An Intuitive Derivation of the Coherence Index Relation in Compressive Sensing

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Abstract—The existence and uniqueness conditions are a prerequisite for reliable reconstruction of sparse signals from reduced sets of measurements within the Compressive Sensing (CS) paradigm. However, despite their underpinning role for practical applications, existing uniqueness relations are either computationally prohibitive to implement (Restricted Isometry Property), or involve mathematical tools that are beyond the standard background of engineering graduates (Coherence Index). This can introduce conceptual and computational obstacles in the development of engineering intuition, design of suboptimal practical solutions, or understanding of limitations. To this end, we introduce a simple but rigorous derivation of the coherence index condition, based on standard linear algebra, with the aim to empower signal processing practitioners with intuition in the design and ease in implementation of CS systems. Given that the coherence index is one of very few CS metrics that admits mathematically tractable and computationally feasible calculation, it is our hope that this work will help bridge the gap between the theory and applications of compressive sensing.

I. INTRODUCTION AND BASIC CS SETTING

Compressive Sensing (CS) is a maturing field which, under appropriate conditions, provides a rigorous framework for efficient data acquisition. Examples include the recovery of sparse signals from vastly reduced sets of measurements, applications which rest upon reliable sensing from the lowest possible number of measurements, and practical solutions when some measurements are physically unavailable or heavily corrupted by disturbance.

Research under an overarching umbrella of sparsity has been a major topic of investigation for about a quarter of century, and has produced solid theory to support the exact reconstruction of sparse signals in compressive sensing scenarios. Among the uniqueness tools in CS, the coherence index is of particular interest for practitioners as it is one of the few supporting theoretical tools in compressive sensing which can be calculated in a computationally feasible way, and is the focus of this work. However, its derivation follows a rather complex and convoluted path which is beyond the standard background of an engineering graduate; this spurred us to revisit the coherence index from a signal processing perspective, in order to equip practitioners with intuition in the design and ease of interpretation in the analysis. We also provide an intuition behind the feasibility of the calculation of the Restricted Isometry Property (RIP) condition.

A. Definitions and Notation

Definition 1: A sequence \( \{X(k)\} \), \( k = 0, 1, \ldots, N-1 \) is referred to as a sparse sequence, if the number \( K \) of its nonzero elements, \( X(k) \neq 0 \), is much smaller than its total length, \( N \), that is,

\[
X(k) \neq 0 \text{ for } k \in \{k_1, k_2, \ldots, k_K\}, \ K \ll N.
\]

Definition 2: A linear combination of elements of \( X(k) \), given by

\[
y(m) = \sum_{k=0}^{N-1} a_m(k)X(k),
\]

is called a measurement, with the weighting coefficients (weights) denoted by \( a_m(k) \).

The above sensing scheme produces measurements, \( y(m) \), \( m = 0, 1, \ldots, M-1 \), and admits a vector/matrix form given by

\[
y = AX,
\]

where \( y = \{y(m)\} \) is an \( M \times 1 \) column vector of the measurements, \( A \) is an \( M \times N \) measurement matrix which comprises the weights \( a_m(k) \) as its elements, and \( X \) is an \( N \times 1 \) sparse column vector with elements \( X(k) \). An illustration of the CS concept is given in Fig. 1.

Without loss of generality, we shall assume that the measurement matrix, \( A \), is normalized, so that the energy of its columns sums up to unity. Consequently, the diagonal elements of its symmetric Gram form, \( A^H A \), are equal to 1, where \( A^H \) is the complex conjugate transpose of \( A \).

While compressive sensing theory states that, under certain mild conditions, it is possible to reconstruct a sparse \( N \)-dimensional vector, \( X \), from a reduced \( M \)-dimensional set of measurements, \( y \), for the applications of CS to become more widespread, the practitioners require physically meaningful, intuitive, and easily interpretable uniqueness tools - a subject of this work.

II. A SOLUTION TO THE CS PARADIGM

Several approaches have been established for the CS paradigm, and we here follow the principles behind the matching-pursuit approach. The simplest case, when the positions of nonzero elements in \( X \) are known, is considered first to provide both the intuition and an example for the detection of unknown positions of the nonzero elements in \( X \). This then serves as a basis for our derivation of the uniqueness relation through coherence index, for a general case of unknown positions of the nonzero elements in \( X \).
In our case $N > M$, the original $N$-dimensional vector $X$ cannot, in general, be recovered from the measurements, $y$ (lack of degrees of freedom). However, the CS paradigm allows for the complete and unique recovery of sparse signals, with the coherence index being a common way to define the corresponding unique recovery conditions. The same $A$ is used in Fig. 2.

