The Homotopy Theory of Cyclotomic Spectra

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Topology Seminar

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Overview

The homotopy theory of cyclotomic spectra

- Joint work with Andrew Blumberg
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- Preprint: arXiv:1303.1694 [math.KT]
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1. How do cyclotomic spectra show up?
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How do cyclotomic spectra show up? 

*THH* and *TC*
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1. How do cyclotomic spectra show up?
   THH and TC

2. What is a cyclotomic spectrum?
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The homotopy theory of cyclotomic spectra

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1. How do cyclotomic spectra show up?
   THH and TC

2. What is a cyclotomic spectrum?
   Some equivariant stable homotopy theory
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The homotopy theory of cyclotomic spectra

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   \( THH \) and \( TC \)

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1. How do cyclotomic spectra show up? 
   \( THH \) and \( TC \)

2. What is a cyclotomic spectrum? 
   Some equivariant stable homotopy theory

3. The homotopy theory of cyclotomic spectra

4. A new interpretation of \( TC \)
Hochschild Homology

Cyclic bar construction

\[ N_{q}^{cy} R = R \otimes \cdots \otimes R \otimes R \]

\[ \text{q factors} \]

Example: \( R = \mathbb{Z}[\pi] \)

\[ \tau_C = \tau_0 \times \]

\[ \times \text{ non-pos curved space} \]

\[ X \cong B \mathbb{K} \]

\[ HH_{\mathbb{K}}(\mathbb{Z}[\pi]) \cong H_{\mathbb{K}}(\wedge X) \]
Hochschild Homology

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Example: \( R = \mathbb{Z}[\pi] \)

Chain complex
Hochschild Homology

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Chain complex

Cyclic structure \( \Rightarrow \) Connes’ \( B \) operator \( \Rightarrow \) cyclic homology
Cyclic bar construction

\[ N_q^{cy} R = \underbrace{R \wedge \cdots \wedge R \wedge R}_{q \text{ factors}} \]

Example: \( R = H\mathbb{Z}[\pi] \) or \( R = \Sigma^\infty \Omega_+ \)

Spectrum
Cyclic structure \( \Rightarrow \)

\[ \pi_1 X = \pi_0 \Omega X \]
\[ \Omega \simeq \Omega X \]
\[ THH(\Sigma^\infty \Omega X+) \simeq \sum^\infty \wedge X_+ \]
Topological Hochschild Homology

Cyclic bar construction

\[ N^c_y \mathcal{R} = \underbrace{\mathcal{R} \wedge \cdots \wedge \mathcal{R}}_{q \text{ factors}} \wedge \mathcal{R} \]

Example: \( \mathcal{R} = H\mathbb{Z}[\pi] \) or \( \mathcal{R} = \Sigma^\infty \Omega_+ \)

Spectrum
Cyclic structure \( \Rightarrow \) circle group action
The Dennis Trace and Goodwillie’s Theorem

Trace map \( K_*(R) \to HH_*(R) \).

Factors through negative cyclic homology

\[
K_*(R) \to HN_*(R)
\]

For a map \( A \to B \), get a map on relative theories

\[
K_*(B, A) \to HN_*(B, A)
\]

Theorem (Goodwillie, 1986)

Let \( A \to \bar{B} \) be a surjection with nilpotent kernel. Then the map on relative theories

\[
K_*(\bar{B}, A) \to HN_*(\bar{B}, A)
\]

is an isomorphism after tensoring with \( \mathbb{Q} \).
The Dennis Trace and Goodwillie's Theorem

Trace map $K_\ast(R) \to HH_\ast(R)$.

Factors through negative cyclic homology

$$K_\ast(R) \to HN_\ast(R)$$

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Factors through negative cyclic homology

$$K_\ast(R) \to HN_\ast(R)$$

For a map $A \to B$, get a map on relative theories

$$K_\ast(B, A) \to HN_\ast(B, A)$$

**Theorem (Goodwillie, 1986)**

Let $A \to \tilde{B}$ be a surjection with nilpotent kernel. Then the map on relative theories

$$K_\ast(\tilde{B}, A) \to HN_\ast(\tilde{B}, A) \simeq H_{C_\ast-1}(B; A)$$

is an isomorphism after tensoring with $\mathbb{Q}$. 
The Cyclotomic Trace and McCarthy/Dundas Theorems

Question (Goodwillie)

Is there a corresponding theory that gets the $p$-adic information?

