Resurgence Analysis of Quantum Invariants: Seifert Manifolds and Surgeries on The Figure Eight Knot

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Abstract

We provide a resurgence analysis of the quantum invariants of all Seifert fibered three manifold, with oriented base of genus zero and at least three exceptional fibers. This generalizes results of Gukov-Putrov-Marinó and Chun. Building on work of the first named author and Hansen, we also give a resurgence analysis of the quantum invariants of a family of hyperbolic three manifolds obtained by surgery along the figure eight knot.

1 Introduction

For a closed oriented three manifold $M$ and a level $k \in \mathbb{N}$, let $\tau_k(M) \in \mathbb{C}$ be the SU(2) quantum invariant constructed by Reshetikhin and Turaev [40, 41, 45]. Let $K = k + 2$ and consider the normalized quantum invariant

$$Z_K(M) = \tau_K(M)/\tau_K(S^2 \times S^1)$$

which was also considered in Witten seminal work on quantum Chern-Simons theory in [47]. In this paper we presents a series of results concerning quantum invariants of three manifolds and their resurgence properties. For an introduction to resurgence, we refer to [13, 14, 33]. Let us recall the basic concept of a resurgent function. Let $\Omega \subset \mathbb{C}$ be a discrete subset, and consider the Riemann surface $\eta: \mathcal{C}_\Omega \to \mathbb{C} \setminus \Omega$, which is the universal cover of $\mathbb{C} \setminus \Omega$. An $\Omega$-resurgent function, is a holomorphic function on $\mathcal{C}_\Omega$. Research on resurgence in TQFT have been pioneered by Garoufalidis [20] and Witten [48]. The main idea in Witten’s work [48] is to propose an analytic continuation of Chern-Simons theory with gauge group SU(2), by formally applying the theory of Laplace integrals with holomorphic phase to the partition function of Chern-Simons theory. This have some remarkable consequences, which relates the TQFT $\tau_K$ to complex Chern-Simons theory. See also the work of Gukov-Marinó-Putrov [21], which verifies some of the implications of Witten’s idea in specific examples.

We engage in a case study of the Seifert manifold $X = \Sigma(0; 0; (p_1/q_1), \ldots, (p_n/q_n))$, with $n$ exceptional fibers and base $S^2$, where $p_j$ and $q_j$ are co-prime integers for all $j$ and the $p_j$’s are pairwise co-prime. We assume $X$ is an integral homology sphere. Let $Z_\infty \in \mathbb{Q}[\{q - 1\}]$ be the Ohtsuki series of $X$ [34, 35, 36]. Let $q = \exp(2\pi i / K)$, and introduce the quantities $P = \prod_{j=1}^n p_j$, and $\phi = -24\lambda(X) - 2 - \sum_{j=1}^n p_j^{-1}$, where $\lambda(X)$ is the Casson-Walker invariant [16]. Define $\tilde{Z}_K(X) = q^{\phi} Z_K(X)$, and $W_\infty = q^{\phi} Z_\infty / \tau_K(S^2 \times S^1)$. By expanding the $q$-series into a $K^{-1}$-series, we get $W_\infty(K) \in K^{-1/2} \mathbb{C}[[K^{-1}]]$. Rozansky and Lawrence proves in [30] the existence of finitely many non-zero rational numbers modulo the integers $R(X)$ and non-vanishing polynomials $W_\theta(z) \in \mathbb{C}[z]$, $\theta \in R(X)$, such that $W_K(X)$ has the following asymptotic expansion (in the Poincaré sense [37])

$$\tilde{Z}_K(X) \sim \sum_{\theta \in R(X)} e^{2\pi i \theta} K^{1/2} W_\theta(K) + W_\infty(K)$$

(1)

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*In Casson’s normalization.
Let \( \text{CS} : \mathcal{A}/\mathcal{G} \to \mathbb{C}/\mathbb{Z} \) be the Chern-Simons action \([4, 13]\), defined on the space of \( \text{SL}(2, \mathbb{C}) \) connections on \( X \) modulo gauge equivalence, and let \( M_{\text{SL}(2, \mathbb{C})}(X) \) be the moduli space of flat \( \text{SL}(2, \mathbb{C}) \) connections. Let \( \text{CS}_C = \text{CS}(M_{\text{SL}(2, \mathbb{C})}^*(X)) \), where \( M_{\text{SL}(2, \mathbb{C})}^*(X) \) is the subset consisting of irreducible flat \( \text{SL}(2, \mathbb{C}) \) connections. The Borel transform \( B : \xi \mapsto \mathbb{C}[\xi^{1/2}] \to \mathbb{C}[[\sqrt{\xi}]] \), is given by the linear extension of \( B(\xi^{-\alpha-1}) = \frac{1}{\Gamma(\alpha+1)} \), where \( \Gamma \) is the Gamma function. We prove the following theorem.

**Theorem 1.1.** We have that
\[
\text{CS}_C \supset R(X).
\]
The Borel transform \( B(W_\infty) \) is a resurgent function, whose set of singularities \( \Omega \) satisfy
\[
\text{CS}_C = \frac{i}{2\pi} \Omega \mod \mathbb{Z}.
\]

We observe that the equality \([3]\) shows that \( W_\infty \) and thus therefore also the quantum invariant \( \hat{Z}_K \) determines all the complex Chern-Simons values \( \text{CS}_C \). It is of course expected that \( R(X) \) is precisely the set of Chern-Simons values assumed on the moduli space of flat \( \text{SU}(2) \)-connections on \( X \).

We establish that \( \hat{Z}_K \) itself and the series \( W_\theta \) can be reconstructed through a process reminiscent of the classical Borel-Laplace resummation from \( W_\infty \) as follows. Let \( Q \in \mathbb{N} \) be the minimal natural number such that \( 4Q\text{CS}_C \subset \mathbb{Z} \). We shall prove that such \( Q \) exists and is equal to \( |P| \).

Introduce
\[
T(\mu) = \{m = 1, ..., 2Q - 1 : -m^2/4Q = \mu \mod \mathbb{Z}\}.
\]

For a meromorphic function \( f \), let \( P_f \) denote the set of its poles. For each \( \xi \in \mathbb{C} \), define \( g_{\xi}(y) = 1/(1 - e^{-\xi y}) \). For each \( x \in \mathbb{Z} \), let \( D_\xi(2\pi ix) \) be a small disc centered at \( 2\pi ix \), with \( (D_\xi(2\pi ix) \setminus \{2\pi ix\}) \cap P_{g_{\xi}} = \emptyset \). Let \( \gamma_{\xi}(x) = \partial D_\xi(2\pi ix) \), oriented anti-clockwise. Introduce the integral operators \( \mathcal{L}_\mu \) and \( \mathcal{L} \), defined on the space of meromorphic functions \( \hat{\phi} \) for which the following integrals exist
\[
\mathcal{L}_\mu(\hat{\phi})(\xi) = -\sqrt{2 \xi} \sum_{x \in T(\mu)} \frac{1}{2\pi i} \int_{\gamma_{\xi}(x)} (i\xi\sqrt{i})^{-1} ye^{\xi y/\sqrt{i}} g_{\xi}(y)\hat{\phi}\left(y^2/8\pi i\right) \, dy,
\]
\[
\mathcal{L}(\hat{\phi})(\xi) = \frac{\sqrt{\xi}}{2\pi i} \int_{C(\xi)} \frac{ye^{x\sqrt{i}}}{i\xi Q} \hat{\phi}\left(y^2/8\pi i\right) \, dy + \sum_{\mu \in \frac{1}{2\pi} P_{\phi} \mod \mathbb{Z}} \mathcal{L}_\mu(\hat{\phi})(\xi),
\]
where \( C(\xi) \) is the contour determined by analytic continuation, which for \( \xi \in \mathbb{R}_{>0} \) satisfy that \(-iC(\xi)\) is the diagonal line contour through the origin, passing from \((-1 + i)\infty \) to \((1 - i)\infty \). Observe that by definition, \( T(\mu) \) is empty for all but finitely many \( \mu \in \mathbb{Q}/\mathbb{Z} \), and therefore \( \mathcal{L}_\mu \) is 0 for all but these finitely many \( \mu \). It follows that the sum giving the second term of \( \mathcal{L} \), is always finite.

**Theorem 1.2.** We can recover \( \hat{Z}_K(X) \) and all the perturbative series from the series \( W_\infty \) by the following formula
\[
\hat{Z}_K(X) = \mathcal{L}(B(W_\infty))(K), \quad e^{2\pi i\theta}K^{-\frac{1}{2}}W_\theta(K) = \mathcal{L}_0(B(W_\infty))(K).
\]

The family of three manifolds we here consider includes the family of Brieskorn homology spheres, which arise in the special case of \( n = 3 \). Theorem \([1, 11]\) generalizes work of Gukov, Putrov and Marinò \([21]\) and work of Chun \([8]\), both of which we shall explain in Subsection \([2.4]\). These works are inspired by Witten’s work \([13]\), and are concerned with a proposal for how to categorify quantum invariants of three manifolds.

