorical Galois theory · Pretorsion theory · Factorization system · Covering.

1 Introduction
In the paper [18], torsion theories and coverings have been studied in the category $\text{PreOrdGrp}$ of preordered groups [11]. A preordered group is a group $(G, +, 0)$ endowed with a preorder $\leq$ which is compatible with the addition law $+$ of the group $G$: for any $a, b, c, d \in G$, $a \leq c$ and $b \leq d$ implies that $a + b \leq c + d$. Notice that, in this context, we do not necessarily assume that the inversion map of the group is monotone, so that a preordered group is not in general an internal group in the category of preordered sets. In other words, a preordered group is given by a group and a relation satisfying some additional conditions (the reflexivity and the transitivity of the given relation as well as its compatibility with the addition law $+$). Now a relation $R$ on a set $X$ can be seen as a function $r: X \times X \to 2 = \{\top, \bot\}$ where,
for any \((x, x') \in X \times X, r(x, x') = \top\) if \((x, x') \in R\) and \(r(x, x') = \bot\) if \((x, x') \notin R\).

This suggests to consider a generalization of the notion of relation: for a commutative and unital \emph{quantale} \(V\) (that is, \(V\) is a complete lattice \((V, \wedge, \vee, \top, \bot)\) equipped with a binary operation \(\otimes\) which is associative, commutative, distributes over arbitrary joins, and has a unit \(k\)), we could consider \(V\)-\emph{relations}, a \(V\)-relation \(r : X \leftrightarrow X\) on a set \(X\) being a function \(r : X \times X \to V\).

Based on this notion of \(V\)-relation, a generalization of the category \(\text{PreOrdGrp}\) of preordered groups has been studied in the article \([12]\). A \(V\)-\emph{group} is a group \((X, +, 0)\) endowed with a \(V\)-relation \(a : X \leftrightarrow X\) satisfying two properties (called the reflexivity and the transitivity axioms), and which is invariant by shifting: for any \(x, x', x'' \in X, a(x', x'') = a(x' + x, x'' + x)\). In this sense, a preordered group is then a particular case of \(V\)-group: it is a 2-group (for the quantale \(2\)) \((\text{in the sense of categorical Galois theory})\). We then prove \(V\)-relations, a \(V\)-relation \(a : X \leftrightarrow X\) satisfying two properties (called the reflexivity and the transitivity axioms), and which is invariant by shifting: for any \(x, x', x'' \in X, a(x', x'') = a(x' + x, x'' + x)\). In this sense, a preordered group is then a particular case of \(V\)-group: it is a 2-group (for the quantale \(2\)) \((\text{in the sense of categorical Galois theory})\). We then prove

The objects in \(\text{V-Grp}_{\text{ind}}\) are \emph{indiscrete} and \emph{separated} \(V\)-groups, respectively. As a direct consequence (up to some results mentioned in Sect. 2), we get that \(\text{V-Grp}_{\text{sep}}\) is a \((\text{normal epi})\)-\emph{reflective} subcategory and that \(\text{V-Grp}_{\text{ind}}\) is \((\text{normal mono})\)-\emph{coreflective} in \(\text{V-Grp}\). The above mentioned torsion theory also induces a factorization system \((\mathcal{E}, \mathcal{M})\) for which we give a fairly simple description. The class \(\mathcal{M}\) of morphisms in \(\text{V-Grp}\) corresponds to the \emph{trivial coverings} (in the sense of categorical Galois theory). We then prove two intermediate propositions. In one of them, we build, for any \((X, a) \in \text{V-Grp}\), a separated \(V\)-group \((Y, b)\) as well as an effective descent morphism

\[
 f : (Y, b) \to (X, a)
\]  

(1.1)

in the category of \(V\)-groups. Thanks to these two propositions, it is possible to conclude that \((\mathcal{E}', \mathcal{M}'^*)\) is a \emph{monotone-light} factorization system \([7]\), where \(\mathcal{E}'\) is the “stabilization” of \(\mathcal{E}\) (i.e. the class of morphisms in \(\text{V-Grp}\) whose pullback along any arrow is in \(\mathcal{E}\) and \(\mathcal{M}'^*\) the “localization” of \(\mathcal{M}\) (i.e. the class of morphisms \(f\) in \(\text{V-Grp}\) for which there is an effective descent morphism \(p\) such that the pullback \(p^*(f)\) of \(f\) along \(p\) is in \(\mathcal{M}\)). The class \(\mathcal{M}'^*\) is then actually the class of coverings in \(\text{V-Grp}\). We are moreover able to characterize the
morphisms in both classes. In particular, when $V$ is an integral quantale, the coverings in $V\text{-Grp}$ are given by the morphisms whose kernel is a separated $V$-group.

In Sect. 4, we then make two observations related to the results of Sect. 3. We first notice that it is possible to classify the coverings in $V\text{-Grp}$ in terms of $\text{Gal}(f)$-actions, where $f$ is the special effective descent morphism as in (1.1) constructed previously and $\text{Gal}(f)$ the Galois groupoid associated with $f$. This can be done by using the notion of locally semisimple covering introduced by Janelidze, Márki and Tholen (see Theorem 3.1 in [26]). Secondly, we observe that $V\text{-Grp}_{\text{sep}}$ is a torsion-free subcategory not only for a torsion theory but also for a pretorsion theory. In this case, the torsion subcategory is given by the full subcategory $V\text{-Grp}_{\text{sym}}$ of symmetric $V$-groups whose objects are $V$-groups $(X, a)$ satisfying the symmetry axiom: for any $x, x' \in X$, $a(x, x') = a(x', x)$. Note that, contrary to the above, for this last result, we do not need to assume that the quantale $V$ is integral. From this pretorsion theory, we deduce that $V\text{-Grp}_{\text{sym}}$ is coreflective in $V\text{-Grp}$. This has already been mentioned in [12], where it is also proven that $V\text{-Grp}_{\text{sym}}$ is reflective.

## 2 Background

### 2.1 The Category $V\text{-Grp}$ of $V$-Groups

We dedicate this first section to the notion of $V$-group: among other things, we define such a notion through the concepts of $V$-relation and $V$-category, we give some examples of $V$-groups (including the example of preordered groups [11]), and we describe some limits and colimits as well as the different kinds of epimorphisms and monomorphisms, and the short exact sequences in this special setting. We then state some properties about the category of $V$-groups which will be useful for our work. For this part, we follow the recent paper [12] where the reader will find more details.

Consider $V$ a commutative and unital quantale, i.e. $V$ is a complete lattice (with bottom element $\bot$ and top element $\top$) endowed with an associative and commutative tensor product $\otimes$ which has a unit $k$ and which preserves arbitrary joins: for any $v \in V$ and any family $\{u_i\}_{i \in I}$ in $V$,

$$v \otimes \left( \bigvee_{i \in I} u_i \right) = \bigvee_{i \in I} (v \otimes u_i) \quad \text{and} \quad v \otimes \bot = \bot.$$  

For the purpose of this paper, we add two assumptions on the quantale $V$. We first suppose that arbitrary joins distribute over finite meets so that, as a lattice, $V$ is a frame. Indeed, following [12, Remark 4.2 (6)], this extra condition allows to check the pullback-stability of regular epimorphisms in $V\text{-Grp}$ and then the regularity of this category (see the end of this subsection). The second additional assumption that we consider is that $V$ is non-trivial, i.e. $\bot \neq \top$. This is in particular used in the proof of Proposition 3.8.

**Remark 2.1** For part of our developments (which will be specified later), we will in addition assume that $V$ is an integral quantale, i.e. that $k = \top$ in $V$.

In order to understand the notion of $V$-group, we first need to define the concepts of $V$-relation and of $V$-category.

A $V$-relation $r : X \rightrightarrows Y$ from a set $X$ to a set $Y$ is a function $r : X \times Y \to V$. As for ordinary relations, a $V$-relation $r : X \rightrightarrows Y$ can be composed with a $V$-relation $s : Y \rightrightarrows Z$,
via a “matricial multiplication”: for any \( x \in X \) and any \( z \in Z \),
\[
(s \cdot r)(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z),
\]
and this defines a \( V \)-relation \( s \cdot r : X \to Z \). The identity for the composition is given by the \( V \)-relation \( 1_X : X \to X \) defined, for any \( x, x' \in X \), by
\[
1_X(x, x') = \begin{cases} k & \text{if } x = x' \\ \bot & \text{if } x \neq x'. \end{cases}
\]
The behaviour of \( V \)-relations under composition explains the terminology of \( V \)-matrices we sometimes use instead of that of \( V \)-relations (see for instance [35] for the use of this terminology). Indeed, the above definition of the composition as well as its properties (associativity and identity) make one think of the matrix product.

We then get the category \( V \)-Rel of \( V \)-relations whose objects are sets and whose arrows are \( V \)-relations. Now there exists a non full, bijective on objects, embedding functor \( \text{Set} \to V \)-Rel which associates with any function \( f : X \to Y \) the \( V \)-relation \( f : X \to Y \) defined, for any \( x \in X \) and any \( y \in Y \), by
\[
f(x, y) = \begin{cases} k & \text{if } f(x) = y \\ \bot & \text{otherwise.} \end{cases}
\]
In addition, for any pair \( (X, Y) \) of sets, we have an order on the set \( V \)-Rel(\( X, Y \)) which is induced by the one on the quantale \( V \): for \( r, r' : X \to Y \) two \( V \)-relations from \( X \) to \( Y \), we say that \( r \leq r' \) if and only if, in \( V \), \( r(x, y) \leq r'(x, y) \) for any \( x \in X \) and \( y \in Y \).

We are now ready for the definition of a \( V \)-category. A \( V \)-category is a pair \( (X, a) \) where \( X \) is a set and \( a : X \to X \) a \( V \)-relation such that
\[
1_X \leq a \quad \text{and} \quad a \cdot a \leq a.
\]
In pointwise notation, these last two inequalities give the reflexivity and transitivity axioms:
(R) \( k \leq a(x, x) \) for any \( x \in X \);
(T) \( a(x, x') \otimes a(x', x'') \leq a(x, x'') \) for any \( x, x', x'' \in X \).

Given two \( V \)-categories \( (X, a) \) and \( (Y, b) \), a \( V \)-functor \( f : (X, a) \to (Y, b) \) is a function \( f : X \to Y \) such that \( f \cdot a \leq b \cdot f \), that is, such that
\[
a(x, x') \leq b(f(x), f(x')) \quad \text{for any } x, x' \in X.
\]
This gives rise to the category \( V \)-Cat of \( V \)-categories and \( V \)-functors.

**Remark 2.2** Pairs \( (X, a) \) with \( X \) a set and \( a : X \to X \) a \( V \)-relation satisfying the reflexivity axiom (R) only are said to be \( V \)-graphs.

