Localization and Toeplitz Operators on Polyanalytic Fock Spaces

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Abstract
The well known conjecture of Coburn [L.A. Coburn, On the Berezin-Toeplitz calculus, Proc. Amer. Math. Soc. 129 (2001) 3331–3338] proved by Lo [M-L. Lo, The Bargmann Transform and Windowed Fourier Transform, Integr. equ. oper. theory, 27 (2007), 397–412.] and Englis [M. Englis, Toeplitz Operators and Localization Operators, Trans. Am. Math Society 361 (2009) 1039–1052.] states that any Gabor-Daubechies operator with window $\psi$ and symbol $a(x,\omega)$ quantized on the phase space by a Berezin-Toeplitz operator with window $\Psi$ and symbol $\sigma(z,\overline{z})$ coincides with a Toeplitz operator with symbol $D\sigma(z,\overline{z})$ for some polynomial differential operator $D$.

Using the Berezin quantization approach, we will extend the proof for polyanalytic Fock spaces. While the generation is almost mimetic for two-windowed localization operators, the Gabor analysis framework for vector-valued windows will provide a meaningful generalization of this conjecture for true polyanalytic Fock spaces and moreover for polyanalytic Fock spaces.

Further extensions of this conjecture to certain classes of Gel’fand-Shilov spaces will also be considered a-posteriori.

Keywords: Localization operators, polyanalytic Fock spaces, Toeplitz operators, Gel’fand-Shilov spaces.

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1. Introduction

1.1. State of art

Localization operators rooted in the works of Berezin [9, 10], Shubin [38], Córdoba & Fefferman [19], Daubechies [20], Wong [42] and Ameur, Makarov & Hedenmalm [1] are

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a broad class of anti-Wick operators with a wide range of applications in signal analysis (cf. \[21, 34, 23, 18\]) and quite recently in random matrix theory (cf. \[2, 32\]).

The very definition of a localization operator in the language of quantum physics (cf. \[26\], pp. 193-221) draws an intuitive construction through coherent states: if we identify each point \((x, \omega)\) on the phase space \(\mathbb{R}^2\) as a point \(z = x + i\omega\) in the complex plane \(\mathbb{C}\), the quantization of a classical observable \(\sigma(z, \tau)\) (which is a ultradistribution at best) with respect to the family of classical states \(\{|z\rangle : z \in \mathbb{C}\}\) on \(L^2(\mathbb{C}, d^2z)\) with duals \(\{\langle z| : z \in \mathbb{C}\}\rangle\) yields as an integral operator in the Bochner sense defined by

\[
S_\sigma = \int_C \sigma(z, \tau) \langle z| d^2z,
\]

where \(d^2z = \frac{dz d\tau}{2i}\) denotes the symplectic 2-form on \(\mathbb{C}\).

Now let \(d\mu(z) = e^{-|z|^2} d^2z\) be the Gaussian measure on \(\mathbb{C}\). The corresponding Hilbert space of square integrable functions with inner product \(\langle \cdot, \cdot \rangle_{d\mu}\) and norm \(\| \cdot \|_{d\mu} = \langle \cdot, \cdot \rangle^{\frac{1}{2}}_{d\mu}\) will be denoted by \(L^2(\mathbb{C}, d\mu)\).

Along this paper we will denote by \(\partial_x = \frac{1}{2} (\partial_x + i\partial_\tau)\) the standard Cauchy-Riemann operator, by \(\partial_\tau = \frac{1}{2} (\partial_x - i\partial_\tau)\) its conjugate and by \(\Delta_x\) the Laplace operator \(4\partial_x \partial_\tau = \partial_x^2 + \partial_\tau^2\). Borrowing from group theoretical backdrop terminology encoded in the Weyl representation \(W_z\):

\[
W_z \Psi(\zeta, \tau) = e^{\pi i \tau^2 x^2} \Psi (\zeta - x, \tau + \pi i z)
\]

one may interpret families of coherent states as orbit spaces of the given group. In concrete, the Weyl representation \(W_z\) is unitary and irreducible on \(L^2(\mathbb{C}, d\mu)\) and gives a projective realization for the Heisenberg group \(\mathbb{H}\) on \(\mathbb{C} \times \mathbb{R}\) endowed with the multiplication rule \((z, t) \ast (\zeta, \tau) = (z + \zeta, t + \tau + \pi i \zeta z)\) (cf. \[42\]).

In this way one may identify \(|z\rangle\) and \(|\zeta\rangle\) as the action of \(W_z\) on the classical states \(\Psi, \Theta \in L^2(\mathbb{C}, d\mu)\), that is \(|z\rangle \leftrightarrow \langle \cdot, W_z \Psi \rangle_{d\mu}\) and \(|\zeta\rangle \leftrightarrow \langle W_z \Theta, \cdot \rangle_{d\mu}\). Under this identification the resulting operator \(S_\sigma\) corresponds to the following Berezin-Toeplitz operator \(L^\sigma_{\Psi, \Theta}\) with windows \(\Psi, \Theta \in L^2(\mathbb{C}, d\mu)\) and symbol \(\sigma(z, \tau)\):

\[
L^\sigma_{\Psi, \Theta} F = \int_C \sigma(z, \tau) \langle F, W_z \Psi \rangle_{d\mu} W_z \Theta d^2z, \quad \forall F \in L^2(\mathbb{C}, d\mu).
\]

This (possibly unbounded) operator defines a localization operator inherit to the Heisenberg group \(\mathbb{H}\) (cf. \[42\], Chapter 17). Further equivalent formulations of \(S_\sigma\) such as wave packets (cf. \[14\]), Gabor-Daubechies (cf. \[21, 18\]) and Gabor-Toeplitz operators (cf. \[23\]) can also be obtained in a similar fashion by replacing the Weyl operator \(W_z\) by time-frequency shifts on the phase space \(\mathbb{R}^2\) and/or the anti-Wick operators \(S_\sigma\) by a suitable Weyl pseudo-differential operator. For an overview of Weyl pseudo-differential operators we refer to the books \[38\], Chapter IV] and \[25\], Chapter 2 and Chapter 3]. For the connection between Weyl pseudo-differential operators and anti-Wick operators we refer to the papers of Daubechies \[20\] and Coburn \[14\].

In case when \(L^\sigma_{\Psi, \Theta}\) acts on a reproducing kernel Hilbert space \(H^2(\mathbb{C}, d\mu)\) of \(L^2(\mathbb{C}, d\mu)\) such that \(Q : L^2(\mathbb{C}, d\mu) \rightarrow H^2(\mathbb{C}, d\mu)\) is a projection operator, it is therefore naturally to ask in which conditions \(L^\sigma_{\Psi, \Theta}\) and the Toeplitz operator \(F \mapsto \text{Toep}_\sigma F := Q(\sigma F)\) are equivalent. The main purpose of this statement consists in to get an amalgamation
between function-theoretical and group theoretical machinery with the purpose of comprising the structure of the Segal-Bargmann space and alike encoded on the structure of the reproducing kernels with the irreducibility and square integrability underlying the Weyl representation \( \mathcal{W} \).

This milestone treated on several papers of Berger & Coburn (cf. [11, 12, 13, 14, 15]) got some remarkable progress on the papers of Bauer [7], Bauer, Coburn & Isralowitz [8] and Coburn, Isralowitz & Li [17]. In [7] the problem of existence of a Toeplitz operator \( \text{Toep}_\sigma \) as a product of two Toeplitz operators initiated on [14, 15] was further extended to several spaces of (possibly unbounded) smooth symbols including the spaces of measurable functions with certain growth at infinity; on the paper [8] the authors used the heat flow framework of Berger and Coburn [13] to study the compactness of Berezin-Toeplitz operators for certain classes of BMO symbols; in [17] the authors fully characterize Gabor-Daubechies operators with BMO symbols using the recent results of Lo [34] and Englis [22].

1.2. The Coburn conjecture for analytic Fock spaces

Let us restrict ourselves to the case when \( H^2(\mathbb{C}, d\mu) \) is the Fock space \( \mathcal{F}(\mathbb{C}) \) and \( Q \) is the projection operator \( P : L^2(\mathbb{C}, d\mu) \to \mathcal{F}(\mathbb{C}) \). For \( \Psi = \Theta = 1 \) and \( \sigma \in L^\infty(\mathbb{C}) \), a short calculation shows that \( \text{Toep}_\sigma \) and \( L^2_{\mathcal{F}} \) coincide. In case when the constant polynomial 1 is replaced by \( \Phi_1(z) = \sqrt{\pi}z \) or \( \Phi_2(z) = \frac{\pi}{2}z^2 \), Coburn’s result (cf. [15]) under the change of variable \( z \mapsto \sqrt{\pi}z \) gives

\[
L^2_{\mathcal{F}} \cdot \Phi_1 = \text{Toep}_{\sigma + \frac{1}{2}\Delta_\sigma}, \quad L^2_{\mathcal{F}} \cdot \Phi_2 = \text{Toep}_{\sigma + \frac{1}{2}\Delta_\sigma + 2(\frac{\pi}{2})^2\sigma}.
\]

The above relations fulfill for every symbol \( \sigma(z, \overline{\sigma}) \) belonging to the algebra of polynomials \( \mathbb{C}[z, \overline{\sigma}] \) or to the algebra \( B_d(\mathbb{C}) \) of Fourier-Stieltjes transforms with compactly supported measures.

Coburn’s most general result conjectured in [15] states that for any \( \Psi \in \mathbb{C}[z, \overline{\sigma}] \cap \mathcal{F}(\mathbb{C}) \) and \( \sigma \in \mathbb{C}[z, \overline{\sigma}] \cup B_d(\mathbb{C}) \) there exists a unique polynomial differential operator \( D \) depending on \( \partial_z, \partial_{\overline{\sigma}} \) and \( \Psi \) such that

\[
L^2_\sigma \cdot \Psi = \text{Toep}_D \sigma.
\]

This conjecture was proved at a first glance by Lo in [34] when \( \text{Toep}_\sigma \) acts solely on analytic polynomials on \( \mathbb{C} \). Moreover, using a mollifier scheme based on the construction of ”cut-off” functions, the author extended relation (3) to a wide class of symbols \( E(\mathbb{C}) \) including \( \mathbb{C}[z, \overline{\sigma}] \) and \( B_d(\mathbb{C}) \) as well, using mainly dominated convergence results. Hereby

\[
E(\mathbb{C}) = \left\{ \sigma \in C^\infty(\mathbb{C}) : \forall k \in \mathbb{N}_0 \exists C, \alpha > 0 \text{ s.t. } |D^k\sigma(z, \overline{\sigma})| \leq C\alpha^{\alpha} |z|, \forall z \in \mathbb{C} \right\}.
\]

An alternative proof of (3) obtained recently by Englis [22] that works for the whole Fock space \( \mathcal{F}(\mathbb{C}) \) is beyond Wick and anti-Wick correspondence (cf. [22, pp.137-142]). In this context \( D \) yields as a Wick ordered operator obtained via the replacements \( z \mapsto -\frac{1}{\sqrt{\pi}}\partial_z \) and \( \overline{\overline{z}} \mapsto -\frac{1}{\sqrt{\pi}}\partial_{\overline{\overline{z}}} \) on the polynomial \( D(z, \overline{\overline{z}}) = \frac{\partial_z \overline{\overline{z}}}{2} |\Psi(z, \overline{\overline{z}})|^2 \).

