Linearized supergravity from Matrix theory

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Abstract

We show that the linearized supergravity potential between two objects arising from the exchange of quanta with zero longitudinal momentum is reproduced to all orders in $1/r$ by terms in the one-loop Matrix theory potential. The essential ingredient in the proof is the identification of the Matrix theory quantities corresponding to moments of the stress tensor and membrane current. We also point out that finite-$N$ Matrix theory violates the equivalence principle.

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1 Introduction

It has by now been firmly established that the matrix quantum mechanics [1, 2, 3] appearing in the BFSS conjecture [4] contains within it a remarkable structure which replicates many features of 11-dimensional supergravity. As emphasized in the lectures of Susskind [5], however, there is as yet no explicit way of deriving supergravity directly from Matrix theory. In this paper we make some progress in this direction by deriving the linearized form of supergravity from a subset of the terms in the one-loop Matrix theory potential.

Supergravity has two propagating bose fields: the metric and the 3-form. The effective potential between an arbitrary pair of classical objects is in principle given by summing all Feynman diagrams connecting the objects. When this summation is truncated to include only single-particle exchange diagrams, the resulting potential corresponds to that of linearized supergravity. For a pair of objects of finite but nonzero size, this potential can be expanded in the inverse separation, giving a power series in $1/r$, where a term of order $1/r^{7+k+l}$ appears for each $k$ and $l$ which is proportional to the product of the $k$th and $l$th moments of the source tensors of the two objects. In this paper we prove that this infinite series of terms corresponds precisely to a series of terms in the one-loop Matrix theory potential. The leading term in this series is the $1/r^7$ term proportional to the product of the integrated source tensors. This term has been shown to be reproduced correctly by Matrix theory in numerous specific cases [4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]; however, to date there has been no general understanding of the structure responsible for this agreement.

It should be emphasized that the higher order terms we consider in this paper are terms of the form $F^4X^n$, which were considered previously in [18]. These should not be confused with the terms of higher order in $F$ in the one-loop potential or the higher loop terms which have been discussed in the context of graviton scattering [19, 20, 21, 22, 23].

The structure of this paper is as follows: In Section 2 we recall the general form of the leading term in the one-loop Matrix theory potential between an arbitrary pair of objects, and show that this term can be rewritten in a fashion which is highly suggestive of supergravity interactions. In Section 3 we derive the leading term in the linearized supergravity potential arising from the exchange of quanta with no longitudinal momentum. In Section 4 we observe that the Matrix theory and supergravity potentials agree precisely if certain Matrix expressions are identified as the stress tensor and currents of a Matrix theory object. We show that in the large-$N$ limit this identification corresponds with known results for the graviton, membrane, and longitudinal 5-brane. Section 5 demonstrates that the subleading terms in the linearized supergravity potential arising from higher moments of the sources are also correctly reproduced to all orders in Matrix theory. Section 6 concludes with a discussion of the relationship of supergravity to finite–$N$ Matrix theory.
2 Matrix theory potential

We will consider the leading term in the long-distance expansion of the Matrix theory potential between an arbitrary pair of localized semiclassical states. Two objects described by matrices \( \hat{X}(t) \) and \( \tilde{X}(t) \) of sizes \( \hat{N} \times \hat{N} \) and \( \tilde{N} \times \tilde{N} \), respectively, correspond to a background in the Matrix gauge theory

\[
\langle \hat{X}^i(t) \rangle = \begin{pmatrix} \hat{X}^i(t) & 0 \\ 0 & \tilde{X}^i(t) \end{pmatrix}.
\]

