WEIGHTED WEAK TYPE ESTIMATES FOR SQUARE FUNCTIONS

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Abstract. For \(1 < p < \infty\) and weight \(w \in A_p\), the following weak-type inequality holds for a Littlewood-Paley square function \(S\),

\[
\|Sf\|_{L^{p,\infty}(w)} \lesssim [w]_{A_p}^{\max\{\frac{1}{2}, \frac{1}{p}\}} \phi([w]_{A_p}) \|f\|_{L^p(w)}.
\]

where \(\phi_p(x) = 1\) for \(1 < p < 2\) and \(\phi_p(x) = 1 + \log x\) for \(2 \leq p\). Up to the logarithmic term, these estimates are sharp.

1. Introduction

Our focus is on weak-type estimates for square functions on weighted \(L^p\) spaces, for Muckenhoupt \(A_p\) weights. Following M. Wilson [16] define the intrinsic square function \(G_\alpha\) as follows.

1.1. Definition. Let \(C_\alpha\) be the collection of functions \(\gamma\) supported in the unit ball with mean zero and such that \(|\gamma(x) - \gamma(y)| \leq |x - y|^{\alpha}\). For \(f \in L^1_{\text{loc}}(\mathbb{R}^n)\) let

\[
A_\alpha f(x, t) = \sup_{\gamma \in C_\alpha} |f \ast \gamma_t(x)|
\]

where \(\gamma_t(x) = t^{-n} \gamma(xt^{-n})\) and take

\[
G_\alpha f(x) = \left( \int_{\Gamma(x)} A_\alpha f(y, t) \frac{dy dt}{t^{n+1}} \right)^{\frac{1}{2}}
\]

where \(\Gamma(x) := \{(y, t) \in \mathbb{R}^{n+1} : |y| < t\}\) is the cone of aperture one in the upper-half plane.

This square function dominates many other square functions. Recall the definition of \(A_p\) weights.

1.2. Definition. Let \(1 < p < \infty\). A weight \(w\) is in \(A_p\) if \(w\) has density \(w(x)\), we have \(w(x) > 0\) a.e., and for \(\sigma(x) := w(x)^{1 - \frac{1}{p}}\) there holds

\[
[w]_{A_p} := \sup_Q w(Q) \left[ \frac{\sigma(Q)}{|Q|} \right]^{\frac{1}{p-1}} < \infty
\]

where the supremum is formed over all cubes \(Q \subset \mathbb{R}^n\).

The main result of this note is as follows.

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1.3. **Theorem.** For $1 < p < 3$, $0 < \alpha \leq 1$, and $w \in A_p$ the following inequality holds.

\[
\|G_\alpha f\|_{L^p(w) \to L^p,\infty(w)} \lesssim [w]_{A_p}^{\max\{\frac{1}{2}, \frac{1}{p}\}} \phi([w]_{A_p}) \|f\|_{L^p(w)},
\]

where $\phi([w]_{A_p}) := \begin{cases} 1 & 1 < p < 2 \\ (1 + \log [w]_{A_p}) & 2 \leq p < 3 \end{cases}$

By example, we will show that the power on $[w]_{A_p}$, but not the logarithmic term, is sharp. This result can be contrasted with these known results. First, for the maximal function $M$, one has the familiar estimate of Buckley [1],

\[
\|M\|_{L^p(w) \to L^p,\infty(w)} \lesssim [w]_{A_p}^{1/p}, \quad 1 < p < \infty.
\]

Thus, the square function estimate equals that for $M$ for $1 < p < 2$, but is otherwise larger. There is also the recent sharp estimate of the strong type norm of $G_\alpha$:

\[
\|G_\alpha\|_{L^p(w) \to L^p(w)} \lesssim [w]_{A_p}^{\max\{\frac{1}{2}, \frac{1}{p-1}\}}.
\]

The weak-type estimate above is smaller for all values of $1 < p < 3$, and is otherwise larger by the logarithmic term. The case of $p = 2$ in the dyadic strong-type inequality was proved by Wittwer [18], also see [4]. The dyadic case, for general $p$, was proved by Cruz-Uribe-Martell-Perez [3], while the inequality as above is the main result of Lerner’s paper [9].

