Abstract

We have recently developed methods for obtaining exact two-point resistance of the complete graph minus $N$ edges. We use these methods to obtain closed formulas of certain trigonometrical sums that arise in connection with one-dimensional lattice, in proving the Scott’s conjecture on permanent of Cauchy matrix, and in the perturbative chiral Potts model. The generalized trigonometrical sums of the chiral Potts model are shown to satisfy recursion formulas that are transparent and direct, and differ from those of Gervois and Mehta. By making a change of variables in these recursion formulas, the dimension of the space of conformal blocks of $SU(2)$ and $SO(3)$ WZW models may be computed recursively. Our methods are then extended to compute the corner-to-corner resistance, and the Kirchhoff index of the first non-trivial two-dimensional resistor network, $2 \times N$. Finally, we obtain new closed formulas for variant of trigonometrical sums, some of which appear in connection with number theory.
1 Introduction

We have recently developed methods for obtaining exact two-point resistance of certain circulant graph namely, the complete graph minus \( N \) edges [1]. In this paper, using similar techniques and ideas, we consider trigonometrical sums that arise in the computation of the two-point resistance of the finite resistor networks [2], in the work of McCoy and Orrick on the chiral Potts model [3], and in the Verlinde dimension formula of the twisted /untwisted space of conformal blocks of the \( SO(3)/SU(2) \) WZW model [4].

Before considering these trigonometrical sums, we test the techniques used in [1], by first deriving the Green’s function of the one-dimensional lattice graphs with free boundaries, and the two-point resistance of the \( N \)-cycle graph [2]. The same techniques is then used to evaluate a trigonometrical sum that played a crucial role to prove R. F Scott’s conjecture on the permanent of the Cauchy matrix [5, 6]. Having tested these techniques, an alternative derivation is then given for certain trigonometrical sum that appeared in the perturbative chiral Potts model [3, 7].

We have also considered the general case studied by Gervois and Mehta [7], Berndt and Yeap [8], here, our results agree with those in [7]. It turns out that the Verlinde’s dimension formulas for the untwisted space of conformal blocks, may be obtained simply by summing over certain parameter of a trigonometrical sum considered in [7]. For the twisted space of conformal blocks, however, the parameter is restricted to take some value. It is shown that the dimension of the conformal blocks on a genus \( g \geq 2 \) Riemann surface may be obtained through a recursion formula that relates different genera. Mathematically speaking, the dimension of the space of conformal blocks is obtained by expanding certain generating function order by order, or using the Hirzebruch-Riemann-Roch theorem [9].

By using the method given in [1], we are able to obtain closed form formula for the two-point resistance of a \( 2 \times N \) resistor network [10]. In this paper, an exact computation of the corner-to-corner resistance as well as the total effective resistance of a \( 2 \times N \) will be given. The total effective effective resistance, also called the Kirchhoff index [11], this is an invariant quantity of the resistor network or graph.

The exact two-point resistance of an \( M \times N \) resistor network is given in terms of a double sum and not in a closed form [2]. Therefore, our computation carried out in this paper, represents the first non-trivial exact results for the two-point resistance of a two-dimensional resistor network.

This method is then used to evaluate variant of trigonometrical sums, some of which are related to number theory, we hope that these trigonometrical sums will have some physical applications. It is interesting to point out that all the computations of the trigonometrical sums in this paper are based on a formula by Schwatt [12] on trigonometrical power sums, and the representation of the binomial coefficients by the residue operator. The Schwatt’s formula is modified slightly, only in the case of the corner-to-corner resistance, the Kirchhoff index and trigonometrical sum given by \( F_1(N, l, 2) \), and \( F_1(N, l, 2) \), see Section 6, this was also the case in our previous paper [1].

This paper is organized as follows; in section 2, we give an explicit computations of the two-point resistance of the \( N \)-cycle graph and the Green’s function of the one-dimensional
lattice, and in Section 3, we give a simple derivation of a trigonometrical sum connected with Scott’s conjecture on the permanent of the Cauchy matrix. In section 4, we consider trigonometrical sums arising in the chiral Potts model, and in the Verlinde formula of the dimension of the conformal blocks. Exact computations of the corner-to-corner and the Kirchhoff index of $2 \times N$ resistor network will be given in section 5. In Section 6, we consider other class of trigonometrical sums some of which are related to number theory, and finally, in Section 7, our conclusions are given.

2 The two-point resistance of one-dimensional lattice using the residue operator

In this Section, we first start with the two-point resistance of the $N$-cycle graph computations, then, move to the trigonometrical sum related to the two-point resistance of the one-dimensional lattice with free boundaries, that is, the path graph. The two-point resistance of the $N$-cycle graph between any two nodes $\alpha$ and $\beta$ is given by the following simple closed formula [2],

$$R(l) = \frac{1}{N} \sum_{n=1}^{N-1} \frac{\sin^2(nl\pi/N)}{\sin^2(n\pi/N)} = \frac{l(N-l)}{N},$$

(1)

where $l = |\alpha - \beta|$, and $1 \leq \alpha, \beta \leq N$. Our derivation for the two-point resistance starts with the following trigonometrical identity

$$\cos(2ln\pi/N) = \sum_{s=0}^{l} (-1)^s \frac{l}{l+s} \frac{(l+s)}{(l-s)} 2^{2s} \sin^{2s}(n\pi/N),$$

from which the above trigonometrical sum may be rewritten as

$$R(l) = \frac{1}{2N} \sum_{s=1}^{l} (-1)^{s+1} \frac{l}{l+s} \frac{(l+s)}{(l-s)} 2^{2s} \sum_{n=1}^{N-1} \sin^{2(s-1)}(n\pi/N).$$

(2)

On the other hand, Schwatt’s formula for trigonometrical power sums [12], gives

$$\sum_{n=1}^{N-1} \sin^{2(s-1)}(n\pi/N) = \frac{1}{2^{2(s-1)-1}} \sum_{t=1}^{s-1} (-1)^{t+1} \binom{2(s-1)}{s-1-t} + \frac{N-1}{2^{2(s-1)}} \binom{2(s-1)}{s-1}. $$

(3)

Therefore, the two-point resistance may be obtained by evaluating the binomial sums in the expression of $R(l)$, based on the residue operator. This operator played a crucial role in evaluating combinatorial sums and proving combinatorial identities [13]. First, let us recall the definition of the residue operator $\text{res}$. To that end, let $G(w) = \sum_{k=0}^{\infty} a_k w^k$ be a generating function for a sequence $\{a_k\}$. Then the k-th coefficient of $G(w)$ may be represented by the formal residue as follows

$$a_k = \text{res}_w G(w) w^{-k-1}.$$
This is equivalent to the Cauchy integral representation of \( a_k \),

\[
    a_k = \frac{1}{2\pi i} \oint_{|z| = \rho} \frac{G(w)}{w^{k+1}} dw,
\]

for coefficients of the Taylor series in a punctured neighborhood of zero. In particular, the generating function of the binomial coefficient sequence \( \binom{n}{k} \) for a fixed \( n \) is given by

\[
    G(w) = \sum_{k=0}^{n} \binom{n}{k} w^k = (1 + w)^n,
\]

and hence

\[
    \binom{n}{k} = \text{res}_w (1 + w)^n w^{-k-1}.
\]

The other binomial coefficient that we need in this paper is the following,

\[
    \binom{2n}{n} = \text{res}_w (1 - 4w)^{-1/2} w^{-n-1}.
\]

Before finishing this brief summary, we should mention one important property of the residue operator \( \text{res} \), namely linearity, this is crucial in doing computations, linearity states that; given some constants \( \alpha \) and \( \beta \), then

\[
    \alpha \text{res}_w G_1(w) w^{-k-1} + \beta \text{res}_w G_2(w) w^{-k-1} = \text{res}_w (\alpha G_1(w) + \beta G_2(w)) w^{-k-1}.
\]

Let us now evaluate the first term in Eq. (2), using the residue operator, namely the following term

\[
    R_1(l) = \frac{2}{N} \sum_{s=1}^{l} (-1)^s \frac{2l}{l+s} \binom{l+s}{l-s} \sum_{t=1}^{s-1} (-1)^t \binom{2(s-1)}{s-1-t} \left( \frac{(1 + w)^{2(s-1)}}{w^{s-t}} \right) \text{res}_w = 0 \frac{(1 + w)^{2(s)}}{(1 + w)^{3} w^{s-1}}.
\]

In obtaining Eq. (4), we discarded an analytic term at the pole \( w = 0 \) of order \( s - 1 \). By making a change of variable \( l - s = k \), then, Eq. (4) may be rewritten as

\[
    R_1(l) = (-1)^{l+1} \frac{2}{N} \text{res}_w = 0 \frac{w}{(1+w)^3} \sum_{k=1}^{l-1} (-1)^k \frac{2l}{2l-k} \binom{2l-k}{k} \left( \frac{1+w}{\sqrt{w}} \right)^{2(l-k)}
\]

\[
    = (-1)^{l+1} \frac{2}{N} \text{res}_w = 0 \frac{w}{(1+w)^3} \left( C_2 l \left( \frac{1+w}{\sqrt{w}} \right) - (-1)^l \right),
\]

