Capacity of the Adini Element for Biharmonic Equations

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Abstract This paper is devoted to the convergence analysis of the Adini element scheme for biharmonic equations in any dimension. In particular, a second order convergence in the energy norm is established provided that the exact solution is in $H^4$. Surprisingly, it is proved that the convergence rate in the $L^2$ norm can not be nontrivially higher than $O(h^2)$ order. The main ingredient for the analysis is a crucial structure of the Adini element space. Numerics are presented to demonstrate the theoretical results.

Keywords Biharmonic equation · Adini element · Error estimate · Arbitrary dimension

1 Introduction

This paper is devoted to the convergence analysis of the Adini element scheme for biharmonic equations in any dimension. The Adini element, (c.f. [1] for 2D, [17] for higher dimensions) is among the earliest finite elements for elliptic problems. It uses the rectangles (2D) and generalised rectangles (higher dimensions) as geometry shapes, and the evaluation and the derivatives of first order on the vertices as nodal parameters. The generation of stiffness matrix is easy and friendly, and this element has become a popular one during the past half
century, and stimulated various works, (see [10,12,13,17]). In this present paper, we discuss the capacity of the Adini element scheme for biharmonic equations, and present a sharp analysis of the upper and lower bounds of the convergence rate in energy and integral norms in any dimension.

When used for second order problems, the Adini element scheme is a conforming one, and the error analysis is straightforward by the fundamental Céa lemma and standard arguments. When used for fourth order problems, however, the Adini element is a nonconforming one, and the convergence analysis is more subtle. In Wang, Shi and Xu [17] where they generalised the Adini element from 2D to arbitrary dimension, the $O(h)$ convergence rate of Adini element in any dimension has been proved for fourth order problems. Meanwhile, a higher accuracy of the scheme is still expected and numerically observed. In 2D, it has been proved by Lascaux and Lesaint [10] that the finite element solution converges to the exact solution with $O(h^2)$ order in the energy norm, provided the rectangular cells in the grid are all the same. Then in 2004, Lin and Luo [12] showed the $O(h^2)$ convergence of the Adini element without assuming the congruence of the cells of the grid. Later in 2006, Mao and Chen [13] showed further the $O(h^2)$ convergence rate for anisotropic grids. So far to our knowledge, the sharp analysis of the convergence rate of Adini element for fourth order problems in higher dimensions is still absent.

In this paper, we study the convergence rate of the Adini element scheme for biharmonic equations in any dimension. Technically, without making use of the nodal interpolation which was done by [10,12,13] and which will bring extra regularity assumption on the exact solution in higher dimensions, our analysis relies on the structure of Adini element space only. We figure out the intrinsic symmetry property of the Adini element space, and show the $O(h^2)$ energy norm convergence rate in a unified way with respect to the dimensions provided the exact solution belongs to $H^4$. The interpolation operator is not present in the analysis, and we only have to reveal and rely on the intrinsic properties of the finite element functions which can be viewed logically as a priori fundamental property of the interpolation operators.

There have been works that study high accuracy nonconforming finite element methods for biharmonic equations. For high accuracy that arises in the context of superconvergence, we refer to [4,6,11]. Several nonconforming finite elements of $O(h^2)$ order have been constructed in, e.g., [5,15,18]. In contrast to these elements, the Adini element space does not possess such moment continuity; the average of the normal derivatives of Adini element function is not continuous across the internal faces. This motivates us to make use of a different way by using the symmetric property inside one cell, and moreover, this unusual property makes it hardly possible to make use of the dual argument to obtain higher order convergence rates in the $H^1$ or the $L^2$ norm. Indeed, in the paper, we further show that the convergence rate in the $L^2$ norm can not be non-trivially higher than $O(h^2)$ order.

The lower bound estimate is a key feature of the capacity of the finite element schemes, especially nonconforming ones. The analysis of the lower bound of the convergence rate of the Adini element scheme in $L^2$ norm is a generalization of Hu–Shi’s work [9], which solved an problem whether the convergent order in $L^2$ norm can always be higher than that in the energy norm. Technically, a decomposition of the residual $(f, v - u_h)$ into a leading term and other higher order terms works crucially, and we estimate the lower bound of the leading term sufficiently. Again, a sharp analysis of the interpolation operator will play a key role. Finally, by the discrete Poincaré inequality, we obtain that the convergence rate of the Adini element scheme for biharmonic equations in the energy norm, the $H^1$ norm and the $L^2$ norm are all of $O(h^2)$ order, and these estimates are all sharp. The lower bound estimate is important to finite element schemes for eigenvalue computation. We refer to [7] for related discussions.
The remaining of the paper is organized as follows. In Sect. 2, we present some preliminaries of the Adini element. In Sect. 3.1, we present the model problem and the Adini finite element discretization. In Sect. 3.2, we show the $O(h^2)$ order convergence rate in the energy norm in any dimension. In Sect. 3.3, we further show the $O(h^2)$ order convergence in the $L^2$ norm in any dimension. In Sect. 4, we provide numerical examples to demonstrate the theoretical results. The paper ends with Sect. 5 which gives some conclusions.

2 Preliminaries: The Adini Element

2.1 The Adini Element

Let $K \subset \mathbb{R}^d$ be a $d$-rectangle, $x_c = (x_{1,c}, x_{2,c}, \ldots, x_{d,c})^T \in \mathbb{R}^d$ be the barycentre of $K$, and $h_i$ the half length of $K$ in $x_i$ direction, $i = 1, 2, \ldots, d$. Then the $d$-rectangle can be denoted by

$$K = \left\{ x = (x_1, x_2, \ldots, x_d)^T | x_i = x_{i,c} + \xi_i h_i, \ -1 \leq \xi_i \leq 1, \ 1 \leq i \leq d \right\}.$$ 

Particularly, the vertices $a_i$, $1 \leq i \leq 2^d$, of $K$ are denoted by

$$a_i = (x_{1,c} + \xi_1 h_1, x_{2,c} + \xi_2 h_2, \ldots, x_{d,c} + \xi_d h_d)^T, \ |\xi_i| = 1, \ 1 \leq j \leq d, \ 1 \leq i \leq 2^d.$$ 

Moreover, denote by $F_{K,i}'$ and $F_{K,i}''$, the two $(d-1)$-dimensional faces of $K$ without the edges parallel to the $x_i$ axe; see Fig. 1.

