How Likely A Coalition of Voters Can Influence A Large Election?

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Abstract
For centuries, it has been widely believed that the influence of a small coalition of voters is negligible in a large election. Consequently, there is a large body of literature on characterizing the asymptotic likelihood for an election to be influenced, especially by the manipulation of a single voter, establishing an $O\left(\frac{1}{\sqrt{n}}\right)$ upper bound and an $\Omega\left(\frac{1}{n^{67}}\right)$ lower bound for many commonly studied voting rules under the i.i.d. uniform distribution, known as Impartial Culture (IC) in social choice, where $n$ is the number of voters.

In this paper, we extend previous studies in three aspects: (1) we consider a more general and realistic semi-random model, where a distribution adversary chooses a worst-case distribution and then a data adversary modifies up to $\psi$ portion of the data, (2) we consider many coalitional influence problems, including coalitional manipulation, margin of victory, and various vote controls and bribery, and (3) we consider arbitrary and variable coalition size $B$. Our main theorem provides asymptotically tight bounds on the semi-random likelihood of the existence of a size-$B$ coalition that can successfully influence the election under a wide range of voting rules. Applications of the main theorem and its proof techniques resolve long-standing open questions about the likelihood of coalitional manipulability under IC, by showing that the likelihood is $\Theta\left(\min\left\{\frac{B}{\sqrt{n}}, 1\right\}\right)$ for many commonly studied voting rules.

The main technical contribution is a characterization of the semi-random likelihood for a Poisson multinomial variable (PMV) to be unstable, which we believe to be a general and useful technique with independent interest.
1 Introduction

For centuries, it has been widely believed that the influence of a small coalition of voters is negligible in a large election. For example, Condorcet commented in 1785, that “In single-stage elections, where there are a great many voters, each voter’s influence is very small” [Con85] (see the translation by McLean and Hewitt [MH94, pp.245]). As another example, Hegel commented in The Philosophy of Right in 1821, that “the casting of a single vote is of no significance where there is a multitude of electors” (see the translation and comments by Buchanan [Buc74]).

Various types of influence were investigated in the literature. For example, coalitional manipulation (CM for short) refers to the phenomenon in which a coalition of voters have incentive to misreport their preferences, so that the winner is more favorable to all of them. The margin of victory (MoV for short) refers to the phenomenon in which a coalition of voters have the power to change the winner by voting differently, regardless of their incentives.

Accurately measuring the influence of voters is highly significant and plays a central role in many other studies. A small influence can be positive news or negative news, depending on the context. For example, a small influence is desirable under various robustness measures, such as decisiveness [Gil77], strategy-proofness [Gib73, Sat75], privacy [LLXZ20]. On the other hand, a large influence is desirable in justifying the voting power of small groups of voters [BI68], and therefore would encourage voter turnout [Dow57, RO68].

Building a realistic model to accurately measure the influence of voters turns out to be a highly challenging mission. Clearly, a sensible and informative measure cannot be completely based on the worst-case analysis. Take coalitional manipulation with a single voter for example: for many voting rules, there always exists a situation where a voter can and has incentive to change the election outcome by changing his/her vote, due to the celebrated Gibbard-Satterthwaite theorem [Gib73, Sat75]. Similarly, unless the voting rule always chooses the same alternative as the winner, there exists a situation where a single voter is highly influential, i.e., MoV = 1.

Consequently, there is a large body of literature on understanding the influence of voters using average-case analysis, especially for coalitiional manipulation. Since the pioneering works by Pazner and Wesley [PW78] and by Peleg [Pel79] in the 1970’s, much of the literature has focused on characterizing the asymptotic likelihood for randomly generated votes to be coalitional manipulable, as the number of voters $n \to \infty$. Previous work has established an $O(\frac{1}{\sqrt{n}})$ upper bound for many commonly-studied voting rules by a coalition of constantly many manipulators, under the i.i.d. uniform distribution over all linear orders, known as the Impartial Culture (IC) in social choice. On the other hand, the quantitative Gibbard-Satterthwaite theorems, e.g., [FKKN11, MR15], established an $\Omega(\frac{1}{\sqrt{n}})$ lower bound for a single manipulator under all voting rules that are constantly far away from dictatorships, under IC.

To the best of our knowledge, the only known matching lower bound is the $\Omega(\frac{1}{\sqrt{n}})$ bound for a single manipulator under the plurality rule [Sh102a]. Little is known for arbitrary sizes of the coalition, other means of coalitional influence, such as margin of victory (MoV), vote controls, and bribery [FR16], and/or other statistical models for generating votes. Specifically, IC has been widely criticized of being unrealistic (see, e.g., [Nur99, p. 30], [Geh06, p. 104], and [LK07]), which means that the conclusions drawn under IC may only have limited implications in practice. See Section 6 for more discussions. Therefore, the following question remains largely open.

**How likely a coalition of voters can influence large elections under realistic models?**

The importance of answering this question has been widely recognized, as Pattanaik [Pat78, p.187] discussed for (coalitional) manipulation soon after the discovery of the Gibbard-Satterthwaite theo-
rem: “For, if the likelihood of such strategic voting is negligible, then one need not be unduly worried about the existence of the possibility as such.”

1.1 Our Contributions

We make three main contributions to address the question on the likelihood of coalitional influence: the main conceptual contribution is a novel and general semi-random model for generating agents’ votes, the main modeling contribution is the conversion of commonly-studied coalitional influence problems to the PMV-instability problems, and the main technical contribution is a powerful theorem for solving general PMV-instability problems, which gives us accurate characterizations of the likelihood of coalitional influence.

The new semi-random model. Our model has two parameters (Π, ψ), where Π is a set of distributions over all rankings over m alternatives and ψ ∈ [0, 1], and two adversaries: the distribution adversary and the data adversary. In our model, the collection of votes for n agents, called a profile, is generated in the following three steps: first, the distribution adversary chooses a (worst-case) distribution for each agent, whose collection is denoted by istration adversary ̄π ∈ Πn; then, agents’ pre 传媒 profile P is generated from ̄π independently; finally, the data adversary modifies P up to ψn votes to obtain a profile P′, denoted by P′ ∼ P.

Given a voting rule r, a budget B ≥ 0, and a profile P, an influence problem is modeled by a binary function X(r, P, B). That is, X(r, P, B) = 1 if there exits a group of no more than B voters who are influential; otherwise X(r, P, B) = 0. Take coalitional manipulation for example, where X = CM. CM(r, P, B) = 1 if and only if there exist a coalition of no more than B voters who have incentive to misreport their preferences to improve the winner.

Given Π and ψ ∈ [0, 1], the max-semi-random likelihood of X under r with n agents and budget B, denoted by XmaxΠ,ψ(r, n, B), is defined as:

\[ X_{Π,ψ}^{\text{max}}(r, n, B) \triangleq \sup_{P \sim ̄π} \Pr_{P \sim ̄π} \left( \exists P' \sim P, \psi \right) \text{ s.t. } X(r, P, B) = 1, \]  

(1)

where P′ ∼ P means that P′ can be obtained from P by modifying no more than ψn votes.

The PMV-instability problem. We show in Section 3 that many coalitional influence problems, including CM and MoV, under commonly studied voting rules can be solved by analyzing the positions of the histogram of the profile w.r.t. certain polyhedra. In our semi-random model, after the distribution adversary choses the distributions, the histogram of the random profile is a Poisson multivariate variable (PMV), whose general version is defined as follows.

Definition 1 (Poisson multivariate variable (PMV)). Given n, q ∈ N and a vector ̄π = (π1, . . . , πn) of n distributions over {1, . . . , q}, an (n, q)-PMV is denoted by Xπ, which represents the histogram of n independent random variables whose distributions are {π1, . . . , πn}, respectively.

We model the goal of influence as a PMV-instability setting, where we are given a source polyhedron ̃Hs, a target polyhedron ̃Ht, a set of vote operations ̃O ⊆ Rq, each of which represents a type of changes to the histogram by the influencers, and a cost vector ̃c for the vote operations in ̃O. Then, the semi-random likelihood of many coalitional influence problems under commonly-studied voting rules can be modeled as PMV-instability problems (Definition 4), where we are additionally given n and B, and we are interested in the semi-random likelihood for a PMV to be unstable, in the sense that the data adversary can change the PMV to be in a set Uα,B, which is
the subset of $H_S$ whose members can further be made into $H_T$ by performing vote operations in $\mathcal{O}$ under budget constraint $B$. More precisely, we are interested in characterizing

$$\sup_{\vec{x} \in \Pi} \Pr \left( \exists \vec{x} \sim \vec{X}_{\vec{x}} \text{ s.t. } \vec{x} \in \mathcal{U}_{n,B} \right)$$

Characterizing the semi-random likelihood. We prove the following characterization of the PMV-instability problem, which is our main technical contribution.

**Theorem 1 (Semi-random instability of PMV for $\psi = 0$, informally put).** For any PMV-instability problem, a large class of $\Pi$, any $n \in \mathbb{N}$, and any $B \geq 0$, the solution to the PMV-instability problem falls into one of the following four cases:

$$0, \exp(-\Theta(n)), \text{ phase transition at } \Theta(\sqrt{n}), \text{ and phase transition at } \Theta(n)$$

In the phase-transition-at-$\Theta(\sqrt{n})$ case, the likelihood increases to its maximum poly-$1(n)$ before $B = \Theta(\sqrt{n})$. In the phase-transition-at-$\Theta(n)$ case, the likelihood increases from $\exp(-\Theta(n))$ to its maximum poly-$1(n)$ around $B = \Theta(n)$. The formal statement of Theorem 1 also characterizes conditions and asymptotically tight bounds for the four cases. The $\psi > 0$ case is addressed by Theorem 5 in Appendix D.4.

Applications. Theorem 1 and Theorem 5 can be easily applied to accurately bound the semi-random likelihood of coalitional influence in a wide range of settings of interest to resolve long-standing open questions. In this paper, we present three examples.

**First,** in Theorem 2 we prove that for CM and MoV, many commonly studied voting rules $r$, a large class of $\Pi$ including IC, any coalition size $B \geq 1$, and any sufficiently large $n$,

$$\tilde{X}_{\Pi,0}^{\max}(r, n, B) = \Theta \left( \min \left\{ \frac{B}{\sqrt{n}}, 1 \right\} \right)$$

A straightforward application of Theorem 2 to CM under IC (Corollary 1) not only closes the $\left( \Omega(\frac{1}{m\sqrt{n}}), O(\frac{1}{\sqrt{n}}) \right)$ gap for these rules with $B = 1$ in previous work, but also provides asymptotically tight bounds on the likelihood of CM for every $B \geq 1$: roughly speaking, each additional manipulator (up to $O(\sqrt{n})$) increases the likelihood of success by $\Theta(\frac{1}{\sqrt{n}})$ under IC.

**Second,** in Theorem 3 we prove an $O \left( \min \left\{ \frac{B}{\sqrt{n}}, 1 \right\} \right)$ upper bound on the likelihood of many types of coalitional influence, including CM and MoV, for all generalized scoring rules (GSRs) [XC08a], which includes all voting rules mentioned in this paper. This supersedes all previous upper bounds we are aware of and extends them to arbitrary $B \geq 1$ and a more general (semi-random) model.

**Third,** we propose and investigate a new coalitional influence problem called coalitional manipulation for the loser, denoted by CML, which requires that a coalition of voters are incentivized to misreport their preferences in order to make the loser win. We prove in Theorem 4, that for any integer positional scoring rules with $m \geq 3$ (except veto), any $\Pi$ from a large class, and any sufficiently large $n$ and $B$,

$$\tilde{CML}^{\max}_{\Pi,0}(r, \tilde{s}, n, B) = \Theta \left( \min \left\{ \frac{B}{\sqrt{n}}, 1 \right\}^{m-1} \right)$$

While CML may be of independent interest, the main purpose of this theorem is to illustrate that the likelihood of coalitional influence can be much smaller than $\Theta(\frac{B}{\sqrt{n}})$ or even $\Omega(\frac{1}{m\sqrt{n}})$, can be non-linear in $B$, the degree of polynomial can depend on $m$, and each additional budget is marginally more powerful. We are not aware of a similar phenomenon in the literature.
2 Preliminaries and Our Semi-Random Model

For any \( q \in \mathbb{N} \), let \([q] = \{1, \ldots, q\}\). Let \( \mathcal{A} = [m] \) denote a set of \( m \geq 3 \) alternatives. Let \( \mathcal{L}(\mathcal{A}) \) denote the set of all linear orders over \( \mathcal{A} \). Let \( n \in \mathbb{N} \) denote the number of agents (voters). Each agent uses a linear order \( R \in \mathcal{L}(\mathcal{A}) \) to represent his or her preferences, called a vote, where \( a \succ_R b \) or \( \{a\} \succ_R \{b\} \) means that the agent prefers alternative \( a \) to alternative \( b \). The vector of \( n \) agents’ votes, denoted by \( P \), is called a (preference) profile, sometimes called an \( n \)-profile. The set of \( n \)-profiles for all \( n \in \mathbb{N} \) is denoted by \( \mathcal{L}(\mathcal{A})^* = \bigcup_{n=1}^{\infty} \mathcal{L}(\mathcal{A})^n \). A fractional profile is a profile \( P \) coupled with a possibly non-integer and/or negative weight vector \( \vec{\omega} = (\omega_R : R \in P) \in \mathbb{R}^n \) for the votes in \( P \). Sometimes the weight vector is omitted when it is clear from the context.

For any (fractional) profile \( P \), let \( \text{Hist}(P) \in \mathbb{R}^m_0 \) denote the anonymized profile of \( P \), also called the histogram of \( P \), which contains the total weight of every linear order in \( \mathcal{L}(\mathcal{A}) \) according to \( P \). An irresolute voting rule \( \vec{r} : \mathcal{L}(\mathcal{A})^* \to (2^A \setminus \{\emptyset\}) \) maps a profile to a non-empty set of winners in \( \mathcal{A} \). A resolute voting rule \( r \) is a special irresolute voting rule that always chooses a single alternative as the (unique) winner. Often a resolute rule is obtained from an irresolute rule by applying a tie-breaking mechanism, e.g., lexicographic tie-breaking, which chooses the co-winner with the smallest index as the unique winner.

### Integer positional scoring rules

An (integer) positional scoring rule \( \vec{r}_\vec{s} \) is characterized by an integer scoring vector \( \vec{s} = (s_1, \ldots, s_m) \in \mathbb{Z}^m \) with \( s_1 \geq s_2 \geq \cdots \geq s_m \) and \( s_1 > s_m \). For any alternative \( a \) and any linear order \( R \in \mathcal{L}(\mathcal{A}) \), we let \( \vec{s}(R, a) = s_i \), where \( i \) is the rank of \( a \) in \( R \).

Commonly studied integer positional scoring rules include plurality, which uses the scoring vector \((1,0,\ldots,0)\), Borda, which uses the scoring vector \((m-1,m-2,\ldots,0)\), and veto, which uses the scoring vector \((1,1,1,0)\).

We recall the definition of generalized scoring rules (GSRs) \([XC08a]\) based on separating hyperplanes \([XC09, MPR13]\) as follows.

**Definition 2.** A generalized scoring rule (GSR) \( r \) is defined by (1) a set of \( K \geq 1 \) hyperplanes \( \vec{H} = (\vec{h}_1, \ldots, \vec{h}_K) \in (\mathbb{R}^m_0)^K \) and (2) a function \( g : \{+,-,0\}^K \to \mathcal{A} \). For any profile \( P \), we let \( r(P) = g(\text{Sign}_\vec{H}(\text{Hist}(P))) \), where \( \text{Sign}_\vec{H}(\vec{x}) = (\text{Sign}(\vec{h}_1 \cdot \vec{x}), \ldots, \text{Sign}(\vec{h}_K \cdot \vec{x})) \) represents the signs of \( \vec{h}_1 \cdot \vec{x}, \ldots, \vec{h}_K \cdot \vec{x} \). When \( \vec{H} \in (\mathbb{Z}^m)^K \), \( r \) is called an integer GSR (int-GSR).

**Example 1 (Borda as a GSR).** Let \( m = 3 \). Borda with lexicographic tie-breaking is a GSR with \( K = m \) and \( \vec{H} = \{\vec{h}_1, \vec{h}_2, \vec{h}_3\} \) defined as follows.

\[
\begin{align*}
\vec{h}_1 &= (1, 2, -1, -2, 1, -1) \\
\vec{h}_2 &= (2, 1, 1, -1, -1, -2) \\
\vec{h}_3 &= (1, -1, 2, 1, -2, -1)
\end{align*}
\]

Let \( \vec{x} = (x_{123}, x_{132}, x_{213}, x_{231}, x_{312}, x_{321}) \) denote the histogram of a profile, where \( x_{123} \) represents the number of \([1 \succ 2 \succ 3]\) votes. It follows that \( \vec{h}_1 \cdot \vec{x} \) is the Borda score of alternative 1 minus the Borda score of 2 in the profile; \( \vec{h}_2 \cdot \vec{x} \) is the Borda score of alternative 1 minus the Borda score of 3 in the profile; and \( \vec{h}_3 \cdot \vec{x} \) is the Borda score of alternative 2 minus the Borda score of 3 in the profile. The \( g \) function chooses the winner based on \( \text{Sign}_\vec{H}(\text{Hist}(P)) \) and the tie-breaking mechanism.

**Coalitional influence problems.** Let \( r \) be a resolute rule, \( P \) be a preference profile, and \( B \) \( \geq 0 \) be a budget. Coalitional manipulation (CM) is defined by a binary function \( \text{CM}(r, P, B) = 1 \) if and only if there exists \( P' \subseteq P \) with \(|P'| \leq B \) and \( P' \) with \(|P'| = |P| \) such that for all \( R \in P' \), \( r(P - P' + P^*) >_R r(P) \). The margin of victory (MoV) is a function \( \text{MoV}(r, P, B) \).
such that \( \text{MoV}(r, P, B) = 1 \) if and only if no more than \( B \) voters can coalesionally change the winner (regardless of their preferences and incentives).

**Semi-random likelihood of coalesional influence.** As discussed in the Introduction, given a coalesional influence problem \( X \), a set \( \Pi \) of distributions over \( \mathcal{L}(A) \), \( \psi \in [0,1] \), a voting rule \( r \), a number of voters \( n \in \mathbb{N} \), and a budget constraint \( B \geq 0 \), the max-semi-random likelihood of \( X \) (max-semi-random \( X \) for short), denoted by \( \tilde{X}^\text{max}_\Pi,\psi(r, n, B) \), is defined in Equation (1).

In the main text we focus on presenting max-semi-random likelihood of CM for Borda, where \( \psi = 0 \). See Appendix B.1 for definitions of some other commonly studied voting rules (which are GSRs), i.e., ranked pairs, Schulze, maximin, Copeland, and STV, and Appendix B.2 for the definitions of some other commonly-studied coalesional manipulation problems, i.e., constructive/destructive control by adding/deleting votes and bribery. Many results in the main text apply to these rules and coalesional influence problems, as stated in their full versions in the Appendix.

**Assumptions on \( \Pi \).** Throughout the paper, we make the following two assumptions on \( \Pi \): (1) **strict positiveness**, which means that there exists a constant \( \epsilon > 0 \) such that the probabilities over all rankings in all \( \pi \in \Pi \) are larger than \( \epsilon \), and (2) **closedness**, which means that \( \Pi \) is a closed set in the Euclidean space.

Clearly, IC corresponds to \( \Pi = \{\pi_\text{uni}\} \), where \( \pi_\text{uni} \) is the uniform distribution over \( \mathcal{L}(A) \). Let us take a look at another example of \( \Pi \) and its corresponding semi-random likelihood.

**Example 2 (Semi-random CM under Borda).** Let \( X = \text{CM} \) and \( r = \text{Borda} \) with lexicographic tie-breaking. Let \( A = \{1, 2, 3\} \) and \( \Pi = \{\pi^1, \pi^2\} \), where \( \pi^1 \) and \( \pi^2 \) are distributions in Table 1.

When \( n = \frac{2}{4} \) and \( B = 1 \), we have

\[
\sup_{\tilde{\pi} \in \{\pi^1, \pi^2\}} \text{Pr}_{P \sim \tilde{\pi}} \text{CM}(\text{Borda}, P).
\]

That is, the adversary has four choices of \( \tilde{\pi} \), i.e., \( \{(\pi^1, \pi^1), (\pi^1, \pi^2), (\pi^2, \pi^1), (\pi^2, \pi^2)\} \).

Each \( \tilde{\pi} \) leads to a distribution over the set of all profiles of two agents, i.e., \( \mathcal{L}(A)^2 \). As we will see later in Example 5, for every sufficiently large \( n \), \( \tilde{X}^{\text{max}}_{\Pi,0}(\text{Borda}, n, 1) = \Theta\left(\frac{1}{\sqrt{n}}\right)\).

### 3 Coalitional Influence Problems as PMV-Instability Problems

We start with an example of modeling CM with \( \psi = 0 \) as systems of linear inequalities, which motivates the study of the more general PMV-instability problem (Definition 4).

**Example 3.** Let \( A = \{1, 2, 3\} \) and let \( r \) be Borda with lexicographic tie-breaking. Let \( \mathcal{U}^{1 \rightarrow 2}_{n,B} \) denote the histograms of all \( n \)-profiles that satisfy the following conditions: (1) the winner before manipulation is 1, (2) a coalition of no more than \( B \) manipulators are motivated to change the winner to 2 by casting different votes. Let \( o_1 \) (respectively, \( o_2 \)) denote the number of voters who change their votes from \( \{2 \succ 1 \succ 3\} \) (respectively, \( \{3 \succ 2 \succ 1\} \)) to \( \{2 \succ 3 \succ 1\} \).

Then, the histogram \( \bar{x} \) of an \( n \)-profile is in \( \mathcal{U}^{1 \rightarrow 2}_{n,B} \) if and only if there exists an integer vector \( \bar{o} = (o_1, o_2) \) such that \( (\bar{x}, \bar{o}) \) is a feasible solution to the following linear program, where the objective
is omitted because only the feasibility matters.

\[
\begin{align*}
-x_{123} - 2x_{132} + x_{213} + 2x_{312} - x_{312} + x_{321} & \leq 0 \\
-2x_{123} - x_{132} - x_{213} + x_{312} + x_{312} + 2x_{321} & \leq 0 \\
x_{123} + 2x_{132} - (x_{213} - o_1) - 2(x_{231} + o_1 + o_2) + x_{312} - (x_{321} - o_2) & \leq -1 \\
-x_{123} + x_{132} - 2(x_{231} - o_1) - (x_{231} + o_1 + o_2) + 2x_{312} + (x_{321} - o_2) & \leq 0 \\
o_1 & \leq 0, -o_2 \leq 0, -(x_{213} - o_1) \leq 0, -(x_{321} - o_2) \leq 0, o_1 + o_2 \leq B
\end{align*}
\]

1 wins before manipulation

\[
\begin{align*}
-x_{123} - 2x_{132} + x_{213} + 2x_{312} - x_{312} + x_{321} & \leq 0 \\
-2x_{123} - x_{132} - x_{213} + x_{312} + x_{312} + 2x_{321} & \leq 0 \\
x_{123} + 2x_{132} - (x_{213} - o_1) - 2(x_{231} + o_1 + o_2) + x_{312} - (x_{321} - o_2) & \leq -1 \\
-x_{123} + x_{132} - 2(x_{231} - o_1) - (x_{231} + o_1 + o_2) + 2x_{312} + (x_{321} - o_2) & \leq 0 \\
-o_1 & \leq 0, -o_2 \leq 0, -(x_{213} - o_1) \leq 0, -(x_{321} - o_2) \leq 0, o_1 + o_2 \leq B
\end{align*}
\]

2 wins after manipulation

In Example 3, the effect of each manipulator can be modeled by its changes to the histogram, and the manipulators aim at manipulating vectors in a source polyhedron, which represents 1 being the winner, into a target polyhedron, which represents 2 being the winner, under the budget constraint $B$. This motivates us to define the general PMV-instability setting as follows.

**Definition 3 (PMV-instability setting).** In a PMV-instability setting $S \triangleq (\mathcal{H}_S, \mathcal{H}_T, \mathcal{O}, \bar{c})$,

- $\mathcal{H}_S$ and $\mathcal{H}_T$ are polyhedra in $\mathbb{R}^n$ for some $q \in \mathbb{N}$, the subscript $S$ and $T$ represent “source” and “target”, respectively. For $Y \in \{S, T\}$, let $\mathcal{H}_Y \triangleq \left\{ \bar{x} \in \mathbb{R}^q : A_Y \times (\bar{x})^\top \leq (b_Y)^\top \right\}$, where $A_Y$ is an integer matrix of $q$ columns;

- $\mathcal{O} \subseteq \mathbb{R}^q$ is a finite set of vote operations, and let $\bar{O}$ denote the $|\mathcal{O}| \times q$ matrix whose rows are the vectors in $\mathcal{O}$;

- $\bar{c} \in \mathbb{R}^{|\mathcal{O}|}$ is the cost vector for the vote operations in $\mathcal{O}$.

W.l.o.g., we assume $\bar{c} > \bar{0}$ and the minimum cost of a single operation is 1, i.e., $\min_{i \leq q} [\bar{c}]_i = 1$. Given a PMV-instability setting $S$, $n \in \mathbb{N}$, and a budget $B \geq 0$, we let $U_{n, B}$ denote the set of non-negative size-$n$ integer vectors that represent unstable histograms w.r.t. vote operations in $\mathcal{O}$ and budget $B$. That is,

\[
U_{n, B} \triangleq \left\{ \bar{x} \in \mathcal{H}_S \cap \mathbb{Z}_n^q : \bar{x} \cdot \bar{1} = n \quad \text{and} \quad \exists \bar{o} \in \mathbb{Z}_q^{|\mathcal{O}|} \quad \text{s.t.} \quad \bar{c} \cdot \bar{o} \leq B \quad \text{and} \quad \bar{x} + \bar{o} \times \bar{O} \in \mathcal{H}_T \right\}
\]

**Example 4.** In the setting of Example 3, $q = m! = 6$. $U_{n, B}^{12}$ is the set of unstable histograms of the PMV-instability setting where $\mathcal{H}_S$ (respectively, $\mathcal{H}_T$) is the polyhedron that represents 1 (respectively, 2) being the winner, $\mathcal{O} = \{(0, 0, -1, 1, 0, 0), (0, 0, 0, 1, 0, -1)\}$ (the indices to rankings are the same as in Example 1), $\bar{O} = \begin{bmatrix} 0, 0, -1, 1, 0, 0 \\ 0, 0, 0, 1, 0, -1 \end{bmatrix}$, $\bar{c} = (1, 1)$.

Notice that Examples 3 and 4 only capture the situations where alternative 1 is manipulated to alternative 2. Similarly, we can define $U_{n, B}^{13}$, $U_{n, B}^{21}$, $U_{n, B}^{23}$, $U_{n, B}^{31}$, and $U_{n, B}^{32}$. Let $\mathcal{M}$ denote the set of the six PMV-instability settings, and define

\[
U_{n, B}^{\mathcal{M}} \triangleq U_{n, B}^{12} \cup U_{n, B}^{13} \cup U_{n, B}^{21} \cup U_{n, B}^{23} \cup U_{n, B}^{31} \cup U_{n, B}^{32}
\]

It follows that

\[
\overline{\text{CM}}_{\Pi, 0}^{\max} (\text{Borda}, n, B) = \sup_{\bar{\pi} \in \Pi_n} \Pr \left( \bar{X}_{\bar{\pi}} \in U_{n, B}^{\mathcal{M}} \right)
\]

It turns out that many other coalitional influence problems can be modeled by the union of multiple PMV-instability settings (formally called the PMV-multi-instability setting in Definition 9 in Appendix) as proved in the following lemma.
Lemma 1 (Coalitional Influence as PMV-instability). For any $X \in \{\text{CM}, \text{MoV}\}$ and any GSR $r$, there exist multiple PMV-instability settings $M = \{S^i : i \leq I\}$ such that for every $n$-profile $P$ and every $B \geq 0$, $X(r, P, B) = 1$ if and only if $\text{Hist}(P) \in U_{n,B}^M$. The full statement of the lemma (which covers other coalitional influence problems) and its proof can be found in Appendix C.2. In light of (2) and Lemma 1, we propose the PMV-instability problem as follows.

Definition 4 (PMV-instability problem). Given a PMV-instability setting $S = \langle H_S, H_T, O, \vec{c} \rangle$, a set $\Pi$ of distributions over $[q]$, $\psi \in [0, 1]$, $n \in \mathbb{N}$, and $B \geq 0$, we are interested in characterizing

\[
\text{max-semi-random instability: } \sup_{\vec{\pi} \in \Pi^n} \Pr \left( \exists \vec{x} \sim \vec{\pi} \ \text{s.t.} \ \vec{x} \in U_{n,B} \right)
\]

That is, the max-semi-random instability is the upper bound on the probability for the PMV to be unstable after the data adversary makes up to $\psi$ fraction of changes, when the underlying probabilities $\vec{\pi}$ are adversarially chosen from $\Pi^n$. Notice that when $\vec{x}, \vec{x}' \in \mathbb{R}_{\geq 0}^q$, $\vec{x} \sim \vec{\pi}$ holds if and only if $|\vec{x}|_1 = |\vec{x}'|_1$ and $|\vec{x} - \vec{x}'|_1 \leq 2\psi|\vec{x}'|_1$.

4 Solving The PMV-Instability Problem

Let us start with some high-level intuitions for solving the PMV-instability problem to motivate the statement and proof of Theorem 1. We will focus on the $\psi = 0$ case in this section for the purpose of clarity, which means that we will focus on characterizing $\sup_{\vec{\pi} \in \Pi^n} \Pr(\vec{X}_{\vec{\pi}} \in U_{n,B})$. The study of the $\psi \in (0, 1]$ case can be found in Appendix D.4.

Intuitions. If $U_{n,B} = \emptyset$, then the max-semi-random instability is 0, and we will refer to this as the 0 case. Suppose the 0 case does not hold, i.e., $U_{n,B} \neq \emptyset$, then we will adopt an approximation of $\Pr(\vec{X}_{\vec{\pi}} \in U_{n,B})$ based on the following two approximations.

- First, let us pretend that all integrality constraints in the linear system (like the one in Example 3) are relaxed. That is, let us pretend that $\vec{X}_{\vec{\pi}}$ can take non-integer values, and the vote operation vector $\vec{o}$ can be fractional (but still need to be non-negative). For simplicity, assume that $\vec{c} = \vec{1}$. Then, the possible changes to the histogram as the result of a unit of budget is characterized by $\text{CH}(\emptyset)$, where $\text{CH}(\cdot)$ represents the convex hull. It follows that $U_{n,B}$ can be approximated by a polyhedron $H_B$, which is the intersection of $H_S$ and the Minkowski addition of $H_T$ and $\bigcup_{0 \leq b \leq B} -b \cdot \text{CH}(\emptyset)$. See Figure 1 (a) for an illustration of $H_B$ in the shaded area, where $\emptyset = \{\vec{\alpha}_1, \vec{\alpha}_2\}$. Notice that $H_B$ is in the $-\text{CH}(\emptyset)$ direction of $H_T$.

- Second, let us pretend that $\vec{X}_{\vec{\pi}}$ is distributed as a $(q-1)$-dimensional Gaussian distribution $\mathcal{N}_{\vec{x}}$ (whose mean is $\sum_{j=1}^n \pi_j \in n \cdot \text{CH}(\Pi)$) in the hyperplane $\{ \vec{x} : \vec{x} \cdot \vec{1} = n \}$. This approximation is justified by various multi-variable central limit theorems, e.g., [Ben05, VV11, DDKT16, DKS16, Rai19].

With these approximations, we adopt the following approximation of $\Pr(\vec{X}_{\vec{\pi}} \in U_{n,B})$:

$$\Pr(\vec{X}_{\vec{\pi}} \in U_{n,B}) \approx \Pr(\mathcal{N}_{\vec{x}} \in H_B)$$  \hspace{1cm} (3)

Now, let us take a look at the max-semi-random PMV instability in light of the approximation in (3). Because the probability mass of $\mathcal{N}_{\vec{x}}$ is mostly centered around an $O(\sqrt{n})$ neighborhood of its mean, to maximize $\Pr(\mathcal{N}_{\vec{x}} \in H_B)$, the adversary aims at choosing $\vec{\pi}$ so that the “volume” of the intersection of an $O(\sqrt{n})$ neighborhood of $\sum_{j=1}^n \pi_j$ and $H_B$ is as large as possible.
When $B = O(\sqrt{n})$, it turns out that $\mathcal{H}_B$ is close to $\mathcal{H}_{S,\leq 0} \cap \mathcal{H}_{T,\leq 0}$, where for any polyhedron $\mathcal{H} \triangleq \{ \bar{x} : A \times (\bar{x})^\top \leq (b)^\top \}$, $\mathcal{H}_{\leq 0} \triangleq \{ \bar{x} : A \times (\bar{x})^\top \leq (0)^\top \}$ denotes its characteristic cone, also known as the recess cone. Therefore, if $\text{CH}(\Pi) \cap \mathcal{H}_{S,\leq 0} \cap \mathcal{H}_{T,\leq 0} = \emptyset$, then the mean of $\bar{X}_\pi$ is $\Theta(n)$ away from $\mathcal{H}_{S,\leq 0} \cap \mathcal{H}_{T,\leq 0}$, which implies that the likelihood is (exponentially) small due to straightforward applications of Hoeffding’s inequality. We call this case the **exponential case**.

When $B = O(\sqrt{n})$ and $\text{CH}(\Pi) \cap \mathcal{H}_{S,\leq 0} \cap \mathcal{H}_{T,\leq 0} \neq \emptyset$, the adversary can choose $\bar{\pi} \in \Pi^n$ so that the mean of $\bar{X}_\bar{\pi}$ is either in $\mathcal{H}_{S,\leq 0} \cap \mathcal{H}_{T,\leq 0}$ or is $O(1)$ away, which means that the likelihood is large. In this case, there is a **phase transition at $B = \Theta(\sqrt{n})$**, as it will be shown that the likelihood reaches its (asymptotic) max at $B = \Theta(\sqrt{n})$.

When $B = \Theta(n)$, again, the adversary aims at choosing $\bar{\pi} \in \Pi^n$ so that the mean of $\bar{X}_\bar{\pi}$ is close to $\mathcal{H}_B$. Let $B_{CH(\Pi)}$ denote the smallest budget so that a “pseudo-conic” approximation to $\mathcal{H}_{B_{CH(\Pi)}}$, denoted by $\mathcal{C}_B$ and is formally defined below in (5), touches $\text{CH}(\Pi)$, which is the convex hull of $\Pi$. See Figure 1 (b) for an illustration of $B_{CH(\Pi)}$ and $\mathcal{C}_{B_{CH(\Pi)}}$ (the shaded area). It is then expected that when $B < B_{CH(\Pi)} \cdot n$, the max-semi-random instability would be small, and when $B > B_{CH(\Pi)} \cdot n$, the max-semi-random instability would be large. In other words, the semi-random stability has a **phase transition at $B = \Theta(n)$**.

Nevertheless, characterizing the conditions and likelihood for each case is still challenging, as the approximation above is only meant to provide a qualitative intuition. Existing multi-variate central limit theorems are often too coarse due to an $\Omega(\frac{1}{\sqrt{n}})$ error, as discussed in [Xia21a].

**Notation.** Now let us define notation to formalize the intuitions discussed above. For any budget $B \geq 0$, we let $\mathcal{H}_B$ denote the relaxation of $\mathcal{U}_{o,B}$ by removing the size constraint and the integrality constraints on $\bar{x}$ and on $\bar{\sigma}$. Recall that $\bar{\sigma}$ still needs to be non-negative. That is,

$$
\mathcal{H}_B \triangleq \left\{ \bar{x} \in \mathcal{H}_S : \exists \bar{\sigma} \in \mathbb{R}_{\geq 0}^{d} \text{ s.t. } \bar{c} \cdot \bar{\sigma} \leq B \text{ and } \bar{x} + \bar{\sigma} \times \mathbf{0} \in \mathcal{H}_T \right\}
$$

(4)

For example, Figure 1 (a) illustrates $\mathcal{H}_B$ in the shaded area. $\mathcal{H}_B$ is a polyhedron because it is the intersection of $\mathcal{H}_S$ and the Minkowski addition of $\mathcal{H}_T$ and the following polyhedron $Q_B$:

$$
Q_B \triangleq \left\{ -\bar{\sigma} \times \mathbf{0} : \bar{\sigma} \in \mathbb{R}_{\geq 0}^{d} \text{ and } \bar{c} \cdot \bar{\sigma} \leq B \right\}
$$

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Specifically, with infinite budget \((B = \infty)\), we have
\[
\mathcal{H}_\infty = \left\{ \vec{x} \in \mathcal{H}_S : \exists \vec{o} \in \mathbb{R}_{\geq 0}^{\mid \Pi \mid} \text{ s.t. } \vec{x} + \vec{o} \times \vec{O} \in \mathcal{H}_T \right\} = \mathcal{H}_S \cap (\mathcal{H}_T + Q_\infty)
\]
For every \(B \geq 0\), we define \(C_B\) to be the polyhedron that consists of all (possibly non-integer) vectors in \(\mathcal{H}_{S,\leq 0}\) that can be manipulated to be in \(\mathcal{H}_{T,\leq 0}\) by using (possibly non-integer) operations \(\vec{o}\) under budget constraint \(B\). That is,
\[
C_B \triangleq \left\{ \vec{x} \in \mathcal{H}_{S,\leq 0} : \exists \vec{o} \in \mathbb{R}_{\geq 0}^{\mid \Pi \mid} \text{ s.t. } \vec{x} + \vec{o} \times \vec{O} \in \mathcal{H}_{T,\leq 0} \right\}
\]
(5)
It is not hard to verify that \(C_B = \mathcal{H}_{S,\leq 0} \cap (\mathcal{H}_{T,\leq 0} + Q_B)\) and \(C_B\) can be viewed as a “pseudo-conic” approximation to \(\mathcal{H}_B\), as \(C_B\) is defined based on the characteristic cones of \(\mathcal{H}_S\) and \(\mathcal{H}_T\), though \(C_B\) itself may not be a cone. Specifically, \(C_0\) and \(C_\infty\) will play a central role in Theorems 1 and 7. It is not hard to verify that
\[
C_0 \triangleq \mathcal{H}_{S,\leq 0} \cap \mathcal{H}_{T,\leq 0}, \quad \text{and}
\]
\[
C_\infty \triangleq \left\{ \vec{x} \in \mathcal{H}_{S,\leq 0} : \exists \vec{o} \in \mathbb{R}_{\geq 0}^{\mid \Pi \mid} \text{ s.t. } \vec{x} + \vec{o} \times \vec{O} \in \mathcal{H}_{T,\leq 0} \right\} = \mathcal{H}_{S,\leq 0} \cap (\mathcal{H}_{T,\leq 0} + Q_\infty)
\]
(6)
Figure 1 (b) illustrates \(C_0\) (which is a line) and \(C_\infty\) (which is the same as \(\mathcal{H}_{S,\leq 0}\)). Both \(C_0\) and \(C_\infty\) are polyhedral cones, because the intersection Minkowski addition of two polyhedral cones is a polyhedral cone. Notice that when \(B \notin \{0, \infty\}\), \(C_B\) may not be a cone.