Bökstedt–Hsiang–Madsen constructed spectrum $TC$ and map

$$K(R) \rightarrow TC(R)$$

Conjecture (1989)

Let $A \rightarrow B$ be a map of rings and assume it is surjective with nilpotent kernel then $K(B, A) \rightarrow TC(B, A)$ is a weak equivalence after $p$-completion.
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Bökstedt–Hsiang–Madsen constructed spectrum $TC$ and map

$$K(R) \rightarrow TC(R)$$

Theorem (McCarthy/Dundas, 1997)
Let $A \rightarrow B$ be a map of ring spectra and assume it is surjective with nilpotent kernel on $\pi_0$ then $K(B, A) \rightarrow TC(B, A)$ is a weak equivalence after $p$-completion.
The table shows the relationships between algebraic and topological theories for Abelian Groups and Spectrum.

| Abelian Groups Spectrum | Algebra | Topology |
|-------------------------|---------|----------|
| Abelian Groups Spectrum | $HH_\ast$, $HH$, $HN_\ast$, $HN$ | $THH_\ast$, $THH$ |

The table indicates that $TC \neq THN$.
### TC is not THN

| Abelian Groups Spectrum | Algebra | Topology |
|-------------------------|---------|----------|
|                         | $\text{HH}_*$ | $\text{THH}_*$ |
|                         | $\text{HH}$   | $\text{THH}$   |
|                         | $\text{HN}_*$ | $\text{HN}$     |
|                         |           |             |

Instead involves fixed points and geometric fixed points for finite subgroups $C_n$. $T^C \neq \text{THH}$
**TC is not THN**

| Abelian Groups Spectrum | Algebra | Topology |
|-------------------------|---------|----------|
|                         | $HH_*$  | $THH_*$  |
|                         | $HH$    | $THH$    |
|                         | $HN_*$  |          |
|                         | $HN$    |          |
TC is not THN

| Algebra          | Topology          |
|------------------|-------------------|
| $HH_*$           | $THH_*$           |
| $HH$             | $THH$             |
| $HN_*$           | $THH_{\mathbb{T}}^{-*}(ET)$ |
| $HN$             | $THH^{h\mathbb{T}}$ |

Instead involves fixed points and geometric fixed points for finite subgroups $C_n$

$\mathbb{T} = \text{circle group}$
TC is not $THN$

| Algebra       | Topology                        |
|---------------|---------------------------------|
| $HH_*$        | $THH_*$                         |
| $HH$          | $THH$                           |
| $HN_*$        | $THH_{\mathbb{T}^*}(E_{\mathbb{T}})$ |
| $HN$          | $THH_{h\mathbb{T}}$             |

Instead involves **fixed points** and **geometric fixed points** for finite subgroups $C_n$

$\mathbb{T} = \text{circle group}$
**TC is not THN**

| Abelian Groups | Algebra | Topology |
|----------------|---------|----------|
| Spectrum       | $HH_*$  | $THH_*$  |
| Homotopy Fixed Points | $HH$ | $THH$ |
| Homotopy Fixed Point Spectrum | $HN_*$ | $THH_T^{-*}(E_T)$ |
|                 | $HN$    | $THH^{h_T}$ |

Instead involves **fixed points** and **geometric fixed points** for finite subgroups $C_n$

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|                           | Algebra    | Topology   |
|---------------------------|------------|------------|
| Abelian Groups Spectrum   | $HH_*$     | $THH_*$    |
| Homotopy Fixed Points     | $HH$       | $THH$      |
| Homotopy Fixed Point Spectrum | $HN_*$   | $THH_T^{-*}(E_T)$ |
|                           | $HN$       | $THH_T^{hT}$ |

$$TC \neq THH_T^{hT}$$

Instead involves **fixed points** and **geometric fixed points** for finite subgroups $C_n$

$T$ = circle group
$TC$ is not $THN$

|                        | Algebra | Topology               |
|------------------------|---------|------------------------|
| Abelian Groups         | $HH_*$  | $THH_*$                |
| Spectrum               | $HH$    | $THH$                  |
| Homotopy Fixed Points  | $HN_*$  | $THH\mathbb{T}^{-\ast}(E\mathbb{T})$ |
| Homotopy Fixed Point Spectrum | $HN$           | $THH_h\mathbb{T}$      |

Instead involves fixed points and geometric fixed points for finite subgroups $C_{p^m}$

$\mathbb{T} = \text{circle group}$
Fixed Points and Geometric Fixed Points

Example

For a $\mathbb{T}$-space $X$, what are the $C_p$-fixed points of $\Sigma^\infty X_+$?

Might want/expect them to be $\Sigma^\infty X_+^{C_p}$

Geometric Fixed Points

The geometric fixed point functor $\Phi^{C_p}$ has the property that

$$\Phi^{C_p} \Sigma^\infty X_+ = \Sigma^\infty X_+^{C_p}$$
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For a $\mathbb{T}$-space $X$, what are the $C_p$-fixed points of $\Sigma_\mathbb{T} \infty X_+$?

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Example

For a $\mathbb{T}$-space $X$, what are the $C_p$-fixed points of $\Sigma^\infty_{\mathbb{T}} X_+$?

Might want/expect them to be $\Sigma^\infty_{\mathbb{T}/C_p} X^C_p$.

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The **geometric fixed point functor** $\Phi^{C_p}$ has the property that

$$\Phi^{C_p} \Sigma^\infty_{\mathbb{T}} X_+ = \Sigma^\infty_{\mathbb{T}/C_p} X^C_p$$

Might want/expect them to be $C_p$-equivariant maps from $S^0$ to $\Sigma_{\mathbb{T}} X_+$

This is a different functor, the **fixed point functor** $(-)^{C_p}$.
Example

For a $\mathbb{T}$-space $X$, what are the $C_p$-fixed points of $\Sigma_\mathbb{T} X$?

