QUASI-MAXIMUM MODULUS PRINCIPLE FOR THE STOKES EQUATIONS

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Abstract. In this paper, we extend the maximum modulus estimate of the solutions of the nonstationary Stokes equations in the bounded $C^2$ cylinders for the space variables in $\mathbb{R}^n$ to time estimate. We show that if the boundary data is $L^\infty$ and the normal part of the boundary data has log-Dini continuity with respect to time, then the velocity is bounded. We emphasize that there is no continuity assumption on space variables in the new maximum modulus estimate. This completes the maximum modulus estimate.

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1. INTRODUCTION

In this paper, we study the maximum modulus principle of the nonstationary Stokes equations:

$$
\begin{align*}
    u_t - \Delta u + \nabla p &= 0 \quad \text{in } \Omega \times (0, T), \\
    \text{div } u &= 0 \quad \text{in } \Omega \times (0, T), \\
    u|_{t=0} &= 0 \quad \text{in } \Omega, \\
    u|_{\partial \Omega \times (0, T)} &= g \quad \text{on } \partial \Omega \times (0, T),
\end{align*}
$$

(1.1)

where $\Omega$ is $C^2$ bounded domain in $\mathbb{R}^n (n \geq 3)$ such that the boundary $\partial \Omega$ of $\Omega$ is connected and $0 < T \leq \infty$. We assume the boundary data $g$ satisfies the compatibility condition:

$$
\int_{\partial \Omega} g(Q, t) \cdot N(Q) dQ = 0
$$

for almost all $t \in (0, T)$, where $N(Q)$ is the outward unit normal vector at $Q \in \partial \Omega$.

The maximum modulus principle for the stationary Stokes equations were studied by many mathematicians (see [2], [4], [5], [6], [9], [13], etc).

But, the maximum modulus estimate of the velocity of the nonstationary Stokes equations, when the boundary data is bounded, are not known. A. V. Solonnikov [12] showed that if $\Omega$ is $C^{2+\alpha}, 0 < \alpha$ smooth convex bounded domain and $g \in C(\partial \Omega \times (0, T))$ with
$g \cdot N = 0$, the solution $u$ of (1.1) is continuous in $\Omega \times (0, T)$ such that

$$\sup_{(x,t) \in \Omega \times (0, T)} |u(x,t)| \leq c \sup_{(Q,t) \in \partial \Omega \times (0, T)} |g(Q,t)|$$

for some positive constant $c$ independent of $g$. In recent, T. Chang and H. Choe\[1\] improve the A. V. Solonnikov’s result. For the self contained presentation and rigorous expressions, we repeat the same symbols and definitions of [1]. We denote $E$ for the fundamental solution to Laplace equation and $\Gamma$ for the fundamental solution to heat equation with unit conductivity. We define the $(n - 1)$-dimensional convolution

$$S(f)(x) = \int_{\partial \Omega} E(x - Q)f(Q)dQ, \quad x \in \Omega$$

for a real-value function $f : \partial \Omega \to \mathbb{R}$ which is just the single layer potential of $f$ on $\partial \Omega$. We introduce a composite kernel which is the core of Poisson kernel. We define a composite kernel function $\kappa(x,t)$ on $\Omega \times (0, \infty)$ by

$$\kappa(x,t) = \int_{\partial \Omega} \frac{\partial}{\partial N(Q)} \Gamma(x - Q,t)E(Q)dQ$$

and a surface potential $T$ for $f$ by

$$T(f)(x,t) = 4 \int_0^t \int_{\partial \Omega} \kappa(x - Q,t - s)f(Q,s)dQds, \quad (x,t) \in \Omega \times (0, T)$$

for real-value function $f : \partial \Omega \times (0, \infty) \to \mathbb{R}$.

For given $x \in \Omega$, $P_x$ is the nearest point of $x$ on $\partial \Omega$ such that $\text{dist}(x, \partial \Omega) = |P_x - x|$ and for a vector valued function $v(x)$, we define the normal component $v_N$ and tangential component $v_T$ to the nearest point $P_x$ by

$$v_N(x) = (v(x) \cdot N(P_x))N(P_x) \quad \text{and} \quad v_T(x) = v(x) - v_N(x).$$

In[1], the essential estimate is stated as the following proposition:

**Proposition 1.1.** Suppose that $\Omega$ is a bounded $C^2$ domain and $u$ is the solution to (1.1) for the bounded boundary data $g$. The normal component $u_N$ of $u$ is bounded and there is also a constant $C(\Omega)$ such that

$$\max_{(x,t) \in \Omega \times (0, T)} |u_N(x,t)| \leq C(\Omega)\|g\|_{L^\infty(\partial \Omega \times (0, T))}.$$ 

Furthermore, the tangential component $u_T$ of the velocity $u$ satisfies that

$$\max_{(x,t) \in \Omega \times (0, T)} |u_T(x,t) - \nabla S(g \cdot N_T)(x,t) - \nabla T(g \cdot N_T)(x,t)| \leq C(\Omega)\|g\|_{L^\infty(\partial \Omega \times (0, T))}.$$ 

For the proof of Proposition [1,1] in $\mathbb{R}^n_+ \times (0, T)$ in [1], the authors used the Riesz operator property of Poisson kernel matrix of Stokes equations obtained by V. A. Solonnikov (see section[2]). The velocity $u$ is represented by convolution of Poisson kernel matrix $(K_{ij})_{1 \leq i,j \leq n}$
and boundary data \( g \). For \( 1 \leq i \leq n - 1 \), the Poisson kernel \( K_{in} \) contains the term \( \delta(t)D_{x_i}E(x' - y', x_n) + B_{in}(x, t) \) which is a kernel of integral operator \( \nabla S(g_n)_T + \nabla T(g_n)_T \), where \( \delta(t) \) is Dirac delta function of time, \( g_n \) is the \( n \)-th element of \( g = (g_1, \cdots, g_n) \) which is the normal component and

\[
B_{in}(x, t) = -\int_{\mathbb{R}^{n-1}} D_{x_n} \Gamma(x' - y', x_n, t) D_{y_i} E(y', 0) dy'.
\]

It is a new observation in this paper that there is a cancelation between \( \nabla S(g_n) \) and \( \nabla T(g_n) \) as an operator of BMO(functions of bounded mean oscillation). Nonetheless, the potential \( \nabla S(g_n)_T + \nabla T(g_n)_T \) could blow up in \( L^\infty \). In fact, we can show that there exists \( g_n \in L^p \cap L^\infty, 1 \leq p < \infty \) such that it blows up. This implies that Proposition 1.1 is optimal.

