BRAISED HOPF ALGEBRAS ARISING FROM MATCHED PAIRS OF GROUPS

NICOLÁS ANDRUSKIEWITSCH AND SONIA NATALE

Abstract. Let $k$ be a field. Let also $(F, G)$ be a matched pair of groups. We give necessary and sufficient conditions on a pair $(\sigma, \tau)$ of 2-cocycles in order that the crossed product algebra and the crossed coproduct coalgebra $kG \# \sigma_kF$ combine into a braided Hopf algebra. We also discuss diagonal realizations of such braided Hopf algebras in the category of Yetter-Drinfeld modules over a finite group.

1. Introduction

Let $k$ be a field and let $F$, $G$ be finite groups. Given a right action $\triangleright$ of $F$ on the set $G$ and a cocycle $\sigma : F \times F \to (k^G)^\times$, one forms the crossed product $kG \# \sigma_kF$. Dually, given a left action $\triangleleft$ of $G$ on the set $F$ and a cocycle $\tau : G \times G \to (k^F)^\times$, one forms the crossed coproduct $kG \# \tau_kF$. In general, $R = kG \# \tau_kF$ is not a Hopf algebra with these multiplication and comultiplication. A necessary condition is that the actions $\triangleleft, \triangleright$ define a matched pair, that is, that they arise from an exact factorization $\Sigma = FG$. Let us assume that this is the case. Then $R$ is a Hopf algebra if and only if $\sigma$ and $\tau$ satisfy a further requirement, which can be expressed as saying that the pair $(\sigma, \tau)$ is a 1-cocycle in certain complex. See for example [M]. The original sources of this construction are [K], [T1], [Mj].

The starting point of this paper is the following observation: even if $(\sigma, \tau)$ is not a 1-cocycle, $R$ might admit a structure of a braided Hopf algebra in the sense of Takeuchi [T2]. That is, under certain conditions, there exists an invertible solution of the braid equation $c : R \otimes R \to R \otimes R$, such that the structure maps of $R$ commute with $c$ and the comultiplication $\Delta$ is an algebra map, with respect to the multiplication in $R \otimes R$ twisted by $c$. The main result of this paper states necessary and sufficient conditions on the pair $(\sigma, \tau)$ in order that $R$ is a braided Hopf algebra with respect to a uniquely determined braiding $c$; cf. Theorem 2.10. It turns out that $c$ is diagonal in the canonical basis of $R$. Furthermore, $R$ is an extension of $kG$ by $kF$ in this case; and the construction by Andruskiewitsch and Sommerh¨ auser explained in [S, Ch. 3] is a particular case of the present one.

A pair $(\sigma, \tau)$, making $R$ into a braided Hopf algebra, will be called a braided compatible datum for the matched pair $\triangleright : G \times F \to F$, $\triangleleft : G \times F \to G$. The main classification result in [S] states that, provided $k$ is algebraically closed of characteristic zero, every cocommutative cosemisimple braided Hopf algebra over a cyclic group of order $p$ fits into this construction, for an appropriate matched pair with $\triangleright$ trivial, and a braided compatible datum $\sigma$ and $\tau = 1$.

The main application we have in view is the construction of new examples of Hopf algebras via bosonization or Radford biproducts. Assume that $R$ is a braided Hopf algebra. By general reasons, there exists always a Hopf algebra $H$ such that $R$, with this braiding, is a braided Hopf algebra in the category of Yetter-Drinfeld modules over $H$. We shall say that $R$ is realizable over $H$. It is interesting to determine all possible Hopf algebras $H$ such that $R$ is realizable over $H$, for a fixed $R$.

Note that, if the characteristic of $k$ does not divide the orders of $F$ and $G$, then $R$ is semisimple and cosemisimple. Thus, if this holds and if $H$ is semisimple and cosemisimple, so is the biproduct $R \# H$.

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There are examples of a braided Hopf algebra $R$ and a Hopf algebra $H$, with $R$ realizable over $H$, but where neither $k^G$ nor $k^F$ are realizable over $H$. We shall say that the extension $k^G \hookrightarrow R \rightarrow k^F$ is realizable if $R$, $k^G$ and $k^F$ are realizable, and the inclusion and projection maps in the extension are morphisms in the category as well.

We study a distinguished class of realizations of the extension $k^G \hookrightarrow R \rightarrow k^F$ over the group algebra $H = kG$ of a finite group $C$; these are the realizations where both the action and the coaction are diagonal in the canonical basis of $R$. Our main result in this direction appears in Theorem 3.5; it allows to avoid the lengthy conditions in Theorem 2.10. The biproduct $R \# k^C$ can be obtained by iterated extensions from group algebras and dual group algebras.

We present explicit examples of the general construction over the field $\mathbb{C}$ of complex numbers. Let $p$ and $q$ be distinct prime numbers. In Proposition 2.18 we give examples of braided compatible data, in the case where both actions $\langle$ and $\triangleright$ are trivial: we obtain noncommutative and noncocommutative braided Hopf algebras of dimension $p^4$. Another family of examples, together with a diagonal realization over the group $\mathbb{Z}_p \oplus \mathbb{Z}_p$, is constructed in Proposition 4.8, as a generalization of the examples in $[S]$; these are in general not commutative and not cocommutative of dimension $p^2q$.

The paper is organized as follows. In Section 2 we present the construction of a braided Hopf algebra as a bicrossed product. We give a cohomological interpretation of the required conditions and consider the problem of equivalences of braided extensions. In Section 3 we look at those braided Hopf algebras arising from our construction which admit a diagonal realization over a finite group $C$; explicit examples of this situation are constructed in Section 4.

By suggestion of the referee, we have also included at the end of the paper an appendix where the main constructions are presented in an alternative language. We point out that this formalism can be interpreted in the language of double categories as defined by Ehresmann.

1.1. **Notation.** All groups are denoted multiplicatively, unless explicitly stated. If $G$ is a finite group, we denote by $k^G$ the algebra of functions on $G$; and by $\delta_g$ the canonical idempotent $\delta_g(h) = \delta_{g,h}$, $h \in G$. These form a basis of $k^G$, the dual basis being the basis $(g)_g \in G$ of the group algebra $kG$. The center of $G$ is denoted by $Z(G)$ and the group of homomorphisms $G \rightarrow k^\times$ is denoted by $\widehat{G}$.

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2. **Extensions of braided Hopf algebras arising from matched pairs.**

2.1. **Matched pairs.** We briefly recall the definition of matched pair of groups. See $[M]$ for further details.

Let $F$ and $G$ be finite groups together with a right action of $F$ on the set $G$, and a left action of $G$ on the set $F$

$$\langle : G \times F \rightarrow G, \quad \triangleright : G \times F \rightarrow F.$$  

We shall assume that these actions satisfy the following conditions:

\begin{align}
(2.1) \quad s\triangleright xy &= (s\triangleright x)(s\langle x)\triangleright y), \\
(2.2) \quad st\langle x &= (s\langle t)\langle x)(t\langle x),
\end{align}

for all $s, t \in G$, $x, y \in F$. It follows that $s\triangleright 1 = 1$ and $1\langle x = 1$, for all $s \in G$, $x \in F$.

Such a data of groups and compatible actions is called a matched pair of groups. Given finite groups $F$ and $G$, providing them with a pair of compatible actions is equivalent to finding a group $\Sigma$ together with an exact factorization $\Sigma = FG$. 

We fix from now on a matched pair of groups \( \triangleleft : G \times F \to G, \triangleright : G \times F \to F \). We note the following consequence of the compatibility conditions (2.1) and (2.2), whose proof is straightforward.

**Lemma 2.1.** We have, for all \( t \in G \) and \( y \in F \),

(i) \( (t \triangleleft y)^{-1} = t^{-1} \triangleright (t \triangleright y) \);

(ii) \( (t \triangleright y)^{-1} = (t \triangleleft y) \triangleright y^{-1} \).

\( \square \)

2.2. Bicrossed products. We consider the associated left action of \( F \) on \( \mathbb{k}^G \), \( (x, \phi)(g) = \phi(g \lhd x) \), \( \phi \in \mathbb{k}^G \); in particular, \( x \delta_g = \delta_{g \lhd x}^{-1} \). Let \( \sigma : F \times F \to (\mathbb{k}^\times)^G \) be a normalized 2-cocycle. If we write \( \sigma = \sum_{g \in G} \sigma_g \delta_g \), then the cocycle and the normalizing conditions read, respectively, as follows:

\[
\begin{align*}
\sigma_{g \lhd x}(y, z)\sigma_g(x, yz) &= \sigma_g(xy, z)\sigma_g(x, y), \\
\sigma_g(x, 1) &= 1 = \sigma_g(1, x),
\end{align*}
\]

We also consider the associated right action of \( G \) on \( \mathbb{k}^F \), \( (\psi, g)(x) = \psi(x \rhd g) \), \( \psi \in \mathbb{k}^F \). Let \( \tau = \sum_{x \in F} \tau_x \delta_x : G \times G \to (\mathbb{k}^\times)^F \) be a normalized 2-cocycle; so that we have

\[
\begin{align*}
\tau_x(gh, k)\tau_{k \lhd x}(g, h) &= \tau_x(h, k)\tau_{x}(g, hk), \\
\tau_x(g, 1) &= 1 = \tau_x(1, g),
\end{align*}
\]

We endow the vector space \( R = \mathbb{k}^G \otimes \mathbb{k}^F \) with the crossed product algebra structure and the crossed coproduct coalgebra structure. By abuse of terminology, we refer to \( R \) as a bicrossed product.

We shall use the notation \( \delta_g x \) to indicate the element \( \delta_g \otimes x \in R \). Then the multiplication of \( R \) is determined by

\[
(\delta_g x)(\delta_h y) = \delta_{g \lhd x, h} \sigma_g(x, y)\delta_g xy, \quad g, h \in G, x, y \in F;
\]

and the comultiplication is determined by

\[
\Delta(\delta_g x) = \sum_{t \in G} \tau_x(t, t^{-1} g) \delta_t(t^{-1} g \lhd x) \otimes \delta_{t^{-1} g} x, \quad g \in G, x \in F.
\]

In the following lemma we give the necessary normalization conditions on \( \sigma \) and \( \tau \) in order that the unit and counit maps preserve the coalgebra and algebra structures, respectively.

**Lemma 2.2.** (i) \( \epsilon \otimes \epsilon : R \to \mathbb{k} \) is an algebra map if and only if

\[
\sigma_1(g, h) = 1, \quad \forall g, h \in G;
\]

(ii) \( \Delta(1) = 1 \otimes 1 \) if and only if

\[
\tau_1(x, y) = 1, \quad \forall x, y \in F.
\]

**Proof.** Straightforward. \( \square \)

We next show that the formula for the antipode still provides the inverse of the identity, even if \( \sigma \) and \( \tau \) do not satisfy any compatibility condition.

