Non-BPS Walls and Their Stability in 5D Supersymmetric Theory

Minoru Eto\textsuperscript{a*} Nobuhito Maru\textsuperscript{b†} and Norisuke Sakai\textsuperscript{a‡}

\textsuperscript{a}Department of Physics, Tokyo Institute of Technology
Tokyo 152-8551, JAPAN
\textsuperscript{b}Theoretical Physics Laboratory, RIKEN
Saitama 351-0198, JAPAN

Abstract

An exact solution of non-BPS multi-walls is found in supersymmetric massive $T^*(\mathbb{C}P^1)$ model in five dimensions. The non-BPS multi-wall solution is found to have no tachyon. Although it is only metastable under large fluctuations, we can give topological stability by considering a model with a double covering of the $T^*(\mathbb{C}P^1)$ target manifold. The $\mathcal{N} = 1$ supersymmetry preserved on the four-dimensional world volume of one wall is broken by the coexistence of the other wall. The supersymmetry breaking is exponentially suppressed as the distance between the walls increases.

\textsuperscript{*}e-mail address: meto@th.phys.titech.ac.jp
\textsuperscript{†}e-mail address: maru@postman.riken.go.jp, Special Postdoctoral Researcher
\textsuperscript{‡}e-mail address: nsakai@th.phys.titech.ac.jp
1 Introduction

Brane-world scenario with extra dimensions\cite{11, 2, 3} have attracted much attention in recent years. It has also been useful to implement supersymmetry (SUSY) to obtain solitons such as walls where particles should be localized. SUSY has been most useful to obtain realistic unified theories\cite{4}, but one of their least understood problems is the origin of the SUSY breaking. Half of SUSY can be preserved by walls\cite{5}. Then they automatically become minimal energy solutions with given boundary conditions\cite{6, 7}. They are called the Bogomol’nyi-Prasad-Sommerfield (BPS) states and are assured of stability by the central charge of the SUSY algebra\cite{7}. It has been found that the coexistence of these walls can break SUSY completely, leading to a possible origin of the SUSY breaking\cite{8}. On the other hand, the stability of such non-BPS configurations are no longer guaranteed by SUSY. If we introduce topological quantum numbers such as the winding number by taking appropriate target manifold and by compactifying the extra dimension, we can have a stable non-BPS walls\cite{9, 10}. By considering the SUSY sine-Gordon model with $\pi_1(S^1) = \mathbb{Z}$, stable non-BPS multi-wall configurations have been obtained in four-dimensional theories \cite{9}. However, models in five dimensions are needed to obtain realistic models for brane-world. In five dimensions or higher, we need to implement at least eight supercharges. Theories with eight SUSY are quite restricted. The simplest multiplet in such theories is called hypermultiplet which contains spin 0 and 1/2 only. In order to obtain interacting hypermultiplets, we need to have nonlinear kinetic terms or gauge interactions \cite{11, 12}. The nonlinear sigma model should have a hyper-Kähler target manifold \cite{13–16} with appropriate potential, which are called massive hyper-Kähler nonlinear sigma models. The BPS wall solutions of hypermultiplets have been obtained for the simplest of such nonlinear sigma models, a massive $T^*(\mathbb{CP}^1)$ model\cite{17, 18}. Even the BPS $n$-wall solutions have been constructed in $T^*(\mathbb{CP}^n)$ for $n \geq 1$ \cite{19}. They have been successfully embedded into supergravity in five dimensions\cite{20, 21}. However, no non-BPS multi-wall solutions have been obtained so far.

The purpose of our paper is to present non-BPS multi-wall solutions in a SUSY theory in five dimensions and to discuss their stability. We show that the $T^*(\mathbb{CP}^1)$ nonlinear sigma model in five dimensions admits exact solutions of non-BPS multi-walls which are identical to those found in four dimensions. We find that the small fluctuations around these solutions have no tachyons. This result implies that the non-BPS walls are stable under small fluctuations. However, the target manifold of the nonlinear sigma model, $T^*(\mathbb{CP}^1)$, does not admit winding number as a topological quantum number, since the
homotopy group is trivial \( (\pi_1(T^*(\mathbb{C}P^1)) = 0) \), contrary to the target manifold of the sine-Gordon nonlinear sigma model in four dimensions. By using a variational approach, we demonstrate that the non-BPS multi-wall configurations can be continuously deformed into an energetically lower configuration with no walls. In conformity with the local stability of the background, the energy of the configuration achieves a local minimum at the non-BPS multi-wall solution and exhibits a maximum before reaching the no wall configuration at large deformations. Therefore the non-BPS multi-wall solutions are only metastable in the \( T^*(\mathbb{C}P^1) \) model. Although the metastability may be sufficient for the brane-world to exist during the finite cosmological lifetime, we can give a topological stability to the non-BPS multi-wall solutions by considering a model with the target space of a double cover of \( T^*(\mathbb{C}P^1) \), which admits a nontrivial homotopy group \( \pi_1(S^1) = \mathbb{Z} \). Since the local properties of the double cover is identical to those of \( T^*(\mathbb{C}P^1) \), it should satisfy all the requirements of hyper-Kähler manifold needed for eight SUSY \[11\]. No singularities or obstructions seem to occur even globally for the double cover. Small fluctuations around a background should not depend on global properties such as a generalization to the double cover. We believe that this may precisely be the reason for the fact that there is no tachyon around the non-BPS multi-wall background, since the small fluctuation analysis for \( T^*(\mathbb{C}P^1) \) should be identical to the case of double cover where the topological stability is guaranteed by the nontrivial homotopy group \( \pi_1 \).

We also obtain spectra of massless bosons or light bosons which become massless in the limit of infinitely separated walls. They correctly form complex scalar fields needed to realize the chiral scalar multiplets of four SUSY. The SUSY breaking due to the coexistence of walls provides a mass difference between bosons and fermions in the chiral scalar multiplets. This is explicitly demonstrated for the supermultiplet with a massless fermion and a slightly massive boson, similarly to the case of four-dimensional models\[8\], \[9\]. The mass splitting is found to decrease exponentially as the distance between walls increases. We can also embed our model and the solution into the five-dimensional supergravity. Similarly to the sine-Gordon model\[22\] and other models\[23\] in four dimensions, we expect that the coupling to gravity\[24\] does not introduce new instability, all the massless fields are absorbed by gauge fields via Higgs mechanism, and the lightest scalar field (radion) is nothing but the lightest massive mode of our solution in the rigid SUSY model, at least at weak gravitational coupling.

In sect\[2\] we describe the non-BPS multi-wall solutions both for the SUSY sine-Gordon model in four dimensions and for the \( T^*(\mathbb{C}P^1) \) model in five dimensions. In sect\[3\], it is shown that the multi-wall solutions of the \( T^*(\mathbb{C}P^1) \) model in five dimensions are stable.
under small fluctuations, but are deformable continuously to no wall configurations, showing their metastability. Double cover of $T^*(\mathbb{C}P^1)$ is also introduced and its topological stability is argued. In sect. SUSY breaking exhibited as mass splitting between light bosons and fermions is discussed. Useful formulas of gamma matrices and spinors in four and five dimensions are summarized in Appendix Massive modes for one of the field in spherical coordinates is worked out in Appendix

2 The non-BPS domain walls in five dimensions

2.1 The BPS and non-BPS solutions in the sine-Gordon model

In this subsection we briefly review BPS and non-BPS domain walls in an $\mathcal{N} = 1$ SUSY complex sine-Gordon model in four dimensions. It contains a chiral superfield

$$A(y^m, \theta) = a(y) + \sqrt{2}\theta \psi(y) + \theta^2 F(y), \quad y^m \equiv x^m + i\theta \sigma^m \bar{\theta}, \quad (2.1)$$

with the sine-Gordon superpotential $P$ and with the minimal kinetic term

$$P(A) = \frac{\Lambda^3}{g^2} \sin \frac{g}{\Lambda} A, \quad K(A, \bar{A}) = \bar{A} A, \quad (2.2)$$

where $K$ is the Kähler potential, $\Lambda$ is a coupling constant of unit mass dimension, and $g$ is a dimensionless coupling constant. The spacetime index $m$ runs from 0 to 3. The bosonic part of the Lagrangian reads

$$\mathcal{L}_{\text{boson}} = -|\partial_m a|^2 - \frac{\Lambda^4}{g^2} \left| \cos \frac{g}{\Lambda} a \right|^2, \quad (2.3)$$

where the auxiliary fields $F$ is eliminated. Defining dimensionless real scalar fields $(\Theta, \Phi)$

$$\Theta - \frac{\pi}{2} + i\Phi \equiv \frac{g}{\Lambda} (\text{Re}[a] + i\text{Im}[a]), \quad (2.4)$$

we can rewrite the above Lagrangian as

$$\mathcal{L}_{\text{boson}} = \frac{2\Lambda^2}{g^2} \left[ -\frac{1}{2} \partial_m \Theta \partial^m \Theta - \frac{1}{2} \partial_m \Phi \partial^m \Phi - \frac{\Lambda^2}{2} \sin^2 \Theta \right] \left( - \frac{\Lambda^2}{2} \sinh^2 \Phi \right). \quad (2.5)$$

This model has infinitely many isolated SUSY vacua at $\Theta = n\pi$, $\Phi = 0$ ($n \in \mathbb{Z}$). The existence of two or more isolated vacua can admit domain wall solutions interpolating between these vacua. The variable $\Theta$ may be regarded as taking any real values. However, the Lagrangian with the sine-Gordon superpotential has the periodicity in $\Theta \simeq \Theta +$
$2\pi$. Therefore the variable $\Theta$ is naturally a periodic variable taking values in $\Theta \in [0, 2\pi)$. On the other hand, $\Phi$ has no periodicity. Then the target space of the Lagrangian is $S^1 \times \mathbb{R}$. In that sense, there are only two isolated vacua at $\Theta = 0, \pi$, $\Phi = 0$ in the fundamental domain ($0 \leq \Theta < 2\pi, -\infty \leq \Phi < \infty$).

Let us assume that the wall has a nontrivial profile in the $y$ coordinate which is identified as the extra dimension. The energy density (tension) of the domain wall is bounded by the Bogomolny bound

$$E = \frac{2\Lambda^2}{g^2} \int_{-\infty}^{\infty} dy \left[ \frac{1}{2} (\Theta' \mp \Lambda \sin \Theta \cosh \Phi)^2 + \frac{1}{2} (\Phi' \pm \Lambda \cos \Theta \sinh \Phi)^2 \right] \pm \Lambda (\Theta' \sin \Theta \cosh \Phi - \Phi' \cos \Theta \sinh \Phi) \right]$$

$$\geq \frac{2\Lambda^2}{g^2} [\mp \Lambda \cos \Theta \cosh \Phi]_{-\infty}^{\infty},$$

where prime denotes the derivative with respect to $y$. The Bogomolny bound is saturated when the following BPS equations are satisfied:

$$\Theta' = \pm \Lambda \sin \Theta \cosh \Phi, \quad \Phi' = \mp \Lambda \cos \Theta \sinh \Phi.$$ (2.7)

Imposing the boundary condition such as $(\Theta, \Phi) = (0, 0)$ or $(\pi, 0)$ at minus (plus) infinity of $y$, we find the above BPS equation becomes simpler $\Theta' = \pm \Lambda \sin \Theta$, $\Phi = 0$ leading to the BPS single wall solutions with $y_0$ as the moduli parameter associated with the center of the mass position of the domain wall:

$$\Theta(y; y_0) = \pm \sin^{-1} (\tanh (\Lambda (y - y_0))) + \frac{\pi}{2},$$ (2.8)

whose tension is given by $E = \frac{4\Lambda^3}{g^2}$ from the boundary conditions and Eq. (2.6). The half of SUSY charges are preserved by the BPS wall solution with plus sign in Eq. (2.8), whereas the other half is preserved by the BPS solution with the minus sign. To distinguish the preserved SUSY charges, we shall call the solution with the minus (plus) sign as anti-BPS (BPS) solution.

