1 Introduction

B. Riemann [19] has constructed a one parameter family of non congruent singly periodic minimal surfaces which are foliated by circles (or straight lines). Each member of this family is a periodic embedded minimal surface in $\mathbb{R}^3$ with infinitely many parallel ends.

Even though the classification of genus zero, embedded minimal surfaces is not complete, W. H. Meeks J. Perez and A. Ros [14], [15], [16] have made progress concerning the question of the uniqueness of the Riemann examples in the class of genus zero embedded minimal surfaces which have an infinite number of ends. They conjecture in [15] that every embedded minimal surface of finite genus and with infinite number of ends is asymptotic (away from a compact piece) to some ”middle” planar end and to two halves of Riemann example which are referred to as ”limit ends”.

In this paper we construct such surfaces. More precisely, we have the :

**Theorem 1.1.** Given $k = 1, \ldots, 37$, there exists a one parameter family of properly embedded minimal surfaces of genus $k$ with two limit ends asymptotic to half Riemann surfaces.

We briefly explain the idea behind the proof, this will give further information about the surfaces constructed. In 1981, C. Costa [2], [3] found a genus one minimal properly embedded surface, with three ends, two of which are asymptotic to catenoidal ends and the third one being asymptotic to a plane. Later, D. Hoffman and W. H. Meeks [7], [8]
have found for every genus $k \geq 2$ a minimal surface with finite topology, two catenoidal ends and one planar end.

Minimal surfaces belonging to Riemann’s family, once they are normalized so that their planar ends are horizontal and at distance 1 one from each other, depend on a parameter (basically the value of the horizontal flux). As this parameter tends to 0, the members of this family can be understood as infinitely many horizontal planes linked by slightly bent catenoid.

The main idea behind our construction is to replace one of these "slightly bent" catenoid by one minimal surface which belongs to the Costa-Hoffman-Meeks family of minimal surfaces. Our main result says that this can be construction is successful provided one can bend the upper and lower end of the genus $k$ Costa-Hoffman-Meeks surface. Thanks to the moduli space theory for minimal surfaces with catenoidal ends and a non-degeneracy result by S. Nayatani [17], we are able to show that the bending of the ends of the genus $k$ Costa-Hoffman-Meeks surface is possible for $1 \leq k \leq 37$.

The paper is organized as follows: In Section 2, we give a description of the Costa-Hoffman-Meeks minimal surfaces and we proceed with the deformation of the top and bottom ends of such surfaces. In Section 3, we describe an isothermal parametrization of Riemann surface, we also obtain some important expansions of pieces of Riemann’s surfaces as the flux becomes vertical. Section 4 is devoted to the study of the mapping properties of the Jacobi operator about a half Riemann surface as the flux becomes vertical. In Section 5, we apply the implicit function theorem to perturb a half Riemann surface, we obtain an infinite dimensional family of minimal surfaces which are asymptotic to a half Riemann surface. In Section 6, we perturb the Costa-Hoffman-Meeks surface using again the implicit function theorem, we again obtain an infinite dimensional family of minimal surfaces which have two boundaries and one horizontal end. In the last section, we explain how the boundary data of the minimal surfaces constructed in Section 4 and Section 5 are chosen so that the union of theses forms a smooth minimal surface with fixed genus and two limit ends.

2 The Costa-Hoffman-Meeks’ family of minimal surfaces

C. Costa [2], [3] and later on D. Hoffman and W. H. Meeks [7], [8] have described, for $k \geq 1$, a properly embedded minimal surface of genus $k$ with three ends. More precisely, for each $k \geq 1$, there exists $M_k$ a complete properly embedded minimal surface of genus $k$ and three ends which, after suitable rotation and translation, enjoys the following properties:
(i) The surface $M_k$ has one planar end $E_m$ asymptotic to the $x_3 = 0$ plane, one top end $E_t$ asymptotic to the upper end of a catenoid with $x_3$-axis of revolution and one bottom end $E_b$ asymptotic to the lower end of a catenoid with $x_3$-axis of revolution. The planar end $E_m$ is located in between the two catenoidal ends.

(ii) The surface $M_k$ is invariant under the action of the rotation of angle $\frac{2\pi}{k+1}$ about the $x_3$-axis, it is also invariant under the action of the symmetry with respect to the $x_2 = 0$ plane. Finally, it is invariant under the action of the composition of a rotation of angle $\frac{\pi}{k+1}$ about the $x_3$-axis and the symmetry with respect to the $x_3 = 0$ plane.

(iii) The surface $M_k$ intersects the $x_3 = 0$ plane in $k+1$ straight lines, which intersect at equal angles $\frac{\pi}{k+1}$ at the origin. The intersection of $M_k$ with the plane $x_3 = \text{cte} \ (\neq 0)$ is a single Jordan curve. The intersection of $M_k$ with the upper half space $x_3 > 0$ (resp. with the lower half space $x_3 < 0$) is topologically an open annulus.

The surface $M_k$ will be referred to as the "genus $k$ Costa-Hoffman-Meeks surface". Observe that, when $k$ is even the surface $M_k$ is also invariant under the action of the rotation of angle $\pi$ about the $x_2$-axis.

The main purpose of this section is to explain how the genus $k$ Costa-Hoffman-Meeks surface $M_k$ can be deformed into a smooth one parameter family of minimal surfaces $M_k(\xi)$, for $\xi \in (-\xi_0, \xi_0)$ and $\xi_0 > 0$ small enough, which are not embedded anymore, are invariant under the action of the symmetry with respect to the $x_2 = 0$ plane, have one horizontal end asymptotic to the $x_3 = 0$ plane and have two catenoidal type ends which are (up to some translations) respectively asymptotic to the upper end and the lower end of a catenoid whose axis of revolution is directed by $\sin \xi e_1 + \cos \xi e_3$. The construction of $M_k(\xi)$ will be a simple consequence of the moduli space theory as described in [13], [11] or [9]. It also relies on a nondegeneracy assumption which is known to be true when $k \leq 37$, thanks to result of S. Nayatani [17].

Given $k \geq 1$, we start with a local description of the surface $M_k$ near its ends and in particular we describe coordinates which will be used to define some weighted spaces of functions on $M_k$. The planar end $E_m$ of the surface $M_k$ can be parameterized by

$$X_m(x) := \left( \frac{x}{|x|^2}, u_m(x) \right) \in \mathbb{R}^3$$
where \( x \in B_{r_0}(0) - \{0\} \subset \mathbb{R}^2 \) and where the function \( u_m \) tends to 0 as \( x \) tends to 0. This reflects the fact that the middle end of \( M_k \) is asymptotic to the horizontal plane. Here \( r_0 > 0 \) is fixed large enough.

Recall that, for surfaces parameterized by

\[
\begin{align*}
  x &\rightarrow \left( \frac{x}{|x|^2}, u(x) \right) \in \mathbb{R}^3
\end{align*}
\]

the minimal surface equation reads

\[
|x|^4 \text{div} \left( \frac{\nabla u}{(1 + |x|^4 |\nabla u|^2)^{1/2}} \right) = 0. \tag{1}
\]

The function \( u_m \) is (by definition) a solution of this equation and it turns out that \( u_m \), which is \textit{a priori} only defined in \( B_{r_0}(0) - \{0\} \), can be extended smoothly to \( B_{r_0} \). We shall make use of this fact, which follows from elliptic regularity theory, without further comment. Observe that \( u_m(x) = \mathcal{O}(|x|) \) near 0, however, given the symmetry with respect to the rotation of vertical axis and angle \( \frac{2\pi}{k+1} \), one checks that \( u_m(x) = \mathcal{O}(|x|^{k+1}) \) near 0. Indeed, since \( u_m \) solves (1), the leading term in the expansion of \( u_m \) in powers of \( |x| \) is necessarily a harmonic function which is invariant under the action of a rotation of angle \( \frac{2\pi}{k+1} \); hence, in polar coordinates, it is a linear combination of the functions \( (r, \theta) \rightarrow r^{k+1} e^{\pm i(k+1)\theta} \).

We now turn to the description of the top end of \( M_k \) (the description of the bottom end will follow at once using the invariance of the surface \( M_k \) by the symmetries which are described in (ii)). As already mentioned, the top end is asymptotic to a catenoid with vertical axis of revolution. We use

\[
X_c(s, \theta) := (\cosh s \cos \theta, \cosh s \sin \theta, s) \in \mathbb{R}^3
\]

as a parametrization of the (standard) catenoid \( C \) with \( x_3 \)-axis of revolution. The unit normal vector field about \( C \) is chosen to be

\[
N_c(s, \theta) := \frac{1}{\cosh s} (\cos \theta, \sin \theta, -\sinh s).
\]

Up to some dilation, we can assume that the top end \( E_t \) of the surface \( M_k \) is asymptotic to some translated copy of the catenoid parameterized by \( X_c \) in the vertical direction. Therefore, \( E_t \) can be parameterized by

\[
X_t := X_c + w_t N_c + \sigma_t e_3
\]

where \( w_t, \sigma_t \) are constants.
for \((s, \theta) \in (s_0, \infty) \times S^1\), where the function \(w_t\) tends to 0 as \(s\) tends to \(\infty\) and \(\sigma_t \in \mathbb{R}\). Again, \(w_t\) tends to 0 as \(s\) tends to \(\infty\), reflecting the fact that the end \(E_t\) is asymptotic to the standard catenoid translated by \(\sigma_t e_3\).

We recall that the surface parameterized by \(X := X_c + w N_c\) is minimal if and only if the function \(w\) satisfies the minimal surface equation which, for normal graphs over a catenoid, can be expanded in powers of \(w\) (and its partial derivatives) as

\[
\frac{1}{\cosh^2 s} \left( \partial_s^2 + \partial_\theta^2 + \frac{2}{\cosh^2 s} \right) w + Q_2 \left( \frac{w}{\cosh s} \right) + \cosh s Q_3 \left( \frac{w}{\cosh s} \right) = 0 \tag{2}
\]

Here \(Q_2\) and \(Q_3\) are nonlinear second order differential operators which satisfy

\[
\|Q_j(v_2) - Q_j(v_1)\|_{C^{2,\alpha}((s,s+1) \times S^1)} \leq c \left( \sup_{i=1,2} \|v_i\|_{C^{2,\alpha}((s,s+1) \times S^1)} \right)^{j-1} \|v_2 - v_1\|_{C^{2,\alpha}((s,s+1) \times S^1)} \tag{3}
\]

for all \(s \in \mathbb{R}\) and all \(v_1, v_2\) such that \(\|v_i\|_{C^{2,\alpha}((s,s+1) \times S^1)} \leq 1\). The important fact is that the constant \(c > 0\) does not depend on \(s\). The proof of this expansion can be easily adapted from the proof of the corresponding expansion for higher dimensional catenoids which is provided in [4], a complete (short) proof is given in the Appendix A.

The function \(w_t\) is (by definition) a solution of (2). Given the symmetry with respect to the rotation of vertical axis and angle \(\frac{2\pi}{k+1}\), one checks that \(w_t\) is in fact bounded by a constant times \(e^{-(k+1)s}\). Indeed, just observe that, in the expansion of \(w_t\) in powers of \(e^{-s}\), the leading term is harmonic (on the cylinder \(\mathbb{R} \times S^1\)) and invariant under the action of the rotation on \(S^1\) by the angle \(\frac{2\pi}{k+1}\), hence it has to be a linear combination of the functions \((s, \theta) \rightarrow e^{-(k+1)s} e^{\pm i(k+1)\theta}\).

Similarly, we define \(X_b\) to parameterize the lower end \(E_b\) of the surface \(M_k\) so that

\[
X_b := X_c - w_b N_c - \sigma_b e_3
\]

for \((s, \theta) \in (-\infty, -s_0) \times S^1\), where the function \(w_b\) tends to 0 as \(s\) tends to \(-\infty\) and \(\sigma_b \in \mathbb{R}\). Again, \(w_b\) tends to 0 as \(s\) tends to \(-\infty\), reflecting the fact that the end \(E_b\) is asymptotic to the standard catenoid translated by \(\sigma_b e_3\). Granted the symmetries of the surface \(M_k\), there is an obvious relation between \(X_t\) and \(X_b\). Indeed, starting from the parametrization of \(E_t\) which we compose by a rotation of angle \(\frac{\pi}{k+1}\) about the \(x_3\)-axis and a symmetry with respect to the \(x_3 = 0\) one finds a parametrization of \(E_b\). This implies that \(\sigma_b = \sigma_t\) and also that

\[
w_b(s, \theta) = -w_t \left( -s, \theta - \frac{\pi}{k+1} \right).
\]
For all \( r < r_0 \) and \( s > s_0 \), we define

\[
M_k(s, r) := M_k - (X_t((s, \infty) \times S^1) \cup X_b((-\infty, -s) \times S^1) \cup X_m(B_{r_0}(0)))
\]  

(4)

The parametrizations of the three ends of \( M_k \) induce a decomposition of \( M_k \) into slightly overlapping components as follows: A compact piece \( M_k(s_0 + 1, r_0/2) \) and three noncompact pieces \( X_t((s_0, \infty) \times S^1) \), \( X_b((-\infty, -s_0) \times S^1) \) and \( X_m(B_{r_0}(0)) \). We are now in a position to define the weighted spaces of functions on \( M_k \).

**Definition 2.1.** Given \( \ell \in \mathbb{N} \), \( \alpha \in (0, 1) \) and \( \delta, \nu \in \mathbb{R} \), the space \( C^{\ell, \alpha}_{\delta, \nu}(M_k) \) is defined to be the space of functions in \( C^{\ell, \alpha}_{\text{loc}}(M_k) \) for which the following norm is finite

\[
\|w\|_{C^{\ell, \alpha}_{\delta, \nu}(M_k)} := \|w\|_{C^{\ell, \alpha}(M_k(s_0 + 1, r_0/2))} + \| |w|^{-\nu} w \circ X_m\|_{C^{\ell, \alpha}(B_{r_0}(0))} + \sup_{s \geq s_0} e^{-\delta s} \left( \|w \circ X_t\|_{C^{\ell, \alpha}((s, s_0 + 1) \times S^1)} + \|w \circ X_b\|_{C^{\ell, \alpha}((-s_0, -s) \times S^1)} \right)
\]

and which are invariant under the action of the symmetry with respect to the \( x_2 = 0 \) plane, i.e. \( w(p) = w(\bar{p}) \) for all \( p \in M_k \), where \( \bar{p} := (x_1, -x_2, x_3) \) if \( p = (x_1, x_2, x_3) \).