For any set \(\Pi^* \subseteq \mathbb{R}^q\), let \(B_{\Pi^*} \in \mathbb{R}\) to be the minimum budget \(B\) such that the intersection of \(\Pi^*\) and \(C_B\) is non-empty. If no such \(B\) exists (i.e., \(\Pi^* \cap C_\infty = \emptyset\)), then we let \(B_{\Pi^*} \triangleq \infty\). Formally,
\[
B_{\Pi^*} \triangleq \inf\{B \geq 0 : \Pi^* \cap C_B \neq \emptyset\}
\]
(7)
Figure 1 (b) illustrates \(B_{\text{CH}(\Pi)}\) and \(B_{\text{CH}(\Pi)^\dagger}\) in the shaded area, where \(\Pi = \{\pi^1, \pi^2\}\) and \(\text{CH}(\Pi)\) is the convex hull of \(\Pi\), which is the line segment between \(\pi^1\) and \(\pi^2\) in this case.

Next, we define notation and conditions used in the statement of the theorem.

**Definition 5.** Given a PMV-instability setting, \(\Pi\), \(B\), and \(n\), define
\[
d_0 = \text{dim}(\mathcal{C}_0), \quad d_\infty = \text{dim}(\mathcal{C}_\infty), \quad \text{and} \quad d_\Delta = d_\infty - d_0,
\]
where \(\text{dim}(\mathcal{C}_0)\) is the dimension of \(\mathcal{C}_0\), which is the dimension of the minimal affine space that contains \(\mathcal{C}_0\). We also define the following three conditions:

| \(\mathcal{U}_{n,B} = \emptyset\) | \(\text{CH}(\Pi) \cap \mathcal{C}_\infty = \emptyset\) | \(\text{CH}(\Pi) \cap \mathcal{C}_0 = \emptyset\) |
|---|---|---|
| \(K_1\) | \(K_2\) | \(K_3\) |

Recall that \(\text{CH}(\Pi)\) is the convex hull of \(\Pi\). Because \(\mathcal{C}_0 \subseteq \mathcal{C}_\infty\), \(d_\Delta \geq 0\). Also notice that \(K_2\) implies \(K_3\), or equivalently, \(\neg K_3\) implies \(\neg K_2\).

**Theorem 1 (Semi-Random PMV-Instability, \(\psi = 0\)).** Given any \(q \in \mathbb{N}\), any closed and strictly positive \(\Pi\) over \([q]\), and any PMV-instability setting \(S = \langle \mathcal{H}_S, \mathcal{H}_T, \mathcal{O}, \vec{c} \rangle\), any \(C_2 > 0\) and \(C_3 > 0\) with \(C_2 < B_{\text{CH}(\Pi)} < C_3\), any \(n \in \mathbb{N}\), and any \(B \geq 0\),

\[
\sup_{\vec{x} \in \Pi^n} \text{Pr}\left( \vec{X}_{\vec{x}} \in \mathcal{U}_{n,B} \right) = \begin{cases} 
\text{Name} & \text{Likelihood} & \text{Condition} \\
\text{0 case} & 0 & K_1 \\
\text{exp case} & \exp(-\Theta(n)) & \neg K_1 \land \neg K_2 \\
\text{PT-}\Theta(\sqrt{n}) & \Theta\left(\frac{\min\{B + 1, \sqrt{n}\}^{d_\Delta}}{\left(\sqrt{n}\right)^{d_0}}\right) & \neg K_1 \land \neg K_3 \\
\text{PT-}\Theta(n) & \begin{align*}
\exp(-\Theta(n)) & \text{ if } B \leq C_2n \\
\Theta\left(\frac{1}{\sqrt{n}}\right)^{d_\infty} & \text{ if } B \geq C_3n
\end{align*} & \text{ otherwise, i.e.,} \neg K_1 \land \neg K_2 \land K_3
\end{cases}
\]
The $B + 1$ value is introduced to handle the $B < 1$ case. For every $B \geq 1$, $\Theta(B + 1) = \Theta(B)$.

**The four cases.** Following the intuition presented at the beginning of this section, we call the first case in Theorem 1 the 0 case, the second case the exponential case, the third case the phase transition at $\Theta(\sqrt{n})$ case (PT-\(\Theta(\sqrt{n})\) for short), and the last case, which contains two subcases, the phase transition at $\Theta(n)$ case (PT-\(\Theta(n)\) for short).

![Figure 2: Illustration of Theorem 1. The x-axis is in log scale.](image)

Figure 2 illustrates the max-semi-random instability as a function of $B$ for the exp case, PT-\(\Theta(\sqrt{n})\) case, and PT-\(\Theta(n)\) case, respectively. Figure 3 (a) illustrates condition $K_2 = [\text{CH(II)} \cap C_\infty = \emptyset]$ for the exp case. Figure 3 (b) illustrates condition $\neg K_3 = [\text{CH(II)} \cap C_0 \neq \emptyset]$ for the PT-\(\Theta(\sqrt{n})\) case. Figure 1 (b) illustrates condition $\neg K_2 \land K_3 = [\text{CH(II)} \cap C_\infty \neq \emptyset] \land [\text{CH(II)} \cap C_0 = \emptyset]$ for the PT-\(\Theta(n)\) case.

![Figure 3: Illustration of the exp case and phase-transition-at-\(\Theta(\sqrt{n})\) case.](image)

(a) The exp case. 
(b) The PT-\(\Theta(\sqrt{n})\) case.

**Limitations and powers of Theorem 1.** Theorem 1 has two major limitations. First, it misses a knife-edge case of $B$, i.e., when $B$ is around $B_{\text{CH(II)}} \cdot n$. Second, the constants in asymptotic bounds may be exponentially large in $m$. Despite these limitations, we believe that Theorem 1 provides a general and useful tool for studying PMV-instability problems, because it converts the complicated PMV-instability problems, which involve reasoning about the likelihood of discrete events (about the PMV) that cannot be easily bounded by standard techniques, to deterministic geometric problems about $\text{CH(II)}$, $C_0$, $C_\infty$, $d_0$, and $d_\infty$. It provides an almost complete characterization of the PMV-instability problem, which can be easily applied to resolve long-standing open...
questions, e.g., in Corollary 1. Practically, \( \text{dim}_0 \) and \( \text{dim}_\Delta \) can still be hard to characterize, but at least Theorem 1 provides a useful guideline about what to look for. See Section 5 for some examples.

**Proof sketch of Theorem 1.** At a high level, the proof follows after the intuitions presented in the beginning of this subsection. The hardest part is the proof of the (asymptotically tight) polynomial bounds in the \( \text{PT-} \Theta(\sqrt{n}) \) case. Take sup and \( B = O(\sqrt{n}) \) for example. To prove the polynomial upper bound, our proof can be viewed as upper-bounding the “volume” of the intersection of an \( O(\sqrt{n}) \) neighborhood of \( \sum_{j=1}^{n} \pi_j \) and \( H_B \). We prove that, in \( d_0 \) dimensions, the volume is large, and each such dimension contributes a multiplicative \( O(1) \) factor to the likelihood; in \( d_\Delta \) dimensions, the volume is \( O(B) \), and each such dimension contributes a multiplicative \( O\left(\frac{B+1}{\sqrt{n}}\right) \) factor to the likelihood; and in the remaining \( q-d_\infty \) dimensions, the volume is \( O(1) \), and each such dimension contributes a multiplicative \( O\left(\frac{1}{\sqrt{n}}\right) \) factor to the likelihood. Putting all together, this proves the desired upper bound

\[
O(1)^{d_0} \times O\left(\frac{B+1}{\sqrt{n}}\right)^{d_\Delta} \times O\left(\frac{1}{\sqrt{n}}\right)^{q-d_\infty} = O\left(\frac{(B+1)^{d_\Delta}}{(\sqrt{n})^{q-d_0}}\right)
\]

To prove the polynomial lower bound, we first pretend that the PMV can take non-integer values, then enumerate (possibly non-integral) vectors that are far away from each other by exploring two directions. The first direction is the convex hull of \( C_0 \), which is a \( (d_0 - 1) \)-dimensional space that represents no budget \( (B = 0) \), and each such dimension contributes an \( \Omega(\sqrt{n}) \) multiplicative factor to the total number of desirable vectors. The second direction is the convex hull of \( C_\infty \), which is a \( d_\Delta \)-dimensional space that represents infinite budget \( (B = \infty) \), and each such dimension contributes an \( \Omega(B+1) \) multiplicative factor to the total number of desirable vectors. Then, we prove that for each such (possibly non-integral) vector, there exists a nearby integer vector, and apply the pointwise concentration bound [Xia21a, Lemma 1] to prove the desired lower bound. The full version of the theorem and its full proof can be found in Appendix D.3.

5 Back to The Coalitional Influence Problems

**The overall approach.** We propose the following two-step polyhedral approach in Procedure 1 for characterizing semi-random \( X \).

**Procedure 1:** The polyhedral approach for semi-random coalitional influence

- **Input:** \( X, r, \Pi, \psi \)
- **Step (i):** Model \( X \) under \( r \) as a union of multiple PMV-instability setting \( \mathcal{M} \).
- **Step (ii):** Characterize the conditions and degree of polynomial for \( \mathcal{M} \) by applying Theorem 1 and combining the results.

Step (i) is often easy. The hardness of Step (ii) is highly problem-dependent, and we see two potential difficulties: first, it might be hard to verify for which \( n, B, \) and PMV-instability problem, the 0 case of Theorem 1 does not happen; and second, sometimes the combination of the results of the applications of Theorem 1 are too complicated.

In this section, we present three applications of Procedure 1 to illustrate its usefulness.

**The first application** is an asymptotically tight characterization for many commonly studied voting rules (defined in Appendix B.1). Recall that \( \pi_{\text{uni}} \) is the uniform distribution over \( \mathcal{L}(\mathcal{A}) \).
Theorem 2 (Max-Semi-Random Coalitional Influence: Commonly Studied Rules). Let \( r \) be an integer positional scoring rule, STV, ranked pairs, Schulze, maximin, or Copeland with lexicographic tie-breaking for any fixed \( m \geq 3 \). For any closed and strictly positive \( \Pi \) with \( \pi_{uni} \in CH(\Pi) \), any \( X \in \{CM, MoV\} \), there exists \( N > 0 \) such that for any \( n > N \) and any \( B \geq 1 \),

\[
\tilde{X}^\max_{\Pi,0}(r, n, B) = \Theta \left( \min \left\{ \frac{B}{\sqrt{n}}, 1 \right\} \right)
\]

Moreover, for any \( \psi > 0 \), \( \tilde{X}^\max_{\Pi,\psi}(r, n, B) = \Theta(1) \).

Proof sketch. We apply Procedure 1 to \( X = CM \) under any integer positional scoring rule \( r_\phi \) to illustrate the idea. For Step (i) of Procedure 1, we adopt the same representation as in the proof of Lemma 1. Due to the \( O \left( \min \left\{ \frac{B}{\sqrt{n}}, 1 \right\} \right) \) upper bound in Theorem 3, which will be proved soon, it suffices to prove the \( \Omega \left( \min \left\{ \frac{B}{\sqrt{n}}, 1 \right\} \right) \) lower bound by identifying a PMV-instability setting \( S \) that represents some unstable histograms \( U_{n,B} \), and then apply Theorem 1 to \( S \) to prove that for all \( B \geq 1 \),

\[
\sup_{\phi \in \Pi^s} \Pr_{P \sim \phi}(Hist(P) \in U_{n,B}) = \Theta \left( \min \left\{ \frac{B}{\sqrt{n}}, 1 \right\} \right)
\]  

(8)

In fact, \( S \) is similar to that defined in Example 3 and 4. Let \( \mathcal{H}_S \) denote the set of vectors where alternative 1’s total score is at least as high as alternative 2’s total score, which is strictly higher than the total score of any other alternative. Let \( \mathcal{H}_T \) denote the set of vectors where 2’s total score is the strictly highest. Let \( \hat{O}^{1\rightarrow 2} \) denote the vote operations where a ranking with 2 > 1 is replaced by another ranking. Let \( S = \langle \mathcal{H}_S, \mathcal{H}_T, \hat{O}^{1\rightarrow 2}, \tilde{1} \rangle \).

For Step (ii) of Procedure 1, we apply Theorem 1 to \( S \). We prove \( K_1 = \text{false} \) for any sufficiently large \( n \) by explicitly constructing a profile where a single manipulator can and has incentive to change the winner from 1 to 2. Then, we show that \( K_3 = \text{false} \), because (1) \( \pi_{uni} \in CH(\Pi) \) (assumption in Theorem 2), and (2) \( \pi_{uni} \in C_0 \) (because \( C_0 = \mathcal{H}_{S, \leq 0} \cap \mathcal{H}_{T, \leq 0} \) represents vectors where 1 and 2 are co-winners, and \( r_\phi(\pi_{uni}) = A \)). This means that the polynomial case of Theorem 1 holds. It is not hard to verify that \( d_0 = \dim(C_0) = m! - 1 \). We then prove that \( d_\infty = m! \) by showing that an interior point of \( \mathcal{H}_{S, \leq 0} \) can be manipulated to an interior point of \( \mathcal{H}_{T, \leq 0} \). Inequality (8) then follows after the polynomial case of sup of Theorem 1, which proves the CM part under \( r_\phi \).

The \( \psi > 0 \) case follows after a straightforward application of Theorem 5.

The full statement of Theorem 2 (including other coalitional influence problems such as the ones defined in Appendix B.2) and its full proof can be found in Appendix E.1. The next example shows an application of Theorem 2 to Borda.

Example 5. In the setting of Example 2, notice that \( \pi_{uni} = \frac{1}{2}(\pi^1 + \pi^2) \), which means that \( \pi_{uni} \in CH(\Pi) \). It follows from Theorem 2 that for all sufficiently large \( n \), \( \text{CM}^\max_{\Pi,0} (\text{Borda}, n, 1) = \Theta \left( \frac{1}{\sqrt{n}} \right) \).

Theorem 2 leads to the following corollary on IC, where \( \Pi = \{\pi_{uni}\} \) and \( \psi = 0 \).

Corollary 1 (Likelihood of Coalitional Influence w.r.t. IC). Let \( r \) be an integer positional scoring rule, STV, ranked pairs, Schulze, maximin, or Copeland with lexicographic tie-breaking for fixed \( m \geq 3 \). For any \( X \in \{CM, MoV\} \), there exists \( N > 0 \) such that for any \( n > N \) and any \( B \geq 1 \),

\[
\Pr_{P \sim IC}(X(r, P, B)) = \Theta \left( \min \left\{ \frac{B}{\sqrt{n}}, 1 \right\} \right)
\]
As discussed in the Introduction, the $X = \text{CM}$ and $B = 1$ case of Corollary 1 proves the $\Omega\left(\min\left\{\frac{1}{\sqrt{n}}, 1\right\}\right)$ matching lower bound for all commonly studied voting rules investigated in the literature, for which only a matching lower bound for plurality was previously known.

**The second application** is an upper bound for all generalized scoring rules and all closed and strictly positive $\Pi$ with $\psi = 0$.

**Theorem 3 (Upper bound on Coalitional Influence under GSRs).** Let $r$ denote any GSR with fixed $m \geq 3$. For any closed and strictly positive $\Pi$, any $X \in \{\text{CM, MoV}\}$, any $n$, and any $B \geq 1$,

$$\tilde{X}_{\Pi,0}^{\max}(r, n, B) = O\left(\min\left\{\frac{B}{\sqrt{n}}, 1\right\}\right)$$

**Proof sketch.** For Step (i) of Procedure 1, we adopt the same representation as in the proof of Lemma 1. Step (ii) relies on the notation and results of a general dichotomy theorem on the semi-random likelihood of influence problems for GSRs (Theorem 6 in Appendix F). The proof can be found in Appendix E.2.

Theorem 3 immediately extends all previous $O\left(\frac{1}{\sqrt{n}}\right)$ upper bound on CM for a single manipulator ($B = 1$) that will be discussed in Section 6 to any coalition size $B \geq 1$, because all rules studied in these works are GSRs.

**The third application** studies a new notion of coalitional manipulation that aims at making the loser win under integer positional scoring rules. For any positional scoring rule, the loser is the alternative with the minimum total score.

**Definition 6 (Coalitional manipulation for the loser).** Given any integer positional scoring rule $r\vec{s}$ with lexicographic tie-breaking, for any profile $P$ and any $B \geq 0$, we define $\text{CML}(r\vec{s}, P, B) = 1$ if and only if a coalition of no more than $B$ voters have incentive to misreport their preferences to make the loser under $P$ win.

Clearly, under veto, no coalition of voters have incentive to misreport their preferences to make the loser win, i.e., $\text{CML}(\text{veto}, P, B) = 0$ for all $P$ and $B$. This is because any voter who prefers the loser to the winner cannot reduce the difference between the total score of the loser and the sum of total score of all other alternatives, which means that no matter how they vote, the loser would not become the winner. The following theorems characterizes the max-semi-random CML under any other integer positional scoring rule for every sufficiently large $n$ and $B$.

**Theorem 4 (Max-Semi-Random Coalitional Manipulation for The Loser).** Let $r\vec{s}$ be an integer positional scoring rule with lexicographic tie-breaking for fixed $m \geq 3$ that is different from veto. For any closed and strictly positive $\Pi$ with $\pi_{uni} \in \text{CH}(\Pi)$, there exists $N > 0$ and $B^* > 0$ such that for any $n > N$ and any $B \geq B^*$,

$$\text{CML}_{\Pi,0}^{\max}(r\vec{s}, n, B) = \Theta\left(\min\left\{\frac{B}{\sqrt{n}}, 1\right\}^{m-1}\right)$$

Moreover, for any $\psi > 0$, $\text{CML}_{\Pi,\psi}^{\max}(r\vec{s}, n, B) = \Theta(1)$.

**Proof sketch.** For Step (i) of Procedure 1, we first define a set of PMV-instability settings, denoted by $\mathcal{M}_{\text{CML}}$, that represents CML under $r\vec{s}$. For every pair of different alternatives $a, b$, define a PMV-instability setting $S_{\text{CML}}^{a\rightarrow b} = (\mathcal{H}_S, \mathcal{H}_T, \mathcal{O}_{a\rightarrow b}, \hat{1})$, where $\mathcal{H}_S$ denotes the set of vectors
where \( a \) is the winner and \( b \) is the loser under \( r \), and \( \mathcal{H}_T \) denotes the set of vectors where \( b \) is the winner under \( r \). Define \( \mathcal{M}_{\text{CML}} = \{ S_{CML}^{a \rightarrow b} : a, b \in A, a \neq b \} \). It is not hard to verify that \( \mathcal{M}_{\text{CML}} \) represents CML under \( r \).

Then, for Step (ii) of Procedure 1, we apply Theorem 1 to every \( S_{\text{CML}}^{a \rightarrow b} \). We first prove \( K_1 = \text{false} \) for any sufficiently large \( n \) and \( B \), by constructing a successful manipulation by \( B \) voters. Then, let \( C_{0}^{a \rightarrow b} = H_{S, \leq 0} \cap H_{T, \leq 0} \). We have \( \pi_{\text{uni}} \in \text{CH(II)} \cap C_0 \), which means that \( K_3 = \text{false} \). Therefore, the polynomial case of Theorem 1 holds. It is then not hard to verify that \( d_0 = \dim(C_{0}^{a \rightarrow b}) = m! - (m - 1) \) and \( d_\infty = \dim(H_{S, \leq 0}) = m! \), which means that \( d_\Delta = m - 1 \). By Theorem 1, we have

\[
\sup_{\vec{\pi} \in \Pi_n} \Pr(\vec{X}_{\vec{\pi}} \in U_{n,B}) = \Theta \left( \frac{\min\{B + 1, \sqrt{n}\}^{m-1}}{(\sqrt{n})^{m! - (m! - (m-1))}} \right) = \Theta \left( \min \left\{ \frac{B}{\sqrt{n}}, 1 \right\}^{m-1} \right)
\]

Theorem 4 follows after applying this bound to all \( a \neq b \). The \( \psi > 0 \) case follows after a straightforward application of Theorem 5. The full proof can be found in Appendix E.3. \( \square \)

6 Related Work and Discussions

Due to the large body of literature, below we briefly discuss some closely related work. More detailed discussions can be found in Appendix A.

**Likelihood of coalitional manipulability: upper bounds w.r.t. IC.** As discussed earlier, a series of work has established an \( O\left(\frac{1}{\sqrt{n}}\right) \) upper bound on the likelihood of single-voter manipulability under commonly studied voting rules w.r.t. IC [Pat75, PW78, Pel79, Nit85, FK89, KR96, Sli02a, Sli02b, Sli02c, BN02, Sli04, MPS07, PW09]. The likelihood of coalitional manipulability was also considered for a few rules with certain sizes of the coalition [PW78, Pel79, Sli04, PW09]. All these results are straightforward corollaries of our Theorem 3, which implies an \( O\left(\frac{1}{\sqrt{n}}\right) \) upper bound for every \( B \geq 1 \) w.r.t. IC under a wide range of voting rules, including (but not limited to) the ones studied in previous work.

**Likelihood of coalitional manipulability: lower bounds w.r.t. IC.** A series of work on quantitative Gibbard-Satterthwaite theorems [FKKN11, DP08, XC08b, IKM10, MR15] established polynomial lower bounds on the likelihood of strategic voting by a single manipulator (\( B = 1 \)) w.r.t. IC. Many of these results applied to any voting rule, and are therefore more general than the CM part of Theorem 2 and Corollary 1. On the other hand, Theorem 2 is stronger and more general in some other aspects: it works for every \( B \geq 1 \), more coalitional influence problems, and a more general and realistic (semi-random) model, and it provides asymptotically tight bounds. Its application to CM w.r.t. IC in Corollary 1 establishes an \( \Omega\left(\frac{1}{\sqrt{n}}\right) \) matching bound for many commonly studied rules, among which this matching lower bound was only known for the plurality rule [Sli02a].

**Likelihood of coalitional manipulability: other distributions.** Results under i.i.d. distributions [Pel79, PR07, XC08a, MPR13, Xia15] demonstrated a phase transition from powerless to powerful at \( \Theta(\sqrt{n}) \) manipulators. There is a large body of literature on the likelihood of CM under the **Impartial Anonymous Culture (IAC)**, which assumes that each histogram happens equally likely and resembles the flat Dirichlet distribution, by theoretical analysis [LM87, Saa90, LM94, HC00, FLS02, FL06, Sli06, WP07, LLS08] and by computer simulations [LV03, PW07, GATC16]. Both IAC and IC are mainly of theoretical interest and “are poor proxies of political electorates” [Nur12].

**Other coalitional influence problems.** All results in this paper also work for MoV, which measures the stability of elections and provides an upper bound on CM. Some previous proofs of
upper bounds on CM were done for MoV [Sli02a, XC08a]. The phase transition behavior for CM also happens for MoV [MPR13, Xia15]. Previous work has also investigated expected MoV [PS06] and MoV for tournament rules [BSKS22]. The likelihood of MoV = 1 has been used to measure the decisiveness of voting, sometimes called voting power, which plays an important role in the paradox of voting [Dow57] and in power indices in cooperative game theory. For two alternatives under the plurality rule, the voting power is equivalent to the likelihood of ties [GM75]. In general, MoV = 1 and the election being tied are different events, as shown in Example 6 and 7 in Appendix G.

Beyond CM and MoV, there is a large body of work on the computational complexity of other types of coalition influence problems, such as constructive/destructive control (making a designated alternative win/lose) by adding/deleting votes, and bribery (different changes in votes have different costs). See [FR16] for a recent survey and see Appendix B.2 for their definitions. Little was known about their likelihood of success, except [Xia15], which does not provide an accurate characterization as discussed in Appendix A. The full versions of our results in the Appendix provide the first asymptotically tight bounds for these coalitional influence problems.

Generality and limitations of our semi-random model. The distribution adversary’s part of our model is the same as the semi-random model proposed in [Xia20], which has two major limitations: the independence of noises among agents’ preferences, and the strict positiveness of Π. Our model addresses the two limitations by introducing the data adversary, who modifies the (semi-randomly) generated data and therefore introduce strong and adversarial correlations. Specifically, our model is more general because the ψ = 0 case of our model degenerates to the model in [Xia20]. How to further generalize our model by introducing (statistical) correlations and relaxing the strict positiveness of Π for the distribution adversary is an important and challenging directions for future research.

Technical novelty. While our Procedure 1 is similar to the “polyhedral approaches” adopted in previous work on semi-random social choice [Xia20, Xia21a, Xia21b], the main technical tool of this paper (Theorem 1 for ψ = 0 and Theorem 5 for ψ > 0) are significant and non-trivial extensions of the main technical theorems in previous work, especially [Xia21a, Theorem 1], which can be viewed as a special case of our Theorem 1 with H_S = H_T, B = 0. The hardest part is the polynomial lower bounds, whose H_S = H_T and B = 0 case was proved in [Xia21a, Theorem 1] by explicitly enumerating sufficiently many target integer vectors for the PMV. However, due to the generality of the PMV-instability problem, we do not see an easy way to perform a similar enumeration. To address this technical challenge, we take a different approach by first pretending that the PMV can take non-integer values, then enumerating (possibly non-integral) vectors that are far away from each other by exploring two directions: the direction that represents no budget (i.e., B = 0) and the direction that represents infinite budget (i.e., B = ∞), and finally proving that for each such (possibly non-integral) vector, there exists a nearby integer target vector for the PMV. We believe that our Theorem 1 and Theorem 5 are useful and general tools for studying likelihood of coalitional influence, as exemplified by its applications to prove Theorems 2–4, Corollary 1, and a general Theorem 6 (Appendix F).

Semi-random analysis and smoothed analysis in general. Semi-random analysis [Blu90, BS95] refers to the analysis under a model where the process of generating instances has adversarial components and random components. For example, in the smoothed complexity analysis [ST09], the input to an algorithm is obtained from an adversarially chosen “ground truth” plus a (small) random perturbation. This can be viewed as the adversary directly choosing a distribution over data (from a set of distributions). See [Fei21] for a recent survey of various semi-random models and complexity results under them.
7 Summary and Future Work

We extend previous studies on average-case likelihood of coalitional manipulation in elections in three aspects: (1) a more general and realistic semi-random model that introduces correlations among agents’ preferences, (2) many other coalitional influence problems, and (3) arbitrary coalition size, by taking the polyhedral approach (Procedure 1). While we do not think that results in this paper are the final answers to the question in Introduction, we do believe that they are a non-trivial step forward, because they address long-standing open questions and expand the scope of previous work to more general and realistic settings.

Moving forward, we see three natural directions for future work. More realistic models: as discussed in Section 6, a natural question is how to relax the statistical independence and strict positiveness constraints on the distribution adversary to build a more realistic yet still tractable model. At a high level, the data adversary may be viewed as another influencer. Therefore, another natural question is how to incorporate different kinds of adversaries and influencers to the model. Stronger theorems: as discussed after Theorem 1, characterizing the constants in the bounds is a natural open question. Additionally, as discussed in the Introduction, Theorem 3 poses a challenge to designing natural rules with lower likelihood of coalitional influence, or improving the $\Omega(\frac{1}{n^{0.7}})$ lower bound (and extending it to $B \geq 2$). More applications: how to develop informative characterizations as done in Section 5 for other rules, other coalition influence problems, or other applications such as matching, resource allocation, fair division [BFGP22], judgement aggregation [LX22], are important and challenging tasks.
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Appendix

A Related Work with More Details

Likelihood of coalitional manipulability: upper bounds under IC. [Pat75] proposed to study the likelihood of strategic voting by single manipulator and conjectured that the likelihood is smaller in larger elections. [PW78] proved that the likelihood of single-voter manipulability goes to 0 as \( n \to \infty \) under plurality. They also noted that the results can be extended to any coalition of \( n^\alpha \) manipulators, where \( 0 \leq \alpha < 1/2 \). [Pel79] proved that the likelihood of single-voter manipulability under any representable voting system, which includes positional scoring rules, goes to 0 under any positive i.i.d. distributions. Peleg also considered coalitional manipulation problem, by noticing in footnote 2 that the result still holds for any coalition of \( o(\sqrt{n}) \) voters. [Nit85] demonstrated that the likelihood decreases when \( n \) is large under Borda, plurality, and range voting. [FK89] proved an \( O\left(\frac{1}{\sqrt{n}}\right) \) upper bound for the convergence to 0 of the likelihood of single-voter manipulability under plurality. [KR96] proved that the likelihood decreases when \( n \) is large under Borda, plurality, and range voting. [Sli02a, Sli02b, Sli02c] proved an \( O\left(\frac{1}{\sqrt{n}}\right) \) upper bound for a single manipulator under plurality with runoff, representable voting system, all voting rules based on the unweighted majority graphs, and Bucklin. [BN02] proved an \( O\left(\frac{1}{\sqrt{n}}\right) \) upper bound for a single manipulator under positional scoring rule, top cycle, and Copeland, when voters’ preferences has small local correlations, which is more general than IC. [Sli04] investigated the likelihood of coalitional manipulation by up to \( Cn^\alpha \) manipulator for any fixed \( 0 \leq \alpha < 1/2 \), and proved an \( O((\frac{1}{n})^{0.5-\alpha}) \) upper bound for any positional scoring rule with strictly decreasing scores. [MPS07] characterized the least manipulable rule by a single manipulator among tops-only, anonymous, and surjective choice rules, to be the unanimity rules with status quo. [PW09] showed that, the likelihood for a coalition of \( v\sqrt{n} \) manipulators to succeed under a positional scoring rule is a function of \( v \), and provided an algorithm based on integer linear program to compute the minimum coalition size.

Likelihood of coalitional manipulability: lower bounds under IC. [Sli02a] proved an \( \Omega\left(\frac{1}{\sqrt{n}}\right) \) lower bound under plurality for a single manipulator. A quantitative Gibbard-Satterthwaite theorem was proved for \( m = 3 \) by [FKKN11], and was subsequently developed in [DP08, XC08b, IKM10], and the general case was proved by [MR15], which implies that any voting rule that is constantly away from any dictatorships, the likelihood of single-voter manipulability under IC is \( \Omega\left(\frac{1}{n^{m\alpha}}\right) \), i.e., \( \Omega\left(\frac{1}{n^{m\alpha}}\right) \) for any fixed \( m \).

Likelihood of coalitional manipulability: simulations. Beyond theoretical work, there is also a large literature on comparing the empirical coalitional manipulability of commonly-studied voting rules, mostly by computer simulations [Cha85, Kel93, Smi99, AK99, PW07, AKSY11, AKSY12, GATC16]. These works confirm that the likelihood for a large election to be manipulable by a single manipulator is low.

Likelihood of coalitional manipulability under other distributions. As discussed above, the convergence-to-0 result for representable voting system in [Pel79] works for any i.i.d. distributions, and the \( O(1/\sqrt{n}) \) upper bound for positional scoring rule, top cycle, and Copeland by [BN02] works for distributions with small local correlations. [PR07] proved that for weighted voters whose preferences are generated independently, positional scoring rules cannot be manipulated by \( o(\sqrt{n}) \) manipulators almost surely as \( n \to \infty \). [XC08a] proved that for a large of class of voting rules and i.i.d. distributions that satisfy certain conditions, a coalition of \( O(n^\alpha) \) manipulators is powerless
when \( \alpha < \frac{1}{2} \), as the likelihood for them to succeed is \( O(n^{\alpha - 0.5}) \); and they are powerful when \( \alpha > \frac{1}{2} \), as the likelihood is lower bounded by \( 1 - \exp(-\Omega(n)) \) in such case. [MPR13] illustrates a smooth transition from powerlessness to powerfulness for a coalition of \( c \sqrt{n} \) manipulators with variable \( c \). [Xia15] proved that for a large class of influence problems including coalitional manipulation, under i.i.d. distributions, with probability that goes to 1 the number of voters needed is 0, \( \Theta(\sqrt{n}) \), \( \Theta(n) \), or impossible. The paper does not characterize the likelihood for each case. [DMN16] used Condorcification to decrease coalitional manipulable profiles, reduces coalitional manipulability under every probability distribution.

There is a large literature on the likelihood of coalitional manipulability under the *Impartial Anonymous Culture (IAC)*, which assumes that each histogram happens equally likely and resembles the flat Dirichlet distribution, based on theoretical analysis [LM87, Saa90, LM94, HC00, FLS02, FL06, Shi06, WP07, LLS08] and computer simulations [LV03, PW07, GATC16]. Both IAC and IC are mainly of theoretical interest and “are poor proxies of political electorates” [Nur12].

**Other coalitional influence problems.** The margin of victory (MoV) of a profile is the smallest coalition of voters who can change the winner by casting different votes (regardless of their preferences). MoV measures the stability of elections and provides an upper bound on CM and some previous proofs of upper bounds on CM are done for MoV, such as [Shi02a, XC08a]. [PS06] proved that for any positional scoring rule, the expected margin of victory under IC is \( \Theta(\sqrt{n}) \) and characterized the voting rules with maximum expected MoV under IC. Results in [MPR13, Xia15] discussed above also apply to MoV. [BSKS22] studies the distribution of MoV for some tournament rules, under a probability distribution on tournament graphs, where the direction of each edge is drawn independently and uniformly.

The likelihood of MoV = 1 has been used to measure the *decisiveness* of voting, sometimes called *voting power*, which plays an important role in the paradox of voting [Dow57] and in definitions of power indices in cooperative game theory. For two alternatives under the plurality rule, the voting power is equivalent to the likelihood of ties [GM75] (see, e.g., [Xia21a] for a semi-random analysis on the likelihood of ties and references therein). In general, the two problems are different, for example as shown in Example 6 and 7.

Beyond CM and MoV, there is a large body of work on the computational complexity of other types of coalition influence problems that based on vote operations, such as constructive/destructive control (making a designated alternative win/lose) by adding/deleting votes, and bribery (different changes in votes have different costs and a total budget is given). See [FR16] for a recent survey. Little work has been done to analyze their likelihood of success, except [Xia15], which as discussed above, does not provide an accurate characterization.

## B Additional Preliminaries

The *min-semi-random likelihood* is defined as:

\[
\tilde{X}^\text{min}_{\Pi,\psi}(r, n, B) \triangleq \inf_{\vec{\pi} \in \Pi_n} \Pr_{P \sim \vec{\pi}} \left( \exists P' \psi \sim P \text{ s.t. } X(r, P, B) = 1 \right), \tag{9}
\]

### B.1 Other commonly studied voting rules

**Weighted Majority Graphs.** For any (fractional) profile \( P \) and any pair of alternatives \( a, b \), let \( P[a > b] \) denote the total weight of votes in \( P \) where \( a \) is preferred to \( b \). Let WMG(\( P \))
denote the *weighted majority graph* of $P$, whose vertices are $A$ and whose weight on edge $a \rightarrow b$ is $w_P(a, b) = P[a \succ b] - P[b \succ a]$. Sometimes a distribution $\pi$ over $L(A)$ is viewed as a fractional profile, where for each $R \in L(A)$ the weight on $R$ is $\pi(R)$. In this case we let $\text{WMG}(\pi)$ denote the weighted majority graph of the fractional profile represented by $\pi$.

A voting rule is said to be *weighted-majority-graph-based (WMG-based)* if its winners only depend on the WMG of the input profile. In this paper we consider the following commonly studied WMG-based rules.