The $d$-dimensional Adini element is defined by the triple $(K, P_A(K), D)$, where

- the geometric shape $K$ is a $d$-rectangle;
- the shape function space is
  $$P_A(K) := \{ p \cdot q : p \in Q_1(K), q \in \{ x_i^2 \}, \ 1 \leq i \leq d \},$$ (1)
  here and throughout this paper, $Q_l(K)$ denotes the space of all polynomials which are of degree $\leq l$ with respect to each variable $x_i$, over $K$;
- the nodal parameters are, for any $v \in C^1(K)$,
  $$D(v) := \left( v(a_i), \ \nabla v(a_i) \right),$$
  where $a_i$ are vertices of $K$, $i = 1, \ldots, 2^d$.

Remark 2.1 Classically, the shape function space of the Adini element was defined as [17]

$$P_A(K) := Q_1(K) + \text{span}\left\{ x_i^2 q \ | \ 1 \leq i \leq d, \ q \in Q_1(K) \right\}.$$
In this paper, we rewrite it into the form of (1), which will bring convenience in calculation.

Let \( \alpha \) denote the multiple-index with \( \alpha = (\alpha_1, \ldots, \alpha_d) \), \( \alpha_i (1 \leq i \leq d) \) are nonnegative integers, and \( |\alpha| = \sum_{i=1}^{d} \alpha_i, \ x^\alpha = \prod_{i=1}^{d} x_i^{\alpha_i} \). The partial derivative operator can be written as

\[
\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.
\]

Let \( e_i \ (1 \leq i \leq d) \) be the \( d \)-dimensional unit multi-index with its \( i \)-th entry equal to 1.

### 2.2 Structural Properties of the Adini Element

Given \( \Omega \) a \( d \)-dimensional domain, \( T_h \) is a triangulation on \( \Omega \), and \( K \in T_h \). Let \( \Pi^1_K \) be the piecewise bilinear interpolation operator on \( K \), namely \( \Pi^1_K v \in Q_1(K) \) and \( (\Pi^1_K v)(P) = v(P) \), for any vertex \( P \) of \( K \), and \( v \in C(K) \). Define on \( C(K) \) the operator \( R^1_K := I d - \Pi^1_K \), with \( I d \) being the identity operator. Define \( \Pi_{0.K} w = \int_K w \, dx \), for any \( w \in L^2(K) \). The global version \( \Pi_0 \) of the interpolation operator \( \Pi_{0.K} \) is defined as

\[
\Pi_0|_K = \Pi_{0.K}, \text{ for any } K \in T_h.
\]

**Lemma 2.2** It holds for \( w_h \in P_A(K) \) that

\[
R^1_K \frac{\partial w_h}{\partial x_i} \bigg|_{F'_K, i} = R^1_K \frac{\partial w_h}{\partial x_i} \bigg|_{F''_K, i}, \quad 1 \leq i \leq d. \tag{2}
\]

**Proof** Given \( w_h \in P_A(K) \), a direct calculation leads to that

\[
\frac{\partial w_h}{\partial x_i} \in Q_1(K) + \text{span} \left\{ (x_j - x_{j,c})^2 \cdot \hat{q}, \ \hat{q} \in Q'_1(K), \ 1 \leq j \leq d \right\}, \tag{3}
\]

where \( Q'_1(K) := \text{span} \{(x - x_c)^a | a_j \leq 1, a_i = 0 \} \). Further,

\[
R^1_K \left( \frac{\partial w_h}{\partial x_i} \right) \in S'_K, \ S'_K = \text{span} \left\{ (x_j - x_{j,c})^2 - h_j^2 \cdot \hat{q}, \ 1 \leq j \leq d, \ \hat{q} \in Q'_1(K) \right\}. \tag{4}
\]

Noting that \((x_j - x_{j,c})^2, \ 1 \leq j \leq d, \) evaluate the same on \( F'_K, i \) and \( F''_K, i \), we obtain (2). This finishes the proof. \( \square \)

For ease of expression, we define the following sets

\[
M_{i,j} = \{(\alpha_1, \ldots, \alpha_d) | \alpha_i = 1, \ 2 \leq \alpha_j \leq 3, \ \alpha_k \leq 1, \ k \neq i, j \},
\]

\[
M'_{i,j} = \{(\alpha_1, \alpha_2, \alpha_3) | \alpha_1 = 1, \ 2 \leq \alpha_j \leq 3, \ \alpha_k \leq 1, \ 2 \leq k \neq j \leq 3 \}.
\]

By means of (4), on \( F'_K, i \), \( F''_K, i \) of the element \( K \), we can get that using the Taylor expansion

\[
\left( R^1_K \frac{\partial w_h}{\partial x_i} \right) (x_1, \ldots, x_{i-1}, x_{i,c} \pm h_i, x_{i+1}, \ldots, x_d) = \sum_{1 \leq j \leq d} \sum_{a \in M_{i,j}} B^K_i(j, a) \Pi_{0.K}(\partial^\alpha w_h), \tag{5}
\]

with \( \alpha \in M_{i,j} \),

\[
B^K_i(j, \alpha) = \frac{1}{\alpha_j!} \left[ (x_j - x_{j,c})^{\alpha_j} - h_j^2 (x_j - x_{j,c})^{\alpha_j - 2} \right] \langle x - x_c \rangle^{\alpha - e_i - \alpha_j e_j}, \tag{6}
\]
\( \partial^\alpha w_h \in \text{span}\{1, x_j\} \), if \( \alpha_j = 2 \), and \( \partial^\alpha w_h \) is constant, if \( \alpha_j = 3 \). \( \quad \) (7)

Notice that \( \partial_i B^K_i (j, \alpha) = 0 \), since \( B^K_i (j, \alpha) \) does not contain the factor \( x_i \), if \( i \neq j \), and \( \Pi_{0, K} (\partial^\alpha w_h) \) are constant.