Let us next consider non-BPS domain wall solutions of the equations of motion. Since the potential monotonically increases as $e^{\Phi}$ for $|\Phi| \to \infty$, we should look for solutions with $\Phi = 0$. Then the Lagrangian and the equations of motion reduce to

$$\mathcal{L}_{\text{boson}} = \frac{\Lambda^2}{g^2} \left[ -\frac{1}{2} (\Theta')^2 - \frac{\Lambda^2}{2} \sin^2 \Theta \right],$$ (2.9)

$$\Theta'' = \Lambda^2 \sin \Theta \cos \Theta.$$ (2.10)
An exact solution of this equation has been found with two parameters $y_0, k$

$$\Theta(y; y_0, k) = \text{am}\left(\frac{\Lambda}{k}(y - y_0), k\right) + \frac{\pi}{2}, \quad 0 < k;$$

(2.11)

where the amplitude function am$(u, k)$ is defined in terms of the Jacobi’s elliptic function sn$(u, k)$ as am$(u, k) = \sin^{-1}\text{sn}(u, k)$. The elliptic functions sn$(u, k)$, cn$(u, k)$ are periodic in $u$ with the period of $4K(k)$, where $K$ is the complete elliptic integral of the first kind. Therefore we can compactify the base space with the radius $L$ by requiring $\frac{1}{2}\pi L = \frac{4kK(k)}{\Lambda}$. When $k = 1$, the radius diverges ($L \to \infty$) and the solution reduces to the BPS wall solution in Eq.(2.8) with $y_0$ as the wall position. For $k \neq 1$, the solution breaks SUSY completely, giving a non-BPS two wall solution with BPS and anti-BPS walls situated at $y_0$ and $y_0 + \pi L$. For $k < 1$, the solution is quasi-periodic and represents a non-BPS two walls with unit winding number in the target space. For $k > 1$, the solution is periodic and represents wall and anti-wall with no winding. These three cases of non-BPS solutions are illustrated in Fig.1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{theta.png}
\caption{The Jacobi’s amplitude function with $k > 1$, $k = 1$, $k < 1$.}
\end{figure}

Since the non-BPS solution (2.11) breaks SUSY completely, its stability is not ensured by the central charge of the SUSY algebra. In fact it has been found that the non-BPS solutions for $k > 1$ is unstable with tachyon, and those for $k < 1$ is stable because of the nontrivial winding number. To see this point, one examines small fluctuations $\theta(x, y), \varphi(x, y)$ around the non-BPS background configurations in Eq.(2.11). The

\footnote{Alternative choices are $2n\pi L = 4kK(k)/\Lambda$, $n = 1, 2, \cdots$, corresponding to $n$ pairs of BPS wall and anti-BPS wall placed with equal interval in the fundamental region $2\pi L$.}
linearized equations of motion for the fluctuation fields are given by

\[
\partial_\mu \partial^\mu \theta = \left[ -\frac{d^2}{dy^2} + \Lambda^2 \left( 2\text{sn}^2 \left( \frac{\Lambda}{k}(y - y_0), k \right) - 1 \right) \right] \theta, \tag{2.12}
\]

\[
\partial_\mu \partial^\mu \varphi = \left[ -\frac{d^2}{dy^2} + \Lambda^2 \right] \varphi, \tag{2.13}
\]

where the world volume coordinates are denoted by \(x^\mu\) with \(\mu\) running from 0 to 2. Mode functions \(\psi^{(l)}_\theta(y)\) \((\psi^{(l)}_\varphi(y))\) for fluctuation field \(\theta\) \((\varphi)\) are defined by

\[
\left[ -\frac{d^2}{dy^2} + \Lambda^2 \right] \psi^{(l)}_\theta = (m^{(l)}_\theta)^2 \psi^{(l)}_\theta, \tag{2.14}
\]

\[
\left[ -\frac{d^2}{dy^2} + \Lambda^2 \right] \psi^{(l)}_\varphi = (m^{(l)}_\varphi)^2 \psi^{(l)}_\varphi. \tag{2.15}
\]

One can expand the fluctuation fields in terms of these mode functions yielding effective fields \(f^{(m)}_\theta(x)\) \((f^{(m)}_\varphi(x))\) on the world volume with mass squared \((m^{(l)}_\theta)^2\) \(((m^{(l)}_\varphi)^2)\)

\[
\theta(x, y) = \sum_m \psi^{(m)}_\theta(y) f^{(m)}_\theta(x), \quad \varphi(x, y) = \sum_m \psi^{(m)}_\varphi(y) f^{(m)}_\varphi(x). \tag{2.16}
\]

Since the Schrödinger-like eigenvalue equation \((2.15)\) for \(\varphi\) obviously has no negative eigenvalues, \(\varphi\) does not have a tachyon. One can obtain three lowest eigenmodes exactly\(^9\) for the eigenvalue equation \((2.14)\) for \(\theta\) :

\[
m^{2}_{\theta,0} = 0, \quad \psi^{(0)}_\theta = \text{dn} \left( \frac{\Lambda}{k}(y - y_0), k \right), \tag{2.17}
\]

\[
m^{2}_{\theta,1} = \frac{1 - k^2}{k^2} \Lambda^2, \quad \psi^{(1)}_\theta = \text{cn} \left( \frac{\Lambda}{k}(y - y_0), k \right), \tag{2.18}
\]

\[
m^{2}_{\theta,2} = \frac{1}{k^2} \Lambda^2, \quad \psi^{(2)}_\theta = \text{sn} \left( \frac{\Lambda}{k}(y - y_0), k \right). \tag{2.19}
\]

The massless mode \(\psi^{(0)}_\theta\) is the Nambu-Goldstone mode for the broken translational invariance. In fact, its profile in the left part of Fig.2 is positive definite corresponding to the derivative of the monotonically increasing background configuration for \(k < 1\). As shown in the left part of Fig.2, the first excited mode \(\psi^{(1)}_\theta\) has a profile of the difference between the translational zero-modes of individual BPS wall and anti-BPS wall. Therefore it corresponds to the fluctuation of relative distance between two walls, so-called breather mode. For the case of \(k < 1\), the mass squared of the breather mode \(\psi^{(1)}_\theta\) is positive, showing the stability of the background non-BPS configuration with nonzero winding number. For the case of \(k > 1\), on the contrary, \(\psi^{(1)}_\theta\) becomes tachyon destabilizing the background without winding number. Since the radius diverges in the limit \(k \to 1\),
the anti-BPS wall at $y = y_0 + \pi L$ goes to infinity, and the solution reduces to the BPS single wall solution. In this limit, $\psi^{(1)}_\theta$ becomes massless, and the sum of $\psi^{(1)}_\theta$ and $\psi^{(0)}_\theta$ is localized on the BPS wall, whereas the difference is localized on the anti-BPS wall, which disappears to infinity.

2.2 BPS and non-BPS Domain walls in five dimensions

2.2.1 Models admitting domain walls

Although a stable non-BPS solutions with a winding number has been obtained for a model in four dimensions, we need a model in five spacetime dimensions to build a realistic brane-world by thick walls. The models should have discrete SUSY vacua for domain walls. This can be achieved either by a SUSY gauge theories interacting with hypermultiplets, or by nonlinear sigma models of hypermultiplets. As a gauge theory, one can take a SUSY $U(N_c)$ gauged theory with $N_f > N_c$ flavors of hypermultiplets in the fundamental representation. If the hypermultiplet masses are nondegenerate and the $U(1)$ factor group of $U(N_c)$ has the Fayet-Iliopoulos (FI) terms, the model exhibits discrete SUSY vacua [25].

For simplicity we will consider the SUSY $U(1)$ gauge theory with $N_f(\geq 2)$ hypermultiplets. The vector multiplet consists of a five-dimensional gauge field $A_M$, a symplectic Majorana fermion $\Lambda^i$ which satisfies the symplectic Majorana condition $\Lambda^i = \epsilon^{ij}C\bar{\Lambda}^T_j$ in
five dimensions \(^2\) and a real adjoint (neutral) scalar field \(\Sigma\). The hypermultiplets consist of charged scalar fields \(H^i_A\) and their fermionic superpartners \(\Psi_A\). The \(SU(2)_R\) doublet index \(i\) runs \(i = 1, 2\) and the flavor index \(A\) runs from 1 to \(N_f\). Notice that the real degrees of freedom is eight both for the symplectic Majorana fermion \(\Lambda^i\) and for the Dirac spinor \(\Psi_A\) in five dimensions. We denote the gauge coupling by \(e\). The Lagrangian is given by

\[
\mathcal{L} = -\frac{1}{4e^2} F^{MN}F_{MN} - \frac{1}{2e^2} \partial_M \Sigma \partial^M \Sigma - \sum_{A=1}^{N_f} \mathcal{D}^M H^*_A \mathcal{D}_M H^i_A - \frac{i}{e^2} \sum_{A=1}^{N_f} \bar{\Psi}_A \Gamma^M \mathcal{D}_M \Psi_A + \sum_{A=1}^{N_f} \left[ i \sqrt{2} \varepsilon_{ij} \bar{\Psi}_A \Lambda^i H^j_A - i \sqrt{2} \varepsilon^{ij} H^*_A \bar{\Lambda}_j \Psi_A - (\Sigma - \mu_A) \bar{\Psi}_A \Psi_A \right] - V,
\]

(2.20)

\[
V = \frac{e^2}{2} \sum_{a=1}^{3} \left( -2 \xi \delta^a_3 + \sum_{A=1}^{N_f} H^*_A (\sigma^a)^j H^j_A \right)^2 + \sum_{A=1}^{N_f} (\Sigma + \mu_A)^2 H^*_A H^i_A,
\]

(2.21)

where \(\mu_A\) is the mass of the \(A\)-th hypermultiplet and the covariant derivatives are

\[
\mathcal{D}_M H^i_A = (\partial_M + i A_M) H^i_A, \quad \mathcal{D}_M \Psi_A = (\partial_M + i A_M) \Psi_A.
\]

(2.22)

The \(SU(2)_R\) triplet FI parameters are chosen to lie in the third direction and is denoted as \(\xi\).

Let us first examine SUSY vacua. We denote the \(SU(2)_R\) components of the hypermultiplets as

\[
H^i_A = \begin{pmatrix} H_A \\ H^*_A \end{pmatrix}, \quad H^i = \begin{pmatrix} H^*_A \\ H_A \end{pmatrix}.
\]

(2.23)

Choosing nondegenerate mass parameters \(\mu_A \neq \mu_B\), we obtain \(N_f\) SUSY vacua in the Higgs phase. The \(A\)-the vacuum is given by

\[
\Sigma = -\mu_A, \quad H^*_{B} = 0, \quad |H_B|^2 = 2\xi \delta^3_B \quad (B = 1, 2, \ldots, N_f).
\]

(2.24)

Therefore we expect the existence of (multi) BPS domain walls which interpolates a pair of these discrete Higgs vacua. The minimal model admitting such a BPS domain wall is the case of \(N_f = 2\), which will be considered from now on.

Even with this simple model, it is generally difficult to obtain exact wall solutions for the case of finite gauge coupling \cite{26}, \cite{27}. Although we will consider also finite

\(^2\)The conventions of gamma matrices and the spinors in five dimensions are given in Appendix A.
gauge coupling $e$ later, it is sufficient to examine the case of infinite gauge coupling to study domain walls. We will see that we can obtain exact solutions in the infinite gauge coupling limit not only for BPS single wall configurations but also for non-BPS multi-wall configurations. As we let gauge coupling to infinity $e \to \infty$, the kinetic term of vector multiplet in the Lagrangian vanishes. At the same time, the scalar potential becomes infinitely steep and the hypermultiplets are constrained to be at the minimum

$$
\sum_{A=1}^{N_f} H_A^* H_A = \sum_{A=1}^{N_f} H_A^* H_A^* = 0, \quad \sum_{A=1}^{N_f} (|H_A|^2 - |H_A^c|^2) = 2\xi.
$$

(2.25)

The gauge field $A_M$ and the adjoint scalar field $\Sigma$ in the vector multiplet become Lagrange multiplier fields which can be eliminated to give the reduced Lagrangian $\mathcal{L}_\infty$ at infinite coupling $e \to \infty$

$$
\mathcal{L}_\infty = - \sum_{A=1}^{N_f} \left( |\partial_M H_A|^2 + |\partial^M H_A^c|^2 \right) + \frac{\left[ \sum_{A=1}^{N_f} \left( H_A^* \overleftrightarrow{\partial_M} H_A + H_A^c \overleftrightarrow{\partial_M} H_A^* \right) \right]^2}{4 \sum_{A=1}^{N_f} (|H_A|^2 + |H_A^c|^2)}
\left. - \sum_{A=1}^{N_f} \left( \mu_A^2 |H_A|^2 + \mu_A^2 |H_A^c|^2 \right) + \frac{\left[ \sum_{A=1}^{N_f} \left( \mu_A |H_A|^2 + \mu_A |H_A^c|^2 \right) \right]^2}{4 \sum_{A=1}^{N_f} (|H_A|^2 + |H_A^c|^2)}, \right.
$$

(2.26)

where we denote $X \overleftrightarrow{\partial_M} Y \equiv X \partial_M Y - Y \partial_M X$. Taking the infinite gauge coupling limit $e \to \infty$ of the SUSY gauge theory with massive hypermultiplets gives a nonlinear sigma model with a potential term as seen above. This is called the massive hyper Kähler quotient method [13, 14, 17, 18].