**Lemma 2.3.** The map \( \mathcal{S} \) defined by

\[
\mathcal{S}(\delta_g x) = \sigma_{(g \lhd x)^{-1}}((g \rhd x)^{-1}) \tau_{x}(g^{-1}, g)^{-1} \delta_{(g \lhd x)^{-1}} (g \rhd x)^{-1}
\]

is the inverse of the identity map with respect to the convolution product in \( \text{End} R \).
Proof. Letting \( x = z = y^{-1} \) in the cocycle condition (2.3), we get
\[
\sigma_{g,xy}(x^{-1}, x) = \sigma_g(x, x^{-1}), \quad g \in G, x \in F.
\]
Combining (2.12) with Lemma 2.1 (i), we have
\[
\sigma_{(t,ax)}^{-1}((t\triangleright x)^{-1}, t\triangleright x) = \sigma_{t^{-1}}(t\triangleright x, (t\triangleright x)^{-1}), \quad t \in G, x \in F.
\]
Let \( X = \delta g x \in R \). We compute
\[
X_1S(X_2) = \sum_{t \in G} t_x(g t^{-1}, t) \delta_{g t^{-1}}(t \triangleright x) S(\delta t x)
\]
\[
= \sum_{t \in G} t_x(g t^{-1}, t) \sigma_{(t,ax)}^{-1}((t\triangleright x)^{-1}, t\triangleright x)^{-1} t_x(t^{-1}, t)^{-1} \delta_{g t^{-1}}(t\triangleright x) \delta_{(t,ax)^{-1}}(t\triangleright x)^{-1}
\]
\[
= \sum_{t \in G} t_x(g t^{-1}, t) \sigma_{(t,ax)}^{-1}((t\triangleright x)^{-1}, t\triangleright x)^{-1} t_x(t^{-1}, t)^{-1} \sigma_{g t^{-1}}(t\triangleright x, (t\triangleright x)^{-1}) \delta_{g t^{-1}}(t\triangleright x) \delta_{(t,ax)^{-1}}(t\triangleright x)^{-1} \delta_{g t^{-1}}
\]
\[
= \delta_{g,1} \sum_{t \in G} \delta t = \delta_{g,1} 1 = \epsilon(X) 1.
\]
In the fourth equality we have used (2.13) and the fact that \( \delta_{g t^{-1},(t,ax)^{-1}}(t\triangleright x)^{-1} = \delta_{g t^{-1},t^{-1}} \), which follows from Lemma 2.1 (i). This proves the lemma.

2.3. Braiding. We shall consider in this subsection 2-cocycles \( \sigma : F \times F \to (k^\times)^G \) and \( \tau : G \times G \to (k^\times)^F \) satisfying the normalization conditions (2.4), (2.6), (2.9) and (2.10).

We shall give necessary and sufficient conditions in order that the resulting algebra and coalgebra structures on \( R \) associated to the data \( < \!, \! \triangleright, \sigma \) and \( \tau \) make it a braided Hopf algebra in the sense of [T2, Section 5]. That is, we shall determine when there exists an invertible linear map \( c : R \otimes R \to R \otimes R \), which satisfies the following conditions:

(i) \( c \) is a solution of the braid equation \((c \otimes \text{id})(\text{id} \otimes c)(c \otimes \text{id}) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c)\);

(ii) the structure maps of \( R \) commute with the braidings. According to the definition in [12], \( m \) commutes with \( c \) if and only if
\[
c(\text{id} \otimes m) = (m \otimes \text{id})c_{1,2}
\]
\[
(\text{id} \otimes m)c_{2,1} = c(m \otimes \text{id}),
\]
where the braidings \( c_{1,2} : R \otimes R^\otimes 2 \to R^\otimes 2 \otimes R \) and \( c_{2,1} : R^\otimes 2 \otimes R \to R \otimes R^\otimes 2 \), are defined by
\[
c_{1,2} := (\text{id} \otimes c)(c \otimes \text{id}), \quad c_{2,1} := (c \otimes \text{id})(\text{id} \otimes c).
\]
Similarly, \( \Delta \) commutes with \( c \) if and only if
\[
(\Delta \otimes \text{id})c = c_{1,2}(\text{id} \otimes \Delta)
\]
\[
(\text{id} \otimes \Delta)c = c_{2,1}(\Delta \otimes \text{id}).
\]
(iii) \( \epsilon : R \to k \) is an algebra map, \( \Delta(1) = 1 \otimes 1 \), and \( \Delta : R \to R \otimes R \) is an algebra map; here the product in \( R \otimes R \) is "twisted" by \( c \): \( m_{R \otimes R} = (m_R \otimes m_R)(\text{id} \otimes c \otimes \text{id}) \). Moreover, the identity map has a convolution inverse \( S \), called the antipode.
Remark 2.4. In order that the product in $R \otimes R$ be associative, we must require that the multiplication map $R \otimes R \to R$ commutes with the braiding.

Definition 2.5. Let $S$ be a braided Hopf algebra and let $H$ be a Hopf algebra. We shall say that $S$ is realizable over $H$ if it can be endowed with a left action $H \otimes R \to R$ and a left coaction $R \to H \otimes R$, such that $S$ is a braided Hopf algebra in the category $\mathcal{H}_H \mathcal{YD}$ of Yetter-Drinfeld modules over $H$, with the braiding $\sigma$ being the corresponding braiding in $\mathcal{H}_H \mathcal{YD}$.

One may also consider the related notion of being realizable over a quasitriangular Hopf algebra; this point of view will not be discussed in this paper.

Let $S$ be a finite dimensional braided Hopf algebra. Recall from [12, Theorem 5.7], that $S$ is realizable over a (non unique) Hopf algebra $H$ if and only if $S$ is a rigid braided Hopf algebra, which means that the braiding $\sigma$ is rigid.

We define $c : R \otimes R \to R \otimes R$ in the form

$$c(\delta_g x \otimes \delta_h y) = Q_{g,h}^{x,y} \delta_h y \otimes \delta_g x, \quad g, h \in G, \ x, y \in F,$$

where $Q : G^2 \times F^2 \to \mathbb{k}^\times$ is a map. Note that $c$ is diagonal and thus automatically satisfies the braid equation. Moreover, since the scalars $Q_{g,h}^{x,y}$ are non-zero by assumption, $c$ is rigid.

The following proposition generalizes [14, Proposition 4.7].

Proposition 2.6. $(R, c)$ is a prebraided Hopf algebra if and only if the following compatibility condition holds, for all $s, t \in G, \ x, y \in F$:

$$\sigma_{ts}(x, y) \tau_{xy}(t, s) = Q_{s,t,q(s\triangleright x)}^{x,(s\triangleright x)} \tau_x(t, s) \tau_y(t \triangleleft (s \triangleright x), s \triangleleft x) \sigma_t(s \triangleleft x, (s \triangleleft x) \triangleright y) \sigma_s(x, y).$$

Proof. Note that, as a consequence of (2.18) and (2.19), the compatibility condition (2.19) implies the following normalization conditions on $Q$:

$$Q_{g,h}^{1,y} = Q_{g,h}^{x,1} = Q_{1,h}^{x,y} = Q_{g,1}^{x,y} = 1.$$ 

We have already established the existence of an antipode. By Lemma 2.2 the counit is a morphism of algebras and $\Delta(1) = 1 \otimes 1$. We shall prove that condition (2.19) is equivalent to the comultiplication $\Delta : R \to R \otimes R$ being a morphism of algebras.

Let $g, h \in G, \ x, y \in F$. We denote by $\bullet$ the product in $R \otimes R$ twisted by $c$. We compute

$$\Delta(\delta_g x) \bullet \Delta(\delta_h y) = \sum_{s, t \in G} \tau_x(t, t^{-1} g) \tau_y(s, s^{-1} h) \left( (\delta_t(t^{-1} g \triangleright x) \otimes \delta_{t^{-1} g x}) \bullet (\delta_s(s^{-1} h \triangleright y) \otimes \delta_{s^{-1} h x y}) \right) \quad \text{for } s, t \in G.$$
using the compatibility conditions (2.1) and (2.2), this equals
\[
\delta_{g,h,x}^{-1} \sum_{t \in G} \tau_x(t, t^{-1} g) \tau_y(t \circ (t^{-1} g \triangleright x), (t \circ (t^{-1} g \triangleright x))^{-1} h) \ Q^{x,(t^{-1} g \triangleright x) \triangleright y}_{t^{-1} g, t \circ (t^{-1} g \triangleright x)} \sigma(t^{-1} g \triangleright x, (t^{-1} g \triangleright x) \triangleright y) \sigma_{t^{-1} g}(x, y) \delta_{t^{-1} g}(x, y) \delta_{t^{-1} gxy}.
\]

On the other hand, we have
\[
\Delta m(\delta_{g,x} \otimes \delta_{h,y}) = \delta_{g,h,x^{-1}} \sigma_{g}(x, y) \sum_{s \in G} \tau_{xy}(s, s^{-1} g) \delta_{s}(s^{-1} g \triangleright (xy)) \otimes \delta_{s^{-1} gxy};
\]
Hence, we find that \( \Delta \) is an algebra map if and only if
\[
\delta_{g,h,x}^{-1} \sigma_{g}(x, y) \tau_{xy}(t, t^{-1} g) = \delta_{g,h,x}^{-1} \tau_{x}(t, t^{-1} g) \tau_{y}(t \circ (t^{-1} g \triangleright x), (t \circ (t^{-1} g \triangleright x))^{-1} h) \ Q^{x,(t^{-1} g \triangleright x) \triangleright y}_{t^{-1} g, t \circ (t^{-1} g \triangleright x)} \sigma(t^{-1} g \triangleright x, (t^{-1} g \triangleright x) \triangleright y) \sigma_{t^{-1} g}(x, y).
\]
Letting \( s = t^{-1} g \), this condition is equivalent to the claimed one. This finishes the proof of the proposition.

\( \square \)

Proposition 2.6 allows us to construct, for any normalized 2-cocycles \( \sigma \) and \( \tau \), a prebraided Hopf algebra structure on \( R \).

**Proposition 2.7.** There exists a unique braiding \( c : R \otimes R \to R \otimes R \) making \( R \) into a prebraided Hopf algebra: it is given by (2.18), where \( Q : G^2 \times F^2 \to k^\times \) is the map defined in the form
\[
Q^{x,y}_{g,h} := \sigma_{(h \circ (g \triangleright x)^{-1}) g} \sigma_{h \circ (g \triangleright x)^{-1}} \sigma_{y} \tau_{x(y \triangleright x)^{-1} y} \tau_{(g \triangleright x)^{-1} y} \tau_{(h \circ (g \triangleright x)^{-1})},
\]
for all \( g, h \in G, x, y \in F \).

In particular, every braided Hopf algebra structure on \( R \) is realizable over some Hopf algebra \( H \).

An alternative proof of the last statement in the proposition is given in Lemma 3.9 below.

**Proof.** It is easy to see that formula (2.22) is equivalent to (2.19). Therefore, if \( c \) is given by (2.22), \( (R, c) \) is a prebraided Hopf algebra.

It follows from [51], that the associativity, coassociativity, unit, counit and antipode axioms on \( R \), together with the condition \( \Delta m = (m \otimes m)(\text{id} \otimes c \otimes \text{id})(\Delta \otimes \Delta) \), uniquely determine the braiding \( c \) by means of the formula \( c = (m \otimes m)(S \otimes \Delta m \otimes S)(\Delta \otimes \Delta) \). Actually, the argument in [51] does not need the associativity of \( R \otimes R \). Therefore, for fixed \( \sigma \) and \( \tau \), the braiding \( c \) making \( R \) into a prebraided Hopf algebra is unique, and has necessarily the prescribed form.

In particular, all such braidings are ‘diagonal’ in the basis \( \delta_{g,x}, g \in G, x \in F \), and they are moreover rigid. It follows that every braided Hopf algebra structure on \( R \) is realizable, as claimed.

\( \square \)

**Remark 2.8.** The normalization conditions (2.9), (2.10) in Lemma 2.2 are equivalent to the normalization conditions on \( Q \) in (2.20), in view of (2.22).

The following lemma gives necessary and sufficient conditions in order that condition (ii) be satisfied.
Lemma 2.9. (i) The multiplication map \( m : R \otimes R \to R \) commutes with \( c \) if and only if
\[
(Q_{g,s}^{x,y})' = Q_{g,s}^{x,y} Q_{g,s}^{x,z}, \\
(2.24)
\]
\[
(Q_{g,s}^{y,z})' = Q_{g,s}^{y,z} Q_{g,s}^{y,s}. \\
(2.25)
\]

(ii) The comultiplication map \( \Delta : R \to R \otimes R \) commutes with \( c \) if and only if
\[
(2.26)
\]
\[
\forall x, y, z \in F, g, s \in G. \\
(2.27)
\]

Proof. (i) Conditions (2.14), (2.17) are equivalent, in this case, to (2.28) and (2.24), respectively.