**Theorem 1.2.** There exists \( g \in L^p \cap L^\infty, 1 \leq p < \infty \) such that the velocity \( u \) blows up.

From the kernel \( \delta(t)D_{x_i}E(x' - y', x_n) + B_{in}(x, t) \), if \( g_n \) has Dini-continuity with respect to space, that is, \( ||f||_{Dini, \partial \Omega} = \int_0^\infty \sup_{x \in \partial \Omega} \omega(f)(r, x) \frac{dr}{r} < \infty \), where \( \omega(f) \) is the modulus of continuity of \( f \) such that \( \omega(f)(r, x) = \sup_{y \in B_r(x) \cap \partial \Omega} |f(y) - f(x)| \), then integral operators \( (\nabla S)_T(f) \) and \( (\nabla T)_T(f) \) are bounded. This is mainly due to boundedness of Riesz operator in BMO. Using this fact, it is obtained the following result:

**Proposition 1.3.** Suppose that the domain \( \Omega \) is bounded \( C^2 \) and \( u \) is a solution to (1.1). Suppose \( g \) is bounded on \( \partial \Omega \times (0, T) \) and the normal component \( g_N \) is Dini-continuous with respect to space. Then, there is a constant \( C(\Omega) \) depending only on \( \Omega \) such that

\[
\max_{(x, t) \in \Omega \times (0, T)} |u(x, t)| \leq C(\Omega) \left( ||g||_{L^\infty(\partial \Omega \times (0, T))} + \sup_{t \in (0, T)} ||g_N(\cdot, t)||_{Dini, \partial \Omega} \right).
\]

Proposition 1.3 means that the maximum modulus principle of the Stokes equations (1.1) can be obtained by the Dini-continuity with respect to the space only. It is important to clarify that the maximum modulus of solution remains bounded under the time continuity of boundary data without continuity assumption in space. In this paper, we answer to the question, in fact, we find a cancelation between \( \delta(t)D_{x_i}E(x' - y', x_n) \) and \( B_{in}(x, t) \) in logDini-continuous functions in time. We say that \( f : \mathbb{R} \rightarrow \mathbb{R} \) is logDini-continuous if \( ||f||_{logDini, (0, T)} := \int_0^\infty \sup_{t \in (0, T)} \omega(f)(r, t) \frac{|\ln r|}{r} dr < \infty \), where \( \omega(f)(r, t) = \sup_{s \in (t-r, t+r) \cap (0, T)} |f(s) - f(t)| \). To handle the initial time \( t = 0 \), we extend \( g_N \) to \( \mathbb{R} \) by

\[
g_N(x, t) = 0 \quad \text{for} \quad x \in \partial \Omega \quad t \leq 0.
\]
Theorem 1.4. Suppose that the domain \( \Omega \) is bounded \( C^2 \) and \( u \) is a solution to (1.1) for bounded boundary data \( g \). Suppose that \( g_N \) is logDini-continuous with respect to time. Then,

\[
\max_{(x,t) \in \Omega \times (0,T)} |\nabla S(g_N)T(x,t) + \nabla T(g_N)T(x,t)| \\
\leq C(\Omega) \left( \|g\|_{L^\infty(\partial\Omega \times (0,T))} + \sup_{x \in \partial\Omega} \|g_N(x,\cdot)\|_{\text{logDini}} \right).
\]

With Proposition 1.1, we get

Theorem 1.5. Suppose that the domain \( \Omega \) is bounded \( C^2 \) and \( u \) is a solution to (1.1) for bounded boundary data \( g \). Suppose that \( g_N \) is logDini-continuous with respect to time. Then, the solution \( u \) of (1.1) satisfies

\[
\|u\|_{L^\infty(\Omega \times (0,T))} \leq C(\Omega) \left( \|g\|_{L^\infty(\partial\Omega \times (0,T))} + \sup_{Q \in \partial\Omega} \|g_N(Q,\cdot)\|_{\text{logDini}} \right).
\]

We present the paper in the following way. In section 2, we discuss about the Poisson kernel of the nonstationary Stokes equations in \( \mathbb{R}^n_+ \times (0,T) \). In section 3, we prove the main Theorem 1.4 in the half space. In section 4, we prove the Theorem 1.4. In section 5, we prove the Theorem 1.2.

2. Kernels on half plane

For notation, we denote \( x = (x', x_n) \), that is, \( x' = (x_1, x_2, \cdots, x_{n-1}) \). Indeed, the symbol \( t \) means the coordinate up to \( n-1 \) and \( \omega_n \) is the surface area of the unit sphere in \( \mathbb{R}^n \). We also denote that \( D_{x_i}u \) are partial derivatives of \( u \) with respect to \( x_i \), \( 1 \leq i \leq n \), that is, \( D_{x_i}u(x) = \frac{\partial}{\partial x_i}u(x) \).

We let \( E \) and \( \Gamma \) be the fundamental solutions to the Laplace equation and the heat equation, respectively, such that

\[
E(x, t) = -\frac{1}{(n-2)\omega_n} \frac{1}{|x|^{n-2}}, \quad \Gamma(x, t) = \begin{cases} \frac{1}{\sqrt{2\pi t}}e^{-\frac{|x|^2}{2t}}, & t > 0 \\ 0, & t \leq 0 \end{cases},
\]

where \( \omega_n \) is the measure of the unit sphere in \( \mathbb{R}^n \).

The Poisson kernel \( (K, \pi) \) for the half space is defined by

\[
K_{ij}(x' - y', x_n, t) = -2\delta_{ij} D_{x_n} \Gamma(x' - y', x_n, t) - L_{ij}(x' - y', x_n, t) \\
+ \delta_{jn}\delta(t)D_{x_i}E(x' - y', x_n),
\]

\[
\pi_{ij}(x' - y', x_n, t) = -2\delta(t)D_{x_j}D_{x_n}E(x' - y', x_n) + 4D_{x_n}D_{x_n}A(x' - y', x_n, t) \\
+ 4D_tD_{x_j}A(x' - y', x_n, t),
\]

where

\[
A(x', y', x_n, t) = \begin{cases} 0, & x_n > y_n \\ 1, & x_n < y_n \end{cases}.
\]
where $\delta(t)$ is the Dirac delta function and $\delta_{ij}$ is the Kronecker delta function. Here, we defined that

$$L_{ij}(x, t) = D_{xj} \int_0^{x_n} \int_{\mathbb{R}^{n-1}} D_{xn} \Gamma(z, t) D_{x} E(x - z) dz,$$

$$A(x, t) = \int_{\mathbb{R}^{n-1}} \Gamma(z', 0, t) E(x' - z', x_n) dz'.$$

We have relations among $L$ such that

$$\sum_{1 \leq i \leq n} L_{ii} = -2D_{xn} \Gamma, \quad L_{in} = L_{ni} + B_{in}, \quad (2.1)$$

where

$$B_{in}(x, t) = \begin{cases} -\int_{\mathbb{R}^{n-1}} D_{xn} \Gamma(x' - y', x_n, t) D_{y} E(y', 0) dy' = \frac{\partial}{\partial x_i} \kappa(x, t) & \text{if } i \neq n, \\ 0 & \text{if } i = n \end{cases} \quad (2.2)$$

(see [3] and [10]).