A. Known Coefficient Positions

Consider the case with $K$ nonzero elements of $X$ located at arbitrary but known positions, that is, $X(k) \neq 0$ for $k \in \{k_1, k_2, \ldots, k_K\}$. Compared to the general form of the measurement relations in (2), this gives rise to the following reduced system of equations

$$
\begin{bmatrix}
y(0) \\
y(1) \\
y(M-1)
\end{bmatrix} =
\begin{bmatrix}
a_0(k_1) & \ldots & a_0(k_K) \\
a_1(k_1) & \ldots & a_1(k_K) \\
a_{M-1}(k_1) & \ldots & a_{M-1}(k_K)
\end{bmatrix}
\begin{bmatrix}
X(k_1) \\
X(k_2) \\
X(k_K)
\end{bmatrix}.
$$

For a successful CS recovery, the matrix form of the above system, given by

$$
y = A_{MK} X_K,
$$

need to be solved for the nonzero elements $X(k)$, located at $k \in \{k_1, k_2, \ldots, k_K\}$, which are here conveniently grouped into a $K \times 1$ vector $X_K$. Observe that the matrix $A_{MK}$ is an $M \times K$ dimensional sub-matrix of the full $M \times N$ measurement matrix $A$ in (2), whereby only the columns that correspond to the positions of the nonzero elements in $X$ are kept (for illustration, see column 2 and column 7 in Fig. 1). The smallest number of measurements needed to recover the $K$-element vector $X_K$ is therefore $M = K < N$. For $M > K$, as in Fig. 1 the system is overdetermined and the solution is found in the Least Squares (LS) sense, to give

$$
X_K = (A_{MK}^H A_{MK})^{-1} A_{MK}^H y = \text{pinv}(A_{MK}) y,
$$

where $\text{pinv}(A_{MK}) = (A_{MK}^H A_{MK})^{-1} A_{MK}^H$ denotes the pseudo-inverse of the matrix $A_{MK}$, while the matrix $A_{MK}^H A_{MK}$ is referred to as the $K \times K$ dimensional Gram matrix of $A_{MK}$.

From (4), the existence of a recovery solution requires that the inverse $(A_{MK}^H A_{MK})^{-1}$ does exist, or in other words that $\text{rank}(A_{MK}^H A_{MK}) = K$. This requirement can be equally expressed via the matrix condition number, as $\text{cond}(A_{MK}^H A_{MK}) < \infty$, which casts the existence condition into a more convenient form dictated by the eigenvalue spread of $A_{MK}^H A_{MK}$. For noisy measurements, the reconstruction error comprises contributions from both input noise and the ill-posedness due to a high matrix condition number (“mathematical artefact”). In this sense, the goal of a successful compressive sensing (sampling) strategy can be interpreted as that of forming the measurement matrix, $A$, in such a way to produce the condition number as close to 1 as possible.

B. Signal Processing (DFT) Framework for CS

The most fertile domains to account for signal sparsity are common linear signal transforms, as these allow for original time-domain samples of the signal $x(n)$ to be considered as measurements (linear combinations) of the representation domain coefficients, $X(k)$. For example, when the Discrete Fourier Transform (DFT) is used as the signal sparsity domain, the measured signal samples can be expressed as

$$
y(m) = x(n_m) = \frac{1}{\sqrt{M}} \sum_{k=0}^{N-1} e^{j2\pi n_m k/N} X(k),
$$

for $m = 0, 1, \ldots, M - 1$. In this case, the measurements, $y(m)$, can be regarded as a reduced set of signal samples, $x(n_m)$. Observe that this expression is conformal with the general CS formulation (1), where the weights $a_m(k) = \exp(j2\pi n_m k/N)/\sqrt{M}$. Since CS employs a random subset of time instants $\{n_1, n_2, \ldots, n_M\} \subset \{0, 1, 2, \ldots, N - 1\}$, the measurement matrix $A$ is obtained from the inverse DFT transform matrix, whereby only the rows corresponding to $\{n_1, n_2, \ldots, n_M\}$ are kept. The matrix $A$, obtained in such a way, is called a partial DFT measurement matrix.

Remark 1: The convenience of the considered DFT representation allows us to consider the Nyquist sub-sampling as a special case of a reduced set of measurements, with the specific positions of $K$ nonzero elements given by

$$
X = [X(0), X(1), \ldots, X(K-1), 0, 0, \ldots, 0]^T.
$$

The classic sub-sampling operation can then be explained starting from the assumption that only the first $K = N/P$ elements are nonzero, with $P > 1$ an integer. Then, the original signal can be sub-sampled at $n_m = mP$, since this yields $a_m(k) = \exp(j2\pi m k/K)/\sqrt{K}$ with $A_{MK}^H A_{MK}$ being an identity matrix.

