Approximate solutions for the ion-laser interaction in the high intensity regime: matrix method perturbative analysis

B. M. Villegas-Martínez · H. M. Moya-Cessa · F. Soto-Eguibar

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Abstract

This work provides an explicit expression for the second-order perturbative solution of a single trapped ion at the high intensity regime. Unlike other perturbative schemes, where the ion-laser dynamics has been explored using unitary transformations and the Lamb-Dicke regime, this analysis relies instead on a direct perturbation method, that may be implemented in simple manner and works especially well without resorting to additional approximations. Based on a matrix method and a final normalization of the perturbed solutions, the second-order perturbative analysis supply the probability to find the ion in its excited state; the resulting perturbative solution renders a high accuracy, comparable to the one based on the small rotation approximation.

Keywords Trapped ion system · Matrix method · Perturbation theory

1 Introduction

Trapped ions interacting with laser beams represent a quantum optical elementary system that has gained considerable attention in quantum information, both experimentally and theoretically, for its potential to realize quantum computation. Their undoubted importance relies on the preparation of non-classical states of the ion’s vibrational motion (Meekhof et al. 1996; Wallentowitz and Vogel 1997; Wallentowitz et al. 1999; Kis et al. 2001; de Matos Filho and Vogel 1996a, b; Moya-Cessa and Tombesi 2000; Casanova et al. 2018), the reconstruction of quasi-probability distribution functions (Leibfried et al. 1996), the production of robust quantum gates and the preparation of entangled Bell states in quantum computers (Ospelkaus et al. 2008; Solano et al. 1999; Barenco et al. 1995), among many others.

Although the interaction of a single ion qubit with a laser light constitutes a fundamental model to study such systems, the trapped-ion Hamiltonian leads to a Schrödinger equation which is extremely difficult to solve exactly. Nevertheless, the theoretical description
of its dynamics can be addressed by suitable approximations; for example, the Lamb-Dicke approximation (Jonathan et al. 2000), in which the ion moves within a region much smaller than the laser wavelength, the vibrational rotating wave approximation (RWA) (Allen and Eberly 1987), where the counter-rotating terms are neglected and the weak excitation regime (Cirac et al. 1993, 1994), in which the Rabi frequency, proportional to the intensity of the laser field, is much smaller than the vibrational frequency of the ion. The latter one has a severe drawback since it ignores the case of an intense laser field that becomes crucial nowadays due its potential applications for faster quantum gates (Duan 2004; Taylor et al. 2017; Puebla et al. 2017).

In the framework of perturbation techniques (Seadawy et al. 2018; Ali et al. 2018; Arshad et al. 2017; Ahmed et al. 2019; Seadawy et al. 2018; Cheemaa et al. 2018, 2019; Özkan et al. 2020; Seadawy and Cheemaa 2019b; Rizvi et al. 2020; Seadawy and Cheemaa 2019a), only a few works achieving perturbative solutions in the strong excitation regime have been proposed, each of them with their own constraints. For instance, (Aniello et al. 2003) developed a perturbative procedure on the ion trap Hamiltonian that allows to get approximated solutions of the evolution operator of the system; however, such process works in the Lamb-Dicke regime and requires a set of unitary transformations to obtain a diagonal balanced Hamiltonian. This method has been also extended to a system of \( N \) equal ions, where the new perturbative parameter is no longer proportional to the laser intensity (Aniello et al. 2004, 2003). Alternatively, Aniello has analyzed the laser-driven trapped ion Hamiltonian through a large and non-trivial perturbative decomposition of the evolution operator in a generalized Magnus expansion (Aniello 2005); nonetheless, such approximated procedure has been addressed also in the Lamb-Dicke regime.

In spite of these theoretical attempts, none of the available perturbation treatments provides a reliable solution in the strong field regime, without imposing an auxiliary unitary transformation or a second approximation to the full ion-laser Hamiltonian; in fact, any of these single techniques may yield highly misleading results with the exact system dynamics behavior when operating outside the Lamb-Dicke approximation. Consequently, an appropriate theoretical approach is necessary in order to obtain reliable solutions of the Schrödinger equation for a single trapped ion, valid for large laser intensities; therefore, this prompts us to explore a straightforward alternative to bridge this knowledge gap. Recently, we have introduced the Normalized Matrix Perturbation Method (NMPM), a general approach to solve any time-independent perturbation problem in quantum mechanics, that is different from those reported previously. Some highlight features of the method are: it enables easy application to any initial condition, because it is based on an approximation of the evolution operator; indeed, it may be used when one may find a unitary evolution operator for the unperturbed Hamiltonian, but it is not possible to find its eigenstates. The procedure also does not distinguish if the unperturbed part of full Hamiltonian is degenerated or not. Further, depending on the free choice of what part of the system represents the unperturbed part and the perturbation, the recursive calculation of the corrections generates a dual Dyson series in matrix form; this offers the possibility to analyze the solution of a quantum system in both regimes of the perturbative parameter; a duality that is very convenient for our purposes to study the single trapped ion system in the high intensity regime. Additionally, the NMPM is capable of providing normalized solutions at any correction order, a property that marks another difference between our approach and those already existing in the literature, where the general formulation of a normalization constant is not easy to obtain. Frasca (1992, 1993, 1998, 2006a, 2006b, 2007).

The manuscript is organized as follows: In Sect. 2, we provide a brief summary relative to the NMPM. Immediately after these background concepts are presented, we jump to the
description of the trapped-ion Hamiltonian, in Sect. 3. Subsequently, in Sect. 4 we solve perturbatively the Schrödinger equation for a single trapped ion up to second order; we assess the validity of our result by comparing it with the reported solution based on the small rotation approximation. As the ion-laser interaction Hamiltonian is similar to the Rabi Hamiltonian (Casanova et al. 2018), we also outline the second order solution of that problem in Sect. 5. Finally, Sect. 6 is dedicated to the conclusions.

2 Outline of the method

The Normalized Matrix Perturbation Method (NMPM) is a time-independent perturbative approach (Martinez-Carranza et al. 2021a, b; Villegas-Martinez et al. 2016, 2017, 2020) based on the Taylor expansion of the evolution operator of the time-dependent Schrödinger equation $|\psi(t)\rangle = e^{-i\hat{H}t}|\psi(0)\rangle$. It is assumed that the Hamiltonian $\hat{H}$ is the sum of an already solved part $\hat{H}_0$ and a perturbation $\lambda \hat{H}_p$. Where $\lambda$ is a real and dimensionless perturbation parameter that establishes the order of the perturbation. For example, an expansion of the propagator truncated to first-order is

$$|\psi(t)\rangle = \left[ e^{-i\hat{H}_0 t} + \lambda \sum_{n=1}^{\infty} \frac{(-it)^n}{n!} \sum_{k=0}^{n-1} \hat{H}_0^{n-1-k} \hat{H}_p^k \right] |\psi(0)\rangle. \quad (2.1)$$

