Abstract

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1 Introduction

By a permutation class of a pointed stable curve over a closed point $\text{Spec}(k)$ we mean a stable curve with marked points, where the labels of the marked points are only determined up to certain permutations.

More formally, let $m \in \mathbb{N}$ be a natural number, and let $\Gamma$ be a subgroup of the group $\Sigma_m$ of permutations on $m$ elements. An $m$-pointed stable curve of genus $g$ over a closed point is given by a tuple $(f : C \to \text{Spec}(k); P_1, \ldots, P_m)$, where $P_1, \ldots, P_m$ define $m$ marked smooth points on an algebraic curve $C$, compare definition 2.2. A permutation class of pointed stable curves with respect to $\Gamma$ is simply an equivalence class of such tuples, where $(f : Browse the document and extract the natural text. Here is a plain text representation of the document:

Moduli stacks
of permutation classes of pointed stable curves

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11.11.2006

Abstract

The notion of $m/\Gamma$-pointed stable curves is introduced. It should be viewed as a generalization of the notion of $m$-pointed stable curves of a given genus, where the labels of the marked points are only determined up to the action of a group of permutations $\Gamma$. The classical moduli spaces and moduli stacks are generalized to this wider setting. Finally, an explicit construction of the new moduli stack of $m/\Gamma$-pointed stable curves as a quotient stack is given.

1 Introduction

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More formally, let $m \in \mathbb{N}$ be a natural number, and let $\Gamma$ be a subgroup of the group $\Sigma_m$ of permutations on $m$ elements. An $m$-pointed stable curve of genus $g$ over a closed point is given by a tuple $(f : C \to \text{Spec}(k); P_1, \ldots, P_m)$, where $P_1, \ldots, P_m$ define $m$ marked smooth points on an algebraic curve $C$, compare definition 2.2. A permutation class of pointed stable curves with respect to $\Gamma$ is simply an equivalence class of such tuples, where $(f :
C \to \text{Spec} (k), P_1, \ldots, P_m) and \((f' : C' \to \text{Spec} (k); P'_1, \ldots, P'_m)\) are called equivalent if \(C = C'\) and if there exists a permutation \(\gamma \in \Gamma\), such that \(P_i = P'_\gamma(i)\) for all \(i = 1, \ldots, m\). Below, such a class will be called an \(m/\Gamma\)-pointed stable curve over \(\text{Spec} (k)\), see definition 3.2 and lemma 3.5.

Over an arbitrary base scheme \(S\) the situation is more subtle. It turns out that an \(m/\Gamma\)-pointed stable curve cannot simply be defined as an equivalence class of \(m\)-pointed stable curves with respect to the analogous equivalence relation. However, the definition of \(m/\Gamma\)-pointed stable curves is made in such a way that locally, with respect to the étale topology, an \(m/\Gamma\)-pointed stable curve \(f : C \to S\) over an arbitrary scheme \(S\) looks like an equivalence class of \(m\)-pointed stable curves over \(S\). The sections of marked points, which exist on an étale cover, are only determined up to permutations in \(\Gamma\), and they need to satisfy certain compatibility conditions, but in general they will not descent to global sections of the original curve.

The classical \(m\)-pointed stable curves are special cases of \(m/\Gamma\)-pointed stable curves, where \(\Gamma\) is the trivial group.

More generally, stable curves with \(m\) distinguished points, where the points are only ordered up to certain permutations, occur on a number of occasions. Consider for example a stable curve \(f : C \to \text{Spec} (k)\) of some genus \(g\), which consists of more than one irreducible component. An irreducible component of \(C\) is a curve \(f' : C' \to \text{Spec} (k)\) of some genus \(g' \leq g\), but it no longer needs to be a stable curve. However, the curve \(f' : C' \to \text{Spec} (k)\) together with the unordered set \(\{P_1, \ldots, P_m\}\), consisting of those closed points where \(C'\) intersects the closure of its complement in \(C\), is stable when considered as an \(m/\Gamma\)-pointed stable curve of genus \(g'\), where \(\Gamma\) is the group of all permutations on \(m\) elements. Note that even if an ordering of the nodes of \(C\) is fixed a priory, there is in general no distinguished ordering of the intersection points \(\{P_1, \ldots, P_m\}\). Because of automorphisms of \(C\), an ordering will be determined only up to certain permutations.

Another example, where permutation classes of pointed stable curves show up naturally, is the moduli space \(\overline{M}_g\) of Deligne-Mumford stable curves of genus \(g\) itself. There is a canonical stratification \(\overline{M}_g = \bigcup_i M_g^{(i)}\), where \(M_g^{(i)}\) denotes the locus of curves with exactly \(i\) nodes, for \(i = 0, \ldots, 3g - 3\). The irreducible components of the subschemes of the stratification can be related to moduli spaces of \(m/\Gamma\)-pointed stable curves of genus \(g'\), where
the number $m$, the group $\Gamma$ and the genus $g'$ varies. The case $i = 3g - 4$ has been discussed in detail in [Zi], with special focus on the relation of the corresponding moduli stacks.

Due to the fact that $m/\Gamma$-pointed stable curves over closed points can be described as equivalence classes of $m$-pointed stable curves, it is possible to construct coarse and fine moduli spaces for them as quotients of moduli spaces of $m$-pointed stable curves. More importantly, because of the étale nature of $m/\Gamma$-pointed stable curves, we can define the corresponding moduli stacks in the sense of [DM]. Finally, generalizing an idea of Edidin [Ed] for Deligne-Mumford stable curves, we describe explicitly how to construct the new moduli stack of $m/\Gamma$-pointed stable curves as a quotient stack.

2 Preliminaries on pointed stable curves

2.1 We want to collect some basics about pointed stable curves first, so that we can refer to them later on. We also generalize a theorem of Edidin [Ed] on the moduli stack of Deligne-Mumford stable curves to the case of pointed stable curves.

Let us first recall the definition of pointed stable curves as it is given in the paper of Knudsen [Kn, def. 1.1].

Definition 2.2 An $m$-pointed stable curve of genus $g$ is a flat and proper morphism $f : C \to S$ of schemes, together with $m$ sections $\sigma_i : S \to C$, for $i = 1, \ldots, m$, such that for all closed points $s \in S$ holds

(i) the fibre $C_s$ is a reduced connected algebraic curve with at most ordinary double points as singularities;

(ii) the arithmetic genus of $C_s$ is $\dim H^1(C_s, \mathcal{O}_{C_s}) = g$;

(iii) for $1 \leq i \leq m$, the point $\sigma_i(s)$ is a smooth point of $C_s$;

(iv) for all $1 \leq i, j \leq m$ holds $\sigma_i(s) \neq \sigma_j(s)$ if $i \neq j$. 

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(v) the number of points where a nonsingular rational component $C'_s$ of $C_s$ meets the rest of $C_s$ plus the number of points $\sigma_i(s)$ which lie on $C'_s$ is at least 3.

