\( \mathcal{M} \) theory as a matrix extension of Chern-Simons theory

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ABSTRACT

We study a new class of matrix models, the simplest of which is based on an \( Sp(2) \) symmetry and has a compactification which is equivalent to Chern-Simons theory on the three-torus. By replacing \( Sp(2) \) with the super-algebra \( Osp(1|32) \), which has been conjectured to be the full symmetry group of \( \mathcal{M} \) theory, we arrive at a supercovariant matrix model which appears to contain within it the previously proposed \( \mathcal{M} \) theory matrix models. There is no background spacetime so that time and dynamics are introduced via compactifications which break the full covariance of the model. Three compactifications are studied corresponding to a hamiltonian quantization in \( D = 10 + 1 \), a Lorentz invariant quantization in \( D = 9 + 1 \) and a light cone gauge quantization in \( D = 11 = 9 + 1 + 1 \). In all cases constraints arise which eliminate certain higher spin fields in terms of lower spin dynamical fields. In the \( SO(9,1) \) invariant compactification we argue that the one loop effective action reduces to the IKKT covariant matrix model. In the light cone gauge compactification the theory contains the standard \( \mathcal{M} \) theory light cone gauge matrix model, but there appears an additional transverse five form field.

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1 Introduction

In this note we present a possible approach to \( \mathcal{M} \) theory based on a simple matrix model, which describes the dynamics of a matrix which is built from the super Lie algebra \( Osp(1|32) \). The original motivation for studying this model came from an attempt to simplify a proposal for a background independent formulation of \( \mathcal{M} \) theory\(^1\), based on a background independent approach to a causal membrane field theory\(^2, 3\). However the model which emerged is quite simple, and so merits a separate presentation. The goal of the present paper is only to initiate a study of this model, much more remains to be done to understand its possible relationship to \( \mathcal{M} \) theory.

The main motivation for this model is that it fully realizes the supersymmetry algebra \( Osp(1|32) \) which has been proposed by different authors as the ultimate symmetry group of \( \mathcal{M} \) theory\(^4\). As we shall argue, various matrix models and string theories may arise from different compactifications of the model. A second motivation was to find an extension of the dWHN-BFSS matrix model\(^5, 6, 7, 8\), which incorporates the full \( 10 + 1 \) dimensional super-Poincare invariance of the super-membrane and the flat space limit of \( 11 \) dimensional supergravity. A third motivation was to find a single model which includes both that light cone gauge theory and the \( SO(9,1) \) supercovariant IKKT matrix model\(^9\) as different reductions.

It is not obvious that such an extension of the matrix model should exist. Among the reasons to think it should not are that such a theory is not likely to be related to either \( 10 \) dimensional super-Yang-Mills theory or the \( 11 \) dimensional membrane, as it can be shown\(^10\) that once light cone gauge has been lifted the gauge group of the membrane is too large to represent manifestly in terms of the \( N \to \infty \) limit of an \( SU(N) \) gauge invariance. A perhaps even more serious issue is that covariantizations of supersymmetric theories which realize the full supersymmetry linearly, and off shell, are normally plagued with ghosts and higher spin fields.

Mindful of these potential pitfalls, we proceed here to invent and study a model. The model differs from perviously studied matrix models in that the action is derived by an extension of a matrix form of Chern-Simons theory which we describe in the next section. This action is cubic in the matrices. One might worry that this leads to instabilities, however as in the case of pure general relativity (with vanishing cosmological constant and in the compact case) the action vanishes on shell. In fact the theory shares several characteristics with first order formulations of general relativity and supergravity, some of which are also cubic in the basic variables\(^11, 12, 13\). In those theories time is only introduced by expanding the theory around a particular classical background. Since the action is cubic this leads to an expression first order in time derivatives. Thus, each choice of time leads to a phase space description. When the canonical theory is analyzed it is found that there are always constraints which resolve the possible problems of ghosts and higher spin fields, leading in the end to a sensible theory, at least classically. Below we will show that the cubic matrix action defines a theory with similar characteristics. In particular time is only introduced when the theory is expanded around particular classical solutions that define a compactification. We
also find that constraints arise which eliminate higher spin fields in terms of lower spin fields.

In the next section we study a simple theory with a cubic action in which the matrices are valued in $Sp(2)$. We show how compactifying it on a circle defines a phase space and that when it is compactified on the three-torus it is equivalent to Chern-Simons theory. In section 3 we extend that model simply by replacing $Sp(2)$ by the superalgebra $Osp(1|32)$. The rest of the paper is then devoted to the study of this model. In section 4 we perform a hamiltonian analysis relevant for a $10 + 1$ dimensional quantization of the theory and we find that there are constraints which eliminate many of the degrees of freedom. In sections 5 and 6 we study respectively compactifications that reduce the symmetry to the super-Poincare group in $9 + 1$ dimensions and the super-Euclidean group in 9 dimensions. Because of the constraints we are unable to make a precise computation of the effective action, however we are able to argue from symmetry that in the first case the covariant matrix model proposed by IKKT is reproduced. In the second case, by going to light cone gauge in $D = 11$ we arrive at a theory that contains the standard dWHN-BFSS matrix model relevant for the light cone gauge description of $M$ theory in flat $10 + 1$ dimensional spacetime. It seems in addition to contain one more field, which is a transverse five form field.

