Dynamics of Nonground-State Bose-Einstein Condensates

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Abstract

Dilute Bose gases, cooled down to low temperatures below the Bose-Einstein condensation temperature, form coherent ensembles described by the Gross-Pitaevskii equation. Stationary solutions to the latter are topological coherent modes. The ground state, corresponding to the lowest energy level, defines the standard Bose-Einstein condensate, while the states with higher energy levels represent nonground-state condensates. The higher modes can be generated by alternating fields, whose frequencies are in resonance with the associated transition frequencies. The condensate with topological coherent modes exhibits a variety of nontrivial effects. Here it is demonstrated that the dynamical transition between the mode-locked and mode-unlocked regimes is accompanied by noticeable changes in the evolutionary entanglement production.

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1 Introduction

Bose-Einstein condensation, as is known, occurs when a macroscopic number of bosons piles down to the same lowest single-particle state. In equilibrium, this is the sole possible type of Bose-Einstein condensate. When the Bose-condensed system is rotated, vortices can appear. The Bose-Einstein condensate with vortices is the most known example of a nonground-state condensate. At low temperatures Bose gases are described by the Gross-Pitaevskii equation (see reviews \cite{1–3}). In the presence of a trapping potential, there arises a whole spectrum of energy levels associated with the stationary solutions to the Gross-Pitaevskii equation. Each of these stationary solutions is, by definition, a topological coherent mode \cite{4}. The lowest energy level corresponds to the standard Bose-Einstein condensate, while the higher states describe various nonground-state condensates. The
latter can be generated by alternating fields, whose frequencies should be in resonance with the related transition frequencies [4–8]. It is feasible to realize an oscillatory modulation of either the trapping potential or of the atomic scattering length. The trapping potential can be either single-well or multiwell.

Nonground-state Bose-Einstein condensates, with generated topological coherent modes, exhibit several interesting effects, such as interference fringes, interference current, mode locking, dynamical transition, critical phenomena, chaotic motion, harmonic generation, parametric conversion, and atomic squeezing [4–8]. The aim of the present communication is to study the behaviour of entanglement production in a trapped nonground-state condensate with topological coherent modes and to demonstrate that the properties of this entanglement production experience noticeable changes under the dynamical transition from the mode-locked regime to mode-unlocked regime.

## 2 Topological Coherent Modes

Dilute Bose-condensed gases at low temperature are described [1–3] by a coherent field $\varphi(r, t)$ satisfying the Gross-Pitaevskii equation

$$i\hbar \frac{\partial}{\partial t} \varphi(r, t) = \left(\hat{H}[\varphi] + \hat{V}\right) \varphi(r, t),$$

in which $\hat{V} = \hat{V}(r, t)$ is a modulating potential and the nonlinear Hamiltonian

$$\hat{H}[\varphi] = -\frac{\hbar^2}{2m} \nabla^2 + U(r) + N A_s |\varphi(r, t)|^2$$

contains a trapping potential $U(r)$ and the interaction intensity $A_s = 4\pi\hbar^2 a_s/m$ with $m$ being atomic mass; $a_s$, scattering length; and $N$, the total number of atoms.

The topological coherent modes are the solutions to the stationary Gross-Pitaevskii equation

$$\hat{H}[\varphi_n] \varphi_n(r) = E_n \varphi_n(r),$$

where $n$ is a labelling multi-index. The generation of topological modes is accomplished by transferring atoms from an occupied mode (say, ground state) to a higher mode by means of an alternating field

$$V(r, t) = \frac{1}{2} B(r)e^{i\omega t} + \frac{1}{2} B^*(r)e^{-i\omega t},$$

whose frequency $\omega$ is in resonance with a chosen transition frequency $\omega_{21} \equiv (E_2 - E_1)/\hbar$ between two selected energy levels, implying that the detuning $\Delta \omega \equiv \omega - \omega_{21}$ is small, $|\Delta \omega| \ll \omega$. Generally, it is admissible to apply several resonant fields of form (4) connecting different modes. The procedure of such a resonant mode generation has been thoroughly explained [4–8].
The solution to Eq. (1) is expressible as

$$\varphi(r, t) = \sum_n c_n(t) \varphi_n(r) \exp \left( -\frac{i}{\hbar} E_n t \right),$$

