Chaotic behavior in a $Z_2 \times Z_2$ field theory

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Abstract

We investigate the presence of chaos in a system of two real scalar fields with discrete $Z_2 \times Z_2$ symmetry. The potential that identify the system is defined with a real parameter $r$ and presents distinct features for $r > 0$ and for $r < 0$. For static field configurations, the system supports two topological sectors for $r > 0$, and only one for $r < 0$. Under the assumption of spatially homogeneous fields, the system exhibits chaotic behavior almost everywhere in parameter space. In particular a more complex dynamics appears for $r > 0$; in this case chaos can decrease for increasing energy, a fact that is absent for $r < 0$.
I. INTRODUCTION

Nonlinearity plays an important role in field theory. It is responsible for the presence of interactions and may enter the game allowing interesting situations, as is the case for instance when the system engenders spontaneous symmetry breaking. In this case the classical equations of motion may give rise to interesting field configurations such as kinks, vortices and monopoles, depending on the particular model in consideration. See for instance Refs. [1–3] for details.

In the standard route to defect formation, one usually search for static field configurations that solve the equations of motion and present finite energy. In the simplest case of a single real scalar field, a system given in terms of a potential that presents $Z_2$ symmetry may support topological solutions in $1 + 1$ dimensions when the potential has at least two degenerate minima. In the case of two real scalar fields, the system is richer and may support distinct types of topological solutions, as we illustrate below.

Nonlinearity also plays an important role on chaotic behavior of systems. For linear systems, the qualitative nature of the behavior does not change when one changes their parameters. However, for systems governed by nonlinear dynamics one may find examples where small change in a parameter can lead to dramatic changes in both the quantitative and qualitative behavior of the system. See for instance Refs. [4–6] for details.

There are distinct routes to chaos in field theory, and here we shall follow the point of view introduced in Ref. [7]. In this case one investigates field theoretical models under the assumption of spatially homogeneous field configurations. We think of field configurations whose space variations are much smaller than the corresponding time variations, and so we treat the fields as depending only on time. This point of view has been explored in several different works, as for instance in Refs. [8–10] and in some works therein. We notice that these works deal with different models, describing Abelian and non Abelian gauge fields in the presence of spontaneous symmetry breaking. In these cases the symmetry to be broken is continuum, local, and the system supports vortices or monopoles. In the present work, however, we shall explore another system, simpler, that presents discrete symmetry and is described by a couple of real scalar fields. Recent investigations have shown that this system presents interesting properties and some soliton solutions have also been found, as we comment on below.

The main motivation of the present work is to investigate the presence of chaos in a system that describes two real scalar fields engendering the discrete $Z_2 \times Z_2$ symmetry. We shall do this by comparing the case where one supposes the field configurations to be spatially homogeneous with the more familiar situation which considers static configurations. The last case is appropriate for searching for soliton solutions, and we comment on that in the next Sec. [II]. We consider the case of spatially homogeneous fields in Sec. [II] where we search for chaotic behavior. As we also comment on, the subject of this paper is related to recent issues [11], which ask for instance whether the chaotic behavior found in flat spacetime persists during the cosmological expansion. In this paper we deal with a model which is defined in flat spacetime and may be seen as the results of an expanding FRW universe, valid under the approximation of very slow expansion rate. Furthermore, the present investigation lands very naturally to the context of hybrid inflation, where one requires real scalars such as the inflaton field and at least another scalar field, which couples to the inflaton field [12] in a
way similar to the one we consider in this work. We end the paper by summarizing the results in Sec. IV.

II. GENERAL CONSIDERATIONS

The system we study in this paper is a $Z_2 \times Z_2$ model of two real fields $\phi(x, t)$ and $\chi(x, t)$. The potential that specifies the system is defined in terms of a single real parameter, $r$, which controls the number of minimum energy states of that potential. As we are going to show, when $r$ changes sign the potential changes form, from the case with four minima ($r > 0$) to the case where only two minima are present ($r < 0$). For $r > 0$, spontaneous symmetry breaking appears in both the $\phi$ and $\chi$ directions in the $(\phi, \chi)$ plane. For $r < 0$ we have spontaneous symmetry breaking only in the $\phi$-field direction, and this last situation is similar to the case of hybrid inflation where a first-order nonthermal phase transition after preheating seems to be present, as recently considered in Ref. [13].

