Periods of singular double octic Calabi–Yau threefolds and modular forms

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Abstract
By the modularity theorem, every rigid Calabi–Yau threefold $X$ has associated modular form $f$ such that the equality of $L$-functions $L(X, s) = L(f, s)$ holds. In this case, period integrals of $X$ are expected to be expressible in terms of the special values $L(f, 1)$ and $L(f, 2)$. We propose a similar interpretation of period integrals of a nodal model of $X$. It is given in terms of certain variants of a Mellin transform of $f$. We provide numerical evidence toward this interpretation based on a case of double octics.

KEYWORDS
Calabi–Yau threefold, $L$-function, modular forms, period integral

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1 | INTRODUCTION

A Calabi–Yau threefold is a smooth complex projective variety of dimension 3 such that

$$\Omega^3_X \simeq \mathcal{O}_X \text{ and } H^1(X, \mathcal{O}_X) = 0.$$ 

In particular, there exists a nonvanishing holomorphic 3-form $\omega$ on $X$, unique up to a constant. Period integrals of a Calabi–Yau threefold $X$ are integrals of the 3-form $\omega$ over integral 3-cycles. We shall denote by $\Lambda_X$ the period lattice of $X$, that is,

$$\Lambda_X := \left\{ \int_\gamma \omega : \gamma \in H_3(X, \mathbb{Z}) \right\} \subset \mathbb{C}.$$ 

As the canonical form $\omega$ is defined up to a constant factor, the lattice $\Lambda_X$ is also defined only up to rescaling. If the Calabi–Yau threefold $X$ is rigid, then $\Lambda_X$ defines an elliptic curve

$$J^2(X) := H^{3,0}(X)^*/H_3(X, \mathbb{Z}) = \mathbb{C}/\Lambda_X,$$

which is a particular case of the Griffiths intermediate Jacobian [8].

Period integrals were computed only for a small number of Calabi–Yau threefolds. In [4], approximations of period integrals of 11 double octics were established by a numeric integration using a very explicit description of the geometry of these varieties. These computations give strong numerical evidence of the proportionality between period integrals
of a rigid Calabi–Yau threefold and special values of the $L$-function of the corresponding cusp form as predicted by the Tate conjecture.

In [1], we proposed a different approach for computing period integrals. If a rigid Calabi–Yau threefold $X$ is a resolution of singularities of a singular element $X_{t_0}$ of a one-parameter family $\mathcal{X} = (X_t)_{t \in \mathbb{C}}$, then we can compute periods of $X$ as limits of certain period integrals of smooth elements of the family $\mathcal{X}$. Since period integrals of the Calabi–Yau threefolds $X_t$ satisfy the Picard–Fuchs equation, we can use this differential equation, especially its monodromy, to determine periods of $X$. This approach has two important advantages. First, it enables computation of much better approximations of the periods. Second, it depends only on the Picard–Fuchs operator and does not require any knowledge of the geometry of the considered variety.

In fact the approach based on the Picard–Fuchs operator computes periods of a singular variety $X_{t_0}$ rather than only those of the rigid Calabi–Yau threefold $X$. In general, the group of period integrals of $X_{t_0}$ has rank 3, while the group of periods of $X$ has rank 2. In [1], we verified that the additional periods computed using the monodromy of the Picard–Fuchs equation agree with additional integrals computed in [4] for polyhedral cycles in the nodal model $X_{t_0}$ that do not lift to a cycle in $X$.

The main goal of this paper is to propose an interpretation of the period integrals of the singular fiber $X_{t_0}$. As showed in [4], period integrals of its smooth, birational model $X$ are proportional to the special values of the $L$-function of the modular form $f$ associated with the rigid Calabi–Yau threefold $X$ by the modularity theorem. We provide numerical evidence for a similar proportionality between periods of $X_{t_0}$ and certain partial integrals $M(f, k)$, which appear naturally in the classical proof of the functional equation for $L(f, s)$. As a consequence, we get strong evidence that the period integrals of $X_{t_0}$ are also determined by the cusp form attached to $X$.

If a rigid Calabi–Yau threefold $X$ has models $X_1, X_2$ isomorphic over $\mathbb{C}$ but not over $\mathbb{Q}$, then the cusp forms $f_1, f_2$ attached to these models can also differ. To remedy this ambiguity of the cusp form, we usually consider the twist of a minimal level. However, the Picard–Fuchs operators of $\mathbb{C}$-isomorphic varieties coincide and so they do not distinguish between different models. Since for different twists of a fixed modular form $f$, the integrals $M(f_1, k), M(f_2, k)$ demonstrate no proportionality (at least on a numerical base with high accuracy), we believe that the period integrals of the singular variety $X_{t_0}$ determine a “preferred” modular form in a more canonical way.

The paper is organized as follows. In Section 2, we introduce basic definitions concerning Calabi–Yau threefolds and their period integrals. We also describe double octics, which are our main source of examples. In Section 3, we proceed to the case of one-dimensional families and associated differential operators. Section 4 provides necessary information on twists of a modular form $f$, and in Section 5 we define the partial integrals $M(f, s)$ associated with $f$. They are supposed to provide a tool for understanding integrals of singular double octic Calabi–Yau threefolds. Finally, in Section 6, we present numerical evidence toward this connection.

## 2 PERIODS OF RIGID DOUBLE OCTIC CALABI–YAU THREEFOLDS

The Bogomolov–Tian–Todorov unobstructedness theorem implies that the universal deformation space of a Calabi–Yau threefold $X$ is a smooth manifold of dimension $h^{2,1}(X)$. In particular, a Calabi–Yau threefold $X$ is rigid, that is, admits no deformations of the complex structure, exactly when $h^{2,1}(X) = 0$. Similarly, a Calabi–Yau threefold has one-parameter universal deformation space if and only if $h^{2,1}(X) = 1$.

By the Hodge decomposition for all Calabi–Yau manifolds $X$, we have the following equality:

$$b_3(X) = 2h^{2,1}(X) + 2.$$

Consequently, for a rigid Calabi–Yau threefold, the group $H_3(X, \mathbb{Z})$ has rank 2. Fixing a (nonzero) canonical form $\omega \in H^{3,0}(X)$, we define the period lattice of $X$ to be

$$\Lambda_X := \left\{ \int_\gamma \omega : \gamma \in H_3(X, \mathbb{Z}) \right\}.$$

Our paper was motivated by a phenomenon exhibited by numerical computations of period integrals of rigid double octic Calabi–Yau threefolds. A double octic is a Calabi–Yau threefold obtained as a resolution of singularities of a double cover of $\mathbb{P}^3$ branched along a union of eight planes $D \subset \mathbb{P}^3$. Double octics defined over $\mathbb{Q}$ with the Hodge number $h^{2,1} \leq 1$ were completely classified in [5]. Among them there are 11 rigid double octics defined over $\mathbb{Q}$.
TABLE 1  Generators of Re \( (\Lambda_f^p) \) and Im \( (\Lambda_f^p) \) for rigid double octics with \( \Lambda_X \subseteq \Lambda_f^p \)

| Arr. | Real integrals          | Imaginary integrals          |
|------|-------------------------|-------------------------------|
| 3    | 14.303841078            | 18.695683053 i               |
| 19   | 12.320533145            | 19.3301891966 i              |
| 32   | 11.13352966             | 12.3280533145 i              |
| 69   | 9.842836120319          | 11.13352966 i               |
| 93   | 8.42836120319           | 11.1335296603 i             |
| 239  | 13.1823084825           | 17.674531944 i              |
| 240  | 3.99263311132           | 6.94406875218 i             |
| 245  | 3.99263311132           | 6.94406875217 i             |

Every rigid Calabi–Yau threefold defined over \( \mathbb{Q} \) is modular. More precisely, the following modularity theorem from [6] is a consequence of the Serre Conjecture proven by Khare and Wintenberger:

**Theorem 2.1.** Let \( X \) be a rigid Calabi–Yau threefold defined over \( \mathbb{Q} \). Then, there exists an integer \( N \) and a Hecke eigenform \( f \in S_4(\Gamma_0(N)) \) such that \( L(X, s) = L(f, s) \).