Moreover, under weak assumptions it was shown that \( \sigma(z, \overline{\overline{z}}) \) belongs to a broader class of symbols including \( BC^\infty(\mathbb{C}) \) (space of all \( C^\infty \)-functions whose derivatives of all orders are bounded) likewise

\[
M_r = \left\{ \sigma \in C^2r(\mathbb{C}) : e^{a \frac{|z|}{|\sigma|}} \left| (\partial_z)^l (\partial_{\overline{\overline{z}}})^m \sigma \right| e^{-\frac{|z|}{r}} \in L^\infty(\mathbb{C}), \forall a > 0, \forall 0 \leq l + m \leq 2r \right\}.
\]
This later function space contains the class of symbols \( \mathbb{C}[z, \overline{z}] \), \( B_\alpha(\mathbb{C}) \) and \( E(\mathbb{C}) \).

1.3. Sketch of Results

In this paper we will provide the generalization of Coburn conjecture given by equation \([3]\) for true polyanalytic Fock spaces/generalized Barymann spaces \( F^j(\mathbb{C}) \) of order \( j \) (cf. \([41, 3]\)), with \( 0 \leq j \leq n \), and moreover for the polyanalytic Fock space \( F^n(\mathbb{C}) = L^2(\mathbb{C}, d\mu) \cap \ker (\partial)^{n+1} \) of order \( n \).

To be more concise, this framework will be centered around the Berezin-Toeplitz operators \([2]\) with windows \( \Psi, \Theta \in F^n(\mathbb{C}) \) and symbol \( \sigma(z, \overline{z}) \) and the family of Toeplitz operators \( \text{Toep}_\sigma^j \) of order \( j \) with symbol \( \sigma(z, \overline{z}) \) defined as being

\[
\text{Toep}_\sigma^j F = \mathcal{P}^j(\sigma \ F) \quad \forall \ F \in F^n(\mathbb{C}).
\]

Hereby \( \mathcal{P}^j : L^2(\mathbb{C}, d\mu) \rightarrow F^j(\mathbb{C}) \) denotes the orthogonal projection operator.

In addition, we will denote by \( \{\Phi_{j,k}\}_{k \in \mathbb{N}_0} \) the corresponding orthonormal basis of \( F^j(\mathbb{C}) \), by \( K^j(\zeta, z) \) the reproducing kernel of \( F^j(\mathbb{C}) \) resp. \( F^n(\mathbb{C}) \). We will write \( K(\zeta, z) \) instead of \( K^0(\zeta, z) = K^0(\zeta, z) \) when we refer to the reproducing kernel of the Fock space \( F(\mathbb{C}) \). The same nomenclature will be used for \( F^0(\mathbb{C}) = F^n(\mathbb{C}) \) when we refer to \( F(\mathbb{C}) \) and analogously to any operator acting on \( F(\mathbb{C}) \).

Notice that \( \text{Toep}_\sigma^j \) maps \( F^n(\mathbb{C}) \) onto \( F^j(\mathbb{C}) \). Moreover for \( \sigma \in L^\infty(\mathbb{C}) \) the boundeness property \( \|\text{Toep}_\sigma^j\| \leq \|\sigma\|_{L^\infty(\mathbb{C})} \) is then immediate from construction while the following explicit formula for \( \text{Toep}_\sigma^j \):

\[
(\text{Toep}_\sigma^j F)(\zeta) = \int_{\mathbb{C}} \sigma(z, \overline{z}) F(z, \overline{z}) K^j(\zeta, z) d\mu(z)
\]

(4)

follows straightforwardly from \([3\) Corollary 7].

The theorem formulated below corresponds to the generalization of Coburn conjecture for true polyanalytic Fock spaces underlying the class of \( BC^\infty(\mathbb{C}) \) symbols:

**Theorem 1.1.** Let \( \Psi \in F^k(\mathbb{C}) \cap \mathbb{C}[z, \overline{z}] \) and \( \Theta \in F^l(\mathbb{C}) \cap \mathbb{C}[z, \overline{z}] \) such that \( \deg(\Psi), \deg(\Theta) < \infty \).

If \( e^{\overline{z}\Delta_k} (\Phi_{k,j}(z, \overline{z}) \Phi_{j,j}(z, \overline{z})) \) divides \( e^{\overline{z}\Delta_k} (\Psi(z, \overline{z}) \Theta(z, \overline{z})) \) then there exists a polynomial \( D_{j,k}(z, \overline{z}) \) of degree \( \deg(D_{j,k}) = \deg(\Psi) + \deg(\Theta) - 2j - 2k \) such that for each \( \sigma \in BC^\infty(\mathbb{C}) \) the operator \( D_{j,k} := D_{j,k} (\frac{1}{\sqrt{\pi}} \partial_{\overline{z}} - \frac{1}{\sqrt{\pi}} \partial_z) \) satisfies \( D_{j,k} \sigma \in L^\infty(\mathbb{C}) \) and

\[
\mathcal{L}_{\sigma, \Theta} = \text{Toep}_{j,k}^j.
\]

Moreover \( D_{j,k} \) is uniquely determined by

\[
D_{j,k} = \frac{e^{\overline{z}\Delta_k} (\Psi - \frac{1}{\sqrt{\pi}} \partial_{\overline{z}} - \frac{1}{\sqrt{\pi}} \partial_z) \Theta (\frac{1}{\sqrt{\pi}} \partial_{\overline{z}} - \frac{1}{\sqrt{\pi}} \partial_z)}{e^{\overline{z}\Delta_k} (\Phi_{k,j} - \frac{1}{\sqrt{\pi}} \partial_{\overline{z}} - \frac{1}{\sqrt{\pi}} \partial_z) \Phi_{j,j} (\frac{1}{\sqrt{\pi}} \partial_{\overline{z}} - \frac{1}{\sqrt{\pi}} \partial_z)}.
\]

Although the method of proof is similar to that of \([22]\), the proof of Theorem 1.1 includes the two-windowed case that can belong to true polyanalytic Fock spaces of
different orders. In addition it highlights more explicitly the interplay between time-frequency analysis and polyanalytic function spaces tactically described on the papers [3, 6] (see Section 2 of this paper) that in turns yields a meaningful generalization of Theorem 1.1 to the polyanalytic Fock space $F^n(C)$ (see Corollary 3.7).

Most of the framework performed in Section 3 uses a two-windowed extension of the Berezin symbol/Berezin transform for generalized Bargmann spaces (cf. [35, 4]). As we will see in Section 4, this kind of symbols allow us to get a constructive proof for the conjecture providing at the same time a natural extension for a wide class of symbols.

Motivated from the modulation spaces framework developed by Janssen & Van Eijndhoven [33], Gröchenig & Zimmermann [29], Teofanov [40] and others we will show in Theorem 4.8 that the Gel’fand-Shilov type spaces resp. tempered ultradistributions introduced in [27] arise naturally as the appropriate symbol classes resp. window classes for studying Berezin-Toeplitz operators on polyanalytic Fock spaces.

2. Bargmann-Fock representations for Polyanalytic Fock spaces

2.1. The true polyanalytic Fock spaces revisited

We will explain the construction of true polyanalytic Fock spaces using the interrelation between the structure of the Heisenberg group and expansions in terms of special functions using the same order of ideas of Thangavelu’s book (see [39, Section 1.2]). Similar constructions can be found of the papers of Askour, Intissar & Mouayn [3], Vasilevski [41] and Haimi & Hedenmalm [32, Section 2]. Let us now turn again our attention to the Weyl representation $W_\zeta$ defined in (1).

The left invariant vector-fields associated to $W_\zeta$ on $\mathbb{H}$ correspond to the generators $I, Z$ and $Z^\dagger$ of the Lie algebra $\mathfrak{h}$ defined viz

$$I : \Psi(z,\bar{z}) \mapsto \Psi(z,\bar{z}),$$

$$Z : \Psi(z,\bar{z}) \mapsto \frac{1}{\sqrt{\pi}} \partial_{\overline{z}} \Psi(z,\bar{z}),$$

$$Z^\dagger : \Psi(z,\bar{z}) \mapsto \frac{1}{\sqrt{\pi}} \partial_{z} \Psi(z,\bar{z}),$$

while the right invariant vector-fields correspond to the generators $\overline{I}, \overline{Z}$ and $\overline{Z}^\dagger$ of $\mathfrak{h}$ with

$$\overline{I} : \Psi(z,\bar{z}) \mapsto \overline{\Psi(z,\bar{z})},$$

$$\overline{Z} : \Psi(z,\bar{z}) \mapsto \frac{1}{\sqrt{\pi}} \partial_{\overline{z}} \overline{\Psi(z,\bar{z})},$$

$$\overline{Z}^\dagger : \Psi(z,\bar{z}) \mapsto \frac{1}{\sqrt{\pi}} \partial_{z} \overline{\Psi(z,\bar{z})}.$$}

Therefore $e^{\pi(\zeta z' - \zeta z)} \Psi(z,\bar{z}) = W_\zeta \Psi(z,\bar{z})$ and $e^{\pi(\zeta \overline{z'} - \zeta \overline{z})} \overline{\Psi(z,\bar{z})} = \overline{W_\zeta \Psi(z,\bar{z})}$ follows from direct combination of Taylor series expansion around the point $(\zeta, \overline{\zeta})$ with the direct application of Baker-Campbell-Hausdorff formula (cf. [43]):

$$e^R e^S = e^{\frac{1}{2}[R,S]} e^{R+S} \quad \text{whenever} \quad [R, [R, S]] = 0 = [S, [R, S]].$$

The properties below underlying $I, Z$ and $Z^\dagger$ resp. $\overline{I}, \overline{Z}$ and $\overline{Z}^\dagger$ follows from construction and direct application of integration by parts:

i) Weyl-Heisenberg relations:

$$[Z, Z^\dagger] = I, \quad [I, Z] = 0, \quad [I, Z^\dagger] = 0,$$

$$[\overline{Z}, \overline{Z}^\dagger] = I, \quad [\overline{I}, \overline{Z}] = 0, \quad [\overline{I}, \overline{Z}^\dagger] = 0.$$
ii) Vacuum vector property: \( Z \Phi(z) = 0 \) whenever \( \Phi \) is anti-analytic on \( \mathbb{C} \) and \( \overline{Z} \Phi(z) = 0 \) whenever \( \Phi \) is analytic on \( \mathbb{C} \).

iii) Adjoint property:

\[
\langle Z \Phi, \Psi \rangle_{d\mu} = \langle \Phi, Z^\dagger \Psi \rangle_{d\mu} \quad \text{and} \quad \langle Z \Phi, \Psi \rangle_{d\mu} = \langle \Phi, Z^\dagger \Psi \rangle_{d\mu}.
\]

Next, for each \( 0 \leq j \leq n \), we define the family of subspaces of \( F^n(\mathbb{C}) \) resp. \( L^2(\mathbb{C}, d\mu) \) using the Fock formalism (cf. [24]):

\[
F^j(\mathbb{C}) = \left\{ \frac{1}{\sqrt{j!}} (Z^\dagger)^j \Phi(z) : \Phi \in \ker Z, \|\Phi\|_{d\mu} = 1 \right\}.
\]

These subspaces are described in terms of the right invariant vector-fields (6) that yield as a direct application of quantum field lemma associated to the second quantization approach (cf. [24]). In particular they are eigenspaces of the magnetic Laplacian \( Z^\dagger Z = \frac{\partial}{\partial z} - \frac{1}{4\pi} \Delta_z \) with eigenvalue \( j \) that include complex Hermite polynomials, complex Laguerre polynomials as well as Fourier expansions of it on \( L^2(\mathbb{C}, d\mu) \). Moreover, the corresponding direct sum decompositions:

\[
F^n(\mathbb{C}) = \bigoplus_{j=0}^n F^j(\mathbb{C}) \quad \text{and} \quad L^2(\mathbb{C}, d\mu) = \bigoplus_{j=0}^\infty F^j(\mathbb{C})
\]

follow from the fact that the family of subspaces \( \{F^j(\mathbb{C})\}_{0 \leq j \leq n} \) are mutually orthogonal and dense in \( L^2(\mathbb{C}, d\mu) \).

