Echoes from Singularity

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Though the cosmic censorship conjecture states that spacetime singularities must be hidden from an asymptotic observer by an event horizon, naked singularities can form as the end product of a gravitational collapse under quite general initial conditions. So the question of how to observationally distinguish such naked singularities from standard black hole spacetimes becomes important. In the present paper, we try to address this question by studying the ringdown profile of the Janis-Newman-Winicour (JNW) naked singularity under axial gravitational perturbation. The JNW spacetime has a surface-like naked singularity that is sourced by a massless scalar field and reduces to the Schwarzschild solution in absence of the scalar field. We show that for low strength of the scalar field, the ringdown profile is dominated by echoes which mellow down as the strength of the field increases to yield characteristic quasinormal frequency of the JNW spacetime.

I. INTRODUCTION

Black holes are undoubtedly one of the most elegant constructs of general relativity that are characterised by a null surface, called the event horizon, which generally conceals a singularity within. Recent observations of gravitational waves by the LIGO-Virgo collaboration[1–7], supplemented by the findings of the Event Horizon Telescope collaboration[8–10] have provided substantial evidence for the existence of these magnificent objects. However, there is still sufficient room, at least theoretically, for more exotic compact objects (ECOs), which can differ from a black hole[11–15]. One such interesting possibility is a naked singularity, where the singularity is not cloaked by an event horizon and is visible to an asymptotic observer.

Though forbidden by the cosmic censorship conjecture[16], naked singularities can form as an end-state of gravitational collapse of massive matter clouds under suitable initial conditions[17–29]. An inevitable question that now arises is how to observationally distinguish such naked singularities from a black hole spacetime.

In the electromagnetic spectrum, there are numerous observations pertaining to gravitational lensing[29–36], accretion disk [29, 37–43], images and shadows[44–46] that highlights the difference in the properties of naked singularities and black hole spacetimes. In the gravitational wave spectrum such observations are also possible, particularly, in the ringdown phase of a compact binary coalescence when the signal is dominated by the quasinormal modes. The quasinormal modes (QNMs) are characterised by damped harmonic oscillations[47, 48] that can also be excited by linear perturbation of the compact object. In general, QNMs of different compact objects will be different and can be used to distinguish ECOs from black holes[11, 14, 49–54]. The horizon-less character of the ECOs may also be manifested by echoes in the ringdown profile [55–63].

In this paper, we aim to investigate the response of the JNW naked singularity-spacetime to axial (odd-parity) gravitational perturbation. We observe that even when the scalar field is weak, the signature of the difference between the spacetime due to a black hole and the naked singularity is quite distinctly elucidated by the existence of echoes for the latter. But as the strength of the scalar field increases, the echoes align and the QNM structure of the JNW ringdown becomes prominent.

The paper is organised as follows. In Sec. II we provide a brief review of the JNW naked singularity. Sec. III discusses the gravitational perturbation of the JNW naked singularity-spacetime and obtains the master equation for

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axial gravitational perturbation. Sec. IV is dedicated to the time domain analysis of the perturbation equation and the evaluation of the associated quasinormal mode frequencies. Finally, in Sec. V we conclude with a brief summary and discussion of the results that we arrived at. Throughout the paper, we employ units in which \( G = c = 1 \).

II. REVIEW OF THE JNW NAKED SINGULARITY

We start with an action in which the Einstein-Hilbert action is minimally coupled to a real massless scalar field \( \Phi \),

\[
S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ \bar{R} - 8\pi \bar{g}^\mu\nu \partial_\mu \Phi \partial_\nu \Phi \right].
\]

The field equations of the above theory,

\[
\bar{R}_{\mu\nu} = 8\pi (\partial_\mu \Phi)(\partial_\nu \Phi),
\]

\[
\Box \Phi = 0,
\]

admits a static, spherically symmetric solution described by the line element,

\[
ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + k(r)\left(d\theta^2 + \sin^2 \theta d\phi^2\right),
\]

where,

\[
f(r) = \left(1 - \frac{b}{r}\right)^\nu \quad \text{and} \quad k(r) = r^2 \left(1 - \frac{b}{r}\right)^{1-\nu}.
\]

