EXTENDING UPPER CLUSTER ALGEBRAS

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Abstract. Let $S$ be an upper cluster algebra, which is a subalgebra of $R$. Suppose that there is some cluster variable $x_e$ such that $R_{x_e} = S[x_e^{\pm 1}]$. We try to understand under which conditions $R$ is an upper cluster algebra, and how the quiver of $R$ relates to that of $S$. Moreover, if the restriction of $(\Delta, W)$ to some subquiver is a cluster model, we give a sufficient condition for $(\Delta, W)$ itself being a cluster model. As an application, we show that the semi-invariant ring of any complete $m$-tuple flags is an upper cluster algebra whose quiver is explicitly given. Moreover, the quiver with its rigid potential is a polyhedral cluster model.

Introduction

The notion of cluster algebra of Fomin-Zelevinsky turned out be ubiquitous in algebraic Lie theory and invariant theory. It was realized later that in some contexts (eg., [13, 11, 12]) the upper cluster algebra [1] is a more useful notion than the cluster algebra. In this paper we try to answer the questions converse to the ones treated in [10].

One of the problems treated in [10] is the following. Suppose that $R$ is an upper cluster algebra $\mathcal{C}(\Delta, x)$ and $e$ is a vertex in the ice quiver $\Delta$. Let $x_e$ be a cluster or a coefficient variable in $x$ corresponding to $e$. Under certain condition on $x_e$, we constructed a subalgebra $S \subset R$ such that the localization $R_{x_e}$ is equal to $S[x_e^{\pm 1}]$ and $S$ is the upper cluster algebra $\mathcal{C}(\Delta_e, x_e)$. Here, $\Delta_e$ is obtained from $\Delta$ by deleting $e$. In terminology of [10] the seed $(\Delta_e, x_e)$ is projected from $(\Delta, x)$ through $e$. In this case, we also say $R$ is a cluster extension of $S$ via $x_e$. Moreover, if $(\Delta, W)$ is a cluster model for some potential $W$, then so is $(\Delta_e, W_e)$ where $W_e$ is the restriction of $W$ to $\Delta_e$. Recall that we say that $(\Delta, W)$ is a cluster model if the generic cluster character maps the set of $\mu$-supported $g$-vectors onto a basis of $\mathcal{C}(\Delta)$. We will explain these notions in Section [1].

It is natural to ask the following converse. We keep assuming that $S$ is an upper cluster algebra $\mathcal{C}(\Delta, x)$ and $S$ is a subalgebra of $R$. Suppose that there is some $x_e \in R$ such that $R_{x_e} = S[x_e^{\pm 1}]$. We ask under which conditions $R$ is an upper cluster algebra, and what is its seed $(\Delta, x)$. Moreover, if $(\Delta, W)$ is a cluster model, when can we make some IQP (ice quiver with potential) $(\Delta, W)$ a cluster model of $\mathcal{C}(\Delta)$? These two converses are clearly harder than the original questions.

For the second question about the cluster model, we obtain a quite satisfactory sufficient condition. This solution is partially motivated by [27] [18]. Recall that a (frozen or mutable) vertex $e$ can be optimized in $\Delta$ if there is a sequence of mutations such that $e$ is a sink or a source of $\Delta$. This notion is related to the
covering pair technique introduced in [27]. We say that it can be optimized in an IQP \((\Delta, W)\) if in addition such a sequence is admissible. In practice, we usually need to deal with a set of vertices \(e\) rather than a single one, but the generalization is straightforward.

**Theorem 0.1.** Let \(W\) be any potential on \(\Delta\) such that its restriction on \(\Delta\) is \(W\). Suppose that \(B(\Delta)\) has full rank, and each vertex in \(e\) can be optimized in \((\Delta, W)\). If \((\Delta, W)\) is a cluster model, then so is \((\Delta, W)\).

Although this is not an “if and only if” solution, it already covers many interesting cases, including the ice quivers in Theorem 0.2 and in [12].

For the first question, there usually exists a natural candidate for the seed \((\Delta, x)\) such that \(\Delta = \Delta_E, x_e = x(e), and R \supseteq \mathcal{C}(\Delta, x)\). However, we are unable to get the equality without help from another subalgebra. To be more precise, what we need is another seed \((\Delta', x')\) mutation equivalent to \((\Delta, x)\) and a vertex \(e'\) such that \(\Delta' = \Delta_E', and R_{e'} = S'[x_{e'}^{\pm 1}]\) for another subalgebra \(S' = \mathcal{C}(\Delta', x')\). A simple observation is that if the common zero locus of \(x_e\) and \(x_{e'}\) has codimension 2 in Spec \(R\), then basic algebraic geometry shows that \(R = \mathcal{C}(\Delta, x)\). We will illustrate this technique in two examples.

Let \(S^m_{\beta}\) be the \(m\)-tuple flag quiver of length \(l\), and \(\beta_l\) be its standard dimension vector (see Example 4.3). One of the main results in [9] is that the semi-invariant ring \(SI_{\beta_l}(S^m_{\beta})\) is the upper cluster algebra \(\mathcal{C}(\Delta_l, s_l)\), where \(\Delta_l\) is the ice hive quiver of size \(l\) and \(s_l\) is a set of Schofield’s semi-invariants. The first example of usefulness of our technique is to give another proof of this result based on results in [9, 10] but independent of Knutson-Tao’s hive model [24]. The new proof is simpler and more conceptual. Motivated by several constructions of Fock-Goncharov in [13], our second example will generalize this result to arbitrary \(m \geq 3\). Let \(D_m\) be the disk with \(m\) marked points on the boundary, and \(\mathbb{T}\) be any ideal triangulation of \(D_m\). We can glue seeds \((\Delta_l, s_l)\) together according to \(\mathbb{T}\), and obtain a new seed denoted by \((\mathcal{T}(\mathbb{T}), s_l(\mathbb{T}))\). We will explain the recipe for gluing in Section 3.

**Theorem 0.2.** The semi-invariant ring \(SI_{\beta_l}(S^m_{\beta})\) is equal to the upper cluster algebra \(\mathcal{C}(\mathcal{T}(\mathbb{T}), s_l(\mathbb{T}))\) for any triangulation \(\mathbb{T}\) of \(D_m\). Moreover, \((\mathcal{T}(\mathbb{T}), W_l(\mathbb{T}))\) is a polyhedral cluster model.

In [13] Fock-Goncharov considered a generic part of the moduli stack \([A^m/G]\) as a cluster variety for \(G = SL_l\) and \(A\) the base affine space \(G/U\). The ring of regular functions on \([A^m/G]\) is the same as \(SI_{\beta_l}(S^m_{\beta})\) (a priori just as multigraded vector spaces). However, we believe that the equality and cluster model established in the above theorem are new.

In the end, we want to mention a “global” solution in [18] to a geometric version of the second question. Let \(A\) be the cluster variety such that \(A\) with principal coefficients has enough global monomial [18, Definition 0.11], then under certain convexity condition, we have that the tropical points in the corresponding Fock-Goncharov’s dual cluster variety parametrize a basis of regular functions on \(A\). Note that the algebra of regular functions on \(A\) is the associated upper cluster algebra. Moreover, let \(\overline{A}\) be a partial compactification of \(A\) coming from a set of frozen variables \(e\). If each vertex in \(e\) can be optimized, then the above holds for \(\overline{A}\) as well. In [18], authors give some sufficient conditions for having enough of global monomials (Proposition 0.14). It would be interesting to compare our
combinatorial condition to theirs, in particular a very general one – Proposition 0.14.(4).

Outline of the paper. In Section 1 we recall the graded upper cluster algebra following [1, 15, 9], and its ice quiver with potential model following [5, 6, 9]. In Section 2 we prove our first main result – Theorem 2.14. We first treat the case when $e$ is frozen in Section 2.1, and then the case when $e$ is mutable in Section 2.2. The approaches to the two cases are quite different. In Section 3 we define how to glue ice hive quivers according to a triangulation of a surface mostly following [13]. Proposition 3.6 is the last statement of Theorem 0.2. In Section 4, we first recall the theory of semi-invariant rings of quiver representations, then reprove the main result in [9]. Finally, we prove our second main result – Theorem 4.16.

1. Graded Upper Cluster Algebras and their IQP model

1.1. Graded Upper Cluster Algebras. In this paper, we will not consider the upper cluster algebras in full generality as introduced in [1]. We are only concerned with those which are (i) of geometric type, (ii) over a field $k$, and (iii) has skew-symmetric exchange matrices. To define such an upper cluster algebra one needs to specify a seed $p \Delta, x \in F \subset k$. Here, $\Delta$ is an ice quiver with no loops or oriented 2-cycles and the extended cluster $x = \{x_1, x_2, \ldots, x_q\}$ is a collection of algebraically independent (over $k$) elements of $F$ attached to each vertex of $\Delta$.

An ice quiver $\Delta = (\Delta_0, \Delta_1)$ is a quiver, where some vertices in $\Delta_0$ are designated as mutable while the rest are frozen. We denote the set of mutable (resp. frozen) vertices of $\Delta$ by $\Delta_m$ (resp. $\Delta_f$). We usually label the vertices of the quiver in such a way that the first $p$ vertices are mutable. If we require no arrows between frozen vertices, then such a quiver is uniquely determined by its $B$-matrix $B(\Delta)$. It is a $p \times q$ matrix given by

$$b_{u,v} = |\text{arrows } u \rightarrow v| - |\text{arrows } v \rightarrow u|.$$

The elements of $x$ associated with the mutable vertices are called cluster variables; they form a cluster. The elements associated with the frozen vertices are called frozen variables, or coefficient variables.

Definition 1.1. Let $u$ be a mutable vertex of $\Delta$. The quiver mutation $\mu_u$ transforms $\Delta$ into the new quiver $\Delta' = \mu_u(\Delta)$ via a sequence of three steps.

1. For each pair of arrows $v \rightarrow u \rightarrow w$, introduce a new arrow $v \rightarrow w$ (unless both $v$ and $w$ are frozen, in which case do nothing);
2. Reverse the direction of all arrows incident to $u$;
3. Remove all oriented 2-cycles.

Definition 1.2. A seed mutation $\mu_u$ at a (mutable) vertex $u$ transforms $(\Delta, x)$ into the seed $(\Delta', x') = \mu_u(\Delta, x)$ defined as follows. The new quiver is $\Delta' = \mu_u(\Delta)$. The new extended cluster is $x' = x \cup \{x'_u\}\backslash\{x_u\}$ where the new cluster variable $x'_u$ replacing $x_u$ is determined by the exchange relation

$$x_u x'_u = \prod_{v \rightarrow u} x_v + \prod_{u \rightarrow w} x_w.$$

We note that the mutated seed $(\Delta', x')$ contains the same coefficient variables as the original seed $(\Delta, x)$. It is easy to check that one can recover $(\Delta, x)$ from $(\Delta', x')$ by...
performing again a seed mutation at \( u \). Two seeds \((\Delta, x)\) and \((\Delta^\dagger, x^\dagger)\) that can be obtained from each other by a sequence of mutations are called \textit{mutation-equivalent}, denoted by \((\Delta, x) \sim (\Delta^\dagger, x^\dagger)\). If \( \Delta \) and \( \Delta^\dagger \) are clear from the context, we may just write \( x \sim x^\dagger \).

Let \( \mathcal{L}(x) \) be the Laurent polynomial algebra in \( x \) over the base field \( k \). If \( \Delta \) is an ice quiver, we denote by \( \mathcal{L}_\Delta(x) \) the Laurent polynomial in \( x \) which is polynomial in \( x(\Delta_0) \), that is, \( \mathcal{L}_\Delta(x) := k \left[ x(\Delta_0)^\pm 1, x(\Delta_0) \right] = k \left[ x_1^\pm 1, \ldots, x_p^\pm 1, x_{p+1}, \ldots, x_q \right] \).

