Gradient estimates for a class of anisotropic nonlocal operators

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Abstract. Using a classical technique introduced by Achi E. Brandt for elliptic equations, we study a general class of nonlocal equations obtained as a superposition of classical and fractional operators in different variables. We obtain that the increments of the derivative of the solution in the direction of a variable experiencing classical diffusion are controlled linearly, with a logarithmic correction. From this, we obtain Hölder estimates for the solution.

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1. Introduction

In this paper, we will consider a general family of nonlocal operators built from classical and fractional Laplacians in different directions. Namely, the whole space $\mathbb{R}^n$ is divided into orthogonal subspaces along which a possibly different order operator acts. These sectional operators can be either classical or fractional, but at least one of them (say, one involving the last coordinate) is assumed to be of classical type. Our aim is to obtain regularity estimates for the solution in this last variable and then to deduce global regularity results.

The mathematical framework in which we work is the following. We denote by $\{e_1, \ldots, e_n\}$ the Euclidean base of $\mathbb{R}^n$. Given a point $x \in \mathbb{R}^n$, we use the notation

$$x = (x_1, \ldots, x_n) = x_1 e_1 + \cdots + x_n e_n,$$

with $x_i \in \mathbb{R}$. 

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We divide the variables of $\mathbb{R}^n$ into $m$ subgroups of variables, that is we consider $m \in \mathbb{N}$ and $N_1, \ldots, N_m \in \mathbb{N}$, with $N_1 + \cdots + N_{m-1} = n-1$ and $N_m = 1$. For $i \in \{1, \ldots, m\}$, we use the notation $N'_i := N_1 + \cdots + N_i$, and we take into account the set of coordinates
\begin{equation}
X_1 := (x_1, \ldots, x_{N_1}) \in \mathbb{R}^{N_1} \\
X_2 := (x_{N_1+1}, \ldots, x_{N_2}) \in \mathbb{R}^{N_2} \\
\vdots \\
X_i := (x_{N'_i-1+1}, \ldots, x_{N'_i}) \in \mathbb{R}^{N_i} \\
\vdots \\
X_{m-1} := (x_{N'_m-2+1}, \ldots, x_{N'_m-1}) \in \mathbb{R}^{N_{m-1}} \\
\text{and } X_m := x_n.
\end{equation}

Given $i \in \{1, \ldots, m - 1\}$ and $s_i \in (0,1]$, in this paper we study the (possibly fractional) $s_i$-Laplacian in the $i$th set of coordinates $X_i$ (the fractional case corresponds to the choice $s_i \in (0,1)$, while the classical case reduces to $s_i = 1$).

To denote these operators acting on subsets of variables, given $y = (y_1, \ldots, y_{N_i}) \in \mathbb{R}^{N_i}$ it is useful to introduce the notion of increment induced by $y$ with respect to the $i$th set of coordinates in $\mathbb{R}^n$, that is one defines
\begin{equation}
y^{(i)} := y_1 \varepsilon_{N'_1-1+1} + \cdots + y_{N_i} \varepsilon_{N'_i} \in \mathbb{R}^n.
\end{equation}

With this notation, one can define the $N_i$-dimensional (possibly fractional) $s_i$-Laplacian in the $i$th set of coordinates $X_i$ as
\begin{equation}
(-\Delta_{X_i})^{s_i} u(x) := \begin{cases} 
-\partial^2_{x_{N'_i-1+1}} u(x) - \cdots - \partial^2_{x_{N'_i}} u(x) & \text{if } s_i = 1, \\
 c_{N_i, s_i} \int_{\mathbb{R}^{N_i}} \frac{2 u(x) - u(x + y^{(i)}) - u(x - y^{(i)})}{|y^{(i)}|^{N_i+2s_i}} \, dy^{(i)} & \text{if } s_i \in (0,1),
\end{cases}
\end{equation}
The quantity $c_{N_i, s_i}$ in (3) is just a positive normalization constant, that is
\begin{equation}
c_{N, s} := \frac{2^{s-1} \Gamma(s + \frac{N}{2})}{\pi^{\frac{N}{2}} \Gamma(-s)},
\end{equation}
where $\Gamma$ is the Euler’s Gamma Function. See for instance [2, 3, 8, 13] and references therein for further motivations and an introduction to fractional operators.

