Trimaximal TM$_1$ neutrino mixing in $S_4$
with spontaneous CP violation

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Abstract

The measurement of the reactor angle by the Daya Bay and RENO experiments in
2012 has ruled out the tri-bimaximal paradigm. Adopting an $S_4$ family symmetry,
we propose direct models of the trimaximal type TM$_1$ in which the tri-bimaximal
Klein symmetry of the neutrino sector is broken to a residual $Z_2$ symmetry. In such
a scenario, the solar mixing angle is decreased compared to its tri-bimaximal value
by about $1^\circ$, thus bringing it in excellent agreement with experimental observation.
The atmospheric mixing angle, on the other hand, depends on the CP violating Dirac
phase $\delta$. Imposing CP conservation in the family symmetry limit, we show how to
break the CP symmetry via flavon VEVs with well-defined complex phases, so that
sizable deviations of the atmospheric angle from maximal mixing, consistent with the
latest global fits, are produced.

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1 Introduction

In spring 2012, the Daya Bay [1] and RENO [2] experiments independently measured the reactor angle $\theta_{13}$, the smallest mixing angle of the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix, to be around $9^\circ$. This remarkable discovery came as a surprise to many physicists who shared a certain degree of prejudice for a smaller angle (as it had happened previously with the solar mixing angle $\theta_{12}$). The paradigm of tri-bimaximal mixing [3–6], which had undeniably dominated the flavour model building landscape, was suddenly overthrown by experimental facts. With it, the whole idea of an underlying family symmetry governing the pattern of fermion masses and mixings was called into question. While neutrino mixing anarchy [7–9] can qualitatively explain large mixing angles, it fails to make quantitative and testable predictions. On the other hand, the family symmetry approach [10–12] – modified to accommodate sizable $\theta_{13}$ [13] – allows to construct predictive models which can be falsified by future experimental data.

In general, non-Abelian family symmetries with triplet representations allow to unify the three chiral families. In order to realistically describe the structure of their masses and mixings, the family symmetry needs to be broken. This is typically achieved by means of Standard Model neutral flavon fields which acquire vacuum expectation values (VEVs) in appropriate directions in flavour space. Depending on these flavon alignments, the family symmetry approach gives rise to particular neutrino mixing patterns in either a direct or an indirect way [14]. In direct models, the flavons appearing in the neutrino sector have to break the underlying family symmetry $\mathcal{G}$ down to the residual $Z_2 \times Z_2$ Klein symmetry of the (Majorana) neutrino mass matrix which is associated with the desired mixing pattern. In indirect models, the flavon VEVs break $\mathcal{G}$ completely, and the special structure of the neutrino mass matrix arises in the context of the type I seesaw [15–18] with sequential dominance [19–23] from the quadratic appearance of the flavon fields in the neutrino Lagrangian or, in some rare cases, accidentally from a combination of the flavon alignments and the group’s Clebsch-Gordan coefficients [24]. Tri-bimaximal mixing can be obtained in the context of constrained sequential dominance [25] where the required flavon alignments can be readily derived from an underlying non-Abelian family symmetry as e.g. $\Delta(27)$ [26], $Z_7 \times Z_3$ [27] and $A_4$ [28–30]. With tri-bimaximal mixing being ruled out by the measurement of $\theta_{13} \approx 9^\circ$, “non-standard” and somewhat more complicated flavon alignments have to be considered, leading to new and predictive versions of constrained sequential dominance [31–34].

In direct models, there are in principle two possible ways of generating a mixing pattern which deviates from tri-bimaximal mixing. The first is solely based on symmetry arguments and requires to consider larger symmetry groups which contain a $Z_2 \times Z_2$ Klein symmetry different from the tri-bimaximal one [35–43]. In this paper, however, we will pursue the second option where a tri-bimaximal setup is augmented by an additional ingredient which breaks the tri-bimaximal structure in a well-defined and controlled way.

As was shown in [44,49], the natural symmetry of tri-bimaximal mixing is the permutation group $S_4$, or any group containing it as a subgroup (e.g. $PSL_2(7)$ [50,52]). Starting 1A related approach adopts non-Abelian groups which contain only half the Klein symmetry of the neutrino sector [44,47].
Table 1: The matrix representation of the $S_4$ generators $S$, $U$ and $T$ for the five irreducible representations in the basis with diagonal $T$. Here $\omega = e^{2\pi i/3}$.

| $S_4$ | $S$ | $U$ | $T$ |
|-------|-----|-----|-----|
| $1, 1'$ | 1 | $\pm 1$ | 1 |
| $2$ | \[
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\] | \[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
\omega & 0 \\
0 & \omega^2
\end{pmatrix}
\] |
| $3, 3'$ | \[
\frac{1}{3} \begin{pmatrix}
-1 & 2 & 2 \\
2 & -1 & 2 \\
2 & 2 & -1
\end{pmatrix}
\] $\mp$ \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\] | \[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \omega^2 & 0 \\
0 & 0 & \omega
\end{pmatrix}
\] |

with the family symmetry $S_4$, it is well known how to construct models of tri-bimaximal neutrino mixing, see e.g. [53–56]. In these models, the flavon fields appearing in the neutrino sector break $S_4$ down to the tri-bimaximal Klein symmetry $Z_2^S \times Z_2^U$, generated by the order-two elements $S$ and $U$. The flavons of the charged lepton sector, on the other hand, break $S_4$ to a residual $Z_3^T$ symmetry, corresponding to the order-three element $T$. Reversely, the three elements $S$, $U$ and $T$ generate the group $S_4$. Their explicit matrix form depends on the basis chosen for the five irreducible representations. It is convenient to work in a basis with diagonal $T$ generator as this automatically yields a diagonal charged lepton mass matrix. Our choice of basis is summarised in Table 1 and corresponds to the basis used e.g. in [57]. In this basis, the Clebsch-Gordan coefficients are real, and we refer the reader to Appendix A of [57] for their explicit values.

A simple way of generating deviations from tri-bimaximal mixing is provided by adding at least one extra term in the neutrino sector which does not share the $Z_2^S \times Z_2^U$ Klein symmetry, see also [58]. However, in order to retain some degree of predictivity, we consider only cases with a residual $Z_2$ symmetry. Out of the three possible cases, one ($Z_2^U$) forces $\theta_{13}$ to vanish, while the other two ($Z_2^S$ and $Z_2^{SU}$) allow to dial a sizable reactor angle. Direct models featuring a remnant $Z_2^S$ symmetry have been studied extensively in the literature, e.g. [57,59–65], and arise straightforwardly in models with an $A_4$ (obtained from $S_4$ by dropping the $U$ generator [66,67]) family symmetry. They are known to lead to the trimaximal TM$_2$ [68,69] neutrino mixing pattern in which the solar angle $\theta_{12}$ retains its tri-bimaximal value to first approximation, while second order corrections lead to a slightly larger angle. On the other hand, direct models with a remnant $Z_2^{SU}$ symmetry have not received a great deal of attention [70,71], despite the fact that the predicted solar angle shows better agreement with data. This case leads to the trimaximal TM$_1$ [68,69,72] neutrino mixing pattern which is characterised by second order corrections to $\theta_{12}$ yielding a slightly smaller solar angle compared to the tri-bimaximal case. In this paper, we propose direct models of trimaximal TM$_1$ mixing which automatically predict a solar angle in excellent agreement with the data and allow to fit the reactor angle to its measured value of about $9^\circ$.

Due to the breaking of the neutrino Klein symmetry to a remnant $Z_2$ in the neutrino
sector, correlations between the mixing parameters of the PMNS matrix ensue. In the physically interesting cases of a residual $Z^S_2$ or $Z^SU_2$ symmetry, the linearised version of these correlations are known as atmospheric mixing sum rules [13] and involve the CP violating Dirac phase $\delta$. The fact that these sum rule predictions involve a CP violating phase motivates us to construct models which are able to make statements about the CP structure of the theory. To this end, it is natural to impose CP conservation at energies above the family symmetry breaking scale. The CP symmetry only gets broken spontaneously by flavon VEVs which pick up specified complex phases. As was discussed recently, it is generally non-trivial to define a CP transformation consistently in the presence of a non-Abelian family symmetry [73,74]. It often requires what is called a generalised (or, more appropriately, general) CP transformation [75–81]. However, the situation is rather simple in the case of an $S_4$ family symmetry formulated in the basis of Table 1, where the general CP transformation maps a field $\psi$ to $[82,83]$

$$\psi(t,x) \xrightarrow{\text{CP}} \rho(g) \psi^*(t,-x).$$

(1.1)

$\rho(g)$ denotes the unitary matrix representation of an $S_4$ element $g$, and the obvious action of CP on the possible spinor indices has been suppressed. In particular we see that the naive CP transformation with $\rho(1) = 1$ is allowed in this case. As a consequence, all coupling constants are real in an $S_4$ model with imposed CP symmetry. Then complex phases and with it CP violation in the Yukawa couplings can only arise from the phase structure of the flavon VEVs. As the CP phase feeds into the correlations of the mixing parameters caused by the residual $Z^SU_2$ symmetry of the neutrino sector, the atmospheric mixing angle $\theta_{23}$ will be a function of the complex flavon VEVs. Taking the hint for deviations from maximal $\theta_{23}$ of the order of approximately $5^\circ$ seriously [84–86], we construct the first direct models of trimaximal TM$_1$ mixing with imposed CP symmetry.

