Periodically time-varying memory static output feedback control design for discrete-time LTI systems

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Abstract

This paper addresses the problem of static output feedback (SOF) stabilization for discrete-time LTI systems. We approach this problem using the recently developed periodically time-varying memory state-feedback controller (PTVMSFC) design scheme. A bilinear matrix inequality (BMI) condition which uses a pre-designed PTVMSFC is developed to design the periodically time-varying memory SOF controller (PTVMSOFC). The BMI condition can be solved by using BMI solvers. Alternatively, we can apply two-steps and iterative linear matrix inequality algorithms that alternate between the PTVMSFC and PTVMSOFC designs. Finally, an example is given to illustrate the proposed methods.

Key words: Static output feedback (SOF) control; linear matrix inequality (LMI); bilinear matrix inequality (BMI); linear time-invariant (LTI) system; periodically time-varying memory controller.

1 Introduction

The design of static output feedback (SOF) controllers has received a significant amount of attention to date since it is common experience in practical control applications that having full access to the state is not always possible. While a wide variety of problems related to controller analysis and design can be recast as convex linear matrix inequality (LMI) problems (Boyd, Ghaoui, Feron, & Balakrishnan, 1994) which are easily tractable by standard convex optimization techniques (Gahinet, Nemirovski, Laub, & Chilali, 1995; Strum, 1999; Löfberg, 2004), this is not the case for the SOF problem (Fu & Luo, 1997) since the most general characterization of the SOF design is bilinear matrix inequalities (BMIs) for which complete and efficient methods to find their global solutions are not available yet. For this reason, the SOF design is one of the most challenging open problems in the control literature. Nowadays, there is immense literature addressing the SOF problem through various approaches, just to name a few:

- A simple method using structural properties of the open-loop system (Garcia, Pardín, & Zeng, 2001);
- Iterative schemes based on the linear quadratic regulator (LQR) theory (Kučera & de Souza, 1995; Rosinová, Veselý, & Kučera, 2003);
- Sufficient LMI conditions using similarly transformations (Prempun & Postlethwaite, 2001; K.H. Lee, J.H. Lee, & Kwon, 2006; Dong & Yang, 2007) and using the elimination lemma (Dong & Yang, 2013);
- Sufficient LMI conditions with linear matrix equality constraints (Crusius & Trofino, 1999);
- Two-steps LMI approaches (Bara & Boutayeb, 2005) using a congruence transformation and fixing the Lyapunov matrix structure;
- Iterative LMI (ILMI) methods based on the LQR theory (Cao, Lam, & Sun, 1998), cone complementarity linearization (El Ghaoui, Oustry, & AitRami, 1997; He & Wang, 2006), quadratic separation concept (Peaucelle & Arzelier, 2005), descriptor system augmentation (Shu, Lam, & Xiong, 2010), and substitutive ILMI algorithm (Fujimori, 2004); ILMI schemes (Peaucelle & Arzelier, 2001) and two-steps LMI approaches (Mehdi, Boukas, & Bachelier, 2004; Agulhari, Oliveira, & Peres, 2010, 2012) alter-
nating between state-feedback (SF) and SOF designs;
• Mixed LMI/randomized methods (Arzelier, Gryazina, Peaucelle, & Polyak, 2010);
• The rank constrained LMI strategy (Orsi, Helmke, & Moore, 2006);
• Nonlinear optimization approaches (Goh, Turan, Safonov, Papavassilopoulos, & Ly, 1994; Kaney, Scherer, Verhaegen, & de Schutter, 2004; Henrion, Lefebvre, Kocvara, & Stingl, 2005; Burke, Henrion, Lewis, & Overton, 2006).

In this paper, we consider the problem of designing a SOF controller for discrete-time LTI systems. Among the important results mentioned earlier, the main idea of this paper is motivated by Peaucelle et al. (2001), where efficient ILMI procedures that alternate between the SF and SOF designs are developed based on the elimination lemma (Boyd et al., 1994). The idea was further developed in Mehdi et al. (2004) for discrete-time LTI systems by introducing new decision variables, in Arzelier et al. (2010) in combination with hit-and-run strategies, and recently in Agulhari et al. (2010, 2012) for reduced-order robust $\mathcal{H}_\infty$ control of continuous-time uncertain LTI systems.

We revisit this idea in a somewhat different direction for discrete-time LTI systems. More specifically, our method is an extension of the work presented in Peaucelle et al. (2001), Mehdi et al. (2004), Agulhari et al. (2010, 2012) to the so-called periodically time-varying memory controller technique, which was developed recently by Ebihara, Kuboyama, Hagiwara, Peaucelle, & Arzelier (2009); Ebihara, Peaucelle, & Arzelier (2011); Trégouët, Arzelier, Peaucelle, Ebihara, Pittet, & Falcoz (2011); Trégouët, Ebihara, Arzelier, Peaucelle, Pittet, & Falcoz (2012); Trégouët, Peaucelle, Arzelier, & Ebihara (2013) for robust control purposes.

In the field of robust control of LTI systems, the development of less conservative robust SF control design has been a fundamental and challenging problem. In the late 1990s, the so-called extended Schur complement and slack variable approaches were developed by the pioneering work in de Oliveira & Peres (1999); Peaucelle, Arzelier, Bachelier, & Bernussou (2000); de Oliveira, Geromel, & Bernussou (2002), which paved the way for the subsequent development of the LMI-based robust analysis and control design approaches (see, e.g, Oliveira, Peres (2007); Oliveira, de Oliveira, & Peres (2008) and references therein). Recently, a new paradigm emerged through a sequence of interesting researches in Ebihara et al. (2009, 2011); Trégouët et al. (2011, 2012, 2013), where the so-called periodically time-varying memory SF controller (PTVMSFC) which makes use of the state information in a periodic manner was proposed and turned out to be effective in reducing the conservatism in the traditional robust SF approaches for discrete-time systems subject to parameter uncertainties. Despite those recent progresses, up to the authors’ knowledge, an extension of the PTVMSFC approach to the SOF problem still remains unresolved.

This paper suggests strategies to design a periodically time-varying memory SOF controller (PTVMSOFC) that stabilizes discrete-time LTI systems. To this end, first, we pay attention for introducing some definitions and notation, which reduce the difficulty of the matrix calculations and their formal expressions. Next, by means of the Finsler’s lemma (Skelton, Iwasaki, & Grioriadis, 1998), a necessary and sufficient condition for designing the PTVMSOFC is derived in terms of BMI problems. Then, following the lines in Peaucelle et al. (2001), we use the elimination lemma (Skelton, Iwasaki, & Grioriadis, 1998) to reduce the structure of the multiplier introduced by the Finsler’s lemma to a special form based on a chosen SF controller, and the BMI problem comes down to solving another BMI. These BMI problems can be treated with PENBMI (Kocvara & Stingl, 2005), a solver for BMIs. Alternatively, at the price of some conservatism, the BMI problem can reduce to an LMI problem, based on which the PTVMSOFC design problem can be solved by applying two-steps LMI and iterative LMI (ILMI) algorithms (Mehdi et al., 2004; Agulhari et al., 2010, 2012; Peaucelle et al., 2001).

Finally, an comparison analysis is given to evaluate the effectiveness of the proposed approaches.