The Jacobi operator about \( M_k \) is defined by

\[
\mathbb{L}_{M_k} := \Delta_{M_k} + |A_{M_k}|^2
\]

where \( |A_{M_k}| \) is the norm of the second fundamental form on \( M_k \). Granted the above defined spaces, one can check that:

\[
L_{\delta} : C^{2, \alpha}_{\delta, \delta}(M_k) \rightarrow C^{0, \alpha}_{\delta-2, \delta-4}(M_k)
\]

\[
w \mapsto \mathbb{L}_{M_k}(w)
\]

is a bounded linear operator. The subscript \( \delta \) is meant to keep track of the weighted space over which the Jacobi operator is acting. Observe that, in the weights of the target space, there is a loss of 2 in the weight parameter at the ends \( E_t \) and \( E_b \), and there is a gain of 4 in the weight parameter at the end \( E_m \). This follows at once from the expression of the Jacobi operator at the ends in the above defined coordinates. Alternatively, this can also be seen by linearizing the nonlinear equation \( u \) at \( u = 0 \) which provides the expression of the Jacobi operator about the plane

\[
\mathbb{L}_{\mathbb{R}^2} := |x|^4 \Delta
\]
and by linearizing the nonlinear equation (2) at \( w = 0 \) which provides the expression of the Jacobi operator about the standard catenoid

\[
\mathbb{L}_C := \frac{1}{\cosh^2 s} \left( \frac{\partial^2_s + \partial^2_\theta + 2}{\cosh^2 s} \right)
\]

Since the Jacobi operator about \( M_k \) is asymptotic to \( \mathbb{L}_{\mathbb{R}^2} \) at \( E_m \) and is asymptotic to \( \mathbb{L}_C \) at \( E_t \) and \( E_b \), this explains the loss of 2 in the weight parameter \( \delta \) and the gain of 4 in the weight parameter \( \nu \).

This being understood, we now recall the notion of nondegeneracy [11] which is classically used in this context:

**Definition 2.2.** The surface \( M_k \) is said to be nondegenerate if \( \mathbb{L}_\delta \) is injective for all \( \delta < -1 \).

The mapping properties of the operator \( \mathbb{L}_\delta \) depends crucially on the choice of \( \delta \). It follows from the general theory of such operators that \( \mathbb{L}_\delta \) has closed range and is Fredholm provided \( \delta \not\in \mathbb{Z} \). Moreover, a duality argument (in weighted Lebesgue spaces !) implies that

\[
(\mathbb{L}_\delta \text{ is injective}) \iff (\mathbb{L}_{-\delta} \text{ is surjective})
\]

provided \( \delta \not\in \mathbb{Z} \). This kind of analysis is by now standard and has been applied to variety of problems. We refer to [13] for references to the general theory and we refer to [9] for references to the theory in the specific context of minimal hypersurfaces with catenoidal type ends. Also, we have the :

**Proposition 2.1.** Assume that \( M_k \) is nondegenerate and \( \delta \in (1, 2) \). Then the operator \( \mathbb{L}_\delta \) is surjective. Moreover the kernel of \( \mathbb{L}_\delta \) is 4-dimensional.

One has to keep in mind that, in the definition of the weighted spaces, we have imposed the invariance under some symmetry and that, in addition, we have implicitly asked that the middle end of the surface remain asymptotic to a horizontal plane. This explains why the dimension of the kernel is only equal to 4 and not equal to \( 9 = 3 \times 3 \) (the number of ends) as is usually the case when no symmetries are imposed.

Recall that a smooth one parameter group of isometries containing the identity generates a Jacobi field i.e. a solution of the homogeneous problem \( \mathbb{L}_{M_k} w = 0 \). We now define 4 of these Jacobi fields and we also provide there expansion at the ends of \( M_k \). Let \( N \) denote a unit normal vector field on \( M_k \) (for example, we agree that the orientation is chosen so that \( N \sim e_3 \) at \( E_m \)). We will denote by

\[
\Phi^{0,+}(p) := N(p) \cdot e_3,
\]
the Jacobi field generated by the one parameter group of vertical translations. Observe that
\[ \Phi_{0,+} = -\tanh s + O((\cosh s)^{-k-2}) \quad \text{at} \quad E_t \]
while \( \Phi_{0,+} = 1 + \mathcal{O}(|x|^{2k+4}) \) at \( E_m \). We will denote by
\[ \Phi_{0,-}(p) := N(p) \cdot p \]
the Jacobi field generated by the one parameter group of dilations. Observe that
\[ \Phi_{0,-} = 1 - s \tanh s + O((\cosh s)^{-k-1}) \quad \text{at} \quad E_t \]
\[ \Phi_{0,-} = s \tanh s - 1 + O((\cosh s)^{-k-1}) \quad \text{at} \quad E_b \]
while \( \Phi_{0,-} = O(|x|^{k+1}) \) at \( E_m \). We denote by
\[ \Phi_{1,+}(p) := N(p) \cdot e_1 \]
the Jacobi field generated by the one parameter group of translations along the \( x_1 \)-axis. Observe that
\[ \Phi_{1,+} = \frac{1}{\cosh s} \cos \theta + O((\cosh s)^{-k-2}) \quad \text{at} \quad E_t \]
\[ \Phi_{1,+} = -\frac{1}{\cosh s} \cos \theta + O((\cosh s)^{-k-2}) \quad \text{at} \quad E_b \]
while \( \Phi_{1,-} = O(|x|^{k+2}) \) at \( E_m \). Finally, we denote by
\[ \Phi_{1,-}(p) := N(p) \cdot (e_2 \times p) \]
the Jacobi field generated by the one parameter group of rotation about the \( x_2 \)-axis. Observe that
\[ \Phi_{1,-} = \left( \frac{s}{\cosh s} + \sinh s \right) \cos \theta + O((\cosh s)^{-k-1}) \quad \text{at} \quad E_t \]
\[ \Phi_{1,-} = -\left( \frac{s}{\cosh s} + \sinh s \right) \cos \theta + O((\cosh s)^{-k-1}) \quad \text{at} \quad E_b \]
while \( \Phi_{1,-} = \frac{s}{|x|^2} + O(|x|^{2k+3}) \) at \( E_m \).

Observe that all these globally defined Jacobi fields are invariant under the action of the symmetry with respect to the \( x_2 = 0 \) plane and that there are in addition three other Jacobi fields which are not invariant under this symmetry, namely the Jacobi field associated to the group of translation along the \( x_2 \)-axis and the Jacobi field corresponding to
the one parameter group of rotation about the $x_1$-axis and the Jacobi field corresponding to the one parameter group of rotation about the $x_3$-axis.

With these notations, we define the deficiency space

$$\mathcal{D} := \text{Span}\{\chi_t \Phi_j^0, \chi_b \Phi_j^+ : j = 0, 1\}$$

where $\chi_t$ is a cutoff function which is identically equal to 1 on $X_t((s_0 + 1, \infty) \times S^1)$, identically equal to 0 on $M_k - X_t((s_0, \infty) \times S^1)$ and which satisfies $\chi_t(p) = \chi_t(\bar{p})$ (so that it is invariant under the action of the symmetry with respect to the $x_2 = 0$ plane). We also define $\chi_b(\cdot) := \chi_t(-\cdot)$. Clearly

$$\tilde{L}_\delta : C^2_{\delta,0}(M_k) \oplus \mathcal{D} \rightarrow C^0_{\delta-2,4}(M_k)$$

$$w \mapsto L_{M_k}(w)$$

is a bounded linear operator.

The linear decomposition Lemma proved in [11] for constant mean curvature surfaces or in [9] for minimal hypersurfaces can be adapted to our situation and we get the:

**Proposition 2.2.** Assume that $M_k$ is nondegenerate and that $\delta \in (-2, -1)$. Then the operator $\tilde{L}_\delta$ is surjective and has a kernel of dimension 4.

We are interested in $\mathcal{M}$ the space of all minimal surfaces (not necessarily embedded) which are close to $M_k$, have 2 catenoidal ends, one horizontal planar end and which are invariant under the action of the symmetry with respect to the $x_2 = 0$ plane. The moduli space theory developed in [11] for constant mean curvature surfaces or in [9] for minimal hypersurfaces can be adapted to our framework and as a corollary of Proposition 2.2 we conclude that, close to $M_k$, the space $\mathcal{M}$ is a smooth manifold of dimension 4, provided $M_k$ is nondegenerate. Moreover, the elements of the kernel of $\tilde{L}_\delta$ span the tangent space to $\mathcal{M}$. Therefore, in order to understand the space $\mathcal{M}$ in a neighborhood of $M_k$, we just need to understand the elements which span the kernel of $\tilde{L}_\delta$ since this will provide the set of parameters which are needed to describe $\mathcal{M}$ in a neighborhood of $M_k$.

It should be clear that the functions $\Phi^0, \Phi^{1,+}$ belong to $C^2_{\delta,0}(M_k) \oplus \mathcal{D}$ and hence we already have 3 linearly independent elements of the kernel of $\tilde{L}_\delta$. Observe that $\Phi^{1,-}$ fails to belong to the kernel of $\tilde{L}_\delta$ since it is not bounded at $E_m$ (and in fact blows up like $|x|^{-1}$ as $x$ tends to 0). Thus, we are left to understand the behavior of a nonzero element $\Phi \in C^2_{\delta,0}(M_k) \oplus \mathcal{D}$ which belongs to the kernel of $\tilde{L}_\delta$ but does not belong to $\text{Span}\{\Phi^0, \Phi^{1,+}\}$, and hence $\Phi \neq 0$. Without loss of generality (i.e. taking suitable
linear combination of $\Phi$ with $\Phi^{0,\pm}$ and $\Phi^{1,\pm}$ we can assume that the expansion of $\Phi$ at $E_t$ is given by
\[ \Phi = a_t \Phi^{1,-} + O((\cosh s)^\delta) \]
and that the expansion of $\Phi$ at $E_b$ is given by
\[ \Phi = a_b \Phi^{1,-} + b_b \Phi^{1,+} + c_b \Phi^{0,+} + d_b \Phi^{0,-} + O((\cosh s)^\delta) \]

Given a function $\Psi$ defined on $M_k$, we set
\[ W(\Psi) := \lim_{s \to \infty} \lim_{r \to 0} \int_{M_k(s,r)} (\Phi L_{M_k} \Psi - \Psi L_{M_k} \Phi) \, d\text{vol}_{M_k} \]
where we recall that $M_k(s,r)$ has been defined in (4).

Since $L_{M_k} \Phi = 0$ and $L_{M_k} \Phi^{0,+} = 0$, we can use the divergence theorem together with the expansions (5)-(6) to get
\[ 0 = W(\Phi^{0,+}) = 2\pi d_b. \]
Similarly, using the fact that $L_{M_k} \Phi^{0,-} = 0$ together with the expansions (5)-(6), we get
\[ 0 = W(\Phi^{0,-}) = -2\pi c_b \]
Next, using the fact that $L_{M_k} \Phi^{1,-} = 0$ together with the expansions (7)-(8), one finds that
\[ 0 = W(\Phi^{1,-}) = -\pi b_b \]
Finally, using the fact that $L_{M_k} \Phi^{1,+} = 0$ together with the expansions (7)-(8), we have
\[ 0 = W(\Phi^{1,+}) = \pi (a_b - a_t). \]
Therefore, we conclude that $b_b = c_b = d_b = 0$ and also that $a_b = a_t$. Now, if we had $a_t = 0$, then we would also have $a_b = 0$ and hence we would conclude that $\Phi \in C^{2,\alpha}_\delta(M_k)$. But, in this case nondegeneracy implies that $\Phi = 0$, which is clearly a contradiction since we have assumed that $\Phi \neq 0$. Therefore, we conclude that $a_t \neq 0$. In other words, there exists an element of the kernel of $\tilde{L}_\delta$ which at $E_t$ (and in fact also at $E_b$) is asymptotic to the Jacobi field associated to the rotation of the catenoidal ends of $M_k$, leaving the middle end horizontal.

Applying (an elaborate version of) the implicit function theorem as in [11] and [9], we see that this Jacobi field is integrable. This shows that there exists in $\mathcal{M}$ a one parameter
family of minimal hypersurfaces \((M_k(\xi))_\xi\), for \(\xi\) close to 0, such that \(M_k(0) = M_k\) and the catenoidal upper end of \(M_k(\xi)\) is asymptotic to the end of a catenoid whose axis of revolution is directed by \(\sin \xi e_1 + \cos \xi e_3\). Observe that, the surface \(M_k(\xi)\) is well defined up to a translation in the \(x_2 = 0\) plane and up to a dilation. In particular, we can require that the upper end of \(M_k(\xi)\) is asymptotic to a translated and rotated version of the (standard) catenoid and also require that the middle end \(E_m(\xi)\) is asymptotic to the \(x_3 = 0\) plane.

If \(R_\xi\) denotes the rotation of angle \(\xi\) about the \(x_2\)-axis, the upper end \(E_t(\xi)\) of \(M_k(\xi)\) can be parameterized by

\[
X_{t,\xi} = R_\xi (X_c + w_{t,\xi} N_c) + \sigma_{t,\xi} e_3 + \varsigma_{t,\xi} e_1
\]

where the function \(w_{t,\xi}\) and \(\sigma_{t,\xi}, \varsigma_{t,\xi} \in \mathbb{R}\) depend smoothly on \(\xi\) and satisfy \(w_{t,0} = w_t\), \(\sigma_{t,0} = \sigma_t\) and \(\varsigma_{t,0} = 0\). More precisely, if follows from the application of the implicit function theorem that

\[
|\sigma_{t,\xi} - \sigma_t| + |\varsigma_{t,\xi}| + \|w_{t,\xi} - w_t\|_{C^2,\alpha((-s_0, +\infty) \times S^1)} \leq c|\xi|
\]

Application of the flux formula [10] shows that the lower end of \(M_k(\xi)\) is, up to a translation, asymptotic to the lower end of the same (standard) catenoid. In particular, the lower end \(E_b(\xi)\) of \(M_k(\xi)\) can be parameterized by

\[
X_{b,\xi} = R_\xi (X_c - w_{b,\xi} N_c) - \sigma_{b,\xi} e_3 - \varsigma_{b,\xi} e_1
\]

where the function \(w_{b,\xi}\) and \(\sigma_{b,\xi}, \varsigma_{b,\xi} \in \mathbb{R}\) depend smoothly on \(\xi\) and satisfy \(w_{b,0} = w_b\), \(\sigma_{b,0} = \sigma_b\) and \(\varsigma_{b,0} = 0\). Again, we also have

\[
|\sigma_{b,\xi} - \sigma_b| + |\varsigma_{b,\xi}| + \|w_{b,\xi} - w_b\|_{C^2,\alpha((-\infty, -s_0) \times S^1)} \leq c|\xi|
\]

Finally, up to now the surface \(M_k(\xi)\) is defined up to a translation along the \(x_1\)-axis but we can help eliminate this confusion by requiring that

\[
\varsigma_{b,\xi} = \varsigma_{t,\xi}
\]

To summarize, we have obtained the :

**Theorem 2.1.** Assume that \(M_k\) is nondegenerate. Then, there exists \(\xi_0 > 0\) and a smooth one parameter family of minimal hypersurfaces \((M_k(\xi))_\xi\) in \(\mathcal{M}\), for \(\xi \in (-\xi_0, \xi_0)\), such that \(M_k(0) = M_k\) and the upper (resp. lower) catenoidal end of \(M_k(\xi)\) is, up to a translation along its axis, asymptotic to the upper (resp. lower) end of the standard catenoid whose axis of revolution is directed by \(\sin \xi e_1 + \cos \xi e_3\).
Observe that, when $k$ is even, the surface $M_k$ is symmetric with respect to the rotation of angle $\pi$ about the $x_2$-axis and one can prove, even if we will not need this information, that the surfaces $M_k(\xi)$ can be defined to enjoy the same symmetry.

