Numerical Solutions of Unsteady Advection-Diffusion Equations by Using EG Iteration with Wave Variable Transformation

Nur Afza Mat Ali¹, Rostang Rahman², Jumat Sulaiman³ and Khadizah Ghazali⁴
¹,²,³,⁴Faculty of Science and Natural Resources, Universiti Malaysia Sabah, 88400 Kota Kinabalu, Sabah, Malaysia
E-mail: afzamatali@yahoo.com

Abstract. The primary goal of this paper is to investigate the effectiveness of the 4-point Explicit Group (4-point EG) iterative method for solving one-dimensional unsteady advection-diffusion problems via similarity transform. By using this transformation approach, the proposed problem can be reduced into the corresponding two-point boundary volume problem. By imposing the second-order central finite difference discretization scheme, then the corresponding approximation equation can be derived to construct a system of linear equations. Having a large linear system, the 4-point EG iterative method has been used to solve the generated system of linear equations. The formulation of the 4-point EG method is also derived. Some numerical experiments are conducted that to verify the 4-point EG method is more effective than the Gauss-Seidel (GS) method.

1. Introduction
Let us consider the one-dimensional advection-diffusion equation (ADE) which is one of the parabolic partial differential equations (PDEs) being given as follows

\[
\frac{\partial u}{\partial t}(x,t) + \beta \frac{\partial u}{\partial x}(x,t) = \alpha \frac{\partial^2 u}{\partial x^2}(x,t), \quad (x,t) \in [a,b] \times [0,T],
\]

(1)

with the initial conditions

\[ u(x,0) = \phi(x), \]

and the boundary conditions

\[ u(a,t) = g_0(t), \quad u(b,t) = g_1(t), \quad t \in [0,T], \]

where \( \alpha \) and \( \beta \) are both constants. Then, in order to develop an approximation equation of problem (1), let us assumed that \([a, b]\) is the solution domain in space and time directions can be uniformly divided into \(m\) subintervals and \(M\) subintervals respectively as follows

\[ \Delta x = \frac{b-a}{m} = h, \quad \Delta t = \frac{T}{M}, \]

where \( \Delta x \) and \( \Delta t \) are defined as an increment of both directions

Parabolic PDEs are primary important to provide mathematical models in wide variety of phenomena such as physical, chemical, biological and engineering. In recent years, many authors have been attracted to study the solutions of problem (1) using various numerical methods such as explicit and implicit
meshless methods [1], finite-difference method [2] and others [3, 4]. Having a linear system generated from the corresponding approximation equation, there are two common methods such as direct methods and iterative methods in order to get the analytical and/or numerical solution of the linear system. However, we intend to discuss a family of iterative methods in which very helpful in order to solve linear systems including a large number of variables. Addition to that, many researches have been used the iterative method such as SOR method [5, 6], block iteration [7] and Newton-block method [8] to solve the linear systems.

As mentioned in the previous paragraph, we investigate the effectiveness of 4-point block iterative method mainly on 4-point Explicit Group (4-point EG) as linear solver of linear system generated from the discretization process of problem (1). Based on the previous studies conducted by Evans [9], Othman and Abdullah [10], Saudi and Sulaiman [11, 12] and Ghazali et al. [8], they pointed out that 4-point EG is one of the effective block iterative methods in solving a large scale linear system. Due to its advantage of using block iteration, the processing of simulation in solving the problem (1) becomes faster in which their iterations number and computational time get smaller.

The fundamental target of this present study is to investigate the effectiveness of the 4-point EG iterative method in solving the problem of ADE (1) followed by finding the similarity reductions of problem (1). The structure of this paper is as follows: section 2 contains reductions of ADE (1) to ordinary differential equations (ODEs) then discretizing it using second-order central finite difference scheme. The formulation of 4-point EG are included in section 3. Next, in section 4 we present some results and discussion. Lastly, in section 5 gives the conclusions and future work.