Some important applications, such as radar signal processing, routinely deal with a very small number of nonzero elements, $X(k)$. However, in contrast to the classic sub-sampling scenario these nonzero transform-domain elements may be located at any position $k \in \{0, 1, \ldots, N - 1\}$. While this makes it impossible to perform classical sub-sampling, this class of applications admits a unique solution with a reduced number of signal samples, within the CS theory framework.

Despite obvious methodological advantages of CS over classical analyses, operation on a reduced set of measurements compromises the uniqueness of the CS solution. It is therefore natural to first examine which conditions should be satisfied.
by the numbers (both their number and properties of the measurement matrix) so that the existence of the CS solution is guaranteed and the solution is unique.

**Example 1:** In order to illustrate how a reduced set of samples can compromise uniqueness of the solution, consider a signal with the total length of \( N \) and the simplest sparsity degree of \( K = 1 \) in the DFT domain, that is, with only one nonzero element in \( \mathbf{X} \). Assume that a significant number of \( M = N/2 \) measurements (signal samples) are available at \( n = 0, 2, \ldots, N - 4, N - 2 \), with an even \( N \), and assume that the measurement values are \( y(m) = x(2m) = 1 \). The solution to this simple problem is then not unique, since \( x(n) = \exp(j2\pi nk_1/N), \ n = 0, 1, \ldots, N-1 \), for both \( k_1 = 0 \) and \( k_1 = N/2 \), satisfies all problem conditions.

**Uniqueness of the CS paradigm:** In general, the question of uniqueness can be considered within the following framework. Consider a \( K \)-sparse vector \( \mathbf{X} \), with the nonzero elements \( X(k) \neq 0 \) at \( k \in \{k_1, k_2, \ldots, k_K\} \), and assume that its vector form, \( \mathbf{X}_K \), is a solution to \( \mathbf{y} = \mathbf{A}_M \mathbf{X}_K \). Assume also that the solution is not unique, so that there exists another vector, \( \mathbf{X}' \), with nonzero elements at different positions \( k \in \{k_{K+1}, k_{K+2}, \ldots, k_{2K}\} \), whose reduced form \( \mathbf{X}'_K \) supports the same measurements \( \mathbf{y} \), that is, \( \mathbf{y} = \mathbf{A}_M \mathbf{X}'_K \). Then, \( \mathbf{A}_M \mathbf{X}_K - \mathbf{A}_M \mathbf{X}'_K = 0 \), and this matrix equation can be combined into \( \mathbf{A}_M \mathbf{X}_2K = 0 \), where \( \mathbf{A}_M \mathbf{X}_2K \) is a \( 2K \times 2K \) dimensional sub-matrix of the measurement matrix \( \mathbf{A} \) and \( \mathbf{X}_2K \) is a \( 2K \)-dimensional vector. Then,

- A nontrivial solution of the matrix equation \( \mathbf{A}_M \mathbf{X}_2K = 0 \) indicates that the CS solution is nonunique. The condition for a nonunique solution is therefore \( \text{rank}(\mathbf{A}_M \mathbf{X}_2K) < 2K \), for at least one combination of \( 2K \) nonzero element positions.
- If \( \text{rank}(\mathbf{A}_M \mathbf{X}_2K) = 2K \), for all possible combinations of \( 2K \) nonzero element positions (out of \( N \)), the scenario of two \( K \)-sparse solutions is not possible and the solution is unique.

The above rationale is a starting point for the definition of common uniqueness criteria in CS. The coherence index relation (whose simple derivation and interpretation is the subject of this paper) is typically derived through the Gershgorin disk theorem.

The maximum robustness of the condition \( \text{rank}(\mathbf{A}_M \mathbf{X}_2K) = 2K \) is achieved if \( \text{cond}(\mathbf{A}_M \mathbf{X}_2K) \) is close to unity. It should be mentioned that the RIP is typically not known. Then, a natural approach to solve the CS problem would be to adopt a two-step strategy as follows:

**Step 1:** Detect the positions of nonzero elements,

**Step 2:** Apply an algorithm for reconstruction with known nonzero element positions.

An intuition for the estimate of the positions of nonzero elements in Step 1 comes from the linear nature of measurements, \( y(m) \), obtained as linear combinations of the sparsity domain elements, \( X(k) \), and the corresponding rows of the measurement matrix, \( \mathbf{A} \).

