On Ergotropic Gap of Tripartite Separable Systems

Ya-Juan Wu\textsuperscript{1}, Shao-Ming Fei\textsuperscript{2,3}, Zhi-Xi Wang\textsuperscript{2} and Ke Wu\textsuperscript{1}

\textsuperscript{1}School of Mathematics, Zhengzhou University of Aeronautics, Zhengzhou 450046, China
\textsuperscript{2}School of Mathematical Sciences, Capital Normal University, Beijing 100048, China
\textsuperscript{3}Max-Planck-Institute for Mathematics in the Sciences, 04103 Leipzig, Germany

Extracting work from quantum system is one of the important areas in quantum thermodynamics. As a significant thermodynamic quantity, the ergotropy gap characterizes the difference between the global and local maximum extractable works. We derive an analytical upper bound of the ergotropic gap with respect to $d \times d \times d$ tripartite separable states. This bound also provides a necessary criterion for the separability of tripartite states. Detailed examples are presented to illustrate the efficiency of this separability criterion.

\textsuperscript{*} Corresponding author: wangzhx@cnu.edu.cn
\textsuperscript{†} Corresponding author: wuke@cnu.edu.cn
I. Introduction

Information plays important roles in thermodynamics [1][2]. Work extraction is a significant aspect in thermodynamics, while quantum correlations are the basic resources in quantum information processing tasks. Their connections have been extensively studied [4][13].

Extracting work from quantum systems has gained renewed interest in quantum thermodynamics [14, 15]. In [16], the authors addressed the concept of ergotropy: the maximum work that can be gained from a quantum state with respect to some reference Hamiltonian and a cyclical unitary transformations. The maximum extractable work can be naturally divided into two parts: the local contribution from each subsystems, named local ergotropy, and the global contribution originating from correlations among the subsystems, named global ergotropy [17]. The work that can be extracted is proportional to the quantum mutual information under the global operations of the subsystems [6, 9]. As the extractable work under cyclic local interaction is strictly less than that obtained under global interaction, the presence of quantum correlations among the subsystems may result in non-vanishing ergotropic gap for the case of non-degenerate energy subspace [18]. This fact brings a new insight in the quantum-to-classical transition in thermodynamics.

As an essential kind of correlations, quantum entanglement plays an important role in quantum teleportation, quantum cryptography, quantum dense coding, quantum secret sharing and the development of quantum computer [19][20]. Distinguishing quantum entangled states from separable ones is of the significant but difficult problems in the theory of quantum entanglement. Numerous entanglement criteria have been proposed, such as the positive partial transpose (PPT) criterion [27, 28], realignment criteria [29–31], correlation matrix or tensor criteria [32–35], entanglement witnesses [28, 38], separability criteria via measurements [39] and so on. In [40], Mir Alimuddin et al. presented a separability criterion (the operational thermodynamic criterion) based on the evaluation of ergotropic gap of bipartite systems.

In this paper, we focus on the thermodynamic quantity ergotropic gap, given by the difference between the global ergotropy and local ergotropy associated with the tripartite quantum system. We present an upper bound on the ergotropic gap for arbitrary dimensional tripartite separable states. The bound presents a sufficient criteria for the entanglement of tripartite states.

II. Bounds on the ergotropic gap of tripartite separable states

Consider a $d \times d \times d$ tripartite state $\rho_{ABC} \in D\{H_A \otimes H_B \otimes H_C\}$, where $H_X (X = A, B, C)$ denotes the Hilbert space corresponding to the subsystem $X$ and $D(\mathcal{X})$ denotes the set of density operator acting on Hilbert space $\mathcal{X}$. The subsystem $X$ is governed by the local Hamiltonian $H_X = \sum_{j=0}^{d-1} j E_j |j\rangle \langle j|$, where $j E_j$ and $|j\rangle$ are the $j$th energy eigenvalue and eigenvector of $H_X$, respectively. The total non-interacting global Hamiltonian is $H_{ABC} = H_A \otimes I_B \otimes I_C + I_A \otimes H_B \otimes I_C + I_A \otimes I_B \otimes H_C$, where $I_X$ denotes the identity operator acting on the Hilbert space $H_X$. Under a cyclic Hamiltonian process, a time-dependent unitary operation $U(\tau) = e^{i \tau \rho \hat{p}} (-i \hbar \int_0^\tau dt [H_{ABC} + V(t)])$ can be applied, which $e^{i \tau \rho \hat{p}}$ denotes the time-ordered exponential and $V(t)$ denotes a time-dependent interaction among the subsystems. The work extraction from an isolated tripartite system under such a process is the change in average work, extracted from the isolated tripartite state $\rho_{ABC}$ by transforming it to the corresponding passive state $\rho^p_{ABC}$, is defined below in (1).

The maximum extractable work, called global ergotropy, is defined by

$$W_e^g = \max_{U \in \mathcal{L}(H_A \otimes H_B \otimes H_C)} \text{Tr}((\rho_{ABC} - U \rho_{ABC} U^\dagger) H_{ABC})$$

$$= \text{Tr}(\rho_{ABC} H_{ABC}) - \min_{U \in \mathcal{L}(H_A \otimes H_B \otimes H_C)} \text{Tr}(U \rho_{ABC} U^\dagger H_{ABC})$$

$$= \text{Tr}(\rho_{ABC} H_{ABC}) - \text{Tr}(\rho^p_{ABC} H_{ABC}),$$

where $\mathcal{L}(X')$ denotes the set of all bounded linear operators on the Hilbert space $X'$ and $\rho^p_{ABC}$ is the passive state with Hamiltonian $H_{ABC}$ for system $ABC$, from which no work can be extracted, of the form $\rho^p_{ABC} = \sum_j \rho_j |j\rangle \langle j|$ with $\rho_j \leq \rho_j > 0$. $\rho_{ABC}$ and $\rho^p_{ABC}$ have the same spectrum, and therefore there exists a unitary operator $U$ transforming the former to the latter.

