QUANTUM CLUSTER CHARACTERS OF HALL ALGEBRAS

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To the memory of Andrei Zelevinsky

Abstract. The aim of the present paper is to introduce a generalized quantum cluster character, which assigns to each object $V$ of a finitary Abelian category $C$ over a finite field $F_q$ and any sequence $i$ of simple objects in $C$ the element $X_{V,i}$ of the corresponding algebra $P_{C,i}$ of $q$-polynomials. We prove that if $C$ was hereditary, then the assignments $V \mapsto X_{V,i}$ define algebra homomorphisms from the (dual) Hall-Ringel algebra of $C$ to the $P_{C,i}$, which generalize the well-known Feigin homomorphisms from the upper half of a quantum group to $q$-polynomial algebras.

If $C$ is the representation category of an acyclic valued quiver $(Q, d)$ and $i = (i_0, i_0)$, where $i_0$ is a repetition-free source-adapted sequence, then we prove that the $i$-character $X_{V,i}$ equals the quantum cluster character $X_V$ introduced earlier by the second author in [29] and [30]. Using this identification, we deduce a quantum cluster structure on the quantum unipotent cell corresponding to the square of a Coxeter element. As a corollary, we prove a conjecture from the joint paper [5] of the first author with A. Zelevinsky for such quantum unipotent cells. As a byproduct, we construct the quantum twist and prove that it preserves the triangular basis introduced by A. Zelevinsky and the first author in [6].

Contents

1. Introduction 2
   1.1. Acknowledgments 4
2. Definitions and main results 4
3. Hall-Ringel algebras and proof of Theorem 2.1 10
4. Special compatible pairs 14
5. Valued Quivers and Proof of Theorem 2.5 21
   5.1. Quantum cluster characters 21
   5.2. Valued quivers, flags, and Grassmannians 22
   5.3. $(i_0, i_0)$-Character is Quantum Cluster Character: conclusion of the proof 24
6. Quantum groups and quantum cluster algebras 25
   6.1. Quantum groups, representations, and generalized minors 25
   6.2. Quantum Cluster Algebras 28
   6.3. Quantum i-seeds and i-characters 30
7. Proof of Theorems 2.9, 2.10, and 2.11 33
   7.1. Proof of Theorem 2.9 33

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7.2. Quantum twist and Proof of Theorem 2.10 37
7.3. Proof of Theorem 2.11 40
8. Example 40
9. Appendix: Twisted Bialgebras in Braided Monoidal Categories 42
References 45

1. INTRODUCTION

The aim of this paper is two-fold:

• Generalize the Feigin homomorphism to Hall algebras of hereditary categories.
• Establish a quantum cluster structure on the image of the homomorphism.

The Feigin homomorphism $\Psi_i : U_+ \rightarrow P_i$ was proposed by B. Feigin in 1992 as an elementary tool for the study of quantized enveloping algebras $U_+ (\mathfrak{g})$, where $\mathfrak{g}$ is a Kac-Moody algebra, $i = (i_1, \ldots, i_m)$ is a sequence of simple roots of $\mathfrak{g}$, and $P_i$ is an appropriate $q$-polynomial algebra in $t_{i_1}, \ldots, t_{i_m}$ (see [1, 20, 21] or Section 2 below for details). Ultimately, Feigin’s homomorphism assigns to each simple generator $E_i$, $i = 1, \ldots, n$ of $U_+$ the linear $q$-polynomial $\sum_{k : i_k = i} t_k$.

It turns out that if $i$ is a reduced word for an element $w$ of the Weyl group $W$, then $I_w := \ker \Psi_i$ depends only on $w$ and $\Psi_i$ defines an isomorphism between the skew-field of fractions of $U_w = U_+/I_w$ and the skew-field of fractions of $P_i$ (see e.g., [1, Theorem 0.5]).

This result can be viewed as an indicator of a possible quantum cluster structure on $U_w$, which, therefore, is more convenient to identify with the quotient of the dual algebra $C_q[N]$, where $N$ is a maximal unipotent subgroup of the corresponding Kac-Moody group $G$. For generic $q$ this identification is done for free because $C_q[N] \cong U_+$, but if $q = 1$, then $C_q[N]$ specializes to the coordinate algebra $C[N]$ and $U_w$ — to the coordinate algebra of the closure of the unipotent cell $N^w = B_\mathfrak{g} \cap N$, where $B_\mathfrak{g}$ is a Borel subgroup of $G$ complementary to $N$ (see [1, 3, 4] for details).

Since the (upper) cluster structure on $C[N^w]$ has been established in [2] by using a “classical” analogue of $\Psi_i$, it is natural to expect an analogous result for each $U_w$.

One of the advantages of Feigin’s homomorphism is that it allows to replace a very complicated algebra $U_w$ with the isomorphic subalgebra $A_i := \Psi_i(C_q[N])$ of $P_i$ and look in a more efficient way for its quantum cluster structure inside the much simpler algebra $P_i$. In fact, we always have coefficients $C_1, \ldots, C_n$ in $A_i$ which are monomials in $t_{i_1}, \ldots, t_{i_m}$. These form an Ore subset which allows to define the localization $\overline{A}_i$ of $A_i$ by $C_1, \ldots, C_n$. Our main result is the following.

**Theorem 1.1.** (Theorem 2.7) For $i = (i_0, i_0')$, where $i_0 = (i_1, \ldots, i_n)$ is any ordering of simple roots of $\mathfrak{g}$, the algebra $\overline{A}_i$ is an (acyclic) upper cluster algebra of rank $n$ with respect to an initial quantum cluster $\mathbf{X} = \{X_1, \ldots, X_n, C_1, \ldots, C_n\}$, where each $X_j$ is a monomial in $t_{i_1}^{-1}, \ldots, t_{i_n}^{-1}$.

The acyclicity of $\overline{A}_{i_0, i_0'}$ will be obvious from the definition of the cluster structure and monomiality of coefficients $C_i$ is known for any $i$ ([1 Theorem 3.1]), but the monomiality of the initial cluster
variables is highly non-trivial, in particular, it implies the following “quantum chamber ansatz” (cf [5]).

**Corollary 1.2.** In the assumptions of Theorem 1.1 for each \( i = 1, \ldots, n \) there exists an element \( \tilde{X}_i \in \mathbb{C}_q[N] \) such that \( \Psi_i(\tilde{X}_i) = X_i C \), where \( C \) is a monomial of \( C_1, \ldots, C_n \).

Using these results, we settle a particular case of Conjecture 10.10 from [5].

**Theorem 1.3.** If \( i = (i_0, i_0) \), then the restriction of the homomorphism in [5, Proposition 10.9] to \( \mathbb{A}_i \) is an isomorphism \( \mathbb{A}_i \to \mathbb{C}_q[N^c] \), where \( c \) is the Coxeter element in \( W \) corresponding to the reduced word \( i_0 \).

We prove Theorem 1.3 (see also Theorem 2.11) in Section 7.3.

It is natural to expect (Conjecture 2.12) that Theorem 1.1 and hence the “quantum chamber ansatz” hold without any assumptions on a reduced word \( i \). It would be interesting to compare the conjecture with the results of [16] where a cluster structure was established (for \( g \) with symmetric Cartan matrix) on quantum Schubert cells \( U_q(w) = \mathbb{C}_q[N \cap wN - w^{-1}] \), which are “close relatives” of quantum unipotent cells \( U_w \).

Our proof of Theorem 1.1 led us to the generalization of Feigin’s homomorphism to Hall-Ringel algebras \( \mathcal{H}(C) \) for most hereditary Abelian categories \( C \), which are natural generalizations and extensions of the quantum algebras \( U_+ \). We first assign to each isomorphism class \([V] \in \mathcal{H}(C)\) a certain element \( X_{V,i} \) of \( \mathcal{P}_i \) (we refer to it as the quantum cluster \( i \)-character of \( V \), formula (2.2)). We prove (Theorem 2.1) that, under certain co-finiteness conditions, for each hereditary category \( C \) the assignment \([V] \mapsto X_{V,i} \) defines a homomorphism of algebras

\[
\Psi_{C,i} : \mathcal{H}(C) \to \mathcal{P}_i
\]

which, in the case when \( C = \text{rep}_F(Q, d) \) for an acyclic valued quiver \((Q, d)\), restricts to Feigin’s homomorphism from \( U_+ \subset \mathcal{H}(C) \) to \( \mathcal{P}_i \) (Corollary 2.2).

Using the homomorphism (1.1), we directly construct all non-initial cluster variables in \( \mathcal{A}_{(i_0, i_0)} \) as images \( \Psi_i([E]) \) where \([E] \in U_+ \) runs over all isomorphism classes of exceptional representations of \((Q, d)\). This is achieved by identifying flags with Grassmannians of subobjects in \( V \) and employing a quantum version of the famous Caldero-Chapoton formula ([7, 8]) developed by the second author in [29, 30] and independently by Qin in [23]. It would be interesting to compare this with a similar bijection between flags and Grassmannians constructed in [15].

We hope to prove Conjecture 2.12 e.g., construct cluster variables in \( \mathcal{A}_i \), in a similar fashion, by identifying our quantum cluster character \( X_{V,i} \) with some quantization of a Caldero-Chapoton type character formula.

As a byproduct, we construct the quantum twist automorphism \( \eta \) of \( \mathcal{A}_i \) in the acyclic case and prove that it preserves the triangular basis of \( \mathcal{A}_i \) (Corollary 2.15). We expect that the twist always exists (Conjecture 2.12(c)) and preserves the canonical basis \( \mathcal{B}_i \) of \( \mathcal{A}_i \) (Conjecture 2.17).
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2. Definitions and main results

Let \( \mathcal{C} \) be a small finitary Abelian category of finite global dimension, that is, \(|\text{Ext}^i(U,V)| < \infty\) for all \( V, W \in \mathcal{C}, i \geq 0 \) and \( \text{Ext}^i(U,V) = 0 \) for all but finitely many \( i \in \mathbb{Z}_{\geq 0} \) (where we follow the convention \( \text{Ext}^0(U,V) = \text{Hom}(U,V) \)).

For \( V, W \in \mathcal{C} \) define the (square root of) the multiplicative Euler-Ringel form \( \langle V, W \rangle \) by:

\[
\langle V, W \rangle = \prod_{i=0}^{\infty} |\text{Ext}^i(U,V)|^{\frac{1}{2}(i-1)}. 
\]

In fact, \( \langle V, W \rangle \) depends only on Grothendieck classes \( |V| \) and \( |W| \) in the Grothendieck group \( K(\mathcal{C}) \) of the category \( \mathcal{C} \) and can be viewed as a bicharacter \( \langle \cdot, \cdot \rangle : K(\mathcal{C}) \times K(\mathcal{C}) \to k^* \), where we fix any field \( k \) of characteristic 0 containing all \( \langle V, W \rangle \) for \( V, W \in \mathcal{C} \).

Let \( S = \{ S_i : i \in I \} \) be a set of pairwise non-isomorphic objects of \( \mathcal{C} \). For any sequence \( i = (i_1, \ldots, i_m) \in I^m \) we define the skew-polynomial algebra \( P_i = P_{C(S, I)} \) over \( k \) to be generated by \( t_1, \ldots, t_m \) subject to the relations

\[
t_k t_{k+1} = \langle S_{i_k}, S_{i_k} \rangle \langle S_{i_k}, S_{i_k} \rangle \cdot t_k t_{k+1} \quad \text{for } k < m.
\]

Furthermore, for each object \( V \in \mathcal{C} \) we define the \( (\text{quantum cluster}) \) \( i \)-character \( X_{V,i} \) of \( V \) in (the completion of) \( P_i \) by

\[
X_{V,i} = \sum_{a=(a_1, \ldots, a_m) \in \mathbb{Z}_{\geq 0}^m} \prod_{1 \leq k \leq m} \left( \langle S_{i_k}, S_{i_k} \rangle / \langle S_{i_k}, S_{i_k} \rangle \right)^{\frac{1}{2}a_k a_{k+1}} \cdot \mathcal{F}_{i,a}(V) \cdot t_1^{a_1} \cdots t_m^{a_m},
\]

where \( \mathcal{F}_{i,a}(V) \) is the set of all flags of subobjects in \( V \) of type \( (i, a) \), that is

\[
\mathcal{F}_{i,a}(V) = \{ \emptyset = V_m \subset V_{m-1} \subset \cdots \subset V_1 \subset V_0 = V : V_{k-1}/V_k \cong S_{i_k}^{a_k}, k = 1, \ldots, m \}.
\]

Note that the sum in (2.2) is finite if \( V \) has finitely many subobjects, otherwise \( X_{V,i} \) belongs to a suitable completion of \( P_i \).

For each object \( V \) of \( \mathcal{C} \) denote by \([V]\) its isomorphism class and let \( \mathcal{H}(\mathcal{C}) \) be the \( k \)-vector space freely spanned by all \([V]\) (this is the Hall-Ringel algebra, see section 3 for details). Denote by \( \mathcal{H}^*(\mathcal{C}) \) the finite dual Hall-Ringel algebra, which is the space of linear functions \( \mathcal{H}(\mathcal{C}) \to k \) with a basis of all delta-functions \( \delta_{[V]} \) labeled by isomorphism classes \([V]\) of objects of \( \mathcal{C} \). It is convenient to slightly rescale the basis of \( \mathcal{H}^*(\mathcal{C}) \) as follows:

\[
[V]^* = \langle V, V \rangle^{-\frac{1}{2}} f([V]) \cdot \delta_{[V]},
\]
where \( f : \mathcal{K}(\mathcal{C}) \to k^\times \) is a homomorphism of groups defined by \( f([S]) = \langle S, S \rangle^{1/2} \) for all simple objects \( S \) of \( \mathcal{C} \). It turns out that the assignment

\[
[V]^* \mapsto X_{V,i}
\]

often defines an algebra homomorphism from \( \mathcal{H}^*(\mathcal{C}) \) to \( P_i \).

**Theorem 2.1.** Assume that \( \mathcal{C} \) is hereditary and cofinitary and let \( S = \{ S_i \mid i \in I \} \) be a set of simple objects having no self-extensions. Then for any sequence \( i = (i_1, \ldots, i_m) \in I^m \) the assignment (2.5) defines an algebra homomorphism

\[
\Psi_{\mathcal{C},i} : \mathcal{H}^*(\mathcal{C}) \to P_i.
\]

We will prove Theorem 2.1 in Section 3 using generalities on bialgebras in braided monoidal categories presented in the Appendix (Section 9). We will also provide some explicit formulas for \( \Psi_{\mathcal{C},i} \) in terms of the \( \mathcal{H}(\mathcal{C}) \)-action on \( \mathcal{H}(\mathcal{C})^* \) (Proposition 3.6). Note that for the category \( \text{rep}_F(\mathcal{Q}, d) \) of finite dimensional \( F \)-linear representations of any finite (valued) quiver \( Q \) is hereditary (see e.g., [19]), so that Theorem 2.1 is applicable to such a category and to any collection \( S \) of simple objects if \( Q \) has no vertex loops.

The homomorphism \( \Psi_{\mathcal{C},i} \) interpolates between a number of known homomorphisms. Denote by \( U_+^* \) the subalgebra of \( \mathcal{H}^*(\mathcal{C}) \) generated by the simple objects \( [S_i]^* \), \( i \in I \).

**Corollary 2.2.** The restriction of \( \Psi_{\mathcal{C},i} \) to the subalgebra \( U_+^* \) is the homomorphism \( \Psi_1 : U_+^* \to P_i \) determined by (for \( j \in I \)):

\[
\Psi_1([S_j]^*) = \sum_{k : i_k = j} t_k.
\]

Note that if \( \mathcal{C} = \text{rep}_F(\mathcal{Q}, d) \) is the representation category over a finite field \( F \) of an acyclic valued quiver \( (\mathcal{Q}, d) \) and \( S \) is the set of simple representations, then \( U_+^* \) is the dual of the quantized enveloping algebra and \( \Psi_1 \) is the Feigin homomorphism used in [1, 20, 21] to relate the skew-fields of fractions of \( U_+^* \) and \( P_i \).

Also note that for \( \mathcal{C} = \text{rep}_F(\mathcal{Q}, d) \), if \( S \) is the set of all simple object in \( \mathcal{C} \), and \( i = i_0 \) is the repetition-free source-adapted sequence, then we obtain the following result (see [25, Lemma 3.3] for details).

**Corollary 2.3.** The homomorphism \( \Psi_{\text{rep}_F(\mathcal{Q}, d),i_0} \) equals the Reineke homomorphism \( f \).

Now we relate the \( i \)-character \( X_{V,i} \) with the quantum cluster character \( X_V \) defined by the second author in [29, 30]. Namely, according to Definition 5.1 of [30], for \( V \in \mathcal{C} \), \( X_V = X_V(\mathcal{C}) \) is an element of a certain based quantum torus \( \mathcal{T}_\mathcal{C} \) given by:

\[
X_V = \sum_{E \subset V} \langle V, V/E \rangle^{-1} \cdot X^{-|E|\cdot|V/E|},
\]

where \(|V|\) denotes the class of \( V \) in the Grothendieck group \( \mathcal{K}(\mathcal{C}) \) and \((\cdot)^*, *\langle \cdot \rangle\) are certain dualities on \( \mathcal{K}(\mathcal{C}) \) (see Section 5.1 for details).
Given a lattice $L$ and a unitary bicharacter $\chi : L \times L \to k^\times$ define the based quantum torus $\mathcal{T}_\chi$ to be a $k$-algebra with the basis $X^e, e \in L$ subject to the relations:

$$X^e X^f = \chi(e, f) X^{e+f}$$

for all $e, f \in \mathbb{Z}^m$. Clearly, if $L = \mathbb{Z}^m$, then $\mathcal{T}_\chi$ is generated by $X_k^{\pm 1} := X^{\pm \varepsilon_k}$, where $\{\varepsilon_1, \ldots, \varepsilon_m\}$ is the standard basis of $\mathbb{Z}^m$, subject to the relations:

$$X_k X_\ell = q_{k\ell} X_\ell X_k$$

for all $1 \leq k < \ell \leq m$, where $q_{k\ell} = \frac{\chi(e_k, e_\ell)}{\chi(e_\ell, e_k)}$. And vice versa, any algebra with a presentation (2.9) is isomorphic to some $\mathcal{T}_\chi$. In what follows, we will always require that $\chi$ is unitary: $\chi(e, e) = 1$ and $\chi(e, f) \chi(f, e) = 1$ for all $e, f \in \mathbb{Z}^m$, so that $\chi$ is uniquely determined by the triangular array $(q_{k\ell})$.

Denote by $\mathcal{L}_I$ the based quantum torus $\mathbb{P} [t_i^{-1}, \ldots, t_m^{-1}]$ with $t_k = t^{\varepsilon_k}$ for all $1 \leq k \leq m$. The following obvious fact will be used several times.

**Lemma 2.4.** For any sequence $i \in I^m$ and any $\mathbb{Z}$-linear automorphism $\varphi$ of $\mathbb{Z}^m$ there exists a unique unitary bicharacter $\chi_{i, \varphi}$ such that $\varphi$ extends to an isomorphism of based quantum tori $\tilde{\varphi}_i : \mathcal{L}_I \cong \mathcal{T}_{\chi_{i, \varphi}}$ defined by

$$\tilde{\varphi}_i(t^a) = X^{\varphi(a)}$$

for $a \in \mathbb{Z}^m$ (e.g., $X_k = X^{\varepsilon_k} = \tilde{\varphi}_i(t^{s_{-1}(s_k)})$ for $k = 1, \ldots, m$).

Let $A$ be any integer $I \times I$-matrix with $a_{ii} = 2$ for $i \in I$. For any sequence $i \in I^m$ we define $\mathbb{Z}$-linear automorphisms $\varphi_i, \rho_i$ of $\mathbb{Z}^m$ by

$$\varphi_i(\varepsilon_k) = -\varepsilon_k - \varepsilon_{k-} - \sum_{\ell : \ell < k < \ell^+} a_{i_{\ell}, i_k} \varepsilon_{\ell}, \quad \rho_i(\varepsilon_k) = \varepsilon_k - \sum_{\ell \leq k : \ell^+ = m + 1} a_{i_k i_{\ell}} \varepsilon_\ell$$

for $1 \leq k \leq m$ (with the convention $\varepsilon_s = 0$ if $s \notin [1, m]$). Here we used the notation from [5]:

$$k^+ = \min\{\ell : \ell > k, i_\ell = i_k\} \cup \{m + 1\}, \quad k^- = \max\{\ell : \ell < k, i_\ell = i_k\} \cup \{0\}$$

for $1 \leq k \leq m$. The inverse of $\varphi_i$ is more complicated, see Lemma 2.2.

**Theorem 2.5.** Let $(Q, \mathbf{d})$ be an acyclic valued quiver with $n$ vertices, $A = (a_{ij})$ be the associated $n \times n$ Cartan matrix, $i_0$ be a source adapted sequence for $Q$, and $i := (i_0, i_0)$. Then:

(a) There exists an extension $(\tilde{Q}, \tilde{\mathbf{d}})$ of $(Q, \mathbf{d})$ on $2n$ vertices such that $\tilde{T}_Q = T_{\chi_{i, -1} \varphi_i}$.

(b) For any representation $V$ of $(Q, \mathbf{d})$ over a finite field $\mathbb{F}$ of $q$ elements one has:

$$\tilde{\rho}_i^{-1} \tilde{\varphi}_i(X_{V,i}) = \tilde{X}_V$$

where $\tilde{X}_V$ denotes the quantum cluster character (2.7) attached to $\tilde{C} = \text{rep}_q (\tilde{Q}, \tilde{\mathbf{d}})$ (here $V$ is viewed as an object of $\tilde{C}$ under the natural embedding).

We prove Theorem 2.5 in Section 5.

**Remark 2.6.** Theorems 2.4 and 2.5 together explain observations of [7, 8] that multiplication of cluster characters resemble the multiplication in the dual Hall-Ringel algebra of $\text{rep}_q(Q, \mathbf{d})$. 
It follows from [10] Theorem 5.1 that if $V$ is a rigid object of $C = \text{rep}_p(Q, d)$ (i.e. $\text{Ext}_C^1(V, V) = 0$) and $Q$ is acyclic, then the basis vector $[V]^*$ of $\mathcal{H}^*(C)$ belongs to the composition algebra $U_C^*$. These observations and Theorem 2.5 imply that the Grassmannians of subobjects of rigid objects often have counting polynomials. To state this result, we need more notation.

For each $e \in K(C)$ and $V \in C$ denote by $\text{Gr}_e(V)$ the set of all subobjects $E \subset V$ such that $|E| = e$, so that the formula (2.7) reads:

$$X_V = \sum_{e \in K(C)} (|V| - e)^{-1} \cdot |\text{Gr}_e(V)| \cdot X^{-e^*(-|V| - e)}.$$  

From now on, until the end of the section, $U_C^*$ will be the generic composition algebra (see Section 5) for details), that is, $U_C^* = U_k^$(A) is a $k(q^2)$-algebra generated by $x_i = [S_i]^*$, $i \in I$ subject to the quantum Serre relations determined by a symmetrizable $I \times I$ Cartan matrix $A$. Accordingly, we view $\mathcal{L}_i$ as a $k(q^2)$-algebra generated by $t_k^+$ subject to the relations (cf (2.11)):

$$t_k t_{k'} = q^{\epsilon_{kk'}} t_{k+k'}$$

for $1 \leq k < k' \leq m$, where $c_{ij} = d_i a_{ij} = a_{ij} d_j$ is the $(ij)$th entry of the symmetrized Cartan matrix. Combining this and (2.12) with [10] Theorem 5.1, we generalize [30] Corollary 1.2.

Corollary 2.7. Let $(Q, d)$ be an acyclic valued quiver and $V \in \text{Rep}_p(Q, d)$ be rigid, i.e., has no self-extensions. Then for any $i \in I^m$, $a \in \mathbb{Z}^m$ where $(i_0, a)$ and $|V| = \sum_{i \in I} v_i |S_i|$, $e = \sum_{i \in I} e_i |S_i|$. Then there exists a polynomial $p^V_{i, a}(x) \in \mathbb{Z}[x]$ such that

$$|\mathcal{F}_{i, a}(V)| = p^V_{i, a}(|V|) .$$

In particular, for any $e \in K(C)$ one has

$$|\text{Gr}_e(V)| = p^V_{i, a}(|V|) ,$$

where $a_e = (v_1 - e_1, \ldots, v_n - e_n, e_1, \ldots, e_n)$ and $|V| = \sum_{i \in I} v_i |S_i|$, $e = \sum_{i \in I} e_i |S_i|$.

