Class of Bell-Clauser-Horne inequalities for testing quantum nonlocality

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Abstract

Quantum nonlocality, one of the most important features of quantum mechanics, is normally connected in experiments with the violation of Bell-Clauser-Horne (Bell-CH) inequalities. We propose effective methods for the rearrangement and linear inequality to prove a large variety of Bell-CH inequalities. We also derive a set of Bell-CH inequalities by using these methods which can be violated in some quantum entangled states.

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I. INTRODUCTION

Quantum nonlocality is one of the most striking aspects of quantum mechanics without a classical analog in reality. It can be characterized as correlated outcomes when we measure two or more entangled quantum systems, even if these systems are spatially separated. Quantum nonlocality originates from a contradiction between local realism and the completeness of quantum mechanics pointed out by Einstein, Podolsky and Rosen in 1935 [1], which was called the “EPR paradox” and led to a great challenge of the concept of “locality” taken for granted by most physicists. To test such a contradiction in a real physical system, Bell, Clauser, and other researchers formulated observable inequalities with correlation functions of measurement outcomes for entangled systems [2–4]. These inequalities were later called Bell-Clauser-Horne-Shimony-Holt (Bell-CHSH) inequalities. They proved that the quantum correlation cannot be fully described by any local hidden-variable theory. Later on, Clauser and Horne proposed the CH inequality in terms of probabilities [5], followed by many CH-like inequalities (we call them all Bell-CH inequalities in this paper). These inequalities have weaker auxiliary assumptions than Bell-CHSH inequalities in experimental considerations.

Violation of Bell-CH inequalities has been widely verified by many experiments in favor of the nonlocal feature of quantum mechanics [6–9]. However, observing the violation of these inequalities is not feasible in every entangled system [10]. One needs to generalize these inequalities in order to test them in experiments. The first attempt to find a systematic way of generalization was made by Froissart [11], and later attempt was made by Pitowsky [12–14], who introduced correlation polytopes to find Bell-CH inequalities inspired by Boole’s method on probabilistic inequalities [15]. After a series of further works, such algorithms were built to find a set of Bell-CH inequalities based on the solution of convex problems [11, 16–21]. A large variety of Bell-CH inequalities have been found [11, 16, 19, 20, 22–29]. In Refs. [27–29], the authors obtained a complete list of Bell-CH inequalities in the case of binary settings up to four measurements on each setting. The main goal of this paper is to prove these inequalities with analytical techniques and construct a class of Bell-CH inequalities. Our inequalities can be used as a substitute for the original ones for testing quantum nonlocality under some circumstances.

We consider a system consisting of two classical or quantum subsystems $x$ and $y$. Alice
chooses \( m \) measurement settings for a quantity that has \( k \) values on \( x \), and Bob chooses \( n \) measurement settings for a quantity that has \( l \) values on \( y \). We denote this case as \( mnkl \).

The system can be measured many times with different selected settings. We define \( P_{x_i} \) as the probability that Alice obtains a certain value \( a \) for measurement setting \( i \) on \( x \), \( P_{y_j} \) as the probability that Bob obtains a certain value \( b \) for measurement setting \( j \) on \( y \), and \( P_{x_iy_j} \) as the probability that Alice obtains \( a \) for \( i \) and Bob obtains \( b \) for \( j \) at the same time. For a classical system labeled \( mn22 \), we have the so-called Bell-CH inequalities of binary settings which hold for certain integer coefficients \( C_{x_i}, C_{y_j}, \) and \( C_{x_iy_j} \),

\[
\sum_{i=1}^{m} C_{x_i} P_{x_i} + \sum_{j=1}^{n} C_{y_j} P_{y_j} + \sum_{i=1}^{m} \sum_{j=1}^{n} C_{x_iy_j} P_{x_iy_j} \leq 0.
\]

In our formulation, we define the corresponding Bell-CH-like inequalities in algebraic forms (we call them algebraic Bell-CH inequalities for short) as

\[
B \sum_{i=1}^{m} C_{x_i} x_i + A \sum_{j=1}^{n} C_{y_j} y_j + \sum_{i=1}^{m} \sum_{j=1}^{n} C_{x_iy_j} x_i y_j \leq 0,
\]

with \( A \) and \( B \) being the upper bounds of two sets of positive variables \( x_i \) and \( y_i \), respectively. The Bell-CH inequalities can be derived from the algebraic ones in local hidden-variable theory. Among these inequalities, there are independent ones which we label \( I_{mn22} \) for Bell-CH inequalities and \( I_{mn22} \) for algebraic ones.

This paper is organized as follows. In this section, we have provided an introduction and the motivation of this work with definitions of Bell-CH inequalities and their algebraic forms. In Sec. II, we derive algebraic Bell-CH inequalities with two methods; one is from the rearrangement inequality, and the other is from the linear inequality. In Sec. III, we derive Bell-CH inequalities and obtain a different class of inequalities. In Sec. IV, we test our inequalities with special quantum entangled states. In Sec. V, we summarize this work and draw conclusions.

II. ALGEBRAIC BELL-CH INEQUALITIES FOR BINARY SETTINGS

A. Method of rearrangement inequality

The rearrangement inequality is a well-known inequality in classical mathematics, from which one can prove many famous inequalities, such as the arithmetic-mean–geometric-mean
inequality, Cauchy inequality, etc. In Appendix A we give a very brief introduction of the rearrangement inequality; readers can see Ref. [30] for a review of this topic.

In this section, we derive two low-order algebraic Bell-CH inequalities by applying the rearrangement inequality: algebraic Bell-CH inequalities $I_{2222}$ and $I_{3322}$. In order to prove these inequalities, we use the maximum and minimum values of variables to enlarge the upper bound of polynomials in intermediate steps. Then we can rearrange the orders of variables in these polynomials and apply the rearrangement inequality to determine their signs. In this way, we find a profound connection between these inequalities and the classical rearrangement inequality.

