ELASTIC SPLINES I: EXISTENCE

ALBERT BORBÉLY & MICHAEL J. JOHNSON
Kuwait University
February 13, 2013

Abstract. Given interpolation points \(P_1, P_2, \ldots, P_n\) in the plane, it is known that there does not exist an interpolating curve with minimal bending energy, unless the given points lie sequentially along a line. We say than an interpolating curve is admissible if each piece, connecting two consecutive points \(P_i\) and \(P_{i+1}\), is an s-curve, where an s-curve is a planar curve which first turns at most \(180^\circ\) in one direction and then turns at most \(180^\circ\) in the opposite direction. Our main result is that among all admissible interpolating curves there exists a curve with minimal bending energy. We also prove, in a very constructive manner, the existence of an s-curve, with minimal bending energy, which connects two given unit tangent vectors.

1. Introduction

Given functional data \((x_i, y_i) \in \mathbb{R}^2\), it is well known that the natural cubic spline interpolant \(y = f(x)\) (satisfying \(y_i = f(x_i)\)) minimizes \(\int_a^b (f''(x))^2 \, dx\) among all smooth functions which interpolate the given data. The functional \(\int_a^b (f''(x))^2 \, dx\) is often viewed as a simple approximation of the curve’s bending energy \(\int_0^L \kappa(s)^2 \, ds\), where \(s\) denotes arclength and \(\kappa\) denotes signed curvature, and it is natural to ask what would happen if one tried to minimize the bending energy amongst all smooth interpolating curves. Unfortunately, such optimal curves do not exist except in the trivial case when the interpolation points lie sequentially along a line. Apparantly, this was first observed by Birkhoff and de Boor [1], along with Birkhoff, Burchard and Thomas [2]. This lack of existence can be understood as a consequence of the effect that scaling has on bending energy: the bending energy of a curve scaled by a factor \(\lambda\), equals \(\frac{1}{\lambda}\) times the original bending energy. As a result, it is always possible to construct smooth interpolating curves with arbitrarily small bending energy. For example, one can do this with a series of ‘concentric circles’ having slowly varying, but very large, radii.

1991 Mathematics Subject Classification. 41A15; 65D17, 41A05.
Key words and phrases. spline, nonlinear spline, elastica, bending energy.

Typeset by \texttt{AMS-\LaTeX}
Subsequent attention was directed towards interpolating curves whose bending energy is locally minimal (i.e., minimal among all nearby interpolating curves). It was reported in [2], and mentioned in [1], that if an interpolating curve \( F \) has a locally minimal bending energy, then each segment of \( F \), connecting two consecutive interpolation points, will be a segment of ‘rectangular elastica’, meaning a planar curve whose signed curvature \( \kappa \) satisfies the differential equation \( 2\frac{d^2\kappa}{ds^2} + \kappa^3 = 0 \). (Rectangular elastica was first described by James Bernoulli (1694) and resides as one species among the nine species of elastica identified by Euler (1750), see [4].) Using a variational calculus and physical reasoning, Lee and Forsyth [6] (see also [3]) have confirmed that each segment of \( F \) is indeed a segment of rectangular elastica, and have moreover shown that the signed curvature of \( F \) is continuous throughout the curve and vanishes at the endpoints. Unfortunately, for a given sequence of interpolation points, interpolating curves with locally minimal bending energy do not always exist and this constitutes a significant deficiency in the theoretical foundation of this interpolation method.

Rather than seeking an interpolating curve with a locally minimal bending energy, an alternate approach is to define a restricted class of ‘admissible’ interpolating curves and then seek a curve with minimal bending energy in the restricted class. Birkhoff proposed a restriction on length and conjectured that among all smooth interpolating curves of length at most \( L_0 \), \( L_0 \) being a prescribed constant, there exists a curve with minimal bending energy. This conjecture was eventually proved by Jerome [5].

Rather than a restriction on length, we propose a restriction on shape. The motivation for our restriction comes from the fact that if a smooth interpolating curve \( F \) has a locally minimal bending energy, then it can be shown that each segment of \( F \), connecting two consecutive interpolation points, is what we call an \( s \)-curve. In brief, an \( s \)-curve is a curve which first turns in one direction (either counter-clockwise or clockwise) at most 180° and then turns in the opposite direction at most 180°. Given a sequence of points \( P_1, P_2, \ldots, P_n \) in \( \mathbb{R}^2 \), an interpolating curve \( F \) is deemed \textit{admissible} if each piece of \( F \), connecting two consecutive interpolation points \( P_i \) and \( P_{i+1} \), is an \( s \)-curve. The family of all admissible interpolating curves is denoted \( A(P_1, P_2, \ldots, P_n) \), and our main result is the following.

\textbf{Theorem 1.1.} Given any sequence of points \( P_1, P_2, \ldots, P_n \) in \( \mathbb{R}^2 \), the family of admissible interpolating curves \( A(P_1, P_2, \ldots, P_n) \) contains a curve with minimal bending energy.

An essential sub-problem which arises in the proof of Theorem 1.1 is that of proving the existence of an \( s \)-curve, with minimal bending energy, which connects two given unit tangent vectors. In addition to facilitating our proof of Theorem 1.1, we anticipate that this sub-problem sits at the core of any numerical algorithm for solving the general problem, and with this in mind, we present a thorough analysis of the sub-problem along with a constructive solution.

An outline of the sequel is as follows. In section 2, we explain the notation used throughout and develop some basic formulae and properties of rectangular elastica. A curve which turns at most 180° in one direction is called a \textit{c-curve} and in section 3, we show the existence of an optimal \( c \)-curve connecting a unit tangent vector to a line as well as connecting two unit tangent vectors. In section 4, the uniqueness, or lack thereof, of the optimal \( c \)-curves found in section 3 is treated. The important sub-problem mentioned above, namely the existence of an optimal \( s \)-curve connecting two unit tangent vectors, is primarily solved.
in section 5, except that one particular case (where the optimal s-curve turns out to be a unique c-curve) is treated in section 6. Finally, in section 7, we prove Theorem 1.1.

2. Notation

A curve is a differentiable function $f : [a, b] \to \mathbb{C}$ whose derivative $f'$ is absolutely continuous and non-zero. The length of $f$ is $\text{len}(f) = \int_a^b |f'(t)| \, dt$. With $L = \text{len}(f)$, let the variables $t \in [a, b]$ and $s \in [0, L]$ be related by $s = \int_t^a |f'(r)| \, dr$ and define $F : [0, L] \to \mathbb{C}$ by $F(s) = f(t)$. It can be shown that $F$ is a curve (ie $F'$ is absolutely continuous) satisfying $|F'| = 1$. The curve $F$ is called the unit speed curve described by $f$ and is denoted $[f]$. Two curves $f$ and $g$ are said to be equivalent, written $f \equiv g$, if $[f] = [g]$. Since $F'$ is absolutely continuous and $|F'| = 1$, it follows that there exists an absolutely continuous function $\tau : [0, L] \to \mathbb{R}$, unique modulo an additive constant in $2\pi\mathbb{Z}$, such that $F' = e^{i\tau}$. We refer to $\tau$ at the direction angle of $F$, while the derivative of $\tau$, denoted $\kappa$, is called the signed curvature of $F$. Since $\tau$ is absolutely continuous, it follows that $\kappa$ is Lebesgue integrable (see [7, pp. 108–112]). The turning angle of $f$, denoted $\Delta(f)$, is defined by $\Delta(f) = \Delta(F) := \int_0^L \kappa(s) \, ds$. Note that the magnitude of the turning angle is bounded by the $L_1$-norm of $\kappa$. If $\kappa \geq 0$ (resp. $\kappa \leq 0$) almost everywhere in $[0, L]$, then $|\Delta(F)| = \|\kappa\|_{L_1}$ and $f$ is called a left-curve (resp. right-curve). A c-curve is a left-curve or a right-curve whose turning angle has magnitude at most $\pi$. A u-turn is a c-curve whose turning angle has magnitude $\pi$.

Given the signed curvature $\kappa$ of $F$ and its initial position and direction, we can recover $F$ as follows:

Step 1. Define $\tau(s) = \tau_0 + \int_0^s \kappa(r) \, dr$, $s \in [0, L]$, where $\tau_0 = \text{arg}(F'(0))$.

Step 2. $F(s) = F(0) + \int_0^s e^{i\tau(r)} \, dr$.

This reconstruction can be used to show that two curves are ‘close’. To see this, suppose $F_1$ is a unit speed curve having the same length and initial position and direction as $F$. It follows from step 1, that $|\tau(r) - \tau_1(r)| \leq \|\kappa - \kappa_1\|_{L_1}$ and then using the Lipschitz continuity of the function $r \mapsto e^{i\tau}$ in step 2, we obtain

\begin{equation}
|F(s) - F_1(s)| \leq \int_0^s |e^{i\tau(r)} - e^{i\tau_1(r)}| \, dr \leq \int_0^s |\tau(r) - \tau_1(r)| \, dt \leq s\|\kappa - \kappa_1\|_{L_1}.
\end{equation}

Whereas the $L_1$-norm of $\kappa$ is necessarily finite, the $L_2$-norm may or may not be finite. When it is finite, we say that $f$ has finite bending energy, where the bending energy of $f$, denoted $\|f\|^2$, is essentially the square of the $L_2$-norm of $\kappa$:

$$\|f\|^2 = \|F\|^2 := \frac{1}{4} \int_0^b |\kappa(s)|^2 \, ds.$$ 

The constant $\frac{1}{4}$ has been inserted for later convenience.

A unit tangent vector is an ordered pair of complex numbers $u = (u_1, u_2) \in \mathbb{C}^2$ such that $|u_2| = 1$ and can be visualized as the directed line segment, of unit length, having position (or base-point) $\text{pos}(u) = u_1$ and direction $\text{dir}(u) = u_2$. For any $t \in [a, b]$, the unit
tangent vector to $f$ at $t$ is $\vec{f}(t) = (f(t), f'(t)/|f'(t)|)$. The unit tangent vectors $u = \vec{f}(a)$ and $v = \vec{f}(b)$ are called, respectively, the initial and terminal unit tangent vectors of $f$, and we say that $f$ connects $u$ to $v$. We also say that $f$ connects $u$ to $v$ if $\ell$ is the line through $f(b)$ which is parallel to $f'(b)$. If $g$ is a curve whose initial unit tangent vector equals the terminal unit tangent vector of $f$, then $[f]$ can be extended by $[g]$ obtaining a unit speed curve, denoted $f \sqcup g$, whose initial and terminal unit tangent vectors equal those of $f$ and $g$, respectively, and whose bending energy satisfies $\|f \sqcup g\|^2 = \|f\|^2 + \|g\|^2$.

A similarity transformation is a mapping $T : \mathbb{C} \to \mathbb{C}$ of the form $T(z) = c_1 z + c_2$ or $T(z) = c_1 z^2 + c_2$, where $c_1, c_2$ are complex constants, $c_1 \neq 0$. The first form preserves the orientation (left or right) of a curve while the second form reverses it. The dilatation factor is $\lambda = |c_1|$, and the effect on a curve $f$ is, as expected, $\text{len}(T \circ f) = \lambda \text{len}(f)$, $\|T \circ f\|^2 = \frac{1}{\lambda} \|f\|^2$ and $|\Delta(T \circ f)| = |\Delta(f)|$. If a curve $g$ is equivalent to $T \circ f$, then we say that $g$ is similar to $f$; in case $\lambda = 1$, $T$ is called a congruency transformation and we say that $g$ is congruent to $f$. Furthermore, we say that $g$ is directly similar (or congruent) to $f$ if $T$ is orientation preserving.

The curves constructed in this article are formed by line segments (denoted $[A, B]$) and various segments of rectangular elasta. For the latter, we employ the parameterization $E(t) = \sin t + i \xi(t)$, where $\xi(t)$ is defined by $\frac{d\xi}{dt} = \frac{\sin^2 t}{\sqrt{1 + \sin^2 t}}$, $\xi(0) = 0$ (see Figure 6.1a). Since $\frac{d\xi}{dt}$ is even and $\pi$-periodic, it follows that $\xi$ is odd and satisfies $\xi(t + \pi) = d + \xi(t)$, where $d := \xi(\pi)$. Since the sine function is odd and $2\pi$-periodic, we conclude that $E(t)$ is odd and satisfies $E(t + 2\pi) = \pi 2d + E(t)$. We use the notation $E_{[a,b]}$ to denote the sub-curve $E(t)$, $t \in [a, b]$, and any curve which is similar to $E_{[a,b]}$ is called a segment of rectangular elasta. For later reference, we mention the following.

$$|E'(t)| = \frac{1}{\sqrt{1 + \sin^2 t}}, \quad \frac{E'(t)}{|E'(t)|} = \cos t \sqrt{1 + \sin^2 t} + i \sin^2 t, \quad \kappa(t) = 2 \sin t,$$

$$\Delta(E_{[a,b]}) = \int_a^b \kappa(t)|E'(t)|\,dt = 2 \cos^{-1} \left( \frac{\cos b}{\sqrt{2}} \right) - \frac{\pi}{2},$$

$$\|E_{[a,b]}\|^2 = \frac{1}{4} \int_a^b \kappa(t)^2|E'(t)|\,dt = \xi(b) - \xi(a).$$

For $t_0 \in (0, \pi]$, the segment $E_{[0,t_0]}$ plays an important role in the sequel. In the following lemma, we establish a connection between the turning angle of $E_{[0,t_0]}$ and the value of $\xi(t_0)$.

**Lemma 2.1.** Let $t_0 \in (0, \pi]$ and put $\tau_0 = \Delta(E_{[0,t_0]})$. Then $\xi(t_0) = \frac{1}{2} \int_0^{\tau_0} \sqrt{\sin \tau} \,d\tau$.