Integrating out the off-diagonal matrix blocks at one loop gives rise to the leading long-distance potential \([24, 25]\)

\[
V_{\text{matrix}} = -\frac{5}{128 \ r^7} \text{STr} \ F
\]

where \( r \) is the separation distance between the objects, \( F \) is defined by

\[
F = 24 F_{0i} F_{0j} F_{0j} + 24 F_{0i} F_{0i} F_{jk} F_{jk} + 96 F_{0i} F_{0j} F_{ik} F_{kj} + 24 F_{ij} F_{jk} F_{kl} F_{li} - 6 F_{ij} F_{ij} F_{kl} F_{kl}
\]

and STr is a symmetrized trace, the average over all orderings in the product of \( F \)'s. The components of \( F \) are defined by

\[
F_{0i} = -F_{i0} = \partial_t K_i, \quad F_{ij} = i [K_i, K_j]
\]

where

\[
K_i = \hat{X}_i \otimes \mathbb{1}_{\hat{N} \times \hat{N}} - \mathbb{1}_{\tilde{N} \times \tilde{N}} \otimes \tilde{X}_i^T.
\]

The potential can be expanded in terms of the field strengths of the individual objects

\[
\hat{F}_{0i} = -\hat{F}_{i0} = \partial_t \hat{X}_i, \quad \hat{F}_{ij} = i [\hat{X}_i, \hat{X}_j]
\]

\[
\tilde{F}_{0i} = -\tilde{F}_{i0} = \partial_t \tilde{X}_i, \quad \tilde{F}_{ij} = i [\tilde{X}_i, \tilde{X}_j].
\]

(In the sequel all variables with a hat (tilde) indicate quantities which depend only on the matrices \( \hat{X}(\tilde{X}) \)). By decomposing the Matrix potential \([\square]\) in terms of \( \hat{F} \) and \( \tilde{F} \), it can be re-expressed as

\[
V_{\text{matrix}} = V_{\text{gravity}} + V_{\text{electric}} + V_{\text{magnetic}}
\]

\[
V_{\text{gravity}} = -\frac{15 R^2}{4 r^7} \left( \hat{T}^{IJ} \tilde{T}_{IJ} - \frac{1}{9} \hat{T}^I_I \hat{T}^J_J \right)
\]

\[
V_{\text{electric}} = -\frac{45 R^2}{r^7} \hat{J}^{IJ} \tilde{J}_{IJ}
\]

\[
V_{\text{magnetic}} = -\frac{45 R^2}{r^7} \hat{M}^{ijkl} \tilde{M}_{ijkl}
\]

\( ^1 \)Notation: \( I, J = 0, \ldots, 10 \) are spacetime indices with metric \( \eta_{IJ} = (- + \cdots +) \), \( x^\pm = \frac{1}{\sqrt{2}} (x^0 \pm x^{10}) \) with \( x^- \approx x^- + 2 \pi R \), and \( i, j = 1, \ldots, 9 \) are transverse spatial indices.
The quantities appearing in this decomposition are defined as follows. (One can verify these relations simply by collecting all terms on both sides with a definite tensor structure in either $\hat{F}$ or $\tilde{F}$.)

$\mathcal{T}^{IJ}$ is a symmetric tensor with components

$$
\mathcal{T}^{--} = \frac{1}{R} \text{Str} \frac{\mathcal{F}}{96}
$$

$$
\mathcal{T}^{-i} = \frac{1}{R} \text{Str} \left( \frac{1}{2} \dot{X}^i \dot{X}^j \dot{X}^j + \frac{1}{4} \dot{X}^i F^{jk} \dot{X}^j + F^{ij} \dot{X}^k \right)
$$

$$
\mathcal{T}^{++} = \frac{1}{R} \text{Str} \frac{1}{2} \dot{X}^i \dot{X}^i + \frac{1}{4} F^{ij} F^{ij}
$$

$$
\mathcal{T}^{ij} = \frac{1}{R} \text{Str} \left( \dot{X}^i \dot{X}^j + F^{ik} F^{kj} \right)
$$

$$
\mathcal{T}^{+i} = \frac{1}{R} \text{Str} \dot{X}^i
$$

$$
\mathcal{T}^{++} = \frac{N}{R}
$$

$\mathcal{J}^{IJK}$ is a totally antisymmetric tensor with components

$$
\mathcal{J}^{-ij} = \frac{1}{6R} \text{Str} \left( \dot{X}^i \dot{X}^k F^{kj} - \dot{X}^j \dot{X}^k F^{ki} - \frac{1}{2} \dot{X}^k \dot{X}^k F^{ij} + \frac{1}{4} F^{ij} F^{kl} F^{kl} + F^{ik} F^{kj} \right)
$$

