SQS-graphs of Solov’eva-Phelps codes

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Dedicated to Charles C. Lindner’s in his 70th anniversary

Abstract

A binary extended 1-perfect code \( C \) folds over its kernel via the Steiner quadruple systems associated with its codewords. The resulting folding, proposed as a graph invariant for \( C \), distinguishes among the 361 nonlinear codes \( C \) of kernel dimension \( \kappa \) obtained via Solov’eva-Phelps doubling construction, where \( 9 \geq \kappa \geq 5 \). Each of the 361 resulting graphs has most of its nonloop edges expressible in terms of lexicographically ordered quarters of products of classes from extended 1-perfect partitions of length 8 (as classified by Phelps) and loops mostly expressible in terms of the lines of the Fano plane.

1 Preliminaries, objectives and plan

We consider the \( n \)-cube \( Q_n \) as the graph with vertex set \( F_2^n = \{0,1\}^n \) in which each two vertices that differ in exactly one coordinate are joined by an edge. A perfect 1-error-correcting code, or 1-perfect code, \( C = C^r \) of length \( n = 2^r - 1 \), where \( 0 < r \in \mathbb{Z} \), is an independent vertex set of \( Q_n \) such that each vertex of \( Q_n \setminus C \) is neighbor of exactly one vertex of \( C \). It follows that \( C \) has distance 3 and \( 2^{n-r} \) vertices.

Each 1-perfect code \( C = C^r \) of length \( n = 2^r - 1 \) can be extended by adding an overall parity check. This yields an extended 1-perfect code \( C = C'^r \) of length \( n + 1 = 2^r \), which is a subspace of even-weight words of \( F_2^{n+1} \). The \( n + 1 \) coordinates of the words of \( F_2^{n+1} \) here are orderly indicated \( 0,1,\ldots,n \).

For every \( n = 2^r - 1 \) such that \( 0 < r \in \mathbb{Z} \) there is at least one linear code \( C^r \) as above and a corresponding linear extension, \( C'^r \). These codes are unique for every \( r < 4 \). The situation changes for \( r \geq 4 \). In fact, there are many nonlinear codes \( C^4 \) and \( C'^4 \) or length 15 and 16, respectively, \[3,6,8,9,10,11\].

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The kernel $\text{Ker}(C)$ of a 1-perfect code $C$ of length $n$ is defined as the largest subset $K \subseteq Q_n$ such that any vector in $K$ leaves $C$ invariant under translations. In other words, $x \in Q_n$ is in $\text{Ker}(C)$ if and only if $x + C = C$. If $C$ contains the zero vector, then $\text{Ker}(C) \subseteq C$. In this case, $\text{Ker}(C)$ is also the intersection of all maximal linear subcodes contained in $C$. The kernel $\text{Ker}(C)$ of an extended 1-perfect code $C$ of length $n + 1$ is defined in a similar fashion in $Q_{n+1}$.

A partition of $F_{2^n}$ into 1-perfect codes $C_0, C_1, \ldots, C_n$ is said to be a 1-perfect partition $\{C_0, C_1, \ldots, C_n\}$ of length $n$. The following result on doubling construction of extended 1-perfect codes of length $2n + 2$ is due to Solov’eva [10] and Phelps [6], so in this work they are called SP-codes.

Theorem 1 [10, 6, 7]

Given two extended 1-perfect partitions $\{C_0, C_1, \ldots, C_n\}$ and $\{D_{n+1}, D_{n+2}, \ldots, D_{2n}\}$ of length $n + 1$ and a permutation $\sigma$ of $[0, n] = \{0, 1, \ldots, n\}$, there exists a 1-perfect code $C$ of length $2n + 2$ given by $C = \bigcup_{i=0}^{n} \{(x, y) | x \in C_i, y \in D_{n+1+\sigma(i)}\}$.

Few invariants for 1-perfect codes $C$ have been proposed toward their classification, namely: the rank of $C$ and the dimension of $\text{Ker}(C)$ [8], the STS-graph $H(C)$ [1] and the STS-graph $H_K(C)$ modulo $K = \text{Ker}(C)$ [2].

In the present work, an invariant for extended 1-perfect codes $C$, referred to as the SQS-graph $H_K(C)$ of $C$ modulo $K = \text{Ker}(C)$, is presented, and computed in particular for the SP-codes of length 16 and kernel dimension $\kappa$ such that $9 \geq \kappa \geq 5$, allowing for us to distinguish successfully between each two of them and to show in Theorem 5 that each such $H_K(C)$ has its links (nonloop edges) expressible in terms of products of classes from extended 1-perfect partitions $\{C_0, \ldots, C_7\}$ and $\{D_8, \ldots, D_{15}\}$, and their loops mostly expressible in terms of the lines of the Fano plane.

In [7], Phelps found that there are exactly eleven 1-perfect partitions of length 7, denoted 0, 1, \ldots, 10. If two such partitions are equivalent, then the corresponding extended partitions are equivalent. However, the converse is false. In fact, puncturing an extended 1-perfect partition at different coordinates may result in nonequivalent 1-perfect partitions. Also, Phelps found that there are just ten nonequivalent extended 1-perfect partitions of length 8. In fact, partitions 2 and 7 in [7] have equivalent extensions.