For co-prime integers \( r, s \) with \( s \neq 0 \), let \( M_{r/s} \) be the three manifold obtained by performing surgery along the figure eight knot \( K \) with surgery coefficient \( r/s \). It is known that \( M_{r/s} \) is hyperbolic if and only if \(|r| > 4 \) or \(|s| > 1 \), see e.g. \([14]\) or \([20]\). By work of the first named author and Hansen \([1]\), there exists functions \( \Phi_{n, \alpha} \) (defined in \([20]\) below) for which the set of critical values is identical (modulo \( \mathbb{Z} \)) to the set of critical \( \text{SL}(2, \mathbb{C}) \) Chern-Simons values on \( M_{r/s} \). In Conjecture
2 in [1], it is stated that the asymptotic of $\tau_K(M_{r/s})$ is governed by a sum of Laplace integrals with phases $\Phi^{\alpha,\beta}$ (see Conjecture 3.1 below). Let $\text{CS}_r = \text{CS}_r(M_{r/s})$ be the Chern-Simons values of flat $\text{SL}(\mathbb{C}, 2)$ connections on $M_{r/s}$, and let $\text{CS} = \text{CS}(M_{r/s})$ be the subset of Chern-Simons values of $\text{SU}(2)$ flat connections. In this paper we show the following theorem.

**Theorem 1.3.** If conjecture 2 in [1] is true, then we have an asymptotic expansion of the form

$$
\tau_K(M_{r/s}) \sim \sum_{\theta \in \text{CS}} e^{2\pi i K \theta} \sum_{a=0}^{\infty} c_{\theta,a} K^{-a/2}.
$$

(5)

For each Chern-Simons value $\theta$, let $Z(\theta)(K) = K^{-1} \sum_{a=0}^{\infty} c_{\theta,a} K^{-a/2}$. The Borel transform $\mathcal{B}(Z(\theta))$ is a resurgent function, and if we let $\Omega(\theta)$ denote its set of singularities, we have

$$
\text{CS}_C - \theta \supset i \frac{1}{2\pi} \Omega(\theta) \mod \mathbb{Z}.
$$

(6)

Each $\mathcal{B}(Z(\theta))$ can be decomposed into a finite sum of resurgent functions

$$
\mathcal{B}(Z(\theta)) = \sum_{\lambda \in \Lambda(\theta)} n_{\lambda} \hat{Z}_\lambda(\theta),
$$

(7)

with the following property. For any two Chern-Simons values $\theta, \theta'$ and $\lambda \in \Lambda(\theta)$, there exists $\mu \in \Lambda(\theta')$ and $n_{\lambda,\mu}$ with

$$
A^{2\pi i (\theta' - \theta)}(\hat{Z}_\lambda(\theta)) = n_{\lambda,\mu} \hat{Z}_\mu(\theta'),
$$

(8)

where $A^\omega$ is Ecalle’s alien derivative, defined relative to a singularity $\omega$.

Let us briefly explain the alien derivative. Let $f$ be an $\Omega$-resurgent function, which is further a simple resurgent function c.f. [33]. Here simple refers to the condition that for any section $\sigma : U \to \Omega$ of $\eta$ defined near $\omega \in \Omega$, we have that

$$
f \circ \sigma(\zeta) = \frac{\mu}{2\pi i (\zeta - \omega)} + \frac{\log(\zeta - \omega)}{2\pi i} \psi(\zeta - \omega) + \vartheta(\zeta - \omega),
$$

where $\psi$, and $\vartheta$ are germs of holomorphic functions near the origin. The alien derivative of $f$ with respect to $(\omega, \sigma)$ is then

$$
A^\sigma_\omega(f)(\zeta) = \mu \delta + \psi(\zeta).
$$

Here $\delta$ is a formal variable as discussed in [33].

1.1 Organization

This article is organized as follows. In Section 2 we give proofs of Theorem 1.1 and Theorem 1.2, and in Section 3 we prove Theorem 1.3.

2 The Seifert fibered case

Consider the following Seifert fibered three manifold $X = \Sigma(0; 0; (p_1/q_1), \ldots, (p_n/q_n))$. This manifold is obtained by surgery on the link given by $n$ unlinked unknots with surgery coefficient $p_1/q_1, \ldots, p_n/q_n$, and one unknot with 0 surgery coefficient, which has linking number 1 with the rest as indicated in the following figure.
Without loss of generality we can assume that $p_2, ..., p_n$ are odd. The homeomorphism type of $X$ is unaltered under a transformation $q_j \mapsto q_j + y_j$ for any choice of integers $y_1, ..., y_n$ such that $(pj, q_j + y_j) = 1$ and

$$\sum_{j=1}^{n} \frac{q_j}{p_j} = \sum_{j=1}^{n} \frac{q_j + y_j}{p_j}.$$  \hspace{1cm} (9)

If $q_j$ is odd for $j > 1$, we perform the transformation $q_j \mapsto q_j + p_j$ and $q_1 \mapsto q_1 - p_1$, which does not change the sum $[9]$. Hence we can assume without loss of generality that $q_2, ..., q_n$ are all even. Finally, changing the sign of $p_j$ and $q_j$ simultaneously if needed, we can assume that $p_j > 0$ for each $j$. The fact that $X$ is assumed to be an integral homology sphere is equivalent to

$$\prod_{j=1}^{n} p_j \sum_{j=1}^{n} \frac{q_j}{p_j} = \pm 1.$$  

Note that this implies that $q_1$ is odd.

Let us now recall some of the details of Lawrence and Rozansky in [30]. We introduce the following functions,

$$F(y) = \left(e^{\frac{i}{2}} - e^{-\frac{i}{2}}\right)^{2-n} \sum_{j=1}^{n} \left(e^{\frac{pi}{j}} - e^{-\frac{pi}{j}}\right), \quad g(y) = \frac{ig^2}{8\pi P},$$

and the constant

$$B = -\frac{\text{sign} P}{4\sqrt{|P|}} \exp\left(\frac{3\pi i}{4} \text{sign}\left(\frac{1}{P}\right)\right).$$

Let $G_0 = G_0(K) := \tau_K(S^2 \times S^1)$. Lawrence and Rozansky shows that

$$\tau_K(M) = \frac{BG_0}{q^{\frac{n}{2}}} \frac{1}{2\pi i} \int_{C} F(y)e^{Kg(y)} \, dy$$

$$= \frac{BG_0}{q^{\frac{n}{2}}} \sum_{m=1}^{2|P|-1} \text{Res}\left(\frac{F(y)e^{Kg(y)}}{1 - e^{-Ky}}, y = 2\pi im\right),$$

where $C = C(K)$ is the steepest descent contour described in the introduction as $C(\xi)$. See also Theorem 1 in section 4.3 of [30]. By the classical theorem on stationary phase approximation for Laplace integrals, the first term of (10) admits an asymptotic expansion giving the contribution from the trivial flat connection (according to [30]) corresponding to Chern-Simons value zero, and this expansion is explicitly given by

$$\left(\frac{BG_0}{q^{\frac{n}{2}}} \frac{1}{2\pi i} \int_{C} F(y)e^{Kg(y)} \, dy \sim \frac{2Bq^{\frac{n}{2}}}{q^{\frac{n}{2}} - q^{-\frac{n}{2}} \sqrt{P}} \sum_{n=1}^{\infty} \frac{n!}{n!} \left(\frac{2\pi iP}{K}\right)^n \right).$$

In our normalization we thus have

$$W_{\infty} = 2B\sqrt{2\pi P} \sum_{n=1}^{\infty} \frac{F^{(2n)}(0)}{n!} \left(\frac{1}{K}\right)^n.$$

We note that Rozansky and Lawrence provide a small discussion of how the phases $R(X)$ in [4] relates to work by Rozansky [33], but no precise statement is proven in this regards.

### 2.1 The moduli space of flat $SL(2, \mathbb{C})$ connections

We have the following presentation of the fundamental group of $X$

$$\pi_1(X) \simeq \langle h, x_1, ..., x_n \mid x_1x_2 \cdots x_n, x_j^{p_j} h^{-q_j}, [x_j, h], j = 1, ..., n \rangle.$$  

Due to work of Fintushel and Stern [17] much is known about the moduli space of flat $SU(2)$ connections on $X$, and we shall now recall a few of their results. Recall that $X$ is an integral
homology sphere by assumption, so the only reducible representation into $SU(2)$ is the trivial one. For an irreducible representation $\rho : \pi_1(X) \to SU(2)$, at most $n - 3$ of the $\rho(x_j)$ are $\pm I$, and if exactly $n - m$ of the $\rho(x_j)$ are equal to $\pm I$, then the component of $\rho$ in $\mathcal{M}_{SU(2)}$ is of dimension $2(n - m) - 6$.

Let

$$L(p_1, \ldots, p_n) \subset \mathbb{N}^n$$

be the the set of $n$-tuples $l = (l_1, \ldots, l_n)$ which satisfies the following condition. We have $0 \leq l_1 \leq p_1 - 1$ and $0 \leq l_j \leq \frac{p_{j-1} - 1}{p_j}$, for $j = 2, \ldots, n$, and there exists a least three distinct $l_{j_1} < l_{j_2} < l_{j_3}$ with $l_{j_t} \neq 0$ for $t = 1, 2, 3$. The following proposition is an adaption of Lemma 2 in [9] and Lemma 2.1, and Lemma 2.2 in [17].

