Some Results on $C$-retractable Modules

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Abstract. An $R$-module $M$ is called $c$-retractable if there exists a nonzero homomorphism from $M$ to any of its nonzero complement submodules. In this paper, we provide some new results of $c$-retractable modules. It is shown that every projective module over a right $SI$-ring is $c$-retractable. A dual Baer $c$-retractable module is a direct sum of a $Z_2$-torsion module and a module which is a direct sum of nonsingular uniform quasi-Baer modules whose endomorphism rings are semi-local quasi-Baer. Conditions are found under which, a $c$-retractable module is extending, quasi-continuous, quasi-injective and retractable. Also, it is shown that a locally noetherian $c$-retractable module is homo-related to a direct sum of uniform modules. Finally, rings over which every $c$-retractable is a $C_4$-module are determined.

2020 Mathematics Subject Classifications: 13B10,13C05,13C13

Key Words and Phrases: Retractable modules, complement submodules, $c$-retractable modules, projective modules

1. Introduction

Throughout all rings are associative with identity and all modules are unitary right module. Let $R$ be a ring. Following [19], we say that an $R$-module $M$ is retractable if $\text{Hom}_R(M,N) \neq \{0\}$ for any nonzero submodules $N$ of $M$. It is shown in [19] that every projective module over a right $V$-ring is retractable. In [19] again, the semisimplicity of retractable modules is studied. M. R. Vedadi [23], introduced the concept of essentially retractable modules and proved that over semiprime right nonsingular rings, a nonsingular essentially retractable module is precisely a module with non-zero dual. In [7], A. Ghorbani and M. R. Vedadi introduced and studied the notion of epi-retractable module, where a module $M$ is called epi-retractable if every submodule of $M$ is a homomorphic image of $M$. They reveal some applications of projective, nonsingular, injective epi-retractable
modules regarding the characterization of Bezout, pri, quasi-Frobenius rings. Note that epi-retractable modules are retractable. Earlier, P. F. Smith and A. Tercan [20] introduced $C_{11}$-module as a generalization of extending modules, where a module $M$ is said to be satisfy $C_{11}$-condition if every submodule of $M$ has a complement which is a direct summand. It is shown in ([20], Theorem 2.7) that a module satisfies $(C_{11}$ if and only if $M = Z_2(M) \oplus K$ for some (nonsingular) $K$ of $M$ and $Z_2(M)$ and $K$ both satisfy $(C_{11})$. Later, the same authors investigated when a direct summand of a $C_{11}$-module inherits the property [21]. Recently, $t$-closed submodules of a module $M$ are defined in [2] as closed submodules of $M$ which contain $Z_2(M)$. In [3], S. H. Asgari, A. Haghighy and A.R. Rezaei studied the modules $M$ for which $C_{11}$-condition holds for $t$-closed submodules ($T_{11}$-type, for short). They showed among others the following results: i) A $T_{11}$-type module is exactly a direct sum of a $Z_2$-torsion module and a nonsingular $C_{11}$-modules. (ii) A $T_{11}^+$-module (modules for which direct summands are $T_{11}$-type) is precisely a direct sum of $Z_2$-torsion and nonsingular $C_{11}^+$-module (modules for which direct summands satisfy $C_{11}$). A. W. Chatters and S. M. Kheuri [4] defined the concept of $c$-retractable module, where an $R$-module $M$ is called $c$-retractable if $\text{Hom}_R(M, C) \neq \{0\}$ for any nonzero complement submodules $C$ of $M$. This notion is a generalization of both the retractable modules and the extending modules. They have shown that if $M$ is a nonsingular $c$-retractable module such that $S_S$ is extending, then $M$ is extending. But the converse is not true in general. On the other hand it is shown in [22] that if $M$ is a retractable $wd$-Rickart module, then every indecomposable submodule of $M$ is a simple direct summand. Motivated by the definition of the modules mentioned above and the results on retractable and $c$-retractable modules, we investigate the $c$-retractibility. Our aim in this paper is to give some new results on $c$-retractable modules. In general, $c$-retractable modules need not be projective and vice versa. Connections between projectivity and $c$-retractibility are investigated. Conditions are found under which, a $c$-retractable module is extending, quasi-continuous, quasi-injective and retractable. With the help of $c$-retractability, we investigated when the notions of $K$-nonsingularity and Baer modules are equivalent. Also, we characterize semisimple artinian rings in terms of $c$-retractable modules. Our paper is structured as follows: In the second section, we are going to give preliminary definitions which we will use throughout this paper. In the third section, we are going to show among others, the following results: (1) Every projective module over a right $SI$-ring is $c$-retractable. (2) Let $M$ be a $wd$-Rickart module in which local summands are summand. Then $M$ is uniform-extending and $c$-retractable if and only if $M$ is extending. (3) Let $M$ be a dual Baer $c$-retractable $R$-module. Then the following hold: (i) $M$ is a direct sum of uniform submodules. (ii) $M = Z_2(M) \oplus (\oplus_{i \in I} M_i)$ with all $M_i$ nonsingular uniform quasi-Baer and $\text{End}(M_i)$ semi-local quasi-Baer. (iii) $M$ is $ADS$ if and only if $M$ is quasi-continuous. (iv) $M$ is auto-invariant if and only if $M$ is quasi-injective. (5) The following conditions are equivalent for a ring $R$:
(a) $R$ is semisimple artinian.
(b) Every $c$-retractable $R$-module is a $C_4$-module.
(c) Every $c$-retractable $R$-module is pseudo-projective.