(ii) Similarly, conditions (2.16), (2.17) correspond to (2.25) and (2.26), respectively. \( \square \)

The following theorem is a consequence of Lemma 2.9 and Proposition 2.7. It gives the necessary and sufficient conditions on \( \sigma \) and \( \tau \) in order that \( R \) be a braided Hopf algebra.

Theorem 2.10. Let \( c : R \otimes R \to R \otimes R \) be given by (2.18), where \( Q : G^2 \times F^2 \to \mathbb{K}^\times \) is the map defined by (2.22). Then \( (R, c) \) is a braided Hopf algebra if and only if the following compatibility conditions hold, for all \( g, s, t \in G, x, y, z \in F \):

\[
\sigma_{(s \circ (g \circ x))^{-1} g}(x, (g' \circ x)^{-1} \circ y) \sigma_{s \circ (g \circ x)^{-1} g}(g' \circ x, y) = \sigma_{g \circ x}(x, (g' \circ x)^{-1} \circ y) \\
(2.23)
\]
\[
\tau_{(g' \circ x)^{-1} \circ y}(s \circ (g \circ x)^{-1} g)
\]
\[
\tau_{g \circ x}(s \circ (g \circ x)^{-1} g) = \tau_{(s \circ (g \circ x)^{-1} g)(x, (g' \circ x)^{-1} \circ y)} \\
(2.24)
\]
\[
\tau_{(s \circ (g \circ x)^{-1} g)}(s \circ (g \circ x)^{-1} g)(g' \circ x, y) = \tau_{s \circ (g \circ x)^{-1} g}(g' \circ x, y) \\
(2.25)
\]
\[
\tau_{s \circ (g \circ x)^{-1} g}(s \circ (g \circ x)^{-1} g)(g' \circ x, y) = \tau_{s \circ (g \circ x)^{-1} g}(g' \circ x, y) \\
(2.26)
\]
\[
\tau_{s \circ (g \circ x)^{-1} g}(s \circ (g \circ x)^{-1} g)(g' \circ x, y) = \tau_{s \circ (g \circ x)^{-1} g}(g' \circ x, y) \\
(2.27)
\]
Remark

Alternate way to construct braided Hopf algebras is indicated in Theorem 3.5 below.

We now give a cohomological interpretation of Proposition 2.6. By [EG], the order of $R$-matrix of $D$ is the canonical $R$-matrix of $D(H)$. By [EG], the order of $R_{21}R$ is finite. On the other hand, we have $c^2(\delta_y x \otimes \delta_h y) = Q_{g,h}^{x,y} Q_{h,g}^{y,x} \delta_h y \otimes \delta_y x$, for all $g, h \in G, x, y \in F$. Thus the claim follows.

Remark 2.12. Suppose that $R$ is realizable over a (finite-dimensional) semisimple Hopf algebra $H$. Then the symmetrizations $Q_{g,h}^{x,y} Q_{h,g}^{y,x}$ are roots of unity, for all $x, y \in F, g, h \in G$.

Proof. The map $c^2 : R \otimes R \to R \otimes R$ is given by the action of $R_{21}R$ on $R \otimes R$, where $R$ is the canonical $R$-matrix of $D(H)$. By [EG], the order of $R_{21}R$ is finite. On the other hand, we have $c^2(\delta_y x \otimes \delta_h y) = Q_{g,h}^{x,y} Q_{h,g}^{y,x} \delta_h y \otimes \delta_y x$, for all $g, h \in G, x, y \in F$. Thus the claim follows.

Remark 2.13. Suppose that $R$ is a braided Hopf algebra. Then, by [L2], the antipode $S$ commutes with $c$; that is, we have $c(S \otimes \text{id}) = (\text{id} \otimes S)c$. This amounts to the condition $Q_{g,h}^{x,y} = Q_{h,g}^{(g \otimes x)^{-1}x, (g \otimes y)^{-1}y}$, for all $x, y \in F, g, h \in G$, which corresponds to the following relationship between $\sigma$ and $\tau$:

$$
\sigma_{(h \circ(g \otimes x)^{-1}y)} (x, (g \otimes x)^{-1}y) \sigma_{h \circ(g \otimes x)^{-1}y} (g \otimes x, y)^{-1} \sigma_y (x, (g \otimes x)^{-1}y)^{-1}
$$

$$
\tau_{h \circ(g \otimes x)^{-1}y} (h \circ(g \otimes x)^{-1}y) \tau_{g \circ x} (h \circ(g \otimes y)^{-1}y)^{-1} \tau_x (h \circ(g \otimes x)^{-1}y)^{-1}
$$

$$
= \sigma_{h \circ(g \otimes x)^{-1}y} \tau_{g \circ x} (h \circ(g \otimes y)^{-1}y)^{-1} \tau_x (h \circ(g \otimes x)^{-1}y)^{-1}
$$

$$
\sigma_{h \circ(g \otimes x)^{-1}y} (x, (g \otimes x)^{-1}y)^{-1} \tau_{g \circ x} (h \circ(g \otimes y)^{-1}y)^{-1} \sigma_{h \circ(g \otimes x)^{-1}y} (h \circ(g \otimes y)^{-1}y)^{-1}
$$

for all $x, y \in F, g, h \in G$.

We now give a cohomological interpretation of Proposition 2.6.
Consider the following double complex:

\[
\begin{array}{ccc}
\vdots & \vdots & \\
\uparrow & \uparrow & \\
(2.27) & C^\cdot := \text{Map}_+(G^2 \times F, k^\times) & \xrightarrow{\delta} \text{Map}_+(G^2 \times F^2, k^\times) \xrightarrow{\delta'} \ldots \\
\delta^\dagger & \delta^\dagger & \\
\text{Map}_+(G \times F, k^\times) & \xrightarrow{\delta} \text{Map}_+(G \times F^2, k^\times) \xrightarrow{\delta'} \ldots ,
\end{array}
\]

where, for all \(n, m \geq 1\), \(\text{Map}_+(G^n \times F^m, k^\times)\) is the abelian group of \(k^\times\)-valued functions \(f\) on \(G^n \times F^m\) with the property that \(f(g_n, \ldots, g_1; x_1, \ldots, x_m) = 1\), if either one of \(g_1, \ldots, g_n\) or \(x_1, \ldots, x_m\) is equal to 1, and the maps \(\delta\) and \(\delta'\) are defined by

\[
\delta f(g_q, \ldots, g_1; x_1, \ldots, x_{p+1}) = f(g_q < (g_{q-1} \ldots g_1) > x_1), \ldots, g_1 < (g_1) > x_2, \ldots, x_{p+1})
\]

\[
\times \prod_{i=1}^{p} f(g_q, \ldots, g_1; x_1, \ldots, x_i x_{i+1}, \ldots, x_{p+1})^{(-1)^i} \times f(g_q, \ldots, g_1; x_1, \ldots, x_p)^{(-1)^{p+1}};
\]

\[
\delta' f(g_{q+1}, \ldots, g_1; x_1, \ldots, x_p)^{(-1)^p} = f(g_{q+1}, \ldots, g_2; g_1 < x_1 > x_1, (g_1) > x_2), \ldots, (g_1 < x_1 \ldots x_{p-1}) > x_p)
\]

\[
\times \prod_{i=1}^{q} f(g_{q+1}, \ldots, g_{i+1} g_i, \ldots, g_1; x_1, \ldots, x_p)^{(-1)^i} \times f(g_q, \ldots, g_1; x_1, \ldots, x_p)^{(-1)^{q+1}}.
\]

It is known that the necessary and sufficient condition for \(R\) to be a (usual) Hopf algebra is that the pair \((\sigma, \tau)\) be a 1-cocycle in the total complex \(\text{Tot}(C^\cdot)\), and moreover that the assignment \((\sigma, \tau) \mapsto R\) defines an isomorphism \(H^1(\text{Tot}(C^\cdot)) \simeq \text{Opext}(k^G, k^F)\). See [M, Proposition 5.2].

Let \(\delta^{\text{tot}}\) denote the coboundary map in the total complex \(\text{Tot}(C^\cdot)\) and let

\[
p_i : \text{Tot}(C^\cdot)^n = \oplus_{i=1}^{n} \text{Map}_+(G^{n-i} \times F^i, k^\times) \to \text{Map}_+(G^{n-i} \times F^i, k^\times),
\]

be the projection on the \(i\)-th. coordinate. Conditions (2.4), (2.6), (2.9) and (2.10) say that \((\tau, \sigma)\) belong to \(\text{Map}_+(G^2 \times F, k^\times) \oplus \text{Map}_+(G \times F^2, k^\times)\), and conditions (2.3), (2.7) amount to \(p_1(\delta^{\text{tot}}(\tau, \sigma)) = p_3(\delta^{\text{tot}}(\tau, \sigma)) = 1\).

**Corollary 2.14.** Let \(c : R \otimes R \to R \otimes R\) be as in (2.18). Then \(R\) is a prebraided Hopf algebra if and only if

\[
Q_{g,h}^{x,y} = p_2(\delta^{\text{tot}}(\tau, \sigma))(g, h < (g \triangleright x)^{-1}; x, (g \triangleright x)^{-1} \triangleright y) = \delta^{\text{tot}}(\tau, \sigma)(g, h < (g \triangleright x)^{-1}; x, (g \triangleright x)^{-1} \triangleright y),
\]

for all \(g, h \in G, \, x, y \in F\).

**Proof.** This is a reformulation of Proposition 2.6. \(\square\)

**Remark 2.15.** It is natural to consider the following question: given \(Q \in \text{Map}_+(G^2 \times F^2, k^\times)\), find all pairs of normalized cocycles \((\tau, \sigma)\) such that \(R\) is a prebraided Hopf algebra with braiding determined by \(Q\) as in (2.18). This (possibly empty) space is a torsor over \(\text{Opext}(k^G, k^F)\).

2.4. **Braided compatible data for trivial actions.** Along this subsection we shall assume that both actions \(\triangleright\) and \(<\) are trivial; that is, \(\Sigma = F \times G\). We discuss the compatibility conditions on the cocycles \(\sigma\) and \(\tau\) in order that the corresponding bicrossed product \(R\) is a braided Hopf algebra with non-trivial braiding.
Let $\sigma : F \times F \to (\mathbb{k}^\times)^G$ and $\tau : G \times G \to (\mathbb{k}^\times)^F$ be 2-cocycles satisfying the normalization conditions $(2.4)$, $(2.6)$, $(2.9)$ and $(2.10)$. We keep the notation and conventions in $(2.3)$.

By triviality of $\triangleright$ and $\triangleleft$, we may regard $\sigma$ and $\tau$ as normalized maps $\sigma : G \to Z^2_+(F, \mathbb{k}^\times)$, $\tau : F \to Z^2_+(G, \mathbb{k}^\times)$, and as such we may take their differentials $\partial \sigma \in Z^2_+(G, Z^2_+(F, \mathbb{k}^\times))$, and $\partial \tau \in Z^2_+(F, Z^2_+(G, \mathbb{k}^\times))$.

Lemma 2.16. (i) Let $x, y \in F$, $g, h \in G$. We have

$\partial \sigma(h, g) (x, y) \partial \tau(x, y) (g, h) = 2$.  

(ii) Suppose that $\tau : F \to Z^2_+(G, \mathbb{k}^\times)$ is a group homomorphism. Then $R$ is a braided Hopf algebra if and only if $\partial \sigma \in \text{Hom}(G/[G, G] \otimes G/[G, G], \text{Hom}(F/[F, F] \otimes F/[F, F], \mathbb{k}^\times))$.

In this case, the braiding $c : R \otimes R \to R \otimes R$ is trivial if and only if $\sigma$ is a group homomorphism.