The field theory that we consider is described by the Lagrangian density

$$
\mathcal{L} = \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi + \frac{1}{2} \partial_\alpha \chi \partial^\alpha \chi - V(\phi, \chi)
$$

We are using standard notation, with $\hbar = c = 1$, and $V(\phi, \chi)$ is the potential, which one supposes to be given by

$$
V(\phi, \chi) = \frac{1}{2} \left( \frac{\partial h}{\partial \phi} \right)^2 + \frac{1}{2} \left( \frac{\partial h}{\partial \chi} \right)^2
$$

where $h = h(\phi, \chi)$ is a smooth function of the two fields $\phi$ and $\chi$. Here it obeys

$$
\frac{1}{\mu} h(\phi, \chi) = \tilde{h}(\phi, \chi) = r \left( \frac{1}{3} \phi^3 - \phi \right) + \phi \chi^2
$$

The parameter $\mu$ is real, with dimension of energy, and $r$ is another real parameter, dimensionless.

In 1+1 dimensions this system presents interesting soliton solutions [14], which has been used in applications in condensed matter [15] and in field theory [16–18]. The equations of motion for $\phi = \phi(x, t)$ and $\chi = \chi(x, t)$ are given by

$$
\begin{align*}
\phi_{tt} - \phi_{xx} + h_\phi h_{\phi\phi} + h_\chi h_{\chi\phi} &= 0 \\
\chi_{tt} - \chi_{xx} + h_\phi h_{\phi\chi} + h_\chi h_{\chi\chi} &= 0
\end{align*}
$$

These are the equations we shall deal with in the following. For simplicity, however, we rewrite them in terms of dimensionless variables $\tilde{t} = \mu t$ and $\tilde{x} = \mu x$ to get

$$
\begin{align*}
\phi_{\tilde{tt}} - \phi_{\tilde{xx}} + \tilde{h}_\phi \tilde{h}_{\phi\phi} + \tilde{h}_\chi \tilde{h}_{\chi\phi} &= 0 \\
\chi_{\tilde{tt}} - \chi_{\tilde{xx}} + \tilde{h}_\phi \tilde{h}_{\phi\chi} + \tilde{h}_\chi \tilde{h}_{\chi\chi} &= 0
\end{align*}
$$

In the case of static fields we have $\phi = \phi(\tilde{x})$ and $\chi = \chi(\tilde{x})$. We change $\tilde{x} \to y$, for simplicity, and now the equations of motion become
\[
\frac{d^2\phi}{dy^2} = 2r^2 (\phi^2 - 1) \phi + 2 (r + 2) \phi \chi^2 \\
\frac{d^2\chi}{dy^2} = 2 (\chi^2 - r) \chi + 2 (r + 2) \phi^2 \chi
\]

which are the equations we deal with when searching for soliton solutions. It is interesting to see that these equations of motion (8) and (9) can be solved by configurations that obey the pair of first-order differential equations

\[
\frac{d\phi}{dy} = r (\phi^2 - 1) + \chi^2 \\
\frac{d\chi}{dy} = 2 \phi \chi
\]

Solutions that obey this pair of first-order equations are BPS solutions [19]. They are stable configurations that minimize the energy, as explicitly shown in [14,15]. See Ref. [18] for further comments on BPS solutions, and for showing explicitly that the system defined by the potential (2) is the bosonic portion of a supersymmetric theory.

