Renormalization of gauge invariant composite operators in light-cone gauge

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Abstract

We generalize to composite operators concepts and techniques which have been successful in proving renormalization of the effective Action in light-cone gauge. Gauge invariant operators can be grouped into classes, closed under renormalization, which is matrix-wise. In spite of the presence of non-local counterterms, an “effective” dimensional hierarchy still guarantees that any class is endowed with a finite number of elements. The main result we find is that gauge invariant operators under renormalization mix only among themselves, thanks to the very simple structure of Lee-Ward identities in this gauge, contrary to their behaviour in covariant gauges.

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I. INTRODUCTION

Composite operators often occur in calculations of physical cross-sections. A celebrated example is deep-inelastic scattering where short-distance products of currents are expressed in terms of local operators by means of a Wilson expansion. But, strictly speaking, the lagrangian density itself is an instance of composite operator.

As quantum fields are distribution-valued operators, one can easily realize that, taking products at the same space-time point, gives rise to singularities. Whence the need of first considering a procedure of regularization and then performing the necessary subtractions in a consistent way to operatively define their finite parts, at least in a perturbative context.

The peculiar phenomenon occurring in composite operator renormalization is their mixing. Locality and polynomiality in the masses of counterterms guarantee the presence of a dimensional hierarchy: counterterms can only have canonical dimensions less than or equal to the ones of the operators we are considering. Therefore, the number of counterterms which mix is finite.

All those concepts and techniques naturally apply to gauge theories with the proviso they have to comply with Ward identities taking care of redundant degrees of freedom. In covariant gauges (typically in generalized Feynman gauges) the relevant Slavnov-Taylor identities involve unphysical operators (Faddeev-Popov ghosts). As a consequence, a deep thorough analysis has shown that gauge invariant operators do mix with unphysical ones under renormalization.

The situation radically changes in the so-called physical gauges $n_\mu A^\mu = 0$, $n_\mu$ being a constant vector, where there is no need of Faddeev-Popov fields and Lee-Ward identities are straightforward. This is the reason why such gauges have been largely adopted in the past for phenomenological applications.

Only recently however a systematic approach has been developed with a sound basis on the axioms of canonical quantum field theory. Effective Action renormalization has been proven, so far, at any order in the loop expansion, only in light-cone (LC) gauge ($n^2 = 0$)
Essential to this goal is to endow the “spurious” singularity, occurring in the vector propagator, with a causal prescription (Mandelstam-Leibbrandt (ML) prescription), as suggested by a careful canonical quantization. This prescription in turn is the source of a potentially serious difficulty: non-local counterterms are needed, already at one loop level, to make one particle irreducible vertices finite.

It is clear that non-locality could in principle destroy dimensional hierarchy. Should the mixing involve an infinite number of independent counterterms, even for a single insertion, the very program of composite operator renormalization would be in jeopardy.

Happily this is not the case. Generalizing concepts and techniques which have been successful in proving renormalization of the effective Action, in next Sections we show that a new kind of “effective” dimensional hierarchy can be established which is enough to prove renormalization at any order in the loop expansion, at least for gauge invariant composite operators, which are the ones directly involved in phenomenological applications. Actually the very simple structure of Lee-Ward identities, which survives renormalization in this case, will allow us to reach a rather strong result: in LC gauge, under renormalization gauge invariant operators mix only among themselves, in classes with finite numbers of elements.

The problems one encounters when treating more general operators, will be briefly discussed in the Conclusions.

In Sect. we introduce our notation, we define the generating functionals with composite operators insertions and derive the Lee-Ward identities they have to satisfy. Sect. is devoted to generalize BPHZ subtraction method to our problem and prove the gauge invariance of renormalized composite operators. In Sect. we discuss power-counting in LC gauge and the need of introducing a more general criterion of superficial degree of divergence, in relation to Weinberg’s theorem. In Sect. we explore all constraints the counterterms have to fulfill and in Sect. we prove that in the mixing of gauge invariant operators a unique independent non-local structure can appear with mass dimension equal to one, the same one encounters when renormalizing the effective Action. Concrete examples of mixing are presented in Sect. while remarks and comments concerning further
II. THE GENERATING FUNCTIONALS WITH COMPOSITE OPERATORS

We start by defining our lagrangian and our notation

\[
\mathcal{L}_{g.i.} = -\frac{1}{2} \text{Tr} (F_{\mu\nu} F^{\mu\nu}) + \bar{\psi} (i \slashed{D} - m) \psi
\]  

where \( F_{\mu\nu} \) is the usual field tensor in the adjoint representation of the algebra \( \text{su}(N) \).

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu],
\]

\[
A_\mu = A_\mu^a \tau^a, \quad a = 1, \ldots, N^2 - 1,
\]

\[
\text{Tr}(\tau^a \tau^b) = \frac{1}{2} \delta^{ab},
\]

\[
[\tau^a, \tau^b] = if^{abc} \tau^c,
\]

\( f^{abc} \) being the structure constants of the group which are completely antisymmetric in this basis. \( \slashed{D} \) is the covariant derivative acting on the fundamental representation,

\[
\slashed{D}_\mu = \partial_\mu - igA_\mu.
\]

The lagrangian density in eq. (1) is invariant under gauge transformations, as is well known. Their infinitesimal form is

\[
\delta^{[\omega]} \psi(x) = ig \omega(x) \psi(x),
\]

\[
\delta^{[\omega]} A_\mu(x) = D_\mu \omega(x),
\]

where \( D_\mu \) is the covariant derivative acting on the adjoint representation

\[
D_\mu = \partial_\mu - ig[A_\mu,] \]

and \( \omega(x) \) are the infinitesimal parameters of the transformation. In order to quantize the theory, we introduce the light-cone gauge fixing

\[
\mathcal{L}_{g.f.} = -\lambda(x) \cdot n_\mu A^\mu(x),
\]

developments are contained in the Conclusions.
\( \lambda(x) \) being Lagrange multipliers and \( n_\mu \) a fixed light-like four-vector \( n^2 = 0 \).