2 Matrix representation of topological field theory

We will take as our starting point a fundamental fact about general relativity and supergravity, which is that they arise by constraining the actions for topological quantum field theories. This suggests the following strategy: find a way to represent some topological quantum field theory as a matrix model, and then find a way to naturally extend it to include the symmetries $M$ theory is expected to have.

What form of an action shall we use? It is clear that if we use a conventional matrix theory action involving quadratic and quartic terms we will not get a representation of a topological field theory. Furthermore, when we extend the symmetry to a covariant superalgebra there will be a great danger of ghost fields and negative norm states coming from the minus signs in the spacetime metric. To avoid this we study instead an action cubic in the matrices. Under compactification to define a time coordinate this can produce an action at most first order in time derivatives. Thus, such as action will define and live on a phase space. This is attractive as first order, phase space actions are a very convenient starting point for analyzing theories with spacetime gauge invariances. They are also the starting point for the discovery of connections between topological quantum field theory and gravitational theories.

To see the effect of using a cubic action, we may consider a very simple model based on $Sp(2)$. Here the field is given by a matrix

$$M_I^J = \begin{bmatrix} x_3 & x_1 \\ x_2 & -x_3 \end{bmatrix}$$

where $x_i, i = 1, 2, 3$ are three $N \times N$ matrices. A simple cubic action is then given by

$$I^{Sp(2)} = Tr\{M_I^J[M_J^K, M_K^I]\} = 6 Tr\{x_1[x_2, x_3]\}$$

$$\text{(2)}$$
where the trace and commutator are in the \( N \times N \) matrix variables. We see that in the classical theory they must all commute with each other. To introduce a time variable we break the \( Sp(2) \) invariance by expanding around a vacuum given by \( x_3 = \hat{D} = \hat{d} + a_3 \), where, using the standard matrix compactification trick\[14, 7\] (to be recalled below), the action reduces in the limit \( N \to \infty \) to

\[
I^{Sp(2)} = 6 \int dt \text{Tr} \left\{ x_2(t) \dot{x}_1(t) + a_3(t) [x_1(t), x_2(t)] \right\}
\]

where \( x_1(t) \) and \( x_2(t) \) are now two one parameter families of \( M \times M \) matrices and \( a_3(t) \) is a one dimensional \( gl(M) \) gauge field. If we require that the matrices be hermitian, we reduce this to a \( U(M) \) gauge invariance. We see that the theory describes a phase space \( \Gamma = (p, x_1) \) with \( p = \delta I / \delta \dot{x}_1 = x_2 \) with the constraint that as \( M \times M \) matrices, \( [p(t), x_1(t)] = 0 \). In this simple case there is no dynamics, the theory is something like a matrix version of a topological field theory.

In fact the cubic form of the action is closely related to topological field theory. To see this let us consider the same \( Sp(2) \) model, but let us make a triple compactification defined by the expansion

\[
x_i = \hat{D}_i = \hat{\partial}_i + a_i
\]

where \( \hat{\partial}_i \) are \( 3 M \times M \) matrices that each give a compactification and \([\hat{\partial}_i, \hat{\partial}_j] = 0\) so that when \( a_i = 0 \) we have a solution to the classical equations of motion. The cubic action is then equal in the limit to

\[
I^{Sp(2)} = 3 \int d^3 x \epsilon^{ijk} \text{Tr} \left\{ a_i \partial_j a_k + \frac{1}{3} a_i [a_j, a_k] \right\}
\]

This is the action for \( U(M) \) (or, with unconstrained matrices, \( gl(M) \)) Chern-Simons theory on the three-torus. The equations of motion are \( F_{ij} = [\hat{\partial}_i, \hat{\partial}_j] = 0 \). Thus the symplectic matrix model suggests that there is a connection between a \( 2M \) dimensional phase space and a \( U(M) \) Chern-Simons theory. The Chern-Simons theory may be regarded as a gauge theory of the symplectic structure, so that the original phase space structure is coded in the Poisson brackets amongst Wilson loops, \( W^n_i = \text{Tr} \left\{ P [e^{\int dx a_i}] \right\} \), for \( i = 1, 2 \) and \( n = 1, \ldots, M \). We will not pursue this further here, but go on to see how an extension of this cubic symplectic matrix model may have something to do with \( \mathcal{M} \) theory.

## 3 The model

We now extend the cubic matrix model by extending the \( Sp(2) \) symmetry to the superalgebra \( Osp(1|32) \) which is believed to be the symmetry group of \( \mathcal{M} \) theory. The degree of freedom of our theory will then be a set of unconstrained \( N \times N \) matrices, each element of which is also valued in the adjoint representation of the superalgebra \( Osp(1|32) \). We first define the notation that we will use to describe the matrices that define the adjoint representation of \( Osp(1|32) \). These are \( 33 \times 33 \) component matrices, \( W^\alpha_\beta \) which are defined to satisfy

\[
W \cdot G = -G \cdot W^T,
\]
where $T$ stands for the matrix transpose and $G$ is the $33 \times 33$ matrix,

$$G_{\alpha \beta} = \begin{bmatrix} 0 & -I & 0 \\ I & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$  \hspace{1cm} (7)

Here the first two rows and columns are $16 \times 16$ dimensional and the third row and column has one component. We may then write for the $33$ fold indices $\alpha, \beta, \ldots, \alpha = A, A', 0$ where $A = 1, \ldots, 16$ and $A' = 1', \ldots, 16'$. The $A$ and $A'$ components have an even grading, so $g(A) = g(A') = 0$ while the $33$'d, $0$ component has an odd grading, $g(0) = 1$.