1. $I_{2222}$

We define two sets of $m$ and $n$ positive real numbers

$$a = \{x_i, i = 1, \cdots , m \},$$
$$b = \{y_j, j = 1, \cdots , n \},$$

with upper bounds $A$ and $B$, respectively,

$$0 \leqslant x_i \leqslant A, \ 0 \leqslant y_j \leqslant B.$$  \hspace{1cm} (4)

The maximum and minimum values of the numbers in set $a$ are denoted $x_+(1, \ldots , m)$ and $x_-(1, \ldots , m)$, respectively, and the maximum and minimum values of the numbers in set $b$ are denoted $y_+(1, \ldots , n)$ and $y_-(1, \ldots , n)$, respectively.

We define $I_2$ as a polynomial of the following form:

$$I_2 = x_1y_1 + x_1y_2 + x_2y_1 - x_2y_2 - x_1B - Ay_1,$$

which involves elements of set $a$ and $b$ in (3) with $m = n = 2$. We now prove the $I_{2222}$ inequality, $I_2 \leq 0$. Using boundary conditions

$$0 \leqslant x_-(1, 2) \leq x_1, x_2 \leq x_+(1, 2) \leq A,$$
$$0 \leq y_-(1, 2) \leq y_1, y_2 \leq y_+(1, 2) \leq B$$  \hspace{1cm} (6)
in (4), we obtain
\[
I_2 \leq x_1(y_1 + y_2) + x_2(y_1 - y_2) - x_1 y_+(1, 2) - x_+(1, 2) y_1 \\
\leq x_1(y_1 + y_2) + x_2(y_1 - y_2) - x_1 y_+(1, 2) - x_+(1, 2) y_1 \\
- [x_1 - x_+(1, 2)] y_-(1, 2) - x_-(1, 2) [y_1 - y_+(1, 2)],
\]
where we have replaced \( A \) and \( B \) with \( x_+(1, 2) \) and \( y_+(1, 2) \) in Eq. (5), respectively, to obtain the first line and added two positive quantities in the second inequality.

Since there are only two variables in each set, we always have
\[
x_+(1, 2) + x_-(1, 2) = x_1 + x_2, \\
y_+(1, 2) + y_-(1, 2) = y_1 + y_2.
\]
Using the above identity, inequality (7) becomes
\[
I_2 \leq x_1(y_1 + y_2) + x_2(y_1 - y_2) - x_1 y_+(1, 2) - x_+(1, 2) y_1 \\
+ x_+(1, 2)y_-(1, 2) + x_-(1, 2)y_+(1, 2) \\
= -(x_1 y_1 + x_2 y_2) + x_+(1, 2)y_-(1, 2) + x_-(1, 2)y_+(1, 2) \\
\equiv I_2^{(0)}.
\]
We see in (9) that \( I_2^{(0)} \) is the difference between the reversed sum and the unordered sum for sets \( a \) and \( b \) in (3) with \( m = n = 2 \). Using the rearrangement inequality, we obtain
\[
I_2 \leq I_2^{(0)} \leq 0.
\]
This concludes the proof of the \( I_{2222} \) inequality by applying the rearrangement inequality.

We can rewrite \( I_2^{(0)} \) with the help of the Heaviside step function,
\[
\theta(x) = \begin{cases} 
1 & x > 0, \\
\frac{1}{2} & x = 0, \\
0 & x < 0.
\end{cases}
\]
The maximum and minimum values can be put into the following form:
\[
x_+(1, 2) = x_1 \theta(x_1 - x_2) + x_2 \theta(-x_1 + x_2), \\
x_-(1, 2) = x_1 \theta(-x_1 + x_2) + x_2 \theta(x_1 - x_2), \\
y_+(1, 2) = y_1 \theta(y_1 - y_2) + y_2 \theta(-y_1 + y_2), \\
y_-(1, 2) = y_1 \theta(-y_1 + y_2) + y_2 \theta(y_1 - y_2).
\]

Inserting the above formula into $I^{(0)}_2$ and using the equation
\[ x\theta(x) = \frac{x + |x|}{2}, \]  
we obtain
\[ I^{(0)}_2 = -(x_1 - x_2)(y_1 - y_2)\left[\theta(x_1 - x_2)\theta(y_1 - y_2) + \theta(-x_1 + x_2)\theta(-y_1 + y_2)\right] \]
\[ = -\frac{1}{2} \left[(x_1 - x_2)(y_1 - y_2) + |(x_1 - x_2)(y_1 - y_2)|\right]. \]  
This form will be used in the next section.