The case $p = 1$ of Theorem 1.3 holds more generally. Chanillo-Wheeden [2], first for the area function, and Wilson [16, 17], showed that for any weight $w$,

\[
w[G_\alpha f > \lambda] \lesssim \frac{1}{\lambda} \int_{\mathbb{R}^n} |f| \cdot Mw \, dx
\]

where $M$ is the Hardy-Littlewood maximal function. In particular, (1.4) holds for $p = 1$.

There are interesting points of comparison with the weak-type estimates for Calderón–Zygmund operators. Hytönen [5] established the strong type estimate. For $T$ an $L^2(\mathbb{R}^n)$ bounded Calderon-Zygmund operator, there holds

\[
\|T\|_{L^p(w) \to L^p(w)} \lesssim [w]_{A_p}^{\max\{\frac{1}{2}, \frac{1}{p}\}} \quad 1 < p < \infty.
\]

Hytönen et. al. [7] the weak-type estimate

\[
\|T\|_{L^p(w) \to L^p,\infty(w)} \lesssim [w]_{A_p}, \quad 1 < p < \infty.
\]

But, the $L^1$-endpoint variant of (1.5) fails, as was shown by Reguera [12], for the dyadic case and Reguera-Thiele [13] for the continuous case. Specializing to the case where $w \in A_1$, Lerner-Ombrosi-Perez [10] have shown that

\[
\|T\|_{L^1(w) \to L^{1,\infty}(w)} \lesssim [w]_{A_1} (1 + \log [w]_{A_1}).
\]

And, in a very interesting twist, some power of the logarithm is necessary, by the argument of Nazarov et al. [11]. It seems entirely plausible to us that in the case of $p = 2$ in (1.4), that some power of the logarithm is required.
Acknowledgment. The question of looking at the weak-type inequalities was suggested to us by A. Volberg [15], and we thank S. Petermichl for conversations about the problem.

2. Proof of Theorem 1.3

Our argument will apply the Lerner median inequality [8]—a widely used technique, see [3, 6, 9], among other papers. To this end, we need some definitions. For a constant \( \rho > 0 \), let us set \( \rho Q \) to be the cube with the same center as \( Q \), and side length \( |\rho Q|^{1/n} = |Q|^{1/n} \). For any cube \( Q \), set \( |Q|(f)_Q := \int_Q f \, dx \). We say that a collection of dyadic cubes \( S \) is sparse if there holds

\[
\left| \bigcup_{Q' \in S} : Q' \subseteq Q \right| < \frac{1}{2}|Q|, \quad Q \in S.
\]

For a sparse collection of cubes \( S \) and \( \rho > 1 \) we define

\[
(T_{S,\rho} f)^2 := \sum_{Q \in S} (f^2)_{\rho Q} 1_Q.
\]

Fix \( f \) supported on a dyadic cube \( Q_0 \). By application to Lerner’s median inequality (compare to [9, (5.8)]), for \( N \to \infty \), there are constants \( m_N \to 0 \) so that there is a sparse collection of cubes \( S_N \) contained in \( NQ_0 \) so that, for \( \rho = 45 \), the following pointwise estimate holds.

\[
|G_{\alpha}f(x)|^2 - m_N \cdot 1_{NQ_0}(x) \leq Mf(x)^2 + T_{S_N}f(x)^2.
\]

Therefore, in order to estimate the \( L^{p,\infty}(w) \) norm of \( G_{\alpha}f \), it suffices to estimate \( L^p(w) \) norm of \( Mf \) and of \( T_S f \), for any sparse collection of cubes \( S \).

Now, by Buckley’s bounds [1], \( \|M\|_{L^p \to L^{p,\infty}(w)} \leq [w]_{A_p}^{1/p} \). As a result, to obtain Theorem 1.3 it suffices to show this Theorem.