The simplest model with $N_f = 2$ hypermultiplets is a nonlinear sigma model with $T^*(\mathbb{C}P^1)$ as target space and with an appropriate potential, which is called the massive $T^*(\mathbb{C}P^1)$ model. To solve the constraint for hypermultiplet scalars in Eq. (2.25), we introduce spherical coordinates $(R, \Omega, \Theta, \Phi)$ as four independent variables [15, 18, 17, 11, 16]

$$
H_1 = g(R) \cos \left( \frac{\Theta}{2} \right) \exp \left( \frac{i}{2} (\Omega + \Phi) \right),
$$

(2.27)

$$
H_2 = g(R) \sin \left( \frac{\Theta}{2} \right) \exp \left( \frac{i}{2} (\Omega - \Phi) \right),
$$

(2.28)

$$
H_1^* = f(R) \sin \left( \frac{\Theta}{2} \right) \exp \left( - \frac{i}{2} (\Omega - \Phi) \right),
$$

(2.29)

$$
H_2^* = - f(R) \cos \left( \frac{\Theta}{2} \right) \exp \left( - \frac{i}{2} (\Omega + \Phi) \right).
$$

(2.30)

where $f(R)$ and $g(R)$ are given by

$$
f(R)^2 = -\xi + \sqrt{R^2 + \xi^2}, \quad g(R)^2 = \xi + \sqrt{R^2 + \xi^2}.
$$

(2.31)
The range of these variables are usually taken as

\[ 0 \leq R < \infty, \quad 0 \leq \Theta \leq \pi, \quad 0 \leq \Phi \leq 2\pi, \quad 0 \leq \Omega \leq 2\pi. \] (2.32)

This is one of the standard parametrizations of \( T^\ast(\mathbb{C}P^1) \) manifold, which is also called Eguchi-Hanson manifold. In terms of these independent variables \((R, \Theta, \Omega, \Phi)\), the Lagrangian (2.26) reads

\[
\mathcal{L}_\infty = \frac{1}{2\sqrt{R^2 + \xi^2}} \left[ -\partial^M R \partial_M R - (R^2 + \xi^2) \partial_M \Theta \partial^M \Theta - (R^2 + \xi^2 \sin^2 \Theta) \partial_M \Phi \partial^M \Phi \\
- R^2 \partial_M \Omega \partial^M \Omega - 2R^2 \cos \Theta \partial_M \Phi \partial^M \Omega - \mu^2 \left( R^2 + \xi^2 \sin^2 \Theta \right) \right].
\] (2.33)

Since a common mass of hypermultiplets can be absorbed by a shift of vector multiplet scalar \( \Sigma \), we set \( \mu_1 = -\mu_2 = \frac{\mu}{2} \) here.

Let us notice that \((R, \Omega)\) parametrize the fiber of \( T^\ast(\mathbb{C}P^1) \) and the submanifold defined by \((R = 0, \Omega = 0)\) is the base space \( \mathbb{C}P^1 \). If we truncate the manifold to the base manifold, it is a Kähler manifold \( \mathbb{C}P^1 \), which is just a sphere \( S^2 \) with the radius \( \xi \). Two coordinates \( \Theta \) and \( \Phi \) correspond to the latitude and the longitude of the sphere, as illustrated in Fig. 3.

The scalar potential on the sphere is given by

\[ V_{CP^1} = \xi \frac{\mu^2}{2} \sin^2 \Theta, \] (2.34)

which has two isolated SUSY vacua at the north and south pole of the sphere.

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Figure 3: The target space of truncated Eguchi-Hanson manifold and the scalar potential on the manifold. The isolated SUSY vacua are at the north pole and south pole.
2.2.2 The BPS and non-BPS domain walls in five dimensions

In this subsection we obtain BPS and non-BPS domain wall configurations in the nonlinear sigma model $T^*({\mathbb{C}P^1})$. Similarly to the sine-Gordon model, the Bogomolny bound can be obtained:

$$E = \int dy \frac{1}{2\sqrt{R^2 + \xi^2}} \left[ (R^2 + \xi^2) (\Theta' \pm \mu \sin \Theta)^2 + (R^2 + \xi^2) \Phi'^2 \sin^2 \Theta \
+ (R' \mp \mu R \cos \Theta)^2 + R^2 (\Omega' + \Phi' \cos \Theta)^2 \
\pm 2\mu R R' \cos \Theta \mp 2\mu (R^2 + \xi^2) \Theta' \sin \Theta \right] \geq \pm \mu \left[ \sqrt{R^2 + \xi^2} \cos \Theta \right]_{-\infty}^{\infty}.$$  (2.35)

This energy bound is saturated when the following BPS equations are satisfied:

$$\Theta' = \pm \mu \sin \Theta, \quad \Phi' \sin \Theta = 0, \quad (2.36)$$

$$R' = \mp \mu R \cos \Theta, \quad R (\Omega' + \Phi' \cos \Theta) = 0, \quad (2.37)$$

where we assume the background configuration to depend only on the extra dimension coordinate $y$. Since we are now interested in BPS wall solutions interpolating between two SUSY vacua with $R = 0, \Omega = 0$, we assume $R = 0, \Omega = 0$ for the wall configurations. Then the BPS equations reduce to

$$\Theta' = \pm \mu \sin \Theta, \quad \Phi' \cdot \sin \Theta = 0. \quad (2.38)$$

From the second equation, $\Phi$ must be a constant. Thus the BPS solution which interpolates the two isolated vacua at the north pole and south pole of $S^2$, is on a great circle $\Phi = \text{const}$. Notice that this BPS equation for $\Theta$ is identical to the BPS equation (2.7) for $\Theta$ in the sine-Gordon model. Therefore we obtain the BPS wall solution:

$$\Theta(y; y_0) = \pm \sin^{-1} (\tanh \mu (y - y_0)) + \frac{\pi}{2}, \quad \Phi = \text{const}., \quad (2.39)$$

where $y_0$ is the position of the wall.

Now we will look for the non-BPS solutions, consisting of a BPS wall and an anti-BPS wall. Inspired by the BPS wall solutions, let us consider solutions of equations of motion with vanishing values and derivatives for $R, \Omega$ as their initial conditions at some $y$. Then the equations of motion dictates $R = \Omega = 0$ for all values of $y$. Therefore we can truncate the $T^*({\mathbb{C}P^1})$ model, and can consider $\mathbb{C}P^1$ model effectively.

$$\mathcal{L}_{\text{boson}}^{\mathbb{C}P^1} = \frac{\xi}{2} \left( -\partial_M \Theta \partial^M \Theta - \sin^2 \Theta \partial_M \Phi \partial^M \Phi - \mu^2 \sin^2 \Theta \right). \quad (2.40)$$
Figure 4: The non-BPS wall solutions are shown. The left figure shows the periodic solution with $k > 1$, the middle shows BPS or anti-BPS solution with $k = 1$ and the right shows quasi-periodic solution with $k < 1$.

Assuming $\Theta$ and $\Phi$ to depend on $y$ only, their equations of motion becomes:

$$\Theta'' - \left(\Phi'^2 + \mu^2\right) \sin \Theta \cos \Theta = 0,$$

(2.41)

$$[\sin^2 \Theta \Phi']' = 0.$$

(2.42)

Similarly to $R, \Omega$, we can consider initial conditions $\Phi' = 0$. Then $\Phi$ becomes constant, and the equation of motion for $\Theta$ reduces to

$$\Theta'' = \mu^2 \sin \Theta \cos \Theta.$$  

(2.43)

This is identical to the equation of motion for the real part of the sine-Gordon model in Eq.(2.10). Therefore we obtain a non-BPS solution

$$\Theta(y; y_0, k) = \text{am} \left(\frac{\mu}{k} (y - y_0), k\right) + \frac{\pi}{2}, \quad \Phi = \text{const}..$$  

(2.44)

We can compactify the base space by $2\pi L = 4kK(k)/\mu$, similarly to the sine-Gordon model.\footnote{Alternative choices are $2n\pi L = 4kK(k)/\Lambda$, $n = 1, 2, \cdots$, corresponding to $n$ pairs of BPS wall and anti-BPS wall placed with equal interval in the fundamental region $2\pi L$.} Besides $y_0$ representing the position of the wall, this solution has one more parameter $k$. In the case of $k > 1$ the solution curve never reaches either vacuum at north and south poles, and oscillates in an interval between them. In the case of $k = 1$ the solution corresponds to the BPS or anti-BPS solution in Eq.(2.39) which interpolates north pole and south pole once. In the case of $k < 1$ the solution passes through both vacua and becomes quasi-periodic. Similarly to the sine-Gordon case, this solution represents the BPS wall and the anti-BPS wall placed at $y_0$ and at $y_0 + \pi L$, respectively.
Table 1: Correspondence between the sine-Gordon model in four dimensions and $T^*(\mathbb{CP}^1)$ model in five dimensions.

| sine-Gordon model | $T^*(\mathbb{CP}^1)$ model |
|-------------------|-----------------------------|
| $\text{Re}(a) = \frac{A}{g} \text{am} \left( \frac{\mu}{k}(y - y_0), k \right)$ | $\Theta = \text{am} \left( \frac{\mu}{k}(y - y_0), k \right) + \frac{\pi}{2}$ |
| $\text{Im}(a) = 0$ | $\Phi = \text{const.}$ |
| nothing | $R = 0$ |
| nothing | $\Omega = 0$ |

depicted in Fig. 4. The solution and the model have similarities and differences with those of the sine-Gordon model in four dimensions, as listed in Table 1. In terms of the original hypermultiplet variables $H_A, H_A^c$, the above non-BPS solution is expressed as

$$H_1 = \sqrt{2} \xi \cos \left[ \frac{1}{2} \text{am} \left( \frac{\mu}{k}(y - y_0), k \right) + \frac{\pi}{4} \right], \quad (2.45)$$

$$H_2 = \sqrt{2} \xi \sin \left[ \frac{1}{2} \text{am} \left( \frac{\mu}{k}(y - y_0), k \right) + \frac{\pi}{4} \right], \quad (2.46)$$

$$H_1^c = H_2^c = 0. \quad (2.47)$$

### 2.3 Domain walls in the finite gauge coupling

Here we review BPS solutions at finite gauge coupling [27], and examine the non-BPS case also, to obtain an idea of how the domain wall solutions are modified at finite gauge coupling. The bosonic Lagrangian with the finite gauge coupling is given by

$$\mathcal{L}_{\text{boson}} = -\frac{1}{4e^2} F^{MN} F_{MN} - \frac{1}{2e^2} \partial^M \Sigma \partial_M \Sigma - \sum_A \left( |D^M H_A|^2 - |D^M H_A^c|^2 \right) - V, \quad (2.48)$$

$$V = 2e^2 \left( \sum_A H_A H_A^c \right)^2 + e^2 \left( \sum_A |H_A|^2 - |H_A^c|^2 - 2\xi \right)^2$$

$$\quad + \sum_A (\Sigma + \mu_A)^2 \left( |H_A|^2 + |H_A^c|^2 \right). \quad (2.49)$$