**Definition 1.3** (Upper Cluster Algebra). The \textit{upper cluster algebra} (or UCA for short) with seed \((\Delta, x)\) is

\[
\mathcal{U}(\Delta, x) := \bigcap_{(\Delta^\dagger, x^\dagger) \sim (\Delta, x)} \mathcal{L}(x^\dagger).
\]

Note that our definition of UCA is slightly different from the original one in [1], where \( \mathcal{L}(x^\dagger) \) is replaced by \( \mathcal{L}(x^\dagger) \). The Laurent Phenomenon [16] says that an UCA contains all cluster and coefficient variables.

In general, there may be infinitely many seed mutations equivalent to \((\Delta, x)\). So the following theorem is very useful to test the membership in a UCA. Following [1], let \( x_u := \mu_u(x) \) be the cluster obtained from \( x \) by applying a single mutation at \( u \). We also set \( x_\circ := x \). We define the \textit{upper bound algebra}

\[
\mathcal{U}(\Delta, x) := \bigcap_{u \in \Delta, \nu \setminus \{0\}} \mathcal{L}(x_u).
\]

**Theorem 1.4** ([1] 21). Suppose that \( B(\Delta) \) has full rank, and \((\Delta, x) \sim (\Delta^\dagger, x^\dagger)\), then \( \mathcal{U}(\Delta, x) = \mathcal{U}(\Delta^\dagger, x^\dagger) \). In particular, \( \mathcal{U}(\Delta, x) = \mathcal{C}(\Delta, x) \).

This theorem is originally proved in [1], Corollary 1.9 for \( \mathcal{U}(\Delta, x) \) and \( \mathcal{C}(\Delta, x) \) with \( \mathcal{L}(x) \) replaced by \( \mathcal{L}(x) \). The version we have taken is proven in [21] Theorem 4.1].

Any UCA, being a subring of a field, is an integral domain (and under our conventions, a \( k \)-algebra). However, it may fail to be Noetherian [34]. Since normality is preserved under localization and intersection, any UCA is normal. The next lemma is useful to identify a UCA as a subalgebra of some given Noetherian normal domain.

Let \( R \) be a finitely generated \( k \)-algebra. We call two elements of \( R \) \textit{coprime} in codimension 1 if the locus of their common zeros has codimension \( \geq 2 \) in \( \text{Spec}(R) \).

**Definition 1.5**. We say that a seed \((\Delta, x)\) is \textit{CR1} in \( R \) if

1. \( x \subset R \) and each \( x'_u \in R \).
2. Each pair of cluster variables in \( x \) and each pair \((x_u, x'_u)\) are coprime in codimension 1 in \( R \).

**Lemma 1.6** ([15] Proposition 3.6]). Let \( R \) be a finitely generated \( k \)-algebra and a normal domain. If \( (\Delta, x) \) is a CR1 seed in \( R \), then \( R \supseteq \mathcal{C}(\Delta, x) \).

One implication of [26] Lemma 4.4.2 is the following lemma.

**Lemma 1.7.** Suppose that \( B(\Delta) \) has full rank and \( \mathcal{C}(\Delta, x) \) is Noetherian. Then \( (\Delta, x) \) is a CR1 seed in \( \mathcal{C}(\Delta, x) \).

**Remark 1.8.** The CR1 and Noetherian conditions are crucial to many proofs later. In view of the above lemma, the CR1 condition for a seed in a UCA seems weaker than the full rank condition of \( B \). However, the full rank condition is more useful because mutations preserve the rank of \( B \) [1] Lemma 3.2].
Let $\Delta$ be a frozen vertex of $\Delta$. Then for any $w \in \overline{C}(\Delta, x)$ such that $x_e \nmid w$ in $L_\Delta(x)$, $(w, x_e)$ is a regular sequence in $\overline{C}(\Delta, x)$.

**Proof.** We first show the following claim. Suppose that $z \in \overline{C}(\Delta, x)$ and $zw/w_e \in L_\Delta(x)$, then $z/x_e \in \overline{C}(\Delta, x)$. It suffices to show that $z/x_e$ is polynomial in $x_e$ when written as a Laurent polynomial in any $\mathbf{x}' \sim x$. This is equivalent to that $z$ has $x_e$ as a factor in $L_\Delta(x')$. Suppose that $\mathbf{x}' = \mathbf{u}(x)$. We will prove the claim by induction on the length of $\mathbf{u} = \mathbf{u}_\mu \cdots \mathbf{u}_2 \mathbf{u}_1$. Since $x_e \nmid w$ and $zw/w_e \in L_\Delta(x)$, $z$ has $x_e$ as a factor in $L_\Delta(x)$. Let $x_k = \mathbf{u}_1 \cdots \mathbf{u}_2 \mathbf{u}_1(x)$. Suppose that $z$ has $x_e$ as a factor in $L_\Delta(x_k)$. Now according to the exchange relation we substitute $x_k(u_{k+1})$ by $\rho/x_k+1(u_{k+1}) \in L_\Delta(x_k+1)$. Since $x_e$ is not a factor of $\rho/x_k+1(u_{k+1})$ in $L_\Delta(x_k+1)$, we conclude that $z$ also has $x_e$ as a factor in $L_\Delta(x_k+1)$.

To show $(w, x_e)$ is a regular sequence, we need to show that $x_e$ is not a zero-divisor in $A/(w)$. Suppose that $x_e y = wz$ for some $y, z \in A$ (i.e., $x_e y \in (w)$), then $y = wz/x_e \in A \subset L_\Delta(x)$. By the claim just proved, we have that $z/x_e \in A$ so that $y \in (w)$. Hence, $x_e$ is not a zero-divisor in $A/(w)$. \qed

Let $e$ be a subset of $\Delta$. We write $x(e)$ for the set $\{x_e \mid e \in e\}$. If $x(e)$ is contained in a ring $R$, then we write $R_{x(e)}$ for the localization of $R$ at the product $\prod_{e \in e} x_e$. Let $\Delta^e$ be the ice quiver obtained from $\Delta$ by freezing every vertex in $e$.

**Lemma 1.10** ([10], Lemma 2.4). Let $e$ be a subset of $\Delta_{\mu}$. Suppose that $(\Delta, x)$ is a CR1 seed in $\overline{C}(\Delta, x)$, which is Noetherian. Then $\overline{C}(\Delta, x)_{x(e)} = \overline{C}(\Delta^e, x)_{x(e)}$.

For a vector $g \in \mathbb{Z}^{\Delta_\mu}$, we write $x^g$ for the monomial $\prod_{e \in \Delta_\mu} x_e^{g(e)}$. For $u \in \Delta_\mu$, we set $y_u = x^{-h_u}$ where $h_u$ is the $u$-th row of the matrix $B(\Delta)$, and let $y = \{y_u \mid u \in \Delta_\mu\}$.

Suppose that an element $z \in \overline{C}(\Delta)$ can be written as

$$z = x^g F(y),$$

where $F$ is a rational polynomial with a constant term, and $g(z) \in \mathbb{Z}^{\Delta_\mu}$. If we assume that the matrix $B(\Delta)$ has full rank, then the elements in $\{y_u \mid u \in \Delta_\mu\}$ are algebraically independent so that the vector $g(z)$ is uniquely determined [17]. We call the vector $g(z)$ the (extended) $g$-vector of $z$. Definition implies at once that for two such elements $z_1, z_2$ we have that $g(z_1 z_2) = g(z_1) + g(z_2)$. So the set $G(\Delta)$ of all $g$-vectors in $\overline{C}(\Delta)$ forms a sub-semigroup of $\mathbb{Z}^{\Delta_\mu}$.

**Lemma 1.11** ([9], Lemma 5.5), cf. [29]. Assume that the matrix $B(\Delta)$ has full rank. Let $Z = \{z_1, z_2, \ldots, z_k\}$ be a subset of $\overline{C}(\Delta)$ with distinct well-defined $g$-vectors. Then $Z$ is linearly independent over $k$.

The proof of the above lemma uses an easy degree argument, which is useful for us later. The full rank condition on $B$ implies that we can assign a grading to $\{x_v \mid v \in \Delta_\mu\}$ such that each $y_u$ has negative total degree. Then the total degree of the monomial $x^g$ is maximal among the total degree of all monomials in $I\overline{C}(\Delta)$.

**Definition 1.12.** We say that $\overline{C}(\Delta)$ is $g$-indexed if it has a basis whose elements have well-defined distinct $g$-vectors in $G(\Delta)$.

**Definition 1.13** ([9]). A weight configuration $\sigma$ of $\mathbb{Z}^n$ on an ice quiver $\Delta$ is an assignment for each vertex $v$ of $\Delta$ a (weight) vector $\sigma(v) \in \mathbb{Z}^n$ such that for each
mutable vertex \( u \), we have that

\[
\sum_{v \rightarrow u} \sigma(v) = \sum_{u \rightarrow w} \sigma(w).
\]

The mutation \( \mu_u \) also transforms \( \sigma \) into a weight configuration \( \sigma' \) on the mutated quiver \( \mu_u(\Delta) \) defined as

\[
\sigma'(v) = \begin{cases} 
\sum_{w \rightarrow v} \sigma(w) - \sigma(u) & \text{if } v = u, \\
\sigma(v) & \text{if } v \neq u.
\end{cases}
\]

We usually write such a weight configuration as a pair \( (\Delta; \sigma) \) and its mutation as \( \mu_u(\Delta; \sigma) \). By slight abuse of notation, we can view \( \sigma \) as a matrix whose \( v \)-th row is the weight vector \( \sigma(v) \). In this matrix notation, the condition (1.2) is equivalent to \( \mathcal{B} \sigma \) being a zero matrix.

Given a weight configuration \( (\Delta; \sigma) \), we can assign a multidegree (or weight) to the UCA \( \mathcal{B}(\Delta, x) \) by setting \( \deg(x_v) = \sigma(v) \) for \( v \in \Delta_0 \). Then mutation preserves multihomogeneity. We say that this UCA is \( \sigma \)-graded, and denote it by \( \mathcal{B}(\Delta, x; \sigma) \). Note that the variables in \( y \) have zero degrees. So if \( z \) has a well-defined \( g \)-vector as in (1.1), then \( z \) is homogeneous of degree \( g \sigma \).

1.2. The Quiver with Potential Model. In [5] and [6], the mutation of quivers with potentials is invented to model the cluster algebras. Following [5], we define a potential \( W \) on an ice quiver \( \Delta \) as a (possibly infinite) linear combination of oriented cycles in \( \Delta \). More precisely, a potential is an element of the trace space \( \text{Tr}(k\Delta) := k\Delta/[k\Delta, k\Delta] \), where \( k\Delta \) is the completion of the path algebra \( k\Delta \) and \([k\Delta, k\Delta]\) is the closure of the commutator subspace of \( k\Delta \). The pair \( (\Delta, W) \) is an ice quiver with potential, or IQP for short. For each arrow \( a \in \Delta_1 \), the cyclic derivative \( \partial_a \) on \( k\Delta \) is defined to be the linear extension of

\[
\partial_a(a_1 \cdots a_d) = \sum_{k=1}^d a^*(a_k) a_{k+1} \cdots a_d a_1 \cdots a_{k-1}.
\]

For each potential \( W \), its Jacobian ideal \( \partial W \) is the (closed two-sided) ideal in \( k\Delta \) generated by all \( \partial_a W \). The Jacobian algebra \( J(\Delta, W) \) is the quotient algebra \( k\Delta/\partial W \). If \( W \) is polynomial and \( J(\Delta, W) \) is finite-dimensional, then the completion is unnecessary to define \( J(\Delta, W) \). This is the situation assumed throughout the paper.