The level \( N \) of the eigenform \( f \) equals the product \( \prod p p^e(p) \) taken over the set of bad primes with \( e(2) \leq 8 \), \( e(3) \leq 5 \), and \( e(p) \leq 2 \) for \( p \geq 5 \) [15, p. 216]. Modular forms for rigid double octics have been computed in [12].

The modular form \( f \) corresponding to a rigid Calabi–Yau threefold \( X \) can be seen as a 2-form on the associated Kuga–Sato variety \( Y \), and the special values \( L(f, 1) \) and \( L(f, 2) \) as its periods. By the Tate conjecture, the equality of \( L \)-functions \( L(X, s) = L(f, s) \) should imply the existence of a correspondence between \( X \) and \( Y \). Consequently, we expect the lattices \( \Lambda_X \) and \( \Lambda_f := (2\pi i)^2 L(f, 1) \mathbb{Z} \oplus (2\pi i) L(f, 2) \mathbb{Z} \) to be commensurable.

Numerical approximations of certain sublattices of \( \Lambda_X \) for rigid double octics were first computed in [4]. If a double octic \( X \) is defined as a resolution of singularities \( \sigma : X \to \mathcal{X} \) of a double covering \( \pi : \mathcal{X} \to \mathbb{P}^3 \) branched along a union of eight planes \( D = P_1 \cup \cdots \cup P_8 \), then the planes \( P_i \) define a decomposition of \( \mathbb{P}^3(\mathbb{R}) \) into polyhedra. The double cover \( \mathcal{C} = \pi^{-1}(C) \longrightarrow C \) of a polyhedron \( C \) from this partition defines a 3-cycle in \( H_3(\mathcal{X}, \mathbb{Z}) \) called a polyhedral cycle.

However, not every polyhedral cycle lifts to a cycle on the resolution of singularities \( X \) of \( \mathcal{X} \). Arrangements of eight planes that define Calabi–Yau threefolds have eight possible types of singularities. A 3-cycle in \( H_3(\mathcal{X}, \mathbb{Z}) \) lifts to a 3-cycle in \( H_3(X, \mathbb{Z}) \) if and only if it satisfies a symmetry condition at singular points of type \( p_0^4 \). A singularity of type \( p_0^4 \) is a point of intersection of four planes \( P_i \), which are generic elsewhere (i.e., this point does not lie on a triple line).

Let \( F(x, y, z, w) \) be the homogeneous equation of the octic arrangement \( D \). Numerical integration of

\[
\iiint_C \frac{dxdydz}{\sqrt{F(x, y, z, 1)}}
\]

over all polyhedra \( C \) gives period integrals of \( X \). Computed integrals generate a subgroup \( \Lambda_X^p \) of the group of period integrals of the singular double cover \( \overline{X} \) of \( \mathbb{P}^3 \):

\[
\Lambda_X^p := \left\{ \int_C \omega : \gamma \in H_3(\overline{X}, \mathbb{Z}) \right\}.
\]

As already mentioned, if a double octic contains a \( p_0^4 \) point, the group \( \Lambda_X^p \) can be larger than \( \Lambda_X \). This happens for 8 out of the 11 rigid double octics. In Table 1, we give real and complex generators of the group \( \Lambda_X^p \) for those cases. The numbering of double octic Calabi–Yau threefolds follows Meyer’s book [12].
Note that for arrangements 19, 240, and 245, the group $\Lambda^p_X$ is of rank 4. This phenomenon is possible because the computations in [4] are carried out on the singular double octic, which contains in these cases several $p_4^0$. Each $p_4^0$ point yields a condition on a cycle in $H_3(X, \mathbb{Z})$ to lift to a cycle in $H_3(X, \mathbb{Z})$. If some $p_4^0$ points define independent conditions, the rank of $\Lambda^p_X$ can be larger than 3.

### 3. ONE-PARAMETER FAMILIES

The main idea behind the method of computing period integrals introduced in [1] is to consider a rigid Calabi–Yau manifold $X$ as a resolution of a degenerate element $X_{t_0}$ of a one-parameter family of smooth Calabi–Yau threefolds $X = (X_t)_{t \in B}$, where $B = \mathbb{P}^1(C) \setminus \Sigma$ is an open subset of $\mathbb{P}^1(C)$.

As an explicit example, let us begin with the case of double octics, which is our main interest in this paper. In this case, the family $X$ of Calabi–Yau threefolds is given by the corresponding family $D_t = \{(x, y, z, w) \in \mathbb{P}^3 : F_t(x, y, z, w) = 0\}$ of arrangements of eight planes inside $\mathbb{P}^3$. Then, the Calabi–Yau threefold $X_t$ is a desingularization of the double cover $\overline{X}_t$ of $\mathbb{P}^3$ branched along $D_t$. A degeneration $X_{t_0}$ is an effect of an additional collision of planes in $D_{t_0}$ that does not happen in a generic fiber $D_t$. The sequence of blow-ups that resolves generic double cover $\overline{X}_t$ gives only a partial resolution $\overline{X}_{t_0}$ of the double cover $\overline{X}_t$ in the degenerate fiber.

To be even more explicit, assume that four planes of the octic arrangement form a tetrahedron that shrinks to a point as $t \to t_0 = 0$. In the appropriate coordinates, we get

$$F_t(x, y, z, w) = xyz(x + y + z - tw)G_t(x, y, z, w),$$

with $G_0(0, 0, 0, 1) \neq 0$. Then, the singular fiber is nodal and the shrinking tetrahedron

$$\{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0, x + y + z \leq t\}$$

defines the vanishing cycle $\delta \in H_3(X_t, \mathbb{Z})$.

The difference between the resolution $X$ of the double cover $\overline{X}_{t_0}$ and its partial resolution $\overline{X}_{t_0}$ is that, in the former case, we first blow-up the $p_4^0$ points and then the double lines, while in the latter, we blow-up only the double lines (see [4] for details). As a consequence, the variety $X_{t_0}$ is nodal with two nodes corresponding to each $p_4^0$ point. These nodes admit a small (crepant) resolution and exceptional lines are equivalent in $H_4(X, \mathbb{C})$. In this situation, we have $b_3(X_t) = 4$, $b_3(X_{t_0}) = 3$, and $b_3(X) = 2$. The homology group $H_3(X_t, \mathbb{Z})$ is spanned by $H_3(X_{t_0}, \mathbb{Z})$ and the class of the vanishing cycle $\delta$.