The main ingredient of the NMPM (Martinez-Carranza et al. 2012b, a; Villegas-Martinez et al. 2017) is the implementation of the upper triangular matrix

$$M = \begin{pmatrix} \hat{H}_0 & \hat{H}_p \\ 0 & \hat{H}_0 \end{pmatrix}, \quad (2.2)$$

whose diagonal elements are conformed by the unperturbed Hamiltonian and the upper triangle by the perturbation. If we multiply the matrix $M$ by itself $n$-times, we found that its upper element contains exactly the same products of $\hat{H}_0$ and $\hat{H}_p$ as the summation in Eq. (2.1). In short, the matrix element $M_{1,2}$ gives us the first order correction; based on this consideration, Eq. (2.1) may be rewritten as

$$|\psi(t)\rangle = \left[ e^{-i\hat{H}_0 t} + \lambda (e^{-iMt})_{1,2} \right] |\psi(0)\rangle. \quad (2.3)$$

Therefore, the approximated solution has been split in two parts; the first one being the solution of the unperturbed system, that is the one we know, while the second one refers to the first-order correction. In order to determine the solution to first order, we have to keep in mind that the problem originally posed must follows the same matrix convention; hence, the approximated solution (2.3) may be conveniently written as

$$|\psi(t)\rangle = |\psi^{(0)}\rangle + \lambda \left( |\psi^{(1)}\rangle \right)_{1,2}. \quad (2.4)$$

Where the superscripts denote the order of the correction and $|\psi^{(1)}\rangle$ is the matrix defined by

$$|\psi^{(1)}\rangle = \begin{pmatrix} |\psi_{1,1}\rangle \\ |\psi_{1,2}\rangle \\ |\psi_{2,1}\rangle \\ |\psi_{2,2}\rangle \end{pmatrix}. \quad (2.5)$$
The solution to first order may be determined by deriving the Eqs. (2.3) and (2.4) with respect to time, equating the corresponding powers of $\lambda$ and performing the algebraic steps outlined in Martinez-Carranza et al. (2012b, 2012a); Villegas-Martinez et al. (2017). Then, we obtain

$$ \left| \psi_{1,2} \right> = -i e^{-iH_0 t} \left[ \int_0^t e^{iH_0 t_1} \hat{H}_\rho e^{-iH_0 t_1} dt_1 \right] \left| \psi(0) \right>. $$

(2.6)

All the information of the second-order correction will be in the element $M_{1,3}$ of a newly defined $3 \times 3$ triangular matrix $M$, completely similar to (2.2). Thus, it becomes clear that the matrix treatment allows to transform the Taylor series of the formal solution of the time-dependent Schrödinger equation in a power series of the matrix $M$, where the kets $\left| \psi^{(k)} \right>$ are obtained iteratively and may be easily handled. Therefore, the corresponding relation that allows us to find perturbative solutions in the Schrödinger equation at $k$-th correction order is given by Martinez-Carranza et al. (2012a, b); Villegas-Martinez et al. (2017)

$$ |\Psi(t)\rangle = N^{(k)}(t) \left[ |\psi^{(0)}\rangle + \sum_{n=1}^{k} \lambda^n \left( |\psi^{(n)}\rangle \right)_{1,n+1} \right], $$

(2.7)

Where $N^{(k)}(t)$ is a normalization factor that preserves the norm at any order and that is calculated to

$$ N^{(k)}(t)^{-2} = 1 + 2 \sum_{n=1}^{k} \lambda^n \Re \left( \langle \psi^{(0)} | \psi_{1,n+1} \rangle \right) + \sum_{n=1}^{k} \lambda^{2n} \langle \psi_{1,n+1} | \psi_{1,n+1} \rangle $$

$$ + 2 \sum_{n=1}^{k-1} \sum_{m=2n+1}^{n+k} \lambda^m \Re \left( \langle \psi_{1,n+1} | \psi_{1,m-n+1} \rangle \right), $$

(2.8)

Where $\Re(z)$ is the real part of $z$. The matrix element $|\psi_{1,n+1}\rangle$ is the relevant solution we are looking for and is expressed in the form

$$ |\psi_{1,n+1}\rangle = (-i)^n e^{-iH_0 t} \int_{0}^{t_n} dt_1 \int_{0}^{t_2} dt_2 \ldots $$

$$ \int_{0}^{t_{n-1}} dt_n e^{i\hat{H}_0 t_1} \hat{H}_\rho e^{-i\hat{H}_0 t_1} e^{i\hat{H}_0 t_2} \hat{H}_\rho e^{-i\hat{H}_0 t_2} \ldots e^{i\hat{H}_0 t_n} \hat{H}_\rho e^{-i\hat{H}_0 t_n} |\psi(0)\rangle. $$

(2.9)

The normalized solution (2.7), that contains $N^{(k)}(t)$, has been obtained and published separately; the reader can consult the reference (Villegas-Martinez et al. 2017) for further details. From here, expressions (2.7), (2.8) and (2.9) are the equations that we will use to obtain the normalized perturbative solutions for the trapped ion-laser system in the high intensity regime.
3 Trapped-ion Hamiltonian

We now consider a simplified model of a single trapped ion interacting with a classical laser field; the Hamiltonian of this system is (Poyatos et al. 1996; Casanova et al. 2018; Wineland et al. 1992; Leibfried et al. 2003; Moya-cessa et al. 2003; Rodríguez-Lara et al. 2005; Cirac et al. 1994)

\[
\hat{H}_{\text{ion}} = v\hat{n} + \frac{\delta}{2}\hat{\sigma}_z + \Omega\left[\hat{\sigma}^+\hat{D}(\eta) + \hat{\sigma}^-\hat{D}^\dagger(\eta)\right],
\]

(3.1)

where \( \hat{D}(\eta) = \exp[\eta(\hat{a} + \hat{a}^\dagger)] \) is the Glauber displacement operator (Gerry and Knight 2005; Puri 2001; Fox 2006; Raymond Chiao 2008; Moya-Cessa and Eguíbar 2011b), \( \hat{a}^\dagger(\hat{a}) \) is the ion’s vibrational creation (annihilation) operator, \( \hat{n} = \hat{a}^\dagger\hat{a} \) is the number operator, \( \eta \) is the Lamb-Dicke parameter (Sørensen and Mølmer 1999; Sackett et al. 2000; Jonathan and Plenio 2001; Wineland and Itano 1979; Morigi et al. 1999), \( \nu \) is the vibrational frequency of the ion, \( \delta = \nu \kappa \) is the laser-ion detuning, \( \Omega \) is the Rabi frequency, and \( \hat{\sigma}^+ = |\uparrow\rangle \langle \downarrow| \) and \( \hat{\sigma}^- = |\downarrow\rangle \langle \uparrow| \) are the atomic raising and lowering operators (Pauli matrices) expressed in terms of the excited \( |\uparrow\rangle = (1, 0) \) and ground \( |\downarrow\rangle = (0, 1) \) states of the two-level ion. Note that the ion-laser coupling strength is the Rabi frequency. We consider that the laser-ion detuning \( \delta \) is a multiple integer of the vibrational frequency of the ion. The atomic raising and lowering operators satisfy the commutation relations \([\hat{\sigma}^+, \hat{\sigma}^-] = \hat{\sigma}_z\) and \([\hat{\sigma}_z, \hat{\sigma}^\pm] = \pm 2\hat{\sigma}^\pm\).