In this paper we consider a pseudo-differential operator, which is the sum of (possibly) fractional Laplacians in the different coordinate directions $X_i$, with $i \in \{1, \ldots, m - 1\}$, plus a local second derivative in the direction $x_n$. The operators involved may have different orders and they may be multiplied by possibly different coefficients. Without loss of generality, we will assume that the last coefficient (that is the one related to the local variable) is normalized to be 1. That is, given $a_1, \ldots, a_{m-1} \geq 0$, we define
\begin{equation}
L := \sum_{i=1}^{m-1} a_i (-\Delta_{X_i})^{s_i} - \partial^2_{x_n}.
\end{equation}
Of course, the operator $\sum_{i=1}^{m-1} a_i (\Delta_{X_i})^{s_i}$ comprises as particular cases the classical Laplacian, the fractional Laplacian, and the sum of fractional Laplacians or fractional derivatives in different directions. Since some of the $a_j$’s may vanish, the case of degenerate operators is also taken into account.

We observe that in many concrete applications, different types of classical/anomalous diffusions may take place in different reference variables: a natural example occurs for instance when classical diffusion involving space variables is considered together with the anomalous diffusion arising from the transmission of genetic information, see e.g. [9,12], and these kinds of phenomena can be fruitfully discussed with the aid of operators such as the one in (5): in particular, though a complete mathematical modelization is still not available, experiments show that the intracellular and molecular transport exhibit anomalous diffusion of anisotropic type, due to the structure of the transmission medium (such as “crowded environments” and “microtubules”).

To state our main result, it is convenient to introduce the following domain notation. Given $r > 0$, we denote by $B^N_r$ the open ball of $\mathbb{R}^N$ centered at the origin and with radius $r$. Also, given $d_1, \ldots, d_m > 0$, we set $d := (d_1, \ldots, d_m)$ and

$$Q_d := B^N_{d_1} \times \cdots \times B^N_{d_{m-1}} \times (-d_m, d_m) = \prod_{i=1}^{m} B^N_{d_i},$$

where in the latter identity we used the convention that $N_m := 1$.

Then, given $\kappa > 0$, we denote by $Q_{d,\kappa}$ the dilation of $Q_d$ of factor $\kappa$ in the last coordinate (leaving the others put), that is

$$Q_{d,\kappa} := B^N_{d_1} \times \cdots \times B^N_{d_{m-1}} \times (-\kappa d_m, \kappa d_m).$$

The main result of this article is a quantitative bound on the continuity of the derivative of the solution with respect to the last coordinate. This quantitative estimate is “almost” of Lipschitz type, in the sense that the increment of the last derivative of the solution is bounded linearly, up to a logarithmic correction. This estimate will also be the cornerstone to prove additional regularity results such as Hölder estimates for the last derivatives and for the solutions in any direction. Thus, the core of the matter is the following result:

**Theorem 1.1.** Let $f : Q_d \to \mathbb{R}$ and $u : \mathbb{R}^n \to \mathbb{R}$ be a solution of $Lu = f$ in $Q_d$. Then, for any $y \in (-\frac{d_m}{4}, \frac{d_m}{4})$,

$$|\partial_{x_n} u(0, y) - \partial_{x_n} u(0, -y)| \leq \frac{8 \|\partial_{x_n} u\|_{L^\infty(\mathbb{R}^n)}}{d_m} |y| + 2\kappa |y| \log \frac{2d_m}{|y|},$$

where

$$\kappa := \frac{4}{3} \left( \|f\|_{L^\infty(Q_d)} + \sum_{i=1}^{m} \|u\|_{L^\infty(\mathbb{R}^n)} \frac{\tilde{c}_{N_i, s_i}}{d_i^{2s_i}} \right)$$

and

$$\tilde{c}_{N,s} := 2^s \Gamma(s+1) \frac{\Gamma(\frac{N}{2} + s)}{\Gamma(\frac{N}{2})}.$$ (6)
Higher regularity results for different types of nonlocal anisotropic operators have been obtained in [10,11]. In particular, in these articles only operators with the same fractional homogeneity were taken into account. Very recently, in [5], a regularity approach for anisotropic operators and for sums of anisotropic fractional Laplacians with different homogeneities has been taken into account, with methods different from the ones exploited in this paper.

Operators as the one studied here have been considered in [7], where a Lipschitz regularity result and a Liouville type theorem have been established (in this sense, Theorem 1.1 can be seen as a higher regularity theorem with respect to formula (7) in [7]).