In the following, dimensional regularization will be understood in the framework of perturbation theory. In \( 2\omega \) dimensions the coupling constant \( g \) will be replaced with \( g\mu^{2-\omega} \) where \( \mu \) is a mass scale.

From eqs. (1) and (9) we can construct the usual functional \( W \) which generates the Green’s functions of the theory

\[
W[J, K, \eta, \bar{\eta}] = \mathcal{N} \int [dA][d\lambda][d\psi][d\bar{\psi}] \exp \left[ i \int d^4x \left( \mathcal{L}_{g.i.} + \mathcal{L}_{g.f.} + \mathcal{L}_s \right) \right]
\]

(10)

where

\[
\mathcal{L}_s = J_\mu \cdot A^\mu + K \cdot \lambda + \bar{\eta} \psi - \bar{\psi} \eta.
\]

(11)

Then we can define in the usual way the functional \( Z = \frac{1}{i} \log W \) which generates the connected Green’s functions; from \( Z \) we get the “classical” fields \( A_\mu, \Lambda, \Psi, \bar{\Psi} \) and, eventually, the functional \( \Gamma \) which generates the proper vertices of the theory

\[
\Gamma[A, \Lambda, \Psi, \bar{\Psi}] = Z[J, K, \eta, \bar{\eta}] - \int d^4x \left( J \cdot A + K \cdot \Lambda + \bar{\Psi} \eta - \bar{\eta} \Psi \right).
\]

(12)

The derivatives with respect to Grassmann variables are understood as left derivatives; with our conventions we get, in particular,

\[
\begin{align*}
\Psi &\equiv \frac{\delta Z}{\delta \eta}, \\
\bar{\Psi} &\equiv \frac{\delta Z}{\delta \bar{\eta}}, \\
\eta &\equiv \frac{\delta \Gamma}{\delta \Psi}, \\
\bar{\eta} &\equiv \frac{\delta \Gamma}{\delta \bar{\Psi}}.
\end{align*}
\]

(13)

We also notice that invariance under a shift in \( \lambda \) of the path-integral entails the condition

\[
nA = K,
\]

(14)

which in turn guarantees that any Green’s function containing \( nA \), but no \( \lambda \), vanishes.

In this section we are mainly concerned with the generalization of such generating functionals to the case in which composite operators are considered. Such a generalization is presented for instance in refs. [15,16]. We denote by \( X = X[A, \psi, \bar{\psi}] \) a polynomial built
from the original fields and their derivatives taken at the same space-time point; Green’s functions with insertions of such operators usually exhibit further singularities. The technique one uses to take those insertions into account is to introduce, in the definition of $W$, a source term related to $X$. In the general case we shall consider a set of operators $X_i$ each associated to a source $\Sigma_i$, by adding the lagrangian density

$$\mathcal{L}_X = \sum_i \Sigma_i \cdot X_i.$$  \hspace{1cm} (15)

In the following we shall not be concerned with composite operators involving Lagrange multipliers as they would affect the equations of motion of the field $\lambda$ that enter in the derivation of the Lee identities. Moreover we shall limit ourselves to gauge invariant composite operators, but in the final section where we shall briefly dwell on possible generalizations.

A crucial point to remark is that the functional $\Gamma$ with insertions is defined by means of a Legendre transformation involving only the classical fields we have already considered

$$\Gamma[A, \Lambda, \Psi, \bar{\Psi}, \Sigma_i] = Z[J, K, \eta, \bar{\eta}, \Sigma_i] - \int d^4x \left( J \cdot A + K \cdot \Lambda + \bar{\Psi} \eta - \bar{\eta} \Psi \right).$$  \hspace{1cm} (16)

As a consequence, one can prove the equality

$$\frac{\delta \Gamma}{\delta \Sigma_i} \bigg|_{A,\Lambda,\Psi,\bar{\Psi},\Sigma_{j\neq i}} = \frac{\delta Z}{\delta \Sigma_i} \bigg|_{J,K,\eta,\bar{\eta},\Sigma_{j\neq i}},$$  \hspace{1cm} (17)

where in the left-hand side (right-hand side) “classical” fields (original sources) are kept fixed besides the sources $\Sigma_{j\neq i}$.

By solving the equations of motion of the Lagrange multiplier it is possible to make explicit the dependence of $\Gamma$ on $\Lambda$

$$\Gamma[A, \Lambda, \Psi, \bar{\Psi}, \Sigma_i] = \tilde{\Gamma}[A, \Psi, \bar{\Psi}, \Sigma_i] - \int d^4x \Lambda \cdot n \cdot A$$  \hspace{1cm} (18)

and to convince oneself that the gauge-fixing term does not renormalize. $\tilde{\Gamma}$ is customarily called the “reduced generating functional”.

As we are concerned with gauge-invariant operators, Lee-Ward identities, which have a very simple form in light-cone gauge, will not entail further difficulties in presence of
insertions. In order to derive the Ward identities, we follow the standard technique of performing a change of variable in the path integral corresponding to an infinitesimal gauge transformation. The related functional determinant in this gauge is independent of the fields, as is well known. As we are here considering gauge-invariant insertions, they cannot affect the form of the Ward identity

\[ D^{ab}_\mu \left[ \frac{\delta}{i\delta J} \right] \left\{ \frac{\delta}{i\delta K^b} n^\mu - J^{ab} \right\} W + ig\mu^{2-\omega} \left( \bar{\eta} \tau^a \frac{\delta W}{i\delta \eta} + \frac{\delta W}{i\delta \bar{\eta}} \tau^a \eta \right) = 0, \tag{19} \]

where \( W \) depends also on sources \( \Sigma_i \) related to composite operators. We can get rid of the term with second order functional derivative in eq. (19), using the equations of motion for the Lagrange multiplier. Then we derive from eq. (19) the following Lee identity for the reduced functional \( \tilde{\Gamma} \)

\[ D^{ab}_\mu [A] \frac{\delta \tilde{\Gamma}}{\delta A^b_\mu} + ig\mu^{2-\omega} \left( \frac{\delta \tilde{\Gamma}}{\delta \Psi} \tau^a \Psi + \bar{\Psi} \tau^a \frac{\delta \tilde{\Gamma}}{\delta \bar{\Psi}} \right) \equiv \Delta^a \tilde{\Gamma} = 0, \tag{20} \]

\( \Delta^a \) being the functional differential operator which describes an infinitesimal gauge transformation of the “classical” fields. We shall use the same symbol also for the analogous operator acting on functionals of elementary fields. Eq. (20) means that \( \tilde{\Gamma} \) is gauge invariant. We stress the fact that \( \tilde{\Gamma} \) depends also on possible sources related to gauge-invariant composite operators.

