Exact Metric Operators as the Ground State functions of the Hermitian Conjugates of a Class of Quasi-Hermitian Hamiltonians

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Abstract

We generalized a class of non-Hermitian Hamiltonians which introduced previously by us in such a way in which every member in the class is non-$PT$-symmetric. For every member of the class, the ground state is a constant with zero energy eigen value. Instead of using an infinite set of coupled operator equations to calculate the metric operator we used a simple realization to obtain the class of closed form metric operators corresponding to the class of non-Hermitian and non-$PT$-symmetric Hamiltonians introduced. The trick is that, if $\psi$ is an eigen function of $H$, then $\phi = \eta \psi$ is an eigen function of $H^\dagger$ with the same eigen value. Thus, knowing any pair $(\psi, \phi)$ one can deduce the form of the exact metric operator. We note that, the class of Hamiltonians generalized in this work has the form of that of imaginary magnetic field which can be absorbed by the quasi-gauge transformations represented by metric operators. Accordingly, it is expected that the $Q$ operators will disappear for the whole members in the class in the path integral formulation. However, the detailed analysis of this issue will appear in another work.

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The recent growing of researchers interest in the quasi-Hermitian Hamiltonians led to the well establishment of such kind of theories \[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\]. Although there exists a Hermitian Hamiltonian which is equivalent to each non-Hermitian Hamiltonian with real spectrum, one can not conclude that such kind of research is redundant. In fact, there is a plenty of benefits in studying the theory in its non-Hermitian representation because in many cases it is simpler than the Hermitian representation. For instance, the non-Hermitian $\phi^4$ scalar field theory is a plausible candidate to play the role of the scalar sector in the standard model for particle interactions \[13\], while its Hermitian equivalent Hamiltonian is not known till now. Also, we have showed that the Hermitian $\phi^6$ field theory with coupling of negative two mass dimension, can be converted to an equivalent non-Hermitian representation in which the coupling has a negative one mass dimension \[14\] which is a very important result toward the solution of the unification problem. Besides, the equivalent non-Hermitian representation is a $\phi^4$-like theory for which the Feynman diagram calculations go more simpler than in the Hermitian $\phi^6$ field theory. According to these benefits, one can say that at least, the quasi-Hermitian representation can be used as a calculational algorithm in both quantum mechanics and quantum field theories. Also, the subject as a whole is needed to discover if a non-Hermitian theory is physically acceptable or not thorough the investigation of the possible existence of a positive definite metric operator.

The metric operator has a vital role in the study of quasi-Hermitian theories \[8, 9\]. However, there exist rare cases for which the metric operator has been obtained in an exact manner \[15, 17\]. Even in this case, one has to solve a set of an infinite number coupled operator equations and one needs to be lucky enough to have a truncation at a small order in the perturbation expansion used. In this letter, we introduce a some how different way to obtain the closed form metric operator for a generalized class of non-Hermitian Hamiltonians. We mean by generalized that, in this work we generalize the form of a new class of non-Hermitian Hamiltonians introduced previously by us. In the generalized form, every member is non-Hermitian as well as non-PT-symmetric rather than the previous new class in Ref.\[16\] for which half of the members are PT-symmetric.

To start, consider a Hamiltonian $H$ which has a set of eigen functions $\{\psi_n\}$. The Hamiltonian $H$ is said to be $\eta$-Pseudo Hermitian if $H^\dagger = \eta H \eta^{-1}$, where $\eta$ is a Hermitian invertible linear operator. In fact, the $\eta$ operator is not unique and if the set $\varepsilon(H)$, the set of all $\eta$’s, includes some $\eta$ such that the inner product defined by $\langle \psi | \eta | \psi \rangle = \langle \langle \psi | \psi \rangle \rangle_{\eta}$ is positive

\[2\]
definite, then $H$ is Hermitian in the Hilbert space endowed with the $\langle \langle \psi | \psi \rangle \rangle_\eta$ inner product and thus the spectrum is real \cite{6, 8, 9}.

To shed light on previous calculational procedures of the the positive definite operator $\eta_+$, consider the Hamiltonian $H$ such that $H|\psi_n\rangle = E_n|\psi_n\rangle$ then $H^\dagger|\phi_n\rangle = E_n|\phi_n\rangle$ and $\langle \phi_n | \psi_m \rangle = \delta_{nm}, \sum_n |\psi_n\rangle\langle \phi_n | = 1$.