The dynamics of a single trapped ion can be studied by solving the time-dependent Schrödinger equation

\[
i\frac{d}{dt}|\Psi(t)\rangle_{\text{ion}} = \hat{H}_{\text{ion}}|\Psi(t)\rangle_{\text{ion}},
\]

(3.2)

We are interested in solving perturbatively the Schrödinger equation through the NMPM. Indeed, our perturbative scheme provides us the great flexibility to choose the unperturbed and perturbed parts of the full Hamiltonian (3.1). Using this freedom, we can get normalized perturbative solutions for the cases when the amplitude \( \Omega \) of the laser is very small compared with the vibrational frequency of the ion, \( \Omega \ll \nu \), and vice versa, i.e. when \( \Omega \gg \nu \); these two approximations correspond to the weak and strong laser intensity regimes, respectively. In addition, the ion-trap system is formally equivalent to the quantum Rabi model when we consider a certain unitary transformation \( \hat{T} \) (Casanova et al. 2018); therefore, we can perform the transformation \( |\phi(t)\rangle_{\text{Rabi}} = \hat{T}|\Psi(t)\rangle_{\text{ion}} \) and also get perturbative solutions of the quantum Rabi model for the weak and strong coupling regimes; this will be done in Sect. 5.

4 High intensity regime

4.1 First order correction

Let us begin our perturbative analysis by solving the high intensity case (\( \Omega \gg \nu \)). In such scenario, we must consider that \( \hat{H}_p = \hat{h} + \frac{\lambda}{\Omega} \hat{\sigma}_z \) is the perturbation with perturbative parameter \( \lambda = \nu / \Omega \), whereas \( \hat{H}_0 = \hat{\sigma}^+\hat{D}(\eta) + \hat{\sigma}^-\hat{D}^\dagger(\eta) \) plays the role of the unperturbed
part. If we re-scale time as $\tau = \Omega t$ and set $k = 1$ into Eqs. (2.7–2.9), we get the approximate solution to first order

$$\langle \Psi(\tau) \rangle = N^{(1)}(\tau) \left( \langle \Psi^0 \rangle + \lambda \langle \Psi^{(1)} \rangle \right), \quad (4.1)$$

Where

$$\langle \Psi^0 \rangle = e^{-iH_0\tau} \langle \Psi(0) \rangle, \quad (4.2a)$$

$$\langle \Psi^{(1)} \rangle = -ie^{-iH_0\tau} \int_0^\tau e^{iH_0\tau_1} \left( \hat{n} + \frac{\kappa}{2} \hat{\sigma}_z \right) e^{-iH_0\tau_1} d\tau \langle \Psi(0) \rangle, \quad (4.2b)$$

$$\left( N^{(1)}(\tau) \right)^{-2} = 1 + 2\lambda \Re \left( \langle \Psi^0 \rangle \langle \Psi^{(1)} \rangle \right) + \lambda^2 \langle \Psi^{(1)} \rangle \langle \Psi^{(1)} \rangle. \quad (4.2c)$$

The integral in Eq. (4.2b) requires to calculate the product of exponential operators with $\hat{n} + \frac{\kappa}{2} \hat{\sigma}_z$. To do this, we first expand in Taylor series the exponential operator $e^{iH_0\tau_1}$ and split the series in even and odd powers of $\hat{H}_0$,

$$e^{iH_0\tau_1} = \sum_{n=0}^\infty \frac{(-1)^n \tau_1^{2n}}{(2n)!} \hat{H}_0^{2n} + i \sum_{n=0}^\infty \frac{(-1)^n \tau_1^{2n+1}}{(2n + 1)!} \hat{H}_0^{2n+1}, \quad (4.3)$$

one can easily check that $\hat{H}_0^{2n} = \hat{1}$ and $\hat{H}_0^{2n+1} = \hat{H}_0$, then equation Eq. (4.3) becomes

$$e^{iH_0\tau_1} = \cos(\tau_1) + i \sin(\tau_1) \left[ \hat{\sigma}^+ \hat{D}(i\eta) + \hat{\sigma}^- \hat{D}^+(i\eta) \right]. \quad (4.4)$$

It is possible to show that

$$e^{iH_0\tau_1} \left( \hat{n} + \frac{\kappa}{2} \hat{\sigma}_z \right) e^{-iH_0\tau_1} = \hat{n} \cos^2(\tau_1) + \frac{i}{2} \sin(2\tau_1) [\hat{H}_0, \hat{n}] + \sin^2(\tau_1) \hat{H}_0 \hat{n} \hat{H}_0$$

$$\quad + \frac{\kappa}{2} \cos(2\tau_1) \hat{\sigma}_z + i \frac{\kappa}{2} \sin(2\tau_1) \hat{H}_0 \hat{\sigma}_z. \quad (4.5)$$

As $[\hat{a}^\dagger, \hat{n}] = -\hat{a}^\dagger$ and $[\hat{a}, \hat{n}] = \hat{a}$, we obtain that

$$[\hat{H}_0, \hat{n}] = \hat{\sigma}^+ \left[ \hat{D}(i\eta), \hat{n} \right] + \hat{\sigma}^- \left[ \hat{D}^+(i\eta), \hat{n} \right]$$

$$\quad = \hat{\sigma}^+ \hat{D}(i\eta) \left[ \hat{n} - \hat{D}^+(i\eta) \hat{D}(i\eta) \right] + \hat{\sigma}^- \hat{D}^+(i\eta) \left[ \hat{n} - \hat{D}(i\eta) \hat{D}^+(i\eta) \right]. \quad (4.6)$$

Using the Hadamard formula Hall (2015); Miller (1984); Rossmann (2002); Moya-Cessa and Eguibar (2011a), $e^{i\hat{A} \hat{B} e^{-i\hat{A}}} = \hat{A} + e [\hat{A}, \hat{B}] + \frac{e^2}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \ldots$, the expression (4.6) is simplified to

$$[\hat{H}_0, \hat{n}] = -\eta \left[ \hat{\sigma}^+ \hat{D}(i\eta) + \hat{\sigma}^- \hat{D}^+(i\eta) \right] \left[ \eta + i(\hat{a} - \hat{a}^\dagger) \hat{\sigma}_z \right]. \quad (4.7)$$

