Entanglement in the adiabatic limit of a two-atom Tavis–Cummings model

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Abstract
We study the adiabatic limit for the sequential passage of atoms through a high-Q cavity, in the presence of frequency chirps. Despite the fact that the adiabatic approximation might be expected to fail, we were able to show that for proper choice of Stark-pulses this is not the case. Instead, a connection to the resonant limit is established, where the robust creation of entanglement is demonstrated. Recent developments in the fabrication of high-Q cavities allow fidelities for a maximally entangled state up to 97%.

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1. Introduction

Entanglement is a special type of correlation between two or more interacting quantum systems [1]. Since they are without any classical analogy, generating such correlations has become important for the purposes of quantum computing and quantum information [2]. Among the experimental demonstrations of entanglement is atomic cavity quantum electrodynamics (QED) in the microwave regime, with which we have had the generation of Einstein–Podolsky–Rosen states, a phase gate and the creation of Schrödinger cats [3].

One of the problems encountered when considering atoms traversing a cavity resonator is that of the spatial effects due to the structure of the resonator mode [4]. As long as the atoms are moving fast enough one can safely assume that the motion of the atomic center of mass is classical, and utilize the spatial dependence of the coupling functions with time-dependent pulses [5]. Based on this assumption, we were recently able to study the adiabatic sequential passage of atoms through a cavity [6]. The main feature of the system is the existence of a pure crossing in the adiabatic limit, which has associated applications in quantum computing, including atomic entanglement.

Here, we study the off-resonant limit for the two-atom time-dependent Tavis–Cummings Hamiltonian [7], where now the atomic transition frequencies are subject to time-dependent chirps. Although one would expect the adiabatic approximation to fail, the main result is that, with a proper choice of Stark-shifting pulses, the system follows a similar adiabatic evolution as in the resonant limit [6]. In addition to this, the use of a chirp is a robust tool for fine tuning the adiabatic phases, and consequently the possible applications. When taking into account the recent developments in the engineering of high-quality cavities [3], we are able to demonstrate the generation of a maximally entangled state with very high fidelity.

The paper is organized as follows. In section 2 the system and the interaction Hamiltonian are introduced and after deriving the evolution matrix, we demonstrate how a maximally entangled state is formed. Section 3 gives a brief summary of the properties of entanglement with respect to the mode structure, and in section 4 we discuss the importance of spontaneous emission and decoherence. We summarize our results in section 5.

2. Time-dependent Tavis–Cummings model

The system under consideration consists of two atoms sequentially crossing an open spherical mirror cavity with velocities \( v_1 = v_2 = v \) (figure 1). The atoms enter the cavity with a time delay \( 2\Delta t \), where in addition to this they can follow different trajectories inside the cavity. The Hamiltonian in the rotating wave picture and within the rotating wave approximation reads (\( \hbar = 1 \))

\[
\hat{H}(\tau) = \sum_{j=1,2} \frac{\Delta_j(\tau)}{2} \hat{a}_j^\dagger \hat{a}_j + \sum_{j=1,2} \eta_j(\tau) \left( \hat{a}_j^\dagger \hat{a}_j^\dagger + \hat{a}_j \hat{a}_j^\dagger \right),
\]

where \( \Delta_j(\tau) \) is the detuning of the \( j \)th atom from the cavity mode.
For a Gaussian mode of width $2w_0$ [4], the coupling functions $\eta_j$ have the following form:

$$\eta_1(\tau) = g_1 \exp\left(-\tau^2/\Delta t^2\right),$$

$$\eta_2(\tau) = g_2 \exp\left(-\left(\tau - \delta\right)^2/\Delta t^2\right),$$

where in general $g_1 \neq g_2$. The dimensionless time $\tau$ and delay $\delta$ are defined in terms of the interaction time $\sigma$ to be

$$\tau = \frac{t}{2\sigma}, \quad \delta = \frac{\Delta t}{2\sigma}, \quad \sigma = \frac{w_0}{v}.$$  

For resonant interactions, i.e. $\Delta_j = 0$, the adiabatic limit for the above Hamiltonian proved to have a rather interesting feature [6]. At a finite time $\tau_c = \ln(g_1/g_2)/(4\delta)$, a pure energy crossing between two of the adiabatic states is observed. This occurs in the vicinity of a temporal degeneracy and results from the absence of coupling between the two degenerate states. As a consequence of this effect, the evolution matrix has a simple form, leading to a conditional entanglement between the atoms and the cavity mode. Furthermore, in the case of equal interactions, $g_1 = g_2$, a number of applications in the field of quantum computing such as logic gates, state mapping and teleportation, can be realized where the output can be tuned by means of a single dynamical phase parameter which defines the entire system evolution.