**Remark 2:** The linearity of the CS paradigm in (2), admits a back-projection of the measurements, \( \mathbf{y} \), to the measurement matrix, \( \mathbf{A} \), defined by

\[
\mathbf{X}_0 = \mathbf{A}^H \mathbf{y} = \mathbf{A}^H \mathbf{A} \mathbf{X}
\]

(6) to be used to estimate the positions of nonzero elements in \( \mathbf{X} \). In an ideal case, the matrix \( \mathbf{A}^H \mathbf{A} \) should ensure that the initial estimate, \( \mathbf{X}_0 \), contains exactly \( K \) elements at positions \( \{k_1, k_2, \ldots, k_K\} \) for which the magnitudes are larger than the biggest magnitude at the remaining positions. Then, by taking the positions of these highest magnitude elements in \( \mathbf{X}_0 \) as the set \( \{k_1, k_2, \ldots, k_K\} \) in (3), the algorithm for the known nonzero element positions, from the previous section, can be applied to reconstruct the signal.

**Remark 3:** Note that if \( \mathbf{A}^H \mathbf{A} \) were an identity matrix, the initial estimate, \( \mathbf{X}_0 \), would correspond to the exact solution, \( \mathbf{X} \). However, with a reduced number of measurements, \( M < N \), this cannot be achieved (due to the Welch lower bound). A pragmatic requirement for the existence of the CS solution would therefore be that the off-diagonal elements of \( \mathbf{A}^H \mathbf{A} \) are as small as possible compared to its unit diagonal elements.

The condition that all \( K \) elements in the initial estimate, \( \mathbf{X}_0 \), which are located at the nonzero element positions in the original sparse vector, \( \mathbf{X} \), are larger than any other element in the initial estimate \( \mathbf{X}_0 \) can be relaxed through an iterative procedure. To be able to find the position \( k_1 \) of the largest nonzero element in \( \mathbf{X}_0 \), its value must be larger than the values \( X_0(k) \) at the original zero-valued element positions. After the largest element position is found and its value is estimated this component can be reconstructed and subtracted from measurements, \( \mathbf{y} \), and the procedure is continued with the remaining \((K-1)\)-sparse elements, in an iterative manner. The stopping criterion then becomes that \( \mathbf{A}_M \mathbf{X}_K = \mathbf{y} \) should hold for the estimated nonzero positions \( \{k_1, k_2, \ldots, k_K\} \) and elements \( X(k) \), as outlined in Algorithm 1.

### III. Unique Reconstruction Condition

The key criterion for signal reconstruction with a reduced set of measurements is the uniqueness of the result. In CS methodology, the uniqueness is commonly defined through the coherence index, as stated below.

**Proposition:** The reconstruction of a \( K \)-sparse signal, \( \mathbf{X} \), is unique if the coherence index, \( \mu \), of the measurement matrix, \( \mathbf{A} \), satisfies (2)

\[
K < \frac{1}{2} \left( 1 + \frac{1}{\mu} \right)
\]

(7)
SIDEBAR 1: DIRECT SEARCH, UNIQUENESS OF THE SOLUTION, AND RESTRICTED ISOMETRY PROPERTY

**Direct Search Solution:** The simplest and intuitive way to extend the solution to CS with unknown positions in $\mathbf{X}$, as in (3), to the general case with $K$ unknown nonzero positions $k_1, k_2, \ldots, k_K$ would be to consider all possible combinations of $K$ nonzero positions (out of $N$) and to solve the system in (3) for each combination dictated by the sequence $k_1, k_2, \ldots, k_K$. After solving this system in the LS sense, the solution should be checked against the given measurement equation (3). A zero error indicates a successful solution of the reconstruction problem. The solution is therefore unique if only one combination of coefficients $k_1, k_2, \ldots, k_K$ produces zero error.

A practical problem with this kind of reconstruction is that it requires the evaluation of $\binom{K}{K}$ combinations, a non-deterministic polynomial-time (NP hard) problem with unaffordable computational burden. For example, for $N = 1024$ and $K = 5$, we would require $10^{13}$ combinations, while for each combination the corresponding overdetermined system (3) needs to be solved.

**Restricted Isometry Property (RIP) Calculation:** A very popular approach to check, in theory, for the uniqueness of the solution is based on the restricted isometry property of the measurement matrix. This can be formalized by stating that a $K$-sparse solution is unique if the measurement matrix $\mathbf{A}$ satisfies the RIP condition with a constant $0 \leq \delta_{2K} < 1$ for a $2K$-sparse signal, whereby in order to obtain the RIP constant, a check of a $2K$ sparse signal requires $\binom{N}{2K}$ combinations of the sub-matrices, $\mathbf{A}_{2K}$, and their eigenvalue decomposition. Therefore, this is an even more complex problem than the direct search solution, which makes it prohibitive for practical use. For example, for a vector with $N = 1024$ entries out of which $K = 5$ nonzero coefficients are considered, the RIP check requires $10^{23}$ combinations and matrix calculations. To put this into perspective, for a given measurement matrix, the RIP is checked only once; however, if the same matrix is used in not more than $10^{10}$ experiments, the direct combinatorial search is still more efficient, although it is equally computationally prohibitive and often not feasible in practice.

