Vortices appearance and diffusion

A M Gaifullin¹,², D A Gadzhiev¹,² and A V Zubtsov¹

¹ Central Aerohydrodynamic Institute, Zhukovskii, Russia
² Moscow Institute of Physics and Technology, Dolgoprudnyi, Russia

E-mail: gaifullin@tsagi.ru

Abstract. The problem of a vortex flow generation by a rotating infinite cylinder is considered. The solution satisfying the two-dimensional unsteady equations describing a viscous gas motion is constructed. Asymptotic and numerical solutions of the problem are obtained. The comparison of the results is given.

1. Introduction

A variety of ways of the vortices formation is known: a vortex sheet shedding from a rigid surface, the flow interaction with the surface, shear layers interaction, etc. The correct description of the vortices diffusion process is important for the solution of a lot of problems. The examination of the point vortex diffusion in viscous incompressible fluid problem can be found in most of books on hydrodynamics. The problem of two opposite vortices diffusion has been solved in [1]. The case in which diffusive and convective processes change in time identically is interesting in a sense of the vortices appearance and diffusion distinctive features investigation. This is the case of the Navier-Stokes equations self-similar solution [2]. The flow interaction with a moving surface is considered in [3].

In this paper, a vortex generation and diffusion by the rotating cylinder for the cases of both incompressible and compressible flows are studied.

2. Rotating cylinder in incompressible fluid

The transverse velocity field

\[ w = \frac{\Gamma}{r}, \]  

where \( \Gamma \) is circulation divided by \( 2\pi \) and \( r \) is the distance from the vortex, is an either Euler equations and Navier - Stokes equations exact solution. However, in the second case the permanent power supply is required to maintain the velocity field (1).

A fluid moving with the velocity (1) has infinite kinetic energy, so to create the field (1) in the whole flow domain is impossible. However, there is a way to create the field (1) in the limited flow domain. Let an infinitely extended cylinder with the radius \( r \) start to rotate around its axis at the time \( t = 0 \) so that its surface speed is \( w = \Gamma / r \). Because of the action of the viscous forces, a fluid near the cylinder starts moving.

We consider that axisymmetric flow with radial velocity equal to zero is implemented. That is, the flow in the absence of disturbances leading possibly to the flow instability is studied. Hence, the flow around the cylinder at \( t \geq 0 \) must satisfy the equation
\[
\frac{\partial \Gamma}{\partial t} = \nu \left( \frac{\partial^2 \Gamma}{\partial r^2} - \frac{1}{r} \frac{\partial \Gamma}{\partial r} \right),
\]  

where \( \Gamma(r,t) = rw \).

We write the boundary conditions at \( t \geq 0 \)

\[
\Gamma(r,t) = \Gamma_* \quad \Gamma(\infty,t) = 0
\]

and the initial condition at \( r > r_* \)

\[
\Gamma(r,0) = 0.
\]

We must pay a special attention for the second condition from (3). This means that for any time \( t \) there is the distance \( r \) where the disturbances due to the cylinder rotation are small and in the limit \( r \to \infty \) they may be considered to tend to zero and, consequently, the circulation \( \Gamma(\infty,t) \) tends to zero too.

Subsequently, for the cases of either incompressible and compressible flows the solution is determined at large times when the scale of the domain in which \( \Gamma(r,t) \sim \Gamma_* \) is much larger than \( r \).

At first we determine the problem solution in the far domain \( r \gg r_* \). Because of the equation (2) and the boundary conditions (3), the solution must depend on the variables \( t, r \) and the value \( \nu \). Only one dimensionless variable \( \eta = r/\sqrt{vt} \) can be composed from these three variables. This dimensionless variable determines the asymptotic domain \( \Omega_2 : r \sim \sqrt{vt} \) in which \( \Gamma(r,t) = \Gamma(\eta) \). The solution is

\[
\Gamma(r,t) = \Gamma_* \exp(-\eta^2/4) = \Gamma_* \exp(-r^2/4\nu t)
\]

The solution (4) satisfies the second boundary condition from (3) but does not satisfy the first one. Consequently, the solution in the inner domain \( \Omega_2 : r \sim r_* \) must be considered. The inner limit of the outer expansion, that is, the solution (4) limit as \( \eta \to 0 \), is

\[
\lim_{\eta \to 0} \Gamma(r,t) = \Gamma_*.
\]

In the domain \( \Omega_3 \), the leading-order behavior of \( \Gamma \) must satisfy the equation (2) and the boundary condition \( \Gamma = \Gamma_* \) as \( r/r_* \to \infty \). The solution is

\[
\Gamma = \Gamma_*.
\]

