Wadati-Konno-Ichikawa-Type Integrable Systems and their Constructions

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Abstract
A standard form of Wadati-Konno-Ichikawa (WKI)-type integrable systems is derived from an $sl(2,\mathbb{R})$-valued spectral problem. Each equation in the resulting hierarchy has a bi-Hamiltonian structure furnished by the trace identity. Then, the higher grading affine algebraic construction of some special cases is proposed. We also show that the generalized short pulse equation arises naturally from the negative WKI flow.

Keywords WKI integrable hierarchy · Bi-Hamiltonian structure · Higher grading structure · Short pulse equation

Mathematics Subject Classification 37K05 · 37K10 · 35Q53

1 Introduction
Integrable systems have attracted extensive attention in natural science because of the successful description and explanation of nonlinear phenomena. The matrix spectral problems associated with Lie algebras are crucial to constructing integrable systems. There has been much work on generating integrable systems from the matrix spectral problems. Interesting examples contain the Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy, the Kaup-Newell (KN) hierarchy, the Wadati-Konno-Ichikawa (WKI) hierarchy, the Korteweg-de Vries (KdV) hierarchy, the modified KdV hierarchy, the Benjamin-Ono hierarchy, the Boiti-Pempinelli-Tu (BPT) hierarchy, the Dirac hierarchy and the coupled Harry-Dym hierarchy [1–15]. These integrable systems usually possess nice properties, such as having hereditary recursion operators, being multi-Hamiltonian, and carrying infinitely many commuting symmetries and conservation laws. The so-called trace identity (or variational identity) provides

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a systematic construction approach for establishing the Hamiltonian structures [16–19].

The WKI-type integrable systems [3, 9–13, 20, 21] not only represent the classical WKI equation [3],

\[ u_t = \left( \frac{u}{\sqrt{1 + u^2}} \right)_{xx}, \]

but also represent a large class of related equations, such as the following short pulse (SP) equation [14],

\[ u_{xt} = u + \frac{1}{6}(u^3)_{xx}. \]

The SP and two-component SP (real or complex) equations are proposed as special integrable cases in the negative WKI hierarchy for the first time in Refs. [9, 10, 14]. Multi-soliton solutions and the Cauchy problem for a two-component SP system are given in Ref. [15]. The SP equation can be used as a model to describe ultra-short optical pulses traversing within a nonlinear media [22–24].

Recently, how to use some specific algebra to systematically construct integrable systems have developed. That aroused great interest of scholars. In Refs. [25–27], the algebraic construction based on Toda field theory was generalized by the addition of fields associated to higher grading operators, yielding the generalized affine Toda models. The higher-grade fields are physically interpreted as matter fields with the usual Toda fields coupled to them. In Ref. [28], authors proposed a general higher grading construction for the zero curvature equation, containing the WKI hierarchy as a particular case. In the construction, the zero-grade Toda fields are completely removed, remaining the higher grade fields only. Based on this method, one can obtain a series of mixed integrable systems such as the mixed mKdV-sine-Gordon equation, the mixed AKNS-Lund-Regge equation and the mixed super symmetric mKdV-sinh-Gordon equation. The integrable systems obtained from negative flows have important physical and mathematical significance, such as the Camassa-Holm (CH) equation, the Degasperis-Procesi (DP) equation and the Vakhnenko equation. A mixed WKI-SP model has been found by combining the positive flows and extending the WKI hierarchy to incorporate the negative flows [28, 29].

The remainder of this paper is organized as follows. In Sect. 2, we would like to construct a standard form of WKI-type integrable hierarchy and the bi-Hamiltonian structure by using the trace identity. In Sect. 3, The higher grading affine algebraic construction method and some special cases in the obtained hierarchy are considered. The local and non-local conserved charges are obtained from the Riccati form. In Sect. 4, the higher order SP equation and the mixed WKI-SP equation are derived by considering the negative flow and mixed flow. The last section is devoted to conclusions and discussions.
2 A Standard form of WKI-Type Integrable Hierarchy

For the sake of readability, let us firstly introduce the three-dimensional real special linear Lie algebra $sl(2, \mathbb{R})$. This algebra consisting of trace-free $2 \times 2$ matrices, has the basis

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

whose nonzero commutation relations are

$$[e_1, e_2] = 2e_2, \quad [e_1, e_3] = -2e_3, \quad [e_2, e_3] = e_1.$$

We can also define the corresponding matrix loop algebra $\tilde{sl}(2, \mathbb{R})$,

$$\tilde{sl}(2, \mathbb{R}) = \left\{ \sum_{j \geq 0} M_j \lambda^{n-j} | M_j \in sl(2, \mathbb{R}), j \geq 0, n \in \mathbb{Z} \right\}.$$

Thus, a brief account of the procedure for building $sl(2, \mathbb{R})$-valued integrable system is described below.

**Step 1**: One needs to select an appropriate spectral matrix $U$ to form a spatial spectral problem $U_{xx} = U_{t} = U(u, \lambda) = U(u, \lambda)$ satisfying the stationary zero curvature equation $W_x = [U, W]$. Then one can prove the localness property for $W$.

**Step 2**: Construct a particular Laurent series solution $W = W(u, \lambda)$ so that the zero curvature equations $U_{tm} - V_{m} + [U, V^{[m]}] = 0$ will generate an integrable hierarchy $u_{tm} = K_{m}(u), m \geq 0$.

**Step 4**: Finally, furnish the Hamiltonian structures $u_{tm} = K_{m}(u) = J \frac{\delta H}{\delta u}, m \geq 0$ by the trace identity.

In Ref. [20], a WKI-type spatial spectral problem, which is associated with $sl(2, \mathbb{R})$, is defined by

$$\phi_x = U\phi = U(u, \lambda)\phi, \quad u = \begin{pmatrix} p \\ q \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad (2.1)$$

where

$$U = (\lambda + ap)e_1 + \lambda pe_2 + \lambda qe_3$$

$$= \begin{pmatrix} \lambda + ap & \lambda p \\ \lambda q & -\lambda - ap \end{pmatrix} \in \tilde{sl}(2, \mathbb{R}). \quad (2.2)$$

When $\alpha = 0$, it is exactly the classical WKI spatial spectral problem [3]. In order to construct the recursion relations, we consider the following standard form of matrix $W.$
Remark 1 In Ref. [20], different from (2.3), the matrix $W$ is taken as

$$W = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \widetilde{sl}(2, \mathbb{R}).$$

which looks very casual. Here the expression of $W$ given by (2.3) is more general and it is helpful to connect the Tu scheme method and the higher grading structure construction method (introduced in the following section).