This is the well known Janis-Newman-Winicour (JNW) spacetime[64]. The JNW spacetime is sourced by a scalar field,

\[
\Phi = \frac{q}{b\sqrt{4\pi}} \ln \left(1 - \frac{b}{r}\right).
\]

The parameter \( b \) is related to the scalar charge \( q \) and the ADM mass \( M \) as \( b = 2\sqrt{q^2 + M^2} \) and \( \nu = 2M/b = M/\sqrt{q^2 + M^2} \) lies in the range, \( 0 \leq \nu < 1 \). For \( q = 0 \) (\( \nu = 1 \)), the scalar field vanishes and one recovers the standard Schwarzschild metric. As the scalar charge increases, \( \nu \) reduces to 0. Thus, the parameter \( \nu \) measures the deformation from the Schwarzschild spacetime. The JNW spacetime has a curvature singularity at \( r = b \) for \( 0 < \nu < 1 \). The absence of an event horizon makes the singularity globally naked. The spacetime also satisfies the weak energy condition[66, 72]. For \( 1/2 < \nu < 1 \), the JNW singularity lies within a photon sphere[34, 36, 73] of radius

\[
r_{\text{ph}} = \frac{b(1+2\nu)}{2},
\]

and the singularity is classified as weakly naked. However, for \( 0 < \nu \leq 1/2 \) the singularity is no longer covered by a photon sphere and the spacetime is classified as strongly singular.

In the present work we will restrict ourselves only to the weakly naked singularity regime of the JNW spacetime.

III. PERTURBATION OF THE JNW SPACETIME

We introduce small perturbations \( h_{\mu\nu} \) to the background metric \( \bar{g}_{\mu\nu} \) such that the resulting perturbed metric \( g_{\mu\nu} \) becomes

\[
g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad \text{where} \quad |h_{\mu\nu}| / |\bar{g}_{\mu\nu}| \ll 1.
\]

The perturbed metric gives rise to the perturbed Christoffel symbols,

\[
\Gamma^\alpha_{\mu\nu} = \bar{\Gamma}^\alpha_{\mu\nu} + \delta\Gamma^\alpha_{\mu\nu},
\]

where, \( \bar{\Gamma}^\alpha_{\mu\nu} \) are the Christoffel symbols due to the unperturbed metric and

\[
\delta\Gamma^\alpha_{\mu\nu} = \frac{1}{2} \bar{g}^{\alpha\beta} (h_{\mu\beta,\nu} + h_{\nu\beta,\mu} - h_{\mu\nu,\beta}).
\]
This results in the perturbed Ricci tensor,

\[ R_{\mu\nu} = \bar{R}_{\mu\nu} + \delta R_{\mu\nu}, \]

where

\[ \delta R_{\mu\nu} = \nabla_\nu \delta \Gamma^\alpha_{\mu\alpha} - \nabla_\alpha \delta \Gamma^\alpha_{\mu\nu}, \]

and \( \nabla_\mu \) is the covariant derivative with respect to the background metric \( \bar{g}_{\mu\nu} \).

