Maximal models up to the first measurable in ZFC

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Abstract

Theorem: There is a complete sentence $\phi$ of $L_{\omega_1,\omega}$ such that $\phi$ has maximal models in a set of cardinals $\lambda$ that is cofinal in the first measurable $\mu$ while $\phi$ has no maximal models in any $\chi \geq \mu$.

In this paper we prove in ZFC the existence of a complete sentence $\phi$ of $L_{\omega_1,\omega}$ such that $\phi$ has maximal models in a set of cardinals $\lambda$ that is cofinal in the first measurable $\mu$ while $\phi$ has no maximal models in any $\chi \geq \mu$. In [BS2x], we proved a theorem with a similar result; the earlier proof required that $\lambda = \lambda^\lambda$, and extended ZFC by requiring an $S \subseteq S_{\lambda_0}$, that is stationary non-reflecting, and $\diamondsuit$ holds. Here, we show in ZFC that the sentence $\phi$ defined in [BS2x] has maximal models cofinally in $\mu$. The additional hypotheses in [BS2x] allow one to demand that if $N$ is a submodel with cardinality $< \lambda$ of the $P_\lambda$-maximal model, $N$ is $K_1$-free (See Remark 4.1); that property fails for the example here. The existence of such a $\phi$ which is not complete is well-known (e.g. [Mag16]).

This paper contributes to the study of Hanf numbers for infinitary logics. Works such as [BKS09, BKS16, BS19, KLH16] study the spectrum of maximal models in the context where the class has a bounded number of models. We list now some properties that are true in every cardinality for first order logic but are true only eventually for complete sentences of $L_{\omega_1,\omega}$ or, more generally, for abstract elementary classes, and compare the cardinalities (the Hanf number) at which the cofinal behavior must begin. Every infinite model of a first order theory has a proper elementary extension and so each theory has arbitrarily large models. Morley [Mor65] showed that every sentence of $L_{\omega_1,\omega}$ that has models up to $\mathfrak{d}_{\omega_1}$ has arbitrarily large models and provided counterexamples showing that cardinal was minimal. Thus he showed the Hanf number for existence of $L_{\omega_1,\omega}$-sentences in a countable vocabulary is $\mathfrak{d}_{\omega_1}$. Hjorth [Hjo02], by a

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much more complicated argument, showed there are complete sentences $\phi_\alpha$ for $\alpha < \omega_1$ such that $\phi_\alpha$ has a model in $\aleph_\alpha$ and no larger so the Hanf number for complete sentences is $\aleph_\omega$. The amalgamation property holds for every complete first order theory. However, [BB17] show that an upper bound on the Hanf number for amalgamation is the first strongly compact; the actual value remains open. Boney and Unger [BU17], building on [She13] show that the Hanf number ‘for all AEC’s are tame’ is the first strongly compact cardinal. They also show the analogous property for various variants on tameness is equivalent to the existence of almost (weakly) compact, measurable, strongly compact). The result here shows in ZFC that the Hanf number for extendability (every model of a complete sentence has a proper $L_{\omega_1, \omega}$-elementary extension) is the first measurable cardinal.

Section 1 provides some background information on Boolean algebras. Section 2 is a set theoretic argument for the existence of a Boolean algebra with certain specified properties in any cardinal $\lambda$ of the form $\lambda = 2^\mu$ that is less than the first measurable; this construction is completely independent of the model theoretic results. Then we make the connection with model theory. In particular, we link the construction here with the complete sentence $\phi$ from [BS2x]. Section 3 builds several approximations to the counterexample. Subsection 3.1 introduces the most basic class of models $K_{-1}$ and explains the connections with [BS2x]. Subsection 3.2 builds on this result to find a $P_0$-maximal model in $K_{-1}$ with cardinality $\lambda$ satisfying certain further restrictions. We recall in Subsection 3.3 the class $K_2$ of models of the complete sentence from [BS2x]. In Section 4, the $P_0$-maximal model from Section 3.2 is converted to the $P_0$-maximal model in $K_2$. From this, it is easy to find a maximal model in $K_2$ of roughly the same cardinality.

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1 Preliminaries

This paper depends heavily on [BS2x] which contains a fuller background and essential material on Boolean algebras. In particular, the incomplete sentence with maximal models cofinal in the first measurable is described there and the construction of the desired complete sentence; in this paper we show in ZFC that sentence has maximal models below the first measurable. We repeat in this section the main slightly non-standard definitions from Boolean algebra that appear in [BS2x] and some immediate consequences.

**Definition 1.1**

1. A Boolean polynomial $p(v_0, \ldots, v_k)$ is a term formed by the compositions of the $\land, \lor, ^{-1}, 0, 1$ on the variables $v_i$; a polynomial over $X$ arises when elements of $X$ are substituted for some of the $v_i$.

2. For $X \subseteq B$ and $B$ a Boolean algebra, $\overline{X} = X_B = \langle X \rangle_B$ denotes the subalgebra of $B$ generated by $X$. 
3. A set $Y$ is independent (or free) over $X$ modulo an ideal $\mathcal{I}$ (with domain $I$) in a Boolean algebra $B$ if and only if for any Boolean polynomial $p(v_0, \ldots, v_k)$ (that is not identically 0, i.e. non-trivial), and any $a \in \langle X \rangle_B - \mathcal{I}$, and distinct $y_i \in Y$, $p(y_0, \ldots, y_k) \land a \notin \mathcal{I}$.

4. A $Y$ which is independent over $X$ modulo $I$ is called a basis for $\langle X \cup Y \cup I \rangle$ over $\langle X \cup I \rangle$.

In this context, ‘independent from’ may sometimes be written ‘independent over’. These are distinct notions for forking independence.

**Observation 1.2** If $\mathcal{I}$ is the 0 ideal, (i.e., $Y$ is independent over $X$),

1. the condition becomes: for any $b \in \langle X \rangle_B - \{0\}$, $B \models p(y_0, \ldots, y_k) \land b > 0$.
   That is, every finite Boolean combination of elements of $Y$ has non-empty meet with each non-zero $a \in \langle X \rangle_B$.

2. or, there is no non-trivial polynomial $q(y, x)$ and $b \subseteq X$ such that $q(y, b) = 0$.

That 2) implies 1) is obvious. For the converse, put a counterexample $q(y, b) = 0$ in disjunctive normal form. Then for each disjunct (i.e. each constituent conjunction) $q'(y, b) = 0$ (not all variables of $q$ may appear in $q'$). We can replace those $b$’s that appear in $q'$ by a single element $b$ of $\langle X \rangle$ to get a $q''(y, b) = 0$; $q''$ contradicts condition 1).

With Observation 1.2 we obtain an analog for Boolean algebras of the notion of dependence in vector spaces in rings or fields: \{y_0, \ldots, y_k\} are dependent over $X$ if some non-trivial polynomial $p(v_0, \ldots, v_k, w_0, \ldots w_m)$ and some $b$ from $X$, $p(y, b) = 0$. This yields that if $B_2$ is freely generated over $B_1$, all atoms in $B_1$ remain atoms in $B_2$. If not, there would be an atom $a$ of $B_1$ and a term $\sigma(b_2, b_1)$ with $0_{B_1} < \sigma(b_2, b_1) < a$ and $\sigma(b_2, b_1) \in B_1$. But then $B_2 \models \sigma(b_2, b_1) \land a = 0$; this contradicts the freeness assumption. This notion of dependence (a depends on $X$ if and only if $a \in \langle X \rangle$) does not satisfy the exchange axiom. See [Grä79, Chapter 5] for the strong consequences if this dependence relation satisfies exchange.

There is no requirement that $\mathcal{I}$ be contained in $X$. Observe the following:

**Observation 1.3** Let $\mathcal{I}$ be an ideal in a Boolean algebra $B$.

1. Let $\pi$ map $B$ to $B/\mathcal{I}$. If ‘$Y$ is independent from $X$ over $\mathcal{I}$’ then the image of $Y$ is free from the image of $X$ (over 0) in $B/\mathcal{I}$. Conversely, if $\pi(Y)$ is independent over $\pi(X)$ in $B/\mathcal{I}$, for any $Y'$ mapping by $\pi$ to $\pi(Y)$, $Y'$ is independent from $X$ over $\mathcal{I}$.

   So, if $X$ is empty, the condition ‘$Y$ is independent over $\mathcal{I}$’ implies the image of $Y$ is an independent subset of $B/\mathcal{I}$.

2. If a set $Y$ is independent (or free) from $X$ over $\mathcal{I}$ in $B$ and $Y_0$ is a subset of $Y$, then $Y - Y_0$ is independent (or free) from $X \cup Y_0$ (\langle X \cup Y_0 \rangle_B) over the ideal $\mathcal{I}$ in the Boolean algebra $B$. 

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2 Set theoretic construction of a Boolean algebra

We define a property \( \boxplus(\lambda) \), which asserts the existence in \( \lambda \) of a Boolean algebra that is 'uniformly \( \aleph_1 \)-incomplete'. We then show certain conditions on \( \lambda \) imply \( \boxplus(\lambda) \). So this section has no model theory. The arguments here are similar to those around page 7 of [GS05]. We connect this construction with our model theoretic approach in Section 3.

**Definition 2.1** (\( \boxplus(\lambda) \)) denotes: There are a Boolean algebra \( B \subset P(\lambda) \) with \( |B| = \lambda \) and a set \( A \subseteq \omega \times B \) such that:

1. \( A \) has cardinality \( \lambda \) and if \( A = \{ A_n : n \in \omega \} \in A \) then for \( \alpha < \lambda \) for all but finitely many \( n, \alpha \notin A_n \).

2. \( B \) includes the finite subsets of \( \lambda \); but is such that for every non-principal ultrafilter \( D \) of \( \lambda \) (equivalently an ultrafilter of \( B \) that is disjoint from \( \lambda^{<\omega} \)) for some sequence \( \langle A_n : n \in \omega \rangle \in A \), there are infinitely many \( n \) with \( A_n \in D \).

We may say that \((B, A)\) witness uniform \( \aleph_1 \)-incompleteness.

**Theorem 2.2 (ZFC)** Assume for some \( \mu, \lambda = 2^\mu \) and \( \lambda \) is less than the first measurable, then \( \boxplus(\lambda) \) from 2.1 holds.

We need the following structure.

**Definition 2.3**
1. Fix the vocabulary \( \tau \) with unary predicates \( P, U \), a binary predicate \( C \), and a binary function \( F \).

2. Let \( \langle C_\alpha : \alpha < \lambda \rangle \) list without repetitions \( P(\mu) \) such that \( C_0 = \emptyset \) and also let \( \langle f_\alpha : \mu \leq \alpha < \lambda \rangle \) list \( ^\mu \omega \).

3. Define the \( \tau \)-structure \( M \) by:
   - (a) The universe of \( M \) is \( \lambda \);
   - (b) \( P^M = \omega; U^M = \mu \);
   - (c) \( C(x, y) \) is binary relation on \( U \times M \) defined by \( C(x, \alpha) \) if and only \( x \in C_\alpha \). Note that \( C \) is extensional. i.e., elements of \( M \) uniquely code subsets of \( U^M \);
   - (d) Let \( F_2^M(\alpha, \beta) \) map \( M \times U^M \to P^M \) by \( F_2^M(\alpha, \beta) = f_\alpha(\beta) \) for \( \alpha < \lambda, \beta < \mu \);
   - (e) \( F_2^M(\alpha, \beta) = 0 \) for \( \alpha < \lambda \) and \( \beta \in [\mu, \lambda) \).

We use the following, likely well-known, fact pointed out to us by Sherwood Hachtman.

**Fact 2.4** Let \( D \subseteq P(X) \) and suppose that for each partition \( Y \subseteq P(X) \) of \( X \) into at most countably many sets, \( |D \cap Y| = 1 \). Then, \( D \) is a countably complete ultrafilter.
Lemma 2.5 If $\lambda$ is less than the first measurable cardinal and $\lambda = 2^\mu$ for some $\mu$ there is a model $M$, with $|M| = \lambda$, and a countable vocabulary with $PM$ denoting the natural numbers such that every first order proper elementary extension $N$ of $M$ properly extends $PM$.

Proof. Fix $M$ as in Definition 2.3. We first show that any proper elementary extension $N$ of $M$ extends $U^M$. Suppose for contradiction there exists $\alpha' \in N - M$ but $U^N = U^M$. By the full listing of the $C_\alpha$, there is a $\beta \in M$ with $\{x : N \models C(x, \beta)\} = \{x : N \models C(x, \alpha')\}$. This contradicts extensionality of the relation $C$ in $N$; but $C$ is extensional in the elementary submodel $M$.

