Abstract: We derive the condition on $f(R)$ gravities that admit Killing spinor equations and construct explicit such examples. The Killing spinor equations can be used to reduce the fourth-order differential equations of motion to the first order for both the domain wall and FLRW cosmological solutions. We obtain exact “BPS” domain walls that describe the smooth Randall-Sundrum II, AdS wormholes and the RG flow from IR to UV. We also obtain exact smooth cosmological solutions that describe the evolution from an inflationary starting point with a larger cosmological constant to an ever-expanding universe with a smaller cosmological constant. In addition, We find exact smooth solutions of pre-big bang models, bouncing or crunching universes. An important feature is that the scalar curvature $R$ of all these metrics is varying rather than a constant. Another intriguing feature is that there are two different $f(R)$ gravities that give rise to the same “BPS” solution. We also study linearized $f(R)$ gravities in (A)dS vacua.

Keywords: Classical Theories of Gravity, Cosmology of Theories beyond the SM, AdS-CFT Correspondence

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1 Introduction

Modifying Einstein gravity with higher-order curvature invariants goes back to early days of General Relativity, notably by Eddington and Weyl simply as an exercise of intellectual curiosity \cite{1, 2}. A natural and perhaps the simplest generalization of Einstein gravity is to replace the Ricci scalar $R$ in the Einstein-Hilbert action with an arbitrary function of $R$ \cite{3-6}. The resulting $f(R)$ theories of gravity can admit Einstein metrics with a positive, negative or zero cosmological constant. This makes $f(R)$ gravities versatile in
both reproducing Newtonian gravity and studying cosmology, although a convincing $f(R)$ theory that fits all the observational data remains elusive. It can be shown \cite{7, 8} that $f(R)$ gravity is equivalent to some special class of the Brans-Dicke gravity/scalar theory \cite{9} by the Legendre transformation; however, the conversion requires to find the inverse function of $f'(R)$ which may not have a close analytical form. Thus in general $f(R)$ gravity should be studied on its own right. Since an inflationary model involving the quadratic Ricci scalar was constructed first time in \cite{10}, there has been ongoing interest in $f(R)$ gravities for the last three decades, and the application has been focused largely on the area of cosmology. (See, e.g., reviews \cite{11–13}.)

The fact that AdS spacetimes arise naturally in $f(R)$ gravities suggests that they can also be used for investigating the AdS/CFT correspondence \cite{14–16}. However, apart from those with constant scalar curvature, exact solutions in $f(R)$ gravities are difficult to come by. When the Maxwell field is introduced in the theory, charged black holes in four dimensions were obtained recently \cite{17–19}. This is possible only because the Maxwell field in four dimensions does not contribute to the trace of the Energy-momentum tensor so that the Ricci scalar $R$ in the four-dimensional charged black holes remains constant. It is easy to see that if $f(R)$ gravity coupled to a matter system whose energy-momentum tensor has vanishing trace, the theory admits analogous solutions as those in Einstein gravity coupled to the matter. Such solutions with constant $R$ can be viewed as somewhat trivial since they do not explore the nature and properties of the function $f$. Non-trivial solutions with varying $R$ have been hitherto unknown in $f(R)$ gravities. With few interesting and non-trivial exact solutions, the effort in applying the AdS/CFT correspondence in $f(R)$ gravities has been severely limited.

In supergravities, owing to the existence of Killing spinor equations, much wider classes of exact BPS solutions have been constructed. This is because the Killing spinor equations can be loosely viewed as the first integral of Einstein’s second-order equations of motion, and hence they significantly simplify the equations. Killing spinor equations are not exclusive for supergravities. Einstein theory of gravity with or without a cosmological constant in any dimension admits a Killing spinor equation.\footnote{The existence of Killing spinor equations in Einstein gravity or even in supergravity does not imply that all background solutions have Killing spinors which are local solutions of the Killing spinor equations. Furthermore, manifolds without spin structure would not admit spinors at all. The backgrounds that admit Killing spinors are called BPS solutions in supergravities, whilst manifolds with Killing spinors in Einstein theory are referred to as ones with the reduced holonomy. The criteria on a theory that admits Killing spinor equations will be discussed in section 2.3.} Killing spinor equations were also known to exist in the gravity/scalar system where the scalar potential is constructed in terms of a superpotential \cite{20}. It has been recently demonstrated that even when involving form fields, some non-supersymmetric gravity theories can admit Killing spinor equations \cite{21–23}. Examples are rare, but the low-energy effective action of the bosonic string up to the $\alpha'$ order was shown to admit Killing spinor equations in any dimension. These Killing spinor equations allow one to find new classes of solutions in these theories.

Obviously, a generic $f(R)$ theory does not admit Killing spinor equations. On the other hand, there must exist subclasses of $f(R)$ gravities that do. The simplest example is
the aforementioned Einstein gravity with/without a cosmological constant. In this paper, we follow the technique developed in [21–23] and propose two Killing spinor equations for \( f(R) \) gravities. (Our paper deals only with the \( f(R) \) theories in the metric formalism.) They involve two functions \( W \) and \( U \) of \( R \). We then derive the condition on \( f(R) \) so that the theory admits these equations. We find that for a given choice of \( W \), the function \( f \) satisfies a second-order linear differential equation, implying that there exists one (non-trivial) parameter family of \( f(R) \) gravities for the same \( W \). The function \( U \) is then fully determined by \( W \) and \( f \). Although there are no analytical solutions to the second-order differential equation in general, we find many explicit examples of \( f(R) \) gravities that do admit Killing spinor equations.

The advantage of having Killing spinor equations in our \( f(R) \) gravities is that they can be used to construct a large class of solutions involving only first-order differential equations. These solutions are analogous to the BPS solutions in supergravities, and hence we refer them as “BPS” even though our \( f(R) \) gravities are not supersymmetric. The focus of our construction is the static domain wall and the FLRW (Friedmann-Lemaître-Robertson-Walker) cosmological solutions, both of which are conformally flat and of cohomogeneity one. It turns out that these “BPS” solutions are solely determined by the function \( W \) in the Killing spinor equations. It follows that there are two different \( f(R) \) gravities that give rise to the same solution. We use explicit examples to compare the pros and cons of such two \( f(R) \) theories. Another characteristic of our \( f(R) \) gravities is that they admit at least two (A)dS vacua, and the solutions we construct are typically smooth, running from one (A)dS vacuum to the other. They provide excellent examples for studying either the AdS/CFT correspondence or cosmology. It should be pointed out that although multiple (A)dS vacua exist also in Lovelock gravities with the Gauss-Bonnet type of topological terms, there is no known flow that links these vacua. This is one major difference between our \( f(R) \) gravities and the Lovelock type of theories.

This paper is organized as follows. In section 2, we first give a quick review of \( f(R) \) gravities and the equations of motion. We then demonstrate a phenomenon with an explicit example that there can be new types of (A)dS vacua with vanishing \( f(R_0) \) but divergent \( f'(R_0) \). The same vacuum can be embedded in a different theory with no such singular behavior. We then propose Killing spinor equations and derive the \( \Gamma \)-matrix projected integrability conditions. These allow us to derive the condition on \( f \) that admits Killing spinor equations. We then give a few examples of such theories. More examples will be given in subsequent sections. In section 3, we consider “BPS” domain wall solutions. We obtain a class of exact solutions that connect two AdS vacua. These solutions include the smooth Randall Sundrum II, AdS wormholes and the RG flow from the IR (infrared) region to the UV (ultraviolet) region. These explicit solutions demonstrate that \( f(R) \) gravities are quite attuned to the investigation of the AdS/CFT correspondence.

In section 4, we examine the FLRW solution with flat spatial directions. We find that Killing spinor equations can also be useful to simplify the time-dependent equations. We obtain large classes of exact cosmological solutions. One type describes the evolution from an inflationary starting point to end with an ever-lasting expanding universe, very much like our Universe. We also find smooth metrics of pre-big bang models, bouncing and crunching universes.
In section 5, we give a couple of examples of converting our $f(R)$ gravities to the Brans-Dicke theory. In these examples, the inverse function of $f'(R)$ can be obtained as simple analytical functions. The majority of our $f(R)$ gravities we obtained in this paper do not give rise to a close form scalar potential in the corresponding Brans-Dicke theory. In section 6, we study the linear spectrum of $f(R)$ gravities in (A)dS vacua. As one would expect, in general the spectrum consists of the massless spin-2 graviton and a massive trace scalar mode. We derive the ghost-free and tachyon-free conditions. We discuss the special circumstance where the spectrum becomes less straightforward with the kinetic terms for the graviton and/or the scalar modes dropped from the linearized action. One intriguing feature of our construction is that there typically exist two $f(R)$ gravities for a given “BPS” solution. For such a solution that connects to two different (A)dS vacua, each of the two theories is suitable for one of the two vacua respectively. We conclude the paper in section 7.

2 $f(R)$ gravities with Killing spinor equations

2.1 Lagrangian and equations of motion

The Lagrangian for $f(R)$ gravity in $D$ dimensions is

$$\mathcal{L}_D = \sqrt{-g} f(R) , \quad (2.1)$$

where $f$ is a generic real function. In this paper, we shall be concerned with only $f(R)$ theories in the metric formalism, and hence the equations of motion from the variation of $g_{\mu\nu}$ are given by

$$G_{\mu\nu} \equiv F(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) F(R) = 0 , \quad (2.2)$$

where $F(R) = f'(R)$. Note that in this paper, we always use a prime to denote a derivative with respect to $R$, unless an explicit new variable is given. Taking the trace, we have

$$\mathcal{R} \equiv R F - \frac{1}{2} D f + (D - 1) \Box F = 0 . \quad (2.3)$$

The equations of motion (2.2) can be equivalently expressed as

$$\mathcal{R}_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{F} \nabla_\mu \nabla_\nu F + \frac{1}{2(D - 1) F} (f - 2 R F) g_{\mu\nu} = 0 . \quad (2.4)$$

Note that although $f(R)$ gravity can be related to the Brans-Dicke theory by the Legendre transformation, the resulting scalar potential involves the inverse function of $F$, which may not have a close analytical expression. Thus for general $f(R)$ gravities, the theories are best studied in their original forms rather than converting them to the corresponding unnatural gravity/scalar system. We shall come back to this point in section 5.
2.2 New (A)dS vacua in \( f(R) \) gravities

The simplest class of solutions for \( f(R) \) gravities are perhaps the metrics with constant \( R \), which we denote as \( R_0 \). In general, the metrics are Einstein, i.e. \( R_{\mu\nu} = \Lambda g_{\mu\nu} \), with the effective cosmological constant \( \Lambda = R_0/D \). It follows from (2.3) that

\[
2 R_0 F(R_0) = Df(R_0).
\]  

(2.5)

Depending on whether \( \Lambda \) is positive, 0, or negative, the vacuum solution is de Sitter (dS), Minkowski or anti-de Sitter (AdS) respectively. In the special case when both \( f(R_0) \) and \( F(R_0) \) vanish, the equations of motion (2.2) are satisfied simply by \( R = R_0 \). This degenerate case allows any metric with constant scalar curvature \( R_0 \) to be a solution, including some Lifshitz black holes \([24, 25]\). As we shall see in section 6, such an (A)dS vacuum has no propagating spin-2 graviton mode, but only a scalar trace mode. (See \([26]\) for a review on solutions in \( f(R) \) theories.)