(5)
where
\[ C_{2l}(x) = 2T_{2l}(x/2) = \sum_{k=0}^{l} (-1)^k \frac{2l}{2l-k} \binom{2l-k}{k} x^{2l-2k}, \]
is the normalized Chebyshev polynomial of the first kind \[14\], and
\[ T_{2l}(x/2) = \frac{1}{2 \sqrt{w}} \left[ \left( \frac{x}{2} + \sqrt{(x/2)^2 - 1} \right)^{2l} + \left( \frac{x}{2} - \sqrt{(x/2)^2 - 1} \right)^{2l} \right]. \]

Using the fact that
\[ C_{2l}(1+w) = \frac{1}{w} + w, \]
then the final expression for the first term \( R_1(l) \), reads
\[ R_1(l) = (-1)^{l+1} \frac{2}{N} \sum_{r=0}^{N_{\text{res}}} \frac{1}{(1+w)^3 w^{l-1}} \]
\[ = - \frac{l(l-1)}{N} \]
(6)

Similarly, the second term may written as
\[ R_2(l) : = \frac{(N-1)}{N} \sum_{s=1}^{l} (-1)^s \frac{2l}{l+s} \left( \frac{l+s}{s-1} \right) \]
\[ = (-1)^{l+1} \frac{(N-1)}{N} \sum_{r=0}^{N_{\text{res}}} \frac{1}{(1+w)^2 w^l} \]
\[ = \frac{l(N-1)}{N} \]
(7)

Adding the contributions given by Eqs. (6), and (7), then, we get
\[ R(l) = \frac{1}{N} \sum_{n=1}^{N-1} \frac{\sin^2(n l \pi/N)}{\sin^2(n \pi/N)} = \frac{l(N-l)}{N}. \]
(8)

Now, we want to evaluate the following trigonometrical sum
\[ F_N(l) = \frac{1}{N} \sum_{n=1}^{N-1} \frac{1 - \cos n l \pi/N}{1 - \cos n \pi/N}, \]
this sum arises in connection with the two-point resistance of a path graph \[2\]. The evaluation of this term may be done as follows;
\[ F_N(l) = \frac{1}{N} \sum_{n=1}^{N-1} \frac{1 - \cos n l \pi/N}{1 - \cos n \pi/N} \]
\[ = \frac{1}{N} \sum_{n=1}^{\frac{N}{2}} \frac{1 - \cos(2n-1) l \pi/N}{1 - \cos(2n-1) \pi/N} + \frac{1}{N} \sum_{n=1}^{\frac{N}{2}-1} \frac{1 - \cos 2n l \pi/N}{1 - \cos 2n \pi/N}, \]
(9)
here, \( N \) is assumed to be even, similar steps may be used for \( N \) odd. It is interesting to note that in evaluating \( F_N(l) \), we only need to compute the first term since the second term is related to the two-point resistance of the \( N \)-cycle graph given by Eq. (8). Then, the first term may be written as

\[
\frac{1}{N} \sum_{n=1}^{N} \frac{1 - \cos(2n - 1)l\pi/N}{1 - \cos(2n - 1)l\pi/N} = \frac{1}{N} \sum_{n=1}^{N} \frac{\sin^2(2n - 1)l\pi/2N}{\sin^2(2n - 1)l\pi/2N} = \frac{1}{2N} \sum_{s=1}^{l} (-1)^{s+1} \frac{l}{s} \frac{l+s}{l-s} \left( \frac{2s}{s-1} \right) \sum_{n=1}^{\frac{N}{2}} \sin^2(s-1) \frac{(2n-1)\pi}{2N}.
\]

(10)

By using the identity

\[
\sum_{n=1}^{N} \sin^2(s-1)(2n - 1)\pi/2N = \frac{1}{2} \left( \sum_{n=1}^{N-1} \sin^2(s-1) n\pi/2N + \sum_{n=1}^{N-1} (-1)^{n-1} \sin^2(s-1) n\pi/2N \right),
\]

and the formulas for trigonometrical power sums given in [12], then, one can show

\[
\sum_{n=1}^{N} \sin^2(s-1)(2n - 1)\pi/2N = \frac{2N}{2s} \left( \frac{2s-1}{s-1} \right),
\]

(11)

which in turn, implies that the formula for the first term should be

\[
\frac{1}{N} \sum_{n=1}^{N} \frac{1 - \cos(2n - 1)l\pi/N}{1 - \cos(2n - 1)l\pi/N} = \frac{1}{2} \sum_{s=1}^{l} (-1)^{s+1} \frac{2l}{s} \frac{l+s}{l-s} \left( \frac{2s}{s-1} \right) \left( \frac{2(s-1)}{s-1} \right)
\]

\[
= \frac{l}{2}.
\]

(12)

To compute the second term given in Eq. (9), we use the following symmetry enjoyed by the two-point resistance of the \( N \)-cycle graph, \( N \) even

\[
\frac{1}{N} \sum_{n=1}^{N-1} \frac{1 - \cos 2n\pi/N}{1 - \cos 2n\pi/N} = \frac{2}{N} \sum_{n=1}^{N-1} \frac{1 - \cos 2n\pi/N}{1 - \cos 2n\pi/N} + \frac{1}{2N} \left( 1 - (-1)^l \right).
\]

(13)

Therefore, the second term may be obtained to give the following closed formula for \( F_N(l) \)

\[
F_N(l) = \frac{1}{N} \sum_{n=1}^{N-1} \frac{1 - \cos n\pi/N}{1 - \cos n\pi/N} = l - \frac{1}{N} \left( \frac{l^2}{2} + \frac{1}{4} \left( 1 - (-1)^l \right) \right)
\]

(14)

This is in a complete agreement with the formula for the Green’s function for the path graph in [2].
3 Trigonometrical sum connected with Scott’s conjecture

In proving R. F Scott’s conjecture on the permanent of the Cauchy matrix, Minc in [5] needed to evaluate the following trigonometrical sum:

\[ \sum_{n=1}^{N} \frac{\cos(2n - 1)l\pi/N}{1 - \cos(2n - 1)\pi/N}. \]  \hspace{1cm} (15)

He obtained a closed-form formula for this sum using induction, and the sum turns out to be equal to \( \frac{N}{2}(N - 2l) \). A short time later, Stembridge and Todd [6], gave a proof for the evaluation for this sum, based on linear algebra. Here, we give a short derivation for this sum using our formula given by Eq. (12), and the well-known identity

\[ \sum_{n=1}^{N-1} \frac{1}{\sin^2(n\pi/N)} = \frac{N^2 - 1}{3}. \]  \hspace{1cm} (16)

Our derivation follows easily by realizing that Eq. (12) is symmetric under the shift \( n \to N - n \), and as a consequence one gets

\[ \sum_{n=1}^{N} \frac{1 - \cos(2n - 1)l\pi/N}{1 - \cos(2n - 1)\pi/N} = Nl. \]  \hspace{1cm} (17)

In order to evaluate the sum in Eq. (15), we need a formula for the sum

\[ \sum_{n=1}^{N} \frac{1}{1 - \cos(2n - 1)\pi/N} = \frac{1}{2} \sum_{n=1}^{N} \frac{1}{\sin^2(2n - 1)\pi/2N}. \]

The latter may be evaluated as follows

\[ \frac{1}{2} \sum_{n=1}^{N} \frac{1}{\sin^2(2n - 1)\pi/2N} = \frac{1}{2} \sum_{n=1}^{2N-1} \frac{1}{\sin^2(n\pi/2N)} - \frac{1}{2} \sum_{n=1}^{N-1} \frac{1}{\sin^2(2n\pi/2N)} = \frac{N^2}{2}. \]  \hspace{1cm} (18)

In obtaining Eq. (18), we used the identity given in Eq. (16), thus, using Eq. (17) and Eq. (18), we may write

\[ \sum_{n=1}^{N} \frac{\cos(2n - 1)l\pi/N}{1 - \cos(2n - 1)\pi/N} = \frac{N^2}{2} - Nl. \]  \hspace{1cm} (19)

This is exactly the result obtained by Minc, Stembridge and Todd [5,6].
4 Trigonometrical sums arising in the chiral Potts model and in the Verlinde’s formula

In this section the trigonometrical sum $T_4(l) := \sum_{n=1}^{N-1} \frac{\sin^2(nl\pi/N)}{\sin(n\pi/N)}$ is evaluated in a closed form using the residue operator. We also give an almost closed formula for the general case $T_{2m}(l) := \sum_{n=1}^{N-1} \frac{\sin^2(nl\pi/N)}{\sin^{2m}(n\pi/N)}$, for $m \geq 1$. The first trigonometrical sum arises in the work of McCoy and Orrick on the chiral Potts model [3], this sum including other trigonometrical identities were proved by Gervois and Mehta [7]. The second sum namely the sum $T_{2m}(l)$, was considered by Gervois and Mehta [7] using a recursion formula. Here, we will obtain recursion formulas for both $T_{2m}(l)$ and

$$T_{2m} := \sum_{n=1}^{N-1} \frac{1}{\sin^{2m}(n\pi/N)}.$$

If, we set $N = k + 2$ and $m = g - 1$, $k, g$ being the level of the $su(2)$ Kac-Moody algebra, and the genus of the Riemann surface respectively. Then, the sum $T_{2m}$ up to to some normalization factor is nothing but the dimension of the space of the conformal blocks of the $SU(2)$ WZW model. As a consequence, the recursion formula derived for $T_{2m}$, may be used to obtain the expression for the dimension of the space of the conformal blocks for a given genus $g$. Similar computations are carried out for the twisted trigonometrical sum

$$T_{2m}' := \sum_{n=1}^{N-1} \frac{(-1)^{n+1}}{\sin^{2m}(n\pi/N)}.$$

This is related to the dimension of the space of the conformal blocks of the $SO(3)$ WZW model.