### 3. Rotating cylinder in compressible fluid

The two-dimensional axisymmetric flow characteristics must satisfy the equations

\[
\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho v^2 + p)}{\partial r} + \frac{P}{r} (v^2 - w^2) = \frac{2}{3} \frac{\partial}{\partial r} \left[ \mu \left( \frac{\partial v}{\partial r} - \frac{v}{r} \right) \right] + \frac{2}{r} \mu \left( \frac{\partial v}{\partial r} - \frac{v}{r} \right),
\]

\[
\frac{\partial (\rho w)}{\partial t} + \frac{\partial (\rho vw)}{\partial r} + 2 \frac{\rho v w}{r} = \frac{\partial}{\partial r} \left[ \mu \left( \frac{\partial w}{\partial r} - \frac{w}{r} \right) \right] + \frac{2}{r} \mu \left( \frac{\partial w}{\partial r} - \frac{w}{r} \right),
\]

\[
\frac{\rho c_p}{\partial t} + \frac{\partial (\rho c_p v)}{\partial r} = \frac{\partial}{\partial r} \left[ \mu \left( \frac{\partial v}{\partial r} - \frac{v}{r} \right) \right] + \frac{2}{r} \mu \left( \frac{\partial v}{\partial r} - \frac{v}{r} \right),
\]

\[
= \frac{c_p \mu}{Pr} \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) \right] + \frac{c_p \mu}{Pr} \frac{\partial T}{\partial r} + \mu \left[ \frac{2}{3} \left( \frac{\partial v}{\partial r} \right)^2 + 2 \left( \frac{\partial v}{r} \right)^2 + \left( \frac{\partial w}{\partial r} - \frac{w}{r} \right)^2 - \frac{2}{3} \left( \frac{\partial v}{\partial r} - \frac{v}{r} \right)^2 \right],
\]

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial r} + \frac{\rho v}{r} = 0,
\]

\[
p = \rho RT.
\]
Here, \( r, \theta \) are polar coordinates, \( v, w \) are corresponding velocity components, \( t \) is time, \( \rho \) is density, \( p \) is pressure, \( c_p \) is specific heat at constant pressure, \( T \) is temperature, \( R \) is gas constant, \( \Pr \) is the Prandtl number, \( \mu \) is dynamic viscosity depending on temperature. In this paper, to be definite, we assume \( \mu/T = \text{const} \), \( c_p = \text{const} \) and \( \Pr = \text{const} \).

The cylinder surface (\( r = r_c \)) boundary conditions associated with equations (5):

\[
v = 0, \quad w = \frac{\Gamma_s}{r_c}, \quad f\left(T, \frac{\partial T}{\partial r}\right) = 0.
\]

(6)

The far-field boundary conditions (\( r \to \infty \)):

\[
v = 0, \quad w = 0, \quad T = T_0, \quad p = p_0, \quad \rho = \rho_0, \quad \mu = \mu_0.
\]

The latter condition from (6) means that either temperature, temperature derivative or their combination can be applied on the cylinder surface. The significant (transverse velocity is not exponentially small) flow disturbances due to the cylinder rotation are propagated to the finite scale expanding in time. The disturbances out of this scale are small, so the flow characteristics as \( r \to \infty \) must satisfy the unperturbed parameters.

As in the case of incompressible flow, we firstly determine the problem solution in the far domain \( \Omega_2 : r \gg r_c \). The study is limited by large Reynolds numbers \( \text{Re} = \Gamma_c \rho_0 / \mu_0 = \Gamma_c / v_0 \gg 1 \) and Mach numbers \( M_c = \Gamma_c / r_c a_0 \sim \mathcal{O}(1) \). Here, \( a_0 = \sqrt{\gamma R T_0} \) is the unperturbed stream sound speed, where \( \gamma \) is the ratio of specific heats, which is assumed to be constant. We note that the number \( M_c \) is introduced as the velocity on the cylinder surface divided by the unperturbed stream sound speed.

The scale of the far domain is \( r \sim \sqrt{\nu_{o} \lambda} \). The gas flow characteristics in this domain can be divided into two types. The first is disturbances due to the cylinder rotation. The typical transverse velocity is estimated as \( w \sim \Gamma_c / r \). The second is disturbances due to the temperature factor defined as the cylinder surface temperature divided by the far field temperature, which can exist even in the absence of the cylinder rotation. Both types of the temperature, pressure and density disturbances are small in the domain \( \Omega_2 \) compared to corresponding magnitudes in the gas being at rest. Consequently, the problem linearization is possible and the disturbances of each type can be studied independently on another.