**Proof.** Fix $t_0 \in (0, \pi]$ and put $\tau = \Delta(E_{[0,t]}), = 2 \cos^{-1} \left( \frac{\cos t}{\sqrt{2}} \right) - \frac{\pi}{2}$, $t \in [0, t_0]$. Then $\frac{d\tau}{dt} = \kappa(t)|E'(t)|$, and since $e^{i\tau} = E'(t)/|E'(t)|$, we have $\sin \tau = \Re E'(t)/|E'(t)| = \sin^2 t$, which implies $\sqrt{\sin \tau} = \sin t$. Hence,

$$\xi(t_0) = \int_0^{t_0} \frac{\sin^2 t}{\sqrt{1 + \sin^2 t}} \,dt = \frac{1}{2} \int_0^{t_0} \sin(t)\kappa(t)|E'(t)|\,dt = \frac{1}{2} \int_0^{\tau_0} \sqrt{\sin \tau} \,d\tau.$$
3. Existence of optimal c-curves

Given a unit tangent vector $u$ and a line $\ell$, let $C_l(u, \ell)$ denote the set of left c-curves which connect $u$ to $\ell$. In this section, we consider the problem of finding a curve in $C_l(u, \ell)$ which has minimal bending energy. We first consider $C_l(u_0, \ell_d)$, where $u_0 = \vec{E}(0)$ and $\ell_d = \{ z \in \mathbb{C} : \exists z = d \}$. We will show that $E_{[0, \pi]}$ has minimal bending energy in $C_l(u_0, \ell_d)$.

Note that which has minimal bending energy. We first consider $C_\ell$, respectively, and let $f \in C_l(u_0, \ell_d)$, put $L = \text{len}(f)$, and let $F = [f]$ denote the unit speed curve described by $f$. Let $\tau$ and $\kappa$ be the direction angle and signed curvature of $F$, respectively, and note that $\int_0^L \kappa(s) \, ds = \pi$ since the turning angle in $C_l(u, \ell)$ is $\pi$. Furthermore, since $F$ originates at $0$ and terminates on $\ell_d$, we have $d = \exists F(L) = \int_0^L \sin \tau \, ds$.

**Lemma 3.1.** If $\kappa$ is continuous and positive, then $\|F\|^2 \geq d$.

**Proof.** We adopt the viewpoint that $\kappa \in (0, \pi]$ and $s \in [0, L]$ are variables related by $\tau = \int_0^s \kappa(r) \, dr$. The assumptions on $\kappa$ ensure that $\tau$ and $s$ are increasing $C^1$ functions of one another. Noting that $\frac{ds}{d\tau} = \kappa(s)$, we observe that $\|F\|^2 = \frac{1}{4} \int_0^\pi \kappa(s)^2 \, ds = \frac{1}{4} \int_0^L \kappa(s) \frac{ds}{d\tau} \, d\tau = \frac{1}{4} \int_0^\pi \kappa(s) d\tau$. Similarly, since $\frac{ds}{d\tau} = 1/\kappa(s)$, we have $\int_0^\pi \frac{\sin \tau}{\kappa(s)} \, d\tau = \int_0^\pi \sin \tau \frac{ds}{\kappa(s)} \, d\tau = \int_0^L \sin \tau \, ds = d$.

Now,

$$d^2 = \left[ \frac{1}{2} \int_0^\pi \sin \tau \, d\tau \right]^2 = \left[ \int_0^\pi \frac{\sin \tau}{\sqrt{\kappa(s)}} \frac{\sqrt{\kappa(s)}}{2} \, d\tau \right]^2 \leq \left( \int_0^\pi \frac{\sin \tau}{\sqrt{\kappa(s)}} \, d\tau \right) \left( \frac{1}{4} \int_0^\pi \kappa(s) \, d\tau \right),$$

by the Cauchy-Schwarz inequality. Hence $d^2 \geq d\|F\|^2$, and therefore $\|F\|^2 \geq d$. \(\square\)

Returning now to the general case, suppose, by way of contradiction, that $\|F\|^2 < d$. Then $\kappa$ is a nonnegative square integrable function satisfying $\int_0^L \kappa(s) \, ds = \pi$. It follows that for every $\varepsilon > 0$, there exists a positive continuous function $\kappa_\varepsilon : [0, L] \rightarrow [0, \infty)$ such that $\int_0^L \kappa_\varepsilon(s) \, ds = \pi$ and $\|\kappa - \kappa_\varepsilon\|_{L^2} < \varepsilon$. Let $F_\varepsilon$ be the unit speed curve having signed curvature $\kappa_\varepsilon$ and initial unit tangent vector $u_0$. Since $F_\varepsilon$ has turning angle $\pi$, it follows that the terminal unit tangent vector of $F_\varepsilon$ is parallel to $\ell_d$, but there is no guarantee that the terminal point $z_\varepsilon = F_\varepsilon(L)$ lies on $\ell_d$. We repair this by multiplying $F_\varepsilon$ with the positive scalar $c_\varepsilon = d/\exists z_\varepsilon$ obtaining the curve $c_\varepsilon F_\varepsilon \in C_l(u_0, \ell_d)$ with $\|c_\varepsilon F_\varepsilon\|^2 = \frac{1}{c_\varepsilon} \|F_\varepsilon\|^2$. Since $\|\kappa - \kappa_\varepsilon\|_{L^2} \rightarrow 0$, it follows that $\|F_\varepsilon\|^2 \rightarrow \|F\|^2$ as $\varepsilon \rightarrow 0$. And since the $L_1$-norm of $\kappa - \kappa_\varepsilon$ is bounded by a constant multiple of its $L_2$-norm, it follows from (2.1) that $c_\varepsilon \rightarrow 1$ as $\varepsilon \rightarrow 0$. Hence $\|c_\varepsilon F_\varepsilon\|^2 < d$ when $\varepsilon > 0$ is sufficiently small, contradicting Lemma 3.1. Therefore, $\|F\|^2 \geq d$. We have thus proved the following.

**Theorem 3.2.** Let $u_0 = \vec{E}(0)$ and $\ell_d = \{ z \in \mathbb{C} : \exists z = d \}$. Then $E_{[0, \pi]}$ has minimal bending energy in $C_l(u_0, \ell_d)$. 
Corollary 3.3. Let \( t_1 \in (0, \pi) \) and put \( u_1 = \vec{E}(t_1) \). Then \( E_{[t_1, \pi]} \) has minimal bending energy in \( C_1(u_1, \ell_d) \).

Proof. If \( f \in C_1(u_1, \ell_d) \) has bending energy less than \( E_{[t_1, \pi]} \), then \( E_{[0,t_1]} \sqcup f \) will be a curve in \( C_1(u_0, \ell_d) \) with bending energy less than \( d \), contradicting Theorem 3.2. \( \square \)

Corollary 3.4. Let \( 0 \leq t_1 < t_2 \leq \pi \) and put \( u_1 = \vec{E}(t_1), u_2 = \vec{E}(t_2) \). Then \( E_{[t_1,t_2]} \) has minimal bending energy in \( C_1(u_1,u_2) \).

Proof. If \( f \in C_1(u_1,u_2) \) has bending energy less than \( E_{[t_1,t_2]} \), then \( E_{[0,t_1]} \sqcup f \sqcup E_{[t_2,\pi]} \) will be a curve in \( C_1(u_0,\ell_d) \) with bending energy less than \( d \), contradicting Theorem 3.2. \( \square \)

Definition 3.5. Let \( \ell \) be a line and \( u \) a unit tangent vector, with \( \text{pos}(u) \not\in \ell \), and assume \( C_1(u, \ell) \) is nonempty. Let \( \delta \in (0,\pi] \) be the common turning angle in \( C_1(u, \ell) \) and let \( t_1 \in [0, \pi) \) be such that \( \Delta(E_{[t_1,\pi]}) = \delta \). There exists a unique similarity transformation \( T \) such that \( T \circ E_{[t_1,\pi]} \) belongs to \( C_1(u, \ell) \). We define \( l(u, \ell) = T \circ E_{[t_1,\pi]} \). In other words, \( l(u, \ell) \) is the unique curve in \( C_1(u, \ell) \) which is similar to \( E_{[t_1, \pi]} \).

Theorem 3.6. Let \( \ell \) be a line and \( u \) a unit tangent vector, with \( \text{pos}(u) \not\in \ell \), and assume \( C_1(u, \ell) \) is nonempty. Then \( l(u, \ell) \) has minimal bending energy in \( C_1(u, \ell) \). Moreover, if \( \delta \in (0,\pi] \) denotes the common turning angle in \( C_1(u, \ell) \) and \( p \) denotes the orthogonal distance from \( \text{pos}(u) \) to \( \ell \), then

\[
\|l(u, \ell)\|^2 = \frac{1}{p} \left[ \frac{1}{2} \int_0^\delta \sqrt{\sin \tau} \, d\tau \right]^2.
\]

Proof. Let \( t_1 \) and \( T \) be as in Definition 3.5 and \( u_1 \) and \( \ell_d \) as in Corollary 3.3. Note that \( T \) maps \( C_1(u_1, \ell_d) \) onto \( C_1(u, \ell) \) and a curve \( f \) has minimal bending energy in \( C_1(u_1, \ell_d) \) if and only if \( T(f) \) has minimal bending energy in \( C_1(u, \ell) \). It therefore follows from Corollary 3.3 that \( l(u, \ell) = T \circ E_{[t_1,\pi]} \) has minimal bending energy in \( C_1(u, \ell) \). In order to compute the bending energy of \( l(u, \ell) \), recall that \( \|l(u, \ell)\|^2 = \frac{1}{p}\|E_{[t_1,\pi]}\|^2 \), where \( \lambda \) is the dilation factor in \( T \). Since the orthogonal distance from \( \text{pos}(u_1) \) to \( \ell_d \) is \( d - \xi(t_1) \), it follows that \( \lambda = p/(d - \xi(t_1)) \). And since \( \|E_{[t_1,\pi]}\|^2 = d - \xi(t_1) \), we have \( \|l(u, \ell)\|^2 = \frac{1}{p}(d - \xi(t_1))^2 \). By Lemma 2.1, \( \xi(t_1) = \int_0^{\pi - \delta} \sqrt{\sin \tau} \, d\tau \) and hence \( d - \xi(t_1) = \frac{1}{2} \int_{\pi - \delta}^\pi \sqrt{\sin \tau} \, d\tau = \frac{1}{2} \int_0^\delta \sqrt{\sin \tau} \, d\tau. \) \( \square \)

Remark 3.7. The definitions and results for right c-curves are analogous to those of left c-curves. In brief, we denote the set of right c-curves connecting \( u \) to \( \ell \) by \( C_r(u, \ell) \), and \( r(u, \ell) \) is defined the same as \( l(u, \ell) \) except that \( \delta \) denotes the magnitude of the common turning angle in \( C_r(u, \ell) \) (right curves have a negative turning angle). Theorem 3.6 then holds with \( C_r(u, \ell) \) and \( r(u, \ell) \) in place of \( C_l(u, \ell) \) and \( l(u, \ell) \), respectively.
4. Uniqueness of optimal c-curves

Having settled the question of existence of an optimal curve in $C_i(u, \ell)$ we now address uniqueness. As with existence we start with $C_i(u_0, \ell_d)$, where $u_0$ and $\ell_d$ are as in Theorem 3.2.

**Theorem 4.1.** For $i = 1, 2$, let $F_i : [0, L_i] \to \mathbb{C}$ be a unit speed curve in $C_i(u_0, \ell_d)$ such that $\|F_i\|^2 = d$ and assume that $F_i$ does not begin or end with a line segment. Then $F_1 = F_2$.

Our proof of this employs the following technical result, which is left as a simple exercise in differential calculus.

**Lemma 4.2.** Let $\lambda_1, \lambda_2 > 0$ and define $H : (0, 1) \to (0, \infty)$ by $H(\mu) = \frac{\lambda_1^2}{\mu} + \frac{\lambda_2^2}{1 - \mu}$. Then $H$ has a unique minimum at $\mu_0 = \lambda_1/(\lambda_1 + \lambda_2)$, where $H(\mu_0) = (\lambda_1 + \lambda_2)^2$.