Similarly to the infinite coupling case, BPS wall solution should be obtained with $H^c = 0$ and $A_M = 0$, since it interpolates between two SUSY vacua with $H^c = 0$ respecting the four-dimensional Poincaré invariance. Assuming that the scalars $H_1$ and $H_2$ depend on $y$ only, we find that the above Lagrangian reduces to

$$\mathcal{L}_{\text{boson}} = -\frac{1}{2e^2} (\Sigma')^2 - |H_1'|^2 - |H_2|^2 - V, \quad (2.50)$$

$$V = \frac{e^2}{2} \left( |H_1|^2 + |H_2|^2 - 2\xi \right)^2 + \left( \Sigma + \frac{\mu}{2} \right)^2 |H_1|^2 + \left( \Sigma - \frac{\mu}{2} \right)^2 |H_2|^2. \quad (2.51)$$
The tension of the domain wall is given by
\[
E = \int dy \left[ \frac{1}{2e^2} (\Sigma')^2 + |H_1'|^2 + |H_2'|^2 + V \right]
\]
\[
= \int dy \left[ \frac{1}{2e^2} \left\{ \Sigma' + e^2 (|H_1|^2 + |H_2|^2 - 2\xi) \right\}^2 + |H_1' + \left( \frac{\mu}{2} \right) H_1|^2 \right]
\]
\[
+ \int dy \left[ |H_2' + \left( \frac{\Sigma}{2} - \frac{\mu}{2} \right) H_2|^2 + \left\{ \left( \Sigma + \frac{\mu}{2} \right) |H_1|^2 + \left( \Sigma - \frac{\mu}{2} \right) |H_2|^2 - 2\xi \right\} \right]^2 \right]
\]
\[
\geq \left[ - (\Sigma + \mu)|H_1|^2 - (\Sigma - \mu)|H_2|^2 + 2\xi \right]_{-\infty}^{\infty}.
\]
(2.52)

The BPS solution saturates this inequality by satisfying the BPS equation:
\[
\frac{1}{e^2} \Sigma' = - (|H_1|^2 + |H_2|^2 - 2\xi),
\]
(2.53)
\[
H_1' = - \left( \Sigma + \frac{\mu}{2} \right) H_1,
\]
(2.54)
\[
H_2' = - \left( \Sigma - \frac{\mu}{2} \right) H_2.
\]
(2.55)

In the limit of infinite gauge coupling \( e \to \infty \), Eq. (2.53) yields a condition
\[
|H_1|^2 + |H_2|^2 = 2\xi,
\]
(2.56)
and Eq. (2.54) and Eq. (2.55) yields
\[
(H_1 - H_2)' = - \frac{\mu}{2} (H_1 - H_2), \quad \Sigma = - \frac{(H_1 + H_2)'}{H_1 + H_2}.
\]
(2.57)

These can be easily solved. In order to show that these BPS equations are equivalent to the BPS equations (2.36) and (2.37) obtained previously, we just have to change variables
\[
H_1 = \sqrt{2\xi} \cos \frac{\Theta}{2}, \quad H_2 = \sqrt{2\xi} \sin \frac{\Theta}{2}.
\]
(2.58)

Then the above BPS equations with infinite gauge coupling are rewritten as
\[
\Theta' = \mu \sin \Theta, \quad \Sigma = - \frac{\mu}{2} \cos \Theta.
\]
(2.59)

We obtain the same BPS solution as (2.39) for the infinite gauge coupling
\[
\Theta = \sin^{-1} \tanh \mu (y - y_0) + \frac{\pi}{2}, \quad \Sigma = \frac{\mu}{2} \tanh \mu (y - y_0),
\]
(2.60)
\[
H_1 = \sqrt{2\xi} \left( \frac{e^{-\mu y}}{e^{\mu y} + e^{-\mu y}} \right)^{\frac{1}{2}}, \quad H_2 = \sqrt{2\xi} \left( \frac{e^{\mu y}}{e^{\mu y} + e^{-\mu y}} \right)^{\frac{1}{2}}.
\]
(2.61)
Let us now turn to the case of finite gauge coupling. We have to solve the above three BPS equations Eq. (2.53), (2.54) and (2.55) for finite gauge coupling $e$. Let us change variables

$$H_1 = \sqrt{2\xi e^\rho} \cos \Theta/2, \quad H_2 = \sqrt{2\xi e^\rho} \sin \Theta/2.$$  

(2.62)

Then the above BPS equations reduce to

$$\frac{1}{e^2} \Sigma' = -2\xi (e^{2\rho} - 1), \quad \rho' = -\Sigma - \frac{\mu}{2} \cos \Theta, \quad \Theta' = \mu \sin \Theta.$$ 

(2.63)

Notice that the equation for $\Theta$ is the same as the equation for the infinite gauge coupling. The solution has already given in Eq. (2.60). Combining the first two equation with this solution (2.60), we obtain an equation for $\rho$:

$$\rho'' - \frac{2}{\alpha^2} \left( e^{2\rho} - 1 \right) = \frac{1}{2 \cosh^2(u - u_0)},$$

(2.64)

where $u \equiv \mu y$, $\alpha^2 \equiv \frac{\mu^2}{\xi e^\rho}$ and a prime denotes a derivative in terms of $u$. The several exact solutions have already been found for several integer $\alpha$ [27]. For example, in the case of $\alpha = 2$ we obtain

$$\rho = \frac{1}{2} \log \left( \frac{\cosh \mu(y - y_0)}{1 + \cosh \mu(y - y_0)} \right).$$

(2.65)

Then we find as shown in Fig. 5

$$H_A^{(\alpha=2)} = \sqrt{\frac{\cosh \mu(y - y_0)}{1 + \cosh \mu(y - y_0)}} \times H_A^{(\alpha=0)},$$

(2.66)

$$\Sigma^{(\alpha=2)} = \Sigma^{(\alpha=0)} - \frac{\mu}{2} \frac{\tanh \mu(y - y_0)}{1 + \cosh \mu(y - y_0)}.$$ 

(2.67)

Let us finally explore non-BPS domain wall solutions in the case of finite gauge coupling. With the parametrization in Eq. (2.60), the Lagrangian can be rewritten as

$$\mathcal{L} = -\frac{1}{2e^2} \Sigma'' - 2\xi e^{2\rho} \left( \rho^2 + \frac{\Theta^2}{4} \right) - V,$$

(2.68)

$$V = 2e^2 \xi^2 (e^{2\rho} - 1)^2 + 2\xi e^{2\rho} \left( \Sigma^2 + \frac{\mu^2}{4} + \mu \Sigma \cos \Theta \right).$$ 

(2.69)

The equations of motion is given by

$$\alpha^2 \Sigma'' - 2e^{2\rho} \left( 2\Sigma + \cos \Theta \right) = 0,$$

(2.70)

$$\rho'' + \rho^2 - \frac{\Theta^2}{4} - \frac{2}{\alpha^2} \left( e^{2\rho} - 1 \right) - \left( \Sigma^2 + \frac{1}{4} + \Sigma \cos \Theta \right) = 0,$$

(2.71)

$$\frac{\Theta''}{2} + \rho' \Theta' + \Sigma \sin \Theta = 0,$$

(2.72)
Figure 5: The solid line is a BPS solution $H_1^{(\alpha=2)}$ with a finite gauge coupling $\alpha = 2$ and the broken line is a BPS solution $H_1^{(\alpha=0)}$ with the infinite gauge coupling $\alpha = 0$.

where we define $\tilde{\Sigma} = \Sigma/\mu$ and a prime denotes a derivative in terms of $u = \mu y$ in the rest of this subsection. One has to solve these coupled equations to obtain non-BPS solutions at finite gauge coupling. So far we have not succeeded to obtain a new solution. Let us finally examine the limit of infinite gauge coupling $\alpha \to 0$ of these equations. In such a limit the first and the second equations yield

$$\tilde{\Sigma} = -\frac{1}{2} \cos \Theta, \quad \rho = 0.$$ (2.73)

Inserting these into the last equation, we obtain the same equation as (2.43)

$$\Theta'' = \sin \Theta \cos \Theta.$$ (2.74)

3 Stability

3.1 Stability under small fluctuation

To examine the stability of the exact non-BPS domain wall solution (2.44) in the massive $T^*(\mathbb{C}P^1)$ nonlinear sigma model in five dimensions, we first study small fluctuations $(r, \theta, \varphi, \omega)$ around the background $\Theta_0(y) = am \left( \frac{A}{k} (y - y_0), k \right) + \frac{\pi}{2}$ and $\Phi_0 = \text{const}$. 

$$\Theta(x^m, y) = \Theta_0(y) + \theta(x^m, y), \quad \Phi(x^m, y) = \Phi_0 + \varphi(x^m, y),$$ (3.1)

$$R(x^m, y) = r(x^m, y), \quad \Omega(x^m, y) = \omega(x^m, y).$$ (3.2)

The part of the Lagrangian quadratic in the fluctuations is decomposed into a sum for each fields

$$\mathcal{L}_{\text{boson}}^{(2)} = \mathcal{L}_{\text{boson}}^{(\theta, 2)} + \mathcal{L}_{\text{boson}}^{(\varphi, 2)} + \mathcal{L}_{\text{boson}}^{(r, 2)}.$$ (3.3)
\[ \mathcal{L}_{\text{boson}}^{(\theta,2)} = \int dy \xi \left\{ -\frac{1}{2} \partial^M \theta \partial_M \theta - \frac{\mu^2}{2} \cos 2\Theta_0 \theta^2 \right\}, \quad (3.4) \]

\[ \mathcal{L}_{\text{boson}}^{(\varphi,2)} = \int dy \xi \left\{ -\frac{1}{2} \sin^2 \Theta_0 \partial^M \varphi \partial_M \varphi \right\}, \quad (3.5) \]

\[ \mathcal{L}_{\text{boson}}^{(r,2)} = \int dy \frac{1}{\xi} \left\{ -\frac{1}{2} \partial^M r \partial_M r - \frac{1}{2} \left( \mu^2 + \frac{1}{2} \Theta_0^\prime r^2 - \frac{\mu^2}{2} \sin^2 \Theta_0 \right) r^2 \right\}. \quad (3.6) \]

The linearized equations of motion read

\[ \left( \partial^m \partial_m + \frac{\partial^2}{\partial y^2} - \mu^2 \cos 2\Theta_0 \right) \theta = 0, \quad (3.7) \]

\[ \sin^2 \Theta_0 \partial^m \partial_m \varphi + \frac{\partial}{\partial y} \left( \sin^2 \Theta_0 \frac{\partial}{\partial y} \varphi \right) = 0, \quad (3.8) \]

\[ \left( \partial^m \partial_m + \frac{\partial^2}{\partial y^2} - \mu^2 - \frac{1}{2} \Theta_0^\prime r^2 + \frac{\mu^2}{2} \sin^2 \Theta_0 \right) r = 0. \quad (3.9) \]

Let us note that fluctuation of \( \Omega \) disappears from the quadratic Lagrangian completely. Although this Lagrangian is sufficient to obtain light modes (those that become massless when radius goes to infinity), massive modes are expected from \( \Omega \) if we wish to respect the four SUSY in the BPS limit of infinite radius. We describe an attempt to recover massive modes from the fluctuations of \( \Omega \) by introducing a composite field \( R \Omega \) in Appendix B.