It is easy to see that the solutions to (6) may be parameterized as

$$W_{\alpha \beta} = \begin{pmatrix} A & B & \Psi \\ C & -A^T & \Phi \\ \Phi^T & -\Psi^T & 0 \end{pmatrix}$$  \hspace{1cm} (8)

where $A$ is a $16 \times 16$ matrix, $B$ and $C$ are $16 \times 16$ symmetric matrices, and $\Psi^A$ and $\Phi^{A'}$ are $16$ component spinors. All quantities are real-Grassman valued, $A, B$ and $C$ are real even Grassman variables while $\Psi^A$ and $\Phi^{A'}$ are real odd Grassmann variables. It will be useful also to decompose $A^{AB}$ into its symmetric and antisymmetric parts,

$$A^{AB} = X^{AB} + Y^{AB},$$  \hspace{1cm} (9)

where $X^{AB} = X^{(AB)}$ and $Y^{AB} = Y^{[AB]}$.

Let us now promote each component of $W_{\alpha \beta}$ to an $N \times N$ matrix, which we will call $Z_{\alpha \beta}^{ab}$, with $a, b, c, \ldots = 1, \ldots, N$.

We then define our theory by the cubic action,

$$I = \frac{1}{g^2} Tr \left\{ Z_{\alpha \beta} \left[ Z_{\gamma \beta}^{\alpha}, Z_{\gamma \alpha}^{\beta} \right] \right\}$$  \hspace{1cm} (10)

The (super)trace and the (super)commutator are both taken in the $N$ component indices. The supercommutator is defined by the usual formula, for two grassman valued $N \times N$ matrices, $X$ and $Y$, $[X, Y] \equiv XY - (-1)^{g(X)g(Y)} YX$. We will call this the cubic action in the rest of the paper.

The action (10) has the following symmetries: a) Global (that is commuting with $GL(N, R)$) supersymmetry in which the components of $Z_{\alpha \beta}^{\gamma}$ transform under the adjoint representation of $Osp(1|32)$. b) Global $GL(N, R)$ symmetry, c) a generalized translation symmetry, under which

$$Z_{aa}^{\beta b} = Z_{\alpha a}^{\beta b} = Z_{aa}^{\beta b} + \delta_a^b V_{\alpha \beta}$$  \hspace{1cm} (11)

Note that the fields and the coupling constant $g$ are all dimensionless, which of course is required as there is nothing in the theory that refers to space or time. We will generally set $g = 1$ for convenience.
One might make the following objections to this model. First the action is not bounded from below or above, second there is no explicit time coordinate, third there is a global translation symmetry. The first two are properties of general relativity so we should perhaps not be surprised to see them in any theory that has general relativity as a limit. Furthermore, we can point out that as in general relativity in the compact case the action vanishes on solutions. To introduce time we will have to expand the theory around a suitably chosen classical background. (This by the way, agrees with some\cite{13}, but not all\cite{16,17}, views on the role of time in quantum gravity.) Once time is defined in this way a Hamiltonian may be constructed. What is required is only that some of the theories defined by these Hamiltonians are stable, for physically interesting choices of backgrounds.

The existence of a global translation symmetry is, however, not a feature of classical general relativity; it suggests that some background dependence has been left in, which should arise only in the presence of certain classical solutions. It does however agree with some proposals concerning $\mathcal{M}$ theory in which the translations symmetry of flat 11 dimensional spacetime is to be absorbed into a larger symmetry group which sometimes has been proposed to be $Osp(1|32)$.\cite{4}

To answer this last criticism one can reduce the translation symmetry by dropping the commutator, so that we have

$$I^{gauged} = \frac{1}{g^2} Tr \left\{ Z^{\beta} Z^{\gamma} Z^{\alpha} \right\}$$

We will refer to this as the gauged cubic action. It has less global symmetry, but a far larger gauge symmetry group, which is given by the possibility of making a valued $Osp(1|32)$ transformations. This makes the model harder to analyze although perhaps more interesting. It will be discussed elsewhere.

Returning to the cubic action, the classical equations of motion that follow from (10) are simply

$$\left[ Z^{\gamma}, Z_{\gamma} \right] = 0 \quad (13)$$

To proceed to analyze the theory we need to decompose the action in terms of (8,9), this gives us

$$I = \frac{1}{g^2} Tr \left\{ 6X_{AB}[B_{BC}, C_{C}] + 6Y_{AB}[X_{BC}, X_{C}] + 2Y_{AB}[Y_{BC}, Y_{C}] + 2X_{AB}\{\Psi^A, \Phi^B\} + B_{AB}\{\Phi^A, \Phi^B\} - C_{AB}\{\Psi^A, \Psi^B\} \right\} \quad (14)$$

We will now discuss how the theory behaves when expanded around three different backgrounds, which are solutions to the classical equations (13).
4 The first compactification and a Hamiltonian formulation

In order to study the dynamics of the model we have to introduce a time coordinate. This can be done by the usual trick of choosing a background which corresponds to an \( S^1 \), which is interpreted as a compactification of the model. This necessarily breaks the symmetry of the model to a subalgebra of \( Osp(1|32) \). But of course in a relativistic theory symmetry reduction is always a consequence of a choice of the time coordinate.

