THE DUAL CHEEGER CONSTANT AND SPECTRA OF INFINITE GRAPHS

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Abstract. In this article we study the top of the spectrum of the normalized Laplace operator on infinite graphs. We introduce the dual Cheeger constant and show that it controls the top of the spectrum from above and below in a similar way as the Cheeger constant controls the bottom of the spectrum. Moreover, we show that the dual Cheeger constant at infinity can be used to characterize that the essential spectrum of the normalized Laplace operator shrinks to one point.

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1. Introduction

While global geometric properties are inherently nonlinear, they nevertheless can often be controlled by linear techniques. A prime example in geometry is the spectrum of the Laplace operator which encodes geometric information about the underlying manifold. This principle has been particularly fertile and most intensively developed in Riemannian geometry, see [Cha84] for an overview. It possesses a more general validity, however. In particular, more recently, spectral methods have been successfully explored in graph theory. In fact, it was observed that, while being very different...
from manifolds, graphs can be investigated by modifying techniques that originally have been developed in Riemannian geometry. This insight was very fruitful and led to many important results in graph theory, such as a discrete version of Courant’s nodal domain theorem \cite{DGLS01}, Sobolev and Harnack inequalities \cite{CY95, DSC96, FY94, CLY12}, heat kernel estimates \cite{Del99, CG98}, and many more. In particular, it turns out that the normalized Laplace operator on graphs is related to the Laplace-Beltrami operator for a Riemannian manifold. However, not only results from the continuous setting triggered new results in the discrete case but also results in graph theory led to new insights in geometric analysis. One example of the mutual stimulation between both fields is for example given in the work of Chung, Grigor’yan and Yau \cite{CGY96, CGY97, CGY00} where a universal approach for eigenvalue estimates on continuous and discrete spaces was developed.

Graph theory, however, also has specific aspects that are different from what occurs in other parts of geometry. Our question then is whether and how these aspects can be explored with spectral methods. For instance, a bipartite graph, i.e. a graph whose vertex set can be split into two subsets such that there are only edges between vertices belonging to different subsets, has no counterpart in Riemannian geometry. Another difference to Riemannian geometry is that for graphs, the spectrum of the normalized Laplace operator is bounded from above by two. For finite graphs, these two properties are connected by the fact that 2 is an eigenvalue of the normalized Laplace operator if and only if the graph is bipartite.

This leads us to the specific purpose of this paper which is to investigate the top of the spectrum of the normalized Laplace operator for infinite graphs. In order to do this we have developed on the one hand new techniques that have no counterpart in Riemannian geometry, and on the other hand have discovered results that are similar to important theorems in the continuous setting.

In Section \ref{section1} we introduce the Dirichlet Laplace operator and its basic properties. Let $\Gamma$ be an infinite graph and $\Omega$ a finite, connected subset of $\Gamma$. Let $\lambda_1(\Omega)$ ($\lambda_{\text{max}}(\Omega)$) be the first (largest) eigenvalue of the Dirichlet Laplace operator on $\Omega$. We then have
\begin{equation}
\lambda_1(\Omega) + \lambda_{\text{max}}(\Omega) \leq 2,
\end{equation}
and we prove in Theorem \ref{thm1} that we have the equality $\lambda_1(\Omega) + \lambda_{\text{max}}(\Omega) = 2$ if and only if $\Omega$ is bipartite. In Section \ref{section2} we introduce a geometric quantity, the so-called dual Cheeger constant $h(\Gamma)$ that roughly speaking measures how close a graph is to a bipartite one. We will show in this section that the dual Cheeger constant $\bar{h}(\Gamma)$ is closely related to the Cheeger constant $h(\Gamma)$. The Cheeger constant $h(\Gamma)$ of a infinite graph $\Gamma$ is one of the most fundamental geometric quantities which gives rise to important analytic consequences such as the first eigenvalue estimate, heat kernel estimates and so on. Also in the setting of finite graphs the Cheeger constant is related to the growth behavior of the graph since it is closely connected to the important notion
of expanders, see [Lub94] and the references therein for more details. The Cheeger constant for a finite subset $\Omega$ of the vertex set $V$ of $\Gamma$ is defined as

$$ h(\Omega) = \inf_{\emptyset \neq U \subset \Omega} \frac{|\partial U|}{\#U < \infty} \text{vol}(U), $$

where the volume of $U$ is the sum of the degrees of its vertices, and $|\partial U|$ measures the edges going from a vertex in $U$ to one outside $U$. A sequence of finite subsets of $\Gamma$ satisfying $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \cdots$ and $\Gamma = \bigcup_{n=1}^{\infty} \Omega_n$, denoted by $\Omega \uparrow \Gamma$, is called an exhaustion of $\Gamma$. Since $h(\Omega)$ is non-increasing when $\Omega$ increases, we can define the Cheeger constant as the limit for subsets exhausting $\Gamma$, i.e. $h(\Gamma) = \lim_{\Omega \uparrow \Gamma} h(\Omega)$, which does not depend on the choice of the exhaustion. The Cheeger constant tells us how difficult it is to carve out large subsets with small boundary. For a finite subset $\Omega \subset V$ we then define the dual Cheeger constant by

$$ \bar{h}(\Omega) = \sup_{V_1, V_2 \subset \Omega, \#V_1, \#V_2 < \infty} \frac{2|E(V_1, V_2)|}{\text{vol}(V_1) + \text{vol}(V_2)}, $$

where $V_1, V_2$ are two disjoint nonempty subsets of $\Omega$ and $|E(V_1, V_2)|$ counts the edges between them. The dual Cheeger constant of $\Gamma$ then is obtained as the limit for exhaustions, i.e. $\bar{h}(\Gamma) = \lim_{\Omega \uparrow \Gamma} \bar{h}(\Omega)$. The dual Cheeger constant tells us how easy it is to find large subsets with few inside connections, but many between them. Observe that a bipartite graph is one that we can divide into two subsets without any internal connections. This is the paradigm behind the dual Cheeger constant.

In Theorem 4.1, we prove

$$ h(\Omega) + \bar{h}(\Omega) \leq 1, $$

with equality $h(\Omega) + \bar{h}(\Omega) = 1$ if the graph is bipartite (the converse for equality is not true, see Example 4.1). There is an obvious analogy between (1) and (2), and their equality cases for bipartite graphs. We shall explore this analogy in this paper. In fact, the dual Cheeger constant was first introduced in [BJ] to effectively estimate the largest eigenvalue of a finite graph. In this paper, we develop this technique for the setting of the Dirichlet Laplace operator on finite subsets $\Omega$ of an infinite graph $\Gamma$. Our first main result is that the largest eigenvalue of the normalized Dirichlet Laplace operator can be controlled from above and below in terms of the dual Cheeger constant, see Theorem 4.5. Thus, we obtain inequalities relating the largest Dirichlet eigenvalue and the dual Cheeger constant $\bar{h}(\Omega)$ that are analogous to the relationship between the first Dirichlet eigenvalue and the Cheeger constant. In Theorem 4.3 we prove that for graphs without self-loops there is another relation between the dual Cheeger constant $\bar{h}(\Omega)$ and $h(\Omega)$,

$$ \frac{1}{2}(1 - h(\Omega)) \leq \bar{h}(\Omega). $$
(2) and (3) thus tell us that \( \bar{h}(\Omega) \) and \( h(\Omega) \) can be controlled by each other. These estimates will be the key tool in Section 10 where we study the essential spectrum of the normalized Laplace operator of an infinite graph. Nevertheless, the analogy between \( \bar{h}(\Omega) \) and \( h(\Omega) \) suggested by (2) and (3) has its limitations, as we shall see in Section 8 (but the analogy will make a surprising comeback in Section 10).

But let us first describe results where the analogy holds. In Section 5 we prove eigenvalue comparison theorems for the largest Dirichlet eigenvalue that are counterparts of the first eigenvalue comparison theorems on graphs by Urakawa \cite{Ura99} which are discrete versions of Cheng’s first eigenvalue comparison for Riemannian manifolds \cite{Che75a, Che75b}. The mostly used comparison models in the literature (see e.g. \cite{Ura99, Fri93}) for graphs are homogeneous trees, denoted by \( T_d \) \((d \in \mathbb{N}, d \geq 2)\). Here we propose a novel comparison model for graphs, the weighted half-line \( R_l \) \((l \in \mathbb{R}, l \geq 2)\) which is inspired by the behavior of the eigenfunctions for the first and largest eigenvalues on balls in the homogeneous tree \( T_l \). Compared to the homogeneous tree \( T_d \), the advantage of the weighted half-line \( R_l \) is that we get better estimates in the comparison theorems since, for our model space, in contrast to the parameter \( d \) for a homogeneous tree, \( l \) need not be an integer. Our main result in this section is that the largest Dirichlet eigenvalue of a ball in a graph can be controlled by that of a ball in the weighted half-line \( R_l \) of the same radius and a quantity that is related to the bipartiteness of the graph.

In Section 6 we show how the eigenvalue comparison theorems from Section 5 can be used to estimate the largest, the second largest and other largest eigenvalues of the normalized Laplace operator for finite graphs. In Section 7 we use the dual Cheeger constant \( \bar{h}(\Gamma) \) to estimate the top of the spectrum of \( \Gamma \). In Section 8 we study the dual Cheeger constant and its geometric and analytic consequences. Let us denote by \( \sigma(\Delta) \) \((\subset [0,2])\) the spectrum of the normalized Laplace operator of an infinite graph \( \Gamma \), by \( \underline{\lambda}(\Gamma) := \lim_{\Omega \uparrow \Gamma} \lambda_1(\Omega) = \inf \sigma(\Delta) \) the bottom of the spectrum of an infinite graph \( \Gamma \) and by \( \overline{\lambda}(\Gamma) := \lim_{\Omega \uparrow \Gamma} \lambda_{\max}(\Omega) = \sup \sigma(\Delta) \) its top. It is well known that the spectral gap for \( \underline{\lambda}(\Gamma) \), i.e. \( \underline{\lambda}(\Gamma) > 0 \), implies \( \Gamma \) has exponential volume growth and a fast heat kernel decay. Moreover, the spectral gap for \( \overline{\lambda}(\Gamma) \) is a rough-isometric invariant (see e.g. \cite{Woe00}). We shall derive the exponential volume growth condition for the spectral gap for \( \overline{\lambda}(\Gamma) \), i.e. \( \overline{\lambda}(\Gamma) < 2 \), if we further assume the graph \( \Gamma \) is, in some sense, close to a bipartite one. More importantly, we present some examples (see Example 8.2 and Example 8.3) to show that the spectral gap for \( \overline{\lambda} \) is not a rough-isometric invariant. Since the dual Cheeger constant \( \bar{h}(\Gamma) \) is closely related to \( \overline{\lambda}(\Gamma) \), it is follows that also \( \bar{h}(\Gamma) < 1 \) is not a rough-isometric invariant. Note that in contrast, \( \bar{h}(\Gamma) > 0 \) is a rough-isometric invariant.

In Section 9 we study the top of the spectrum of infinite graphs with certain symmetries. A graph \( \Gamma \) is called quasi-transitive if it has only finitely many orbits by the action of the group of automorphisms, \( \Aut(\Gamma) \), i.e. \( \# \Gamma/ \)
It is obvious that Cayley graphs are quasi-transitive. We prove in Theorem 9.1 that for a non-bipartite quasi-transitive graph $\Gamma$, $\bar{h}(\Gamma) \leq 1 - \delta$, where $\delta = \delta(\Gamma) > 0$. As a corollary (see Corollary 9.1), we obtain that for a quasi-transitive graph $\Gamma$, $\lambda(\Gamma) = 2$ implies that $\Gamma$ is bipartite. This is a generalization of a result for Cayley graphs in [dlHRV93]. The proof in [dlHRV93] is based on techniques from functional analysis, in particular C*-algebras. Our proof is completely different, since we use a combinatorial argument to estimate the geometric quantity $\bar{h}(\Gamma)$. From the geometric point of view this proof has the advantage that it can easily be extended to quasi-transitive graphs, which is not true for the proof technique used in [dlHRV93].

In Section 10 we investigate the essential spectrum of infinite graphs. The main result is that the dual Cheeger constant at infinity can be used to characterize that the essential spectrum shrinks to one point. Let $\sigma^{\text{ess}}(\Gamma)$ denote the essential spectrum of an infinite graph $\Gamma$. Note that the essential spectrum, which is related to the geometry at infinity, cannot be empty since the normalized Laplace operator is bounded. Fujiwara [Fuj96b] discovered (see also [Kel10]) that the Cheeger constant at infinity, denoted by $h_\infty$, is equal to 1 if and only if the essential spectrum is smallest possible (i.e. $\sigma^{\text{ess}}(\Gamma) = \{1\}$). We show the following criteria of concentration of the essential spectrum by the dual Cheeger constant at infinity, denoted by $\bar{h}_\infty$.

**Theorem 1.1** (see Theorem 10.4). Let $\Gamma$ be an infinite graph without self-loops. Then

$$\bar{h}_\infty(\Gamma) = 0 \iff h_\infty(\Gamma) = 1 \iff \sigma^{\text{ess}}(\Gamma) = \{1\}.$$ 

The typical example here is a rapidly branching tree. This result is somehow surprising since our results in Section 8 suggest that the dual Cheeger constant is weaker than the Cheeger constant. However it turns out that at infinity (in the extreme case $\bar{h}_\infty = 0$ and $h_\infty = 1$) both quantities contain the same information. This is a new phenomenon for discrete structures that has no analogue for Riemannian manifolds. For graphs with self-loops, we give an example which has $\bar{h}_\infty = h_\infty = 0$ and $\sigma^{\text{ess}} = \{0\}$. In another direction, we generalize the results about the essential spectrum in [Ura99] by the comparison method, see Theorem 10.7, 10.8 and Corollary 10.2. These results are discrete analogues of results in [Don81, DL79].

2. **Preliminaries**

Let $\Gamma = (V, E)$ denote a locally finite, connected graph with infinitely many vertices. Here $V$ denotes the vertex and $E \subset V \times V$ the edge set of $\Gamma$. In this article we study weighted graphs, i.e. we consider a positive symmetric weight function $\mu$ on the edge set $\mu : E \to \mathbb{R}_+$. In particular, for the edge $e = (x, y)$ connecting $x$ and $y$ (also denoted by $x \sim y$) we write $\mu(e) = \mu_{xy}$ and we extend $\mu$ to the whole of $V \times V$ by setting $\mu_{xy} = 0$ if $(x,y) \notin E$. We point out that we do allow self-loops in the graph, that
is $\mu_{xx} > 0$ is possible for all $x \in V$. Moreover, we consider a measure $\mu$ (by abuse of notation, we will also denote this measure by $\mu$ but this should not lead to any confusion) on the vertex set $\mu : V \to \mathbb{R}_+$ defined by $\mu(x) = \sum_{y \in V} \mu_{xy}$ for all $x \in V$. In this paper, for simplicity, we always assume that $\Omega$ is a finite, connected subset of $\mathbb{R}^2$ (otherwise one can easily extend the results to the connected components of $\Omega$), and the cardinality of $\Omega$ satisfies $\sharp \Omega \geq 2$ (if $\sharp \Omega = 1$, then the Dirichlet Laplace operator is trivial, i.e. has one single eigenvalue which is equal to 1 - $\frac{\mu_{xx}}{\mu(x)}$ - see below for more details). We define the volume of $\Omega$ as $\text{vol}(\Omega) := \sum_{x \in \Omega} \mu(x)$. The boundary $\partial \Omega$ of $\Omega$ is defined as the set of all vertices $y \notin \Omega$ for which there exists a vertex $x \in \Omega$ such that $x \sim y$ and we define $|\partial \Omega| = \sum_{x \in \Omega} \sum_{y \in \Gamma^c} \mu_{xy}$, where $\Gamma^c$ denotes the complement of $\Omega$.

For any two subsets $\Omega_1, \Omega_2 \subset V$ we denote $E(\Omega_1, \Omega_2) = \{(x, y) \in E : x \in \Omega_1, y \in \Omega_2\}$ and $|E(\Omega_1, \Omega_2)| = \sum_{x \in \Omega_1} \sum_{y \in \Omega_2} \mu_{xy}$. If $\Omega_1$ and $\Omega_2$ are disjoint subsets of a simple ($\mu_{xx} = 0$ for all $x$) and unweighted (i.e. $\mu_{xy} = 1$ for any $x \sim y$) graph $\Gamma$, then $|E(\Omega_1, \Omega_2)| = \sharp E(\Omega_1, \Omega_2)$ and $|E(\Omega_1, \Omega_1)| = 2 \sharp E(\Omega_1, \Omega_1)$ where $\sharp E(\Omega_1, \Omega_2)(\sharp E(\Omega_1, \Omega_1))$ is the number of edges in $E(\Omega_1, \Omega_2)(E(\Omega_1, \Omega_1))$. In particular, we have $|\partial \Omega| = |E(\Omega, \Omega^c)|$.

The following formula connecting the volume and the boundary of a subset $\Omega \subset V$ will be repeatedly used throughout this paper

$$\text{vol}(\Omega) = |E(\Omega, \Omega)| + |E(\Omega, \Omega^c)| = |E(\Omega, \Omega)| + |\partial \Omega|. \tag{4}$$

An important concept in this article is that of an exhaustion of an infinite graph by finite subsets. A sequence of finite subsets of an infinite graph $\Gamma$ satisfying $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_n \subset \cdots$ and $\Gamma = \cup_{n=1}^\infty \Omega_n$, denoted by $\Omega \uparrow \Gamma$, is called an exhaustion of $\Gamma$. For quantities that are monotone in $\Omega$, i.e. if $\Omega_1 \subset \Omega_2$ we have $f(\Omega_1) \leq f(\Omega_2)$ (or $f(\Omega_1) \geq f(\Omega_2)$) we write $\lim_{\Omega \uparrow \Gamma} f(\Omega) := \lim_{n \to \infty} f(\Omega_n) = f(\Gamma)$. Note that for monotone functions in $\Omega$ this limit exists and it does not depend on our choice of the exhaustion.

Now we are going to introduce the main object of interest of this article - the normalized Laplace operator of a graph. We introduce the Hilbert space,

$$\ell^2(V, \mu) := \{f : V \to \mathbb{R} \mid (f, f)_\mu < \infty\},$$

where we denote the inner product on $\ell^2(V, \mu)$ by $(f, g)_\mu = \sum_{x \in V} \mu(x) f(x) g(x)$. Note that the space of functions with finite support denoted by $C_0(V)$ is dense in $\ell^2(V, \mu)$, i.e. $C_0(V) = \ell^2(V, \mu)$. The normalized Laplace operator $\Delta : \ell^2(V, \mu) \to \ell^2(V, \mu)$ is pointwise defined by

$$\Delta f(x) = f(x) - \frac{1}{\mu(x)} \sum_{y \in V} \mu_{xy} f(y).$$

It is well known (see e.g. [DK86]) that $\Delta$ is a nonnegative, self-adjoint operator whose spectrum is bounded from above by two. We use the convention that the bottom and the top of the spectrum of $\Delta$ are denoted by $\underline{\lambda}(\Gamma) = \inf \sigma(\Delta)$ and $\overline{\lambda}(\Gamma) = \sup \sigma(\Delta)$, respectively.
Besides the normalized Laplace operator $\Delta$ one can also study the normalized Laplace operator with Dirichlet boundary conditions. Let $\Omega$ be a finite subset of $V$ and $\ell^2(\Omega, \mu)$ be the space of real-valued functions on $\Omega$. Note that every function $f \in \ell^2(\Omega, \mu)$ can be extended to a function $\tilde{f} \in \ell^2(V, \mu)$ by setting $\tilde{f}(x) = 0$ for all $x \in \Omega^c$. The Laplace operator with Dirichlet boundary conditions $\Delta_\Omega$ is defined as $\Delta_\Omega : \ell^2(\Omega, \mu) \to \ell^2(\Omega, \mu)$,

$$\Delta_\Omega f = (\Delta \tilde{f})|_{\Omega}.$$ 

Thus for $x \in \Omega$ the Dirichlet Laplace operator is pointwise defined by

$$\Delta_\Omega f(x) = f(x) - \frac{1}{\mu(x)} \sum_{y \in \Omega} \mu_{xy} f(y) = \tilde{f}(x) - \frac{1}{\mu(x)} \sum_{y \in V} \mu_{xy} \tilde{f}(y).$$

A simple calculation shows that $\Delta_\Omega$ is a positive self-adjoint operator. We arrange the eigenvalues of the Dirichlet Laplace operator $\Delta_\Omega$ in increasing order, i.e.

$$\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \ldots \leq \lambda_N(\Omega),$$

where $N$ is the cardinality of the set $\Omega$, i.e. $N = |\Omega|$. In the following we will also denote the largest Dirichlet eigenvalue $\lambda_N(\Omega)$ by $\lambda_{\max}(\Omega)$.