The method of proof of Theorem 1.1 relies on an elementary, but very deep, technique introduced in [1] for the classical case of the Laplacian. Roughly speaking, this method is based on dealing with a family of additional variables and an operator in this extended space. These additional variables are chosen to take into account the increments of the solution and the extended operator to preserve the right notion of solutions. Then, one constructs barriers for this new operator, which in turn provide the desired estimate on the original solution.

Of course, in the nonlocal case one has to construct new barriers for the extended operator, since this operator also possesses nonlocal features, and the new barriers, differently from the classical case, must control the original solution on the whole of the complement of the domain, and not only along the boundary.

From Theorem 1.1, and the fact that

$$\lim_{y \to 0} |y|^\alpha \log |y| = 0 \quad \text{for any } \alpha \in (0, 1),$$

we deduce that, for any $\alpha \in (0, 1)$, any Lipschitz solution $u$ is $C^{1,\alpha}$ in the interior with respect to the variable $x_n$, as stated explicitly in the next result:

**Corollary 1.2.** Let $u : \mathbb{R}^n \to \mathbb{R}$ be a solution of $Lu = f$ in $B_1$. Then, for any $\alpha \in (0, 1)$,

$$\|\partial_{x_n} u\|_{C^{\alpha}(B_{1/2})} \leq C \left( \|f\|_{L^\infty(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)} \right),$$

for some $C > 0$, that depends on $\alpha, a_1, \ldots, a_{m-1}$ s and $N_1, \ldots, N_{m-1}$.

Using the results of this paper, it is also possible to deduce regularity results in all the variables. As an example, we consider the operator $L$ in the case in which $s_1 = \cdots = s_{m-1} =: s \in (0, 1)$, namely

$$L_* := \sum_{i=1}^{m-1} a_i (-\Delta X_i)^s - \partial_{x_n}^2,$$

and we give the following result:
Corollary 1.3. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a solution of $L_* u = f$ in $B_1$ and $	ilde{u}(x_1, \ldots, x_{n-1}) := u(x_1, \ldots, x_{n-1}, 0)$. Let $\alpha \in (0, 1)$ with $\alpha + 2s \notin \mathbb{N}$. Then
\[ \|\tilde{u}\|_{C^{\alpha+2s}(B_1^{n-1})} \leq C \left( \|f\|_{C^\alpha(B_1)} + \|\partial_{x_n} f\|_{L^\infty(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)} + \|u\|_{C^{\alpha}(B_1)} \right), \]
for some $C > 0$.

We notice that when $\alpha + 2s > 1$, the estimate in Corollary 1.3 provides continuity of the first derivative of the solution. The organization of the rest of this paper is the following. In Sect. 2, we introduce an auxiliary function which serves as barrier for our solution. Interestingly, following an idea in [1], it is convenient to construct this barrier in an extended space. The additional variable plays the role of a translation for the original solution and the first barrier is constructed by superposing appropriate one-side translations of the original solution, while the second barrier is a power-like function that solves the equation with constant right-hand-side with a logarithmic modification of a harmonic function in the translation coordinates.

The proof of Theorem 1.1 is completed in Sect. 3, using the previous barrier and the maximum principle. Corollaries 1.2 and 1.3 are proved in Sects. 4 and 5, respectively.

2. Building barriers

We use the notation $X' := (X_1, \ldots, X_{m-1})$ and define
\[ Q' := \left\{ (x', y, z) \in \mathbb{R}^{n+1} \text{ s.t. } x' \in B_{x_1}^{N_1} \times \cdots \times B_{x_{m-1}}^{N_{m-1}}, y \in \left( 0, \frac{d_m}{4} \right) \right\}. \]

Then, we consider the extended operator defined as
\[ L := \sum_{i=1}^{m-1} a_i (-\Delta X_i)^{s_i} - \frac{1}{2} \partial_y^2 - \frac{1}{2} \partial_z^2. \]

