The unfair consequences of equal opportunities: comparing exchange models of wealth distribution

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Simple agent based exchange models are a commonplace in the study of wealth distribution of artificial societies. Generally, each agent is characterized by its wealth and by a risk-aversion factor, and random exchanges between agents allow for a redistribution of the wealth. However, the detailed influence of the amount of capital exchanged has not been fully analyzed yet. Here we present a comparison of two exchange rules and also a systematic study of the time evolution of the wealth distribution, its functional dependence, the Gini coefficient and time correlation functions. In many cases a stable state is attained, but, interesting, some particular cases are found in which a very slow dynamics develops. Finally, we observe that the time evolution and the final wealth distribution are strongly dependent on the exchange rules in a nontrivial way.

PACS numbers:

I. INTRODUCTION

Empirical studies of the distribution of income of workers, companies and countries were first presented, a little more than a century ago, by Italian economist Vilfredo Pareto. He asserted that in different countries and times the distribution of income and wealth follows a power law behaviour, i.e. the cumulative probability $P(w)$ of agents whose income is at least $w$ is given by $P(w) \propto w^{-\alpha}$. Today, this power law distribution is known as Pareto distribution, and the exponent $\alpha$ is named Pareto index. However, recent data indicates that, even though Pareto’s distribution provides a good fit to the distribution of high range of income, it does not agree with observed data over the middle and low range of income. For instance, data from Japan, Italy, India, the United States of America and the United Kingdom are fitted by a log-normal or Gibbs distribution with a maximum in middle range plus a power law for the high income strata. The existence of these two regimes may be justified in a qualitative way by stating that in the low and middle income class the process of accumulation of wealth is additive, causing a Gaussian-like distribution, while in the high income class the wealth grows in a multiplicative way, generating the power law tails.

Different models of capital exchange among economic agents have been proposed trying to explain these empirical data. Most of these models consider an ensemble of interacting economic agents that exchange a fixed or random amount of a quantity called “wealth”. The wealth represents the welfare of the agents. The exact choice of this quantity is not straightforward. For instance, in the model of Drăgulescu and Yakovenko the wealth is associated with the amount of money a person has available to exchange. Within this model the amount of money corresponds to a kind of economic “energy” that may be exchanged by the agents in a random way and the resulting wealth distribution is – unsurprisingly – a Gibbs exponential distribution. An exponential distribution but as a function of a poverty line with finite wealth is also obtained, describing a way to diminish inequality in real societies.

Aiming to obtain distributions with power law tails, several methods have been proposed. Keeping the constraint of wealth conservation a detailed studied proposition is that each agent saves a fraction –constant or random– of their resources. Numerical results, as well as recent analytical calculations, indicate that one possible result of this model is condensation, i.e. concentration of all available wealth in just one or a few agents. To overcome this situation, different rules of interaction have been applied, for example increasing the probability of favoring the poorer agent in a transaction. However, to our knowledge, there are few detailed studies comparing the effect of these two parameters, the risk-aversion parameter and the probability of favoring the poorer agent. Besides, most of the previous works do not consider the time evolution of the system, moreover many of them do not guarantee that a steady state was attained indeed.

We present here a systematic study of an agent based model where exchanges are made by pairs of agents chosen at random, so it is model with no underlying lattice. Each agent, $i$, is characterized by a wealth, $w_i$ and a risk-aversion factor $\beta_i$, while in the exchange there is a probability of favoring the poorer partner given by

$$p = \frac{1}{2} + f \times \frac{|w_i(t) - w_j(t)|}{w_i(t) + w_j(t)},$$

where $f$ is a factor going from 0 (equal probability for both agents) to 1/2 (highest probability of favoring the poorer
agent). Thus, in each interaction the poorer agent has probability $p$ of earning a quantity $dw$, whereas the richer one has probability $1-p$. We focus on the choice of the quantity $dw$ transferred from the loser to the winner. In most of the previous work a kind of fair lottery is used: the amount of money exchanged corresponds to the minimum stake among the partners: $dw = \min[(1 - \beta_i)w_i(t); (1 - \beta_j)w_j(t)]$, so it is the same amount for both agents. However this equal opportunity rule produces an evil after-effect: in the $f = 0$ case, “condensation” occurs: all available wealth goes to one (or very few) agent, i.e. the Gini coefficient converges to 1. Also, for $f \neq 0$, even if the poorer partner has bigger chances of winning, its gains are as negligible as its own capital, so chances of improving are very low. For this reason we also investigate an alternative rule, where $dw$ is just the amount risked by the loser $-(1-\beta_j)w_j(t)$ — being $j$ the loser agent. We call this rule loser rule. Actually, variations of this loser rule have been used in some of the papers quoted above, but there is no a good reason why a rich agent will risk more than its poorer partner. Possible examples are marriage followed by divorce, the parties do combine their holdings and later divide them, or, in the corporate world, mergers followed by spin-offs.

In what follows we compare the results between the two rules in terms of the following quantities that we define thereafter: wealth distribution $H(w)$, Gini index vs. time, and wealth temporal correlation function. The wealth distribution is probably the most important quantity for the global description of an economic system. $H(w)$ vs. $dw$ gives the fraction of the population that have wealth between $w$ and $w + dw$. However, this distribution is obtained at a given time, both in real situations as well as in simulations, and in the case of simulations it is important to know whether the results are stable or not. With this purpose we measure the Gini coefficient as a function of time. It represents a practical way to verify the time dependence of the economic parameters. Finally, we also present the wealth temporal autocorrelation function. For one side this is another possible measure of time dependence in a system. Besides, it is more sensitive because it depends on two times, so aging properties, if there is any, could be grasped.