It is well known that the spectra of the Laplace operator $\Delta$ and the Laplace operator with Dirichlet boundary conditions $\Delta_\Omega$ are connected to each other. In particular, for an exhaustion $\Omega \uparrow \Gamma$ we have \cite{DK86}

$$\lim_{\Omega \uparrow \Gamma} \lambda_1(\Omega) = \lambda(\Gamma) \text{ and } \lim_{\Omega \uparrow \Gamma} \lambda_{\max}(\Omega) = \overline{\lambda}(\Gamma).$$

Note that these limits are well defined since $\lambda_1(\Omega)$ and $\lambda_{\max}(\Omega)$ are monotone in $\Omega$. Because of this connection our strategy is to first study the eigenvalues of the Dirichlet Laplace operator and then use an exhaustion of the graph to estimate the top and the bottom of the spectrum of the Laplace operator $\Delta$. In particular, we obtain estimates for the essential spectrum of $\Delta$.

## 3. The Dirichlet Laplace Operator

The Dirichlet Laplace operator has the following basic properties:

**Lemma 3.1** (Basic properties of $\Delta_\Omega$).

(i) $0 < \lambda_1(\Omega) \leq 1 - \frac{1}{\mu} \sum_{x \in \Omega} \frac{\mu_{xx}}{\mu(x)} \leq 1$.

(ii) $\lambda_1(\Omega)$ is a simple eigenvalue.

(iii) The eigenfunction $f_1$ corresponding to $\lambda_1(\Omega)$ satisfies $f_1(x) > 0$ or $f_1(x) < 0$ for all $x \in \Omega$.

(iv) $\lambda_1(\Omega) + \lambda_{\max}(\Omega) \leq 2 - 2 \min_{x \in \Omega} \frac{\mu_{xx}}{\mu(x)} \leq 2$.

(v) $\lambda_1(\Omega)$ ($\lambda_{\max}(\Omega)$) is nonincreasing (nondecreasing) when $\Omega$ increases.

(vi) The first eigenvalue is given by the Rayleigh quotient

$$\lambda_1(\Omega) = \inf_{f \in \ell^2(\Omega, \mu) \neq 0} \frac{(\Delta_\Omega f, f)_\mu}{(f, f)_\mu}$$

$$= \inf_{f \neq 0} \frac{\frac{1}{2} \sum_{x, y \in V} \mu_{xy} (\tilde{f}(x) - \tilde{f}(y))^2}{\sum_{x \in V} \mu(x) \tilde{f}(x)^2}.$$
and the largest eigenvalue is given by

\[ \lambda_{\text{max}}(\Omega) = \sup_{f \in \ell^2(\Omega, \mu) \neq 0} \frac{(\Delta \Omega f, f)_\mu}{(f, f)_\mu} \]

(8)

\[ = \sup_{f \neq 0} \frac{1}{2} \sum_{x,y \in V} \mu_{xy} (\tilde{f}(x) - \tilde{f}(y))^2}{\sum_{x \in V} \mu(x)f^2(x)} \]

(9)

Proof. All these facts are well-known and can for instance be found in [Dod84, Fri93, Gri09]. □

Before we continue we make the following two technical remarks.

Remark 3.1. (i) From the definition of the Dirichlet Laplace operator one immediately observes that in the trivial case \( \sharp \Omega = 1 \), the Dirichlet Laplace operator has one single eigenvalue which is equal to \( 1 - \frac{\mu_{xx}}{\mu(x)} \). Because of this we restrict ourselves from now on to subsets \( \Omega \subset V \) with \( \sharp \Omega > 1 \).

(ii) Often it is convenient to sum over edges instead of vertices or vice versa. However if there are loops in the graph we have

\[ \frac{1}{2} \sum_{x,y \in V} \mu_{xy} \neq \sum_{e = (x,y) \in E} \mu_{xy}. \]

This is the case because in the first sum the loops are only counted once. Hence in order to replace a sum over vertices by a sum over edges we have to adjust the edge weights. We define a new weight function \( \theta : E \to \mathbb{R}_+ \) on the edge set \( E \) by \( \theta_{xy} = \mu_{xy} \) for all \( x \neq y \in V \) and \( \theta_{xx} = \frac{1}{2} \mu_{xx} \) for all \( x \in V \). For these new edge weights we have

\[ \frac{1}{2} \sum_{x,y \in V} \mu_{xy} = \sum_{e = (x,y) \in E} \theta_{xy}. \]

Note that \( \theta : E \to \mathbb{R}_+ \) coincides with \( \mu : E \to \mathbb{R}_+ \) in the case \( \Gamma \) has no loops.

Lemma 3.2 (Green’s formula). Let \( f, g \in \ell^2(\Omega, \mu) \) then,

\[ (\Delta \Omega f, g)_\mu = \frac{1}{2} \sum_{x,y \in V} \mu_{xy} (\nabla_{xy} \tilde{f})(\nabla_{xy} \tilde{g}) \]

(10)

\[ = \frac{1}{2} \sum_{x,y \in V} \mu_{xy} (\tilde{f}(x) - \tilde{f}(y))(\tilde{g}(x) - \tilde{g}(y)) \]

(11)

\[ = \sum_{e = (x,y) \in E} \theta_{xy} (\tilde{f}(x) - \tilde{f}(y))(\tilde{g}(x) - \tilde{g}(y)), \]

(12)

where \( \nabla \) is the co-boundary operator. More precisely, if we fix an orientation on the edge set the co-boundary operator of an edge \( e = (x,y) \) from \( x \) to \( y \) is given by \( \nabla_{xy} f = f(x) - f(y) \).

Proof. See [Dod84, Gri09]. □
We recall here the well-known eigenvalue interlacing theorem sometimes also referred to as the inclusion principle (cf. [HJ06]).

**Lemma 3.3.** Let $A$ be a $N \times N$ Hermitian matrix, let $r$ be an integer with $1 \leq r \leq N - 1$ and let $A_{N-r}$ denote the $(N-r) \times (N-r)$ principle submatrix of $A$ obtained by deleting $r$ rows and the corresponding columns from $A$. For each integer $k$ such that $1 \leq k \leq N - r$ the eigenvalues of $A$ and $A_{N-r}$ satisfy
\[ \lambda_k(A) \leq \lambda_k(A_{N-r}) \leq \lambda_{k+r}(A). \]

Lemma 3.3 yields the following generalization of Lemma 3.1 (v).

**Corollary 3.1.** Let $\Omega_2 \subset \Omega_1 \subset V$ such that $|\Omega_1| = N$ and $|\Omega_2| = N - r$. Then the Dirichlet eigenvalues of $\Delta_{\Omega_1}$ and $\Delta_{\Omega_2}$ interlace, i.e.
\[ \lambda_k(\Omega_1) \leq \lambda_k(\Omega_2) \leq \lambda_{k+r}(\Omega_1) \]
for all integers $1 \leq k \leq N - r$. In particular,
\[ \lambda_1(\Omega_1) \leq \lambda_1(\Omega_2) \text{ and } \lambda_{\max}(\Omega_2) \leq \lambda_{\max}(\Omega_1). \]

**Proof.** Looking at a matrix representation of the Dirichlet Laplace operator of a subset $\Omega$ (also denoted by $\Delta_\Omega$) we observe that $\Delta_\Omega = I_\Omega - D_\Omega^{-1}W_\Omega$ is not Hermitian or in this case real symmetric. Here $I_\Omega$, $D_\Omega$ and $W_\Omega$ are the identity matrix, the diagonal matrix of vertex degrees and the weighted adjacency matrix of $\Omega$, respectively. Note that since $\Omega$ is connected, $\mu(x) > 0$ for all $x \in \Omega$ and hence $D^{-1}$ always exists. So we cannot directly apply Lemma 3.3 to our Dirichlet Laplace operator $\Delta_\Omega$. However, Chung's version of the Dirichlet Laplace operator [Chu97] $L_\Omega = I_\Omega - D_\Omega^{-\frac{1}{2}}W_\Omega D_\Omega^{-\frac{1}{2}}$ is real symmetric. Hence we can apply Lemma 3.3 and obtain that the eigenvalues of $L_{\Omega_1}$ and $L_{\Omega_2}$ interlace. Now we observe that for any $\Omega$ the Laplacians $\Delta_\Omega$ and $L_\Omega$ satisfy
\[ \Delta_\Omega = D_\Omega^{-\frac{1}{2}}L_\Omega D_\Omega^{-\frac{1}{2}}, \]
i.e. $\Delta_\Omega$ and $L_\Omega$ are similar to each other and hence have the same spectrum. Since this holds for all $\Omega$ it follows that if the eigenvalues of $L_{\Omega_1}$ and $L_{\Omega_2}$ satisfy (13), then the same is true for the eigenvalues of $\Delta_{\Omega_1}$ and $\Delta_{\Omega_2}$. \(\square\)

For the rest of this section we have a closer look at the Dirichlet eigenvalues of subsets $\Omega$ that are bipartite. We say a subset $\Omega \subset V$ is bipartite if $\Omega$ is a bipartite induced subgraph of $\Gamma$. Recall that a graph (subgraph) is bipartite if and only if it does not contain a cycle of odd length.

**Lemma 3.4.** Let $\Omega$ be bipartite. Then the eigenvalues of the Dirichlet Laplace operator satisfy: with $\lambda(\Omega)$, $2 - \lambda(\Omega)$ is also an eigenvalue of $\Delta_\Omega$, i.e. the spectrum is symmetric about 1.

**Proof.** The proof is very simple - it relies on the observation that if $f$ is an eigenfunction for $\lambda(\Omega)$, then for a bipartition $U, \overline{U}$ of $\Omega$
\[ g(x) = \begin{cases} f(x) & \text{if } x \in U \\ -f(x) & \text{if } x \in \overline{U} \end{cases} \]
is also an eigenfunction for $\Delta_{\Omega}$ corresponding to the eigenvalue $2 - \lambda(\Omega)$. □

For this result we obtain immediately the following useful corollaries.

**Corollary 3.2.** The largest eigenvalue $\lambda_{\text{max}}(\Omega)$ is simple if $\Omega$ is bipartite.

*Proof.* This is a direct consequence of Lemma 3.1 (ii) and Lemma 3.4. □

**Corollary 3.3.** Let $\Omega$ be bipartite. Then the eigenfunction $f_{\text{max}}$ corresponding to the largest eigenvalue satisfies $f_{\text{max}}(x) \neq 0$ for all $x \in \Omega$.

*Proof.* From the proof of the last lemma we know that if $f_1$ is the eigenfunction for the smallest eigenvalue then the eigenfunction for the largest eigenvalue is given by

$$f_{\text{max}}(x) = \begin{cases} f_1(x) & \text{if } x \in U \\ -f_1(x) & \text{if } x \in \overline{U} \end{cases}.$$ (14)

By Lemma 3.1 (iii) we have $f_1(x) \neq 0$ for all $x \in \Omega$ and hence $f_{\text{max}}(x) \neq 0$ for all $x \in \Omega$. □

**Theorem 3.1.** We have $\lambda_1(\Omega) + \lambda_{\text{max}}(\Omega) = 2$ iff $\Omega$ is bipartite.

*Proof.* By Lemma 3.4, it suffices to prove that $\lambda_1(\Omega) + \lambda_{\text{max}}(\Omega) = 2$ implies $\Omega$ is bipartite. It is well known (see c.f. [Gri09]) that $\lambda_1(\Omega) + \lambda_{\text{max}}(\Omega) \leq 2$.

Let $f$ be an eigenfunction for $\lambda_{\text{max}}(\Omega)$. Then we have

$$\lambda_{\text{max}}(\Omega) = \frac{\frac{1}{2} \sum_{x,y} \mu_{xy}(f(x) - f(y))^2}{\sum_x \mu(x) f(x)^2}$$

and

$$\lambda_1(\Omega) = \inf_g \frac{\frac{1}{2} \sum_{x,y} \mu_{xy}(g(x) - g(y))^2}{\sum_x \mu(x) g(x)^2} \leq \frac{\frac{1}{2} \sum_{x,y} \mu_{xy}(|f|(x) - |f|(y))^2}{\sum_x \mu(x) |f|(x)^2}. $$ (15) (16)

We have

$$((f(x) - f(y))^2 + (|f|(x) - |f|(y))^2) = 2f(x)^2 + 2f(y)^2 - 2f(x)f(y) - 2|f|(x)|f|(y) \leq 2f(x)^2 + 2f(y)^2.$$ (17)

Thus we have

$$\lambda_1(\Omega) + \lambda_{\text{max}}(\Omega) \overset{(*)}{\leq} \frac{\frac{1}{2} \sum_{x,y} \mu_{xy}[(f(x) - f(y))^2 + (|f|(x) - |f|(y))^2]}{\sum_x \mu(x) f(x)^2}$$

$$\overset{(**)}{\leq} \frac{\sum_{x,y} \mu_{xy} f(x)^2 + f(y)^2}{\sum_x \mu(x) f(x)^2} = \frac{2 \sum_{x,y} \mu_{xy} f(x)^2}{\sum_x \mu(x) f(x)^2} = 2.$$ (18)
Now we have a closer look at equation (17). We have strict inequality in (17) and hence in (***) of (18) if $f(x)f(y) > 0$ for some $x \sim y$. Suppose that $\Omega$ is not bipartite, then $\Omega$ contains a cycle $C$ of odd length with no repeated vertices (called circuit) and hence there exists at least one pair of neighbors $x \sim y$ such that $f(x)f(y) \geq 0$, (if not, $f$ has alternating signs along the cycle $C$ which contradicts to the odd length of $C$). If $f(x)f(y) > 0$ we are done. Otherwise there exists at least one vertex $x \in \Omega$ such that $f(x) = 0$ and hence $|f(x)| = 0$. From Lemma 3.1 (iii) it follows that $|f|$ is not an eigenfunction for $\lambda_1(\Omega)$. Hence we have strict inequality in (16) and then also in (*) of (18). Thus $\lambda_1(\Omega) + \lambda_{\max}(\Omega) < 2$ if $\Omega$ is not bipartite. It remains to show that $\lambda_1(\Omega) + \lambda_{\max}(\Omega) = 2$ if $\Omega$ is bipartite. This follows immediately from Lemma 3.4 since the eigenvalues of the Dirichlet Laplace operator $\Delta_{\Omega}$ are symmetric about 1 if $\Omega$ is bipartite. □

4. The Cheeger and the dual Cheeger estimate

In this section we introduce the Cheeger constant $h$ and the dual Cheeger constant $\bar{h}$ of a graph and show how they can be used to estimate the smallest and the largest eigenvalue of the Dirichlet Laplace operator from above and below. Moreover, we discuss in detail the connection between $h$ and $\bar{h}$. This will be particularly important in Section 10 when we study the essential spectrum of the Laplace operator $\Delta$.

**Definition 4.1** (Cheeger constant). For any subset $\Omega \subset V$ we define the Cheeger constant $h(\Omega)$ by

$$h(\Omega) = \inf_{\emptyset \neq U \subset \Omega} \frac{\|\partial U\|}{\text{vol}(U)}.$$ 

**Definition 4.2** (Dual Cheeger constant). For any subset $\Omega \subset V$ we define the dual Cheeger constant $\bar{h}(\Omega)$ by

$$\bar{h}(\Omega) = \sup_{V_1,V_2 \subset \Omega} \frac{2|E(V_1,V_2)|}{\text{vol}(V_1) + \text{vol}(V_2)},$$

where $V_1, V_2$ are two disjoint nonempty subsets of $\Omega$.

We observe that the dual Cheeger constant can only be defined if $\sharp \Omega > 1$. However, as discussed above we exclude the case $\sharp \Omega = 1$ since it is trivial anyway. The Cheeger constant and the dual Cheeger constant are related to each other in the following way:

**Theorem 4.1.** We have

$$\bar{h}(\Omega) \leq 1 - h(\Omega)$$

and equality holds if $\Omega$ is bipartite.
\textbf{Proof.} Let $V_1, V_2 \subset \Omega$ be two nonempty disjoint subsets of $\Omega$. Note that the volume of the subset $V_1$, can be written in the form (cf. eq. (4))

$$\text{vol}(V_1) = |E(V_1, V_2)| + |E(V_1, V_1)| + |E(V_1, (V_1 \cup V_2)')|$$

and a similar expression holds for the volume of $V_2$. Hence we have

$$\text{vol}(V_1) + \text{vol}(V_2) = 2|E(V_1, V_2)| + |E(V_1, V_1)| + |E(V_2, V_2)| + |\partial(V_1 \cup V_2)|$$

(20)

$$\geq 2|E(V_1, V_2)| + |\partial(V_1 \cup V_2)|.$$

From this it follows that

$$\frac{2|E(V_1, V_2)|}{\text{vol}(V_1) + \text{vol}(V_2)} \leq 1 - \frac{|\partial(V_1 \cup V_2)|}{\text{vol}(V_1 \cup V_2)}.$$

Taking the supremum over all nonempty disjoint subsets $V_1, V_2 \subset \Omega$ we get

(21) \hspace{1cm} \tilde{h}(\Omega) \overset{(*)}{=} \sup_{V_1, V_2 \subset \Omega} \frac{2|E(V_1, V_2)|}{\text{vol}(V_1) + \text{vol}(V_2)} \leq \sup_{V_1, V_2 \subset \Omega} \left(1 - \frac{|\partial(V_1 \cup V_2)|}{\text{vol}(V_1 \cup V_2)}\right)

= 1 - \inf_{U \subset \Omega, \#U \geq 2} \frac{|\partial U|}{\text{vol}(U)} \overset{(**)}{=} 1 - \inf_{U \subset \Omega} \frac{|\partial U|}{\text{vol}(U)} = 1 - h(\Omega),$$

where (**) follows from the fact that $\inf_{U \subset \Omega} \frac{|\partial U|}{\text{vol}(U)}$ cannot be achieved on singletons, i.e. $\#U = 1$. More precisely, if $\#U = 1$ then clearly $\frac{|\partial U|}{\text{vol}(U)} = 1$. However, since $\#\Omega > 1$ and $\Omega$ is connected, we can find $W \subset \Omega$ such that $|E(W,W)| > 0$. Thus we have

$$\frac{|\partial W|}{\text{vol}(W)} = \frac{|\partial W|}{|\partial W| + |E(W,W)|} < 1$$

which contradicts that $U$ achieves the infimum.

Now if $\Omega$ is bipartite, we claim that in (*) of (21) the supremum is obtained for two subsets $V_1, V_2 \subset \Omega$ that satisfy $|E(V_1, V_1)| = |E(V_2, V_2)| = 0$. If it is not the case, there exists $V_1', V_2' \subset \Omega$ that achieve the supremum in (*) of (21) and satisfy $|E(V_1', V_1')| \neq 0$ or $|E(V_2', V_2')| \neq 0$. Since $\Omega$ is bipartite (i.e. in particular $\mu_{xx} = 0$ for any $x \in V$), we can find nonempty disjoint subsets $V_1, V_2 \subset \Omega$ that satisfy $V_1 \cup V_2 = V_1' \cup V_2'$ and $|E(V_1, V_1)| = |E(V_2, V_2)| = 0$. Then we have

$$\frac{1}{2} \sum_{x,y \in V_1 \cup V_2} \mu_{xy} = |E(V_1, V_2)| + \frac{1}{2}|E(V_1, V_1)| + \frac{1}{2}|E(V_2, V_2)| = |E(V_1, V_2)|$$

and

$$\frac{1}{2} \sum_{x,y \in V_1' \cup V_2'} \mu_{xy} = |E(V_1', V_2')| + \frac{1}{2}|E(V_1', V_1')| + \frac{1}{2}|E(V_2', V_2')| > |E(V_1', V_2')|,$$

where we used in the last equation that $|E(V_1', V_1')| \neq 0$ or $|E(V_2', V_2')| \neq 0$. By construction we have $V_1 \cup V_2 = V_1' \cup V_2'$ which implies $|E(V_1', V_2')| < |E(V_1, V_2)|$ and $\text{vol}(V_1) + \text{vol}(V_2) = \text{vol}(V_1') + \text{vol}(V_2')$. This is a contradiction to the assumption that $V_1', V_2'$ achieve the supremum in (*) of (21).
Using (20) and the claim in [21], we have

\[
\bar{h}(\Omega) = \sup_{V_1, V_2 \subseteq \Omega} \frac{2|E(V_1, V_2)|}{\text{vol}(V_1) + \text{vol}(V_2)} = \sup_{V_1, V_2 \subseteq \Omega} \left(1 - \frac{|\partial(V_1 \cup V_2)|}{\text{vol}(V_1 \cup V_2)}\right)
\]

(22)

\[
= 1 - \inf_{U \subset \Omega} \frac{|\partial U|}{\text{vol}(U)} = 1 - h(\Omega),
\]

where in (22) we use the fact (since \(\Omega\) is bipartite) that for any disjoint \(V_1, V_2\) there exist disjoint \(U_1\) and \(U_2\) such that \(V_1 \cup V_2 = U_1 \cup U_2\) and \(|E(U_1, U_1)| = |E(U_2, U_2)| = 0\).