**Proposition 2.1.** Let $l = (l_1, l_2, \ldots, l_n) \in L(p_1, \ldots, p_n)$. Then there exists matrices $Q_j \in SL(2, \mathbb{C})$ and a representation $\rho_l$

$$\rho_l : \pi_1(X) \to SL(2, \mathbb{C}),$$

with

$$\rho_l(x_1) = Q_1 \begin{pmatrix} e^{\frac{\pi i l_1}{p_1}} & 0 \\ 0 & e^{-\frac{\pi i}{p_1}} \end{pmatrix} Q_1^{-1}, \quad \rho_l(x_j) = Q_j \begin{pmatrix} e^{\frac{2\pi i l_j}{p_j}} & 0 \\ 0 & e^{-\frac{2\pi i l_j}{p_j}} \end{pmatrix} Q_j^{-1},$$

for $j = 2, \ldots, n$. For any non-trivial representation $\rho : \pi_1(X) \to SL(2, \mathbb{C})$, there exists $l' \in \mathbb{N}^n$ with at most $n - 3$ of the elements of the set $\{l_j\}$ being divisible by $p_j$, such that $\rho$ is of the form

$$\rho(x_1) = S_1 \begin{pmatrix} e^{\frac{\pi i l_1}{p_1}} & 0 \\ 0 & e^{-\frac{\pi i l_1}{p_1}} \end{pmatrix} S_1^{-1}, \quad \rho(x_j) = S_j \begin{pmatrix} e^{\frac{2\pi i l_j}{p_j}} & 0 \\ 0 & e^{-\frac{2\pi i l_j}{p_j}} \end{pmatrix} S_j^{-1}. \quad (12)$$

for some $S_1, \ldots, S_n \in SL(2, \mathbb{C})$.

For the representation $\rho_l$, we can in fact choose $Q_j = I$ for $j \neq j_2, j_3$. Before commencing the proof, let us introduce the following notation

$$\exp(x) = \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix},$$

which should not cause any ambiguities as long as the context shows that we are dealing with a matrix.

**Proof.** We start with the construction of $\rho_l$. Introduce

$$X_j = \exp(\pi i l_j/p_j), \quad X_j = \exp(2\pi i l_j/p_j)$$

for $j \in \{2, \ldots, n\} \setminus \{j_2, j_3\}$. Rewrite the relation

$$\prod_{j=1}^{n} x_j = 1$$

as the equivalent relation

$$x_{j_3+1} \cdots x_n x_1 \cdots x_{j_1} \cdots x_{j_2} \cdots x_{j_3-1} = x_{j_3}^{-1}.$$ 

Assume we have chosen $Q_{j_2}, Q_{j_3} \in SL(2, \mathbb{C})$ such that

$$X_{j_3+1} \cdots X_n X_1 \cdots X_{j_1} \cdots Q_{j_2} X_{j_2} Q_{j_2}^{-1} \cdots X_{j_3-1} = Q_{j_3}^{-1} X_{j_3}^{-1} Q_{j_3}. \quad (13)$$

Taking $Q_j = I$ for $j \notin \{j_2, j_3\}$, we can define $\rho : \pi_1(X) \to SL(2, \mathbb{C})$ by

$$\rho(x_j) = Q_j X_j Q_j^{-1}, \quad \rho(h) = X_1^{p_1}.$$
To see this, observe that $H := X^p_j = (-I)^j$ is central, and as $q_1$ is odd whereas $q_j$ is even for $j \geq 2$ we also have

$$X^p_j = H^q_j,$$

for all $j$. The last relation in $\pi_1(X)$ is ensured by (13). Observe that it will suffice to choose $Q \in \text{SL}(2, \mathbb{C})$ with

$$\text{tr} \left( X_{j+1} \cdots X_n X_1 \cdots X_{j_i} \cdots Q X_j Q^{-1} \cdots X_{j-1} \right) = 2 \cos \left( \frac{2\pi j}{p_{j_i}} \right) < 2,$$

(14)

because this will ensure that there exists some $T \in \text{SL}(2, \mathbb{C})$ with

$$TX_{j+1} \cdots X_n X_1 \cdots X_{j_1} \cdots Q X_j Q^{-1} \cdots X_{j-1} T^{-1} = X_{j_2}.$$

For (14) we used our assumption on $j_i$. Write

$$X_{j+1} \cdots X_n X_1 \cdots X_{j_1} \cdots X_{j-1} = \exp(ia),$$

$$X_{j_2} = \exp(ib),$$

$$X_{j_3} \cdots X_{j_{i-1}} = \exp(ic),$$

$$X_{j_1} = \exp(id).$$

Define

$$Q = \begin{pmatrix} u & -v \\ 1 & 1 \end{pmatrix},$$

for $u, v$ to be chosen below. Assume $u + v = 1$ so that $Q \in \text{SL}(2, \mathbb{C})$. We compute

$$X_{j+1} \cdots X_n X_1 \cdots X_{j_1} \cdots Q X_j Q^{-1} \cdots X_{j-1}$$

$$= \left( \exp(ia) \begin{pmatrix} u & -v \\ 1 & 1 \end{pmatrix} \right) \left( \exp(ib) \begin{pmatrix} 1 & v \\ -1 & u \end{pmatrix} \exp(ic) \right)$$

$$= \begin{pmatrix} u e^{ia} & -v e^{ia} \\ e^{-ia} & e^{-ia} \end{pmatrix} \begin{pmatrix} e^{ib} & v e^{-ic} \\ -e^{ib} & u e^{-ic} \end{pmatrix}$$

$$= \begin{pmatrix} u e^{i(b+c)} & v e^{i(b-c)} \\ e^{-i(c-a)} & u e^{-i(b+c)} \end{pmatrix}$$

$$= \begin{pmatrix} u e^{i(a+b+c)} + v e^{i(a-b+c)} \\ e^{i(b+a-c)} - e^{i(c-a-b)} \end{pmatrix}.$$

We have

$$\text{tr} \left( \frac{u e^{i(a+b+c)} + v e^{i(a-b+c)}}{e^{i(b+a-c)} - e^{i(c-a-b)}} u v e^{i(a-b-c)} - u v e^{i(a-b-c)} \right)$$

$$= 2u \cos(a + b + c) + 2v \cos(a + c - b).$$

It follows that we must solve

$$\begin{pmatrix} \cos(a + b + c) & \cos(a + c - b) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 \cos(d) \\ 1 \end{pmatrix}.$$

(15)

Using the trigonometric identity

$$\cos(x + y) = \cos(x) \cos(y) - \sin(x) \sin(y)$$

we get

$$\det \begin{pmatrix} \cos(a + b + c) & \cos(a + c - b) \\ 1 & 1 \end{pmatrix} = \cos((a + c) + b) - \cos((a + c) - b)$$

$$= -2 \sin(a + c) \sin(b).$$
Thus it remains to argue \( a + c \notin \pi \mathbb{Z} \) and, \( b \notin \pi \mathbb{Z} \). Assume towards a contradiction that \( a + c = \pi m \) for some \( m \in \mathbb{Z} \). Hence we would have \( P(a + c) = Pm\pi \) which would imply

\[
l_{j_1} 2^\ell \prod_{i \neq j_1} p_i = 0 \mod p_{j_1},
\]

for \( \epsilon \in \{0, 1\} \), with \( \epsilon = 0 \) for \( j_1 = 1 \). This is a contradiction, as \( 2^\ell \prod_{i \neq j_1} p_i \) is invertible in \( \mathbb{Z}/p_{j_1}\mathbb{Z} \), and \( 1 \leq l_{j_1} \leq (p_{j_1} - 1)/2^\ell \). Here we use co-primality. Similarly one sees that \( b \notin \pi \mathbb{Z} \). Thus we can solve \([15]\), and this concludes the first part of the proposition.

Now let \( \rho : \pi_1(X) \to \text{SL}(2, \mathbb{C}) \), be an arbitrary non-trivial representation. As remarked before any non-trivial representation is irreducible since \( X \) is integral homology three sphere. Since \( \rho(h) \) computes with the image of \( \rho \), we see that \( \rho(h) = \pm I \). Hence the relation \( x_1^{p_1} = h^{q_1} \) implies that \( \rho(x_1)^{p_1} = \pm I \), and for \( j = 2, \ldots, n \) we must have \( \rho(x_j)^{p_j} = I \), as \( q_j \) is even. Hence \( \rho \) must be of the form \([12]\) for some \( l' \in \mathbb{N}^n \). It only remains to argue, that at most \( n - 3 \) of the \( \rho(x_j) \) are \( \pm I \). If not, the relation \( x_1 x_2 \cdots x_k = 1 \) implies that there is \( j_1 < j_2 \) with \( \rho(x_{j_1}) \rho(x_{j_2}) = \pm I \). As \( p_{j_1} \) and \( p_{j_2} \) are relatively coprime, this is only possible if \( \rho(x_{j_1}) = \pm 1 \) and \( \rho(x_{j_2}) = \pm 1 \). This would imply that \( \rho(\pi_1(X)) \subset \{ \pm 1 \} = Z(\text{SU}(2)) \), which contradicts the fact that \( \rho \) is irreducible, since it was assumed non-trivial.

### 2.2 Chern-Simons values

Let \( P \to M \) be a principal \( \text{SL}(2, \mathbb{C}) \) bundle. Let \( \mathcal{A} \) be the space of connections on \( P \), and let \( \mathcal{G} \) be the group of gauge transformations. The Chern-Simons functional \([7, 18]\) \( \mathcal{A}/\mathcal{G} \to \mathbb{C}/\mathbb{Z} \), is defined for a connection \( A \) by

\[
\text{CS}(\mathcal{A}) = \frac{1}{8\pi} \int_M \text{tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right),
\]

where we have used a trivialization of \( \mathcal{A} \). To identify \( \mathcal{A} \simeq \Omega^1(M, \text{sl}(2, \mathbb{C})) \).