(6) Let $M$ be a locally noetherian $c$-retractable $R$-module. Then $M$ is homo-related to a direct sum $\oplus_{i \in I} U_i$ of uniform submodules of $M$.

For an $R$-module $M$, $S = \text{End}_R(M)$ denotes the endomorphism ring of $M$. For $\phi \in S$, $\text{Im}\phi$ stands for image of $\phi$. The notations $N \leq M$, $N \leq_e M$ and $N \leq^{\oplus} M$ mean that $N$ is a submodule of $M$, an essential submodule and a direct summand of $M$, respectively.

2. Preliminaries

In this section, we are going to give preliminary definitions which we will use throughout this paper.

**Definition 1.** Let $S$ be a submodule of an $R$-module $M$. A submodule $C$ of $M$ is said to be complement to $S$ in $M$ if $C$ is maximal with respect to the property that $C \cap S = \{0\}$.

**Definition 2.** A submodule $C$ of an $R$-module is a complement in $M$ ($C \subseteq_c M$, for short) if there exists a submodule $S$ of $M$ such that $C$ is complement to $S$ in $M$.

**Definition 3.** 1. An $R$-module $M$ is called extending module if every complement submodule of $M$ is a direct summand.
2. An $R$-module $M$ is called continuous if it is extending and satisfies the following condition: (C2) Every submodule of $M$ that is isomorphic to a direct summand $M$ is itself a direct summand of $M$.
3. An $R$-module $M$ is called quasi-continuous if it is extending and satisfies the following condition: (C3) If $N$ and $K$ are direct summands of $M$ with $N \cap K = 0$, then $N \oplus K$ is a direct summand of $M$.

**Definition 4.** Let $M$ be an $R$-module, put $Z(M) = \{m \in M : \text{ann}_R(m) \leq_e R\}$. $M$ is called nonsingular if $Z(M) = \{0\}$, and singular if $Z(M) = M$. The Goldie torsion submodule $Z_2(M)$ of $M$ is defined by $Z(M/Z(M)) = Z_2(M)/Z(M)$. $M$ is $Z_2$-torsion if, $Z_2(M) = M$.

**Definition 5.** A module $M$ has finite uniform dimension $n$ (written $U\text{dim}(M) = n$) if there is an essential submodule $V \leq_e M$ that is a direct sum of $n$ uniform submodules.

3. Main results

**Definition 6.** An $R$-module is called $c$-retractable if $\text{Hom}_R(M,C) \neq 0$ for each $0 \neq C \subseteq_c M$.