Remark 2.3 Condition (v) guarantees that the group $\text{Aut}(C_s)$ of automorphisms of a fibre $C_s$ is finite. Because of condition (iv), the permutation group $\Sigma_m$ acts faithfully on the set of sections $\{\sigma_1, \ldots, \sigma_m\}$, as well as on the set of marked points $\{\sigma_1(s), \ldots, \sigma_m(s)\}$ of each fibre $C_s$.

2.4 Let $f : C \to S$ be an $m$-pointed stable curve of genus $g$ such that $2g - 2 + m > 0$. The sections $\sigma_1, \ldots, \sigma_m$ determine effective Cartier divisors $S_1, \ldots, S_m$ on $C$.

There is a canonical invertible dualizing sheaf on $C$, which is denoted by $\omega_{C/S}$. In [Kn, cor. 1.9, cor 1.11] it is shown that for $n \geq 3$ the sheaf $(\omega_{C/S}(S_1 + \ldots + S_m))^\otimes n$ is relatively very ample, and furthermore that $f_*( (\omega_{C/S}(S_1 + \ldots + S_m))^\otimes n)$ is locally free of rank $(2g - 2 + m)n - g + 1$.

Consider the Hilbert scheme $\text{Hilb}_{P^N}^{P_{g,n,m}}$ of curves $C \to \text{Spec}(k)$ embedded in $P^N$ with Hilbert polynomial $P_{g,n,m} := (2g - 2 + m)nt - g + 1$, where $N := (2g - 2 + m)n - g$.

The Hilbert scheme of a simple point in $P^N$ is of course $P^N$ itself. The incidence condition of $m$ points lying on a curve $C$ defines a closed subscheme

$I \subset \text{Hilb}_{P^N}^{P_{g,n,m}} \times (P^N)^m$.

There is an open subset $U \subset I$ parametrizing curves, where the $m$ points are pairwise different, and smooth points of $C$. Finally, there is a closed subscheme $\overline{H}_{g,n,m}$ of $U$, which represents such embedded $m$-pointed stable curves $C \to \text{Spec}(k)$ of genus $g$, where the embedding is determined by an isomorphism

$\left(\omega_{C/\text{Spec}(k)}(Q_1 + \ldots + Q_m)\right)^\otimes n \cong \mathcal{O}_{P^N}(1)|C$.

Here, $Q_1, \ldots, Q_m$ denote the marked points on $C$. The scheme constructed in this way is in fact a quasi-projective subscheme

$\overline{H}_{g,n,m} \subset \text{Hilb}_{P^N}^{P_{g,n,m}} \times (P^N)^m$. 

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of the Hilbert scheme. There is also a subscheme $H_{g,n,m} \subset \overline{H}_{g,n,m}$ corresponding to non-singular stable curves. Compare [FP] section 2.3] for the more general situation of moduli of stable maps.

There is a one-to-one correspondence between morphisms $\vartheta : S \to \overline{H}_{g,n,m}$ and $m$-pointed stable curves $f : C \to S$ together with global trivializations of Grothendieck’s associated projective bundle

$$\mathbb{P}f_*\left( (\omega_{C/S}(S_1 + \ldots + S_m))^\otimes n \right) \sim \mathbb{P}^N \times S.$$  

In particular, the curve $f : C \to S$ can be considered as embedded into $\mathbb{P}^N \times S$, such that the diagram

$$C \xleftarrow{f} \mathbb{P}^N \times S \xrightarrow{pr_2} S$$

commutes.

An element of the group $\text{PGL}(N + 1)$ acts on $\mathbb{P}^N$, changing embeddings of $C$ by an isomorphism. Hence there is a natural action of $\text{PGL}(N + 1)$ on $\text{Hilb}_{\mathbb{P}^N}^{P_{g,n,m}}$, and thus on $\text{Hilb}_{\mathbb{P}^N}^{P_{g,n,m}} \times (\mathbb{P}^N)^m$, which restricts to an action on $\overline{H}_{g,n,m}$. Furthermore, the symmetric group $\Sigma_m$ acts on $(\mathbb{P}^N)^m$ by permutation of the coordinates, which is equivalent to the permutation of the $m$ sections of marked points. Hence $\Sigma_m$ acts freely on $\overline{H}_{g,n,m} \times (\mathbb{P}^N)^m$. The combined action of $\text{PGL}(N + 1) \times \Sigma_m$ on $\overline{H}_{g,n,m}$ shall be written as an action from the right.

2.5 Let $f : C_0 \to \text{Spec}(k)$ be an $m$-pointed stable curve over an algebraically closed field $k$, so in particular an algebraic curve of genus $g$ with nodes as its only singularities, and $m$ marked points $P_1, \ldots, P_m$. If we fix one (arbitrary) embedding of $C_0$ into $\mathbb{P}^N$ via an isomorphism

$$\mathbb{P}f_*((\omega_{C_0/k}(P_1 + \ldots + P_m))^\otimes n) \cong \mathbb{P}^N,$$

then this distinguishes a $k$-valued point $[C_0] \in \overline{H}_{g,n,m}$. For the group of automorphisms of $C_0$ respecting the marked points, i.e. automorphisms which map each marked point to itself, there is a natural isomorphism

$$\text{Aut}(C_0) \cong \text{Stab}_{\text{PGL}(N+1)}([C_0]),$$
that is, an isomorphism with the subgroup of those elements in $\text{PGL}(N+1)$, which stabilize the point $[C_0]$. Recall that $\Sigma_m$ acts on $\tilde{\mathcal{P}}_{g,n,m}$ without fixed points, so the stabilizer of $[C_0]$ in $\text{PGL}(N+1)$ can be identified with the stabilizer of $[C_0]$ in $\text{PGL}(N+1) \times \Sigma_m$.

In general, elements $\gamma \in \text{PGL}(N+1)$ are in one-to-one correspondence with isomorphisms

$$\gamma : C_0 \to C_\gamma$$

of embedded $m$-pointed stable curves, where $C_\gamma$ is the curve represented by the point $[C_\gamma] := [C_0] \cdot \gamma$ in $\tilde{\mathcal{P}}_{g,n,m}$.

2.6 Generalizing the construction of $\overline{\mathfrak{M}}_g$ as described by Gieseker, the moduli space $\overline{\mathfrak{M}}_{g,m}$ of $m$-pointed stable curves of genus $g$ is constructed as the GIT-quotient of the action of $\text{PGL}(N+1)$ on $\tilde{\mathcal{P}}_{g,n,m}$, see [GS], [DM] and [HM] for the case of $m = 0$, and compare [FP, Remark 2.4] for the more general case of stable maps. Note that the notion of stability of embedded $m$-pointed stable curves is compatible with the notion of stability of the corresponding points in $\tilde{\mathcal{P}}_{g,n,m}$ with respect to the group action. In particular, there is a canonical quotient morphism

$$\tilde{\mathcal{P}}_{g,n,m} \to \overline{\mathfrak{M}}_{g,m}.$$

The construction of the moduli space is independent of the choice of $n$, provided it is large enough, and hence independent of $N$. For a stable algebraic curve $C_0 \to \text{Spec}(k)$, the fibre over the point in $\overline{\mathfrak{M}}_{g,m}$ representing it is isomorphic to the quotient $\text{Aut}(C_0) \setminus \text{PGL}(N+1)$.

