Spectral properties of general hypergraphs

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Abstract

In this paper, we investigate spectral properties of the adjacency tensor, Laplacian tensor and signless Laplacian tensor of general hypergraphs (including uniform and non-uniform hypergraphs). We obtain some bounds for the spectral radius of general hypergraphs in terms of vertex degrees, and give spectral characterizations of odd-bipartite hypergraphs.

Keywords: Adjacency tensor, Laplacian tensor, hypergraph, Spectrum

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1. Introduction

Let $V(H)$ and $E(H)$ denote the vertex set and the edge set of a hypergraph $H$, respectively. A hypergraph $G$ satisfying $V(G) \subseteq V(H), E(G) \subseteq E(H)$ is called a sub-hypergraph of $H$. If $G$ is a sub-hypergraph of $H$ and $G \neq H$, then $G$ is said to be a proper sub-hypergraph of $H$. The degree of a vertex $i$ of $H$ is defined as $d_i = |E_i|$, where $E_i$ denotes the set of edges containing $i$. The rank and co-rank of $H$ is the maximum and minimum cardinality of an edge in $H$, respectively \cite{1}. Let $r(H)$ and $cr(H)$ denote the rank and co-rank of $H$, respectively. If $r(H) = cr(H) = k$, then $H$ is called $k$-uniform. 2-uniform hypergraphs are ordinary graphs. A path of length $l$ in a hypergraph $H$ is defined to be an alternating sequence $u_1e_1u_2 \cdots u_le_1u_{l+1}$, where $u_1, \ldots, u_{l+1}$ are distinct vertices of $H$, $e_1, \ldots, e_l$ are distinct edges of $H$ and $u_i, u_{i+1} \in e_i$ for $i = 1, \ldots, l$. If there exists a path between any two vertices of $G$, then $G$ is called connected.

Hypergraphs and tensors are generalizations of graphs and matrices, and there is a natural one-to-one correspondence between uniform hypergraphs

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and tensors. In 2005, the concept of eigenvalues of tensors was posed by Qi [11] and Lim [8], independently. Due to the developments of spectral theory of tensors, there have been many attempts to extend spectral graph theory to hypergraphs via tensors. We first introduce some concepts and notations on tensor eigenvalues.

An order $m$ dimension $n$ tensor $A = (a_{i_1i_2...i_m})$ is a multidimensional array with $n^m$ entries ($i_j \in \{1, \ldots, n\}, j = 1, \ldots, m$). $A$ is called symmetric if $a_{i_1i_2...i_k} = a_{i_{\sigma(1)}i_{\sigma(2)}...i_{\sigma(k)}}$ for any permutation $\sigma$ on $\{1, \ldots, k\}$. For $A = (a_{i_1i_2...i_m}) \in \mathbb{C}^{n \times n \times \cdots \times n}$ and $x = (x_1, \ldots, x_n)^T \in \mathbb{C}^n$, $Ax^{m-1}$ is a vector in $\mathbb{C}^n$ whose $i$-th component is

$$ (Ax^{m-1})_i = \sum_{i_2,...,i_m=1}^n a_{i_1i_2...i_m}x_{i_2} \cdots x_{i_m}. $$

A number $\lambda \in \mathbb{C}$ is called an eigenvalue of $A$, if there exists a nonzero vector $x \in \mathbb{C}^n$ such that $Ax^{m-1} = \lambda x^{m-1}$, where $x^{m-1} = (x_1^{m-1}, \ldots, x_n^{m-1})^T$. In this case, $x$ is an eigenvector of $A$ associated with $\lambda$. If $\lambda$ is a real eigenvalue with a real eigenvector, then $\lambda$ is called an $H$-eigenvalue of $A$. An $H$-eigenvalue with positive eigenvector is called an $H^{++}$-eigenvalue. Let $\lambda(A)$ denote the largest $H$-eigenvalue of $A$. The spectral radius of $A$ is defined as $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$, where $\sigma(A)$ is the set of all eigenvalues of $A$.