**Remark 2.3** The notions of \( V \)-categories and \( V \)-functors we are considering are particular cases of the broader concepts of \( \mathcal{V} \)-categories and \( \mathcal{V} \)-functors, where \( \mathcal{V} \) is more generally a symmetric monoidal closed category [15].

Now, a \( V \)-group \( (X, a, +) \) is a \( V \)-category \( (X, a) \) endowed with a group structure
\[
(X, + : X \times X \to X, - : X \to X, 0 : 0 \to X)
\]
such that \( + : (X, a) \otimes (X, a) \to (X, a) \) is a \( V \)-functor, where we define \((X, a) \otimes (X, a) \) by \((X \times X, a \otimes a) \) with \((a \otimes a)((x_1, x_1'), (x_2, x_2')) = a(x_1, x_2) \otimes a(x_1', x_2') \). Note that,
even though we are using the additive notation, the group structure involved in the definition is not necessarily abelian. A \( V \)-homomorphism is a \( V \)-functor which is also a group homomorphism. \( V \)-groups and \( V \)-homomorphisms define the category \( V \)-\text{Grp} of \( V \)-groups.

When we want to check that a given pair \((X, a)\), with \((X, +)\) a group, is a \( V \)-group, it is sometimes easier to use the following proposition:

**Proposition 2.4** [12, Proposition 3.1] Let \((X, a)\) be a \( V \)-graph and \((X, +)\) be a group. Then the following conditions are equivalent:

1. \(+\) : \((X, a) \otimes (X, a) \rightarrow (X, a)\) is a \( V \)-functor;
2. \((X, a)\) is a \( V \)-category and \( a \) is invariant by shifting, i.e., for any \( x, x', x'' \in X\),
   \[ a(x', x'') = a(x' + x, x'' + x). \]

If we consider the quantale \( 2 = (\{\bot, \top\}, \wedge, \top) \) (i.e. \( \otimes = \wedge \) and \( k = \top \)), then any \( 2 \)-group is in fact a preordered group, and the converse also holds. Indeed, given a \( 2 \)-group \((X, a)\), we can get a preordered group given by \((X, \leq)\), where \( x \leq x' \) in \( X \) if and only if \( a(x, x') = \top \). Conversely, any preordered group \((X, \leq)\) gives rise to a \( 2 \)-group \((X, a)\) by defining \( a : X \rightarrow X\), for any \( x, x' \in X\), by

\[
a(x, x') = \begin{cases} 
\top & \text{if } x \leq x' \\
\bot & \text{otherwise.}
\end{cases}
\]

As a consequence, the category \( 2 \)-\text{Grp} of \( 2 \)-groups (and \( 2 \)-homomorphisms) is isomorphic to the category \( \text{PreOrdGrp} \) of preordered groups (and preorder preserving group homomorphisms), and this latter category is then an example of a category of \( V \)-groups. Other examples of categories of \( V \)-groups are given by the categories \( \text{MetGrp} \) of Lawvere (generalized) metric groups and non expansive group homomorphisms (with \( V = ([0, \infty], \geq) \), \( \otimes = + \) and \( k = 0 \)), \( \text{UMetGrp} \) of Lawvere (generalized) ultrametric groups and non expansive group homomorphisms (with \( V = ([0, \infty], \geq) \), \( \otimes = \) the max for the usual order on real numbers and \( k = 0 \)), \( \text{ProbMetGrp} \) of probabilistic metric groups (with

\[
V = \left\{ \phi : [0, \infty] \rightarrow [0, 1] \mid \phi \text{ is monotone and } \phi(x) = \bigvee_{y < x} \phi(y) \right\},
\]

with the pointwise order, and with \( \otimes \) defined by \( (\phi \otimes \psi)(u) = \bigvee_{x + y \leq u} \phi(x) \times \psi(y) \), etc.

We now mention some interesting full subcategories of the category \( V \)-\text{Grp} of \( V \)-groups. First of all, we can consider the subcategory \( V \)-\text{Grp}_{\text{sym}} \) of symmetric \( V \)-groups, i.e. \( V \)-groups \((X, a, +)\) which coincide with their dual \((X, a^\circ, +)\), where \( a^\circ \) is the \( V \)-relation on \( X \) defined, for any \( x, x' \in X \) by \( a^\circ(x, x') = a(x', x) \). We also have the subcategory \( V \)-\text{Grp}_{\text{sep}} \) of separated \( V \)-groups, that is of \( V \)-groups \((X, a, +)\) satisfying the following axiom: for \( x, x' \in X \),

\[
a(x, x') \geq k \quad \text{and} \quad a(x', x) \geq k \quad \implies \quad x = x'.
\]

Note that, since \( a \) is invariant by shifting, the above property is equivalent to the following: for \( x \in X \),

\[
a(x, 0) \geq k \quad \text{and} \quad a(0, x) \geq k \quad \implies \quad x = 0.
\]

We borrow this terminology (which, to our knowledge, does not exist in the literature yet) from the more general notion of separated \( V \)-category (mentioned, for instance, in the Appendix of [23] and in [24]). Note that, for \( V = ([0, \infty], \geq) \) (with \( \otimes = + \)), symmetric \( V \)-groups...
coincide with the usual symmetric Lawvere metric groups (which are Lawvere metric groups \((X, d)\) with \(d(x, x') = d(x', x)\) for any \(x, x' \in X\), while separated \(V\)-groups correspond to the usual separated Lawvere metric groups (i.e. those Lawvere metric groups \((X, d)\) satisfying the separation axiom: \(d(x, x') = 0 = d(x', x) \implies x = x'\)). We can next consider \(V\)-groups \((X, a, +)\) with the indiscrete \(V\)-category structure: \(a(x, x') = \top\) for any \(x, x' \in X\). This structure gives rise to the category \(V\)-\text{Grp}_{\text{ind}}\) of indiscrete \(V\)-groups which is a full subcategory of \(V\)-Grp.

The category \(V\)-Grp of \(V\)-groups is complete and cocomplete. Before describing the limits and some colimits in this particular setting, we first remark that the initial object in \(V\)-Grp is given by the \(V\)-group \((\{\ast\}, \kappa)\) where \(\kappa(\ast, \ast) = k\) while the terminal object is \((\{\ast\}, c)\) where \(c(\ast, \ast) = \top\), so that the category \(V\)-Grp of \(V\)-groups is pointed if and only if \(k = \top\), i.e. \(V\) is integral.

In \(V\)-Grp, the binary product of two \(V\)-groups \((X, a)\) and \((Y, b)\) is given by \((X \times Y, a \land b)\), where

\[
(a \land b)((x_1, y_1), (x_2, y_2)) = a(x_1, x_2) \land b(y_1, y_2),
\]

for any \((x_1, y_1), (x_2, y_2) \in X \times Y\). The equalizer of a pair

\[
f, g: (X, a) \longrightarrow (Y, b)
\]

of arrows in \(V\)-Grp is given by \((E, \tilde{a}) \xrightarrow{e} (X, a)\) where \((E, e)\) is the equalizer of \(f\) and \(g\) in the category \(\text{Grp}\) of groups and \(\tilde{a}\) is the \(V\)-category structure induced by the one of \(X\):

\[
\tilde{a} = e^\ast \cdot a \cdot e \text{ or, equivalently, } \tilde{a}(x, x') = a(e(x), e(x')) \text{ for any } x, x' \in E.
\]

So the pullback of two arrows \(f: (X, a) \rightarrow (Z, c)\) and \(g: (Y, b) \rightarrow (Z, c)\) is \(((X \times_Z Y, a \land b), \pi_1, \pi_2)\), where \((X \times_Z Y, \pi_1, \pi_2)\) is the pullback of \(f\) and \(g\) in the category \(\text{Grp}\) of groups. In particular, the kernel of \(f\) is given by \((K, \tilde{a}) \xrightarrow{\kappa} (X, a)\) where \((K, \kappa)\) is the kernel of \(f\) in \(\text{Grp}\) and \(\tilde{a}\) is the \(V\)-category structure induced by the one of \(X\).

It is also possible to describe the colimits in \(V\)-Grp. The coequalizer of two parallel arrows \(f\) and \(g\) as in (2.1) is obtained by computing the coequalizer \((Q, q)\) of \(f\) and \(g\) in \(\text{Grp}\) and then by equipping the group \(Q\) with the final \(V\)-category structure \(\tilde{b} = q \cdot b \cdot q^\circ\), that is

\[
\tilde{b}(z_1, z_2) = \bigvee_{q(y_l) = z_l} b(y_1, y_2)
\]

for any \(z_1, z_2 \in Q\). In particular, the cokernel of \(f\) is given by \((Y, b) \xrightarrow{\bar{q}} (Q, \bar{b})\) where \((Q, q)\) is the cokernel of \(f\) in \(\text{Grp}\) and \(\bar{b}\) is the final \(V\)-category structure. The description of coproducts in \(V\)-Grp is more complex but it will not be needed for our paper so that we do not give details about it here and refer the interested reader to [12].

Let us now study the different kinds of epimorphisms and monomorphisms which will be of interest for our work. An arrow \(f: (X, a) \rightarrow (Y, b)\) in \(V\)-Grp is an epimorphism if and only if it is surjective. It is a regular epimorphism whenever it is in addition final in the sense that \(b = f \cdot a \cdot f^\circ\). Moreover, the category of \(V\)-groups has the particularity that any regular epimorphism is a normal epimorphism (whenever \(V\) is integral). The morphism \(f\) is a monomorphism if and only if it is injective, and it is a normal monomorphism whenever it is a normal monomorphism in \(\text{Grp}\) and \(a(x, x') = b(f(x), f(x'))\) for any \(x, x' \in X\).

In the next proposition we gather the information from [12] which is useful to describe short exact sequences in the category \(V\)-Grp of \(V\)-groups:

\[\text{Springer}\]
Proposition 2.5 A pair of composable arrows

\[
(X, a) \xrightarrow{\kappa} (Y, b) \xrightarrow{f} (Z, c)
\]  

is a short exact sequence in \( V\text{-Grp} \) if and only if

\[
0 \longrightarrow X \xrightarrow{\kappa} Y \xrightarrow{f} Z \longrightarrow 0
\]

is a short exact sequence in \( \text{Grp} \), \( a(x, x') = b(\kappa(x), \kappa(x')) \) for any \( x, x' \in X \) (i.e. \( a = \kappa^\circ \cdot b \cdot \kappa \)) and \( c(z_1, z_2) = \bigvee_{f(y_1) = z_1} b(y_1, y_2) \) for any \( z_1, z_2 \in Z \) (i.e. \( c = f \cdot b \cdot f^\circ \)).