For example, in the two-dimensional case,

\[
\mathcal{R}_K^1 \frac{\partial w_h}{\partial x_1} (x_{1,c} \pm h_1, x_2) = \frac{1}{2} [(x_2 - x_{2,c})^2 - h_2^2] \Pi_{0, K} \frac{\partial^3 w_h}{\partial x_1 \partial x_2^2} + \frac{1}{6} [(x_2 - x_{2,c})^3 - h_2^2 (x_2 - x_{2,c})] \frac{\partial^4 w_h}{\partial x_1 \partial x_2^3},
\]

and in the three-dimensional case

\[
\mathcal{R}_K^1 \frac{\partial w_h}{\partial x_1} (x_{1,c} \pm h_1, x_2, x_3)
= \sum_{j=2}^3 \sum_{\alpha' \in M^j_{0, j}} \frac{1}{\alpha'_j!} \left[ (x_j - x_{j,c})^{\alpha'_j} - h_j^2 (x_j - x_{j,c})^{\alpha'_j - 2} \right] (x - x_c)^{\alpha_e - \alpha_j \epsilon_j} \Pi_{0, K} (\partial^{\alpha'} w_h)
\]

Given \( K \in \mathcal{T}_h \), we define the canonical interpolation operator \( \Pi_K : C^1 (K) \to P_A (K) \) by, for any \( v \in C^1 (K) \),

\[
(\Pi_K v)(P) = v(P) \quad \text{and} \quad (\nabla \Pi_K v)(P) = \nabla v(P),
\]

for any vertex \( P \) of \( K \). The interpolation operator \( \Pi_K \) has the following error estimates:

\[
| v - \Pi_K v |_{l, K} \leq C h^{4-l} | v |_{4, K}, \quad l = 0, 1, 2, 3, 4, \quad (8)
\]

provided that \( v \in H^s (K) \), where \( s \geq 4 \) and \( s > \frac{d}{2} + 1 \) such that \( H^s (K) \subset C^1 (K) \), see Remark 3.7.

**Lemma 2.3** For any \( u \in P_4 (K) \) and \( v \in P_A (K) \), it holds that

\[
(\nabla^2 (u - \Pi_K u), \nabla^2 v)_{L^2 (K)} = - \sum_{i=1}^d \sum_{1 \leq j \leq d \atop j \neq i} \frac{h_j^2}{3} \int_K \frac{\partial^4 u}{\partial x_i^2 \partial x_j^2} \frac{\partial^2 v}{\partial x_i^2} \, dx. \quad (9)
\]

**Proof** It follows from the definition of \( P_A (K) \) that

\[
\frac{\partial^2 v}{\partial x_i^2} \in Q_1 (K),
\]

\[
\frac{\partial^2 v}{\partial x_i \partial x_j} \in \left\{ p \cdot \tilde{q} \mid p \in P_1 (x_i, x_j), \tilde{q} \in Q_i^{k,j} (K) \right\}
+ \text{span} \left\{ x_k^2 \cdot \tilde{q}, \tilde{q} \in Q_1^{k,j} (K), \ 1 \leq k \leq d, i \neq j \right\}, \quad (10)
\]

where \( P_1 (x_i, x_j) := \text{span} \{1, x_i, x_j\}, Q_i^{k,j} (K) := \text{span} \{ x^\alpha \}_{|\alpha| \leq 1, \alpha_i = \alpha_j = 0} \).

Since \( u \in P_4 (K) \), we have, with \( \xi_i = \frac{x_i - x_{i,c}}{h_i} \),

\[
u = u_1 + \frac{h_1^4}{4!} \sum_{i=1}^d \frac{\partial^4 u}{\partial x_i^4} \xi_i^4 + \frac{h_2^2 h_j}{4} \sum_{i=1}^d \sum_{1 \leq j \leq d \atop j \neq i} \frac{\partial^4 u}{\partial x_i^2 \partial x_j^2} \xi_i^2 \xi_j^2, \quad (11)
\]

where \( u_1 \in P_A (K) \).
The Taylor expansion and the definition of the operator $\Pi_K$ yield
\[ u - \Pi_K u = \frac{h_i^4}{4!} \sum_{i=1}^d \frac{\partial^4 u}{\partial x_i^4} (x_i^2 - 1)^2 + \frac{h_i^2 h_j^2}{4} \sum_{1 \leq i < j \leq d} \frac{\partial^4 u}{\partial x_i^2 \partial x_j^2} (x_i^2 - 1)(x_j^2 - 1). \] (12)

Thus
\[ \frac{\partial^2 (u - \Pi_K u)}{\partial x_i^2} = \frac{h_i^2}{4!} \frac{\partial^4 u}{\partial x_i^4} (12x_i^2 - 4) + \frac{h_j^2}{2} \frac{\partial^4 u}{\partial x_i^2 x_j^2} (x_j^2 - 1), \quad 1 \leq i < j \leq d, \]
\[ \frac{\partial^2 (u - \Pi_K u)}{\partial x_i \partial x_j} = h_i h_j \frac{\partial^4 u}{\partial x_i^2 x_j^2} x_i x_j, \quad 1 \leq i < j \leq d. \]

Since
\[ \int_K (12x_i^2 - 4) q_k dx = 0, \quad q_k \in Q_1(K), \quad 1 \leq k \neq i \leq d. \]
and
\[ \int_K (x_j^2 - 1) x_i dx = 0, \quad \int_K (x_j^2 - 1) x_j dx = 0, \quad \int_K (x_j^2 - 1) x_i x_j dx = 0, \quad 1 \leq i \neq j \leq d, \]
a combination of (10) and (12) and some elementary calculation yield
\[ \int_K \frac{\partial^2 (u - \Pi_K u)}{\partial x_i^2} \frac{\partial^2 v}{\partial x_i^2} dx = -\frac{h_i^2}{3} \int_K \frac{\partial^4 u}{\partial x_i^2 \partial x_j^2} \frac{\partial^2 v}{\partial x_i^2} dx, \quad 1 \leq i \neq j \leq d. \] (13)

By the same argument, it yields
\[ \int_K \frac{\partial^2 (u - \Pi_K u)}{\partial x_i \partial x_j} \frac{\partial^2 v}{\partial x_i \partial x_j} dx = 0, \] (14)
which completes the proof.