2.1. Theorem. For \( 1 < p < 3 \), weight \( w \in A_p \), any sparse collection of cubes \( S \) and any \( \rho \geq 1 \), there holds

\[
\|T_S\|_{L^p(w) \to L^{p,\infty}(w)} \leq [w]_{A_p} \phi([w]_{A_p}).
\]

We turn to the proof of this estimate. With \( \rho > 1 \) fixed, it is clear that it suffices to consider collections \( S \) which satisfy this strengthening of the definition of sparseness: On the one hand,

\[
\left| \bigcup_{Q' \in S} : Q' \subseteq Q \right| < \frac{|Q|}{8\rho^2}, \quad Q \in S.
\]

and on the other, if \( Q \neq Q' \in S \) and \( |Q| = |Q'| \), then \( \rho Q \cap \rho Q' = \emptyset \). We can assume these conditions, as a sparse collection is the union of \( O(\rho^{n+1}) \) subcollections which meet these conditions, and we are not concerned with the effectiveness of our estimates in \( \rho \).

Let \( S^1 \) consist of all \( Q \) such that \( (f)_{\rho Q} > 1 \). Then if \( Q \in S^1 \) we have \( Q \subseteq \{Mf > 1\} \) so that

\[
w\left\{ \sum_{Q \in S^1} (f^2)_{\rho Q} 1_Q > 1 \right\} \leq w\left\{ \bigcup_{Q \in S^1} Q \right\} \leq w\{Mf > 1\} \leq [w]_{A_p} \|f\|_{L^p(w)}^p.
\]

We split the remaining cubes into disjoint collections setting

\[
S_\ell := \{Q \in S : 2^{-\ell-1} < (f)_{\rho Q} \leq 2^{-\ell}\}, \quad \ell = 0, 1, \ldots,
\]
Now let $E(Q) = \rho Q \setminus R(Q)$ where and $R(Q) = \bigcup\{\rho Q' : Q' \subseteq \rho Q, \ Q' \in S_t\}$. Notice, that $|R(Q)| < \frac{1}{2}|\rho Q|$, whence

$$
(f_1^{E(Q)})_{\rho Q} = (f)_{\rho Q} - (f_1^{R(Q)})_{\rho Q}
$$

$$
\geq (f)_{\rho Q} - 8^{-1}(f)_{R(Q)}
$$

$$
\geq 2^{-\ell - 1} - 8^{-1}2^{-\ell} \geq 2^{-\ell}.
$$

That is, we have good lower bound on these averages, and moreover the sets $E(Q)$ are pairwise disjoint in $Q \in S_t$. We will estimate

$$
(2.2) \sum_{Q \in S_t} 2^{-2\ell}1_Q \leq \sum_{Q \in S_t} (f_1^{E(Q)})_{\rho Q}^2 1_Q.
$$

The following lemma is elementary.

2.3. Lemma. Let $T$ be a collection of cubes. We have, for $1 < p < \infty$, and sequences $\{g_Q : Q \in T\}$ of non-negative functions,

$$
\left\| \left[ \sum_{Q \in T} (g_Q)^p 1_Q \right]^{1/p} \right\|_{L^p(w)} \leq [w]_{A_p}^{1/p} \left\| \left[ \sum_{Q \in T} g_Q^p \right]^{1/p} \right\|_{L^p(w)}
$$

Proof. This is a well-known estimate on the $A_p$ norm of simple averaging operators. Writing $\sigma(x) = w(x)^{1-p}$, we will exchange out an average over Lebesgue measure for an average over $\sigma$-measure. Thus, set

$$
\langle \psi \rangle_\sigma := \sigma(Q)^{-1}\int_Q \psi \ d\sigma.
$$

We can estimate as follows.