**Proof of Theorem 4.1.** Let $\tau_i$ and $\kappa_i$ be the direction angle and signed curvature of $F_i$, respectively. Since $F_i$ does not begin or end with a line segment, we have $0 < \tau_i(s) < \pi$ for all $s \in (0, L_i)$, and it follows that $F_i$ can be reparameterized as $t \mapsto g_i(t) + it$, where $g_i$ is continuous on $[0, d]$ and continuously differentiable on $(0, d)$. Fix $\gamma \in (0, \pi)$ and let $t \in (0, \pi)$ be such that $\Delta(E_{[0,t]}(\mu_i)) = \gamma$. Let $s_i \in (0, L_i)$ be such that $\tau_i(s_i) = \gamma$, and put $v_i = \vec{F_i}(s_i)$ and $t_i = \exists F_i(s_i)$. We claim that $t_1 = \xi(t) = t_2$. Noting that the turning angle in $C_i(v_i, \ell_d)$ is $\pi - \gamma$ and the orthogonal distance from $\text{pos}(v_i)$ to $\ell_d$ is $d - t_i$, and since $F_{i[s_i,L]}$ belongs to $C_i(v_i, \ell_d)$, we obtain from Theorem 3.6 and Lemma 2.1 that $\|F_{i[s_i,L]}\|^2 \geq \frac{1}{d-t_i}(d - \xi(t))^2$. By a similar argument (using right c-curves) we obtain $\|F_{i[0,s]}\|^2 \geq \frac{1}{t_i} \xi(t)^2$. Therefore,

$$d = \|F_i\|^2 = \|F_{i[0,s]}\|^2 + \|F_{i[s,L]}\|^2 \geq \frac{1}{t_i} \xi(t)^2 + \frac{1}{d-t_i}(d - \xi(t))^2.$$

With $\lambda_1 = \xi(t)$, $\lambda_2 = d - \xi(t)$, $\mu = t_i/d$, and with $H(\mu)$ as in Lemma 4.2, we can express the above inequality as $d \geq \frac{1}{d} H(\mu)$, or equivalently, $d^2 \geq H(\mu)$. By Lemma 4.2, $H$ has a unique minimum at $\mu_0 = \xi(t)/d$ where $H(\mu_0) = d^2$. But since $d^2 \geq H(\mu)$, it must be the case that $\mu = \mu_0$; therefore $t_i = \xi(t)$ as claimed. In terms of the functions $g_1$ and $g_2$, we have proved that if $g_i'(t_1) = \cot \gamma = g_2'(t_2)$, then $t_1 = \xi(t) = t_2$. Since, for $i = 1, 2$, $g_i'$ is continuous and decreasing on $(0, d)$, with range $(-\infty, \infty)$, we conclude that $g_1' = g_2'$ on $(0, d)$. Since $g_1(0) = 0 = g_2(0)$, we have $g_1 = g_2$ on $[0, d]$. From this we conclude that $F_1$ and $F_2$ are equivalent, but since both are unit speed curves, they must be equal. \(\square\)

As an immediate corollary, we have the following.

**Corollary 4.3.** If $f \in C_i(u_0, \ell_d)$ has minimal bending energy, then $f$ contains a subcurve which is equivalent to $c + E_{[0,\pi]}$ for some constant $c \geq 0$.

Imitating the proof of Corollary 3.3 and Theorem 3.6, one easily obtains the following.
Corollary. Let \( \ell \) be a line and \( u \) a unit tangent vector, with \( \text{pos}(u) \notin \ell \), and assume \( C_l(u, \ell) \) is nonempty. Let \( \delta \in (0, \pi] \) denote the common turning angle in \( C_l(u, \ell) \) and let \( f \in C_l(u, \ell) \) have minimal bending energy.

(i) If \( \delta = \pi \), then \( f \) contains a subcurve which is congruent to \( l(u, \ell) \).

(ii) If \( \delta < \pi \), then either \( f \equiv l(u, \ell) \) or \( f \equiv l(u, \ell) \sqcup [A, B] \) for some line segment \([A, B]\).

We have seen in Corollary 3.4 that \( E_{[t_1, t_2]} \) has minimal bending energy in \( C_l(u_1, u_2) \). Using the same technique as above, one easily obtains the following.

Theorem 4.4. Let \( 0 \leq t_1 < t_2 \leq \pi \), put \( u_1 = \vec{E}(t_1) \), \( u_2 = \vec{E}(t_2) \) and assume \( f \in C_l(u_1, u_2) \) has minimal bending energy in \( C_l(u_1, u_2) \). If \( f \) is not equivalent to \( E_{[t_1, t_2]} \), then \([t_1, t_2] = [0, \pi] \) and a sub-curve of \( f \) is congruent to \( E_{[0, \pi]} \) (i.e. \( f \) is equivalent to \([0, c] \sqcup (c + E_{[0, \pi]} ) \sqcup [c + i d, i d] \) for some \( c > 0 \)).

5. Optimal s-curves, part I

An s-curve is either a c-curve (considered a degenerate s-curve) or a curve of the form \( f = f_1 \sqcup f_2 \), where \( f_1 \) and \( f_2 \) are c-curves which turn in opposite directions. Let \( u \) and \( v \) be two unit tangent vectors and let \( S(u, v) \) denote the set of all s-curves which connect \( u \) to \( v \). In this section and the next, we will prove the following.

Theorem 5.1. Let \( u \) and \( v \) be two unit tangent vectors with distinct positions. If \( S(u, v) \) is nonempty, then there exists a curve in \( S(u, v) \) with minimal bending energy.

In addition to proving existence, our proof of Theorem 5.1 will actually describe all optimal curves in \( S(u, v) \). Expecting that the numerical problem of finding an optimal curve in \( S(u, v) \) lies at the heart of future algorithms, we have structured our proof so that it easily translates into a numerical algorithm.

To begin, let \( u \) and \( v \) be two unit tangent vectors with distinct positions. By applying a similarity transformation, if necessary, and possibly a direction reversal (i.e. \( S(-v, -u) \) in place of \( S(u, v) \)), we can assume without loss of generality that \( u = (0, e^{i \alpha}) \) and \( v = (1, e^{i \beta}) \), where \( \alpha \in [0, \pi] \) and \( |\beta| \leq \alpha \) (see Figure 5.1).

![Fig. 5.1](image1)

We leave it to the reader, as a worthwhile exercise, to verify that \( S(u, v) \) is non-empty if and only if \( \alpha < \pi \) and \( \beta \geq \alpha - \pi \). With that in mind, we proceed assuming that \( \alpha \in [0, \pi) \), \( |\beta| \leq \alpha \) and \( \beta \geq \alpha - \pi \).
If \( \alpha = 0 \), then \( \beta = 0 \) as well and the line segment \([0, 1]\) is the unique curve (modulo equivalence) in \( S(u, v) \) having minimal bending energy. Having dispensed with the trivial case, we assume henceforth that \( \alpha > 0 \).

Our proof of existence will show that there exists an optimal curve in \( S(u, v) \) having one of the following two forms.

**Definition 5.2.** A curve \( f \) is of

(i) **first form** if there exist \(-\pi < t_0 < t < \pi\) such that \( f \) is directly similar to \( E_{[t_0, t]} \),

(ii) **second form** if there exists \( c \geq 0 \) and \( t \in [0, \pi] \) such that \( f \) is directly similar to \( E_{[-\pi, 0]} \sqcup [0, c] \sqcup (c + E_{[0, t]}). \)

Note that curves of first form do not contain u-turns, while curves of second form do.

While studying right-left s-curves in \( S(u, v) \), the following quantities will gradually take on significance, but for easy reference we gather and define them here. In relation to a generic right-left s-curve \( f \) in \( S(u, v) \), the angle \( \gamma \) is illustrated in Fig. 5.3, and the set of all possible angles \( \gamma \) is denoted \( \Gamma \).

**Definition 5.3.** For \( \gamma \) in \( \Gamma := \begin{cases} [\alpha - \pi, \beta] & \text{if } \beta < 0 \\ [\alpha - \pi, 0) & \text{if } \beta \geq 0 \end{cases} \), we define the following:

\[
y_1 := y_1(\gamma) := \frac{1}{2} \int_0^{\alpha - \gamma} \sqrt{\sin \tau} \, d\tau \quad \text{(bending energy of } E_{[0, t_1]} )
\]

\[
y_2 := y_2(\gamma) := \frac{1}{2} \int_0^{\beta - \gamma} \sqrt{\sin \tau} \, d\tau \quad \text{(bending energy of } E_{[0, t_2]} )
\]

\[
G(\gamma) := \frac{1}{-\sin \gamma} (y_1 + y_2)^2 \quad \text{(lower bound on } \| f \|^2 \text{)}
\]

\[
\sigma(\gamma) := \cos \gamma + \frac{\sin \gamma}{y_1 + y_2} (\sqrt{\sin(\alpha - \gamma)} + \sqrt{\sin(\beta - \gamma)}) \quad \text{(signed distance)}
\]

\[
\lambda(\gamma) := -\frac{\sin \gamma}{y_1 + y_2} \quad \text{(dilation factor)}
\]

Note that, by Lemma 2.1 (see Fig. 2.1), \( y_1 \) and \( y_2 \) can also be expressed as \( y_1 = \xi(t_1) = \| E_{[0, t_1]} \| \) and \( y_2 = \xi(t_2) = \| E_{[0, t_2]} \| \), where \( t_1, t_2 \in [0, \pi] \) are determined by \( \Delta(E_{[0, t_1]}) = \alpha - \gamma \) and \( \Delta(E_{[0, t_2]}) = \beta - \gamma \).

We mention further that \( G(\gamma) \) (see Theorem 5.6) is a lower bound on the bending energy of our generic curve \( f \), and \( \sigma(\gamma) \) is a signed distance, which is illustrated in Fig. 5.4. Regarding \( \lambda(\gamma) \), we mention that the curves \( r(u, \ell) \) and \( l(\ell, v) \), shown in Fig. 5.4, are similar to \( E_{[0, t_1]} \) and \( E_{[0, t_2]} \), respectively, with common dilation factor \( \lambda(\gamma) \). The crucial identity relating \( G(\gamma), \sigma(\gamma) \) and \( \lambda(\gamma) \) is given in Lemma 5.10.

Our constructive proof that \( S(u, v) \) contains an optimal curve is broken into several cases which depend on \( \alpha \) and \( \beta \). To help the reader track these cases, we give here a short description of each case and where in this section or the next it is treated.
Summary 5.4. We assume $\alpha \in (0, \pi)$, $|\beta| \leq \alpha$ and $\beta \geq \alpha - \pi$.

(a) $\beta = \alpha - \pi$. This case is treated just above Remark 5.5 and results in an optimal curve of second form.

(b) $\beta \geq 0$ or $(\alpha - \pi < \beta < 0$ and $\sigma(\beta) \geq 0)$. It is shown in Lemma 5.11 that the function $G$ has a minimum value $G_{\min}$, and in Corollary 5.12 $(v)$ it is shown that $G_{\min}$ equals the minimum bending energy in $S(u, v)$. Each $\gamma \in \Gamma$, where $G$ is minimized, gives rise to an optimal curve in $S(u, v)$, but the form of the optimal curve depends on whether or not $\gamma$ is the left endpoint of $\Gamma$. If $G$ is minimized at the left endpoint $\gamma = \alpha - \pi$, then it is shown in Corollary 5.12 $(iii)$, that the curve $f_{\alpha - \pi}$, which is of second form, is an optimal curve in $S(u, v)$. If $G$ is minimized at any other point $\gamma > \alpha - \pi$, then it is shown in Corollary 5.12 $(iv)$, $(v)$ that the curve $f_\gamma$, which is of first form, is an optimal curve in $S(u, v)$.

(c) $\alpha - \pi < \beta < 0$ and $\sigma(\beta) \leq 0$. In Theorem 6.2, it is shown that the unique curve (modulo equivalence) in $S(u, v)$ having minimal bending energy is a c-curve of first form.

The case $\alpha - \pi < \beta < 0$ and $\sigma(\beta) = 0$, which is included in both cases (b) and (c) above, serves as a bridge between these two cases; the first half of section 6, culminating in Corollary 6.8, is dedicated to showing that the function $G$ is uniquely minimized at $\gamma = \beta$ when $\alpha - \pi < \beta < 0$ and $\sigma(\beta) = 0$.

Our analysis employs an initial partitioning $S(u, v) = S'_{lr}(u, v) \cup S_{rl}(u, v)$, where $S'_{lr}(u, v)$ (which is non-empty if and only if $\alpha - \pi < \beta < 0$) denotes the set of all non-degenerate left-right s-curves in $S(u, v)$ and $S_{rl}(u, v)$ denotes the set of all right-left s-curves in $S(u, v)$. The set $S_{rl}(u, v)$ will be further partitioned into subsets $s_\gamma(u, v)$, $\gamma \in \Gamma$, and for each $\gamma \in \Gamma$ a distinguished subset $s^*_\gamma(u, v) \subset s_\gamma(u, v)$ will arise. The case $\gamma = \beta < 0$ is exceptional, so we address it first.

When $\beta < 0$, $C_r(u, v)$ is non-empty and we define $s^*_\beta(u, v) := s_\beta(u, v) := C_r(u, v)$. Let $\ell_\beta$ denote the line through the point 1 with direction $e^{i\beta}$. Since $C_r(u, v) \subset C_r(u, \ell_\beta)$, it follows that $\|f\|^2 \geq \|r(u, \ell_\beta)\|^2$ for all $f \in C_r(u, v)$. Since the turning angle in $r(u, \ell_\beta)$ is $-(\alpha - \beta)$ and the orthogonal distance from 0 to $\ell_\beta$ is $-\sin \beta$, it follows from Theorem 3.6 and Remark 3.7 that $\|r(u, \ell_\beta)\|^2 = G(\beta)$ and that $r(u, \ell_\beta)$ is directly similar to $E_{[-t, 0]}$. It can be shown, in a manner similar to the proof of Theorem 3.6, that $r(u, \ell_\beta)$ is directly congruent to $\lambda(\beta)E_{[-t, 0]}$. Furthermore, if $P_1$ denotes the terminal point of $r(u, \ell_\beta)$, then it can be shown that the signed distance, in the direction $e^{i\beta}$, from $P_1$ to 1 is $\sigma(\beta)$. We omit the details of this calculation since a very similar calculation will be performed below. The case $\beta = \alpha - \pi$ is particularly easy to handle since $\Gamma = \{\alpha - \pi\}$, rather than an interval. Assuming $\beta = \alpha - \pi$ (which implies $\alpha \geq \pi/2$), we note that the turning angle in $r(u, \ell_\beta)$ equals $-\pi$ and therefore $S(u, v) = C_r(u, v)$. One easily verifies that $\sigma(\beta) = -\cos \alpha \geq 0$ and it follows from the results of section 4 that $r(u, \ell_\beta) \sqcup [P_1, 1]$ is the unique curve, modulo equivalence and elongation of u-turns, in $S(u, v)$ with minimal bending energy. Moreover, $r(u, \ell_\beta) \sqcup [P_1, 1]$ is of second form with $c = \sigma(\beta)/\lambda(\beta)$ and $t = 0$.