Due to spherical symmetry of the background, we can decompose the perturbations \( (h_{\mu\nu}) \) into odd- \( (h_{\mu\nu}^{\text{odd}}) \) and even-type \( (h_{\mu\nu}^{\text{even}}) \) perturbation, based on their parity under two dimensional rotation[74, 75]. (For excellent reviews we refer to Refs. [48, 76].) In the present work we will concentrate on the odd-parity or axial perturbation in which the perturbation \( \delta \Phi \) of the background scalar field \( \Phi \) does not contribute[77]. Thus, the evolution of the axial perturbation is governed by the field equation,

\[ \delta R_{\mu\nu} = 0. \] (13)

The perturbation variables \( h_{\mu\nu} \) can be expanded in a series of spherical harmonics. The components of the axial perturbation \( (h_{\mu\nu}^{\text{odd}}) \) can be further simplified by utilising the residual gauge freedom to choose a proper gauge. A preferred choice in this case is the “Regge-Wheeler” gauge[74] in which the axial perturbation can be represented in terms of only two unknown functions \( h_0(t, r) \) and \( h_1(t, r) \),

\[ h_{\mu\nu}^{\text{odd}} = \begin{pmatrix} 0 & 0 & 0 & h_0(t, r) \\ 0 & 0 & 0 & h_1(t, r) \\ h_0(t, r) & h_1(t, r) & 0 & 0 \end{pmatrix} \sin \theta \partial_\theta P_\ell(\cos \theta)e^{im\phi}, \] (14)

where \( P_\ell(\cos \theta) \) is the Legendre polynomial of order \( \ell \) and \( m \) is the azimuthal harmonic index.

The \( (t, \phi) \), \( (r, \phi) \) and \( (\theta, \phi) \) components of Eq. (13) can be explicitly written in a simplified form as,

\[ \delta R_{t\phi} = - \frac{1}{2} \left( 1 - \frac{b}{r} \right)^\nu \frac{b^2 \nu (\nu + 1) - 2b (2\nu + \ell^2 + \ell) + r^2 (\ell + 1)}{2r^2(b - r)^2} \]

\[ \frac{1}{2r(b - r)} \left( 1 - \frac{b}{r} \right)^\nu \left( r(b - r) \partial_r^2 h_0(t, r) - (b\nu + b - 2r) \partial_r h_1(t, r) + r(r - b) \partial_r \partial_r h_1(t, r) \right) = 0, \] (15)

\[ \delta R_{r\phi} = - \frac{1}{2} \left( 1 - \frac{b}{r} \right)^{\nu - 1} (r - b)^{-\nu - 1} \left( r(b - r) \left( \partial_r \partial_t h_0(t, r) - \partial_r^2 h_1(t, r) \right) \right) - 

\frac{(b\nu + b - 2r) \partial_r h_0(t, r) - (\ell^2 + \ell - 2) (1 - \frac{b}{r})^\nu h_1(t, r)}{2r(b - r)} = 0, \] (16)

\[ \delta R_{\theta\phi} = \frac{1}{2} \left( 1 - \frac{b}{r} \right)^{\nu - 1} \left( \partial_{t} h_0(t, r) - \left( 1 - \frac{b}{r} \right)^{2\nu} \partial_{r} h_1(t, r) \right) + \frac{b\nu r^{-\nu - 1}(r - b)^{\nu - 1} h_1(t, r)}{2b - 2r} = 0. \] (17)

Using Eq. (17), we eliminate \( h_0(t, r) \) from a combination of Eqs. (15) and (16) and obtain a Schrödinger-like form,

\[ \frac{\partial^2}{\partial \tau^2} \Psi(t, r) - \frac{\partial^2}{\partial r^2} \Psi(t, r) + V_{eff}(r) \Psi(t, r) = 0, \] (18)

where \( \Psi(t, r) = \frac{h_1(t, r)}{r} (1 - b/r)^{\frac{3\nu - 1}{2}} \) and the effective potential is given by

\[ V_{eff}(r) = \frac{1}{4} r^{-2(\nu + 1)} (r - b)^2 (\nu - 1)(3b^2 (\nu + 1)^2 - 4br (3\nu + \ell^2 + \ell) + 4r^2 \ell (\ell + 1)). \] (19)