$$
\mathcal{J}^{+i} = \frac{1}{6R} \text{Str} \left( F^{ij} \dot{X}^j \right)
$$

$$
\mathcal{J}^{ijk} = -\frac{1}{6R} \text{Str} \left( \dot{X}^i F^{jk} + \dot{X}^j F^{ki} + \dot{X}^k F^{ij} \right)
$$

$$
\mathcal{J}^{+ij} = -\frac{1}{6R} \text{Str} F^{ij}
$$

Note that we retain some quantities – in particular $\mathcal{J}^{+-i}$ and $\mathcal{J}^{+ij}$ – which vanish at finite $N$ (by the Gauss constraint and antisymmetry of $F^{ij}$, respectively). These terms represent BPS charges (for longitudinal and transverse membranes) which are only present in the large $N$ limit [26]. We define higher moments of these terms in Section 5 which can be non-vanishing at finite $N$.

$\mathcal{M}^{IJKLMN}$ is a totally antisymmetric tensor with

$$
\mathcal{M}^{+ijk} = \frac{1}{12R} \text{Str} \left( F^{ij} F^{kl} + F^{ik} F^{lj} \right)
$$

At finite $N$ this vanishes by the Jacobi identity, but we shall retain it because it represents the charge of a longitudinal 5-brane. The other components of $\mathcal{M}^{IJKLMN}$ do not appear in the Matrix potential, and we are unable to determine expressions for them. This is closely related to the difficulty of constructing a transverse 5-brane in Matrix theory.

We have thus expressed the leading term in the one-loop Matrix theory potential in an algebraic form which is highly suggestive of supergravity. This potential is a time-dependent
instantaneous interaction potential which depends upon various tensor expressions in the Matrix variables describing the two objects. We now compare this with the form of the linearized supergravity potential between an arbitrary pair of objects.

3 Supergravity potential

We begin with a few general remarks on the supergravity potential in light-front coordinates. According to the BFSS conjecture [4], the leading term in the one-loop matrix theory potential should agree with the leading long-distance supergravity interaction between two objects when no longitudinal momentum is transferred. Thus, it should be compared to the interaction expected from linearized supergravity, in which the exchanged quantum (metric or 3-form) has zero longitudinal momentum. In light-front coordinates, this leads to a rather peculiar propagator, as emphasized in [27]. For example, the scalar Green’s function is

\[
\square^{-1}(x) = \frac{1}{2\pi R} \sum_n \int \frac{dk^0 d^9 k_\perp}{(2\pi)^{10}} \frac{e^{-i\pi^2 x^- - ik^+ + ik_\perp x_\perp}}{2n R k^- - k_\perp^2}
\]

Keeping just the \( n = 0 \) term in the sum leads to the propagator at zero longitudinal momentum

\[
\square^{-1}(x - y) = \frac{1}{2\pi R} \delta(x^+ - y^+) \frac{-15}{32\pi^4 |x_\perp - y_\perp|^7}
\]

Note that the exchange of quanta with zero longitudinal momentum gives rise to interactions that are instantaneous in light-front time, precisely the type of instantaneous interactions that we find at one loop in Matrix theory. Such action-at-a-distance potentials are allowed by the Galilean invariance manifest in light-front quantization.

An object with internal dynamics (such as an oscillating membrane) gives rise to gravitational radiation, which appears as time-dependent fluctuations in the metric tensor seen by a distant observer. An observer using light-front coordinates will see part of this radiation as instantaneous potential, even though the radiation respects causality. This instantaneous potential carries no energy away from the radiating object; to see a loss of energy due to outgoing radiation it would be necessary to calculate the rate of emission of gravitons with non-zero longitudinal momentum.

