Research Article

Existence and Uniqueness of Solutions for a Third-Order Three-Point Boundary Value Problem via Measure of Noncompactness

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In this paper, we consider a nonlinear third-order three-point boundary value problem and give the existence and uniqueness of solutions by constructing Green’s function and using its properties. The methods used here are based on Darbo’s fixed point theorem combined with the technique of measure of noncompactness. Finally, as applications, two examples are given to illustrate our main results.

1. Introduction

Multipoint boundary value problem (BVP) for the third- and higher-order differential equations plays an important part in various fields, such as fluid mechanics, physics, engineering, and many other branches of applied mathematics (see [1–14]). In the past decades, third-order three-point boundary value problems have been widely investigated. In particular, Anderson in [1] considered a right focal problem

\[
\begin{align*}
\begin{cases}
\quad x^{(3)}(t) = 0, \\ x(t_1) = x'(t_2) = 0, \\ x(t_3) = 0,
\end{cases}
\end{align*}
\]  

(1)

where \( g \geq 0, \delta > 0, t_1 < t_2 < t_3 \) are real numbers and \( t_2 - t_1 > t_3 - t_2 \), and the Krasnoselskii, Leggett, and Williams fixed point theorems were used to prove the existence of at least three solutions of (1).

In [2], the authors solved the existence of the solutions of the BVP

\[
\begin{align*}
\begin{cases}
\quad x^{(3)}(t) = f(t, x(t), x'(t), x''(t)) + e(t), \\ x(0) = \sum_{i=1}^{m-2} a_i x(\xi_i), \\ x'(0) = 0, \\ x(1) = \beta x(\eta),
\end{cases}
\end{align*}
\]  

(2)

where \( f: [0, 1] \times \mathbb{R}^3 \to \mathbb{R} \) is a continuous function, \( e \in L^1[0, 1], \quad a_i \in \mathbb{R}, 1 \leq i \leq m-2 \), \( \beta \geq 0 \), \( 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1 \), \( \eta \in [0, 1] \). The coincidence degree theory of Mawhin is the fundamental tool to deduce the existence results.

In 2019, the authors in [3] studied the existence and uniqueness of solutions of the BVP

\[
\begin{align*}
\begin{cases}
\quad x^{(3)}(t) + f(t, x(t)) = 0, \\ x(a) = x'(a) = 0, \\ x(b) = kx(\eta),
\end{cases}
\end{align*}
\]  

(3)

where \( \eta \in (a, b), k \in \mathbb{R}, f: C[a, b] \to \mathbb{R}, f(t, 0) \neq 0 \), and its primary tools are contracting mapping theorem and variation of parameters formula; namely, it first deals with

\[
\begin{align*}
\begin{cases}
\quad u^{(3)}(t) + h(t) = 0, \\ u(a) = u'(a) = 0, \\ u(b) = 0.
\end{cases}
\end{align*}
\]  

(4)

And then, the solution of BVP (3) can be expressed as

\[
x(t) = u(t) + (\lambda_0 + \lambda_1 t + \lambda_2 t^2) u(\eta),
\]  

(5)

where \( \lambda_0, \lambda_1, \lambda_2 \) are constants determined by its boundary value conditions.
Motivated greatly by above-mentioned works, in this paper, we consider the existence and uniqueness of solutions to the following BVP

\[
x^{(3)}(t) + f(t, x(t)) = 0, \quad a \leq t \leq b,
\]

\[
x(b) = x_{n}(a) = 0, x(a) = kx(\eta),
\]

where \(\eta \in (b - (b - a/k), b), \ k > 1\), \(f: [a, b] \times \mathbb{R} \rightarrow \mathbb{R}\) is continuous and \(f(t, 0) \neq 0\). The methods used are a measure of noncompactness and Darbo’s fixed point theorem, which prove the existence of solutions, and at the same time, its uniqueness also holds by Banach contraction principle, especially in proving the existence of solutions for classes of nonlinear equations (see [15–30]).

Comparing with other papers, our advantage is that the compactness was opened up in 1930 by Kuratowski, which is different from [3]. A new direction of research with various integral, differential equations, integro-differential equations as well as their systems. As an important application, our results can be applied to prove the existence results related to compactness for the set \(X\) is compact if and only if any sequence of Banach space \(E\) is totally bounded if and only if it has a finite \(\varepsilon\)-net.