**Remark 1.** Clearly, every retractable module is $c$-retractable. The converse is not true in general. For example: $\mathbb{Q}$ as a $\mathbb{Z}$-module is $c$-retractable while it is not retractable..
Example 1. Every extending module is c-retractable.

Remark 2. If $R = \mathbb{Z}[x]$, then $R$ is c-retractable by ([4], Example 2.4). Clearly, $R \oplus R$ is a c-retractable $R$-module. However, $R \oplus R$ is not extending. (see [4], Example 2.4).

Remark 3. ([4], Example 3.2) Let $R$ be the ring of all $2 \times 2$ upper triangular matrices which have arbitrary real numbers on the diagonal and an arbitrary complex number in the $(1, 2)$-position and let $e_{ij}$ be the element of $R$ with 1 in the $(i, j)$-position and 0 elsewhere. Set $P = e_{11}R$ and let $K$ denote the field of real numbers. Hence, $P$ is a nonsingular projective $R$-module which is not a c-retractable $R$-module while the $R$-module $M = R \oplus P$ is c-retractable. This shows that a direct summand (hence a submodule or a factor module) of a c-retractable module need not be c-retractable.

Proposition 1. Let $M$ be a c-retractable $R$-module. Then $M/N$ is c-retractable for any fully invariant complement submodule $N \leq M$.

Proof. Let $K/N \leq c M/N$ where $N \leq K \leq M$ and $N$ is a fully invariant complement submodule of $M$. Then, $K \leq c M$ by Proposition 6.28 in [11]. Thus, there exists a nonzero homomorphism $f : M \rightarrow K$. Now, $f(N) \leq N$ by hypothesis, and so $\overline{f} : M/N \rightarrow K/N$ defined by $\overline{f}(m + N) = f(m) + N$ for all $m \in M$ is a nonzero homomorphism.

Proposition 2. Let $M$ be a c-retractable $R$-module such that $\text{Hom}_R(M/C, C)$ contains a monomorphism for any $C \subseteq c M$. Then $M/C$ is c-retractable.

Proof. Let $N/C \subseteq c M/C$. By the c-retractable condition on $M$, there is a nonzero homomorphism $g : M \rightarrow N$. From this and by our assumption, $\text{Hom}_R(M/C, N/C) \neq 0$.

Proposition 3. Let $M$ be a c-retractable $R$-module. If $M = L \oplus N$ such that $\text{Hom}_R(L, N) = 0$, then $N$ is a c-retractable $R$-module.

Proof. Note that $\text{End}_R(M) = \begin{bmatrix} \text{End}_R(L) & \text{Hom}_R(N, L) \\ 0 & \text{End}_R(N) \end{bmatrix}$. Hence, $\text{End}_R(M) \begin{bmatrix} L \\ 0 \end{bmatrix} \subseteq \begin{bmatrix} L \\ 0 \end{bmatrix}$.

Proposition 4. Let $M$ be a c-retractable $R$-module and $0 \neq C \subseteq c M$. If $\text{Hom}_R(M/C, C) = 0$, then $C$ is c-retractable.

Proof. Let $0 \neq K \subseteq c C$. Thus, there exists $0 \neq f \in S$ such that $\text{Im} f \subseteq K$.

If $f(C) = 0$, then the rule $m + n \rightarrow m + \text{Ker} f$ yields a nonzero homomorphism $M/C \rightarrow M/\text{Ker} f \cong \text{Im} f$ which is in contradiction with our assumption $\text{Hom}_R(M/C, C) = 0$. Thus, $f(C) \neq 0$, hence $f|_C$ is a nonzero endomorphism of $C$ with image in $K$. 
Proposition 5. If an arbitrary direct sum of copies of $M$ is $c$-retractable, then $M$ is $c$-retractable.

Proof.
This follows from ([17], Proposition 2.10).

Remark 4. A projective module need not be $c$-retractable and vice-versa. In fact a simple is $c$-retractable but not be projective. Moreover, by Remark 3, there is a projective module which is not $c$-retractable. In the following, we show that certains classes of projective modules are $c$-retractable.

Following [24], we call an $R$-module $SI$ if every singular module is $M$-injective. Recall that a ring $R$ is called right $SI$, if every singular $R$-module is injective.