$$
\int_{\mathbb{R}^n} (g)^p 1 \ dw = \left( (g\sigma^{-1})_{\rho Q} \right)^p \left( \frac{\sigma(\rho Q)}{\rho Q} \right)^p w(\rho Q)
$$

$$
\leq [w]_{A_p} \sigma(Q) \left( (g\sigma^{-1})_{\rho Q} \right)^p
$$

$$
= [w]_{A_p} \int_{\mathbb{R}^n} g^p \sigma^{-p} \ d\sigma = [w]_{A_p} \int_{\mathbb{R}^n} g^p \ dw.
$$

And the Lemma is a trivial extension of this inequality. \hfill \Box

2.1. The Case of $1 < p < 2$. We let $k_\epsilon \simeq \epsilon^{-1}$ be a constant such that

$$
w\left\{ \sum_{Q \in S_t} (f)_{\rho Q}^2 1_Q > k_\epsilon \right\} = w\left\{ \sum_{\ell=0}^{\infty} \sum_{Q \in S_t} 2^{-2\ell}1_Q > \sum_{\ell=0}^{\infty} 2^{-\ell} \right\} \leq \sum_{\ell=0}^{\infty} w\left\{ \sum_{Q \in S_t} 2^{-2\ell}1_Q > 2^{-\ell} \right\}.
$$

For fixed $\ell$ we may estimate, using (2.2), and exchanging out a square for a $p$th power,

$$
w\left\{ \sum_{Q \in S_t} 2^{-2\ell}1_Q > 2^{-\ell} \right\} \leq w\left\{ \sum_{Q \in S_t} (f_1^{E(Q)})_{\rho Q}^p 1_Q \geq 2^{(2-p)\ell} \right\}
$$

$$
\leq [w]_{A_p} 2^{-[(2-p)\ell]/2} \|f\|_{L^p(w)}^p.
$$
where in the last inequality we have used Lemma 2.3. Choosing \( \epsilon = 1 - p/2 \) and summing over \( \ell \) gives the result.

2.2. The case of \( p = 2 \). Of course the estimate is a bit crude. For a large constant \( C \), take \( \ell_0 \) to be the integer part of \( C(1 + \log[w]_{A_2}) \). Estimate

\[
\mathrm{w}\left\{ \sum_{Q \in S_\ell} \langle f \rangle^2_{\rho_Q} 1_Q > 2 \right\} \leq \mathrm{w}\left\{ \sum_{\ell = 0}^{\ell_0 - 1} \sum_{Q \in S_\ell} \langle f \rangle^2_{\rho_Q} 1_Q > 1 \right\} + \mathrm{w}\left\{ \sum_{\ell = \ell_0}^{\infty} \sum_{Q \in S_\ell} \langle f \rangle^2_{\rho_Q} 1_Q > 2^{-\ell/8} \right\}
\]

\[
\leq \sum_{\ell = 0}^{\ell_0 - 1} \mathrm{w}\left\{ \sum_{Q \in S_\ell} \langle f \rangle^2_{\rho_Q} 1_Q > \frac{1}{\ell_0} \right\} + \sum_{\ell = \ell_0}^{\infty} \mathrm{w}\left\{ \sum_{Q \in S_\ell} \langle f \rangle^2_{\rho_Q} 1_Q > 2^{-\ell/8} \right\}
\]

Recall the the \( A_\infty \) property for \( A_2 \) weights: For any cube \( Q \) and \( E \subset Q \), with \( |E| < \frac{1}{2} |Q| \), there holds

\[
\mathrm{w}(E) \leq \left(1 - \frac{c}{|w|_{A_2}}\right)\mathrm{w}(Q),
\]

for absolute choice of constant \( c \). Applying this in an inductive fashion, we see that

\[
\mathrm{w}\left\{ \sum_{Q \in S_\ell} \langle f \rangle^2_{\rho_Q} 1_Q > 2^{-\ell/8} \right\} \leq \mathrm{w}\left\{ \sum_{Q \in S_\ell} 1_Q \gtrsim 2^{15\ell/8} \right\}
\]

\[
\lesssim \exp((-c2^{15\ell/8})/[w]_{A_2})\mathrm{w}([Q : Q \in S_\ell])
\]

\[
\lesssim [w]_{A_2} 2^\ell \exp(-c2^{15\ell/8}) \|f\|^2_{L^2(w)}.
\]

where \( 0 < c < 1 \) is a fixed constant. This is summable in \( \ell \geq \ell_0 \) to at most a constant, for \( C \) sufficiently large.

