Besov Spaces Induced by Doubling Weights

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Abstract
Let $1 \leq p < \infty$, $0 < q < \infty$, and $\nu$ be a two-sided doubling weight satisfying

$$\sup_{0 \leq r < 1} \frac{(1-r)^q}{\int_r^1 \nu(t) \, dt} \int_0^r \frac{\nu(s)}{(1-s)^q} \, ds < \infty.$$ 

The weighted Besov space $B_{\nu}^{p,q}$ consists of those $f \in H^p$ such that

$$\int_0^1 \left( \int_0^{2\pi} |f'(re^{i\theta})|^p \, d\theta \right)^{q/p} \nu(r) \, dr < \infty.$$

Our main result gives a characterization for $f \in B_{\nu}^{p,q}$ depending only on $|f|$, $p$, $q$, and $\nu$. As a consequence of the main result and inner-outer factorization, we obtain several interesting by-products. For instance, we show the following modification of a classical factorization by F. and R. Nevanlinna: If $f \in B_{\nu}^{p,q}$, then there exist $f_1, f_2 \in B_{\nu}^{p,q} \cap H^\infty$ such that $f = f_1/f_2$. Moreover, we give a sufficient and necessary condition guaranteeing that the product of $f \in H^p$ and an inner function belongs to $B_{\nu}^{p,q}$. Applying this result, we make some observations on zero sets of $B_{\nu}^{p,q}$.

Keywords Doubling weight · Besov space · Hardy space · Inner-outer factorization · Mixed norm space · Zero set

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1 Introduction and Characterizations

Let $D$ be the open unit disc of the complex plane $\mathbb{C}$ and $T$ be the boundary of $D$. The set of all analytic functions in $D$ is denoted by $\mathcal{H}(D)$. For $0 < p < \infty$, the Hardy space $H^p$ consists of those $f \in \mathcal{H}(D)$ such that

$$\|f\|_{H^p} = \sup_{0 \leq r < 1} M_p(r, f) < \infty,$$

where

$$M_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p}.$$ 

The Hardy space $H^\infty$ is the set of all bounded functions in $\mathcal{H}(D)$. Moreover, we recall that a measurable function $f$ on $T$ belongs to $L^p(T)$ for some $p \in (0, \infty)$ if

$$\|f\|_{L^p} = \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta < \infty.$$ 

Alternatively, the Hardy space $H^p$ for $0 < p < \infty$ can be characterized as follows: $f \in H^p$ if and only if $f \in \mathcal{H}(D)$, the nontangential limit $f(e^{i\theta})$ exists almost everywhere on $T$, and $f(e^{i\theta}) \in L^p(T)$. In particular, $\|f\|_{H^p} = \|f\|_{L^p}$ for $0 < p < \infty$ and $f \in H^p$. This is due to Hardy’s convexity and the mean convergence theorems. These results and much more can be found in the classic book [8] by P. Duren.

A function $\nu : D \to [0, \infty)$ is called a (radial) weight if it is integrable over $D$ and $\nu(z) = \nu(|z|)$ for all $z \in D$. For $0 < p, q < \infty$, and a weight $\nu$, the weighted mixed norm space $A^{p,q}_\nu$ consists of those $f \in \mathcal{H}(D)$ such that

$$\|f\|_{A^{p,q}_\nu}^q = \int_0^1 M^q_p(r, f) \, \nu(r) \, dr < \infty.$$ 

If $\nu(z) = (1 - |z|)^\alpha$ for $-1 < \alpha < \infty$, then the notation $A^{p,q}_\alpha$ is used for $A^{p,q}_\nu$. In this paper, we study class $\mathcal{D}$ of so-called two-sided doubling weights, which originates from the work of J. A. Peláez and J. Rättyä [19,20]. For the definition of $\mathcal{D}$ we have to define two wider classes. For a weight $\nu$, set

$$\widehat{\nu}(z) = \widehat{\nu}(|z|) = \int_{|z|}^{1} \nu(s) \, ds, \quad z \in D.$$ 

If a weight $\nu$ satisfies the condition $\widehat{\nu}(r) \leq C\widehat{\nu}(\frac{1+r}{2})$ for all $0 \leq r < 1$ and some $C = C(\nu) > 0$, then we write $\nu \in \mathcal{D}$. Correspondingly, $\nu \in \mathcal{D}$ if there exist $K = K(\nu) > 1$ and $C = C(\nu) > 1$ such that

$$\widehat{\nu}(r) \geq C\widehat{\nu}\left(1 - \frac{1-r}{K}\right), \quad 0 \leq r < 1.$$
Class $\mathcal{D}$ is the intersection of $\hat{\mathcal{D}}$ and $\hat{\mathcal{D}}$. In addition, we define the following subclass of $\hat{\mathcal{D}}$: $\nu \in \hat{\mathcal{D}}_p$ for some $p \in (0, \infty)$ if the condition
\[
\hat{\mathcal{D}}_p(\nu) = \sup_{0 \leq r < 1} \frac{(1 - r)^p}{\nu(r)} \int_0^r \frac{\nu(s)}{(1 - s)^p} \, ds < \infty \tag{1.1}
\]
is satisfied. As a concrete example, we mention that $\nu_1(z) = (1 - |z|)^{\alpha}$ and $\nu_2(z) = (1 - |z|)^{\alpha} \left( \log \frac{r}{1 - |z|} \right)^{\beta}$ for any $\beta \in \mathbb{R}$ belonging to $\mathcal{D} \cap \hat{\mathcal{D}}_p$ if and only if $-1 < \alpha < p - 1$. Additional information about weights can be found in [18–20]. Some basic properties are recalled also in Sect. 2.

Define the weighted Besov space $B^{p,q}_\nu$ by $B^{p,q}_\nu = \{ f : f' \in A^{p,q}_\nu \} \cap H^p$. For $-1 < \alpha < \infty$ and $\nu(z) = (1 - |z|)^{\alpha}$, the notation $B^{p,q}_\nu$ is used for $B^{p,q}_\nu$. The space $B^{p,q}_\nu$ is the main research objective of this paper. Hence it is worth pointing out that the definition is rational, which means that $H^p$ is not a subset of $\{ f : f' \in A^{p,q}_\nu \}$ in general, or conversely. The family of Blaschke products offers examples for the case where $f \in H^\infty$ and $f' \notin A^{p,q}_\nu$; see for instance [24]. Moreover, it would be natural that certain lacunary series $g$ lie out of $H^p$, while $g' \in A^{p,q}_\nu$. Arguments for these kinds of examples can be found in M. Pavlović’s book [16], which contains numerous important observations on the topic of this paper. The existence of both examples, of course, depends on $p, q, \nu$. In other words, under certain hypotheses for $p, q, \nu$, an inclusion relation between $\{ f : f' \in A^{p,q}_\nu \}$ and $H^p$ might be valid. However, this is not the case in general.

For $0 < p < \infty$ and $f \in L^p(\mathbb{T})$, the $L^p$ modulus of continuity $\omega_p(t, f)$ is defined by
\[
\omega_p(t, f) = \sup_{0 < h < t} \left( \int_0^{2\pi} |f(e^{i(\theta + h)}) - f(e^{i\theta})|^p \, d\theta \right)^{1/p}, \quad 0 < t \leq 2\pi.
\]
We interpret $\omega_p(t, f) = \omega_p(2\pi, f)$ for $t > 2\pi$. It is a well-known fact that, for $0 < p, q < \infty$, $-1 < \alpha < q - 1$, and $f \in H^p$, the derivative of $f$ belongs to $A^{p,q}_\alpha$ if and only if
\[
\int_0^\infty \frac{\omega_p(t, f)^q}{t^{q-\alpha}} \, dt < \infty.
\]
This result originates from E. M. Stein’s book [26, Chapter V, Section 5], and the complete version is a consequence of [17, Theorems 2.1 and 5.1] or [12, Theorem 1.2] by M. Pavlović and M. Jevtić. Our first theorem is a partial generalization of the result. Its proof uses some ideas from [8,21,22].

**Theorem 1** Let $1 \leq p < \infty$, $0 < q < \infty$, and $\nu \in \mathcal{D}$. Then $\nu \in \hat{\mathcal{D}}_q$ if and only if there exists a constant $C = C(p, q, \nu) > 0$ such that
\[
\int_{1/2}^1 \omega_p(1 - r, f)^q \frac{\nu(r)}{(1 - r)^q} \, dr \leq C \| f' \|_q^{q} A^{p,q}_\nu \tag{1.2}
\]
for all $f \in H^p$. 

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Note that (1.2) is valid also if $0 < p, q < \infty$, $\nu$ is a weight, and there exists $\beta = \beta(q, \nu) < q - 1$ such that $\nu(r)/(1 - r)^\beta$ is increasing for $0 \leq r < 1$. This is due to [17, Theorem 5.1] and its proof. Even though the result is valid also for $0 < p < 1$, Theorem 1 is more useful for our purposes. In particular, it is worth underlining that the hypothesis $1 \leq p < \infty$ in Theorems 2 and 3 below is natural.