II. WEALTH DISTRIBUTIONS

All the simulations have been performed for a system of 1000 agents and the results have been averaged over 1000 samples. Wealth distributions are evaluated at the final stage, while at the initial condition, both wealth ($w_i$) and risk-aversion factor ($\beta_i$) are uniformly distributed in the $[0,1]$ interval. Final stage means not further — or very small — changes are observed as time goes by. Support for that assumption will be presented in the next section. At first both rules seem to be qualitatively similar (one of them, the fair rule, was already discussed in ref. 17). The main features of the wealth distributions emerging of the present model are: a) an almost uniform region for very low incomes ($w < 3 \times 10^{-1}$), being more uniform as $f$ is bigger, b) a power law region extended for less than three decades with an $f$-dependent exponent. The distinctive feature is in the intermediate income region where appears less populated, at low $f$ values, in the fair rule, when compared to the loser rule. The straight line depicted in both graphs, with slope equal to two, is a guide to the eye, and it follows approximately the slope of the distributions in the intermediate income region, with $f = 0.5$, for both rules. We would like to emphasize the $f = 0$ case, where both distributions exhibit a power law, but the loser case has a bigger exponent ($-2.17$ compared to $-1.93$ for the fair case) so indicating a less unequal distribution.

![FIG. 1: Wealth distribution for several values of the parameter $f$, with fair (left) and loser (right) rule. The straight line with slope 2 corresponds to the slope of the $f = 0.5$ case, in the intermediate income region, for both rules.](image-url)
III. GINI COEFFICIENT

The Gini coefficient is a measure of the inequality of a distribution and is often used to measure income inequality by economists and statistical organizations because it gives a raw picture of the inequalities with a single number between 0 and 1. It is defined as the ratio of the area enclosed by the Lorenz curve of the distribution (or cumulative distribution function) and the curve of the uniform distribution, to the area under the uniform distribution. In a operational way we define the Gini coefficient as:

$$G = \frac{1}{2} \sum_{i,j} |w_i - w_j|$$

and it is evident that it varies between 0, which corresponds to perfect equality (i.e. everyone has the same income), and 1, that corresponds to perfect inequality (i.e. one person has all the income, while everyone else has zero income). Here we will use the Gini coefficient to measure the degree of inequality, but also to determine the stability of the wealth distribution. With this objective in mind, we show in Fig. 2 the time evolution of the Gini index, where we can observe that the systems converge rather fast to a fixed value, indicating that, after a transient, the systems arrive to an almost stable wealth distribution. A further analysis (measuring the slope) reveals that in some cases stabilization is not complete, and that a slow dynamics is still present. This point is confirmed by the analysis of the time correlation function presented in the next section. The effect of $f$ on the Gini coefficient is the expected one, the bigger the value of $f$, the bigger the probability of favoring the poorer agent in each transaction, and the lower the Gini index. However, the loser rule produces, for low values of $f$, a considerable lower Gini index than the fair rule, suggesting that an unfair lottery produces a more equal society than a fair one. Particularly, for $f = 0$ the fair rule delivers a Gini coefficient equal to 1 (full condensation of the economy), while the loser rule produces a finite Gini coefficient near 0.8. This is a result that can induce some second thoughts about the concept of equal opportunity.

IV. WEALTH TIME AUTOCORRELATION FUNCTION

We define a time autocorrelation function for the wealth as follows:

$$\text{corr}(\tau, t) = \frac{\sum_{i=1}^{N} w_i(\tau) w_i(\tau + t)}{\sum_{i=1}^{N} w_i(\tau) w_i(\tau)}$$

The behavior depicted by the time correlation functions is very rich, in a way that is not perceive neither by the distribution or the Gini index. The behaviour in the rules are very disparate, but in both cases clearly signs of aging...
are observed (we checked this varying the initial time \( \tau \)). While in the *fair* case the time correlation decreases, in the *loser* case, after an initial and short decrease, the correlation function increases to values bigger than one. We remark that in the *fair* case and for \( f = 0 \) the time correlation is stable and equal to 1, because the system condensates very fast and the changes in the wealth of the agents are very small. But when increasing the value of \( f \) the time correlation function decreases, suggesting a higher degree of social mobility as time evolves. On the other hand, for the *loser* rule the time correlation exhibits slow dynamics (glassy behaviour, also observed for the *fair* rule in the case \( f = 0.1 \)); that means that the system requires a large time-period to attain an almost stable situation (as that represented in Fig.1), and this characteristic time increases when increasing the size of the system, as expected. Another interesting point is that, with the exception of \( f = 0.5 \), the value of the autocorrelation function is bigger than one, indicating a kind of bias in the evolution: on average rich agents are becoming richer and poor agents poorer. This result, combined with the conservation of the total wealth, explains the fact that the correlation function increases as time goes by. A full discussion of these results, considering also different values of \( \tau \) will be presented elsewhere.

V. CONCLUSIONS

Our results clearly show that in many cases the system does not arrive to a steady configuration (a remarkable exception is the case \( f = 0.5 \), when the probability has its strongest bias to the poorer partner). This point is not decisive for the description of economic systems as usually they are not in a steady state neither they are conservative, but the implications could be interesting for possible physical systems that behaves in a similar glassy way.

Probably the most relevant result is the fact that the *loser* rule appears to produce a less unequal wealth distributions than the one we call *fair* rule. That is valid for values of the \( f < 0.3 \), which represent situations more close to real economic systems. Thus, the *loser* rule behaves somehow like an *unfair* lottery, in the sense that the richer agent risks more – in average – than its poorer partner, but has less chances to win. As a consequence, this bias attenuates the inequalities induced by low \( f \) values. It seems to us that this result is an indication that the best way to diminish inequality does not pass only through equal opportunity (fair rule) but through some kind of positive action increasing
the odds of poorer strata of the society.