Remark 2.8. In [23] Qin proved that for an equally valued $Q$ all counting polynomials $p^V_{i, a}$ for $\text{Gr}_e(V)$ belong to $\mathbb{Z}_{\geq 0}[x]$ (this was announced earlier by Caldero and Reineke in [9], but their proof was incomplete). We expect that all $p^V_{i, a}$ for rigid $V$ have nonnegative coefficients. This has been recently confirmed by the second author in [31] when $Q$ is any acyclic valued quiver with two vertices and $i = (i_0, i_0)$. Moreover, based on numerous examples and results of [10], we expect that all $p^V_{i, a}$ for rigid $V$ are unimodular, i.e., $p^V_{i, a} \in \mathbb{Z}[x]$ for some $n \geq 0$, where $(k)_q := \frac{q^n - 1}{q - 1}$.

For each sequence $i = (i_1, \ldots, i_m)$ we define the upper cluster algebra $\mathcal{U}_i = \mathcal{U}(X_i, \tilde{B}_i) \subset \mathcal{L}_i$ of the seed $(X_i, \tilde{B}_i)$, where $X_i = \{t^{\epsilon_{i-1}^{-1}(e_1)}, \ldots, t^{\epsilon_{i-1}^{-1}(e_m)}\}$ in the notation (2.10) and $\tilde{B}_i$ is the exchange matrix defined in [3] Section 8.2 (see also Section 4).

Main Theorem 2.9. Let $A$ be a symmetrizable $I \times I$ Cartan matrix and let $i = (i_0, i_0)$, where $i_0$ is an ordering of $I$. Then

(a) $\Psi_i(U_C^*) \subset \mathcal{U}_i$.

(b) The assignment $V \mapsto |\mathcal{F}_i(V)^*|$ is a bijection between isomorphism classes of exceptional representations of the associated valued quiver $(Q, d)$ and non-initial cluster variables in $\mathcal{U}_i$.

(c) If $i$ is reduced, then the localization of $\Psi_i(U_C^*)$ by the cluster coefficients is equal to $\mathcal{U}_i$. 
We prove Theorem 2.9 in Section 7.

**Theorem 2.10.** Assume that \( i = (i_0, i_0) \) is reduced. Then there exists an isomorphism \( \eta : \mathcal{U}_i \to \mathcal{U}_i \) such that

\[ \eta(t^{\nu_i^{-1}(e_k)}) = \Psi_i([V_k]^*), \quad \eta(t^{\nu_i^{-1}(e_{k+1})}) = t^{-\omega_i^{-1}(e_{k+1})} \]

for \( k = 1, \ldots, |I| \), where \( V_k \) is the exceptional object of \( \text{rep}_{\mathcal{E}}(Q, \mathbf{d}) \) such that \( |V_k| = s_{i_1} \cdots s_{i_{k-1}} |S_k| \).

We prove Theorem 2.10 in Section 7.2. Using these results, we settle a particular case of Conjecture 10.10 from [5]. Indeed, following [5, Section 9.3], for any extremal weight \( \gamma \) of a fundamental simple \( U_q(\mathfrak{g}) \)-module \( V_\gamma \), \( i \in I \) one defines an extremal vector \( \Delta_\gamma \in V_\omega \subset U_+^* \) (it is sometimes called a “generalized minor”), where \( \omega_i \) is the \( i \)-th fundamental weight of \( \mathfrak{g} \). Using \( q \)-commutation relations between various \( \Delta_\gamma \) (see e.g., [5, Theorem 10.1]), one has an injective homomorphism of algebras (for each reduced word \( i = (i_1, \ldots, i_m) \) for an element \( w \) in the Weyl group \( W \) of \( \mathfrak{g} \)):

\[ (2.14) \quad \mathcal{L}_i \to \text{Frac}(k_q[N^w]) \]

by:

\[ t^{\nu_i^{-1}(e_k)} \mapsto \Delta_{i_1 \cdots i_k} \omega_{i_k} \]

for \( k = 1, \ldots, m \), where \( x \) is the image of \( x \in U_+^* \) under the canonical projection \( U_+^* \to k_q[N^w] \).

A particular case of Conjecture 10.10 from [5] asserts that the restriction of \( (2.14) \) to \( \mathcal{U}_i \) is an isomorphism of algebras

\[ \mathcal{U}_i \cong k_q[N^w]. \]

It follows from the results of [1] that \( \ker \Psi_i \) is the defining ideal of \( k_q[N^w] \) in \( U_+^* \) for any reduced word \( i \) for \( w \in W \), therefore, \( \Psi_i \) factors through the injective homomorphism

\[ \Psi : k_q[N^w] \to \mathcal{L}_i \]

(see Section 7.3 for details). The following result settles the above conjecture for \( i = (i_0, i_0) \).

**Theorem 2.11.** In the assumptions of Theorems 2.9(c) and 2.10 the restriction of \( (2.14) \) to \( \mathcal{U}_i \) is an isomorphism

\[ \Psi_i^{-1} \circ \eta : \mathcal{U}_i \to k_q[N^{c^2}], \]

where \( c \) is the Coxeter element in \( W \) corresponding to the reduced word \( i_0 \).

We prove Theorem 2.11 in Section 7.3. It is natural to expect that both Theorem 2.9 and Theorem 2.11 hold for all reduced words \( i \).

**Conjecture 2.12.** Let \( A \) be a symmetrizable \( I \times I \) Cartan matrix and let \( i = (i_1, \ldots, i_m) \) be a reduced word for an element \( w \in W \). Then:

(a) \( \Psi_i(U_+^*) \subset \mathcal{U}_i \). Moreover, the localization of \( \Psi_i(U_+^*) \) by the cluster coefficients is \( \mathcal{U}_i \).

(b) For each exceptional representation \( V \) of \( (Q, \mathbf{d}) \) the element \( \Psi_i([V]^*) \in \mathcal{U}_i \) is a cluster variable.

(c) The assignment \( t^{\nu_i^{-1}(e_k)} \mapsto \Psi_i(\Delta_{i_1 \cdots i_k} \omega_{i_k}) \) defines an isomorphism \( \eta_i : \mathcal{U}_i \to k_q[N^w] \).

**Remark 2.13.** The “classical” version of the conjecture is known: it follows from [2, Theorem 2.10] and the existence of the “classical” twist \( \eta_{\text{tot}} \), a certain automorphism of the unipotent cell \( N^w \) ([3, Theorem 1.2]) which interpolates between two embeddings of tori \( (k^\times)^m \hookrightarrow N^w, (k^\times)^m \hookrightarrow N^w \).
Remark 2.14. Similar to Theorem 2.11 Conjecture 2.12(c) implies [5, Conjecture 10.10], i.e., for any reduced word $i$ for $w \in W$, the restriction of (2.14) to $U_i$ is an isomorphism $\Psi_{i}^{-1} \circ \eta_{i}: U_{i} \rightarrow \mathbb{Q}[N^w]$.

We conclude the section with the relationship between the twist $\eta: U_i \rightarrow U_i$ from Theorem 2.10 and canonical basis in $U_i$.

Recall that in [6] A. Zelevinsky and the first author constructed a triangular basis $B(\Sigma)$ in the upper cluster algebra $U(\Sigma)$ for each acyclic quantum seed $\Sigma$ in $U_i$ and proved that $B(\Sigma)$ does not depend on the choice of $\Sigma$ ([6, Theorem 1.6]). This basis consists of bar-invariant elements and has a triangular transition matrix (in terms of powers of $q^{1/2}$) to the initial standard monomial basis $E(\Sigma) = \{E_a, a \in \mathbb{Z}^m\}$. Since $\eta$ commutes with the bar-involution and sends the initial standard monomial basis $E(\Sigma_i) \rightarrow E(\Sigma'_i)$ for some other seed $\Sigma'_i$ of $U_i$, the following is immediate.

Corollary 2.15. In the assumptions of Theorem 2.10 one has

$$\eta(B(\Sigma_i)) = B(\Sigma_i).$$

The basis $B(\Sigma)$ is an analogue of the dual canonical basis (see e.g., discussion in [6, Remark 1.8]). To make this analogy precise, we define the canonical basis $B'_i$ in $\Psi_i(U_+)$ and relate it to the twist $\eta_i$ from Conjecture 2.12(c).

Indeed, let $B^*$ be the dual canonical basis of $U_+$. For each reduced $i \in I^m$ we define $B'_i \subset L_i$ by:

$$B'_i = \Psi_i(B^*) \setminus \{0\}.$$ 

The following is immediate (part (a) follows from [24, Proposition 4.2], part (b) follows from that $\Psi_i$ commutes with the bar-involutions and part (c) – from [22, Theorem 6.18]).

Proposition 2.16. For each reduced $i \in I^m$ one has:

(a) The set $B'_i$ is a basis of $\Psi(U_+)$.

(b) For each $b \in B'_i$ one has $\tilde{b} = b$, where the bar-involution $t \mapsto \tilde{t}$ on $L_i$ is the only $\mathbb{Q}$-linear anti-automorphism such that $\tilde{t}_k = t_k$, $q^{1/2} = q^{-1/2}$.

(c) For any $b \in B'_i$ and each $k \in [1, m] \setminus \text{ex}$ there is $r \in \mathbb{Z}$ such that $q^{r}b \cdot X_k \in B'_i$.

Proposition 2.16(c), in particular, implies that the set $B_i$ of all bar-invariant elements of the form:

$$q^{r} \cdot b \prod_{k \in [1, m] \setminus \text{ex}} X_k^{a_k},$$

where $r, a_k \in \mathbb{Z}$, $b \in B'_i$ is a basis of the Ore localization of $\Psi_i(U_+)$ by all $X_k^{-1}$, $k \in [1, m] \setminus \text{ex}$.

Thus, Conjecture 2.12(a) asserts that $B_i$ is a basis of $U_i$. Now we (yet conjecturally) relate the bases with the twist $\eta_i$.

Conjecture 2.17. In the assumptions of Conjecture 2.12 one has:

(a) $\eta_i(B_i) = B_i$.

(b) If $i = (i_0, i_0)$, then $B_i$ is the triangular basis $B(\Sigma_i)$.
3. Hall-Ringel algebras and proof of Theorem 2.1

In this section we recall some basic facts about Hall-Ringel algebras of finitary categories and present some results on generalizations of Feigin homomorphisms.

Let \( \mathcal{C} \) be a small finitary Abelian category of finite global dimension. For an object \( V \in \mathcal{C} \) we will write \( [V] \) for the isomorphism class of \( V \) and write \( |V| \) for the class of \( V \) in the Grothendieck group \( K(\mathcal{C}) \). Let \( \mathcal{H}(\mathcal{C}) = \bigoplus \mathbf{k} \cdot [V] \) be the free \( K(\mathcal{C}) \)-graded \( \mathbf{k} \)-vector space spanned by the isomorphism classes of objects of \( \mathcal{C} \) with the natural grading via class in \( K(\mathcal{C}) \). For \( U, V, W \in \mathcal{C} \) define the (finite) set \( \mathcal{F}_{U,W}^V \) by:

\[
\mathcal{F}_{U,W}^V = \{ R \subset V \mid R \cong W, V/R \cong U \}.
\]

**Proposition 3.1** ([27]). The assignment

\[
[U][W] := \langle [U], [W] \rangle \sum_{[V]} |\mathcal{F}_{U,W}^V| \cdot |V|
\]

defines an associative multiplication on \( \mathcal{H}(\mathcal{C}) \).

The algebra \( \mathcal{H}(\mathcal{C}) \) is known as the Hall-Ringel algebra. The following formula for iterated multiplication is well-known and follows easily from the definitions:

\[
[U_1] \cdots [U_m] = \sum \prod \langle [U_k], [U_\ell] \rangle \cdot |\mathcal{F}_{U_1,\ldots,U_m}^V| \cdot |V|
\]

for \( U_1, \ldots, U_m, V \in \mathcal{C} \), where

\[
\mathcal{F}_{U_1,\ldots,U_m}^V = \{ 0 = V_m \subset V_{m-1} \subset \cdots \subset V_1 \subset V_0 = V : V_{k-1}/V_k \cong U_k \}.
\]

We will consider \( \mathcal{H}(\mathcal{C}) \otimes \mathcal{H}(\mathcal{C}) \) as an algebra with the twisted multiplication:

\[
\langle [U_1] \otimes [U_2], [V_1] \otimes [V_2] \rangle = \langle U_2, V_1 \rangle \langle V_1, U_2 \rangle [U_1][V_1] \otimes [U_2][V_2].
\]

**Proposition 3.2** ([18]). The map \( \Delta : \mathcal{H}(\mathcal{C}) \to \mathcal{H}(\mathcal{C}) \otimes \mathcal{H}(\mathcal{C}) \) given by

\[
\Delta([V]) = \sum_{[V], [W]} \langle [U], [W] \rangle \cdot |\mathcal{F}_{U,W}^V| \cdot \frac{|Aut(U)| \cdot |Aut(W)|}{|Aut(V)|} \cdot [U] \otimes [W]
\]

defines a coalgebra structure on \( \mathcal{H}(\mathcal{C}) \). Moreover, if \( \mathcal{C} \) is hereditary and cofinitary, then \( \Delta \) is an algebra homomorphism \( \mathcal{H}(\mathcal{C}) \to \mathcal{H}(\mathcal{C}) \otimes \mathcal{H}(\mathcal{C}) \).

Note that if the objects of \( \mathcal{C} \) do not have finitely many subobjects it will be necessary to consider the codomain of \( \Delta \) to be the completion \( \hat{\mathcal{H}}(\mathcal{C}) \otimes \hat{\mathcal{H}}(\mathcal{C}) \), see [22] for more details.

For each simple object \( S \in \mathcal{C} \) define the exponential

\[
E_S := \exp_q([S]) = \sum_{n=0}^{\infty} \frac{1}{(n)_q!} [S]^n \in \hat{\mathcal{H}}(\mathcal{C})
\]

where \( q = \langle S, S \rangle^2 \) and \( (n)_q! = (1)_q \cdots (n)_q \) for \( (k)_q = \frac{k^q - 1}{q - 1} \).

**Proposition 3.3.** For each simple object \( S \) with no self-extensions one has:

(a) \( E_S \) is a grouplike element of \( \hat{\mathcal{H}}(\mathcal{C}) \).

(b) \( \frac{1}{(n)_q!} [S]^n = \langle S, S \rangle^{\frac{n(n-1)}{2q+2}} [S^{\otimes n}] \), where \( q = |\text{End}(S)| \).
Proof. To prove (a) note that $S$ is primitive, i.e., $\Delta([S]) = [S] \otimes 1 + 1 \otimes [S]$, hence the assertion follows from Lemma 9.2.

For (b) we proceed by induction in $n$. Indeed, if $n = 0$, we have nothing to prove. Let $n \geq 1$. Then:

$$[S^\oplus n^{-1}][S] = \langle S^\oplus n^{-1}, S \rangle F_{S^\oplus n^{-1}, S}^{S^\oplus n} = \langle S, S \rangle^{n-1}(q^n + \cdots + q + 1)[S^\oplus n].$$

Using the inductive hypothesis, this is equivalent to:

$$\frac{1}{(n-1)q}(\langle S, S \rangle - (n-1)(n-2)[S]^{n-1} \cdot [S] = \langle S, S \rangle^{n-1}(n_1^n)[S^\oplus n].$$

This proves (b). The proposition is proved. \qed

We are now ready to prove Theorem 2.4. Assume in addition that $C$ is hereditary and cofinitary. The algebra $U := H(C)$ is naturally graded by $\Gamma := K(C)$ and according to Proposition 3.2, $U$ is a braided bialgebra in $C^\chi$ (see Section 9), where $\chi : \Gamma \times \Gamma \to \mathbb{k}^\times$ is the bicharacter given by the symmetrized Euler-Ringel form:

$$\chi([V], [W]) = \langle V, W \rangle \cdot \langle W, V \rangle.$$ (3.3)

Let $S = \{S_i : i \in I\}$ be the set of all (up to isomorphism) simple objects of $C$. Let $i = (i_1, \ldots, i_m) \in I^m$ be such that each $S_{i_k}$ has no self-extensions. We set $P := L_i$, where $L_i$ is the quantum torus as in Section 2. Write $E = (E_{i_1}, \ldots, E_{i_m})$ for the exponentials $E_i := ES_i$ (see 3.2). By Proposition 3.3[a], each member of the family $E$ is a group-like element in $U \otimes U$. It is easy to see that $E$ is $P$-adapted (in terminology of Section 9) via the homomorphisms $\tau_k : \mathbb{Z} \cdot |S_{i_k}| \to P$ defined by $\tau_k(r|S_{i_k}|) = t_k^r$, $k = 1, \ldots, m, r \in \mathbb{Z}$. Thus all hypotheses of Theorem 9.3 are satisfied and the assignment

$$x \mapsto \sum_{n \in \mathbb{Z}_{\geq 0}^m} x \left( [S_{i_1}]^{(a_1)} \cdots [S_{i_m}]^{(a_m)} \right) \cdot t_{i_1}^{a_1} \cdots t_{i_m}^{a_m}$$

for $x \in H^*(C)$ is an algebra homomorphism $\Psi_{C,i} : H^*(C) \to L_i$, where we abbreviate $[S]^{(n)} := \frac{1}{(n)!}[S]^n$, and $q = |\operatorname{End}(S)|$. It remains to show that

$$\Psi_{C,i}([V]^*) = X_{V,i}$$

for all objects $V$ of $C$.

To verify (3.5), we need the following fact.

**Lemma 3.4.** For $a = (a_1, \ldots, a_m) \in \mathbb{Z}_{\geq 0}^m$ one has:

$$[S_{i_1}]^{(a_1)} \cdots [S_{i_m}]^{(a_m)} = \sum_{[V]} \left( \prod_{k=1}^m \langle S_{i_k}, S_{i_k} \rangle^{(a_k - 1)/2} \right) \cdot \left( \prod_{k < \ell} \langle S_{i_k}, S_{i_\ell} \rangle^{a_ka_\ell} \right) \cdot |F_{i,a}(V)| \cdot [V],$$

where $F_{i,a}(V)$ is defined in 2.3.

**Proof.** Indeed, by Proposition 3.3(b) and 3.1, we obtain:

$$[S_{i_1}]^{(a_1)} \cdots [S_{i_m}]^{(a_m)} = \left( \prod_{k=1}^m \langle S_{i_k}, S_{i_k} \rangle^{(a_k - 1)/2} \right) [S_{i_1}^{a_1}] \cdots [S_{i_m}^{a_m}].$$
\[
\sum_{[V]} \left( \prod_{k=1}^{m} \langle S_{i_k}, S_{i_k} \rangle^{a_k(a_k-1)_2} \right) \cdot \left( \prod_{k<\ell} \langle S_{i_k} \otimes a_{\ell} \rangle \right) \cdot \left| \mathcal{F}_{V}^{V_{i_1}, \ldots, V_{i_m}} \right| \cdot [V].
\]

This proves the lemma because \( \mathcal{F}_{V}^{V_{i_1}, \ldots, V_{i_m}} = \mathcal{F}_{i,a}(V) \) and \( \langle S_{i_k} \otimes a_{\ell} \rangle = \langle S_{i_k}, S_{i_k} \rangle^{a_k a_{\ell}} \).

Using definition (2.4) of \( [V]^* \) and Lemma 3.3, we compute:
\[
\Psi_{C,i}([V]^*) = \langle V, V \rangle^{\frac{1}{2}} f([V]) \Psi_{C,i}(\delta[V]) = \langle V, V \rangle^{\frac{1}{2}} f([V]) \sum_{a \in \mathbb{Z}_{\geq 0}^m} \delta[V] \left( \langle S_{i_1} \rangle^{(a_1)} \cdots \langle S_{i_m} \rangle^{(a_m)} \right) \cdot t_1^{a_1} \cdots t_m^{a_m}
\]
\[
= \langle V, V \rangle^{\frac{1}{2}} f([V]) \sum_{a \in \mathbb{Z}_{\geq 0}^m} \left( \prod_{k=1}^{m} \langle S_{i_k}, S_{i_k} \rangle^{a_k(a_k-1)} \right) \cdot \left( \prod_{k<\ell} \langle S_{i_k}, S_{i_\ell} \rangle^{a_k a_{\ell}} \right) \cdot |\mathcal{F}_{i,a}(V)| \cdot t_1^{a_1} \cdots t_m^{a_m}.
\]

Furthermore, note that \( \mathcal{F}_{i,a}(V) \neq \emptyset \) implies that \( |V| = \sum_{k=1}^{m} a_k \cdot |S_{i_k}| \), hence
\[
f([V]) = \sum_{k=1}^{m} f(\langle S_{i_k} \rangle^{a_k}) = \prod_{k=1}^{m} \langle S_{i_k}, S_{i_k} \rangle^{\frac{1}{2} a_k}.
\]

Also,
\[
\langle V, V \rangle^{-\frac{1}{2}} = \prod_{k, \ell} \langle S_{i_k}, S_{i_\ell} \rangle^{-\frac{1}{2} a_k a_\ell} = \prod_{k=1}^{m} \langle S_{i_k}, S_{i_k} \rangle^{-\frac{1}{2} a_k^2} \prod_{k<\ell} \langle S_{i_k}, S_{i_\ell} \rangle^{-\frac{1}{2} a_k a_{\ell}} \prod_{k<\ell} \langle S_{i_\ell}, S_{i_k} \rangle^{-\frac{1}{2} a_k a_{\ell}}.
\]

Therefore,
\[
\Psi_{C,i}([V]^*) = \sum_{a \in \mathbb{Z}_{\geq 0}^m} \langle V, V \rangle^{-\frac{1}{2}} f([V]) \left( \prod_{k=1}^{m} \langle S_{i_k}, S_{i_k} \rangle^{\frac{1}{2} a_k(a_k-1)} \right) \cdot \left( \prod_{k<\ell} \langle S_{i_k}, S_{i_\ell} \rangle^{a_k a_{\ell}} \right) \cdot |\mathcal{F}_{i,a}(V)| \cdot t_1^{a_1} \cdots t_m^{a_m}
\]
\[
= \sum_{a \in \mathbb{Z}_{\geq 0}^m} \prod_{1 \leq k < \ell \leq m} \left( \frac{\langle S_{i_k}, S_{i_\ell} \rangle}{\langle S_{i_\ell}, S_{i_k} \rangle} \right)^{\frac{1}{2} a_k a_{\ell}} \cdot |\mathcal{F}_{i,a}(V)| \cdot t_1^{a_1} \cdots t_m^{a_m} = X_{V,i}.
\]

This verifies (3.5) and thus finishes the proof of Theorem 2.1. \(\square\)

We conclude the section with a formula for \( \Psi_{C,i} \) in terms of \( \mathcal{H}(C) \)-actions on \( \mathcal{H}^*(C) \). We need some notation.

Given a finitary Abelian category \( C \), for each \( U \in C \) define linear maps \( \partial_U, \partial_{U}^{op} : \mathcal{H}^*(C) \to \mathcal{H}^*(C) \) (in the notation (2.4) by):
\[
\partial_U(\delta[V]) = \sum_{[W]} \langle W, U \rangle |\mathcal{F}_{W,U}^V| \cdot \delta[W], \quad \partial_{U}^{op}(\delta[V]) = \sum_{[W]} \langle U, W \rangle |\mathcal{F}_{U,W}^V| \cdot \delta[W].
\]

The following fact is obvious.

**Lemma 3.5.** For any \( U, V \in C, x \in \mathcal{H}^*(C) \) one has:
\[
\partial_U(x([W])) = x([W][U]), \quad \partial_{U}^{op}(x([W])) = x([U][W])
\]
hence \([U] \otimes x \mapsto \partial_U(x)\) and \([U] \otimes x \mapsto \partial_{U}^{op}(x)\) are respectively left and right \( \mathcal{H}(C) \)-actions on \( \mathcal{H}^*(C) \).
For each $U \in \mathcal{C}$ define a linear transformation $K_U : \mathcal{H}^* (\mathcal{C}) \to \mathcal{H}^* (\mathcal{C})$ by

$$K_U (\delta_V) = \chi(|U|, |V|)^{-1} \cdot \delta_V$$

for any $V \in \mathcal{C}$, where $\chi(\cdot, \cdot)$ is as in (3.3).