To see how to choose a time direction in the \( Osp(1|32) \) version of the theory we may choose a coordinatization which describes an embedding of the 11 dimensional Super-Poincare algebra in \( Osp(1|32) \). We do this by choosing a real 32 dimensional representation of \( Clif(10,1) \), given by the following choice:

\[
\Gamma^i = \begin{pmatrix}
\gamma^i & 0 & 0 \\
0 & -\gamma^i & 0 \\
0 & 0 & 0
\end{pmatrix}; \quad \Gamma^{10} = \Gamma^{9} = \begin{pmatrix}
0 & I & 0 \\
I & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}; \quad \Gamma^{0} = \begin{pmatrix}
0 & -I & 0 \\
I & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

where \( \gamma^i \) are \( 16 \times 16 \), symmetric, real, nine dimensional \( \gamma \)-matrices normalized by \( \gamma^i \gamma^j + \gamma^j \gamma^i = +2 \delta^{ij} \), with \( i = 1, \ldots , 9 \). It will be useful to note also the corresponding representation of \( Spin(10,1) \in Clif_0(10,1) \),

\[
\Gamma^{ij} = \Gamma^i \Gamma^j = \begin{pmatrix}
\gamma^{ij} & 0 & 0 \\
0 & \gamma^{ij} & 0 \\
0 & 0 & 0
\end{pmatrix}; \quad \Gamma^{#i} = \begin{pmatrix}
0 & -\gamma^i & 0 \\
\gamma^i & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
\Gamma^{0#} = \begin{pmatrix}
I & 0 & 0 \\
0 & -I & 0 \\
0 & 0 & 0
\end{pmatrix}; \quad \Gamma^{0i} = \begin{pmatrix}
0 & \gamma^i & 0 \\
\gamma^i & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Note that \( \gamma_{AB}^{ij} \) is real and antisymmetric so that \( \Gamma^{ij} \in Sp(32) \).

In order to understand the physical content of the theory it is useful to understand the decomposition of the adjoint rep of \( Sp(32) \) into irreps of \( Spin(9) \). This helps because \( Spin(9) \) governs the degrees of freedom of the light cone gauge of the 11 dimensional theory, which is where the degrees of freedom should be manifest and we expect to make contact with the standard \( \mathcal{M} \) theory matrix model. We have,

\[
\text{Adj}ont_{Sp(32)} = 3R \oplus 3V \oplus 3V^4 \oplus V^2 \oplus V^3
\]

where \( V = R^9 \) is the vector representation of \( Spin(9) \) and \( V^p \) is the antisymmetric \( p \)-fold product. The three vectors are then represented by \( \Gamma^i, \Gamma^{#i} \) and \( \Gamma^{0i} \), and the scalars by \( \Gamma^0, \Gamma^# \) and \( \Gamma^{0#} \). These live, respectively, in the vector and trace parts of \( X^{AB} \) and \( B^{AB}_\pm = B^{AB} \pm C^{AB} \).

To see what spin content to expect from the theory we may consider the decomposition of the symmetric \( 16 \times 16 \) tensor:

\[
X^{AB} = \delta^{AB} R + \Gamma_i^{AB} V^i + \Gamma_{i4}^{AB} V^4
\]
where we use a notation $i_p = [i_1 \ldots i_p]$ for the antisymmetric combination. The antisymmetric tensor is

$$Y^{AB} = \Gamma^{ij}_{AB} V^{ij} + \Gamma^{i_3}_{AB} V^{i_3}$$

The three scalar’s, three vectors and the $V^{ij}$ parameterize the embedding of the 11 dimensional DeSitter algebra, $SO(10,2)$ in $Sp(32)$. The remaining $V^3$ and the three $V^4$’s represent elements of $Sp(32)$ that do not come from $SO(10,2)$. In the contraction of $Osp(1|32)$ that becomes the 11 dimensional Super-Poincare algebra they become the central charges.

The 32 supersymmetry charges decompose into two 16’s of $Spin(9)$ which may be parameterized as

$$\epsilon_A \hat{Q}^A + \chi_A' \hat{Q}'^A = \begin{pmatrix} 0 & 0 & \epsilon \\ 0 & 0 & \chi \\ \chi^T & -\epsilon^T & 0 \end{pmatrix}$$

We now introduce a time coordinate in the direction parameterized by $\Gamma^0$ by the usual trick [1] of a matrix compactification. This means that we expand around a background given by all fields vanishing except

$$B^{AB}_{ab} = -C^{AB}_{ab} = \delta^{AB} D_{0ab}$$

where $D_0$ has the property as an $N \times N$ matrix that as $N$ tends to infinity

$$Tr W[D_0,M] = \frac{1}{T} \int dt Tr \left\{ W(t) \left( t \frac{\partial M(t)}{\partial t} + [A_0, M] \right) \right\}$$