Phelps also found in [7] that there are exactly 963 extended 1-perfect codes of length 16 obtained via Theorem 1 applied to the cited partitions. The members in the corresponding list of 963 SP-codes in [7] are referred below according to their order of presentation, with numeric indications (using three digits), from 001 (corresponding to the linear code) to 963, (or from 1 to 963). This numeric indication is presented in an additional final column in a copy of the listing of [7] that can be retrieved from http://home.coqui.net/dejterij/963.txt. It contains, for each one of its 963
lines: a reference number, rank, dimension of the kernel, two numbers in \(\{0, 1, 2, 3, 4, 5, 6, 8, 9, 10 = a\}\) representing corresponding Phelps’ 1-perfect source partition \(\{C_0, \ldots, C_7\}\) and target partition \(\{D_8, \ldots, D_{15}\}\) and a permutation \(\sigma\) as in Theorem 1.

In Section 2, \(H_K(C)\) is defined and subsequently applied to the nonlinear SP-codes of length 16 and kernel dimension \(\kappa \geq 5\), according to their classification in [7]. We note that there are 361 nonlinear SP-codes of length 16 with \(\kappa \geq 5\), namely 2, 10, 18, 86 and 245 respectively with \(\kappa = 9, 8, 7, 6\) and 5. Section 3 accounts for the participating STS(15)-types defined in Section 2 and in accordance with [4, 5, 12]. Section 4 accounts for those SP-codes that show homogeneous behavior with respect to the involved Steiner quadruple and triple systems, according to the results of Theorem 5, (Section 5), with which we deal in the the rest of the paper.

## 2 Definition of \(H_K(C)\)

A *Steiner quadruple system*, (or SQS), is an ordered pair \((V, B)\), where \(V\) is a finite set and \(B\) is a set of quadruples of \(V\) such that every triple of \(V\) is a subset of exactly one quadruple in \(B\). A subset of \(B\) will be said to be an *SQS-subset*.

The minimum-distance graph \(M(C)\) of an extended 1-perfect code \(C\) of length \(n\) has \(C\) as its vertex set and exactly one edge between each two vertices \(v, w \in C\) whose Hamming distance is \(d(v, w) = 4\). Each edge \(vw\) of \(M(C)\) is naturally labeled with the quadruple of coordinate indices \(i \in \{0, \ldots, n\}\) realizing \(d(v, w) = 4\). As a result, the labels of the edges of \(M(C)\) incident to any particular vertex \(v\) constitute a Steiner quadruple system \(S(C, v)\) formed by \(n(n+1)(n-1)/24\) quadruples on the \(n+1\) coordinate indices which in our case, namely for \(n+1 = 16\), totals 140 quadruples.

Given an edge \(vw\) of \(M(C)\), its labeling quadruple is denoted \(s(vw)\). Each codeword \(v \in C\) is labeled by the equivalence class \(S[v]\) of Steiner quadruple systems on \(n\) elements corresponding to \(S(C, v)\), called for short the *SQS(n)-type* \(S[v]\). We say that \(M(C)\) with all these vertex and edge labels is the *SQS-graph* of \(C\). Let \(L \subseteq K = \text{Ker}(C)\) be a linear subspace of \(C\). Clearly, \(L\) partitions \(C\) into classes \(v + L\). \((w \in C\) is in \(v + L\) if and only if \(v - w \in L\). These classes \(v + L\) are said to be the *classes* of \(C\) mod \(L\). The set they form can be taken as a quotient set \(C/L\) of \(C\). The following three results and accompanying comments are similar in nature to corresponding results in [2], but now the code \(C\) in their statements is assumed to be an extended 1-perfect code.

**Lemma 2** Each \(v+L \in C/L\) can be assigned a well-defined Steiner quadruple system \(S(C, v)\).

\(\square\)
Lemma 2 suggests the following ‘foldability’ condition. If for any two classes \( u + L \) and \( v + L \) of \( C \mod L \) with \( d(u, v) = 4 \) realized by \( s(uv) \) holds that for any \( u' \in u + L \) there is a \( v' \in v + L \) with \( d(u', v') = 4 \) and realized exactly by \( s(u'v') = s(uv) \), then we say that \( C \) is foldable over \( L \) via the Steiner quadruple systems \( S(C, v) \) associated to the codewords \( v \) of \( C \). In this case, we can take \( C/L \) as the vertex set of a quotient graph \( H_L(C) \) of \( M(C) \) by setting an edge between two classes \( u + L \) and \( v + L \) of \( C/L \) if and only if \( uv \) is an edge of \( M(C) \).

**Proposition 3** Every extended 1-perfect code \( C \) is foldable over any linear \( L \subseteq K \).

Recall that a covering graph map is a graph map \( \phi : G \rightarrow G' \) for which there is a nonnegative integer \( s \) such that the inverse image \( \phi^{-1} \) of each vertex and of each edge of \( G' \) has cardinality \( s \).

**Corollary 4** If \( C \) is foldable over a linear subspace \( L \) of \( K \), then the natural projection \( C \rightarrow C/L \) is extendible to a covering graph map \( \phi_L : M(C) \rightarrow H_L(C) \). Moreover, if \( C \) is foldable over \( K \), then it is also foldable over \( L \).

In the setting of Corollary 4, given an edge \( e = (v+L)(w+L) \) of \( H_L(C) \), its multiplicity is the cardinality of the set of labeling quadruples of edges in the SQS-subset \( \phi_L^{-1}(e) \). Note that the sum of the multiplicities of the edges incident to any fixed vertex \( v \) of \( H_K(C) \) must equal 140, being this the cardinality of the SQS induced by \( C \) at \( v \). Of these 140 quadruples, 28 will be treated in Theorem 5, item 1, and Section 6 in relation to the loops in each \( H_K(C) \), where \( C \) is and SP-code with \( 9 \geq \kappa \geq 5 \). The remaining 112 edges will be treated in Theorem 5, item 2, and Section 7 as seven bunches of 16 quadruples formed as products of classes from partitions \( \{C_0, \ldots, C_7\} \) and \( \{D_8, \ldots, D_{15}\} \).