The key notion introduced in [5] is the mutation of quivers with potentials and their decorated representations. Since we do not need the mutation in an explicit way, we refer readers to the original text. Unlike the quiver mutation, the mutation of IQP is not always defined for any sequence of (mutable) vertices because 2-cycles may be created along the way. A sequence of vertices is call admissible for an IQP if its mutation along this sequence is defined. In this case the mutation of IQP in certain sense “lifts” the quiver mutation. If all sequences are admissible for \( (\Delta, W) \), then we call \( (\Delta, W) \) nondegenerate.

**Definition 1.14 ([5]).** A potential \( W \) is called rigid on a quiver \( \Delta \) if every potential on \( \Delta \) is cyclically equivalent to an element in the Jacobian ideal \( \partial W \). Such a QP \( (\Delta, W) \) is also called rigid.
It is known \cite[Proposition 8.1, Corollary 6.11]{9} that every rigid QP is 2-acyclic, and the rigidity is preserved under mutations. In particular, any rigid QP is nondegenerate.

**Definition 1.15.** A decorated representation of a Jacobian algebra $J := J(\Delta, W)$ is a pair $\mathcal{M} = (M, M^+)$, where $M$ is a finite-dimensional $J$-module and $M^+$ is a finite-dimensional $k^\Delta_0$-module.

Let $\mathcal{R}ep(J)$ be the set of decorated representations of $J(\Delta, W)$ up to isomorphism. Let $K^b(\text{proj-} J)$ be the homotopy category of bounded complexes of projective representations of $J$, and $K^2(\text{proj-} J)$ be the subcategory of 2-term complexes in fixed degrees (say $-1$ and $0$). There is a bijection between two additive categories $\mathcal{R}ep(J)$ and $K^2(\text{proj-} J)$ mapping any representation $M$ to its minimal presentation in $\mathcal{R}ep(J)$, and the simple representation $S^+_0$ of $k^\Delta_0$ to $P_u \rightarrow 0$. We use the notation $P(\beta)$ for $\bigoplus_{e \in \Delta_0} \beta(e)P_v$, where $\beta \in \mathbb{Z}_{\geq 0}^\Delta$ and $P_v$ is the indecomposable projective representation corresponding to the vertex $v$. The weight vector of a projective presentation $P(\beta_1) \rightarrow P(\beta_0)$ is equal to $\beta_1 - \beta_0$.

**Definition 1.16.** The $g$-vector $g(\mathcal{M})$ of a decorated representation $\mathcal{M}$ is the weight vector of its image in $K^2(\text{proj-} J)$.

**Definition 1.17** \cite[2]{2}. To any $g \in \mathbb{Z}_{\geq 0}^\Delta$ we associate the reduced presentation space $\text{PHom}_J(g) := \text{Hom}_J(P([g]_+), P([-g]_+))$, where $[g]_+$ is the vector satisfying $[g]_+(u) = \max(g(u), 0)$. We denote by $\text{Coker}(g)$ the cokernel of a general presentation in $\text{PHom}_J(g)$.

Reader should be aware that $\text{Coker}(g)$ is just a notation rather than a specific representation. If we write $M = \text{Coker}(g)$, this simply means that we take a presentation general enough (according to context) in $\text{PHom}_J(g)$, then let $M$ to be its cokernel.

**Definition 1.18** \cite[9]{9}. A representation is called $\mu$-supported if its supporting vertices are all mutable. A weight vector $g \in K_0(\text{proj-} J)$ is called $\mu$-supported if $\text{Coker}(g)$ is $\mu$-supported. Let $G(\Delta, W)$ be the set of all $\mu$-supported vectors in $K_0(\text{proj-} J)$.

**Definition 1.19** \cite[29]{29}. We define the generic character $C_W : G(\Delta, W) \rightarrow \mathbb{Z}(x)$ by
\begin{equation}
C_W(g) = x^g \sum_e \chi(\text{Gr}^e(\text{Coker}(g)))y^e,
\end{equation}
where $\text{Gr}^e(M)$ is the variety parameterizing $e$-dimensional quotient representations of $M$, and $\chi(-)$ denotes the topological Euler-characteristic.

It is known \cite[Lemma 5.3]{9} that $C_W(g)$ is an element in $\overline{\mathcal{C}}(\Delta)$. Note that $C_W(g)$ has a well-defined $g$-vector $\mathbf{g}$, so we have that $G(\Delta, W) \subseteq G(\Delta)$.

**Theorem 1.20** \cite[Corollary 5.14]{9}, cf. \cite[29]{29} Theorem 1.1]. Suppose that IQP $(\Delta, W)$ is nondegenerate and $B(\Delta)$ has full rank. The generic character $C_W$ maps $G(\Delta, W)$ (bijectively) to a set of linearly independent elements in $\overline{\mathcal{C}}(\Delta)$ containing all cluster monomials.

**Definition 1.21.** We say that an IQP $(\Delta, W)$ models an algebra $\mathcal{A}$ if the generic cluster character maps $G(\Delta, W)$ onto a basis of $\mathcal{A}$. If $\mathcal{A}$ is the upper cluster algebra $\overline{\mathcal{C}}(\Delta)$, then we simply say that $(\Delta, W)$ is a cluster model. If in addition $G(\Delta, W)$ is given by lattice points in some polyhedron, then we say that the model is polyhedral.
This definition itself does not require the IQP to be nondegenerate. [9, Proposition 5.15] implies that being a (polyhedral) cluster model is mutation-invariant.

2. Extensions

2.1. Extension Through Coefficient Variables. Let $\Delta$ be an ice subquiver of $\Delta$ such that $\Delta_0 = \Delta_0 \cup e$. Unless otherwise stated, we assume in this subsection that $e$ is a set of frozen vertices. Suppose that $(\Delta, x)$ and $(\Delta, y)$ are two seeds satisfying that $x(u)$ is equal to $x(u)$ up to a monomial factor in $x(e)$ for each $u \in \Delta_0$, that is,

$$x(u) = x(u)x(e)^{\Theta(u)} \text{ for some } \Theta(u) \in \mathbb{Z}^e.$$  

Here, by convention we set $x(e) = 1$ for each $e \in e$ so that $\Theta(e) = e$.

Definition 2.1. We say that $(\Delta, x)$ is a balanced extension of $(\Delta, y)$ via $x(e)$ if $\Theta : \Delta_0 \rightarrow \mathbb{Z}^e$ is a weight configuration. This is equivalent to say that

$$x(u) = x(u)x(e)^{\Theta(u)} \text{ for some } \Theta(u) \in \mathbb{Z}^e.$$  

Lemma 2.2. For any balanced extension $(\Delta, x)$ of $(\Delta, y)$ we have that $x^g = x^g e^{\Theta}$ and $y^d = y^d$. 

Proof. We have seen from (2.3) that each $x'(u)$ differs from $x'(u)$ by a monomial factor in $x(e)$. The claim follows easily from the definition of the upper cluster algebra. $\square$

It is a trivial fact that balanced extensions always exist if $B : = B(\Delta)$ has full rank. Indeed, we put $e$ as last frozen vertices so the $B$-matrix $B : = B(\Delta)$ decomposes in blocks $(B, B_e)$. We write $\Theta$ in corresponding blocks $\Theta = (\Theta^0_{11}, \Theta^0_{12})$. The condition $B\Theta = 0$ is equivalent to that

$$B\Theta = -B_e.$$  

This is an overdetermined linear equation (for solving $\Theta$) if $B$ has full rank. For the rest of this subsection, we always assume that $(\Delta, x)$ is a balanced extension of $(\Delta, y)$ via $x(e)$ and some $\Theta$.

Let $y$ be the set of $y$-variables for the seed $(\Delta, x)$.

Lemma 2.3. We have that $x^g = x^g e^{\Theta}$ and $y^d = y^d$. 

$$x(u) = x(u)x(e)^{\Theta(u)} \text{ for some } \Theta(u) \in \mathbb{Z}^e.$$  

Here, by convention we set $x(e) = 1$ for each $e \in e$ so that $\Theta(e) = e$.
Proof. The statement for $x$-variables is a rephrasing of (2.1). For $y$-variables, it suffices to show that $y_u = y_u$ for each $u \in \Delta_i$. The equality (2.3) implies that the $u$-th row $b_u$ and $b_v$ of $B$ and $\overline{B}$ are related by $b_u = (b_u, -b_u \Theta)$. We have that
\[ y_u = x^{-b_u} = x^{-b_u, \Theta} = x^{-b_u} = y_u. \]
\[ \square \]

Now we are ready to study the IQP models. Let $(\Delta, W)$ be an IQP as in Section 1.2. We first consider a special case when $e$ is a single sink or source $e$. In this case, we put the same potential $W$ on $\Delta_i$. Let $P_e$ be the indecomposable projective representation of $(\Delta, W)$ corresponding to $u \in \Delta_i$.

Lemma 2.4. Suppose that $e$ is a frozen source of $(\Delta, W)$, then
\begin{enumerate}
  \item $G(\Delta, W) = G(\Delta, W) + \mathbb{Z}_{\geq 0} e$;
  \item $C_W((g, h)) = C_W(g) \mathbb{Z}^{h + g \Theta}$ for any $g \in G(\Delta, W)$ and $h \in \mathbb{Z}_{\geq 0}$.
\end{enumerate}

Proof. (1). Let $g := (g, h)$ be any vector in $G(\Delta, W)$. In particular, $\text{Coker}(g)$ is not supported on $e$. Since $e$ is a source, $h$ must be nonnegative. Moreover, there is no morphism from $P_e$ to other indecomposable projectives. It follows that $g$ must be in $G(\Delta, W)$. The other direction is more obvious.

(2). Since $e$ is a source, we have that $\text{Coker}(g, h)$ is the extension by a zero from $\text{Coker}(g)$. The equality follows from (1.1) and Lemma 2.3. \[ \square \]

Lemma 2.5. Suppose that $e$ is a frozen source or sink of $(\Delta, W)$ and (2.4) has a solution. Then $(\Delta, W)$ is a (polyhedral) cluster model if and only if so is $(\Delta, W)$.

Proof. We only prove the source case because the sink case can be proved by considering the opposite quiver of $(\Delta, W)$ and the dual version of (1.4).

Suppose that $(\Delta, W)$ is a cluster model. Let $(\Delta, x)$ be a balanced extension of $(\Delta, x)$ via $x(e)$ and some $\Theta$. Let $t$ be any $\Theta$-degree $d$ element in $C(\Delta, x)$. Recall from (2.1) that any element in $C(\Delta, x)$ has $\Theta$-degree zero. By Lemma 2.2, $t = s \omega^d$ for some $s \in C(\Delta, x)$. We can write $s$ as a sum $s = \sum a_i g_i$ for some $a_i \in k$ and $g_i \in G(\Delta, W)$. By Lemma 2.3, each $C_W(g_i)$ is equal to $C_W((g_i, 0)) \mathbb{Z}^{g_i, \Theta}$. Note that $C_W((g_i, 0))$ contains a "leading" term $x(g_i, 0)$. The meaning of "leading" is explained after Lemma 1.1. Since $t$ is a polynomial in $x$, and the $g_i$ are distinct, we conclude that $d - g_i, \Theta \geq 0$ for each $i$. By Lemma 2.3, we have that $t = \sum a_i C_W((g_i, d - g_i, \Theta))$, so $(\Delta, W)$ is a cluster model. The proof shows that if in addition $(\Delta, W)$ is polyhedral, then so is $(\Delta, W)$.