Theorem 2 gives a practical estimate for $\|f'\|_{A^p_v,q}^q + \|f\|_{H^p}^q$, when $f \in H^p$. As a by-product of its argument, we deduce that also the converse inequality of (1.2) holds if the norm $\|f\|_{H^p}^q$ is added into the right-hand side. Setting

$$d\mu_z(\theta) = \frac{1 - |z|^2}{|e^{i\theta} - z|^2} \frac{d\theta}{2\pi}, \quad z \in \mathbb{D}, \quad 0 \leq \theta < 2\pi,$$

Theorem 2 reads as follows.

**Theorem 2** Let $1 \leq p < \infty$, $0 < q < \infty$, and $\nu \in \mathcal{D} \cap \hat{\mathcal{D}}_q$. Then there exist positive constants $C_1$ and $C_2$ depending only on $p$, $q$, and $\nu$ such that

$$\|f'\|_{A^p_v,q}^q \leq \left( \int_0^1 \left( \int_0^{2\pi} \left| f(e^{i\theta}) - f(r e^{i\theta}) \right| d\mu_{r e^{i\theta}}(\theta) \right)^p \frac{\nu(r)}{(1 - r)^q} dr \right)^{q/p} \leq C_1 \left( \int_0^1 \omega_p(1 - s, f)^q \frac{\nu(s)}{(1 - s)^q} ds + \|f\|_{H^p}^q \right) \leq C_2 \left( \|f'\|_{A^p_v,q}^q + \|f\|_{H^p}^q \right) \leq C_2 (\|f'\|_{A^p_v,q}^q + \|f\|_{H^p}^q)$$

for all $f \in H^p$.

By studying the classical weight $\nu(z) = (1 - |z|)^\alpha$, where $-1 < \alpha < q - 1$, we obtain K. M. Dyakonov’s [9, Proposition 2.2(a)] as a direct consequence of Theorem 2. Hence it does not come as a surprise that the proofs of [9, Theorem 2.1] and Theorem 2 have some similarities. Nonetheless, it is worth mentioning that the presence of general weights complicates the argument, and consequently, our proof is quite technical. Note also that Theorem 1 plays an essential role in the proof.

Our main result below gives a characterization for functions $f$ in $\mathcal{B}_{v}^{p,q}$ depending only $|f|, p, q,$ and $\nu$. This result improves B. Bøe’s [6, Theorem 1.1], which concentrates only on the case where $1 \leq p, q < \infty, -1 < \alpha < q - 1,$ and $\nu(z) = (1 - |z|)^\alpha$. It also generalizes the essential contents of [2, Proposition 2.4] and [9, Proposition 2.2(b)] made by A. Aleman and K. M. Dyakonov, respectively.

**Theorem 3** Let $1 \leq p < \infty$, $0 < q < \infty$, and $\nu \in \mathcal{D} \cap \hat{\mathcal{D}}_q$. Then there exist positive constants $C_1$ and $C_2$ depending only on $p$, $q$, and $\nu$ such that

$$\|f'\|_{A^p_v,q}^q \leq C_1 (F_1(f) + F_2(f)) \leq C_2 \left( \|f'\|_{A^p_v,q}^q + \|f\|_{H^p}^q \right), \quad f \in H^p,$$

where

$$F_1(f) = \int_0^1 \left( \int_0^{2\pi} \left| f(e^{i\theta}) \right| d\mu_{r e^{i\theta}}(\theta) \left| f(r e^{i\theta}) \right| \right)^p \frac{\nu(r)}{(1 - r)^q} dr$$

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and

\[
F_2(f) = \int_0^1 \left( \int_0^{2\pi} \left( \int_0^{2\pi} |f(e^{i\theta})| - \int_0^{2\pi} |f(e^{i\theta})| d\mu_{r e^{i\theta}}(s) \right) d\mu_{r e^{i\theta}}(\theta) \right)^p \frac{q/p}{(1-r)^q} dr.
\]

Before we talk about the argument of Theorem 3, recall the inner-outer factorization. An inner function is a member of \(H^\infty\) having unimodular radial limits almost everywhere on \(T\). For \(0 < p \leq \infty\), an outer function for \(H^p\) takes the form

\[
O_\phi(z) = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log \phi(e^{i\theta}) d\theta \right), \quad z \in \mathbb{D},
\]

where \(\phi\) is a non-negative function in \(L^p(T)\) and \(\log \phi \in L^1(T)\). The inner-outer factorization asserts that \(f \in H^p\) can be represented as the product of an inner and outer function; see for instance [8, Theorem 2.8]. It is worth noting that the factorization is unique, and

\[
|f(\xi)| = |O_\phi(\xi)| = \phi(\xi) \tag{1.5}
\]

for almost every \(\xi \in T\) if \(O_\phi\) is the outer function from the factorization of \(f\). Equation (1.5) is due to the definition of inner functions, the Poisson integral formula, the harmonicity of \(\log |O_\phi(z)|\) and the fact that

\[
|O_\phi(z)| = \exp \left( \int_0^{2\pi} \log \phi(e^{i\theta}) d\mu_z(\theta) \right), \quad z \in \mathbb{D}.
\]

The last inequality in (1.4) can be proved by applying Theorem 2. In the argument of the first inequality, the inner-outer factorization, the Schwarz–Pick lemma, and an upper estimate for \(|O_\phi'|\) from [6] are the main tools. It is worth underlining that Bøe’s idea to make an upper estimate for \(|f'|\) by using the factorization seems to be quite effective. Another way to prove results like Theorem 3 is to use a modification of Theorem 2 together with the well-known equation

\[
\int_0^{2\pi} |f(e^{i\theta}) - f(z)|^2 d\mu_z(\theta) = \int_0^{2\pi} |f(e^{i\theta})|^2 d\mu_z(\theta) - |f(z)|^2, \quad z \in \mathbb{D};
\]

but this method of Dyakonov has the obvious defect that it works only when \(f \in H^2\). The advantage of this method in the case where \(2 \leq p < \infty\), \(0 < q < \infty\), \(q/2 - 1 < \alpha < q - 1\), and \(\nu(z) = (1 - |z|)^\alpha\) is that \(F_1(f) + F_2(f)\) in Theorem 3 can be replaced by

\[
\int_0^1 \left( \int_0^{2\pi} \left( \int_0^{2\pi} |f(e^{i\theta})|^2 d\mu_{r e^{i\theta}}(\theta) - |f(r e^{i\theta})|^2 \right)^{p/2} dt \right)^{q/p} \frac{\nu(r)}{(1-r)^q} dr;
\]
see [9, Proposition 2.2(b)]. It is an open problem to prove a corresponding estimate for general weights.

Next we give an example that shows that the hypothesis $\nu \in D \cap \hat{D}_q$ in Theorem 3 for $p \geq 2$ is sharp in a certain sense. Note that the example is a modification of [22, Example 8]. Before the statement we fix some notation. Write $f \lesssim g$ if there exists a constant $C > 0$ such that $f \leq Cg$, while $f \gtrsim g$ is understood analogously. If $f \lesssim g$ and $f \gtrsim g$, then we write $f \sim g$.

**Example 4** Let $2 \leq p < \infty$, $q = p$, $\nu(z) = (1 - \vert z \vert)^{p-1}$, and $dA(z)$ be the two-dimensional Lebesgue measure $dxdy$. Let $f$ be an inner function such that

$$
\int_{\{z \in \mathbb{D} : \vert f(z) \vert < \epsilon \}} \frac{dA(z)}{1 - \vert z \vert} = \infty
$$

for some $\epsilon \in (0, 1)$. The existence of such $f$ is guaranteed by [7, Theorem 5]. Then

$$
F_1(f) + F_2(f) = F_1(f) \lesssim \int_{\mathbb{D}} (1 - \vert f(z) \vert)^p (1 - \vert z \vert)^{-1} dA(z) = \infty,
$$

while

$$
\Vert f' \Vert_{A_p^q}^q + \Vert f \Vert_{H_p}^q = \Vert f' \Vert_{A_{p-1}^p}^p + 1 < \infty
$$

by the well-known inclusion

$$
H_p \subset \{ g : g' \in A_{p-1}^p \}, \quad 2 \leq p < \infty,
$$

which originates from [13].

We close the section by explaining how the remainder of this paper is organized. Auxiliary results on weights are recalled in the next section. The utility of Theorem 3 is demonstrated in Sects. 3 and 4. More precisely, in Sect. 3, we prove the factorization that states that, for any $f \in B_v^{p,q}$, there exist $f_1, f_2 \in B_v^{p,q} \cap H^{\infty}$ such that $f = f_1 / f_2$. Section 4 begins with a result giving a sufficient and necessary condition guaranteeing that the product of $f \in H^p$ and an inner function belongs to $B_v^{p,q}$. As a consequence of this theorem, we obtain some results on zero sets of $B_v^{p,p}$. Sections 5, 6, and 7 consist of the proofs of Theorems 1, 2, and 3, respectively.

## 2 Auxiliary Results on Weights

In this section, we recall some basic properties of weights in $\hat{D}$ and $\hat{D}$. These properties are needed in subsequent sections. Another reason for these results is to help the reader to understand the nature of weights in $\mathbb{D}$. We begin with a result that is essentially [19, Lemma 3]; see also [18, Lemma 2.1].

**Lemma A** Let $\nu$ be a weight. Then the following statements are equivalent:

[\[ Springer]
(i) \( \nu \in \hat{D} \).

