Analysis of the reliability and efficiency of the technical system with preemptive recovery of a component

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Abstract. One of the methods to improve the reliability and efficiency of technical systems is preemptive recovery, which allows to reduce the effects of system failures. The article discusses the semi-Markov model for monitoring latent failures of a technical system, taking into account preemptive recovery for the remaining time until the next latent failure. The characteristics of the reliability and efficiency of the system are found. The results can be used to assess the effectiveness of the use of preemptive recovery.

1. Introduction

One of the methods to improve the reliability and efficiency of technical systems is preemptive recovery [1 - 5]. Preemptive recovery allows reducing the effects of aging and deterioration of system elements until latent failures appear. Latent failure is a failure that can only be detected as a result of monitoring. When preemptive recovery is used, the problem arises of determining the optimal period for monitoring the system, taking into account this factor [6].

The theory of semi-Markov processes is currently widely used for modeling systems for various purposes [7-11].

In [5], a semi-Markov model for monitoring latent failures of a technical system with preemptive recovery is considered, taking into account the time elapsed since the last latent system failure. This article creates a semi-Markov model of the technical system, which takes into account the control of latent failures and the conduct of preemptive recovery, taking into account the time remaining until the next latent failure. The stationary distribution of the embedded Markov chain is found. The stationary characteristics of the reliability and efficiency of the system are obtained, which allow analyzing the effect of preemptive recovery on the functioning of the system.

2. Description of the system

The system $S$ consists of one functional component and a control unit. The system operates as follows. At the initial instant the component starts to operate, control is activated. Component time to failure is random variables (RV) $\alpha$ with DF $F(t) = P(\alpha \leq t)$ and distribution density (DD) $f(t)$. Control is performed through random time intervals $\delta$ with DF $R(t) = P(\delta \leq t)$ and DD $r(t)$. Failure is detected after control execution only (latent failure). While control execution, component does not operate. Control execution time is RV $\gamma$ with DF $V(t) = P(\gamma \leq t)$ and DD $v(t)$. If, as a result of carrying-out of the control, it was determined that the component is operable and its residual time to failure is less
than prescribed level $\mu > 0$, then the preemptive recovery is being carried out. When residual time to latent failure is more than prescribed level $\mu > 0$, the component continues to operate. If, as a result of carrying-out of the control, a latent failure of the component was determined, then its ordinary recovery is carried out; the time of preemptive or ordinary recovery is $RV_\beta$ with DF $G(t) = P[\beta \leq t]$ and DD $g(t)$. It is supposed that $RV_\alpha$, $RV_\beta$, $RV_\delta$, $RV_\gamma$ are independent and have finite expectations.

3. Semi-Markov model construction

Let us describe the operation of Semi-Markov system $S$ by the Semi-Markov process $\xi(t)$ with discrete-continuous phase space of states. Let us introduce the following set $E$ of Semi-Markov states of the system

$$E = \{11, 210x, 211x, 2\bar{2}, 101x, 200, 222\}.$$

The informative meaning of the states encoding is the next:

- $11$ – the component is renewed and has started to operate, the control is on;
- $210x$ – the control has started, the component is operable and off, the time remaining to failure of the component is $x > 0$;
- $211x$ – the control has been performed, the time to failure of the component $x > 0$, is more than prescribed level $\mu$, operable component continued the operation;
- $2\bar{2}$ – the control has been performed, the time to failure of the component $x > 0$, is less than prescribed level $\mu$, the preemptive recovery has been started, the control is suspended;
- $101x$ – the failure has occurred, the time $x > 0$ is left until control beginning;
- $200$ – the control has been started, the failed component is off;
- $222$ – the control has been performed, the failure has been detected and ordinary recovery has been started, the control is suspended;

Time diagram of the system is depicted in the figure 1.

![Figure 1. Time diagram of the system’s operation.](image)

Let us determine sojourn times in the states of the system. For example, the sojourn time in the state $211x$ is defined by two factors: residual time $x$ until the latent failure and the time $\delta$ of control periodicity, hence $\theta_{211x} = \delta \land x$. Sojourn times in other states are defined in the same way:

$$\theta_{11} = \alpha \land \delta, \quad \theta_{210x} = \gamma, \quad \theta_{22} = \beta, \quad \theta_{101x} = x, \quad \theta_{200} = \gamma, \quad \theta_{222} = \beta.$$ (1)

Let us describe system transition events. Transition events from the states $11$, $211x$ are defined by expressions (2), (3) respectively:

$$\begin{align*}
\{11 \to 210x\} &= \{\alpha = \delta + x, \ x > 0\}; \\
\{111 \to 101dx\} &= \{\delta - \alpha \in dx, \ x > 0\};
\end{align*}$$ (2)

$$\begin{align*}
\{211x \to 210y\} &= \{\delta = x - y, \ x > \mu, \ y < x\}; \\
\{211x \to 101x\} &= \{\delta = x + y, \ x > \mu, \ y > 0\}. 
\end{align*}$$ (3)