Clearly, $K_U = K_U \circ K_{U/U'}$ for any sub-object $U'$ of $U$. It is also easy to see that

$$(3.7) \quad \partial_U K_U = \chi(|U|, |U'|)^{-1} K_U \partial_U, \quad \partial_U^{op} K_U = \chi(|U|, |U'|)^{-1} K_U \partial_U^{op}$$

Then denote

$$(3.8) \quad \partial_U := K_U^{\#} \partial_U, \quad \partial_U^{op} := K_U^{\#} \partial_U^{op}$$

for all $U \in \mathcal{C}$.

The following is an immediate corollary of Lemma 9.6

**Lemma 3.6.** Let $\mathcal{C}$ be a hereditary cofinitary category. Then for any simple object $S \in \mathcal{C}$ and any $x, y \in \mathcal{H}^* (\mathcal{C})$ one has:

$$\partial_S (xy) = \partial_S (x) K_S^{-1} (y) + x \partial_S (y), \quad \partial_S^{op} (xy) = \partial_S^{op} (x) y + K_S^{-1} (x) \partial_S^{op} (y),$$

$$\partial_S (xy) = \partial_S (x) K_S^{-\frac{1}{2}} (y) + K_S^{\frac{1}{2}} (x) \partial_S (y), \quad \partial_S^{op} (xy) = \partial_S^{op} (x) K_S^{-\frac{1}{2}} (y) + K_S^{\frac{1}{2}} (x) \partial_S^{op} (y).$$

Using this we can refine (9.3), (9.4) as follows.

**Proposition 3.7.** In the assumptions of Theorem 2.1 for any sequence $i = (i_1, \ldots, i_m) \in I^m$ and any homogeneous $x \in \mathcal{H}^* (\mathcal{C})$ of degree $\gamma$ has

$$(3.9) \quad \Psi_{C, i} (x) = \sum \partial_S^{[a_1]} \cdots \partial_S^{[a_m]} (x) \cdot t^a = \sum (\partial_S^{[a]} [a_1] \cdots \partial_S^{[a]} [a_m]) \partial_S^{[a]} [a_1] \cdots \partial_S^{[a]} [a_m] x \cdot t^a,$$

where the summation is over all $a = (a_1, \ldots, a_m) \in \mathbb{Z}_{\geq 0}^m$ such that $a_1 |S_{i_1}| + \cdots + a_m |S_{i_m}| = \gamma$. (where we abbreviated $X_i^\ell := |\text{End}(S_i)| \frac{\ell^{\gamma - 1}}{\ell^{\gamma - 1}} X_i^\ell$).

**Proof.** Indeed, taking into account that

$$(K_U^{\#} \partial_U)^a = \chi(|U|, |U|)^{- \frac{a(a-1)}{2}} K_U^{\#} \partial_U^{op}, \quad (K_U^{\#} \partial_U^{op})^a = \chi(|U|, |U|)^{- \frac{a(a-1)}{2}} K_U^{\#} \partial_U^{op}$$

for all $U \in \mathcal{C}$, $a \geq 0$, we obtain after repeatedly applying (3.7):

$$\partial_S^{[a]} \cdots \partial_S^{[a]} = K_S^{\frac{a_1}{2}} \partial_S^{[a_1]} \cdots K_S^{\frac{a_m}{2}} \partial_S^{[a_m]} = \chi \cdot K_U \partial_S^{[a_1]} \cdots \partial_S^{[a_m]},$$

$$(\partial_S^{[a]} [a_1] \cdots \partial_S^{[a]} [a_m]) = K_S^{\frac{a_1}{2}} (\partial_S^{[a]} [a_1]) \cdots K_S^{\frac{a_m}{2}} (\partial_S^{[a]} [a_m]) = \chi \cdot K_U (\partial_S^{[a]} [a_1]) \cdots (\partial_S^{[a]} [a_m]),$$

where $U = a_1 S_{i_1} + \cdots + a_m S_{i_m}$ and $\chi = \prod_{1 \leq k < \ell \leq m} \chi(|S_{i_k}|, |S_{i_j}|)^{- \frac{a_k a_j}{2}}$. Therefore,

$$\partial_S^{[a]} \cdots \partial_S^{[a]} (x) = \chi \cdot \partial_S^{[a]} \cdots \partial_S^{[a]} (x), \quad (\partial_S^{[a]} [a_1] \cdots \partial_S^{[a]} [a_m]) (x) = \chi \cdot (\partial_S^{[a]} [a_1] \cdots \partial_S^{[a]} [a_m]) (x)$$

for all $x \in \mathcal{H}^*$ of degree $\gamma = a_1 |S_{i_1}| + \cdots + a_m |S_{i_m}|$.

Note that (2.1) in the form $t_k t_k = \chi(|S_{i_k}|, |S_{i_k}|)^{\frac{a_k a_k}{2}} t_k^2 \cdots t_m^a$ for $k < \ell$ and (2.8) imply that

$$t^a = \prod_{1 \leq k < \ell \leq m} \chi(|S_{i_k}|, |S_{i_j}|)^{\frac{a_k a_j}{2}} t_k^a \cdots t_m^a = \chi(|S_{i_k}|, |S_{i_k}|)^{\frac{a_k a_k}{2}} t_k^2 \cdots t_m^a$$
for any \(a = (a_1, \ldots, a_m) \in \mathbb{Z}^m\). Therefore, in the notation \(\ref{3.4}\), we have:

\[
x \left( [S_{i_1}]^{(a_1)} \ldots [S_{i_m}]^{(a_m)} \right) t_{i_1}^{a_1} \ldots t_{i_m}^{a_m} = \partial_{S_{i_1}}^{(a_1)} \ldots \partial_{S_{i_m}}^{(a_m)}(x) \cdot t_{i_1}^{a_1} \ldots t_{i_m}^{a_m} = \partial_{S_{i_1}}^{[a_1]} \ldots \partial_{S_{i_m}}^{[a_m]}(x) \cdot t^{a} = (\partial_{S_{i_1}}^{(a_1)}) (\partial_{S_{i_2}}^{(a_2)}) \ldots (\partial_{S_{i_m}}^{(a_m)}) (x) \cdot t^{a}.
\]

This verifies \(\ref{3.9}\). Proposition \(\ref{3.7}\) is proved.

\[
\square
\]

## 4. Special compatible pairs

Following and generalizing \[5\] Section 8, here we construct compatible pairs \((\Lambda_i, \tilde{B}_i)\) for all sequences \(i \in I^m\) and certain symmetrizable \(I \times I\) matrices. These will later serve as the exchange data for the initial seed in \(\mathcal{L}_1\).

Let \(A = (a_{ij})\) be a symmetrizable \(I \times I\) integer matrix such that \(a_{ii} = 2\) for \(i \in I\) and let \(D = \text{diag}(d_i, i \in I)\) be a diagonal matrix with all \(d_i \in \mathbb{Z}_{>0}\) such that \(C = DA = (d_i a_{ij}) = (c_{ij})\) is symmetric. Denote by \(Q^\vee\) the lattice with the free basis \(a_i^\vee, i \in I\). For each \(i \in I\) define \(a_i := d_i a_i^\vee\) and denote by \(Q\) the sublattice of \(Q^\vee\) (freely) spanned by \(a_i, i \in I\).

Let \(P := Q \oplus \text{Hom}(Q^\vee, \mathbb{Z})\) be the corresponding weight lattice. By definition, \(P\) has a free basis \(\{a_i, \omega_i | i \in I\}\), where \(\omega_i : Q^\vee \to \mathbb{Z}\) determined by \(\omega_i(a_j^\vee) = \delta_{ij}\). We extend the canonical evaluation pairing \(Q^\vee \times \text{Hom}(Q^\vee, \mathbb{Z}) \to \mathbb{Z}\) to the pairing \((\cdot, \cdot) : Q^\vee \times P \to \mathbb{Z}\) by

\[
(a_i^\vee, \alpha_j) = a_{ij} \quad \text{for } i, j \in I.
\]

In particular, since \(Q \subset Q^\vee\), one has a paring \(Q \times P \to \mathbb{Z}\) whose restriction to \(Q \times Q\) is symmetric. Note that modulo the right kernel of \((\cdot, \cdot)\) we may identify \(a_j, j \in I\) with \(\sum_{i \in I} a_{ij} \omega_i\).

For \(i \in I\) define a linear endomorphism \(s_i\) of \(P\) by

\[
s_i a_j = a_j - a_{ij} a_i \quad \text{and} \quad s_i \omega_j = \omega_j - \delta_{ij} a_i \quad \text{for } j \in I.
\]

Denote by \(W\) the group of automorphisms of \(P\) generated by \(s_i, i \in I\). Clearly, \(s_i^2 = 1\) for all \(i \in I\) and \(W(Q) \subset Q\). The following result is easy to check.

**Lemma 4.1.** For any symmetrizable \(I \times I\) matrix \(A\) as above one has:

(a) There is a unique action of \(W\) on \(Q^\vee\) such that:

\[
s_i a_j^\vee = a_j^\vee - a_{ij} a_i^\vee \quad \text{for all } i, j \in I.
\]

(b) The pairing \((\cdot, \cdot)\) is \(W\)-invariant.

To each sequence \(i = (i_1, \ldots, i_m) \in I^m\) we assign a sequence \(w_k \in W, k = 1, \ldots, m\) by \(w_k = s_{i_1} s_{i_2} \cdots s_{i_k}\) (with the convention that \(w_0 = 1\)).

**Proposition 4.2.** For each \(i \in I^m\) the inverse of \(\varphi_1 : \mathbb{Z}^m \to \mathbb{Z}^m\) defined in \(\ref{2.10}\) is given by

\[
\varphi_1^{-1}(\varepsilon_i) = -\sum_{k=1}^{\ell} (w_k a_{ik}^\vee, w_{\ell} \omega_{i_k}) \varepsilon_k \quad \text{for } \ell = 1, \ldots, m.
\]
Proof. Define the $i$-derivative and $i$-integral $\partial_i, \int_i : \mathbb{Z}^m \to \mathbb{Z}^m$ respectively by

$$\partial_i \varepsilon_k = \varepsilon_k - \varepsilon_{k-} \quad \text{and} \quad \int_i \varepsilon_k = \varepsilon_k + \varepsilon_{k-} + \cdots .$$

for $1 \leq k \leq m$, where $k^{-}$ is defined in (2.11). Clearly, these maps are mutually inverse. We need the following fact.

Lemma 4.3. For any sequence $i \in I^m$ one has:

$$\int_i \varphi_i (\varepsilon_k) = -\varepsilon_k - \sum_{\ell=1}^{k-1} a_{ii_{i\ell}} \varepsilon_{\ell}$$

for $k = 1, \ldots, m$,

$$\varphi_i^{-1} (\partial_i \varepsilon_{\ell}) = -\varepsilon_{\ell} - \sum_{k=1}^{\ell-1} (w_k a_{ik}^\vee, w_{\ell} a_{i\ell}) \varepsilon_k$$

for $k = 1, \ldots, m$.

Proof. We obtain (4.4) by a direct computation:

$$\int_i \varphi_i (\varepsilon_k) = - (\varepsilon_k + \varepsilon_{k-} + \cdots) - \sum_{\ell < k < \ell+} a_{ii_{i\ell}} (\varepsilon_{\ell} + \varepsilon_{\ell-} + \cdots)$$

$$= -\varepsilon_k - 2\varepsilon_{k-} - 2\varepsilon_{k-} - \cdots - \sum_{\ell < k, i\ell \neq i_p} a_{ii_{i\ell}} \varepsilon_{\ell} = -\varepsilon_k - \sum_{\ell < k} a_{ii_{i\ell}} \varepsilon_{\ell} .$$

To prove (4.5) apply $\int_i \varphi_i$ to the right hand side of (4.6) for $\ell := p$, we obtain for all $p \in [1, m]$:

$$\int_i \varphi_i (-\varepsilon_p - \sum_{k=1}^{p-1} (w_k a_{ik}^\vee, w_p a_{i\ell}) \varepsilon_k) = - \int_i \varphi_i (\varepsilon_p) - \sum_{k=1}^{p-1} (w_k a_{ik}^\vee, w_p a_{i\ell}) \int_i \varphi_i (\varepsilon_k)$$

$$= \varepsilon_p - \sum_{k=1}^{p-1} (w_k a_{ik}^\vee, w_p a_{i\ell}) \varepsilon_k + \sum_{k=1}^{p-1} (w_k a_{ik}^\vee, w_p a_{i\ell}) \varepsilon_k + \sum_{k=1}^{p-1} (w_k a_{ik}^\vee, w_p a_{i\ell}) \varepsilon_k$$

$$= \varepsilon_p + \sum_{k=1}^{p-1} a_{ii_{i\ell}} \varepsilon_{\ell} + \sum_{k=1}^{p-1} (w_k a_{ik}^\vee, w_p a_{i\ell}) \varepsilon_k + \sum_{k=1}^{p-1} (w_k a_{ik}^\vee, w_p a_{i\ell}) \varepsilon_k$$

$$= \varepsilon_p + \sum_{k=1}^{p-1} a_{ii_{i\ell}} \varepsilon_{\ell} + \sum_{k=1}^{p-1} (w_k a_{ik}^\vee, w_p a_{i\ell}) \varepsilon_k + \sum_{k=1}^{p-1} (w_k a_{ik}^\vee, w_p a_{i\ell}) \varepsilon_k$$

$$= \varepsilon_p + \sum_{k=1}^{p-1} a_{ii_{i\ell}} \varepsilon_{\ell} + \sum_{k=1}^{p-1} (w_k a_{ik}^\vee, w_p a_{i\ell}) \varepsilon_k + \sum_{k=1}^{p-1} (w_k a_{ik}^\vee, w_p a_{i\ell}) \varepsilon_k$$

$$= \varepsilon_p + \sum_{k=1}^{p-1} a_{ii_{i\ell}} \varepsilon_{\ell} + \sum_{k=1}^{p-1} (w_k a_{ik}^\vee, w_p a_{i\ell}) \varepsilon_k + \sum_{k=1}^{p-1} (w_k a_{ik}^\vee, w_p a_{i\ell}) \varepsilon_k$$

where we used $w_k (a_{ik_{i\ell}}, a_{ik_{i\ell}}^\vee) = w_k a_{ik_{i\ell}}^\vee - w_{k-1} a_{ik_{i\ell}}^\vee$ in the third equality and $(w_p a_{i\ell}, w_p a_{i\ell}) = -(a_{i\ell}^\vee, a_{i\ell}) = -a_{ii_{i\ell}}$ for the last equality. The lemma is proved.

Denote by $\varphi_i'$ the $\mathbb{Z}$-linear map given by (4.2). To prove (4.2), it suffices to check that $\varphi_i' \partial_i$ is given by (4.5). Indeed,

$$\varphi_i' (\partial_i \varepsilon_{\ell}) = \varphi_i' (\varepsilon_{\ell} - \varepsilon_{\ell-}) = - \sum_{k=1}^{\ell} (w_k a_{ik}^\vee, w_{\ell} a_{i\ell}) \varepsilon_k + \sum_{k=1}^{\ell} (w_k a_{ik}^\vee, w_{\ell} a_{i\ell}) \varepsilon_k$$

$$= - \sum_{k=1}^{\ell} (w_k a_{ik}^\vee, w_{\ell} a_{i\ell}) \varepsilon_k + \sum_{k=1}^{\ell} (w_k a_{ik}^\vee, w_{\ell} a_{i\ell}) \varepsilon_k$$

$$= - \sum_{k=1}^{\ell} (w_k a_{ik}^\vee, w_{\ell} a_{i\ell}) \varepsilon_k + \sum_{k=1}^{\ell} (w_k a_{ik}^\vee, w_{\ell} a_{i\ell}) \varepsilon_k$$

$$= - \sum_{k=1}^{\ell} (w_k a_{ik}^\vee, w_{\ell} a_{i\ell}) \varepsilon_k + \sum_{k=1}^{\ell} (w_k a_{ik}^\vee, w_{\ell} a_{i\ell}) \varepsilon_k.$$
\[\Lambda(4.6) \quad \Lambda_i(\varepsilon_k, \varepsilon_{k+}) = \text{sgn}(k-\ell)c_{i_k,i_{\ell}}.\]

Denote \(\mathbf{ex} = \mathbf{ex}_i := \{k \in [1, m] | k^+ \leq m\}\) and call it the set of \textit{exchangeable indices} of \(i\). Following [5] Section 8, for \(k \in \mathbf{ex}\) define the vector \(\mathbf{b}^k = \mathbf{b}^k(i) = \sum_{p=1}^{m} b_{pk}\varepsilon_p \in \mathbb{Z}^m\) by:

\[b_{pk} = \begin{cases} 
-1 & \text{if } p = k^- \\
-a_{i_p,i_k} & \text{if } p < k < p^+ < k^+ \\
\alpha_{p,i_k} & \text{if } k < p < k^+ < p^+. \\
1 & \text{if } p = k^+ \\
0 & \text{otherwise} \end{cases}\]

Denote by \(\tilde{B}_i\) the \(m \times \mathbf{ex}\) matrix with columns \(\mathbf{b}^k, k \in \mathbf{ex}\).

Our objective in this section is to prove the following compatibility condition which refines and generalizes [5] Theorem 8.3] for single words \(i\) (the double word version can be stated verbatim).

\textbf{Theorem 4.4.} For any \(k \in \mathbf{ex}\) and \(1 \leq \ell \leq m\), one has
\[\Lambda_i(\varphi_i^{-1}(\mathbf{b}^k), \varphi_i^{-1}(\varepsilon_{\ell})) = 2d_{ik}\delta_{kk}.\]

\textbf{Proof.} We need the following results.

\textbf{Lemma 4.5.} For any sequence \(i \in I^m\) and \(k \in \mathbf{ex}\) one has:
\[\int_i \mathbf{b}^k = \varepsilon_k + \varepsilon_{k+} + \sum_{p=k+1}^{k^+ - 1} a_{i_p,i_k}\varepsilon_p.\]

\textbf{Proof.} By definition, for any \(\mathbf{a} = \sum_{p=1}^{m} a_p\varepsilon_p\), one has
\[\int_i \mathbf{a} = \sum_{p=1}^{m} a_p(\varepsilon_p + \varepsilon_{p-} + \cdots) = \sum_{p=1}^{m} (a_p + a_{p+} + \cdots)\varepsilon_p.\]

The result follows easily from this and the characterization of \(b_{pk}\) from [5] Remark 8.8:
\[b_{pk} = s_{pk} - s_{p,k^-} - s_{p^+,k} + s_{p^+,k^+}\]
for \(p \in [1, m]\) and \(k \in \mathbf{ex}\), where \(s_{pk} = \frac{1}{2}\text{sgn}(p-k)a_{i_p,i_k}\) (with the convention \(s_{p^+,k} = \alpha_{i_p,i_k}\) if \(p^+ = m + 1\)). Indeed, we obtain:
\[\int_i \mathbf{b}^k = \sum_{p=1}^{m} (b_{pk} + b_{p^+,k} + \cdots)\varepsilon_p = \sum_{p=1}^{m} (s_{pk} - s_{p,k^+})\varepsilon_p = \varepsilon_k + \varepsilon_{k+} + \sum_{p=k+1}^{k^+ - 1} a_{i_p,i_k}\varepsilon_p\]
Quantum cluster characters of Hall algebras

because $s_{pk} - s_{p^+, k} = \frac{1}{2}(\text{sgn}(p - k) - \text{sgn}(p - k^+))a_{ip, ik} = \begin{cases} 1 & \text{if } p = k \text{ or } p = k^+ \\ a_{ip, ik} & \text{if } k < p < k^+ \\ 0 & \text{otherwise} \end{cases}$. □

Proposition 4.6. For any $i \in I^n$ and $1 \leq k, \ell \leq m$ one has:

\begin{equation}
\Lambda_i(\varphi_i^{-1}(\partial_i \varepsilon_k), \varphi_i^{-1}(\partial_i \varepsilon_\ell)) = \text{sgn}(\ell - k)(w_k \alpha_{ik}, w_\ell \alpha_{i\ell})
\end{equation}

\begin{equation}
\Lambda_i(\varphi_i^{-1}(\varepsilon_k), \varphi_i^{-1}(\varepsilon_\ell)) = \begin{cases} (w_k \omega_{ik} - \omega_{ik}, \omega_\ell + w_\ell \omega_i) & \text{if } k \leq \ell \\ (\omega_i - w_i \omega_{i\ell}, w_k \omega_{ik} + \omega_{ik}) & \text{if } k > \ell \\
\end{cases}
\end{equation}

Proof. To prove (4.10) we need the following easy but powerful fact.

Lemma 4.7. Let $\Phi$ be a bilinear form on $k^m$, $\{\varepsilon_1, \ldots, \varepsilon_m\}$ be a basis of $k^m$, and $c_1, \ldots, c_m \in k \setminus \{0\}$. Define a linear map $\varphi : k^m \rightarrow k^m$ by:

$\varphi(\varepsilon_k) = \sum_{\ell=1}^m c_\ell \Phi(\varepsilon_k, \varepsilon_\ell) \varepsilon_\ell$ .

If $\varphi$ is invertible, then:

$\varphi^{-1}(\varepsilon_\ell) = \sum_{k=1}^m c_k \Phi(\varphi^{-1}(\varepsilon_k), \varphi^{-1}(\varepsilon_\ell)) \varepsilon_k$ .

Proof. Since $\varphi$ is invertible we have

$\varphi(\varphi^{-1}(\varepsilon_k)) = \sum_{\ell=1}^m c_\ell \Phi(\varphi^{-1}(\varepsilon_k), \varepsilon_\ell) \varepsilon_k = \varepsilon_k$ .

In particular, this implies $\Phi(\varphi^{-1}(\varepsilon_k), \varepsilon_\ell) = c_k^{-1} \delta_{k\ell}$. Now composing $\varphi$ with the map

$\varphi'(\varepsilon_\ell) := \sum_{k=1}^m c_k \Phi(\varphi^{-1}(\varepsilon_k), \varphi^{-1}(\varepsilon_\ell)) \varepsilon_k$

gives

$\varphi'(\varphi(\varepsilon_\ell)) = \sum_{k=1}^m c_k \Phi(\varphi^{-1}(\varepsilon_k), \varepsilon_\ell) \varepsilon_k = \varepsilon_\ell$ .

Thus $\varphi' = \varphi^{-1}$. The lemma is proved. □

We apply Lemma 4.7 with $\varphi := \int_i \varphi_i$ and $c_\ell := d_{i\ell}^{-1}$ for $\ell = 1, \ldots, m$. That is, by (4.3) and (4.5), the bilinear form $\Phi_i$ on $\mathbb{Z}^m$ defined by

\begin{equation}
\Phi_i(\varepsilon_k, \varepsilon_\ell) = \begin{cases} -d_{ik} & \text{if } k = \ell \\ -c_{i\ell, ik} & \text{if } \ell < k \\ 0 & \text{otherwise} \end{cases}
\end{equation}
satisfies

\[
\Phi_i(\varphi_i^{-1}(\partial_i \varepsilon_k), \varphi_i^{-1}(\partial_i \varepsilon_\ell)) = \begin{cases} 
-d_{ik} & \text{if } k = \ell \\
-(w_k \alpha_{ik}, w_\ell \alpha_{i_\ell}) & \text{if } k < \ell \\
0 & \text{otherwise}
\end{cases}.
\]

The identity (4.10) follows from this and the fact that \( \Lambda_i \) is a skew-symmetrization of \( \Phi_i \), i.e.,

\[
\Lambda_i(a, b) = \Phi_i(b, a) - \Phi_i(a, b)
\]

for all \( a, b \in \mathbb{Z}^m \).

To prove (4.11), we need the following result.