We remark that Brandt’s original barriers are modeled on second degree polynomials, while the ones exploited here are algebraically more complicated. Nevertheless, we believe that there is a heuristic idea that can link the classical barriers to the new ones. Philosophically, the quadratic part of a second degree polynomial takes care of a “constant” right hand side of a second order equation, which is somehow “the worst term” in the class of bounded right hand sides. On the other hand, the linear part of a second order polynomial can be used to provide additional symmetries with respect to a section that divides the cube into two equal parts. The linear term is also responsible for the final estimate, since the quadratic part is negligible near the origin. The barriers constructed in this papers are based on these heuristic considerations: they recover the original barriers by Brandt as $s \nearrow 1$ and they somehow preserve the geometric structures that we have discussed.
Also, we use the standard notation \( r_+ := \max\{r, 0\} \) for any \( r \in \mathbb{R} \) and, for any \((X', y, z) \in \mathbb{R}^{n+1}\), we define
\[
\phi(X', y, z) := \frac{1}{4} \left[ u(X', y_+ + z_+) - u(X', y_+ - z_+) \right.
\]
\[-u(X', -y_+ + z_+) + u(X', -y_+ - z_+) \right].
\]

The main properties of this barrier are listed below:

**Lemma 2.1.** For any \((X', y, z) \in Q'\), we have that
\[
\mathcal{L}\phi(x', y, z) = \frac{1}{4} \left[ Lu(X', y + z) - Lu(X', y - z) \right.
\]
\[-Lu(X', -y + z) + Lu(X', -y - z) \] (7)
and
\[
\|\mathcal{L}\phi\|_{L^\infty(Q')} \leq \|Lu\|_{L^\infty(Q_d)}. \tag{8}
\]
Also,
\[
\text{if either } y \leq 0 \text{ or } z \leq 0, \text{ then } \phi(X', y, z) = 0. \tag{9}
\]
Furthermore,
\[
|\phi(X', y, z)| \leq \|\partial_{x_n} u\|_{L^\infty(\mathbb{R}^n)} \min\{y_+, z_+\}. \tag{10}
\]

**Proof.** A direct calculation gives (7), which in turn implies (8). Formula (9) also follows by inspection.

As for (10), since the roles of \( y \) and \( z \) are the same, we just prove the estimate when \( y_+ \leq z_+ \). To this aim, we use (9) and we see that
\[
|\phi(x', y, z)| = |\phi(x', y, z) - \phi(x', 0, z)|
\]
\[\leq \frac{1}{4} \left[ |u(X', y_+ + z_+ - u(X', z_+)| + |u(X', y_+ - z_+) - u(X', -z_+)| + |u(X', -y_+ + z_+) - u(X', z_+)| + |u(X', -y_+ - z_+) - u(X', -z_+)| \right]
\]
\[\leq \|\partial_{x_n} u\|_{L^\infty(\mathbb{R}^n)} y_+,
\]
which implies (10). \(\square\)

Now, for any \((X', y, z) \in \mathbb{R}^{n+1}\), we define
\[
\psi(X', y, z) := \sum_{i=1}^{m-1} \|u\|_{L^\infty(\mathbb{R}^n)} \left( 1 - \frac{|X'|^2}{d_i^2} \right) \left( 1 - \frac{|X_i|^2}{d_i^2} \right) +
\]
\[
+ \frac{4 \|\partial_{x_n} u\|_{L^\infty(\mathbb{R}^n)} y_+ z_+}{d_m} + \kappa y_+ z_+ \left| \log \frac{2d_m}{y_+ + z_+} \right|,
\]
where \( \kappa \) and \( \tilde{c}_{N_i,s_i} \) are as in (6).

We notice that the choice of \( \tilde{c}_{N_i,s_i} \) is made in such a way that
\[
(-\Delta X_i)^{s_i} \left( 1 - \frac{|X_i|^2}{d_i^2} \right) \left( 1 - \frac{|X_i|^2}{d_i^2} \right) = \tilde{c}_{N_i,s_i} \frac{d_{2s_i}}{d_{2s_i}}.
\tag{11}
\]
see e.g. Table 3 in [4]. On the other hand, the definition of \( \kappa \) makes it sufficiently large to let \( \psi \) dominate \( \phi \), as stated in the following result:
Lemma 2.2. We have that
\[ \psi \pm \phi \geq 0 \text{ in } \mathbb{R}^{n+1} \setminus Q' \] \tag{12}
and \[ \mathcal{L}(\psi \pm \phi) \leq 0 \text{ in } Q'. \] \tag{13}