2 Preliminaries

2.1 Notation

The adopted notation is as follows: $\mathbb{N}$ and $\mathbb{N}_+$: sets of nonnegative and positive integers, respectively; $\mathbb{Z}_{[k_1, k_2]}$: set of integers $\{k_1, k_1 + 1, \ldots, k_2\} \subseteq \mathbb{N}$; $\mathbb{R}^n$: $n$-dimensional Euclidean space; $\mathbb{R}^{n \times m}$: set of all $n \times m$ real matrices; $S_+^n$: set of all $n \times n$ real symmetric positive definite matrices; $A^T$: transpose of matrix $A$; $\text{He}(A) := A^T + A$; $\rho(A)$: spectral radius of matrix $A$; $A_+ \leq B_+:$ any matrices whose columns form bases of the right null-space of matrix $A$, $A \otimes B$: Kronecker’s product of matrices $A$ and $B$, $A \succ 0$ ($A \succeq 0$, $A \preceq 0$, and $A \preceq 0$, respectively): symmetric positive definite (negative definite, positive semi-definite, and negative semi-definite, respectively) matrix $A$; $0_n$: zero matrix of appropriate dimensions; $0_{n \times m}$ and $0_n$: zero matrix and zero vector of dimensions $n \times m$ and $n$, respectively; $I_n$: $n \times n$ identity matrix; $\mathcal{L}_N := [I_N \ 0_N] \in \mathbb{R}^{N \times (N+1)}$; $R_N := [0_N \ I_N] \in \mathbb{R}^{N \times (N+1)}$; $e_{i}(N, 0)$: unit vector of dimension $N$ with a 1 in the $i$-th component and 0’s elsewhere; for given two integers $k$ and $N$, $[k]_N$: remainder
of $k$ divided by $N$;
\[
T_N := \begin{bmatrix}
0 & \cdots & 0 & 1 \\
\vdots & \ddots & \vdots & \ddots \\
0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0
\end{bmatrix} \in \mathbb{R}^{N \times N}.
\]

### 2.2 Problem formulation

Consider the discrete-time LTI system described by
\[
\begin{cases}
x(k+1) = Ax(k) + Bu(k) \\
y(k) = Cx(k)
\end{cases}
\]
where $k \in \mathbb{N}$; $x(k) \in \mathbb{R}^n$ is the state; $u(k) \in \mathbb{R}^m$ is the control input; $y(k) \in \mathbb{R}^p$ is the measured output; $\Sigma := (A, B, C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n}$ is a tuple of constant matrices. Inspired by the recently developed PTVMSFC (Ebihara et al., 2009, 2011), we suggest the PTVMSOFC (or $N$-PTVMSOFC) of the following form:
\[
u(k) = \left\lfloor \frac{k}{N} \right\rfloor F_{\text{SOF}}^{N, i} y(k-i),
\]
where $N \in \mathbb{N}_+$ is the period of the controller and $F_{\text{SOF}}^{N, i} \in \mathbb{R}^{m \times p}$, $(\left\lfloor \frac{k}{N} \right\rfloor, i) \in \mathbb{Z}_{[0,N-1]} \times \mathbb{Z}_{[0,N-1]}$ are the SOF gains to be designed. In the case $N = 1$, this is the classical SOF controller. Substituting (2) into (1), the $N$-periodic control system (closed-loop system) can be written as
\[
x(k+1) = Ax(k) + B \sum_{i=0}^{\left\lfloor \frac{k}{N} \right\rfloor} F_{\text{SOF}}^{N, i} C x(k-i). \tag{3}
\]

The problem addressed in this paper is to seek the $N$-PTVMSOFC (2) such that the $N$-periodic control system (3) is asymptotically stable.

### 3 Main result

To streamline notation, for two integers $k_1, k_2 \in \mathbb{N}$, $k_1 \leq k_2$, $x(k_1 : k_2)$ and $x(k_2 : k_1)$, respectively, denote the vectors $x(k_2 : k_1)^T := [x(k_2)^T \; x(k_2-1)^T \; \cdots \; x(k_1)^T]$ and $x(k_1 : k_2)^T := [x(k_1)^T \; x(k_1+1)^T \; \cdots \; x(k_2)^T].$

#### 3.1 Augmented system representation

As stated in Ebihara et al. (2011), for any $k \in \{ k \in \mathbb{N} : \left\lfloor \frac{k}{N} \right\rfloor = 0 \}$, the input of the PTVMSOFC (2) can be expressed in the augmented form:
\[
u(k+N-1 : k) = \mathcal{F}_{\text{SOF}}^{N, \uparrow}(I_N \otimes C)x(k+N-1 : k),
\]
where
\[
\mathcal{F}_{\text{SOF}}^{N, \uparrow} := \begin{bmatrix} F_{\text{SOF}}^{(0,0)} & F_{\text{SOF}}^{(0,0)} & \cdots & 0 \\
F_{\text{SOF}}^{(1,0)} & F_{\text{SOF}}^{(1,0)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
F_{\text{SOF}}^{(N-1,0)} & F_{\text{SOF}}^{(N-1,1)} & \cdots & F_{\text{SOF}}^{(N-1,0)}
\end{bmatrix}.
\]

In light of this, $N$-periodic control system (3) can be formulated as
\[
x(k+N : k+1) = A_{\text{AUG}}^{(N, \downarrow)}x(k+N-1 : k),
\]
where $A_{\text{AUG}}^{(N, \downarrow)} := (I_N \otimes A) + (I_N \otimes B)\mathcal{F}_{\text{SOF}}^{N, \uparrow}(I_N \otimes C).$ Alternatively, based on the transformation $x(k+N-1 : k+1) = (T_N \otimes I_m)x(k+N-1 : k)$, (4) can be expressed as
\[
x(k+1 : k+N) = A_{\text{AUG}}^{(N, \downarrow)}x(k+N-1 : k),
\]
where $A_{\text{AUG}}^{(N, \downarrow)} := (I_N \otimes A) + (I_N \otimes B)\mathcal{F}_{\text{SOF}}^{N, \downarrow}(I_N \otimes C)$ and gain matrix $\mathcal{F}_{\text{SOF}}^{(N, \downarrow)}$ takes the form
\[
\mathcal{F}_{\text{SOF}}^{(N, \downarrow)} := (T_N \otimes I_m)\mathcal{F}_{\text{SOF}}^{(N, \uparrow)}(T_N \otimes I_p)
\]
\[
= \begin{bmatrix} F_{\text{SOF}}^{(0,0)} & 0 & \cdots & 0 \\
F_{\text{SOF}}^{(1,0)} & F_{\text{SOF}}^{(1,0)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
F_{\text{SOF}}^{(N-1,0)} & F_{\text{SOF}}^{(N-1,1)} & \cdots & F_{\text{SOF}}^{(N-1,0)}
\end{bmatrix}.
\]
with the state variables $\phi(t) = x(Nt)$, where $\Sigma := (A, B, C)$. Based on the augmented system representation (5), let
\[
\mathcal{F}_{\text{SOF}}^{(\delta, i)} := [I_m \ 0_{dm \times (N-\delta)m}] \mathcal{F}_{\text{SOF}}^{(N, i)} [I_p \ 0_{dp \times (N-\delta)p}]^T
\]
and
\[
A_{\text{AUG}}^{(\delta, i)} := I_\delta \otimes A + (I_\delta \otimes B) \mathcal{F}_{\text{SOF}}^{(\delta, i)} (I_\delta \otimes C), \quad \forall \delta \in \mathbb{Z}[1, N].
\]
Furthermore, define $A_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(N, i)}, \Sigma), i \in [1, N-1]$ as matrices satisfying
\[
x(k + i) = A_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(N, i)}, \Sigma) x(k),
\forall (i, k) \in \mathbb{Z}[1, N-1] \times \{k \in \mathbb{N} : [k]_N = 0\}.
\]
Then, taking into account (5), it is straightforward to see that
\[
x(k + 1 : k + N) = A_{\text{AUG}}^{(N, i)} x(k : k + N - 1)
\]
and hence, we can obtain the following expression of $A_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(N, i)}, \Sigma)$:
\[
A_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(N, i)}, \Sigma)
= \begin{bmatrix} I_n \\ A_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(1, i)}, \Sigma) \\ \vdots \\ A_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(N-1, i), \Sigma}) \end{bmatrix} x(k)
\]
\forall k \in \{k \in \mathbb{N} : [k]_N = 0\},
(8)
and or equivalently, from (4), we have
\[
A_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(N, i), \Sigma}) = A_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(N, i)}, \Sigma)
\]
\[
= \begin{bmatrix} I_n \\ 0_{n \times (N-1)n} \ I_n \\ \vdots \\ A_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(N-1, i)}, \Sigma) \end{bmatrix} x(k)
\]
Based on the observation, $A_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(N, i)}, \Sigma)$ can be constructed using the recursion in Algorithm 1.