Might want/expect them to be $\Sigma_{\mathbb{T}/C_p} X^C_p$.

Geometric Fixed Points

The geometric fixed point functor $\Phi^C_p$ has the property that

$$\Phi^C_p \Sigma_{\mathbb{T}} X = \Sigma_{\mathbb{T}/C_p} X^C_p$$

Might want/expect them to be $C_p$-equivariant maps from $S^0$ to $\Sigma_{\mathbb{T}} X$.

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Geometric Fixed Points

The geometric fixed point functor $\Phi^{C_p}$ has the property that

$$\Phi^{C_p} \Sigma_\mathbb{T} X_+ = \Sigma_{\mathbb{T}/C_p} X^C_p$$

Might want/expect them to be $C_p$-equivariant maps from $S^0$ to $\Sigma_\mathbb{T} X_+$

This is a different functor, the fixed point functor $(-)^{C_p}$. 
Look at the case of $S = \Sigma_{\mathbb{T}}^\infty S^0$.
$S^{C_p} = \text{spectrum of } C_p\text{-equivariant maps from } S \text{ to } S.$
Fixed Points

Look at the case of \( S = \Sigma_T S^0 \).

\( S^{C_p} \) = spectrum of \( C_p \)-equivariant maps from \( S \) to \( S \).

Non-equivariant map example
Fixed Points

Look at the case of $S = \Sigma_{\mathbb{T}} S^0$.

$S^{C_p} = \text{spectrum of } C_p\text{-equivariant maps from } S \text{ to } S$.

$C_2\text{-equivariant map example}$
Look at the case of $S = \Sigma_T^\infty S^0$.

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$C_2\text{-equivariant map examples}$
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$C_{2}\text{-equivariant map examples}$
Fixed Points and Geometric Fixed Points

Canonical map

\[ T^{C_p} \to \Phi^{C_p} T \]

For suspension spectra, this map is split (tom Dieck splitting). Other piece is suspension spectrum of \( C_p \) homotopy orbits.

Summary

For \( T = \sum_{T}^\infty X_+ \)

- Geometric fixed points \( \Phi^{C_p} T = \sum_{T/C_p}^\infty X^{C_p}_+ \)
- Fixed points \( T^{C_p} = \sum_{T/C_p}^\infty (X^{C_p} \amalg (E_{T/C_p} \times X))_+ \)
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- Fixed points \( T^{C_p} = \sum_{\mathbb{T}/C_p} \infty (X^{C_p} \amalg (E^{\mathbb{T}} \times_{C_p} X))_+ \)
Fixed Points and Geometric Fixed Points

Canonical map

$$T^{C_p} \rightarrow \Phi^{C_p} T$$

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For $T = \sum_{T}^\infty X_+$

- Geometric fixed points $\Phi^{C_p} T = \sum_{T/C_p}^\infty X_{C_p}^{C_p}$
- Fixed points $T^{C_p} = \sum_{T/C_p}^\infty (X^{C_p} \amalg (E_{T/C_p} \times_{C_p} X))_+$
Fixed Points and Geometric Fixed Points

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- Geometric fixed points \( \Phi^{C_p} T = \Sigma_{\mathbb{T}/C_p} X^{C_p}_+ \)
- Fixed points \( T^{C_p} = \Sigma_{\mathbb{T}/C_p} (X^{C_p} \amalg (E_{\mathbb{T}} \times_{C_p} X))_+ \)
Fixed Points and Geometric Fixed Points

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- Geometric fixed points \( \Phi^{C_p} T = \Sigma_{\mathbb{T}/C_p}^\infty X_{C_p}^+ \)
- Fixed points \( T^{C_p} = \Sigma_{\mathbb{T}/C_p}^\infty (X_{C_p}^+ \amalg (E_{\mathbb{T}} \times_{C_p} X))_+ \)

\[ T^{C_p^2} = \Sigma_{\mathbb{T}/C_p^2}^\infty (X_{C_p^2}^+ \amalg (E_{\mathbb{T}/C_p} \times_{C_p^2/C_p} X_{C_p}^+) \amalg (E_{\mathbb{T}} \times_{C_p^2} X))_+ \]
The Structure of $THH$

Example

$R = \sum_{\infty}^{\infty} \Omega_+, \Omega = \Omega X. \quad THH(R) \simeq \sum_{T}^{\infty} \Lambda X$,

$\Phi_{Cp} THH \simeq \sum_{T/Cp}^{\infty} (\Lambda X)_{Cp}$
The Structure of \( THH \)

**Example**

\[
R = \Sigma^\infty \Omega_+, \ \Omega = \Omega X.
\]

\[
THH(R) \simeq \Sigma^\infty \Lambda X_+.
\]

\[
\Phi^{C_p} THH \simeq \Sigma^\infty_{\mathbb{T}/C_p} (\Lambda X)^{C_p}_+.
\]
The Structure of $THH$