This is done by breaking each $N \times N$ matrix, $M_{AB}$ up into a very large number $2F + 1$ of $P \times P$ blocks, $\tilde{M}_{\tilde{a}\tilde{b}}(n)$ such that

$$Tr_{N\times N} M = \sum_{n=-F}^F Tr_{P \times P} \tilde{M}(n)$$

$D_{0ab}$ is then defined as $D_{0ab} = K_{ab} + A_{0ab}$ where

$$Tr W[K,M] = \sum_n Tr_{P \times P} W(n) n \tilde{M}(n)$$

If we introduce a fundamental scale $l_{Pl}$ then we can fourier transform to define

$$M(t) = \sum_{n=-F}^F e^{i2\pi nt/T} \tilde{M}(n)$$

where $T = (2F + 1)l_{Pl}$, leading in the limit $F \to \infty$, with $T$ held fixed and $l_{Pl} \to 0$ to (23).

It is easy to see that (22) is a solution to the equations of motion (13). Note that all dimensional quantities will be proportional to some power of $l_{Pl}$, by definition. The notion of a dimensional scale is just a convenient device to compare with known physics, the physics actually depends only on $P$ which we interpret as the ratio of the compactification scale and
$l_{Pl}$. Since in physics we expect both the compactification scale and $l_{Pl}$ to be finite we regard expressions such as (23) as shorthand for the more exact expressions of the form of (27) with $F$ very large but finite.

After some algebra the cubic action (10) takes the form

$$I = \frac{1}{T} \int dt T r \left\{ \Phi_A[\partial_0, \Phi^A] + \Psi_A[\partial_0, \Psi^A] + X_{AB}[\partial_0, B^{AB}_+] \right.$$ \nonumber
$$\left. - \mathcal{H}[X, B_+, \Psi, \Phi] + A_{0ab} G^{ab} + C(Y, B^-; X, B_+, \Psi, \Phi) \right\}$$

(27)

We have made several redefinitions,

$$C^{AB} = -\delta^{AB} D_0 + \tilde{C}^{AB}, \quad B^{AB} = +\delta^{AB} D_0 + \tilde{B}^{AB}.$$  

(28)

and

$$B^{\pm AB} = \tilde{B}^{AB} \pm \tilde{C}^{AB}.$$  

(29)

We see that the fields have split into a dynamical set, consisting of $X, B_+, \Psi, \Phi$ and a remaining non-dynamical set consisting of $Y$ and $B^-$. $X^{AB}$ are the momenta conjugate to $B^{AB}_+$, while the $B^{\pm AB}_-$ are constrained fields. The hamiltonian density in terms of the dynamical fields is,

$$\mathcal{H}[X, B_+, \Psi, \Phi] = -B^{+ AB}_A \left( \{ \Phi^A, \Phi^B \} - \{ \Psi^A, \Psi^B \} \right) - 2X_{AB} \{ \Psi^A, \Phi^B \} - \frac{3}{2} X[B_+, B_+]$$

(30)

The Gauss’s law constraint is

$$G^{ab} = [\Psi_A, \Psi^A] + [\Phi_A, \Phi^A] + [X_{AB}, B^{AB}_+]$$

(31)

This is first class and generates local $GL(N, R)$ transformations on the dynamical fields.

We see the very interesting fact that $Y^{AB}$ and $B^{- AB}_-$ have no conjugate momenta and are then constrained in terms of the dynamical fields $X, P, \Psi^\pm$. The potential energy density for these constrained fields is

$$C(Y, B^-; X, B_+, \Psi, \Phi) = \{6Y[X, X] + 2Y[Y, Y] + \frac{3}{2} X([B_-, B_-] - 2[B_+, B_-] \right.$$ \nonumber
$$+ \frac{1}{2} B^{- AB}_A (\{ \Phi^A, \Phi^B \} - \{ \Psi^A, \Psi^B \}) \}$$

(32)

Varying with respect to $Y$ and $B^-$, respectively, we have constraint equations

$$E^{[AB]} = [X^{\{A}_C, X^{B\}C}] + [Y^{[A}_C, Y^{B]}C] = 0$$

(33)

$$J^{(AB)} = \frac{1}{2} \left( \{ \Phi^A, \Phi^B \} + \{ \Psi^A, \Psi^B \} \right) - 3[X, B_-] + 3[X, B_+] = 0$$

(34)

These define quadratic surfaces in the space of matrices, and can be solved to express $Y^{AB}$ and $B^{AB}$ in terms of the dynamical fields $X, B_+, \Psi, \Phi$. The result is that the total hamiltonian density is

$$\mathcal{H}^{total}(X, P, \Psi^\pm) = \mathcal{H}[X, B_+, \Psi, \Phi] - C(Y(X), B^- (X, B_+, \Psi, \Phi), X, B_+, \Psi, \Phi)$$

(35)
This has not yet been done explicitly. Unless there is some miracle the result will be non-polynomial, but this is not surprising given that this is the case also for general relativity for most choices of variables. The further analysis of the hamiltonian theory requires careful consideration of the space of solutions of the constraints, which has not yet been carried out.