3 The Capacity of Adini Element for Biharmonic Equations

3.1 Model Problem and Finite Element Discretisation

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. In this paper, We consider the model problem:
\[ \begin{aligned}
\Delta^2 u &= f, \quad \text{in } \Omega, \\
u &= \frac{\partial u}{\partial n} = 0, \quad \text{on } \partial \Omega.
\end{aligned} \] (15)

The variational formulation is, given $f \in H^{-2}(\Omega)$, to find $u \in V := H_0^2(\Omega)$, such that
\[ a_\Omega(u, v) = (f, v), \quad \text{for any } v \in V, \] (16)
where $a_\Omega(u, v) := \sum_{i,j=1}^d \int_{\Omega} \partial_{ij} u \partial_{ij} v$ for $u, v \in H^2(\Omega)$ and $(f, v) := \int_{\Omega} f v dx$. 
Let $T_h$ be a regular $d$-rectangle triangulation of the domain $\Omega$. Define the Adini element space in a standard way by

$$ V_h := \{ v_h \in L^2(\Omega) : v_h|_K \in P_A(K), \forall K \in T_h, \text{ and } \nabla v_h \text{ is continuous at all internal vertices} \}, $$

and associated with the boundary condition,

$$ V_{h0} := \{ v_h \in V_h : v_h \text{ and } \nabla v_h \text{ vanishes at all boundary vertices} \}. $$

Evidently, $V_h \subset H^1(\Omega)$ and $V_{h0} \subset H^1_0(\Omega)$ [16]. However, $V_h \not\subset H^2(\Omega)$, and $V_{h0} \not\subset H^2_0(\Omega)$.

Evidently, $P_3(K) \subset P_A(K)$ for any $K \in T_h$. By the standard technique,

$$ \inf_{v_h \in V_{h0}} |v - v_h|_{l,h} \leq C h^{4-l} |v|_{4,\Omega}, \ l = 0, 1, 2, 3, 4, $$

for any $v \in H^4(\Omega)$. Herein and throughout this paper, $C$ denotes a generic positive constant which is independent of the meshsize and may be different at different places.

Associated with the model problem, the Adini finite element problem is to find $u_h \in V_{h0}$, such that

$$ a_h(u_h, v_h) = (f, v_h)_{L^2(\Omega)}, \text{ for any } v_h \in V_{h0}, $$

where $a_h(u_h, v_h) := \sum_{K \in T_h} a_K(u_h, v_h)$. The problem (18) is well defined by [17].

Define a semi-norm over $V_h$ by $|v_h|_h^2 := \sum_{K \in T_h} \| \nabla^2 u_h \|_0^2, K$. By Poincaré inequality, $|\cdot|_h$ is a norm on $V_{h0}$, and it is equivalent to $\| \cdot \|_h$, while the latter denotes the piecewise $H^2$ norm.

### 3.2 Error Analysis in the Energy Norm

In this section, we present the upper bound of the energy norm of the error of the finite element scheme (18). First of all, the following estimate is standard and well known, (c.f., e.g., [17]) and we include it here for completeness.

**Theorem 3.1** Let $u$ and $u_h$ be the solutions of (16) and (18), respectively. If $u \in H^2_0(\Omega) \cap H^3(\Omega)$, then

$$ \|u - u_h\|_{2,h} \leq C h \|u\|_{3,\Omega}. $$

Now we turn to the estimate where the solution is less or more regular.

#### 3.2.1 Error Estimate Without Full Regularity

Herein, we consider the regularity of the solution $u \in H^{2+s}(\Omega)$ with $0 < s < 1$. We refer to, e.g., [2] for related discussion.

For $f \in L^2(\Omega)$, define

$$ \text{osc}(f) := \left( \sum_{T \in T_h} h^4 \left( \inf_{f \in P_{r-2}(T)} \| f - \bar{f} \|_{0,T}^2 \right) \right)^{1/2}, $$

here, $r \geq m$ is arbitrary. It holds that

$$ \text{osc}(f) \leq C h^2 \| f \|_{0,\Omega}. $$

In order to obtain our main result, we need two important lemmas from [8] listed below, which can be extended to any dimension by the same techniques.
Lemma 3.2 Let $s_h \in V_h$. Then
\[
\left( \sum_{T \in T_h} h^4 ||f - \Delta^2 s_h||^2_{0,T} \right)^{1/2} \leq C ||\nabla^2 u - \nabla^2 s_h||_{0,\Omega} + \text{osc}(f).
\] (22)

Lemma 3.3 The Adini element passes the F-E-M Test from [8] and [14], which implies
\[
|T_k(s_h, v_h)|_0 \leq C (||\nabla^2 u - \nabla^2 s_h||_0 + \text{osc}(f) + ||\nabla^2 u - \Pi^0 \nabla^2 u||_0)|v_h|_{2,h},
\] (23)
where
\[
T_k(s_h, v_h) := \sum_{T \in T_h} \int_{\partial T} \frac{\partial}{\partial n} \left( \frac{\partial s_h}{\partial x_k} \right) \frac{\partial (v_h - \Pi^c v_h)}{\partial x_k} \, ds, \quad k = 1, \ldots, d,
\] (24)
with $s_h, v_h \in V_h$ and $\Pi^c$ be the projection average interpolation operator as defined in [8].

Now, we give our main result as follows

Theorem 3.4 Let $u \in H^{2+s}(\Omega)$, $0 < s < 1$ and $u_h \in V_h$ be the solutions of (16) and (18), respectively. It holds that
\[
||\nabla^2 u - u_h||_0 \leq Ch^s|u|_{2+s}.
\] (25)

Proof Given $v_h, s_h \in V_h$, the decomposition of the consistency error can be written as, by integrating by parts
\[
(\nabla^2 u, \nabla^2 v_h) - (f, v_h)
\]
\[
= (\nabla^2 u, \nabla^2 (v_h - \Pi^c v_h)) - (f, v_h - \Pi^c v_h)
\]
\[
= (\nabla^2 u - \nabla^2 s_h, \nabla^2 (v_h - \Pi^c v_h)) - (f, v_h - \Pi^c v_h)
\]
\[
+ (\nabla^2 s_h, \nabla^2 (v_h - \Pi^c v_h))
\]
\[
= (\nabla^2 u - \nabla^2 s_h, \nabla^2 (v_h - \Pi^c v_h)) - (f, v_h - \Pi^c v_h)
\]
\[
- \sum_{T \in T_h} \int_T (\nabla \cdot \nabla^2 s_h) \cdot \nabla (v_h - \Pi^c v_h) \, dx
\]
\[
+ \sum_{k=1}^d \sum_{T \in T_h} \int_{\partial T} \frac{\partial}{\partial n} \left( \frac{\partial s_h}{\partial x_k} \right) \frac{\partial (v_h - \Pi^c v_h)}{\partial x_k} \, ds
\]
\[
= (\nabla^2 u - \nabla^2 s_h, \nabla^2 (v_h - \Pi^c v_h)) + \sum_{T \in T_h} \int_T (\Delta^2 s_h - f)(v_h - \Pi^c v_h) \, dx
\]
\[
- \sum_{T \in T_h} \int_{\partial T} \frac{\partial (\Delta s_h)}{\partial n} (v_h - \Pi^c v_h) \, ds + \sum_{k=1}^d T_k(s_h, v_h)
\]
\[
:= I_1 + I_2 + I_3 + I_4.
\] (26)
The first term on the right-hand side of (26) can be bounded as
\[
|I_1| \leq ||\nabla^2 u - \nabla^2 s_h||_0|v_h|_{2,h}.
\] (27)
The usual inverse estimate implies that
\[
\sum_{T \in T_h} \sum_{j=0}^1 \left( h^{2(j-2)} ||v_h - \Pi^c v_h||_j^2 \right) + ||\Pi^c v_h||^2_{2,h} \leq C |v_h|_{2,h}^2.
\] (28)
It follows from Lemma 3.2 and (28) that
\[ |I_2| \leq C \left( \| \nabla^2 u - \nabla^2_h s_h \|_0 + \text{osc}(f) \right) |v_h|_{2,h}. \] (29)