Proof. We remark that the complement of \( Q' \) can be written as \( P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5 \), where
\[ P_1 := \left\{ |X_i'| \geq \frac{d_i}{2} \text{ for some } i \in \{1, \ldots, m - 1\} \right\}, \]
\[ P_2 := \{ y \leq 0 \}, \]
\[ P_3 := \{ z \leq 0 \}, \]
\[ P_4 := \{ y \geq \frac{d_m}{4} \}, \]
and \[ P_5 := \{ z \geq \frac{d_m}{4} \}. \]

Now, on \( P_1 \),
\[ \psi \geq \left\| u \right\|_{L^\infty(\mathbb{R}^n)} \left( 1 - \left( 1 - \frac{1}{4} \right)^s \right) = \left\| u \right\|_{L^\infty(\mathbb{R}^n)} \geq |\phi|. \]

Also, on \( P_2 \cup P_3 \), using (9) we see that \( \psi \geq 0 = |\phi| \). In addition, recalling (10), in \( P_4 \) we have that
\[ |\phi| \leq \left\| \partial_x u \right\|_{L^\infty(\mathbb{R}^n)} z \leq \frac{4 \left\| \partial_x u \right\|_{L^\infty(\mathbb{R}^n)} y_+ z_+ \leq \psi; \]
and a similar computation holds in \( P_5 \). By collecting these estimates, the claim in (12) plainly follows.

Now we observe that in \( Q' \) we have that
\[ \frac{2d_m}{y_+ + z_+} = \frac{2d_m}{y + z} \geq 1 \]
and consequently
\[ \left| \log \frac{2d_m}{y_+ + z_+} \right| = \log \frac{2d_m}{y + z}. \]

Also,
\[ \frac{\partial^2}{\partial y^2} \left( yz \log \frac{2d_m}{y + z} \right) = - \frac{2z}{y + z} + \frac{yz}{(y + z)^2} \]
and therefore
\[ \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left( yz \log \frac{2d_m}{y + z} \right) = -2 + \frac{2yz}{(y + z)^2}. \] \tag{14}
Now, for any \( y, z > 0 \), we have that \( (y - z)^2 \geq 0 \), and, as a result,
\[ \frac{yz}{(y + z)^2} \leq \frac{1}{4}. \] \tag{15}
From (14) and (15) we obtain that
\[
\left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left( yz \log \frac{2d_m}{y + z} \right) \leq -2 + \frac{1}{2} = -\frac{3}{2}.
\]
Therefore, in view of (11), in \( Q' \) it holds that
\[
L \psi \leq \sum_{i=1}^{m} \| u \|_{L^\infty(\mathbb{R}^n)} \frac{\bar{c}_{N_i,s_i}}{d^2 s_i} - \frac{3}{4} \kappa.
\]
Hence, our choice of \( \kappa \) in (6) implies that
\[
L \psi \leq \| Lu \|_{L^\infty(Q_d)}.
\]
This, together with (8), proves (13). \( \square \)

3. Completion of the proof of Theorem 1.1

By Lemma 2.2 and the maximum principle (see e.g. formula (22) in [7]), we have that \( \psi \pm \phi \geq 0 \) in \( Q' \), that is \( |\phi| \leq \psi \) in \( Q' \). We write this inequality at \( X' = 0, y > 0 \), divide by \( z > 0 \) and pass to the limit: we find that
\[
\frac{1}{2} |\partial_{x_n} u(0,y) - \partial_{x_n} u(0,-y)|
\]
\[
= \lim_{z \searrow 0} \frac{1}{4z} |u(0,y+z) - u(0,y-z) - u(0,-y+z) + u(0,-y-z)|
\]
\[
\leq 4 \| \partial_{x_n} u \|_{L^\infty(\mathbb{R}^n)} \frac{2d_m}{d_m} \frac{y + \kappa y \log \frac{2d_m}{y}}{y}.
\]
This establishes Theorem 1.1.

4. Proof of Corollary 1.2

To prove Corollary 1.2, we first give a preliminary result that follows directly from Theorem 1.1:

Corollary 4.1. Let \( u : \mathbb{R}^n \to \mathbb{R} \) be a solution of \( Lu = f \) in \( B_1 \). Then, for any \( \alpha \in (0,1) \),
\[
\| \partial_{x_n} u \|_{C^\alpha(B_{1/2})} \leq C \left( \| f \|_{L^\infty(B_1)} + \| u \|_{L^\infty(\mathbb{R}^n)} + \| \partial_{x_n} u \|_{L^\infty(\mathbb{R}^n)} \right), \tag{16}
\]
for some \( C > 0 \), that depends on \( \alpha, a_1, \ldots, a_{m-1}, s_1, \ldots, s_{m-1} \) and \( N_1, \ldots, N_{m-1} \).