We now calculate the light-front supergravity interaction potential at leading order in \( 1/r \). We take the bulk action for 11-dimensional supergravity to be\(^2\)

\[
S_{\text{SUGRA}} = -\frac{1}{2\kappa^2} \int d^{11}x \sqrt{-g} \left( R + \frac{1}{24} F_{IJKL} F^{IJKL} + \cdots \right)
\]

where \( F \) is the field strength of the 3-form potential \( C \). Integrating out the metric and 3-form in the linearized approximation gives rise to the effective action

\[
S = \int d^{11}x d^{11}y \left( -\frac{1}{8} T_{IJ}(x) D_{\text{graviton}}^{IJKL}(x - y) T_{KL}(y) - \frac{1}{2} J_{IJK}(x) D_{3\text{-form}}^{IJKLMN}(x - y) J_{LMN}(y) \right)
\]

\(^2\)Conventions: The Planck length is defined by \( 2\pi \kappa^2 = (2\pi)^8 l_p^3 \). We adopt units in which \( 2\pi l_p^3 = R \). The membrane tension is \( T_2 = \frac{1}{(2\pi)^2 l_p^3} \), and \( F_{IJKL} \equiv \frac{1}{6} (\partial_I C_{JKL} \pm 23 \text{ terms}) \).
where the gauge-fixed graviton and 3-form propagators are
\[
D_{\text{graviton}}^{IJ,KL} = 2\kappa^2 \left( \eta^{IK} \eta^{JL} + \eta^{IL} \eta^{JK} - \frac{2}{9} \eta^{IJ} \eta^{KL} \right) \square^{-1}(x - y)
\]
\[
D_{3\text{-form}}^{IJK,LMN} = 2\kappa^2 \left( \eta^{IL} \eta^{JM} \eta^{KN} \pm 5 \text{ terms} \right) \square^{-1}(x - y)
\]
and where $T^{IJ}$ and $J^{IJK}$ are the stress tensor and ‘electric’ 3-form current. To compare with Matrix theory, we use the zero longitudinal momentum propagator (10) to find the potential between two objects arising from graviton and electric 3-form exchange:
\[
V_{\text{sugra}} = V_{\text{gravity}} + V_{\text{electric}}
\]
\[
V_{\text{gravity}} = -\frac{15R^2}{4r^7} \left( \hat{T}^{IJ} \hat{T}_{IJ} - \frac{1}{9} \hat{T}^I \hat{T}_I \right)
\]
\[
V_{\text{electric}} = -\frac{45R^2}{r^7} \hat{J}^{IJK} \hat{J}_{IJK}
\]
where we define, for example, $\hat{T}^{IJ} \equiv \int dx^- d^9 x_\perp T^{IJ}(x)$.

The 3-form field also has ‘magnetic’ couplings to the M-theory 5-brane. Although it is not known how to describe a theory containing both electric and magnetic sources in terms of an action principle, in the linearized theory we can simply consider interactions between magnetic sources to be mediated by a quantum of the 6-form field dual to $C$. This leads to an additional term in the supergravity potential
\[
V_{\text{magnetic}} = -\frac{3R^2}{2r^7} \hat{M}^{IJKLMN} \tilde{M}_{IJKLMN}
\]
where $\hat{M}^{IJKLMN}$ is the ‘electric’ current for the 6-form field.

4 Correspondence of potentials

The one-loop Matrix potential (3) clearly has a structure identical to the linearized supergravity potential derived in the last section. At a formal level, this shows that if we define the stress tensor and currents of objects in Matrix theory to be given by (7), (8) and (9) then there is a precise correspondence between the Matrix theory and supergravity potentials between any pair of objects. This formal correspondence is valid even at finite $N$. However, to verify that this equivalence is physically meaningful, we must check that the Matrix theory stress tensor and currents we have defined correspond in a sensible way with those of supergravity. In this section we show that this correspondence does indeed hold in the large-$N$ limit; finite-$N$ effects are discussed in Section 6.