We end this section by stating some properties which will turn out to be useful later on. The category \( V\text{-Grp} \) is first of all regular, but not Barr-exact [1], as observed in [12, Proposition 4.3]. In particular, every \( V\text{-homomorphism} f : (X, a) \to (Y, b) \) factorizes as

\[
(X, a) \xrightarrow{e} (Y, b) \xrightarrow{m} (f(X), c)
\]

with \( e \) a regular epimorphism in \( V\text{-Grp} \), \( m \) a monomorphism in \( V\text{-Grp} \) and \( c = e \cdot a \cdot e^\circ \).

It is moreover normal (in the sense of the following section) whenever the quantale \( V \) is integral, since any regular epimorphism is then a normal epimorphism, and effective descent morphisms (see Sect. 4 for a short explanation) exactly coincide with regular epimorphisms. The detailed proofs of these assertions are due to Clementino and Montoli [12].

2.2 Normal Categories

A finitely complete category \( \mathcal{C} \) is said to be normal [31, Definition 3.11] when

1. \( \mathcal{C} \) has a zero object, denoted by 0;
2. any arrow \( f : A \to B \) in \( \mathcal{C} \) factorizes as a normal epimorphism (i.e. a cokernel) followed by a monomorphism;
3. normal epimorphisms are stable under pullbacks: in a pullback diagram

\[
\begin{array}{ccc}
E \times_B A & \xrightarrow{\pi_2} & A \\
\pi_1 \downarrow & & \downarrow f \\
E & \xrightarrow{p} & B
\end{array}
\]

\( \pi_2 \) is a normal epimorphism whenever \( p \) is a normal epimorphism.

Categories of groups, abelian groups, rings and Lie algebras are all examples of normal categories, as well as many other algebraic categories. Any semi-abelian category [27] and likewise any homological category [3,4] is normal. As examples of semi-abelian categories, we can mention the categories of cocommutative Hopf algebras over a field [21], of compact groups [4] and of \( C^* \)-algebras [19]. In connection with the present article, we also have that the category \( V\text{-Grp} \) of \( V \)-groups is normal whenever \( V \) is an integral quantale (see [12, Proposition 4.3]). In particular, \( \text{PreOrdGrp} \) [11], \( \text{MetGrp} \), \( \text{UMetGrp} \) and \( \text{ProbMetGrp} \) are normal categories. This information will be crucial for this work.

We now state two fundamental properties of normal categories:
Lemma 2.6 Let \( \mathcal{C} \) be a normal category.

(1) Any regular epimorphism \( f : A \to B \) is the cokernel of its kernel, so that the pair \((\ker(f), f)\) forms a short exact sequence in \( \mathcal{C} \):

\[
0 \longrightarrow \text{Ker}(f) \overset{\ker(f)}{\longrightarrow} A \overset{f}{\longrightarrow} B \longrightarrow 0.
\]

(2) Given a commutative diagram of short exact sequences in \( \mathcal{C} \)

\[
0 \longrightarrow A \overset{\kappa}{\longrightarrow} B \overset{f}{\longrightarrow} C \longrightarrow 0
\]

\[
0 \longrightarrow A' \overset{\kappa'}{\longrightarrow} B' \overset{f'}{\longrightarrow} C' \longrightarrow 0
\]

the left-hand square is a pullback if and only if the arrow \( c \) is a monomorphism.

The proof of this last assertion essentially follows from the fact that a morphism \( f : A \to B \) in a normal category \( \mathcal{C} \) is a monomorphism if and only if its kernel \( \text{Ker}(f) \) is trivial [31, Proposition 3.12].

2.3 Torsion and Pretorsion Theories

One of the main goals of this paper is to find a torsion theory in the category \( V\text{-Grp} \) of \( V \)-groups. This is why we devote this section to the concept of pretorsion theory in an arbitrary category, which is an extension of that of torsion theory (in a normal category). Note that there are many different approaches to non-abelian torsion theories, which are detailed for example in [6,10,13,30] (and in the references therein).

Let \( \mathcal{C} \) be an arbitrary category, and let us then recall the notion of pretorsion theory in the sense of [16]. For a thorough study of this topic, we refer to [17] and, for a closely related approach, to [9,22,33].

Let \( \mathcal{Z} \) denote a (non-empty) class of objects of \( \mathcal{C} \), and then write \( \mathcal{N} \) for the class of morphisms in \( \mathcal{C} \) that factorize through an object of \( \mathcal{Z} \).

Given an arrow \( f : A \to B \) in \( \mathcal{C} \), one says that \( \kappa : K \to A \) is a \( \mathcal{Z} \)-kernel of \( f \) when

- \( f \cdot \kappa \in \mathcal{N} \);
- for any morphism \( \alpha : X \to A \) such that \( f \cdot \alpha \in \mathcal{N} \), there exists a unique arrow \( \phi : X \to K \) such that \( \kappa \cdot \phi = \alpha \).

Dually, an arrow \( c : B \to C \) is a \( \mathcal{Z} \)-cokernel of \( f : A \to B \) when

- \( c \cdot f \in \mathcal{N} \);
- for any morphism \( \alpha : B \to X \) such that \( \alpha \cdot f \in \mathcal{N} \), there exists a unique arrow \( \phi : C \to X \) such that \( \phi \cdot c = \alpha \).

Remark that, by definition, any \( \mathcal{Z} \)-kernel is a monomorphism and any \( \mathcal{Z} \)-cokernel is an epimorphism.

Definition 2.7 Let \( f : A \to B \) and \( g : B \to C \) be two composable arrows in \( \mathcal{C} \). The sequence

\[
A \overset{f}{\longrightarrow} B \overset{g}{\longrightarrow} C
\]
Torsion Theories and Coverings...

is said to be a short $\mathcal{Z}$-exact sequence when $f$ is a $\mathcal{Z}$-kernel of $g$ and $g$ is a $\mathcal{Z}$-cokernel of $f$.

Note that, when the category $\mathcal{C}$ is pointed and $\mathcal{Z}$ is reduced to the zero object, we recover the classical notions of kernel, cokernel and short exact sequence.

**Definition 2.8** A $\mathcal{Z}$-pretorsion theory in the category $\mathcal{C}$ is given by a pair $(\mathcal{T}, \mathcal{F})$ of full replete subcategories of $\mathcal{C}$, with $\mathcal{Z} = \mathcal{T} \cap \mathcal{F}$, such that:

1. any morphism in $\mathcal{C}$ from $T \in \mathcal{T}$ to $F \in \mathcal{F}$ belongs to $N$;
2. for any object $C$ of $\mathcal{C}$ there exists a short $\mathcal{Z}$-exact sequence $T \xrightarrow{\epsilon_C} C \xrightarrow{\eta_C} F$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

We use the terms torsion subcategory and torsion-free subcategory to refer to the subcategories $\mathcal{T}$ and $\mathcal{F}$, respectively, by analogy with the terminology of the classical torsion theory $(\text{Ab}_t, \text{Ab}_{t,f})$ in the category $\text{Ab}$ of abelian groups, where $\text{Ab}_t$ is the category of torsion abelian groups and $\text{Ab}_{t,f}$ the category of torsion-free abelian groups.

Note that the short $\mathcal{Z}$-exact sequence in Definition 2.8 (2) is actually unique up to isomorphism (see [17, Proposition 3.1]). Consider in addition a morphism $\phi: C \to C'$ in the category $\mathcal{C}$, and the two (unique up to isomorphism) short $\mathcal{Z}$-exact sequences associated with $C$ and $C'$:

\[
\begin{array}{cccc}
T & \xrightarrow{\epsilon_C} & C & \xrightarrow{\eta_C} & F \\
T(\phi) & \downarrow \phi & & \downarrow \phi & \xrightarrow{\epsilon_{C'}} \xrightarrow{F(\phi)} C' & \xrightarrow{\eta_{C'}} & F' \\
\end{array}
\]

Using the universal properties of $\mathcal{Z}$-kernels and $\mathcal{Z}$-cokernels, we find two morphisms $T(\phi): T \to T'$ and $F(\phi): F \to F'$ such that $\epsilon_{C'} \cdot T(\phi) = \phi \cdot \epsilon_C$ and $F(\phi) \cdot \eta_C = \eta_{C'} \cdot \phi$.

This construction gives rise to two functors, $F: \mathcal{C} \to \mathcal{F}$ and $T: \mathcal{C} \to \mathcal{T}$. The first one is a left adjoint for the inclusion functor $U: \mathcal{T} \to \mathcal{C}$ and, for any $C \in \mathcal{C}$, the $C$-component of the unit of the adjunction $F \dashv U$ is given by the epimorphism $\eta_C: C \to F = UF(C)$ in Definition 2.8. As a consequence, the functor $F$ is an epi-reflector. The dual statement also holds: the functor $T$ is a mono-coreflector. Indeed, $T$ is a right adjoint of the inclusion functor $V: \mathcal{F} \to \mathcal{C}$ and any $C$-component $\epsilon_C: T = VT(C) \to C$ of the counit of the adjunction $V \dashv T$, which corresponds to the arrow $\epsilon_C$ in Definition 2.8, is a monomorphism.

Now, if the category $\mathcal{C}$ is normal (so in particular pointed), we find the notion of torsion theory by considering $\mathcal{Z}$ the class with the zero object only:

**Definition 2.9** A torsion theory in a normal category $\mathcal{C}$ is given by a pair $(\mathcal{T}, \mathcal{F})$ of full (replete) subcategories of $\mathcal{C}$ such that:

1. the only arrow from any $T \in \mathcal{T}$ to any $F \in \mathcal{F}$ is the zero arrow;
2. for any object $C$ of $\mathcal{C}$ there exists a short exact sequence $0 \longrightarrow T \xrightarrow{\epsilon_C} C \xrightarrow{\eta_C} F \longrightarrow 0$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

In this particular situation, we also find the results mentioned above: the short exact sequence of Definition 2.9 (2) is unique up to isomorphism, and any torsion theory gives rise to two functors $F: \mathcal{C} \to \mathcal{F}$ and $T: \mathcal{C} \to \mathcal{T}$ which are (normal epi)-reflection and (normal mono)-coreflection, respectively.
2.4 Monotone-Light Factorization Systems and Coverings

Another goal of this article is to characterize the coverings (in the sense of categorical Galois theory) in the category $V$-$\text{Grp}$ of $V$-groups. Let us then recall a few notions and results which will be helpful for our future developments. For this part, we mainly follow [7,8,13,25], where the reader will find more information about (monotone-light) factorization systems and coverings.