The paper is organised as follows. In Section 2 we revisit all three possibilities of breaking the tri-bimaximal Klein symmetry down to a residual $Z_2$. The available flavon vacuum configurations are collected and the resulting predictions for $\theta_{12}$ and $\theta_{23}$ are given. Section 3 together with Appendix A, discusses the trimaximal case TM$_1$ in detail. Requirements on the phase structure of the input parameters (i.e. flavon VEVs) are identified and the possible neutrino mass spectra are presented. The neutrino phenomenology of models based on the type II [87–90] as well as the type I seesaw [15–18] is scrutinised in Section 4. The derivation of the flavon VEV configurations, together with their phase structure is given in Section 5. Finally, we conclude in Section 6.

2 Residual $Z_2$ symmetries from neutrino flavon VEVs

In the tri-bimaximal limit, the neutrino mass matrix arises from the structure

$$\nu\nu(\alpha_1 \phi_1 + \alpha_2 \phi_2 + \alpha_3 \phi_3^*),$$

(2.1)

where the neutrinos $\nu$ transform in the triplet representation $3$ of $S_4$. The coupling constants $\alpha_r$ parameterise their interaction with the flavons $\phi_r$ (living in the representation $r$
of $S_4$), whose VEVs are aligned as

$$\langle \phi_1 \rangle = \varphi_1, \quad \langle \phi_2 \rangle = \varphi_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \langle \phi_3' \rangle = \varphi_3' \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \quad (2.2)$$

The resulting mass matrix reads

$$M_{TB} = x_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + x_3' \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad (2.3)$$

where $x_r = \alpha_r \varphi_r$. This is most general matrix symmetric under the triplet matrix representation of $S$ and $U$ as given in Table 1 and as a consequence it is diagonalised by the tri-bimaximal mixing matrix

$$U_{TB} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \quad (2.4)$$

Adding a term which breaks $Z^S_2 \times Z^U_2$ to a residual $Z_2$ symmetry, adds one new (complex) degree of freedom to the neutrino mass matrix,

$$M = M_{TB} + \Delta M. \quad (2.5)$$

It is straightforward to work out the form of $\Delta M$ in each of the three cases. For residual $Z_2$ symmetries generated by $U$, $S$ and $SU$, we obtain

$$\Delta M_U = y \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \Delta M_S = y \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad \Delta M_{SU} = y \begin{pmatrix} 0 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & -2 \end{pmatrix}, \quad (2.6)$$

respectively. Here $y$ denotes the magnitude of the Klein symmetry breaking contribution to the neutrino mass matrix. Clearly, there is some ambiguity in defining the explicit form of $\Delta M$ as any linear combination with the $S$ and $U$ preserving tri-bimaximal mass matrices of Eq. (2.3) yields a structure with the same unbroken $Z_2$ symmetry. We have chosen the form of $\Delta M$ given in Eq. (2.6) so that the coefficient $y$ is directly related to the term arising from coupling $\nu \nu$ to one of the Klein symmetry breaking flavon fields of Table 2 which however respect one of the three possible $Z_2$ subgroups generated by $U$, $S$ and $SU$, respectively. Again, the VEV alignments of Table 2 are ambiguous in the case of the representations 2 and 3’ for which there exist vacuum alignments which respect both $S$ and $U$, so that any linear combination of such alignments would have identical symmetry properties. For instance, in the case of the $S$ preserving doublet alignment of Table 2 actually any vacuum alignment that is different from $(1,1)^T$ breaks $Z^S_2 \times Z^U_2$ down to $Z^S_2$.

Neutrino mass matrices which are symmetric under a residual $Z_2$ generated by $U$, $S$ or $SU$ have the useful property that they are diagonalised by a unitary matrix which shares
residual
symmetry | $U$  | $S$  | $SU$
---|---|---|---
1  | −  | −  | −  
1′ | −  | 1  | −  
2  | −  | $(1, -1)^T$ | −  
3  | $(0, 1, -1)^T$ | $(1, 1, 1)^T$ | $(2, -1, -1)^T$ 
3′ | $(1, 0, 0)^T$ | −  | $(0, 1, -1)^T$

Table 2: All available vacuum configurations which break the $Z_2^S \times Z_2^U$ Klein symmetry of the neutrino sector to the residual $Z_2$ symmetries generated by $U$, $S$ and $SU$, respectively.

one of the columns of the tri-bimaximal mixing matrix given in Eq. (2.4). For unbroken $U$, $S$ and $SU$, these are the third, second and first columns, respectively. This can be seen by realising that any $U$ symmetric mass matrix has an eigenvector $(0, 1, -1)^T$, while $(1, 1, 1)^T$ and $(2, -1, -1)^T$ are eigenvectors of mass matrices which are symmetric under $S$ and $SU$, respectively. Alternatively, one can apply the tri-bimaximal mixing matrix $U_{TB}$ on $M$ of Eq. (2.5),

$$M' = U_{TB}^T (M_{TB} + \Delta M) U_{TB} = M_{TB}^{\text{diag}} + \Delta M', \quad (2.7)$$

with

$$M_{TB}^{\text{diag}} = U_{TB}^T M_{TB} U_{TB} = \begin{pmatrix} x_1 - x_2 + 3x_3' & 0 & 0 \\ 0 & x_1 + 2x_2 & 0 \\ 0 & 0 & -x_1 + x_2 + 3x_3' \end{pmatrix}, \quad (2.8)$$

and

$$\Delta M'_U = y \begin{pmatrix} 1 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Delta M'_S = \sqrt{3} y \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \Delta M'_{SU} = \sqrt{6} y \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (2.9)$$

This shows that the full mixing matrix which diagonalises $M$ has the form $U_{TB} U_{ij}$, where the second factor denotes a unitary transformation involving only generations $i$ and $j$, hence leaving one column of the tri-bimaximal mixing matrix unchanged.

The fact that one column of the mixing matrix is exactly known, allows to formulate two predictions in each case.

- $U$ symmetry. In this case, the third column of the mixing matrix takes the form $(0, 1, -1)/\sqrt{2}$. Adopting the PDG parameterisation, we immediately find

$$\theta_{13} = 0^\circ, \quad \theta_{23} = 45^\circ, \quad (2.10)$$

while the solar mixing angle $\theta_{12}$ remains undetermined.
• S symmetry. This is the trimaximal case TM$_2$ \[68][69], where the second column of the mixing matrix takes the form (1, 1, 1)/\sqrt{3}, and we obtain the relation $\sin \theta_{12} \cos \theta_{13} = \frac{1}{\sqrt{3}}$. To first order in $\theta_{13}$, the resulting solar angle takes the tri-bimaximal value of about $35.3^\circ$. However, the second order correction yields a shift to slightly larger angles. Inserting the measured values of $\theta_{13} \approx 9^\circ$, we numerically find

$$\theta_{12} \approx 35.8^\circ,$$  \hspace{1cm} (2.11)

which lies outside of the $1\sigma$ allowed regions of all three global fits to three neutrino mixing \[84–86\]. In fact, this value is only barely consistent at the $3\sigma$ level according to \[86\]. Concerning the atmospheric angle, it has been shown that, to first order in $\theta_{13}$, the following mixing sum rule holds \[13\]

$$\theta_{23} \approx 45^\circ - \frac{1}{\sqrt{2}} \theta_{13} \cos \delta,$$ \hspace{1cm} (2.12)

which involves the CP violating Dirac phase $\delta$. Deviations from maximal mixing of the order of $6^\circ$ can be obtained in the case of CP conservation, that is if $\delta = 0, \pi$.