The definition of the Hausdorff measure of noncompactness \(\chi(\cdot)\) defined on bounded set \(S\) of Banach space \(E\) is

\[
\chi(S) = \inf\{\varepsilon > 0 : S \subset \bigcup_{i=1}^{m} B(x_i, r_i), x_i \in S, r_i < \varepsilon (i = 1, 2 \ldots m), m \in \mathbb{N}\},
\]

where

\[
B(x_i, r_i) = \{y \in X, d(y, x_i) < r_i\},
\]

The definition of the Hausdorff measure of noncompactness for the set \(S\) can equivalently be stated as follows:

\[
\chi(S) = \inf\{\varepsilon > 0 : S\ has\ a\ finite\ \varepsilon\ -\ net\ in\ E\}.
\]

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\[
\chi(S) = \inf\{\varepsilon > 0 : S\ has\ a\ finite\ \varepsilon\ -\ net\ in\ E\}.
\]
\[ \beta(S) = \sup{\varepsilon > 0: S \text{ contains an infinite } \varepsilon \text{- discrete set}}. \]  

Almost all known measures of noncompactness possess the property that they are equal to zero on the family of all relatively compact sets in a given space.

**Lemma 1** (see [32]). Let \( E \) be a Banach space, \( S \) and \( T \) be bounded subsets of \( E, a \in R \). Then,

1. \( \alpha(S) = 0 \) if and only if \( S \) is a relatively compact
2. the family ker \( \mu = \{ Y \in M_E: \mu(Y) = 0 \} \) is nonempty and ker \( \mu \subset N_E \)
3. \( S \subset T \Rightarrow \alpha(S) \leq \alpha(T) \)
4. \( \alpha(S \cup T) = \max\{\alpha(S), \alpha(T)\} \)
5. \( \alpha(S + T) \leq \alpha(S) + \alpha(T) \) where \( S + T = \{ x: x = y + z, y \in S, z \in T \} \)
6. \( \alpha(aS) = |a|\alpha(S) \), where \( aS = \{ x: x = ay, y \in S \} \)
7. \( \bar{\alpha}(S) = \alpha(S) \)
8. \( \alpha(S + x) = \alpha(S) \) for any \( x \in E \), where \( S + x = \{ z = s + x: s \in S \} \)
9. \( |\alpha(S) - \alpha(T)| \leq 2\delta_b(S, T) \)

where \( \delta_b(S, T) \) denotes the distance of Hausdorff between \( S \) and \( T \); i.e.,

\[ d_b(S, T) = \max\left\{ \sup_{x \in S} d(x, T), \sup_{x \in T} d(x, S) \right\}. \]

10. \( X_n \subset M_E, \overline{X_n} = X_n, X_{n+1} \subset X_n \) for \( n = 1, 2, \ldots \) if \( \lim_{n \to \infty} \alpha(X_n) = 0 \); then, \( X_{\infty} = \cap_{i=1}^{\infty} X_i \neq \emptyset \)

In this article, we work on infinite space \( C[a, b] \), which denotes the space of all continuous functions defined on the interval \( [a, b] \). For \( x \in C[a, b] \), we define the usual norm as follows:

\[ \|x\| = \max\{|x(t)|: t \in [a, b]\}. \]

It is well known that \( (C[a, b], \| \cdot \|) \) is complete.

The measure of noncompactness in \( C[a, b] \) can be formulated as follows ([33]).

Let \( X \) be a nonempty and bounded subset of the space \( C[a, b] \) and for \( x \in X, \varepsilon > 0, T > 0 \), let

\[ \omega^T(x, \varepsilon) = \sup\{|x(t) - x(s)|: t, s \in [0, T], |t - s| < \varepsilon\}, \]

\[ \omega^T(x, \varepsilon) = \sup\{\omega^T(x, \varepsilon): x \in X\}, \]

\[ \omega^T_0(X) = \lim_{\varepsilon \to 0} \omega^T(x, \varepsilon), \]

\[ \omega^T_0(X) = \lim_{T \to \infty} \omega^T_0(X), \]

\[ X(t) = \{x(t): x \in X, t \in [0, T]\}, \]

and

\[ \alpha(X) = \omega^T_0(X) + \limsup_{\varepsilon \to \infty} \text{diam } X(t), \]

where

\[ \text{diam } X(t) = \sup\{|x(t) - y(t)|: x, y \in X\}. \]

**Lemma 2** (Agarwala and O’Regan [34]). Let \( Z \) be a closed, convex subset of a Banach space \( E \). Then, every compact, continuous map \( T: Z \to Z \) has at least one fixed point.