In this section, $\mathcal{C}$ will denote an arbitrary category. Consider the particular case where we have a full reflective subcategory $\mathcal{F}$ of $\mathcal{C}$:

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{F} \\
\downarrow & \searrow U & \\
\mathcal{E} & \xrightarrow{\eta} & \mathcal{F}
\end{array}
$$

(2.4)

Then something interesting happens when the reflector $F : \mathcal{C} \to \mathcal{F}$ is semi-left-exact:

Definition 2.10 [8] A reflector $F : \mathcal{C} \to \mathcal{F}$ as in (2.4) is semi-left-exact when it preserves pullbacks of the form

$$
\begin{array}{ccc}
P & \xrightarrow{U(C)} & U(F(B)) \\
\downarrow & \searrow \downarrow \text{U(f)} & \\
B & \xrightarrow{\eta_B} & U(F(B)),
\end{array}
$$

where $\eta_B : B \to UF(B)$ is the $B$-component of the unit of the adjunction (2.4) and $f : C \to F(B)$ is an arrow in the subcategory $\mathcal{F}$ of $\mathcal{C}$.

In fact, a reflection is semi-left-exact if and only if it is admissible in the sense of categorical Galois theory [25] (with respect to the classes of all morphisms, as explained in [7]).

Note that there exists a natural property, for a reflector $F : \mathcal{C} \to \mathcal{F}$, that is stronger than being semi-left-exact:

Definition 2.11 [8] A reflector $F : \mathcal{C} \to \mathcal{F}$ as in (2.4) has stable units when it preserves pullbacks of the form

$$
\begin{array}{ccc}
P & \xrightarrow{f} & C \\
\downarrow & \searrow \downarrow \text{f} & \\
B & \xrightarrow{\eta_B} & UF(B),
\end{array}
$$

where $\eta_B : B \to UF(B)$ is the $B$-component of the unit of the adjunction (2.4) and $f : C \to UF(B)$ is any arrow in the category $\mathcal{C}$.

In the semi-left-exact context, we then naturally get a factorization system $(\mathcal{E}, \mathcal{M})$ defined as follows [8]:

- $\mathcal{E} = \{ f \in \mathcal{C} \mid F(f) \text{ is an isomorphism}\}$;
- $\mathcal{M} = \{ f \in \mathcal{C} \mid \text{the following square (2.5) is a pullback }\}$:

$$
\begin{array}{ccc}
A & \xrightarrow{\eta_A} & UF(A) \\
\downarrow \text{f} & & \downarrow \text{U(f)} \\
B & \xrightarrow{\eta_B} & UF(B),
\end{array}
$$

(2.5)

where $\eta$ is the unit of the adjunction (2.4), and the morphisms in the class $\mathcal{M}$ are called trivial coverings.
Remark 2.12 When we have a torsion theory \((\mathcal{T}, \mathcal{F})\) in a normal category \(\mathcal{C}\), the (induced) reflector \(F : \mathcal{C} \to \mathcal{F}\) then gives rise to a factorization system \((\mathcal{E}, \mathcal{M})\) as defined above since any reflector induced by a torsion theory in a normal category has stable units [14, Theorem 1.6].

Given the reflection (2.4), we now consider the following two subclasses of morphisms in \(\mathcal{C}\):

- \(\mathcal{E}' = \{ f \in \mathcal{C} \mid \text{the pullback of } f \text{ along any morphism in } \mathcal{C} \text{ is in } \mathcal{E}\}\);
- \(\mathcal{M}^* = \{ f \in \mathcal{C} \mid \text{there exists an effective descent morphism } p \text{ such that } p^*(f) \text{ is in } \mathcal{M} \}\).

Morphisms in \(\mathcal{M}^*\) are called coverings (in the sense of categorical Galois theory). As already mentioned, one of the main goals of this paper is to describe these coverings in the category \(V^{-}\text{Grp}\) of \(V\)-groups, and also to prove that, in this context, the pair \((\mathcal{E}', \mathcal{M}^*)\) is a monotone-light factorization system in the following sense:

Definition 2.13 [7] A factorization system is said to be monotone-light when it is of the form \((\mathcal{E}', \mathcal{M}^*)\) for some factorization system \((\mathcal{E}, \mathcal{M})\).

3 Torsion Theory and Coverings in the Category of \(V\)-Groups (for an Integral Quantale \(V\))

We first of all state a lemma which will turn out to be useful not only in the present section but also in the next one.

Lemma 3.1 Let \((X, a)\) be a \(V\)-group. Then:

1. the subset 
   \[ N_X = \{ x \in X \mid a(0, x) \geq k \text{ and } a(x, 0) \geq k \} \]
   of \(X\) is a normal subgroup of \(X\);
2. the pair \((X/N_X, \bar{a})\) is a separated \(V\)-group, where \(\bar{a} = \eta_X \cdot a \cdot \eta_X^a\) with 
   \[ X \xrightarrow{\eta_X} X/N_X \]
   the quotient group morphism.

Proof (1) We first remark that the subset \(N_X\) is a subgroup of \(X\) since

- \(0 \in N_X: a(0, 0) \geq k\);
- \(x \in N_X \implies -x \in N_X: a(0, -x) = a(x, 0) \geq k \text{ and } a(-x, 0) = a(0, x) \geq k\);
- \(x, y \in N_X \implies x + y \in N_X:\)
  \[ a(0, x + y) \geq a(0, x) \otimes a(0, y) \geq k \otimes k = k \]
  and
  \[ a(x + y, 0) \geq a(x, 0) \otimes a(y, 0) \geq k \otimes k = k \]
  since \(+ : (X, a) \otimes (X, a) \to (X, a)\) is a \(V\)-functor.

This subgroup \(N_X\) is actually normal in \(X\). Indeed, by invariance of \(a\) by shifting, for \(x \in X\) and \(n \in N_X\), we have that 
\[ a(0, x + n - x) = a(-x + x, n) = a(0, n) \geq k \]
and
\[ a(x + n - x, 0) = a(n, -x + x) = a(n, 0) \geq k, \]
which implies that \( x + n - x \in N_X \), as desired.

(2) Assume that \( \vec{a}(y, 0) \geq k \) and that \( \vec{a}(0, y) \geq k \) for \( y \in X/N_X \). Then
\[
k \leq \vec{a}(y, 0) = \bigvee_{\eta_X(x_1) = y, \eta_X(x_2) = 0} a(x_1, x_2) \tag{3.1}
\]
and
\[
k \leq \vec{a}(0, y) = \bigvee_{\eta_X(x'_1) = 0, \eta_X(x'_2) = y} a(x'_1, x'_2). \tag{3.2}
\]

Now, we observe that, for any \( x_1, x_2, x'_1, x'_2 \in X \) such that \( \eta_X(x_1) = \eta_X(x'_1) \) and \( \eta_X(x_2) = \eta_X(x'_2) \), we have
\[
a(x_1, x_2) = a(x_1 - x'_1 + x'_1, x_2 - x'_2 + x'_2) \\
\geq a(x_1 - x'_1, 0) \otimes a(0, x_2 - x'_2) \otimes a(x'_1, x'_2) \\
\geq k \otimes k \otimes a(x'_1, x'_2) = a(x'_1, x'_2)
\]
since \(+ : (X, a) \otimes (X, a) \to (X, a)\) is a \( V \)-functor and since \( x_1 - x'_1 \in N_X \) and \( x_2 - x'_2 \in N_X \). Symmetrically, we also show that \( a(x'_1, x'_2) \geq a(x_1, x_2) \) for such \( x_1, x_2, x'_1, x'_2 \), and this proves that \( a(x_1, x_2) = a(x'_1, x'_2) \) for any \( x_1, x_2, x'_1, x'_2 \in X \) such that \( \eta_X(x_1) = \eta_X(x'_1) \) and \( \eta_X(x_2) = \eta_X(x'_2) \). Since the morphism \( \eta_X : X \to X/N_X \) is surjective, there exists an \( x \in X \) such that \( \eta_X(x) = y \). Equations (3.1) and (3.2) then imply, by idempotence of the operation \( \vee \), that \( a(x, 0) \geq k \) and that \( a(0, x) \geq k \). In other words, \( x \in N_X \). This shows that \( (X/N_X, \vec{a}) \in V\text{-Grp}_{\text{sep}} \) since
\[
y = \eta_X(x) = 0. \quad \Box
\]

As previously announced, from now on, the commutative and unital quantale \( V \) will be assumed to be also integral. Under this additional hypothesis, the reflexivity axiom from the definition of a \( V \)-category now becomes:
\[
a(x, x) = k = \top \quad \text{for any } x \in X.
\]

### 3.1 A Torsion Theory in \( V\text{-Grp} \)

We first prove that there is a torsion theory in the normal category \( V\text{-Grp} \), where the torsion subcategory is the category of indiscrete \( V \)-groups while the torsion-free subcategory is that of separated \( V \)-groups.

**Proposition 3.2** The pair of full and replete subcategories \( (V\text{-Grp}_{\text{ind}}, V\text{-Grp}_{\text{sep}}) \) of \( V\text{-Grp} \) is a torsion theory in the normal category \( V\text{-Grp} \).

**Proof** Let us first show that the only arrow in \( V\text{-Grp} \) from an object of \( V\text{-Grp}_{\text{ind}} \) to an object of \( V\text{-Grp}_{\text{sep}} \) is the zero morphism. For this, consider an arrow \( f : (X, a) \to (Y, b) \) in \( V\text{-Grp} \), with \( (X, a) \in V\text{-Grp}_{\text{ind}} \) and \( (Y, b) \in V\text{-Grp}_{\text{sep}} \). Then, since \( f \) is a \( V \)-homomorphism, for any \( x \in X \),
\[
b(f(x), 0) \geq a(x, 0) = \top \implies b(f(x), 0) = \top = k
\]
and, similarly,
\[ b(0, f(x)) \geq a(0, x) = \top \implies b(0, f(x)) = \top = k, \]
which implies that \( f(x) = 0 \) since \( f(x) \in Y \) with \((Y, b)\) a separated \( V \)-group, and then \( f = 0 \).