• SU symmetry. This is the trimaximal case TM$_1$ \[68][69][72], where the first column of the mixing matrix is of the form (2, $-1$, $-1$)/\sqrt{6}, and we obtain the relation

$$\cos \theta_{12} \cos \theta_{13} = \sqrt{\frac{2}{3}}.$$ 

To first order in $\theta_{13}$, the resulting solar angle again retains its tri-bimaximal value. However, this time, the second order correction shifts its value to slightly smaller angles. Numerically, using $\theta_{13} \approx 9^\circ$, we get

$$\theta_{12} \approx 34.2^\circ,$$ \hspace{1cm} (2.13)

which falls inside the $1\sigma$ allowed regions of \[84–86\] with the exception of the “free flux” fit in \[86\], where the $1\sigma$ region is just narrowly missed. Similar to the case with conserved S symmetry, the atmospheric angle satisfies a mixing sum rule which now reads \[13\]

$$\theta_{23} \approx 45^\circ + \sqrt{\frac{2}{3}} \theta_{13} \cos \delta.$$ \hspace{1cm} (2.14)

In order to generate deviations from maximal mixing which are of the order of $6^\circ$ a non-trivial CP phase is required, with $|\cos \delta| \approx 0.5$ and therefore $\delta \approx \pm 120^\circ$ for solutions in the first $\theta_{23}$ octant and $\delta \approx \pm 60^\circ$ for solutions in the second octant. It is interesting to note that these phases are identical to the phases of $\omega = e^{2\pi i/3}$ and $\omega^2$ for the first octant, while the phases required for solutions in the second octant are identical to those of $-\omega^2$ and $-\omega$.

3 Trimaximal TM$_1$ mixing and CP phases

We have seen in the previous section that the most general $Z^S_{3\text{SU}}$ invariant mass matrix is diagonalised by a tri-bimaximal mixing matrix followed by an additional unitary 2-3 transformation,

$$U_{23} = \begin{pmatrix} 1 & 0 \\ 0 & u_{23} \end{pmatrix},$$ \hspace{1cm} (3.1)
where we parameterise the 2-3 mixing by \( \kappa, \vartheta \in \mathbb{R} \),
\[
 u_{23} = \begin{pmatrix} e^{-i\kappa} & 0 \\ 0 & e^{i\kappa} \end{pmatrix} \begin{pmatrix} \cos \vartheta & \sin \vartheta \\ -\sin \vartheta & \cos \vartheta \end{pmatrix}. \tag{3.2}
\]

Then the complete matrix diagonalising \( M_{SU} = M_{TB} + \Delta M_{SU} \), see Eqs. (2.3,2.6), takes the form
\[
 U_{\nu} = U_{TB} U_{23} = \begin{pmatrix} \frac{2}{\sqrt{6}} & \cos \vartheta e^{-i\kappa} \sqrt{\frac{2}{3}} & \sin \vartheta e^{-i\kappa} \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{6}} \cos \vartheta e^{-i\kappa} \sqrt{\frac{2}{3}} & -\sin \vartheta e^{-i\kappa} \sqrt{\frac{2}{3}} & \sin \vartheta e^{-i\kappa} \sqrt{\frac{2}{3}} \\ \frac{1}{\sqrt{6}} \cos \vartheta e^{-i\kappa} \sqrt{\frac{2}{3}} & \sin \vartheta e^{-i\kappa} \sqrt{\frac{2}{3}} & -\cos \vartheta e^{-i\kappa} \sqrt{\frac{2}{3}} \end{pmatrix}. \tag{3.3}
\]

The reactor angle \( \theta_{13} \) is obtained from the 1-3 entry. As the parameter \( \vartheta \) can be both positive and negative, we write
\[
 \sin \theta_{13} = \frac{\text{sign} \vartheta}{\sqrt{2}} \sin \vartheta, \tag{3.4}
\]

where \( \text{sign} \vartheta \) denotes the sign of \( \vartheta \). The atmospheric angle is similarly given by the ratio of the 2-3 and 3-3 entries of \( U_{\nu} \),
\[
 \tan \theta_{23} = \left| \frac{\cos \vartheta + \sqrt{\frac{2}{3}} e^{-2i\kappa} \sin \vartheta}{\cos \vartheta - \sqrt{\frac{2}{3}} e^{-2i\kappa} \sin \vartheta} \right|. \tag{3.5}
\]

This relation can be easily solved for \( \theta_{23} \), leading to the expansion
\[
 \theta_{23} = 45^\circ + \sqrt{\frac{2}{3}} \vartheta \cos(2\kappa) + \mathcal{O}(\vartheta^3). \tag{3.6}
\]

It is worth emphasising that this simple expression is correct up to second order in \( \vartheta \). Note, however, that \( 2\kappa \) is not identical to the physical CP violating oscillation phase \( \delta \). Therefore, the atmospheric sum rule of Eq. (2.14) only holds up to linear order in the reactor angle. With \( \theta_{13} \approx 9^\circ \), Eq. (3.6) can be written as
\[
 \theta_{23} \approx 45^\circ + 12.8^\circ \cdot \text{sign} \vartheta \cos(2\kappa). \tag{3.7}
\]

Deviations from maximal atmospheric mixing of the order of about \( 6^\circ \) are possible if
\[
 2\kappa \approx \pm 120^\circ, \quad \text{or} \quad 2\kappa \approx \pm 60^\circ. \tag{3.8}
\]

This intriguing observation motivates us to relate the phase \( 2\kappa \) to flavon fields which acquire complex VEVs with phases \( \omega^0 \) or \( \omega^1 \) or \( \omega^2 \).

In a first step, we investigate how the phase \( 2\kappa \) arises from the input parameters \( x_1, x_2, x_3', \text{ and } y \) of the most general \( Z_2^{SU} \) invariant neutrino mass matrix \( M_{SU} = M_{TB} + \Delta M_{SU} \). To this end, we apply a tri-bimaximal rotation to \( M_{SU} \) yielding, see Eqs. (2.8,2.9)
\[
 M'_{SU} = \begin{pmatrix} x_1 - x_2 + 3x_3' & 0 & 0 \\ 0 & x_1 + 2x_2 & \sqrt{6}y \\ 0 & \sqrt{6}y & -x_1 + x_2 + 3x_3' \end{pmatrix}. \tag{3.9}
\]
In order to determine $U_{23}$, we consider the complex 2-3 submatrix of $M'_{SU}$

$$m'_{SU} = \begin{pmatrix} x_1 + 2x_2 & \sqrt{6}y \\ \sqrt{6}y & -x_1 + x_2 + 3x_3' \end{pmatrix},$$

(3.10)

and diagonalise

$$m'_{SU}m'_{SU}^\dagger = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix},$$

with

$$A = |x_1 + 2x_2|^2 + 6|y|^2,$$
$$B = \sqrt{6}[(x_1 + 2x_2)y^* - y(x_1 - x_2 - 3x_3')]^*,$$
$$D = |x_1 - x_2 - 3x_3'|^2 + 6|y|^2,$$

(3.11)

such that $u_{23}^*m'_{SU}m'_{SU}^\dagger u_{23}$ becomes real and diagonal. This requires

$$e^{2i\kappa} = \frac{B}{\sqrt{BB^*}} \quad \text{and} \quad \tan(2\vartheta) = \frac{2\sqrt{BB^*}}{D - A}.$$  \hspace{1cm} (3.12)

A simple relation between the phase $2\kappa$ and the phases of the input parameters can be realised in the case where one of the two terms in $B$, see Eq. (3.11), dominates over the other i.e. either

(i) $|x_1 + 2x_2| \gg |x_1 - x_2 - 3x_3'|$, or (ii) $|x_1 + 2x_2| \ll |x_1 - x_2 - 3x_3'|$.  \hspace{1cm} (3.13)

Clearly, such a situation requires some amount of tuning, which, however, is typically unavoidable in direct family symmetry models which accommodate realistic neutrino masses with $\frac{\Delta m^2_{\text{atm}}}{\Delta m^2_{\text{sol}}} \approx \pm 32$ \cite{84,85,86}. Let us dwell a little bit on the size of the singular values $M_i$ of $M_{SU}$. Ignoring the effect of the parameter $y$ in Eq. (3.9) and assuming no further tuning among the input parameters $x_1, x_2, x_3'$, which would suppress $M_1$, we find the following approximate mass ratios

$$M_1 : M_2 : M_3 \sim \begin{cases} 1 : 1 : \epsilon, & \text{for case (i)}, \\ 1 : \epsilon : 1, & \text{for case (ii)}, \end{cases}$$