On each $M_k(\xi)$ one can define weighted spaces as in Definition 2.1 and also define the corresponding notion of nondegeneracy. The Jacobi operator about $M_k(\xi)$ will be denoted by $L_{M_k(\xi)}$. We define

$$L_{\xi,\delta} : C^{2,\alpha}_{\delta,0}(M_k(\xi)) \rightarrow C^{0,\alpha}_{\delta-2,4}(M_k(\xi))$$

$$w \mapsto L_{M_k(\xi)}(w)$$

It is easy to check that, reducing $\xi_0$ if this is necessary, all the surfaces $M_k(\xi)$ are non-degenerate and hence we have the :

**Proposition 2.3.** Assume that $M_k$ is nondegenerate and choose $\delta \in (1,2)$. Then (reducing $\xi_0$ if this is necessary) the operator $L_{\xi,\delta}$ is surjective and has a kernel of dimension 4. Moreover, there exists $G_{\xi,\delta}$ a right inverse for $L_{\xi,\delta}$ which depends smoothly on $\xi$ and in particular whose norm is bounded uniformly as $|\xi| < \xi_0$.

For example, a right inverse $G_{\xi,\delta}$ which depends smoothly on $\xi$ can be obtained by a simple perturbation argument starting from a right inverse $G_{\delta,0}$ and reducing $\xi_0$ if this is necessary.

The purpose of the next Lemma is to write a portion of the upper and lower ends of the surface $M_k(\xi)$ as vertical graphs over the horizontal plane. It is clear that the ends $E_t$ and $E_b$ of $M_k$ can be written, at least away from a compact set, as vertical graphs over the horizontal plane $x_3 = 0$. This is not true anymore for the ends $E_t(\xi)$ and $E_b(\xi)$ of $M_k(\xi)$, when $\xi \neq 0$ but this property will remain true for the piece of $M_k(\xi)$ we are interested in, namely the piece of $M_k(\xi)$ which corresponds to $s \sim \pm \frac{1}{2} \log \varepsilon$ in the parametrization given in [9] and [10]. We have the :

**Lemma 2.1.** There exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0,\varepsilon_0)$ and all $|\xi| \leq \varepsilon$ an annular part of $E_t(\xi)$ and $E_b(\xi)$ in $M_k(\xi)$ can be written as vertical graphs over the horizontal plane for the functions

$$\bar{U}_t^0(r, \theta) = \sigma_t + \ln(2r) + \xi r \cos \theta + O(\varepsilon)$$

$$\bar{U}_b^0(r, \theta) = -\sigma_b + \ln(2r) + \xi r \cos \theta + O(\varepsilon)$$

Here $(r, \theta)$ are polar coordinates in the $x_3 = 0$ plane. The functions $O(\varepsilon)$ are defined in the annulus $B_{4\varepsilon^{-1/2}} - B_{\varepsilon^{-1/2/4}}$ and are bounded in $C^\infty_b$ topology by a constant (independent
of \(\varepsilon\) times \(\varepsilon\), where partial derivatives are computed with respect to the vector fields \(r \partial_r\) and \(\partial_\theta\).

**Proof**: Elementary computations writing

\[ X_{t,\xi}(s,\tilde{\theta}) = (r \cos \theta, r \sin \theta, U_{t}^{0}(r, \theta)) \]

when \(s \sim -\frac{1}{2} \log \varepsilon\).

We end this section by recalling the result of S. Nayatani [17] which states that \(M_k\) is nondegenerate for all \(k \leq 37\). More precisely we have the :

**Theorem 2.2.** Assume that \(k \leq 37\), then any bounded Jacobi field on \(M_k\) is a linear combination of \(N \cdot e_j\) and \(N \cdot (p \times e_3)\).

In order to apply this result, just observe that, according to this result, when \(k \leq 37\), the only Jacobi field which decays at all ends is \(N \cdot (p \times e_3)\). However, this Jacobi field is not invariant with respect to the action of \(p \rightarrow \tilde{p}\), hence \(M_k\) is nondegenerate in the sense defined in Definition 2.2.

### 3 Riemann minimal surface

B. Riemann [19] has discovered a one parameter family of periodic minimal surfaces embedded in \(\mathbb{R}^3\) which are foliated by circles (and straight lines). Each element of this family has infinitely many planar ends, is topologically a cylinder \(\mathbb{R} \times S^1\) and in fact is conformal to the cylinder \(\mathbb{R} \times S^1\) with infinitely many points \((p_i)_{i \in \mathbb{Z}}\) removed in a periodic way, each of these points corresponds to one of the planar ends of the surface.

Recall that (up to some dilation and some rigid motion) we can parameterize a fundamental piece of Riemann’s surface by

\[ (t, \theta) \rightarrow (c(t) + R(t) \cos \theta, R(t) \sin \theta, t) \]

where \(t \in (-t_\varepsilon, t_\varepsilon)\), \(\theta \in S^1\) and where the functions \(c\) and \(R\) are determined by

\[ (\partial_t R)^2 + 1 = R^2 + \varepsilon^2 R^4 \]

and

\[ \partial_t c = \varepsilon R^2 \]

Here \(\varepsilon > 0\) is a parameter. We shall normalize the solutions of these ordinary differential equations by asking that \(R(0) > 0\), \(\partial_t R(0) = 0\) and \(c(0) = 0\), and naturally, \(R\) is a
nonconstant smooth solution of (13). Even though $R$ and $c$ both depend on $\varepsilon$, we shall not make this dependence explicit in the notation. It is easy to check that the functions $R$ and $c$ blow up in finite time $t_\varepsilon < \infty$ and that
\[ \ell_\varepsilon = \lim_{t \to t_\varepsilon} (c(t) - R(t)) \]
exists. Riemann surface $R_\varepsilon$ is then obtained by translation of the fundamental piece by $2 (\ell_\varepsilon e_1 + t_\varepsilon e_3) \mathbb{Z}$.

A conformal parametrization of Riemann surfaces had already been considered by M. Shiffman [20] and has been generalized by L. Hauswirth [5]. Granted the above parametrization, in order to define this conformal parametrization, it is enough to look for a function $(t, y) \to \psi(t, y)$ such that
\[ X_\varepsilon(t, y) := (c(t) + R(t) \cos(\psi(t, y)), R(t) \sin(\psi(t, y)), t) \] (14)
is a conformal parametrization. This leads to the first order differential system
\[ \partial_t \psi = \varepsilon R \sin \psi \]
which come from the equation $\partial_t X_\varepsilon \cdot \partial_y X_\varepsilon = 0$ and
\[ (\partial_y \psi)^2 = 1 + \varepsilon^2 R^2 (1 + \cos^2 \psi) + 2 \varepsilon \partial_t R \cos \psi \]
which comes from the requirement that $|\partial_t X_\varepsilon|^2 = |\partial_y X_\varepsilon|^2$. One checks easily (from a direct computation) that the integrability condition $\partial_y (\partial_t \psi) = \partial_t (\partial_y \psi)$ is fulfilled.

We define the real valued function $\phi$ by
\[ \partial_t X_\varepsilon = (\sinh \phi \cos \psi, \sinh \phi \sin \psi, 1) \quad \text{and} \quad \partial_y X_\varepsilon = (-\cosh \phi \sin \psi, \cosh \phi \cos \psi, 0) \]
In particular
\[ \cosh \phi = R \partial_y \psi \] (15)

With these notations, the first fundamental form about the surface parameterized by $X_\varepsilon$ reads
\[ ds^2 = \cosh^2 \phi (dt \otimes dt + dy \otimes dy) \]
and, if we define the normal vector field by
\[ N_\varepsilon := \frac{1}{\cosh \phi} (\cos \psi, \sin \psi, -\sinh \phi) \] (16)
the second fundamental form about the surface parameterized by $X_\varepsilon$ is then given by

$$h = \partial_t \phi \, dt \otimes dt - \partial_y \psi \, dy \otimes dy - 2 \partial_t \psi \, dt \otimes dy$$

Observe that $\partial_y (\partial_t X) = \partial_t (\partial_y X)$ and hence

$$\partial_y \phi = -\partial_t \psi \quad \text{and} \quad \partial_t \phi = \partial_y \psi \quad (17)$$

It will be convenient to define

$$a := \frac{\partial_t \psi}{\cosh \phi} \quad \text{and} \quad b = \frac{\partial_y \psi}{\cosh \phi} \quad (18)$$

With these notations, the Jacobi operator about Riemann’s surface $R_\varepsilon$ is given by

$$\mathbb{L}_{R_\varepsilon} = \frac{1}{\cosh^2 \omega} \left( \partial_t^2 + \partial_y^2 + 2 \left( a^2 + b^2 \right) \right)$$

Observe that it follows from (15) that $b = \frac{1}{R}$ and hence the equation satisfied by $b$ reads

$$(\partial_t b)^2 + b^4 = \varepsilon^2 + b^2$$

Moreover, $\partial_y b = 0$ since $R$, and hence $b$, does not depend on $y$.

It should be clear that (18) together with (17) yields $\partial_t a + \partial_y b = 0$ and hence the function $a$ does not depend on $t$. It remains to find the ordinary differential equation satisfied by $a$. We have

$$\partial_y (\cosh \phi a) - \partial_t (\cosh \phi b) = 0.$$ 

Hence

$$(a^2 + b^2) \sinh \phi = \partial_y a - \partial_t b \quad (19)$$

Taking the derivative of (19) with respect to $y$ and using the fact that the function $b$ does not depend on $t$, we get

$$\partial_y^2 a = -a (a^2 + b^2) + \frac{a}{a^2 + b^2} \left( (\partial_y a)^2 - (\partial_t b)^2 \right) \quad (20)$$

In other words

$$\partial_y \left( \frac{(\partial_y a)^2 - (\partial_t b)^2}{a^2 + b^2} + a^2 \right) = 0$$

Hence, we conclude that the function

$$(t, y) \rightarrow \frac{(\partial_y a)^2 - (\partial_t b)^2}{a^2 + b^2} + a^2$$
only depends on \( t \). Taking this information into account in \((20)\) we conclude that

\[
\partial_y^2 a + 2 a^3 + \alpha(t) a = 0
\]

where \( \alpha \) is a function which only depends on \( t \), and hence \( \alpha \) has to be constant. Therefore

\[
(\partial_y a)^2 + a^4 + \alpha a^2 - \beta = 0
\]

for some fixed constant \( \beta \). Inserting these into \((20)\), we conclude that

\[
(\partial_y a)^2 + a^4 + a^2 - \varepsilon^2 = 0
\]

(21)

The function \( y \rightarrow a(y) \) and \( t \rightarrow b(t) \) are defined up to some translation in the \( y \) or \( t \) variables. In particular, we can require that \( a \) (resp. \( b \)) takes its maximal value at \( y = 0 \) (resp. \( t = 0 \)). Finally, observe that the functions \( a \) is periodic, we will denote by \( y_\varepsilon \) its least period. Finally, we extend the function \( b \) to be a \( 2t_\varepsilon \)-periodic function.

It is also easy to check that, as \( \varepsilon \) tends to 0 the functions \( \varepsilon^{-1} a, b \) and their derivatives remain uniformly bounded. Indeed, we have on the one hand

\[
(\partial_y a)^2 \leq \varepsilon^2 \quad \text{and} \quad 2 a^2 \leq \sqrt{1 + 4 \varepsilon^2} - 1
\]

(22)

and on the other hand

\[
(\partial_t b)^2 \leq \varepsilon^2 + \frac{1}{4} \quad \text{and} \quad 2 b^2 \leq 1 + \sqrt{1 + 4 \varepsilon^2}.
\]

(23)

A simple application of Ascoli-Arzela’s Theorem implies the:

**Lemma 3.1.** As \( \varepsilon \) tends to 0, the sequence of functions \( (b)_{\varepsilon > 0} \) converges uniformly on compacts to the function

\[
t \longrightarrow \frac{1}{\cosh t}
\]

and the sequence of functions \( (\varepsilon^{-1} a)_{\varepsilon > 0} \) converges uniformly to the function

\[
y \longrightarrow \cos y
\]

We can also obtain the expansion of the period \( 2t_\varepsilon \) of the function \( b \). Indeed, we have the formula

\[
t_\varepsilon = \int_0^{\zeta_\varepsilon} \frac{d\zeta}{\sqrt{1 + \zeta^2 - \varepsilon^2 \zeta^4}}
\]

16
where $0 < \zeta_{\varepsilon}$ is the largest root of $\varepsilon^2 \zeta^4 = \zeta^2 + 1$. It is easy to check that

$$t_{\varepsilon} = -\log \varepsilon + O(1)$$

as $\varepsilon$ tends to 0.

We claim that

$$\frac{1}{\sqrt{1 + 4\varepsilon^2}} \leq \left( \frac{y_{\varepsilon}}{2\pi} \right)^2 \leq 1$$

Indeed, write $a(y) = a(0) \cos v$ where $2a(0)^2 + 1 = \sqrt{1 + 4\varepsilon^2}$ and using (22) we get

$$(\partial_y v)^2 = 1 + a(0)^2 (1 + \cos^2 v)$$

from which it follows that $1 \leq \partial_y v \leq 1 + 2a(0)^2$. Integration of this inequalities from 0 to $y_{\varepsilon}$ yields the required inequalities since $v(y_{\varepsilon}) = 2\pi$.

In the next Lemma, we give precise expansions of the functions $R$ and $c$ when $t \in (-t_{\varepsilon}, t_{\varepsilon})$.

**Lemma 3.2.** For $\varepsilon > 0$ small enough, we have

$$R(t) = \cosh t + O(\varepsilon^2 \cosh^3 t)$$

and

$$c(t) = \varepsilon \left( \frac{t}{2} + \frac{\sinh(2t)}{4} \right) + O(\varepsilon^3 \cosh^4 t)$$

when $t \in [-t_{\varepsilon} + 1, t_{\varepsilon} - 1]$.

**Proof:** We define the function $t \rightarrow v(t)$ such that $R(t) = R(0) \cosh v(t)$ and $v(0) = 0$. It follows from (13) that

$$(\partial_t v)^2 = 1 + \varepsilon^2 R^2(0) (1 + \cosh^2 v)$$

Now, as long as $t \leq v(t) \leq t + c$ (where $c > 0$ is some fixed constant), we can estimate $(\partial_t v)^2 = 1 + O(\varepsilon^2 \cosh^3 t)$ and hence we conclude that $v(t) = t + O(\varepsilon^2 \cosh^2 t)$. We remark 	extit{a posteriori} that $t \leq v(t) \leq t + c$ holds for $t \in [0, -\ln \varepsilon - 1]$ provided $c > 0$ is chosen large enough. The first estimate then follows at once. The second estimate follows directly from

$$\partial_t a = \varepsilon R^2$$

once the first estimate has been established. \qed
Remark 3.1. As a Corollary of the proof of this Lemma, observe that

\[ b(t) \cosh t \leq \frac{1}{R(0)} \]

for all \( t \in [-t_\varepsilon, t_\varepsilon] \) since we always have \( v(t) \geq t \). Since \( R(0) \) converges to 1 as \( \varepsilon \) tends to 0 and using the fact that \( b \) is even, this yields a uniform upper bound for \( b \) as \( \varepsilon \) tends to 0.

