REMARKS ON BLOW-UP PHENOMENA IN p–LAPLACIAN HEAT EQUATION WITH INHOMOGENEOUS NONLINEARITY

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ABSTRACT. We investigate the $p$–Laplace heat equation $u_t - \Delta_p u = \zeta(t)f(u)$ on a bounded smooth domain $\Omega \subset \mathbb{R}^N$. Using differential inequalities arguments, we prove blow-up results under suitable conditions on $\zeta$, $f$, and the initial data $u_0$. We also give an upper bound for the blow-up time in each case.

1. INTRODUCTION

In the past decade a strong interest in the phenomenon of blow-up of solutions to various classes of nonlinear parabolic problems has been assiduously investigated. We refer the reader to the books [16, 15] as well as to the survey paper [1]. Problems with constant coefficients were investigated in [13], and problems with time-dependent coefficients under homogeneous Dirichlet boundary conditions were treated in [11]. See also [12] for a related system. The question of blow-up for nonnegative classical solutions of $p$–Laplacian heat equations with various boundary conditions has attracted considerable attention in the mathematical community in recent years. See for instance [2, 9, 10, 8].

There are two effective techniques which has been employed to prove non-existence of global solutions: the concavity method ([7]) and the eigenfunction method ([6]). The later one was firstly used for bounded domains but can be adapted to the whole space $\mathbb{R}^N$. The concavity method and its variants were used in the study of many nonlinear evolution partial differential equations (see e.g. [3, 4, 14]).

In the present paper, we investigate the blow-up phenomena of solutions to the following nonlinear $p$–Laplacian heat equation:

\[
\begin{aligned}
    u_t - \Delta_p u &= \zeta(t)f(u), \quad x \in \Omega, \quad t > 0, \\
    u(t, x) &= 0, \quad x \in \partial\Omega, \quad t > 0, \\
    u(0, x) &= u_0(x), \quad x \in \Omega,
\end{aligned}
\]  

(1.1)

where $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$–Laplacian operator, $p \geq 2$, $\Omega$ is a bounded sufficiently smooth domain in $\mathbb{R}^N$, $\zeta(t)$ is a nonnegative continuous function. The nonlinearity $f(u)$ is assumed to be continuous with $f(0) = 0$. More specific assumptions on $f$, $\zeta$ and $u_0$ will be made later.

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The case of $p = 2$ has been studied in [13] for $\zeta(t) \equiv 1$, and in [11] for $\zeta$ being a non-constant function of $t$. Concerning the case $p > 2$, Messaoudi [10] proved the blow-up of solutions with vanishing initial energy when $\zeta(t) \equiv 1$. See also [9] and references therein.

Recently, a $p$−Laplacian heat equations with nonlinear boundary conditions and time-dependent coefficient was investigated in [2]. This note may be regarded as a complement, and in some sense an improvement, of [11, 10].

Let us now precise the assumptions on $f$ and $\zeta$. If $p = 2$, we suppose either

\begin{equation}
 f \in C^1(\mathbb{R}) \quad \text{is convex with} \quad f(0) = 0; \tag{1.2}
\end{equation}

\begin{equation}
 \exists \quad \lambda > 0 \quad \text{such that} \quad f(s) > 0 \quad \text{for all} \quad s \geq \lambda; \tag{1.3}
\end{equation}

\begin{equation}
 \int_{\infty}^{\infty} \frac{ds}{f(s)} < \infty; \tag{1.4}
\end{equation}

\begin{equation}
 \inf_{t \geq 0} \left( \int_{0}^{t} (\zeta(s) - 1) ds \right) := m \in (-\infty, 0]. \tag{1.5}
\end{equation}

or

\begin{equation}
 sf(s) \geq (2 + \epsilon) F(s) \geq C_0 |s|^\alpha, \tag{1.6}
\end{equation}

for some constants $\epsilon, C_0 > 0$, $\alpha > 2$, and

\begin{equation}
 \zeta \in C^1([0, \infty)) \quad \text{with} \quad \zeta(0) > 0 \quad \text{and} \quad \zeta' \geq 0. \tag{1.7}
\end{equation}

Here $F(s) = \int_{0}^{s} f(\tau) d\tau$.

Our first main result concerns the case $p = 2$ and reads as follows.