**Example**

$$R = \Sigma^\infty \Omega_+ , \Omega = \Omega X .$$

$$THH(R) \simeq \Sigma^\infty \Lambda X$$

$$\Phi^C_p \ THH \simeq \Sigma^\infty \frac{\Lambda X}{C_p}$$

$$\rho : \mathbb{T} \cong \mathbb{T}/C_p$$
The Structure of $THH$

**Example**

$$R = \Sigma^\infty \Omega_+, \ \Omega = \Omega X.$$  

$$THH(R) \simeq \Sigma^\infty \Lambda X$$

$$\Phi^{C_p} THH \simeq \Sigma^\infty_{T/C_p} (\Lambda X)^{C_p}$$

$$\rho^* (\Lambda X)^{C_p} \cong \Lambda X$$  

$$\implies \rho^* \Sigma^\infty_{T/C_p} (\Lambda X)^{C_p} \simeq \Sigma^\infty T \Lambda X$$

$$\rho: T \cong T/C_p$$
Example

\[ R = \Sigma^\infty \Omega_+, \ \Omega = \Omega X. \]

\[ \text{THH}(R) \simeq \Sigma^\infty \Lambda X \]

\[ \Phi^C_p \text{THH} \simeq \Sigma^\infty_{\mathbb{T}/C_p} (\Lambda X)^C_p \]

\[ \rho^* (\Lambda X)^C_p \simeq \Lambda X \quad \Rightarrow \quad \rho^* \Sigma^\infty_{\mathbb{T}/C_p} (\Lambda X)^C_p \simeq \Sigma^\infty \Lambda X_{\perp} \]

\[ \rho: \mathbb{T} \simeq \mathbb{T}/C_p \]
The Structure of \( THH \)

**Example**

\[
R = \Sigma^\infty \Omega_+ , \quad \Omega = \Omega X .
\]

\[
THH(R) \simeq \Sigma^\infty \Lambda X
\]

\[
\Phi^C_p \text{THH} \simeq \Sigma^\infty_{\mathbb{T}/C_p} (\Lambda X)^{C_p}
\]

\[
\rho^*(\Lambda X)^{C_p} \simeq \Lambda X \quad \implies \quad \rho^* \Sigma^\infty_{\mathbb{T}/C_p} (\Lambda X)^{C_p} \simeq \Sigma^\infty \Lambda X
\]

**Cyclotomic Structure**

\[
\rho^*: \Phi^C_p \text{THH}(R) \xrightarrow{\simeq} \text{THH}(R)
\]

\[
\rho: \mathbb{T} \simeq \mathbb{T}/C_p
\]
Construction of $TC$

The maps $R$ and $F$

\[
R : THH^{C_{p^m}} \rightarrow (\rho^* \Phi^{C_{p^m}} THH)^{C_{p^{m-1}}} \rightarrow THH^{C_{p^{m-1}}}
\]

\[
F : THH^{C_{p^m}} \rightarrow THH^{C_{p^{m-1}}}
\]

Definition

\[
TR = \text{holim}(\cdots \xrightarrow{R} THH^{C_{p^m}} \xrightarrow{R} \cdots \xrightarrow{R} THH)
\]

\[
TF = \text{holim}(\cdots \xrightarrow{F} THH^{C_{p^m}} \xrightarrow{F} \cdots \xrightarrow{F} THH)
\]

\[
TC = TR^{hF} \simeq TF^{hR}
\]
Construction of $TC$

The maps $R$ and $F$

$$R: \text{THH}^{C_{p^m}} \rightarrow (\rho^* \Phi^C_p \text{THH})^{C_{p^{m-1}}} \rightarrow \text{THH}^{C_{p^{m-1}}}$$

$$F: \text{THH}^{C_{p^m}} \rightarrow \text{THH}^{C_{p^{m-1}}}$$

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$$TR = \text{holim}(\cdots \xrightarrow{R} \text{THH}^{C_{p^m}} \xrightarrow{R} \cdots \xrightarrow{R} \text{THH})$$

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$$TC = TR^{hF} \simeq TF^{hR}$$
Construction of \( TC \)

The maps \( R \) and \( F \)

\[
R : \text{THH}^{C_p m} \to (\rho^* \Phi^{C_p} \text{THH})^{C_p m-1} \to \text{THH}^{C_p m-1}
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F : \text{THH}^{C_p m} \to \text{THH}^{C_p m-1}
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Construction of $TC$

The maps $R$ and $F$

$$R : \text{THH}^{C_p m} \to (\rho^* \Phi^{C_p} \text{THH})^{C_p m-1} \to \text{THH}^{C_p m-1}$$

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Definition

$$TR = \text{holim}(\cdots R \to \text{THH}^{C_p m} R \to \cdots R \to \text{THH})$$

$$TF = \text{holim}(\cdots F \to \text{THH}^{C_p m} F \to \cdots F \to \text{THH})$$

$$TC = TR^{hF} \simeq TF^{hR}$$
Construction of \( TC \)