Using the measure of noncompactness, Darbo generalized Lemma 2, namely, Darbo’s fixed point theorem.

**Lemma 3** (Darbo [35]). Let \( Z \) be a nonempty, bounded, closed, and convex subset of a Banach space \( E \) and \( T: Z \to Z \) be a continuous mapping. Assume that there is a constant \( \zeta \in (0, 1) \) such that

\[ \alpha(\lambda T) \leq \zeta \alpha(M), M \subset Z. \]

Then, \( T \) has a fixed point.

**Lemma 4** (Ahmad et al. [36]). Let \( Z \) be a closed subset of Banach space \( E \) and \( T: Z \to Z \) be a strictly contractive operator; i.e., there exists a constant \( \gamma \in (0, 1) \) such that

\[ \|T(u) - T(v)\| \leq \gamma \|u - v\|, \forall u, v \in Z. \]

Then, \( T \) has a unique fixed point.

### 3. The Construction of Green’s Function

First of all, let us construct the Green’s function for the BVP

\[ \begin{aligned}
&x^{(3)}(t) + h(t) = 0, \quad a \leq t \leq b, \\
&x(b) = x(\eta) = 0, x(a) = kx(\eta),
\end{aligned} \]

where \( \eta \in (b - (b - a/k), b), k > 1, h \in C[a, b] \).

**Lemma 5.** The above BVP (22) has the solution

\[ x(t) = \frac{k}{2m} \int_a^\eta (t - s)^2 h(s)ds + \int_a^b G(t, s)h(s)ds, \]

where

\[ m = (1 - k)b - (a - k\eta), \]

and

\[ G(t, s) = \frac{1}{2m} \left\{ \begin{array}{ll}
[(1 - k)t - (a - k\eta)](b - s)^2 - m(t - s)^2, & a \leq s \leq t \leq b, \\
[(1 - k)t - (a - k\eta)](b - s)^2, & a \leq t \leq s \leq b.
\end{array} \right. \]

Proof. We easily know that \( x(t) = c_1 + c_2t + c_3t^2 - (1/2) \int_a^t (s - t)^2h(s)ds \), where \( c_1, c_2, c_3 \) are constants. As a result,

\[ x''(t) = 2c_3 - \int_a^t h(s)ds, \]

by using the boundary condition \( x(\eta) = 0 \), which yields that \( c_3 = 0 \), and thus,
\[ x(t) = c_1 + c_2t - (1/2) \int_a^t (t-s)^2 h(s) ds. \] (27)

Substituting the values of \( a, b \) and \( \eta \) into the above equation, one has

\[
\begin{aligned}
(1-k)c_1 + (a-k\eta)c_2 &= -\frac{k}{2} \int_a^\eta (\eta-s)^2 h(s) ds, \\
c_1 + bc_2 &= \frac{1}{2} \int_a^b (b-s)^2 h(s) ds.
\end{aligned}
\] (28)

Applying the Gaussian formula, we have

\[
c_1 = -\frac{1}{2m} \left[ kb \int_a^\eta (\eta-s)^2 h(s) ds + (a-k\eta) \int_a^b (b-s)^2 h(s) ds \right],
\] (29)

and

\[
c_2 = \frac{1}{2m} \left[ k \int_a^\eta (\eta-s)^2 h(s) ds + (1-k) \int_a^b (b-s)^2 h(s) ds \right].
\] (30)

Therefore,

\[
x(t) = -\frac{1}{2m} \left[ kb \int_a^\eta (\eta-s)^2 h(s) ds + (a-k\eta) \int_a^b (b-s)^2 h(s) ds \right] \\
+ \frac{t}{2m} \left[ k \int_a^\eta (\eta-s)^2 h(s) ds + (1-k) \int_a^b (b-s)^2 h(s) ds \right] - \frac{1}{2} \int_a^t (t-s)^2 h(s) ds \\
= \frac{k}{2m} \int_a^\eta (t-b)(\eta-s)^2 h(s) ds + \frac{1}{2m} \int_a^b [(1-k)t - (a-k\eta)] (b-s)^2 h(s) ds \\
- \frac{1}{2m} \int_a^t m(t-s)^2 h(s) ds \\
= \frac{k}{2m} \int_a^\eta (t-b)(\eta-s)^2 h(s) ds + \int_a^b G(t,s) h(s) ds.
\]