### III. GAUGE INVARIANCE OF RENORMALIZED OPERATORS

In order to renormalize either the Action or a composite operator, we adopt the graph-by-graph subtraction method (or BPHZ method) summarized by the Bogoliubov’s \( \bar{R} \) operator on Feynman graphs \[11–13\]. We just stress the fact here that in presence of diagrams with operator insertions the definition of 1PI diagram remains the same if the operator vertices are treated just like ordinary interaction vertices.

In the following we shall work in the minimal subtraction scheme (MS) on dimensionally regularized diagrams: we denote with \( KG \) the singular part of the Laurent expansion of the
graph $G$ in a neighborhood of $\omega = 2$. The renormalized graph $RG$ is obtained by subtracting the singular part from the subdivergence-free diagram $\overline{RG}$

$$RG = (1 - K)\overline{RG}.$$  \hfill (21)

We shall also use the notation

$$CG = -K\overline{RG}$$ \hfill (22)

to indicate the specific counterterm necessary to renormalize the graph $G$. $CG$ is different from zero if and only if $G$ is 1PI and superficially divergent. In this case a specific counterterm chosen to produce $CG$ as an additional Feynman rule, has to be added to the lagrangian. If $G$ involves composite operator vertices it contributes to the renormalization of the $\Sigma$-dependent terms, otherwise it renormalizes the original lagrangian.

By performing this procedure for every 1PI diagram up to order l-loops, one builds an Action $S^{(l)}$ renormalized to this order. A synthetic and completely equivalent description of this method is given through the generating functional $\Gamma$. We define

$$S^{(0)}[A, \psi, \bar{\psi}, \Sigma_i] = \int d^4x \left( \mathcal{L}_{g.i.}[A, \psi, \bar{\psi}] + \mathcal{L}_X[A, \psi, \bar{\psi}, \Sigma_i] \right),$$ \hfill (23)

as unrenormalized Action. In this definition the gauge-fixing term is excluded as it does not renormalize; hence $S^{(0)}$ is gauge invariant. We denote by $\tilde{\Gamma}^{(l)}$ the reduced generating functional obtained from the Action $S^{(l)}$ and perform the loopwise expansion

$$\tilde{\Gamma}^{(l)} = \sum_{m=0}^{\infty} \tilde{\Gamma}^{(l)}_m;$$ \hfill (24)

$\tilde{\Gamma}^{(l)}_m$ represents the m-loops contribute to $\tilde{\Gamma}^{(l)}$. Now we are able to define iteratively the renormalized Action

$$S^{(l)}[A, \psi, \bar{\psi}, \Sigma_i] = S^{(l-1)}[A, \psi, \bar{\psi}, \Sigma_i] - K\tilde{\Gamma}^{(l-1)}_i[A, \psi, \bar{\psi}, \Sigma_i],$$ \hfill (25)

where $K$ picks up just the singular part at $\omega = 2$ of the regularized expression $\tilde{\Gamma}^{(l-1)}_i[A, \psi, \bar{\psi}, \Sigma_i]$; in this functional, the fields $A$, $\psi$ and $\bar{\psi}$ take the place of the corresponding classical fields.
In general, even in a covariant theory, if the dimension of $X_i$ is $\geq 4$, an infinite number of counterterms of arbitrarily high degree in $\Sigma_i$ are introduced in the renormalized Action by eq. (25). The “renormalized operator” $[X_k]^{(l)}$ is defined by

$$ [X_k]^{(l)}(x) \equiv \frac{\delta S^{(l)}[A, \psi, \bar{\psi}, \Sigma_i]}{\delta \Sigma_k(x)} \bigg|_{\Sigma_i=0 \forall i} ; \tag{26} $$

the operator is renormalized only in the sense that Green’s functions with at most one single insertion of $[X_k]^{(l)}$ are finite in the renormalized theory. If finite Green’s functions with more operator insertions are needed one has to consider the whole renormalized Action $S^{(l)}$ whose functional $W^{(l)}[J, K, \eta, \bar{\eta}, \Sigma_i]$ is finite up to order $l$ at any degree in $\Sigma_i$.

A “weak” form of renormalization will also be considered in which only counterterms at most linear in the sources $\Sigma_i$ are introduced. To this purpose we define

$$ \tilde{\Gamma}^L \equiv \tilde{\Gamma}|_{\Sigma_i=0} + \int d^4x \sum_j \left( \frac{\delta \tilde{\Gamma}}{\delta \Sigma_j(x)} \bigg|_{\Sigma_i=0} \cdot \Sigma_j(x) \right) \tag{27} $$

the part of $\tilde{\Gamma}$ linear in the sources $\Sigma_i$. Then we define recursively as weakly renormalized Action

$$ S^{(l)}_w[A, \psi, \bar{\psi}, \Sigma_i] = S^{(l-1)}_w[A, \psi, \bar{\psi}, \Sigma_i] - K \tilde{\Gamma}^{L(l-1)}_L[A, \psi, \bar{\psi}, \Sigma_i] . \tag{28} $$

Of course one gets

$$ S^{(l)}_w[A, \psi, \bar{\psi}, \Sigma_i] = S^{(l)}[A, \psi, \bar{\psi}] + \sum_i \Sigma_i \cdot [X_i]^{(l)} , \tag{29} $$

where the first term in the r.h.s. is the renormalized Action one would obtain if operator insertions were absent. Only the linear parts in the $\Sigma_i$s of the generating functionals obtained from $S_w$ are finite.