Firstly, we consider the disturbances due to the cylinder rotation. The typical Mach number \( M_1 \) in the domain \( \Omega_2 \) is defined as the typical transverse velocity \( w \sim \Gamma_c / \sqrt{\nu_{o} \lambda} \) divided by the unperturbed stream sound speed:

\[
M_1 = \frac{\Gamma_c}{a_0 \sqrt{\nu_{o} \lambda}} << 1.
\]

The equations (5) are followed by the asymptotic form of the variables

\[
w = M_c a_0 w_{1,1}(\eta, t), \quad v = \frac{M_c^2 a_0}{\text{Re}} v_{1,1}(\eta, t), \quad T = T_0 \left(1 + M_c^2 T_{1,1}(\eta, t)\right),
\]

\[
p = p_0 \left(1 + M_c^2 p_{1,1}(\eta, t)\right), \quad \rho = \rho_0 \left(1 + M_c^2 \rho_{1,1}(\eta, t)\right), \quad \mu = \mu_0 \left(1 + M_c^2 \mu_{1,1}(\eta, t)\right),
\]

where \( \eta = r / \sqrt{\nu_{o} \lambda} \).

For the dimensionless variables, the leading-order equations following from the equations (5) are

\[
\frac{\partial \tilde{p}_{1,1}}{\partial \eta} = \frac{\gamma \tilde{w}_{1,1}^2}{\eta},
\]

where
\[
\frac{\partial^2 w_{1,1}}{\partial \eta^2} + \frac{\partial w_{1,1}}{\partial \eta} \left( \frac{1 + \eta}{2} \right) + w_{1,1} \left( \frac{1 - \frac{1}{2}}{\eta} \right) = t \frac{\partial w_{1,1}}{\partial t},
\]

\[
\frac{\gamma}{\gamma - 1} \left[ \frac{1}{\text{Pr}} \left( \frac{\partial^2 T_{1,1}}{\partial \eta^2} + \frac{1}{\eta} \frac{\partial T_{1,1}}{\partial \eta} \right) + T_{1,1} + \frac{\eta}{2 \gamma} \frac{\partial T_{1,1}}{\partial \eta} - t \frac{\partial^2 \rho_{1,1}}{\partial \eta^2} \right] - \left( \rho_{1,1} + \frac{\eta}{2} \frac{\partial \rho_{1,1}}{\partial \eta} - t \frac{\partial \rho_{1,1}}{\partial t} \right) + \frac{1}{\gamma} \left( \frac{\partial w_{1,1}}{\partial \eta} - \frac{w_{1,1}}{\eta} \right)^2 = 0,
\]

(7)

The self-similar solution of the second equation from (7) according to the incompressible flow solution is

\[
w_{1,1} = \frac{A}{\eta} \exp \left(-\eta^2 / 4 \right).
\]

To require that the circulation as \( \eta \to 0 \) must be equal to the circulation on the cylinder surface is not appropriate because the solution is obtained in the domain \( \Omega \), whereas the cylinder surface boundary condition must be satisfied when the solution in the near-cylinder domain is constructed. So the value of the coefficient \( A \) is determined later. The equations (7) using the expression (8) result in

\[
p_{1,1} = \gamma A \int_{-\infty}^{\infty} \frac{\exp \left(-\eta^2 / 2 \right)}{\eta^3} d\eta.
\]

The asymptotic behaviour of the equations solution as \( \eta \to 0 \) is

\[
w_{1,1} \to \frac{A}{\eta}, \quad p_{1,1} \to -\frac{\gamma A^2}{2\eta^2}, \quad T_{1,1} \to \frac{1}{\eta} \text{Pr} \frac{A^2}{\eta^2}, \quad \rho_{1,1} \to A \left[ \frac{1}{\eta} \text{Pr} - \frac{\gamma}{2} \right], \quad v_{1,1} \to 0.
\]

The disturbances of the second type are determined in the presence of heat convection in the absence of the transverse velocity. In that case, the asymptotic expansion of the variables is written as follows:

\[
T = T_0 \left[ 1 + \varepsilon T_{1,2} (\eta,t) \right], \quad \rho = \rho_0 \left[ 1 + \varepsilon \rho_{1,2} (\eta,t) \right], \quad v = \sqrt{\frac{\nu_0}{t}} v_{1,1} (\eta,t), \quad p = p_0 \left[ 1 + \varepsilon \frac{\rho_0 \nu_0}{p_0 t} \rho_{1,2} (\eta,t) \right],
\]

\[
\mu = \mu_0 \left[ 1 + \varepsilon T_{1,2} (\eta,t) \right].
\]