The total achievable work, called local ergotropy, is given by

$$W_e^l = W_e^A + W_e^B + W_e^C,$$
where $W^A_e, W^B_e$ and $W^C_e$ are the maximum local extractable works from systems $A$, $B$ and $C$, respectively,

$$W^A_e = \text{Tr}(\rho_{ABC} H_A \otimes I_B \otimes I_C) - \min_{\rho \in \mathcal{L}(H_A)} \text{Tr}((U_A \otimes I_B \otimes I_C)\rho_{ABC}(U_A \otimes I_B \otimes I_C)^\dagger H_A \otimes I_B \otimes I_C)$$

$$= \text{Tr}(\rho_{ABC} H_A \otimes I_B \otimes I_C) - \text{Tr}(\rho_A^0 H_A),$$

(3)

and, similarly, $W^B_e = \text{Tr}(\rho_{ABC} I_A \otimes H_B \otimes I_C) - \text{Tr}(\rho_B^0 H_B)$, $W^C_e = \text{Tr}(\rho_{ABC} I_A \otimes I_B \otimes H_C) - \text{Tr}(\rho_C^0 H_C)$ with $\rho_A^0$, $\rho_B^0$ and $\rho_C^0$ the passive states associated with the subsystems $A$, $B$ and $C$, respectively. Hence,

$$W^d_e = \text{Tr}(\rho_{ABC} H_{ABC}) - \{\text{Tr}(\rho_A^0 H_A) + \text{Tr}(\rho_B^0 H_B) + \text{Tr}(\rho_C^0 H_C)\}. \quad (4)$$

The difference between the global ergotropy and the local ergotropy is called the ergotropic gap $\Delta_{EG}$,

$$\Delta_{EG} = W^g_e - W^d_e = \{\text{Tr}(\rho_A^0 H_A) + \text{Tr}(\rho_B^0 H_B) + \text{Tr}(\rho_C^0 H_C)\} - \text{Tr}(\rho_{ABC} H_{ABC}). \quad (5)$$

Indeed, $\Delta_{EG} \geq 0$ as global unitary operations are capable of extracting work from subsystems as well as from correlations among the subsystems. Clearly the ergotropic gap depends on various kinds of correlations presented in a tripartite quantum system. It is generally a challenging problem to compute $\Delta_{EG}$ analytically. In the following we derive analytic upper bounds of the ergotropic gap.

**Theorem 1.** Consider a $d \times d \times d$ tripartite state $\rho_{ABC}$ with spectrum $\lambda(\rho_{ABC}) = \{x_0, x_1, \cdots, x_{d-1}\}$ in nonincreasing order. Let the subsystems be governed by the same Hamiltonian $H_A = H_B = H_C = \sum_{j=0}^{d-1} j E|j\rangle\langle j|$. If $\rho_{ABC}$ is separable, then the ergotropic gap is bounded by

$$\Delta_{EG} \leq \min \left\{(Y - Z)E, M(d)E\right\}, \quad (6)$$

where

$$Y = 3 \sum_{i=0}^{d-1} i x_i + 3(d - 1) \sum_{i=d}^{d^2 - 1} x_i,$$

$$Z = \sum_{i=1}^{d-1} \sum_{j'=0}^{\frac{(i+2)(i+1)}{6} - 1} x_{D_i + j'} + \sum_{i=1}^{d-1} (d-1+i) \sum_{k'=0}^{\frac{(d+1)(d+2)}{6} - 1} x_{D_{d+1-1} - 3D_{i-1} + k'},$$

and

$$D = \frac{(2d - 1)2d(2d + 1)}{6} - \frac{(d - 1)d(d + 1)}{3},$$

$$D_i = \frac{i(i+1)(i+2)}{6},$$

$$M(d) = \frac{3(d-1)}{2} - \frac{l \cdot l^3 + 2l^2 - 5l + 2}{8} + m + 1.$$

The integers $l$ and $m$ are uniquely determined by the constraint $\frac{(l-1)(l+1)(l+2)}{6} + m = d - 1$, where $0 \leq m \leq \frac{(l+1)(l+2)}{2}$.

**Proof:** Let the spectra of the reduced sub-states $\rho_A$, $\rho_B$ and $\rho_C$ be $\lambda(\rho_A) = \{p_0, p_1, \cdots, p_{d-1}\}$, $\lambda(\rho_B) = \{q_0, q_1, \cdots, q_{d-1}\}$ and $\lambda(\rho_C) = \{r_0, r_1, \cdots, r_{d-1}\}$, respectively. Without loss of generality, we assume that the spectra are arranged in nonincreasing order.

i) **Proof of $\Delta_{EG} \leq (Y - Z)E$**

From (5), the ergotropic gap of the system can be written in the following form:

$$\Delta_{EG} = \sum_{i=0}^{d-1} i(p_i + q_i + r_i)E - \text{Tr}(\rho_{ABC}^0 H_{ABC}). \quad (7)$$
where the first three terms are the local passive state energies of subsystems $A$, $B$ and $C$, respectively, and the last term is the global passive state energy.