### 3 Participating STS(15)-types

We denote the type of a Steiner triple system of length 15, or STS(15)-type, by its associated integer \( t = 1, \ldots, 80 \) in the common-order lists of the 80 existing STS(15)-types in [3] [5] [12]. Pasch configurations, nicknamed as fragments in [4], are present in varying proportions in these STS(15)-types. An algorithmic approach, (see also [1] [2]), uses these fragments in order to determine the STS-graph invariant \( H_K(C) \). This yields the STS(15)-types \( S[v] \) associated with each one of the \( 2^{21} = 2048 \) codewords \( v \in C \).

In the case of extended 1-perfect codes \( C \) of length 16, first we obtain the STS-graphs modulo \( K \) of the 16 punctured codes \( C_i, (i = 1, \ldots, 16) \), of

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*References*: [1], [2], [3], [5], [12].
length 15 that can be obtained from each such code $C$. Each such punctured code yields a collection of 2048 16-tuples, determining, as mentioned, an integer in $[1, 80]$ representing its $STS(15)$-type.

The integers in $[1, 80]$ representing the punctured codes of $SP$-codes with $9 \geq \kappa \geq 5$ are: $2, 3, 4, 5, 6, 7, 8, 13, 14, 16$. According to [4], they have numbers of fragments accompanied by corresponding 15-tuples of numbers of fragments containing each a specific coordinate index, but given in non-decreasing order, as follows, (where the first line, cited just for reference, stands for the linear code):

1: 105(42,42,42,42,42,42,42,42,42,42,42,42,42,42,42,42);
2: 73(42,30,30,30,30,30,30,30,30,30,30,30,30,30,30,30);
3: 57(26,26,26,24,24,24,24,24,24,24,24,18,18,18,18,18);
4: 49(30,26,22,20,20,20,20,20,18,18,18,18,18,18,18,14,14);
5: 49(26,26,20,20,20,20,18,18,18,18,18,18,18,18,18,18,18);
6: 37(22,22,22,14,14,14,14,14,14,14,14,14,14,14,14,14,10);
7: 33(18,18,18,12,12,12,12,12,12,12,12,12,12,12,12,12,12);
8: 37(18,18,18,15,15,15,15,14,14,14,14,14,14,14,14,14,10);
13: 33(20,16,16,14,14,12,12,12,12,12,12,12,12,12,12,12,10);
14: 37(24,16,16,15,15,15,14,14,14,14,14,14,14,14,14,14,12,12);
16: 49(21,21,21,21,21,21,21,18,18,18,18,18,18,18,18,18,18,18,18,18,18,18,18,18).

A list of the 16-tuples representing the vertices of the SQS-graphs for the treated $SP$-codes can be found in [http://home.coqui.net/dejterij/tuples.txt](http://home.coqui.net/dejterij/tuples.txt).

4 Which $SP$-codes $C$ are homogeneous?

An extended 1-perfect code $C$ is $SQS$-homogeneous if and only if the SQSs determined by its codewords are all equivalent: in case $C$ is of length 16, the 16 punctured codes of $C$ have the same distribution of $STS(15)$-types.

An $SQS$-homogeneous 1-perfect code of length 16 is $STS$-homogeneous if and only if each one of its punctured codes is homogeneous via a common $STS(15)$. Among the $SP$-codes, we calculated that those behaving in this fashion are, for each $\kappa$ such that $9 \geq \kappa \geq 5$, the following ones:

- $\kappa=9$: (007,2), (008,2);
- $\kappa=8$: (114,5), (115,3), (963,6);
- $\kappa=7$: (002,2), (003,2), (004,2);
- $\kappa=6$: (064,1), (917,8), (918,8);
- $\kappa=5$: (708,4);

where each code denomination is accompanied (between parentheses) by the $STS(15)$-type, from 1 to 80, common as induced $STS(15)$ to the corresponding 2048 codewords, and the types 13, 14 and 16 are respectively represented by the letters $c$, $d$ and $g$.

We also calculated that the remaining $SQS$-homogeneous $SP$-codes such that $9 \geq \kappa \geq 5$ and having 16 $STS$-homogeneous punctured codes are:
\[\kappa = 8: \quad (112, 3333777733337777), \quad (113, 55555555775775), \quad (117, 5555555522222222), \]
\[\kappa = 7: \quad (101, 4444422224444444), \quad (102, 4444444433223322), \quad (104, 4422422255225525), \quad (105, 4444444422322225), \quad (959, 8855885585858585). \]
\[\kappa = 6: \quad (960, 8888888833333333), \quad (961, 8888888822222222), \quad (962, 8888888811111111), \quad (963, 8888888811111111), \quad (964, 8888888811111111). \]

where each code denomination is accompanied between parenthesis by the common 16-tuple of STS(15)-types formed from the 16 punctured codes.

We also checked that there are still some SQS-homogeneous SP-codes with \(9 \geq \kappa \geq 5\) whose punctured codes are not STS-homogeneous:

\[\kappa = 6: \quad (914, 33338888888888gggg), \quad (915, 33338888888888gggg), \quad (926, 223333ddddddddgg), \quad (928, 224444455588888dd), \quad (929, 33334445555666677). \]
\[\kappa = 5: \quad (705, 44444444ddddeee), \quad (706, 44444444ddddeee), \quad (711, 44444444ddddeee), \quad (712, 44444444ddddeee). \]

where STS(15)-type denomination numbers are given in non-decreasing order, but their actual order differs coordinate by coordinate. For example, SP-code \(C = 914\), which is SQS-homogeneous, has \(H_K(C)\) holding 16 vertices yielding the 16-tuple 888888883g3g3g3 and 16 vertices yielding the 16-tuple 88888888g3g3g3g3 ≠ 888888883g3g3g3.