2.2. The time-frequency approach

Now we will summarize how the time-frequency analysis framework enters into account in the description of (true) polyanalytic Fock spaces. Most of this results can be found on the papers of Gröchenig & Lyubarskii [30, 31] and Abreu [5, 6]. Most of the time-frequency setting that we will use here and elsewhere is based on the book of Gröchenig [28].

For each \( \psi \in L^2(\mathbb{R}) \), let us denote by \( T_x \psi(t) = \psi(t - x) \) a translation by \( x \in \mathbb{R} \) by \( M_\omega \psi(t) = e^{2\pi i\omega t} \psi(t) \) a modulation by \( \omega \in \mathbb{R} \) and by \( M_u T_x \psi(t) = e^{2\pi i\omega t} \psi(t - x) \) a time-frequency shift by \((x, \omega) \in \mathbb{R}^2\). The short-time Fourier transform (shortly, STFT) with window \( \psi \in L^2(\mathbb{R}) \) corresponds to

\[
(V_\psi f)(x, \omega) = (f, M_\omega T_x \psi)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(t) \overline{\psi(t-x)} e^{-2\pi i\omega t} dt.
\]

This transform possess many structural properties underlying the phase space \( \mathbb{R}^2 \). In particular, the following ones will be useful on the sequel:

**Covariance property** (cf. [28, Lemma 3.1.3]) Whenever \( V_\psi \) is defined, for any \( (x, \omega), (u, \eta) \in \mathbb{R}^2 \), we have

\[
V_\psi(T_u M_\eta f)(x, \omega) = e^{-2\pi i\omega \eta} V_\psi f(x - u, \omega - \eta).
\]

In particular \( |V_\psi(T_u M_\eta f)(x, \omega)| = |V_\psi f(x - u, \omega - \eta)| \).
Orthogonality relations (cf. [28, Theorem 3.2.1]) Let \( f, g, \phi, \psi \in L^2(\mathbb{R}) \). Then

\[
(V_\phi f, V_\psi g)_{L^2(\mathbb{R}^2)} = (f, g)_{L^2(\mathbb{R})} (\psi, \phi)_{L^2(\mathbb{R})}.
\]

(13)

Let us restrict ourselves to the STFT underlying a (normalized) Hermite function of order \( N \) (cf. [28, Theorem 3.2.1]) Let

\[
\text{Orthogonality relations}
\]

corresponds to a meaningful generalization of the Bargmann transform

The properties below correspond to a generalization of the results obtained in [28] (see Proposition 3.4.1) and follow straightforwardly by few calculations and by a direct application of the orthogonality relations [19] (cf. [30, 31, 5, 6]).

**Proposition 2.1.** If \( f \) is a function on \( \mathbb{R} \) such that for each \( t \in \mathbb{R} \) \(|f(t)| = O(|t|^N)\) holds for \( N \) sufficiently large, then:

1. \( \mathcal{B}^j f \) is given componentwise by

\[
(\mathcal{B}^j f)(z) = (\pi^j j!)^{-\frac{1}{2}} \sum_{l=0}^{j} \binom{j}{l} (-\pi z)^{j-l} (\partial_z)^l (\mathcal{B} f)(z).
\]

2. The function \( z \mapsto (\mathcal{B}^j f)(z) \) is polyanalytic of order \( j + 1 \):

\[
(\partial_z)^{j+1} (\mathcal{B}^j f)(z) = 0.
\]

3. For any \( f, g \in L^2(\mathbb{R}) \) we then have \( (f, g)_{L^2(\mathbb{R})} = (\mathcal{B}^j f, \mathcal{B}^j g)_{d\mu} \).

Thus \( \mathcal{B}^j : L^2(\mathbb{R}) \to \mathcal{F}^j(\mathbb{C}) \) is an isometry.

As a direct consequence of Proposition 2.1 \( \mathcal{B}^j [L^2(\mathbb{R})] = \mathcal{F}^j(\mathbb{C}) \) and moreover, the collection of polynomials \( \{\Phi_k\}_{k \in \mathbb{N}_0} \) and \( \{\Phi_{j,k}\}_{k \in \mathbb{N}_0} \) defined as

\[
\Phi_k(z) = \left(\frac{\pi^k}{k!}\right)^{\frac{1}{2}} z^k, \quad \Phi_{j,k}(z, \overline{z}) = (\pi^j j!)^{-\frac{1}{2}} \sum_{l=0}^{j} \binom{j}{l} (-\pi z)^{j-l} (\partial_z)^l (\Phi_k(z))
\]

(17)
provide a natural basis to the spaces $\mathcal{F}(\mathbb{C})$ and $\mathcal{F}^j(\mathbb{C})$, respectively. Moreover they satisfy $\Phi_k(z) = (Bh_k)(z)$, $\Phi_{0,k}(z,\overline{z}) = \Phi_k(z)$, $\Phi_{j,k}(z,\overline{z}) = (B^j h_k)(z)$ and the following raising/lowering properties:

\[
\begin{align*}
\left(\sqrt{\pi}z - \frac{1}{\sqrt{\pi}}\partial_{\overline{z}}\right) \Phi_{j,k}(z,\overline{z}) &= \sqrt{k+1} \Phi_{j,k+1}(z,\overline{z}) \\
\left(\sqrt{\pi}z - \frac{1}{\sqrt{\pi}}\partial_z\right) \Phi_{j,k}(z,\overline{z}) &= -\sqrt{j+1} \Phi_{j+1,k}(z,\overline{z}) \\
\frac{1}{\sqrt{\pi}}\partial_{\overline{z}} \Phi_{j,k}(z,\overline{z}) &= \sqrt{k} \Phi_{j,k-1}(z,\overline{z}) \\
\frac{1}{\sqrt{\pi}}\partial_z \Phi_{j,k}(z,\overline{z}) &= -\sqrt{j} \Phi_{j-1,k}(z,\overline{z}).
\end{align*}
\]

(18) (19)

Combining the Taylor series expansion of the operator $e^{-\frac{1}{4}\Delta_z}$ with the lowering properties (19), one can recast $\Phi_{j,k}(z,\overline{z})$ as a series expansion in terms of the basis functions $\{\Phi_k\}_{k \in \mathbb{N}_0}$ of $\mathcal{F}(\mathbb{C})$:

\[
\Phi_{j,k}(z,\overline{z}) = (\pi^j j!)^{-\frac{1}{4}} \sum_{l=0}^{j} \frac{1}{l!(\pi^j)^l} (\partial_{\overline{z}})^l (-\pi z)^l (\partial_z)^l (\Phi_k(z)) \\
= \sum_{l=0}^{\infty} \frac{1}{l!(\pi^j)^l} (\partial_{\overline{z}})^l (\partial_z)^l (\Phi_j(\overline{z})\Phi_k(z)) \\
= e^{-\frac{1}{4}\Delta_z} (\Phi_j(\overline{z})\Phi_k(z)).
\]

(20)

**Remark 2.2.** The true polyanalytic Bargmann transform of order $j$ [13] is only onto when restricted to the subspace $\mathcal{F}^j(\mathbb{C})$. Moreover the inverse for $B^j : L^2(\mathbb{R}) \rightarrow \mathcal{F}^j(\mathbb{C})$ is given by the adjoint mapping $(B^j)\dagger : \mathcal{F}^j(\mathbb{C}) \rightarrow L^2(\mathbb{R})$.

From the border view of representation theory (see [28, Section 9.2] and references given there) this follows from the fact that the square integrable representation $z \mapsto W_z$ of $L^2(\mathbb{C},d\mu)$ is reducible on $\mathbb{F}^n(\mathbb{C})$ but irreducible on each $\mathcal{F}^j(\mathbb{C})$.

The mutual orthogonality relations (13) underlying the (normalized) Hermite functions [13] together with (10) shows that for each $F,G \in \mathbb{F}^n(\mathbb{C})$ the inner product $(F,G)_{d\mu}$ is uniquely determined by the inner product between the vector-valued functions $\overrightarrow{f} = (f_0, f_1, \ldots, f_n)$ and $\overrightarrow{g} = (g_0, g_1, \ldots, g_n)$ on the Hilbert module $L^2(\mathbb{R}; \mathbb{C}^{n+1})$ such that $P^j F = B^j f_j$ and $P^j G = B^j g_j$, that is

\[
\langle F,G \rangle_{d\mu} = \sum_{j=0}^{n} \langle B^j f_j, B^j g_j \rangle_{d\mu} = \langle \overrightarrow{f}, \overrightarrow{g} \rangle_{L^2(\mathbb{R}; \mathbb{C}^{n+1})}.
\]

Therefore the isometry $B^n : L^2(\mathbb{R}; \mathbb{C}^{n+1}) \rightarrow \mathbb{F}^n(\mathbb{C})$ defined viz

\[
B^n \overrightarrow{f} = \sum_{j=0}^{n} B^j f_j.
\]

(21)

is rather natural and corresponds to a superposition of the true polyanalytic Bargmann transforms [13].
A short calculation shows that the intertwining properties underlying the time-frequency shifts \( M_\eta T_u \) and the Bargmann shifts \( \beta_{u+\eta} = e^{iu\eta}W_{u-\eta} \) for any \( 0 \leq j \leq n \) can be extended from linearity to \( L^2(\mathbb{R}; C^{n+1}) \). This corresponds to

\[
\beta_{u+\eta}(B^\eta j') = B^\eta(M_\eta T_u j'), \quad \forall j' \in L^2(\mathbb{R}; C^{n+1}).
\] (22)

Next, we define the Gabor-Daubechies localization operator \( A^\psi_\alpha \) with windows \( \psi, \theta \in L^2(\mathbb{R}) \) and symbol \( a(x, \omega) \) as being

\[
A^\psi_\alpha f = \int_{\mathbb{R}^2} a(x, \omega)(f, M_\omega T_x \psi)^{(2)} L^2(\mathbb{R}) \, d\omega
d = \int_{\mathbb{R}^2} a(x, \omega)(V_\psi f)(x, \omega) \, d\omega.
\] (23)

We will end this section by showing the interplay between the Gabor-Daubechies operator \( A^\psi_\alpha \) and the Berezin-Toeplitz operator \( B^\eta \) likewise the boundeness properties for \( (2) \) as well.