The next example shows that the converse of the second assertion in Theorem 4.1 is in general not true, i.e. \(h(\Omega) + \bar{h}(\Omega) = 1\) does not imply that \(\Omega\) is bipartite.

**Example 4.1.** Let \(G\) be the standard lattice \(\mathbb{Z}^2\) with one more edge, \(((0, 1), (1, 0))\), and \(\Omega'\) a finite subset of \(G\) containing the origin and the additional edge, i.e. \((0, 0) \in \Omega'\) and \(((0, 1), (1, 0)) \in E(\Omega', \Omega')\). Denote by \(M := \text{vol}(\Omega')\) the volume of \(\Omega'\). Moreover, let \(K_{m,n}\) be a large complete bipartite graph such that \(K := \text{vol}(K_{m,n}) \geq M\). By adding an edge that connects the origin in \(\Omega\) to a vertex in \(K_{m,n}\), we obtain an infinite graph, \(\Gamma = G \cup K_{m,n}\), see Figure 4. Let \(\Omega := \Omega' \cup K_{m,n}\). First of all we note that \(\Omega\) is not bipartite. However, we will show that \(h(\Omega) + \bar{h}(\Omega) = 1\) holds. We claim that \(U_0 := K_{m,n}\) achieves the Cheeger constant in \(\Omega\), i.e. \(\frac{|\partial U_0|}{\text{vol}(U_0)} \leq \frac{|\partial U|}{\text{vol}(U)}\), for any \(U \subset \Omega\). Note that by construction of \(\Omega\), \(U_0\) is the only subset of \(\Omega\) s.t. \(|\partial U| = 1\), i.e. \(|\partial U| = 1\) implies that \(U = U_0\). Thus for all \(U \neq U_0\)

\[
\frac{|\partial U|}{\text{vol}(U)} \geq \frac{2}{\text{vol}(U)} \geq \frac{2}{\text{vol}(\Omega)} \geq \frac{2}{K + M + 2}.
\]

Since \(M \leq K\),

\[
\frac{|\partial U|}{\text{vol}(U)} \geq \frac{1}{K + 1} = \frac{|\partial U_0|}{\text{vol}(U_0)}.
\]

This proves our claim that \(U_0\) achieves the Cheeger constant. Moreover by choosing \(V_1, V_2\) to be the bipartition of \(K_{m,n}\) we have

\[
\bar{h}(\Omega) = \sup_{V_1, V_2 \subseteq \Omega} \frac{2|E(V_1, V_2)|}{\text{vol}(V_1) + \text{vol}(V_2)} \geq \frac{K}{K + 1}
\]

and thus

\[
h(\Omega) + \bar{h}(\Omega) \geq \frac{1}{K + 1} + \frac{K}{K + 1} = 1.
\]

Together with Theorem 4.1 this implies that \(h(\Omega) + \bar{h}(\Omega) = 1\) although \(\Omega\) is not bipartite.
Remark 4.1. The last example showed that $h(\Omega) + \bar{h}(\Omega) = 1$ does not imply that $\Omega$ is bipartite. However, if $h(\Omega) + \bar{h}(\Omega) = 1$ we can show that the partition $V_1, V_2$ that achieves $\bar{h}(\Omega)$ is bipartite. This can be seen as follows: Let $V_1, V_2$ be the partition that achieves $\bar{h}(\Omega)$ and let $U = V_1 \cup V_2$. Then by definition $\frac{|\partial U|}{\text{vol}(U)} \geq h(\Omega)$ and thus we have

$$1 = h(\Omega) + \bar{h}(\Omega) \leq \frac{2|E(V_1, V_2)| + |\partial(V_1 \cup V_2)|}{2|E(V_1, V_2)| + |E(V_1, V_1)| + |E(V_2, V_2)| + |\partial(V_1 \cup V_2)|}. $$

This implies that $|E(V_1, V_1)| = |E(V_2, V_2)| = 0$ which yields that the partition $V_1, V_2$ is bipartite.

In order to give a lower bound for $\bar{h}(\Omega)$ in terms of $(1 - h(\Omega))$ we recall the following theorem from [BJ]:

**Theorem 4.2.** Let $\Gamma$ be a graph without self-loops. Then for any finite $U \subset V$ there exists a partition $V_1 \cup V_2 = U$ such that

$$|E(V_1, V_2)| \geq \max\{|E(V_1, V_1)|, |E(V_2, V_2)|\}. $$

**Proof.** For completeness, we include a proof here. Suppose the assertion is not true, that is for any partition $V_1 \cup V_2 = U$ we have

$$|E(V_1, V_2)| < \max\{|E(V_1, V_1)|, |E(V_2, V_2)|\}. $$
Theorem 4.3. If there are no self-loops in the graph, then we assume \(|E(V_1, V_2)| < |E(V_1, V_1)|\), i.e.

\[
\sum_{x \in V_1} \sum_{y \in V_2} \mu_{xy} < \sum_{x \in V_1} \sum_{z \in V_1} \mu_{xz}.
\]

Then there exists a vertex \(x \in V_1\) such that

\[
0 \leq \sum_{y \in V_2} \mu_{xy} = |E(\{x\}, V_2)| < \sum_{z \in V_1} \mu_{xz} = |E(\{x\}, V_1)|.
\]

Since \(|E(\{x\}, V_1)| > 0\) it follows that \(\exists V_1 \geq 2\) because otherwise \(V_1 = \{x\}\) and hence \(E(\{x\}, V_1) = 0\). We define a new partition, \(V_1' \cup V_2' = U\), as

\[
V_1' = V_1 \setminus \{x\} \neq \emptyset, \quad V_2' = V_2 \cup \{x\}.
\]

Then it is evident that

\[
|E(V_1', V_2')| = |E(V_1, V_2)| + |E(\{x\}, V_1)| - |E(\{x\}, V_2)| \geq |E(V_1, V_2)| + \epsilon_0,
\]

where

\[
\epsilon_0 := \min\{|E(\{x\}, V_1)| - |E(\{x\}, V_2)| : V_1 \cup V_2 = U, V_1 \cap V_2 = \emptyset, x \in U, |E(\{x\}, V_1)| - |E(\{x\}, V_2)| > 0\}.
\]

Note that since \(U\) is finite and by our assumption [24], it follows that \(\epsilon_0 > 0\).

Since [24] holds for all partitions, we may carry out this process for infinitely many times. That is we may obtain arbitrary large \(|E(V_1, V_2)|\) which contradicts that \(\text{vol}(U) < \infty\).

\(\square\)

**Theorem 4.3.** If there are no self-loops in the graph, then

\[
(25) \quad \frac{1}{2}(1 - h(\Omega)) \leq \tilde{h}(\Omega).
\]

**Proof.** We have

\[
\tilde{h}(\Omega) = \sup_{U \subset \Omega} \sup_{V_1, V_2 : U \geq 2, V_1 \cup V_2 = U} \frac{2|E(V_1, V_2)|}{\text{vol}(V_1) + \text{vol}(V_2)}
\]

\[
= \sup_{U \subset \Omega} \sup_{V_1, V_2 : U \geq 2, V_1 \cup V_2 = U} \left(\frac{2|E(V_1, V_2)| + \frac{1}{2} |\partial(V_1 \cup V_2)|}{\text{vol}(V_1) + \text{vol}(V_2)} - \frac{1}{2} \frac{|\partial(V_1 \cup V_2)|}{\text{vol}(V_1) + \text{vol}(V_2)}\right).
\]

\[
= \sup_{U \subset \Omega} \sup_{V_1, V_2 : U \geq 2, V_1 \cup V_2 = U} \left(\frac{2|E(V_1, V_2)| + \frac{1}{2} |\partial(V_1 \cup V_2)|}{2|E(V_1, V_2)| + |E(V_1, V_1)| + |E(V_2, V_2)| + |\partial(V_1 \cup V_2)|}
\]

\[
- \frac{1}{2} \frac{|\partial(V_1 \cup V_2)|}{\text{vol}(V_1) + \text{vol}(V_2)}\right).
\]
Using Theorem 4.2 we obtain
\[
\bar{h}(\Omega) \geq \sup_{\substack{U \subseteq \Omega \\mu \geq 2}} \left( \frac{1}{2} - \frac{1}{2} \frac{|\partial U|}{\text{vol}(U)} \right)
\]
\[
= \frac{1}{2} - \frac{1}{2} \inf_{U \subseteq \Omega} \frac{|\partial U|}{\text{vol}(U)}
\]
\[
= \frac{1}{2}(1 - h(\Omega)),
\]
where we again observe that the infimum of \( \frac{|\partial U|}{\text{vol}(U)} \) cannot be achieved by a singleton. \( \square \)

**Remark 4.2.**

(i) It is clear that for weighted graphs with self-loops \( \bar{h}(\Omega) \) can be arbitrarily close to zero. Consider for instance a subgraph consisting of two vertices connected by an edge. If one of the two vertices (say vertex \( x \)) has a self-loop with weight \( \mu_{xx} \), then both \( \bar{h}(\Omega) \) and \( h(\Omega) \rightarrow 0 \) as \( \mu_{xx} \rightarrow \infty \).

(ii) If we allow self-loops in the graph the best lower bound for \( \bar{h}(\Omega) \) that we can obtain is \( \bar{h}(\Omega) \geq 2 \min_{x,y \in \Omega} \frac{\mu_{xy}}{\text{vol}(\Omega)} \). This estimate is sharp for the graph discussed in (i) as \( \mu_{xx} \rightarrow \infty \).

(iii) Using an argument by Alon [Alo96] and Hofmeister and Lefmann [HL98] (see also Scott [Sco05]) we can show that even strict inequality holds in (25).

We introduce the following notation:

**Definition 4.3.** For a function \( g \in \ell^2(\Omega, \mu) \) we define
\[
P(g) := \{ x \in \Omega : g(x) > 0 \}
\]
and
\[
N(g) := \{ x \in \Omega : g(x) < 0 \}.
\]

**Definition 4.4 (Auxiliary Cheeger constant).** For a function \( g \in \ell^2(\Omega, \mu) \) the auxiliary Cheeger constant \( h(\Omega, g) \) is defined by
\[
(26) \quad h(\Omega, g) := \min_{\emptyset \neq U \subseteq P(g)} \frac{|E(U, U^c)|}{\text{vol}(U)}.
\]

**Remark 4.3.** Since for every function \( g \in \ell^2(\Omega, \mu) \) we have \( P(g) \subset \Omega \) it is obvious that \( h(\Omega, g) \geq h(\Omega) \) for all \( g \in \ell^2(\Omega, \mu) \).

We recall the following two lemmata from [DS91], see also [BJ].

**Lemma 4.1.** For every function \( g \in \ell^2(\Omega, \mu) \) we have
\[
1 + \sqrt{1 - h^2(\Omega, g)} \geq \sum_{e=(x,y)} \mu_{xy}(\tilde{g}_+(x) - \tilde{g}_+(y))^2 \geq \sum_x \mu(x)\tilde{g}_+(x)^2 \geq 1 - \sqrt{1 - h^2(\Omega, g)},
\]
where \( g_+ \) is the positive part of \( g \), i.e.
\[
g_+(x) = \begin{cases} g(x) & \text{if } x \in P(g) \\ 0 & \text{else}. \end{cases}
\]
Proof. First, we write

\[ W := \frac{\sum_{e=(x,y)} \mu_{xy} (\tilde{g}_+(x) - \tilde{g}_+(y))^2}{\sum_x \mu(x) \tilde{g}_+(x)^2} = \frac{\sum_{e=(x,y)} \theta_{xy} (\tilde{g}_+(x) - \tilde{g}_+(y))^2}{\sum_x \mu(x) \tilde{g}_+(x)^2} = \frac{\sum_{e=(x,y)} \theta_{xy} (\tilde{g}_+(x) - \tilde{g}_+(y))^2 \sum_{e=(x,y)} \theta_{xy} (\tilde{g}_+(x) + \tilde{g}_+(y))^2}{\sum_x \mu(x) \tilde{g}_+(x)^2} =: I/II. \]

Using the Cauchy-Schwarz inequality we obtain

\[ I \geq \left( \sum_{e=(x,y)} \theta_{xy} |\tilde{g}_+(x)^2 - \tilde{g}_+(y)^2| \right)^2 = \left( \sum_{e=(x,y)} \mu_{xy} |\tilde{g}_+(x)^2 - \tilde{g}_+(y)^2| \right)^2. \]

Now we have

\[ \sum_{e=(x,y)} \mu_{xy} |\tilde{g}_+(x)^2 - \tilde{g}_+(y)^2| = \sum_{e=(x,y): \tilde{g}_+(x) > \tilde{g}_+(y)} \mu_{xy} (\tilde{g}_+(x)^2 - \tilde{g}_+(y)^2) = 2 \sum_{e=(x,y): \tilde{g}_+(x) > \tilde{g}_+(y)} \mu_{xy} \int_{\tilde{g}_+(y)}^{\tilde{g}_+(x)} t dt = 2 \int_0^\infty \sum_{e=(x,y): \tilde{g}_+(x) \leq t \leq \tilde{g}_+(y)} \mu_{xy} t dt. \]

Note that \( \sum_{e=(x,y): \tilde{g}_+(y) \leq t < \tilde{g}_+(x)} \mu_{xy} = |E(P_t, P_c^c)| \) where \( P_t := \{ x : \tilde{g}_+(x) > t \} \). Using (26) we obtain,

\[ \sum_{e=(x,y)} \mu_{xy} |\tilde{g}_+(x)^2 - \tilde{g}_+(y)^2| \geq 2h(\Omega, g) \int_0^\infty \text{vol}(P_t) dt = 2h(\Omega, g) \int_0^\infty \sum_{x: \tilde{g}_+(x) > t} \mu(x) t dt = 2h(\Omega, g) \sum_{x \in V} \mu(x) \int_{\tilde{g}_+(x)}^{\infty} t dt = h(\Omega, g) \sum_{x} \mu(x) \tilde{g}_+(x)^2 \]

and so it follows that

\[ I \geq h^2(\Omega, g)(\sum_{x} \mu(x) \tilde{g}_+(x)^2)^2. \]
\[ II = \sum_x \mu(x) \bar{g}_+(x)^2 \sum_{e=(x,y)} \theta_{xy} (\bar{g}_+(x) + \bar{g}_+(y))^2 \]
\[ = \sum_x \mu(x) \bar{g}_+(x)^2 \left( \sum_x \mu(x) \bar{g}_+(x)^2 + \sum_{x,y} \mu_{xy} \bar{g}_+(x) \bar{g}_+(y) \right) \]
\[ = \sum_x \mu(x) \bar{g}_+(x)^2 \left( 2 \sum_x \mu(x) \bar{g}_+(x)^2 - \sum_{e=(x,y)} \mu_{xy} (\bar{g}_+(x) - \bar{g}_+(y))^2 \right) \]
\[ = (2 - W) \left( \sum_x \mu(x) \bar{g}_+(x)^2 \right)^2. \]

Combining everything we obtain,
\[ W \geq \frac{h^2(\Omega, g)}{2 - W} \]
and consequently
\[ 1 + \sqrt{1 - h^2(\Omega, g)} \geq W \geq 1 - \sqrt{1 - h^2(\Omega, g)}. \]

The second observation that we need to prove the Cheeger inequality is the following lemma, see [DS91]:

Lemma 4.2. For every non-negative real number \( \lambda \) and \( g \in \ell^2(\Omega, \mu) \) we have
\[ \lambda \geq \frac{\sum_{e=(x,y)} \mu_{xy} (\bar{g}_+(x) - \bar{g}_+(y))^2}{\sum_x \mu(x) \bar{g}_+(x)^2} \]
if \( \Delta_\Omega g(x) \leq \lambda g(x) \) for all \( x \in P(g) \).

Proof. On the one hand we have
\[ (\Delta_\Omega g, g_+)_\mu = \sum_{x \in \Omega} \mu(x) \Delta_\Omega g(x) g_+(x) \]
\[ = \sum_{x \in P(g)} \mu(x) \Delta_\Omega g(x) g_+(x) \]
\[ \leq \lambda \sum_{x \in P(g)} \mu(x) g_+(x) g_+(x) \]
\[ = \lambda \sum_{x \in V} \mu(x) \bar{g}_+(x) \bar{g}_+(x), \]
where we used our assumption. On the other hand we have
\[ (\Delta_\Omega g, g_+)_\mu = \sum_{e=(x,y) \in E} \theta_{xy} (\bar{g}(x) - \bar{g}(y)) (\bar{g}_+(x) - \bar{g}_+(y)) \]
\[ \geq \sum_{e=(x,y) \in E} \theta_{xy} (\bar{g}_+(x) - \bar{g}_+(y))^2, \]
where we used the Green’s formula, see Lemma 3.2. \( \square \)
Theorem 4.4 (Cheeger inequality cf. [DK88]). We have

\[ 1 - \sqrt{1 - h^2(\Omega)} \leq \lambda_1(\Omega) \leq h(\Omega). \]

Proof. For completeness we give a proof here.

First we show that \( \lambda_1(\Omega) \leq h(\Omega) \). We consider the following function:

\[
f(x) = \begin{cases} 
1 & \text{if } x \in U \subset \Omega \\
0 & \text{else.}
\end{cases}
\]

Using this function in the Rayleigh quotient \([6]\) yields

\[
\lambda_1(\Omega) \leq \frac{1}{2} \sum_{x,y} \mu_{xy}(f(x) - f(y))^2 \sum_x \mu(x) f^2(x) = \frac{\sum_{x \in U} \sum_{y \in U^c} \mu_{xy}}{\text{vol}(U)} = \frac{|\partial U|}{\text{vol}(U)}.
\]

Since this holds for all \( U \subset \Omega \) we have

\[
\lambda_1(\Omega) \leq \inf_{\emptyset \neq U \subset \Omega} \frac{|\partial U|}{\text{vol}(U)} = h(\Omega).
\]

The inequality \( 1 - \sqrt{1 - h^2(\Omega)} \leq \lambda_1(\Omega) \) follows from Lemma 4.1, Lemma 4.2 and Remark 4.3 by taking \( \lambda = \lambda_1(\Omega) \) and \( g = u_1 \) an eigenfunction for \( \lambda_1(\Omega) \).

The next theorem is the main result of this section.

