Self-organization in BML Traffic Flow Model: Analytical Approaches *

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Analytical investigations are made on BML two-dimensional traffic flow model with alternative movement and exclude-volume effect. Several exact results are obtained, including the upper critical density above which there are only jamming configurations asymptotically, and the lower critical density below which there are only moving configurations asymptotically. The jamming transition observed in the ensemble average velocity takes place at another critical density \( p_c(N) \), which is dependent on the lattice size \( N \) and is in the intermediate region between the lower and upper critical densities. It is suggested that \( p_c(N) \) is proportional to a power of \( N \), in good agreement with the numerical simulation. The order parameter of this jamming transition is identified.

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I. INTRODUCTION

In recent years much attention is paid on cellular automaton models for the investigations on complex systems. These models can be viewed as statistical models with dynamics. Some models of traffic flow are under intensive studies. A two-dimensional model was introduced by Biham, Middleton and Levine (BML) [1]. It is defined on a square lattice with periodic boundary condition. Each site contains either an arrow pointing upwards or to the right, or is empty. The dynamics is controlled by the traffic light, such that the right arrows move only on even time steps and the up arrows move on odd time steps. On even time steps, each right arrows moves one lattice constant to the right unless the site on its right-hand side is occupied by an arrow, which is either up or right. If it is blocked, it does not move, even if during the same time step the blocking arrow moves out of that site. Similar rules apply to the up arrows, which move upwards. The velocity \( v \) of a right (up) arrow is defined to be the number of moves it makes within a number of even (odd) time steps divided by this number of time steps. It has a maximal value \( v = 1 \), indicating that the arrow is never stopped. The minimal value \( v = 0 \) represents that the arrow is stopped during the entire time duration. The average velocity \( \overline{v} \) for the system is obtained by averaging \( v \) over all the arrows in the system. If it is further averaged over many asymptotic configurations, one obtains the ensemble average.

BML model is fully deterministic. It is called to be self-organized because whatever the initial condition is, one often (but not always) finds in the asymptotic configurations that, all the arrows move freely in their turns hence the velocity averaged over all the arrows is \( \overline{v} = 1 \), or they are all stopped, with \( \overline{v} = 0 \). These two types of configurations are referred to as moving and jamming ones, respectively. In the language of dynamics, they are the biggest basins of attraction. Which asymptotic configuration is finally reached depends on both the density of arrows and the initial condition. Is there any asymptotic configuration in which some arrows are moving while others are blocked? The answer is yes. Consider a column occupied by less than \( N/2 \) up arrows, where \( N \) is the number of the lattice points on each column, while its two neighboring columns are both full of up arrows. Then asymptotically the arrows in this column are moving forever, independent of other arrows, which are all blocked. But such configurations are rare, that is, they occupy a very small volume in the phase space, compared with those of the moving or jamming configurations. This is indicated by the simulation result that there is a sharp moving-to-jamming transition with the increase of arrow density. The simulation result (see Fig.3 of Ref. [1]) also tells us that the fraction of phase space volume occupied by the moving or jamming configurations increases with the size of the lattice.

Here we make some analytical approaches. We give exact results on the lower critical density, below which there are only moving configurations asymptotically, and an upper critical density, above which there are only jamming configurations asymptotically. Between these two critical densities, the asymptotic configuration can be moving or jamming, or even with both moving and blocked arrows, depending on the initial configurations. As indicated in the simulation, there is another critical density above which the asymptotic configurations are typically (but not always) jamming. This is the sharp (but not absolutely stepwise) jamming transition discov-

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The content of this article is as follows. For convenience of discussions, we introduce some notations in Sec. II. In Secs. III and IV, we give some exact results, the upper and lower critical densities are determined. In Sec. V, by considering the typical pattern formation of the jamming cluster, we obtain, in a heuristic way, the critical density for the jamming transition. The dependence on the lattice size is determined, and the order parameter is identified. Sec. VI is a summary.