(ii) There exist \( C = C(\nu) > 0 \) and \( \beta = \beta(\nu) > 0 \) such that
\[
\hat{\nu}(r) \leq C \left( \frac{1-r}{1-s} \right)^\beta \hat{\nu}(s), \quad 0 \leq r \leq s < 1.
\]

(iii) There exist \( C = C(\nu) > 0 \) and \( \gamma = \gamma(\nu) > 0 \) such that
\[
\int_0^r \left( \frac{1-r}{1-s} \right)^\gamma \nu(s) \, ds \leq C \hat{\nu}(r), \quad 0 \leq r < 1.
\]

(iv) The estimate
\[
\int_0^1 s^x \nu(s) \, ds \asymp \hat{\nu} \left( 1 - \frac{1}{x} \right), \quad 1 \leq x < \infty,
\]
is satisfied.

From the point of view of our main results, Lemma A(iii) is interesting because it states that \( \nu \in \hat{D} \) if and only if \( \nu \in \hat{D}_p \) for some \( p > 0 \). This means that \( \hat{D} = \bigcup_{p > 0} \hat{D}_p \). Nevertheless, Lemma A(ii) gives maybe the most interesting description for \( \hat{D} \). Together with its counterpart below it offers a very practical characterization for weights in \( D \). Essentially this characterization says that \( \hat{\nu} \) is normal in the sense of A. L. Shields and D. L. Williams [25].

**Lemma B** Let \( \nu \) be a weight. Then \( \nu \in \tilde{D} \) if and only if there exist \( C = C(\nu) > 0 \) and \( \alpha = \alpha(\nu) > 0 \) such that
\[
\hat{\nu}(s) \leq C \left( \frac{1-s}{1-r} \right)^\alpha \hat{\nu}(r), \quad 0 \leq r \leq s < 1.
\]

Lemma B originates from [20], and it can be proved in a corresponding manner as Lemma A(ii). See in particular the proof of [18, Lemma 2.1].

By the definition of class \( \hat{D}_p \), it is clear that \( \hat{D}_p \subset \hat{D}_{p+\varepsilon} \) for any \( \varepsilon > 0 \). Next we state [21, Lemma 3], which shows that also the converse inclusion is true for sufficiently small \( \varepsilon = \varepsilon(\nu, p) > 0 \). The proof of this result is based on integration by parts. Note that \( \hat{D}_p(\nu) \) in the statement is defined by (1.1).

**Lemma C** If \( 0 < p < \infty \) and \( \nu \in \tilde{D}_p \), then \( \nu \in \tilde{D}_{p-\varepsilon} \) for any \( \varepsilon \in \left( 0, \frac{p}{\hat{D}_p(\nu)+1} \right) \).

The last result of this section is [22, Lemma 5], which shows that \( \nu \in D \) in the norm \( \| f \|_{A^p(q)} \) can be replaced by \( \hat{\nu}(z)/(1 - |z|) \) without losing any essential information.

**Lemma D** Let \( 0 < p, q < \infty \) and \( \nu \) be a weight.
(i) If \( \nu \in \hat{\mathcal{D}} \), then there exists \( C = C(\nu) > 0 \) such that
\[
\| f \|^q_{A^p_v, q} \geq C \int_0^1 M^q_p(r, f) \frac{\hat{\nu}(r)}{1-r} \, dr, \quad f \in \mathcal{H}(\mathbb{D}).
\]
(ii) If \( \nu \in \hat{\mathcal{D}} \), then there exists \( C = C(\nu) > 0 \) such that
\[
\| f \|^q_{A^p_v, q} \leq C \int_0^1 M^q_p(r, f) \frac{\hat{\nu}(r)}{1-r} \, dr, \quad f \in \mathcal{H}(\mathbb{D}).
\]

For \( \nu \in \mathcal{D} \), Lemmas A(ii) and B yield
\[
\hat{\nu}(r) \approx \int_r^1 \frac{\hat{\nu}(s)}{1-s} \, ds, \quad 0 \leq r < 1. \tag{2.1}
\]

In [22], Lemma D is proved by applying this fact together with partial integrations. An alternative way to prove results like Lemma D is to split the integral with respect to \( dr \) into infinitely many parts by using a dyadic partition, and then apply (2.1) together with the monotonicity of \( M^q_p(r, f) \). An advantage of the last method is that \( M^q_p(r, f) \) can be easily replaced by a certain monotonic function \( g(r) \). This observation will be utilized several times in the argument of Theorem 2.

### 3 Quotient Factorization

Recall that if \( f \in H^p \) for some \( p > 0 \), then there exist \( f_1, f_2 \in H^\infty \) such that \( f = f_1 / f_2 \). This is an important consequence of classical factorization [8, Theorem 2.1] by F. and R. Nevanlinna. The main purpose of this section is to give the following \( B^p_v, q \) counterpart to the above-mentioned result.

**Theorem 5** Let \( 1 \leq p < \infty \), \( 0 < q < \infty \), and \( \nu \in \mathcal{D} \cap \hat{\mathcal{D}}_q \). If \( f \in B^p_v, q \), then there exist \( f_1, f_2 \in B^p_v, q \cap H^\infty \) such that \( f = f_1 / f_2 \) and \( f_2 \) is an outer function.

It is worth mentioning that [16, Theorem 9.19] is a similar type of result as Theorem 5 with a different hypothesis for \( \nu \). Moreover, we note that Theorem 5 generalizes [2, Corollary 2.7], [6, Theorem 3.4] and [9, Corollary 3.4]. For its argument we need an extension of [6, Theorem 3.3]. Note that a part of what follows is really inspired by [6].

**Proposition 6** Let \( 1 \leq p < \infty \), \( 0 < q < \infty \), \( \nu \in \mathcal{D} \cap \hat{\mathcal{D}}_q \) and \( f \in H^p \) be the product of an inner function \( I \) and an outer function \( O_\phi \). Then there exists a constant \( C = C(p, q, \nu) > 0 \) such that
\[
\| O_{\max(\phi, 1)} \|^q_{A^p_v, q} + \| (I O_{\min(\phi, 1)})' \|^q_{A^p_v, q} + \| O_{\max(\phi, 1)} \|^q_{H^p} \\
\leq C \left( \| f' \|^q_{A^p_v, q} + \| f \|^q_{H^p} + 1 \right).
\]
Before the proof of Proposition 6, we note that the quantities $F_1(f)$ and $F_2(f)$ in Theorem 3 are used repeatedly hereafter.

**Proof** Let us begin by noting that $|O_{\phi}(e^{i\theta})| = \phi(e^{i\theta})$, $|O_{\max\{\phi,1\}}(e^{i\theta})| = \max\{\phi(e^{i\theta}), 1\}$, and

$$\max\{\phi(e^{i\theta}), 1\} - \phi(e^{i\theta}) = \frac{\max\{\phi(e^{i\theta}), 1\} - \phi(e^{i\theta})}{\max\{\phi(e^{i\theta}), 1\}} \leq |O_{\max\{\phi,1\}}(z)| \left(1 - \frac{\phi(e^{i\theta})}{\max\{\phi(e^{i\theta}), 1\}}\right)$$

for all $z \in \mathbb{D}$ and almost every $\theta \in [0, 2\pi)$. Using these facts together with Jensen’s inequality [10, Chapter I, Lemma 6.1] and the definition of outer functions, we obtain

$$\int_0^{2\pi} |O_{\max\{\phi,1\}}(e^{i\theta})| d\mu_z(\theta) - \int_0^{2\pi} |O_{\phi}(e^{i\theta})| d\mu_z(\theta) \leq |O_{\max\{\phi,1\}}(z)| \left(1 - \frac{\phi(e^{i\theta})}{\max\{\phi(e^{i\theta}), 1\}}\right) \exp \left(\log \phi(e^{i\theta}) - \log \max\{\phi(e^{i\theta}), 1\}\right) d\mu_z(\theta)$$

$$\leq |O_{\max\{\phi,1\}}(z)| \left(1 - \frac{|O_{\phi}(z)|}{|O_{\max\{\phi,1\}}(z)|}\right), \quad z \in \mathbb{D}.$$

Consequently, the obvious inequality $|f(z)| \leq |O_{\phi}(z)|$ yields

$$\int_0^{2\pi} |O_{\max\{\phi,1\}}(e^{i\theta})| d\mu_z(\theta) - |O_{\max\{\phi,1\}}(z)| \leq \int_0^{2\pi} |f(e^{i\theta})| d\mu_z(\theta) - |f(z)|, \quad z \in \mathbb{D}. \quad (3.1)$$

Write $z = re^{it}$. Raising both sides of (3.1) to power $p$, integrating from 0 to $2\pi$ with respect to $dt$, then raising both sides to power $q/p$, and finally integrating from 0 to 1 with respect to $v(r)dr/(1 - r)^q$, we obtain $F_1(O_{\max\{\phi,1\}}) \leq F_1(f)$.