Lemma 1. ([24], Proposition 2.2)
Every homomorphic image of a $SI$-module is a $SI$-module.

Lemma 2. ([24], Proposition 2.7)
The following conditions are equivalent for a ring $R$.
(1) $R$ is a right $SI$-ring.
(2) Every $R$-module is a $SI$-module.

Theorem 1. Let $R$ be any ring. Then every projective $SI$ $R$-module is retractable and hence $c$-retractable.

Proof.
Let $M$ be a nonzero projective $SI$ $R$-module. Let $0 \neq m \in M$. For a given submodule $A$ of $mR$, there exists a submodule $C$ of $mR$ such that $C \oplus A \leq mR$. Thus, $mR/(C \oplus A)$ is singular. Since $M$ is a $SI$-module, $M/(C \oplus A)$ is a $SI$-module by Lemma 1. Hence, $mR/(C \oplus A)$ is $M/(C \oplus A)$-injective and hence a direct summand of $M/(C \oplus A)$. It follows that $M$ has a submodule $B$ such that $M/B$ is isomorphic to $mR/(C \oplus A)$. Hence there exists a nonzero homomorphism $f : M \rightarrow mR/(C \oplus A)$. By the projective condition on $M$, $f$ can be lifted to a nonzero of homomorpism $g : M \rightarrow mR$. Therefore, $M$ is $c$-retractable.

Corollary 1. Let $R$ be a right $SI$-ring. Then every projective $R$-module is $c$-retractable.

Theorem 2. Let $R$ be a right perfect ring. Then the following statements are equivalent for a hereditary $R$-module:
(1) $M$ is $c$-retractable.
(2) $Hom_R(M, C)$ contains an epimorphism for any $0 \neq C \subseteq M$.
(3) $M$ is extending.

Proof.
(1) $\Rightarrow$ (2) follows a similar argument to the one used in ([14], Theorem 2.2).
(2) $\Rightarrow$ (3). Let $0 \neq C \subseteq M$. By (2), there exists an epimorhism $f : M \rightarrow C$. Then $I_C : C \rightarrow C$ can be lifted to a nonzero homomorphism $g : C \rightarrow M$, and hence $C \leq M$. 


Therefore, $M$ is extending.

(3) $\Rightarrow$ (1) It is easy to see.

Recall that a family $\{N_i\}_I$ of independent submodules of a module $M$ is said to be a local summand, if for any finite subset $A \subseteq I$, $\oplus_{A}N_{\alpha}$ is a direct summand of $N$. An $R$-module $M$ is called uniform-extending if every uniform submodule is essential in a direct summand of $M$.

Recall that a module $M$ is called $wd$-Rickart if the image any endomorphism of $M$ contains a nonzero direct summand.

**Lemma 3.** If $M$ is an $R$-module such that every nonzero complement submodule contains a nonzero direct summand, then $M$ is $c$-retractable.

**Proof.**
This is clear.

**Lemma 4.** Let $M$ be a $wd$-Rickart $R$-module. Then $M$ is $c$-retractable if and only if every nonzero complement submodule of $M$ contains a nonzero direct summand.

**Proof.**
The sufficiency follows from Lemma 3. Conversely, assume that $M$ is any $wd$-Rickart $c$-retractable module. Let $0 \neq C \subseteq c M$. Since $M$ is $c$-retractable, there is a nonzero endomorphism $\varphi$ of $M$ such that $Im\varphi \subseteq C$. Thus, the $wd$-Rickart property of $M$ implies that $C$ contains a nonzero direct summand.

**Lemma 5.** If $M$ is any $wd$-Rickart $c$-retractable $R$-module, then every indecomposable complement submodule of $M$ is uniform.

**Proof.**
Let $M$ be any $wd$-Rickart $c$-retractable module. Let $C$ be an indecompsable complement submodule of $M$. Let $D$ any nonzero complement submodule of $C$. Since $D \subseteq c M$, we infer from Lemma 4 that $D$ contains a nonzero direct summand $E$ of $M$. As $E \leq C \leq M$ and $E \leq c M$, $E \leq c C$. Since $C$ is indecomposable, $C = E = D$. It follows that $D$ is a direct summand of $C$, and hence $C$ is an extending module. Since $C$ is indecomposable, $C$ is uniform.