We shall now prove the

**Proposition 1** Let $S^{(0)}$ be the Action with insertions of gauge invariant operators $X_i$, defined by eq. (23) and $S^{(l)}$ the Action renormalized up to $l$-loops according to eq. (25), then
1. $S^{(l)}$ is gauge invariant $\forall l$

$$\Delta^a S^{(l)}[A, \psi, \bar{\psi}] = 0. \quad (30)$$

2. the renormalized operators $[X_i]^{(l)}$ are gauge invariant $\forall l$

$$\Delta^a [X_i]^{(l)}[A, \psi, \bar{\psi}] = 0. \quad (31)$$

**proof:** 2. follows directly from 1. and eq. (29). Let us show 1.: by induction on $l$; as it obviously holds if $l = 0$, we have just to prove the inductive step. Let us start from eq. (30).

The form of Lee identities is not affected by renormalization

$$\Delta^a \Gamma^{(l)}[A, \Psi, \bar{\Psi}] = 0. \quad (32)$$

The same equation must hold for the singular part of the Laurent expansion in $\omega = 2$ and for each contribution in the loopwise expansion

$$\Delta^a K^{(l)}_m[A, \Psi, \bar{\Psi}] = 0. \quad (33)$$

From eq. (25) we finally obtain the desired result

$$\Delta^a S^{(l+1)}[A, \psi, \bar{\psi}] = 0. \quad (34)$$

$\blacksquare$

Of course, from eq. (28) it also follows that the weakly renormalized Action $S_w^{(l)}$ is gauge invariant.

**IV. POWER COUNTING IN LIGHT CONE GAUGE**

The main feature of Feynman graphs in light-cone gauge is the presence of spurious poles introduced by the particular form of the free gauge field propagator

$$\langle 0 | TA^a_\mu(x)A^b_\nu(y)|0 \rangle_{g=0} = \int \frac{d^2k}{(2\pi)^4} e^{ik(x-y)} \frac{-i\delta^{ab}}{k^2 + i\epsilon} \left[ g^{\mu\nu} - \frac{n^\mu k^\nu + n^\nu k^\mu}{[nk]} \right]. \quad (35)$$
The prescription of the spurious pole is the so-called Mandelstam-Leibbrandt (ML) prescription

\[ \frac{1}{[nk]} \overset{Man}{=} \frac{1}{nk + i\epsilon \sigma(n^*k)} \overset{Lei}{=} \frac{n^*k}{(nk)(n^*k) + i\epsilon} , \]  

(36)

being \( \sigma(\cdot) \) the sign function and \( n^*_\mu \) a new four-vector on the light-cone independent from \( n_\mu \). The choice of \( n^*_\mu \) represents a further violation of Lorentz covariance. We choose \( n_\mu = \frac{1}{\sqrt{2}} (1, 0, 0, 1) \) and \( n^*_\mu = \frac{1}{\sqrt{2}} (1, 0, 0, -1) \) in a particular Lorentz frame. Therefore

\[ \frac{1}{[nk]} = \frac{1}{k^+ + i\epsilon \sigma(k^-)} \]  

(37)

where we have introduced the light-cone coordinates (LCC)

\[ k^\pm = k_\mp = \frac{k^0 \pm k^3}{\sqrt{2}} . \]  

(38)

One can prove that the derivative of the ML distribution with respect to \( p^\mu \) is given by

\[ \frac{\partial}{\partial p^\mu} \frac{1}{[np]} = -\frac{n_\mu}{[np]^2} - 2\pi i n^*_\mu \delta(np) \delta(n^*p) . \]  

(39)

This result will be useful when discussing the form of non-polynomial counterterms in section V.

The ML distribution has also correct homogeneity properties with respect to both \( n_\mu \) and \( n^*_\mu \); this can be seen observing that

\[ n^\mu \frac{\partial}{\partial n^\mu} \frac{1}{[np]} = -\frac{1}{[np]} ; \]  

(40)

while eq. (39) is manifestly invariant under dilation of the vector \( n^*_\mu \). As a consequence, the homogeneity degrees of a composite operator with respect to both gauge vectors are preserved under renormalization.

One can show that using the ML prescription, the euclidean UV power counting is a good convergence criterion for the corresponding minkowskian integrals. On the other hand, the spurious poles behave as convergence factors only for the “longitudinal” variables \( k^0 \) and \( k^3 \) and not for the “transverse” ones \( k^1 \) and \( k^2 \). It follows that in light-cone gauge
a diagram may have a divergence associated to certain proper subsets of the integration variables yet involving all integration momenta. From an analytical point of view — i.e. as for Weinberg’s theorem — these divergences are subdivergences, but from a graphical point of view they are to be considered as overall divergences because they are not related to subdiagrams and therefore they are not removed by counterterms in the graph-by-graph subtraction method. Hence we shall call “superficially divergent” a graph $G$ if it exhibits positive power counting on some subset (proper or not) of its integration variables not limited to a proper subdiagram of $G$. In the following we introduce an appropriate superficial degree of divergence consistent with this definition.

First we consider a 1-loop diagram $G^{(1)}$. We denote with $\delta_{\forall}(G^{(1)})$ the usual degree of divergence one obtains by a dilation of all the variables $\{k_{\forall}\} = \{k^0, k^1, k^2, k^3\}$ and we define the analogous quantity $\delta_{\perp}(G^{(1)})$ obtained considering just the transverse variables $\{k_{\perp}\} = \{k^1, k^2\}$. $\delta_{\perp}$ differs from $\delta_{\forall}$ on differentials

$$\delta_{\perp}(d^4k) = 2 \quad (41)$$

or in the cases

$$\delta_{\perp}(nk) = \delta_{\perp}(n^*k) = \delta_{\perp}\left(\frac{1}{[nk]}\right) = 0 \quad, \quad (42)$$

while we shall keep for a single component

$$\delta_{\perp}(k^\mu) = \delta_{\forall}(k^\mu) = 1 \quad, \quad (43)$$

as the result of integrals will always be written in four-vector notation.