At this point, one can prove that

$$\hat{H}_0 \hat{n} \hat{H}_0 = \left( \begin{array}{cc} \hat{D}(i\eta) \hat{n} \hat{D}^+(i\eta) & 0 \\ 0 & \hat{D}^+(i\eta) \hat{n} \hat{D}(i\eta) \end{array} \right) = \hat{n} + \eta^2 + i\eta(\hat{a} - \hat{a}^\dagger) \hat{\sigma}_z, \quad (4.8)$$
and when this result is substituted in Eq. (4.5), together with Eq (4.7), yields to
\[
\hat{\Psi}(\tau) = \hat{n} + \frac{\eta}{2} \left[ \sigma^{+} D(i\eta) + \sigma^{-} \hat{D}^{\dagger}(i\eta) \right] \left[ \eta + i(\hat{a} - \hat{a}^{\dagger}) \hat{\sigma}_{z} - \frac{\kappa}{\eta} \hat{\sigma}_{z} \right] \sin(2\tau)
\]
\[+ \frac{\kappa}{2} \cos(2\tau) \hat{\sigma}_{z}
\]
\[+ \eta \left[ \eta + i(\hat{a} - \hat{a}^{\dagger}) \hat{\sigma}_{z} \right] \sin^{2}(\tau), \tag{4.9}
\]

that can be easily integrated to give
\[
\int_{0}^{\tau} e^{iH_{0}^{*} \tau_{1}} \left( \hat{H} + \frac{\kappa}{2} \hat{\sigma}_{z} \right) e^{-iH_{0} \tau_{1}} \, d\tau_{1}
\]
\[= \left\{ 2\hat{n} + \eta \left[ \eta + i(\hat{a} - \hat{a}^{\dagger}) \hat{\sigma}_{z} \right] \right\} \frac{\tau}{2} - \frac{\eta}{4} \left[ \eta + i(\hat{a} - \hat{a}^{\dagger}) \hat{\sigma}_{z} - \frac{\kappa}{\eta} \hat{\sigma}_{z} \right] \sin(2\tau) \tag{4.10}
\]
\[= \left\{ \hat{n} + \frac{\eta}{2} \left[ \eta + i(\hat{a} - \hat{a}^{\dagger}) \hat{\sigma}_{z} \right] \right\} \left[ \eta + i(\hat{a} - \hat{a}^{\dagger}) \hat{\sigma}_{z} - \frac{\kappa}{\eta} \hat{\sigma}_{z} \right] \sin^{2}(\tau);
\]

substituting in Eq. (4.2b) and after some algebra, the first order term is obtained,
\[
|\psi^{(1)}\rangle = -i \cos(\tau) \left\{ \hat{n} + \frac{\eta}{2} \left[ \eta + i(\hat{a} - \hat{a}^{\dagger}) \hat{\sigma}_{z} \right] \right\} |\psi(0)\rangle
\]
\[- \tau \sin(\tau) \left[ \hat{\sigma}^{+} D(i\eta) + \hat{\sigma}^{-} \hat{D}^{\dagger}(i\eta) \right] \left\{ \hat{n} + \frac{\eta}{2} \left[ \eta + i(\hat{a} - \hat{a}^{\dagger}) \hat{\sigma}_{z} \right] \right\} |\psi(0)\rangle. \tag{4.11}
\]

Now, the normalization constant \(N^{(1)}(\tau)\) of Eq. (4.2c) is obtained doing the inner product of \(|\psi^{(1)}\rangle\) with itself, and once it is calculated and being substituted in Eq. (4.1), give us
\[
|\Psi(\tau)\rangle_{\text{ion}} \approx N^{(1)}(\tau) \cos(\tau) \left\{ 1 - i\lambda \tau \hat{n} - i \lambda \frac{\eta}{2} \left[ \tau - \tan(\tau) \right] \left[ \eta + i(\hat{a} - \hat{a}^{\dagger}) \hat{\sigma}_{z} \right] \right\}
\]
\[= -i \frac{\lambda \kappa}{2} \tan(\tau) \hat{\sigma}_{z} \right\} |\psi(0)\rangle \tag{4.12}
\]
\[= -i \lambda N^{(1)}(\tau) \sin(\tau) \left[ \hat{\sigma}^{+} D(i\eta) + \hat{\sigma}^{-} \hat{D}^{\dagger}(i\eta) \right] \left[ 1 + \lambda \frac{\eta}{2} (\hat{a} - \hat{a}^{\dagger}) \hat{\sigma}_{z} \right]
\]
\[-i \lambda \tau \left( \hat{n} + \frac{\eta^{2}}{2} \right) |\psi(0)\rangle.
\]

The above expression is the first order approximated solution of Eq. (3.2) and the normalization constant is
Let us consider as initial state $|\psi(0)\rangle = |n\rangle |g\rangle$, which represents $n$ vibrational quanta and the ion in the ground internal state $|g\rangle$; with this initial state the solution to first-order is

$$
|\Psi(\tau)\rangle_{ion,n,g} \approx N_{n,g}(\tau) \cos(\tau) \left\{ 1 - i \lambda \tau n - i \lambda \eta^2 2 \frac{[\tau - \tan(\tau)] + i \frac{\lambda K}{2} \tan(\tau)}{2} \right\} |n\rangle |g\rangle 
$$

$$
- \frac{\lambda \eta}{2} N_{n,g}(\tau) \cos(\tau) [\tau - \tan(\tau)] \left[ \sqrt{n}|n-1\rangle - \sqrt{n+1}|n+1\rangle \right] |g\rangle 
$$

$$
- i \eta N_{n,g}(\tau) \sin(\tau) \left[ 1 - i \lambda \tau (n + \eta^2/2) \right] |e\rangle 
$$

$$
+ \frac{\lambda \eta}{2} N_{n,g}(\tau) \sin(\tau) \left[ \sqrt{n}|in;:n-1\rangle - \sqrt{n+1}|in;:n+1\rangle \right] |e\rangle, 
$$

(4.14)

Where $|\alpha;m\rangle \equiv \hat{D}(\alpha)|m\rangle$ is a displaced number state (Gerry and Knight 2005; Puri 2001; Fox 2006; Raymond Chiao 2008; Vogel 2006; Moya-Cessa and Eguibar 2011b), whereas the normalization constant $N_{n,g}(\tau)$ is given by

$$
\left[ N_{n,g}(\tau) \right]^{-2} = 1 + \frac{\lambda^2 \eta^2}{8} \left\{ 4\tau^2 n (\eta^2 + 2n) + 2 [\eta^2 (\eta^2 + 2n + 1) + \kappa^2] \sin^2(\tau) + 2\tau \eta^2 (\eta^2 + 4n + 1) [\tau - \sin(2\tau)] \right\} 
$$

$$
- \frac{\lambda^2 \kappa}{4} \sin(2\tau) \left\{ 2\tau + \eta^2 [\tau - \tan(\tau)] \right\}. 
$$

(4.15)