Now let us for a moment return to a general situation of a family $X = (X_t)_{t \in B}$. If we fix a holomorphic family of 3-forms $\omega_t \in H^{3,0}(X_t)$ and a cycle $\delta \in H_3(X_{t_0}, \mathbb{Z})$, in a punctured neighborhood of $t_0 \in \Sigma$, we can consider a (locally) holomorphic function $y(t) := \int_\delta \omega_t$, called the period function of this family. For any loop $\gamma \in \pi_1(B, b)$, where $X \to B$ is the total space of the family and $b \in B$ is some base point close to $t_0$, we can continue $y$ analytically along $\gamma$ and obtain a new function, which we denote by $M_{\gamma}(y)$. It turns out that the periods of $X$ can be recovered from the values $M_{\gamma}(y(t_0))$.

A period function $y$ satisfies a fourth-order differential equation called the Picard–Fuchs operator of the family $X_t$ (see, e.g., [9]). For any regular point $b$ the space of solutions of $P = 0$ near $b$ is four-dimensional and the fundamental group $\pi_1(B, b)$ acts on it by analytical continuation. After a choice of basis, this action defines the monodromy group $\text{Mon}(P) \subset GL(4, \mathbb{C})$ of the operator $P$. Every boundary point $s \in B$ of the family $X$ has associated local monodromy operator $M_s \in \text{Mon}(P)$, given by a small loop encircling $s$ counterclockwise. If the local monodromy $M_s$ has the Jordan form

$$\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},$$

the singular point $s$ is called a conifold point. Local exponents of a Picard–Fuchs operator at a conifold point equal $(0, 1, 1, 2)$. By direct inspection, we verify that the local exponents $(0, 1, 1, 2)$ of the Picard–Fuchs operators of one-parameter families of double octics correspond exactly to a degeneration of the described type (introducing a new $p_4^0$ from a shrinking tetrahedron).
Remark 3.1. Picard–Fuchs operators of a one-parameter family of double-octic Calabi–Yau threefolds have also singular points with local exponents \(\left(0, \frac{1}{2}, \frac{1}{2}, 1\right)\) and \(\left(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\). The geometry of the degenerate fiber in this case can be more complicated. After a quadratic or quartic base-change, totally ramified at such singularity, we can get an equation with local exponents \((0, 1, 1, 2)\) and \((0, 1, 4, 1, 4, 1, 2)\). The geometry of the degenerate fiber in this case can be more complicated. After a quadratic or quartic base-change, totally ramified at such singularity, we can get an equation with local exponents \((0, 1, 1, 2)\). As a consequence, the monodromy behavior in this case is much better understood (see [2]). On the geometric side, we expect that a singular point of type \(\frac{1}{n}C\), that is, with local exponents \(\left(0, \frac{1}{n}, \frac{1}{n}, 2\right)\), corresponds to a nodal Calabi–Yau threefold after a semistable base-change. Here, for simplicity, we focus on singularities of type \(C\).

The main advantage of the approach via Picard–Fuchs operators is that it does not depend on a geometric description of the considered rigid Calabi–Yau threefold. Assume that \(\mathcal{P}\) is a Fuchsian differential operator of order 4 such that \(t_0\) is a conifold singularity. The image \(\text{Im}(M_{t_0} - \text{Id})\) is one-dimensional. A generator of this subspace is called the conifold period and denoted by \(f_c\); in the case of Picard–Fuchs operators, it corresponds to the integral over the vanishing cycle. Now we define

\[
\mathcal{L}_{P,t_0} := \langle \{M(f_c)(t_0) : M \in \text{Mon}(\mathcal{P})\} \rangle.
\]  

(3.1)

The group \(\mathcal{L}_{P,t_0}\) is only defined up to scaling, since in the definition, we have to choose a specific conifold period \(f_c\). In order to avoid this ambiguity, we therefore normalize it by choosing the conifold period satisfying the condition \(f_c(t) = (t-t_0) + O((t-t_0)^2)\). The group \(\mathcal{L}_{P,t_0}\) contains limits of period integrals of fibers \(X_t\) as \(t \to t_0\). They are periods of the singular variety \(X_{t_0}\) but not necessarily periods of \(X\). Consequently, we get only the inclusion \(\mathcal{L}_{P,t_0} \subset \Lambda_{X,t_0}\).

Assume that \(t_0 \in \mathbb{Q}\) and that the Picard–Fuchs operator \(P\) of the family \(X\) has rational coefficients:

\[
P = P_4(t)D^4 + P_3(t)D^3 + \cdots + P_0(t), \quad P_i \in \mathbb{Q}[t].
\]

In this situation, the Frobenius basis of solutions of \(P\) at \(t_0\) has the form

\[
f_1(t-t_0), \quad f_2(t-t_0), \quad f_3(t-t_0) + f_2(t-t_0) \cdot \log(t-t_0), \quad f_4(t-t_0)
\]

with \(f_1 \in \mathbb{Q}[t], f_2, f_3 \in t\mathbb{Q}[t]\), and \(f_4 \in t^2\mathbb{Q}[t]\). Consequently, the space \(\mathcal{L}_{P,t_0}\) is invariant under complex conjugation and

\[
2 \cdot (\text{Re}(\mathcal{L}_{P,t_0}) \oplus \text{Im}(\mathcal{L}_{P,t_0})) \subset \mathcal{L}_{P,t_0} \subset \text{Re}(\mathcal{L}_{P,t_0}) \oplus \text{Im}(\mathcal{L}_{P,t_0})i,
\]

where

\[
\text{Re}(\mathcal{L}_{P,t_0}) := \{\text{Re}(v) : v \in \mathcal{L}_{P,t_0}\}, \quad \text{Im}(\mathcal{L}_{P,t_0}) := \{\text{Im}(v) : v \in \mathcal{L}_{P,t_0}\}
\]

denote the real and complex parts of \(\mathcal{L}_{P,t_0}\).

In general, it is difficult to compute the monodromy group of a Picard–Fuchs operator. Assume that the family \(X\) has a point of Maximal Unipotent Monodromy (MUM) at \(t = 0\). A choice of a path connecting points 0 and \(t_0\), while avoiding other singularities of the Picard–Fuchs operator \(P\), gives a subgroup

\[
\mathcal{L}_{P,t_0}^0 := \langle \{\text{Re}(M^0_n) : n \in \mathbb{Z}\} \rangle + \langle \{\text{Im}(M^0_n)i : n \in \mathbb{Z}\} \rangle \subset \mathcal{L}_{P,t_0}
\]  

(3.2)

defined by a local monodromy \(M_0\) of \(P\) around the MUM point \(t = 0\).