Firstly, we solve the stationary zero curvature equation

$$W_x = [U, W],$$

which becomes

$$\begin{cases}
  a_x = \lambda (pc - qb), \\
  b_x = 2\lambda b - 2\lambda pa + 2\alpha pb, \\
  c_x = -2\lambda c + 2\alpha qa - 2\alpha pc.
\end{cases}$$

(2.4)

Substituting the following Laurent series expansion

$$W = \sum_{k=0}^{\infty} W_k \lambda^k, \quad W_k = \begin{bmatrix} a_k & b_k \\ c_k & a_k \end{bmatrix}, \quad k \geq 0$$

(2.5)

into (2.4), we have

$$\begin{cases}
  a_{kx} = pc_{k+1} - qb_{k+1}, \\
  b_{k+1} = \frac{1}{2}(b_{kx} + 2pa_{k+1} - 2\alpha pb_k), \quad k \geq 0, \\
  c_{k+1} = \frac{1}{2}(-c_{kx} + 2qa_{k+1} - 2\alpha pc_k),
\end{cases}$$

(2.6)

and

$$pc_0 - qb_0 = 0, \quad b_0 = pa_0, \quad c_0 = qa_0.$$  

(2.7)

To determine the recursion relation between $\{b_{k+1}, c_{k+1}\}$ and $\{b_k, c_k\}$, we need to represent $a_{k+1}$ by $\{b_k, c_k\}$. In order to achieve this purpose, we rewrite $a_{kx}$ as
Thus we change $a_{k+1,x}$ to

$$a_{k+1,x} = -\frac{1}{2} pc_{k+1,x} - \frac{1}{2} qb_{k+1,x} - ap^2 c_k + apqb,$$

and

$$a_{k+1} = \frac{1}{\sqrt{pq+1}} \partial^{-1} \frac{1}{\sqrt{pq+1}} \left[ \frac{1}{4} pc_{kxx} - \frac{1}{4} qb_{kxx} + \frac{1}{2} ap(pc)_k + \frac{1}{2} aq(pb)_k ight]$$

Let us rewrite the above equation again as

$$\left( \sqrt{pq+1} a_{k+1} \right)_x = \frac{1}{\sqrt{pq+1}} \left[ \frac{1}{4} pc_{kxx} - \frac{1}{4} qb_{kxx} + \frac{1}{2} ap(pc)_k + \frac{1}{2} aq(pb)_k ight]$$

Then we arrive at

$$a_{k+1} = \frac{1}{\sqrt{pq+1}} \partial^{-1} \frac{1}{\sqrt{pq+1}} \left[ \frac{1}{4} pc_{kxx} - \frac{1}{4} qb_{kxx} + \frac{1}{2} ap(pc)_k + \frac{1}{2} aq(pb)_k ight]$$

So we can compute $\{a_k, b_k, c_k, k \geq 1\}$ recursively from the following initial values

$$a_0 = \frac{1}{\sqrt{pq+1}}, \quad b_0 = \frac{p}{\sqrt{pq+1}}, \quad c_0 = \frac{q}{\sqrt{pq+1}}$$

by using Eq. (2.9) and the last two equations of (2.6). Here $\{a_0, b_0, c_0\}$ are determined by the initial conditions (2.7). To guarantee the uniqueness of $\{a_k, b_k, c_k\}$, we also need impose the integration conditions

$$a_k |_{x=0} = b_k |_{x=0} = c_k |_{x=0} = 0, \quad k \geq 1.$$
Here we use the Maple software to deal with the complicated symbolic computations. The first two sets are listed as follows:

\[
\begin{align*}
a_1 &= \frac{1}{4} \left( \frac{pq_x - qp_x}{pq + 1} \right)^{3/2} + \alpha \left( \frac{p^2 q}{pq + 1} \right)^{3/2}, \\
b_1 &= \frac{1}{2} \left( \frac{pq_x}{pq + 1} \right)^{3/2} - \alpha \left( \frac{p^2}{pq + 1} \right)^{3/2}, \\
c_1 &= -\frac{1}{2} \left( \frac{q_x}{pq + 1} \right)^{3/2} - \alpha \left( \frac{pq}{pq + 1} \right)^{3/2}, \\
a_2 &= -\frac{1}{32} \left( \frac{pq + 1}{pq} \right)^{3/2} \left( 4q^2 q_{xx} - 5p^2 q_x^2 - 14pq_p q_x + 4pq^2 p_{xx} - 5q^2 p_x^2 + 4pq_{xx} 
-4pq_x + 4pq_{xx} \right) - \frac{3}{4} \alpha \left( \frac{pq_x - pq}{pq + 1} \right)^{3/2} - \frac{3}{2} \alpha \left( \frac{pq}{pq + 1} \right)^{3/2}, \\
b_2 &= -\frac{1}{32} \left( \frac{pq + 1}{pq} \right)^{3/2} \left( -4p^2 q^2 p_{xx} + 4p^2 q q_{xx} - 5p^3 q_x^2 - 2p^2 q_p q_x + 7pq^2 p_x^2 - 12pq_p q_x 
+4p^2 q_{xx} + 8pp_p q_x + 12pq_x - 8p_q \right) - \frac{3}{2} \alpha \left( \frac{pp_x}{pq + 1} \right)^{5/2} - \frac{1}{2} \alpha \left( \frac{pq - 2}{pq + 1} \right)^{5/2}, \\
c_2 &= \frac{1}{32} \left( \frac{pq + 1}{pq} \right)^{3/2} \left( -4pq^3 p_{xx} + 5q^3 p_x^2 + 4p^2 q^2 q_{xx} + 2pq^2 p_x q_x - 7pq^2 q_x^2 - 4q^2 q_{xx} \right) 
+12pqq_{xx} - 8pq_p q_x - 12pq_x + 8q_q \right) - \frac{1}{2} \alpha \left( \frac{pq_{xx} - pq_x}{pq + 1} \right)^{3/2} - \frac{2}{pq - 2} - \frac{2}{pq + 1} \right)^{5/2}, \\
c_{k+1} &= \frac{1}{2\sqrt{pq + 1}} \sum_{i+j=k+1 \atop i,j \geq 1} (a_i a_j + b_i c_j) + \frac{pc_{xx} - qb_{xx}}{4(pq + 1)} + \alpha \left( \frac{pc_k + qb_k}{pq} \right), \\
which is derived from \(a^2 + bc = (a^2 + bc)|_{u=0} = 1.\)