2.1. Frequency chirps

An interesting problem that arises when considering applications based on this system is that of the fine tuning of the output state. In this paper, we show how the Stark-shift technique, which is used to tune the interaction time in experiments with Rydberg atoms [3], can be used to deliver a robust control over the output state.

The whole idea requires the use of smooth Gaussian frequency chirps, such that each atom experiences a detuning $\Delta_j(\tau)$ at different times:

$$\Delta_1(\tau) = \Delta_0 \exp\left(-\left(\tau + \tau_0\right)^2/\sigma_1^2\right),$$

$$\Delta_2(\tau) = -\Delta_0 \exp\left(-\left(\tau - \tau_0\right)^2/\sigma_2^2\right).$$

These atomic detunings can be produced with Gaussian EM pulses. When the first atom enters the cavity, a weak EM pulse with a spatial distribution $E_1^2(x)$ (figure 1), is used to Stark-shift the atom, while crossing through the narrow region of $E_1^2(x)$. Once the first atom has crossed that region, the EM pulse is turned off, long before the second atom enters the cavity. In a similar way, a second pulse $E_2^2(x)$ is used after the first atom exits the cavity to shift the atomic transition frequency of the second atom (figure 1). If both pulses have a width $2L$ and they have a peak at $x = -x_0$ and $x_0$, respectively, then the time $\tau_0$ and the width $\sigma_i$ will be

$$\tau_0 = \frac{v\Delta t + x_0}{2w_0}, \quad \sigma_i = \frac{L}{w_0}.$$  

In the adiabatic limit, the system evolution will be described by the time-dependent eigenfunctions of the Hamiltonian (1). Although the analytic expressions for the adiabatic states can be derived, the whole process is lengthy [6] and beyond the scope of this paper. Here, we will demonstrate the basic features of the system by means of numerical simulations with the Schrödinger equation and along with simple qualitative arguments explain the main features of the system.

Assuming that the Stark-chirps have a short duration, $\sigma_i \ll 1 \left(L \ll w_0\right)$, and that their location $\tau = \pm \tau_0$ is away from the crossing point $\tau_c$ for the resonant limit, the propagator for symmetric interactions, $g_1 \approx g_2 = g_0$, will have a similar form as in the resonant limit [6], i.e.

$$|n; e_1, e_2\rangle \rightarrow |n; e_1, e_2\rangle,$$  

$$(6a)$$

$$|n + 1; g_1, e_2\rangle \rightarrow -|n + 1; e_1, g_2\rangle,$$  

$$(6b)$$

$$|n + 1; e_1, g_2\rangle \rightarrow \cos(\phi_\delta)|n + 1; g_1, e_2\rangle - i \sin(\phi_\delta)|n + 2; g_1, g_2\rangle,$$  

$$(6c)$$

$$|n + 2; g_1, g_2\rangle \rightarrow -i \sin(\phi_\delta)|n + 1; e_1, g_2\rangle + \cos(\phi_\delta)|n + 2; g_1, g_2\rangle,$$  

$$(6d)$$

where in terms of the resonant dynamical phase $\phi_\delta$, the corresponding off-resonant adiabatic phase $\phi_0$ reads

$$\phi_0 = \phi_\delta + 2\sigma \int_{-\infty}^{\infty} \sum_{j=1}^{2} \sqrt{\Delta_j(\tau)^2 + 4(n + 2)\eta_j(\tau)^2} \frac{d\tau}{2},$$

$$-2\sigma \int_{-\infty}^{\infty} \sum_{j=1}^{2} \eta_j(\tau) \sqrt{n + 2}. \quad (7)$$

The dynamical phase $\phi_\delta$ is the integral over all time for the time-dependent energy of one of the adiabatic states in the resonant limit [6].