Coming back to the case of double octics, in [1], we observed (numerically) that for all one-parameter families of double octics, the group of real periods \(\text{Re}(\mathcal{L}_{P,t_0}^0)\) has rank 1, while in nine cases the subgroup of imaginary periods \(\text{Im}(\mathcal{L}_{P,t_0}^0)\) has rank 2. Thus, in this case, the inclusion \(\Lambda_X \subset \mathcal{L}_{P,t_0}^0\) is strict; note that due to results from [2], this cannot happen for a singularity of type \(\frac{1}{2}C\). Moreover, the generators of \(\mathcal{L}_{P,t_0}^0\) (Table 2) and the generators of \(\Lambda_X^C\) (Table 1) can be expressed in terms of each other.
TABLE 2  Generators of \( L^0_{P,t_0} \) for singular points \( t_0 \) with rank \( (L^0_{P,t_0}) = 3 \).

| Operator | Conifold point | Rigid Arr. | Form | Generators of \( \text{Im} L^0_{P,t_0} \) |
|----------|----------------|------------|------|--------------------------------------|
| 5        | 0              | 3          | 32/2 | 3.78853747194184773010686231258i     |
|          |                |            |      | 61.0738845852922464400038239965i     |
| 5        | 2              | 3          | 32/2 | 0.9471346798546193252671571987i      |
|          |                |            |      | 3.90285969880676941968001329994i     |
| 20       | -2             | 19         | 32/1 | 0.9792788247157944816600593885i      |
|          |                |            |      | 4.45674355709314111413743112i        |
| 95       | \( -\frac{1}{2} \) | 93         | 8/1  | 2.996830787050846536143680729i       |
|          |                |            |      | 34.0238543159967756814545903982i     |
| 244      | \( \frac{1}{2} \) | 240        | 6/1  | 2.5882359080556184578157001028i      |
|          |                |            |      | 128.1993497301613578977414i          |
| 244      | 2              | 240        | 6/1  | 10.352943632222478307260280020i      |
|          |                |            |      | 128.1993497301613578977414i          |
| 253      | -2             | 245        | 6/1  | 6.2684709434912100335907942945i      |
|          |                |            |      | 7.96011334055139325749281017005i     |
| 274      | \( -\frac{1}{2} \) | 245        | 6/1  | 0.839792675513409448977564026085i    |
|          |                |            |      | 8.12249522907253404678014309610i     |
| 274      | -2             | 245        | 6/1  | 1.4929647566181062877124575360i      |
|          |                |            |      | 14.4399915143798181025349619989i     |

Remark 3.2. In Table 2, as well as in the rest of the paper, we use the notation for modular forms of weight 4 from the supplement to [12] available online at https://www.fields.utoronto.ca/publications/supplements

In the symbol \( N/M \), the number \( N \) denotes the level of the corresponding form. In particular, forms 6/1, 8/1, and 12/1 are the unique forms of weight 4 and the appropriate level, while 32/1 and 32/2 are the only new forms of level 32. These forms can be also found in the database [11], where they are denoted 6.4.\( \text{a.a} \), 8.4.\( \text{a.a} \), 12.4.\( \text{a.a} \), 32.4.\( \text{a.b} \), and 32.4.\( \text{a.a} \).

Obviously the group \( L_{P,t_0} \) depends not only on the smooth double octic \( X \) but also on the choice of a one-parameter smoothing. In fact, birational models of rigid double octic can be realized as specializations of several one-parameter families. Up to commensurability, we always have inclusions \( \Lambda_X \subset L_{P,t_0} \subset \Lambda^p_{X,t_0} \subset \Lambda^p_X \) As we already mentioned, there are cases when \( \Lambda_X = L_{P,t_0} \) but there are also examples in which rank\( (L_{P,t_0}) = 3 \). Additional period integrals in \( \Lambda^p_X \) are related to singular points of type \( p^0_4 \). The classification in [5] shows that \( \Lambda^p_X \) is the sum of \( \Lambda^p_{X,t_0} \) taken over all one-parameter smoothings \( X_t \) of \( X \).

Results of [1] have two important consequences: Using Maple implementation of algorithms for solving differential equations and numerical approximations to construct an analytic continuation along any polyline path, we can compute the elements of \( L_{P,t_0} \). The error in this computation depends only on the order \( N \) to which we solve the equation and in fact can be seen to be bounded by \( C^{-N} \) for an explicit constant \( C > 1 \) (see [10]). Hence, the elements of \( L_{P,t_0} \) can be computed with very high precision. This is in striking contrast with computations in [4], where only precision of 10 digits could be obtained. Moreover, the definition (3.1) is given purely in terms of the differential equation and thus allows us to assign an analog of \( \Lambda_X \) to any smooth Calabi–Yau threefold \( X \), which is birational to a degeneration of a family of Calabi–Yau threefolds \( X_t \) with \( h^{2,1}(X_t) = 1 \). The question of understanding period integrals of singular models \( \overline{X} \) of a rigid Calabi–Yau threefold \( X \) is therefore replaced with a more general problem of describing the elements of \( L_{P,t_0} \), where \( P \) is the Picard–Fuchs operator of a one-parameter smoothing of a singular model \( X_{t_0} \) of \( X \). We want to accomplish it in terms of the modular form associated to \( X \) by the modularity theorem.

4 TWISTS BY A DIRICHLET CHARACTER

A rigid Calabi–Yau threefold can have models, which are isomorphic over a number field but not isomorphic over \( \mathbb{Q} \). In this situation, the associated modular form is not uniquely determined by its model over complex numbers. Since a double octic Calabi–Yau threefold is hyperelliptic, it admits a quadratic twist by any square-free integer. Quadratic twists
exist for a large class of Calabi–Yau threefolds including double octics and Schoen’s fiber products. However, existence of quadratic twists for an arbitrary Calabi–Yau threefold is an open question. In [7], Gouëva, Kiming, and Yui proposed an abstract definition of a quadratic twist.

To be more explicit, if a rigid double octic \( X \) is given as a resolution of the hypersurface

\[ \{u^2 = f(x)\} \subset \mathbb{P}(1^4, 4), \]

then there exists a quadratic twist \( X_d \) by a square-free integer \( d \) given by a resolution of

\[ \{u^2 = d \cdot f(x)\} \subset \mathbb{P}(1^4, 4). \]

Threefolds \( X \) and \( X_d \) are obviously isomorphic over \( \mathbb{Q}[\sqrt{d}] \), but they are not isomorphic over \( \mathbb{Q} \), unless the corresponding modular form is of CM-type.

If \( X \) is a rigid Calabi–Yau manifold defined over \( \mathbb{Q} \) with attached modular form

\[ f(z) = \sum_{n=1}^{\infty} a_n q^n \in \Gamma_0(N), \quad q = \exp(2\pi iz), \]

then the modular form associated to a quadratic twist \( X_d \) by \( d \) is

\[ f_{\chi_d}(z) = \sum_{n=1}^{\infty} a_n \chi_d(n) q^n \in \Gamma_0(N), \quad q = \exp(2\pi iz). \]

Thus, it is the quadratic twist of \( f \) by the Dirichlet character \( \chi_d \) [7, Theorem 1].