4.1 Trigonometrical sums and the perturbative chiral Potts model

Let us first start with the trigonometrical sums arising in the perturbative treatment of the chiral Potts model [3]. Techniques of the previous section, may be used to evaluate the sum $T_4(l)$, as follows

$$T_4(l) = \sum_{n=1}^{N-1} \frac{\sin^2(nl\pi/N)}{\sin^4(n\pi/N)}$$

$$= l^2 \sum_{n=1}^{N-1} \frac{1}{\sin^2(n\pi/N)} + \frac{1}{2} \sum_{s=2}^{l} (-1)^{s+1} \frac{l}{l+s} \left( \frac{l+s}{l-s} \right)^2 2^{2s} \sum_{n=1}^{N-1} \sin^{2(s-2)}(n\pi/N)$$

$$= \frac{l^2}{3} (N^2 - 1) + 8 \sum_{s=2}^{l} (-1)^{s} \frac{2l}{l+s} \left( \frac{l+s}{l-s} \right)^{s-2} \sum_{t=1}^{l} (-1)^t \frac{2(s-2)}{s-2-t}$$

$$+ 4(N-1) \sum_{s=2}^{l} (-1)^{s+1} \frac{2l}{l+s} \left( \frac{l+s}{l-s} \right) \frac{2(s-2)}{s-2},$$

(20)
the first term in the above equation follows from the well-known identity

\[ \sum_{n=1}^{N-1} \frac{1}{\sin^2(n\pi/N)} = \frac{N^2 - 1}{3}, \]

while the second and the third terms may computed using the residue operator as in the previous section to give,

\[ \sum_{s=2}^{l} (-1)^{s+1} \frac{2l}{l+s} \frac{1}{l-s} \left( \sum_{t=1}^{s-2} (-1)^t \left( \frac{2(s-2)}{s-2-t} \right) \right) = (-1)^{l+1} \text{res}_{w=0} \frac{1}{(1+w)^5 w^{l-2}} = \frac{1}{4!} (l+1)l(l-1)(l-2), \quad (21) \]

and

\[ \sum_{s=2}^{l} (-1)^{s+1} \frac{2l}{l+s} \frac{1}{l-s} \left( \frac{2(s-2)(l+s)}{l-s} \right) = (-1)^{l+1} \text{res}_{w=0} \frac{1}{(1+w)^4 w^{l-1}} = \frac{1}{3!} (l+1)l(l-1). \quad (22) \]

Therefore, the closed formula for the sum given in Eq. (20), reads

\[ \sum_{n=1}^{N-1} \frac{\sin^2(nl\pi/N)}{\sin^4(n\pi/N)} = \frac{l^2}{3} (N^2 - 1) + \frac{1}{3} (l+1)l(l-1)(l-2) - \frac{2(N-1)}{3} (l+1)l(l-1) \]

\[ = \frac{l^2}{3} (N-l)(N-l) + \frac{2}{3} (N-l). \quad (23) \]

This is exactly the result obtained by Gervois and Mehta using a recursion formula satisfied by \( T_{2m}(l) \) \[5\]. Next, we will give another recursion formula for the sum \( T_{2m}(l) \). Now, \( T_{2m}(l) \), may be written as

\[ T_{2m}(l) = \sum_{n=1}^{N-1} \frac{\sin^2(nl\pi/N)}{\sin^{2m}(n\pi/N)} \]

\[ = \frac{1}{2} \sum_{m=1}^{N-1} \frac{1}{s} \left( \sum_{s=1}^{l-s} (-1)^{s+1} \frac{l}{l+s} \frac{1}{l-s} \right) 2^{2s} \sum_{n=1}^{N-1} \frac{1}{\sin^{2m-s}(n\pi/N)} \]

\[ + \frac{1}{2} \sum_{s=m}^{l} (-1)^{s+1} \frac{l}{l+s} \frac{1}{l-s} 2^{2s} \sum_{n=1}^{N-1} \sin^{2(s-m)}(n\pi/N). \quad (24) \]

The first term on the right-hand side, is written in terms of the sum

\[ T_{2k} = \sum_{n=1}^{N-1} \frac{1}{\sin^{2k}(n\pi/N)}. \]
This may be computed [7], using

\[ T_{2k} = \sum_{n=1}^{N-1} \left( \cot^2 \left( \frac{n\pi}{N} \right) + 1 \right)^k = \sum_{l=1}^{k} \binom{k}{l} S_l, \]

where \( S_l = \sum_{n=1}^{N-1} \left( \cot^2 \left( \frac{n\pi}{N} \right) \right)^l \) and a recurrence relation satisfied by the power sums \( S_l \).

It turns out that \( T_{2k} \), may also be obtained using a recursion formula, this will be shown shortly. Now, the second term may be written as follows

\[ \tilde{T}_{2m}(l) = \frac{1}{2} \sum_{s=m}^{l} (-1)^{s+1} \frac{l + s}{l + s} \left( \frac{l + s}{l - s} \right) \sum_{t=1}^{s-m} (-1)^t \left( \frac{2(s-m)}{s-m-t} \right) \]

\[ = 2^{2m-1} \sum_{s=m}^{l} (-1)^{s} \frac{2l}{l + s} \left( \frac{l + s}{l - s} \right) \sum_{t=1}^{s-m} (-1)^t \left( \frac{2(s-m)}{s-m-t} \right) \]

\[ + 2^{2m-2}(N-1) \sum_{s=m}^{l} (-1)^{s+1} \frac{2l}{l + s} \left( \frac{l + s}{l - s} \right) \left( \frac{2(s-m)}{s-m} \right) \]

\[ = (-1)^{t+1} 2^{2m-1} \sum_{s=m}^{l} (-1)^{s+1} \frac{2l}{l + s} \left( \frac{l + s}{l - s} \right) \sum_{t=1}^{s-m} (-1)^t \left( \frac{2(s-m)}{s-m-t} \right) \]

\[ = \frac{1}{(1 + w)^{2m+1} w^{l-m}} \]

\[ + (-1)^{t+1} 2^{2m-2}(N-1) \sum_{s=m}^{l} (-1)^{s+1} \frac{2l}{l + s} \left( \frac{l + s}{l - s} \right) \left( \frac{2(s-m)}{s-m} \right) \]

\[ = \frac{1}{(1 + w)^{2m+1} w^{l-m}} \sum_{s=m}^{l} (-1)^{s+1} \frac{2l}{l + s} \left( \frac{l + s}{l - s} \right) \sum_{t=1}^{s-m} (-1)^t \left( \frac{2(s-m)}{s-m-t} \right) \]

\[ = (-1)^{m} 2^{2m-1} \frac{(l + m - 1)!}{(l - m - 1)! (2m)!} + (-1)^{m+1}(N-1) 2^{2m-2} \frac{(l + m - 1)!}{(l - m)! (2m - 1)!} \]

\[ \text{(25)} \]

Therefore, we succeeded in writing \( \tilde{T}_{2m}(l) \) in a closed form formula, One can check easily that our results agree with those given in [7], and so the formula for \( T_{2m}(l) \) becomes

\[ T_{2m}(l) = \sum_{n=1}^{N-1} \frac{\sin^2 \left( \frac{n\pi}{N} \right)}{\sin^2 \left( \frac{m\pi}{N} \right)} \]

\[ = \frac{1}{2} \sum_{s=1}^{m-1} (-1)^{s+1} \frac{l + s}{l + s} \left( \frac{l + s}{l - s} \right) 2^{2s} T_{2(m-s)} \]

\[ + (-1)^{m+1} 2^{2m-1} \frac{(l + m - 1)!}{(l - m)! (2m)!} (mN - l). \]

\[ \text{(26)} \]

setting \( m = 1, 2 \), then, our previous results given by Eq’s. (8) and (23) respectively are recovered. From Eq. (26), it is clear that in order to have a closed formula for \( T_{2m}(l) \), one needs also, the exact expressions for \( T_{2k} \), \( k = 1 \cdots m - 1 \). Next, we will show that \( T_{2k} \) satisfies a recursion formula that involves the \( T_{2k} \)'s. The expression for \( T_{2m} \) may be obtained
from \( T_{2m}(l) \) from the following simple formula;

\[
\sum_{l=1}^{N-1} T_{2m}(l) = \sum_{l=1}^{N-1} \sum_{n=1}^{N-1} \sin^2(nl\pi/N) \sin^{2m}(n\pi/N) \\
= \frac{N}{2} \sum_{n=1}^{N-1} \frac{1}{\sin^{2m}(n\pi/N)},
\]