The maps \( R \) and \( F \)

\[
R: \, \text{THH}^{C_{p^m}} \to (\rho^* \Phi^{C_p} \text{THH})^{C_{p^{m-1}}} \to \text{THH}^{C_{p^{m-1}}}
\]

\[
F: \, \text{THH}^{C_{p^m}} \to \text{THH}^{C_{p^{m-1}}}
\]

Definition

\[
TR = \text{holim}(\cdots \overset{R}{\to} \text{THH}^{C_{p^m}} \overset{R}{\to} \cdots \overset{R}{\to} \text{THH})
\]

\[
TF = \text{holim}(\cdots \overset{F}{\to} \text{THH}^{C_{p^m}} \overset{F}{\to} \cdots \overset{F}{\to} \text{THH})
\]

\[
TC = TR^{hF} \cong TF^{hR}
\]
Cofiber sequences

Hesselholt-Madsen: Computation of $K$-theory of local fields and proof of the Quillen-Lichtenbaum conjecture

Ausoni-Rognes: Program for understanding $A(\ast)$

Blumberg-Mandell: Localization sequence, Mayer-Vietoris, blow-up theorem, and projective bundle theorem for $TC$ of schemes.

What are maps of cyclotomic spectra?

Strictly commuting structure maps?

Homotopy commuting structure maps?

What is the set of homotopy classes of cyclotomic maps?

What is the homotopy type of the space/spectrum of cyclotomic maps?
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The Homotopy Theory of Cyclotomic Spectra

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- Homotopy commuting structure maps?
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- What is the homotopy type of the space/spectrum of cyclotomic maps?
Consists of a category $C$ having all finite limits and colimits
Together with three classes of maps, called cofibrations, fibrations, and weak equivalences
Such that:

1. Weak equivalences satisfy the 2-out-of-3 property
2. All three classes of maps are closed under retracts
3. Cofibrations have the left lifting property with respect to acyclic fibrations and fibrations have the left lifting property with respect to acyclic cofibrations
4. Every map factors as a cofibration followed by an acyclic fibration and as an acyclic cofibration followed by a fibration

$\Rightarrow$ abstract homotopy theory / homotopy category / good theory of derived functors / etc., etc., etc.
**Model Category**

Consists of a category $\mathcal{C}$ having all finite limits and colimits
Together with three classes of maps, called **cofibrations**, **fibrations**, and **weak equivalences**
Such that:

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A pre-cyclotomic spectrum is an orthogonal $\mathbb{T}$-spectrum $T$ together with a map of orthogonal $\mathbb{T}$-spectra

$$\rho^* \Phi \mathcal{C}_p T \to T.$$ 

A cyclotomic spectrum is a pre-cyclotomic spectrum for which the structure map is* a weak equivalence.

A map of pre-cyclotomic spectra is a map of orthogonal $\mathbb{T}$-spectra that commutes with the structure map.
The Model* Category of Cyclotomic Spectra

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$$\exists \ X \rightarrow X_{n}$$
The Model* Category of Cyclotomic Spectra

Definition

A weak equivalence of pre-cyclotomic spectra is a map that is an $\mathcal{F}_p$-equivalence of the underlying orthogonal $T$-spectra.

This is precisely a map that is a weak equivalence of non-equivariant spectra on $C_{p^m}$ geometric fixed points for all $m \geq 0$.

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$\mathcal{F}_p = \mathbb{Z}[C_{p^m}]$
The Model* Category of Cyclotomic Spectra

**Observation**

\[ FX = X \lor \rho^* \Phi^C_X \lor \rho^* \Phi^C_{\rho^* \Phi^C_X} \lor \rho^* \Phi^C_{\rho^* \Phi^C_{\rho^* \Phi^C_X}} \lor \cdots \]

is a monad on the category of orthogonal \( T \)-spectra.

The category of pre-cyclotomic spectra is the category of algebras over the monad \( F \).
The Model* Category of Cyclotomic Spectra

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\[ FX = X \lor \rho^* \Phi^C_p X \lor \rho^* \Phi^C_p (\rho^* \Phi^C_p X) \lor \rho^* \Phi^C_p (\rho^* \Phi^C_p (\rho^* \Phi^C_p X)) \lor \ldots \]

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The Model* Category of Cyclotomic Spectra

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is a monad on the category of orthogonal \( \mathbb{T} \)-spectra.

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Model* structure on pre-cyclotomic spectra created by \( \mathbb{F} \) from the \( \mathbb{F}_p \)-local model structure on orthogonal \( \mathbb{T} \)-spectra.

**Translation**

Cofibrations are built by attaching cells of the form

\[ \mathbb{F} \Sigma_+^\infty (S^{n-1} \times \mathbb{T} / C_p^m) \]

Fibrations are fibrations of the underlying orthogonal \( \mathbb{T} \)-spectra.
The Model* Category of Cyclotomic Spectra

**Observation**

\[ FX = X \lor \rho^* \Phi^C_p X \lor \rho^* \Phi^C_p (\rho^* \Phi^C_p X) \lor \rho^* \Phi^C_p (\rho^* \Phi^C_p (\rho^* \Phi^C_p X)) \lor \cdots \]

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Model* structure on pre-cyclotomic spectra created by \( F \) from the \( F_p \)-local model structure on orthogonal \( \mathbb{T} \)-spectra.