- **Copeland.** The Copeland rule is parameterized by a number $0 \leq \alpha \leq 1$, and is therefore denoted by $\text{Copeland}_\alpha$, or $\text{Cd}_\alpha$ for short. For any fractional profile $P$, an alternative $a$ gets 1 point for each other alternative it beats in their head-to-head competition, and gets $\alpha$ points for each tie. $\text{Copeland}_\alpha$ chooses all alternatives with the highest total score as the winners.

- **Maximin.** For each alternative $a$, its min-score is defined to be $\min_{b \in A} w_P(a, b)$. Maximin, denoted by MM, chooses all alternatives with the max min-score as the winners.

- **Ranked pairs.** Given a profile $P$, an alternative $a$ is a winner under ranked pairs (denoted by RP) if there exists a way to fix edges in $\text{WMG}(P)$ one by one in a non-increasing order w.r.t. their weights (and sometimes break ties), unless it creates a cycle with previously fixed edges, so that after all edges are considered, $a$ has no incoming edge. Ties between edges are broken lexicographically. For example, if $1 \rightarrow 2$ and $2 \rightarrow 3$ have the same weight, then $1 \rightarrow 2$ is chosen first. If $1 \rightarrow 2$ and $1 \rightarrow 3$ have the same weight, then $1 \rightarrow 2$ is chosen first.

- **Schulze.** For any directed path in the WMG, its strength is defined to be the minimum weight on any single edge along the path. For any pair of alternatives $a, b$, let $s[a, b]$ be the highest weight among all paths from $a$ to $b$. Then, we write $a \succeq b$ if and only if $s[a, b] \geq s[b, a]$, and [Sch11] proved that the strict version of this binary relation, denoted by $\succ$, is transitive. The Schulze rule, denoted by Sch, chooses all alternatives $a$ such that for all other alternatives $b$, we have $a \succeq b$.

**STV with lexicographic tie-breaking mechanism.** The (single-winner) STV with lexicographic tie-breaking chooses winners in $m - 1$ rounds. In each round, the loser of plurality under lexicographic tie-breaking is removed from the election. We note that this rule is different from first computing STV winners under parallel universe tie-breaking [CRX09] and then breaking ties among the co-winners.

**Plurality with runoff.** The plurality with runoff rule with lexicographic tie-breaking, denoted by Pro, chooses the winner in two rounds. In the first round, the two alternatives with highest plurality scores are chosen (ties are broken lexicographically), and all other alternatives are removed. In the second round, the majority rule with lexicographic tie-breaking is applied to choose the winner.

**Proposition 1.** Any representable voting system with lexicographic tie-breaking is a GSR.

*Proof.* The proof is similar to the proof that shows positional scoring rules with lexicographic tie-breaking are GSRs illustrated in Example 1. Formally, we define the following score difference vector that is similar to the score difference vector defined for positional scoring rules [Xia20].

**Definition 7 (Score difference vector for representable voting system).** For any scoring function $s : L(A) \times A \rightarrow \mathbb{R}_{\geq 0}$ and any pair of different alternatives $a, b$, let $\text{Score}^s_{a, b}$ denote the $m!$-dimensional vector indexed by rankings in $L(A)$: for any $R \in L(A)$, the $R$-element of $\text{Score}^s_{a, b}$ is $s(R, a) - s(R, b)$. 

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Let $K = \binom{m}{n}$ and the hyperplanes are score difference vectors $\{\text{Score}_{a,b}^\pi : a \in A, b \in A, a \neq b\}$. For any profile $P$, $\text{Sign}_P(\text{Hist}(P))$ contains information about the comparisons of total scores of all pairs of alternatives, from which $g$ chooses a winner and applies the tie-breaking mechanism when needed.

### B.2 Other commonly studied coalitional influence problems

In the constructive control by adding votes (CCAV) (respectively, destructive control by adding votes (DCAV)) problem, we are given a distinguished alternative $d$, and we let $\text{CCAV}_d(r, P, B) = 1$ (respectively, $\text{DCAV}_d(r, P, B) = 1$), if there exists a preference profile $P^*$ with $|P^*| \leq B$ such that $r(P + P^*) = \{d\}$ (respectively, $r(P + P^*) \neq \{d\}$).

In the constructive control by deleting votes (CCDV) (respectively, destructive control by deleting votes (DCDV)) problem, we are given a distinguished alternative $d$, and we let $\text{CCDV}_d(r, P, B) = 1$ (respectively, $\text{DCDV}_d(r, P, B) = 1$), if there exists $P' \subseteq P$ with $|P'| \leq B$ such that $r(P - P^*) = \{d\}$ (respectively, $r(P - P^*) \neq \{d\}$).

For convenience, we let $\text{CONTROL}$ denote the control problems introduced above, and let $\text{E-CONTROL}$ denote their effective variants, formally defined as follows.

**Definition 8** (Effective control problems). Define

$$\text{E-CONTROL} = \{\text{E-CCAV}, \text{E-CCDV}, \text{E-DCAV}, \text{E-DCDV}\}$$ and

$$\text{CONTROL} = \{\text{CCAV}, \text{CCDV}, \text{DCAV}, \text{DCDV}\}$$

### C Materials for Section 3

#### C.1 The PMV-multi-instability problem

**Definition 4** (PMV-instability problem, full version). Given a PMV-instability setting $S = (\mathcal{H}_S, \mathcal{H}_T, \emptyset, \bar{e})$, a set $\Pi$ of distributions over $[q]$, $n \in \mathbb{N}$, and $B \geq 0$, we are asked to bound

- **max-semi-random instability**: $\sup_{\bar{\pi} \in \Pi^n} \text{Pr}\left( \exists \vec{x} \overset{\psi}{\sim} \vec{X}_{\bar{\pi}} \text{ s.t. } \vec{x} \in \mathcal{U}_{n,B} \right)$, and

- **min-semi-random instability**: $\inf_{\bar{\pi} \in \Pi^n} \text{Pr}\left( \exists \vec{x} \overset{\psi}{\sim} \vec{X}_{\bar{\pi}} \text{ s.t. } \vec{x} \in \mathcal{U}_{n,B} \right)$

**Definition 9** (PMV-multi-instability setting). A PMV-multi-instability setting, denoted by $\mathcal{M} = \{\mathcal{S}^i : i \leq I\}$, is a set of $I \in \mathbb{N}$ PMV-instability settings, where $\mathcal{S}^i = (\mathcal{H}_S^i, \mathcal{H}_T^i, \emptyset^i, \bar{e}^i)$, whose unstable histograms are denoted by $\mathcal{U}_{n,B}^\mathcal{M}$. Let $\mathcal{U}_{n,B}^\mathcal{M} = \bigcup_{i \leq I} \mathcal{U}_{n,B}^i$.

**Definition 10** (The PMV-multi-instability problem). Given a PMV-multi-instability setting $\mathcal{M} = \{\mathcal{S}^i : i \leq I\}$, a set $\Pi$ of distributions over $[q]$, $n \in \mathbb{N}$, and $B \geq 0$, we are asked to bound

- **max-semi-random multi-instability**: $\sup_{\bar{\pi} \in \Pi^n} \text{Pr}\left( \vec{X}_{\bar{\pi}} \in \mathcal{U}_{n,B}^\mathcal{M} \right)$, and

- **min-semi-random multi-instability**: $\inf_{\bar{\pi} \in \Pi^n} \text{Pr}\left( \vec{X}_{\bar{\pi}} \in \mathcal{U}_{n,B}^\mathcal{M} \right)$

Like PMV-instability problems, the max- (respectively, min-) semi-random multi-instability represents the upper bound (respectively, lower bound) on the likelihood for the PMV to be unstable w.r.t. any $\mathcal{S}^i$ in $\mathcal{M}$, when the underlying probabilities $\bar{\pi}$ is adversarially chosen from $\Pi^n$. 

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C.2 Full Version of Lemma 1 and Its Proof

Lemma 1. (Coalitional Influence as PMV-multi-instability, Full Version). For any coalitional influence problem $X \in \{\text{CM}, \text{MoV}, \text{CB}_{d,e}, \text{DB}_{d,e}, \text{e-CB}_{d,e}, \text{e-DB}_{d,e}\}$ and any GSR $r$, there exist a set $\mathcal{M} = \{S^i : i \leq I\}$ of $I$ PMV-instability settings such that for every $n$-profile $P$ and every $B \geq 0$, $X(r, P, B) = 1$ if and only if $\text{Hist}(P) \in U_{n,B}^{\mathcal{M}}$.

Proof. We first recall some formal notation about GSR. For any real number $x$, let $\text{Sign}(x) \in \{+,-,0\}$ denote the sign of $x$. Given a set of $K$ hyperplanes in the $q$-dimensional Euclidean space, denoted by $\mathcal{H} = (\vec{h}_1, \ldots, \vec{h}_K)$, for any $\vec{x} \in \mathbb{R}^q$, we let $\text{Sign}_{\mathcal{H}}(\vec{x}) = (\text{Sign}(\vec{x} \cdot \vec{h}_1), \ldots, \text{Sign}(\vec{x} \cdot \vec{h}_K))$. In other words, for any $k \leq K$, the $k$-th component of $\text{Sign}_{\mathcal{H}}(\vec{x})$ equals to $0$, if $\vec{p}$ lies in hyperplane $\vec{h}_k$; and it equals to $+$ (respectively, $-$) if $\vec{p}$ lies in the positive (respectively, negative) side of $\vec{h}_k$. Each element in $\{+, -, 0\}^K$ is called a signature.

Definition 11 (Feasible signatures). Given integer $\tilde{H}$ with $K = |\tilde{H}|$, let $S_{\tilde{H}} = \{+, -, 0\}^K$. A signature $\vec{i} \in S_{\tilde{H}}$ is feasible, if there exists $\vec{x} \in \mathbb{R}^m$ such that $\text{Sign}_{\tilde{H}}(\vec{x}) = \vec{i}$. Let $S_{\tilde{H}} \subseteq S_{H}$ denote the set of all feasible signatures.

The domain of any GISR $\tau$ can be naturally extended to $\mathbb{R}^m$ and to $S_{\tilde{H}}$. Specifically, for any $\vec{i} \in S_{\tilde{H}}$ we let $\tau(\vec{i}) = g(\vec{i})$. It suffices to define $g$ on the feasible signatures, i.e., $S_{\tilde{H}}$.

See [Xia21b, Section D.2] for the GSR representations of some commonly-studied voting rules, especially the rules defined in Appendix B.1.

Next, given $\tilde{H}$ and a feasible signature $\vec{i}$, we recall from [Xia21b, Section D.4] the definition of $\mathcal{H}_{\tilde{H}, \vec{i}}$ that represents profiles whose signatures are $\vec{i}$.

Definition 12 ($\mathcal{H}_{\tilde{H}, \vec{i}}$ (\(\mathcal{H}_{\tilde{H}, \vec{i}}\) in short)). For any $\tilde{H} = (\vec{h}_1, \ldots, \vec{h}_K) \in (\mathbb{R}^d)^K$ and any $\vec{i} \in S_{\tilde{H}}$, we let $A_{\vec{i}} = \begin{bmatrix} A^+_{\vec{i}} \\ A^-_{\vec{i}} \\ A^0_{\vec{i}} \end{bmatrix}$, where

- $A^+_{\vec{i}}$ consists of a row $-\vec{h}_i$ for each $i \leq K$ with $t_i = +$.
- $A^-_{\vec{i}}$ consists of a row $\vec{h}_i$ for each $i \leq K$ with $t_i = -$.
- $A^0_{\vec{i}}$ consists of two rows $-\vec{h}_i$ and $\vec{h}_i$ for each $i \leq K$ with $t_i = 0$.

Let $b_{\vec{i}} = \begin{bmatrix} -\vec{i}^- \\ -\vec{i}^+ \\ \vec{i}^0 \end{bmatrix}$. The corresponding polyhedron is denoted by $\mathcal{H}_{\tilde{H}, \vec{i}}$, or $\mathcal{H}_{\tilde{H}, \vec{i}}$ in short when $\tilde{H}$ is clear from the context.

Then, we formally define some vote operations that will be used in the proof.

Definition 13. We define four vote operations as follows.

- **Vote change**: $O_{\pm} = \{\text{Hist}(R_2) - \text{Hist}(R_1) : R_1, R_2 \in \mathcal{L}(A)\}$.

- **Motivated vote change**: for any pair of different alternatives $a, b$, let $O_{+, -}^{a, b} = \{\text{Hist}(R_b) - \text{Hist}(R_a) : R_b, R_a \in \mathcal{L}(A) \text{ and } b \succ_R a\}$

- **Generalized vote change**: for any pair of different alternatives $a, b$, let $O_{\pm}^* = \{\text{Hist}(R_1), \text{Hist}(R_2), \text{Hist}(R_2) - \text{Hist}(R_1) : R_1, R_2 \in \mathcal{L}(A)\}$.
We are now ready to define the PMV-instability settings whose union models the coalitional influence problems described in the statement of the proposition.

- \( X = \text{CM} \). For every pair of different alternatives \( a, b \), and every pair of feasible signatures \( \vec{t}_a, \vec{t}_b \) such that \( r(\vec{t}_a) = \{a\} \) and \( r(\vec{t}_b) = \{b\} \), \( M \) contains
  \[ \langle H^{\vec{t}_a}, H^{\vec{t}_b}, O^{a \rightarrow b}_\pm, \vec{1} \rangle \]

- \( X = \text{MoV} \). For every pair of different alternatives \( a, b \), and every pair of feasible signatures \( \vec{t}_a, \vec{t}_b \) such that \( r(\vec{t}_a) = \{a\} \) and \( r(\vec{t}_b) = \{b\} \), \( M \) contains
  \[ \langle H^{\vec{t}_a}, H^{\vec{t}_b}, O_{\pm}, \vec{1} \rangle \]

- \( X = \text{CB}_{d, \vec{c}} \). For every pair of feasible signatures \( \vec{t}, \vec{t}_a \) such that \( r(\vec{t}_a) = \{a\} \), \( M \) contains
  \[ \langle H^{\vec{t}}, H^{\vec{t}_a}, O^\ast_{\pm}, \vec{c} \rangle \]

- \( X = \text{DB}_{d, \vec{c}} \). For every alternative \( b \neq a \), and every pair of feasible signatures \( \vec{t}, \vec{t}_b \) such that \( r(\vec{t}_b) = \{b\} \), \( M \) contains
  \[ \langle H^{\vec{t}}, H^{\vec{t}_b}, O^\ast_{\pm}, \vec{c} \rangle \]

- \( X = \text{e-CB}_{d, \vec{c}} \). For every pair of feasible signatures \( \vec{t}, \vec{t}_a \) such that \( r(\vec{t}_a) \neq \{a\} \) and \( r(\vec{t}) = \{a\} \), \( M \) contains
  \[ \langle H^{\vec{t}}, H^{\vec{t}_a}, O^\ast_{\pm}, \vec{c} \rangle \]

- \( X = \text{e-DB}_{d, \vec{c}} \). For every alternative \( b \neq a \), and every pair of feasible signatures \( \vec{t}_a, \vec{t}_b \) such that \( r(\vec{t}_a) = \{a\} \) and \( r(\vec{t}_b) = \{b\} \), \( M \) contains
  \[ \langle H^{\vec{t}_a}, H^{\vec{t}_b}, O^\ast_{\pm}, \vec{c} \rangle \]

\( \square \)

D Materials for Section 4

D.1 Definitions and Conditions for Min-Semi-Random Likelihood

For any set \( \Pi^* \subseteq \mathbb{R}^q \), we define \( B_{\Pi^*} \in \mathbb{R} \) to be the minimum budget \( B \) such that \( \Pi^* \) is completely contained in \( C_B \). If no such \( B \) exists, then we let \( B_{\Pi^*} \triangleq \infty \). Formally,

\[ B_{\Pi^*} \triangleq \inf\{ B \geq 0 : \Pi^* \subseteq C_B \} \quad (10) \]

For example, \( B_{\text{CH}(\Pi)}^- = \infty \) in Figure 1 (b), because no matter how large \( B \) is, \( C_B \subseteq \mathcal{H}_{S_{\leq 0}} \), and \( \mathcal{H}_{S_{\leq 0}} \) does not contain all vectors in \( \text{CH}(\Pi) \).

Next, we define notation and conditions used in the statement of the theorem.

**Definition 5 (Full version).** Given a PMV-instability setting, \( \Pi, B, \) and \( n \), define

\[ d_0 = \dim(C_0), d_\infty = \dim(C_\infty), \text{ and } d_\Delta = d_\infty - d_0, \]

where \( \dim(C_0) \) is the dimension of \( C_0 \), which is the dimension of the minimal affine space that contains \( C_0 \). We also define the following five conditions:
Recall that $\text{CH}(\Pi)$ is the convex hull of $\Pi$. Because $C_0 \subseteq C_\infty$, $d_\Delta \geq 0$. Also notice that $K_2$ implies $K_3$, or equivalently, $-K_3$ implies $-K_2$. Similarly, $K_4$ implies $K_5$, or equivalently, $-K_5$ implies $-K_4$.

### D.2 Properties of $B_{\Pi^*}$ and $B_{\Pi^-}$

**Claim 1.** For any convex and compact set $\Pi^*$, if $B_{\Pi^*} \neq \infty$ then $\Pi^* \cap C_{B_{\Pi^*}} \neq \emptyset$.

**Proof.** Let $\{B_j : j \in \mathbb{N}\}$ denote a sequence that converges to $B_{\Pi^*}$, such that for all $j \in \mathbb{N}$, $\Pi^* \cap C_{B_j} \neq \emptyset$. For any $i \in \mathbb{N}$, let $\vec{y}_i \in \Pi^* \cap C_{B_j}$ denote an arbitrary vector. Because $\Pi^*$ is compact, a subsequence of $\{\vec{y}_j : j \in \mathbb{N}\}$, denoted by $\{\vec{y}_j : i \in \mathbb{N}\}$ converges to a vector $\vec{y}^* \in \Pi^*$. Notice that $\vec{y}_i$ is in $\Pi^*$, $\mathcal{H}_{S_i} \leq 0$, and both are closed sets. Therefore, $\vec{y}^* \in \Pi^* \cap \mathcal{H}_{S_i} \leq 0$.

Next, we prove that $\vec{y}^* \in C_{B_{\Pi^*}}$. For every $i \in \mathbb{N}$, let $\vec{\sigma}_j \in \mathbb{R}_0^{[0]}$ denote the operation vector such that $\vec{\sigma} \cdot \vec{\sigma}_j \leq B_j$, and $\vec{y}_i + \vec{\sigma}_j \times \mathbf{O} \in \mathcal{H}_{T_i} \leq 0$. Let $\vec{x}_i = \vec{y}_i + \vec{\sigma}_j \times \mathbf{O}$. Because $\vec{\sigma}_j$'s are bounded ($\vec{\sigma} \cdot \vec{\sigma}_j \leq B_1$), there exists a subsequence $\{\vec{y}_j : i \in \mathbb{N}\}$ of $\{\vec{y}_j : j \in \mathbb{N}\}$ such that $\vec{y}_j$ converges to a vector $\vec{y}^*$. It is not hard to verify that $\vec{x}_i = \vec{y}_i + \vec{\sigma}_j \times \mathbf{O}$. Because for all $i \in \mathbb{N}$, $\vec{x}_i \in \mathcal{H}_{T_i} \leq 0$ and $\mathcal{H}_{T_i} \leq 0$ is closed, we have $\vec{y}^* + \vec{\sigma} \times \mathbf{O} \in \mathcal{H}_{T_i} \leq 0$ as well. This proves that $\vec{y}^* \in C_{B_{\Pi^*}}$, which completes the proof of Claim 1.

**Claim 2.** For any bounded set $\Pi^*$, if $\Pi^* \subseteq C_\infty$ then $B_{\Pi^-} \neq \infty$.

**Proof.** It suffices to prove that there exists $B^* \geq 0$ so that $\Pi^* \subseteq C_{B^*}$. Because $\Pi^*$ is bounded, let $Q$ denote any cube that contains $\Pi^*$. Because $Q$ is a polytope and $C_\infty$ is a polyhedral cone, $Q \cap C_\infty$ is a polytope that contains $\Pi^*$. Let the V-representation of $Q \cap C_\infty$ be $\text{CH}(\{\vec{x}_1, \ldots, \vec{x}_k\})$ for some $k \in \mathbb{N}$. For every $j \leq k$, because $\vec{x} + j \in C_\infty$, there exists $B_j \geq 0$ such that $\vec{x}_j \in C_{B_j}$. Let $B^* = \max\{B_1, \ldots, B_k\}$. It follows that $\Pi^* \subseteq Q \cap C_\infty \subseteq C_{B^*}$, which proves Claim 2.

### D.3 Full Proof of Theorem 1 and Its Proof

**Theorem 1 (Semi-Random PMV-Instability, $\psi = 0$, full version).** Given any $q \in \mathbb{N}$, any closed and strictly positive $\Pi$ over $[q]$, and any PMV-instability setting $(\mathcal{H}_S, \mathcal{H}_T, \emptyset, \vec{c})$, any $C_2 > 0$ and $C_3 > 0$ with $C_2 < B_{\text{CH}(\Pi)} < C_3$, any $n \in \mathbb{N}$, and any $B \geq 0$,

\[
\sup_{\pi \in \Pi} \Pr(\vec{X}_\pi \in U_{n,B}) = \begin{cases} 
0 & \text{if } B < C_2 n \\
\exp(-\Theta(n)) & \text{if } B \leq C_2 n \\
\Theta\left(\frac{\min\{B + 1, \sqrt{n}\}^{d_\Delta}}{(\sqrt{n})^{q-d_\Delta}}\right) & \text{if } B \geq C_3 n \end{cases} \]

- **Name**: Likelihood
- **Condition**: $0$ case
- $\exp$ case
- $PT-\Theta(\sqrt{n})$
- $PT-\Theta(n)$

- $\exp(-\Theta(n))$
- $\Theta\left(\frac{\min\{B + 1, \sqrt{n}\}^{d_\Delta}}{(\sqrt{n})^{q-d_\Delta}}\right)$

- $K_1$
- $-K_1 \land -K_2 \land K_3$

- $-K_1 \land -K_3$

- $-K_1 \land -K_2 \land K_3$

- otherwise, i.e.,
For any $C_2^- < B_{\text{CH}(\Pi)}^- < C_3^-$, any $n \in \mathbb{N}$, and any $B \geq 0$, we define the likelihood function and the condition:

\[
\inf_{\vec{\pi} \in \Pi^n} \Pr \left( \vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B} \right) = \begin{cases} 
0 & \text{0 case}, \\
\exp(-\Theta(n)) & \text{exp case}, \\
\Theta \left( \frac{\min\{B+1, \sqrt{n}\}^d}{(\sqrt{n})^{q-d_0}} \right) & \text{PT-\Theta(\sqrt{n}) case}, \\
\exp(-\Theta(n)) & \text{PT-\Theta(n) case}
\end{cases}
\]

\[
\Theta \left( \frac{\min\{\sqrt{n}\}^{q-d_0}}{(\sqrt{n})^{q-d_0}} \right)
\]

\[
\Theta \left( \frac{\min\{B+1, \sqrt{n}\}^d}{(\sqrt{n})^{q-d_0}} \right)
\]

\[
B \leq C_2 n, \\
B \geq C_3 n
\]

**Proof.** We first prove the sup part of the theorem, then leverage the techniques to prove the inf part.

**Proof for the sup part.** For convenience, hyperlinks (in red) to the proofs of four cases are provided as follows.

\[
\sup_{\vec{\pi} \in \Pi^n} \Pr \left( \vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B} \right) = \begin{cases} 
0 & \text{0 case}, \\
\exp(-\Theta(n)) & \text{exponential case}, \\
\Theta \left( \frac{\min\{B+1, \sqrt{n}\}^d}{(\sqrt{n})^{q-d_0}} \right) & \text{PT-\Theta(\sqrt{n})-sup case}, \\
\Theta \left( \frac{\min\{\sqrt{n}\}^{q-d_0}}{(\sqrt{n})^{q-d_0}} \right) & \text{PT-\Theta(\sqrt{n})-sup case}
\end{cases}
\]

\[
\text{PT-\Theta(n)-sup} \\
\text{PT-\Theta(\sqrt{n})-sup}
\]

\[
\Theta \left( \frac{\min\{B, \sqrt{n}\}^d}{(\sqrt{n})^{q-d_0}} \right)
\]

\[
\Theta \left( \frac{\min\{\sqrt{n}\}^{q-d_0}}{(\sqrt{n})^{q-d_0}} \right)
\]

\[
B \leq C_2 n, \\
B \geq C_3 n
\]

**Proof for the 0 case of sup** is straightforward, because $\mathcal{U}_{n,B} = \emptyset$ states that the PMV-instability problem does not have a size-$n$ non-negative integer solution. In the rest of the proof for sup, it suffices prove the exponential case and the polynomial case for all $n$ that are larger than a constant $N$. This is because for any $n$ such that the 0 case does not hold (which means that $\mathcal{U}_{n,B} \neq \emptyset$), and for every $\vec{\pi} \in \Pi^n$,

\[
\Pr \left( \vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B} \right) \in [\epsilon^n, 1]
\]

Therefore, for every $n$ below a constant $N$, we have

\[
\sup_{\vec{\pi} \in \Pi^n} \Pr \left( \vec{X}_{\vec{\pi}} \in \mathcal{U}_{n,B} \right) \in [\epsilon^N, 1],
\]

**Proof for the exponential case of sup.** Because CH(\Pi) $\cap C_\infty = \emptyset$, CH(\Pi) is convex and compact, and $C_\infty$ is convex, due to the strict hyperplane separation theorem, for every $\vec{\pi} \in \Pi^n$, $\sum_{i=1}^n \pi_i$ is $\Omega(n)$ away from any vector in $C_\infty$, which means that $\sum_{i=1}^n \pi_i$ is $\Omega(n)$ away from any vector in $H_\infty$, because $C_\infty$ is the characteristic cone of $H_\infty$ as proved in the following claim.

**Claim 3.** The characteristic cone of $H_\infty$ is $C_\infty$.

**Proof.** We prove two general observations about characteristic cones. For each $i \in \{1, 2\}$, let $\mathcal{H}^i$ denote a polyhedron whose V-representation is $\mathcal{V}_i + \mathcal{H}^i_{\leq 0}$, where $\mathcal{V}_i$ is a convex polytope and $\mathcal{H}^i_{\leq 0}$ is the characteristic cone of $\mathcal{H}^i$. 

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Recall that for the \( i \)th dimension of \( \vec{x} \), which is \( \vec{x}_i \), the proof proceeds in the following three steps:

- **Observation 1.** The characteristic cone of \( \mathcal{H}^1 + \mathcal{H}^2 \) is \( \mathcal{H}^1 \leq_0 + \mathcal{H}^2 \leq_0 \). This is because
  \[
  \mathcal{H}^1 + \mathcal{H}^2 = (\mathcal{V}_1 + \mathcal{V}_2) + (\mathcal{H}^1 \leq_0 + \mathcal{H}^2 \leq_0)
  \]
  Here \( \mathcal{H}^1 \leq_0 + \mathcal{H}^2 \leq_0 \) is indeed a finitely generated cone, because suppose for \( i \in \{1, 2\} \), \( \mathcal{H}^i \leq_0 \) is the convex cone generated from \( \mathcal{B}_i \), then it is not hard to verify that \( \mathcal{H}^1 \leq_0 + \mathcal{H}^2 \leq_0 \) is a cone generated by \( \mathcal{B}_1 \cup \mathcal{B}_2 \).

- **Observation 2.** If \( \mathcal{H}^1 \cap \mathcal{H}^2 \neq \emptyset \), then its characteristic cone is \( \mathcal{H}^1 \leq_0 \cap \mathcal{H}^2 \leq_0 \). This is proved by the \( H \)-representations of \( \mathcal{H}^1 \) and \( \mathcal{H}^2 \). Suppose for each \( i \in \{1, 2\} \), \( \mathcal{H}^i = \left\{ \vec{x} : A_i \times (\vec{x})^\top \leq (b_i)^\top \right\} \), which means that \( \mathcal{H}^i_\leq_0 = \left\{ \vec{x} : A_i \times (\vec{x})^\top \leq (b_i)^\top \right\} \). Then, we have
  \[
  \mathcal{H}^1 \cap \mathcal{H}^2 = \left\{ \vec{x} : \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \times (\vec{x})^\top \leq (b_1, b_2)^\top \right\},
  \]
  whose characteristic cone is \( \mathcal{H}^1_\leq_0 \cap \mathcal{H}^2_\leq_0 \).

Then, the upper bound in the exponential case of \( \text{sup} \) follows after a straightforward application of Hoeffding’s inequality and the union bound (applied to all \( q \) dimension). More precisely, recall that \( \vec{\pi} \) is strictly positive, then Hoeffding’s inequality implies that for every \( i \in [q] \), the probability for the \( i \)th dimension of \( \vec{X}_{\vec{\pi}} \) to be more than \( \Omega(n) \) away from the \( i \)th dimension of \( \sum_{j=1}^n \pi_j \) is exponentially small. Therefore, according to the union bound, the probability for the \( L_\infty \) distance between \( \vec{X}_{\vec{\pi}} \) and \( \sum_{j=1}^n \pi_j \) to be \( \Omega(n) \) is exponentially small, which implies that the probability for \( \vec{X}_{\vec{\pi}} \) to be in \( \mathcal{U}_{n,B} \) is exponentially small. The lower bound in the exponential case is straightforward, because for any \( \vec{\pi} \in \Pi^n \) (recall that all distributions in \( \vec{\pi} \) are strictly positive) and any \( \vec{x} \in \mathcal{U}_{n,B} \), \( \Pr(\vec{X}_{\vec{\pi}} = \vec{x}) = \exp(-\Theta(n)) \).

**Proof for the phase transition at \( \Theta(\sqrt{n}) \) case of \( \text{sup} \) (PT-\( \Theta(\sqrt{n}) \)-sup for short).** The proof proceeds in the following three steps:

- **Prove the polynomial upper bound for** \( B \leq \sqrt{n} \), i.e., \( O\left(\frac{(B + 1)\Delta}{(\sqrt{n})^{q-d_0}}\right) \).
- **Prove the polynomial upper bound for** \( B > \sqrt{n} \), i.e., \( O\left(\frac{1}{\sqrt{n}}\right)^{q-d_0} \).
- **Prove the asymptotically matching polynomial lower bound**, i.e., \( \Omega\left(\frac{\min\{B + 1, \sqrt{n}\}\Delta}{(\sqrt{n})^{q-d_0}}\right) \)

We first introduce some notation and assumptions that will be used in the proofs. Let \( A_0 = \begin{bmatrix} A_S \\ A_T \end{bmatrix} \) and let \( A_\infty \) denote an integer matrix that characterizes \( C_\infty \). That is,

\[
C_\infty = \left\{ \vec{x} \in \mathbb{R}^q : A_\infty \times (\vec{x})^\top \leq (0)^\top \right\}
\]
The existence of such $\mathbf{A}_\infty$ is due to [CCZ14, Proposition 3.12], which states that any polyhedron that has a rational H-representation has a rational V-representation, and vice versa. More precisely, because $\mathcal{H}_{T,\leq 0}$ has a rational H-representation, it has a rational V-representation, denoted by $\text{Cone}(\{\bar{x}_1, \ldots, \bar{x}_k\})$, where $\{\bar{x}_1, \ldots, \bar{x}_k\} \subseteq \mathbb{Q}^d$. Then, we have

$$\mathcal{H}_{T,\leq 0} + \mathcal{Q}_\infty = \text{Cone}(\{\bar{x}_1, \ldots, \bar{x}_k\} \cup \mathbb{O}),$$

which means that $\mathcal{H}_{T,\leq 0} + \mathcal{Q}_\infty$ can be represented by a set of linear inequalities with rational coefficients, due to [CCZ14, Proposition 3.12]. Consequently, $\mathcal{H}_{S,\leq 0} \cap (\mathcal{H}_{T,\leq 0} + \mathcal{Q}_\infty)$ can be represented by a set of linear inequalities with rational coefficients, by combining the linear inequalities for $\mathcal{H}_{S,\leq 0}$ and the linear inequalities for $\mathcal{H}_{T,\leq 0} + \mathcal{Q}_\infty$.

Let $\mathbf{A}_0^\equiv$ and $\mathbf{A}_\infty^\equiv$ denote the implicit equalities of $\mathbf{A}_0$ and $\mathbf{A}_\infty$, respectively. We have $\text{Rank}(\mathbf{A}_0^\equiv) = q - d_0$ and $\text{Rank}(\mathbf{A}_\infty^\equiv) = q - d_\infty$ ([CCZ14, Theorem 3.17]). Next, we show that, without loss of generality, in the rest of the proof for PT-$(\sqrt{n})$-sup, we can assume that $\bar{I}$ cannot be represented as a linear combination of rows in $\mathbf{A}_0^\equiv$ or a linear combination of rows in $\mathbf{A}_\infty^\equiv$. Formally,

**Assumption 1.** $\text{Rank}\left(\left[\begin{array}{c} \mathbf{A}_0^\equiv \\ \mathbf{I} \end{array}\right]\right) = q - d_0 + 1$.

**Assumption 2.** $\text{Rank}\left(\left[\begin{array}{c} \mathbf{A}_\infty^\equiv \\ \mathbf{I} \end{array}\right]\right) = q - d_\infty + 1$.

To see that we can assume Assumption 1, suppose for the sake of contradiction that Assumption 1 does not hold, which means that $\bar{I}$ is a linear combination of rows in $\mathbf{A}_0^\equiv$. Then, for every $\bar{x} \in \mathcal{C}_0 = \mathcal{H}_{S,\leq 0} \cap \mathcal{H}_{T,\leq 0}$, we have $\bar{x} \cdot \mathbf{1} = 0$. Therefore, $\text{CH}(\bar{I}) \cap \mathcal{C}_0 = \emptyset$, which contradicts $-K_3$. Similarly, if Assumption 2 does not hold, then we have $\text{CH}(\bar{I}) \cap \mathcal{C}_\infty = \emptyset$, which again contradicts $-K_3$, because $\mathcal{C}_0 \subseteq \mathcal{C}_\infty$.

**Proof for the polynomial upper bound of PT-$(\sqrt{n})$-sup, $B \leq \sqrt{n}$.**

**Overview of proof.** The proof proceeds in three steps. In Step 1, we use $\mathbf{A}_0^\equiv$ and $\mathbf{A}_\infty^\equiv$ to define a partition of $[q]$ into three sets $I_{0+}$, $I_{0-}$, and $I_1$, which contain $q - d_0 + 1$, $d_\infty - d_0$, and $d_\infty - 1$ numbers, respectively. For convenience, we rename the coordinates so that

$$1, \ldots, q - d_\infty + 1, q - d_\infty + 2, \ldots, q - d_0 + 1, q - d_0 + 2, \ldots, q.$$

Let $I_{1+} = I_{0-} \cup I_1$ and let $I_0 = I_{0+} \cup I_{0-}$. Step 2 proves two properties of the partition. Let $\mathcal{H}_{B,n} = \{\bar{x} \in \mathcal{H}_B : \bar{x} \cdot \bar{I} = n\}$ denote the subset of $\mathcal{H}_B$ that consists of all size-$n$ vectors. First, in Step 2.1, we prove that given the $I_{1+}$ coordinates of vectors in $\mathcal{H}_{B,n}$, each of its remaining coordinates (in $I_{0+}$) can take no more than $O(1)$ integer values. Second, in Step 2.2, we prove that given the $I_1$ coordinates of vectors in $\mathcal{H}_{B,n}$, each of its remaining coordinates (in $I_0$) can take no more than $O(B)$ integer values.

In light of Step 2.1 and Step 2.2, we can enumerate integer vectors $\bar{y}$ in $\mathcal{H}_{B,n}$ as follows: first, we fix the $I_1$ coordinates of $\bar{y}$; second, each of the $d_\Delta = d_\infty - d_0$ coordinates in $I_{0-}$ takes no more than $O(1)$ integer values; and finally, each of the $q - d_\infty + 1$ coordinates in $I_{0+}$ takes no more than $O(1)$ integer values. Then in Step 3, we leverage this enumeration method with the Bayesian network representation and the point-wise anti-concentration bound in [Xia21a] to prove the upper bound.

**Step 1 of poly upper bound:** Define the partition $[q] = I_{0+} \cup I_{0-} \cup I_1$. Let $\mathcal{P}_0$ and $\mathcal{P}_\infty$ denote the affine hulls of $\mathcal{C}_0$ and $\mathcal{C}_\infty$, respectively (which are the same as the linear spaces generated
by \( C_0 \) and \( C_\infty \), because both contains \( \emptyset \). It follows from [CCZ14, Theorem 3.17] that
\[
\mathcal{P}_0 = \left\{ \bar{x} \in \mathbb{R}^q : \mathbf{A}_0^\infty \times (\bar{x})^\top = (\bar{0})^\top \right\} \quad \text{and} \quad \mathcal{P}_\infty = \left\{ \bar{x} \in \mathbb{R}^q : \mathbf{A}_\infty^\infty \times (\bar{x})^\top = (\bar{0})^\top \right\}
\] (11)

That is, \( \mathcal{P}_0 \) (respectively, \( \mathcal{P}_\infty \)) consists of vectors that satisfies of the implicit equalities of \( \mathbf{A}_0 \) (respectively, \( \mathbf{A}_\infty \)).