With this result, an elementary, but useful, cut-off\(^2\) argument, gives that:

\(^2\)The main difference between (16) and (17) is that the norm of \( \partial_{x_n} u \) gets localized in the second formula.
Corollary 4.2. Let $u : \mathbb{R}^n \rightarrow \mathbb{R}$ be a solution of $Lu = f$ in $B_1$. Then, for any $\alpha \in (0, 1)$,
\[
\|\partial_{x_n} u\|_{C^\alpha(B_{1/4})} \leq C \left( \|f\|_{L^\infty(B_{1/2})} + \|u\|_{L^\infty(\mathbb{R}^n)} + \|\partial_{x_n} u\|_{L^\infty(B_{1/2})} \right),
\]
for some $C > 0$, that depends on $\alpha$, $a_1, \ldots, a_{m-1}$ and $N_1, \ldots, N_{m-1}$.

Proof. Let $\tau_o \in C^\infty(\mathbb{R})$ be such that $\tau_o(r) = 1$ if $|r| \leq 3/4$ and $\tau_o(r) = 0$ if $|r| \geq 4/5$. For any $i \in \{1, \ldots, m\}$ and any $X_i \in \mathbb{R}^{N_i}$, we set $\tau_i(X_i) := \tau_o(|X_i|)$. Let also
\[
\tau(x) = \tau(X_1, \ldots, X_m) := \tau_1(X_1) \cdots \tau_m(X_m)
\]
and $v(x) := \tau(x) u(x)$.

Notice that $v = u$ in $B_{3/4}$. Also, if $x = (X_1, \ldots, X_m) \in B_{1/2}$ and $|y^{(i)}| \leq \frac{1}{10}$, we have that $|X^{(i)} + y^{(i)}| \leq \frac{3}{4}$ and $|X^{(j)}| \leq 3/4$, that gives $\tau(x) = 1$. Thus, for any $x \in B_{1/2}$ and for any $i \in \{1, \ldots, m-1\}$, if $s_i \in (0, 1)$,
\[
\int_{\mathbb{R}^{N_i}} \frac{v(x) - v(x + y^{(i)})}{|y^{(i)}|^{N_i + 2s_i}} dy^{(i)} = \int_{\mathbb{R}^{N_i}} \frac{u(x) - v(x + y^{(i)})}{|y^{(i)}|^{N_i + 2s_i}} dy^{(i)} + g_i(x)
= \int_{\mathbb{R}^{N_i}} \frac{u(x) - u(x + y^{(i)})}{|y^{(i)}|^{N_i + 2s_i}} dy^{(i)} + g_i(x)
\quad \text{where } g_i(x) := \int_{|y^{(i)}| \geq \frac{1}{10}} \frac{(1 - \tau)u(x + y^{(i)})}{|y^{(i)}|^{N_i + 2s_i}} dy^{(i)}.
\]
Notice that, for any $x \in B_{1/2}$ and for any $i \in \{1, \ldots, m-1\}$,
\[
|g_i(x)| \leq \int_{|y^{(i)}| \geq \frac{1}{10}} \frac{\|u\|_{L^\infty(\mathbb{R}^n)}}{|y^{(i)}|^{N_i + 2s_i}} dy^{(i)} \leq C \|u\|_{L^\infty(\mathbb{R}^n)},
\]
for some $C > 0$. In addition, in $B_{1/2}$,
\[
\partial^2_{x_n} v = \partial^2_{x_n} u.
\]
As a consequence of this and (18), in $B_{1/2}$ we have that
\[
Lv = g,
\]
with
\[
g(x) := f(x) + \sum_{\substack{1 \leq i \leq m-1 \\ s_i \in (0, 1)}} b_i g_i(x),
\]
for suitable $b_1, \ldots, b_{m-1}$. We remark that
\[
\|g\|_{L^\infty(B_{1/2})} \leq C \left( \|f\|_{L^\infty(B_{1/2})} + \|u\|_{L^\infty(\mathbb{R}^n)} \right),
\]
up to renaming $C > 0$, thanks to (19). From this, (21) and Corollary 4.1 we obtain that
\[
\|\partial_{x_n} u\|_{C^\alpha(B_{1/4})} = \|\partial_{x_n} v\|_{C^\alpha(B_{1/4})}
\leq C \left( \|g\|_{L^\infty(B_{1/2})} + \|v\|_{L^\infty(\mathbb{R}^n)} + \|\partial_{x_n} v\|_{L^\infty(\mathbb{R}^n)} \right)
\leq C \left( \|f\|_{L^\infty(B_{1/2})} + \|u\|_{L^\infty(\mathbb{R}^n)} + \|\partial_{x_n} u\|_{L^\infty([-1,1]^n)} \right),
\]
up to renaming $C > 0$. This is the desired result, up to resizing balls. □