II. NOTATIONS

For convenience of discussions, we introduce some notations here. There are $N \times N$ lattice points, the density of up (right) arrows is $p^\uparrow = n^\uparrow/N^2$ ($p_\to = n_\to/N^2$), where $n^\uparrow$ ($n_\to$) is the number of up (right) arrows. The total density of arrows is $p = p^\uparrow + p_\to$. The number of empty lattice points is denoted as $n_0$. The empty sites can be regarded as occupied by holes. Each lattice point is given a coordinate $(i, j)$. $i$ and $j$ each runs from 1 to $N$, hence the lower-left corner is $(1, 1)$. The periodic boundary condition can be expressed as

$$(i + N, j) = (i, j + N) = (i + N, j + N) = (i, j). \tag{1}$$

Because of the periodic boundary condition, the lattice can be transformed in the way shown in Fig. 1, so that it can be viewed as a parallelogram, also with the periodic boundary condition. This parallelogram is made up of $N$ lines parallel to the left-falling diagonal of the original square, on each of these lines there are also $N$ lattice points. For convenience, we say that the lattice is composed of $N$ "left-falling diagonals" (LFD). For example, this transformation translates the line linking $(1, i)$ and $(i, 1)$ $N$ units upwards to be connected with the line linking $(i + 1, N)$ and $(N, i + 1)$, composing a LFD. Since the arrows are right or up, this viewpoint is very useful in our discussions.

To avoid confusion, the word "state" is used for the lattice points, while "configuration" is for the whole system. The state of $(i, j)$ is denoted as $|i, j\rangle$. $|i, j\rangle = \uparrow, \to$ or $0$ if $(i, j)$ is occupied by an up arrow, a right arrow or a hole, respectively. $|i, j\rangle$ is, of course, dependent on time, so it can be written as $|i, j\rangle(t)$ if necessary. Obviously, in a moving configuration, $|i, j\rangle(t) = |i + \delta, j\rangle(t + \delta)$ if $|i, j\rangle(t) = \to$, $|i, j\rangle(t) = |i, j + \delta\rangle(t + \delta)$ if $|i, j\rangle(t) = \uparrow$.

FIG. 1. Equivalent transformation of the appearance of the lattice. The square PORQ can be transformed to the parallelogram $PQRS'$, because of periodic boundary condition. The coordinates of the marked points are $O(1, 1)$, $R(N, 1)$, $Q(N, N)$, $P(1, N)$, $S(1, N - 1)$, $T(N - 1, 1)$, $O'(1, N + 1)$, $T'(N - 1, N + 1)$, $S'(1, 2N - 1)$. $O$ and $O'$, $T$ and $T'$, $S$ and $S'$ are equivalently the same points, respectively.

III. EXACT RESULTS ON MOVING CONFIGURATION

First we point out that not only a jamming configuration, but also a moving configuration is stationary, in the sense that all arrows of the same type move simultaneously and thus form a rigid body.

The exact results are stated in the form of theorems.

Theorem 1.—In a moving configuration, a LFD always consists of a same type of arrows, as well as holes.

Proof.—Suppose $|i, j\rangle(t) = \uparrow$, while $|i - \delta, j + \delta\rangle(t) = \to$, where $\delta$ is a positive integer. If $t$ is an odd (even) time step, then after $\delta$ odd (even) time steps, the right (up) arrow is blocked by the up (right) arrow. This should not
there cannot be both up and right arrows on a same LFD. Q.E.D.

Theorem 2.—In a moving configuration where there are both right and up arrows, there is at least one empty LFD at any instant.

Proof.—Without lose of generality, consider an odd time step. For a LFD of up arrows, there cannot be any right arrow on its upside LFD, seen as follows. Suppose there are right arrows on this upside LFD. If a right arrow is just above an up arrow, the latter is blocked, which is not permitted in a moving configuration. If a right arrow is above a hole on the LFD composed of up arrows and holes, at the next time step, this right arrow will fill the hole (which has moved one step upwards), and join the up arrows (which have moved one step upwards) on a same LFD. Hence there appears a LFD where there are both up and right arrows. This is forbidden in a moving configuration, according to Theorem 1. If there are both up and right arrows on the lattice, because of periodic boundary condition, there must be right arrows “before” the up arrows, even though on the original square lattice they are “after” the up arrows. Therefore there is at least one empty LFD. On the other hand, at this odd time step, for a LFD of right arrows, it is not necessary for its righthand side (just the upside) LFD to be empty, since the right arrows do not move at this time step. In conclusion, the least number of empty LFD is only 1. Q.E.D.

Theorem 3.—Consider N > 2. There is an upper critical density, above which there is no moving configuration. The upper critical density is 1/2 if N is odd, and is 1/2 − 1/2N if N is even.