Proof. Part (i) is straightforward. If $\tau$ is a homomorphism, then $Q^x y = \partial \sigma(h, g)(x, y)$. Thus, part (ii) is a consequence of (i) and Lemma 2.9. Clearly, $c$ is trivial if and only if $Q = 1$, if and only $\sigma$ is a group homomorphism.

Remark 2.17. Let us fix $\tau : F \to Z^2_+(G, \mathbb{k}^\times)$ a group homomorphism. If we start by taking $\sigma : G \to \text{Hom}(G/[G, G] \otimes G/[G, G], \text{Hom}(F/[F, F] \otimes F/[F, F], \mathbb{k}^\times)$, then $\sigma$ will be a 2-cocycle on $F$, for all $g \in G$ (every bicharacter is), and the image of $\partial \sigma$ will be contained in $\text{Hom}(F/[F, F] \otimes F/[F, F], \mathbb{k}^\times)$. In order that the data be compatible, but $c$ be not trivial, we need to have that $\sigma$ is a bicharacter, but $\sigma$ is not a group homomorphism.

Let now $\mathbb{k}$ be the field $\mathbb{C}$ of complex numbers. Let $p$ be an odd prime number and let $G = F = \mathbb{F}_p \oplus \mathbb{F}_p$ be 2-dimensional vector spaces over the field $\mathbb{F}_p$ with $p$ elements. We use additive notation in both $G$ and $F$. The elements of $F$ will be denoted with roman letters $(x, y), (x', y'), \ldots$, and the elements of $G$ will be denoted with greek letters $(\alpha, \beta), (\alpha', \beta'), \ldots$.

We have $H^2(F, \mathbb{C}^\times) \cong \text{Hom}(\Lambda^2 F, \mathbb{C}^\times)$ is of order $p$, and it consists of the classes of the cocycles

$((x, y), (x', y')) \mapsto \exp \frac{2\pi i n}{p} (xy' - x'y)$,

where $n$ runs over the integers modulo $p$.

Proposition 2.18. Let $a$ and $b$ be integers modulo $p$ such that $ab \neq 0 \mod p$. Let $\tau \in \text{Hom}(F, Z^2_+(G, \mathbb{C}^\times))$ be given by

$\tau_{(x, y)} ((\alpha, \beta), (\alpha', \beta')) = \exp \left( \frac{2\pi i}{p} (x + y)(\alpha \beta' - \alpha' \beta) \right)$, \hspace{1em} $x, y, \alpha, \beta, \alpha', \beta' \in \mathbb{F}_p,$

and let $\sigma : G \to \text{Hom}(\Lambda^2 F, \mathbb{C}^\times)$ be given by

$\sigma_{(x, y)} ((x, y), (x', y')) = \exp \left( \frac{2\pi i}{p} (a \alpha^2 + b \beta^2)(xy' - x'y) \right)$, \hspace{1em} $x, y, x', y', \alpha, \beta \in \mathbb{F}_p.$

Then the associated bicrossed product $R$ is a braided Hopf algebra, with non-trivial braiding $c$ given by $(2.18)$, and

$Q^{(x, y), (x', y')}_{(\alpha, \beta), (\alpha', \beta')} = \exp \left( \frac{4\pi i}{p} (a \alpha \alpha' + b \beta \beta')(xy' - x'y) \right)$, \hspace{1em} $x, y, x', y', \alpha, \beta, \alpha', \beta' \in \mathbb{F}_p.$

Moreover, $R$ is not commutative and not cocommutative.
Proof. The proof follows from Lemma 2.10. Note that, for instance, $Q_{(1,0),(1,0)}^{(1,0),(1,0)} = \exp\left(\frac{4\pi i p}{a}\right)$, which is not equal to 1 since $p$ is odd and $a \neq 0 \mod p$. Finally, it is not difficult to see that $R$ is not commutative and not cocommutative.

Remark 2.19. Observe that $Q = \partial \sigma$ is the symmetric bilinear map associated with the quadratic map $\sigma$. Also, $\tau$ is obtained as the composition of the epimorphism $F \to \mathbb{F}_p$, $(x, y) \mapsto x + y$, with the natural isomorphism $(\mathbb{F}_p, +) \simeq \text{Hom}(\Lambda^2 F, \mathbb{C}^\times)$.

2.5. A categorical exact sequence. We shall assume in this subsection that $(R, c)$ is a braided Hopf algebra. By Proposition 2.7, the braiding $c$ is as described in Proposition 2.7, where in addition, the conditions in Theorem 2.10 are satisfied.

Let $(V, c)$ be a braided vector space. Recall [T2] that a subspace $W$ of $V$ is called categorical if $c(V \otimes W) \subseteq W \otimes V$ and $c(W \otimes V) \subseteq V \otimes W$. In particular, if $W$ is a categorical subspace, then it is a braided subspace with respect to $c|_{W \otimes W} : W \otimes W \to W \otimes W$.

A quotient space $p : V \to U$ will be called categorical if the kernel of $p$ is a categorical subspace of $V$. In this case, $U$ is a quotient braided space with respect to the braiding $(p \otimes p)(c) : U \otimes U \to U \otimes U$.

The definition of extension of Hopf algebras may be generalized to braided Hopf algebras as follows. We shall assume in this subsection that $(R, c)$ is a braided Hopf algebra, which can be constructed from the matched pair arising from the exact factorization $\Sigma = GF$, and it fits into an exact sequence of braided Hopf algebras

\begin{equation}
1 \longrightarrow S \quad \overset{\iota}{\longrightarrow} \quad R \quad \overset{\pi}{\longrightarrow} \quad T \quad \longrightarrow \quad 1
\end{equation}

where the braiding in $k^G$ and $k^F$ is the usual flip.

\begin{equation}
1 \longrightarrow k^G \quad \overset{\iota}{\longrightarrow} \quad R \quad \overset{\pi}{\longrightarrow} \quad k^F \quad \longrightarrow \quad 1
\end{equation}

Proof. The kernel of $\pi$ is equal to the span of all elements $\delta_g x$, where $g \in G \setminus \{1\}$, and $x \in F$. Note that since the braiding $c$ is diagonal in the basis $\delta_g x$, $g \in G$, $x \in F$, it follows that $\iota$ and $\pi$ are categorical.

By construction, $\iota$ is an algebra inclusion and $\pi$ is a coalgebra surjection. Also, it is not difficult to see that condition (2.3) is equivalent to $\iota : k^G \to R$ being a coalgebra map, while condition (2.10) is equivalent to $\pi : R \to k^F$ being an algebra map. The rest of the proposition follows easily.

Remark 2.21. $R^*$ is also a braided Hopf algebra, which can be constructed from the matched pair arising from the exact factorization $\Sigma = GF$, and it fits into an exact sequence of braided Hopf algebras

\[1 \longrightarrow k^F \quad \overset{\pi^*}{\longrightarrow} \quad R^* \quad \overset{\iota^*}{\longrightarrow} \quad k^G \quad \longrightarrow \quad 1.\]

Definition 2.22. We shall say that the extension of braided Hopf algebras (2.33) is realizable over $H$, whenever the braided Hopf algebras $k^G$, $R$ and $k^F$, as well as the maps $\iota$ and $\pi$, are in the category $\mathcal{YD}$. It follows from Proposition 2.21 and the results in [T2, Section 6] that there exists a Hopf algebra $H$ such that (2.33) is realizable over $H$. Indeed, there is a Hopf algebra $H$ such that $\iota$ is categorical by [T2, Proposition 6.6]; but then $\pi$ is also categorical.
But it is not true that if $R$ is realizable over any Hopf algebra $K$ then $k^G$ and $kF$ also are. For instance, assume that $k$ is algebraically closed of characteristic zero and let $R$ be the group algebra of a finite group $L$, $N$ a normal abelian subgroup, $G$ the group of characters of $N$ and $F = L/N$. Let $\theta$ be a non-trivial automorphism of $L$, say of finite order, and let $C$ be the subgroup of $\text{Aut} L$ generated by $\theta$. Then $R$ is a Yetter-Drinfeld module over $kC$ with trivial coaction, but $k^G$ is not a Yetter-Drinfeld submodule of $R$ unless $N$ is $\theta$-stable. For a concrete example, let $N$ be a finite abelian group, $L = N \times N$ and $\theta$ the transposition.

Let $S$ be any braided Hopf algebra. Consider the left braided adjoint action of $S$ on itself, given by
\begin{equation}
\text{ad}_c(a)(b) = m((\text{id} \otimes m(\text{id} \otimes S))c)(\Delta \otimes \text{id})(a \otimes b), \quad a, b \in S.
\end{equation}
Let $H$ be a Hopf algebra such that $S$ is realizable over $H$. Then the left braided adjoint action $\text{ad}_c$ of $S$ coincides with the restriction to $S$ of the left adjoint action of the corresponding Radford biproduct $S \# H$: $\text{ad}_c(a)(b) = \text{ad}_{S \# H}(a)(b)$, for all $a, b \in S$.

We come back to our situation. The next lemma shows that $k^G$ is ‘braided normal’ in $R$.

**Lemma 2.23.** We have $\text{ad}_c(\delta_gx)(\delta_h) = \delta_{g,1}\delta_{h^{-1},x^{-1}}^{-1}$, for all $g, h \in G$, $x \in F$.

In particular, the categorical braided Hopf subalgebra $k^G \subseteq R$ is stable under the left braided adjoint action.

**Proof.** Straightforward. □

### 2.6. Equivalences

Let $R$ and $R'$ be braided Hopf algebras. A linear map $\Theta : R \rightarrow R'$ is called a **morphism of braided Hopf algebras** if it preserves the multiplication, comultiplication, unit and counit maps. Since the antipode is the convolution inverse of the identity, it follows that any morphism $\Theta$ of braided Hopf algebras preserves also the antipode. Hence, by [Sb], $\Theta$ commutes with the braiding; that is, $(\Theta \otimes \Theta)c_R = c_R'((\Theta \otimes \Theta))$.

Let $\xymatrix{1 \ar[r]^i & S \ar[r]_\iota & R \ar[r]_\pi & T \ar[r] & 1}$ and $\xymatrix{1 \ar[r]^i & S \ar[r]_{\iota'} & R' \ar[r]_{\pi'} & T \ar[r] & 1}$ be two extensions of braided Hopf algebras. An isomorphism $\Theta : R \rightarrow R'$ of braided Hopf algebras is an **isomorphism of extensions** if the following diagram commutes
\[
\begin{array}{ccc}
1 & \xrightarrow{i} & S & \xrightarrow{\iota} & R & \xrightarrow{\pi} & T & \xrightarrow{\Theta} & 1 \\
\downarrow{\text{id}} & & \downarrow{\Theta} & & \downarrow{\text{id}} & & \downarrow{\Theta} & & \downarrow{\text{id}} \\
1 & \xrightarrow{\iota'} & S & \xrightarrow{\iota'} & R' & \xrightarrow{\pi'} & T & \xrightarrow{\Theta} & 1
\end{array}
\]

**Proposition 2.24.** Let $R = k^G \#_{\sigma} kF$ and $R' = k^{G'} \#_{\sigma'} kF$ be braided Hopf algebras and consider the corresponding extensions as in $(2.33)$. Let $\nu \in \text{Map}_+(G \times F, k^X)$ and define $\Theta : R \rightarrow R'$ in the form
\[
\Theta(\delta_gx) = \nu(g, x)\delta_gx, \quad \text{for all } g \in G, \ x \in F.
\]
Then $\Theta$ is an isomorphism of extensions if and only if $(\tau, \sigma) = (\tau', \sigma')\delta^{\text{tot}}\nu$ in the complex $(3.15)$.

Furthermore, any isomorphism of extensions $\Theta : R \rightarrow R'$ arises in this way for a unique $\nu$.