In 2012, Cooper and Dutle [3] investigated spectral properties of uniform hypergraphs via adjacency tensor. The adjacency tensor of a $k$-uniform hypergraph $H$ with $n$ vertices, denoted by $A_H$, is an order $k$ dimension $n$ symmetric tensor with entries

$$ a_{i_1i_2...i_k} = \begin{cases} \frac{1}{(k-1)!}, & \{i_1, \ldots, i_k\} \in E(G), \\ 0, & \text{otherwise.} \end{cases} $$

In 2014, Qi [12] defined the Laplacian tensor and signless Laplacian tensor of $H$ as $L_H = D_H - A_H$ and $Q_H = D_H - A_H$, respectively, where $D_H$ is the diagonal tensor of vertex degrees of $H$. Recently, the research on spectral properties of $A_H, L_H$ and $Q_H$ has attracted extensive attention [3,4,6,9,12-16].

In many literatures on spectral hypergraph theory, only uniform hypergraphs are considered. There are few work on spectral properties of non-uniform hypergraphs. In order to investigate spectra of non-uniform hypergraphs, we first extend the concept of the adjacency tensor, Laplacian
tensor and signless Laplacian tensor from uniform hypergraphs to general hypergraphs as follows.

**Definition 1.1.** The adjacency tensor of a hypergraph \( H \) with \( r(\mathcal{H}) = k \), denoted by \( \mathcal{A}_H \), is an order \( k \) dimension \( |V(\mathcal{H})| \) symmetric tensor with entries

\[
a_{i_1i_2\ldots i_k} = \begin{cases} \frac{1}{(k-1)!} & \{i_1, \ldots, i_k\} \in E(\mathcal{H}), \\ \frac{1}{(k-s+1)!} & \{i_1, \ldots, i_k\} = \{j_1^{(k-s+1)}, j_2, \ldots, j_s\}, \{j_1, \ldots, j_s\} \in E(\mathcal{H}), \\ 0 & \text{otherwise}, \end{cases}
\]

where \( s < k \) and \( j_1^{(k-s+1)} \) means that the multiplicity of \( j_1 \) is \( k-s+1 \). Let \( \mathcal{D}_H \) denote the order \( k \) dimension \( |V(\mathcal{H})| \) diagonal tensor whose diagonal entries are vertex degrees of \( \mathcal{H} \). The tensors \( \mathcal{L}_H = \mathcal{D}_H - \mathcal{A}_H \) and \( \mathcal{Q}_H = \mathcal{D}_H + \mathcal{A}_H \) are the Laplacian tensor and the signless Laplacian tensor of \( \mathcal{H} \), respectively.

The following is an example for the tensor representations of a non-uniform hypergraph.

**Example.** Let \( H \) be a hypergraph whose vertex set and edge set are \( V(\mathcal{H}) = \{1, 2, 3, 4, 5\} \) and \( E(\mathcal{H}) = \{1234, 45\} \), respectively. Then \( r(\mathcal{H}) = 4 \) and \( \text{cr}(\mathcal{H}) = 2 \). By Definition 1.1, \( \mathcal{A}_H = (a_{i_1i_2i_3i_4}) \) is a symmetric tensor of dimension 5, where \( a_{i_1i_2i_3i_4} = \frac{1}{3!} = \frac{1}{6} \) if \( \{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\} \), \( a_{i_1i_2i_3i_4} = \frac{3!}{4!} = \frac{3}{4} \) if \( \{i_1, i_2, i_3, i_4\} \in \{\{4, 4, 4, 5\}, \{4, 5, 5, 5\}\} \), and \( a_{i_1i_2i_3i_4} = 0 \) otherwise.

In this paper, we investigate spectral properties of the adjacency tensor, Laplacian tensor and signless Laplacian tensor of general hypergraphs. We obtain some bounds for the spectral radius of general hypergraphs in terms of vertex degrees, and give spectral characterizations of odd-bipartite hypergraphs.

### 2. Preliminaries

For a tensor \( \mathcal{A} = (a_{i_1\ldots i_m}) \in \mathbb{C}^{n\times \ldots \times n} \), we associate with \( \mathcal{A} \) a digraph \( \Gamma_\mathcal{A} \) as follows. The vertex set of \( \Gamma_\mathcal{A} \) is \( V(\mathcal{A}) = \{1, \ldots, n\} \), the arc set of \( \Gamma_\mathcal{A} \) is \( E(\mathcal{A}) = \{(i, j)|a_{i_1\ldots i_m} \neq 0, j \in \{i_2, \ldots, i_m\} \neq \{i, \ldots, i\}\} \). \( \mathcal{A} \) is said to be weakly irreducible if \( \Gamma_\mathcal{A} \) is strongly connected \([2, 10]\). We can get the following lemma from Definition 1.1.