To show that our definitions of the Matrix theory stress tensor and currents are sensible, we now proceed to show that they correspond with the definitions expected from supergravity for three classes of semiclassical states: gravitons, membranes, and longitudinal 5-branes.
4.1 Gravitons

We begin by discussing graviton states. A classical graviton with transverse coordinates \( x^i(t) \) and \( N \) units of longitudinal momentum is described in Matrix theory by matrices \( X^i = x^i \mathbb{1}_{N \times N} \). This is sufficient for our purposes because at the one-loop level the graviton wavefunction does not enter: all we need is a background which satisfies the classical Matrix equations of motion. Plugging this background into the Matrix definitions (7), (8) and (9) one finds that, as expected, the 3-form and 6-form currents vanish, while the stress tensor is given by

\[
\mathcal{T}^{ij} = \frac{N}{R} \dot{x}^i \dot{x}^j \\
\mathcal{T}^{++} = \frac{N}{2R} \dot{t}^2 \\
\mathcal{T}^{-i} = \frac{N}{2R} \dot{x}^i \dot{x}^2 \\
\mathcal{T}^{--} = \frac{N}{4R} \left( \dot{x}^2 \right)^2
\]

This matches the supergravity result: the classical expression for the stress tensor of a point particle is

\[
\int dx^- d^9 x_\perp T^{IJ}(x) = \frac{p^I p^J}{p^+}
\]

where for a massless particle \( p^+ = N/R \), \( p^i = p^+ \dot{x}^i \), and \( p^- = p_\perp^2/2p^+ \).

4.2 Membranes

The bosonic part of the supermembrane action is

\[
S_{\text{membrane}} = \int d^3 \xi \left\{ -T_2 \sqrt{-\text{det} \partial_\alpha X^I \partial_\beta X^J g_{IJ}} + \sqrt{2} T_2 \frac{1}{3!} \epsilon^{\alpha \beta \gamma} \partial_\alpha X^I \partial_\beta X^J \partial_\gamma X^K C_{IJK} \right\} \tag{11}
\]

The resulting stress tensor and 3-form current are given by

\[
T^{IJ}(x) = -T_2 \int d^3 \xi \delta^{11} (x - X(\xi)) \sqrt{-g} g^{\alpha \beta} \partial_\alpha X^I \partial_\beta X^J \\
J^{IJK}(x) = T_2 \int d^3 \xi \delta^{11} (x - X(\xi)) \frac{1}{3!} \epsilon^{\alpha \beta \gamma} \partial_\alpha X^I \partial_\beta X^J \partial_\gamma X^K \tag{12, 13}
\]

where \( g_{\alpha \beta} = \partial_\alpha X^I \partial_\beta X^J g_{IJ} \) is the induced metric on the membrane.

There is a remarkable connection between supermembrane theory in light-front gauge and Matrix theory [28, 29, 30, 31]. In light-front gauge, with coordinates \( \xi^a = (t, \sigma^a) \) on the membrane worldvolume, the dynamics may be expressed in terms of a Poisson bracket \( \{f, g\} \equiv \epsilon^{ab} \partial_a f \partial_b g \). The fields \( X^+, X^- \) are constrained by

\[
X^+ = t \\
\dot{X}^- = \frac{1}{2} \dot{X}^i \dot{X}^i + \frac{1}{N^2} \{X^i, X^j\} \{X^i, X^j\} \\
\partial_a X^- = \dot{X}^i \partial_a X^i
\]
hence the integrated sources can be expressed as

\[
\int dx d^9 x_\perp T^{IJ} = T_2 N \int d^2 \sigma \left( \frac{1}{2} \dot{X}^I \dot{X}^J - \frac{2}{N^2} \gamma^{ab} \partial_a X^I \partial_b X^J \right) \tag{14}
\]

\[
\int dx d^9 x_\perp J^{IJK} = T_2 \int d^2 \sigma \left( \frac{1}{3!} \epsilon^{\alpha \beta \gamma} \partial_\alpha X^I \partial_\beta X^J \partial_\gamma X^K \right) \tag{15}
\]

where \( \gamma_{ab} \equiv g_{ab} \). Expressions in terms of the light-front membrane fields can be translated into their counterparts in Matrix theory by replacing Poisson brackets with matrix commutators, and integrals with traces:

\[
\{ \cdot, \cdot \} \leftrightarrow -iN^2 \left[ \cdot, \cdot \right] N^4 \pi \int d^2 \sigma \leftrightarrow \text{Tr} . \tag{16}
\]

Using this correspondence and the constraints on \( X^+ \) and \( X^- \) it is straightforward to show that the expressions (14) and (15) agree with the Matrix theory definitions (7) and (8), and that the 6-form current (9) vanishes; the identity \( \{ X^i, X^k \} \{ X^j, X^k \} = \gamma^{ab} \partial_a X^i \partial_b X^j \) is useful in proving these relations.