In the case of spatially homogeneous fields we have \( \phi = \phi(\tilde{t}) \) and \( \chi = \chi(\tilde{t}) \). We change \( \tilde{t} \to t \), for simplicity, and here we get the equations of motion

\[
\frac{d^2\phi}{dt^2} = 2r^2 (1 - \phi^2) \phi - 2 (r + 2) \phi \chi^2 \\
\frac{d^2\chi}{dt^2} = 2 (r - \chi^2) \chi - 2 (r + 2) \phi^2 \chi
\]

These are the equations we have to deal with when searching for a chaotic behavior in the time evolution. It is interesting to see that the above equations have some analogies with the equations investigated in Ref. [9]. However, in Ref. [9] the system under consideration is the Abelian-Higgs model. This is the relativistic generalization of the Ginzburg-Landau theory of superconductivity, and it is well known that it presents vortex solutions [20]. The reason for the similarity between eqs. (12) and (13) and the equations investigated in [9] seems to rely on the assumptions introduced in Ref. [9].

We follow the aim of this paper, which is to investigate the chaotic behavior of eqs. (12) and (13), together with the study of the static solutions of eqs. (8) and (9). We start presenting some general considerations regarding the static solutions. Our system is identified by the potential \( V(\phi, \chi) = \mu^2 \tilde{V}(\phi, \chi) \), where

\[
\tilde{V}(\phi, \chi) = \frac{1}{2} \mu^2 (\phi^2 - 1)^2 + r (\phi^2 - 1) \chi^2 + \frac{1}{2} \chi^4 + 2 \phi^2 \chi^2
\]
to say that in the two particular cases when \( r = 1 \) and when \( r = -2 \) the two fields decouples \[17\] and so we expect no chaotic behavior in these two cases. First we show that, from the point of view of the topological soliton solutions admitted by Eqs. (8) and (9), the case \( r > 0, r \neq 1 \) is richer than the case of \( r < 0, r \neq -2 \). When \( r \) is positive, \( r \neq 1 \), the four vacuum states are given by

\begin{align*}
v_1 &= (-1, 0), \quad v_2 = (1, 0) \\
v_3 &= (0, -\sqrt{r}), \quad v_4 = (0, \sqrt{r})
\end{align*}

(15)

For \( r \) negative, \( r \neq -2 \), we have the two vacuum states

\begin{align*}
\bar{v}_1 &= (-1, 0), \quad \bar{v}_2 = (1, 0)
\end{align*}

(17)

As it was already shown \[14,15\], the energy of the stable BPS solutions are obtained according to the values of \( \tilde{h}(\phi, \chi) \) in the vacuum states.

For \( r \) positive, \( r \neq 1 \) we name \( h_i \) as the value of \( h \) at the vacuum state \( v_i \), and so we have

\begin{align*}
h_1 &= \frac{2}{3} \mu r \\
h_2 &= -(2/3) \mu r \\
h_3 &= h_4 = 0
\end{align*}

(18)

(19)

(20)

This means that we have two type of BPS solutions, with energies \( E_i = |\mu| \varepsilon_i, i = 1, 2 \), where

\begin{align*}
\varepsilon_1 &= \frac{4}{3} r, \quad (r > 0, r \neq 1) \\
\varepsilon_2 &= \frac{2}{3} r, \quad (r > 0, r \neq 1)
\end{align*}

(21)

(22)

In the first case the solutions connect the vacuum states \((\pm 1, 0)\) and in the second case pair of vacuum states belonging to different axes. In the topological sector defined by the two vacuum states \((-1, 0)\) and \((1, 0)\) there are pairs of analytical solutions given by \[14\]

\begin{align*}
\phi(y) &= -\tanh[r(y - \bar{y})], \quad \chi(y) = 0
\end{align*}

(23)

and by

\begin{align*}
\phi(y) &= -\tanh[2(y - \bar{y})] \\
\chi(y) &= \pm \frac{\sqrt{r - 2}}{\cosh[2(y - \bar{y})]}
\end{align*}

(24)

(25)

Here \( \bar{y} \) stands for the center of the topological solution. The first pair (23) is valid for \( r > 0 \) and the second one for \( r > 2 \). We see that the second pair of solutions (24) and (25) gets to the first pair (23) in the limit \( r \to 2 \).