The “superficial degree of divergence” of $G^{(1)}$ is then defined as

$$\delta(G^{(1)}) = \max\left\{\delta_{\forall}(G^{(1)}), \delta_{\perp}(G^{(1)})\right\} \quad. \quad (44)$$

It is easy to show that $\delta(G^{(1)})$ is the maximum degree among the ones related to all possible subsets of integration variables.

Now we consider a graph $G$ with $l$ integration momenta $k_1, \ldots, k_l$. We still define
\[
\{k_i^\gamma\} = \{k_i^0, k_i^1, k_i^2, k_i^3\} \\
\{k_i^\perp\} = \{k_i^1, k_i^2\} \quad i = 1, \ldots, l, ,
\] (45) (46)

and denote by

\[
\delta_{v_1, \ldots, v_l}(G) \quad v_i \in \{\forall, \perp\},
\] (47)

the degree of divergence of \(G\) related to the variables

\[
\{k_1^{v_1}\} \cup \{k_2^{v_2}\} \cup \cdots \cup \{k_l^{v_l}\}.
\] (48)

The superficial degree of divergence of \(G\) is now defined as

\[
\delta(G) = \max_{v_i \in \{\forall, \perp\}} \{\delta_{v_1, \ldots, v_l}(G)\}.
\] (49)

It is easy to realize that this definition leads to a sufficient condition for convergence. To show that it is not too cautious, we look at the following two-loop example: for the integral

\[
I = \int d^2\omega_1 d^2\omega_2 \frac{k_1^\mu k_1^\nu k_2^\rho}{(k_1 - q)^2(k_1 - k_2)^4k_2^2 \left[\left|n(k_1 + k_2 + s)\right| \left|n(k_1 + p)\right| \left|nk_1\right|^2\right]},
\] (50)

one finds

\[
\delta(I) = \max \begin{cases} 
\delta_{\forall\forall} = -1 \\
\delta_{\perp\forall} = 0 \\
\delta_{\forall\perp} = -3 \\
\delta_{\perp\perp} = -1 
\end{cases} = 0,
\] (51)

and hence \(I\) may diverge in the variables \(\{k_1^\perp\} \cup \{k_2^\forall\}\). We remark that \(I\) has negative mass dimension and no subdivergences.

We say that a diagram \(G\) is superficially convergent (divergent) if \(\delta(G) < 0 (\delta(G) \geq 0)\). We say that \(G\) has a subdivergence if it has a superficially divergent 1PI proper subdiagram.

It is crucial to notice that, while in covariant gauges the usual degree \(\delta(G)\) has a dimensional meaning because it equals the dimension of the 1PI momentum-space graph, in light-cone gauge \(\delta(G)\) depends on the particular topological structure of the graph. As
a consequence, different graphs contributing to the same proper vertex have in general a
different degree and any proper vertex, whatever its dimension, can have superficially di-
vergent graphs. Therefore, in light-cone gauge power counting arguments do not limit the
type of counterterms entering the lagrangian or a composite operator under renormalization.
In particular, as even diagrams with negative mass dimension may be superficially diver-
gent, non-local — i.e. non polynomial in external momenta — counterterms are generally
expected.

V. GENERAL FORM OF NON-POLYNOMIAL COUNTERTERMS

In a covariant field theory and in particular in Yang-Mills theories with covariant gauges,
the so called BPH theorem holds. The counterterm $C_G$ of a 1PI graph $G$ is polynomial in
the external momenta and thereby the locality of the lagrangian or a composite operator
is preserved under renormalization. The theorem does not hold in light-cone gauge: the
lacking argument in the proof is that in a covariant theory, the action of the derivative with
respect to an external momentum on a graph $G$, lowers its degree of divergence. In light-
cone gauge this is not true. Consider for instance a graph $G$ with $l$ integration momenta
$k_1, \ldots, k_l$. Suppose that $G$ has a spurious pole of the form

$$\frac{1}{[[n(k_1 + r + \text{other momenta})]]} \quad (52)$$

$r$ being an external momentum. The degree $\delta(G)$ is not necessarily lowered by a differenti-
ation with respect to $r^\mu$, as the degrees $\delta_{\perp, v_2, \ldots, v_l}$ are not. As a consequence, $C_G$ is not in
general a polynomial in $r^\mu$. However it is easy to see that a suitable number of derivatives

$$\frac{\partial}{\partial p^\alpha} \quad \alpha \in \{-, 1, 2\} \quad (53)$$

acting on a graph $G$, does indeed make it converge (see eq. (39)). Therefore the BPH
theorem is modified as follows:

**Proposition 2** Let $G$ be a 1PI diagram in light-cone gauge and $p$ an external momentum.
If $\delta(G) \geq 0$, then $C_G$ is a polynomial in the components $p^\alpha, \alpha \in \{-, 1, 2\}$. 

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Possible non-localities of counterterms are therefore limited to non polynomial functions of $p_i^+ = np_i$. We shall see that the non localities can only appear as spurious poles $[[np]]^{-1}$ in the external momenta.

By the same arguments one can show that in light cone gauge, as in covariant gauges, for a 1PI graph $G$, the counterterm $CG$ is polynomial in the fermionic masses.

