ON DERIVATIONS OF PARABOLIC LIE ALGEBRAS

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ABSTRACT. Let \( K \) be an algebraically-closed, characteristic-zero field. Let \( g \) be a reductive Lie algebra over \( K \) or over \( \mathbb{R} \). Let \( q \) be a parabolic subalgebra of \( g \). We characterize the derivations of \( q \) by decomposing the derivation algebra \( \text{Der} q \) as the direct sum of ideals \( \mathcal{L} \oplus \text{ad} q \), where \( \mathcal{L} \) consists of all linear transformations on \( q \) that map into \( q \), and map the derived algebra \([q,q]\) to 0. Additionally, we include a brief literature review and several examples to provide context.

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1. INTRODUCTION

The study of derivations belongs to the classical theory of Lie algebras. We begin with the well-known result that if \( g \) is a semisimple Lie algebra over a field of characteristic not equal to two, then \( g \) admits only inner derivations \( [g,g] \), in which case \( g \cong \text{Der} g \). By 1972, Leger and Luks extended this result to the Borel algebras of \( g \) \( [8] \). More generally, their result applies to the class of Lie algebras \( b \) that can be expressed as the semidirect product \( b = a \ltimes b' \) where the subalgebra \( b' \) is nilpotent and the ideal \( a \) is abelian and acts diagonally on \( b' \) \( [8] \). This wider class of Lie algebras includes Borel subalgebras of a semisimple \( g \) but does not include parabolic subalgebras. Working independently, Tolpygo arrived at the same result for the parabolic subalgebras \( q \) of the semisimple algebra \( g \), but only in special case that the scalar field is the complex numbers \( \mathbb{C} \) \( [14] \).

The recent direction that work on derivations has taken has been to relax the definition of Lie algebra to include consideration of Lie algebras that draw scalars from commutative rings rather than from fields, characterizing derivations of specific classes of such Lie algebras \( [11, 15, 16] \). Zhang in 2008 takes a different approach, defining a new class of solvable Lie algebras over \( \mathbb{C} \) and characterizing their derivation algebras \( [18] \).

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Other work has been in the direction of considering certain maps that are similar to but potentially fail to be derivations \cite{2,3,17}. Wang et al. recently defined a product zero derivation of a Lie algebra \( \mathfrak{g} \) as a linear map \( f : \mathfrak{g} \rightarrow \mathfrak{g} \) satisfying \( [f(x), y] + [x, f(y)] = 0 \) whenever \( [x, y] = 0 \) \cite{17}. The authors go on to characterize the product zero derivations of parabolic subalgebras \( \mathfrak{q} \) of simple Lie algebras over an algebraically-closed, characteristic-zero field, ultimately showing all product zero derivations of \( \mathfrak{q} \) to be sums of inner derivations and scalar multiplication maps \cite{17}. In papers appearing in 2011 and 2012, Chen et al. consider nonlinear maps satisfying derivability and nonlinear Lie triple derivations \cite{2,3}. The authors characterize all such maps on parabolic subalgebras of a semisimple Lie algebra over \( \mathbb{C} \) as the sums of inner derivations and certain maps called quasi-derivations that may fail to be linear \cite{2,3}.

The purpose of this paper is to extend the classical results of Leger and Luks \cite{8} and Tolpygo \cite{14} to the case where \( \mathfrak{q} \) is a parabolic subalgebra of a reductive algebra \( \mathfrak{g} \). We prove the following theorem.

**Theorem.** Let \( \mathfrak{q} \) be a parabolic subalgebra of a reductive Lie algebra \( \mathfrak{g} \) over an algebraically-closed, characteristic-zero field \( \mathbb{K} \) or over \( \mathbb{R} \). Let \( \mathfrak{L} \) be the set of all linear transformations mapping \( \mathfrak{q} \) into its center \( \mathfrak{q}_Z \) and sending \( [\mathfrak{q}, \mathfrak{q}] \) to 0. Then \( \mathfrak{L} \) is an ideal of \( \text{Der} \mathfrak{q} \) and \( \text{Der} \mathfrak{q} \) decomposes as the Lie algebra direct sum

\[
\text{Der} \mathfrak{q} = \mathfrak{L} \oplus \text{ad} \mathfrak{q}.
\]

The proof of the algebraically-closed case is constructive: given a derivation \( D \) on \( \mathfrak{q} \), we explicitly construct a linear map \( L \) and an element \( x \in \mathfrak{q} \) such that \( D = L + \text{ad} x \), after which we prove that our construction satisfies the stated properties. The proof the real case, in contrast, is abstract, appealing to the complex case as applied to \( \mathfrak{q} \), the complexification of the real parabolic subalgebra \( \mathfrak{q} \).

This is largely due to the following consideration. A Lie algebra over an algebraically-closed field support a more regular structural decomposition then a real Lie algebra affords. In particular, Langland’s decomposition—an important tool in the proof of the real case—is not needed for the algebraically-closed case.

The method of proof for the algebraically-closed relies on utilization of the root system \( \Phi \) of \( \mathfrak{g} \). In order to motivate the methods employed, we offer the following example. The reader is encouraged to keep this example in mind throughout the sequel.

We consider the parabolic subalgebra \( \mathfrak{q} \) of \( \mathfrak{g} = \mathfrak{gl}_6(\mathbb{C}) \) consisting of block upper triangular matrices in block sizes 3, 2, 1 (see figure \ref{fig:parabolic}). We write \( \mathfrak{gl}_6(\mathbb{C}) = \mathfrak{g}_Z \oplus \mathfrak{g}_S \), where the center \( \mathfrak{g}_Z = C \mathfrak{I} \) and maximal semisimple ideal \( \mathfrak{g}_S = \mathfrak{s}\mathfrak{l}_6(\mathbb{C}) \). We decompose \( \mathfrak{q} \) similarly: \( \mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{q}_S \), where \( \mathfrak{q}_S = \mathfrak{q} \cap \mathfrak{g}_S \).

\( \mathfrak{g}_S \) has root space direct sum decomposition

\[
\mathfrak{g}_S = \mathfrak{h} + \sum_{i \neq j} \mathbb{C} e_{i,j}
\]

where \( \mathfrak{h} \) consists of traceless diagonal 6 \times 6 matrices. It is well known that the coroots \( h_1 = e_{ii} - e_{i+1,i+1} \) form a basis of \( \mathfrak{h} \). We further decompose \( \mathfrak{h} \) into \( t + c \), where \( t = \text{Span}\{h_1, h_2, h_4\} \) and \( c = \text{Span}\{h_3, h_5\} \) (see figure \ref{fig:root}). It follows that \( t = \mathfrak{h} \cap [\mathfrak{q}, \mathfrak{q}] \) and that \( \mathfrak{q} \) has the vector space direct sum decomposition

\[
\mathfrak{q} = \mathfrak{g}_Z + c + [\mathfrak{q}, \mathfrak{q}].
\]

In light of this decomposition (and noting that \( \mathfrak{g}_Z = \mathfrak{q}_Z \)), a linear transformation that sends \( \mathfrak{q} \) to \( \mathfrak{q}_Z \) and sends \( [\mathfrak{q}, \mathfrak{q}] \) to 0 has the block matrix form illustrated by figure \ref{fig:linear}.
The claim of the theorem—that \( \text{Der} \mathfrak{q} = \mathfrak{L} \oplus \text{ad} \mathfrak{q} \)—may then be explicitly verified via computation in this special case. The proofs of the theorem in general will rely on carrying out the same decomposition of \( \mathfrak{q} \) and the accompanying computations in abstract.

We give a brief outline of this paper. Section 2 provides the necessary background definitions and tools needed to understand the results in the sequel. Except where noted in subsection 2.2, this section does not contain original results and may be skimmed or even skipped by the expert. Section 3 treats the algebraically-closed case and section 4 treats the real case. Section 5 contains several corollaries and a short discussion of possible directions in which to generalize the results of this paper. Appendix A contains data on several classical and sporadic Lie algebras.

Before we begin, we shall make note of some conventions of terminology and notation. \( \mathbb{K} \) will always denote an algebraically-closed, characteristic-zero field. The notation \( e_{i,j} \) is used to denote the matrix with 1 in the \( i \)-th row, \( j \)-th column entry and zeros elsewhere. The notation \( I_n \) (or simply \( I \) if \( n \) is understood) denotes the \( n \times n \) identity matrix. If \( \mathfrak{g} \) is a Lie algebra, we will denote its center by \( \mathfrak{g}Z \). If \( \mathfrak{g} \) is reductive, we denote its unique maximal semisimple ideal by \( \mathfrak{g}S \). If \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) are subspaces of \( \mathfrak{g} \), intersect trivially, and together span \( \mathfrak{g} \), we write \( \mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2 \). If \( \mathfrak{a} \) is a subalgebra (denoted \( \mathfrak{a} \leq \mathfrak{g} \)) and \( \mathfrak{b} \) is an ideal (denoted \( \mathfrak{b} \unlhd \mathfrak{g} \)), we will write \( \mathfrak{g} = \mathfrak{a} \times \mathfrak{b} \) or \( \mathfrak{g} = \mathfrak{b} \rtimes \mathfrak{a} \) interchangeably. The notation \( \mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b} \) is reserved for the special case where both \( \mathfrak{a} \) and \( \mathfrak{b} \) are ideals of \( \mathfrak{g} \).

2. Preliminaries

This section develops the basic facts of the classical theory of Lie algebras which are required for an understanding of the subsequent discussion. Proofs are included as space permits. Where a particular definition, theorem, or proof is not cited in the line of the text, the reader is referred to any of the standard texts on the subject, e.g., [1, 5, 6, 7, 9, 12].
Proposition 2.1 (Ado’s Theorem). Let \( \mathbb{F} \) be a characteristic-zero field. Let \( \mathfrak{g} \) be a (finite-dimensional) Lie algebra over \( \mathbb{F} \). Then, \( \mathfrak{g} \) is isomorphic to a space of matrices with entries in \( \mathbb{F} \) and bracket \([M,N] = MN - NM\) [1, Ch. I, §7.3].

Proposition 2.2. An inner derivation maps \( \mathfrak{g} \) into \([\mathfrak{g}, \mathfrak{g}]\) and stabilizes ideals.