(27)

Therefore, we may write

\[
T_{2m} = \sum_{n=1}^{N-1} \frac{1}{\sin^{2m}(n\pi/N)} \\
= \frac{1}{N} \sum_{l=1}^{N-1} \sum_{s=1}^{m-1} (-1)^{s+1} \frac{l}{l+s} \left( \frac{l+s}{l-s} \right) 2^{2s} T_{2(m-s)} \\
+ \frac{2}{N} \sum_{l=1}^{N-1} (-1)^{l+1} 2^{2m-1} \text{res}_{w=0} \frac{1}{(1+w)^{2m+1} w^{l-m}} \\
+ \frac{2}{N} \sum_{l=1}^{N-1} (-1)^{l+1} 2^{2m-2}(N-1) \text{res}_{w=0} \frac{1}{(1+w)^{2m} w^{l+1-m}} \\
= \frac{1}{N} \sum_{l=1}^{N-1} \sum_{s=1}^{m-1} (-1)^{s+1} \frac{l}{l+s} \left( \frac{l+s}{l-s} \right) 2^{2s} T_{2(m-s)} \\
+ \frac{(-1)^{m+1} 2^{2m-1}}{N(N-m-1)(2m+1)!} \left( \frac{N+m-1}{(2mN+1)!(2m+1)!} \right) (2mN + 1 - N).
\]

(28)

As a result, from our recursion formula, the different \( T_{2m} \)'s may be obtained directly, we do not have to use the recurrence relation satisfied by the power sum \( S_l \) [7]. For \( m = 1 \) the first term in Eq. (28) does not contribute and the second term gives the well known formula \( T_2 = \frac{N^2-1}{3} \). Now, for \( m = 2 \), the first term may be computed to give \( \frac{(N^2-1)(N-1)(2N-1)}{9} \), while the second term gives \( -\frac{(N^2-1)(N-2)(3N+1)}{15} \), and hence, \( T_4 = \frac{(N^2-1)(N^2+1)}{45} \) in a full agreement with [7], [8].

4.2 The Verlinde dimension formula

The verlinde dimension formula may be obtained simply by setting \( m = g-1, N = k+2 \) in the expression of \( T_{2m} \), where \( g \geq 2, k \) are the genus of the Riemann surface, and the level of the lie algebra \( SU(2) \), respectively. Then, \( T_{2(g-1)} \) up to some normalization factor, is the
dimension of the space of conformal blocks $V_g$ of the $SU(2)$ WZW model \[4\],

$$
dim V_{g,k} = \left( \frac{k+2}{2} \right)^{g-1} \sum_{n=1}^{k+1} \frac{1}{\sin^{2g-2} \frac{n\pi}{(k+2)}}$$

$$
= \left( \frac{k+2}{2} \right)^{g-1} \frac{1}{(k+2)} \sum_{l=1}^{k+1} \sum_{s=1}^{g-2} (-1)^{s+1} \frac{l}{l+s} T_{2g-2s}^{2sT_{2g-2s}}
+ (-1)^{g} 2^{2g-3} \left( \frac{k+2}{2} \right)^{g-1} \frac{1}{(k+2) (2g-1)} \left( \frac{k+g}{2g-2} \right) (k+2)(2g-3) + 1) \tag{29}
$$

As a consequence, the dimension of the space of the conformal blocks of the $SU(2)$ WZW model, may be computed using our recursion formula for $T_{2k}$. Our formula Eq. \[29\] may be used to give

$$
dim V_{2,k} = \frac{(k+1)(k+2)(k+3)}{6}
$$

$$
dim V_{3,k} = \frac{1}{5} \left[ \frac{(k+1)(k+2)(k+3)}{6} \left( \frac{(k+1)(k+2)(k+3)}{6} + 2(k+2) \right) \right]
$$

$$
dim V_{4,k} = \frac{1}{7} \left[ \frac{(k+1)(k+2)(k+3)}{6} \left( \frac{2(k+1)(k+2)(k+3) + 27(k+2)}{6} \right) \right] + 6(k+2)^2 \tag{30}
$$

Our first two expressions for dimension of the space of conformal blocks agree with those computed using conformal field theory\[15\]. For $g = 4$, our formula for $dim V_{4,k}$ is identical to the formula given by Zagier \[9\] provided the shift $k \to k+2$ is taken. This shift is natural, since Zagier defined the dimension of the space conformal blocks as $dim V_{g,k-2}$.

For the WZW model based on $SO(3)$, the level $k$ must be even \[16\], and the formula for the dimension of the twisted space of the conformal blocks $V_{g,k}^t$, may be written as

$$
dim V_{g,k}^t = \left( \frac{k+2}{2} \right)^{g-1} \sum_{n=1}^{k+1} (-1)^{n+1} \frac{1}{\sin^{2g-2} \frac{n\pi}{(k+2)}}
$$

In order to derive a recursion formula for the dimension $dim V_{g,k}^t$, we first note that the

\[1\]The last term $k+2$ in the expression of $dim V_{3,k}$ should be corrected in \[15\] in order to make it a positive integer.
expression for the twisted version of the trigonometrical sum $T_{2m}(l)$ given by Eq. (24) is

$$T^t_{2m}(l) = \sum_{n=1}^{N-1} (-1)^{n+1} \frac{\sin^2(nl\pi/N)}{\sin^{2m}(n\pi/N)}$$

$$= \frac{1}{2} \sum_{s=1}^{m-1} (-1)^{s+1} \frac{l}{l+s} \frac{l+s}{l-s} 2^{2s} T^t_{2(m-s)}$$

$$+ \frac{1}{2} \sum_{s=m}^{l} (-1)^{s+1} \frac{l}{l+s} \frac{l+s}{l-s} 2^{2s} \sum_{n=1}^{N-1} (-1)^{n+1} \sin^2(s-m)(n\pi/N),$$

(31)

where

$$T^t_{2m} = \sum_{n=1}^{N-1} (-1)^{n+1} \frac{1}{\sin^{2m}(n\pi/N)}$$

The trigonometrical sum

$$\sum_{n=1}^{N-1} (-1)^{n+1} \sin^2(s-m)(n\pi/N),$$

is non-vanishing only if $N$ is even [12] and is given by

$$\sum_{n=1}^{N-1} (-1)^{n+1} \sin^2(s-m)(n\pi/N) = 2^{2(m-s)+1} \sum_{t=1}^{s-m} (-1)^t \binom{2(s-m)}{s-m-t}$$

$$+ 2^{2(m-s)} \binom{2(s-m)}{s-m}$$

(32)

This formula shows clearly that the dimension of the twisted space of the conformal blocks is non-vanishing only if $k$ is even, $N = k + 2$ in agreement with the algebro-geometrical argument [16]. Then, the expression for $T^t_{2m}$ may be written as

$$T^t_{2m}(l) = \sum_{n=1}^{N-1} (-1)^{n+1} \frac{\sin^2(nl\pi/N)}{\sin^{2m}(n\pi/N)}$$

$$= \frac{1}{2} \sum_{n=1}^{N-1} (-1)^{n+1} \sum_{s=1}^{m-1} (-1)^{s+1} \frac{l}{l+s} \frac{l+s}{l-s} 2^{2s} \sum_{n=1}^{N-1} (-1)^{n+1} \sin^2(s-m)(n\pi/N)$$

$$+ 2^{2m-1} \sum_{s=m}^{l} (-1)^{s+1} \frac{l}{l+s} \frac{l+s}{l-s} \sum_{t=1}^{s-m} (-1)^t \binom{2(s-m)}{s-m-t}$$

$$+ 2^{2m-2} \sum_{s=m}^{l} (-1)^{s+1} \frac{l}{l+s} \frac{l+s}{l-s} \binom{2(s-m)}{s-m}$$

$$= \frac{1}{2} \sum_{s=1}^{m-1} (-1)^{s+1} \frac{l}{l+s} \frac{l+s}{l-s} 2^{2s} T^t_{2(m-s)}$$

$$+ (-1)^{m+1} 2^{m-1} \frac{(l + m - 1)!}{(l - m)! (2m)!}.$$
The recursion formula for the twisted trigonometrical sum $T_{2m}^t$, may be derived by setting $l = N/2$ in Eq. (33)

$$
T_{2m}^t = \sum_{n=1}^{N-1} (-1)^{n+1} \frac{1}{\sin^2 m(n\pi/N)}
$$

$$
= \sum_{s=1}^{m-1} (-1)^{s+1} \frac{N/2}{N/2 + s} \left(\frac{N/2 + s}{N/2 - s}\right) 2^{2s} T_{2(m-s)}^t
$$