If we suppose an initial condition with the ion in the excited state, $|\psi(0)\rangle = |n\rangle |e\rangle$, we get

$$
|\Psi(\tau)\rangle_{ion,n,e} \approx N_{n,e}(\tau) \cos(\tau) \left\{ 1 - i \lambda \tau n - i \lambda \eta^2 2 \frac{[\tau - \tan(\tau)] + i \frac{\lambda K}{2} \tan(\tau)}{2} \right\} |n\rangle |e\rangle 
$$

$$
+ \frac{\lambda \eta}{2} N_{n,e}(\tau) \cos(\tau) [\tau - \tan(\tau)] \left[ \sqrt{n}|n-1\rangle - \sqrt{n+1}|n+1\rangle \right] |e\rangle 
$$

$$
- i \eta N_{n,e}(\tau) \sin(\tau) \left[ 1 - i \lambda \tau (n + \eta^2/2) \right] |in;:n|g\rangle 
$$

$$
- i \frac{\lambda \eta}{2} N_{n,e}(\tau) \sin(\tau) \left[ \sqrt{n}\left|in;:n-1\right\rangle - \sqrt{n+1}\left|in;:n+1\right\rangle \right] |g\rangle, 
$$

(4.16)

Where

$$
\left[ N_{n,e}(\tau) \right]^{-2} = \left[ N_{n,g}(\tau) \right]^{-2} + \frac{\lambda \kappa}{2} \sin(2\tau) \left\{ 2\tau + \eta^2 [\tau - \tan(\tau)] \right\}. 
$$
As a third initial condition, we assume \(|\psi(0)\rangle = |\alpha\rangle |e\rangle\), which means an initial vibrational coherent state for the field and the excited state for the ion; then Eq. (4.12) becomes

\[
|\Psi(\tau)\rangle_{\text{ion},ia,e} = N^{(1)}_{ia,e}(\tau) \left\{ 1 + i \frac{\lambda}{2} (\alpha - \eta)[\tan(\tau) - i \frac{\lambda}{2} \tan(\tau)] \right\} |\alpha\rangle |e\rangle
\]

\[
- i \frac{\lambda}{2} N^{(1)}_{ia,e}(\tau) \left\{ 2 \alpha \tau - \eta[\tan(\tau)] \right\} \left( \frac{\partial}{\partial \alpha} + \alpha \right) |\alpha\rangle |e\rangle
\]

\[
- \frac{2}{\lambda} \tau N^{(1)}_{ia,e}(\tau) (2 \alpha - \eta) \sin(\tau) \left[ \frac{\partial}{\partial(\alpha - \eta)} + \alpha - \eta \right] |i(\alpha - \eta)\rangle |g\rangle
\]

\[
- i \frac{\lambda}{2} N^{(1)}_{ia,e}(\tau) \sin(\tau) \left[ 1 - i \frac{a \eta \lambda \tau}{2} \right] |i(\alpha - \eta)\rangle |g\rangle,
\]

\[(4.17)\]

with

\[
\left[ N^{(1)}_{ia,e}(\tau) \right]^{-2} = 1 + \frac{\lambda^2}{2} \left\{ a \eta (\alpha - \eta)(\eta - 4 \alpha^2) + 2 \alpha \eta^2 [\alpha (\alpha^2 + 1) - \eta (\eta^2 + 1)] \right\} + \frac{\lambda^2}{4} \sin^2(\tau)
\]

\[- \frac{\lambda^2}{8} \left\{ \eta^2 [(2 \alpha - \eta)^2 + 1] \cos(2 \tau) - 2 \tau \eta (2 \alpha - \eta) [\eta^2 + 2 \alpha (\alpha - \eta) + 1] \sin(2 \tau) \right\}
\]

\[+ \frac{\lambda^2 k}{4} \sin(2 \tau) \left\{ 2 \alpha \tau - (2 \alpha - \eta) \eta [\tau - \tan(\tau)] \right\} + \frac{\lambda^2}{8} \eta^2 (\eta^2 + 1) (2 \tau^2 + 1)
\]

\[(4.18)\]

Where we have used the coherent states properties \( \hat{a} |\alpha\rangle = \alpha |\alpha\rangle \), \( \hat{a}^\dagger |\alpha\rangle = \left( \frac{\partial}{\partial \alpha} + \frac{\alpha^*}{\alpha} \frac{\partial}{\partial \alpha^*} + \alpha^* \right) |\alpha\rangle \) and \( \hat{D}^\dagger(\alpha) \hat{D}(\alpha) = \hat{a}^\dagger + \alpha^* \) (Gerry and Knight 2005; Puri 2001; Fox 2006; Raymond Chiao 2008; Vogel 2006; Moya-Cessa and Eguibar 2011a). For simplicity, we have taken \( \alpha \) as a real number, but all calculation can be done with a \( \alpha \) complex.

### 4.2 Second order correction

Let us now turn to get the second perturbative order solution by using again the general solution (2.7), but now running \( k = 2 \); we obtain the equation

\[
|\Psi(\tau)\rangle_{\text{ion}} \approx N^{(2)}(\tau) \left[ |\psi(0)\rangle + \lambda |\psi^{(1)}\rangle + \lambda^2 |\psi^{(2)}\rangle \right],
\]

\[(4.19)\]

Where

\[
|\psi^{(2)}\rangle = -e^{-i\hat{H}_0 \tau} \int_0^\tau e^{i\hat{H}_0 \tau_1} \left( \hat{n} + \frac{\kappa}{2} \hat{\sigma}_z \right) e^{-i\hat{H}_0 \tau_1} \int_0^{\tau_1} e^{i\hat{H}_0 \tau_2} \left( \hat{n} + \frac{\kappa}{2} \hat{\sigma}_z \right) e^{-i\hat{H}_0 \tau_2} d\tau_2 d\tau_1 |\psi(0)\rangle,
\]

\[(4.20a)\]

\[
N^{(2)}(\tau)^{-2} = 1 + \lambda^2 \left[ 2 \Re \left( \langle \psi^{(1)} | \psi^{(2)} \rangle \right) + \langle \psi^{(1)} | \psi^{(1)} \rangle \right] + 2 \lambda^3 \Re \left( \langle \psi^{(1)} | \psi^{(2)} \rangle \right)
\]

\[+ \lambda^4 \left( \langle \psi^{(2)} | \psi^{(2)} \rangle \right).
\]

\[(4.20b)\]

We insert Eq. (4.9) into (4.20a) and after integration, one gets...
\[
|\psi^{(2)}\rangle = \frac{e^{-i\hat{H}_0 \tau}}{4} \left\{ -4 \sin^4(\tau) + \sin^2(2\tau) \hat{\Omega}_7 + (\hat{\Omega}_5 - \hat{\Omega}_2 + \hat{\Omega}_8) \sin(2\tau) + (4\hat{\Omega}_4 + 2\hat{\Omega}_6 - \hat{\Omega}_3) \sin^2(\tau) + \tau [-2\hat{\Omega}_8 - 2\cos(2\tau)\hat{\Omega}_2 + \sin(2\tau)\hat{\Omega}_3 - \hat{\Omega}_6) |\psi(0)\rangle \right\},
\]

(4.21)

with

\[
\hat{\Omega}_1 = \hat{n}^2 + \frac{\eta}{2} \hat{n} [\eta + i(\hat{\alpha} - \hat{\alpha}^\dagger) \hat{\sigma}_z],
\]