Remark 5.5. If a curve $f \in S(u, v)$ contains a u-turn (eg. the curve $r(u, \ell_\beta) \sqcup [P_1, 1]$ above), then it is always possible to elongate the u-turn by inserting a pair of congruent line segments before and after the u-turn. Although longer, the resulting curve still belongs to $S(u, v)$ and has the same bending energy as $f$.

Having treated both the trivial case $\alpha = 0$ and the extreme case $\beta = \alpha - \pi$, we proceed
assuming that

\[(5.1)\quad \alpha \in (0, \pi), \quad |\beta| \leq \alpha, \quad \beta > \alpha - \pi.\]

Note that \(\Gamma\) is now an interval with right end point \(\beta_0 := \min\{\beta, 0\}\).

**Proposition 5.6.** If \(\beta < 0\), then \(\|f\|^2 > G(\beta)\) for all \(f \in S'_{lr}(u,v)\).

**Proof.** Let \(f : [0, L] \to \mathbb{C}\) be a non-degenerate left-right s-curve in \(S(u,v)\). There exist \(0 < s_0 < s_2 < L\) such that \(A = f(s_0)\) is an inflection point (i.e. \(f_{[0,s_0]}\) is a left c-curve and \(f_{[s_0,L]}\) is a right c-curve) and \(C = f(s_2)\) lies on the line through the point 0 with direction \(e^{i\alpha}\) (see Figure 5.2). It can be shown that \(\arg f(s_0) < \arg f'(s_0)\) while \(\arg f(s_2) > \arg f'(s_2)\), and it follows by continuity that there exists \(s_1 \in (s_0, s_2)\) such that \(\arg f(s_1) = \arg f'(s_1)\).

This equality implies that \(g := [0, B] \sqcup f_{[s_1,L]}\) is a right c-curve, where \(B = f(s_1)\). With \(\alpha_1 = \arg f(s_1)\) and \(u_1 = (0, e^{i\alpha_1})\), we note that \(g \in C_r(u_1, \ell_\beta)\) with bending energy \(\|g\|^2 = \|f_{[s_0,L]}\|^2 < \|f\|^2\). Since \(s_1 < s_2\), it follows that \(\alpha_1 > \alpha\), and therefore

\[
\|f\|^2 > \|g\|^2 \geq \|r(u_1, \ell_\beta)\|^2 = \frac{1}{-\sin \beta} \left( \frac{1}{2} \int_0^{\alpha_1 - \beta} \sqrt{\sin \tau} \, dt \right)^2 > \frac{1}{-\sin \beta} \left( \frac{1}{2} \int_0^{\alpha - \beta} \sqrt{\sin \tau} \, dt \right)^2 = \|r(u, \ell_\beta)\|^2 = G(\beta). \]

\(\square\)

Let \(f\) be a non-degenerate right-left s-curve in \(S(u,v)\). Then \(f\) has a well-defined inflection line \(\ell\) and inflection direction \(\gamma\), and it is easy to verify that \(\gamma \in [\alpha - \pi, \beta_0)\) (see Figure 5.3).

Let \(a\) denote the orthogonal distance from 0 to \(\ell\). Since \(\ell\) necessarily passes between 0 and 1, it follows that \(a\) belongs to the interval \((0, b)\), where \(b = -\sin \gamma\) denotes the orthogonal distance from 0 to the line through 1 with direction \(e^{i\gamma}\). Let us write \(f = f_r \sqcup f_l\), where \(f_r\) terminates (and \(f_l\) originates) at an inflection point \(I\) of \(f\). Since \(f_r \in C_r(u, \ell)\) and \(f_l \in C_l(\ell, v)\), it follows that

\[(5.2)\quad \|f\|^2 = \|f_r\|^2 + \|f_l\|^2 \geq \|r(u, \ell)\|^2 + \|l(\ell, v)\|^2 = \frac{y_1^2}{a} + \frac{y_2^2}{b - a}.\]
Let $H$ be the function defined in Lemma 4.2 with $\lambda_1 = y_1, \lambda_2 = y_2$ and $\mu = \frac{a}{b} \in (0, 1)$. Then (5.2) can be expressed as $\|f\|^2 \geq \frac{1}{b}H(\mu)$, and it follows from Lemma 4.2 that $H(\mu) \geq (y_1 + y_2)^2$, with equality if and only if $\mu = y_1/(y_1 + y_2)$; that is, if and only if $a = a_\gamma := -\sin \gamma \frac{y_1}{y_1 + y_2}$. We therefore conclude that $\|f\|^2 > G(\gamma)$ if $a \neq a_\gamma$ and $\|f\|^2 \geq G(\gamma)$ if $a = a_\gamma$.

Let $s_\gamma(u, v)$ denote the set of all non-degenerate right-left $s$-curves $f \in S(u, v)$ with inflection direction $\gamma$ and let $s_\gamma^*(u, v)$ be the subset of $s_\gamma(u, v)$ where $a = a_\gamma$. Let $\ell_\gamma$ be the inflection line of a curve in $s_\gamma^*(u, v)$, which is well-defined since all curves in $s_\gamma^*(u, v)$ have the same inflection line. We summarize the above discussion, along with the discussion preceding (5.1), in the following.

**Theorem 5.7.** Assume (5.1). For $\gamma \in \Gamma$ and $f \in s_\gamma(u, v)$, the following hold.

(i) $\|f\|^2 \geq G(\gamma)$.

(ii) If $\|f\|^2 = G(\gamma)$, then $f \in s_\gamma^*(u, v)$.

The significance of the quantity $\lambda(\beta)$, when $\beta < 0$, has been seen in that $r(u, \ell_\beta)$ is directly congruent to $\lambda(\beta)E_{[-t_1, 0]}$; we now reveal the significance of $\lambda(\gamma)$.

**Proposition 5.8.** Assume (5.1). For $\gamma \in [\alpha - \pi, \beta_0)$, $r(u, \ell_\gamma)$ is directly congruent to $\lambda(\gamma)E_{[-t_1, 0]}$ and $l(\ell_\gamma, v)$ is directly congruent to $\lambda(\gamma)E_{[0, t_2]}$.

**Proof.** Let $T_1$ and $T_2$ be the similarity transformations such that $r(u, \ell_\gamma) = T_1 \circ E_{[-t_1, 0]}$ and $l(\ell_\gamma, v) = T_2 \circ E_{[0, t_2]}$. Since the orthogonal distance from $E(-t_1)$ to the real axis is $\xi(t_1)$ and the orthogonal distance from 0 to $\ell_\gamma$ is $a_\gamma$, it follows that the dilation parameter of $T_1$ equals $\frac{a_\gamma}{\xi(t_1)} = \frac{1}{y_1} \frac{y_1}{y_1 + y_2} (-\sin \gamma) = \lambda(\gamma)$. By similar reasoning, the dilation parameter of $T_2$ equals $\frac{b - a_\gamma}{\xi(t_2)} = \frac{1}{y_2} (-\sin \gamma + y_1 \sin \gamma/(y_1 + y_2)) = \lambda(\gamma)$. \(\square\)

It was mentioned above that $\sigma(\beta)$ corresponds to the signed distance, in the direction $e^{i\beta}$, from $P_1$ to 1. The following reveals the significance of $\sigma(\gamma)$.

**Proposition 5.9.** Assume 5.1. For $\gamma \in [\alpha - \pi, \beta_0)$, let $P_1$ and $P_2$ denote the terminal and initial points of $r(u, \ell_\gamma)$ and $l(\ell_\gamma, v)$, respectively. Then $\sigma(\gamma)$ equals the signed distance, in the direction $e^{i\gamma}$, from $P_1$ to $P_2$.

**Proof.** Let $h$ denote the signed distance in question, and put $B = P_2 - P_1 = he^{i\gamma}$ (see Figure 5.4). It follows from Proposition 5.8 that $f = r(u, \ell_\gamma) \cup (l(\ell_\gamma, v) - B)$ is directly congruent to $\lambda(\gamma)E_{[-t_1, t_2]}$. Since the projected distance, in the direction $e^{i0}$, from $E(-t_1)$ to $E(t_2)$ equals $\sin t_1 + \sin t_2$, it follows that the projected distance, in the direction $e^{i\gamma}$, from the initial point to the terminal point of $f$ equals $\lambda(\gamma)(\sin t_1 + \sin t_2)$. Noting that the projected distance, in the direction $e^{i\gamma}$, from 0 to 1 equals $\cos \gamma$, we deduce that $\lambda(\gamma)(\sin t_1 + \sin t_2) = \cos \gamma - h$. Solving for $h$ and then employing the identity $\sin^2 \delta = \sin t$, when $\delta = \Delta(E_{[0, t]})$ and $t \in [0, \pi]$, yields the desired conclusion $h = \sigma(\gamma)$. \(\square\)

**Remark 5.10.** Two important consequences of Proposition 5.8 and Proposition 5.9 are:

1. Let $\gamma \in \Gamma$, with $\gamma > \alpha - \pi$. If $\sigma(\gamma) = 0$, then $f_\gamma := r(u, \ell_\gamma) \cup l(\ell_\gamma, v)$ has bending energy $G(\gamma)$ and is directly congruent to $\lambda(\gamma)E_{[-t_1, t_2]}$. It follows from the latter that $f_\gamma$ is of first form with $t_0 = -t_1$ and $t = t_2$. 

2. If $\sigma(\alpha - \pi) \geq 0$, then $f_{\alpha - \pi} := r(u, \ell, \gamma) \sqcup [P_1, P_2] \sqcup l(\ell, v)$ has bending energy $G(\alpha - \pi)$ and is of second form with $c = \sigma(\alpha - \pi)/\lambda(\alpha - \pi)$ and $t = t_2$.

In the following result, we see that $\sigma(\gamma)$ appears as a factor in the derivative $G'(\gamma)$.

**Lemma 5.11.** Assume \((5.1)\). The function $G : \Gamma \to (0, \infty)$ is continuously differentiable, has a minimum value $G_{min}$, and satisfies $\frac{d}{d\gamma} G(\gamma) = \frac{1}{\lambda(\gamma)^2} \sigma(\gamma)$ for all $\gamma \in \Gamma$.

**Proof.** For $\gamma \in \Gamma$, we have

$$G'(\gamma) = \frac{\cos \gamma}{\sin^2 \gamma} (y_1 + y_2)^2 - \frac{2}{\sin \gamma} (y_1 + y_2)(y'_1(\gamma) + y'_2(\gamma)) = \frac{1}{\lambda(\gamma)^2} \left( \cos \gamma - \frac{2\sin \gamma}{y_1 + y_2} \left( \frac{1}{2} \sqrt{\sin(\alpha - \gamma)} - \frac{1}{2} \sqrt{\sin(\beta - \gamma)} \right) \right) = \frac{1}{\lambda(\gamma)^2} \sigma(\gamma),$$

and we note that both $\lambda$ and $\sigma$ are continuous on $\Gamma$ and $\lambda$ is positive. If $\beta < 0$, then $\Gamma = [\alpha - \pi, \beta]$ and it is clear that $G$ has a minimum value. On the other hand, if $\beta \geq 0$, then $\Gamma = [\alpha - \pi, 0)$, but we note that $G(\gamma) \to \infty$ as $\gamma \to 0^-$; hence $G$ has a minimum value. \(\square\)

In preparation for the main result of this section, we remind the reader that $S(u, v)$ has been partitioned as

\[(5.3) \quad S(u, v) = S'_l(u, v) \cup \bigcup_{\gamma \in \Gamma} S_\gamma(u, v),\]

where $S'_l(u, v)$ is non-empty only when $\beta < 0$.

**Corollary 5.12.** Let \((5.1)\) be in force, and in case $\beta < 0$, assume $\sigma(\beta) \geq 0$. The following hold.

(i) If $\beta < 0$, then $\|f\|^2 > G_{min}$ for all $f \in S'_l(u, v)$.

(ii) If $\gamma \in \Gamma$ and $G(\gamma) > G_{min}$, then $\|f\|^2 > G_{min}$ for all $f \in s_\gamma(u, v)$.

(iii) If $G(\alpha - \pi) = G_{min}$, then $\sigma(\alpha - \pi) \geq 0$ and the curve $f_{\alpha - \pi}$, defined in Remark 5.10, is the unique curve, modulo equivalence and elongation of $u$-turns, in $s_{\alpha - \pi}(u, v)$ with bending energy $G_{min}$.

(iv) If $\gamma \in (\alpha - \pi, \beta_0)$ and $G(\gamma) = G_{min}$, then $\sigma(\gamma) = 0$ and the curve $f_\gamma$, defined in Remark 5.10, is the unique curve (modulo equivalence) in $s_\gamma(u, v)$ with bending energy $G_{min}$.

(v) If $\beta < 0$ and $G(\beta) = G_{min}$, then $\sigma(\beta) = 0$ and $f_\beta := r(u, \ell, \beta)$, which is of first form with $t_0 = -t_1$ and $t = 0$, is the unique curve (modulo equivalence) in $s_\beta(u, v)$ with bending energy $G_{min}$.

(vi) The minimum bending energy in $S(u, v)$ is $G_{min}$.