For \( r \) negative, \( r \neq -2 \), there is just one type of BPS solution, with energy \( \bar{E}_1 = |\mu| \varepsilon_1 \), where

\begin{align*}
\varepsilon_1 &= \frac{4}{3} |r|, \quad (r < 0, r \neq -2)
\end{align*}

(26)
The first pair of solutions given by Eq. (23) is also a pair of solutions in the case $r < 0$.

It is clear that for $r > 0, r \neq 1$ the situation is richer than in the case $r < 0, r \neq -2$, at least from the point of view of the above topological soliton solutions. In the first case the system admits two type of topological BPS solutions, while in the second case it has only one. We also expect a difference between the two $r > 0$ and $r < 0$ regimes to be present in the chaotic behavior of the eqs. (12) and (13).

Another interesting feature of this model concerns the masses of the $\phi$ and $\chi$ fields. From the potential $V(\phi, \chi)$ we can introduce the matrix

$$
\begin{pmatrix}
V_{\phi\phi}^v & V_{\phi\chi}^v \\
V_{\chi\phi}^v & V_{\chi\chi}^v
\end{pmatrix}
$$

(27)

where $V_{\phi\phi} = \partial^2 V / \partial \phi \partial \phi$ and so forth. The superscript $v$ stands for substituting the fields for their vacuum values once the derivatives are done. For the potential introduced in Eq. (14) we see that

$$V_{\phi\chi}^v = V_{\chi\phi}^v = 4 \mu^2 (r + 2) \phi \chi$$

(28)

and so $V_{\phi\phi}^v = V_{\chi\chi}^v$ vanishes for every vacuum state, despite the sign of $r$. This means that we can read the (square) mass of each field directly from the matrix (27). We have at most two different mass values. They are

$$
m_\phi^2 = 4 \mu^2 r^2 \quad m_\chi^2 = 4 \mu^2 (r > 0, \phi \text{ axis})
$$

(29)

$$
m_\phi^2 = 4 \mu^2 r \quad m_\chi^2 = 4 \mu^2 r (r > 0, \chi \text{ axis})
$$

(30)

and also

$$\bar{m}_\phi^2 = 4 \mu^2 r^2 \quad \bar{m}_\chi^2 = 4 \mu^2 (r < 0, \phi \text{ axis})$$

(31)

where we inform the axis of the vacuum state used to obtain the respective masses. We note that for $r < 0$ the $\chi$ field develops no symmetry breaking, and its mass does not depend on $r$. This is the way this specific $Z_2 \times Z_2$ system behaves, and this behavior may have interesting connections with models of hybrid inflation [12,13,21–23]. We remark that there are other $Z_2 \times Z_2$ systems, as for instance the ones investigated recently in Ref. [18], which are also defined with a single real parameter, but that present distinct behavior, with different sets of masses and vacuum states. They certainly lead to richer scenarios, and may give other informations on the role the parameter $r$ plays in such models.

### III. CHAOTIC BEHAVIOR

In this Section we investigate the chaotic behavior of the system introduced in Sec. II.

We focus our attention to the pair of Eqs. (12) and (13). These equations can be seen as the equations of motion that follow from the Hamiltonian

$$H = \frac{1}{2} p_\phi^2 + \frac{1}{2} p_\chi^2 + \tilde{V}(\phi, \chi)$$

(32)
They describe the motion of a classical particle in the bidimensional potential \( \tilde{V}(\phi, \chi) \). We remark that this choice of Hamiltonian makes the energy dimensionless. The equations of motion can be written in first order form, in terms of the canonical variables \((\phi, p_\phi)\) and \((\chi, p_\chi)\),

\[
\begin{align*}
\dot{\phi} &= \frac{\partial H}{\partial p_\phi} = p_\phi \\
\dot{p}_\phi &= -\frac{\partial H}{\partial \phi} = 2r^2 (1 - \phi^2) \phi - 2(r + 2) \phi \chi^2 \\
\dot{\chi} &= \frac{\partial H}{\partial p_\chi} = p_\chi \\
\dot{p}_\chi &= -\frac{\partial H}{\partial \chi} = 2(r - \chi^2) \chi - 2(r + 2) \phi^2 \chi
\end{align*}
\]

(33) \hspace{1cm} (34) \hspace{1cm} (35) \hspace{1cm} (36)

In the rest of the paper we study for which values of the parameter \( r \) and of the energy \( E \) the trajectories \( \phi(t), \chi(t), p_\phi(t), \) and \( p_\chi(t) \) show a regular or a chaotic behavior.