**Translation**

Cofibrations are built by attaching cells of the form

\[ F \Sigma^\infty_V (S^{n-1} \times \mathbb{T} / C_p^m)_+ \rightarrow F \Sigma^\infty_V (B^{n-1} \times \mathbb{T} / C_p^m)_+ \]

Fibrations are fibrations of the underlying orthogonal \( \mathbb{T} \)-spectra.
Do (pre-)cyclotomic spectra have mapping spectra?

Is $\Phi^C p$ a spectral functor: $F^T(T, U) \rightarrow F^T(\Phi^C p T, \Phi^C p U)$?

Spectrum of maps is an equalizer

$$F_{Cy}(T, U) \rightarrow F^T(T, U) \Rightarrow F^T(\rho^* \Phi^C p T, U)$$
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Mapping Spectra

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$$F_{Cy}(T, U) \xrightarrow{\rho^* \Phi^{C_p}} F^T(T, U) \xrightarrow{\sim} F^T(\rho^* \Phi^{C_p} T, U)$$

Theorem

$F_{Cy}$ plays nice with cofibrations, fibrations and weak equivalences (satisfies the analogue of SM7)

$\Rightarrow$ derived mapping spectrum functor $\mathbb{R} F_{Cy}$
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$F_{Cy}$ plays nice with cofibrations, fibrations and weak equivalences (satisfies the analogue of SM7)

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Calculating Mapping Spectra

Spectrum of maps is an equalizer

$$F_{Cy}(T, U) \rightarrow F^T(T, U) \Rightarrow F^T(\rho^* \Phi^Cp T, U)$$

Structure map commuting up to homotopy is a homotopy equalizer

$$F_{Cy}^{ho}(T, U) \xrightarrow{ho} F^T(T, U) \Rightarrow F^T(\rho^* \Phi^Cp T, U)$$

Theorem

If $T$ is a cofibrant cyclotomic or pre-cyclotomic spectrum then

$$F_{Cy}(T, U) \rightarrow F_{Cy}^{ho}(T, U)$$

is a level equivalence.
Calculating Mapping Spectra

Spectrum of maps is an equalizer

\[ F_{Cy}(T, U) \rightarrow F^T(T, U) \Rightarrow F^T(\rho^*\Phi^C_p T, U) \]

Structure map commuting up to homotopy is a homotopy equalizer

\[ F_{ho}^{ho}(T, U) \xrightarrow{ho} F^T(T, U) \Rightarrow F^T(\rho^*\Phi^C_p T, U) \]

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\[ F_{Cy}^{ho}(T, U) \xrightarrow{ho} F^T(T, U) \Rightarrow F^T(\rho^* \Phi^C_p T, U) \]

\[ \rho^* \Phi^C_p T \to \rho^* \Phi^C_p U \]

\[ T \to U \]

Theorem

If \( T \) is a cofibrant cyclotomic or pre-cyclotomic spectrum then

\[ F_{Cy}(T, U) \to F_{Cy}^{ho}(T, U) \]

is a level equivalence.
Let $S_{TR} = \Sigma_T^\infty (T \amalg (T/C_p) \amalg (T/C_{p^2}) \amalg \cdots)_+$

$$
\Phi^{C_p} \Sigma_T^\infty (T/C_{p^m})_+ = \Sigma_{T/C_p}^\infty (T/C_{p^m})_+
\quad \Rightarrow \quad \rho^* \Phi^{C_p} \Sigma_T^\infty (T/C_{p^m})_+ = \Sigma_T^\infty (T/C_{p^{m-1}})_+
$$

Quick Computation 1

$$
\rho^* \Phi^{C_p} S_{TR} = S_{TR}
$$

Use this for cyclotomic structure for $S_{TR}$. 
TR is corepresentable

Let $S_{TR} = \Sigma_T^\infty (T \amalg (T/C_p) \amalg (T/C_{p^2}) \amalg \cdots)_+$

\[
\Phi^{C_p} \Sigma_T^\infty (T/C_{p^m})_+ = \Sigma_{T/C_p}^\infty (T/C_{p^m})_+ \\
\Rightarrow \rho^* \Phi^{C_p} \Sigma_T^\infty (T/C_{p^m})_+ = \Sigma_T^\infty (T/C_{p^{m-1}})_+
\]

Quick Computation 1

$\rho^* \Phi^{C_p} S_{TR} = S_{TR}$

Use this for cyclotomic structure for $S_{TR}$. 
$TR$ is corepresentable

Let $S_{TR} = \sum_{T}^{\infty} (T \amalg (T/C_p) \amalg (T/C_{p^2}) \amalg \cdots)_{+}$

$$\Phi^{C_p} \sum_{T}^{\infty} (T/C_{p^m})_{+} = \sum_{T/C_p}^{\infty} (T/C_{p^m})_{+}$$

$$\Rightarrow \rho^* \Phi^{C_p} \sum_{T}^{\infty} (T/C_{p^m})_{+} = \sum_{T}^{\infty} (T/C_{p^{m-1}})_{+}$$