A state $\rho$ is said to be majorized by a state $\sigma$, $\lambda(\rho) < \lambda(\sigma)$, if $\sum_{i=1}^{k} p_i^{k} \leq \sum_{i=1}^{k} q_i^{k}$ for $1 \leq k \leq n - 1$ and $\sum_{i=1}^{n} p_i^{n} = \sum_{i=1}^{n} q_i^{n}$, where $\lambda(\rho) \equiv \{ p_i^{k} \}$ and $\lambda(\sigma) \equiv \{ q_i^{k} \}$ are the spectra of $\rho$ and $\sigma$, respectively, arranged in nonincreasing order. By convention one appends zeros to make the two vectors $\lambda(\rho)$ and $\lambda(\sigma)$ have the same dimensions. From the Nielsen-Kempe separability criterion \cite{11}, if $\rho_{ABC}$ is separable, one has $\lambda(\rho_A) > \lambda(\rho_{ABC}) \land \lambda(\rho_B) > \lambda(\rho_{ABC}) \land \lambda(\rho_C) > \lambda(\rho_{ABC})$. Namely, e.g., for $\rho_A$,

$$
p_0 \geq x_0, \quad p_0 + p_1 \geq x_0 + x_1, \quad \cdots, \quad p_0 + \cdots + p_i \geq x_0 + \cdots + x_i, \quad \cdots,
$$

$$
\sum_{i=0}^{d-2} p_i \geq \sum_{i=0}^{d-2} x_i, \quad \cdots, \quad \sum_{i=0}^{d-2} p_i \geq \sum_{i=0}^{d-2} x_i, \quad \cdots, \quad \sum_{i=0}^{d-1} p_i = \sum_{i=0}^{d-1} p_i = \sum_{i=0}^{d-1} x_i,
$$

(8)

by substituting $p_i$ for $q_i$ and $r_i$, similar results are obtained for $\rho_B$ and $\rho_C$, respectively.

Subtracting the last term by 1st term, 2nd term, $\cdots$, $(d^3 - 1)$-th term in \cite{53}, respectively, we get

$$
\sum_{i=1}^{d-1} p_i \leq \sum_{i=1}^{d-1} x_i, \quad \sum_{i=2}^{d-1} p_i \leq \sum_{i=2}^{d-1} x_i, \quad \cdots, \quad \sum_{i=j}^{d-1} p_i \leq \sum_{i=j}^{d-1} x_i, \quad \cdots, \quad \sum_{i=d-1}^{d-1} p_i \leq \sum_{i=d-1}^{d-1} x_i.
$$

(9)

Similar inequalities can be obtained for $\rho_B$ and $\rho_C$, respectively.

Summing over the above inequalities for $\rho_A$, $\rho_B$ and $\rho_C$, we obtain

$$
\sum_{i=0}^{d-1} i(p_i + q_i + r_i) \leq 3 \sum_{i=0}^{d-1} i x_i + 3(d - 1) \sum_{i=d}^{d^3 - 1} x_i.
$$

(10)

From \cite{10} into \cite{7}, we get the bound of the ergotropy gap,

$$
\Delta_ECG \leq 3 \sum_{i=0}^{d-1} i x_i E + 3(d - 1) \sum_{i=d}^{d^3 - 1} x_i E - \text{Tr}(\rho^\rho_{ABC} H_{ABC}) \triangleq (Y - Z)E,
$$

(11)

where $Y \equiv 3 \sum_{i=0}^{d-1} i x_i + 3(d - 1) \sum_{i=d}^{d^3 - 1} x_i$ and $ZE = \text{Tr}(\rho^\rho_{ABC} H_{ABC})$.

To evaluate $ZE = \text{Tr}(\rho^\rho_{ABC} H_{ABC}) = \min_{U \in L(H_A \otimes H_B \otimes H_C)} \text{Tr}(U \rho_{ABC} U^\dagger H_{ABC})$, note that the total non-interacting global Hamiltonian

$$
H_{ABC} = H_A \otimes I_B \otimes I_C + I_A \otimes H_B \otimes I_C + I_A \otimes I_B \otimes H_C
$$

$$
= \text{diag}\{0, \cdots, d - 1\} E \otimes \text{diag}\{1, \cdots, 1\} \otimes \text{diag}\{1, \cdots, 1\} + \text{diag}\{0, \cdots, d - 1\} E
$$

$$
\otimes \text{diag}\{1, \cdots, 1\} + \text{diag}\{1, \cdots, 1\} \otimes \text{diag}\{0, \cdots, d - 1\} E
$$

$$
= \text{diag}\{0, \cdots, d - 1, \cdots, d - 1, \cdots, 2d - 2, \cdots, 2d - 2, \cdots, 2d - 2, \cdots, 3d - 3\} E.
$$

(12)

To obtain the minimum value, we need to designate the corresponding spectrum of the passive state $\rho^\rho_{ABC}$:

Energy 0: $x_0 \rightarrow |000\rangle$.

Energy 1: $x_1 \rightarrow |100\rangle$; $x_2 \rightarrow |010\rangle$; $x_3 \rightarrow |001\rangle$.

Energy 2: $x_4 \rightarrow |200\rangle$; $x_5 \rightarrow |110\rangle$; $x_6 \rightarrow |020\rangle$; $x_7 \rightarrow |011\rangle$; $x_8 \rightarrow |002\rangle$; $x_9 \rightarrow |101\rangle$.