In this subsection, we demonstrate that new classes of (A)dS solutions can emerge in \( f(R) \) gravity, which are not solutions of (2.5), but characterized by the divergent \( F(R_0) \).

To illustrate this, let us consider the following \( f(R) \) theory

\[
L_4 = \sigma_1 \sqrt{-g} R \sqrt{48 \beta^2 - R},
\]  

(2.6)

It follows from (2.5) that the theory has two vacua with

\[
R_0 = 0 \quad \text{and} \quad R_0 = 96 \beta^2.
\]  

(2.7)

It is clear that the two vacua cannot be connected, since for a given \( \sigma_1 \), the Lagrangian can only be real either at the vicinity of \( R_0 = 0 \) or \( R_0 = 96 \beta^2 \), but not at both. As we shall see later, this theory admits Killing spinor equations which enable us to find an exact cosmological solution with varying \( R \):

\[
ds_4^2 = -dt^2 + a^2(dx_1^2 + dx_2^2 + dx_3^2),
\]

\[
a^2 = 1 + e^{4\beta(t-t_0)}.
\]  

(2.8)

It is a straightforward exercise to verify that the metric satisfies (2.2). Let us consider the case with \( \beta > 0 \). The metric is Minkowski when \( t \to -\infty \), and it runs to the de Sitter spacetime with \( R_0 = 48 \beta^2 \) when \( t \to +\infty \). The Ricci scalar increases monotonically with respect to \( t \). The derivation of this solution can be found in section 4.4.

The solution (2.8) describes a pre-big bang model without singularity. Long before the inflation starts, the universe is Minkowski under the perturbative Lagrangian of Einstein gravity with higher-order Ricci curvature terms

\[
L_4 = 4\sqrt{3} \beta \sigma \sqrt{-g} \left( R - \frac{R^2}{96 \beta^2} + \cdots \right).
\]  

(2.9)

The universe bursts into inflation around \( t = t_0 \) when the non-perturbative effect takes place.
What we would like to draw attention to here is that if we rescale\( x_i \rightarrow e^{2\beta t_0} x_i \) and then send the integration constant \( t_0 \rightarrow -\infty \), we arrive at a dS metric
\[
d s^2 = -dt^2 + e^{4\beta t}(dx_1^2 + dx_2^2 + dx_3^2), \tag{2.10}
\]
with \( R_0 = 48\beta^2 \). It is clear that this is a limiting solution of our \( f(R) \) gravity; however, it is not a solution of (2.5), which gives only (2.7). This solution is characterized by \( f(R_0) = 0 \) and \( F(R_0) = \infty \), and hence the equation (2.3) breaks down when \( R \rightarrow R_0 \).

In this paper, we are able to uncover such (A)dS solutions because our Killing spinor equations allow us to find exact solutions with non-constant \( R \). New (A)dS metrics emerge when we let \( R \) run to such \( R_0 \). It is not obvious to us how to find such a solution in the situation when we cannot obtain an exact solution with running \( R \). It is a subject worth further investigation. It should be pointed out that from the point of view of the Brans-Dicke theory, the solution should be considered as singular since the \( F(R) \) corresponds to the scalar mode and it blows up at \( R = R_0 \). One can also take a different point of view that such a theory is intrinsically pure gravity.

Interestingly, we find that there is another quite different \( f(R) \) theory that can give rises to exactly the same cosmological solution (2.8), namely
\[
L_4 = \sigma_2 \sqrt{-g} R \left[ 12\beta - \sqrt{3(48\beta^2 - R)} \arctanh \left( \frac{\sqrt{48\beta^2 - R}}{4\sqrt{3} \beta} \right) \right]. \tag{2.11}
\]
This somewhat more complicated theory is convergent at \( R_0 = 48\beta^2 \), namely
\[
F(R_0) = 24\beta \sigma_2, \quad R_0 f''(R_0) = 16\beta \sigma_2. \tag{2.12}
\]
Thus in this case, the equation (2.5) is satisfied. As we shall discuss in section 6, this theory is ghost free in this de Sitter vacuum. On the other hand, the theory becomes singular at \( R = 0 \), since \( f(R) \sim R \log R \) when \( R \rightarrow 0 \). Thus, \( F \) is divergent at \( R = 0 \) in this case.

Thus we find an interesting phenomenon in \( f(R) \) gravities. The same cosmology (2.8) can be generated by two different classical actions, but neither theory can smoothly describe the full evolution. In the perturbative flat region, the Lagrangian (2.6) is a better theory. The later inflationary epoch is better studied by the theory (2.11). This is similar to the phenomenon in differential geometry that a manifold typically requires multiple different but overlapping coordinate patches to cover it. More detailed analysis will be given in section 6.

### 2.3 Killing spinor equations

In the previous subsection, we present two \( f(R) \) gravities that give rise to the same cosmological solution with varying \( R \). Such an exact solution is possible because the two theories are special in that they admit Killing spinor equations. In this subsection, we derive the condition on \( f \) so that the \( f(R) \) theories admit Killing spinor equations.

It is clear that for a generic function \( f \) there can be no consistent Killing spinor equations. However, for some classes of functions, the theories can admit Killing spinor
equations. The simplest example is that \( f = R - (D - 2)\Lambda_0 \), whose Killing spinor equation is well established and given by

\[
\hat{D}_\mu \epsilon \equiv \left( D_\mu + \frac{1}{2} \sqrt{-\frac{\Lambda_0}{D - 1}} \Gamma_\mu \right) \epsilon = 0, \tag{2.13}
\]

where \( D_\mu \) is a covariant derivative on a spinor, defined by

\[
D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4} \omega^{ab}_\mu \Gamma_{ab} \epsilon. \tag{2.14}
\]

Whilst the equation \( D_\mu \epsilon = 0 \) has a clear geometric interpretation that \( \epsilon \) is a covariant constant spinor, the extra \( \Gamma_\mu \) term in (2.13) lacks an immediate explanation. One defining property for the Killing spinor equation is that the \( \Gamma \)-matrix projected integrability condition gives rise to the Einstein equations of motion, namely

\[
0 = \Gamma_\mu [\hat{D}_\mu, \hat{D}_\nu] \epsilon = \frac{1}{2} \Gamma_\mu (R_{\mu \nu} - \Lambda_0 g_{\mu \nu}) \epsilon. \tag{2.15}
\]

Thus we see that the projected integrability condition is satisfied provided that the metric is Einstein with cosmological constant \( \Lambda_0 \). It follows that without the extra \( \Gamma_\mu \) term in (2.13), the Killing spinor equation would become irrelevant to the theory with the cosmological constant. Conversely, the significance of the above projected integrability condition is the following. Although the existence of Killing spinor (2.13) for a background does not in general imply that it always satisfies the equations of motion \( R_{\mu \nu} = \Lambda_0 g_{\mu \nu} \), if one of the Killing vectors constructed from the Killing spinors is time-like, the background then indeed satisfies. This provides a powerful tool for constructing exact solutions, since first-order equations are much more manageable than the second-order ones. (The integrability condition without the \( \Gamma \)-matrix projection is related to Riemann tensor, and hence not related directly to the Einstein equations of motion. This implies that not all solutions have Killing spinors, even if the theory admits the Killing spinor equation.)

We are now in the position to derive the condition on \( f \) so that the theory admits Killing spinor equations, whose \( \Gamma \)-matrix projected integrability conditions are analogous to (2.15). Such Killing spinor equations were constructed recently for non-supersymmetric theories involving the metric, a dilaton and a form field [21–23]. It turns out the condition is very restrictive and the only known non-trivial examples are either the low-energy effective action of the bosonic string up to the \( \alpha' \) order [21, 22] or the Kaluza-Klein theory with a non-trivial scalar potential [23]. In \( f(R) \) gravity, we propose

\[
D_\mu \epsilon \equiv \left( D_\mu + W(F(R)) \Gamma_\mu \right) \epsilon = 0, \quad \left( \Gamma^\mu \nabla_\mu F + U(F(R)) \right) \epsilon = 0, \tag{2.16}
\]

where the functions \( W \) and \( U \) are to be determined. The first equation is the natural generalization of (2.13). The second equation is inspired by the fact that \( f(R) \) gravities are effectively a special, albeit inconvenient, class of Brans-Dicke theories. In supergravities, the left-hand sides of equations in (2.16) would be the supersymmetric variations for the gravitino and dilatino fields, and \( F, W \) are analogous to the scalar and the superpotential.
for the scalar. If \( f \) is linear in \( R \) and hence \( F \) is a constant, we must set \( U = 0 \) and we recover the previous example.