Returning to our Seifert fibered three manifold \( X \), we now give a formula for the Chern-Simons values of the flat \( \text{SL}(2, \mathbb{C}) \) connections constructed above, and compute the range of the Chern-Simons action on the subset of the non-trivial flat \( \text{SL}(2, \mathbb{C}) \) connections, \( \mathcal{M}^\ast_{\text{SL}(2, \mathbb{C})}(X) \).

**Proposition 2.2.** For \( l = (l_1, \ldots, l_n) \in L(p_1, \ldots, p_n) \), let \( \rho_l : \pi_1(X) \to \text{SL}(2, \mathbb{C}) \) be the associated representation defined in Proposition 2.1. We have that

\[
\text{CS}(\rho_l) = \left( P \left( \frac{l_1}{p_1} + \frac{\sum_{j=2}^{n} 2l_j}{p_j} \right) \right)^2 \mod \mathbb{Z}. \tag{16}
\]

Moreover, we have

\[
\text{CS} \left( \mathcal{M}^\ast_{\text{SL}(2, \mathbb{C})}(X) \right) = \left\{ \frac{-m^2}{4P} \mod \mathbb{Z} : m \in \mathbb{Z} \text{ and } m \text{ is divisible by at most } n - 3 \text{ of the } p_j \right\}. \tag{17}
\]

Here, and in the rest of this paper, at most means less than or equal to.

**Remark 2.1.** Let \( \rho : \pi_1(X) \to \text{SL}(2, \mathbb{C}) \) be a homomorphism, and let \( l \in \mathbb{N}^n \) be the associated \( n \)-tuple, as in the second part of Proposition 2.2. We have

\[
\text{CS}(\rho) = \left( P \left( \frac{l_1}{p_1} + \frac{\sum_{j=2}^{n} 2l_j}{p_j} \right) \right)^2 \mod \mathbb{Z}.
\]

This will be shown in the proof below.

The formula \([16]\) was proven for the \( \text{SU}(2) \) connections by Kirk and Klassen, and it is stated in Theorem 5.2 in \([20]\). There is a difference of sign coming from a different choice of orientation of \( \Sigma(p, q, r) \). Their proof uses Theorem 4.2 in \([20]\), which we now recall.
Let $M$ be a closed oriented three manifold containing a knot $K$. Let $Y$ be the complement of a tubular neighborhood of $K$ in $M$. The boundary of $Y$ is a torus. With respect to an identification $M \setminus Y \simeq D^2 \times S^1$, choose simple closed curves $\mu, \lambda$ on $\partial Y$ intersecting in a single point, such that $\mu$ bounds a disc. Let $\rho_t : \pi_1(Y) \to \text{SU}(2)$ be a path of representations such that $\rho_0(\mu) = \rho_1(\mu) = 1$, and for which there exists piecewise continuous differentiable functions $\alpha, \beta : I \to \mathbb{R}$ with

$$
\rho_t(\mu) = \begin{pmatrix}
0 & e^{2\pi i \alpha(t)} \\
e^{-2\pi i \alpha(t)} & 0
\end{pmatrix}, \quad \rho_t(\lambda) = \begin{pmatrix}
0 & e^{2\pi i \beta(t)} \\
e^{-2\pi i \beta(t)} & 0
\end{pmatrix}.
$$

Thinking of $\rho_1, \rho_0$ as flat connections on $M$, we have

$$
\text{CS}(\rho_0) - \text{CS}(\rho_1) = -2 \int_0^1 \beta(t) \alpha'(t) \, dt \mod \mathbb{Z}. \tag{18}
$$

Notice that the formula (18) differs from the corresponding formula in [26] by a sign. This discrepancy was already discussed by Freed and Gompf in [19], and is due to a sign convention. See the footnote on page 98 in [19].

Kirk and Klassen remarks in [26] that (18) is also valid for a path of $\text{SL}(2, \mathbb{C})$ connections, as long as the path $\rho_t$ stays away from parabolic representations. This is to ensure that $\rho_t$ is conjugate to a path which maps $\lambda, \mu$ to the maximal $\mathbb{C}^*$ torus of diagonal matrices. Using this, we shall show that one can easily extend (16) to flat $\text{SL}(2, \mathbb{C})$ by adapting the proof of Theorem 5.2. We include a proof for the sake of completeness.

**Proof of Proposition 2.2.** Let $K \subset X$ be the $n$’th exceptional fiber. Let $Y$ be the complement of a tubular neighborhood of $K$ in $X$. Removing $K$ has the effect on $\pi_1$ of removing the relation $x_n^{p_n} = h^{-q_n}$, i.e. we have a presentation

$$
\pi_1(Y) \simeq \langle h, x_1, \ldots, x_n \mid [x_j, h], x_1^{p_1} h^{-q_1}, \ldots, x_n^{p_n} h^{-q_n} \rangle. \tag{19}
$$

As meridian and longitude of $\partial Y$, we can take $\mu = x_n^{p_n} h^{q_n}$ and $\lambda = x_n^{-p_1 \cdots p_n} h^c$ respectively, where $c = \sum_{j=1}^{n-1} \frac{p_1 \cdots p_{j-1}}{p_j}$. Let $\rho : \pi_1(X) \to \text{SL}(2, \mathbb{C})$ be any irreducible representation. Let $l \in \mathbb{N}^n$ be the $n$-tuple, such that (12) holds. To prove formula (16), it will suffice to show

$$
\text{CS}(\rho) = \left( \frac{l_1}{p_1} + \sum_{j=2}^{n} \frac{2l_j}{p_j} \right)^2 \mod \mathbb{Z}.
$$

As the $p_j$ are pairwise co-prime, this will also imply that $\text{CS} \left( \mathcal{M}_{\text{SL}(2, \mathbb{C})}^*(X) \right)$ is included in the set

$$
\mathcal{W}(l_1, \ldots, l_n) \coloneqq \left\{ \frac{-m^2}{4P} \mod \mathbb{Z} : m \in \mathbb{Z} \text{ and } m \text{ is divisible by at most } n-3 \text{ of the } p_j \text{'s} \right\}.
$$

Introduce

$$
\epsilon = P \left( \frac{l_1}{p_1} + \sum_{j=2}^{n} \frac{2l_j}{p_j} \right),
$$

and

$$
\eta = \frac{\epsilon}{P}.
$$

The proof of (16) presented here, consists analogously with the proof of Theorem 5.2 in [26] of two parts. In the first part we find a path of $\text{SL}(2, \mathbb{C})$ connections on $X$ connecting $\rho_1$ to an abelian representation $\rho_0$. In fact $\rho_0$ will be an $\text{SU}(2)$ connection on $X$. In the second part we then find a path from $\rho_0$ to the trivial representation $\rho^{\text{triv}}$, and we then apply Kirk and Klassen formula (18).
After conjugating by $S_n^{-1}$ we have $\rho(x_n) = \exp_M \left( \frac{2\pi i l_j}{p_n} \right)$. Consider the subset

$$S \subset \text{Hom}(\pi_1(Y), \text{SL}(2, \mathbb{C}))$$

of representations $\hat{\rho}$ satisfying

$$\hat{\rho}(h) = \rho(h), \quad \text{tr}(\hat{\rho}(x_1)) = 2 \cos \left( \frac{\pi i l_1}{p_1} \right), \quad \text{tr}(\hat{\rho}(x_j)) = 2 \cos \left( \frac{2\pi i l_j}{p_j} \right), \quad \text{for } j \geq 2.$$ 

By considering the presentation $[19]$, we see that $S$ is naturally homeomorphic to the product of $n - 1$ conjugacy classes

$$S \simeq \left[ \exp_M \left( \frac{\pi i l_1}{p_1} \right) \right] \times \prod_{j=2}^{n-1} \left[ \exp_M \left( \frac{2\pi i l_j}{p_j} \right) \right].$$

Here $[Q]$ denotes the $\text{SL}(2, \mathbb{C})$ conjugacy class of $Q \in \text{SL}(2, \mathbb{C})$. Therefore the connectedness of $\text{SL}(2, \mathbb{C})$ implies that $S$ is connected. Write $\rho = \rho_1$. Choose a smooth path $\rho_t$ in $S$ connecting $\rho_1$ to $\rho_0 \in S$ given by

$$\rho_0(x_1) = \exp_M \left( -\frac{\pi i l_1}{p_1} \right), \quad \rho_0(x_j) = \exp_M \left( -\frac{2\pi i l_j}{p_j} \right), \quad j = 2, \ldots, n - 1$$

and

$$\rho_0(x_n) = \exp_M \left( \frac{\pi i l_1}{p_1} + \sum_{j=2}^{n-1} \frac{2\pi i l_j}{p_j} \right).$$