The value of \( \varepsilon \ll 1 \) is determined later. For the dimensionless temperature, density and viscosity disturbances, we obtain the following equations:

\[
\frac{\partial T_{1,2}}{\partial t} = \frac{\partial^2 T_{1,2}}{\partial \eta^2} + \frac{1 + \eta}{2 \gamma} \frac{\partial T_{1,2}}{\partial \eta},
\]

(9)

\[
\rho_{1,2} = -T_{1,2}.
\]

The self-similar solution of the first equation from (9)

\[
T_{1,2} = \int_{-\infty}^{\eta} \frac{\nu_{1,2}}{\eta} \exp \left(-\eta t \right) d\eta.
\]

The asymptotic behaviour of the solution as \( \eta \to 0 \)

\[
T_{1,2} \to \ln \eta + c_1,
\]

where \( c_1 = \frac{C + \ln \text{Pr}}{2} - \ln 2 \), \( C \) is the Euler’s constant.

Since the problem in the domain \( \Omega \) is linear, we can write the solution as \( \eta \to 0 \):

\[
w \to \frac{Ma_0 A}{\eta} = \Gamma A \frac{1}{r}, \quad p \to p_0 \left( 1 - M_1^2 \frac{\gamma A^2}{2\eta^2} \right) = p_0 \left( 1 - \frac{\Gamma^2 \gamma A^2}{2a_0^2 r^2} \right),
\]

where \( r = \frac{C + \ln \text{Pr}}{2} - \ln 2 \), \( \Gamma = \frac{Ma_0 A}{\eta} \), \( M_1 = \frac{a_0}{r} \), \( a_0 \) is the initial velocity, \( \gamma \) is the adiabatic index, \( \text{Pr} \) is the Prandtl number.
\[ T \rightarrow T_0 \left[ 1 - M_i^2 \left( \frac{\gamma - 1}{\eta} \right) Pr A^2 + \varepsilon \ln \eta + \varepsilon c_1 \right] = T_0 \left[ 1 - \frac{\Gamma_i^2 \left( \frac{\gamma - 1}{\eta} \right) Pr A^2}{a_0^2 r^2} + \varepsilon \ln \frac{r}{\sqrt{\nu_0 t}} + \varepsilon c_1 \right]. \] (10)

The expression (10) is followed that transverse velocity and pressure tend to functions independent on time whereas temperature has both the disturbance independent on time and the disturbance dependent on time.

When the temperature \( T = T_0 \) close to the temperature \( T_0 \) and the Mach number \( M_i = \Gamma_i / \rho a_0 \ll 1 \) are given on the cylinder surface, that is, disturbances are small, it is easily shown that the solution (10) can be extended to the cylinder surface. In this case
\[ A = 1, \quad \varepsilon = \left[ \frac{T - T_0}{T_0} + M_i^2 \left( \frac{\gamma - 1}{\eta} \right) Pr \right] \left( \ln \frac{r}{\sqrt{\nu_0 t}} + c_1 \right). \]

In the case of the thermally-insulated cylinder surface (\( \partial T / \partial r = 0 \)) for small disturbances we obtain
\[ A = 1, \quad \varepsilon = -2M_i^2 \left( \frac{\gamma - 1}{\eta} \right) Pr, \]
\[ T = T_0 \left[ 1 - 2M_i^2 \left( \frac{\gamma - 1}{\eta} \right) Pr \left( \ln \frac{r}{\sqrt{\nu_0 t}} + c_1 + \frac{1}{2} \right) \right]. \] (11)

The expression (11) is followed that the gas temperature on the cylinder surface is larger than the unperturbed gas temperature and increases in time slowly (logarithmically).

In the \( M_i \sim O(1) \) case, the temperature disturbances reach the order of unity on the scale \( r \sim M_i r = \Gamma_i / \rho a_0 \), according to the expression (10). Then, we study the flow characteristics in the strong disturbances domain \( \Omega_2 \) turning from the independent variables \( r, t \) to the new independent variables \( r_2, t \), where \( r_2 = r a_0 / \Gamma_i \).