Let now \((X, a)\) be an object of \( V \)-Grp, and consider the normal subgroup
\[ N_X = \{ x \in X | a(0, x) \geq k \text{ and } a(x, 0) \geq k \} \]
of \( X \) (see Lemma 3.1). So the sequence
\[ N_X \xrightarrow{\kappa_X} X \xrightarrow{\eta_X} X/N_X, \]
where \( \kappa_X \) is the inclusion of \( N_X \) in \( X \) and \( \eta_X \) the quotient morphism, is a short exact sequence in \( \text{Grp} \). Now, we endow \( N_X \) with the \( V \)-category structure \( \bar{a} \) induced by the one of \( X \), i.e. \( \bar{a} = \kappa_X^0 \cdot a \cdot \kappa_X \), and we equip the quotient \( X/N_X \) with the final structure \( \bar{a} \), that is \( \bar{a} = \eta_X \cdot a \cdot \eta_X^0 \). By Proposition 2.5, the sequence
\[ (N_X, \bar{a}) \xrightarrow{\kappa_X} (X, a) \xrightarrow{\eta_X} (X/N_X, \bar{a}) \]
is a short exact sequence in the category \( V \)-Grp of \( V \)-groups. It remains to show that \((N_X, \bar{a}) \in V \)-Grp}\text{ind} and that \((X/N_X, \bar{a}) \in V \)-Grp}\text{sep}. Thanks to Lemma 3.1, it is first of all clear that \((X/N_X, \bar{a}) \in V \)-Grp}\text{sep}. We then compute that, for any \( n, n' \in N_X \),
\[ \bar{a}(n, n') = a(\kappa_X(n), \kappa_X(n')) = a(n, n') = a(0, n' - n) \geq k = \top, \]
since \( n' - n \in N_X \) and \( V \) is integral, which implies that \( \bar{a}(n, n') = \top \), and this shows that \((N_X, \bar{a}) \in V \)-Grp}\text{ind}.

\begin{example}
In all the examples of \( V \)-groups discussed in the Background section, the quantale \( V \) is integral. Let us then take a look at what Proposition 3.2 means in these different cases:
\end{example}

1. If we consider the quantale \( 2 = (\{\bot, \top\}, \land, \top) \) (i.e. the context of preordered groups), then Proposition 3.2 states that (Grp, ParOrdGrp) forms a torsion theory in the category PreOrdGrp of preordered groups, where Grp is the full subcategory of PreOrdGrp whose objects are given by preordered groups endowed with the indiscrete relation and ParOrdGrp the subcategory of partially ordered groups. Proposition 3.2 is then a generalization of a result described in [18, Proposition 3.1].

2. Another consequence of Proposition 3.2 is that (MetGrp}\text{ind}, MetGrp}\text{sep}) is a torsion theory in the category MetGrp of Lawvere (generalized) metric groups (i.e. in \( V \)-Grp, where \( V = ([0, \infty], \geq), \otimes = + \) and \( k = 0 \)). The objects of its torsion subcategory MetGrp}\text{ind are the Lawvere metric groups \((X, d)\) whose distance \( d \) is always 0, while the objects of the torsion-free subcategory MetGrp}\text{sep correspond to the usual separated Lawvere metric groups, i.e. those Lawvere metric groups \((X, d)\) satisfying the separation axiom: \( d(x, x') = 0 = d(x', x) \implies x = x' \).}

3. Similarly to (2), by Proposition 3.2 there exists a torsion theory in the category UMetGrp of Lawvere (generalized) ultrametric groups (case where \( V = ([0, \infty], \geq), \otimes = \) the max for the usual order on real numbers and \( k = 0 \)) which is given by (UMetGrp}\text{ind}, UMetGrp}\text{sep}).

4. By Proposition 3.2, (ProbMetGrp}\text{ind}, ProbMetGrp}\text{sep) is a torsion theory in the category ProbMetGrp of probabilistic metric groups.

As a direct consequence of Proposition 3.2, we get the following result:
Corollary 3.4

- The category \( V\text{-Grp}_{\text{sep}} \) is reflective in the category \( V\text{-Grp} \)

\[
V\text{-Grp} \xleftarrow{\mathcal{F}} \xrightarrow{\mathcal{U}} V\text{-Grp}_{\text{sep}} \tag{3.4}
\]

and each component of the unit \( \eta \) of the adjunction (as in (3.3)) is a normal epimorphism.
- The category \( V\text{-Grp}_{\text{ind}} \) is coreflective in \( V\text{-Grp} \) and each component of the counit \( \kappa \) of the adjunction (as in (3.3)) is a normal monomorphism.

**Proof** This follows from Proposition 3.2 and the (only) Proposition in [30] (see also [6], and [10]). \( \square \)

### 3.2 Monotone-Light Factorization System and Coverings in \( V\text{-Grp} \)

As recalled in Sect. 2, by Remark 2.12, the reflection (3.4) gives rise to a factorization system \((\mathcal{E}, \mathcal{M})\) since the reflective subcategory \( V\text{-Grp}_{\text{sep}} \) is a torsion-free subcategory in \( V\text{-Grp} \) (as a consequence of Proposition 3.2). The next proposition characterizes its two classes of morphisms \( \mathcal{E} \) and \( \mathcal{M} \) as follows:

**Proposition 3.5** Consider, in \( V\text{-Grp} \), the commutative diagram

\[
\begin{array}{ccc}
(N_X, \bar{a}) & \xrightarrow{\kappa_X} & (X, a) & \xrightarrow{\eta_X} & (X/N_X, \bar{a}) \\
\phi \downarrow & & (1) & & (2) \downarrow \\
(N_Y, \bar{b}) & \xrightarrow{\kappa_Y} & (Y, b) & \xrightarrow{\eta_Y} & (Y/N_Y, \bar{b})
\end{array}
\]

where, as before, \( \bar{a} = \kappa_X \cdot a \cdot \kappa_X, \bar{b} = \kappa_Y \cdot b \cdot \kappa_Y, \) \( \bar{a} = \eta_X \cdot a \cdot \eta_X, \) and \( \bar{b} = \eta_Y \cdot b \cdot \eta_Y, \) where \( \eta_X \) (respectively \( \eta_Y \)) is the \( (X, a) \)-component (respectively the \( (Y, b) \)-component) of the unit of the adjunction (3.4), \( \phi \) is the restriction of \( f \) to \( (N_X, \bar{a}) \) and where we write \( \alpha \) for \( F(f) \).

The adjunction (3.4) induces a factorization system \((\mathcal{E}, \mathcal{M})\) in \( V\text{-Grp} \) where:

- \( f : (X, a) \to (Y, b) \) is in the class \( \mathcal{E} \) if and only if the following conditions hold:
  - (a) the square (1) is a pullback;
  - (b) \( \eta_Y \cdot f \) is a regular epimorphism;
- \( f : (X, a) \to (Y, b) \) is in the class \( \mathcal{M} \) if and only if the restriction \( \phi : N_X \to N_Y \) of \( f \) to \( N_X \) is a group isomorphism and \( a = b \land \bar{a} \).

**Proof**

- Assume that \( f : (X, a) \to (Y, b) \) is in the class \( \mathcal{E} \), so that \( \alpha \) is an isomorphism in \( V\text{-Grp} \). Since \( V\text{-Grp} \) is a normal category, by Lemma 2.6, the fact that \( \alpha \) is a monomorphism implies that the square (1) in the above diagram is a pullback. Now, knowing that \( \alpha \) is a regular epimorphism, we have that the composite \( \alpha \cdot\eta_X \) is also a regular epimorphism (since \( \eta_X \) is itself a regular epimorphism and the category \( V\text{-Grp} \) is regular), that is \( \eta_Y \cdot f \) is a regular epimorphism by commutativity of the square (2).

Conversely, if the square (1) in the above diagram is a pullback, then it implies, by Lemma 2.6, that \( \alpha \) is a monomorphism since \( V\text{-Grp} \) is a normal category. The assumption (b) then implies that \( \alpha \cdot\eta_X \) is a regular epimorphism (by commutativity of the square (2)).
Knowing that the category $V$-$\text{Grp}$ is regular, it follows that $\alpha$ is a regular epimorphism. As a conclusion, $\alpha$ is an isomorphism, that is $f$ is in the class $\mathcal{E}$.

- Assume that $f : (X, a) \to (Y, b)$ is in the class $\mathcal{M}$. It means that the square (2) in the above diagram is a pullback. It is then easily seen that $a = b \land \bar{a}$ and that the arrow $\phi$ is an isomorphism in the category of $V$-groups, so in particular in the category of groups. Conversely, if we suppose that $\phi$ is a group isomorphism, then the square $XX/N_X \rightarrow YY/N_Y$ is a pullback in the category $\text{Grp}$ of groups since this category is indeed protomodular [5]. By assumption, we also know that $a = b \land \bar{a}$ so that the square (2) in the above diagram is a pullback in $V$-$\text{Grp}$. It follows that $f$ is in the class $\mathcal{M}$.

In particular, the previous proposition provides us with a description of the trivial coverings (i.e. the $V$-homomorphisms in the class $\mathcal{M}$). In order to find all the coverings, we use the following result from [13, Theorem 1.6] (see also [7]), which not only gives a very simple characterization of the morphisms in the class $\mathcal{M}^*$ but also states that the pair $(\mathcal{E}', \mathcal{M}^*)$ is a monotone-light factorization system.

**Theorem 3.6** Let $\mathcal{C}$ be a normal category. Let $(\mathcal{T}, \mathcal{P})$ be a torsion theory in $\mathcal{C}$ such that, for any normal monomorphism $\kappa : K \to A$, the monomorphism $\kappa \cdot \epsilon_K : T(K) \to A$ is normal in $\mathcal{C}$, where $\epsilon_K : T(K) \to K$ is the $K$-component of the counit $\epsilon$ of the coreflection $\mathcal{C} \to \mathcal{T}$. We write $(\mathcal{E}, \mathcal{M})$ for the factorization system associated with the reflector $F : \mathcal{C} \to \mathcal{T}$, which has stable units.

If for any object $C$ in $\mathcal{C}$ there is an effective descent morphism $p : X \to C$ with $X \in \mathcal{P}$, then $(\mathcal{E}', \mathcal{M}^*)$ is a monotone-light factorization system, and moreover

- $\mathcal{E}'$ is the class of normal epimorphisms in $\mathcal{C}$ whose kernel is in $\mathcal{T}$;
- $\mathcal{M}^*$ is the class of morphisms in $\mathcal{C}$ whose kernel is in $\mathcal{F}$.

**Remark 3.7** As already mentioned, in the context of $V$-groups, the effective descent morphisms are exactly the regular epimorphisms. We refer the reader to the beginning of Sect. 4 for a short explanation about effective descent morphisms in categories with pullbacks.

It only remains to check two assumptions in order to be allowed to apply Theorem 3.6 to our context. This is the aim of the following two propositions.

**Proposition 3.8** For any object $(X, a)$ in the category $V$-$\text{Grp}$ of $V$-groups, there exist an object $(Y, b)$ in the subcategory $V$-$\text{Grp}_{\text{sep}}$ of separated $V$-groups and an effective descent morphism

$$f : (Y, b) \to (X, a)$$

from $(Y, b)$ to $(X, a)$.

**Proof** Let $(X, a) \in V$-$\text{Grp}$. Define $(Y, b)$ in the following way:

- $Y = \mathbb{Z} \times X$ as a group;
- for any $z, z' \in \mathbb{Z}$ and any $x, x' \in X$,

$$b((z, x), (z', x')) = \begin{cases} a(x, x') & \text{if } z < z' \\ \top & \text{if } z = z' \text{ and } x = x' \\ \perp & \text{otherwise.} \end{cases}$$
It is then clear that $(Y, b)$ is a $V$-graph: for any $(z, x) \in Y$,
\[ b((z, x), (z, x)) = T = k. \]

Let us now check that $+: (Y, b) \otimes (Y, b) \to (Y, b)$ is a $V$-functor. We recall that we have to prove the following: for any $(z_1, x_1), (z'_1, x'_1), (z_2, x_2), (z'_2, x'_2) \in Y$,
\[ b((z_1, x_1), (z'_1, x'_1)) \otimes b((z_2, x_2), (z'_2, x'_2)) \leq b((z_1 + z_2, x_1 + x_2), (z'_1 + z'_2, x'_1 + x'_2)). \]

For this, we consider the different possible cases.