### Algorithm 1: Construct $A_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(N, i)}, \Sigma)$

1. $\Phi \leftarrow I_n$
2. for $\delta \leftarrow \{1, 2, \ldots, N\}$ do
3. \[ A_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(\delta, i)}, \Sigma) \leftarrow [0_{n \times (\delta-1)n} \ I_n] A_{\text{AUG}}^{(\delta, i)} \]
4. \[ \Phi \leftarrow \begin{bmatrix} \Phi \\ A_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(\delta, i)}, \Sigma) \end{bmatrix} \]
5. end for
6. return $A_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(N, i)}, \Sigma)$

#### 3.3 N-PTVMSOFC synthesis

We start with the following necessary and sufficient BMI condition so that (7) is asymptotically stable:

**Theorem 1** There exists $\mathcal{F}_{\text{SOF}}^{(N, i)} \in \mathbb{R}^{Nm \times Np}$ defined in (6) such that $N$-periodic control system (3) or equivalent LTI system (7) is asymptotically stable if and only if there exists $P = P^T \in \mathbb{R}^n$ and $M \in \mathbb{R}^{(N+m) \times ((N-1)n + Nm)}$ such that the following problem is satisfied with $\mathcal{F}_{\text{SOF}}^{(N, i)} \in \mathbb{R}^{Nm \times Np}$:
\[
P > 0,
\]
\[
\Pi_N^T \chi_N(P, 1) \Pi_N + \text{He}(MC(\mathcal{F}_{\text{SOF}}^{(N, i)})) < 0,
\]
where
\[
\chi_N(P, \gamma) :=\begin{pmatrix} -\gamma e_{(N+1, 1)} e_{(N+1, 1)}^T + e_{(N+1, \delta+1)} e_{(N+1, \delta+1)}^T \\
-I_n \otimes I_m \end{pmatrix} \otimes P;
\]
\[
\Pi_N :=\begin{pmatrix} e_{(N, 1)}^T \otimes I_n & 0_{n \times Nm} \\
I_n \otimes A & I_n \otimes B \end{pmatrix}
\]
\[
\mathcal{C}(\mathcal{F}_{\text{SOF}}^{(N, i)}) := \mathcal{F}_{\text{SOF}}^{(N, i)} (I_n \otimes C) - I_n \otimes I_m
\]
\[
\mathcal{L}_{N-1} \otimes A - \mathcal{L}_{N-1} \otimes I_n \mathcal{L}_{N-1} \otimes B
\]
**Proof.** By the Lyapunov argument, $N$-periodic SOF control system (3) is asymptotically stable if and only if there exists $P \in \mathbb{R}^{n \times n}$ such that (9) and $A_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(N, i)})^T P A_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(N, i)}) - P < 0$ hold. After some algebraic manipulations and using the relation (8), one can prove that
\[
A_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(N, i), \Sigma})^T P A_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(N, i), \Sigma}) - P
\]
\[
= \begin{bmatrix} I_n \\ A_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(1, i), \Sigma}) \end{bmatrix} \chi_N(P, 1) \begin{bmatrix} I_n \\ \vdots \\ A_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(N, i), \Sigma}) \end{bmatrix}
\]
\[
= Q_N^T \Pi_N^T \chi_N(P, 1) \Pi_N Q_N
\]
and $C(F_{\text{SOF}}^{(N, i)}) Q_N = 0_{((N-1)n+Nm) \times n}$, where

$$Q_N := \begin{bmatrix} (I_N \otimes I_n) & I_{(N-1)n} \\
F_{\text{SOF}}^{(N, i)} (I_N \otimes C) & A_{\text{LTI}} (F_{\text{SOF}}^{(1, i)}, \Sigma) \\
\vdots & \vdots \\
F_{\text{SOF}}^{(N, i)} (I_N \otimes C) & A_{\text{LTI}} (F_{\text{SOF}}^{(N-1, i)}, \Sigma) \end{bmatrix}.$$ 

Now, note that $Q_N$ has full column rank, and $\text{rank}(Q_N) = n$. Moreover, to use the Finsler’s lemma (Skelton et al., 1998), we need to show that $C(F_{\text{SOF}}^{(N, i)})$ is of full row rank. To see this, multiplying $C(F_{\text{SOF}}^{(N, i)})$ by the nonsingular matrix

$$\begin{bmatrix} \mathcal{L}_{N-1} \otimes B & I_{(N-1)n} \\
I_{Nm} & 0_{Nm \times (N-1)n} \end{bmatrix}$$

on the left yields

$$\begin{bmatrix} (\mathcal{L}_{N-1} \otimes B) F_{\text{SOF}}^{(N, i)} (I_N \otimes C) + \mathcal{L}_{N-1} \otimes A - \mathcal{C}_{N-1} \otimes I_n \\
F_{\text{SOF}}^{(N, i)} (I_N \otimes C) & -I_N \otimes I_m \end{bmatrix},$$

which clearly has full row rank. Therefore, $\text{rank}(F_{\text{SOF}}^{(N, i)}) = (N-1)n + Nm$ and $C(F_{\text{SOF}}^{(N, i)})$ has a right null-space of dimension $n$. This implies that $C(F_{\text{SOF}}^{(N, i)})^\perp = Q_N$, and it follows from (11) and

$$A_{\text{LTI}} (F_{\text{SOF}}^{(N, i)}, \Sigma)^T P A_{\text{LTI}} (F_{\text{SOF}}^{(N, i)}, \Sigma) - P < 0$$

that

$$A_{\text{LTI}} (F_{\text{SOF}}^{(N, i)}, \Sigma)^T P A_{\text{LTI}} (F_{\text{SOF}}^{(N, i)}, \Sigma) - P = C(F_{\text{SOF}}^{(N, i)})^T \Pi_N \chi_N (P, 1) \Pi_N C(F_{\text{SOF}}^{(N, i)})^\perp < 0.$$

(12)

Applying the Finsler’s lemma to (12), we have that (12) holds if and only if there exists $M$ such that (10) is satisfied. This completes the proof.