**Lemma 2.3** (see Appendix A). For any \( 0 \leq j, k \leq n \) the Gabor-Daubechies operator \( A^\psi_\alpha \) with symbol \( a(x, \omega) \) and windows \( \psi, \theta \in L^2(\mathbb{R}) \) and the Berezin-Toeplitz operator defined in \( (2) \) are interrelated by

\[
L^\psi_\sigma B^\psi_\theta = B^k A^\psi_\alpha (B^j)^\dagger, \quad \text{with } \sigma(z, \tau) = a(\mu(z), \Im(\tau)).
\]

Moreover for \( \overrightarrow{\psi} = (\psi_0, \psi_1, \ldots, \psi_n) \) and \( \overrightarrow{\theta} = (\theta_0, \theta_1, \ldots, \theta_n) \) have

\[
L^\psi_\sigma B^\psi_\theta = \sum_{k=0}^n B^k A^\psi_\alpha (B^j)^\dagger \quad \text{and} \quad L^\psi_\sigma B^\psi_\theta = \sum_{j,k=0}^n B^k A^\psi_\alpha (B^j)^\dagger.
\]

**Proposition 2.4.** For any \( \Psi, \Theta \in \mathcal{F}^n(\mathbb{C}) \), and \( \sigma \in L^\infty(\mathbb{C}) \) the operator \( L^\psi_\sigma \) satisfies the boundeness condition:

\[
\|L^\psi_\sigma\| \leq \|\sigma\|_{L^\infty(\mathbb{C})}\|\Psi\|_{d\mu}\|\Theta\|_{d\mu}.
\]

**Proof:** From Lemma 2.3 and Proposition 2.1 it is equivalent to show the following boundeness condition for \( A^\psi_\alpha \):

\[
\|A^\psi_\alpha\| \leq \|a\|_{L^\infty(\mathbb{R}^2)}\|\psi\|_{L^2(\mathbb{R})}\|\theta\|_{L^2(\mathbb{R})}.
\] (24)

By applying Cauchy-Schwartz inequality to the right-hand side of (23) we get

\[
\frac{|\langle A^\psi_\alpha f, g \rangle_{L^2(\mathbb{R})}|}{\|f\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}} \leq \|a\|_{L^\infty(\mathbb{R}^2)} \frac{\|V_\psi f\|_{L^2(\mathbb{R})}}{\|f\|_{L^2(\mathbb{R})}} \frac{\|V_\theta g\|_{L^2(\mathbb{R})}}{\|g\|_{L^2(\mathbb{R})}},
\]

and hence, from the orthogonality property 2.3 the right-hand side of the above inequality is equal to \( \|a\|_{L^\infty(\mathbb{R}^2)}\|\psi\|_{L^2(\mathbb{R})}\|\theta\|_{L^2(\mathbb{R})} \), and thus (24) follows from definition.
3. Main results

3.1. Reproducing kernels and Berezin symbols

Let us now turn our attention to the reproducing kernel property arising in $F^j(\mathbb{C})$.

For each $\psi \in L^2(\mathbb{R})$ and $(x, \omega), (u, \eta) \in \mathbb{R}^2$ it follows straightforward from the orthogonality property \((13)\) that $K_\psi(x + i\omega, u + i\eta) = \|\psi\|_{L^2(\mathbb{R})} V_\psi(M_T u \psi)(x, \omega)$ is a reproducing kernel for the Hilbert space $V_\psi(L^2(\mathbb{R}))$:

\[ (V_\psi f)(u, \eta) = (V_\psi f, K_\psi(\cdot, u + i\eta))_{L^2(\mathbb{R}^2)}. \]  \((25)\)

In the case when $\psi$ corresponds to the (normalized) Hermite function $h_j$ \((14)\), the relation \((15)\) together with the intertwining property \((16)\) regarding the time-frequency shifts $M_{-\eta}T_u$ and the Bargmann shifts $\beta_u = e^{-i\pi u \eta} W_u + i \eta = e^{-i\pi u \eta} W_u + i \eta$ enables to reformulate the reproducing kernel property \((25)\) for $F_j(\mathbb{C})$ in terms of the action of $W_\zeta$ on $\Phi_{j,j}(z, \bar{z})$. Namely, the relation

\[ e^{i\pi u \eta} e^{-\frac{1}{2}(u^2 + \eta^2)} (B^j f)(u + i\eta) = e^{i\pi u \eta} (B^j f, W_{u+i\eta} \Phi_{j,j})_{d\mu} \]

combined with the direct sum decompositions \((10)\) yields

\[ K_j(\zeta, z) = e^{\pi z \bar{z}} W_{\zeta} \Phi_{j,j}(\zeta, \bar{\zeta}) = e^{\pi \zeta \bar{\zeta}} \Phi_{j,j}(\zeta - \bar{\zeta}). \]  \((26)\)

**Remark 3.1.** Formula \((26)\) provides a meaningful description for the reproducing kernels obtained in [3, Theorem 3.1], [6, Corollary 5] and [32, Proposition 2.2] using solely the group representation theory framework underlying the Heisenberg group $\mathbb{H}$.

**Remark 3.2.** From \((26)\) and the direct sum decompositions \((10)\) the reproducing kernel for $F^n(\mathbb{C})$ is given by

\[ K^n(z, \zeta) = \sum_{j=0}^n \sum_{j=0}^n \Phi_{j,j}(\zeta - \bar{\zeta}). \]

Next we turn our attention for the interplay between the Berezin-Toeplitz operators \((2)\) and the Toeplitz operators \((4)\). The relations below hold in the weak sense for any $F \in F^k(\mathbb{C})$ and $G \in F^j(\mathbb{C})$:

\[ \langle \text{Toep}_D F, G \rangle_{d\mu} = \int_{\mathbb{C}} \sigma(z, \bar{z}) F(z, \bar{z}) G(z, \bar{z}) d\mu(z) \]

\[ = \int \sigma(z, \bar{z}) (F, K^k(\cdot, z), G)_{d\mu}(z) d\mu(z) \]

\[ = \int \sigma(z, \bar{z}) (F, W_{\zeta} \Phi_{k,k}, G)_{d\mu}(z) d\mu(z) \]

\[ = \langle L_{\Phi_{k,k}} \Phi_{j,j} F, G \rangle_{d\mu}. \]

This combined with \((10)\) give the following identities:

\[ \text{Toep}_D^j F = L_{\Phi_{k,k}} \Phi_{j,j} F \quad \text{for any} \quad F \in F^k(\mathbb{C}) \]

\[ \text{Toep}_D^n F = \sum_{k=0}^n L_{\Phi_{k,k}} \Phi_{j,j} F \quad \text{for any} \quad F \in F^n(\mathbb{C}). \]  \((27)\)
Next, let $\text{Op}$ a bounded linear operator on $F^n(\mathbb{C})$ and set $K^j_{\zeta}(z) = K^j(\zeta, \zeta) - \frac{i}{2} K^j(\zeta, \zeta)$ for any $0 \leq j \leq n$. We define $\widehat{\text{Op}}(\zeta)$ as the $n \times n$ matrix whose entries are given by the Berezin symbols $(\text{Op}(\zeta))_{j,k} = \left\langle \text{Op} K^k_{\zeta}, K^j_{\zeta} \right\rangle_{d\mu}$.

From (20) we get $K^j(\zeta, \zeta) = e^{\pi|\zeta|^2}$ for any $0 \leq j \leq n$. This gives

$$K^j_{\zeta}(z) = W_{\zeta} \Phi_{j,j}(z, \overline{\zeta}).$$

On the other hand, since $W^j_z = W_{-z}$ is the adjoint of $W_z$ on $L^2(\mathbb{C}, d\mu)$, it is easy to check from the Baker-Campbell-Haussdorff formula (7) the above relations for any $0 \leq j \leq n$:

$$W^j_{-z} \Phi^j_{\zeta} = e^{i\pi \bar{\zeta} z} \Phi^j_{\zeta}.$$

The following characterizations for the matrix coefficients defined in (3.3) will be important on the sequel:

**Lemma 3.3 (see Appendix A).** For any $\Psi \in F^k(\mathbb{C})$ and $\Theta \in F^j(\mathbb{C})$ we have

$$\left( L^\sigma_{\Psi, \Theta}(\zeta) \right)_{j,k} = \left[ \sigma \ast \left( |\Psi| e^{-\pi |\cdot|^2} \right) \right] (\zeta, \overline{\zeta}).$$

Moreover $\left( L^\sigma_{\Psi, \Phi}(\zeta) \right)_{j,k} = \left( \text{Top}_\sigma(\zeta) \right)_{j,k}$.

**Proposition 3.4.** Let $\text{Op}$ a bounded linear operator on $F^n(\mathbb{C})$. Then the following statements hold:

1. $\left( \widehat{\text{Op}}(\zeta) \right)_{j,k} \leq \|\text{Op}\|$ for each $\zeta \in \mathbb{C}$.
2. $\left( \text{Op} F, K^j_{\zeta} \right)_{d\mu} = e^{-\frac{1}{2} |\zeta|^2} \left( P^j \text{Op} F \right)(\zeta, \overline{\zeta})$ for any $F \in F^n(\mathbb{C})$.
3. $\text{Op}$ is uniquely determined by $\sum_{j,k=0}^{n} \left( \widehat{\text{Op}}(\zeta) \right)_{j,k}$.
4. For any $z \in \mathbb{C}$ we have $\left( W^j_{-z} \text{Op} W_z \right) (\zeta) = \left( \text{Op}(\zeta + z) \right)_{j,k}$.

**Proof:** For the proof of statement (1), we start to recall that $W_{\zeta}$ is unitary on $L^2(\mathbb{C}, d\mu)$ while $B^j : L^2(\mathbb{R}) \to F^j(\mathbb{C})$ is a unitary operator.

Therefore the relations (28) and $\Phi_{j,j} = B^j h_j$ gives $\left\| K^j_{\zeta} \right\|_{d\mu} = \| W_{\zeta} \Phi_{j,j} \|_{d\mu} = 1$, and hence, for any $0 \leq j \leq n$ the Cauchy-Schwartz inequality gives

$$\left| \left( \widehat{\text{Op}}(\zeta) \right)_{j,k} \right| \leq \|\text{Op}\| \left\| K^j_{\zeta} \right\|_{d\mu} \left\| K^j_{\zeta} \right\|_{d\mu} = \|\text{Op}\|.$$

The proof of statement (2) follows straightforwardly from the identity $\left( \text{Op} F, K^j(\cdot, z) \right)_{d\mu} = (P^j \text{Op} F)(\zeta, \overline{\zeta})$ and from (28).