Transitions $210x \to 222$, $210x \to 211x$, $222x \to 111$, $101x \to 200$, $200 \to 222$, $222 \to 111$ occur with unity probability.
Formulas (2), (3) give transient probabilities of embedded Markov chain (EMC) $\{\xi_n; n \geq 0\}$:

$$p_{111}^{210x} = P[\alpha = \delta + x] = \int_0^\infty f(x + t)r(t)dt, \ x > 0;$$

$$p_{111}^{101dx} = P[\alpha - \delta \in dx] = \int_0^\infty r(x + t)f(t)dt, \ x > 0;$$

$$p_{211}^{210y} = P[\delta = x - y] = r(x - y), \ x > \mu, \ y < x;$$

$$p_{211}^{211y} = P[\delta = x + y] = r(x + y), \ x > \mu, \ y > 0;$$

$$p_{210x}^{222} = 1, \text{if } 0 < x < \mu; \quad p_{211x}^{211x} = 1, \text{if } x > \mu;$$

$$P_{111}^{111} = P_{200}^{200} = P_{222}^{222} = 1$$

4. Definition of embedded Markov Chain stationary distribution

Let us denote by $\rho(111), \rho(222), \rho(200), \rho(222)$ the values of EMC $\{\xi_n; n \geq 0\}$ stationary distribution in states 111, 222, 200, 222. Assume stationary densities $\rho(210x), \rho(211x)$ and $\rho(101x)$ existence for states $210x, 211x$ and $101x$ correspondingly.

Using probabilities and probability densities of EMC $\{\xi_n; n \geq 0\}$ transition (4), let us construct the system of integral equations for the stationary distribution definition:

$$\begin{cases}
\rho(111) = \rho(222) + \rho(222), \\
\rho(210x) = \rho(111) \int_0^\infty f(x + t)r(t)dt + \int_{x>\mu}^\infty r(y - x)\rho(211y)dy, \ x < y, \ y > \mu, \\
\rho(211x) = \rho(210x), \ x > \mu, \\
\rho(222) = \int_0^\mu \rho(210x)dx, \\
\rho(101x) = \rho(111) \int_0^\infty r(x + t)f(t)dt + \int_{y>\mu}^\infty \rho(211y)r(x + y)dy, \ y > \mu, \\
\rho(200) = \int_0^\infty \rho(101y)dy, \\
\rho(222) = \rho(200), \\
2\rho(111) + \rho(200) + \int_0^\mu (\rho(101x) + \rho(210x))dx + \int_{y>\mu}^\infty \rho(211x)dx = 1.
\end{cases}$$

The last equation in the system (5) is a normalization requirement. By means of the method of successive approximations [12], the system of equations (5) is proved to have the following solution:
\( \rho_0 = (111), \)
\( \rho(2\hat{1}0x) = \rho_0 \int_0^\infty h_y(y-x)f(y)dy, \quad x > \mu, \)
\( \rho(2\hat{1}0x) = \rho_0 \left( \int_0^\infty f(x+t)r(t)dt + \rho_0 \int_0^\infty f(t)dt \int_0^\infty h_y(t-y)r(y-x)dy \right), \quad 0 < x < \mu, \)
\( \rho(211x) = \rho(2\hat{1}0x), \quad x > \mu, \)
\( \rho(222) = \rho_0 \left( \int_0^\infty f(t)R(t)dt - \int_\mu^\infty f(t)dt \int_0^\infty h_y(t-y)\overline{R}(y)dy \right), \)
\( \rho(101x) = \rho_0 \left( \int_0^\infty r(x+t)f(t)dt + \int_\mu^\infty f(t)dt \int_0^\infty r(x+y)h_y(t-y)dy \right), \quad y > \mu, \)
\( \rho(2\hat{0}0) = \rho_0 \left( \int_0^\infty f(t)\overline{R}(t)dt + \int_\mu^\infty f(t)dt \int_0^\infty h_y(t-y)\overline{R}(y)dy \right), \)
\( \rho(222) = \rho(2\hat{0}0). \)

Where \( h_y(t) = \sum_{n=1}^\infty r^{(n)}(t) \) is the density of renewal function of renewal process, generated by RV \( \delta \), \( r^{(n)}(t) \) is the \( n \)-th fold convolution of the distribution density \( r(t) \).