Now we recall that an estimate for $\| \partial_{x_n} u \|_{L^\infty(B_{1/2})}$ has been given in formula (8) of [7]. From this and (17), the claim in Corollary 1.2 plainly follows, up to resizing balls.

5. Proof of Corollary 1.3

The argument combines some techniques from Corollary 1.3 in [7] and Theorem 1.1(b) in [10], together\(^3\) with Corollary 1.2 here.

To prove Corollary 1.3 we start with a preliminary and global version of it:

**Lemma 5.1.** Let $\alpha \in (0, 1)$ with $\alpha + 2s \notin \mathbb{N}$. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a solution of $L_* u = f$ in $B_1$.

Let $\bar{u}(x_1, \ldots, x_{n-1}) := u(x_1, \ldots, x_{n-1}, 0)$.

Then

$$\| \bar{u} \|_{C^{\alpha + 2s}(B_{3/4})} \leq C \left( \| f \|_{C^\alpha(B_{4/5})} + \| \partial_{x_n} f \|_{L^\infty(B_1)} + \| \partial_{x_n} u \|_{L^\infty(\mathbb{R}^n)} \right),$$

for some $C > 0$.

**Proof.** Given $\tau \in \mathbb{R}$, with $|\tau|$ sufficiently small, we set

$$u^{(\tau)}(x) := \frac{u(x + \tau e_n) - u(x)}{\tau} \quad \text{and} \quad f^{(\tau)}(x) := \frac{f(x + \tau e_n) - f(x)}{\tau}.$$

We remark that one can bound $\| u^{(\tau)} \|_{L^\infty(\mathbb{R}^n)}$ with $\| \partial_{x_n} u \|_{L^\infty(\mathbb{R}^n)}$. Similarly, one can bound $\| \partial_{x_n} u^{(\tau)} \|_{L^\infty(\mathbb{R}^n)}$ with $\| \partial_{x_n}^2 u \|_{L^\infty(\mathbb{R}^n)}$.

Notice also that $L_* u^{(\tau)} = f^{(\tau)}$ in $B_1$, and thus Corollary 1.2 implies that, for any $\alpha \in (0, 1)$,

$$\| \partial_{x_n} u^{(\tau)} \|_{C^\alpha(B_{4/5} \times [-1/100, 1/100])} \leq C \left( \| f^{(\tau)} \|_{L^\infty(B_{3/4} \times [-1/10, 1/10])} + \| u^{(\tau)} \|_{L^\infty(\mathbb{R}^n)} \right),$$

for some $C > 0$.\(^3\)

---

\(^3\)We take this opportunity to correct a flaw in the statement of Corollary 1.3 in [7]. Namely, the condition

$$\gamma := \begin{cases} 2s & \text{if } s \neq 1/2, \\
1 - \epsilon & \text{if } s = 1/2 \end{cases}$$

has to be replaced by

$$\gamma := \begin{cases} 2s & \text{if } s < 1/2, \\
1 - \epsilon & \text{if } s \geq 1/2 \end{cases}$$