$$
+ (-1)^{m+1} 2^{2m} \frac{(N/2 + m - 1)!}{(N/2 - m)! (2m)!} \frac{N}{2}
$$

$$
- \sum_{n=1}^{N-1} \frac{1}{\sin^2 m(n\pi/N)}.
$$

For $m = 1$ and $m = 2$ the twisted trigonometrical sums are

$$
T_2^t = \frac{N^2 + 2}{6}
$$

and

$$
T_4^t = \frac{7N^4 + 40N^2 + 88}{360},
$$

respectively. In obtaining these results we used the expressions for $T_2$ and $T_4$. These twisted trigonometrical sums appeared earlier as coefficients of certain generating function \[9\]. Using the recursion formula Eq. (34), the twisted trigonometrical sum $T_6^t$ is

$$
T_6^t = \frac{31N^6 + 294N^4 + 1344N^2 + 3056}{15120}.
$$

The twisted trigonometrical sum formula given by Eq. (34), implies that the dimension of the twisted space of the conformal blocks is may be deduced for any genus $g \geq 2$, through the following formula

$$
dim V_{g,k}^t = \left(\frac{k + 2}{2}\right)^{g-1} \sum_{n=1}^{g-1} (-1)^{n+1} \frac{1}{\sin^2 g-2 n\pi/(k+2)}
$$

$$
= \left(\frac{k + 2}{2}\right)^{g-1} \sum_{s=1}^{g-2} (-1)^{s+1} \frac{(k + 2)/2}{(k + 2)/2 + s} \left(\frac{(k + 2)/2 + s}{(k + 2)/2 - s}\right) 2^{2s} T_{2g-2-2s}^t
$$

$$
+ (-1)^g 2^{2g-2} \frac{k + 2}{2} \left(\frac{(k + 2)/2 + g - 2)!}{((k + 2)/2 - g + 1)!(2g - 2)!} (k + 2)/2
$$

$$
- \dim V_{g,k}.
$$

Note that, the relation between $\dim V_{g,k}^t$ and $\dim V_{g,k}$ is expected from the simple identity

$$
\dim V_{g,k-2}^t = \dim V_{g,k-2} - 2^g \dim V_{g,k/2-1},
$$
where $k$ even. The formula by Zagier [9] for $\dim V_{g,k-2}$, may be obtained using the following generating function

$$\sum_{g=1}^{\infty} \dim V_{g,k-2} \left( \frac{2}{k} \sin^2 x \right)^{g-1} = \frac{k \sin(k-1)x}{\sin kx \cos x}.$$
In order to compute the corner-to-corner resistance of a $2 \times N$ resistor network, we set $M = 2$, $r_1 = (0, 0)$ and $r_2 = (1, N - 1)$ into Eq. (39), and so the double sum of the above equation is reduced to a single sum, and the corner-to-corner resistance may be written as

$$R_{\{2\times N\}}^{\text{free}}((0, 0), (1, N - 1)) = \frac{1}{N} + \frac{N - 1}{2} + \frac{1}{3N} \sum_{n=1}^{N-1} \frac{(1 + (-1)^n)(1 + \cos n\pi/N)}{2(1 - 2/3 \cos^2 n\pi/2N)}.$$  (40)

For $N$ even, the sum over $n$ may be reduced to

$$\frac{2}{3N} \sum_{n=1}^{N/2-1} \frac{\cos^2 n\pi/N}{(1 - 2/3 \cos^2 n\pi/N)} = \frac{1}{3N} \sum_{n=1}^{N-1} \frac{\cos^2 n\pi/N}{(1 - 2/3 \cos^2 n\pi/N)} = \frac{1}{3N} \sum_{j=0}^{\infty} (2/3)^j \sum_{n=1}^{N-1} \cos^{2(j+1)} n\pi/N,$$  (41)

to evaluate this sum, we follow closely the method developed by the author in [1]. As explained in [1], the formula for the sum $\sum_{n=1}^{N-1} \cos^2 n\pi/N$, given by Schwatt [12] is not the right one to use, we use instead the formula

$$\sum_{n=1}^{N-1} \cos^2 n\pi/N = -1 + \frac{N}{2^{2j-1}} \sum_{p=1}^{[J/N]} \left( \frac{2J}{J - pN} \right) + \frac{1}{2^j} \left( \frac{2J}{J} \right),$$  (42)

thus, the sum contribution to the corner-to-corner resistance using the residue representation of binomials is

$$\frac{1}{3N} \sum_{j=0}^{\infty} (2/3)^j \sum_{n=1}^{N-1} \cos^{2(j+1)} n\pi/N = -\frac{1}{N} + \sum_{j=0}^{\infty} \text{res}_w \frac{(1 + w)^{2j}}{(6w)^j} \frac{w^N}{w(1 - w^N)}$$

$$+ \frac{1}{2} \left[ \text{res}_w (1 - 4w)^{-1/2} \sum_{j=0}^{\infty} (1/6w)^j w^{-1} - 1 \right]$$

$$= -\frac{1}{N} + \sqrt{3} \frac{(2 - \sqrt{3})^N}{1 - (2 - \sqrt{3})^N} + \frac{1}{2} (\sqrt{3} - 1).$$  (43)

Finally, the corner-to-corner resistance of $2 \times N$ resistor network becomes,

$$R_{\{2\times N\}}^{\text{free}}((0, 0), (1, N - 1)) = \frac{N - 1}{2} + \sqrt{3} \frac{(2 - \sqrt{3})^N}{1 - (2 - \sqrt{3})^N} + \frac{1}{2} (\sqrt{3} - 1).$$  (44)

It is not difficult to see that this formula is also valid for $N$ odd.

Examples. For $N = 2, 3, 4$ our formula Eq. (44), gives

$$R_{\{2\times 2\}}^{\text{free}}((0, 0), (1, 1)) = 1$$

$$R_{\{2\times 3\}}^{\text{free}}((0, 0), (1, 2)) = 1.4$$

$$R_{\{2\times 4\}}^{\text{free}}((0, 0), (1, 3)) = 1.875,$$  (45)

these results are in a full agreement with Eq. (40).
5.2 The Kirchhoff index

The computation of the total effective resistance of a $2 \times N$ resistor network, that is, the Kirchhoff index, may be computed in two ways. It may be evaluated by summing over all effective resistances between nodes of a given resistor network, or alternatively by summing over all eigenvalues of a Laplacian associated with resistor network [20]. So, we do not have to know the effective resistance between each node to compute the total effective resistance of a resistor network. The formula that gives the Kirchhoff index of a resistor network in terms of the eigenvalues is

$$Kf(G) = N \sum_{n=1}^{N-1} \frac{1}{\lambda_n},$$

where $\lambda_n$ are the eigenvalues of the Laplacian of the network, or the graph $G$ made of nodes and edges considered as unit resistors. Our network is given by the cartesian product $2 \times N$, that is, made of two path lines with $N$ nodes, and $N$ path lines with two nodes. Now, the Kirchhoff index of a path line is

$$Kf(P_n) = N \sum_{n=1}^{N-1} \frac{1}{4 \sin^2(n\pi/2N)} = \frac{N}{8} \left[ \sum_{n=1}^{2N-1} \frac{1}{\sin^2(n\pi/2N)} - 1 \right] = \frac{N^3 - N}{6}.$$ 

Thus, the contribution from these path lines is $N + \frac{N^3 - N}{3}$, by connecting the system together, then the corresponding eigenvalues of the laplacian are $\lambda_{1,n} = 3(1 - 2/3 \cos^2 n\pi/2N)$. As a consequence, the Kirchhoff index of a $2 \times N$, resistor network can be written as

$$Kf(2 \times N) = N + \frac{N^3 - N}{3} + N \sum_{n=1}^{N-1} \frac{1}{3(1 - 2/3 \cos^2 n\pi/2N)}, \quad (46)$$

Note that, our simple deduction of this expression gives the same value of the Kirchhoff index given by theorem 4.1 in [21]. The above sum seems difficult to evaluate, however, using a simple trick, we will be able to get a closed form for the Kirchhoff index. To that end, let us write

$$\sum_{n=1}^{N-1} \frac{1}{(1 - 2/3 \cos^2 n\pi/2N)} = \sum_{n=1}^{N/2-1} \frac{1}{(1 - 2/3 \cos^2 n\pi/N)} + \sum_{n=1}^{N/2} \frac{1}{(1 - 2/3 \cos^2 (2n - 1)\pi/2N)}, \quad (47)$$

where $N$ is assumed to be even. The first sum may be carried out using the following trick;

$$\sum_{j=0}^{\infty} (2/3)^j \sum_{n=1}^{N-1} \cos^{2(j+1)} n\pi/N = \frac{3}{2} \left[ \sum_{j=0}^{\infty} (2/3)^j \sum_{n=1}^{N-1} \cos^{2j} n\pi/N - (N - 1) \right]. \quad (48)$$
Now, the sum on the left-hand side was computed before, see Eq. (43), then one may deduce

\[ \sum_{n=1}^{N-1} \frac{1}{(1 - 2/3 \cos^2 n\pi/N)} = \sum_{j=0}^{\infty} (2/3)^j \sum_{n=1}^{N-1} \cos^{2j} n\pi/N \]

\[ = -3 + \frac{6N(2 - \sqrt{3})^N}{\sqrt{3}(1 - (2 - \sqrt{3})^N)} + \sqrt{3}N. \]  