Some results about the most general form of non-local counterterms prove to be very important in renormalization theory. The following proposition states that the only possible non-localities of a counterterm are spurious poles $\frac{1}{np}$ in the external momenta. In the proof, the following “splitting formula” holding for the ML distribution, is used,

$$
\frac{1}{[[n(p_1 + k)]] [[n(p_2)]]} = \left( \frac{1}{[[n(p_2 - p_1)]]} - \frac{1}{[[n(p_2 + k)]]} \right) .
$$

(54)

**Proposition 3** let $G$ be a 1PI graph in light-cone gauge. Without loss of generality we can consider

$$
G = \int \frac{d^2\omega_{k_1} \ldots d^2\omega_{k_{\beta}} f(p, k, n, n^*, g_{\mu\nu})}{\Pi_{j=1}^\beta \left( t_j^2(k, p) - m_j^2 \right) \Pi_k [[ns_k(k, \tilde{p})]]},
$$

(55)

where:

- $p_i$ are the external momenta;
- $\tilde{p}_j$ ($j = 1, \ldots, \beta$) are linear combination of the $p_i$;
- $f(\hat{f})$ is a polynomial in its arguments;
- $t_j(\hat{t}_j)$ are linear combinations of the $p_i$ and $k_i$;
- $s_k(\hat{s}_k)$ are linear combinations of the $k_i$ and $\tilde{p}_i$.
- $m_j(\hat{m}_j)$ are possible fermionic masses.

Then $G$ can be expressed as a sum.
where each $I_k$ is of the form:

$$I = \prod_{r=1}^{\gamma} \frac{1}{n^s_r(\bar{p})} \int \frac{d^{2\omega}k_1 \ldots d^{2\omega}k_l}{\prod_{j=1}^{\alpha} (\hat{t}^2_j(k,p) - \hat{m}^2_j)} \frac{\hat{f}(p,k,n,g_{\mu\nu})}{\prod_{m=1}^{l}[nk_m]^{\beta_m}},$$

with $\beta_m \geq 0 \forall m$ and

$$\sum_{m=1}^{l} \beta_m + \gamma = \beta \Rightarrow \gamma \leq \beta. \quad (58)$$

**Corollary 4** $CG$ is a meromorphic function in the variables $p_i^+ = np_i$ with poles at most of order $\beta$.

**proof:** (corollary) it follows directly from the form of the integrals $\hat{I}$ observing that $C\hat{I}$ is polynomial in the external momenta as the spurious poles are $p_i$-independent. □

**proof:** (proposition) by induction on $l$. We first show the thesis for $l = 1$. If no spurious poles depend on $k_1$, then $G$ is already of the form $I$ with $\beta_1 = 0$ and $\gamma = \beta$. Otherwise, by using formula (54), one can factor out of the integrand all spurious poles but one $[[n(\bar{p} + k_1)]]^{-\beta_1}$ that can possibly be of higher order ($\beta_1 > 1$) if multiple poles were originally present in $G$. By shifting $k_1$ these poles become $[[nk_1]]^{-\beta_1}$ and therefore $G$ is decomposed as in eq. (56).

Let us now assume that the thesis holds for $l - 1$ loops; we can apply to the integral in $d^{2\omega}k_l$ the same procedure above considering as “external momenta” also the variables $k_i$ $i = 1, \ldots, l - 1$. $G$ is therefore reduced to a sum of terms of the form

$$\int \frac{d^{2\omega}k_1 \ldots d^{2\omega}k_{l-1}}{\prod_{r=1}^{\beta_1} [[n\hat{s}_r(\bar{p}, k_1, \ldots, k_{l-1})]]} \int \frac{d^{2\omega}k_l \hat{f}(p,k,n,g_{\mu\nu})}{\prod_{j=1}^{\alpha} (\hat{t}^2_j(k,p) - \hat{m}^2_j)} \frac{[[nk_l]]^{\beta_l}}{\prod_{m=1}^{l}[nk_m]^{\beta_m}} =$$

$$= \int \frac{d^{2\omega}k_l}{[[nk_l]]^{\beta_l}} \int \frac{d^{2\omega}k_1 \ldots d^{2\omega}k_{l-1}}{\prod_{j=1}^{\alpha} (\hat{t}^2_j(k,p) - \hat{m}^2_j)} \frac{\hat{f}(p,k,n,g_{\mu\nu})}{\prod_{r=1}^{\beta_1} [[n\hat{s}_r(\bar{p}, k_1, \ldots, k_{l-1})]]}.$$

We can now apply the inductive hypothesis to the multiple integral in the r.h.s. considering $p_i$ and $k_l$ as external momenta but remembering that the variables $\bar{p}_i$ do not depend on $k_l$. 

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As a consequence, the spurious poles extracted from the multiple integral can be factorized out of the integral in $d^{2ω}k_l$ giving only terms of the form \((57)\).

We now discuss a feature of Feynman integrals in light cone gauge that is fundamental in selecting the possible structures involved in operator renormalization. To this purpose let us consider the following peculiar property of the ML prescription under the algebraic splitting

$$
\frac{1}{nk + iεσ(n^*k)} \frac{1}{n(k - s) + iεσ(n^*(k - s))} = \frac{1}{ns + iεσ(n^*s)} \left[ \frac{1}{n(k - s) + iεσ(n^*(k - s))} - \frac{1}{nk + iεσ(n^*k)} \right].
$$

(59)

In the limit $s_μ → 0$, no singularity is present in either term of the equality; in particular at the left hand side we have a double pole at $nk = 0$ with causal prescription and at the right hand side the pole at $ns = 0$ is cancelled by a corresponding zero of the quantity in square brackets. However, would we consider the limit $ns → 0$ with $n^*s ≠ 0$, a singularity at $ns = 0$ would persist owing to the dependence on $n^*$.

This is at variance with the behaviour of a prescription involving only one gauge vector, after the disposal of Poincaré-Bertrand terms \([17]\), and is at the root of the non-local behaviour of some counterterms in light-cone gauge with the ML prescription.

It is however clear that, should we restrict the spurious denominators to the subregions $n^*s_i = -ns_i$, $s_i$ being any generic four-vector, we would get

$$
\frac{1}{ns + iεσ(n^*s)} \rightarrow \frac{ns}{(ns)^2 - iε} = CPV \frac{1}{ns},
$$

(60)

and would recover locality by the very same argument which is used in the proof in the one-vector space-like case \([17]\). We stress that this restriction must be understood in the sense of the theory of distributions. In particular multiple poles should always be interpreted as derivatives, otherwise one immediately runs into powers of CPV prescription, i.e. meaningless quantities.

Having the above discussed property in mind, we now require that all acceptable non-local structures in counterterms, have to become local when $n^*∂$ is replaced by $-n∂$. 