**Complexity of Coherence Index Calculation:** The coherence index check requires only $\binom{K}{K}$ combinations, which is both computationally feasible and sparsity-invariant. This makes it a preferred choice in the optimization of measurement matrices.

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**Algorithm 1.** Matching Pursuit Based Reconstruction

**Input:**
- Measurement vector $\mathbf{y}$
- Measurement matrix $\mathbf{A}$
- Required precision $\varepsilon$

**Output:**
- Reconstructed signal elements $\mathbf{X}$

```
1: $\mathcal{K} \leftarrow \emptyset$
2: $\mathbf{e} \leftarrow \mathbf{y}$
3: while $\|\mathbf{e}\|_2 > \varepsilon$ do
4:    $k \leftarrow$ position of the highest value in $\mathbf{A}^H \mathbf{e}$
5:    $\mathcal{K} \leftarrow \mathcal{K} \cup k$
6:    $\mathbf{A}_K \leftarrow$ columns of matrix $\mathbf{A}$ selected by set $\mathcal{K}$
7:    $\mathbf{X}_K \leftarrow \text{pinv}(\mathbf{A}_K)\mathbf{y}$
8:    $\mathbf{y}_K \leftarrow \mathbf{A}_K \mathbf{X}_K$
9:    $\mathbf{e} \leftarrow \mathbf{y} - \mathbf{y}_K$
10: end while
11: $\mathbf{X} \leftarrow \begin{cases} 0, & \text{for the positions not in } \mathcal{K}, \\ \mathbf{X}_K, & \text{for the positions in } \mathcal{K}. \end{cases}$
```

where for a normalized measurement matrix, $\mathbf{A}$, the coherence index is equal to the maximum absolute off-diagonal element of $\mathbf{A}^H \mathbf{A}$.

The above proposition is usually proven based on the Gershgorin disk theorem [9], a topic not covered in engineering curricula; this is an obstacle which prevents both more widespread engagement of engineers in the CS field and success of eventual applications. To this end, we shall now derive the reconstruction condition in a self-contained and intuitive way which does not require advanced mathematics.

As an example, consider a $7 \times 14$ measurement matrix $\mathbf{A}$; for clarity we employ the so called equiangular tight frame (ETF) matrix, for which the absolute value of the off-diagonal elements of $\mathbf{A}^H \mathbf{A}$ is constant and equal to the coherence index, $\mu$. The matrix $\mathbf{A}^H \mathbf{A}$ is visualized in Fig. 1 (left). Its diagonal elements are, by definition, equal to 1, while its off-diagonal elements that have values $\pm \mu$ can be treated as the disturbances in the detection of nonzero element positions in $\mathbf{X}$. Observe that the off-diagonal column elements of this matrix represent the normalized contribution of the corresponding nonzero element in the sparse vector $\mathbf{X}$ to the cumulative disturbance.

Consider now the case with only one nonzero element in $\mathbf{X}$, say at position $k_1 = 2$, with its initial estimate, $\mathbf{X}_0$, designated by the dark blue bar-plot in Fig. 2 (top-middle). Therefore, for only one nonzero element in $\mathbf{X}$, the condition for the correct detection of its position in the estimate $\mathbf{X}_0$ would be that the maximum possible disturbance value, $\mu$, is smaller than the value of the diagonal, which is in this case $\mu < 1$.

Assume next that the signal sparsity index is $K = 2$, with nonzero elements in $\mathbf{X}$ at $X(2) = 1$ and $X(7) = 1$, as indicated in Fig. 2 by the "yellow" and "blue" column in matrix $\mathbf{A}^H \mathbf{A}$. The set of $M = 7$ measurements, $\mathbf{y} = \mathbf{A} \mathbf{X}$, is then used to calculate the initial estimate in (6) as $\mathbf{X}_0 = \mathbf{A}^H \mathbf{y} = \mathbf{A}^H \mathbf{A} \mathbf{X}$, as shown in the stacked bar-plot in Fig. 2 (top-right). More specifically, the multiplication of $\mathbf{A}^H \mathbf{A}$ by $\mathbf{X}$ results in two components in the initial estimate:

(i) the component whose values are shown in dark blue bars represents the contribution of $X(2)$ to the initial estimate
(ii) the component designated by yellow bars represents the corresponding contribution of \(X(7)\).