(3.14)

with $\epsilon \ll 1$. In the case where $M_{SU}$ corresponds directly to the Majorana mass matrix of the left-handed neutrinos, for instance in the context of the type II seesaw mechanism, the pattern of case (i) suggests an inverted neutrino mass hierarchy, while the mass ratios of case (ii) are incompatible with the experimental data. The situation is somewhat more model dependent if the type I seesaw is at work, and $M_{SU}$ corresponds to the right-handed Majorana neutrino mass matrix. In the simplest scenario where the Dirac neutrino Yukawa matrix $Y_\nu$ is proportional to

$$Y_\nu \propto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

(3.15)

\footnote{We point out that there are other special cases where simple phase relations can be obtained. For instance, if $x_r, y \in \mathbb{R}$, then $2\kappa = 0$ or $\pi$. Another such simple scenario would be to have real $x_r$ and purely imaginary $y$, in which case one would find $2\kappa = \pm \frac{\pi}{2}$, see e.g. \cite{82}.}
the light neutrinos will have mass ratios as in Eq. \((3.14)\) with \(\epsilon\) replaced by \(\epsilon^{-1}\). We immediately see, that the resulting pattern for case \((i)\) suggests a normal neutrino mass hierarchy, while the pattern for case \((ii)\) is again not viable.

In the remainder of this paper, we therefore focus on scenarios of type \((i)\) in Eq. \((3.13)\), where the phase \(2\kappa\) is approximately identical to the argument of \((x_1 + 2x_2)y^*\). The idea is then to construct models with spontaneous CP violation in which the parameters \(x_r\) are effectively, i.e. after absorbing an overall phase, real, while the parameter \(y\) has the phase \(\omega\) or \(\omega^2\). As discussed above, such a situation will drive the atmospheric angle \(\theta_{23}\) away from its maximal value by about \(6^\circ\). The direction of this shift, into the first or second octant, will however depend on the signs of the involved parameters, which remain beyond the reach of pure model building arguments.

Before discussing concrete model realisations in the following sections, we comment on possible strategies for obtaining the alignments presented in Table 2. One option would be to construct them effectively from combining two flavons in a specific way. In the \(Z_{SU}^2\) symmetric case, one could, for instance, consider a flavon triplet \(\chi_3\) with alignment \((1, 0, 0)^T\) and multiply it with the doublet flavon \(\phi_2\) of Eq. \((2.2)\) to generate an effective \(3'\) flavon with the \(SU\) preserving alignment \((0, 1, -1)^T\). However, as \(\langle \chi_3 \rangle\) breaks \(Z_{SU}^2\), the TM\(_1\) scenario arises accidentally, and can be easily violated by other contributions. Indeed, coupling \(\chi_3\) to the flavon \(\phi_{3'}\) of Eq. \((2.2)\) can generate the effective doublet alignment \((1, -1)^T\) which breaks \(Z_{SU}^2\). Even though it is usually possible to construct ultraviolet completions such that only the desirable contractions are produced, it is generally advantageous to generate the flavon alignments in Table 2 directly from a suitable flavon potential.

### 4 Neutrino phenomenology

In this section we consider supersymmetric models in which the charged lepton mass matrix is diagonal by construction. This can be readily achieved along the lines of already existing \(S_4\) models of lepton flavour, e.g. \([57, 82]\). In the neutrino sector, we employ the tri-bimaximal flavon fields \(\phi_2\) and \(\phi_{3'}\) of Eq. \((2.2)\) together with a \(Z_{SU}^2\) preserving flavon \(\tilde{\phi}_{3'}\) which is aligned as

\[
\langle \tilde{\phi}_{3'} \rangle = \tilde{\phi}_{3'} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \tag{4.1}
\]

see Table 2. Notice that we do not include the flavon \(\phi_1\) as this choice proves to be a good starting point for generating an inverted (normal) neutrino mass spectrum in the context of the type II (type I) seesaw.

Tri-bimaximal mixing is obtained from coupling the neutrinos to \(\phi_2\) and \(\phi_{3'}\) as done in Eq. \((2.1)\). Adding in the flavon \(\tilde{\phi}_{3'}\) breaks the tri-bimaximal to the trimaximal TM\(_1\) pattern, which can either happen at the same or at higher order. In the following we will choose the former option since numerically there is no pronounced hierarchy between the effective parameters \(x_r\) and \(y\). As will be shown below, this is due to Eq. \((3.4)\) which

\[3\text{With } x_1 = 0 \text{ and } y = 0 \text{ in Eq. } (3.9), \text{ the requirement } (i) \text{ of Eq. } (3.13) \text{ yields } x_2 \approx -3x_{3'}, \text{ which in turn gives } M_1 \approx M_2.\]
$$\nu \nu (2 \phi_2 + 3' \phi_3' + \xi_1 + \xi_3') A_2^\theta A_2^\phi A_3^\phi A_3^{\tilde\phi} C_1^{\tilde\phi} C_1^{\phi} D_1 D_1'$$

Table 3: The particle content required in the neutrino sector of the $S_4 \times Z_3 \times Z'_3 \times Z_6^\theta$ models.

translates the measured reactor angle $\theta_{13} \approx 9^\circ$ to a relatively large value for $|\bar{\theta}| \approx 16^\circ$. With this assumption, the coupling of the flavon fields to the neutrinos would take the form

$$\nu \nu (\alpha_2 \phi_2 + \alpha_3' \phi_3' + \tilde{\alpha}_3 \tilde{\phi}_3') .$$

As such a structure would require identical quantum numbers for $\phi_3'$ and $\tilde{\phi}_3'$, it is not possible to distinguish these two flavons. As a consequence, the desired flavon alignment could not be achieved by means of symmetries alone. In order to avoid this conclusion, we introduce the $S_4$ singlet flavons $\xi_1$ and $\tilde{\xi}_1$ (with VEVs $\xi$ and $\tilde{\xi}$, respectively), and modify the coupling of the flavons to the neutrinos to

$$\nu \nu \left( (\alpha_2 \phi_2 + \alpha_3' \phi_3') \xi_1 + \tilde{\alpha}_3 \tilde{\phi}_3' \tilde{\xi}_1 \right) .$$

This structure can be readily enforced by a $Z_3 \times Z'_3$ symmetry, where the former distinguishes $\phi_3'$ and $\tilde{\phi}_3'$, while the latter forbids terms with only one flavon coupling to the neutrinos. The explicit charge assignments of the fields in the neutrino sector are listed in Table 3 including two auxiliary flavons $\theta_3'$ and $\tilde{\theta}_3'$ (together with an associated $Z_6^\theta$ symmetry) which are relevant for generating the $Z_{SU}^2$ preserving alignment of $\tilde{\phi}_3'$. Furthermore, the driving fields introduced in this setup are shown and can be identified by their $U(1)_R$ charge of 2. Before discussing the resulting flavon potential in Section 5, we wish to illustrate how Eq. (4.2) can give rise to phenomenologically viable neutrino masses and mixings.

### 4.1 Inverted mass hierarchy from type II seesaw

We first present the case where the structure of Eq. (4.2) arises from a type II seesaw, with $\nu$ representing the left-handed neutrinos $\nu_L$. The Higgs triplet $\Delta_H$ is neutral under all symmetries of Table 3 and does not play any role in the discussion of the neutrino mixing.