There is only one subtlety in this correspondence. Taking the membrane expression (15) for the component \( J^{-ij} \) and translating into Matrix theory language through (16) we find that the resulting current is not precisely equal to (8). We find that the Matrix theory current \( J^{-ij} \) contains an extra term of the form

\[
\frac{1}{6R} \text{STr} \left( \frac{1}{2} F^{\epsilon i j} F^{kl} F^{\epsilon kl} + F^{ik} F^{kl} F^{lij} \right) .
\]

Translating this back into membrane language, we see that it vanishes identically. Thus, this term in the Matrix theory expression is unaffected by a single membrane source. We have not found a physical interpretation for this term.

One final note of interest with regard to the membrane is that the correspondence (16) is only exact in the large-\( N \) limit. Thus, the Matrix theory stress tensor and currents are only identified with the membrane stress tensor and currents up to \( 1/N \) corrections. We will return to this issue in the final section.

4.3 Longitudinal 5-branes

To complete our comparison of the Matrix theory expressions for the stress tensor and source terms with supergravity, we consider the only other extended object which has been found in Matrix theory: the longitudinal 5-brane. Flat L5-branes were found in [32, 26], where it was shown that the L5-brane charge is proportional to \( \mathcal{M}_{-ijkl} \). Finite size L5-branes with the transverse geometry of a 4-sphere were constructed in [33]. We will now discuss the matrix expressions for the stress tensor and the sources for an L5-brane sphere of radius \( r \); to get the results for a flat L5-brane one can take \( r \rightarrow \infty \) and focus on a region of the sphere which is locally flat.

We assume that our L5-brane sphere is extended in directions 1 − 5 and has radius \( r \), with \( \dot{r} = 0 \) for simplicity. For \( n \) superimposed L5-brane spheres we have \( N = (n + 1)(n + \)
Using the explicit expressions for this object from (33), it is easy to calculate the various tensors. As for all Matrix theory objects, we have $T^{++} = N/R$ and $T^{+-} = E$ where $E$ is the light-front (Matrix theory) energy. The only other nonzero components of the stress tensor are $T^{ij}$ and $T^{--}$. The first of these is equal to $-4E\delta^{ij}/5$ in the five directions in which the brane is extended. The component $T^{--}$ vanishes at leading order in $1/N$, as discussed in (33). This is in accord with the fact that a static 0-brane (graviton) should feel no force from a 4-brane (L5-brane). However, there are contributions at lower order which make this term non-vanishing at finite $N$. All components of $J$ vanish, as one would expect since the L5-brane should not be an electric source for the 3-form field. Finally, although the net L5-brane charge is zero and the trace giving the source term $M^{+-ijkl}$ vanishes, the higher moments of this term (discussed in the following section) are non-vanishing, and can be seen to give the correct values (up to $1/N$ corrections) from the fact that the L5-brane charge locally has the correct values, as discussed in (33).

Thus, we see that the stress tensor and currents defined in (7), (8) and (9) agree with the values we expect from the correspondence between the known Matrix theory objects and their supergravity counterparts. It is interesting that we have seen no sign of other components of the 6-form current $M$. This is presumably related to the absence of transverse 5-brane charge in the superalgebra (26) and is a reflection of the well-known difficulty of constructing a transverse 5-brane in Matrix theory.

5 Higher order terms

In the supergravity potential between two extended objects, there are terms arising from single particle exchange which appear at higher orders in $1/r$. These terms arise from higher moments of the stress-energy and 3-form source tensors of the two objects. In this section, we show that these terms are precisely replicated by an infinite series of terms in the one-loop Matrix theory potential. This result was shown in the special case of the membrane-graviton potential in (18); here we generalize to the case of two extended objects and general source terms. The terms we consider in the one-loop Matrix theory potential were calculated using the quasi-static approximation. It is known (34) that the results of such a calculation can differ from those calculated using the phase-shift method; however, the quasi-static approximation should be correct for the terms of the form $F^4X^n$ which we consider here.