Now, by taking

\[
V^{[m]} = \lambda^2 (m^m W)_x + \Delta_m
= \lambda^{m+2} W_0 + \lambda^{m+1} W_1 + \cdots + \lambda^2 W_m + \left[ h_m \lambda^{m+2-k} \right],
\]

the zero curvature equations

\[
U_m - V^{[m]} + [U, V^{[m]}] = 0, \quad n \geq 0
\]
give

\[ f_m = \frac{1}{2}(b_{mx} - 2\alpha pb_m), \]

\[ g_m = -\frac{1}{2}(c_{mx} + 2\alpha pc_m), \]

\[ a_{mx} = pg_m - qf_m, \quad (2.11) \]

\[ p_{tm} = f_{mx} - 2\alpha pf_m + 2ph_m, \]

\[ q_{tm} = g_{mx} - 2qh_m + 2apg_m, \]

\[ h_{mx} = ap_{tm}. \]

Due to (2.6), we can easily see

\[ f_m = \frac{1}{2}(b_{mx} - 2\alpha pb_m) = b_{m+1} - pa_{m+1}, \]

\[ g_m = -\frac{1}{2}(c_{mx} + 2\alpha pc_m) = c_{m+1} - qa_{m+1}, \]

\[ a_{mx} = pc_{m+1} - qb_{m+1}, \]

and

\[ h_{mx} = ap_{tm} = a(f_{mx} - 2\alpha pf_m + 2ph_m) \]

\[ = a[b_{m+1,x} - (pa_{m+1})_x - 2\alpha p(b_{m+1} - pa_{m+1}) + 2ph_m]. \]

Thus we can introduce

\[ h_m = -\alpha pa_{m+1} + \alpha b_{m+1}, \]

and then \( p_{tm}, q_{tm} \) can be expressed as

\[ p_{tm} = f_{mx} - 2\alpha pf_m + 2ph_m \]

\[ = \frac{1}{2}b_{mxx} - \alpha (pb_m)_x - \alpha (b_{mx} - 2\alpha pb_m) + 2p(-\alpha pa_{m+1} + \alpha b_{m+1}) \]

\[ = \frac{1}{2}b_{mxx} - \alpha (pb_m)_x, \]

\[ q_{tm} = -\frac{1}{2}c_{mxx} - \alpha (pc_m)_x + 2aa_{mx}. \]

Therefore, we have obtained a WKI-type integrable hierarchy associated with the Lie algebra \( sl(2, \mathbb{R}) \):

\[
\begin{bmatrix}
p \\
q
\end{bmatrix}_{tm} = K_m = \begin{bmatrix}
\frac{1}{2}b_{mxx} - \alpha (pb_m)_x \\
-\frac{1}{2}c_{mxx} - \alpha (pc_m)_x + 2aa_{mx}
\end{bmatrix}.
\]  

(2.12)

When \( \alpha = 0 \), it is just the classical WKI integrable hierarchy [3].
Next, we construct the Hamiltonian structures of the above WKI-type integrable hierarchy (2.12), which are furnished by using the following trace identity [16–19],

\[
\frac{\delta}{\delta u} \int \text{tr} \left( \frac{\partial U}{\partial \lambda} W \right) \, dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^\prime \text{tr} \left( \frac{\partial U}{\partial u} W \right), \quad \gamma = -\frac{\lambda}{2} \frac{d}{d\lambda} \ln |\text{tr}(W^2)|,
\]

where \( W \) solves the stationary zero curvature equation \( W_x = [U, W] \). Thus, the corresponding trace identity becomes

\[
\frac{\delta}{\delta u} \int \left( -\frac{2a_m + qb_m + pc_m}{m - 1} \right) \, dx = \left( c_m + 2aa_{m-1} \right) b_m. \quad (2.13)
\]

By means of Eq. (2.12), we can compute

\[
p_{tm} = \frac{1}{2} b_{m,xx} - \alpha(p b_m)_x = \frac{1}{2} \partial^2 b_m - \alpha \partial(p b_m),
\]

\[
q_{tm} = -\frac{1}{2} c_{m,xx} - \alpha(p c_m)_x + 2aa_{px}
\]

\[
= -\frac{1}{2} \partial^2 c_m - \alpha \partial(p c_m) + 2\alpha(-\frac{1}{2} pc_{m,x} - \frac{1}{2} q b_{m,x} - \alpha^2 c_m + \alpha pq b_m)
\]

\[
= -\frac{1}{2} \partial^2 (c_m + 2aa_{m-1}) + \alpha \partial^2 a_{m-1} - \alpha \partial(p c_m + 2aa_{m-1})
\]

\[
+ 2\alpha^2 \partial p c_{m+1} + 2aa_{m-1})
\]

\[
+ 2\alpha^2 \partial p a_{m-1} - \alpha q \partial b_m - 2\alpha^2 p^2 (c_m + 2aa_{m-1}) + 4\alpha^3 p^2 a_{m-1} + 2\alpha^2 pq b_m
\]

\[
= \left( -\frac{1}{2} \partial^2 + \alpha \partial \partial - \alpha \partial^2 - 2\alpha^2 p^2 \right) (c_m + 2aa_{m-1}) + (-\alpha q \partial - \alpha \partial q) b_m.
\]

Thus, the above integrable hierarchy (2.12) can be represented as the following Hamiltonian forms

\[
\frac{d}{dt} u_m = K_m = \int \frac{\delta H_m}{\delta u},
\]

with the Hamiltonian operator

\[
J = \begin{pmatrix}
0 & \frac{1}{2} \partial^2 - \alpha \partial p \\
-\frac{1}{2} \partial^2 - \alpha \partial \partial & -\alpha q \partial - \alpha \partial q
\end{pmatrix}
\]

and the Hamiltonian functionals
It is now a direct computation that all members in Eq. (2.12) are bi-Hamiltonian. We compute

\[
\mathcal{H}_m = \int \left( -\frac{2a_m + qb_m + pc_m}{m - 1} \right) dx, \quad m \geq 2.
\]