Now we show that if $U^M \subset N$ and $PM = PN$, then there is a countably complete non-principal ultrafilter on $\mu$, contradicting that $\mu$ is not measurable. Note that the sequence $\langle f_\alpha : \mu \leq \alpha < \lambda \rangle$ can be viewed as a list of all non-trivial partitions of $\mu$ into at most countably many pieces. Let $\nu^* \in U^N - U^M$. For $\alpha \in N$, denote $F^N(\alpha, \nu^*)$ by $n_{\alpha}$. Since $PM = PN$, $n_\alpha \in M$. By elementarity, for $\alpha \in M, \eta \in U^M$, $F^\alpha(\alpha, \eta) = F^M(\alpha, \eta) = f_\alpha(\eta)$. Now, let

$$D = \{x \subseteq U^M : x \neq \emptyset \land (\exists \alpha \in M) x \supseteq f_\alpha^{-1}(n_\alpha)\}.$$

We show $D$ satisfies the conditions from Fact 2.4. Let $W$ be a partition, indexed by $f_\alpha$. Then $f_\alpha^{-1}(n_\alpha) \neq \emptyset$ and is in $D$. Suppose for contradiction there are $x_0 \neq x_1$ in $W$ that are both in $D$. Then, there are $\alpha_i \in M$ such that $x_i \in W \cap D$ contains $f_\alpha^{-1}(n_\alpha_i)$ for $i = 0, 1$. So, $N \models F(\alpha_i, \nu^*) = n_{\alpha_i}$ for $i = 1, 2$. Since $\alpha_i \in M$ and $M \prec N$, $M \models \exists x(F(\alpha_0, x) = n_{\alpha_0} \land F(\alpha_1, x) = n_{\alpha_1}$. So, by Definition 2.3 (d), for any witness $a$ in $M$ for this formula, $a \in x_0 \cap x_1$; but $x_0 \cap x_1 = \emptyset$ since $W$ is a partition.

Finally, $D$ is non-principal on $U^M$ since if it were generated by an $a \in U^M$,

$$D = \{x \subseteq U : (\exists \alpha) x \supseteq f_\alpha^{-1}(n_\alpha)\} = \{x \subseteq U : a \in x\}.$$

Since $\{a\} \in D$, for some $\alpha_0 \in M$, $\{a\} = f_{\alpha_0}^{-1}(n_{\alpha_0})$. Note that $\alpha_0 \in M$, because the definition of $D$ is about the model $M$. That is, $M \models \exists x F(\alpha_0, y) = n_{\alpha_0}$. But $N \models F(\alpha_0, a) = n_{\alpha_0} \land F(\alpha_0, \nu^*) = n_{\alpha_0}$. This contradicts the assumption $M \prec N$ and completes the proof. $\square_{2.5}$

The following claim completes the proof of Theorem 2.2

Claim 2.6 If $B$ is the Boolean algebra of definable formulas in the $M$ defined in Definition 2.3, there is an $A$ such that $(B, A)$ is uniformly $\aleph_1$-incomplete so $\aleph(\lambda)$ holds.

Proof. We may assume $\tau$ has Skolem functions for $M$ and then define $B$ and $A$ as follows to satisfy $\aleph(\tau)$ (ii). Let $B$ be the Boolean algebra of definable subsets of $M$. I.e.,

$$B = \{X \subseteq M : \text{ for some $\tau$-formula } \phi(x, y) \text{ and } b \in \downarrow \phi(x, y) \subseteq M, \phi(M, b) = X\}$$
Note $\mathbb{B}$ is a Boolean algebra of cardinality $\lambda$ with the normal operations. We define the Skolem functions a little differently than usual: as maps $\sigma_\phi = \sigma_\phi(x,w,y)$ from $M^{n+1}$ to $M$ for formulas $\phi(x,w,y)$ such that $\phi(\sigma_\phi(b,a),b,a)$. Here $\lg(y) = n$. Then, we specialize the Skolem functions by considering the unary function arising from fixing the $y$ entry of $\sigma_\phi(w,y)$ to obtain $\sigma_\phi(w,a)$.

\[ A_n^{\sigma_\phi(w,a)} = \{ \alpha < \lambda : \phi(\sigma_\phi^M(\alpha,a),\alpha,a) \land P(\sigma_\phi^M(\alpha,a)) \land \sigma_\phi^M(\alpha,a) \not\in n \} \]

\[ \cup \{ \alpha < \lambda : n = 0 \land \neg P(\sigma_\phi^M(\alpha,a)) \}. \]

Then let $A_{\sigma_\phi}(w,a) = (A_n^{\sigma_\phi(w,a)} : n < \omega)$ and

\[ (\ast) \ A = \{ A_{\sigma_\phi}(w,a) : \text{for some $\tau_M$-term $\sigma_\phi(w,y)$ and $a \in \lg(y)M.$} \} \]

Note $|A| = \lambda = \lambda^\omega$ as for each $a \in M$ and each of the countably many terms $\sigma_\phi(w,a), A_{\sigma_\phi(x,w,a)}$ is a map from $\omega$ into $\mathbb{B}$. For each $\alpha$, for each $0 < m < \omega$ and $A = A_{\sigma_\phi(\alpha,b)}$, the set $\{ m : \alpha \in A_m \}$ is finite, bounded by $\sigma_\phi(\alpha,a)$. Thus, clause i) of $\mathbb{B}$ is satisfied.

We now show Clause ii) of $\mathbb{B}$. Let $D$ be an arbitrary non-principal ultrafilter on $\lambda$ and where $\phi(v,y)$ varies over first order $\tau$-formulas such that $y$ and $a$ have the same length, define the type $p(x) = p_D(x)$ as:

\[ p(x) = \{ \phi(x,a) : \{ \alpha \in M : M \models \phi(\alpha,a) \} \in D \}. \]

Since $D$ is an ultrafilter, $p$ is a complete type over $M$. So there is an elementary extension $N$ of $M$ where an element $d$ realizes $p$. Let $N$ be the Skolem hull of $M \cup \{d\}$. Since $D$ is non-principal, so is $p$; thus, $N \neq M$. By Lemma 2.5, we can choose a witness $c \in P^N - P^M$. Since $N$ is the Skolem hull of $M \cup \{d\}$ there is a Skolem term $\sigma(w,y) = \sigma_\phi(w,y)$ and $a \in M$ such that $c = \sigma_\phi(d,a)$. Since $c \not\in M$, for each $n \in P^M, N \models \bigwedge_{k < n} c \neq k$ so $N \models \bigwedge_{k < n} \sigma(d,a) \neq k$ so $\bigwedge_{k < n} \sigma(x,a) \neq k$ is in $p$.

That is, for each $\sigma_\phi$ and each $n, A_n^{\sigma_\phi(w,a)}$ is in $D$.

\[ \Box_{2.6} \]

3 Three Classes of Models and an Approximate Counterexample

In this section we define the model theoretic classes that produce first an amalgamation class $K_{-1}$ of finite structures (Section 3.1), then the class $K_2$ (Definition 3.3.2) of models of a complete $L_{\omega_1,\omega}$-sentence. Using Theorem 2.2, we build in Subsection 3.2 a model $M_1$ in $K_{-1}$ with cardinality $\lambda$, which is $P_0$-maximal. Subsection 3.3 defines the classes $K_1$ and $K_2$ which give us the complete sentence. In Section 4 we modify $M_1$ to a $P_0$-maximal model in $K_2$ and then construct the required maximal model in $K_2$.  

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3.1 Finitely generated models

We include, as needed, definitions of the classes of model $K_{-1}, K_1, K_2$ introduced in [BS2x]. For each $i$, $K_{<\aleph_0}^i$ denotes the class of finitely generated members of $K_i$.

**Definition 3.1.1** $\tau$ is a vocabulary with unary predicates $P_0, P_1, P_2, P_4$, binary $R, \land, \lor, \leq$ unary functions $\tau$, constants $0, 1$ and unary functions $F_n$, for $n < \omega, \leq$ is a partial order on $P_1^2$ and the Boolean algebra can be defined from it.

We occasionally use the notations $(\forall^\infty n)$ and $(\exists^\infty n)$ to mean ‘for all but finitely many’ and ‘for infinitely many’ respectively. It is easy to see that $K_{-1}$ is $L_{\omega_1, \omega}$-axiomatizable but far from complete.

**Definition 3.1.2** $(K_{-1}) M \in K_{<\aleph_0}^{-1}$ is the class of finitely generated structures $M$ satisfying the following conditions.

1. $P_0^M, P_1^M, P_2^M$ partition $M$.

2. $(P_1^M, 0, 1, \land, \lor, \leq, \tau)$ is a Boolean algebra ($\tau$ is complement). We also consider ideals and restrictions to them of the relations/operations except for complement.

3. $R \subseteq P_0^M \times P_1^M$ with $R(M, b) = \{ a : P^M_1(a, b) \}$ and the set of $\{ R(M, b) : b \in P_1^M \}$ is a Boolean algebra. $f^M : P_1^M \to P(P_0^M)$ by $f^M(b) = R(M, b)$ is a Boolean algebra homomorphism into $P(M)$.

Note that $f$ is not$^1$ in $\tau$; it is simply a convenient abbreviation for the relation between the Boolean algebra $P_1^M$ and the set algebra on $P_0$ by the map $b \mapsto R(M, b)$.

4. $P_4^M$ is the set containing each join of $n$ distinct atoms from $P_4^M$, $P_4^M$ is the union of the $P_4^M$ and so is an ideal. That is, $P_4^M$ is the set of all finite joins of atoms.

There is an element $b^* \in P_4^M$ such that $P_4^M = \{ c : c \leq b^* b_n \}$. Note that $b_n$ is not a function symbol in $\tau$.

5. $G_1^M$ is a bijection from $P_0^M$ onto $P_4^M$ such that $R(M, G_1^M(a)) = \{ a \}$. (Note that $P_0^M = \emptyset$ is allowed.

6. $P_2^M$ is finite (and may be empty). Further, for each $c \in P_2^M$ the $F_n^M(c)$ are functions from $P_2^M$ into $P_1^M$. Note that it is allowed that for all but finitely many $n$, $F_n^M(c) = 0_{P_1^M}$.

7. (countable incompleteness) If $a \in P_4^M$ and $c \in P_2^M$ then $(\forall^\infty n) a \not\subseteq M F_n^M(c)$.

Since $a \land F_n^M(c) = 0$ and $a$ is an atom, this implies $\bigwedge_{n \in \omega} \{ x : (G_1(x) \in F_n^M(c)) \} = 0$.

8. $P_1^M$ is generated as a Boolean algebra by $P_4^M \cup \{ F_n^M(c) : c \in P_2^M, n \in \omega \} \cup X$ where $X$ is a finite subset of $P_1^M$.

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$^1$The subsets of $P_0^M$ are not elements of $M$. 

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Definition 3.1.3  
1. $K_{-1}$ is the class of $\tau$ structures $M$ such that every finitely generated substructure of $M$ is in $K_{<\aleph_0}$. $K_{\mu}^{-1}$ is the members of $K_{-1}$ with cardinality $\mu$.

2. We say $M \in K_{-1}$ is atomic if $P^M_1$ is atomic as a Boolean algebra. That is, $P^M_1$ is dense in $B_M$.

3.2 A $P_0$-maximal model in $K_{-1}$

In this section we invoke Theorem 2.2 to show (Theorem 3.2.6) that we can construct $P_0$-maximal structures in the class $K_{-1}$ of appropriate cardinality below the first measurable.

Definition 3.2.1 We say $M \in K_{-1}$ is $P_0$-maximal (in $K_{-1}$) if $M \subseteq N$ and $N \in K_{-1}$ implies $P^M_0 = P^N_0$.

The notion $uf(M)$ is the crucial link between Section 2 and $P_0$-maximality. Lemma 3.2.4 is central for Theorem 3.2.6 and is applied in Theorem 4.9.

Definition 3.2.2 ($uf(M)$) For $M \in K_{-1}$, let $uf(M)$ be the set of ultrafilters $D$ of the Boolean Algebra $P^M_1$ such that $D \cap P^M_{4,1} = \emptyset$ and for each $c \in P^M_2$ only finitely many of the $F^M_n(c)$ are in $D$.

For applications we rephrase this notion with the following terminology. For any $M \in K_{-1}$ and $d \in P^M_2$, let $S^M_d(D) = \{n : F^M_n(d) \in D\}$. So $uf(M) = \emptyset$ if and only if for every ultrafilter $D$ on $P^M_1$, there exists a $d \in P^M_2$ such that $S^M_d(D)$ is infinite.

We use the following standard properties of a Boolean algebra $B$ and ideal $I$ in proving Lemma 3.2.4 and deducing Claim 3.2.9 from Definition 3.2.8.

Fact 3.2.3  
1. $b \wedge c \in I$ implies $b/I$ and $c/I$ are disjoint.

2. $b \Delta c \in I$ implies $b/I = c/I$.

3. $b - c \in I$ implies $b/I \leq c/I$.

For our collection of structures $K_{-1}$, we can characterize $P_0$-maximality in terms of ultrafilters.

Lemma 3.2.4 An $M \in K_{-1}$ is $P_0$-maximal if and only if $uf(M) = \emptyset$.

Proof. Suppose $M$ is not $P_0$-maximal and $M \subseteq N$ with $N \in K_{-1}$ and $d^* \in P^N_0 - P^M_0$. Then $\{b \in M : R^N(d^*, b)\}$ is a non-principal ultrafilter $D_0$ of the Boolean algebra $P^M_1$ [BS2x, 3.3.11]. To see $D_0$ is non-principal suppose there is a $b_0 \in P^M_1$ such that $D_0 = \{b \in M : b_0 \leq b\}$. Note $b_0 = G^N_1(a)$ for some $a \in P^N_0$. But $N \models G^N_1(d^*) \not\leq b_0$, contradicting $\{d^*\} \in D_0$.