The scheme $\overline{\mathfrak{M}}_{g,m}$ is at the same time a moduli space for the stack $\overline{\mathfrak{M}}_{g,m}$ of $m$-pointed stable curves of genus $g$.

Definition 2.7 The moduli stack $\overline{\mathfrak{M}}_{g,m}$ of $m$-pointed stable curves of genus $g$ is the stack defined as the category fibred in groupoids over the category of schemes, where for a scheme $S$ the objects in the fibre category $\overline{\mathfrak{M}}_{g,m}(S)$ are the $m$-pointed stable curves of genus $g$ over $S$.

Morphisms in $\overline{\mathfrak{M}}_{g,m}$ are given as follows. Let $f : C \to S$ and $f' : C' \to S'$ be $m$-pointed stable curves of genus $g$, i.e. objects of $\overline{\mathfrak{M}}_{g,m}(S)$ and $\overline{\mathfrak{M}}_{g,m}(S')$,
respectively, with sections $\sigma_i : S \to C$ and $\sigma'_i : S' \to C'$ for $i = 1, \ldots, m$. A morphism from $f : C \to S$ to $f' : C' \to S'$ in $\overline{M}_{g,m}$ is given by a pair of morphisms of schemes

$$(g : S' \to S, \overline{g} : C' \to C),$$

such that the diagram

$$\begin{array}{ccc} C' & \xrightarrow{\overline{g}} & C \\ \downarrow{f'} & & \downarrow{f} \\ S' & \xrightarrow{g} & S \end{array}$$

is Cartesian, and for all $i = 1, \ldots, m$ holds

$$\overline{g} \circ \sigma'_i = \sigma_i \circ g.$$

**Remark 2.8** It was shown by Knudsen [Kn] that the moduli stack $\overline{M}_{g,m}$ is a smooth, irreducible Deligne-Mumford stack, which is proper over $\text{Spec}(\mathbb{Z})$. Compare also [DM] and [Ed].

The following proposition was proven for the case $m = 0$ in [Ed], but also holds true in the case of arbitrary $m$. Here, as usual, square brackets are used to denote quotient stacks.

**Proposition 2.9** There are isomorphisms of stacks

$$\overline{M}_{g,m} \cong \left[ \overline{H}_{g,n,m}/\text{PGL}(N + 1) \right],$$

and

$$M_{g,m} \cong \left[ H_{g,n,m}/\text{PGL}(N + 1) \right].$$

**Proof.** We will only give the proof in the case of the closed moduli stack, the second case being completely analogous. Recall that for any given scheme $S$ an object of the fibre category $\left[ \overline{H}_{g,n,m}/\text{PGL}(N + 1) \right] (S)$ is a triple $(E, p, \phi)$, where $p : E \to S$ is a principal $\text{PGL}(N + 1)$-bundle, and $\phi : E \to \overline{H}_{g,n,m}$ is a $\text{PGL}(N + 1)$-equivariant morphism.
(i) An isomorphism of stacks from $\overline{M}_{g,m}$ to $[\overline{M}_{g,n,m}/\text{PGL}(N+1)]$ is constructed as follows. Let an $m$-pointed stable curve $f: C \rightarrow S \in \text{Ob}(\overline{M}_{g,m})$ be given, with sections $\sigma_1, \ldots, \sigma_m: S \rightarrow C$ defining divisors $S_1, \ldots, S_m$ on $C$. We denote by $p: E \rightarrow S$ the principal $\text{PGL}(N+1)$-bundle associated to the projective bundle $\mathbb{P} f^*(\omega_{C/S}(S_1 + \ldots + S_m))^\otimes n$ over $S$. Consider the Cartesian diagram

$$
\begin{array}{ccc}
C \times_S E & \xrightarrow{\mathcal{I}} & E \\
\downarrow \pi & & \downarrow p \\
C & \xrightarrow{f} & S.
\end{array}
$$

The pullback of $\mathbb{P} f^*(\omega_{C/S}(S_1 + \ldots + S_m))^\otimes n$ to $E$ has a natural trivialization as a projective bundle. Denote by $\hat{S}_1, \ldots, \hat{S}_m$ the divisors on $C \times_S E$ defined by the sections $\hat{\sigma}_i: E \rightarrow C \times_S E$, with $\hat{\sigma}_i(e) := [\sigma_i \circ p(e), e]$ for $e \in E$ and $i = 1, \ldots, m$. Then by the universal property of the pullback there is a unique isomorphism

$$
p^* \mathbb{P} f^*(\omega_{C/S}(S_1 + \ldots + S_m))^\otimes n \cong \mathbb{P} \mathcal{I}_* \left( \omega_{C \times_S E/E}(\hat{S}_1 + \ldots + \hat{S}_m) \right)^\otimes n
$$

of projective bundles over $E$. So we have an $m$-pointed stable curve $\mathcal{I}: C \times_S E \rightarrow E$, together with a natural trivialization of the projective bundle $\mathbb{P} \mathcal{I}_*(\omega_{C \times_S E/E}(\hat{S}_1 + \ldots + \hat{S}_m))^\otimes n$. This is equivalent to specifying a morphism

$$
\phi: E \rightarrow [\overline{M}_{g,n,m}]
$$

by the universal property of the Hilbert scheme. By construction, the morphism $\phi$ is $\text{PGL}(N+1)$-equivariant, so we obtain an object $(E, p, \phi) \in [\overline{M}_{g,n,m}/\text{PGL}(N+1)](S)$.

To define a functor from $\overline{M}_{g,m}$ to $[\overline{M}_{g,n,m}/\text{PGL}(N+1)]$ we need also to consider morphisms $(g: S' \rightarrow S, \overline{\varphi}: C' \rightarrow C)$ between stable curves $f: C \rightarrow S$ and $f': C' \rightarrow S'$. Because for the respective sections $\sigma_1, \ldots, \sigma_m: S \rightarrow C$ and $\sigma'_1, \ldots, \sigma'_m: S' \rightarrow C'$ holds $\sigma_i \circ g = \overline{\varphi} \circ \sigma_i'$ for all $i = 1, \ldots, m$ by definition, there is an induced morphism

$$
\mathbb{P} f'_*(\omega_{C'/S'}(S'_1 + \ldots + S'_m))^\otimes n \rightarrow \mathbb{P} f_*(\omega_{C/S}(S_1 + \ldots + S_m))^\otimes n,
$$

where $S'_1, \ldots, S'_m$ denote the divisors of the marked points on $C'$. Hence there is an induced morphism of the associated principal $\text{PGL}(N+1)$-bundles
\[ \tilde{g} : E' \to E, \] which fits into a commutative diagram

\[ \begin{array}{ccc}
E' & \xrightarrow{\tilde{g}} & E \\
\downarrow{\phi'} & & \downarrow{\phi} \\
S' & \xrightarrow{g} & S.
\end{array} \]

One easily verifies that this assignment is functorial. In this way we obtain a functor from the fibred category \( \mathcal{M}_{g,m} \) to the quotient fibred category \( \bH_{g,n,m}/\text{PGL}(N+1) \) over the category of schemes, and thus a morphism of stacks.