The solution $(u, p)$ of the Stokes system (1.1) in $\mathbb{R}^n$ with boundary data $g$ is expressed by

$$u^i(x, t) = \sum_{j=1}^n \int_0^t \int_{\mathbb{R}^{n-1}} K_{ij}(x' - y', x_n, t - s) g_j(y', s) dy' ds, \quad (2.3)$$

$$p(x, t) = \sum_{j=1}^n \int_0^t \int_{\mathbb{R}^{n-1}} \pi_j(x' - y', x_n, t - s) g_j(y', s) dy' ds.$$

(See [3] and [10]).

3. Maximum modulus estimate in the half space

In this section, we consider the maximum modulus estimate in the half space. In [1], the authors proved the following proposition.

**Proposition 3.1.** Let $1 \leq i \leq n$ and $1 \leq j \leq n - 1$. Then

$$\int_0^\infty \int_{\mathbb{R}^{n-1}} |L_{ij}(x', x_n, t)| dx' dt < C, \quad (3.1)$$

where $C > 0$ is independent of $x_n > 0$ and hence it follows that

$$\int_0^\infty \int_{\mathbb{R}^{n-1}} |L_{in}(x', x_n, t) - B_{in}(x', x_n, t)| dx' dt < C, \quad (3.2)$$

where $C > 0$ is independent of $x_n > 0$.

The maximum modulus theorem for the half space follows from proposition 3.1 (see [1]).

**Proposition 3.2.** Let $g = (g_1, g_2, \cdots, g_n) \in L^\infty(\mathbb{R}^{n-1} \times (0, T))$ and $(u, p)$ is represented by (2.3). Then,

$$\|u_T - \nabla S_T(g_n) - \nabla T_T(g_n)\|_{L^\infty(\mathbb{R}^n_+ \times (0, T))} \leq C \|g\|_{L^\infty(\partial \mathbb{R}^n_+ \times (0, T))} \quad (3.3)$$
for some $C > 0$ independent of $T$. Furthermore, the normal component of the velocity $u$ is bounded and there is also a constant $C$ such that
\[
\|u_n\|_{L^\infty(\mathbb{R}^n_+ \times (0,T))} \leq C\|g\|_{L^\infty(\partial \mathbb{R}^n_+ \times (0,T))}.
\]

Now, to prove the main theorem 1.4 we prove the following lemma.

**Lemma 3.3.** Let $g_n \in L^\infty(\mathbb{R}^{n-1} \times (0,T))$ and $g_n$ satisfies the logDini-continuity with \( \text{supp} g_n \subset B_M' = \{ |x'| < M \} \) for some $M > 0$, then
\[
\|(\nabla S_T + \nabla T_T)(g_n)\|_{L^\infty(\mathbb{R}^n_+ \times (0,T))} \leq C_M \left( \|g_n\|_{L^\infty(\mathbb{R}^{n-1} \times (0,T))} + \| \max_{x' \in \mathbb{R}^{n-1}} g_n(x', \cdot) \|_{\text{logDini}} \right)
\]
for some $C_M > 0$.

By Proposition 3.2 and lemma 3.3, we obtain

**Theorem 3.4.** Let $g = (g_1, g_2, \ldots, g_n) \in L^\infty(\mathbb{R}^{n-1} \times (0,T))$ and $g_n$ satisfies the logDini-continuity with \( \text{supp} g_n(\cdot, t) \subset B_M' = \{ |x'| < M \} \) for all $0 < t < \infty$. Then the $u$ represented by 2.3 satisfies
\[
\|u_T\|_{L^\infty(\mathbb{R}^n_+ \times (0,T))} \leq C_M \left( \|g_n\|_{L^\infty(\mathbb{R}^{n-1} \times (0,T))} + \| \max_{x' \in \mathbb{R}^{n-1}} g_n(x', \cdot) \|_{\text{logDini}} \right)
\]
for some $C_M > 0$.

To show the lemma 3.3, we study the kernel of \( (\nabla S_T + \nabla T_T)(x, y', t, s) \). Note that the kernel of \( (\nabla S_T + \nabla T_T)_i \) is $\delta(t)D_{x_i}E(x' - y', x_n) + B_{in}(x, t)$. Note that
\[
B_{in}(x, t) = -D_{x_n} \Gamma_1(x_n, t) \int_{\mathbb{R}^{n-1}} \Gamma'(y', t)D_{y_i}E(x' - y', 0)dy'
\]
where $\Gamma_1$ and $\Gamma'$ are Gaussian kernels in $\mathbb{R}$ and $\mathbb{R}^{n-1}$, respectively.

**Lemma 3.5.** For $1 \leq i \leq n - 1$, we get
\[
(3.5) \quad \int_{|y'| \leq \frac{1}{2}|x'|} \Gamma'(y', t)D_{y_i}E(x' - y', 0)dy' \leq C_0^{-\frac{n-1}{2}}e^{-c|y'|^2} + C|x'|^{-n+1} \int_{|y'| \leq \frac{1}{2}|x'|} |y'|^2 e^{-c|y'|^2} dy'
\]
\[
(3.6) \quad \int_{\frac{1}{2}|x'| \leq |y'| \leq 2|x'|, |x'-y'| \geq \frac{1}{2}|x'|} \Gamma'(y', t)D_{y_i}E(x' - y', 0)dy' \leq C_0^{-\frac{n+1}{2}}e^{-c|x'|^2},
\]
\[
(3.7) \quad \int_{|x'-y'| \leq \frac{1}{2}|x'|} \Gamma'(y', t)D_{y_i}E(x' - y', 0)dy' \leq C_0^{-\frac{n+1}{2}}|x'|^2 e^{-c|x'|^2}
\]
\[
(3.8) \quad \int_{|y'| \geq 2|x'|} \Gamma'(y', t)D_{y_i}E(x' - y', 0)dy' \leq C_0^{-\frac{n+1}{2}} \int_{|y'| \leq |y'| \leq 2|y'|} |y'|^{-n+1} e^{-c|y'|^2} dy',
\]
where $c, C > 0$ are independent of $x', y_n$ and $t$. 

Proof. Using integration by parts, we get
\[
\int_{|y'|\leq\frac{1}{2}|x'|} \Gamma'(y', t) D_{y'} E(x' - y', 0) dy'
\]
(3.6) 
\[
= \int_{|y'|=\frac{1}{2}|x'|} \frac{y_i}{|y'|} \Gamma'(y', t) E(x' - y', 0) \sigma(dy') - \int_{|y'|\leq\frac{1}{2}|x'|} D_{y_i} \Gamma'(y', t) E(x' - y', 0) dy'.
\]

For \(y'\) with \(|y'| = \frac{1}{2}|x'|\), we get \(|\Gamma'(y', t)| \leq C t^{-\frac{n-1}{2}} e^{-c|y'|^2} t^\frac{n-1}{2} \) and \(|E(x' - y', 0)| \leq C t^{-\frac{1}{n-2}} \).