Energy 3: $x_{10} \rightarrow |300\rangle$; $x_{11} \rightarrow |210\rangle$; $x_{12} \rightarrow |120\rangle$; $x_{13} \rightarrow |030\rangle$; $x_{14} \rightarrow |021\rangle$; $x_{15} \rightarrow |012\rangle$; $x_{16} \rightarrow |003\rangle$; $x_{17} \rightarrow |102\rangle$; $x_{18} \rightarrow |201\rangle$; $x_{19} \rightarrow |111\rangle$.

$\cdots$.

Energy $i$: $x_{6(i+1)+2} \triangleq x_{D_i} \rightarrow |i00\rangle$; $x_{D_i+1} \rightarrow |(i-1)10\rangle$; $\cdots$; $x_{D_i+i} \rightarrow |0i0\rangle$; $x_{D_i+i+1} \rightarrow |0(i-1)1\rangle$; $\cdots$; $x_{D_i+2i} \rightarrow |00i\rangle$; $\cdots$; $x_{D_i+j_i} \rightarrow |j_i1j_i2j_i3\rangle$; $\cdots$; $x_{D_i+i-1}$, where $0 \leq j_i \leq \frac{(i+1)(i+2)}{2} - 1$ and $j_i' + j_i' + j_i'' = i$. $\cdots$.
Energy \( d - 1 \): \( x_{D_d-1} \to |(d-1)00\rangle; x_{D_{d-1}+1} \to |(d-2)10\rangle; \ldots; x_{D_d-1+j_{d-1}'} \to |j_{(d-1)1}'j_{(d-1)2}'j_{(d-1)3}'\rangle; \ldots; x_{D_d-1}, \)
where \( 0 \leq j_{d-1}' \leq \frac{d(d+1)}{2} - 1 \) and \( j_{(d-1)1}' + j_{(d-1)2}' + j_{(d-1)3}' = d - 1. \)

Energy \( d \): \( x_{D_d} \to |(d-1)10\rangle; x_{D_{d+1}} \to |(d-2)20\rangle; \ldots; x_{D_d+k_1}' \to |k_{(d-1)1}'k_{(d-1)2}'k_{(d-1)3}'\rangle; \ldots; x_{D_{d+1}-3D_1-1}, \)
where \( 0 \leq k_1' \leq \frac{(d+1)(d+2)}{2} - \frac{3}{2}(1 \times 2) - 1 \) and \( k_{(d-1)1}'+k_{(d-1)2}'+k_{(d-1)3}' = d \) with \( k_{(d-1)1}',k_{(d-1)2}',k_{(d-1)3}' \leq d - 1. \)

Energy \( d+i-1 \): \( x_{D_{d+i-1}-3D_{d-1}} \to |(d-1)i0\rangle; x_{D_{d+i-1}-3D_{d-1}+1} \to |(d-2)(i+1)0\rangle; \ldots; x_{D_{d+i-1}-3D_{d-1}+k_i'} \to |k_{(d-1)1}'k_{(d-1)2}'k_{(d-1)3}'\rangle; \ldots; x_{D_{d+i-1}-3D_{d-1},} \)
where \( 0 \leq k_i' \leq \frac{(d+i)(d+i+1)}{2} - \frac{3}{2}(i+1) - 1 \) and \( k_{(d-1)1}'+k_{(d-1)2}'+k_{(d-1)3}' = d+i-1 \) with \( k_{(d-1)1}',k_{(d-1)2}',k_{(d-1)3}' \leq d - 1. \)

Energy \( 2d-2 \): \( x_{D_{2d-2}-3D_{d-2}} \to |(d-1)(d-1)0\rangle; x_{D_{2d-2}-3D_{d-2}+1} \to |(d-2)d0\rangle; \ldots; x_{D_{2d-2}-3D_{d-2}+k_{i-1}'} \to |k_{(d-1)1}'k_{(d-1)2}'k_{(d-1)3}'\rangle; \ldots; x_{D_{2d-2}-3D_{d-2}-1}, \)
where \( D = \frac{(2d-1)2d(2d+1)}{6} - \frac{(d-1)d+1}{3}, 0 \leq l_1' \leq \frac{(d-1)d}{2} - 1 \) and \( l_{(d-1)1}'+l_{(d-1)2}'+l_{(d-1)3}' = 2d-2 \) with \( l_{(d-1)1}',l_{(d-1)2}',l_{(d-1)3}' \leq d - 1. \)

Energy \( 2d+i-2 \): \( x_{D_{2d+i-2}} \to |(d-1)(d-1)i\rangle; x_{D_{2d+i-2}+1} \to |(d-2)(d-1)(i+1)\rangle; \ldots; x_{D_{2d+i-2}+l_i'} \to |l_{(d-1)1}'l_{(d-1)2}'l_{(d-1)3}'\rangle; \ldots; x_{D_{2d+i-2}-1}, \)
where \( 0 \leq l_i' \leq \frac{(d-i)(d-i+1)}{2} - 1 \) and \( l_{(d-1)1}'+l_{(d-1)2}'+l_{(d-1)3}' = 2d+i-2 \) with \( l_{(d-1)1}',l_{(d-1)2}',l_{(d-1)3}' \leq d - 1. \)

Energy \( 3d-3 \): \( x_{D_{3d-3}} \to |(d-1)(d-1)(d-1)\rangle. \)