The argument for the converse is rather trivial. Indeed, suppose that $(\Delta, W)$ is a cluster model. Then by Lemma 2.2, for any $s \in C(\Delta, x)$, $t := s \omega^d \in C(\Delta, x)$ for sufficiently large $d$. We write $t$ as $t = \sum a_i C_W((g_i, h_i))$. Then $s = \sum a_i C_W(g_i) \omega^h - g_i, \Theta - d$ by Lemma 2.3. A similar degree argument shows that each $h_i + g_i, \Theta - d$ must vanish so that $s = \sum a_i C_W(g_i)$. \[ \square \]

Remark 2.6. The direction "=\rightarrow" of the above lemma is true without the sink-source assumption. We conjecture that this direction is true for $e$ mutable as well. A weaker statement (with an additional condition on $\omega_i$) was already proved in [10, Theorem 5.6]. We now sketch a proof of the above claim. We check that assignment $x(u) \mapsto x(u)$ and $x(e) \mapsto 1$ induces an epimorphism $\ell_e : C(\Delta, x) \rightarrow C(\Delta, x)$. Then...
it suffices to show the following diagram commutes:

\[
\begin{array}{c}
G(\Delta, W) \xrightarrow{C_W} \mathcal{C}(\Delta, \mathbf{x}) \\
\downarrow l_e \downarrow \downarrow t_e \\
G(\Delta, W) \xrightarrow{C_W} \mathcal{C}(\Delta, \mathbf{x})
\end{array}
\]

where \(l_e\) is the map forgetting the \(e\)-th coordinate \((g, h) \mapsto g\). Using Lemma 2.3 and the restriction and induction functor (see [10, Section 5.1 and Lemma 5.4]), we can show that the diagram indeed does commute.

**Definition 2.7 ([18]).** We say that a (frozen or mutable) vertex \(e\) can be **optimized** in \(\Delta\) if there is a sequence of mutations away from \(e\) making \(e\) into a sink or source of \(\Delta\). It can be optimized in an IQP \((\Delta, W)\) if in addition such a sequence is admissible.

Finally we come back to the more general case when \(e\) can be more than one vertex. Let \(W\) be any potential on \(\Delta\) such that its restriction on \(\Delta\) is \(W\). For any admissible sequence of mutations \(\mu\), we set \((\Delta', W') := \mu(\Delta, W)\) and \((\Delta', W') := \mu(\Delta, W)\). Since mutations are away from \(e\), \(\Delta'\) is a subquiver of \(\Delta\) and we have that \(W'|_{\Delta'} = W'\).

**Theorem 2.8.** Suppose that \(B(\Delta)\) has full rank and each vertex in \(e\) can be optimized in \((\Delta, W)\). If \((\Delta, W)\) is a (polyhedral) cluster model, then so is \((\Delta, W)\).

**Proof.** The proof goes by induction on the number of vertices in \(e = \{e_1, e_2, \ldots, e_n\}\). The statement is trivially true if \(e\) is empty. We assume that it is true for \(n = k\). Let \((\Delta_k, x_k, W_k)\) be the restriction of \((\Delta, x, W)\) to its full subquiver containing \(\Delta_0 \cup \{e_1, e_2, \ldots, e_k\}\). Then each \(B(\Delta_k)\) has full rank.

Suppose that \((\Delta_k, W_k)\) is a cluster model. Let \((\Delta_{k+1}, x_{k+1})\) be any balanced extension of \((\Delta_k, x_k)\) via \(x(e_{k+1})\). Let \(\mu\) be a sequence of mutations such that \(e_{k+1}\) is a sink or source of \(\mu(\Delta_k)\). Recall that being a cluster model is mutation-invariant so \(\mu(\Delta_k, W_k)\) is a cluster model. By the discussion above, we have that \(\mu(W_{k+1})\) restricted on \(\mu(\Delta_k)\) is \(\mu(W_k)\). So by Lemma 2.5 \(\mu(\Delta_{k+1}, W_{k+1})\) is a cluster model and so is \((\Delta_{k+1}, W_{k+1})\). The induction completes the proof. \(\square\)

### 2.2. Extension Through Cluster Variables

Let \(\Delta\) still be an ice subquiver of \(\Delta\) such that \(\Delta_0 = \Delta_0 \cup e\). In this subsection we assume that vertices in \(e\) are all mutable. Recall that \(\Delta'\) is obtained from \(\Delta\) by freezing \(e\). We assume that \(\mathcal{C}(\Delta', x)\) is a balanced extension of \(\mathcal{C}(\Delta, x)\) through \(e\). Recall that such an extension always exists if \(B(\Delta)\) has full rank. To study the relation between \(\mathcal{C}(\Delta, x)\) and \(\mathcal{C}(\Delta', x)\), we employ \(\mathcal{C}(\Delta', x)\) as an intermediate object.

As before, we first treat the case when \(e\) is a single vertex \(e\). Recall the notation that \(x_u := \rho_u(x)\) and \(x_e := x\). We write the initial exchange relation at \(u\) as \(x_u x_u' = \rho_u\). Let \(\Gamma\) be an ice quiver. We shall apply Proposition 2.10 below for \((\Gamma, x) = (\Delta, x)\) later.

**Lemma 2.9.** Let \(e\) be a sink or a source of \(\Gamma\). Suppose that \((\Gamma, x)\) is CR1 and its UCA is Noetherian. Then \(\mathcal{C}(\Gamma^e, x)[x_e']\) is a normal domain.

**Proof.** Let \(A = \mathcal{C}(\Gamma^e, x)\) and \(B = \mathcal{C}(\Gamma, x)\). We have that \(x_e' \in A_{x_e}\) so \(A[x_e']_{x_e} = A_{x_e}\). Recall that every UCA is normal so \(A[x_e']_{x_e}\) is normal. Let \(w = \prod_{u \rightarrow e} x_u \in A\).
Consider
\[ A[x'_e]_w = A_w[x'_e] = \overline{\mathcal{C}}(\Gamma^d, x)|x'_e], \quad \text{where} \quad d = \{u \mid u \to e\} \cup \{e\}. \]

The second equality is due to Lemma 1.10 (we apply it for \( e = \{u \mid u \to e\} \)). We notice that after freezing \( d \), we can view \( \overline{\mathcal{C}}(\Gamma^d, x)|x'_e] \) as a UCA with coefficients in \( R := k[x(d), x'_e] \). To be more precise,
\[(2.5) \quad \overline{\mathcal{C}}(\Gamma^d, x)|x'_e] = \bigcap_{(\Gamma^d', x') \sim (\Gamma^d, x)} R[x'(\Gamma_p')^{-1}].\]

It is easy to see that \( R \) is normal. Indeed, \( R \) is a complete intersection because it can be presented as \( R = k[x(d), z]/(x_e z - 1 - w) \). We can easily check by the Jacobian criterion [8, Theorem 16.9] that \( R \) is regular in codimension 1. Then the normality of \( R \) follows from Serre’s criterion [8, Theorem 11.5]. By [8, Exercise 4.17] any Laurent polynomial ring over \( R \) is also normal. As an intersection of normal domains, \( A[x'_e]_w \) is normal. Now to show \( A[x'_e] \) is normal, it suffices to show that the complements of zero sets \( Z(x_e) \) and \( Z(w) \) cover \( \text{Spec}(A[x'_e]) \). This follows from the exchange relation: \( x_e x'_e - w = 1 \).

**Proposition 2.10.** Let \( e \) be a sink or a source of \( \Gamma \). Suppose that \( (\Gamma^e, x) \) is CR1 and its UCA is Noetherian. Then so is \( (\Gamma, x) \) and we have that
\[ \overline{\mathcal{C}}(\Gamma, x) = \overline{\mathcal{C}}(\Gamma^e, x)|x'_e]. \]

**Proof.** We recall the notation that \( A = \overline{\mathcal{C}}(\Gamma^e, x) \) and \( B = \overline{\mathcal{C}}(\Gamma, x) \). It is showed in the proof of [10, Lemma 2.4] that \( A \subseteq B \). Since \( x'_e \in \overline{\mathcal{C}}(\Gamma, x) \), we proved “\( \supseteq \)”.

To show the other direction, we are going to apply Lemma 1.10. We already have that \( x_u, x'_u \in \overline{\mathcal{C}}(\Gamma^e, x)|x'_e] \) for any \( u \in \Gamma^e \). It remains to verify the coprime condition (Definition 1.3 (2)) for the pairs \( (x_u, x_v) \) and \( (x_u, x'_v) \) in \( A[x'_e] \). Since the quotient fields of \( A \) and \( A[x'_e] \) have the same transcendence degree, \( A \) and \( A[x'_e] \) have the same Krull dimension. After localizing at \( x_e \), \( A_{x_e} \to A_{x'_e} \) is an integral extension by the exchange relation of \( x_e \). Our assumption says that \( (\Gamma^e, x) \) is CR1 in \( A \). It follows from the going-up property (see also [8, Proposition 9.2]) that the coprime condition holds for those pairs with \( x_u \neq x_e \). Hence it suffices to look at the pairs \( (x_e, x_e) \) and \( (x_e, x'_e) \).

We note that for any \( u \to e \), \( x_u \) and \( x_e \) cannot both vanish due to the exchange relation. So we localize at \( w = \prod_{u \to e} x_u \) as before. Since \( x_e \) is in the coefficient ring \( R \) (see (2.5)), the codimension 2 statement follows easily. For \( (x_e, x'_e) \), we observe that \( A[x'_e]/(x_e, x'_e) = A/I \) where \( I \) is an ideal containing \( x_e \) and \( 1 + w \). By Lemma 1.9, the common zero locus of \( x_e \) and \( 1 + w \) has codimension 2 in \( A \). Therefore \( (x_e, x'_e) \) is coprime in codimension 1 in \( A[x'_e] \). \( \square \)

We now come back to the setting at the beginning of this subsection. We assume that \( B(\Delta) \), and thus \( B(\Delta^e) \), has full rank so that the conditions in Proposition 2.10 are satisfied for \( \Gamma = \Delta \).

**Corollary 2.11.** If \( \overline{\mathcal{C}}(\Delta^e) \) is \( g \)-indexed by \( G(\Delta^e) \), then \( \overline{\mathcal{C}}(\Delta) \) is \( g \)-indexed by \( G(\Delta^e) + \mathbb{Z}_{\geq 0} \text{g}(S_e) \) where \( \text{g}(S_e) \) is the \( g \)-vector of the simple representation at \( e \). If \( \overline{\mathcal{C}}(\Delta) \) is \( g \)-indexed by \( G(\Delta) \) and \( e \) is a sink, then \( \overline{\mathcal{C}}(\Delta) \) is \( g \)-indexed by \( G(\Delta) + \mathbb{Z} \cdot e_e \).

**Proof.** We observe that \( \text{g}(S_e) \) is the same as the \( g \)-vector of \( x'_e \). The first statement follows from Proposition 2.10. For the second statement, we see from Lemmas.
that $\mathcal{C}(\Delta^e, \mathbf{x})_{\mathcal{C}}$ is $g$-indexed by $G(\Delta) + Ze_e$. It follows that $\mathcal{C}(\Delta^e)$ is $g$-indexed by some subset

$$H := \{(g, h) \mid g \in G(\Delta), h \geq h(g) \text{ for some } h(g)\}.$$  

When $e$ is sink, $g(S_e) = -e_e$, then we conclude from the first statement that $\mathcal{C}(\Delta)$ is $g$-indexed by $H - Ze_0e_e = G(\Delta) + Ze_e.$ \hfill $\square$

Lemma 2.12. Suppose that $e$ is a mutable sink of $\Delta$.

1. $G(\Delta^e, W) = G(\Delta, W) + Ze_e$;
2. $C_W(g - he_e) = C_W(g)x_e^h$ for any $h \geq 0$.

Proof. (1) is straightforward. Since $e$ is a sink, a general presentation of weight $g - he_e$ is a direct sum of a general presentation of weight $g$ and $h$ copies of $P_e = S_e$.