Here, the first term of the right hand side in (3.6) is dominated by
\[
\int_{|y'|=\frac{1}{2}|x'|} |\Gamma'(y', t)||E(x' - y', 0)| \sigma(dy') \leq C t^{-\frac{n-1}{2}} e^{-c|y'|^2} t^\frac{n-1}{2} \leq C t^{-\frac{n-1}{2}} e^{-c|y'|^2}.
\]

Since \(\int_{|y'|\leq\frac{1}{2}|x'|} D_{y_i} \Gamma'(y', t) dy' = 0\) for all \(t > 0\), using the Mean-value theorem, the second term of the right hand side of (3.6) satisfies
\[
|\int_{|y'|\leq\frac{1}{2}|x'|} D_{y_i} \Gamma'(y', t) \left( D_{y'} E(x' - y', 0) - D_{y_i} E(x', 0) \right) dy'|
\]
(3.7) 
\[
\leq C |x'|^{-n-1} \int_{|y'|\leq\frac{1}{2}|x'|} t^{-\frac{n-1}{2} - 1} |y'| e^{-c|y'|^2} dy' 
\]
\[
\leq C |x'|^{-n-1} \int_{|y'|\leq\frac{1}{2}|x'|} |y'| e^{-c|y'|^2} dy'.
\]

By (3.6) and (3.8), we obtain (3.5).1.

For (3.5)2, note that for \(y'\) satisfying \(\frac{1}{2}|x'| \leq |y'| \leq 2|x'|\) we have \(|x' - y'| \geq \frac{1}{2}|x'|\). We have \(|\Gamma'(y', t)| \leq C t^{-\frac{n-1}{2}} e^{-c|y'|^2} t^\frac{n-1}{2} \) and \(D_{y_i} E(x' - y', 0) \leq C |x'|^{-n+1} \), and thus we get
\[
|\int_{\frac{1}{2}|x'| \leq |y'| \leq 2|x'|, |x' - y'| \geq \frac{1}{2}|x'|} \Gamma'(y', t) D_{y_i} E(x' - y', 0) dy' | \leq C t^{-\frac{n}{2} + \frac{1}{2}} e^{-c|y'|^2} t^\frac{n}{2}.
\]

Hence, we obtain (3.5)2.

For \(y'\) satisfying \(|x' - y'| \leq \frac{1}{2}|x'|\), we have \(|\nabla \Gamma'(y', t)| \leq C t^{-\frac{n-1}{2}} |x'| e^{-c|y'|^2} t^\frac{n-1}{2} \). Hence, since \(\int_{|x' - y'| \leq \frac{1}{2}|x'|} D_{y_i} E(x' - y', 0) dy = 0\), using Mean-value Theorem (3.5)3 is proved by
\[
|\int_{|x' - y'| \leq \frac{1}{2}|x'|} \left( \Gamma'(y', t) - \Gamma'(x', t) \right) D_{y_i} E(x' - y', 0) dy' |
\]
\[
\leq C t^{-\frac{n+1}{2}} |x'| e^{-c|y'|^2} t^\frac{n+1}{2} \int_{|x' - y'| \leq \frac{1}{2}|x'|} |x' - y'|^{-n+2} dy' 
\]
\[
\leq C t^{-\frac{n+1}{2}} |x'|^2 e^{-c|y'|^2} t^\frac{n+1}{2}.
\]

Finally, (3.5)4 follows by
\[
|\int_{|y'| \geq 2|x'|} \Gamma'(y', t) D_{y_i} E(x' - y', 0) dy' | \leq C t^{-\frac{n+1}{2}} \int_{2|x'| \leq |y'| \geq 2|x'|} |y'|^{-n+1} e^{-c|y'|^2} dy' 
\]
(3.7) 
\[
= C t^{-\frac{n+1}{2}} \int_{2|x'| \leq |y'| \leq |y'|} |y'|^{-n+1} e^{-c|y'|^2} dy'.
\]
Lemma 3.6. Let $M > 0$. Then, for $x' \in \mathbb{R}^{n-1}$ and $s > 0$, we get
\[
\int_{|y'| \leq M} \left| \int_{\mathbb{R}^{n-1}} \Gamma'(x' - z', s) D_{z_i} E(y' - z', 0) dz' | dy' \right| \leq C(1 + a(M, s)),
\]
where $a(M, s) := \ln M - \ln \min(\sqrt{s}, M)$.

Proof. Using the change of variables, we have
\[
\int_{|y'| \leq M} \left| \int_{\mathbb{R}^{n-1}} \Gamma'(x' - z', s) D_{z_i} E(y' - z', 0) dz' | dy' \right| = \int_{|x' - y'| \leq M} \left| \int_{\mathbb{R}^{n-1}} \Gamma'(z', s) D_{z_i} E(y' - z', 0) dz' | dy' \right|.
\]

For fixed $y'$ satisfying $|x' - y'| \leq M$, let us
\[
A_1 = \{ z' \in \mathbb{R}^{n-1} \mid |z'| \leq \frac{1}{2} |y'| \},
\]
\[
A_2 = \{ z' \in \mathbb{R}^{n-1} \mid |y' - z'| \leq \frac{1}{2} |y'| \},
\]
\[
A_3 = \{ z' \in \mathbb{R}^{n-1} \mid \frac{1}{2} |y'| \leq |z'| \leq 2 |z'|, \ |y' - z'| \geq \frac{1}{2} |y'| \},
\]
\[
A_4 = \{ z' \in \mathbb{R}^{n-1} \mid |y' - z'| \geq 2 |y'| \}.
\]

We divide the integral of the left-side of (3.9) by four parts;
\[
\int_{|x' - y'| \leq M} \left| \int_{\mathbb{R}^{n-1}} \Gamma'(z', s) D_{z_i} E(y' - z', 0) dz' | dy' \right| = I_1 + I_2 + I_3 + I_4,
\]
where
\[
I_1 = \int_{|x' - y'| \leq M} \left| \int_{A_1} \Gamma'(z', s) D_{z_i} E(y' - z', 0) dz' | dy' \right|,
\]
\[
I_2 = \int_{|x' - y'| \leq M} \left| \int_{A_2} \Gamma'(z', s) D_{z_i} E(y' - z', 0) dz' | dy' \right|,
\]
\[
I_3 = \int_{|x' - y'| \leq M} \left| \int_{A_3} \Gamma'(z', s) D_{z_i} E(y' - z', 0) dz' | dy' \right|,
\]
\[
I_4 = \int_{|x' - y'| \leq M} \left| \int_{A_4} \Gamma'(z', s) D_{z_i} E(y' - z', 0) dz' | dy' \right|.
\]