The set $\{|\psi_n\rangle, \langle \phi_n |\}$ form a biorthonormal system and $H = \sum_n E_n|\psi_n\rangle\langle \phi_n |$, $H^\dagger = \sum_n E_n|\phi_n\rangle\langle \psi_n |$. In Ref. \cite{9}, it is deduced that the operator $\eta_+$ can be represented as

$$\eta_+ = \sum_n |\phi_n\rangle\langle \phi_n |,$$  \hspace{1cm} (1)

which is a positive-definite operator, satisfies the condition $H^\dagger = \eta_+H\eta_+^{-1}$ and is a difficult infinite sum too.

In Bender’s regime, the inner product is defined through the introduction of a $C$ operator, which is represented in the coordinate space as the sum

$$C(x, y) = \sum_n \phi_n(x)\phi_n(y),$$  \hspace{1cm} (2)

where $\{\phi_n(x)\}$ are the coordinate-space eigen functions of the Hamiltonian \cite{10}. Because it is impossible to use the sum in Eq. (2) for the calculation of the $C$ operator, Bender et.al introduced a powerful method for calculating $C$ by seeking an operator representation of $C$ in the from $C = \exp(-Q(x, p)P$, where $P$ is the parity operator. The calculation procedure is based on the observations that

$$[C, PT] = 0, \quad C^2 = 1 \quad \text{and} \quad [C, H] = 0.$$  \hspace{1cm} (3)

Accordingly, if we assume the Hamiltonian to take the form

$$H = H_0 + \epsilon H_1,$$  \hspace{1cm} (4)

and $Q(x, p)$ has the perturbative expansion

$$Q(x, p) = \epsilon Q_1(x, p) + \epsilon^3 Q_3(x, p) + \ldots$$  \hspace{1cm} (5)
then the first three operator equations are given by

\begin{align}
[H_0, Q_1] &= -2H_1, \quad (6) \\
[H_0, Q_3] &= -\frac{1}{6}[Q_1, [Q_1, H_1]], \quad (7) \\
[H_0, Q_5] &= \frac{1}{360}[Q_1, [Q_1, [Q_1, H_1]]] - \frac{1}{6}[Q_1, [Q_3, H_1]] - \frac{1}{6}[Q_3, [Q_1, H_1]]. \quad (8)
\end{align}

Using such a representation they were able to calculate $C$ up to $\epsilon^7$ for the non-Hermitian model

\[ H = \frac{1}{2}p^2 + \frac{1}{2}\mu^2 x^2 + i\epsilon x^3, \quad (9) \]

and up to $g^1$ for the quantum field version of the form

\[ H = \frac{1}{2} (\partial \phi)^2 + \frac{1}{2}m^2 \phi^2 + ig\phi^3. \quad (10) \]

For the $(-g\phi^4)$, we need a non-perturbative tool (WKB, for instance).

So, for most of the theories, one need to duplicate the effort to obtain a well established formulation of a non-Hermitian theory with real spectra. At one hand, we calculate the wave functions up to some order and on the other hand one has to calculate the metric operator or the $C$ operator up to the same order. Here we suggest another route for the calculation of the positive definite metric operator. In fact, a simple realization is that if we have a non-Hermitian Hamiltonian such that there exists a linear invertible operator $\eta$ satisfying

\[ \eta H \eta^{-1} = H^\dagger, \quad (11) \]

and if $\psi_n$ is an eigen function of $H$ with eigen value $E_n$ then

\[ H^\dagger (\eta \psi) = \eta H \eta^{-1} (\eta \psi) \quad (12) \]
\[ = \eta H \psi = E_n (\eta \psi), \quad (13) \]

i.e $\eta \psi$ is an eigen function of $H^\dagger$ with the same eigen value. Accordingly, by solving the Shrödinger equations for both $H$ and $H^\dagger$, and the through the relation

\[ \phi = \eta \psi, \quad (14) \]

where $\phi$ is the eigen function of $H^\dagger$, one can deduce the form of $\eta$. To clarify the point, consider the non-Hermitian Hamiltonian of the form
\[-\frac{d^2\psi}{dx^2} + (2ixp) \psi = E\psi, \quad (15)\]

where \(p\) is the momentum operator. Clearly, the constant \(c\) is an eigen function of zero eigen value. Now consider the the corresponding Shrödinger equation for \(H^\dagger\)

\[-\frac{d^2\phi}{dx^2} - (2ipx) \phi = E\phi, \quad (16)\]

which has the eigen function \(c \exp(-x^2)\). Accordingly, \(\eta = \exp(-x^2)\). To check that this really the metric operator consider

\[
\exp(-x^2) H \exp(x^2) = p^2 - 2ixp - 2 = H^\dagger. \quad (16)
\]

Also, consider the relation

\[
\exp\left(-\frac{x^2}{2}\right) H \exp\left(\frac{x^2}{2}\right) = p^2 + x^2 - 1, \quad (17)
\]

and thus the Hermitian Hamiltonian \(h = p^2 + x^2 - 1\) is equivalent to the non-Hermitian Hamiltonian \(p^2 + 2ixp\) which shows that \(\eta = \exp(-x^2)\) passed all the tests as a positive definite metric operator.