If $F_{\text{SOF}}^{(N, i)}$ should be determined by Theorem 1, due to the product of multiplier $M$ introduced by the Finsler’s lemma and controller parameter $F_{\text{SOF}}^{(N, i)}$ (10) is a BMI problem. There are several iterative algorithms to obtain a local solution to BMI problems; for instance, the alternating minimization algorithm (Goh et al., 1994) is one of the simplest methods. The BMI problem can be also solved locally by using the BMI solver, PENBMI (Kočvara, 2005). It is important to note that the quality of their solutions depends on initial parameters of non-convex variables. Therefore, a reasonable initial guess of the solution can improve the results. In this context, a very promising result was presented in Peaucelle et al. (2001), where based on the a priori selection of a suitable SF controller and using elimination lemma (Boyd et al., 1994), a necessary condition for $M$ to satisfy (10) was derived, and based on this, ILMI algorithms that alternate between the SF and SOF designs were proposed. Inspired by the idea in Peaucelle et al. (2001), we suggest an alternative BMI problem which can be viewed as an extension of those in Peaucelle et al. (2001); Mehdi et al. (2004); Arzelier et al. (2010); Agulhari et al. (2010, 2012). To this end, we need some preliminary results on the following PTVMSFC (or $N$-PTVMSFC) proposed in Ebihara et al. (2009, 2011):

$$u(k) = \sum_{i=0}^{[k]_N} F_{\text{SF}}^{(k, i)} x(k - i),$$

(13)

where $N \in \mathbb{N}_+$ is the period of the controller and $F_{\text{SF}}^{(k, i)} \in \mathbb{R}^{m \times n}$, $([k]_N, i) \in \mathbb{Z}_{[0, N-1]} \times \mathbb{Z}_{[0, N-1]}$ are the SF gains to be designed. Similarly to (3), substituting (13) into (1) leads to the $N$-periodic SF control system:

$$x(k + 1) = Ax(k) + B \sum_{i=0}^{[k]_N} F_{\text{SF}}^{(k, i)} x(k - i).$$

(14)

Following the same line as in the PTVMSOFC case, let

$$\xi(t + 1) = A_{\text{LTI}} (F_{\text{SF}}^{(N, i)}, \Sigma) \xi(t), \quad t \in \mathbb{N},$$

(15)

be the equivalent LTI representation of the $N$-periodic SF control system (14) corresponding to PTVMSFC gain matrix $F_{\text{SOF}}^{(N, i)}$. Then, by using a descriptor-like form of (1), the system-theoretic concept of duality (Ebihara et al., 2011), and the Finsler’s lemma (Skelton et al., 1998), a necessary and sufficient LMI condition to design (13) was established in Ebihara et al. (2011). For the sake of completeness, it is presented below.

**Lemma 1 (Ebihara et al. (2011))** $N$-periodic control system (14) or equivalent LTI system (15) is asymptotically stable if and only if there exists matrices $P = P^T \in \mathbb{R}^{n \times n}$, $G(i, j) \in \mathbb{R}^{n \times n}$, and $J(i, j) \in \mathbb{R}^{m \times n}$ such that the following LMI problem is satisfied:

$$\chi_N (P, 1) + \text{He} \begin{bmatrix} (L_N^T \otimes A) G(R_N \otimes I_n) \\
(\mathcal{L}_N^T \otimes B) J(R_N \otimes I_n) \\
-(R_N^T \otimes I_n) G(R_N \otimes I_n) \end{bmatrix} < 0,$$

(16)

where

$$G := \begin{bmatrix} G(1, 1) & \cdots & G(1, N) \\
0 & \ddots & \vdots \\
0 & \ddots & G(N, N) \end{bmatrix} \in \mathbb{R}^{nN \times NN},$$
Based on the definitions and using the idea that stems from Peaucelle et al. (2001), we establish the following theorem:

**Theorem 2** Suppose that \((\hat{P}, \hat{J}, \hat{G})\) is a solution to (16), and let \(\hat{F}^{(N, i)}_{\text{SF}} = (T_N \otimes I_n) \hat{J} \hat{G}^{-1} (T_N \otimes I_n)\). Then, there exist matrices \(V \in \mathbb{R}^{((N-1)n+Nm) \times ((N-1)n+Nm)}\) and \(\hat{F}^{(N, i)}_{\text{SOF}} \in \mathbb{R}^{Nm \times Np}\) defined in (6) such that BMIs (9) and (10) in Theorem 1 with \(M = \mathcal{H}(\hat{F}^{(N, i)}_{\text{SF}})TV\) have a solution if and only if

\[
P(\mathcal{A}_{\text{LTI}}(\hat{F}^{(N, i)}_{\text{SF}}, \Sigma)) \cap S^{(N)}_{\text{SOF}}(\Sigma) \neq \emptyset \tag{17}
\]

holds, where

\[
\mathcal{H}(\hat{F}^{(N, i)}_{\text{SF}}) := \begin{bmatrix}
\hat{F}^{(N, i)}_{\text{SF}} & -I_N \otimes I_m \\
-I_{N-1} \otimes A & -R_{N-1} \otimes I_n
\end{bmatrix}.
\]

**Proof.** (Sufficiency) If (17) holds, then there exists a pair \((P, F^{(N, i)}_{\text{SOF}})\) such that \(P \in \mathbb{S}^n_+\),

\[
\mathcal{A}_{\text{LTI}}(\hat{F}^{(N, i)}_{\text{SOF}}, \Sigma) \mathcal{P} \mathcal{A}_{\text{LTI}}(\hat{F}^{(N, i)}_{\text{SOF}}, \Sigma) - P < 0
\]

and \(\mathcal{A}_{\text{LTI}}(\hat{F}^{(N, i)}_{\text{SOF}}, \Sigma) \mathcal{P} \mathcal{A}_{\text{LTI}}(\hat{F}^{(N, i)}_{\text{SOF}}, \Sigma) - P < 0\) hold. Following similar lines to the proof of Theorem 1, we have that

\[
\begin{align}
\mathcal{A}_{\text{LTI}}(\hat{F}^{(N, i)}_{\text{SOF}}, \Sigma)^T P \mathcal{A}_{\text{LTI}}(\hat{F}^{(N, i)}_{\text{SOF}}, \Sigma) - P &< 0, \\
\mathcal{A}_{\text{LTI}}(\hat{F}^{(N, i)}_{\text{SOF}}, \Sigma)^T P \mathcal{A}_{\text{LTI}}(\hat{F}^{(N, i)}_{\text{SOF}}, \Sigma) - P &< 0,
\end{align}
\]

where

\[
\mathcal{H}(\hat{F}^{(N, i)}_{\text{SF}}) := \begin{bmatrix}
\hat{F}^{(N, i)}_{\text{SF}} & -I_N \otimes I_m \\
-I_{N-1} \otimes A & -R_{N-1} \otimes I_n
\end{bmatrix}.
\]

Then, relying on the elimination lemma (Boyd et al., 1994), we prove that both (18) and (19) are satisfied if and only if there exists \(V\) such that (10) holds with \(M = \mathcal{H}(\hat{F}^{(N, i)}_{\text{SF}})TV\). This proves the sufficiency.