For the proof of statement (3), recall that from (28)

$$\left\langle \text{Op} K^{j,k}_{\zeta} K^{j,k}_{\overline{\zeta}} \right\rangle_{d\mu} = e^{-\frac{1}{2} (|\zeta|^2 + |z|^2)} (P^j \text{Op} K^k)(\cdot, z)(\zeta, \overline{\zeta})$$

$$= e^{-\frac{1}{2} (|\zeta|^2 + |z|^2)} (P^k \text{Op} K^j)(\cdot, \overline{\zeta})(z, \overline{\zeta}).$$
On the other hand notice that \((P^{i} \text{Op} K^{k}(.\zeta))(\zeta, \zeta) = (\overline{P^{k} \text{Op}^{\dagger} K^{j}(.\zeta)})(z, \overline{z})\) is true polyanalytic of order \(j\) resp. \(k\) in the variable \(\zeta\) resp. \(\overline{\zeta}\).

Now take \(u = \frac{1}{2}(\zeta + \overline{\zeta})\), \(\eta = \frac{1}{2}(\zeta - \overline{\zeta})\) and set \(G(u, \eta) = (P^{i} \text{Op} K^{k}(.\zeta))(\zeta, \zeta)\), for some function \(G\). Then \(G\) can be expanded as a Taylor series in the variables \(u\) and \(\eta\) whenever \(u, \eta \in \mathbb{R}\) (cf. [25, Proposition 1.69]). This implies \(z = \zeta\) and hence \(\langle \text{Op} K^{k}_{z}, K^{j}_{\zeta} \rangle_{d\mu}\) is uniquely determined by \(\langle \text{Op}(\zeta) \rangle_{j,k}\).

Now recall that each function \(F(z, \overline{z})\) belonging to \(\mathcal{F}^{n}(\mathbb{C})\) can be rewritten in terms of the reproducing kernel \(K^{n}(\zeta, z)\) (see Remark 3.1) and moreover, as a Fourier-Hermite series expansion in terms of the basis functions (17) i.e.

\[
F(z, \overline{z}) = (F, K^{n}(\cdot, z))_{d\mu} = \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} (F, \Phi_{j,l})_{d\mu} \Phi_{j,l}(z, \overline{z}).
\]

Thus the normalization of each \(K^{j}(\cdot, z)\) provided by (28) shows that \(\text{Op}\) is uniquely determined by \(\sum_{j,k=0}^{\infty} (\text{Op}(\zeta))_{j,k}\), as desired (cf. [25, Corollary 1.70]).

Finally, the proof of statement (1) yields from direct application of the property (29) in terms of \(-z\). since \(W_{z}^{\dagger} = W_{-z}\) is the adjoint of \(W_{z}\) on \(L^{2}(\mathbb{C}, d\mu)\).

\[\square\]

**Remark 3.5.** From Lemma 3.6, the coefficients \(\mathcal{L}_{\sigma}^{\Psi, \Theta}(\zeta)\) of the \(n \times n\) matrix \(\mathcal{L}_{\sigma}^{\Psi, \Theta}(\zeta)\) correspond to the two-windowed generalization of the magnetic Berezin transform attached to the true polyanalytic Fock spaces \(\mathcal{F}^{j}(\mathbb{C})\) (cf. [25, 2]) whereas Proposition 3.4 gives a generalization of [28, Proposition 3] for \(\mathcal{F}^{n}(\mathbb{C})\).

### 3.2. Proof of Coburn Conjecture for Polyanalytic Fock spaces

For the proof of Theorem 1.1 the following lemma will be required a-posteriori:

**Lemma 3.6 (see Appendix A).** For each \(m \in \mathbb{N}\) we have the following intertwining properties on \(L^{2}(\mathbb{C}, d\mu)\):

\[(\pi z - \partial_{r})^{m} = e^{-\frac{i}{2} \pi \Delta z} (\pi z)^{m} e^{\frac{i}{2} \pi \Delta z}, \quad (\pi \overline{z} - \partial_{l})^{m} = e^{-\frac{i}{2} \pi \Delta \overline{z}} (\pi \overline{z})^{m} e^{\frac{i}{2} \pi \Delta \overline{z}}\]

**Proof:** [Proof of Theorem 1.1] From (2), we obtain by (29) the relation

\[
\mathcal{L}_{\sigma \{z \pm \xi, \zeta \mp \overline{\xi}\}}^{\Psi, \Theta}(\zeta) = \int_{\mathbb{C}} \sigma(\zeta + z, \overline{\zeta} + \overline{z}) \langle \cdot, W_{z} \Psi \rangle_{d\mu} W_{z} \Theta d^{2}\zeta = \int_{\mathbb{C}} \sigma(\zeta, \overline{\zeta}) \langle \cdot, W_{z} \overline{\Psi} \rangle_{d\mu} W_{z} \Theta d^{2}\zeta = W_{z}^{\dagger} \mathcal{L}_{\sigma}^{\Psi, \Theta}(\zeta) W_{z}.
\]

In particular, from (27) we then have \(\text{Toep}_{\sigma(\cdot + z, \cdot \mp \overline{z})}^{j} = W_{z}^{\dagger} \text{Toep}_{\sigma}^{j} W_{z}\).

From Proposition 2.4 and statements (1) and (4) of Proposition 3.4, for each \(\sigma \in L^{\infty}(\mathbb{C})\) the Berezin symbols \(\mathcal{L}_{\sigma}^{\Psi, \Theta}(\zeta)\) and \(\text{Toep}_{\sigma}^{j}(\zeta)\) are bounded above by bounded true polyanalytic functions of order \(j\) which are invariant under translations.
Then for each $\Psi \in \mathcal{F}^k(\mathbb{C}) \cap \mathbb{C}[z, \overline{z}]$ and $\Theta \in \mathcal{F}^j(\mathbb{C}) \cap \mathbb{C}[z, \overline{z}]$ the distributions $\overline{\Psi(z, \overline{z})}\Theta(z, \overline{z})e^{-|z|^2}$ and moreover $\Phi_{k,k}(z, \overline{z})\Phi_{j,j}(z, \overline{z})e^{-|z|^2}$ satisfying Lemma 3.3 are uniquely determined (cf. [37, Theorem 6.33]).

Now let $D_{j,k}(z, \overline{z})$ be a polynomial of degree $N = \deg(\Psi) + \deg(\Theta) - 2j - 2k$ written in terms of sequence of polynomials $\{\Phi_k\}_{k \in \mathbb{N}_0}$ of $\mathcal{F}(\mathbb{C})$ defined in (17):

$$D_{j,k}(z, \overline{z}) = \sum_{l+m=0}^{N} d_{l,m} \Phi_l(z)\Phi_m(\overline{z}).$$

Notice that the adjoint properties (9) on $L^2(\mathbb{C}, d\mu)$ combined with Lemma 3.7 gives the sequence of identities

$$\left\langle (-\partial_\sigma)^m (-\partial_\zeta)^l \sigma(\zeta - \cdot, \zeta - \cdot), \Phi_{k,k}\Phi_{j,j} \right\rangle_{d\mu} =$$

$$= \left\langle \sigma(\zeta - \cdot, \zeta - \cdot), (\pi z - \partial_\sigma)^l (\pi \overline{z} - \partial_\zeta)^m (\Phi_{k,k}(z, \overline{z})\Phi_{j,j}(z, \overline{z})) \right\rangle_{d\mu},$$

Using linearity arguments we then have

$$\left\langle D_{j,k} \left( \frac{1}{\sqrt{\pi}} \partial_\sigma - \frac{1}{\sqrt{\pi}} \partial_\zeta \right) \sigma(\zeta - \cdot, \zeta - \cdot), \Phi_{k,k}(z, \overline{z})\Phi_{j,j}(z, \overline{z}) \right\rangle_{d\mu} =$$

$$= \left\langle \sigma(\zeta - \cdot, \zeta - \cdot), e^{-\frac{1}{\sqrt{\pi}}\Delta_\zeta} D_{j,k}(z, \overline{z}) e^{-\frac{1}{\sqrt{\pi}}\Delta_\zeta} (\Phi_{k,k}(z, \overline{z})\Phi_{j,j}(z, \overline{z})) \right\rangle_{d\mu}.$$
Corollary 3.7. Let $Ψ, Θ ∈ Fⁿ(ℂ) \cap ℂ[z, \overline{z}]$ and $P^j : L²(ℂ, dµ) → F^j(ℂ)$, $P^k : L²(ℂ, dµ) → F^k(ℂ)$ the corresponding projection operators.

If $e^{\frac{1}{D}Δ_z} (Φ_{k,l}(z, \overline{z})Ψ_{j,l}(z, \overline{z}))$ divides $e^{\frac{1}{D}Δ_z} \left( (P^kΨ)(z, \overline{z}) (P^jΘ)(z, \overline{z}) \right)$ then there exists a unique polynomial differential operator $D_j := D_j \left( -\frac{1}{\sqrt{\partial}} \partial_z - \frac{1}{\sqrt{\partial}} \partial_{\overline{z}} \right)$ such that

1. $D_j(z, \overline{z})$ has degree
   \[ \deg(D_j) = \max_{0 \leq k \leq n} \left( \deg(P^kΨ) + \deg(P^jΘ) - 2j - 2k \right). \]
2. $D_j, Σ ∈ L^{\infty}(ℂ)$.
3. $L_{σ, Θ}^Ψ = \sum_{j=0}^n \text{Toep}_j D_j, σ$.

Proof: Let $Ψ, Θ ∈ Fⁿ(ℂ) \cap ℂ[z, \overline{z}]$. The finite expansion in terms of the projection operators $P^j : L²(ℂ, dµ) → F^j(ℂ)$ resp. $P^k : L²(ℂ, dµ) → F^k(ℂ)$ yielding from the direct sum decompositions gives

\[ L_{σ, Θ}^Ψ = \sum_{j,k=0}^n L_{σ, Θ}^P P^j Ψ \cdot P^k Θ. \]

From hypothesis $e^{\frac{1}{D}Δ_z} (Φ_{k,l}(z, \overline{z})Ψ_{j,l}(z, \overline{z}))$ divides $e^{\frac{1}{D}Δ_z} \left( (P^kΨ)(z, \overline{z}) (P^jΘ)(z, \overline{z}) \right)$, under the hypothesis of Theorem 1.1 there exists a unique polynomial differential operator $D_{j,k}$ with symbol

\[ D_{j,k}(z, \overline{z}) = e^{\frac{1}{D}Δ_z} \left( (P^kΨ)(z, \overline{z}) (P^jΘ)(z, \overline{z}) \right) \]

such that $D_{j,k} Σ ∈ L^{\infty}(ℂ)$ has degree $\deg(D_{j,k}) = \deg(P^kΨ) + \deg(P^jΘ) - 2j - 2k$ and satisfies

\[ L_{σ, Θ}^P P^j Ψ \cdot P^k Θ = \text{Toep}_j D_{j,k}, σ. \]

Thus for $D_j := \sum_{k=0}^n D_{j,k}$, the polynomial $D_j(z, \overline{z}) := \sum_{k=0}^n D_{j,k}(z, \overline{z})$ has degree $\deg(D_j) = \max_{0 \leq k \leq n} \deg(D_{j,k})$ and the later equation is equivalent to

\[ L_{σ, Θ}^Ψ = \sum_{j=0}^n \text{Toep}_j D_j, σ, \quad \text{with } D_j, σ ∈ L^{\infty}(ℂ). \]

The next corollary which is then immediate from Theorem 1.1 a mimic generalization of a result obtained by Engliš (cf. [22, Corollary 4]) to the polyanalytic Fock space $Fⁿ(ℂ)$ following also from the same order of ideas used on the proof of Corollary 3.7.