Now consider the non-Hermitian and non-PT-Symmetric Hamiltonian of the form

\[
H = \frac{1}{2}p^2 + i (gx^2 + \omega x + d) p, \quad (18)
\]

Clearly, the constant \(c\) represents the ground state function of \(H\). Moreover, the function \(\phi = c \exp(-\frac{2gx^3}{3} - \omega x^2 - 2dx)\) represents the ground state function for \(H^\dagger\) and the corresponding Hermitian Hamiltonian \(h\) takes the form

\[
h = \frac{1}{2}p^2 + \left(gd + \frac{1}{2}\omega^2\right) x^2 + \frac{1}{2}g^2x^4 + gm x^3 + (\omega d - g) x - \frac{1}{2}\omega + \frac{1}{2}d^2. \quad (19)
\]

To assure our result, let us consider one more example for the Hamiltonian

\[
H = \frac{1}{2}p^2 + i (fx^3 + gx^2 + \omega x + d) p, \quad (20)
\]

with the ground state function \(c\) and the the ground state

\[
\phi = c \exp(-\frac{1}{2}fx^4 - \frac{2}{3}gx^3 - \omega x^2 - 2dx). \quad (21)
\]

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for $H^\dagger$ and
\[
\begin{align*}
h &= \exp\left(-\frac{1}{2} f x^4 - \frac{2}{3} g x^3 - \omega x^2 - 2 dx\right) H \exp\left(-\frac{1}{2} f x^4 - \frac{2}{3} g x^3 - \omega x^2 - 2 dx\right) \\
&= \frac{1}{2} p^2 + \left(\frac{1}{2} \omega^2 + gd - \frac{3}{2} f\right) x^2 + (fd + g\omega) x^3 + \left(\frac{1}{2} g^2 + f\omega\right) x^4 + fg x^5 \\
&\quad + \frac{1}{2} f^2 x^6 + (\omega d - g) x - \frac{1}{2} \omega + \frac{1}{2} d^2.
\end{align*}
\]

In general, a non-Hermitian Hamiltonian of the form
\[
H_M = \frac{1}{2} p^2 + i \left(\sum_{n=0}^M a_{\pm n} x^{\pm n}\right) p,
\]
where $M$ is a positive integer, has a real spectrum and the exact metric operator for this Hamiltonian is given by
\[
\eta = \exp\left(-\sum_{n=0}^M \frac{2a_{\pm n} x^{\pm n+1}}{\pm n + 1}\right),
\]
and the corresponding Hermitian Hamiltonian takes the form;
\[
h_M = \exp\left(-\sum_{n=0}^M a_{\pm n} x^{\pm n+1}\right) H_M \exp\left(\sum_{n=0}^M \frac{a_{\pm n} x^{\pm n+1}}{\pm n + 1}\right).
\]

For example, when $M = 6$ and we take positive $n$ only in $H_M$ we get the equivalent Hermitian Hamiltonian of the form
\[
h_6 = \frac{1}{2} a_6^2 x^{12} + a_5 x^{11} a_6 + \left(\frac{1}{2} a_5^2 + a_4 a_6\right) x^{10} + (a_4 a_5 + a_3 a_6) x^9 + \left(a_2 a_6 + \frac{1}{2} a_4^2 + a_3 a_5\right) x^8
\]
\[
+ (a_3 a_4 + a_2 a_5 + a_1 a_6) x^7 + \left(a_0 a_6 + a_2 a_4 + \frac{1}{2} a_3^2 + a_1 a_5\right) x^6
\]
\[
+ (-3a_6 + a_0 a_5 + a_1 a_4 + a_2 a_3) x^5 + \left(\frac{1}{2} a_2^2 - \frac{5}{2} a_5 + a_1 a_3 + a_0 a_4\right) x^4
\]
\[
+ (a_1 a_2 + a_0 a_3 - 2a_4) x^3 + \left(-\frac{3}{2} a_3 + a_0 a_2 + \frac{1}{2} a_1^2\right) x^2 + (-a_2 + a_0 a_1) x
\]
\[
- \frac{1}{2} a_1 + \frac{1}{2} a_0^2 + \frac{1}{2} p^2.
\]