Next we show $F_2(O_{\max\{\phi,1\}}) \leq F_2(f)$. Set

$$\Gamma_1 = \Gamma_1(z, \phi) = \left\{\theta \in [0, 2\pi) : \int_0^{2\pi} \max\{\phi(e^{is}), 1\} d\mu_z(s) \leq \phi(e^{i\theta})\right\}$$

and

$$\Gamma_2 = \Gamma_2(z, \phi) = \left\{\theta \in [0, 2\pi) : \int_0^{2\pi} \phi(e^{is}) d\mu_z(s) \leq \phi(e^{i\theta})\right\}, \quad z \in \mathbb{D}.$$
Then elementary calculations yield

\[
\begin{align*}
\int_0^{2\pi} \max\{\phi(e^{i\theta}), 1\} &- \int_0^{2\pi} \max\{\phi(e^{is}), 1\}d\mu_z(s) \, d\mu_z(\theta) \\
&= 2 \int_{\Gamma_1} \left( \max\{\phi(e^{i\theta}), 1\} - \int_0^{2\pi} \max\{\phi(e^{is}), 1\}d\mu_z(s) \right) \, d\mu_z(\theta) \\
&\quad + \int_0^{2\pi} \left( \int_0^{2\pi} \max\{\phi(e^{is}), 1\}d\mu_z(s) - \max\{\phi(e^{i\theta}), 1\} \right) \, d\mu_z(\theta) \\
&= 2 \int_{\Gamma_1} \left( \phi(e^{i\theta}) - \int_0^{2\pi} \max\{\phi(e^{is}), 1\}d\mu_z(s) \right) \, d\mu_z(\theta) \\
&\leq 2 \int_{\Gamma_2} \left( \phi(e^{i\theta}) - \int_0^{2\pi} \phi(e^{is})d\mu_z(s) \right) \, d\mu_z(\theta) \\
&= \int_0^{2\pi} \left| \phi(e^{i\theta}) - \int_0^{2\pi} \phi(e^{is})d\mu_z(s) \right| \, d\mu_z(\theta), \quad z \in \mathbb{D}.
\end{align*}
\]

(3.2)

Consequently, we obtain \( F_2(O_{\max\{\phi, 1\}}) \leq F_2(O_{\phi}) = F_2(f) \) by doing a similar integral procedure as above. Now Theorem 3 together with the inequalities for \( F_1(f) \) and \( F_2(f) \) gives

\[
\| O'_{\max\{\phi, 1\}} \|_{A^p_v, q}^q + \| O_{\max\{\phi, 1\}} \|_{H^p}^q \lesssim \| f' \|_{A^p_v, q}^q + \| f \|_{H^p}^q + 1.
\]

(3.3)

By (3.3) it suffices to show

\[
\|(IO_{\min\{\phi, 1\}})' \|_{A^p_v, q}^q \lesssim \| f' \|_{A^p_v, q}^q + \| f \|_{H^p}^q.
\]

(3.4)

Since

\[
\phi(e^{i\theta}) - \min\{\phi(e^{i\theta}), 1\} \geq |O_{\min\{\phi, 1\}}(z)| \left( \frac{\phi(e^{i\theta})}{\min\{\phi(e^{i\theta}), 1\}} - 1 \right),
\]

we obtain

\[
\begin{align*}
\int_0^{2\pi} |O_{\phi}(e^{i\theta})|d\mu_z(\theta) - |O_{\phi}(z)| \\
&\geq \int_0^{2\pi} |O_{\min\{\phi, 1\}}(e^{i\theta})|d\mu_z(\theta) - |O_{\min\{\phi, 1\}}(z)|, \quad z \in \mathbb{D},
\end{align*}
\]
by arguing as above using Jensen’s inequality. It follows that

\[
\int_0^{2\pi} |f(e^{i\theta})|d\mu_z(\theta) - |f(z)| = (\int_0^{2\pi} |O_{\phi}(e^{i\theta})|d\mu_z(\theta) - |O_{\phi}(z)|) + |O_{\phi}(z)|(1 - |I(z)|)
\]

\[
\geq (\int_0^{2\pi} |O_{\min\{\phi,1\}}(e^{i\theta})|d\mu_z(\theta) - |O_{\min\{\phi,1\}}(z)|) + |O_{\min\{\phi,1\}}(z)|(1 - |I(z)|)
\]

\[
= \int_0^{2\pi} |IO_{\min\{\phi,1\}}(e^{i\theta})|d\mu_z(\theta) - |IO_{\min\{\phi,1\}}(z)|, \quad z \in \mathbb{D}.
\]

Hence it is easy to deduce \( F_1(IO_{\min\{\phi,1\}}) \leq F_1(f) \). Since

\[
F_2(IO_{\min\{\phi,1\}}) = F_2(O_{\min\{\phi,1\}}) \leq F_2(O_{\phi}) = F_2(f)
\]

can be shown by using a modification of (3.2), the desired estimate (3.4) follows from Theorem 3. This completes the proof. \( \Box \)

Now we can easily prove Theorem 5 by using Proposition 6.

**Proof of Theorem 5** By the inner-outer factorization, there exist an inner function \( I \) and an outer function \( O_{\phi} \) such that \( f = IO_{\phi} \). Since \( O_{\phi} = O_{\min\{\phi,1\}}O_{\max\{\phi,1\}} \), we have \( f = f_1/f_2 \), where \( f_1 = IO_{\min\{\phi,1\}} \) and \( f_2 = 1/O_{\max\{\phi,1\}} \). Applying Proposition 6 together with the inequalities

\[
|O_{\min\{\phi,1\}}(z)| \leq 1 \leq |O_{\max\{\phi,1\}}(z)|
\]

and

\[
|f_2'(z)| \leq |O_{\max\{\phi,1\}}(z)|^2 |f_2'(z)| = |O_{\max\{\phi,1\}}'(z)|, \quad z \in \mathbb{D},
\]

we can check that \( f_1 \) and \( f_2 \) belong to \( B^{p,q}_{v} \cap H^\infty \). Moreover, it is obvious that \( f_2 \) is an outer function. Hence the proof is complete. \( \Box \)

### 4 Product of \( f \in H^p \) and an Inner Function in \( B^{p,q}_{v} \)

Theorem 7 below gives a sufficient and necessary condition guaranteeing that the product of \( f \in H^p \) and an inner function belongs to \( B^{p,q}_{v} \). This result generalizes [6, Corollary 3.2], the essential contents of [6, Corollary 3.1] and [9, Theorem 3.2].

**Theorem 7** Let \( 1 \leq p < \infty, 0 < q < \infty, v \in \mathcal{D} \cap \hat{\mathcal{D}}_q, f \in H^p, \) and \( I \) be an inner function. Then \( fI \in B^{p,q}_{v} \) if and only if \( f \in B^{p,q}_{v} \) and

\[
\int_0^1 \left( \int_0^{2\pi} \left( \frac{|f(re^{it})|(1 - |I(re^{it})|)}{1 - r} \right)^p \frac{dt}{d\theta} \right)^{q/p} v(r) dr < \infty.
\]
Proof We have
\[
\int_0^{2\pi} |f_I(e^{i\theta})|d\mu_z(\theta) - |f_I(z)| = \left( \int_0^{2\pi} |f(e^{i\theta})|d\mu_z(\theta) - |f(z)| \right) + |f(z)|(1 - |I(z)|) \tag{4.1}
\]
for all \(z \in \mathbb{D}\). Write \(z = re^{it}\). Raising both sides of (4.1) to power \(p\), integrating from 0 to \(2\pi\) with respect to \(dt\), then raising both sides to power \(q/p\), integrating from 0 to 1 with respect to \(v(r)dr/(1 - r)^q\) and finally splitting the right-hand side into two parts by using well-known inequalities, we obtain
\[
F_1(f_I) \asymp F_1(f) + \int_0^1 \left( \int_0^{2\pi} \left( \frac{|f(re^{it})|(1 - |I(re^{it})|)}{1 - r} \right)^p dt \right)^{q/p} v(r)dr.
\]
Since
\[
F_2(f_I) + \|f_I\|_{H^p}^q = F_2(f) + \|f\|_{H^p}^q,
\]
the assertion follows from Theorem 3.

Recall that a subspace \(X\) of \(H^p\) satisfies the \(F\)-property if the hypothesis \(f_I \in X\), where \(f \in H^p\) and \(I\) is an inner function, implies \(f \in X\). The \(F\)-property for \(B^p_{v,q}\) is a direct consequence of Theorem 7. However, it is worth mentioning that if one just aims to prove the \(F\)-property for \(B^p_{v,q}\), our argument is maybe not the simplest one, taking into account the length of proofs of Theorem 3 and its auxiliary results. Ideas for an alternative proof can be found, for instance, in [16, Section 5.8.3].

A sequence \(\{z_n\} \subset \mathbb{D}\) is said to be a zero set of \(B^p_{v,q}\) if there exists \(f \in B^p_{v,q}\) such that \(\{z : f(z) = 0\} = \{z_n\}\). Here each zero \(z_n\) is repeated according to its multiplicity and function \(f\) is not identically zero. Applying Theorem 7, we make some observations on zero sets of \(B^p_{v,q}\). More precisely, we concentrate on the case where \(\{z_n\}\) is separated, which means that there exists \(\delta = \delta(\{z_n\}) > 0\) such that \(d(z_n, z_k) > \delta\) for all \(n \neq k\), where
\[
d(z, w) = \left| \frac{z - w}{1 - \overline{z}w} \right|, \quad z, w \in \mathbb{D},
\]
is the pseudo-hyperbolic distance between points \(z\) and \(w\). Before these results some basic properties of Hardy spaces are recalled.