**Proof.** It is easy to see that $\Theta$ is an algebra map if and only if $\sigma_g(x, y)\nu(g, xy) = \sigma'_g(x, y)\nu(g, x)\nu(ga, y)$, and $\nu(g, 1) = 1$, for all $g \in G, \ x, y \in F$. Also, $\Theta$ is a coalgebra map if and only if $\tau'_g(g, h)\nu(gh, x) = \tau_g(g, h)\nu(g, h \triangleright x)\nu(h, x)$, and $\nu(1, x) = 1$, for all $g, h \in G, \ x \in F$. This proves the first claim.

Let now $\Theta : R \rightarrow R'$ be an isomorphism of extensions of braided Hopf algebras. Since $\pi = \pi'\Theta$, it can be seen that $\Theta(x)x^{-1} \in k^G$, for all $x \in F$. Define $\nu(g, x)$ by $\Theta(x)x^{-1} = \sum_{g \in G} \nu(g, x)\delta_g$. Then $\Theta(\delta_gx) = \delta_g\Theta(x) = \nu(g, x)\delta_gx$, and the conclusion follows from the first claim. □
Corollary 2.25. The group of automorphisms of the extension \((2.33)\) is isomorphic to \(Z^0(\text{Tot}(C^\cdot))\). Any such automorphism is categorical.

Remark 2.26. Suppose that the 2-cocycle \(\sigma : F \times F \to (k^\times)^G\) is a coboundary. Then \(R\) is isomorphic to a bicrossed product \(k^G \rtimes_{\sigma} k^F\). In particular, this always happens if all the Sylow subgroups of \(F\) are cyclic; see [N, Lemma 1.2.5].

2.7. Commutativity. We shall say that a braided Hopf algebra \(R\) is braided commutative if \(m = mc : R \otimes R \to R\); respectively, \(R\) is called braided cocommutative if \(\Delta = c\Delta : R \to R \otimes R\).

Let \(R = k^G \rtimes_{\sigma} k^F\). The verification of the following claims is straightforward:

(a) \(R\) is braided commutative if and only if \(F\) is abelian, \(\triangleleft\) is trivial and
\[(2.35) \quad Q_{g,y}^{x,y} = \sigma_g(x,y)\sigma_y(y,x)^{-1}, \quad \forall x, y \in F, g \in G.\]

(b) \(R\) is braided cocommutative if and only if \(G\) is abelian, \(\triangleright\) is trivial and
\[(2.36) \quad Q_{g,h}^{x,x} = \tau_x(h,g)\tau_x(g,h)^{-1}, \quad \forall x \in F, g, h \in G.\]

3. Diagonal realizations over finite groups

We shall consider in this section 2-cocycles \(\sigma : F \times F \to (k^\times)^G\) and \(\tau : G \times G \to (k^\times)^F\) satisfying the normalization conditions \((2.4), (2.6), (2.9)\) and \((2.10)\). We discuss a particular but important class of realizations.

We fix a finite group \(C\) and we let \(H = kC\). We fix functions \(z : G \times F \to Z(C)\) and \(\chi : G \times G \to \hat{C}\), and we define a structure of left Yetter-Drinfeld module on \(R\) by imposing \(\delta_g x \in R^{\chi(g,x)}\), \(g \in G, x \in F\). That is, the action and the coaction of \(H\) on \(R\) are given, respectively, by
\[(3.1) \quad u, \delta_g x = \langle \chi(g, x), u \rangle \delta_g x, \quad u \in C; \quad \rho(\delta_g x) = z(g,x) \otimes \delta_g x.\]
In particular, the braiding \(c : R \otimes R \to R \otimes R\) is given in this case by
\[(3.2) \quad c(\delta_g x \otimes \delta_h y) = \langle \chi(h, y), z(g,x) \rangle \delta_h y \otimes \delta_g x.\]

Lemma 3.1. (i). The multiplication of \(R\) given by \((2.7)\) is a morphism of \(H\)-modules if and only if
\[(3.3) \quad \chi(g, xy) = \chi(g, x)\chi(g \triangleright x, y), \quad g \in G, x, y \in F.\]

(ii). The comultiplication of \(R\) given by \((2.8)\) is a morphism of \(H\)-modules if and only if
\[(3.4) \quad \chi(gh, x) = \chi(g, h \triangleright x)\chi(h, x), \quad g, h \in G, x \in F.\]

(iii). The multiplication of \(R\) is a morphism of \(H\)-comodules if and only if
\[(3.5) \quad z(g, xy) = z(g, x)z(g \triangleright x, y), \quad g \in G, x, y \in F.\]

(iv). The comultiplication of \(R\) is a morphism of \(H\)-comodules if and only if
\[(3.6) \quad z(gh, x) = z(g, h \triangleright x)z(h, x), \quad g, h \in G, x \in F.\]

Proof. Straightforward.
Remark 3.2. The following normalization properties follow from (3.3)–(3.6):

\begin{equation}
(3.7) \quad z(1,x) = 1, \quad z(g,1) = 1, \quad \chi(1,x) = 1, \quad \chi(g,1) = 1, \quad x \in F, \ g \in G.
\end{equation}

It is not difficult to see that these amount to the unit and counit maps being morphisms of Yetter-Drinfeld modules.

Example 3.3. The action $\triangleright$ induces a right action of $G$ on $\text{Map}_+(F, Z(C))$ in the form $(\phi \leftarrow g)(x) = f(g \triangleright x)$. Let $\psi \in \text{Map}_+(F, Z(C))$ and consider the function $z_\psi : G \times F \rightarrow Z(C)$ given by $z_\psi(g, x) = \partial_\psi(g)(x) = \psi(g \triangleright x)\psi^{-1}(x)$. Then $z_\psi$ satisfies the cocycle condition (3.6) by construction: indeed, $z_\psi$ is the 1-coboundary of $\psi$ 'in the first variable'.

Lemma 3.4. A sufficient condition for $z_\psi$ to satisfy condition (3.5) is that $\psi : F \rightarrow Z(C)$ be a group homomorphism.

Proof. We compute, for all $x, y \in F, g \in G$,

\begin{align*}
z_\psi(g, xy) &= \psi(g \triangleright (xy))\psi^{-1}(xy), \\
z_\psi(g, x)z_\psi(g \triangleright y) &= \psi(g \triangleright x)\psi((g \triangleright x) \triangleright y))\psi^{-1}(x)\psi^{-1}(y);
\end{align*}

In view of (2.1), both expressions are equal whenever $\psi$ is a group homomorphism. \hfill \Box

We give now an alternative approach to Theorem 2.10. The following theorem is a consequence of Proposition 2.6.

Theorem 3.5. Suppose that $\sigma : G \times F \rightarrow Z(C)$ and $\chi : G \times F \rightarrow \hat{C}$ satisfy conditions (3.3)–(3.6) in Lemma 2.1. Then $R$ is a braided Hopf algebra over $kC$ if and only if

\begin{equation}
(3.8) \quad \sigma_{ts}(x, y)\tau_{xy}(t, s) = \chi(t \triangleright (s \triangleright x), (s \triangleright x) \triangleright y), \quad z(s, x)) \tau_x(t, s) \tau_y(t \triangleright (s \triangleright x), s \triangleright y) \sigma_s(x, y),
\end{equation}

for all $s, t \in G, x, y \in F$. If this holds, we shall say that $(\sigma, \chi)$ is a diagonal realization of $R$ over $kC$. \hfill \Box

Remark 3.6. Consider the conditions

\begin{align}
(3.9) \quad \sigma_{ts}(x, y) &= \chi(t \triangleright (s \triangleright x), (s \triangleright x) \triangleright y), \quad z(s, x)) \sigma_t(s \triangleright x, (s \triangleright x) \triangleright y) \sigma_s(x, y), \\
(3.10) \quad \tau_{xy}(t, s) &= \tau_x(t, s) \tau_y(t \triangleright (s \triangleright x), s \triangleright y),
\end{align}

for all $s, t \in G, x, y \in F$. It is clear that any two among (3.8), (3.9) and (3.10) imply the third. This observation will be used later in order to systematically produce examples of diagonal realizations.

A similar observation applies if one considers instead the conditions

\begin{align}
(3.11) \quad \sigma_{ts}(x, y) &= \sigma_t(s \triangleright x, (s \triangleright x) \triangleright y) \sigma_s(x, y), \\
(3.12) \quad \tau_{xy}(t, s) &= \chi(t \triangleright (s \triangleright x), (s \triangleright x) \triangleright y), \quad z(s, x)) \tau_x(t, s) \tau_y(t \triangleright (s \triangleright x), s \triangleright x).
\end{align}

Suppose that $R = kG^r \#_k kF$ is a braided Hopf algebra and the 2-cocycle $\sigma : F \times F \rightarrow (k^r)^G$ is a coboundary. In view of Remark 2.26, $R$ is isomorphic to a bicrossed product $kG^{r'} \#_k F$. Since $\sigma' = 1$ satisfies (3.11), then $\tau'$ must satisfy (3.12).

This simplifies the search of braided compatible data in many cases, for instance, in the case where all Sylow subgroups of $F$ are cyclic.
Suppose that $R$ admits a diagonal realization over $\mathbb{k}C$, and consider the Radford biproduct $R \# \mathbb{k}C$.

**Proposition 3.7.** (i) The extension $1 \to \mathbb{k}^G \to R \to \mathbb{k}F \to 1$ is realizable over $\mathbb{k}C$; (ii) there are exact sequences of Hopf algebras

$$1 \to \mathbb{k}^G \to R \# \mathbb{k}C \to \mathbb{k}F \otimes \mathbb{k}C \to 1,$$

$$1 \to \mathbb{k}^G \otimes \mathbb{k}C \to R \# \mathbb{k}C \to \mathbb{k}F \to 1,$$

where all maps are canonical.

**Proof.** (i). Consider the trivial action and coaction of $\mathbb{k}C$ on $\mathbb{k}^G$ and $\mathbb{k}F$, making them Yetter-Drinfeld modules. It follows from the normalization conditions (3.7), that the canonical maps $\iota$ and $\pi$ are morphisms of Yetter-Drinfeld modules. This proves (i).

Note that for the corresponding biproducts we have $\mathbb{k}^G \# \mathbb{k}C = \mathbb{k}^G \otimes \mathbb{k}C$, and $\mathbb{k}F \# \mathbb{k}C = \mathbb{k}F \otimes \mathbb{k}C$.

(ii). Conditions (3.7) also imply that the action and coaction of $\mathbb{k}C$ on elements $\delta_g$, $g \in G$, and also on elements $x \in F$, are both trivial. Using this plus part (i), one sees that the maps in (3.13) and (3.14) are Hopf algebra maps. The exactness follows easily. \qed

We now give an interpretation of the conditions in Lemma 3.1 in the terms of the cohomology of a complex closely related to that considered in (2.27). Let $M$ be an abelian group. Let $n, m \geq 1$, and let $\text{Map}_+(G^n \times F^m, M)$ be the abelian group of $M$-valued functions $f$ on $G^n \times F^m$ with the property that $f(g_1, \ldots, g_n; x_1, \ldots, x_m) = 1$, if either one of $g_1, \ldots, g_n$ or $x_1, \ldots, x_m$ is equal to 1. Consider the double complex

$$
\begin{array}{ccc}
\vdots & \vdots \\
\uparrow & \uparrow \\
\text{C}^\cdot\cdot(M) := \text{Map}_+(G^2 \times F, M) & \xrightarrow{\delta} & \text{Map}_+(G^2 \times F^2, M) \xrightarrow{\delta'} \cdots \\
\delta' & \delta & \\
\text{Map}_+(G \times F, M) & \xrightarrow{\delta} & \text{Map}_+(G \times F^2, M) \xrightarrow{\delta'} \cdots 
\end{array}
$$

where the maps $\delta$ and $\delta'$ are defined as for the complex (2.27).