Proof. Let \( D \) be an inner derivation, so \( D = \text{ad}x \) for some \( x \in \mathfrak{g} \). Let \( y \in \mathfrak{g} \) be arbitrary and notice \( D(y) = \text{ad}x(y) = [x,y] \in [\mathfrak{g}, \mathfrak{g}] \), verifying the first assertion. Next, let \( a \leq \mathfrak{g} \). \( D(a) = \text{ad}x(a) = [x,a] \subseteq a \) by the definition of ideal.

An outer derivation does not necessarily stabilize ideals; however, the derived algebra and center of \( \mathfrak{g} \) are stabilized by outer derivation.

Proposition 2.3. Let \( D \) be a derivation on an arbitrary Lie algebra \( \mathfrak{g} \). \( D \) stabilizes \([\mathfrak{g}, \mathfrak{g}]\) and \( \mathfrak{g}z \).

Proof. Let \( x, y \in \mathfrak{g} \).

\[
D([x, y]) = D(x, y) + [x, D(y)] \in [\mathfrak{g}, \mathfrak{g}],
\]
so \( D \) stabilizes \([\mathfrak{g}, \mathfrak{g}]\) as desired.

Next, let \( z \in \mathfrak{g}z \). We want \( D(z) \in \mathfrak{g}z \). Let \( x \in \mathfrak{g} \) and consider \( D([z, x]) = 0 \).

\[
0 = D([z, x]) = [D(z), x] + [z, D(x)] = [D(z), x],
\]
so \([D(z), x] = 0\) for all \( x \in \mathfrak{g} \), as desired.

Let \( i \) denote the imaginary unit. Let \( \mathfrak{g} \) be a Lie algebra over \( \mathbb{R} \). In light of Ado’s Theorem (proposition 2.1), \( \mathfrak{g} \) is isomorphic to a real Lie algebra consisting of matrices with real entries, and we think of \( \mathfrak{g} \) in this way as we proceed in order to avoid several issues with notation. \( \hat{\mathfrak{g}} \) will denote the complexification of \( \mathfrak{g} \). Since \( \mathfrak{g} \) is real, we have

\[
\hat{\mathfrak{g}} = \mathfrak{g} + i\mathfrak{g} = \{x + iy \mid x, y \in \mathfrak{g}\}.
\]

We note that the bracket on \( \hat{\mathfrak{g}} \) is given by

\[
[x + iy, u + iv] = [x, u] - [y, v] + i([x, v] + [y, u]).
\]

Proposition 2.4. Let \( \mathfrak{g} \) be real, let \( \hat{\mathfrak{g}} = \mathfrak{g} + i\mathfrak{g} \) be the complexification of \( \mathfrak{g} \). Then the center of \( \hat{\mathfrak{g}} \) is the complexification of the center of \( \mathfrak{g} \), namely \( \hat{\mathfrak{g}}z = \mathfrak{g}z + i\mathfrak{g}z \).

Proof. Let \( z \in \mathfrak{g}z \). Write \( z = x + iy \) with \( x, y \in \mathfrak{g} \). Now, for arbitrary \( w = u + iv \in \hat{\mathfrak{g}} \) with \( u, v \in \mathfrak{g} \) we have

\[
0 = [z, w] = [x + iy, u + iv] = [x, u] - [y, v] + i([x, v] + [y, u])
\]

and by direct sum decomposition \([x, u] = [y, v] \) and \([x, v] = -[y, u]\). Adding these equations gives

\[
\forall u, v \in \mathfrak{g}, \quad [x, u + v] = [y, v - u].
\]

Setting \( v = u \) in equation (1) produces \([x, 2u] = 0\) for all \( u \in \mathfrak{g} \), so \( x \in \mathfrak{g}z \). Similarly, setting \( u = -v \) in equation (1) produces \( 0 = [y, 2v] \) for all \( v \in \mathfrak{g} \), so \( y \in \mathfrak{g}z \), giving \( \hat{\mathfrak{g}}z \subseteq \mathfrak{g}z \).

The reverse inclusion is clear.

Proposition 2.5. Let \( \mathfrak{g} \) be a semisimple (res. reductive) Lie algebra over \( \mathbb{R} \). The complexification \( \hat{\mathfrak{g}} \) of \( \mathfrak{g} \) is semisimple (res. reductive) [2, Ch. VI, §9].

Proposition 2.6. Let \( D \) be a derivation of the real Lie algebra \( \mathfrak{g} \). Then \( \hat{D} \) defined by \( \hat{D}(x + iy) = D(x) + iD(y) \) is a derivation of \( \hat{\mathfrak{g}} \). Moreover, \( \hat{D} \) stabilizes \( \mathfrak{g} \).
Proof. Let \( z = x + iy, w = u + iv \) be arbitrary elements of \( \hat{g} \).
\[
\hat{D}([z,w]) = \hat{D}([x+iy,u+iv]) = D([x,u] - [y,v] + i([x,v] + [y,u])) = D([x,u] - [y,v]) + iD([x,v] + [y,u]) = D([x,u]) - D([y,v]) + iD([x,v]) + D([y,u])
\]
\[
= [D(x),u] + [x,D(u)] - [D(y),v] - [y,D(v)] + i[[D(x),v] + [x,D(v)] + [D(y),u] + [y,D(u)]
\]
\[
= [D(x),u + iv] + [x + iy,D(u)] + i[x + iy,D(v)] + i[D(y),u + iv]
\]
\[
= [D(x),w] + i[D(y),w] + [z,D(u)] + i[z,D(v)]
\]
\[
= [D(x) + iD(y),w] + [z,D(u) + iD(v)]
\]
\[
= [\hat{D}(z),w] + [z,\hat{D}(w)]
\]

So \( \hat{D} \) is a derivation on \( \hat{g} \).
\( \hat{D} \) stabilizes \( \mathfrak{g} \) by definition. Indeed, if \( x \in \mathfrak{g} \), then \( \hat{D}(x) = \hat{D}(x + 0i) = D(x) \in \mathfrak{g} \). □

2.1. Langland’s decomposition. What follows is developed more completely in chapter V, section 7 of [7] in case \( \mathfrak{g} \) is over \( \mathbb{K} \) and in chapter VII, section 7 of [7] in case \( \mathfrak{g} \) is over \( \mathbb{R} \).

Let \( \mathfrak{g} \) be semisimple over \( \mathbb{K} \) or over \( \mathbb{R} \) and let \( \mathfrak{q} \subseteq \mathfrak{g} \) be a parabolic subalgebra. Without loss of generality, we may assume that \( \mathfrak{q} \) arises as a standard parabolic subalgebra from a (restricted) root space decomposition of \( \mathfrak{g} \).

In the algebraically-closed case, we have the following situation:

\[
\mathfrak{g} = \mathfrak{h} + \sum_{\beta \in \Phi} \mathfrak{g}_\beta, \text{ where}
\]
\( \mathfrak{h} \) is a Cartan subalgebra of \( \mathfrak{g} \),
\( \Phi \) is the root system of \( \mathfrak{g} \) relative to \( \mathfrak{h} \),
\( \Delta \) is a base of \( \Phi \),
\( \Delta' \subseteq \Delta \) is the subset of \( \Delta \) corresponding to \( \mathfrak{q} \), and
\( \Phi' = \Phi^+ \cup (\Phi \cap \text{Span} \Delta') \).

Then \( \mathfrak{q} = \mathfrak{h} + \sum_{\beta \in \Delta'} \mathfrak{g}_\beta \).

Considering the case where \( \mathfrak{g} \) is real, we have the analogous situation:

\[
\mathfrak{g} = \mathfrak{a} + \mathfrak{m} + \sum_{\lambda \in \Phi} \mathfrak{g}_\lambda, \text{ where,}
\]
\( \mathfrak{g} = \mathfrak{t} + \mathfrak{p} \) is the Cartan decomposition of \( \mathfrak{g} \),
\( \mathfrak{a} \) is a maximal abelian subspace of \( \mathfrak{p} \),
\( \mathfrak{m} = Z_\mathfrak{g}(\mathfrak{a}) \) is the centralizer of \( \mathfrak{a} \) in \( \mathfrak{g} \),
\( \Phi \) is the restricted root system of \( \mathfrak{g} \) relative to \( \mathfrak{a} \),
\( \Delta \) is a set of simple restricted roots of \( \Phi \),
\( \Delta' \subseteq \Delta \) is the subset of \( \Delta \) corresponding to \( \mathfrak{q} \), and
\( \Phi' = \Phi^+ \cup (\Phi \cap \text{Span} \Delta') \),
so that \( \mathfrak{q} = \mathfrak{a} + \mathfrak{m} + \sum_{\lambda \in \Phi'} \mathfrak{g}_\lambda \).

\( \Phi' \) may be partitioned into two subsets, \( \Phi' \cap -\Phi' \) and \( \Phi' \setminus -\Phi' \). This partition of \( \Phi' \)
results in a vector space direct sum decomposition of \( \mathfrak{q} \) as

\[
\mathfrak{q} = \mathfrak{t} + \mathfrak{n}
\]

where

\[
\mathfrak{t} = \mathfrak{h} + \sum_{\beta \in \Phi \cap -\Phi'} \mathfrak{g}_\beta
\]
in case \( g \) is over an algebraically-closed field, or
\[
I = a + m + \sum_{\beta \in \Phi \cap -\Phi'} g_\beta
\]
in case \( g \) is over \( \mathbb{R} \), and
\[
n = \sum_{\beta \in \Phi \setminus -\Phi'} g_\beta.
\]

**Proposition 2.7.** Notation as above, \( n \) is an ideal of \( q \), \( l \) is a subalgebra of \( q \), and \( I \) is reductive. \( \square \)

\( I \) is called the Levi factor and \( n \) is called the nilradical of \( q \). The decomposition \( q = l \ltimes n \) is referred to as Langland’s decomposition.

We extend this terminology and notion to the case where \( g = g_Z \oplus g_S \) is reductive and \( q = g_Z \oplus q_S \), by simply writing
\[
q = g_Z \oplus (l \oplus n)
\]
where \( l \) is the Levi factor and \( n \) is the nilpotent radical of \( q_S \). In such case, we say \( I \) (res. \( n \)) is the Levi factor (res. nilradical) of \( q \) and of \( q_S \) interchangeably.