Corollary 3.8. Let $Ψ, Ψ^*, Θ, Θ^* ∈ Fⁿ(ℂ) \cap ℂ[z, \overline{z}]$. Then the following statements are equivalent:
(a) There exist a unique sequence of polynomial differential operators \( \{ D_{j,k} \}_{0 \leq j, k \leq n} \) with Wick symbols \( D_{j,k}(\tau, z) \) such that

\[
\mathcal{L}^\Psi_{\sigma} \Theta = \sum_{j,k=0}^{n} \mathcal{L}_{D_{j,k}\sigma}^{P^j \Psi^*, P^j \Theta^*}.
\]  

(b) \( e^{\frac{1}{2} \Delta_2} \left( (P^k \Psi^*)(z, \tau)(P^j \Theta^*)(z, \tau) \right) \) divides \( e^{\frac{1}{2} \Delta_2} \left( (P^k \Psi)(z, \tau)(P^j \Theta)(z, \tau) \right) \).

Whence, if (a) or (b) fulfills the Wick symbol \( D_{j,k}(\tau, z) \) has degree \( \deg(D_{j,k}) = \deg(P^j \Theta) + \deg(P^k \Psi) - \deg(P^j \Theta^*) - \deg(P^k \Psi^*) \) and (31) holds for every \( \sigma \in BC^\infty(\mathbb{C}) \).

4. Extension to a Wide Class of Symbols

4.1. Gel’fand-Shilov type Spaces

According to the proof of Theorem 3.1 and subsequent corollaries in Subsection 3.2, \( D_{j,k} \) is an anti-Wick ordered operator constructed as bijective mapping of the set of polynomials \( \mathbb{C}[\tau, z] \) onto the set of differential operators with polynomial coefficients whereas the condition \( \sigma \in BC^\infty(\mathbb{C}) \) assures that for each \( 0 \leq j, k \leq n \) the symbols \( D_{j,k} \sigma \) belongs to \( L^\infty(\mathbb{C}) \).

Motivated by the framework described by Lo (cf. [34]) for the spaces \( B_\alpha(\mathbb{C}) \) and \( E(\mathbb{C}) \) and by Englis (cf. [22]) for the spaces \( M_r \), we will introduce a new family of function spaces that constitute a rich class of symbols including \( E(\mathbb{C}) \) and \( M_r \) as well.

For \( a > 0 \), \( 1 \leq p \leq \infty \) and \( \frac{1}{2} \leq \alpha \leq 1 \), we introduce the function spaces \( W^{p,n}_{a,\alpha} \), \( G^{(\alpha)}_n \) and \( G^{(\alpha)}_{n,a} \) as follows:

i) \( \sigma \in W^{p,n}_{a,\alpha} \) if and only if for every \( 0 \leq j \leq n \) and \( l, m \in \mathbb{N}_0 \) we have

\[
e^{a|\tau|} e^{-\frac{|z|}{2}} P^j \sigma \in L^p(\mathbb{C}).
\]

ii) \( \sigma \in G^{(\alpha)}_{n,a} \) if and only if there exists \( a > 0 \) such that \( \sigma \in W^{p,n}_{a,\alpha} \).

iii) \( \sigma \in G^{(\alpha)}_n \) if and only if \( \sigma \in W^{p,n}_{a,\alpha} \) for every \( a > 0 \).

In case when \( \sigma \in W^{p,n}_{a,\alpha} \), the quantity

\[
\|\sigma\|_{W^{p,n}_{a,\alpha}} := \sum_{j=0}^{n} \left\| e^{a|\tau|} e^{-\frac{|z|}{2}} P^j \sigma \right\|_{L^p(\mathbb{C})}
\]

is a quasi-norm for \( W^{p,n}_{a,\alpha} \) while \( G^{(\alpha)}_n \) resp. \( G^{(\alpha)}_{n,a} \) are the complex analogues of the Gel’fand-Shilov spaces \( S^{(\alpha)}(\mathbb{R}) \) resp. \( \Sigma^{(\alpha)}(\mathbb{R}) \) whereas its dual spaces \( G^{(\alpha)}_{n,a} \) resp. \( G^{(\alpha)}_n \) are the complex analogues of the spaces of tempered ultradistributions \( S^{(\alpha)}(\mathbb{R})' \) resp. \( \Sigma^{(\alpha)}(\mathbb{R})' \) of Beurling resp. Romieu type (cf. [27]).

From the following reformulation of a result of Gröchenig and Zimmermann ([29, Proposition 4.3]) for the classes of Gel’fand-Shilov spaces \( S^{(\alpha)}(\mathbb{R}) \) resp. \( \Sigma^{(\alpha)}(\mathbb{R}) \) and tempered ultradistributions \( S^{(\alpha)}(\mathbb{R})' \) resp. \( \Sigma^{(\alpha)}(\mathbb{R})' \) one can prove that they are indeed isomorphic. In terms of the true polyanalytic Bargmann transforms [16], this theorem is stated as follows:
Theorem 4.1. Let $\frac{1}{2} \leq \alpha \leq 1$. Then for each $f \in S_0^\alpha(\mathbb{R})'$ (resp. for each $f \in \Sigma_0^\alpha(\mathbb{R})'$) and for each $j \in \mathbb{N}$, the following two conditions are equivalent:

i) $f \in S_0^\alpha(\mathbb{R})$ (resp. $f \in \Sigma_0^\alpha(\mathbb{R})$)

ii) There exists $b, c > 0$ (resp. for every $b, c > 0$) such that

$$e^{b|x|^\frac{\alpha}{2} + c|\omega|^\frac{\alpha}{2}}(B^j f)(x + i\omega)e^{-\frac{\alpha}{2}(x^2 + \omega^2)} \in L^\infty(\mathbb{R}^2).$$

Proposition 4.2. We have the following isometric isomorphisms:

$$G^{(\alpha)} \cong \bigotimes_{0 \leq j \leq n} S_0^\alpha(\mathbb{R}) \quad \text{and} \quad G^{(\alpha)} \cong \bigotimes_{0 \leq j \leq n} \Sigma_0^\alpha(\mathbb{R})$$

$$G^{(\alpha)'} \cong \bigotimes_{0 \leq j \leq n} S_0^\alpha(\mathbb{R})' \quad \text{and} \quad G^{(\alpha)'} \cong \bigotimes_{0 \leq j \leq n} \Sigma_0^\alpha(\mathbb{R})'.$$

Proof: Using the fact that for any $\sigma \in L^\infty(\mathbb{C})$ such that $(P^1(\sigma))(z) = (B^j f_j)(x + i\omega)$, the quantities $\|e^{a|\cdot|^\frac{\alpha}{2} + b|\cdot|^\frac{\alpha}{2}} (P^1(\sigma))\|_{L^\infty(\mathbb{C})}$ and $\|e^{a|\cdot|^\frac{\alpha}{2} + b|\cdot|^\frac{\alpha}{2}} ((B^j f_j)(\cdot) + i)\|_{L^\infty(\mathbb{R}^2)}$ endow equivalent norms, we show that $\sigma \in \mathcal{G}_{\mathcal{N}}^{(\alpha)}$ resp. $\sigma \in \mathcal{G}_{\mathcal{N}}^{(\alpha)}$ if and only if the vector $\mathcal{F} = (f_0, f_1, \ldots, f_n)$ belongs to $\bigotimes_{0 \leq j \leq n} S_0^\alpha(\mathbb{R})$ resp. $\bigotimes_{0 \leq j \leq n} \Sigma_0^\alpha(\mathbb{R})$. This shows the isometric isomorphisms

$$G^{(\alpha)} \cong \bigotimes_{0 \leq j \leq n} S_0^\alpha(\mathbb{R}) \quad \text{and} \quad G^{(\alpha)} \cong \bigotimes_{0 \leq j \leq n} \Sigma_0^\alpha(\mathbb{R}).$$

Moreover, the isometric isomorphisms

$$G^{(\alpha)'} \cong \bigotimes_{0 \leq j \leq n} S_0^\alpha(\mathbb{R})' \quad \text{and} \quad G^{(\alpha)'} \cong \bigotimes_{0 \leq j \leq n} \Sigma_0^\alpha(\mathbb{R})'$$

yield from duality arguments underlying Banach spaces.

Let us now make a short parenthesis about the concept of modulation space in time-frequency analysis: Accordingly to [28, Chapter 11], for each $1 \leq p \leq \infty$ the modulation space $M^p_{m_{a,\alpha}}$ with weight $m_{a,\alpha}(x, \omega)$ consists on the space of all tempered distributions $f \in S(\mathbb{R})$ such that $\|f\|_{M^p_{m_{a,\alpha}}} = \|m_{a,\alpha}(\cdot, \cdot)V_\alpha f\|_{L^p(\mathbb{R}^2)}$ is finite and independent of the choice of $\psi$. When $m_{a,\alpha}(x, \omega) = e^{a(|x|^2 + |\omega|^2)}\frac{1}{\sqrt{2\pi}}$ one can get a weaker characterization for $M^p_{m_{a,\alpha}}$ in terms of $f \in S_0^\alpha(\mathbb{R})'$ (cf. [29, Proposition 4.1] and [10, Section 4]).

In this way, choosing $\psi$ in the range of (normalized) Hermite functions defined in [14], the characterization of $\mathcal{G}_{\mathcal{N}}^{(\alpha)}$ resp. $\mathcal{G}_{\mathcal{N}}^{(\alpha)}$ and its duals as inductive/projective limits involving $\mathcal{W}_{\mathcal{S}}^{p,\alpha}$ resp. $\mathcal{W}_{\mathcal{S},a,\alpha}$ can be obtained by mimicking the result obtained by Teofanov (cf. [40, Theorem 4.3]).