5 What do the edges of \(H_K(C)\) stand for?

In what follows we present Theorem 5, accounting for the structure of the SQS-subsets \(\phi^{-1}_K(e)\) represented by the edges \(e\) of \(H_K(C)\), for the 361 SP-codes \(C\) with \(9 \geq \kappa \geq 5\). (Recall that \(|\phi^{-1}_K(e)\|\) is the multiplicity of \(e\).

To express the coordinate indices in codewords of \(C\), we use hexadecimal notation: these indices constitute the set \([0, f] = \{0, 1, \ldots, 9, a, b, \ldots, f\}\).

Let \(0 < s \in \mathbb{Z}\). If \(Y\) is a set of quadruples of \([0, s]\), let the \(s\)-supplement of \(Y\) be the set of quadruples \(\{x_1, x_2, x_3, x_4\} \in [0, s]\) such that \(\{s - x_1, s - x_2, s - x_3, s - x_4\} \in Y\).
Let $S = \{s_1, \ldots, s_t\}$ be a partition of number 7 into positive integers $s_i$ such that $s = s_1 + \ldots + s_t$ and $s_1 \leq \ldots \leq s_t$. Let $Y$ be a $t$-set of quadruples of $[0,7]$. We define a descending (resp. an ascending), $S$-partition $P^1_{Y}$, (resp. $P^1_{Y}$), to be a partition $\{Y_1, \ldots, Y_t\}$ of $Y$ such that $|Y_i| = s_{i}$, for each $i \in [1,t]$, and if $(w_i, w_j) \in Y_i \times Y_j$, where $i, j \in [1, t]$ and $i < j$, then $w_i < w_j$, (resp. $w_i > w_j$), lexicographically.

If $S$ has $s_1$, (resp. $s_t$), equal to $\min \{2^{\kappa - 5} - 1, 7\}$ and has every other $s_i$ equal to $\min \{2^{\kappa - 5}, 7\}$, then we say that $P^1_{Y}$, (resp. $P^1_{Y}$), is a descending, (resp. ascending), $(\kappa - 5)$-partition.

Associated to the Fano plane on vertex set $[1,7]$ and line set $\{123, 145, 167, 247, 256, 346, 357\}$, we have the following three sets of quadruples:

- $X = \{0123, 0145, 0167, 0247, 0256, 0346, 0357\}$
- $Y = \{1457, 1678, 1467, 1578, 1567, 1256, 1257, 1246\}$
- $Z = \{cdef, abef, 89ef, 8bd, 9def, 9be, 8acf, 89ab, 89cd, abed, 9ace, 8bce, 8ace, 9bce\}$

where $Z = f$-supplement of $X \cup Y$.

**Theorem 5** Let $C$ be an $SP$-code of length 16 with $9 \leq \kappa \leq 5$. Then:

1. each vertex $v$ of $H_K(C)$ has a loop $\ell_v$ of multiplicity $|\phi_K^{-1}(\ell_v)|$, with its $SQS$-subset $\phi_K^{-1}(\ell_v)$ formed as the union of:
   - (a) $Z$; (b) the last set $Y_i$ in the descending $(\kappa - 5)$-partition $P^1_{Y}$, where $t = 2^{\max\{0,7-\kappa\}}$; (c) $X$, if $\kappa \geq 8$; (d) a specific product $C_{io} \times D_{8+j0}$ of partition classes $C_{io}$ and $D_{8+j0}$, if $\kappa = 9$;

2. each link $e$ of $H_K(C)$ has $\phi_K^{-1}(e)$ formed as the union of:
   - (a) a union of lexicographically ordered quarters (LOQs) of products $C_i \times D_{8+j}$ ($\neq C_{io} \times D_{8+j0}$ in item 1 above, if $\kappa = 9$), namely: (i) the eight LOQs in two such products, if $\kappa = 9$; (ii) the four LOQs in one such product, if $\kappa = 8$; (iii) between one and three LOQs in one such product, if $\kappa \leq 7$; (b) at most either one set $\neq Y_i$ in the descending $(\kappa - 5)$-partition $P^1_{Y}$ or one set in the ascending $(\kappa - 5)$-partition $P^1_{X}$, if $\kappa \leq 7$.

**Proof.** The properties of the loops, resp. links, of $C$, establishing the statement of the theorem for the five treated kernel dimensions $\kappa$ are considered in Section 6, resp. 7, with data details for the 361 cases considered present in [http://home.coqui.net/dejterij/details.tex](http://home.coqui.net/dejterij/details.tex) In Section 6, the diagonal and nondiagonal elements in the tables closing the three final subsections, (that is, for $\kappa \leq 7$), allow to establish items 1(b) and 2(b), respectively, in the statement of the theorem.

$\square$
6 Vertices and loops of $H_K(C)$

6.1 Case $\kappa = 9$

For each SP-code $C$ with $\kappa = 9$, namely for $C = 007$ and 008, there is a subspace $L$ of index 2 in $K = \text{Ker}(C)$ such that the vertices of $H_L(C)$ are given by eight classes mod $L$ that we denote $k = 0, \ldots, 7$, leading to four classes mod $K$ formed by the union of classes $2j$ and $2j + 1$ mod $L$, for $j = 0, 1, 2, 3$. This graph $H_L(C)$ has a loop of multiplicity 28 at each vertex of $H_L(C)$ represented by $X \cup Y \cup Z$. This loop together with an edge of multiplicity 16 obtained from a product as in the table of Subsection 7.1 below, for each vertex of $H_L(C)$, project onto a loop of multiplicity $28 + 16 = 44$ in $H_K(C)$.