2. $I_{3322}$

Like $I_2$ in (5), we can define $I_3$ as
\[ I_3 = x_1y_2 + x_1y_3 + x_2y_1 - x_2y_2 + x_2y_3 + x_3y_1 + x_3y_2 - x_3y_3 \]
\[ -(x_1 + x_2)B - A(y_1 + y_2), \]  
which involves elements of sets $a$ and $b$ in (3) with $m = n = 3$. We now prove the $I_{3322}$ inequality, $I_3 \leq 0$. We use boundary conditions (14), or, explicitly,
\[ 0 \leq x_-(1, 2, 3) \leq x_1, x_2, x_3 \leq x_+(1, 2, 3) \leq A, \]
\[ 0 \leq y_-(1, 2, 3) \leq y_1, y_2, y_3 \leq y_+(1, 2, 3) \leq B, \]  
and replace $A$ and $B$ in (15) with $x_+(1, 2, 3)$ and $y_+(1, 2, 3)$, respectively, to obtain
\[ I_3 \leq x_1(y_2 + y_3) + x_2(y_1 + y_3) + x_3(y_1 + y_2) - x_2y_2 - x_3y_3 \]
\[ -(x_1 + x_2)y_+(1, 2, 3) - x_+(1, 2, 3)(y_1 + y_2). \]  
Using the inequalities
\[ -x_2y_2 \leq -x_2y_-(1, 2, 3) - x_-(1, 2, 3)y_2 + x_-(1, 2, 3)y_-(1, 2, 3), \]
\[ -x_3y_3 \leq -x_3y_+(1, 2, 3) - x_+(1, 2, 3)y_3 + x_+(1, 2, 3)y_+(1, 2, 3), \]  
we can enlarge the right-hand side of (17) to obtain
\[ I_3 \leq x_1(y_2 + y_3) + x_2(y_1 + y_3) + x_3(y_1 + y_2) \]
\[ -x_2y_-(1, 2, 3) - x_-(1, 2, 3)y_2 + x_-(1, 2, 3)y_-(1, 2, 3) \]
\[ -x_3y_+(1, 2, 3) - x_+(1, 2, 3)y_3 + x_+(1, 2, 3)y_+(1, 2, 3) \]
\[ -(x_1 + x_2)y_+(1, 2, 3) - x_+(1, 2, 3)(y_1 + y_2). \]  
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The inequalities in (18) are true because they can be put into the following form:

\[- [x_2 - x_-(1, 2, 3)] [y_2 - y_-(1, 2, 3)] \leq 0,\]
\[- [x_3 - x_+(1, 2, 3)] [y_3 - y_+(1, 2, 3)] \leq 0.\]  \hspace{1cm} (20)

After adding additional positive terms to enlarge the right-hand side of (19), we can derive

\[I_3 \leq x_1(y_2 + y_3) + x_2(y_1 + y_3) + x_3(y_1 + y_2)\]
\[\quad - (x_1 + x_2 + x_3)y_+(1, 2, 3) - x_+(1, 2, 3)(y_1 + y_2 + y_3)\]
\[\quad - [x_1 - x_+(1, 2, 3)] y_-(1, 2, 3) - x_2y_-(1, 2, 3) - [x_3 - x_+(1, 2, 3)] y_-(1, 2, 3)\]
\[\quad - x_-(1, 2, 3) [y_1 - y_+(1, 2, 3)] - x_-(1, 2, 3)y_2 - x_-(1, 2, 3) [y_3 - y_+(1, 2, 3)]\]
\[\quad + x_-(1, 2, 3)y_-(1, 2, 3) + x_+(1, 2, 3)y_+(1, 2, 3)\]
\[\quad = -(x_1y_1 + x_2y_2 + x_3y_3) + x_+(1, 2, 3)y_-(1, 2, 3) + x_-(1, 2, 3)y_+(1, 2, 3)\]
\[\quad + [x_1 + x_2 + x_3 - x_+(1, 2, 3) - x_-(1, 2, 3)]\]
\[\quad \times [y_1 + y_2 + y_3 - y_+(1, 2, 3) - y_-(1, 2, 3)].\]  \hspace{1cm} (21)

Finally, we define \(I_3^{(0)}\) as the last polynomial of (21) for later use,

\[I_3^{(0)} = -(x_1y_1 + x_2y_2 + x_3y_3) + x_+(1, 2, 3)y_-(1, 2, 3)\]
\[\quad + x_-(1, 2, 3)y_+(1, 2, 3) + x_ry_r,\]  \hspace{1cm} (22)

where \(x_r\) and \(y_r\) denote the middle terms in sets \(a\) and \(b\) as

\[x_r = x_1 + x_2 + x_3 - x_+(1, 2, 3) - x_-(1, 2, 3),\]
\[y_r = y_1 + y_2 + y_3 - y_+(1, 2, 3) - y_-(1, 2, 3).\]  \hspace{1cm} (23)

It is obvious that \(I_3^{(0)}\) in Eq. (22) is the difference between the reversed sum and the unordered sum for sets \(a\) and \(b\) in (3) with \(m = n = 3\). From the rearrangement inequality, we have

\[I_3 \leq I_3^{(0)} \leq 0.\]  \hspace{1cm} (24)

This concludes the proof of the \(I_{3322}\) inequality using the method of the rearrangement inequality.
B. Method of linear inequality

In Sec. [II A], we derived two algebraic inequalities in connection with low-order Bell-CH and rearrangement inequalities. But it is not easy to generalize this method to higher-order cases. In Ref. [16], the authors obtained a specific type of Bell-CH inequalities and found the relationship between lower-order and higher-order inequalities of this type. Following Ref. [16], there has been a lot of discussion on the perspective of mathematics along this line [17, 18, 31, 32]. Inspired by these works, we find a general method which we call the method of linear inequality to prove higher-order inequalities \( I_{mn22} \) from the lowest-order one systematically. As a by-product, this method can be used to construct higher-order inequalities more effectively than the traditional method. We will prove a lemma using the property of monotonicity. Then we will apply the lemma to linear functions and obtain a theorem. Finally, we will use this method to prove a large variety of inequalities that have been obtained.

We consider two sets \( a \) and \( b \) defined by Eq. (3) with conditions (4). Starting from positivity inequalities

\[
-x_i y_j \leq 0, \\
-x_i (y_j - B) \leq 0, \\
(x_i - A)y_j \leq 0,
\]

we present a theorem for a specific type of algebraic Bell-CH inequalities.