Let \( \mathbf{A}_* = \begin{bmatrix} \mathbf{A}_\infty^\infty \\ \mathbf{A}_0^\infty \end{bmatrix} \). We will define a partition of \([q]\) as \( I_{0+} \cup I_{0-} \cup I_1 \), where \( |I_{0+}| = q - d_\infty + 1 \), \( |I_{0-}| = d_\Delta \), and \( |I_1| = d_0 - 1 \), and the partition satisfies the following two conditions.

- **Condition 1.** The \( I_{0+} \) columns of \( \begin{bmatrix} \mathbf{A}_\infty^\infty \\ \mathbf{I} \end{bmatrix} \) are linearly independent.

- **Condition 2.** The \( I_{0+} \cup I_{0-} \) columns of \( \begin{bmatrix} \mathbf{A}_0^\infty \\ \mathbf{I} \end{bmatrix} \) are linearly independent.

We first define two sets \( I_{0+} \) and \( I_{0-} \) as follows.

Define \( I_{0+}' \). Recall that \( \text{Rank}(\mathbf{A}_0^\infty) = q - d_\infty \). Therefore, \( \mathbf{A}_0^\infty \) contains a set of \( q - d_\infty \) linearly independent column vectors, whose indices are denoted by \( I_{0+}' \subseteq [q] \). W.l.o.g. let \( I_{0+}' = \{1, \ldots, q - d_\infty\} \)—if this is not the case, then we shift the \( I_{0+}' \) columns in \( \mathbf{A}_0^\infty \) to be the first \( q - d_\infty \) columns and rename the coordinates.

Define \( I_{0-}' \). Notice that the \( I_{0+}' \) columns of \( \mathbf{A}_* \) are linearly independent (because their \( \mathbf{A}_\infty^\infty \) parts are already linearly independent). Because \( C_0 \subseteq C_\infty \), we have \( \mathcal{P}_0 \subseteq \mathcal{P}_\infty \), which means that
\[
\left\{ \bar{x} \in \mathbb{R}^q : \mathbf{A}_* \times (\bar{x})^\top = (\bar{0})^\top \right\} = \mathcal{P}_\infty \cap \mathcal{P}_0 = \mathcal{P}_0
\]

This means that \( \text{Rank}(\mathbf{A}_*) = \text{Rank}(\mathbf{A}_0^\infty) = q - d_0 \). Consequently, there exist a set of \( q - d_0 - (q - d_\infty) = d_\infty - d_0 = d_\Delta \) columns of \( \mathbf{A}_* \), whose indices are denoted by \( I_{0-}' \subseteq ([q] \setminus I_{0+}') \), such the \( I_{0+}' \cup I_{0-}' \) columns of \( \mathbf{A}_* \) are linearly independent. W.l.o.g., let \( I_{0-}' = \{q - d_\infty + 1, \ldots, q - d_0\} \).

Notice that the \( I_{0+}' \) columns of \( \begin{bmatrix} \mathbf{A}_\infty^\infty \\ \mathbf{I} \end{bmatrix} \) are linearly independent (because their \( \mathbf{A}_\infty^\infty \) parts are already linearly independent). Let \( J \) denote the indices to columns of \( \begin{bmatrix} \mathbf{A}_\infty^\infty \\ \mathbf{I} \end{bmatrix} \) that are linearly independent with the \( I_{0+}' \) columns. That is,
\[
J = \left\{ i_+ \in ([q] \setminus I_{0+}') : \text{the } I_{0+}' \cup \{i_+\} \text{ columns of } \begin{bmatrix} \mathbf{A}_\infty^\infty \\ \mathbf{I} \end{bmatrix} \text{ are linearly independent} \right\}
\] (12)

By definition, we have \( J \neq \emptyset \), because according to Assumption 2,
\[
\text{Rank} \left( \begin{bmatrix} \mathbf{A}_\infty^\infty \\ \mathbf{I} \end{bmatrix} \right) = \text{Rank} (\mathbf{A}_0^\infty) + 1 = q - d_\infty + 1 > q - d_\infty = |I_{0+}'|
\]

Next, we define two specific columns: \( i_+ \in J \) and \( i_- \) in the following two cases ( \( i_+ = i_- \) in case 2), prove that the \( I_{0+}' \cup I_{0-}' \cup \{i_-\} \) columns of \( \begin{bmatrix} \mathbf{A}_* \\ \mathbf{I} \end{bmatrix} \) are linearly independent (in Claim 4), and then use them to define \( I_{0+}, I_{0-} \) and \( I_1 \).
• **Case 1:** $J \cap I_0^- \neq \emptyset$. Let $i_+$ denote an arbitrary number in $J \cap I_0^-$ and let $i_- \in ([q] \setminus (I_0^+ \cup I_0^-))$ denote an arbitrary number such that the $I_0^+ \cup I_0^- \cup \{i_-\}$ columns of 
\[
\begin{bmatrix}
A_{\infty} \\
I
\end{bmatrix}
\] are linearly independent. The existence of such $i_-$ is guaranteed by the following two observations. First, according to the definitions of $I_0^+$ and $I_0^-$, the $I_0^+ \cup I_0^-$ columns of 
\[
\begin{bmatrix}
A_{\infty} \\
I
\end{bmatrix}
\] are linearly independent. Second, according to Assumption 1,

\[
\text{Rank} \left( \begin{bmatrix} A_{\infty} \\ I \end{bmatrix} \right) \geq \text{Rank} \left( \begin{bmatrix} A_0^- \\ I \end{bmatrix} \right) = \text{Rank} (A_0^-) + 1 = q - d_0 + 1 > q - d_0 = |I_0^+ \cup I_0^-|
\]

• **Case 2:** $J \cap I_0^- = \emptyset$. Choose any $i_+ \in J \subseteq I_1$ and let $i_- = i_+$. We now prove that the $I_0^+ \cup I_0^- \cup \{i_-\}$ columns of 
\[
\begin{bmatrix}
A_{\infty} \\
I
\end{bmatrix}
\] are linearly independent. Suppose for the sake of contradiction this is not true, which means that column $i_-$ of 
\[
\begin{bmatrix}
A_{\infty} \\
I
\end{bmatrix}
\] can be written as an affine combination of the $I_0^+ \cup I_0^-$ columns of 
\[
\begin{bmatrix}
A_0^- \\
I
\end{bmatrix}
\]. This mean that column $i_-$ of 
\[
\begin{bmatrix}
A_{\infty} \\
I
\end{bmatrix}
\] can be written as the same affine combination of the $I_0^+ \cup I_0^-$ columns of 
\[
\begin{bmatrix}
A_{\infty} \\
I
\end{bmatrix}
\]. Recall that $J \cap I_0^- = \emptyset$, which means that in matrix 
\[
\begin{bmatrix}
A_{\infty} \\
I
\end{bmatrix}
\], each column in $I_0^-$ is an affine combination of the $I_0^+$ columns of 
\[
\begin{bmatrix}
A_0^- \\
I
\end{bmatrix}
\]. Therefore, in 
\[
\begin{bmatrix}
A_{\infty} \\
I
\end{bmatrix}
\], column $i_-$ is linearly dependent with the $I_0^+$ columns. This contradicts the definition of $i_-$, which is the same as $i_+ \in J$.

Notice that in both cases, the following claim holds.

**Claim 4.** The $I_0^+ \cup I_0^- \cup \{i_-\}$ columns of 
\[
\begin{bmatrix}
A_{\infty} \\
I
\end{bmatrix}
\] are linearly independent.

**Define $I_{0+}$, $I_{0-}$, and $I_1$.** Given $I_0^+$, $I_0^-$, $i_+$, and $i_-$ defined above, we are now ready to define $I_{0+}$, $I_{0-}$, and $I_1$ as follows. Let

\[
I_{0+} = I_0^+ \cup \{i_+\}, \quad I_{0-} = I_0^- \cup \{i_-\} \setminus \{i_+\}, \quad \text{and} \quad I_1 = [q] \setminus (I_{0+} \cup I_{0-})
\]

By definition, we have $|I_{0+}| = q - d_\infty + 1$, $|I_{0+}| = d_0 - d_\infty$, and $|I_1| = d_0 - 1$. For convenience, we rename the coordinates so that

\[
I_{0+} = \{1, \ldots, q - d_\infty + 1\}, \quad I_{0-} = \{q - d_\infty + 2, \ldots, q - d_0 + 1\} \quad \text{and} \quad I_1 = \{q - d_0 + 2, \ldots, q\}
\]

**Verify Condition 1.** Because $i_+ \in J$, the $I_{0+} = I_0^+ \cup \{i_+\}$ columns of 
\[
\begin{bmatrix}
A_{\infty} \\
I
\end{bmatrix}
\] are linearly independent (due to the definition of $J$ in (12)), which means that **Condition 1** is satisfied.

**Verify Condition 2.** Recall that when defining $i_-$, we proved that the $I_{0+} \cup I_{0-} = I_0^+ \cup I_0^- \cup I_0^- \cup \{i\}$ columns of 
\[
\begin{bmatrix}
A_{\infty} \\
I
\end{bmatrix}
\] are linearly independent. Because 
\[
\text{Rank} \left( \begin{bmatrix} A_{\infty} \\ A_0^- \end{bmatrix} \right) = \text{Rank}(A_\ast) =
\]
$\text{Rank}(A_0^{-}) = q - d_0$, each row in $A_0^{-}$ can be written as an affine combination of rows in $A_0^{-}$. Therefore, if some linear combination of the $I_0+ \cup I_0- = I_0+ \cup I_0- \cup \{i\}$ columns of $[A_0^{-} \hspace{1cm} I]$ equals to $\vec{0}$, then the same linear combination of the $I_0+ \cup I_0- = I_0+ \cup I_0- \cup \{i\}$ columns of $[A_0^{+} \hspace{1cm} I]$ equals to $\vec{0}$ as well, which is a contradiction to Claim 4. This verifies Condition 2.

In the remainder of the proof, we let

$$I_0 = I_0+ \cup I_0- \text{ and } I_1 = I_1 \cup I_0-.$$ 

This leads to two partitions of $[q]$, i.e., $[q] = I_0 \cup I_1 = I_0+ \cup I_1+$, which will be used in the next step.

**Step 2 of poly upper bound:** Bound the width of coordinates in $I_{0+}$ and $I_0$. For any polyhedron $H \subseteq \mathbb{R}^n$, any $I \subseteq [q]$, any $\vec{y} \in \mathbb{R}^I$, and any $i \in ([q] \setminus I)$, let $\text{width}_i(H, \vec{y})$ denote the difference between the maximum value of the $i$-th component of vectors in $H$ whose $I$-components are $\vec{y}$ and the minimum value of the $i$-th component of vectors in $H$ whose $I$-components are $\vec{y}$. Formally,

$$\text{width}_i(H, \vec{y}) = \max_{x \in H, [x]_I = \vec{y}} [x]_i - \min_{x \in H, [x]_I = \vec{y}} [x]_i.$$ 

Recall that $H_{B,n} = \{ \vec{x} \in H_B : \vec{x} \cdot \vec{1} = n \}$. In Step 2.1 and Step 2.2, we bound width$_i(H_{B,n}, \vec{y})$ for $I = I_{1+} = I_0- \cup I_1$ and $I = I_1$, respectively.

**Step 2.1 of poly upper bound:** Bound the width of coordinates in $I_{0+}$. In this step, we prove that there exists a constant $C^*$ such that for any $B$, any $n$, any $\vec{y}_{I_{1+}} \in \mathbb{R}^{I_{1+}}$, and any $i \in I_{0+},$

$$\text{width}_i(H_{B,n}, \vec{y}_{I_{1+}}) \leq C^*$$

Notice that $H_{B,n} \subseteq H_{\infty,n} = \{ \vec{x} \in H_\infty : \vec{x} \cdot \vec{1} = n \}$. Therefore, it suffices to prove the following stronger inequality.

$$\text{width}_i(H_{\infty,n}, \vec{y}_{I_{1+}}) \leq C^* \quad (13)$$

According to the V-representation of $H_\infty$, for any $\vec{y} = (\vec{y}_{I_{0+}}, \vec{y}_{I_{1+}}) \in H_{\infty,n} \subseteq H_\infty$, we can write $\vec{y} = \vec{v} + \vec{x}$, where $\vec{v} = (\vec{v}_{I_{0+}}, \vec{v}_{I_{1+}})$ is in a convex polytope and $\vec{x} = (\vec{x}_{I_{0+}}, \vec{x}_{I_{1+}})$ is in the characteristic cone of $H_\infty$. Let $n' = \vec{x} \cdot 1$. Next, we use Gauss-Jordan elimination to define a matrix $D_\infty$ based on $\left[ \begin{array}{c|c} A_\infty^{-} & \vec{1} \end{array} \right]$, such that

$$\vec{x}_{I_{0+}} = (\vec{x}_{I_{1+}}, n') \times D_\infty \quad (14)$$

By Claim 3, we have $\vec{x} \in C_\infty \subseteq P_\infty$. Recall from Condition 1 that the first $q - d_\infty + 1$ columns (i.e., the $I_{0+}$ columns) of $\left[ \begin{array}{c|c} A_\infty^{-} & \vec{1} \end{array} \right]$ are linearly independent, and recall from Assumption 2 that $\text{Rank} \left( \begin{array}{c|c} A_\infty^{-} & \vec{1} \end{array} \right) = q - d_\infty + 1$. Therefore, Gauss-Jordan elimination on $\left[ \begin{array}{c} A_\infty^{-} \\ \vec{1} \end{array} \right] \times (\vec{x})^\top = (\vec{0}, n')^\top$ leads to a $d_\infty \times (q - d_\infty + 1)$ matrix $D_\infty$ such that

$$\left[ \begin{array}{c} A_\infty^{-} \\ \vec{1} \end{array} \right] \times (\vec{x})^\top = (\vec{0}, n')^\top \text{ if and only if } \vec{x}_{I_{0+}} = (\vec{x}_{I_{1+}}, n') \times D_\infty,$$

which proves (14).
Let $C_{\text{max}}$ denote the maximum $L_\infty$ norm of vectors in $\mathcal{V}$, which means that $|\vec{v}|_\infty \leq C_{\text{max}}$. Then,

$$|n - n'| = |\vec{y} \cdot \vec{l} - \vec{x} \cdot \vec{l}| = |\vec{v} \cdot \vec{l}| \leq qC_{\text{max}}$$

Let $\hat{C}$ denote the maximum absolute value of entries in $D_\infty$ and let $C^* = C_{\text{max}} + 2qC_{\text{max}}\hat{C}$. We prove that $\vec{y}_{I_0}$ is in a $C^*$ neighborhood of $(\vec{y}_{I_1}, n) \times D_\infty$ in $L_\infty$ as follows.

$$|\vec{y}_{I_0} - (\vec{y}_{I_1}, n) \times D_\infty|_\infty = |\vec{y}_{I_0} + \vec{x}_{I_0} - (\vec{y}_{I_1} + \vec{x}_{I_1}, n) \times D_\infty|_\infty$$

$$= |\vec{y}_{I_0} + (\vec{x}_{I_1}, n') \times D_\infty - (\vec{y}_{I_1} + \vec{x}_{I_1}, n) \times D_\infty|_\infty$$

$$= |\vec{y}_{I_0} - (\vec{y}_{I_1} + n - n') \times D_\infty|_\infty$$

by (14)

$$\leq C_{\text{max}} + 2qC_{\text{max}}\hat{C} = C^*$$

This proves (13) and completes Step 2.1.

**Step 2.2 of poly upper bound: Bound the width of coordinates in $I_0$.** In this step, we prove that there exists a constant $C^*$ such that for any $B \geq 0$, any $n$, any $\vec{y}_{I_1} \in \mathbb{R}^n$, and any $i \in I_0$,

$$\text{width}_i(H_{B,n}, \vec{y}_{I_1}) \leq C^*(B + 1) \quad (15)$$

We first prove that for any $\vec{x} \in H_B$, there exists $\vec{x}' \in C_0$ that is $O(B + 1)$ away from $\vec{x}$ in $L_\infty$.

**Claim 5.** There exists $C$ such that for any $\vec{x} \in H_B$, there exists $\vec{x}' \in C_0$ such that $|\vec{x} - \vec{x}'|_\infty \leq C(B + 1)$.

**Proof.** The proof is done by analyzing the following two linear programs, denoted by $LP^B_H$ and $LP^B_{\text{Cone}}$ whose variables are $\vec{x}$ and $\vec{\sigma}$.

| $LP^B_H$ | $LP^B_{\text{Cone}}$ |
|---|---|
| max 0 | max 0 |
| s.t. $A_S \times (\vec{x})^\top \leq (b_S)^\top$ | s.t. $A_S \times (\vec{x})^\top \leq (\vec{0})^\top$ |
| $A_T \times (\vec{x} + \vec{\sigma} \times O)^\top \leq (b_T)^\top$ | $A_T \times (\vec{x} + \vec{\sigma} \times O)^\top \leq (\vec{0})^\top$ |
| $-\vec{\sigma} \leq \vec{0}$ | $-\vec{\sigma} \leq \vec{0}$ |
| $\vec{c} \cdot \vec{\sigma} \leq B$ | $\vec{c} \cdot \vec{\sigma} \leq 0$ |

Because $\vec{x} \in H_B$, there exists $\vec{\sigma} \geq \vec{0}$ such that $\vec{x} + \vec{\sigma} \times O \in H_T$ and $\vec{c} \cdot \vec{x} \leq B$. Therefore, $(\vec{x}, \vec{w})$ is a feasible solution to $LP^B_H$. Notice that $LP^B_{\text{Cone}}$ is feasible (for example, $\vec{0}$ is a feasible solution) and $LP^B_H$ and $LP^B_{\text{Cone}}$ only differ on the right hand side of the inequalities. Therefore, due to [CGST86, Theorem 5 (i)], there exists a feasible solution $(\vec{x}', \vec{w}')$ to $LP^B_{\text{Cone}}$ that is no more than $q\Delta \max\{|b_S - \vec{0}|_\infty, |b_T - \vec{0}|_\infty, B\} = O(B + 1)$ away from $(\vec{x}, \vec{w})$ in $L_\infty$, where $\Delta$ is the maximum absolute value of determinants of square sub-matrices of the left hand side of $LP^B_H$ and $LP^B_{\text{Cone}}$.

i.e.,

$$\begin{bmatrix}
A_S & 0 \\
A_T & A_T \times (O)^\top
\end{bmatrix}$$

Recall that $\vec{c} \geq \vec{0}$. This means that the $-\vec{\sigma} \leq \vec{0}$ constraint and the $\vec{c} \cdot \vec{\sigma} \leq 0$ constraint in $LP^B_{\text{Cone}}$ imply that $\vec{\sigma} = \vec{0}$, which means that $\vec{x}' \in C_0$. This completes the proof of Claim 5. \qed
Like in Step 2.1, we define $D_0$ to be the matrix obtained from applying Gauss-Jordan elimination on $\begin{bmatrix} A & \bar{e} \\ \bar{1} & 1 \end{bmatrix}$. That is, for every $\bar{x}' = (\bar{x}'_{I_0}, \bar{x}'_{I_1}) \in C_0$, let $n' = \bar{x}' \cdot \bar{1}$, we have

$$\bar{x}'_{I_0} = (\bar{x}'_{I_1}, n') \times D_0 \tag{16}$$

Next, we use Claim 5 to prove that for any $\bar{y} = (\bar{y}_{I_0}, \bar{y}_{I_1}) \in H_{B,n}$, $|\bar{y} - ((\bar{y}_{I_1}, n) \times D_0, \bar{y}_{I_1})|_\infty = O(B + 1)$, which would prove (15).

For any $\bar{y} = (\bar{y}_{I_0}, \bar{y}_{I_1}) \in H_{B,n}$, let $\bar{x}' = (\bar{x}'_{I_0}, \bar{x}'_{I_1}) \in C_0$ denote the vector in $C_0$ that is no more than $C(B + 1)$ away from $\bar{y}$ guaranteed by Claim 5. This means that $\bar{x}'_{I_1}$ and $n'$ are $O(B + 1)$ away from $\bar{y}_{I_1}$ and $n$, respectively, which implies that $\bar{x}'_{I_0} = (\bar{x}'_{I_1}, n') \times D_0$ is $O(B + 1)$ away from $(\bar{y}_{I_1}, n) \times D_0$. Also because $\bar{x}'_{I_0}$ is $O(B + 1)$ away from $\bar{y}_{I_0}$, we have that $\bar{y}_{I_0}$ is $O(B + 1)$ away from $(\bar{y}_{I_1}, n) \times D_0$. Formally, let $d_{\text{max}}$ denote the maximum absolute value of entries in $D_0$, we have the following bound.

$$|\bar{y} - ((\bar{y}_{I_1}, n) \times D_0, \bar{y}_{I_1})|_\infty = |\bar{y}_{I_0} - (\bar{y}_{I_1}, n) \times D_0|_\infty$$

$$\leq |\bar{y}_{I_0} - \bar{x}'_{I_0}|_\infty + |\bar{x}'_{I_0} - (\bar{x}'_{I_1}, n') \times D_0|_\infty + |(\bar{x}'_{I_1}, n') \times D_0 - (\bar{y}_{I_1}, n) \times D_0|_\infty$$

$$\leq C(B + 1) + 0 + |(\bar{x}'_{I_1} - \bar{y}_{I_1}, n' - n) \times D_0|_\infty \tag{16}$$

by (16)

$$\leq C(B + 1)(2q d_{\text{max}} + 1)$$

The last inequality holds because $|\bar{x}'_{I_1} - \bar{y}_{I_1}|_\infty \leq C(B + 1)$ and $|n' - n| \leq qC(B + 1)$. This completes the proof of Step 2.2.

**Step 3 of poly upper bound: Upper-bound the probability.** Recall that $\vec{X}_{\bar{\pi}} = \text{Hist}(P)$, where $P$ consists of $n$ independent random variables $Y_1, \ldots, Y_n$ distributed as $\bar{\pi}$. Like [Xia21a], we represent each $Y_j$ as two random variables $Z_j$ and $W_j$ and a simple Bayesian network based on the partition $[q] = I_0 \cup I_1$.

**Definition 14 (Alternative representation of $Y_1, \ldots, Y_n$ [Xia20]).** For each $j \leq n$, we define a Bayesian network with two random variables $Z_j \in \{0, 1\}$ and $W_{j} \in [q]$, where $Z_j$ is the parent of $W_j$. The conditional probabilities are defined as follows.

- For each $\ell \in \{0, 1\}$, let $\Pr(Z_j = \ell) \triangleq \Pr(Y_j \in I_\ell)$.
- For each $\ell \in \{0, 1\}$ and each $t \leq q$, let $\Pr(W_j = t|Z_j = \ell) \triangleq \Pr(Y_j = t|Y_j \in I_\ell)$.

In particular, if $t \notin I_\ell$ then $\Pr(W_j = t|Z_j = \ell) = 0$. It is not hard to verify that for any $j \leq n$, $W_j$ has the same distribution as $Y_j$. For any $\vec{z} \in \{0, 1\}^n$, we let $Id_0(\vec{z}) \subseteq [n]$ denote the indices of components of $\vec{z}$ that equal to 0. Given $\vec{z}$, we define the following random variables.

- Let $\vec{W}_{Id_0}(\vec{z}) \triangleq \{W_j : j \in Id_0(\vec{z})\}$. That is, $\vec{W}_{Id_0}(\vec{z})$ consists of random variables $\{W_j : z_j = 0\}$.
- Let $\text{Hist}(\vec{W}_{Id_0}(\vec{z}))$ denote the vector of the $q - d_0 + 1 = |I_0|$ random variables that correspond to the histogram of $\vec{W}_{Id_0}(\vec{z})$ restricted to $I_0$. Technically, the domain of every random variable in $\vec{W}_{Id_0}(\vec{z})$ is $[q]$, but since they only receive positive probabilities on $I_0$, they are treated as random variables over $I_0$ when $\text{Hist}(\vec{W}_{Id_0}(\vec{z}))$ is defined.
- Similarly, let $\vec{W}_{Id_1}(\vec{z}) \triangleq \{W_j : j \in Id_1(\vec{z})\}$ and let $\text{Hist}(\vec{W}_{Id_1}(\vec{z}))$ denote the vector of $|I_1| = d_0 - 1$ random variables that correspond to the histogram of $\vec{W}_{Id_1}(\vec{z})$.

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Let $\mathcal{H}_{B,n}^{Z} \triangleq \mathcal{H}_{B,n} \cap \mathbb{Z}^n$. For any $\vec{y}_1 \in \mathbb{Z}^{d_0-1}_{\geq 0}$, we let $\mathcal{H}_{B,n}^{Z} \mid \vec{y}_1$ denote the $I_0$ components of $\vec{y} \in \mathcal{H}_{B,n}^{Z}$ whose $I_1$ components are $\vec{y}_1$. Formally,

$$\mathcal{H}_{B,n}^{Z} \mid \vec{y}_1 \triangleq \{ \vec{y}_0 \in \mathbb{Z}^{d_0-1}_{\geq 0} : (\vec{y}_0, \vec{y}_1) \in \mathcal{H}_{B,n}^{Z} \}$$

We recall the following calculations in [Xia21a] for any $\vec{a} \in \Pi^n$, which is done by first separating the $|\text{Id}_0(\vec{y})| \geq 0.9en$ case (which happens with $1-\exp(-\Omega(n))$ probability) from the $|\text{Id}_0(\vec{y})| < 0.9en$ (which happens with exponentially small probability), then applying the law of total probability conditioned on $\vec{z}$, and finally using the conditional independence in the Bayesian network (i.e., $\vec{W}$’s are independent given $\vec{Z}$) to simplify the formula.

$$\Pr_{P \sim \vec{a}}(\text{Hist}(P) \in \mathcal{H}_{B,n}^{Z}) \leq \sum_{\vec{z} \in \{0,1\}^n : |\text{Id}_0(\vec{z})| \geq 0.9en} \Pr(\vec{Z} = \vec{z}) \sum_{\vec{y}_1 \in \mathbb{Z}^{d_0-1}_{\geq 0}} \Pr(\text{Hist}(\vec{W}_{\text{Id}_1}(\vec{z})) = \vec{y}_1 \mid \vec{Z} = \vec{z})$$

$$\times \Pr(\text{Hist}(\vec{W}_{\text{Id}_0}(\vec{z})) \in \mathcal{H}_{B,n}^{Z} \mid \vec{y}_1) \mid \vec{Z} = \vec{z}) + \Pr(|\text{Id}_0(\vec{y})| < 0.9en) \quad (17)$$

To upper-bound (17), we will show that for any $\vec{z}$ with $|\text{Id}_0(\vec{z})| \geq 0.9en$ and any $\vec{y}_1 \in \mathbb{Z}^{d_0-1}_{\geq 0}$,

$$\Pr(\text{Hist}(\vec{W}_{\text{Id}_0}(\vec{z})) = \vec{y}_1 \mid \vec{Z} = \vec{z}) = O((B + 1)^{d_\Delta}) \times O \left( n^{\frac{d_0-q}{2}} \right) \quad (18)$$

Conditioned on $\vec{Z} = \vec{z}$, Hist($\vec{W}_{\text{Id}_0}(\vec{z})$) can be viewed as a PMV of $|\text{Id}_0(\vec{z})|$ strictly positive independent variables over $\text{Id}_0 = [q - d_0 + 1]$. Therefore, according to the point-wise anti-concentration bound [Xia20, Lemma 3 in the Appendix], for any $\vec{z}$ with $|\text{Id}_0(\vec{z})| \geq 0.9en$ and any $\vec{y}_0 \in \mathcal{H}_{B,n}^{Z} \mid \vec{y}_1$,

$$\Pr(\text{Hist}(\vec{W}_{\text{Id}_0}(\vec{z})) = \vec{y}_0 \mid \vec{Z} = \vec{z}) = O \left( |\text{Id}_0(\vec{z})|^{\frac{d_0-q}{2}} \right) = O \left( n^{\frac{d_0-q}{2}} \right)$$

Then, to prove (18), it suffices to prove $|\mathcal{H}_{B,n}^{Z} \mid \vec{y}_1| = O((B + 1)^{d_\Delta})$. This is done by enumerating vectors in $\mathcal{H}_{B,n}^{Z} \mid \vec{y}_1$ as follows: According to Step 2.2 of the poly upper bound, each $I_{0-}$ component of vectors in $\mathcal{H}_{B,n}^{Z} \mid \vec{y}_1$ has no more than $[C^*(B + 1) + 1]$ choices, and given the $I_{0-}$ components, each $I_{0+}$ component has no more than $[C^* + 1]$ choices, where $C^*$ is the maximum value of the constants in Step 2.1 and 3.2. Therefore,

$$|\mathcal{H}_{B,n}^{Z} \mid \vec{y}_1| \leq (C^* + 1)^{q - d_\infty + 1}(C^*(B + 1) + 1)^{d_\infty - d_0} = O((B + 1)^{d_\infty - d_0}) = O((B + 1)^{d_\Delta})$$

This proves (18). Then, combining (17) and (18), and following a similar argument as in [Xia21a], we have

$$\Pr_{P \sim \vec{a}}(\text{Hist}(P) \in \mathcal{U}_{n,B}) \leq \Pr_{P \sim \vec{a}}(\text{Hist}(P) \in \mathcal{H}_{B,n}^{Z})$$

$$\leq \sum_{\vec{z} \in \{0,1\}^n : |\text{Id}_0(\vec{z})| \geq 0.9en} \Pr(\vec{Z} = \vec{z}) \sum_{\vec{y}_1 \in \mathbb{Z}^{d_0-1}_{\geq 0}} \Pr(\text{Hist}(\vec{W}_{\text{Id}_1}(\vec{z})) = \vec{y}_1 \mid \vec{Z} = \vec{z})$$

$$\times O((B + 1)^{d_\Delta}) \times O \left( n^{\frac{d_0-q}{2}} \right) + \exp(-\Theta(n))$$

$$= \Pr(|\text{Id}_0(\vec{Z})| \geq 0.9en) \times O((B + 1)^{d_\Delta}) \times O \left( n^{\frac{d_0-q}{2}} \right) + \exp(-\Theta(n))$$

$$= O \left( (B + 1)^{d_\Delta} \cdot \left( \frac{1}{\sqrt{n}} \right)^{q-d_0} \right)$$

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This proves the polynomial upper bound for $B \leq \sqrt{n}$.

**Proof for the polynomial upper bound of sup for $B > \sqrt{n}$**. Notice that for any $B$ and $n$, $\mathcal{H}_{B,n} \subseteq \mathcal{H}_\infty$. Therefore, it suffices to prove the following stronger claim, which holds for any $B$, any $n$, and any $\bar{\pi} \in \Pi^n$.

**Claim 6.** For any $n$ and $\bar{\pi} \in \Pi^n$, $\Pr \left( X_{\bar{\pi}} \in \mathcal{H}_\infty \right) = O \left( \left( \frac{1}{\sqrt{n}} \right)^{q-d_\infty} \right)$.

**Proof.** The proof is similar to Step 3 of the $B \leq \sqrt{n}$ case above. We will define a Bayesian network for the partition $[q] = I_{0+} \cup I_{1+}$, and then apply (13) to enumerate vectors in $\mathcal{H}_{\infty,n}^{Z \mid \bar{y}_{1+}}$.

**Definition 15 (Alternative representation of $Y_1, \ldots, Y_n$).** For each $j \leq n$, we define a Bayesian network with two random variables $Z_j^+ \in \{0, 1\}$ and $W_j^+ \in [q]$, where $Z_j^+$ is the parent of $W_j^+$. The conditional probabilities are defined as follows.

- For each $\ell \in \{0, 1\}$, let $\Pr(Z_j^+ = \ell) \triangleq \Pr(Y_j \in I_{\ell+})$.
- For each $\ell \in \{0, 1\}$ and each $t \leq q$, let $\Pr(W_j^+ = t \mid Z_j^+ = \ell) \triangleq \Pr(Y_j = t \mid Y_j \in I_{\ell+})$.

In particular, if $t \notin I_{\ell+}$ then $\Pr(W_j^+ = t \mid Z_j^+ = \ell) = 0$. It is not hard to verify that for any $j \leq n$, $W_j^+$ follows the same distribution as $Y_j$. For any $\bar{z} \in \{0, 1\}^n$, we let $\text{Id}_{0+}(\bar{z}) \subseteq [n]$ denote the indices of components of $\bar{z}$ that equal to 0. Given $\bar{z}$, we define the following random variables.

- Let $\bar{W}_{\text{id}_{0+}+}(\bar{z}) \triangleq \{ W_j^+ : j \in \text{Id}_{0+}(\bar{z}) \}$. That is, $\bar{W}_{\text{id}_{0+}+}(\bar{z})$ consists of random variables $\{ W_j^+ : z_j = 0 \}$.
- Let $\text{Hist}(\bar{W}_{\text{id}_{0+}+}(\bar{z}))$ denote the vector of the $|I_{0+}| = q-d_\infty+1$ random variables that correspond to the histogram of $\bar{W}_{\text{id}_{0+}+}(\bar{z})$ restricted to $I_{0+}$.
- Similarly, let $\bar{W}_{\text{id}_{1+}+}(\bar{z}) \triangleq \{ W_j^+ : j \in \text{Id}_{1+}(\bar{z}) \}$ and let $\text{Hist}(\bar{W}_{\text{id}_{1+}+}(\bar{z}))$ denote the vector of $|I_{1+}| = d_\infty-1$ random variables that correspond to the histogram of $\bar{W}_{\text{id}_{1+}+}(\bar{z})$.

For any $\bar{y}_{1+} \in \mathbb{Z}_{\geq 0}^{I_{1+}}$, we let $\mathcal{H}_{\mathcal{Z}_{\infty,n}^{Z \mid \bar{y}_{1+}}}$ denote the $I_{0+}$ components of $\bar{y} \in \mathcal{H}_{\mathcal{Z}_{\infty,n}^{Z}}$ whose $I_{1+}$ components are $\bar{y}_{1+}$. Formally,

$$
\mathcal{H}_{\mathcal{Z}_{\infty,n}^{Z \mid \bar{y}_{1+}}} \triangleq \left\{ \bar{y}_{0+} \in \mathbb{Z}_{\geq 0}^{q-d_\infty+1} : (\bar{y}_{0+}, \bar{y}_{1+}) \in \mathcal{H}_{\mathcal{Z}_{\infty,n}^{Z}} \right\}
$$

Like Step 3 of the $B \leq \sqrt{n}$ case, for any $\bar{\pi} \in \Pi^n$,

$$
\Pr_{P \sim \bar{\pi}}(\text{Hist}(P) \in \mathcal{H}_{\mathcal{Z}_{\infty,n}^{Z}}) \leq \sum_{\bar{z} \in \{0, 1\}^n : |\text{Id}_{0+}(\bar{z})| \geq 0.9en} \Pr(\bar{Z}^+ = \bar{z}) \sum_{\bar{y}_{1+} \in \mathbb{Z}_{\geq 0}^{d_\infty-1}} \Pr \left( \text{Hist}(\bar{W}_{\text{id}_{1+}+}(\bar{z})) = \bar{y}_{1+} \mid \bar{Z}^+ = \bar{z} \right)
$$

$$
\times \Pr \left( \text{Hist}(\bar{W}_{\text{id}_{0+}+}(\bar{z})) \in \mathcal{H}_{\mathcal{Z}_{\infty,n}^{Z \mid \bar{y}_{1+}}} \mid \bar{Z}^+ = \bar{z} \right) + \Pr(|\text{Id}_{0+}(\bar{z})| < 0.9en)
$$

(19)

Following (13) and the point-wise anti-concentration bound [Xia20, Lemma 3 in the Appendix], for any $\bar{z}$ with $|\text{Id}_{0+}(\bar{z})| \geq 0.9en$ and any $\bar{y}_{1+} \in \mathbb{Z}_{\geq 0}^{d_\infty-1}$, we have

$$
\Pr \left( \text{Hist}(\bar{W}_{\text{id}_{0+}+}(\bar{z})) \in \mathcal{H}_{\mathcal{Z}_{\infty,n}^{Z \mid \bar{y}_{1+}}} \mid \bar{Z}^+ = \bar{z} \right) = O(1) \times O \left( n^{\frac{d_\infty-q}{2}} \right)
$$

(20)
Then, combining (19) and (20) and recalling that Hist(\(P\)) is a size-\(n\) non-negative integer vector, we have

\[
\Pr_{\tilde{P} \sim \pi}(\text{Hist}(P) \in \mathcal{U}_{n,B}) \leq \Pr_{\tilde{P} \sim \pi}(\text{Hist}(P) \in \mathcal{H}_{\infty,n}^Z)
\]

\[
\leq \sum_{\tilde{z} \in \{0,1\}^n:|\text{Id}_0(\tilde{z})| \geq 0.9\epsilon n} \Pr(\tilde{Z}^+ = \tilde{z}) \sum_{\tilde{y}_1 \in \mathbb{Z}^{d_{\infty}-1}_{\geq 0}} \Pr\left(\text{Hist}(\tilde{W}^+_{\text{Id}_1(\tilde{z})}) = \tilde{y}_1 \mid \tilde{Z}^+ = \tilde{z}\right)
\]

\[
\times O\left(n^{d_{\infty}-2} \epsilon n\right) + \exp(-\Theta(n))
\]

\[
= \Pr(|\text{Id}_0(\tilde{Z}^+) | \geq 0.9\epsilon n) \times O\left(n^{d_{\infty}-2} \epsilon n\right) + \exp(-\Theta(n)) = O\left(\left(\frac{1}{\sqrt{n}}\right)^{q-d_{\infty}}\right)
\]

This proves Claim 6.