**Lemma 2.1.** For any hypergraph \( \mathcal{H} \), the following are equivalent:

1. \( \mathcal{H} \) is connected.
(2) $A_H$ is weakly irreducible.
(3) $L_H$ is weakly irreducible.
(4) $Q_H$ is weakly irreducible.

A tensor $A$ is said to be nonnegative if all entries of $A$ are nonnegative.

**Lemma 2.2.** [12] Let $A$ be a weakly irreducible nonnegative tensors of dimension $n$. Then $\rho(A) = \lambda(A) > 0$ is the unique $H^+$-eigenvalue of $A$.

**Lemma 2.3.** [7] Let $A$ be a symmetric nonnegative tensors of order $k$ and dimension $n$. Then $\rho(A) = \lambda(A) = \max \left\{ x^T (A x^{k-1}) | x \in \mathbb{R}^n_+, \sum_{i=1}^n x_i^k = 1 \right\}$.

Furthermore, $x \in \mathbb{R}^n_+$ with $\sum_{i=1}^n x_i^k = 1$ is an eigenvector of $A$ corresponding to $\rho(A) = \lambda(A)$ if and only if $\rho(A) = \lambda(A) = x^T (A x^{k-1})$.

For two nonnegative tensors $A = (a_{i_1i_2...i_m})$ and $B = (b_{i_1i_2...i_m})$ of dimension $n$, $B \leq A$ means that $b_{i_1i_2...i_m} \leq a_{i_1i_2...i_m}$ for all $i_1, i_2, \ldots, i_m \in \{1, \ldots, n\}$.

**Lemma 2.4.** [5] Let $A$ and $B$ be two nonnegative tensors of order $m$ and dimension $n$, and $A$ is weakly irreducible. If $B \leq A$ and $B \neq A$, then $\rho(B) < \rho(A)$.

For $A = (a_{i_1...i_m}) \in \mathbb{C}^{n^m}$, the principal subtensor of $A$ with respect to $\alpha \subseteq \{1, \ldots, n\}$ is defined as $A[\alpha] = (a_{i_1...i_m}) \in \mathbb{C}^{(|\alpha|^n \times |\alpha|^n)}$, where $i_1, \ldots, i_m \in \alpha$. If $|\alpha| < n$, then $A[\alpha]$ is called the proper principal subtensor of $A$.

**Lemma 2.5.** [5] Let $A$ be a weakly irreducible nonnegative tensors of dimension $n$. If $A[\alpha]$ is a proper principal subtensor of $A$, then $\rho(A[\alpha]) < \rho(A)$.

In [13], Shao defined the following product of tensors, which is a generalization of the matrix multiplication.

**Definition 2.6.** [13] Let $A$ and $B$ be order $m \geq 2$ and order $k \geq 1$, dimension $n$ tensors, respectively. The product $AB$ is the following tensor $C$ of order $(m-1)(k-1) + 1$ and dimension $n$ with entries

$$c_{i\alpha_1...\alpha_m-1} = \sum_{i, i_2,..., i_m \in [n]} a_{i_1i_2...i_m} b_{i_2\alpha_1...i_m\alpha_{m-1}},$$

where $i \in \{1, \ldots, n\}, \alpha_1, \ldots, \alpha_{m-1} \in \{1, \ldots, n\}$.
Lemma 2.7. \[ \text{Let } A = (a_{i_1 \cdots i_k}) \text{ be an order } k \geq 2 \text{ dimension } n \text{ tensor, and let } P = (p_{ij}), Q = (q_{ij}) \text{ be } n \times n \text{ matrices. Then} \\
(PAQ)_{i_1 \cdots i_k} = \sum_{j_1, \ldots, j_k \in [n]} a_{j_1 \cdots j_k} p_{i_1 j_1} q_{j_2 i_2} \cdots q_{j_k i_k}. \]

By Theorems 2.3 and 2.5 in [13], we have the following lemma.