The purpose of the next Lemma is to write the pieces of \( R_\varepsilon \) at height \( t \sim -\frac{1}{2} \log \varepsilon \) (resp. at height \( t \sim \frac{1}{2} \log \varepsilon \)) as a vertical graph over the horizontal plane for some function \( t \) (resp. \( t_b \)). But before doing so we first dilate the surface \( R_\varepsilon \) by some factor \((1 + \gamma)\) and we next translate this dilated surface along the \( x_1 \)-axis by \( \varsigma \), so that the fundamental piece of this surface is now parameterized by

\[(t, y) \rightarrow (\varsigma + (1 + \gamma) (c(t) + R(t) \cos \psi(t, y)), (1 + \gamma) R(t) \sin \psi(t, y), (1 + \gamma) t)\]

We consider the change of coordinates:

\[(r \cos \theta, r \sin \theta) = (\varsigma + (1 + \gamma) (c(t) + R(t) \cos \psi(t, y)), (1 + \gamma) R(t) \sin \psi(t, y))\]

where as before, \((r, \theta)\) are polar coordinates in the \( x_3 = 0 \) plane. Obviously this change of coordinates is not valid everywhere but we are only interested in the range \( t \sim \pm \frac{1}{2} \log \varepsilon \) where the change of coordinates holds.

Lemma 3.3. Assume that \(|\varsigma| \leq 1\) and also assume that \(|\gamma| \leq \frac{1}{2}\). Then the following expansion holds

\[ U_t^0(r, \theta) = (1 + \gamma) \log \left( \frac{2r}{1 + \gamma} \right) - \frac{\varepsilon}{2} r \cos \theta - (1 + \gamma) \frac{\varsigma}{r} \cos \theta + O(\varepsilon) \] (25)

and

\[ U_b^0(r, \theta) = -(1 + \gamma) \log \left( \frac{2r}{1 + \gamma} \right) - \frac{\varepsilon}{2} r \cos \theta + (1 + \gamma) \frac{\varsigma}{r} \cos \theta + O(\varepsilon) \] (26)

when \( r \sim \varepsilon^{-1/2} \). Here the functions \( O(\varepsilon) \) are smooth functions which are defined in the annulus \( B_{1\varepsilon^{-1/2}} - B_{\varepsilon^{-1/2}/4} \) and are bounded by a constant (independent of \( \varepsilon \)) times \( \varepsilon \) in \( C^0_b \) topology, where partial derivatives are understood with respect to the vector fields \( r \partial_r \) and \( \partial_\theta \). In addition all these estimates hold uniformly in \( \sigma \) and \( \gamma \), provided \(|\varsigma| \leq 1\) and \(|\gamma| \leq \frac{1}{2}\).
4 The Jacobi operator about Riemann’s surface

We keep the notations of the previous section. Recall that the Jacobi operator about Riemann’s surface is given by

\[ \mathbb{L}_{R_\varepsilon} := \frac{1}{\cosh^2 \omega} \left( \partial_t^2 + \partial_y^2 + 2 \left( a^2 + b^2 \right) \right) \]  

(27)

Obviously the mapping properties of this operator translate into the mapping properties of the operator

\[ L_\varepsilon = \partial_t^2 + \partial_y^2 + 2 \left( a^2 + b^2 \right) \]  

(28)

We define, for all \( \varepsilon \geq 0 \) the operator

\[ D_\varepsilon := \partial_y^2 + 2 a^2 \]

which acts on functions of \( y \) which are \( y_\varepsilon \) periodic and even. This operator is clearly elliptic and self-adjoint and hence has discrete spectrum \((\lambda_i)_{i \geq 0}\). Since we only consider even functions, each eigenvalue is simple and we can arrange the eigenvalues so that \( \lambda_i < \lambda_{i+1} \). The corresponding eigenfunctions are denoted by \( f_i \) and are normalized so that

\[ \int_0^{y_\varepsilon} f_i^2 \, dy = 1 \]

Even though we have not made this explicit in the notations, the eigendata of \( D_\varepsilon \) do depend on \( \varepsilon \) since \( a \) does. It is easy to check that, as \( \varepsilon \) tends to 0, the \( \lambda_i \) converge to \( i^2 \).

We will not need this result but rather the simpler:

**Lemma 4.1.** The following estimate holds

\[ \lambda_i \geq i^2 + 1 - \sqrt{1 + 4\varepsilon^2}. \]  

(29)

**Proof:** The assertion follows from the variational characterization of the eigenvalues

\[ \lambda_i = \sup_{\text{codim } E = i} \inf_{f \in E, \|f\|_{L^2} = 1} \int_0^{y_\varepsilon} \left( (\partial_y f)^2 - 2 a^2 f^2 \right) \, dy \]

together with the fact that \( y_\varepsilon \leq 2\pi \) and \( 0 \leq 2a^2 \leq \sqrt{1 + 4\varepsilon^2} - 1 \). \( \square \)

For each \( \varepsilon > 0 \), the family \( \{f_i\}_{i \in \mathbb{N}} \) is a Hilbert basis of the space \( H \) of \( L^2 \)-integrable functions which are even and \( y_\varepsilon \)-periodic. We consider the eigenfunction decomposition of a function \( (t, y) \mapsto v(t, y) \), which is \( y_\varepsilon \)-periodic and even in the \( y \) variable,

\[ v(t, y) = \sum_{i=0}^{\infty} v_i(t) \, f_i(y). \]
This decomposition induces a decomposition of the operator $L_\varepsilon$ into the sequence of ordinary differential operators

$$L_{\varepsilon,i} = \partial_y^2 + 2b^2 - \lambda_i.$$  

It follows from the result of Lemma 4.1 and from the estimate (29) that

$$2b^2 - \lambda_i \leq 2 \sqrt{1 + 4\varepsilon^2} - i^2 \quad (30)$$

Let $S^1(\tau)$ denote the circle of radius $\tau$. The previous estimate immediately implies the following injectivity result, by maximum principle:

**Lemma 4.2.** Assume that $\varepsilon \in (0, \sqrt{\frac{3}{4}})$ and $t_0 < t_1$. Let $v$ be a solution of

$$L_\varepsilon v = 0$$

which is defined on $[t_0, t_1] \times S^1(\tau_\varepsilon)$ and satisfies $v(t_0, \cdot) = v(t_1, \cdot) = 0$. Further assume that

$$\int_0^{\tau_\varepsilon} v(t, y) f_i(y) \, dy = 0$$

for all $t \in [t_0, t_1]$ and $i = 0, 1$. Then $v = 0$.

The constant $\sqrt{\frac{3}{4}}$ is not optimal but this will be sufficient for our purposes since our aim is to use the result for small values of $\varepsilon$. We now define weighted H"older spaces which will turn to be useful for the understanding the mapping properties of the operator $L_\varepsilon$ as the parameter $\varepsilon$ tends to 0.

**Definition 4.1.** Given $\ell \in \mathbb{N}$, $\alpha \in (0, 1)$, $\mu \in \mathbb{R}$ and a closed interval $I \subset \mathbb{R}$, we define the space $C^{\ell,\alpha}_\mu(I \times S^1(\tau))$ to be the space of functions $u \in C^{\ell,\alpha}_{loc}(I \times S^1(\tau))$ for which the following norm

$$\|u\|_{C^{\ell,\alpha}_\mu} = \|e^{-\mu t} u\|_{C^{\ell,\alpha}(I \times S^1(\tau))},$$

is finite.

We set

$$\tau_\varepsilon = \frac{\varepsilon}{2\pi}$$

It should be obvious that

$$C^{2,\alpha}_\mu([t_0, \infty) \times S^1(\tau_\varepsilon)) \longrightarrow C^{0,\alpha}_\mu([t_0, \infty) \times S^1(\tau_\varepsilon))$$

$$w \longmapsto L_\varepsilon w$$

20
for any \( \mu \in \mathbb{R} \) and \( t_0 \in \mathbb{R} \). We prove that, provided the parameter \( \mu \) is suitably chosen, there exists a right inverse for \( L_\varepsilon \) whose norm is uniformly bounded as \( \varepsilon \) tends to 0 and independently of \( t_0 \in \mathbb{R} \). This is the content of the following:

**Proposition 4.1.** Fix \( \mu \in (-2, -1) \). Then, there exists \( \varepsilon_0 > 0 \) and, for all \( \varepsilon \in (0, \varepsilon_0) \), for all \( t_0 \in \mathbb{R} \), there exists an operator

\[
G_{\varepsilon, t_0} : C_{\mu}^{0, \alpha}([t_0, \infty) \times S^1(\tau_\varepsilon)) \rightarrow C_{\mu}^{2, \alpha}(\{t_0, \infty\} \times S^1(\tau_\varepsilon)),
\]

such that for all \( g \in C_{\mu}^{0, \alpha}([t_0, \infty) \times S^1(\tau_\varepsilon)) \), the function \( v := G_{\varepsilon, t_0}(g) \) solves

\[
\begin{align*}
L_\varepsilon v &= g \quad \text{in} \quad [t_0, \infty) \times S^1(\tau_\varepsilon) \\
v &\in \text{Span} \{f_0, f_1\} \quad \text{on} \quad \{t_0\} \times S^1(\tau_\varepsilon)
\end{align*}
\]

Moreover,

\[
||G_{\varepsilon, t_0}(g)||_{C_{\mu}^{2, \alpha}} \leq c ||g||_{C_{\mu}^{0, \alpha}},
\]

for some constant \( c > 0 \) which is independent of \( \varepsilon \in (0, \varepsilon_0) \) and also independent of \( t_0 \in \mathbb{R} \).

**Proof:** We decompose \( f \) into

\[
g = g_0 f_0 + g_1 f_1 + \overline{g}
\]

where \( \overline{g}(t, \cdot) \) is \( L^2 \) orthogonal to \( f_0 \) and \( f_1 \) for each \( t \). For the sake of simplicity in the notations, we shall not mention the parameter \( \tau_\varepsilon \) and write \( S^1 \) instead of \( S^1(\tau_\varepsilon) \). Observe that, as \( \varepsilon \) tends to 0, \( \tau_\varepsilon \) tends to 1.

**Step 1.** We show that, for each \( t_1 > t_0 + 1 \) it is possible to solve

\[
L_\varepsilon \overline{v} = \overline{g},
\]

on \( S^1 \times [t_0, t_1] \) with \( \overline{v}(t_0, \cdot) = \overline{v}(t_1, \cdot) = 0 \). This just follows from the result of Lemma 4.2 which states that, restricted to the set of functions \( L^2 \) orthogonal to \( f_0 \) and \( f_1 \) for each \( t \), the operator \( L_\varepsilon \) is injective.

We claim that, provided \( \varepsilon \) is chosen small enough, there exists a constant \( c > 0 \) such that

\[
\sup_{[t_0, t_1] \times S^1} e^{-\mu t} |\overline{v}| \leq c \sup_{[t_0, t_1] \times S^1} e^{-\mu t} |\overline{g}|
\]

21
The proof of this fact is by contradiction. If this were false, there would exist a sequence $(\varepsilon_n)_n$ tending to 0, sequences $(t_{0,n})_n$ and $(t_{1,n})_n$ such that $t_{0,n} \leq t_{1,n}$, a sequence of functions $(g_n)_n$ and a sequence of solutions $(\bar{v}_n)_n$ such that

$$
\sup_{[t_{0,n},t_{1,n}] \times S^1} e^{-\mu t} |\bar{g}_n| = 1
$$

and

$$
\lim_{n \to +\infty} A_n := \sup_{[t_{0,n},t_{1,n}] \times S^1} e^{-\mu t} |\bar{v}_n| = \infty
$$

We denote by $(t_n,y_n) \in [t_{0,n},t_{1,n}] \times S^1$ a point where $A_n$ is achieved. We define the function $\tilde{v}_n$ by

$$
\tilde{v}_n(t,y) = \frac{1}{A_n} e^{-\mu t_n} \bar{v}_n(t_n+t,y).
$$

Observe that elliptic estimates imply that

$$
\sup e^{-\mu t} |\nabla \tilde{v}_n| \leq c (1 + A_n)
$$

and, since $\bar{v}_n$ vanishes on the boundaries of $[t_{0,n},t_{1,n}] \times S^1$, this in turn implies that the sequences $(t_n-t_{0,n})_n$ and $(t_{1,n} - t_n)_n$ remain bounded away from 0.

Without loss of generality, we can assume that the sequence $(t_n-t_{0,n})_n$ (resp. $(t_{1,n} - t_0)_n$) converges to $\tilde{t}_0 \in ([-\infty,0)$ (resp. to $\tilde{t}_1 \in (0,+\infty]$). We denote by $I = (\tilde{t}_0, \tilde{t}_1)$.

Up to a subsequence, we can assume, without loss of generality that the sequence of functions $(\tilde{v}_n)_n$ converges on compacts to a nontrivial function $\tilde{v}$ defined on $I \times S^1$. This follows at once from Ascoli-Arzela theorem, once it is observed that the sequence of functions $(\tilde{v}_n)_n$ is uniformly bounded (by $t \to e^{\mu t}$) and, by elliptic regularity theory, the sequence of functions $(\nabla \tilde{v}_n)_n$ is also uniformly bounded (by $t \to c e^{\mu t}$). We now derive some properties of the limit function $\tilde{v}$. These properties are all inherited by $\tilde{v}$ from similar properties which hold for the functions $\bar{v}_n$.

First, $\tilde{v}(t,\cdot)$ is $L^2$ orthogonal to the constant function and the function $y \to \cos y$ for each $t \in I$. Next, $\tilde{v}$ is equal to 0 on $\{\tilde{t}_0\} \times S^1$ and on $\{\tilde{t}_1\} \times S^1$ if either $\tilde{t}_0 > -\infty$ or $\tilde{t}_1 < +\infty$. Also

$$
\sup_{I \times S^1} e^{-\mu t} |\tilde{v}| = 1
$$

Finally, $\tilde{v}$ is either a solution of

$$
\left( \partial_t^2 + \partial_y^2 + \frac{2}{\cosh^2(\cdot + \tilde{t})} \right) \tilde{v} = 0
$$
for some $\tilde{y} \in \mathbb{R}$ or is a solution of

$$(\partial_t^2 + \partial_y^2) \tilde{v} = 0.$$ 

To reach a contradiction we consider the eigenfunction decomposition of $\tilde{v}$

$$\tilde{v}(t, y) = \sum_{j=2}^{\infty} a_j(t) \cos(jy).$$

When $\tilde{t}_0 = -\infty$, observe that the function $a_j$ is either blowing up like $t \rightarrow e^{-jt}$ or decaying like $y \rightarrow e^{jt}$. The choice of $\mu \in (-2, -1)$ implies that $a_j$ decays exponentially at $-\infty$. Multiplying the equation (33) by $a_j e^{jt}$ and integrating by parts over $I$ (all integrations are justified because $a_j$ decays exponentially at both $\pm \infty$ if either $\tilde{t}_0 = -\infty$ or $\tilde{t}_1 = +\infty$), we get either

$$\int_{-\infty}^{+\infty} (|\partial_t a_j|^2 + j^2 a_j^2) dt = \int_{-\infty}^{+\infty} \frac{2}{\cosh^2(t + \bar{t})} a_j^2 dt$$

or

$$\int_{-\infty}^{+\infty} (|\partial_t a_j|^2 + j^2 a_j^2) dt = 0$$

In either case, we obtain $a_j = 0$ which clearly contradicts (32).