(Necessity) Assume that BMIs (9) and (10) with \(M = \mathcal{H}(\hat{F}^{(N, i)}_{\text{SF}})TV\) admit a solution \((P, F^{(N, i)}_{\text{SOF}})\). By means of the elimination lemma, we have that (18) and (19)
hold. This implies $P \in \mathcal{P}(\mathcal{A}_{\text{LTI}}(\mathcal{F}_{\text{SF}}^{(N,1)}, \Sigma))$ and $P \in \mathcal{S}_{\text{SOF}}^{(N)}(\Sigma)$, so (17) is satisfied. This completes the proof. ■

Theorem 2 tells us that if $(\hat{P}, \hat{\mathcal{J}}, \hat{\mathcal{G}})$ is a solution to (16), then $M = \mathcal{H}(\mathcal{F}_{\text{SF}}^{(N,1)})^T \mathcal{V}$ with appropriately selected $\mathcal{V}$ can be a reasonable choice of $M$ so that (9) and (10) in Theorem 1 become feasible. The following corollary can be immediately obtained from Theorems 1 and 2:

**Corollary 1** Suppose that

a) $(\hat{P}, \hat{\mathcal{J}}, \hat{\mathcal{G}})$ is a solution to (16), and $\mathcal{F}_{\text{SF}}^{(N,1)} = (T_N \otimes \mathbb{I}_m)\hat{\mathcal{J}}\hat{\mathcal{G}}^{-1}(T_N \otimes \mathbb{I}_n)$;

b) (17) is satisfied.

Then, there exists $\mathcal{F}_{\text{SOF}}^{(N,1)} \in \mathbb{R}^{Nm \times Np}$ defined in (6) such that $N$-periodic control system (3) is asymptotically stable if and only if there exist matrices $P = PT^T \in \mathbb{R}^{n \times n}$ and $\mathcal{V} \in \mathbb{R}^{((N-1)n+nM)((N-1)n+nM)}$ such that the following problem is satisfied with $\mathcal{F}_{\text{SOF}}^{(N,1)} \in \mathbb{R}^{Nm \times Np}:

\[
P > 0,
\]

\[
\Pi_N^T \mathcal{X}_N(P, 1)\Pi_N + \text{He}(\mathcal{H}(\mathcal{F}_{\text{SF}}^{(N,1)})^T \mathcal{V} \mathcal{C}(\mathcal{F}_{\text{SOF}}^{(N,1)})) < 0.
\]

\[
\text{Proof.} \quad \text{The sufficiency follows immediately from Theorem 1. To prove the necessity, suppose that there exists } \mathcal{F}_{\text{SOF}}^{(N,1)} \text{ defined in (6) such that } N\text{-periodic control system (3) is asymptotically stable. Since (17) is satisfied by assumption, one can select } \mathcal{F}_{\text{SOF}}^{(N,1)} \text{ so that } \mathcal{P}(\mathcal{A}_{\text{LTI}}(\mathcal{F}_{\text{SF}}^{(N,1)}, \Sigma)) \cap \mathcal{P}(\mathcal{A}_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(N,1)}, \Sigma)) \neq \emptyset. \text{ The rest of the proof then follows the same line as in the sufficient part of the proof of Theorem 2. ■}
\]

**Remark 2** In the case $N = 1$, (20) and (21) reduce to Theorem 1 in Peaucelle et al. (2001) and Arzelier et al. (2010).

It should be kept in mind that there is no guarantee that $\mathcal{F}_{\text{SF}}^{(N,1)} \in \mathcal{L}_{\text{SF}}^{(N)}$ obtained by solving LMI (16) satisfies condition (17). In addition, since (17) is used only in the necessity part of the proof of Corollary 1, in a practical implementation, Corollary 1 should be regarded as only a sufficient condition. Note also that the condition of Corollary 1 is still a BMI problem. However, as in Peaucelle et al. (2001), one can expect that solving the BMI of Corollary 1 gives better results than solving the BMI of Theorem 1, since an initial guess of $M$ in Theorem 1 is used in Corollary 1. Unfortunately, if $\mathcal{F}_{\text{SF}}^{(N,1)} \in \mathcal{L}_{\text{SF}}^{(N)}$ does not satisfy (17), the BMI of Corollary 1 has no solution even when the solution set of the original BMI of Theorem 1 is nonempty. In this respect, it can be said that another source of conservatism is introduced in the BMI of Corollary 1. Conceptually, we conjecture that this conservatism can be reduced by increasing $N$. To give an intuitive perspective on how increasing $N$ reduces this kind of conservatism, let us introduce the following lemmas:

**Lemma 2** Assume $\mathcal{S}_{\text{SF}}^{(1)}(\Sigma) \neq \emptyset$. The following statements are true:

a) $\mathcal{S}_{\text{SF}}^{(1)}(\Sigma) \subseteq \mathcal{S}_{\text{SF}}^{(N)}(\Sigma), \forall N \in \mathbb{N}_+$.

b) $\lim_{N \to \infty} \mathcal{S}_{\text{SF}}^{(N)}(\Sigma) = \mathcal{S}_{\text{SF}}^{(1)}$.

**Proof.** a) For any $P \in \mathcal{S}_{\text{SF}}^{(1)}(\Sigma)$, assume that $F = \mathcal{F}_{\text{SF}}^{(1,1)}$ satisfies $\mathcal{A}_{\text{LTI}}(\mathcal{F}_{\text{SF}}^{(1,1)}, \Sigma)^T P \mathcal{A}_{\text{LTI}}(\mathcal{F}_{\text{SF}}^{(1,1)}, \Sigma) - P = (A + BF)^T P (A + BF) - P < 0$. Then, $P \in \mathcal{S}_{\text{SF}}^{(N)}(\Sigma)$ because $(A + BF)^N P (A + BF)^N - P = \mathcal{A}_{\text{LTI}}(\mathcal{F}_{\text{SF}}^{(1,1)}, \Sigma)^N P \mathcal{A}_{\text{LTI}}(\mathcal{F}_{\text{SF}}^{(1,1)}, \Sigma) - P < 0$ holds with $\mathcal{F}_{\text{SF}}^{(N,1)} = I_N \otimes F$. This implies a) is true.

b) For any $\mathcal{F}_{\text{SF}}^{(1,1)} \in \mathcal{F}_{\text{SF}}^{(1,1)}$ and $P \in \mathcal{S}_{\text{SF}}^{(1)}$, it holds that $\lim_{N \to \infty} (\mathcal{A}_{\text{LTI}}(\mathcal{F}_{\text{SF}}^{(1,1)}, \Sigma)^N P \mathcal{A}_{\text{LTI}}(\mathcal{F}_{\text{SF}}^{(1,1)}, \Sigma) - P) = -P < 0$. Since $\mathcal{A}_{\text{LTI}}(\mathcal{F}_{\text{SF}}^{(1,1)}, \Sigma)^N = \mathcal{A}_{\text{LTI}}(\mathcal{F}_{\text{SF}}^{(N,1)}, \Sigma)$ with $\mathcal{F}_{\text{SF}}^{(N,1)} = I_N \otimes \mathcal{F}_{\text{SF}}^{(1,1)}$ , we have

\[
\lim_{N \to \infty} (\mathcal{A}_{\text{LTI}}(\mathcal{F}_{\text{SF}}^{(1,1)}, \Sigma)^T P \mathcal{A}_{\text{LTI}}(\mathcal{F}_{\text{SF}}^{(N,1)}, \Sigma) - P) < 0
\]

for any $P \in \mathcal{S}_{\text{SF}}^{(1)}$. This implies b) is satisfied, and the proof is completed. ■

**Lemma 3** Assume $\mathcal{S}_{\text{SF}}^{(1)}(\Sigma) \neq \emptyset$. The following statements are true:

a) $\mathcal{S}_{\text{SO}}^{(1)}(\Sigma) \subseteq \mathcal{S}_{\text{SO}}^{(N)}(\Sigma), \forall N \in \mathbb{N}_+$.

b) $\lim_{N \to \infty} \mathcal{S}_{\text{SO}}^{(N)}(\Sigma) = \mathcal{S}_{\text{SO}}^{(1)}$.

c) $\mathcal{S}_{\text{SOF}}^{(1)}(\Sigma) \subseteq \mathcal{S}_{\text{SOF}}^{(N)}(\Sigma), \forall N \in \mathbb{N}_+$.