Similarly, we have

\[
c_m + 2aa_{m-1} = -\frac{1}{2}c_{m-1,x} - apc_{m-1} + \frac{q}{\sqrt{pq + 1}} \vartheta^{-1} \left( \frac{1}{4}pc_{m-1,x} - \frac{1}{4}qb_{m-1,x} \right)
+ \frac{1}{2}ap(pc_{m-1})_x + \frac{1}{2}aq(pb_{m-1})_x + \frac{1}{2}ap^2c_{m-1,x}
+ \frac{1}{2}apqb_{m-1,x} + \alpha^2 p^3 c_{m-1}
- \alpha^2 p^2 qb_{m-1} + 2a\vartheta^{-1} \left( -\frac{1}{2}pc_{m-1,x} - \frac{1}{2}qb_{m-1,x} - ap^2 c_{m-1} + apqb_{m-1} \right)
= \Psi_{11}(c_{m-1} + 2aa_{m-2}) + \Psi_{12}b_{m-1},
\]

with

\[
\Psi_{11} = -\frac{1}{2} \partial + \frac{q}{\sqrt{pq + 1}} \vartheta^{-1} \left( \frac{1}{4}p\vartheta^2 + \frac{1}{2}ap^2\vartheta \right) - a\vartheta^{-1} p\vartheta,
\]

\[
\Psi_{12} = \frac{q}{\sqrt{pq + 1}} \vartheta^{-1} \left( \frac{1}{4}q\vartheta^2 + \frac{1}{2}aq\vartheta + \frac{1}{2}apq\vartheta + \frac{1}{2}ap\vartheta q \right) - a\vartheta^{-1} q\vartheta - aq.
\]

Similarly, we have

\[
b_m = \Psi_{21}(c_{m-1} + 2aa_{m-2}) + \Psi_{22}b_{m-1},
\]

where

\[
\Psi_{21} = \frac{p}{\sqrt{pq + 1}} \vartheta^{-1} \left( \frac{1}{4}p\vartheta^2 + \frac{1}{2}ap^2\vartheta \right),
\]

\[
\Psi_{22} = \frac{1}{2} \partial + \frac{p}{\sqrt{pq + 1}} \vartheta^{-1} \left( \frac{1}{4}q\vartheta^2 + \frac{1}{2}aq\vartheta + \frac{1}{2}apq\vartheta + \frac{1}{2}ap\vartheta q \right) - a\vartheta.
\]

So we arrive at

\[
U_{tm} = K_m = J \frac{\delta H_m}{\delta u} = \Psi \frac{\delta H_{m-1}}{\delta u}, \quad (2.14)
\]

where

\[
\Psi = \begin{pmatrix}
\Psi_{11} & \Psi_{12} \\
\Psi_{21} & \Psi_{22}
\end{pmatrix}.
\]

Therefore, it is easy to see that the WKI-type hierarchy (2.12) is Liouville integrable.
3 The Higher Grading Construction

Firstly, we give a brief description for the higher grading construction method. A more detailed description of the method is given in Ref. [28]. Let \( \hat{G} \) be an affine Kac-Moody algebra and \( Q \) an operator decomposing the algebra into the graded subspaces

\[
\hat{G} = \bigoplus_{j \in \mathbb{Z}} \hat{G}^{(j)}, \quad [Q, \hat{G}^{(j)}] = j \hat{G}^{(j)}.
\]

As a consequence of the Jacobi identity, \([\hat{G}^{(i)}, \hat{G}^{(j)}] \subset \hat{G}^{(i+j)}\). Let \( E \) be a semi-simple element, with a definite grade, defining the kernel subspace \( \mathcal{K} \). The image subspace, \( \mathcal{M} \), is its complement and \( \hat{G} = \mathcal{M} \oplus \mathcal{K} \). Then we have the relations \([\mathcal{K}, \mathcal{K}] \subset \mathcal{K}, [\mathcal{K}, \mathcal{M}] \subset \mathcal{M} \) and we assume the symmetric space structure \([\mathcal{M}, \mathcal{M}] \subset \mathcal{K}\).

Integrable systems can be constructed from the zero curvature equation

\[
[D_x + U, D_t + V] = 0, \tag{3.1}
\]

where \( U \) and \( V \) lied on \( \hat{G} \) and have the following forms

\[
U = E^{(0)} + E^{(1)} + A^{(1)}[\phi],
\]

\[
V = \sum_{i=-m}^{n} D^{(i)}[\phi]. \tag{3.2}
\]

Here \( E^{(0)} \in \hat{G}^{(0)} \) and \( E^{(1)} \in \hat{G}^{(1)} \) is a constant semi-simple element. \( A^{(1)}[\phi] \in \mathcal{M}^{(1)} \), where \( \mathcal{M}^{(1)} = \mathcal{M} \bigcap \hat{G}^{(1)} \) is the operator containing the grade one field functions.

Then substituting Eq. (3.2) into Eq. (3.1), the zero curvature representation becomes

\[
[D_x + E^{(0)} + E^{(1)} + A^{(1)}, D_t + D^{(-m)} + D^{(-m+1)} + \ldots + D^{(-n)} + D^{(n)}] = 0.
\]

With this algebraic structure, this zero curvature equation can be solved nontrivially. The projection into each graded subspace yields the following set of equations.
Now we consider a concrete Kac-Moody algebra $\hat{A}_1 = \{H_j, E_j^+, E_j^-, \hat{c}, \hat{d}\}$ with the following commutation relations:


d\partial_x [E^{(1)} + A^{(1)}, D^{(n)}] = 0, \\
d\partial_x D^{(n)} + [E^{(0)}, D^{(n)}] + [E^{(1)} + A^{(1)}, D^{(n-1)}] = 0, \\
\vdots \\
d\partial_x D^{(1)} + [E^{(0)}, D^{(1)}] + [E^{(1)} + A^{(1)}, D^{(0)}] - \partial_t (E^{(1)} + A^{(1)}) = 0, \\
d\partial_x D^{(0)} + [E^{(0)}, D^{(0)}] + [E^{(1)} + A^{(1)}, D^{(-1)}] - \partial_t E^{(0)} = 0, \\
\vdots \\
d\partial_x D^{(-m+1)} + [E^{(0)}, D^{(-m+1)}] + [E^{(1)} + A^{(1)}, D^{(-m)}] = 0, \\
d\partial_x D^{(-m)} + [E^{(0)}, D^{(-m)}] = 0.