For the case of \( 0 \leq \ell < \ell_0 \), we use the estimate of Lemma 2.3 to obtain

\[
\sum_{\ell = 0}^{\ell_0 - 1} \mathrm{w}\left\{ \sum_{Q \in S_\ell} \langle f \rangle^2_{\rho_Q} 1_Q > \frac{1}{\ell_0} \right\} \leq \ell_0^2 [w]_{A_2} \|f\|^2_{L^2(w)} = [w]_{A_2} (1 + \log[w]_{A_2})^2 \|f\|^2_{L^2(w)}
\]

concluding the proof of this case.

2.3. The case of \( 2 < p \). The case of \( p = 2 \) is the critical case, and so the case of \( p \) larger than 2 follows from extrapolation. However, here we are extrapolating weak-type estimates. It is known that this is possible, with estimates on constants. We outline the familiar argument as found in [10].

We have

\[
\mathrm{w}\left\{ \left[ \sum_{Q \in S} \langle f \rangle^2_{\rho_Q} 1_Q \right]^{1/2} > 1 \right\} \lesssim \left( \mathrm{w}\left\{ \left[ \sum_{Q \in S} \langle f \rangle^2_{\rho_Q} 1_Q \right]^{1/2} > 1 \right\} \right)^{p/2}
\]

\[
= \left( \mathrm{hw}\left\{ \left[ \sum_{Q \in S} \langle f \rangle^2_{\rho_Q} 1_Q \right]^{1/2} > 1 \right\} \right)^{1/2}
\]
for \( h \in L^q(w) \) with norm 1, where \( q = \frac{p}{2} \). Now by the Rubio de Francia algorithm there is a function \( H \) such that

i. \( h \leq H \)

ii. \( \|H\|_{L^q(w)} \leq \|h\|_{L^q(w)} \)

iii. \( Hw \in A_1 \)

iv. \( \|Hw\|_{A_1} \leq [w]_{A_p} \).

We can continue,

\[
(hw\{\left(\sum_{Q \in S} \langle f \rangle_Q^2 1_Q \right)^{1/2} > 1\})^{1/2} \lesssim (Hw\{\left(\sum_{Q \in S} \langle f \rangle_Q^2 1_Q \right)^{1/2} > 1\})^{1/2} \\
\lesssim \left([Hw]_{A_2} \left(1 + \log [Hw]_{A_2}\right) \right)^{1/2} \int_R f^2 Hw \\
\lesssim [Hw]_{A_2}^{1/2} \left(1 + \log [Hw]_{A_2}\right) \|f\|_{L^p(w)} \|H\|_{L^q(w)} \\
\lesssim [w]_{A_p}^{1/2} \left(1 + \log [w]_{A_p}\right) \|f\|_{L^p(w)}.
\]

2.4. Remark. The estimate in Theorem 2.1 is a weighted estimate for a vector-valued dyadic positive operator. One of us [14] has characterized such inequalities in terms of testing conditions. Using this condition, we could not succeed in eliminating the logarithmic estimate in the case of \( p = 2 \). It did however suggest one of the examples in the next section.

3. Examples

The usual example of a power weight in one dimension \( w(x) = |x|^{-\epsilon} \), with \( 0 < \epsilon < 1 \) has \([w]_{A_p} \simeq \epsilon^{-1} \). It is straightforward to see that for \( \sigma(x) = w(x)^{1-p'} \), and appropriate constant \( c = c(p) \), we have

\[ c^p w(S1_{[0,1]} > c) = c^p w([0,1]) \simeq \epsilon^{-1} \]

whereas \( \|1_{[0,1]}\|_{L^p(\sigma)} \simeq 1 \). Hence, the smallest power on \([w]_{A_p}\) that we can have is \( \frac{1}{p} \).