By the trace inequality, the inverse estimate and the triangle inequality, it yields
\[ |I_3| \leq C \left( \| \nabla^2 u - \nabla^2_h s_h \|_0 + \| \nabla^2 u - \Pi^0 \nabla^2 u \|_0 \right) |v_h|_{2,h}. \] (30)
A combination of (21), (23), (27), (29) and (30) completes the proof. □

3.2.2 Error Estimate with High Regularity

Now we consider the case where the regularity of the solution is high.

**Theorem 3.5** Let \( u \) and \( u_h \) be the solutions of (16) and (18), respectively. Assume that \( u \in H^4(\Omega) \). Then
\[ |u - u_h|_h \leq C h^2 |u|_{4,\Omega}. \] (31)

**Proof** By the second Strang Lemma, we have
\[ |u - u_h|_h \leq C \left( \inf_{v \in V_0} |u - v|_h + \sup_{w_h \in V_0} \frac{|E_h(u, w_h)|}{|w_h|_h} \right). \] (32)
where
\[ E_h(u, w_h) := a_h(u, w_h) - (f, w_h) = \sum_{K \in T_h} \int_{\partial K} \frac{\partial^2 u}{\partial n^2} \frac{\partial w_h}{\partial n} ds. \] (33)
The first term of (32) is the approximation error and the second one is the consistency error.

We shall consider separately the faces orthogonal to the \( x_i \) axes (1 ≤ i ≤ d), namely we rewrite the consistency error to
\[ E_h(u, w_h) = \sum_{i=1}^d E_{x_i}(u, w_h), \] (34)
with
\[ E_{x_i}(u, w_h) = \sum_{K \in T_h} \int_{\partial K} \frac{\partial^2 u}{\partial n^2} \frac{\partial w_h}{\partial x_i} n_{x_i} ds \]
\[ = \sum_{K \in T_h} \int_{\partial K} \frac{\partial^2 u}{\partial n^2} R_{1}^{K} \frac{\partial w_h}{\partial x_i} n_{x_i} ds \]
\[ = \sum_{K \in T_h} \left( \int_{F_{K,i}^w} - \int_{F_{K,i}^w} \right) \frac{\partial^2 u}{\partial x_i^2} R_{1}^{K} \frac{\partial w_h}{\partial x_i} ds \]
\[ := \sum_{K \in T_h} I_i^{K} \left( \frac{\partial^2 u}{\partial x_i^2} R_{1}^{K} \frac{\partial w_h}{\partial x_i} \right), \] (35)
where \( n_{x_i} \) is the unit outward normal parallel to the \( x_i \) axis.
Let $K \in \mathcal{T}_h$ and denote $g = \frac{\partial^2 u}{\partial x_i^2} |_{K}$. It holds that, by (5),

$$I^K_i(\frac{\partial w_h}{\partial x_i}, R^K_1) = \left( \int_{F'_{K,i}} - \int_{F_{K,i}} \right) g R^K_1 \frac{\partial w_h}{\partial x_i} \, ds = \left( \int_{F'_{K,i}} - \int_{F_{K,i}} \right) g \sum_{1 \leq j \leq d} \sum_{\alpha \in M_{i,j}} B^K_i(\alpha) \Pi_{0,K} \partial^\alpha w_h \, ds$$

$$= \int_K \frac{\partial g}{\partial x_i} \sum_{1 \leq j \leq d} \sum_{\alpha \in M_{i,j}} B^K_i(\alpha) \Pi_{0,K} \partial^\alpha w_h \, dx$$

$$= \int_K \frac{\partial g}{\partial x_i} \sum_{1 \leq j \leq d} \sum_{\alpha \in M_{i,j}} B^K_i(\alpha) \partial^\alpha w_h \, dx$$

$$+ \int_K \frac{\partial g}{\partial x_i} \sum_{1 \leq j \leq d} \sum_{\alpha \in M_{i,j}} B^K_i(\alpha) (\Pi_{0,K} - I) \partial^\alpha w_h \, dx$$

$$:= L^K_{i,1} + L^K_{i,2}.$$ 

Integrating by parts yields

$$L^K_{i,1} = \int_K \frac{\partial g}{\partial x_i} \sum_{1 \leq j \leq d} \sum_{\alpha \in M_{i,j}} B^K_i(\alpha) \partial^\alpha w_h \, dx$$

$$= - \int_K \frac{\partial^2 g}{\partial x_i^2} \sum_{1 \leq j \leq d} \sum_{\alpha \in M_{i,j}} B^K_i(\alpha) \partial^{\alpha - \ell_i} w_h \, dx$$

$$+ \left( \int_{F'_{K,i}} - \int_{F_{K,i}} \right) \frac{\partial g}{\partial x_i} \sum_{1 \leq j \leq d} \sum_{\alpha \in M_{i,j}} B^K_i(\alpha) \partial^{\alpha - \ell_i} w_h \, ds.$$ 

Since $u \in H^4(\Omega)$, $w_h \in H^1(\Omega)$ and $\partial^{\alpha - \ell_i} w_h$, $(\alpha \in M_{i,j})$ are tangential derivatives of the faces that orthogonal to the axe $x_i$, thus $\frac{\partial^3 u}{\partial x_i^3}$ and $\partial^{\alpha - \ell_i} w_h$, $(\alpha \in M_{i,j})$ are continuous across faces $F'_{K,i}$, $F''_{K,i}$, we obtain

$$\sum_{K \in \mathcal{T}_h} L^K_{i,1} = - \sum_{K \in \mathcal{T}_h} \int_K \frac{\partial^4 u}{\partial x_i^4} \sum_{1 \leq j \leq d} \sum_{\alpha \in M_{i,j}} B^K_i(\alpha) \partial^{\alpha - \ell_i} w_h \, dx$$

$$\leq C h^{|\alpha|-1} \sum_{K \in \mathcal{T}_h} |u|_{4,K} |\partial^{\alpha - \ell_i} w_h|_{0,K},$$

where we have used the fact that $\max_j \max_{\alpha \in K} B^K_i(\alpha) \leq C h^{|\alpha|-1}$.