We can choose the arc $\rho_t$ such that $\rho_t(x_n) = \exp_M (2\pi i f(t))$ for a smooth function $f(t)$. In particular, we must have

$$f(0) = \frac{l_1}{2p_1} + \sum_{j=1}^{n-1} \frac{l_j}{p_j}$$

and

$$f(1) = \frac{l_n}{p_n}.$$ 

Notice that

$$f(0) = \frac{n}{2} - f(1).$$

As $q_n$ is even and $c$ is odd we have

$$\rho_t(\mu) = \rho_t(x_n)^{p_n} \rho_t(h)^{q_n} = \exp_M(2\pi i p_n f(t)), \quad \rho_t(\lambda) = \rho_t(x_n)^{-p_1 \cdots p_{n-1}} \rho_t(h)^{c} = \exp_M(-2\pi i p_1 \cdots p_{n-1} f(t) + \pi i).$$

Define

$$\alpha_1(t) = p_n f(t), \quad \beta_1(t) = -\frac{P}{p_n} f(t) + \frac{1}{2}.$$ 

We have that

$$-2 \int \alpha_1 \beta_1 = -2 \int_0^1 p_n f(t) \left( -\frac{p}{p_n} f(t) + \frac{1}{2} \right) \, dt$$

$$= -2 \int_{f(0)}^{f(1)} \left( -pu + \frac{p_n}{2} \right) \, du$$

$$= -2 \left[ \left( -\frac{pu^2}{2} + \frac{p_n u}{2} \right) \bigg|_{u=f(0)}^{u=f(1)} \right]$$

$$= P(f(1))^2 - p_n f(1) - P(f(0))^2 + p_n f(0)$$

$$= P(f(1))^2 - p_n f(1) - P \left( \frac{n}{2} - f(1) \right)^2 + p_n \left( \frac{n}{2} - f(1) \right)$$

$$= -\frac{Pn^2}{4} + Pnf(1) + p_n \left( \frac{n}{2} - 2Pf(1) \right)$$

$$= -\frac{c^2}{4P} + \frac{Pn\epsilon}{2P} + \frac{cl_n}{p_n} \mod \mathbb{Z}.$$
For the second part, we use the fact that \( H_1(Y) \cong \mathbb{Z} \) with generator \( \mu \) to conclude that the abelian SU(2) connection \( \rho_0 \) can be connected to the trivial representation \( \rho^{\text{triv}} \) by a path of SU(2) representations \( \sigma_t \) with

\[
\sigma_t(\mu) = \exp_M(2\pi it\alpha_1(0)), \quad \sigma_t(\lambda) = \exp_M(2\pi i\beta(0)).
\]

Let \( \alpha_0(t) = t\alpha_1(0) \) and \( \beta_0(t) = \beta(0) \). As \( \text{CS}(\rho^{\text{triv}}) = 0 \), we can apply Kirk and Klassen’s formula \[18\] to obtain

\[
-\text{CS}(\rho) = \text{CS}(\rho^{\text{triv}}) - \text{CS}(\rho)
\]

\[
= -2 \int_0^1 \alpha_0'(t)\beta_0(t) \, dt - 2 \int_0^1 \alpha_1'(t)\beta_1(t) \, dt.
\]

We have

\[
-2 \int_0^1 \alpha_1'(t)\beta_1(t) \, dt = -2 \int_0^1 \alpha_1(0)\beta_1(0) \, dt
\]

\[
= -2\alpha_1(0)\beta_1(0)
\]

\[
= -2p_\alpha f(0) \left( -\frac{Pf(0)}{p_\alpha} + \frac{1}{2} \right)
\]

\[
= 2Pf(0)^2 - p_\alpha f(0)
\]

\[
= 2P \left( \frac{\eta}{2} - f(1) \right) - p_\alpha \left( \frac{\eta}{2} - f(1) \right)
\]

\[
= 2\frac{\epsilon^2}{4P} + 2P(f(1))^2 - \frac{p_\alpha \epsilon}{2P} - 2\epsilon f(1) + l_n.
\]

Comparing this with \[20\] we get that

\[
-\text{CS}(\rho) = - \frac{\epsilon^2}{4P} + \frac{p_\alpha \epsilon}{2P} + \frac{l_n}{p_\alpha}
\]

\[
+ 2\frac{\epsilon^2}{4P} + 2P(f(1))^2 - \frac{p_\alpha \epsilon}{2P} - 2\epsilon f(1) \mod \mathbb{Z}
\]

\[
= \frac{\epsilon^2}{4P} + 2f(1)(Pf(1) - \epsilon/2) \mod \mathbb{Z}
\]

\[
= \frac{\epsilon^2}{4P} + 2f(1)P \left( - \frac{l_1}{2p_1} - \sum_{j=2}^{n-1} \frac{l_j}{p_j} \right) \mod \mathbb{Z}
\]

\[
= \frac{\epsilon^2}{4P} \mod \mathbb{Z}.
\]

This is what we wanted.

We now turn to the proof of \[17\]. We have already shown that \( \text{CS}(\mathcal{M}_n^{\text{SL}(2,\mathbb{C})}(X)) \subset \mathcal{W}(p_1,\ldots,p_n) \) so it will suffice to show the reverse inclusion. It will suffice to show that for any \( y \in \mathbb{Z} \) which is not divisible by more than at most three of the \( p_j \), we can find \( l = (l_1,\ldots,l_n) \in L(p_1,\ldots,p_n) \) which solves the congruence equation

\[
y^2 = -\left( P \left( \frac{\pi_1}{p_1} + \sum_{j=2}^{n} \frac{2\pi l_j}{p_j} \right) \right) \mod \mathbb{Z}
\]

(21)

For \( x \in \mathbb{Z} \) and \( d \in \mathbb{N} \) let \( [x]_d \) denote the congruence class of \( x \) in the quotient ring \( \mathbb{Z}/d\mathbb{Z} \). Since \( p_j \) is odd for \( j \geq 2 \), it follows that \( 4p_1, p_2, \ldots, p_n \) are also pairwise co-prime. Hence the Chinese remainder theorem applies, and the natural ring homomorphism \( q : \mathbb{Z} \rightarrow \mathbb{Z}/4p_1\mathbb{Z} \oplus \bigoplus_{j=2}^{n} \mathbb{Z}/p_j\mathbb{Z} \), given by \( x \mapsto ([x]_{4p_1},\ldots,[x]_{p_n}) \), descends to an isomorphism of rings

\[
\eta : \mathbb{Z}/4P\mathbb{Z} \cong \mathbb{Z}/4p_1\mathbb{Z} \bigoplus \bigoplus_{j=2}^{n} \mathbb{Z}/p_j\mathbb{Z}.
\]
It follows that \([21]\) is in fact equivalent to the following \(n\) congruence equations

\[
[y]_{4p_1}^2 = \left[ l_{j} \prod_{j=2}^{n} p_j + 2p \left( \sum_{j=2}^{n} l_{j} \prod_{l \neq j} p_t \right) \right]_{4p_1}^2,
\]

\((22)\)

\[
[y]_{p_j}^2 = \left[ 2l_{j} \prod_{t \neq j} p_t \right]_{p_j}^2, \quad \forall j \geq 2.
\]

The coprimality conditions ensures that \(2 \prod_{l \neq j} p_l\) is an invertible element in \(\mathbb{Z}/p_j\mathbb{Z}\) and therefore solving the last \(n - 1\) of the equations in \((22)\) can indeed be done with \(0 \leq l_{j} \leq (p_j - 1)/2\). It remains only to consider the first of the equations in \((22)\). We note that

\[(cp_1 + j)^2 - (dp_1 - j)^2 = (2(c + d) + (c^2 - d^2)p_1)p_1,
\]

and if \(c\) and \(d\) have the same parity, this is divisible by \(4p_1\). Here we use that \([2(c+d)]_4 = [c^2 - d^2] = 0\), if \(c\) and \(d\) have the same parity. It follows that the squares \([x]_{4p_1}^2\) occur in a repeating pattern, which is symmetric around multiples of \(p_1\)

\[
x : \quad p_1 - j \ldots \quad p_1 - 1 \quad p_1 \quad p_1 + 1 \ldots \quad p_1 + j
\]

\([x]_{4p_1}^2 : \quad [p_1 - j]_{4p_1}^2 \ldots \quad [p_1 - 1]_{4p_1}^2 \quad [p_1]_{4p_1}^2 \quad [p_1 + 1]_{4p_1}^2 \ldots \quad [p_1 + j]_{4p_1}^2
\]

In particular \([y]_{4p_1}^2 \in \{[0]_{4p_1}, [1]_{4p_1}, [2]_{4p_1}, \ldots, [p_1 - 2]_{4p_1}, [p_1 - 1]_{4p_1}\}\), and equation \((22)\) is reduced to

\[
[y]_{4p_1}^2 = \left[ l_{j} \prod_{t=2}^{n} p_t \right]_{4p_1}^2,
\]

which is independent of \(l_2, \ldots, l_n\). As \(\prod_{t=2}^{n} p_t\) is invertible modulo \(4p_1\), we can for every \(m = 1, \ldots, p_1 - 1\) find a unique \(x_m \in \{1, \ldots, p_1 - 1\}\) and \(d_m \in \mathbb{N}\) with

\[
x_m \prod_{t=2}^{n} p_t = m + d_m p_1.
\]

As multiplication is linear, we must have

\[
x_{p_1 - m} = p_1 - x_m,
\]

hence

\[
(p_1 - x_m) \prod_{t=2}^{n} p_t = p_1 - m + d_{p_1 - m} p_1.
\]

Thus

\[
p_1(d_m + d_{p_1 - m} + 1) = \left( x_m \prod_{t=2}^{n} p_t - m \right)
\]

\[
+ \left( p_1 - x_m \prod_{t=2}^{n} p_t - (p_1 - m) \right) + p_1
\]

\[
= p_1 \prod_{t=2}^{n} p_t = P.
\]

As \(p_2, \ldots, p_n\) are all odd this implies that \(d_m\) and \(d_{p_1 - m}\) have the same parity which imply

\[
\left\{ \left[ x_m \prod_{t=2}^{n} p_t \right]_{4p_1}^2, \left[ x_{p_1 - m} \prod_{t=2}^{n} p_t \right]_{4p_1}^2 \right\} = \left\{ [m]_{4p_1}^2, [p_1 - m]_{4p_1}^2 \right\}.
\]

It follows that we can in fact solve \((22)\) with \(l_1 \in \{0, \ldots, p_1 - 1\}\). This finishes the proof. \(\square\)
2.3 The Borel transform

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We start by giving a characterization of which of the phases in (10) give a non-zero contribution.