**Theorem 3.** Let $M$ be a $wd$-Rickart $R$-module for which local summands are summand. Then $M$ is uniform-extending and $c$-retractable if and only if $M$ is extending.

**Proof.**
Suppose that $M$ is a uniform-extending $c$-retractable module. Since local summands of $M$ are summand, $M$ is a direct sum of indecomposable modules (see [14], Theorem 2.17). Thus by Lemma 5, $M$ is direct sum of uniform modules. Therefore, by ([6], 8.5), $M$ is extending. The converse implication is clear.
Corollary 2. Let $M$ be a $wd$-Rickart quasi-discrete $R$-module. Then $M$ is uniform-extending and $c$-retractable if and only if $M$ is extending.

Proof. This follows from Theorem 3 and the fact that any local summand of a quasi-discrete module is a summand (see [6], Corollary 4.13).

Remark 5. By Lemma 5, an indecomposable $wd$-Rickart $c$-retractable module is uniform.

Recall that a module $M$ is called simple radical, if $M \neq 0$ such that $\text{Rad}(M) = M$ and $M$ has no proper nonzero submodules $N$ with $\text{Rad}(N) = N$. Hence a simple radical $c$-retractable module is uniform.

Let $M$ be an $R$-module and $N \leq M$. Put $D(N) = \{ \varphi \in S : \text{Im} \varphi \subseteq N \}$. $M$ is called dual Baer if for every $N \leq M$, there is $e^2 = e \in S$ such that $D(N) = eS$.

Recall that an $R$-module $M$ is said to be ADS if for every decomposition $M = S \oplus T$ and every complement $T'$ of $S$, we have $M = S \oplus T'$. Recall that an $R$-module $M$ is called quasi-Baer if, for all fully invariant submodules $N \leq M$, $L_S(N) = Se$, with $e^2 = e \in S$.

Proposition 6. Let $M$ be a dual Baer $c$-retractable $R$-module. Then the following statements hold:

1. $M$ is a direct sum of uniform submodules.
2. $M = Z_2(M) \oplus \bigoplus_{i \in I} M_i$ with all $M_i$ nonsingular uniform quasi-Baer and $\text{End}(M_i)$ semi-local quasi-Baer.
3. If $R$ is a right self-injective ring, then $M = Z_2(M) \oplus M'$ where $M'$ is nonsingular semisimple.

Proof.

1. Suppose $M$ is dual-Baer $c$-retractable. By Corollary 2.6(i) in [10], $M$ is a direct sum of indecomposable submodules. By ([24], Theorem 3.1), $M$ is $wd$-Rickart. Thus, according to Lemma 5, $M$ is a direct sum of uniform submodules.

2. Suppose $M$ has the stated condition. Then by (1), $M$ is a direct sum of uniform modules. Hence by ([2], Corollary 2.3, Theorems 3.2 and 3.9), $M = Z_2(M) \oplus M'$ where $M'$ is quasi-Baer. Since $M$ is dual Baer, we infer from Corollaries 2.5 and 2.6 in [2] that $M' = \bigoplus_{i \in I} M_i$ with all $M_i$ indecomposable. Thus, $M = Z_2(M) \oplus \bigoplus_{i \in I} M_i$ where each $M_i$ is indecomposable. Consequently, each $M_i$ is nonsingular uniform by Lemma 5. On the other hand since $M'$ is quasi-Baer, it follows from ([18], Theorem 3.17) that each $M_i$ is quasi-Baer for each $i \in I$. The last part follows from ([10], Corollary 2.5 and Proposition 2.17) and ([18], Theorem 4.1).

3. By (2), $M = Z_2(M) \oplus \bigoplus_{i \in I} M_i$ with all $M_i$ nonsingular uniform. Let $M' = \bigoplus_{i \in I} M_i$. Thus, since $R$ is right self-injective, all $M_i$ are simple, proving the result.