For convenience, we give a diagram to represent the above specification for the corresponding spectrum of the passive state \( \rho^p_{ABC} \). In this diagram, elements in each row have the same energy, as follows:

\[
\begin{pmatrix}
0 & x_0 \\
1 & x_1 & \cdots & x_3 \\
& \vdots \\
i & x_{D_i} & \cdots & x_{D_i+j_i'} & \cdots & x_{D_{i+1} \cdots 1} \\
& \vdots \\
d-1 & x_{D_{d-1} \cdots 1} & \cdots & x_{D_{d-1}} & \cdots & x_{D_{d+1}-3D_1 \cdots 1} \\
& \vdots \\
d & x_{D_d \cdots 1} & \cdots & x_{D_d \cdots 1} & \cdots & x_{D_{d+i-1}-3D_{d-1} \cdots 1} \\
& \vdots \\
d+i-1 & x_{D_{d+i-1}-3D_{d-1} \cdots 1} & \cdots & x_{D_{d+i-1}-3D_{d-1}+k_i'} & \cdots & x_{D_{d+i-1}-3D_{d-1} \cdots 1} \\
& \vdots \\
2d-2 & x_{D_{2d-2}-3D_{d-2} \cdots 1} & \cdots & x_{D_{2d-2}-3D_{d-2} \cdots 1} & \cdots & x_{D_{2d-2}-3D_{d-2}-1} \\
& \vdots \\
2d-1 & x_{D_{2d-1}-3D_{d-1} \cdots 1} & \cdots & x_{D_{2d-1}-3D_{d-1} \cdots 1} & \cdots & x_{D_{2d-1}-3D_{d-2} \cdots 1} \\
& \vdots \\
2d+i-2 & x_{D_{2d+i-2}-3D_{d-2} \cdots 1} & \cdots & x_{D_{2d+i-2}-3D_{d-2} \cdots 1} & \cdots & x_{D_{2d+i-2}-3D_{d-2}-1} \\
& \vdots \\
3d-3 & x_{D_{3d-3}} & \cdots & x_{D_{3d-3}+l_i'} & \cdots & x_{D_{3d-3}-1} \\
& \vdots \\
\end{pmatrix}
\]

Therefore,

\[
ZE = \text{Tr}(\rho^p_{ABC} H_{ABC}) = \sum_{i=1}^{d-1} \frac{(i+1)(i+2)-1}{d} \sum_{j_i'=0}^{d-1} x_{D_i+j_i'} + \sum_{i=1}^{d-1} \frac{(d+i)(d+i+1)}{2} - \frac{3}{2}(i+1) - 1 \sum_{k_i'=0}^{d-i-1} x_{D_{d+i-1}-3D_{d-1}+k_i'} + \sum_{l_i'=0}^{(d-2)(d-1)-1} (2d-2+i) x_{D_{D_{d-1}+l_i'}}
\]

(13)
which proves $\Delta_{EG} \leq (Y - Z)E$.

ii) Proof of $\Delta_{EG} \leq M(d)E$

Similar to the approach used in (10), we rewrite (7) as,

$$\Delta_{EG} = \sum_{i=1}^{d-1} i(p_i + q_i + r_i)E - \sum_{i=1}^{l-1} iE \sum_{k'=0}^{l(i+1)(i+2)-1} r_{D_i+k'} - lE \sum_{k'=0}^{m} r_{D_i+k'}$$

$$+ \sum_{i=1}^{l-1} iE \sum_{k'=0}^{l(i+1)(i+2)-1} r_{D_i+k'} + lE \sum_{k'=0}^{m} r_{D_i+k'} - \Tr(p_{ABC}H_{ABC}),$$

where $l$ and $m$ are integers determined uniquely by the constraint

$$D_i + m = \frac{l(l+1)(l+2)}{6} + m = d - 1, \quad 0 \leq m \leq \frac{(l+1)(l+2)}{2}.$$

Replacing $p_i$ with $r_i$ in (9), we obviously have

$$\sum_{i=1}^{l-1} iE \sum_{k'=0}^{l(i+1)(i+2)-1} r_{D_i+k'} + lE \sum_{k'=0}^{m} r_{D_i+k'} \leq \sum_{i=1}^{l-1} iE \sum_{k'=0}^{l(i+1)(i+2)-1} x_{D_i+k'} + lE \sum_{k'=0}^{m} x_{D_i+k'}.$$

Putting the expressions of (14) and (18) into (16), we get

$$\Delta_{EG} \leq \sum_{i=0}^{d-1} i(p_i + q_i + r_i)E - \sum_{i=1}^{l-1} iE \sum_{k'=0}^{l(i+1)(i+2)-1} r_{D_i+k'} - lE \sum_{k'=0}^{m} r_{D_i+k'}$$

$$+ \sum_{i=1}^{l-1} iE \sum_{k'=0}^{l(i+1)(i+2)-1} x_{D_i+k'} + \sum_{i=1}^{d-1} (d - 1 + i)E \sum_{k'=0}^{l(i+1)(i+2)-1} x_{D_{D_i+i-3D_{i-1}+k'}}$$

$$+ \sum_{i=1}^{d-1} (2d - 2 + i)E \sum_{k'=0}^{l(i+1)(i+2)-1} x_{D_{D_i+i-3D_{i-1}+k'}}.$$