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Actually a replacement of the kind $n^* \partial \to \kappa n \partial$, with any constant $\kappa$, would do the job. Our previous choice is reminiscent of the condition $n^* \to -n$, advocated in ref. [6] to recover locality in analogy with the spacelike planar gauge. We stress however that the present condition is imposed in a form of a phase-space restriction while standing on the light-cone; in this sense it is closer to the spirit of the discussion in ref. [18].

We shall show in the next Section that this criterion is extremely efficient in selecting among a priori possible non-local structures the only acceptable one: $\Omega$, the same already present when renormalizing the effective Action [6].

VI. CONSTRUCTION OF RENORMALIZATION CLASSES

In covariant Yang-Mills theories, the renormalization of gauge invariant composite operators is governed by the BPH theorem; the locality of counterterms guarantees that a composite operator can only mix with operators of lower or equal canonical dimension. This dimensional hierarchy automatically limits the number of renormalization constants needed by a single renormalized operator. Nevertheless, in covariant gauges, the renormalization of composite operators is a very complicated matter because of the presence of non physical degrees of freedom that contribute non trivially to renormalized operators. For this reason gauge invariance of a composite operator is generally lost under renormalization [3].

In light cone gauge the situation is opposite. Renormalization preserves gauge invariance of the operator, but the presence of non local counterterms could allow in principle an infinite number of independent structures to appear.

Our aim is to show that on the contrary the renormalization of a gauge invariant composite operator involves only a finite number of renormalization constants and that non local terms do not affect physical quantities. From now on we will focus on weak renormalization (a single insertion) because, in the more general case, an infinite number of counterterms is expected on general grounds also in covariant theories if the operator has dimension $\geq 4$.

Let us consider a local gauge invariant operator $X$ being a Lorentz tensor of rank $i$ ($i$ free
Lorentz indices), with homogeneity degrees $O_n$ and $O_n^*$ with respect to the gauge vectors and mass dimension $d_m$. The most general form of the renormalized operator $[X]$ is a structure having the same characteristics of $X$ mentioned above but locality since poles of the form $n\partial^{-1}$ may be present; however the structure has to become local if the substitution $n^*\partial \rightarrow n\partial$ is performed. Such structures will be called quasi-local. We observe that the canonical dimension of the field $A_\mu$ cannot be defined in light-cone gauge as the UV behaviour of its propagator does depend on the gauge vector; the only well-defined dimension of operators in light-cone gauge is mass dimension. As a consequence in the expression of the renormalized operator $[X]$ we shall consider mass parameters as part of mixed operators and shall work with dimensionless renormalization constants. Hence we can state the following:

**Proposition 5**  
Local or quasi-local gauge invariant composite operators with the same mass dimension $d_m$, the same homogeneity degrees $O_n$ and $O_n^*$, and the same tensorial rank form a class that is closed under renormalization.

We want to show that each of these renormalization classes contain a finite number of independent operators. In the construction of quasi-local gauge invariant structures, besides the usual covariant tensors, spinors and derivatives, the following covariant non local integral operator can be used as a building block:

$$\left[nD^{-1}\right]^{ab} = \delta^{ab} \frac{1}{[n\partial]} - g f^{acd} \frac{1}{[n\partial]} \left\{n A^c [nD^{-1}]^{db}\right\}; \quad (61)$$

the formula has to be understood recursively and can easily be expanded in powers of $g$. Because of the negative mass dimension of $nD^{-1}$, for any given composite operator, an infinite set of possible independent counterterms can be obtained still satisfying the requirements of correct mass dimension, homogeneity and tensorial structure. On the contrary, only very few structures containing $nD^{-1}$ become local when $n^*\partial \rightarrow n\partial$; by explicit construction one realizes that the only acceptable ones are those in which $nD^{-1}$ is carried by the following combination transforming in the adjoint representation:

$$\Omega^a = \frac{n^\mu n^\nu}{n^*n} [nD^{-1}]^{ab} F_{\mu\nu}^b. \quad (62)$$
This structure is peculiar since $n_\mu n_\nu F^{\mu \nu}$ develops a factor $nD$ in the numerator when $n^* \partial \rightarrow n \partial$:

$$
\Omega|_{n^* \partial \rightarrow n \partial} = \frac{1}{n^* n} \frac{1}{nD} \left( n \partial n^* A - n^* n \partial nA - ig[nA, n^* A] \right) = 
= \frac{1}{n^* n} \frac{1}{nD} \left( nD n^* A - nD nA \right) = \frac{1}{n^* n} (n^* A - nA).
$$

(63)

Such a condition is indeed stronger than the one considered in ref. [6] as structures like

$$
n_\mu n_\nu F^{\mu \nu} \times \text{(non local)},
$$

(65)
or

$$
\frac{1}{nD} n^* D \times \text{(anything)},
$$

(66)

become local if one replaces $n^* \rightarrow n$ but not if $n^* \partial \rightarrow n \partial$.

The crucial point here to observe is that $\Omega$ has positive mass dimension; hence, for any given operator the number of possible independent gauge invariant counterterms built from local covariant objects and $\Omega$, is automatically limited by dimensionally arguments. Equivalently, each renormalization class is finite. Moreover, by directly inspecting the expansion of $\Omega$,

$$
\Omega^a = \Omega_0^a + \sum_{k=1}^{\infty} g^k \Omega_k^a,
$$

(67)

$$
\Omega_0^a = n^* A^a - \frac{n^* \partial}{n \partial} nA^a,
$$

$$
\Omega_k^a = (-1)^{k+1} f_{abk} h_{k-1} f^{h_{k-1} b_{k-2}} \ldots f_{h_2 b_1} f^{h_1 c}.
$$

one learns that all non local terms appearing in the renormalized operator $[X]$ will be proportional to the field $nA$; therefore only the local part of the renormalized operator $[X]$ will contribute to $\lambda$-independent Green’s functions (see eq. (14)).