The coordinate \( r_* \) is known as the tortoise coordinate and is defined by the relation,

\[ \frac{dr_*}{dr} = \left( 1 - \frac{b}{r} \right)^{-\nu}. \] (20)
For $\nu \in (0,1)$, the tortoise coordinate maps the singularity at $r = b$ to $r_\ast = 0$. The effective potential vanishes as $r_\ast \to \infty$, whereas, near the singularity it rises to an infinite wall,

$$V_{\text{eff}}(r \to b) \to \infty \quad \text{for} \quad 0 < \nu < 1.$$  

The effective potential as a function of the tortoise coordinate $r_\ast$ for different value of the parameter $\nu$ is depicted in Fig. 1. For numerical simplicity the origin of the tortoise coordinate in Fig. 1 has been shifted from $r = b$ to $r = b + \epsilon$ ($\epsilon \ll 1$). We have chosen $b = 2$, i.e., $q^2 + M^2 = 1$ for a better control over the numerical work. This implies we actually work on the basis of the relative strength of the scalar charge $q$ to the mass $M$, for $\nu = 1$, $q = 0$ and the system reduces to a Schwarzschild geometry, whereas for $\nu = \frac{1}{2}$, $\frac{q}{M}$ is as high as $\sqrt{3}$. Away from the singularity, for $\nu$ in the range $(1/2, 1)$, the effective potential is characterised by a peak. As $\nu$ decreases from $\nu \approx 1$ to $\nu \approx 1/2$, the maximum of the secondary peak increases and it moves closer to the $r_\ast = 0$ surface. For $0 < \nu \leq 1/2$, i.e., in the strongly naked singularity regime, the secondary peak vanishes and the potential profile is solely characterised by a potential wall, gradually rising to infinity at $r_\ast = 0$.

IV. TIME-DOMAIN PROFILE AND QUASINORMAL MODES

In order to study the time-evolution of the perturbation, we rewrite the wave equation (18) in terms of the light-cone (null) coordinates, $u = t - r_\ast$ and $v = t + r_\ast$, as,

$$4 \frac{\partial^2}{\partial u \partial v} \Psi(u, v) + V_{\text{eff}}(u, v)\Psi(u, v) = 0.$$  

The appropriate discretisation scheme to integrate Eq. (22) as proposed in Ref. [78] is,

$$\Psi(N) = \Psi(W) + \Psi(E) - \Psi(S) - \Delta^2 \frac{V_{\text{eff}}(W)\Psi(W) + V_{\text{eff}}(E)\Psi(E)}{8} + O(\Delta^4),$$

where we have used the following designations for the points in the $u - v$ plane with step-size $\Delta$: $N = (u + \Delta, v + \Delta)$, $W = (u + \Delta, v)$, $E = (u, v + \Delta)$ and $S = (u, v)$. In the linear regime the eigenfrequencies of the JNW spacetime are not sensitive to the choice of the initial condition, so, we model the initial perturbation by a Gaussian pulse of width $\sigma$ centred around $v = v_c$,