**Proof.** Proofs for statements a) and b) follow immediately from those of Lemma 2. For statement c), assume that $P \in \mathcal{S}_{\text{SOF}}^{(N)}(\Sigma)$, which means there exists $\mathcal{F}_{\text{SOF}}^{(N,1)} \in \mathcal{L}_{\text{SOF}}^{(N)}$ satisfying $\mathcal{A}_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(N,1)}, \Sigma)^T P \mathcal{A}_{\text{LTI}}(\mathcal{F}_{\text{SOF}}^{(N,1)}, \Sigma) - P < 0$. Then, $P \in \mathcal{S}_{\text{SOF}}^{(N)}(\Sigma)$ because

\[
\mathcal{A}_{\text{LTI}}(\mathcal{F}_{\text{SF}}^{(N,1)}, \Sigma)^T P \mathcal{A}_{\text{LTI}}(\mathcal{F}_{\text{SF}}^{(N,1)}, \Sigma) - P < 0
\]

holds with $\mathcal{F}_{\text{SF}}^{(N,1)} = \mathcal{F}_{\text{SOF}}^{(N,1)}(I_N \otimes C)$. This completes the proof. ■

Again, recall that $(\hat{P}, \hat{\mathcal{J}}, \hat{\mathcal{G}})$ is a solution to (16), and $\mathcal{F}_{\text{SF}}^{(N,1)} = (T_N \otimes I_m)\hat{\mathcal{J}}\hat{\mathcal{G}}^{-1}(T_N \otimes I_n) \in \mathcal{L}_{\text{SF}}^{(N)}$. Let us assume $\mathcal{S}_{\text{SOF}}^{(1)}(\Sigma) \neq \emptyset$, Then, in view of Lemmas 2
and 3, it is true that \( S_{SOF}^{(N)}(\Sigma) \subseteq S_{SF}^{(N)}(\Sigma) \), \( \forall N \in \mathbb{N}_+ \),
\( \lim_{N \to \infty} S_{SF}^{(N)}(\Sigma) \cap S_{SOF}^{(N)}(\Sigma) = S_{SF}^{(N)}(\Sigma) \), and \( \lim_{N \to \infty} \{ P \in \mathbb{R}^n : P \in S_{SF}^{(N)}(\Sigma), P \notin S_{SOF}^{(N)}(\Sigma) \} = \emptyset \). In addition, let us suppose that \( \hat{P}^{-1} \in \mathcal{P} \) is a random matrix within \( S_{SF}^{(N)}(\Sigma) \). Then, we can expect that as \( N \) gets larger, set \( \{ P \in \mathbb{R}^n : P \in S_{SF}^{(N)}(\Sigma), P \notin S_{SF}^{(N)}(\Sigma) \} \) tends to shrink and eventually become the empty set as \( N \to \infty \). Thus, it is more likely that \( \hat{P}^{-1} \in \mathcal{P} \) lies within \( S_{SF}^{(N)}(\Sigma) \) as \( N \to \infty \). In other words, as \( N \) increases, there is a more possibility that (17) holds, and thus, the solution set of the BMI problem of Corollary 1 is nonempty.

To determine \( F_{SOF}^{(N,1)} \), the problem of Corollary 1 is still a BMI problem (not an LMI in \( \mathcal{V} \) and \( F_{SOF}^{(N,1)} \)). Local solutions to the BMI problem of Corollary 1 can be obtained by using PENBMI (Kočvara, 2005). Alternatively, with a suitable choice of particular \( \mathcal{V} \), the BMI can reduce to a convex LMI at the price of some conservatism. For instance, letting
\[
\mathcal{V} = \begin{bmatrix} \mathcal{V}_{11} & \mathcal{V}_{12} \\ 0 & \mathcal{V}_{22} \end{bmatrix},
\]
where
\[
\mathcal{V}_{11} := \begin{bmatrix} V_{11}^{(1,1)} & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ V_{11}^{(N,1)} & \ldots & V_{11}^{(N,N)} \end{bmatrix} \in \mathbb{R}^{Nm \times Nm},
\]
\( V_{11}^{(i,j)} \in \mathbb{R}^{m \times m}, \mathcal{V}_{12} \in \mathbb{R}^{Nm \times (N-1)n}, \) and \( \mathcal{V}_{22} \in \mathbb{R}^{(N-1)n \times (N-1)n} \), the following result is obtained:

\[
\mathcal{V}_{11} \text{ is defined in (23),}
\]
\[
\mathcal{M} := \begin{bmatrix} M^{(1,1)} & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & M^{(N,N)} \end{bmatrix} \in \mathbb{R}^{Nm \times Np},
\]
\( \mathcal{D}(\mathcal{M}, \mathcal{V}_{11}, \mathcal{V}_{12}, \mathcal{V}_{22}) \) is defined in (28) at the top of the next page. Moreover, an admissible \( N \)-PTVMSOFC gain matrix is given by \( \hat{F}_{SOF}^{(N,1)} = \mathcal{V}_{11}^{-1} \mathcal{M} \).

**Proof.** Noting that (26) ensures the invertibility of \( \mathcal{V}_{11} \), substituting (22) into (21), and using the change of variables \( \mathcal{M} = \mathcal{V}_{11}^{(N,1)} \), we have that (25) is equivalent to (21).

**Remark 3** LMI (26) guarantees that \( \mathcal{V}_{11} \) is nonsingular. If it is eliminated, then LMIIs (24) and (25) can yield less conservative results although the invertibility of \( \mathcal{V}_{11} \) is not guaranteed. Therefore, instead of (24)-(26), we can use only (24) and (25), and when they are feasible, the invertibility of \( \mathcal{V}_{11} \) should be checked to ultimately determine the feasibility of the control design problem.

Based on Corollary 2, the two-steps algorithms suggested in Mehdi et al. (2004); Agulhari et al. (2010, 2012) can be adopted to design the \( N \)-PTVMSOFC.

**Algorithm 2. Two-Steps LMI Algorithm.**

Step 1. Solve LMI (16) for \( (P, \mathcal{G}, \mathcal{J}) \) and let \( \hat{F}_{SF}^{(N,1)} = (\mathcal{T}_N \otimes \mathcal{I}_m)\hat{\mathcal{J}}\hat{\mathcal{G}}^{-1}(\mathcal{T}_N \otimes \mathcal{I}_n) \) with \( (\hat{P}, \hat{\mathcal{G}}, \hat{\mathcal{J}}) \in \{P, \mathcal{G}, \mathcal{J} : \text{LMI (16)}\} \).

Step 2. With \( \hat{F}_{SF}^{(N,1)} \) obtained from the previous step, solve for \( \Lambda := (P, \mathcal{M}, \mathcal{V}_{11}, \mathcal{V}_{12}, \mathcal{V}_{22}) \) LMIIs (24)-(26) with \( \gamma = 1 \):

\[
(\hat{P}, \mathcal{M}, \mathcal{V}_{11}, \mathcal{V}_{12}, \mathcal{V}_{22})
\]
\( \in \{ \Lambda : \text{LMIIs (24) - (26), } \gamma = 1 \} \).