In case the reader is unfamiliar with Langland’s decomposition, we submit a short example for consideration. Let \( g = gl_6(\mathbb{C}) = Cl_6 \oplus sl_6(\mathbb{C}) \), let \( h \) consist of trace-0, diagonal matrices. Then \( h \) has dimension 5. The root system \( \Phi \) will embed into a five-dimensional Euclidean space \( \subseteq h^* \). Write \( h_i = e_i - e_{i+1}, e_i \) for \( 1 \leq i \leq 5 \). Then \( \{h_i\} \) spans \( h \), and the dual functionals \( \{h_i^*\} \) span \( h^* \). By computing \( [h_i, e_{j, j+1}] \) for each pair \( (i, j) \in \{1, \ldots, 5\}^2 \) we find five simple roots \( \Delta = \{\alpha_1, \ldots, \alpha_5\} \) and their associated root spaces, recorded in table 1.

| Root \( \alpha_i \) | \( \alpha_i \) in terms of \( \{h_i^*\} \) | Root space \( \mathfrak{g}_{\alpha_i} \) |
|---------------------|--------------------------------|------------------|
| \( \alpha_1 \)      | \( (2, -1, 0, 0, 0) \)        | \( \mathfrak{e}_{1,2} \) |
| \( \alpha_2 \)      | \( (-1, 2, -1, 0, 0) \)        | \( \mathfrak{e}_{2,3} \) |
| \( \alpha_3 \)      | \( (0, -1, 2, -1, 0) \)        | \( \mathfrak{e}_{3,4} \) |
| \( \alpha_4 \)      | \( (0, 0, -1, 2, -1) \)        | \( \mathfrak{e}_{4,5} \) |
| \( \alpha_5 \)      | \( (0, 0, 0, -1, 2) \)         | \( \mathfrak{e}_{5,6} \) |

**Table 1.** Simple roots of \( sl_6(\mathbb{C}) \)

The root spaces of \( sl_6(\mathbb{C}) \) are listed in table 2. We enumerate each (positive) root \( \beta \in \Phi \) as a vector with respect to the basis \( \Delta = \{\alpha_1, \ldots, \alpha_5\} \) and also with respect to the basis \( \{h_1^*, \ldots, h_5^*\} \).

Take \( \Delta' = \{\alpha_1, \alpha_2, \alpha_4\} \). The standard parabolic subalgebra \( q \) corresponding to \( \Delta' \) consists of block upper triangular matrices corresponding to the partition \( 3 + 2 + 1 \) of 6, as illustrated in figure 3.

\[
q = \begin{pmatrix}
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & *
\end{pmatrix}
\]

**Figure 3.** The parabolic subalgebra \( q \leq gl_6(\mathbb{C}) \) corresponding to \( \Delta' = \{\alpha_1, \alpha_2, \alpha_4\} \)

Consideration of the root system \( \Phi \) shows that \( \Phi' \setminus -\Phi' = \{\pm \alpha_1, \pm \alpha_2, \pm (\alpha_1 + \alpha_2), \pm \alpha_4\} \). \( \Phi' \setminus -\Phi' \) consists of the remaining positive roots.
Lemma 2.8. Let \( q = \mathfrak{g}_Z \oplus \mathfrak{h}_S \) be a parabolic subalgebra of the reductive Lie algebra \( \mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S \) over \( K \) or over \( \mathbb{R} \). The center of \( q \) is \( \mathfrak{g}_Z \).

Proof. We consider first the special case where \( \mathfrak{g} = \mathfrak{h} + \sum_{\beta \in \Phi} \mathfrak{g}_\beta \) is semisimple over \( K \). We assume without loss of generality that \( q \) is a standard parabolic subalgebra, and \( \mathfrak{b} = \mathfrak{h} + \sum_{\beta \in \Phi^+} \mathfrak{g}_\beta \) is the standard Borel subalgebra of \( \mathfrak{g} \) such that \( \mathfrak{b} \subseteq \mathfrak{q} \). Then

\[
\mathfrak{q}_Z = Z_\mathfrak{q}(\mathfrak{q}) \subseteq Z_\mathfrak{g}(\mathfrak{q}) \subseteq Z_\mathfrak{g}(\mathfrak{b}).
\]
We prove that $Z_q(b) = 0$, so that $q_Z = 0$. Choose $x_\alpha \in g_\alpha \setminus \{0\}$ for every $\alpha \in \Phi$. Let $z = z_h + \sum_{\alpha \in \Phi} c_\alpha x_\alpha$ be an arbitrary element of $Z_q(b)$, where $z_h \in h$, $c_\alpha \in \mathbb{K}$.

1. For every $\beta \in \Phi^+$, $x_\beta \in b$ so that

$$0 = [z, x_\beta] = [z_h, x_\beta] + \sum_{\alpha \in \Phi} c_\alpha [x_\alpha, x_\beta] \in \beta(z_h)x_\beta + \left(h + \sum_{\alpha \in \Phi \setminus \{\beta\}} g_\alpha\right).$$

Therefore, $\beta(z_h) = 0$ for every $\beta \in \Phi^+$, so that $z_h = 0$.

2. Every $h \in h$ is also in $b$. So

$$0 = [z, h] = \sum_{\alpha \in \Phi} c_\alpha [x_\alpha, h] = -\sum_{\alpha \in \Phi} \alpha(h)c_\alpha x_\alpha.$$

For every $\alpha \in \Phi$, we may choose $h \in h$ such that $\alpha(h) \neq 0$. Then $c_\alpha = 0$.

The above argument shows that $z = 0$. Therefore, $q_Z = Z_q(b) = 0$.

Having established that $q_Z = 0$ when $g$ is semisimple over $\mathbb{K}$, that $q_Z = q_Z$ when $g$ is reductive over $\mathbb{K}$ follows from the Lie algebra direct sum decomposition $q = q_Z \oplus q_S$. We now consider the case where $q$ is a parabolic subalgebra of a real reductive $g$. We have $\hat{q}$ is a parabolic subalgebra of $\hat{g}$ by definition. Then

(2) $g_Z + iq_Z = (g_Z) = (\hat{q})_Z = (\hat{q}_Z) = q_Z + iq_Z$.

by above case

Finally, by Ado’s Theorem (proposition 2.7), we may assume that $g$ consists of real matrices, so that we may separate the real and imaginary part in equation 2 giving $g_Z = q_Z$, as desired. \hfill \Box

3. THE ALGEBRAICALLY-CLOSED CASE

Throughout this section we use the following notational conventions:

- $\mathbb{K}$ denotes an algebraically-closed, characteristic-zero field;
- $g = g_Z \oplus g_S$ denotes a reductive Lie algebra over $\mathbb{K}$, where
- $g_Z$ is the center of $g$, and
- $g_S$ is the maximal semisimple ideal of $g$;
- $q = g_Z \oplus q_S$ is a given parabolic subalgebra of $g$, where
- $q_S = q \cap g_S$ is a parabolic subalgebra of $g_S$.

We choose a Cartan subalgebra $h$, a root system $\Phi$, and a base $\Delta$ compatible with $q_S$ in the sense that $q_S$ is a standard parabolic subalgebra of $g_S$ relative to $(h, \Phi, \Delta)$ and corresponds to a subset $\Delta' \subseteq \Delta$. Then

$$q_S = h + \sum_{\alpha \in \Phi'} \mathbb{K}x_\alpha,$$

where

$$\Phi' = \Phi'^+ \cup (\Phi \cap \text{Span}\Delta')$$

and where each $x_\alpha$ is chosen arbitrarily from the one-dimensional root space it spans.

Define $t$ and $\mathfrak{c}$ by

$t = h \cap [q, q]$ and
$\mathfrak{c} = \text{Span}\{[x_\alpha, x_{-\alpha}] | \alpha \in \Delta \setminus \Delta'\}.$

Claim. $h$ decomposes as $h = \mathfrak{c} + t$.

Proof. Notice that $h = \text{Span}\{[x_\alpha, x_{-\alpha}] | \alpha \in \Delta\}$ and that $t = \text{Span}\{[x_\alpha, x_{-\alpha}] | \alpha \in \Delta'\}$. From these observations, we see that $\mathfrak{c} \cap t = 0$ and that $\text{Span}(\mathfrak{c} \cup t) = h$. \hfill \Box
Noting that \([q, q] = t + \sum_{\alpha \in \Phi} Kx_\alpha\), we arrive at the desired vector space direct-sum decompositions of \(q\):

\[
q = g_Z + h + \sum_{\alpha \in \Phi'} Kx_\alpha \\
= g_Z + c + t + \sum_{\alpha \in \Phi'} Kx_\alpha \\
[\alpha, q]
= g_Z + c + [q, q].
\]

We take a moment to note that alternatively \(c\), as a direct sum complement of \([q, q]\) in \(q_S\), may have been chosen so that it coincides with the center of \(l\) in Langland’s decomposition \(q_S = l + n\). This approach is not required in order to prove the algebraically-closed case, but it is taken in order to simplify the proof of theorem 4.2 in the real case.

For the remainder of the section, we assume all of the notational conventions described above without further mention, starting with a restatement of the central theorem in terms of the adopted notation.

**Theorem 3.1.** For a parabolic subalgebra \(q = g_Z \oplus q_S\) of a reductive Lie algebra \(g = g_Z \oplus g_S\) over \(K\), the derivation algebra \(\text{Der}_q q\) decomposes as the direct sum of ideals

\[
\text{Der}_q q = \Sigma \oplus \text{ad}_q,
\]

where \(\Sigma\) consists of all \(K\)-linear transformations on \(q\) mapping into \(q_Z\) and mapping \([q, q]\) to 0.