**Lemma 4.8.** The form \( \Phi_i \) given by (4.12) satisfies:

\[
\Phi_i(\varphi_i^{-1}(\varepsilon_k), \varphi_i^{-1}(\varepsilon_\ell)) = \begin{cases} 
(\omega_{ik} - w_k \omega_{ik}, w_\ell \omega_{i_\ell}) & \text{if } k < \ell \\
(w_\ell \omega_{i_\ell} - \omega_i, \omega_{ik}) & \text{if } k \geq \ell
\end{cases}.
\]

**Proof.** Indeed, let us compute using (4.13):

\[
\Phi_i(\varphi_i^{-1}(\varepsilon_k), \varphi_i^{-1}(\varepsilon_\ell)) = \Phi_i(\varphi_i^{-1}(\partial_i \varepsilon_k), \varphi_i^{-1}(\partial_i \varepsilon_\ell)) = \Phi_i(\varphi_i^{-1}(\varepsilon_k + \varepsilon_\ell + \cdots), \varphi_i^{-1}(\partial_i \varepsilon_\ell))
\]

\[
= \sum_{p \leq \min(k, \ell)} \Phi_i(\varphi_i^{-1}(\partial_i \varepsilon_p), \varphi_i^{-1}(\partial_i \varepsilon_\ell)) = -\varepsilon_k \ell d_{ik} - \sum_{p \leq \min(k, \ell-1)} (w_p \alpha_{ik}, w_\ell \alpha_{i_\ell})
\]

\[
= -\varepsilon_k \ell d_{ik} - \sum_{p \leq \min(k, \ell-1)} (w_p \omega_{ik} - w_\ell \omega_{ik}, w_\ell \alpha_{i_\ell}) = -\varepsilon_k \ell d_{ik} + (\omega_{ik} - w_k \omega_{ik}, w_\ell \alpha_{i_\ell})
\]

where \( \varepsilon_k, \ell = 1 \) if \( k \geq \ell \), \( i_k = i_\ell \) and 0 otherwise and \( k_0 = \max\{p \leq \min(k, \ell - 1) : i_p = i_k\} \).

Note that if \( k \geq \ell \), then \( w_k \omega_{ik} = w_{\ell-1} \omega_{ik} \) and so \( -\varepsilon_k \ell d_{ik} + (\omega_{ik} - w_k \omega_{ik}, w_\ell \alpha_{i_\ell}) = (\omega_{ik}, w_\ell \alpha_{i_\ell}) \).

Therefore,

\[
\Phi_i(\varphi_i^{-1}(\varepsilon_k), \varphi_i^{-1}(\varepsilon_\ell)) = (\omega_{ik}, w_\ell \alpha_{i_\ell}) - \begin{cases} 
(w_k \omega_{ik}, w_\ell \alpha_{i_\ell}) & \text{if } k < \ell \\
0 & \text{if } k \geq \ell
\end{cases}.
\]

Using this, we obtain:

\[
\Phi_i(\varphi_i^{-1}(\varepsilon_k), \varphi_i^{-1}(\varepsilon_\ell)) = \Phi_i(\varphi_i^{-1}(\varepsilon_k), \varphi_i^{-1}(\varepsilon_\ell)) = \Phi_i(\varphi_i^{-1}(\varepsilon_k), \varphi_i^{-1}(\partial_i \varepsilon_\ell + \cdots)) = \Phi_i(\varphi_i^{-1}(\varepsilon_k), \varphi_i^{-1}(\partial_i \varepsilon_\ell))
\]

\[
= \sum_{p \leq \ell, i_p = i_k} \Phi_i(\varphi_i^{-1}(\varepsilon_k), \varphi_i^{-1}(\partial_i \varepsilon_p)) = \sum_{p \leq \ell, i_p = i_k} (\omega_{ik}, w_p \alpha_{i_\ell}) - \begin{cases} 
(w_k \omega_{ik}, w_\ell \alpha_{i_\ell}) & \text{if } k < \ell \\
0 & \text{if } k \geq \ell
\end{cases}
\]

\[
= \sum_{p \leq \ell, i_p = i_k} (\omega_{ik}, w_p \alpha_{i_\ell}) - \begin{cases} 
\sum_{k < p \leq \ell, i_p = i_k} (w_k \omega_{ik}, w_p \alpha_{i_\ell}) & \text{if } k < \ell \\
0 & \text{if } k \geq \ell
\end{cases}
\]

\[
= (\omega_{ik}, w_\ell \omega_{i_\ell} - \omega_{i_k}) - \begin{cases} 
(w_k \omega_{ik}, w_\ell \omega_{i_\ell} - \omega_{i_0} \omega_{i_k}) & \text{if } k < \ell \\
0 & \text{if } k \geq \ell
\end{cases}.
\]
where $\ell_0 = \max\{p \leq \min(k, \ell) : i_p = i_\ell\}$. Here we used the identities $w_p \alpha_i = w_p \omega_{i\ell} - w_p \omega_i$, whenever $i_p = i_\ell$ and $w_{i_p} \omega_{i\ell} = w_k \omega_{i\ell}$. The lemma is proved.

Finally, using (4.14) and Lemma 4.8, we obtain for all $k, \ell \in [1, m]$:

$$\Lambda_i(\varphi_i^{-1}(\varepsilon_k), \varphi_i^{-1}(\varepsilon_\ell)) = \Phi_i(\varphi_i^{-1}(\varepsilon_k), \varphi_i^{-1}(\varepsilon_\ell)) - \Phi_i(\varphi_i^{-1}(\varepsilon_k), \varphi_i^{-1}(\varepsilon_\ell))$$

$$= \left\{\begin{array}{ll}
(\omega_{i\ell} - w_{i\ell} \omega_{i\ell}, w_k \omega_{i\ell}) & \text{if } \ell < k
\\
(w_k \omega_{i\ell} - \omega_{i\ell}, \omega_{i\ell}) & \text{if } \ell \geq \ell
\end{array}\right.$$

This proves (4.11) and thus completes the proof of Proposition 4.6.

To prove the compatibility conditions (4.8), we compute using (4.9) and (4.10):

$$\sum_{\ell<k} \alpha_{i\ell} \cdot \sum_{p<k} a_{i_p,i_\ell} \Phi_i(\omega_i, \omega_\ell) = 0$$

The relation $a_{i_p,i_\ell} = \alpha_i - s_{i_p} \alpha_i$ implies $a_{i_p,i_\ell} = w_p \alpha_i - w_{i_p} \alpha_i$, hence the sum above telescopes and we obtain:

$$\sum_{p=k+1}^{k+1} \text{sgn}(\ell - p)a_{i_p,i_\ell} \Phi_i(\omega_i, \omega_\ell) = 0$$

Therefore, taking into account that $(w_{i_p} \alpha_i, w_{i_\ell} \alpha_i, w_{i_\ell} \alpha_{i_\ell}) = (\alpha_i, s_{i_p} \alpha_i + \alpha_{i_\ell}) = 0$, we obtain:

$$\Lambda_i(\varphi_i^{-1}(\omega_i), \varphi_i^{-1}(\omega_\ell)) = \left\{\begin{array}{ll}
\delta_{k\ell}(w_k \alpha_i, w_{i_\ell} \alpha_i) & \text{if } \ell \leq k
\\
-\delta_{k+},(w_k \alpha_i, w_{i_\ell} \alpha_i) & \text{if } \ell \geq k^+
\end{array}\right.$$

Using this, we obtain the desired result:

$$\Lambda_i(\varphi_i^{-1}(\omega_i), \varphi_i^{-1}(\omega_\ell)) = \Lambda_i(\varphi_i^{-1}(\omega_i), \varphi_i^{-1}(\omega_\ell)) + \cdots + \Lambda_i(\varphi_i^{-1}(\omega_i), \varphi_i^{-1}(\omega_\ell)) = \cdots = 2d_{i_k}(\delta_{k\ell} - \delta_{k^+,\ell})$$

This completes the proof of (4.8) and therefore Theorem 4.4 is proved.

Generalizing second equation (4.11), we define $\mathbb{Z}$-linear automorphisms $\rho_i^+, \rho_i^-$ of $\mathbb{Z}^m$ by:

$$\rho_i^+(\varepsilon_k) = \varepsilon_k + \sum_{\ell<k} a_{i\ell} \varepsilon_\ell, \quad \rho_i^-(\varepsilon_k) = \varepsilon_k - \sum_{\ell<k} a_{i\ell} \varepsilon_\ell$$

(4.15)
Proposition 4.9. Let \( k \in \mathbf{ex} \) and \( 1 \leq \ell \leq m \), one has:
\[
\Lambda_i(((\rho_1^b)^{-1} \varphi_1)^{-1}(b_\ell^k)), ((\rho_1^b)^{-1} \varphi_1)^{-1}(\varepsilon_\ell)) = 2d_{i\ell} \delta_{k\ell} ,
\]
where \( b_\ell^k \) and \( b_\ell^\pm_k \), \( k \in \mathbf{ex} \), are truncations of \( b^k \) given by
\[
b_\ell^k = \sum_{p \in \mathbf{ex}} b_{pk} \varepsilon_p + \delta_k \varepsilon_{k^+}
\]
and \( \delta_\ell = \begin{cases} 
0 & \text{if } \ell \in \mathbf{ex} \\
1 & \text{if } \ell \notin \mathbf{ex} 
\end{cases} \)

Proof. The following fact is obvious.

Lemma 4.10. Under the hypotheses of Proposition 4.9 one has:
\[
\rho_1^+(\varepsilon_k) = \varepsilon_k + \delta_k \sum_{\ell < k; \ell \not\in \mathbf{ex}} a_{i_\ell i_k} \varepsilon_\ell, \quad \rho_1^-(\varepsilon_k) = \varepsilon_k - \delta_k \sum_{\ell \leq k; \ell \not\in \mathbf{ex}} a_{i_\ell i_k} \varepsilon_\ell .
\]

We also need the following fact.

Lemma 4.11. Under the hypotheses of Proposition 4.9 one has \( \rho_1^\pm_k(b_\ell^\pm_k) = b^k \) for \( k \in \mathbf{ex} \).

Proof. Indeed, using Lemma 4.10 for each \( k \in \mathbf{ex} \) we have:
\[
\rho_1^+(b_\ell^k) = \rho_1^+(b_\ell^k - \delta_k \varepsilon_{k^+} + \varepsilon_{k^+}) = b_\ell^k - \delta_k \varepsilon_{k^+} + \rho_1^+(\varepsilon_{k^+}) = b_\ell^k + \delta_k \varepsilon_{k^+} \cdot \sum_{\ell < k^+, \ell \not\in \mathbf{ex}} a_{i_\ell i_k} \varepsilon_\ell = b^k
\]
\[
\rho_1^-(b_\ell^k) = \rho_1^-(b_\ell^k + \delta_k \varepsilon_{k^+} - \varepsilon_{k^+}) = b_\ell^k + \delta_k \varepsilon_{k^+} - \rho_1^-(\varepsilon_{k^+}) = b_\ell^k + \delta_k \varepsilon_{k^+} \cdot \sum_{\ell \leq k^+, \ell \not\in \mathbf{ex}} a_{i_\ell i_k} \varepsilon_\ell = b^k
\]
because, by definition (4.7), \( b_{ik} = \begin{cases} 
a_{i_\ell i_k} & \text{if } k < \ell < k^+ \\
1 & \text{if } \ell = k^+ \quad \text{whenever } \ell \in [1, m] \setminus \mathbf{ex} .
\end{cases} \)

Now we are ready to prove 4.10. Indeed, using Lemmas 4.10 and 4.11 we obtain for all \( k \in \mathbf{ex} \), \( \ell \in [1, m] \):
\[
\Lambda_i(((\rho_1^b)^{-1} \varphi_1)^{-1}(b_\ell^k)), ((\rho_1^b)^{-1} \varphi_1)^{-1}(\varepsilon_\ell)) = \Lambda_i((\varphi_1^{-1} \rho_1^b)^{-1}(b_\ell^k), (\varphi_1^{-1} \rho_1^b)^{-1}(\varepsilon_\ell)) = \Lambda_i((\varphi_1^{-1}(b_\ell^k), \varphi_1^{-1}(\varepsilon_\ell))
\]
\[
= \Lambda_i((\varphi_1^{-1}(b_\ell^k), \varphi_1^{-1}(\varepsilon_\ell) + \varphi_1^{-1}(\rho_1^b(\varepsilon_\ell) - \varepsilon_\ell)) = \Lambda_i((\varphi_1^{-1}(b_\ell^k), \varphi_1^{-1}(\varepsilon_\ell)) = 2d_{i\ell} \delta_{k\ell}
\]
by Theorem 4.3 because \( \Lambda_i((\varphi_1^{-1}(b_\ell^k), \varphi_1^{-1}(\varepsilon_\ell)) = 0 \) whenever \( \ell' \in [1, m] \setminus \mathbf{ex} \). The proposition is proved.

For each sequence \( i \in I^m \) and \( \lambda \in \mathcal{P} \) define the vector \( a_\lambda \in \mathbb{Z}^m \) by
\[
a_\lambda := - \sum_{k=1}^{m} (w_k a_{ik}^\gamma, w_m \lambda) \varepsilon_k .
\]
This notation is justified by the following collection of easily verifiable facts.
Lemma 4.12. For any $i \in I^m$ and $\lambda \in P$ one has:

(a) $a_\lambda = \{0\}$ for all $\lambda$ in the right kernel of the pairing $(\cdot, \cdot) : Q^\vee \times P \to \mathbb{Z}$.

(b) $a_{i_j} = \sum_{i \in I} a_{ij} a_{i_j}$ for all $j \in I$.

(c) The set $L_0 := \{a_\lambda | \lambda \in P\}$ is a sub-lattice of $\mathbb{Z}^m$ spanned by $a_{\omega_i}$, $i \in I$.

(d) For any $w \in W$ the lattice $L_0$ is spanned by $a_{w\omega_i}$, $i \in I$.

(e) $\varphi^{-1}(e_i) = a_{\omega_i}$ for each $i \in [1, m] \setminus \{\text{ex.}\}$.

(f) $|a_\lambda| = w_m \lambda - \lambda$ for $\lambda \in P$, where we abbreviated $|a| := \sum_{k=1}^m a_k \alpha_k$.

Proposition 4.13. For each $i \in I^m$ and $a = (a_1, \ldots, a_m) \in \mathbb{Z}^m$ one has:

$$\Lambda_i(a, a_\lambda) = (|a|, w_m \lambda + \lambda)$$

Proof. It suffices to verify the result for $a = e_k$:

$$\Lambda_i(e_k, a_\lambda) = -\sum_{\ell=1}^m \text{sgn}(k - \ell) c_{ik, \ell} (w_\ell \alpha_i^\vee, w_m \lambda) = -\sum_{\ell=1}^m \text{sgn}(k - \ell) (w_\ell \alpha_i^\vee - w_{\ell-1} \alpha_i, w_m \lambda)$$

$$= (\alpha_i, w_m \lambda) - (w_{k-1} \alpha_i, w_m \lambda) - (w_k \alpha_i, w_m \lambda) + (w_m \alpha_i, w_m \lambda) = (\alpha_i, w_m \lambda + \lambda).$$

Since $|e_k| = \alpha_i$, this completes the proof. \qed

5. Valued Quivers and Proof of Theorem 2.5

5.1. Quantum cluster characters. Let $C$ be a finitary Abelian category with finitely many simple objects $S = \{S_i \mid i \in I\}$ with no self-extensions. Denote by $K(C)$ the Grothendieck group of $C$. Clearly, $K(C) = \bigoplus_{i \in I} \mathbb{Z} \cdot |S_i|$, where $|S_i| \in K(C)$ is the class of $S_i$.

Note that $k_i := \text{End}_C(S_i)$ is a finite field for all $i \in I$.

Define the $I \times I$ Cartan matrix $A = A_{k_i} = (a_{ij})$ of $C$ by

$$a_{ij} = 2\delta_{ij} - \dim_{k_j}(\text{Ext}^1(S_i, S_j)) - \dim_{k_i}(\text{Ext}^1(S_j, S_i)).$$

From now on, we assume that for each $i \neq j$: $\text{Ext}^1(S_i, S_j) = 0$ or $\text{Ext}^1(S_j, S_i) = 0$ and define an $I \times I$ matrix $B = B_C = (b_{ij})$ by:

$$b_{ij} = \begin{cases} 
\dim_{k_i}(\text{Ext}^1(S_i, S_j)) & \text{if } \text{Ext}^1(S_i, S_j) \neq 0 \\
-\dim_{k_i}(\text{Ext}^1(S_j, S_i)) & \text{if } \text{Ext}^1(S_j, S_i) \neq 0 \\
0 & \text{otherwise}
\end{cases}$$

Clearly, $a_{ij} = -|b_{ij}|$ for $i \neq j$.

Following [30], for $e \in K(C)$ denote by $e \mapsto *e$ and $e \mapsto e^*$ respectively the endomorphisms of $K(C)$ given by

$$|S_j| = |S_j| - \sum_{k \in I} [b_{kj}]_+ \cdot |S_k|, \quad |S_j|^* = |S_j| - \sum_{k \in I} [-b_{kj}]_+ \cdot |S_k|.$$ 

Define $b^j$, $j \in I$ by

$$b^j := |S_j|^* - |S_j|$$

Clearly, $b^j = \sum_{k \in I} b_{kj} \cdot |S_k|$ for all $j \in J$, so it can be identified with the $j$-th column of $B$.

Fix a subset $\mathbf{ex}$ of $I$ and a unitary bicharacter $\chi = \chi_C$ on $\mathcal{K}(\mathcal{C})$ such that

$$\chi(b^j, |S_i|) = |k_i|^{b_{ij}}$$

for all $i \in I$, $j \in \mathbf{ex}$.

Denote by $\mathcal{T}_\chi$ the based quantum torus with the basis $X^a$, $a \in \mathcal{K}(\mathcal{C})$ subject to the relation $28$:

$$X^a X^b = \chi_C(a, b) X^{a+b}$$

for $a, b \in \mathcal{K}(\mathcal{C})$.

Then the quantum cluster character $X_V \in \mathcal{T}_\chi$ of $V \in \mathcal{C}$ is given by (2.13):

$$X_V = \sum_{e \in \mathcal{K}(\mathcal{C})} (|V|, |V| - e)^{-1} \cdot |Gr_e(V)| \cdot X^{-e^\tau}(-|V| - e).$$

5.2. Valued quivers, flags, and Grassmannians. Let $\mathbb{F}$ be a finite field with $q$ elements. Fix an algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$ and for each $a > 0$ denote by $\mathbb{F}_a$ the degree $a$ extension of $\mathbb{F}$ in $\overline{\mathbb{F}}$. Note that the largest subfield of $\overline{\mathbb{F}}$ contained in both $\mathbb{F}_a$ and $\mathbb{F}_b$ is $\mathbb{F}_{gcd(a, b)}$. If $a | b$, then $\mathbb{F}_a \subset \mathbb{F}_b$ and $F_b \cong (F_a)^{n_b}$ as a vector space over $\mathbb{F}_a$.

Fix an integer $n$ and let $Q = \{Q_0, Q_1, s, t\}$ be a quiver with vertices $Q_0 = \{1, 2, \ldots, n\}$ and arrows $Q_1$, where we denote by $s(\alpha)$ and $t(\alpha)$ the source and target of an arrow $\alpha : s(\alpha) \to t(\alpha)$. Given a map $d : Q_0 \to \mathbb{Z}_{>0}$ ($i \mapsto d_i$), we refer to the pair $(Q, d)$ as a valued quiver.

Define a representation $V = \{\{V_i\}_{i \in Q_0}, \{\varphi_\alpha\}_{\alpha \in Q_1}\}$ of $(Q, d)$ by assigning an $\mathbb{F}_{d_i}$-vector space $V_i$ to each vertex $i \in Q_0$ and an $\mathbb{F}_{gcd(d(s(\alpha)), d(t(\alpha)))}$-linear map $\varphi_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$ to each arrow $\alpha \in Q_1$. Morphisms, direct sums, kernels, and co-kernels are defined as in the well-known representation theory of quivers. Denote by $\text{rep}_p(Q, d)$ the (Abelian) category of all finite dimensional representations of $(Q, d)$.

Following [30] to each skew-symmetricizable $n \times n$ matrix $B$, i.e., such that $(DB)^T = -B^T D$, for some diagonal matrix $D = \text{diag}(d_1, \ldots, d_n)$, $d_i \in \mathbb{Z}_{>0}$ we assign a valued quiver $(Q_B, d)$ having no loops or two-cycles with:

- $Q_0 = \{1, \ldots, n\}$, $d = (d_1, \ldots, d_n)$;
- $gcd(|b_{ij}|, |b_{ji}|)$ arrows in $Q_1$ from $i$ to $j$ for distinct $i, j \in Q_0$ whenever $b_{ij} > 0$.

One can easily recover the matrix $B$ from a valued quiver $(Q, d)$ having no loops or two-cycles by $B = B_Q := B_{\text{rep}_p(Q, d)}$ via (5.2), where $S = \{S_1, \ldots, S_m\}$ is the set of all simple representations of $(Q, d)$ (see also [30] for details).

The following fact is obvious.

Lemma 5.1. Let $(Q, d)$, $(Q', d')$ be valued quivers with $Q_0 = \{1, 2, \ldots, n\}$ and $Q'_0 = \{1, 2, \ldots, r\}$ respectively and $C$ be an integer $r \times n$ matrix such that

$$D' C = CD,$$

where $D = \text{diag}(d_1, \ldots, d_n)$, $D' = \text{diag}(d'_1, \ldots, d'_r)$. 
Then there exists a valued quiver \((\tilde{Q}, \tilde{d})\) with \(\tilde{Q}_0 = \{1, 2, \ldots, n + r\}\) such that

\[
\tilde{d} = (d, d'), \quad B_{\tilde{Q}} = \begin{pmatrix} B_Q & -D^{-1}CD' \\ C & B_{Q'} \end{pmatrix}.
\]

We will call \(Q\) the principal subquiver of \(\tilde{Q}\). Our default choice of extension \((\tilde{Q}, \tilde{d})\) in Lemma 5.1 is with \(r = n\), \((Q', d') = (Q, d)\) and \(C = \pm I_n\) (where \(I_n\) is the identity \(n \times n\) matrix) so that

\[
(5.7) \quad \tilde{d} = (d, d), \quad B_{\tilde{Q}}^\pm = \begin{pmatrix} B_Q & \mp I_n \\ \pm I_n & B_{Q'} \end{pmatrix}.
\]

For a vertex \(i \in Q_0\) write \(\mu_i Q\) for the quiver obtained from \(Q\) by reversing all arrows \(a \in Q_1\) with \(s(a) = i\) or \(t(a) = i\). Let \(i_0 = (i_1, \ldots, i_n) \in Q_0^n\) be a repetition-free source adapted sequence in \(Q\), i.e. \(i_k\) (\(k \geq 1\)) is a source in the quiver \(\mu_{i_{k−1}} \cdots \mu_{i_1} Q\).

Recall that for \(a \in \mathbb{Z}_{\geq 0}^n\) and any sequence \(i \in Q_0^n\) the flag variety \(\mathcal{F}_{i,a}(V)\) is given by

\[
\mathcal{F}_{i,a}(V) = \{0 = V_m \subset V_{m−1} \subset \cdots \subset V_1 \subset V_0 = V : V_{k−1}/V_k \cong a_k S_{i_k}\}
\]

and for each \(k \in [1, m]\) denote by \(\pi_k : \mathcal{F}_{i,a}(V) \to Gr(V)\) the map which assigns to each flag \((0 = V_m \subset V_{m−1} \subset \cdots \subset V_1 \subset V_0 = V) \in \mathcal{F}_{i,a}(V)\) the subobject \(V_k\) of \(V\).

In what follows, we identify the Grothendieck group \(K(Q, d)\) with \(\mathbb{Z}^n\) via \(|S_{i_k}| \to \varepsilon_k\), \(k = 1, \ldots, n\).

**Theorem 5.2.** Let \(V \in \text{rep}_F(Q, d)\), \(i = (i_0, i_0)\). Then for any \(e \in \mathbb{Z}_{>0}^n\) the map \(\pi_n\) defines a bijection

\[
\mathcal{F}_{i,(|V|−e, e)}(V) \cong Gr_e(V).
\]

**Proof.** We will construct the inverse \(\iota : Gr_e(V) \to \mathcal{F}_{i,(|V|−e, e)}(V)\) of \(\pi_n\). Since \(i_0\) is a source adapted sequence in \(Q\), given \(W \subset V \in Gr_e(V)\) there is a unique flag in \(\mathcal{F}_{i_0,e}(W)\) which we denote by \((0 = V_{2n} \subset V_{2n−1} \subset \cdots \subset V_n = W)\). This gives the lower half of the flag associated to \(W\) by \(\iota\).

Now we recursively construct the remaining subobjects \(V_k, k = 0, \ldots, n−1\). Set \(V_0 = V\) and suppose that \(V_0 \supset \cdots \supset V_{k−1}\) are already constructed for some \(k \geq 1\). Since \(i_k\) is a source in the support of \(V_{k−1}/V_n\), there is a unique surjective map from \(V_{k−1}/V_n\) to \((v_{i_k} − e_{i_k})S_{i_k}\). Denote by \(V_k \subset V_{k−1}\) the kernel of the composition \(V_{k−1} \to V_{k−1}/V_n \to (v_{i_k} − e_{i_k})S_{i_k}\), so that \(V_{k−1}/V_k \cong (v_{i_k} − e_{i_k})S_{i_k}\).