Lemma 2.8. \text{Let } A \text{ and } B \text{ be two order } m \text{ dimension } n \text{ tensors. If there exists nonsingular diagonal real matrix } D \text{ such that } B = D^{-(m-1)} AD, \text{ then } A \text{ and } B \text{ have the same spectrum and } H\text{-spectrum.} \]

3. Main results

Let \( H \) be a hypergraph with \( n \) vertices and \( r(H) = k \). For \( x \in \mathbb{C}^n \) and a vertex subset \( \alpha \) of \( H \), let \( x^\alpha = \prod_{i \in \alpha} x_i \). By Definition 1.1, we have

\[
(A_H x^{k-1})_i = \sum_{i_2, \ldots, i_k=1}^n a_{i_1 i_2 \cdots i_k} x_{i_2} \cdots x_{i_k} = \frac{1}{k} \sum_{e \in E_i} [(k - |e|) x_{e \setminus \{i\}} x_i^{k-|e|} + x^e \sum_{j \in e} x_j^{k-|e|}], \quad i = 1, \ldots, n,
\]

\[
x_i (A_H x^{k-1})_i = \frac{1}{k} \sum_{e \in E_i} [(k - |e|) x_{e} x_i^{k-|e|} + x^e \sum_{j \in e} x_j^{k-|e|}], \quad i = 1, \ldots, n,
\]

\[
A_H x^k = x^T (A x^{k-1}) = \sum_{i_1, \ldots, i_k=1}^n a_{i_1 i_2 \cdots i_k} x_{i_1} \cdots x_{i_k} = \sum_{e \in E(H)} x^e \sum_{j \in e} x_j^{k-|e|}.
\]

Moreover, if \( H \) is \( k \)-uniform, then

\[
x^T (A x^{k-1}) = k \sum_{e \in E(H)} x^e, \quad (A_H x^{k-1})_i = \sum_{e \in E\setminus\{i\}} x^e, \quad x_i (A_H x^{k-1})_i = \sum_{e \in E_i} x^e
\]

for \( i = 1, \ldots, n. \)

From the above equalities, we obtain the following result.
Proposition 3.1. Let $H$ be a hypergraph with $n$ vertices and $r(H) = k$. If $\lambda$ is an eigenvalue of $H$ with an eigenvector $x$, then

$$\lambda x_i^{k-1} = \frac{1}{k} \sum_{e \in E_i} [(k - |e|)x^{e \setminus \{i\}}_i x^{k-|e|}_j + x^{e \setminus \{i\}}_j \sum_{j \in e} x^{k-|e|}_j], \ i = 1, \ldots, n,$$

$$\lambda x_i^k = \frac{1}{k} \sum_{e \in E_i} [(k - |e|)x^e x^{k-|e|}_i + x^e \sum_{j \in e} x^{k-|e|}_j], \ i = 1, \ldots, n,$$

$$\lambda \sum_{i=1}^{n} x_i^k = \sum_{e \in E(H)} x^e \sum_{j \in e} x^{k-|e|}_j.$$

Moreover, if $H$ is $k$-uniform, then

$$\lambda \sum_{i=1}^{n} x_i^k = k \sum_{e \in E(H)} x^e, \ \lambda x_i^{k-1} = \sum_{e \in E_i} x^{e \setminus \{i\}}, \ \lambda x_i^k = \sum_{e \in E_i} x^e, \ i = 1, \ldots, n.$$

Theorem 3.2. Let $H$ be a connected hypergraph. Then the following hold:
(1) $\rho(A_H) = \lambda(A_H) > 0$ is the unique $H^{++}$-eigenvalue of $A_H$.
(2) $\rho(Q_H) = \lambda(Q_H) > 0$ is the unique $H^{++}$-eigenvalue of $Q_H$.
(3) If $G$ is a proper sub-hypergraph of $H$ with $r(G) = r(H)$, then

$$\rho(A_G) < \rho(A_H), \ \rho(Q_G) < \rho(Q_H).$$

Proof. Since $H$ is connected, by Lemma 2.1, $A_H$ and $Q_H$ are weakly irreducible. By Lemma 2.2, we know that (1) and (2) hold.

Since $G$ is a proper sub-hypergraph of $H$ with $r(G) = r(H)$, one of the following cases holds:

(a) $A_G \leq A_H, Q_G \leq Q_H$ and $A_G \neq A_H, Q_G \neq Q_H$.

(b) $A_G$ and $Q_G$ are proper principal subdents of $A_H$ and $Q_H$, respectively.

Since $A_H$ is weakly irreducible, by Lemmas 2.4 and 2.5, we know that (3) holds.

Theorem 3.3. Let $H$ be a hypergraph with average degree $\bar{d}$. Then $\rho(A_H) \geq \bar{d}$, with equality if and only if $H$ is regular.