(49)

and so,

\[ \sum_{n=1}^{N/2-1} \frac{1}{(1 - 2/3 \cos^2 n\pi/N)} = \frac{1}{2} \left[ \sum_{n=1}^{N-1} \frac{1}{(1 - 2/3 \cos^2 n\pi/N)} - 1 \right] \]

\[ = -2 + \frac{3N(2 - \sqrt{3})^N}{\sqrt{3}(1 - (2 - \sqrt{3})^N)} + \frac{\sqrt{3}}{2}N. \]  

(50)

Using the identity [1],

\[ \sum_{n=1}^{N/2} \cos^{2j}(2n-1)\pi/N = \frac{N}{2^{2j+1}} \binom{2j}{j} + \frac{N}{2^{2j}} \sum_{p=1}^{\lceil j/2N \rceil} \left( 2j \right) \left( j - 2pN \right) \]

\[ - \frac{N}{2^{2j}} \sum_{p=1}^{\lceil j/2N \rceil} \left( j - (2p - 1)N \right), \]  

(51)

and by following similar steps as in the above computations, then, one may show

\[ \sum_{n=1}^{N/2} \frac{1}{(1 - 2/3 \cos^2 (2n-1)\pi/2N)} = \frac{3N(2 - \sqrt{3})^{2N}}{\sqrt{3}(1 - (2 - \sqrt{3})^{2N})} - \frac{3N(2 - \sqrt{3})^N}{\sqrt{3}(1 - (2 - \sqrt{3)^{2N}})} + \frac{\sqrt{3}}{2}N. \]  

(52)

Finally, the exact expression of the Kirchhoff index of a 2 × N reeds

\[ Kf(2 \times N) = N + \frac{N^3 - N}{3} + \frac{N}{3} \left[ -2 + \frac{6N(2 - \sqrt{3})^{2N}}{\sqrt{3}(1 - (2 - \sqrt{3)^{2N}})} + \sqrt{3}N \right]. \]  

(53)

One can show that, the above formula for the Kirchhoff formula is valid for N odd as well.

Example, For N = 1, 2, 3, 4, 5, the Kirchhoff indices are respectively,

\[ Kf(2 \times 2) = 5 \]

\[ Kf(2 \times 3) = 14.2 \]

\[ Kf(2 \times 4) = 30.57142857 \]

\[ Kf(2 \times 5) = 56.10047847 \]  

(54)

These results are in complete agreement with those obtained using formula given by Eq. (46), or theorem 4.1 of reference [21].
6 Some trigonometrical sums related to number theory

In this section other class of trigonometrical sums will be evaluated using similar techniques as in the previous sections. Some of these trigonometrical sums are related to number theory. We will start with the following sum

\[ S(l) := \sum_{n=1}^{N-1} (-1)^n \frac{\sin^2(nl\pi/N)}{\sin^2(n\pi/N)}, \]

which is the alternating sum associated with the sum \( R(l) \) given in Eq. (1). This sum has the following closed formula

**Proposition 6.1**

\[ S(l) = \sum_{n=1}^{N-1} (-1)^n \frac{\sin^2(nl\pi/N)}{\sin^2(n\pi/N)} = -l^2 \]  \( (55) \)

To derive the above formula, we follow similar computations carried out for \( R(l) \), except that this time the sum over \( n \) is non-vanishing only if \( N \) is even

\[ \sum_{n=1}^{N-1} (-1)^{s-1} \sin^2(nl\pi/N) = \frac{1}{2^{2(s-1)-1}} \sum_{t=1}^{s-1} (-1)^{t+1} \left( \frac{2(s-1)}{s-1-t} \right) + \frac{-1}{2^{2(s-1)}} \left( \frac{2(s-1)}{s-1} \right), \]  \( (56) \)

Comparing Eq. (56) and Eq. (3), and using the previous results, then, without any further computations, the formula for the trigonometrical sum \( S(l) \) is obtained. Due to the the symmetry enjoyed by \( S(l) \), \( S(N-l) \) the right hand side of equation (55) should be read with this constraint, that is for both \( l \), and \( N-l \), \( S(l) = -l^2 \). Next, let us consider the sums \( S_1(l) := \sum_{n=1}^{N-1} \frac{\sin(nl\pi/N)}{\sin(n\pi/N)} \) and \( S_2(l) := \sum_{n=1}^{N-1} (-1)^n \frac{\sin(nl\pi/N)}{\sin(n\pi/N)} \) that are closely related. We will prove that closed formulas for the sums \( S_1(l) \) and \( S_2(l) \) are given by

**Theorem 6.1**

\[ S_1(l) = \sum_{n=1}^{N-1} \frac{\sin(nl\pi/N)}{\sin(n\pi/N)} = \begin{cases} N-l & \text{for } l \text{ odd} \\ 0 & \text{for } l \text{ even} \end{cases}, \]  \( (57) \)

\[ S_2(l) = \sum_{n=1}^{N-1} (-1)^n \frac{\sin(nl\pi/N)}{\sin(n\pi/N)} = \begin{cases} -(2l-1) & \text{for } l \text{ odd and } N \text{ even} \\ 0 & \text{for } l \text{ odd and } N \text{ odd} \\ -2l & \text{for } l \text{ even and } N \text{ odd} \\ 0 & \text{for } l \text{ even and } N \text{ even} \end{cases}. \]  \( (58) \)

To prove the first formula, we use the following trigonometrical identity \[22] \[ \frac{\sin(nl\pi/N)}{\sin(n\pi/N)} = \sum_{s \geq 0} (-1)^s \left( \frac{l-s-1}{s} \right) 2^{l-2s-1} \cos^{l-2s-1}(n\pi/N), \]  \( (59) \)
thus, for \( l \) odd, one has

\[
S_1(l) = \sum_{n=1}^{N-1} \frac{\sin((2l-1)n\pi/N)}{\sin n\pi/N} = \sum_{s \geq 0} (-1)^s \binom{2l-2-s}{s} 2^{l-2-2s} \sum_{n=1}^{N-1} \cos^{2l-2-2s}(n\pi/N).
\]

(60)

The sum over \( n \), may be computed from Schwatt’s book \[12\], see Eq. (107), page 221 to give

\[
\sum_{n=1}^{N-1} \cos^{2l-2-2s}(n\pi/N) = \frac{-2}{2^{l-2-2s}} \sum_{t=1}^{l-1-s} \binom{2l-2-2s}{l-1-s-t} + \frac{N-1}{2^{l-2-2s}} \binom{2l-2-2s}{l-1-s}.
\]

(61)

then, the first contribution to the sum given in Eq. (60) is

\[
S_1(l)' = -2 \sum_{s=0}^{l-1} (-1)^s \binom{2l-2-s}{s} \sum_{t=1}^{l-1-s} \binom{2l-2-2s}{l-1-s-t} = -2 \text{res}_{w=0} \left( \frac{1 + w}{\sqrt{w}} \right) \frac{1}{1 - w},
\]

(62)

in obtaining the last line of the above equation we used the expression for the normalized Chebyshev polynomial of the second kind \( U_n(x_2) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} x^{n-2k} \). The residue may be evaluated using

\[
U_n(x_2) = \frac{(x + \sqrt{x^2 - 1})^{n+1} - (x - \sqrt{x^2 - 1})^{n+1}}{2\sqrt{x^2 - 1}},
\]

to obtain

\[
S_1(l)' = -2 \text{res}_{w=0} \frac{1}{w^{l-1}} \frac{1}{(1 - w)^2} = -2(l - 1).
\]

(63)

Similarly, the second contribution reads

\[
S_1(l)'' = (N-1) \text{res}_{w=0} U_{2l-2} \left( \frac{1 + w}{2\sqrt{w}} \right) \frac{1}{w} = N - 1.
\]

(64)

Therefore, combining these contributions, the closed formula for \( S_1(l) \) is

\[
S_1(l) = \sum_{n=1}^{N-1} \frac{\sin((2l-1)n\pi/N)}{\sin n\pi/N} = N - (2l - 1).
\]