Remark 4: Compared to the case with sparsity \(K = 1\), in the case when the sparsity degree is \(K = 2\) we can observe two differences:

1. The disturbances arising from each nonzero element in \(X\) combine, so that the maximum possible disturbance is increased to \(2\mu\).
2. The unit value of the original nonzero element in \(X\) is also affected by the disturbing values of the other nonzero element in \(X\), with the maximum possible amplitude reduction of this nonzero element of \(1 - \mu\), as shown in the stacked bar plot in Fig. 2 (top-right).

Remark 5: From the absolute value of the initial estimate in Fig. 2 (bottom-right), that is, \(|\text{"blue bars" + "yellow bars"}|\), we can conclude that, for \(K = 2\), the correct nonzero element position in \(X\) will always be detected if the original unit amplitude, reduced by the maximum possible disturbance, \(\mu\), is greater than the maximum possible disturbance value, \(2\mu\), at the original zero-valued positions in \(X\). In other words, for \(K = 2\), the reconstruction condition is given by \(1 - \mu > 2\mu\). If this condition is met, then the position of the nonzero element will always be correctly detected, which is precisely the aim of uniqueness analysis.

Note also that, for rigor, we assumed the worst case scenario for the largest position detection of \(X(2) = X(7) = 1\), whereby the disturbance from the other nonzero element is the strongest possible. If \(|X(2)| > |X(7)|\), this would relax the condition for detection of the \(|X(2)|\) position, and vice versa.

More specifically, the coherence index for this ETF matrix is \(\mu = 0.2774\). In the worst case scenario, the initial estimate values at the nonzero positions \(k_1 = 2\) and \(k_2 = 7\), would be \(1 - \mu = 1 - 0.2774 = 0.7226\), which is greater than the largest possible value \(2\mu = 2 \times 0.2774 = 0.5547\) of the initial estimate at the positions where the original vector \(X\) is zero-valued, \(k \notin \{2, 7\}\). Therefore, the coherence index condition \(K < 0.5(1 + 1/\mu) = 2.3\) is satisfied for \(K = 2\). Observe also that the matrix \(A\) defined above cannot be used with \(K \geq 3\), since in that case the maximum possible disturbance of \(3\mu\) would be larger than a maximally reduced unit value, at the nonzero element position, in the initial estimate. The amount
while the remaining elements are not greater than unity, that is

disturbed by the influence of other nonzero coefficients,

possible absolute value of

where

can be written, by definition, as

malized, that is,

\[ \mu, -\mu \]

Without loss of generality, assume that the element

three unit-valued nonzero elements in \( X \) that is

disturbance, elements in \( X \) \( X \in \{2, 5, 19\} \), the condition for the correct detection of the nonzero element positions, with \( K = 3 \), is therefore \( 1 - 2\mu > 3\mu \), that is \( 1 - (K - 1)\mu > K\mu \), which straightforwardly generalizes to any \( K \).

### Sidebar 2: Proposed Derivation of the Coherence Index Relation in (7)

The formal definition of the coherence index is:

\[ \mu = \max_{k, i, k \neq i} \sum_{m=0}^{M-1} a_m(k) a_m^*(l) \]

where it is assumed that the measurement matrix is normalized, that is, \( \sum_{m=0}^{M-1} |a_m(k)|^2 = 1 \) for each \( k \).

A measurement (or signal sample) of a \( K \)-sparse signal can be written, by definition, as

\[ y(m) = \sum_{i=1}^{K} X(k_i) a_m(k_i) \]

Its initial estimate in (6), for one coefficient \( X_0(k) \), is

\[ X_0(k) = \sum_{i=1}^{K} X(k_i) \sum_{m=0}^{M-1} a_m(k) a_m^*(k_i) = \sum_{i=1}^{K} X(k_i) \mu(k, k_i) \]

where \( \mu(k, k_i) = \sum_{m=0}^{M-1} a_m(k) a_m^*(k_i) \). The maximum possible absolute value of \( \mu(k, k_i) \) is then equal to the coherence index, that is, \( \mu = \max_{k, k_i} |\mu(k, k_i)| \).

Without loss of generality, assume that the element \( X(k_1) \) at position \( k_1 \) is of unit amplitude, \( X(k_1) = 1 \), while the remaining elements are not greater than unity, \( |X(k_i)| \leq 1, i = 2, \ldots, K \). The unit value at \( k_1 \) will be disturbed by the influence of other nonzero coefficients, that is

\[ X_0(k_1) = X(k_1) + \sum_{i=2}^{K} X(k_i) \mu(k, k_i) \]

In the worst case scenario, for the detection of the coefficient at position \( k_1 \), all \( X(k_i)\mu(k, k_i) \) should assume their lowest possible value of \( -\mu \). Then,

\[ |X_0(k_1)| \geq 1 - \sum_{i=2}^{K} |X(k_i)\mu(k, k_i)| \geq 1 - (K - 1)\mu \]

and the minimum possible value of \( |X_0(k_1)| \) is therefore

\[ \min |X_0(k_1)| = 1 - (K - 1)\mu \]