Theorem 4.5 (Dual Cheeger inequality). We have

\[ 2h(\Omega) + h(\Omega) \leq \lambda_{\text{max}}(\Omega) \leq 1 + \sqrt{1 - (1 - h(\Omega))^2}. \]

Proof. Let \( V_1, V_2 \) be two disjoint nonempty subsets of \( \Omega \). We consider the following function:

\[
f(x) = \begin{cases} 
1 & \text{if } x \in V_1 \\
-1 & \text{if } x \in V_2 \\
0 & \text{else.}
\end{cases}
\]
Using this function in the Rayleigh quotient \[ f \] yields

\[
\lambda_{\text{max}}(\Omega) \geq \frac{\frac{1}{2} \sum_{x,y} \mu_{xy}(\bar{f}(x) - \bar{f}(y))^2}{\sum_{x} \mu(x)f^2(x)} = \frac{\sum_{x \in V_1} \sum_{y \in V_2} \mu_{xy}^4 + \sum_{x \in V_1 \cup V_2} \sum_{y \in (V_1 \cup V_2)^c} \mu_{xy}}{\text{vol}(V_1) + \text{vol}(V_2)} \geq 2 \frac{2|E(V_1, V_2)|}{\text{vol}(V_1) + \text{vol}(V_2)} + \frac{|\partial(V_1 \cup V_2)|}{\text{vol}(V_1) + \text{vol}(V_2)} \geq 2 \frac{2|E(V_1, V_2)|}{\text{vol}(V_1) + \text{vol}(V_2)} + \inf_{\mu \in U \supseteq 2} \frac{|\partial U|}{\text{vol}(U)},
\]

where we again used the observation that the infimum is not achieved by a singleton. Since this holds for all disjoint nonempty subsets \( V_1, V_2 \subset \Omega \) we have

\[
2 \tilde{h}(\Omega) + h(\Omega) \leq \lambda_{\text{max}}(\Omega).
\]

Now we prove the remaining inequality \( \lambda_{\text{max}}(\Omega) \leq 1 + \sqrt{1 - (1 - \tilde{h}(\Omega))^2} \). When one studies the largest eigenvalue of \( \Delta_{\Omega} \) it is convenient to introduce the operator \( Q_{\Omega} = 2I_{\Omega} - \Delta_{\Omega} = I_{\Omega} + P_{\Omega} \) where \( I_{\Omega} \) is the identity operator on \( \Omega \) and \( P_{\Omega} \) the transition probability operator. If \( \lambda(\Omega) \) is an eigenvalue of \( \Delta_{\Omega} \) and corresponding eigenfunction \( f \) then \( f \) is also an eigenfunction for \( Q_{\Omega} \) and corresponding eigenvalue \( \xi(\Omega) = 2 - \lambda(\Omega) \). Thus, controlling the largest eigenvalue \( \lambda_{\text{max}}(\Omega) \) of \( \Delta_{\Omega} \) from above is equivalent to controlling the smallest eigenvalue \( \xi_1(\Omega) \) of \( Q_{\Omega} \) from below. The smallest eigenvalue \( \xi_1(\Omega) \) of \( Q_{\Omega} \) is given by

\[
\xi_1(\Omega) = \inf_{\mu \neq 0} \frac{\frac{1}{2} \sum_{x,y \in V} \mu_{xy}(\bar{f}(x) + \bar{f}(y))^2}{\sum_{x \in V} \mu(x)f(x)^2} = \inf_{\mu \neq 0} \frac{\sum_{x=(x,y) \in E} \theta_{xy}(\bar{f}(x) + \bar{f}(y))^2}{\sum_{x \in V} \mu(x)f(x)^2},
\]

where as above \( \theta_{xy} = \mu_{xy} \) for all \( x \neq y \) and \( \theta_{xx} = \frac{1}{2} \mu_{xx} \) for all \( x \in V \). This simply follows from the standard minmax characterization of the eigenvalues

\[
\xi_1(\Omega) = \inf_{\mu \neq 0} \frac{(Q_{\Omega} f, f)_\mu}{(f, f)_\mu},
\]

and the following formula for any \( f, g \in \ell^2(\Omega, \mu) \)

\[
(Q_{\Omega} f, g)_\mu = \sum_{x \in \Omega} \mu(x)Q_{\Omega} f(x)g(x) = \sum_{x \in V} \sum_{y \in V} \mu_{xy}(\bar{f}(x) + \bar{f}(y))\bar{g}(x) = \sum_{y \in V} \sum_{x \in V} \mu_{yx}(\bar{f}(y) + \bar{f}(x))\bar{g}(y).
\]
where we just exchanged $x$ and $y$. Adding the last two lines and setting $\tilde{f} = \tilde{g}$ yields

$$(Q_{\Omega}f, f)_{\mu} = \frac{1}{2} \sum_{x,y} \mu_{xy}(\tilde{f}(x) + \tilde{f}(y))^2.$$ 

In order to prove the lower bound for $\xi_1(\Omega)$ we will use a technique developed in [DR94]. The idea is the following: Construct a graph $\Gamma'$ out of $\Gamma$ s.t. the quantity $h'(\Omega, g)$ defined in [26] for the new graph $\Gamma'$ controls $\xi_1(\Omega)$ from below. In a second step, we show that $h'(\Omega, g)$ in turn can be controlled by the quantity $1 - \overline{h}(\Omega)$ of the original graph. This then yields the desired estimate.

Let $f$ be an eigenfunction for the eigenvalue $\xi_1(\Omega)$ of $Q_\Omega$ and define as above $P(f) = \{x \in \Omega : f(x) > 0\}$ and $N(f) = \{x \in \Omega : f(x) < 0\}$. Then the new graph $\Gamma' = (V', E')$ is constructed from $\Gamma = (V, E)$ in the following way. Duplicate all vertices in $P(f) \cup N(f)$ and denote the copies by a prime, e.g. if $x \in P(f)$ then the copy of $x$ is denoted by $x'$. The copies of $P(f)$ and $N(f)$ are denoted by $P'(f)$ and $N'(f)$ respectively. The vertex set $V'$ of $\Gamma'$ is given by $V' = V \cup P'(f) \cup N'(f)$. Every edge $(x, y) \in E(P(f), P(f))$ in $\Gamma$ is replaced by two edges $(x', y')$ and $(y, x')$ in $\Gamma'$ s.t. $\mu_{xy} = \mu'_{xy} = \mu'_{yx'}$. Similarly, if the edge is a loop, then $e = (x, x)$ is replaced by one edge $(x, x')$ s.t. $\mu_{xx} = \mu_{xx'}$. The same is done with edges in $E(N(f), N(f))$. All other edges remain unchanged, i.e. if $(k, l) \in E(\Gamma \setminus (E(P(f), P(f)) \cup E(N(f), N(f))))$ then $(k, l) \in E'$ and $\mu_{kl} = \mu'_{kl}$. It is important to note that this construction does not change the degrees of the vertices in $V = V' \setminus (P'(f) \cup N'(f))$.

Consider the function $\tilde{g} : V' \to \mathbb{R}$,

$$\tilde{g}(x) = \begin{cases} |f(x)| & \text{if } x \in P(f) \cup N(f) \\ 0 & \text{else.} \end{cases}$$

It can easily be checked that by construction of $\Gamma'$ we have

$$\xi_1(\Omega) \geq \frac{\sum_{e=(x,y) \in E} \theta_{xy}(\tilde{f}(x) + \tilde{f}(y))^2}{\sum_{x \in V} \mu(x) \tilde{f}(x)^2} \geq \frac{\sum_{e'=(x,y) \in E'} \mu'_{xy}(\tilde{g}(x) - \tilde{g}(y))^2}{\sum_{x \in V'} \mu'(x) \tilde{g}(x)^2} \geq 1 - \sqrt{1 - (h'(\Omega, g))^2}$$

where we used Lemma [4.1] to obtain the last inequality. For any non-empty subset $W \subseteq P(g) = P(f) \cup N(f)$ we define $P(W) = W \cap P(f)$ and $N(W) =$
Let \( \emptyset \neq U \subseteq P(g) \) the subset that realizes the infimum, i.e.

\[
h'(\Omega, g) = \inf_{\emptyset \neq W \subseteq P(g)} \frac{|E'(W, W^c)|}{\text{vol}(W)} = \frac{|E'(U, U^c)|}{\text{vol}(U)}
\]

\[
= \frac{|E(P(U), P(U))| + |E(N(U), N(U))| + |\partial(P(U) \cup N(U))|}{\text{vol}(P(U)) + \text{vol}(N(U))}
\]

\[
= 1 - \frac{2|E(P(U), N(U))|}{\text{vol}(P(U)) + \text{vol}(N(U))}
\]

\[
\geq 1 - \sup_{V_1, V_2 \subseteq \Omega} \frac{2|E(V_1, V_2)|}{\text{vol}(V_1) + \text{vol}(V_2)}
\]

\[
= 1 - \overline{h}(\Omega),
\]

where we used that \( P(U), N(U) \subseteq \Omega \). Thus we have

\[
2 - \lambda_{\text{max}}(\Omega) = \xi_1(\Omega) \geq 1 - \sqrt{1 - (1 - \overline{h}(\Omega))^2}
\]

and so

\[
\lambda_{\text{max}}(\Omega) \leq 1 + \sqrt{1 - (1 - \overline{h}(\Omega))^2}.
\]

\[\square\]

**Remark 4.4.**

(i) Note that if \( \Omega \) is bipartite, then the upper bound for \( \lambda_{\text{max}}(\Omega) \) follows directly from Lemma 3.1 (iv), the Cheeger inequality, and Theorem 4.1.

(ii) For finite graphs it is also possible to obtain a Cheeger and a dual Cheeger estimate for the smallest nonzero and the largest eigenvalue, respectively (see for instance [Chu97] and [BJ]). For finite graphs the Cheeger together with the dual Cheeger estimate are very powerful since many graph properties such as the diameter, the independence number or the convergence of a random walk to its stationary distribution are controlled by the maximal difference from the smallest nonzero and the largest eigenvalue from 1, see [Chu97, Lub94].

5. Eigenvalue comparison theorems

In this section we prove some eigenvalue comparison theorems for the largest eigenvalue of the Dirichlet Laplace operator that do not have analogues in Riemannian geometry. Similar eigenvalue comparison theorems for the smallest eigenvalue were obtained by Urakawa [Ura99]. Urakawa’s results are discrete versions of Cheng’s eigenvalue comparison theorem for Riemannian manifolds [Che75a, Che75b]. In the literature (see e.g. [Ura99, Fri93]), the mostly used comparison models for graphs are homogeneous trees, since regular trees are considered to be discrete analogues of simply connected space forms. Here we propose a novel comparison model for graphs, the weighted half-line \( R_l \ (l \in \mathbb{R}, \ l \geq 2) \) which is defined by \( R_l := (V, E), V = \mathbb{N} \cup \{0\}, E = \{(i, i + 1), i \geq 0, i \in V\} \) and the edge weights are defined as \( \mu_{(i, i+1)} = (l - 1)^i \), see Figure 2. More precisely, we
want to compare the largest Dirichlet eigenvalue of a ball in any graph with
the largest Dirichlet eigenvalue of a ball with the same radius in $R_l$ (in
contrast to the existing literature, where homogenous trees $T_d$ were used
instead). The advantage of the weighted half-line $R_l$ is that we get better
estimates in the comparison theorems since we can overcome the restriction
that $d$ must be an integer if we use the homogenous tree $T_d$ as a compari-
son model. In particular, the results using $R_l$ or $T_d$ as a comparison model
coincide if $l = d$ are integers.

The next lemma shows that the eigenvalues and eigenfunctions of the
Dirichlet operator of $R_l$ behave like those on infinite $d$-homogenous trees $T_d$,
since a similar result was obtained for $T_d$ by Friedmann [Fri93] and Urakawa
[Ura99].

**Lemma 5.1.** Let $V_l(r) = \{ x \in R_l : d(x, 0) \leq r \}$ and let $l \geq 2$. The first
eigenvalue of the Dirichlet Laplace operator on $V_l(r)$ in $R_l$ is

$$\lambda_1(V_l(r)) = 1 - \frac{2\sqrt{l-1}}{l} \cos \theta_{l,r},$$

where $\theta_{l,r}$ is the smallest positive solution $\theta$ of $g_r(\theta) = \frac{l}{2(l-1)}$, where

$$g_r(\theta) = \frac{\sin((r+1)\theta) \cos \theta}{\sin(r\theta)}.$$

We have $\theta_{2,r} = \frac{\pi}{2(r+1)}$ and for $l > 2$ we have

$$\max\{\frac{\pi}{r+\eta}, \frac{\pi}{2(r+1)}\} \leq \theta_{l,r} \leq \frac{\pi}{r+1},$$

where $\eta = \eta(l) = \frac{2(l-1)}{l-2}$. The first eigenfunction $f_1$ is

$$f_1(x) = (l-1)^{-\frac{r(x)}{2}} \sin(r + 1 - r(x)) \theta_{l,r},$$

which is positive and decreasing (as a function of one variable) on $V_l(r)$
where $r(x) = d(x, 0)$.

**Proof.** We omit the proof of this lemma since it is straightforward and similar
to the proof in the case of $d$-regular trees $T_d$, see [Ura99] (Lemma 3.1) and
[Fri93] (Proposition 3.2).\hfill $\square$

The following corollary is a straightforward consequence of the previous
lemma by the bipartiteness of $R_l$, see Theorem 3.1.
Lemma 5.2. For \( l > 2 \),
\[
1 - \frac{2\sqrt{l-1}}{l} \cos \frac{\pi}{r + \eta} \leq \lambda_1(V_l(r)) \leq 1 - \frac{2\sqrt{l-1}}{l} \cos \frac{\pi}{r + 1},
\]
\[
1 + \frac{2\sqrt{l-1}}{l} \cos \frac{\pi}{r + 1} \leq \lambda_{\text{max}}(V_l(r)) \leq 1 + \frac{2\sqrt{l-1}}{l} \cos \frac{\pi}{r + \eta},
\]
where \( \eta = \frac{2(l-1)}{l-2} \), and for \( l = 2 \)
\[
\lambda_1(V_l(r)) = 1 - \cos \frac{\pi}{2(r + 1)} \quad \text{and} \quad \lambda_{\text{max}}(V_l(r)) = 1 + \cos \frac{\pi}{2(r + 1)}.
\]

Now we study the eigenfunction \( f_{\text{max}} \) for the largest Dirichlet eigenvalue of a ball in \( R_l \).

Lemma 5.3. The eigenfunction \( f_{\text{max}} \) for the largest Dirichlet eigenvalue \( \lambda_{\text{max}}(V_l(r)) \) satisfies:

(i) \( |f_{\text{max}}(s)| \) is decreasing in \( s \), i.e. \( |f_{\text{max}}(s)| > |f_{\text{max}}(s+1)| \) \( \forall 0 \leq s = r(x) \leq r \).

(ii) \( f_{\text{max}}(s)f_{\text{max}}(s+1) < 0 \) for all \( 0 \leq s < r - 1 \).

Proof. Since the weighted half line \( R_l \) is bipartite, we know from the proof of Lemma 5.1 that the eigenfunction for the largest eigenvalue \( f_{\text{max}} \) is given by
\[
f_{\text{max}}(x) = \begin{cases} f_1(x) & \text{if } x \in U \\ -f_1(x) & \text{if } x \in \overline{U} \end{cases}
\]
where \( U \cup \overline{U} = V_l(r) \) is a bipartition of \( V_l(r) \) and \( f_1 \) is the first eigenfunction. Since by Lemma 5.1 the first eigenfunction \( f_1 \) is positive and decreasing both statements follow immediately. \( \square \)

In order to prove the eigenvalue comparison theorem we also need the following Barta-type theorem for the largest Dirichlet eigenvalue.

Lemma 5.4 (Barta-type theorem for \( \lambda_{\text{max}}(\Omega) \)). Let \( f \in \ell^2(\Omega, \mu) \) be any function s.t. \( f(x) \neq 0 \) for all \( x \in \Omega \). Then we have
\[
\lambda_{\text{max}}(\Omega) \geq \inf_{x \in \Omega} \frac{\Delta_\Omega f(x)}{f(x)}.
\]

Proof. For all \( f \in \ell^2(\Omega, \mu) \) with \( f(x) \neq 0 \) for all \( x \in \Omega \) we have
\[
(\Delta_\Omega f, f)_\mu = \sum_{x \in \Omega} \mu(x) \Delta_\Omega f(x)f(x)
= \sum_{x \in \Omega} \mu(x) \frac{\Delta_\Omega f(x)}{f(x)} f(x)
\geq \sum_{x \in \Omega} \mu(x) \inf_{x \in \Omega} \frac{\Delta_\Omega f(x)}{f(x)} f(x)
= \inf_{x \in \Omega} \frac{\Delta_\Omega f(x)}{f(x)} (f, f)_\mu.
\]
Now the lemma follows since for all such functions $f$

$$
\lambda_{\text{max}}(\Omega) = \sup_{g \in \ell^2(\Omega, \mu)} \frac{(\Delta g, g)_\mu}{(g, g)_\mu} \geq \frac{(\Delta f, f)_\mu}{(f, f)_\mu} \geq \inf_{x \in \Omega} \frac{\Delta f(x)}{f(x)}.
$$

We introduce the following notation:

**Definition 5.1.** For a graph $\Gamma = (V, E)$ we define

$$
V_+(x) = \{ y \in V : y \sim x, d(x_0, y) = d(x_0, x) + 1 \}
$$

$$
V_-(x) = \{ y \in V : y \sim x, d(x_0, y) = d(x_0, x) - 1 \}
$$

$$
V_0(x) = \{ y \in V : y \sim x, d(x_0, y) = d(x_0, x) \}
$$

for some fixed $x_0 \in V$. Moreover, we define

$$
\mu_+(x) := \sum_{y \in V_+(x)} \mu_{xy}, \quad \mu_-(x) := \sum_{y \in V_-(x)} \mu_{xy} \quad \text{and} \quad \mu_0(x) := \sum_{y \in V_0(x)} \mu_{xy}.
$$

Clearly, $\mu(x) = \mu_+(x) + \mu_-(x) + \mu_0(x)$.

For the sequel, let us observe that for the weighted half-line $R_l$ and reference point $x_0 = 0$, at every vertex, $\frac{\mu(x)}{\mu_-(x)} = l$ and $\mu_0(x) = 0$.

**Theorem 5.1** (Eigenvalue comparison theorem I). Let $B(x_0, r)$ be a ball in $\Gamma$ centered at $x_0$ with radius $r$ and $2 \leq l := \sup_{x \in B(x_0, r)} \frac{\mu(x)}{\mu_-(x)} < \infty$. Then the largest Dirichlet eigenvalue $\lambda_{\text{max}}(B(x_0, r))$ satisfies

$$
\lambda_{\text{max}}(B(x_0, r)) \geq \lambda_{\text{max}}(V_l(r)) - 2\kappa(x_0, r),
$$

where $V_l(r)$ is a ball of radius $r$ centered at $0$ in the weighted half-line $R_l$ and $\kappa(x_0, r) := \sup_{x \in B(x_0, r)} \frac{\mu_0(x)}{\mu_-(x)}$.

Note that if $x_0$ is not contained in a circuit of odd length $\leq 2r + 1$ then $\kappa(x_0, r) = 0$. In particular, if the ball $B(x_0, r)$, considered as an induced subgraph in $\Gamma$, is bipartite, then $\kappa(x_0, r) = 0$. In particular we have the following corollary:

**Corollary 5.1.** Let $B(x_0, r)$ be a ball on $\Gamma$ that does not contain any cycle of odd length, then under the same assumptions as in the last theorem we have

$$
\lambda_{\text{max}}(B(x_0, r)) \geq \lambda_{\text{max}}(V_l(r)).
$$

**Remark 5.1.** The choice of $l$ in the last theorem yields the best possible results. However, all we need is that $l \geq \frac{\mu(x)}{\mu_-(x)}$ for all $x \in B(x_0, r)$. Thus it is sufficient to choose $l \geq \sup_{x \in V} \frac{\mu(x)}{\mu_-(x)}$ in Theorem 5.1. This also shows why we restrict ourselves to $R_l$ for $l \geq 2$. If $l < 2$ and $l \geq \sup_{x \in V} \frac{\mu(x)}{\mu_-(x)}$, then this immediately implies that the graph is finite - in fact the graph consists of two vertices connected by one edge, provided the graph is connected.
Before we prove this theorem, we show that the quantity \( \kappa(x_0, r) \) is related to a local clustering coefficient through the following quantity

\[
C(x_0, r) := \sup_{x \in B(x_0, r)} \frac{\sum_{y \in C(x, r)} \mu_{xy}}{\mu(x)}
\]

where \( C(x, r) := \{ y \in V : y \sim x \text{ and } y \text{ are both contained in a circuit of odd length } \leq 2r+1 \} \) and a circuit is a cycle without repeated vertices. Recall that a graph is bipartite if and only if there are no cycles of odd length in the graph. Hence \( C(x_0, r) \) expresses how different a graph is from a bipartite graph. Moreover, \( C(x_0, r) \) is related to a local clustering coefficient since \( C(x_0, r) = 0 \) implies that all \( x \in B(x_0, r) \) are not contained in any triangle. Estimates for the largest eigenvalue of the Laplace operator for finite graphs in terms of a local clustering coefficient were obtained in [BJ].