**Theorem 1.1.** Suppose that assumptions $(1.2)$-$(1.3)$-$(1.4)$-$(1.5)$ are fulfilled. Let $0 \leq u_0 \in L^\infty(\Omega)$ satisfy $\int_{\Omega} u_0 \phi_1$ is large enough. Then the solution $u(t, x)$ of problem $(1.1)$ blows up in finite time.

**Remark 1.2.**

(i) The function $\phi_1$ stands for the eigenfunction of the Dirichlet-Laplace operator associated to the first eigenvalue $\lambda_1 > 0$, that is

\begin{equation}
 \Delta \phi_1 = -\lambda_1 \phi_1, \quad \phi_1 > 0, \quad x \in \Omega; \quad \phi_1 = 0, \quad x \in \partial\Omega, \quad \int_{\Omega} \phi_1 = 1.
\end{equation}

(ii) The assumptions $(1.2)$-$(1.3)$-$(1.4)$-$(1.5)$ on $f$ and $\zeta$ cover the example

\begin{equation}
 f(u) = e^u - 1 \quad \text{and} \quad \zeta(t) = e^{t^2}. \tag{1.8}
\end{equation}

Note that this example is not studied in [11], and Theorem 1.1 can be seen as an improvement of Theorem 1 of [11].
(iii) As it will be clear in the proof below, an upper bound of the maximal time of existence is given by

\[ T^* = -m + 2 \int_{y_0}^{\infty} \frac{ds}{f(s)}, \quad (1.9) \]

where \( m \) is as in (1.5) and \( y_0 = e^{\mu \lambda_1} \int u_0 \phi_1 \).

(iv) The conclusion of Theorem 1.1 remains valid for \( \Omega = \mathbb{R}^N \) if we replace \( \phi_1 \) by \( \varphi(x) = \pi^{-N/2} e^{-|x|^2} \).

In order to state our next result (again for \( p = 2 \)), we introduce the energy functional

\[ E(u(t)) := \frac{1}{2} \int_{\Omega} |\nabla u(t, x)|^2 dx - \zeta(t) \int_{\Omega} F(u(t, x)) dx. \quad (1.10) \]

Using (1.7), we see that \( t \mapsto E(u(t)) \) is nonincreasing along any solution of (1.1). This leads to the following.

**Theorem 1.3.** Suppose that assumptions (1.6)-(1.7) are fulfilled. Assume that either \( E(u_0) \leq 0 \) or \( E(u_0) > 0 \) and \( \|u_0\|_2 \) is large enough. Then the corresponding solution \( u(t, x) \) blows up in finite time.

**Remark 1.4.** An upper bound for the blow-up time is given by

\[ T^* = \begin{cases} 
\frac{(2+\epsilon)|\Omega|^{\alpha/2-1}||u_0||_2^{2-\alpha}}{\epsilon \zeta(0) C_0 (\alpha-2)} & \text{if } E(u_0) \leq 0, \\
\int \frac{dz}{Az^{\alpha/2} - 2E(u_0)} & \text{if } E(u_0) > 0, 
\end{cases} \quad (1.11) \]

where

\[ A = \frac{2^{\alpha/2} C_0 \epsilon \zeta(0)}{(2+\epsilon)|\Omega|^{\alpha/2-1}}. \]

We turn now to the case \( p > 2 \). In [5], the author studied (1.1) when \( \zeta(t) \equiv 1 \). He established:

- local existence when \( f \in C^1(\mathbb{R}) \);
- global existence when \( uf(u) \lesssim |u|^q \) for some \( q < p \);
- nonglobal existence under the condition

\[ \frac{1}{p} \int_{\Omega} |\nabla u_0|^p dx - \int_{\Omega} F(u_0) dx < 0. \quad (1.12) \]

Later on Messaoudi [10] improved the condition (1.12) by showing that blow-up can be obtained for vanishing initial energy. Note that by adapting the arguments used in [5], we can show a local existence result as stated below.
**Theorem 1.5.** Suppose \( \zeta \in C([0, \infty]) \) and \( f \in C(\mathbb{R}) \) satisfy \(|f| \leq g\) for some \( C^1 \)-function \( g \). Then for any \( u_0 \in L^\infty(\Omega) \cap W^{1,p}_0(\Omega) \), the problem (1.1) has a local solution

\[
u \in L^\infty((0, T) \times \Omega) \cap L^p((0, T); W^{1,p}_0(\Omega)), \quad \nu_t \in L^2((0, T) \times \Omega).
\]

The energy of a solution \( u \) is

\[
E_p(u(t)) = \frac{1}{p} \int_{\Omega} |\nabla u(t, x)|^p dx - \zeta(t) \int_{\Omega} F(u(t, x)) dx.
\]

We also define the following set of initial data

\[
\mathcal{E} = \left\{ u_0 \in L^\infty(\Omega) \cap W^{1,p}_0(\Omega); \quad u_0 \not\equiv 0 \quad \text{and} \quad E_p(u_0) \leq 0 \right\}.
\]

Our main result concerning \( p > 2 \) ca be stated as follows.