**Proof.** Items (i) and (ii) are immediate consequences Proposition 5.6 and Theorem 5.7 (i), respectively. For (iii), assume $G(\alpha - \pi) = G_{\min}$. If $\sigma(\alpha - \pi) < 0$, then it follows from Theorem 5.11 that $G'(\alpha - \pi) < 0$, which contradicts the assumption that $G$ attains its minimum at $\alpha - \pi$; therefore, $\sigma(\alpha - \pi) \geq 0$. It now follows from Remark 5.10
that \( f_{\alpha - \pi} \) has bending energy \( G_{\min} \). Now, suppose \( f \in s_{\alpha - \pi}(u, v) \) has bending energy \( G(\alpha - \pi) = G_{\min} \). By Theorem 5.7 (ii), \( f \) belongs to \( s_{\alpha - \pi}^*(u, v) \) and writing \( f = f_1 \cup f_2 \), as in the discussion preceding (5.2), it follows from (5.2) that \( \|f_1\|^2 = \|r(u, \ell_{\alpha - \pi})\|^2 \) and \( \|f_2\|^2 = \|l(\ell_{\alpha - \pi}, v)\|^2 \). It can then be deduced from the results of section 4 that \( f \) is equivalent to \( f_{\alpha - \pi} \) or can be obtained (equivalently) from \( f_{\alpha - \pi} \) by elongation of u-turns. We have thus proved (iii). Turning next to (iv), assume \( \gamma \in (\alpha - \pi, \beta_0) \) and \( G(\gamma) = G_{\min} \). Then \( G'(\gamma) = 0 \) and by Theorem 5.11, we have \( \sigma'(\gamma) = 0 \). It now follows from Remark 5.10 that \( f_\gamma \) has bending energy \( G_{\min} \) and the previous argument can be applied to show that if \( f \in s_\gamma(u, v) \) has bending energy \( G_{\min} \), then \( f \) is equivalent to \( f_\gamma \) (elongation of u-turns is ruled out since curves in \( s_\gamma(u, v) \) do not have u-turns). This proves (iv). For (v), assume \( \beta < 0 \) and \( G(\beta) = G_{\min} \). If \( \sigma(\beta) > 0 \), then \( G'(\beta) > 0 \), by Theorem 5.11, which contradicts the assumption that \( G \) is minimized at \( \beta \). Therefore, \( \sigma(\beta) = 0 \) and it follows that \( f_\beta := r(u, \ell_\beta) \) belongs to \( s_\beta(u, v) = C_r(u, v) \) (ie \( P_1 = 1 \)). From the discussion following Definition 5.3, we know that \( \|r(u, \ell_\beta)\|^2 = G(\beta) \) and that \( r(u, \ell_\beta) \) is directly congruent to \( \lambda(\beta)E_{[-t_1, 0]} \). From these it follows that \( \|f_\beta\|^2 = G_{\min} \) and that \( r(u, \ell_\beta) \) is of first form, with \( t_0 = -t_1 \) and \( t = 0 \). Since the turning angle in \( r(u, \ell_\beta) \) has magnitude less than \( \pi \), it easily follows from the results of section 4 (more precision, please) that \( f_\beta \) is the unique curve (modulo equivalence) in \( s_\beta(u, v) \) with bending energy \( G_{\min} \) and the proof of (v) is complete. We now prove (vi). It follows from (i), Theorem 5.7 and (5.3) that \( \|f\|^2 \geq G_{\min} \) for all \( f \in S(u, v) \). Since the function \( G \) has a minimum, there exists \( \gamma \in \Gamma \) such that \( G(\gamma) = G_{\min} \), and it then follows from items (iii), (iv) and (v) that \( f_\gamma \) is a curve in \( S(u, v) \) with bending energy \( G_{\min} \). This proves (vi). \( \square \)

The remaining case, \( \beta < 0 \) with \( \sigma(\beta) < 0 \), will be addressed in the following section.

6. Optimal s-curves, part II

The purpose of this section is to prove the following two results.

**Theorem 6.1.** Let \( 0 \leq t_1 < t_2 \leq \pi \) satisfy \( t_2 - t_1 < \pi \). Then \( E_{[t_1, t_2]} \) is the unique curve (modulo equivalence) in \( S(E(t_1), E(t_2)) \) with minimal bending energy.

**Theorem 6.2.** In the notation of section 5, let \( \alpha \in (0, \pi) \) and \( \beta < 0 \) satisfy (5.1) and suppose \( \sigma(\beta) \leq 0 \). Then there exist \( -\pi < t_1 < t_2 \leq 0 \) such that \( E_{[t_1, t_2]} \) is directly similar to a curve \( f \in S(u, v) \). Moreover, the curve \( f \) is the unique curve (modulo equivalence) in \( S(u, v) \) with minimal bending energy.

For \( t \in (0, \pi] \), let \( \psi \) and \( \theta \), as shown in Figure 6.1b, be the positive angles made by the chord [0, \( E(t) \)] and the segment \( E_{[0, t]} \). With \( \ell \) denoting the tangent line to \( E \) at \( E(t) \), let \( p(t) \) denote the orthogonal distance from 0 to \( \ell \).
Lemma 6.3. For $t \in (0, \pi)$, $\theta(t) > \psi(t)$.

Proof. We will first show, by way of contradiction, that $\theta(t) \neq \psi(t)$. Assume $\theta(t) = \psi(t)$ for some $t \in (0, \pi)$. Let $T(z) = c_1 z + c_2$ be the congruency transformation which interchanges $E(t)$ and 0, and set $W = T \circ E_{[t,0]}$, where $E_{[t,0]}$ denotes the reversal of $E_{[0,t]}$. Since $\theta(t) = \psi(t)$, it follows that $W$ belongs to $C_1(\hat{E}(0), \hat{E}(t))$. But Theorem 4.4 asserts that $E_{[0,t]}$ is the unique curve (modulo equivalence) in $C_1(\hat{E}(0), \hat{E}(t))$ with minimal bending energy. Therefore, since $W$ and $E_{[0,t]}$ have the same bending energy, they must be equivalent. However, they cannot be equivalent because $W$ begins with a nonzero curvature, namely $2 \sin t$, while $E_{[0,t]}$ begins with curvature 0. This proves that $\theta(t) \neq \psi(t)$ for all $t \in (0, \pi)$. While $\theta(\pi)$ and $\psi(\pi)$ both equal $\pi/2$, a simple computation shows that their derivatives satisfy $-\theta'(\pi) = \psi'(\pi) = 1/d$, and it follows that $\theta(t) > \psi(t)$ for $t \in (0, \pi)$ sufficiently close to $\pi$. Since $\theta$ and $\psi$ are continuous, we conclude that $\theta(t) > \psi(t)$ for all $t \in (0, \pi)$. □

Lemma 6.4. For $t \in [0, \pi)$, $p(t) \xi(t) < (2d - \xi(t))^2$.

Proof. The orthogonal distance $p(t)$ can be formulated as the magnitude of the cross product $E(t) \times \frac{E'(t)}{|E'(t)|}$ which yields

$$p(t) = \det \left[ \begin{array}{c} \sin t  \\ \cos t \sqrt{1 + \sin^2 t} \end{array} \right] \frac{\xi(t)}{\sin^2 t} = \sin^3 t - \xi(t) \cos t \sqrt{1 + \sin^2 t}, \quad 0 \leq t \leq \pi.$$

We therefore have

$$p(t) \xi(t) - (2d - \xi(t))^2 = (\sin^3 t + 4d) \xi(t) - 4d^2 - \left(1 + \cos t \sqrt{1 + \sin^2 t}\right) \xi(t)^2$$

$$\leq (\sin^3 t + 4d) \xi(t) - 4d^2 =: g(t),$$

where the inequality holds since $-1 \leq \cos t \sqrt{1 + \sin^2 t} \leq 1$. We note that $g(\pi) = 0$ and $g'(t) = \sin^2 t \left(3 \xi(t) \cos t + \frac{\sin^3 t + 4d}{\sqrt{2 - \cos t}}\right)$. It is clear that $g'(t) > 0$ for $t \in (0, \pi/2]$, and for $t \in (\pi/2, \pi)$ (where $-\cos t > 0$), we have

$$g'(t) = (-3 \cos t) \sin^2 t \left(-\xi(t) + \frac{\sin^3 t + 4d}{-3 \cos t \sqrt{2 - \cos t}}\right) \geq (-3 \cos t) \sin^2 t \left(-\xi(t) + \frac{4d}{3}\right),$$
as \( 0 < -\cos t \sqrt{2 - \cos \xi} < 1 \) on \((\pi/2, \pi)\). Since \(0 \leq \xi(t) \leq d\), it follows that \(g'(t) > 0\) for all \(t \in (0, \pi)\) and hence \(g\) is increasing on \([0, \pi]\). For \(t \in [0, \pi]\), we therefore have \(p(t)\xi(t) - (2d - \xi(t))^2 \leq g(t) < g(\pi) = 0\), which completes the proof. \(\square\)

In the following, we again use the notation \(S'_r(u, v)\) (resp. \(S'_l(u, v)\)) for the set of all non-degenerate left-right (resp. right-left) s-curves connecting \(u\) to \(v\).

**Lemma 6.5.** For \(t \in (0, \pi)\), the following hold:

(i) If \(f \in S'_r(\vec{E}(0), \vec{E}(t))\), then \(\|f\|^2 > \|E_{[0,t]}\|^2\).

(ii) If \(f \in S'_l(\vec{E}(0), \vec{E}(t))\) ends with a left u-turn, then \(\|f\|^2 > \|E_{[0,t]}\|^2\).

Proof. We will employ the notation and results of the previous section, so in order to minimize confusion, we will actually prove the following equivalent formulations:

(i') If \(f \in S'_r(\vec{E}(-t), \vec{E}(0))\), then \(\|f\|^2 > \|E_{[-t,0]}\|^2\).

(ii') If \(f \in S'_l(\vec{E}(-t), \vec{E}(0))\) begins with a right u-turn, then \(\|f\|^2 > \|E_{[-t,0]}\|^2\).

Let \(T(z) = c_1z + c_2\) be the similarity transformation determined by \(T(E(-t)) = 0\) and \(T(0) = 1\), and note that \(T\) brings the configuration \((\vec{E}(-t), \vec{E}(0))\) to the canonical form \((u, v)\) (see Figure 6.2), where \(u = (0, e^{i\alpha})\), \(v = (1, e^{i\beta})\) with \(\alpha = \theta(t)\), \(\beta = -\psi(t)\). Since \(0 < \psi(t) < \theta(t) < \pi\), it follows that (5.1) holds. Noting that \(r(u, \ell_{\beta}) = T \circ E_{[-t,0]}\), we see that \(\sigma(\beta) = 0\) and \(G(\beta) = \|r(u, \ell_{\beta})\|^2\). For (i'), suppose \(f \in S'_r(\vec{E}(-t), \vec{E}(0))\). Then \(T \circ f \in S'_r(u, v)\), and it follows from Proposition 5.6 that \(\|T \circ f\|^2 > \|r(u, \ell_{\beta})\|^2\). Since \(r(u, \ell_{\beta}) = T \circ E_{[-t,0]}\), we immediately obtain (i'). Now suppose \(f \in S'_l(\vec{E}(-t), \vec{E}(0))\) begins with a right u-turn. Then \(T \circ f\) belongs to the set \(s_{\alpha-\pi}(u, v)\) defined just above Theorem 5.7, and it follows from this theorem that \(\|T \circ f\|^2 \geq G(\alpha - \pi)\). Since \(G(\beta) = \|r(u, \ell_{\beta})\|^2\) and \(r(u, \ell_{\beta}) = T \circ E_{[-t,0]}\), in order to establish (ii'), it suffices to show that \(G(\alpha - \pi) > G(\beta)\).

From Definition 5.3, we have \(G(\beta) = \xi(t)^2/\sin \psi(t)\) and \(G(\alpha - \pi) = (2d - \xi(t))^2/\sin \theta(t)\). Referring to Figure 6.1b, we see that \(\sin \psi(t) = \xi(t)/|E(t)|\) and \(\sin \theta(t) = p(t)/|E(t)|\). Hence

\[
G(\alpha - \pi) - G(\beta) = \frac{|E(t)|}{p(t)} \left((2d - \xi(t))^2 - p(t)\xi(t)\right) > 0,
\]

by the previous lemma, and this completes the proof of (ii'). \(\square\)

**Proposition 6.6.** Let \(t \in (0, \pi)\). Then \(E_{[0,t]}\) is the unique curve (modulo equivalence) in \(S(\vec{E}(0), \vec{E}(t))\) having minimal bending energy.
Proof. With Corollary 3.4, Theorem 4.4 and Lemma 6.5 in view, it suffices to show that $\|f\|^2 > \|E_{[0,t]}\|^2$ whenever $f \in S'_{[t]}(\vec{E}(0), \vec{E}(t))$ does not end with a left u-turn. Let $f = f_r \sqcup f_l$ be as stated, where $f_r$ is a right c-curve and $f_l$ is a left c-curve (see Figure 6.3). Since $f_r$ originates on $\vec{E}(0)$, it follows that there exists $t_0 \in (t, \pi)$ such that $f_l \sqcup E_{[t,t_0]}$ is a left u-turn. Thus $f \sqcup E_{[t,t_0]}$ belongs to $S'_{[t]}(\vec{E}(0), \vec{E}(t_0))$ and ends with a left u-turn. By Lemma 6.5 (ii),

$$\|f\|^2 + \|E_{[t,t_0]}\|^2 = \|f \sqcup E_{[t,t_0]}\|^2 > \|E_{[0,t_0]}\|^2 = \|E_{[0,t]}\|^2 + \|E_{[t,t_0]}\|^2,$$

whence we obtain $\|f\|^2 > \|E_{[0,t]}\|^2$. □

Remark 6.7. By symmetry, it follows from Proposition 6.6 that $E_{[t,\pi]}$ is the unique curve (modulo equivalence) in $S(\vec{E}(t), \vec{E}(\pi))$ having minimal bending energy.

In the context of the previous section, Proposition 6.6 asserts the following.