Let \( g(\chi_d) \) be the Gauss sum of a Dirichlet character \( \chi_d \) modulo \( d \):

\[ g(\chi_d) := \sum_{a=1}^{d} \chi_d(a) \exp \left( \frac{2\pi i a}{d} \right). \]

If the character \( \chi_d \) is primitive, then \( |g(\chi_d)| = \sqrt{d} \). For a Dirichlet character \( \chi \), denote by \( K_\chi \) the field of definition of \( \chi \). We have the following formulas for the special values of a twist \( f_\chi \) of the modular form \( f \) by a Dirichlet character:

**Theorem 4.1** [16, Theorem 1]. Let \( f \in S_k(\Gamma_0(N)) \) be a cusp form of weight \( k \) for the group \( \Gamma_0(N) \) with rational coefficients. There exist complex numbers \( u^+ \) and \( u^- \) such that for any Dirichlet character \( \chi \) and any positive integer \( m < k \), we have

\[
(2\pi i)^{-m} g(\chi)^{-1} L(f_\chi, m) \in \begin{cases} u^+ K_\chi, & \text{if } \chi(-1) = (-1)^m \\ u^- K_\chi, & \text{if } \chi(-1) = -(-1)^m \end{cases}.
\]

In the special case of a quadratic twist \( X_d \) of a Calabi–Yau threefold and the attached modular forms \( f \) and \( f_d \), we get

if \( d > 0 \), then \( L(1, f_d) \in \sqrt{d} \cdot L(1, f) \cdot \mathbb{Q} \); \( L(2, f_d) \in \sqrt{d} \cdot L(2, f) \cdot \mathbb{Q} \);

if \( d < 0 \), then \( L(1, f_d) \in \frac{\sqrt{-d}}{2\pi i} \cdot L(2, f) \cdot \mathbb{Q} \); \( L(2, f_d) \in \sqrt{-d} \cdot 2\pi i \cdot L(1, f) \cdot \mathbb{Q} \).

These formulas agree with the behavior of period integrals of a double octic Calabi–Yau threefold under a quadratic twists, since the period integrals of \( X_d \) equal period integrals of \( X \) divided by \( \sqrt{d} \). In particular, when \( d \) is negative, real periods of \( X \) correspond to complex periods of \( X_d \) and vice versa.

As we mentioned in Section 1, period integrals of a rigid Calabi–Yau threefold \( X \) defined over \( \mathbb{Q} \) are expected to be proportional to the special values of the \( L \)-function of the corresponding modular form and thus we want to interpret the elements of \( \Lambda_X \) in a similar way.
However, we have to take into account that Calabi–Yau threefolds isomorphic over \( \mathbb{C} \) need not have equal associated modular forms, as they can fail to be isomorphic over \( \mathbb{Q} \). This phenomenon can occur also for one-parameter families of Calabi–Yau threefolds. Consequently, given a differential operator \( P \), it may happen that we find two families \( \mathcal{X} \) and \( \mathcal{Y} \) having \( P \) as the Picard–Fuchs operator, yet such that the smooth, rigid models of singular fibers at a conifold point have associated modular forms equal only up to a twist. It is then not a priori clear which modular form should the elements of \( \Lambda_\mathcal{X} \) be compared with.

In a similar manner, the distinction between different twists of the same manifold is not visible from the differential operator. The Picard–Fuchs operator \( P \) of a family \( \mathcal{X}_t^3 \) given by the equation \( x_0^2 = a F_t(x_1, x_2, x_3, x_4) \) is independent of the choice of \( a \in \mathbb{C}^* \). Indeed, the preferred choices of the period function

\[
\omega^a_t = \int_\gamma \sum_{i=1}^4 (-1)^i x_i dx_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots dx_4 \sqrt{a F_t(x_1, x_2, x_3, x_4)},
\]

differ only by a scalar and thus satisfy the same differential equation. Therefore, if we are only given the Picard–Fuchs operator and not the family itself, it is not possible to determine from which of the families \( \mathcal{X}_t^3 \) it comes.

What we may do, however, is to normalize the conifold period \( f_c \) so that

\[
f_c(t) = (t-t_0) + O((t-t_0)^2).
\]

This normalization is usually used in the descriptions of the Frobenius method, it was also in place for computations in [1] and in our definition (3.1). Note that this choice happens on the level of the differential equation and not on the level of the family. Then, inside \( \mathcal{L}_{P,t_0} \otimes \mathbb{Q} \), we can identify the lattice \( \lambda \Lambda_f \) for some \( \lambda \in \mathbb{C}^* \), where \( f \) is a modular form associated to some rigid birational model of \( \mathcal{X}_{t_0} \). It is then natural to assume that a modular form \( f_{t_0} \) associated with the point \( t_0 \) is the one for which \( \Lambda_{f_{t_0}} \) and \( \lambda \Lambda_f \) are commensurable. One may consult table 4 from [1] to see examples for which \( \lambda \neq 1 \), and thus the modular form of the minimal level associated to the rigid model of \( \mathcal{X}_{t_0} \) by Meyer is not the one for the singular model in the sense just described.

## 5 Invariants of the Modular Form

We now consider the associated modular forms in order to check whether we can describe elements of \( \mathcal{L}_{P,t_0} \) in a way similar as we describe periods of a rigid Calabi–Yau threefold in terms of \( L(f, 1) \) and \( \frac{L(f, 2)}{2\pi i} \). To this end, we have to embed \( \Lambda_f \) into some intrinsically defined group of greater rank, and therefore it is natural to try and write the special values generating \( \Lambda_f \) as sums of some invariants of \( f \).

Let \( W_N \) be the Fricke involution on the space of modular cusp forms \( S_k(\Gamma_0(N)) \) of weight \( k \) and level \( N \). It is a linear operator defined by \( W_N(f)(z) := i^k N^{-k/2} z^{-k} f \left( \frac{1}{Nz} \right) \). One easily checks that \( W_N \) is an involution, that is, \( W_N^2 \) is \( \text{Id} \). If \( f \) is a modular form associated with a rigid Calabi–Yau threefold, it is also an eigenvector of \( W_N \) and hence \( W_N(f) = \varepsilon \cdot f \), where \( \varepsilon = \pm 1 \) is the Fricke sign of \( f \).

The completed \( L \)-function \( \Lambda(s) := \left( \frac{\sqrt{N}}{2\pi} \right)^s \Gamma(s)L(f, s) \) of a Hecke eigenform \( f \) satisfies the functional equation \( \Lambda(s) = \varepsilon \Lambda(k-s) \). The completed \( L \)-function can be also defined in terms of the Mellin transform of \( f \) as \( \Lambda(s) := \sqrt{N} \int_0^\infty f(iz)z^{s-1} dz \). The decomposition of a cycle on a smooth Calabi–Yau threefold into a sum of cycles in its nodal model is given by splitting at the node. Motivated by this description, we define a "partial" \( L \)-function of a modular form:

**Definition 5.1.** Let \( f \in S_k(\Gamma_0(N)) \) be a Hecke eigenform. Then, we define

\[
M(f, s) := \left( \frac{2\pi}{\Gamma(s)} \right)^{2s-k} \int_0^\infty f(iz)z^{s-1} dz.
\]

From the standard proof of the functional equation for \( \Lambda(s) \), we can deduce the following property of the function \( M(f, s) \), fundamental to our goal:

**Theorem 5.2.** If \( f \in S_k(\Gamma_0(N)) \) is an eigenform of the Fricke involution, then

\[
L(f, s) = M(f, s) + \varepsilon \left( \frac{2\pi}{\sqrt{N}} \right)^{2s-k} \frac{\Gamma(k-s)}{\Gamma(s)} M(f, k-s).
\]
Proof. We have the following chain of equalities:

\[
\Lambda(s) = \frac{\gamma}{2} \int_0^\infty f(iz)z^{s-1}dz = \frac{\gamma}{2} \int_1^{\sqrt{N}} f(iz)z^{s-1}dz + \frac{\gamma}{2} \int_{\sqrt{N}}^\infty f(iz)z^{s-1}dz
\]