For \(\{z_n\} \subset \mathbb{D}\) satisfying the Blaschke condition \(\sum_n (1 - |z_n|) < \infty\) and a point \(\theta \in [0, 2\pi)\), the Blaschke product with zeros \(\{z_n\}\) is defined by
\[
B(z) = e^{i\theta} \prod_n \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n}z}, \quad z \in \mathbb{D}.
\]
For \(z_n = 0\), the interpretation \(|z_n|/z_n = -1\) is used. By factorization [8, Theorem 2.5] made by F. Riesz, we know that any \(f \in H^p\) for some fixed \(p \in (0, \infty]\) can be
represented in the form \( f = Bg \), where \( B \) is a Blaschke product and \( g \in H^p \) does not vanish in \( \mathbb{D} \). More precisely, Beurling factorization [8, Theorem 2.8] asserts that \( g \) is the product of an outer function and a singular inner function

\[
S(z) = \exp \left( \int_{\mathbb{T}} \frac{z + \xi}{z - \xi} \, d\sigma(\xi) + i\theta \right), \quad z \in \mathbb{D},
\]

where \( \theta \in [0, 2\pi) \) is a constant and \( \sigma \) is a positive measure on \( \mathbb{T} \), singular with respect to the Lebesgue measure. Consequently, every zero set of \( B^{p,q} \) satisfies the Blaschke condition. With these preparations we are ready to state and prove the following result.

**Corollary 8** Let \( 1 \leq p < \infty \), \( \nu \in \mathcal{D} \cap \hat{\mathcal{D}}_p \), and assume that \( \{z_n\} \) is a finite union of separated sequences and zero set of \( B^{p,p}_\nu \). Then there exists an outer function \( O_\phi \in B^{p,p}_\nu \) such that

\[
\sum_n |O_\phi(z_n)|^p \frac{\hat{\nu}(z_n)}{(1 - |z_n|)^{p-1}} < \infty.
\]

**Proof** Let \( \{z_n\} = \bigcup_{j=1}^M \{z_{n,j}\} \), where \( M \in \mathbb{N} \) and each \( \{z_{n,j}\} \) is separated. Let \( B \) be the Blaschke product with zeros \( \{z_n\} \), \( S \) a singular inner function and \( O_\phi \) an outer function such that \( BS O_\phi \in B^{p,q}_\nu \). By Theorem 7, we know that \( O_\phi \) and \( BO_\phi \) belong to \( B^{p,p}_\nu \).

For \( w \in \mathbb{D} \) and \( 0 < r < 1 \), set

\[
\Delta(w, r) = \{ z : d(z, w) < r \}
\]

and

\[
\Lambda(w, r) = \{ z : |w - z| < r(1 - |w|) \}.
\]

Since each \( \{z_{n,j}\} \) is separated, we find \( R_j, \delta_j \in (0, 1) \) such that, for a fixed \( j \), discs \( \Lambda(z_{n,j}, R_j) \) are pairwise disjoint and the inclusion \( \Delta(z_{n,j}, \delta_j) \subseteq \Lambda(z_{n,j}, R_j) \) is valid for every \( n \). Hence \( \hat{\nu} \) is essentially constant in each disc \( \Delta(z_{n,j}, \delta_j) \) by Lemma A(ii). Moreover,

\[
|B(z)| \leq \left| \frac{z_{n,j} - z}{1 - \overline{z_{n,j}} z} \right| \leq \delta_j, \quad z \in \Delta(z_{n,j}, \delta_j).
\]

Using these facts together with the subharmonicity of \( |O_\phi|^p \), we obtain

\[
\sum_n |O_\phi(z_n)|^p \frac{\hat{\nu}(z_n)}{(1 - |z_n|)^{p-1}} = \sum_{j=1}^M \sum_n |O_\phi(z_{n,j})|^p \frac{\hat{\nu}(z_{n,j})}{(1 - |z_{n,j}|)^{p-1}}
\]

\[
\lesssim \sum_{j=1}^M \sum_n \int_{\Delta(z_{n,j}, \delta_j)} |O_\phi(z)|^p \, dA(z) \frac{\hat{\nu}(z_{n,j})}{(1 - |z_{n,j}|)^{p+1}}
\]

\[
\lesssim \sum_{j=1}^M \sum_n \int_{\Delta(z_{n,j}, \delta_j)} |O_\phi(z)|^p \frac{\hat{\nu}(z)}{(1 - |z|)^{p+1}} \, dA(z)
\]
\[
\sum_{j=1}^{M} (1 - \delta_j)^{-p} \sum_{n} \int_{\Delta(z_j, \delta_j)} \left( \frac{|O_\phi(z)| (1 - |B(z)|)}{1 - |z|} \right)^p \frac{\hat{\nu}(z)}{1 - |z|} dA(z)
\]

where \(dA(z)\) is the two-dimensional Lebesgue measure. Now it suffices to show that the last integral in (4.2) is finite.

Set \(\psi(z) = \hat{\nu}(z)/(1 - |z|)\) for \(z \in \mathbb{D}\). Note that \(\hat{\nu}(r) \asymp \hat{\psi}(r)\) for \(0 \leq r < 1\) by Lemmas A(ii) and B. Moreover, integrating by parts, one can show that \(\nu \in \hat{D}_p\) if and only if

\[
(1 - r)^p \int_0^r \frac{\hat{\nu}(s)}{(1 - s)^{p+1}} ds \asymp 1, \quad r \to 1^-.
\]

In particular, \(\psi \in D \cap \hat{D}_p\) by the hypotheses of \(\nu\). Since Lemma D implies \(BO_\phi\) in \(B_{\psi}^{p,p}\), Theorem 7 gives

\[
\int_{\mathbb{D}} \left( \frac{|O_\phi(z)| (1 - |B(z)|)}{1 - |z|} \right)^p \frac{\hat{\nu}(z)}{1 - |z|} dA(z) < \infty.
\]

This completes the proof.

Recall that a sequence \(\{z_n\} \subset \mathbb{D}\) is said to be uniformly separated if

\[
\inf_{n \in \mathbb{N}} \prod_{k \neq n} \left| \frac{z_k - z_n}{1 - \overline{z}_k z_n} \right| > 0;
\]

and a finite union of uniformly separated sequences is called a Carleson–Newman sequence. It is worth mentioning that any Carleson–Newman sequence is a finite union of separated sequences satisfying the Blaschke condition, but the converse statement is not true. For \(1 < p < \infty\), \(p - 2 < \alpha < p - 1\), and a Carleson–Newman sequence \(\{z_n\}\), we can give a sufficient and necessary condition for \(\{z_n\}\) to be a zero set of \(B_\alpha^{p,p}\). This is a straightforward consequence of Theorem 7, Corollary 8, and the reasoning made in paper [4] by N. Arcozzi, D. Blasi and J. Pau.

**Corollary 9** Let \(1 < p < \infty\), \(p - 2 < \alpha < p - 1\), and \(\{z_n\}\) be a Carleson–Newman sequence. Then \(\{z_n\}\) is a zero set of \(B_\alpha^{p,p}\) if and only if there exists an outer function \(O_\phi \in B_\alpha^{p,p}\) such that

\[
\sum_n |O_\phi(z_n)|^p (1 - |z_n|)^{\alpha+2-p} < \infty.
\]

**Proof** Let \(B\) be the Blaschke product with zeros \(\{z_n\}\) and \(O_\phi \in B_\alpha^{p,p}\) an outer function satisfying (4.4). Then [14, Theorem 3.5] together with some elementary calculations.

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gives
\[
\int_D \left( \frac{|O_\phi(z)|(1 - |B(z)|)}{1 - |z|} \right)^p (1 - |z|)^\alpha dA(z)
\]
\[
\leq 2 \int_D |O_\phi(z)|^p \sum_n \frac{1 - |z_n|^2}{|1 - \overline{z}_n z|^2} (1 - |z|)^{\alpha + 1 - p} dA(z)
\]
\[
\lesssim \sum_n \int_D |O_\phi(z_n)|^p \frac{1 - |z_n|^2}{|1 - \overline{z}_n z|^2} (1 - |z|)^{\alpha + 1 - p} dA(z)
\]
\[
+ \sum_n \int_D |O_\phi(z) - O_\phi(z_n)|^p \frac{1 - |z_n|^2}{|1 - \overline{z}_n z|^2} (1 - |z|)^{\alpha + 1 - p} dA(z)
\]
\[
=: I_1 + I_2.
\]

Following the reasoning in the proof of [4, Proposition 3.2], it is easy to check that $I_1$ and $I_2$ are finite. More precisely, estimating in a natural manner, one can show
\[
I_1 \lesssim \sum_n |O_\phi(z_n)|^p (1 - |z_n|)^{\alpha + 2 - p} < \infty.
\]

In the argument of $I_2 \lesssim \|O_\phi\|_{\mathcal{B}_{\alpha}^{p,p}} < \infty$, [5, Lemma 2.1] and the hypothesis that \{z\} is a Carleson–Newman sequence play key roles.