Theorem 4. Let $M$ be a dual Baer $R$-module. Then the following statements are equivalent:

1. $M$ is ADS and $c$-retractable.
2. $M$ is continuous.
3. $M$ is quasi-continuous.
Proof.
(1) ⇒ (2) Suppose $M$ is ADS and c-retractable. Since $M$ is dual Baer, we infer from Proposition 6(1) that $M = \bigoplus_{i \in I} M_i$ is a direct sum of uniform modules. Thus, every $M_i$ is quasi-continuous for every $i \in I$. On the other hand since $M$ is ADS, we infer from Lemma 3.1 in [1] that $\bigoplus_{i \neq j \in I} M_j$ is $M_i$-injective for every $i \in I$. Therefore $M$ is quasi-continuous by ([14], Theorem 2.13). Now, let $\varphi$ be an essential monomorphism of $M$. Then $\text{Im} \varphi \leq_e M$. Since $M$ is dual Baer, $\text{Im} \varphi \leq \bigoplus M$. Hence, $\text{Im} \varphi = M$. Therefore, according to ([14], Lemma 3.14), $M$ is continuous.

(3) ⇒ (1) This implication is clear.

Corollary 3. Let $M$ be a dual Baer $c$-retractable $R$-module such that every nonsingular summand is ADS. Then $M = Z_2(M) \oplus M'$ where $M'$ is nonsingular quasi-continuous.

Proof.
By Proposition 6(2), $M = Z_2(M) \oplus (\bigoplus_{i \in I} M_i)$ with all $M_i$ nonsingular uniform. Let $M' = \bigoplus_{i \in I} M_i$. Thus, by our assumption, $M'$ is ADS. Therefore, applying the same techniques as in the proof of Theorem 4, one can show easily that $M'$ is quasi-continuous.

Proposition 7. Let $M$ be a $d$-Rickart $R$-module with $S$ is left $T$-nilpotent. Then the following statements are equivalent:
(1) $M$ is ADS and $c$-retractable.
(2) $M$ is quasi-continuous.

Proof.
(1) ⇒ (2) Since $M$ is $d$-Rickart and $S$ is left $T$-nilpotent, it follows from Proposition 3.4.11 in [12] that $M = \bigoplus^n M_i$ with all $M_i$ indecomposable. Since $d$-Rickart modules are $wd$-Rickart, we infer from Lemma 5 that $M = \bigoplus^n M_i$ with all $M_i$ uniform. On the other hand since $M$ is ADS, we infer from Lemma 3.1 in [1] that $\bigoplus_{i \neq j} M_j$ is $M_i$-injective for every $1 \leq i \leq n$. Thus $M$ is quasi-continuous by ([14], Lemma 2.14).

(2) ⇒ (1) This implication is clear.

Let $M$ be an $R$-module. The left annihilator of $N \leq M$ in $S = \text{End}_R(M)$ is denoted by $L_S(N) = \{\phi \in S : \phi N = \{0\}\}$.

Let $M$ be a module. A submodule $N$ of $M$ is said to be an automorphism-invariant submodule if $\varphi N \subseteq N$ for automorphism $\varphi$ of $M$. $M$ is called auto-invariant if it is an automorphism-invariant submodule of its injective hull.

Proposition 8. Let $M$ be a dual Baer $R$-module. Then $M$ is auto-invariant and $c$-retractable if and only if $M$ is quasi-injective.

Proof.
Suppose $M$ is auto-invariant and $c$-retractable. Since $M$ is dual Baer, we infer from Proposition 6(1) that $M = \bigoplus_{i \in I} M_i$ is a direct sum of extending modules. Thus, by Corollary 15 in [13], $M$ is quasi-injective. The converse implication is clear.
Recall that an $R$-module $M$ is called $C_4$ if, whenever $A$ and $B$ are submodules of $M$ with $M = A \oplus B$ and $f : A \rightarrow B$ is an homomorphism with $\ker f \subseteq A$, we have $\text{Im} f \subseteq B$.

**Proposition 9.** If every $2$-generated $R$-module is a $C_4$-module, then every dual Baer $c$-retractable $R$-module is semisimple.