Note that

$$\sum_{i=0}^{d-1} ir_iE - \sum_{i=1}^{l-1} iE \sum_{k'=0}^{l(i+1)(i+2)-1} r_{D_i+k'} - lE \sum_{k'=0}^{m} r_{D_i+k'} = \sum_{i=1}^{d-1} (i - l')r_iE + \sum_{i=1}^{d-1} (i - l)r_iE,$$

where $(l', m')$ are determined by $i = \frac{l'(l'+1)(l'+2)}{6} + m'$, $0 \leq m' \leq \frac{(l'+1)(l'+2)}{2}$. Substituting (20) in (19), we have

$$\Delta_{EG} \leq \sum_{i=0}^{d-1} ip_iE + \sum_{i=1}^{l-1} iq_iE + \sum_{i=1}^{d-1} (i - l')r_iE + \sum_{i=1}^{d-1} (i - l)r_iE - \Delta_{EG},$$

where $\Delta_{EG}$ is given by (15). This completes the proof of (16).
where

\[
\delta \triangleq \left[ \sum_{i=1}^{d-1} \sum_{j'=0}^{i(i+2) - 1} x_{D_i+j'} \right] + \sum_{i=1}^{d-1} (d-1+i) \sum_{k'=0}^{(d+1)(d+1) - \frac{3}{2} i(i+1) - 1} x_{2D_{i-1}+3D_{i-1}+k'} + \sum_{i=1}^{d-1} (2d-2+i) \sum_{l'=0}^{(d-1)(d-1) - 1} x_{D_{D_i-l}+l'} - \left( \sum_{i=1}^{l-1} \sum_{k'=0}^{(i+1)(i+2) - 1} x_{D_i+k'} + l \sum_{k=0}^{m} x_{D_i+k'} \right) \geq 0.
\]

(22)

Maximizing the right hand side of (21) we have \(\min(\delta) = 0, p_i = q_i = r_i = 1/d\) for \(i = 1, \ldots, d - 1\), and

\[
\Delta_{EG} \leq M(d) E = \left( \frac{d-1}{2} + \frac{d-1}{2} + \frac{d-1}{2} - \frac{1}{d} \sum_{i=1}^{l-1} \frac{i(i+1)(i+2)}{2} - \frac{l(m+1)}{d} \right) E
\]

\[
= \left( \frac{3(d-1)}{2} - \frac{l^3 + 2l^2 - 5l + 2}{8} + m + 1 \right) E.
\]

(23)

 Altogether we have \(\Delta_{EG} \leq \min \left\{ (Y-Z) E, M(d)E \right\} \).

From the Theorem, we have the following separability criterion

**Corollary 1.** A tripartite state \(\rho_{ABC}\) is entangled if

\[
\Delta_{EG} > \min \left\{ (Y-Z) E, M(d)E \right\}.
\]

(24)

In particular, for three-qubit case, the reduced states are just qubit ones. The local qubit systems are governed by the same two-energy levels Hamiltonian \(H = E |1 \rangle \langle 1 |\). For separable three-qubit states \(\rho_{ABC}\) with the spectrum \(|x_0, x_1, \ldots, x_7\rangle\) in nonincreasing order, the ergotropy gap is bounded by

\[
\Delta_{EG} \leq \min \left\{ \left| 2(x_1 + x_2 + x_3) + x_4 + x_5 + x_6 \right| E, E \right\}.
\]

(25)

Let us consider several examples.

**Example 1.** The superposition of W and GHZ states \([42]\): \(|\psi\rangle = \sqrt{p} (\text{GHZ}) + \sqrt{1-p} |W\rangle\), \(0 \leq p \leq 1\), where \(|\text{GHZ}\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)\) and \(|W\rangle = \frac{1}{\sqrt{4}} (|000\rangle + |010\rangle + |001\rangle)\). For this state, we have \(\Delta_{EG} = \frac{3-\sqrt{1+4p-5p^2}}{2} E\), \(Y-Z = 0\) and \(M(d) = 1\). From \([24]\) we have that \(|\psi\rangle\) is entangled for \(0 \leq p \leq 1\).

The entanglement criterion for GHZ states \([26]\) claims that any separable 3-qubit state \(\rho\) satisfies the following inequalities

\[
|\langle A_1 \rangle_\rho + \langle A_2 \rangle_\rho + \langle A_3 \rangle_\rho| \leq 1, \quad |\langle A_1 \rangle_\rho - \langle A_2 \rangle_\rho - \langle A_3 \rangle_\rho| \leq 1,
\]

where \(A_1 = \sigma_x \otimes \sigma_x \otimes \sigma_x, A_2 = I \otimes \sigma_z \otimes \sigma_z, A_3 = \sigma_y \otimes \sigma_y \otimes \sigma_x\), and \(\langle A_i \rangle_\rho = Tr(A_i \rho)\). The inequalities may be violated by entangled states. It has been shown that \(|\psi\rangle\) is entangled for \(\frac{p}{4} < p \leq 1\).

And the entanglement criterion for W states \([27]\) claims that any separable 3-qubit state \(\rho\) satisfies the following inequalities

\[
|\langle B_1 \rangle_\rho + \langle B_2 \rangle_\rho + \langle B_3 \rangle_\rho| \leq 1, \quad |\langle B_1 \rangle_\rho - \langle B_2 \rangle_\rho - \langle B_3 \rangle_\rho| \leq 1,
\]

where \(B_1 = I \otimes \sigma_x \otimes \sigma_z, B_2 = I \otimes \sigma_y \otimes \sigma_y, B_3 = I \otimes \sigma_z \otimes \sigma_z,\) and \(\langle B_i \rangle_\rho = Tr(B_i \rho)\). The inequalities may be violated by entangled states. It has been shown that \(|\psi\rangle\) can be identified as an entangled state for \(0 \leq p < \frac{4}{7}\). It is obvious that our result is an improvement.