For \( \theta \) and \( r \), we can immediately define Shrödinger-type equations for mode functions \( \psi_A^{(n)} \) with mass squared \( m_{A,n}^2 \) of effective fields on world volume as eigenvalues

\[ \left[ -\frac{\partial^2}{\partial y^2} + \mathcal{V}_A(y) \right] \psi_A^{(n)}(y) = m_{A,n}^2 \psi_A^{(n)}(y), \quad A = \theta, r, \quad (3.10) \]

\[ \mathcal{V}_\theta(y) = \mu^2 \cos 2\Theta_0 = \mu^2 \left\{ 2\sin^2 \left( \frac{\mu}{k} (y - y_0), k \right) - 1 \right\}, \quad (3.11) \]

\[ \mathcal{V}_r(y) = \mu^2 + \frac{1}{2} \Theta_0^\prime r^2 - \frac{\mu^2}{2} \sin^2 \Theta_0 = \frac{1 + k^2}{2k^2} \mu^2. \quad (3.12) \]

For \( \varphi \), we need to eliminate the first derivative of \( \varphi \) in Eq.(3.8) in order to obtain a Shrödinger-type equation. This is achieved by defining a field \( \tilde{\varphi} \)

\[ \tilde{\varphi}(x, y) = \varphi(x, y) \sin \Theta_0(y), \quad (3.13) \]

\[ \left( \partial^m \partial_m - \frac{\partial^2}{\partial y^2} - \frac{\cos \Theta_0 \Theta_0^\prime}{\sin \Theta_0} \Theta_0^\prime + \left( \Theta_0^\prime \right)^2 \right) \tilde{\varphi} = 0. \quad (3.14) \]
We now define mode functions $\psi^{(n)} \equiv \psi^{(n)} \sin \Theta_0$ for the potential $V_\phi(y)$ yielding mass squared $m^2_{\phi,n}$ of the $n$-th effective fields of $\phi$

$$\left[-\frac{\partial^2}{\partial y^2} + V_\phi(y)\right] \psi^{(n)}(y) = m^2_{\phi,n} \psi^{(n)}(y), \quad (3.15)$$

$$V_\phi(y) = \frac{\cos \Theta_0 \Theta''_0 - (\Theta'_0)^2}{\sin \Theta_0} = \mu^2 \left\{ 2\sin^2 \left(\frac{\mu k}{k} (y - y_0), k\right) - \frac{1}{k^2} \right\}. \quad (3.16)$$

We will first solve these eigenvalue equations (3.10) and (3.15), and later study their normalizability in order to determine the physical modes among these solutions. If we replace $\mu$ by $\Lambda$, the Schrödinger potential (3.11) is identical to the potential (2.12) for $\theta$ in the sine-Gordon model in four dimensions. Therefore we obtain the same exact solutions for low-lying mode functions with normalization factors $N$’s:

$$m^2_{\theta,0} = 0, \quad \psi^{(0)}_\theta = N^{(0)}_\theta \text{dn} \left(\frac{\mu k}{k} (y - y_0), k\right), \quad (3.17)$$

$$m^2_{\theta,1} = \frac{1 - k^2}{k^2} \mu^2, \quad \psi^{(1)}_\theta = N^{(1)}_\theta \text{cn} \left(\frac{\mu k}{k} (y - y_0), k\right), \quad (3.18)$$

$$m^2_{\theta,2} = \frac{1}{k^2} \mu^2, \quad \psi^{(2)}_\theta = N^{(2)}_\theta \text{sn} \left(\frac{\mu k}{k} (y - y_0), k\right). \quad (3.19)$$

Eq. (3.16) shows that the potential for $\phi$ is identical to that for $\theta$ except a constant shift: $V_\phi = V_\theta - \frac{1 - k^2}{k^2} \mu^2$. Therefore the same eigenfunctions as $\theta$ solve the eigenvalue problem for $\phi$ and the corresponding mass squared are shifted accordingly

$$m^2_{\phi,-1} = -\frac{1 - k^2}{k^2} \mu^2, \quad \psi^{(-1)}_\phi = N^{(-1)}_\phi \text{dn} \left(\frac{\mu k}{k} (y - y_0), k\right), \quad (3.20)$$

$$m^2_{\phi,0} = 0, \quad \psi^{(0)}_\phi = N^{(0)}_\phi \text{cn} \left(\frac{\mu k}{k} (y - y_0), k\right), \quad (3.21)$$

$$m^2_{\phi,1} = \mu^2, \quad \psi^{(1)}_\phi = N^{(1)}_\phi \text{sn} \left(\frac{\mu k}{k} (y - y_0), k\right). \quad (3.22)$$

In contrast to the case of $\theta$, the solution (3.20) at first sight appears to indicate instability of the background solution for $k < 1$ (with unit winding number), contrary to our expectations. However, we will see below that the possible tachyonic mode $\psi^{(-1)}_\phi$ is not normalizable and unphysical in our case of $T^*(\mathbb{C}P^1)$ model in five dimensions. The mode functions for $r$ can be completely obtained by plane waves with mass squared spectra

$$m^2_{r,n} = \frac{1 + k^2}{2k^2} \mu^2 + \left(\frac{n}{R}\right)^2, \quad n = 0, 1, 2, \ldots. \quad (3.23)$$

Now let us determine physical modes by requiring the normalizability of these modes in the effective action. We expand the fluctuations in the quadratic Lagrangian (3.3)
by means of mode functions \( \psi^{(n)}_{A}(y) \) with effective fields \( f^{(n)}_{A}(x^{m}) \) as their coefficients \( (A = \theta, \varphi, r) \)

\[
\theta(x^m, y) = \sum_{n} \psi^{(n)}_{\theta}(y) f^{(n)}_{\theta}(x^m), \quad \varphi(x^m, y) = \sum_{n} \psi^{(n)}_{\varphi}(y) f^{(n)}_{\varphi}(x^m), \tag{3.24}
\]

\[
r(x^m, y) = \sum_{n} \psi^{(n)}_{r}(y) f^{(n)}_{r}(x^m), \tag{3.25}
\]

where we use \( \psi^{(n)}_{\varphi}(y) = \psi^{(n)}_{\varphi}(y) / \sin \Theta_0(y) \). Apart from a trivial renaming of parameters, the quadratic Lagrangian \((3.4)\) for \( \theta \) is identical to that for \( \theta \) in sine-Gordon model in four dimensions \([9]\). Therefore we conclude that these modes of \( \theta \) fluctuations are all physical.

In the case of \( k \leq 1 \) where we have quasi-periodic solution, we obtain no tachyonic mode, so that the background configuration is stable for the fluctuation of \( \theta \). In the \( k > 1 \) case, there is a tachyonic mode which destabilizes the background configuration. These results are identical to the four-dimensional case \([9]\).

The quadratic Lagrangian for \( \varphi \) consists of kinetic and mass terms

\[
\mathcal{L}^{(\varphi, 2)}_{\text{boson}} = \mathcal{L}^{(\varphi)}_{\text{kin}} + \mathcal{L}^{(\varphi)}_{\text{mass}} \tag{3.26}
\]

By means of the expansion \((3.24)\), the kinetic term is given by

\[
\mathcal{L}^{(\varphi)}_{\text{kin}} = -\frac{\xi}{2} \sum_{k,l} \int dy \, \partial^m f^{(k)}_{\varphi}(x) \partial_m f^{(l)}_{\varphi}(x) \psi^{(k)}_{\varphi}(y) \psi^{(l)}_{\varphi}(y). \tag{3.27}
\]

This gives the canonically normalized kinetic terms for effective fields

\[
\mathcal{L}^{(\varphi, 2)}_{\text{boson}} = -\frac{1}{2} \sum_{n} \partial^m f^{(n)}_{\varphi}(x) \partial_m f^{(n)}_{\varphi}(x), \tag{3.28}
\]

if the following normalization condition is satisfied

\[
\int dy \, \psi^{(n)}_{\varphi}(y) \psi^{(l)}_{\varphi}(y) = \frac{1}{\xi} \delta^{nl}. \tag{3.29}
\]

The mass term, however, requires partial integrations in order to be transformed into the Schrödinger-type operator \((3.10)\),

\[
\mathcal{L}^{(\varphi)}_{\text{mass}} = -\frac{\xi}{2} \int dy \, \left( \sin \Theta_0 \partial_y \varphi \right)^2 = -\frac{\xi}{2} \int dy \, \left( \frac{\partial_y \varphi - \left( \frac{\sin \Theta_0}{\sin \Theta_0} \right) \varphi}{\sin \Theta_0} \right)^2
\]

\[
= -\frac{\xi}{2} \int dy \, \varphi \left( -\partial_y^2 + V_{\varphi}(y) \right) \varphi - \frac{\xi}{2} \left[ \frac{\partial \varphi}{\partial y} - \left( \frac{\sin \Theta_0}{\sin \Theta_0} \right) \varphi \right]^2 \pi L. \tag{3.30}
\]

The first term of the last equation reduces to the mass terms for effective fields by using the eigenvalue equation \((3.10)\) and the orthonormality condition \((3.29)\)

\[
\mathcal{L}^{(\varphi)}_{\text{mass}} = -\frac{1}{2} \sum_{n} \left( m_n f^{(n)}_{\varphi}(x) \right)^2. \tag{3.31}
\]
Therefore the normalizability of the mode functions is equivalent to the vanishing of the surface term in Eq. (3.30). The solutions (3.20)–(3.22) for lower mass squared eigenvalues satisfy the orthonormality conditions. We find, however, that the surface term diverges for \( \psi_{\varphi}^{(-1)} \) in Eq. (3.20). Note that one has to evaluate the integral carefully by separating the integration region at \( \mu(y - y_0)/k = K(k), 3K(k) \) in performing the partial integrations, since the integrand has singularities there. Therefore the possible tachyonic mode \( \psi_{\varphi}^{(-1)} \) is not normalizable and hence unphysical. Other modes turn out to be normalizable and physical.

Let us give a more intuitive explanation for the absence of the tachyonic modes. The contribution of the fluctuation \( \varphi \) to the tension of the wall is nonnegative and vanishes if and only if \( \varphi = \text{const} \).

\[
E_{\varphi} = \sin^2 \Theta_0 \left( \dot{\varphi}^2 + (\vec{\nabla} \varphi)^2 \right) \geq 0. \tag{3.32}
\]

Therefore any fluctuation \( \varphi \) around our background configuration \( \Phi_0 = \text{const} \) increases the tension. This explains the classical stability of our non-BPS background configuration against the small fluctuation \( \varphi \).

Since the mode functions (3.23) of the remaining fluctuations \( r \) are obviously normalizable and consist of massive modes only, we conclude that our non-BPS two-wall solution is stable against small fluctuations.

### 3.2 Large fluctuations and topological aspect

In this subsection we will show the instability of our non-BPS solution with respect to large fluctuations, by considering the topology of the model. Especially we would like to clarify differences between the four SUSY sine-Gordon model in four dimensions and the eight SUSY \( T^\ast(\mathbb{C}P^1) \) model in five dimensions. We will also propose to use a model with the double cover of \( T^\ast(\mathbb{C}P^1) \) manifold to assure the topological stability of our non-BPS solution.

Let us first recall the situation of the four-SUSY sine-Gordon model. In the sine-Gordon model, the real part \( \Theta \) of the chiral scalar field is naturally a compact variable, taking values on \( \Theta \in [0, 2\pi) \simeq S^1 \). Then the model acquires a topological quantum number \( \pi_1(S^1) \) if we compactify the base space \( y \sim y + 2\pi L \) with the radius \( 2\pi L \equiv 4nkK(k)/\Lambda, (n \in \mathbb{N}) \). Since the non-BPS solution (2.11) is quasi-periodic for \( k < 1 \), the nontrivial topological quantum number \( n \in \pi_1(S^1) \) assures the stability of the non-BPS solution (2.11) for \( k < 1 \) even under large fluctuations. On the other hand, the
non-BPS solution is periodic for \( k > 1 \) and has a vanishing topological quantum number corresponding to \( n \) pairs of walls and anti-walls. They are unstable even under small fluctuations. The cases \( k > 1, k = 1, k < 1 \) are illustrated and compared in Fig. 4. The \( k < 1 \) solution with \( n \in \pi_1(S^1) \) has \( n \) BPS walls and \( n \) anti-BPS walls alternately. Since non-BPS configurations should have higher energy than the sum of two BPS walls, the BPS wall and the anti-BPS wall tend to exert repulsive force each other, resulting in wall positions at equal intervals on \( S^1 \). This intuitive explanation is in accord with our result that the non-BPS multi-wall configuration with \( k < 1 \) is stable under large as well as small fluctuations.

Let us turn our attention to the stability of the non-BPS solution (2.44) in \( T^*(\mathbb{C}P^1) \) model. Although the field \( \Theta \) in \( T^*(\mathbb{C}P^1) \) model parametrize a circle \( S^1 \), similarly to the \( \Theta \) in the sine-Gordon model, this circle is just a great circle \( S^2 \simeq \mathbb{C}P^1 \subset T^*(\mathbb{C}P^1) \). Unlike \( S^1 \), \( S^2 \) is homotopically trivial \( \pi_1(S^2) = 0 \). Therefore the stability of our non-BPS solution (2.44) is not supported by topological quantum numbers in the case of \( T^*(\mathbb{C}P^1) \) model. Although we have already verified that our non-BPS solution (2.44) is stable against the small fluctuations, we still need to examine a possibility that the solution may be unstable under large fluctuations.

To verify the instability under large deformations, we will examine a continuous deformation which makes the wall path shrinking to a point on \( S^2 \), in the spirit of variational approach. We will verify below that the energy of such a configuration shows local minimum around our non-BPS solution but eventually leading to true vacuum configuration without walls after passing over a maximum. This at least shows the existence of a continuous deformation of our non-BPS solution leading to no walls at all.