Now we consider a concrete Kac-Moody algebra $\hat{A}_1 = \{H_j, E_j^+, E_j^-, \hat{c}, \hat{d}\}$ with the following commutation relations:

\[
[H^k, H^l] = 2k\delta_{k+j,0}\hat{c}, \quad [H^k, E^j_\pm] = \pm 2kE^k_\pm, \quad [E^k_+, E^l_-] = H^{k+j} + k\delta_{k+j,0}\hat{c},
\]

\[
[\hat{d}, T^j] = jT^j, \quad [\hat{c}, T^j] = 0,
\]

where $T^j \in \{H^j, E^j_+, E^j_-\}$ and $j$ is an integer.

Then by setting $\hat{c} = 0$, we use the loop algebra to construct integrable systems. The homogeneous gradation $Q = \hat{d}$ yields the grading subspaces

\[
\mathcal{G}^{(j)} = \{H^j, E^j_+, E^j_-\}.
\]

Fix the semi-simple element as $E = \alpha p H^0 + H^1$. We have $\mathcal{K}^{(j)} = \{H^j\}$ and $\mathcal{M}^{(j)} = \{E^j_+, E^j_-\}$. Thus the operator containing functions $p \equiv p(x, t), q \equiv q(x, t)$ has the form

\[
A^{(1)} = pE^1_+ + qE^1_-.
\]

By setting the Lax operator $V$

\[
D^{(j)} = a_j E^j_+ + b_j E^j_- + c_j H^j,
\]

we have the following zero curvature equation of positive flow

\[
[\partial_x + \alpha p H^0 + H^1 + pE^1_+ + qE^1_-, \partial_t + D^{(n)} + D^{(n-1)} + \cdots + D^{(0)}] = 0. \tag{3.3}
\]
Here the coefficients $a_j, b_j, c_j$ will be determined in terms of the field functions $p$ and $q$.

For $n = 2$, the grad-by-grad decomposing of Eq. (3.3) leads to

$$
[H^1 + pE_+^1 + qE_-^1, D^{(2)}] = 0,
$$

$$
\partial_\xi D^{(2)} + [aH^0, D^{(2)}] + [H^1 + pE_+^1 + qE_-^1, D^{(1)}] = 0,
$$

$$
\partial_\xi D^{(1)} + [aH^0, D^{(1)}] + [H^1 + pE_+^1 + qE_-^1, D^{(0)}] - \partial_\xi (H^1 + pE_+^1 + qE_-^1) = 0,
$$

$$
\partial_\xi D^{(0)} + [aH^0, D^{(0)}] - \partial_\xi (aH^0) = 0.
$$

Therefore, we can get the following WKI-type integrable system

$$
\partial_\xi p + \partial_\xi^2 \left( \frac{p}{2(1 + pq)^{1/2}} \right) + \partial_\xi \left( \frac{ap^2}{(1 + pq)^{1/2}} \right) = 0,
$$

$$
\partial_\xi q - \partial_\xi^2 \left( \frac{q}{2(1 + pq)^{1/2}} \right) + \partial_\xi \left( \frac{a(pq - 2)}{(1 + pq)^{1/2}} \right) = 0.
$$

(3.4)

with the Lax pair

$$
U = aH^0 + H^1 + pE_+^1 + qE_-^1,
$$

$$
V = \frac{p}{(1 + pq)^{1/2}} E_+^2 + \frac{q}{(1 + pq)^{1/2}} E_-^2 + \frac{1}{(1 + pq)^{1/2}} H^2
$$

$$
- \left[ \partial_\xi \left( \frac{p}{2(1 + pq)^{1/2}} \right) + \frac{ap^2}{(1 + pq)^{1/2}} \right] E_+^1 + \left[ \partial_\xi \left( \frac{q}{2(1 + pq)^{1/2}} \right) - \frac{apq}{(1 + pq)^{1/2}} \right] E_-^1
$$

$$
- \left[ \partial_\xi \left( \frac{ap}{2(1 + pq)^{1/2}} \right) + \frac{a^2 p^2}{(1 + pq)^{1/2}} \right] H^0.
$$

Equation (3.4) is just the first equation in the WKI-type hierarchy (2.12) with $m = 0$ by replacing $-U, -V$ with $U, V$ respectively. In fact, we can find that there is a correspondence between the method in Sect. 2 and this method. In other words, this method gives a Kac-Moody algebraic interpretation of the Tu scheme method.

Similarly, for $n = 3$, we can construct the following WKI-type integrable system

$$
\partial_\xi p - \partial_\xi^2 \left( \frac{p_x + 2ap^2}{4(1 + pq)^{3/2}} \right) - \partial_\xi \left( \frac{ap(p_x + 2ap^2)}{2(1 + pq)^{3/2}} \right) = 0,
$$

$$
\partial_\xi q - \partial_\xi^2 \left( \frac{q_x - 2apq}{4(1 + pq)^{3/2}} \right) + \partial_\xi \left( \frac{ap(q_x - 2apq)}{2(1 + pq)^{3/2}} \right)
$$

$$
+ 2a \partial_\xi \left( \frac{ap_x - pq_x + 4ap^2 q}{4(1 + pq)^{3/2}} \right) = 0,
$$

(3.5)

with the Lax pair
\[ U = \alpha p H^0 + H^1 + p E_1^1 + q E_1^1, \]
\[ V = \frac{p}{(1 + pq)^{1/2}} E_3^3 + \frac{q}{(1 + pq)^{1/2}} E_3^3 + \frac{1}{(1 + pq)^{1/2}} H^3 \]
\[ - \frac{p_x + 2\alpha p}{2(1 + pq)^{3/2}} E_3^2 + \frac{q_x - 2\alpha pq}{2(1 + pq)^{3/2}} E_3^2 + \frac{qp_x - pq_x + 4\alpha^2 q}{4(1 + pq)^{3/2}} H^2 \]
\[ + \left[ \partial_x \left( \frac{p_x + 2\alpha p}{4(1 + pq)^{3/2}} \right) + \frac{\alpha p}{2(1 + pq)^{3/2}} \right] E_1^1 \]
\[ + \left[ \partial_x \left( \frac{q_x - 2\alpha pq}{4(1 + pq)^{3/2}} \right) - \frac{\alpha p}{2(1 + pq)^{3/2}} \right] E_1^1 \]
\[ + \left[ \partial_x \left( \frac{\alpha p}{4(1 + pq)^{3/2}} \right) + \frac{\alpha^2 p}{2(1 + pq)^{3/2}} \right] H^0. \]

Equation (3.5) is just the second equation in the WKI-type hierarchy (2.12) with \( m = 1 \) by replacing \(-U, -V\) with \( U, V \) respectively.