There is a finer example, expressed through the dual inequality, which shows that the power on \([w]_{A_p}\) can never be less than \( \frac{1}{p} \). Consider the Haar square function inequality \( \|S(\sigma f)\|_{L^{p,\infty}(w)} \lesssim \|f\|_{L^{p}(\sigma)} \), where \( \sigma(x) = w(x)^{1-p'} \) is the dual measure. Viewing this as a map from \( L^p(\sigma) \) to \( L^{p,\infty}(w; \ell^2) \), the dual map takes \( L^{p',1}(w; \ell^2) \) to \( L^{p',1}(\sigma) \), and the inequality is

\[
(3.1) \quad \left\| \sum_Q |Q|^{1/2} \langle a_Q \cdot w \rangle_Q h_Q \right\|_{L^{p',1}(\sigma)} \lesssim \left\| \left[ \sum_Q a_Q^2 \right]^{1/2} \right\|_{L^{p',1}(w)}.
\]

In the inequality, \( \{a_Q\} \) are a sequence of measurable functions. We show that the implied constant is at least \( C_p [w]_{A_p}^\beta \), for any \( 0 < \beta < \frac{1}{2} \).
In the inequality \((3.1)\), the right hand side is independent of the signs of the functions \(a_Q\). Hence it, with the standard Khintchine estimate, implies the inequality

\[
\left\| \sum_Q \langle a_Q \cdot w \rangle_Q^2 1_Q \right\|_{L^{p'}(\sigma)}^{1/2} \leq \left\| \sum_Q a_Q^2 \right\|_{L^{p',1}(w)}^{1/2}
\]

As the left hand side is purely positive, we prefer this form. Indeed, we specialize the inequality above to one which is of testing form. Take the functions \(\{a_k\}\) to be

\[
a_k(x) := c \sum_{j=k+1}^{\infty} \frac{1}{(j-k)^{\alpha}} 1_{[2^{-j},2^{-j+1})}, \quad \frac{1}{2} < \alpha < 1, \ k \in \mathbb{N}.
\]

For appropriate choice of constant \(c = c_\alpha\), there holds \(\sum_{k=1}^{\infty} a_k(x)^2 \leq 1_{[0,1]}\), whence we have

\[
\left\| \sum_{k=1}^{\infty} a_k(x)^2 \right\|_{L^{p',1}(w)}^{1/2} \leq w([0,1])^{1/p'}.
\]

The next and critical point, concerns the terms \(\langle a_k \cdot w \rangle_{[0,2^{-k})}\). Recalling \(w(x) = |x|^{\epsilon-1}\), we have

\[
\langle a_k \cdot w \rangle_{[0,2^{-k})} \simeq 2^k \sum_{j=k+1}^{\infty} \frac{1}{(j-k)^{\alpha}} 2^{-ej} = 2^{k(1-\epsilon)} \sum_{j=1}^{\infty} \frac{1}{j^\alpha} 2^{-ej} \simeq 2^{k(1-\epsilon)} \int_1^{\infty} \frac{1}{x^\alpha} 2^{-ex} dx \simeq e^{-1+\alpha} 2^{k(1-\epsilon)}.
\]

From this we conclude that the testing term is

\[
\int_{[0,1]} \left[ \sum_{k=1}^{\infty} \langle a_k \cdot w \rangle_{[0,2^{-k})}^2 1_{[0,2^{-k})} \right]^{p'/2} \ d\sigma \simeq e^{(-1+\alpha)p'} \int_{[0,1]} w(x)^{p'} \ d\sigma(x) \simeq e^{(-1+\alpha)p'} \int_{[0,1]} w(x)^{p'} w(x)^{1-p'} dx \simeq [w]_{A_p}^{(1-\alpha)p'} w([0,1]).
\]

Therefore, the power on \([w]_{A_p}\) in the implied constant in \((3.1)\) can never be strictly less than \(\frac{1}{2}\), since \(\alpha > 1/2\) is arbitrary.

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