Explicitly, for any root system \(\Phi\) with respect to which \(q\) is a standard parabolic subalgebra, \(q\) decomposes as \(q = g_Z + c + [q, q]\) and the ideal \(\Sigma\) consists of all \(K\)-linear transformations on \(q\) that map \(g_Z + c\) into \(g_Z\) and map \([q, q]\) to 0, whereby

\[
\text{Der}_q q \cong \text{Hom}_K(g_Z + c, g_Z) \oplus q_S \quad \text{as Lie algebras}.
\]

We must explain what is meant by \(\text{Hom}_K(g_Z + c, g_Z)\) as a Lie algebra, since it is merely a space of linear maps and does not come equipped with a Lie bracket by default. For vector spaces \(V_1, V_2\), we consider the space \(\text{Hom}_K(V_2, V_1)\) an abelian Lie algebra. Then, \(\text{Hom}_K(V_1 + V_2, V_1)\) may be realized as the Lie algebra semidirect sum

\[
\text{Hom}_K(V_1 + V_2, V_1) = \mathfrak{gl}(V_1) \ltimes \text{Hom}_K(V_2, V_1)
\]

with the action of \(\mathfrak{gl}(V_1)\) on \(\text{Hom}_K(V_2, V_1)\) defined by

\[
f \cdot g = f \circ g \quad \forall f \in \mathfrak{gl}(V_1), g \in \text{Hom}_K(V_2, V_1).
\]

This definition is intrinsic in the sense that if we fix bases for \(V_1\) and \(V_2\), then we may identify \(\text{Hom}_K(V_1 + V_2, V_1)\) with the subalgebra of \(\mathfrak{gl}(V_1 + V_2)\) consisting of block matrices of the form illustrated in figure 5 (compare to figure 2), and the Lie bracket defined by the action above coincides with the standard Lie bracket on matrices, i.e., \([M, N] = MN - NM\).

\[
\begin{pmatrix}
V_1 & V_2 \\
V_1 \ast \ast V_2 \\
V_1 \ast \ast V_2
\end{pmatrix}
\]

**Figure 5.** Embedding of \(\text{Hom}_K(V_1 + V_2, V_1)\) in \(\mathfrak{gl}(V_1 + V_2)\)
Proof of theorem 3.1. For clarity, the proof of the theorem is organized into a progression of claims. The first three claims establish that an arbitrary derivation may be written as a sum of an inner derivation and a derivation mapping $g_{Z} + c$ to $g_{Z}$ and $[q, q]$ to 0. To this end, let $D$ be an arbitrary derivation of $q$.

Claim 1. There is an $x \in q$ such that $D - ad_x$ maps $c$ to $g_{Z}$, annihilates $t$, and stabilizes each root space $\mathbb{K}_{x_{\alpha}}$.

Let $h, k \in \mathfrak{h}$ be arbitrary and write $D(h) = z + h + \sum_{\gamma} a_{\gamma}(h)x_{\gamma}$ and $D(k) = c + k + \sum_{\gamma} a_{\gamma}(k)x_{\gamma}$ with $z, c \in g_{Z}$ and $h', k' \in \mathfrak{h}$ and $a_{\gamma}(h), a_{\gamma}(k) \in \mathbb{R}$. Recall $[h, k] = 0$ since $h, k \in \mathfrak{h}$ and consider $D([h, k])$.

\[
0 = D([h, k]) \\
= [h, D(k)] - [k, D(h)] \\
= \left[ h, c + k + \sum_{\gamma} a_{\gamma}(k)x_{\gamma} \right] - \left[ k, z + h + \sum_{\gamma} a_{\gamma}(h)x_{\gamma} \right] \\
= \left[ h, \sum_{\gamma} a_{\gamma}(k)x_{\gamma} \right] - \left[ k, \sum_{\gamma} a_{\gamma}(h)x_{\gamma} \right] \\
= \sum_{\gamma} a_{\gamma}(k)[h, x_{\gamma}] - \sum_{\gamma} a_{\gamma}(h)[k, x_{\gamma}] \\
= \sum_{\gamma} (a_{\gamma}(k)\gamma(h) - a_{\gamma}(h)\gamma(k))x_{\gamma}.
\]

So

\[ a_{\gamma}(k)\gamma(h) - a_{\gamma}(h)\gamma(k) = 0 \text{ for all } \gamma \in \Phi', h, k \in \mathfrak{h}. \]

Furthermore, for any pair $h, k$ for which $\gamma(h) \neq 0$ and $\gamma(k) \neq 0$, we have that

\[ \frac{a_{\gamma}(h)}{\gamma(h)} = \frac{a_{\gamma}(k)}{\gamma(k)}. \]

This observation, along with the fact that $\gamma(h) \neq 0$ for at least one $h \in \mathfrak{h}$, allows us to associate with each $\gamma \in \Phi'$ the numerical invariant

\[ d_{\gamma} = \frac{a_{\gamma}(h)}{\gamma(h)} \]

independently of our choice of $h$. Notice that $a_{\gamma}(h) - d_{\gamma}\gamma(h) = 0$ by definition when $\gamma(h) \neq 0$. If $\gamma(h) = 0$, the same equality still holds, as equation 3 becomes $a_{\gamma}(h)\gamma(k) = 0$ for all $k \in \mathfrak{h}$. Since at least one $k \in \mathfrak{h}$ satisfies $\gamma(k) \neq 0$ we have $a_{\gamma}(h) = 0$ in case $\gamma(h) = 0$, giving

\[ a_{\gamma}(h) - d_{\gamma}\gamma(h) = 0 \text{ for all } h \in \mathfrak{h}. \]

Now, set $x = \sum_{\gamma} -d_{\gamma}x_{\gamma}$. Write $D' = D - ad_x$. We will show that $D'$ maps $c$ to $g_{Z}$, annihilates $t$, and stabilizes each root space $\mathbb{K}_{x_{\alpha}}$. 
We first show that $D'$ maps $\mathfrak{h}$ to $\mathfrak{g}_Z + \mathfrak{h}$. Let $h \in \mathfrak{h}$ be arbitrary and again write $D(h) = z + h' + \sum_\gamma a_\gamma(h)x_\gamma$. We have that

$$D'(h) = D(h) - \text{ad}x(h)$$

$$= z + h' + \sum_\gamma a_\gamma(h)x_\gamma - \sum_\gamma -d_\gamma \text{ad}x_\gamma(h)$$

$$= z + h' + \sum_\gamma a_\gamma(h)x_\gamma - \sum_\gamma d_\gamma \gamma h x_\gamma$$

$$= z + h' + \underbrace{\sum_\gamma (a_\gamma(h) - d_\gamma \gamma h)}_{= 0 \text{ by (4)}}$$

$$= z + h',$$

affirming the assertion.

Having established that $D'$ maps $\mathfrak{h}$ into $\mathfrak{g}_Z + \mathfrak{h}$, we have left to show that $D'$ annihilates $\mathfrak{t}$ and stabilizes each $\mathbb{K}x_\alpha$. Let $h \in \mathfrak{h}$ and $\alpha \in \Phi'$ be arbitrary, and write $D'(h) = z + h'$ and $D'(x_\alpha) = c + k + \sum_\gamma b_\gamma x_\gamma$ with $z, c \in \mathfrak{g}_Z$ and $h', k \in \mathfrak{h}$ and $b_\gamma \in \mathbb{K}$. Consider $D'([h, x_\alpha])$. On one hand,

$$D'( [h, x_\alpha] ) = D'( \alpha(h)x_\alpha )$$

$$= \alpha(h)D'(x_\alpha)$$

$$= \alpha(h)c + \alpha(h)k + \sum_\gamma \alpha(h)b_\gamma x_\gamma. \tag{5}$$

On the other hand,

$$D'( [h, x_\alpha] ) = [ D'(h), x_\alpha ] + [ h, D'(x_\alpha) ]$$

$$= [z + h', x_\alpha] + \left[ h, c + k + \sum_\gamma b_\gamma x_\gamma \right]$$

$$= [h', x_\alpha] + \sum_\gamma b_\gamma [h, x_\gamma]$$

$$= \alpha(h')x_\alpha + \sum_\gamma \gamma(h)b_\gamma x_\gamma$$

$$= (\alpha(h') + \alpha(h)b_\alpha)x_\alpha + \sum_{\gamma \neq \alpha} \gamma(h)b_\gamma x_\gamma. \tag{6}$$

By equating lines (5) and (6) and by direct sum decomposition of $q$ we obtain

$$\alpha(h)c = 0, \tag{7}$$

$$\alpha(h)k = 0, \tag{8}$$

$$\alpha(h)b_\gamma = \gamma(h)b_\gamma \quad \text{for } \gamma \neq \alpha, \text{ and} \tag{9}$$

$$\alpha(h)b_\alpha = \alpha(h') + \alpha(h)b_\alpha. \tag{10}$$

Since $h$ is arbitrary, equations (7) and (8) give $c = 0$ and $k = 0$ respectively. Second, equations (9) give us $b_\gamma (\gamma - \alpha)(h) = 0$ for all $\gamma \neq \alpha$. If any one $b_\gamma \neq 0$, then we would have $\gamma = \alpha$, a contradiction, so each $b_\gamma = 0$, whence $D'$ stabilizes each root space.

Next, equation (10) gives us $0 = \alpha(h')$. Since $\alpha$ is arbitrary in $\Phi'$ and $\Phi'$ contains a basis of $\mathfrak{h}^*$, $h' = 0$, so $D'(\mathfrak{h}) \subseteq \mathfrak{g}_Z$. Since derivations in general stabilize $[q, q]$, $D'(t) \subseteq \mathfrak{g}_Z \cap [q, q] = 0$, so $D'$ annihilates $\mathfrak{t}$. The claim is verified.
Claim 2. There is an $h \in \mathfrak{h}$ whereby $D - \text{ad} x - \text{ad} h$ annihilates $[q, q]$.

We have the $D' = D - \text{ad} x$ maps $\varepsilon$ to $\mathfrak{g}_Z$, annihilate $\mathfrak{t}$, and stabilize each root space $\mathfrak{k} x_\alpha$. For each $\gamma \in \Phi'$ write
\[ D'(x_\gamma) = c_\gamma x_\gamma \]
with $c_\gamma \in \mathbb{K}$. Taking each $\alpha \in \Delta$, the scalars $c_\alpha$ define a linear functional
\[ \tilde{c} : \mathfrak{h}^* \rightarrow \mathbb{K}. \]
We first verify that for each $\gamma \in \Phi'$, $c_\gamma = \tilde{c}(\gamma)$.