The motion is always bounded, i.e. confined to a finite region whose size changes according to the energy \( E \) of the particle. In Fig. 1 we report the potential contour plot defined by \( \tilde{V}(\phi, \chi) = E \); the different values of the energy \( E \) are reported in the figure. Fig. 1a refers to \( r = -5 \) while Fig. 1b and Fig. 1c refer to \( r = 1 \) and \( r = 3 \), respectively.

For \( r < 0 \) we note the presence of two minima \((\pm 1, 0)\), corresponding to the two vacuum states previously discussed, and a saddle point at the origin. The potential vanishes at the two minima, while at the origin \((0, 0)\) it assumes the value \( r^2/2 \) (12.5 for case \( r = -5 \) in Fig. 1a). Dynamical trajectories, e.g. solutions of Eqs. (33)-(36) are confined to one of the two minima regions if \( E < 12.5 \), while they can jump from one side to the other when \( E > 12.5 \). For \( r = -2 \) the two fields decouple and the numerical investigation shows no chaotic behavior.

The situation for \( r > 0 \) is qualitatively different because we have four minima \( \bar{V} = 0 \) at \((\pm 1, 0)\) and at \((0, \pm \sqrt{r})\), corresponding to the four vacuum states previously discussed, and a local maximum \( \bar{V} = r^2/2 \) at the origin \((0, 0)\). There are also four saddle points with \( \phi \neq 0 \) and \( \chi \neq 0 \); they are given by

\[
\phi^2 = \frac{r}{2(r + 1)}, \quad \chi^2 = \frac{r^2}{2(r + 1)}
\]

(37)

In the saddle points the potential gets the value

\[
\bar{V} = \frac{r^2}{2(r + 1)}
\]

(38)

The case \( r = 1 \) is a particular case because the potential presents \( Z_4 \) symmetry \[17\] (see Fig. 1b); the two fields decouple and the numerical simulations show no chaotic behavior. When \( r \neq 1 \) trajectories can have different behaviors according to the energy \( E \). We discuss the case \( r = 3 \) in Fig. 1c: when \( E \leq 1.125 \) the four minima are unconnected regions and the motion remains confined around the minimum in which we start the trajectory. As soon as \( E > 1.125 \) (see Fig. 1c) the trajectory crosses the saddle point [see Eq. (38)] between the minima and wonders from one minimum to another. The region around the relative
maximum at the origin \((0,0)\) is still not allowed. When \(E > r^2/2 = 4.5\) the dynamical trajectories can cross the origin and the four minima are all directly interconnected.

In order to study the dynamical behavior of our system for different values of \(r\) and energy \(E\), we integrate numerically Eqs. (33)-(36) using a fourth order symplectic algorithm with a time step \(\Delta t = 0.001\). The time step has been determined in order to keep the error in energy conservation below \(\Delta E/E = 10^{-8}\) for any value of \(r\) and \(E\); the results are stable respect to a further reduction of \(\Delta t\). For any conservative system such as the one we are considering, the Hamiltonian is an integral of motion, and the energy conservation restricts trajectories to lie on a three-dimensional surface \(H(\phi, \chi, p_\phi, p_\chi) = E\) in the four-dimensional phase space \((\phi, \chi, p_\phi, p_\chi)\). We can obtain a graphical information about the system by plotting the intersection of this three-dimensional surface with a plane. These kind of plot is called Poincaré surface of section, and gives an indication about the dynamical behavior of the system. To define the surface of section in our case we follow a trajectory and we plot \(\phi\) and \(\chi\) each time \(p_\phi = 0\). Regular regions will appear as a series of points (a mapping) which lie on a one dimensional curve (invariant KAM curve), while chaotic regions will appear as a scatter of points limited to a finite area due to energy conservation [4,5].