Therefore, we have constructed a unique flag \(0 = V_{2n} \subset V_{2n−1} \subset \cdots \subset V_1 \subset V_0 = V\) in \(\mathcal{F}_{(i_0, i_0),(|V|−e, e)}(V)\) with \(V_n = W\). Clearly \(\pi_n \circ \iota\) and \(\iota \circ \pi_n\) are identity maps respectively.

**Remark 5.3.** In [15], Geiss, Leclerc, and Schröer constructed a bijection between flags in representations of the preprojective algebra and Grassmannians for certain quivers. We will study such generalized bijections in terms of quantum cluster characters in our future work.

**Corollary 5.4.** For \(i = (i_0, i_0)\) and \(V \in \mathcal{C}\), one has:

\[
(5.8) \quad X_{V, i} = \sum_{e \in \mathbb{Z}_{>0}^n} \langle e, |V|−e \rangle^{-1} |Gr_e(V)| \cdot t^{(|V|−e, e)}
\]
Proof. By definition (2.2) of $X_{V,i}$ we have

$$X_{V,i} = \sum_{a \in \mathbb{Z}_{\geq 0}^n} \prod_{k \neq \ell} \langle (S_{ik}, S_{i\ell}) \rangle \frac{t^a}{a_k a_\ell} |F_{i,a}(V)| \cdot t^a,$$

where $t^a = \prod_{k < \ell} \langle (S_{ik}, S_{i\ell}) \rangle \frac{1}{a_k a_\ell} t_1^{a_1} \cdots t_m^{a_m}$ is a basis element of the based quantum torus $\mathbb{L}_1$.

Note that $|F_{i,a}(V)| = 0$ unless $|V| = \sum_{k=1}^{2n} a_k |S_{ik}|$ and thus we will restrict the sum to these $a \in \mathbb{Z}_{\geq 0}^n$.

Since $i = (i_0, i_0)$, we have $\langle S_{i\ell}, S_{ik} \rangle = 1$ if $1 \leq k < \ell \leq n$ or $1 \leq k - n < \ell - n \leq n$. Therefore,

$$\prod_{k < \ell} \langle (S_{ik}, S_{i\ell}) \rangle \frac{1}{a_k a_\ell} \langle S_{i\ell}, S_{ik} \rangle = \prod_{1 \leq k < \ell \leq 2n} \langle a_\ell S_{i\ell}, a_k S_{ik} \rangle^{-1} = \left( \sum_{\ell = n+1}^{2n} a_\ell S_{i\ell}, \sum_{k=1}^{n} a_k S_{ik} \right)^{-1} = \langle e, |V| - e \rangle^{-1},$$

where $e = \sum_{\ell = n+1}^{2n} a_\ell S_{i\ell}$ determines $a = (|V| - e, e)$. Combining with Theorem 5.2 we obtain:

$$X_{V,i} = \sum_{e \in \mathbb{Z}_{\geq 0}} \langle e, |V| - e \rangle^{-1} |F_{i,|V| - e,e}(V)| \cdot t^{(|V| - e,e)} = \sum_{e \in \mathbb{Z}_{\geq 0}} \langle e, |V| - e \rangle^{-1} |Gr_e(V)| \cdot t^{(|V| - e,e)}.$$

This proves (5.8).

5.3. $(i_0, i_0)$-Character is Quantum Cluster Character: conclusion of the proof. Denote by $(\tilde{Q}, \tilde{d})$ the valued quiver on $2n$ vertices given by (5.7) with the “minus” sign, that is, $B_{\tilde{Q}} = \begin{pmatrix} B_Q & I_n \\ -I_n & B_Q \end{pmatrix}$, where $I_n$ is the $n \times n$ identity matrix.

Proposition 5.5. Let $i = (i_0, i_0)$ be twice a source adapted sequence. Then

$$\rho_i^{-1} \varphi_i(X_{i,V}) = \tilde{X}_V,$$

for any $V \in \text{rep}_D(Q, d)$, where $\tilde{X}_V$ stands for the quantum cluster character attached to $\tilde{C} = \text{rep}_D(\tilde{Q}, \tilde{d})$ with $\text{ex} = \{1, \ldots, n\}$ (and $V$ is regarded as an object of $\tilde{C}$).

Proof. In what follows, we identify the Grothendieck group $K(\tilde{Q}, \tilde{d})$ with $\mathbb{Z}^{2n}$ via $|S_i| \mapsto \varepsilon_i$ for $i = 1, \ldots, 2n$. We need the following result.

Lemma 5.6. Let $i = (i_0, i_0)$. Then, in the notation (2.10) one has:

$$\rho_i^{-1} \varphi_i(f, e) = -(e, 0)^* - *(f, 0)$$

for all $f, e \in \mathbb{Z}^n$.

Proof. It suffices to show that

$$(5.11) \quad \rho_i^{-1} \varphi_i(\varepsilon_k) = \begin{cases} -\varepsilon_k & \text{if } 1 \leq k \leq n \\ -\varepsilon_{k-n} & \text{if } n+1 \leq k \leq 2n \end{cases}$$

for $k = 1, \ldots, 2n$, where the elements $\varepsilon_k, \varepsilon_k^n$ are defined by (5.3).
Indeed, the choice of \( B_{\frac{k}{q}} \) implies that the elements \( *\varepsilon_k, \varepsilon_k^* \), \( k = 1, \ldots, n \) are equal to:

\[
*\varepsilon_k = \varepsilon_k - \sum_{\ell=1}^{2n} [b_{\ell k}]_+ \varepsilon_\ell = \varepsilon_k + \sum_{\ell=1}^{k-1} a_{i\ell i} \varepsilon_\ell, ~ \varepsilon_k^* = \varepsilon_k - \sum_{\ell=1}^{2n} [-b_{\ell k}]_+ \varepsilon_\ell = \varepsilon_k - \varepsilon_{k+n} + \sum_{\ell=k+1}^{n} a_{i\ell i} \varepsilon_\ell
\]

because \( b_{\ell k} = \text{sgn}(\ell - k)a_{i\ell i} \) if \( 1 \leq k, \ell \leq n \) and \( b_{\ell k} = -\delta_{\ell-n,k} \) if \( n < \ell \leq 2n, k = 1, \ldots, n \). Therefore, using Lemma 4.10 we obtain \( \rho_i(*\varepsilon_k) = *\varepsilon_k \) and:

\[
\rho_i(\varepsilon_k) = \varepsilon_k - \rho_i(\varepsilon_{k+n}) + \sum_{\ell=k+1}^{n} a_{i\ell i} \varepsilon_\ell = \varepsilon_k - \varepsilon_{k+n} + \sum_{\ell=k+1}^{n} a_{i\ell i} \varepsilon_\ell
\]

Furthermore, by definition (2.10) one has:

\[
\varphi_i(\varepsilon_k) = \begin{cases} -\varepsilon_k - \sum_{\ell=1}^{k-1} a_{i\ell i} \varepsilon_\ell & \text{if } 1 \leq k \leq n \\ -\varepsilon_k - \varepsilon_{k-n} - \sum_{\ell=k+1}^{k-1} a_{i\ell i} \varepsilon_\ell & \text{if } n + 1 \leq k \leq 2n \end{cases}
\]

which proves (5.11). The lemma is proved.

Now we are ready to finish the proof of Proposition 5.5. Applying \( \rho_i^{-1}\varphi_i \) to (5.8) and taking into account that \( \rho_i^{-1}\varphi_i(\mathbf{0}) = X\rho_i^{-1}\varphi_i(\mathbf{a}) \) for \( \mathbf{a} \in \mathbb{Z}^{2n} \) by Lemma 2.4 we obtain using Lemma 5.6

\[
\rho_i^{-1}\varphi_i(X_{V, i}) = \sum_{e \in \mathbb{Z}_{\geq 0}^n} \langle e, |V| - e \rangle^{-1} [Gr_e(V)] \cdot X\rho_i^{-1}\varphi_i(|V| - e, e)
\]

\[
= \sum_{e \in \mathbb{Z}_{\geq 0}^n} \langle e, |V| - e \rangle^{-1} [Gr_e(V)] \cdot X^{-\langle e, 0 \rangle} X^{-\langle |V| - e, 0 \rangle} = X_V.
\]

Proposition 5.5 is proved.

Therefore, Theorem 2.5 is proved.

6. Quantum groups and quantum cluster algebras

6.1. Quantum groups, representations, and generalized minors. Let \( A = (a_{ij}) \) be an \( I \times I \) symmetrizable Cartan matrix with symmetrizing matrix \( D = \text{diag}(d_i, i \in I) \). Denote by \( U_q(\mathfrak{g}) \) the quantized enveloping algebra associated to \( \mathfrak{g} \). It is generated over \( k \) by \( E_i, F_i, K_i, K_i^{-1}, i \in I \) subject to the relations (we retain notation of section 4):

- For each \( i \in I \) the quadruple \( E_i, F_i, K_i, K_i^{-1} \) are standard generators of \( U_q(sl_2) \), where \( q_i = q^{d_i} \);
- For each \( i \in I \) the element \( K_i \) acts by conjugation on each graded component \( U_q(\mathfrak{g})_{\gamma}, \gamma \in \mathbb{Q} \) with the eigenvalue \( q_i^{(\alpha_i^*|\gamma)} = q^{(\alpha_i, \gamma)} \);
- Quantum Serre relations for \( i \neq j \):

\[
\sum_{k=0}^{1-a_{ij}} (-E_i)^{[k]} E_i^{[1-a_{ij}-k]} = \sum_{k=0}^{1-a_{ij}} (-F_i)^{[k]} F_i^{[1-a_{ij}-k]} = 0
\]

(Here \( x_i^{[\ell]} := \frac{1}{[\ell]_{q_i}} x_i^\ell \) is an \( i \)-th divided power, where \( [\ell]_{q_i} = [1]_{q_i} \cdots [\ell]_{q_i} \), and \( k_{q_i} = \frac{q^k - q^{-k}}{q_i - q_i^{-1}} \).)
Let $U_q(b_+)^*$ be the algebra generated by $U_q^+$ and $v_\lambda$, $\lambda \in \mathcal{P}$ (in the notation of Section 3) subject to the relations:

\[(6.1) \quad v_\lambda x_i = q^{-(\alpha_i, \lambda)} x_i v_\lambda \quad \text{and} \quad v_\lambda v_\mu = v_{\lambda+\mu}\]

for $i \in I$, $\lambda, \mu \in \mathcal{P}$, where we abbreviated $x_i := [S_i]^* \in U_q^+$. 

It is well-known that $U_q(b_+)^*$ is a finite dual Hopf algebra of the quantized universal enveloping algebra $U_q(b_+)$, where $b_+$ is a Borel subalgebra of $\mathfrak{g}$. The algebra $U_q(b_+)^*$ is naturally graded by $\mathcal{P}$ via $|v_\lambda| = \lambda$ and $|x_i| = -\alpha_i$ for $i \in I$. Therefore, for a homogeneous $x \in U_q^+$ and $\lambda \in \mathcal{P}$ one has:

\[(6.2) \quad v_\lambda x = q^{(\lambda, x)} x v_\lambda .\]

The following result is well-known (see e.g., [1, Section 3]).

**Lemma 6.1.** There is an action of $U_q(\mathfrak{g})$ on $U_q(b_+)^*$ given by the following formulas (for $i \in I$):

- $K_i(y) = q^{(\alpha_i, y)} y = q_{i}^{(\alpha_i, y)} y$ for all homogeneous elements $y \in U_q(b_+)^*$.
- $F_i(y) = \frac{K_i^{-1}(q_{i}^{-1}) x_i}{q_{i}^{-1} - 1}$ for all $y \in U_q(b_+)^*$.
- $E_i$ is a $K_i$-derivation of $U_q(b_+)^*$ determined by:
  - $E_i(x_j) = \delta_{ij}$ and $E_i(v_\lambda) = 0$ for all $i, j \in I$, $\lambda \in \mathcal{P}$.
  - the Leibniz rule: $E_i(x) = E_i(x) K_i(y) + x E_i(y)$ for all $x, y \in U_q(b_+)^*$.

It is easy to see that $V_\lambda := U_q(\mathfrak{g})(v_\lambda) \subset U_q(b_+)^*$ is a simple $U_q(\mathfrak{g})$-module with the highest weight $\lambda$ for each $\lambda \in \mathcal{P}$. Denote by $\mathcal{P}^+ \subset \mathcal{P}$ the cone of dominant integral weights, i.e., all $\lambda \in \mathcal{P}$ such that $(\alpha_i^\vee, \lambda) \in \mathbb{Z}_{\geq 0}$ for all $i \in I$.

In particular, if $\lambda \in \mathcal{P}^+$, then for each $w \in W$ the module $V_\lambda \subset U_q(b_+)^*$ contains an extremal vector $v_{w\lambda}$ which is the unique (up to a scalar) element of $V_\lambda$ such that $|v_{w\lambda}| = w\lambda$. This $v_{w\lambda}$ can be computed recursively by:

\[(6.3) \quad v_{w\lambda} = (F_i K_i^{-\frac{1}{2}})^{|N|} (v_{s_i w\lambda})\]

for any $i$ such that $\ell(s_i w) = \ell(w) - 1$, where $N = (\alpha_i^\vee, s_i w\lambda)$ and $\ell(w)$ is the the Coxeter length $w$.

Assume that $k$ has an involutive automorphism $c \mapsto \overline{c}$ such that $q^\frac{t}{2} = q^{-\frac{t}{2}}$. We extend it uniquely to $U_q^+$ by $\overline{x_i} = x_i$ for $i \in I$ and

$$\overline{x \cdot y} = \overline{y} \cdot \overline{x}$$

for $x, y \in U_q^+$. We further extend the bar-involution to $U_q(b_+)^*$ uniquely by

$$\overline{v_\lambda \cdot x} = x \cdot \overline{v_\lambda}$$

for each $\lambda \in \mathcal{P}$, $x \in U_q^+$.

The following fact is a direct consequence of Lemma 6.1.

**Lemma 6.2.** For each $y \in U_q(b_+)^*$ and $i \in I$ one has:

$$\overline{F_i K_i^{-\frac{1}{2}}(y)} = F_i K_i^{-\frac{1}{2}}(\overline{y}) \quad \text{and} \quad \overline{K_i^{-\frac{1}{2}} E_i(y)} = K_i^{-\frac{1}{2}} E_i(\overline{y}) .$$

Furthermore, we define the generalized quantum minor $\Delta_{w\lambda} \in U_q^+$ by

$$\Delta_{w\lambda} := q^{-\frac{1}{2}(w\lambda, -\lambda, \lambda)} v_{w\lambda} \cdot v_{\lambda}^{-1} .$$
Lemma 6.3. For any dominant \( \lambda \in \mathcal{P} \) and \( w \in W \), both \( v_w \lambda \) and \( \Delta_w \lambda \) are bar-invariant.

Proof. We will proceed by induction in the length \( \ell(w) \) of \( w \in W \). Indeed, for \( \ell(w) = 0 \), i.e. \( w = 1 \), we have nothing to prove as \( \overline{v_\lambda} = v_\lambda \) by definition. Suppose that \( \ell(w) > 1 \), fix \( i \in I \) such that \( \ell(s_i w) < \ell(w) \). We proceed by induction in \( \ell(w) \). By (6.3) and Lemma 6.2,

\[
\overline{v_{s_i w} \lambda} = (F^\pm_i K^\pm_i)^{[N]}(v_{s_i w} \lambda) = (F^\pm_i K^\pm_i)^{[N]}(\overline{v_{s_i w} \lambda}) = (F^\pm_i K^\pm_i)^{[N]}(v_{s_i w} \lambda) = v_{s_i w} \lambda.
\]

Finally,

\[
\Delta_{w \lambda} = q^{\frac{1}{2}((w \lambda - \lambda) \cdot \overline{v_{w \lambda}})} v_{w \lambda} = q^{\frac{1}{2}((w \lambda - \lambda) \cdot \overline{v_{w \lambda}})} v_{w \lambda} = q^{\frac{1}{2}((w \lambda - \lambda) \cdot v_{w \lambda} \cdot \overline{v_{w \lambda}})} = \Delta_{w \lambda}
\]

because \( v_{\lambda}^{-1} = v_{-\lambda} \), \( |v_{w \lambda}| = w \lambda \), and \( v_{\lambda} \cdot v_{w \lambda} = q^{(w \lambda - \lambda) \cdot \overline{v_{w \lambda}} \cdot \overline{v_{w \lambda}}} \) by (6.2). □

The following is an analogue of [3, (10.2)], it also follows from [3 Lemma 10.3].

Lemma 6.4. If \( w, w' \in W \) such that \( \ell(w w') = \ell(w) + \ell(w') \), then for any \( \lambda, \mu \in \mathcal{P}^+ \) one has:

\[
v_{w \mu} \cdot v_{w' \lambda} = q^{(w' \lambda - \lambda, \mu)} v_{w w' \lambda} \cdot v_{w \mu}.
\]

We conclude the section with an explicit recursion on generalized quantum minors.

Proposition 6.5. For each \( \lambda \in \mathcal{P}^+ \), \( w \in W \), and \( i \in I \) such that \( \ell(s_i w) < \ell(w) \), one has:

\[
v_{w \lambda} = \frac{q_i^N}{(q_i - q_i^{-1})^N} \sum_{k=0}^{N} x_i^{[N-k]} \cdot v_{s_i w \lambda} \cdot (-q_i^{-1} x_i)^{[k]},
\]

(6.4)

\[
\Delta_{w \lambda} = \frac{q_i^N}{(q_i - q_i^{-1})^N} \sum_{k=0}^{N} (q_i^{(\alpha_i^\vee, \lambda)} - x_i^{[N-k]}) \Delta_{s_i w \lambda} \cdot (-q_i^{-1} x_i)^{[k]},
\]

(6.5)

where \( N = -(\alpha_i^\vee, w \lambda) \).

Proof. This is a direct consequence of the following result.

Lemma 6.6. For each \( i \in I \), \( y \in U_q(\mathfrak{b}_+)^* \) and \( N \geq 0 \) one has:

\[
F^N_i(y) = (q_i - q_i^{-1})^{-N} \sum_{k=0}^{N} x_i^{[N-k]} \cdot K_i^{-k}(y) \cdot (-q_i^{-1} x_i)^{[k]},
\]

(6.6)

In particular, if \( y \) is a homogeneous element of \( U_q(\mathfrak{b}_+)^* \), then

\[
F^N_i(y) = (q_i - q_i^{-1})^{-N} \sum_{k=0}^{N} x_i^{[N-k]} \cdot y \cdot (-q_i^{-M-1} x_i)^{[k]},
\]

where \( M = (\alpha_i^\vee, |y|) \).

Proof. We proceed by induction in \( N \). If \( N = 0 \) we have nothing to prove. If \( N = 1 \), then the assertion follows from Lemma 6.1. The induction step follows easily from the binomial identity

\[
q_i^{-k} \binom{N}{k} + q_i^{N+1-k} \binom{N}{k-1} = q_i^{N+1} \binom{N+1}{k},
\]
where \[ \binom{N}{k} = \frac{[N]_{w\lambda}!}{[k]_{w\lambda}![N-k]_{w\lambda}!}. \]

In particular, if \( \ell(s_j w) < \ell(w) \), then taking \( y = v_{s_j w\lambda} \) and \( N = M = (\alpha^\vee, s_j w\lambda) = - (\alpha^\vee, w\lambda) \) in \((6.6)\), we obtain:

\[
(6.7) \quad F_i^{[N]}(v_{s_j w\lambda}) = (q_i - q_i^{-1})^{-N} \sum_{k=0}^{N} x^{[N-k]}_i \cdot v_{s_j w\lambda} \cdot (-q_i^{-1} x_i)^{[k]}.
\]

Let \( F'_i := F_i K_i^{\frac{1}{2}} \). Taking into account that \( F_i^{[N]} = (F'_i K_i^{\frac{1}{2}})^N = q_i^{\frac{N(N+1)}{2}} F'_i^{N} K_i^{-\frac{N}{2}} \), \((6.3)\) becomes:

\[
F_i^{[N]}(v_{s_j w\lambda}) = q_i^{\frac{N(N+1)}{2}} F'_i^{[N]} K_i^{-\frac{N}{2}} (v_{s_j w\lambda}) = q_i^{-\frac{N}{2}} F'_i^{[N]} (v_{s_j w\lambda}) = q_i^{-\frac{N}{2}} v_{w\lambda}.
\]

Combining this with \((6.7)\), we obtain \((6.4)\).

Prove \((6.5)\) now. Indeed, using \((6.4)\) we obtain

\[
\Delta_{w\lambda} = q^{-\frac{1}{2}(w\lambda - s_j w\lambda)} v_{w\lambda} \cdot v_{\lambda}^{-1} = \frac{q_i^{\frac{N}{2}}}{(q_i - q_i^{-1})^N} \sum_{k=0}^{N} x^{[N-k]}_i \cdot v_{s_j w\lambda} \cdot (-q_i^{-1} x_i)^{[k]} v_{\lambda}^{-1}
\]

\[
= \frac{q_i^{\frac{N}{2}}}{(q_i - q_i^{-1})^N} \sum_{k=0}^{N} x^{[N-k]}_i \cdot q^{-\frac{1}{2}(w\lambda - s_j w\lambda)} \Delta_{s_j w\lambda} \cdot (-q_i^{-1} x_i)^{[k]} v_{\lambda}^{-1}
\]

\[
= \frac{q_i^{\frac{N}{2}}}{(q_i - q_i^{-1})^N} \sum_{k=0}^{N} x^{[N-k]}_i \cdot q^{-\frac{1}{2}(w\lambda - s_j w\lambda + 2k\alpha_i, \lambda)} \Delta_{s_j w\lambda} \cdot (-q_i^{-1} x_i)^{[k]} v_{\lambda}^{-1}
\]

\[
= \frac{q_i^{\frac{N}{2}}}{(q_i - q_i^{-1})^N} \sum_{k=0}^{N} \left( q_i^{\frac{(\alpha^\vee, \lambda)}{2}} x_i \right)^{[N-k]} \Delta_{s_j w\lambda} \cdot (-q_i^{-1} \cdot q_i^{\frac{(\alpha^\vee, \lambda)}{2}} x_i)^{[k]} v_{\lambda}^{-1}
\]

where we used the fact that \( w\lambda - s_j w\lambda + N\alpha_i = 0 \). This proves \((6.5)\).

The proposition is proved. \(\square\)

### 6.2. Quantum Cluster Algebras

Fix a field \( k \) of characteristic zero, a natural number \( m \) and an \( n \)-element subset \( \text{ex} \) of \( \{1, \ldots, m\} \) \( (n \leq m) \) and consider a pair \((\chi, \tilde{B})\), where \( \chi : \mathbb{Z}^m \times \mathbb{Z}^m \to k^\times \) is a unitary bicharacter of \( \mathbb{Z}^m \), and \( B = (b^k \mid k \in \text{ex}) \) is an \( m \times \text{ex} \) matrix. Following [4], we say that the pair \((\chi, \tilde{B})\) is compatible if (cf. [5.3]):

\[
\chi(b^k, e_i) = q_k^{\delta_{ik}}
\]

for all \( k \in \text{ex}, \ i = 1, \ldots, m \) where \( q_k \in k^\times, \ k \in \text{ex} \) are some non-roots of unity.

Fix a skew-field \( F \). A quantum seed in \( F \) is a pair \((X, \tilde{B})\), where

- \( X = \{X_1, \ldots, X_m\} \) is (an algebraically independent) generating set of \( F \) such that:

\[
X_i X_j = q_{ij} X_j X_i
\]

for all \( i, j = 1, \ldots, m \) for some \( q_{ij} \in k^\times \) satisfying \( q_{ii} = 1 \), \( q_{ij} q_{ji} = 1 \) for all \( i, j = 1, \ldots, m \);
• \( \tilde{B} = (b_{ij}) \) is an integer \( m \times n \) matrix such that the pair \((\chi, \tilde{B})\) is a compatible, where \( \chi \) is the unique unitary bicharacter of \( \mathbb{Z}^m \) such that \( \chi(\varepsilon_i, \varepsilon_j) = q_{ij} \) for all \( i, j \in [1, m] \).