Corollary 6.8. Let direction angles $\alpha \in (0, \pi)$, $\beta < 0$ satisfy (5.1) and suppose $\sigma(\beta) = 0$. Then $G(\gamma) > G(\beta)$ for all $\gamma \in [\alpha - \pi, \beta)$; that is, $G(\gamma)$ is uniquely minimized at $\gamma = \beta$.

Proof of Theorem 6.1. The extreme cases $t_1 = 0$ and $t_2 = \pi$ have been settled in Proposition 6.6 and Remark 6.7, respectively, so assume $0 < t_1 < t_2 < \pi$. By symmetry, and with Corollary 3.4 and Theorem 4.4 in view, it suffices to show that $\|f\|^2 > \|E_{[t_1,t_2]}\|^2$ whenever $f$ belongs to $S'_{[t]}(\vec{E}(t_1), \vec{E}(t_2))$. Let $f$ be as stated, and let $\gamma \in (-\pi, \pi)$ be the direction angle of $f$ at an inflection point $I$.

Case 1: $\gamma \in [0, \pi)$.

Then $f \sqcup E_{[t_2,\pi]}$ belongs to $S'_{[t]}(\vec{E}(t_1), \vec{E}(\pi))$, and it follows from Remark 6.7 that $\|f \sqcup E_{[t_2,\pi]}\|^2 > \|E_{[t_1,\pi]}\|^2$, which implies $\|f\|^2 > \|E_{[t_1,t_2]}\|^2$.

Case 2: $\gamma \in (-\pi, 0)$

Since $f$ begins at $E(t_1)$ with a direction angle in $(0, \pi)$, there exists a point $B$ on $f$, between $E(t_1)$ and $I$, where $f$ has direction angle $0$ (see Figure 6.4). Let us write $f = f_1 \sqcup f_2$, where $f_1$ terminates (and $f_2$ originates) at $B$. Let $\ell$ be the (horizontal) tangent line to $f$ at $B$, and put $q = l(\ell, \vec{E}(t_2))$. Since $q$ and $E_{[0,t_2]}$ are similar and terminate at the same unit tangent vector, it follows that $q$ originates at the point of intersection $A$ between $\ell$ and the line segment $[0, E(t_2)]$. Moreover, since $q$ is at a smaller scale than $E_{[0,t_2]}$, we have $\|q\|^2 > \|E_{[0,t_2]}\|^2$. Now, it follows from Proposition 6.6 that $\|[A, B] \sqcup f_2\|^2 > \|q\|^2$, and therefore

$$\|f\|^2 > \|[A, B] \sqcup f_2\|^2 > \|q\|^2 > \|E_{[0,t_2]}\|^2 > \|E_{[t_1,t_2]}\|^2.$$

□

Remark 6.9. By symmetry, Theorem 6.1 remains valid when $-\pi < t_1 < t_2 < \pi$. 
Lemma 6.10. Define $J : [\pi, \infty) \to \mathbb{C}$ by $J(t) = \begin{cases} E(t) & \text{if } t \in [\pi, 0] \\ t & \text{if } t > 0 \end{cases}$. Given positive angles $\alpha \geq \delta > 0$, with $\alpha + \delta < \pi$, there exists $t_1 \in (\pi, 0)$ and $t_2 > t_1$ such that the chord $[J(t_1), J(t_2)]$ intersects $J$ with interior angles $\alpha$ and $\delta$ at $J(t_1)$ and $J(t_2)$, respectively.

Proof. We refer to Figure 6.5. For $t \in [\pi, 0]$, let $\tau(t)$ denote the direction angle of $\vec{E}(t)$. As $t$ ranges from $\pi$ to $0$, $\tau(t)$ decreases continuously from $\pi$ to $0$, and it follows that there exists $b \in (\pi, 0)$ such that $\tau(b) = \alpha$. For $t \in [\pi, b)$, let $R_t$ denote the ray emanating from $J(t)$ with direction angle $\tau(t) - \alpha$ and note that since the direction angle is positive, $R_t$ intersects $J$ at a unique point $J(\mu(t))$, where $\mu(t) > t$. Let $\omega(t)$ denote the interior angle, at $J(\mu(t))$, made when the chord $[J(t), J(\mu(t))]$ intersects $J$ (the interior angle at $J(t)$ equals $\alpha$ by construction). It is clear that $\omega(t)$ depends continuously on $t \in [\pi, b)$ and tends to $0$ as $t \to b^-$. We claim that $\omega(-\pi) > \delta$. If $\mu(-\pi) \geq 0$ (ie $J(\mu(t))$ lies on $[0, \infty)$), then $\omega(-\pi) = \pi - \alpha$ and the claim follows immediately from the assumption that $\alpha + \delta < \pi$. On the other hand, if $\mu(-\pi) < 0$, then $\alpha = \psi(-\mu(t))$ and $\omega(-\pi) = \theta(-\mu(t))$; hence, by Lemma 6.3, $\omega(-\pi) > \alpha$ and now the claim follows from the assumption $\alpha \geq \delta$.

By the intermediate value property of continuous functions, there exists $t_1 \in [-\pi, b)$ such that $\omega(t_1) = \delta$, and the lemma is proved with $t_2 = \mu(t_1)$. \qed

Proof of Theorem 6.2. Put $\delta = -\beta > 0$ and note that the hypothesis of Lemma 6.10 follows from (5.1), and we obtain the conclusion of the lemma. We claim that $t_2 \leq 0$. To see this, assume to the contrary that $t_2 > 0$. Let $T(z) = c_1 z + c_2$ be the similarity transformation determined by $T(J(t_1)) = 0$ and $T(J(t_2)) = 1$. It follows from the discussion following Definition 5.3 that $T \circ E_{[t_1, 0]} = r(u, \ell \beta)$ and that $\sigma(\beta) = |c_1| (t_2 - 0) > 0$, which contradicts the assumption that $\sigma(\beta) \leq 0$. Therefore, $t_2 \leq 0$ and we conclude, from Theorem 6.1 and Remark 6.9, that $f = T \circ E_{[t_1, t_2]}$ is the unique curve (modulo equivalence) in $S(u, v)$ with minimal bending energy. \qed

7. Proof of Theorem 1.1

Given any sequence of points $P_1, ..., P_m \in \mathbb{R}^2$ we denote by $A(P_1, ..., P_m)$ the family of admissible curves through $P_1, ..., P_m$ such that they are s-curves between any two consecutive points $P_i$ and $P_{i+1}$.

The main theorem of this section is as follows.
Theorem 7.1. Given any sequence of distinct points \( P_1, \ldots, P_m \in \mathbb{R}^2 \) there is a curve \( c \in A(P_1, \ldots, P_m) \), which minimizes the bending energy in the family and between any two consecutive points \( P_i \) and \( P_{i+1} \) it is an s-curve.

First we have to show that

Proposition 7.2. Let \( P_1, \ldots, P_m \in \mathbb{R}^2 \) be distinct points, then \( A(P_1, \ldots, P_m) \neq \emptyset \).

Proof of Prop. 7.2. We have to construct a curve through the points \( P_1, \ldots, P_m \) such that it is an s-curve between any two consecutive points \( P_i \) and \( P_{i+1} \). Actually, we will prove a slightly stronger statement, namely we will construct a closed curve passing through the points \( P_1, P_2, \ldots, P_m \) and back to \( P_1 \) again. The constructed curve will be a \( C^1 \) curve which is \( C^2 \) between the points \( P_i \) and \( P_{i+1} \), therefore it will satisfy our definition for curves in Section 2.

Let \( v_i \) be the unit vector at \( P_i \) that is orthogonal to the line bisecting the angle \( \angle(P_{i-1}P_iP_{i+1}) \) and satisfies the inequality \( \alpha_i = \angle(v_i, \overrightarrow{P_iP_{i+1}}) \leq \pi/2 \) (see Figure 7.1). The vector \( v_i \) is uniquely determined except in the case when the point \( P_{i+1} \) is on the half-line starting at \( P_i \) and passing through \( P_{i-1} \). In this case we have two choices for \( v_i \) and we choose one of them. For this to make sense for \( v_1 \) and \( v_m \) as well we set \( P_{m+1} = P_1 \) and \( P_0 = P_m \).

From the construction it follows that \( |\angle(v_{i-1}, \overrightarrow{P_{i-1}P_{i+1}})| \leq \pi/2 \) and \( |\angle(v_i, \overrightarrow{P_{i-1}P_i})| \leq \pi/2 \) for all \( i = 1, 2, \ldots, m \) (again we set \( v_0 = v_m \) and from this it is not hard to see that \( v_{i-1}, v_i \) is s-feasible, that is there is an s-curve \( c_{i-1} \) connecting \( v_{i-1} \) to \( v_i \) for all \( i = 1, 2, \ldots, m \)).

The union of these curves will be the desired curve. This completes the proof of the proposition.

Let \( u, v \) be s-feasible \( (S(u, v) \neq \emptyset) \) unit tangent vectors. Denote by \( ||u, v||^2 \) the minimum of the bending energy in \( S(u, v) \). By the previous sections (Theorem 5.1) this minimum is assumed by a curve in \( S(u, v) \).

We will need the following proposition which shows that the limit of s-feasible vectors is also s-feasible assuming that the bending energy is bounded.

Proposition 7.3. Let \( P_u \neq P_v \) be different points of \( \mathbb{R}^2 \). Let \( u_n, v_n \) be s-feasible unit tangent vectors with base-points \( P_u \) and \( P_v \) respectively, such that \( \lim u_n = u \) and \( \lim v_n = v \). If \( ||u_n, v_n||^2 \) are bounded then \( u, v \) is also s-feasible.

Proof. Without loss of generality we can assume that \( P_u = 0 \in \mathbb{C} \) and \( P_v = 1 \in \mathbb{C} \). Let \( \alpha, \alpha_n \) be the angle of \( u \) and \( u_n \), respectively, measured from the positive \( x \)-axis and similarly \( \beta, \beta_n \) be the angle of \( v, v_n \), respectively.
It is easy to see that if $u_n, v_n$ are s-feasible configurations with $\alpha_n \to \pm \pi$ then $||u_n, v_n||^2 \to \infty$. By reversing the directions of the curves we can show the same way that if $\beta_n \to \pm \pi$, then $||u_n, v_n||^2 \to \infty$.

This implies not only that $\alpha_n, \beta_n \in (-\pi, \pi)$ for all $n \in \mathbb{N}$ but from the assumption that the bending energies of $||u_n, v_n||^2$ were bounded it follows also that $\alpha, \beta \in (-\pi, \pi)$.

If $\alpha = 0$ then $\beta \in (-\pi, \pi)$ and it is an s-feasible configuration.

If $\alpha \neq 0$ we may assume, without loss of generality, that $\alpha > 0$ and $\alpha_n > 0$ for all $n \in \mathbb{N}$. It is not hard to see that $u_n, v_n$ are s-feasible if and only if $\beta_n \in [\alpha_n - \pi, \pi)$. Taking into consideration that $\beta \in (-\pi, \pi)$, we conclude that $\beta \in [\alpha - \pi, \pi)$, therefore the limit configuration is s-feasible. This concludes the proof of the proposition.

Next, we show that the bending energy $||u, v||^2$ is continuous in $u$ and $v$. For this we will need some preparation.

Let $f : [0, L] \to \mathbb{C}$ be a unit speed right c-curve with initial and terminal directions $u, v$, where $P_u = 0 \in \mathbb{C}$ and $P_v = 1 \in \mathbb{C}$. Let $\bar{u}, \bar{v}$ be c-feasible unit tangent vectors with the same base-points as $u$ and $v$, respectively. Treating them as complex numbers set $\alpha = \arg(u), \bar{\alpha} = \arg(\bar{u}), \beta = \arg(v)$ and $\bar{\beta} = \arg(\bar{v})$, where the argument is in the interval $(-\pi, \pi)$. If any one of the angles were to be equal to $\pi$ or $-\pi$, the configuration $u, v$ or $\bar{u}, \bar{v}$ would not be c-feasible.

Since we assumed that $f$ was a right c-curve we have $\alpha \geq 0$ and $\beta \leq 0$. We have the following.

**Proposition 7.4.** With the notations introduced above let us assume that $|f(t)| < M$ for some $M > 1$ and $\alpha, |\beta| > \eta$ for some $\eta > 0$. Then, for every $\epsilon > 0$ there is a $\delta(\epsilon, M, \eta) > 0$ (depending only on $\epsilon, M, \eta$) such that if $\bar{u}, \bar{v}$ are c-feasible unit tangent vectors with the same base-points as $u$ and $v$, respectively, and $|\bar{\alpha} - \alpha|, |\bar{\beta} - \beta| < \delta$, then there is a c-curve $c \in C(\bar{u}, \bar{v})$ with $||c||^2 < ||f||^2 + \epsilon$.

**Proof of Proposition 7.4.** We will modify the curve $f$ at both ends to obtain the curve $c$. The modifications are essentially independent from one another.

Although it is not true that if $\bar{u}, \bar{v}$ are c-feasible, then $u, v$ (or $\bar{u}, \bar{v}$) will also be c-feasible (a counterexample can be found easily by considering the case when $f$ is a u-turn) but it is easy to see that one of them $\bar{u}, v$ or $u, \bar{v}$ will be c-feasible. Let us assume that $\bar{u}, v$ is c-feasible. The other case can be treated similarly.

First, we will describe how to modify the curve near the base point of $u$ to obtain a new curve $c$, whose angles with its chord will be $\bar{\alpha}$ and $\beta$. We can make the bending energy of the new curve as close as we want to the bending energy of $f$ if $\bar{\alpha}$ is sufficiently close to $\alpha$.