2. Similarities Transform via Finite Difference Approximation

In this section, we apply similarity reductions of ADE (1) in which solutions of the PDEs which may either be expressed in terms of lower-order PDEs or an ODEs. Therefore, this study intends to reduce ADE into the corresponding ODE by using similarity transformation. Then, the corresponding ODEs can be discretized by applying the second-order central finite difference scheme to form the approximation equation. By considering the similarity reduction, any function of two variables \( u(x,t) \) in equation (1) can be reduced to one variable function \( u(\xi) \) where \( \xi = (x-ct) \) is called wave variable [13, 14].

To apply this reduction, we consider the general linear PDE as follows

\[
P(u, u_x, u_t, u_{xx}, u_{tt}, \ldots) = 0. \tag{2}
\]

By using a transformation of wave variable \( \xi = (x-ct) \) where \( c \) is constant, therefore we can rewrite equation (2) in the following linear ODE

\[
Q(u, u_\xi, u_{\xi\xi}, \ldots) = 0. \tag{3}
\]

We can derive equation (3) via the wave variable transformation by using the following terms

\[
\begin{align*}
\frac{\partial}{\partial t} &= -c \frac{\partial}{\partial \xi}, \\
\frac{\partial^2}{\partial t^2} &= c^2 \frac{\partial^2}{\partial \xi^2}, \\
\frac{\partial}{\partial x} &= \frac{\partial}{\partial \xi}, \\
\frac{\partial^2}{\partial x^2} &= \frac{\partial^2}{\partial \xi^2}.
\end{align*}
\tag{4}
\]

By substituting equation (4) into equation (1), the corresponding linear ODE can be formed as follows

\[
\alpha \frac{d^2 u}{d\xi^2} + \lambda \frac{du}{d\xi} = 0. \tag{5}
\]
Before discretizing equation (5), let us consider the following second-order central difference schemes

\[
\frac{du}{d\xi} = \frac{u_{i+1} - u_{i-1}}{2h}
\]

\[
\frac{d^2u}{d\xi^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}
\]  \hspace{1cm} (6)

By imposing the second-order central difference scheme (6) to equation (5), the discretization process can be implemented in order to derive the second-order finite difference approximation equation as follow

\[
\alpha \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \lambda \frac{u_{i+1} - u_{i-1}}{2h} = 0.
\]  \hspace{1cm} (7)

For convenience, we can approximation equation (7) and rewrite in following single equation as generally stated in the following equation

\[
r u_{i-1} + s u_i + t u_{i+1} = 0,
\]  \hspace{1cm} (8)

where

\[
r = \frac{\alpha}{h^2} - \frac{c - \beta}{2h}, \quad s = \frac{-2\alpha}{h^2}, \quad t = \frac{\alpha}{h^2} + \frac{c - \beta}{2h}
\]

The derivation of linear system (9) which can be developed from approximation equation (8) can be written in matrix form as follows

\[
AU = F
\]  \hspace{1cm} (9)

where

\[
A = \begin{bmatrix}
\alpha & s & t & & & \\
r & \alpha & s & t & & \\
& \ddots & \ddots & \ddots & \ddots & \\
& & r & s & t & \\
& & & r & s & \alpha
\end{bmatrix}_{m \times m},
\]

\[
U = \begin{bmatrix}
U_1 & U_2 & U_3 & \ldots & U_{m-2} & U_{m-1}
\end{bmatrix}^T,
\]

\[
F = [-rU_0 \quad 0 \quad 0 \quad \ldots \quad 0 \quad -tU_m]^T.
\]

3. Formulation of Four-Point EG Iteration

Referring to the main characteristics of the coefficient matrix \(A\) in equation (9), it can be observed that the coefficient matrix \(A\) has been categorized as large-scale and sparse. Basically, this linear system can be called as large-scale tridiagonal linear system subjected to the value of \(m\). To speed up the iteration process, we intend to deal with the 4-point block method namely 4-EG iterative method. Actually the 4-point block iterative method has been proposed by Evans [9] via the explicit group iterative method in order to solve the sparse linear systems. In this study, the formulation of the 4 Point-EG iterative methods can be constructed based on the approximation equation (8).