We begin with $\gamma \in \Phi' \cap \Phi^+$. Let $\gamma = \alpha_1 + \ldots + \alpha_k$ with each $\alpha_i \in \Delta$ and where each sequential partial sum $\alpha_1 + \ldots + \alpha_i \in \Phi'$. Then
\[ ax_\gamma = \left[ \ldots, \left[ x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3}, \ldots, x_{\alpha_k} \right] \right] \]
for some $0 \neq a \in \mathbb{K}$. Apply $D'$ to both sides. Since $D'$ is a derivations, we have
\[
c_\gamma ax_\gamma = D' \left[ \ldots, \left[ x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3}, \ldots, x_{\alpha_k} \right] \right]
\]
\[
= \left[ \ldots, \left[ D'(x_{\alpha_1}), x_{\alpha_2}, x_{\alpha_3}, \ldots, x_{\alpha_k} \right] \right]
\]
\[
+ \left[ \ldots, \left[ x_{\alpha_1}, D'(x_{\alpha_2}), x_{\alpha_3}, \ldots, x_{\alpha_k} \right] \right]
\]
\[
+ \ldots + \left[ \ldots, \left[ x_{\alpha_1}, x_{\alpha_2}, D'(x_{\alpha_3}), \ldots, x_{\alpha_k} \right] \right]
\]
\[
+ \ldots + \left[ \ldots, \left[ x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3}, \ldots, D'(x_{\alpha_k}) \right] \right]
\]
\[
= c_{\alpha_1} \left[ \ldots, \left[ x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3}, \ldots, x_{\alpha_k} \right] \right]
\]
\[
+ c_{\alpha_2} \left[ \ldots, \left[ x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3}, \ldots, x_{\alpha_k} \right] \right]
\]
\[
+ c_{\alpha_3} \left[ \ldots, \left[ x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3}, \ldots, x_{\alpha_k} \right] \right]
\]
\[
+ \ldots + c_{\alpha_k} \left[ \ldots, \left[ x_{\alpha_1}, x_{\alpha_2}, x_{\alpha_3}, \ldots, x_{\alpha_k} \right] \right]
\]
\[
= c_{\alpha_1} ax_\gamma + \ldots + c_{\alpha_k} ax_\gamma
\]
\[
= \tilde{c}(\alpha_1 + \ldots + \alpha_k) ax_\gamma
\]
\[
= \tilde{c}(\gamma) ax_\gamma
\]
whereby $c_\gamma = \tilde{c}(\gamma)$ for all $\gamma \in \Phi' \cap \Phi^+$.

Next let $\gamma \in \Phi' \cap \Phi^-$. Consider $[x_\gamma, x_{-\gamma}] \in \mathfrak{t}$, and apply $D'$.\[
0 = D'([x_\gamma, x_{-\gamma}])
\]
\[
= [D'(x_\gamma), x_{-\gamma}] + [x_\gamma, D'(x_{-\gamma})]
\]
\[
= c_\gamma [x_\gamma, x_{-\gamma}] + c_{-\gamma} [x_\gamma, x_{-\gamma}]
\]
\[
= (c_\gamma + c_{-\gamma}) [x_\gamma, x_{-\gamma}].
\]
Since $[x_\gamma, x_{-\gamma}] \neq 0$, we have $c_\gamma + c_{-\gamma} = 0$ so
\[
c_\gamma = -c_{-\gamma}
\]
\[
= -\tilde{c}(-\gamma)
\]
since $-\gamma \in \Phi' \cap \Phi^+$
\[
= \tilde{c}(\gamma)
\]
as desired.

Next, we use the canonical isomorphism $\Psi : \mathfrak{h}^* \rightarrow \mathfrak{h}$ [10, Ch. VII, §4] to produce $\Psi(\tilde{c}) = h \in \mathfrak{h}$. Notice that for each $\gamma \in \Phi'$ we have the identity
\[
(11) \quad \tilde{c}(\gamma) - \gamma(h) = 0
\]
by the definition of the canonical isomorphism.
The claim is that $D' - \text{ad} h$ annihilates $[q, q]$. Since $[h, t] = 0$, we need only check that $D' - \text{ad} h$ maps each $x_\gamma$ to 0, for $\gamma \in \Phi^* \cap -\Phi^*$.

$$(D' - \text{ad} h)(x_\gamma) = \bar{c}(\gamma)x_\gamma - \gamma(h)x_\gamma = (\bar{c}(\gamma) - \gamma(h))x_\gamma = 0,$$

verifying the claim.

Claim 3. $D = L + \text{ad} p$ for some $p \in q$ and some derivation $L$ which maps $g_Z + c$ to $g_Z$ and maps $[q, q]$ to 0.

Set $p = x + h$ as above and set $L = D - \text{ad} p = D' - \text{ad} h$. Then $D = L + \text{ad} p$ as desired. We note that since $L$ is the difference of two derivations, $L$ is itself a derivation. We know from claim 2 that $L$ annihilates $[q, q]$. We must check that $L$ maps $g_Z + c$ to $g_Z$.

We have already seen that $g_Z$ is the center of $q$, and more, that a derivation of $q$ must stabilize the center of $q$. What is left to verify claim 3 is to check that $L$ maps $c$ into $g_Z$. Let $c \in \mathfrak{c}$ be arbitrary. We have

$$L(c) = (D' - \text{ad} h)(c) = D'(c) - [h, c] = D'(c) \in g_Z$$

by claim 2.

Since $D$ was arbitrary, we now have that $\text{Der} q$ is spanned by $\text{ad} q$ and the subset of $\text{Der} q$ consisting of derivations that map $g_Z + c$ to $g_Z$ and $[q, q]$ to 0. The next three claims establish facts about the relationship between these two sets.

Claim 4. $\mathfrak{L} \subseteq \text{Der} q$.

$\mathfrak{L}$ is defined as the set of $\mathbb{K}$-linear endomorphisms of $q$ mapping into the center of $q$ and mapping $[q, q]$ to 0. We will show that any such linear map is indeed a derivation of $q$. Suppose $L : q \rightarrow q$ is any $\mathbb{K}$-linear map satisfying $L(q) \subseteq g_Z$ and $L([q, q]) = 0$. Then, for any $x, y \in q$ we have

$$[L(x), y] + [x, L(y)] = 0 = L([x, y])$$

so $L$ is a derivation.

Claim 5. $\mathfrak{L}$ is an ideal of $\text{Der} q$.

First we note that $\mathfrak{L}$ is linearly closed: Indeed, if $L_1, L_2 \in \mathfrak{L}$, then $L_1 + kL_2$ maps into $g_Z$ and maps $[q, q]$ to 0. Second, let $L \in \mathfrak{L}$ and $D \in \text{Der} q$. We must show $[D, L] \in \mathfrak{L}$. Recall (from proposition 2.3) that $D(g_Z) \subseteq g_Z$ and $D([q, q]) \subseteq [q, q]$. Let $x \in q$. Consider $[D, L](x)$.

$$[D, L](x) = D(L(x)) - L(D(x)) \in g_Z$$

so $[D, L]$ maps $q$ into $g_Z$. Now, let $y \in [q, q]$ and consider $[D, L](y)$.

$$[D, L](y) = D(L(y)) - L(D(y)) = 0$$

so $[D, L] \in \mathfrak{L}$, verifying the claim.

Claim 6. $\mathfrak{L}$ and $\text{ad} q$ intersect trivially.
Suppose \( D \in \mathcal{L} \cap \text{ad} \mathfrak{q} \). Since \( D \in \mathcal{L} \), \( D \) maps \( \mathfrak{q} \) into \( \mathfrak{g}_Z \). Since \( D \in \text{ad} \mathfrak{q} \), \( D \) maps \( \mathfrak{q} \) into \([\mathfrak{q}, \mathfrak{q}]\). So, \( D \) maps \( \mathfrak{q} \) into \([\mathfrak{g}_Z \cap [\mathfrak{q}, \mathfrak{q}]] = 0\), whereby \( D = 0 \), completing the proof of the theorem.

As a simple application, we will use theorem 3.1 to derive a formula for the dimension of Der \( \mathfrak{q} \) in terms of \( \mathfrak{q} \) and \( \mathfrak{q} \) and their invariants.

**Corollary 3.2.** For \( \mathfrak{q} \) a parabolic subalgebra of the reductive Lie algebra \( \mathfrak{g} \cong \mathbb{K}^n \oplus \mathfrak{g}_S \) over \( \mathbb{K} \), and with notation as above, the dimension of Der \( \mathfrak{q} \) is given by

\[
\dim \text{Der} \mathfrak{q} = (n + |\Delta| - |\Delta'|)n + \dim \mathfrak{q}_S.
\]

**Proof.** The corollary follows from the isomorphism Der \( \mathfrak{q} \cong \text{Hom}_\mathbb{K} (\mathfrak{g}_Z + \mathfrak{c}, \mathfrak{g}_Z) \oplus \text{ad} \mathfrak{q} \). We have \( \dim \mathfrak{c} = |\Delta| - |\Delta'| \), and \( \dim \text{ad} \mathfrak{q} = \dim \mathfrak{q}_S \) because \( \text{ad} \mathfrak{q} \cong \mathfrak{q}/I_{\mathfrak{q}} = \mathfrak{q}_S \). □

The statement and proof of corollary 3.2 rely heavily on the explicit description of the ideal \( \mathcal{L} \) and knowledge of the dimension of the subspace \( \mathfrak{c} \subseteq \mathfrak{q} \), which in turn rely on properties of the root space decomposition for Lie algebras over algebraically-close, characteristic-zero fields. Analogous properties fail to hold in general for the restricted root space decomposition of a real Lie algebra; however, the dimension of Der \( \mathfrak{q} \) in the real case may be calculated on a case-by-case basis.

### 4. The Real Case

In this section, \( \mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S \) denotes a reductive Lie algebra over \( \mathbb{R} \) with center \( \mathfrak{g}_Z \) and maximal semisimple ideal \( \mathfrak{g}_S \). \( \mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{q}_S \) is a parabolic subalgebra of \( \mathfrak{g} \), where \( \mathfrak{q}_S = \mathfrak{q} \cap \mathfrak{g}_S \) is a parabolic subalgebra of \( \mathfrak{g}_S \).

We begin by proving a limited sense of the central theorem in the context of real Lie algebras. The proof will rely heavily on the complexification \( \hat{\mathfrak{g}} \) of \( \mathfrak{g} \), to which we will apply theorem 3.1. Afterwards, we consider the restricted root space decomposition of \( \mathfrak{g} \) and expand upon the central theorem.

**Theorem 4.1.** For a parabolic subalgebra \( \mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{q}_S \) of a reductive Lie algebra \( \mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S \) over \( \mathbb{R} \), the derivation algebra Der \( \mathfrak{q} \) decomposes as the sum of ideals

\[
\text{Der} \mathfrak{q} = \mathcal{L} \oplus \text{ad} \mathfrak{q},
\]

where \( \mathcal{L} \) consists of all \( \mathbb{R} \)-linear transformations on \( \mathfrak{q} \) mapping into \( \mathfrak{q}_Z \) and mapping \([\mathfrak{q}, \mathfrak{q}]\) to 0.