**Theorem 1.** \( I_{kk}(x_1, \ldots, x_k|y_1, \ldots, y_k) \leq 0 \) is an algebraic Bell-CH inequality for \( k \geq 2 \) with variables \( x_i \) and \( y_j \) in sets \( a \) and \( b \), respectively, where \( I_{kk}(x_1, \ldots, x_k|y_1, \ldots, y_k) \) is defined as

\[
I_{kk}(x_1, \ldots, x_k|y_1, \ldots, y_k) = \sum_{j=1}^{k} \sum_{i=1}^{k+1-j} x_i y_j - \sum_{i=2}^{k} x_i y_{k+2-i} - \sum_{i=1}^{k-1} (k-i)x_i B - Ay_1. \tag{26}
\]

**Proof.** Let \( f(x_i, \{ c_j^y \}) = x_i (\sum_j c_j^y y_j) \) and \( g(y_i, \{ c_j^x \}) = (\sum_j c_j^x x_j) y_i \), where \( x_i \) and \( y_j \) are variables in sets \( a \) and \( b \) and \( c_j^x \) and \( c_j^y \) are coefficients. Then we have the following linear inequalities:

\[
f(x_i, \{ c_j^y \}) \leq \theta (\sum_j c_j^y y_j) f(A, \{ c_j^y \}) + \theta (-\sum_j c_j^y y_j) f(0, \{ c_j^y \}),
\]

\[
g(y_i, \{ c_j^x \}) \leq \theta (\sum_j c_j^x x_j) g(B, \{ c_j^x \}) + \theta (-\sum_j c_j^x x_j) g(0, \{ c_j^x \}). \tag{27}
\]
We can prove \( I_{kk}(x_1, \cdots, x_k|y_1, \cdots, y_k) \leq 0 \) with the help of (27) by the induction method. First, we should prove the case for \( k = 2 \), which is just the algebraic version of the original Bell-CH inequality. We have

\[
I_{22}(x_1, x_2|y_1, y_2) = (x_1 + x_2)y_1 + (x_1 - x_2)y_2 - x_1B - Ay_1
\]

\[
\leq \theta(x_1 - x_2)I_{22}(x_1, x_2|y_1, B) + \theta(-x_1 + x_2)I_{22}(x_1, x_2|y_1, 0)
\]

\[
= \theta(x_1 - x_2)[(x_1 + x_2)y_1 - x_2B - Ay_1]
\]

\[
+ \theta(-x_1 + x_2)[(x_1 + x_2)y_1 - x_1B - Ay_1],
\]

(28)

where we have used (27) for \( g(y_2, \{c_j^x\}) = (x_1 - x_2)y_2 \) with \( \{c_j^x\} = \{1, -1\} \). Then we can rewrite the right-hand side of the above inequality and obtain

\[
I_{22}(x_1, x_2|y_1, y_2) \leq \theta(x_1 - x_2)[(x_1 - A)y_1 + x_2(y_1 - B)]
\]

\[
+ \theta(-x_1 + x_2)[(x_2 - A)y_1 + x_1(y_1 - B)]
\]

\[
\leq 0.
\]

(29)

Thus, the inequality holds for \( k = 2 \).

Then we assume the inequality holds for \( k - 1 \) with

\[
I_{k-1,k-1}(x_1, x_3, \cdots, x_k|y_1, \cdots, y_{k-1}) \leq 0,
\]

\[
I_{k-1,k-1}(x_2, x_3, \cdots, x_k|y_1, \cdots, y_{k-1}) \leq 0.
\]

(30)

According to the induction method, the inequality should hold for \( I_{kk}(x_1, \cdots, x_k|y_1, \cdots, y_k) \) in (28). We now apply (27) to \( I_{kk} \) for \( g(y_k, \{c_j^x\}) = (x_1 - x_2)y_k \), with \( \{c_j^x, j = 1, 2, \cdots, k\} = \{1, -1, 0, \cdots, 0\} \), as

\[
I_{kk}(x_1, \cdots, x_k|y_1, \cdots, y_k)
\]

\[
\leq \theta(x_1 - x_2)I_{kk}(x_1, \cdots, x_k|y_1, \cdots, y_{k-1}, B) + \theta(-x_1 + x_2)I_{kk}(x_1, \cdots, x_k|y_1, \cdots, y_{k-1}, 0)
\]

\[
= \theta(x_1 - x_2)[(\sum_{i=1}^{k} x_i)y_1 + (\sum_{i=1}^{k-1} x_i - x_k)y_2 + \cdots + (x_1 + x_2 - x_3)y_{k-1}]
\]

\[
+ (x_1 - x_2)B - \sum_{i=1}^{k-1} (k - i)x_i B - Ay_1]
\]

\[
+ \theta(-x_1 + x_2)[(\sum_{i=1}^{k} x_i)y_1 + (\sum_{i=1}^{k-1} x_i - x_k)y_2 + \cdots + (x_1 + x_2 - x_3)y_{k-1}]
\]

\[
- \sum_{i=1}^{k-1} (k - i)x_i B - Ay_1].
\]

(31)
We can then rewrite the right-hand side of the above inequality and obtain

\[
I_{kk}(x_1, \cdots, x_k|y_1, \cdots, y_k) \\
\leq \theta(x_1 - x_2)\{I_{k-1,k-1}(x_1, x_3, \cdots, x_k|y_1, \cdots, y_{k-1}) + \sum_{i=1}^{k-1}[x_2(y_i - B)]\} \\
+ \theta(-x_1 + x_2)\{I_{k-1,k-1}(x_2, x_3, \cdots, x_k|y_1, \cdots, y_{k-1}) + \sum_{i=1}^{k-1}[x_1(y_i - B)]\} \\
\leq I_{kk}^{(0)}(x_1, \cdots, x_k|y_1, \cdots, y_k) \\
\leq 0. \tag{32}
\]

Here, \(I_{kk}^{(0)}\) is given by

\[
I_{kk}^{(0)}(x_1, \cdots, x_k|y_1, \cdots, y_k) \\
= \max\{I_{k-1,k-1}(x_1, x_3, \cdots, x_k|y_1, \cdots, y_{k-1}) + \sum_{i=1}^{k-1}[x_2(y_i - B)], \\
I_{k-1,k-1}(x_2, x_3, \cdots, x_k|y_1, \cdots, y_{k-1}) + \sum_{i=1}^{k-1}[x_1(y_i - B)]\}. \tag{33}
\]

So we prove in (32) that the inequality holds for \(k\). This concludes the proof of the theorem.