(65)
It is not difficult to show that there is no contribution to the sum $S_1(l)$ for $l$ even. In proving the second formula for $S_2(l)$ Eq. \((58)\), one notes that in evaluating the sum over $n$ in

$$ S_2(l) = \sum_{s \geq 0} (-1)^s \binom{l-s-1}{s} 2^{l-2s-1} \sum_{n=1}^{N-1} (-1)^n \cos^{l-2s-1}(n\pi/N), $$

turns out to depend on both $l$, $N$ unlike the previous case. By using formulas given by Eq.\((113)\), and Eq.\((114)\) in [12], we have

$$ \sum_{n=1}^{N-1} (-1)^n \cos^{l-2s-1}(n\pi/N) = -\frac{2}{2^{l-2-2s}} \sum_{t=1}^{l-1-s} \left(2l - 2 - 2s - t\right) - \frac{1}{2^{l-2-2s}} \left(2l - 2 - 2s\right), \tag{66} $$

for $l$ odd, $N$ even, and the sum vanishes for $l$ odd, $N$ odd. If $l$ is even, then, the above sum is non-vanishing only for $N$ odd. Therefore, for $l$ odd it follows from Eq. \((66)\) that we have

$$ S_2(l) = \sum_{n=1}^{N-1} (-1)^n \frac{\sin(2l-1)n\pi/N}{\sin(n\pi/N)} $$

$$ = -2 \text{res}_{w=0} \frac{1}{w^{l-1}} \frac{1}{1-w^2} - \text{res}_{w=0} \frac{1}{w^l} \frac{1}{1-w} = -(2l-1), \tag{67} $$

for $l$ even and $N$ odd, one has the following identity

$$ \sum_{n=1}^{N-1} (-1)^n \cos^{l-2s-1}(n\pi/N) = -\frac{2}{2^{l-1-2s}} \sum_{t=0}^{l-1-s} \left(2l - 1 - 2s - t\right), \tag{68} $$

from which

$$ S_2(l) = \sum_{n=1}^{N-1} (-1)^n \frac{\sin(2nl\pi/N)}{\sin(n\pi/N)} = -2 \text{res}_{w=0} \frac{1}{w^l} \frac{1}{1-w^2} = -2l, \tag{69} $$

these results were recently verified by simulations without proof in connection with number theory\(^2\) It is interesting to note that the closed formula for $S_2(l)$ may be expected since if we let $l$ go to $N-l$ in $S_1(l)$, then, $S_2(l) = -l$. Now, we will consider the following non-trivial and interesting trigonometrical sums,

$$ F_1(N, l, 2) := \sum_{n=1}^{N-1} \frac{\sin(nl\pi/N) \sin(2nl\pi/N)}{\sin(n\pi/N) \sin(2n\pi/N)}, $$

and

$$ F_2(N, l, 2) := \sum_{n=1}^{N-1} (-1)^n \frac{\sin(nl\pi/N) \sin(2nl\pi/N)}{\sin(n\pi/N) \sin(2n\pi/N)}, $$

where $F_1(N, N-l, 2) = F_2(N, l, 2)$, and $N$ is assumed to be odd. So, if $F_2(N, l, 2)$ is known, then, $F_2(N, l, 2)$ may be obtained and vice-versa. In the rest of this paper, we show that both the trigonometrical sums may be evaluated to give the following closed formulas

\(^2\)Anonymous author working on characters of a finite field and the Polya-Vinogradov inequality.
Theorem 6.2

\[ F_1(N, l, 2) = \sum_{n=1}^{N-1} \frac{\sin(nl\pi/N) \sin(2nl\pi/N)}{\sin(n\pi/N) \sin(2n\pi/N)} \]

\[ = -\frac{1}{2}(3l-2)(3l-3) + \frac{1}{2}(l-1)(l-2) - l + \frac{1}{2}(1 - (-1)^l) \]

\[ + \frac{N-1}{2} \left(2l - 1 - (-1)^l\right) + N \left(3l - 2 - N + \frac{1}{2}(1 - (-1)^{l-N})\right) \]

(70)

where \( l \) is odd and the sum vanishes for even \( l \), also, note that the last term namely the coefficient of \( N \) is different from zero only if \( 3l - 2 > N \). The closed formula for \( F_2(N, l, 2) \) reads

\[ F_2(N, l, 2) = \sum_{n=1}^{N-1} (-1)^n \frac{\sin(nl\pi/N) \sin(2nl\pi/N)}{\sin(n\pi/N) \sin(2n\pi/N)} \]

\[ = -\frac{1}{2}3l(3l-1) + \frac{1}{2}(l-1) - l + N \left(3l - \frac{(N+1)}{2} + \frac{1}{2}(1 - (-1)^{l-N})\right) \]

(71)

where \( l \) is even and the sum vanishes for odd \( l \), also, note that the last term whose coefficient is \( N \) is different from zero only if \( 3l > \frac{N+1}{2} \).

To prove the first formula, we note that \( F_1(N, 2l, 2) = 0 \), and hence the only sum to consider is the sum \( F_1(N, 2l-1, 2) \). The latter may be written as

\[ F_1(N, 2l-1, 2) = \sum_{n=1}^{N-1} \frac{\sin(n(2l-1)\pi/N) \sin(2n(2l-1)\pi/N)}{\sin(n\pi/N) \sin(2n\pi/N)} \]

\[ = \sum_{s,k \geq 0} (-1)^{s+k} \binom{2l-2-s}{s} \binom{2l-2-k}{k} 2^{2(l-2)-2(s+k)} \]

\[ \times \sum_{j=0}^{2l-2-2k} (-1)^j 2^j \binom{2l-2-2k}{k} \sum_{n=1}^{N-1} \cos^{2l-2-2(s-j)}(n\pi/N). \]

(72)

The sum over \( n \), formally looks like that given in Eq. (61), however, the variable \( t \), may be a multiple of \( N \) and in that case the Schwatt’s formula given by Eq (107), does not work it has to be modified slightly. The formula that takes into account this fact may be shown to
be given by

\[
\sum_{n=1}^{N-1} \cos^{2l-2-2(s-j)}(n\pi/N) = \frac{-2}{2^{l-2-2(s-j)}} \sum_{t=1}^{l-1-s} \left( 2l - 2 - 2(s-j) \right) \\
\sum_{t=1}^{l-1-s} \left( 2l - 2 - (s-j) - t \right)
\]

\[
+ \frac{N-1}{2^{l-2-2(s-j)}} \left( 2l - 2 - 2(s-j) \right) \\
\left( l - 1 - (s-j) \right)
\]

\[
+ \frac{2N}{2^{l-2-2(s-j)}} \sum_{p=1}^{l-1-(s-j)/N} \left( 2l - 2 - 2(s-j) \right) \\
\left( l - 1 - (s-j) - pN \right), \quad (73)
\]

where the first two terms in the above formula are those expected from Eq. (61), while the third term is precisely the correction to the formula Eq. (61) for \( t \) congruent to \( N \). Therefore, there are three contributions to the sum given in Eq. (72), the first of which reads

\[
\sum_{n=1}^{N-1} \cos^{2l-2-2(s-j)}(n\pi/N) = \frac{-2}{2^{l-2-2(s-j)}} \sum_{t=1}^{l-1-s} \left( 2l - 2 - 2(s-j) \right) \\
\sum_{t=1}^{l-1-s} \left( 2l - 2 - (s-j) - t \right)
\]

\[
+ \frac{N-1}{2^{l-2-2(s-j)}} \left( 2l - 2 - 2(s-j) \right) \\
\left( l - 1 - (s-j) \right)
\]

\[
+ \frac{2N}{2^{l-2-2(s-j)}} \sum_{p=1}^{l-1-(s-j)/N} \left( 2l - 2 - 2(s-j) \right) \\
\left( l - 1 - (s-j) - pN \right), \quad (73)
\]

where the first two terms in the above formula are those expected from Eq. (61), while the third term is precisely the correction to the formula Eq. (61) for \( t \) congruent to \( N \). Therefore, there are three contributions to the sum given in Eq. (72), the first of which reads

\[
F_1'(N, 2l - 1, 2) = -2 \sum_{s,k \geq 0}^{l-1} (-1)^{s+k} \left( 2l - 2 - s \right) \left( 2l - 2 - k \right) \\
2^{l-2-2k} \sum_{j=0}^{2l-2-2k} (-1)^{j} \frac{1}{2^j} \left( 2l - 2 - 2k \right) \\
\left( l - 1 - (s-j) - t \right)
\]

\[
= -2 \text{res}_{w=0} U_{2l-2} \left( 1 + w \right) \\
U_{2l-2} \left( 1 + w^2 \right) \frac{1}{1-w}
\]

\[
= -2 \text{res}_{w=0} \left( \frac{1}{w^{l-3}} - \frac{1}{w^{l-2}} \right) \frac{1}{(1-w)w} \\
= -\frac{1}{2} (3l-2)(3l-3) + \frac{1}{2} (l-1)(l-2) - l + \frac{1}{2} (1 - (-1)^l) \quad (74)
\]

while the second contribution is

\[
F_1''(N, 2l - 1, 2) = (N-1) \sum_{s,k \geq 0}^{l-1} (-1)^{s+k} \left( 2l - 2 - s \right) \left( 2l - 2 - k \right) \\
2^{l-2-2k} \sum_{j=0}^{2l-2-2k} (-1)^{j} \frac{1}{2^j} \left( 2l - 2 - 2k \right) \\
\left( l - 1 - (s-j) \right)
\]

\[
= (N-1) \text{res}_{w=0} U_{2l-2} \left( 1 + w \right) \\
U_{2l-2} \left( 1 + w^2 \right) \frac{1}{w}
\]

\[
= (N-1) \text{res}_{w=0} \left( \frac{1}{w^{l-3}} - \frac{1}{w^{l-1}} \right) \frac{1}{1-w^2} \frac{1}{1-w} \\
= \frac{N-1}{2} \left( 2l - 1 - (-1)^l \right). \quad (75)
\]
To obtain the last contribution we write the sum over \( p \), in Eq. (73), as