If feasible, then \( \hat{F}_{SOF}^{(N,1)} = \hat{\mathcal{V}}_{11}^{-1} \hat{\mathcal{M}} \) is a stabilizing \( N \)-PTVMSOFC gain matrix.

Moreover, versions of the ILMI algorithm that alternates between the SF and the SOF designs developed in Peaucelle et al. (2001) can be also applied as less conservative alternatives.

**Algorithm 3. ILMI Algorithm.**

Step 1. (Initialization). Set \( i = 1 \), the maximum number of iterations \( N_{iter} \in \mathbb{N}_+ \), and a sufficiently small positive real number \( \delta \). Solve LMI (16) for \( (P, \mathcal{G}, \mathcal{J}) \) and let \( \mathcal{F}_{SF}^{(N,1)} = (\mathcal{T}_N \otimes \mathcal{I}_m)\hat{\mathcal{J}}\hat{\mathcal{G}}^{-1}(\mathcal{T}_N \otimes \mathcal{I}_n) \) with \( (\hat{P}, \hat{\mathcal{G}}, \hat{\mathcal{J}}) \in \{P, \mathcal{G}, \mathcal{J} : \text{LMI (16)}\} \).
Step 2. With $\mathcal{F}_{\text{SF}}^{(N, i)}$ obtained from the previous step, solve for $\Lambda := (\gamma_1, P, M, V_{11}, V_{12}, V_{22})$ the following optimization problem:

$$
(\gamma_i, P, M, V_{11}, V_{12}, V_{22}) := \arg\min_{\Lambda} (\gamma \in \mathbb{R} : \text{LMIs (24) - (26)}).
$$

(29)

Step 3. If $\gamma_i \leq 1$, then $\mathcal{F}_{\text{SOF}}^{(N, i)} = V_{11}^{-1} \hat{M}$ is a stabilizing N-PTVMSOFC gain matrix. STOP. Otherwise, if $i \geq 2$ and $|\gamma_{i-1} - \gamma_i| \leq \delta$ or $\gamma_i < \gamma_i$ or $i = N_{\text{iter}}$, then this algorithm cannot get a feasible solution. STOP.

Step 4. With $(\gamma_i, P, M, V_{11}, V_{12}, V_{22})$ obtained from the previous step, solve the LMI problem

$$
(P, \mathcal{F}_{\text{SF}}^{(N, i)})
$$

$$
\in \left\{ (P, \mathcal{F}_{\text{SF}}^{(N, i)}) : P \succ 0, \Pi_{N}^{T}X_{N}(P, \gamma_i)\Pi_{N} \right\} + \text{He}(\mathcal{H}(\mathcal{F}_{\text{SF}}^{(N, i)})^{T} D(M, V_{11}, V_{12}, V_{22})) < 0,
$$

set $i = i + 1$, and go to Step 2.

Remark 4 The optimization problem (29) is a unidimensional minimization subject to LMI constraints, and for fixed $\gamma$, conditions (20) and (21) are LMIs tractable via LMI solvers (Gahinet et al., 1995; Löfberg, 2004; Strum, 1999). Thus, the optimization problem can be solved by means of a sequence of LMI problems, i.e., a line search or a bisection process over $\gamma$. Moreover, the optimization problem belongs to the class of eigenvalue problems, which are convex optimizations (Boyd et al., 1994), and hence, can be directly treated with the aid of the LMI solver (Gahinet et al., 1995).

Remark 5 It is not difficult to show that, at least theoretically, if the LMI problem at Step 2 and $i = 1$ is feasible, then all the subsequent LMIs are also feasible for all $i > 1$, and $\{\gamma_1, \gamma_2, \ldots\}$ is a converging and non-increasing sequence. However, in practice, the LMIs after Step 2 can fail to find a feasible solution or $\gamma_i$ can increase and fluctuate irregularly in many cases. This phenomenon may be common to many other ILMI schemes and may be due to the fact that as solution spaces of the LMIs become narrower, the feasibility of the LMIs tends to be more sensitive to small numerical errors of the solutions computed at the previous steps. In this case, the algorithm can be deemed not to be able to get a solution.

Example 1 For a statistical comparison analysis of the proposed results with existing ones, we randomly generated thousand systems with $(n, m, p) = (3, 1, 1)$ whose open-loop systems were unstable. Each system was computed using the following procedure: 1) triplet $(A, B, C)$ is generated with matrices whose entries are real numbers uniformly distributed in the interval $[-2, 2]$; 2) $A$ is replaced with $(1.2/\rho(A))A$ so that the spectral radius of $A$ becomes $1.2$; 3) if $(A, B)$ is stabilizable and $(C, A)$ is detectable, then add the triplet to the list of test systems. Else, discard it and go to step 1). Since the PTVMSOFC can be interpreted as a sort of dynamic output feedback (DOF) controller, the proposed approaches are also compared with the full-order DOF design (Iwasaki et al., 1994; Scherer et al., 1997). The number of stabilizable systems, denoted by $N_{\text{stable}}$, in the context of feasibility of several approaches are listed in Table 5 with the average computational time (in seconds) spent by each test, where for Algorithm 3, we set $(N_{\text{iter}}, \delta) = (10, 10^{-4})$, and for optimization (29), a bisection algorithm over $\gamma$ was used. In addition, for PENBMI, we used the BMI condition

$$
\begin{bmatrix}
-P \\
(P(A + BFC)^{T}P) -P
\end{bmatrix} < 0,
$$

From Table 5, the following observation can be made:

a) The results show that at the price of a higher computational cost, the proposed method offers improvement over the previous approaches except for the full-order DOF design. The number of parameters of the controller is $mnN(N + 1)/2$ for the PTVMSOFC while $n^{2} + np + mn + mp$ for the full-order DOF controller.

In order to compare and evaluate the on-line computational burden, we will check the number of operations including multiplication and addition. The total multiplication and addition during period $k \in \{0, 1, \ldots, N - 1\}$ are summarized in Table 1 for the full-order DOF and Table 2 for the PTVMSOFC.

It might not be an easy task to perform the quali-
Example 2 In this example, we consider the discrete-time two-mass-spring system from Kothare et al. (1996)
Example 1. Number of stabilizable systems, $N_{\text{stable}}$, and the average computational time.