1. If either $b((z_1, x_1), (z'_1, x'_1)) = \perp$ or $b((z_2, x_2), (z'_2, x'_2)) = \perp$, then the identity (3.5) is obviously satisfied.
2. If both $b((z_1, x_1), (z'_1, x'_1)) = T$ and $b((z_2, x_2), (z'_2, x'_2)) = T$, then $z_1 + z_2 = z'_1 + z'_2$ (since $z_1 = z'_1$ and $z_2 = z'_2$) and $x_1 + x_2 = x'_1 + x'_2$ (since $x_1 = x'_1$ and $x_2 = x'_2$), which implies that
\[ b((z_1 + z_2, x_1 + x_2), (z'_1 + z'_2, x'_1 + x'_2)) = T. \]

In this case, the identity (3.5) is then also verified.
3. If $b((z_1, x_1), (z'_1, x'_1)) = T$ and $b((z_2, x_2), (z'_2, x'_2)) = a(x_2, x'_2)$, then
\[ b((z_1, x_1), (z'_1, x'_1)) \otimes b((z_2, x_2), (z'_2, x'_2)) = T \otimes a(x_2, x'_2) = a(x_1, x_1) \otimes a(x_2, x'_2) \leq a(x_1 + x_2, x_1 + x'_2) = a(x_1 + x_2, x'_1 + x'_2) \]
since $(X, a)$ is a $V$-group and $x_1 = x'_1$. This leads to (3.5) because, in this case,
\[ b((z_1 + z_2, x_1 + x_2), (z'_1 + z'_2, x'_1 + x'_2)) = a(x_1 + x_2, x'_1 + x'_2) \]
since $z_1 + z_2 = z'_1 + z'_2 < z'_1 + z'_2$. The symmetric situation $(b((z_1, x_1), (z'_1, x'_1)) = a(x_1, x'_1)$ and $b((z_2, x_2), (z'_2, x'_2)) = T)$ is treated in a similar way.
4. If $b((z_1, x_1), (z'_1, x'_1)) = a(x_1, x'_1)$ (i.e. $z_1 < z'_1$) and $b((z_2, x_2), (z'_2, x'_2)) = a(x_2, x'_2)$ (i.e. $z_2 < z'_2$), then
\[ b((z_1, x_1), (z'_1, x'_1)) \otimes b((z_2, x_2), (z'_2, x'_2)) = a(x_1, x'_1) \otimes a(x_2, x'_2) \leq a(x_1 + x_2, x'_1 + x'_2) = b((z_1 + z_2, x_1 + x_2), (z'_1 + z'_2, x'_1 + x'_2)) \]
since $+: (X, a) \otimes (X, a) \to (X, a)$ is a $V$-functor and $z_1 + z_2 < z'_1 + z'_2$.

As a conclusion, thanks to Proposition 2.4, we deduce that $(Y, b)$ is a $V$-group. In particular, $(Y, b) \in V\text{-Grp}_{\text{sep}}$. Indeed, for $(z, x) \in Y$, if
\[ b((z, x), (0, 0)) = T = b((0, 0), (z, x)), \]
then $(z, x) = (0, 0)$ since
- if $z < 0$, then
  \[ b((0, 0), (z, x)) = \perp \neq T; \]
- if $z > 0$, then
  \[ b((z, x), (0, 0)) = \perp \neq T; \]
- if $z = 0$ with $x \neq 0$, then
  \[ b((z, x), (0, 0)) = b((0, 0), (z, x)) = \perp \neq T. \]

\[ \text{Springer} \]
Let us now consider the function $f : Y \to X$ defined, for any $(z, x) \in Y$, by

$$f(z, x) = x.$$  

It is clear that $f$ is a group homomorphism. We now prove that, in addition, $f : (Y, b) \to (X, a)$ is a $V$-functor, i.e., for any $(z, x), (z', x') \in Y$,

$$b((z, x), (z', x')) \leq a(f(z, x), f(z', x')),$$

i.e.

$$b((z, x), (z', x')) \leq a(x, x').$$

Again, we consider the different possible cases:

- if $z < z'$, then
  $$b((z, x), (z', x')) = a(x, x');$$
- if $z = z'$ and $x = x'$, then
  $$b((z, x), (z', x')) = \top = a(x, x) = a(x, x');$$
- in the other cases,
  $$b((z, x), (z', x)) = \bot \leq a(x, x').$$

We can then conclude that $f$ is a $V$-homomorphism. It just remains to show that $f$ is an effective descent morphism in $V$-Grp, which is the same as showing that it is a regular epimorphism. It is first of all clear that $f$ is surjective. Let us next prove that $f$ is also final, i.e. that, for any $x_1, x_2 \in X$,

$$a(x_1, x_2) = \bigvee_{f(y_i) = x_i} b(y_1, y_2).$$

We compute, for any $x_1, x_2 \in X$, that

$$\bigvee_{f(y_i) = x_i} b(y_1, y_2) = \bigvee_{z_1, z_2 \in \mathbb{Z}} b((z_1, x_1), (z_2, x_2)).$$

Accordingly,

- if $x_1 = x_2$, then
  $$\bigvee_{f(y_i) = x_i} b(y_1, y_2) = \top = a(x_1, x_2)$$
  since $b((z, x_1), (z, x_2)) = \top$ for any $z \in \mathbb{Z}$;
- if $x_1 \neq x_2$, then
  $$\bigvee_{f(y_i) = x_i} b(y_1, y_2) = \left( \bigvee_{z_1 < z_2} b((z_1, x_1), (z_2, x_2)) \right) \lor \left( \bigvee_{z_1 \geq z_2} b((z_1, x_1), (z_2, x_2)) \right)$$
  $$= a(x_1, x_2) \lor \bot$$
  $$= a(x_1, x_2).$$

This completes the proof. \qed
Proposition 3.9 For any normal monomorphism \( i : (K, b) \to (X, a) \) in \( V\text{-Grp} \), the monomorphism \( i \cdot \kappa : (NK, \tilde{b}) \to (X, a) \) is normal, where \( \kappa : (NK, \tilde{b}) \to (K, b) \) is the \((K, b)\)-component of the counit of the coreflection \( T : V\text{-Grp} \to V\text{-Grp}_\text{ind} \).

Proof Since \( i \) is a normal monomorphism, there exists an arrow \( f : (X, a) \to (Z, c) \) in \( V\text{-Grp} \) such that \( i = \ker(f) \). It means that \( K \) is a normal subgroup of \( X \) and that, for any \( x, x' \in K \),
\[
\begin{align*}
b(x, x') &= a(i(x), i(x')).
\end{align*}
\]
Let us show that \( NK \) is a normal subgroup of \( X \):

\[
\begin{align*}
NK &= \{ x \in K \mid b(0, x) \geq k \text{ and } b(x, 0) \geq k \} \\
&= K \cap \{ x \in X \mid a(0, x) \geq k \text{ and } a(x, 0) \geq k \} \\
&= K \cap NX
\end{align*}
\]
with \( K \) and \( NX \) two normal subgroups of \( X \). Hence, \( NK \) is normal in \( X \). One then observes that, for any \( n, n' \in NK \),
\[
\begin{align*}
a((i \cdot \kappa)(n), (i \cdot \kappa)(n')) &= a(i(\kappa(n)), i(\kappa(n'))) = b(\kappa(n), \kappa(n')) = \tilde{b}(n, n').
\end{align*}
\]
We conclude that the monomorphism \( i \cdot \kappa \) is normal. \( \square \)

It is now possible to apply Theorem 3.6:

Theorem 3.10 Let us consider the following classes of morphisms in \( V\text{-Grp} \):

- \( \mathcal{E}' = \{ f \in V\text{-Grp} \mid f \text{ is a normal epimorphism such that } \ker(f) \in V\text{-Grp}_\text{ind} \} \);
- \( \mathcal{M}^* = \{ f \in V\text{-Grp} \mid \ker(f) \in V\text{-Grp}_\text{sep} \} \).

Then, \((\mathcal{E}', \mathcal{M}^*)\) is a monotone-light factorization system.

Proof This result follows from Theorem 3.6, which can be applied to the reflection (3.4) thanks to the two previous propositions. \( \square \)

As a consequence, the coverings in \( V\text{-Grp} \) with respect to the adjunction (3.4) are the \( V\)-homomorphisms \( f : (X, a) \to (Y, b) \) such that \( \ker(f) \in V\text{-Grp}_\text{sep} \).

Example 3.11 (1) The coverings in \( \text{PreOrdGrp} \) with respect to the adjunction (3.4) (where \( V = (\{\bot, \top\}, \land, \lor) \)) are the preorder preserving group morphisms whose kernel is a partially ordered group. As for the morphisms in the class \( \mathcal{E}' \), they are the normal epimorphisms in \( \text{PreOrdGrp} \) whose kernel is in the subcategory \( \text{Grp} \). We therefore recover, from Theorem 3.10, the result developed in [18, Theorem 3.12] for preorder groups.

(2) When \( V = ([0, \infty], \geq), \otimes = + \) and \( k = 0 \), the coverings are given by those non expansive group homomorphisms with kernel in \( \text{MetGrp}_\text{sep} \), while the elements of the class \( \mathcal{E}' \) are the normal epimorphisms in \( \text{MetGrp} \) whose kernel is in \( \text{MetGrp}_\text{ind} \).

(3) In the setting of Lawvere (generalized) ultrametric groups, we get a result analogous to (2).

(4) By Theorem 3.10, the coverings with respect to the adjunction (3.4) (for \( V \) the suitable quantale) in the category \( \text{ProbMetGrp} \) of probabilistic metric groups are the morphisms whose kernel is in \( \text{ProbMetGrp}_\text{sep} \), and the morphisms in this context which are in the class \( \mathcal{E}' \) coincide with those whose kernel is in \( \text{ProbMetGrp}_\text{ind} \).
4 Two Observations

We now make two observations related to the previous section. The first one concerns the coverings we have just characterized above, while the second one is about the torsion theory \((V \text{-Grp}_{\text{ind}}, V \text{-Grp}_{\text{sep}})\) that we have in the category \(V \text{-Grp}\) when \(V\) is an integral quantale.

4.1 Coverings of \(V\)-Groups Classified as Internal Actions

We first remark that it is possible to classify the coverings in the category \(V \text{-Grp}\) of \(V\)-groups in terms of internal actions (whenever, as before, the quantale \(V\) is integral). This can be done by means of a result of [26] that we state below (see Theorem 4.1). For the reader’s convenience, we remind a few notions needed for the understanding of that theorem. Note that, with respect to what is developed in [26], the context is slightly adapted in order to include the example of the category \(V \text{-Grp}\).