To establish the relevance of these two Killing spinor equations (2.16) to \( f(R) \) gravities, we follow the procedure developed in [21–23]. We first act on the second equation with \( \Gamma^\mu \nabla_\nu \), which gives

\[
\left( \Box F - U(\dot{U} + 2(D - 1)W) \right) \epsilon = 0.
\]  

(2.17)

In this paper, a dot is always denoted as a derivative with respect to \( F \). Note that for any function \( X \), we have \( \dot{X} = \frac{X'}{f''} \), by the virtue of the Leibnitz’ chain rule. The \( \Gamma \)-matrix projected integrability condition for the first equation in (2.16) is somewhat more involved, and we find it is given by

\[
\Gamma^\mu [\mathcal{D}_\mu, \mathcal{D}_\nu] \epsilon = \frac{1}{2} \Gamma^\mu \mathcal{R}_{\mu\nu} \epsilon + \frac{1}{2} \Gamma^\mu X_{\mu\nu},
\]  

(2.18)

where \( \mathcal{R}_{\mu\nu} \) is given by (2.4) and \( X_{\mu\nu} \) is given by

\[
X_{\mu\nu} = \left( 2\dot{W}U + 4(D - 1)W^2 + \frac{4(D - 2)W\dot{W}U}{2W - \dot{U}} - \frac{1}{2(D - 1)F}(f - 2RF) \right) g_{\mu\nu}
\]  

\[
+ \left( \frac{1}{F} - \frac{2(D - 2)W}{2W - \dot{U}} \right) \nabla_\mu \nabla_\nu F.
\]  

(2.19)

Thus for an \( f(R) \) theory to admit consistent Killing spinor equations (2.16), we must have

\[
F = \frac{2W - \dot{U}}{2(D - 2)W}, \quad U(\dot{U} + 2(D - 1)W) = \frac{1}{2(D - 1)}(Df - 2RF),
\]  

\[
2\dot{W}U + 4(D - 1)W^2 + \frac{2UW}{F} - \frac{1}{2(D - 1)F}(f - 2RF) = 0.
\]  

(2.20)

These equations can be reduced, giving rise to

\[
U = -\frac{4D(D - 1)W^2 + R}{4(D - 1)W}, \quad \dot{W} = \frac{(4D(D - 1)W^2 + R)W}{(4D(D - 1)(D - 2)W^2 + R)F - f}.
\]  

(2.21)

Thus, once \( W \) and \( f \) are known, \( U \) comes out straightforwardly. It is advantageous to express all functions in terms of the variable \( R \), in which case, the second equation above becomes

\[
f'' - \frac{(4D(D - 1)(D - 2)W^2 + R)W'}{(4D(D - 1)W^2 + R)W} f' + \frac{W'}{(4D(D - 1)W^2 + R)W} f = 0.
\]  

(2.22)

One way to view this equation is that it is a non-linear differential equation of \( W \) for a given \( f \). Owing to the non-linearity, however, a solution for \( W \) does not always exist for a generic function \( f \). An alternative view point is that (2.22) is a second-order linear differential equation for \( f \). Thus (2.22) must have solutions of \( f \) for any given \( W \), even though the analytical expression for \( f \) may not exist. This is very much parallel to the gravity/scalar system, where a generic scalar potential may not admit a superpotential, but any superpotential can yield a potential. The difference is that here there is no simple
expression directly for \( f \) in terms of function \( W \), but instead \( f \) has to be solved via the second-order linear differential equation. What is curious is that for a given \( W \), there can exist a two-parameter (one non-trivial) family of \( f(R) \) theories, since second-order linear differential equations tend to give two solutions associated with two integration constants. In other words, there are two different \( f(R) \) gravities for one \( W \). Of course the function \( U \) is also different. This is different from the usual gravity/scalar system whose scalar potential can be expressed simply in terms of a superpotential when the theory admits Killing spinor equations. In section 2.2, we have demonstrated such two different \( f(R) \) gravities can give rise to the same solution. Furthermore, the cosmology at two different evolution epochs is better studied by either one or the other \( f(R) \) theories.

To summarize, we give a simple way of constructing \( f(R) \) gravity with Killing spinor equations. We can begin with a function \( W \), from which the \( f(R) \) is determined by (2.22), and the expression for \( U \) follows straightforwardly from (2.21). The corresponding Killing spinor equations are then (2.16).

It should be emphasized that the Killing spinor equations we have discussed are for the theory with generic backgrounds rather than for a specific solution. For an \( f(R) \) theory with such Killing spinor equations, we can check whether a specific solution preserves Killing spinors and obtain the fraction of the maximally allowed Killing spinors that survive. For example, we can establish whether the (A)dS vacua satisfying (2.5) admit any Killing spinor that satisfies (2.16). Furthermore, armed with these Killing spinor equations, we can construct new solutions. As mentioned earlier, the Killing spinor equations (2.16) can be viewed as the first integrals of the Einstein equations of motion. For a background with a Killing spinor \( \epsilon \) satisfying (2.16), the integrability conditions imply, as we have shown,

\[
\mathcal{R}\epsilon = 0, \quad \Gamma^\mu\mathcal{R}_{\mu\nu}\epsilon = 0, \quad (2.23)
\]

where \( \mathcal{R} \) and \( \mathcal{R}_{\mu\nu} \) are defined in (2.3) and (2.4) respectively. Thus we see immediately that the trace equation (2.3) is automatically satisfied, and the second equation in (2.23) takes the same form as (2.15). Note that the first equation above is implied by the second equation, and hence it is not independent. If the Killing vector constructed from the Killing spinor \( \epsilon \), namely \( K^\mu(\epsilon) = \bar{\epsilon}\Gamma^\mu\epsilon \), is e.g. time-like, it is straightforward to demonstrate that the equations of motion (2.4) are satisfied as well. Thus for our \( f(R) \) gravity with Killing spinor equations, we can obtain a class of solutions by solving only the Killing spinor equations (2.16) with the corresponding time-like Killing vector, instead of solving the more difficult Einstein equations directly. This class of solutions are analogous to the BPS solutions in supergravities, and we shall refer them as “BPS” even though our \( f(R) \) gravities are not supersymmetric. Note that in practice, our task may not be about constructing the most general “BPS” solutions, but only some special solutions with some simple ansatz. In this case, the existence of some Killing spinors that satisfy the Killing spinor equations can determine the ansatz fully. We can then simply substitute the result into (2.2) to verify whether it is a solution or not.
2.4 A few examples

Here we give some simple explicit examples of $f(R)$ gravities that admit Killing spinor equations (2.16) and give the corresponding $W$ and $U$ functions. An obvious search is to consider quadratic $f(R)$ gravity in $D$ dimensions, namely

$$ f(R) = \sigma R - (D - 2)\Lambda_0 + \alpha R^2, \quad (2.24) $$

where $\sigma, \alpha$ and $\Lambda_0$ are constants. For these general parameters, we find that the Killing spinor equations do not exist. However, if the three parameters satisfy the following constraint

$$ \alpha \Lambda_0 = -\frac{4(D - 1)^2\sigma^2}{(D - 2)(5D - 2)^2}, \quad (2.25) $$

we find that they do exist, with

$$ W = \frac{R}{2\sqrt{D(D - 1)\beta}}, \quad U = \frac{(D - 2)R((D - 1)R + \beta D)}{D\sqrt{D(D - 1)\beta} (R + \beta)^{2/D}}. \quad (2.28) $$

Another example is also a $D$-dimensional theory:

$$ f(R) = \sigma R (R + \beta)^{D-2}. \quad (2.27) $$

We find that the corresponding $W$ and $U$ are given by

$$ W = \frac{R}{2\sqrt{D(D - 1)\beta}}, \quad U = \frac{(D - 2)R((D - 1)R + \beta D)}{D\sqrt{D(D - 1)\beta} (R + \beta)^{2/D}}. \quad (2.28) $$

In section 2.2, we have shown that the two four-dimensional $f(R)$ gravities (2.6) and (2.11) give rise to the same cosmological solution (2.8). Both theories admit Killing spinor equations, with the same $W$, i.e.

$$ W = \frac{R}{48\beta}. \quad (2.29) $$

However, these two theories have different $U$ functions. For the theory (2.6), the function $U$ is

$$ U = \frac{\sigma_1 R(R - 64\beta^2)}{16\beta\sqrt{48\beta^2 - R}}. \quad (2.30) $$

For the theory (2.11), we have

$$ U = \frac{\sigma_2 \left(12\beta(32\beta^2 - R)\sqrt{48\beta^2 - R} + \sqrt{3} R(64\beta^2 - R)\arccoth\left(\frac{4\sqrt{3}\beta}{\sqrt{48\beta^2 - R}}\right)\right)}{16\beta\sqrt{48\beta^2 - R}}. \quad (2.31) $$

More examples of $f(R)$ gravities that admit Killing spinor equations will be given in subsequent sections.
3 “BPS” domain wall solutions

3.1 General properties

As discussed in section 2, Einstein metrics arise naturally in \( f(R) \) gravities. However, for a generic \( f(R) \) theory, any exact solution beyond constant \( R \) is more or less impossible to construct, since one has to handle in general fourth-order non-linear differential equations. However, for our \( f(R) \) gravities with Killing spinor equations, exact solutions with varying \( R \) can be obtained. In particular, we shall consider static domain wall solutions, with the ansatz

\[
d s^2_D = dr^2 + e^{2A(r)} d\mathbf{x}^\mu d\mathbf{x}_\mu. \tag{3.1}
\]

Making a natural choice of vielbein, \( e^r = dr, e^i = e^A dx^i \), we find that the non-vanishing spin connection is \( \omega^i_r = A^r, e^i \). The Ricci tensor and scalar are given by

\[
R_{rr} = -(D - 1)(A_{rr} + A^2_r), \quad R_{\mu\nu} = -(A_{rr} + (D - 1)A^2_r)g_{\mu\nu},
\]

\[
R = -2(D - 1)A_{rr} - D(D - 1)A^2_r. \tag{3.2}
\]

For the metric ansatz (3.1), the Killing spinor equations (2.16) can be easily solved. The Killing spinors are given by

\[
\epsilon = e^{\frac{1}{2}A} \epsilon_0, \quad \Gamma_r \epsilon_0 = \epsilon_0, \tag{3.3}
\]

where \( \epsilon_0 \) is a constant spinor in the \((D - 1)\)-dimensional world-volume. The full set of equations of motion is now reduced to simply

\[
A_{rr} = -2W. \tag{3.4}
\]

This is a tremendous simplification of the Einstein equation (2.2). We see that the structure of domain wall is completely determined by the function \( W \). This is analogous to supergravities, where the BPS domain walls are solely determined by the superpotential.