**Lemma 3.8.** (i) Conditions (3.3) and (3.4) are equivalent to $\chi \in Z^0(\text{Tot}(\text{C}^\cdot\cdot(C)))$.

(ii) Conditions (3.5) and (3.6) are equivalent to $c \in Z^0(\text{Tot}(\text{C}^\cdot\cdot(Z(C))))$.

**Proof.** Straightforward. \qed

We close this section by showing that diagonal realizations over abelian groups always exist. Suppose that $R$ is a braided Hopf algebra; so that the conditions in Theorem 2.10 are satisfied. Let $Q^{x,y}_{g,h}$ be given by formula (2.22); thus $Q$ satisfies the conditions in Lemma 2.9. Let $\Gamma$ be either $\mathbb{Z}$, or $\mathbb{Z}/N$ provided that the order of $Q^{x,y}_{g,h}$ divides $N$ for all $x, y \in F$, $g, h \in G$. Consider the group $\Gamma^{G \times F}$, with the canonical elements $e(g, x)$. We then define

$$C := \Gamma^{G \times F}/\{e(g, xy) - e(g, x) - e(gx, y); e(gh, x) - e(g, h \triangleright x) - e(h, x) : g, h \in G, x, y \in F\},$$

$$z(g, x) := \text{the class of } e(g, x) \text{ in } C.$$
Lemma 3.9. There are characters $\chi(g,x)$ of $C$ defined by
\[ \chi(g,x) \cdot (z(h,y)) = Q^{x,y}_{g,h}. \]
Furthermore, $(z, \chi)$ is a diagonal realization of $R$ over $\mathbb{k}C$.

Proof. The characters are well-defined by (2.23) and (2.25); this is a diagonal realization by the definition of $z$, (2.24) and (2.26). \hfill \square

4. Examples of diagonal realizations

We discuss in the next subsections, under additional assumptions on the matched pair $(G,F)$, some reductions in order to determine maps $c: G \times F \to Z(C)$ and $\chi : G \times F \to \hat{C}$ satisfying the conditions in Lemma 3.1. We shall assume in this section that $\mathbb{k}$ is algebraically closed of characteristic zero.

4.1. Semidirect products. Consider the case where the action $\triangleright$ is trivial; so that $\triangleleft : G \times F \to G$ is an action by group automorphisms and the group $\Sigma = FG$ is isomorphic to the associated semidirect product $F \rtimes G$.

The action $\triangleleft$ induces by transposition left actions of $F$ on $\text{Hom}(G,Z(C))$ and on $\text{Hom}(G,\hat{C})$; for instance, we have $(x \cdot \phi)(g) = \phi(g \triangleleft x)$, for all $x \in F$, $g \in G$, $\phi \in \text{Hom}(G,Z(C))$.

Lemma 4.1. (i) The set of maps $\chi : G \times F \to \hat{C}$ satisfying (3.3) and (3.4) is in bijective correspondence with $Z^1(F,\text{Hom}(G,\hat{C}))$.

(ii) The set of maps $z : G \times F \to Z(C)$ satisfying (3.5) and (3.6) is in bijective correspondence with $Z^1(F,\text{Hom}(G,Z(C)))$.

Proof. We prove (i), the proof of (ii) being similar. The correspondence is given by $\chi \mapsto \tilde{\chi} : F \to \text{Hom}(G,\hat{C})$, $\tilde{\chi}(x) = \chi(g,x)$. Condition (3.4) amounts to $\tilde{\chi}(x) \in \text{Hom}(G,\hat{C})$, for all $x \in F$, and condition (3.3) says exactly that $\tilde{\chi}$ is a 1-cocycle. \hfill \square

Corollary 4.2. Suppose that $|G|$ and $|Z(C)|$ are relatively prime. If $z : G \times F \to Z(C)$ satisfies (3.5) and (3.6), then $z(g,x) = 1$, for all $x \in F$, $g \in G$.

Proof. In this case we have $\text{Hom}(G,Z(C)) = 1$. Therefore the claim follows from Lemma 4.1. \hfill \square

Example 4.3. Let $p$ be a prime number and suppose that $\dim R = p^3$. In other words, $\Sigma$ has order $p^3$. Up to passing to the dual, we may assume that $|G| = p^2$ and thus that $G$ is normal in $\Sigma$. Assume also that $p$ does not divide the order of $C$. Then Corollary 4.2 implies that $c = 1$. Thus $Q^{x,y}_{g,h} = 1$, for all $x,y \in F$, $g,h \in G$. Hence every $R$ arising from this setup is trivial, i.e., is a usual Hopf algebra.

We now fix $\tilde{\chi} \in Z^1(F,\text{Hom}(G,Z(C)))$ and $\tilde{\chi} \in Z^1(F,\text{Hom}(G,\hat{C}))$; they are the 1-cocycles corresponding to maps $z : G \times F \to Z(C)$ and $\chi : G \times F \to \hat{C}$, respectively, as in Lemma 4.1. We look for $\sigma, \tau$ satisfying the conditions in Theorem 3.5.

Consider the action of $F$ on $Z^2(G,\mathbb{k}^\times)$ given by $(x.f)(g,h) = f(g \triangleleft x, h \triangleleft x)$, $g,h \in G$, $x \in F$; this action is well-defined because $F$ acts by group automorphisms on $G$.

The map $\tau : G \times G \to (\mathbb{k}^\times)^F$ can be regarded as a map $F \to \text{Map}(G \times G, \mathbb{k}^\times)$; we shall write $\tilde{\tau}$ to indicate this latter map. Note that $\tau$ is a 2-cocycle if and only if the image of $\tilde{\tau}$ is contained in $Z^2(G,\mathbb{k}^\times)$. 
Proposition 4.4. Let \( \bar{\zeta} \in Z^1(F, \text{Hom}(G, Z(C))) \) and \( \bar{\chi} \in Z^1(F, \text{Hom}(G, \hat{C})) \). Let also \( \sigma \in Z^2(F, (k^\times)G) \) be a normalized 2-cocycle such that the following compatibility condition holds:

\[
\sigma_{ts}(x, y) = \langle (x \rightarrow \bar{\chi})(y)(s), \bar{\chi}(x)(t) \rangle \sigma_s(x, y) \sigma_t(x, y).
\]

Let \( \tau : G \times G \to (k^\times)^F \) be a normalized 2-cocycle. Assume that the normalization conditions (2.9) and (2.10) hold.

Then \( R \) is a braided Hopf algebra over \( kC \) if and only if \( \bar{\tau} : F \to Z^2(G, k^\times) \) is a 1-cocycle.

Proof. This is a special instance of Remark 3.6. In view of Theorem 3.3, and using (4.1), \( R \) is a braided Hopf algebra if and only if

\[
\tau_{xy}(t, s) = \tau_x(t, s) \tau_y(t \triangleleft x, s \triangleleft x) = \tau_x(t, s) (x.\tau_y)(t, s),
\]

for all \( s, t \in G, x, y \in F \), which is exactly the 1-cocycle condition on \( \bar{\tau} \). This proves the proposition. \( \square \)

Example 4.5. If \( \tau \) is the trivial 2-cocycle, then \( \bar{\tau} \) is the trivial 1-cocycle, and we get a braided Hopf algebra \( R \); note that \( R \) is the tensor product \( k^G \otimes kF \) as a coalgebra. This braided Hopf algebra structure of \( R \) is due to Andruskiewitsch and Sommerhăuser. Its construction appears in [S, 3.2].

The proposition above can be used to construct examples of non-trivial braided Hopf algebras, using the data in [S, Ch. 3]. These examples are not commutative and also not cocommutative. A particular case of this construction is done in the next subsection.

4.2. An example from finite fields. We first recall the construction in [S, 3.3]. Let \( K \) be a finite ring, let \( F \) be a finite group, let \( G \) denotes the additive group of \( K \) written additively and let there be given the following data:

(4.2) a group homomorphism \( \nu : F \to K^\times \)

(we endow \( G \) with the \( F \)-action defined by \( g \triangleleft x := g\nu(x), x \in F, g \in G \)),

(4.3) two 1-cocycles \( \alpha, \beta \in Z^1(F, G) \),

(4.4) a normalized 2-cocycle \( \phi \in Z^2(F, G) \),

(4.5) two characters \( \eta, \lambda : G \to k^\times \), such that \( \langle \lambda, ghs \rangle = \langle \lambda, hgs \rangle, g, h, s \in K \).

\((F, G)\) is a matched pair with respect to the action \( \triangleleft : G \times F \to G \) and the trivial action \( \triangleright : G \times F \to F \). With respect to the above data, we define \( z : G \times F \to K, \chi : G \times F \to K \) and \( \sigma : F \times F \to (k^\times)^G \) in the form

(4.6) \( z(g, x) = g\beta(x) \),

(4.7) \( \langle \chi(g, x), h \rangle = \langle \lambda, h\alpha(x) \rangle^2 \),

(4.8) \( \sigma_g(x, y) = \langle \eta, g\phi(x, y) \rangle \langle \lambda, g^2\nu(x)\beta(x)\alpha(y) \rangle \),

for all \( x, y \in F, g, h \in G \). The result in [S, 3.3], combined with Lemma 4.4, implies that for any normalized 1-cocycle \( \bar{\tau} \in Z^1(F, Z^2(G, k^\times)) \), the associated bicrossed product \( R \) is a braided Hopf algebra over \( kC \), where \( C \) is the additive group of \( K \).
We construct now explicit examples of 1-cocycles \( \tilde{\tau} \) in this situation. For notational simplicity, we assume that \( k \) is the field \( \mathbb{C} \) of complex numbers. Let \( p \) and \( q \) be prime numbers such that \( p = 1 \mod q \), let \( K = \mathbb{F}_p^2 \) be the field with \( p^2 \) elements, and let \( F = \mathbb{Z}_q \) be the cyclic group of order \( q \).

The assumption on \( q \) implies that there exists a unit modulo \( p \), \( \nu \in \mathbb{F}_p^\times \subseteq K^\times \), of order \( q \); by abuse of notation, we let \( \nu : F \to K^\times \) be the group homomorphism given by \( \nu(x) = \nu^x \).

As before, let \( G \) denote the additive group of \( K \), and consider the right action of \( F \) on \( G \) given by \( g \cdot x = \nu^x g \).

**Lemma 4.6.** Let \( 0 \neq \alpha : F \to G \) be a map. Then \( \alpha \) is a normalized 1-cocycle if and only of there exists \( r \in K^\times \) such that \( \alpha \) has the form

\[
\alpha(x) = r[x]_\nu, \quad x \in F - 0, \quad \alpha(0) = 0.
\]

Here, \([x]_\nu \) denotes the \( \nu \)-number: \([x]_\nu := 1 + v + \cdots + v^{x-1} \in \mathbb{F}_p \).

**Proof.** The 1-cocycle condition on \( \alpha \) says that \( \alpha(x + y) = \alpha(x) + \nu^x \alpha(y) \), for all \( x, y \in F \). Using that \( F \) is cyclic, generated by 1, one can show by induction that \( \alpha(x) = \alpha(1)[x]_\nu \), for all \( x \in F \). Putting \( r = \alpha(1) \), which is non-zero by assumption, the lemma follows. \( \square \)

Let \( a \in K \) such that \( K = \mathbb{F}_p(a) \); so that every element \( g \in K \) writes uniquely in the form \( g = j + la \), with \( j, l \in \mathbb{F}_p \). This determines an isomorphism between the additive group of \( K \) and \( \mathbb{F}_p \oplus \mathbb{F}_p \). We shall denote \( \det_a : K \times K \to \mathbb{F}_p \) the function defined by \( \det_a(g, h) = jl' - lj', \) for \( g = j + la, \) \( h = j' + l'a \in K \).

**Lemma 4.7.** Let \( x \in F, g, h \in G \). The formula

\[
\tau_x(g, h) = \exp \left( \frac{2\pi i}{p} [x]_\nu \det_a(g, h) \right),
\]

defines a 1-cocycle \( \tilde{\tau} \in Z^1(F, Z^2(G, \mathbb{C}^\times)) \).