Now, the definition of \(M(f, s)\) gives the assertion:

\[
L(f, s) = \left( \frac{2\pi}{\sqrt{N}} \right)^s \frac{1}{\Gamma(s)} \Lambda(s)
\]

\[
= (2\pi)^s \int_{\sqrt{N}}^\infty f(iz)z^{s-1}dz + \epsilon \left( \frac{2\pi}{\sqrt{N}} \right)^{2s-k} \frac{\Gamma(k-s)}{\Gamma(s)} \int_{\sqrt{N}}^\infty f(iz)z^{k-s-1}dz
\]

\[
= M(f, s) + \epsilon \left( \frac{2\pi}{\sqrt{N}} \right)^{2s-k} \frac{\Gamma(k-s)}{\Gamma(s)} M(f, k-s). \quad \Box
\]

For our purposes, crucial is the application of Theorem 5.2 to the special values of the \(L\)-function of a cusp form \(f\) of weight 4 at the critical points 1 and 2. For \(L(f, 1)\), we obtain the decomposition

\[
L(f, 1) = M(f, 1) + \epsilon N \frac{M(f, 3)}{2\pi^2},
\]

which suggests that \(M(f, 1)\) and \(\frac{M(f, 3)}{\pi^3}\) might be the additional elements in \(\Lambda_X\) needed to decompose integrals of \(X\) as previously described. On the other hand, for the special value \(L(f, 2)\), the situation is different. In the geometric context, the imaginary period is computed as an integral over a cycle in \(H_3(X)\) that is not decomposed into a sum of two cycles in \(H_3(X)\). On the modular side, Theorem 5.2 in this case yields

\[
L(f, 2) = (1 + \epsilon)M(f, 2).
\]

When \(\epsilon = -1\), this obviously implies \(L(f, 2) = 0\). Thus, in this case, \(\Lambda_f\) is not a lattice but a group of rank 1 and the proportionality of the special \(L\)-value \(L(f, 2) = 0\) with the period integral is trivial, hence meaningless. If \(\epsilon = 1\), then \(M(f, 2) = \frac{L(f, 2)}{2\pi i}\), which means that the Fricke involution divides the 3-cycle computing imaginary period into two subsets of equal \(\omega\)-volume. Similarly, adding \(\frac{M(f, 2)}{2\pi i}\) to \(\Lambda_f\) results in a commensurable lattice.

Thus, let us define

\[
\Lambda_f^C := \langle M(f, 1), \frac{L(f, 2)}{2\pi i}, \frac{M(f, 3)}{2\pi^2} \rangle.
\]

As we have seen, \(\Lambda_f \subset \Lambda_f^C\), and we hope that \(\Lambda_f^C\) can play the role of a “modular” analog of \(\mathcal{L}_{P, t_0}\). The last section will present numerical evidence supporting this hypothesis, as well as a comment on certain problems with its direct application to the case of the form \(6/1\).

### 6 MODULAR INTERPRETATION OF ADDITIONAL INTEGRALS

In this section, we shall propose a conjectural relation between the integrals \(M(f, 1)\) and \(\frac{M(f, 3)}{2\pi^2}\) and the period integrals of the degenerate element of a one-dimensional family. As in Section 3, we shall consider a one-parameter family \(\mathcal{X} = (X_t)_{t \in \Delta}\).
of projective varieties such that for \( t \not\in \{0, t_0\} \), the variety \( X_t \) is a (smooth) Calabi–Yau threefold with \( h^{1,2}(X_t) = 1 \). We also assume that \( t = 0 \) is a point of MUM, \( t = t_0 \) is a conifold point, and that the degeneration \( X_{t_0} \) admits a crepant resolution of singularities \( X \), which is a rigid Calabi–Yau manifold defined over \( \mathbb{Q} \).

Recall that there exist 11 rigid double octics defined over \( \mathbb{Q} \). Among them, there are eight examples for which integrals over polyhedral cycles on the singular double cover of \( \mathbb{P}^3 \) generate a group of rank greater than 2; they correspond to the arrangements \( 3, 19, 32, 69, 93, 239, 240, \) and \( 245 \). Twists of minimal level of modular forms associated with these examples are:

- \( 6/1 \) : Arr. 240, 245,
- \( 8/1 \) : Arr. 32, 69, 93,
- \( 12/1 \) : Arr. 239,
- \( 32/1 \) : Arr. 19,
- \( 32/2 \) : Arr. 3.

For all rigid double octics, except those with the modular form \( 6/1 \), we have the following relations between additional integrals (listed in Table 1) and the invariants \( M(f, 1) \) and \( M(f, 3) \) of the corresponding modular form (listed in Table 3):

**Proposition 6.1.** Up to the precision of computations in [4], we have the following relations between additional period integrals for the singular double cover \( \overline{X} \) and the invariants \( M(f, 1) \) and \( M(f, 3) \) of the modular form associated to the corresponding double octic Calabi–Yau threefold \( X \):

\[
\begin{align*}
\text{Arr.} 3 & : \\
& \begin{cases}
14.30384107 = -12\pi^2 M(f_{32/2}, 1) + 64M(f_{32/2}, 3) \\
18.69568305 = 20\pi^2 M(f_{32/2}, 1) - 64M(f_{32/2}, 3)
\end{cases} \\
\text{Arr.} 19 & : \\
& \begin{cases}
12.32805331 = -6\sqrt{2}\pi^2 M(f_{32/1}, 1) + 32\sqrt{2}M(f_{32/1}, 3) \\
19.33018919 = 4\sqrt{2}\pi^2 M(f_{32/1}, 1)
\end{cases} \\
\text{Arr.} 32, 69 & : \\
& \begin{cases}
11.13352966 = -8\pi^2 M(f_{8/1}, 1) + 32M(f_{8/1}, 3) \\
16.85672240 = 16\pi^2 M(f_{8/1}, 1)
\end{cases} \\
\text{Arr.} 93 & : \\
& \begin{cases}
8.428361203 = 8\pi^2 M(f_{8/1}, 1) \\
11.13352966 = -8\pi^2 M(f_{8/1}, 1) + 32M(f_{8/1}, 3)
\end{cases} \\
\text{Arr.} 239 & : \\
& \begin{cases}
13.18230848 = 8\pi^2 M(f_{12/1}, 1) \\
17.67145319 = 24M(f_{12/1}, 3)
\end{cases}
\]

Consequently in all those cases, \( \Lambda_c^f = \Lambda_c^\overline{X} \), up to commensurability.

The precision of period integrals in Proposition 6.1 is limited due to the method used in [4]. However, by [1], these period integrals agree (up to their exactness) with period integrals \( \mathcal{L}^{0}_{p_{J_0}} \) computed via the analytic continuation of a conifold period. Consequently, the groups \( \mathcal{L}^{0}_{p_{J_0}} \) and \( \Lambda_c^f \) are commensurable as well.