Since $O_\phi \in \mathcal{B}_{\alpha}^{p,p}$ and the first integral in (4.5) is finite, $BO_\phi$ belongs to $\mathcal{B}_{\alpha}^{p,p}$ by Theorem 7. Consequently, the implication $\Leftarrow$ is valid. The converse implication is a direct consequence of Corollary 8. Hence the proof is complete. \qed

It is an open problem to prove a $\mathcal{B}_{\alpha}^{p,p}$ counterpart of Corollary 9. One could try to prove such a result, for instance, assuming $\nu \in \mathcal{D} \cap \hat{\mathcal{D}}_p$ and
\[
\sup_{0 \leq r < 1} \frac{(1 - r)^{p-1}}{\hat{\nu}(r)} \int_r^1 \frac{\nu(s)}{(1 - s)^{p-1}} ds < \infty.
\]

In this case, the implication $\Leftarrow$ is the problematic part. An idea to approach this problem is to follow the argument of [4, Proposition 3.2] and aim to apply therein [3, Theorem 3.1] instead of [5, Lemma 2.1]. The downside of this method is that it leads to laborious computations of Bekollé–Bonami weights.

Corollaries 8 and 9 are related to some main results in [15] by J. Pau and J. A. Peláez. In particular, the equivalence (i) $\iff$ (ii) in [15, Theorem 1] follows from Corollary 9 by setting $p = 2$. Moreover, Corollary 8 shows that the implication (i) $\Rightarrow$ (ii) in [15, Theorem 1] is valid also if \{z\} in the statement is a finite union of separated sequences. Applying the last observation, we can also replace a Carleson–Newman sequence in [15, Corollary 1] by a finite union of separated sequences: If $0 < \alpha < 1$, \{z\} is a finite union of separated sequences and zero set of $\mathcal{B}_{\alpha}^{2,2}$, then
\[
\int_0^{2\pi} \log \left( \sum_n \frac{(1 - |z_n|)^{\alpha+1}}{|e^{i\theta} - z_n|^2} \right) d\theta < \infty.
\]
This result offers a practical way to construct Blaschke sequences that are not zero sets of \( B^{2,2}_α \); see [15, Theorem 2] and its proof.

Note that (4.2) and (4.5) together with the estimates for \( I_1 \) and \( I_2 \) are valid also if outer function \( O_φ \) is replaced by an arbitrary \( f \in H^p \). Using this observation and Theorem 7, we can rewrite Corollary 9 in the following form.

**Corollary 10** Let \( 1 < p < \infty \), \( p - 2 < \alpha < p - 1 \), \( f \in H^p \), and \( B \) be a Blaschke product associated with a Carleson–Newman sequence \( \{z_n\} \). Then \( fB \in B^{p,p}_α \) if and only if \( f \in B^{p,p}_α \) and

\[
\sum_n |f(z_n)|^p (1 - |z_n|)^{\alpha + 2 - p} < \infty.
\]

Corollary 10 is a partial improvement of the main result in M. Jevtić’s paper [11]. More precisely, this paper contains an extended counterpart of Corollary 10 (in the sense of \( p \) and \( q \)) with the defect \( f \equiv 1 \). It is also worth mentioning that Corollary 10 is not valid if the Carleson–Newman sequence \( \{z_n\} \) is replaced by an arbitrary Blaschke sequence. This can be shown by studying the case where \( f \equiv 1 \) and \( B \) is a Blaschke product with zeros on the positive real axis. More precisely, the counter example follows from [23, Theorem 1], which asserts that all such Blaschke products belong to \( B^{p,p}_α \) for \( 1/2 < p < \infty \) and \( p - 3/2 < \alpha < \infty \).

Theorem 7 for \( f \equiv 1 \) (or Theorem 3 for inner functions) also has the extended counterpart [22, Theorem 1].

**Theorem E** Let \( 0 < p, q < \infty \), and \( v \in \mathcal{D} \). Then \( v \in \hat{\mathcal{D}}_q \) if and only if

\[
\|I\|_{A^{p,q}}^q \asymp \int_0^1 \left( \int_0^{2\pi} \left( \frac{1 - |I(re^{i\theta})|}{1 - r} \right)^p d\theta \right)^{q/p} v(r) \, dr
\]

for all inner functions \( I \). Here the comparison constants may depend only on \( p, q, \) and \( v \).

Theorem E confirms that the hypothesis \( v \in \hat{\mathcal{D}}_q \) in Theorems 3 and 7 is sharp in a certain sense. Studying the argument of this result in [22], we can also deduce that the proof of Theorem 3 is more straightforward when \( f \) is an inner function, and the statement is valid for all \( 0 < p < \infty \). It is also worth mentioning that results like Theorem E have turned out to be useful in the theory of inner functions. Several by-products of Theorem E can be found in [22,24].

**5 Proof of Theorem 1**

Before the proof of Theorem 1 we recall [22, Lemma 6], which is a modification of [1, Lemma 5].
Lemma F If \(0 < p \leq 1\) and \(g : [0, 1) \to [0, \infty)\) is measurable, then
\[
\left( \int_r^1 g(s)ds \right)^p \leq 2 \int_r^1 \sup_{0 \leq x \leq s} g(x)^p (1 - s)^{p-1} ds
\]
for \(0 \leq r < 1\).

Proof of Theorem 1 Let \(\frac{4}{5} \leq s < 1\) and choose \(n = n(s) \in \mathbb{N} \setminus \{1, 2, 3, 4\}\) such that \(1 - \frac{1}{n} \leq s < 1 - \frac{1}{n+1}\). Set \(f_n(z) = z^n\) for \(z \in \mathbb{D}\). Since
\[
|e^{in(\theta+h)} - e^{in\theta}|^2 = |1 - e^{inh}|^2 = 2(1 - \cos(nh)) = 2n^2h^2 \sum_{k=1}^{\infty} (-1)^{k-1} (nh)^{2(k-1)} (2k)!
\]
we have
\[
\int_{1/2}^1 \omega_p(1 - r, f_n)q \frac{v(r)}{(1 - r)^q} dr \leq \left( \int_{1/2}^{1-2/n} + \int_{1-2/n}^1 \right) \sup_{0 < h < 1 - r} |1 - e^{inh}|q \frac{v(r)}{(1 - r)^q} dr
\]
\[
\geq \int_{1/2}^{1-2/n} |1 - e^{2i\theta}|q \frac{v(r)}{(1 - r)^q} dr + \int_{1-2/n}^s |1 - e^{in(1-r)}|q \frac{v(r)}{(1 - r)^q} dr
\]
\[
\geq \int_{1/2}^{1-2/n} \frac{v(r)}{(1 - r)^q} dr + \int_{1-2/n}^s (n + 1)^q v(r) dr
\]
\[
\geq \int_{1/2}^s \frac{v(r)}{(1 - r)^q} dr + (1 - s)^{-q} \int_{1-2/n}^s v(r) dr
\]
\[
\geq \int_{1/2}^s \frac{v(r)}{(1 - r)^q} dr \geq \int_{0}^s \frac{v(r)}{(1 - r)^q} dr.
\]

Using the hypothesis \(v \in \hat{D}\) together with Lemma A(iv)(ii) in a similar manner as in the proof of [21, Theorem 1], we obtain
\[
\|f_n\|^q_{A_p} \leq n^q \int_0^1 r^{q(n-1)+1} v(r) dr \leq n^q \int_1^{1/r_{n(n-1)+1}} v(r) dr
\]
\[
\leq n^q \int_1^{s/r_{n(n-1)+1}} v(r) dr \leq \frac{\hat{v}(s)}{(1 - s)^q}.
\]

Finally combining the estimates above and using the inequality
\[
\int_0^t \frac{v(r)}{(1 - r)^q} dr \leq 1 \times \frac{\hat{v}(t)}{(1 - t)^q}, \quad 0 < t < \frac{4}{5},
\]
we deduce that if \( \nu \in \hat{D} \) and (1.2) is satisfied for all \( f \in H^p \), then \( \nu \in \hat{D}_p \). Hence it suffices to prove the converse statement.

Let \( f \in H^p, 0 \leq \theta < 2\pi, \frac{1}{2} < r < 1 \), and \( 0 < h < \frac{1}{2} \). Set \( \rho = r - h \) and let \( \Gamma \) be the contour that goes first rapidly from \( re^{i\theta} \) to \( \rho e^{i\theta} \), then along the circle \( \{ z : |z| = \rho \} \) to \( \rho e^{i(\theta+h)} \), and finally rapidly to \( re^{i(\theta+h)} \). Since

\[
f(re^{i(\theta+h)}) - f(re^{i\theta}) = \int_{\Gamma} f'(z) \, dz,
\]

we have

\[
|f(re^{i(\theta+h)}) - f(re^{i\theta})| \leq \int_{\rho}^{r} |f'(\rho e^{it})| \, dt + \int_{\rho}^{r} |f'(se^{i(\theta+h)})| \, ds.
\]

Consequently, the discrete and continuous forms of Minkowski’s inequality, a change of variable, and Hardy’s convexity theorem yield

\[
\left( \int_{0}^{2\pi} |f(re^{i(\theta+h)}) - f(re^{i\theta})|^p \, d\theta \right)^{1/p} \leq \left( \int_{0}^{2\pi} \left( \int_{\rho}^{r} |f'(se^{i\theta})| \, ds \right)^p \, d\theta \right)^{1/p}
\]

\[
+ \left( \int_{0}^{2\pi} \left( \int_{0}^{h} |f'(\rho e^{i(x+\theta)})| \, dx \right)^p \, d\theta \right)^{1/p}
\]

\[
+ \left( \int_{0}^{2\pi} \left( \int_{\rho}^{r} |f'(se^{i(\theta+h)})| \, ds \right)^p \, d\theta \right)^{1/p}
\]

\[
\leq 2 \int_{\rho}^{r} M_p(s, f') \, ds + hM_p(\rho, f') \leq 3 \int_{r-h}^{r} M_p(s, f') \, ds.
\]

Note that the deduction above can be found, for instance, in the proof of [8, Theorem 5.4].