The same method can also be applied to prove general inequalities \(I_{mn22}\) recursively. For example, we can prove \(I_3\) in (15) as the symmetric version of \(I_{33}(x_1, x_2, x_3|y_1, y_2, y_3)\): \(I_3\) can be obtained from \(I_{33}(x_1, x_2, x_3|y_1, y_2, y_3)\) by the transformation \(x_1 \rightarrow A - x_1, y_2 \rightarrow B - y_2, y_3 \rightarrow B - y_3\) and then relabeling indices of \(\{x_i\}\) and \(\{y_j\}\) by interchanging \(1 \leftrightarrow 3\). Hence, \(I_3 \leq 0\) can be proved by using lower-order inequalities with the help of (27).

As another example, we can reconstruct and prove \(I_{5322}\) by applying linear inequalities. We use the polynomial \(I_{53}(x_1, x_2, x_3, x_4, x_5|y_1, y_2, y_3)\) to represent \(I_{5322}\) as

\[
I_{53}(x_1, x_2, x_3, x_4, x_5|y_1, y_2, y_3) \\
= x_1y_1 - x_1y_2 + x_1y_3 + x_2y_2 + x_2y_3 + x_3y_1 + x_3y_2 + x_4y_1 - x_4y_3 \\
- x_5y_1 + x_5y_2 - x_5y_3 - (x_1 + x_2 + x_3)B - A(y_1 + y_2) \\
\leq 0. \tag{34}
\]

To prove inequality (34), we can apply linear inequalities (27) for \(f(x_4, \{c^y_j\}) = x_4(y_1 - y_3),\)
with \(\{c^y_j, j = 1, 2, 3\} = \{1, 0, -1\}\), and obtain
\[
I_{53}(x_1, x_2, x_3, x_4, x_5|y_1, y_2, y_3) \\
\leq \theta(y_1 - y_3)I_{53}(x_1, x_2, x_3, A, x_5|y_1, y_2, y_3) + \theta(-y_1 + y_3)I_{53}(x_1, x_2, x_3, 0, x_5|y_1, y_2, y_3) \\
= \theta(y_1 - y_3)[I_{22}(x_2, x_1|y_3, y_2) + I_{22}(x_3, x_5|y_2, y_1) + x_1(y_1 - B) - x_5y_3] \\
+ \theta(-y_1 + y_3)[I_{22}(x_2, x_5|y_2, y_3) + I_{22}(x_3, x_1|y_1, y_2) + x_1(y_3 - B) - x_5y_1] \\
\leq 0, \tag{35}
\]
where we have used inequality (29) for \(I_{22}(x_2, x_1|y_3, y_2)\), \(I_{22}(x_3, x_5|y_2, y_1)\), \(I_{22}(x_3, x_1|y_1, y_2)\), and \(I_{22}(x_3, x_5|y_2, y_3)\). It was shown in Refs. [33, 34] that there is only one \(I_{5322}\), and the explicit form was first found by [26]. Here, we give the proof of the corresponding algebraic one using the linear inequality method.

In the Supplemental Material [35], we summarize proofs of 257 algebraic Bell-CH inequalities using the linear inequality method. In these proofs, we can show that all inequalities can be reduced to second- and third-order ones.

III. BELL-CH INEQUALITIES FOR BINARY SETTINGS

According to “objective local theories” introduced by Clauser and Horne [5], any physical system, either a classical or quantum-mechanical one, can be considered as a state. The state is labeled by some local hidden variables \(\lambda\) without any other assumptions. For example, in a bipartite correlated system, \(P(x, \lambda)\) describes the probability density of some certain measurement outcome \(x\) in one subsystem, and \(P(x, y, \lambda)\) is the correlation probability density of measurement outcomes \(x\) and \(y\) in two subsystems.

From the definition of the state and local hidden variables, we can derive Bell-CH inequalities from algebraic ones. In a theory of local hidden variables, the physically detectable probability density \(P(x)\) is related to a hidden variable \(\lambda\), which is assumed to be drawn from the probability distribution \(\rho(\lambda) \in [0, 1]\)
\[
P(x) = \int P(x, \lambda)\rho(\lambda)d\lambda. \tag{36}
\]
Likewise, the joint probability density \(P(x, y)\) is obtained with
\[
P(x, y) = \int P(x, \lambda)P(y, \lambda)\rho(\lambda)d\lambda. \tag{37}
\]
Let us take the derivation of the CH inequality as an example. From the $I_{2222}$ inequality, replacing $x_i/A$ and $y_i/B$ with $P(x_i, \lambda)$ and $P(y_i, \lambda)$ for $i = 1, 2$, respectively, we obtain

$$ P(x_1, \lambda) [P(y_1, \lambda) + P(y_2, \lambda)] + P(x_2, \lambda) [P(y_1, \lambda) - P(y_2, \lambda)] - P(x_1, \lambda) - P(y_1, \lambda) \leq 0. $$

(38)