**Lemma 5.5.** We have for all \( x_0 \in V \) and \( r \geq 0 \),

\[
\kappa(x_0, r) \leq C(x_0, r)
\]

**Proof.** It suffices to show that if \( z \in V_0(x) \) for some fixed \( x_0 \), then \( z \in C(x, r) \), i.e. \( V_0(x) \subseteq C(x, r) \). We assume that for a fixed vertex \( x_0 \in V \) we have \( d(x_0, x) = d(x_0, z) = N \leq r \) and \( z \sim x \). Let \( P_x \) and \( P_z \) denote a shortest path from \( x_0 \) to \( x \) and \( z \), respectively. We label the vertices in the path \( P_x \) by \( x_0, x_1, \ldots, x_N \) and the vertices in \( P_z \) by \( x_0, x_1, \ldots, x_N \). Let \( 0 \leq k \leq N - 1 \) be such that \( x_k = z \) for all \( l > k \), i.e. \( x_k = z_k \) is the last branching point of the two paths \( P_x \) and \( P_z \). We claim that \( x_k, x_{k+1}, \ldots, x_N = x, z = z_N, z_{N-1}, \ldots, z_k = x_k \) is a circuit of odd length. Clearly, this is a cycle of length \( 2(N - k) + 1 \), i.e. a cycle of odd length. Moreover, by construction the vertices in this cycle are all different from each other. Thus \( z \) and \( x \) are contained in a circuit of odd length \( \leq 2(N - k) + 1 \leq 2r + 1 \) and hence \( z \in C(x, r) \). This completes the proof. 

Using techniques developed by Urakawa [Ura99] we now give a proof of Theorem 5.1.

**Proof of Theorem 5.1.** Let \( B(x_0, r) \) be a ball centered at \( x_0 \) in \( \Gamma \) and \( V_l(r) \) be a ball in \( R_l \) centered at 0 with the same radius \( r \). Using the eigenfunction \( f_{\max} \) on \( V_l(r) \) we define a function (denoted by the same letter) on the ball \( B(x_0, r) \) in \( \Gamma \) by

\[
f_{\max}(x) = f_{\max}(r(x)) \quad \text{for all } x \in B(x_0, r),
\]

where \( r(x) = d(x, x_0) \). Now let \( \Delta \) and \( \Delta_l \) be the Laplace operators on \( B(x_0, r) \) and \( V_l(r) \) respectively. For all \( x \in B(x_0, r) \) and \( z \in V_l(r) \) with \( 0 < s = r(x) = r_l(z) \leq r \) where \( r_l(z) = d(z, 0) = |z| \) we obtain

\[
\Delta f_{\max}(x) = f_{\max}(x) - \frac{\mu_0(x)f_{\max}(x) + \mu_+(x)f_{\max}(x + 1) + \mu_-(x)f_{\max}(x - 1)}{\mu(x)}
\]

\[
= \frac{\mu_-}{\mu(x)}(f_{\max}(x) - f_{\max}(x - 1)) + \frac{\mu_+}{\mu(x)}(f_{\max}(x) - f_{\max}(x + 1))
\]
and
\[ \Delta^l f_{\text{max}}(z) = \frac{1}{l} (f_{\text{max}}(x) - f_{\text{max}}(x - 1)) + \frac{l - 1}{l} (f_{\text{max}}(x) - f_{\text{max}}(x + 1)) . \]

Putting these two equations together we obtain:

\[ \frac{\Delta^l f_{\text{max}}(z)}{f_{\text{max}}(z)} - \frac{\Delta f_{\text{max}}(x)}{f_{\text{max}}(x)} = \left( \frac{1}{l} \cdot \frac{\mu_-(x)}{\mu(x)} \right) \frac{f_{\text{max}}(s) - f_{\text{max}}(s - 1)}{f_{\text{max}}(s)} \]

\[ + \left( \frac{l - 1}{l} \cdot \frac{\mu_+(x)}{\mu(x)} \right) \frac{f_{\text{max}}(s) - f_{\text{max}}(s + 1)}{f_{\text{max}}(s)} . \]  

(34)

Moreover, in the case \( r(x) = r_l(z) = 0 \) we obtain

\[ \frac{\Delta^l f_{\text{max}}(0)}{f_{\text{max}}(0)} - \frac{\Delta f_{\text{max}}(x_0)}{f_{\text{max}}(x_0)} = 0 . \]

(35)

It is important to note that by Lemma \ref{lem:5.3}

\[ 1 < \frac{f_{\text{max}}(s) - f_{\text{max}}(s + 1)}{f_{\text{max}}(s)} \leq \frac{f_{\text{max}}(s) - f_{\text{max}}(s - 1)}{f_{\text{max}}(s)} \text{ for all } 0 \leq s \leq r \]

and by our assumption it holds that \( \left( \frac{1}{l} - \frac{\mu_-(x)}{\mu(x)} \right) \leq 0 \) for all \( x \in B(x_0, r) \).

Hence we obtain

\[ \frac{\Delta^l f_{\text{max}}(z)}{f_{\text{max}}(z)} - \frac{\Delta f_{\text{max}}(x)}{f_{\text{max}}(x)} \leq \left( \frac{1}{l} \cdot \frac{\mu_-(x)}{\mu(x)} + \frac{l - 1}{l} \cdot \frac{\mu_+(x)}{\mu(x)} \right) \frac{f_{\text{max}}(s) - f_{\text{max}}(s + 1)}{f_{\text{max}}(s)} \]

\[ = \frac{\mu_0(x)}{\mu(x)} \left( 1 - \frac{f_{\text{max}}(s + 1)}{f_{\text{max}}(s)} \right) \]

\[ \leq 2 \frac{\mu_0(x)}{\mu(x)} . \]

where used again Lemma \ref{lem:5.3} Using the Bartia-type estimate in Lemma \ref{lem:5.4} it follows

\[ \lambda_{\text{max}}(B(x_0, r)) \geq \inf_{x \in B(x_0, r)} \frac{\Delta f_{\text{max}}(x)}{f_{\text{max}}(x)} \]

\[ \geq \inf_{z \in V_l(r)} \frac{\Delta^l f_{\text{max}}(z)}{f_{\text{max}}(z)} - 2 \sup_{x \in B(x_0, r)} \frac{\mu_0(x)}{\mu(x)} \]

\[ = \lambda_{\text{max}}(V_l(r)) - 2 \kappa(x_0, r) . \]
Corollary 5.2. Let $B(x_0, r)$ be a ball in $\Gamma$ and $2 \leq l = \sup_{x \in B(x_0, r)} \frac{\mu(x)}{\mu_-(x)} < \infty$. Then the largest Dirichlet eigenvalue $\lambda_{\max}(B(x_0, r))$ satisfies for $l > 2$

$$\lambda_{\max}(B(x_0, r)) \geq 1 + \frac{2\sqrt{l-1}}{l} \cos\left(\frac{\pi}{r+1}\right) - 2\kappa(x_0, r),$$

and for $l = 2$

$$\lambda_{\max}(B(x_0, r)) \geq 1 + \cos\left(\frac{\pi}{2(r+1)}\right) - 2\kappa(x_0, r).$$

By a similar argument as in Theorem 5.1 we obtain a lower bound estimate of $\lambda_1$ which implies the upper bound estimate for $\lambda_{\max}$. The following theorem generalizes (ii) of Theorem 3.3 in [Ura99].

Theorem 5.2 (Eigenvalue comparison theorem II). Let $B(x_0, r)$ be a ball in $\Gamma$. Suppose $2 \leq l := \inf_{x \in B(x_0, r)} \frac{\mu(x)}{\mu_+(x)} < \infty$, then

$$\lambda_1(B(x_0, r)) \geq \lambda_1(V_l(r)) - \kappa(x_0, r),$$

(36)

$$\lambda_{\max}(B(x_0, r)) \leq \lambda_{\max}(V_l(r)) + \kappa(x_0, r),$$

(37)

where $\kappa(x_0, r) := \sup_{x \in B(x_0, r)} \frac{\mu_0(x)}{\mu(x)}$.

Proof. By (iv) of Lemma 3.1 the bipartiteness of the weighted half-line $R_l$ and Theorem 3.1 it suffices to prove (36). Let $f_1$ be the first eigenfunction of the Dirichlet Laplace operator on $V_l(r)$ in $R_l$. By Lemma 5.1, $f_1$ is a positive, decreasing function on $V_l(r)$. We define a function on $B(x_0, r)$ in $\Gamma$ as $f_1(x) = f_1(r(x))$ where $r(x) = d(x, x_0)$. The same calculation as (34) and (35) yields for any $x \in B(x_0, r)$ and $z \in V_l(r)$ with $0 < s = r(x) = r_1(z) \leq r$ where $r_1(z) = d(z, 0)$

$$\frac{\Delta l f_1(z)}{f_1(z)} - \frac{\Delta f_1(x)}{f_1(x)} = \left(\frac{1 - \mu_-(x)}{l} - \frac{\mu_+(x)}{\mu(x)}\right) \frac{f_1(s) - f_1(s - 1)}{f_1(s)} + \left(\frac{l - 1}{l} - \frac{\mu_+(x)}{\mu(x)}\right) \frac{f_1(s) - f_1(s + 1)}{f_1(s)},$$

and

$$\frac{\Delta l f_1(0)}{f_1(0)} - \frac{\Delta f_1(x_0)}{f_1(x_0)} = 0.$$
Hence
\[
\frac{\Delta^l f_1(z)}{f_1(z)} - \frac{\Delta f_1(x)}{f_1(x)} \leq \left( \frac{l-1}{l} - \frac{\mu_+(x)}{\mu(x)} \right) \frac{f_1(s) - f_1(s+1)}{f_1(s)}
\]
\[
= \left( \frac{\mu_0(x)}{\mu(x)} + \frac{\mu_-(x)}{\mu(x)} - \frac{1}{l} \right) \left( 1 - \frac{f_1(s+1)}{f_1(s)} \right)
\]
\[
\leq \frac{\mu_0(x)}{\mu(x)} \leq \kappa(x_0, r).
\]

Then by the Barta’s theorem for the first eigenvalue, see Theorem 2.1 in [Ura99],

\[
\lambda_1(B(x_0, r)) \geq \inf_{x \in B(x_0, r)} \frac{\Delta f_1}{f_1} \geq \inf_{z \in V_l(r)} \frac{\Delta^l f_1}{f_1} - \kappa(x_0, r)
\]
\[
= \lambda_1(V_l(r)) - \kappa(x_0, r).
\]

By Lemma 5.2 we obtain the following corollary which is the counterpart to Corollary 5.2 (we omit the easier case of \( l = 2 \)).

**Corollary 5.3.** Let \( B(x_0, r) \) be a ball in \( \Gamma \) and \( 2 < l = \inf_{x \in B(x_0, r)} \frac{\mu(x)}{\mu_-(x)} < \infty \). Then

\[
\lambda_1(B(x_0, r)) \geq 1 - \frac{2\sqrt{l-1}}{l} \cos \frac{\pi}{r + \eta} - \kappa(x_0, r),
\]
\[
\lambda_{\text{max}}(B(x_0, r)) \leq 1 + \frac{2\sqrt{l-1}}{l} \cos \frac{\pi}{r + \eta} + \kappa(x_0, r)
\]

where as before \( \eta = \frac{2(l-1)}{l-2} \), and \( \kappa(x_0, r) = \sup_{x \in B(x_0, r)} \frac{\mu_0(x)}{\mu(x)} \).

### 6. Eigenvalue estimates for finite graphs

In this section, we show how the eigenvalue comparison theorem obtained in the last section can be used to estimate the largest eigenvalues of finite graphs.

**Definition 6.1.** We say two balls \( B_1 \) and \( B_2 \) are edge disjoint if \( |E(B_1, B_2)| = 0 \).

**Lemma 6.1.** Let \( G = (V, E) \) be a finite graph with \( \#V = N \) and let \( B_1, \ldots, B_m \) be \( m \geq 1 \) edge disjoint balls in \( \Gamma \). Then

\[
\theta_{N-m} \geq \min_{i=1, \ldots, m} \lambda_{\text{max}}(B_i),
\]

where \( \theta_i, i = 0, \ldots, N - 1 \) are the eigenvalues of the Laplace operator of the finite graph and \( \lambda_i \) are the eigenvalues of the Laplace operator with Dirichlet boundary conditions.
Proof. We use the same technique as Quenell [Que96] who proved a similar result for the smallest eigenvalue of a finite graph. By the variational characterization of the eigenvalues we have

\[ \theta_{N-1} = \sup_{g \neq 0} \frac{(\Delta g, g)_G}{(g, g)_G}, \]

and

\[ \theta_{N-m} = \sup_{g \perp g_{N-1}, \ldots, g_{N-m+1}} \frac{(\Delta g, g)_G}{(g, g)_G}, \]

where \((f, g)_G = \sum_{x \in V} \mu(x) f(x) g(x)\) is the scalar product on the finite graph \(G\) and \(g_i\) is an eigenfunction for the eigenvalue \(\theta_i\). From now on we use the following notation: \(\Delta_i = \Delta_{B_i}\) denotes the Dirichlet Laplace operator on the ball \(B_i\).

Let \(f_i : B_i \to \mathbb{R}\) be the eigenfunction corresponding to the largest eigenvalue \(\lambda_{\text{max}}(B_i)\) of the Dirichlet Laplace operator \(\Delta_i\). We extend this to a function \(\tilde{f}_i : V \to \mathbb{R}\) on the whole of \(V\) by setting

\[ \tilde{f}_i(x) = \begin{cases} f_i(x) & \text{if } x \in B_i \\ 0 & \text{if } x \notin B_i. \end{cases} \]

Because the balls \(B_1, \ldots, B_m\) are disjoint the functions \(\tilde{f}_i, \tilde{f}_j\) satisfy \(\text{supp} \tilde{f}_i \cap \text{supp} \tilde{f}_j = \emptyset\) and hence \((\tilde{f}_i, \tilde{f}_j)_G = 0\) for all \(i \neq j\). From this it follows that the functions \(\tilde{f}_1, \ldots, \tilde{f}_m\) span a \(m\)-dimensional subspace in \(\ell^2(G, \mu)\). Hence there exists coefficients \(a_i\) such that the function

\[ \tilde{f} := \sum_{i=1}^{m} a_i \tilde{f}_i \]

is orthogonal to the \(m-1\) dimensional subspace spanned by the \(g_i\), i.e. \((\tilde{f}, g_i) = 0\) for all \(i = N-1, \ldots, N-m+1\). From the variational principle \(38\) we immediately obtain

\[ \theta_{N-m} \geq \frac{(\Delta \tilde{f}, \tilde{f})_G}{(\tilde{f}, \tilde{f})_G}. \]

We wish to show that

\[ \frac{(\Delta \tilde{f}_i, \tilde{f}_i)_G}{(\tilde{f}_i, \tilde{f}_i)_G} = \frac{(\Delta_i f_i, f_i)_{B_i}}{(f_i, f_i)_{B_i}}. \]
This can be seen from the following straightforward calculation:

\[
\begin{align*}
(\Delta \tilde{f}_i, \tilde{f}_i)_G &= \sum_{x \in V} \mu(x) \Delta \tilde{f}_i(x) \tilde{f}_i(x) \\
&= \sum_{x \in V} \mu(x)[(\tilde{f}_i(x) - \frac{1}{\mu(x)} \sum_{y \in V} \mu_{xy} \tilde{f}_i(y)) \tilde{f}_i(x)] \\
&= \sum_{x \in B_i} \mu(x)[(f_i(x) - \frac{1}{\mu(x)} \sum_{y \in B_i} \mu_{xy} f_i(y)) f_i(x)] \\
&= \sum_{x \in B_i} \mu(x) \Delta_i f_i(x) f_i(x) \\
&= (\Delta_i f_i, f_i)_{B_i}.
\end{align*}
\]

Together with \((\tilde{f}_i, \tilde{f}_i)_G = \sum_{x \in V} \mu(x) \tilde{f}_i(x)^2 = \sum_{x \in B_i} \mu(x) f_i(x)^2 = (f_i, f_i)_{B_i}\), we conclude

\[
\frac{(\Delta \tilde{f}_i, \tilde{f}_i)_G}{(f_i, \tilde{f}_i)_G} = \frac{(\Delta_i f_i, f_i)_{B_i}}{(f_i, f_i)_{B_i}} = \lambda_{\max}(B_i).
\]

Next we show that

\[
(41) \quad (\Delta \tilde{f}_i, \tilde{f}_j)_G = \sum_{x \in V} \mu(x)[\tilde{f}_i(x) - \frac{1}{\mu(x)} \sum_{y \in V} \mu_{xy} \tilde{f}_i(y)] \tilde{f}_j(x) = 0 \text{ if } i \neq j.
\]

We have a look at each term

\[
(42) \quad [\tilde{f}_i(x) - \frac{1}{\mu(x)} \sum_{y \in V} \mu_{xy} \tilde{f}_i(y)] \tilde{f}_j(x)
\]

in the sum separately: We distinguish the following two cases: (i) If \(x \notin B_j\) equation (42) is obviously equal to zero. (ii) If \(x \in B_j\) it follows from the edge disjointness of the balls that \(x \notin B_i\) and that all neighbors \(y \sim x\) are not contained in \(B_i\). Hence also in the latter case equation (42) is equal to zero. Hence if \(i \neq j\), then \((\Delta \tilde{f}_i, \tilde{f}_j)_G = 0\).
Using (40) and (41) we compute

\[(\Delta \tilde{f}, \tilde{f})_G = (\Delta \sum_{i=1}^m a_i \tilde{f}_i, \sum_{j=1}^m a_j \tilde{f}_j)_G\]

\[
= \sum_{i,j=1}^m a_i a_j (\Delta \tilde{f}_i, \tilde{f}_j)_G \\
= \sum_{i=1}^m a_i^2 (\Delta \tilde{f}_i, \tilde{f}_i)_G \\
= \sum_{i=1}^m a_i^2 (\Delta_i \tilde{f}_i, f_i)_B_i \\
\geq \min_{i=1, \ldots, m} \lambda_{\max}(B_i) \sum_{i=1}^m a_i^2 (f_i, f_i)_B_i \\
= \min_{i=1, \ldots, m} \lambda_{\max}(B_i) (\tilde{f}, \tilde{f})_G.
\]

Combining this with (39) completes the proof. □

**Theorem 6.1.** Let \( G = (V, E) \) be a finite graph and suppose that \( 2 < l = \sup_{x \in V} \frac{\mu(x)}{\mu_-(x)} < \infty \). The \((N - m)\)-th eigenvalue \( \theta_{N-m} \), \( m = 1, 2, \ldots, \left\lfloor \frac{D}{2} \right\rfloor \) of the Laplace operator on \( G \) satisfies

\[
\theta_{N-m} \geq 1 + 2 \sqrt{l - 1} \cos \left( \frac{\pi}{\left\lfloor \frac{D}{2m} \right\rfloor} \right) - 2 \kappa(G),
\]

where \( D \geq 2 \) is the diameter of the graph, \( \kappa(G) = \sup_{x_0 \in G} \kappa(x_0, G) \) and \( \kappa(x_0, G) = \sup_{x \in G} \frac{\mu_0(x)}{\mu(x)} \).

The case \( D = 1 \) is not interesting since the graph is then a complete graph and hence all eigenvalues are precisely known anyway.

**Proof.** Let \( m = 1, 2, \ldots, \left\lfloor \frac{D}{2} \right\rfloor \) be given. We can find \( m \) vertices \( x_1, \ldots, x_m \) in \( G \) that satisfy

\[
d(x_i, x_{i+1}) \geq 2r,
\]

where \( r = \left\lfloor \frac{D}{2m} \right\rfloor \geq 1 \). Then the balls \( B_i := B(x_i, r - 1) \), \( i = 1, \ldots, m \) are edge disjoint. Indeed if we assume that there exists \( x \in B_i \) and \( y \in B_{i+1} \) such that \( x \sim y \), then we obtain from the triangle inequality

\[
2r \leq d(x_i, x_{i+1}) \leq d(x_i, x) + d(x, y) + d(y, x_{i+1}) \leq r - 1 + 1 + r - 1 = 2r - 1
\]

which is a contradiction. Hence we have shown that we can find \( m = 1, \ldots, \left\lfloor \frac{D}{2} \right\rfloor \) edge disjoint balls \( B(x_i, \left\lfloor \frac{D}{2m} \right\rfloor - 1) \) in \( G \). Using Corollary 5.2 and
Lemma 6.1 we obtain
\[
\theta_{N-m} \geq \min_{i=1,\ldots,m} \lambda_{\text{max}}(B(x_i, \lfloor \frac{D}{2m} \rfloor - 1)) \\
\geq 1 + \frac{2\sqrt{l-1}}{l} \cos \left( \frac{\pi}{\lfloor \frac{D}{2m} \rfloor} \right) - 2 \max_{i=1,\ldots,m} \kappa(x_i, G) \\
\geq 1 + \frac{2\sqrt{l-1}}{l} \cos \left( \frac{\pi}{\lfloor \frac{D}{2m} \rfloor} \right) - 2\kappa(G).
\]

\[\square\]

Remark 6.1. 
(i) Under the same assumptions as in the last theorem, Urakawa [Ura99] showed that
\[\theta_m \leq 1 - 2 \frac{\sqrt{l-1}}{l} \cos \left( \frac{\pi}{\lfloor \frac{D}{2m} \rfloor} \right),\]
for all \(m = 1, 2, \ldots, \lfloor \frac{D}{2} \rfloor\). Thus Theorem 6.1 is a counterpart of Urakawa’s result.