Proof. Let $x = (\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})^T$, where $k = r(H), n = |V(H)|$. By Lemma 2.3, we have

$$\rho(A_H) \geq x^T (A_H x^{k-1}) = \sum_{e \in E(H)} x^e \sum_{j \in e} x^{k-|e|}_j = \frac{1}{n} \sum_{e \in E(H)} |e| = \bar{d},$$
with equality if and only if $A_H x^{k-1} = \rho(A_H) x^{k-1}$, i.e., $H$ is regular. \hfill \Box

**Theorem 3.4.** Let $H$ be a connected hypergraph with maximum degree $\Delta$. Then $\rho(A_H) \leq \Delta$, with equality if and only if $H$ is regular.

**Proof.** Let $x = (x_1, \ldots, x_n)^T$ be the positive eigenvector corresponding to $\rho(H)$, and let $x_i = \max_{1 \leq j \leq n} x_j$. By Proposition 3.1 we have

$$\rho(A_H) x_i^k = \frac{1}{k} \sum_{e \in E_i} [(k - |e|) x_i^e x_i^{k-|e|} + x_i^e \sum_{j \in e} x_i^{k-|e|}] \leq d_i x_i^k,$$

with equality if and only if $x_j = x_i$ for any $j \in e \in E_i$. Since $H$ is connected, we have $\rho(A_H) \leq \Delta$, with equality if and only if $H$ is regular. \hfill \Box

**Theorem 3.5.** Let $H$ be a connected hypergraph with $r(H) = k$. Then

$$\rho(A_H) \leq \max \left\{ \sqrt[k]{d_{i_1}^{k-s+1} d_{i_2} \cdots d_{i_s}} \mid \{i_1, \ldots, i_s\} \in E(H), d_{i_1} \geq \cdots \geq d_{i_s} \right\}.$$

**Proof.** Let $x$ be the positive eigenvector corresponding to $\rho(A_H)$. Suppose that $x_{i_1} \cdots x_{i_k} = \max_{(A_H)_{j_1 \cdots j_k} \neq 0} x_{j_1} \cdots x_{j_k}$. By Proposition 3.1 we have

$$\rho(A_H) x_{i_j}^k = \frac{1}{k} \sum_{e \in E_{i_j}} [(k - |e|) x_{i_j}^e x_{i_j}^{k-|e|} + x_{i_j}^e \sum_{l \in e} x_{i_j}^{k-|e|}] \leq d_{i_j} x_{i_1} \cdots x_{i_k}$$

for all $j = 1, \ldots, k$. Then

$$\rho(A_H)^k \prod_{j=1}^k x_{i_j}^k \leq d_{i_1} \cdots d_{i_k} (x_{i_1} \cdots x_{i_k})^k,$$

$$\rho(A_H) \leq \sqrt[k]{d_{i_1} \cdots d_{i_k}}.$$

By Definition 1.1 we have

$$\rho(A_H) \leq \max \left\{ \sqrt[k]{d_{i_1}^{k-s+1} d_{i_2} \cdots d_{i_s}} \mid \{i_1, \ldots, i_s\} \in E(H), d_{i_1} \geq \cdots \geq d_{i_s} \right\}.$$

We can obtain the following result from the above Theorem.
Corollary 3.6. Let $H$ be a connected $k$-uniform hypergraph. Then
\[
\rho(A_H) \leq \max_{\{i_1, \ldots, i_k\} \in E(H)} \sqrt{d_{i_1}d_{i_2} \cdots d_{i_k}}.
\]

Remark. In [15], Yuan et al. proved that
\[
\rho(A_H) \leq \max_{e \in E(H)} \max_{\{i, j\} \subseteq e} \sqrt{d_i d_j}.
\]
Corollary 3.6 is a refinement of this upper bound.

A hypergraph $H$ is called odd-bipartite, if its vertex set has a partition $V(H) = V_1 \cup V_2$ such that each edge of $H$ contains odd number of vertices in $V_1$ and odd number of vertices in $V_2$. Spectral characterizations of odd-bipartite uniform hypergraphs are given in [4, 14]. We extend these work to general odd-bipartite hypergraphs as follows.

Theorem 3.7. Let $G$ be a connected hypergraph. Then the following are equivalent:

1. $G$ is odd-bipartite.
2. $A_G$ and $-A_G$ have the same spectrum and H-spectrum.
3. $-\rho(A_G)$ is an H-eigenvalue of $A_G$.