\[
\begin{bmatrix}
\alpha & s & t & 0 & 0 \\
r & \alpha & s & t & 0 \\
0 & r & \alpha & s & t \\
0 & 0 & r & \alpha & s
\end{bmatrix}
\begin{bmatrix}
U_i \\
U_{i+1} \\
U_{i+2} \\
U_{i+3}
\end{bmatrix}
= \begin{bmatrix}
L_1 \\
L_2 \\
L_3 \\
L_4
\end{bmatrix}
\]  \hspace{1cm} (10)
where

\[
L_1 = F_i - r U_{i-1}, \\
L_2 = F_{i+1}, \\
L_3 = F_{i+2}, \\
L_4 = F_{i+3} - U_{i+4}.
\]

Referring to Figure 1, let us consider a set of 4-point group to form the follow linear system of order four.

![Figure 1. Implementation of 4-point EG iteration method over the solution domain at \(m=16\).](image)

By determining the inverse matrix of the coefficient matrix in equation (10), the general form of 4-point EG iterative method can be easily stated as

\[
\begin{bmatrix}
U_i^{(k+1)} \\
U_{i+1}^{(k+1)} \\
U_{i+2}^{(k+1)} \\
U_{i+3}^{(k+1)}
\end{bmatrix} = \frac{1}{\det} \begin{bmatrix}
s^3 - 2rs & r - s^2 & s & -1 \\
rs & s - r^2 & -s^2 & s \\
r^2s & r & s^3 - rs & r - s^2 \\
r^2s & r & rs & s^3 - 2rs
\end{bmatrix} \begin{bmatrix}
L_1 \\
L_2 \\
L_3 \\
L_4
\end{bmatrix}
\]

where \(\det = r^2 - 3rs^2 + s^4\). In order to calculate the current value of \(u_p^{(k+1)}\), \(p = i, i+1, i+2, i+3\), the general scheme of the 4-EG iterative method can be simplified as

\[
\begin{align*}
U_i^{(k+1)} &= \frac{\left(s^3 - 2rs\right)L_1 + L_a}{\det}, \\
U_{i+1}^{(k+1)} &= \frac{\left(r - s^2\right)L_4 - sL_a}{\det}, \\
U_{i+2}^{(k+1)} &= \frac{\left(sL_b + \left(r - s^2\right)L_4\right)}{\det}, \\
U_{i+3}^{(k+1)} &= \frac{\left(s^3 - 2rs\right)L_4 + rL_a}{\det},
\end{align*}
\]

where

\[
L_a = \left(r - s^2\right)L_2 + sL_3 - L_4, \\
L_b = r^2L_1 - rsL_2 + \left(s^2 - r\right)L_3.
\]

For \(i = 1, 5, 9, \ldots, m - 7\), other value of the remaining interior point will be calculated based on the ungroup case [15]. Based on equation (12), the general algorithm for the implementation of 4-point EG iterative method in solving the linear system (9) may be demonstrated in Algorithm 1

**Algorithm 1: 4-Point EG iteration**

i. Set all the parameters.

ii. Calculate the value of \(u_i^{(k+1)}\)

For \(i = 1, 5, 9, \ldots, m - 7\), calculate using following equations
\[ \det = r^2 - 3rs^2 + s^4 \]
\[ L_a = (r - s^2)L_2 + sL_3 - L_4, \]
\[ L_b = r^2L_4 - rsL_2 + (s^2 - r)J_3. \]
\[ U_i = \left( \frac{s^3 - 2rs}{\det} \right) L_1 + L_a, \]
\[ U_{i+1} = \left( \frac{r(r - s^2)}{\det} \right) L_1 - sL_a, \]
\[ U_{i+2} = \left( \frac{sL_b + (r - s^2)L_4}{\det} \right), \]
\[ U_{i+3} = \left( \frac{s^3 - 2rs}{\det} \right) L_4 + rL_a. \]

for \( i = m - 6 \), calculate an ungroup of three points.

iii. Perform the convergence test, \( \left| U^{(k+1)} - U^{(k)} \right| \leq \varepsilon = 10^{-10} \). If yes, move to step (vi). Otherwise go to step (iv).

iv. Stop.