For simplicity, we consider a path on \( S^2 \) which cuts off our non-BPS solution at \( \Theta \), turns around a circle of \( \Phi \) rotating by \( \pi \) with the constant \( \Theta \), and going to back through our non-BPS solution reflected at \( \Theta = \pi \):

\[
\Theta_1(u; u_1) = \begin{cases} 
\text{am}(u, k) + \frac{\pi}{2} & -K < u \leq u_1, \\
\Theta_* \equiv \text{am}(u_1, k) + \frac{\pi}{2} & u_1 < u \leq 2K - u_1, \\
\frac{3\pi}{2} - \text{am}(u, k) & 2K - u_1 < u \leq 3K,
\end{cases}
\]

\[
\Phi_1(u; u_1) = \begin{cases} 
0 & -K < u \leq u_1, \\
\frac{\pi}{2} \frac{u - K}{K - u_1} + \frac{\pi}{2} & u_1 < u \leq 2K - u_1, \\
\pi & 2K - u_1 < u \leq 3K,
\end{cases}
\]
where we have defined a variable

$$u \equiv \frac{\mu}{k}(y - y_0),$$

(3.35)

and $u_*$ denotes the position in extra dimension corresponding to the value $\Theta = \Theta_*$. This path is shown in Fig. 6. This path $(\Theta_1, \Phi_1)$ interpolates the energy of our non-BPS solution $(\Theta_0, \Phi_0 = 0)$ at $\Theta_* \to \pi$ and the true vacuum $(\Theta = 0, \Phi = 0)$ without walls at $\Theta_* \to 0$. This is because the rotation of $\Phi$ by $\pi$ becomes irrelevant when $\Theta = 0, \pi$, since all values of $\Phi$ corresponds to a single point at north or south poles.

The energy of the configuration is given by

$$E = \int_{-K}^{3K} du \frac{\mu \xi}{2k} \left[ (\partial_u \Theta)^2 + ((\partial_u \Phi)^2 + k^2) \sin^2 \Theta \right].$$

(3.36)

The energy of the above trial function is given by

$$E(u_*) = 2E_0(u_*) + 2E_1(u_*),$$

(3.37)

where we define

$$E_0(u_*) = \frac{\mu \xi}{2k} \int_{-K}^{u_*} du \left[ (\partial_u \Theta_0)^2 + ((\partial_u \Phi_0)^2 + k^2) \sin^2 \Theta_0 \right],$$

$$= \frac{\mu \xi}{2k} \int_{-K}^{u_*} du \left[ 2dn^2(u, k) + k^2 - 1 \right],$$

$$= \frac{\mu \xi}{2k} \left[ E(u_*) - E(-K) + (k^2 - 1)(u_* + K) \right],$$

(3.38)
Figure 7: The energy $E$ of trial function as a function of $\ell \equiv K - u_*$.  

$$E_1(u_*) \equiv \frac{\mu \xi}{2k} \int_{u_*}^{K} du \left[ \Theta_1^2 + \left( \Phi_1'^2 + k^2 \right) \sin^2 \Theta_1 \right]$$

$$= \frac{\mu \xi}{2k} \int_{u_*}^{K} du \left[ \left( \frac{\pi}{2K - u_*} \right)^2 + k^2 \right] \text{cn}^2(u_*, k)$$

$$= \frac{\mu \xi}{2k} \left[ \frac{\pi^2}{4} \frac{1}{K - u_*} + k^2(K - u_*) \right] \text{cn}^2(u_*, k),$$

which is shown in Fig. 7. We observe that the energy of the path of the continuous deformation has a local minimum at our non-BPS solution, in accordance with our result of no tachyon under small fluctuations. It then shows a maximum before reaching to the absolute minimum at the true vacuum. We regard this result as an evidence for the instability under large fluctuations.

Although it may be enough to have metastability of our non-BPS solution with sufficiently long lifetime compared to the lifetime of our universe, it is certainly desirable to obtain topological stability under large fluctuations. We can give the topological stability if we consider a double cover of the manifold $T^* (\mathbb{C}P^1)$. For simplicity, let us explain in the case of the $\mathbb{C}P^1 \sim S^2$ model. If we take two spheres $S^2$, and identify the north (south) pole of one sphere $S^2_{(1)}$ with the south (north) pole of the other sphere $S^2_{(2)}$, we obtain a double cover of $S^2$. This construction is illustrated in the right of Fig. 8. The fundamental region and our non-BPS solution are shown in the left upper of Fig. 8 for $\mathbb{C}P^1$. 

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model and the left lower for the double cover of $\mathbb{CP}^1$. In this model, we can go over to the other sphere only through poles. Moreover, the path leading to the original sphere after going through the second sphere wind around the two spheres and cannot be deformed to nothing. Therefore we obtain the topological quantum number $\pi_1 = \mathbb{Z}$. On the other hand, all the local properties such as curvature are unaffected by such an identification. We believe that there is no global obstructions either. This consideration can be applied not only to the $\mathbb{CP}^1$ model, but also to the $T^*(\mathbb{CP}^1)$ model. An interesting by-product of this double-cover model is that we can understand the reason why we have found no tachyons under small fluctuations of our non-BPS solution in the $T^*(\mathbb{CP}^1)$ model. This is because the quadratic Lagrangian of small fluctuations depends on only local properties of the model and should be identical to our stable model of double cover of $T^*(\mathbb{CP}^1)$.

4 SUSY breaking

If we take a limit $L \to \infty$ with fixed $y_0$, we obtain a single BPS wall placed at $y_0$. On the other hand, if we take a limit $L \to \infty$ with fixed $y_0 + \pi L$, we obtain a single anti-BPS wall placed at $y_0 + \pi L$. The BPS wall solution preserves half of SUSY, say $Q_1$, and breaks the other half, $Q_2$. The anti-BPS wall solution preserves $Q_2$, and breaks $Q_1$. Therefore the coexistence of these two walls in our non-BPS two solution breaks eight SUSY completely [8, 9]. From the brane world viewpoint, it is interesting to study how SUSY breaking effects are generated on a wall by the existence of the other walls. In usual SUSY breaking scenarios in the brane world, there are two 3-branes called the...
“hidden brane” and the “visible brane” in the higher dimensional spacetime. Once SUSY is broken by the vacuum expectation values of auxiliary fields of some supermultiplets in the hidden brane, SUSY breaking effects are mediated to the visible brane by bulk fields interacting with both branes. Then, soft SUSY breaking terms of MSSM fields on the visible brane are generated. In this framework, various fields have to be added on the hidden brane and/or in the bulk by hand to break SUSY and to transmit the SUSY breaking effects to our world. Furthermore, mechanisms of radius stabilization have to be specified to be phenomenologically viable. On the other hand, we have no need to add extra fields mentioned above since the non-BPS configuration itself breaks SUSY and the fields forming the non-BPS wall are responsible for SUSY breaking and its transmission to our world. As shown in the previous section, our non-BPS wall configuration is stable at least under small fluctuations. Therefore there is no need to introduce an additional mechanism to stabilize the radius. Moreover it acquires a topological stability under large fluctuations as well if we consider double cover of $T^\ast(\mathbb{C}P^1)$ model. In the light of these facts, it is worth while studying how SUSY breaking arises in our model.

Let us first understand the symmetry reason for the low-lying bosonic KK modes. In particular we are interested in those modes obtained in Eqs. (3.17), (3.18), (3.21), that are massless in the limit of large radius. Two zero modes $\theta(0)$ and $\varphi(0)$ are the Nambu-Goldstone modes corresponding to the broken spacetime and internal rotation symmetry, respectively. On the other hand, $\theta(1)$ represents fluctuations of relative distance of two walls, which we call the breather mode. Similarly to the four-dimensional case, the wave function of $\theta(0)$ and $\theta(1)$ are peaked at two walls, but they have opposite sign around the anti-BPS wall located at $y = y_0 + \pi L$. Therefore the sum $(\theta(0) + \theta(1))/\sqrt{2}$ is localized at the BPS wall, and the difference $(\theta(0) - \theta(1))/\sqrt{2}$ at the anti-BPS wall. When we take the infinite radius limit of $k \to 1$, the mass of breather mode vanishes.

There are also fermionic zero modes which are dictated by symmetry reason. In the limit of $L \to \infty$ with $y_0 + \frac{\pi L}{2}$ fixed, we obtain the BPS wall which breaks half of SUSY, $Q_2$. Therefore the corresponding Nambu-Goldstone fermion $f_0^{(2)}$ is localized on the BPS wall. Similarly, the anti-BPS wall is obtained by taking the $L \to \infty$ limit with $y_0 + \frac{3\pi L}{2}$ fixed, and the corresponding Nambu-Goldstone fermion $f_0^{(1)}$ is localized on the anti-BPS wall. Since our non-BPS solution breaks all eight SUSY, we have two fermionic zero modes $f_0^{(1)}$ and $f_0^{(2)}$. The Nambu-Goldstone fermion $f_0^{(2)}$ is localized on the BPS wall, and another Nambu-Goldstone fermion $f_0^{(1)}$ is localized on the anti-BPS wall.

If we fix $y_0 + \frac{\pi L}{2}$ in taking the $L \to \infty$ limit, we find the mode $(\theta(0) + \theta(1))/\sqrt{2}$ as the surviving massless bosonic mode. This becomes the superpartner of the massless
Nambu-Goldstone fermion $f^{(2)}_0$ corresponding to the SUSY broken by the BPS wall. On the other hand, if we fix $y_0 + \frac{3\pi L}{2}$ in taking the $L \to \infty$ limit, the mode $(\theta^{(0)} - \theta^{(1)})/\sqrt{2}$ is the surviving massless boson which becomes the superpartner of the Nambu-Goldstone fermion $f^{(1)}_0$ corresponding to the SUSY broken by the anti-BPS wall.

These situations are precisely analogous to those of $\theta$ in the four SUSY sine-Gordon model in four dimensions [9]. In that case, the BPS wall (anti-BPS wall) preserves two SUSY. Since the representation of two SUSY requires only a real scalar field, the real scalar field $(\theta^{(0)} + \theta^{(1)})/\sqrt{2}$ is sufficient as a scalar component of the low energy effective theory on the $1 + 2$ dimensional world volume. On the contrary, four SUSY should be preserved by the BPS wall in our five-dimensional model. The minimal representation of four SUSY requires a chiral scalar multiplet which contains a complex scalar field. Thus there must be one more real scalar mode in addition to $\theta^{(0)}$. This is supplied by $\varphi^{(0)}$, which is the massless Nambu-Goldstone boson corresponding to the broken internal rotation symmetry, since SUSY demands equal mass for the real and imaginary parts of the scalar component of the chiral multiplet. Another supporting evidence comes from the fact that the wave function $\tilde{\varphi}^{(0)}$ becomes identical to that of the wave function $(\theta^{(0)} + \theta^{(1)})/\sqrt{2}$ or $(\theta^{(0)} - \theta^{(1)})/\sqrt{2}$ (apart from the sign), in the $L \to \infty$ limit with $y_0$ or $y_0 + \pi L$ fixed. It is interesting to observe that the wave functions become identical to $(\theta^{(0)} + \theta^{(1)})/\sqrt{2}$ or $(\theta^{(0)} - \theta^{(1)})/\sqrt{2}$, only if the redefined field $\tilde{\varphi}$ is used instead of the original field $\varphi$. It is not easy to recognize the localization in terms of the original field $\varphi$, since its wave function $\psi^{(0)}_\varphi$ is constant. However, we can easily see that the wave function $\psi^{(0)}_{\tilde{\varphi}}$ in Eq. (3.21) is identical to that for $\psi^{(0)}_\theta$ in Eq. (3.17) and is localized, because of the nontrivial weight for the inner product of $\varphi$.

In the limit of the infinite distance of walls $L \to \infty$ ($k \to 1$), both fermionic and bosonic light modes become massless. However, as the walls approach each other, the bosonic field $\theta^{(1)}$ acquires a nonvanishing mass squared because of SUSY breaking. The mass splitting $\Delta m^2 \equiv m^2_{\text{boson}} - m^2_{\text{fermion}}$ is simply given by $\Delta m^2 = m^2_{\theta,1} = \frac{1-k^2}{k^2} \mu^2$. This mass splitting can be related to the distance between the walls by noting $2\pi \mu L = 4kK(k)$ where $K(k)$ is the complete elliptic integral of first kind. In the limit $k \to 1$, we obtain $K(k) \to \frac{1}{2} \log \left(\frac{1}{1-k^2}\right)$, leading to

$$\Delta m^2 = \mu^2 \frac{e^{-\pi \mu L}}{1-e^{-\pi \mu L}} \approx \mu^2 e^{-\pi \mu L}. \quad (4.1)$$

The mass splitting is exponentially suppressed as a function of the distance $\pi L$ between walls. If one considers the case with $L \to \infty$, the mass splitting vanishes, one recovers the single wall case which preserves the four SUSY. In this way, the result (4.1) is consistent
with our physical understanding. This result is also phenomenologically fascinating in
that the low SUSY breaking scale can be naturally generated from the five-dimensional
Planck scale $\mu \sim O(M_5)$ without an extreme fine-tuning of parameters.