For instance, the relation

\[
14.303841078 \approx \frac{\pi^2}{16} \cdot (61.073884585292464400038239965 - 10 \cdot 3.78853747194184773010686231258)
\]
suggests equalities

\[61.073884585292464400038239965 = 208M(f, 1) - 256 \frac{1}{\pi^2} M(f, 3),\]
\[3.7885374719418773010686231258 = 40M(f, 1) - 128 \frac{1}{\pi^2} M(f, 3),\]
\[M(f, 1) = \frac{1}{128} 61.073884585292464400038239965 - \frac{1}{64} 3.7885374719418773010686231258,\]
\[M(f, 3) = \frac{5\pi^2}{2048} 61.073884585292464400038239965 - \frac{13\pi^2}{1024} 3.7885374719418773010686231258.\]

Since both elements of \(L^0_{P,t_0}\) and the partial integrals \(M(f, k), k = 1, 2\), can be computed with very high precision, these equalities can be easily verified with accuracy \(10^{-100}\) and higher, unlike those in Proposition 6.1.

The case of Arr. No. 19 is exceptional, because we have commensurability of groups \(L^0_{P,t_0}\) and \(\sqrt{2}\Lambda_c f_{32/1}\). In this situation, we expect that the double octic corresponds to the modular form \(f_{64/3}\) (64.4.a.c in [11]), which is the twist of \(f_{32/1}\) by Dirichlet character \(\chi_{8,5}\) or \(\chi_{8,3}\). Modular form \(f_{32/1}\) has complex multiplication by \(\mathbb{Q}[\sqrt{-1}]\), hence it is invariant under the twist by the character \(\chi_{4,3}\). As a consequence, we cannot distinguish twists of \(f_{32/1}\) by odd and even character.

\[M(f_{64/3}, 1) \approx 0.366733368496185708303364416057\]
\[M(f_{64/3}, 3) \approx 0.909804035050076969996940010381\]

The Frick sign for modular form \(f_{64/3}\) equals \(-1\). In particular, \(L(f_{64/3}, 2) = 0\) and in this situation, we do not predict that groups \(L^0_{P,t_0}\) and \(N'_f\) are commensurable.

**Corollary 6.2.** For octic arrangements Nos. 3, 32, 69, 93, 239, and modular forms 32/2, 8/1, 8/1, 8/1, 12/1, respectively, groups \(L^0_{P,t_0}\) and \(N'_f\) are commensurable. For the octic arrangement No. 19, groups \(L^0_{P,t_0}\) and \(\sqrt{2}\Lambda_c f_{32/1}\) are commensurable.

In the above corollary, commensurability means that the generators of one group can be expressed as integral linear combinations of generators of the second group with very high accuracy.

### 6.1 Operator No. 8.62

Consider the differential operator no. 8.62 in the online database [3]:

\[P = \theta^4 + x (578\theta^4 - 572\theta^3 - 359\theta^2 - 73\theta - 6)\]
\[+ 3^2 x^2 (4673\theta^4 + 1892\theta^3 + 31601\theta^2 + 11514\theta + 1728)\]
\[-2^3 3^4 x^3 (9185\theta^4 - 134298\theta^3 - 35420\theta^2 - 22329\theta - 5544)\]
\[+ 2^4 3^8 x^4 (19051\theta^4 + 118466\theta^3 + 114678\theta^2 + 65939\theta + 14290)\]
\[-2^6 3^{12} x^5 (7540\theta^4 + 80683\theta^3 - 6459\theta^2 - 7907\theta - 2300)\]
\[-2^6 3^{16} x^6 (3919\theta^4 + 27744\theta^3 + 29957\theta^2 + 14208\theta + 2556)\]
\[+ 2^9 3^{20} 5^2 x^7 (199\theta^4 + 590\theta^3 + 744\theta^2 + 449\theta + 106)\]
\[-2^{12} 3^{24} 5^2 x^8 (\theta + 1)^4]\]

It is the Picard–Fuchs operator of the one-parameter family of Calabi–Yau manifolds constructed as a resolution of singularities of fiber products of semistable rational elliptic surfaces (see [13]) with singular fibers matched as in the following diagram:

\[
\begin{array}{cccccccc}
I_5 & I_3 & I_2 & I_1 & I_1 & - \\
I_3 & I_6 & I_2 & - & - & I_1
\end{array}
\]
For the special value of the parameter $t_0 = -\frac{1}{81}$, fibers $I_5$ and $I_1$ in the first surface collide producing a fiber of type $I_6$. Consequently, the family contains degeneration at the conifold point given by a fiber product of the following Beauville surfaces:

$$
\begin{array}{cccccc}
I_6 & I_3 & I_2 & I_1 & I_1 - \\
I_3 & I_6 & I_2 & - & I_1
\end{array}
$$

Since a generic fiber product in this family has 37 nodes while the special one has 40 nodes, the degenerate element $X_{t_0}$ of the family of smooth Calabi–Yau manifolds has three nodes. A small resolution of $X_{t_0}$ is the Calabi–Yau manifold $W_2$ constructed by Schütt and its associated modular form $f$ is the form $21/2$ (21.4.1.1 in [11]) of weight 4 and level 21 (see [14]).

In this case, the group

$$L^0 \big| \mathcal{P} = -\frac{1}{81} = \begin{pmatrix}0.079041901426502594058424764412257593, 0.13670990041323305298936699557707682i\end{pmatrix}$$

has rank 2 and

$$0.079041901426502594058424764412257593 \approx \frac{4}{27} L(f_{21/2}, 1),$$

so we expect that the real generator should be expressible by the invariants of the modular form $f_{336/7}$, which is a twist of $f_{21/2}$ by $X_{4,3}$.

$$\begin{array}{cc}
L(f_{21/2}, 1) \approx 0.53353283462889250989436715978273 & L(f_{21/2}, 2) = 0 \\
M(f_{336/7}, 1) \approx 0.415965257835022165771740339742 & M(f_{336/7}, 3) \approx 0.98553633711659774489352376439
\end{array}$$

However, the Frick sign of $f_{21/2}$ is $-1$ and consequently $L(f_{21/2}, 2) = 0$ and again we cannot predict commensurability.

### 6.2 Operator No. 8.67

Consider the differential operator

$$P = 5^2 \theta^4 + 5x \left( 477 \theta^4 + 978 \theta^3 + 769 \theta^2 + 280 \theta + 40 \right) - 2^3 x^2 \left( 46 \theta^4 - 2582 \theta^3 - 5689 \theta^2 - 4120 \theta - 1040 \right) + 2^3 x^2 \left( 772 \theta^4 - 4872 \theta^3 - 11765 \theta^2 - 7335 \theta - 1480 \right) + 2^3 x^4 \left( 140 \theta^4 + 500 \theta^3 - 672 \theta^2 - 1313 \theta - 512 \right) - 2^6 x^5 \left( 31 \theta^4 + 154 \theta^3 - 596 \theta^2 - 729 \theta - 227 \right) + 2^7 x^6 \left( 32 \theta^4 - 264 \theta^3 - 500 \theta^2 - 303 \theta - 58 \right) + 2^8 x^7 \left( 12 \theta^4 + 72 \theta^3 + 121 \theta^2 + 85 \theta + 22 \right) - 2^{12} x^8 ((\theta + 1)^4).$$