By (3.5) of the lemma [3.5] we get
\[
I_1 \leq C \int_{|x' - y'| \leq M} \left( s^{-\frac{n}{2} + \frac{1}{2}} e^{-\frac{2|x'|^2}{s}} + |y'|^{-n+1} \int_{|z'| \leq \frac{1}{2}|y'|} |z'|^2 e^{-|z'|^2} dz' \right) dy':
\]
\[
:= I_{11} + I_{12}.
\]
Here,

\begin{equation}
I_{11} \leq C \int_{\mathbb{R}^{n-1}} s^{-\frac{n+1}{2}} e^{-c \frac{|y'|^2}{s}} dy' = C.
\end{equation}

And

\begin{equation}
I_{12} \leq C \int_{|x'-y'| \leq M, \frac{1}{2} \sqrt{s} \geq |y'|} |y'|^{-n+1} \left( \frac{|y'|}{\sqrt{s}} \right)^{n+1} dy' + C \int_{|x'-y'| \leq M, \frac{1}{2} \sqrt{s} \leq |y'|} |y'|^{-n+1} dy'.
\end{equation}

If \(|x'| \leq 2M\), then from (3.12), we have

\begin{equation}
I_{12} \leq C \int_{|y'| \leq \min \left( \frac{1}{2} \sqrt{s}, 3M \right)} |y'|^{-n+1} \left( \frac{|y'|}{\sqrt{s}} \right)^{n+1} dy' \leq C(1 + a(M, s)).
\end{equation}

If \(|x'| \geq 2M\) and \(s \geq (3|x'|)^2\), then from (3.12), we have

\begin{equation}
I_{12} \leq C \int_{|y'| \leq \min \left( \frac{1}{2} \sqrt{s}, 3M \right)} |y'|^{-n+1} \left( \frac{|y'|}{\sqrt{s}} \right)^{n+1} dy' \leq C.
\end{equation}

If \(|x'| \geq 2M\) and \(|x'|^2 \leq s \leq (3|x'|)^2\), then from (3.10), we have

\begin{equation}
I_{12} \leq C |x'|^{-n+1} \int_{|x'-y'| \leq M} dy' \leq C.
\end{equation}

Hence, by (3.10) and (3.16), we have

\begin{equation}
I_1 \leq C(1 + a(M, s)).
\end{equation}

By the (3.5)\textsubscript{2} and (3.5)\textsubscript{3} of lemma 3.5, we get

\begin{equation}
I_2 + I_3 \leq C \int_{\mathbb{R}^{n-1}} \left( s^{-\frac{n+1}{2}} e^{-c \frac{|y'|^2}{s}} + s^{-\frac{n+1}{2}} |y'|^2 e^{-c \frac{|y'|^2}{s}} \right) dy' \leq C.
\end{equation}
By the \((3.5)\) of lemma \(3.5\) we get

\[
I_4 \leq C \int_{|x' - y'| \leq 3M} s^{\frac{n+1}{2}} \int_{\frac{2|y'|}{\sqrt{s}} \leq |x'|} |z'|^{-n+1} e^{-|z'|^2} d'z' dy' \\
\leq C \int_{|y'| \leq \frac{1}{2}\sqrt{s}} s^{\frac{n+1}{2}} dy' + C \int_{\frac{1}{2}\sqrt{s} \leq |y'|} s^{\frac{n+1}{2}} e^{-\frac{y'^2}{\sqrt{s}}} dy'
\]

\leq C.

Hence, we completed the proof of lemma. 

\[
\square
\]

**Proof of Lemma** \(3.3\) Note that for \(i \neq n\), by \((2.1)\), we have

\[
(\nabla S_T + \nabla T_T)_i(g_n)(x, t) = B_{in}g_n(x, t) - \frac{\partial}{\partial x_i} \mathcal{E}g_n(x, t),
\]

where

\[
\mathcal{E}g_n(x, t) = \int_{\mathbb{R}^{n-1}} E(x' - y', x_n)g_n(y', t)dy',
\]

\[
B_{in}g_n(x, t) = \int_0^t \int_{\mathbb{R}^{n-1}} B_{in}(x' - y', x_n, t - s)g_n(y', s)dy' ds.
\]

Note that

\[
B_{in}g_n(x, t) - \frac{\partial}{\partial x_i} \mathcal{E}g_n(x, t)
\]

\[
= - \int_0^t \int_{\mathbb{R}^{n-1}} \mathcal{G}(x' - y', x_n, t - s)D_{x_n} E(y' - z', 0)D_{y_i} E(x' - y', x_n)dy' ds
\]

\[
- \int_{\mathbb{R}^{n-1}} \mathcal{G}(y', t)D_{y_i} E(x' - y', x_n)dy'
\]

\[
= - \int_0^t \int_{\mathbb{R}^{n-1}} (g_n(y', s) - g_n(y', t)) \int_{\mathbb{R}^{n-1}} D_{x_n} \mathcal{G}(x' - z', x_n, t - s)D_{z_i} E(y' - z', 0)dz' dy' ds
\]

\[
- \int_0^t \int_{\mathbb{R}^{n-1}} \mathcal{G}(y', t) \int_{\mathbb{R}^{n-1}} D_{x_n} \mathcal{G}(x' - z', x_n, t - s)D_{z_i} E(y' - z', 0)dz' dy' ds
\]

\[
- \int_{\mathbb{R}^{n-1}} \mathcal{G}(y', t)D_{y_i} E(x' - y', x_n)dy'.
\]

Since \(\int_0^\infty \mathcal{G}(x', x_n, s)ds = -E(x', x_n)\), we have \(\int_0^\infty D_{x_n} \mathcal{G}(x', x_n, s)ds = -\frac{\partial}{\partial x_n} E(x', x_n)\). Furthermore, we have \(D_{x_n} E(\cdot, x_n) \ast' D_{x_i} E(\cdot, 0)(x') = D_{x_i} E(x', x_n)\), where \(\ast'\) is a convolution.
in \( \mathbb{R}^{n-1} \). Hence, we get

\[
B_{\text{in}} g_n(x, t) - \frac{\partial}{\partial x_i} \mathcal{E} g_n(x, t)
= - \int_0^t \int_{\mathbb{R}^{n-1}} (g_n(y', s) - g_n(y', t)) \int_{\mathbb{R}^{n-1}} D_{x_n} \Gamma(x' - z', x_n, t - s) D_{z_i} E(y' - z', 0) dz' dy' ds
- \int_0^\infty \int_{\mathbb{R}^{n-1}} g_n(y', t) \int_{\mathbb{R}^{n-1}} D_{x_n} \Gamma(x' - z', x_n, s) D_{z_i} E(y' - z', 0) dz' dy' ds
- \int_t^\infty \int_{\mathbb{R}^{n-1}} g_n(y', t) \int_{\mathbb{R}^{n-1}} D_{x_n} \Gamma(x' - z', x_n, s) D_{z_i} E(y' - z', 0) dz' dy' ds
- \int_{\mathbb{R}^{n-1}} g_n(y', t) D_{y_i} E(x' - y', x_n) dy'
= \int_0^t \int_{\mathbb{R}^{n-1}} (g_n(y', s) - g_n(y', t)) \int_{\mathbb{R}^{n-1}} D_{x_n} \Gamma(x' - z', x_n, t - s) D_{z_i} E(y' - z', 0) dz' dy' ds
- \int_t^\infty \int_{\mathbb{R}^{n-1}} g_n(y', t) \int_{\mathbb{R}^{n-1}} D_{x_n} \Gamma(x' - z', x_n, s) D_{z_i} E(y' - z', 0) dz' dy' ds.
\]