5. Definition of stationary characteristics

Let us split phase state space \( E \) into the following two subsets:
\( E_+ = \{111,211x\} \) — the system is in up-state;
\( E_- = \{2\hat{1}0x,222,101x,2\hat{0}0,222\} \) — the system is in down-state.

Let us find average stationary operating time to failure \( T_+ \) and average stationary restoration time \( T_- \) with the help of the next formulas [9,10]:
\[
T_+ = \frac{\int_{E_+} m(e)\rho(de)}{\int_{E_-} P(e,E_-)\rho(de)}, \quad T_- = \frac{\int_{E_-} m(e)\rho(de)}{\int_{E_+} P(e,E_+)\rho(de)},
\]
(7)

Where \( \rho(de) \) is the EMC stationary distribution \( \{\xi_n; n \geq 0\} \); \( m(e) \) are the average values of system sojourn times in states; \( P(e,E_-) \) are the probabilities of EMC transitions \( \{\xi_n; n \geq 0\} \) from up into down states.

Let us determine system average sojourn times in states with the help of formulas (1):
\[
m(111) = \int_0^\infty \overline{F}(t)\overline{R}(t)dt, \quad m(211x) = \int_0^\infty \overline{R}(t)dt,
\]
\[
m(2\hat{1}0x) = M_T, \quad m(222) = M_\beta, \quad m(101x) = x, \quad m(2\hat{0}0) = M_\gamma, \quad m(222) = M_\beta.
\]
(8)

Taking into account (3), (5) and (8), let us write down the expressions included in (7).
\[
\int_{E_+} m(e)\rho(de) = m(111)\rho(111) + \int_{E_-} m(211x)\rho(211x)dx =
\]
\[ F(t) = \rho_0 \left( \int_0^\infty \overline{F}(t \overline{R}(t))dt + \int_0^\infty dx \int_0^\infty \overline{R}(t)dt \int_0^t h_r(y-x)f(y)dy \right). \] (9)

After performing the necessary transformations of expression (9), we obtain:
\[ \int m(e)\rho(de) = \rho_0 \left( M\alpha + \mu \int_0^\infty \overline{F}(y+\mu)d\overline{H}_r(\gamma) - \mu \int_0^\infty \overline{R}(t)dt \int_0^\infty \overline{F}(y+t)d\overline{H}_r(y) \right) \] (10)

Next,
\[ \int m(e)\rho(de) = m(222)\rho(222) + m(200)\rho(200) + m(222)\rho(222) + \]
\[ + \int m(210x)\rho(210x)dx + \int m(101x)\rho(101x)dx = \]
\[ = \rho_0 \left( \beta E \int_0^\infty f(t)R(t)dt - \int_0^\infty f(t)\mu dt + \int_0^\infty f(t)dt \int_0^t h_r(t-y)(R(y)-R(y-\mu))dy \right) + \]
\[ + \gamma E \int_0^\infty f(t)\overline{R}(t)dt + \int_0^\infty f(t)dt \int_0^t h_r(t-y)\overline{R}(y)dy \right) + \]
\[ + \beta E \int_0^\infty f(t)\overline{R}(t)dt + \int_0^\infty f(t)dt + \int_0^t h_r(t-y)\overline{R}(y)dy \right) + \]
\[ + \gamma E \int_0^\infty dx f(x+t)r(t)dt + \int_0^\infty dx \int_0^t h_r(t-y)r(y-x)dy + \]
\[ + \int_0^\infty dx \int_0^\infty h_r(y-x)f(y)dy \right) + \int_0^\infty dx \int_0^\infty r(x+t)f(t)dt + \int_0^\infty dx \int_0^\infty f(t)dt \int_0^\infty r(x+y)h_r(t-y)dy \right). \] (11)

After performing the necessary transformations of expression (11), we obtain:
\[ \int m(e)\rho(de) = \rho_0 \left( E\beta + (E\gamma + E\delta) \int_0^\infty \overline{F}(t+\mu)d\overline{H}_r(t) - \int_0^\infty \overline{R}(x)dx \int_0^\infty \overline{F}(t+\mu)d\overline{H}_r(t) - E\alpha + \int_0^\infty \overline{R}(x)dx \int_0^\infty \overline{F}(t+x)d\overline{H}_r(t), \right) \] (12)

where \( \overline{H}_r(t) = \sum_{n=1}^{\infty} R^{(n)}(t) \) is the renewal function of the renewal process, generated by RV \( \delta \), \( R^{(n)}(t) \) is the \( n \)-th fold convolution of the distribution function \( R(t) \).