**Theorem 1.6.** Suppose that assumption (1.7) is fulfilled. Let \( f \in C(\mathbb{R}) \) satisfy \(|f| \leq g\) for some \( C^1 \)-function \( g \) and

\[
0 \leq \kappa F(u) \leq uf(u), \quad \kappa > p > 2.
\]

Then for any \( u_0 \in \mathcal{E} \) the solution \( u(t, x) \) of (1.1) given in Theorem 1.5 blows up in finite time.

**Remark 1.7.** Although the proof uses the Poincaré inequality in a crucial way, we believe that a similar result can be obtained for \( \Omega = \mathbb{R}^N \). This will be investigated in a forthcoming paper.

We stress that the set \( \mathcal{E} \) is non empty as it is shown in the following proposition.

**Proposition 1.8.** Suppose that assumption (1.15) is fulfilled and \( \zeta(0) > 0 \). Then \( \mathcal{E} \neq \emptyset \).

2. Proofs

This section is devoted to the proof of Theorems 1.1-1.3-1.6 as well as Proposition 1.8. 2.1. **Proof of Theorem 1.1.** The main idea in the proof is to define a suitable auxiliary function \( y(t) \) and obtain a differential inequality leading to the blow-up. Define the function \( y(t) \) as

\[
y(t) = a(t)^{\int_{\Omega} u(t, x) \phi_1(x) dx},
\]

where

\[
a(t) = e^{\lambda_1(m-\Theta(t))},
\]

and

\[
\Theta(t) = \int_0^t (\zeta(s) - 1) \, ds.
\]
We compute
\[
y'(t) = \frac{a'(t)}{a(t)} y(t) - \lambda_1 y(t) + a(t) \zeta(t) \int_{\Omega} f(u(t, x)) \phi_1(x) \, dx
\]
\[
= -\lambda_1 \zeta(t) y(t) + a(t) \zeta(t) \int_{\Omega} f(u(t, x)) \phi_1(x) \, dx,
\]
where we have used \( \frac{a'}{a} - \lambda_1 = -\lambda_1 \zeta \). By using (1.2) and the fact that \( 0 \leq a \leq 1 \), we easily arrive at
\[
y'(t) \geq \zeta(t) (-\lambda_1 y(t) + f(y(t))). \tag{2.4}
\]
Since \( f \) is convex and due to (1.4), there exists a constant \( C \geq \lambda \) such that \( f(s) \geq 2\lambda_1 s \) for all \( s \geq C \). Suppose \( y(0) > C \). It follows from (2.4) that, as long as \( u \) exists, \( y(t) \geq C \). Therefore
\[
y(t) \geq \frac{\zeta(t)}{2} f(y(t)).
\]
Hence
\[
\frac{t + m}{2} \leq \frac{1}{2} \int_0^t \zeta(s) \, ds \leq \int_{y(0)}^\infty \frac{ds}{f(s)} < \infty.
\]
This means that the solution \( u \) cannot exist globally and leads to the upper bound given by (1.9).

2.2. **Proof of Theorem 1.3.** Let \( y(t) \) be the auxiliary function defined as follows
\[
y(t) = \frac{1}{2} \int_{\Omega} u^2(t, x) \, dx.
\]
We compute
\[
y'(t) = \int_{\Omega} u (\Delta u + \zeta(t) f(u)) \, dx
\]
\[
= -\int_{\Omega} |\nabla u|^2 \, dx + \zeta(t) \int_{\Omega} u f(u) \, dx
\]
\[
= -2E(u(t)) + \zeta(t) \int_{\Omega} (uf(u) - F(u)) \, dx,
\]
where \( E(u(t)) \) is given by (1.13). Taking advantage of (1.6), we obtain that
\[
y'(t) \geq -2E(u(t)) + \frac{\epsilon C_0}{2 + \epsilon} \zeta(t) \int_{\Omega} |u|^\alpha \, dx. \tag{2.5}
\]
Moreover, we compute
\[
E'(u(t)) = -\int_{\Omega} u_t^2 \, dx - \zeta'(t) \int_{\Omega} F(u) \, dx \leq 0, \tag{2.6}
\]
thanks to (1.7). It then follows that \( E(u(t)) \) is non-decreasing in \( t \) so that we have

\[
E(u(t)) \leq E(u(0)) = E(u_0), \quad t \geq 0. \tag{2.7}
\]