The polynomial upper bound for \(B > \sqrt{n}\) follows after Claim 6.

**Proof for the \(\Omega\left(\min\{B + 1, \sqrt{n}\}^{d_{\Delta}} / (\sqrt{n})^{q-d_0}\right)\)** lower bound of PT-\(\Theta(\sqrt{n})\)-sup. It suffices to prove the lower bound for \(B \leq \sqrt{n}\), because when \(B \geq \sqrt{n}\), we have

\[
\sup_{\pi \in \Pi^0} \Pr(X_{\pi}^0 \in \mathcal{U}_{n,B}) \geq \sup_{\pi \in \Pi^0} \Pr(X_{\pi}^0 \in \mathcal{U}_{n,\sqrt{n}}) = \Omega\left(\left(\frac{1}{\sqrt{n}}\right)^{q-d_{\infty}}\right),
\]

which is the desired lower bound.

The proof for \(B \leq \sqrt{n}\) proceeds in three steps. In Step 1, for any strictly positive \(\pi^* \in \mathcal{C}_0\), we identify an \(O(\sqrt{n})\) neighborhood of \(n \cdot \pi^*\) that contains \(\Omega\left((B + 1)^{d_\Delta} \cdot (\sqrt{n})^{d_0-1}\right)\) integer vectors in \(\mathcal{U}_{n,B}\). In Step 2, for any \(\pi^* \in \text{CH}(\Pi)\), we define \(\tilde{\pi}^0 = (\pi_1^0, \ldots, \pi_n^0)\) \(\in \Pi^n\) such that \(\sum_{j=1}^n \pi_j^0\) is \(O(\sqrt{n})\) away from \(n \cdot \pi^*\). The lower bound is then proved in Step 3. Among the three steps, Step 1 is the most involved part, and Steps 2 and 3 follow after similar proofs in [Xia21a].

**Step 1 of poly lower bound: Identify \(\Omega\left((B + 1)^{d_\Delta} \cdot (\sqrt{n})^{d_0-1}\right)\) vectors in \(\mathcal{U}_{n,B}\).**

**Overview of Step 1.** The proof proceeds in four sub-steps. In Step 1.1, we prove in Claim 7 that there exists a constant \(C'\) such that for every (possibly non-negative) integer vector in \(\mathcal{H}_{B,n}\), there is an “accompany” integer vector in \(\mathcal{U}_{n,B}\) that is at most \(C'\) away in \(L_\infty\). Then, we identify a set \(\mathcal{R}_{B,n}\) of possibly non-negative integer vectors in \(\mathcal{H}_{B,n}\) that are at least \(2C'\) away from each other, and apply Claim 7 to obtain a set \(\mathcal{R}_{B,n}^0\) of integer vectors in \(\mathcal{U}_{n,B}\) of the same size (as \(\mathcal{R}_{B,n}\)). To define \(\mathcal{R}_{B,n}\), we explore two directions in an \(O(\sqrt{n})\) neighborhood of \(n \cdot \pi^*\): the \(\mathcal{P}_0\) direction and the \(\mathcal{P}_\infty\) direction (which we recall from (11) are affine hulls of \(\mathcal{C}_0\) and \(\mathcal{C}_\infty\), respectively). More precisely, we will first enumerate \(\Omega((\sqrt{n})^{d_0-1})\) vectors in a \(\mathcal{P}_0\) neighborhood of \(n \cdot \pi^*\) (defined as \(\mathcal{R}_{n}^{0}\) in Step 1.2), and then enumerate \(\Omega((B + 1)^{d_\Delta})\) vectors in a neighborhood of \(n \cdot \pi^*\) (defined as \(\mathcal{R}_{B,n}^{\Delta}\) in Step 1.3) that is a complement of \(\mathcal{P}_0\) in \(\mathcal{P}_\infty\). Finally, in Step 1.4, we formally define \(\mathcal{R}_{B,n}\) and \(\mathcal{R}_{B,n}^0\) and prove that \(|\mathcal{R}_{B,n}| = |\mathcal{R}_{B,n}^0| = \Omega((B + 1)^{d_\Delta} \cdot (\sqrt{n})^{d_0-1})\) and \(\mathcal{R}_{B,n} \subseteq \mathcal{H}_{B,n} \cap \mathbb{R}_{\geq 0}^d\) in Claim 8 and Claim 9, respectively.

**Step 1.1: Define \(C'\).** We prove the following claim.

**Claim 7.** There exists a constant \(C'\) that does not depend on \(n\) or \(B\), such that for any \(\bar{x} \in \mathcal{H}_{B,n} \cap \mathbb{R}_{\geq 0}^d\), there exists a (non-negative integer) vector \(\bar{x}' \in \mathcal{U}_{n,B}\) such that \(|\bar{x} - \bar{x}'|_\infty < C'\).
Proof. Consider the following linear program $LP_{B,n}$ whose variables are \( \bar{x} \) and \( \bar{\sigma} \):

\[
LP_{B,n} = \left\{ \begin{array}{l}
\max \quad 0 \\
\text{s.t.} \quad A_S \times (\bar{x})^\top \leq (b_S)^\top \\
\quad A_T \times (\bar{x} + \bar{\sigma} \times \mathbf{O})^\top \leq (b_T)^\top \\
\quad \bar{x} \cdot \mathbf{1} = n \\
\quad \bar{c} \cdot \bar{\sigma} \leq B \\
\quad \bar{x} \geq \bar{0}, \quad \bar{\sigma} \geq \bar{0}
\end{array} \right.
\]

It is not hard to verify that, because \( \bar{x} \in H_{B,n} \cap \mathbb{R}_+^n \), \( LP_{B,n} \) has a feasible solution \((\bar{x}, \bar{\sigma})\) for some \( \bar{\sigma} \geq \bar{0} \) (which may not be an integer vector). Recall that \( U_{n,B} \neq \emptyset \), which means that \( LP_{B,n} \) has a feasible integer solution. Therefore, by [CGST86, Theorem 1], \( LP_{B,n} \) has an integer solution \((\bar{x}^*, \bar{\sigma}^*)\) whose \( L_\infty \) distance to \((\bar{x}, \bar{\sigma})\) is no more than \((q + |\mathcal{O}|)\Delta\), where \( \Delta \) is the maximum absolute determinant of square submatrices of the matrix that defines \( LP_{B,n} \), i.e.,

\[
\begin{bmatrix}
A_S & 0 \\
A_T & A_T \times (\mathbf{O})^\top \\
\mathbf{1} & \bar{0} \\
-\mathbf{1} & \bar{0} \\
\bar{0} & \bar{c} \\
\bar{0} & -\mathbf{1}
\end{bmatrix}.
\]

Notice that this matrix does not depend on \( B \) or \( n \). The claim follows after letting \( C' \triangleq (q + |\mathcal{O}|)\Delta \)

\[
\begin{align*}
\text{Step 1.2: Define } \mathcal{R}_n^0. \text{ Let } \mathcal{P}_0^* \text{ denote the size-0 vectors of } \mathcal{P}_0. \text{ That is,} \\
\mathcal{P}_0^* = \mathcal{P}_0 \cap \left\{ \bar{x} : \bar{x} \cdot \mathbf{1} = 0 \right\} = \left\{ \bar{x} \in \mathbb{R}^q : \begin{bmatrix} A_0^- & A_0^+ \end{bmatrix} \times (\bar{x})^\top = (\bar{0})^\top \right\}
\end{align*}
\]

Recall from Assumption 1 that \( \text{Rank} \left( \begin{bmatrix} A_0^- & A_0^+ \end{bmatrix} \right) \) = \( \text{Rank}(A_0^-) + 1 = q - d_0 + 1 \), which means that \( \text{dim}(\mathcal{P}_0^*) = q - \text{Rank} \left( \begin{bmatrix} A_0^- & A_0^+ \end{bmatrix} \right) = d_0 - 1 \). Therefore, \( \mathcal{P}_0^* \) contains a basis of \( d_0 - 1 \) linearly independent vectors, denoted by \( \mathcal{B}^0 = \{ \bar{p}_1^0, \ldots, \bar{p}_{d_0-1}^0 \} \). Let \( \mathbb{L}^0 \) denote the lattice generated by \( \mathcal{B}^0 \) excluding \( \bar{0} \). That is,

\[
\mathbb{L}^0 = \left\{ \sum_{i=1}^{d_0-1} \gamma_i \circ \bar{p}_i^0 : \forall i \leq d_0 - 1, \gamma_i \in \mathbb{Z} \text{ and } \exists i \leq d_0 - 1 \text{ s.t. } \gamma_i \neq 0 \right\}
\]

Let \( C^0 \) denote the minimum \( L_\infty \) norm of all vectors in \( \mathbb{L}^0 \), i.e.,

\[
C^0 = \inf \{ \| \bar{x} \|_\infty : \bar{x} \in \mathbb{L}^0 \}
\]

Next, we prove \( C^0 > 0 \). Suppose for the sake of contradiction that \( C^0 = 0 \). Then, there exist a sequence of vectors

\[
\left\{ \bar{x}^j = \sum_{i=1}^{d_0-1} \gamma_i^j \circ \bar{p}_i^0 : j \in \mathbb{N} \right\}
\]

such that for all \( j \in \mathbb{N} \), \( \| \bar{x}^j \|_\infty \leq \frac{1}{j} \). Let \( \bar{\gamma}^j = (\gamma_1^j, \ldots, \gamma_{d_0-1}^j) \). Because \( \| \bar{\gamma}^j \|_\infty \geq 1 \), we have \( \| \bar{\gamma}^j \|_\infty \leq \frac{1}{j} \). Notice that \( \bar{\gamma} \in \mathbb{R}^{d_0-1} : \| \bar{\gamma} \|_\infty = 1 \) is closed and compact, there exists a subsequence
of $\left\{ \frac{\vec{y}}{||\vec{y}||} \right\}$ that converges to a vector $\vec{\gamma}^*$ with $|\vec{\gamma}^*|_\infty = 1$. Let $\vec{x}^* = \sum_{i=1}^{d_0-1} \gamma_i^* \cdot \vec{p}_i^0$. It follows that $\left\{ \frac{\vec{y}_i}{||\vec{y}_i||} : j \in \mathbb{N} \right\}$ converges to $\vec{x}^*$, which means that $|\vec{x}^*|_\infty = \lim_{j \to \infty} \left| \frac{\vec{y}_i}{||\vec{y}_i||} \right|_\infty = 0$. This contradicts the linear independence of vectors in $B^0$.

Then, we use $C^0$ and $B^0$ to define a subset of $\frac{2C'}{\sqrt{n}} \cdot L^0 \cup \{0\}$ as follows.

$$R^0_n = \left\{ \sum_{i=1}^{d_0-1} \frac{2C'}{\sqrt{n}^0} \gamma_i \cdot \vec{p}_i^0 : \forall i \leq d_0 - 1, \gamma_i \in \{0, 1, \ldots, \lfloor \sqrt{n} \rfloor\} \right\}$$

By definition, $R^0_n \subseteq P_0^* \subseteq P_0$, and for every vector $\vec{x} \in R^0_n$, we have $\vec{x} \cdot \vec{1} = 0$ and $|\vec{x}| = O(\sqrt{n})$. The value $\frac{2C'}{\sqrt{n}}$ is chosen so that the $L_\infty$ distance between any pair of different vectors in $R^0_n$ is at least $2C'$.

**Step 1.3: Define $R^\infty_B$.** Recall that $P_0 \subseteq P_\infty$, $d_0 = \dim(P_0)$ and $d_\infty = \dim(C_\infty) = \dim(P_\infty)$. In the following procedure, we define a set of $d_\Delta$ linear independent vectors $B^\infty \subseteq C_\infty$, which are the basis of a complement of $P_0$ in $P_\infty$.

**Procedure for defining $B^\infty$**. Let $\{\vec{p}_1, \ldots, \vec{p}_{d_\Delta}\}$ denote an arbitrary set of $d_\Delta$ linearly independent vectors in $C_\infty$, whose existence is guaranteed by the fact that $\dim(C_\infty) = d_\Delta$. Start with $B^\infty = \emptyset$.

For every $1 \leq j \leq d_\Delta$, we add $\vec{p}_j$ to $B^\infty$ if and only if it is linearly independent with $P_0 \cup B^\infty$. At the end of procedure we have $|B^\infty| = d_\infty - d_0 = d_\Delta$. W.l.o.g., let $B^\infty = \{\vec{p}_1^\infty, \ldots, \vec{p}_{d_\Delta}^\infty\}$.

Let $P_0 = \text{Span}(B^\infty)$. It follows that $\dim(P_0) = d_\Delta$ and

$$P_0 \cap \overline{P_0} = \{0\} \text{ and } P_0 + \overline{P_0} = P_\infty$$

For each $i \leq d_\Delta$, because $\vec{p}_i^\infty \in C_\infty$, we have $\vec{p}_i^\infty \in H_{S, \leq 0}$ and we can write $\vec{p}_i^\infty = \vec{y}_i^\infty - \vec{\sigma}_i^\infty \times \vec{0}$, where $\vec{y}_i^\infty \in H_{T, \leq 0}$ and $\vec{\sigma}_i^\infty \geq 0$. For every $i \leq d_\Delta$, we must have $\vec{w}_i^\infty \neq 0$, because otherwise $\vec{p}_i^\infty \in C_0 \subseteq P_0$, which contradicts (22), because $\vec{p}_i^\infty \in \overline{P_0}$. Recall that $\vec{c} > 0$. W.l.o.g. we can assume $\vec{c} \cdot \vec{w}_i^\infty = 1$, otherwise we divide $\vec{p}_i^\infty$ by $\vec{c} \cdot \vec{w}_i^\infty$.

Let $L^\infty$ denote the lattice generated by $B^\infty$ excluding $\vec{0}$. That is,

$$L^\infty = \left\{ \sum_{i=1}^{d_\Delta} \eta_i \cdot \vec{p}_i^\infty : \forall i \leq d_\Delta, \eta_i \in \mathbb{Z} \text{ and } \exists i \leq d_\Delta \text{ s.t. } \eta_i \neq 0 \right\}$$

Let $C^\infty$ denote the minimum distance between $L^\infty$ and $P_0$. That is,

$$C^\infty = \inf \left\{ |\vec{x}^\infty - \vec{y}|_\infty : \vec{x}^\infty \in L^\infty, \vec{y} \in P_0 \right\}$$

Following an argument that is similar to the proof of $C^0 > 0$, we have $C^\infty > 0$. Formally, for the sake of contradiction suppose $C^\infty = 0$. Then, there exist two sequences of vectors

$$\left\{ \vec{x} = \sum_{i=1}^{d_\Delta} \eta_i \cdot \vec{p}_i^\infty : j \in \mathbb{N} \right\} \text{ and } \left\{ \vec{y}_i \in P_0 : j \in \mathbb{N} \right\}$$

such that for all $j \in \mathbb{N}$, $|\vec{x}_j - \vec{y}_j|_\infty \leq \frac{1}{j}$. Let $\vec{y}_j = (\eta_1, \ldots, \eta_{d_\Delta})$. Because $|\vec{y}_j|_1 \geq 1$ and $P_0$ includes $\vec{0}$, the distance between $\frac{\vec{y}_j}{||\vec{y}_j||}$ and $P_0$ is at most $\frac{1}{\sqrt{j}}$. Notice that $\{\vec{y}_j \in \mathbb{R}^{d_\Delta} : |\vec{y}_j|_1 = 1\}$ is closed and compact, there exists a subsequence of $\left\{ \frac{\vec{y}_j}{||\vec{y}_j||} \right\}$ that converges to a vector $\vec{\gamma}^*$ with $|\vec{\gamma}^*|_1 = 1$. Let $\{j_i : i \in \mathbb{N}\}$ denote the indices of the subsequence. Let $\vec{x}^* = \sum_{i=1}^{d_\Delta} \gamma_i \cdot \vec{p}_i^\infty$. Because $\vec{p}_i^\infty$'s are linearly independent, we have $\vec{x}^* \neq \vec{0}$. Then, $\{\vec{x}_{j_i} : i \in \mathbb{N}\}$ and $\{\vec{y}_{j_i} : i \in \mathbb{N}\}$ both converge to $\vec{x}^*$. Because $P_0$ is closed, we have $\vec{x}^* \in P_0$, which means that $\vec{x}^* \in \overline{P_0} \cap P_0 = \{\vec{0}\}$. This contradicts (22).
Because $U_{a,B} \neq \emptyset$, the infimum of all $B^* \geq 0$ such that there exists $n^* \in \mathbb{N}$ so that $U_{B^*,n^*} \neq \emptyset$ is well-defined, formally defined as $B^#$ as follows.

$$B^# = \inf \{ B^* \geq 0 : \exists n^* \in \mathbb{N} \text{ s.t. } U_{B^*,n^*} \neq \emptyset \}$$  \hspace{1cm} (24)

We note that there exists $n^* \in \mathbb{N}$ such that $U_{B^*,n^#} \neq \emptyset$, because there are finite number of combinations of operations whose total budget is under $B$, and $B^#$ is the minimum budget among the successful combinations (at a non-negative integer vector whose size is $n^#$). Clearly $B \geq B^#$. Define

$$\mathcal{R}_B^\infty = \left\{ \sum_{i=1}^{d_\Delta} \frac{2C'}{C^\infty} \eta_i \cdot \tilde{p}_i^\infty : \forall i \leq d_\Delta, \eta_i \in \{0,1,\ldots, \left\lfloor \frac{(B-B^#)C^\infty}{2d_\Delta C'} \right\rfloor \} \right\}$$  \hspace{1cm} (25)

where we recall that $C'$ is the constant guaranteed by Claim 7. Intuitively, $\mathcal{R}_B^\infty$ consists of some “grids” in $\frac{2C'}{C^\infty} \cdot L^\infty \cup \{\vec{0}\}$, which is a subset of $C_\infty$, because for all $i \leq d_\Delta, \tilde{p}_i^\infty \in C_\infty$.

**Step 1.4:** Define $\mathcal{R}_{B,n}$ and $\mathcal{R}_{B,n}^\infty$. Recall the definition of $B^#$ from (24) that $U_{B^#,n^#} \neq \emptyset$. Fix $\vec{x}^# \in U_{B^#,n^#}$. Let $\vec{x}^0 \in C_0$ denote an arbitrary interior point of $C_0$. That is, let $A_0^S$ denote the remaining rows of $A_0 = \begin{bmatrix} A_S \\ A_T \end{bmatrix}$ after removing the implicit equalities $A_0^\infty$, there exists a constant $\epsilon^0 > 0$ such that

$$A_0^+ \times (\vec{x}^0)^\top < - (\epsilon^0 \cdot \vec{1})^\top \text{ and } A_0^\infty \times (\vec{x}^0)^\top = (\vec{0})^\top$$

Let $C^\alpha > 0$ denote an arbitrary constant such that

$$C^\alpha > \frac{2C'}{C^0} \cdot \sum_{i=1}^{d_0} |A_0^+ \times (\vec{p}_i)\top|_\infty$$  \hspace{1cm} (26)

The constraint on $C^\alpha$ in (26) guarantees that for any $\vec{x}^0 \in R_0^n$, we have $C^\alpha \sqrt{n} \vec{x}^0 + \vec{x}^0 \in C_0$, which will be formally proved and used in the proof of Claim 9 below.

For any $n \in \mathbb{N}$ and any $\vec{x}^\infty \in \mathcal{R}_B^\infty$, define

$$\mathcal{R}_{\vec{x}^\infty} = (n-n^# - C^\alpha \sqrt{n} \cdot (\vec{x}^0 \cdot \vec{1}) - \vec{x}^\infty \cdot \vec{1}) \cdot \pi^* + \vec{x}^# + C^\alpha \sqrt{n} \vec{x}^0 + \mathcal{R}_n^0 + \vec{x}^\infty$$

It follows that the size of every vector in $\mathcal{R}_{\vec{x}^\infty}$ is $n$. Let $\mathcal{R}_{B,n} = \bigcup_{\vec{x}^\infty \in \mathcal{R}_B^\infty} \mathcal{R}_{\vec{x}^\infty}$. The following claim states that vectors in $\mathcal{R}_{B,n}$ are $2C'$ away from each other in $L^\infty$, where we recall that $C'$ is the constant guaranteed by Claim 7.

**Claim 8 (Sparsity of $\mathcal{R}_{B,n}$).** For any pair of vectors $\vec{x}^1, \vec{x}^2 \in \mathcal{R}_{B,n}$ whose $R_0^n$ or $R_B^\infty$ components are different, we have $|\vec{x}^1 - \vec{x}^2|_\infty \geq 2C'$.

**Proof.** For $j \in \{1,2\}$, we write

$$\vec{x}^j = \ell^j \pi^* + \vec{x}^# + C^\alpha \sqrt{n} \vec{x}^0 + \sum_{i=1}^{d_0} \frac{2C'}{C^0} \gamma_i^j \cdot \vec{p}_i^0 + \sum_{i=1}^{d_\Delta} \frac{2C'}{C^\infty} \eta_i^j \cdot \tilde{p}_i^\infty,$$

where $\ell^1$ and $\ell^2$ guarantee that $\vec{x}^1 \cdot \vec{1} = \vec{x}^2 \cdot \vec{1} = n$. Then,

$$\vec{x}^1 - \vec{x}^2 = \left( (\ell^1 - \ell^2) \pi^* + \sum_{i=1}^{d_0} \frac{2C'}{C^0} (\gamma_1^1 - \gamma_1^2) \cdot \vec{p}_i^0 + \sum_{i=1}^{d_\Delta} \frac{2C'}{C^\infty} (\eta_1^1 - \eta_1^2) \cdot \tilde{p}_i^\infty \right) \in \frac{2C'}{C^0} \cdot L^n \cup \{\vec{0}\}.$$

For $j \in \{1,2\}$, let $\vec{\gamma}^j = (\gamma_{d_0}^j, \ldots, \gamma_{d_\Delta}^j)$ and $\vec{\eta}^j = (\eta_{d_0}^j, \ldots, \eta_{d_\Delta}^j)$. Let $\vec{\eta}^\Delta = \vec{\eta}^1 - \vec{\eta}^2$ and $\vec{\eta}^\Delta = \vec{\eta}^1 - \vec{\eta}^2$. Claim 8 is proved in the following two cases.
• $\bar{\eta}^\Delta \neq \bar{0}$. Notice that $\sum_{i=1}^{d_\Delta} (\eta_i^1 - \eta_i^2) \cdot \bar{p}_i^\infty \in \mathbb{L}^\infty$. Recall that $\pi^* \in \mathcal{C}_0 \subseteq \mathcal{P}_0$, which means that $(\ell^2 - \ell^1) \pi^* - \sum_{i=1}^{d_\Delta} \frac{2C'}{C_0} (\gamma_i^1 - \gamma_i^2) \cdot \bar{P}_i^0 \in \mathcal{P}_0$. According to (23), the $L_\infty$ distance between $\sum_{i=1}^{d_\Delta} (\eta_i^1 - \eta_i^2) \cdot \bar{p}_i \in \mathbb{L}^\infty$ and any vector in $\mathcal{P}_0$ is at least $C^\infty$. Therefore,

$$|x_1^1 - x_2^2|_\infty = \left| (\ell^1 - \ell^2) \pi^* + \sum_{i=1}^{d_0-1} \frac{2C'}{C_0} (\gamma_1^1 - \gamma_1^2) \cdot \bar{P}_i^0 + \sum_{i=1}^{d_\Delta} \frac{2C'}{C_0^\infty} (\eta_i^1 - \eta_i^2) \cdot \bar{p}_i^\infty \right| = \frac{2C'}{C_0} \left| \sum_{i=1}^{d_\Delta} (\eta_i^1 - \eta_i^2) \cdot \bar{p}_i^\infty - \frac{C^\infty}{2C'} \left( (\ell^2 - \ell^1) \pi^* - \sum_{i=1}^{d_0-1} \frac{2C'}{C_0} (\gamma_i^1 - \gamma_i^2) \cdot \bar{P}_i^0 \right) \right| \geq \frac{2C'}{C_0} \cdot C^\infty = 2C'$$

• $\bar{\eta}^\Delta = \bar{0}$. In this case we must have $\ell^1 = \ell^2$ and $\bar{\gamma}^\Delta = \bar{0}$. Notice that $\sum_{i=1}^{d_0-1} (\gamma_i^1 - \gamma_i^2) \cdot \bar{P}_i^0 \in \mathbb{L}^0$, and recall that the $L_\infty$ norm of any vector in $\mathbb{L}^0$ is at least $C^0 > 0$, we have

$$|x^1_1 - x^2|_\infty = \left| \sum_{i=1}^{d_0-1} (\gamma_i^1 - \gamma_i^2) \cdot \bar{P}_i^0 \right| \geq \frac{2C'}{C_0} \cdot C^0 = 2C'$$

This completes the proof of Claim 8.

By Claim 8, $|\mathcal{R}_{B,n}| = \left( 1 + \left( \frac{(B-B^\#) C^\infty}{2d_\Delta C'} \right)^{\frac{d_\Delta}{d_\Delta}} \right) C^\Delta \times (\lfloor \sqrt{n} \rfloor)^{d_0-1}$, which is $\Omega((B+1)^d \Delta \cdot (\sqrt{n})^{d_0-1})$. Next, we prove that for all sufficiently large $n$, $\mathcal{R}_{B,n} \subseteq \mathcal{H}_{B,n} \cap \mathbb{R}^q_{\geq 0}$.

**Claim 9 ($\mathcal{R}_{B,n} \subseteq \mathcal{H}_{B,n} \cap \mathbb{R}^q_{\geq 0}$).** There exists $N \in \mathbb{N}$ that does not depend on $B$ or $n$, such that when $n \geq N$, $\mathcal{R}_{B,n} \subseteq \mathcal{H}_{B,n} \cap \mathbb{R}^q_{\geq 0}$.

**Proof.** For any $\bar{x} \in \mathcal{R}_{B,n}$, we can write

$$\bar{x} = \ell \pi^* + \bar{x}^\# + C^0 \sqrt{n} \bar{x}^\circ + \bar{x}^0 + \bar{x}^\infty,$$

where $\bar{x}^0 \in \mathcal{R}^0_{\mathcal{H}_{B,n}}$, $\bar{x}^\infty \in \mathcal{R}^\infty_{\mathcal{H}_{B,n}}$, and $\ell$ guarantees that $\bar{x} \cdot \bar{1} = n$. Recall that $\pi^* \in \mathcal{C}_0$ is strictly positive. Therefore, $\bar{x} \geq \bar{0}$ for any sufficiently large $n$, which means that $\mathcal{R}_{B,n} \subseteq \mathbb{R}^q_{\geq 0}$. The rest of the claim proves $\bar{x} \in \mathcal{H}_{B,n}$ in the following three steps.

1. $C^\infty \sqrt{n} \bar{x}^\circ + \bar{x}^0 \in \mathcal{C}_0$. Let $\bar{x}'' = C^\infty \sqrt{n} \bar{x}^\circ + \bar{x}^0$, we prove the following.

   (1.1) $A^\circ_0 \times (\bar{x}'')^T = \begin{pmatrix} 0 \end{pmatrix}^T$. Recall that $\bar{x}^\circ$ is an interior point of $\mathcal{C}_0 \subseteq \mathcal{H}_{S_{\leq 0}}$, which means that $A^\circ_0 \times (\bar{x}^\circ)^T = \begin{pmatrix} 0 \end{pmatrix}^T$. Also recall that $\mathcal{H}^0_\mathcal{H} \subseteq \mathcal{P}_0^0$, which means that $A^\infty_0 \times (\bar{x}^0)^T = \begin{pmatrix} 0 \end{pmatrix}^T$. Therefore,

   $$A^\circ_0 \times (\bar{x}'')^T = C^\infty \sqrt{n} A^\circ_0 \times (\bar{x}^\circ)^T + A_0^\infty \times (\bar{x}^0)^T = \begin{pmatrix} 0 \end{pmatrix}^T$$

   (1.2) $A_0^+ \times (\bar{x}'')^T < \begin{pmatrix} 0 \end{pmatrix}^T$. Let $\bar{x}^0 = \sum_{i=1}^{d_0-1} \frac{2C'}{C_0} \gamma_i^j \cdot \bar{P}_i^0$. Recall that $A_0^+ \times (\bar{x}^\circ)^T < \begin{pmatrix} -\epsilon^\circ \cdot \bar{1} \end{pmatrix}^T$, and recall the lower bound of $C^\circ$ from (26). We have

   $$A_0^+ \times (\bar{x}'')^T = C^\infty \sqrt{n} A_0^+ \times (\bar{x}^\circ)^T + \sum_{i=1}^{d_0-1} \frac{2C'}{C_0} \gamma_i^j \cdot A_0^+ \times (\bar{P}_i^0)^T < \begin{pmatrix} 0 \end{pmatrix}^T$$
(2) $\vec{x} \in \mathcal{H}_S$. Recall that $\pi^* \in \mathcal{C}_0$, $\vec{x}^\# \in \mathcal{U}_{B^*, n^*} \subseteq \mathcal{H}_S$, and $\vec{x}^\infty \in \mathcal{C}_\infty \subseteq \mathcal{H}_{S, \leq 0}$. Therefore,

$$
\begin{align*}
A_S \times (\vec{x})^T &= \ell A_S \times (\pi^*)^T + A_S \times (\vec{x}^\#)^T + A_S \times (C^0 \sqrt{n} \vec{x}^\# + \vec{x}^0)^T + A_S \times (\vec{x}^\infty)^T \\
&\leq (\vec{0})^T + (b_S)^T + (\vec{0})^T + (\vec{0})^T = (b_S)^T
\end{align*}
$$

This proves $\vec{x} \in \mathcal{H}_S$.

(3) $\vec{x}$ is $B$-manipulable. Because $\vec{x}^\# \in \mathcal{U}_{B^*, n^*}$, there exists $\vec{o}^\# \in \mathbb{Z}_{\geq 0}^n$ such that $\vec{c} \cdot \vec{o}^\# \leq B^#$ and $\vec{x}^\# + \vec{o}^\# \times \mathbf{O} \in \mathcal{H}_T$. Let $\vec{x}^\infty = \sum_{i=1}^{d_\Delta} \frac{2C'}{C_\infty} \eta_i \cdot \vec{o}^\infty_i$. Also recall that for all $i \leq d_\Delta$, $\vec{p}^\infty_i = \vec{y}^\infty_i - \vec{o}^\infty_i \times \mathbf{O}$, where $\vec{y}^\infty_i \in \mathcal{H}_{T, \leq 0}$, $\vec{o}^\infty_i \geq \vec{0}$, and $\vec{c} \cdot \vec{o}^\infty_i = 1$. Let $\vec{o}^* = \vec{o}^\# + \sum_{i=1}^{d_\Delta} \frac{2C'}{C_\infty} \eta_i \cdot \vec{o}^\infty_i$. We prove that $\vec{c} \cdot \vec{o}^* \leq B$ as follows.

$$
\begin{align*}
\vec{c} \cdot \vec{o}^* &= \vec{c} \cdot \vec{o}^\# + \sum_{i=1}^{d_\Delta} \frac{2C'}{C_\infty} \eta_i \cdot \vec{c} \cdot \vec{o}^\infty_i \\
&\leq B^# + \sum_{i=1}^{d_\Delta} \frac{2C'}{C_\infty} \cdot \frac{(B - B^#) C_\infty}{2d_\Delta C'} = B^# + B - B^# = B
\end{align*}
$$

Then,

$$
\begin{align*}
A_T \times (\vec{x} + \vec{o}^* \times \mathbf{O})^T &= \ell A_T \times (\pi^*)^T + A_T \times (\vec{x}^\# + \vec{o}^\# \times \mathbf{O})^T + A_T \times (C^0 \sqrt{n} \vec{x}^\# + \vec{x}^0)^T \\
&\quad+ \sum_{i=1}^{d_\Delta} \frac{2C'}{C_\infty} \eta_i \cdot \mathbf{A}_T \times (\vec{p}^\infty_i + \vec{o}^\infty_i \times \mathbf{O})^T \\
&\leq (\vec{0})^T + (b_T)^T + (\vec{0})^T + (\vec{0})^T = (b_T)^T
\end{align*}
$$

This proves that $\vec{x} + \vec{o}^* \times \mathbf{O} \in \mathcal{H}_T$ and completes the proof of Claim 9.

By Claim 7, for every $\vec{x} \in \mathcal{R}_{B,n}$, there exists $\vec{x}' \in \mathcal{U}_{n,B}$ that is no more than $C'$ away from $\vec{x}$ in $L_\infty$. Because vectors in $\mathcal{R}_{B,n}$ are at least $2C'$ away from each other, their corresponding vectors in $\mathcal{U}_{n,B}$ are different, denoted by $\mathcal{R}_{B,n}'$ and is formally defined as follows.

**Definition 16 ($\mathcal{R}_{B,n}'$).** Let $\mathcal{R}_{B,n}'$ denote the integer vectors corresponding to the vectors in $\mathcal{R}_{B,n}$ guaranteed by Claim 7.

It follows that $|\mathcal{R}_{B,n}'| = |\mathcal{R}_{B,n}| = \Omega \left((B+1)^{d_\Delta} \cdot (\sqrt{n})^{d_0-1}\right)$. This completes Step 1 for poly lower bound.

**Step 2 for poly lower bound:** Define $\vec{\pi}^o$ for any $\pi^* \in \text{CH}(\Pi)$. In this step, we define $\vec{\pi}^o = (\pi^o_1, \ldots, \pi^o_n) \in \Pi^n$ such that $\sum_{j=1}^n \pi^o_j$ is $O(\sqrt{n})$ away from $n\pi^*$ in a way that is similar to [Xia21a].

More precisely, because $\pi^* \in \text{CH}(\Pi)$, by Carathéodory’s theorem for convex/conic hulls (see e.g., [LP09, p. 257]), we can write $\pi^*$ as the convex combination of $1 \leq t \leq q$ distributions in $\Pi$, i.e.,

$$
\pi^* = \sum_{i=1}^t \alpha_i \pi^*_i, \quad \text{where } \sum_{i=1}^t \alpha_i = 1 \text{ and } \pi^*_i \in \Pi
$$

40
We note that \( \pi^* \geq \epsilon \cdot \tilde{1} \), because \( \Pi \) is strictly positive (by \( \epsilon \)). For each \( i \leq t-1 \), let \( \pi_i^t \) denote the vector of \( \beta_i = [\ell \alpha_i] \) copies of \( \pi_i^* \). Let \( \pi_k^t \) denote the vector of \( \beta_i = n - \sum_{i=1}^{t-1} \beta_i \) copies of \( \pi_i^* \). It follows that for any \( i \leq t-1 \), \( |\beta_i - \ell \alpha_i| \leq 1 \), and \( |\beta_i - \ell \alpha_i| \leq t + (\tilde{y}^n \cdot \tilde{1} - \ell \pi^* \cdot \tilde{1}) = O(\sqrt{t}) = O(\sqrt{n}) \).

Let \( \pi^0 = (\pi_1^t, \ldots, \pi_t^t) \), or equivalently

\[
\pi^0 = \left( \pi_1^t, \ldots, \pi_1^*, \pi_2^t, \ldots, \pi_2^*, \ldots, \pi_t^t, \ldots, \pi_t^* \right)
\]

**Step 3 for poly lower bound: Lower-bound** \( \Pr_{\pi \sim \pi}(\tilde{X}_\pi \in \mathcal{H}_B) \).

**Claim 10.** For any PMV-instability setting \( S \), any strictly positive set of distributions \( \Pi \) (by \( \epsilon > 0 \)), and any \( \alpha > 0 \), there exist \( C_S > 0 \) and \( N > 0 \) such that for any \( n \geq N \), any \( 0 \leq B \leq \sqrt{n} \), and any \( \pi \in \Pi^n \) such that the \( L_\infty \) distance between \( \sum_{j=1}^n \pi_j \) and \( C_0 \) is no more than \( \alpha \sqrt{n} \),

\[
\Pr(X_{\pi} \in \mathcal{U}_{n,B}) \geq C_S \cdot (B + 1)^d \Delta \cdot \left( \frac{1}{\sqrt{n}} \right)^{q - d_0}
\]

**Proof.** Let \( \pi^* \in \mathcal{C}_0 \) denote an arbitrary vector such that \( |\sum_{j=1}^n \pi_j - n \pi^*|_\infty < 2\alpha \sqrt{n} \). Let \( \mathcal{R}_{B,n}^\pi \) denote the set of integer vectors for \( \pi^* \) (Definition 16 in Step 1.4 above). Because \( B = O(\sqrt{n}) \), each vector in \( \mathcal{R}_{B,n}^\pi \) is \( O(\sqrt{n}) \) away from \( n \pi^* \), which is \( O(\sqrt{n}) \) away from \( \sum_{j=1}^n \pi_j \). By the point-wise concentration bound ([Xia21a, Lemma 1]), for every \( \tilde{x} \in \mathcal{R}_{B,n}^\pi \), we have \( \Pr_{\pi \sim \pi}(\text{Hist}(\pi) = \tilde{x}) = \Omega((\sqrt{n})^{1-q}) \). Therefore,

\[
\Pr(X_{\pi} \in \mathcal{U}_{n,B}) \geq \Pr_{\pi \sim \pi}(\text{Hist}(\pi) \in \mathcal{R}_{B,n}^\pi) \geq |\mathcal{R}_{B,n}^\pi| \times \Omega((\sqrt{n})^{1-q}) = \Omega((B + 1)^d \Delta \cdot (\sqrt{n})^{d_0 - 1}) \times \Omega((\sqrt{n})^{1-q}) = \Omega((B + 1)^d \Delta \cdot \left( \frac{1}{\sqrt{n}} \right)^{q - d_0})
\]

This completes proof of Claim 10. \( \square \)

The lower bound for PT-\( \Theta(\sqrt{n}) \)-sup follows after applying Claim 10 to any \( \pi^* \in \text{CH}(\Pi) \cap \mathcal{C}_0 \) and \( \pi^0 \) defined in Step 2 for poly lower bound above.