The second and third terms in (7) are derived after taking into account the fact that when $\Delta_1(\tau)$ or $\Delta_2(\tau)$ is on, then $\eta_2(\tau) = 0$ or $\eta_1(\tau) = 0$, respectively. Thus the system will
correspond to a single atom interacting with the cavity mode, with a time-dependent detuning between the atomic transition and the mode frequency. Because of the choice made for the chirp, the adiabatic energies at this stage of the evolution will be pushed away from each other, and the system will adiabatically evolve during the time interval for which the atoms go through the fields $E^j(t)$.

When the chirp is off, the atom has crossed the region of the field $E^j(t)$, the system returns to the initial two-atom resonant adiabatic state, with a phase shift emerging from the chirped interaction of each atom with the mode. These phase shifts are the two integral terms in (7). Between the two chirps the system evolves as in the resonant limit, resulting in an additional phase $\phi_n$, first term in equation (7).

Thus in the adiabatic limit and for symmetric interactions, the system evolves according to (6a)–(6d), provided the previously derived conditions for using the adiabatic approximation are satisfied [6]. More specifically, the coupling between the atoms and the cavity mode must be strong, $g_0 \gg \nu / \omega_0$, and the delay time between the atoms must be of the order of $4 \omega_0 / \nu$. In addition to these conditions, the EM fields used to produce the Stark-chirps must be far from the cavity center, $x_0 \gg \omega_0$, and their width must be smaller than the mode width, $L \ll \omega_0$. It is also important to note that the pulses should be weaker than the atom–cavity coupling, i.e. $\Delta_0 \ll g_0$. As we see later on, violation of this condition is not detrimental for the system.

2.2. Maximally entangled atoms

Equations (6a)–(6d) describe the conditional entanglement of the second atom with the cavity mode. In order for this to be the case, the atom must be initially in its ground state. Instead of entangling the cavity mode with one of the atoms, one could generate a maximally entangled state of the two atoms. Preparing the system in the factored state

$$\psi = \frac{1}{\sqrt{2}} |0\rangle \left( |g_1\rangle + |e_1\rangle \right) \left( |g_2\rangle + |e_2\rangle \right)$$

before sending the atoms through the cavity, will result in a maximally entangled state of the two atoms

$$\psi_{ee} = \frac{1}{\sqrt{2}} |g_1\rangle \left( |g_1\rangle - |e_1\rangle \right) + \frac{1}{\sqrt{2}} |e_2\rangle \left( |g_1\rangle + |e_1\rangle \right)$$

if $\phi_n = 2m\pi$ with $m$ being an integer. Thus, a maximally entangled state of the two atoms can be generated and the output can be fine tuned by means of the Stark-chirps.

In figure 2 the fidelity for a maximally entangled state [2] is given for different values of the chirp amplitude $\Delta_0$. The first important feature is the periodic reappearance of a maximally entangled state even for $\Delta_0 \geq g_0$. This is due to the fact that the system still evolves adiabatically even if the Stark fields are stronger than the atom–cavity coupling. For $\Delta_0 < g_0$, the phase $\phi_n \propto \Delta_0^2$ and the corresponding error for the fidelity

$$F (\Delta_0) = |3 + \cos(\phi_n)| / 4$$

will be a linear function of $\Delta_0$. Thus the Stark-shift technique is expected to be robust for weak chirps $\Delta_0 \ll g_0$.

The robustness is demonstrated in figure 2 where the variations for the fidelity around the optimum value $\Delta_{max} = 0.44 g_0$ are plotted as a function of the ratio $\Delta_0 / \Delta_{max}$. From this we can see that variations of the order of 10% around the optimum value $\Delta_0 = \Delta_{max}$ have a very small impact on the fidelity (<1%). Thus generating a maximally entangled state, or realizing the previously proposed applications [6], can be robust by means of weak EM fields used to detune the atomic transition frequencies from the cavity frequency. This in return will induce a second-order shift in the dynamical phases (7), which can be used to fine tune the output of the evolution under the Hamiltonian (1) in the adiabatic limit.