Multiplying the above inequality by $\rho(\lambda)$ and taking an integration over $\lambda$, we obtain the CH inequality

$$ I_{2,CH} \leq 0, $$

(39)

where $I_{2,CH}$ is defined as

$$ I_{2,CH} = P(x_1, y_1) + P(x_1, y_2) + P(x_2, y_1) - P(x_2, y_2) - P(x_1) - P(y_1). $$

(40)

In correspondence to (10), we obtain the inequality

$$ I_{2,CH} \leq I_{2,CH}^{(0)} \leq 0, $$

(41)

where $I_{2,CH}^{(0)}$ is defined as

$$ I_{2,CH}^{(0)} = \frac{1}{2} \int \left\{ [P(x_1, \lambda) - P(x_2, \lambda)] [P(y_1, \lambda) - P(y_2, \lambda)] 
+ |[P(x_1, \lambda) - P(x_2, \lambda)] [P(y_1, \lambda) - P(y_2, \lambda)]| \right\} \rho(\lambda) d\lambda. $$

(42)

Here, we use Eq. (14) and make the replacements $x_i/A \rightarrow P(x_i, \lambda)$ and $y_i/B \rightarrow P(y_i, \lambda)$ for $i = 1, 2$, respectively, multiplied by $\rho(\lambda)$ and integrated over $\lambda$. We apply Jensen’s inequality in Appendix B assuming the convex function $\varphi(x) = |x|$, which leads to an upper bound for $I_{2,CH}^{(0)}$ and then for $I_{2,CH}$,

$$ I_{2,CH} \leq I_{2,CH}^{(0)} \leq -\frac{1}{2} \left\{ [P(x_1, y_1) - P(x_1, y_2) - P(x_2, y_1) + P(x_2, y_2)] 
+ |P(x_1, y_1) - P(x_1, y_2) - P(x_2, y_1) + P(x_2, y_2)| \right\}, $$

(43)

where we have employed inequality (41). We can easily see that the upper bound of $I_{2,CH}^{(0)}$ is less than or equal to zero.

Another approach to our class of inequalities is through the method of linear inequalities in deriving algebraic inequalities $I_{nn22}$. We take a type of Bell-CH inequality as an example. In correspondence to inequality (32), we obtain the inequality

$$ I_{kk;Q} \leq I_{kk;Q}^{(0)} \leq 0, $$

(44)
in which we have defined

\[ I_{kk,Q} \equiv (AB)^{-1} \int d\lambda \rho(\lambda) I_{kk}[AP(x_1, \lambda), \cdots, AP(x_k, \lambda)|BP(y_1, \lambda), \cdots, BP(y_k, \lambda)], \]

\[ I_{kk,Q}^{(0)} \equiv (AB)^{-1} \int d\lambda \rho(\lambda) I_{kk}^{(0)}[AP(x_1, \lambda), \cdots, AP(x_k, \lambda)|BP(y_1, \lambda), \cdots, BP(y_k, \lambda)]. \]

(45)

Here, we have made the replacements in (32) \( x_i/A \rightarrow P(x_i, \lambda) \) and \( y_i/B \rightarrow P(y_i, \lambda) \) for \( i = 1, \ldots, k \), respectively, multiplied by \( \rho(\lambda) \) and integrated over \( \lambda \). The inequality \( I_{kk,Q}^{(0)} \leq 0 \) leads to the following alternative inequality:

\[ \max \left\{ I_{k-1,k-1;Q}^{(1)} + \sum_{i=1}^{k-1} [P(x_2, y_i) - P(x_2)], I_{k-1,k-1;Q}^{(2)} + \sum_{i=1}^{k-1} [P(x_1, y_i) - P(x_1)] \right\} \leq 0, \]

(46)

where \( I_{k-1,k-1;Q}^{(1)} \) and \( I_{k-1,k-1;Q}^{(2)} \) are defined as

\[ I_{k-1,k-1;Q}^{(1)} \equiv \sum_{j=1}^{k-1} \sum_{i=1, i \neq 2}^{k-j} P(x_i, y_j) - \sum_{i=3}^{k-1} P(x_i, y_{k+1-i}) - \sum_{i=1, i \neq 2}^{k-2} (k - 1 - i)P(x_i) - P(y_1), \]

\[ I_{k-1,k-1;Q}^{(2)} \equiv \sum_{j=1}^{k-1} \sum_{i=2}^{k-j} P(x_i, y_j) - \sum_{i=3}^{k-1} P(x_i, y_{k+1-i}) - \sum_{i=2}^{k-2} (k - 1 - i)P(x_i) - P(y_1). \]

(47)

Note that this inequality can be tested in physical systems.

Applying this method to \( Imm22 \), we can obtain a set of Bell-CH inequalities which is summarized in the Supplemental Material [35]. In the next section, we will test our inequalities with special quantum entangled states.

IV. TESTING OUR BELL-CH INEQUALITIES WITH QUANTUM ENTANGLED STATES

To test our class of Bell-CH inequalities we consider a simple quantum system with two qubits. The entangled states of two qubits we will use in the test are parameterized as

\[ |\psi(\theta)\rangle = \cos \theta |00\rangle + \sin \theta |11\rangle. \]

(48)
For such quantum states, the correlation probabilities can be obtained from expectation values of operators acting on the Hilbert space,

\[ P(x_i) = \langle \psi(\theta) | x_i \otimes I_y | \psi(\theta) \rangle, \]

\[ P(y_j) = \langle \psi(\theta) | I_x \otimes y_j | \psi(\theta) \rangle, \]

\[ P(x_i, y_j) = \langle \psi(\theta) | x_i \otimes y_j | \psi(\theta) \rangle, \]  

with \( x_i \) (\( y_j \)) being the projectors measured by Alice (Bob) on \( x \) (\( y \)), and \( I_x \) (\( I_y \)) being the unit operators acting on \( x \) (\( y \)).