**Proof.**
Let $M$ be any dual Baer $c$-retractable $R$-module. Thus, as in the proof of Theorem 4, $M = \bigoplus_{i \in I} M_i$ where each $M_i$ is uniform. Now, we have to show that each $M_i$ is semisimple. For any $0 \neq m \in E(M_i)$, let $0 \neq N \leq mR$ and take $0 \neq n \in N$. By our assumption, $mR \oplus nR$ is a $C_4$-module. Consider the inclusion map $i : nR \rightarrow mR$. Thus $i(nR) = nR \leq \bigoplus mR$. Since $mR$ is indecomposable, $nR = mR$, and hence $N = mR$. Thus, every cyclic submodule of $mR$ is a direct summand. It follows that $mR$ is semisimple. Hence, $E(M_i)$ is semisimple. Consequently, $M_i$ is semisimple. Therefore, $M$ is semisimple.

**Theorem 5.** The following conditions are equivalent for a ring $R$:
(1) $R$ is semisimple artinian.
(2) Every $c$-retractable $R$-module is a $C_4$-module.
(3) Every $c$-retractable $R$-module is pseudo-projective.

**Proof.**
(1) $\Rightarrow$ (2) is clear.
(2) $\Rightarrow$ (1) Let $I$ be a right ideal of $R$. Clearly, $I \oplus R$ is $c$-retractable, and hence a $C_4$-module by (2). Consider the inclusion map $i : I \rightarrow R$. Therefore, $i(I) = I \leq R$. Hence, $R_R$ is semisimple. Thus, $R$ is semisimple artinian.
(3) $\Rightarrow$ (1) Let $S$ be a simple $R$-module. Then there is a free $R$-module $F$ and an epimorphism $f : F \rightarrow S$. Hence, $S \oplus F$ is $c$-retractable by ([19], Proposition 1.4). By our assumption, $S \oplus F$ is pseudo-projective. Now, Consider the exact sequence $0 \rightarrow \ker f \rightarrow M \rightarrow S \rightarrow 0$. So, by the proof of ([15], Proposition 3.9), this sequence splits. Consequently, $S \leq F$, and hence $S$ is projective. Therefore, $R$ is semisimple.

**Remark 6.** Theorem 5 shows that the condition "right V-ring" in ([15], Proposition 3.9) is superfluous.

Recall that an $R$-module is called Baer if, for all $N \leq M$, $L_S(N) = Se$, with $e^2 = e \in S$. A module $M$ is called $K$-nonsingular if, $\forall \varphi \in \text{End}(M)$, $\ker \varphi \subseteq e \implies \varphi = 0$.

**Proposition 10.** Let $M$ be a $K$-nonsingular $c$-retractable $R$-module. Then $S$ is right nonsingular.

**Proof.**
See proof of ([16], Proposition 3.6).

**Proposition 11.** Let $M$ be a $c$-retractable $R$-module such that $S_S$ is extending. Then $M$ is $K$-nonsingular if and only if $M$ is Baer.
Proof.
Suppose $M$ is $K$-nonsingular. By Proposition 10, $S$ is right nonsingular. Let $N$ be a submodule of $M$. Thus, $L_S(N)$ is a complement right ideal in $S$. Because $S_S$ is extending, then $L_S(N) = S(1 - e)$ for some $e = e^2 \in S$, and hence $M$ is Baer. The converse implication follows from ([18], Lemma 2.15).

Recall that a module is locally noetherian if any of its finitely generated submodules is noetherian. An $R$-module $M$ is said to be homo-related to an $R$-module $L$ if there are $\alpha : M \rightarrow L$ and $\beta : L \rightarrow M$ such that $\beta \alpha \neq 0$.

**Theorem 6.** Let $M$ be a locally noetherian c-retractable $R$-module. Then $M$ is homo-related to a direct sum $\bigoplus_{i \in I} U_i$ of uniform submodules of $M$.