The qualitative features of SUSY breaking is the same for the sine-Gordon model
in four dimensions \cite{9}. The exponentially suppressed mass splitting has already been
obtained in the sine-Gordon model, which seems to be generic in this kind of models.
The difference is the multiplet structure on a wall. In the sine-Gordon model in four
dimensions, the multiplet is real since the three-dimensional theory on a wall preserves
only two SUSY. In the present $T^*(\mathbb{C}P^1)$ model in five dimensions, the multiplet should
contains a complex scalar since the four-dimensional theory on a wall preserves four SUSY.

The exponentially suppressed mass splitting has also been discussed in a model of
SUSY warped compactification \cite{28}. The twisted boundary condition at $y = \pi L$ brane
generates the tree level mass splitting of order $e^{-\pi L/l}/l$ where $l$ is a length scale of the
$AdS_5$ and $L$ is a compactification radius. The suppression factor originates from the warp
factor of the metric in the model \cite{28} and the mass splitting vanishes in the flat limit
$l \rightarrow \infty$. On the other hand, the suppression factor in our case comes from the nontrivial
nature of the background configuration and the SUSY breaking effects are present already
in the purely rigid SUSY theory.

We can embed our model with rigid SUSY into five-dimensional supergravity. In
fact, we have considered a similar problem in the case of four-dimensional SUSY sine-
Gordon model and found that the non-BPS two-wall solution is stable even when it is
embedded into supergravity at least for weak gravitational coupling\cite{22}. The massless
Nambu-Goldstone modes are absorbed by a Higgs mechanism into massive gauge fields.
The first massive boson (the breather mode) of the rigid SUSY model becomes the lightest
scalar field, which is usually called radion, in the supergravity model. Since we found
that massless modes are precisely those expected from the spontaneous breaking of global
symmetries, and the remaining light fields are the breather modes corresponding to the
fluctuations of the distance between the walls. This situation is completely analogous to
the above four-dimensional sine-Gordon model. Therefore we anticipate that the same
reasoning will be applicable for the stability of our solution embedded into supergravity:
our non-BPS solution is stable even in the presence of gravity, and the breather mode
gives the lightest scalar field, the radion if the gravitational coupling is weak. The only
difference compared to four-dimensional model is that the field content is richer in order
to represent the more symmetry, such as twice larger numbers of SUSY.
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A Gamma matrix and spinor in \( d = 4, 5 \) dimensions

A.1 Four dimensions

A.1.1 The Clifford algebra and the Lorentz algebra

The Clifford algebra in four dimensions is of the form:

\[
\{\gamma_m, \gamma_n\} = -2\eta_{mn} \times 1, \tag{A.1}
\]

where \( m, n \) run from 0 to 3 and \( \eta_{mn} = \text{diag.}(-1, 1, 1, 1) \). One of the representation of the Clifford algebra is given by

\[
\gamma_m = \begin{pmatrix}
0 & \sigma_m \\
\bar{\sigma}_m & 0
\end{pmatrix}, \tag{A.2}
\]

where \( \sigma_m = (1, \sigma_i), \bar{\sigma}_m = (1, -\sigma_i) \). The chirality matrix is defined as

\[
\gamma \equiv \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix}
-i & 0 \\
0 & i
\end{pmatrix}. \tag{A.3}
\]

This chirality matrix anticommute with all the gamma matrices.

A representation of the Lorentz algebra is given by these gamma matrices:

\[
M_{mn} \equiv \frac{i}{4} [\gamma_m, \gamma_n]. \tag{A.4}
\]

These satisfy the Lorentz algebra

\[
[M_{kl}, M_{mn}] = i (\eta_{km} M_{ln} - \eta_{kn} M_{lm} - \eta_{lm} M_{kn} + \eta_{ln} M_{km}). \tag{A.5}
\]

Notice that if \( \gamma_m \) satisfies the above Clifford algebra, \(-\gamma_m, \gamma_m^*, -\gamma_m^*, \gamma_m^T, -\gamma_m^T, \gamma_m, -\gamma_m\) also satisfy the same Clifford algebra. These belong to the same equivalence class of the
Clifford algebra, so there exist non singular matrices which intertwine $\gamma_m$ and others. Let us define these intertwiners as

\begin{align*}
(4) A \gamma_m A^{-1} &= \gamma_m^\dagger, & (4) B \gamma_m B^{-1} &= -\gamma_m, & (4) C^{-1} \gamma_m C &= -\gamma_m^T, & (4) D^{-1} \gamma_m D &= -\gamma_m^*. \tag{A.6}
\end{align*}

We can find other intertwiners by combining $A, B, C$ and $D$. In our representation (A.2) we obtain

\begin{align*}
(4) A &= \gamma_0, & (4) B &= \gamma, & (4) C &= \begin{pmatrix} i \sigma_2 & 0 \\ 0 & i \sigma_2 \end{pmatrix}, & (4) D &= (4) (4) \left( \begin{array}{cc} 0 & i \bar{\sigma}_2 \\ i \sigma_2 & 0 \end{array} \right). \tag{A.7}
\end{align*}

### A.1.2 Four component spinors: Dirac, Majorana, Weyl spinors

Dirac spinor $\Psi$ is defined as a representation vector of the spinor representation of the Lorentz group:

$$
\Psi \rightarrow \exp \left( -\frac{i}{2} \theta^{mn} M_{mn} \right) \Psi. \tag{A.8}
$$

Let us also define Dirac conjugate spinor of $\Psi$ by

$$
\bar{\Psi} \equiv \Psi^\dagger A = \Psi^\dagger \gamma_0. \tag{A.9}
$$

Because of the relation such as $(4) M_{mn} A^{-1} = M_{mn}^\dagger$, the Dirac conjugate spinor transforms as follows:

$$
\bar{\Psi} \rightarrow \bar{\Psi} \exp \left( \frac{i}{2} \theta^{mn} M_{mn} \right). \tag{A.10}
$$

So we can easily find that $\bar{\Psi} \Psi$ is scalar under the Lorentz transformation.

The above representation $M_{mn}$ of the Lorentz algebra is not irreducible. There are two ways to obtain irreducible representations from this representation. One of them is called Majorana spinor representation. To define the Majorana spinor, we first need to define charge conjugate of the Dirac spinor by

$$
\Psi^C \equiv (4) D \psi^* = (4) (4) \left( A \Psi^T \right)^* = (4) C \bar{\Psi}^T. \tag{A.11}
$$

Notice that the transformation law of the charge conjugate spinor $\Psi^C$ is the same as $\Psi$ because of the relation such as $(4) D^{-1} M_{mn} D = -M_{mn}^*$:

$$
\Psi^C \rightarrow \exp \left( -\frac{i}{2} \theta^{mn} M_{mn} \right) \Psi^C. \tag{A.12}
$$
Taking this property into account, we can consistently define the Majorana spinor by

\[ \Psi = \Psi^C. \]  \hspace{1cm} (A.13)

Another possibility of the irreducible representation of the Lorentz algebra is called Weyl spinor representation. In order to define the Weyl spinor, we first introduce a projection operator:

\[ \mathcal{P}^{(\pm)} = \frac{1 \pm iB}{2} = \frac{1 \pm i\gamma}{2}. \]  \hspace{1cm} (A.14)

We can decompose the Dirac spinor into two kinds of Weyl spinors:

\[ \Psi^{(+)} \equiv \mathcal{P}^{(+)}\Psi, \quad \Psi^{(-)} \equiv \mathcal{P}^{(-)}\Psi. \]  \hspace{1cm} (A.15)

The representation matrix of the Lorentz group is given by

\[ M^{(\pm)}_{mn} \equiv \mathcal{P}^{(\pm)}M_{mn}. \]  \hspace{1cm} (A.16)

Each of them also forms a representation of the Lorentz algebra. More explicitly we obtain

\[ M^{(+)}_{mn} = \begin{pmatrix} \frac{i}{4} (\sigma_m \bar{\sigma}_n - \sigma_n \bar{\sigma}_m) & 0 \\ 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} \Sigma^{(+)}_{mn} & 0 \\ 0 & 0 \end{pmatrix}, \]  \hspace{1cm} (A.17)

\[ M^{(-)}_{mn} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{i}{4} (\bar{\sigma}_m \sigma_n - \bar{\sigma}_n \sigma_m) \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ 0 & \Sigma^{(-)}_{mn} \end{pmatrix}. \]  \hspace{1cm} (A.18)

### A.1.3 Two component spinor: Weyl spinor

In terms of the four component spinor notation, the Weyl spinors are represented as

\[ \Psi^{(+)} = \begin{pmatrix} \psi_1 \\ 0 \\ 0 \end{pmatrix}, \quad \Psi^{(-)} = \begin{pmatrix} 0 \\ 0 \\ \psi_1 \end{pmatrix}, \]  \hspace{1cm} (A.19)

where both \( \psi_1 \) and \( \psi_1 \) are two component complex spinors. It is obvious that the upper half components of \( \Psi^{(+)} \) or the lower half components of \( \Psi^{(-)} \) are enough for a representation of the Lorentz group. Transformation law of these Weyl spinors is given by

\[ \psi_1 \rightarrow \exp \left( -\frac{i}{2} \bar{\sigma}_m \sigma_n \Sigma^{(+)} \right) \psi_1, \]  \hspace{1cm} (A.20)

\[ \psi_1 \rightarrow \exp \left( -\frac{i}{2} \bar{\sigma}_m \sigma_n \Sigma^{(-)} \right) \psi_1. \]  \hspace{1cm} (A.21)
Notice that $\Sigma^{(+)}{}^m = \Sigma^{(-)}{}^m$. So $\psi_\uparrow{}^\dagger \psi_\uparrow$ and $\psi_\downarrow{}^\dagger \psi_\downarrow$ are scalar under the Lorentz transformation. Let us introduce a useful notation: dotted and undotted spinorial index. First we define index of $\sigma_m$ and $\bar{\sigma}_m$ as follows:

$$\sigma_m \equiv \sigma_{ma\dot{\alpha}}, \quad \bar{\sigma}_m \equiv \bar{\sigma}_{m\dot{\alpha}}. \quad \text{(A.22)}$$

This implies index structure of $\Sigma^{(+)}{}^m$ and $\Sigma^{(-)}{}^m$ as follows

$$\begin{align*}
(S^{(+)}{}^m)_{\alpha}^\beta &= \frac{i}{4} \left( \sigma_{m\alpha\beta}\bar{\sigma}_{n\dot{\alpha}\dot{\beta}} - \sigma_{n\alpha\dot{\beta}}\bar{\sigma}_{m\dot{\alpha}\beta} \right), \\
(S^{(-)}{}^m)_{\dot{\alpha}}^\dot{\beta} &= \frac{i}{4} \left( \bar{\sigma}_{m\alpha\dot{\beta}}\sigma_{n\beta\dot{\alpha}} - \bar{\sigma}_{n\beta\dot{\alpha}}\sigma_{m\alpha\dot{\beta}} \right). 
\end{align*} \quad \text{(A.23, A.24)}$$

We use a convention where undotted spinor indices are contracted like $\Downarrow$, and dotted spinor indices like $\Uparrow$. Thus we should take the index structure of $\psi_\uparrow$ and $\psi_\downarrow$ as

$$\psi_\uparrow = \psi_\uparrow{}^\alpha, \quad \psi_\downarrow = \psi_\downarrow{}^{\dot{\alpha}}. \quad \text{(A.25)}$$

The Lorentz algebra is isomorphic to $SL(2, C)$. The invariant tensor of $SL(2, C)$ is an anti-symmetric tensor $\epsilon^{\alpha\beta}$:

$$\epsilon^{\alpha\beta} \exp \left( -\frac{i}{2} \Theta^{kl} \Sigma^{(+)}{}_{kl} \right) \gamma^\alpha \exp \left( -\frac{i}{2} \Theta^{mn} \Sigma^{(+)}{}_{mn} \right) \gamma^\beta = \epsilon^{\gamma\delta}. \quad \text{(A.26)}$$

Therefore we find that

$$\epsilon^{\alpha\beta} \chi_\uparrow{}^\gamma \psi_\uparrow{}^\alpha \equiv \chi_\uparrow{}^\gamma \psi_\uparrow{}^\alpha \quad \text{(A.27)}$$

is a Lorentz scalar. Comparing $(\psi_\downarrow{}^\alpha)^\dagger \psi_\uparrow{}^\alpha$ and $\chi_\uparrow{}^\alpha \psi_\uparrow{}^\alpha$, we can identify

$$(\psi_\downarrow{}^\alpha)^\dagger = \psi_\uparrow{}^\alpha. \quad \text{(A.28)}$$

The Dirac spinor and Majorana spinor are represented by Weyl spinors as follows:

$$\Psi^D = \begin{pmatrix} \psi_\uparrow{}^\alpha \\ \chi_\downarrow{}^{\dot{\alpha}} \end{pmatrix}, \quad \Psi^M = \begin{pmatrix} \psi_\uparrow{}^\alpha \\ \bar{\psi}_\downarrow{}^{\dot{\alpha}} \end{pmatrix}, \quad \text{(A.29)}$$

where $\bar{\psi}_\uparrow$ is complex conjugate of $\psi_\uparrow$.