Proof.—For N = 2, there cannot be any moving configuration with the presence of both up and right arrows. Hence we only consider N > 2. Without lose of generality, consider an odd time step. At this time step, there can be an arrow on the righthand side of a right arrow, but there cannot be any arrow on the upside of an up arrow. The most crowded configuration is the following: an empty LFD is on the upside of a block of LFD-s consisting of up arrows and holes, which is followed by a block of LFD-s consisting of right arrows and holes. Hence the number of up arrows n↑ and the holes in the upside block, n↑, should satisfy n↑ ≤ n↑ if there are even number of LFD-s in this block, and n↑ ≤ n↑ + N if there are odd number of LFD-s in this block. Similarly, the number of right arrows n→ and the holes in the right block, n→, should satisfy n→ ≤ n→ if there are even number of LFD-s in this block, and n→ ≤ n→ + N if there are odd number of LFD-s in this block. In addition, the total number of the holes on the lattice should satisfy n0 ≥ n↑ + n→ + N, since there is at least one empty LFD, according to Theorem 2. Therefore if N is even, at the most crowded case, we have odd number of non-empty LFD-s, hence we have either an odd number of up LFD-s and an even number of right LFD-s, or an even number of up LFD-s and an odd number of right LFD-s. In either case, we have n = n↑ + n→ ≤ n↑ + n→ + N ≤ n0 = N2 − n, hence p = n/N2 ≤ 1/2. If N is odd, at the most crowded case, we have even number of non-empty LFD-s, hence we have either an odd number of up LFD-s and an odd number of right LFD-s, or an even number of up LFD-s and an even number of right LFD-s. The odd-odd case, we have n = n↑ + n→ ≤ n↑ + n→ + 2N ≤ no + N = N2 + N − n, hence p = n/N2 ≤ 1/2 + 1/2N. In the even-even case, we have n = n↑ + n→ ≤ n↑ + n→ ≤ no − N = N2 − N − n, hence p = n/N2 ≤ 1/2 − 1/2N. Combining these two cases we have p ≤ 1/2 − 1/2N if N is even. Q.E.D.

IV. EXACT RESULTS ON JAMMING CONFIGURATION

Because an up arrow can only be blocked by an arrow (which can be right or up) above it, while a right arrow can only be blocked by an arrow on its righthand side, these arrows form a directed path in a jamming configuration. All directed paths point upwards or to the right. Considering the periodic boundary condition, one may obtain the following theorems.

Theorem 4.—In a jamming configuration, starting from an arbitrary arrow, one can obtain a directed path which returns to either the starting arrow or another arrow on this path.

We call such a path a closed path. If it returns to the starting arrow, it is a circular path. Each closed path contains a circular path as a part.

Theorem 5.—In a jamming configuration, there must be at least one circular path.

Clearly this is a necessary condition for a configuration to be jamming.

Theorem 6.—The length of a circular path is N if it is composed of only one type of arrows, and is 2N if it is composed of both types of arrows. Here the unit of the length is the lattice constant. For example, the length of an edge of the square is N − 1.

Proof.—Obviously the circular path made up of one type of arrows is parallel to the edge of the square, hence its length is N. If the circular path is made up of both types of arrows, because it is directed, generally it appears as two parts on the square lattice. For example, one part is a directed path connecting (1, J) and (I, N), another part is a directed path connecting (I, 1) and (N, J). Note that (N, J) is a nearest neighbor of (1, J) ≡ (N + 1, J), and (I, N) is a nearest neighbor of (I, 1) ≡ (I, N + 1). Clearly the total length of such a
The circular path is $2N$. Q.E.D.

**Theorem 7.**—There is a lower critical density, below which there is no jamming configuration. The lower critical density is $(1 + p_s/p_l)/N$, where $p_s$ and $p_l$ are respectively the smaller and larger one of $p_{\uparrow}$ and $p_{\downarrow}$.

**Proof.**—Suppose there is a circular path made up of only the arrows with the larger density, and there are no other arrows of this type. The arrows with the smaller density are blocked by this circular path. Therefore $N_s = N$, $N_l = (p_s/p_l)N$. Thus $(1 + p_s/p_l)/N$ is the smallest possible density for the jamming configuration if the circular path is made up of only one type of arrows. If all the up and right arrows take part in composing the circular path, the density is $2/N$. Since $2/N \geq (1 + p_s/p_l)/N$, the theorem holds in general. Q.E.D.