From the Killing spinors (3.3), we can construct the Killing vectors, namely \( K^M(\epsilon_0) = e^A \epsilon_0 \Gamma^M \epsilon_0 \), and hence \( K^r = e^A \epsilon_0 \epsilon_0, K^i = e^A \epsilon_0 \Gamma^i \epsilon_0 \) and \( K^t = e^A \epsilon_0 \Gamma^t \epsilon_0 \). Whether this Killing vector can be time-like or not depends on dimensions and the \( \Gamma \)-matrix properties. Since \( \epsilon_0 \) can be any constant spinor in the \( D - 1 \) dimensional spacetime \( d\mathbf{x}^\mu d\mathbf{x}_\mu \), we would expect that we could choose some appropriate \( \epsilon_0 \) such that \( K^r = 0 = K^i, \) but \( K^t \neq 0 \). In the case when \( D \) is odd, and hence \((D - 1)\) is even, such an \( \epsilon_0 \) can be easily found. For example, let us consider the convention that \( \bar{\epsilon} = e^{\frac{1}{2}A} \) with Hermitian \( \Gamma^i \) and anti-Hermitian \( \Gamma^t \), the Killing spinors \( \epsilon_0 \) satisfying the above conditions are given by

\[
\epsilon_0 = (1 + \gamma) \eta_0, \tag{3.5}
\]

where \( \gamma = a \prod_i \Gamma^i \) with \( a \) so chosen that \( \gamma^2 = 1 \). It is straightforward to verify that the corresponding \( K \) is a time-like Killing vector. Following the discussion in the previous section, the equation (3.4) must satisfy (2.2). In even \( D \) dimensions, the discussion is somewhat more complicated. Let us consider two constant spinors \( \epsilon^\pm_0 = (1 \pm \Gamma_1) \eta_0 \), and denote \( K^\pm \) as the corresponding Killing vectors. It is clear that \( K^\pm \) are both null vectors
and hence we can choose a convention such that $K = K^+ + K^-$ is time-like. Since we must have $\mathcal{R}_{\mu\nu}K^\nu = 0$, it is then straightforward to show that $\mathcal{R}_{\mu\nu} = 0$. Note that this demonstration works in odd $D$ dimensions as well.

In fact, for such a simple background, it is quite easy to demonstrate that (3.4) indeed satisfies all the equations of motion by simply substituting (3.4) directly into (2.2). From the expression for the Ricci scalar in (3.2), we obtain the first-order equation

$$W,_{r} = W' R,_{r} = \frac{R + 4D(D - 1)W^2}{4(D - 1)}. \quad (3.6)$$

Together with (2.22), we find that

$$F,_{r} = \frac{(R + 4(D - 1)(D - 2)W^2)F - f}{4(D - 1)V}. \quad (3.7)$$

It is now straightforward to establish that the full set of equations of motion are all satisfied.

Now let us consider the general properties of the solution. We use $X$ to denote the right-hand side of (3.6), namely

$$X(W) = \frac{R + 4D(D - 1)W^2}{4(D - 1)}. \quad (3.8)$$

Here we treat $R$ as a function of $W$. The equation (3.6) is of the first order, and can be solved as

$$r - r_0 = \int \frac{dW}{X(W)}. \quad (3.9)$$

If $X(W)$ has a zero, such that

$$X(W) = X'(W_0)(W - W_0) + \cdots, \quad (3.10)$$

we find that near the region of $W_0$, the solution is given by

$$W = W_0 + e^{X'(W_0)r}, \quad e^A = \exp\left(-W_0 r - \frac{e^{X'(W_0)r}}{X'(W_0)}\right). \quad (3.11)$$

Thus if $X'(W_0) > 0$, the metric becomes AdS when $r \to -\infty$. The resulting metric is AdS horizon if $W_0 < 0$, and it is AdS asymptotic boundary if $W_0 > 0$. If on the other hand we have $X'(W_0) < 0$, the metric becomes AdS when $r \to +\infty$. The resulting metric is AdS horizon if $W_0 > 0$, and it is AdS asymptotic boundary if $W_0 < 0$. Thus we see that near the region of $W = W_0$, the solution is regular, approaching either the AdS horizon or the AdS boundary. Thus, If $X(W)$ has at least two roots, we can expect smooth solutions that run from one AdS to the other, associated with two adjacent roots.

### 3.2 A class of exact solutions

As discussed above, in order to construct smooth solutions, it is necessary that the function $X(W)$ has two roots. In this subsection, we consider a class of $W$, which is given by

$$R = 4D(D - 1)((a - 1)W^2 + bW + c). \quad (3.12)$$
where \(a, b, c\) are constants. It follows that \(X(W)\) is quadratic:

\[
X(W) = D(aW^2 + bW + c),
\]

which has two roots, given by

\[
2W_\pm = \lambda_\pm \equiv \frac{b \pm \sqrt{\Delta}}{a}.
\]

We require that the discriminant \(\Delta \equiv b^2 - 4ac > 0\) so that the two roots are real. The \(W\) equation (3.6) implies that

\[
W = \frac{1}{2a} \left( b + \sqrt{\Delta} \tanh \left( \frac{1}{2} D \sqrt{\Delta} (r + r_0) \right) \right).
\]

It follows from (3.4) that we have an explicit solution

\[
e^{2A} = \left( e^{bD_\tau} \cosh ^2 \left( \frac{1}{2} D \sqrt{\Delta} (r - r_0) \right) \right)^{\frac{2}{\sqrt{\Delta}}}.
\]

The metric is smooth with \(r\) running from \(-\infty\) to \(+\infty\). In both limits, the metric approaches AdS, namely

\[
e^{2A} \to e^{2\lambda_\pm r}, \quad \text{for} \quad r \to \pm \infty.
\]

The resulting effective cosmological constants in the AdS limits are \(\Lambda_\pm = -(D - 1)\lambda_\pm^2\).

For the choice of \(W\) given by (3.12), it follows from (2.22) that \(f\) can be determined by the following differential equation

\[
D(aW^2 + bW + c)(2(a - 1)W + b)W f_{,WWW}
\]

\[- 4(a - 1)W((aD - 1)W^2 + cD) + b((5aD - 4D - 2)W^2 + cD) + b^2 DW)f_{,W}
\]

\[+ D(2(a - 1)W + b)^2 f = 0. \quad (3.18)
\]

For generic parameters \((a, b, c)\), there is no analytical solution for \(f\); however, for some special choices of these constants, we obtain explicit \(f(R)\) gravities. One example is that \(b = 0\). In this case, the \(f(R)\) is given by two hypergeometric functions

\[
f = \sigma_1 W^3 \, _2 F_1(x_-,x_+; -\frac{1}{2}, -\frac{aW^2}{c}) + \sigma_2 \, _2 F_1(y_-,y_+; \frac{5}{2}, -\frac{aW^2}{c}),
\]

(3.19)

where \(\sigma_1\) and \(\sigma_2\) are integration constants properly chosen so that the function is real, and

\[
x_\pm = \frac{2 - 3aD \pm \sqrt{4 + aD((a + 8)D - 12)}}{4aD},
\]

\[
y_\pm = \frac{2 + 3aD \pm \sqrt{4 + aD((a + 8)D - 12)}}{4aD}.
\]

The expression for \(W\) in terms of \(R\) can be obtained from (3.12). Another example is provided that \(a = 1\) and \(c = 0\), in which case, we have

\[
f(R) = \sigma R \left( R + 4D(D - 1)b^2 \right)^{\frac{D-2}{D}}.
\]

(3.21)

Note that here we have not presented the other choice for \(f\) which involves hypergeometric functions. This simple \(f(R)\) theory were presented earlier in section 2.4.
3.3 Randall-Sundrum II

The Randall-Sundrum (RS) II scenario is characterized by the metric profile of the type $e^{2A} = e^{-2kr}$, with positive constant $k$ [27]. In other words, the metric approaches the AdS horizons at both $r \to \pm \infty$ limits, with one maximum in the middle. (Although $r$ is a non-compact coordinate, the volume integration is finite with respect to $r$.) This can be achieved in our domain wall solution by requiring $a < 0$, $c < 0$, which ensures that

$$\lambda_+ < 0, \quad \lambda_- > 0.$$ (3.22)

Note that the constant $b$ can be arbitrary in this case. Our solution describes in general the asymmetric RS II scenario when $\lambda_+ + \lambda_-$ does not vanish, since the cosmological constants $\Lambda_+ \neq \Lambda_-$ are not equal at the two AdS horizons. This of course is not crucial for trapping gravity on the wall. A symmetric RS II can be obtained by further requiring $b = 0$, whose $f(R)$ theory is given by (3.19).

To study the trapping of gravity on the wall, it is advantageous to express first the metric in the conformally-flat frame, namely

$$ds^2 = e^{2A(z)}(dz^\mu dx_\mu + dz^2),$$ (3.23)

where the coordinate $z$ is related to $r$ as follows

$$-\lambda_- z = e^{-kr} \left( 1 + \exp(D\sqrt{\Delta} r) \right)^{\frac{1}{2\Delta}} \, \frac{2}{\Delta} F_1 \left( \frac{\sqrt{\Delta} - b}{aD\sqrt{\Delta}}, \frac{2}{aD}; 1 + \frac{\sqrt{\Delta} - b}{aD\sqrt{\Delta}}; -e^{D\sqrt{\Delta} r} \right).$$ (3.24)

Note that we have set the inessential $r_0$ to zero. The linear fluctuation of the graviton modes in the $(D-1)$-dimensional flat world-volume in the context of $f(R)$ theory has not been studied yet except for $D = 5$ [28]. Thus we shall focus our attention on five dimensions. Following the procedure outlined in [28], we let $h_{\mu\nu} = e^{-3A/2}F^{-1/2}n_{\mu\nu}\phi(z)$, where $n_{\mu\nu}$ is transverse and traceless, we find that

$$-\frac{1}{2} \phi_{zz} + V \phi = 0,$$ (3.25)

where the Schrödinger potential $V = V_0 + V_1$ contains two parts. The first part is given by

$$V_0 = \frac{1}{2} k^2 + \frac{1}{4}(D-2)A_{zz} + \frac{1}{8}(D-2)^2 A_{zz}^2,$$ (3.26)

which is the same as that in [27]. For our general domain wall solution, we find

$$V_0 = \frac{1}{2} k^2 + \frac{D - 2}{8a^2} \exp \left[ \frac{2}{a} (br + \frac{2}{D} \log \left( \cosh \left( \frac{1}{2} D\sqrt{\Delta} (r - r_0) \right) \right) \right] \times \left( \frac{aD}{\Delta} \right)^2 \left( b + \sqrt{\Delta} \tanh \left( \frac{1}{2} D\sqrt{\Delta} r \right) \right).$$ (3.27)

The second part is the contribution from $F(R)$, and it was established only for $D = 5$ [28]. It is given by

$$V_1 = \frac{3A_{zz} F_{zz}}{4F} - \frac{F_{zz}^2}{8F^2} + \frac{F_{zz}}{4F}.$$ (3.28)
Note that the conversion to the coordinate $z$ does not give a close form for general parameters and one can appeal to the numerical approach. For some choice of parameters, the explicit $V$ as a function of $z$ can be obtained. For example, let $a = -2/D, b = 0$ and $c = D/2$, in which case, we have

$$e^A = -\frac{1}{\cosh(Dr)}.
$$

(3.29)