In Fig. 2 and in Fig. 3 we show the surface of section in the plane \((\phi, \chi)\) for \(r = -5\) and for \(r = 3\), respectively. Each figure is obtained from 100 different trajectories followed for a time \(t = 500\). The trajectories are generated from random uniformly-distributed initial conditions \(\phi_0, p_\phi, \chi_0, p_\chi\).

For \(r = -5\) we plot the two cases \(E = 10\) and \(E = 20\) (Fig. 2a and Fig. 2b respectively). When \(E = 10\) the two minima are separated and the system exhibits only regular behavior (the surface of section shows only invariant KAM curves). When \(E = 20\) the two minima are connected and the surface of section contains both regular and stochastic regions. The volume of the stochastic region increases with increasing energy, as it will be clear when we calculate Lyapunov exponents.

For \(r = 3\) the situation is richer. We report the surface of section for three different energies \(E = 0.5, E = 2,\) and \(E = 6\). When \(E = 0.5\) (Fig. 3a) the four minima are not interconnected and we only have invariant KAM curves. When \(E = 2\) (Fig. 3b) neighbour minima regions are connected. Regular and stochastic regions coexist in the surface of section and are interwined in a very complicated way. The scatter of points representing the chaotic region fill the area between regular curves, and then smaller regular islands are imbedded in the chaotic sea. The white regions correspond to regular region, and these would be filled by regular curves if we could increase even more the number of initial conditions considered in the figure. For energy \(E = 6\) (Fig. 3c) the trajectory can cross the center and the resulting surface of section shows an increase of the chaotic area respect to the case \(E = 2\).

The cases \(r = 1\) and \(r = -2\) (which we do not report in figure) are two very particular cases. The two equations decouples, and indeed the Poincaré surface of section show only KAM curves for every value of the energy \(E\).

The information we can get from Poincaré surface of section is qualitative. A way to quantify the chaotic behavior of a system is by calculating its Lyapunov exponents. Chaos is defined in terms of the dynamical behavior of pairs of orbits which initially are close together in the phase space. The Lyapunov exponents are given by the rate of exponential divergence of close orbits and are defined from the long term evolution of an initial infinitesimal volume.
We consider separately the two cases \( r < 0 \) and \( r > 0 \) for which we expect to find a qualitative difference in the chaotic properties of Eqs. (12) and (13). We discussed above the topological soliton solutions of Eqs. (8) and (9) for these two cases and we found that case \( r > 0 \) is richer than case \( r < 0 \). In fact, for \( r > 0, r \neq 1 \) the system supports two topological sectors, admitting two different type of topological BPS solutions; for \( r < 0, r \neq -2 \) there is just one topological sector. This difference is due to the presence of four or two minima in the potential \( V(\phi, \chi) \), for \( r > 0 \) or \( r < 0 \) respectively. We expect the difference in the shape of the potential to have important implications also on the chaotic behavior of our system. In Fig. 5 and Fig. 6 we plot the ratio \( R = N_e/N \) and \( \lambda_1 \), as function of energy for the two cases \( r < 0, r \neq -2 \) and \( r > 0, r \neq 1 \), respectively. The system exhibits an order to chaos transition as a function of energy for any value of \( r \), but for \( r = -2 \) and \( r = 1 \). These two cases are particular cases because the two field decouple and the system degenerates to two systems of a single field each one; \( R \) and \( \lambda_1 \) are identically equal to zero in the whole energy range and are not reported in figures.

We show the typical behavior for \( r \) negative in Fig. 5. We report \( R \) and \( \lambda_1 \) vs. energy for \( r = -5 \); other values of \( r \) have the same qualitative behavior. The onset of chaos is for...
energy equal to $r^2/2$ (12.5 in figure), when both minima can be visited by a single trajectory and $R$ and $\lambda_1$ start to be different from zero. For larger values of energy both $R$ and $\lambda_1$ increase with the energy.