Let \(C\) be any normal category in which normal epimorphisms and effective descent morphisms coincide. Let us consider a fixed class \(X\) of objects in \(C\), called a generalized semisimple class, which is such that the following two properties hold for any pullback:

1. \(E \in X\) and \(A \in X\) implies that \(E \times_B A \in X\);
2. \(B \in X\), \(E \in X\) and \(E \times_B A \in X\) implies that \(A \in X\).

The notion of locally semisimple covering is then defined relatively to a generalized semisimple class \(X\) in a category \(C\): a morphism \(\alpha : A \to B\) is a locally semisimple covering in \(C\) when there is a normal epimorphism \(p : E \to B\) such that the pullback \(p^*(\alpha)\) of \(\alpha\) along \(p\) lies in the corresponding full subcategory \(X\) of \(C\). For a fixed \(B \in C\), we write \(\text{LocSSimple}_X(B)\) for the full subcategory of the slice category \(C \downarrow B\) over \(B\) whose objects are pairs \((A, \alpha)\), where \(\alpha : A \to B\) is a locally semisimple covering.

We now briefly recall the concept of effective descent morphism and refer the interested reader to [28], for instance, for a more complete presentation of the subject.

Consider a morphism \(p : E \to B\) in a category \(C\) with pullbacks. We write \(p^* : C \downarrow B \to C \downarrow E\) for the induced pullback functor along \(p\), with \(C \downarrow B\) and \(C \downarrow E\) the usual slice categories. We say that the morphism \(p : E \to B\) is an effective descent morphism when the pullback functor \(p^*\) is monadic. Now, there exists a different (and equivalent) way of defining such a morphism, which is expressed in terms of internal actions. Let us remind this notion before introducing the alternative definition of effective descent morphisms.

Consider an internal category \(C\) in \(C\), represented by a diagram of the form

\[
\begin{array}{c}
C \quad C_1 \times C_0 \quad C_1 \\
\downarrow \quad \downarrow p_1 \quad \downarrow \xi \\
C_0 \quad C_1 \times C_0 \quad C_0
\end{array}
\]

where \(C_0\) is the “object of objects”, \(C_1\) the “object of arrows”, and \(C_1 \times C_0 \times C_1\) the “object of composable pairs of arrows” (see [2], for instance, for more information about internal categories). An internal \(C\)-action is a triple \((A_0, \pi, \xi)\) as in the diagram.
where $C_1 \times_{C_0} A_0$ is the pullback of the “domain” arrow $d : C_1 \to C_0$ and $\pi : A_0 \to C_0$, making the following diagram commute:

\[
\begin{array}{ccc}
C_1 \times_{C_0} A_0 & \overset{\xi}{\longrightarrow} & A_0 \\
\downarrow m \times 1 & & \downarrow \pi \\
C_1 \times_{C_0} A_0 & \overset{\xi}{\longrightarrow} & A_0 \\
\downarrow \pi_1 & & \downarrow \pi \\
C_1 & \overset{c}{\longrightarrow} & C_0
\end{array}
\]

where $\pi_1 : C_1 \times_{C_0} A_0 \to C_1$ is the first projection in the pullback of $d$ and $\pi$. There is a natural notion of morphisms of $C$-actions [29], so that we get a category, the category $\mathcal{C}^C$ of internal $C$-actions.

One example of internal category is particularly interesting when studying effective descent morphisms. For a morphism $p : E \to B$ in $\mathcal{C}$, consider its kernel pair \((Eq(p), p_1, p_2)\). Then

\[
Eq(p) \xrightarrow{p_1} E \xleftarrow{p_2} \eta
\]

is an internal category. To be more precise, it is an internal (effective) equivalence relation. A morphism $p : E \to B$ is called an effective descent morphism if and only if the comparison functor

\[
K_p : \mathcal{C} \downarrow B \to \mathcal{C}^{Eq(p)},
\]

sending an object $\alpha : A \to B$ of $\mathcal{C} \downarrow B$ to the $Eq(p)$-action

\[
Eq(p) \times_E (E \times_B A) \xrightarrow{p_1 \times \zeta_2} E \times_B A \xrightarrow{\xi_1} E
\]

where $\xi_1 : E \times_B A \to E$ and $\zeta_2 : E \times_B A \to A$ are the projections of the pullback $E \times_B A$, is an equivalence of categories. This characterization of effective descent morphisms turns out to be useful in the proof of the following theorem.

**Theorem 4.1** [26, Theorem 3.1] Consider a normal category $\mathcal{C}$ where normal epimorphisms are effective descent morphisms, and $\mathcal{X}$ a generalized semisimple class in $\mathcal{C}$. If $p : E \to B$ is a normal epimorphism in $\mathcal{C}$ such that $E \in \mathcal{X}$, there exists an equivalence of categories

\[
\text{LocSSimple}_{\mathcal{X}}(B) \cong \mathcal{X}^{Eq(p)},
\]

where $\mathcal{X}^{Eq(p)}$ denotes the full subcategory of $\mathcal{C}^{Eq(p)}$ whose objects are the internal $Eq(p)$-actions $(A_0, \pi, \xi)$ with $A_0 \in \mathcal{X}$.

The idea behind the proof of the above theorem is that, since the normal epimorphisms are exactly the effective descent morphisms in $\mathcal{C}$, $p$ is then an effective descent morphism so that the comparison functor

\[
K_p : \mathcal{C} \downarrow B \to \mathcal{C}^{Eq(p)}
\]

is an equivalence of categories. It is then possible to prove that, since $E \in \mathcal{X}$, when restricted to $\text{LocSSimple}_{\mathcal{X}}(B)$, this equivalence corestricts to $\mathcal{X}^{Eq(p)}$ (see [26] for a detailed proof).
We consider here the particular case where $\mathcal{C} = V\text{-Grp}$ (with $V$ an integral quantale so that $V\text{-Grp}$ is a normal category) and $\mathcal{X} = V\text{-Grp}_{\text{sep}}$ which is easily seen (by using for instance an extension, from the homological to the normal setting, of Lemma 2.7 in [20]) to be a generalized semisimple class. By means of Theorem 4.1 and also of the following lemma, we will see that it is then possible to characterize the coverings of $V$-groups as internal actions.

**Lemma 4.2** A morphism $h : (Z, c) \to (X, a)$ in $V\text{-Grp}$ is a locally semisimple covering (relatively to the subcategory $V\text{-Grp}_{\text{sep}}$) if and only if its kernel is a separated $V$-group.

We refer to [18, Lemma 4.3] for a proof of this lemma in the particular case of preordered groups since it is exactly the same idea for an arbitrary integral quantale $V$.

**Remark 4.3** The previous lemma is a particular case of a more general fact observed in [26, Proposition 2.3] where the role of the kernel of an arrow was played by the “fibers” (as defined in [26]).

**Theorem 4.4** Let $(X, a) \in V\text{-Grp}$. Consider the effective descent morphism

$$f : (Y, b) \to (X, a)$$

from Proposition 3.8. Then there exists an equivalence of categories

$$\mathcal{M}^* \downarrow (X, a) \cong V\text{-Grp}_{\text{sep}}^{E^q(f)}$$

where $\mathcal{M}^* \downarrow (X, a)$ is the category of coverings over $(X, a)$.

**Proof** Since the $V$-homomorphism $f : (Y, b) \to (X, a)$ from Proposition 3.8 is an effective descent morphism such that $(Y, b) \in V\text{-Grp}_{\text{sep}}$, by Theorem 4.1, we have the following equivalence of categories:

$$\text{LocSSimple}_{V\text{-Grp}_{\text{sep}}} (X, a) \cong V\text{-Grp}_{\text{sep}}^{E^q(f)}.$$

The proof is now complete thanks to the previous lemma and Theorem 3.10 since both the locally semisimple coverings and the coverings in $V\text{-Grp}$ are given by the $V$-homomorphisms having their kernel in the subcategory of separated $V$-groups. ☐

**Remark 4.5** By definition, the Galois groupoid $\text{Gal}(f)$ associated with $f : (Y, b) \to (X, a)$ (where, as before, $f$ denotes the effective descent morphism from Proposition 3.8) is the image of $E^q(f)$ by the reflector $F : V\text{-Grp} \to V\text{-Grp}_{\text{sep}}$. But since the diagram

$$\begin{array}{c}
(E^q(f), b \land b) \\
\downarrow
\end{array} \quad (Y, b)$$

lies in $V\text{-Grp}_{\text{sep}}$, the image of $E^q(f)$ by $F$ is $E^q(f)$ itself. Accordingly, $E^q(f)$ coincides with the Galois groupoid associated with $f$, and

$$\mathcal{M}^* \downarrow (X, a) \cong V\text{-Grp}_{\text{sep}}^{\text{Gal}(f)}. \quad (4.1)$$

This equivalence classifies the coverings in $V\text{-Grp}$ as internal $\text{Gal}(f)$-actions.

**Example 4.6** (1) When $V = (\bot, T, \land, T)$, Equation (4.1) results in

$$\mathcal{M}^* \downarrow (G, \leq) \cong \text{ParOrdGrp}^{\text{Gal}(f)},$$

for any preordered group $(G, \leq)$. This is precisely the result obtained in [18, Section 4].
(2) When $V = ([0, \infty], \geq), \otimes = +$ and $k = 0$, Equation (4.1) results in

$$\mathcal{M}^* \downarrow (X, d) \cong \text{MetGrp}_{\text{Gal}}^\text{Gal}(f),$$

for any Lawvere (generalized) metric group $(X, d)$.

(3) For Lawvere (generalized) ultrametric groups, we have a result which is similar to (2).

(4) In the context of probabilistic metric groups, we get from Equation (4.1) that

$$\mathcal{M}^* \downarrow (X, d) \cong \text{ProbMetGrp}_{\text{Gal}}^\text{Gal}(f),$$

for any probabilistic metric group $(X, d)$.

### 4.2 A Pretorsion Theory in $V$-Grp

Next, we show that the reflection (3.4) also gives rise to a pretorsion theory in the category $V$-Grp of $V$-groups. Note that, for this section, we do not need to assume that the quantale $V$ is integral.

**Lemma 4.7** The subcategories $V$-Grp$_{\text{sym}}$ and $V$-Grp$_{\text{sep}}$ are closed under regular epimorphisms and monomorphisms, respectively.

**Lemma 4.8** For any $V$-group $(X, a)$, the pair $(X, \hat{a})$, where $\hat{a} : X \rightarrow X$ is the $V$-relation on $X$ defined, for any $x, x' \in X$, by

$$\hat{a}(x, x') = a(x, x') \wedge a(x', x),$$

is a symmetric $V$-group.