For entangled states \((48)\), we define \( Q, Q_a, \) and \( Q_b \) as the maximum violations of \( Imn22 \) and the corresponding Bell-CH inequalities \( Imn22^a \) and \( Imn22^b \) presented in the Supplemental Material \([35]\), respectively. For the test, we have calculated the maximum violations of our inequalities for all \( Imn22 \) listed in \([35]\). For states with the maximum violation characterized by \( \theta_{\text{max}} \), we have also calculated the resistance to noise \( \lambda_{\text{max}} \) defined through the mixture state

\[ \rho = \lambda |\psi(\theta_{\text{max}})\rangle \langle \psi(\theta_{\text{max}})| + (1 - \lambda) I_4, \]  

so that it does not violate the inequality marginally, where \( 1 - \lambda \) is the parameter for white noise. In Table 1, we give some selected results of 32 \( Imn22 \) and our corresponding Bell-CH inequalities from the complete table in the Supplemental Material \([35]\).

| Name  | \( Q \) | \( \theta_{\text{max}}/\pi \) | \( \lambda_{\text{max}} \) | \( Q_a \) | \( \theta_{\text{max}}^a/\pi \) | \( \lambda_{\text{max}}^a \) | \( Q_b \) | \( \theta_{\text{max}}^b/\pi \) | \( \lambda_{\text{max}}^b \) |
|-------|-------|----------------|----------------|-------|----------------|----------------|-------|----------------|----------------|
| \( I_3^{22} \) | 0.25  | 0.25           | 0.8            | 0.2071| 0.25           | 0.7836         | -     | -               | -               |
| \( I_4^{422} \) | 0.056 | 0.1316         | 0.9728         | 0.2361| 0.2332         | 0.864          | 0.4142| 0.25           | 0.7071         |
| \( I_4^{1422} \) | 0.4103| 0.238          | 0.8298         | 0.4282| 0.2377         | 0.8034         | 0.3793| 0.2304         | 0.7981         |
| \( I_6^{422} \) | 0.2407| 0.219          | 0.8791         | 0.226 | 0.2362         | 0.8691         | 0.2071| 0.25           | 0.8579         |
| \( I_8^{422} \) | 0.1812| 0.168          | 0.9508         | 0.2983| 0.2195         | 0.9096         | 0.5436| 0.2278         | 0.7863         |
| Name   | Q   | $\theta_{\text{max}}/\pi$ | $\lambda_{\text{max}}$ | $Q_a$ | $\theta_{\text{max}}^a/\pi$ | $\lambda_{\text{max}}^a$ | $Q_b$ | $\theta_{\text{max}}^b/\pi$ | $\lambda_{\text{max}}^b$ |
|--------|-----|----------------|----------------|-------|----------------|----------------|-------|----------------|----------------|
| $J_3^{4422}$ | 0.7249 | 0.2448 | 0.838 | 0.7337 | 0.2419 | 0.8267 | - | - | - |
| $J_4^{4422}$ | 0.6862 | 0.2452 | 0.8138 | 0.6579 | 0.2426 | 0.7917 | - | - | - |
| $J_5^{4422}$ | 0.5007 | 0.2404 | 0.8749 | 0.4434 | 0.2303 | 0.8494 | - | - | - |
| $J_7^{4422}$ | 0.3642 | 0.2304 | 0.8917 | 0.6057 | 0.2376 | 0.7675 | - | - | - |
| $J_8^{4422}$ | 0.2657 | 0.19 | 0.9186 | 0.2772 | 0.2057 | 0.9002 | - | - | - |
| $J_{12}^{4422}$ | 0.4198 | 0.25 | 0.9147 | 0.5797 | 0.2407 | 0.8381 | - | - | - |
| $J_{13}^{4422}$ | 0.5629 | 0.2422 | 0.8766 | 0.5541 | 0.2213 | 0.8633 | 0.636 | 0.2421 | 0.8251 |
| $J_{15}^{4422}$ | 0.6133 | 0.2433 | 0.8412 | 0.5668 | 0.2416 | 0.8411 | - | - | - |
| $J_{16}^{4422}$ | 0.5441 | 0.2397 | 0.8655 | 0.5717 | 0.2255 | 0.8279 | - | - | - |
| $J_{21}^{4422}$ | 0.2459 | 0.1832 | 0.9242 | 0.4201 | 0.2398 | 0.8264 | - | - | - |
| $J_{22}^{4422}$ | 0.2133 | 0.1847 | 0.9336 | 0.3796 | 0.2432 | 0.8556 | - | - | - |
| $J_{24}^{4422}$ | 0.4075 | 0.2377 | 0.8804 | 0.3925 | 0.2318 | 0.8515 | - | - | - |
| $J_{28}^{4422}$ | 0.6012 | 0.2362 | 0.8331 | 0.6722 | 0.2462 | 0.7881 | - | - | - |
| $J_{30}^{4422}$ | 0.6678 | 0.2419 | 0.8046 | 0.675 | 0.2466 | 0.7874 | - | - | - |
| $J_{34}^{4422}$ | 0.616 | 0.2468 | 0.7851 | 0.5956 | 0.2446 | 0.7705 | - | - | - |
| $J_{39}^{4422}$ | 0.8196 | 0.2399 | 0.83 | 0.8484 | 0.2448 | 0.793 | - | - | - |
| $J_{42}^{4422}$ | 0.5972 | 0.2496 | 0.8543 | 0.5695 | 0.2426 | 0.8404 | - | - | - |
| $J_{45}^{4422}$ | 0.7399 | 0.2445 | 0.8352 | 0.8259 | 0.2404 | 0.7841 | - | - | - |
| $J_{49}^{4422}$ | 0.5 | 0.25 | 0.8333 | 0.4331 | 0.2449 | 0.822 | - | - | - |
| $A_{10}$ | 0.4154 | 0.229 | 0.8082 | 0.3944 | 0.2388 | 0.7918 | - | - | - |
| $A_{11}$ | 0.4561 | 0.2379 | 0.7933 | 0.3944 | 0.2388 | 0.7918 | - | - | - |
| $A_{13}$ | 0.4031 | 0.2375 | 0.8128 | 0.4142 | 0.25 | 0.7836 | 0.4142 | 0.25 | 0.7836 |
Table I: (Continued.)