**Proof.** We may assume without loss of generality that \( \mathfrak{g} \) is realized as a set of real matrices by Ado’s Theorem (proposition 2.1). We fix the following notation:

- \( \mathfrak{i} \) denotes the imaginary unit;
- \( \hat{\mathfrak{g}} = \mathfrak{g} + i\mathfrak{g} = (\mathfrak{g}_Z + i\mathfrak{g}_S) \oplus (\mathfrak{g}_S + i\mathfrak{g}_S) \) denotes the complexification of \( \mathfrak{g} \);
- \( \mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{q}_S \) denotes a parabolic subalgebra of \( \hat{\mathfrak{g}} \);
- \( \hat{\mathfrak{q}} = \mathfrak{q} + i\mathfrak{q} = (\mathfrak{g}_Z + i\mathfrak{g}_Z) \oplus (\mathfrak{q}_S + i\mathfrak{q}_S) \) is a parabolic subalgebra of \( \hat{\mathfrak{g}} \); and
- \( \mathfrak{g}_Z = \mathfrak{g}_Z + i\mathfrak{g}_Z = \hat{\mathfrak{g}}_Z \) denotes the center of \( \hat{\mathfrak{g}} \).

Given a derivation \( D \) of \( \mathfrak{q} \), we have a corresponding derivation \( \hat{D} \) of \( \hat{\mathfrak{q}} \) given by \( \hat{D}(x + iy) = D(x) + iD(y) \). As a derivation of \( \hat{\mathfrak{q}} \), \( \hat{D} \) decomposes as \( \hat{D} = L + \text{ad}(x + iy) \) with \( L \) mapping \( \hat{\mathfrak{q}} \) into \( \mathfrak{g}_Z \) and mapping \([\hat{\mathfrak{q}}, \hat{\mathfrak{q}}]\) to 0 and with \( x, y \in \mathfrak{q} \). Note that \( L \) sends \([\mathfrak{q}, \mathfrak{q}]\) to 0, since \([\mathfrak{q}, \mathfrak{q}] \subseteq [\hat{\mathfrak{q}}, \hat{\mathfrak{q}}]\).

Let \( u \in \mathfrak{q} \). \( \hat{D} \) stabilizes \( \mathfrak{q} \), so we have

\[
D(u) = \hat{D}(u) = L(u) + \text{ad}(x + iy)(u) = L(u) + [x, u] + i[y, u] \in \mathfrak{q}
\]
Now, \( L(u) \in \mathfrak{g}_Z = \mathfrak{g}_Z + i\mathfrak{z}_Z \), so we may write \( L(u) = v_1 + iv_2 \) with \( v_1, v_2 \in \mathfrak{g}_Z \). Then

\[
D(u) = v_1 + iv_2 + [x, u] + i[y, u] = (v_1 + [x, u]) + i(v_2 + [y, u]) \in \mathfrak{g}
\]

so \( v_2 + [y, u] = 0 \), and by the direct-sum decomposition \( \mathfrak{q} = \mathfrak{g}_Z + [\mathfrak{q}, \mathfrak{q}] \), \( v_2 = 0 \) and \( [y, u] = 0 \).

In particular, we have \( L(u) = v_1 \), so \( L \) maps \( \mathfrak{q} \) into \( \mathfrak{g}_Z \). Furthermore, since \( u \) was arbitrary and \( [y, u] = 0 \), we have \( y \in \mathfrak{g}_Z \). Since \( y \in \mathfrak{g}_Z \), we have for any arbitrary \( z = u + iv \in \mathfrak{g} \),

\[
\text{ad}(x + iy)(z) = \text{ad}(x)(z) + i\text{ad}(y)(z) = \text{ad}(x(z)) + i[y, u] - [y, v] = \text{ad}(x(z)),
\]

thus we have \( \text{ad}(x + iy) = \text{ad}x \).

We now have \( D = L|_{\mathfrak{q}} + \text{ad}x \) with \( x \in \mathfrak{q} \) and \( L|_{\mathfrak{q}} \) an \( \mathbb{R} \)-linear transformation mapping \( \mathfrak{q} \) to \( \mathfrak{g}_Z = \mathfrak{q}_Z \) and \([\mathfrak{q}, \mathfrak{q}]\) to \( 0 \), as desired. We have left to check that arbitrary \( \mathbb{R} \)-linear maps sending \( \mathfrak{q} \) to \( \mathfrak{g}_Z \) and \([\mathfrak{q}, \mathfrak{q}]\) to \( 0 \) are derivations, that \( \mathfrak{L} \) as described is an ideal of \( \mathfrak{q} \), and that \( \mathfrak{L} \) and \( \text{ad} \mathfrak{q} \) intersect trivially, the proofs of which are identical to the proofs given of claims 4, 5, and 6 of theorem 3.1 respectively. \( \square \)

We now examine the relationship between the direct sum decomposition of Der\( \mathfrak{q} \) and the restricted root space decomposition of \( \mathfrak{g} \). Given a parabolic subalgebra \( \mathfrak{q} = \mathfrak{g}_Z \oplus \mathfrak{q}_S \) of a reductive real Lie algebra \( \mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S \), we may choose a restricted root space decomposition of \( \mathfrak{g} \) that is compatible with \( \mathfrak{q} \) in the sense that \( \mathfrak{q} \) is a standard parabolic subalgebra of \( \mathfrak{g} \). We may then decompose \( \mathfrak{q} \) into the sum of \( \mathfrak{q} = \mathfrak{g}_Z + c + [\mathfrak{q}, \mathfrak{q}] \) where \( c \) is an appropriately-chosen complimentary subspace, similar to the algebraically-closed case. To achieve this decomposition, we rely on Langland’s decomposition of \( \mathfrak{q}_S \), described in chapter 2. We fix the following notation pertaining to the restricted root space decomposition of \( \mathfrak{g} \):

- \( \mathfrak{g} = \mathfrak{g}_Z + a + m + \sum_{\alpha \in \Phi \mathfrak{g}_\alpha} \)
- \( \Delta \) a base of \( \Phi \)
- \( \Delta' \subseteq \Delta \) corresponding to \( \mathfrak{q}_S \)
- \( \Phi' = \Phi^+ \cup (\Phi \cap \text{Span} \Delta') \)
- \( q = \mathfrak{g}_Z + a + m + \sum_{\gamma \in \Phi'} \mathfrak{g}_\gamma \)

Write Langland’s decomposition of \( \mathfrak{q}_S \):

\[
\mathfrak{l} = a + m + \sum_{\gamma' \in \Phi' \setminus -\Phi'} \mathfrak{g}_\gamma
\]

so that \( \mathfrak{q}_S = \mathfrak{l} \oplus \mathfrak{n} \) with \( \mathfrak{l} \) reductive and \( \mathfrak{n} \) nilpotent. Write \( c \) for the center of \( \mathfrak{l} \) and \( \mathfrak{l}_S \) for the unique semisimple ideal of \( \mathfrak{l} \).

**Claim.** \( [\mathfrak{q}, \mathfrak{q}] = \mathfrak{l}_S + \mathfrak{n} \)

**Proof.** Let \( x, y \in \mathfrak{q} \). We must show \([x, y] \in \mathfrak{l}_S + \mathfrak{n}\). Without loss of generality, we may assume \( x, y \in \mathfrak{q}_S \), since their projections onto \( \mathfrak{g}_Z \) are lost upon applying bracket.

Write \( x = x_1 + x_n \) and \( y = y_1 + y_n \) with \( x_1, y_1 \in \mathfrak{l} \) and \( x_n, y_n \in \mathfrak{n} \). Then

\[
[x, y] = [x_1 + x_n, y_1 + y_n] = [x_1, y_1] + [x_1, y_n] + [x_n, y_1] + [x_n, y_n] \in \mathfrak{l}_S + \mathfrak{n}
\]

since \( \mathfrak{l} \) is reductive and \( \mathfrak{n} \) is an ideal of \( \mathfrak{q}_S \). \( \square \)
Thus we arrive at the desired decomposition,

\[ q = g_Z + l + n \]

\[ = g_Z + c + l_S + n \]

\[ = g_Z + c + [q, q]. \]

**Theorem 4.2.** For any root system \( \Phi \) with respect to which \( q \) is a standard parabolic subalgebra, \( q \) decomposes as \( q = g_Z + c + [q, q] \) (where \( c \) is the center of the Levi factor \( l \) of \( q \)) and the ideal \( \mathcal{L} \) of \( \text{Der} q \) consists of all \( R \)-linear transformation on \( q \) that map \( g_Z + c \) to \( g_Z \) and map \( [q, q] \) to 0, whereby

\[ \text{Der} q \cong \text{Hom}_R (g_Z + c, g_Z) \oplus \text{ad} q. \]

**Proof.** The proof is essentially done. The majority is merely the description of the decomposition of \( q \), already done above. We have left to show only that \( \mathcal{L} \cong \text{Hom}_R (g_Z + c, g_Z) \), which is obvious in light of the decomposition \( q = g_Z + c + [q, q]. \) \( \square \)

Because of the coarseness of the restricted root space decomposition, the dimension of \( c \) is not readily available in the real case, in contrast to the algebraically-closed case. \( \dim c \) may be calculated if given a specific real Lie algebra \( g \) and a specific standard parabolic subalgebra \( q \).

### 5. Further Remarks and Corollaries

The following three corollaries represent extremal cases of the central theorem. Corollary 5.1 applies to arbitrary parabolic subalgebras of a semisimple Lie algebra (ie, the case \( g_Z = 0 \)). Corollaries 5.2 and 5.3 apply specifically to minimal parabolic subalgebras (ie, Borel subalgebras) and maximal parabolic subalgebras (ie, the entire Lie algebra \( g \)), respectively.

**Corollary 5.1.** For a parabolic subalgebra \( q \) of a semisimple Lie algebra \( g \) over the field \( K \) or over \( R \), the derivation algebra \( \text{Der} q \) satisfies

\[ \text{Der} q = \text{ad} q. \]

**Proof.** By the central theorem, \( \text{Der} q = \mathcal{L} \oplus \text{ad} q \), and since \( g_Z = 0 \), we have \( \mathcal{L} = 0. \) \( \square \)

Corollary 5.1 was proven for Borel subalgebras of semisimple Lie algebras over an arbitrary field by Leger and Luks in [8]. Tolpygo found the same result for parabolic subalgebras of complex Lie algebras [14]. Our results show that the same is true for real Lie algebras.