$$\Psi(u = 0, v) = e^{-\frac{(v-v_c)^2}{2\sigma^2}},$$

and assume the perturbation to be constant near the singularity,

$$\Psi(r = b + \epsilon, t) = \Psi(u = v - v_0, v) = \text{Constant}, \quad \forall t; \quad \epsilon \ll 1.$$
shows the time evolution of axial perturbation of the JNW space time for different values of the parameter $\nu$. We see from Table I that the numbers along the axes of the plots are in arbitrary units. But the qualitative features are clearly described by the plots. We find that close to the Schwarzschild limit ($\nu = 1$), when scalar charge $q$ is small, the response of the JNW spacetime is dominated by damped harmonic oscillations, which soon gives way to distinctive echoes. The echoes cannot be characterised by a single dominant frequency. However, with the decrease in $\nu$, as the peak of the effective potential moves closer to the wall, the echoes become less prominent and finally, the enveloping oscillation of the echoes align to yield characteristic frequencies of the JNW spacetime. To extract the characteristic quasinormal frequencies from the time-domain profile we use the Prony method of fitting the data via superposition of damped exponentials with some excitation factors.\[\nu \text{ waves at spatial infinity. The quasinormal mode frequency for the standard black hole - boundary condition of completely ingoing waves at the horizon and completely outgoing waves at spatial infinity. The quasinormal mode frequency for } \nu = 1 \text{ matches with that obtained in Refs. [79]. Table I shows the fundamental quasinormal frequencies of the weakly naked JNW spacetime. The quasinormal modes for the Schwarzschild case (} \nu = 1 \text{) corresponds to the standard black hole - boundary condition of completely ingoing waves at the horizon and completely outgoing waves at spatial infinity. The quasinormal mode frequency for } \nu = 1 \text{ matches with that obtained in Refs. [80–82]. It deserves mention that the usual more accurate methods of extracting the quasinormal modes, such as the WKB approximation, do not work in this case. However, the method adopted here yields the qualitative features quite clearly. We see from Table I that for a given multipole index, } \ell, \text{ once the scalar charge becomes significantly large}

\begin{table}
\centering
\begin{tabular}{|c|c|c|}
\hline
$\nu$ & $\omega (\ell = 2)$ & $\omega (\ell = 3)$ \\
\hline
1.00 & 0.3730 $-0.0891i$ & 0.5993 $-0.0927i$ \\
0.99 & 0.3473 $-0.1001i$, echo & 0.6078 $-0.0934i$, echo \\
0.95 & 0.3716 $-0.1049i$, echo & 0.6348 $-0.0968i$, echo \\
0.86 & I.O., 0.3405 $-0.0004i$ & 0.7284 $-0.1077i$, echo \\
0.83 & I.O., 0.4166 $-0.0032i$ & 0.8492 $-0.1951i$, echo \\
0.80 & I.O., 0.4903 $-0.0123i$ & I.O., 0.6324 $-0.0004i$ \\
0.78 & I.O., 0.5371 $-0.0235i$ & I.O., 0.7103 $-0.0019i$ \\
0.75 & I.O., 0.6044 $-0.0493i$ & I.O., 0.8229 $-0.0104i$ \\
0.73 & I.O., 0.6503 $-0.0706i$ & I.O., 0.8947 $-0.0234i$ \\
\hline
\end{tabular}
\caption{Fundamental quasinormal mode frequencies for axial gravitational perturbation of the JNW spacetime for $\ell = 2, 3$. (I.O. stands for initial outburst.)}
\end{table}

Using the above integration scheme, we study the time evolution of the field $\Psi$ along a line of constant $r_*$. Fig. 2 shows the time evolution of axial perturbation of the JNW space time for different values of the parameter $\nu$. One should note that the numbers along the axes of the plots are in arbitrary units. But the qualitative features are quite clearly described by the plots. We find that close to the Schwarzschild limit ($\nu = 1$), when scalar charge $q$ is small,

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{Time-domain profiles for axial perturbation of the JNW spacetime as extracted at $r_* = 100$ with $b = 2$ for $\nu = 0.99, 0.95, 0.75$ (from left to right) and $\ell = 2, 3$ (from top to bottom).}
\end{figure}
and the echoes align, both the oscillation frequency and the damping rate of the fundamental quasinormal mode (the real and imaginary parts of the QNM frequency respectively) increase with the decrease of \( \nu \). With increasing values of the multipole index the echoes persists for even smaller values of \( \nu \).

An important parameter in the analysis of ringdown signal is the quality factor which is defined as the ratio of the real and imaginary parts of the quasinormal mode frequency

\[
Q \sim \frac{\text{Real}(\omega)}{|\text{Im}(\omega)|}.
\]

The quality factor for the characteristic quasinormal modes of the Schwarzschild black hole and the JNW spacetime is shown in Table II. We observe that for a given multipole index, the quality factor decreases with the parameter \( \nu \). For a given value of the parameter \( \nu \), the real part of the quasinormal mode frequency increases with the multipole index whereas the imaginary part decreases, thereby increasing the quality factor.