(ii) Using our results in this section together with the results in [Ura99] one immediately obtains a proof of the famous Alon-Boppana-Serre theorem for the smallest nonzero eigenvalue and a Alon-Boppana-Serre-type theorem for the largest eigenvalue if there are no short circuits of odd length in the graph, see [Fri93, Lub94, DSV03]. In fact one even obtains slightly stronger results since we do not need the assumption that the graph is regular nor that \(l\) in the above theorems is an integer, but we only need that the degrees are bounded from above. A generalization of the Alon-Boppana-Serre theorem (for both the smallest non-zero and the largest eigenvalue) for non-regular graphs with bounded degree but integer \(l\) were obtained recently in [Moh10], by using a different method based on the eigenvalue interlacing property of induced subgraphs.

7. The Top of the Spectrum of \(\Delta\)

In Section 2 we discussed that the spectrum of the Laplace operator is connected to the spectrum of the Dirichlet Laplace operator, see (5). Now we are going to use the results obtained for the Dirichlet Laplace operator in the previous sections to estimate the top of \(\sigma(\Delta)\).

The following simple observation will be often used throughout this section.

Lemma 7.1. Let \(K\) be a finite subset of \(\Gamma\) and \(\Omega \uparrow \Gamma \setminus K\) be an exhaustion of \(\Gamma \setminus K\). We have
\[
\lim_{\Omega \uparrow \Gamma \setminus K} \lambda_{\text{max}}(\Omega) = \overline{\lambda}(\Gamma \setminus K),
\]
where \(\overline{\lambda}(\Gamma \setminus K) = \sup \sigma(\Delta_{\Gamma \setminus K})\) and \(\Delta_{\Gamma \setminus K}\) is the Laplace operator with Dirichlet boundary condition on \(\Gamma \setminus K\).
Proof. First we note that the limit on the l.h.s. exists. This follows from the monotonicity of the largest eigenvalue (see Lemma 3.1 (v)).

For any $f \in C_0(\Gamma \setminus K)$ (as before $C_0$ denotes the space of finitely supported functions) there exists a finite $\Omega(f) \subset \Gamma \setminus K$ such that $\text{supp} f \subset \Omega(f)$ and consequently

$$\frac{(df, df)}{(f, f)} \leq \sup_{g \in C_0(\Omega(f))} \frac{(dg, dg)}{(g, g)}.$$ 

If $\Omega \uparrow \Gamma \setminus K$ is an exhaustion of $\Gamma \setminus K$, we obtain for all $f \in C_0(\Gamma \setminus K)$

$$\frac{(df, df)}{(f, f)} \leq \lim_{\Omega \uparrow \Gamma \setminus K} \sup_{g \in C_0(\Omega)} \frac{(dg, dg)}{(g, g)}.$$

Since $\overline{C_0(\Gamma \setminus K)} = \ell^2(\Gamma \setminus K)$ one can show that [DK88]

$$\overline{\lambda}(\Gamma \setminus K) = \sup_{f \in C_0(\ell^2(\Gamma \setminus K))} \frac{(df, df)}{(f, f)} = \sup_{f \in C_0(\ell^2(\Gamma \setminus K))} \frac{(df, df)}{(f, f)} = \lim_{\Omega \uparrow \Gamma \setminus K} \lambda_{\max}(\Omega).$$

Altogether we conclude that

$$\overline{\lambda}(\Gamma \setminus K) = \sup_{f \in C_0(\Gamma \setminus K)} \frac{(df, df)}{(f, f)} \leq \lim_{\Omega \uparrow \Gamma \setminus K} \sup_{g \in C_0(\Omega)} \frac{(dg, dg)}{(g, g)} = \lim_{\Omega \uparrow \Gamma \setminus K} \lambda_{\max}(\Omega).$$

On the other hand, for any finite $\Omega \subset \Gamma \setminus K$ it follows from the monotonicity of the largest Dirichlet eigenvalue that

$$\lambda_{\max}(\Omega) \leq \overline{\lambda}(\Gamma \setminus K).$$

Taking an exhaustion $\Omega \uparrow \Gamma \setminus K$ we obtain

$$\lim_{\Omega \uparrow \Gamma \setminus K} \lambda_{\max}(\Omega) \leq \overline{\lambda}(\Gamma \setminus K).$$

This completes the proof. $\square$

**Lemma 7.2.** Similarly, the smallest eigenvalue satisfies

$$\lim_{\Omega \uparrow \Gamma \setminus K} \lambda_1(\Omega) = \underline{\lambda}(\Gamma \setminus K),$$

where $\underline{\lambda}(\Gamma \setminus K) = \inf \sigma(\Delta_{\Gamma \setminus K}).$

Proof. The proof is similar to the one of the last lemma, so we omit it here. $\square$

The last two lemmata will be very useful in the following since they allows us to transfer results form finite to infinite subsets.

**Lemma 7.3.** For any finite $K \subset \Gamma$ we have

$$2\bar{h}(\Gamma \setminus K) + h(\Gamma \setminus K) \leq \overline{\lambda}(\Gamma \setminus K) \leq 1 + \sqrt{1 - (1 - \bar{h}(\Gamma \setminus K))^2},$$

where $\bar{h}(\Gamma \setminus K) = \lim_{\Omega \uparrow \Gamma \setminus K} \bar{h}(\Omega)$ and $h(\Gamma \setminus K) = \lim_{\Omega \uparrow \Gamma \setminus K} h(\Omega)$. 

$$\text{(44)}$$
Proof. From Lemma 7.1 and (28) we immediately obtain for an exhaustion \( \Omega \uparrow \Gamma \setminus K \):

\[
\overline{\lambda}(\Gamma \setminus K) = \lim_{\Omega \uparrow \Gamma \setminus K} \lambda_{\max}(\Omega)
\leq \lim_{\Omega \uparrow \Gamma \setminus K} \left( 1 + \sqrt{1 - (1 - \tilde{h}(\Omega))^2} \right)
= 1 + \sqrt{1 - (1 - \tilde{h}(\Gamma \setminus K))^2}
\]

and

\[
\overline{\lambda}(\Gamma \setminus K) = \lim_{\Omega \uparrow \Gamma \setminus K} \lambda_{\max}(\Omega)
\geq \lim_{\Omega \uparrow \Gamma \setminus K} (2\tilde{h}(\Omega) + h(\Omega))
= 2\tilde{h}(\Gamma \setminus K) + h(\Gamma \setminus K).
\]

\(\Box\)

Now we obtain an estimate for the top of the spectrum by the Cheeger and the dual Cheeger constant.

**Theorem 7.1.** The top of the spectrum satisfies:

\[
2\tilde{h}(\Gamma) + h(\Gamma) \leq \overline{\lambda}(\Gamma) \leq 1 + \sqrt{1 - (1 - \tilde{h}(\Gamma))^2}
\]

**Proof.** Let \( K = \emptyset \) in (44). This completes the proof. \(\Box\)

For completeness, we include the estimate of the bottom of the spectrum which is an easy consequence of Theorem 4.4 and Lemma 7.2.

**Theorem 7.2** (cf. [Fuj96b]). For any finite \( K \subset \Gamma \) we have

(45) \( 1 - \sqrt{1 - h^2(\Gamma \setminus K)} \leq \underline{\lambda}(\Gamma \setminus K) \leq h(\Gamma \setminus K) \),

(46) \( 1 - \sqrt{1 - h^2(\Gamma)} \leq \underline{\lambda}(\Gamma) \leq h(\Gamma) \).

8. THE LARGEST EIGENVALUE AND GEOMETRIC PROPERTIES OF GRAPHS

For an infinite graph \( \Gamma \) with positive spectrum (called a nonamenable graph), i.e. \( \lambda(\Gamma) > 0 \) which is equivalent to \( h(\Gamma) > 0 \) by (46) in Theorem 7.2 it is known that the graph has exponential volume growth and very fast heat kernel decay, (see (10.4) and Lemma 8.1 in [Woe00]). In addition, nonamenability of graphs is a rough-isometric invariant property (see Theorem 4.7 in [Woe00]). In this section we ask the question what the top of the spectrum can tell us about the graph. More precisely we are asking what we can infer from \( \bar{\lambda}(\Gamma) < 2 \) about the graph. In contrast to \( \underline{\lambda}(\Gamma) > 0 \) we will present some examples (see Example 8.2 and Example 8.3) which show that \( \bar{\lambda}(\Gamma) < 2 \) is not rough-isometric invariant. Before giving these examples, we prove some affirmative results which indicate that the top of the spectrum controls the geometry of the graph to the same extent as the
first eigenvalue does if the graph is in some sense close to a bipartite one. Note that Theorem 7.1 implies that $\bar{\lambda}(\Gamma) < 2$ is equivalent to $\bar{h}(\Gamma) < 1$.

In the sequel, we fix some vertex $x_0 \in \Gamma$. We denote the geodesic sphere of radius $r$ ($r \in \mathbb{N} \cup \{0\}$) centered at $x_0$ by $S_r := S_r(x_0) = \{ y \in \Gamma : d(y, x_0) = r \}$. Let $p_r := |E(S_r, S_{r+1})|$, $q_r := |E(S_r, \Gamma)|$, $P_r := \sum_{i=0}^{r} p_i$, $Q_r := \sum_{i=0}^{r} q_i$ and $p_{-1} = 0$, $p_0 = \mu(x_0) - \mu_{x_0,x_0}$, $q_0 = \mu_{x_0,x_0}$. Then $\text{vol}(B_r) := \text{vol}(B_r(x_0)) = P_{r-1} + P_r + Q_r$.

**Theorem 8.1.** Let $\Gamma$ be an infinite graph with $\bar{h}(\Gamma) \leq 1 - \epsilon_0$ for some $0 < \epsilon_0 < 1$. If

$$\limsup_{r \to \infty} \frac{Q_r}{\text{vol}(B_r)} < \epsilon_0,$$

then

$$\text{vol}(B_r) \geq C_1 e^{C_2 r},$$

for some $C_1, C_2 > 0$ and any $r \geq 1$.

**Remark 8.1.** The condition (47) means that the graph $\Gamma$ is close to a bipartite graph in the sense that $Q_r$ is dominated by $\text{vol}(B_r)$ (or equivalently $P_r$). Of course condition (47) is trivially satisfied for bipartite graphs since $(Q_r = 0)$ in this case. Obviously $\lim_{r \to \infty} \frac{Q_r}{\text{vol}(B_r)} = 0$ is stronger than (47) which will be used in Corollary 8.1 and 8.2.

**Proof.** By (47) there exists $r_0 > 0$ and $\theta < \epsilon_0$ such that $Q_r \leq \theta \text{vol}(B_r)$ for any $r \geq r_0$. Then we have

$$\text{vol}(B_r) = P_{r-1} + P_r + Q_r \leq P_{r-1} + P_r + \theta \text{vol}(B_r).$$

This yields

$$2\theta \text{vol}(B_r) \leq P_{r-1} + P_r \leq 2P_r. \tag{48}$$

We introduce an alternating partition of $B_r$ as follows. Let $V_1 = S_0 \cup S_2 \cup \cdots \cup S_r$ and $V_2 = S_1 \cup S_3 \cup \cdots \cup S_{r-1}$ if $r$ is even, or $V_1 = S_0 \cup S_2 \cup \cdots \cup S_{r-2}$ and $V_2 = S_1 \cup S_3 \cup \cdots \cup S_r$ if $r$ is odd. Then $\bar{h}(\Gamma) \leq 1 - \epsilon_0$ implies that

$$2|E(V_1, V_2)| \leq (1 - \epsilon_0) \text{vol}(B_r).$$

That is

$$2P_{r-1} \leq (1 - \epsilon_0) \text{vol}(B_r) \leq \frac{2(1 - \epsilon_0)}{1 - \theta} P_r,$$

where the last inequality follows from (48). Then we have for any $r \geq r_0 + 1$

$$p_r = P_r - P_{r-1} \geq \left( \frac{1 - \theta}{1 - \epsilon_0} - 1 \right) P_{r-1} \geq \left( \frac{1 - \theta}{1 - \epsilon_0} - 1 \right) \left( \frac{1 - \theta}{2} \right) \text{vol}(B_r) \geq \delta(\epsilon_0) \text{vol}(B_{r-1}),$$
where $\delta(\epsilon_0) = \left(\frac{1-\theta}{1-\epsilon_0} - 1\right) \frac{(1-\theta)}{2} > 0$ since $\theta < \epsilon_0$. Then for $r \geq r_0 + 1$

$$\text{vol}(B_r) = \text{vol}(B_{r-1}) + p_r + p_{r-1} + q_r \geq \text{vol}(B_{r-1}) + p_r \geq (1 + \delta(\epsilon_0))\text{vol}(B_{r-1}).$$

The iterations imply that

$$\text{vol}(B_r) \geq (1 + \delta)^{r-r_0}\text{vol}(B_{r_0}),$$

for any $r \geq r_0$. The theorem follows.$\square$

Next we give some sufficient conditions for the previous theorem. We assume in the next two corollaries that $\text{vol}(B_r) \uparrow \infty$ ($r \to \infty$). A sufficient condition for this is for example that $\mu_{xy} \geq C_0 > 0$ for any $x \sim y$.

**Corollary 8.1.** Let $\Gamma$ be an infinite graph with $\tilde{h}(\Gamma) \leq 1 - \epsilon_0$ for some $0 < \epsilon_0 < 1$ and

$$\sum_{r=0}^{\infty} \frac{q_r}{\text{vol}(B_r)} < \infty,$$

where we assume that $\text{vol}(B_r) \uparrow \infty$ ($r \to \infty$). Then we have

$$\text{vol}(B_r) \geq C_1e^{C_2r},$$

for some $C_1, C_2 > 0$ and any $r \geq 1$.

**Proof.** Kronecker’s lemma, which is well-known in probability theory (see [Dur96]), (49), and $\text{vol}(B_r) \uparrow \infty$ imply that

$$\frac{Q_r}{\text{vol}(B_r)} \to 0, \quad r \to \infty.$$ 

Then the corollary follows from Theorem 8.1.$\square$

**Corollary 8.2.** Let $\Gamma$ be an infinite graph with $\text{vol}(B_r) \uparrow \infty$ ($r \to \infty$), $\tilde{h}(\Gamma) \leq 1 - \epsilon_0$ for some $0 < \epsilon_0 < 1$ and

$$p_r \to \infty, \quad \frac{q_r}{p_r} \to 0, \quad r \to \infty.$$ 

Then we have

$$\text{vol}(B_r) \geq C_1e^{C_2r},$$

for some $C_1, C_2 > 0$ and any $r \geq 1$.

**Proof.** The corollary follows from

$$\frac{Q_r}{\text{vol}(B_r)} \leq \frac{Q_r}{P_r} \to 0, \quad r \to \infty,$$

since $p_r \to \infty$ and $\frac{q_r}{p_r} \to 0$ (L’Hospital’s Rule).$\square$

We recall the definition of Cayley graphs. Let $G$ be a group. A subset $S \subset G$ is called a generating set of $G$ if any $x \in G$ can be written as $x = s_1s_2 \cdots s_n$ for some $n \geq 1$ and $s_i \in S$, $1 \leq i \leq n$. 


**Definition 8.1.** For a group $G$ and a finite symmetric generating set $S$ of $G$ (i.e. $S = S^{-1}$), there exists a graph $\Gamma = (V,E)$ associated with the pair $(G,S)$ where the set of vertices $V = G$ and $(x,y) \in E$ iff $x = ys$ for some $s \in S$. It is called the Cayley graph associated with $(G,S)$.

In the following examples, we calculate the largest eigenvalues of Dirichlet Laplace operator. These calculations will show that $\bar{\lambda} < 2$ is not a rough-isometric invariant.

**Example 8.1.** Let $P_\infty$ be the graph of an infinite line which is the Cayley graph of $(\mathbb{Z},\{\pm 1\})$, see Figure 3. Denote $\Omega^0_n := \{k_0 + i : 0 \leq i \leq n - 1\}$. By the symmetry, $\lambda(\Omega^0_n) = \lambda(\Omega^1_n)$ for any $k_0, k_1 \in \mathbb{Z}$. Without loss of generality, we consider the Dirichlet eigenvalues of $\Omega_n := \Omega^0_n$. Then the eigenvalues of $\Delta_{\Omega_n}$ are

$$1 - \cos \frac{j\pi}{n + 1}, \quad j = 1, 2, \cdots, n.$$  

Hence

$$\lambda_1(\Omega_n) = 1 - \cos \frac{\pi}{n + 1} \to 0,$$

and

$$\lambda_{\max}(\Omega_n) = 1 + \cos \frac{\pi}{n + 1} \to 2,$$

as $n \to \infty$. By (5) this implies that $\bar{\lambda}(\Gamma) = 0$ and $\overline{\lambda}(\Gamma) = 2$.

**Example 8.2.** Let $\Gamma = (V,E)$, $V = \{i \in \mathbb{Z}\} \cup \{i' \in \mathbb{Z}\} = V_1 \cup V_2$, i.e. the disjoint union $\mathbb{Z} \sqcup \mathbb{Z}$, and $E = E(V_1,V_1) \cup E(V_2,V_2) \cup E(V_1,V_2)$ where

$$E(V_1,V_1) = \{(i,j) : i,j \in V_1, |i-j| = 1\},$$

$$E(V_2,V_2) = \{(i',j') : i',j' \in V_2, |i'-j'| = 1\},$$

and

$$E(V_1,V_2) = \{(i,j') : i \in V_1, j' \in V_2, |i-j'| = 0 \text{ or } 1\},$$

see Figure 4. Let $\Omega_n = \{i \in V_1 : 0 \leq i \leq n - 1\} \cup \{j' \in V_2 : 0 \leq j' \leq n - 1\}$. Then

$$\lambda_1(\Omega_n) = \frac{4}{5}(1 - \cos \frac{\pi}{n + 1}) \to 0,$$

and

$$\lambda_{\max}(\Omega_n) = \frac{4}{5}(1 + \cos \frac{\pi}{n + 1}) \to \frac{8}{5} < 2,$$

as $n \to \infty$. Again by (5) we have $\bar{\lambda}(\Gamma) = 0$ and $\overline{\lambda}(\Gamma) = \frac{8}{5}$.
Example 8.3. Let $\Gamma$ be the Cayley graph of $(\mathbb{Z} \times \mathbb{Z}_3, \{(\pm 1, \pm 1)\})$ and $\Omega_n = \{i \in \mathbb{Z} : 0 \leq i \leq n - 1\} \times \mathbb{Z}_3$, see Figure 5. Then
\[
\lambda_1(\Omega_n) = \frac{1}{2}(1 - \cos \frac{\pi}{n + 1}) \to 0,
\]
and
\[
\lambda_{\text{max}}(\Omega_n) = \frac{5}{4} + \frac{1}{2} \cos \frac{\pi}{n + 1} \to \frac{7}{4} < 2,
\]
n $\to \infty$. Again by \(\frac{5}{4}\) we have $\lambda(\Gamma) = 0$ and $\bar{\lambda}(\Gamma) = \frac{7}{4}$.

Remark 8.2. Example 8.2 and Example 8.3 show that in general without additional conditions like (47) one cannot hope that $\Gamma$ has exponential volume growth under the assumption of $\bar{h}(\Gamma) < 1$ (or, by Theorem 7.1, equivalently $\bar{\lambda}(\Gamma) < 2$). The heat kernels $p_n(x, y)$ in Example 8.2 and Example 8.3 decay as $\frac{1}{\sqrt{n}}$ which is the slowest possible rate of heat kernel on infinite graph, see [Cou99]. Hence the spectral gap of $\bar{\lambda}(\Gamma)$ ($\bar{\lambda}(\Gamma) < 2$) does not imply any nice heat kernel estimate.