A further application of the inverse estimate yields

$$\sum_{K \in \mathcal{T}_h} L^K_{i,1} \leq C h^2 \sum_{K \in \mathcal{T}_h} |u|_{4,K} \nabla^2 w_h |_{0,K}$$

$$\leq C h^2 |u|_{4,\Omega} w_h |_{2,h}. \quad (39)$$
Then, we estimate the second term of (36) \(L_{i,2}^K\).

\[
L_{i,2}^K = \int_K \frac{\partial g}{\partial x_i} \sum_{1 \leq j \leq d} \sum_{\alpha \in M_{i,j}} B^K_i (j, \alpha)(\Pi_{0,K} - Id) \partial^\alpha w_h \, dx
\]

\[
= \int_K (Id - \Pi_0^K) \frac{\partial g}{\partial x_i} \sum_{1 \leq j \leq d} \sum_{\alpha \in M_{i,j}} B^K_i (j, \alpha)(\Pi_{0,K} - Id) \partial^\alpha w_h \, dx
\]

\[
+ \int_K \Pi_{0,K}(\frac{\partial g}{\partial x_i}) \sum_{1 \leq j \leq d} \sum_{\alpha \in M_{i,j}} B^K_i (j, \alpha)(\Pi_{0,K} - Id) \partial^\alpha w_h \, dx.
\]

According to (7), since

\[
\int_K [(x_j - x_{j,c})^2 - h_j^2](\Pi_{0,K} - Id)(c_1 + c_2 x_j) \, dx = 0, \quad c_1, \ c_2 \ are \ constant \ coefficients,
\]

and

\[
\int_K [(x_j - x_{j,c})^3 - h_j^2(x_j - x_{j,c})](\Pi_{0,K} - Id)c_3 \, dx = 0, \quad c_3 \ is \ a \ constant,
\]

we can get that

\[
\int_K \Pi_{0,K}(\frac{\partial g}{\partial x_i}) \sum_{1 \leq j \leq d} \sum_{\alpha \in M_{i,j}} B^K_i (j, \alpha)(\Pi_{0,K} - Id) \partial^\alpha w_h \, dx = 0. \tag{40}
\]

The interpolation error estimate and the inverse estimate yield

\[
\int_K (Id - \Pi_{0,K}) \frac{\partial g}{\partial x_i} \sum_{1 \leq j \leq d} \sum_{\alpha \in M_{i,j}} B^K_i (j, \alpha)(\Pi_{0,K} - Id) \partial^\alpha w_h \, dx
\]

\[
\leq Ch \|g\|_{2,K} h^{|\alpha|-1} \|\partial^\alpha w_h\|_{0,K} \leq Ch^{|\alpha|} \|u|_{4,K} \|\partial^\alpha w_h\|_{0,K}
\]

\[
\leq Ch^2 \|u|_{4,K} \|\nabla^2 w_h\|_{0,K}.
\]

Then, we can get

\[
L_{i,2}^K \leq Ch^2 \|u|_{4,K} \|\nabla^2 w_h\|_{0,K}. \tag{41}
\]

A combination of (39) and (41) leads to

\[
E_{x_i}(u, w_h) = \sum_{K \in T_h} I^K_i \left( \frac{\partial^2 u}{\partial x_i^2}, R^K_i \frac{\partial w_h}{\partial x_i} \right) \leq Ch^2 \|u|_{4,\Omega} \|w_h\|_{2,h}.
\]

Similarly we obtain further,

\[
E_h(u, w_h) \leq Ch^2 \|u|_{4,\Omega} \|w_h\|_{2,h}. \tag{42}
\]

This, combined with the approximation error estimate, finishes the proof.
3.3 Error Analysis of the Adini Element in the $L^2$ Norm

In this section, we present the lower bound estimate of the error in the $L^2$ norm. This is a generalisation of the result in [9] to arbitrary dimension. The main result of this section is the theorem below.

**Theorem 3.6** Let $u$ and $u_h$ be solutions of problem (16) and (18), respectively. Suppose that $u \in H^2_0(\Omega) \cap H^s(\Omega)$, $s \geq 4$ and $s > \frac{d}{2} + 1$. Then, provided $||f||_{L^2(\Omega)} \neq 0$,

$$||u - u_h||_{L^2(\Omega)} \geq \beta h^2,$$

where $\beta = \delta/||f||_{L^2(\Omega)}$, $\delta$ is a positive constant, which is independent of the mesh size $h$.

**Remark 3.7** By the embedding theorem of the Sobolev space, [3, 16], we need higher regularity of the solution in higher dimensions in order to guarantee $H^s(K) \subset C^1(K)$. It ensures the continuity of interpolation operators.

**Remark 3.8** For the $d$-dimensional domain $\Omega$, the condition $||f||_{L^2(\Omega)} \neq 0$ implies that $|\frac{\partial^2 u}{\partial x_i \partial x_j}|_{H^1(\Omega)} \neq 0$, $1 \leq i \neq j \leq d$. In fact, if $|\frac{\partial^2 u}{\partial x_i \partial x_j}|_{H^1(\Omega)} = 0$, $1 \leq i \neq j \leq d$, then $u$ is of the form

$$u = \sum_{i=1}^{d} \sum_{1 \leq j \leq d, j \neq i} c_{ij} x_i x_j + \sum_{i=1}^{d} g(x_i),$$

for some function $g(x_i)$ with respect to $x_i$. Then the boundary condition indicates $u \equiv 0$, which contradicts $u \neq 0$.

We postpone the proof of Theorem 3.6 after several technical lemmas.