**Remark 2** Furthermore, by setting \( E = \alpha \sqrt{pq + 1} H^0 + H^1 \) (proposed in Ref. [21]) instead of \( E = \alpha p H^0 + H^1 \), the method introduced in this section is also valid and the similar results will be obtained.

Next we shall derive the local and nonlocal charges from the Riccati form of the spectral problem
\[ (\partial_x + U)\Psi = 0, \quad (\partial_x + V)\Psi = 0. \]  
(3.6)

By using the matrix representation, we have
\[ U = \begin{pmatrix} \lambda + \alpha p & \lambda p \\ \lambda q & \lambda - \alpha p \end{pmatrix}, \quad V = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & -A(\lambda) \end{pmatrix}, \quad \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \]  
(3.7)

where \( \lambda \) is the spectral parameter. By introducing the variables
\[ \Gamma = \frac{\Psi_2}{\Psi_1}, \quad \Gamma^{-1} = \frac{\Psi_1}{\Psi_2}, \]  
(3.8)

we can write the Riccati form of the spectral problem (3.6). Its compatibility yields the conservation laws
\[ \partial_x (\lambda p \Gamma + \alpha p) = \partial_x (A + B \Gamma), \]
\[ \partial_x (\lambda q \Gamma^{-1} - \alpha p) = \partial_x (\Gamma^{-1} - A + C \Gamma^{-1}). \]  
(3.9)

Therefore, we can construct an infinite number of conserved charges by using \( p \Gamma \) and \( q \Gamma^{-1} \) and assuming a power series in \( \lambda \). Let \( F = p \Gamma \) and \( G = q \Gamma^{-1} \). We obtain
which are the generating equations for the conserved densities.

To get the local density, we expand $F$ in the power series of $1/\lambda$,

$$F = \sum_{n=0}^{\infty} f_n \lambda^{-n}. \quad (3.11)$$

By substituting (3.11) into (3.10) and equating the terms of the same powers of $1/\lambda$, we obtain

$$f_0 = -1 + (1 + pq)^{1/2},$$
$$f_1 = -\frac{1}{2} \partial_x \left( \ln \frac{p}{(1 + pq)^{1/2}} \right) + \frac{1}{2} \frac{\partial_x p + 2ap}{p(1 + pq)^{1/2}} - ap,$$
$$\vdots$$

The charges associated to these densities are

$$H_0 = \int_{-\infty}^{\infty} \left( -1 + (1 + pq)^{1/2} \right) dx,$$
$$H_1 = \int_{-\infty}^{\infty} \left( \frac{1}{2} \frac{\partial_x p + 2ap}{p(1 + pq)^{1/2}} - ap \right) dx,$$
$$\vdots$$

These charges are the Hamiltonians which will generate the WKI-type integrable systems within the positive flows.

Furthermore, we set the most general expansion

$$F = \sum_{n=-1}^{\infty} f_n \lambda^n, \quad (3.12)$$

which would generate the nonlocal densities. Substituting (3.12) into (3.10) leads to

$$\partial_x f_1 - (\partial_x \ln p + 2ap)f_1 = f_1^2,$$
$$\partial_x f_0 - (\partial_x \ln p + 2f_1 + 2ap)f_0 = 2f_1,$$
$$\partial_x f_{-1} - (\partial_x \ln p + 2f_1 + 2ap)f_{-1} = f_0^2 + 2f_0 - pq,$$
$$\partial_x f_{-2} - (\partial_x \ln p + 2f_1 + 2ap)f_{-2} = 2f_0f_{-1} + 2f_{-1},$$
$$\partial_x f_{-3} - (\partial_x \ln p + 2f_1 + 2ap)f_{-3} = f_{-1}^2 + 2f_0f_{-2} + 2f_{-2},$$
$$\vdots$$

Using the similar method [28], we get
Here \( P \) and \( Q \) are defined as
\[
P = pe^{2\alpha x^{-1}p}, \quad Q = qe^{-2\alpha x^{-1}p}.
\]
The respective conserved charges are given by
\[
H_n = \int_{-\infty}^{\infty} f_n dx, \quad n = 1, 2, 3 \ldots
\]  
(3.13)
Thus we believe that the charges are conserved and can be explicitly checked, either from the positive or the negative flows.

**Remark 3** For the spectral problem (3.6) with \( U = \alpha \sqrt{pq} + H^0 + H^1 + pE^1_+ + qE^1_- \), the local and nonlocal charges can be worked out in the same way.

### 4 The SP-Type Integrable Systems

In this section, we study the SP-type integrable systems by using the Lax operator
\[
V = \sum_{i=-n}^{1} D[\phi].
\]
Here we introduce the operator \( \partial_x^{-1} f(x) = \int_{-\infty}^{x} f(y)dy \). Assume that the fields and their derivatives decay sufficiently fast when \( |x| \to \infty \) under the condition
\[
\partial_x \partial_x^{-1} f(x) = \partial_x^{-1} \partial_x f(x) = f(x).
\]

According to our construction, the negative flows can be constructed from the zero curvature equation
\[
[\partial_x + H^1 + pE^1_+ + qE^1_-, \partial_x + D(-n) + D(-n+1) + \cdots + D(0) + D(1)] = 0.
\]
When \( n = 1 \), we can obtain the following two-component SP equation \([9, 28]\)
\[
u_{xt} = 4v + 2\partial_x(\nu \nu_x),
\]
with the corresponding Lax pair
\[
U = H^1 + u_x E^1_+ + v_x E^1_-, \\
V = H^{-1} + 2u E^0_+ - 2v E^0_- + 2\nu \nu_x E^1_+ + 2\nu \nu E^1_- + 2\nu H^1.
\]

Contrary to the positive flows of the WKI-type equations, the SP equation does not seem do describe the large amplitude solutions. A multi-component generalization of the above equation with the same structure, has also been proposed in Ref. [30]. This generalization can also be obtained by considering the untwisted algebra \( \hat{A}_{n-1} \sim \hat{s}(n) \). These conclusions have been proposed in Ref. [28]. Now we continue to consider the higher order SP-type integrable systems.