6.2 Case $\kappa = 8$

The vertices of each one of the eight existing $H_K(C)$ here, namely for

$C = 005, 006, 112, 113, 114, 115, 116, 963$,

are given by 8 classes mod $K$ that we denote $k = 0, \ldots, 7$. Each such class has a loop of multiplicity 28, represented by $X \cup Y \cup Z$.

6.3 Case $\kappa = 7$

The vertices of each one of the 18 existing $H_K(C)$ here, namely for

$C = 002, \ldots, 004, 101, \ldots, 111, 959, \ldots, 962$,

are given by 16 classes mod $K$ that we denote $k_n$, with $k = 0, \ldots, 7$ and $n = 0, 1$. Each $H_K(C)$ presents a loop of multiplicity 21 represented by $X' = Y \cup Z$ and an edge of multiplicity 7 represented by $X$. The following contributive table for SQS-subsets $\phi^{-1}_K(\ell)$ of loops and links $\ell$ of $H_K(C)$ holds, with multiplicities indicated between parenthesis:

|       | $k_0$          | $k_1$          |
|-------|----------------|----------------|
| $k_0$ | $X'(21)$       | $X(7)$         |
| $k_1$ | $X(7)$         | $X'(21)$       |

6.4 Case $\kappa = 6$

The vertices of each one of the 86 existing $H_K(C)$ here, namely for

$C = 063, 064, 065, \ldots, 098, 099, 100, 911, 912, 913, \ldots, 956, 957, 958,$
are given by 32 classes mod $K$ that we denote $k_n$, where $k = 0, \ldots, 7$ and $n = 0, 1, 2, 3$. Let

$$A = \{0123, 0145, 0167\}, \quad B = \{0247, 0256, 0346, 0357\},$$

$A' = [0, 7]$-supplement of $A$, $B' = [0, 7]$-supplement of $B$

and $Z' = Z \cup A'$. The following contributive table for SQS-subsets $\phi_K^{-1}(\ell)$ of loops and links $\ell$ of $H_K(C)$ holds, with multiplicities indicated between parenthesis:

|   | $k_0$ | $k_1$ | $k_2$ | $k_3$ |
|---|-------|-------|-------|-------|
| $k_0$ | $Z'(17)$ | $B'(4)$ | $B(4)$ | $A(3)$ |
| $k_1$ | $B'(4)$ | $Z'(17)$ | $A(3)$ | $B(4)$ |
| $k_2$ | $B(4)$ | $A(3)$ | $Z'(17)$ | $B'(4)$ |
| $k_3$ | $A(3)$ | $B(4)$ | $B'(4)$ | $Z'(17)$ |

### 6.5 Case $\kappa = 5$

The vertices of each one of the 244 existing $H_K(C)$ here, namely for

$$C = 029, \ldots, 060, 061, 062, 701, 702, 703, 704, \ldots, 906, 907, 908, 909, 910,$$

are given by 64 classes mod $K1$ that we denote $k_n$, where $k, n \in \{0, \ldots, 7\}$. Let $A_0 = \{0123\}$, $A_1 = \{0145, 0167\}$, $B_0 = \{0247, 0256\}$, $B_1 = \{0346, 0357\}$; $A'_i = [0, 7]$-supplement of $A_i$, $B'_i = [0, 7]$-supplement of $B_i$, for $i = 0, 1$, and $Z_0 = Z \cup A'_0$. The following contributive table for SQS-subsets $\phi_K^{-1}(\ell)$ of loops and links $\ell$ of $H_K(C)$ holds, with multiplicities indicated between parenthesis and $f = 15$:

|   | $k_0$ | $k_1$ | $k_2$ | $k_3$ | $k_4$ | $k_5$ | $k_6$ | $k_7$ |
|---|-------|-------|-------|-------|-------|-------|-------|-------|
| $k_0$ | $Z_0(f)$ | $A'_1(2)$ | $B'_0(2)$ | $B'_1(2)$ | $B(2)$ | $B'_0(2)$ | $A_1(2)$ | $A_0(1)$ |
| $k_1$ | $A'_1(2)$ | $Z_0(f)$ | $B'_1(2)$ | $B'_0(2)$ | $B(2)$ | $B_1(2)$ | $A_0(1)$ | $A_1(2)$ |
| $k_2$ | $B'_0(2)$ | $B'_1(2)$ | $Z_0(f)$ | $A'_1(2)$ | $A_1(2)$ | $A_0(1)$ | $A_1(2)$ | $B_0(2)$ |
| $k_3$ | $B'_1(2)$ | $B'_0(2)$ | $A'_1(2)$ | $Z_0(f)$ | $A_0(1)$ | $A_1(2)$ | $B_0(2)$ | $B_1(2)$ |
| $k_4$ | $B_0(2)$ | $B_1(2)$ | $A_0(1)$ | $Z_0(f)$ | $A_1(2)$ | $B'_0(2)$ | $B'_1(2)$ | $B'_1(2)$ |
| $k_5$ | $B'_0(2)$ | $B'_1(2)$ | $A_0(1)$ | $A_1(2)$ | $A'_1(2)$ | $Z_0(f)$ | $B'_1(2)$ | $B'_1(2)$ |
| $k_6$ | $A'_1(2)$ | $A_0(1)$ | $B_0(2)$ | $B_1(2)$ | $B'_0(2)$ | $B'_1(2)$ | $Z_0(f)$ | $A'_1(2)$ |
| $k_7$ | $A_0(1)$ | $A'_1(2)$ | $B_0(2)$ | $B_1(2)$ | $B'_0(2)$ | $B'_1(2)$ | $A'_1(2)$ | $Z_0(f)$ |