In the quasi-static approach to calculating the matrix theory potential described in (25), there is an infinite series of terms which are of order $F^4$ but which have extra factors of $K/r$ inserted into the trace. This series of terms was calculated in the membrane-graviton case in (18). To generalize to the case of two extended objects we will use the convention that $\hat{X}$ and $\tilde{X}$ describe the two objects in their respective centers of mass at time $t$, while $r_i$ denotes the separation at this time. The full series of terms contributing to the potential at order...
can then be expressed as

\[ V_{\text{matrix}} = \sum_{n=0}^{\infty} \sum_{k \leq n/2} \frac{(-1)^{k+1} (5 + 2n - 2k)!}{3 \cdot 2^{2k} k! (n - 2k)! \cdot 2^{n-2k}} \text{STr} \left( (r \cdot K)^{n-2k} K^{2k} \mathcal{F} \right). \]  

(17)

The trace is symmetrized with respect to the matrices \( r \cdot K, K^2 \) and \( F \). Just as the leading term was separated in (3) into terms having different forms of Lorentz index contraction between the two traces, the same decomposition can be performed in all the subleading terms. For example, at the first subleading order we have

\[ -\frac{35}{128} \frac{1}{r^9} \text{STr} \left[ (r \cdot \hat{X} - r \cdot \hat{X}^T) \mathcal{F} \right] = \frac{245 R^2}{6} \frac{1}{r^9} \left( \hat{T}^{+-(i)} \hat{T}^{+-} - \hat{T}^{-+(i)} \right) + \ldots \]  

(18)

where we define the moments of \( \hat{T}^{+-} \) through

\[ \hat{T}^{+-(i_1i_2\ldots i_n)} = \frac{1}{R} \text{STr} \left[ \left( \frac{1}{2} \hat{X}^i \hat{X}^i + \frac{1}{4} F^{ij} F^{ij} \right) X^{i_1} X^{i_2} \cdots X^{i_n} \right]. \]  

(19)

The moments of all of the components of the stress tensor and currents can be defined in an analogous fashion, and all the symmetrized moments of \( \text{STr} \mathcal{F} \) can be shown to decompose in an analogous fashion to (3). Note that there is a subtlety here when a pair of indices in the moment are contracted; in this case, the trace is symmetrized with respect to the square \( X^2 \) so that the separate \( X \)'s cannot be separated, as in (17). This unusual property of the moments is presumably related to the effects of noncommutative geometry at finite \( N \) discussed in the final section.

In order to compare this matrix theory calculation with supergravity, we need to determine the effects of higher moments in the sources associated with \( \mathcal{T}, \mathcal{J} \) and \( \mathcal{M} \). Generalizing the argument of [18] to two-body systems, we have for the stress tensor

\[ V_{\text{gravity}} = \sum_{m \leq n=0}^{\infty} -\frac{15 R^2}{4} \frac{1}{r^9} \left[ \frac{(-1)^{n-m}}{(n-m)!m!} \hat{T}^{IJ(i_1i_2\ldots i_{n-m})} \left( \eta_{IK} \eta_{JL} - \frac{1}{9} \eta_{IJK} \eta_{KL} \right) \hat{T}^{KL(j_1j_2\ldots j_m)} \right] \]  

\[ \times \partial_{i_1} \partial_{i_2} \cdots \partial_{i_{n-m}} \partial_{j_1} \partial_{j_2} \cdots \partial_{j_m} \left( \frac{1}{r^9} \right) \]  

(20)

where the moments of the stress tensor in the supergravity theory are defined through

\[ \hat{T}^{IJ(i_1i_2\ldots i_n)} \equiv \int dx^- d^9 x_\perp \left( T^{IJ}(x) x^{i_1} x^{i_2} \cdots x^{i_n} \right). \]

It is a straightforward exercise in combinatorics to verify that (20) reproduces the terms in (17) associated with higher moments of the terms in (3). The proof of the analogous statement for the higher moments of the source terms in (3) and (3) follows in exactly the same fashion.