**Proof.**
Suppose $M$ is a c-retractable locally noetherian module. Hence, every submodule of $M$ contains a uniform submodule. Thus, by Zorn’s Lemma, $M$ contains a maximal local direct summand $N = \bigoplus_{i \in I} U_i$ where each $U_i$ is uniform. Also by the locally noetherian condition on $M$ again, $R/r(m) \cong mR$ is noetherian for any element $m$ in $M$. Hence, $R$ satisfies ACC on right ideals of the form $r(m)$ where $m \in M$. Thus, according to ([6], 8.1), $N$ is a complement submodule of $M$. Since $M$ is c-retractable, there exists a nonzero homomorphism $f : M \rightarrow N$. It follows that $M$ is homo-related to $N$.

**Corollary 4.** Let $R$ be a right noetherian ring. Then every c-retractable $R$-module is homo-related to a direct sum $\bigoplus_{i \in I} U_i$ of uniform submodules of $M$.

**Theorem 7.** Let $M$ be a nonsingular c-retractable $R$-module such that every $U \operatorname{dim}(mR) < \infty$ for every element $m \in M$. Then $M$ is homo-related to a direct sum $\bigoplus_{i \in I} U_i$ of indecomposable nonsingular submodules of $M$.

**Proof.**
Suppose $M$ has the stated condition. By Zorn’s Lemma, $M$ contains a maximal local direct summand $N = \bigoplus_{i \in I} U_i$ where each $U_i$ is indecomposable nonsingular. Let $m \in M$. Then $R/r(m)$ is a nonsingular $R$-module which has finite uniform dimension. By ([6], Section 5.10), $R$ has ACC on right ideals of the form $r(m)$ where $m \in M$. Thus, according to ([6], 8.1), $N$ is a complement submodule of $M$. Since $M$ is c-retractable, there exists a nonzero homomorphism $f : M \rightarrow N$. It follows that $M$ is homo-related to $N$.

**Proposition 12.** Let $M$ be a c-retractable $R$-module with $U \operatorname{dim}(M) \geq 2$. Then $M$ is retractable.

**Proof.**
Suppose $M$ has the stated condition. Let $0 \neq N \leq M$. Since $U \operatorname{dim}(N) < \infty$, $N$ contains a uniform submodule $U$. After replacing $U$ by an essential closure, we may assume that $U$ is a complement submodule of $M$. By our assumption, there is a nonzero homomorphism $M \rightarrow U$. Therefore, $M$ is retractable.
Corollary 5. Let $M$ be an $R$-module with $U\text{dim}(M) \geq 2$. Then $M$ is $wd$-Rickart c-retractable if and only if $M$ is semisimple.

Proof. Suppose $M$ is $wd$-Rickart c-retractable. Since $U\text{dim}(M) \geq 2$, $M$ is a finite direct sum of indecomposable submodules. By Proposition 12, $M$ is retractable. Therefore, according to ([23], Proposition 2.17), $M$ is semisimple. The converse implication is clear.

Proposition 13. The following statements are equivalent for an $R$-module $M$ with $u\text{dim}(M) = n \geq 2$.

1. $M$ is c-retractable.
2. $\text{Hom}_R(M,U) \neq 0$ for every uniform submodule $U$ of $M$.
3. $\text{Hom}_R(M,U) \neq 0$ for every cyclic uniform submodule $U$ of $M$.
4. $M$ is retractable.

Proof. 
(1) $\Rightarrow$ (2) follows from Proposition 12.
(2) $\Rightarrow$ (3) Clear.
(3) $\Rightarrow$ (4) Let $0 \neq N \leq M$. Let $0 \neq m \in N$. By hypothesis, $mR$ has finite uniform dimension and hence $mR$ contains a uniform submodule $U$. Let $0 \neq u \in U$. By (3), $\text{Hom}_R(M,uR) \neq 0$. Hence, $M$ is retractable.
(4) $\Rightarrow$ (1) is clear.

Remark 7. By Proposition 12, every c-retractable module $M$ with $u\text{dim}(M) = n \geq 2$ is retractable. Note that the condition $u\text{dim}(M) = n \geq 2$ can not be dropped. In fact, $\mathbb{Q}$ as a $\mathbb{Z}$-module is c-retractable uniform but it is not retractable.

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