5 Compactification to an $SO(9,1)$ covariant theory

We next study a compactification which breaks the symmetry down to the $D = 9 + 1$ superPoincare algebra. To do this we compactify in the 11’th dimension, which is the degree of freedom generated by $\Gamma^\#$. We do this by writing

$$B_{\pm AB} = \delta_{AB}(D_\# - T + b_\pm + \tilde{D}_{AB}; \quad C_{\pm AB} = \delta_{AB}(D_\# + T + b_\pm + \tilde{C}_{AB})$$

where $\tilde{D}_{AB}$ and $\tilde{C}_{AB}$ are now tracefree. We expand around a classical solution in which all fields except $D_\#$ vanish and we impose conditions on $D_\#$ identical to those imposed in the last section on $D_0$. $b_\pm$ carries the fluctuations around the compactification radius. $T$ is the field in the direction $\Gamma^0$.

The cubic action is now most simply expressed in terms of redefined fields, $\tilde{B}_{\pm AB} = \tilde{B}_{AB} \pm \tilde{C}_{AB}, X_{\pm AB} = \tilde{X}_{AB} + \delta_{AB} x$ and $\Phi_\pm = \Phi^A \pm \Psi^A$. We have

$$I_\#^2 = -Tr \left\{ \Phi_+^A[D_\#, \Phi_+^A] + 12 \tilde{D}_{AB}[D_\#, \tilde{X}_{AB}] + 12 x[D_\#, T] - \mathcal{H}_\# + \mathcal{C}_\# \right\}$$

where the hamiltonian is now

$$\mathcal{H}_\# = \frac{1}{4} \tilde{B}^- (\{\Phi^+, \Phi^+\} + \{\Phi^-, \Phi^-\}) + \frac{1}{2} \tilde{X} (\{\Phi^+, \Phi^+\} - \{\Phi^-, \Phi^-\} - 2\{\Phi^+, \Phi^-\})$$

$$- \frac{3}{2} X[B_{-}, B_{-}] - \frac{1}{2} T (\{\Phi^+, \Phi^+\} + \{\Phi^-, \Phi^-\}) + \frac{1}{2} x (\{\Phi^+, \Phi^+\} - \{\Phi^-, \Phi^-\} - 2\{\Phi^+, \Phi^\})$$

and the constraints come from

$$\mathcal{C}_\# = \frac{2}{3} X \left( [B^+, B^+] + 2[B^+, B^-] \right) + \frac{1}{2} B^+ \{\Phi^+, \Phi^-\} + 2 Y[Y, Y] + 6 Y[X, X]$$

The quadratic term tells us how to perform the quantization with respect to the Euclidean time $X^\#$. We see that we again have a division into dynamical and non-dynamical fields. The dynamical fields now are $\tilde{B}_-, \tilde{X}, x, T, \Phi^\pm$. The non-dynamical fields are $B_+, b_+$ and $Y_{AB}$. These will be determined by constraints analogous to (33) and (34) as a result of which we will have

$$Y_{AB} = Y_{AB}[X]; \quad B_{+AB} = B_{+AB}[\tilde{B}_-, \tilde{X}, x, T, \Phi^\pm]$$

We note that these constrained fields make up an $SO(9,1)$ scalar, $b_+$, two form, $V^{\mu \nu}$, and four form, $W^{\mu \nu \lambda \sigma}$ (with $9 + 1$ dimensional indices $\mu, \nu = (0, i)$). These are given, in terms of 9D gamma matrices, by

$$B_{+AB} = \gamma_{AB} V^{0i} + \gamma_{AB} W^{i4}$$
\[ Y^{AB} = \gamma^{AB}_{ij} V_{ij} + \gamma^{AB}_{i3} W_{0i3} \]  

(42)

We will not here solve the constraints and compute the resulting hamiltonian for the unconstrained fields. As a result, we cannot commute the one-loop effective potential of the dynamical fields precisely. But we can use the unbroken symmetry to determine its form. We first organize the dynamical fields in terms of \(9 + 1\) dimensional tensors. The \(T\) component combines with \(X_i\), where

\[ X^{AB} = \gamma^{AB}_i X^i + \gamma^{AB}_{i3} X^i_3 \]

(43)
to make the \(9 + 1\) vector of matrices.

\[ X^\mu = (T, X^i) \]

(44)

The remaining fields are the \(X^i_4\), the fields in \(B_{-AB}\), given by

\[ B_{-AB} = \gamma^{AB}_i B^i - \gamma^{AB}_{i4} B^i_4 \]

(45)
and the scalar \(x\). These do not combine to form any more \(SO(9,1)\) tensors, although they play the role of canonical momenta (in the \(D_#\) time) to fields that are parts of \(SO(9,1)\) tensors. They must then be eliminated in the computation of the one loop effective potential, which then will have, at least to lowest order in the fields, the \(SO(9,1)\) invariant form \[9\]

\[ S^# = Tr \left\{ [X^\mu, X^\nu][X^\mu, X^\nu] + \Psi_\bar{\alpha}[X^\mu, \Psi_\bar{\beta}]\Gamma_{\mu}^{\bar{\alpha}\bar{\beta}} \right\} \]

(46)

Here \(\bar{\alpha}, \bar{\beta} = (A, A')\) is a 32 component \(SO(9,1)\) spinor index and \(\Psi_\bar{\alpha} = (\Psi^A, \Phi^{A'})\). The spinor may be decomposed into chiral eigenstates \(\Psi^\pm_\bar{\alpha} = (\Psi^A, \pm \Phi^{A'})\). Under supersymmetry transformations generated by \(Q^\pm_A = Q^1_A \pm Q^2_A\) we have

\[ \delta \Psi^\pm_\bar{A} = \pm D_# Q^\pm_\bar{A} \]