For correct detection, the amplitude of \( X_0(k_1) \) should be greater than the maximum possible disturbance at positions where the original coefficients are zero-valued, \( k \neq k_i \), which is equivalent to

\[ \max_{k \neq k_i} \sum_{i=1}^{K} X(k_i)\mu(k, k_i) = K\mu \]

The condition that the coefficient position, \( k_1 \), is correctly detected then becomes

\[ \min |X_0(k_1)| \geq \max_{k \neq k_i} \sum_{i=1}^{K} X(k_i)\mu(k, k_i) \]

From the last two relations it immediately follows that

\[ 1 - (K - 1)\mu > K\mu \]

which, as desired, produces the coherence index relation given in (7).
Sidebar 3: Coherence Index: From Theory to Practice

To establish a link with common CS practice, we now statistically compare the reconstruction performance based on the following measurement matrices: the ETF, the frequently used in practice Gaussian (real-valued and complex-valued with i.i.d. real and imaginary part) matrix, and partial DFT measurement matrix, with $N = 258$ and $M = 129$.

For the Gaussian and the partial DFT matrices we chose the best measurement matrix (with smallest $\mu$) among 1000 random realizations. The corresponding values of $\mu$ were 0.3527, 0.2737, 0.0624, and 0.1484 for the Gaussian (real and complex-valued) matrix, the ETF, and the partial DFT matrix, respectively. The lowest value for the coherence index, $\mu = \sqrt{\frac{N-M}{MN(M-1)}} = 0.0624$, was obtained for the ETF, as expected from the Welch bound theory. The theoretic limits for the sparsity degree, $K$, that guarantee the reconstruction with probability 1 are $K < 0.5(1+1/\mu) = 8.5$, $K < 3.9$, $K < 2.3$, and $K < 1.9$, for the ETF, the partial DFT, and Gaussian (complex and real-valued) measurement matrices, respectively.

In the experiment, all four matrices were used to reconstruct signals for a range of sparsity degrees, $K = 1, 2, 3, \ldots, 64$. For each sparsity level, $K$, the problem was solved 100,000 times with random positions of nonzero coefficients. For each $K$, the solution was checked against the known positions and values of the nonzero elements in $X(k)$, and the number of mis-detections was recorded. The number of mis-detections for each $K$ was then divided by the total number of realizations.

The results for the mis-detection probability are shown in the Figure. For a sufficient practical probability value of mis-detection, for example that lower than $10^{-5}$, in this statistical experiment the partial DFT performed significantly better than the ETF. The Figure shows that it produced reconstruction without any error with probability 0.99999, for $K < 21$, while for the ETF this holds for $K < 15$.

This can be explained by the probability distribution of the off-diagonal elements of $A^H A$ being the worst possible in the ETF, $p(\xi) = 0.5(\delta(\xi - \mu) + \delta(\xi + \mu))$, with the variance $\sigma^2 = 2\mu^2 = 2\frac{N-M}{MN(M-1)}$. For the partial DFT matrix, the off-diagonal elements of $A^H A$ were approximately Gaussian distributed, with the variance $\sigma^2 = \frac{N-M}{MN(M-1)}$, [6], while in the Gaussian measurement matrix the off-diagonal elements were also Gaussian distributed, with $\sigma^2 = 1/M$.

The probabilistic approach can also be used to derive a relation between $K$, $N$, and $M$, for a successful reconstruction of $X$ with a given probability, as thoroughly shown in [6]. We here present a brief practical analysis.

The real and imaginary parts of the off-diagonal elements of $A^H A$ (disturbances at original zero-valued positions in $X$) combine within the initial estimate for each nonzero element in $X$. For a large sparsity degree $1 < K < N$, the resulting distribution of disturbance is approximately Gaussian $\sim N(0, K\sigma^2/2)$, where $\sigma^2$ is the variance of a single complex-valued off-diagonal element of $A^H A$. A successful reconstruction may be expected if the amplitude of disturbance (Rayleigh distributed) is below the nonzero unit-amplitude element in the initial estimate with a high probability. For example, if $K\sigma^2$ is denoted by $1/\kappa$ then the amplitude in at least one position of zero-valued elements in $X$ will be above 1, with a probability of $(N-K)e^{-\kappa} \sim Ne^{-\kappa}$. For a partial DFT matrix, assuming $\kappa = 16$ when $Ne^{-\kappa} \sim 10^{-5}$, the reconstruction is achieved with high probability for $K < \frac{1}{16}M(N-1)$.