For each $c \in P^M_2$, since $N \in K_{-1}$, by countable incompleteness (clause 7 of Definition 3.1.2), for all $a \in P^N_0$ and all but finitely many $n$, $G^N_1(a) \not\leq F^N_n(c)$. Since
$F_n^N(c) = F_n^M(c)$, only finitely many of the $F_n^M(c)$ can be in $D_0$, which implies $D_0 \in \text{uf}(M)$. By contraposition we have the right to left.

Conversely, if $D \in \text{uf}(M)$, we can construct an extension by adding an element $d \in P_0^N$ satisfying $R^N(d, b)$ iff $b \in D$. Let $P_0^N$ be the Boolean algebra generated by $P_0^M \cup \{G_1(d)\}$ modulo the ideal generated by $\{G_1^N(d) - b : b \in D\}$; this implies that in the quotient $G_1^N(d) \leq b$. (Compare Fact 3.2.3). Let $P_2^N = P_2^M$ and $F_n^N(c) = F_n^M(c)$.

Since $D \in \text{uf}(M)$, it is easy to check that $N \in K_{\omega_1}$.

We now introduce the requirement that the Boolean algebras constructed will, when the atoms are factored out, be free. Moreover, there is a set $Y \subseteq P_2^N$ with $|Y| = \lambda$ such that different $c \in Y$ generate coinitially disjoint collections of $F_n^N(c)$ as $c$ varies. This strong requirement is used inductively in this section to construct an approximation to the counterexample. The correction in Section 4 loses this disjointness (and thus freeness).

**Definition 3.2.5 (Nicely Free)** We say $M \in K_{\omega_1}$ is nicely free when $|P_1^M| = \lambda$ and there is a sequence $b = (b_\alpha : \alpha < \lambda)$ such that

(a) $b_\alpha \in P_1^M - P_4^M$;

(b) $\langle b_\alpha / P_4^M : \alpha < \lambda \rangle$ generate $P_1^M / P_4^M$ freely;

(c) there is a set $Y \subseteq P_2^M$ of cardinality $\lambda$ such that $\{F_n(c) : n < \omega; c \in Y\}$ without repetition is a subset of the basis $\{b_\alpha : \alpha < \lambda\}$ mod atoms. For $c \in Y$, we write $u_c = \{F_n^M(c) : n < \omega\}$.

Nicely free is quite distinct from the notion $K_1$-free introduced in [BS2x]. There are maximal nicely free models but there are no maximal $K_1$-free models. Note that condition Definition 3.2.5.c asserts that a subset of $P_2^M$ partitions a subset of the basis.

Here is the main theorem of Section 3. The hypotheses $\lambda = 2^\mu$ and $\lambda$ is less than the first measurable cardinal were used essentially as the hypotheses for proving $\mathbb{H}(\lambda)$, the existence of a uniformly $\aleph_1$-incomplete Boolean algebra. But here we use $\mathbb{H}(\lambda)$ and don’t rely again on $\lambda$ being less than the first measurable cardinal. The argument here does depend on $\lambda = \lambda^{\aleph_0}$, which follows from $\lambda = 2^\mu$. By constructing a nicely free model, we introduce at this stage the independence requirements, needed in Section 4 to satisfy Definition 3.3.1.6, on the $F_n^M(c)$.

**Theorem 3.2.6** If for some $\mu$, $\lambda = 2^\mu$ and $\lambda$ is less than the first measurable cardinal then there is a $P_0$-maximal model $M_\ast$ in $K_{\omega_1}$ such that $|P_i^{M_\ast}| = \lambda$ (for $i = 0, 1, 2$), $P_i^{M_\ast}$ is an atomic Boolean algebra, $\text{uf}(M_\ast) = \emptyset$, and $M_\ast$ is nicely free.

Proof. We first construct by induction a $P_0$-maximal model in $K_{\omega_1}^\omega$. The property $\mathbb{H}(\lambda)$ (Definition 2.1) appears in the construction to satisfy Specification (f) and is used in the proof that the construction works in considering possibility 2. We choose $M_\ast$, $D_\ast$, and other auxiliaries by induction for $\epsilon \leq \omega + 1$ to satisfy the following specifications of the construction.

**Construction 3.2.7 (Specifications)** (a) For $\epsilon \leq \omega + 1$, $M_\epsilon$ is a continuous increasing chain of members of $K_{\omega_1}^\epsilon$ with each $P_i^{M_\epsilon}$ atomic and $P_1^{M_{\omega + 1}} = P_1^{M_\omega}$.
(b) For all \( \epsilon \leq \omega, |P_i^M| = \lambda \) and \( P_i^{M_\omega} = P_i^{M_{\omega + 1}} \) for \( i = 0, 1 \).

(c) For all \( \epsilon \leq \omega + 1, P_1^{M_\epsilon} / P_4^{M_\epsilon} \) is a free Boolean algebra:

(d) (i) If \( \epsilon < \omega, D_\epsilon \in \text{uf}(M_\epsilon) \).

(ii) If \( \epsilon = 0 \), then \( b_{-1} = \langle b_{-1, \alpha} : \alpha < \lambda \rangle \) is a free basis of \( P_1^{M_0}, P_4^{M_0} \), listed without repetition, and \( \langle F_n^{M_0}(c) : n < \omega, c \in P_2^{M_0} \rangle \) lists \( \langle b_{-1, \alpha} : \alpha < \lambda \rangle \) without repetition.

(iii) If \( \epsilon = \zeta + 1 < \omega \) then there is a free basis \( b_\zeta = \langle b_{\zeta, \alpha} / P_4^{M_\zeta} : \alpha < \lambda \rangle \) of \( P_1^{M_{\zeta + 1}} / P_4^{M_{\zeta + 1}} \). Note \( b_{\zeta, \alpha} \in P_1^{M_{\zeta + 1}} - P_4^{M_{\zeta + 1}} \).

(e) If \( \epsilon = \omega + 1 \), for each \( \mathbf{a} \in \omega(P_1^{M_{\omega + 1}} - P_4^{M_{\omega + 1}}) \) such that for each \( a \in P_4^{M_{\omega + 1}} \), satisfying that all but finitely many \( n, a \in R(M_{\omega + 1}, d_n) \), there is a \( c \in P_2^{M_{\omega + 1}} \), \( F_{n+1}^{M_{\omega + 1}}(c) = d_n \): (We will in fact have that \( P_1^{M_{\omega + 1}} = P_4^{M_{\omega + 1}} \) and \( P_4^{M_{\omega + 1}} = P_4^{M_{\omega + 1}} \)).

(f) \( \epsilon = \zeta + 1 < \omega \):

Let \( B \) and \( A \) be as in Definition 2.1. There is a 1-1 function \( f_\epsilon \) from \( \lambda \) onto \( P_4^{M_{\omega + 1}} \) such that:

i) for every \( X \in B \) (from \( \mathbb{B} \)) there is a \( b = b_X \in P_1^{M_{\omega + 1}} \) such that

\[ \{ \alpha < \lambda : f_\epsilon(\alpha) \leq_M b_X \} = X; \]

ii) for each \( \mathbf{a} = \langle A_n : n < \omega \rangle \in A \) there is a \( c \in P_2^{M_{\omega + 1}} \) such that for each \( n \):

\[ A_n = \{ \alpha < \lambda : f_\epsilon(\alpha) \leq_{P_4^{M_{\omega + 1}}} F_n^{M_{\omega + 1}}(c) \}. \]

Carrying out the construction.

Case 1: When \( \epsilon = 0 \), take \( P_1^{M_0} \) as the Boolean algebra generated by a set \( P_1^{M_0} \) of cardinality \( \lambda \) along with a set \( \{ b_{-1, \alpha} : \alpha < \lambda \} \) of independent subsets of \( \mathcal{P}(\lambda) \). Let \( G_1 \) be a bijection between a set \( P_1^{M_0} \) and \( P_4^{M_0} \). Set \( P_4^{M_0} \) as the ideal generated by the image of \( G_1 \). For \( a \in P_1^{M_0} \) and \( b \in P_4^{M_0} \), define \( R^{M_0}(a, b) \) to hold if \( G_1(a) \leq b \). Set \( P_4^{M_0} \) as a set of cardinality of \( \lambda \) and let \( \langle F_n^{M_0}(c) : n < \omega, c \in P_2^{M_0} \rangle \) list \( \langle b_{-1, \alpha} : \alpha < \lambda \rangle \) without repetition. Thus, any non-principal ultrafilter on \( P_1^{M_0} \) is in \( \text{uf}(M_0) \).

Case 2: For \( \epsilon = \omega, M_\omega = \bigcup_{n \in \omega} M_n \). Since the set of free generators is extended at the step, the union is also free mod \( P_4^M \).

Case 3: If \( \epsilon = \zeta + 1 < \omega \), the main effort is to verify clauses (c), (d), and (f) of Specification 3.2.7. The element \( b_{\zeta, \alpha} \) is the \( b_{A_n} \) from Specification 3.2.7.f.(i).

Now, to construct \( M_\zeta \):

(i) Recall that \( D_\zeta \in \text{uf}(M_\zeta) \).

(ii) choose as the new atoms introduced at this stage a set \( B_\epsilon \subseteq \mathcal{P}(\lambda) \) with \( B_\epsilon \cap M_\zeta = \emptyset \) and \( |B_\epsilon| = \lambda \).
(iii) Let $f$ be a one-to-one function from $\lambda$ onto $B \cup P_{4.1}^{M_\zeta}$.

(iv) Let $\langle X_\gamma : \gamma < \lambda \rangle$ list the elements of $B$ (definable subsets of $M$) from $B.ii$ with $X_0 = \emptyset$.

(v) Fix a sequence $\{b_{\zeta,\alpha} : \alpha < \lambda\}$, which are distinct and not in $M_\zeta \cup B_\zeta$, and let $B'_\zeta$ be the Boolean Algebra generated freely by

$$P_{1}^{M_\zeta} \cup \{b_{\zeta,\alpha} : \alpha < \lambda\} \cup \{f_\zeta(\alpha) : \alpha < \lambda\}.$$ 

Using Lemma 3.2.3, we apply the following definition at the successor stage. Here we take an abstract Boolean algebra $B'_\zeta$ and impose relations to embed $P_{1}^{M_\zeta}$ in a quotient $B''_\zeta$ of $B'_\zeta$.

**Definition 3.2.8 (Ideal)** Let $I_\zeta$ be the ideal of $B'_\zeta$ generated by:

1. $\sigma(a_0, \ldots, a_m)$ when $\sigma(x_0, \ldots, x_m)$ is a Boolean term, $a_0, \ldots, a_m \in P_{1}^{M_\zeta}$ and $P_{1}^{M_\zeta} \models \sigma(a_0, \ldots, a_m) = 0$.

   The next two clauses aim to show that in $M/I_\zeta$, the element $b_{\zeta, \gamma}$ is the $b_{X_\gamma}$ from Specification 3.2.7 f.ii). That is, $\{\alpha < \lambda : f_\zeta(\alpha) \leq_{M_\zeta} b_{\zeta, \gamma}\} = X_\gamma$. Recall (Definition 2.1) that the $X_\gamma$ enumerate $B$ and are subsets of $\lambda$.

2. $f_\zeta(\alpha) - b_{\zeta, \gamma}$ when $\alpha \in X_\gamma$ and $\alpha, \gamma < \lambda$.

3. $b_{\zeta, \gamma} \wedge f_\zeta(\alpha)$ when $\alpha \in \lambda - X_\gamma$ and $\alpha, \gamma < \lambda$.

   To show the $f_\zeta(\gamma)$ are disjoint atoms we add:

4. For any $f_\zeta(\gamma)$ and any $b \in B'_\zeta$ either $(f_\zeta(\gamma) \wedge b) \in I_\zeta$ or $(f_\zeta(\gamma) - b) \in I_\zeta$.

5. $f_\zeta(\gamma_1) \wedge f_\zeta(\gamma_2)$ when $\gamma_1 < \gamma_2 < \lambda$;

6. $f_\zeta(\alpha) - b$ when $\alpha < \lambda$, $f_\zeta(\alpha) \notin P_{4.1}^{M_\zeta}$ and $b \in D_\zeta$.

This asserts: Every new atom is below each $b \in D_\zeta$ and is used at the end of case 3 of the construction.

Let $B''_\zeta = B'_\zeta/I_\zeta$. Applying Fact 3.2.3, we see from Definition 3.2.8:

**Claim 3.2.9** The structure $P_{1}^{M_\zeta}$ is embedded as a Boolean algebra into $B''_\zeta$ by the map $b \mapsto b/I_\zeta$ and

1. For $\gamma < \lambda$, $f_\zeta(\gamma)/I_\zeta$ is an atom of $B''_\zeta$;

2. If $b \in P_{1}^{M_\zeta}$ is non-zero, then $b/I_\zeta \geq_{B''_\zeta} f_\zeta(\gamma)$ for some $\gamma < \lambda$. (Since $f_\zeta^{-1}$ induces an isomorphism of $B''_\zeta$ into $P(\lambda)$)
We take a further quotient of $\mathbb{B}_\zeta'$. Let
\[ J_\zeta = \{ b \in \mathbb{B}_\zeta': b/I_\zeta \wedge \mathbb{B}_\zeta', f_\alpha(\gamma) = 0 \text{ for every } \gamma < \lambda \}. \]

Then $J_\zeta$ is an ideal of $\mathbb{B}_\zeta'$ extending $I_\zeta$ so $b \mapsto b/J_\zeta$ is a homomorphism. Further, $f_\alpha(\gamma)$ is an atom of $\mathbb{B}_\zeta'/J_\zeta$ for $\gamma < \lambda$. These atoms are distinct and dense in $\mathbb{B}_\zeta'/J_\zeta$. That is, $\mathbb{B}_\zeta$ is an atomic Boolean algebra.