(4.22a)

\[
\hat{\Omega}_2 = i [\hat{\sigma}^x \hat{D}(i\eta) + \hat{\sigma}^z \hat{D}^\dagger(i\eta)] \left\{ -\eta \hat{n} + \frac{\eta^2}{4} (\hat{\alpha}^2 + \hat{\alpha}^{12}) - \frac{\eta^4}{4} - \frac{\eta^2}{4} - i \frac{\eta}{2} (\hat{\alpha} - \hat{\alpha}^\dagger) (\hat{n} + \eta^2) \hat{\sigma}_z + \left[ \frac{\kappa}{2} \hat{n} + \frac{\eta^2 \kappa}{4} + \frac{i \eta \kappa}{4} (\hat{\alpha} - \hat{\alpha}^\dagger) \right] \hat{\sigma}_z \right\},
\]

(4.22b)

\[
\hat{\Omega}_3 = \frac{\eta^2}{2} [4\hat{n} - (\hat{\alpha}^2 + \hat{\alpha}^{12}) + \eta^2 + 1] + i \eta (\hat{\alpha} - \hat{\alpha}^\dagger) (\hat{n} + \eta^2) \hat{\sigma}_z,
\]

(4.22c)

\[
\hat{\Omega}_4 = \frac{\eta}{4} \hat{n} [\eta + i(\hat{\alpha} - \hat{\alpha}^\dagger) \hat{\sigma}_z - \frac{\kappa \eta}{\eta} \hat{\sigma}_z].
\]

(4.22d)

\[
\hat{\Omega}_5 = \hat{\Omega}_2 + [\hat{\sigma}^x \hat{D}(i\eta) + \hat{\sigma}^z \hat{D}^\dagger(i\eta)] \left\{ i \frac{\eta^2}{2} \left[ \eta + i(\hat{\alpha} - \hat{\alpha}^\dagger) \hat{\sigma}_z - \frac{\kappa \eta}{\eta} \hat{\sigma}_z \right]^2 - \frac{i \eta^2}{4} \left[ \eta + i(\hat{\alpha} - \hat{\alpha}^\dagger) \hat{\sigma}_z - \frac{\kappa \eta}{\eta} \hat{\sigma}_z \right] \left[ \eta + i(\hat{\alpha} - \hat{\alpha}^\dagger) \hat{\sigma}_z \right] - \frac{\eta}{2} (\hat{\alpha} + \hat{\alpha}^\dagger) \hat{\sigma}_z \right\},
\]

(4.22e)

\[
\hat{\Omega}_6 = \frac{\kappa}{4} \left\{ 2\hat{n} + \eta [\eta + i(\hat{\alpha} - \hat{\alpha}^\dagger) \hat{\sigma}_z] \right\} \hat{\sigma}_z.
\]

(4.22f)

\[
\hat{\Omega}_7 = \frac{\kappa}{8} \left\{ \kappa \hat{\sigma}_z - \eta [\eta + i(\hat{\alpha} - \hat{\alpha}^\dagger) \hat{\sigma}_z] \right\} \hat{\sigma}_z.
\]

(4.22g)

\[
\hat{\Omega}_8 = -2i [\hat{\sigma}^x \hat{D}(i\eta) + \hat{\sigma}^z \hat{D}^\dagger(i\eta)] \hat{\Omega}_7.
\]

(4.22h)

Applying the unperturbed evolution operator \( e^{-i\hat{H}_0 \tau} \) and after some algebraic manipulation, we arrive to
\[
|\psi^{(2)}(\tau)\rangle = -\frac{\tau}{8} \cos(\tau) \left\{ 4\tau \hbar^2 + \eta^2 [\tau - \tan(\tau)](6\hbar + \eta^2 + 1) - \eta^2 [\tau - \tan(\tau)](\hat{a}^2 + \hat{a}^\dagger^2) \right\} |\psi(0)\rangle \\
- i\frac{\eta^2}{8} \cos(\tau) [\tau - \tan(\tau)(1 + \tau^2)] \left[ \hat{\sigma}^+ \hat{D}(i\eta) + \hat{\sigma}^- \hat{D}^\dagger(i\eta) \right] (2\hbar + \eta^2 + 1) |\psi(0)\rangle \\
+ i\frac{\eta^2}{8} \cos(\tau) [\tau - \tan(\tau)(1 + \tau^2)] \left[ \hat{\sigma}^+ \hat{D}(i\eta) + \hat{\sigma}^- \hat{D}^\dagger(i\eta) \right] (\hat{a}^2 + \hat{a}^\dagger^2) |\psi(0)\rangle \\
- i\frac{\lambda}{8} \cos(\tau) [\tau - \tan(\tau)] \left[ \hat{\sigma}^+ \hat{D}(i\eta) + \hat{\sigma}^- \hat{D}^\dagger(i\eta) \right] \left[ \eta + i(\hat{a} - \hat{a}^\dagger) \hat{\sigma}_z \right] |\psi(0)\rangle \\
- \frac{\eta^3}{4} \cos(\tau) [\tau - \tan(\tau)(1 + \tau^2)] \left[ \hat{\sigma}^+ \hat{D}(i\eta) - \hat{\sigma}^- \hat{D}^\dagger(i\eta) \right] (\hat{a} - \hat{a}^\dagger)|\psi(0)\rangle \\
\]

Once the second order term has been calculated, the normalization constant and the complete solution can be obtained using Eq. (4.20b) and Eq. (4.19). For practical purpose, let us consider as initial condition the field in a coherent state and the ion in the excited state, i.e., \(|\psi(0)\rangle = |i\alpha\rangle|e\rangle\); the solution to second order is given by

\[
|\Psi(\tau)\rangle_{\text{ion},i\alpha,e} = N_{i\alpha,e}^{(2)} \cos(\tau) \left\{ 1 + i \frac{\lambda\eta}{2} \left( (\alpha - \eta)[\tau - \tan(\tau)] - \frac{\kappa}{\eta} \tan(\tau) \right) \right\} |i\alpha\rangle|e\rangle \\
- \lambda^2 N_{i\alpha,e}^{(2)} F_1(\alpha, \tau)|i\alpha\rangle|e\rangle \\
- i\frac{\lambda}{2} N_{i\alpha,e}^{(2)}(\tau) \left\{ \cos(\tau)(2\alpha - \eta)\tau + \eta \tan(\tau) \right\} - 2i\lambda F_2(\alpha, \tau) \right\} |i\alpha\rangle|e\rangle \\
\left( \frac{\partial}{\partial \alpha} + \alpha \right) |i\alpha\rangle|e\rangle \\
- \frac{\lambda}{2} N_{i\alpha,e}^{(2)}(\tau) \left[ (2\alpha - \eta)\tau \sin(\tau) + 2i\lambda F_3(\alpha, \tau) \left[ \frac{\partial}{\partial (\alpha - \eta)} + \alpha - \eta \right] \right] |i\alpha - \eta\rangle|g\rangle \\
- i\lambda^2 N_{i\alpha,e}^{(2)}(\tau) F_6(\alpha, \tau) \left[ \frac{\partial^2}{\partial^2 (\alpha - \eta)} + 2(\alpha - \eta) \frac{\partial}{\partial (\alpha - \eta)} + (\alpha - \eta) \right] \left[ \frac{\partial}{\partial (\alpha - \eta)} + \alpha - \eta \right] \right] |i\alpha - \eta\rangle|g\rangle \\
- i\lambda^2 N_{i\alpha,e}^{(2)}(\tau) F_8(\alpha, \tau) \left[ \sin(\tau) \left( 1 - i\frac{\alpha\eta\lambda\tau}{2} \right) - \lambda^2 F_4(\alpha, \tau) \right] |i\alpha - \eta\rangle|g\rangle \\
\right] \left[ \frac{\partial^2}{\partial^2 \alpha} + 2\alpha \frac{\partial}{\partial \alpha} + \alpha^2 \right] |i\alpha\rangle|e\rangle.
\]