(47)

This tells us that if we keep both fermion fields in the dynamics, we have two rigid supersymmetries, with \([D_#, Q^\pm_A] = 0\). However, if we decouple one of the fields, say \(\Psi^+\) then we need only require \([D_#, Q^-_\bar{A}] = 0\) so the theory will be invariant under one global and one local supersymmetry. This suggests that supersymmetry will protect one, but not both spinor fields, so we are left with the field content of the IKKT model \[9\]. It is not hard to see that if we ignore the constrained fields completely the one-loop effective potential is exactly of this form. But a precise calculation cannot be done until the constraints have been properly dealt with.

**6 Triple compactification and the discrete light cone quantization**

We next consider a different compactification, which is suitable for extracting the infinite momentum frame description of the theory in \(10 + 1\) dimensions. This should have as the
explicit symmetry only the super-Euclidean group in 9 dimensions. To construct this limit we study a triple compactification on all three of the $Spin(9)$ scalar modes of $Z_\alpha^\beta$. These correspond to $X^\pm = X^0 \pm X^\#$ and the longitudinal boost $\Gamma^0\#$. The idea is then to compute the one loop effective potential that follows from integrating out the modes of the fields in the time coordinate generated by $\Gamma^0\#$. This gives a theory expressed in terms of $SO(9)$ transverse degrees of freedom and the light cone coordinates and momenta defined in terms of $X^\pm$.

Again we cannot make an exact calculation as there are constraints in the $\Gamma^0\#$ time, analogous to those we encountered before. But we can use group theory to constrain the possible form of the one-loop effective potential, and we can also verify that the terms in it do appear in a version of the calculation in which the constrained degrees of freedom are ignored rather than solved for.

We begin by compactifying only the longitudinal boost direction, which is given by the background in which all fields vanish except

$$X^{AB} = \delta^{AB} D_\tau$$

(48)

where $D_\tau$ is defined similarly to $D_0$ above. We find an expression similar to the previous one, differing of course because we are introducing a different time coordinate,

$$I = -\frac{1}{T} \int d\tau Tr \left\{ \Psi_A [D_0, \Phi^A] + B_{AB} [D_0, C^{AB}] \\ - \mathcal{H}_\tau [B, C, \Psi, \Phi] + + A_{\alpha\beta} G^{\alpha\beta}_\tau + C^\tau (Y, X, \text{other fields}) \right\}$$

(49)

We find the unconstrained hamiltonian density is now simply

$$\mathcal{H}_\tau [B, C, \Psi, \Phi] = B_{AB} \{ \Phi^A, \Phi^B \} - C_{AB} \{ \Psi^A, \Psi^B \}$$

(50)

Now it is $X^{AB}$ along with $Y^{AB}$ which is to be determined by the solution to constraints. The new constrained potential energy is

$$C^\tau (Y, X, \text{other fields}) = 6 \bar{X} [B, C] - 2Y [Y, Y] - 6X [Y, Y] + 2 \bar{X}_{AB} \{ \Psi^A, \Phi^B \}$$

(51)

We first consider what happens if we simply ignore these constrained fields, $X^{AB}$ and $Y^{AB}$ and study the theory defined by (49) with the term $C^\tau$ ignored. The dynamical fields are only $C^{AB}, B^{AB}, \Phi^A, \Psi^A$. Keeping in mind the fact that $\tau$ is a Euclidean time coordinate, we can integrate out over the modes which propagate in $\tau$. The effective potential for the unconstrained fields is then, to lowest order, of the form,

$$I = I^0 + \hbar I^1$$

(52)

where

$$I^0 = Tr \mathcal{H}_\tau$$

(53)
and the one loop effective potential has the form

\[ I^1 = Tr \left\{ [B, C]^2 + \Phi[B, \Phi] + \Psi[C, \Psi] \right\} \] (54)

We next compactify the \( x^+ \) and \( x^- \) directions, which are generated by \( \Gamma^\pm = \Gamma^0 \pm \Gamma^\# \). We do this by writing

\[ B^{AB} = \delta^{AB} D_+ + \tilde{B}^{AB}, \quad C^{AB} = \delta^{AB} D_- + \tilde{C}^{AB} \] (55)

and expand around the background whose only non-zero fields are \( D^{\tau}, D_+^{\pm}, D_- \), where \( D_+, D_- \) are defined as in the cases of the other time coordinates. The compactification radii are \( R^\pm \). The result is a 1 + 1 field theory defined on the torus, whose effective action contains the terms

\[ S = \frac{1}{R^+ R^-} \int d^4x \int d^4x' \left\{ [D_-, \tilde{C}]^2 + [D_+, \tilde{B}]^2 + \Phi[D_-, \Phi] + \Phi[D_+, \Phi] + \Psi[D_-, \Psi] + \Psi[D_+, \Psi] + [\tilde{B}, \tilde{C}]^2 + \Phi[A, \Phi] + \Phi[A, \Phi] + \Psi[A, \Psi] + \Psi[A, \Psi] \right\} \] (56)