Since \( \text{supp} g_n(\cdot, t) \subset B'_M \) for all \( t \in (0, \infty) \), we get

\[
|B_{\text{in}} g_n(x, t) - \frac{\partial}{\partial x_i} \mathcal{E} g_n(x, t)|
\leq c \int_0^t (t - s)^{-\frac{3}{2}} x_n e^{-\frac{t^2}{2s}} \int_{|y'| \leq M} |g_n(y', s) - g_n(y', t)|
\times \int_{\mathbb{R}^{n-1}} \Gamma'(x' - z', t - s) D_{z_i} E(y' - z', 0) dz' dy' ds
+ c \int_t^\infty s^{-\frac{3}{2}} x_n e^{-\frac{t^2}{2s}} \int_{|y'| \leq M} |g_n(y', t)| \int_{\mathbb{R}^{n-1}} \Gamma'(x' - z', s) D_{z_i} E(y' - z', 0) dz' dy' ds
\]
\[
:= J_1 + J_2.
\]

First, we estimate \( J_1 \). Note that \( |g_n(y', t) - g_n(y', s)| \leq \omega(g_n)(t - s, t) \). Hence, using the lemma \[3.6\] if \( t \leq M^2 \), we have

\[
J_1 \leq C \int_0^t \omega(g_n)(s, t) s^{-\frac{3}{2}} x_n e^{-\frac{t^2}{2s}} (1 + a(M, s)) ds
\leq C \int_0^t \omega(g_n)(s, t) s^{-1} (1 + |\ln M| + |\ln s|) ds
\leq C_M (\|g_n\|_{L^\infty} + \|g_n\|_{\log Dini}).
\]

(3.20)
Here, we used the fact that for any $m > 0$, $e^{-a} \leq c_m a^{-m}$ for the second inequality. If $t \geq M^2$, then we have

$$J_1 \leq C \int_0^{M^2} \omega(g_n)(s,t)s^{-1}(1 + |\ln M| + |\ln s|)ds$$

$$+ C\|g_n\|_{L^\infty} \int_{M^2}^t s^{-\frac{3}{2}}x_n e^{-\frac{s^2}{2}} (1 + |\ln M|)ds$$

(3.21)

$$\leq C_M(\|g_n\|_{L^\infty} + \|g_n\|_{logDini}).$$

Hence, by (3.20) and (3.21), we get

(3.22)

$$J_1 \leq C_M(\|g_n\|_{L^\infty} + \|g_n\|_{logDini}).$$

Next, we estimate $J_2$. If $t \geq M^2$, then

$$J_2 \leq C\|g_n\|_{L^\infty} \int_t^{\infty} s^{-\frac{3}{2}}x_n e^{-\frac{s^2}{2}} (1 + |\ln M|)ds$$

$$= C_M\|g_n\|_{L^\infty} \int_t^{\infty} s^{-\frac{3}{2}}x_n e^{-\frac{s^2}{2}} ds$$

(3.23)

$$\leq C_M\|g_n\|_{L^\infty}.$$

If $t \leq M^2$, then, we get

(3.24)

$$J_2 \leq C\|g_n\|_{L^\infty} \int_{M^2}^\infty s^{-\frac{3}{2}}x_n e^{-\frac{s^2}{2}} (1 + |\ln M|)ds$$

$$+ C|g_n(t)| \int_{M^2}^{\infty} s^{-\frac{3}{2}}x_n e^{-\frac{s^2}{2}} (1 + \ln M + |\ln s|)ds$$

$$\leq C_M\Big(\|g_n\|_{L^\infty} + |g_n(t)| \int_{M^2}^{\infty} s^{-1}|\ln s|ds\Big).$$

Note that by compatibility condition, we have $g_n(0) = 0$. Hence, for $t \leq s$, we have $|g_n(t)| = |g_n(t) - g_n(0)| \leq w(g_n)(t,t) \leq w(g_n)(s,t)$. Hence, we get

(3.25)

$$J_2 \leq C\Big(\|g_n\|_{L^\infty} + \|g_n\|_{logDini}\Big).$$

Hence, by from (3.23) to (3.25), we get

(3.26)

$$J_2 \leq C\Big(\|g_n\|_{L^\infty} + \|g_n\|_{logDini}\Big)$$

for all $x \in \mathbb{R}^n_+$ and $t > 0$. Hence, by (3.24) and (3.26), we have that for $x' \in \mathbb{R}^n_+$ and $0 < t$, we have

$$|\mathcal{B}_{in}g_n(x,t) - \frac{\partial}{\partial x_i} \mathcal{E}g_n(x,t)| \leq C\Big(\|g_n\|_{L^\infty} + \|g_n\|_{logDini}\Big).$$

Hence, we completed the proof of lemma 3.3.
4. Proof of Theorem 1.4

For the interior $L^\infty$ bound estimate, the authors in [1] showed the boundedness using the layer potential method in [5].

**Proposition 4.1.** Suppose the boundary data $g \in L^\infty(0, T; L^2(\partial \Omega))$. If $\text{dist}(x, \partial \Omega) \geq r_0 > 0$, $\epsilon > 0$ and $t < T$, then there is $C$ such that

$$|u(x, t)| \leq \frac{C}{r_0^{n-1-\epsilon}||g||_{L^\infty(0, T; L^2(\partial \Omega))}}.$$  

(See Corollary 4.2 in [1]).