**Proof.** Every cohomology class in \( H^2(G, \mathbb{C}^\times) \simeq \mathbb{F}_p \) can be represented by one of the 2-cocycles \( \kappa_n : (g, h) \mapsto \exp \frac{2\pi i}{p} n \det_a(g, h) \), where \( n \) runs over the integers modulo \( p \), giving a group isomorphism \( \mathbb{F}_p \simeq H^2(G, \mathbb{C}^\times) \).

We have

\[
(x, \kappa_n)(g, h) = \exp \frac{2\pi i}{p} n \det_a(g \cdot x, h \cdot x) = \exp \frac{2\pi i}{p} n \det_a(g \nu^x, h \nu^x) = \exp \frac{2\pi i}{p} n \nu^{2x} \det_a(g, h),
\]

the last equality because we have chosen \( \nu \in \mathbb{F}_p \). Thus the action of \( x \in F \) on \( \mathbb{F}_p \simeq H^2(G, \mathbb{C}^\times) \) is given by multiplication by \( \nu^{2x} \).

The argument in the proof of Lemma 4.9 shows that every 1-cocycle \( \tilde{\tau} : F \to H^2(G, \mathbb{C}^\times) \) is of the form \( \tau(x) = \kappa_{[x]_\nu^2} \), for some \( r \in \mathbb{F}_p^\times \). This implies the lemma. \( \square \)
Proposition 4.8. Let $C$ be the additive group of $\mathbb{F}_p^2$. Let $\nu \in \mathbb{F}_p^\times$, and consider the matched pair $(F, G)$ as above. Let also $z : G \times F \to C$, $\chi : G \times F \to C$, $\sigma : F \times F \to (\mathbb{K}^\times)^G$ and $\tau : G \times G \to (\mathbb{K}^\times)^F$, be defined by

\begin{align*}
(4.11) \quad z(g, x) &= g[x]_\nu, \\
(4.12) \quad \langle \chi(g, x), h \rangle &= \exp \left( \frac{4\pi i}{p} \operatorname{tr}(hg)[x]_\nu \right), \\
(4.13) \quad \sigma_g(x, y) &= \exp \left( \frac{2i}{k} \operatorname{tr}(g^2) \nu^x[y]_\nu \right), \\
(4.14) \quad \tau_g(h) &= \exp \left( \frac{2i}{k} [x]_\nu \det a(g, h) \right),
\end{align*}

for all $x, y \in F, g, h \in G$, where $\operatorname{tr} : \mathbb{F}_p^2 \to \mathbb{F}_p$ is the trace map.

Then the associated bicrossed product $R$ is a braided Hopf algebra over $\mathbb{C}C$. The braiding on $R$ is given by (2.18), where

\begin{align*}
(4.15) \quad Q_{g,h}^{x,y} &= \exp \left( \frac{4\pi i}{p} \operatorname{tr}(hg)[x]_\nu [y]_\nu \right), \quad g, h \in x, y \in F.
\end{align*}

Note that $R$ is non-trivial and also not commutative and not cocommutative. The dimension of $R$ is $p^2 q$, and the dimension of the biproduct $R\#\mathbb{C}C$ is $p^4 q$.

Proof. The proof follows from [8, 3.3] and Lemma 4.7, using Lemma 4.1. Take $\phi = 1, \eta = 1; \nu : F \to G$, group homomorphism given by $\nu(x) = \nu^x$; 1-cocycles $\alpha = \beta : F \to G$, with $\alpha(x) = [x]_\nu$; and let $\lambda : G \to \mathbb{C}^\times$ be the group homomorphism defined by $\langle \lambda, g \rangle = \exp(\frac{2i}{k} \operatorname{tr}(g))$, for all $x, g, \in G$. \hfill \Box

4.3. Direct products. Suppose that both actions $\triangleright$ and $\triangleleft$ are trivial, i.e., that $\Sigma \simeq F \times G$. In this case, the maps $z : G \times F \to Z(C)$ satisfying (3.3) and (3.6) correspond bijectively to group homomorphisms $z : F/[F, F] \otimes G/[G, G] \to Z(C)$, and the maps $\chi : G \times F \to \hat{C}$ satisfying (3.3) and (3.4) correspond bijectively to group homomorphisms $z : F/[F, F] \otimes G/[G, G] \to \hat{C}$.

As in Remark 3.6, we look for $\sigma \in \operatorname{Hom}(G, Z^2(F, \mathbb{K}^\times))$ and $\tau : F \to Z^2(G, \mathbb{K}^\times)$, such that $\tau_1 = 1$ and

\begin{align*}
(4.16) \quad \tau_{xy}(t, s) &= \langle \chi(t, g), z(s, x) \rangle \tau_x(t, s) \tau_y(t, s),
\end{align*}

for all $x, y \in F, s, t \in G$.

A fairly complete answer can be given in the case when $F$ and $G$ are cyclic. So assume that $F = \langle a \rangle$ has order $N$ and $G = \langle b \rangle$ has order $M$. Let $C = \langle u \rangle$ be of order $(M, N) = \gcd(M, N)$. Let also $\zeta \in \mathbb{K}^\times$ be a primitive $(M, N)$-th. root of unity. Define $z : G \otimes F \to C$ and $\chi : G \otimes F \to \hat{C}$ by

\begin{align*}
z(b^h \otimes a^j) &= u^{bj}, \\
\langle \chi(b^h \otimes a^j), u^l \rangle &= \zeta^{hj};
\end{align*}

for all $0 \leq h \leq M - 1, 0 \leq j \leq N - 1, 0 \leq l \leq (N, M) - 1$.

We first determine the possible $\sigma$’s. Since $F$ is cyclic, $Z^2(F, \mathbb{K}^\times) = B^2(F, \mathbb{K}^\times)$; hence giving $\sigma \in \operatorname{Hom}(G, Z^2(F, \mathbb{K}^\times))$ is equivalent to choosing $\sigma_b \in B^2(F, \mathbb{K}^\times)$ such that $\sigma_b^M = 1$.

Concrete examples can be given as follows. Let $\omega \in \mathbb{K}^\times$ be such that $\omega^{MN} = 1$. We define $f : F \to \mathbb{K}^\times$ in the form $f(a^h) = \omega^h$, and let $\sigma_b = \partial f$ be the coboundary of $f$. So that

\begin{align*}
(4.17) \quad \sigma_b(a^j, a^h) &= \omega^{Nq},
\end{align*}

for all $0 \leq j, h \leq N - 1$, where $j + h = Nq + r, 0 \leq r \leq N - 1$. 

We next consider the possibilities for $\tau$’s.

**Lemma 4.9.** The following are equivalent:

(i) $\tau : F \to Z^2(G, k^\times)$ satisfies $\tau_1 = 1$ and \((4.16)\).

(ii) There exists $\tau_a \in Z^2(G, k^\times)$ satisfying

\begin{equation}
\zeta^{st\frac{N(N-1)}{2}} \tau_a(b^s, b^t)^N = 1,
\end{equation}

for all $0 \leq s, t \leq M - 1$, such that

\begin{equation}
\tau_a^n(b^s, b^t) = \zeta^{st\frac{m(m-1)}{2}} \tau_a(b^s, b^t)^m, \quad 0 \leq s, t \leq M - 1, \quad 0 \leq m \leq N - 1.
\end{equation}

**Proof.** (i) $\implies$ (ii). Condition \((4.18)\) follows from $\tau_1 = 1$, and condition \((4.19)\) follows from \((4.16)\) by induction on $m$.

(ii) $\implies$ (i). Left to the reader. $\square$

**Remark 4.10.** Let $\zeta$ be a square root of $\zeta$, and let $\eta : G \to k^\times$ be given by $\eta(b^s) = \zeta^{st\frac{N(N-1)}{2}}$. Condition \((4.18)\) can be rephrased as saying that $\tau_a^N = \partial \eta$ in $Z^2(G, k^\times)$.

If $(M, N)$ divides $\frac{N(N-1)}{2}$, then condition \((4.18)\) amounts to $\tau_a^N = 1$.

Let us give some concrete examples.

**Example 4.11.** Let $N, M \geq 1$ be integers such that $(M, N)$ divides $\frac{N(N-1)}{2}$. Let $\zeta \in k^\times$ be a primitive $(M, N)$-th. root of unity, and let $\omega, \mu \in k^\times$ be such that $\omega^M = \mu^N = 1$. Then there exists a braided Hopf algebra $R = k^{G \times \#_\omega kF}, F$ and $G$ as above, where $\prec, \succ$ are trivial and

$$
\sigma_a (a^j, a^h) = \omega^{Nqs}, \quad \text{if} \quad j + h = Nq + r, 0 \leq r \leq N - 1,
$$

$$
\tau_a (b^s, b^t) = \zeta^{st\frac{m(m-1)}{2}} \mu^{M\tilde{q}m}, \quad \text{if} \quad s + t = M\tilde{q} + \tilde{r}, 0 \leq \tilde{r} \leq M - 1.
$$

These braided Hopf algebras are commutative and cocommutative.

Note that we have $\sigma_a (a^j, a^h) = \nu(b^s, a^j)\nu(b^s, a^h)\nu(b^s, a^{j+h})^{-1}$, for all $0 \leq s \leq M - 1, 0 \leq j, h \leq N - 1$, where $\nu(b^s, a^j) = \omega^{s\tilde{j}}$. Hence, the braided Hopf algebra corresponding to the pair $(\omega, \mu)$ is isomorphic to the braided Hopf algebra corresponding to the pair $(1, \omega\mu)$. See Proposition 2.24.

The Examples given by Kashina in [Ka] fit into the present construction. Indeed, take $M = 2, N = 2^n$, with $n > 1, \zeta = -1$ and $\mu = 1$.

If $\omega = 1$, we get the braided Hopf algebra $R_{n+1}^+$ in [Ka]. This is dual to one of the examples in 4.5.

If $\omega^N = -1$, we get the braided Hopf algebra $R_{n+1}$ in [Ka]. Again, this is dual to one of the examples in 4.5.

4.4. **Cyclic groups.** We shall now consider the case where $F$ is a cyclic group of order $N$. Write $F = \langle a : a^N = 1 \rangle$. Let $f : G \to \mathbb{S}_{N-1}$ be the group homomorphism associated to the action $\triangleright : G \times F \to F$. By abuse of notation, we shall use the same symbol to indicate an element $g \in G$ and its image under $f$; so that we have $g \triangleright a^i = a^{f(i)}$, $g \in G, 1 \leq i \leq N - 1$. 
Lemma 4.12. Let $A$ be a finite abelian group. The following collections of data are in bijective correspondence

(a) maps $\alpha : G \times F \to A$ satisfying

\begin{align}
\alpha(g, xy) &= \alpha(g, x)\alpha(g,x, y), \quad g \in G, x, y \in F, \\
\alpha(gh, x) &= \alpha(g, h \triangleright x)\alpha(h, x), \quad g, h \in G, x \in F,
\end{align}

and

(b) maps $\gamma : G \to A$ satisfying

\begin{align}
1 &= \gamma(g)\gamma(g \triangleleft a) \ldots \gamma(g \triangleleft a^{N-1}), \\
\gamma(gh) &= \gamma(g)\gamma(g \triangleleft a) \ldots \gamma(g \triangleleft a^{i-1})\gamma(g \triangleleft a^i) \ldots \gamma(g \triangleleft a^{i+j-1})
\end{align}

The bijection is given by

\begin{align}
\alpha(g, a^i) &= \gamma(g)\gamma(g \triangleleft a) \ldots \gamma(g \triangleleft a^{i-1}), \quad g \in G, 0 \leq i \leq N - 1, \\
\alpha(1, a) &= 1.
\end{align}

Proof. Let $\gamma : G \to A$ be a function satisfying (4.22) and (4.23). Let $g \in G$, $0 \leq i, j \leq N - 1$, and write $i + j = Nq + r$, where $0 \leq r \leq N - 1$ and $q = 0, 1$. By definition, we have

$$\gamma(g \triangleleft a^r) = 1$$

and on the other hand,

$$\alpha(g, a^i)\alpha(a^i, a^j) = \gamma(g)\gamma(g \triangleleft a) \ldots \gamma(g \triangleleft a^{i-1})\gamma(g \triangleleft a^i) \ldots \gamma(g \triangleleft a^{i+j-1})$$

the second equality because of condition (4.22). This shows that $\alpha$ verifies (4.20).