**Notation 3.2.10** Let $\mathbb{B}_\zeta$ be $\mathbb{B}_\zeta'/J_\zeta$ with quotient map, $j_\zeta(b) = b/J_\zeta$.

Now we define $M_\zeta$ by setting $P_{1M_\zeta}^M = \mathbb{B}_\zeta$ which contains $P_1^{M_\zeta}$; $P_{4_1M_\zeta}$ is the injective image in $P_{4_1M_\zeta}$ of $P_{1M_\zeta}^{M_\zeta} \cup B_\zeta$. For $a \in P_{1M_\zeta}^{M_\zeta}$ and $b \in P_{4_1M_\zeta}^{M_\zeta}$, set $R_{\mathbb{M}_\zeta}(a, b)$ if for some $\gamma$, $a = f_\alpha(\gamma)/J_\zeta$ and $f_\gamma(\gamma)/J_\zeta \leq \mathbb{B}_\zeta$, $b/J_\zeta$. Finally, let $D_\zeta$ be the ultrafilter on $P_{1M_\zeta}^{M_\zeta}$ generated by
\[ D_\zeta \cup \{ j_\zeta(-b_{\mathbb{M}_\zeta}; \gamma < \lambda) \cup \{ j_\zeta(-f_\alpha(\gamma)); \gamma < \lambda \}. \]

We verify $M_\zeta \in \mathbb{K}_{-1}$ below. By Claim 3.2.9, we have the cardinality and atomicity conditions of Specification 3.2.7(a) and (b); the definition of $I_\zeta$ guarantees, (c) and (d). (ii), (d). (iii). The elements $b_{\mathbb{M}_\zeta}; \gamma$ along with (our later) definition of $F_{nM_\zeta}(c)$ show d. (i), $D_\zeta \in u\mathfrak{f}(M_\zeta)$, (as no new $F_n(c)$ is in $D_\zeta$); the elements of $B_\zeta$ show $D_\zeta$ is non-principal as each complement of an atom is in the ultrafilter. Note that Specification 3.2.7(e) does not apply except in the $\omega + 1$st stage of the construction.

For Specification 3.2.7(f) (i), let $X \in \mathbb{B}$ be a set of atoms of $M_\zeta$ and note that we can choose $b_X$ by conditions ii) and iii) in Definition 3.2.8 of $I_\zeta$.

We can choose $P_{1M_\zeta}^{M_\zeta}$ and $F_{nM_\zeta}$ to satisfy Specification 3.2.7(f) (ii). Fix an $A \in \mathcal{A}$ (as given by $\boxtimes$). Fix a $c = c_{\mathcal{A}} \zeta$ and define, using the last paragraph, the $F_{nM_\zeta}(c)$ as $b_{A_n}$, so that for each $n$, $A_n = \{ a < \lambda : f_\alpha(a) \leq P_{1M_\zeta}^{M_\zeta}, F_{nM_\zeta}(c) \}$. These are the only new $c \in P_{1M_\zeta}^{M_\zeta}$.

Thus, it remains only to show that $M_\zeta \in \mathbb{K}_{-1}$. Most of the cases are obvious. E.g. for Definition 3.1.2.(8), just look at where the generators can be and recall countable free algebras are atomless. Showing $M_\zeta$ satisfies countable incompleteness, Definition 3.1.2.(7), is a bit more complex but we do so now.

(♣) If $a \in P_{4_1M_\zeta}^{M_\zeta}$ and $c \in P_{2M_\zeta}^{M_\zeta}$ then $(\forall n) a \neq_M F_{nM_\zeta}(c)$.

If $c \in P_{2M_\zeta}^{M_\zeta}$, $F_{nM_\zeta}(c) \neq_M F_{nM_\zeta}(c) \in P_{2M_\zeta}^{M_\zeta}$ and we know by induction that ♣ holds for $a \in P_{4_1M_\zeta}^{M_\zeta})$. For $a \in P_{4_1M_\zeta}^{M_\zeta} - P_{4_1M_\zeta}^{M_\zeta},$ Definition 3.1.2.5, and condition (vi) on $I_\zeta$ (from Definition 3.2.8) imply $a \leq_M b$ for every $b \in D_\zeta$. As $c \in P_{2M_\zeta}^{M_\zeta}$ and $D_\zeta \in u\mathfrak{f}(M_\zeta)$, all but finitely many $n$, $e_n = F_{nM_\zeta}(c)$, are not in $D_\zeta$. So for all but finitely many $n$, the complement $e_n \in D_\zeta$. That is, $a \leq_M e_n; \text{ so } a \wedge_M e_n = 0$ as required.

If $c \in P_{2M_\zeta}^{M_\zeta} - P_{2M_\zeta}^{M_\zeta}$ then by our choice of $P_{2M_\zeta}^{M_\zeta}$ and the $F_{nM_\zeta}$, there is an $A_c$ that is enumerated by the $F_{nM_\zeta}(c)$ and satisfies ♣ by (i) of $\boxtimes$ (Definition 2.1.(i)). This completes the verification of ♣ at stage $\epsilon$ and so $M_\epsilon$ satisfies all the specifications of the induction.

**Case 4:** $\epsilon = \omega + 1$: 12
Only clauses (c) and (e) of Specification 3.2.7 are relevant. Define \( P^M_2 \) and \( F^M_n \) to satisfy clause (e). Since \( P^M_i = P^M_{i\omega} \) for \( i = 0, 1 \), specification c) is immediate. This completes the construction.

The construction suffices.

Having completed the induction, let \( M = M_{\omega+1} \). Using specifications d) and a) of 3.2.7, it is straightforward to verify that \( M \in K_{-1} \) and the Boolean algebra is atomic. By (b), \( P^M_i \) for \( i = 0, 1 \) have cardinality \( \lambda \). And by (f), the same holds for \( P^M_{\omega+1} \).

We now show \( M \) is nicely free. Let \( b = \{ b_{n,\alpha} : \beta < \lambda \} \) enumerate \( \{ b_{n,\alpha} : n < \omega, \alpha < \lambda \} \) without repetition and such that \( \{ b_{-1,\alpha} : \alpha < \lambda \} = \{ b_{2n} : \alpha < \lambda \} \). So this picks out a first level of \( P^M_0 \) which is enumerated by the \( F^M_n(c) \) for \( c \in P^M_2 \) and \( n < \omega \) by case 1 of the construction.

Now, \( b \) satisfies the requirements in Definition 3.2.5 of nicely free. As, by Specifications 3.2.7. (c), (d) and since \( P^M_1 \) is constructed as the union of the \( P^M_{1n} \), \( P^M_1 \) is generated freely by \( b/P^M_1 \). Finally, clause c) of Definition 3.2.5 holds by clause (d.ii) of Specification 3.2.7.

The crux is to show \( M = M_{\omega+1} \) is \( P_0 \)-maximal. For this, assume for a contradiction:

\[(*) \quad P^M_0 \] is not maximal; by Lemma 3.2.4, there is a \( D \in \text{uf}(M_{\omega+1}) = \text{uf}(M_{\omega}) \).

For every \( n < \omega \), is there a \( d \in P^M_2 \) such that \( M \in \text{uf}(M_{\omega+1}) = \text{uf}(M_{\omega}) \)?

Possibility 1: For every \( n < \omega \), the answer is yes, exemplified by \( d_n \in P^M_2 \). Now for each \( a \in P^M_{1n} \), \( a \notin \text{uf}(M_{\omega+1}, d_n) \) for all \( m \geq n \). So the sequence \( a \in (d_n : n < \omega) \) satisfies the hypothesis of Specification 3.2.7(e) and so there is a \( c \in P^M_2 \) such that for each \( n < \omega \), \( F^M_n(c) = d_n \). Thus, recalling Definition 3.2.2, \( D \notin \text{uf}(M) \).

Possibility 2: For some \( n < \omega \), there is no such \( d_n \); without loss of generality, assume \( n > 0 \). We apply specification f) with \( \epsilon = n \). Recall that \( f_n \) is a 1-1 map from \( \lambda \) onto \( P^M_{1n} \). Let \( g_1 \) be the following homomorphism from the Boolean algebra \( P^M_{1\omega+1} = P^M_1 \) into \( \mathcal{P}(\lambda) : g_1(b) = \{ \alpha < \lambda : f_n(\alpha) \in B_{M,\omega} \} \). By Specification f.i) of 3.2.7, the Boolean algebra \( B \) provided by \( \mathbb{B} \) is contained in the range of \( g_1 \).

Let \( T_n \) denote the ideal of \( P^M_1 \) generated by \( P^M_{1n} \). Since \( D \) is non-principal, \( T_n \cap D = \emptyset \). Now, \( g_1 \) maps any \( b \in P^M_{1\omega+1} - P^M_{1\omega} \) (and, thus, any \( b \in P^M_{1n} - T_n \)) to a nonempty subset of \( \lambda \). Recalling \( T_n \cap D = \emptyset \), \( D_1 = g_1(D) \) is an ultrafilter of the Boolean Algebra \( \text{rg}(g_1) \) and so \( D_2 = D_1 \cap \mathbb{B} \) is an ultrafilter of the Boolean algebra \( \mathbb{B} \). We show, \( D_2 \) is non-principal, i.e., for any \( \alpha < \lambda \), \( \{ \alpha \} \notin D_2 \). As, \( f_n(\alpha) \in P^M_{1n} \) and so \( f_n(\alpha) \notin D_1 \). Thus, \( \lambda - \{ \alpha \} \notin D_1 \) and so \( \lambda - \{ \alpha \} \notin D_2 \). So \( \{ \alpha \} \notin D_2 \) as promised.

Now we apply the second clause of \( \mathbb{B} \) to the ultrafilter \( D_2 \). Since we satisfied specification f.ii) in the construction, we can conclude there is \( \mathbb{A} = \{ A_n : N < \omega \} \) such that for infinitely many \( k \), \( A_k \) is in \( D_2 \). Thus, \( u = \{ k : A_k \in D \} \) is infinite. We will finish the proof by showing there is a \( c \) such that \( u = u_c \) (Definition 3.2.5) is the set of images of the \( F^M_n(c) \).

Since we are in possibility 2), each \( A_k \in \mathbb{B} \), \( A_k \in \text{rg}(g_1) \). So we can choose \( d_k \in P^M_{1\omega} \) with \( g_1(d_k) = A_k \). As \( A_k \in D_2 \), by the choice of \( D_1, D_2 \) we have \( d_k \) is in the ultrafilter \( D \) from the hypothesis for contradiction: (*)

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We show the sequence $d = \langle d_k : k < \omega \rangle$ satisfies the hypothesis of clause e of Specification 3.2.7. First, $d_k \in P_{1}^{M_\omega} - P_{4}^{M_\omega}$ as $D$ is a non-principal ultrafilter on $P_{1}^{M_\omega}$ so the first hypothesis is satisfied. Further, for every $\alpha \in P_{0}^{M_\omega}$ all but finitely many $k$, $G_{1}^{M_\omega}(a) \not\leq_{M_\omega} d_k$ because $\overline{A} \in \mathcal{A}$, which implies by $\oplus$ ii) that for every $\alpha < \lambda$, for some $k_\alpha$, we have $k \geq k_\alpha$ implies $\alpha \not\in A_k$. Now by the definition of $g_1$, recalling $g_1(d_k) = A_k$, we have $k \geq k_\alpha$ implies $f_k(\alpha) \not\in d_k$ (in $P_{1}^{M_\omega}$). So by Specification 3.2.7. f.ii), there is a $c \in P_{2}^{M_\omega}$ such that if for all $k < \omega$, $F_{k}^{M_\omega}(c) = d_k$. So, for each finite $k$, $d_k \in D$ and $F_{k}^{M_\omega+1}(c) = d_k$. This contradicts $D \in \text{uf}(M_{\omega+1})$ and we finish.

\[\square_{3.2.6}\]

### 3.3 $K_1$ and $K_2$

We now introduce some terminology from [BS2x]. We first describe three subclasses of $K_{-1}$: $K_{<\omega_0}^1$, the finitely generated models, their direct limits $K_1$ and then the extension to $K_2$, the models of the complete sentence.

**Definition 3.3.1 ($K_{<\omega_0}^1$ Defined)** $M$ is in the class of structures $K_{<\omega_0}^1$ if $M \in K_{<\omega_0}^{-1}$ and there is a witness $\langle n_*, B, b_* \rangle$ such that:

1. $b_* \in P_{1}^{M}$ is the supremum of the finite joins of atoms in $P_{1}^{M}$. Further, for some $k$, $\bigcup_{j \leq k} P_{4}^{M,j} = \{ c : c \leq b_* \}$ and for all $n > k$, $P_{4,n}^{M} = \emptyset$.