For $N = 258$ and $M = 129$, it then follows that $K < M/8 = 16.1$, as suggested for the practical sparsity limit in [10]. This corresponds to the above statistical results, with a probability of wrong reconstruction below $10^{-5}$.

Similar practical relations can be obtained for other desired probabilities and measurement matrices, based on their variances $\sigma^2$, as given in the main text.
of this maximum reduction would be $2\mu$, thus resulting in the initial estimate value of $1 - 2\mu$ and a miss-detection of the nonzero element position in $X$.

As an example of the tightness of the coherence index condition, Fig.3 presents the CS recovery based on a $12 \times 16$ partial DFT measurement matrix $A$, with $\mu = 0.2455$, for the case of $K = 2$. Observe that, unlike in the ETF case, here the disturbing terms are not equal. However, all conclusions regarding the worst case scenario remain valid here and the nonzero element positions in $X(k)$ will be correctly detected based on the presented initial estimate, $X_0(k)$, if $1 - \mu > 2\mu$.

To provide further intuition, the illustration in Fig.2 is next repeated for a $15 \times 30$ measurement matrix $A$ and $K = 3$, with the results shown in Fig.4. Following the same reasoning as in the previous case, we can conclude from the absolute value of the initial estimate that the detection condition now becomes $1 - 2\mu > 3\mu$. The coherence index for this matrix is therefore $\mu = 0.1857$, with the corresponding coherence index condition $K < 0.5(1 + 1/\mu) = 3.2$. Notice that this matrix cannot be used for $K = 4$, since $1 - 3\mu > 4\mu$ would not hold, meaning that a disturbance would be misdected as a nonzero component.

**Remark 6:** Following the above simple and inductive approach, it becomes immediately obvious that, for a general case of a $K$-sparse $X$, the position of the largest element position in $X$ will be correctly detected in the initial estimate, $X_0$, if

$$1 - (K - 1)\mu > K\mu.$$  

The above bound directly yields the coherence index condition in [7], with the derivation obtained in a natural and practically relevant way.

**Remark 7:** After the position of the first nonzero component in a sparse $X$ is successfully detected and this component is reconstructed and removed, the same procedure and relations can be iteratively applied to the remaining “deflated” signal which now exhibits a reduced $(K - 1)$-sparsity level, thus guaranteeing a unique solution.

Sidebar 2 provides a simple analytic derivation to support the intuition behind the proposed proof of the condition for unique reconstruction, as illustrated in Fig. 2, Fig. 3 and Fig. 4.

The case with small measurement matrices of the ETF type, considered so far, fully supports the proposed derivation of the coherence index and its appropriateness in theory and practice, as it produces a tight bound on the existence and uniqueness of reconstruction. The corresponding performance for problems of large dimensions is illustrated in Sidebar 3.

In summary, the coherence index has been derived over several independent worst case scenarios, to provide a rigorous and easily interpretable bound for practical application scenarios, where the uniqueness condition is typically further relaxed. This this end, in our approach, we considered the case where:

- **Amplitudes of all nonzero components in $X$ are equal.** If this is relaxed to the general case of different values of nonzero elements in $X$, we can expect a successful, more relaxed and unique reconstructions even if the coherence index condition may be violated.

- **Only one element in the initial estimate, $X_0$, was compared to the maximum possible disturbance.** If any of the original nonzero components in the initial estimate at a position of nonzero element in $X$ is above the maximum disturbance, this would guarantee a successful reconstruction, thus further relaxing the reconstruction condition.

- **All disturbances at both the nonzero and zero-valued element positions are assumed to be in phase.** That is, they are summed up with maximum possible magnitudes. This is a very low probability event, again relaxing the practical sparsity limit for the reconstruction.

- **The distribution of the off-diagonal elements of $A^HA$ plays an important role.** While, for several nonzero elements in $X$, these combine to an approximately Gaussian distributed variable, the resulting disturbance may obey different distribution, thus yielding different results.

### IV. Conclusion

The coherence index condition for unique sparse signal reconstruction, a prerequisite for the successful compressive sensing paradigm, has been derived using simple signal processing tools and through an intuitive example. Our perspective has first demonstrated that this index provides a tight uniqueness bound for relatively small and moderate dimension of CS problems, followed by a clear interpretation of its conservative nature for large scale problems. This has been achieved for a range of high probabilities of obtaining the correct result and avoiding mis-detection. It has also been shown that general measurement matrices, like the frequently used partial DFT matrix, are likely to outperform the results based on the considered ETF based measurement matrix, if the analysis is restricted to pragmatically high probabilities of the correct solution - a practical relaxation of theoretical probability of one for avoiding mis-detection.

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