Let now $g, h \in G$, $0 \leq i \leq N - 1$. Observe that $g \triangleleft a^r = (g \triangleleft (h \triangleright a^s))(h \triangleright a^s) = (g \triangleleft a^{h(s)})(h \triangleright a^s)$. Whence, in view of (4.23),

$$\gamma(g \triangleleft a^r) = \gamma((g \triangleleft a^{h(s)}))(h \triangleright a^s)) = \gamma(g \triangleleft a^{h(s)}) \ldots \gamma(g \triangleleft a^{h(s)+(h \triangleright a^r)(1)-1})\gamma(h \triangleright a^s).$$

Thus,

$$\alpha(gh, a^i) = \gamma(gh)\gamma(g \triangleleft a) \ldots \gamma(g \triangleleft a^{i-1})$$

the first equality by (4.23) and the third equality because of the following claim:

Claim 4.1. We have $(h \triangleleft a^i)(1) = h(i + j) - h(i)$, for all $i$.

Proof. The compatibility condition (2.1) implies that $(h \triangleleft a^i) \triangleright a^j = (h \triangleleft a^i)^{-1}(h \triangleleft a^{i+j}) = a^{h(i+j)-h(i)}$, whence the claim follows. □
Therefore, \( \alpha \) satisfies (1.21).

Conversely, assume that the map \( \alpha \) given by (1.24) satisfies (1.20) and (1.21). Using (1.20) and \( a^N = 1 \) to compute \( \alpha(g, a^N) \), one sees that \( \gamma \) satisfies (4.22). Similarly, putting \( x = a \) in (4.21), the relation (4.23) follows. This finishes the proof of the lemma.

\[ \text{Proposition 4.13.} \text{ Let } \gamma : G \to \widehat{\mathcal{C}} \text{ and } \eta : G \to Z(C) \text{ be maps satisfying (1.22) and (1.23). Let } \tau : G \times G \to (k^\times)^F \text{ be a normalized 2-cocycle satisfying} \]

\[
\tau_{a^i+j}(t, s) = \langle \gamma(t < a^{s(i)}), \gamma(t < a^{s(i)+1}) \ldots \gamma(t < a^{s(i+j)-1}), \eta(s) \eta(s < a) \ldots \eta(s < a^{i-1}) \rangle \tau_{a^i}(t, s) \tau_{a^j}(t < a^{s(i)}, s < a^i),
\]

and \( \tau_1(s, t) = 1 \), for all \( s, t \in G, 0 \leq i, j \leq N - 1 \), Then the bicrossed product \( R = k^{G \tau} \# kF \) is a braided Hopf algebra over \( kC \), with respect to the maps \( \chi : G \times F \to \widehat{\mathcal{C}} \) and \( z : G \times F \to Z(C) \) given by

\[
\chi(g, a^i) = \gamma(g) \gamma(g < a) \ldots \gamma(g < a^{i-1}), \quad \chi(1, a) = 1, \quad g \in G, 0 \leq i \leq N - 1,
\]

\[
z(g, a^i) = \eta(g) \eta(g < a) \ldots \eta(g < a^{i-1}), \quad z(1, a) = 1, \quad g \in G, 0 \leq i \leq N - 1.
\]

Moreover, all braided Hopf algebras admitting a diagonal realization over \( kC \) are of this form.

\[ \text{Proof.} \text{ The first statement follows from Theorem 3.5 and Lemma 4.12. See Remark 3.6. It follows also from Remark 3.6 that every braided Hopf algebra admitting a diagonal realization over } kC \text{ has this form.} \]

5. Appendix

The contents of this appendix have been suggested by the referee. It presents an alternative language which seems appropriate in discussions about matched pairs of groups. This can be found for instance in [12]; see also [13]. The main constructions of the paper are translated into this language.

5.1. Notation. Let \( \mathcal{S} \) be the set of all diagrams

\[
\begin{array}{c}
\begin{array}{c}
g \\
\downarrow_t \\
v \end{array}
\rightleftharpoons
\begin{array}{c}
x \\
\downarrow_t \end{array}
\end{array}
\]

where \( g, t \in G, x, v \in F \) are such that \( gx = vt \).

Thus \( v = g \triangleright x \) and \( t = g \triangleleft x \). Sometimes, we shall simply write \( \begin{array}{c}
g \\
\downarrow_t \end{array} x = v \begin{array}{c}
g \\
\downarrow_t \end{array} \). A horizontal identity is an element of the form \( \begin{array}{c}
g \\
\downarrow_t \end{array} x \); a vertical identity is an element of the form \( \begin{array}{c}
1 \\
\downarrow_t \end{array} \).

Let \( A = \begin{array}{c}
g \\
\downarrow_t \end{array} x, B = \begin{array}{c}
h \\
\downarrow_s \end{array} y \) be in \( \mathcal{S} \). We shall write

\[ (1.1) \; A \triangleright B \text{ if } x = w. \text{ Then } AB := \begin{array}{c}
gh \\
\downarrow_{ts} \end{array} y \text{ is in } \mathcal{S} \text{ (horizontal product).} \]

\[ (1.2) \; \frac{A}{B} \text{ if } t = h. \text{ Then } \frac{A}{B} := vw \begin{array}{c}
g \\
\downarrow_s \end{array} xy \text{ is in } \mathcal{S} \text{ (vertical product).} \]
Remark 5.1. The notation $\begin{array}{c} A \\ C \end{array} \begin{array}{c} B \\ D \end{array}$ means that all possible horizontal and vertical products are allowed; this implies that $\begin{array}{c} AB \\ CD \end{array}$, $\begin{array}{c} A \\ B \\ C \\ D \end{array}$ and there is no ambiguity in the expression $\begin{array}{c} AB \\ CD \end{array}$.

5.2. Cocycles. Let $\sigma$ and $\tau$ be as in 2.2. We define a function, that we still denote $\sigma$, on the set of all pairs $(A, B)$ with $A \parallel B$, and a function $\tau$ on the set of all pairs $(A, B)$ with $A \parallel B$, by means of the formulas:

$$\sigma(A, B) = \sigma(g(x, y)),$$

$$\tau(A, B) = \tau_x(g, h),$$

where $A = \begin{array}{c} g \\ x \end{array}$ and $B = \begin{array}{c} h \\ y \end{array}$ are in $\mathcal{S}$. The cocycle and normalization conditions (2.3), (2.4), (2.5) and (2.6) translate, respectively, as follows:

$$\text{If } A \parallel B \text{ or } B \parallel C, \text{ then } \sigma(A, B)\sigma(A, C) = \sigma(B, C)\sigma(A, C).$$

$$\text{If } A \text{ or } B \text{ is a vertical identity, then } \sigma(A, B) = 1.$$”

5.3. Operations. The bicrossed product $R = k^G \otimes kF$ has $\mathcal{S}$ as a basis with identification $\delta_g x = \begin{array}{c} g \\ x \end{array}$. In this basis the operations of $R$ are determined by the formulas

- $A.B = \tau(A, B)\begin{array}{c} A \\ B \end{array}$, if $A \parallel B$, and 0 otherwise.
- $\Delta(A) = \sum \sigma(B, C)B \otimes C$, where the sum is over all pairs $(B, C)$ with $B \parallel C$ and $A = BC$.

The unitary conditions (2.9) and (2.10) translate into

$$\text{If } A = A^h = \begin{array}{c} g \\ h \end{array} \text{ is in } \mathcal{S}, \text{ we put}$$

$$A^h = \begin{array}{c} y^{-1} \\ h \end{array} \begin{array}{c} x \\ h^{-1} \end{array} \quad \text{(horizontal inverse)},$$

$$A^v = \begin{array}{c} h^{-1} \\ y^{-1} \end{array} \begin{array}{c} g \\ x \end{array} \quad \text{(vertical inverse)},$$

$$A^{-1} = (A^h)^v = (A^v)^h.$$
The formula (2.11) for the antipode is then
\[
S(A) = \sigma(A^{-1}, A^h)^{-1} \tau(A^h, A)^{-1} A^{-1}.
\]

**Remark 5.2.** Note that \(A^{-1} A^v A^h A^{-1} A^v\). We have also \(\sigma(A, A^v) = \sigma(A^v, A)\) from (2.3) and (2.4), and \(\tau(A, A^h) = \tau(A^h, A)\) similarly.

### 5.4. Braiding

The braiding (2.18) takes the form \(c(A \otimes B) = Q_{A,B} A \otimes B\), where \(Q_{A,B} := Q_{x,y}^{g,h}\), for \(A = \begin{array}{c} \text{g} \\ \text{x} \end{array}\) and \(B = \begin{array}{c} \text{h} \\ \text{y} \end{array}\) in \(\mathcal{G}\).

The compatibility condition (2.19) and the normalization condition (2.20) read, respectively, as follows:

\[(5.11) \text{If } \begin{array}{c} A \\ C \end{array} | \begin{array}{c} B \\ D \end{array}, \text{ then } \sigma(AB, CD) \tau \left( \begin{array}{c} A \\ C \end{array}, \begin{array}{c} B \\ D \end{array} \right) = Q_{B,C} \tau(A, B) \tau(C, D) \sigma(A, C) \sigma(B, D).\]

\[(5.12) \text{If } A \text{ or } B \text{ is a horizontal or vertical identity, then } Q_{A,B} = 1.\]

**Proposition 5.3.** For any \(B, C\) in \(\mathcal{G}\), there are unique \(A, D\) in \(\mathcal{G}\) with \(\begin{array}{c} A \\ C \end{array} \begin{array}{c} B \\ D \end{array}\). If we define \(Q_{B,C}\) by (5.11), then \(R\) becomes a pre-braided Hopf algebra.

### 5.5. Braided Hopf algebra

The compatibility conditions (2.23), (2.24), (2.25), (2.26) in Lemma 2.9 translate, respectively, as follows:

\[(5.13) \text{If } \begin{array}{c} B \\ C \end{array}, \text{ then } Q_{A,B} Q_{A,C}.\]

\[(5.14) \text{If } \begin{array}{c} A \\ D \end{array}, \text{ then } Q_{A,B} Q_{D,B}.\]

\[(5.15) \text{If } B | C, \text{ then } Q_{A,B,C} = Q_{A,B} Q_{A,C}.\]

\[(5.16) \text{If } D | A, \text{ then } Q_{D,A,B} = Q_{D,B} Q_{A,B}.\]

### 5.6. Realization

In Section 3, the maps \(z: \mathcal{G} \to Z(C)\) and \(\chi: \mathcal{G} \to \hat{C}\) should satisfy the following conditions:

\[(5.17) \text{If } \begin{array}{c} A \\ B \end{array}, \text{ then } \chi \left( \begin{array}{c} A \\ B \end{array} \right) = \chi(A) \chi(B).\]

\[(5.18) \text{If } A | B, \text{ then } \chi(AB) = \chi(A) \chi(B).\]

\[(5.19) \text{If } \begin{array}{c} A \\ B \end{array}, \text{ then } z \left( \begin{array}{c} A \\ B \end{array} \right) = z(A) z(B).\]

\[(5.20) \text{If } A | B, \text{ then } z(AB) = z(A) z(B).\]

Finally, equation (3.2) is now \(Q_{A,B} = \langle \chi(B), z(A) \rangle\).
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