Define the global interpolation operator $\Pi_h$ to $V_h$ by

$$\Pi_h|_K = \Pi_K \text{ for any } K \in T_h.$$ 

By means of Lemma 2.3, we can obtain the following crucial result.

**Lemma 3.9** For $u \in H^2_0(\Omega) \cap H^s(\Omega)$, $s \geq 4$ and $s > \frac{d}{2} + 1$, it holds that,

$$(\nabla^2_h(u - \Pi_h u), \nabla^2_h u)_{L^2(\Omega)} \geq \alpha h^2,$$

for some positive constant $\alpha$, which is independent of the mesh size $h$ provided that $||f||_{L^2(\Omega)} \neq 0$ and that the mesh size is small enough.

**Proof** Given any element $K$, we follow [9] to define $P_K v \in P_4(K)$ by

$$\int_K \nabla^l P_K v \, dx = \int_K \nabla^l v \, dx, \text{ } l = 0, 1, 2, 3, 4,$$

for any $v \in H^s(\Omega)$, $(s \geq 4$ and $s > \frac{d}{2} + 1)$. Note that the operator $P_K$ is well-defined. The interpolation operator $P_K$ has the following error estimates:

$$|v - P_K v|_{j,K} \leq Ch^{4-j}|v|_{4,K}, \text{ } j = 0, 1, 2, 3, 4,$$

$$|v - P_K v|_{j,K} \leq C h|v|_{j+1,K}, \text{ } j = 0, 1, 2, 3,$$

provided that $v \in H^s(\Omega)$, $(s \geq 4$ and $s > \frac{d}{2} + 1)$. It follows from the definition of $P_K$ in (45) that

$$\nabla^4 P_K v = \Pi_{0,K} \nabla^4 v.$$
By the aid of $P_K$, we have the following decomposition

\[
(\nabla_h^2 (u - \Pi_h u), \nabla_h^2 \Pi_h u)_{L^2(\Omega)} = \sum_{K \in T_h} (\nabla_h^2 (P_K u - \Pi_K P_K u), \nabla_h^2 \Pi_K u)_{L^2(K)} \\
+ \sum_{K \in T_h} (\nabla_h^2 (I - \Pi_K)(I - P_K) u, \nabla_h^2 \Pi_K u)_{L^2(K)} = J_1 + J_2. \tag{48}
\]

We first analyze the first term $J_1$ on the right-hand side of (48). By means of Lemma 2.3, the first term $J_1$ on the right-hand side of (48) can be rewritten as

\[
J_1 = - \sum_{K \in T_h} \sum_{i=1}^d \sum_{1 \leq j \leq d, j \neq i} h_j^2 \int_K \frac{\partial^4 P_K u}{\partial x_i^2 \partial x_j^2} \frac{\partial^2 \Pi_K u}{\partial x_i^2} \, dx \\
- \sum_{K \in T_h} \sum_{i=1}^d \sum_{1 \leq j \leq d, j \neq i} h_j^2 \int_K \frac{\partial^4 u}{\partial x_i^2 \partial x_j^2} \frac{\partial^2 u}{\partial x_i^2} \, dx \\
+ \sum_{K \in T_h} \sum_{i=1}^d \sum_{1 \leq j \leq d, j \neq i} h_j^2 \int_K \frac{\partial^4 (I - P_K) u}{\partial x_i^2 \partial x_j^2} \frac{\partial^2 \Pi_K u}{\partial x_i^2} \, dx \\
+ \sum_{K \in T_h} \sum_{i=1}^d \sum_{1 \leq j \leq d, j \neq i} h_j^2 \int_K \frac{\partial^4 (I - \Pi_K) u}{\partial x_i^2 \partial x_j^2} \frac{\partial^2 (I - P_K) u}{\partial x_i^2} \, dx.
\]

Since \( \frac{\partial u}{\partial x_j} \bigg|_{F_{K,j}^e} = 0, \frac{\partial u}{\partial x_j} \bigg|_{F_{K,j}^{e'}} = 0 \), and \( \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_j} \bigg|_{F_{K,j}^e} = 0, \frac{\partial^3 u}{\partial x_i \partial x_j \partial x_j} \bigg|_{F_{K,j}^{e'}} = 0 \), integrating by parts yields

\[
\sum_{K \in T_h} \sum_{i=1}^d \sum_{1 \leq j \leq d, j \neq i} h_j^2 \int_K \frac{\partial^4 u}{\partial x_i^2 \partial x_j^2} \frac{\partial^2 u}{\partial x_i^2} \, dx = - \sum_{K \in T_h} \sum_{i=1}^d \sum_{1 \leq j \leq d, j \neq i} h_j^2 \int_K \left( \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \right)^2 \, dx \\
= - \sum_{K \in T_h} \sum_{i=1}^d \sum_{1 \leq j \leq d, j \neq i} h_j^2 \left\| \frac{\partial^3 u}{\partial x_i^2 \partial x_j} \right\|_{L^2(K)}^2.
\]

By the commuting property of (47),

\[
\frac{\partial^4 (I - P_K) u}{\partial x_i^2 \partial x_j^2} = (I - \Pi_{0,K}) \frac{\partial^4 u}{\partial x_i^2 \partial x_j^2}, \quad 1 \leq i \neq j \leq d.
\]

Note that

\[
\sum_{i=1}^d \left\| \frac{\partial^2 \Pi_K u}{\partial x_i^2} \right\|_{L^2(K)} \leq C |u|_{3,K}.
\]
Hence, a combination of (52)–(56) leads to
\[ J_1 = \sum_{K \in \mathcal{T}_h} \sum_{i=1}^{d} \sum_{1 \leq j \leq d, j \neq i} \frac{h_j^2}{3} \left\| \frac{\partial^3 u}{\partial x_i \partial x_j} \right\|_{L^2(K)}^2 + O(h^2) \left\| (I - \Pi_0, \Pi) \nabla_h^4 u \right\|_{L^2(K)}^2 \|u|_{3,K}. \] (49)

We turn to the second term \( J_2 \) on the right-hand side of (48). By the Poincare inequality, and the commuting property of (47),
\[ |J_2| \leq \chi h^2 \sum_{K \in \mathcal{T}_h} \| \nabla_h^4 (I - \Pi_0) u \|_{L^2(K)}^2 \|u|_{3,\Omega} \]
\[ \leq \chi h^2 \| (I - \Pi_0) \nabla_h^4 u \|_{L^2(\Omega)}^2 \|u|_{3,\Omega}. \] (50)