Since we have reached a contradiction, the proof of the claim is complete. Once the claim is proven, we can use once more elliptic estimates and Ascoli-Arzela theorem to pass to the limit as $t_1$ tends to $+\infty$ in a sequence of solutions which are defined on $[t_0, t_1] \times S^1$. This proves the existence of a solution of

$$L_\varepsilon \tilde{v} = \tilde{f}$$

which is defined in $S^1 \times [t_0, +\infty)$ and which satisfies $\tilde{v}(t_0, \cdot) = 0$. In addition, we know that

$$\sup_{[t_0, +\infty) \times S^1} e^{-\mu t} |\tilde{v}| \leq c \sup_{[t_0, +\infty) \times S^1} e^{-\mu t} |\tilde{f}|$$

Using a last time elliptic estimates, we complete the proof of the result in the case where the eigenfunction decomposition of $f$ does not involve $e_{\varepsilon, 0}$ or $e_{\varepsilon, 1}$.

**Step 2.** Now we consider the case where the function $g$ is collinear to $e_0$ and $e_1$, namely

$$g(t, y) = g_0(t) f_0(y) + g_1(t) f_1(y)$$

23
We extend the function $g$ to be equal to 0 when $t \leq t_0$, keeping the same notation. Given $t_1 > t_0$, we consider the equation

$$(\partial_t^2 + 2b^2 - \lambda_j) v_j = g_j$$

in $(-\infty, t_1)$ with boundary data $v_j(t_1) = \partial_t v_j(t_1) = 0$. The existence of $v_j$ is standard. We claim that

$$\sup_{(-\infty,t_1)} e^{-\mu t} |v_j| \leq c \sup_{R} e^{-\mu t} |g_j|$$

for some constant which does not depend on $t_1$, provided $\varepsilon$ is chosen small enough. As before, we argue by contradiction. Assume that the claim is not true, there would exist a sequence $(\varepsilon_n)_n$ tending to 0, sequences $(t_0,n)_n$ and $(t_1,n)_n$ such that $y_0,n \leq t_1,n$, a sequence of functions $(g_{j,n})_n$ and a sequence of solutions $(v_{j,n})_n$ such that

$$\sup_{(-\infty,t_1,n] \times S^1} e^{-\mu t} |g_{j,n}| = 1$$

and

$$\lim_{n \to +\infty} A_n := \sup_{(-\infty,y_1,n] \times S^1} e^{-\mu t} |v_{j,n}| = \infty$$

We denote by $(t_n, y_n) \in (-\infty, t_1, n] \times S^1$ a point where $A_n$ is achieved. Observe that, the solution $v_{j,n}$ is a linear combination of the two solutions of the homogeneous problem $L_{\varepsilon,j} w = 0$ and these are known to be at most linearly growing thanks to Jacobi fields coming from isometries. Hence the above supremum is achieved. We define the function $\tilde{v}_{j,n}$ by

$$\tilde{v}_{j,n}(t,y) = \frac{1}{A_n} e^{-\mu t_n} v_{j,n}(t_n + t, y).$$

As above, one shows that the sequence $(t_{1,n} - t_n)_n$ remains bounded away from 0.

Without loss of generality, we can assume that the sequence $(t_{1,n} - t_0)_n$ converges to $\bar{t}_1 \in (0, +\infty]$. We denote by $I = (-\infty, \bar{t}_1)$.

As in Step 1, we can also assume, without loss of generality that the sequence of functions $(\tilde{v}_{j,n})_n$ converges on compacts to a nontrivial function $\tilde{v}_j$ defined on $I \times S^1$. We now derive some properties of the limit function $\tilde{v}_j$.

First, $\tilde{v}_j(t, \cdot)$ is $\bar{v}$ is equal to 0 on $\{\bar{t}_1\} \times S^1$ if $\bar{t}_1 < +\infty$. Also

$$\sup_{I \times S^1} e^{-\mu t} |\tilde{v}_j| = 1$$

(34)
Finally, $\tilde{v}_j$ is either a solution of
\[
\left( \partial^2_t - j^2 + \frac{2}{\cosh^2(\cdot + \tilde{t})} \right) \tilde{v}_j = 0
\] (35)
for some $\tilde{t} \in \mathbb{R}$ or is a solution of
\[
(\partial^2_t - j^2) \tilde{v}_j = 0.
\] (36)

When $n = 0$, the solutions of (35) are linear combinations of the functions
\[
t \longrightarrow \tanh(t + \tilde{t}) \quad \text{and} \quad t \longrightarrow (t + \tilde{t}) \tanh(t + \tilde{t}) - 1
\]
and when $n = 1$ they are linear combinations of the following functions
\[
t \longrightarrow \frac{1}{\cosh(t + \tilde{t})} \quad \text{and} \quad t \longrightarrow \frac{(t + \tilde{t})}{\cosh(t + \tilde{t})} + \sinh(t + \tilde{t})
\]

Finally, when $n = 0$ the solutions of (36) are linear combinations of the functions
\[
t \longrightarrow 1 \quad \text{and} \quad t \longrightarrow t
\]
and when $n = 1$ they are linear combinations of the functions
\[
t \longrightarrow e^t \quad \text{and} \quad t \longrightarrow e^{-t}
\]

To reach a contradiction we observe that all the solutions of these two equations, when $j = 0, 1$ are explicitly known and that none of them satisfies (34) since we have chosen $\mu \in (-2, -1)$.

Since we have reached a contradiction, the proof of the claim is complete. Once the claim is proven, we pass to the limit as $t_1$ tends to $+\infty$ in a sequence of solutions which are defined on $(-\infty, t_1] \times S^1$. This proves the existence of a solution of
\[
L_{\varepsilon,j} v_j = g_j
\]
which is defined in $[t_0, +\infty) \times S^1$. In addition, we know that
\[
\sup_{[t_0, +\infty) \times S^1} e^{-\mu t} |v_j| \leq c \sup_{[t_0, +\infty) \times S^1} e^{-\mu t} |g_j|
\]
The estimate for the derivatives follow from standard elliptic estimates. \hfill \Box

The following result is standard and left to the reader (a proof can be found in [4]).
Lemma 4.3. There exists an operator

\[ P : C^{2, \alpha}(S^1) \to C^{2, \alpha}_{-2}([0, +\infty) \times S^1), \]

such that for all \( \varphi \in C^{2, \alpha}(S^1) \), with \( \varphi \) orthogonal to 1 and \( \theta \to \cos \theta \) in the \( L^2 \)-sense and is an even function of \( y \in S^1 \) the function \( w = P(\varphi) \) solves

\[
\begin{cases}
(\partial^2_t + \partial^2_\theta) w = 0 & \text{in } [0, +\infty) \times S^1 \\
 w = \varphi & \text{on } \{0\} \times S^1
\end{cases}
\]

Moreover,

\[ \|P(\varphi)\|_{C^{2, \alpha}_{-2}} \leq c \|\varphi\|_{C^{2, \alpha}}, \]

for some constant \( c > 0 \).

5 An infinite dimensional family of minimal surfaces which are close to a half Riemann surface

In this section we are interested in minimal surfaces which are close to a half of Riemann’s surface and have prescribed boundary. We consider surfaces which are normal graphs over Riemann’s surface. More precisely, we consider the surface parameterized by

\[ Z_{\varepsilon, u} := X_{\varepsilon} + u N_{\varepsilon}, \]

where \( X_{\varepsilon} \) and \( N_{\varepsilon} \) have been defined in (14) and (16) and, in the following Proposition, we give an expansion of the mean curvature operator for this surface in terms of the function \( u \) and its partial derivatives.

Proposition 5.1. The surface parameterized by \( Z_{\varepsilon, u} := X_{\varepsilon} + u N_{\varepsilon} \) is minimal if and only if the function \( u \) is a solution of

\[
L_{\varepsilon} u = (\cosh \varphi)^2 Q_{\varepsilon} \left( \frac{u}{\cosh \varphi}, \frac{\nabla u}{\cosh \varphi}, \frac{\nabla^2 u}{\cosh \varphi} \right)
\]

where \( L_{\varepsilon} \) is the operator which has already been defined in (28) and the nonlinear operator \( Q_{\varepsilon} \) satisfies

\[
|Q_{\varepsilon}(v_2) - Q_{\varepsilon}(v_1)|_{C^{0, \alpha}((t, t+1) \times S^1(\tau_{\varepsilon}))} \leq c \sup_{i=1,2} |v_i|_{C^{2, \alpha}((t, t+1) \times S^1(\tau_{\varepsilon}))} |v_2 - v_1|_{C^{2, \alpha}((t, t+1) \times S^1(\tau_{\varepsilon}))}
\]

for all \( v_1, v_2 \) such that \( |v_i|_{C^{2, \alpha}((t, t+1) \times S^1(\tau_{\varepsilon}))} \leq 1 \). Here the constant \( c > 0 \) does not depend on \( t \in \mathbb{R} \), nor on \( \varepsilon \in (0, 1) \).
**Proof:** We omit the indices \( \varepsilon \) and \( u \) for the sake of simplicity in the notations. We have
\[
\partial_t Z = \partial_t X + \partial_t u N + u \partial_t N \quad \text{and} \quad \partial_y Z = \partial_y X + \partial_y u N + u \partial_y N
\]
A simple computation shows that the coefficients of \( g_u \), the first fundamental form of the surface parameterized by \( Z_u \), are given by
\[
|\partial_t Z|^2 = \cosh^2 \varphi + 2b \cosh \varphi u + (\partial_t u)^2 + (a^2 + b^2) u^2
\]
\[
\partial_t Z \cdot \partial_y Z = 2a \cosh \varphi u + \partial_t u \partial_y u
\]
\[
|\partial_y Z|^2 = \cosh^2 \varphi - 2b \cosh \varphi u + (\partial_y u)^2 + (a^2 + b^2) u^2
\]
Collecting these, we have the expansion of the determinant of \( g_u \)
\[
|g_u| = \cosh^4 \varphi \left( 1 + \frac{1}{\cosh^2 \varphi} \left( |\partial_t u|^2 + |\partial_y u|^2 - 2 (a^2 + b^2) u^2 \right) \right)
+ P_3 \left( \frac{u}{\cosh \varphi}, \frac{\nabla u}{\cosh \varphi} \right) + P_4 \left( \frac{u}{\cosh \varphi}, \frac{\nabla u}{\cosh \varphi} \right)
\]
where \( P_i \) has coefficients which are bounded independently of \( \varepsilon \in (0,1) \) are is homogeneous of degree \( i \). Here we have implicitly used the fact that the functions \( a \) and \( b \) are uniformly bounded when \( \varepsilon \in (0,1) \).

We consider the area energy
\[
A(u) := \int \sqrt{|g_u|} dt \, dy
\]
and the surface parameterized by \( Z \) will be minimal if and only if the first variation if 0. This can be written as
\[
2 DA_u(v) = \int \frac{1}{\sqrt{|g_u|}} D_u |g_u| (v) dt \, dy
\]
Observe that
\[
\frac{1}{\sqrt{|g_u|}} D|g_u|_u (v) = 2 (\partial_t u \partial_t v + \partial_y u \partial_y v - 2 (a^2 + b^2) u v)
+ \cosh \varphi Q \left( \frac{u}{\cosh \varphi}, \frac{\nabla u}{\cosh \varphi} \right) v + \cosh \varphi \hat{Q} \left( \frac{u}{\cosh \varphi}, \frac{\nabla u}{\cosh \varphi} \right) \partial_t v
+ \cosh \varphi \hat{Q} \left( \frac{u}{\cosh \varphi}, \frac{\nabla u}{\cosh \varphi} \right) \partial_y v
\]
(37)
where the operator $Q$, $\tilde{Q}$ and $\hat{Q}$ enjoy properties similar to the one enjoyed by $Q_\varepsilon$ in the statement of the result.

The result then follows at once provided one notices that

$$|\partial_t \cosh \phi| + |\partial_y \cosh \phi| \leq (|a| + |b|) \cosh \phi \leq c \cosh^2 \phi,$$

for some constant $c > 0$ which does not depend on $\varepsilon \in (0, 1)$. This explains the $\cosh^2 \phi$ in front of the nonlinearity $Q_\varepsilon$ whereas (37) would only suggest a $\cosh \phi$. □

We consider the surface parameterized by $X_\varepsilon$ which we first dilate by a factor $(1 + \gamma)$ and then translate by $\frac{\varsigma}{1 + \gamma}$ along the $x_1$ axis and by $(1 + \gamma) \log(1 + \gamma) + \sigma$ along the $x_3$-axis. This surface, which will be referred to as $R_\varepsilon(\gamma, \sigma, \varsigma)$ is parameterized by

$$(1 + \gamma) X_\varepsilon + \frac{\varsigma}{1 + \gamma} e_1 + ((1 + \gamma) \log(1 + \gamma) + \sigma) e_3.$$  (38)

The parameters $\gamma$ and $\varsigma$ are now chosen to satisfy

$$|\gamma| + \varepsilon^{1/2} |\varsigma| \leq \kappa \varepsilon$$

for some constant $\kappa > 0$ which will be fixed later on.

Using the result of Lemma 3.3, we see that part of this surface (basically the one at height $t \sim -\frac{1 + \gamma}{2} \log \varepsilon$ is a graph over the annulus $B_{4\varepsilon - 1/2} - B_{\varepsilon - 1/4}$ in the $x_3 = 0$ plane for the function

$$(r, \theta) \rightarrow (1 + \gamma) \log(2r) + \sigma - \frac{\varepsilon}{2} r \cos \theta - \frac{\varsigma}{r} \cos \theta + O(\varepsilon)$$  (39)

which has been expanded in (25). We now truncate the surface $R_\varepsilon(\gamma, \sigma, \varsigma)$ at the graph of the curve $r = \frac{1}{2} \varepsilon^{-1/2}$ by the function defined in (39) and consider only the upper half of this surface which we will refer to as $R^l_\varepsilon(\gamma, \sigma, \varsigma)$.

We are interested in normal graphs over the surface $R^l_\varepsilon(\gamma, \sigma, \varsigma)$ which are minimal surfaces and are asymptotic to $R^l_\varepsilon(\gamma, \sigma, \varsigma)$. Thanks to Proposition 5.1, we can state that the surface parameterized by

$$(1 + \gamma) (X_\varepsilon + u N_\varepsilon) + \frac{\varsigma}{1 + \gamma} e_1 + ((1 + \gamma) \log(1 + \gamma) + \sigma) e_3$$

is minimal, if and only if the function $u$ is a solution of

$$L_\varepsilon u = (1 + \gamma)^{-1} \cosh^2 \phi Q_\varepsilon \left( (1 + \gamma) \frac{u}{\cosh \phi}, (1 + \gamma) \frac{\nabla u}{\cosh \phi}, (1 + \gamma) \frac{\nabla^2 u}{\cosh \phi} \right)$$  (40)
The fact that the perturbed surface is asymptotic to the non-perturbed one can then be translated into the fact that the function $u$ tends to 0 at $\infty$.