### V. CONCLUSION

The detection of gravitational waves from compact binary coalescence have opened a new window of opportunity to probe the strong field regime of gravity and hence to test the existence of exotic compact objects. One such horizonless compact object is a naked singularity which can form as a result of gravitational collapse under suitable initial conditions.

In the present work, we studied the ringdown profile of the Janis-Newman-Winicour naked singularity-spacetime\(^{64-66} \) for an axial gravitational perturbation. We specifically investigated the weakly naked singularity regime where the spacetime is characterised by a photon sphere. At the singularity at \( r = b \), the effective potential becomes infinite. To study the time evolution of the perturbation, we considered a surface close to the singularity at \( r = b + \epsilon \) (\( \epsilon << 1 \)) and assumed the perturbation to be constant on this surface. For the QNM pertaining to a black hole one has an advantage of picking up boundary conditions more elegantly, pure incoming waves at the horizon and pure outgoing waves at large distances. The presence of a singular surface does not provide any natural choice of such a boundary condition. The present work assumes a constant boundary condition, which one can have at least as a limiting condition (Eq. (25)).

Near the Schwarzschild limit, the initial response of the spacetime is characterised by damped oscillations, reminiscent of the potential peak at \( r > b \). At later times, these damped oscillations give rise to distinct echoes. However, as the parameter \( \nu \) is increased, the echoes die down and characteristic quasinormal modes emerge. In the extreme right plots of Fig. 2, both up and down, when \( \nu = 0.75 \), corresponding to \( \frac{\sqrt{\nu}}{\sqrt{\ell}} = \frac{\sqrt{7}}{3} \), there are no echoes. Certainly the QNMs are different from the black holes. As the echoes align, the QNM spectrum is dominated by modes with very low damping rate, hence, high quality factor (as evident from Tables I and II for \( \nu = 0.86 \) for \( \ell = 2 \) and \( \nu = 0.80 \) for \( \ell = 3 \)). Thus, the JNW spacetime in this range is an excellent oscillator. With further increase in \( \nu \), both the oscillation frequency and the damping rate of the quasinormal modes increase which in turn reduces the quality factor of the oscillation. It is also important to note that in the entire analysis we do not observe any unstable quasinormal modes with frequency having positive imaginary part.

The existence of echoes in the ringdown signal of the JNW naked singularity is a novel result which categorically differentiates it from a black hole. It deserves mention that Chirenti, Saa and Skákalá\(^{71} \) showed the absence of asymptotically highly damped modes in the Wyman naked singularity\(^{65, 66} \), but their work was based on the perturbation of a test scalar field whereas the present work is based on the tensor perturbation of the metric.

Further investigations may be able to compare the ringdown profile of the JNW spacetime with that of other exotic compact objects such as wormholes which produce similar echoes in the ringdown phase\(^{54-62} \). We also plan to

### Table II: Quality factor for the characteristic fundamental quasinormal mode frequencies for axial gravitational perturbation of the JNW spacetime with \( \ell = 2, 3 \).

| \( \nu \) | \( Q (\ell = 2) \) | \( Q (\ell = 3) \) |
|---|---|---|
| 1.00 | 4.19 | 6.46 |
| 0.86 | 851.37 | — |
| 0.83 | 130.25 | — |
| 0.80 | 39.86 | 1581 |
| 0.78 | 22.86 | 373.84 |
| 0.75 | 12.25 | 79.125 |
| 0.73 | 9.211 | 38.23 |
extend our analysis to polar perturbations (even parity) as well. In this regard it is also important to note that the perturbation of the scalar field $\Phi$, which does not contribute in the present odd-parity case, will give rise to breathing modes\cite{83} in the QNM spectrum, which results purely from $\delta\Phi$.

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