Thus, we have shown that all terms in the supergravity potential arising from the propagation of a single graviton or 3-form quantum between a pair of extended objects are precisely matched by terms in the one-loop matrix theory potential of the form \( F^4 X^n \), where the terms with an insertion of \( X^n \) correspond to higher moments of the source tensors associated with the extended objects.
6 Finite $N$ and the equivalence principle

As was shown in section 2, the Matrix theory potential can be written in a form (3) which is highly reminiscent of supergravity. The sources $T$, $J$ and $M$ which appear in the potential are traces of polynomials in the Matrix theory variables; if we take these quantities to be the definitions of the stress tensor and the 3-form and 6-form currents then the Matrix potential is formally identical to the supergravity potential.

In the large-$N$ limit we showed that this definition is sensible; these quantities agree with the corresponding supergravity expressions. But the Matrix expressions for the sources are perfectly well-defined at finite $N$. Is finite-$N$ Matrix theory also related to supergravity? A number of recent arguments \[35, 36, 37, 38\] in support of the DLCQ conjecture \[39\] show that finite-$N$ Matrix theory can be identified with DLCQ M-theory. The question is then whether DLCQ M-theory is described at low energies by DLCQ supergravity. But these two theories may well be different \[40, 27\], and indeed a number of explicit calculations point to discrepancies \[41, 21, 42, 43, 44\].

We now wish to point out that finite-$N$ Matrix theory, unlike supergravity, does not obey the equivalence principle. This may be seen in the context of a simple example: consider the interaction of two Matrix theory objects, a graviton initially located at the origin with momentum

$$\tilde{p}^+ = \tilde{N}/R \quad \tilde{p}^i = 0 \quad \tilde{p}^- = 0$$

(described semiclassically by $\tilde{X}^i = 0_{N \times \tilde{N}}$), and a spherical membrane \[25\] initially located at transverse position $x^i$ with momentum

$$p^+ = N/R \quad p^i = 0 \quad p^- = \frac{8r_0^4}{RN^2c_2}.$$

The membrane is described semiclassically by

$$X^i = \begin{cases} x^i \mathbb{1}_{N \times N} + \frac{2}{N} r_0 J^i & i = 1, 2, 3 \\ x^i \mathbb{1}_{N \times N} & \text{otherwise} \end{cases}$$

where $r_0$ is the maximal radius of the sphere, $J^i$ are generators of the $N$-dimensional representation of $SU(2)$, and $c_2 = \frac{N^2-1}{4}$ is the corresponding quadratic Casimir. We take $\tilde{N} \gg N$, so that the graviton acts as a static gravitational source, and study the resulting motion of the membrane.

The initial velocity of the membrane is

$$\dot{x}^+ = 1 \quad \dot{x}^i = 0 \quad \dot{x}^- = \frac{\dot{p}^-}{p^+} = \frac{8r_0^4 c_2}{N^4}.$$

Consider changing $r_0$ while preserving the initial velocity. That is, consider scaling $r_0$ with $N$ according to $r_0^4 \sim \frac{N^2}{c_2}$. The equivalence principle predicts that spheres with different radii but identical initial velocities should fall at the same rate in a uniform gravitational field: the
initial acceleration of the sphere should be independent of its mass, \textit{i.e.} of $r_0$ and therefore $N$.

But this is not the case. Using the one-loop Matrix potential one finds that for large separation the initial acceleration is

$$\ddot{x}^i = -\frac{R}{N} \frac{\partial V_{\text{matrix}}}{\partial x^i} = -1680 R \tilde{N} \frac{x^i}{|x|^9} \frac{r_0^8}{N^8} \left( \frac{c_2^2}{3} - \frac{1}{3} c_2 \right).$$

For large $N$, with the above scaling, the acceleration is indeed independent of the mass. But there are subleading terms of order $1/N^2$ which violate the equivalence principle.

3The expressions for $\dot{x}^i$ and $\ddot{x}^i$ receive quantum corrections, but these corrections are suppressed by a factor of $\frac{\tilde{N}^3}{N r_d^3}$ relative to the leading term and hence cannot resolve this violation.

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