2. $B = \langle B_n : n \geq n_* \rangle$ is an increasing sequence of finite Boolean subalgebras of $P_{1}^{M}$. Further, for some $P_{2}^{M}$ is a free Boolean algebra.

3. $B_{n_*} \supseteq \{ a \in P_{1}^{M} : a \leq b_* \} = P_{4}^{M}$; the subset

$$P_{4}^{M} \cup \{ F_{n}^{M}(c) : n < n_*, c \in P_{2}^{M} \}$$

generates $B_{n_*}$.

Moreover, the Boolean algebra $B_{n_*}$ is free over the ideal $P_{4}^{M}$ (equivalently, $B_{n_*} / P_{4}^{M}$ is a free Boolean algebra\(^2\)).

4. $\bigcup_{n \geq n_*} B_n = P_{1}^{M}$. \(\square\)

5. $P_{2}^{M}$ is finite and not empty. Further, for each $c \in P_{2}^{M}$ the $F_{n}^{M}(c)$ for $n < \omega$ are independent over $P_{4}^{M}$.

6. The set $\{ F_{m}^{M}(c) : m \geq n_*, c \in P_{2}^{M} \}$ (the enumeration is without repetition) is free from $B_{n_*}$ over $P_{4}^{M}$, $B_{n_*} \supseteq P_{4}^{M}$ and $F_{m}^{M}(c) \wedge b_* = 0$ for $m \geq n_*$. (In this definition, $0 = 0_{P_{4}^{M}}$.)

In detail, let $\sigma(\ldots x_{c_i} \ldots)$ be a Boolean algebra term in the variables $x_{c_i}$ (where the $c_i$ are in $P_{2}^{M}$ which is not identically 0. Then, for finitely many $n_1 \geq n_*$ and a finite sequence of $c_i \in P_{2}^{M}$:

$$\sigma(\ldots F_{n_1}^{M}(c_i) \ldots) > 0.$$\(^2\)

\(\square\)

A further equivalence: $|\text{Atom}(B_{n_*})| / |P_{4}^{M,1}|$ is a power of two.

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Further, for any non-zero \( d \in B_n \), with \( d \land b^* = 0 \), (i.e. \( d \in B_n - P^M_n \)),

\[
\sigma(...F^M_m(c_i)...) \land d > 0.
\]

7. For every \( n \geq n^* \), \( B_n \) is generated by \( B_{n^*} \cup \{F^M_m(c) : n \geq m \geq n^*, c \in P^M_n\} \).

Thus \( P^M_1 \) and so \( M \) is generated by \( B_{n^*} \cup P^M_2 \).

Recall some terminology from [BS2x].

**Definition 3.3.2 (\( K_1, K_2 \) Defined)**

1. \( K_1 \) denotes the collection of all direct limits of models in \( K_{<\aleph_0} \).

2. We say a model \( M \) in \( K_1 \) is rich if for any \( N_1, N_2 \in K_{<\aleph_0} \) with \( N_1 \subseteq N_2 \) and \( N_1 \subseteq M \), there is an embedding of \( N_2 \) into \( M \) over \( N_1 \).

3. \( K_2 \subseteq K_1 \) is the class of rich models.

Note that the free generation in item 6 of Definition 3.3.1 is not preserved by arbitrary direct limits and so is not a property of each model in \( K_1 \). In particular, as \( M^*_m \) is corrected to a model of \( K_1 \), we check the freeness only for finitely generated submodels as it will be false in general.

Since \( K_{<\aleph_0} \) has joint embedding, amalgamation and only countably many finitely generated models, we construct in the usual way a generic model; thus \( K_2 \) is not empty.

**Fact 3.3.3** There is a countable generic model \( M \) for \( K_1 \) (Corollary 3.2.18 of [BS2x]). We denote its Scott sentence by \( \phi \). \( K_2 \) is the class of models of this \( \phi \).

### 4 Correcting \( M^*_m \) to a model of \( K_2 \)

We now ‘correct’ the \( P_0 \)-maximal model of \( K_{-1}, M^*_m \), constructed in Section 3, to obtain a \( P_0 \)-maximal model \( M \) (Definition 3.2.1) of the complete sentence constructed in [BS2x], i.e. \( M \in K_2 \). In Theorem 4.18 we modify \( M^*_m \), to construct a model \( M \in K_2 \) with \( P^M_2 \subseteq P^M_1 \) and redefining the \( F_n \), but retaining \( M \upharpoonright (P^M_0 \cup P^M_1) = M_m \upharpoonright (P^M_0 \cup P^M_1) \). The old values of \( F^M_n \) will be used to divide the work of ensuring each ultrafilter \( D \) is not in \( \text{uf}(M) \) by for each \( D \), attending one by one to only those \( c \) with infinitely many \( F^M_n(c) \) in \( D \).

We now describe some of the salient properties of the model \( M \) obtained by ‘correcting’ the \( M^*_m \) of Section 3.

**Remark 4.1 (The Corrections)**

1. The domains of the structures constructed in this section are subsets of \( M^*_m \); the \( F_n \) are redefined so the new structures are substructures only of the reduct of \( M^*_m \) to \( \tau - \{F_n : n < \omega\} \).

2. In particular, for all the \( M \) considered in Section 4, \( P^M_1 = P^M_1 \) and these Boolean algebras have the same set of ultrafilters. However, \( \text{uf}(M) \neq \text{uf}(M^*_m) \) as the definition of \( \text{uf} \) depends on properties of the \( F_n \).
3. The set \( \{ R_n^M(c) : c \in P^M_2 \} \) is not required to be an independent subset to put \( M \in K_{-1} \).

4. Lemma 4.13 demands a sequence of finite Boolean algebras \( B_n \) to witness finitely generated substructures belong to \( K_1 \) (not required for \( K_{-1} \)). The stronger class of \( K_1 \)-free structures [BS2x, Definition 3.2.11], which is closed under extension by members of \( K_1 \) and so has no maximal models plays no active role in this paper. In particular, the final counterexample, Theorem 4.18, is in \( K_1 \) but is not \( K_1 \)-free.

5. The proof is in ZFC. The proof in [BS2x] that a non-maximal model in \( \lambda \) makes \( \lambda \) measurable depends on \( \diamond \).

The main task of this section is to prove:

**Theorem 4.2** If \( \lambda \) is less than the first measurable cardinal, \( 2^{\aleph_0} < \lambda \), and for some \( \mu \), \( 2^\mu = \lambda \) (whence \( \lambda^\omega = \lambda \)), then there is a \( P_0 \)-maximal model in \( K_2 \) of cardinality \( \lambda \).

Conclusion 4.3, summarises the results of the construction in Theorem 3.2.6, specifically to fix our assumptions for this section.

**Conclusion 4.3** If \( \lambda \) is as in Theorem 4.2 then there is a model \( M_* \) with \( |M_*| = \lambda \) satisfying:

1. \( P_1^{M_*} \) is an atomic Boolean algebra and \( M_* \) is \( P_0 \)-maximal. Further, \( |P_i^{M_*}| = \lambda \) for \( i = 0, 1 \).
2. \( P_4^{M_*} \) is the set of atoms of \( M_* \).
3. \( M_* \) is nicely free (Definition 3.2.5); in particular, \( P_4^{M_*} / P_4^{M_*} \) is a free Boolean algebra of cardinality \( \lambda \).

In order to ‘correct’ \( M_* \) to a model in \( K_2 \), we lay out some notation for the indexing of the tasks performed in the construction, the generating set of \( P_1^{M_*} \), and the free basis of the Boolean algebra \( P_1^{M_*} / P_4^{M_*} \).

**Notation 4.4** We define a family of trees of sequences:

1. For \( \alpha < \lambda \), let \( T_\alpha = \{ \} \cup \{ \alpha \hat{\eta}; \eta \in \omega^3 \} \) and \( T = \bigcup_{\alpha < \lambda} T_\alpha \).
2. \( \lim(T_\alpha) \) is the collection of paths through \( T_\alpha \).

Combining the requirements for constructing \( M_* \) (Specification 3.2.7) and the Definition 3.2.5 of nicely free, we have

**Claim 4.5 (Fixing Notation)** Since \( M_* \) is nicely free, without loss of generality, we may assume:

1. The universe of \( M_* \) is \( \lambda \) and the 0 of \( P_4^{M_*} \) is the ordinal 0.
2. We can choose sequences of elements of $P_1^{M_*}$, $b = \langle b_\eta; \eta \in T \rangle$ so that their images in the natural projection of $P_1^{M_*}$ on $P_2^{M_*}$ freely generate $P_2^{M_*} / P_2^{M_*}$.

3. For every $a \in P_4^{M_*}$ and the even ordinals $\alpha < \lambda$, there is an $n$ such that for any $\nu \in \mathcal{T}_\alpha$, $\lg(\nu) \geq n$ implies $a \land b_\nu = 0$.

Proof. The only difficulty is deducing from c) of Definition 3.2.5 (nicely free) that 3) holds. For that, we can insist that for each even $\alpha$, for some $c \in P_4^{M_*}$, $\{b_\omega^n; n < \omega \}$ enumerates $u_c = \{F_n^M(c) : n < \omega \}$ (from Definition 3.2.5.c). Now for $\alpha > 0$, let $\langle b_\eta; \eta \in \mathcal{T}_\alpha \setminus \{\langle \rangle \} \rangle$ list $\{b_\omega^n; n < \omega \}$ without repetition and $\langle b_\eta; \eta \in \mathcal{T}_0 \rangle$ list $\{b_0^n; n < \omega \}$. By Definition 3.1.2.7 $(K_{-1})$ we have: for every $a \in P_4^{M_*}$ for all but finitely many $n$, $a \land b_\omega^n = 0_{P_4^{M_*}}$; whence for even $\alpha$ all but finitely many of the $\nu \in \mathcal{T}_\alpha$ satisfy $a \land b_\nu = 0_{P_4^{M_*}}$. \(\square_{4.5}

Note that Claim 4.5 provides a 1-1 map from $P_2^{M_*}$ to ordinals less than $\lambda$. We introduce the collection of models that is the starting point for the following construction.

**Definition 4.6 (M_1 Defined)** Let $M_1 = M_1(\lambda)$ be the set of $M \in K_{-1}$ such that the universe of $M$ is contained in $\lambda$, the universe of $M_*$, and for $i < 2$, (or $i = 4$ or $(4, 1)$)

$P_i^M = P_i^{M_*}$, $M \upharpoonright (P_0^M \cup P_1^M) = M_* \upharpoonright (P_0^{M_*} \cup P_1^{M_*})$ while $P_2^M$ will not equal $P_2^{M_*}$.

The posited $M_*$ differs from any $M \in M_1$ only in that $P_2^M$ is, in general, a proper subset of $P_2^{M_*}$ and the newly defined $F_n^M(c)$ (usually) do not equal the $F_n^{M_*}(c)$. We now spell out the tasks which must be completed to correct $M_*$ to the required member of $K_2$. The $F_n^{M_*}(c)$ are used as oracles.

**Definition 4.7 (Tasks)** 1. Let $T_1$, the set of 1-tasks, be the set of pairs $(N_1, N_2)$ such that:

(a) $N_1 \subseteq N_2 \subseteq \lambda$

(b) $N_1, N_2 \in K^1_{<\eta_0}$

(c) $N_1 \subseteq M$ for some $M \in M_1$. More explicitly, $P_2^M \subseteq P_2^{M_*}$ and $N_1 \upharpoonright (P_0^M \cup P_1^M) \subseteq M_*$ and $(F_n^M \upharpoonright P_2^{N_1}) = F_n^{N_1}$ for each $n$.

2. Let $T_2$, the set of 2-tasks, be the set of $c \in P_2^{M_*}$.

3. $T = T_1 \cup T_2$.

4. Let $\{t_\alpha; \alpha < \lambda \}$ enumerate $T$.

Note $|T_1| = |T_2| = |T|$.

**Definition 4.8 (Task Satisfaction)** The task $t$ is relevant to the structure $M$ if $M \in M_1$ and i) if $t$ is 1-task $(N_1, N_2)$ and $N_1 \subseteq M$ or ii) if $t$ is a 2-task $c$ and $c \in P_2^{M_*}$.

We say $M \in M_1$ satisfies the task $t$ if either:

A) $t = (N_1, N_2) \in T_1$ (so $N_1 \subseteq M$) and there exists an embedding of $N_2$ into $M$ over $N_1$. 

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Recall Definition 3.2.2 of \( \text{uf}(M) \) and Lemma 3.2.4 connecting \( \text{uf}(M) \) with \( P_0 \)-maximality of \( M \).

**Claim 4.9** If \( M \in \mathbb{M}_1 \) satisfies all tasks in \( T \) and is in \( K_1 \) then from satisfying the \( T_2 \) tasks, \( M \) is \( P_0 \)-maximal and satisfying the tasks in \( T_1 \) guarantees it is in \( K_2 \).

Proof. For \( P_0 \)-maximality of \( M \), it suffices, by Lemma 3.2.4 (since \( \mathbb{M}_1 \subseteq K_{-1} \)), to show \( \text{uf}(M) = \emptyset \). But, since \( \text{uf}(M_e) = \emptyset \), for every ultrafilter \( D \) on \( P_1^{M_e} \) there is \( c \in P_2^{M_e} \) with \( S^M_c(D) \) infinite (Definition 3.2.2); satisfying task \( e \) means there is \( d \in P_2^{M_e} \) such that \( S^M_d(D) \) is infinite and so \( D \) is not in \( \text{uf}(M) \). Since \( M \) and \( M_e \) have the same ultrafilters, this implies \( \text{uf}(M) = \emptyset \), as required. Since we have assumed \( M \in K_1 \), the second assertion follows by realizing that satisfying all the tasks in \( T_1 \) establishes the model is rich, which suffices by Fact 3.3.3. \( \square \).