| Name | $Q$   | $\theta_{\text{max}}/\pi$ | $\lambda_{\text{max}}$ | $Q_a$ | $\theta_{\text{max}}^a/\pi$ | $\lambda_{\text{max}}^a$ | $Q_b$ | $\theta_{\text{max}}^b/\pi$ | $\lambda_{\text{max}}^b$ |
|------|-------|-----------------------------|-------------------------|-------|-----------------------------|-------------------------|-------|-----------------------------|-------------------------|
| $A_{16}$ | 0.416 | 0.2402                      | 0.8278                  | 0.453 | 0.2447                      | 0.7751                  | 0.3944 | 0.2388                      | 0.7601                  |
| $A_{34}$ | 0.514 | 0.2461                      | 0.7956                  | 0.535 | 0.2476                      | 0.7659                  | -     | -                           | -                      |
| $A_{69}$ | 0.3304 | 0.2245                      | 0.8833                  | 0.4459 | 0.2393                      | 0.8177                  | 0.5148 | 0.25                        | 0.7727                  |
| $A_{83}$ | 0.6962 | 0.2438                      | 0.798                   | 0.62  | 0.2424                      | 0.784                   | -     | -                           | -                      |
| $A_{88}$ | 0.0768 | 0.1575                      | 0.9702                  | 0.197 | 0.2356                      | 0.9103                  | 0.25  | 0.25                        | 0.8571                  |

For all $Imn22$, we find that our Bell-CH inequalities have positive violation by entangled states [18]. According to the preceding section, our inequalities can be decomposed into groups of inequalities formed by combinations of some original low-order $Imn22$; the nonlocality implied by the violation of high-order $Imn22$ can be replaced by that of our inequalities. Moreover, the resistance to noise for some of these inequalities is lower than that of the original ones; hence, our inequalities could be better candidates for testing nonlocality in some physical systems.

V. SUMMARY AND CONCLUSIONS

In this work effective methods for the rearrangement and linear inequality were employed to prove the Bell-CH inequalities. Alternative types of Bell-CH inequalities were found to be violated by entangled states. The main results are summarized as follows. First, a large variety of $Imn22$ ($m, n \leq 5$) can be easily derived through the rearrangement inequality and the linear inequality. Second, along with $I2222$ or the CH inequality, we gave an inequality in [13] using the rearrangement inequality method. Third, all original $Imn22$ can be replaced by the maximum of lower-order $Imn22$ combinations using the linear inequality method, which can be violated by some entangled states in $Q_a$ and $Q_b$.

This work can help us understand the mathematical structures of Bell-CH inequalities. A number of interesting topics are open for future studies. One could investigate a set of Bell-CH inequalities for multipartite systems using the method of the linear inequality. Appropriate ways might be found to test our Bell-CH inequalities, especially inequality [13],
in systems of optics, high-energy physics, or condensed matter.

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[19] T. Ito, H. Imai, and D. Avis, *Phys. Rev. A* **73**, 042109 (2006).
The rearrangement inequality (or the permutation inequality) is

\[ x_n y_1 + \cdots + x_1 y_n \leq x_{\sigma(1)} y_1 + \cdots + x_{\sigma(n)} y_n \leq x_1 y_1 + \cdots + x_n y_n \]  

for every permutation of \( \sigma(i) \) \( (i = 1, \ldots, n) \), with \( n \) real numbers \( x_1, \ldots, x_n \) which satisfy

\[ x_1 \leq \cdots \leq x_n \]  

and \( n \) real numbers \( y_1, \ldots, y_n \) which satisfy

\[ y_1 \leq \cdots \leq y_n. \]
If the numbers are different, that is to say, \( x_1 < \cdots < x_n \) and \( y_1 < \cdots < y_n \), then the lower bound is obtained only for the permutation which reverses the order, i.e., \( \sigma(i) = n - i + 1 \) for all \( i = 1, \ldots, n \), and the upper bound is obtained only for the identity permutation, i.e., \( \sigma(i) = i \) for all \( i = 1, \ldots, n \).

**Appendix B: Jensen’s inequality**

Suppose \( f(x) \) is a non-negative measurable function satisfying

\[
\int_{-\infty}^{\infty} f(x)dx = 1, \tag{B1}
\]

which is a probability density function in the probabilistic view. Jensen’s inequality about convex integrals is

\[
\varphi \left( \int_{-\infty}^{\infty} g(x)f(x)dx \right) \leq \int_{-\infty}^{\infty} \varphi(g(x))f(x)dx \tag{B2}
\]

for any real-valued measurable function \( g(x) \) and \( \varphi \) is convex over \( g(x) \). If \( g(x) = x \), then Jensen’s inequality reduces to

\[
\varphi \left( \int_{-\infty}^{\infty} xf(x)dx \right) \leq \int_{-\infty}^{\infty} \varphi(x)f(x)dx. \tag{B3}
\]