We next perform a very large boost in the positive \# direction, which in the limit will take us to the infinite momentum frame. In the limit all terms proportional to \( D_- \) decouple as those backwards moving modes have in the limit infinite energy. The degrees of freedom which survive the limit are only those with kinetic energies proportional to \( D_+ \), their dynamics is described by the action containing the terms,

\[ I^+ = \frac{1}{R^+} \int d^4x Tr \left\{ \Phi_A[D_+, \Phi^A] + \Psi_A[D_+, \Psi^A] + [D_+, \bar{B}^{AB}][D_+, \bar{B}^{AB}] + \Phi_A[\tilde{B}^{AB}, \Phi_A] + \Psi_A[\tilde{B}^{AB}, \Psi_A] \right\} \] (57)

This is close to the standard DLCQ action for \( \mathcal{M} \) theory. There is again one fermion field too many for there to remain a local supersymmetry. When we integrate this out this leaves us with an effective action of the form

\[ I^{IMF} = \frac{1}{R^+} \int d^4x Tr \left\{ \Phi_A[D_+, \Phi^A] + [D_+, \bar{B}^{AB}][D_+, \bar{B}^{AB}] + \Phi_A[\tilde{B}^{AB}, \Phi_A] + [\tilde{B}^{AB}, \tilde{B}^{CD}][\tilde{B}^{AB}, \tilde{B}^{CD}] \right\} \] (58)

Our infinite momentum frame action (58) is almost, but not quite the matrix model for \( \mathcal{M} \) theory described in [5, 6, 7] which is simultaneously a description of the supermembrane in light cone gauge and the reduction to one dimension of \( D = 10 \) supersymmetric Yang-Mills theory. The difference is that the tracefree part of the \( B^{AB} \) field contains a five form as well as a vector, which is given by the decomposition (19) of the symmetric trace free spinor \( B^{AB} \). Thus the theory is an extension of the usual matrix model with

\[ X_i^A \gamma_i^{AB} \to B^{AB} = \gamma_i^{AB} X_i + \gamma_{ijkl} V_{ijkl} \] (59)

The additional degree of freedom may be interpreted to be a transverse five-form field \( A_{jklmn} = V^{ijkl}_{ijkl} \). The chief consequence of its addition is that the supersymmetry algebra now contains central terms. This will be discussed in more detail elsewhere.
What we have reported here is just the first step in the analysis of the model given by the cubic action. The most important technical problem to be resolved is the correct way to handle the constraints which arise in the different quantizations. Once this is done the effective action can be calculated exactly to any order desired, and the results compared with the IKKT and dWHN-BFSS forms of the matrix theory. What we have argued here is that by expanding around the appropriate classical solutions those theories will be reproduced, with the possible addition of a transverse five form field in the light cone gauge case.

If the theory passes this test then it will be of interest to investigate whether all the known consistent perturbative string theories, together with the web of dualities, may be understood as arising from expanding the present model around different classical solutions. It is known that the several different string theories can be gotten by compactifying the IKKT and dWHN-BFSS matrix models\cite{7,8,9}; it will be of interest to see if others may be found. It would also be interesting to see if there are compactifications of this theory which reduce to the proposals presented in \cite{20,21}.

Another set of questions to explore arise from the relationship between the simplest symplectic matrix model and Chern-Simons theory we described in section 2. This suggests that the triple compactification of the $Osp(1|32)$ theory, one limit of which we argued gives rise to the light cone gauge matrix model, may be studied also as a $2 + 1$ dimensional topological quantum field theory. A closely related set of structures are the basis of the connection between this model and the background independent approaches to membrane and $\mathcal{M}$ theory described in \cite{1,2,3}. This will be discussed elsewhere.

Beyond this there are several deep questions. The first is the question of what the right quantization procedure should be for the full theory. In this model time is only introduced by expanding around a classical solution, given by an appropriate compactification. It is not at all clear if a quantum theory can be defined in the absence of any time variable, for in that case there is no canonical formulation to base the quantization on.

It is possible that there may be an unconventional answer to this question, in which quantum statistics emerges for the local observables when the matrices are thermalized. This is suggested by the fact that in the absence of the choice of a time variable no clear distinction can be made between thermal and quantum fluctuations, as that depends on the signature of the action. General arguments tell us that in quantum gravity the distinction between quantum and thermal statistics should exist only relative to local inertial reference frames in spacetimes with lorentzian signature\cite{18}. A matrix model in which quantum statistics emerged from a large $N$ limit of ordinary statistics was described in \cite{19}. It was found that such models are able to evade the experimental limits on local hidden variables theories because only the eigenvalues of the matrices are associated with local observables, while the matrix elements themselves are non-local. Alternatively it may be that there is an algebraic approach to the quantization of such systems defined by an appropriate triple product. Such theories have been studied by \cite{22}.

\footnote{I would like to thank Miao Li for pointing out to me the latter work.}
Another set of questions arises from the fact that the time coordinates introduced via compactifications are periodic. It is of interest to understand if this is fundamental or if there are ways to introduce time and space coordinates which are not compact.

Finally, we note that there are a number of other models which might be studied, which have many features in common with the present one. By complexifying the degrees of freedom of our model we may arrive at a model based on $SU(16,16|1)$. It is interesting to consider this as an extension of twistor theory, as that is based on an $SU(2,2)$ symmetry.

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