### A.2 Five dimensions

#### A.2.1 The Clifford algebra and the Lorentz algebra

Let us begin with the Clifford algebra in five dimensions

$$\{\Gamma_M, \Gamma_N \} = -2\eta_{MN} \times 1 \quad (M, N = 0, 1, 2, 3, 4). \quad \text{(A.30)}$$
where $\eta_{MN} = \text{diag.} \, (-1, 1, 1, 1, 1)$. We can easily construct a representation of this algebra by using a representation of the Clifford algebra in four dimensions as follows:

$$\Gamma_{M=m} = \gamma_m, \quad \Gamma_{M=4} = \gamma.$$  \hfill (A.31)

There is no chirality matrix in five-dimensional Clifford algebra, since the corresponding $\Gamma$ matrix is proportional to the unit matrix:

$$\Gamma \equiv \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 = -\gamma^2 = 1.$$  \hfill (A.32)

A representation of the Lorentz algebra is given by

$$M_{MN} = i/4 \left[ \Gamma_M, \Gamma_N \right].$$  \hfill (A.33)

Notice that there are 2 equivalence classes in $d=5$ Clifford algebra, since there are no regular matrices which intertwine $\Gamma_M$ and $-\Gamma_M$. The components of such 2 equivalence classes are

$$\left\{ \Gamma_M, \Gamma^\dagger_M, \Gamma^T_M, \Gamma^*_M \right\}, \quad \left\{ -\Gamma_M, -\Gamma^\dagger_M, -\Gamma^T_M, -\Gamma^*_M \right\}. \hfill (A.34)$$

Let us define intertwiners as follows:

$$(5)^{(5)} \Gamma_M A^{-1} = \Gamma^\dagger_M, \quad (5)^{(5)} \Gamma_M ^{-1} C = \Gamma^T_M, \quad (5)^{(5)} \Gamma_M ^{-1} D = \Gamma^*_M. \hfill (A.35)$$

For our representation (A.31) with (A.2) we find

$$(5)^{(5)} A = (4)^{(4)} A, \quad (5)^{(5)} C = iB^{-1}C, \quad (5)^{(5)} D = iB^{-1}D. \hfill (A.36)$$

### A.2.2 Dirac spinor

Similarly to the four-dimensional case, the Dirac spinor can be defined as

$$\Psi \rightarrow \exp \left( -i \frac{1}{2} \theta^{MN} M_{MN} \right) \Psi,$$  \hfill (A.37)

where $\Psi$ has four complex components, namely eight real degrees of freedom. Dirac conjugate of $\Psi$ is also defined by

$$\Psi \equiv \Psi^\dagger A. \hfill (A.38)$$

Because of $A = \hat{A}$ in Eq. (A.36), we can easily verify the following transformation law

$$\Psi \rightarrow \Psi \exp \left( i \frac{1}{2} \theta^{MN} M_{MN} \right). \hfill (A.39)$$
### A.2.3 Symplectic Majorana spinor

Unlike the case of four dimensions, neither Majorana nor Weyl condition can be imposed in the case of five dimensions. However, we can impose other condition, so-called symplectic Majorana condition. Let us consider 2 Dirac spinor $\Psi^i (i = 1, 2)$ and write its Dirac conjugate as $\bar{\Psi}_i$. We can define charge conjugate spinor by

$$\Psi^C_i \equiv C \bar{\Psi}^T_i.$$  \hfill (A.40)

Using the property

$$C M^T_{MN} C^{-1} = -M_{MN},$$  \hfill (A.41)

we can easily verify that the transformation law of $\Psi^C$ is the same as that of $\Psi$. So we can define the Symplectic Majorana spinor by

$$\Psi^i = \varepsilon^{ij} \Psi^C_j = \varepsilon^{ij} C \bar{\Psi}^T_j.$$  \hfill (A.42)

In terms of the two component spinors, the symplectic Majorana spinors are expressed as

$$\Psi^1 = \begin{pmatrix} \psi_\uparrow \alpha \\ \bar{\psi}_\downarrow \dot{\alpha} \end{pmatrix}, \quad \Psi^2 = \begin{pmatrix} \psi_\downarrow \alpha \\ -\bar{\psi}_\uparrow \dot{\alpha} \end{pmatrix}.$$  \hfill (A.43)

The following relations can also be verified:

$$\bar{\Psi} \Gamma^M \partial_M \Psi = \frac{1}{2} \bar{\Psi} \Gamma^M \partial_M \Psi^i;$$  \hfill (A.44)
$$\bar{\Psi}_i \Psi^i = 0;$$  \hfill (A.45)
$$\bar{\Psi}_i \Gamma^M \Psi^i = 0.$$  \hfill (A.46)

In terms of the symplectic Majorana spinor, the internal $SU(2)$ symmetry is manifest.

### B Massive modes of $\Omega$

As discussed in subsect 3.1, the field $\Omega$ disappears from the quadratic Lagrangian if $(R, \Phi, \Theta, \Omega)$ are used as independent fields for fluctuations. To respect the preserved four SUSY, we should have even number of real scalar fields as fluctuations. Therefore we consider a composite field $\omega \equiv R \Omega$ as elementary excitations, instead of $\Omega$ itself. This may perhaps be related to the fact that $\Omega$ becomes meaningful only when $R \neq 0$. In the limit of infinite radius $L \to \infty$, the non-BPS walls for the $T^*(\mathbb{C} \mathbb{P}^1)$ model should recover...
four SUSY. Therefore there should be four fields in the case of the $T^*\left(\mathbb{CP}^1\right)$ model, which become degenerate at least in the limit of $L \to \infty$.

Let us recall that the relevant part of Lagrangian $\mathcal{L}^\Omega$ for $\Omega$ gives a quadratic part $\mathcal{L}^{(2,\omega)}$ for the redefined field $\omega \equiv R\Omega$

$$\mathcal{L}^\Omega = -\frac{1}{2\xi} \left[ R^2 \partial_M \Omega \partial^M \Omega + 2R^2 \cos\Theta \partial_M \Phi \partial^M \Omega \right]$$

$$\rightarrow \mathcal{L}^{(2,\omega)} = -\frac{1}{2\xi} \left[ \partial_m \omega \partial^m \omega + \left(-\frac{R_0}{R_0} \omega + \omega' \right)^2 \right],$$

where $R_0, \Phi_0$ denote the classical solutions of equations of motion for $R$ and $\Phi$, and we used the fact that $\Phi_0$ is a constant. Let us note that our background solution may give nontrivial $R'/R_0$, since $R_0 = R'_0 = 0$. The linearized equation for $\omega$ is obtained from Eq.(B.2)

$$\left( \partial^m \partial_m + \frac{\partial^2}{\partial y^2} - \left(\frac{R'_0}{R_0}\right)^2 - \left(\frac{R'}{R_0}\right)' \right) \omega = 0.$$  

We will evaluate the potential term in Eq.(B.3) by analyzing the equation of motion for $R$. Since we are interested in solutions close to $R_0 \approx 0$, we can use the linearized equation of motion (B.9) together with Eq.(3.12) for $R_0$ and multiply it by $R'_0$ to obtain by Eq.(3.9)

$$R'_0 \partial^m \partial_m R_0 + \left(\frac{1}{2}(R'_0)^2\right)' - \frac{1 + k^2}{2k^2} \mu^2 \left(\frac{1}{2}R_0^2\right)' = 0.$$  

(B.4)

Since $R_0$ depends only on $y$, we obtain

$$(R'_0)^2 - \frac{1 + k^2}{2k^2} \mu^2 R_0^2 = \text{const} \equiv A.$$  

(B.5)

Using this expression, it is easy to calculate the potential for $\omega$ in Eq.(B.3),

$$\frac{R'_0}{R_0} = \pm \sqrt{\frac{1 + k^2}{2k^2} \mu^2 + \frac{A}{R_0^2}},$$

$$\left(\frac{R'_0}{R_0}\right)' = -\frac{A}{R_0^2},$$

(B.6)

(B.7)

leading to

$$\left(\frac{R'_0}{R_0}\right)^2 + \left(\frac{R'}{R_0}\right)' = \frac{1 + k^2}{2k^2} \mu^2.$$  

(B.8)

By inserting the result (B.8) into Eq.(B.3) shows that the linearized equation of motion for $\omega \equiv R\Omega$ completely agrees with the linearized equation of motion (3.9) together with (3.12) for $R$ in non-BPS case. Therefore, the mass spectra for $\omega$ and $R$ are identical. Note also that this result is valid irrespective of the value of the integration constant $A$. 

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Noting that our solution is $R_0 = 0$, we have to check whether the solution obtained from Eq. (B.5) is consistent with $R_0 = 0$. It is easy to solve Eq. (B.5) as

$$R_0(y) = \sqrt{-\frac{2A k^2}{(1 + k^2)\mu^2}} \cosh \left[ \pm \sqrt{\frac{1 + k^2}{2k^2} \mu^2} (y + C) \right] \quad (C : \text{const}). \quad \text{(B.9)}$$

When $A = 0$, $R_0 = 0$ is reproduced. Therefore the choice of the background solution $R_0 = R'_0 = 0$ corresponds to the integration constant $A = 0$.

Before closing Appendix B, we comment on BPS case. BPS case is obtained from the $k \to 1$ limit of non-BPS case. The agreement of the potential and the mass spectrum for $R$ and $\omega \equiv R\Omega$ in BPS case is obvious since the potential of $R$ and $R\Omega$ is $\frac{1 + k^2}{2k^2} \mu^2 \to \mu^2 \ (k \to 1)$. On the other hand, the consistency of BPS solution with $R_0 = 0$ can also be seen from BPS equations. Recall that BPS equations for $\Theta, R$ have already been given by the first equation of Eqs. (2.36) and (2.37),

$$\Theta'_0 = \pm \mu \sin \Theta_0, \quad R'_0 = \mp \mu R_0 \cos \Theta_0. \quad \text{(B.10)}$$

The BPS solution for $R_0$ is easily obtained by using the BPS solution (2.39) for $\Theta_0$

$$R_0(y; y_0) = C_0 \cosh[\mu(y - y_0)] \quad (C_0 : \text{const}). \quad \text{(B.11)}$$

Substituting this BPS solution into Eq. (B.5), the integration constant $A$ is fixed as

$$A = R_0^2(\tanh^2[\mu(y - y_0)] - 1) = \frac{-R_0^2}{\cosh^2[\mu(y - y_0)]} = -\mu^2 C_0^2. \quad \text{(B.12)}$$

To satisfy $R_0 = 0$, we find $C_0 = 0$ from (B.11), which again gives $A = 0$. This result is consistent with Eq. (B.5).

Summarizing this appendix, we have shown that the mass spectra of $R$ and $\omega \equiv R\Omega$ agree completely. It is also shown that the BPS and non-BPS solutions of $R$ obtained in this appendix is consistent with our solution $R_0 = 0$.

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