Since the piecewise constant functions are dense in the space \( L^2(\Omega) \),
\[ \| (I - \Pi_0) \nabla_h^4 u \|_{L^2(\Omega)} \to 0, \text{ when } h \to 0. \] (51)

Summation of (49), (50) and (51) completes the proof. \( \square \)

Lemma 3.10 Let \( u \) and \( u_h \) be solutions of problem (16) and (18), respectively. Then,
\[-f, u - u_h \|_{L^2(\Omega)} = a_h(u, \Pi_h u - u_h) - (f, \Pi_h u - u_h)_{L^2(\Omega)} + a_h(u - \Pi_h u, u - \Pi_h u) + a_h(u - \Pi_h u, u_h - \Pi_h u) \]
\[+ 2(f, \Pi_h u - u)_{L^2(\Omega)} + 2a_h(u - \Pi_h u, \Pi_h u). \] (52)

Proof of Theorem 3.6 It follows from (42) that
\[ a_h(u, \Pi_h u - u_h) - (f, \Pi_h u - u_h)_{L^2(\Omega)} \leq \chi h^2 \|u|_{4,\Omega} \|\Pi_h u - u_h\|_{h} \leq \chi h^4 |u|_{4,\Omega}^2. \] (53)

By the Cauchy–Schwarz inequality and the error estimate (17), it yields
\[ a_h(u - \Pi_h u, u - \Pi_h u) + 2(f, \Pi_h u - u)_{L^2(\Omega)} \leq \chi h^4 (|u|_{4,\Omega} + \|f\|_{L^2(\Omega)}) |u|_{4,\Omega}. \] (54)
\[ a_h(u - \Pi_h u, u_h - \Pi_h u) \leq \chi h^4 |u|_{4,\Omega}^2. \] (55)

The error estimate of the last term of (52) by Lemma 3.9 gives
\[ a_h^2 \leq a_h(u - \Pi_h u, \Pi_h u). \] (56)

Hence, a combination of (52)–(56) leads to
\[-f, u - u_h \|_{L^2(\Omega)} \geq \delta h^2. \]
for some positive constant \( \delta \), which is independent of the mesh size \( h \) and the mesh size is small enough.

Therefore,
\[ \|u - u_h\|_{L^2(\Omega)} = \sup_{0 \neq w \in L^2(\Omega)} \frac{(w, u - u_h)_{L^2(\Omega)}}{\|w\|_{L^2(\Omega)}} \geq \frac{(-f, u - u_h)_{L^2(\Omega)}}{\| - f \|_{L^2(\Omega)}} \geq \delta |\|f\|_{L^2(\Omega)} h^2. \]

This finishes the proof. \( \square \)
Fig. 2 The errors in $L^2$ and $H^2$ norms for $u_1(x, y, z)$ and $u_2(x, y, z)$ by uniform cubic meshes

Fig. 3 The errors in $L^2$ and $H^2$ norms for $u_1(x, y, z)$ and $u_2(x, y, z)$ by non-congruence meshes

4 Numerical Examples

In this section, we present some numerical results of the three-dimensional Adini element by congruence partition of cubic meshes and non-congruence partition of domain $\Omega$ to demon-
The errors of \( u_3(x,y,z) \)

![Graph showing log(errors) vs log(N) for L2err and H2err](image)

**Fig. 4** The errors in \( L^2 \) and \( H^2 \) norms for \( u_3(x,y,z) \) in three-dimensional L-shaped domain

strate our theoretical results. Herein, we give \( u_1(x, y, z) = \sin^2(\pi x)\sin^2(\pi y)\sin^2(\pi z) \) and \( u_2(x, y, z) = x^2(1-x)^2y^2(1-y)^2z^2(1-z)^2 \) as the exact solution of problem (15), respectively. One can see the errors and the rate of convergence computed by uniform cubic meshes with the meshsize \( h = \frac{1}{N} \) for some integer \( N \) in Fig. 2. One can also see the errors and the rate of convergence computed by non-congruence meshes in Fig. 3 for the logarithmic plot. Furthermore, we also give the numerical results in the three-dimensional L-shaped domain using the exact solution \( u_3(x, y, z) = \sin^2(2\pi x)\sin^2(2\pi y)\sin^2(2\pi z) \) listed in Fig. 4.

5 Concluding Remarks

In this paper, we studied the accuracy of the Adini element as a discretization scheme for biharmonic equations in any dimension. We showed that the convergence rate is of \( O(h^2) \) order in the energy norm in any dimension, and moreover, we show that the convergence rate can not be non-trivially higher than \( O(h^2) \) order in the \( L^2 \) norm in any dimension. By the Poincaré inequality, we arrive at the conclusion that the convergence rate of the Adini
Table 1  Numbers of DOFs and convergence order for Adini, rectangular Morley, and BFS elements

|                  | Adini | Rectangular Morley | B–F–S |
|------------------|-------|-------------------|-------|
| #(DOFs)/#(DOFs of Adini) | 1     | 1                 | 2^{d}/(d+1) |
| r                | 2     | 1                 | 2     |

The ratio is in the asymptotic sense.

element for discretising biharmonic equations is $O(h^2)$ in $L^2$, $H^1$ and energy norms. This presents a complete exploration of the capacity of the scheme.

The results provided in this paper are optimal in two-folded. On one hand, the full convergence rate of the energy norm is established under the assumption $u \in H^4(\Omega)$, which is standard and of the lowest regularity. Besides, we do not need the grid to be a uniform one. On the other hand, combining the two results together illustrates that neither of these two can be improved. The analysis of this paper has been sharp and economic.

The high accuracy makes the Adini element a competitive element, especially for high dimensions. Indeed, let us compare the degree of freedoms and the convergence rate of the Adini element with those of other famous elements, for example, the rectangular Morley element, and the BFS element, [3,16]. We list the number of degrees of freedom (DOFs for breivation) and order of convergence (denoted by $r$) of each element in Table 1. It can be seen that the convergent rate of Adini element is one order higher than that of the rectangular Morley element with asymptotically a same number of DOFs. As to the BFS element, it has the same convergence rate as that of the Adini element, but it needs much more DOFs, especially in high dimensions.

It is somehow surprising to observe the convergence rates in different norms are of the same order. This is because the Adini element function is not moment continuous across the edges, but it possesses internal symmetry on every cell when the grid is of tensor type. This point of view will motivate us on other elements.

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