**Corollary 5.2.** For a Borel subalgebra \( b = g_Z + g_0 + \sum_{\alpha \in \Phi^+} g_\alpha \) of the reductive Lie algebra \( g = g_Z \oplus g_S \) over \( K \) or over \( R \), the derivation algebra \( \text{Der} b \) satisfies

\[ \text{Der} b \cong \text{Hom}_R (g_Z + (g_0)_Z, g_Z) \oplus \text{ad} b. \]

**Proof.** Write \( b_S = g_0 + \sum_{\alpha \in \Phi^+} g_\alpha \). Since \( \sum_{\alpha \in \Phi^+} g_\alpha \) is clearly the nilpotent radical of \( b_S \), the Levi factor \( l = g_0 \). Applying the central theorem gives the result. \( \square \)

Farnsteiner proved corollary 5.2 for semisimple algebras over an algebraically-closed field in [4]. As with corollary 5.1, our results establish this fact for real Lie algebras, as well as establishing the result for reductive algebras.
**Corollary 5.3.** For a reductive Lie algebra \( \mathfrak{g} = \mathfrak{g}_Z \oplus \mathfrak{g}_S \) over \( \mathbb{K} \) or over \( \mathbb{R} \), the derivation algebra \( \text{Der} \mathfrak{g} \) satisfies

\[
\text{Der} \mathfrak{g} \cong \mathfrak{gl}(\mathfrak{g}_Z) \oplus \text{ad} \mathfrak{g}.
\]

**Proof.** \( \mathfrak{g}_S \) is its own Levi factor. Being semisimple, the center of \( \mathfrak{g}_S \) is trivial, so \( \mathfrak{L} \) consists of linear maps stabilizing \( \mathfrak{g}_Z \) and sending \( \mathfrak{g}_S = [\mathfrak{g}, \mathfrak{g}] \) to 0, which are exactly the linear maps on \( \mathfrak{g}_Z \) direct sum with the zero map on \( \mathfrak{g}_S \). \( \square \)

The next corollary provides a high-level abstract description of \( \text{Der} \mathfrak{q} \) useful for dimension-counting arguments. It is also satisfying on a theoretical level, since it relies on simple constructions that can be carried out on any Lie algebra, suggesting that the result here for parabolic subalgebras of reductive Lie algebras might be generalized to a larger classes of Lie algebras.

**Corollary 5.4.** For a parabolic subalgebra \( \mathfrak{q} \) of a reductive Lie algebra \( \mathfrak{g} \) over \( \mathbb{K} \) or over \( \mathbb{R} \), we have

\[
\text{Der} \mathfrak{q} \cong \text{Hom}(\mathfrak{q}/[\mathfrak{q}, \mathfrak{q}], \mathfrak{g}_Z) \oplus (\mathfrak{q}/\mathfrak{q}_Z).
\]

**Proof.** Recall that \( \mathfrak{q}/[\mathfrak{q}, \mathfrak{q}] \) is the minimal abelian quotient of \( \mathfrak{q} \). Since \( \mathfrak{g}_Z + \mathfrak{c} \) is abelian and since \( \mathfrak{q} = \mathfrak{g}_Z + \mathfrak{c} + [\mathfrak{q}, \mathfrak{q}] \), we have \( \mathfrak{g}_Z + \mathfrak{c} \cong \mathfrak{q}/[\mathfrak{q}, \mathfrak{q}] \). Also, \( \mathfrak{g}_Z = \mathfrak{q}_Z \), and \( \text{ad} \mathfrak{q} \cong \mathfrak{q}/\mathfrak{q}_Z \), thus the corollary. \( \square \)

Our work largely follows the results of Leger and Luks and Tolpygo. Leger’s and Luks’s results imply that all derivations of a Borel subalgebra of a simple Lie algebra are inner (over any field with characteristic not 2) \([8]\), and similarly Tolpygo’s results (applicable specifically over the complex field) imply that all derivations of a parabolic subalgebra of a semisimple Lie algebra are inner \([14]\). The full results in these papers are somewhat more general and stated in the language of cohomology: The authors prove that all cohomology group \( H^n(\mathfrak{g}, \mathfrak{g}) \) are trivial for their respective classes of Lie algebras \( \mathfrak{g} \) under consideration \([8, 14]\).

For instance, the first cohomology group \( H^1(\mathfrak{g}; \mathfrak{g}) \) of a Lie algebra \( \mathfrak{g} \) satisfies the isomorphism

\[
H^1(\mathfrak{g}; \mathfrak{g}) \cong \text{Der} \mathfrak{g}/\text{ad} \mathfrak{g},
\]

From this isomorphism, it follows that \( H^1(\mathfrak{g}; \mathfrak{g}) = 0 \) implies that all derivations of \( \mathfrak{g} \) are inner. A further application of cohomology is to extensions of a Lie algebra \( \mathfrak{b} \) by an ideal \( \mathfrak{a} \). We have the isomorphism

\[
H^2(\mathfrak{b}; \mathfrak{a}) \cong \text{Ext}(\mathfrak{b}; \mathfrak{a})
\]

so the second cohomology group \( H^2(\mathfrak{b}; \mathfrak{a}) \) parametrizes the extensions of \( \mathfrak{b} \) by \( \mathfrak{a} \).

In light of these two isomorphisms, the language and methods of cohomology provide a strong framework for discovering structural properties of \( \text{Der} \mathfrak{g} \) as they relate to properties of \( \mathfrak{g} \). Our results on derivations apply to reductive Lie algebras, trivial extensions of semisimple Lie algebras by an abelian Lie algebra. A consideration of \( H^2 \) might be employed to study the derivations of general extensions of Lie algebras.

A second vehicle for future research that we will discuss deals with the abstract form of the decomposition of \( \text{Der} \mathfrak{q} \) given in corollary 5.4. If we denote by \( \mathfrak{g} \) a parabolic subalgebra, we have that the derivation algebra \( \text{Der} \mathfrak{g} \) decomposes as

\[
\text{Der} \mathfrak{g} \cong \text{Hom}(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}_Z) \oplus \text{ad} \mathfrak{g}.
\]

The necessary constructions are completely general, motivating the following question: for which Lie algebras \( \mathfrak{g} \) does isomorphism \([12]\) hold?
We remind the reader that a Lie algebra $\mathfrak{g}$ is called complete if $\mathfrak{g}_Z = 0$ and $\mathfrak{g}$ has only inner derivations. Analogously, we propose the following definition: a Lie algebra $\mathfrak{g}$ is almost complete if isomorphism 12 holds, and as an area for future investigation we may wish to characterize the class of almost complete Lie algebras.

**APPENDIX A. DATA ON REDUCTIVE EXTENSIONS OF SIMPLE LIE ALGEBRAS**

As an aid to the reader, we providing worked examples and tabular data for reductive Lie algebras of types $A_5$, $G_2$, and $F_4$. For each example, we provide an algorithmic method for computing the center $l_Z$ of the Levi factor in the Langland’s decomposition of a parabolic subalgebra corresponding to any given subset of the base $\Delta$ of the root system. We then enumerate all standard parabolic subalgebras and give the dimensions of $\mathcal{L}$, $\mathfrak{q}_S$ (which, recall, is isomorphic to $\text{ad}\mathfrak{q}$), and $\text{Der}\mathfrak{q}$ in tabular form.

**A.1. Type $A_5$.** Let $\mathfrak{g} \cong \mathbb{C}^n \oplus \mathfrak{g}_5$ where $\mathfrak{g}_5 \cong \mathfrak{sl}_6(\mathbb{C})$. $\mathfrak{g}_5$ has the root space decomposition

$$\mathfrak{g}_5 = \mathfrak{h} + \sum_{i \neq j} \mathbb{C}e_{i,j}$$

where $\mathfrak{h}$ consists of traceless diagonal $6 \times 6$ complex matrices. Choose $\Delta = \{\alpha_1, \ldots, \alpha_5\}$ as a base where $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_{i,i+1}$. Then $\Phi^+ = \{\sum_{i=1}^5 a_i \alpha_i \mid a_i \in \{0, 1\}\}$ and $\Phi = \Phi^+ \cup -\Phi$.

Write $x_i = e_{i,i+1}$, $y_i = e_{i+1,i}$, and $h_i = [x_i, y_i] = e_{i,i} - e_{i+1,i+1}$. For each $i$, let $t_i$ be the coroot dual to $\alpha_i$, so $\alpha_i(t_j) = \delta_{i,j}$. $\mathcal{T} = \{t_1, \ldots, t_5\}$ is a basis for $\mathfrak{h}$. Partial multiplication tables for $\mathfrak{g}$ in terms of $\mathcal{H} = \{h_1, \ldots, h_5\}$ and $\mathcal{T}$ are provided in tables 3 and 4 respectively.

|   | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|
| $h_1$ | $2x_1$ | $-x_2$ | 0 | 0 | 0 |
| $h_2$ | $-x_1$ | $2x_2$ | $-x_3$ | 0 | 0 |
| $h_3$ | 0 | $-x_2$ | $2x_3$ | $-x_4$ | 0 |
| $h_4$ | 0 | 0 | $-x_3$ | $2x_4$ | $-x_4$ |
| $h_5$ | 0 | 0 | 0 | $-x_4$ | $2x_5$ |

**Table 3.** Partial multiplication table for $\mathfrak{sl}_6(\mathbb{C})$ in terms of $\mathcal{H}$

|   | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ |
|---|---|---|---|---|---|
| $t_1$ | $x_1$ | 0 | 0 | 0 | 0 |
| $t_2$ | 0 | $x_2$ | 0 | 0 | 0 |
| $t_3$ | 0 | 0 | $x_3$ | 0 | 0 |
| $t_4$ | 0 | 0 | 0 | $x_4$ | 0 |
| $t_5$ | 0 | 0 | 0 | 0 | $x_5$ |

**Table 4.** Partial multiplication table for $\mathfrak{sl}_6(\mathbb{C})$ in terms of $\mathcal{F}$

For any $\Delta' \subseteq \Delta$ with corresponding parabolic subalgebra $\mathfrak{q} = \mathfrak{g}_Z + h + \sum_{\beta \in \Phi'} \mathfrak{g}_\beta$, we make three observations. First, the derived algebra $[\mathfrak{q}, \mathfrak{q}]$ is determined by $\Delta'$ as

$$[\mathfrak{q}, \mathfrak{q}] = \text{Span} \{h_i \mid \alpha_i \in \Delta'\} + \sum_{\beta \in \Phi'} \mathfrak{g}_\beta.$$  