**Proof for the phase transition at \( \Theta(n) \) case of sup (PT-\( \Theta(n) \)-sup for short).**

**\( B \leq C_2 n \).** Let \( c^* = \frac{1}{2}(C_2 + B_{\text{CH}(\Pi)}) \). We first show that \( n \cdot \text{CH}(\Pi) \) is \( O(n) \) away from \( C_{nc^*} \), which is equivalent to \( \text{CH}(\Pi) \) being \( \Omega(1) \) away from \( C_{c^*} \). Due to the minimality of \( B_{\text{CH}(\Pi)} \), we have \( \text{CH}(\Pi) \cap C_{c^*} = \emptyset \). Notice that \( \text{CH}(\Pi) \) is convex and compact and \( C_{c^*} \) is convex. By the strict separating hyperplane theorem, the distance between \( \text{CH}(\Pi) \) and \( C_{c^*} \) is strictly positive, denoted by \( c^* \). Let \( c_2 > 0 \) denote any fixed constant that is smaller than \( c^* \).

Next, we prove that \( n \cdot \text{CH}(\Pi) \) is \( O(n) \) away from \( \mathcal{H}_B \). To this end, we prove the following claim, which states that for any sufficiently large \( B' \geq 0 \), \( \mathcal{H}_{B'} \) is \( O(1) \) away from \( C_{B' + O(1)} \).

**Claim 11.** Given a PMV-instability setting \( S \), there exists a constant \( C \) such that for any \( B' \geq C \), any \( n \), and any \( \tilde{x} \in \mathcal{H}_{B'} \), there exists \( \tilde{x}' \in C_{B' + C} \) such that \( |\tilde{x} - \tilde{x}'|_\infty \leq C \).

**Proof.** The proof is done by analyzing feasible solutions to the following two linear programs \( \text{LP}_H \) and \( \text{LP}_{\text{Cone}} \), whose variables are \( \tilde{x} \) and \( \tilde{o} \).
Because \( \bar{x} \in \mathcal{H}_B \), there exists \( \delta \geq \bar{\delta} \) such that \( \bar{x} + \delta \cdot \bar{\vec{y}} \in \mathcal{H}_T \) and \( \bar{c} \cdot \bar{\delta} \leq B' \). Therefore, \((\bar{x}, \bar{w})\) is a feasible solution to \( \mathcal{L}_H \). Notice that \( \mathcal{L}_{\text{Cone}} \) is feasible (for example \( \bar{0} \) is a feasible solution) and \( \mathcal{L}_H \) and \( \mathcal{L}_{\text{Cone}} \) only differ on the right hand side of the inequalities. Therefore, due to \[ \text{CGST86, Theorem 5 (i)}, \] there exists a feasible solution \((\bar{x}', \bar{w}')\) to \( \mathcal{L}_{\text{Cone}} \) that is no more than \( q \Delta \max\{|b_S - \bar{0}|_\infty, |b_T - \bar{0}|_\infty\} = O(1) \) away from \((\bar{x}, \bar{w})\) in \( L_\infty \), where \( \Delta \) is the maximum absolute value of determinants of square sub-matrices of the left hand side of \( \mathcal{L}_H \) and \( \mathcal{L}_{\text{Cone}} \), i.e.,

\[
\begin{bmatrix}
A_S & 0 \\
A_T & A_T \times (O)^T \\
0 & -I
\end{bmatrix}
\]

This implies that

\[ \bar{c} \cdot \bar{\delta} \leq \bar{c} \cdot \bar{w}' + (\bar{c} \cdot \bar{I})q \Delta \max\{|b_S|_\infty, |b_T|_\infty\} \]

The claim follows after letting \( C = (\bar{c} \cdot \bar{I})q \Delta \max\{|b_S|_\infty, |b_T|_\infty\} \).

As proved above, \( n \cdot \text{CH}(\Pi) \in \Omega(n) \) away from \( C_{nc} \) for any sufficiently large \( n \). Notice that for any sufficiently large \( n \), we have \((c^* - C_2)n > C\), where \( C \) is the constant guaranteed by Claim 11. Therefore, by Claim 11, every vector \( \bar{x} \in \mathcal{H}_B \) is \( O(1) \) away from a vector \( \bar{x}' \) in \( \mathcal{C}_{B + O(1)} \subseteq C_{nc} \). Recall from above that \( n \cdot \text{CH}(\Pi) \in \Omega(n) \) away from \( C_{nc} \). Therefore, \( n \cdot \text{CH}(\Pi) \in \Omega(n) \) away from \( \mathcal{H}_B \).

Finally, for any \( \bar{\pi} \in \Pi^n \), let \( \pi' = \frac{1}{n} \sum_{j=1}^n \pi_j \). Because \( \pi' \in \text{CH}(\Pi) \), the \( L_\infty \) distance between \( \pi' \), which is the mean vector of \( \bar{X}_\varphi \), and \( \mathcal{H}_B \) is \( \Omega(n) \). The (exponential) upper bound for the \( B \leq C_2n \) case is proved by a straightforward application of Hoeffding’s inequality and the union bound to all \( q \) dimensions as in the proof of the exponential case of \( \sup \). The exponential lower bound trivially holds.

\[ B \geq C_3n. \]

The \( \Omega\left(\frac{(\frac{1}{\sqrt{n}})^{q-d_\infty}}{}\right) \) upper bound of this case follows after Claim 6. At a high level, the proof of the \( \Omega\left(\frac{(\frac{1}{\sqrt{n}})^{q-d_\infty}}{}\right) \) lower bound is similar to the proof of the polynomial lower bound of \( \text{PT-}\Theta(\sqrt{n})\)-sup. The main difference is that the condition is weaker now \( \left(\text{CH}(\Pi) \cap C_{\infty} \neq \emptyset \right) \) compared to \( C_0 \cap \text{CH}(\Pi) \neq \emptyset \). Therefore, we will identify two different sets \( R_{B,n}^+ \subseteq \mathcal{H}_{B,n} \) and \( R_{B,n}^\varphi \subseteq \mathcal{U}_{B,n} \) for any strictly positive \( \pi^+ \in \mathcal{C}_{B^+} \) with \( \pi^+ \cdot \bar{I} = 1 \), where \( B^+ \) is a fixed number such that \( C_3 > B^+ > B_{\text{CH}(\Pi)} \). It follows that \( \pi^+ \in \mathcal{H}_{S,\leq0} \) and we can write \( \pi^+ = \bar{\pi}^+ - \bar{\delta}^+ \cdot \bar{O} \), where \( \bar{\pi}^+ \in \mathcal{H}_{T,\leq0} \), \( \bar{\delta}^+ \geq \bar{\delta} \), and \( \bar{c} \cdot \bar{\delta}^+ \leq B^+ \).

In the following procedure, we define a basis \( B^+ \) of \( \mathcal{P}_{\infty} \) that is similar to \( B^\infty \). Let \( \{\bar{p}_0^+, \ldots, \bar{p}_{d_\infty}^+\} \) denote a set of \( d_\infty \) linearly independent vectors in \( C_{\infty} \subseteq \mathcal{P}_{\infty} \). Without loss of generality, suppose for every \( j \leq d_\infty \), \( \bar{p}_j^+ \cdot \bar{I} \in \{-1, 0, 1\} \) — otherwise we divide \( \bar{p}_j^+ \) by \( |\bar{p}_j^+ \cdot \bar{I}| \). For every \( j \leq d_\infty \), let \( \bar{p}_j^+ = \bar{y}_j^+ - \bar{\delta}_j^+ \cdot \bar{O} \), where \( \bar{y}_j^+ \in \mathcal{H}_{T,\leq0} \) and \( \bar{\delta}_j^+ \geq \bar{\delta} \). For convenience, let \( \bar{p}_0^+ = \pi^+ \).

**Procedure.** Start with \( B^+ = \{\bar{p}_0^+\} \). For every \( 1 \leq j \leq d_\infty \), we add \( \bar{p}_j^+ \) to \( B^+ \) if and only if it is linearly independent of existing vectors in \( B^+ \). At the end of the procedure, we have \( |B^+| = d_\infty \). Without loss of generality, let \( B^+ = \{\bar{p}_0^+, \bar{p}_1^+, \ldots, \bar{p}_{d_\infty-1}^+\} \).
Next, we define $\mathbb{L}^+, C^+$, and $R^+_n$ that are similar to $\mathbb{L}^0$, $C^0$, and $R^0_n$ in Step 1.2 of the proof of the polynomial lower bound of PT-$\Theta(\sqrt{n})$-sup, respectively.

Let $\mathbb{L}^+$ denote the lattice generated by $\mathbb{B}^+$ excluding $\vec{0}$. That is,

$$\mathbb{L}^+ = \left\{ \sum_{i=0}^{d_\infty - 1} \lambda_i \cdot \vec{p}_i^+ : \forall 0 \leq i \leq d_\infty - 1, \lambda_i \in \mathbb{Z} \text{ and } \exists 0 \leq i \leq d_\infty - 1 \text{ s.t. } \lambda_i \neq 0 \right\}$$

Let $C^+$ denote the minimum $L_\infty$ norm of vectors in $\mathbb{L}^+$. That is,

$$C^+ = \inf \left\{ |\vec{x}|_\infty : \vec{x} \in \mathbb{L}^+ \right\}$$

Following a similar argument that proves $C^0 > 0$, we have $C^+ > 0$. Then, define

$$R^+_n = \left\{ \sum_{i=1}^{d_\infty - 1} \left[ \frac{2C'}{C^+} \right] \cdot \lambda_i \cdot \vec{p}_i^+ : \forall 1 \leq i \leq d_\infty - 1, \lambda_i \in \{0, 1, \ldots, \lceil \sqrt{n} \rceil \} \right\}$$

Notice that vectors in $R^+_n$ do not have $\vec{p}_0 = \pi^+$ components. Recall that $\vec{x}^\# \in U_{B^+, n^+}$, which means that $\vec{x}^\# \cdot \vec{1} = n^\#$. For any $n \in \mathbb{N}$, define

$$R^+_{B,n} = \left\{ (n - n^\# - \vec{x}^\# \cdot \vec{1}) \cdot \pi^+ + \vec{x}^\# + \vec{x}^+ : \vec{x}^+ \in R^+_n \right\}$$

Like Claim 8, in the following claim we prove that vectors in $R^+_{B,n}$ are at least $2C'$ from each other in $L_\infty$.

**Claim 12 (Sparsity of $R^+_{B,n}$).** For any pair of vectors $\vec{x}^1, \vec{x}^2 \in R^+_{B,n}$ whose $R^+$ components are different, we have $|\vec{x}^1 - \vec{x}^2|_\infty \geq 2C'$.

**Proof.** For $j \in \{1, 2\}$, we write

$$\vec{x}^j = \ell^j \cdot \pi^+ + \vec{x}^\# + \sum_{i=1}^{d_\infty - 1} \left[ \frac{2C'}{C^+} \right] \cdot \lambda_i^j \cdot \vec{p}_i^+,$$

where $\ell^1$ and $\ell^2$ guarantee that $\vec{x}^1 \cdot \vec{1} = \vec{x}^2 \cdot \vec{1} = n$. Then, because $(\vec{x}^1 - \vec{x}^2) \cdot \vec{1} = 0$ and $\pi^+ \cdot \vec{1} = 1$, we have

$$\ell^1 - \ell^2 = \sum_{i=1}^{d_\infty - 1} \left[ \frac{2C'}{C^+} \right] (\lambda_i^2 - \lambda_i^1) \cdot (\vec{p}_i^+ \cdot \vec{1})$$

Therefore,

$$\vec{x}^1 - \vec{x}^2 = (\ell^1 - \ell^2) \pi^+ + \sum_{i=1}^{d_\infty - 1} \left[ \frac{2C'}{C^+} \right] (\lambda_i^2 - \lambda_i^1) \cdot \vec{p}_i^+$$

$$= \sum_{i=1}^{d_\infty - 1} \left[ \frac{2C'}{C^+} \right] (\lambda_i^2 - \lambda_i^1) \cdot (\vec{p}_i^+ \cdot \vec{1}) \cdot \pi^+ + \sum_{i=1}^{d_\infty - 1} \left[ \frac{2C'}{C^+} \right] (\lambda_i^1 - \lambda_i^2) \cdot \vec{p}_i^+$$

Recall that for all $i \leq d_\infty - 1$, $\vec{p}_i^+ \cdot \vec{1}$ is an integer, which means that $\sum_{i=1}^{d_\infty - 1} (\lambda_i^2 - \lambda_i^1) \cdot (\vec{p}_i^+ \cdot \vec{1})$ is an integer. Also because the $R^+$ components of $\vec{x}^1$ and $\vec{x}^2$ are different, there exists $i \leq d_\infty - 1$ such that $\lambda_i^1 - \lambda_i^2 \neq 0$. Therefore,

$$\sum_{i=1}^{d_\infty - 1} (\lambda_i^2 - \lambda_i^1) \cdot (\vec{p}_i^+ \cdot \vec{1}) \cdot \pi^+ + \sum_{i=1}^{d_\infty - 1} (\lambda_i^1 - \lambda_i^2) \cdot \vec{p}_i^+ \in \mathbb{L}^+$$

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Therefore,

\[ |\tilde{x}^1 - \tilde{x}^2|_\infty = (\ell^1 - \ell^2)\pi^+ + \sum_{i=1}^{d_{\infty}-1} \left[ \frac{2C'}{C^+} \right] (\lambda^+_i - \lambda_i^2) \cdot \tilde{p}_i^+ \]

\[ = \sum_{i=1}^{d_{\infty}-1} \left[ \frac{2C'}{C^+} \right] (\lambda^+_i - \lambda_i^2) \cdot (\tilde{p}_i^+ \cdot \vec{1}) \cdot \pi^+ + \sum_{i=1}^{d_{\infty}-1} \left[ \frac{2C'}{C^+} \right] (\lambda^+_i - \lambda_i^2) \cdot \tilde{p}_i^+ \]

\[ = \left[ \frac{2C'}{C^+} \right] \sum_{i=1}^{d_{\infty}-1} (\lambda^+_i - \lambda_i^2) \cdot (\tilde{p}_i^+ \cdot \vec{1}) \cdot \pi^+ + \sum_{i=1}^{d_{\infty}-1} (\lambda^+_i - \lambda_i^2) \cdot \tilde{p}_i^+ \]

\[ \geq \left[ \frac{2C'}{C^+} \right] \cdot C^+ = 2C' \]

This completes the proof of Claim 12.

Then, we prove the counterpart of Claim 9 for \( R^+_B,n \) in the following claim.

**Claim 13** (\( R^+_B,n \subset H_{B,n} \cap \mathbb{R}_{\geq 0}^q \)). There exists \( N \in \mathbb{N} \) that does not depend on \( B \) or \( n \), such that when \( n \geq N \), \( R^+_B,n \subset H_{B,n} \cap \mathbb{R}_{\geq 0}^q \).

**Proof.** Let \( \vec{x} = (n - n^# - \tilde{x}^+ \cdot \vec{1}) \cdot \pi^+ + \tilde{x}^# + \tilde{x}^\# \) denote any vector in \( R^+_B,n \), where \( \tilde{x}^+ = \sum_{i=1}^{d_{\infty}-1} \left[ \frac{2C'}{C^+} \right] \cdot \lambda_i \cdot \tilde{p}_i^+ \). Clearly \( R^+_B,n \subset \mathbb{R}_{\geq 0}^q \) for sufficiently large \( n \), because \( \pi^+ \) is strictly positive. It is not hard to verify that \( \pi^+ \in H_{S,\leq 0}, \tilde{x}^# \in H_S, \) and \( \tilde{x}^+ \in H_{S,\leq 0} \). Let

\[ \vec{\sigma} = (n - n^# - \tilde{x}^+ \cdot \vec{1}) \cdot \vec{\sigma}^+ + \vec{\sigma}^# + \sum_{i=1}^{d_{\infty}-1} \left[ \frac{2C'}{C^+} \right] \cdot \lambda_i \cdot \vec{\sigma}_i^+ \]

It follows that \( \vec{e} \cdot \vec{w} \leq B^+ n + B^# + O(\sqrt{n}) \), which is smaller than \( B \geq C_3 n \) for any sufficiently large \( n \), because \( C_3 > B^+ \). Then,

\[ \vec{x} + \vec{w} \times \vec{O} = (n - n^# - \tilde{x}^+ \cdot \vec{1}) \cdot \left( \vec{y}^+ \right)_{\text{in } H_{T,\leq 0}} + \left( \vec{y}^# \right)_{\text{in } H_T} + \sum_{i=1}^{d_{\infty}-1} \left[ \frac{2C'}{C^+} \right] \cdot \lambda_i \cdot \left( \vec{y}_i^+ \right)_{\text{in } H_{T,\leq 0}} \in H_T \]

Therefore, \( R^+_B,n \subset H_{B,n} \). This completes the proof of Claim 13.

Let \( \mathcal{R}^+_B,n \subset \mathcal{U}_{n,B} \) denote the integer vectors corresponding to vectors in \( R^+_B,n \) guaranteed by Claim 7. Then, \( |\mathcal{R}^+_B,n| = \Omega((\sqrt{n})^{d_{\infty}-1}) \). The \( \Omega \left( \left( \frac{1}{\sqrt{n}} \right)^{d_{\infty}} \right) \) lower bound follows after similar steps as Step 2 and Step 3 for the polynomial lower bound of PT-\( \Theta(\sqrt{n}) \)-sup. Specifically, we have the following counterpart to Claim 10 with a similar proof.

**Claim 14.** For any PMV-instability setting \( S \), any strictly positive set of distributions \( \Pi \) such that \( B_{CH(\Pi)}^- < \infty \), and any \( C_3^> > B_{CH(\Pi)}^- \), there exist \( C_S > 0 \) and \( N > 0 \) such that for any \( n \geq N \), any \( B \geq C_3^> n \), and any \( \vec{\pi} \in \Pi^n \),

\[ \Pr(\tilde{X}_{\vec{\pi}} \in \mathcal{U}_{n,B}) \geq C_S \cdot \left( \frac{1}{\sqrt{n}} \right)^{q-d_{\infty}} \]

**Proof for inf.** Like the sup part, we call the four cases in inf part the 0 case, the exponential case, the phase transition at \( \Theta(\sqrt{n}) \) case, and the phase transition at \( \Theta(n) \) case.
The 0 case is straightforward. To prove the exponential upper bound, we need to prove that there exists $\bar{\pi} \in \Pi^n$ such that $\Pr(\bar{X}_\bar{\pi} \in U_{n,B}) = \exp(-\Omega(n))$. Let $\pi'$ denote an arbitrary vector in $CH(\Pi) \setminus C_{\infty}$. Because $C_{\infty}$ is a closed cone, the distance between $\pi'$ and $C_{\infty}$ is strictly positive, which means that the distance between $n \cdot \pi'$ and $C_{\infty}$ is $O(n)$. Then, let $\bar{\pi} \in \Pi^n$ denote an arbitrary vector such that $\frac{1}{n} \sum_{j=1}^n \pi_j$ is $O(1)$ from $\pi'$ in $L_{\infty}$. Therefore, $\sum_{j=1}^n \pi_j$, which is the mean vector of $\bar{X}_\bar{\pi}$, is $\Theta(n)$ away from $C_{\infty} \supseteq C_B$. Then, similar to the proof for the exponential case of sup, by Claim 3, $\sum_{j=1}^n \pi_j$ is $\Theta(n)$ away from $H_B \supseteq U_{n,B}$. The exponential upper bound of inf then follows after Hoeffding’s inequality and the union bound. The exponential lower bound trivially holds.

**Proof for the phase transition at $\Theta(\sqrt{n})$ case of inf.** The (polynomial) upper bound follows after the upper bound of the phase transition at $\Theta(\sqrt{n})$ case of sup. The polynomial lower bound follows after applying Claim 10 to every $\bar{\pi} = (\pi_1, \ldots, \pi_n) \in \Pi^n$ and letting $\pi^* = \frac{1}{n} \sum_{j=1}^n \pi_j \in C_0$.

**Proof for the phase transition at $\Theta(n)$ case of inf.** When $B \leq C_2^{-1} n$, the (exponential) lower bound is straightforward. The proof for the (exponential) upper bound is similar to the proof of the $B \leq C_2 n$ case of PT-$\Theta(n)$-sup. More precisely, let $c^* = \frac{1}{4}(C_2^{-1} + B_{CH(\Pi)})$. According to the minimality of $B_{CH(\Pi)}^-$, we have $CH(\Pi) \nsubseteq C_{c^*}$. Let $\pi' \in CH(\Pi) \setminus C_{c^*}$, which means that $\pi'$ is $O(1)$ away from $C_{c^*}$, because $C_{c^*}$ is a closed set. Let $\bar{\pi} \in \Pi^n$ denote an arbitrary vector such that $\sum_{j=1}^n \pi_j$ is $O(1)$ from $n \cdot \pi'$ in $L_{\infty}$. Then, $\sum_{j=1}^n \pi_j$ is $\Omega(n)$ away from $C_{c^*}$. By Claim 11, $\sum_{j=1}^n \pi_j$, which is the mean vector of $\bar{X}_\bar{\pi}$, is $\Omega(n)$ away from $H_B$ in $L_{\infty}$. The exponential upper bound follows after Hoeffding’s inequality and the union bound.

When $B \geq C_2^{-1} n$, the polynomial upper bound follows after the sup case. To prove the polynomial lower bound, notice that $C_3^+ > B_{CH(\Pi)}^- \supseteq B_{CH(\Pi)}$. Let $B^+$ be any number such that $B_{CH(\Pi)}^- < B^+ < C_3^-$. This means that $CH(\Pi) \subseteq C_{B^+}$. Because $B_{CH(\Pi)}^- \geq B_{CH(\Pi)}$, we have $B^+ > B_{CH(\Pi)}$. The proof for the lower bound is similar to the proof for the lower bound in the phase transition at $\Theta(n)$ case of inf: notice that in the proof of the $B \geq C_3 n$ case of PT-$\Theta(n)$-sup, the proof works for any $\pi^+ \in CH(\Pi) \cap C_{B^+} = CH(\Pi)$.

**D.4 Theorem 5: The $\psi > 0$ Case of Semi-Random PMV-Instability**

**Definition 17.** For any $\psi \in [0, 1]$, any n-profile $P$, any vector $\bar{x} \in \mathbb{R}^q$, and any set $S$, let Nb$(P, \psi)$ denote the set of n profiles that are obtained from $P$ by changing no more than $\psi n$ votes, and let

$\text{Nb}^*(\bar{x}, \psi) \triangleq \{ \bar{x}' : |\bar{x}'|_1 = |\bar{x}|_1 \text{ and } |\bar{x}' - \bar{x}|_1 \leq 2\psi|\bar{x}|_1\}$

$\text{Nb}(\bar{x}, \psi) \triangleq \{ \bar{x}' : |\bar{x}'|_1 = |\bar{x}|_1, |\bar{x}' - \bar{x}|_1 \leq 2\psi|\bar{x}|_1, \text{ and } |\bar{x}' - \bar{x}|_1 \in \mathbb{Z}^q\}

Nb(S, \psi) \triangleq \bigcup_{\bar{x} \in S} \text{Nb}(\bar{x}, \psi)$

**Theorem 5** (Semi-Random PMV-Instability, $\psi > 0$). Given any $q \in \mathbb{N}$, any closed and strictly positive $\Pi$ over $[q]$, and any PMV-instability setting $S = (H_S, H_T, \emptyset, c)$, any $C_1 > 0$, any $n \in \mathbb{N}$, and any $0 \leq B \leq C_1 \sqrt{n}$,

$\sup_{\bar{x} \in \Pi^n} \Pr(\text{Nb}(\bar{X}_{\bar{x}}, \psi) \cap U_{n,B} \neq \emptyset) = \begin{cases} 0 & \text{if } U_{n,B} = \emptyset \\ \exp(-\Theta(n)) & \text{otherwise, if } CH(\Pi) \cap \text{Nb}^*(C_0, \psi) = \emptyset \\ \Theta(1) & \text{otherwise} \end{cases}$

$\inf_{\bar{x} \in \Pi^n} \Pr(\text{Nb}(\bar{X}_{\bar{x}}, \psi) \cap U_{n,B} \neq \emptyset) = \begin{cases} 0 & \text{if } U_{n,B} = \emptyset \\ \exp(-\Theta(n)) & \text{otherwise, if } CH(\Pi) \cap \text{Nb}^*(C_0, \psi) = \emptyset \\ \Theta(1) & \text{otherwise} \end{cases}$
Proof. The 0 case of sup is straightforward. The proof of the exponential case of sup is similar to the proof of the exponential case of sup for \( \psi = 0 \). We will show that there is a linear separation between CH(II) and Nb(\( U_{n,B} \), \( \psi \)). Because \( U_{n,B} \subseteq H_B \), it suffices to show the separation between CH(II) and Nb(\( H_B \), \( \psi \)). To this end, we first recall from Claim 5 that every vector \( \bar{x} \in H_B \) is no more than \( \Theta(B + 1) \) away from \( C_0 \). Let \( \psi^* > \psi \) be an arbitrary number such that CH(II) \( \cap \) Nb\(*(C_0, \psi^*) = \emptyset \). Then, as \( B = O(\sqrt{n}) \), Nb(\( U_{n,B} \), \( \psi \)) \( \subseteq \) Nb\(*(C_0, \psi^*)\).

Next, we prove that Nb\(*(C_0, \psi^*)\) is a polyhedron. We first define LP\(_{\psi}\), whose variables are \( \bar{\chi}, \bar{k}, \) and \( \bar{x} \). Let \( K \) be the \( q(q - 1) \times q \) matrix that represents the changes that the data adversary can make to a single vote, i.e., every row contains a 1 and a \(-1\). \( \bar{k} \) represents the modification by the data adversary.

\[
\text{LP}_{\psi} \triangleq \begin{cases}
\text{max} & 0 \\
\text{s.t.} & A_S \times (\bar{x})^\top \leq (\bar{0})^\top \\
& A_T \times (\bar{x})^\top \leq (\bar{0})^\top \\
& (\bar{\chi})^\top = (\bar{x})^\top + (\bar{k} \times K)^\top \\
& \bar{k} \cdot \bar{I} \leq \psi \cdot \bar{x} \cdot \bar{I} \\
& \bar{x} \geq \bar{0}, \bar{\chi} \geq \bar{0}, \bar{k} \geq \bar{0}
\end{cases}
\]  

Clearly, Nb\(*(C_0, \psi^*)\) is the projection of the feasible solutions of LP\(_{\psi}\) (which is a polyhedron in the \((\bar{\chi}, \bar{k}, \bar{x})\) space) to \( \bar{\chi}'\)'s. The rest of proof for the exponential case is the same as the exponential case of sup for \( \psi = 0 \).

The proof of the \( \Theta(1) \) case is similar to the proof of the lower bound of the polynomial case of sup for \( \psi = 0 \). The main difference and challenge is to define the counterpart of Step 1.4: we will specify \( \Theta(n^{(q-1)/2}) \) integer vectors in an \( O(\sqrt{n}) \) neighborhood of the average of some \( \bar{\pi}^* \in \Pi^n \), such that each such vector can be modified by a data adversary to a vector \( \bar{x} \in C_0 \).

Let \( \pi^* \in \text{CH}(\Pi) \cap \text{Nb}*(C_0, \psi) \). Clearly \( n \pi^* \in \text{Nb}*(C_0, \psi) \), which means that \( C_0 \cap \text{Nb}^*(n \pi^*, \psi) \neq \emptyset \). Choose \( \bar{x} \in C_0 \cap \text{Nb}^*(n \pi^*, \psi) \). Recall from the proof of Claim 5 that \( C_0 \) are the feasible solutions to LP\(_{\text{Cone}}\), which means that \( \bar{x} \) is a feasible solution to LP\(_{\text{Cone}}\). Then, as proved in the proof of Claim 5, solutions to LP\(_{\text{Cone}}\) and solutions to LP\(_{H}^B\) are close to each other (as an application of [CGST86, Theorem 5 (i)]), there exists a solution \( \bar{x}' \) to LP\(_{H}^B\) such that \( |\bar{x} - \bar{x}'|_1 = O(B + 1) \). It follows that \( \bar{x}' \in H_B \). Define

\[
\bar{\chi}^0 \triangleq n \pi^* + (\bar{x}' - \bar{x})
\]

It follows that \( \bar{\chi}^0 \in \text{Nb}^*(H_B, \psi) \). Let \( \bar{\chi}^* \) denote the vector obtained from \( \bar{\chi}^0 \) by moving towards the \( \bar{x}' \) direction so that the \( L_1 \) distance between \( \bar{\chi}^0 \) and \( \bar{\chi}^* \) is \( 2\sqrt{n} \). That is,

\[
\bar{\chi}^0 \triangleq \bar{\chi}^0 + 2\sqrt{n} \cdot \frac{\bar{x}' - \bar{\chi}^0}{|\bar{x}' - \bar{\chi}^0|_1}
\]

Let \( B_{\sqrt{n}^\prime}(\bar{\chi}^*) \) denote the set of all vectors whose \( L_1 \) distance to \( \bar{\chi}^* \) is no more than \( \sqrt{n} \). When \( n \) is sufficiently large, all vectors in \( B_{\sqrt{n}^\prime}(\bar{\chi}^*) \) are in \( \mathbb{R}^n_{>0} \). In such cases,

\[
B_{\sqrt{n}^\prime}(\bar{\chi}^*) \subseteq \text{Nb}^*(H_B, \psi),
\]

because for any \( \bar{\chi} \in B_{\sqrt{n}^\prime}(\bar{\chi}^*) \), \( |\bar{\chi}|_1 = n \) and \( |\bar{\chi} - \bar{x}'|_1 \leq 2n \psi \). Moreover, the dimension of \( B_{\sqrt{n}^\prime}(\bar{\chi}^*) \) is \( q - 1 \). Therefore, we can choose and fix a (sparse) subset \( R_{B,n,\psi} \) of \( \Theta(n^{(q-1)/2}) \) of \( B_{\sqrt{n}^\prime}(\bar{\chi}^*) \) whose vectors are \( \Theta(1) \) away from each other.
Next, like Claim 7, for every \( \bar{\chi} \in B_{\sqrt{n}}(\bar{\chi}^*) \), we will identify a vector \( \chi' \in \text{Nb}(U_{n,0}, \psi) \) that is \( O(1) \) away by considering the following LP.

\[
\text{LP}_{n,B,\psi} \triangleq \begin{cases} 
\max & 0 \\
\text{s.t.} & A_S \times (\bar{x})^\top \leq (b_S)^\top \\
& A_T \times (\bar{x} + \bar{\sigma} \times O)^\top \leq (b_T)^\top \\
& \bar{x} \cdot \bar{1} = n \\
& \bar{c} \cdot \bar{\sigma} \leq B \\
& (\bar{\chi})^\top = (\bar{x})^\top + (\bar{\kappa} \times K)^\top \\
& \bar{\kappa} \cdot \bar{1} \leq \psi \cdot \bar{x} \cdot \bar{1} \\
& \chi' \geq \bar{0}, \bar{\kappa} \geq \bar{0}, \bar{x} \geq \bar{0}, \bar{\sigma} \geq \bar{0}, \\
\end{cases} \tag{28}
\]

Because \( \bar{\chi} \in B_{\sqrt{n}}(\bar{\chi}^*) \subseteq \text{Nb}^*(H_B, \psi) \), there exist \( \bar{\kappa}, \bar{x}, \) and \( \bar{\sigma} \) such that \((\bar{\chi}, \bar{\kappa}, \bar{x}, \bar{\sigma})\) is a feasible solution to \( \text{LP}_{n,B,\psi} \). Because \( U_{n,B} \neq \emptyset \), \( \text{LP}_{n,B,\psi} \) has a feasible integer solution. Therefore, like in the proof of Claim 7 and following [CGST86, Theorem 1], \( \text{LP}_{n,B,\psi} \) has an integer solution \( \chi' \) that is \( \Theta(1) \) away from \( \bar{\chi} \).

The rest of the proof for the \( \Theta(1) \) case of sup is the same as the proof of the lower bound of the polynomial case of sup for \( \psi = 0 \). The inf part is similar to the inf part for \( \psi = 0 \) with similar modifications with the sup part above.

\[\square\]

E Materials for Section 5

E.1 Full Version of Theorem 2 and Its Proof

**Theorem 2** (Max-Semi-Random Coalitional Influence). Let \( r \) be an integer positional scoring rule, STV, ranked pairs, Schulze, maximin, Copeland, plurality with runoff, or Bucklin with lexicographic tie-breaking. For any closed and strictly positive \( \Pi \) with \( \pi_{uni} \in \text{CH}(\Pi) \), any \( X \in \{\text{CM}, \text{MoV}\} \cup \text{E-CONTROL} \) (except \( X \in \text{E-CONTROL} \) when \( r = \text{Cd}_0 \)), there exists \( N > 0 \) such that for all \( n > N \) and all \( B \geq 0 \),

\[
\tilde{X}_{\Pi,0}^\text{max}(r,n,B) = \begin{cases} 
0 & \text{if } B = 0 \\
\Theta \left( \min \left\{ \frac{B}{\sqrt{n}}, 1 \right\} \right) & \text{if } B \geq 1 \\
\end{cases}
\]

For any \( X \in \text{CONTROL} \), any \( n > N \) and any \( B \geq 0 \), \( \tilde{X}_{\Pi,0}^\text{max}(r,n,B) = \Theta(1) \). Moreover, for any \( \psi > 0 \), \( \tilde{X}_{\Pi,\psi}^\text{max}(r,n,B) = \Theta(1) \) for the non-zero cases above.

**Proof.** We first prove the \( B = 0 \) case. For any \( X \in \{\text{CM}, \text{MoV}\} \cup \text{E-CONTROL} \), it is not hard to verify that if no budget is given, then the goal under \( X \) cannot be reached, which requires the winner to be changed. Therefore, \( \tilde{X}_{\Pi,0}^\text{max}(r,n,B) = 0 \). For any \( X \in \text{CONTROL} \), it suffices to prove that for any alternative \( a \),

\[
\sup_{\bar{x} \in \Pi^n} \Pr_{P \sim \bar{x}}(r(P) = \{a\}) = \Theta(1) \tag{29}
\]

It is not hard to see that for all voting rules mentioned in this statement of the theorem, there exists a polyhedron \( \mathcal{H}_a \) such that for all \( \bar{x} \in \mathcal{H}_a \), \( r(\bar{x}) = \{a\} \), \( \pi_{uni} \in \mathcal{H}_{\leq 0}^a \), and \( \dim(\mathcal{H}_{\leq 0}^a) = m! \). Therefore, (29) follows after [Xia21a, Theorem 1] (or equivalently, Theorem 1 with \( \mathcal{H}_S = \mathcal{H}_T = \mathcal{H}_a \) and \( B = 0 \)).

In the rest of the proof, we assume that \( X \in \{\text{CM}, \text{MoV}\} \cup \text{E-CONTROL} \) and \( B \geq 1 \).
Overview. Due to Theorem 3, it suffices to prove the \( \Theta \left( \min \left\{ \frac{B}{\sqrt{n}}, 1 \right\} \right) \) matching lower bound by identifying a PMV-instability setting \( S \) that represents some unstable histograms \( U_{n,B} \), such that for all \( B \geq 1 \),
\[
\bar{X}_{H,S}^{\max}(r,n,B) \geq \sup_{\bar{s} \in \Pi^n} \Pr_{P \sim \bar{s}}(\text{Hist}(P) \in U_{n,B}),
\]
and
\[
\sup_{\bar{s} \in \Pi^n} \Pr_{P \sim \bar{s}}(\text{Hist}(P) \in U_{n,B}) = \Theta \left( \min \left\{ \frac{B}{\sqrt{n}}, 1 \right\} \right)
\]  \( (30) \)

Notice that for any constant \( C_1 \) and any \( B \geq C_1 \sqrt{n} \), the right hands side of (30) is \( \Theta(1) \). Therefore, it suffices to prove (30) for all \( 1 \leq B \leq C_1 \sqrt{n} \). For each \( X \) and each voting rule \( r \) in the statement of the theorem, we define \( S = (H_S, H_T, \emptyset, \bar{1}) \), prove that \( U_{n,1} \neq \emptyset \) for any sufficiently large \( n \) by construction, and prove that \( d_0 = m! - 1 \) (which if often obvious), \( d_\infty = m! \) (by applying Claim 15).

We now prove Theorem 2 for \( X = \text{CM} \), and then comment on how to modify the proof for other \( X \in \{ \text{MoV} \} \cup \text{E-Control} \).