Proposition 4.3. Let $1 \leq p \leq \infty$. Then we have

$$\mathcal{G}_n^{(\alpha)} = \bigcap_{a>0} \mathcal{W}_{a,\alpha}^{p,n}, \quad \mathcal{G}_n^{(\alpha)'} = \bigcup_{a>0} \mathcal{W}_{a,\alpha}^{p,n}$$

$$\mathcal{G}_n^{(\alpha)} = \bigcup_{a>0} \mathcal{W}_{a,\alpha}^{p,n}, \quad \mathcal{G}_n^{(\alpha)'} = \bigcap_{a>0} \mathcal{W}_{a,\alpha}^{p,n}.$$
Proof: Since for any $\sigma \in L^\infty(\mathbb{C})$ such that $(P^j \sigma)(x + i\omega, x - i\omega) = e^{-\pi x^2} e^{2(x^2 + \omega^2)}(V_{h_j} f_j)(x, -\omega)$ (see equation (11)) the quantities $\sum_{j=0}^{n} \|m_{a,n}(\cdot , \cdot )V_{h_j} f_j\|_{L^p(\mathbb{R}^2)}$ and $\|\sigma\|_{W_{\alpha}^{a,n}}$ coincide, the proof of Theorem 4.3 follows straightforwardly from Theorem 4.2 and 40. Theorem 4.3. \[\square\]

Remark 4.4. It is clear from the above proposition that these spaces satisfy the following quadruple of imbeddings $G_n^{(\alpha)} \hookrightarrow G_n^{(\alpha)}' \hookrightarrow G_n^{(\alpha)}''$. Among this weighted function spaces employed, we will take $W_{-a,\alpha}^{\infty,n}$ for the class of symbols and $W_{1,a,\alpha}^{1,n}$ for the class of windows. From the triplet of imbeddings $W_{-a,\alpha}^{\infty,n} \hookrightarrow F_{1,n}(\mathcal{C}) \hookrightarrow W_{1,-a,\alpha}^{1,n}$ we are now able to get a weaker formulation of Proposition 3.4. This corresponds to the following result:

Proposition 4.5. For any $\Psi, \Theta \in W_{1,-a,\alpha}^{1,n}$ and $\sigma \in W_{a,\alpha}^{\infty,n}$ there exists $C > 0$ such that $L^\sigma_{\Psi, \Theta}$ satisfies the boundeness condition:

$$\|L^\sigma_{\Psi, \Theta}\| \leq C \|\sigma\|_{W_{-a,\alpha}^{\infty,n}} \|\Psi\|_{W_{1,-a,\alpha}^{1,n}} \|\Theta\|_{W_{1,-a,\alpha}^{1,n}}.$$ 

Proof: In order to prove the boundeness condition for $L^\sigma_{\Psi, \Theta}$, recall first the following boundeness result for the Gabor-Daubechies operator $A^{(\psi, \theta)}_a$ obtained by Cordero and Gröchenig in [18] in terms of modulation spaces $M_{-a,\alpha}^{\infty}$ and $M_{1,m_{a,\alpha}}^{1}$ (cf. [13] Theorem 3.2):

For each $a \in M_{-a,\alpha}^{\infty}$ and $\psi_k, \theta_j \in M_{1,m_{a,\alpha}}^{1}$ there exists a constant $C_{j,k} > 0$ such that

$$\|A^{(\psi, \theta)}_a\| \leq C_{j,k} \|a\|_{M_{-a,\alpha}^{\infty}} \|\psi_k\|_{M_{1,m_{a,\alpha}}^{1}} \|\theta_j\|_{M_{1,m_{a,\alpha}}^{1}}.$$ 

Therefore for any $\Psi, \Theta \in W_{1,-a,\alpha}^{1,n}$ and $\sigma \in W_{-a,\alpha}^{\infty,n}$ such that $P^j \Psi = B_k^j \psi_k$, $P^j \Theta = B^j \theta_j$, and $\sigma(z, \tau) = a(\psi(z), \psi(\tau))$, Theorem 4.3 gives the following isometry relation:

$$L^\sigma_{\Psi, \Theta} = \sum_{j,k=0}^{n} B_k^j A^{(\psi, \theta)}_a B^j \theta_j (B^j)^{\dagger}.$$ 

Finally, the identities $\|P^j \Psi\|_{W_{1,-a,\alpha}^{1,n}} = \|\psi_k\|_{M_{1,m_{a,\alpha}}^{1}}$, $\|P^j \Theta\|_{W_{1,-a,\alpha}^{1,n}} = \|\theta_j\|_{M_{1,m_{a,\alpha}}^{1}}$ triangle's inequality gives

$$\|L^\sigma_{\Psi, \Theta}\| \leq \sum_{j,k=0}^{n} C_{j,k} \|a\|_{M_{-a,\alpha}^{\infty}} \|\psi_k\|_{M_{1,m_{a,\alpha}}^{1}} \|\theta_j\|_{M_{1,m_{a,\alpha}}^{1}}$$

$$= \sum_{j,k=0}^{n} C_{j,k} \|\sigma\|_{W_{-a,\alpha}^{\infty,n}} \|P^j \Psi\|_{W_{1,-a,\alpha}^{1,n}} \|P^j \Theta\|_{W_{1,-a,\alpha}^{1,n}}$$

$$\leq C \|\sigma\|_{W_{-a,\alpha}^{\infty,n}} \|\Psi\|_{W_{1,-a,\alpha}^{1,n}} \|\Theta\|_{W_{1,-a,\alpha}^{1,n}},$$

with $C = \max_{0 \leq j,k \leq n} C_{j,k}$. \[\square\]
4.2. Coburn Conjecture Revisited

Now we will extend the framework obtained in Section 3.2 using for the class of windows the space of tempered ultradistributions \( \mathcal{G}_n^{(\alpha)^c} \) of Romieu type and for the class of symbols the Gel’fand-Shilov type space \( \mathcal{G}_n^{(\alpha)} \).

First we will start the condition of Lemma 4.6 (see Appendix A). This is indeed a consequence of the following lemma:

**Lemma 4.6** (see Appendix A). For each \( \sigma \in \mathcal{W}_{\alpha,\alpha}^\infty \), \( \Psi, \Theta \in \mathcal{G}_n^{(1/2)} \) and 0 \( \leq j, k \leq n \) the operator \( D_{j,k} \) defined on Theorem 4.4. satisfies the following convolution formula on \( \mathbb{C} \):

\[
D_{j,k} \sigma * \left( \mathcal{P}^k \Psi \mathcal{P}^j \Theta e^{-|\cdot|^2} \right) = \sigma * D_{j,k} \left( \mathcal{P}^k \Psi \mathcal{P}^j \Theta e^{-|\cdot|^2} \right).
\]

**Remark 4.7.** Accordingly to [33, Section 2], the condition \( \Psi, \Theta \in \mathcal{G}_n^{(1/2)} \) is equivalent to the characterization of \( \otimes_{0\leq j \leq n} S_{1/2}^1(\mathbb{R}) \) in terms of the Fourier-Hermite coefficients \( \langle \psi, \phi_{j,m} \rangle_{d\mu} \) resp. \( \langle \phi, \phi_{j,m} \rangle_{d\mu} \) of \( \Psi \) resp. \( \Theta \).

Indeed each (normalized) Hermite function \( h_m \) defined in \([33]\) belongs to \( S_{1/2}^1(\mathbb{R}) \) assures that \( \sum_{j=0}^n \phi_{j,m} = \sum_{j=0}^n \mathcal{B}^j h_m \) belongs to \( \mathcal{G}_n^{(1/2)} \) \( \cong \otimes_{0\leq j \leq n} S_{1/2}^1(\mathbb{R}) \).

The next theorem corresponds to a weaker version of Theorem 1.1 and Corollary 3.7.

**Theorem 4.8.** Let \( \Psi, \Theta \in \mathcal{G}_n^{(1/2)} \cap \mathbb{C}[z, \overline{z}] \) with \( \deg(\Psi), \deg(\Theta) < \infty \). Under the assumptions of Corollary 3.7 underlying \( \Psi \) and \( \Theta \) let us assume that for each \( a > 0 \) the symbol \( \sigma(z, \overline{z}) \) belongs to \( \mathcal{W}_{\alpha,\alpha}^\infty \). Then for any \( F \in \mathcal{W}_{\alpha,a}^1 \)

\[
\sum_{j=0}^n \text{Toep}_{D_{j,k}} \sigma \mathcal{P}^k F = \text{Toep}_{D_{j,k}} \sigma \mathcal{P}^k F = \mathcal{L}_\sigma^{\mathcal{P}^k \Psi \mathcal{P}^j \Theta} (\mathcal{P}^k F)
\]

Moreover \( D_{j,k} \sigma, D_j \sigma \in \mathcal{G}_n^{(\alpha)} \) and \( F \in \mathcal{G}_n^{(\alpha)^c} \).

**Proof:** Since Lemma 4.4 fulfills for any \( \sigma \in \mathcal{W}_{\alpha,a}^\infty \), from Cauchy-Schwarz inequality \( \langle D_{j,k} \sigma \mathcal{P}^k F, (D_j \sigma) F \rangle \in L^2(\mathbb{C}, d\mu) \) and hence

\[
\text{Toep}_{D_{j,k}} \sigma \mathcal{P}^k F \in \mathcal{F}^j(\mathbb{C}) \quad \text{and} \quad \text{Toep}_{D_{j,k}} \sigma F \in \mathcal{F}^j(\mathbb{C}).
\]

Applying the sequence of ideas used on the proof of Theorem 1.1 we obtain from Lemma 4.5 that we are under the conditions of Proposition 4.4. Then the operators \( D_{j,k} \) and \( D_j \) determined by Theorem 1.1 and Corollary 3.7 respectively, satisfy \( D_{j,k} \sigma, D_j \sigma \in \mathcal{G}_n^{(\alpha)} \) and also the set of equations

\[
\text{Toep}_{D_{j,k}} \sigma \mathcal{P}^k F = \mathcal{L}_\sigma^{\mathcal{P}^k \Psi \mathcal{P}^j \Theta} \mathcal{P}^k F \quad \text{and} \quad \sum_{j=0}^n \text{Toep}_{D_{j,k}} \sigma F = \mathcal{L}_\sigma^{\Psi \mathcal{P}^j \Theta} F.
\]

Moreover the constraint \( F \in \mathcal{G}_n^{(\alpha)^c} \) follows straightforwardly from Proposition 3.3. \( \square \)
Remark 4.9. For a general $F \in \mathcal{G}_n^{(\alpha)}$, the functions $(D_{j,k}\sigma)P^{j}\sigma$ and $(D_{j}\sigma)F$ do not belong to $L^2(\mathbb{C},d\mu)$ in general, and thus, Toeplitz operators $P^{j}F$ likewise Toeplitz operators $F$ are not necessarily bounded on $\mathcal{F}^j(\mathbb{C})$.

However for $F = K^n_{\mathbf{a}} = \sum_{j=0}^{n} K_j^n$, where $K_j^n$ is the normalized reproducing kernel of $\mathcal{F}^j(\mathbb{C})$ obtained in \cite{22}, we are under the conditions of Theorem 4.8 since by construction $\sigma K^n_{\mathbf{a}} \in \mathcal{G}_n^{(\alpha)}$ and $\mathcal{G}_n^{(\alpha)} \subset L^2(\mathbb{C},d\mu)$. Thus Theorem 4.8 also fulfills for $F \in \mathcal{G}_n^{(\alpha)}$ whenever $F$ is a linear combination in terms of $K^n_{\mathbf{a}} = \sum_{k=0}^{n} K^n_{z^k}$, with $z \in \mathbb{C}$.