### 7 Links of $H_K(C)$

In this section, we specify the form of the products claimed in Theorem 5. The actual denomination numbers in $\{0, \ldots, 6, 8, 9, 10\}$ for the partitions $\{C_0, \ldots, C_7\}$ and $\{D_8, \ldots, D_f\}$ of Theorem 1, which are used in those products, for each SP-code $C$ with $9 \geq \kappa \geq 5$, are shown explicitly in [http://home.coqui.net/dejterij/details.pdf](http://home.coqui.net/dejterij/details.pdf).
7.1 Case $\kappa = 9$

Consider the following partitions of length 7:

$$1_a = \frac{1}{3}, \quad 2_a = \frac{2}{3}, \quad 3_a = \frac{3}{3}, \quad 4_a = \frac{4}{3}, \quad 5_a = \frac{5}{3}, \quad 6_a = \frac{6}{3}, \quad 7_a = \frac{7}{3},$$

$$1_b = \frac{1}{6}, \quad 2_b = \frac{2}{6}, \quad 3_b = \frac{3}{6}, \quad 4_b = \frac{4}{6}, \quad 5_b = \frac{5}{6}, \quad 6_b = \frac{6}{6}, \quad 7_b = \frac{7}{6},$$

where two notations for each partitions are used. The first notation, $\alpha_p$, where $\alpha = 0, \ldots, 7$ and $p$ is a letter, is a shorthand used in the tables below. The second notation, $\alpha_p^\ell$, represents the partition with a lexicographically ordered form $\{0\alpha, x\ell, ym, zw\}$, as in the subsequent example, where the symbol $\alpha_p\beta_q (= 1_a1_a)$, with $\alpha, \beta \in [0, 7]$ and $p, q \in \{a, b\}$, represents the product $\alpha_p \times (\beta + 8)_q$ of the two partitions $\alpha_p$ and $(\beta + 8)_q$, in each case:

$$1_a1_a = 1^2_13^5_3 = \{01, 23, 45, 67\} \times \{89, ab, cd, ef\} = \{0189, \ldots, 67ef\}. \quad (1)$$

In case $C = 007$, we have the following contributive table for SQS-subsets $\phi_{e^{-1}}(e)$ of links $e$ of $H_L(C)$, where $L$ is as in Subsection 6.1:

| $k \backslash \ell$ | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   |
|---------------------|-----|-----|-----|-----|-----|-----|-----|-----|
| 0                   | ....| 1a  | 2a  | 3a  | 4a  | 5a  | 6a  | 7a  |
| 1                   | 1a  | ....| 2a  | 3a  | 4a  | 5a  | 6a  | 7a  |
| 2                   | 3a  | 2a  | ....| 1a  | 2a  | 3a  | 4a  | 5a  |
| 3                   | 2a  | 3a  | 1a  | 2a  | ....| 6a  | 7a  | 5a  |
| 4                   | 5a  | 4a  | 6a  | 7a  | 6a  | 5a  | 4a  | 6a  |
| 5                   | 4a  | 7a  | 5a  | 6a  | 7a  | 5a  | 4a  | 6a  |
| 6                   | 5a  | 6a  | 7a  | 5a  | 6a  | 7a  | 5a  | 4a  |
| 7                   | 7a  | 5a  | 6a  | 7a  | 5a  | 6a  | 7a  | 5a  |

In this table, the 16 quadruples corresponding to each sub-diagonal entry form the product $C_i \times C_j$ contributing to the SQS-subset $\phi^{-1}(\ell)$ of a corresponding loop $\ell$ of $H_K(C)$ as in item 1 of Theorem 5. The SQS-subsets $\phi_{e^{-1}}(e)$ for links $e$ of $H_K(C)$ are obtained by considering that the vertices of $H_K(C)$ are unions of the classes $2j$ and $2j + 1 \mod L$, for $j = 0, 1, 2, 3$.

A similar disposition for the case 008 is shown in tabulated format in http://home.coqui.net/dejterij/xyzPAT.txt where $xyz = 008$. By replacing $xyz$ by any other 3-string of an SP-code with $9 \geq \kappa \geq 5$, a corresponding file may be downloaded.

7.2 Case $\kappa = 8$

We deal here with 8 classes mod $K$, (instead of 8 classes mod $L$, as above). For $C = 005$, we have the following contributive table for SQS-subsets $\phi_{e^{-1}}(e)$ of links $e$ of $H_K(C)$, (otherwise, we refer to the last comment in Subsection 7.1):
7.3 Case $\kappa = 7$

In addition to the partitions mentioned in the subsections above, we need the following ones:

\[
\begin{align*}
1_c &= 1_5^3, & 2_c &= 2_5^3, & 3_c &= 3_2^5, & 4_c &= 4_5^5, & 4_d &= 4_6^7, & 4_e &= 4_6^9, & 5_c &= 5_7^4, \\
5_d &= 5_7^2, & 5_e &= 5_8^7, & 6_c &= 6_7^1, & 6_d &= 6_7^5, & 6_e &= 6_7^5, & 7_c &= 7_8^5, & 7_d &= 7_8^5, \\
7_c &= 7_8^4.
\end{align*}
\]

Codes 002, 003, 004, 102, 103, 104, 106, 107, 109, 959, 960, 961, 962, (resp. 101, 105), [resp. 108, 110, 111], use partitions of the form $\alpha_a, \alpha_b$, (resp. $\alpha_c, \alpha_d$), [resp. $\alpha_e, \alpha_\ell$], where $\alpha = 0, \ldots, 7$.