and the two lines after the statement can be deleted. The correct statement of Corollary 1.3 in [7] is the one in the arxiv version [6].
Accordingly, sending $\tau \to 0$,
\[
\|\partial^2_{x_n} u\|_{C^\alpha(B_{4/5}^{n-1} \times [-1/100,1/100])} \leq C \left( \|\partial_{x_n} f\|_{L^\infty(B_1)} + \|\partial_{x_n} u\|_{L^\infty(\mathbb{R}^n)} \right). \tag{22}
\]
In addition, given $x_n \in \mathbb{R}$, with $|x_n| \leq \frac{1}{100}$, we define
\[
\tilde{f}(x_1, \ldots, x_{n-1}) := f(x_1, \ldots, x_{n-1}, 0) + \partial^2_{x_n} u(x_1, \ldots, x_{n-1}, 0).
\]
Notice that
\[
\|\tilde{u}\|_{C^\alpha(\mathbb{R}^{n-1})} \leq \|u\|_{C^\alpha(\mathbb{R}^{n-1} \times [-1/100,1/100])}
\]
and
\[
\tilde{f} \cdot \|C^\alpha(B_{4/5}^{n-1}) \leq \|f\|_{C^\alpha(B_{4/5}^{n-1} \times [-1/100,1/100])}
\]
\[
+ \|\partial^2_{x_n} u\|_{C^\alpha(B_{4/5}^{n-1} \times [-1/100,1/100])},
\]
for some $C > 0$, provided that $\alpha + 2s \notin \mathbb{N}$. From this and (23) we obtain that
\[
\|\tilde{u}\|_{C^\alpha+2s(B_{3/4}^{n-1})} \leq C \left( \|\tilde{u}\|_{C^\alpha(\mathbb{R}^{n-1})} + \|\tilde{f}\|_{C^\alpha(B_{4/5}^{n-1})} \right),
\]
up to renaming $C > 0$. This and (22) imply the desired result, again, up to renaming $C > 0$. □

Now we complete the proof of Corollary 1.3 by using a cut-off argument as in the proof of Corollary 4.2. Indeed, in that notation, from (18) and (20), we can write
\[
Lv = g \quad \text{in } B_{1/2}, \tag{24}
\]
with $g$ as in (21) and
\[
g_i(x) := \int_{|x - z(i)| \geq \frac{1}{100}} \frac{(1 - \tau)u(z(i))}{|x - z(i)|^{N_i+2s_i}} \, dz(i).
\]
Notice that we can take derivatives in $x$ inside the integral, hence, for any $j \in \mathbb{N}$ and $x \in B_{1/2},$
\[
|D^j g_i(x)| \leq C_j \int_{|x - z(i)| \geq \frac{1}{100}} \frac{\|u\|_{L^\infty(\mathbb{R})}}{|x - z(i)|^{N_i+2s_i+j}} \, dz(i) \leq C_j \|u\|_{L^\infty(\mathbb{R}^n)},
\]
for suitable $C_j > 0$. In particular, we have that
\[
\|g\|_{C^\alpha(B_{4/5})} + \|\partial_{x_n} g\|_{L^\infty(B_1)} \leq C \left( \|f\|_{C^\alpha(B_{4/5})} + \|\partial_{x_n} f\|_{L^\infty(B_1)} + \|u\|_{L^\infty(\mathbb{R}^n)} \right).
\]
Hence, writing $\tilde{v}(x_1, \ldots, x_{n-1}) := v(x_1, \ldots, x_{n-1}, 0)$, from (24) and Lemma 5.1, we obtain that
\[ \| \tilde{u} \|_{C^{\alpha+2\epsilon}(B_{3/4}^n)} = \| \tilde{u} \|_{C^{\alpha+2\epsilon}(B_{3/4}^n)} \]
\[ \leq C \left( \| g \|_{C^{\alpha}(B_{1/100}^n)} + \| \partial_{x_n} g \|_{L^\infty(B_i)} + \| v \|_{C^{\alpha}(\mathbb{R}^n \times [-1/100, 1/100])} + \| \partial_{x_n} v \|_{L^\infty(\mathbb{R}^n)} \right) \]
\[ \leq C \left( \| f \|_{C^{\alpha}(B_{1/100}^n)} + \| \partial_{x_n} f \|_{L^\infty(B_1)} + \| u \|_{L^\infty(\mathbb{R}^n)} + \| \partial_{x_n} u \|_{L^\infty(Q)} \right), \]

where \( Q \) is the support of the cut-off \( \tau \).

Now, up to resizing balls, we may suppose that the original equation was satisfied in some domain \( Q' \), with \( Q' \supseteq Q \). Hence, from formula (8) of [7], we know that

\[ \| \partial_{x_n} u \|_{L^\infty(Q)} \leq C \left( \| f \|_{L^\infty(Q')} + \| u \|_{L^\infty(\mathbb{R}^n)} \right), \]

and hence the result in Corollary 1.3 follows.

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