In conclusion, this approach is a refinement of Lo’s (see \cite[Theorems 4.5 & Corollary 4.6]{34}) and Enghil's approach (see \cite[Theorem 5]{22}) since the imbedding argument $L^\infty(\mathbb{C}) \hookrightarrow L^2_{\text{loc}}(\mathbb{C}) \hookrightarrow C^\infty(\mathbb{C})$ (cf. \cite[Theorem 7.25]{37}) assures that the symbol classes $E(\mathbb{C})$ and $\mathcal{M}_r$ belong to $\mathcal{G}_n^{(\alpha)}$ for any $r \in \mathbb{N}$. Moreover this approach also includes an intriguing characterization for the window classes in terms of Gel’fand-Shilov type spaces of order $\frac{1}{2} \leq \alpha \leq 1$.

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Appendix A. Proof of Technical Lemmata

Appendix A.1. Proof of Lemma 2.3

Proof: Recall that for each $0 \leq j, k \leq n$ the operator $B^j$ resp. $B^k$ maps isometrically $L^2(\mathbb{R})$ onto $\mathcal{F}^j(\mathbb{C})$ resp. $\mathcal{F}^k(\mathbb{C})$. Combining this with the change of variable $(x, \omega) \mapsto (x, -\omega)$ we can recast $A_{n}^{\mathbf{a},\mathbf{b}}$ in the weak form as follows:

$$
\langle A_{n}^{\mathbf{a},\mathbf{b}}, f \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}^2} a(x, -\omega) \langle f, M_{\omega}T_z \phi \rangle_{L^2(\mathbb{R})} \langle M_{-\omega}T_z \theta, g \rangle_{L^2(\mathbb{R})} - dxd\omega
$$

$$
= \int_{\mathbb{R}^2} a(x, -\omega) \langle B^k f, B^k (M_{-\omega}T_z \phi) \rangle_{L^2(\mathbb{R})} \langle B^j (M_{\omega}T_z \theta), B^j g \rangle_{L^2(\mathbb{R})} dxd\omega
$$

$$
= \int_{\mathbb{R}^2} a(x, -\omega) \langle B^k f, \beta_{x-\omega} (B^k \psi) \rangle_{L^2(\mathbb{R})} \langle \beta_{x+\omega} (B^j \theta), B^j g \rangle_{L^2(\mathbb{R})} dxd\omega.
$$

Now set $z = x + i\omega$, $\sigma(z, \overline{z}) = a(x, -\omega)$ and take $F = B^k f$, $\Psi = B^k \psi$, $\Theta = B^j \theta$, $G = B^j g$.

From the relations $z = R(z)$, $-\omega = \Im(\overline{z})$ and $d\omega dx = \frac{dz}{2\pi}$ the right-hand side of the above formula can be expressed as the following integration formula over $\mathbb{C}$ with respect to $d^2 z$:

$$
\langle A_{n}^{\mathbf{a},\mathbf{b}}, f \rangle_{L^2(\mathbb{R})} = \int_{\mathbb{C}} \sigma(z, \overline{z}) \langle F, \beta_{z} \Psi \rangle_{d\mu} \langle \beta_{\overline{z}} \Theta, G \rangle_{d\mu} d^2 z
$$

$$
= \int_{\mathbb{C}} \sigma(z, \overline{z}) e^{-i\pi R(z) \Im(\overline{z})} \langle F, W_z \Psi \rangle_{d\mu} e^{i\pi R(z) \Im(\overline{z})} \langle W_{\overline{z}} \Theta, G \rangle_{d\mu} d^2 z.
$$

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The above equation is equivalent to \([A^θ_ψ f, g]_{L^2(\mathbb{R})} = [L^θ_ψ F, G]_{dμ}\) and therefore \([B^k A^θ_ψ (B^l)]^\dagger = L^θ_ψ B^l φ\), as desired.

Finally, the proof of relations \(L^θ_ψ B^k = \sum_{k=0}^n B^k A^θ_ψ (B^l)]^\dagger\) and \(L^θ_ψ B^k = \sum_{j,k=0}^n B^k A^θ_ψ (B^l)]^\dagger\) follows from combination of definition (21) with linearity arguments.

Appendix A.2. Proof of Lemma 3.3

Proof: Starting from definition, straightforward computations combining property (20) with the change of variable \(z \mapsto ζ - z\) results into

\[
\left(\mathcal{L}^θ_ψ(φ(ζ))\right)_{j,k} = \int_\mathbb{C} σ(z, ζ)(K^j_ζ W_ζ Ψ)_{dμ}(W_ζ Ψ, K^j_ζ)_{dμ} d^2z
\]

\[
= \int_\mathbb{C} σ(z, ζ)(Ψ, W^k_ζ K^j_ζ)_{dμ}(W_ζ Ψ, K^j_ζ)_{dμ} d^2z
\]

\[
= \int_\mathbb{C} σ(z, ζ)(Ψ, K^j_ζ)_{dμ}(W_ζ Ψ, K^j_ζ)_{dμ} d^2z
\]

Finally, the reproducing kernel property (20) shows that the latter integral coincides with \([φ * (Ψ, e^{-|ζ|^2}θ)]\) and consequently

Moreover, the proof of relation \(L^θ_ψ(φ(ζ))\) follows straightforwardly from (20).

Appendix A.3. Proof of Lemma 3.6

Proof: From the Weyl-Heisenberg relations (25) it follows straightforwardly that \([ζ, \frac{1}{σ} Δ_ζ] = Ω_ζ\) and \([Ψ, \frac{1}{σ} Δ_ζ] = Ω_ζ\) holds on \(L^2(\mathbb{C}, dμ)\). This leads to

\[
[ζ, e^{-\frac{1}{σ} Δ_ζ}] = Ω_ζ e^{-\frac{1}{σ} Δ_ζ}\quad\text{and}\quad[Ψ, e^{-\frac{1}{σ} Δ_ζ}] = Ω_ζ e^{-\frac{1}{σ} Δ_ζ},
\]

or equivalently,

\[
(ζ - Ω_ζ) e^{-\frac{1}{σ} Δ_ζ} = e^{-\frac{1}{σ} Δ_ζ}(ζ) \quad\text{and}\quad (Ψ - Ω_ζ) e^{-\frac{1}{σ} Δ_ζ} = e^{-\frac{1}{σ} Δ_ζ}(Ψ).
\]

Multiplying both sides of the above identities on the right by the operator \(e^{-\frac{1}{σ} Δ_ζ}\), induction over \(m ∈ \mathbb{N}\) completes the proof of Lemma 3.6

Appendix A.4. Proof of Lemma 4.6

Proof: Recall that raising resp. lowering properties (18) resp. (19) shows that \((Ω_ζ - ζ) \frac{1}{σ} Δ_ζ (φ(ζ)) = \frac{1}{σ} \frac{1}{Ω_ζ - ζ} Ω_ζ - ζ (φ(ζ))\) while \((1 + ζ)^m < e^{ζ|ζ|} ≤ e^{ζ|ζ|^{1/2}}\) shows that \((1 + ζ)^m (P^θ) (ζ, ζ)\) belongs to \(W^∞_{n-π, ζ}\).
Therefore \((\partial_z - \pi \zeta)^r (\partial_z)^j K^j(z, \zeta) = \sqrt{\frac{(j - r)!}{(j - l)!}} K^{j-m+1}(z, \zeta)\), and hence, to the sequence of identities

\[
(\partial_z)^j (\partial_z)^m (P^j \sigma)(z, \zeta) = \sum_{r=0}^{m} \binom{m}{r} (\pi \zeta)^{m-r} (\partial_z - \pi \zeta)^r (\partial_z)^j (P^j \sigma)
\]

\[
= \sum_{r=0}^{m} \binom{m}{r} (\pi \zeta)^{m-r} \int_{C} \sigma(\zeta, \zeta)(\partial_z - \pi \zeta)^r (\partial_z)^j K^j(z, \zeta) d\mu(\zeta)
\]

\[
= \sum_{r=0}^{m} \binom{m}{r} (\pi \zeta)^{m-r} \sqrt{\frac{(j - r)!}{(j - l)!}} (P^{j-m+1}\sigma)(z, \zeta).
\]

Thus, the sequence of estimates

\[
|\langle \partial_z \rangle^j (\partial_z)^m \sigma(z, \zeta)| \leq \sum_{r=0}^{m} \binom{m}{r} (\pi |z|)^{m-r} \sqrt{\frac{j!}{(j - l)!}} |(P^{j-m+1}\sigma)(z, \zeta)|
\]

\[
\leq \sum_{j=m-l}^{m} \sqrt{\frac{j!}{(j - l)!}} (1 + \pi |z|)^m |(P^{j-m+1}\sigma)(z, \zeta)|.
\]

yields \((\partial_z)^j (\partial_z)^m \sigma \in W^{a,\sigma}_{\alpha(a)}\) for any \(l, m \in \mathbb{N}_0\).

Now let us assume the constraint \(\Psi, \Theta \in C^1_0(\mathbb{C})\). Then for any \(l, m \in \mathbb{N}_0\) and for some \(b, d, C > 0\) we obtain the following upper estimate:

\[
|\langle \partial_z \rangle^j (\partial_z)^m \sigma(z - \zeta, \zeta - \zeta) \overline{P^k \Psi(\zeta, \zeta)} P^j \Theta(z, \zeta) e^{-\pi |z|^2}| \leq C e^{\frac{\pi}{2}|z|^2} |z| e^{-(a\pi)|\zeta - \zeta|^2} |\zeta|^2 - (b + d)|z|^2.
\]

The term \(e^{\frac{\pi}{2}|z|^2} |z| e^{-(a\pi)|\zeta - \zeta|^2} |\zeta|^2 - (b + d)|z|^2\) is integrable on \(\mathbb{C}\) and satisfies the limit condition \(\lim_{|z| \to \infty} e^{\frac{\pi}{2}|z|^2} |z| e^{-(a\pi)|\zeta - \zeta|^2} |\zeta|^2 - (b + d)|z|^2 = 0\).

This combined with integration by parts gives

\[
\int_{\mathbb{C}} \partial_z \left( e^{\frac{\pi}{2}|z|^2} |z| e^{-(a\pi)|\zeta - \zeta|^2} |\zeta|^2 - (b + d)|z|^2 \right) d^2z = 0
\]

\[
\int_{\mathbb{C}} \partial_{\zeta} \left( e^{\frac{\pi}{2}|z|^2} |z| e^{-(a\pi)|\zeta - \zeta|^2} |\zeta|^2 - (b + d)|z|^2 \right) d^2z = 0,
\]

and hence, induction over \(l, m \in \mathbb{N}_0\) results into the convolution formula

\[
(\partial_z)^j (\partial_z)^m \sigma \ast \left( \overline{P^k \Psi} P^j \Theta e^{-\pi |z|^2} \right) = \sigma \ast \left( (\partial_z)^j (\partial_z)^m \overline{P^k \Psi} P^j \Theta e^{-\pi |z|^2} \right).
\]

Finally, from linearity arguments, the operators \(D_{j,k}\) defined on the last section satisfy

\[
D_{j,k} \sigma \ast \left( \overline{P^k \Psi} P^j \Theta e^{-\pi |z|^2} \right) = \sigma \ast D_{j,k} \left( \overline{P^k \Psi} P^j \Theta e^{-\pi |z|^2} \right).
\]

\[\square\]
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