Furthermore, we refer to \( X \) as a quantum cluster, and for each \( a = (a_1, \ldots, a_m) \in \mathbb{Z}^m \) denote:

\[
X^a := \prod_{1 \leq i < j \leq m} q_{ij}^{|a_i - a_j|} X_1^{a_1} \cdots X_m^{a_m}.
\]

If \( a_1, \ldots, a_n \) are nonnegative integers, we refer to \( X^a \) as cluster monomial. By construction,

\[
X^a \cdot X^b = \chi(a, b)X^{a+b},
\]

for all \( a, b \in \mathbb{Z}^m \), that is, the subalgebra of \( \mathcal{F} \) generated by \( X_1, \ldots, X_m \) is isomorphic to the based quantum torus \( \mathcal{T}_\chi \) defined in (2.8).

To each quantum seed we associate the upper quantum cluster algebra \( \mathcal{U}(X, \tilde{B}) \) by the formula:

\[
\mathcal{U}(X, \tilde{B}) = \bigcap_{j \in \text{ex}} k[X_1^{\pm 1}, \ldots, X_{j-1}^{\pm 1}, X_j, X_j', X_j^{\pm 1}, \ldots, X_m^{\pm 1}],
\]

(here \( k[S] \) denotes the subalgebra of \( \mathcal{F} \) generated by \( S \) where \( X_i = X^{\varepsilon_i} \) for all \( i \) and

\[
X_i' = X_i^{b_i^j + \varepsilon_k} + X_i^{b_i^j - \varepsilon_k}
\]

for \( k \in \text{ex} \).

Given a quantum seed \((X, \tilde{B})\), a \( k \)-th mutation \( \mu_k(X, \tilde{B}) \) is a pair \((X', \tilde{B}')\), where

• \( X' = X \setminus \{X_k\} \cup \{X_k'\} \), where \( X_k' \) is given by (6.11);

• \( \tilde{B}' = \mu_k(\tilde{B}) \) is given by \( b_{ij}' = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{|b_{ik}b_{kj} + b_{ik}b_{kj}|}{2} & \text{otherwise} \end{cases} \).

It is easy to show that each mutation \( \mu_k(X, \tilde{B}) \) is also a quantum seed.

**Theorem 6.7 (\([5]\) Theorem 5.1).** For any quantum seed \((X, \tilde{B})\) and \( k \in \text{ex} \) one has:

\[
\mathcal{U}(\mu_k(X, \tilde{B})) = \mathcal{U}(X, \tilde{B}),
\]

i.e., the upper quantum cluster algebra does not depend on the choice of a quantum seed in a mutation-equivalence class.

In particular, one immediately obtains the quantum Laurent phenomenon: each cluster \( X' \) of each quantum seed \( \mu_{j_1} \cdots \mu_{j_l}(\Sigma) \) belongs to \( \mathcal{U} \).

In fact, there is another way to modify a seed \((X, B)\) such that the upper cluster algebra does not change. The following is obvious.

**Lemma 6.8.** Let \( \rho \) be a \( \mathbb{Z} \)-linear automorphism of \( \mathbb{Z}^m \) such that \( \rho(\varepsilon_k) = \delta_{k\ell} \varepsilon_\ell \) for all \( k \in \text{ex} \) and \( \ell \in [1, m] \). Then for any quantum seed \((X, \tilde{B})\) one has:

(a) The pair \((\rho(X), \rho^{-1}(\tilde{B}))\) is a quantum seed, where \( \rho(X) := \{X^{\rho(\varepsilon_1)}, \ldots, X^{\rho(\varepsilon_m)}\} \) and \( \rho^{-1}(\tilde{B}) \) is obtained by applying \( \rho^{-1} \) to each column \( b^k \) of \( \tilde{B} \).

(b) \( \mathcal{U}(\rho(X), \rho^{-1}(\tilde{B})) = \mathcal{U}(X, \tilde{B}) \).

Given a quantum seed \( \Sigma = (X, \tilde{B}) \), define the lower cluster algebra \( \mathcal{A}_\Sigma \), to be the subalgebra of \( \mathcal{U} \) generated by \( X \) and by the neighboring cluster variables \( X_j', j \in \text{ex} \) given by (6.11).
Theorem 6.9 ([5]). \( A_c = \mathcal{U}(\Sigma) \) if and only if the seed \( \Sigma \) is acyclic, i.e., there exists an ordering \((j_1, \ldots, j_n)\) of \( \mathbf{e} \) such that \( b_{j_p, j_p'} \geq 0 \) for all \( 1 \leq p < p' \leq n \).

Although the quantum Laurent phenomenon guarantees that each cluster variable is an element of \( \mathcal{T}_\chi \), it is a non-trivial task to compute their initial cluster expansions.

The main result from [30] solves this Laurent problem in some cases via the quantum cluster characters (5.6). To state the result, we need the following obvious fact.

Lemma 6.10. Given a finitary abelian category \( C \), a collection \( S = \{S_i, i \in 1, \ldots, m\} \) of simple objects without self-extensions, a subset \( \mathbf{e} \) of \( \{1, \ldots, m\} \), and a compatible unitary bicharacter \( \chi_C \) of \( \mathcal{K}(C) \) in the sense of (5.5). Then the pair \( \Sigma_C = (X, \tilde{B}) \) is a quantum seed in \( \mathcal{T}_{\chi_C} \), where:

- \( X = \{X^{[S_1]}, \ldots, X^{[S_m]}\} \) is an initial cluster in the quantum torus \( \mathcal{T}_{\chi_C} \),
- \( \tilde{B} \) is the restriction of the matrix \( B_C \) given by (5.2) to \( m \times \mathbf{e} \).

By definition, for each \( V \in C \) its cluster character \( X_V \) given by (5.6) belongs to \( \mathcal{T}_{\chi_C} \). The following result is essentially [30, Theorem 1.1].

Theorem 6.11. In the notation of Lemma 6.10, for each valued quiver \( (\tilde{Q}, \tilde{d}) \) whose restriction to \( \mathbf{e} \) is acyclic, the quantum cluster character \( V \mapsto X_V \) is a bijection between indecomposable exceptional representations \( V \in C \) supported in \( \mathbf{e} \) and non-initial quantum cluster variables of \( \mathcal{U}(\Sigma_C) \).

6.3. Quantum i-seeds and i-characters. We construct quantum i-seeds in \( \mathcal{L}_i \) for symmetrizable \( I \times I \) Cartan matrices \( A = (a_{ij}) \) with \( a_{ii} = 2 \) and all sequences \( i \in I^m \). Indeed, given a non-root of unity \( q^{\frac{1}{k}} \in \mathbb{k} \), define the unitary bicharacter \( \chi_i : \mathbb{Z}^m \times \mathbb{Z}^m \to \mathbb{k}^\times \) by:

\[
\chi_i(a, b) = q^{\Lambda_i(a, b)},
\]

where \( \Lambda_i \) is the bilinear form given by (4.6). The following fact is obvious (cf. (2.1)).

Lemma 6.12. For any \( A \) as above and any \( i \in I^m \) the algebra \( \mathcal{L}_i \) generated by \( t_1, \ldots, t_m \) subject to the relations

\[
t_{\ell}t_k = q^{\epsilon_{i_{\ell}}t_{\epsilon_{k}}}t_kt_{\ell},
\]

for all \( 1 \leq k < \ell \leq m \), is a based quantum torus with the bicharacter \( \chi_i \). In particular,

\[
t_a^bt^n = q^{\Lambda_i(a, b)}t^b t^a
\]

for all \( a, b \in \mathbb{Z}^m \).

The following result is a direct corollary of Theorem 4.4.

Corollary 6.13. For any \( A \) as above and any \( i \in I^m \) the pair \( \Sigma_i = (X_i, \tilde{B}_i) \) is a quantum seed in \( \mathcal{L}_i \), where:

- \( X_i := \{t^{\epsilon_{1}}(\varepsilon_1), \ldots, t^{\epsilon_{m}}(\varepsilon_m)\} \subset \mathcal{L}_i \) is a cluster in \( \mathcal{L}_i \),
- \( \tilde{B}_i \) is the \( m \times \mathbf{e} \) matrix given by (4.7) (and \( \mathbf{e} = \mathbf{ex}_i = \{k \in [1, m] \mid k^+ \leq m\} \)).
In what follows, we denote by $\mathcal{U}_i$ the corresponding upper cluster algebra $\mathcal{U}(\Sigma_i)$. We will mostly deal with the case when $A$ is a Cartan matrix and $i$ is reduced.

**Proposition 6.14.** For any sequence $i = (i_1, \ldots, i_m) \in I^m$ and any homogeneous $x \in U^+_i$ one has (in the notation of Lemma 6.1):

\[(6.12) \quad \Psi_i(x) = \sum (K_m^{-\frac{\overline{a}}{2}} E_1)^{[a_m]} \cdots (K_1^{-\frac{\overline{a}}{2}} E_1)^{[a_1]}(x) \cdot t^a,
\]
where the summation is over all $a = (a_1, \ldots, \gamma_m) \in \mathbb{Z}_{\geq 0}^m$ such that $a_1 \alpha_{i_1} + \cdots + a_m \alpha_{i_m} = -|x|$.

**Proof.** Indeed, Lemmas 3.6 and 6.1 imply that $K_{S_i}|_{U^+_i} = K_i$ and $\mathcal{O}_{S_i}|_{U^+_i} = K_i^{-\frac{\overline{a}}{2}} E_i$. Therefore, (6.12) directly follows from the second equation (3.9).

In the notation (6.12) for each $i \in I^m$ and $\lambda \in \mathcal{P}$ define the monomial $t_\lambda = t_{\lambda, i} \in \mathcal{L}_i$ by

\[(6.13) \quad t_\lambda := t^a_\lambda.
\]

Lemma 4.12 and Proposition 4.13 imply the following corollary.

**Lemma 6.15.** In the notation of Lemma 4.12, for each $i \in I^m$ one has:

(a) For each $\ell \in [1, m] \setminus \text{ex}$ the coefficient $X_\ell$ of the quantum cluster $X_i$ equals $t_{\omega_{i, \ell}}$.

(b) $t^a : t_\lambda = v(|a|, \omega_{\lambda, \lambda}) t_{\lambda} : t^a$ for all $\lambda \in \mathcal{P}$ and $a = (a_1, \ldots, a_m) \in \mathbb{Z}^m$.

(c) $t^a : t_\lambda = v(w_{m, \mu}, w_{m, \lambda}) t_{\lambda} : t^a$ for all $\lambda, \mu \in \mathcal{P}$.

In view of (3.9) and (6.12), the following result essentially coincides with [1] Theorem 3.1.

**Proposition 6.16.** $\Psi_i(\Delta_{w, \lambda}) = t_\lambda^{-1}$ for any reduced word $i$ of $w \in W$ and any $\lambda \in \mathcal{P}^+$.

This and Lemma 6.15(b) imply that $\{t_\lambda, \lambda \in \mathcal{P}^+\} \subset \Psi_i(U^+_i)$ and $\{t_\lambda^{-1}, \lambda \in \mathcal{P}^+\}$ is an Ore set in $\Psi_i(U^+_i)$. It is also clear from Lemma 6.15(c) that $t_\lambda \in \mathcal{U}_i$ for all $\lambda \in \mathcal{P}$.

This and the following result provide some partial evidence to Conjecture 2.12(a).

**Proposition 6.17.** For each reduced word $i$ of $w \in W$ such that $m^\sim \not= 0$ the cluster variables $X_1$ and $X_{m^-}$ of the seed $\Sigma_i$ belong to the localization of $\Psi_i(U^+_i)$ by the Ore set $\{t_\lambda^{-1}, \lambda \in \mathcal{P}^+\}$.

**Proof.** The following version of [1] Proposition 2.5 is immediate.

**Lemma 6.18.** Let $i \in I^m$ and $x \in U^+_i$ be such that $\Psi_i(x) \in k^x \cdot t_1^{a_1} \cdots t_m^{a_m}$ for some $a_1, \ldots, a_m \geq 0$.

• If $a_1 > 0$, then (in the notation of (3.6)) $\Psi_i(\partial_{S_{i, m}}^p(x)) \in k^x \cdot t_1^{a_1-1} \cdots t_m^{a_m}$.

• If $a_m > 0$, then $\Psi_i(\partial_{S_{i, m}}^p(x)) \in k^x \cdot t_1^{a_1} \cdots t_m^{a_m-1}$.

Proposition 6.16 guarantees that the lemma is applicable to each $x = \Delta_{w, \lambda}$ for $w \in W$, $\lambda \in \mathcal{P}^+$ so that $(a_1, \ldots, a_m) = -a_{\lambda, \lambda}$, e.g., $a_1 = -(\alpha_{i, \lambda}, w_{w, \lambda}) = -(w^{-1}a_{\lambda, \lambda})$ and $a_m = (\alpha_{i, \lambda}, \lambda)$. Clearly, one can choose $\lambda$ in such a way that $a_1 > 0$ and $a_m > 0$. Then, multiplying both $\Psi_i(\partial_{S_{i, m}}^p(\Delta_{w, \lambda}))$ and $\Psi_i(\partial_{S_{i, m}}(\Delta_{w, \lambda}))$ by $t_\lambda = t^a$, we obtain

$$ t_1^{-1}, t_m^{-1} \in \Psi_i(U^+_i)[t_{\lambda}, \lambda \in \mathcal{P}^+] $$
This, in particular, proves the first assertion of the proposition. To prove the second one, note that
\[
\varphi_k^{-1}(\varepsilon_m) = -\sum_{k=1}^{m-1} (w_k\alpha_k^\vee, w_{m-k}\omega_i^\vee)\varepsilon_k = aw_m - \omega_i + \sum_{k=m+1}^{m} (\alpha_k^\vee, w_{k-1}^{-1}w_{m-1}\omega_i^\vee)\varepsilon_k = aw_{s_1\omega_i^\vee} - \varepsilon_m
\]
by (4.2) and (4.17), where we used that \(w_{m-k}\omega_i^\vee = w_{m-k-1}\omega_i = ws_i\omega_i^\vee\) and \(w_{k-1}^{-1}w_{m-1}\omega_i^\vee = \omega_i\) for \(m < k < m\). Therefore
\[
X_{m-} = t^{a w_{s_1\omega_i^\vee} - \varepsilon_m} = q^r t_m^{-1} t_{w_{s_1\omega_i^\vee}}
\]
for some \(r \in \mathbb{Z}\) hence \(X_{m-} \in \Psi_1(U_+^+)_{[\lambda, \lambda] \in \mathcal{P}}\).

The proposition is proved. 

We conclude the section by constructing another important seed \(\hat{\Sigma}_i = (\hat{X}_i, \hat{B}_i)\) in \(\mathcal{F} = \text{Frac}(\mathcal{L}_i)\).

In the notation of Section 6.1 for any \(i \in \ell\) define the subset \(\hat{X}_i = \{\hat{X}_1, \ldots, \hat{X}_m\}\) of \(\ell\) by

\[
(6.14) \quad \hat{X}_k := \Psi_{1}(\Delta_{s_1 \cdots s_k \omega_k})
\]

for \(k = 1, \ldots, m\). It follows from Proposition (6.10) and Lemma (4.12e) that the coefficients \(\hat{X}_\ell\) \((\ell \in [1, m] \setminus \mathbf{ex})\) are inverse to the coefficients of the initial seed \(\Sigma_i\).

**Lemma 6.19.** If \(i\) is a reduced word for \(w \in \mathcal{W}\), then \(\hat{X}_i\) is a quantum cluster for \(\text{Frac}(\mathcal{L}_i)\) with

\[
(6.15) \quad \hat{X}_k \hat{X}_\ell = q^{X_i(\varphi_i^{-1}(\varepsilon_k), \varphi_i^{-1}(\varepsilon_\ell))} \hat{X}_\ell \hat{X}_k
\]

for all \(1 \leq k, \ell \leq m\). In particular, the assignment \(X_i \mapsto \hat{X}_i\), \(k = 1, \ldots, m\) defines a homomorphism

\[
(6.16) \quad \hat{\eta}_i : \mathcal{L}_i \rightarrow \text{Frac}(\mathcal{L}_i)
\]

**Proof.** It follows from Lemma (6.4) that for any \(w, w' \in \mathcal{W}\) such that \(\ell(ww') = \ell(w) + \ell(w')\) and for any \(\lambda, \mu \in \mathcal{P}\) one has:

\[
\Delta_{w_\mu} \cdot v_{ww'} = q^{-\frac{1}{2}(ww' - \lambda, \lambda)} q^{(ww' - \lambda, \lambda)} \Delta_{w_\mu}.
\]

Since \(|\Delta_{w_\mu}| = w_\mu - \mu\), multiplying both sides on the right by \(q^{-\frac{1}{2}(ww' - \lambda, \lambda)} v_{\lambda}^{-1}\) and applying (6.2) we obtain:

\[
\Delta_{w_\mu} \cdot \Delta_{ww'} = q^{-\frac{1}{2}(ww' - \lambda, \lambda)} q^{(ww' - \lambda, \lambda)} \Delta_{ww'} \Delta_{w_\mu} = q^{(ww' - \lambda, \lambda, \mu - \lambda)} \Delta_{ww'} \Delta_{w_\mu}.
\]

For \(w = w_k, w_k = w_\ell, \mu = w_\ell, k \leq \ell\), this, taken together with (4.11) gives (6.15). The second assertion follows from the fact that \(\mathcal{L}_i\) is generated by \(X_i = \{X_1, \ldots, X_m\}\) subject to the relations (6.15). The lemma is proved. 

The following is a particular case of [5, Conjecture 10.10].

**Conjecture 6.20.** Under the hypotheses of Lemma 6.19 the restriction of \(\hat{\eta}_i \circ \hat{\mathcal{U}}_{\mathcal{L}_i}\) is an isomorphism of algebras \(\hat{\eta}_i : \hat{\mathcal{U}}_{\mathcal{L}_i} \rightarrow \hat{\mathcal{L}}_i\).

The following obvious fact gives a possible line of attack on the conjecture.
Lemma 6.21. Let \( \Sigma = (\mathbf{X}, \hat{\mathbf{B}}) \) be a quantum seed in a skew-field \( \mathcal{F} \) and let \( \hat{\Sigma} = (\hat{\mathbf{X}}, \hat{\mathbf{B}}) \) be a quantum seed in a skew-field \( \hat{\mathcal{F}} \). Assume that the corresponding unitary bicharacters \( \chi, \hat{\chi} \) are equal. Then:

(a) The assignment \( X_k \mapsto \hat{X}_k, k = 1, \ldots, m \) defines an isomorphism of algebras \( \eta : \mathcal{U}(\Sigma) \to \mathcal{U}(\hat{\Sigma}) \).

(b) For any other seed \( \Sigma' = (\mathbf{X}', \hat{\mathbf{B}}') \) the seed \( \eta(\Sigma') = (\eta(\mathbf{X}'), \hat{\mathbf{B}}') \) satisfies \( \mathcal{U}(\eta(\Sigma')) = \mathcal{U}(\Sigma) \).

We finish the section with the brief listing of properties of \( \Psi_i \) from [1], see also [5, Section 9.3].

Lemma 6.22. For any reduced word \( \mathbf{i} = (i_1, \ldots, i_m) \) for \( w \in W \) one has:

(a) the kernel of \( \Psi_i \) is the orthogonal complement \( \ker \Psi_i \) of all monomials \( [S_{i_1}]^{a_1} \cdots [S_{i_m}]^{a_m} \), \( a_1, \ldots, a_m \in \mathbb{Z}_{\geq 0} \).

(b) \( \mathcal{E}_i = \mathcal{E}_i' \) for any other reduced word \( \mathbf{i}' \) for \( w \).

(c) The images \( \Delta_{\eta^\mathbf{i}} \) of \( \Delta_{\eta^\mathbf{i}} \lambda, \lambda \in \mathcal{P}^+ \) in \( U^+_\mathbf{Q} / \ker \Psi_i \) form an Ore set.

(d) The localization \( (U^+_\mathbf{Q}/\ker \Psi_i)[\Delta_{\eta^\mathbf{i}}^{-1}, \lambda \in \mathcal{P}^+] \) is \( \mathbb{K}_q [N^w] \).

(e) \( \Psi_i(\Delta_{\eta^\mathbf{i}}) \) is \( \mathbb{K}^\times \cdot t_{\lambda}^{-1} \) for \( \lambda \in \mathcal{P}^+ \), where \( t_{\lambda} \in \mathcal{L}_1 \) is defined by (6.13).

This implies that \( \Psi_i \) defines an injective homomorphism of algebras:

\[
(6.17) \quad \Psi_i : \mathbb{K}_q [N^w] \to \mathcal{L}_1.
\]

7. Proof of Theorems 2.9, 2.10, and 2.11

7.1. Proof of Theorem 2.9. Let \( A \) be an \( n \times n \) symmetrizable Cartan matrix with symmetrizing matrix \( D \). Let \( (Q, \mathbf{d}) \) be any acyclic valued quiver \( \mathcal{H}(\mathcal{C}) \) and \( \mathcal{H}^\ast(\mathcal{C}) \) of the \( \mathbb{K} \)-linear span \( \mathcal{E} \) of all monomials \( [S_{i_1}]^{a_1} \cdots [S_{i_m}]^{a_m}, a_1, \ldots, a_m \in \mathbb{Z}_{\geq 0} \).

- \( A = A_{\mathbf{S}} \) as in (5.1), where \( \mathbf{S} = \{S_1, \ldots, S_n\} \) are simple representations of \( (Q, \mathbf{d}) \);
- \( \mathbf{i}_0 = (1, 2, \ldots, n) \) is a repetition-free source adapted sequence for \( (Q, \mathbf{d}) \);
- \( |\mathcal{E}| = q^{n^2} \).

This, in particular, implies that \( A = 2 \cdot I_n - [B_Q]_+ - [-B_Q]_+ \).

It is convenient to slightly modify the initial seed \( \Sigma_i = (\mathbf{X}_i, \hat{\mathbf{B}}_i) \) (defined in Corollary 6.13) for \( i = (\mathbf{i}_0, \mathbf{i}_0) \) as follows.

Denote \( \Sigma_i^{\pm} = (\mathbf{X}_i^{\pm}, \hat{\mathbf{B}}_i^{\pm}) \), where \( \mathbf{X}_i^{\pm} = \rho_i^{\pm} (\mathbf{X}_i) = \{ t^{\epsilon_1 - \epsilon_i} \rho_i^{\pm} (\epsilon_1), \ldots, t^{\epsilon_n - \epsilon_i} \rho_i^{\pm} (\epsilon_n) \} \) and \( \hat{\mathbf{B}}_i^{\pm} = \left( \begin{array}{c} \mathbf{B}_Q \\ \pm I_n \end{array} \right) \).

Therefore, Lemma 6.8 guarantees that both \( \Sigma_i^{\pm} \) and \( \Sigma_i^{-} \) are quantum seeds for \( \mathcal{U}_i \), i.e., \( \mathcal{U}_i = \mathcal{U}(\Sigma_i^{\pm}) = \mathcal{U}(\Sigma_i^{-}) \).

We will now prove Theorem 2.9(a), i.e., show that \( \Psi_{(\mathbf{i}_0, \mathbf{i}_0)}(U^+_\mathbf{Q}) \subset \mathcal{U}_i \).

Recall that \( U^+_\mathbf{Q} \) is generated by \( [S_i]^+ \), \( i \in [1, n] \) and it follows from Lemma 6.9 that \( \Psi_{(\mathbf{i}_0, \mathbf{i}_0)}([S_k]^+) = t_k + t_{k+n} \) for \( k = 1, \ldots, n \). Thus it suffices to prove the following result.

Lemma 7.1. Let \( \mathbf{i} = (\mathbf{i}_0, \mathbf{i}_0) \). Then \( t_k + t_{k+n} \in \mathcal{U}_i \) for \( 1 \leq k \leq n \).

Proof. Since each coefficient \( X_{n+k}^{\pm} \) belongs to \( \mathcal{U}_i \) by Lemma 4.12(c) and Proposition 6.18, it suffices to show that the \( k \)-th mutation of the quantum seed \( \Sigma_i^{-} \) results in:

\[
(7.1) \quad X_k^{\pm} = t_k + t_{k+n}
\]
for \( k = 1, \ldots, n \).