---

4We note, however, that the flavons could be separated in an extra dimensional setup.
with

\[ x_1 = 0 , \quad x_2 = \alpha_2 \varphi_2 \xi \frac{(\Delta m)^2}{\Lambda^2} , \quad x_3' = \alpha_3' \varphi_3' \xi \frac{(\Delta m)^2}{\Lambda^2} , \quad y = \tilde{\alpha}_3' \tilde{\varphi}_3' \tilde{\xi} \frac{(\Delta m)^2}{\Lambda^2} , \]

where \( \Lambda \) denotes a high mass scale. Imposing CP conservation renders all coupling constants real, and CP is only violated spontaneously by complex flavon VEVs. The desired phase structure depends on the flavon potential. Suitable flavon VEVs are those leading to identical phases for \( \varphi_2 \xi \) and \( \varphi_3' \xi \), while the phase of \( \tilde{\varphi}_3' \tilde{\xi} \) has to be shifted relative to these by \( \pm \omega \) or \( \pm \omega^2 \). Anticipating the results of Section 3, the flavons can develop VEVs with phases

\[
\frac{\varphi_{3'}}{|\varphi_{3'}|} = \pm \frac{\varphi_2}{|\varphi_2|} = \pm \omega^k , \quad \frac{\xi}{|\xi|} = \pm \omega^l , \quad \frac{\tilde{\varphi}_{3'}^{\prime}}{|\tilde{\varphi}_{3'}^{\prime}|} = \pm \omega^{2l} , \quad \frac{\tilde{\xi}}{|\tilde{\xi}|} = \pm \omega^{\tilde{l}} ,
\]

where \( k, l, \tilde{l} = 0, 1, 2 \) and the signs depend on the undetermined signs of the real coupling constants of the flavon potential. The common phase of \( \varphi_2 \xi \) and \( \varphi_{3'} \xi \) can be absorbed by a redefinition of the neutrino fields \( \nu_L \) in Eq. (4.2). This generates the following phases for the parameters \( x_r \) and \( y \) in Eq. (4.3), namely

\[ x_r \in \mathbb{R} , \quad \frac{y}{|y|} = \pm \omega^{l+k} = \pm \omega^m , \]

where we have introduced \( m = 0, 1, 2 \). We now assume that Nature has chosen one of the two CP violating cases \( m = 1 \) or \( m = 2 \), thus entailing a relative phase between \( x_r \) and \( y \) of either \( \pm 120^\circ \) or \( \pm 60^\circ \). Together with the assumption that \( x_2 \approx -3x_3' \), such a phase structure leads to the result \( 2\kappa \approx \pm 120^\circ \) or \( 2\kappa \approx \pm 60^\circ \), cf. Eqs. (3.11,3.12). Hence, in this model, the atmospheric angle will be shifted away from its maximal value by an angle of about \( 6^\circ \), provided the second relation in Eq. (3.12) yields a value of \( \vartheta \approx -16^\circ \) which is consistent with a reactor angle of about \( 9^\circ \) and an inverted neutrino mass hierarchy. In the limit \( x_2 = -3x_3' \), in which the phase factor \( e^{2i\kappa} \) is given exactly by either \( \pm \omega \) or \( \pm \omega^2 \), this requirement translates to

\[ |y| = -\frac{1}{\sqrt{6}} \tan(2\vartheta) |x_2| \approx 0.25 |x_2| . \]

The approximate values of the effective parameters are then related as

\[ |x_2| : |x_3'| : |y| \approx 3 : 1 : 0.75 , \]

which, as anticipated above, does not feature any clear hierarchical structure, and so motivates the structure of Eq. (4.2) where all terms enter at the same order.

In Appendix A we sketch how the three effective parameters can be directly determined by demanding that they give rise to the physically viable values \( \theta_{13} \approx 9^\circ \), \( \frac{\Delta m^2_{\text{atm}}}{\Delta m^2_{\text{sol}}} \approx -32 \), and \( \Delta m^2_{\text{atm}} \approx -2.43 \cdot 10^{-3} \) (eV)². Fixing the phase of \( x_3' \) at zero, we obtain

\[ x_2 \approx -0.0228 \text{ eV} , \quad x_3' \approx 0.0086 \text{ eV} , \quad y \approx (-1)^p \omega^m \cdot 0.0055 \text{ eV} , \]

where \( p \) is an integer.
where $p = 0,1$ and $m = 1,2$. Having fixed these input parameters, all other physical parameters of the neutrino sector are predicted. The four choices of the phase of $y$, see Eq. (4.6), cause a discrete ambiguity in the obtained mixing parameters,

$$\theta_{12} \approx 34.2^\circ, \quad \theta_{23} \approx 45^\circ - (-1)^p \cdot 5.7^\circ, \quad \delta \approx [1 + (-1)^p] \cdot 90^\circ - (-1)^m \cdot 66^\circ, \quad (4.7)$$
or more explicitly

$$p = 0 : \quad \theta_{23} \approx 39.3^\circ, \quad \delta \approx \left\{ \begin{array}{ll}
-114^\circ, & m = 1 \\
+114^\circ, & m = 2
\end{array} \right., \quad (4.8)$$

$$p = 1 : \quad \theta_{23} \approx 50.7^\circ, \quad \delta \approx \left\{ \begin{array}{ll}
+66^\circ, & m = 1 \\
-66^\circ, & m = 2
\end{array} \right.. \quad (4.9)$$

This shows that solutions of the atmospheric mixing angle in the first octant are predicted for $p = 0$, while the choice $p = 1$ gives $\theta_{23} > 45^\circ$. We emphasise that these are predictions for $\theta_{23}$ and $\delta$ which, for a suitable choice of $p$ and $m$, happen to be consistent with all three global fits at the 1σ level. The allowed regions of the fit in [86] are met for $(p,m) = (0,1)$ and $(p,m) = (1,2)$, while the global fit in [85] requires $(p,m) = (0,1)$. The analysis of [84] only yields a 1σ solution in the second octant for inverted neutrino mass ordering and does not constrain the Dirac phase at all, hence, it can be described consistently by the choice $(p,m) = (1,1)$ and $(p,m) = (1,2)$. The linear sum rule of Eq. (2.14) agrees well with the more accurate result in Eq. (4.7): for $p = 0$ ($p = 1$), Eq. (4.7) gives a Dirac phase $\delta \approx \pm 114$ ($\delta \approx \pm 66$), which in turn yields the linear sum rule $\theta_{23} \approx 45^\circ - 5.2^\circ$ ($\theta_{23} \approx 45^\circ + 5.2^\circ$).

The neutrino masses, on the other hand, are independent of $p$ and $m$,

$$m_{\nu_1} \approx 0.0486 \text{ eV}, \quad m_{\nu_2} \approx 0.0494 \text{ eV}, \quad m_{\nu_3} \approx 0.0033 \text{ eV}. \quad (4.10)$$

Likewise, one can determine the effective mass $m_{\beta \beta}$ relevant for neutrinoless double beta decay without ambiguity, yielding

$$m_{\beta \beta} = |(M_{SU})_{11}| = |2x_3'| \approx 0.017 \text{ eV}. \quad (4.11)$$

### 4.2 Normal mass hierarchy from type I seesaw

In the case of a type I seesaw model, the structure of Eq. (4.12) arises for the right-handed neutrinos $\nu = \nu_R$. The matrix $M_{SU}$ of Section 3, which depends on the effective parameters

$$x_1 = 0, \quad x_2 = \alpha_2 \varphi_2 \frac{1}{\Lambda}, \quad x_3' = \alpha_3' \varphi_3' \frac{1}{\Lambda}, \quad y = \tilde{\alpha}_3' \tilde{\varphi}_3' \frac{1}{\Lambda}, \quad (4.12)$$

then corresponds to the right-handed neutrino mass matrix $M_R$. In order to find the effective light neutrino mass matrix, we need to fix the structure of the Dirac Yukawa coupling. In the simplest case, the $S_4 \times Z_3 \times Z'_3 \times Z_6^\theta$ charge assignments of the lepton doublet $L$ are chosen such that they allow for the trivial coupling $y_D L_{\nu_R} H_u$, with $H_u$ denoting the (flavour blind) up-type Higgs doublet. The Dirac neutrino mass matrix then takes the form

$$m_D = y_D v_u \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (4.13)$$
where $v_u$ is the VEV of $H_u$. Application of the seesaw formula yields the light neutrino mass matrix

$$m^\text{eff}_\nu = m_D M^{-1} m_D^T = (y_D v_u)^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} U_\nu (M_{SU}^{\text{diag}})^{-1} U_\nu^T \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (4.14)$$

where $U_\nu$ is given in Eq. (3.3) and can be obtained as discussed in Section 3. Knowing $U_\nu$ one trivially finds the unitary matrix which diagonalises $m^\text{eff}_\nu$, that is the PMNS mixing matrix

$$U_{\text{PMNS}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} U_\nu^* . \quad (4.15)$$