3. Entanglement spatial properties

Up to this point, one of the main assumptions was that both atoms are coupled to the cavity field via time-dependent coupling functions with equal amplitudes $g_1 = g_2$. In the most general case and for a spherical mirror resonator [4], these amplitudes will be functions of the coordinates $y_j$ and $z_j$. This means that when $y_1 \neq y_2$ and $z_1 \neq z_2$, the propagator (6a)–(6d) will have a different form [6]. Although the main feature is that of conditional entanglement between the second atom and the cavity mode, entanglement will be a function of the spatial coordinates $y_j$ and $z_j$. It is now interesting to consider the entanglement properties for the two atoms, as their positions $z_j$ are varied. In order to take into account the possibility of forming a tripartite entangled state of the atoms with the cavity mode, we quantify entanglement in terms of the multipartite pure state concurrence [8], which for our case is

$$C_3 (|\psi\rangle) = \sqrt{3 - \text{tr} \rho_1^2 - \text{tr} \rho_2^2 - \text{tr} \rho_3^2}$$

where $\rho_j$ is the reduced density matrix for atom $j$ and $\rho_3$ is the reduced density matrix for the cavity mode. When $C_3 = 0$, no entanglement exists between the atoms and the mode, whereas for $C_3 > 1$ tripartite entanglement is formed. For $C_3 < 1$, additional information is required to characterize the type of entanglement. Here, we assume that $y_j = 0$, and that the first atom moves along an anti-node of the standing wave.
where the concurrence \( \rho \) and \( \lambda \) obtain their minimum value. Around the local minima where a separation \( z \) for the concurrence \( (z_2-z_1)\lambda \), the importance of both effects, we solved the master equation for the density matrix of the entire system, the first atom entangles with the cavity mode, forming sharp dips in figure 3 where the concurrence obtains its minimum value. Around the local minima where a maximally entangled state of the two atoms is formed, symmetric spikes with \( C_3(|\psi_i>) \) > 1 signify the tripartite entanglement of the atoms and the cavity mode. For values of \( C_3(|\psi_i>) \) < 1 and arbitrary atomic separations, the system could be in either a tripartite or a bipartite entangled state.

4. Spontaneous emission and cavity losses

Spontaneous emission from the atoms to modes other than the cavity mode and photon losses through the cavity mirrors are both detrimental for entanglement. In an attempt to quantify the importance of both effects, we solved the master equation for the density matrix of the entire system,

\[
\frac{d\rho}{dt} = -i[\hat{H}(t), \rho] + \mathcal{L}_s(\rho) + \mathcal{L}_c(\rho). \tag{12}
\]

The Liouvilleian terms \( \mathcal{L}_s(\rho) \) and \( \mathcal{L}_c(\rho) \), respectively, describe the atomic spontaneous emission with a rate \( \Gamma \), and the decay of the cavity with a rate \( \gamma \) into different thermal reservoirs at zero temperature [9],

\[
\mathcal{L}_s(\rho) = -\frac{\Gamma}{2} \sum_{j=1,2} \left( \rho \hat{a}_j^\dagger \hat{a}_j + \hat{a}_j^\dagger \hat{a}_j \rho - 2 \hat{a}_j^\dagger \hat{a}_j \rho \hat{a}_j^\dagger \right), \tag{13}
\]

\[
\mathcal{L}_c(\rho) = -\frac{\gamma}{2} \left( \hat{a}^\dagger \hat{a}^\dagger + \rho + \rho \hat{a}^\dagger \hat{a} - 2 \hat{a} \rho \hat{a}^\dagger \right). \tag{14}
\]