Proof. (1)$\Rightarrow$(2). If $G$ is odd-bipartite, then by Lemma 2.7, there exists a diagonal matrix $P$ with diagonal entries $\pm 1$ such that $A_G = -P^{-(k-1)}A_GP$, where $k = r(G)$. By Lemma 2.8, we know that $A_G$ and $-A_G$ have the same spectrum and H-spectrum.

(2)$\Rightarrow$(3). If $A_G$ and $-A_G$ have the same H-spectrum, then $-\rho(A_G)$ is an H-eigenvalue of $A_G$.

(3)$\Rightarrow$(1). By Lemma 1.2 in [14], there exits real diagonal matrix $P = \text{diag}(x_1, \ldots, x_n)$ such that $A_G = -P^{-(k-1)}A_GP$ and $|x_1| = \cdots |x_n| = 1$. By Lemma 2.7, we have
\[
(A_G)_{i_1i_2\cdots i_k} = -(A_G)_{i_1i_2\cdots i_k}x_{i_1}^{-(k-1)}x_{i_2}x_{i_3} \cdots x_{i_k}.
\]
For any $(A_G)_{i_1i_2\cdots i_k} \neq 0$, we have
\[
x_{i_1}x_{i_2} \cdots x_{i_k} = x_{i_1}^k = \cdots = x_{i_k}^k.
\]
If $k$ is odd, then $x_{i_1} = \cdots = x_{i_k}$, a contradiction to $x_{i_1}x_{i_2} \cdots x_{i_k} = -x_{i_1}^k$. So $k$ is even and $x_{i_1}x_{i_2} \cdots x_{i_k} = -1$ for any $(A_G)_{i_1i_2\cdots i_k} \neq 0$. Let $V_1 = \{u | u \in V(G), x_u = -1\}$. By Definition 1.1, we know that each edge $e$ of $G$ contains odd number of vertices in $V_1$ and $|e|$ is even. Hence $G$ is odd-bipartite. \qed
Theorem 3.8. Let $G$ be a connected hypergraph. Then the following are equivalent:

(1) $G$ is odd-bipartite.

(2) $L_G$ and $Q_G$ have the same spectrum and $H$-spectrum.

(3) 0 is an $H$-eigenvalue of $Q_G$.

Proof. (1)$\Rightarrow$(2). If $G$ is odd-bipartite, then by Lemma 2.7, there exists a diagonal matrix $P$ with diagonal entries $\pm 1$ such that $L_G = P^{-(k-1)}Q_GP$, where $k = r(G)$. By Lemma 2.8, we know that $L_G$ and $Q_G$ have the same spectrum and $H$-spectrum.

(2)$\Rightarrow$(3). Let $x = (1, \ldots, 1)^T$. Since $L_G x^{k-1} = 0$, 0 is always an $H$-eigenvalue of $L_G$. If $L_G$ and $Q_G$ have the same $H$-spectrum, then 0 is an $H$-eigenvalue of $Q_G$.

(3)$\Rightarrow$(1). Let $x = (x_1, \ldots, x_n)^T$ be a real eigenvector of $Q_G$ corresponding to the $H$-eigenvalue 0. Then

$$\frac{1}{k} \sum_{e \in E} [(k - |e|)x^{k-|e|}_e + x^{e\{i\}} \sum_{j \in e} x^{k-|e|}_j] = -d_i x^{k-1}_i, \quad i = 1, \ldots, n.$$

Let $|x_j| = \max_{1 \leq i \leq n} |x_i|$. Then

$$-d_i x^k_j = \frac{1}{k} \sum_{e \in E_j} [(k - |e|)x^{k-|e|}_e + x^{e\{i\}} \sum_{i \in e} x^{k-|e|}_i],$$

$$d_i |x^k_j| \leq \frac{1}{k} \sum_{e \in E_j} [(k - |e|)|x^k_j| + |e||x^k_j|] = d_i |x^k_j|.$$

Hence $x^e x^{k-|e|}_i = -x^k_i$ for any $i \in e$ and any $e \in E(G)$. If $k$ is odd, then $x_1 = \cdots = x_n$, a contradiction to $x^e x^{k-|e|}_i = -x^k_i$. So $k$ is even and $x^e x^{k-|e|}_i = -|x^e_j|$ for any $i \in e$ and any $e \in E(G)$. Let $V_1 = \{u \mid u \in V(G), x_u = -|x_j|\}$, then each edge $e$ of $G$ contains odd number of vertices in $V_1$ and $|e|$ is even. Hence $G$ is odd-bipartite.

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