From (2.5), (2.7), and the Hölder inequality, we find that

\[
y'(t) \geq -2E(u_0) + \frac{\epsilon \zeta(0) C_0 2^{\alpha/2}}{(2 + \epsilon)|\Omega|^{\alpha/2-1}} y(t)^{\alpha/2}. \tag{2.8}
\]

To conclude the proof we use the following result.

**Lemma 2.1.** Let \( y : [0, T) \to [0, \infty) \) be a \( C^1 \) function satisfying

\[
y'(t) \geq -C_1 + C_2 y(t)^p, \tag{2.9}
\]

for some constants \( C_1 \in \mathbb{R}, C_2 > 0, p > 1 \). Then

\[
T \leq \begin{cases}
\frac{y_1 - y(0)}{C_2^{p-1}} & \text{if } C_1 \leq 0, \\
\int_{y(0)}^{\infty} \frac{dz}{C_2 z^p - C_1} & \text{if } C_1 > 0.
\end{cases} \tag{2.10}
\]

2.3. **Proof of Theorem 1.6.** We define

\[
H(t) = \zeta(t) \int_{\Omega} F((u(t, x)) dx - \frac{1}{p} \int_{\Omega} |\nabla u(t, x)|^p dx, \tag{2.11}
\]

and

\[
L(t) = \frac{1}{2} \|u(t)\|_2^2. \tag{2.12}
\]

By using (1.1), we obtain that

\[
H'(t) = \int_{\Omega} u_t^2(t, x) dx + \zeta'(t) \int_{\Omega} F(u(t, x)) dx
= \frac{\zeta'(t)}{\zeta(t)} H(t) + \int_{\Omega} u_t^2(t, x) dx + \frac{\zeta'(t)}{p \zeta(t)} \int_{\Omega} |\nabla u(t, x)|^p dx
\geq \frac{\zeta'(t)}{\zeta(t)} H(t).
\]

Hence \( H(t) \geq H(0) \geq 0 \) by virtue of (1.7).
Recalling (1.1), (2.11), and (1.15), we compute

\[ L'(t) = - \int_\Omega |\nabla u(t, x)|^p \, dx + \zeta(t) \int_\Omega u(t, x) f(u(t, x)) \, dx \]

\[ \geq - \int_\Omega |\nabla u(t, x)|^p \, dx + \kappa \zeta(t) \int_\Omega F(u(t, x)) \, dx \]

\[ \geq \kappa H(t) + \left( \frac{\kappa}{p} - 1 \right) \int_\Omega |\nabla u(t, x)|^p \, dx. \]

Applying Hölder inequality and then Poincaré inequality yields

\[ L(t) \leq |\Omega|^{1-2/p} \left( \int_\Omega |u(t, x)|^p \, dx \right)^{2/p} \leq C \left( \int_\Omega |\nabla u(t, x)|^p \, dx \right)^{2/p}, \]

where \( C > 0 \) is a constant depending only on \( \Omega \) and \( p \). Hence

\[ L'(t) \geq \frac{\kappa - p}{p} L^{p/2}(t). \]  \hspace{1cm} (2.13)

Integrating the differential inequality (2.13) leads to

\[ t \leq \frac{2pC^{p/2}L^{1-p/2}(0)}{(p-2)(\kappa - p)} < \infty. \]

Therefore \( u \) blows up at a finite time \( T^* \leq \frac{2pC^{p/2}L^{1-p/2}(0)}{(p-2)(\kappa - p)}. \)

2.4. Proof of Proposition 1.8. Recalling (1.15), we obtain that

\[ F(u) \geq C u^\kappa, \quad \text{for all} \quad u \geq 1, \]  \hspace{1cm} (2.14)

for some constant \( C > 0 \). Let \( K \subset \Omega \) be a compact nonempty subset of \( \Omega \). Fix a smooth cut-off function \( \phi \in C^\infty(\Omega) \) such that

\[ \phi(x) = 1 \quad \text{for} \quad x \in K. \]