Then,
\[ \int m(e)\rho(de) = \rho_0 \left( E\beta + (E\delta + E\gamma) \int_0^\infty \overline{F}(t+\mu)d\overline{H}_r(t) \right). \] (13)

Next,
\[ \int P(e, E, \rho(de) = P(111, E, \rho(111) + \int_0^\infty P(211x,E, \rho(211x)dx = \]
After performing the necessary transformations of expression (13), we obtain:

$$
\int_0^\infty P(e, E_\cdot)\rho(de) = \rho_0 \left[ \tilde{F}(t + \mu)d\tilde{H}_r(t) \right].
$$

Therefore, taking into account formulas (10), (15), average stationary operating time to failure $T_+$ is:

$$
T_+ = \frac{E\alpha + \int_0^\infty \tilde{R}(t)dt \int_0^\infty \tilde{F}(y + \mu)dH_r(y) - \int_0^\infty \tilde{R}(t)dt \int_0^\infty \tilde{F}(y + t)dH_r(y)}{\int_0^\infty \tilde{F}(y + \mu)d\tilde{H}_r(y)},
$$

(16)
taking into account formulas (13), (15), average stationary restoration time $T_-$ is:

$$
T_- = \left\{ \begin{array}{l}
E\beta - E\alpha + (E\gamma + E\delta) \int_0^\infty \tilde{F}(t + \mu)d\tilde{H}_r(t) + \int_0^\mu \tilde{R}(x)dx \int_0^\infty \tilde{F}(t + x)dH_r(t) - \\
- \int_0^\mu \tilde{R}(x)dx \int_0^\infty \tilde{F}(t + \mu)dH_r(t) \end{array} \right\} \int_0^\infty \tilde{F}(t + \mu)d\tilde{H}_r(t),
$$

(17)

One should note the characteristics stationary availability factor $K_a$, mean stationary operating time to failure $T_+$, mean stationary restoration time $T_-$ relate like this:

$$
K_a = \frac{T_+}{T_+ + T_-},
$$

(18)

Let us find stationary availability factor from the ratio (18):

$$
K_a = \frac{E\alpha + \int_0^\infty \tilde{R}(t)dt \int_0^\infty \tilde{F}(y + \mu)dH_r(y) - \int_0^\infty \tilde{R}(t)dt \int_0^\infty \tilde{F}(y + t)dH_r(y)}{E\beta + (E\delta + E\gamma) \int_0^\infty \tilde{F}(t + \mu)d\tilde{H}_r(t)}
$$

(19)

Let us define system stationary efficiency characteristics: average specific income per calendar time unit $S$ and average specific expenses per time unit of up state $C$ [6]. To determine them, we use the formulas:

$$
S = \frac{\int_E m(e)f_s(e)\rho(de)}{\int_E m(e)\rho(de)}, \quad C = \frac{\int_E m(e)f_c(e)\rho(de)}{\int_E m(e)\rho(de)},
$$

(20)

where $f_s(e), f_c(e)$ are the functions, defining income and expenses per time unit in each state respectively.

Let $c_1$ be the income time unit of up state; $c_2$ be expenses for the system failure per time unit.

For the system considered $f_s(e), f_c(e)$ are:

$$
f_s(e) = \begin{cases}
  c_1, & e \in \{111, 211x\}, \\
  -c_2, & e \in \{210x, 222, 101x, 200, 222\}.
\end{cases}
$$
f_c(e) = \begin{cases} 
0, & e \in \{111, \ 211x\}, \\
c_2, & e \in \{210x, 222, 2101x, 200, 222\}. 
\end{cases} \tag{21}

Taking into account formulas (4), (6), (8), (21), average specific income per calendar time unit is defined by the ratio:

\[
S = \frac{(c_1 + c_2) \left( \mu \rho + \int_0^\infty \tilde{F}(y + \mu) dH_r(y) - \int_0^\infty \tilde{F}(y + t) dH_r(y) \right)}{\mu \beta + \left( M \delta + M \gamma \right) \int_0^\infty \tilde{F}(t + \mu) d\tilde{H}_r(y)} - c_2. \tag{22}
\]

Average specific expenses per time unit of system up-state are as follows:

\[
C = \frac{c_2 \left( M \beta + \left( M \gamma + M \delta \right) \int_0^\infty \tilde{F}(t + \mu) d\tilde{H}_r(y) \right)}{\mu \rho + \int_0^\infty \tilde{F}(y + \mu) dH_r(y) - \int_0^\infty \tilde{F}(y + t) dH_r(y)} - c_2. \tag{23}
\]

6. Conclusion

The formulas obtained (16), (17), (19), (22), (23) allow to calculate the values of average stationary time to failure, stationary availability factor, average income and expenses for different initial data and to solve the problem of finding the optimal period for preemptive recovery carrying-out. It is proposed to use this technique in the future for modeling of multicomponent systems taking into account preemptive recovery.

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