Lawrence and Rozansky give an explicit formula for the residue $\text{Res} \left( \frac{F(y)e^{K_{\phi}(y)}}{1 - e^{-K_y}}, y = 2\pi i\tilde{m} \right)$ which is valid for integral $K$. From this we get an expression

$$\sum_{\tilde{m} \in \mathcal{T}(-m^2/4P)} \text{Res} \left( \frac{F(y)e^{K_{\phi}(y)}}{1 - e^{-K_y}}, y = 2\pi i\tilde{m} \right) = e^{2\pi iK\left(\frac{-m^2}{4P}\right)} \tilde{H}_m(K),$$

with $\tilde{H}_m(K)$ being a polynomial in $K$ of degree at most $n - 3$, and where $\mathcal{T}(z)$ is as defined in (11) in the introduction, and $K$ is a positive integer. From their formula for $\text{Res} \left( \frac{F(y)e^{K_{\phi}(y)}}{1 - e^{-K_y}}, y = 2\pi i\tilde{m} \right)$, it appears that there is a term $K^{-1}$, but when we sum over all $\tilde{m} \in \mathcal{T}(-m^2/4P)$, these contributions cancel out, and thus the coefficient of $K^{-1}$ is 0. According to [30] it is the expression $e^{2\pi iK\left(\frac{-m^2}{4P}\right)} H_m(K)$, we should consider also away from integers, rather than the left hand side of the above equality, when we compute $W_g$.

Now the set of phases $R(X)$ in (1) consists of the values $\frac{m^2}{4P}$, $m = 1, ..., 2P - 1$, for which

$$\sum_{x \in \mathcal{T}(-m^2/4P)} \text{Res} \left( \frac{F(y)e^{K_{\phi}(y)}}{1 - e^{-K_y}}, y = 2\pi ix \right) \neq 0. \quad (23)$$

Thus we must prove that if (23) holds, then we have

$$\text{CS}(\rho) = \frac{-m^2}{4P} \mod \mathbb{Z},$$

for some homomorphism $\rho : \pi_1(X) \to \text{SL}(2, \mathbb{C})$. We start by noting that

$$\mathcal{P}_F = \{ \pi im | m \in \mathbb{Z} and m \text{ is divisible by at most } n - 3 \text{ of the } p_j \text{'s} \}. \quad (24)$$

It follows that if $\tilde{m}$ is divisible by at least $n - 2$ of the $p_j$, then $F(y)$ does not have a pole at $y = 2\pi i\tilde{m}$, and we get for integral $K$

$$\text{Res} \left( \frac{F(y)e^{K_{\phi}(y)}}{1 - e^{-K_y}}, y = 2\pi i\tilde{m} \right) = F(2\pi i\tilde{m})e^{K_{\phi}(2\pi i\tilde{m})} \text{Res} \left( \frac{1}{1 - e^{-K_y}}, y = 2\pi i\tilde{m} \right) = F(2\pi i\tilde{m})e^{K_{\phi}(2\pi i\tilde{m})} \text{Res} \left( \frac{1}{1 - (e^{-y}K)}, y = 2\pi i\tilde{m} \right) = F(2\pi i\tilde{m})e^{K_{\phi}(2\pi i\tilde{m})} \frac{1}{K}.$$

As the coefficient of $K^{-1}$ in $\tilde{H}_m(K)$ is zero, it follows that

$$\sum_{\tilde{m} \in \mathcal{T}(-m^2/4P), \text{ and } \tilde{m} \text{ is divisible by at least } n - 2 \text{ of the } p_j} \text{Res} \left( \frac{F(y)e^{K_{\phi}(y)}}{1 - e^{-K_y}}, y = 2\pi i\tilde{m} \right) = 0.$$

Therefore we see that if (23) holds, then there is some $\tilde{m} \in \mathcal{T}(-m^2/4P)$, which is divisible by at most $n - 3$ of the $p_j$. By Proposition 2.2 this implies that there exists some $l \in L(p_1, ..., p_n)$ with

$$\frac{-m^2}{4P} = \frac{-\tilde{m}^2}{4P} = \text{CS}(\rho_l),$$

where the first equality is by definition of $\mathcal{T}(-m^2/4P)$. This establishes (2).

We now show that $B(W_\infty)$ defines a resurgent function, and compute its set of singularities. We start by noting that as $F(-y) = F(y)$, we have that

$$F(y) = \sum_{n=0}^{\infty} \frac{F(2n)(0)}{(2n)!} y^{2n}. \quad (25)$$
Recall the classical duplication formula for the Gamma function, which can be found for instance in [37]

\[ \Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma \left( z + \frac{1}{2} \right). \]

In particular

\[ \Gamma \left( n + \frac{1}{2} \right) = \frac{\sqrt{\pi} (2n)!}{4^n n!}, \]

and

\[ B(z^{-n-\frac{1}{2}}) = \frac{4^n n! \zeta^n}{(2n)! \sqrt{\pi \zeta}}. \]  (26)

By combining equation (11), equation (25) and equation (26) we compute an exact expression for the Borel transform

\[ B(W_\infty) = B \left( 2 B \sqrt{2iP} \sum_{n=0}^{\infty} \frac{F(2n)(0)}{(2n)!} \left( \frac{1}{K} \right)^{n+\frac{1}{2}} \right) \]  (27)

\[ = 2 B \frac{2 B \sqrt{2iP}}{\sqrt{\pi \zeta}} \sum_{n=0}^{\infty} \frac{F(2n)(0)}{(2n)!} \left( \sqrt{8\pi i \zeta} \right)^{2n} \]

\[ = 2 B \frac{2 B \sqrt{2iP}}{\sqrt{\pi \zeta}} F(\sqrt{8\pi i \zeta}). \]

As \( F(-y) = F(y) \) we note that the factor \( F(\sqrt{8\pi i \zeta}) \) gives a well-defined meromorphic function. Thus \( B(W_\infty)(\zeta) \) is multi-valued meromorphic function with a square root singularity at 0 and with singularities for \( \sqrt{8\pi i \zeta} \in \mathcal{T}_F \), where \( \mathcal{T}_F \) is the set of poles of \( F(y) \). This set was computed above, see equation (24), and we conclude that the poles of \( B(W_\infty)(\zeta) \) occur at

\[ \zeta_m = \frac{\pi m^2}{8P}, \]

with \( m \in \mathbb{Z} \) being divisible by less than or equal to \( n - 3 \) of the \( p_j \)'s. It follows that

\[ \frac{i}{2\pi} \zeta_m \in \text{CS}(\mathcal{M}_{\text{SL}(2,\mathbb{C})}(X)). \]

Thus we conclude the proof by appealing to Proposition 2.2. \( \square \)

We now prove Theorem 1.2.

**Proof of Theorem 1.2.** It is an easy consequence of Proposition 2.2 that \( Q \) exists and is equal to \( P \).

We observe that by (27) we have that

\[ BF(y) = \frac{y}{8P} B(W_{\infty}) \left( \frac{y^2}{8P \pi i} \right). \]  (28)

From (10) we see

\[ \frac{W_\theta(K)}{\sqrt{K}} = -\frac{\sqrt{2}}{\sqrt{K} 2\pi i} \sum_{x \in T(\theta)} \int_{\gamma(x,K)} F(y) e^{Kg(y)} \frac{1}{1-e^{-Ky}} \, dy. \]

It immediately follows that

\[ \frac{W_\theta(K)}{\sqrt{K}} e^{2\pi i a} = -\frac{\sqrt{2}}{\sqrt{K} 2\pi i} \sum_{x \in T(\theta)} \int_{\gamma(x,K)} \frac{y e^{Kg(y)}}{iP (1-e^{-K})} B(W_{\infty}) \left( \frac{y^2}{8P \pi i} \right) \, dy \]

\[ = \mathcal{L}_\theta(B(W_{\infty}))(K). \]

The equation

\[ W_K(X) = \mathcal{L}(B(W_{\infty}))(K), \]

is proven similarly using (28) and (10). \( \square \)
2.4 Brieskorn integral homology spheres

We now turn to the case of a Brieskorn integral homology sphere. The family of Brieskorn integral homology spheres is very special due to the fact that the moduli space of flat $\text{SL}(2, \mathbb{C})$ connections $\mathcal{M}_{\text{SL}(2, \mathbb{C})}(\Sigma(p_1, p_2, p_3))$ is finite with cardinality given by the $\text{SL}(2, \mathbb{C})$ Casson invariant introduced by Curtis \cite{9, 10}

$$\lambda_{\text{SL}(2, \mathbb{C})}(\Sigma(p_1, p_2, p_3)) = (p_1 - 1)(p_2 - 1)(p_3 - 1)/4.$$ 