What we have said before can be summarized as follows:

**Proposition 6** Let $X$ be a local or quasi-local gauge invariant composite operator; then
1. [X] involves a finite number of renormalization constants

2. possible non local terms of [X] are proportional to nA and therefore do not contribute to physical quantities.

Let us see how to build a basis of independent operators for a given renormalization class characterized by the mass dimension $d_m$, the homogeneity degrees $O_n$ and $O^*_n$, and the tensor rank $i$ of its gauge invariant operators. In the following table we list the “blocks” that can be used to build a local or quasi-local operator:

- $f = \#[\bar{\psi} \ldots \psi]$
- $q = \#[F_{\mu\nu} = F^a_{\mu\nu} \tau^a]$
- $p = \#[D^{ab}_\mu \text{ or } D_\mu]$
- $\omega = \#[\Omega = \Omega^a \tau^a]$
- $j = \#[n_\mu]$
- $k = \#[n^*_\mu]$
- $l = \#[(n^* n)^{-1}]$
- $g = \#[\gamma^\mu]$
- $r = \#[g^{\mu\nu}]$
- $m = \#[\text{masses } m \text{ or derivatives } \partial_\mu]$ 

where of course $g^{\mu\nu}$ is understood only with free indices. The positive integer variables $f, q, \ldots, r$ denote the multiplicity of a single factor inside a given operator. Of course, as already anticipated in (68), derivatives acting on gauge invariant quantities are also allowed, each one entailing a unit dimension in mass. The values the variables can assume are subject to the constraints due to mass dimension and homogeneity respect to $n$ and $n^*$:

\[ 3f + 2q + p + \omega + m = d_m, \quad (69a) \]
\[ j - \omega - l = O_n, \quad (69b) \]
\[ k - l = O^*_n, \quad (69c) \]
Eq. (69a) gives an upper limit to all the variables in the l.h.s.; the remaining variables are always limited by imposing the correct rank \( i \) of the operators and their independence.

Of course more operators can correspond to the same combination of variables. Starting from the table above, one has first to build all possible combinations, whose number is however \textit{finite} for a given composite operator, and then to check their independence. Some examples are discussed in the next Section.

\textbf{VII. EXAMPLES}

In this Section we discuss two simple examples of mixing.

The first gauge invariant composite operator we consider is \( \bar{\psi}\psi \). One can easily realize that the only allowed counterterms are

\[ [\bar{\psi}\psi] = \zeta_1 \bar{\psi}\psi + \frac{\zeta_2}{2n^*} \bar{\psi}\gamma^5 \psi + \zeta_3 m^3. \]  

(70)

In this case renormalization involves only local gauge invariant fermionic bilinears, besides the mass term. An explicit one-loop calculation gives

\[ \zeta_1 = 1 - \frac{g^2}{8\pi^2} \frac{N^2 - 1}{2N} \frac{1}{2 - \omega}, \]

\[ \zeta_2 = 0, \]

\[ \zeta_3 = \frac{N}{4\pi^2} \frac{1}{2 - \omega}. \]  

(71)

As a second example we consider the fermionic U(1) conserved current \( \bar{\psi}\gamma_\mu \psi \). There are several independent gauge invariant structures with mass dimension 3 it can \textit{a priori} mix with:

\[ [\bar{\psi}\gamma_\mu \psi] = \zeta_1 \bar{\psi}\gamma_\mu \psi + \zeta_2 \frac{n_\mu}{nn^*} \bar{\psi}\eta^5 \psi + \zeta_3 \frac{n_\mu^*}{nn^*} \bar{\psi}\eta^5 \psi + \]

\[ + \zeta_4 \frac{\eta^5}{nn^*} \bar{\psi}\gamma_\mu \psi + \zeta_5 n^\nu F^a_{\nu\mu} \Omega^a + \zeta_6 n_\mu n_\nu n_\sigma^* F^{\nu\sigma,a} \Omega^a. \]  

(72)

Some of them are non local, i.e. involve \( \Omega \). However, this current is related to the fermion propagator by the U(1) Ward identity which sets constraints between \( \zeta_i \)'s and the wave
function renormalization constants $Z_2$ and $\tilde{Z}_2$. An explicit one-loop calculation fully confirms this result and reproduces the values

$$
\zeta_1 = Z_2^{-1} = 1 - \frac{g^2}{16\pi^2} \frac{N^2 - 1}{2N} \frac{1}{2 - \omega},
$$

$$
\zeta_2 = -\zeta_3 = \tilde{Z}_2^{-1} - 1 = \frac{g^2}{8\pi^2} \frac{N^2 - 1}{2N} \frac{1}{2 - \omega}.
$$

(73)

Finally a last example, definitely requiring non-local counterterms, is the lagrangian density itself, as discussed in [6]; of course further structures involving total derivatives must be considered in this case.

**VIII. CONCLUSIONS**

In this paper we have solved the problem of renormalizing at any order in the loop expansion gauge invariant composite operators in Yang-Mills theories quantized in light-cone gauge with the correct causal ML prescription in vector propagator. We have here generalized the treatment developed in refs. [5,17], concerning effective Action.

Main results are:

- the proof that renormalization preserves the gauge invariance of composite operators (Sect. [II]).

- the full characterization of admissible non-local structures in counterterms, which can be only carried by the quantity $\Omega$, and therefore cannot contribute to physical quantities; hence the proof that the renormalization of a composite operator always entails a finite number of renormalization constants (Sect. [VI]).

Specific examples of mixing under renormalization are presented in Sect. [VII]; in particular we have found quite instructive the behaviour of the U(1) conserved fermionic current, which is endowed with a direct physical interest.
Generalizations to gauge dependent operators are *a priori* possible, however one is immediately faced with a basic difficulty concerning Lee-Ward identities, which are no longer form-invariant under renormalization. Besides, physical applications are usually concerned with gauge invariant composite operators.

Few preliminary results have already appeared in the literature [19], while a pedagogical review on the whole subject will be reported elsewhere [20].
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* We take here the opportunity of correcting eq. (A6.2) of reference [17].

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