This completes the proof. \( \square \)

**Lemma 6.** The Green’s function \( G(t,s) \) from Lemma 5 satisfies the following properties:

1. \( G(t,s) \) is continuous
2. \( 0 \leq G(t,s) \leq (1/2m)k(\eta-a)(b-s)^2 \) for \( t \in [a,b] \)

**Proof.** The continuity of \( G(t,s) \) is obvious, and next, we consider (2). First, we note that

\[
G_t(t,s) = \frac{1}{2m} \left\{ \begin{array}{ll}
(1-k)(b-s)^2 - 2m(t-s), a \leq s \leq t \leq b, \\
(1-k)(b-s)^2, a \leq t \leq s \leq b.
\end{array} \right.
\] (32)

(i) When \( a \leq t \leq s \leq b \),

\[
G_t(t,s) = \frac{1}{2m} (1-k)(b-s)^2 < 0.
\] (33)

(ii) When \( a \leq s \leq t \leq b \),

\[
G_t(t,s) = \frac{1}{2m} [(1-k)(b-s)^2 - 2m(t-s)] < 0.
\] (34)

Therefore, \( G(t,s) \) is decreasing with respect to \( t \) for fixed \( s \in [a,b] \), so

\[
\min_{a \leq t \leq b} G_t(t,s) = G(b,s) = 0,
\] (35)

and \( \max_{a \leq t \leq b} G(t,s) = G(a,s) = \frac{1}{2m}k(\eta-a)(b-s)^2. \) (36)

This completes the proof.

Now, we shall present our main results concerning the existence and uniqueness of solutions for problem (6). Let us introduce the following conditions:

\( (C_1) \) There exists a nonnegative constant \( L \) such that \( |f(t,x) - f(t,y)| \leq L|x - y|, x, y \in \mathbb{R} \). \hspace{1cm} (37)

\( (C_2) \) \( \zeta_1 = kl/6m(b-a)(\eta-a)(\eta-a)^2 + (b-a)^2 \leq 1 \).

\( (C_3) \) There exists a positive number \( d_0 \) satisfying the following inequality:

\[
\frac{\zeta}{L} (Ld_0 + \gamma) \leq d_0,
\] (38)

where \( \gamma = \max_{a \leq t \leq b} |f(t,0)|. \) \( \square \)

**4. Main Results**

We now give the existence and uniqueness of solutions for BVP (6).

**Theorem 1.** Under the conditions \( (C_1) - (C_3) \), BVP (6) has at least a solution.
Proof. First of all, define an operator $H$ on $C[a, b]$ by

$$Hx(t) = \frac{k}{2m} \int_a^\eta (t - b)(\eta - s)^2 f(s, x(s))ds + \int_a^b G(t, s)f(s, x(s))ds, \quad t \in [a, b].$$

(39)

Then, $x(t)$ is a solution of BVP (6) if and only if it is a fixed point of $H$. Next, we are going to divide the progress into three steps.

Step 1. We need to show the operator $H$ is continuous.

According to Lemma 6 and condition $(C_1)$, for any $\varepsilon > 0$, there exists $\delta = \varepsilon/\zeta$ such that, for any $x, y \in B_{d_0}$ and $\|x - y\| < \delta$, we have

$$\|Hx - Hy\| < \varepsilon.$$  \hspace{1cm} (40)

In fact,

$$|Hx(t) - Hy(t)| = \left| \frac{k}{2m} \int_a^\eta (t - b)(\eta - s)^2 f(s, x(s))ds \right|$$

$$+ \left| \int_a^b G(t, s)f(s, x(s))ds - \frac{k}{2m} \int_a^\eta (t - b)(\eta - s)^2 f(s, y(s))ds - \int_a^b G(t, s)f(s, y(s))ds \right|$$

$$\leq \frac{k}{2m} (b - a) \int_a^\eta (\eta - s)^3 |f(s, x(s)) - f(s, y(s))|ds$$

$$+ \frac{k}{2m} (\eta - a) \int_a^b (b - s)^3 |f(s, x(s)) - f(s, y(s))|ds$$

$$\leq \frac{kL}{6m} (b - a)(\eta - a)[(\eta - a)^2 + (b - a)^2]\|x - y\|$$

$$= \zeta\|x - y\|.$$  \hspace{1cm} (41)

So,

$$\|Hx - Hy\| = \sup_{t \in [a, b]} |Hx(t) - Hy(t)| \leq \zeta\|x - y\| < \varepsilon.$$  \hspace{1cm} (42)

Step 2. We prove $H: B_{d_0} \rightarrow B_{d_0}$, where $B_{d_0}$ denotes a closed sphere of $C[a, b]$ which center is 0 and radius is $d_0$.