It follows that $r = \frac{1}{D}\arcsinh(Dz)$, and hence

$$V_0 = \frac{1}{2}k^2 + \frac{D^2(D - 2)(D^3z^2 - 2)}{8(D^2z^2 + 1)^2}.
$$

(3.30)

This potential profile is very much like the one obtained in [30] and is capable of trapping gravity on the wall, but ours is realized by $f(R)$ gravity, with

$$f(R) = \sigma_1|W|^3_2F_1\left(\frac{1}{2}(1 - \sqrt{2 - D}), \frac{1}{2}(1 + \sqrt{2 - D}; \frac{5}{2}; \frac{4W^2}{D^2})\right)
+ \sigma_2_2F_1\left(-1 - \frac{1}{2}\sqrt{2 - D}, -1 + \frac{1}{2}\sqrt{2 - D}; -\frac{1}{2}; \frac{4W^2}{D^2}\right),
$$

$$W^2 = \frac{2D^2(D - 1) - R}{4(D - 1)(D + 2)},
$$

(3.31)

The alarming-looking complex arguments in the hypergeometric functions do not prevent the $f(R)$ from being real provided that $|W| \leq D/2$. Of course, in order to demonstrate the trapping of gravity in the $f(R)$ theory, we also need to look at the contribution from $V_1$. Let us consider the simpler example associated with $\sigma_2$ in (3.31) in $D = 5$. We find that

$$V_1 = \frac{25}{8(1 + 25z^2)^2}\left[6 - 3\tanh^2\left(\sqrt{3}\arcsin\frac{5z}{\sqrt{1 + 25z^2}}\right)
- 50z\sqrt{\frac{3 + 75z^2}{1 + 25z^2}}\tanh\left(\sqrt{3}\arcsin\frac{5z}{\sqrt{1 + 25z^2}}\right)\right].
$$

(3.32)

The Schrödinger potential $V = V_0 + V_1$ becomes a bit more complicated. It has a local maximum $V = 0$ at $z = 0$, and two negatives minimums when we increase $|z|$, and it becomes positive until it hit a maximum before it approaches zero at $|z| = \infty$. Thus we demonstrate that $f(R)$ gravity can easily reproduce the smooth RS II scenario with thick domain walls. Note that we have added in an absolute value symbol on $W^3$ so that the $f(R)$ is a symmetric function of $W$. This does not create a discontinuity in (3.18) at $W = 0$ since it involves only up to the second-order derivatives. The Ricci scalar for these solutions runs from $R_0 = -D^3(D - 1)$ on the AdS horizon at $r = -\infty$ to the maximum value of $-2D^2(D - 1)$, and then decreases and approaches $R_0$ again on the AdS horizon at $r = +\infty$, and correspondingly $W$ runs from $-D/2$ to $D/2$. It is easy to verify that for (3.31), the constant $R_0 = -D^3(D - 1)$ solution indeed satisfy the equation (2.5).

Furthermore, we have

$$F(R_0) = -\frac{3D \sinh(\frac{1}{2}\sqrt{D - 2}\pi)}{16(D + 2)(D - 1)\sqrt{D - 2}}\sigma_1 + \frac{\cosh(\frac{1}{2}\sqrt{D - 2}\pi)}{2D^2(D - 1)}\sigma_2.
$$

(3.33)
The $f''(R_0)$ is divergent unless the constants $\sigma_1$ and $\sigma_2$ are specifically related. Since the expression becomes very complicated, we shall only give the result for $D = 5$. In this case, we have

$$\sigma_2 = \frac{125\sqrt{3}}{56} \coth \left( \frac{1}{2} \sqrt{3} \pi \right) \sigma_1,$$  

(3.34)

and

$$F(R_0) = \frac{5\sqrt{3}}{448} \csc \left( \frac{1}{2} \sqrt{3} \pi \right) \sigma_1,$$

$$R_0 f''(R_0) = -\frac{75\sqrt{3}}{6272} \csc \left( \frac{1}{2} \sqrt{3} \pi \right) \sigma_1,$$  

(3.35)

where $R_0 = -500$.

More generally, the RS II scenario arises in any $X(W)$ given in (3.8) with a profile that it has one positive, one negative root and a maximum in between.

Note that an analytical domain-wall solution in quadratic $f(R)$ theory together with additional scalar with $\phi^4$ potential was obtained in [29].

3.4 AdS wormholes

The situation is quite different if we have

$$\lambda_+ > 0, \quad \lambda_- < 0.$$  

(3.36)

This can be achieved by requiring $a > 0$ and $c < 0$, while $b$ can be arbitrary. In this case, the solutions describe smooth AdS wormholes that connect two AdS boundaries at $r \to \pm \infty$, with no bulk singularity and horizon in between. For non-vanishing $b$, the wormhole connects two asymmetric AdS boundaries with different $\Lambda_\pm$. For $b = 0$, the AdS boundaries are symmetric. Let us present a relative simple example with $a = 2/D, b = 0$ and $c = -D/2$. In this case the theory is given by

$$f = \sigma_1 |W|^3_2 F_1 \left( -\frac{1}{2}(1 + \sqrt{D - 1}), -\frac{1}{2}(1 - \sqrt{D - 1}); -\frac{1}{2}; -\frac{4}{D^2} W^2 \right)$$

$$+ \sigma_2 F_1 \left( 1 - \frac{1}{2} \sqrt{D - 1}, 1 + \frac{1}{2} \sqrt{D - 1}; \frac{5}{2}; \frac{4}{D^2} W^2 \right),$$

$$W^2 = -\frac{2D^2(2(D - 1) + R)}{4(D - 1)(D - 2)}.$$  

(3.37)

We add an absolute sign in $W$ for the same reason explained in the previous subsection. The solution is quite simple:

$$ds^2 = \cosh^2(Dr) dx^\mu dx_\mu + dr^2.$$  

(3.38)

The metric reaches AdS boundaries with $R_0 = -D^3(D - 1)$. For $D = 1 + k^2$, the function $f$ is given by simple functions. For example, when $D = 5$, we have

$$f(R) = \sigma_1 \sqrt{-(R + 200)^3} + \sigma_2 (R + 50) \sqrt{R + 500}.$$  

(3.39)

For the $f(R) = \sigma_1 \sqrt{-(R + 200)^3}$ theory, we have $F(R_0) = -15\sqrt{3}\sigma_1$, and hence the (2.5) is satisfied. For the $f(R) = \sigma_2 (R + 50) \sqrt{R + 500}$ theory, on the other hand, the $F(R_0)$ is divergent. Such a situation was discussed in section 2.2.
It is worth commenting that although there is a no-go theorem in the usual Einstein theory that the configuration with two AdS boundaries connected in the bulk without a horizon separating them will violet the energy condition [34]. This no-go theorem can be easily circumvented in higher-order derivative theories. Smooth wormholes with two AdS boundaries were constructed in Einstein gravity with the Gauss-Bonnet term [31–33]. Our examples demonstrate that wormholes arise naturally in \( f(R) \) gravities as well. More generally, wormhole solutions occur in any \( X(W) \), given in (3.8), with a profile that it has one positive and one negative roots with a minimum in between. Note that our wormhole solutions are brane-like and static. Only stationary brane-like wormholes were known to exist in Einstein gravity and supergravities in higher dimensions [35–38].

We can also consider a different parametrization. We set \( a = 1 \) and \( c = 0 \), but with non-vanishing \( b \). In this case, one \( f \) is given in (3.21). The domain wall solution is given by

\[
e^{2A} = \left( 1 + e^{Db(r-r_0)} \right)^{\frac{4}{D}}.
\]

(3.40)

For \( b > 0 \), this solution describes a wormhole that connects a flat spacetime at \( r \to -\infty \) to the AdS boundary at \( r \to +\infty \) with \( R_0 = -4D(D-1)b^2 \). Note that the solution with \( b < 0 \) is equivalent to \( b > 0 \), by reversing the sign of \( r \). Similar solutions that connect the AdS boundary in one asymptotic region to the flat spacetime in another have also been found in supergravities [36–38]; however, these solutions are stationary rather than static.

As discussed earlier, there is a different \( f(R) \) theory that would give rise to the same solution (3.40). It is much more complicated, given by

\[
f = \sigma W(W+b) \left( 1 - \frac{2b}{(n-2)(W+b)} + \left( \frac{W}{W+b} \right)^{\frac{2}{n}} 2F_1 \left( -\frac{2}{D}, -\frac{2}{D}; \frac{D-2}{D}; -\frac{b}{W} \right) \right).
\]

(3.41)

where \( W = R/(4D(D-1)b) \). In the case of \( D = 4 \), the expression is simpler, given by

\[
f(R) = \sigma R \left[ 12b - \sqrt{3}\left( R + 48b^2 \right) \arctanh \left( \frac{\sqrt{R + 48b^2}}{4\sqrt{3}b} \right) \right].
\]

(3.42)

This theory satisfies that \( F'(R_0) = 24b\sigma \) and \( R_0 f''(R_0) = 16b\sigma \).

### 3.5 RG flow from IR to UV

In this case, we have \( \lambda_+\lambda_- > 0 \), which can be achieved by requiring \( b^2 > 4ac > 0 \). It is clear that \( \lambda_\pm \) being both positive is equivalent to the case with both being negative, by merely reversing the sign of the coordinate \( r \). Let us thus discuss the case with both being positive. If \( a > 0 \) and hence \( b > 0 \), we then have

\[
\lambda_+ > \lambda_- > 0.
\]

(3.43)

The metric describes a flow running from the AdS horizon with \( \Lambda_- \) at \( r \to -\infty \) to the AdS boundary with \( \Lambda_+ \) at \( r \to +\infty \), which corresponds to the IR and the UV regions in the dual conformal field theory respectively. The cosmological constant \( \Lambda_- \) in the IR
region is smaller than the $\Lambda^+$ in the UV region. This type of behavior is similar to the domain wall solutions in supergravities [39]. Note that such solutions can be obtained in any $X(W)$ that has two adjacent positive roots with a minimum in between. If instead, $a < 0$ and hence $b < 0$, the cosmological constant in the IR region is bigger than that in the UV region. Such a solution occurs in any $X(W)$ that has two adjacent positive roots with a maximum in between.