\[
(4.24)
\]

where
\[
\frac{N_{\text{av}}^{(2)}}{N^2} = 1 - \frac{\tau^2 \lambda^2}{4} a \eta (\eta^2 + 1) \sin^2(\tau) + \lambda^4 \left\{ F_1^2(\alpha, \tau) + F_2^2(\alpha, \tau) + 2 \left[ F_3^2(\alpha, \tau) + F_6^2(\alpha, \tau) \right] \\
+ F_4^2(\alpha, \tau) + F_5^2(\alpha, \tau) \right\} \\
+ \alpha^2 \lambda^4 \left\{ 2 \left[ F_1(\alpha, \tau) + 2 F_3(\alpha, \tau) \right] F_3(\alpha, \tau) + F_5^2(\alpha, \tau) \right\} \\
+ \alpha^3 \lambda^4 \left[ 2 F_2(\alpha, \tau) + \alpha F_3(\alpha, \tau) \right] F_3(\alpha, \tau) \\
+ 2 \alpha \lambda^4 \left[ F_1(\alpha, \tau) + 2 F_3(\alpha, \tau) \right] F_2(\alpha, \tau) + 2(\alpha - \eta) \lambda^4 \left[ 2 F_6(\alpha, \tau) - F_4(\alpha, \tau) \right] \\
F_5(\alpha, \tau) \\
+ (\alpha - \eta)^2 \lambda^4 \left\{ F_2^2(\alpha, \tau) - 2 [F_4(\alpha, \tau) - 2 F_6(\alpha, \tau)] F_6(\alpha, \tau) \right\} \\
+ (\alpha - \eta)^3 \lambda^4 \left[ 2 F_5(\alpha, \tau) + (\alpha - \eta) F_6(\alpha, \tau) \right] F_6(\alpha, \tau),
\]
\]
(4.25)

with

\[
F_1(\alpha, \tau) = \frac{\tau \eta}{8} \cos(\tau) \left\{ (\tau - \tan(\tau)) \left[ \alpha^2 \eta + (\eta^2 + 1)(\eta - 2\alpha) \right] + \frac{\kappa^2}{\eta} \tan(\tau) \right\},
\]
(4.26a)

\[
F_2(\alpha, \tau) = \frac{\tau}{4} \cos(\tau) \left\{ 2\alpha \tau + \eta \left[ 3\eta \alpha - 2\alpha^2 - (\eta^2 + 1) \right] (\tau - \tan(\tau)) + 2\alpha \kappa \tan(\tau) \right\},
\]
(4.26b)

\[
F_3(\alpha, \tau) = \frac{\tau}{8} \cos(\tau) \left\{ 4\tau \alpha^2 + \eta(\eta - 4\alpha) \left[ \tau - \tan(\tau) \right] \right\},
\]
(4.26c)

\[
F_4(\alpha, \tau) = \frac{\eta}{8} \cos(\tau) \left\{ \tau^2 [\alpha \eta (\alpha + \eta) + 3\alpha - \eta \tan(\tau)] \\
- \left[ 2(\alpha \kappa + \eta - \alpha) + \eta \left( \alpha^2 + \frac{\kappa^2}{\eta^2} + 1 \right) \right] \left[ \tau - \tan(\tau) \right] \right\},
\]
(4.26d)

\[
F_5(\alpha, \tau) = \frac{\cos(\tau)}{4} \left\{ \eta(\alpha \eta + \kappa + 1) \left[ \tau - \tan(\tau) \right] - (2\alpha - \eta)(\alpha \eta + 1) \tau^2 \tan(\tau) \right\},
\]
(4.26e)

\[
F_6(\alpha, \tau) = \frac{\cos(\tau)}{8} \left\{ \tau^2 [4\alpha(\eta - \alpha) - \eta^2] \tan(\tau) + \eta^2 [\tau - \tan(\tau)] \right\}.
\]
(4.26f)

### 4.3 Comparison of the perturbative solution with the small rotation approximation solution

In order to verify the validity and accuracy of our perturbative solution, we calculate the probability to find the ion in its excited state, \(P_e(\tau) = \langle \Psi(\tau) | |e\rangle \langle e| \Psi(\tau) \rangle\), and compare it with the expression
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\[ P_e(\tau)_{\text{SRA}} = \frac{1}{2} \left\{ 1 + \exp \left[ -2(\alpha - \eta/2)^2 \sin^2 \left( \tau \chi^{\text{high}} \right) \right] \cos \left[ \tau (2 - \chi^{\text{high}}) - (\alpha - \eta/2)^2 \sin \left( 2 \tau \chi^{\text{high}} \right) \right] \right\}, \]

(4.27)

where \( \chi^{\text{high}} = -\lambda^2 \eta^2 / 2 \) in the case of high intensity regime. The probability \( P_e(\tau)_{\text{SRA}} \) was calculated using the small rotation approximation (SRA) by Zuñiga-Segundo et al in reference (Zuñiga-Segundo et al. 2011), and is established in Eqs. (16) and (18) of their article. Also, it should be emphasized that does not exist other solution to compare it, since our is the first perturbative solution in this regime without consider the Lamb-Dicke approximation; i.e., \( \eta \ll 1 \).

