Well-Posedness of Boundary Layer Equations for Time-Dependent Flow of Non-Newtonian Fluids

Michael Renardy and Xiaojun Wang

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Abstract. We consider the flow of an upper convected Maxwell fluid in the limit of high Weissenberg and Reynolds number. In this limit, the no-slip condition cannot be imposed on the solutions. We derive equations for the resulting boundary layer and prove the well-posedness of these equations. A transformation to Lagrangian coordinates is crucial in the argument.

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1. Introduction

Classical fluid mechanics is based on the Navier-Stokes equations supplemented by a no-slip boundary condition on walls. In the limit of zero viscosity, the Euler equations are obtained. However, the Euler equations do not allow for a no-slip boundary condition, and only the normal component of the velocity can be prescribed to be zero on a wall. It was Prandtl’s fundamental insight more than a century ago [15] that, for many high Reynolds number flows, the Euler equations provide an adequate description except in a thin layer close to the boundary, which is called a boundary layer. By taking advantage of the thinness of this layer, the Navier-Stokes equations can formally be reduced to the system which is now known as the Prandtl equations. For two-dimensional flow, and a boundary placed at $y = 0$, these equations take the form

$$
\begin{align*}
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= \nu \frac{\partial^2 u}{\partial y^2} - \frac{\partial p}{\partial x}, \\
\frac{\partial p}{\partial y} &= 0, \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\
u(x, y, 0) &= u_0(x, y), \\
0(x, y, 0, t) &= v(x, 0, t) = 0, \\
0(x, \infty, t) &= u^\infty(x, 0, t).
\end{align*}
$$

(1)

Here $u, v$ are velocities in $x, y$ directions, $p$ is pressure, $\nu$ is viscosity, and $u^\infty(x, y, t)$ represents the given flow in the core region.

One might hope to obtain a simplified procedure for solving high Reynolds number flow problems by solving the Euler equations (or even the simpler special case of potential flow) in the core of the flow domain, and then solving the Prandtl equations near the boundary. This program, however, runs into

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difficulties related to the question of well-posedness of the Prandtl equations. Oleinik [14] established a well-posedness result under the assumption that the velocity profile in the boundary layer is monotone. Sammartino and Caflisch [19] established an existence result for analytic initial data. We also refer to the review article of Weinan [1] for further work prior to 2000. Recently, Gérard-Varet and Dormy [4] established that, for general initial data, the Prandtl equations are not well-posed in Sobolev spaces.

Viscoelastic flows exhibit phenomena of instability which share many characteristics of turbulence [5]. In this paper, we shall focus on the upper convected Maxwell model for the viscoelastic flow. In the limit of high elasticity, a limiting equation can be derived which is similar to the system of ideal magnetohydrodynamics and, like the Euler equations, does not allow the imposition of a no-slip boundary condition. The well-posedness of this system has been established in [22]. The goal of this manuscript is to supplement this analysis with a study of the well-posedness of the accompanying boundary layer equations.

We note that the ill-posedness of the Prandtl equations [4] is linked to shear flow instabilities. Elasticity has a stabilizing effect on high Reynolds number flow instabilities [6,7,12,13,17]. This stabilizing effect can restore well-posedness of the hydrostatic approximation in situations where this approximation is ill-posed for the Euler equations [18]. We may therefore hope to establish well-posedness in the boundary layer system as well. Indeed, this turns out to be the case. We shall show that a transformation into Lagrangian coordinates transforms the boundary layer system into a semilinear wave equation for which well-posedness can be readily established.

2. The High Weissenberg Number Limit and the Issue of Boundary Layers

We start with the upper convected Maxwell model in dimensionless form:

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \frac{1}{R} \nabla \cdot \mathbf{T} - \nabla p, \\
\frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{T} - (\nabla \mathbf{v})^T \mathbf{T} = \frac{1}{W} (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) + \frac{1}{W} \mathbf{T},
\]

where \( R \) is the Reynolds number and \( W \) is the Weissenberg number measuring the elasticity. Since normal stresses in shear flow are of order \( W \) rather than order 1, we shall also scale the stresses with an additional factor \( W \) and obtain

\[
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = E \nabla \cdot \mathbf{T} - \nabla p, \\
\frac{\partial \mathbf{T}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{T} - (\nabla \mathbf{v})^T \mathbf{T} = \frac{1}{W} (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) + \frac{1}{W^2} \nabla (\nabla \mathbf{v} + (\nabla \mathbf{v})^T). 
\]

Here \( E = W/R \) is the elasticity number. We are interested in a limit where \( W \) and \( R \) tend to infinity simultaneously, but \( E \) remains fixed.

If we formally set \( W = \infty \) above, we obtain the limiting system

\[
\frac{\partial \mathbf{v}^0}{\partial t} + (\mathbf{v}^0 \cdot \nabla) \mathbf{v}^0 = E \nabla \cdot \mathbf{T}^0 - \nabla p^0, \\
\frac{\partial \mathbf{T}^0}{\partial t} + (\mathbf{v}^0 \cdot \nabla) \mathbf{T}^0 - (\nabla \mathbf{v}^0)^T \mathbf{T}^0 = \mathbf{T}^0 (\nabla \mathbf{v}^0)^T = 0, \\
\nabla \cdot \mathbf{v}^0 = 0.
\]

In [22], we proved the well-posedness of this system. We considered the initial-boundary value problem in a smooth domain \( \Omega \), subject to initial conditions for \( \mathbf{v}^0 \) and \( \mathbf{T}^0 \), and the boundary condition \( \mathbf{v}^0 \cdot \mathbf{n} = 0 \). A crucial assumption was that \( \mathbf{T}^0 \cdot \mathbf{n} = 0 \); it can be shown that the equations preserve this condition if it is satisfied initially. The physical background behind this assumption is that the local flow near a solid wall is always a shear flow, and at high Weissenberg number the extra stress is dominated by the first normal stress, i.e. the stress component tangent to the wall.