Let $x \in \Omega$. Assume $\text{dist}(x, \partial \Omega) < r_0$ for some fixed small $r_0 > 0$. Since Stokes equations is translation and rotation invariant, we assume that $P = 0$ and $x = (0, x_n), \ x_n > 0$. If $x$ is close enough to $\partial \Omega$, there is a ball $B_{r_0}(0)$ centered at origin and $C^2$ function $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\Omega \cap B_{r_0}(0) = \{y_n > \Phi(y')\} \cap B_{r_0}(0)$ and $\partial \Omega \cap B_{r_0}(0) = \{y_n = \Phi(y')\} \cap B_{r_0}(0)$. Furthermore, $\Phi$ satisfies that

$$\text{(4.1) } |\Phi(y')| \leq C|y'|^2, \ |\nabla'\Phi(y')| \leq C|y'|, \ |\nabla'\nabla'\Phi(y')| \leq C$$

for $y' \in B_{r_0}'(0) = \{y' \in \mathbb{R}^{n-1} \mid |y'| < r_0\}$ and the outward unit normal vector $N(Q)$ at $Q = (y', \Phi(y')) \in \partial \Omega \cap B_{r_0}(0)$ is

$$N(y', \Phi(y')) = \frac{1}{\sqrt{1 + |\nabla'\Phi(y')|^2}}(\nabla'\Phi(y'), -1).$$

In particular, $N(x', \Phi(x')) = N(0, 0) = (0, -1)$. Hence, the $i$-th component of $\nabla S(g \cdot N)_T(x, t) + \nabla T(g \cdot N)_T(x, t)$ is

$$\begin{align*}
\frac{\partial}{\partial x_i} S(g \cdot N) + \frac{\partial}{\partial x_i} T(g \cdot N) - \sum_{1 \leq k \leq n} \frac{\partial}{\partial x_k} S(g \cdot N) N_k N_i - \sum_{1 \leq k \leq n} \frac{\partial}{\partial x_k} T(g \cdot N) N_k N_i \\
= \begin{cases} \\
\frac{\partial}{\partial x_i} S(g \cdot N) + \frac{\partial}{\partial x_i} T(g \cdot N) & 1 \leq i \leq n - 1, \\
0 & i = n.
\end{cases}
\end{align*}$$

Note that for $1 \leq i \leq n - 1$, we have

$$\frac{\partial}{\partial x_i} S(g \cdot N)(x, t)$$

$$= \int_{\partial \Omega \cap B_{r_0}} \frac{\partial}{\partial x_i} E(x - y)(g \cdot N)(y, t)d\sigma(y) + \int_{\partial \Omega \Delta B_{r_0}} \frac{\partial}{\partial x_i} E(x - y)(g \cdot N)(y, t)d\sigma(y)$$

$$:= A_{i1} + A_{i2}.$$
Similarly, we have
\[
\frac{\partial}{\partial x_i} T(g \cdot N)(x,t) = 4 \int_0^t \int_{\partial \Omega \cap B_{r_0}} \frac{\partial}{\partial x_i} \kappa(x-y, t-s)(g \cdot N)(y,s) d\sigma(y) ds + 4 \int_0^t \int_{\partial \Omega \cap B_{r_0}^c} D_{x_i} \kappa(x-y, t-s)(g \cdot N)(y,s) d\sigma(y) ds
\]
\[
:= B_{i1} + B_{i2}.
\]
Since \(|x-y| \geq r_0\) for \(y \in \partial \Omega \cap B_{r_0}^c\), we get
\[
|A_{i2}|, |B_{i2}| \leq C \|g \cdot N\|_{L^\infty(\partial \Omega \times (0,T))}.
\]
Let \(G(y',t) := (g \cdot N)(y', \Phi(y'), t) \sqrt{1 + |\nabla \Phi(y')|^2}\). Then, from \(A_{i1}\) and \(B_{i1}\), we have
\[
A_{i1} + B_{i1} = \int_{B_{r_0}'} \frac{\partial}{\partial x_i} \left( E(y', x_n - \Phi(y')) - E(y', x_n) \right) G(y', t) dy'
\]
\[
+ 4 \int_0^t \int_{B_{r_0}'} \frac{\partial}{\partial x_i} \left( \kappa(y', x_n - \Phi(y'), t-s) - \kappa(y', x_n, t-s) \right) G(y', s) dy' ds
\]
\[
:= C_{i11} + C_{i12} + C_{i13},
\]
where
\[
C_{i11} := \int_{B_{r_0}'} \left( \frac{\partial}{\partial x_i} E(y', x_n - \Phi(y')) - \frac{\partial}{\partial x_i} E(y', x_n) \right) G(y', t) dy'
\]
\[
C_{i12} := 4 \int_0^t \int_{B_{r_0}'} \left( \frac{\partial}{\partial x_i} \kappa(y', x_n - \Phi(y'), t-s) - \frac{\partial}{\partial x_i} \kappa(y', x_n, t-s) \right) G(y', s) dy' ds
\]
\[
C_{i13} := \int_{B_{r_0}'} \frac{\partial}{\partial x_i} E(y', x_n) G(y', t) dy'
\]
\[
+ 4 \int_0^t \int_{B_{r_0}'} \left( \frac{\partial}{\partial x_i} \kappa(y', x_n, t-s) \right) G(y', s) dy' ds.
\]
Note that \(C_{i13}\) is the \(i\)-th component of \(\nabla S(G)_T(x,t) + \nabla T(G)_T(x,t)\). Hence, by the lemma 3.3, we get
\[
(4.2) \quad |C_{i13}| \leq C(\|G\|_{L^\infty} + \|G\|_{\log Dini}).
\]
Since \(|\Phi(y')| \leq c|y'|^2\) for \(y' \in B_{r_0}'\), using the Mean-value theorem, we have
\[
|C_{i11}| \leq C \int_{B_{r_0}'} \frac{1}{|y'|^n} |\Phi(y')| |G(y', t)| dy'
\]
\[
\leq C \int_{B_{r_0}'} \frac{1}{|y'|^{n-2}} |G(y', t)| dy'
\]
\[
\leq C \|G\|_{L^\infty(B_{r_0}' \times (0,T))}.
\]
and, similarly, we have
\[
|c_{12}| \leq 4 \int_0^t \int_{B_t'(0)} \left| \frac{\partial}{\partial x_i} R(y', x_n - \Phi(y'), t - s) - \frac{\partial}{\partial x_i} R(y', x_n, t - s) \right| |G(y', s)| dy' ds
\leq C\|G\|_{L^\infty(D_t'(0) \times B_t'(0))}.
\]
Since \(\|G\|_{L^\infty(D_t'(0) \times B_t'(0))} \leq C\|g \cdot N\|_{L^\infty(B_t'(0) \times (0, T))} \) and \(\|G\|_{logDini} \leq C\|g \cdot N\|_{L^\infty} + \|g \cdot N\|_{logDini}\), we get
\[
|\nabla S(g \cdot N)[T(x, t) + \nabla T(g \cdot N)[T(x, t)]| \leq C\|g \cdot N\|_{L^\infty} + \|g \cdot N\|_{logDini}.
\]
Therefore, we completed the proof of our main Theorem 1.2.
\(\square\)