Quick Computation 1

$$\rho^* \Phi^{C_p} S_{TR} = S_{TR}$$

Use this for cyclotomic structure for $S_{TR}$. 
Let $S_{TR} = \Sigma_{T}^{\infty} (\mathbb{T} \amalg (\mathbb{T}/C_p) \amalg (\mathbb{T}/C_{p^2}) \amalg \cdots)_{+}$

$$
\Phi^C_p \Sigma_{T}^{\infty} (\mathbb{T}/C_{p^m})_{+} = \Sigma_{T/C_p}^{\infty} (\mathbb{T}/C_{p^m})_{+}
$$

$$
\Rightarrow \rho^* \Phi^C_p \Sigma_{T}^{\infty} (\mathbb{T}/C_{p^m})_{+} = \Sigma_{T}^{\infty} (\mathbb{T}/C_{p^{m-1}})_{+}
$$

Quick Computation 1

$$
\rho^* \Phi^C_p S_{TR} = S_{TR}
$$

Use this for cyclotomic structure for $S_{TR}$. 
**TR** is corepresentable

\[ S_{TR} = \Sigma_{T}^{\infty} (\mathbb{T} \amalg (\mathbb{T}/C_p) \amalg (\mathbb{T}/C_{p^2}) \amalg \cdots ) \]

**Quick Computation 2**

\[ F^{T}(S_{TR}, X) = X \times X^{C_p} \times X^{C_{p^2}} \times \cdots \]

\[ S_{TR} \overset{\rho^* \Phi^{C_p}}{\longrightarrow} T \]

\[ S_{TR} \longrightarrow \rho^* \Phi^{C_p} T \]

**Conclusion**

\[ \mathbb{R}F_{Cy}(S_{TR}, T) = (T \times T^{C_p} \times T^{C_{p^2}} \times \cdots )^{hR} \]

\[ = \text{holim}(\cdots \rightarrow T^{C_{p^m}} \overset{R}{\rightarrow} \cdots \overset{R}{\rightarrow} T) \]

\[ = TR(T) \]
$TR$ is corepresentable

\[ S_{TR} = \Sigma_T^\infty (T \amalg (T/C_p) \amalg (T/C_{p^2}) \amalg \cdots)_+ \]

Quick Computation 2

\[ F^T(S_{TR}, X) = X \times X^{C_p} \times X^{C_{p^2}} \times \cdots \]

\[ S_{TR} \twoheadrightarrow \rho^* \Phi^{C_p} T \]

\[ \Downarrow \]

\[ S_{TR} \twoheadrightarrow T \]

Conclusion

\[ RF_{Cy}(S_{TR}, T) = (T \times T^{C_p} \times T^{C_{p^2}} \times \cdots)^{hR} \]

\[ = \text{holim}(\cdots \to T^{C_{p^m}} \xrightarrow{R} \cdots \xrightarrow{R} T) \]

\[ = TR(T) \]
**$TR$ is corepresentable**

$$S_{TR} = \Sigma_T^\infty (T \amalg (T/C_p) \amalg (T/C_{p^2}) \amalg \cdots )_+$$

**Quick Computation 2**

$$F^T(S_{TR}, X) = X \times X^{C_p} \times X^{C_{p^2}} \times \cdots$$

\[
\begin{align*}
S_{TR} & \longrightarrow \rho^* \Phi^{C_p} T \\
\Downarrow & \\
S_{TR} & \longrightarrow T
\end{align*}
\]

**Conclusion**

$$\mathbb{R}F_{Cy}(S_{TR}, T) = (T \times T^{C_p} \times T^{C_{p^2}} \times \cdots)^{hR}$$

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$$= TR(T)$$
TR is corepresentable

\[ S_{TR} = \Sigma_{\mathbb{T}} \mathbb{T} \amalg (\mathbb{T}/C_p) \amalg (\mathbb{T}/C_{p^2}) \amalg \cdots \]

Quick Computation 2

\[ F^\mathbb{T}(S_{TR}, X) = X \times X^{C_p} \times X^{C_{p^2}} \times \cdots \]

\[ \begin{array}{ccc}
S_{TR} & \rightarrow & \rho^* \Phi^{C_p} T \\
\downarrow & & \downarrow \\
S_{TR} & \rightarrow & T
\end{array} \]

Conclusion

\[ \mathbb{R}F_{Cy}(S_{TR}, T) = (T \times T^{C_p} \times T^{C_{p^2}} \times \cdots)^{hR} \]

\[ = \text{holim}(\cdots \rightarrow T^{C_{p^m}} \xrightarrow{R} \cdots \xrightarrow{R} T) \]

\[ = TR(T) \]
"TR is corepresentable"

\[ S_{TR} = \Sigma_{T}^{\infty} (T \amalg (T/C_p) \amalg (T/C_{p^2}) \cdots ) + \]

Quick Computation 2

\[ F^T(S_{TR}, X) = X \times X^{C_p} \times X^{C_{p^2}} \times \cdots \]

\[ S_{TR} \rightarrow \rho^* \Phi^{C_p} T \]

\[ \cong \]

\[ S_{TR} \rightarrow T \]