\[
\sum_{p=1}^{[l-1-(s-j)/N]} \left( \frac{2l - 2 - 2(s-j)}{l - 1 - (s-j) - pN} \right) = \text{res}_{w=0} \left( \frac{(1 + w)^{2l - 2 - 2(s-j)}w^N}{w^{l-(s-j)}(1 - w^N)} \right),
\]

and using the fact that \( l \leq N - 1 \), then, the third contribution may be computed to give

\[
F''_1(N, 2l - 1, 2) = 2N \sum_{s,k \geq 0}^{2l-2k} (-1)^{s+k} \left( \frac{2l - 2 - s}{s} \right) \left( \frac{2l - 2 - k}{k} \right) 2^{2l-2-2k} \times \sum_{j=0}^{2l-2k} (-1)^j \frac{1}{2j} \left( \frac{2l - 2 - 2k}{k} \right) \text{res}_{w=0} \left( \frac{(1 + w)^{2l - 2 - 2(s-j)}w^N}{w^{l-(s-j)}(1 - w^N)} \right)
\]

\[
= 2N \text{res}_{w=0} \left( \frac{1}{1 - w^N}U_{2l-2} \left( \frac{1 + w}{2\sqrt{w}} \right) U_{2l-2} \left( \frac{1 + w^2}{2w} \right) w^{N-1} \right)
\]

\[
= 2N \text{res}_{w=0} \left( \frac{1}{1 - w^N}w^{3l-2-N} \left( \frac{1}{1 - w} \right) \frac{1}{(1 - w^2)} \right)
\]

\[
= N \left( 3l - 2 - N + \frac{1}{2} (1 - (-1)^{l-N}) \right). \quad (76)
\]

Note that this will contribute only for \( 3l - 2 > N \), and as a result the formula for the sum \( F_1(N, 2l - 1, 2) \) is

\[
F_1(N, 2l - 1, 2) = \sum_{n=1}^{N-1} \frac{\sin(n(2l - 1)\pi/N)}{\sin(n\pi/N)} \frac{\sin(2n(2l - 1)\pi/N)}{\sin(2n\pi/N)}
\]

\[
= \frac{1}{2} (3l - 2)(3l - 3) + \frac{1}{2} (l - 1)(l - 2) - l + \frac{1}{2} (1 - (-1)^l)
\]

\[
+ \frac{N - 1}{2} \left( 2l - 1 - (-1)^l \right) + N \left( 3l - 2 - N + \frac{1}{2} (1 - (-1)^{l-N}) \right). \quad (77)
\]

Having obtained a closed formula for the sum \( F_1(N, 2l - 1, 2) \), we now wish to prove the formula for the alternating sum \( F_2(N, l, 2) \). First, we note that for \( N \) odd, the sum is non-vanishing only for \( l \) is even. Therefore, the formula for \( F_2(N, l, 2) \) becomes

\[
F_2(N, 2l, 2) = \sum_{n=1}^{N-1} (-1)^n \frac{\sin((2l)n\pi/N)}{\sin(n\pi/N)} \frac{\sin((2l)2n\pi/N)}{\sin(2n\pi/N)}
\]

\[
= \sum_{s,k \geq 0}^{2l-2k} (-1)^{s+k} \left( \frac{2l - 1 - s}{s} \right) \left( \frac{2l - 1 - k}{k} \right) 2^{2(2l-1)-2(s+k)} \times \sum_{j=0}^{2l-2-2k} (-1)^j \frac{2j}{k} \sum_{n=1}^{N-1} (-1)^n \cos^{2l-1-2(s-j)}(n\pi/N). \quad (78)
\]
The sum over $n$ may carried out using Eq. (114), in [12] with the slight modification as explained before, then, it is not difficult to show
\[
\sum_{n=1}^{N-1} (-1)^n \cos^{2l-1-2(s-j)-1} (n\pi/N) = \frac{-2}{2^{2l-1-2(s-j)}} \sum_{t=0}^{l-s} \left( \frac{2l - 1 - 2(s - j)}{l - 1 - (s - j) - t} \right) + \frac{2N}{2^{2l-1-2(s-j)}} \sum_{p \geq 1} \left( \frac{2l - 1 - 2(s - j)}{l - 1 - (s - j) - \frac{(2p-1)N-1}{2}} \right),
\]

(79)

where the sum over $p$, may be written as
\[
\sum_{p \geq 1} \left( \frac{2l - 1 - 2(s - j)}{l - 1 - (s - j) - \frac{(2p-1)N-1}{2}} \right) = \text{res}_{w=0} \left( \frac{(1 + w)^{2l-1-2(s-j)} w^{N/2}}{w^{l-(s-j)+1/2} (1 - w^N)} \right).
\]

By using Eq. (78), computations show that the closed formula for the sum $F_2(N, 2l, 2)$ is
\[
F_2(N, 2l, 2) = \sum_{n=1}^{N-1} (-1)^n \sin((2l)n\pi/N) \sin((2l)2n\pi/N) \sin(n\pi/N) \sin(2n\pi/N)
\]
\[
= -2 \text{res}_{w=0} \left( U_{2l-2} \left( \frac{1 + w}{2\sqrt{w}} \right) U_{2l-2} \left( \frac{1 + w^2}{2w} \right) \frac{1}{\sqrt{w}(1 - w)} \right)
+ 2N \text{res}_{w=0} \frac{1}{1 - w^N} \left( U_{2l-1} \left( \frac{1 + w}{2\sqrt{w}} \right) U_{2l-1} \left( \frac{1 + w^2}{2w} \right) \frac{w^{N/2}}{w} \right)
\]
\[
= -2 \text{res}_{w=0} \left( \frac{1}{w^{3l-1}} - \frac{1}{w^{l-1}} \right) \frac{1}{(1 - w)^2} \frac{1}{1 - w^2}
+ 2N \text{res}_{w=0} \frac{1}{1 - w^N} \frac{1}{w^{3l-(N+1)/2}} \frac{1}{(1 - w)^2} \frac{1}{1 - w^2}
\]
\[
= -\frac{1}{2} 3l(3l - 1) + \frac{1}{2} l(l - 1) - l + N \left( 3l - \frac{(N + 1)}{2} \right) + \frac{1}{2} (1 - (-1)^{l-(N+1)/2} \right),
\]

(80)

where the last term whose coefficient is $N$, contributes only for $3l > \frac{(N+1)}{2}$. Let us now, check that the formulas $F_1(N, 2l-1, 2)$, $F_2(N, 2l, 2)$ are consistent with symmetry discussed earlier, that is, $F_1(N, N - l, 2) = F_2(N, l, 2)$, this in turns implies that the correctness of the formulas. To do so, we will give some explicit examples, from the expression of $F_1(N, 2l - 1, 2)$ given in Eq. (77), it is clear that the sum should be $N - 1$, for $l = 1$ and to check this, one has to take into account that when substituting $l = 1$ in the formula, the last term of Eq. (77) does not contribute. From the symmetry that relates the two sums, we should have $F_2(N, N - 1, 2) = N - 1$. Indeed, this is the case, we simply let $l = \frac{N-1}{2}$ into Eq. (80), this time, however, the last term of this equation does contribute. An explicit computation shows that $F_2(N, 2, 2) = -4$, for $N > 3$, and $F_2(N, 2, 2) = 2$, for $N = 3$, it is interesting to note that these two cases for $l = 1$ are contained in the last term of Eq. (79),
since for $N > 3$, the last term is equal to 0, and hence $F_2(N, 2, 2) = -4$, while for $N = 3$, the last term is equal to 6, that is, our formula gives the right answer. Using the symmetry, we obtain $F_1(N, N - 2, 2) = -4$, this can be easily checked using our formula given by Eq. (77), and $l = \frac{N-1}{2}$.

7 Conclusion

To conclude, in this paper we used our method in [1], to give alternative derivations to closed formulas for trigonometrical sums that appear in one-dimensional lattice, and in the proof of the conjecture of F. R Scott on Permanent of the Cauchy matrix. A new derivation of certain trigonometrical sum of the perturbative chiral Potts model is given as well as new recursion formulas of certain trigonometrical sums [7]. By using these recursion formulas, then, one is able deduce the Verlinde dimension formulas for the untwisted (twisted) space of conformal blocks of $SU(2)$ $(SO(3))$WZW. In this paper, we reported closed-form formulas for the corner-to-corner resistance and the Kirchhoff index of the first non-trivial two-dimensional resistor network, $2 \times N$. We have also, considered other class of trigonometrical sums, some of which appear in number theory. Here, we followed similar formalism as in [1], as a consequence the non-trivial circulant electrical networks (the cycle and complete graphs are not included) are related to non-trivial trigonometrical sums in number theory. For example in [1], we had to introduce certain numbers that we called the Bejaia and Pisa numbers with well known properties so that the trigonometrical sums that arise in the computation of the two-point resistance are written in terms of these numbers nicely. By using the well known connection between the electrical networks and the random walks [23], one may hope to give interpretations to some of the trigonometrical sums in number theory other than those associated with the two-point resistance of a given electrical network, since the latter provides an alternative way to compute the basic quantity relevant to random walks known as the first passage time, the expected time to hit a target node for the first time for a walker starting from a source node [24].

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