The physical mixing parameters are therefore identical to those of $U_\nu$, with the exception of the 2-3 mixing angle which changes the octant $\theta_{23} \to 90^\circ - \theta_{23}$ as well as the CP phase $\delta$ for which $\delta \to -\delta + \pi$. The light neutrino masses $m_{\nu_i}$ are given by the singular values $M_i$ of $M_{SU}$, see Eqs. (A.7-A.9), and the product $y_D v_u$,

$$m_{\nu_i} = \frac{(y_D v_u)^2}{M_i} . \quad (4.16)$$

Analogous to the type II setup, the relative size of the input parameters $x_2$, $x_3'$ and $y$ is determined by requiring $\theta_{13} \approx 9^\circ$ and $\frac{\Delta m^2_{\text{atm}}}{\Delta m^2_{\text{sol}}} \approx +32$, while their absolute magnitude cannot be fixed uniquely due to the factor $(y_D v_u)^2$ in Eq. (4.16). Demanding $\Delta m^2_{\text{atm}} \approx 2.47 \cdot 10^{-3} \text{ (eV)}^2$, we obtain

$$\frac{x_2}{(y_D v_u)^2} \approx -32.74 \text{ (eV)}^{-1}, \quad \frac{x_3'}{(y_D v_u)^2} \approx 18.33 \text{ (eV)}^{-1}, \quad \frac{y}{(y_D v_u)^2} \approx (-1)^p \omega^m \cdot 5.99 \text{ (eV)}^{-1} . \quad (4.17)$$

The resulting predictions for the neutrino mixing parameters, $\theta_{12} \approx 34.2^\circ$ and

$$p = 0 : \quad \theta_{23} \approx 48.4^\circ , \quad \delta \approx \begin{cases} -76^\circ, & m = 1, \\
+76^\circ, & m = 2,
\end{cases} \quad (4.18)$$

$$p = 1 : \quad \theta_{23} \approx 41.6^\circ , \quad \delta \approx \begin{cases} +104^\circ, & m = 1, \\
-104^\circ, & m = 2,
\end{cases} \quad (4.19)$$

are again consistent with the global fits for $p = 1$ and a suitable choice of $m$ (at the 1$\sigma$ level for [81,82] and at the 2$\sigma$ level for [83]). The light neutrino masses take the values

$$m_{\nu_1} \approx 0.0114 \text{ eV} , \quad m_{\nu_2} \approx 0.0144 \text{ eV} , \quad m_{\nu_3} \approx 0.0510 \text{ eV} , \quad (4.20)$$

and the effective mass $m_{\beta\beta}$ of neutrinoless double beta decay becomes

$$m_{\beta\beta} = (y_D v_u)^2 \left| (M_{SU}^{-1})_{11} \right| = (y_D v_u)^2 \left| \frac{(x_2 + x_3')(x_2 + 3x_3') - 4y^2}{2(x_2 - 3x_3')(x_2^2 + 3x_2 x_3' - 3y^2)} \right| \approx 0.0023 \text{ eV} . \quad (4.21)$$
5 Flavon sector

In the flavon sector, we have to generate both the alignments of the flavon VEVs as well as their phases. To achieve this we adopt the $F$-term alignment mechanism \[91, 92\].

Introducing the driving fields $A_2^\theta$, $A_2^\phi$, $A_3^\phi$ and $A_3^{\tilde{\phi}}$ produces the alignments of the flavons $\theta_3'$, $\phi_2$, $\phi_3'$ and $\tilde{\theta}_3'$, respectively. The alignment of the flavon $\tilde{\phi}_3'$ which breaks the tribimaximal pattern while respecting the $Z_{SU}^2$ symmetry of $S_4$, uses orthogonality conditions obtained from the driving fields $O_1^{\tilde{\phi}}$ and $O_1^{\phi}$. Subsequently, the VEVs of the flavons $\phi_2'$, $\phi_3'$, $\xi_1$, $\tilde{\xi}_1$, $\theta_3'$ and $\tilde{\theta}_3'$ are driven to non-zero values with fixed phases due to the presence of (five copies of) the driving field $D_1$. Finally, the driving field $D_1'$ is responsible for fixing the VEV of the flavon $\tilde{\phi}_3'$.

With the charges listed in Table 3, the leading order terms of the effective flavon superpotential are

\[
W_{\text{flavon}}^{\text{eff}} \sim A_2^\theta (\theta_3' \theta_3') + A_2^\phi (\phi_2' \phi_2 + \phi_3' \phi_3') + A_3^\phi (\phi_2' \phi_3') \\
+ A_3^{\tilde{\phi}} (\phi_2' \tilde{\phi}_3' + \phi_3' \tilde{\phi}_3') + O_1^{\tilde{\phi}} (\theta_3' \tilde{\phi}_3') + O_1^{\phi} (\tilde{\theta}_3' \tilde{\phi}_3') \\
+ D_1 \left[ \frac{(\phi_2')^3 + (\phi_3')^3 + \phi_2 (\phi_3')^2 + (\xi_1)^3 + (\tilde{\phi}_3')^3 + (\tilde{\xi}_1)^3 + (\theta_3')^3 + (\tilde{\theta}_3')^2 - M^2}{\Lambda} \right] \\
+ D_1' \left[ (\tilde{\phi}_3')^2 - M \xi_1 \right],
\]

where we have suppressed the dimensionless coupling constants. $M$ and $\Lambda$ denote high mass scales, with $\Lambda$ being related to the mass of certain messenger fields which generate the respective non-renormalisable operators. We emphasise that these will generally be different for different operators. In the following we discuss the individual terms of the flavon potential in turn.

Starting with the driving field $A_2^\theta$, the derived $F$-term condition reads

\[
\begin{pmatrix}
\langle \theta_3' \rangle_2^2 + 2 \langle \theta_3' \rangle_3 \langle \theta_3' \rangle_1 \\
\langle \theta_3' \rangle_3^2 + 2 \langle \theta_3' \rangle_1 \langle \theta_3' \rangle_2
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

It is straightforward to show that the most general solution to this equation takes the form

\[
\langle \theta_3' \rangle = \theta \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},
\]

as well as alignments obtained from this by applying any of the 24 $S_4$ transformations. With $S_4$ being a symmetry of the theory, we can choose the alignment given in Eq. (5.3) without loss of generality.
The $F$-term conditions obtained from the driving fields $A_2^\phi$ and $A_3^\phi$ take the form

\[
\left( \frac{\langle \phi_2 \rangle^2}{\langle \phi_2 \rangle_1} \right) + f \left( \frac{\langle \phi_3' \rangle^2}{\langle \phi_3' \rangle_1} + 2\langle \phi_3' \rangle_3 \langle \phi_3' \rangle_1 \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right),
\]

or

\[
\left( \frac{\langle \phi_2' \rangle_2}{\langle \phi_2' \rangle_1} \right) - \langle \phi_2' \rangle_3 \langle \phi_2' \rangle_1 = \left( \begin{array}{c} 0 \\ 0 \end{array} \right),
\]

where a relative coupling constant $f$ has been introduced in Eq. (5.4) since this condition arises from two independent terms in the effective flavon superpotential of Eq. (5.1). The most general solution to Eq. (5.5) with non-zero VEVs $\varphi_2$ and $\varphi_3'$ is given by

\[
\langle \phi_2 \rangle = \varphi_2 \left( \omega^{-k} \omega^k \right), \quad \langle \phi_3' \rangle = \varphi_3' \left( \begin{array}{c} 1 \\ \omega^{-k} \omega^k \end{array} \right),
\]

with $k = 0, 1, 2$. Application of the $S_4$ transformation $T^k$ brings these alignments into the standard form of Eq. (2.2). Notice that such a $T$ transformation does not change the alignment of the flavon $\theta_3'$ in Eq. (5.3). The VEVs $\varphi_2$ and $\varphi_3'$ are related via Eq. (5.4), yielding

\[
\varphi_3'^2 = - \frac{1}{3f} \varphi_2^2.
\]

As we are interested in the case where the phases of $\varphi_2$ and $\varphi_3'$ are identical (up to a possible minus sign), we impose that the relative coupling constant $f$ be negative.