Indeed, since \( \tilde{B}^-_1 = (b_1^-, \ldots, b_n^-) \) is the \([1, 2n] \times [1, n]\) submatrix of \( B^-_Q \), the equation \((7.1)\) under the identification \( |S_k| := \varepsilon_k \) implies that

\[
\varepsilon_k = \varepsilon_k - [b_k^+]_+, \quad \varepsilon_k^* = \varepsilon_k - [-b_k^-]_+
\]

for \( k = 1, \ldots, n \). Combining this with Lemma 5.6 we obtain:

\[
\rho_i^{-1} \varphi_i(\varepsilon_k) = -\varepsilon_k = [b_k^+]_+ - \varepsilon_k, \quad \rho_i^{-1} \varphi_i(\varepsilon_{n+k}) = -\varepsilon_k^* = [-b_k^-]_+ - \varepsilon_k.
\]

Hence

\[
(\rho_i^{-1} \varphi_i)^{-1}([b_k^+]_+ - \varepsilon_k) = \varepsilon_k, \quad (\rho_i^{-1} \varphi_i)^{-1}([-b_k^-]_+ - \varepsilon_k) = \varepsilon_{n+k}
\]

for \( k = 1, \ldots, n \). In turn, this, implies

\[
X_k' = \rho_i^{-1} \varphi_i([b_k^+]_+ - \varepsilon_k) + \rho_i^{-1} \varphi_i([-b_k^-]_+ - \varepsilon_k) = t_k + t_{n+k}.
\]

This proves \((7.1)\) and the lemma.

Therefore, Theorem 2.9(a) is proved.

Prove Theorem 2.9(b) now. Indeed, \( \Sigma^- = \Sigma^- \) in the notation of Lemma 6.10 where \( \tilde{C} = \text{rep}_F(\tilde{Q}, \tilde{d}) \) and \( (\tilde{Q}, \tilde{d}) \) is the valued quiver on \( 2n \) vertices as in \((5.7)\).

Since \( (Q, d) \) is the (acyclic) restriction of \( (\tilde{Q}, \tilde{d}) \) to \( \text{ex} = \{1, \ldots, n\} \), both Theorem 2.5 and Theorem 6.11 are applicable and combining them we obtain that the assignment

\[
V \mapsto \Psi_1([V]^*)
\]

is a bijection between all exceptional representations of \( (Q, d) \) and all non-initial quantum cluster variables in \( \mathcal{U}_i = \mathcal{U}(\Sigma^{-}_i) \). Therefore, Theorem 2.9(b) is proved.

It remains to prove Theorem 2.9(c).

We need the following result.

**Proposition 7.2.** Assume that \( i = (i_0, i_0) \) is reduced. Then there exists an acyclic quantum seed \( \Sigma' \) such that all cluster variables of \( \Sigma' \) and \( \mu_j(\Sigma') \), \( j = 1, \ldots, n \) belong to \( \Psi_1(U_i^+) \).

**Proof.** The essential ingredient is contained in the following result.

**Lemma 7.3.** In the assumptions of Proposition 7.2, for \( k \in [1, n - 1] \) let

\[
\Sigma^{(k)} = \mu_k \cdots \mu_1 \mu_n \cdots \mu_1(\Sigma^-_i)
\]

Then for \( j \in [1, n] \) one has:

(a) The \( j \)-th cluster variable \( X_j^{(k)} \), \( j \in [1, 2n] \) of \( \Sigma^{(k)} \) is homogeneous and

\[
|X_j^{(k)}| = \begin{cases} 
  cs_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}) & \text{if } 1 \leq j \leq k \\
  s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}) & \text{if } k + 1 \leq j \leq n \\
  \alpha_{i_j} + c\alpha_{i_j} & \text{if } j > n
\end{cases}
\]

where \( c = s_{i_1} \cdots s_{i_n} \) is the corresponding Coxeter element of \( W \).
(b) The $j$-th cluster variable $X_j^{(k)}$, $j \in [1,n]$ of $\mu_j(\Sigma^{(k)})$ is homogeneous and

\begin{equation}
|X_j^{(k)}| = \begin{cases} 
- s_{i_1} \cdots s_{i_k} (\alpha_{i_j}) & \text{if } s_{i_1} \cdots s_{i_k} (\alpha_{i_j}) < 0 \\
-c s_{i_1} \cdots s_{i_k} (\alpha_{i_j}) & \text{if } s_{i_1} \cdots s_{i_k} (\alpha_{i_j}) > 0
\end{cases},
\end{equation}

**Proof.** The first two cases of part (a) are a direct application of the main result from \cite{29}. To see the last case we note that $X_{n+j} = t^{a_{n+j}}$ for $1 \leq j \leq n$ (see Lemma \ref{lem:cluster}), where $X_{n+j}$ is the coefficient of $\Sigma_j$ hence $|X_{n+j}| = |a_{\omega_{i_j}}| = c^2 \omega_{i_j} - \omega_{i_j}$ according to Lemma \ref{lem:cluster}(f). Since $X_{n+j}^\pm = X_{n+j}^\mp$ for all $k$, we obtain:

$$
|X_{n+j}| = |\varphi^{-1}_i (\varepsilon_{n+j})| = c^2 \omega_{i_j} - \omega_{i_j} - \sum_{i=1}^j a_{i,i_j}(c^2 \omega_{i} - \omega_{i}) = c^2 \mu - \mu = (1+c)(c-1)\mu,
$$

where $\mu = \omega_{i_j} - \sum_{i=1}^j a_{i,i_j} \omega_{i_i}$. Thus, it suffices to show that $(c-1)\mu = \alpha_{i_j}$.

Indeed, $(c-1)\mu = (s_{i_1} \cdots s_{i_j} - 1)\mu$. Furthermore, let $\theta_j = -\alpha_{i_j} + \sum_{i=1}^j a_{i,i_j} \omega_{i_i}$. Clearly, $\theta_j$ is $W$-invariant and $\mu + \theta_j = s_{i_1} \cdots s_{i_j} \omega_{i_i} - \sum_{i=1}^j a_{i,i_j} \omega_{i_i}$. Combining these, we obtain

$$(c-1)\mu = (s_{i_1} \cdots s_{i_j} - 1)\mu = (s_{i_1} \cdots s_{i_j} - 1)(\mu + \theta_j) = (s_{i_1} \cdots s_{i_j} - 1)(s_{i_1} \omega_{i_1}) = \omega_{i_j} - s_{i_1} \omega_{i_1} = \alpha_{i_j}.$$ 

This finishes the proof of part (a).

Prove (b) now. Since $X_j^{(k)} = X_j^{(k-1)}$, in view of part (a) it suffices to verify \eqref{eq:cluster} only for $j \in [1,n] \setminus \{k\}$.

Since $\bar{B_i}^{-} = \begin{pmatrix} B_Q \\ -I_n \end{pmatrix}$, it is easy to see that

\begin{equation}
\mu_n \cdots \mu_1 (\bar{B_i}^{-}) = \begin{pmatrix} B_Q \\ I_n \end{pmatrix} = \bar{B_i}^{+}.
\end{equation}

Without loss of generality we assume for the rest of the proof that $i_0 = (1, \ldots, n)$. Since $\bar{B}^{(k)} = \mu_k \cdots \mu_1 (\bar{B_i}^{+})$, it is easy to show that its $j$-th column $b_j^{(k)}$, $j \neq k$ is given by:

$$
b_j^{(k)} = \sum_{i=1}^n \text{sgn}(j-k) \delta_{i}^{j-k} a_{ij} \varepsilon_i + d_i^{j,k} \varepsilon_{n+i},
$$

where $s_1 \cdots s_k a_{ij} = \sum_{i=1}^n d_i^{j,k} a_{ij}$ and $\delta_{i}^{j,k} = \begin{cases} 
0 & \text{if } i = j \\
-1 & \text{if } i \in [\min(j,k), 1, \max(j,1-k)] \\
1 & \text{otherwise}
\end{cases}.$

In particular, $|\text{sgn}(j-k) \cdot b_j^{(k)}|_{+} = \sum_{i=1}^n |[\delta_{i}^{j,k} a_{ij}]_{+} \cdot \varepsilon_i + [\text{sgn}(j-k) d_i^{j,k}]_{+} \cdot \varepsilon_{n+i}|$. Using this, we can compute $|X_j^{(k)}|$, $j \in [1,n] \setminus \{k\}$ by:

\begin{equation}
|X_j^{(k)}| = |X^{(k)}| \text{sgn}(j-k) b_j^{(k)} - \varepsilon_j = -|X_j^{(k)}| + \sum_{i=1}^n |[\delta_{i}^{j,k} a_{ij}]_{+} \cdot |X_j^{(k)}| + |\text{sgn}(j-k) d_i^{j,k}]_{+} \cdot |X_{n+i}^{(k)}|.
\end{equation}
Furthermore, if \( \alpha := s_1 \cdots s_k \alpha_j < 0 \), i.e. all \( d_i^j \leq 0 \), then necessarily \( j \leq k \), \( a_{j+1,j} = \cdots a_{kj} = 0 \) hence \( s_1 \cdots s_j \alpha_j = \alpha \) and we have:

\[
|X_j^{(k)}| = -|X_j^{(k)}| - \sum_{i=1}^{k} a_{ij} |X_i^{(k)}| - \sum_{i=1}^{n} d_i^{j,k} |X_{n+i}^{(k)}| = cs_1 \cdots s_j \alpha_j - (\alpha + \alpha \alpha) = -\alpha
\]

where we used the identity

\[
(7.5) \quad \sum_{i=1}^{n} d_i^{j,k} |X_{n+i}^{(k)}| = s_1 \cdots s_k \alpha_j + cs_1 \cdots s_k \alpha_j .
\]

It remains to consider the case when \( \alpha := s_1 \cdots s_k \alpha_j > 0 \), then i.e. all \( d_i^j \geq 0 \). If \( j < k \), then

\[
|X_j^{(k)}| = -|X_j^{(k)}| - \sum_{i \in [j+1,k]} a_{ij} |X_i^{(k)}| = cs_1 \cdots s_j \alpha_j - \sum_{i \in [j+1,k]} a_{ij} cs_1 \cdots s_i \alpha_i
\]

\[
= cs_1 \cdots s_j \alpha_j - \sum_{i \in [j+1,k]} cs_1 \cdots s_i-1 (\alpha_j - s_i \alpha_j) = cs_1 \cdots s_j \alpha_j - (s_1 \cdots s_j \alpha_i - c_\alpha) = c_\alpha
\]

by the telescopic summation argument since \( a_{ij} \alpha_i = \alpha_j - s_i \alpha_j \).

Finally, if \( \alpha = s_1 \cdots s_k \alpha_j > 0 \) and \( j > k \), then

\[
|X_j^{(k)}| = -|X_j^{(k)}| - \sum_{i \in [k+1,j-1]} a_{ij} |X_i^{(k)}| + \sum_{i=1}^{n} d_i^{j,k} |X_{n+i}^{(k)}|
\]

\[
= cs_1 \cdots s_j \alpha_j - \sum_{i \in [k+1,j-1]} a_{ij} s_1 \cdots s_i-1 \alpha_i + \alpha + \alpha \alpha = cs_1 \cdots s_j \alpha_j - (\alpha - s_1 \cdots s_j \alpha_j) + (\alpha + \alpha) = c_\alpha
\]

again by the telescopic summation argument and (7.5).

The lemma is proved. \( \square \)

Theorem 2.9(b) implies that a given cluster variable \( X \) of \( U \) equals \( \tilde{X}_V \) for some exceptional \( V \in \text{rep}_G(Q,d) \) if and only if \( |X| \) is a positive root. Moreover, [10] Theorem 5.1 asserts that for each exceptional object \( V \in \text{rep}_G(Q,d) \), the corresponding element \( [V]^* \in H^*(\mathcal{C}) \) belongs to \( U^*_+ \) and hence \( \tilde{X}_V = \Psi_1([V]^*) \) belongs to \( \Psi_1(U^*_+) \). This and Lemma 7.3(a) imply that \( X_j^{(k)} \in \Psi_1(U^*_+) \) for all \( k \in [1,n-1], j \in [1,n] \) because \( |X_j^{(k)}| > 0 \). In view of the above and Lemma 7.3(b), it remains to find \( k \in [1,n-1] \) such that \( |X_j^{(k)}| > 0 \) for all \( j \in [1,n] \setminus \{k,k+1\} \).

To do so we need some more notation.

Denote

\[
I_0 = \{ k \in [1,n-1] \mid c_\alpha k > 0 \}
\]

and fix a source adapted sequence \( \mathbf{i}_0^0 \) for \( Q \) such that its prefix is any appropriate ordering of \( I_0 \) and the remainder - any appropriate ordering of \( [1,n] \setminus I_0 \).

Without loss of generality, we may relabel \( Q \) and all cluster variables in such a way that \( \mathbf{i}_0^0 = (1, \ldots, n) \). Then, clearly, \( I_0 = [1,k], \) i.e.,

\[
c_\alpha_1 > 0, \ldots, c_\alpha_k > 0, \ c_\alpha_{k+1} < 0, \ldots, c_\alpha_n < 0 ,
\]

e.g., \( s_r \alpha_\ell = \alpha_\ell \) for all \( r > \ell > k \).
Therefore, Lemma 7.5(b) implies that for \( j \in [1, n] \setminus \{k, k + 1\} \), one has
\[
|X_j^{(k)}| = \begin{cases} 
-s_1 \cdots s_k(\alpha_j) & \text{if } s_1 \cdots s_k(\alpha_j) < 0 \\
-cs_1 \cdots s_k(\alpha_j) & \text{if } s_1 \cdots s_k(\alpha_j) > 0 
\end{cases}
\]

If \( j > k \), then \( s_1 \cdots s_k(\alpha_j) = -c\alpha_j > 0 \) and \( cs_1 \cdots s_k(\alpha_j) = cs_1 \cdots s_{j-1}(\alpha_j) > 0 \) hence \( |X_j^{(k)}| > 0 \).
Suppose that \( j < k \) now. If \( s_1 \cdots s_k(\alpha_j) < 0 \), then \( |X_j^{(k)}| = cs_1 \cdots s_k(\alpha_j) > 0 \) and we are done. In the remaining case, we have \( s_1 \cdots s_k(\alpha_j) > 0 \) and \( |X_j^{(k)}| = cs_1 \cdots s_k(\alpha_j) \). Taking into account that
\[
s_1 \cdots s_k(\alpha_j) = \sum_{i=1}^{k} d_i \alpha_i
\]
where all \( d_i \in \mathbb{Z}_{\geq 0} \), we obtain:
\[
|X_j^{(k)}| = c(\sum_{i=1}^{k} d_i \alpha_i) = \sum_{i=1}^{k} d_i c\alpha_i > 0.
\]

The proposition is proved. \( \Box \)

Let \( \Sigma' \) be as in Proposition 7.2. Since \( \mathfrak{U}_i = \mathcal{U}(\Sigma') \) and \( \Sigma' \) is acyclic, Theorem 6.9 guarantees that \( \mathfrak{U}_i \) is generated by \( \mathbf{X}' \cup \bigcup_{n=1}^{n} \mu_j(\mathbf{X}') \) and the inverses of coefficients \( \{X_{n+1}^{-1}, \ldots, X_{2n}^{-1}\} \). On the other hand, Proposition 7.2 guarantees that \( \mathbf{X}' \subset \Psi(U^*_+) \) and \( \mu_j(\mathbf{X}') \subset \Psi(U^*_+) \) for \( j = 1, \ldots, n \). In turn, this implies the containment
\[
\mathfrak{U}_i \subset \Psi(U^*_+) [X_{n+1}^{-1}, \ldots, X_{2n}^{-1}]
\]
in \( \mathcal{L}_i \). But Theorem 2.9(a) implies the opposite containment, which proves Theorem 2.9(c).

Therefore, Theorem 2.9 is proved. \( \Box \)

7.2. Quantum twist and Proof of Theorem 2.10

Let \( (Q, \mathbf{d}) \) be an acyclic valued quiver on vertices \( Q_0 = \{1, \ldots, n\} \). Without loss of generality, we assume that \( \mathbf{i}_0 = (1, \ldots, n) \) is a complete source adapted sequence.

Definition 7.4. For \( 1 \leq i \leq j \leq n \) denote by \( V_{ij} \) the unique (up to isomorphism) indecomposable representation of \( (Q, \mathbf{d}) \) with \( |V_{ij}| = s_i \cdots s_j - 1 \alpha_j \).

It is well-known that each \( V_{ij} \) is exceptional.

Proposition 7.5. For \( 1 \leq i \leq j \leq n \) one has, under specialization \( q := |F|^{1/2} \):
\[
|V_{ij}|^* = \Delta_{s_i \cdots s_j} \omega_j.
\]

Proof. We start with the following corollary of [10] Proposition 4.3.3.

Lemma 7.6. For \( 1 \leq i < j \leq n \), one has the recursion in \( \mathcal{H}^*(\text{rep}_F(Q, \mathbf{d})) \):
\[
|V_{ij}|^* = \frac{v_{ij}}{(v_i - v^{-1}_j)^N} \sum_{k=0}^{N} x_i^{[N-k]} |V_{i+1,j}|^* (-v^{-1}_j x_i)^{[k]},
\]
where \( N := (\alpha_i^*, |V_{i+1,j}|^*) = -(\alpha_i^*, |V_{i+1,j}|) \).
Proof. Indeed, since \((\alpha_i^\vee, [V_{i+1,j}]) \leq 0\), [10] Proposition 4.3.3 implies that each \([V_{ij}]\) belongs to \(U_+\) or, more precisely, one has the following recursion in \(\mathcal{H}(\text{rep}_\Sigma(Q,d))\) for \(1 \leq i < j \leq n\):

\[
(7.7) \quad [V_{ij}] = v_i^N \sum_{k=0}^{N} [S_i]^{[N-k]} \cdot [V_{i+1,j}] \cdot (-v_i^{-1}[S_i])^k,
\]

where \(N = -(\alpha_i^\vee, [V_{i+1,j}])\) and \(v_i = \langle S_i, S_i \rangle\).

Using the isomorphism \(\mathcal{H}(\text{rep}_\Sigma(Q,d)) \cong \mathcal{H}^*(\text{rep}_\Sigma(Q,d))\) given by \([V] \mapsto [\text{Aut}(V)] \cdot \delta_{|V|}\) for \(V \in \text{rep}_\Sigma(Q,d)\), we obtain a similar recursion for \(\delta_{|V_{ij}|} \in \mathcal{H}^*(\text{rep}_\Sigma(Q,d))\):

\[
\delta_{|V_{ij}|} = \frac{v_i^N [\text{Aut}(V_{i+1,j})]}{[\text{Aut}(V_j)] \cdot [\text{Aut}(S_i)]} \sum_{k=0}^{N} x_i^{[N-k]} \cdot [V_{i+1,j}] \cdot (-v_i^{-1}x_i)^k.
\]

hence

\[
(7.8) \quad [V_{ij}]^* = c_{ij}' \sum_{k=0}^{N} x_i^{[N-k]} \cdot [V_{i+1,j}]^* \cdot (-v_i^{-1}x_i)^k,
\]

where \(c_{ij}' = \frac{v_i^N [\text{Aut}(V_{i+1,j})]}{[\text{Aut}(V_j)] [\text{Aut}(S_i)]} \cdot \frac{\langle V_{i+1,j}, V_{i+1,j} \rangle}{\langle V_{i+1,j}, V_{i+1,j} \rangle} \cdot \frac{f([V_{ij}])}{f([V_{i+1,j}])}\). Since \([V_{ij}] = s_i[V_{i+1,j}] = |V_{i+1,j}| + N\alpha_i\)

we have

\[
f([V_{ij}]) = f([V_{i+1,j}]) \langle S_i, S_i \rangle^{\frac{N}{2}}.
\]

Then using the fact that \(|\text{Aut}(V_{i+1,j})| = |\text{Aut}(V_j)|\) (see e.g., [11] Proposition 2.1) and \(\langle V_{ij}, V_{ij} \rangle = \langle V_{i+1,j}, V_{i+1,j} \rangle\), we obtain:

\[
c_{ij}' = \frac{v_i^N (S_i, S_i)^{\frac{N}{2}}}{[\text{Aut}(S_i)]^{\frac{N}{2}}} = \frac{v_i^N}{(v_i - v_i^{-1})^N}.
\]

because \(\langle S_i, S_i \rangle = v_i\) and \(|\text{Aut}(S_i)| = v_i^2 - 1\).

This proves the lemma. \(\square\)

We are now in a position to prove (7.6) by induction in \(j - i\). Indeed, for \(j - i = 0\) we will set \(w = s_i = s_j\) and \(\lambda = s_i w \lambda = \omega_j\) in (6.5) where \(N = (\alpha_i^\vee, \omega_i) = 1\). Taking into account that \(\Delta_{\omega_j} = 1\), we have

\[
\Delta_{s_j \omega_j} = \frac{\frac{q_i^2}{q_i - q_i^{-1}}}{q_i - q_i^{-1}} \sum_{k=0}^{1} \left(\frac{q_i}{q_i - q_i^{-1}}\right)^{1-k} \cdot (-q_i^{1-k} x_i)^k = \frac{q_i^2}{q_i - q_i^{-1}} \left(\frac{q_i}{q_i - q_i^{-1}}\right)^{1-k} x_i = [V_{jj}]^*.
\]

Now assume that \(j - i > 0\), then specializing (6.5) at \(q_i = v_i, \lambda = \omega_j, w = s_i \cdots s_j\) and using the fact that \((\alpha_i^\vee, \omega_j) = 0\) if \(i \neq j\), we obtain the following recursive formula for \(\Delta_{s_i \cdots s_j \omega_j}\):

\[
(7.9) \quad \Delta_{s_i \cdots s_j \omega_j} = \frac{v_i^N}{(v_i - v_i^{-1})^N} \sum_{k=0}^{N} x_i^{[N-k]} \cdot \Delta_{s_{i+1} \cdots s_j \omega_j} \cdot (-v_i^{-1}x_i)^k.
\]

Combining Lemma (7.6) with the inductive hypotheses for \([V_{i+1,j}]^*\), we obtain by (7.9):

\[
[V_{ij}]^* = \frac{v_i^N}{(v_i - v_i^{-1})^N} \sum_{k=0}^{N} x_i^{[N-k]} \cdot \Delta_{s_{i+1} \cdots s_j \omega_j} \cdot (-v_i^{-1}x_i)^k = \Delta_{s_i \cdots s_j \omega_j}.
\]

The proposition is proved. \(\square\)
To complete the proof of Theorem 2.10 we need to show that \( \hat{\eta}(U_i) = U_i \), where \( \hat{\eta} : L_i \rightarrow \text{Frac}(L_i) \) is defined in Lemma 6.19.

First note that Lemma 6.19 implies that \( \hat{\Sigma}_i = (\hat{X}_i, \hat{B}_i) \) is a quantum seed in \( L_i \). Denote \( \hat{\Sigma}_i^+ = (\hat{X}_i^+, \hat{B}_i^+) \), where \( \hat{X}_i^+ = \rho_i^+(X_i) \) in the notation of Lemma 6.8. By Lemma 6.8(a) it is a quantum seed for \( L_i \).

**Lemma 7.7.** \( \hat{\Sigma}_i^+ = \mu_n \cdots \mu_1(\Sigma_i^-) \).

**Proof.** In the notation of Section 7.1 denote 
\[
\Sigma' = (X', \hat{B}') := \mu_n \cdots \mu_1(\Sigma_i^-)
\]
Clearly, \( \hat{B}' = \hat{B}_i^+ \) by (7.3).

Furthermore, repeating the argument from the proof of Proposition 7.2, we obtain 
\[
X'_j = \Psi_i([V_{1j}]^*)
\]
for \( 1 \leq j \leq n \), where \( V_{1j} \) is defined in Definition 7.4.

It follows from Proposition 7.5 that 
\[
X'_j = \Psi_i([V_{1j}]^*) = \Psi_i(\Delta_\omega i \cdots s_j \omega_j) = \hat{X}_j
\]
for \( j = 1, \ldots, n \). Clearly, the coefficients \( X'_{n+1}, \ldots, X'_{2n} \) of \( X' \) are inherited from the cluster \( X_i^- = \{X_1^-, \ldots, X_{2n}^-\} \) therefore
\[
(7.10) X'_{n+k} = X_{n+k}^- = t^{\varphi_i^{-1}(\varepsilon_{n+k})} t^{-\varphi_i^{-1}(\varepsilon_{n+k})},
\]
where we use that \( \rho_i^-(\varepsilon_{n+k}) = -\rho_i^+(\varepsilon_{n+k}) \).

Following Proposition 6.16 and Lemma 4.12(e) we have
\[
(7.11) \hat{X}_{n+k} = \Psi_i(\Delta \omega_{i \cdots n+k}) = t^{-1}_{\omega_{i \cdots n+k}} = t^{-\varphi_i^{-1}(\varepsilon_{n+k})}
\]
for \( 1 \leq k \leq n \). Combining (7.10) and (7.11) we see that
\[
X'_{n+k} = \rho_i^+(\hat{X}_i)_{n+k} = t^{-\varphi_i^{-1}(\varepsilon_{n+k})} = \hat{X}_{n+k}^+.
\]
This proves that \( X' = \hat{X}_i^+ \). The proposition is proved. 