3.6 On holographic $c$-theorems

As we see in our explicit constructions of domain wall solutions, $f(R)$ gravities are quite suitable for investigating the AdS/CFT correspondence. One natural question is to examine the holographic $c$-theorem. One may view that $f(R)$ gravity is simply Einstein gravity with an effective energy-momentum tensor built from the Ricci scalar, with the equations of motion (2.2) expressed as

$$G_{\mu\nu} = T_{\mu\nu}^{\text{eff}} = G_{\mu\nu} - G_{\mu\nu}, \quad \text{(3.44)}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is the Einstein tensor. If one takes this point of view, one can follow the procedure in [39] and define

$$a(r) \equiv \frac{\pi^{(D-1)/2}}{\Gamma\left(\frac{1}{2}(D-1)\right) A_{r}^{D-2}}. \quad \text{(3.45)}$$

Its variation with respect to the co-moving coordinate $r$ is given by

$$a_{,r} = -\frac{\pi^{(D-1)/2}}{\Gamma\left(\frac{1}{2}(D-1)\right) A_{r}^{D-1}} \left( (T_{\mu\nu}^{\text{eff}})^{t}_{t} - (T_{\mu\nu}^{\text{eff}})^{r}_{r} \right). \quad \text{(3.46)}$$

The holographic $c$-theorem follows provided that $a_{,r} \geq 0$, which implies that the cosmological constant at the IR is smaller than that in the UV. It is clear that whether the $c$-theorem holds or not depends on the specific choice of $f(R)$ gravities, and we have examples that both support and violate the $c$-theorem.

A different approach is to treat $f(R)$ gravities as Brans-Dicke theory, and the $c$-theorem is then dictated by the corresponding scalar potential. The third approach is treating $f(R)$ theory as a pure gravity theory that can coupled to additional matter so that the equations are now given by

$$G_{\mu\nu} = T_{\mu\nu}^{\text{mat}}. \quad \text{(3.47)}$$

This follows the same approach of [40] where all ghost-free curvature squared and cubic terms were considered. A monotonic function $a$ can be found in these higher-order theories [40]. It is of great interest to investigate the constraints on $f$ so that the holographic $c$-theorem also holds and whether such constraints are consistent with the conditions for Killing spinor equations.
4 “BPS” cosmology

4.1 The set up

It is well-known that the de Sitter spacetimes also admit Killing spinors, even though Einstein gravity with a positive cosmological constant cannot be supersymmetrized. The Killing spinor equation is given by

$$\hat{\nabla}_\mu \epsilon \equiv \left( D_\mu + \frac{i}{2} \sqrt{\frac{\Lambda_0}{D-1}} \Gamma_\mu \right) \epsilon = 0,$$  \hspace{1cm} (4.1)

This property of de Sitter space was exploited in constructing de Sitter “supergravities” [41, 42] which are effectively the analytical continuation of AdS supergravities. The function $W$ in Killing spinor equations (2.16) is pure imaginary in this case. We would like to assume implicitly that $W$ and $U$ in Killing spinor equations are real. Thus for the purpose of studying cosmology, we would like to rewrite the Killing spinor equations as follows

$$\mathcal{D}_\mu \epsilon \equiv \left( D_\mu + i W \Gamma_\mu \right) \epsilon = 0, \hspace{1cm} \left( \Gamma^\mu \nabla_\mu F + i U \right) \epsilon = 0,$$  \hspace{1cm} (4.2)

where $U$ is given by

$$U = \frac{R - 4D(D-1)W^2}{4(D-1)W}.$$  \hspace{1cm} (4.3)

It is important to note that our procedure of sending $W$ and $U$ to imaginary values does not affect the reality of the function $f$, which now satisfies

$$f'' - \frac{(R - 4(D-1)(D-2)W^2)W'}{(R - 4D(D-1)W^2)W} f' + \frac{W'}{(R - 4D(D-1)W^2)W} f = 0.$$  \hspace{1cm} (4.4)

We now construct “BPS” cosmological solutions that admit Killing spinors. The ansatz is the FLRW metric with flat spatial directions

$$ds^2 = -dt^2 + a^2 dx^i dx^i.$$  \hspace{1cm} (4.5)

Requiring that the solution admit Killing spinors, the full set of Einstein equations of motion is reduced to

$$\frac{a_t}{a} = 2W,$$  \hspace{1cm} (4.6)

which implies the following first-order equation

$$W_t = Y(W) \equiv \frac{R - 4D(D-1)W^2}{4(D-1)}.$$  \hspace{1cm} (4.7)

We verify that (4.6) indeed satisfies (2.2). Smooth cosmology emerges when $Y(W)$ has two adjacent roots corresponding to two de Sitter spaces. The cosmological evolution runs from one de Sitter to the other.

As an illustrative example, let us consider

$$R = 4D(D-1)((\alpha + 1)W^2 + \beta W + \gamma),$$  \hspace{1cm} (4.8)
such that \( Y(W) = D(\alpha W^2 + \beta W + \gamma) \). The corresponding \( f(R) \) can be determined by the following second-order linear differential equation

\[
\frac{4(\alpha + 1)(\alpha D + 1)W^3 + \beta(5\alpha D + 4D + 2)W^2 + D(4(\alpha + 1)\gamma + \beta^2)W + \beta D\gamma}{D W(2(\alpha + 1)W + \beta)(\alpha W^2 + \beta W + \gamma)} f,W
+ f,W W + \frac{2(\alpha + 1)W + \beta}{W(\alpha W^2 + \beta W + \gamma)} f = 0. \tag{4.9}
\]

When \( \beta = 0 \), the equation can be solved explicitly, giving

\[
f = \sigma_1 W^3 \, _2F_1 \left( \frac{1}{2}; \frac{5}{2}; -\frac{\alpha W^2}{2}; \right) + \sigma_2 \, _2F_1 \left( \frac{1}{2}; \frac{1}{2}; -\frac{1}{2}; -\frac{\alpha W^2}{2}; \right). \tag{4.10}
\]

where \( \sigma_1 \) and \( \sigma_2 \) are integration constants properly chosen so that the function is real, and

\[
\tilde{x}_+ = \frac{3\alpha D - 2 \pm \sqrt{4 + \alpha D((\alpha - 8)D + 12)}}{4\alpha D},
\]

\[
\tilde{y}_+ = \frac{-3\alpha D - 2 \pm \sqrt{4 + \alpha D((\alpha - 8)D + 12)}}{4\alpha D}. \tag{4.11}
\]

The general “BPS” cosmological solution for \((4.8)\) is given by

\[
a = \left( e^{\beta D t} \cosh \left( \frac{1}{2} D \sqrt{\Delta} (t - t_0) \right) \right)^{-\frac{\Delta}{2}}, \tag{4.12}
\]

where \( \Delta = \beta^2 - 4\alpha \gamma > 0 \). The solution approaches de Sitter spaces in both \( t \to \pm \infty \) limits, with \( a \sim e^{\lambda_{\pm} t} \), where

\[
\lambda_{\pm} = -\frac{\beta \pm \sqrt{\Delta}}{\alpha}. \tag{4.13}
\]

The corresponding cosmological constants of the de Sitter spaces in these limits are \( \Lambda_{\pm} = (D-1)\lambda_{\pm}^2 \). Depending on the sign and values of \( \lambda_{\pm} \), various cosmological scenarios emerge.

If we take a view that \( f(R) \) gravity is simply Einstein gravity with an effective energy-momentum tensor, as in \((3.44)\), the effective energy density and pressure for our cosmology are given by

\[
\rho_{\text{eff}} = \frac{\rho}{a^2} = \frac{\left( \beta + \sqrt{\Delta} \tanh \left( \frac{1}{2} D \sqrt{\Delta} t \right) \right)^2}{\alpha^2},
\]

\[
p_{\text{eff}} = -\frac{1}{D - 1} \left( \rho + (D - 2) \frac{a_{,tt}}{a} \right) = \frac{1}{2(D - 1)\alpha^2} \left[ (\alpha D(D - 2) + 2(D - 1))\Delta \left( \frac{\Delta}{2(D - 1)} \tanh \left( \frac{1}{2} D \sqrt{\Delta} t \right) \right)^2 \right. 
- 2(D - 1) \left( \beta^2 + \Delta + 2\sqrt{\Delta} \tanh \left( \frac{1}{2} D \sqrt{\Delta} t \right) \right). \tag{4.14}
\]

Then we have

\[
w_{\text{eff}}(t) = \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = -1 - \frac{\alpha D(D - 2)\Delta}{2(D - 1) \left( \beta \cosh \left( \frac{1}{2} D \sqrt{\Delta} t \right) + \sqrt{\Delta} \sinh \left( \frac{1}{2} D \sqrt{\Delta} t \right) \right)^2}. \tag{4.15}
\]
Note that in this discussion, we have set the inessential \( t_0 \) to zero. In the limits of \( t \to \pm \infty \), we have \( w_{\text{eff}}(\pm \infty) = -1 \) as one would have expected. Since \( \Delta > 0 \), the sign choice of \((w_{\text{eff}} + 1)\) for the “dark energy” depends solely on the parameter \( \alpha \). An extremum occurs at

\[
    t = -\frac{2}{D\sqrt{\Delta}} \text{arctanh} \left( \frac{\sqrt{\Delta}}{\beta} \right),
\]

(4.16)
corresponding to

\[
    (w_{\text{eff}} + 1)_{\text{extremum}} = \frac{D(D - 2)\Delta}{8(D - 1)\gamma}.
\]

(4.17)

### 4.2 From inflation to ever-expanding universe

The parameters \( \alpha, \beta \) and \( \gamma \) should be chosen such that \( \lambda_+ \) and \( \lambda_- \) are both positive. Furthermore we must have \( \lambda_+ < \lambda_- \). In this model, the universe starts an inflation with a bigger cosmological constant \( \Lambda_- \) at \( t \to -\infty \) and end with an ever-expanding de Sitter universe with a smaller cosmological constant \( \Lambda_+ \). Since we have

\[
    \lambda_+ - \lambda_- = -\frac{2\sqrt{\Delta}}{\alpha}.
\]

(4.18)

It follows that we must have \( \alpha > 0 \). This implies that for this model the sign choice of \((w_{\text{eff}} + 1)\) for the dark energy is always positive throughout the evolution. Furthermore, \( \beta \) must be negative and \( \gamma \) must be positive. As a semi-realistic model of our universe, we require

\[
    \frac{\Lambda_+}{\Lambda_-} \ll 1.
\]

(4.19)

This can be achieved by requiring \( 4\alpha\gamma/\beta^2 \ll 1 \), in which case we have

\[
    \lambda_- \sim -\frac{2\beta}{\alpha}, \quad \lambda_+ \sim -\frac{4\alpha\gamma}{\beta}.
\]

(4.20)

It can be shown that in the later part of the evolution, \( f(R) \sim R - R^2/\beta + \mathcal{O}(R^3) \). Note that this type of semi-realistic solutions emerge as long as \( Y(W) \) has two positive roots with a positive maximum in between.