We start with the case when $0 < \bar{\alpha} < \alpha$. We modify our curve as follows (see Figure 7.2).
First, draw a line with angle $\bar{\alpha}$ through a point on the negative $x$-axis such that this line is disjoint from our curve $f$, then translate it horizontally towards $f$ until it makes first contact. Denote by $P$ the point on the negative $x$-axis where the line intersects the $x$-axis and by $f(t_0)$ the point (or one of the points) on the curve where the line makes contact with the curve. Let $c_1$ be the c-curve that goes from $P$ to $f(t_0)$ along the tangent line then follows the curve $f$ from $f(t_0)$ to 1.

It is clear from the construction that $||c_1||^2 \leq ||f||^2$. Moreover, it is easy to see that there is a $\delta_1(\epsilon, \eta, M)$ depending only on $\epsilon, \eta$ and $M$ such that if $\eta/2 < \alpha - \delta_1 < \bar{\alpha} < \alpha$, then $l = dist(P, 1) < \frac{||f||^2 + \epsilon/2}{||f||^2}$. Re-scale the curve $c_1$ with center 1 and with scaling factor $1/l$ to obtain the curve $c$. Since the bending energy changes inversely with the scaling factor we obtain

$$||c||^2 = ||c_1||^2 l \leq ||f||^2 \frac{||f||^2 + \epsilon/2}{||f||^2} = ||f||^2 + \epsilon/2.$$ 

In case $\bar{\alpha} > \alpha$ we modify our curve $c$ in two steps (see Figure 7.3).

First, we draw the tangent lines $l_u$ and $l_v$ to $f$ at 0 and 1, respectively and append line segments to $f$ at both ends such that the chord of the resulting curve $c_1$ is parallel to the $x$-axis. Denote the distance of the chord of the new curve and the $x$-axis by $d$. The endpoints of the curve $c_1$ are denoted by $P \in l_u$ and by $Q \in l_v$. Notice that the bending energy of $c_1$ is the same as the bending energy of $f$.

Second, replace the line segment $[P, 0]$ by an arc of a circle that is tangent to $c_1$ at 0 and intersects the segment $[P, Q]$ at an angle $\bar{\alpha}$. Let us denote the intersection point on $[P, Q]$ by $\bar{P} \in [P, Q]$. One can construct such an arc easily and we leave the details to the reader. Denote the resulting curve by $c_2$. Let us remark that $c_2$ is a c-curve since $\bar{u}, v$ was c-feasible. For the bending energy of the curve $c_2$ we have $||c_2||^2 = ||f||^2 + ||\text{arc of the circle}||^2$. There
are two parameters to be determined: $d$ and the radius of the circle.

It is easy to see that there is a $1/2 > d(\eta) > 0$, such that if $d = d(\eta)$, then $\text{dist}(P, Q) < \frac{||f||^2 + \epsilon/2}{||f||^2 + \epsilon/4}$. Once $d$ is fixed one can find a $\delta_2(d, \eta, \epsilon) > 0$ such that if $\alpha < \bar{\alpha} < \alpha + \delta_2 < \pi - \eta/2$, then there is an arc of a circle (simply by choosing the radius large enough) with bending energy less than $\epsilon/4$. We can also ensure (by choosing $\delta_2$ sufficiently small) that $P$ lies between $P$ and $Q$.

At this point we have $||c_2||^2 < ||f||^2 + \epsilon/4$ and $l = \text{dist}(P, Q) < \frac{||f||^2 + \epsilon/2}{||f||^2 + \epsilon/4}$.

As before, re-scale the curve $c_2$ with scaling factor $1/l$ and translate it to obtain the curve $c$ connecting the points 0 and 1. Since the bending energy changes inversely with the scaling factor we obtain

$$||c||^2 = ||c_2||^2l \leq (||f||^2 + \epsilon/4) \frac{||f||^2 + \epsilon/2}{||f||^2 + \epsilon/4} = ||f||^2 + \epsilon/2.$$ 

Applying the same procedure to the other end of the curve concludes the proof of Proposition 7.4.

For given $\alpha, \beta \in (-\pi, \pi)$ let $u, v$ be unit tangent vectors, with base points 0 and 1, respectively, such that $\alpha = \text{arg}(u), \beta = \text{arg}(v)$. It will be useful to introduce the following notations: $C(\alpha, \beta) := C(u, v)$, $S(\alpha, \beta) := S(u, v)$ and $||\alpha, \beta||^2 := ||u, v||^2$, which we will call the bending energy of the pair $\alpha, \beta$. We say $\alpha, \beta$ are s-feasible if and only if $u, v$ are s-feasible.

**Proposition 7.5.** For any $\epsilon > 0$ there is a $\mu(\epsilon) > 0$ such that if $|\alpha|, |\beta| < \mu(\epsilon)$, then $||\alpha, \beta||^2 < \epsilon$.

**Proof of Proposition 7.5.** All one needs to do is to construct s-curves with small bending energy with initial and terminal directions $u$ and $v$. The construction is easy and we will leave the details to the reader.

In what follows we will rely heavily on the results of the previous sections, mainly sections 5 and 6. In those sections it was always assumed that $\alpha$ and $\beta$ were in ”canonical” arrangement, that is, $\alpha \geq |\beta| \geq 0$. However, if we modify $\alpha$ and $\beta$ by a small amount the new pair $\bar{\alpha}$ and $\bar{\beta}$ may no longer be in ”canonical” arrangement. The following two propositions will help dealing with this situation.

**Proposition 7.6.**

(i) $||\alpha, \beta||^2 = ||-\beta, -\alpha||^2$

(ii) $||\alpha, \beta||^2 = ||-\alpha, -\beta||^2$

(iii) $||\alpha, \beta||^2 = ||\beta, \alpha||^2$.

**Proof of Proposition 7.6.** For any curve in $S(\alpha, \beta)$ if we reflect the curve with respect to the $x = 1/2$ line and reverse its orientation, we obtain a curve in $S(-\beta, -\alpha)$ with the same bending energy. This means, that there is a bijection between $S(\alpha, \beta)$ and $S(-\beta, -\alpha)$, which preserves the bending energy. This implies (i). Similarly, reflection across the $x$-axis gives a bending energy preserving bijection between $S(\alpha, \beta)$ and $S(-\alpha, -\beta)$, which yields (ii). Combining (i) and (ii) we obtain (iii).
**Proposition 7.7.** Let us assume that \( \alpha \geq |\beta| \geq 0 \) but \( 0 < \bar{\alpha} < |\bar{\beta}| \).

Set \((\bar{\alpha}, \bar{\beta}) = \begin{cases} (\bar{\beta}, \bar{\alpha}) & \text{if } \bar{\alpha} < \bar{\beta} \\ (-\bar{\beta}, -\bar{\alpha}) & \text{if } \bar{\alpha} < -\bar{\beta} \end{cases} \)

If \( |\alpha - \bar{\alpha}|, |\beta - \bar{\beta}| < \delta \), then \( |\alpha - \bar{\alpha}|, |\beta - \bar{\beta}| < \delta \).

Proof of Proposition 7.7. The proof is elementary and we will leave it to the reader.

Let us indicate how we will use the previous two propositions in the proof of the next one. Assume that \( \alpha \geq |\beta| \), and \( |\alpha - \bar{\alpha}|, |\beta - \bar{\beta}| < \delta \) with \( \alpha > \delta > 0 \). If \( \bar{\alpha} < |\bar{\beta}| \), then we will replace \( \bar{\alpha}, \bar{\beta} \) with a new pair \( \hat{\alpha}, \hat{\beta} \) as in Proposition 7.7. Then the new pair will be in "canonical" arrangement \( (\hat{\alpha} \geq |\hat{\beta}|), |\hat{\alpha} - \bar{\alpha}|, |\hat{\beta} - \bar{\beta}| < \delta \) and from Proposition 7.6 we have \( ||\alpha, \beta||^2 = ||\hat{\alpha}, \hat{\beta}||^2 \).

**Proposition 7.8.** With the notations introduced above let us assume that \( \alpha, \beta \) are s-feasible satisfying \( \alpha \geq |\beta| \geq 0 \). If \( \pi - \eta > \alpha > \eta \), then for every \( \epsilon > 0 \) there is a \( \delta(\epsilon, \eta) > 0 \) (depending only on \( \epsilon, \eta \)) such that if \( |\alpha - \bar{\alpha}|, |\beta - \bar{\beta}| < \delta \) and \( \hat{\alpha}, \hat{\beta} \) are s-feasible, then \( ||\hat{\alpha}, \hat{\beta}||^2 < ||\alpha, \beta||^2 + \epsilon \).

Before we start the proof of Proposition 7.8 let us recall some quantities defined in Section 5 (Definition 5.3). Assuming that \( \gamma \in [\alpha - \pi, \beta] \cap (-\infty, 0) \) we have

\[
y_1(\alpha, \gamma) = \frac{1}{2} \int_0^{\alpha - \gamma} \sqrt{\sin \tau} d\tau, \quad y_2(\beta, \gamma) = \frac{1}{2} \int_0^{\beta - \gamma} \sqrt{\sin \tau} d\tau, \quad G(\alpha, \beta, \gamma) = \frac{(y_1 + y_2)^2}{-\sin \gamma}.
\]

\( G_{\min}(\alpha, \beta) = \min\{G(\alpha, \beta, \gamma) : \gamma \in [\alpha - \pi, \beta] \cap (\beta - \pi, 0)\} \).

Recall from Section 3 that \( d = \xi(\pi) = \frac{1}{2} \int_0^\pi \sqrt{\sin \tau} d\tau \) and define the quantity \( \gamma_0 \) by \( \gamma_0 = -\sin^{-1}\left( \sin \eta (1 - \cos \eta)^2 / (16d^2) \right) \). Since \( \pi - \alpha \geq \eta \), from the formulas above one can verify immediately that

\[
G(\alpha, \beta, \alpha - \pi) \leq \frac{4d^2}{\sin \eta} < G(\alpha, \beta, \gamma) \quad \text{if} \quad \gamma_0 < \gamma < \beta.
\]

This implies that if \( G(\alpha, \beta, \gamma) = G_{\min}(\alpha, \beta) \), then \( \gamma \leq \beta^* = \min\{\beta, \gamma_0\} \).

It will be convenient to extend the domain of \( G(\alpha, \beta, \gamma) \) to include any \( \gamma \in [-\pi, 0] \) without changing the minimum \( G_{\min}(\alpha, \beta) \) or the value of \( \gamma \) at which the minimum is assumed. Let us define the set \( K_\eta \) to be \( K_\eta = \{(\alpha, \beta) : \eta \leq \alpha \leq \pi - \eta, \quad |\beta| \leq \alpha, \quad \alpha - \pi \leq \beta\} \). For \( (\alpha, \beta) \in K_\eta \), \( \beta^* = \min\{\beta, \gamma_0\} \) and \( \gamma \in [-\pi, 0] \) we set

\[
\hat{G}(\alpha, \beta, \gamma) = \begin{cases} G(\alpha, \beta, \beta^*) + \gamma - \beta^* & \text{if } \beta^* < \gamma \leq 0 \\ G(\alpha, \beta, \gamma) & \text{if } \alpha - \pi \leq \gamma \leq \beta^* \\ G(\alpha, \beta, \alpha - \pi) + \alpha - \pi - \gamma & \text{if } -\pi \leq \gamma < \alpha - \pi \end{cases}.
\]
From the remark following inequality (7.1) and from the construction of \( \hat{G} \) it is clear that \( G_{\text{min}}(\alpha, \beta) = \hat{G}_{\text{min}}(\alpha, \beta) = \min\{\hat{G}(\alpha, \beta, \gamma) : \gamma \in [-\pi, 0]\} \). Moreover \( G \) and \( \hat{G} \) assumes their minimum at the same points, that is \( G_{\text{min}}(\alpha, \beta) = \hat{G}(\alpha, \beta, \gamma) \) if and only if \( \hat{G}_{\text{min}}(\alpha, \beta) = \hat{G}(\alpha, \beta, \gamma) \).

The quantity \( \sigma(\gamma) \) will be interesting for us only in the case when \( \gamma = \beta \). Therefore we have

\[
\sigma(\alpha, \beta) = \cos \beta + \frac{\sin \beta}{y_1(\alpha, \beta)} \sqrt{\sin(\alpha - \beta)}, \quad \alpha - \pi \leq \beta < 0.
\]

It is easy to see that \( \sigma(\alpha, \beta) \to 1 \) as \( \beta \to 0 \). Therefore we can extend the domain of \( \sigma(\alpha, \beta) \) to the region \( 0 \leq \beta \) by setting

\[
\sigma(\alpha, \beta) = 1, \quad \text{if} \quad \beta \geq 0.
\]

We can summarize the results of the previous sections as follows.

If \( \sigma(\alpha, \beta) \leq 0 \), then there is a c-curve in \( S(\alpha, \beta) \) that minimizes the bending energy.

If \( \sigma(\alpha, \beta) > 0 \), then there is an s-curve in \( S(\alpha, \beta) \) that minimizes the bending energy and in that case \( ||\alpha, \beta||^2 = G_{\text{min}}(\alpha, \beta) \).

**Proof of Proposition 7.8.**

Case 1. Let us assume that \( \sigma(\alpha, \beta) \leq 0 \). Then \( \beta < 0 \) and there is a c-curve \( f \in C(\alpha, \beta) \), which is a segment of rectangular elastica minimizing the bending energy in the family \( C(\alpha, \beta) \).

Let us remark that the condition that \( f \) is bounded (\( |f(t)| < M \) for some \( M > 1 \)) will be fulfilled for \( M = 10 \), since the ratio of length over breadth for any segment of rectangular elastica is bounded by 10.