Conclusion

\[ R F_{Cy}(S_{TR}, T) = (T \times T^{C_p} \times T^{C_{p^2}} \times \cdots )^{hR} \]

\[ = \text{holim}(\cdots \rightarrow T^{C_{p^m}} \xrightarrow{R} \cdots \xrightarrow{R} T) \]

\[ = TR(T) \]
$TC$ is corepresentable (in pre-cyclotomic spectra)

Let $S_{TC,m} = \Sigma^\infty_T (\mathbb{T}/C_{p^m})_+$ with pre-cyclotomic structure

$$\rho^* \Phi^C p S_{TC,m} = \Sigma^\infty_T (\mathbb{T}/C_{p^{m-1}})_+ \rightarrow \Sigma^\infty_T (\mathbb{T}/C_{p^m})_+ = S_{TC,m}$$

$$F^T(S_{TC,m}, X) = X^{C_{p^m}}$$

Quick Computation

$\mathbb{R}F_{Cy}(S_{TC,m}, T)$ is the homotopy equalizer of $R, F: T^{C_{p^m}} \rightarrow T^{C_{p^{m-1}}}$.

Let $S_{TC} = \text{hocolim } S_{TC,m}$

Theorem

$\mathbb{R}F_{Cy}(S_{TC}, T) = TC(T)$
TC is corepresentable (in pre-cyclotomic spectra)

Let \( S_{TC,m} = \Sigma_\mathbb{T}(\mathbb{T}/C_{p^m})_+ \) with pre-cyclotomic structure

\[
\rho^* \Phi_{Cp} S_{TC,m} = \Sigma_\mathbb{T}(\mathbb{T}/C_{p^{m-1}})_+ \rightarrow \Sigma_\mathbb{T}(\mathbb{T}/C_{p^m})_+ = S_{TC,m}
\]

\[
F^\mathbb{T}(S_{TC,m}, X) = X^{C_{p^m}}
\]

Quick Computation

\( \mathbb{R}F_{Cy}(S_{TC,m}, T) \) is the homotopy equalizer of \( R, F : T^{C_{p^m}} \rightarrow T^{C_{p^{m-1}}} \).

Let \( S_{TC} = \text{hocolim} S_{TC,m} \)

Theorem

\( \mathbb{R}F_{Cy}(S_{TC}, T) = TC(T) \)
$TC$ is corepresentable (in pre-cyclotomic spectra)

Let $S_{TC,m} = \Sigma_{T}^{\infty} (\mathbb{T}/C_{p^m})_+$ with pre-cyclotomic structure

$$\rho^* \Phi^C p S_{TC,m} = \Sigma_{T}^{\infty} (\mathbb{T}/C_{p^{m-1}})_+ \to \Sigma_{T}^{\infty} (\mathbb{T}/C_{p^m})_+ = S_{TC,m}$$

$F^T(S_{TC,m}, X) = X^{C_{p^m}}$

Quick Computation

$\mathbb{R}F_{Cy}(S_{TC,m}, T)$ is the homotopy equalizer of $R, F: T^{C_{p^m}} \to T^{C_{p^{m-1}}}$.

Let $S_{TC} = \text{hocolim} \ S_{TC,m}$

Theorem

$\mathbb{R}F_{Cy}(S_{TC}, T) = TC(T)$
**TC is corepresentable (in pre-cyclotomic spectra)**

Let \( S_{TC,m} = \sum_{T}^{\infty} (T/C_{p^m})_{+} \) with pre-cyclotomic structure

\[
\rho^* \Phi^{C_p} S_{TC,m} = \sum_{T}^{\infty} (T/C_{p^{m-1}})_{+} \to \sum_{T}^{\infty} (T/C_{p^m})_{+} = S_{TC,m}
\]

\[
F^T(S_{TC,m}, X) = X^{C_{p^m}}
\]

**Quick Computation**

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**Theorem**

\( \mathbb{R}F_{Cy}(S_{TC}, T) = TC(T) \)
Let $S_{TC,m} = \Sigma_{\mathbb{T}}^\infty (\mathbb{T}/C_{p^m})_+$ with pre-cyclotomic structure

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$F^\mathbb{T}(S_{TC,m}, X) = X^{C_{p^m}}$

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Is $TC$ corepresentable in cyclotomic spectra?

**Conjecture (Kaledin ICM 2010)**

There is a homotopy theory of cyclotomic spectra and in it, $TC$ is corepresented as maps out of the sphere spectrum.
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$$(S_T^C)_r^\wedge \simeq S^\wedge_r$$

**Theorem**

$TC(T)^\wedge \simeq UF_{Cy}(S, T)^\wedge$
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There is a homotopy theory of cyclotomic spectra and in it, $TC$ is corepresented as maps out of the sphere spectrum after finite completion.

$$\left( S_{TC} \right)^{\wedge} \simeq S^{\wedge}$$

Theorem

$$TC(T)^{\wedge} \simeq RF_{Cy}(S, T)^{\wedge}$$

$$THN \simeq THK$$