By raising both sides of (5.1) to power \( q \), adding \( \sup_{0<h<1-t} \), and then integrating from \( 1/2 \) to \( r \) with respect to \( \nu(t) \, dt/(1-t)^q \), we obtain

\[
\int_{1/2}^{r} \sup_{0<h<1-t} \left( \int_{0}^{2\pi} |f(re^{i(\theta+h)}) - f(re^{i\theta})|^p \, d\theta \right)^{q/p} \frac{\nu(t)}{(1-t)^q} \, dt
\]

\[
\lesssim \int_{1/2}^{r} \left( \int_{r-(1-t)}^{r} M_p(s, f') \, ds \right)^{q} \frac{\nu(t)}{(1-t)^q} \, dt.
\]

Letting \( r \to 1^- \) and using the monotone and mean convergence theorems together with the hypothesis \( f \in H^p \), we deduce

\[
\int_{1/2}^{1} \omega_p(1-t, f)^q \frac{\nu(t)}{(1-t)^q} \, dt \lesssim \int_{1/2}^{1} \left( \int_{1}^{1} M_p(s, f') \, ds \right)^{q} \frac{\nu(t)}{(1-t)^q} \, dt =: \mathcal{I}.
\]
Hence it suffices to show $I \lesssim \|f'\|_{A^p_v}^q$. Note that the argument of this estimate uses ideas from [22].

If $q \leq 1$, then Lemma F with the choice $g(s) = M_p(s, f')$, Hardy’s convexity theorem, Fubini’s theorem, the hypothesis $\nu \in \hat{D}_q$, and Lemma D give

$$I \lesssim \int_0^1 \frac{v(t)}{(1-t)^q} \int_t^1 \sup_{0 \leq x \leq s} M_p^q(x, f')(1-s)^{q-1} \, ds \, dt$$

$$= \int_0^1 \frac{v(t)}{(1-t)^q} \int_t^1 M_p^q(s, f')(1-s)^{q-1} \, ds \, dt$$

$$= \int_0^1 M_p^q(s, f')(1-s)^{q-1} \int_0^s \frac{v(t)}{(1-t)^q} \, dt \, ds$$

$$\lesssim \int_0^1 M_p^q(s, f') \frac{\tilde{\nu}(s)}{1-s} \, ds \asymp \|f'\|_{A^p_v}^q$$

for all $f \in H(D)$. Hence the assertion for $q \leq 1$ is proved. If $q > 1$, $0 < \epsilon < q/(\hat{D}_q(\nu) + 1)$, and $h(s) = (1-s)^{\frac{q-1-\epsilon}{q}}$, then Hölder’s inequality and Fubini’s theorem yield

$$I \lesssim \int_0^1 \int_t^1 M_p^q(s, f') h(s)^q \, ds \left( \int_t^1 h(r)^{-\frac{q}{q-\epsilon}} \, dr \right)^{q-1} \frac{v(t)}{(1-t)^q} \, dt$$

$$\times \int_0^1 \frac{v(t)}{(1-t)^{q-\epsilon}} \int_t^1 M_p^q(s, f')(1-s)^{q-1-\epsilon} \, ds \, dt$$

$$= \int_0^1 M_p^q(s, f')(1-s)^{q-1-\epsilon} \int_0^s \frac{v(t)}{(1-t)^{q-\epsilon}} \, dt \, ds$$

for all $f \in H(D)$. Since $\nu \in \hat{D}_{q-\epsilon}$ by Lemma C, the assertion for $q > 1$ follows from Lemma D. This completes the proof.

Since

$$\int_0^{1/2} \omega_p(1-t, f)^q \frac{v(t)}{(1-t)^q} \, dt \leq 2^{2q} \|f\|_{H^p}^q \int_0^{1/2} v(t) \, dt$$

by Minkowski’s inequality, Theorem 1 has the following consequence.

**Corollary 11** Let $1 \leq p < \infty$, $0 < q < \infty$, and $\nu \in D \cap \hat{D}_q$. Then there exists a constant $C = C(p, q, \nu) > 0$ such that

$$\int_0^1 \omega_p(1-r, f)^q \frac{v(r)}{(1-r)^q} \, dr \leq C \left( \|f'\|_{A^p_v}^q + \|f\|_{H^p}^q \right)$$

for all $f \in H^p$.

Note that Corollary 11 is a part of Theorem 2. We state it here as an independent result because it is needed for the proof of Theorem 2.
6 Proof of Theorem 2

We go directly to the proof of Theorem 2.

Proof of Theorem 2 Let $0 \leq r < 1$ and $0 \leq t < 2\pi$. Since

$$\int_0^{2\pi} \frac{e^{i\theta} d\theta}{(e^{i\theta} - re^{it})^2} = 0,$$

Cauchy’s integral formula gives

$$|f'(re^{it})| = \frac{1}{2\pi(1 - r^2)} \left| \int_0^{2\pi} \left( f(e^{i\theta}) - f(re^{it}) \right) \frac{e^{i\theta} (1 - r^2)}{(e^{i\theta} - re^{it})^2} d\theta \right| \leq \frac{1}{1 - r} \int_0^{2\pi} \left| f(e^{i\theta}) - f(re^{it}) \right| d\mu_{re^{it}}(\theta), \ f \in H^1.$$

Raising both sides to power $p$, integrating from 0 to $2\pi$ with respect to $dt$, then raising both sides to power $q/p$, and finally integrating from 0 to 1 with respect to $\nu(r) \, dr$, we obtain

$$\|f'\|_{A^q_{p,q}}^q \leq \int_0^1 \left( \int_0^{2\pi} \left( \int_0^{2\pi} \left| f(e^{i\theta}) - f(re^{it}) \right| d\mu_{re^{it}}(\theta) \right)^p \, dt \right)^{q/p} \frac{\nu(r)}{(1 - r)^q} \, dr$$

(6.1)

for all $f \in H^p$.

Let $f \in H^p$, $0 < q \leq 1$, and set

$$I(r) = \left( \int_0^{2\pi} \left( \int_0^{2\pi} \left| f(e^{i\theta}) - f(re^{it}) \right| d\mu_{re^{it}}(\theta) \right)^p \, dt \right)^{q/p}.$$

By the proof of [9, Theorem 2.1], we know that

$$I(r) \lesssim \left( \sum_{k=0}^{\infty} 2^{-k} \omega_p(2^k(1 - r), f) \right)^q.$$

(6.2)

Hence the sub-additivity of $g(x) = x^q$ for $x \geq 0$ and Fubini’s theorem give

$$\int_0^1 I(r) \nu(r) \frac{dr}{(1 - r)^q} \lesssim \sum_{k=0}^{\infty} 2^{-qk} \int_0^1 \omega_p(2^k(1 - r), f)^q \frac{\nu(r)}{(1 - r)^q} \, dr.$$

(6.3)

Next we show that the weight $\nu(r)$ in the right-hand side can be replaced by $\hat{\nu}(r)/r$ without losing any essential information.

Set $\psi(z) = \hat{\nu}(z)/(1 - |z|)$ for $z \in \mathbb{D}$, and recall that $\hat{\nu}(r) \asymp \hat{\psi}(r)$ for $0 \leq r < 1$ by Lemmas A(ii) and B. In particular, $\psi$ belongs to class $\mathcal{D}$, and thus, there exist
\( K = K(\psi) > 1 \) and \( C = C(\psi) > 1 \) such that
\[
\hat{\psi}(r) \geq C \hat{\psi}\left(1 - \frac{1-r}{K}\right), \quad 0 \leq r < 1.
\] (6.4)

Let \( k \in \mathbb{N} \cup \{0\} \) and \( r_n = 1 - K^{-n} \) for \( n \in \mathbb{N} \cup \{0\} \). Using (6.4) together with Lemma A(ii), we obtain
\[
(C - 1) \hat{\psi}(r_{n+1}) = C \hat{\psi}\left(1 - \frac{1-r_n}{K}\right) - \hat{\psi}(r_{n+1}) \leq \hat{\psi}(r_n) - \hat{\psi}(r_{n+1})
\]
\[
= \int_{r_n}^{r_{n+1}} \psi(r) \, dr \leq \hat{\psi}(r_n) \times \hat{\psi}(r_{n+1}).
\] (6.5)

Now Minkowski’s inequality, the monotonicity of \( \omega_p(s, f) \) with \( s \) and (6.5) yield
\[
\int_0^1 \omega_p(2^k(1-r), f)^q \frac{v(r)}{(1-r)^q} \, dr
\]
\[
\lesssim \sum_{n=1}^{\infty} \int_{r_n}^{r_{n+1}} \omega_p(2^k(1-r), f)^q \frac{v(r)}{(1-r)^q} \, dr + \|f\|_H^q
\]
\[
\leq \sum_{n=1}^{\infty} \omega_p(2^k(1-r_n), f)^q \frac{\hat{\nu}(r_n)}{(1-r_n)^q} + \|f\|_H^q
\]
\[
\times \sum_{n=1}^{\infty} \omega_p(2^k(1-r_n), f)^q \frac{\hat{\psi}(r_n) - \hat{\psi}(r) \, dr}{(1-r)^q} + \|f\|_H^q
\]
\[
\leq \sum_{n=0}^{\infty} \int_{r_n}^{r_{n+1}} \omega_p(2^k(1-r), f)^q \frac{\hat{\psi}(r_n) - \hat{\psi}(r) \, dr}{(1-r)^q} + \|f\|_H^q
\]
\[
= \int_0^1 \omega_p(2^k(1-r), f)^q \frac{\hat{\nu}(r)}{(1-r)^{q+1}} \, dr + \|f\|_H^q.
\] (6.6)

It is worth noting that a similar deduction works also in the opposite direction.