We set
\[
\tilde{t}_\varepsilon := -\frac{1}{2} \log \varepsilon
\]

Two modifications are now required. First, even though the surface $R^l(\gamma, \sigma, \varsigma)$ can be parameterized by (38), its boundary does not correspond to the curve $t = \tilde{t}_\varepsilon$. We therefore modify the above parametrization so that the part of $R^l(\gamma, \sigma, \varsigma)$ corresponding to $t \geq \tilde{t}_\varepsilon + \log 4$ is still parameterized by (38), while over the graph over the annulus $B_{2\varepsilon - \frac{1}{2}} - B_{\varepsilon - \frac{1}{2}}$ the function defined in (39), we change coordinates
\[
(r, \theta) = \left( \frac{1}{2} e^t, \tau_\varepsilon y \right)
\]

Finally, we interpolate (smoothly) between the two parameterizations in the graph over the annulus $B_{3\varepsilon - \frac{1}{2}} - B_{\varepsilon - \frac{1}{2}}$ by the function (39).

The next modification we need to do is concerned with the normal vector field about $R^l(\gamma, \sigma, \varsigma)$ since we would like this vector field to be vertical near the boundary of this surface. This can be achieved by modifying the normal vector field into a transverse vector field $\tilde{N}_\varepsilon$ which agrees with the unit normal vector field $N_\varepsilon$ for all $t \geq \tilde{t}_\varepsilon + \log 4$ and which agrees with $e_3$ for all $t \in [\tilde{t}_\varepsilon, \tilde{t}_\varepsilon + \log 2]$.

Now, we consider a graph over this surface for some function $u$, using the modified vector field $\tilde{N}_\varepsilon$. This graph will be minimal if and only if the function $u$ is a solution of some nonlinear elliptic equation which is not exactly equal to (40) because of the above two modifications. Indeed, starting from (40) and taking into account the effects of the change of parametrization and the change in the vector field $N_\varepsilon$ into $\tilde{N}_\varepsilon$, we see that the minimal surface equation now reads
\[
L_\varepsilon u = \tilde{L}_\varepsilon u + \cosh^2 \phi \tilde{Q}_\varepsilon \left( \frac{u}{\cosh \phi}, \frac{\nabla u}{\cosh \phi}, \frac{\nabla^2 u}{\cosh \phi} \right)
\]

The nonlinear operator $\tilde{Q}_\varepsilon$ enjoys the same properties as $Q_\varepsilon$ in Proposition 5.1. We will write for short
\[
\tilde{Q}_\varepsilon(u) := \tilde{Q}_\varepsilon \left( \frac{u}{\cosh \phi}, \frac{\nabla u}{\cosh \phi}, \frac{\nabla^2 u}{\cosh \phi} \right).
\]

Observe that $\tilde{Q}_\varepsilon$ is explicitly given by
\[
\tilde{Q}_\varepsilon(\cdot) = (1 + \gamma)^{-1} Q_\varepsilon((1 + \gamma) \cdot)
\]
when \( t \geq \tilde{t}_\varepsilon + \log 4 \).

The operator \( \tilde{L}_\varepsilon \) is a linear second order operator whose coefficients are supported in \([\tilde{t}_\varepsilon, \tilde{t}_\varepsilon + \log 4] \times S^1(\tau_\varepsilon)\) and are bounded by a constant times \( \varepsilon^{1/2} \), in \( C^\infty \) topology, where partial derivatives are computed with respect to the vector fields \( \partial_t \) and \( \partial_y \). Let us briefly comment on the estimate of the coefficients of \( \tilde{L}_\varepsilon \). If we were only taking into account the effect of the change from \( N_\varepsilon \) into \( \tilde{N}_\varepsilon \), we would obtain, applying the result of Appendix B, a similar formula where the coefficients of the corresponding operator \( \tilde{L}_\varepsilon \) are bounded by a constant times \( \varepsilon \) since \( \tilde{N}_\varepsilon \cdot N_\varepsilon = 1 + \mathcal{O}(\varepsilon) \) when \( t \in [\tilde{t}_\varepsilon, \tilde{t}_\varepsilon + \log 2] \). If we were only taking into account the effect of the change in the parametrization, we would obtain a similar formula where the coefficients of the corresponding operator \( \tilde{L}_\varepsilon \) are bounded by a constant times \( \varepsilon^{1/2} \), this basically follows from (25) which shows that

\[
U^0_t(r, \theta) = \log(2r) + \mathcal{O}(\varepsilon^{1/2}),
\]

where the change of coordinates takes place. The estimate of the coefficients of \( \tilde{L}_\varepsilon \) follows from these considerations.

Now, assume that we are given a function \( \varphi \in C^{2,\alpha}(S^1) \) which is even with respect to \( y \), \( L^2 \)-orthogonal to 1 and \( y \rightarrow \cos(y) \) and which satisfies

\[
\|\varphi\|_{C^{2,\alpha}} \leq \kappa \varepsilon.
\]

We set

\[
w_\varphi(\cdot, \cdot) := \mathcal{P}(\varphi)(\cdot - \tilde{t}_\varepsilon, /\tau_\varepsilon)
\]

In order to solve (41), we choose

\[
\mu \in (-2, -1)
\]

and look for \( u \) of the form

\[
u = w_\varphi + v
\]

where \( v \in C^{2,\alpha}_\mu([\tilde{t}_\varepsilon, \infty) \times S^1(\tau_\varepsilon)) \). Using the result of Proposition 4.1 we can rephrase this problem as a fixed point problem

\[
v = S(v)
\]

where the nonlinear mapping \( S(= S_{\varepsilon, T_1, \varphi}) \) (which depends on \( \varepsilon, \gamma, T_1 \) and \( \varphi \)) is defined by

\[
S(v) := G_{\varepsilon, \tilde{t}_\varepsilon} \left( \tilde{L}_\varepsilon(w_\varphi + v) - L_\varepsilon w_\varphi + \cosh^2 \phi \tilde{Q}_\varepsilon(w_\varphi + v) \right).
\]
where the operator $G_{\varepsilon, \hat{t}_\varepsilon}$ is the one defined in Proposition 4.1. The existence of a fixed point of (42) is an easy consequence of the following technical:

**Lemma 5.1.** There exist constants $c_\kappa > 0$ and $\varepsilon_\kappa > 0$, such that

$$\|S(0)\|_{C_{\mu}^{2, \alpha}} \leq c_\kappa (\varepsilon^{(3+\mu)/2} + \varepsilon^{4+2\mu})$$

(43)

and, for all $\varepsilon \in (0, \varepsilon_\kappa)$

$$\|S(v_2) - S(v_1)\|_{C_{\mu}^{2, \alpha}} \leq \frac{1}{2} \|v_2 - v_1\|_{C_{\mu}^{2, \alpha}}$$

for all $v_1, v_2 \in C_{\mu}^{2, \alpha}([\hat{t}_\varepsilon, \infty) \times S^1(\tau_\varepsilon))$ such that $\|v_i\|_{C_{\mu}^{2, \alpha}} \leq 2 c_\kappa (\varepsilon^{(3+\mu)/2} + \varepsilon^{4+2\mu})$.

**Proof:** Using the properties of $w_\varphi$ given in Lemma 4.3 together with the properties of $\hat{L}_\varepsilon$, we immediately get

$$\|\hat{L}_\varepsilon (w_\varphi)\|_{C_{\mu}^{0, \alpha}} \leq c_\kappa \varepsilon^{(3+\mu)/2}$$

Next, we use the fact that

$$L_\varepsilon w_\varphi = 2(a^2 + b^2) w_\varphi$$

However, we have proved in (22) that $a^2 \leq \varepsilon^2$. Furthermore, $b^2$ is an even function and, thanks to Remark 3.1, we know that $b^2 \leq c (\cosh t)^{-2}$ for all $t \in [0, t_\varepsilon]$ and some constant $c > 0$ independent of $\varepsilon$ small enough. Therefore, we conclude (with little work) that

$$\|L_\varepsilon w_\varphi\|_{C_{\mu}^{0, \alpha}} \leq c_\kappa (\varepsilon^{2+\mu/2} + \varepsilon^{4+2\mu})$$

Observe that the norm on the left hand side is achieved when $t \sim 2 t_\varepsilon$.

While the last term is easily estimated by

$$\|\cosh^2 \varphi \hat{Q}_\varepsilon (w_\varphi)\|_{C_{\mu}^{0, \alpha}} \leq c_\kappa \varepsilon^{2+\mu/2}$$

This completes the proof of the first estimate. The second estimate follows from similar considerations and is left to the reader.

The previous Lemma shows that, provided $\varepsilon$ is chosen small enough, the nonlinear mapping $S$ is a contraction mapping from the ball of radius $2 c_\kappa (\varepsilon^{(3+\mu)/2} + \varepsilon^{4+2\mu})$ in $C_{\mu}^{2, \alpha}([\hat{t}_\varepsilon, \infty) \times S^1(\tau_\varepsilon))$ into itself. Consequently $S$ has a unique fixed point $v$ in this ball. This provides a minimal surface $R^t_\varepsilon(\gamma, \sigma, \varsigma, \varphi)$ which is asymptotic to a half Riemann
surface \( R^t_\varepsilon(\gamma, \sigma, \varsigma) \). Observe that near its boundary, this surface is a vertical graph over the annulus \( B_{\varepsilon-1/2} - B_{\varepsilon-1/2}/2 \) for some function \( U_t \) which can be expanded as

\[
U_t(r, \theta) = (1 + \gamma) \log(2r) + \sigma - \frac{\varepsilon}{2} r \cos \theta - \frac{\varsigma}{r} \cos \theta + \mathcal{P}(\varphi)(\log(2r) - \tilde{t}_\varepsilon, \theta) + V_t(r, \theta)
\]

in which case the boundary of the surface corresponds to \( r = \frac{1}{2} \varepsilon^{-1/2} \). Here the function \( V_t = V_t(\varepsilon, \gamma, \varsigma, \varphi) \) depends nonlinearly on \( \gamma, \varsigma \) and \( \varphi \) and satisfies the following

\[
\|V_t(\varepsilon, \gamma, \varsigma, \varphi)\|_{C^{2,\alpha}_b} \leq c \varepsilon
\]

and

\[
\|V_t(\varepsilon, \gamma, \varsigma, \varphi) - V_t(\varepsilon, \gamma, \varsigma, \varphi')\|_{C^{2,\alpha}_b} \leq c \left( \varepsilon^{1/2} + \varepsilon^{3+3\mu/2} \right) \|\varphi - \varphi'\|_{C^{2,\alpha}}
\]

where the constant \( c > 0 \) does not depend on \( \varepsilon \) or \( \kappa \) and \( c_\kappa \) only depends on \( \kappa \) but not on \( \varepsilon \). The space \( C^{2,\alpha}_b \) is the space of \( C^{2,\alpha} \) function where partial derivatives are taken with respect to the vector fields \( r \partial_r \) and \( \partial_\theta \).

A similar analysis can be carried over starting from the lower end of Riemann surface to obtain a minimal surface, which will be referred to as \( R^b_\varepsilon(\gamma, \sigma, \varsigma, \varphi) \), which is asymptotic to a half Riemann surface and which, near its boundary is a vertical graph over the annulus \( B_{\varepsilon-1/2} - B_{\varepsilon-1/2}/2 \) for some function \( U_b \) which can be expanded as

\[
U_b(r, \theta) = -(1 + \gamma) \log(2r) - \sigma - \frac{\varepsilon}{2} r \cos \theta + \frac{\varsigma}{r} \cos \theta + \mathcal{P}(\varphi)(\log(2r) - \tilde{t}_\varepsilon, \theta) + V_b(r, \theta)
\]

in which case the boundary of the surface corresponds to \( r = \frac{1}{2} \varepsilon^{-1/2} \). The function \( V_b \) satisfies exactly the same properties as the function \( V_t \). Equivalently one can apply a rotation of angle \( \pi \) about the \( x_2 \)-axis to the surface \( R^t_\varepsilon(\gamma, \sigma, \varsigma, \tilde{\varphi}) \), where \( \tilde{\varphi}(\cdot) := -\varphi(\cdot + \pi) \).

### 6 An infinite dimensional family of minimal surfaces which are close to \( M_k \)

We perform an analysis close to the one performed in the previous section, starting this time from the minimal surface \( M_k(\xi) \) defined in Section 2, for \( \xi \) small enough. Recall that the surface \( M_k(\xi) \) has two ends \( E_t(\xi) \) and \( E_b(\xi) \) which can be parameterized as in \([10]\) and \([10]\). Also recall that, according to the result of Lemma 2.1, a portion of these ends can be written as a graph over the \( x_3 = 0 \) plane for functions \( U_{\xi,t} \) and \( U_{\xi,b} \) which are defined in the annulus \( B_{\varepsilon-1/2} - B_{\varepsilon-1/2}/4 \).
Recall that we have defined
\[ \tilde{t}_\varepsilon = -\frac{1}{2} \log \varepsilon \]
As in the previous section, we modify the parametrization of the end \( E_t(\xi) \) which is given by (9), say when \( s \in [\tilde{t}_\varepsilon - \log 8, \tilde{t}_\varepsilon + \log 4] \), so that, when \( r \in [\varepsilon^{-1/2}/4, 2 \varepsilon^{-1/2}] \) the curve corresponding to the image of
\[ \theta \mapsto (r \cos \theta, r \sin \theta, U_t, \xi(r, \theta)) \]
corresponds to the curve \( s = \log(2r) \). We perform a similar task for the parametrization of \( E_b(\xi) \) so that, when \( r \in [\varepsilon^{-1/2}/4, 2 \varepsilon^{-1/2}] \) the curve corresponding to the image of
\[ \theta \mapsto (r \cos \theta, r \sin \theta, U_b, \xi(r, \theta)) \]
corresponds to the curve \( s = -\log(2r) \).

This being understood, as in the previous section, we modify the unit normal vector field on \( M_k(\xi) \) to produce a transverse unit vector field \( \tilde{N}_\xi \) which coincides with the normal vector field \( N_\xi \) on \( M_k(\xi) \), is equal to \( e_3 \) on the graph over \( B_{2\varepsilon^{-1/2}} - B_{3\varepsilon^{-1/2}/8} \) of the functions \( U_t, \xi \) and \( U_b, \xi \) and interpolate smoothly in between the different definitions of \( \tilde{N}_\varepsilon \) in different subsets of \( M_k(\xi) \).