For any $x \in C[a, b]$, $t \in [a, b]$, from $(C_1), (C_2)$ and Lemma 6 (2), we have

$$|Hx(t)| = \left| \frac{k}{2m} \int_a^\eta (\eta - s)^2 (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|)ds \right|$$

$$+ \frac{k}{2m} (\eta - a) \int_a^b (b - s)^2 (|f(s, x(s)) - f(s, 0)| + |f(s, 0)|)ds$$

$$\leq \frac{k(b - a)}{2m} \int_a^\eta (\eta - s)^2 (L|x(s)| + \gamma)ds + \frac{k(\eta - a)}{2m} \int_a^b (b - s)^2 (L|x(s)| + \gamma)ds$$

$$\leq \frac{k(b - a)(\eta - a)}{6m} \left[ (\eta - a)^2 + (b - a)^2 \right] (L\|x\| + \gamma)$$

$$= \frac{\zeta}{L} (L\|x\| + \gamma).$$  \hspace{1cm} (43)

This implies

$$\|Hx\| \leq \frac{\zeta}{L} (L\|x\| + \gamma).$$  \hspace{1cm} (44)
Combining condition $(C_2)$ with (44), we know that there exists $d_0 > 0$ such that $H$ maps $B_{d_0}$ into itself.

**Step 3.** Now, we prove $\alpha(HB_{d_0}) \leq \zeta \alpha(B_{d_0})$.

First, for all $x, y \in B_{d_0}$, $t \in [a, b]$, based on the proof process of inequality (40), we can obtain

$$|Hx(t) - Hy(t)| \leq \zeta \sup_{s \in [a, b]} |x(s) - y(s)|,$$

and it follows that

$$|Hx(t_1) - Hx(t_2)| = \frac{k}{2m} \int_a^b (t_1 - b) (\eta - s)^3 f(s, x(s)) ds + \int_a^b G(t_1, s) f(s, x(s)) ds - \frac{k}{2m} \int_a^b (t_2 - b) (\eta - s)^3 f(s, y(s)) ds - \int_a^b G(t_2, s) f(s, y(s)) ds \leq \frac{k}{2m} (b - a) \int_a^b (\eta - s)^3 |f(s, x(s)) - f(s, y(s))| ds + \int_a^b \|G(t_1, s) - G(t_2, s)\| f(s, x(s)) - f(s, y(s))| ds + \int_a^b G(t_2, s) f(s, x(s)) - f(s, y(s))| ds \leq \frac{kL}{6m} (b - a) (\eta - a)^3 \sup_{s \in [a, b]} |x(s) - y(s)| + \frac{1}{2m} (\eta - a) \int_a^b (b - s)^3 |f(s, x(s)) - f(s, y(s))| ds \leq \frac{kL}{6m} (b - a) (\eta - a)^3 \sup_{s \in [a, b]} |x(s) - y(s)| + \epsilon (b - a) (L\|x\| + \gamma) + \frac{kL}{6m} (b - a)^3 (\eta - a) \sup_{s \in [a, b]} |x(s) - y(s)| = \zeta \sup_{s \in [a, b]} |x(s) - y(s)| + \epsilon (b - a) (L\|x\| + \gamma),$$

which implies that

$$\omega^T(Hx, \epsilon) \leq \zeta \omega^T(x, \epsilon).$$

Consequently,

$$\omega^T_0(Hx) \leq \zeta \omega^T_0(x).$$

Let $T \to \infty$, we obtain

$$\omega_0(Hx) \leq \zeta \omega_0(x).$$

In view of (45) and (50), we easily obtain

$$\alpha(HB_{d_0}) \leq \zeta \alpha(B_{d_0}).$$

Thus, by Lemma 3, we conclude that $H$ has at least a solution in $B_{d_0} \subseteq C[a, b]$; that is, BVP (6) has at least a solution in $B_{d_0} \subseteq C[a, b]$. This completes the proof.