We recall the definition of rough isometries between metric spaces, also called quasi-isometries (see Gro81, Woe00, BBI01).

Definition 8.2. Let $(X, d^X), (Y, d^Y)$ be two metric spaces. A rough isometry is a mapping $\phi : X \to Y$ such that
\[
a^{-1}d^X(x, y) - b \leq d^Y(\phi(x), \phi(y)) \leq ad^X(x, y) + b,
\]
where $a, b \geq 0$.
for all \(x,y \in X\), and
\[
d^Y(z,\phi(x)) \leq b,
\]
for any \(z \in Y\), where \(a \geq 1, b \geq 0\) and \(d^Y(z,\phi(x)) := \inf\{d^Y(z,\phi(x)) : x \in X\}\). It is called an \((a,b)\)-rough isometry.

For infinite graphs, we consider the metric structure defined by the graph distance. It is easy to see that the graphs in Example 8.1, Example 8.2, and Example 8.3 are rough-isometric to each other, but in the first example \(\bar{\lambda}(\Gamma) = 2\) whereas in the other two examples \(\bar{\lambda}(\Gamma) < 2\). Hence the spectral gap of \(\bar{\lambda}(\Gamma)\) is not a rough-isometric invariant although this is true for the spectral gap of \(\lambda(\Gamma)\), see \[Woe00\]. We summarize these insights in the following corollary:

**Corollary 8.3.** For infinite graphs, the property that \(\bar{\lambda}(\Gamma) < 1\), or equivalently \(\bar{\lambda}(\Gamma) < 2\), is not a rough-isometric invariant.

### 9. The Largest Eigenvalue of Graphs with Certain Symmetries

In this section, we shall consider an upper bound estimate for the top of the spectrum which comes from the geometric properties of graphs with certain symmetries, see quasi-transitive graphs in \[Woe00\].

Let \(\Gamma\) be a locally finite, connected, unweighted (infinite) graph with the graph distance \(d\). An automorphism of \(\Gamma\) is a self-isometry of the metric space \((\Gamma,d)\). The group of all automorphisms of \(\Gamma\) is denoted by \(\text{Aut}(\Gamma)\). For any \(x \in \Gamma\), as a subset of \(\Gamma\), \(\text{Aut}(\Gamma)x := \{gx : g \in \text{Aut}(\Gamma)\}\) is called an orbit of \(x\) with respect to \(\text{Aut}(\Gamma)\). It is easy to see that all orbits, denoted by \(\Gamma/\text{Aut}(\Gamma) = \{O_i\}_{i \in I}\), compose a partition of \(\Gamma\).

**Definition 9.1.** A graph \(\Gamma\) is called quasi-transitive if there are finitely many orbits for \(\text{Aut}(\Gamma)\), i.e. \(\#I < \infty\). It is called vertex-transitive if there is only one orbit, i.e. \(\#I = 1\).

By the definition, vertex-transitive graphs are quasi-transitive. In particular, Cayley graphs (see Definition 8.1) are vertex-transitive and hence quasi-transitive. Figure 6 is an example of a quasi-transitive graph which is not vertex-transitive.

Let \(\Gamma\) be a quasi-transitive graph and \(\Gamma/\text{Aut}(\Gamma) = \{O_i\}_{i \in I}\). Denote by \(d(\Gamma) := \max\{d_x : x \in \Gamma\}\) the maximal degree of \(\Gamma\) (in accordance with the existing literature we denote in this section the degree of an unweighted graph by \(d_x\) instead of \(\mu(x)\)). For any (infinite) graph \(\Gamma\), a cycle of length \(n\), is a closed path as \(x_0 \sim x_1 \sim \cdots \sim x_{n-1} \sim x_0\). It is called a circuit (or simple cycle) if there are no repeated vertices on the cycle. Let us denote by \(OG(\Gamma)\) the odd girth of \(\Gamma\), i.e. the length of the shortest odd cycles in the graph \(\Gamma\) (\(OG(\Gamma) = \infty\) if there is no odd cycles). It is easy to see that there is an odd-length circuit attaining this number if \(OG(\Gamma) < \infty\). In addition, \(OG(\Gamma) = \infty\) if and only if \(\Gamma\) is bipartite.

In case \(\Gamma\) is a finite graph, we consider the ordinary normalized Laplace operator (not the Dirichlet one) defined on \(\Gamma\), see \[Chu97, Gri09, BJ\]. Then
the largest eigenvalue of the Laplace operator on $\Gamma$ is equal to 2 if and only if $\Gamma$ is bipartite. We want to prove a similar result for the largest eigenvalue of the Dirichlet Laplace operator on infinite graphs. However, for infinite graphs, 2 is not always contained in the spectrum if $\Gamma$ is bipartite (think of the homogenous trees), nor is the graph bipartite if 2 is in the spectrum (see the example in Remark 9.1). However, if the graph is a quasi-transitive graph we have the following theorem which is the main result of this section:

**Theorem 9.1.** Let $\Gamma$ be a quasi-transitive graph which is not bipartite. Then

$$\bar{\lambda}(\Gamma) \leq 1 - \delta,$$

where $\delta = \delta(d(\Gamma), \#\mathcal{I}, \text{OG}(\Gamma))$.

Recall that by Theorem 7.1, $\bar{\lambda}(\Gamma) = 2$ is equivalent to $\bar{h}(\Gamma) = 1$. As a direct corollary of Theorem 9.1, we obtain

**Corollary 9.1.** Let $\Gamma$ be a quasi-transitive graph. If $\bar{\lambda}(\Gamma) = 2$, then $\Gamma$ is bipartite.

**Remark 9.1.** This corollary is known in the case of Cayley graphs, see [dlHRV93]. Instead of using C*-algebra techniques, we estimate the geometric quantity, the dual Cheeger constant, to prove this result so that it can be easily extended to graphs with symmetry, e.g. quasi-transitive graphs. Note that this corollary is not true for general graphs. For instance, the standard lattice $\mathbb{Z}^2$ with one more edge $((0,0), (1,1))$ is a counterexample. In addition, the converse of the assertion is obviously not true (e.g. for the infinite $d$-regular tree $T_d$, we have $\bar{\lambda}(\Gamma) = 1 + \frac{2\sqrt{d-1}}{d}$).

In order to illustrate the main idea of the proof, we first consider the case of Cayley graphs which is much easier. Let $\Gamma$ be the Cayley graph of a group
and a generating set \((G, S)\). For simplicity, we assume the Cayley graph \(\Gamma\) is simple (i.e. the unit element \(e \notin S\)) and unweighted (i.e. \(\mu_{xy} = 1\) for any \(x \sim y\)). Cayley graphs are regular graphs with homogeneous structure. If there exists a circuit in \(\Gamma\), then there are isomorphic circuits passing through each vertex of \(\Gamma\).

The idea of the proof comes from the upper bound estimate of the maximum cut of finite graphs.

**Definition 9.2.** The maximum cut of a finite graph \(\Gamma\) is defined as

\[
Mc(\Gamma) := \max_{V = V_1 \cup V_2, V_1 \cap V_2 = \emptyset} |E(V_1, V_2)|.
\]

**Theorem 9.2.** Let \(\Gamma = (V, E)\) be a finite Cayley graph of \((G, S)\). If it is not bipartite, then

\[
Mc(\Gamma) \leq m(1 - \delta),
\]

where \(m\) is the number of edges in \(\Gamma\) and \(\delta = \delta(d(\Gamma), OG(\Gamma))\).

**Proof.** Let us denote by \(n = \sharp V, m = \sharp E\) and \(d = \sharp S\) the number of vertices, the number of edges and the degree of \(\Gamma\), respectively.

Case 1. \(OG(\Gamma) = 3\).

Then each vertex of \(\Gamma\) is contained in at least one triangle. We denote by \(\Delta\) the set of all triangles in \(\Gamma\) and by \(|\Delta| := \sharp \Delta\) the number of triangles in \(\Gamma\). Since each vertex is contained in at least one triangle, \(3|\Delta| \geq n\). For any partition of \(V = V_1 \cup V_2\) (\(V_1 \cap V_2 = \emptyset\)), we define a (single-valued) mapping as

\[
T : \Delta \to E(V_1, V_2)^c
\]

\[
\Delta \ni \Delta_1 \mapsto T(\Delta_1),
\]

where \(E(V_1, V_2)^c = E \setminus E(V_1, V_2)\) and \(T(\Delta_1)\) is one of the edges of the triangle \(\Delta_1\) which does not lie in \(E(V_1, V_2)\). Note that there always exists at least one such edge in each triangle \(\Delta_1\).

We shall bound the multiplicity of the mapping \(T\), i.e. the universal upper bound of \(\sharp T^{-1}(e)\) for any \(e \in E(V_1, V_2)^c\). If \(\Delta_1, \ldots, \Delta_k \in T^{-1}(e)\) for some \(e \in E(V_1, V_2)^c\), then the triangles \(\Delta_1, \ldots, \Delta_k\) share one edge \(e\). It is easy to see that \(k \leq d - 1\) by local finiteness. Hence

\[
|E(V_1, V_2)^c| \geq \sharp T(\Delta) \geq \frac{|\Delta|}{d - 1}.
\]

It follows from \(m = \frac{nd}{2}\) and \(3|\Delta| \geq n\) that

\[
|E(V_1, V_2)^c| \geq \frac{n}{3(d - 1)} = \frac{2m}{3d(d - 1)}.
\]

This yields

\[
(50) \quad |E(V_1, V_2)| \leq m - \frac{2m}{3d(d - 1)} = m \left(1 - \frac{2}{3d(d - 1)}\right).
\]

Case 2. \(OG(\Gamma) = 2s + 1\) for \(s \geq 2\).
Let $\bigcirc$ denote the set of circuits of length $2s + 1$ in $\Gamma$ and $|\bigcirc| := \sharp \bigcirc$. Then there exists at least one circuit of length $2s + 1$ passing through each vertex which implies $(2s + 1)|\bigcirc| \geq n$. Given any partition of $\Gamma = V_1 \cup V_2$ ($V_1 \cap V_2 = \emptyset$), we may define a mapping as

$$T : \bigcirc \rightarrow E(V_1, V_2)^c$$

where $T(\bigcirc_1)$ is one of edges of the circuit $\bigcirc_1$ which does not lie in $E(V_1, V_2)$. By the odd length of the circuit $\bigcirc_1$, the mapping $T$ is well defined.

The key point is to estimate the multiplicity of the mapping $T$. Suppose $\bigcirc_1, \cdots, \bigcirc_k \in T^{-1}(e)$ for some $e \in E(V_1, V_2)^c$, then each of them is contained in the geodesic ball $B_s(a)$ where $a$ is one of the vertices of $e$. Since the number of vertices in $B_s(a)$ is bounded above by $d^s$, the number of circuits of length $2s + 1$ in $B_s(a)$ is bounded above by $(2s+1)^s$. That is $k = \sharp T^{-1}(e) \leq (2s+1)^s$. Hence

$$|E(V_1, V_2)^c| \geq \sharp T(\bigcirc) \geq \frac{|\bigcirc|}{(2s+1)^s} \geq \frac{2m}{d(2s + 1)(2s+1)} = \frac{m}{C(d, s)}$$

This implies that

$$|E(V_1, V_2)| \leq m - \frac{m}{C(d, s)} = m(1 - \delta(d, s)). \tag{51}$$

The theorem follows from (50) and (51). \hfill \square

The ingredient of Theorem 9.2 is that the ratio $\frac{M_G(V_1, V_2)}{m} \leq 1 - \delta$ has an upper bound which is independent of the number of vertices of the finite Cayley graph. This suggests that we may obtain similar estimates for $\bar{h}(\Gamma)$ for infinite Cayley graphs. Indeed, we now prove the following theorem for infinite Cayley graphs which is the special case of Theorem 9.1.

**Theorem 9.3.** Let $\Gamma$ be an infinite Cayley graph of $(G, S)$. If it is not bipartite, then

$$\bar{h}(\Gamma) \leq 1 - \delta,$$  \tag{52}

where $\delta = \delta(d(\Gamma), OG(\Gamma)) > 0$.

**Remark 9.2.** The ingredient of the proof is that once there is an odd-length circuit in the Cayley graph $\Gamma$, there is at least one odd-length circuit passing through each vertex which makes the graph systematically different from a bipartite one. Then it becomes possible to estimate $\bar{h}(\Gamma)$ from above.

**Proof.** By the definition of $\bar{h}(\Gamma) = \lim_{\Omega \nearrow \Gamma} \bar{h}(\Omega)$ and (19), it suffices to show that for any finite disjoint subsets $V_1, V_2 \subset \Gamma$,

$$|E(V_1, V_2)| \leq C(|E(V_1, V_1)| + |E(V_2, V_2)| + |\partial(V_1 \cup V_2)|),$$

where $C = C(d(\Gamma), OG(\Gamma))$. We denote $V_3 := (V_1 \cup V_2)^c = \Gamma \setminus (V_1 \cup V_2)$ and $F(V_1, V_2) := E(V_1, V_1) \cup E(V_2, V_2) \cup \partial(V_1 \cup V_2)$. Let $d = d(\Gamma)$.

Case 1. $OG(\Gamma) = 3$.
Then there is a triangle passing through each vertex of $\Gamma$. We define a mapping as

$$T : E(V_1, V_2) \rightarrow F(V_1, V_2)$$

$$E(V_1, V_2) \ni e \mapsto T(e).$$

Given any $e \in E(V_1, V_2)$, we choose a vertex $a \in e$ and a triangle $\triangle_1$ containing the vertex $a$. Since there exists at least one edge $e_1$ of $\triangle_1$ which lies in $F(V_1, V_2)$, we define $T(e) = e_1$.

The key point is to estimate the multiplicity of the mapping $T$. For any $e_1 \in F(V_1, V_2)$ such that $T^{-1}(e_1) \neq \emptyset$, any $e \in T^{-1}(e_1)$ lies in the geodesic ball $B_2(b)$ where $b$ is one of vertices of $e_1$. Then

$$\sharp T^{-1}(e_1) \leq \text{the number of edges in } B_2(b) \leq \binom{d^2}{2}.$$  

Hence

$$\sharp F(V_1, V_2) \geq \sharp T(E(V_1, V_2)) \geq \frac{|E(V_1, V_2)|}{\binom{d^2}{2}}.$$  

Since $|E(V_1, V_1)| = 2\sharp E(V_1, V_1)$, we have

$$\sharp F(V_1, V_2) \leq |E(V_1, V_1)| + |E(V_2, V_2)| + |\partial(V_1 \cup V_2)|.$$  

This yields

$$|E(V_1, V_1)| \leq C(d)(|E(V_1, V_1)| + |E(V_2, V_2)| + |\partial(V_1 \cup V_2)|).$$  

Case 2. $OG(\Gamma) = 2s + 1 \ (s \geq 2)$.

Then there exists at least one circuit of length $2s + 1$ passing through each vertex. We claim that for any circuit $C$ of length $2s + 1$ which intersects $V_1 \cup V_2$, $C \cap F(V_1, V_2) \neq \emptyset$, i.e. there exists at least one edge of $C$ contained in $F(V_1, V_2)$. If not, all the edges of $C$ are contained in $E(V_1, V_2)$ since the set of edges $E = E(V_1, V_2) \cup F(V_1, V_2) \cup E(V_3, V_3)$ and $C \cap (V_1 \cup V_2) \neq \emptyset$. Then $C$ is a bipartite subgraph which contradicts to the odd length of $C$. Then the claim follows.

We define a mapping as

$$T : E(V_1, V_2) \rightarrow F(V_1, V_2)$$

$$E(V_1, V_2) \ni e \mapsto T(e).$$

For any $e \in E(V_1, V_2)$ and a vertex $a \in e$, there is a circuit $C$ of length $2s + 1$ passing through $a$. By the claim, we may choose one of edges of $C$, $e_1$, which lies in $F(V_1, V_2)$ and define $T(e) = e_1$.

We shall estimate the multiplicity of the mapping $T$. For any $e_1 \in T(E(V_1, V_2))$ and $e \in T^{-1}(e_1)$, the edge $e$ lies in the geodesic ball $B_{s+1}(b)$ where $b$ is a vertex of $e_1$. Then $\sharp T^{-1}(e_1) \leq \binom{d^{s+1}}{2}$. Hence

$$\sharp F(V_1, V_2) \geq \frac{|E(V_1, V_2)|}{\binom{d^{s+1}}{2}}.$$  

Then

$$|E(V_1, V_2)| \leq C(d, s)(|E(V_1, V_1)| + |E(V_2, V_2)| + |\partial(V_1 \cup V_2)|).$$
Combining the case 1 and 2, we prove the theorem. □

Now we prove the main theorem of this section.

Proof of Theorem 10.1. Let $\Gamma$ be a quasi-transitive graph and $\Gamma/\text{Aut}(\Gamma) = \{O_i\}_{i \in \mathcal{I}}$ ($\sharp \mathcal{I} < \infty$). Suppose $\text{OG}(\Gamma) = 2s + 1$ ($s \geq 1$), then there exists a circuit of length $2s + 1$, $C_{2s+1}$, passing through a vertex in $O_j$ for some $j \in \mathcal{I}$. By the group action of $\text{Aut}(\Gamma)$, there exists at least one circuit of length $2s + 1$ passing through each vertex in $O_j$. Since $\Gamma$ is connected, for any $x \in \Gamma$, there is a path $P = \bigcup_{i=0}^{s-1}\{(x_i, x_{i+1})\}$, i.e. $x = x_0 \sim x_1 \sim \cdots \sim x_l$ such that $x_l \in O_j$ and $l \leq \sharp \mathcal{I}$.

For any two disjoint subset $V_1$ and $V_2$, we define a mapping

$$T : E(V_1, V_2) \rightarrow F(V_1, V_2)$$

where $F(V_1, V_2) := E(V_1, V_1) \cup E(V_2, V_2) \cup \partial(V_1 \cup V_2)$. For any $e \in E(V_1, V_2)$ and a vertex $x_0 \in e$, there is a path $P = \bigcup_{i=0}^{s-1}\{(x_i, x_{i+1})\}$ such that $x_l \in O_j$ and $l \leq \sharp \mathcal{I}$. Let $C_{2s+1}$ be a circuit of length $2s + 1$ passing through $x_l$.

The same argument in Theorem 9.3 implies that $(P \cup C_{2s+1}) \cap F(V_1, V_2) \neq \emptyset$. We choose one edge $e_1 \in (P \cup C_{2s+1}) \cap F(V_1, V_2)$ and define $T(e) = e_1$.

We estimate the multiplicity of the mapping $T$. For any $e_1 \in T(E(V_1, V_2))$ and $e \in T^{-1}(e_1)$, the edge $e$ lies in the geodesic ball $B_{\sharp \mathcal{I} + 1}(b)$ where $b$ is a vertex of $e_1$. Since $d(\Gamma) = \max\{d_x : x \in \Gamma\}$, $\sharp T^{-1}(e_1) \leq (d(\Gamma)\sharp + 1).$ Hence

$$\sharp F(V_1, V_2) \geq \frac{|E(V_1, V_2)|}{(d(\Gamma)\sharp + 1)}. $$

Then

$$|E(V_1, V_2)| \leq C(d(\Gamma), \sharp \mathcal{I}, s)(|E(V_1, V_1)| + |E(V_2, V_2)| + |\partial(V_1 \cup V_2)|)$$

which proves the theorem. □

10. THE ESSENTIAL SPECTRUM OF $\Delta$

In this section, we use the results in the previous sections to estimate the essential spectrum of infinite graphs. We denote by $\sigma^{\text{ess}}(\Gamma)$ the essential spectrum of normalized Laplace operator $\Delta$ of an infinite graph $\Gamma$ and define the bottom and the top of the essential spectrum as $\Lambda^{\text{ess}}(\Gamma) := \inf \sigma^{\text{ess}}(\Gamma)$ and $\lambda^{\text{ess}}(\Gamma) = \sup \sigma^{\text{ess}}(\Gamma)$. Recall that the spectrum is given by $\sigma(\Gamma) := \sigma^{\text{disc}}(\Gamma) \cup \sigma^{\text{ess}}(\Gamma)$, where $\sigma^{\text{disc}}(\Gamma)$ contains isolated eigenvalues of finite multiplicity.

We recall the following theorem due to Fujiwara [Fuj96b] which is also known as the decomposition principle in the continuous setting, see [DL79].