By \cite{6} Proposition 3.2 we have that $C_W(g - he_e) = C_W(g)x_e^h$. \hfill $\square$

Corollary 2.13. Let $e$ be a mutable sink of $\Delta$. We assume that $B(\Delta)$ has full rank and $\mathcal{C}(\Delta)$ is Noetherian. If $(\Delta, W)$ is a (polyhedral) cluster model, then so is $(\Delta^e, W)$.

Proof. By Corollary 2.11, $G(\Delta) = G(\Delta^e) + Ze_e$. Since $(\Delta, W)$ is a cluster model, $G(\Delta) = G(\Delta, W)$. We have that $G(\Delta^e, W) \supseteq G(\Delta, W) + Ze_e$ and $G(\Delta) \supseteq G(\Delta^e, W)$. Putting these (in)equalities together, we get that $G(\Delta, W) \supseteq G(\Delta) \supseteq G(\Delta^e, W)$. So $G(\Delta) = G(\Delta^e, W)$. Hence, $(\Delta, W)$ is a cluster model by Corollary 2.11. \hfill $\square$

More generally, we should allow $e$ to contain more than one vertex. The proof of this more general case is exactly the same as that of Theorem 2.8. Combining this with Theorem 2.8, we get the following theorem where $e$ can contain both frozen and mutable vertices.

Theorem 2.14. Let $W$ be any potential on $\Delta$ such that its restriction on $\Delta$ is $W$. Suppose that $B(\Delta)$ has full rank, $\mathcal{C}(\Delta)$ is Noetherian, and each vertex in $e$ can be optimized in $(\Delta, W)$. If $(\Delta, W)$ is a (polyhedral) cluster model, then so is $(\Delta^e, W)$.

Definition 2.15. A sink-source extension of an IQP $(\Delta, W)$ through a (mutable or frozen) vertex $e$ is a new IQP $(\Delta^e, W)$ such that $e$ is a sink or source, $\Delta^e = \Delta_0 \sqcup \{e\}$, and $\Delta \mid_{\Delta^e} = \Delta$.

We denote by $\mathfrak{d}$ the class of IQPs obtained from a single vertex by sink-source extensions and mutations. Since the rigidity is preserved in this procedure, any IQP in $\mathfrak{d}$ is rigid. By abuse of language we also say such a quiver (with rigid potential) is in $\mathfrak{d}$.

Corollary 2.16. Any IQP in $\mathfrak{d}$ is a polyhedral cluster model if the $B$-matrix of the IQP has full rank throughout the procedure.

Remark 2.17. Recall that any extension through a frozen vertex and any mutation preserve the rank of $B$-matrix. The full rank condition can only be broken when we extend through a mutable vertex. We even conjecture that the full rank condition can be dropped in this corollary.
Remark 2.18. We believe that the class $\mathfrak{d}$ does not cover all cluster models. Here may be an example. Let us consider the following quiver $\Delta$

It is easy to show that up to quiver automorphisms, there are only 6 quivers mutation-equivalent to this one. None of them contains a sink or source. This quiver is obtained from the torus with disc cut out and two marked points on the boundary (see Section 3, also [27, 11.4]). We can show that this quiver with generic potential $W$ has finite-dimensional Jacobian algebra. Since the quiver is coefficient-free, the set $G_{\Delta, W}$ is the whole lattice $\mathbb{Z}^5$. On the other hand, [28, Corollary 4.9] implies that $\mathcal{C}_{\Delta}$ also has a basis parametrized by $\mathbb{Z}^5$. This provides some evidence that the image of $G_{\Delta, W}$ under the generic character forms a basis in $\mathcal{C}_{\Delta}$.

3. Gluing Hives

We fix an integer $l \geq 2$. To the oriented triangle $t$ we can associate an *ice hive quiver* $\Delta_l$ of size $l$ [9]. The vertices of $\Delta_l$ are labeled by $(i, j, k)$ such that $(i, j, k) \in \mathbb{Z}_{\geq 0}^3$, $i + j + k = l$ and at most one of $i, j, k$ is 0. If one of $i, j, k$ is 0, then $(i, j, k)$ is frozen, and lies on an *edge* of $\Delta_l$. An edge of $\Delta_l$ corresponds to an edge $ab$ ($a, b = 1, 2, 3$) of $t$ if the vertices on it have nonzero $a, b$-th coordinates in their labeling. The direction of arrows are thus determined by the orientation of $t$. For example, $\Delta_5$ is depicted in Figure 2 (the dashed lines are edges).
Later we will consider oriented triangles with vertices labeled by letters, say $a, b, c$. Such an oriented triangle $t$ can be represented by a cyclically ordered triple, eg. $[a, b, c] = [b, c, a] = [c, a, b]$. We define $\Delta_i$ as a vertex relabeling of $\Delta_l$ as follows. Fixing a representative of $t$ (say $t = [a, b, c]$), we relabel the vertex $(i, j, k)$ in $\Delta_l$ as the cyclic triple $[a, b, c]_{i, j, k}$. If the representative of $t$ is given from the context, we may write $(i, j, k)$ for that vertex.

Given a pair of oriented triangles, we can glue them along a pair of edges.

![Figure 3](image)

This gluing corresponds to the gluing of two ice hive quivers of the same size by identifying frozen vertices along the common edge. After the identification, we unfreeze those frozen vertices, and add additional arrows depending on the orientations as illustrated in Figure 4. The additional arrows are in red (please ignore the color of vertices).

![Figure 4](image)

More generally, let us recall that an ideal triangulation of a marked bordered surface $S$ is a triangulation of $S$ whose vertices are the marked points of $S$. All triangulations in this paper will be ideal. Let $T$ be a triangulation of a marked bordered surface $S$. Then we can glue ice hive quivers according to such a triangulation. We denote the resulting ice quiver by $\hat{Q}_l(T)$. We allow self-fold triangles and two triangles glued along more than one edge. Note that they happen only when there are punctures (=marked points on the interior of $S$). All surfaces in this paper have no punctures, but this flexibility is necessary to incorporate and others.
A triangulation is called consistent if any pair of glued edges are pointing in the opposite direction. If the surface is orientable, then we always have a consistent triangulation. In this case the construction was considered by Fock and Goncharov in [13] when they study the cluster structure of certain $\text{SL}_2$-character varieties. Readers may be more familiar with the degenerate case when $l = 2$. For example, when $l = 2$ Figure 4 (left) degenerates to

![Figure 4](image1)

We observe that when $l = 2$, the quiver $\hat{\Delta}_l(T)$ is the same as the adjacency quiver of $T$ according to the recipe of [14].

One of the main results of [14] says that the flip of the triangulation corresponds to the mutation of its adjacency quiver. Let $d = bc$ be the common edge of two adjacent triangles in $T$ as in Figure 5 (left). A flip along $d$ is an operation on $T$ sending the pair to Figure 5 (right), and keeping the remaining triangles unchanged.

![Figure 5](image2)

We denote the new triangulation by $T'$. It is well-known that any two consistent triangulations of $S$ are related by a sequence of flips. We warn readers that the flip is not defined for a pair of triangles as in Figure 3 (right).

More generally for any $l \geq 2$, Fock and Goncharov defined in [13, 10.3] a sequence of mutations $\mu^d$ such that $\mu^d(\hat{\Delta}_l(T)) = \hat{\Delta}_l(T')$. Let us briefly recall the definition. The definition is local in the sense that we only need to mutate at interior vertices in the two hives to be flipped. By a maximal central rectangle (wrt. the common edge), we mean a maximal rectangle symmetric about the common edge with vertices in $\hat{\Delta}_l(T)_\mu$. There are $l - 1$ maximal central rectangles (including two degenerated to lines). For example in Figure 4 the blue (resp. green) vertices are in the second (resp. third) maximal central rectangle. Let $\mu_k$ be the sequence of mutations along the $k$-th rectangle (in whatever orders). Then $\mu^d$ is the composition $\mu_{k-1} \cdots \mu_2 \mu_1$.

Consider the following hive strip of length $n + 1$. The vertex 0 can be frozen or mutable.

![Figure 6](image3)
Lemma 3.1. The sequence of mutations $(n, n-1, \ldots, 1)$ transforms the quiver into

In particular, the sequence optimizes the vertex 0. If we delete 0 and apply additional sequence of mutations $(1, \ldots, n-1, n)$, then the quiver becomes the original one with 0 deleted.

Proof. The proof is by an easy induction on $n$. A variation was proved in [25, Proposition 19].

Example 3.2. Let $\Delta^{(p)}_5$ (resp. $\Delta^{(q)}_5$) be the quiver obtained from $\Delta_5$ by forgetting frozen vertices on any two (resp. one) edges. It is shown in [9, 10, 11] using quite different methods that with the rigid potentials they (including $\Delta^{(p)}_5$) are all polyhedral cluster models. There are similar result for $\Delta^{(p)}_5$ and $\Delta_5$ in a slightly different context in [25]. Using the above lemma and Theorem 2.14, it is easy to show this for $\Delta^{(q)}_5$ and $\Delta_5$ assuming the result for $\Delta^{(p)}_5$. For example, applying the mutation at $(2, 1, 2)$ then $(2, 2, 1)$ optimizes the frozen vertices $(2, 0, 3)$ and $(2, 3, 0)$ in $\Delta_5$ of Figure 2. We strongly recommend readers to play with a few examples using [22].

The same method can prove that $\Delta^{(p)}_5$ are in $\partial$ (so $\Delta^{(q)}_5$ and $\Delta_5$ are in $\partial$ as well). However, to apply Lemma 3.1 to $\Delta^{(q)}_5$, we have to remove frozen vertices of $\Delta^{(p)}_5$ first. At some point, this will violates the full rank condition of Theorem 2.14. Interested readers can check that similar sequences of Lemma 3.1 cannot optimize any mutable vertex without removing frozen ones first.

To obtain the full condition for some $\diamondsuit_1(T)$, we need the following lemmas.

Lemma 3.3. Let $\diamondsuit$ be obtained from $\Delta$ and $\nabla$ by gluing along a set of frozen vertices $c$ (and unfreezing $c$). Let $a$ and $a'$ (resp. $b$ and $b'$) be the set of rest frozen vertexesc (resp. mutable) in $\Delta$ and $\nabla$. Suppose that the submatrix of $B(\Delta)$ consisting of $b \cup c$-rows and $a \cup b$-columns is of full rank, and $B$-matrix of $\nabla$ is also of full rank. Then so is the $B$-matrix of $\diamondsuit$.

Proof. The $B$-matrix of $\diamondsuit$ is given by the block matrix on the left,

\[
\begin{pmatrix}
a & b & c & b' & a' \\
b & * & * & * & 0 & 0 \\
c & * & * & * & * \\
b' & 0 & 0 & * & * & * \\
a & b & c & b' & a'
\end{pmatrix}
\sim
\begin{pmatrix}
a & b & c & b' & a' \\
b & * & * & 0 & 0 \\
c & * & * & 0 & 0 \\
b' & 0 & 0 & * & * & *
\end{pmatrix}
\]

which is equivalent to the one on the right by elementary column transformations. The statement is now clear.

Lemma 3.4. Let $a$ (resp. $c$) be the set of frozen vertices on one (resp. the other) edge of $\Delta^{(q)}_5$. Let $\Delta^{(q)}_5$ be an ice quiver obtained from $\Delta^{(p)}_5$ by unfreezing $c$. Then the submatrix of $B(\Delta^{(q)}_5)$ consisting of $b \cup c$-rows and $a \cup b$-columns has full rank as in Lemma 3.3.
Proof. Suppose that \( a \) is on the edge 12 and \( c \) is on the edge 23 (see Figure 1). It is easy to see that if we rearrange the rows and columns of the submatrix according to the lexicographic order, then we get a triangular matrix with 1’s on the diagonal. □

To give a simpler proof of the next proposition, we will introduce a new involutive operation on triangulated surfaces called twist. To describe this operation, we need to specify a triangle \( t \) and one of its edges \( e \). The twist consists of 3 steps.