CM for integer positional scoring rules. Let \( r = r_{\bar{s}} \) denote the positional scoring rule with scoring vector \( \bar{s} \). Let \( H_S \) denote the set of vectors where 1’s total score is at least as high as 2’s total score, which is strictly higher than the total score of any other alternative. Let \( H_T \) denote the set of vectors where 2’s total score is strictly the highest. To formally define \( H_S \) and \( H_T \), we first recall the definition of score difference vectors.

Definition 18 (Score difference vector [Xia20]). For any scoring vector \( \bar{s} = (s_1, \ldots, s_m) \) and any pair of different alternatives \( a, b \), let \( \text{Score}_{a,b}^\bar{s} \) denote the \( m! \)-dimensional vector indexed by rankings in \( L(A) \): for any \( R \in L(A) \), the \( R \)-element of \( \text{Score}_{a,b}^\bar{s} \) is \( s_{j_1} - s_{j_2} \), where \( j_1 \) and \( j_2 \) are the ranks of \( a \) and \( b \) in \( R \), respectively.

In words, \( \text{Score}_{a,b}^\bar{s} \) is the score vector of \( a \) (under all linear orders) minus the score vector of \( b \). Then, we define
\[
H_S \triangleq \left\{ \bar{x} : \forall i \geq 3, \, \text{Score}_{i,1}^\bar{x} \cdot \bar{x} \leq 0 \right\}, \quad H_T \triangleq \left\{ \bar{x} : \forall i \neq 2, \, \text{Score}_{i,2}^\bar{x} \cdot \bar{x} \leq -1 \right\}, \quad \text{and}
\]
\[
S_{\bar{s}} = (H_S, H_T, \emptyset_1, \bar{1})
\]

It is not hard to verify that for any \( \bar{y} \in H_S \) and any \( \bar{x} \in H_T \), we have \( r_{\bar{s}}(\bar{y}) = \{1\} \) (1 has the highest score and wins due to tie-breaking if 2 also has the highest score) and \( r_{\bar{s}}(\bar{x}) = \{2\} \) (2 has the strictly highest score).

Next, we show that \( U_{n,B} \neq \emptyset \) for any sufficiently large \( n \) by constructing a successful instance of manipulation by a single voter. We first define some profiles and rankings that will be used in the rest of the proof. For any \( a \in A \), let \( \sigma_a \) denote a cyclic permutation among \( A \setminus \{a\} \). Let \( P_a \) denote the following \( (m-1) \)-profile.
\[
P_a \triangleq \{ \sigma_a^i(a \succ \text{others}) : 1 \leq i \leq m-1 \} \cup 3 \times L(A),
\]
where alternatives in “others” are ranked alphabetically. Let
\[
P_* \triangleq 2m \times (P_1 \cup P_2) \cup \bigcup_{i=3}^m 2(m-i) \times P_i
\]

It follows that the \( \bar{s}(P_1, 1) = \bar{s}(P_2, 2) > \bar{s}(P_3, 3) > \cdots > \bar{s}(P_m, m) \).
Let \( R_1 \) (respectively, \( R_2 \)) denote the ranking where 1 (respectively, 2) is ranked at the top, 2 (respectively, 1) is ranked at the bottom, and the remaining alternatives are ranked alphabetically. That is,
\[
R_1 \triangleq [1 > 3 > \cdots > m > 2] \quad \text{and} \quad R_2 \triangleq [2 > 3 > \cdots > m > 1]
\]
Let \( \ell \leq m - 1 \) denote the index to the minimum value of \( s_\ell - s_{\ell+1} \). Let \( R'_2 \) denote the ranking where 2 and 1 are ranked at the \( \ell \)-th and the \((\ell + 1)\)-th positions respectively, and the remaining alternatives are ranked alphabetically. That is,
\[
R'_2 \triangleq 3 > \cdots > \ell + 1 > 2 > 1 > \ell + 2 > \cdots > m
\]
Next, we define \( P_S \) and \( P_T \). We first define \( P'_S \) to be the \( n \)-profile that consists of as many copies of \( P_* \) as possible, and the remaining rankings are \( R_1 \). That is, let \( n' \triangleq \lceil \frac{n}{|P_*|} \rceil \times |P_*| \), and
\[
P'_S \triangleq \frac{n'}{|P_*|} \times P_* \cup (n - n') \times \{R_1\}
\]
Let \( P_S \) denote the profile obtained from \( P'_S \) by replacing \( \frac{(n-n')(s_1-s_m)}{s_1 + s_{\ell+1} - s_{\ell+2} - s_m} \) copies of \( R'_2 \) by \( R_2 \). It follows that for any sufficiently large \( n \) (so that \( P'_S \) contains enough copies of \( R'_2 \) and the score difference between 1 and any alternative \( i \geq 3 \) is sufficiently large), \( P_S \) is well-defined and \( r_S(P_S) = \{1\} \). Let \( P_T \) be obtained from \( P_S \) by replacing an \( R'_2 \) vote by an \( R_2 \) vote. It follows that \( r_S(P_T) = \{2\} \). This proves that \( U_{n,1} \neq \emptyset \) for any sufficiently large \( n \).

It is not hard to verify that \( d_0 = m! - 1 \) (the only implicit equality represents 1 and 2 have the same score). Let \( P'_S \) be the profile obtained from \( P_S \) by replacing an \( R_2 \) vote by an \( R'_2 \) vote. It follows that \( \text{Hist}(P'_S) \) and \( \text{Hist}(P_T) \) are interior points of \( \mathcal{H}_{S,\leq 0} \) and \( \mathcal{H}_{T,\leq 0} \), respectively. Therefore, By Claim 15, \( d_\infty = m! \). The lower bound (30) follows after the polynomial case of sup of Theorem 1 (applied to \( S_\ell \)). This completes the proof of Theorem 2 for CM under integer positional scoring rules.

**CM for STV.** Let \( \mathcal{H}_S \) consists of the histograms (for which the STV winner is 1) where the execution of STV satisfies the following conditions:

- for every \( 1 \leq i \leq m - 4 \), in round \( i \), alternative \( m + 1 - i \) has the strictly lowest plurality score among the remaining alternatives;
- In round \( m - 3 \), alternative 3 has the highest score, and the score of 2 is no more than the score of 1;
- if 1 loses in round \( m - 3 \), then 2 would become the winner; and if 2 loses in round \( m - 3 \), then 1 would become the winner.

Formally, let us recall the score difference vector (for a pair of alternatives \( a,b \), after a set of alternatives \( B \) is removed) to define \( \mathcal{H}_S \) and \( \mathcal{H}_T \).

**Definition 19** ([Xia21b]). For any pair of alternatives \( a,b \) and any subset of alternatives \( B \subset (A \setminus \{a,b\}) \), we let \( \text{Score}_{B,a,b}^\Delta \) denote the vector, where for every \( R \in \mathcal{L}(A) \), the \( R \)-th component of \( \text{Score}_{B,a,b}^\Delta \) is the plurality score of \( a \) minus the plurality score of \( b \) in \( R \) after alternatives in \( B \) are removed.

Then, \( \mathcal{H}_S \) consists of vectors \( \vec{x} \) that satisfies the following linear constraints.
For every $i \leq m - 4$ and every $i' < i$,

$$\text{Score}_{\{m+2-i',...,m\}}^{\Delta} \cdot \bar{x} \leq -1$$

- Let $B_3 = \{4, \ldots, m\}$. There are two constraints: $\text{Score}_{B_3,2,1}^{\Delta} \cdot \bar{x} \leq 0$ and $\text{Score}_{B_3,1,3}^{\Delta} \cdot \bar{x} \leq -1$.
- $\text{Score}_{B_3 \cup \{1\}, 3, 2}^{\Delta} \cdot \bar{x} \leq -1$ and $\text{Score}_{B_3 \cup \{2\}, 3, 1}^{\Delta} \cdot \bar{x} \leq -1$.
- For every $R \in L(A)$, there is a constraint $-x_R \leq 0$.

Let $\mathcal{H}_T$ denote the polyhedron that differs from $\mathcal{H}_S$ in round $m - 3$, where 1 has the lowest plurality score and drops out, which means that 2 is the STV winner. It is not hard to verify that for all $\tilde{y} \in \mathcal{H}_S$ and all $\tilde{x} \in \mathcal{H}_T$, we have STV($\tilde{y}$) = $\{1\}$ and STV($\tilde{x}$) = $\{2\}$.

Let $\mathcal{S}_{\text{STV}} \triangleq (\mathcal{H}_S, \mathcal{H}_T, \emptyset, 1, 2, 1)$. Next, we construct profiles $P_S$ and $P_T$ to show that $\mathcal{U}_{n,B} \neq \emptyset$ for any sufficiently large $n$ and $B \geq 1$. For any $a \in A$, let $P^*_a$ denote the $(m - 1)!$-profile that is obtained from $L(A \setminus \{a\})$ by putting $a$ at the top. Let

$$P^* \triangleq \bigcup_{i=4}^{m} (m - i) \times P^*_a \text{ and } n^* \triangleq |P^*|$$

Let

$$P_S \triangleq \left[ \frac{n - n^*}{3} - 1 \right] \times \{[1 \succ 2 \triangleright \text{others}], [2 \succ 1 \triangleright \text{others}]\}$$

$$+ \left( n - n^* + 2 \left[ \frac{n - n^*}{3} \right] \right) \times \{3 \succ 2 \triangleright 1 \triangleright \text{others} \} + P^*$$

It follows that $|P_S| = n$, and for all $n \geq n^* + 3m(m - 1)!$, Hist($P_S) \in \mathcal{H}_S$. Let $P_T$ be the profile obtained from $P_S$ by replacing one vote of $[3 \succ 2 \triangleright 1 \triangleright \text{others}]$ by $[2 \succ 3 \triangleright 1 \triangleright \text{others}]$. It is not hard to verify that Hist($P_T) \in \mathcal{H}_T$. Therefore, $\mathcal{U}_{n,B} \neq \emptyset$ for every $B \geq 1$.

It is not hard to verify that $d_\infty = \dim(\mathcal{H}_{S,\leq 0} \cap \mathcal{H}_{T,\leq 0}) = m! - 1$, where the only implicit equality requires 1 and 2 are tied for the last place in round $m - 3$. To see $d_\infty = m!$, let $P^*_S$ denote the profile obtained from $P_S$ by adding one vote of $[1 \succ 2 \triangleright \text{others}]$. Let $\tilde{y} = \text{Hist}(P^*_S)$ and let $\bar{x}$ be the histogram of the profile obtained from $P^*_S$ by changing two votes of $[3 \succ 2 \triangleright 1 \triangleright \text{others}]$ to $[2 \succ 3 \triangleright 1 \triangleright \text{others}]$. It is not hard to verify that $\tilde{y}$ is an interior point of $\mathcal{H}_{S,\leq 0}$, $\tilde{y}$ is an interior point of $\mathcal{H}_{T,\leq 0}$, $\dim(\mathcal{H}_{S,\leq 0}) = \dim(\mathcal{H}_{T,\leq 0}) = m!$. By Claim 15, we have $d_\infty = m!$.

Then, (30) follows after the application of the sup part of Theorem 1 to the PMV-instability setting $\mathcal{S}_{\text{STV}}$ for CM under STV.

**CM for Ranked Pairs, Schulze, and maximin.** The proof for the three rules share the same construction. Let $\mathcal{H}_S$ denote the polyhedron that consists of vectors $\bar{x}$ whose WMG satisfies the following conditions.

- The weights on the following edges are strictly positive: $1 \to 2$, $2 \to 3$, $3 \to 1$, $\{1, 2, 3\} \to \{4, \ldots, m\}$.
- For all $i \geq 4$, the weight on $1 \to i$ is strictly larger than the weight on $1 \to 2$.
- $w_{\bar{x}}(2 \to 3) > w_{\bar{x}}(1 \to 2) \geq w_{\bar{x}}(3 \to 1)$.

See Figure 4 (a) for an example of WMG that satisfies these conditions. Formally, we first recall the pairwise difference vectors as follows.
Definition 20 (Pairwise difference vectors [Xia20]). For any pair of different alternatives \(a, b\), let \(Pair_{a,b}\) denote the \(m!\)-dimensional vector indexed by rankings in \(\mathcal{L}(A)\): for any \(R \in \mathcal{L}(A)\), the \(R\)-component of \(Pair_{a,b}\) is \(1\) if \(a \succ_R b\); otherwise it is \(-1\).

Then, let \(\mathcal{H}_S\) be characterized by the following linear inequalities/constraints:

- For each edge \(a \to b \in \{1 \to 2, 2 \to 3, 3 \to 1\} \cup \{(1, 2, 3) \to \{4, \ldots, m\}\}\), there is a constraint \(Pair_{b,a} \cdot \vec{x} \leq -1\).
- For all \(i \geq 4\), \((Pair_{1,2} - Pair_{1,i}) \cdot \vec{x} \leq -1\).
- \((Pair_{1,2} - Pair_{2,3}) \cdot \vec{x} \leq -1\) and \((Pair_{3,1} - Pair_{1,2}) \cdot \vec{x} \leq 0\).
- For all linear order \(R \in \mathcal{L}(A)\), there is a constraint \(-x_R \leq 0\).

Let \(\mathcal{H}_T\) denote the polyhedron that consists of vectors \(\vec{x}\) whose WMG satisfies the same conditions as \(\mathcal{H}_T\), except that now it is required that \(w_n(2 \to 3) > w_n(3 \to 1) > w_n(1 \to 2)\). See Figure 4 (b) for an example of WMG that satisfies these conditions for odd \(n\). We have dim\((\mathcal{H}_{T, \leq 0}) = m! - 1\) (the implicit equality is \((Pair_{3,1} - Pair_{2,3}) \cdot \vec{x} = 0\)) and dim\((\mathcal{H}_{T, < 0}) = m!\).

![Figure 4: CM under RP, Sch, and MM.](image)

Let \(\mathcal{S} = (\mathcal{H}_S, \mathcal{H}_T, O^{1 \to 2, 1}_1, \mathbf{1})\). It follows that for any \(\vec{y} \in \mathcal{H}_S\) and any \(\vec{x} \in \mathcal{H}_T\), RP\((\vec{y}) = \{1\}\) (1 \(\to 2\) is fixed before 3 \(\to 1\) due to tie-breaking) and RP\((\vec{x}) = \{2\}\) (3 \(\to 1\) is fixed before 1 \(\to 2\)); Sch\((\vec{y}) = \{1\}\) (1 and 2 are co-winners, so 1 wins due to tie-breaking) and Sch\((\vec{x}) = \{2\}\) (2 is the unique winner); and MM\((\vec{y}) = \{1\}\) (1 and 2 are co-winners, so 1 wins due to tie-breaking) and MM\((\vec{x}) = \{2\}\) (2 is the unique winner).

Next, we define \(n\)-profiles \(P_S\) and \(P_T\) to show that \(U_{n,1} \neq \emptyset\). The construction depends on the parity of \(n\). If \(n\) is odd, let \(P_S\) denote a profile whose WMG is the figure shown in Figure 4 (a), where the weights on 1 \(\to 2\) and 3 \(\to 1\) are the same. Due to McGarvey’s theorem [McG53], such \(P_S\) exists for all sufficiently large odd number \(n\), and we can assume that \(P_S\) contains two copies of \(\mathcal{L}(A)\). Let \(P_T\) denote the profile obtained from \(P_S\) by replacing a [2 \(\succ 1 \succ 3 \succ \text{others}\)] by [\(2 \succ 3 \succ 1 \succ \text{others}\)], which means that the WMG of \(P_T\) is Figure 4 (b). If \(n\) is even, then let the positive weights on edges in the WMGs of \(P_S\) and \(P_T\) be one more than those for odd \(n\), so that all weights become even numbers. In either case, it is not hard to verify that \(U_{n,B} \neq \emptyset\) for every sufficiently large \(n\) and every \(B\).

It is not hard to verify that \(d_0 = m! - 1\). To see \(d_{\infty} = m!\), let \(\vec{y}\) be any vector such that WMG\((\vec{x})\) is the same as Figure 4 (c). Let \(\vec{x}\) denote the vector obtained from \(\vec{y}\) by replacing two votes of [2 \(\succ 1 \succ 3 \succ \text{others}\)] by [2 \(\succ 3 \succ 1 \succ \text{others}\)]. It follows that the WMG\((\vec{x})\) is
Figure 4 (b). Notice that $\vec{y}$ is an interior point of $H_{S, \leq 0}$; $\vec{x}$ is an interior point of $H_{T, \leq 0}$; and $\dim(H_{S, \leq 0}) = \dim(H_{T, \leq 0}) = m!$. By Claim 15, we have $d_\infty = m!$. Then, (30) follows after the application of the sup part of Theorem 1 to the PMV-instability setting for CM under ranked pairs, Schulze, and maximin.

CM for Copeland.

CM, odd $n$. Let $H_S$ denote the polyhedron that consists of vectors $\vec{x}$ whose UMG has the following edges: $1 \to 3$, $3 \to 2$, $2 \to 1$, $\{1, 2, 3\} \to \{4, \ldots, m\}$. See Figure 5 (a) for an example for $m = 4$. Formally, $H_S$ is characterized by the following linear inequalities/constraints:

- For each edge $a \to b \in \{1 \to 3, 3 \to 2, 2 \to 1\} \cup (\{1, 2, 3\} \to \{4, \ldots, m\})$, there is a constraint $\text{Pair}_{b,a} \cdot \vec{x} \leq -1$.

- For all linear order $R \in L(A)$, there is a constraint $-x_R \leq 0$.

Let $H_T$ denote the polyhedron that consists of vectors $\vec{x}$ whose UMG has the same edges $H_S$, except that the direction between 2 and 3 is flipped, i.e., $w_\vec{x}(3 \to 2) \leq -1$. See Figure 5 (b) for an example of the UMG for $m = 4$.

![Figure 5: CM under Cd$_{\alpha}$, odd $n$.](image)

Let $S_{Cd_{\alpha}} = \langle H_S, H_T, O_{1-2}^1, \vec{1} \rangle$. It follows that for any $\vec{y} \in H_S$ and any $\vec{x} \in H_T$, $Cd_{\alpha}(\vec{y}) = \{1\} \cup \{2\}$ (1, 2, 3 have the same highest Copeland score, and then 1 wins due to tie-breaking) and $Cd_{\alpha}(\vec{x}) = \{2\}$ (2 is the Condorcet winner).

Next, we define $n$-profiles $P_S$ and $P_T$ to show that $U_{n,1} \neq \emptyset$. Let $P_S$ denote a profile whose WMG is the figure shown in Figure 5 (c), where the weight on $3 \to 2$ is 1. Due to McGarvey’s theorem [McG53], such $P_S$ exists for all sufficiently large odd number $n$, and we can assume that $P_S$ contains $L(A)$. Let $P_T$ denote the profile obtained from $P_S$ by replacing a $[3 \succ 2 \succ 1 \succ \text{others}]$ vote by $[2 \succ 3 \succ 1 \succ \text{others}]$, which means that the UMG of $P_T$ is Figure 5 (b).

Recall that $P_T$ is obtained from $P_S$ by replacing a vote. Therefore, Hist($P_S$) $\in U_{n,1}$, which means that $U_{n,1} \neq \emptyset$. It is not hard to verify that $d_0 = m! - 1$ (the implicit equality corresponds to the tie between 2 and 3). Notice that Hist($P_S$) and Hist($P_T$) are interior points of $H_{S, \leq 0}$ and $H_{T, \leq 0}$, respectively, and $\dim(H_{S, \leq 0}) = \dim(H_{T, \leq 0}) = m!$. By Claim 15, we have $d_\infty = m!$. Also notice that $\tau_{\min} \in H_{S, \leq 0} \cap H_{T, \leq 0}$. Therefore, by Theorem 1, for any $B \geq 1$ and any sufficiently large odd $n$, we have

$$
\sup_{\vec{y} \in \Pi^n} \Pr_{R \sim \vec{y}}(\text{Hist}(P) \in U_{n,B}) = \Theta \left( \min \left\{ \frac{B}{\sqrt{n}}, 1 \right\} \right)
$$

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CM, even \( n, \alpha > 0 \). Let \( \mathcal{H}_S \) be the same as defined for the odd \( n \) case above (the UMG of all vectors in \( \mathcal{H}_S \) is illustrated in Figure 6 (a)). Let \( \mathcal{H}_T \) be the polyhedron that consists of vectors \( \vec{x} \) whose UMG satisfies the following conditions: the weights on the following edges are strictly positive: \( 1 \rightarrow 3, 2 \rightarrow 1, \{1,2,3\} \rightarrow \{4,\ldots,m\} \). In addition, we require that WMG(\( \vec{x} \)) does not contain the edge \( 3 \rightarrow 2 \), i.e., we require \( w_{x}(2 \rightarrow 3) \geq 0 \). See Figure 6 (b) for the UMG for \( m = 4 \) (where the dashed edge from 2 to 3 means that either there is no edge between 2 and 3, or there is an edge \( 2 \rightarrow 3 \)). Formally, \( \mathcal{H}_T \) is characterized by the following linear inequalities/constraints:

- For each edge \( a \rightarrow b \in \{1 \rightarrow 3, 2 \rightarrow 1\} \cup \{\{1,2,3\} \rightarrow \{4,\ldots,m\}\} \), there is a constraint \( \text{Pair}_{b,a} \cdot \vec{x} \leq -1 \).
- \( \text{Pair}_{3,2} \cdot \vec{x} \leq 0 \).
- For all linear order \( R \in L(A) \), there is a constraint \( -x_R \leq 0 \).

\[
\begin{align*}
\text{(a) UMG for } \mathcal{H}_S & \quad \text{(b) UMG for } \mathcal{H}_T & \quad \text{(c) WMG}(P_S) \\
\end{align*}
\]

Figure 6: e-CCAV under \( C_{d_\alpha} \), even \( n, \alpha > 0 \).

Let \( S_{C_{d_\alpha}} = \langle \mathcal{H}_S, \mathcal{H}_T, O_{\pm}^{1 \rightarrow 2}, \vec{1} \rangle \). It follows that for any \( \vec{y} \in \mathcal{H}_S \) and any \( \vec{x} \in \mathcal{H}_T \), \( C_{d_\alpha}(\vec{y}) = \{1\} \) (1,2,3 have the same highest Copeland score, and then 1 wins due to tie-breaking) and \( C_{d_\alpha}(\vec{x}) = \{2\} \) (2 has the highest Copeland score, which is at least \( m - 2 + \alpha \)).

Next, we define \( n \)-profiles \( P_S \) and \( P_T \) to show that \( U_{n,1} \neq \emptyset \). Let \( P_S \) denote any \( n \)-profile that contains \( L(A) \), whose UMG is as in Figure 6 (a), and its existence is guaranteed by McGarvey’s theorem [McG53]. Let \( P_T \) denote the profile obtained from \( P_S \) by replacing a \([3 \succ 2 \succ 1 \succ \text{others}] \) vote by \([2 \succ 3 \succ 1 \succ \text{others}] \), which means that the UMG of \( P_T \) is consistent Figure 6 (b) (and there is no edge between 2 and 3 in UMG(\( P_T \))).

To characterize \( d_\infty \), we prove the following convenient claim for general PMV-instability problems (for general \( q \)) that will be frequently used in the proofs of this paper.

**Claim 15.** Suppose \( \dim(\mathcal{H}_{T \leq 0}) = q \) and \( \mathcal{H}_{S \leq 0} \) has an interior point that can be manipulated to an interior point of \( \mathcal{H}_{S \leq 0} \), then \( d_\infty = \dim(\mathcal{H}_{S \leq 0}) \).

**Proof.** Because \( C_{\infty} \subseteq \mathcal{H}_{S \leq 0} \), we have \( d_\infty \leq \dim(\mathcal{H}_{S \leq 0}) \). Next, we prove that \( d_\infty \geq \dim(\mathcal{H}_{S \leq 0}) \). Let \( \vec{y} = \vec{x} - \vec{\sigma} \times \vec{0} \) denote an interior point \( \mathcal{H}_{S \leq 0} \), where \( \vec{x} \) is an interior point of \( \mathcal{H}_{T \leq 0} \) and \( \vec{\sigma} \geq 0 \) represents successful operations (without budget constraints). Let \( C > 0 \) denote an arbitrary number so that the \( C \) neighborhood of \( \vec{x} \) in \( L_\infty \) s contained in \( \dim(\mathcal{H}_{T \leq 0}) \). Therefore, the \( C \) neighborhood of \( \vec{y} \) in \( L_\infty \) is contained in \( \mathcal{H}_{T \leq 0} + Q_{\vec{\sigma} \times \vec{0}} \). It follows that any vector \( \vec{y} \in \mathcal{H}_{S \leq 0} \) that is at most \( C \) away from \( \vec{y} \) in \( L_\infty \) is in \( C_{\infty} \). Because \( \vec{y} \) is an interior point of \( \mathcal{H}_{S \leq 0} \), we have \( d_\infty \geq \dim(\mathcal{H}_{S \leq 0}) \), which proves Claim 15. □
It is not hard to verify that \( d_0 = m! - 1 \) and \( d_\infty = m! \) (by Claim 15). Notice that \( \pi_{\text{uni}} \in \mathcal{H}_{S,\leq 0} \cap \mathcal{H}_{T,\leq 0} \). The even \( n \) and \( \alpha > 0 \) case follows after Theorem 1.

CM, even \( n \), \( \alpha = 0 \). Let \( \mathcal{H}_S \) be the polyhedron that consists of vectors \( \vec{x} \) whose UMG contains 1 \( \rightarrow \) 3, 2 \( \rightarrow \) 1, \( \{1,2,3\} \rightarrow \{4,\ldots,m\} \). In addition, we require that WMG(\( \vec{x} \)) does not contain 2 \( \rightarrow \) 3, that is, \( w_{\vec{x}}(3 \rightarrow 2) \geq 0 \). See Figure 7 (a) for the UMG for \( m = 4 \) (where the dashed edge from 3 to 2 means that either there is no edge between 2 and 3, or there is an edge 3 \( \rightarrow \) 2). Formally, \( \mathcal{H}_S \) is characterized by the following linear inequalities/constraints:

- For each edge \( a \rightarrow b \in \{1 \rightarrow 3, 2 \rightarrow 1\} \cup \{\{1,2,3\} \rightarrow \{4,\ldots,m\}\} \), there is a constraint \( \text{Pair}_{a,b} \cdot \vec{x} \leq -1 \).
- \( \text{Pair}_{2,3} \cdot \vec{x} \leq 0 \).
- For all linear order \( R \in \mathcal{L}(\mathcal{A}) \), there is a constraint \( -x_R \leq 0 \).

Then, we let \( \mathcal{H}_T \) be the same as \( \mathcal{H}_T \) in the odd \( n \) case above (as illustrated in Figure 7 (b) for \( m = 4 \)). Let \( \mathcal{S}_{Cd_a} = (\mathcal{H}_S, \mathcal{H}_T, \mathcal{O}_{\pm}^{2}, \bar{1}) \). It follows that for any \( \vec{y} \in \mathcal{H}_S \) and any \( \vec{x} \in \mathcal{H}_T \), \( \text{Cd}_a(\vec{y}) = \{1\} \) (1 and 2 have the same highest Copeland score, so 1 wins due to tie-breaking) and \( \text{Cd}_a(\vec{x}) = \{2\} \) (2 has the strictly highest Copeland score \( m - 1 \)).

Let \( P_S \) denote an arbitrary \( n \)-profile whose WMG is as shown in Figure 7 (c) and it contains two copies of \( \mathcal{L}(\mathcal{A}) \). Let \( P_T \) denote the \( n \)-profile obtained from \( P_S \) by replacing a \( [3 \succ 2 \succ 1 \succ \text{others}] \) vote by \( [2 \succ 3 \succ 1 \succ \text{others}] \), which means that the UMG of \( P_T \) is like Figure 7 (b). It follows that \( \text{Hist}(P_S) \in \mathcal{H}_S \) and \( \text{Hist}(P_T) \in \mathcal{H}_T \), which proves that \( U_{n,1} \neq \emptyset \).

![Figure 7: e-CCAV under Cd_a, even n, \( \alpha = 0 \).](image)

It is not hard to verify that \( d_0 = m! - 1 \). To see \( d_\infty = m! \), let \( \vec{y} \) be any vector such that WMG(\( \vec{x} \)) is the same as Figure 7 (d) and it contains two copies of \( [3 \succ 2 \succ 1 \succ \text{others}] \). Let \( \vec{x} \) denote the vector obtained from \( \vec{y} \) by replacing two votes of \( [3 \succ 2 \succ 1 \succ \text{others}] \) vote by \( [2 \succ 3 \succ 1 \succ \text{others}] \). It follows that the UMG(\( \vec{x} \)) is Figure 7 (b). Notice that \( \vec{y} \) is an interior point of \( \mathcal{H}_{T,\leq 0} \); \( \vec{x} \) is an interior point of \( \mathcal{H}_{S,\leq 0} \); and \( \dim(\mathcal{H}_{S,\leq 0}) = \dim(\mathcal{H}_{T,\leq 0}) = m! \). By Claim 15, we have \( d_\infty = m! \). The even \( n \) and \( \alpha = 0 \) case follows after Theorem 1.

Other coalitional influence problems.

The proof for MoV is based on the same constructions of \( P_S \), \( P_T \), \( \mathcal{H}_S \), and \( \mathcal{H}_T \). The only difference is that the set of vote operations in \( \mathcal{S} \) is \( \mathcal{O}_\pm \).

The proofs for e-CCAV and e-CCDV are based on similarly constructions of \( P_S \), \( P_T \), \( \mathcal{H}_S \), and \( \mathcal{H}_T \). The main differences are, first, the set of vote operations in \( \mathcal{S} \) is \( \mathcal{O}_+ \) and \( \mathcal{O}_- \) for e-CCAV and e-CCDV, respectively. Second, the added (respectively, deleted) votes correspond to the new (respectively, old) votes in CM. We add \( 3 \times \mathcal{L}(\mathcal{A}) \) to \( P_S \) so that there is enough votes to be deleted.
for e-CCDV. Below we take constructive control \( \{d\} = r(P_T) \) for example (where \( P_T \) depends on the problem and will be specified soon). Other cases can be proved similarly.

- **Integer positional scoring rules.** If \( d \neq 1 \), then \( P_S, H_S, \) and \( H_T \) are similar to their counterparts in the proof of CM under integer positional scoring rules. Take \( d = 2 \) for example, for e-CCAV, the added votes are \( RM_2 \); and for e-CCDV, the deleted votes are \( RM_1 \). If \( d = 1 \), then we switch the roles of \( H_S, \) and \( H_T, \) and swap the roles of \( P_S \) and \( P_T \) in the proof of CM under integer positional scoring rules. Then, for e-CCAV, the added votes are \( RM_1 \); and for e-CCDV, the deleted votes are \( RM_2 \).

- **STV.** If \( d \neq 1 \), then \( P_S, H_S, \) and \( H_T \) are similar to their counterparts in the proof of CM under STV. Take \( d = 2 \) for example, for e-CCAV, for e-CCDV, the added votes are \([2 \succ 3 \succ 1 \succ \text{others}] \); and for e-CCDV, the deleted votes are \([4 \succ 1 \succ 2 \succ 3 \succ \text{others}] \). If \( d = 1 \), then we switch the roles of \( H_S, \) and \( H_T, \) and switch the roles of \( P_S \) and \( P_T \) in the proof of CM under integer STV.

- **Ranked pairs, Schulze, and maximin.** If \( d \neq 1 \), then \( P_S, H_S, \) and \( H_T \) are similar to their counterparts for CM. Take \( d = 2 \) for example, for e-CCAV, the added votes are \([2 \succ 3 \succ 1 \succ \text{others}] \); for e-CCDV, the deleted votes are \([1 \succ 3 \succ 2 \succ \text{others}] \). When \( n \) is even, the weights in WMG(\( P_S \)) are all even, for example the positive weights can be one plus the weights in Figure 4 (a). If \( d = 1 \), then we switch the roles of \( H_S, \) and \( H_T, \) and switch the roles of \( P_S \) and \( P_T \) in the proof of their counterparts for CM.

- **Copeland.** The proof for Copeland (\( \alpha \neq 0 \)) is slightly more complicated than the proof for other rules, as \( H_S, H_T, P_S, \) and \( P_T \) depend on the parity of \( n \). We prove e-CCAV with \( d = 2 \) (changed from a profile \( H_S \) where 1 is the winner) for odd and even \( n \) respectively, then comment on how to modify it for other cases.

**When \( n \) is odd**, let \( H_S \) and \( H_T \) be the same as those for CM, even \( n, \alpha > 0 \) (Figure 6 (a) and (b)). Let \( P_S \) be any \( n \)-profile whose WMG is as shown in Figure 8 (a), and let \( P_T \) be the \((n + 1)\)-profile obtained from \( P_S \) by adding one vote of \([2 \succ 1 \succ 3 \succ \cdots \succ m] \). The WMG of \( P_T \) is shown in Figure 8 (b).

![Figure 8: e-CCAV under Cd_α, odd n.](image)

Let \( S = \langle H_S, H_T, S_+, \bar{1} \rangle \). It follows that for any \( \vec{y} \in H_S \) and any \( \vec{x} \in H_T, Cd_\alpha(\vec{y}) = \{1\} \) (1, 2, and 3 have the same highest Copeland score, so 1 wins due to tie-breaking) and \( Cd_\alpha(\vec{x}) = \{2\} \) (2 has the strictly highest Copeland score \( m - 2 + \alpha \)). It is not hard to verify that \( d_0 = m! - 1 \) (the only implicit equality is the tie between 2 and 3). Moreover, let \( \vec{y} = Hist(H_S) \) and let
\( \vec{x} = \text{Hist}(\mathcal{H}_S \cup 2 \times [2 \succ 1 \succ 3 \succ \cdots \succ m]), \) whose WMG is illustrated in Figure 8 (c). Then we have that \( \vec{y} \) and \( \vec{x} \) are the interior points of \( \mathcal{H}_{S,\leq 0} \) and \( \mathcal{H}_{T,\leq 0} \), respectively, and it follows from Claim 15 that \( d_\infty = m! \). The case of \( d = 2 \), odd \( n \) follows after Theorem 1.

**When \( n \) is even**, let \( \mathcal{H}_S \) and \( \mathcal{H}_T \) be the same as those for CM, even \( n, \alpha = 0 \) (Figure 7 (a) and (b)). Let \( P_S \) be any \( n \)-profile whose WMG is as shown in Figure 9 (a) (which is the same as 7 (c)), and let \( P_T \) be the \((n + 1)\)-profile obtained from \( P_S \) by adding one vote of \([2 \succ 1 \succ 3 \succ \cdots \succ m]\). The WMG of \( P_T \) is shown in Figure 9 (b) (which is the same as Figure 9 (c)).

\[
\begin{array}{ccc}
1 & 4 & 2 \\
12 & 16 & 6 \\
4 & 20 & 3
\end{array}
\quad
\begin{array}{ccc}
1 & 5 & 2 \\
13 & 17 & 7 \\
4 & 21 & 3
\end{array}
\quad
\begin{array}{ccc}
1 & 3 & 2 \\
11 & 15 & 5 \\
4 & 19 & 3
\end{array}
\]

(a) WMG(\( P_S \))  
(b) WMG(\( P_T \))  
(c) WMG(\( \vec{y} \))

**Figure 9**: e-CCAV under Cd_{\alpha}, even \( n \).

Let \( S = \langle \mathcal{H}_S, \mathcal{H}_T, S_+, \tilde{1} \rangle \). It follows that for any \( \vec{y} \in \mathcal{H}_S \) and any \( \vec{x} \in \mathcal{H}_T \), \( \text{Cd}_{\alpha}(\vec{y}) = \{1\} \) \( (1, 2, \) and \( 3 \) have the same highest Copeland score, so 1 wins due to tie-breaking) and \( \text{Cd}_{\alpha}(\vec{x}) = \{2\} \) \( (2 \) has the strictly highest Copeland score \( m - 2 + \alpha \)). It is not hard to verify that \( d_0 = m! - 1 \) \( (the only implicit equality is the tie between \( 2 \) and \( 3 \)). Moreover, let \( \vec{y} \) denote the histogram of \( P_S \) subtracting one vote of \([2 \succ 1 \succ 3 \succ \cdots \succ m]\) \( (whose WMG \ is \ illustrated \ in \ Figure \ 9 \ (c)), \) \( which \ is \ the \ same \ as \ Figure \ 8 \ (a)), \) and let \( \vec{x} = \text{Hist}(\mathcal{H}_T) \). Then we have that \( \vec{y} \) and \( \vec{x} \) are the interior points of \( \mathcal{H}_{S,\leq 0} \) and \( \mathcal{H}_{T,\leq 0} \), respectively, and it follows from Claim 15 that \( d_\infty = m! \). The case of \( d = 2 \), odd \( n \) follows after Theorem 1.

The proof for e-CCAV and any \( d \geq 3 \) is similar, which is done by simply switching the role 2 and \( d \) in the proof for \( d = 2 \). The proof for e-CCAV and \( d = 1 \) is done by (1) switching the role of \( \mathcal{H}_S \) and \( \mathcal{H}_T \), (2) switching the role of \( P_S \) and \( P_T \), and (3) the added vote is the inverse of \([2 \succ 1 \succ 3 \succ \cdots \succ m]\). The proof for e-CCDV is similar, by noticing that adding \([2 \succ 1 \succ 3 \succ \cdots \succ m]\) is equivalent to subtracting its inverse.