This is shown by Boden and Curtis \cite{6}. Prior to this and in relation to Floer homology, Fintuschel and Stern \cite{17} analyzed the $\text{SU}(2)$ moduli space $\mathcal{M}_{\text{SU}(2)}(X)$, of the Seifert fibered three manifold $X$ considered in this paper, and their work shows that the components are even dimensional manifolds with top dimension $2n - 6$. This is in stark contrast to the finiteness of the moduli space $\mathcal{M}_{\text{SL}(2, \mathbb{C})}(\Sigma(p_1, p_2, p_3))$. Moreover; the situation at hand is also special because all the Chern-Simons values of flat $\text{SL}(2, \mathbb{C})$ connections we encounter are in $\mathbb{R}/\mathbb{Z}$. In the three fibered case, this is naturally explained by work of Kitano and Yamaguchi \cite{28} which gives a decomposition of $\mathcal{M}_{\text{SL}(2, \mathbb{C})}(\Sigma) (\Sigma = \Sigma(p_1, p_2, p_3))$ as a union

$$\mathcal{M}_{\text{SL}(2, \mathbb{C})}(\Sigma) = \mathcal{M}_{\text{SL}(2, \mathbb{R})}(\Sigma) \bigcup_{\mathcal{M}_{\text{SU}(2)}(\Sigma)} \mathcal{M}_{\text{SU}(2)}(\Sigma).$$

Thus we pay special attention to the class of Brieskorn integral homology spheres, and we obtain the following corollary, where $\mathcal{M}^*_{\text{SL}(2, \mathbb{C})}$ denotes the moduli space of irreducible flat $\text{SL}(2, \mathbb{C})$-connections.

**Corollary 2.1.** For co-prime integers $p_1, p_2, p_3$ and $X = \Sigma(p_1, p_2, p_3)$, the inclusion \( \mathcal{M}^*_{\text{SL}(2, \mathbb{C})}(X) \) is an equality of sets and we have

$$\text{CS}\left(\mathcal{M}^*_{\text{SL}(2, \mathbb{C})}(X)\right) = \left\{ \frac{-y^2}{4p_1p_2p_3} \mod \mathbb{Z} : y \in \mathbb{Z}, p_j \nmid y, j = 1, 2, 3 \right\}. $$

If $p_1, p_2, p_3$ are odd primes, the action $\text{CS} : \mathcal{M}_{\text{SL}(2, \mathbb{C})}(X) \to \mathbb{R}/\mathbb{Z}$ is injective.

**Proof.** Comparing the number of representations found in Proposition 2.1 shows that this number is equal to $(p_1 - 1)(p_2 - 1)(p_3 - 1)/4$. By the work of Boden and Curtis \cite{9, 10}, and the proof of Theorem \cite{14} this shows the first part. Assume now that $p_1, p_2, p_3$ are odd primes. The second part follows from counting the number of squares in the ring $\mathbb{Z}/4p_1p_2p_3\mathbb{Z} \simeq \mathbb{Z}/4\mathbb{Z} \oplus_{j=1}^3 \mathbb{Z}/p_j\mathbb{Z}$, whose components in the last three factors are invertible. \( \square \)

As mentioned in the introduction, Theorem \cite{14} generalizes work by Gukov, Putrov and Marinò \cite{21} in which the quantum invariant of the Poincaré sphere $\Sigma(2,3,5)$ and the Brieskorn homology sphere $\Sigma(2,3,7)$ was treated, as well as work by Chun \cite{5} which treated the Brieskorn sphere $\Sigma(2,5,7)$. These works proves an inequality in \( \mathcal{M}^*_{\text{SL}(2, \mathbb{C})}(X) \) for these special three cases. They also show an analog of Theorem \cite{14} although their formulas are found by quite different means, in particular their formula for the $\bar{W}_0(K)$ are found by using certain regularization procedures for infinite sums. Moreover; their results are phrased in the context of categorification of quantum invariants.

Both of these works builds on and relies on the connection between quantum invariants and modular forms, which was first discovered by Lawrence and Zagier \cite{31} for the Poincaré sphere and later generalized by Hikami \cite{22, 23} to all Brieskorn integral homology three spheres. These works by Hikami also show that the phases $R(X)$ in \( \mathcal{M}^*_{\text{SL}(2, \mathbb{C})}(X) \) match with the Chern-Simons value of flat $\text{SU}(2)$ connections, in the case of a Brieskorn integral homology sphere.

3 The hyperbolic case

We now turn to the hyperbolic three manifolds $M_{r/s}$ with surgery link giving by the figure eight knot with framing $r/s$. 
We start by introducing in detail the phase functions $\Phi_{\alpha,\beta}^n$, mentioned in the introduction, and state conjecture 2 of [1]. We use Faddeev’s quantum dilogarithm [3] [15] [16] [25] with parameter $\kappa \in (0, 1)$. This is given by

$$S_\kappa(z) = \exp \left( \frac{1}{4} \int_C \frac{e^{zy}}{\sinh(\pi y) \sinh(\kappa y)} \, dy \right),$$

for $|\text{Re}(z)| < \kappa + \pi$, and $C$ is the contour $(-\infty, -1/2) \cup \Delta \cup (1/2, \infty)$, where $\Delta$ is the half-circle from $-1/2$ to $1/2$ in the upper half-plane. We shall always take $\kappa$ to be $K$-dependent and given by

$$\kappa = \frac{\pi}{K}.$$

We have the following functional equation valid for $|\text{Re}(\zeta)| < \pi$

$$(1 + e^{i\zeta})S_\kappa(\zeta + \kappa) = S_\kappa(\zeta - \kappa). \quad (29)$$

From (29), one can deduce that $x \mapsto S_\kappa(-\pi + 2\pi x)$ admits an analytic extension to $\mathbb{C} \setminus \{ \frac{m}{K} + \frac{1}{2K} : m = K, K + 1, \ldots \}$. If $m \in \mathbb{N}$, then the set $\{ \frac{m}{K} + \frac{1}{2K} : n = mK, mK + 1, \ldots, (m+1)K - 1 \}$, consists of poles of order $m$ whereas the set $\{ \frac{m}{K} + \frac{1}{2K} : n = -mK, -mK + 1, \ldots, -mK + K - 1 \}$, consists of zeroes of order $m$.

### 3.1 Conjecture 2

Choose $c, d \in \mathbb{Z}$ with $rd - cs = 1$. Introduce the function

$$\chi_{n,K}(x, y) = \sin \left( \frac{\pi}{8} (x - nd) \right) e^{2\pi i K \left( \frac{ax^2 + bx + c}{2} - \frac{z}{y} - xy \right)} S_\kappa(-\pi + 2\pi(x - y)) S_\kappa(-\pi + 2\pi(x + y)). \quad (30)$$

The first named author and Hansen in [1] established the following formula

$$\tau_K(M_r/s) = \nu K q^\mu \sum_{n \in \mathbb{Z}/|s| \mathbb{Z}} \int_{C_1(K) \times C_2(K)} \cot(\pi K x) \tan(\pi K y) \chi_{n,K}(x, y) \, dy \, dx,$$

where $\nu, \mu \in \mathbb{C}^*$ and $C_j(K)$ is a simple closed contour which encircles the set $\{ m/K : m = 1, 2, \ldots, K - 1 \}$, and $C_2(K)$ is a simple closed contour encircling $\{ (m+1/2)/K : m = 0, 1, \ldots, K - 1 \}$. Both contours are oriented anti-clockwise.

The small $\kappa$ asymptotic of the quantum dilogarithm is given by Euler’s dilogarithm, which for $|\text{Re}(\zeta)| < \pi$ is given by

$$\text{Li}_2(z) = -\int_0^z \frac{\log(1 - y)}{y} \, dy.$$

Euler’s dilogarithm satisfies

$$S_\kappa(z) = \exp \left( \frac{1}{2i\kappa} \text{Li}_2(-e^{iz}) + I_\kappa(z) \right), \quad (31)$$

where the error term

$$I_\kappa(z) = \frac{1}{4} \int_{\Delta} \frac{e^{zy}}{y \sinh(\pi y)} \left( \frac{1}{\sinh(\kappa y)} - \frac{1}{\kappa y} \right) \, dy,$$
is small for small $\kappa$. See Appendix A in [1].