**Proposition 4.9** The pair of full and replete subcategories $(V$-Grp$_{\text{sym}}, V$-Grp$_{\text{sep}})$ of $V$-Grp is a $\mathcal{Z}$-pretorsion theory in $V$-Grp, where $\mathcal{Z} = V$-Grp$_{\text{sym}} \cap V$-Grp$_{\text{sep}}$ is given by

$$\mathcal{Z} = \{(X, a) \in V$-Grp $| a(x, x') = a(x', x) \forall x, x' \in X \text{ and } a(x, x') \geq k \implies x = x'\}.$$

**Proof** Let us first prove that any arrow

$$f : (X, a) \rightarrow (Y, b),$$

with $(X, a) \in V$-Grp$_{\text{sym}}$ and $(Y, b) \in V$-Grp$_{\text{sep}}$, belongs to $\mathcal{N}$. Consider then, in $V$-Grp, the (regular epimorphism, monomorphism)-factorization of $f$:

$$\xymatrix{ (X, a) \ar[rr]^f \ar[dr]_e & & (Y, b) \ar[dl]^m \\ & (f(X), \bar{a}). }$$

By Lemma 4.7, we have that $(f(X), \bar{a})$ is in both $V$-Grp$_{\text{sym}}$ and $V$-Grp$_{\text{sep}}$. Accordingly, $f$ factorizes in $V$-Grp through an object of $\mathcal{Z}$, and we have proved the first point of Definition 2.8.

Consider now any $V$-group $(X, a)$ and let us show that the sequence

$$\xymatrix{ (X, \hat{a}) \ar[r]^{1_X} & (X, a) \ar[r]^{\eta_X} & (X/N_X, \bar{a}) }$$
is a short $\mathscr{A}$-exact sequence where, as before, we write $X \xrightarrow{\eta_X} X/N_X$ for the quotient morphism, and where $(X, \hat{a})$ is a symmetric $V$-group (by Lemma 4.8) and $(X/N_X, \hat{a})$ a separated $V$-group (by Lemma 3.1).

We begin by showing that $\eta_X$ is the $\mathscr{A}$-cokernel of the arrow $1_X$. Let us consider a morphism $f: (X, a) \rightarrow (Y, b)$ in $V\text{-Grp}$ such that $f \cdot 1_X \in \mathscr{M}$, i.e. such that $f \cdot 1_X$ factorizes through an object $(Z, c)$ of $\mathscr{A}$: $f \cdot 1_X = \alpha \cdot \beta$.

\[
\begin{array}{ccc}
(X, \hat{a}) & \xrightarrow{1_X} & (X, a) \\
\downarrow{\beta} & & \downarrow{f} \\
(Z, c) & \xrightarrow{\alpha} & (Y, b)
\end{array}
\]

Let $x \in N_X$. Then,

\[
a(0, x) \geq k \quad \text{and} \quad a(x, 0) \geq k
\]

which implies that

\[
k \leq a(0, x) \land a(x, 0) = \hat{a}(0, x) \leq c(0, \beta(x)),
\]

since $\beta$ is a $V$-homomorphism, so that

\[
c(0, \beta(x)) \geq k.
\]

It follows that $\beta(x) = 0$ since $(Z, c) \in \mathscr{A}$ and then that

\[
f(x) = (\alpha \cdot \beta)(x) = 0
\]

for any $x \in N_X$. Accordingly, by the universal property of the cokernel $\eta_X$, there exists a unique arrow $\phi: X/N_X \rightarrow Y$ in the category $\text{Grp}$ of groups such that $\phi \cdot \eta_X = f$:

\[
\begin{array}{ccc}
N_X & \xrightarrow{k_X} & X \\
\downarrow{f} & & \downarrow{\eta_X} \\
& & X/N_X
\end{array}
\]

Let us show that this group morphism $\phi$ is in fact a $V$-functor. Consider $w_1, w_2 \in X/N_X$ and $x_1, x_2 \in X$ such that $\eta_X(x_i) = w_i$ for any $i = 1, 2$. Then, we have that

\[
a(x_1, x_2) \leq b(f(x_1), f(x_2)) = b((\phi \cdot \eta_X)(x_1), (\phi \cdot \eta_X)(x_2)) = b(\phi(w_1), \phi(w_2))
\]

since $f$ is a $V$-homomorphism, which implies that

\[
\hat{a}(w_1, w_2) = \bigvee_{\eta_X(x_i) = w_i} a(x_1, x_2) \leq b(\phi(w_1), \phi(w_2))
\]

and this shows that the group morphism $\phi$ is a $V$-functor and so a $V$-homomorphism. As a conclusion, there exists a unique morphism $\phi: (X/N_X, \hat{a}) \rightarrow (Y, b)$ in $V\text{-Grp}$ such that $\phi \cdot \eta_X = f$.

Next, we show that $1_X$ is the $\mathscr{A}$-kernel of $\eta_X$. Consider $f: (Y, b) \rightarrow (X, a)$ in $V\text{-Grp}$ such that $\eta_X \cdot f \in \mathscr{M}$, i.e. $\eta_X \cdot f$ factorizes through an object $(Z, c)$ of $\mathscr{A}$: $\eta_X \cdot f = \alpha \cdot \beta$.

\[
\begin{array}{ccc}
(X, \hat{a}) & \xrightarrow{1_X} & (X, a) \\
\downarrow{\phi} & & \downarrow{\eta_X} \\
(Y, b) & \xrightarrow{f} & (Z, c)
\end{array}
\]
Let us take \( \phi = f \), since this is the only possible arrow in \( \text{Grp} \) such that \( 1_X \cdot \phi = f \). It remains to show that \( \phi \) is a \( V \)-functor, i.e. that, for any \( y, y' \in Y \),

\[
b(y, y') \leq \hat{a}(\phi(y), \phi(y')),
\]

with

\[
\hat{a}(\phi(y), \phi(y')) = a(f(y), f(y')) \wedge a(f(y'), f(y)).
\]

It is clear that

\[
b(y, y') \leq a(f(y), f(y'))
\]

for any \( y, y' \in Y \) since \( f \) is a \( V \)-homomorphism. Let us then prove that we also have the following identity:

\[
b(y, y') \leq a(f(y'), f(y)).
\]

Let \( y, y' \in Y \). We compute that

\[
b(y, y') \leq c(\beta(y), \beta(y')) = c(\beta(y'), \beta(y))
\]

\[
\leq \hat{a}((\alpha \cdot \beta)(y'), (\alpha \cdot \beta)(y))
\]

\[
= \hat{a}((\eta_X \cdot f)(y'), (\eta_X \cdot f)(y)) = \bigvee_{\eta_X(x')=(\eta_X \cdot f)(y')} a(x', x)
\]

since \((Z, c) \in \mathcal{E}\). But, for any \( x, x' \in X \) such that \( \eta_X(x') = (\eta_X \cdot f)(y') \) and \( \eta_X(x) = (\eta_X \cdot f)(y) \), we have that

\[
a(f(y'), f(y)) = a(f(y') - x' + x, f(y) - x + x)
\]

\[
\geq a(f(y') - x', 0) \otimes a(0, f(y) - x) \otimes a(x', x)
\]

\[
\geq k \otimes k \otimes a(x', x) = a(x', x)
\]

since \( + : (X, a) \otimes (X, a) \rightarrow (X, a) \) is a \( V \)-functor and since \( f(y') - x' \in N_X \) and \( f(y) - x \in N_X \). As a consequence, for any \( y, y' \in Y \),

\[
b(y, y') \leq \bigvee_{\eta_X(x')=(\eta_X \cdot f)(y')} a(x', x) \leq a(f(y'), f(y)),
\]

and this completes the proof: there exists a unique morphism \( \phi : (Y, b) \rightarrow (X, \hat{a}) \) in \( V\text{-Grp} \) such that \( 1_X \cdot \phi = f \).

**Example 4.10**

(1) If we consider the quantale \( 2 = (\bot, \top, \wedge, \vee) \), then Proposition 4.9 states that \((\text{ProtoPreOrdGrp}, \text{ParOrdGrp})\) forms a pretorsion theory in \( \text{PreOrdGrp} \) where the objects of the torsion subcategory are the preordered groups whose preorder is an equivalence relation. This shows that Proposition 4.9 is a generalization of a particular result proved in [18, Proposition 5.3]. Remark that, as shown in [11], the objects of \( \text{ProtoPreOrdGrp} \) are the so-called protomodular objects \([34]\) of \( \text{PreOrdGrp} \).

(2) Another application of Proposition 4.9 is that \((\text{MetGrp}_{\text{sym}}, \text{MetGrp}_{\text{sep}})\) is a pretorsion theory in \( \text{MetGrp} \). The objects of its torsion subcategory \( \text{MetGrp}_{\text{sym}} \) coincide with the usual symmetric Lawvere metric groups, which are Lawvere metric groups \((X, d)\) with \( d(x, x') = d(x', x) \) for any \( x, x' \in X \).

(3) Similarly to (2), by Proposition 4.9 there exists a pretorsion theory in \( \text{UMetGrp} \), which is given by \((\text{UMetGrp}_{\text{sym}}, \text{UMetGrp}_{\text{sep}})\).
According to Proposition 4.9, \((\text{ProbMetGrpsym}, \text{ProbMetGrpsep})\) is a pretorsion theory in \(\text{ProbMetGrp}\).

From Proposition 4.9, we deduce in particular that the subcategory \(\text{V-Grpsym}\) of symmetric \(\text{V}\)-groups is mono-coreflective in \(\text{V-Grp}\):

**Corollary 4.11** The subcategory \(\text{V-Grpsym}\) of symmetric \(\text{V}\)-groups is mono-coreflective in the category \(\text{V-Grp}\) of \(\text{V}\)-groups

\[
\begin{array}{ccc}
\text{V-Grp} & \xleftarrow{W} & \text{V-Grpsym} \\
\downarrow{R} & & \downarrow{R} \\
\text{V-Grp} & & \text{V-Grpsym}
\end{array}
\]

with its coreflection defined, for any \((X, a) \in \text{V-Grp}\), by \(R(X, a) = (X, \hat{a})\).

**Remark 4.12** The coreflectivity of the subcategory \(\text{V-Grpsym}\) of symmetric \(\text{V}\)-groups has already been observed and proved in [12, Proposition 3.5]. We mention this result here because we view it as a consequence of our pretorsion theory (which has not, to our knowledge, been discovered before) and also because we want to give the generalization of a proposition presented in [18, Corollary 5.4] in the particular case of preordered groups. Note that Proposition 3.5 in [12] states that \(\text{V-Grpsym}\) is reflective in \(\text{V-Grp}\) as well.

**Remark 4.13** Similar results to the ones presented in this section can be obtained in the more general context of the category \(\text{V-Cat}\) of \(\text{V}\)-categories. We shall present these developments in a forthcoming article.

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