The situation is different for $r$ positive. In Fig. 6 we plot $R$ and $\lambda_1$ vs. $E$ for two different values of $r$, $r = 1.5$ (dashed line) and $r = 3$ (dashed-dotted line). The behavior for $r = 1.5$ and $r = 3$ is qualitatively similar, although shifted in energy. Chaos starts when the energy overcomes the barrier between two minima [which is given in accordance with Eq. (12.5)], and it keep increasing as function of the energy until a new back bending appears, both in $R$ and in $\lambda_1$. This behavior is to be connected to a stabilization effect occurring when the trajectory can cross the origin since both $R$ and $\lambda_1$ have local minima at $E = r^2/2$. Although the chaotic behavior for $r = 1.5$ and for $r = 3$ are qualitatively similar, the absolute value of $R$ and $\lambda_1$ is as larger as closer $r$ gets to unity. This result can be of interest to application since the absolute value of $r$ also controls the masses of the two fields and may be used in models of hybrid inflation.

The behavior at $r > 0$ is therefore different from the behavior at $r < 0$. The most important qualitative difference between these two cases is that for $r > 0$ we can find localized regions in energy where suppression of chaotic behavior appears for increasing energies. Such behavior is not present for $r$ negative, and this is directly related to the reduction of the number of minima in this last case. Similarly, we have shown that for static solutions the number of topological sectors changes from two to one when $r$ changes from positive to negative values, respectively.

IV. COMMENTS AND CONCLUSIONS

In this work we have investigated the presence of chaotic behavior in a system of two real scalar fields that engenders the $Z_2 \times Z_2$ symmetry. The system is defined by a potential that is controlled by a single real parameter, $r$. This parameter can be positive or negative, and each case leads to different behavior. For static field configurations for instance, the system supports two distinct topological sectors for $r$ positive, $r \neq 1$, and only one when $r$ is negative, $r \neq -2$.

Within the context of chaotic behavior, we have shown that chaos is present almost everywhere in parameter space, with distinct qualitative behavior for $r > 0$ and for $r < 0$. The system is richer for $r$ positive, and this is directly related to the presence of the four minima when $r > 0$. For $r > 0$ we can find localized regions in energy where suppression of chaotic behavior appears for increasing energy values, a fact that is absent for $r$ negative. On the other hand, in the case of static solution there are two topological sectors for $r > 0$, in contraposition with the single sector that the system supports for $r < 0$.

There are other models that engender the $Z_2 \times Z_2$ symmetry, and that are also governed by a similar real parameter $r$. Some examples appear in Ref. [18], and they can also be studied within the context of chaotic behavior, under the assumption of spatially homogeneous field configurations. Such investigations can introduce further light toward a better understanding of the connection between chaotic behaviors and the parameter $r$ that specifies the physical properties of the system.

As we have already commented on, the present investigations may be of interest to inflationary cosmology, in connection with issues raised in the recent works [11-13]. The
present results seem to be valid in an expanding FRW universe when the rate of expansion is very small. The chaotic behavior that we have found may have been present in early times and may have played some role in the cosmic evolution. A study in which the coupling to gravity is fully considered, and other related issues are presently under consideration.

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FIGURE CAPTIONS

Fig. 1. Potential contours plot for different values of E. a), b), and c) refer to $r = -5, 1$ and 3, respectively.

Fig. 2. Poincaré surface of section for $r = -5$ in $(\phi, \chi)$ plane. $E = 10$ in a) and $E = 20$ in b).

Fig. 3. Poincaré surface of section for $r = 3$ in $(\phi, \chi)$ plane. $E = 0.5$ in a), $E = 2$ in b), and $E = 6$ in c).

Fig. 4. Lyapunov Exponents $\lambda_1(t)$ and $\lambda_2(t)$ vs. time for $r = 3$ and $E = 5.5$. Two chaotic trajectories are plotted in the top panel, while a regular trajectory is plotted in the bottom panel.

Fig. 5. Fraction R of the phase space which is chaotic and largest Lyapunov exponent $\lambda_1$ vs. $E$ for $r = -5$

Fig. 6. Same as in Fig.5 for two positive values of $r$, namely $r = 1.5$ (dashed) and $r = 3$ (dotted-dashed).