Theorem 10.1 (Fujiwara [Fuj96b]). Let $\Gamma$ be an infinite graph and $K$ be a finite subgraph. Then $\sigma^{\text{ess}}(\Delta(\Gamma)) = \sigma^{\text{ess}}(\Delta(\Gamma \setminus K))$ where $\Delta(\Gamma \setminus K)$ denotes the Laplace operator with Dirichlet boundary conditions.
This theorem shows that the essential spectrum of a graph is invariant under compact perturbations of the graph. Hence the essential spectrum does not depend on local properties of graph but rather on the behavior of the graph at infinity. In order to estimate the essential spectrum of $\Delta$ we are going to define the Cheeger and the dual Cheeger constant at infinity.

**Definition 10.1** (cf. [Bro84, Fuj96b]). The Cheeger constant at infinity $h_\infty$ is defined as

$$h_\infty = \lim_{K \uparrow \Gamma} h(\Gamma \setminus K).$$

**Definition 10.2.** The dual Cheeger constant at infinity $\tilde{h}_\infty$ is defined as

$$\tilde{h}_\infty = \lim_{K \uparrow \Gamma} \tilde{h}(\Gamma \setminus K).$$

**Theorem 10.2.** For the top of the essential spectrum we obtain

$$2\tilde{h}_\infty + h_\infty \leq \lambda^{\text{ess}}(\Gamma) \leq 1 + \sqrt{1 - (1 - \tilde{h}_\infty)^2}$$

**Proof.** Take an exhaustion $K \uparrow \Gamma$ in (44). This completes the proof. □

For completeness, we write down the estimate of the bottom of the essential spectrum which is a consequence of (45) in Theorem 7.2, see also [Fuj96b].

**Theorem 10.3** (cf. [Fuj96b]). For the bottom of the essential spectrum we have

$$1 - \sqrt{1 - h_\infty^2} \leq \lambda^{\text{ess}}(\Gamma) \leq h_\infty.$$

**Remark 10.1.** Remark 8.2 and Corollary 8.3 suggest that the dual Cheeger constant $\tilde{h}$ is weaker than the Cheeger constant $h$. Nevertheless, in the next theorem we prove the somehow surprising results that at infinity both quantities contain the same information.

The next theorem is the main result of this section.

**Theorem 10.4.** Let $\Gamma$ be a graph without self-loops. The following statements are equivalent:

(i) $\tilde{h}_\infty = 0$

(ii) $h_\infty = 1$

(iii) $\sigma^{\text{ess}}(\Gamma) = \{1\}$

**Proof.** The equivalence of (ii) and (iii) was proved by Fujiwara [Fuj96b]. It suffices to show the equivalence of (i) and (ii). From Theorem 4.1 and Theorem 4.3 it follows that for finite subsets $\Omega \subset V$

$$\frac{1}{2}(1 - h(\Omega)) \leq \tilde{h}(\Omega) \leq 1 - h(\Omega).$$
This immediately translates to results for infinite subsets, i.e. for finite \( K \subset \Gamma \)

\[
\frac{1}{2} (1 - h(\Gamma \setminus K)) = \lim_{\Omega \uparrow \Gamma \setminus K} \frac{1}{2} (1 - h(\Omega)) \leq \lim_{\Omega \uparrow \Gamma \setminus K} \tilde{h}(\Omega) = \tilde{h}(\Gamma \setminus K) \\
\leq \lim_{\Omega \uparrow \Gamma \setminus K} (1 - h(\Omega)) = 1 - h(\Gamma \setminus K).
\]

Taking an exhaustion \( K \uparrow \Gamma \) yields

\[
\frac{1}{2} (1 - h_\infty) \leq \overline{h}_\infty \leq 1 - h_\infty.
\]

This proves that \( \overline{h}_\infty = 0 \) if and only if \( h_\infty = 1 \). \(\square\)

**Remark 10.2.**

(i) The implication \( h_\infty = 1 \Rightarrow \overline{h}_\infty = 0 \) is also true for graphs with self-loops by Theorem 4.1.

(ii) The reason why the proof of Theorem 10.4 does not work for graphs with self-loops is that for such graphs one cannot always find a partition that satisfies (23) and hence the lower bound for \( h(\Omega) \) in Theorem 4.3 is not true.

(iii) In general, for graphs with self-loops, one cannot even expect that \( h_\infty > 0 \) if \( \overline{h}_\infty = 0 \). The next example shows a graph with self-loops for which \( h_\infty = \overline{h}_\infty = 0 \) holds.

**Example 10.1.** Consider the graph \( \Gamma \) in Figure 7. It is easy to verify that for this graph \( \overline{h}_\infty = h_\infty = 0 \). Moreover, for this graph we have \( \sigma^{ess} = \{0\} \).

This can be seen as follows: We know that \( \lim_{\Omega \uparrow \Gamma \setminus K} \lambda_{max}(\Omega) = \lambda(\Gamma \setminus K) \), where \( \Omega \uparrow \Gamma \setminus K \) is an exhaustion of \( \Gamma \setminus K \) and \( \lambda_{max}(\Omega) \) is the largest eigenvalue of the Laplace operator with Dirichlet boundary conditions. Moreover, \( \lim_{K \uparrow \Gamma} \lambda(\Gamma \setminus K) = \lambda^{ess}(\Gamma) \) which yields \( \lim_{K \uparrow \Gamma} \lim_{\Omega \uparrow \Gamma \setminus K} \lambda_{max}(\Omega) = \lambda^{ess}(\Gamma) \).

By abuse of notation we denote in the following the closed ball centered at 0 with radius \( K \) by \( K \), i.e. \( K := B(0, K) \). By considering the trace of \( \Delta_\Omega \) where \( \Omega \uparrow \Gamma \setminus B(0, K) \) we have

\[
\lambda_{max}(\Omega) \leq \sum_{i=1}^{2^\Omega} \lambda_i(\Omega) = \text{tr}(\Delta_\Omega) \leq \sum_{k=K+1}^{2^\Omega+K} \frac{2}{2k+2} \leq \sum_{k=K+1}^{\infty} \frac{1}{2k-1} = 2^{1-K}.
\]

Taking the limit \( \Omega \uparrow \Gamma \setminus B(0, K) \) and the limit \( K \uparrow \Gamma \) (which corresponds to \( K \to \infty \)) on both sides we arrive at

\[
\overline{\lambda}^{ess}(\Gamma) = \lim_{K \to \infty} \lim_{\Omega \uparrow \Gamma \setminus B(0, K)} \lambda_{max}(\Omega) \leq 0.
\]

Since \( \sigma^{ess}(\Gamma) \subset [0, 2] \) and \( \sigma^{ess}(\Gamma) \neq \emptyset \) it follows that \( \sigma^{ess}(\Gamma) = \{0\} \). Together with Theorem 10.2 this also yields a proof for the statement \( h_\infty = h_\infty = 0 \).

**Theorem 10.5.** \( \overline{h}_\infty + h_\infty = 1 \) if \( \Gamma \) is bipartite.

**Proof.** As above one can show that for bipartite graphs Theorem 4.1 implies that \( h_\infty + h_\infty = 1 \). This completes the proof. \(\square\)
Figure 7. Counterexample for Theorem 10.4 if we allow loops in the graph.

Remark 10.3. In particular, Theorem 10.2, Theorem 10.3 and Theorem 10.5 imply that for bipartite graphs with $h_\infty = 0$ (or equivalently $\bar{h}_\infty = 1$) the essential spectrum is not concentrated at all since both 0 and 2 are contained in the essential spectrum.

We need the following consequence of the spectral theorem, see [Gla65] Theorem 13 or [Don81] Proposition 2.1:

Theorem 10.6. The interval $[\lambda, \infty)$ intersects the essential spectrum of a self-adjoint operator $A$ if and only if for all $\epsilon > 0$ there exists an infinite dimensional subspace $H_\epsilon \subset D(A)$ of the domain of $A$ such that $(A f - \lambda f + \epsilon f, f) > 0$ for all $f \in H_\epsilon$.

Theorem 10.7. Let $2 \leq l = \sup_{x \in V} \frac{\mu(x)}{\mu_{\infty}(x)} < \infty$, then

$$\sigma_{ess}(\Delta) \cap \left[ 1 + \frac{2\sqrt{l-1}}{l} - 2\kappa_\infty, 2 \right] \neq \emptyset,$$

where $\kappa_\infty := \lim_{K \uparrow \Gamma} \kappa(\Gamma \setminus K)$, $\Lambda(\Gamma \setminus K) = \sup_{x_i \in \Gamma \setminus K} \kappa(x_i, \Gamma \setminus K)$ and $\kappa(x_i, \Gamma \setminus K) = \sup_{x \in \Gamma \setminus K} \frac{\mu(x)}{\mu_{\infty}(x)}$ where we calculate $\mu_0$ with respect to $x_i$.

Proof. Since $2 \leq l = \sup_{x \in V} \frac{\mu(x)}{\mu_{\infty}(x)} < \infty$ we have from Theorem 5.1 that for all $r \geq 0$

$$\lambda_{\max}(B(x_0, r)) \geq \lambda_{\max}(V_i(r)) - 2\kappa(x_0, r).$$

Moreover since $\lim_{r \to \infty} \lambda_{\max}(V_i(r)) = \lambda_{\max}(R_i)$ we know that for all $\epsilon > 0$ there exists a $r(\epsilon) > 0$ such that

$$\lambda_{\max}(V_i(r(\epsilon))) > \lambda_{\max}(R_i) - \epsilon.$$

Thus for sufficiently large $r(\epsilon)$ we have

$$\lambda_{\max}(B(x_0, r(\epsilon))) > \lambda_{\max}(R_i) - 2\kappa(x_0, r(\epsilon)) - \epsilon.$$

Since $\Gamma \setminus K$ is an infinite graph we can find infinitely many points $x_i \in \Gamma \setminus K$ such that $B(x_i, r(\epsilon)) \subset \Gamma \setminus K$ are disjoint balls. Hence there exists infinitely many functions $f_i$ with finite disjoint support, $\text{supp} f_i \subset B(x_i, r(\epsilon)) \subset \Gamma \setminus K$ such that

$$\frac{(\Delta f_i, f_i)}{(f_i, f_i)} > \lambda_{\max}(R_i) - 2\kappa(\Gamma \setminus K) - \epsilon = 1 + \frac{2\sqrt{l-1}}{l} - 2\kappa(\Gamma \setminus K) - \epsilon.$$

Taking an exhaustion $K \uparrow \Gamma$ it follows that there exists an infinite dimensional subspace of $\ell^2(V, \mu)$ such that

$$(\Delta f - (1 + \frac{2\sqrt{l-1}}{l}) f + \kappa_\infty f + \epsilon f, f)_\mu > 0.$$
By Theorem 10.6 we conclude that
\[ \sigma^{\text{ess}}(\Delta) \cap \left[ 1 + \frac{2\sqrt{l - 1}}{l} - 2\kappa_{\infty}, 2 \right] \neq \emptyset. \]
□

From Lemma 5.5 we immediately obtain the following corollary:

**Corollary 10.1.** Let \( 2 \leq l = \sup_{x \in V} \frac{\mu(x)}{\mu_{-}(x)} < \infty \), then
\[ \sigma^{\text{ess}}(\Delta) \cap \left[ 1 + \frac{2\sqrt{l - 1}}{l} - 2C(\Gamma)_{\infty}, 2 \right] \neq \emptyset, \]
where \( C(\Gamma)_{\infty} := \lim_{K \uparrow \Gamma} \sup_{x \in \Gamma \setminus K} \sum_{y \in C(x)} \frac{\mu_{xy}}{\mu(x)} \) and \( C(x) := \{ y : y \sim x, x \text{ and } y \text{ are both contained in a circuit of odd length} \}. \)

**Remark 10.4.** In the special case of unweighted graphs with vertex degrees uniformly bounded from above by \( l \) where \( l \) is an integer, Fujiwara [Fuj96b] (Theorem 6) showed that \( \sigma^{\text{ess}} \cap \left[ 1 + \frac{2\sqrt{l - 1}}{l}, 2 \right] \neq \emptyset \). Even in the non-optimal case when we restrict ourselves to integer valued \( l \) our results improve the ones by Fujiwara [Fuj96b] if \( \kappa_{\infty} \leq \frac{\sqrt{l - 1}}{l - 1} \). Moreover, for bipartite graphs Fujiwara obtained \( \sigma^{\text{ess}}(\Delta) \cap \left[ 1 + \frac{2\sqrt{l - 1}}{l}, 2 \right] \neq \emptyset \). This is a special case of our result since we obtain the same estimate under the weaker assumption \( \kappa_{\infty} = 0 \) and we allow non-integer values for \( l \).

For some fixed reference point \( x_{0} \), we give the following definitions which are introduced in [Ura00].

**Definition 10.3.** For any (possibly infinite) subset \( U \subset \Gamma \) we define
\[ M_{-}(x_{0}, U) := \sup_{x \in U} \frac{\mu_{-}(x)}{\mu(x)}, \]
(53)\[ M_{-}\infty(x_{0}, \Gamma) := \lim_{K \uparrow \Gamma} M_{-}(x_{0}, \Gamma \setminus K), \]
and
\[ \kappa(x_{0}, U) := \sup_{x \in U} \frac{\mu_{0}(x)}{\mu(x)}, \]
(54)\[ \kappa_{\infty}(x_{0}, \Gamma) := \lim_{K \uparrow \Gamma} \kappa(x_{0}, \Gamma \setminus K). \]

For the decomposition principle (Theorem 10.1) we immediately obtain the following:

**Lemma 10.1.** Let \( \Gamma \) be an infinite graph and \( \Gamma' \) be the graph obtained from \( \Gamma \) by adding or deleting finitely many edges. Then \( \sigma^{\text{ess}}(\Delta(\Gamma)) = \sigma^{\text{ess}}(\Delta(\Gamma')) \).

**Proof.** Since we only change finitely many edges, there exists a finite subgraph \( K \subset \Gamma \) and a finite subgraph \( K' \subset \Gamma' \) such that \( \Gamma \setminus K = \Gamma' \setminus K' \). Thus by the last theorem the essential spectra of \( \Delta(\Gamma) \) and \( \Delta(\Gamma') \) coincide. □
A similar result to Lemma 10.1 was obtained by Mohar [Moh82] for the adjacency operator.

By using Lemma 10.1 we obtain the following estimates for the essential spectra of graphs in terms of \( M_{-\infty}(x_0, \Gamma) \) and \( \kappa_{\infty}(x_0, \Gamma) \).

**Theorem 10.8.** Let \( \Gamma \) be an infinite graph. Then

\[
\lambda^{\text{ess}}(\Gamma) \geq 1 - \frac{2\sqrt{I - 1}}{l} - \kappa_{\infty}(x_0, \Gamma),
\]

\[
\bar{\lambda}^{\text{ess}}(\Gamma) \leq 1 + \frac{2\sqrt{I - 1}}{l} + \kappa_{\infty}(x_0, \Gamma),
\]

where \( l = (M_{-\infty}(x_0, \Gamma))^{-1} \) and \( x_0 \) is an arbitrary vertex in \( \Gamma \).

**Proof.** Since it is well known [Fuj96a, DK88] that \( \lambda^{\text{ess}}(\Gamma) + \bar{\lambda}^{\text{ess}}(\Gamma) \leq 2 \), it suffices to show (55). By the definitions of \( M_{-\infty}(x_0, \Gamma) \) and \( \kappa_{\infty}(x_0, \Gamma) \), (53) and (54), for any \( \epsilon > 0 \), there exists \( r_0 > 0 \) such that

\[
M_{-\infty}(x_0, \Gamma) \leq M_{-}(x_0, \Gamma \setminus B(x_0, r_0)) \leq M_{-\infty}(x_0, \Gamma) + \epsilon
\]

and

\[
\kappa_{\infty}(x_0, \Gamma) \leq \kappa(x_0, \Gamma \setminus B(x_0, r_0)) \leq \kappa_{\infty}(x_0, \Gamma) + \epsilon.
\]

Let \( \Gamma \setminus B(x_0, r_0) = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_k \) where \( \Gamma_i (1 \leq i \leq k) \) are the connected components of \( \Gamma \setminus B(x_0, r_0) \). Then by Lemma 10.1

\[
\sigma^{\text{ess}}(\Gamma) = \sigma^{\text{ess}}(\Gamma \setminus B(x_0, r_0)) = \bigcup_{i=1}^k \sigma^{\text{ess}}(\Gamma_i).
\]

We want to use Corollary 5.3 to estimate \( \sigma^{\text{ess}}(\Gamma_i) \). But the reference point \( x_0 \) is not contained in \( \Gamma_i \) and thus the \( \frac{\mu_+(x)}{\mu(x)} \) and \( \frac{\mu_0(x)}{\mu(x)} \) will change if we choose some reference point in \( \Gamma_i \). In the following, we add a new reference point to each \( \Gamma_i \) which preserves the quantities \( \frac{\mu_+(x)}{\mu(x)} \) and \( \frac{\mu_0(x)}{\mu(x)} \).

We add a new vertex \( p_i \) to each \( \Gamma_i \) and let \( \Gamma_i' = \Gamma_i \cup \{p_i\} \). For each \( z \in \Gamma_i \) with \( d(z, x_0) = r_0 + 1 \), i.e. \( z \in S_{r_0+1}(x_0) \cap \Gamma_i \), we add an new edge \( zp_i \) in \( \Gamma_i' \) with the edge weight \( \mu_{zp_i} = |E(z, B(x_0, r_0))| \), that is, we identify the ball \( B(x_0, r_0) \) as a new vertex \( p_i \) and preserve the edge between \( \Gamma_i \) and \( B(x_0, r_0) \). Other edges in \( \Gamma_i \) remain unchanged in \( \Gamma_i' \). Then by this construction, for any \( x \in \Gamma_i \subset \Gamma_i' \) the quantities \( \frac{\mu_-(x)}{\mu(x)} \) and \( \frac{\mu_0(x)}{\mu(x)} \) w.r.t. \( x_0 \) in \( \Gamma \) are the same as those w.r.t. \( p_i \) in \( \Gamma_i' \). Then

\[
P_{-}(p_i, \Gamma_i') = M_{-}(x_0, \Gamma_i), \quad \kappa(p_i, \Gamma_i') = \kappa(x_0, \Gamma_i)
\]

since \( \frac{\mu_-(p_i)}{\mu(p_i)} = \frac{\mu_0(p_i)}{\mu(p_i)} = 0 \) w.r.t. \( p_i \) in \( \Gamma_i' \). Hence by (38) in Corollary 5.3 (letting \( r \to \infty \)), we have

\[
\lambda(\Gamma_i') \geq 1 - \frac{2\sqrt{(M_{-}(p_i, \Gamma_i'))^{-1} - 1}}{(M_{-}(p_i, \Gamma_i'))^{-1}} - \kappa(p_i, \Gamma_i').
\]

Since \( \sigma^{\text{ess}}(\Gamma_i') \subset \sigma(\Gamma_i') \),

\[
\lambda^{\text{ess}}(\Gamma_i') \geq \lambda(\Gamma_i').
\]
Hence
\[
\lambda_{\text{ess}}(\Gamma) = \min_{1 \leq i \leq k} \lambda_{\text{ess}}(\Gamma_i) \\
= \min_{1 \leq i \leq k} \lambda_{\text{ess}}(\Gamma'_i) \\
\geq \min_{1 \leq i \leq k} \left\{ 1 - \frac{2\sqrt{(M_-(p_i, \Gamma'_i))^{-1} - 1}}{(M_-(p_i, \Gamma'_i))^{-1}} - \kappa(p_i, \Gamma'_i) \right\} \\
\geq 1 - \frac{2\sqrt{(M_{-,\infty}(x_0, \Gamma) + \epsilon)^{-1} - 1}}{(M_{-,\infty}(x_0, \Gamma) + \epsilon)^{-1}} - \kappa_{x_0,\infty}(\Gamma) - \epsilon,
\]
where we use Lemma 10.1, (57), (58), (59), (60) and (61). Let \( \epsilon \to 0 \), we prove the theorem. \( \square \)

Theorem 10.8 yields a more general sufficient condition for the concentration of the essential spectrum than Theorem 5.4 in [Ura99]. It is a special case of a class of graphs with concentrated essential spectrum discovered by Higuchi, see [Fuj96b].

**Corollary 10.2.** Let \( \Gamma \) be an infinite graph. If \( M_{-,\infty}(x_0, \Gamma) = 0 \) and \( \kappa_{\infty}(x_0, \Gamma) = 0 \), then \( \sigma_{\text{ess}}(\Gamma) = \{1\} \).

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