A graph of the function \( u \), using the vector field \( \tilde{N}_\xi \), will be a minimal surface if and only if \( u \) is a solution of a second order nonlinear elliptic equation of the form
\[ \mathbb{L}_{M_k(\xi)} u = \tilde{L}_{\xi,\varepsilon} u + Q_{\xi,\varepsilon}(u) \]
where \( \mathbb{L}_{M_k(\xi)} \) is the Jacobi operator about \( M_k(\xi) \), \( Q_{\xi,\varepsilon} \) is a nonlinear second order differential operator which collects all the nonlinear terms and \( \tilde{L}_{\xi,\varepsilon} \) is a linear operator which take into account the change of parametrization and the change of the normal vector field \( N_\xi \) into \( \tilde{N}_\xi \), which are described above. Now, we can be more precise and, at the ends \( E_t(\xi) \) and \( E_b(\xi) \). For example at \( E_t(\xi) \), and granted the above parametrization, the nonlinear operator \( Q_{\xi,\varepsilon} \) can be expanded as
\[ Q_{\xi,\varepsilon}(u) = Q_{2,\xi,\varepsilon} \left( \frac{w}{\cosh s} \right) + \cosh s Q_{3,\xi,\varepsilon} \left( \frac{w}{\cosh s} \right) \]
where \( Q_{2,\xi,\varepsilon} \) and \( Q_{3,\xi,\varepsilon} \) are nonlinear second order differential operators which satisfy (8), uniformly in \( \xi \) and \( \varepsilon \).

The operator \( \tilde{L}_{\xi,\varepsilon} \) is a linear operator which is supported in \([\tilde{t}_\varepsilon - \log 8, \tilde{t}_\varepsilon + \log 4] \times S^1\) and has coefficients which are bounded by a constant times \( \varepsilon^{3/2} \), uniformly in \( \xi \) and \( \varepsilon \).
(The rational for this estimate is that \( \varepsilon^{3/2} = \varepsilon^{1/2} \), the first \( \varepsilon \) comes from the conformal factor \( (\cosh s)^{-2} \) and the \( \varepsilon^{1/2} \) comes from the modification in the parametrization and the vector field as in the previous section).

Finally, observe that still in \( E_t(\xi) \) the difference

\[
(cosh s)^4 \left( \mathbb{L}_{M_k}(\xi) - \frac{1}{\cosh^2 s} (\partial_s^2 + \partial_\theta^2) \right)
\]

is a second order differential operator in \( \partial_s \) and \( \partial_\theta \) whose coefficients are bounded in \( C^\infty \) topology (where partial derivatives are taken with respect to the vector fields \( \partial_s \) and \( \partial_\theta \)) uniformly in \( \xi \) and \( \varepsilon \). All these facts follow from the expansion provided in (2).

Now, assume that we are given two functions \( \varphi_t, \varphi_b \in C^2(S^1) \) which is even with respect to \( \theta \) and \( L^2 \) orthogonal to \( 1 \) and \( \theta \rightarrow \cos \theta \) and satisfy

\[
\|\varphi_t\|_{C^2,\alpha} + \|\varphi_b\|_{C^2,\alpha} \leq \kappa \varepsilon.
\]

We set \( \Phi := (\varphi_t, \varphi_b) \) and we define \( w_\Phi \) to be the function which is equal to \( \chi_t \mathcal{P}(\varphi_t)(\cdot - t\varepsilon, \cdot) \) on the image of \( X_{t,\xi} \) where \( \chi_t \) is a cutoff function equal to 0 for \( s \leq s_0 + 1 \) and identically equal to 1 for \( s \geq s_0 + 2 \), and is equal to \( \chi_b \mathcal{P}(\varphi_b)(\cdot + t\varepsilon, \cdot) \) on the image of \( X_{b,\xi} \) where \( \chi_b \) is a cutoff function equal to 0 for \( s \geq -s_0 - 1 \) and identically equal to 1 for \( s \leq -s_0 - 2 \).

We define \( M_k(\xi, \varepsilon) \) to be equal to \( M_k(\xi) \) with the image of \( (\tilde{t}_\varepsilon, +\infty) \times S^1 \) by \( X_{t,\xi} \) and the image of \( (-\infty, -\tilde{t}_\varepsilon) \times S^1 \) by \( X_{b,\xi} \) removed. We would like to solve the equation

\[
\mathbb{L}_{M_k}(\xi)(w_\Phi + v) = \tilde{L}_{\xi,\varepsilon}(w_\Phi + v) + Q_{\xi,\varepsilon}(w_\Phi + v)
\]

on \( M_k(\xi, \varepsilon) \), so that the graph of \( w_\Phi + v \) will be a minimal surface.

We choose

\[
\delta \in (1, 2)
\]

and use the result of Proposition 2.3 so that we can rephrase the above problem as a fixed point problem

\[
v = T(v)
\]

where

\[
T(v) = G_{\xi,\delta} \circ \mathcal{E}_\varepsilon \left( \tilde{L}_{\xi,\varepsilon}(w_\Phi + v) + \mathbb{L}_{M_k}(\xi) w_\Phi + Q_{\xi,\varepsilon}(w_\Phi + v) \right)
\]

where \( \mathcal{E}_\varepsilon \) is an extension (linear) operator

\[
\mathcal{E}_\varepsilon : C^0_{\delta-2,\alpha}(M_k(\xi, \varepsilon)) \rightarrow C^0_{\delta-2,\alpha}(M_k(\xi)),
\]

34
defined by $E_\varepsilon v = v$ in $M_k(\xi, \varepsilon)$, $E_\varepsilon v = 0$ on the image of $[\tilde{\ell}_\varepsilon + 1, +\infty) \times S^1$ by $X_{t,\xi}$ and the image of $(-\infty, -\tilde{\ell}_\varepsilon - 1] \times S^1$ by $X_{b,\xi}$ and $E_\varepsilon v$ interpolate between these so that, for example,

$$(E_\varepsilon v) \circ X_{t,\xi}(t, \theta) = (1 + \tilde{\ell}_\varepsilon - t) v \circ X_{t,\xi}(\tilde{\ell}_\varepsilon, \theta)$$

for $(t, \theta) \in [\tilde{\ell}_\varepsilon, \tilde{\ell}_\varepsilon + 1] \times S^1$. Here $C^{2,2}_\delta(M_k(\xi))$ is the space of restrictions of elements of $C^{0,2}_\delta(M_k(\xi))$ to $M_k(\xi, \varepsilon)$, endowed with the induced norm.

As in Section 5, the existence of a fixed point $v \in C^{2,2}_\delta(M_k(\xi))$ for (45) follows at once from the technical:

**Lemma 6.1.** There exist constants $c_\kappa > 0$ and $\varepsilon_\kappa > 0$, such that

$$\|T(0)\|_{C^{2,2}_\delta} \leq c_\kappa \varepsilon^2$$

(46)

and, for all $\varepsilon \in (0, \varepsilon_\kappa)$

$$\|T(v_2) - T(v_1)\|_{C^{2,2}_\delta} \leq \frac{1}{2} \|v_2 - v_1\|_{C^{2,2}_\delta}$$

for all $v_1, v_2 \in C^{2,2}_\delta(M_k(\xi))$ and satisfying $\|v\|_{C^{2,2}_\delta} \leq 2 c_\kappa \varepsilon^2$.

**Proof.** The proof is similar to the one in the proof of Lemma 5.1. Again, we use the result of Lemma 4.3 to obtain the estimate

$$\|E_\varepsilon(L_{M_k(\xi)} w\Phi)\|_{C^{0,2}_\delta} \leq c_\kappa \varepsilon^2$$

and, using the properties of $\tilde{L}_{\xi,\varepsilon}$, we obtain

$$\|E_\varepsilon(\tilde{L}_{\xi,\varepsilon} w\Phi)\|_{C^{0,2}_\delta} \leq c_\kappa \varepsilon^{\frac{3+\delta}{4}}$$

Finally, we have

$$\|E_\varepsilon(Q_{\xi,\varepsilon}(w\Phi))\|_{C^{0,2}_\delta} \leq c_\kappa \varepsilon^{3+\delta/4}$$

We leave the details to the reader. $\square$

The previous Lemma shows that, provided $\varepsilon$ is chosen small enough, the nonlinear mapping $T$ is a contraction mapping from the ball of radius $2 c_\kappa \varepsilon^2$ in $C^{2,2}_\delta(M_k(\xi))$ into itself. Consequently $T$ has a unique fixed point $v$ in this ball. This provides a minimal
surface $M_k(\xi, \varepsilon, \varphi_t, \varphi_b)$ which is close to $M_k(\xi, \varepsilon)$, has one horizontal end and two boundaries. This surface is, close to its upper boundary, a vertical graph over the annulus $B_{\varepsilon^{-1/2}} - B_{\varepsilon^{-1/2}/4}$ for some function $U_t$ which can be expanded as

$$U_t(r, \theta) = \sigma_t, \xi + \log(2r) + \xi r \cos \theta + P(\varphi_t)(\tilde{t}_\varepsilon - \log(2r), \theta) + \tilde{V}_t(r, \theta)$$

and this surface is, close to its lower boundary, a vertical graph over the annulus $B_{\varepsilon^{-1/2}} - B_{\varepsilon^{-1/2}/4}$ for some function $U_b$ which can be expanded as

$$U_b(r, \theta) = -\sigma_b, \xi - \log(2r) + \xi r \cos \theta + P(\varphi_b)(\tilde{t}_\varepsilon - \log(2r), \theta) + \tilde{V}_b(r, \theta)$$

where $\tilde{V}_t = \tilde{V}_t(\xi, \Phi)$ and $\tilde{V}_b = \tilde{V}_b(\xi, \Phi)$ depend nonlinearly on $\varepsilon, \xi$ and $\Phi$ and satisfy

$$\|\tilde{V}(\varepsilon, \Phi)\|_{C^2} \leq c \varepsilon$$

and

$$\|\tilde{V}(\varepsilon, \Phi) - \tilde{V}(\varepsilon, \Phi')\|_{C^2} \leq c \varepsilon^{1 - \delta/2} \|\Phi - \Phi'\|_{C^2} \quad (47)$$

where the constant $c > 0$ does not depend on $\varepsilon$ or $\kappa$ and $c_\kappa$ only depends on $\kappa$ but not on $\varepsilon$. The boundaries of the surface corresponds to $r = \frac{1}{2} \varepsilon^{-1/2}$.

7 The matching of Cauchy data and the proof of the main result

We collect the results we have obtained in Section 5 and Section 6. In Section 5, we have obtained two surfaces which are perturbations of the upper (rep. the lower end) of Riemann’s surface. The first surface

$$R_{\varepsilon^t}^{t} (\gamma_t, \sigma_t, \xi + \sigma_t, \varsigma_t, \varphi_t)$$

depends on the parameters $\eta_t, \gamma_t, \sigma_t, \varsigma_t$ and the function $\varphi_t$ and can be parameterized, close to its boundary as the vertical graph of

$$U_t(r, \theta) := (1 + \gamma_t) \log(2r) + \sigma_t, \xi + \sigma_t - \frac{\varepsilon + \eta_t}{2} r \cos \theta - \frac{\varsigma_t}{r} \cos \theta + P(\varphi_t)(\log r - \tilde{t}_\varepsilon, \theta) + O(\varepsilon)$$

The second surface

$$R_{\varepsilon^b}^{b} (\gamma_b, \sigma_b, \xi + \sigma_b, \varsigma_b, \varphi_b)$$

36
depends on the parameters \( \eta_b, \gamma_b, \sigma_b, \varsigma_b \) and the function \( \varphi_b \) and can be parameterized, close to its boundary as the vertical graph of

\[
U_b(r, \theta) := -(1 + \gamma_b) \log(2r) - \sigma_b \xi - \sigma_b - \frac{\varepsilon + \eta_b}{2} r \cos \theta + \frac{\varsigma_b}{r} \cos \theta + \mathcal{P}(\varphi_b)(\log r - \tilde{t}_\varepsilon, \theta) + \mathcal{O}(\varepsilon)
\]

Now, collecting the result of Section 6, we have a surface

\[
M_k(\xi, \varepsilon, \tilde{\varphi}_b, \tilde{\varphi}_t)
\]

which has two boundaries, one end asymptotic to a horizontal plane and can be parameterized, close to its upper boundary as the vertical graph of

\[
\bar{U}_t(r, \theta) := \log(2r) + \sigma_t \xi + \xi r \cos \theta + \mathcal{P}(\tilde{\varphi}_t)(\tilde{t}_\varepsilon - \log r, \theta) + \mathcal{O}(\varepsilon)
\]

while it can be parameterized close to its lower boundary as the vertical graph of

\[
\bar{U}_b(r, \theta) := -\log(2r) - \sigma_b \xi + \xi r \cos \theta + \mathcal{P}(\tilde{\varphi}_b)(\tilde{t}_\varepsilon - \log r, \theta) + \mathcal{O}(\varepsilon)
\]

We set

\[
\xi = -\frac{\varepsilon}{2}
\]

and assume that the parameters and the boundary functions are chosen so that

\[
e^{-1/2} (|\eta_t| + |\eta_b|) + \varepsilon^{1/2} (|\gamma_t| + |\gamma_b|) + |\log \varepsilon|^{-1} (|\sigma_t| + |\sigma_b|)
\]

\[
+ |\gamma_t| + |\gamma_t| + \|\varphi_t\|_{C^2,\alpha} + \|\varphi_b\|_{C^2,\alpha} + \|\tilde{\varphi}_t\|_{C^2,\alpha} + \|\tilde{\varphi}_b\|_{C^2,\alpha} \leq \kappa \varepsilon
\]

where the constant \( \kappa > 0 \) is fixed large enough. Recall that the functions \( \varphi_t, \varphi_b, \tilde{\varphi}_t \) and \( \tilde{\varphi}_b \) are assumed to be even and \( L^2 \) orthogonal to the functions \( 1 \) and \( \theta \to \cos \theta \).

The functions \( \mathcal{O}(\varepsilon) \) do depend nonlinearly on the different parameters and boundary data functions but are bounded by a constant (independent of \( \kappa \) and \( \varepsilon \)) times \( \varepsilon \) in \( C^{2,\alpha} \) topology, when partial derivatives are taken with respect to the vector fields \( r \partial_r \) and \( \partial_\theta \).

It remains to show that, for all \( \varepsilon \) small enough, it is possible to choose the parameters and boundary functions in such a way that the surface

\[
R_{\varepsilon+\eta_t}(\gamma_t, \sigma_t \xi + \sigma_t, \varsigma_t, \varphi_t) \cup M_k(\xi, \varepsilon, \tilde{\varphi}_b, \tilde{\varphi}_t) \cup R_{\varepsilon+\eta_b}(\gamma_b, \sigma_b \xi + \sigma_b, \varsigma_b, \varphi_b)
\]

is a \( C^1 \) surface across the boundaries of the different summands. Regularity theory will then ensure that this surface is in fact smooth and by construction is has the desired properties. This will therefore complete the proof of the main theorem.
Granted the description of the surfaces close to their respective boundaries it is enough to fulfill that following system of equations

\[ U_t = \bar{U}_t \quad U_b = \bar{U}_b \quad \partial_t U_t = \partial_t \bar{U}_t \quad \partial_t U_b = \partial_t \bar{U}_b \]
on \( S^1(\frac{1}{2} \varepsilon^{-1/2}) \).