Second, the center $l_Z$ of the Levi factor $l$ is given by

$$l_Z = \text{Span} \{t_i \mid \alpha_i \in \Delta \setminus \Delta'\}.$$
Third, the matrix whose columns are the members of $\mathcal{F}$ written as vectors in terms of the basis $\mathcal{H}$ is the inverse of the transpose of the Cartan matrix of $\mathfrak{g}$. Figure 6 gives the Cartan matrix and the inverse transpose of $\mathfrak{g}$, and Table 5 gives members of $\mathcal{F}$ in terms of $\mathcal{H}$ and as matrices.

| $t_i$ | $t_i$ in terms of $\mathcal{H}$ | $t_i$ as a diagonal matrix |
|-------|---------------------------------|-----------------------------|
| $t_1$ | (5/6, 0, 0)                     | diag(5/6, 0, 0)             |
| $t_2$ | (2/3, 4/3, 1/2)                 | diag(2/3, 4/3, 1/2)         |
| $t_3$ | (1/2, 1, 3/2)                   | diag(1/2, 1, 3/2)           |
| $t_4$ | (1/3, 2, 1/4, 3/2)              | diag(1/3, 2, 1/4, 3/2)      |
| $t_5$ | (1/6, 1/3, 1/2, 2/3, 5/6)       | diag(1/6, 1/3, 1/2, 2/3, 5/6) |

Table 5. $\mathcal{F}$ in terms of $\mathcal{H}$ and as matrices

Utilizing the three above observations allows one to explicitly compute a basis for the ideal $\mathfrak{z}$ of $\text{Der} \mathfrak{g}$. Table 6 contains data on for all standard parabolic subalgebras of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \quad (A^T)^{-1} = \begin{bmatrix} 5/6 & 2/3 & 1/2 & 1/3 & 1/6 \\ 2/3 & 4/3 & 1 & 2/3 & 1/3 \\ 1/2 & 1 & 3/2 & 1 & 1/2 \\ 1/3 & 2/3 & 1 & 4/3 & 2/3 \\ 1/6 & 1/3 & 1/2 & 2/3 & 5/6 \end{bmatrix}$$

**Figure 6.** Cartan matrix and transpose inverse for Type $A_5$

A.2. **Type $G_2$.** Let $\mathfrak{g} \cong \mathbb{C}^2 \oplus \mathfrak{g}_5$ where $\mathfrak{g}_5$ is simple of type $G_2$. The same observations in the previous example apply to any parabolic subalgebra corresponding to a $\Delta' \subseteq \Delta$. In particular, $\mathfrak{t}_2 = \text{Span} \{ t_i \mid \alpha_i \in \Delta \setminus \Delta' \}$. Figure 7 gives the Cartan matrix for Type $G_2$ and gives the inverse transpose, whose columns are used to calculate $t_i$ in terms of the $h_i$. Table 7 gives data for the four standard parabolic subalgebras of $\mathfrak{g}$.

$$A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \quad (A^T)^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

**Figure 7.** Cartan matrix and transpose inverse for Type $G_2$

A.3. **Type $F_4$.** Let $\mathfrak{g} \cong \mathbb{C}^4 \oplus \mathfrak{g}_5$ where $\mathfrak{g}_5$ is simple of type $F_4$. Again, to any parabolic subalgebra corresponding to a $\Delta' \subseteq \Delta$ we have $\mathfrak{t}_2 = \text{Span} \{ t_i \mid \alpha_i \in \Delta \setminus \Delta' \}$. Figure 8 gives the Cartan matrix for Type $F_4$ and gives the inverse transpose, whose columns are $t_i$ in terms of the $h_i$. Table 8 gives data for all the standard parabolic subalgebras of $\mathfrak{g}$.

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad (A^T)^{-1} = \begin{bmatrix} 2 & 3 & 2 & 1 \\ 3 & 6 & 4 & 2 \\ 4 & 8 & 6 & 3 \\ 2 & 4 & 3 & 2 \end{bmatrix}$$

**Figure 8.** Cartan matrix and transpose inverse for Type $F_4$
| $\Lambda'$ | dim $\mathcal{L}$ | $\dim \mathcal{L}$ | $\dim \mathcal{Q}_S$ | $\dim \operatorname{Der}_\mathcal{Q}$ |
|---|---|---|---|---|
| $\emptyset$ | 5 | $n^2 + 5n$ | 20 | $n^2 + 5n + 20$ |
| 10000 | 4 | $n^2 + 4n$ | 21 | $n^2 + 4n + 21$ |
| 01000 | 4 | $n^2 + 4n$ | 21 | $n^2 + 4n + 21$ |
| 00100 | 4 | $n^2 + 4n$ | 21 | $n^2 + 4n + 21$ |
| 00010 | 4 | $n^2 + 4n$ | 21 | $n^2 + 4n + 21$ |
| 00001 | 4 | $n^2 + 4n$ | 21 | $n^2 + 4n + 21$ |
| 11000 | 3 | $n^2 + 3n$ | 23 | $n^2 + 3n + 23$ |
| 10100 | 3 | $n^2 + 3n$ | 22 | $n^2 + 3n + 22$ |
| 10010 | 3 | $n^2 + 3n$ | 22 | $n^2 + 3n + 22$ |
| 10001 | 3 | $n^2 + 3n$ | 22 | $n^2 + 3n + 22$ |
| 01100 | 3 | $n^2 + 3n$ | 23 | $n^2 + 3n + 23$ |
| 01010 | 3 | $n^2 + 3n$ | 22 | $n^2 + 3n + 22$ |
| 01001 | 3 | $n^2 + 3n$ | 22 | $n^2 + 3n + 22$ |
| 00110 | 3 | $n^2 + 3n$ | 23 | $n^2 + 3n + 23$ |
| 00101 | 3 | $n^2 + 3n$ | 22 | $n^2 + 3n + 22$ |
| 00011 | 3 | $n^2 + 3n$ | 23 | $n^2 + 3n + 23$ |
| 11100 | 2 | $n^2 + 2n$ | 26 | $n^2 + 2n + 26$ |
| 11010 | 2 | $n^2 + 2n$ | 24 | $n^2 + 2n + 24$ |
| 11001 | 2 | $n^2 + 2n$ | 24 | $n^2 + 2n + 24$ |
| 10110 | 2 | $n^2 + 2n$ | 24 | $n^2 + 2n + 24$ |
| 10101 | 2 | $n^2 + 2n$ | 23 | $n^2 + 2n + 23$ |
| 10011 | 2 | $n^2 + 2n$ | 24 | $n^2 + 2n + 24$ |
| 01110 | 2 | $n^2 + 2n$ | 26 | $n^2 + 2n + 26$ |
| 01101 | 2 | $n^2 + 2n$ | 24 | $n^2 + 2n + 24$ |
| 01011 | 2 | $n^2 + 2n$ | 24 | $n^2 + 2n + 24$ |
| 00111 | 2 | $n^2 + 2n$ | 26 | $n^2 + 2n + 26$ |
| 11110 | 1 | $n^2 + n$ | 30 | $n^2 + n + 30$ |
| 11101 | 1 | $n^2 + n$ | 27 | $n^2 + n + 27$ |
| 11011 | 1 | $n^2 + n$ | 26 | $n^2 + n + 26$ |
| 10111 | 1 | $n^2 + n$ | 27 | $n^2 + n + 27$ |
| 01111 | 1 | $n^2 + n$ | 30 | $n^2 + n + 30$ | 0 | $n^2$ | 35 | $n^2 + 35$ |

TABLE 6. Parabolic subalgebras of type $A_5$

| $\Lambda'$ | $\mathcal{L}$ | dim $\mathcal{L}$ | $\dim \mathcal{L}$ | $\dim \mathcal{Q}_S$ | $\dim \operatorname{Der}_\mathcal{Q}$ |
|---|---|---|---|---|---|
| $\emptyset$ | $\emptyset$ | $n^2 + 2n$ | 8 | $n^2 + 2n + 8$ |
| $\{ \alpha_1 \}$ | $\text{Span}\{h_1 + 2h_2\}$ | $n^2 + n$ | 9 | $n^2 + n + 9$ |
| $\{ \alpha_2 \}$ | $\text{Span}\{2h_1 + 3h_2\}$ | $n^2 + n$ | 9 | $n^2 + n + 9$ | 0 | $n^2$ | 14 | $n^2 + 14$ |

TABLE 7. Parabolic subalgebras of type $G_2$

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\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$\Delta$ & dim $\mathcal{L}_1$ & dim $\mathcal{L}_2$ & dim $\mathcal{Q}_1$ & dim Der $\mathcal{Q}_1$ \\
\hline
0 & 4 & $n^2 + 4n$ & 28 & $n^2 + 4n + 28$ \\
1000 & 3 & $n^2 + 3n$ & 29 & $n^2 + 3n + 29$ \\
0100 & 3 & $n^2 + 3n$ & 29 & $n^2 + 3n + 29$ \\
0010 & 3 & $n^2 + 3n$ & 29 & $n^2 + 3n + 29$ \\
0001 & 3 & $n^2 + 3n$ & 29 & $n^2 + 3n + 29$ \\
1100 & 2 & $n^2 + n$ & 31 & $n^2 + 2n + 31$ \\
1010 & 2 & $n^2 + n$ & 31 & $n^2 + 2n + 31$ \\
1001 & 2 & $n^2 + n$ & 31 & $n^2 + 2n + 31$ \\
0110 & 2 & $n^2 + n$ & 31 & $n^2 + 2n + 31$ \\
1110 & 1 & $n^2 + n$ & 37 & $n^2 + n + 37$ \\
1101 & 1 & $n^2 + n$ & 32 & $n^2 + n + 32$ \\
1011 & 1 & $n^2 + n$ & 32 & $n^2 + n + 32$ \\
0111 & 1 & $n^2 + n$ & 37 & $n^2 + n + 37$ \\
\hline
$\Delta$ & 0 & $n^2$ & 54 & $n^2 + 54$ \\
\hline
\end{tabular}
\caption{Parabolic subalgebras of type $F_4$}
\end{table}

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