Note that, since the exact metric operator for the above mentioned Hamiltonians is a function of $x$ only then the more general Hamiltonian of the form
\[
H_M = \frac{1}{2} p^2 + i \left(\sum_{n=0}^M a_{\pm n} x^{\pm n}\right) p + f(x),
\]
where \( f(x) \) is a polynomial in \( x \) is quasi-Hermitian and have the metric operator

\[
\eta = \exp \left( - \sum_{n=0}^{M} \frac{2a_{\pm n}x^{\pm n+1}}{(\pm n + 1)} \right),
\]

(34)

and the equivalent Hermitian Hamiltonian is given by

\[
h_M = \exp \left( - \sum_{n=0}^{M} \frac{a_{\pm n}x^{\pm n+1}}{(\pm n + 1)} \right) H_M \exp \left( \sum_{n=0}^{M} \frac{a_{\pm n}x^{\pm n+1}}{(\pm n + 1)} \right).
\]

(35)

Accordingly, knowing only one wave function out of the spectra of both \( H \) and \( H^\dagger \) yields in an automatic way the closed form positive definite metric operator provided that both \( H \) and \( H^\dagger \) have the same real spectra.

The above formulations shows that, the non-Hermitian representations in quantum mechanics is a powerful calculational tool for certain cases as it turns out a Hermitian Hamiltonian which is a polynomial of order \( 2M \) in the creation and annihilation operators into an equivalent non-Hermitian Hamiltonian which is a polynomial of degree \( M \) in \( a \) and \( a^\dagger \). As we know, in perturbation calculations, the number of non-vanishing matrix elements increases as the highest power in \( a \) and \( a^\dagger \) increases and thus working in the non-Hermitian representation will lower the number of non-vanishing matrix elements and thus simplifies the calculations.

For the generalized class in this work, the ground state function of the non-Hermitian Hamiltonian is a constant for each member in the class with the ground state energy is zero. Accordingly, the ground state function for the corresponding Hermitian Hamiltonian can be taken as 

\[
\rho = \sqrt{\eta} = \exp \left( - \sum_{n=0}^{M} \frac{2a_{\pm n}x^{\pm n+1}}{2(\pm n + 1)} \right).
\]

Thus, for \( M \) even and for positive \( n \) values only, the ground state function is not square integrable and thus does not represent a true Physical state. On the other hand, for \( M \) odd, the ground state is square integrable.

What is amazing in this work that it relates Hermitian Hamiltonians of polynomial form to a non-Hermitian Hamiltonians with velocity dependent potentials. In fact, the non-Hermitian Hamiltonians in this work have the form of a theory of imaginary magnetic field. Moreover, the metric operator represents a quasi-quage transformation which adds a terminal term to the action and thus it is expected that the \( Q \) operator will disappear in the path integral formulation. However, we postpone such kind of investigations to another work.

To conclude, we generalized a form of a class of quasi-Hermitian theories in such a way that each member in the class is non-PT-symmetric. We used a simple method to obtain
the cosed form metric operator for each member in the class. In this method, since we realized that for each member the ground state is a constant, we solved the corresponding Schrödinger equation for $H^\dagger$ with $E = 0$, which then is nothing but the metric operator $\eta$. In fact, the Schrödinger equation for $H^\dagger$ have a simple shape of the well known Fokker-Plank equations for which one can get the exact solution for $E = 0$.

We assert that all the members of the class are real line theories and thus makes the calculations simple. Also, the interaction terms has the form of a particle in an imaginary magnetic field and the operator operator as well as the operator $\rho = \sqrt{\eta}$ represent a quasi-gauge transformations which means that one can work with any of the operator $H$, $H^\dagger$ and $\h$ in the path integral formulations with the results stay the same. We will make the explicit calculations regarding this issue and present it in another work.

The ground states for the class of Hermitian Hamiltonians $\h$ are not all physically acceptable. in fact, the ground state function for the corresponding Hermitian Hamiltonian can be taken as $\rho = \sqrt{\eta} = \exp \left( - \sum_{n=0}^{M} \frac{2a_+ x^n + 1}{2(x+1)} \right)$. Thus, for $M$ even and for positive $n$ values only, the ground state function are not square integrable and thus does not represent a true Physical state. On the other hand, for $M$ odd, the ground states are square integrable. Thus, it is not correct to consider each non-Hermitian with real spectrum to be Physically acceptable.

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