(ii) Conversely, consider a triple \((E, p, \phi) \in \bH_{g,n,m}/\text{PGL}(N+1)(S)\). The morphism \( \phi : E \to \bH_{g,n,m} \) determines an \( m \)-pointed stable curve \( f' : C' \to E \) of genus \( g \), together with a trivialization \( \mathbb{P}f'_*(\omega_{C'/E}(S'_1 + \ldots + S'_m)) \cong \mathbb{P}^N \times E \), where \( S'_1, \ldots, S'_m \) denote the divisors on \( C' \) determined by the \( m \) sections \( \sigma'_1, \ldots, \sigma'_m : E \to C' \). The group \( \text{PGL}(N+1) \) acts diagonally on \( \mathbb{P}^N \times E \). By the \( \text{PGL}(N+1) \)-equivariance of \( \phi \), we have for all \( e \in E \) and all \( \gamma \in \text{PGL}(N+1) \) the identity of fibres \( C'_e \cdot \gamma = C'_{\gamma(e)} \) as embedded \( m \)-pointed curves. Therefore there is an induced action of \( \text{PGL}(N+1) \) on the embedded curve \( C' \subset \mathbb{P}^N \times E \), which respects the sections \( \sigma'_1, \ldots, \sigma'_m \). Taking quotients we obtain

\[ C := C'/\text{PGL}(N+1) \to E/\text{PGL}(N+1) \cong S, \]

which defines an \( m \)-pointed stable curve \( C \to S \in \mathcal{M}_{g,m}(S) \).

The same construction assigns to a morphism in \( \bH_{g,n,m}/\text{PGL}(N+1) \) a morphism between \( m \)-pointed stable curves. This defines a functor between fibred categories, and thus a morphism of stacks from \( \bH_{g,n,m}/\text{PGL}(N+1) \) to \( \mathcal{M}_{g,m} \).

(iii) It remains to show that the two functors defined above form an equivalence of categories. In fact, the composition

\[ \mathcal{M}_{g,m} \to \bH_{g,n,m}/\text{PGL}(N+1) \to \mathcal{M}_{g,m} \]

is just the identity on \( \mathcal{M}_{g,m} \).
Conversely, let $(E, p, \phi) \in \overline{H}_{g,n,m}/\text{PGL}(N+1)(S)$ for some scheme $S$. This defines an $m$-pointed stable curve $f : C \to S$ in $\overline{M}_{g,m}(S)$, which in turn defines an object $(E', p', \phi') \in \overline{H}_{g,n,m}/\text{PGL}(N+1)(S)$. We claim that the triples $(E, p, \phi)$ and $(E', p', \phi')$ are isomorphic.

By construction we have a commutative diagram

\[
\begin{array}{ccc}
C' & \xrightarrow{f'} & E \\
\downarrow p & & \downarrow p \\
C := C'/\text{PGL}(N+1) & \xrightarrow{f} & E/\text{PGL}(N+1) = S,
\end{array}
\]

where $f' : C' \to E$ is the $m$-pointed stable curve over $E$ defined by the morphism $\phi : E \to \overline{H}_{g,n,m}$, and the vertical arrows represent the natural quotient morphisms. Hence there is an isomorphism of projective bundles

\[p^*\mathbb{P}f_*(\omega_{C/S}(S_1 + \ldots + S_m))^{\otimes n} \cong \mathbb{P}f'_*(\omega_{C'/E}(S'_1 + \ldots + S'_m))^{\otimes n},\]

together with a trivialization of the bundle $\mathbb{P}f'_*(\omega_{C'/E}(S'_1 + \ldots + S'_m))^{\otimes n}$ determined by $\phi : E \to \overline{H}_{g,n,m}$. Thus the pullback of the projective bundle $\mathbb{P}f_*(\omega_{C/S}(S_1 + \ldots + S_m))^{\otimes n}$ to the principal $\text{PGL}(N+1)$-bundle $p : E \to S$ is trivial, and hence $E$ must be isomorphic to the principal $\text{PGL}(N+1)$-bundle associated to $\mathbb{P}f_*(\omega_{C/S}(S_1 + \ldots + S_m))^{\otimes n}$.

It is straightforward to give the corresponding arguments for the morphisms in both categories to show that the two constructions are inverse to each other, up to isomorphism. \qed

**Remark 2.10** Since $\overline{M}_{g,m}$ is a categorical quotient of $\overline{H}_{g,n,m}$, there is a natural morphism of stacks $[\overline{H}_{g,n,m}/\text{PGL}(N+1)] \to \overline{M}_{g,m}$. In fact, the diagram

\[
\begin{array}{ccc}
\overline{M}_{g,m} & \xrightarrow{\sim} & [\overline{H}_{g,n,m}/\text{PGL}(N+1)] \\
\downarrow \text{M}_{g,m} & & \\
& &
\end{array}
\]

commutes.
3 Moduli of $m/\Gamma$-pointed stable curves

3.1 In the introduction above we said that an $m/\Gamma$-pointed stable curve of genus $g$ over a closed point should be understood as an equivalence class of $m$-pointed stable curves $(f : C \to \text{Spec}(k), \sigma_1, \ldots, \sigma_m)$, where $f : C \to \text{Spec}(k)$ is an $m$-pointed stable curve with its marked points given by sections $\sigma_i : \text{Spec}(k) \to C$ for $i = 1, \ldots, m$, and where the equivalence relation is taken with respect to permutations $\gamma \in \Gamma$ of the tuple $(\sigma_1, \ldots, \sigma_m)$.

However, in order to provide a notion which is useful in applications, our definition needs to be more technical. An $m/\Gamma$-pointed stable curve over an arbitrary scheme $S$ cannot simply be defined as an equivalence class of pointed stable curves $f : C \to S$, together with sections of marked points $\sigma_1, \ldots, \sigma_m : S \to C$. In fact, an $m/\Gamma$-pointed stable curve will in general not admit sections of marked points. For example, choose a smooth stable curve $(f_0 : C_0 \to \text{Spec}(k), P_1, \ldots, P_4)$ of genus $g = 1$ with four marked points on it, for which there exists an automorphism $\tau : C_0 \to C_0$, which interchanges the points $P_1$ and $P_2$, as well as $P_3$ and $P_4$. Let $U_1$ and $U_2$ be two Zariski open affine subschemes covering $\mathbb{P}^1$. Obviously the four distinguished points $P_1, \ldots, P_4$ on $C_0$ determine sections of the two direct products $p_1 : U_1 \times C_0 \to U_1$ and $p_2 : U_2 \times C_0 \to U_2$.