When \( n = 3 \), the zero curvature representation reads
\[ \partial_x + H^1 + pE^1 + qE^1, \partial_y + D^{(-3)} + D^{(-2)} + D^{(-1)} + D^{(0)} + D^{(1)} = 0, \]

and decomposes into six independent equations:

\[
\begin{align*}
[H^1 + pE^1 + qE^1, D^{(1)}] &= 0, \\
\partial_x D^{(1)} + [H^1 + pE^1 + qE^1, D^{(0)}] - \partial_y (H^1 + pE^1 + qE^1) &= 0, \\
\partial_x D^{(0)} + [H^1 + pE^1 + qE^1, D^{(-1)}] &= 0, \\
\partial_x D^{(-1)} + [H^1 + pE^1 + qE^1, D^{(-2)}] &= 0, \\
\partial_x D^{(-2)} + [H^1 + pE^1 + qE^1, D^{(-3)}] &= 0, \\
\partial_x D^{(-3)} &= 0.
\end{align*}
\]

We solve this system step by step. The projection into \( \hat{\mathcal{G}}^{(-3)} \) implies that \( a_{-3}, b_{-3} \) and \( c_{-3} \) are all constants. Furthermore, by setting \( a_{-3} = b_{-3} = 0 \), we have \( c_{-2} = \text{constant} \) and

\[
a_{-2} = 2c_{-3} \partial_x^{-1} p, \quad b_{-2} = -2c_{-3} \partial_x^{-1} q,
\]

by calculating the \( \hat{\mathcal{G}}^{(-2)} \) projection. Then the \( \hat{\mathcal{G}}^{(-1)} \) projection yields

\[
\begin{align*}
a_{-1} &= 2 \partial_x^{-1} (pc_{-2} - 2c_{-3} \partial_x^{-1} p), \\
b_{-1} &= -2 \partial_x^{-1} (qc_{-2} + 2c_{-3} \partial_x^{-1} q), \\
c_{-1} &= 2c_{-3} \partial_x^{-1} (p \partial_x^{-1} q + q \partial_x^{-1} p).
\end{align*}
\]

Similarly, we can obtain the following equation from the \( \hat{\mathcal{G}}^{(0)} \) projection

\[
\begin{align*}
a_0 &= 4 \partial_x^{-1} [pc_{-3} \partial_x^{-1} (p \partial_x^{-1} q + q \partial_x^{-1} p) - \partial_x^{-1} (pc_{-2} - 2c_{-3} \partial_x^{-1} p)], \\
b_0 &= -4 \partial_x^{-1} [qc_{-3} \partial_x^{-1} (p \partial_x^{-1} q + q \partial_x^{-1} p) + \partial_x^{-1} (qc_{-2} + 2c_{-3} \partial_x^{-1} q)], \\
c_0 &= 2 \partial_x^{-1} [q \partial_x^{-1} (pc_{-2} - 2c_{-3} \partial_x^{-1} p) + p \partial_x^{-1} (qc_{-2} + 2c_{-3} \partial_x^{-1} q)].
\end{align*}
\]

The projection into \( \hat{\mathcal{G}}^{(2)} \) implies that \( a_1 = pc_1 \) and \( b_1 = qc_1 \). The \( \hat{\mathcal{G}}^{(1)} \) projection yields the field equations plus one constraint

\[
\begin{align*}
\partial_x p &= \partial_x^{-1} a_1 + 2(a_0 - pc_0), \\
\partial_x q &= \partial_x^{-1} b_1 + 2(qc_0 - b_0), \\
\partial_x c_1 &= qa_0 - pb_0.
\end{align*}
\]

Substituting (4.2) into the third equation of (4.3) and choosing \( c_{-2} = 0 \), we can obtain
\[ c_1 = \partial_x^{-1}(qa_0 - pb_0) \]
\[ = 4c_{-3}\partial_x^{-1}[(q\partial_x^{-1}p + p\partial_x^{-1}q)\partial_x^{-1}(q\partial_x^{-1}p + p\partial_x^{-1}q)] \]
\[ + 8c_{-3}\partial_x^{-1}(q\partial_x^{-3}p + p\partial_x^{-3}q). \]

By fixing \( c_{-3} \), we have the following nonlocal equations
\[
\partial_t p = \partial_x(pc_1) + 2(a_0 - pc_0) \\
= 4c_{-3}\partial_x[p\partial_x^{-1}(q\partial_x^{-1}p + p\partial_x^{-1}q) \\
+ p\partial_x^{-1}(q\partial_x^{-1}p + p\partial_x^{-1}q) + 2q\partial_x^{-3}p + 2p\partial_x^{-3}q] \\
+ 8c_{-3}[\partial_x^{-1}p\partial_x^{-1}(q\partial_x^{-1}p + p\partial_x^{-1}q) + p\partial_x^{-1}(q\partial_x^{-2}p - p\partial_x^{-2}q) + 2\partial_x^{-3}p],
\]
\[
\partial_t q = \partial_x(qc_1) + 2(qc_0 - b_0) \\
= 4c_{-3}\partial_x[q\partial_x^{-1}(q\partial_x^{-1}p + p\partial_x^{-1}q) \\
+ p\partial_x^{-1}(q\partial_x^{-1}p + p\partial_x^{-1}q) + 2q\partial_x^{-3}p + 2p\partial_x^{-3}q] \\
+ 8c_{-3}[\partial_x^{-1}q\partial_x^{-1}(q\partial_x^{-1}p + p\partial_x^{-1}q) + q\partial_x^{-1}(p\partial_x^{-2}q - q\partial_x^{-2}p) + 2\partial_x^{-3}q].
\]

Introduce a new field function defined through \( \rho = -q = u_{xxx} \). We get the following model
\[
u_{xxxx} = \frac{2}{15}c_{-3}(u_{xx}^5)_{xx} - 16c_{-3}\left[u_{xxx}\left(u_{xx}u_x - \frac{1}{2}u_x^2\right)\right]_{x} - \frac{8}{3}c_{-3}u_{xx}^3 + 16c_{-3}u, \quad (4.4)
\]
which is the higher order \( \text{SP} \) equation. Here \( c_{-3} \) is an arbitrary constant. The Lax pair of this model reads
\[
U = H^1 + u_{xxxx}E_+^1 - u_{xxx}E_1, \\
V = H^{-3} + 2u_{xx}E_{-}^2 + 2u_{xx}E_{-}^{-2} - 4u_xE_{-}^1 + 4u_xE_{-}^{-1} - 2u_{xx}H^{-1} \\
+ \left(-\frac{4}{3}u_{xx}^3 + 8u\right)E_0^0 + \left(-\frac{4}{3}u_{xx}^3 + 8u\right)E_0^0 + u_{xxx}\left(\frac{2}{3}u_{xx}^4 - 16u_{xx}u_x + 8u_x^2\right)E_+^1 \\
- u_{xxx}\left(\frac{2}{3}u_{xx}^4 - 16u_{xx}u_x + 8u_x^2\right)E_0^1 + \left(\frac{2}{3}u_{xx}^4 - 16u_{xx}u_x + 8u_x^2\right)H^1
\]
by setting \( c_{-3} = 1 \) for the convenience of writing.