The alignment of the auxiliary flavon field $\tilde{\theta}_3'$ arises due to the $F$-term equations of $A_3^\phi$. Inserting the already determined flavon alignments $\langle \phi_2 \rangle$ and $\langle \phi_3' \rangle$, see Eq. (2.2), yields

\[
\left( \frac{\langle \tilde{\theta}_3' \rangle_2}{\langle \tilde{\theta}_3' \rangle_3} \right) + f \left( \frac{\langle \tilde{\theta}_3' \rangle_3}{\langle \tilde{\theta}_3' \rangle_1} + 2\langle \tilde{\theta}_3' \rangle_2 \langle \tilde{\theta}_3' \rangle_1 \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right),
\]

where the scales of the flavon VEVs $\varphi_2$ and $\varphi_3'$ have been absorbed into the relative coupling constant $g$. As this constant is generically not equal to $\pm 1$, Eq. (5.7) implies the alignment

\[
\langle \tilde{\theta}_3' \rangle = \tilde{\theta} \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right).
\]

Now that $\theta_3'$, $\phi_2$, $\phi_3'$ and $\tilde{\theta}_3'$ are aligned, we can derive the alignment of the flavon $\tilde{\phi}_3'$ using the driving fields $O_1^\phi$ and $O_1^{\tilde{\phi}}$. The resulting $F$-term conditions demand orthogonality of $\langle \tilde{\phi}_3' \rangle$ with $\langle \theta_3' \rangle$ as well as with $\langle \tilde{\theta}_3' \rangle$. This immediately produces the $Z_2^{SU}$ preserving alignment of Eq. (5.1).

The third line the flavon superpotential in Eq. (5.1) generates constraints on only the magnitude and phase of the flavon VEVs. Due to the imposed $Z_3$ symmetries, most flavons
couple cubically to the driving field $D_1$, which is neutral under all discrete symmetries. Only the auxiliary flavon $\tilde{\theta}_3$ is allowed to couple quadratically to $D_1$. Furthermore, we get exactly one mixed term involving a product of $\phi_2$ and $\phi_3'$ flavons which can be contracted to an $S_4$ singlet. Due to the correlation between $\varphi_2$ and $\varphi_3'$ of Eq. (5.6), the first three cubic terms in the third line of Eq. (5.1) can be combined into one effective term. A further simplification is achieved by noting that the contraction of $\langle \tilde{\varphi}_3' \rangle^3$ to an $S_4$ singlet vanishes for the vacuum alignment of Eq. (4.1). The $F$-term condition originating from the driving field $D_1$ can therefore be cast into the simple form

$$h_1\varphi_2^3 + h_2\xi^3 + h_3\tilde{\xi}^3 + h_4\theta^3 + h_5\tilde{\theta}^2 - M^2 = 0,$$

where $h_i$ are new coupling constants. Thus the flavon VEVs $\varphi_2$, $\xi$, $\tilde{\xi}$, $\theta$ and $\tilde{\theta}$ get separately driven to non-zero values. Due to the imposed CP symmetry, and the resulting real coupling constants, the phases of the first three flavon VEVs are fixed as given in Eq. (4.4). We remark that the obtained phase predictions for the VEVs of the auxiliary flavon fields $\theta_3'$ and $\tilde{\theta}_3'$ do not have any effect on the neutrino mixing parameters.

The remaining flavon $\tilde{\varphi}_3'$ is driven to a non-zero VEV using the $F$-term equation of the driving field $D_1'$ in Eq. (5.1). Inserting the flavon VEVs $\tilde{\varphi}_3'$ and $\xi = \pm|\xi|\omega^l$, we find

$$h_6\tilde{\varphi}_3'^2 = M|\xi|\omega^l,$$

where the sign ambiguity of the flavon VEV $\xi$ has been absorbed into the coupling constant $h_6$. In the case where the real parameters $h_6$ and $M$ have the same sign, the phase of $\tilde{\varphi}_3'$ is related to the phase of $\xi$ as given in Eq. (4.4). With opposite signs for $h_6$ and $M$, an additional factor of $i$ would arise, however, we shall not consider this option in this paper.

Having shown how the flavon VEV configurations with the phase structure given in Eq. (4.4) can be derived from the effective flavon potential of Eq. (5.1), the question of higher order corrections to the flavon alignment arises. At the purely effective level, we indeed find several (higher) non-renormalisable terms which obey all imposed symmetries. However, not all of these potentially dangerous terms arise in concrete ultraviolet (UV) completions of an effective model [93]. In such UV completed models, the non-renormalisable terms of the effective theory arise by integrating out heavy messenger fields. If no messenger field exists to mediate a particular non-renormalisable term, this term will simply not get generated.
Table 4: The messenger fields required to generate (some of) the non-renormalisable operators of the flavon superpotential of Eq. (5.1).

In order to obtain the flavon superpotential of Eq. (5.1) it is mandatory to introduce some messenger fields which induce the non-renormalisable terms of the third line. All the other terms of $W_{\text{eff flavon}}$ are already renormalisable, and their existence is therefore not subject to the presence of messenger fields. As we have seen, there are only four non-renormalisable terms which are relevant for driving the flavon VEVs to non-vanishing values with fixed phases,

$$D_1 \frac{1}{\Lambda} \left[ (\phi_2)^3 + (\xi_1)^3 + (\tilde{\xi}_1)^3 + (\theta_{3'})^3 \right].$$

Each of these terms requires its own pair of messenger fields denoted by $\Sigma, \Sigma^c$. Their charge assignments are listed in Table 4. With these charges, the renormalisable superpotential involving these messenger fields reads

$$W_{\Sigma} \sim D_1 \phi_2 \Sigma_2 + \Sigma_2^c (\phi_2 \phi_2 + \phi_3 \phi_3') + \Lambda \Sigma_2 \Sigma_2^c + D_1 \xi_1 \Sigma_1 + \Sigma_1^c \xi_1 \xi_1 + \Lambda \Sigma_1 \Sigma_1^c + D_1 \tilde{\xi}_1 \tilde{\Sigma}_1 + \tilde{\Sigma}_1^c \tilde{\xi}_1 \tilde{\xi}_1 + \Lambda \tilde{\Sigma}_1 \tilde{\Sigma}_1^c + D_1 \theta_{3'} \Sigma_{3'} + \Sigma_{3'}^c \theta_{3'} \theta_{3'} + \Lambda \Sigma_{3'} \Sigma_{3'}^c$$

$$+ A_2^\phi (M \Sigma_2) + D'_1 (\Sigma_1 \Sigma_1) .$$

The first four lines of Eq. (5.13) give rise to the four effective non-renormalisable terms of Eq. (5.12), plus the extra but harmless operator $D_1 \phi_2 (\phi_3')^2 / \Lambda$, cf. Eq. (5.1). The two operators in the fifth line of Eq. (5.13) yield additional contributions to the effective flavon superpotential,

$$A_2^\phi \frac{M}{\Lambda} (\phi_2 \phi_2 + \phi_3' \phi_3') + D'_1 \frac{\xi_1^4}{\Lambda^2} .$$

which can be easily verified by systematically integrating out the $\Sigma, \Sigma^c$ messengers. The two terms of Eq. (5.14) arise from the renormalisable theory involving the messengers fields of Table 4. We emphasise that they are the only non-renormalisable operators which have to be added to the effective flavon superpotential $W_{\text{eff flavon}}$ of Eq. (5.1). Their presence, however, does not change the discussion of the flavon alignment nor the phase structure of the flavon VEVs. This can be seen by noting that the first term of Eq. (5.14) can be
absorbed into the corresponding and already existing renormalisable term of Eq. (5.1). The second term of Eq. (5.14) modifies the couplings of the driving field $D'_1$ to

$$D'_1 \left[ (\bar{\phi}_3)'^2 - M\xi_1 + M\xi_1 \frac{\xi_1^3}{MA^2} \right].$$  (5.15)

As the cube of the VEV $\xi$ is real, see Eq. (5.10), the resulting $F$-term condition is of the same form as in Eq. (5.11), where the real mass parameter $M$ is slightly corrected due to the presence of the non-renormalisable term. This shows that it is possible to generate the effective flavon superpotential of Eq. (5.1) without higher order corrections other than the two harmless operators of Eq. (5.14). Therefore, the desired flavon alignment, together with a particular phase structure for the flavon VEVs, can be achieved in a UV completed model involving only a few messenger fields.

6 Conclusion

The idea of an underlying family symmetry which, together with its breaking, dictates the structure of the fermion masses and mixings has not been ruled out by the measurement of a sizable reactor mixing angle $\theta_{13}$ of about 9°. However, tri-bimaximal neutrino mixing (and other simple patterns which predict vanishing $\theta_{13}$), are dead. Successful models must necessarily involve deviations from tri-bimaximal mixing. In the context of direct models, where the family symmetry is intimately linked to the symmetries of the mass matrices, there exist two ways to produce new mixing patterns. The first is based on “large” family symmetries which allow for non-standard $Z_2 \times Z_2$ Klein symmetries of the neutrino mass matrix. The second approach is based on models with a tri-bimaximal $Z_2^S \times Z_2^U$ Klein symmetry which, however, gets broken to a residual $Z_2$ symmetry in the neutrino sector.