Definition 4.11 lays out the use of the generating elements \( b_i \) in correcting the \( F^M_n \) to require independence while maintaining that infinite intersections of members of the ultrafilter under consideration are empty. The infinite sequence \( \eta_d \) will guide the choice of \( F^M_n(d) \).

The following facts about the relation of symmetric difference and ultrafilters are central for calculations below.

**Remark 4.10** Recall that the operation of symmetric difference is associative.

1. Suppose \( \mathbb{B}_1 \subseteq \mathbb{B}_2 \) are Boolean algebras with \( a \in \mathbb{B}_1 \), and \( b_1 \neq c_1 \) are in \( \mathbb{B}_2 \), and \( \{b_1, c_1\} \) is independent over \( \mathbb{B}_1 \) in \( \mathbb{B}_2 \).

   The element \((b_1 \triangle c_1) \triangle a \in \mathbb{B}_2\) is independent over \( \mathbb{B}_1 \). More generally, if \( \{b_i, c_i : i < \omega\} \) are independent over \( \mathbb{B}_1 \), \( \{a_i : i < \omega\} \subseteq \mathbb{B}_2 \), \( e_i = b_i \triangle c_i \triangle a_i \), and \( f_i = b_i \triangle c_i \) then each of \( \{e_i : i < \omega\} \) and \( \{f_i : i < \omega\} \) are independent over \( \mathbb{B}_1 \).

2. Let \( D \) be an ultrafilter on a Boolean algebra \( \mathbb{B} \).

   (a) For \( a, b \in D \),
   \[
   (a \in D \iff b \in D) \text{ if and only if } a \triangle b \notin D.
   \]

   (b) If \( a_0, a_1, a_2 \in \mathbb{B} \) are distinct then at least one of \( a_i \triangle a_j \notin D \).

   (c) More importantly for our use later, it is easy to check:
   \[
   (a_0 \in D \text{ iff } a_1 \in D)
   \]
   \[
   \iff (a_0 \triangle a_1 \triangle a_2) \in D \iff a_2 \in D.
   \]

3. If \( a \) is an atom, \( a \land b_0 = 0 \) and \( a \land b_1 = 0 \), then \( a \land (b_0 \triangle b_1) = 0 \).
Proof. 1) If the element \((b \triangle c) \triangle a \in \mathbb{B}_2\) is not independent over \(\mathbb{B}_1\) there is a polynomial \(p\) over \(\mathbb{B}_1\) with \(p((b \triangle c) \triangle a) \in \mathbb{B}_1\). But then, by Observation 1.2, \(p(x, y) = p((x \triangle y) \triangle a)\) is also a polynomial over \(\mathbb{B}_1\) witnessing \(\{b, c\}\) is dependent over \(\mathbb{B}_1\). In the more general case any polynomial witnessing dependence in \(n\) of the \(e_i\) give a polynomial in \(2n\) of the \(a_i, b_i, c_i\) witnessing dependence of the original set.

2) For a), if, say \(a \in D\) and \(b \notin D\), then \(a - b\) and hence \(a \triangle b \in D\) so we have \(\text{‘left to right’} by contraposition. If both are in \(D\), so is their meet which is disjoint from \(a \triangle b\) so \(a \triangle b \notin D\). Since \(a^- \triangle b^- = a \triangle b\), we have the result if neither is in \(D\).

b) holds since the intersection over all pairs \(i, j < 3\) of the \(a_i \triangle a_j\) is empty. And c) is propositional logic from a) and b).

3) \(a \leq (b^- \land b^-) \leq (b^- \land b^-) \leq (b_0 \land b_1)^-=\). As \(a\) is an atom, \(a \land (b_0 \land b_1) = 0\).

\(\square\)

We define a class \(\mathcal{M}_2 \subseteq \mathcal{M}_1\) such that for each \(d \in P_2^M \subseteq \mathcal{M}_2\) there is an ordinal \(\alpha_d\), a tree of elements of \(P_1^M\), indexed by sequences in \((T_{\alpha_d}) \subseteq <\omega 3\), a target path \(\eta_d\) through that tree and a sequence \(a_{d,n}\), whose indices are not in \(T_{\alpha_d}\), but which satisfy that each \(a \in P_{4,1}^M = P_4^M\) is in at most finitely many \(a_{d,n}\). In the construction (Theorem 4.18) of a model in \(\mathcal{M}_2\), \(\eta_d\) guides definition of the sequence \(F_k^M(b_{n_k})\). The \(a_{d,n}\) are introduced to make Definition 4.11B uniform. In cases 2 and 3 of Theorem 4.18 \(a_{d,n}\) is always 0. In case 4, where the \(F_n^M(d)\) are defined as \(M\) is corrected from \(M_s\), \(a_{d,n} = F_n^M(d)\). The result is the values of the \(F_n^M(d)\) are both independent over a finite initial segment and satisfy \(\bigmeet_{n<\omega} F_n^M(d) = \emptyset\). The next definition abstracts from this construction to identify the key ideas of the proof that if \(M \in \mathcal{M}_2\) then \(M \in \mathcal{K}_1\) (Lemma 4.13) and further that there are \(M \in \mathcal{M}_2\) that are in \(\mathcal{K}_2\). The notation \(\langle Z\rangle\) denotes the Boolean subalgebra of \(P_1^M\) generated by \(Z\).

**Definition 4.11 (\(\mathcal{M}_2\) Defined)** Let \(\mathcal{M}_2\) be the set of \(M \in \mathcal{M}_1\) such that there is a sequence \(w = \langle (\alpha_d, \eta_d, a_{d,n}) : d \in P_2^M, n < \omega \rangle\) witnessing the membership, which means:

**A**

(a) For each \(d \in P_2^M\), \(\alpha_d < \lambda\) is even and \(d_1 \neq d_2\) implies \(\eta_{d_1} \neq \eta_{d_2}\). (In case 4 of Lemma 4.18, many \(d_n\) have the same \(\alpha_n\).)

(b) \(\langle \alpha_d \rangle \triangle < \eta_d \in \lim(T_{\alpha_d})\).

**B**

For each \(n < \omega\), there are\(^3\) \(a_{d,n}\) in \(P_{4,1}^M = P_4^M\) such that for each \(d \in P_2^M\), there are distinct\(^4\) \(\nu_1[n, d]\) and \(\nu_2[n, d]\) that extend \(\eta_d[n, \nu_i(0) = \alpha_d]\), and have length \(n + 1\) such that:

(a) For every \(n\), \(F_n^M(d) = (b_{\nu_1[n, d]} \triangle b_{\nu_2[n, d]}) \triangle a_{d,n}\).

(b) for each \(a \in P_{4,1}^M\) and each \(d \in P_2^M\), there are only finitely many \(n\) with \(a \leq F_n^M, a_{d,n}\).

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\(^3\)In applications, the \(a_{d,n}\) are either 0 or \(F_n^M(c)\) (for an appropriate \(c \in P_{4,1}^M\)).

\(^4\)I.e., \(\nu_i[n, d]\) depends on \(d\) and \(n\).
C (K₁) For each finite Y ⊆ P₂ᴺ there is a list {dₜ : ℓ < |Y|} of Y such that:

(a) The dₜ list Y without repetition and αₜ = αₜₖ.
(b) If i₁ < i₂ < i₃ < |Y| and αᵢ₁ = αᵢ₂ then αᵢ₃ = αᵢ₂.
(c) There is a k₁ ∈ K₁ such that
   i. For i ≠ j, both less than n, ηᵢ | κ₁ ≠ ηⱼ | κ₁
   ii. Set W ⊆ P₁ᴹ as:
      \[ W = \{ aₜₙ : k < |Y| ∧ n < ω \} \]
      \[ \cup \{ F_M^{κ}(dₙ) : k < |Y|, i < κ₁ \} \] (1)

Then W is included in the subalgebra \( B[V] \) of \( P₁ᴹ \) generated by
\[ \{ b_ν : \bigwedge_{i<|Y|} (η_i|κ₁) \not= ν \} \cup \{ b_0 \} \cup P_{4,1}^M \]
where \( η_i \) abbreviates \( η₂ᵢ \).

Note that the \( B[V] \) is a cocountable subset of \( P₁ᴹ \) (the complement is contained in finite set of countable trees).

We will apply the following lemma three times to show that for \( M \in \mathbb{M}_2 \), the set \( \{ F_n^{κ}(c) \} \) is countably incomplete (witnessing Definition 3.1.2.7). It is a straightforward application of Remark 4.10 to Definition 4.11.2.

**Lemma 4.12** Let \( M \in \mathbb{M}_1 \). For any \( (α_d, η_d, a_d,n) \) as in Definition 4.11, (in particular \( α_d \) is even) and any atom \( a \in P_{4,1}^M \) for all but finitely many \( n \)
\[ a \wedge (b_ν \uplus b_ρ \uplus a_d,n) = 0. \]

Proof. Recall from 4.5.3, that for every \( a \in P_{4,1}^M \) and the even ordinals \( α < λ \), there is an \( n \), such that for any \( ν, ρ \in T_α \) with \( lg(ν) \geq n \) and \( lg(ρ) \geq n \), \( a \wedge b_ν = 0 \) and \( a \wedge b_ρ = 0 \). Definition 4.11.B.b asserts each \( d \) and for sufficiently large \( n \), \( a \wedge a_d,n = 0. \) Apply Remark 4.10.3 twice.

We will show in Lemma 4.13 that members of \( \mathbb{M}_2 \) are in \( K_1 \) and then in Theorem 4.18 that there are structures in \( \mathbb{M}_2 \) that are in \( K_2 \). Two main features distinguish \( K_1 \) from \( K_{-1} \). The \( F_n(d) \) retain the ‘countable incompleteness’ property from \( K_{-1} \) but also must be independent; \( M \in K_1 \) when \( M \) is a direct limit of members of \( K_{<ω₀}^1 \).

**Lemma 4.13** If \( M \in \mathbb{M}_2 \), then \( M \in K_1 \).

Proof. Suppose \( M \in \mathbb{M}_2 \). Let \( Y ⊆ P₂ᴺ \) and \( X ⊆ P₁ᴹ \) be finite; we shall find \( N = N_N \in K_{<ω₀}^1 \) such that \( Y \cup X ⊆ N \subseteq M \); this suffices. As, \( K_1 \) is defined to be the collection of direct limits of finitely generated structures⁶ in \( K_{<ω₀}^1 \).

Our two main jobs in proving Lemma 4.13 are to find an \( N, n_a,b_a \) in which

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⁵See proof of goal in Lemma 4.18.
⁶The proof of Lemma 53 shows there is a common substructure of \( M \) containing any finite collection of finitely generated (as in this argument) substructures of \( M \).
Lemma 4.16

There is a

Fact 4.15

tion 4.14; we now show the other conditions are satisfied.

\[ \eta \]

Proof. Recall (Claim 4.5) that

(\text{without repetition and denote, for } i < n)

\[ \text{Definition 4.14 (proved in Fact 4.15).} \]

\[ \text{The existence of } k_{1}^{XY} \text{ satisfying the conditions of Definition 4.14 is proved in Fact 4.15.} \]

**Definition 4.14** \( k_{1}^{XY} \) Let the sequence \( \langle (\alpha_{d}, \eta_{d}, a_{d,k}) : d \in P_{2}^{M}, k < \omega \rangle \) witness \( M \in \mathcal{M}_{2} \) as in Definition 4.11. Let \( X \subset P_{2}^{M} \) and \( \langle d_{i} : i < n \rangle \) enumerate \( Y \subset P_{2}^{M} \) without repetition and denote, for \( i < n \), \( \eta_{d_{i}} \) by \( \eta_{i} \) and \( \alpha_{d_{i}} \) by \( \alpha_{i} \). Without loss, the \( \langle \eta_{i}(0) : i < n \rangle \) are non-decreasing:

A Fix \( k_{1} = k_{1}^{XY} \) such that

(a) \( k_{1}^{XY} \geq Y \) (see Definition 4.11.B);

(b) \( \langle \eta_{i} k_{1}^{XY} : i < n \rangle \) are distinct for \( i < n \);

(c) \( k_{1}^{XY} \geq \max \{ \lg(\nu) : b_{\nu} \in \langle X \cup \{ F_{k}^{M}(d) : i < |Y| \}, k < k_{1}^{XY} \} \}

B We consider the following sets determined by \( X \cup Y \) and the \( \eta_{i} \).

(a) \( F_{\leq 0} = F_{0} = X \cup \{ F_{k}^{M}(d) : i < |Y|, k \leq k_{1}^{XY} \} \);

(b) For \( 1 \leq \ell < |Y| \), \( F_{\ell} = \{ F_{k}^{M}(d_{\ell}) : k \geq k_{1}^{XY} \} \);

(c) \( F_{\leq \ell+1} = F_{\leq \ell} \cup F_{\ell+1} ; \)

(d) \( \mathbb{P}_{\ell} = \langle F_{\leq \ell} \rangle_{M} \).