**Example 2.** The GHZ state mixed with a colored noise \([43, 44]\).

\[
\rho = \frac{p}{2} (|000\rangle \langle 000| + |111\rangle \langle 111|) + (1-p) |\text{GHZ}\rangle \langle \text{GHZ}|,
\]

\(0 \leq p \leq 1\). For this state, we have \(\Delta_{EG} = \left( \frac{3}{2} - \frac{p}{2} \right) E\), \(Y-Z = p \) and \(M(d) = 1\). Therefore \(\rho_{ABC}\) is entangled for \(p < 1\), which is the same result obtained in \([24]\).

**Example 3.** The GHZ state mixed with the white noise,

\[
\rho = (1-p) \frac{1}{8} + p |\text{GHZ}\rangle \langle \text{GHZ}|, \quad 0 \leq p \leq 1.
\]

For this state, we have \(\Delta_{EG} = \frac{3}{2} p E\), \(Y-Z = \frac{p}{2} (1-p)\) and \(M(d) = 1\). Hence \(\rho_{ABC}\) is entangled for \(p > \frac{4}{7}\), as the result obtained in \([43, 44]\).
III. Conclusion

We have investigated the ergotropic gap, the difference between the global ergotropy and local ergotropy for tri-partite systems. The upper bounds of the ergotropic gap have been analytically derived. The violation of the bound provides a sufficient criterion for entanglement. The ergotropic gap is tightly related to the work extraction and correlations among the subsystems. Our results may highlight further studies on ergotropic gaps for multipartite systems the detection of bi-separability and the genuine multipartite entanglement.

IV. Acknowledgement

This work is supported by the National Natural Science Foundation of China under Grant Nos. 11871350, 12075159 and 12171044; Beijing Natural Science Foundation(Grant No. Z190005); Academy for Multidisciplinary Studies, Capital Normal University; the Academician Innovation Platform of Hainan Province; Shenzhen Institute for Quantum Science and Engineering, Southern University of Science and Technology(No. SIQSE202001).