The first two equations lead to the system

\begin{align*}
-\frac{1}{2} \log \varepsilon \gamma_t + \sigma_t - \left( \frac{1}{4} \varepsilon^{-1/2} \eta_t + 2 \varepsilon^{1/2} s_t \right) \cos \theta + \varphi_t - \tilde{\varphi}_t &= \mathcal{O}(\varepsilon) \\
\frac{1}{2} \log \varepsilon \gamma_b - \sigma_b - \left( \frac{1}{4} \varepsilon^{-1/2} \eta_b - 2 \varepsilon^{1/2} s_b \right) \cos \theta + \varphi_b - \tilde{\varphi}_b &= \mathcal{O}(\varepsilon)
\end{align*} \hspace{1cm} (48)

while the last two equations read

\begin{align*}
\gamma_t - \left( \frac{1}{4} \varepsilon^{-1/2} \eta_t - 2 \varepsilon^{1/2} s_t \right) \cos \theta + \partial_t (\mathcal{P}(\varphi_t + \tilde{\varphi}_t)) &= \mathcal{O}(\varepsilon) \\
-\gamma_b - \left( \frac{1}{4} \varepsilon^{-1/2} \eta_b + 2 \varepsilon^{1/2} s_b \right) \cos \theta + \partial_t (\mathcal{P}(\varphi_b + \tilde{\varphi}_b)) &= \mathcal{O}(\varepsilon)
\end{align*} \hspace{1cm} (49)

Projection of every equation of this system over the \( L^2 \)-orthogonal complement of Span\{1, \cos\}, we obtain the system

\[ \varphi_t - \tilde{\varphi}_t = \mathcal{O}(\varepsilon) \quad \varphi_b - \tilde{\varphi}_b = \mathcal{O}(\varepsilon) \]

\[ \partial_t \mathcal{P}(\varphi_t + \varphi_t) = \mathcal{O}(\varepsilon) \quad \partial_t \mathcal{P}(\varphi_b + \varphi_b) = \mathcal{O}(\varepsilon) \]

Observe that the operator

\[ \mathcal{C}^{2,\alpha} \longrightarrow \mathcal{C}^{1,\alpha} \]

\[ \varphi \longmapsto \partial_t \mathcal{P}(\varphi) \]

is invertible, and hence the last system can be rewritten as

\[ (\varphi_t, \tilde{\varphi}_t, \varphi_b, \tilde{\varphi}_b) = \mathcal{O}(\varepsilon) \] \hspace{1cm} (50)

Recall that the right hand side depends nonlinearly on \( \varphi_t, \tilde{\varphi}_t, \varphi_b, \tilde{\varphi}_b \). Thanks to (44) and (47) we can use a fixed point theorem for contraction mapping in the ball of radius \( \kappa \varepsilon \) in \( (\mathcal{C}^{2,\alpha})^4 \) to obtain, for all \( \varepsilon \) small enough, a solution (50) which depends at least continuously (and in fact smoothly) on the parameters \( \gamma_t, \gamma_b, \sigma_t, \sigma_b, s_t, s_b, \eta_t, \eta_b \).

Inserting this solution into (48) and (49), we see that it remains to solve a system of the form

\begin{align*}
-\frac{1}{2} \log \varepsilon \gamma_t + \sigma_t - \left( \frac{1}{4} \varepsilon^{-1/2} \eta_t + 2 \varepsilon^{1/2} s_t \right) \cos \theta &= \mathcal{O}(\varepsilon) \\
\frac{1}{2} \log \varepsilon \gamma_b - \sigma_b - \left( \frac{1}{4} \varepsilon^{-1/2} \eta_b - 2 \varepsilon^{1/2} s_b \right) \cos \theta &= \mathcal{O}(\varepsilon) \\
\gamma_t - \left( \frac{1}{4} \varepsilon^{-1/2} \eta_t - 2 \varepsilon^{1/2} s_t \right) \cos \theta &= \mathcal{O}(\varepsilon) \\
-\gamma_b - \left( \frac{1}{4} \varepsilon^{-1/2} \eta_b + 2 \varepsilon^{1/2} s_b \right) \cos \theta &= \mathcal{O}(\varepsilon)
\end{align*} \hspace{1cm} (51)
where the right hand sides depend nonlinearly on \( \gamma_t, \gamma_b, \sigma_t, \sigma_b, \varsigma_t, \varsigma_b, \eta_t \) and \( \eta_b \).

Projecting this system over the constant function and the function \( \theta \rightarrow \cos \theta \), we see that this system can be rewritten as

\[
(\gamma_t, \gamma_b, \sigma_t, \sigma_b, \varsigma_t, \varsigma_b, \eta_t, \eta_b) = \mathcal{O}(\varepsilon)
\]

where we have set

\[
(\tilde{\varsigma}_t, \tilde{\varsigma}_b) := \varepsilon^{1/2} (\varsigma_t, \varsigma_b), \quad (\tilde{\sigma}_t, \tilde{\sigma}_b) := |\log \varepsilon|^{-1} (\sigma_t, \sigma_b)
\]

and

\[
(\tilde{\eta}_t, \tilde{\eta}_b) := \varepsilon^{-1/2} (\eta_t, \eta_b)
\]

This time we can use Leray-Schauder degree theory in the ball of radius \( \varepsilon \) in \( \mathbb{R}^8 \) to solve (52), for all \( \varepsilon \) small enough. This completes the proof of a solution of (48)-(49) and hence the proof of the main theorem.

Remark 7.1. Alternatively, with more work, one can use a fixed point argument for contraction mapping to solve (52).

8 Appendix A

We consider the surface parameterized by

\[
X = X_c + w N_c
\]

The coefficients of \( g_w \), the first fundamental form of this surface, are given by

\[
|\partial_s X|^2 = \cosh^2 s - 2 w + \frac{1}{\cosh^2 s} w^2 + (\partial_s w)^2
\]

\[
|\partial_b X|^2 = \cosh^2 s + 2 w + \frac{1}{\cosh^2 s} w^2 + (\partial_b w)^2
\]

and

\[
\partial_s X \cdot \partial_b X = \partial_s w \partial_b w
\]

It follows from these that the determinant of the metric \( g_w \) can be expanded as

\[
|g_w| = \cosh^4 s \left(1 + \frac{1}{\cosh^2 s} \left((\partial_s w)^2 + (\partial_b w)^2 - \frac{2}{\cosh^2 s} w^2\right)\right)
\]

\[
+ \left(\frac{1}{\cosh s} P_3 \left(\frac{w}{\cosh s}, \frac{\nabla w}{\cosh s}\right) + P_4 \left(\frac{w}{\cosh s}, \frac{\nabla w}{\cosh s}\right)\right)
\]
where the $P_i$ are homogeneous polynomials of degree $i$, whose coefficients are bounded smooth functions of $s$ and $\theta$.

We consider the area energy

$$A(w) := \int \sqrt{|g_w|} \, ds \, d\theta$$

The surface parameterized by $X_w$ is minimal if and only if the first variation of $A$ at $w$, is 0. This can be written as

$$2 \partial A|_w(v) = \int \frac{1}{\sqrt{|g_w|}} D_w|g_w|(v) \, ds \, d\theta$$

Observe that

$$\frac{1}{\sqrt{|g_w|}} D_w|g_w|(v) = \partial_s w \partial_s v + \partial_\theta w \partial_\theta v - \frac{2}{\cosh^2 s} w v$$

$$+ \left( \tilde{Q}_2 \left( \frac{w}{\cosh s} \cdot \frac{\nabla w}{\cosh s} \right) + \cosh s \tilde{Q}_3 \left( \frac{w}{\cosh s} \cdot \frac{\nabla w}{\cosh s} \right) \right) v$$

$$+ \left( \tilde{Q}'_2 \left( \frac{w}{\cosh s} \cdot \frac{\nabla w}{\cosh s} \right) + \cosh s \tilde{Q}'_3 \left( \frac{w}{\cosh s} \cdot \frac{\nabla w}{\cosh s} \right) \right) \partial_s v$$

$$+ \left( \tilde{Q}''_2 \left( \frac{w}{\cosh s} \cdot \frac{\nabla w}{\cosh s} \right) + \cosh s \tilde{Q}''_3 \left( \frac{w}{\cosh s} \cdot \frac{\nabla w}{\cosh s} \right) \right) \partial_\theta v$$

(53)

where the operator $Q_2, \ldots$ and the operators $\tilde{Q}_3, \ldots$ enjoy properties similar to the one enjoyed by $Q_2$ and $Q_3$ in the statement of the result. The result then follows at once.

9 Appendix B

This appendix is essentially a generalization of the corresponding analysis in [12]. Let be $\Sigma$ be a smooth surface embedded in a Riemannian manifold $(M,g)$. We denote by $N$ the unit normal vector field compatible with the orientation of $\Sigma$. Suppose that $\tilde{N}$ is another unit vector field transverse to $\Sigma$, the implicit function theorem implies that, given $p_0 \in \Sigma$, there exist neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of $(p_0,0) \in \Sigma \times \mathbb{R}$ and a diffeomorphism $(p,s) \rightarrow (\varphi(p,s),\psi(p,s))$ from $\mathcal{U}$ to $\mathcal{V}$ such that

$$\text{Exp}_p^M (s \tilde{N}(p)) = \text{Exp}_{\varphi(p,s)}^M (\psi(p,s) N(\varphi(p,s)))$$

(54)

where $\text{Exp}^M$ denotes the exponential map in $(M,g)$. In addition $\varphi(p,0) = p$ and $\psi(p,0) = 0$. 

40
Differentiation of (54) with respect to \( s \) at \( s = 0 \) yields

\[
\tilde{N}(p) = \partial_s \varphi(p, 0) + \partial_s \psi(p, 0) N(p)
\]  
(55)

Taking the scalar product with \( N(p) \) we conclude that

\[
g(\tilde{N}(p), N(p)) = \partial_s \psi(p, 0)
\]  
(56)

This immediately implies that \( \psi(p, s) = g(\tilde{N}(p), N(p)) s + \mathcal{O}(s^2) \). On the other hand, projection of (55) over \( T_p \Sigma \) yields

\[
\tilde{N}^t(p) = \partial_s \varphi(p, 0)
\]  
(57)

where \( \tilde{N}^t(p) \) is the tangential component of \( \tilde{N} \).

Next any surface \( \tilde{\Sigma} \) sufficiently close to \( \Sigma \) can be either parameterized as a graph of the function \( w \) over \( \Sigma \) using the vector field \( \tilde{N} \) or the graph of the function \( \bar{w} \) for the normal vector field \( N \). Thanks to the above analysis we can write

\[
\bar{w}(\varphi(p, w(p))) = \psi(p, w(p))
\]

Now, the mean curvature of the surface \( \Sigma \) at the point \( \text{Exp}_p^M(w(p) \tilde{N}(p)) \) and at the point \( \text{Exp}_\bar{p}^M(\bar{w}(\bar{p}) N(\bar{p})) \) are the same if \( \bar{p} = \varphi(p, w(p)) \). We phrase this property as

\[
H_{\tilde{N}, \bar{w}}(p) = H_{N, \bar{w}}(p)
\]

Differentiation with respect to \( w \) at \( w = 0 \) yields

\[
DH_{\tilde{N}, 0} u = DH_{N, 0}(\partial_s \psi) + \nabla_{\partial_s \varphi} H_{N, 0}
\]

Taking into account the partial derivatives of \( \varphi \) and \( \psi \), which are given in (57) and (56), we conclude that

\[
DH_{\tilde{N}, 0} u = DH_{N, 0}(g(\tilde{N}(p), N(p)) u) + \left( \nabla_{\tilde{N}^t(p)} H_{N, 0} \right) u
\]

for any smooth function \( u \) defined on \( \Sigma \). In the special case where \( \Sigma \) has constant mean curvature, we simply get

\[
DH_{\tilde{N}, 0}(u) = \mathcal{L}_\Sigma(g(\tilde{N}(p), N(p)) u)
\]

which gives the relation between \( \mathcal{L}_\Sigma \) the Jacobi operator about \( \Sigma \) and \( DH_{\tilde{N}, 0} \) the linearized mean curvature operator when the normal vector field \( N \) is changed into a transverse vector field \( \tilde{N} \).
References

[1] S. Alinhac and P. Gérard, *Opérateurs pseudo-différentiels et théorème de Nash-Moser*, InterEditions/Editions du CNRS (1991).

[2] C. J. Costa, *Imersões minimas en \( \mathbb{R}^3 \) de gênero un e curvatura total finita*, PhD thesis, IMPA, Rio de Janeiro, Brasil, 1982.

[3] C. J. Costa, *Example of a complete minimal immersion in \( \mathbb{R}^3 \) of genus one and three embedded ends*, Bol. Soc. Brasil. Mat. 15 (1984), no. 1-2, 47-54.

[4] S. Fakhi and F. Pacard, *Existence of complete minimal hypersurfaces with finite total curvature*, Manuscripta Mathematica. 103, (2000), 465-512.

[5] L. Hauswirth, *Generalized Riemann examples in three-dimensional manifolds*, To appear in Pacific Journal of Math. series 1, 29 (1993), 77-84.

[6] D. Hoffman and H. Karcher, *Complete embedded minimal surfaces of finite total curvature*, Geometry, V, 5-93, 267-272, Encyclopaedia Math. Sci.,90, Springer, Berlin, 1997.

[7] D. Hoffman and W.H. Meeks III, *The asymptotic behavior of properly embedded minimal surfaces of finite topology*, Journal of the AMS, (2) 4 (1989),667-681.

[8] D. Hoffman and W.H. Meeks III, *Embedded minimal surfaces of finite topology*, Annals of Mathematics, 131 (1990), 1-34.

[9] M. Jleli, *Constant mean curvature hypersurfaces*, PhD Thesis, University Paris 12 (2004).

[10] N. Korevaar, R. Kusner and B. Solomon, *The Structure of Complete Embedded n. Surfaces with Constant Mean Curvature*, J. of Differential Geometry, 30 (1989), 465-503.

[11] R. Kusner, R. Mazzeo and D. Pollack, *The moduli space of complete embedded constant mean curvature surfaces*, Geom. Funct. Anal. 6 (1996) 120–137.

[12] R. Mazzeo and F. Pacard, *Constant mean curvature surfaces with Delaunay ends*, Comm. Analysis and Geometry. 9, 1, (2001), 169-237.
[13] R. Melrose, The Atiyah-Patodi-Singer index theorem. Research Notes in Mathematics. (1993).

[14] W. H. Meeks III, J. Perez and A. Ros, W. H. Meeks III, J. P., Uniqueness of the Riemann minimal examples, Invent. Math., 131(1998), 107-132.

[15] W. H. Meeks III, J. Perez and A. Ros, The geometry of minimal surfaces of finite genus I; curvature estimates and quasiperiodicity. Preprint (2004).

[16] W. H. Meeks III, J. Perez and A. Ros, The geometry of minimal surfaces of finite genus II; non existence of one limit end examples, Invent. Math. (to appear).

[17] S. Nayatani, Morse index and Gauss maps of complete minimal surfaces in Euclidean 3-space, Comment. Math. Helv. 68(1993), Number 4, 511-537.

[18] J. Perez and A. Ros, The space of properly embedded minimal surfaces with finite total curvature, Indiana Univ. Math. Journal, 45 (1996), no. 1, 177–204.

[19] B. Riemann, Oeuvres mathématiques de Riemann, Gauthiers-Villards, Paris 1898.

[20] M. Shiffman, On surfaces of stationary area bounded by two circles, or convex curves, in parallel planes, Annals of Mathematics, (2) 63 (1956), 77-90.