Thus the two products are both 4-pointed stable curves of genus 1 in the classical sense. We now glue the two trivial products with a twist, which is provided by applying the automorphism $\tau$ to the fibres over the intersection of $U_1$ and $U_2$. What we obtain is a family $f : C \to \mathbb{P}^1$ of curves of genus $g = 1$, each fibre with $m = 4$ distinguished points, which cannot be labeled using global sections. In particular, this is no longer a 4-pointed stable curve in the sense of Knudsen. However, there are two well-defined classes of distinguished points: those coming from points labeled $P_1$ or $P_2$ on the covering, and those coming from points $P_3$ or $P_4$. This is an example of an $m/\Gamma$-pointed stable curve of genus $g = 1$ over $\mathbb{P}^1$ as defined below, where $m = 4$, and $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2$ is the subgroup of the permutation group $\Sigma_4$, which is generated by the two transpositions interchanging “1” and “2”, and “3” and “4”, respectively. The labels of the four distinguished points in each fibre are only determined up to permutations in $\Gamma$.

In general, global sections of marked points exist only on an étale cover...
of the curve $f : C \to S$. Conversely, if there exist global sections on an étale cover, which satisfy a certain compatibility condition (*), which will be discussed in remark 3.4 in detail, then there will be well-defined classes of labels of the distinguished points.

In the case where $S = \text{Spec}(k)$, lemma 3.5 shows that the formal definiton agrees with our geometric intuition.

**Definition 3.2** Let $\Gamma$ be a subgroup of the symmetric group $\Sigma_m$.

(i) A charted $m/\Gamma$-pointed stable curve of genus $g$ is a tuple

$$(f : C \to S, \ u : S' \to S, \ \sigma_1, \ldots, \sigma_m : S' \to C'),$$

where $f : C \to S$ is a flat and proper morphism of schemes, with reduced and connected algebraic curves as its fibres, $u : S' \to S$ is an étale covering, defining the fibre product $C' := C \times_S S'$, such that the induced morphism $f' : C' \to S'$, together with the sections $\sigma_1, \ldots, \sigma_m : S' \to C'$, is an $m$-pointed stable curve of genus $g$. Furthermore, we require that for all closed points $s, s' \in S'$ with $u(s) = u(s')$, there exists a permutation $\gamma_{s,s'} \in \Gamma$, such that

$$(*) \quad \overline{u} \circ \sigma_i(s) = \overline{u} \circ \sigma_{\gamma_{s,s'}(i)}(s')$$

holds for all $i = 1, \ldots, m$, where $\overline{u} : C' \to C$ is the morphism induced by $u : S' \to S$.

(ii) Two charted $m/\Gamma$-pointed stable curves are called equivalent, if there exists a third charted $m/\Gamma$-pointed stable curve dominating both of them, in the sense of remark 3.3 below.

(iii) An $m/\Gamma$-pointed stable curve of genus $g$ is an equivalence class of charted $m/\Gamma$-pointed stable curves of genus $g$.

**Remark 3.3** Let a charted $m/\Gamma$-pointed stable curve $(f : C \to S, \ u' : S' \to S, \ \sigma_1, \ldots, \sigma_m : S' \to C')$ be given. A second charted $m/\Gamma$-pointed stable curve $(f : C \to S, \ u'' : S'' \to S, \ \tau_1, \ldots, \tau_m : S'' \to C'')$ with the same underlying curve $f : C \to S$ is said to dominate the first one, if the morphism $u'' : S'' \to S$ factors as $u'' = u' \circ v$, where $v : S'' \to S'$ is an étale covering, which induces an isomorphism

$$C'' \cong C' \times_{S'} S''.$$
and such that for all closed points \( s \in S'' \) there exists a permutation \( \gamma_s \in \Gamma \), with

\[
\text{(**)} \quad \tau \circ \tau_i(s) = \sigma_{\gamma_s(i)} \circ v(s)
\]

holding for all \( i = 1, \ldots, m \). Here \( \tau : S'' \rightarrow C' \) denotes the morphism induced by \( v : S'' \rightarrow S' \).

**Remark 3.4** Let us analyse what the definition of an \( m/\Gamma \)-pointed stable curve means on the fibres of a curve, i.e. let us consider curves \( f : C \rightarrow S \) for \( S = \text{Spec}(k) \).

(i) Let \((f : C \rightarrow \text{Spec}(k), u : S' \rightarrow \text{Spec}(k), \sigma_1, \ldots, \sigma_m : S' \rightarrow C')\) be a charted \( m/\Gamma \)-pointed stable curve of genus \( g \) over a simple point. Then, by the compatibility condition (***) of definition \( \ref{def:chart} \), the sections \( \sigma_1, \ldots, \sigma_m \) distinguish \( m \) distinct points on \( C \), which are necessarily different from any nodes \( C \) might have.

Let \([i]\) denote the equivalence class of an element \( i \in \{1, \ldots, m\} \) with respect to the action of \( \Gamma \). By the compatibility condition (**), the sections of \( f' : C' \rightarrow S' \) associate to each distinguished point of \( C \) a unique label \([i]\), for some \( i \in \{1, \ldots, m\} \). Formally we define for a distinguished point \( q \in C \)

\[
\text{class}(q) := [i],
\]

if \( q = \tau \circ \sigma_i(s) \) for some \( i \in \{1, \ldots, m\} \) and some \( s \in S' \). In other words, a charted \( m/\Gamma \)-pointed stable curve of genus \( g \) defines an \( m \)-pointed stable curve of genus \( g \), but where the labels of the marked points are only determined up to a permutation in \( \Gamma \). We will make this precise in lemma \( \ref{lem:chart} \) below.

(ii) Let \((f : C \rightarrow S, u'' : S'' \rightarrow S, \tau_1, \ldots, \tau_m : S'' \rightarrow C'')\) be a second charted \( m/\Gamma \)-pointed stable curve with the same underlying curve \( f : C \rightarrow S \), which dominates the first one. The compatibility condition (**) of remark \( \ref{rem:compat} \), together with condition (***) of definition \( \ref{def:chart} \) ensures that the distinguished points on \( C \) are the same in both cases. Even more, the classes of the distinguished points as defined above remain the same.

(iii) The notion of a charted \( m/\Gamma \)-pointed stable curve makes it necessary to specify one étale cover of \( f : C \rightarrow S \) by an \( m \)-pointed stable curve. The definition of an \( m/\Gamma \)-pointed stable curve is independent of such a choice.
Lemma 3.5 There is one-to-one correspondence between \(m/\Gamma\)-pointed stable curves of genus \(g\) over \(\text{Spec}(k)\) and equivalence classes of \(m\)-pointed stable curves of the same genus over \(\text{Spec}(k)\) in the sense of remark \ref{rem:3.1}.