[1] K. Maruyama, F. Morikoshi, and V. Vedral, Thermodynamical Detection of Entanglement by Maxwells Demons, Phys. Rev. A 71, 012108 (2005).
[2] R. Landauer, Irreversibility and heat generation in the computing process, IBM J. Res. Dev. 5, 183 (1961).
[3] C. H. Bennett, The thermodynamics of computational A review, Int. J. Theor. Phys. 21, 905 (1982).
[4] R. Dillenschneider and E. Lutz, Energetics of Quantum Correlations, Europhys. Lett. 88, 50003 (2009).
[5] K. Funo, Y. Watanabe, and M. Ueda, Thermodynamic Work Gain from Entanglement, Phys. Rev. A 88, 052319 (2013).
[6] J. Oppenheim, M. Horodecki, P. Horodecki, and R. Horodecki, Thermodynamical Approach to Quantifying Quantum Correlations, Phys. Rev. Lett. 89, 180402 (2002).
[7] R. Alicki, M. Horodecki, P. Horodecki, and R. Horodecki, Thermodynamics of Quantum Information Systems Hamiltonian Description, Open Syst. Inf. Dyn. 11, 205 (2004).
[8] W. H. Zurek, Quantum Discord and Maxwell’s Demons, Phys. Rev. A 67, 012320 (2003).
[9] S. Jevtic, D. Jennings, and T. Rudolph, Maximally and Minimally Correlated States Attainable within a Closed Evolving System, Phys. Rev. Lett. 108, 110403 (2012).
[10] O. C. O. Dahlsten, R. Renner, E. Rieper, and V. Vedral, Inadequacy of von Neumann Entropy for Characterizing Extractable Work, New J. Phys. 13, 053015 (2011).
[11] V. Viguie, K. Maruyama, and V. Vedral, Work Extraction from Tripartite Entanglement, New J. Phys. 7, 195 (2005).
[12] H. C. Braga, C. C. Rulli, T. R. de Oliveira, and M. S. Sarandy, Maxwell’s Demons in Multiparticle Quantum Correlated Systems, Phys. Rev. A 90, 042338 (2014).
[13] K. Maruyama, F. Nori, and V. Vedral, Colloquium: The physics of Maxwell’s demon and information, Rev. Mod. Phys. 81, 1 (2009).
[14] L. D. Landau and E. M. Lifshitz, Statistical Physics, part I (Pergamon, Oxford, 1978).
[15] H. B. Callen, Thermodynamics and an Introduction to Thermostatics (Wiley, New York, 1985), 2nd ed.
[16] G. Francica, J. Goold, F. Plastina, and M. Paternostro, Daemonic ergotropy: enhanced work extraction from quantum correlations, NPJ Quantum Information 3, Article number: 12 (2017).
[17] M. Perarnau-Llobet, K. V. Hovhannisyan, M. Huber, P. Skrzypczyk, N. Brunner, and A. Acín, Extractable Work from Correlations, Phys. Rev. X 5, 041011 (2015).
[18] A. Mukherjee, A. Roy, S. S. Bhattacharya, and M. Banik, Presence of quantum correlations results in a nonvanishing ergotropic gap, Phys. Rev. E 93, 052140 (2016).
[19] C. H. Bennett, G. Brassard, C. CrSpeau, R. Jozsa, A. Peres, and W. K. Wootters, Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels, Phys. Rev. Lett. 70, 1895 (1993).
[20] C. H. Bennett, Quantum cryptography using any two nonorthogonal states, Phys. Rev. Lett. 68, 3121 (1992).
[21] C. H. Bennett and S. J. Wiesner, Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states, Phys. Rev. Lett. 69, 2881 (1992).
[22] C. Bai, Z. Li, C. Liu, and Y. Li, Quantum secret sharing using orthogonal multiqudit entangled states, Quantum Inf. Process. 16, 304 (2017).
[23] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Rev. Mod. Phys. 81, 865 (2009).
[24] Z. Zhang, Y. Luo, and Y. Li, Tighter monogamy and polygamy relations in multi-qubit systems, Eur. Phys. J. D, 73, 13 (2019).
[25] D. Gottesman, Measurement-based quantum computation on cluster states, Phys. Rev. A 54, 1862 (1996).
[26] Y. Akbari-Kourbolagh, Entanglement criteria for the three-qubit states, Int. J. Quantum Inf. 15, 1750049 (2017).
[27] A. Peres, Separability Criterion for Density Matrices, Phys. Rev. Lett. 77, 1413 (1996).
[28] M. Horodecki, P. Horodecki, and R. Horodecki, Separability of mixed states: necessary and sufficient conditions, Phys. Lett. A 223, 1 (1996).
[29] O. Rudolph, Some properties of the computable cross-norm criterion for separability, Phys. Rev. A 67, 032312 (2003); O. Rudolph, Further results on the cross norm criterion for separability, Quantum Inf. Process. 4, 219 (2005).
[30] K. Chen and L. A. Wu, A matrix realignment method for recognizing entanglement, Quantum Inf. Comput. 3, 193 (2003).
[31] C. J. Zhang, Y. S. Zhang, S. Zhang, and G. C. Guo, Entanglement detection beyond the computable cross-norm or realignment criterion, Phys. Rev. A 77, 060301(R) (2008).
[32] J. I. de Vicente, Separability criteria based on the Bloch representation of density matrices, Quantum Inf. Comput. 7, 624 (2007).
[33] J. I. de Vicente, Further results on entanglement detection and quantification from the correlation matrix criterion, J. Phys. A: Math. Theor. 41, 065309 (2008); C. Klöckl and M. Huber, Characterizing multipartite entanglement without shared reference frames, Phys. Rev. A 91, 042339 (2015); M. Li, J. Wang, S. M. Fei, and X. Li-Jost, Quantum separability criteria for arbitrary-dimensional multipartite states, Phys. Rev. A 89, 022325 (2014).
[34] A. S. M. Hassan and P. S. Joag, Separability criterion for multipartite quantum states based on the Bloch representation of density matrices, Quantum Inf. Comput. 8, 773 (2008).
[35] J. I. de Vicente and M. Huber, Multiparticle entanglement detection from correlation tensors, Phys. Rev. A 84, 062306 (2011).
[36] O. Gühne, P. Hyllus, O. Gittsovich, and J. Eisert, Covariance Matrices and the Separability Problem, Phys. Rev. Lett. 99, 130504 (2007); O. Gittsovich, O. Gühne, P. Hyllus, and J. Eisert, Unifying several separability conditions using the covariance matrix criterion, Phys. Rev. A 78, 052319 (2008).
[37] O. Gittsovich, P. Hyllus, and O. Gühne, Multiparticle covariance matrices and the impossibility of detecting graph-state entanglement with two-particle correlations, Phys. Rev. A 82, 032306 (2010).
[38] B. M. Terhal, Bell Inequalities and The Separability Criterion, Phys. Lett. A 271, 319 (2000); D. Chruściński and G. Sarbicki, Entanglement witnesses: construction, analysis and classification, J. Phys. A: Math. Theor. 47, 483001 (2014).
[39] C. Spengler, M. Huber, S. Brierley, T. Adaktylos, and B. C. Hiesmayr, Entanglement detection via mutually unbiased bases, Phys. Rev. A 86, 022311 (2012); B. Chen, T. Ma, and S. M. Fei, Influence of the single-particle structure on the nuclear surface and the neutron skin, Phys. Rev. A 89, 064302 (2014); S. Q. Shen, M. Li, and X. F. Duan, Entanglement detection via some classes of measurements, Phys. Rev. A 91, 012326 (2015).
[40] M. A. Nielsen and J. Kempe, Separable States are More Disordered Globally than Locally, Phys. Rev. Lett. 86, 5184 (2001).
[41] R. Lohmayer, A. Osterloh, J. Siewert, and A. Uhlmann, Entangled Three-Qubit States without Concurrence and Three-Tangle, Phys. Rev. Lett. 97, 260502 (2006).
[42] A. Cabello, A. Feito, and A. Lamas-Linares, Bell’s inequalities with realistic noise for polarization-entangled photons, Phys. Rev. A 72, 052112 (2005).
[43] M. Seevinck, Parts and wholes, Ph.D. Thesis, Utrecht University, arXiv:0811.1027.
[44] O. Gühne and M. Seevinck, Separability criteria for genuine multipartite entanglement, New J. Phys. 12, 053002 (2010).
[45] S. Szalay, Separability criteria for mixed three-qubit states, Phys. Rev. A 83, 062337 (2011).