C

\[ \mathbb{B}_{XY}^{\ell} = \{ b_{\nu} : \bigwedge_{i < n} (\eta_{i}[k_{1}^{XY}] \not\equiv \nu \text{ for } i < \ell+1) \} \cup \{ b_{0} \} \cup P_{4,1}^{M} . \]

For each \( \ell \), \( \mathbb{B}_{XY}^{\ell} \supseteq \mathbb{B}_{XY}^{\ell} \) since \( k_{1}^{XY} \geq k_{1}^{Y} \) and \( \mathbb{B}_{XY}^{\ell+1} \supseteq \mathbb{B}_{XY}^{\ell} \). In the proof of Lemma 4.16 \( B_{n_{r}} \) will be \( \mathbb{P}_{0}^{0} \) and \( N \) will be \( \mathbb{P}_{n-1}^{0} \).

Since \( X \) and \( Y \) are finite we now choose \( k_{1}^{XY} \) to satisfy conditions 1-3 of Definition 4.14; we now show the other conditions are satisfied.

**Fact 4.15** There is a \( k_{1} = k_{1}^{XY} \) such that for each \( \ell \), \( F_{\ell} \) is contained in \( \mathbb{B}_{XY}^{\ell} \).

Proof. Recall (Claim 4.5) that \( M_{s} \) is free on the \( \{ b_{\eta} : \eta \in \mathcal{T} \} \) modulo the \( P_{4}^{M_{+}} \). Choose \( k_{1}^{XY} \) larger than the length of any \( \nu \) such that for some \( x \in X \), \( b_{\nu} \) is a generator in a minimal representation of \( x \) or \( \nu(0) \in \overline{\alpha} = \{ \alpha_{0}, \ldots, \alpha_{n-1} \} \). Then,
\[ F_0 \subseteq \{b_\nu : \nu \in \tau, \lg(\nu) < k_1^{XY}\} \cup \{b_0\} \cup P^M_4 \subseteq \mathbb{B}_{XY}^\ell. \]

Recall from Definition 4.11.D. that as \( \ell \) increases \( F^M_k(d_i) \) for \( i < \ell \) and all \( k \) are admitted to \( \mathbb{B}_{XY}^\ell \) and so \( F_\ell \subseteq \mathbb{B}_{XY}^\ell. \)

To establish job 1) we need the following claim.

**Lemma 4.16** For each \( 1 \leq \ell < n \), \( F_\ell \) is independent over \( \mathbb{B}_{XY}^0 \) mod \( P^M_4 \).

Proof. We prove this claim by showing by induction on \( \ell \leq |Y| = n \):

\[ (\oplus \ell) \quad F_\ell = \{F^M_k(d_i) : k \geq k_1^{XY} \text{ and } i < \ell\} \]

is independent in \( P^M_4 \) over \( \mathbb{B}_{XY}^{\ell-1} \) mod \( P^M_4 \).

For \( 1 \leq \ell < |Y| \), the induction on \( \ell \) shows incrementally, at stage \( \ell + 1 \), the independence of the \( b_{n_i}|r \) with \( r \geq k_1^{XY} \) over \( \mathbb{B}_{XY}^\ell \). By Claim 4.5.2 and the choice of \( r \geq k_1^{XY} \), the \( \{b_{\nu_1[d,r]} : r \geq k_1^{XY}\} \) are independent mod \( P^M_4 \). Thus (using the \( f_i \) from Remark 4.10) the infinite set \( \{b_{\nu_1[d,n]} \Delta b_{\nu_2[d,n]} : i \in \{0, 1\}, n \geq k_1^{XY}\} \) is independent over \( \mathbb{B}_Y^{\ell-1} \). By Definition 4.11.C the \( \{a_{d_4,k} : k \geq k_1^{XY}\} \) are in \( \mathbb{B}_Y^0 \subseteq \mathbb{B}_{XY}^\ell \).

Further, by Definition 4.11.B) for all \( n \):

\[ F^M_n(d_\ell) = (b_{\nu_1[d_\ell,n]} \Delta b_{\nu_2[d_\ell,n]}) \Delta a_{d_\ell,n}. \]

So, Lemma 4.10.2 (now using the \( c_i \)) implies \( F_\ell \) is independent over \( \mathbb{B}_Y^\ell \). Since independence is transitive (Lemma 1.3.3) \( F_\ell \) is independent over \( \mathbb{B}_Y^0). \)

We continue the proof of Lemma 4.13. By Lemma 4.12, for sufficiently large \( n \), \( a \not\in F^M_n(d_\ell) \). So the countable incompleteness condition in the definition of \( K_{n-1} \) is satisfied. This completes job 1). To accomplish job 2) and finish the proof of Lemma 4.13 by satisfying conditions 2-4 of Definition 3.3.1, we must define appropriate \( P^N_1 \) and find a sequence of finite Boolean algebras \( B_n \) witnessing that \( N \in K_{1_{<\kappa_0}} \).

Let \( P^N_1 = \mathbb{F}^{n-1} \). We have \( P^N_1 \) is freely generated (modulo the ideal generated by the atoms of \( B_{n_1} \)) by the countable set \( F_{|Y|} \) over \( B_{n_\ast} = \mathbb{F}^0 \). Let \( b_\ast \) be the supremum of the atoms in \( B_{n_\ast} \), and \( P^N_4 \) the predecessors of \( b_\ast \).

For \( m \geq n_\ast \), let \( B_m \) be generated by \( B_{n_\ast} \) and the first \( m \) elements of this generating set. Then, \( P^N_1 = \cup_{n_1 \leq m < \omega} B_m \) and \( P^N_1/P^N_4 \) is atomless. Set\(^7\) \( P^N_2 = Y \) and \( P^N_0 = \langle (G^M_1)^{\ast}(a) : a \in P^N_1 \cap P^N_1 \rangle \); thus \( P^N_1 \subseteq B_{n_\ast} \). Boolean algebras are locally finite and we can recognize whether \( (X) \) is free if by whether it has \( 2^{|X|} \) atoms. Thus, we can refine the sequence \( B_m \) to finite free algebras to witness that \( N \in K_{1_{<\kappa_0}} \). Since \( X \) and \( Y \) were arbitrary, \( M \in K_{1} \).

This completes the proof of Lemma 4.13. Now we show \( M_{2_\beta} \) is non-empty and at least one member satisfies all the tasks. In case 4) of this argument we address the requirement that \( uf(M_{\alpha}) = \emptyset \) and so \( uf(M) = \emptyset \) as well. We need the following observation because as the construction proceeds, an \( N_{1} \) may become a substructure of \( M_{\beta} \) because some value of an \( F_n \) is newly defined on a point of \( P^M_{2\beta} \).

\(^7 G^M_1 \) is from Definition 3.1.2.5.
Notation 4.17 We can enumerate $T$ as $\langle t_\alpha : \alpha < \lambda \rangle$ such that each task appears $\lambda$ times, as we assumed in Hypothesis 4.3 that $\lambda = \lambda^{\aleph_0}$.

For Theorem 4.18, to realize all the tasks, $\lambda > 2^\aleph_0$ would suffice; the requirement in Lemma 2.5 that $\lambda = 2^\mu$ is used to get maximal models. The object of case 3) is to ensure that the final model is rich (existentially complete); case 4) shows $\text{uf}(M) = \text{uf}(M_\lambda) = \emptyset$. After satisfying each task a final section labeled goal verifies that each $M_\alpha \in \mathcal{M}_2$ and so $M \in \mathcal{M}_2$.

Theorem 4.18 There is an $M \in \mathcal{M}_2$ and in $K_1$ that satisfies all the tasks, Thus, by Claim 4.9 $M \in K_2$, and is $P_0$-maximal.

Proof. As we construct $M$, we show at appropriate stages that tasks from $T_1$ and $T_2$ are satisfied. Further, we show at each stage $\alpha$ the goal: $M_\alpha \in \mathcal{M}_2$. We choose $M_\alpha$ by induction on $\alpha \leq \lambda$ such that:

1. $w_\alpha$ witnesses $M_\alpha \in \mathcal{M}_2$ (Definition 4.11). And for $\beta < \alpha$, $w_\alpha$ extends $w_\beta$. That is, for $d \in P_2^{M_{\beta+1}}$, $\alpha_d[w_\alpha] = \alpha_d[w_\beta]$, $\eta_d[w_\alpha] = \eta_d[w_\beta]$, and $a_{d,n}[w_\alpha] = a_{d,n}[w_\beta]$.

2. $P_2^{M_\alpha} \subseteq P_2^{M_\beta}$ has cardinality at most $|\alpha| + 2^{\aleph_0}$.

3. if $\alpha = \beta + 1$ and $t_\beta$ is relevant to $M_{\beta}$, $M_{\alpha}$ satisfies task $t_\beta$.

case 1 If $\alpha = 0$, set $M_0 = M_*\upharpoonright (P_0^{M_*} \cup P_1^{M_*})$.

This condition will be preserved by the induction for all $\alpha$.

case 2 Take unions at limits.

At the successor stage, we now verify task $t_{\beta+1}$ for each of two different types of task. Then, we will consider the two cases together to show the goal that $M = \bigcup_{\alpha < \lambda} M_\alpha \in \mathcal{M}_2$.

case 3 $\alpha = \beta + 1$ and say, $t_\beta \in T_1$, say $t_\beta = (N_1, N_2)$.(Definition 4.7)

Choose $M_\alpha$:

If $N_1$ is not a subset of $M_\beta$ then the task is irrelevant and let $M_\alpha = M_\beta$ and $w_\alpha = w_\beta$. If it is, let $\langle a_\ell : \ell < m \rangle$ enumerate $P_{2}^{N_2} - P_{2}^{N_1}$ and $\langle a'_\ell : \ell < m \rangle$ enumerate the first $m$ elements of $P_{2}^{M_*} - P_{2}^{M_{\beta}}$. Let $M_\alpha$ extend the $P_2^{M_\beta}$ by adding $\langle a'_\ell : \ell < m \rangle$ from $P_2^{M_*}$ to form $P_2^{M_\alpha}$. It remains to define the $w_\alpha$ and $F^{M_\alpha}(a'_\ell)$.

Let $U_\alpha = \{ \delta : (\exists b_\nu \in M_\beta)[\nu(0) = \delta] \}$. Clearly $|U_\alpha| \leq |\alpha| + 2^{\aleph_0}$ and

\[ (*) \{ a_{d,k} : k < \omega, d \in P_2^{M_\beta} \} \cup \{ b_\nu : (\exists d \in P_2^{M_\beta}) \nu \in T_{\alpha,\nu} \} \cup P_4^{M_*} \]

is included in the subalgebra of $M_\alpha$ generated by the

\[ \{ b_\nu : \exists \beta \in U_\alpha, \nu(0) = \beta \} \cup \{ b_{0\nu} \} \cup P_4^{M_*} \].

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By induction, since $M_β \in M_2$ there are witnesses $w_β = \langle α_d, η_d, a_d, k \rangle$ (formally $⟨α_β, η_β, a_β, d_β⟩$) for each $d \in P^M_β$. For the new $α'_β$, let $w_α(f) = (γ_ℓ, η_ℓ, 0^M_α)$ be chosen with the $γ_ℓ$ as the first $m$ even elements of $λ - U_α$ and with $η_ℓ(w_α) = η_κ$ chosen⁸ so that $η_ℓ(0) = γ_ℓ$. We complete the definition of $M_α$ below by choosing the $F^M_κ$ to satisfy the task.

**Task:** We now verify task $t_{β+1}$. By showing in two stages that $N_2$ can be embedded over $N_1$ into $M_α$. First we show there is an embedding of the Boolean algebras; then we define the $F_κ$ on the image to put $M_α$ in $K^1_{β+1}$. Since $N_2 \in K^1_{β+1}, P^N_1$ is decomposed as a union of the finite free Boolean algebras⁹ $⟨B^N_1 : i ≥ n^N_2⟩$ where¹⁰, writing $n_*$ for $n^N_2$, $N_2$ is freely generated over $B^N_1$ modulo $P^N_1$ by $\{F^N_κ(f) : k ≥ n^N_2, f \in P^N_2\}$. Similarly, we decompose $P^N_1$ by $⟨B^N_1 : i ≥ n_*⟩$.

Since $N_1 \subseteq M_α$ and $N_1 \subseteq N_2$, for each element $e \in P^N_1$ and any $s$,

$$P^M_α(a) ⇔ P^N_1(α) ⇔ P^N_2(a).$$

So no atom in $N_2 - N_1$ is below any element of $N_1$.

Let $c = ⟨c_0, \ldots, c_{p-1}⟩$ enumerate the atoms of $N_2$ with the $c_i$ for $i < r$ enumerating those in $N_2 - N_1$; they are all in $B^N_1$. We set $c'_i = c_i$ if $r ≤ i < p$, choose any $r$ atoms $c'_i$ from $M_α - N_1$ and by Claim 4.5, we can find a $t$ (depending on all of the $c'_i$) such that for all $i$ if $ν(0) = γ_ℓ$ and $k > t$, $b_{ν[k]} ∧ c'_i = 0$.