(5)

where $c_n(t)$ are functions of time to be found. Substituting expansion (5) into Eq. (1) makes it possible to obtain the equations for $c_n(t)$. The dynamics of two selected modes, connected by the resonant field (4), is described by the temporal dependence of the corresponding coefficients $c_1(t)$ and $c_2(t)$, whose moduli squared, $|c_n|^2$, define the fractional mode populations. Let us introduce the population difference

$$s \equiv |c_2|^2 - |c_1|^2$$

and the phase difference $x \equiv i \ln(c_1|c_2|/|c_1|^2)$. The dynamical properties of a given atomic system, prepared in a state with fixed initial conditions, can be governed by varying the amplitude $|\beta| \equiv (\varphi_1, B\varphi_2)/\hbar$ and the detuning $\Delta \omega \equiv \omega - \omega_{21}$ of the modulating field (4), with $(\varphi_1, \varphi_2)$ being a scalar product.

It is convenient to deal with dimensionless parameters normalizing $|\beta|$ and $\Delta \omega$ by the value $\alpha \sim A_s N/\hbar$ characterizing the atomic interaction strength. Then one deals with the parametric manifold $\{b, \delta\}$, in which $b \equiv |\beta|/\alpha$ is the dimensionless modulating-field amplitude and $\delta$ is the dimensionless detuning. On this parametric manifold, there is a critical line $\{b_c, \delta_c\}$ separating two qualitatively different regimes of motion. From one side of the critical line, there occurs the \textit{mode-locked regime}, when the fractional mode populations are locked in their initial half-planes, such that for all times $t$ one has either $0 \leq |c_n(t)|^2 \leq 1/2$ or $1/2 \leq |c_n(t)|^2 \leq 1$, depending on the initial value $c_n(0)$. And from another side of the critical line, the mode populations become unlocked, oscillating in the whole region $0 \leq |c_n(t)|^2 \leq 1$, which corresponds to the \textit{mode-unlocked regime}. In this way, by varying either $b$ or $\delta$, and crossing the critical line $\{b_c, \delta_c\}$, one can realize the \textit{dynamic transition} from one regime of motion to another, between the mode-locked and mode-unlocked regimes. On the plane of the variables $s$ and $x$, the dynamic transition happens when the initial point of a trajectory crosses a saddle separatrix given by the equation $s^2/2 - b\sqrt{1 - s^2}\cos x + \delta s = |b|$. Thus in Fig. 1 the initial point $\{-1, 0\}$, for the parameter $b < b_c$, is below the saddle separatrix, because of which the total related trajectory lies below the separatrix, which corresponds to the mode-locked regime of motion. In Fig. 2 the same initial point, but for $b > b_c$, is already above the separatrix, which results in the mode-unlocked regime.

**Figure 1:** Phase portrait on the plane $\{s, x\}$ for a two-mode condensate, with the parameters $\delta = -0.1, b = 0.35; b_c = 0.39821$.

**Figure 2:** Phase portrait on the plane $\{s, x\}$ for a two-mode condensate, with the parameters $\delta = -0.1, b = 0.49; b_c = 0.39821$.