Assuming an underlying $S_4$ family symmetry, we have presented all available flavon alignments which give rise to such a scenario. We focus on the trimaximal TM$_1$ case as this is preferred over the trimaximal TM$_2$ case due to its excellent agreement of the predicted solar mixing angle $\theta_{12}$ with the measured value ($\theta_{12}^{\text{exp}} \approx 34°$ compared to the predictions $\theta_{12}^{\text{TM}_1} \approx 34.2°$ and $\theta_{12}^{\text{TM}_2} \approx 35.8°$, respectively). Enforcing the TM$_1$ case by means of a remnant $Z_2^S \times SU_2$ symmetry, we propose two explicit supersymmetric models which respect CP symmetry in the family symmetry limit. The CP symmetry gets broken in a controlled way when the flavon fields acquire VEVs with well-defined complex phases.

The first model is based on the type II seesaw mechanism and generates an inverted neutrino mass spectrum. Fitting the reactor angle to its measured value, the remaining parameters of the PMNS mixing matrix are predicted within their 1σ experimentally allowed regions. In particular, the atmospheric angle $\theta_{23}$ deviates from its maximal value by about 5.7°. The CP violating Dirac phase $\delta$ is predicted to be $\mp 114°$ for solutions in the first octant, while it becomes $\pm 66°$ for solutions in the second octant. Imposing the constraints from the measured neutrino mass squared differences, the model predicts all neutrino masses $m_{\nu_i}$ as well as the effective mass $m_{\beta\beta}$ of neutrinoless double beta decay, see Eqs. (4.10,4.11).

The second model is based on the type I seesaw and generates a normal neutrino mass hierarchy. As in the previous model, fitting the reactor angle leads to very good agreement...
of the remaining predicted mixing parameters with their measured values. In particular, the atmospheric mixing angle $\theta_{23}$ deviates from its maximal value by about $3.4^\circ$. Solutions in the first octant entail $\delta \approx \pm 104^\circ$, while solutions in the second octant have $\delta \approx \mp 76^\circ$.

As before, the neutrino masses are completely fixed after matching the input parameters to the solar and atmospheric mass squared differences, see Eqs. (4.20, 4.21).

To arrive at these predictions, the assumed flavon alignments have to be justified. Here, we make use of the so-called $F$-term alignment mechanism available in a supersymmetric context. Imposing a $U(1)_R$ symmetry as well as two $Z_3$ and one $Z_6$ symmetry, we have studied the flavon potential in detail. We have shown how to derive the required flavon VEV configurations as well as the VEVs’ complex phases. In order to guarantee that higher order terms do not spoil the successful results achieved for the flavon alignments and their phases, we have formulated a UV completion of the flavon sector.

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**Appendix**

A Fixing the effective parameters $x_2$, $x_3'$ and $y$

In Section 4 we have presented realisations of trimaximal TM$_1$ neutrino mixing which rely on the presence of several flavon fields. Their VEVs give rise to three effective parameters $x_2$, $x_3'$ and $y$. It is convenient to reparameterise these as

$$x_2 = a b, \quad x_3' = a, \quad y = \omega^m a c, \quad a, b, c \in \mathbb{R}, \quad (A.1)$$

with $m = 1, 2$. In this notation, the entries of $m'_{SU} m'^\dagger_{SU}$, see Eq. (3.11), can be rewritten as

$$A = a^2 \left[ 4b^2 + 6c^2 \right], \; (A.2)$$
$$D = a^2 \left[ (3 + b)^2 + 6c^2 \right], \; (A.3)$$
$$B = \sqrt{6} a^2 c \left[ (3 + b)\omega^m + 2b\omega^{-m} \right], \; (A.4)$$

and the absolute value of $B$ simplifies to

$$|B| = 3\sqrt{2} a^2 c|\sqrt{3 + b^2}|. \; (A.5)$$

Notice that $A$, $D$ and $|B|$ do not depend on $m = 1, 2$ nor on the sign of $c$. Plugging these expressions into the second relation of Eq. (3.12) relates the real parameters $b$ and $c$ to $\tan(2\vartheta)$,

$$\tan(2\vartheta) = \frac{2\sqrt{2} \sqrt{3 + b^2}}{3 + 2b - b^2} |c|. \; (A.6)$$
As the numerical value of $\vartheta$ is fixed by $\theta_{13} \approx 9^\circ$ via Eq. (3.4), we obtain an expression for $|c|$ as a function of the yet to be determined parameter $b$. In order to pin down the physically viable value of $b$, we have to consider the neutrino masses. The eigenvalues of $M_{SU}M_{SU}^\dagger$ are given by

$$M_1^2 = a^2(3 - b)^2,$$

$$M_2^2 = A \cos^2 \vartheta + D \sin^2 \vartheta - |B| \sin(2\vartheta),$$

$$M_3^2 = A \sin^2 \vartheta + D \cos^2 \vartheta + |B| \sin(2\vartheta).$$

In the type II seesaw model, these masses correspond directly to the light neutrino masses

$$m_{\nu_i}^2 = M_i^2.$$  \hfill (A.10)

Replacing $A$, $D$ and $|B|$ using Eqs. (A.2,A.3,A.5), we find that the squared neutrino masses are functions of $a^2$, $b$ and $|c|$. The absolute scale is fixed by the common factor $a^2$, which drops out if we calculate the ratio of the atmospheric and the solar mass squared differences. The parameter $|c|$ can be eliminated by means of Eq. (A.6). Then

$$\frac{\Delta m_{\text{atm}}^2}{\Delta m_{\text{sol}}^2} = \frac{M_3^2 - M_2^2}{M_2^2 - M_1^2},$$

becomes a function of a single parameter, $b$, which can be calculated numerically by setting $\frac{\Delta m_{\text{atm}}^2}{\Delta m_{\text{sol}}^2} \approx -32$. Notice the minus sign which appears due to the requirement of an inverse mass ordering, i.e. negative $\Delta m_{\text{atm}}^2$. Of the two possible solutions only one is consistent with positive $\Delta m_{\text{sol}}^2$. This solution requires $b \approx -2.6515$; as a consequence of Eq. (A.6), we directly get $|c| \approx 0.6370$. Finally, the overall scale $a$ is obtained by setting $\Delta m_{\text{atm}}^2 \approx -2.43 \times 10^{-3} \text{ (eV)}^2$, leading to the value $a \approx 0.0086 \text{ eV}$. The effective parameters of Eq. (A.1) are then uniquely determined up to the discrete ambiguity related to $m = 1, 2$ and the sign of $c$, which is parameterised by the factor $(-1)^p$ in Eq. (4.6).

In the type I seesaw model, the mass matrix $M_{SU}$ of Section 3 corresponds to $M_R$, the mass matrix of the right-handed neutrinos $\nu_R$. With a trivial Dirac mass matrix of the form as given in Eq. (4.13), the PMNS mixing matrix is almost identical to the unitary matrix which diagonalises $M_{SU}$, see Eq. (4.15). The mixing angle $\vartheta$ of Eq. (A.6) is still related to the reactor angle $\theta_{13} \approx 9^\circ$ via Eq. (3.4). Therefore, $|c|$ can be expressed as a function of the parameter $b$. The latter will be fixed by considering the light neutrino masses, which are obtained from Eqs. (A.7-A.9) by

$$m_{\nu_i}^2 = \frac{(y_D v_u)^4}{M_i^2}.$$ \hfill (A.12)

As a result, the ratio of the mass squared splittings is given by

$$\frac{\Delta m_{\text{atm}}^2}{\Delta m_{\text{sol}}^2} = \frac{M_1^2 - M_3^2}{M_1^2 - M_2^2} \cdot \frac{M_2^2}{M_3^2}.$$ \hfill (A.13)

Equating this to $+32$ for a normal neutrino mass hierarchy determines $b \approx -1.7857$, which in turn results in $|c| \approx 0.3267$. The absolute mass scale $a$ can finally be derived.
from $\Delta m_{\text{atm}}^2 \approx 2.47 \cdot 10^{-3} \text{(eV)}^2$, leading to a value of $a \approx 18.33 \frac{(y_D v_a)^2}{eV}$. The parameters of Eq. (A.1) are then uniquely determined for the type I seesaw setup, up to a discrete ambiguity which is expressed in terms of $m = 1, 2$ and the sign of $c$; the latter corresponding to the factor $(-1)^p$ in Eq. (4.17).

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