For example, in the case 002, we have the following contributive table for SQS-subsets $\phi^{-1}_m(e)$ of links $e$ of $H_K(C)$, where $m = 0, 1$:

| $k_n \, \backslash \, \ell_m$ | 0_m | 1_m | 2_m | 3_m | 4_m | 5_m | 6_m | 7_m |
|-----------------------------|-----|-----|-----|-----|-----|-----|-----|-----|
| 0_0                        |     | 1_1 | 2_1 | 3_1 | 4_1 | 5_1 | 6_1 | 7_1 |
| 0_1                        |     | 1_1 | 2_1 | 3_1 | 4_1 | 5_1 | 6_1 | 7_1 |
| 0_2                        |     | 1_1 | 2_1 | 3_1 | 4_1 | 5_1 | 6_1 | 7_1 |
| 0_3                        |     | 1_1 | 2_1 | 3_1 | 4_1 | 5_1 | 6_1 | 7_1 |
| 0_4                        |     | 1_1 | 2_1 | 3_1 | 4_1 | 5_1 | 6_1 | 7_1 |
| 0_5                        |     | 1_1 | 2_1 | 3_1 | 4_1 | 5_1 | 6_1 | 7_1 |
| 0_6                        |     | 1_1 | 2_1 | 3_1 | 4_1 | 5_1 | 6_1 | 7_1 |
| 0_7                        |     | 1_1 | 2_1 | 3_1 | 4_1 | 5_1 | 6_1 | 7_1 |

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The LOQs in which the products $\alpha_p\beta_q$ in the table above divide are the destinations of the classes $k_n$ in $M(C)$ that yield the contributions to the SQS-subsets $\phi_K^{-1}(e)$ of the edges $e$ of $H_K(C)$. A similar second table can be set with a symbol $\epsilon_1\epsilon_2\epsilon_3\epsilon_4$ in each non-diagonal entry, each $\epsilon_i$ representing a LOQ of an $\alpha_p\beta_q$. In fact, for $k_n$ with $n = 0$, $(n = 1)$, we have: $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 1000$, $(\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 0111)$, where $k \in [0, 7]$. For example, $1\frac{2}{3}1\frac{2}{3}$ in position $(k_n, \ell_m) = (0_0, 1_m)$ in the table above has $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 = 1000$ in this second table, meaning that $0_0$ assigns

$$\begin{align*}
\{018a, 019b, 01cd, 01ef\} & \text{ to } \ell_m = \ell_{n_1} = 1_0; \\
\{238a, 239b, 23cd, 23ef\} & \text{ to } \ell_m = \ell_{n_2} = 0_0; \\
\{458a, 459b, 45cd, 45ef\} & \text{ to } \ell_m = \ell_{n_3} = 0_0; \\
\{678a, 679b, 67cd, 67ef\} & \text{ to } \ell_m = \ell_{n_4} = 0_0.
\end{align*}$$

A listing showing the combination of the file $xyz$PAT.txt mentioned in Subsection 7.1 and the $\epsilon_1\epsilon_2\epsilon_3\epsilon_4$ above can be retrieved from http://home.coqui.net/dejterij/xyzTEST.txt, where $xyz$ is 002 or the value of $xyz$ corresponding to any SP-code with $\kappa = 7, 6, 5$.

### 7.4 Case $\kappa = 6$

Consider the partitions of length 7 given above together with:

$$\begin{align*}
1_d = 1\frac{2}{3}, & \quad 1_e = 1\frac{2}{3}, & \quad 1_f = 1\frac{2}{3}, & \quad 2_d = 2\frac{3}{5}, & \quad 2_e = 2\frac{3}{5}, & \quad 2_f = 2\frac{3}{5}, \\
3_d = 3\frac{4}{5}, & \quad 3_e = 3\frac{4}{5}, & \quad 3_f = 3\frac{4}{5}, & \quad 4_d = 4\frac{5}{7}, & \quad 4_e = 4\frac{5}{7}, & \quad 4_f = 4\frac{5}{7}, \\
5_f = 5\frac{6}{7}, & \quad 5_g = 5\frac{6}{7}, & \quad 6_f = 6\frac{1}{3}, & \quad 6_g = 6\frac{1}{3}, & \quad 7_f = 7\frac{2}{3}, & \quad 7_g = 7\frac{2}{3}.
\end{align*}$$

For each SP-code $C$ with $\kappa = 6$, we can assign a product $\alpha_p\beta_q$ to each class $k_n = 0_0, \ldots, 7_3$ in eight tables, each with rows headed by $k_n$, where $k \in [0, 7]$ is fixed and $n$ varies in $[0, 3]$, and with columns headed by all values of $k_m$, where $m \in [0, 3]$ is fixed and $k$ varies in $[0, 7]$. Each of these eight tables expresses the needed products $\alpha_p\beta_q$. We exemplify the values of $p$ for the case $C = 066$ in a table with each entry $(k, n) \in [0, 7] \times [0, 3]$ containing a literal 7-tuple $(p_1, p_2, p_3, p_4, p_5, p_6, p_7)$ which is the 7-tuple of subindexes in a corresponding expression $(1_{p_1}, 2_{p_2}, 3_{p_3}, 4_{p_4}, 5_{p_5}, 6_{p_6}, 7_{p_7})$:

| $k \setminus n$ | 0   | 1   | 2   | 3   |
|-----------------|-----|-----|-----|-----|
| 0               | abacdb | abacdb | abacdb | abacdb |
| 1               | abacdb | abacdb | abacdb | abacdb |
| 2               | abacdb | abacdb | abacdb | abacdb |
| 3               | bacabca | bacabca | bacabca | bacabca |
| 4               | bacabca | bacabca | bacabca | bacabca |
| 5               | bcaabb | bcaabb | bcaabb | bcaabb |
| 6               | bcaabb | bcaabb | bcaabb | bcaabb |
| 7               | bcaabb | bcaabb | bcaabb | bcaabb |