### 3 Evolutional Entanglement Production

The dynamic transition, with changing regimes of motion, should also be accompanied by noticeable changes in all dynamical characteristics of the system, for instance, in
the behaviour of atomic squeezing [6]. Here we concentrate on the temporal behaviour of entanglement production in the considered system. Entanglement is an important characteristic that is assumed to be exploited for quantum information processing and quantum computation. We consider here the entanglement generated by the second-order density matrix $\hat{\rho}_2$ for the given coherent two-mode condensate. The entanglement generated by $\hat{\rho}_2$ can be described [9,10] by the entanglement-production measure $\varepsilon_2(t) \equiv \varepsilon(\hat{\rho}_2) = \log\left(\frac{||\hat{\rho}_2||_D}{||\hat{\rho}_2^\otimes 2||_D}\right)$, in which the logarithm is to base 2, $\hat{\rho}_2^\otimes$ is the nonentangling counterpart of $\hat{\rho}_2$ and $D$ is the set of disentangled states formed of the products $\varphi_m(r_1)\varphi_n(r_2)$. For the density matrix $\hat{\rho}_2$, we obtain $\varepsilon_2(t) = -\log \sup_n |c_n(t)|^2$.

Numerical calculations for the entanglement-productions measure $\varepsilon_2(t)$ are presented in Figs. 3 and 4 for the initial conditions $c_1(0) = 1$ and $c_2(0) = 0$. Fixing the zero detuning, we vary the modulating-field amplitude $b$. In Fig. 3, the latter is below the critical value $b_c$, while in Fig. 4, it is above $b_c$. As is seen, the entanglement production is rather different in the mode-locked regime, when $b < b_c$, and in the mode-unlocked regime, when $b > b_c$. The considered entanglement, generated in a trap, quantifies the amount of correlations between each pair of atoms. If atoms are released from the trap, say in the time-of-flight regime, then the studied measure $\varepsilon_2(t)$ will change in time, with the entanglement produced in a trap serving as the initial condition. Depending on the latter, the pair correlations of outcoupled atoms will be different and could be regulated by preparing the required entanglement production of trapped atoms. A two-mode system is mathematically similar to two-level atoms or to $1/2$ spins. Therefore such a system can also be employed for realizing quantum gates, which can be done by switching on and off the external field. Thus, by varying the amplitude of the modulating field one can essentially modify the characteristics of entanglement production, which could be employed for information processing.

Figure 1: Evolution of the entanglement produced by the second-order density matrix for $\delta = 0$, $b = 0.3$. Time is measured in units of $\alpha$.

In conclusion, the nonground-state Bose-Einstein condensates, with resonantly generated topological coherent modes, are the objects whose dynamical properties can be straightforwardly regulated by external modulating fields. By means of such fields, it is feasible to switch the dynamic behaviour between two qualitatively different regimes, the
Figure 2: Entanglement production of the second-order density matrix as a function of time, measured in units of $\alpha$, for $\delta = 0, b = 0.7$.

mode-locked and mode-unlocked regimes. This change-over from one regime to another allows one to radically alter the system evolution, with the related changes in the behaviour of the fractional mode populations, atomic squeezing, and entanglement production.

References

[1] A.S. Parkins and D.F. Walls, Phys. Rep. 303, 1 (1998).
[2] P.W. Courteille, V.S. Bagnato, and V.I. Yukalov, Laser Phys. 11, 659 (2001).
[3] L.P. Pitaevskii and S. Stringari, Bose-Einstein Condensation (Oxford University, Oxford, 2003).
[4] V.I. Yukalov, E.P. Yukalova, and V.S. Bagnato, Phys. Rev. A 56, 4845 (1997).
[5] K.P. Marzlin and W. Zhang, Phys. Rev. A 57, 4761 (1998).
[6] V.I. Yukalov, E.P. Yukalova, and V.S. Bagnato, Phys. Rev. A 66, 043602 (2002).
[7] V.I. Yukalov and E.P. Yukalova, J. Phys. A 35, 8603 (2002).
[8] V.I. Yukalov, K.P. Marzlin, and E.P. Yukalova, Phys. Rev. A 69, 023620 (2004).
[9] V.I. Yukalov, Phys. Rev. Lett. 90, 167905 (2003).
[10] V.I. Yukalov, Phys. Rev. A 68, 022109 (2003).