If instead we have \( \lambda_+ > \lambda_- > 0 \), which can be achieved by requiring \( \alpha < 0, \beta > 0 \) and \( \gamma < 0 \), the universe would start with a mild inflation, and inflates faster and faster. Such a solution arises in general when \( Y(W) \) has two positive roots with a negative minimum in between. Such a theory provides a model for the multi-stage inflationary scenario.

### 4.3 Bouncing universe

The universe bounces when \( \lambda_- < 0 \), but \( \lambda_+ > 0 \). This occurs when \( \alpha < 0 \) and \( \gamma > 0 \). The minimum \( a \) occurs when \( t = t_{\text{min}} \), given by

\[
    t_{\text{min}} - t_0 = -\frac{2}{D\sqrt{\Delta}} \text{arctanh} \left( \frac{\beta}{\sqrt{\Delta}} \right).
\]

(4.21)
The minimum scale factor is given by

\[ a_{\text{min}} = \left(1 - \frac{\beta^2}{\Delta} \right)^{\frac{1}{\alpha D}} \left(\frac{\sqrt{\Delta} + \beta}{\sqrt{\Delta} - \beta} \right)^{\frac{\beta}{\alpha D \sqrt{\Delta}}}. \]  

(4.22)

This type of bouncing universe emerges when \( Y(W) \) has one positive and one negative adjacent roots with a positive maximum in between.

### 4.4 Pre-big bang model

In the special limit, namely \( \gamma = 0 \) and \( \alpha = -1 \), the solution is simple, given by

\[ a = \left(1 + e^{\beta D(t-t_0)}\right)^{\frac{2}{D}}. \]  

(4.23)

In this case, one \( f(R) \) gravity takes a simple form, i.e.

\[ f(R) = R \left(4D(D-1)\beta^2 - R\right)^{\frac{D-2}{D}}. \]  

(4.24)

We shall not present the other \( f(R) \) gravity that gives rise to the exact same solution. The solution connects the flat \( R = 0 \) region to the \( R = 4D(D-1)\beta^2 \) de Sitter space. It can be used to model the singularity-free inflation scenarios. In particular it predicts a “pre-big bang” flat universe which bursts into inflation by the non-perturbative effect of the higher-order curvatures. The four-dimensional case was discussed in section 2.

### 4.5 Smooth crunching universe

If we have \( \lambda_- > 0 \), but \( \lambda_+ < 0 \), the universe starts with an inflation, but end with a big crunch. What is interesting is that usually such model encounters a curvature singularity at the crunch. But in our solution, the universe shrinks in the manner of a de Sitter space, and hence there is no singularity.

### 5 Relating to the Brans-Dicke theory

It is well-known that \( f(R) \) gravity can be cast into the form of Brans-Dicke theory by the Legendre transformation. To see this, one starts with the Lagrangian

\[ \mathcal{L} = \sqrt{-g} \left(f(\chi) + f,\chi(\chi)(R - \chi)\right). \]  

(5.1)

Variation with respect to \( \chi \) gives rise to

\[ f,\chi(R - \chi) = 0. \]  

(5.2)

Thus provided that \( f,\chi \neq 0 \), it follows that \( \chi = R \), and hence (5.1) is the usual \( f(R) \) theory. Alternatively, one can define

\[ \varphi = f,\chi(\chi), \]  

(5.3)
and hence the $f(R)$ gravity is equivalent to the Brans-Dicke theory of the type

$$\mathcal{L} = \sqrt{-g} \left( \varphi R + f(\varphi) - \varphi \chi(\varphi) \right). \quad (5.4)$$

This is a special class of Brans-Dicke theory with no manifest kinetic term for $\varphi$. The conversion of $f(R)$ gravity to the Brans-Dicke theory requires finding the inverse function of $F = f'$, which in general does not have explicit analytical form. In most of our examples that admit Killing spinor equations discussed in this paper, the $f(R)$ theories are better discussed on their original form, rather than converting to the corresponding Brans-Dicke theories. There are couple of examples we find that can be converted into the gravity/scalar system, where the scalar potentials are expressed in terms of simple functions.

### 5.1 A quadratic $f(R)$ theory

The first example is the quadratic $f(R)$ gravity given in section (2.4). This is a particular simple example, since $F$ is a linear function with a simple inverse. Using the procedure above, we find that the gravity/scalar theory in Einstein frame is given by

$$\mathcal{L} = \sqrt{-g} \left( R - \frac{1}{2} (\partial \phi)^2 - V \right), \quad (5.5)$$

where the scalar potential can be expressed in terms of a superpotential, namely

$$V = \widetilde{W}^2 \phi - \frac{D - 1}{2(D - 2)} \widetilde{W}^2. \quad (5.6)$$

We find that

$$\widetilde{W} = \frac{4\sqrt{2} (D - 1) c}{3(5D - 2)} \left( 3(D + 2) \sigma - (5D - 2) e^{a_1 \phi} \right)e^{a_2 \phi},$$

$$a_1 = \sqrt{\frac{D - 2}{2(D - 1)}}, \quad a_2 = -\frac{D}{2\sqrt{2(D - 1)(D - 2)}}. \quad (5.7)$$

The domain wall solution in the $f(R)$ gravity can be obtained by treating $R$ as the coordinate, rather than $z$. In other words, we have

$$dz = \frac{dz}{dR} dR = \frac{W_R}{X(R)} dR, \quad (5.8)$$

where

$$X(R) = \frac{R + 4D(D - 1)W(R)^2}{4(D - 1)}. \quad (5.9)$$

The function $A$ is now given by

$$A = - \int \frac{2W W_R}{X(R)} dR. \quad (5.10)$$

The domain wall approaches the AdS boundary at $R = R_0$ where $X(R_0) = 0$. 




5.2 Another example

Another example is provided by (2.27). In $D = 4$, we have

$$L_4 = R\sqrt{R + \beta}.$$  \hfill (5.11)

The resulting scalar/gravity system (5.5) has a complicate scalar potential, given by

$$V = -\frac{2}{27}\Phi^{-2}\left(\Phi^2 - 3\beta + \sqrt{\Phi^4 + 3\beta^2}\right)\left(-3\Phi + \sqrt{2\Phi^2 + 3\beta + \sqrt{\Phi^4 + 3\beta^2}}\right), \hfill (5.12)$$

where $\Phi = e^{2\phi/\sqrt{3}}$.

It should be pointed out that for the majority of our $f(R)$ gravities that admit Killing spinor equations, it is unnatural to convert them to the Brans-Dicke theory. If one insists on doing so, the philosophy should be applied to Einstein gravity with a Gauss-Bonnet term where the $R + \alpha R^2$ part should be converted to the Brans-Dicke theory as well. The consequence is that Einstein gravity with Gauss-Bonnet term should be viewed as the Brans-Dicke theory coupled with the Ricci and Riemann tensor square terms. This formalism is clearly less elegant than the original pure gravity formalism.

6 Linear spectrum in (A)dS

As was discussed in section 3, $f(R)$ gravity admits (A)dS metrics as its vacuum solutions. We have constructed a large number of “BPS” domain wall and cosmological solutions that run from one (A)dS to the other. It is a formidable task to examine the stability of these solutions. In this section, we study the linear fluctuation of $f(R)$ gravity in such a (A)dS vacuum instead. As has been discussed in section 2, there are two types of (A)dS vacua that could arise in $f(R)$ gravities. The first type is the usual one that satisfies (2.5). The second type is the one we discovered in this paper and it is characterized by the divergent $F(R_0)$. In this section, we shall be only concerned with the linearization $f(R)$ gravities around the (A)dS vacua of the first type. Linearized $f(R)$ gravity in such AdS$_4$ were studied in [43]. For the linear perturbation $g_{\mu\nu} \rightarrow g_{\mu\nu} + h_{\mu\nu}$, we impose the gauge condition

$$\nabla^\mu h_{\mu\nu} = \nabla_\nu h.$$ \hfill (6.1)

This gauge condition is different from the usual de Donder gauge, but it is more effective to use in theories with a cosmological constant since it implies the vanishing of the trace scalar mode in Einstein gravity with a cosmological constant. It has been adopted in recent studies in critical gravities [44–46]. We find that the linearized equation of (2.2) becomes

$$-\frac{1}{2} f'(R_0)\left(\Box - \frac{2\Lambda}{D - 1}\right)H_{\mu\nu} = 0,\hfill (6.2)$$

$$-(D - 1)\Lambda f''(R_0)\left(\Box - m^2\right)h = 0,\hfill (6.3)$$
where
\[ H_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{D}g_{\mu\nu}h - \frac{2(D-1)f''(R_0)}{(D-2)f'(R_0)}J_{\mu\nu}, \]
\[ J_{\mu\nu} \equiv \left( \nabla_\mu \nabla_\nu - \frac{1}{D} \Box \right) h, \quad m^2 = -\frac{R_0 f''(R_0)}{(D-1)f''(R_0)} - \frac{1}{2} \frac{(D-2)f'(R_0)}{(D-1)f''(R_0)} \tag{6.4}. \]

To derive the above equations, we have made use of the following formulae
\[ [\Box, \nabla_\mu]h = \Lambda \nabla_\nu h, \quad [\Box, \nabla_\mu \nabla_\nu]h = \frac{2}{D-1} J_{\mu\nu}. \tag{6.5} \]

It is clear that \( H_{\mu\nu} \) is traceless; it is also transverse by the virtue of the equation of motion for \( h \). In the above, we assume that \( R_0, f'(R_0), f''(R_0) \) and \( f'''(R_0) \) are all non-vanishing. Thus we see that in general, in addition to the massless spin-2 graviton mode, there is also a massless scalar trace mode. However, there is no higher-order propagator for both modes, unlike the case in theories with more general higher curvature invariants. This is consistent with the fact that in terms of physical degrees of freedom, \( f(R) \) gravity is equivalent to a special class of the Brans-Dicke theory. The lacking of higher-order propagators implies that there is no critical phenomenon as those discussed in [44–46].

In the special case, where \( f''(R_0) = 0 \), the equation (6.3) implies that \( h = 0 \), and the theory contains only the massless graviton, as in the case of Einstein gravity. If \( f'(R_0) = 0 \), graviton \( H_{\mu\nu} \) no longer has its kinetic term. If \( f''(R_0) = 0 = f'''(R_0) \), the theory has no propagating mode at all. For example, the theory
\[ \mathcal{L}_4 = \sqrt{-g}(R - R_0)^3 \tag{6.6} \]
satisfies the criteria. What is interesting is that although such a theory does not have any perturbative propagating degrees of freedom, it nevertheless admits the (A)dS Schwarzschild black hole solution. This particular aspect of the theory is similar to three-dimensional Einstein gravity with a cosmological constant.