1. Cut along the other two edges of the chosen triangle (see Figure 6 left);
2. Change the orientation of the triangle and the identification of the two edges (the new identification is indicated by arrows, see Figure 6 right);
3. Glue according to the new identification.

![Figure 6.](image)

We have a few remarks. In Step (1), if any of two edges is a part of boundary, then we do nothing for that edge. Readers should be aware that the new identification in Step (2) is not just a naive “interchange”. For example, let \( a \) be the vertex opposite to the chosen edge. According to our definition, the vertices of adjacent triangles previously glued to \( a \) will not be glued to \( a \). Finally, we need to warn readers that this operation may alter the topology of the surface (see [12]). However, this will not happen in this paper. We denote the new triangulation (of a possibly new surface) by \( T_{t,e} \).

**Lemma 3.5.** There is sequence of mutations \( \mu_{1:e} \) such that

\[
\mu_{1:e}(\varpi_l(T)) = \varpi_{l}(T_{t,e}).
\]

**Proof.** The sequence to be defined is local in the sense that we only need to perform mutations at non-edge vertices in the chosen triangle \( t \). So we will define it for a single hive. Such a sequence of mutations is defined in [10] and [1] B.2 in a boarder context (The notation for this operation there is \( \mu_\sqrt{l} \), but unfortunately, the \( l \) under the square root is not the integer but has other meaning).

Let us briefly recall the definition. Suppose that the hive corresponds to the triangle in Figure 1 and \( 23 \) is the chosen edge (For other choices of edges, the operation can be defined using the cyclic symmetry). Then let \( t_k \) be the triangle with vertex 1 and its opposite side parallel to \( 23 \) of \( k \)-arrow length. Denote by \( \mu_k \) the composition of mutations at each (mutable) vertex in \( t_k \) in the reverse lexicographic ordering for the labeling. We define the sequence of mutations as \( \mu := \mu_2\mu_3 \cdots \mu_{l-1} \). Let us take the hive \( \Delta_5 \) as an example. Then we have that

\[
\mu_4 = \mu(1,1,3)\mu(1,2,2)\mu(1,3,1)\mu(2,1,2)\mu(2,2,1)\mu(3,1,1), \quad \mu_3 = \mu(2,1,2)\mu(2,2,1)\mu(3,1,1), \quad \text{and} \quad \mu_2 = \mu(3,1,1).
\]

This lemma is almost a reformulation of [1] Corollary B.9 in our setting of triangulation except that we need to take care of arrows between edge vertices. Recall that the twist changes the orientation of the triangle. So arrows between its
edge vertices will change (see Figure 4). But at least for the edge $bc$ this is implied in [25, Lemma 9], where author considered the subsequence $\mu_{l-1}$ of $\mu$. The reason is that other subsequences have no effect on arrows between vertices on $\overrightarrow{23}$. The proof goes similarly for the other two edges. \hfill \square

Let $D_m$ be the disk with $m$ marked points on the boundary.

**Proposition 3.6.** Let $T$ be any (consistent) triangulations of $D_m$. Then $\diamondsuit_l(T)$ is in $\mathfrak{d}$, and with the rigid potential $W_l(T)$ is a polyhedral cluster model.

**Proof.** Since all consistent triangulations are related by flips, it suffices to prove for a particular $T$. Consider the following non-consistent alternating triangulation.

\[
\begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

For each even (clockwise oriented) triangle with any of its edges, we apply the twist. Then we can easily see by induction that it transforms to a consistent triangulation of $D_m$ (The alternating triangulation in the above picture transforms to $D_9$). Since the twist is involutive, by Lemma 3.5 it suffices to prove the statement for the quivers corresponding to these alternating triangulations.

The proof will go backwards, that is, instead of using sink-source extension we use sink-source deletion. Using Lemma 3.1 we show that each frozen vertex can be optimized. So let us remove all frozen vertices except for ones corresponding to the leftmost edge. The purpose of preserving those frozen vertices is that this will guarantee the full rank requirement of Theorem 2.14 throughout later operations. This can be verified by Lemma 3.3 and 3.4. For any $\diamondsuit_l(T)$ if we delete a set of frozen vertices corresponding to an edge in $T$, then pictorially we make this edge a dashed line. So after deleting frozen vertices, the quiver corresponds to the one in Figure 7 (ignoring arrows).

\[
\begin{array}{cccc}
\text{Figure 7.} & \end{array}
\]

We perform the operation defined in Lemma 3.1 as follows. We apply the sequence of mutations in the direction of arrows (see Figure 7), then delete the sink and apply the same sequence of mutations backwards. In this way we can one by one delete rows parallel to the common edge as shown above. Below is a schematic diagram. The number indicates the order for deletion (reader can check that the order does matter). The gray part is the part already deleted, so the current one to be deleted is vertex 5. We are going apply the sequence of Lemma 3.1 for the
In the end, what left over is the quiver represented by

which is nothing but the quiver $\Delta^\triangledown$. We have seen that this quiver is in $\mathfrak{d}$, and is a cluster model. We conclude from Theorem 2.14 that $\mathfrak{d}$ with the rigid potential is a polyhedral cluster model.

**Conjecture 3.7.** Let $\mathbb{T}$ be a triangulation of a (not necessarily orientable) surface such that there is a rigid potential $W_l$ on $\hat{\mathbb{T}}$. Then whether $(\hat{\mathbb{T}}, W_l)$ is a cluster model does not depend on $l$.

**Remark 3.8.** Whether $\hat{\mathbb{T}} \in \mathfrak{d}$ will depend on $l$ (See [12] for an example).

### 4. Application to Semi-invariant Ring of Quiver Representations

#### 4.1. Some Generalities

Let us briefly recall the semi-invariant rings of quiver representations [32]. Readers can find any unexplained terminology in [4]. Let $Q$ be a finite quiver without oriented cycles. For an arrow $a$, we denote by $t(a)$ and $h(a)$ its tail and head. For a dimension vector $\beta$ of $Q$, let $V$ be a $\beta$-dimensional vector space $\prod_{i \in Q_0} k^{\beta(i)}$. We write $V_i$ for the $i$-th component of $V$. The space of all $\beta$-dimensional representations is

$$\text{Rep}_\beta(Q) := \bigoplus_{a \in Q_1} \text{Hom}(V_{t(a)}, V_{h(a)}).$$

The product of general linear group

$$\text{GL}_\beta := \prod_{i \in Q_0} \text{GL}(V_i)$$

acts on $\text{Rep}_\beta(Q)$ by the natural base change. Define $\text{SL}_\beta \subset \text{GL}_\beta$ by

$$\text{SL}_\beta = \prod_{i \in Q_0} \text{SL}(V_i).$$

We are interested in the rings of semi-invariants

$$\text{SI}_\beta(Q) := k[\text{Rep}_\beta(Q)]^{\text{SL}_\beta}.$$ 

The ring $\text{SI}_\beta(Q)$ has a weight space decomposition

$$\text{SI}_\beta(Q) = \bigoplus_{\sigma} \text{SI}_\beta(Q)_\sigma,$$
where $\sigma$ runs through the multiplicative characters of $\text{GL}_\beta$. We refer to such a decomposition the $\sigma$-grading of $\text{SI}_\beta(Q)$. Recall that any character $\sigma : \text{GL}_\beta \rightarrow k^*$ can be identified with a weight vector $\sigma \in \mathbb{Z}^{Q_0}$

$$\tag{4.1} (g(i))_{i \in Q_0} \rightarrow \prod_{i \in Q_0} \left( \det g(i) \right)^{\sigma(i)}.$$ 

Since $Q$ has no oriented cycles, the degree zero component is the field $k$ [23].

Let us understand these multihomogeneous components

$$\text{SI}_\beta(Q)_\sigma := \{ f \in k[\text{Rep}_\beta(Q)] \mid g(f) = \sigma(g)f, \ \forall g \in \text{GL}_\beta \}.$$ 

For any projective presentation $f : P_1 \rightarrow P_0$, we view it as an element in the homotopy category $K^b(\text{proj}-Q)$. Let $f \in \mathbb{Z}^{Q_0}$ be the weight vector of $f$. From now on, we will view a weight vector as an element in the dual $\text{Hom}_\mathbb{Z}(\mathbb{Z}^{Q_0}, \mathbb{Z})$ via the usual dot product. We apply the functor $\text{Hom}_Q(-, N)$ to $f$ for $N \in \text{Rep}_\beta(Q)$

$$\tag{4.2} \text{Hom}_Q(P_0, N) \xrightarrow{\text{Hom}_Q(f, N)} \text{Hom}_Q(P_1, N).$$

If $f(\beta) = 0$, then $\text{Hom}_Q(f, N)$ is a square matrix. Following Schofield [32], we define

$s(f, N) := \det \text{Hom}_Q(f, N).$

We set $s(f)(-)=s(f, -)$ as a function on $\text{Rep}_\beta(Q)$. It is proved in [32] that $s(f) \in \text{SI}_\beta(Q)_f$. In fact,

**Theorem 4.1** ([31] [33] [27]). $\{s(f)\}_{t=\sigma}$ spans $\text{SI}_\beta(Q)_\sigma$ over the base field $k$.

Let $\Sigma_\beta(Q)$ be the set of all weights $\sigma$ such that $\text{SI}_\beta(Q)_\sigma$ is non-empty. It is known that such a weight must correspond to some dimension vector $\alpha$, that is, $\sigma(-) = -\langle \alpha, - \rangle$ where $\langle - , - \rangle$ is the Euler form of $Q$. A weight is called extremal in $\Sigma_\beta(Q)$ if it lies on an extremal ray of $\mathbb{R}_+ \Sigma_\beta(Q)$.

**Lemma 4.2** ([21] Lemma 1.8). If $\sigma$ is an indivisible extremal weight, then any semi-invariant function of weight $\sigma$ is irreducible.

**Example 4.3.** Let $S_l^m$ be the $m$-tuple flag quiver of length $l$.

\begin{align*}
(1) & \quad \begin{array}{cccccc}
1 & \rightarrow & 2 & \rightarrow & \cdots & \rightarrow & l - 1
\end{array} \\
(2) & \quad \begin{array}{cccccc}
1 & \rightarrow & 2 & \rightarrow & \cdots & \rightarrow & l - 1
\end{array} \\
& \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
(m-1) & \quad \begin{array}{cccccc}
1 & \rightarrow & 2 & \rightarrow & \cdots & \rightarrow & l - 1
\end{array} \\
(m) & \quad \begin{array}{cccccc}
1 & \rightarrow & 2 & \rightarrow & \cdots & \rightarrow & l - 1
\end{array}
\end{align*}

The $i$-th vertex of the $a$-th flag of $S_l^m$ is denoted by a pair $(i, \beta)$ (with the convention that $(1, \beta) = l$ for any $a$). The standard dimension vector $\beta$ is the one such that $\beta_l = i$. For any $(i, j, k) \in \mathbb{Z}^{2}_+ \mathbb{Z}$ with $i + j + k = l$, let $s_{i,j,k}^{a,b,c}$ be the Schofield’s semi-invariant function on $\text{Rep}_{\beta}(S_l^m)$ associated to the presentation

$$P_l \xrightarrow{(p_i, p_j, p_k)} P_l^a \oplus P_j^b \oplus P_k^c,$$
where \( p^a_i \) is the unique path from the \( i \)-th vertex of the \( a \)-th flag to the vertex \( l \). Each \( s_{i,j,k}^{a,b,c} \) has weight equal to \( \sigma_{i,j,k}^{a,b,c} := e_l - e_i^a - e_j^b - e_k^c \), which is clearly extremal. By our convention we ignore \( P^a_i, P^a_i \), and \( e_i^a \) if \( i = 0 \). Moreover, each weight \( \sigma := \sigma_{i,j,k}^{a,b,c} \) corresponds to a real Schur root of \( S^m_l \) so the weight space \( \text{SI}_{\beta}(S^m_l) \) is 1-dimensional [9, Lemma 1.7].