Besides, for all $|t_1 - t_2| < \epsilon, t_1, t_2 \in [a, b]$, by the continuity of $G(t, s)$, we know that

$$|G(t_1, s) - G(t_2, s)| < \epsilon, \text{ for all } \epsilon \in [a, b],$$

combining (46) with conditions $(C_1)$ and $(C_2)$, for any $x, y \in B_{d_0}$, we have

$$\limsup_{t \to \infty} \text{diam}(HX(t)) \leq \zeta \limsup_{t \to \infty} \text{diam} X(t).$$

**Corollary 1.** In fact, Theorem 1 not only proves the existence of solutions of BVP (6) but also illustrates its uniqueness.

**Proof.** According to the proof of inequality (40) and condition $(C_2)$, we know, namely, operator $H : C [a, b] \to C[a, b]$ is a contraction mapping, so $H$ has a unique solution in $C[a, b]$ by construction mapping theorem in Lemma 4; that is, BVP (6) has a unique solution in the whole space $C[a, b]$. This completes the proof.

To be honest, if we only need to know the BVP (6) has an unique solution in its domain, the Lipschitz conditions $(C_1)$ and $(C_2)$ are enough. But, it is better to find the more accurate range of the solution in practical matters and concrete situations, so we could choose Theorem 1 to identify the scope of the solution.

In the following, we give two concrete examples to illustrate our main results. □
Example 1. Consider the following problem:

\[
\begin{aligned}
  x''''(t) + 1 + t + \sin x(t) &= 0, t \in [1, 2], \\
  x''(1) &= 0, x(2) = 0, x(1) = \frac{3}{2} x\left(\frac{5}{3}\right).
\end{aligned}
\]  

(53)

Here, we have

\[
  f(t, x) = 1 + t + \sin x, a = 1, b = 2, k = \frac{3}{2}, \eta = \frac{5}{3}
\]

(54)

By easy calculation, we get

\[
  m = (1 - k)b - (a - k\eta) = \frac{1}{2},
\]

(55)

\[
  \frac{\partial f}{\partial x}(t, x) = |\cos x| \leq 1 = L,
\]

and

\[
  y = \max_{0 \leq t \leq 2} |1 + t| = 3, \zeta = \frac{13}{27} < 1,
\]

(56)

choose \( d_0 = 2.8 \), with the help of Theorem 1, the problem (51) has at least a solution \( x(t) \) in \( B_{d_0} \).

Example 2. Consider the following problem:

\[
\begin{aligned}
  x''''(t) + e^t \cos x(t) + \sin x(t) &= 0, t \in [0, 1], \\
  x''(0) &= 0, x(1) = 0, x(0) = 2x\left(\frac{3}{2}\right).
\end{aligned}
\]  

(57)

Here, we have

\[
  f(t, x) = e^t \cos x + \sin x, a = 0, b = 1, k = 2, \eta = \frac{3}{4}
\]

(58)

By easy calculation, we get

\[
  m = (1 - k)b - (a - k\eta) = \frac{1}{2},
\]

(59)

\[
  \frac{\partial f}{\partial x}(t, x) = |\cos x| \leq 1 = L,
\]

and

\[
  y = \max_{0 \leq t \leq 1} |e^t \cos x| = e \cos 1 \approx 2.718, \zeta = \frac{25}{32} < 1,
\]

(60)

choose \( d_0 = 9.8 \), with the help of Theorem 1, the problem (55) has a solution \( x(t) \) in \( B_{d_0} \).

5. Conclusion

In this paper, we study the nonlinear third-order three-point boundary value problem (6). First, we construct Green’s function for the third-order three-point boundary value problem (22) and discuss its properties. Second, based on Green’s function and its properties, we define a solution operator \( H \). Then, we prove that \( H \) has at least a fixed point by using Darbo’s fixed point theorem combined with the technique of measure of noncompactness. The fixed point is unique by Banach fixed point theorem. Therefore, the existence and uniqueness of solutions for (6) have been established. Finally, as applications, two examples are given to illustrate our main results. It should be pointed out that the method used here can be applied to impulsive differential equation boundary value problems, such as [37].

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares no conflicts of interest.

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