The expression (10) determines the asymptotic form of the flow characteristics in the domain \( \Omega_2 \) as follows:
\[ w = a_0 w_2 \left( r_2, t \right), \quad p = p_0 p_2 \left( r_2, t \right), \quad T = T_0 T_2 \left( r_2, t \right), \quad \rho = \frac{\rho_0 p_2}{T_2}, \quad \mu = \mu_0 T_2. \]

The leading-order behaviour of the dimensionless variables \( w_2, p_2, T_2 \) is determined from the following equations:
\[ \frac{\partial}{\partial r_2} \left[ \mu \left( \frac{\partial w_2}{\partial r_2} - \frac{w_2}{r_2} \right) \right] + 2\mu \left( \frac{\partial^2 w_2}{\partial r^2} \right) = 0, \]
\[ \frac{\partial p_2}{\partial r_2} = \frac{\gamma}{T_2} \frac{\rho_0 p_2}{T_2} \frac{w_2^2}{r_2}, \]
\[ \left( \frac{\partial^2 T_2}{\partial r_2^2} + \frac{1}{r_2^2} \frac{\partial T_2}{\partial r_2} \right) T_2 + \frac{\partial T_2}{\partial r_2} + \left( \frac{\partial T_2}{\partial r_2} \right)^2 + Pr \left( \frac{\gamma - 1}{\eta} \right) T_2 \left( \frac{\partial^2 w_2}{\partial r_2^2} - \frac{w_2}{r_2} \right)^2 = 0. \] (12)

The first equation from (12) results in
\[ \frac{\partial w_2}{\partial r_2} - \frac{w_2}{r_2} = \frac{\alpha}{\mu r_2^2}. \] (13)

The constant \( \alpha \) is determined from the condition of matching the solution in the domain \( \Omega_1 \)
\[ w_2 = \frac{A}{r_2} \text{ as } r_2 \rightarrow \infty. \] (14)

Therefore, \( \alpha = -2A\mu_0 \).
The equation (13) substitution in the third equation from (12) and the definition \( u = T_z^2 \) give
\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{8A^2 \Pr (\gamma - 1)}{r^4 \sqrt{u}} = 0
\] (15)
with the corresponding condition as \( r_z \to \infty \)
\[
u = 1 - 2 \frac{A^2 \Pr (\gamma - 1)}{r_z^2} + 2 \varepsilon \left( \ln \frac{r_z \Gamma}{\alpha_{\nu} \sqrt{\nu t}} + c_1 \right).
\] (16)

Thus, the equations (13), (15) with the boundary conditions (14), (16) as \( r_z \to \infty \) and two boundary conditions on the cylinder surface (at \( r_z = 1/M_z \)) must be solved. The first cylinder surface boundary condition is the non-slip condition for the transverse velocity
\[
w_z = M_z.
\] (17)
The second is the condition for the cylinder surface temperature
\[
f_z \left( T_z, \frac{\partial T_z}{\partial r_z} \right) = 0.
\] (18)

The system of two ordinary differential equations (13), (15) with the boundary conditions (14), (16) - (18) has been solved numerically. Two unknown constants \( A \) and \( \varepsilon \) were fitted using iterative method so that the cylinder surface boundary conditions (17), (18) were fulfilled.

Using the solutions in the domains \( \Omega_1 \) and \( \Omega_2 \), the composite solution of the problem has been constructed.

In addition to the asymptotic study, the numerical solution of the cylinder rotating in a compressible fluid problem using the commercial program ANSYS Fluent 16.2.0 has been carried out. The comparison of asymptotic and numerical results is represented in the Figure 1 and Figure 2.

**Figure 1.** The circulation radial distribution \( \frac{\Gamma(r_z)}{\Gamma_*} \) for the case \( M_z = 1 \) at time \( r_z / \sqrt{\nu t} = 5 \times 10^{-3} \). The solid line is the composite asymptotic solution; the dashed line is the numerical solution using ANSYS Fluent.

**Figure 2.** The temperature radial distribution \( \frac{T(r_z)}{T_0} \) for the case \( M_z = 1 \) at time \( r_z / \sqrt{\nu t} = 5 \times 10^{-3} \). The solid line is the composite asymptotic solution; the dashed line is the numerical solution using ANSYS Fluent.

**Acknowledgements**
The work was supported by RFBR (the project № 16-01-00128).

**References**
[1] Gaifullin A M and Zubtsov A V 2004 Diffusion of two vortices *J. Fluid Mech.* 39 112
[2] Gaifullin A M 2005 Self-similar unsteady viscous flow *J. Fluid Mech.* 40 526
[3] Gaifullin A M 2006 Flow past a plate with an upstream-moving surface *J. Fluid Mech.* 41 375