Let \( \epsilon \) be a real Schur root, and \( E \) be the general representation in \( \text{Rep}_A(Q) \). We recall that \( \epsilon \) is a Schur root means that \( \text{Hom}_Q(E, E) = k \), and it is called real if \( \langle \epsilon, \epsilon \rangle = 1 \). In [31] Schofield defined a quiver \( Q_{\epsilon} \) with dimension vector \( \beta_{\epsilon} \) associated to \( \epsilon \). The semi-invariant ring \( \text{SI}_{\beta_{\epsilon}}(Q_{\epsilon}) \) can be identified as a subalgebra \( S := \bigoplus_{(\epsilon, \alpha) = 0} \text{SI}_{\beta_{\epsilon}}(Q_{\epsilon}) \). Let \( s \) be any semi-invariant function spanning the weight space \( \text{SI}_{\beta_{\epsilon}}(Q_{\epsilon}) \). In connection with Lemma 2.2, we have the following proposition.

**Proposition 4.4.** Assume that the weight corresponding to \( \epsilon \) spans an extremal ray in \( \Sigma_{\beta_{\epsilon}}(Q) \). Then the algebra morphism defined by

\[
\varphi : \text{SI}_{\beta_{\epsilon}}(Q_{\epsilon})[x^{\pm 1}] \to \text{SI}_{\beta_{\epsilon}}(Q_{\epsilon}) \quad \text{by} \quad rx^d \mapsto \iota(r)s^d.
\]

gives an isomorphism

\[
\text{SI}_{\beta_{\epsilon}}(Q_{\epsilon}) \cong \text{SI}_{\beta_{\epsilon}}(Q_{\epsilon})[x^{\pm 1}].
\]

**Proof.** The injectivity is proved in [9, Lemma 8.2]. To show surjectivity, by Theorem 1.1 it suffices to show that for any \( s(f) \in \text{SI}_{\beta_{\epsilon}}(Q_{\epsilon}) \), we can always rewrite it as \( s(f)s^c \) such that \( s(f)s^c \in S \). Let \( M \) be the cokernel of \( f \). Since the weight corresponding to \( \epsilon \) is extremal, the universal homomorphism \( hE \to M \) must be injective (see the proof of [10, Proposition 3.6]), where \( h = \dim \text{Hom}_Q(E, M) \). Let \( C \) be the cokernel of \( hE \to M \), then we consider the universal extension \( 0 \to C \to M \to eE \to 0 \), where \( e = \dim \text{Ext}_Q(E, M) \). By construction, we have that \( M \in E^+ \). Let \( f \) be the minimal resolution of \( M \). By Lemma 1.1, up to a scalar \( s(f) = s(f)s^c \in S \), where \( c = e - h = -\langle \epsilon, \alpha \rangle \).

4.2. From \( A \) to Complete Triple Flags. We denote the linear quiver \( S^k_l \) by \( A_k \). Let \( \text{Rep}_{\beta_{\epsilon}}(A_k) \) be the open subset of \( \text{Rep}_{\beta_{\epsilon}}(A_k) \) where each linear map is injective. Let \( F^- \) (resp. \( F^+ \)) be the representation of \( A_k \) given by \( (I_k, 0) \) (resp. \( (0, I_k) \)) for the arrow \( k \to k + 1 \), where \( I_k \) is the \( k \times k \) identity matrix and 0 is the zero column vector.

Throughout we set \( G := \text{SL}_k \). Let \( U^- \) (resp. \( U^+ \)) and \( H \) be the subgroups of lower (resp. upper) triangular and diagonal matrices in \( G \). Let \( W \) (resp. \( \hat{W} \)) be the Weyl group of \( G \) (resp. \( GL_k \)), and \( w_0 \) be the longest element in \( \hat{W} \). We can realize \( \hat{W} \) inside \( \text{GL}_k \) as the set of permutation matrices. For any \( m \in \mathbb{N} \), we decompose \( \text{SL}_{\gamma} \) as \( \text{SL}_{\gamma} \times \text{SL}_k \). We denote by \( A \) the categorical quotient \( \text{Rep}_{\beta_{\epsilon}}(A_k)/\text{SL}_{\gamma} \). The group \( \text{SL}_{\gamma} \) acts freely on \( \text{Rep}_{\beta_{\epsilon}}(A_k) \) so \( A^0 := \text{Rep}_{\beta_{\epsilon}}(A_k)/\text{SL}_{\gamma} \) is a geometric quotient. By abuse of notation, we still write \( F^- \) and \( F^+ \) for the corresponding points on \( A \). Since the action of \( \text{GL}_k \) commutes with that of \( \text{SL}_{\gamma} \), \( \text{GL}_k \) acts on \( A \) as well.

**Lemma 4.5.** \( GF^- \) is the only \( G \)-orbit in \( A^0 \) and the stabilizer of \( F^- \) (in \( G \)) is \( U^- \). Moreover, \( F^+ = w_0 F^- \) so its stabilizer is \( U^+ \).

**Proof.** It is an easy exercise on matrix manipulation to show that \( G \) acts transitively on \( A^0 \). So \( GF^- \) is the only \( G \)-orbit on \( A^0 \). For any \( u \in U^- \) we have that
\( \hat{u}^{-1}(I_{l-1}, 0)u = (I_{l-1}, 0) \) where \( \hat{u} \) is the upper left \( l-1 \times l-1 \) submatrix of \( u \). We see by induction that \( U^- \) stabilizes \( F^- \). On the other hand, there is no \( g \in \text{SL}_{l-1} \) such that \( \hat{g}^{-1}(I_{l-1}, 0)g = (I_{l-1}, 0) \) if \( g \in WHU^+\{e\} \). Hence, the stabilizer of \( F^- \) is \( U^- \). It is another easy exercise to show that \( F^+ = w_0F^- \).

Let \( s_{i,j}^{1,2} \) \((i + j = l)\) be the semi-invariant function on \( \text{Rep}_{\beta_l}(S_T^2) \) defined in Example 4.3. Let \( D = \prod_{i,j} s_{i,j}^{1,2} \). We denote \( \text{Rep}_{\beta_l}(S_T^2) \) by the complement of \( Z(D) \) in \( \text{Rep}_{\beta_l}(S_T^2) \). Recall the determinantal definition of each \( s_{i,j}^{1,2} \). It is not hard to see that \( \text{Rep}_{\beta_l}(S_T^2) \) consists of those representations equivalent to \( (F^-, F^+) \), that is

\[
\text{Rep}_{\beta_l}(S_T^2) = \text{GL}_{\beta_l}(F^-, F^+).
\]

Here, \( (F^-, F^+) \) is the representation whose restriction to the first and second flags is \( F^- \) and \( F^+ \) respectively. Let \( D \) be the zero locus of \( D \) on \( A \times A \). Since localization with respect to an invariant commutes with taking invariants, we have that

\[
(\Delta \subseteq \Delta) \setminus D = \text{Rep}_{\beta_l}(S_T^2)/\text{SL}_{\gamma_l} = (\Delta^\circ \times \Delta^\circ) \setminus D.
\]

A pair \((A_1, A_2) \in A \times A \) is called generic if it is not in \( D \).

**Corollary 4.6.** \( G \) acts freely on \((\Delta \subseteq \Delta) \setminus D \) and its quotient is isomorphic to \( H \).

**Proof.** Since \( \text{GL}_{\gamma_l} \cong \text{SL}_{\gamma_l} \times (k^*)^{2l-2} \cong \text{SL}_{\gamma_l} \times H^2 \), by (4.3) and (4.4) we can pick without lose of generality a point \( x \in (\Delta \subseteq \Delta) \setminus D \) represented by \((h_1F^-, h_2F^+)\) for some \((h_1, h_2) \in H^2 \). Since \( hU^\pm h^{-1} = U^\pm \), the action of \( H^2 \) does not change the stabilizer. So by Lemma 4.3, the stabilizer of \( x \) for the \( G \)-action is equal to \( U^- \cap U^+ = \{e\} \). We do not need the result on the quotient so we leave it as an easy exercise.

We denote \( T_l := S_T^3 \). Let \( s_l(i, j, k) = s_{i,j,k}^{1,2,3} \) for \((i, j, k) \in (\Delta)_{\alpha} \). A main result in \( \ref{10} \) is Theorem 4.9 below. Now we reprove this assuming a few results in \( \ref{9} \) but without \( \ref{24} \). We first recall that

**Lemma 4.7 (\ref{9}).** The upper cluster algebra \( \overline{\text{C}}(\Delta_l, s_l) \) is a subalgebra of \( \text{SI}_{\beta_l}(T_l) \).

Let \( v_c \) \((c = 1, 2, 3)\) be the set of frozen vertices on the edge opposite to the vertex \( c \) of \( t \). Let \( D_c := \prod_{v \in v_c} s_l(v) \), and \( H_c \) be the torus generated by \( s_l(v_c) \). It follows from \( \ref{10} \) Example 3.10 and Proposition 4.3 that

**Lemma 4.8.** The localization of \( \text{SI}_{\beta_l}(T_l) \) at any \( D_c \) can be identified with \( k[A \times H_c] \). Moreover, \( k[A] \) is the upper cluster algebra \( \overline{\text{C}}(\Delta_l, s_l) \).

Readers can find explicit definition of the seed \((\Delta_l, s_l)\) in [10] Example 3.10]. All we need to know here is that \((\Delta_l', s_l')\) is obtained from \((\Delta_l, s_l)\) via the projection defined there. That in particular implies that \( \overline{\text{C}}(\Delta_l, s_l) \) is a balanced extension of \( \overline{\text{C}}(\Delta_l', s_l') \) via \( s_l(v_c) \). So by Lemma 2.2 we have that \( \overline{\text{C}}(\Delta_l, s_l)_{D_c} = \overline{\text{C}}(\Delta_l', s_l')[s_l(v_c)^{\pm 1}] \).

**Theorem 4.9.** The semi-invariant ring \( \text{SI}_{\beta_l}(T_l) \) is equal to the upper cluster algebra \( \overline{\text{C}}(\Delta_l, s_l) \). Moreover, \((\Delta_l, W_l)\) is a polyhedral cluster model.

**Proof.** By Lemma 4.8 and the above discussion, we have that \( \text{SI}_{\beta_l}(T_l)_{D_c} = \overline{\text{C}}(\Delta_l, s_l)_{D_c} \) for \( c = 1, 2, 3 \). Let \( X := \text{Spec}(\text{SI}_{\beta_l}(T_l)) \) and \( Y := \text{Spec}(\overline{\text{C}}(\Delta_l, s_l)) \). We have a regular birational map \( X \to Y \) induced by the inclusion of Lemma 4.7. To show it is in fact an isomorphism, it suffices to show according to the following lemma that
Lemma 4.10 ([31 Corollary 1 of II.4.4]). Let \( f : X \to Y \) be a regular birational map, which is not an isomorphism. If \( Y \) is locally factorial, then \( f \) has an exceptional subvariety (i.e., a codimension 1 subvariety \( Z \subset X \) such that \( \text{codim}(f(Z)) \geq 2 \)).

We remark that the condition for \( Y \) required in the original text is being nonsingular, but we see from the proof that it can be weakened as being locally factorial.