For general case with non-vanishing \( f'(R_0) \) and \( f''(R_0) \), the ghost-free conditions are
\[ f'(R_0) > 0, \quad R_0 f''(R_0) > 0. \tag{6.7} \]

The tachyon-free Breitenlohner-Freedman (BF) condition in AdS is given by
\[ m^2 \geq \frac{D-1}{4D} R_0, \tag{6.8} \]

We now examine the stability of some of our \( f(R) \) gravities in the (A)dS vacua. The first example to consider is the quadratic Ricci-scalar action (2.24). We have demonstrated in section 5 that this is equivalent to (5.5). We choose a convention that the AdS fixed point for the superpotential (5.7) occurs at \( \phi = 0 \), which implies that
\[ \sigma = -\frac{(D-4)(5D-2)}{3D(D+2)}. \tag{6.9} \]

Expanding the scalar potential \( V \) around \( \phi = 0 \), we find that
\[ V = \frac{D-2}{D-1} R_0 + \frac{1}{2} M^2 \phi^2 + \cdots, \tag{6.10} \]
where
\[ \alpha R_0 = \frac{4(D-1)^2}{3D(D+2)}, \quad M^2 = \frac{(D-4)(3D^2-4D+4)R_0}{16(D-1)^3}. \] (6.11)
It follows from (2.26) that the reality condition requires that \( \alpha < 0 \), and hence the vacuum is AdS. Furthermore, we have
\[ M^2 - \frac{D-1}{4D} R_0 = -\frac{(D^2+2)^2}{16D(D-1)^3} R_0 > 0, \] (6.12)
hence the BF bound is satisfied. Thus we find that the scalar/gravity theory is both tachyon and ghost free in the AdS vacuum with \( R = R_0 \). Now let us examine the corresponding \( f(R) \) gravity. For \( D \neq 4 \), there are two AdS vacua, namely
\[ R = R_0 \quad \text{and} \quad R = \tilde{R}_0 \equiv \frac{D-4}{3(D+2)\alpha}, \] (6.13)
where \( R_0 \) is given by (6.11). The \( R = R_0 \) vacuum is “BPS”, whilst the \( R = \tilde{R}_0 \) one is not. It is easy to verify that \( F(R_0) = 1 \) and \( R_0 f''(R_0) = 2\alpha R_0 > 0 \). The \( m^2 \) calculated from (6.4) is exactly the same as \( M^2 \). The situation is quite different for the non-“BPS” vacuum with \( R = R_0 \). Although there is no tachyon, the spin-2 graviton is a ghost field. If we reverse the overall sign of the action, the spin-0 trace mode becomes a ghost. That the “BPS” vacuum is stable whilst the non-“BPS” vacuum is unstable is consistent with our expectation.

The second example we would like to examine is the two theories given in section 2.2. Both theories (2.6) and (2.11) can give rise to the same cosmological solution (2.8) which describes an evolution from the flat spacetime to the inflationary de Sitter vacuum. It is clear that in the flat region, (2.6) is a good perturbative theory. However, in the region where \( R = R_0 \equiv 48\beta^2 \), the theory becomes singular with divergent \( F(R_0) \) and \( f''(R_0) \). On the other hand, theory (2.11) is opposite. In the \( R = 0 \) region, the theory is singular, but it is well behaved in the \( R = R_0 \) region, with \( F(R_0) \) and \( f''(R_0) \) given in (2.12), and hence the vacuum is ghost free provided that \( \beta \sigma_2 > 0 \). Note that the same conclusion also holds for the pair of theories (3.42) and (3.21) in \( D = 4 \). Furthermore the BF bound for the theory (3.42) in the “BPS” AdS vacuum is satisfied, and hence the theory is both ghost and tachyon free.

Thus we see an interesting phenomenon in our \( f(R) \) gravities, which we have mentioned in section 2. For a giving Killing spinor equation, and hence one “BPS” domain wall or cosmological solution, there can be two \( f(R) \) gravities. For a solution that connects two different AdS vacua with \( \Lambda_+ \) and \( \Lambda_- \), one \( f(R) \) theory is well-defined in the \( \Lambda_+ \) vacuum with no ghost and tachyon, but becomes singular at \( \Lambda_- \), and vice versa for the other \( f(R) \).

The third example we consider is the five-dimensional \( f(R) \) theory (3.39) that admits the AdS wormhole solution (3.38). The AdS wormhole is symmetric with both AdS boundaries having the same \( R_0 = -500 \). It is clear that the theory associated with \( \sigma_2 \) is singular. On the other hand, the theory associated with \( \sigma_1 \), namely
\[ f(R) = \sigma_1\sqrt{-(R + 200)^3}, \] (6.14)
is well defined in the vacuum. We have $F(R_0) = 15\sqrt{3} \sigma_1$ and $R_0 f''(R_0) = 25\sqrt{3} \sigma_1/2$, and hence the vacuum fluctuation is ghost free for positive $\sigma_1$. The mass square of the spin-0 mode is given by $m^2 = -100$, which precisely saturates the BF bound. This theory is different from the second example, in that both AdS boundaries have the same $R_0$ and hence one theory is needed instead of having to have both theories to patch different regions.

The last example we shall examine is the smooth Randall-Sundrum II solutions discussed in section 3. For the simpler case (3.31) in $D = 5$, both $F(R_0)$ and $R_0 f''(R_0)$ are given in (3.35). They cannot be both positive and hence the AdS vacuum suffers from having a ghost field. (It is easy to obtain $m^2 = 300$ for the scalar mode, and hence it is not a tachyon.) Of course, this is only one example of many possible RS II solutions, and it is of interest to investigate whether such a ghost problem of the “BPS” RS II in $f(R)$ gravities is generic or not. Furthermore, we have imposed that $f''(R_0)$ be finite, which is not entirely clear to be necessary.

7 Conclusion

In this paper, we follow the procedure outlined in [21–23] and obtain the condition on the subclass of $f(R)$ theories that admit Killing spinor equations. We present many examples of such $f(R)$ gravities. One advantage of our theories is that the Killing spinor equations reduce the fourth-order Einstein equations for the domain wall and FLRW ansatze to very simple first-order equations, and hence exact solutions can be constructed.

For domain wall solutions, we find exact smooth examples that describe the RS II scenario, AdS wormholes and the RG flow from the IR to the UV. In all these solutions, the metric runs from one AdS to another. This is very different from other higher-derivative theories such as Lovelock gravities with the Gauss-Bonnet term, which also have multiple AdS vacua, but have no known flow running from one to the other. Our examples demonstrate that $f(R)$ is a fruitful arena to investigate and apply the AdS/CFT correspondence.

Rich classes of exact and smooth cosmological solutions also emerge in our $f(R)$ gravities. We find a semi-realistic cosmological solution that evolves from an inflationary starting point to end with an ever-lasting expanding universe with a much smaller cosmological constant. We also find a pre-big bang model where a flat universe bursts into inflation by the non-perturbative effect of the higher-order curvature terms. In addition, we find smooth bouncing and crunching universes.

Since the cosmological evolution in our $f(R)$ gravities is solely governed by the equation (4.6), it is a matter of finding the right profile of $W(R)$ in order to fit the observational data. However, one technical drawback is that for a given $W$, the $f(R)$ is not determined directly, but via a second-order linear differential equation, which may not have a close-form solution. Nevertheless we have obtained many explicit examples in this paper. Classically, it can be argued that this is not essential since $W$ gives all the information. At the quantum level, the exact form of $f(R)$ is likely to become much more important. It is of interest to investigate whether it is possible to compute the quantum effect on the information given by the $W$ alone.
The full analysis of the stability of our “BPS” solutions is beyond the scope of this paper. Instead, we investigate the stability of the (A)dS vacua that these metrics connect to. We study the linearized gravity around the (A)dS vacua. We adopt the gauge that was used previously for studying critical gravities [44, 45]. In general, $f(R)$ gravity consists of one massless spin-2 and one-massive spin-0 modes. We also obtain the condition for which the spin-0 mode decouples so that the spectrum is identical to that of Einstein gravity. More exotic situation can arise where a theory has no propagating degree of freedom, yet it admits the Schwarzschild (A)dS black hole as a solution, analogous to Einstein gravity with a cosmological constant in three dimensions. We obtain the conditions for $f(R)$ theories to be absent from the ghost and tachyon fields, and give a detail analysis for a few examples.

There is an intriguing phenomenon in our $f(R)$ gravities. As we have mentioned, for a giving $W$ in the Killing spinor equations, and hence one “BPS” domain wall or cosmological solution, there can be two $f(R)$ gravities. For the solution that connects two different AdS vacua with $\Lambda_+$ and $\Lambda_-$. We find examples that one $f(R)$ theory is well-defined in the $\Lambda_+$ vacuum with no ghost and tachyon, but becomes singular at $\Lambda_-$, and vice versa for the other $f(R)$. This suggests that there can exist multiple classical $f(R)$ gravities that give the same full cosmological evolution; however, different stages of the evolution may select different specific theories for the quantum description. This is similar to the common phenomenon in differential geometry that a typical manifold requires multiple different but overlapping coordinate patches in order to cover it.

It was shown in [47, 48] that non-supersymmetric theories that admit Killing spinor equations can be pseudo-supersymmetrized by introducing pseudo fermionic partners. In these theories, it can be shown that the Lagrangian is invariant under the pseudo-supersymmetric transformation rules up to the quadratic order in fermions. This suggests that there should be pseudo-supersymmetric versions of our $f(R)$ gravities. It is of great interest to construct such $f(R)$ pseudo-supergravities.

To conclude, our construction of $f(R)$ gravities that admit Killing spinor equations allows us to find exact “BPS” domain wall and FLRW cosmological solutions with varying Ricci scalar $R$. The significance of these solutions is that they explore the function $f(R)$ in contrast to the previously known solutions with fixed $R$. This opens a new door to study both the AdS/CFT correspondence and cosmology in the context of $f(R)$ gravities. Our construction is based on $f(R)$ theories in the metric formalism, and hence it is natural to extend our discussion to the Palatini formalism where both the metric and the connection are assumed to be independent variables. It is also of interest to investigate whether the Killing spinor equations of our $f(R)$ gravities can be extended to include matter.

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