Each $e \in B^N_2 - (P^N_1 ∪ c)$ is a finite join of $c_i$. (Note $P^N_2$ is an alias of $B^N_1$.)

Recall $\{F^N_κ(f) : k ≥ n^N_2, f \in P^N_2\}$ is the pre-image of a basis of $P^N_1/P^N_2$.

For $f \in P^N_2$, each $F^N_κ(f)$ has $e ≤ b^N_2$. Now define $h_β$ mapping $N_2$ into $M_α$ by

1. $h_β | P^N_1$ is the identity
2. $h_β(c_i) = c'_i$
3. For $e \in B^N_2 - P^N_1, h_β(e) = e' = ∨_{c_i ≤ e} c'_i$.
4. The $b_{η_κ}(t + k)$ for $k ≥ n_*$ are independent mod $P^N_1$; for $a_κ$ in $P^N_2 - P^N_1$ set
   $$h_β(F^N_κ(a_κ)) = b_{η_κ(t+k)} ∨ b_{η_κ(t+k')} \therefore e' = F^M_κ(a_κ)$$
   where $e' = h_β(e)$ and $e = F^N_κ(a_κ)$.
5. Since the $F^N_κ(a_κ)$ freely generate $N_2/N_1$ modulo the atoms, $h_β$ extends to an embedding of $N_2$ into $M_α$.

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⁸In case 3, we need choose only a single $η_κ$ for each $ℓ < m$. In case 4, we choose $2^{ℵ_α}$ distinct $d_η$.

¹⁰We do not choose $n^M_α$. But the value is set once and for all at stage $α$ so we just call it by the final name.
Check using Claim 4.10.3 that step 4) is a homomorphism.

We now show $M_\alpha \in \mathcal{M}_2$. To clarify notation, by setting\(^1\) $a_{d,t,k} = 0$ for $i < m$, we declared:

$$F_k^{M_\alpha}(d_t) = (b_{n_i,|k'0} \triangle b_{n_i,|k'1}) \triangle a_{d,t,k}.$$  

By Lemma 4.12, for some $n$, for all $k \geq n$, $a \not\in P'_{M_\alpha}$, $F_k^{M_\alpha}(d_t)$ so condition 4.11.B.2 holds.

Finally, applying Remark 4.10.1 to $f_i = b_{n_i,|k'0} \triangle b_{n_i,|k'1}$ the $\{F_k^{M_\alpha}(d_t) : k_1^Y \leq k < \omega\}$ are independent for each $i$ and form a basis for a subalgebra $N'_{\beta}$ of $P_{M_\alpha}$ over $N_1$. Thus, $N'_{\beta} \in K_{<\kappa_0}$ and we have verified that task $t_{\beta+1}$ is satisfied.

case 4 $\alpha = \beta + 1$ and $t_\beta \in T_2$; say, $t_\beta = c$.

We define $M_\alpha$. Define $U_\alpha$ as in Case 3, but extending $U_\alpha$ to $U'_\alpha$ by adding the ordinal named by $c$ if $c \not\in M_\beta$. This extension guarantees that the $F_k^{M_\alpha}(c)$ are in $\mathcal{B}_Y$. Now choose an even ordinal $\gamma$ in $\lambda - U_\alpha$ such that

$$\{b_\eta : \eta(0) = \gamma\} \cap \{b_\eta : \eta(0) \in U'_\alpha\} = \emptyset.$$  

Extend $P_2^{M_\alpha}$ by adding a $d_\eta \in P_2^{M_\alpha}$ to $P_2^{M_\alpha}$ for each $\eta$ with $\eta(0) = \gamma$.

To define $F_{k_1}^{M_\alpha}(d_\eta)$, for each $\eta \in \lim T_2$ and $k < \omega$, choose $i_0 < i_1 \leq 2$ that are different from $\eta(k)$. Recalling $c = t_\beta$, let

$$F_{k_1}^{M_\alpha}(d_\eta) = (b_{\eta,|k'i_0} \triangle b_{\eta,|k'i_1}) \triangle (F_k^{M_\alpha}(c)).$$  

Since $M_\alpha \in K_{<\kappa_1}$ for each $a \in P_1^{M_\alpha}$, for all but finitely many $n$, $a \wedge F_k^{M_\alpha}(c) = 0$. Thus, for the $d \in P_2^{M_\alpha} - P_2^{M_\beta}$, chosen towards satisfying $t_\beta = c$, we have set $(a_{d,t}, \eta_1, a_{d,t}) = (\gamma, d_\eta, F_k^{M_\alpha}(c))$. That is, $a_{d,t} = F_k^{M_\alpha}(c)$. Thus, by Lemma 4.12 for any atom $a$ and all but finitely many $n$, $a \wedge F_{k_1}^{M_\alpha}(c) = 0$ and the countable incompleteness requirement is satisfied.

**Task:** We must show $M_\alpha$ satisfies task $t_\beta$. Since $uf(M_\alpha) = \emptyset$, for any non-principal ultrafilter $D$, there is an $e \in P_1^{M_\alpha}$ such that the set $S_\alpha^e(D) = \{n : F_n^{M_\alpha}(e) \in D\}$ is infinite (Definition 3.2.2). By the definition of the task $t_\beta = c$, there is a $D$ where the given $c$ witnesses for $D$ in $uf(M_\alpha)$. We show task $t_\beta$ is satisfied for $D$ by one of the $d_\eta$, which thus is a witness to $D \not\in uf(M_\alpha)$.

Define $\eta^D \in \lim(T_\alpha)$ by induction\(^2\): $\eta^D(0) = \gamma$. By Remark 4.10.2 one of the three elements $b_{(\gamma,i)} \triangle b_{(\gamma,j)}$, for $i \neq j$ and $i, j < 3$, must not be in $D$. Let $\eta^D(1)$ be the other member of $\{0, 1, 2\}$. For $k \geq 1$, suppose $\nu = \eta^D|k$ has been defined. Again, by Remark 4.10.2 one of the three elements $b_{\nu,i} \triangle b_{\nu,j}$, for $i \neq j$ and $i, j < 3$, must not be in $D$. Let $\eta^D(k)$ be third of the symmetric differences, which by Remark 4.10.2.c must be in $D$. For the infinitely many $n$ with $F_n^{M_\alpha}(c) \in D$, we have $F_{n\beta}^{M_\alpha}(d_{\eta,n}) \in D$.

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\(^1\)The $a_{d,t}$ are dummies in this case to provide uniformity with case 4 in proving Lemma 4.13.

\(^2\)This argument is patterned on the simple black box in Lemma 1.5 of [She], but even simpler.
Now we establish the goal for both cases.

**Goal: M_{\alpha} \in \mathbb{M}_2:** To show \( M \in K_{-1} \) (and so in \( \mathbb{M}_1 \), Definition 4.6) note that countable incompleteness (Definition 3.1.2.vii) in each case separately. For \( M_{\alpha} \in \mathbb{M}_2 \), we show \( M_{\alpha} \) satisfies Definition 4.11. The descriptive portions of Conditions A and B.i) of Definition 4.11 are clearly satisfied by the construction; Condition B.ii) was shown in the proof of each case.

For condition C, choose any finite \( Y \subset P_{2}^{M_{\alpha}} \) and partition \( Y \) into \( Y_1 = Y \cap P_{2}^{M_{\beta}} \) and \( Y_2 = Y - Y_1 \). Set \( k_1 = k_{1Y}^1 \) as the least integer\(^{13} \) such that for all \( \eta_d \neq \eta_e \) with \( d, e \in Y, \eta_d|k^1 \neq \eta_e|k^1 \).

For those \( d \in Y_1 \), we just leave \( w_{\alpha} = w_{\beta} \). For \( d \in Y_2 \), the two cases differ slightly.

In case 3, \( d \in P_{2}^{M_{\alpha}} - P_{2}^{M_{\beta}} = Y_2 \), we (implicitly) defined \( w_d(\alpha) = \langle \alpha_d, \eta_d, 0 \rangle \).

In case\(^{14} \) 4 the elements of \( Y_2 \) are among the \( 2^{M_0} d_{\eta} \) with \( \eta(0) = \gamma \). For them, \( w_d(\alpha) = \langle \gamma, \eta_d, F_{n}^{M_{\alpha}}(c) \rangle \).

For Condition 4.11.C, we show every element of \( W \) is in the \( \langle \{ b_\nu; \nu(0) \in U_\alpha \} \rangle \) and so in \( B^0_1 \). By the first line of the proof of case 4, we only need to consider the \( F_{i}^{M}(d) \) for \( d \in Y \) and \( i < |Y| \).

\[ \{ F_{i}^{M}(d_k); i < |Y|, k \leq k_{1Y} \} \] For \( d \in Y_1 \) this follows since \( d \in P_{2}^{M_{\beta}} \) implies \( \alpha_d \in U_\alpha \). In case 3, for \( d \in Y_2 \) the \( a_{d,n} \) are all 0 and the \( F_{n}^{M}(d) \) for \( i < n \) and \( n < \omega \) are all Boolean combinations of elements \( b_\nu \) with \( \nu \leq \eta_i \) \( k_{1} \).

The difference for case 4 is in verifying the \( a_{d,n} \) are in \( U_\alpha \). Now if \( a_{d,n}|w_\alpha = F_{n}^{M_{\alpha}}(c) \) is \( b_\nu \), then \( \nu(0) \in U_\alpha \) by the revised definition of \( U_\alpha \) in case 4 and so \( a_{d,n} \in B^0_1 \).

Now, let \( M = \bigcup_{\alpha < \lambda} M_{\alpha} \). Then, \( M \in \mathbb{M}_2, |P_2^M| = \lambda \). By Lemma 4.13, \( M \in K_{1} \) and each task has been satisfied, so by Claim 4.9, \( M \in K_2 \).

\( \square_{4.18} \)

This yields.

**Conclusion 4.19** The \( M \in K_2 \) constructed in Theorem 4.18 \( P_0 \)-maximal and all \( |P_1^M| = \lambda \). As in [BS2x, Corollary 3.3.14], for every \( \lambda \) less than the first measurable, since \( M \in K_2 \) implies \( |M| \leq 2^{P_0^M} \), there is a maximal model \( M \in K_2 \) with \( 2^\lambda \leq |M| < 2^{2^\lambda} \).

**Question 4.20**

1. Is there a \( \kappa < \mu \), where \( \mu \) is the first measurable, such that if a complete sentence has a maximal model in cardinality \( \kappa \), it has maximal models in cardinalities cofinal in \( \mu \)?

2. Is there a complete sentence that has maximal models cofinally in some \( \kappa \) with \( \sum_{\omega_1} \kappa < \kappa < \mu \) where \( \mu \) is the first measurable, but no larger models are maximal. Could the first inaccessible be such a \( \kappa \)?

\(^{13} \) Naturally this is only relevant when \( \alpha_d = \alpha_e \) but than can happen in case 3 and must happen in case 4.

\(^{14} \) Note that in case 3, \( a_{d,n} \) is constant, while in case 4 it depends on \( n \).
References

[BB17] John T. Baldwin and William Boney. Hanf numbers and presentation theorems in AEC. In Jose Iovino, editor, Beyond First Order Model Theory, pages 81–106. Chapman Hall, 2017.

[BKS09] John T. Baldwin, A. Kolesnikov, and S. Shelah. The amalgamation spectrum. Journal of Symbolic Logic, 74:914–928, 2009.

[BKS16] John T. Baldwin, M. Koerwien, and I. Souldatos. The joint embedding property and maximal models. Archive for Mathematical Logic, 55:545–565, 2016.

[BS19] John T. Baldwin and Ioannis Souldatos. Complete $L_{\omega_1,\omega}$ with maximal models in multiple cardinalities. Mathematical Logic Quarterly, 65(4):444–452, 12 2019.

[BS2x] John T. Baldwin and S. Shelah. Hanf numbers for extendibility and related phenomena. submitted: Shelah number 1092; first posted 2016; http://homepages.math.uic.edu/˜jbaldwin/pub/ahanfmaxoct3118.pdf, 202x.

[BU17] W. Boney and S. Unger. Large cardinal axioms from tameness in AECs. Proceedings of the American Mathematical Society, 145:4517–4532, 2017.

[GS05] R. Göbel and S. Shelah. How rigid are reduced products. Journal of Pure and Applied Algebra, 202:230–258, 2005.

[Grä79] George Grätzer. Universal Algebra. Springer-Verlag, 1979.

[Hjo02] Greg Hjorth. Knight’s model, its automorphism group, and characterizing the uncountable cardinals. Journal of Mathematical Logic, pages 113–144, 2002.

[KLH16] Alexei Kolesnikov and Christopher Lambie-Hanson. The Hanf number for amalgamation of coloring classes. Journal of Symbolic Logic, 81:570–583, 2016.

[Mag16] M. Magidor. Large cardinals and strong logics: CRM tutorial lecture 1. http://homepages.math.uic.edu/˜jbaldwin/pub/MagidorBarc.pdf, 2016.

[Mor65] M. Morley. Omitting classes of elements. In Addison, Henkin, and Tarski, editors, The Theory of Models, pages 265–273. North-Holland, Amsterdam, 1965.

[She] S. Shelah. Black boxes. paper 309 archive.0812.0656.

[She13] S. Shelah. Maximal failures of sequence locality in a.e.c. preprint on archive: Sh index 932, 2013.