Proof Let \((f : C \to \text{Spec}(k), \sigma_1, \ldots, \sigma_m)\) represent an equivalence class of \(m\)-pointed stable curves with respect to the action of \(\Gamma\). Then \((f : C \to \text{Spec}(k), \text{id}_{\text{Spec}(k)}, \sigma_1, \ldots, \sigma_m)\) is a charted \(m/\Gamma\)-pointed stable curve. If \((f : C \to \text{Spec}(k), \tau_1, \ldots, \tau_m)\) is a different representative of the above class, then by definition there exists a permutation \(\gamma \in \Gamma\), such that \(\sigma_i = \tau_{\gamma(i)}\) for all \(i = 1, \ldots, m\). The disjoint union \(f \coprod f' : C \coprod C' \to \text{Spec}(k) \coprod \text{Spec}(k)\) defines a charted \(m/\Gamma\)-pointed stable curve, dominating both \((f : C \to \text{Spec}(k), \text{id}_{\text{Spec}(k)}, \sigma_1, \ldots, \sigma_m)\) and \((f : C \to \text{Spec}(k), \text{id}_{\text{Spec}(k)}, \tau_1, \ldots, \tau_m)\). Hence to each equivalence class of \(m\)-pointed stable curves there is associated a well-defined \(m/\Gamma\)-pointed stable curve.

Conversely, let an \(m/\Gamma\)-pointed stable curve be given by a representative \((f : C \to \text{Spec}(k), u : S' \to \text{Spec}(k), \sigma_1, \ldots, \sigma_m : S' \to C')\). Choose some closed point \(s \in S'\). Then the fibre \(f'_s : C'_s \to \text{Spec}(k)\) of \(f' : C' \to S'\) over \(s\), together with the sections \(\tau_i := \sigma_i\{s\}\), for \(i = 1, \ldots, m\), defines an \(m\)-pointed stable curve \((f'_s : C'_s \to \text{Spec}(k), \tau_1, \ldots, \tau_m)\). Let \((f''_s : C''_s \to \text{Spec}(k), \tau'_1, \ldots, \tau'_m)\) denote the \(m\)-pointed curve obtained from the fibre over a different point \(s' \in S'\). We clearly have \(C'_{s'} \cong C \cong C'_{s}\). Because of condition (* of definition \ref{def:3.2} there exists a permutation \(\gamma \in \Gamma\), such that \(\tau_i = \tau_{\gamma(i)}\) for all \(i = 1, \ldots, m\). Hence the equivalence class of the \(m\)-pointed stable curve is well defined.

The two constructions are inverse to each other, which concludes the proof of the lemma. \(\square\)

Remark 3.6 Note that the permutations \(\gamma_{s,s'} \in \Gamma\) in definition \ref{def:3.2} are necessarily uniquely determined, and the same is true for the permutations \(\gamma_s\) in remark \ref{rem:3.3}.

Definition 3.7 (i) Let \((f_k : C_k \to S_k, u_k : S'_k \to S_k, \sigma_1^{(k)}, \ldots, \sigma_m^{(k)})\) be two charted \(m/\Gamma\)-pointed stable curves of genus \(g\) for \(k = 1, 2\). A morphism between charted \(m/\Gamma\)-pointed stable curves is a morphism of schemes

\[h : S_1 \to S_2,\]
which induces an isomorphism $C_1 \cong C_2 \times_{S_1} S_1$, and which satisfies the condition (⋄) below.

Put $S''_1 := S'_1 \times_{S_2} S'_2$. Then the natural composed morphism $u'' : S''_1 \to S_1$ is an étale covering of $S_1$, and for $C''_1 := C_1 \times_{S_1} S''_1$ the tuple $(f_1 : C_1 \to S_1, u'' : S''_1 \to S_1, \overline{\sigma}^{(1)}, \ldots, \overline{\sigma}^{(m)} : S''_1 \to C''_1)$ is a charted $m/\Gamma$-pointed stable curve dominating $(f_1 : C_1 \to S_1, u' : S'_1 \to S_1, \sigma^{(1)}, \ldots, \sigma^{(m)} : S'_1 \to C'_1)$. Here $\overline{\sigma}^{(i)}$ denotes the section induced by $\sigma^{(i)}$, for $i = 1, \ldots, m$. There are induced morphisms $\overline{h} : S''_1 \to S'_2$ and $h'' : C''_1 \to C'_2$, such that the diagram

\[
\begin{array}{c}
\begin{array}{ccc}
C''_1 & \xrightarrow{h''} & C'_2 \\
\downarrow & & \downarrow \\
C_1 & \xrightarrow{h} & C_2 \\
\downarrow & & \downarrow \\
S''_1 & \xrightarrow{\overline{h}} & S'_2 \\
\end{array}
\end{array}
\]

commutes, and all quadrangles are Cartesian. We require that there exists a permutation $\gamma \in \Gamma$, such that

\[
(\diamond) \quad h'' \circ \overline{\sigma}^{(i)} = \sigma^{(2)}_{\gamma(i)} \circ \overline{h}
\]

holds for all $i = 1, \ldots, m$.

(ii) A morphism between $m/\Gamma$-pointed stable curves is an equivalence class of morphisms between charted $m/\Gamma$-pointed stable curves.

**Remark 3.8** (i) If $h : S_1 \to S_2$ is a morphism between a pair of charted $m/\Gamma$-pointed stable curves, then it is also a morphism for any other pair equivalent to it. This can easily be seen by drawing the appropriate extensions of the above commutative diagram. In particular, the notion of morphisms between $m/\Gamma$-pointed stable curves is well-defined.

(ii) A morphism $h : S_1 \to S_2$ between $m/\Gamma$-pointed stable curves is an isomorphism, if and only if it is an isomorphism of schemes $S_1 \cong S_2$. Note however that not any isomorphism of schemes defines an isomorphism of
$m/\Gamma$-pointed stable curves, it might not even define a morphism between them.

(iii) Two $m/\Gamma$-pointed stable curves over the same underlying curve $f : C \to S$ are isomorphic if and only if they admit étale coverings by $m$-pointed stable curves, which are equivalent up to a permutation $\gamma$ of the labels of their sections, with $\gamma$ contained in $\Gamma$, compare remark 3.1.

**Remark 3.9** A morphism $h : S_1 \to S_2$ is a morphism of $m/\Gamma$-pointed stable curves if and only if there is a Cartesian diagram

$$
\begin{array}{ccc}
C_1 & \xrightarrow{h'} & C_2 \\
\downarrow & & \downarrow \\
S_1 & \xrightarrow{h} & S_2
\end{array}
$$

such that for all closed points $s \in S_1$ and all distinguished points $q \in C_{1,s}$ in the fibre over $s$ holds

$$\text{class}(q) = \text{class}(h'(q)).$$

See remark 3.4 for the notation.

**Remark 3.10** Recall that the symmetric group $\Sigma_m$ acts naturally on the set of $m$-pointed stable curves by permutation of the labels of the marked points. Since the $m$ sections of marked points are disjoint, the action of $\Sigma_m$, and of any subgroup $\Gamma \subset \Sigma_m$, on $\overline{M}_{g,n,m}$ is free.