Some readers may complain that Lemma 4.10 is proved in [10] with Theorem 4.9. However, Lemma 4.10 can also be proved in a similar way using [10, Example 3.11] if we assume a classical result on the cluster algebra structure of \( U \).

4.3. Generalization to Complete \( m \)-tuple flags. Let \( T \) be any consistent triangulation of \( D_m \). Given a triangle \( \alpha = [a,b,c] \in T \), there is a natural \( G \)-equivariant projection

\[
A^m \to A^3, \quad (A_1, A_2, \ldots, A_m) \mapsto (A_a, A_b, A_c).
\]

By the universal property of the categorical quotient, we get a map

\[
\pi_\alpha : A^m / G \to A^3 / G.
\]

We denote \( X_m := A^m / G \). For each such a triangulation \( T \), we thus get a map

\[
\pi_T = \prod_{\alpha \in T} \pi_\alpha : X_m \to \prod_{\alpha \in T} X^\alpha.
\]

Let \( \Delta^\alpha \) be the ice hive subquiver of \( \mathcal{Q}_G(T) \) corresponding to \( \alpha \). By pulling back the seed \((\Delta^\alpha, s^\alpha)\) of \( k[X_3] = S\mathcal{L}_G(T) \) via \( \pi_\alpha \), we get a set of semi-invariant functions \( s^\alpha_i \) in \( S\mathcal{L}_G(S^n) \) indexed by vertices of \( \Delta^\alpha \). The function \( s^\alpha_i(i,j,k) \) is nothing but \( s^\alpha_{a,b,c} \) defined in Example 4.3. The assignment clearly does not depend on the representation of \( \alpha \).

We call a common edge in \( T \) a diagonal. It is clear from the construction that all elements in \( s^\alpha_i \) (\( \alpha \in T \)) agree on diagonals. So we obtain a set \( s^\alpha_i(T) \subset S\mathcal{L}_G(S^n) \) indexed by the vertices of \( \mathcal{Q}_G(T) \). Let \( d := d(T) \) be the set of all vertices on the diagonals of \( T \), and \( d|_T \) be the subset of \( d \) where vertices lie on some edge of \( T \). We set \( D(T) := \prod_{v \in d} s^\alpha_i(T)(v) \) and \( D_t := \prod_{v \in d} s^\alpha_i(v) \). Let \( X_m^\alpha \) (resp. \( X_m^\alpha \)) be the complement of the zero set \( Z(D(T)) \) in \( X_m \) (resp. the complement of \( Z(D_t) \) in \( X_m^\alpha \)).

Lemma 4.11. The restriction of the map \( \pi_T : X_m \to \prod_{\alpha \in T} X^\alpha \) to the open set \( X_m^\alpha \) is a closed embedding to \( \prod_{\alpha \in T} (X^\alpha)^\circ \).

Proof. We only consider the special case when \( T \) consisting of two triangles because the general case is similar. By definition \( \pi_T \) maps \( X_m^\alpha \) to \( \prod_{\alpha \in T} (X^\alpha)^\circ \). We first show that \( \pi_T \) is injective when restricted to \( X_m^\alpha \). Let \( \alpha_1 = [a,b,c] \) and \( \alpha_2 = [b,d,c] \) be the two triangles in \( T \) meeting at a diagonal, say \( bc \). We will explicitly construct a section from the image \( \pi_T(X_m^\alpha) \). The image \( \pi_T(X_m^\alpha) \) in \( X_3^\alpha \times X_3^\beta \) can be represented by \( (A_a, A_b, A_c), (A_b, A_d, A_c) \in A^3 \) such that \( (A_a, A_c) \) is generic. By Corollary 4.6 the \( G \)-stabilizer of \( (A_a, A_c) \) is trivial. So the map \( ((A_a, A_b, A_c), (A_b, A_d, A_c)) \mapsto (A_a, A_b, A_d, A_c) \) descends to a well-defined morphism \( \pi_T(X_m^\alpha) \to X_m \), which is clearly a section of \( \pi_T \).
Lemma 4.12. The set $s_1(\mathbb{T})$ is algebraically independent over $k$.  

**Proof.** By Lemma 4.11, the functional field of $X_m$ is isomorphic to that of the image of $\pi_T$. So $k(X_m) = \pi^*k(\bigcap_{l \in T} X^3_3)$ is generated by $\bigcup_{l \in T} s_1^0 = s_1(\mathbb{T})$ as a field. The Krull dimension of the integral domain $k[X_m]$ is equal to 

$$m \dim A - \dim G = \frac{1}{2}(l-1)(l+1)(m-2) + (l-1)m.$$  

By a simple counting, this is exactly the cardinality of $s_1(\mathbb{T})$. Hence $s_1(\mathbb{T})$ must be algebraically independent over $k$. \qed

Lemma 4.13. The upper cluster algebra $\overline{\mathcal{C}}(\mathcal{O}_I(\mathbb{T}), s_1(\mathbb{T}))$ is a subalgebra of $SI_{\beta_l}(S^m_1)$.  

**Proof.** We will apply Lemma 4.10. For the first condition of CR1, we need to check that $s_1(\mathbb{T})(u)'$ lies in $SI_{\beta_l}(S^m_1)$ for each $u \in \mathcal{O}_I(\mathbb{T})_m$. For $u$ not on a diagonal, this is the content of [30] Lemma 8.10.  

For $u$ on a diagonal, we observe that all functions involved are supported on a quadruple flag, say labeled by $a, b, c, d$ with common edge $bc$ in $\mathbb{T}$. We claim that $(s^0_{a,b,c})'$ is the Schofield’s semi-invariant associated to the presentation 

$$P_1 \xrightarrow{(p^a_1, p^b_1, p^c_1)} P_1^a \oplus P_{j-1}^b \oplus P_{k-1}^c.$$  

We need to verify the exchange relation 

$$s^0_{a,b,c}(s^0_{a,b,c})' = s^0_{a,b,c} s^1_{1,j-1,k} s^1_{j-1,k-1} s^0_{1,j-1,k} + s^0_{a,b,c} s^1_{1,k-1} s^0_{1,j-1,k}.$$

It suffices to verify on the dense set, where the $b, c$-th flags are represented by $F^-$ and $F^+$. Let $M$ be such a general representation. Suppose that $M(p^a_1) = u$ and $M(p^d_1) = v$ for some row vectors $u$ and $v$ of length $l$. Then 

$$s^{b,d,c}_{1,j-1,k-1} = v(j + 1), \quad s^{a,b,c}_{1,j-1,k} = (-1)^{j-1}u(j),$$  

$$s^{b,d,c}_{j-1,k-1} = v(j), \quad s^{a,b,c}_{1,k-1} = (-1)^j u(j + 1),$$  

$$s^{a,b,c}_{0,j,k} = 1, \quad (s^{a,b,c}_{0,j,k})' = \det \begin{pmatrix} 1 & u & 0 \\ v & 0 & \ell_k \end{pmatrix} = (-1)^{j-1}(u(j)v(j + 1) - u(j + 1)v(j)).$$

We conclude that the exchange relation holds.  

Finally we need to verify the coprime condition. Each $s^{a,b,c}_{i,j,k}$ or $(s^{a,b,c}_{i,j,k})'$ has an extremal weight, and thus irreducible by Lemma 4.2. Since $SI_{\beta_l}(S^m_1)$ is factorial [30] Theorem 3.17, the coprime condition follows. \qed

Corollary 4.14. For any triangulation $\mathbb{T}$ of $D_m$, we have that 

$$SI_{\beta_l}(S^m_1)_{D(\mathbb{T})} = \overline{\mathcal{C}}(\mathcal{O}_I(\mathbb{T}), s_1(\mathbb{T}))_{D(\mathbb{T})}.$$
Proof. We already have from Lemma 4.13 the containment “⊆”. By Lemma 4.10 \( \mathcal{C}(\Delta_i(T), s_i(T))_{D(T)} \) is the same as \( \mathcal{C}(\Delta_i(T)^d, s_i(T))_{D(T)} \) where \( \Delta_i(T)^d \) is obtained from \( \Delta_i(T) \) by freezing all diagonal vertices. Then \( \mathcal{C}(\Delta_i(T), s_i(T))_{D(T)} \) contains all \( \mathcal{C}(\Delta_i^j, s_j^i) \) (\( t \in T \)). Recall from Theorem 4.9 that \( \mathcal{C}(\Delta_i^j, s_j^i) \) is equal to \( k[X_j] = \text{SI}_{\beta_i}(S_j^i) \). By Lemma 4.11 \( \text{SI}_{\beta_i}(S_j^m)_{D(T)} \) is generated by all \( \text{SI}_{\beta_i}(S_j^i) \) and inverse cluster variables on the diagonal \( d(T) \). We thus get the other containment “⊇”. □

**Lemma 4.15.** Let \( \mu := \mu^d \) be the sequence of mutations corresponding to the flip along a diagonal \( d \). Then up to a sign we have that \( \mu(s_i(T)) = s_i(T') \).

Proof. It suffices to prove for two adjacent triangles, say the left one of Figure 5. Let \( \sigma \) be another triangulation of \( D_m \) with \( s_i(T) \) and \( s_i(T') \). We already have from Lemma 4.13 the containment “⊆”. By Lemma 4.10 \( \mathcal{C}(\Delta_i(T), s_i(T))_{D(T)} \) is the same as \( \mathcal{C}(\Delta_i(T)^d, s_i(T))_{D(T)} \) where \( \Delta_i(T)^d \) is obtained from \( \Delta_i(T) \) by freezing all diagonal vertices. Then \( \mathcal{C}(\Delta_i(T), s_i(T))_{D(T)} \) contains all \( \mathcal{C}(\Delta_i^j, s_j^i) \) (\( t \in T \)). Recall from Theorem 4.9 that \( \mathcal{C}(\Delta_i^j, s_j^i) \) is equal to \( k[X_j] = \text{SI}_{\beta_i}(S_j^i) \). By Lemma 4.11 \( \text{SI}_{\beta_i}(S_j^m)_{D(T)} \) is generated by all \( \text{SI}_{\beta_i}(S_j^i) \) and inverse cluster variables on the diagonal \( d(T) \). We thus get the other containment “⊇”. □

**Theorem 4.16.** The semi-invariant ring \( \text{SI}_{\beta_i}(S_j^m) \) is equal to the upper cluster algebra \( \mathcal{C}(\Delta_i(T), s_i(T))_{D(T)} \) for any triangulation \( T \) of \( D_m \). Moreover, \( \langle \Delta_i(T), W_i(T) \rangle \) is a polyhedral cluster model.

Proof. Let \( T' \) be another triangulation of \( D_m \) such that \( T \) and \( T' \) do not share any diagonal. By Corollary 4.14 we have that \( \text{SI}_{\beta_i}(S_j^m)_{D(T)} = \mathcal{C}(\Delta_i(T), s_i(T))_{D(T)} \). The same statement holds if we replace \( T \) by \( T' \). By Lemma 4.15 we have that \( \mathcal{C}(\Delta_i(T), s_i(T))_{D(T)} = \mathcal{C}(\Delta_i(T'), s_i(T')) \). Let \( X := \text{Spec} (\text{SI}_{\beta_i}(S_j^m)) \) and \( Y := \text{Spec} (\mathcal{C}(\Delta_i(T), s_i(T))) \). We have a regular birational map \( X \to Y \) induced by the inclusion of Lemma 4.13. To show it is in fact an isomorphism, it suffices to show according to Lemma 4.10 that \( Z(D(T)) \cap Z(D(T')) \) has codimension 2 in \( X \). Recall that each function in \( s_i(T) \) and \( s_i(T') \) is irreducible. Since \( X \) is factorial \((\text{Kos})\) and all irreducible factors in \( D(T) \) and \( D(T') \) are distinct, we conclude that the intersection has codimension 2. The last statement was proved in Proposition 3.6. □

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