Calculating multipartite concurrence for a mixed state is not an easy task [8]. One would have to perform an optimization for the concurrence \( C_3(|\psi_i>) \) (11) over all pure state ensembles \( \{|\psi_i>\} \), which equally represent the mixed state \( \rho \) that corresponds to the solution of equation (12). Instead of this approach, we were able to show after a number of simulations that the final state of the system has a simple form, \( \rho(\infty) \approx |0\rangle \otimes |\gamma(\infty)\rangle \), where \( \rho(\infty) \) is the reduced density matrix for the atoms for \( t = \infty \). This result is for the initial state (8), and is rather accurate since the average photon number \( \langle n(\infty)\rangle \) is very small, \( \langle n(\infty)\rangle < 10^{-3} \), whereas the correlations between the mode and the atoms are negligible. This result allows the calculation of the concurrence for the atomic pair, by means of the two-qubit mixed state concurrence [8]

\[
c(\rho) = \max \left\{ 0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \right\}, \tag{15}
\]

where \( \lambda_i \) are the eigenvalues of the matrix \( R(\rho) = \rho \otimes (\sigma_j \otimes \sigma_j) \rho^* \), with \( \sigma_j \) being the Pauli matrix and the eigenvalues \( \lambda_i \) in increasing order i.e. \( \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 \).

The results from numerical simulations with equation (12) and (14) are plotted in figure 4, where we see that entanglement between the two atoms exponentially decays with respect to both decay rates. Furthermore, the most important detrimental effect is that of spontaneous emission from the atoms. This is because the effective decay rate for the subsystem of the two atoms is greater than the single-atom decay rate \( \Gamma \). That is, entanglement is more sensitive to decay due to spontaneous emission than decay due to cavity losses.

4.1. Experimental feasibility

For experiments with circular Rydberg states of Rb85 atoms, with lifetimes \( T_{\text{rot}} = 30 \text{ ms} \) [3], the main requirement for adiabatic evolution is for the system to be in the strong coupling regime, \( g_0/2\pi = 50 \text{ kHz} \) and a mode waist \( 2w_0 = 6 \text{ mm} \), the atomic
velocities must be $v \approx 60–95 \text{ m s}^{-1}$. These speeds are large enough to avoid reflection of the atoms from the cavity field. In addition to this the initial atomic displacement must be $2v\Delta t \approx (4–5)\frac{v}{w_0} = (12–15) \text{ mm}$.

This distance is bigger than the width of the Stark fields, which is $2L \approx 0.4\frac{v}{w_0} = 1.2 \text{ mm}$. This allows the Stark fields to be turned on and off at proper times so that they interact with only one atom and consequently produce the desired chirps (4). The location of the Stark pulses can vary between 4.5 and 6 mm with no impact on the robustness of the system. In addition to all these, a recent development was the engineering of a cavity with $Q = 4.2 \times 10^{10}$ [3]. For this quality factor and the corresponding lifetimes for the circular Rydberg atoms, the fidelity for a maximally entangled state (9) is reduced by about 3% varying between 96 and 97%, (figure 2), which is a very high fidelity.

5. Conclusion

In this work we have studied a time-dependent, off-resonant two-atom Tavis–Cummings Hamiltonian. Considering pairs of slowly moving atoms, sequentially crossing a Gaussian mode resonator, we examine the adiabatic limit for the system. However, we were able to show, for proper choice of atomic frequency chirps, the evolution of the system bears strong similarities to the resonant limit. The system propagator has the same structure as in the resonant limit, and one can realize the exact same applications [6].

The only and also crucial difference from the resonant limit is that the adiabatic phase is now shifted due to frequency chirps. Because of the quadratic dependence of this shift with respect to the chirps’ amplitude, all the proposed applications can be robust. As an example, the creation of a maximally entangled state of the atoms can be achieved with fidelities higher than 99%. Despite potential detrimental effects, such as atomic spontaneous emission or photon losses from the resonator mirrors, recent developments in the fabrication of high-quality cavities [3] allow fidelities as high as 97%. In addition to this, the adiabatic condition can be easily satisfied for experiments with Rydberg atoms in the microwave regime [3].

Although time-dependent frequency chirps could be detrimental for the adiabatic approximation, our results show that proper timing of these chirps could enable adiabatic evolution even if they are stronger than the coupling between the atoms and the field. One could again coherently tune the output of the evolution, but this would be less robust. This is due to the sinusoidal dependence of the error with respect to the chirp amplitude for strong Stark fields. On the other hand, for weak chirps the error in the fidelity is very small making the Stark-shift technique highly robust.

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