**Remark 3.11** A subgroup $\Gamma$ of $\Sigma_m$ acts on $\overline{M}_{g,m}$. The quotient

$$\overline{M}_{g,m}/\Gamma := \overline{M}_{g,m}/\Gamma$$

is a coarse moduli space for $m/\Gamma$-pointed curves. Let $f : C \to S$ be an $m/\Gamma$-pointed stable curve of genus $g$. By lemma 3.5, the closed points of $\overline{M}_{g,m}/\Gamma$ are in one-to-one correspondence with isomorphism classes of $m/\Gamma$-pointed stable curves of genus $g$ over $\text{Spec}(k)$.

Let $(f : C \to S, u : S' \to S, \sigma_1, \ldots, \sigma_m : S' \to C')$ be a charted $m/\Gamma$-pointed stable curve of genus $g$. Since $f' : C' \to S'$ is an $m$-pointed stable
curve, there is an induced morphism \( \vartheta' : S' \to \overline{M}_{g,m} \), and by composition a morphism \( \vartheta_{f'} : S' \to \overline{M}_{g,m}/\Gamma \). By definition, for any closed point \( s \in S \), and for any pair of points \( s', s'' \in S'' \) with \( u(s') = u(s'') = s \), holds the identity \( \vartheta_{f'}(s') = \vartheta_{f'}(s'') \). Thus there is an induced morphism \( \vartheta : S \to \overline{M}_{g,m}/\Gamma \), such that \( \vartheta \circ u = \vartheta_{f'} \). Clearly any other charted \( m/\Gamma \)-pointed stable curve dominating this one induces the same morphism from \( S \) into \( \overline{M}_{g,m}/\Gamma \). Hence there is a well defined morphism \( \vartheta_f : S \to \overline{M}_{g,m}/\Gamma \), for any \( m/\Gamma \)-pointed stable curve \( f : C \to S \). Furthermore, by referring back to \( \overline{M}_{g,m} \) again, one easily sees that \( \overline{M}_{g,m}/\Gamma \) dominates any other scheme with this universal property, so it is indeed a coarse moduli space for \( m/\Gamma \)-pointed stable curves.

**Remark 3.12** If \( \Gamma = \{ \text{id} \} \) is the trivial subgroup of \( \Sigma_m \), then the glueing condition \((*)\) for \( m/\Gamma \)-pointed stable curves implies that there are \( m \) induced global sections of marked points of \( f : C \to S \). Thus in this case an \( m/\Gamma \)-pointed stable curve is just an \( m \)-pointed stable curve in the classical sense.

**Remark 3.13** Let \( \Gamma \subset \Sigma_m \) be a subgroup. There is a free action of \( \Gamma \) on \( \overline{H}_{g,n,m} \). The quotient shall be denoted by

\[ \overline{\mathcal{P}}_{g,n,m}/\Gamma := \overline{\mathcal{P}}_{g,n,m}/\Gamma. \]

Let \( u_{g,n,m} : \mathcal{C}_{g,n,m} \to \overline{H}_{g,n,m} \) denote the universal curve over \( \overline{H}_{g,n,m} \). Note that fibres of \( u_{g,n,m} \) over such points of \( \overline{\mathcal{P}}_{g,n,m} \), which correspond to each other under the action of \( \Gamma \), are identical as embedded curves in \( \mathbb{P}^N \), with the same distinguished points on them. The only difference is in the labels of the marked points, which are permuted by elements of \( \Gamma \). Thus the action of \( \Gamma \) on \( \overline{\mathcal{P}}_{g,n,m} \) extends to an action on \( \mathcal{C}_{g,n,m}/\Gamma \), by identifying fibres over corresponding points. The quotient

\[ u_{g,n,m}/\Gamma : \mathcal{C}_{g,n,m}/\Gamma := \mathcal{C}_{g,n,m}/\Gamma \to \overline{\mathcal{P}}_{g,n,m}/\Gamma = \overline{\mathcal{P}}_{g,n,m}/\Gamma \]

exists as an \( m/\Gamma \)-pointed stable curve over \( \overline{H}_{g,n,m}/\Gamma \). The original universal curve \( u_{g,n,m} : \mathcal{C}_{g,n,m} \to \overline{\mathcal{P}}_{g,n,m} \), together with the étale morphism \( \overline{\mathcal{P}}_{g,n,m} \to \overline{\mathcal{P}}_{g,n,m}/\Gamma \) defines a charted \( m/\Gamma \)-pointed stable curve representing it.

**3.14** Let \( f : C \to S \) be an \( m/\Gamma \)-pointed stable curve of genus \( g \), which is given together with an embedding of the underlying curve \( f : C \to S \) into
For any étale covering $u : S' \to S$, there is an induced embedding of $f' : C' = C \times_S S' \to S'$ into $\text{pr}_2 : \mathbb{P}^N \times S' \to S'$. Hence, for a representing charted $m/\Gamma$-pointed curve, the curve $f' : S' \to S$ is an embedded $m$-pointed stable curve, and thus induces a morphism

$$\vartheta_f : S' \to \overline{\mathcal{M}}_{g,n,m}.$$ 

For any closed point $s \in S$, the embedding of the fibre $C'_s$ into $\mathbb{P}^N$ is the same for all points $s' \in S'$ with $u(s') = s$, as it is nothing else but the embedding of $C_s$ into $\mathbb{P}^N$. Note that the points in $\mathcal{M}_{g,n,m}$ representing different fibres $C'_s$ need not be the same, as the labels of the marked points may differ by a permutation in $\Gamma$. However, the morphism $\vartheta_f : S' \to \mathcal{M}_{g,n,m}/\Gamma$ obtained by composition with the quotient map, factors through a morphism $\vartheta_f : S \to \mathcal{M}_{g,n,m}/\Gamma$. This morphism is even independent of the chosen charted $m/\Gamma$-pointed stable curve representing $f : C \to S$.

By the universal property of the Hilbert scheme, there is an isomorphism of $m$-pointed stable curves between $\vartheta_f^* \mathcal{C}_{g,n,m}$ and $C'$ over $S'$. As a scheme over $S'$, the curve $\vartheta_f^* \mathcal{C}_{g,n,m}$ is isomorphic to $\overline{\mathcal{C}}_{g,n,m}/\Gamma$, even though the latter has no natural structure as an $m$-pointed curve. Therefore, by the definition of $C' = u^* C$, and since $\vartheta_f^* = \vartheta_f \circ u$, one can construct an isomorphism of schemes over $S$ between $C$ and $\vartheta_f^* \mathcal{C}_{g,n,m}/\Gamma$. Together with the isomorphisms between the covering $m$-pointed stable curves, this shows that $f : C \to S$ is isomorphic to $\vartheta_f^* \mathcal{C}_{g,n,m}/\Gamma \to S$ as an $m/\Gamma$-pointed stable curve.