Using Eq. (4.24) to calculate \( P_e(\tau) \), yields the following expression

\[
P_e(\tau) = \frac{N^{(2)}_{\text{ia},e}^{-2}}{\lambda^4} \left\{ - \lambda^2 \cos (\tau) [2\alpha^2 F_3(\alpha, \tau) + \alpha F_2(\alpha, \tau) + 2 F_1(\alpha, \tau)] + \lambda^4 [F_3^2(\alpha, \tau) + F_2^2(\alpha, \tau) + 2 F_1^2(\alpha, \tau)] \right\}
\]

(4.28)

\[
+ \alpha^2 \lambda^4 \left\{ 2 F_2^2(\alpha, \tau) [F_1(\alpha, \tau) + 2 F_3(\alpha, \tau)] \right\}
\]

where

\[
G_1(\alpha, \tau) = \frac{\eta \cos (\tau)}{2} \left\{ (\alpha - \eta)[\tau - \tan (\tau)] - \frac{\kappa}{\eta} \tan (\tau) \right\},
\]

(4.29a)

\[
G_2(\alpha, \tau) = \frac{\cos (\tau)}{2} \left\{ 2\alpha \tau - \eta[\tau - \tan (\tau)] \right\}. \]

(4.29b)

The \( P_e(\tau) \) obtained in our approach, Eq. (4.28), and that from the small rotation approximation are plotted in Fig. 1, for several values of the perturbative parameter \( \lambda \). It is clear that the perturbative results, indicated by the red dashed line, are sufficiently accurate to reproduce the small rotation approximation solution denoted by the black dotted line, provided the condition \( \lambda \tau \ll 1 \) holds. Otherwise, it is logical to expect that when \( \lambda \tau \gg 1 \) a substantial difference will arise between the perturbative solution and the small rotation approximation solution.

5 Perturbative solution for the Rabi model

As we commented at the end of Sect. 3, it has been shown in Casanova et al. (2018) that the ion-trap system is formally equivalent to the quantum Rabi model when we consider the unitary transformation
and we take $\Omega \rightarrow 0$ and $\eta \rightarrow \frac{\omega_0}{2}$ and $\frac{\eta g}{2} \rightarrow g$; when this transformation is done, we get the transformed Schrödinger equation

$$\hat{T} = \frac{1}{2} \sqrt{2} \left[ \hat{D}^\dagger (i\eta/2) + \hat{D}(i\eta/2) \right] + \frac{1}{2} \sqrt{2} \left[ \hat{D}^\dagger (i\eta/2) - \hat{D}(i\eta/2) \right] \hat{\sigma}_z$$

$$+ \frac{1}{2} \left[ \hat{\sigma}_+ \hat{D}(i\eta/2) - \hat{\sigma}_- \hat{D}^\dagger (i\eta/2) \right],$$

and we perform the transformation $|\phi(t)\rangle_{\text{Rabi}} = \hat{T} |\Psi(t)\rangle_{\text{ion}}$, and also get perturbative solutions of the quantum Rabi model for the weak $(g/\omega) \ll 1$ and strong $(g/\omega) \gg 1$ coupling regime. As the low intensity regime case has been considered extensively (Meekhof et al. 1996; Wallentowitz and Vogel 1997; Wallentowitz et al. 1999; Kis et al. 2001; de Matos Filho and Vogel 1996a, b; Moya-Cessa and Tombesi 2000; Casanova et al. 2018), we focus now on the high intensity regime. Taking advantage of this equivalence and applying $\hat{T}$ to expression (4.24), one gets

$$i \frac{d}{dt} |\phi(t)\rangle_{\text{Rabi}} = \hat{\mathcal{H}}_{\text{Rabi}} |\phi(t)\rangle_{\text{Rabi}},$$

where

$$\hat{\mathcal{H}}_{\text{Rabi}} = \hat{T} \hat{\mathcal{H}}_{\text{ion}} \hat{T}^\dagger = \omega \hat{n} + \frac{\omega_0}{2} \hat{\sigma}_z + ig(\hat{a} - \hat{a}^\dagger) \left( \hat{\sigma}_+ + \hat{\sigma}_- \right) - \frac{\omega_0}{2} \left( \hat{\sigma}_+ + \hat{\sigma}_- \right) + \frac{\eta g}{2}.$$
\[ |\phi(\tau)\rangle_{\text{Rabi}} \approx -N^{(2)}_{ia,e} \left\{ \cos(\tau) + i\lambda \left[ G_1(\alpha, \tau) - \frac{n}{2} G_2(\alpha, \tau) \right] \right. \\
- \lambda^2 \left[ F_1(\alpha, \tau) + \frac{n}{2} F_2(\alpha, \tau) + \frac{n^2}{4} F_3(\alpha, \tau) \right] \left| -\right\rangle |\gamma\rangle \\
+ N^{(2)}_{ia,e} \left\{ i\lambda^2 \left[ F_4(\alpha, \tau) + \frac{n}{2} F_5(\alpha, \tau) - \frac{n^2}{4} F_6(\alpha, \tau) \right] - i \sin(\tau) - \lambda \left[ G_3(\alpha, \tau) - \frac{n}{2} G_4(\alpha, \tau) \right] \right\} \left| +\right\rangle |\gamma\rangle \\
+ N^{(2)}_{ia,e} \left\{ \lambda^2 \left[ F_2(\alpha, \tau) + \eta F_3(\alpha, \tau) \right] + i\lambda G_2(\alpha, \tau) \right\} \left( \frac{\partial}{\partial \gamma} + \gamma \right) |\gamma\rangle \\
- N^{(2)}_{ia,e} \left\{ \lambda G_4(\alpha, \tau) + i\lambda^2 \left[ F_5(\alpha, \tau) - \eta F_6(\alpha, \tau) \right] \right\} \left( \frac{\partial}{\partial \gamma} + \gamma \right) |\gamma\rangle \\
+ \lambda^2 N^{(2)}_{ia,e} \left[ F_3(\alpha, \tau) |\gamma\rangle - i F_6(\alpha, \tau) |\gamma\rangle \right] \left( \frac{\partial^2}{\partial \gamma^2} + 2\gamma \frac{\partial}{\partial \gamma} + \gamma^2 \right) |\gamma\rangle \right) \\
(5.4) \]

which is the second order perturbative solution for the Rabi model and where \(|\pm\rangle = \frac{1}{\sqrt{2}} [|g\rangle \pm |e\rangle], \gamma = i(\alpha - \eta/2), G_3(\alpha, \tau) = \frac{\tan(\tau)}{2} \sin(\tau)\) and \(G_4(\alpha, \tau) = \frac{i}{2}(2\alpha - \eta) \sin(\tau)\).

6 Conclusions

We have shown that by using the Normalized Matrix Perturbation Method allowed us to produce an appropriate perturbative treatment of a single trapped ion interacting with a laser field in the high intensity regime. In fact, two features make the procedure presented here particularly attractive; the first one is that the NMPM makes the problem amenable to be solved directly without falling in the use of auxiliary unitary transformations or the Lamb–Dicke approximation; the second appealing aspect lies in the fact that the perturbative solution is capable of reproducing, with high accuracy and self-consistency, the already known results of the small rotation approximation. Albeit in a technical level the normalized perturbative solutions appear too long and complicated, such solutions generate reliable results; this is a noticeable advantage over others perturbative approaches, with rather cumbersome algebra and based on special assumptions, that produce too complicated solutions that could lead to imprecise results. We believe that our work could pave the way to address a more complicated, but physically important, generalization, which is the study of the perturbative solution of a system of \(N\) equal ions; indeed, the case of many ions is a very promising scenario, which could be useful to study the trapped-ion quantum logic operation outside the Lamb–Dicke regime.

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