This operator has no. 8.67 in [3], it is the Picard–Fuchs operator of a family of resolutions of fiber products of the same elliptic surfaces as in the case of operator 8.62 but with different matching of singular fibers

$$\begin{array}{cccccc}
I_5 & I_3 & I_2 & I_1 & I_1 - \\
I_6 & I_2 & I_3 & - & - & I_1
\end{array}$$

For a special value of the parameter, fibers $I_5$ and $I_1$ in the first surface collide producing a fiber of type $I_6$. Degeneration at the conifold point $t_0 = -1$ is a resolution of the fiber product

$$\begin{array}{cccccc}
I_6 & I_3 & I_2 & I_1 - \\
I_3 & I_6 & I_2 & - & I_1
\end{array}$$

A small resolution of $X_{t_0}$ is a Calabi–Yau manifold $W_1$ from [14] and its associated modular form $f$ is the form $f_{17/1}$ (17.4.a.a in [11]) of weight 4 and level 17.
In this case, \( \text{rank}(\mathcal{L}_{P,-1}) = 3 \). More precisely, a finite index subgroup \( \mathcal{L}_{P,-1} \) has one real and two imaginary generators.

| Real generator | Imaginary generators |
|----------------|----------------------|
| 42.906578481269266425208768540874910763200659 | 10.19666107517043760275253892389042409717592 |
| 15.92910087995971902859587308255012478345243 | |

In particular, the real generator equals:

\[
42.906578481269266425208768540874910763200659 = -108L(f_{17/1}, 1) = \frac{36}{\pi} L(f_{272/1}, 2),
\]

the form \( f_{272/1} \) (272.4.a.d in [11]) is the twist of \( f_{17/1} \) by \( \chi_{4,3} \). Consequently,

\[
\begin{array}{cc}
L(f_{17/1}, 1) & L(f_{17/1}, 2) = 0 \\
M(f_{272/1}, 1) \approx 0.12059537134699121108836815988156741 & M(f_{272/1}, 3) \approx 0.6013243252344910630079471213129171 \\
\end{array}
\]

In this case, the Fricke sign of \( f_{17/1} = -1 \), hence \( L(f_{17/1}, 2) = 0 \) and consequently we do not predict commensurability.

### 6.3 Arrangement nos. 240 and 245

Finally, we shall go back to the most involved cases of rigid double octic Calabi–Yau threefolds with modular form 6/1. In both cases, the group \( \Lambda_\mathcal{X} \) of periods of the singular double cover \( \mathcal{X} \) has rank at least 4 with real and imaginary parts of rank at least 2. The generator of \( \text{Re}(\mathcal{L}_{P,0}) \) for four conifold points \( \frac{1}{2}, 2 \) and \( -\frac{1}{2}, -2 \) appearing in for Calabi–Yau operators \( P \) of arrangement nos. 244 and 274 equals, respectively (see [1]),:

\[
\begin{align*}
20\sqrt{2L(f_{6/1}, 1)}, & \quad 80\sqrt{2L(f_{6/1}, 1)}, \quad 10\sqrt{2L(f_{6/1}, 1)}, \quad 40\sqrt{2L(f_{6/1}, 1)}, \\
\end{align*}
\]

while for the conifold point \( -2 \) in arrangement no. 253, it is equal to

\[
36\sqrt{2L(f_{6/1}, 1)}.
\]

Consequently, we consider the modular forms \( f_{192/4} \) (192.4.a.i in [11]) and \( f_{192/7} \) (192.4.a.c in [11]), which are twists of \( f_{6/1} \) by Dirichlet characters \( \chi_{8,5} \) and \( \chi_{8,3} \), respectively. As

\[
L(f_{192/4}, 1) = -30\sqrt{2L(f_{6/1}, 1)} \quad \text{and} \quad L(f_{192/7}, 1) = -36\sqrt{2L(f_{6/1}, 2)}.
\]

generators of \( \text{Re}(\mathcal{L}_{P,0}) \) equal

\[
\begin{align*}
-\frac{2}{3}L(f_{192/4}, 1), & \quad -\frac{8}{3}L(f_{192/4}, 1), \quad -L(f_{192/7}, 1), \quad -\frac{1}{3}L(f_{192/4}, 1), \quad -\frac{16}{27}L(f_{192/4}, 1).
\end{align*}
\]

The Fricke sign of both these forms is \( -1 \) and, as in the examples above, we do not predict that the generators of \( \text{Im}(\mathcal{L}_{P,0}) \) can be expressed in terms of \( M(f, 1) \) and \( M(f, 3) \) (and we were not able to find such an expression numerically).

Recall that the number \( M(f, s) \) is defined by a partial integral

\[
M(f, s) = \frac{(2\pi)^s}{\Gamma(s)} \int_1^\infty f(i\zeta)\zeta^{s-1}d\zeta.
\]

We can consider a more general integral

\[
M(f, s; t) := \frac{(2\pi)^i}{\Gamma(s)} \int_1^\infty f(i\zeta)\zeta^{s-1}d\zeta,
\]

where \( t \in (0, +\infty) \) is a nonnegative number and \( s \) belongs to the half-plane of convergence; in particular \( M(f, s) = M(f, s, N) \).
Let $f := f_{6/1}$, let $P$ be the Picard–Fuchs operator for arrangement 253, and let $t_0 := -2$. Since the Atkin–Lehner signs of the modular form $f_{192/4}$ equal

\[
\frac{p}{\text{sign}} = 2 \quad \frac{3}{1}
\]

and

\[
M(f, 1) = M(f_{6/1}, 1; 6) = 0.0705795645108305472255139009133496203387610115943 \ldots,
\]
\[
M(f, 3) = M(f_{6/1}, 3; 6) = 0.49691599739243476804039800218000443074993876412363 \ldots,
\]
\[
M\left(f_{6/1}, 1; \frac{3}{2}\right) = 0.00588018380632647168784781079042205384110122041895 \ldots,
\]
\[
M\left(f_{6/1}, 3; \frac{3}{2}\right) = 0.11354370318430276251965275943335072038142945788569 \ldots,
\]

comparing generators of $\text{Im}(L^0_{P,0})$ (cf. Table 2) with $\sqrt{2}M(f_{192/4}, s; \frac{3}{2})$, we get

\[
6.26847094349121003359079492495 \approx 20\sqrt{2}M(f, 1; 6) + 60\sqrt{2}\frac{\pi^2}{M(f, 3; 6)},
\]
\[
7.9601134055139325749281017005 \approx 48\sqrt{2}M(f, 1; 6) + 48\sqrt{2}\frac{\pi^2}{M(f, 3; 6)} + 64\sqrt{2}M\left(f, 1; \frac{3}{2}\right) - 48\sqrt{2}\frac{\pi^2}{M\left(f, 3; \frac{3}{2}\right)}.
\]

This suggests that, in general to identify additional periods, one has to consider partial integrals $M(f, s; t)$ for different values of the parameter $t$.

However, even this approach did not yield results for the remaining operators 244 and 274, since (unlike in all other cases) the rank 2 part $\text{Im}(L^0_{P,0})$ of $L^0_{P,0}$ consists of imaginary integrals and contains $\frac{L(f, 2)}{2\pi i}$. Perhaps a different way of decomposing imaginary integrals, and consequently $L(f, 2)$, is necessary.

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