| Methods                                                                 | $N_{\text{stable}}$ | Time (s) |
|------------------------------------------------------------------------|----------------------|----------|
| Cone complementarity linearization algorithm in El Ghaoui et al. (1997) | 509                  | 34.35    |
| Discrete $P$-problem in Crusius et al. (1999)                          | 248                  | 0.10     |
| Discrete $W$-problem in Crusius et al. (1999)                          | 243                  | 0.10     |
| Algorithm A in Rosinová et al. (2003) with $(R, Q) = (0.01I_m, I_n)$   | 306                  | 0.04     |
| Two-steps LMI approach of Theorem 3.1 in Mehdi et al. (2004) with constraint $-G - G^T \prec 0$ | 427                  | 0.18     |
| Two-steps LMI approach of Theorem 3.1 in Mehdi et al. (2004) with $F_1 = F_2 = 0$ | 427                  | 0.18     |
| Lemma 3 in Dong et al. (2007) with $T = [C^T (C C^T)^{-1} C]_n$ (Method in de Oliveira et al. (2002)) | 287                  | 0.10     |
| Theorems 3.1 and 3.3 in Bara et al. (2005) with $T = [C^T (C C^T)^{-1} C]_n$ | 222                  | 0.14     |
| Algorithm 1 in Shu et al. (2010)                                       | 309                  | 9.90     |
| PENBMI (Kočvara, 2005)                                                | 484                  | 0.08     |
| Full-order DOF design (discrete-time version of Scherer et al. (1997)) | 1000                 | 0.10     |
| Theorem 1 solved with PENBMI for $N = 1$                                | 355                  | 0.09     |
| Theorem 1 solved with PENBMI for $N = 2$                                | 495                  | 0.25     |
| Theorem 1 solved with PENBMI for $N = 3$                                | 508                  | 1.05     |
| Corollary 1 solved with PENBMI for $N = 1$                             | 425                  | 0.18     |
| Corollary 1 solved with PENBMI for $N = 2$                             | 791                  | 0.23     |
| Corollary 1 solved with PENBMI for $N = 3$                             | 925                  | 0.49     |
| Algorithm 2 with $N = 1$                                               | 427                  | 0.18     |
| Algorithm 2 with $N = 2$                                               | 642                  | 0.21     |
| Algorithm 2 with $N = 3$                                               | 842                  | 0.3      |
| Algorithm 2 with $N = 1$ and without constraint (26)                   | 427                  | 0.17     |
| Algorithm 2 with $N = 2$ and without constraint (26)                   | 673                  | 0.20     |
| Algorithm 2 with $N = 3$ and without constraint (26)                   | 998                  | 0.28     |
| Algorithm 3 with $N = 1$                                               | 513                  | 5.46     |
| Algorithm 3 with $N = 2$                                               | 822                  | 8.24     |
| Algorithm 3 with $N = 3$                                               | 959                  | 12.47    |
| Algorithm 3 with $N = 1$ and without constraint (26)                   | 513                  | 5.21     |
| Algorithm 3 with $N = 2$ and without constraint (26)                   | 825                  | 9.86     |
| Algorithm 3 with $N = 3$ and without constraint (26)                   | 999                  | 6.72     |

Based on Corollary 2, we can readily arrive at the following result, which is presented without the proof:

**Corollary 3** Suppose that $(\hat{P}, \hat{J}, \hat{G})$ is a solution to (16), and let $\hat{F}^{(N, i)}_{SF} = (\hat{T}_N \otimes I_m) \hat{J} \hat{G}^{-1}(\hat{T}_N \otimes I_n)$. Then, system (1) is stabilizable via $N$-PTVMSOFC (2) and $\rho(\mathcal{A}_{\text{LTI}}(\hat{F}^{(i, i)}_{\text{SOF}}, \Sigma)) < \beta$, $\beta \in \mathbb{Z}[1, N-1]$ are guaranteed if there exists matrices $P = P^T \in \mathbb{R}^{n \times n}$, $S = S^T \in \mathbb{R}^{n \times n}$, $M^{(i, j)} \in \mathbb{R}^{m \times p}$, $V^{(i, j)}_{11} \in \mathbb{R}^{m \times m}$, $V^{(i, j)}_{12} \in \mathbb{R}^{Nm \times (N-1)n}$, and $V^{(i, j)}_{22} \in \mathbb{R}^{(N-1)n \times (N-1)n}$ such that (24), (25), (26) with $\gamma = 1$, and the following LMI problem is satisfied:

$$S > 0,$$

$$\Pi_i^X \mathcal{A}_i(S, \beta) \Pi_X + \mathcal{H}(\hat{F}^{(N, i)}_{SF})^T \mathcal{D}(\mathcal{M}, V^{(i, j)}_{11}, V^{(i, j)}_{12}, V^{(i, j)}_{22}) > 0, i \in \mathbb{Z}[1, N-1],$$

where $V^{(i, j)}_{11} \in \mathbb{R}^{Nm \times Nm}$ and $\mathcal{M} \in \mathbb{R}^{Nm \times Np}$ are defined in (23) and (27), respectively. Moreover, an admissible PTVMSOFC gain matrix is given by $\hat{F}^{(N, i)}_{\text{SOF}} = V^{-1}_{11} \mathcal{M}$. 

might not be so good since the asymptotic stability is guaranteed only for the states $x(k), \forall k \in \{k \in \mathbb{N} : [k]_N = 0\}$. This property can cause chattering problems as we can see from Fig. 1. In order to alleviate the problem, we propose a simple procedure which may be helpful to some degree in reducing the chattering effect. Specifically, once a solution to the LMI problem, we propose a simple procedure which may be helpful to some degree in reducing the chattering effect.

Specifically, once a solution to the problem, we propose a simple procedure which may be helpful to some degree in reducing the chattering effect.
Example 3  Let us consider Example 2 again. For \( \tilde{F}^{(2, 1)}_{\text{SOF}} \) given in Example 2, we applied an ILMI algorithm to reduce \( \beta \) and obtained gain matrix

\[
\tilde{F}^{(2, 1)}_{\text{SOF}} = \begin{bmatrix}
0.0018 & 0 \\
365.0515 & -428.3888
\end{bmatrix}
\]

with \( \rho(\mathcal{A}_{\text{LTI}}(\tilde{F}^{(2, 1)}_{\text{SOF}}, \Sigma)) = 0.9535 \). The eigenvalues of \( \mathcal{A}_{\text{LTI}}(\tilde{F}^{(2, 1)}_{\text{SOF}}, \Sigma) \) were \( (0.5094 \pm 0.3349i, 0.9501 \pm 0.806i) \) and \( \rho(\mathcal{A}_{\text{LTI}}(\tilde{F}^{(1, 1)}_{\text{SOF}}, \Sigma)) = 1.0025 \), while in Example 2, \( \rho(\mathcal{A}_{\text{LTI}}(\tilde{F}^{(1, 1)}_{\text{SOF}}, \Sigma)) = 1.2141 \). The simulation result under the same initial condition is plotted in Fig. 2, which clearly shows that the amplitude of oscillation was mitigated in comparison with that of Fig. 1.

Remark 6  An extension of our methods to the LQR formulation is straightforward. Let us consider the following cost function:

\[
J_\infty(x, u) := \sum_{k \in \mathbb{N} : [k]_N = 0} \begin{bmatrix} x(k : k + N - 1) \\ u(k : k + N - 1) \end{bmatrix}^T W \begin{bmatrix} x(k : k + N - 1) \\ u(k : k + N - 1) \end{bmatrix},
\]

where \( W := \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \geq 0 \) is a given weighting matrix.

Then, with only a little modification, it is easy to see that if the LMIs of Corollary 2 with (25) replaced by

\[
\Pi_N^T \mathcal{X}_N(P, 1) \Pi_N + W + \text{He} \{ \mathcal{H}(\tilde{F}^{(N, 1)}_{\text{SOF}})^T D(\mathcal{M}, \mathcal{V}_{11}, \mathcal{V}_{12}, \mathcal{V}_{22}) \} < 0,
\]

is satisfied, then \( N \)-periodic control system (3) with \( \tilde{F}^{(N, 1)}_{\text{SOF}} = \mathcal{V}_{11}^{-1} M \) is asymptotically stable, and the cost function satisfies the bound \( J_\infty(x, u) < x(0)^T P x(0) \).

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