In other words, $\mathcal{M}_{g,n,m}/\Gamma$ is in fact a fine moduli space for embedded $m/\Gamma$-pointed stable curves of genus $g$, with universal curve $u_{g,n,m}/\Gamma : \mathcal{C}_{g,n,m}/\Gamma \to \overline{\mathcal{M}}_{g,n,m}/\Gamma$.

**Remark 3.15** The action of $\text{PGL}(N + 1)$ on $\mathcal{M}_{g,n,m}$ induces an action of $\text{PGL}(N + 1)$ on $\overline{\mathcal{M}}_{g,n,m}/\Gamma$. Indeed, by the construction of the universal embedded curve over $\overline{\mathcal{M}}_{g,n,m}$, the embedding of one of its fibres, considered as an $m$-pointed stable curve of genus $g$, does not depend on the ordering of the labels of its marked points. So the action of $\text{PGL}(N + 1)$ on $\mathcal{M}_{g,n,m}$ commutes with the action of any subgroup $\Gamma \subset \Sigma_m$ on $\overline{\mathcal{M}}_{g,n,m}$. The coarse moduli space $\overline{\mathcal{M}}_{g,m}/\Gamma$ is a GIT-quotient of $\overline{\mathcal{M}}_{g,n,m}/\Gamma$ by the action of $\text{PGL}(N + 1)$.

**Definition 3.16** The moduli stack $\mathcal{M}_{g,m}/\Gamma$ of $m/\Gamma$-pointed stable curves of genus $g$ is the stack defined as the category fibred in groupoids over the
category of schemes, where for a scheme $S$ the objects in the fibre category $\mathcal{M}_{g,m/\Gamma}(S)$ are the $m/\Gamma$-pointed stable curves of genus $g$ over $S$. Morphisms in $\mathcal{M}_{g,m/\Gamma}$ are morphisms between $m/\Gamma$-pointed stable curves.

**Proposition 3.17** There is an isomorphism of stacks

$$\mathcal{M}_{g,m/\Gamma} \cong \left[ \mathcal{H}_{g,n,m/\Gamma} / \text{PGL}(N + 1) \right].$$

**Remark 3.18** Note that by the general theory of quotient stacks, and the fact that the action of $\Gamma$ on $\mathcal{H}_{g,n,m}$ is free, there is an isomorphism of stacks

$$\left[ \mathcal{H}_{g,n,m/\Gamma} / \text{PGL}(N + 1) \right] \cong \left[ \mathcal{H}_{g,n,m} / \Gamma \times \text{PGL}(N + 1) \right].$$

In particular, the quotient stack $\mathcal{M}_{g,m/\Gamma}$ can be viewed as a stack quotient of the moduli stack $\mathcal{M}_{g,m}$ with respect to the group $\Gamma$.

**Proof of the proposition.** The proof is analogous to that of proposition 2.9, so we may be brief here, and concentrate on those parts of the proof, which need to be adapted.

(i) Let $f : C \to S \in \text{Ob}(\mathcal{M}_{g,m/\Gamma})$ be an $m/\Gamma$-pointed stable curve. For each étale cover $u : S' \to S$, the $m$ disjoint sections $\sigma_1, \ldots , \sigma_m : S' \to C'$ define hypersurfaces $S_1, \ldots , S_m$ in $C'$. Their images in $C'$ define a divisor in $C$, which will be denoted by $S_m/\Gamma$. Note that this divisor is independent of the chosen étale cover.

Let $p : E \to S$ be the principal PGL($N + 1$)-bundle associated to the projective bundle $\mathbb{P}_{f*}(\omega_{C/S}(S_m/\Gamma))^\otimes n$. Let $E' := E \times_S S'$ be the pullback of $E$ to some étale covering $u : S' \to S$ of $S$. In the same way as in the proof of proposition 2.9, there is a natural embedding of the $m$-pointed stable curve $C' \times_S E' \to E'$, and this induces a morphism

$$\phi' : E' \to \mathcal{H}_{g,n,m}.$$
For all closed points \( e, e' \in E' \) which project to the same point in \( E \), we have \( \overline{\phi}(e) = \overline{\phi}(e') \). Therefore \( \overline{\phi} \) factors through a morphism \( \phi : E \to \overline{H}_{g,n,m/\Gamma} \), which is also \( \text{PGL}(N+1) \)-equivariant. Thus we obtain an object \( (E, p, \phi) \in \overline{\mathcal{M}}_{g,m/\Gamma} / \text{PGL}(N+1)(S) \).

The construction of the functor from \( \overline{\mathcal{M}}_{g,m/\Gamma} \) to \( \overline{H}_{g,n,m/\Gamma} / \text{PGL}(N+1) \) on morphisms is straightforward.

(ii) Conversely, consider a triple \( (E, p, \phi) \in \overline{H}_{g,n,m/\Gamma} / \text{PGL}(N+1)(S) \) for some scheme \( S \). The morphism \( \phi : E \to \overline{H}_{g,n,m/\Gamma} \) determines an embedded \( m/\Gamma \)-pointed stable curve \( f' : C' \to E \) of genus \( g \), together with an isomorphism between \( C' \) and the pullback of the universal curve \( \mathcal{C}_{g,n,m/\Gamma} \) as schemes over \( E \). The quotient \( f : C := C'/\text{PGL}(N+1) \to E/\text{PGL}(N+1) \cong S \) exists, and it is by construction an \( m/\Gamma \)-pointed stable curve.

One verifies that the two functors, whose construction we outlined above, are inverse to each other as morphisms of stacks.\( \square \)

**Remark 3.19** From proposition 3.17 it follows in particular that \( \overline{\mathcal{M}}_{g,m/\Gamma} \) is a smooth Deligne-Mumford stack, with \( \overline{\mathcal{M}}_{g,m/\Gamma} \) as its moduli space. There is a canonical commutative diagram

\[
\begin{array}{ccc}
\overline{\mathcal{M}}_{g,m} & \longrightarrow & \overline{\mathcal{M}}_{g,m/\Gamma} \\
\downarrow \cong & & \downarrow \cong \\
[\overline{H}_{g,n,m}/\text{PGL}(N+1)] & \longrightarrow & [\overline{H}_{g,n,m}/\Gamma \times \text{PGL}(N+1)] \\
\downarrow & & \downarrow \\
\overline{\mathcal{M}}_{g,m} & \longrightarrow & \overline{\mathcal{M}}_{g,m/\Gamma}.
\end{array}
\]

All horizontal arrows represent finite morphisms of degree equal to the order of \( \Gamma \). The morphisms between the stacks are unramified, while in general the morphisms between the moduli spaces are not.
Remark 3.20 Note that for everything we said above analogous statements hold true if we replace $\mathcal{H}_{g,n,m}$ and $\mathcal{M}_{g,m}$ by the open subschemes $H_{g,n,m}$ and $M_{g,m}$, as well as $\mathcal{H}_{g,n,m}/\Gamma$ and $\mathcal{M}_{g,m}/\Gamma$ by the open subschemes $H_{g,n,m}/\Gamma$ and $M_{g,m}/\Gamma$, respectively.

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