Now we can finish the proof of Theorem 2.10. Indeed, by Lemma 6.8(b), \( U(\hat{\Sigma}_i^+) = U(\hat{\Sigma}_i) \). On the other hand, \( U(\hat{\Sigma}_i^+) = U(\hat{\Sigma}_i^+) = U_i \) by Lemma 7.7 Therefore, Theorem 2.10 is proved. 

\[ \square \]
7.3. **Proof of Theorem 2.11** For simplicity, we set \( i_0 := (1, \ldots, n) \) as above. Then Theorem 2.9 and (6.17) guarantee that if \( i = (i_0, i_0) \) is a reduced word for an element \( c^2 \), where \( c = s_1 \cdots s_n \), then \( \Psi_i \) is an isomorphism

\[ \Psi_i : k_q[Nc] \cong \mathcal{U}_i. \]

On the other hand, by Theorem 2.10, we obtain:

\[ \eta(X_j) = \Psi_i(\Delta_{s_1 \cdots s_j}) = \Psi_i(\Delta_{s_1 \cdots s_j \omega_j}) \]

for \( j = 1, \ldots, n \).

Therefore, combining this with Lemma 6.22(e) we see that if \( i = (i_0, i_0) \) is a reduced word for \( c^2 \), the assignment

\[ X_j \mapsto \Psi_j^{-1}(\eta(X_j)) = \begin{cases} \Delta_{i_1 \cdots i_{s_j}} & \text{if } j \leq n \\ \Delta_{i_1 \cdots i_{s_j} i_{s_j+1} \cdots i_{s_n} \omega_j} & \text{if } j > n \end{cases} \]

for \( j = 1, \ldots, 2n \) defines an injective homomorphism

\[ \mathcal{L}_i \hookrightarrow Frac(k_q[Nc^2]) \]

whose restriction to \( \mathcal{U}_i \subset \mathcal{L}_i \) is an isomorphism \( \mathcal{U}_i \cong k_q[Nc^2] \). This proves Theorem 2.11.

8. **Example**

In this section we compute a complete example to illustrate our main results. Consider the Cartan matrix \( A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \) with symmetrizing matrix \( D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \). Our example will be built on the word \( i = (1, 2, 1, 2) \). In this case \( \mathcal{L}_i \) is the algebra over \( \mathbb{Z}[v^\pm 1] \) generated by \( t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, \) and \( t_4^{\pm 1} \) subject to the commutation relations:

\[ t_2t_1 = v^{-3}t_1t_2, \quad t_3t_1 = v^2t_1t_3, \quad t_4t_1 = v^{-3}t_1t_4, \quad t_3t_2 = v^{-3}t_2t_3, \quad t_4t_2 = v^6t_2t_4, \quad t_4t_3 = v^{-3}t_3t_4. \]

We may compute the initial exchange matrix as \( \tilde{B}_i = \begin{pmatrix} 0 & 3 \\ -1 & 0 \\ 1 & -3 \end{pmatrix} \) and the initial cluster

\[ X_i = (X_1, X_2, X_3, X_4) \subset \mathcal{L}_i \]

given by

\[ X_1 = t^{-\epsilon_1}, \quad X_2 = t^{-2\epsilon_1 - \epsilon_2}, \quad X_3 = t^{-2\epsilon_1 - \epsilon_2 - \epsilon_3}, \quad X_4 = t^{-3\epsilon_1 - 2\epsilon_2 - 3\epsilon_3 - \epsilon_4}. \]

We will also need another choice of coefficients \( X_3^- \) and \( X_4^- \) computed by:

\[ X_3^- = X^{-\epsilon_3} = t^{2\epsilon_1 + \epsilon_2 + \epsilon_3}, \quad X_4^- = X^{3\epsilon_4 - \epsilon_4} = t^{-3\epsilon_1 - \epsilon_2 + \epsilon_4}. \]

The variables of \( X_i \) or \( X_i^- \) (and their inverses) form another generating set for \( \mathcal{L}_i \) and we may use them to express the generators \( t_1, t_2, t_3, \) and \( t_4 \) as follows:

\[ t_1 = X^{-\epsilon_1}, \quad t_2 = X^{3\epsilon_1 - \epsilon_2}, \quad t_3 = X^{-\epsilon_1 + \epsilon_2 - \epsilon_3} = X_1^{-\epsilon_1 + \epsilon_2 + \epsilon_3}, \quad t_4 = X^{-\epsilon_2 + 3\epsilon_3 - \epsilon_4} = X_3^{-\epsilon_2 + \epsilon_4}, \]
where we write $X^a$ for bar-invariant monomials in the cluster $X^-_i$. It is easy to see that the commutation matrix of $X_i$ is given by
\[
\begin{pmatrix}
0 & 3 & 1 & 3 \\
-3 & 0 & 0 & 3 \\
-1 & 0 & 0 & 3 \\
-3 & -3 & -3 & 0
\end{pmatrix}
\]
and this is compatible with $\tilde{B}_i$, verifying Theorem 4.4. Similarly the commutation matrix of $X^-_i$ is
\[
\begin{pmatrix}
0 & 3 & -1 & 0 \\
-3 & 0 & 0 & -3 \\
1 & 0 & 0 & 3 \\
0 & 3 & -3 & 0
\end{pmatrix}
\]
compatible with the exchange matrix $\tilde{B}_i^- = \begin{pmatrix} 0 & 3 \\ -1 & 0 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$.

Inside $k_v[N]$ we have the generalized quantum minors $\Delta_{s_1\omega_1}$, $\Delta_{s_1s_2\omega_2}$, $\Delta_{s_1s_2s_1\omega_1}$, and $\Delta_{s_1s_2s_1s_2\omega_2}$ which provide the cluster variables and coefficients of the cluster $(X_i, \tilde{B}_i)$. These generalized quantum minors can easily be computed using (6.5) as follows:

\[
\Delta_{s_1\omega_1} = x_1,
\]
\[
\Delta_{s_1s_2\omega_2} = \frac{v^3 x_1 x_2 - v^3 [3] x_1^2 x_2 x_1 + v^3 [3] x_1 x_2^2 - v^3 x_2 x_1^3}{(v^3 - v^{-3})(v^2 - v^{-2})(v - v^{-1})},
\]
\[
\Delta_{s_1s_2s_1\omega_1} = \frac{-v^3 x_1^2 x_2 + (v^3 + v^{-3} + v^{-3}) x_1^2 x_2 x_1 - (v^3 + v^3 + v^{-3}) x_1 x_2 x_1^2 + v^3 x_2 x_1^3}{(v^3 - v^{-3})(v^2 - v^{-2})(v - v^{-1})},
\]
\[
\Delta_{s_1s_2s_1s_2\omega_2} = \frac{v^3 x_1 \Delta_{s_2s_1s_2\omega_2} - v^3 [3] x_1^2 \Delta_{s_2s_1s_2\omega_2} x_1 + v^3 [3] x_1 \Delta_{s_2s_1s_2\omega_2} x_1^2 - v^3 \Delta_{s_2s_1s_2\omega_2} x_1^3}{(v^3 - v^{-3})(v^2 - v^{-2})(v - v^{-1})},
\]

where we have used the notation $[3]_v = v^2 + 1 + v^{-2}$.

Applying the Feigin homomorphism $\Psi_{(1,2,1,2)}$ we obtain the cluster variables as follows:

\[
\check{X}_1 = \Psi_{(1,2,1,2)}(\Delta_{s_1\omega_1}) = t^{e_1} + t^{e_3},
\]
\[
\check{X}_2 = \Psi_{(1,2,1,2)}(\Delta_{s_1s_2\omega_2}) = t^{3e_1 + e_2} + t^{3e_3 + e_4} + [3]_v t^{2e_1 + e_2 + e_3 + e_4} + [3]_v t^{e_1 + 2e_3 + e_4} + t^{3e_3 + e_4},
\]
\[
\check{X}_3 = \Psi_{(1,2,1,2)}(\Delta_{s_1s_2s_1\omega_1}) = t^{2e_1 + e_2 + e_3} = X^{-e_3},
\]
\[
\check{X}_4 = \Psi_{(1,2,1,2)}(\Delta_{s_1s_2s_1s_2\omega_2}) = t^{3e_3 + 2e_2 + 3e_3 + e_4} = X^{-e_4}.
\]

Applying the monomial change $\tilde{X}_i$ we see that

\[
\check{X}_1 = \Psi_{(1,2,1,2)}(\Delta_{s_1\omega_1}) = X^{-e_1} + X^{-e_1 + e_2 + e_3}
\]
is the new variable obtained by mutating the seed $(X^-_i, \tilde{B}_i^-)$ in direction 1 and that mutating further in direction 2 produces the new cluster variable:

\[
\check{X}_2 = \Psi_{(1,2,1,2)}(\Delta_{s_1s_2\omega_2}) = X^{-e_2} + X^{-3e_1 + e_2 + e_4} + (v^2 + 1 + v^{-2}) X^{-3e_1 + e_3 + e_4}
+(v^2 + 1 + v^{-2}) X^{-3e_1 + 2e_2 + e_3 + e_4} + X^{-3e_1 + 2e_2 + 3e_3 + e_4}.
\]
9. Appendix: Twisted Bialgebras in Braided Monoidal Categories

Let \( \mathbb{k} \) be a field and \( \Gamma \) an additive monoid. For any unitary bicharacter \( \chi : \Gamma \times \Gamma \rightarrow \mathbb{k}^\times \) let \( \mathcal{C}_\chi \) be the tensor category of \( \Gamma \)-graded vector spaces \( V = \bigoplus_{\gamma \in \Gamma} V(\gamma) \) such that each component \( V(\gamma) \) is finite-dimensional. Clearly, this category is braided via \( \Psi_{U,V} : U \otimes V \rightarrow V \otimes U \) given by

\[
\Psi_{U,V}(u_\gamma \otimes v_\gamma) = \chi(\gamma, \gamma') \cdot v_{\gamma'} \otimes u_\gamma
\]

for any \( u_\gamma \in U(\gamma), v_{\gamma'} \in V(\gamma') \).

Let \( \mathcal{U} = \bigoplus_{\gamma \in \Gamma} \mathcal{U}(\gamma) \) be a bialgebra in \( \mathcal{C}_\chi \). Denote by \( \hat{\mathcal{U}} \) the completion of \( \mathcal{U} \) with respect to the grading, that is, the space of all formal series \( \hat{\mathcal{U}} = \sum_{\gamma \in \Gamma} u_\gamma \) where \( u_\gamma \in \mathcal{U}(\gamma) \). For each such \( \hat{\mathcal{U}} \) denote by \( \text{Supp}(\hat{\mathcal{U}}) \) the submonoid of \( \Gamma \) generated by \( \{ \gamma : u_\gamma \neq 0 \} \).

From now on we assume that for any \( \gamma \in \Gamma \) the set

\[
A_\gamma = \{ (\gamma', \gamma'') : \gamma' + \gamma'' = \gamma \}
\]

of two-part partitions of \( \gamma \) is finite. It is easy to see that, under this assumption, \( \hat{\mathcal{U}} \) has a well-defined multiplication. The coproduct on \( \mathcal{U} \) extends to \( \hat{\Delta} : \hat{\mathcal{U}} \rightarrow \hat{\mathcal{U}} \otimes \hat{\mathcal{U}} \) so that \( \hat{\mathcal{U}} \) becomes a complete bialgebra. The following fact is obvious.

**Lemma 9.1.** Let \( E = \sum_{\gamma \in \Gamma} E(\gamma) \) be a formal series, where each \( E(\gamma) \in \mathcal{U}(\gamma) \). Then \( E \) is grouplike in \( \hat{\mathcal{U}} \) (i.e., \( \hat{\Delta}(E) = E \otimes E \)) if and only if

\[
\Delta(E(\gamma)) = \sum_{(\gamma', \gamma'') \in A_\gamma} E(\gamma') \otimes E(\gamma'')
\]

for each \( \gamma \in \Gamma \).

As a corollary of Lemma 9.1 we have the following well-known result.

**Lemma 9.2.** If \( x \in \mathcal{U} \) is primitive, i.e. \( \Delta(x) = x \otimes 1 + 1 \otimes x \) and \( \Psi_{\mathcal{U},\mathcal{U}}(x \otimes x) = qx \otimes x \) for some non-root of unity \( q \in \mathbb{k}^\times \), then the braided exponential

\[
\exp_q(x) = \sum_{k=0}^{\infty} \frac{1}{(k)_q!} k^k
\]

of \( x \) is grouplike in \( \hat{\mathcal{U}} \), where \( (k)_q! = (1)_q \cdots (k)_q \) and \( (\ell)_q = \frac{q^{\ell}-1}{q-1} \).

However, the product of grouplike elements is not always grouplike. We can sometimes restore the grouplike property of a product by twisting the factors with elements of an appropriate non-commutative algebra \( \mathcal{P} \) in \( \mathcal{C}_\chi \). This, contained in Proposition 9.24, is the main idea behind the forthcoming theorem.

Now we define the restricted dual algebra \( \mathcal{A} \) of \( \mathcal{U} \). As a vector space, \( \mathcal{A} \) is the set of all \( \mathbb{k} \)-linear forms \( x : \mathcal{U} \rightarrow \mathbb{k} \) such that \( x \) vanishes on \( \mathcal{U}(\gamma) \) for all but finitely many \( \gamma \in \Gamma \). In other words, \( \mathcal{A} = \bigoplus_{\gamma \in \Gamma} \mathcal{A}(\gamma) \) where \( \mathcal{A}(\gamma) = \text{Hom}_\mathbb{k}(\mathcal{U}(\gamma), \mathbb{k}) \). Clearly, \( \mathcal{A} \) is an algebra in \( \mathcal{C}_\chi \) with the product (resp. unit) adjoint of the coproduct \( \Delta \) (resp. counit) on \( \mathcal{U} \).
Let $E = (E_1, \ldots, E_m)$ be a family of grouplike elements

$$E_k = \sum_{\gamma \in \Gamma} E_k^{(\gamma)}$$

in $\hat{U}$. We say that $E$ is $P$-adapted if for each $k = 1, \ldots, m$ there exists a homomorphism $\tau_k$ from the monoid $\text{Supp}(E_k)$ to the multiplicative monoid of $P$ such that

$$\tau_k(\gamma_k) \tau_k(\gamma_k) = \chi(\gamma_k, \gamma_k) \cdot \tau_k(\gamma_k) \tau_k(\gamma_k)$$

for all $k < \ell$ and $\gamma_k \in \text{Supp}(E_k)$. For every $P$-adapted family $E$ we define a map $\Psi_E : A \to P$ by the formula

$$\Psi_E(x) = \sum_{\gamma_1 \in \text{Supp}(E_1), \ldots, \gamma_m \in \text{Supp}(E_m)} x \left( E_1^{(\gamma_1)} \cdots E_m^{(\gamma_m)} \right) \tau_1(\gamma_1) \cdots \tau_m(\gamma_m),$$

where we denote by $(x, u) \mapsto x(u)$ the natural non-degenerate evaluation pairing $A \times U \to \mathbb{k}$. Note that the sum in (9.2) is always finite because $x$ vanishes on all but finitely many homogeneous components of $U$.

**Theorem 9.3.** For any $P$-adapted family $E$ of grouplike elements the map $\Psi_E : A \to P$ defined by (9.2) is a homomorphism of $\Gamma$-graded algebras.

**Proof.** For any $k$-algebra $P$ denote $U_P := U \otimes P$ and view it as an algebra with the standard (NOT braided!) algebra structure. We will often abbreviate $u \cdot t := u \otimes t$ for $u \in U$, $t \in P$.

Denote by $\hat{U}_P$ the completion of $U_P$, i.e., $\hat{U}_P = \hat{U} \otimes P$ is the space of formal series of the form $\sum_{\gamma \in \Gamma} u_\gamma \cdot t_\gamma$, where $u_\gamma \in U(\gamma)$ and $t_\gamma \in P$. Consider the tensor square $V_P = U_P \otimes U_P$ where the left factor is regarded as a right $P$-module and the right factor as a left $P$-module. Note that $V_P$ is a $P$-bimodule in which we can write $t(u \otimes v) = (tu) \otimes v = u \otimes (tv) = (u \otimes v)t$ for any $u, v \in U, t \in P$.

Under the standard identification $V_P \cong (U \otimes U) \otimes P$, this bimodule $V_P$ becomes an algebra.

We will also need the completed tensor square $\hat{V}_P = U_P \otimes \hat{U}_P$. There is a natural morphism of $P$-bimodules

$$\hat{\Delta}_P : \hat{U}_P \to \hat{V}_P$$

which is the $P$-linear extension of the coproduct $\Delta$ on $\hat{U}$. Clearly, $\hat{\Delta}_P$ is an algebra homomorphism.

For each $P$-adapted family $E$ define an element $\tilde{E} \in \hat{U}_P$ as follows:

$$\tilde{E} = E_1 \cdots E_m,$$

where

$$\tilde{E}_k = \sum_{\gamma \in \text{Supp}(E_k)} E_k^{(\gamma)} \cdot \tau_k(\gamma).$$

**Proposition 9.4.** For any $P$-adapted family of grouplike elements $E$ the element $\tilde{E} \in \hat{U}_P$ is grouplike, i.e.,

$$\Delta_P(\tilde{E}) = \hat{E} \otimes \hat{E}.$$
Proof. We need the following fact.

Lemma 9.5. In the assumptions of Proposition 9.4 one has:
(a) each $\tilde{E}_k$ is a grouplike element in $\tilde{U}_P$.
(b) $(1 \otimes \tilde{E}_k)(\tilde{E}_l \otimes 1) = (\tilde{E}_l \otimes 1)(1 \otimes \tilde{E}_k)$ for any $1 \leq k < l \leq m$.

Proof. To prove (a), note that by Lemma 9.1 we have

$$\Delta_P(\tilde{E}_k) = \sum_{\gamma \in \text{Supp}(E_k)} \Delta(E_k^{(\gamma)}) \tau_k(\gamma) = \sum_{\gamma', \gamma'' \in \text{Supp}(E_k)} E_k^{(\gamma')} \otimes E_k^{(\gamma'')} \tau_k(\gamma' + \gamma'')$$

where we have used the multiplicativity of $\tau_k$. To prove (b), abbreviate $\tilde{E}_k^{(\gamma)} := E_k^{(\gamma)} \cdot \tau_k(\gamma)$ for $k = 1, \ldots, m$, $\gamma \in \text{Supp}(E_k)$. Then for $k < \ell$ and $\gamma_k \in \text{Supp}(E_k)$, $\gamma_k \in \text{Supp}(E_\ell)$ we deduce the following commutation relation using (9.1):

$$(1 \otimes \tilde{E}_k^{(\gamma_k)})(\tilde{E}_\ell^{(\gamma_\ell)} \otimes 1) = (1 \otimes E_k^{(\gamma_k)})(E_\ell^{(\gamma_\ell)} \otimes 1) \cdot \tau_k(\gamma_k) \tau_\ell(\gamma_\ell) = (E_k^{(\gamma_k)} \otimes 1)(1 \otimes E_\ell^{(\gamma_\ell)}) \cdot \tau_k(\gamma_k) \tau_\ell(\gamma_\ell)$$

Since $\tilde{E}_k = \sum_{\gamma_k \in \text{Supp}(E_k)} \tilde{E}_k^{(\gamma_k)}$ and $\tilde{E}_\ell = \sum_{\gamma_\ell \in \text{Supp}(E_\ell)} \tilde{E}_\ell^{(\gamma_\ell)}$, we obtain the desired result:

$$(1 \otimes \tilde{E}_k)(\tilde{E}_\ell \otimes 1) = \sum_{\gamma_k \in \text{Supp}(E_k), \gamma_\ell \in \text{Supp}(E_\ell)} (1 \otimes \tilde{E}_k^{(\gamma_k)})(\tilde{E}_\ell^{(\gamma_\ell)} \otimes 1)$$

Lemma 9.5 is proved.

Now we are ready to finish the proof of Proposition 9.3. Using Lemma 9.5 and the identities $u \otimes v = (\tilde{u} \otimes 1)(1 \otimes \tilde{v})$, $(\tilde{u} \otimes 1)(\tilde{v} \otimes 1) = \tilde{u} \tilde{v} \otimes 1$ and $(1 \otimes \tilde{u})(1 \otimes \tilde{v}) = 1 \otimes \tilde{u} \tilde{v}$, for any $\tilde{u}, \tilde{v} \in \tilde{U}_P$, we compute

$$\Delta_P(\tilde{E}) = \Delta_P(\tilde{E}_1 \cdots \tilde{E}_m) = \Delta_P(\tilde{E}_1) \cdots \tilde{E}_m = (\tilde{E}_1 \otimes \tilde{E}_1) \cdots (\tilde{E}_m \otimes \tilde{E}_m)$$

Finally, we define the pairing $A \times \tilde{U}_P \to P$ by:

$$x(\sum u_\gamma \cdot t_\gamma) = \sum x(u_\gamma) t_\gamma$$

Clearly, the pairing is well-defined because all but finitely many terms $x(u_\gamma)$ are 0 for each $u \in A$. For every $P$-adapted family $E$ we see that the map $\Psi_E : A \to P$ defined in (9.2) is given by the formula $\Psi_E(x) := x(\tilde{E})$. 

The definition of the multiplication in $\mathcal{A}$ implies that $(xy)(\tilde{u}) = (x \otimes y)(\tilde{\Delta}(\tilde{u}))$ for all $x, y \in \mathcal{A}$ and $\tilde{u} \in \mathcal{U}_P$, where

$$(x \otimes y)(\tilde{u}_1 \otimes \tilde{u}_2) := x(\tilde{u}_1)y(\tilde{u}_2)$$

for any $\tilde{u}_1, \tilde{u}_2 \in \mathcal{U}_P$. Thus, we have

$$\Psi_E(xy) = (xy)(\tilde{E}) = (x \otimes y)(\tilde{\Delta}(\tilde{E})) = (x \otimes y)(\tilde{E} \otimes \tilde{E}) = x(\tilde{E})y(\tilde{E}) = \Psi_E(x)\Psi_E(y),$$

which finishes the proof of Theorem 9.3. \hfill \Box

For each $u \in \mathcal{U}$ define the linear operators $x \mapsto u(x)$ and $x \mapsto u^{op}(x)$ on $\mathcal{A}$ by:

$$u(x)(u') = x(u'u), \quad u^{op}(x)(u') = x(uu')$$

for all $u' \in \mathcal{U}$, $x \in \mathcal{A}$.

Clearly, the operators $x \mapsto u(x)$ and $x \mapsto u^{op}(x)$ define respectively the left and the right $\mathcal{U}$-action on $\mathcal{A}$ and $u(x), u^{op}(x) \in \mathcal{A}_{\gamma' - \gamma}$ for each homogeneous $u \in \mathcal{U}$, and $x \in \mathcal{A}_{\gamma'}$.

Using this in the form $x(u_1 \cdots u_m) = (u_1 \cdots u_m(x))(1) = u_m^{op} \cdots u_1^{op}(x)(1)$, we rewrite (9.2) for any homogeneous $x \in \mathcal{A}_\gamma$ as:

$$\Psi_E(x) = \sum E_{1}^{(\gamma_1)} \cdots E_{m}^{(\gamma_m)}(x) \cdot \tau_1(\gamma_1) \cdots \tau_m(\gamma_m),$$

$$\Psi_E(x) = \sum E_{m}^{(\gamma_m)^{op}} \cdots E_{1}^{(\gamma_1)^{op}}(x) \cdot \tau_1(\gamma_1) \cdots \tau_m(\gamma_m),$$

where the summation is over all $(\gamma_1, \ldots, \gamma_m) \in \text{Supp}(E_1) \times \cdots \times \text{Supp}(E_m)$ such that $\gamma_1 + \cdots + \gamma_m = \gamma$.

We finish with the following obvious, however, useful fact.

**Lemma 9.6.** Let $E \in \mathcal{U}_\alpha$ be any homogeneous primitive element. Then for any $x \in \mathcal{A}_\gamma$ and $y \in \mathcal{A}$ one has

$$E(yx) = \chi(\gamma, \alpha) \cdot E(y)x + yE(x), \quad E^{op}(xy) = E^{op}(x)y + \chi(\alpha, \gamma) \cdot xE^{op}(y).$$

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