It is possible to combine a positive flow with a negative flow. So we consider the mixed WKI-SP integrable systems by using the following zero curvature equation
\[
[\partial_x + H^1 + pE_+^1 + qE_-^1, \partial_t + D^{(3)} + D^{(2)} + D^{(1)} + D^{(0)} + D^{(-1)} + D^{(-2)} + D^{(-3)}] = 0.
\]
We can decompose it into eight independent equations.
We can exactly solve each grade projection starting from the highest to the lowest. From the $\hat{G}(0)$ to the $\hat{G}(4)$ projection, the process is similar to the positive flows in the case $n = 3$. Thus, in the $\hat{G}(0)$ projection, we have

$$\begin{align*}
\partial_x p &= \partial_x a_1 + 2(a_0 - pc_0), \\
\partial_x q &= \partial_x b_1 + 2(qc_0 - b_0).
\end{align*}$$

From the $\hat{G}(4)$ projection, we have $a_3 = pc_3$ and $b_3 = qc_3$. The $\hat{G}(3)$ projection gives

$$a_2 = -\frac{1}{2} \partial_x a_3 + pc_2, \quad b_2 = \frac{1}{2} \partial_x b_3 + qc_2.$$

The $\hat{G}(2)$ projection gives

$$a_1 = -\frac{1}{2} \partial_x a_2 + pc_1, \quad b_1 = \frac{1}{2} \partial_x b_2 + qc_1, \quad \partial_x c_1 = qa_0 - pb_0.$$

So we can obtain

$$a_2 = -\frac{1}{2} \left( \frac{p_x}{(1 + pq)^{3/2}} \right), \quad b_2 = \frac{1}{2} \left( \frac{q_x}{(1 + pq)^{3/2}} \right), \quad c_2 = \frac{1}{4} \left( \frac{qp_x - pq_x}{(1 + pq)^{3/2}} \right).$$

Now we let some coefficients, that were previously considered constants, to depend on time $t$, thus providing the non-autonomous ingredient. Those coefficients were not interesting for the individual flows because they come as a global factor in the final equation. From the $\hat{G}(-3)$ to $\hat{G}(-1)$ projection, the results are the same as (4.4). Therefore, we get
after choosing $p = -q = u_{xxx}$. Here $a(t)$ and $b(t)$ are arbitrary functions of $t$. Equation (4.7) is just a higher order mixed WKI-SP equation. Due to $a(t)$ and $b(t)$, the dispersion relation will have a time dependent velocity and the solitons will accelerate [28]. Equation (4.7) may be nice candidates in applications having accelerated ultra-short optical pulses [28].

5 Conclusions and Discussions

This paper investigates a WKI-type integrable hierarchy derived from an $sl(2, \mathbb{R})$-valued spectral problem. In particular, a trace identity is exploited for the construction of Hamiltonian structures, and some equations of the hierarchy are considered from the point of view of a Kac-Moody algebraic approach. In addition, the higher grading construction method is adopted to generate more novel integrable systems. The local and nonlocal charges from the Riccati form of the spectral problem are presented. Furthermore, the WKI-type integrable hierarchy is extended to the negative flow, which yields a higher order SP-type integrable system (4.4). A novel integrable non-autonomous WKI-SP equation (4.7) is also proposed, by mixing a positive with a negative flow. This mixed model may have many applications in nonlinear optics, specially concerning accelerated ultra-short optical pulses [28].

The SP-type equations and their properties are a subject of current interest in nonlinear optics and electrodynamics, both theoretically and experimentally [31]. In Refs. [24, 30, 32], based on the hodograph transformation and the KP reduction technology, the authors developed a systematic procedure for constructing exact solutions for the SP-type equations. We might construct the following multi-component SP-type system

\[
q_{ix} = q_i + \left[ \left( \sum_{j=1}^{N} \sigma_j |q_j|^2 \right) q_{ix} \right]_{x} - \left( \sum_{j=1}^{N} \sigma_j |q_{j,x}|^2 \right) q_i, \quad i = 1, 2, \ldots, N.
\]  

Specifically, when $n = 2$ and $\sigma_1 = \sigma_2 = 1$, we have

\[
q_{1,x} = q_1 + \left( |q_1|^2 + |q_2|^2 \right) q_{1,x} - \left( |q_{1,x}|^2 + |q_{2,x}|^2 \right) q_1,\\
q_{2,x} = q_2 + \left( |q_1|^2 + |q_2|^2 \right) q_{2,x} - \left( |q_{1,x}|^2 + |q_{2,x}|^2 \right) q_2,
\]  

with the $4 \times 4$ Lax pair
Here $I_2$ is the $2 \times 2$ identity matrix and 
\[
\begin{align*}
U &= \lambda \left( (1 - |q_{1x}|^2 - |q_{2x}|^2)I_2 \begin{pmatrix} 2 \bar{q}_x \\ - (1 - |q_{1x}|^2 - |q_{2x}|^2)I_2 \end{pmatrix}, \\
V &= \begin{pmatrix} \lambda(|q_1|^2 + |q_2|^2)(1 - |q_{1x}|^2 - |q_{2x}|^2)I_2 + \frac{1}{4\lambda}I_2 \\
2\lambda(|q_1|^2 + |q_2|^2) \hat{q}_x - \hat{q} \\
- \lambda(|q_1|^2 + |q_2|^2)(1 - |q_{1x}|^2 - |q_{2x}|^2)I_2 - \frac{1}{4\lambda}I_2 \end{pmatrix}
\end{align*}
\]

The exact solutions and the physical applications of these equations need to be further studied.

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