5. Proof of Theorem 1.2

For the simplicity, we assume that \(n = 2\). Let \(g_1 = 0\) and \(g_2(x_1, t) = \chi_{(-1,1)}(x_1)\chi_{\frac{1}{2} < t < 1}(t)\). We will show that for \(x_n < \frac{1}{2}\) there is a positive constant \(c > 0\) such that
\[
(5.3) \quad u^1(1, x_2, 1 + x_2^2) \leq c(\ln x_n + 1).
\]
Hence, we proved the theorem 1.2. By the representation (2.3) of solution and by the second identity of (2.1), we have
\[
u^1(x, t) = \int_0^t \int_{R^{n-1}} K_1(x_1 - y_1, x_2, t - s) g_2(y_1, s) dy_1 ds
\]
\[
= - \int_0^t \int_{R} L_2(x_1 - y_1, x_2, t - s) g_2(y_1, s) dy_1 ds
+ \int_{R} D_{y_1} E(x_1 - y_1, x_2) g_2(y_1, t) dy_1
\]
\[
= - \int_0^t \int_{R} L_2(x_1 - y_1, x_2, t - s) g_2(y_1, s) dy_1 ds
- \int_0^t \int_{R} B_1(x_1 - y_1, x_2, t - s) g_2(y_1, s) dy_1 ds
+ \int_{R} D_{y_1} E(x_1 - y_1, x_2) g_2(y_1, t) dy_1 := u^1_1 + u^1_2 + u^1_3.
\]
By the (3.1), \(u^1_1\) is bounded in \(R^{n-1} \times (0, \infty)\) and \(u^1_3 = 0\) for \(t > 1\). Hence, for \(t > 1\), we get
\[
(5.4) \quad u^1(x, t) \leq u^1_2(x, t) + c.
\]
Note that
\[
u^1_2(x, t) = \int_0^t \int_{R} D_2 \Gamma(x_1 - y_1, x_2, t - s) H g_2(y_1, s) dy_1 ds,
\]
where $H$ is Hilbert transform. By the direct calculation, we have

\[
Hg_2(x_1, t) = \frac{1}{\pi} \chi_{\{t<1\}}(t) \int_{-1}^{1} \frac{1}{x_1 - y_1} dy_1
= \frac{1}{\pi} \chi_{\{t<1\}}(t) \begin{cases}
\ln(x_1 + 1) - \ln(1 - x_1), & |x_1| < 1 \\
\ln(1 + x_1) - \ln(x_1 - 1), & x_1 > 1 \\
\ln(-1 - x_1) - \ln(1 - x_1), & x_1 < -1.
\end{cases}
\]

We take $x_1 = 1$ and $t = 1 + x_2^2$ for small $x_2 > 0$. Then, we get

\[
(5.5) \quad u_2^1(1, x_2, 1 + x_2^2) = \int_{\frac{1}{2}}^{1} \int_{0}^{1} D_2 \Gamma(1 - y_1, x_2, 1 + x_2^2 - s) Hg_2(y_1, s) dy_1 ds = I_1 + I_2 + I_3,
\]

where

\[
I_1 := \int_{\frac{1}{2}}^{1} \int_{-\infty}^{0} D_2 \Gamma(1 - y_1, x_2, 1 + x_2^2 - s) Hg_2(y_1, s) dy_1 ds,
I_2 := \int_{\frac{1}{2}}^{1} \int_{0}^{1} D_2 \Gamma(1 - y_1, x_2, 1 + x_2^2 - s) Hg_2(y_1, s) dy_1 ds,
I_3 := \int_{\frac{1}{2}}^{1} \int_{1}^{\infty} D_2 \Gamma(1 - y_1, x_2, 1 + x_2^2 - s) Hg_2(y_1, s) dy_1 ds.
\]

Since $Hg_2$ is bounded in the intervals $(-\infty, 0)$ and $(2, \infty)$, $I_1$ and $I_3$ are bounded. And

\[
I_2 = \frac{1}{\pi} \int_{\frac{1}{2}}^{1} \int_{0}^{1} D_2 \Gamma(1 - y_1, x_2, 1 + x_2^2 - s) (\ln(y_1 + 1) - \ln(1 - y_1)) dy_1 ds
+ \frac{1}{\pi} \int_{\frac{1}{2}}^{1} \int_{1}^{2} D_2 \Gamma(1 - y_1, x_2, 1 + x_2^2 - s) (\ln(1 + y_1) - \ln(y_1 - 1)) dy_1 ds
= \frac{1}{\pi} \int_{\frac{1}{2}}^{1} \int_{0}^{1} D_2 \Gamma(1 - y_1, x_2, 1 + x_2^2 - s) (-\ln(1 - y_1)) dy_1 ds
+ \frac{1}{\pi} \int_{\frac{1}{2}}^{1} \int_{1}^{2} D_2 \Gamma(1 - y_1, x_2, 1 + x_2^2 - s) (-\ln(y_1 - 1)) dy_1 ds + Bd,
\]
where \( Bd \) is bounded such that \(|Bd| \leq c\) for some positive constant \( c > 0\). Hence, from (5.4) and (5.5), we get

\[
u^1(1, x_2, 1 + x_2^2) \leq I_2 + c
\]

\[
= \frac{1}{2\pi^2} \int_{1/2}^{1} \int_{0}^{1} D_2 \Gamma(1 - y_1, x_2, 1 + x_2^2 - s)(-\ln(1 - y_1))dy_1 ds + c
\]

\[
= \frac{1}{2\pi^2} e^{-\frac{1}{2}} \int_{1/2}^{1} \int_{1-x_2}^{1} e^{-\frac{x_2^2}{2(1+x_2^2-s)}} e^{-\frac{-1-\ln(1-y_1)}{1+x_2^2-s}} \ln(1 - y_1)dy_1 ds + c
\]

\[
= \frac{1}{2\pi^2} e^{-\frac{1}{2}} \int_{1/2}^{1} \int_{1-x_2}^{1} e^{-\frac{x_2^2}{2(1+x_2^2-s)}} (1 + x_2^2 - s)^2 e^{-\frac{x_2^2}{2(1+x_2^2-s)}} x_2 (\ln(x_2) + 1)ds + c
\]

\[
= \frac{1}{2\pi^2} e^{-\frac{1}{2}} \int_{1/2}^{1} x_2^2 e^{-\frac{x_2^2}{2(1+x_2^2-s)}} (1 + x_2^2 - s)^2 e^{-\frac{x_2^2}{2(1+x_2^2-s)}} x_2 (\ln(x_2) + 1)ds + c
\]

\[
= \frac{1}{2\pi^2} e^{-\frac{1}{2}} \int_{1/2}^{1} e^{-\frac{x_2^2}{2(1+x_2^2-s)}} (1 + x_2^2 - s)^2 e^{-\frac{x_2^2}{2(1+x_2^2-s)}} x_2 (\ln(x_2) + 1)ds + c
\]

This implies (5.3). \( \Box \)

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