V. FORMATION OF A JAMMING CONFIGURATION

The so-called jamming transition observed in the simulation refers to a sharp, but not stepwise, transition in the ensemble average velocity. There are both moving and jamming configurations between the upper and lower critical densities, depending on the initial condition. In [1], the critical density for the jamming transition was defined to be at the center of the transition region. Here we define the critical density of jamming transition as the value of the density at which the ensemble average velocity starts to be nearly zero. We approximate the non-stepwise transition in the ensemble average by a typical process of formation of a jamming configuration. Consequently the critical density of jamming transition is approximated as the density above which there forms a typical jamming cluster.

A jamming configuration is formed soon after the appearance of a circular path, which usually consists of both up and right arrows. When an arrow meet the circular path, it is blocked. The circular path blocks the right arrows on its lefthand side and up arrows on its downside. Blocked arrows block other arrows, and so on. Consequently, a global cluster with directed branching structure emerges. The skeleton of this cluster is a circular path.

Now note that in the final jamming cluster, there are some arrows which are the ends of the cluster. If the end-arrow is an up (right) one, its upside (righthand side) must be occupied, while there must be neither right arrow on its lefthand side nor up arrow on its downside. So the density of end-arrows are

$$p_e(p) = p^2(1 - p_{\downarrow})(1 - p_{\uparrow}) = p^2 - p^3 + p^2 p_{\downarrow} p_{\uparrow} \approx p^2.$$  \hspace{1cm} (2)

Hence the number of ends is $p^2 N^2$. Since the length of the circular path is $N$, it is very reasonable to assume that the number of the ends is a power of $N$. Consequently the critical density for the “jamming transition” is a power of $N$, that is,

$$p_c(N) = CN^\alpha,$$  \hspace{1cm} (3)

where $C$ is a coefficient, while $\alpha$ is the exponent.

The simulation results can be used to test this heuristic argument, and determine $\alpha$ and $C$. With the approximate values of $p_c(N)$ for $N = 16, 32, 64, 128, 512$ obtained from Fig. 3 in Ref. [1], we obtain a good fit to Eq. (3) with $\alpha \approx -0.14$ and $C \approx 0.76$, as shown in Fig. 2.

![FIG. 2. Log-log plot of $p_c(N)$ with the lattice size $N$. $p_c(N)$ is the density above which typical jamming configurations form, and thus the ensemble average velocity is nearly 0. The circles are the results observed from Ref. [3]. The straight line is the least square fit yielding $p_c(N) \approx CN^\alpha$, with $C \approx 0.76$, $\alpha \approx -0.14$.](image)

Eq. (3) suggests that the jamming cluster at $p_c$ is a fractal with dimensionality $2 + \alpha = 1.86$, which is close to $91/48$, the fractal dimension of the infinite cluster of two-dimensional percolation. This is understandable since the jamming cluster forms soon after the circular path forms, which is similar to the formation of an infinite cluster in percolation. The order parameter of percolation is the probability that an arbitrarily chosen occupied site or bond belongs to an infinite cluster. Likewise, we identify the order parameter in jamming transition as the probability that an arbitrarily chosen arrow belongs to a closed path.

VI. SUMMARY

We have studied BML two-dimensional traffic flow model analytically. In particular we gives exact results
on the two most typical asymptotic configurations, the moving configuration and jamming configuration. In a moving configuration, all arrows keep moving, while in a jamming configuration, all arrows are blocked. Theorems 1 and 2 give two basic properties of a moving configuration. Based on these, Theorem 3 provides the upper critical density, above which there is no moving configuration asymptotically. The upper critical density is $1/2$ if $N$ is odd, and is $1/2 - 1/N$ if $N$ is even. Theorems 4, 5 and 6 give basic properties of a jamming configuration. The crucial thing is the formation of a so-called circular path which cuts the lattice into two parts. Theorem 7 then gives the lower critical density, below which there is no jamming configuration asymptotically. The so-called jamming transition observed in the ensemble average velocity happens in a short region between the upper and lower critical density. We define the critical density for the jamming transition as that on which the ensemble average velocity begins to be close to zero. We investigate it approximately by considering the formation of a typical jamming cluster. The $N$-dependent critical density $p_c(N)$ is claimed to be $CN^\alpha$, which fits the numerical data very well, with $C \approx 0.76$ and $\alpha \approx -0.14$.

The jamming transition turns out to be quite similar to percolation, and we identify the order parameter as the probability that an arbitrarily chosen arrow belongs to a closed path. Further investigations on possible criticality are interesting.

[1] O. Biham, A. A. Middleton and D. Levine, Phys. Rev. A 46 (1992) 3290.
[2] D. Stauffer, *Introduction to Percolation Theory* (Taylor & France, London, 1995).