4. Numerical Results

We consider the four tested examples of advection-diffusion problems to investigate the effectiveness of 4-point EG and GS iterative methods. For the numerical simulations, three criteria are used to make the comparative analysis mainly on number of iterations, execution time and maximum absolute error. The following four examples are used in this paper.

Example 1 [1]
Let us consider the one-dimensional problem as follows
\[ \frac{\partial^2 u}{\partial t^2} = \alpha \frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial u}{\partial x}, \quad 0 < x < 1, t > 0, \]  \hfill (13)
with the exact solution is of the form
\[ u(x, t) = ae^{(bt-cx)}, \quad c = \frac{\sqrt{\nu^2 + 4kb}}{2k} > 0. \]
The implementation of numerical experiment for this example has considered \( \alpha = 0.1, \beta = 0.1, a = 1.0, b = 0.1 \).

Example 2 [2]
Let us consider the following PDE which is found in many transport phenomena
\[ \frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} - \alpha \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 1, t > 0, \]  \hfill (14)
with the exact solution is given by
\[ u(x, t) = \frac{1}{\sqrt{4t + 1}} e^{-\frac{(x-0.2-\beta t)^2}{\alpha(4t+1)}}. \]
The implementation of numerical experiment for this example has considered \( \alpha = 0.1, \beta = 1.0 \).
Example 3 [3]
The linear convection-diffusion equation is considered in the following
\[
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2} \quad 0 < x < 1, t > 0,
\]
with the exact solution
\[
u(x,t) = \frac{1}{\sqrt{s}} e^{-\frac{50(x-t)^2}{s}} \quad s = 1 + 200\alpha t.
\]
The implementation of numerical experiment for this example has considered \(\alpha = 1.0\).

Example 4 [4]
We consider the following advection-diffusion equation
\[
\frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2} (x,t),
\]
with the exact solution is of the form
\[
u(x,t)=e^{\left\{\frac{x}{4} - \frac{t}{2}\right\}} e^{\left\{-\frac{\pi^2}{40} \left[\cos \left(\frac{\pi}{2} x\right) - 0.25 \sin \left(\frac{\pi}{2} x\right)\right]\right\}}.
\]
The implementation of numerical experiment for this example has considered \(\alpha = 0.1, \beta = 1.0\).

Through the simulations, we have set up the tolerance error, \(\varepsilon = 10^{-10}\) as a convergence test at different sizes grid which are \(m = 512, 1024, 2048, 4096\) and \(8192\). We have used maximum absolute error as stopping error in which very helpful when the convergence is very slow to converge. The following Table 1 show the comparison of the three criteria by using 4-point EG and GS iterative methods. Meanwhile, Table 2 represents the percentage of reduction for the 4-point EG method as compared to GS iterative method.

Table 1 demonstrate the numerical results of four tested examples which consist of iterations number, execution time in seconds and maximum absolute error. These three parameters have been required to compare the effectiveness of both iterative methods which are GS and 4-point EG iterative methods in solving the ADE problems. Clearly, Table 2 shows that the 4-point EG iterative method involves the least number of iterations by approximately 66.83% – 72.39% compared to GS iterative method. While in terms of execution time, 4-point EG proved to be very faster by approximately 59.01% - 70.74% compared to GS iterative method.
The results have showed that 4-point EG iterative method is drastically better than GS iterative method in aspects of that these two criteria. Overall, 4-point EG iterative method is a best solver compare to the GS method for solving advection-diffusion problem. For the future work, other 4-point block iterative families such as 4-point EDGAOR method [16], 4-point MEGMSOR method [17] and 4-point Newton-EGMSOR method [18] also can be considered as linear solvers to solve this problem.