Introduce the following functions

$$
\Phi_n(x, y) = \frac{1}{4\pi^2} \left( \text{Li}_2(e^{2\pi i(x+y)}) - \text{Li}_2(e^{2\pi i(x-y)}) \right) - \frac{dn^2}{s} - \frac{r}{4s} x^2 + \frac{n}{s} x + xy,
$$

and consider the following refinement, indexed by $\alpha, \beta \in \{0, 1\}$ and $n \in \mathbb{Z}/|s|\mathbb{Z}$

$$
\Phi_n^{\alpha, \beta}(x, y) = \alpha(x + y) + \beta(x - y) + \Phi_n(x, y).
$$

Let

$$
S = \{(x, y) \in \mathbb{R} \times \mathbb{C} : e^{2\pi i} \in (-\infty, 0)\}.
$$

A comparison of [31] with [30] naturally leads to consider the functions [32] as phase functions in a stationary phase approximation of the quantum invariant, and this is one major part of the following conjecture

**Conjecture 3.1** (Andersen-Hansen, [1] conjecture 2). There exists chains $\Gamma_{n, \alpha, \beta} \subset \mathbb{C}^2$ of real dimensions 2 meeting only non-degenerate stationary points of $\Phi_n^{\alpha, \beta}$ in $S$, and holomorphic 2-forms $\chi_{n, \alpha, \beta}$ indexed by $\alpha, \beta \in \{0, 1\}$, and $n \in \mathbb{Z}/|s|\mathbb{Z}$ such that the leading order large $K$ asymptotics of the quantum invariant is given by

$$
\tau_K(M_{r/s}) \sim K \sum_n \sum_{\alpha, \beta} \int_{\Gamma_{n, \alpha, \beta}} e^{2\pi i K \Phi_n^{\alpha, \beta}} \chi_{n, \alpha, \beta}.
$$

### 3.2 The moduli space $\mathcal{M}_{\text{SL}(2, \mathbb{C})}(M_{r/s})$

We now recall the parametrization of the moduli space of flat $\text{SL}(2, \mathbb{C})$ connections on $M_{r/s}$. As mentioned in the introduction, this builds on work by Riley [42], Klassen [29] and Kirk-Klassen [26][27]. We have the following presentation of the fundamental group of $M_{r/s}$

$$
\pi_1(M_{r/s}) \simeq \langle x, y | [x^{-1}, y]x = y[x^{-1}, y], x^r(yx^{-1}y^{-1}x^2y^{-1}x^{-1}y)^s = 1 \rangle,
$$

where $\mu = x$ and $\lambda = yx^{-1}y^{-1}x^2y^{-1}x^{-1}y$ correspond to the preferred meridian and longitude. Consider the following two equations

$$
\begin{aligned}
  v^{-r} &= \left(\frac{w-v^2}{1-v^2w}\right)^s, \\
v^2w &= (1-v^2w)(w-v^2).
\end{aligned}
$$

Given a solution $(v, w)$ to (33) with $v^2 \neq 1$, one can define for $(s, u+1) = (v, w)$ a representation $\rho_{(s, u+1)} : \pi_1(M_{r/s}) \rightarrow \text{SL}(2, \mathbb{C})$ where $\rho_{(t, l)}$ is given by

$$
\rho_{(t, l)}(x) = \begin{pmatrix} t & -t^{-1} \\ 0 & t^{-1} \end{pmatrix}, \quad \rho_{(t, l)}(y) = \begin{pmatrix} l & 0 \\ -tl & l \end{pmatrix}.
$$

The first named author and Hansen proves the following result.

**Theorem 3.1** (Andersen-Hansen, [1] Theorem 2). The map

$$
(x, y) \mapsto [\rho_{(e^{2\pi i}, e^{2\pi i} - 1)}],
$$

gives a surjection from the set of critical points $(x, y)$ of the phase functions $\Phi_n^{\alpha, \beta}$ with $x \notin \mathbb{Z}$ onto $\mathcal{M}_{\text{SL}(2, \mathbb{C})}(M_{r/s})$, and $[\rho_{(e^{2\pi i}, e^{2\pi i} - 1)}]$ is conjugate to an $\text{SU}(2)$ representation if and only if $(x, y) \in S$. 

---

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3.3 Laplace integrals with holomorphic phase

The starting point of Witten’s article [48] is the mathematical framework of resurgence and Picard-Lefschetz theory for Laplace integrals of the form

\[ I_\Delta (k) = \int_\Delta e^{kf(z)} \chi(z) \]  

(34)

where \( \Delta \subset \mathbb{C}^d \) is a chain of real dimension \( d \), \( f(z) \) is holomorphic, and \( \chi(z) \) is a holomorphic \( d \)-form. This was developed by Dingle, Malgrange, Howls, Berry, Pham and Delabaere [5, 11, 12, 24, 32, 38]. The idea is to allow \( k \) to become complex valued and Witten formally applies this to the partition function of Chern-Simons theory. The proof of Theorem 1.3 is a formality using the standard theory of Laplace integrals of this form (34). See the monograph by Arnold et al. [4] for a good introduction to Picard-Lefschetz theory and vanishing cycles, and how this is used in the context of Laplace integrals.

**Proof of Theorem 1.3.** We start by showing how the asymptotic expansion comes about. By conjecture 2 of [2]

\[ K^{-1} \tau_K (M_{/s}) \sim \sum_n \sum_{\alpha, \beta} \int_{\Gamma_{\alpha, \beta}^n} e^{2\pi i K \Phi_{\alpha, \beta}^n} \chi_{\alpha, \beta}^n. \]

Introduce the normalization

\[-F_{\alpha, \beta}^n = 2\pi i \Phi_{\alpha, \beta}^n\]

Let \( \{(x_j, y_j)\} \) denote the isolated set of critical points of \( F_{\alpha, \beta}^n \). We shall assume that the chain \( \Gamma_{\alpha, \beta}^n \) satisfies a generic condition specified in the work by Pham [38], which allows us to give a decomposition

\[ \int_{\Gamma_{\alpha, \beta}^n} e^{-K F_{\alpha, \beta}^n} \chi_{\alpha, \beta}^n \approx \sum_j n_j \int_{\Gamma_{\alpha, \beta}^n(j)} e^{-K F_{\alpha, \beta}^n} \chi_{\alpha, \beta}^n \]

where \( \approx \) means equality up to addition of a function which is \( O(K^{-M}) \) for all \( M \in \mathbb{N} \), and for each critical point \( (x_j, y_j) \) the chain \( \Gamma_{\alpha, \beta}^n(j) \) is the associated Picard-Lefschetz thimble containing \( (x_j, y_j) \), and \( n_j \in \mathbb{Z} \). The Picard-Lefschetz thimble is characterized by the fact that \( \text{Im}(F_{\alpha, \beta}^n) \) is constant on \( \Gamma_{\alpha, \beta}^n(j) \), and there exists so-called vanishing cycles \( \gamma_{\alpha, \beta}^n(j, t) = \gamma(j, t) \) of real dimension 1 giving a foliation

\[ \Gamma_{\alpha, \beta}^n(j) = \bigcup_{t \geq 0} \gamma(j, t), \]

such that \( \text{Re}(\Phi_{\alpha, \beta}^n)|_{\gamma(j, t)} = t \). In our context, the \( \gamma(j, t) \) will be circles. If \( \theta \) is the Chern-Simons value satisfying

\[ \theta = \Phi_{\alpha, \beta}^n(x_j, y_j) \mod \mathbb{Z}, \]

(35)

this means that we can rewrite the integral as follows

\[ \int_{\Gamma_{\alpha, \beta}^n(j)} e^{-K F_{\alpha, \beta}^n} \chi_{\alpha, \beta}^n = e^{2\pi i K \theta} \int_0^\infty e^{-tK G_{\alpha, \beta}^n} \chi_{\alpha, \beta}^n \]

where \( G(j, t) = G_{\alpha, \beta}^n(j, t) \) is the Gelfand-Leray form given by

\[ G(j, t) = \int_{\gamma(j, t)} \chi_{\alpha, \beta}^n \]

Near \( t = 0 \) we have a convergent expansion

\[ G(j, t) = \sum_{l=0}^\infty c_l(j) t^l. \]

Integrating this up against \( e^{-tK} \) and using the standard formula

\[ \int_0^\infty e^{-tK} t^l \ dt = \frac{\Gamma(l + 1)}{K^{l+1}}, \]
gives the asymptotic expansion. [5].

All of this was first shown by Pham for Laplace integrals with polynomial phase function with isolated stationary points [38], building upon work by Malgrange [32], which provided the asymptotic expansion given a suitably nice integration chain. Howls extended the work of Pham in [24] to the context of a general holomorphic phase function with non-degenerate stationary points.

We now turn to the Borel transforms. From the description of the asymptotic expansion given above, it is immediate that \( B(Z(\theta))(\zeta) \) is a linear combination of the Gelfand-Leray transforms

\[
B(Z(\theta))(\zeta) = \sum n_j G_{\alpha,\beta}^n(j, \zeta)
\]

with \( \alpha, \beta, n, j \) ranging over all solutions to (35). In [24], Howls goes on to show that \( G_{\alpha,\beta}^n(j, \zeta) \) is a resurgent function in our termininology, and that it set of singularities are of the form \( 2\pi i(\theta - \theta') \), where \( \theta' \) range over the set of critical values of \( \Phi_{\alpha,\beta}^n \). This gives [5]. Howls also gives a detailed analysis of the kind of singularities \( G_{\alpha,\beta}^n(j, \zeta) \) has. If \( \theta' \) denotes another critical level of \( \Phi_{\alpha,\beta}^n \), then there exists some \( j' \), depending on the choice of path \( \gamma \) from 0 to \( \theta - \theta' \) such that

\[
A_\gamma^{2\pi i(\theta - \theta')}(G_{\alpha,\beta}(j)) = n_{j,j'} G_{\alpha,\beta}^n(j', \zeta),
\]

for some and a constant \( n_{j,j'} \). This constant can be computed as a certain intersection number of the associated Picard-Lefschetz thimbles. This imply (7) and (8). These last two points concerning resurgence properties and the alien operators are also discussed in detail by Delabaere and Howls, in the case of a polynomial phase with non-degenerate stationary points. See Section 4 in [11].

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