The proofs for e-DCAV and e-DCDV are similar to the proofs for e-CCAV and e-CCDV, as the proof essentially works for control of changing any source winner to any target winner. \qed

**E.2 Full Version of Theorem 3 and Its Proof**

**Theorem 3** (Upper bound on Coalitional Influence Problems under GSRs). Let \( r \) denote any GSR with fixed \( m \geq 3 \). For any closed and strictly positive \( \Pi \), any \( X \in \{\text{CM, MoV}\} \cup \text{CONTROL} \cup \text{e-CONTROL} \), any \( n \), and any \( B \geq 0 \),

\[
\bar{X}_{\Pi,0}^{\max}(r, n, B) = O \left( \min \left\{ \frac{B + 1}{\sqrt{n}}, 1 \right\} \right)
\]
Proof. We prove the theorem as a corollary of Theorem 6 in Appendix F. Let $X$ be any coalitional influence problem described in the lemma and let its PMV multi-instability representation be $\mathcal{M} = \left\{ S^i = \langle \mathcal{H}^i_S, \mathcal{H}^i_T, \emptyset, \overline{c}^i \rangle : i \leq I \right\}$ due to Lemma 1. For every $i \leq I$, recall that there exist a pair of feasible signatures $\overline{t}_1$ and $\overline{t}_2$ so that $\pi(\overline{t}_1) \neq \pi(\overline{t}_2)$ and $\mathcal{H}^i_S = \mathcal{H}^i_{\overline{t}_1}$ and $\mathcal{H}^i_T = \mathcal{H}^i_{\overline{t}_2}$. Therefore, at least one component of $\overline{t}_1 \oplus \overline{t}_2$ is zero. This means that $d^i_0 = \dim(\mathcal{H}_{\overline{t}_1 \oplus \overline{t}_2}) \leq m! - 1$. Also notice that $d^i_\infty \leq m!$, which means that $d_\Delta \leq m! - d^i_0$. Therefore, according to Theorem 1, we have

$$\sup_{\pi \in \Pi^n} \Pr_{\pi \sim \pi}(\text{Hist}(P) \in U^i_{n,B}) = O\left(\min\left\{B + 1, \sqrt{n}\right\}^{d^i_\Delta} \cdot \left(\frac{1}{\sqrt{n}}\right)^{m! - d^i_0}\right)$$

This proves Theorem 3 because $I$ is finite. $\square$

E.3 Proof of Theorem 4

**Theorem 4 (Max-Semi-Random Coalitional Manipulation for The Loser).** Let $r_\overline{a}$ be an integer positional scoring rule with lexicographic tie-breaking for fixed $m \geq 3$ that is different from veto. For any closed and strictly positive $\Pi$ with $\pi_{uni} \in \text{CH}(\Pi)$, there exists $N > 0$ and $B^* > 0$ such that for any $n > N$ and any $B \geq B^*$,

$$\sup_{\pi \in \Pi^n} \Pr_{\pi \sim \pi}(\text{Hist}(P) \in U^i_{n,B}) = \Theta\left(\min\left\{B \left(\frac{1}{\sqrt{n}}\right)^{m! - d^i_0}\right\}\right)$$

Moreover, for any $\psi > 0$, $\sup_{\pi \in \Pi^n} \Pr_{\pi \sim \pi}(\text{Hist}(P) \in U^i_{n,B}) = \Theta(1)$

**Proof.** The proof proceeds in the following two steps.

**Define the PMV-multi-instability setting $\mathcal{M}_{\text{CML}}$.** For every pair of different alternatives $a, b$, we define a PMV-instability setting $\mathcal{S}^{a \rightarrow b}_{\text{CML}} = (\mathcal{H}_S, \mathcal{H}_T, \emptyset \rightarrow a \rightarrow b, \overline{1})$, where

- $\mathcal{H}_S$ denote the set of vectors where $a$ is the winner and $b$ is the loser under $r_\overline{a}$.
- $\mathcal{H}_T$ denote the set of vectors where $b$ is the winner under $r_\overline{a}$.

Take $a = 1$ and $b = 2$ for example. Recall that $\text{Score}^{\overline{a}}_{a,b}$ is the score vector of $b$ (under all linear orders) minus the score vector of $a$ (Definition 18). Then, we have:

$$\begin{align*}
\mathcal{H}_S &\triangleq \left\{ \overline{x} : \forall i \geq 2, \text{Score}^{\overline{1}}_{i,1} \cdot \overline{x} \leq 0 \right\}, \\
\mathcal{H}_T &\triangleq \left\{ \overline{x} : \forall i \geq 3, \text{Score}^{\overline{2}}_{i,2} \cdot \overline{x} \leq -1, \overline{x} \cdot \overline{x} \leq -1 \right\},
\end{align*}$$

and

$$\mathcal{M}_{\text{CML}} = \left\{ \mathcal{S}^{a \rightarrow b}_{\text{CML}} : a, b \in \mathcal{A}, a \neq b \right\}$$

**Apply Theorem 1.** In this step, we prove that for every $\mathcal{S}^{a \rightarrow b}_{\text{CML}}$ (with corresponding $U^i_{n,B}$),

$$\sup_{\pi \in \Pi^n} \Pr_{\pi \sim \pi}(\overline{X} \in U^i_{n,B}) = \Theta\left(\min\left\{B \left(\frac{1}{\sqrt{n}}\right)^{m! - d^i_0}\right\}\right) \quad (31)$$

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We first prove $K_1 = \text{false}$ (i.e., $U_{n,B}^{a-b} \neq \emptyset$) for any sufficiently large $n$ and $B$, by constructing a successful manipulation by $B$ voters. Recall from the proof of Theorem 2 that for any $a \in A$, $\sigma_a$ denotes a cyclic permutation among $A \setminus \{a\}$. Then, we define two profiles, each of which consists of $m - 1$ votes as follows.

$$P_{a}^{\top} \triangleq \{\sigma_a^i(a \succ \text{others}) : 1 \leq i \leq m - 1\} \quad \text{and} \quad P_{a}^{\bot} \triangleq \{\sigma_a^i(\text{others} \succ a) : 1 \leq i \leq m - 1\}$$

We further define the following “cyclic” profile of $m$ votes: let $\sigma$ denote any cyclic permutation among $A$, e.g., $1 \rightarrow 2 \rightarrow \cdots \rightarrow m \rightarrow 1$.

$$P_{\text{cyc}} \triangleq \{\sigma^i(1 \succ \cdots \succ m) : 1 \leq i \leq m\}$$

Let $P_S$ denote the $n$-profile that consists of $P_{a}^{\top}$, $P_{a}^{\bot}$, as many copies of $P_{\text{cyc}}$ as possible, and the remaining rankings are $[a \succ \text{others} \succ b]$. Let $P_T$ denote the profile obtained from $P_S$ by replacing $\left\lceil \frac{(m-1)(s_1-s_m)}{(m-2)(s_1-s_m-1)} \right\rceil$ + 1 copies of $[\text{others} \succ b \succ a]$ to $[b \succ \text{others} \succ a]$. $P_T$ is well-defined for any sufficiently large $n$. It is not hard to verify that $a$ has the strictly highest score in $P_S$, $b$ has the strictly lowest score in $P_S$, and $b$ has the strictly highest score in $P_T$. This means that for any sufficient large $B$, $U_{n,B}^{a-b} \neq \emptyset$.

Next, let $C_0^{a-b} = H_{S,\leq 0} \cap H_{T,\leq 0}$. We have $\pi_{\text{uni}} \in CH(\Pi) \cap C_0$, which means that $K_3 = \text{false}$. Therefore, the polynomial case of Theorem 1 holds for $S_{\text{CML}}^{a-b}$. Notice that

$$C_0^{a-b} = H_{S,\leq 0} \cap H_{T,\leq 0} = \left\{ \begin{array}{l}
\forall i \not= b, \ \text{Score}_{i \cdot \cdot \cdot}^b \cdot \vec{x} \leq 0 \\
\vec{x} : \forall i \not= b, \ \text{Score}_{i \cdot \cdot \cdot}^b \cdot \vec{x} \leq 0 \\
-\vec{x} \leq 0 \end{array} \right\}$$

Therefore, we have $d_0 = \dim(C_0^{a-b}) = m! - (m - 1)$ (because the implicit equalities represent the scores of all alternatives are the same, which are characterized by $m - 1$ equations). Notice that $\text{Hist}(P_S)$ is an interior point of $H_{S,\leq 0}$, $\text{Hist}(P_T)$ is an interior point of $H_{T,\leq 0}$, and $\dim(H_{S,\leq 0}) = \dim(H_{T,\leq 0}) = m!$. Therefore, by Claim 15, we have $d_\infty = \dim(H_{S,\leq 0}) = m!$, which means that $d_\Delta = m - 1$. It follows from Theorem 1 that

$$\sup_{\vec{x} \in \Pi^n} \Pr \left( X_{\vec{x}} \in U_{n,B} \right) = \Theta \left( \min \left\{ B + 1, \frac{\sqrt{n}}{m! - (m - 1)} \right\}^{m-1} \right) = \Theta \left( \min \left\{ \frac{B}{\sqrt{n}}, 1 \right\}^{m-1} \right),$$

which proves Equation (31). Then, Theorem 4 follows after applying Equation (31) to all $a \not= b$. \qed

F A General Theorem on Semi-Random Coalitional Influence

F.1 Constructive/Destructive Generalized Bribery with Anonymous Prices

In this section, we define two large classes of bribery problems that include some commonly studied control problems as special cases.

**Definition 21.** A constructive generalized bribery with anonymous prices problem is denoted by $\text{CB}_{d,c}(r,P,B)$, where $r$ is a voting rule, $P$ is a profile, $a$ is a distinguished alternative, $\vec{c} > \vec{0}$ is a strictly positive cost vector, where each component is indexed by a pair $(R,R') \in (\mathcal{L}(A) \cup \{\emptyset\}) \times (\mathcal{L}(A) \cup \{\emptyset\})$ that represents the price for the briber to convert an $R$ vote to an $R'$ vote, and $B \geq 0$ is the total budget. We are asked whether the briber can make a win by changing the votes in
the profile under the budget constraint \( B \)—if so then we let \( \text{CB}_{d,\varepsilon}(r, P, B) = 1 \), otherwise we let \( \text{CB}_{d,\varepsilon}(r, P, B) = 0 \).

Destructive bribery with anonymous price problem, denoted by \( \text{DB}_{d,\varepsilon}(r, P, B) \), is defined similarly, and the only difference is that the goal of the briber is to make a not the winner.

Specifically, when \( R = \emptyset \), performing an \((R, R')\) bribery is effectively the same as adding an \( R' \) vote to \( P \); and if \( R' = \emptyset \), performing an \((R, R')\) bribery is effectively the same as removing an \( R \) vote from \( P \). Moreover, we allow the price of an \((R, R')\) operation to be \( \infty \), which means that this operation is not allowed in the problem.

For each constructive/destructive control/bribery problem, we also study its “effective” variant, which requires that the influencers’ goal is not achieved in the original profile. We will add “e-” to the name to denote this variant. For example, e-\( \text{CB}_{d,\varepsilon}(r, P, B) = 1 \) if \( r(P) \neq \{a\} \) and \( a \) can be made the winner under budget \( B \).

**Proposition 2.** CCAV\(_d\) and CCDV\(_d\) are special cases of CB\(_{d,\varepsilon}\). e-CCAV\(_d\) and e-CCDV\(_d\) are special cases of e-CB\(_{d,\varepsilon}\). DCAV\(_d\) and DCDV\(_d\) are special cases of DB\(_{d,\varepsilon}\). e-DCAV\(_d\) and e-DCDV\(_d\) are special cases of e-DB\(_{d,\varepsilon}\).

**Proof.** It is not hard to verify that CCAV\(_d\) (respectively, DCAV\(_d\)) is equivalent to CB\(_{d,\varepsilon}\) (respectively, DB\(_{d,\varepsilon}\)), where for any \((R, R') \in (\mathcal{L}(A) \cup \{\emptyset\}) \times (\mathcal{L}(A) \cup \{\emptyset\})\), the \((R, R')\) component of \( \varepsilon \), denoted by \([\varepsilon]_{(R, R')}\), is \( \begin{cases} 1 & \text{if } R = \emptyset \\ 0 & \text{otherwise} \end{cases} \).

CCDV\(_d\) (respectively, DCDV\(_d\)) is equivalent to CB\(_{d,\varepsilon}\) (respectively, DB\(_{d,\varepsilon}\)), where for any \((R, R') \in (\mathcal{L}(A) \cup \{\emptyset\}) \times (\mathcal{L}(A) \cup \{\emptyset\})\),

\[
[\varepsilon]_{(R, R')} = \begin{cases} 1 & \text{if } R' = \emptyset \\ 0 & \text{otherwise} \end{cases}
\]

The proofs for e-variants are similar. \(\square\)

### F.2 Theorem 6: Semi-Random Coalitional Influence for GSRs

**Theorem 6.** Let \( r \) denote any int-GSR with fixed \( m \geq 3 \). For any closed and strictly positive \( \Pi \) and any \( X \in \{\text{CM}, \text{MoV}\} \), there exists a constant \( C_1 > 0 \), such that for any \( n \in \mathbb{N} \) and any \( B \geq 0 \) with \( B \leq C_1 n \), there exist \( \{d^\text{max}_0, d^\text{max}_\Delta, d^\text{min}_0, d^\text{min}_\Delta\} \subseteq [m!] \) such that

\[
\tilde{X}_{\Pi, 0}^\text{max}(r, n, B) = 0, \exp(-\Theta(n)), \text{ or } \Theta\left(\frac{\min\{B + 1, \sqrt{n}\}^d_{\text{max}}}{(\sqrt{n})^{m! - d^\text{max}_0}}\right), \text{ and }
\]

\[
\tilde{X}_{\Pi, 0}^\text{min}(r, n, B) = 0, \exp(-\Theta(n)), \text{ or } \Theta\left(\frac{\min\{B + 1, \sqrt{n}\}^d_{\text{min}}}{(\sqrt{n})^{m! - d^\text{min}_0}}\right)
\]

As explained in the Introduction, the main merit of Theorem 6 is conceptual, as it illustrates a separation between 0, exponential, and polynomial cases of different degrees. The most interesting part is the asymptotically tight polynomial lower bounds, because little was known about them even under CM, IC, and \( B = 1 \), as discussed in Section 6. The theorem works for many other commonly-studied coalitional influence problem such as those defined in Appendix B.2.

**Constants in the theorem.** \( C_1 \) and the constants in the asymptotic notation \( \Theta(\cdot) \) depend on \( r \), \( X \), and \( \Pi \). In other words, they do not depend on \( n \) or \( B \), which means that there exists
\( \beta > 1 \) that does not depend on \( n \) or \( B \), such that for every \( n \) and every \( B \geq 0 \),

\[
X_{1,0}^{\max}(r, n, B) \in \{0\} \cup \exp(-\beta \cdot n) \exp\left( -\frac{1}{\beta} \cdot n \right) \cup \left( \frac{1}{\beta}, \beta \right] \cdot \min\{B + 1, \sqrt{n}\} \frac{d_{\Delta}^{\max}}{(\sqrt{n})^{m!-d_{\max}}} 
\]

A similar argument holds for \( X_{1,0}^{\min}(r, n, B) \). We note that \( \{d_0^{\max}, d_{\Delta}^{\max}, d_0^{\min}, d_{\Delta}^{\min}\} \) may depend on \( n \) and \( B \), yet they are integers that are no more than \( m! \) as stated in the theorem.

### F.3 Proof of Theorem 6

#### Proof overview.
Recall that the full version of Lemma 1 in Appendix C.2 states that the coalition influence problems can be represented as PMV-multi-instability problems (Definition 10). Next, we extend Theorem 1 to solve the PMV-multi-instability problem in Theorem 7, and then apply it to the PMV-multi-instability problems defined in the proof of Lemma 1 to prove Theorem 6.

For every \( i \leq I \), we use superscript \( i \) to denote the notation defined for \( S^i \). For example, \( d_0^i \), and \( d_{\Delta}^i \) denote \( d_0 \) and \( d_{\Delta} \) for \( S^i \). To present the result, it is convenient to define the following graph.

**Definition 22 (Activation graph for PMV-multi-instability).** Given a PMV multi-instability setting \( M = \{S^i : i \leq I\} \), \( n \), and \( B \), we define a weighted undirected bipartite graph, called activation graph and is denoted by \( A_{n,B} \), as follows.

- **Vertices.** There are two sides of vertices: \( CH(\Pi) \) and \( \{S^1, \ldots, S^I\} \).

- **Edges and weights.** For any \( \pi \in CH(\Pi) \) and any \( S^i \in M \), the weight on the edge between \( \pi \) and \( S^i \), denoted by \( w_{n,B}(\pi, S^i) \), is defined as follows: for every PMV-instability setting \( S \), define

\[
w_{n,B}(\pi, S) \triangleq \begin{cases} 
-\infty & \text{if } U_{n,B} = \emptyset \\
-\frac{2n}{\log n} & \text{if } U_{n,B} \neq \emptyset \text{ and } \pi \notin C_0 \\
d_0 + d_{\Delta} \cdot \min \left\{ \frac{2 \log (B+1)}{\log n}, 1 \right\} & \text{otherwise}
\end{cases}
\]  

(32)

Notice that while the conditions in (32) depend on \( \pi \), the values of \( w_{n,B}(\pi, S) \) do not depend on \( \pi \), and they are chosen so that \((\sqrt{n})^{w_{n,B}}\) corresponds to the values in the exponential cases and polynomial cases of Theorem 1. Specifically, the \(-\frac{2n}{\log n}\) value in the second case is chosen so that \((\sqrt{n})^{\frac{2n}{\log n}} = \exp(-n)\), which corresponds to the exponential cases.

Given the PMV multi-instability setting \( M = \{S^i : i \leq I\} \), \( B \), and \( n \), let \( w_{\max} \) denote the maximum weight on edges in \( A_{n,B} \) and let \((\pi_{\max}, i_{\max})\) denote an arbitrary edge with the max weight. That is,

\[
w_{\max} \triangleq \max_{\pi \in CH(\Pi), i \leq I} \{ w_{n,B}(\pi, S^i) \} \quad \text{and} \quad (\pi_{\max}, i_{\max}) \triangleq \arg \max_{\pi \in CH(\Pi), i \leq I} w_{n,B}(\pi, S^i)
\]

Let \( w_{\min} \) denote the weight of the minimax weighted edge in \( A_{n,B} \), denoted by \((\pi_{\min}, i_{\min})\), where \( \min \) is taken over all \( \pi \in CH(\Pi) \) and max is taken over all edges connected to \( \pi \). That is,

\[
w_{\min} \triangleq \min_{\pi \in CH(\Pi)} \max_{i \leq I} \{ w_{n,B}(\pi, S^i) \} \quad \text{and} \quad (\pi_{\min}, i_{\min}) \triangleq \arg \min_{\pi \in CH(\Pi)} \max_{i \leq I} w_{n,B}(\pi, S^i)
\]

Notice that \( i_{\max} \) and \( i_{\min} \) are both in \([q]\) and \( w_{\max}, w_{\min}, i_{\max}, \) and \( i_{\min} \) depend on \( B \) and \( n \), which are clear from the context.
Theorem 7 (Semi-Random PMV-multi-instability, $B = O(n)$). Given any $q \in \mathbb{N}$, any closed and strictly positive $\Pi$ over $[q]$, and any PMV multi-instability setting $\mathcal{M} = \{S^i : i \leq I\}$, there exists a constant $C_1 > 0$ so that for any $n \in \mathbb{N}$ and any $B \geq 0$ with $B \leq C_1 n$,

$$
\sup_{\mathcal{M}} \Pr \left( \bar{X}_{\pi} \in \mathcal{U}_{n,B}^\mathcal{M} \right) = \begin{cases} 
0 & \text{if } w_{\text{max}} = -\infty \\
\exp(-\Theta(n)) & \text{if } w_{\text{max}} = -\frac{2n}{\log n} \\
\Theta \left( \frac{1}{\sqrt{n}} \right)^{q \cdot w_{\text{max}}} & \text{otherwise}
\end{cases}
$$

$$
\inf_{\mathcal{M}} \Pr \left( \bar{X}_{\pi} \in \mathcal{U}_{n,B}^\mathcal{M} \right) = \begin{cases} 
0 & \text{if } w_{\text{min}} = -\infty \\
\exp(-\Theta(n)) & \text{if } w_{\text{min}} = -\frac{2n}{\log n} \\
\Theta \left( \frac{1}{\sqrt{n}} \right)^{q \cdot w_{\text{min}}} & \text{otherwise}
\end{cases}
$$

Proof. The 0 cases of sup and inf are straightforward. Like the proof of Theorem 1, it suffices to prove the other cases of sup and inf for every sufficiently large $n$.

Proof for the sup part. We first prove a convenient corollary of Theorem 1, which uses the weight in the activation graph to represent conditions of the sup part of Theorem 1.

Corollary 2 (Activation graph representation of the sup part of Theorem 1). Given any $q \in \mathbb{N}$, any closed and strictly positive $\Pi$ over $[q]$, and any PMV-instability setting $\mathcal{S} = \langle \mathcal{H}, \mathcal{H}_T, \mathcal{O}, \mathcal{C} \rangle$, any $C_2$ with $C_2 < B_{\mathcal{CH}(\Pi)}$, any $n \in \mathbb{N}$, and any $0 \leq B \leq C_2 n$, let $w^* = \sup_{\pi \in \mathcal{CH}(\Pi)} w_{n,B}(\pi, \mathcal{S})$,

$$
\sup_{\pi \in \Pi^n} \Pr \left( \bar{X}_{\pi} \in \mathcal{U}_{n,B} \right) = \begin{cases} 
0 & \text{if } w^* = -\infty \\
\exp(-\Theta(n)) & -\infty < w^* < 0 \\
\Theta \left( \frac{1}{\sqrt{n}} \right)^{q \cdot w^*} & \text{otherwise}
\end{cases}
$$

Proof. The 0 case is straightforward. Like in the proof of Theorem 1, for the non-zero cases it is without loss of generality to assume that $n$ is larger than a constant.

Exponential case. In this case we have $w^* = -\frac{2n}{\log n}$, which means that $\mathcal{CH}(\Pi) \cap \mathcal{C}_0 = \emptyset$. Therefore, the exponential case of Corollary 2 follows after the exponential case and the $B \leq C_2 n$ case of in Theorem 1.

Polynomial case. Because $\mathcal{CH}(\Pi)$ is bounded and closed, it is compact. Therefore, there exists $\pi^* \in \mathcal{CH}(\Pi)$ such that $w_{n,B}(\pi^*, \mathcal{S}) = w^* = d_0 + d_\Delta \cdot \min \left\{ \frac{2 \log(B+1)}{\log n}, 1 \right\}$. It follows from Theorem 1 that

$$
\sup_{\pi \in \Pi^n} \Pr \left( \bar{X}_{\pi} \in \mathcal{U}_{n,B} \right) = \Theta \left( \frac{\min \{B + 1, \sqrt{n} \}^{d_\Delta}}{(\sqrt{n})^{q \cdot d_0}} \right)
$$

Notice that

$$
\log \left( \frac{\min \{B + 1, \sqrt{n} \}^{d_\Delta}}{(\sqrt{n})^{q \cdot d_0}} \right) = d_\Delta \min \left\{ \log(B + 1), \frac{\log n}{2} \right\} + (d_0 - q) \frac{\log n}{2}
$$

$$
= \frac{\log n}{2} \cdot \left( d_0 + d_\Delta \cdot \min \left\{ \frac{2 \log(B + 1)}{\log n}, 1 \right\} - q \right) = \log \left( \left( \frac{1}{\sqrt{n}} \right)^{q \cdot w^*} \right)
$$

This completes the proof of Corollary 2. \qed
Define $C_1$ for sup. Let $C_1 > 0$ denote any positive number that is smaller than any strictly positive $B^i_{\text{CH(Π)}}$. That is,

$$0 < C_1 < \min\{B^i_{\text{CH(Π)}} : B^i_{\text{CH(Π)}} > 0, i \leq I\}$$

The rest of the proof for the sup part of Theorem 7 is done by combining the results of the applications of Corollary 2 to all PMV-instability settings and the following inequality.

$$\max_{i \in I} \sup_{\pi \in \Pi^n} \Pr\left(\bar{X}_\pi \in U_{n,B}^i\right) \leq \sup_{\pi \in \Pi^n} \Pr\left(\bar{X}_\pi \in U_{n,B}^M\right) \leq I \cdot \max_{i \in I} \sup_{\pi \in \Pi^n} \Pr\left(\bar{X}_\pi \in U_{n,B}^i\right) \quad (33)$$

Exponential case of sup. In this case $w_{\text{max}} = -\frac{2n}{\log n}$. Notice that for all $\pi \in \text{CH(Π)}$ and all $i \leq I$, we have $w_{n,B}(\pi, S^i) \leq -\frac{2n}{\log n}$, and there exists $\pi^* \in \text{CH(Π)}$ and $i^* \leq I$ such that $w_{n,B}(\pi^*, S^{i^*}) = -\frac{2n}{\log n}$. Therefore, by Corollary 2, $\sup_{\pi \in \Pi^n} \Pr\left(\bar{X}_\pi \in U_{n,B}^i\right) = -\frac{2n}{\log n}$, which means that

$$\max_{i \in I} \sup_{\pi \in \Pi^n} \Pr\left(\bar{X}_\pi \in U_{n,B}^i\right) = \exp(-\Theta(n))$$

The exponential case follows after (33).

Polynomial case of sup. The proof for the polynomial case is similar. To prove the polynomial upper bound, notice that for all $\pi \in \text{CH(Π)}$ and all $i \leq I$, we have $w_{n,B}(\pi, S^i) \leq w_{\text{max}}$. By Corollary 2, when $n$ is sufficiently large, for all $i \leq I$, we have

$$\max_{i \in I} \sup_{\pi \in \Pi^n} \Pr\left(\bar{X}_\pi \in U_{n,B}^i\right) = \max_{i \in I} \Theta\left(\left(\frac{1}{\sqrt{n}}\right)^{q-w^{i^*}}\right) = O\left(\left(\frac{1}{\sqrt{n}}\right)^{q-w_{\text{max}}}\right),$$

where $w^{i^*}$ is the weight $w^*$ in Corollary 2 applied to $S = S^i$.

To prove the polynomial lower bound, let $\pi^* \in \text{CH(Π)}$ and $i^* \leq I$ be such that $w_{n,B}^{i^*}(\pi^*, S^{i^*}) = w_{\text{max}}$. According to the polynomial case of Corollary 2, we have

$$\max_{i \in I} \sup_{\pi \in \Pi^n} \Pr\left(\bar{X}_\pi \in U_{n,B}^i\right) = \Omega\left(\left(\frac{1}{\sqrt{n}}\right)^{q-w_{\text{max}}}\right)$$

The polynomial lower bound of sup follows after (33). This proves the sup part of Theorem 7.

Proof for the inf part. The hardness in proving the inf part is that any $\bar{\pi} \in \Pi^n$ that achieves inf for one PMV-instability setting $S^i$ may not achieve inf for another PMV-instability setting $S^{i'}$, and therefore may not achieve inf of the PMV-multi-instability problem. This is different from the sup part, where the $\bar{\pi} \in \Pi^n$ with the highest value of sup under some $S^{i'}$ achieves sup of the union-manipulation problem. Consequently, even though the inf counterpart of Corollary 2 can be proved for inf, it is cannot be leveraged to prove the inf part of Theorem 7.

To prove the inf part, we first define some distributions that will be used to prove the upper bounds for inf.

Definition 23 ($\pi_\mathcal{I}$’s). Given a PMV-multi-instability setting $\mathcal{M}$, for every non-empty set $\mathcal{I} \subseteq [I]$ such that $\text{CH(Π)} \notin \bigcup_{i \in \mathcal{I}} C^i_0$, we choose $\pi_\mathcal{I} \in \text{CH(Π)} \setminus \left(\bigcup_{i \in \mathcal{I}} C^i_0\right)$.

Because every $C^i_0$ is a closed set, the distance between $\pi_\mathcal{I}$ and $\bigcup_{i \in \mathcal{I}} C^i_0$ is strictly positive and is denoted by $\delta_\mathcal{I} > 0$. Due to Claim 5, for every $i \in \mathcal{I}$ there exists a constant $c_i$ such that each vector
in $H_B$ is no more than $c_i(B + 1)$ away from a vector in $C_0$. It follows that for any $n > \frac{4\delta_i}{\delta_2}$ and any $B \leq \frac{\delta_i}{4\delta_1}n$, the distance between $\pi_I$ and $H_B$ is at least $\frac{\delta_2}{2}n$.

**Define $C_1$ for inf.** Let $C_1$ denote the minimum $\frac{\delta_2}{\delta_1}$ for all well-defined $\delta_2$ and all $i \in I$. Let $\delta$ denote the minimum $\frac{\delta_2}{\delta_1}$ for all well-defined $\delta_2$. It follows that for any sufficiently large $n$ (that is larger than all $\frac{4\delta_i}{\delta_2}$) and any $B \leq C_1n$, the distance between any well-defined $\pi_I$ and any $i \in I$ is at least $\delta n$.

**Exponential case of inf.** In this case $w_{\text{min}} = -\frac{2n}{\log n}$. The exponential lower bound trivially holds, because there exists an active $S^i$. To prove the exponential upper bound, let $\pi_{MM} \in \text{CH}(\Pi)$ be an arbitrary distribution such that for all $i \leq I$, $w_{n,B}(\pi_{MM}, S^i) \leq -\frac{2n}{\log n}$. Let $C_{MM}$ denote the inactive indices to the active PMV-instability settings (whose $c_i$’s do not contain $\pi_{MM}$), that is,

$$C_{MM} \triangleq \left\{ i \leq I : w_{n,B}(\pi_{MM}, S^i) = -\frac{2n}{\log n} \right\}$$

Because $\pi_{MM} \in \text{CH}(\Pi) \setminus (\bigcup_{i \in I} C_0^i)$, we have that $\pi_CMM$ is well-defined. Therefore, for every $i \in C_{MM}$, the distance between $\pi_{CMM}$ and $C_0^i$ is at least $\delta n$. The exponential upper bound follows after applying Claim 11, Hoeffding’s inequality, and the union bound to any $n, B > \delta n$. As done in the B $\leq C_2n$ case of the proof for PT-$\Theta(n)$-sup of Theorem 1.

**Polynomial case of inf.** To prove the polynomial lower bound, notice that for every $\bar{\pi} \in \Pi^n$, there exists $i \leq I$ such that $w_{n,B}(\text{avg}(\bar{\pi}), S^i) = d_{i,B} \geq w_{\text{min}}$. It follows from Claim 10 (applied to $S^i$, and $\bar{\pi}$) that, for every $B \leq \sqrt{n}$,

$$\Pr(\bar{\pi} \notin U_{n,B}^i) \geq \Theta \left( \left( \frac{1}{\sqrt{n}} \right)^{q - d_{i,B}} \right) = \Omega \left( \left( \frac{1}{\sqrt{n}} \right)^{q - w_{\text{min}}} \right)$$

Notice that for every $B > \sqrt{n}$, the inequality still holds, because we have $\Pr(\bar{\pi} \notin U_{n,B}^i) \geq \Pr(\bar{\pi} \notin U_{n,\sqrt{n}}^i)$.

To prove the polynomial upper bound, let

$$\pi_{MM} \triangleq \arg \min_{\pi \in \text{CH}(\Pi)} \max_{i \leq I} \left\{ w_{n,B}(\pi, S^i) \right\}$$

Like in the proof of the exponential upper bound of inf above, define

$$C_{MM} \triangleq \left\{ i \leq I : w_{n,B}(\pi_{MM}, S^i) < 0 \right\}$$

Because $\pi_{MM} \in \text{CH}(\Pi) \setminus (\bigcup_{i \in I} C_0^i)$, we have that $\pi_{CMM}$ is well-defined. Therefore, for every $i \in C_{MM}$, the distance between $\pi_{CMM}$ and $C_0^i$ is at least $\delta n$. Choose any $\bar{\pi} \in \Pi^n$ such that $|\sum_{j=1}^n \pi_j - n \cdot \pi_{CMM}| = O(1)$. Like the exponential case above, for every $i \in C_{MM}$, we have $\Pr(\bar{\pi} \notin U_{n,B}^i) \leq \exp(-\Omega(n))$. Moreover, for every $i \notin C_{MM}$ such that $S^i$ is active, we have $\pi_{MM} \in C_0^i$, which means that $C_0^i \cap \text{CH}(\Pi) \neq \emptyset$. Recall that $d_{i,B} \leq w_{\text{min}}$. By Corollary 2, we have

$$\Pr(\bar{\pi} \notin U_{n,B}^i) \leq \sum_{i \leq I} \Pr(\bar{\pi} \notin U_{n,B}^i) \leq O \left( \left( \frac{1}{\sqrt{n}} \right)^{q - w_{\text{min}}} \right)$$

Therefore,

$$\Pr(\bar{\pi} \notin U_{n,B}^\mathcal{M}) \leq \sum_{i \leq I} \Pr(\bar{\pi} \notin U_{n,B}^i) = O \left( \left( \frac{1}{\sqrt{n}} \right)^{q - w_{\text{min}}} \right),$$

which proves the polynomial upper bound of inf.
Notice that in Theorem 7, when \( w_{\text{max}} > 0 \) and \( w_{\text{min}} > 0 \), we have

\[
\left( \frac{1}{\sqrt{n}} \right)^{q-w_{\text{max}}} = \min \{ B + 1, \sqrt{n} \} d_{\text{max}}^{\Delta} \quad \text{and} \quad \left( \frac{1}{\sqrt{n}} \right)^{q-w_{\text{min}}} = \min \{ B + 1, \sqrt{n} \} d_{\text{min}}^{\Delta}
\]

\( (34) \)

Recall from Lemma 1 that any coalitional influence problem \( X \in \{ \text{CM}, \text{MoV} \} \) under any GSR \( r \) can be represented by a PMV-multi-instability problem. Therefore, Theorem 6 follows immediately after Theorem 7 and (34). This completes the proof of Theorem 6.

**G  Ties \( \not\Leftrightarrow [\Theta(1) \text{ instability}] \)**

**Example 6 (Ties \( \not\Leftrightarrow \Theta(1) \text{ instability})**. Consider \( C_{d_{\alpha}} \) with four alternatives. Let \( P' \) denote an arbitrary profile whose UMG is the same as Figure 10 (a). For any \( n' \in \mathbb{N} \), we let \( P = n'P' \). It is not hard to verify that \( C_{d_{\alpha}}(P) = \{1, 2\} \). The winner under \( P \) is stable with \( \Theta(n) \) changes in votes, because the UMG of any profile whose histogram is \( \Theta(1) \) away from Hist(\( P \)) is the same as Figure 10 (a).

\begin{figure}[h]
\centering
\begin{subfigure}{0.4\textwidth}
\centering
\begin{tikzpicture}
\node (1) at (0,0) {1};
\node (2) at (2,0) {2};
\node (3) at (0,-2) {3};
\node (4) at (2,-2) {4};
\draw[->] (1) -- (2);
\draw[->] (1) -- (3);
\draw[->] (1) -- (4);
\draw[->] (2) -- (3);
\draw[->] (2) -- (4);
\draw[->] (3) -- (4);
\end{tikzpicture}
\caption{UMG for a tied stable profile.}
\end{subfigure}
\hspace{1cm}
\begin{subfigure}{0.4\textwidth}
\centering
\begin{tikzpicture}
\node (1) at (0,0) {1};
\node (2) at (2,0) {2};
\node (3) at (0,-2) {3};
\node (4) at (2,-2) {4};
\draw[->] (1) -- (2);
\draw[->] (1) -- (3);
\draw[->] (1) -- (4);
\draw[->] (2) -- (3);
\draw[->] (2) -- (4);
\draw[->] (3) -- (4);
\end{tikzpicture}
\caption{UMG for a untied unstable profile.}
\end{subfigure}
\caption{Figure 10: Graphs used in Example 6 and 7.}
\end{figure}

**Example 7 (Ties \( \not\Leftrightarrow \Theta(1) \text{ instability})**. Let \( \tau \) denote a biased Copeland\( _{0} \) rule for four alternatives, which differs from Copeland\( _{0} \) in that if \( 1 \rightarrow 2 \), then alternative 1 gets 2 points (instead of 1). Let \( P' \) denote an arbitrary profile whose UMG is the same as Figure 10 (b). For any \( n' \in \mathbb{N} \), we let \( P = n'P' \). Notice that \( \tau(P) = \{2\} \). To see that \( P \) is \( \Theta(1) \) unstable, let \( R \) denote any vote in \( P \) where \( 2 \succ 1 \). Replace \( R \) by \([1 \succ 2 \succ \text{others}]\), the winner becomes 1.

Let \( P^* \) denote any profile that is \( \Theta(1) \) away from \( P \). It is not hard to see that UMG(\( P^* \)) contains the same edges as the graph in Figure 10 (b) except the edge between 1 and 2. This means that either \( \tau(P^*) = \{2\} \) (if there is no edge between 1 and 2 or there is an edge \( 2 \rightarrow 1 \) in UMG(\( P^* \))), or \( \tau(P^*) = \{1\} \) (if \( 1 \rightarrow 2 \) in UMG(\( P^* \))). This means that \( P \) is not close to any tied profile under \( \tau \).
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