We look for initial data \( u_0 = \lambda \phi \) where \( \lambda > 0 \) to be chosen later. Clearly \( u_0 \in L^\infty(\Omega) \cap W_0^{1,p}(\Omega) \), and for \( \lambda \geq 1 \) we have using (2.14)

\[ E_p(u_0) = \frac{1}{p} \int_\Omega |\nabla u_0|^p - \zeta(0) \int_\Omega F(u_0), \]

\[ = \frac{\lambda^p}{p} \int_\Omega |\nabla \phi|^p - \zeta(0) \int_K F(\lambda) - \zeta(0) \int_{\Omega \setminus K} F(u_0), \]

\[ \leq \frac{\lambda^p}{p} \int_\Omega |\nabla \phi|^p - \tilde{C} \lambda^\kappa, \]
for some constant $\tilde{C} > 0$. Since $\frac{\lambda^p}{p^p} \int_{\Omega} |\nabla \phi|^p - \tilde{C} \lambda^p \leq 0$ for $\lambda \geq \left( \frac{\|\nabla \phi\|_p^p}{pC} \right)^{1/(\kappa-p)}$, we deduce that $u_0 \in \mathcal{E}$ for $\lambda$ large enough. This finishes the proof of Proposition 1.8.

References

[1] C. Bandle and H. Brunner, *Blow-up in diffusion equations: a survey*, J. Computat. Appl. Math., 97 (1998), 3–22.

[2] J. Ding and X. Shen, *Blow-up in p–Laplacian heat equations with nonlinear boundary conditions*, Z. Angew. Math. Phys. 67, 125 (2016).

[3] V. A. Galaktionov and S. I. Pohozaev, *Existence and blow-up for higher-order semilinear parabolic equations: majorizing order-preserving operators*, Indiana Univ. Math. J., 51 (2002), 1321–1338.

[4] V. A. Galaktionov and S. I. Pohozaev, *Blow-up and critical exponents for nonlinear hyperbolic equations*, Nonlinear Anal., 53 (2003), 453–466.

[5] Z. Junning, *Existence and nonexistence of solutions for $u_t = \text{div}(|\nabla u|^{p-2} \nabla u) + f(\nabla u, u, x, t)$*, J. Math. Anal. Appl. 172 (1993) 130–146.

[6] S. Kaplan, *On the growth of solutions of quasi-linear parabolic equations*, Comm. Pure Appl. Math. 16 (1963), 305–330.

[7] H. A. Levine, *Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $P_{\lambda}u_t = -Au + F(u)$*, Arch. Rational Mech. Anal. 51 (1973), 371–386.

[8] M. Liao, Q. Liu and H. Ye, *Global existence and blow-up of weak solutions for a class of fractional p–Laplacian evolution equations*, 1

[9] W. Liu and M. Wang, *Blow-up of the Solution for a p–Laplacian Equation with Positive Initial Energy*, Acta Appl. Math., 103 (2008), 141–146.

[10] S. A. Messaoudi, *A note on blow up of solutions of a quasilinear heat equation with vanishing initial energy*, J. Math. Anal. Appl., 273 (2002), 243–247.

[11] L. E. Payne and G. A. Philippin, *Blow-up phenomena in parabolic problems with time dependent coefficients under Dirichlet boundary conditions*, Proceedings of the American Mathematical Society, 141 (2013), 2309–2318.

[12] L. E. Payne and G. A. Philippin, *Blow-up phenomena for a class of parabolic systems with time-dependent coefficients*, Applied Mathematics, 3 (2012), 325–330.

[13] L. E. Payne and P. W. Schaefer, *Lower bound for blow-up time in parabolic problems under Neumann conditions*, Applic. Analysis, 85 (2006), 1301–1311.

[14] P. Pucci and J. Serrin, *Global nonexistence for abstract evolution equations with positive initial energy*, J. Differential Equations, 150 (1998), 203–214.

[15] P. Quittner and P. Souplet, *Superlinear parabolic problems*, Birkhäuser Verlag, Basel (2007), xii+584.

[16] B. Straughan, *Explosive instabilities in mechanics*, Springer (1998).
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