Since \( \sigma(\alpha, \beta) \) is continuous and \( \pi - \eta > \alpha > \eta > 0 \), one can see that there is an \( \eta_1(\eta) > 0 \), such that \( |\beta| > \eta_1(\eta) \). Applying Proposition 7.4 to the curve \( f \), it implies that there is a \( \delta(\eta_1, \eta, \epsilon) > 0 \) such that if \( |\alpha - \alpha_1|, |\beta - \beta| < \delta \), then \( ||\alpha, \beta||^2 < ||\alpha, \beta||^2 + \epsilon/2 \).

Case 2. Let us assume that \( \sigma(\alpha, \beta) > 0 \). In this case \( ||\alpha, \beta||^2 = G_{\text{min}}(\alpha, \beta) = G(\alpha, \beta, \gamma) \), for some not necessarily unique \( \gamma = \gamma(\alpha, \beta) \). From the remark following inequality (7.1) we have \( \gamma(\alpha, \beta) \in [\alpha - \pi, \beta^*] \), where \( \beta^* = \min\{\beta, \gamma_0\} \).

From the definition of \( G \) and \( \hat{G} \) one can see that \( \hat{G}(\alpha, \beta, \gamma) \) is continuous, hence uniformly continuous on the region \( K_\eta \times [-\pi, 0] \). Therefore, there is a \( \delta_1(\eta_0) > 0 \) such that for \( (\alpha, \beta), (\alpha', \beta') \in K_\eta \) we have

\[
|\hat{G}(\alpha, \beta, \gamma) - \hat{G}(\alpha', \beta', \gamma')| < \frac{\epsilon}{2}, \quad \text{if} \quad |\alpha - \alpha'|, |\beta - \beta'|, |\gamma - \gamma'| < \delta_1.
\]

Let us assume that \( |\alpha - \alpha_1|, |\beta - \beta| < \delta \), where \( \delta < \min\{\delta_1, \frac{\eta}{4}\} \) is determined later. We can further assume, without loss of generality, that \( \alpha \geq |\beta| \). Otherwise, replace \( \alpha, \beta \) with \( \tilde{\alpha}, \tilde{\beta} \) defined in Proposition 7.7. Then we have, according to Proposition 7.7, \( |\tilde{\alpha} - \alpha_1|, |\tilde{\beta} - \beta| < \delta \) and we can prove the statement for \( \tilde{\alpha}, \tilde{\beta} \). Since Proposition 7.6 implies \( ||\tilde{\alpha}, \tilde{\beta}||^2 = ||\alpha, \beta||^2 \) the statement is proved for \( \tilde{\alpha}, \tilde{\beta} \).

Part (a). If \( \sigma(\tilde{\alpha}, \tilde{\beta}) \geq 0 \) define \( \tilde{\gamma} = \gamma(\tilde{\alpha}, \tilde{\beta}) \) to be an angle where \( G(\tilde{\alpha}, \tilde{\beta}, \gamma) \) assumes its minimum. Since \( G \) and \( \hat{G} \) assumes their minimum at the same points (see the remarks following the definition of \( \hat{G} \)) we have

\[
||\tilde{\alpha}, \tilde{\beta}||^2 = G_{\text{min}}(\tilde{\alpha}, \tilde{\beta}) = G(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) = \hat{G}(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})
\]
and

\[ \|\alpha, \beta\|^2 = G_{\text{min}}(\alpha, \beta) = G(\alpha, \beta, \gamma) = \tilde{G}(\alpha, \beta, \gamma). \]

Taking into consideration (7.2) we have

\[ \|\tilde{\alpha}, \tilde{\beta}\|^2 = G_{\text{min}}(\tilde{\alpha}, \tilde{\beta}) = \tilde{G}(\tilde{\alpha}, \tilde{\beta}, \gamma) \leq \tilde{G}(\alpha, \beta, \gamma) < \tilde{G}(\alpha, \beta, \gamma) + \frac{\epsilon}{2} = \|\alpha, \beta\|^2 + \frac{\epsilon}{2}. \]

This concludes Part (a).

Part (b). Assume that \( \sigma(\tilde{\alpha}, \tilde{\beta}) < 0 \). Since \( \delta < \eta/4 \) we have \( (\alpha, \beta), (\tilde{\alpha}, \tilde{\beta}) \in K_{\eta/2} \), which is a convex set. Therefore the line segment \( [(\alpha, \beta), (\tilde{\alpha}, \tilde{\beta})] \) is also a subset of \( K_{\eta/2} \). Since \( \sigma \) is a continuous function of \( \alpha, \beta \) there is a pair \( (\alpha_1, \beta_1) \in [(\alpha, \beta), (\tilde{\alpha}, \tilde{\beta})] \) with \( \sigma(\alpha_1, \beta_1) = 0 \).

Applying the previous argument for \( \alpha_1, \beta_1 \) instead of \( \tilde{\alpha}, \tilde{\beta} \) we obtain

\[ \|\alpha_1, \beta_1\|^2 < \|\alpha, \beta\|^2 + \frac{\epsilon}{2}. \]

Since \( \sigma(\alpha_1, \beta_1) = 0 \) there is a segment of rectangular elastica with \( \|f\|^2 = \|\alpha_1, \beta_1\|^2 \). Moreover we have \( \alpha_1 > \eta \), therefore we have, as in Case 1, \( |\beta_1| > \eta_1(\eta/2) \). By applying Proposition 7.4 we obtain

\[ \|\tilde{\alpha}, \tilde{\beta}\|^2 < \|\alpha_1, \beta_1\|^2 + \frac{\epsilon}{2}, \]

if \( \delta < \delta(\epsilon/2, 10, \min\{\eta/2, \eta_1(\eta/2)\}) \) as mentioned in Proposition 7.4.

Combining this with the previous inequality we obtain

\[ \|\tilde{\alpha}, \tilde{\beta}\|^2 < \|\alpha, \beta\|^2 + \epsilon. \]

This concludes the proof of Proposition 7.8.

Next we show that:

**Lemma 7.9.** The bending energy \( \|u,v\|^2 \) is continuous in \( u \) and \( v \).

**Proof of Lemma 7.9.** Let \( \epsilon > 0 \) be arbitrary and \( u, v \) be s-feasible. Set \( \alpha, \beta \) as before and assume that \( \alpha \geq |\beta| \geq 0 \).

Case 1. If \( \alpha < \mu(\epsilon)/2 \), where \( \mu(\epsilon) \) was given in Proposition 7.5, then choose \( \delta < \mu(\epsilon)/2 \). If \( |\tilde{\alpha} - \alpha|, |\tilde{\beta} - \beta| < \delta \), then \( \tilde{\alpha}, |\tilde{\beta}| < \mu(\epsilon) \). Applying Proposition 7.5 implies \( \|\tilde{\alpha}, \tilde{\beta}\|^2, \|\alpha, \beta\|^2 < \epsilon \), therefore

\[ \|\tilde{\alpha}, \tilde{\beta}\|^2 - \|\alpha, \beta\|^2 < \epsilon. \]

Case 2. If \( \pi > \alpha > \mu(\epsilon)/2 \), then there is a \( \mu_1 > 0 \) such that \( \pi - 2\mu_1 > \alpha > 2\mu_1 \). Let \( \delta = \min\{\mu_1, \delta(\epsilon, \mu_1)\} \), where \( \delta(\epsilon, \mu_1) \) comes from Proposition 7.8. If \( |\tilde{\alpha} - \alpha|, |\tilde{\beta} - \beta| < \delta \), then \( \pi - \mu_1 > \tilde{\alpha}, \alpha > \mu_1 \). If \( \tilde{\alpha} \geq |\tilde{\beta}| \geq 0 \), then applying Proposition 7.8 twice to \( \alpha, \beta \) and to \( \tilde{\alpha}, \tilde{\beta} \) we obtain

\[ \|\tilde{\alpha}, \tilde{\beta}\|^2 < \|\alpha, \beta\|^2 + \epsilon \quad \text{and} \quad \|\alpha, \beta\|^2 < \|\tilde{\alpha}, \tilde{\beta}\|^2 + \epsilon, \]

hence

\[ \|\tilde{\alpha}, \tilde{\beta}\|^2 - \|\alpha, \beta\|^2 < \epsilon. \]
If \( \bar{\alpha} < |\bar{\beta}| \), then, as in the previous proof, replace \( \bar{\alpha}, \bar{\beta} \) with \( \tilde{\alpha}, \tilde{\beta} \) defined in Proposition 7.7. Then we have, according to Proposition 7.7, \(|\bar{\alpha} - \alpha|, |\bar{\beta} - \beta| < \delta\) and we can prove the statement for \( \tilde{\alpha}, \tilde{\beta} \). Since Proposition 7.6 implies \(||\alpha, \beta|||^2 = ||\alpha, \beta|||^2\) the statement is proved for \( \tilde{\alpha}, \tilde{\beta} \).

This completes the proof of Lemma 7.9.

**Proof of Theorem 7.1.** Let \( c^n \in \mathcal{A}(P_1, ..., P_m) \) be a sequence of unit speed curves that minimizes the bending energy. Let \( t^n_i \in \mathbb{R} \) be the points where \( c^n(t^n_i) = P_i \) and set \( v^n_i = \vec{c}^n(t^n_i) \) to be the tangent vectors of \( c^n \) at \( P_i \).

Passing on to a subsequence if necessary we can assume without loss of generality that \( v^n_i \) is convergent for every \( i = 1, 2, ..., m \) as \( n \) tends to infinity. Set \( v_i = \lim_{n \to \infty} v^n_i \).

Since \( c^n \) is a minimizing sequence, \( ||c^n||^2 \) is bounded. Proposition 7.3 implies that \( v_i, v_{i+1} \) are s-feasible for all \( i = 1, 2, ..., m - 1 \).

For all \( i = 1, 2, ..., m - 1 \) let \( s_i \in S(v_i, v_{i+1}) \) be a unit speed curve with minimal bending energy (see Theorem 5.1), and let \( s \) be the union of these curves. By construction it is a \( C^1 \) curve, and it is a piecewise s-curve, therefore, \( s \in \mathcal{A}(P_1, ..., P_m) \).

We will show that it has minimal bending energy in the family \( \mathcal{A}(P_1, ..., P_m) \).

On the one hand it is clear that

\[
||s||^2 = \sum_{i=1}^{m} ||s_i||^2 .
\]

On the other hand for \( i = 1, 2, ..., m - 1 \) denote by \( s^n_i \in S(v^n_i, v^n_{i+1}) \) a unit speed curve with minimal bending energy in the family \( S(v^n_i, v^n_{i+1}) \) (see Theorem 5.1). We know that

\[
||s^n_i||^2 \leq ||c^n[t^n_i, t^n_{i+1}]||^2,
\]

therefore,

\[
\sum_{i=1}^{m-1} ||s^n_i||^2 \leq \sum_{i=1}^{m-1} ||c^n[t^n_i, t^n_{i+1}]||^2 = ||c^n||^2 .
\]

From Lemma 7.9 we conclude that

\[
\lim_{n \to \infty} \sum_{i=1}^{m-1} ||s^n_i||^2 = \sum_{i=1}^{m} ||s_i||^2 = ||s||^2 .
\]

Combining this with the previous inequality we get

\[
||s||^2 \leq ||c^n||^2 ,
\]

which proves the claim and the theorem as well.

Remark. It will be shown in a subsequent paper, that if each of the curves \( s_i \) are of form one (see Definition 5.2), then the resulting optimal curve \( s \) is \( C^2 \).

Remark. Let us denote by \( \mathcal{A}_{closed}(P_1, ..., P_m) \) the set of closed curves passing through the points \( P_1, ..., P_m \) such that they are s-curves between any two consecutive points. Notice, that the proof of Theorem 7.1 will work equally well for closed admissible curves, provided there is at least one closed curve in \( \mathcal{A}_{closed}(P_1, ..., P_m) \). This is exactly what we showed in the proof of Proposition 7.2. Therefore we have the following extension of Theorem 7.1:
Theorem 7.10. Given any sequence of points \( P_1, \ldots, P_m \in \mathbb{R}^2 \) there is a closed curve \( c \in A_{\text{closed}}(P_1, \ldots, P_m) \), which minimizes the bending energy in the family.

Acknowledgements. The authors are very grateful to Hakim Johnson (Kuwait English School) for writing the computer program Curve Ensemble, based on elastic splines, which was used to make the figures. We are also grateful to Aurelian Bejancu for discussions on variational calculus which lead to a clean proof of Theorem 3.2.

References

1. G. Birkhoff & C.R. de Boor, Piecewise polynomial interpolation and approximation, Approximation of Functions, Proc. General Motors Symposium of 1964, H.L. Garabedian ed., Elsevier, New York and Amsterdam, 1965, pp. 164-190.
2. G. Birkhoff, H. Burchard & D. Thomas, Nonlinear interpolation by splines, pseudosplines, and elastica, Res. Publ. 468, General Motors Research Laboratories, Warren, Mich., 1965.
3. G.H. Brunnett, Properties of minimal-energy splines, Curve and surface design, SIAM, Philadelphia PA, 1992, pp. 3-22.
4. V.G.A. Goss., Snap buckling, writhing and loop formation in twisted rods, PhD. Thesis, University College London (2003).
5. J.W. Jerome., Smooth interpolating curves of prescribed length and minimum curvature, Proc. Amer. Math. Soc. 51 (1975), 62–66.
6. E.H. Lee & G.E. Forsyth, Variational study of nonlinear spline curves, SIAM Rev. 15 (1975), 120–133.
7. H.L. Royden, Real Analysis, 3rd ed., Prentice Hall, New Jersey, 1988.