5. Conclusions
This paper attempts to show the effectiveness of the 4-point block iterative method namely 4-point EG by doing the comparative analysis based on the iterations number and execution time. The results have showed that 4-point EG iterative method is drastically better than GS iterative method in aspects of that these two criteria. Overall, 4-point EG iterative method is a best solver compare to the GS method for solving advection-diffusion problem. For the future work, other 4-point block iterative families such as 4-point EDGAOR method [16], 4-point MEGMSOR method [17] and 4-point Newton-EGMSOR method [18] also can be considered as linear solvers to solve this problem.

Table 1. Comparison of the three criteria using 4-point EG and GS iterative methods.

| Example | M   | Iterations Number | Time(second) | Maximum Absolute Error |
|---------|-----|-------------------|--------------|------------------------|
|         |     | GS    | 4EG | GS    | 4EG | GS    | 4EG |
| 1       | 512 | 352650 | 97373 | 8.27 | 2.43 | 3.1578e-06 | 5.1498e-06 |
|         | 1024 | 1263318 | 352655 | 59.19 | 17.32 | 4.8105e-06 | 3.1577e-06 |
| 2       | 2048 | 4464138 | 1263323 | 418.19 | 123.98 | 3.6693e-05 | 4.8105e-06 |
|         | 4096 | 15500010 | 4463412 | 2912.91 | 877.64 | 1.6417e-04 | 3.6683e-05 |
|         | 8192 | 52573871 | 15500115 | 19998.65 | 6096.53 | 6.7414e-04 | 1.6418e-04 |
| 3       | 512 | 317427 | 88567 | 6.07 | 2.2 | 9.4640e-04 | 9.4441e-04 |
|         | 1024 | 1122422 | 317431 | 41.02 | 15.59 | 9.5437e-04 | 9.4640e-04 |
| 4       | 2048 | 3900542 | 1122424 | 285.46 | 110.06 | 9.8624e-04 | 9.5437e-04 |
|         | 4096 | 13245607 | 3900541 | 1943.82 | 770.34 | 1.1137e-03 | 9.8624e-04 |
|         | 8192 | 43556212 | 13245601 | 16333.84 | 5210.69 | 1.6237e-03 | 1.1137e-03 |
| 5       | 512 | 260024 | 74215 | 6.09 | 1.85 | 5.3321e-04 | 5.3128e-04 |
|         | 1024 | 892767 | 260016 | 41.66 | 12.8 | 5.4094e-04 | 5.3322e-04 |
| 6       | 2048 | 2981836 | 892746 | 278.68 | 87.61 | 5.7193e-04 | 5.4095e-04 |
|         | 4096 | 9570609 | 2981791 | 1821.26 | 586.29 | 6.9675e-04 | 5.7194e-04 |
|         | 8192 | 28855877 | 9570516 | 10830.73 | 3761.16 | 1.2029e-03 | 6.9676e-04 |
| 7       | 512 | 279909 | 79187 | 5.1 | 1.98 | 3.4938e-02 | 3.4940e-02 |
|         | 1024 | 972111 | 279853 | 35.27 | 13.7 | 3.4930e-02 | 3.4938e-02 |
| 8       | 2048 | 3298823 | 971993 | 239.74 | 94.97 | 3.4899e-02 | 3.4930e-02 |
|         | 4096 | 10837780 | 3298584 | 1580.08 | 645.6 | 3.4773e-02 | 3.4899e-02 |
|         | 8192 | 33923001 | 10837297 | 10349.3 | 4242.02 | 3.4272e-02 | 3.4773e-02 |

Table 2. Percentage reduction for the 4-point EG compared to GS iterative method.

| Example | Iterations Number (%) | Execution Time (second) (%) |
|---------|-----------------------|-----------------------------|
| 1       | 70.52-72.39           | 69.52-70.74                 |
| 2       | 69.59-72.1            | 60.37-68.1                  |
| 3       | 66.83-71.46           | 65.27-69.62                  |
| 4       | 68.05-71.71           | 59.01-61.18                  |
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