The components $\beta_q$ of products $\alpha_p\beta_q$ here are constant-partition 4-tuples. The information for case $C = 066$ can be condensed as follows:
where \( n, m \in [0, 3] \). An observation on LOQs of the \( \alpha_p \beta_q \)'s similar to the one in Subsection 7.3 holds here. In fact, an accompanying table for the resulting 4-tuples \( \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 \) of LOQs can be composed by replacing the symbols \( A \) and \( B \) in the simplified table on the left side below (accompanying the one above) by the sub-tables on its right side:

\[
\begin{array}{ccccccccccc}
\hline
k \backslash \ell & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & A & B \\
\hline
0 & A & B & B & B & B & B & B & B & k_0 & k_0 \ \\
1 & A & B & B & B & B & B & B & B & k_1 & k_1 \ \\
2 & B & B & A & B & B & B & B & B & k_2 & k_2 \ \\
3 & B & B & B & B & B & B & B & B & k_3 & k_3 \ \\
4 & B & B & B & B & A & B & B & B & & & \\
5 & B & B & B & B & B & B & A & B & & & \\
6 & B & B & B & B & B & B & B & A & & & \\
7 & B & B & B & B & B & B & B & A & & & \\
\hline
\end{array}
\]

### 7.5 Case \( \kappa = 5 \)

In addition to the partitions given above, consider:

\[
1_g = \frac{15}{4}, \quad 1_h = \frac{15}{4}, \quad 1_i = \frac{15}{4}, \quad 1_j = \frac{15}{4}, \quad 1_k = \frac{15}{4}, \quad 2_g = \frac{26}{4}, \quad 2_h = \frac{26}{4}, \quad 2_i = \frac{26}{4}, \quad 2_j = \frac{26}{4}, \quad 2_k = \frac{26}{4},
\]

\[
3_g = \frac{36}{4}, \quad 3_h = \frac{36}{4}, \quad 3_i = \frac{36}{4}, \quad 3_j = \frac{36}{4}, \quad 3_k = \frac{36}{4},
\]

\[
4_g = \frac{46}{4}, \quad 4_h = \frac{46}{4}, \quad 4_i = \frac{46}{4}, \quad 4_j = \frac{46}{4}, \quad 4_k = \frac{46}{4},
\]

\[
5_g = \frac{56}{4}, \quad 5_h = \frac{56}{4}, \quad 5_i = \frac{56}{4}, \quad 5_j = \frac{56}{4}, \quad 5_k = \frac{56}{4},
\]

\[
6_g = \frac{66}{4}, \quad 6_h = \frac{66}{4}, \quad 6_i = \frac{66}{4}, \quad 6_j = \frac{66}{4}, \quad 6_k = \frac{66}{4},
\]

For each SP-code \( C \) with \( \kappa = 5 \), we can assign a product \( \alpha_p \beta_q \) to each class \( k_n = 0_0, \ldots, 7_7 \) in eight tables, each with rows headed by \( k_n \), where \( k \in [0, 7] \) is fixed and \( n \) varies in \([0, 7] \), and with columns headed by all the values of \( k_m \), where \( m \in [0, 7] \) is fixed and \( k \) varies in \([0, 7] \). Each of these eight tables expresses the needed products \( \alpha_p \beta_q \). A condensed form of these eight tables exists as in Subsection 7.4 and can be obtained from the sources cited at the end of Subsections 7.1 and 7.3.

An observation on LOQs of the \( \alpha_p \beta_q \)'s similar to those in Subsections 7.3-4 holds. An accompanying table for the resulting 4-tuples \( \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 \) of LOQs can be composed by replacing the symbols \( A, B, C, D \) in the simplified table below, to the left (accompanying the one above) by the sub-tables.
on its right side, where the column headers $A, B, C, D$ stand for the corresponding 4-tuples $\epsilon_1\epsilon_2\epsilon_3\epsilon_4$ of LOQs:

| $k\backslash \ell$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $k_0$ | $k_1$ | $k_2$ | $k_3$ | $k_4$ | $k_5$ | $k_6$ | $k_7$ |
|------------------|---|---|---|---|---|---|---|---|------|------|------|------|------|------|------|------|
| 0                | A | B | C | D | D | D | D | D | 6100 | 4300 | 5200 | 4110 | 4210 | 5310 | 4311 | 4311 |
| 1                | B | C | A | D | D | D | D | D | 7110 | 5211 | 6311 | 5311 | 5311 | 6320 | 4320 | 4320 |
| 2                | C | B | A | D | D | D | D | D | 7110 | 5211 | 6311 | 5311 | 5311 | 6320 | 4320 | 4320 |
| 3                | D | D | D | D | A | B | C | B | 5332 | 7342 | 6314 | 7454 | 4654 | 6314 | 7454 | 4654 |
| 4                | D | D | D | D | A | B | C | B | 5332 | 7342 | 6314 | 7454 | 4654 | 6314 | 7454 | 4654 |
| 5                | D | D | D | D | C | B | A | .  | 5354 | 7564 | 4654 | 6314 | 7454 | 4654 | 6314 | 7454 |
| 6                | D | D | D | D | C | B | A | .  | 5354 | 7564 | 4654 | 6314 | 7454 | 4654 | 6314 | 7454 |
| 7                | D | D | D | D | B | C | A | .  | 1776 | 3774 | 2775 | 3764 | 2764 | 3775 | 2764 | 3764 |

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