Using (6.3) and (6.6), we obtain
\[
\int_0^1 I(r) v(r) \frac{(1-r)^q}{(1-r)^q} \, dr \lesssim \sum_{k=0}^{\infty} 2^{-qk} \int_0^{1-2^{-k}} \omega_p(2^k(1-r), f)^q \frac{\hat{\nu}(r)}{(1-r)^{q+1}} \, dr + \|f\|_H^q
\]
\[
+ \sum_{k=0}^{\infty} 2^{-qk} \int_{1-2^{-k}}^1 \omega_p(2^k(1-r), f)^q \frac{\hat{\nu}(r)}{(1-r)^{q+1}} \, dr
\]
\[
=: I_1 + I_2.
\]
Minkowski’s inequality, (4.3) with $p$ being replaced by $q$, and Lemma B yield

$$I_1 \lesssim \|f\|_{H^p}^q \sum_{k=0}^{\infty} 2^{-qk} \int_0^{1-2^{-k}} \frac{\widehat{\nu}(r)}{(1-r)^{q+1}} \, dr$$

$$\lesssim \|f\|_{H^p}^q \sum_{n=0}^{\infty} \widehat{\nu}(1-2^{-k})$$

$$\lesssim \widehat{\nu}(0) \|f\|_{H^p}^q \sum_{n=0}^{\infty} 2^{-\alpha k} \lesssim \|f\|_{H^p}^q$$

for some $\alpha = \alpha(\nu) > 0$. The continuity of $\widehat{\nu}$, changes of variables, Fubini’s theorem, and the hypothesis $\nu \in D$ give

$$I_2 \asymp \int_0^{1} \frac{\omega_p(1-s, f)^q}{(1-s)^{q+1}} \int_0^{\infty} \widehat{\nu}(1-2^{-k}(1-s)) \, dk \, ds$$

$$= \frac{1}{\log 2} \int_0^{1} \frac{\omega_p(1-s, f)^q}{(1-s)^{q+1}} \int_s^{1} \frac{\widehat{\nu}(x)}{1-x} \, dx \, ds$$

$$\asymp \int_0^{1} \omega_p(1-s, f)^q \frac{\widehat{\nu}(s)}{(1-s)^{q+1}} \, ds.$$

Summarizing, we have shown

$$\int_0^{1} \frac{I(r)\nu(r)}{(1-r)^q} \, dr \lesssim \int_0^{1} \omega_p(1-s, f)^q \frac{\widehat{\nu}(s)}{(1-s)^{q+1}} \, ds + \|f\|_{H^p}^q. \quad (6.7)$$

Applying a similar argument as in (6.6), we can replace $\widehat{\nu}(s)$ in the right-hand side of (6.7) by $\nu(s)(1-s)$. Consequently, (6.1) and Corollary 11 imply (1.3) for all $f \in H^p$. Hence the assertion for $q \leq 1$ is proved.

Let $1 < q < \infty$. Then (6.2), the continuous form of Minkowski’s inequality, (6.6), and well-known inequalities give

$$\int_0^{1} \frac{I(r)\nu(r)}{(1-r)^q} \, dr \lesssim \int_0^{1} \left( \sum_{k=0}^{\infty} 2^{-k} \omega_p(2^k(1-r), f) \right)^q \frac{\nu(r)}{(1-r)^q} \, dr$$

$$\lesssim \left( \sum_{k=0}^{\infty} 2^{-k} \left( \int_0^{1} \omega_p(2^k(1-r), f)^q \frac{\nu(r)}{(1-r)^q} \, dr \right)^{1/q} \right)^q$$

$$\lesssim \left( \sum_{k=0}^{\infty} 2^{-k} \left( \int_0^{1} \omega_p(2^k(1-r), f)^q \frac{\widehat{\nu}(r)}{(1-r)^{q+1}} \, dr \right)^{1/q} \right)^q + \|f\|_{H^p}^q$$

$$\times \left[ \left( \sum_{k=0}^{\infty} 2^{-k} \left( \int_0^{1-2^{-k}} \omega_p(2^k(1-r), f)^q \frac{\widehat{\nu}(r)}{(1-r)^{q+1}} \, dr \right)^{1/q} \right)^q + \|f\|_{H^p}^q \right]$$
\[ + \left( \sum_{k=0}^{\infty} 2^{-k} \left( \int_{1-2^{-k}}^{1} \omega_p(2^k(1-r), f)^q \frac{\hat{\nu}(r)}{(1-r)^q+1} \, dr \right) \right)^{1/q} \]

\[ =: I_3 + I_4. \]

Minkowski’s inequality, (4.3) with \( p \) being replaced by \( q \), and Lemma B yield

\[ I_3 \lesssim \| f \|_q^{H_p} \left( \sum_{k=0}^{\infty} 2^{-k} \left( \int_{0}^{1-2^{-k}} \frac{\hat{\nu}(r)}{(1-r)^q+1} \, dr \right) \right)^{1/q} \]

\[ \lesssim \| f \|_q^{H_p} \left( \sum_{k=0}^{\infty} \hat{\nu}(1-2^{-k})^{1/q} \right)^q \propto \| f \|_q^{H_p}. \]

By Lemma B, there exists a constant \( \alpha = \alpha(\nu) > 0 \) such that

\[ \hat{\nu} \left( 1 - 2^{-k}(1 - s) \right) \lesssim 2^{-\alpha k} \hat{\nu}(s), \quad 0 \leq s \leq 1 - 2^{-k}(1 - s) < 1. \]

Using this together with a change of variable and modification of (6.6), we get

\[ I_4 = \left( \sum_{k=0}^{\infty} \left( \int_{0}^{1} \omega_p(2^k(1-s), f)^q \frac{\hat{\nu}(1-2^{-k}(1-s))}{(1-s)^q+1} \, ds \right) \right)^{1/q} \]

\[ \lesssim \left( \sum_{k=0}^{\infty} 2^{-\alpha k/q} \left( \int_{0}^{1} \omega_p(1-s, f)^q \frac{\hat{\nu}(s)}{(1-s)^q+1} \, ds \right) \right)^{1/q} \]

\[ \times \int_{0}^{1} \omega_p(1-s, f)^q \frac{\nu(s)}{(1-s)^q} \, ds \]

\[ \lesssim \int_{0}^{1} \omega_p(1-s, f)^q \frac{\nu(s)}{(1-s)^q} \, ds + \| f \|_q^{H_p}. \]

Finally (6.1), (6.8), and Corollary 11 imply (1.3) for all \( f \in H^p \). This completes the proof. \( \square \)

### 7 Proof of Theorem 3

Before the proof of Theorem 3 we recall the following result, which is a part of the argument of [6, Theorem 1.1].

**Lemma G** If \( O_\phi \) is an outer function, then

\[
|O'_\phi(z)| \leq \frac{4}{1-|z|} \left( \int_{0}^{2\pi} \left| \phi(e^{i\theta}) - \int_{0}^{2\pi} \phi(e^{is}) d\mu_z(s) \right| \, d\mu_z(\theta) \right) + \int_{0}^{2\pi} \phi(e^{ih}) d\mu_z(h) - |O_\phi(z)|
\]

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for all $z \in \mathbb{D}$.

**Proof of Theorem 3** Let $f \in H^p$. Then there exist an inner function $I$ and an outer function $O_\phi$ such that $f = IO_\phi$. Hence the Schwarz–Pick lemma, Lemma G, and the fact that $\phi(\xi) = |f(\xi)|$ for almost every $\xi \in \mathbb{T}$ yield

\[
|f'(z)|(1 - |z|) \leq \left( |I(z)O_\phi'(z)| + |I'(z)O_\phi(z)| \right) (1 - |z|)
\]

\[
\leq |O_\phi'(z)|(1 - |z|) + 2|O_\phi(z)|(1 - |I(z)|)
\]

\[
\leq 4 \int_0^{2\pi} |f(e^{i\theta})| - \int_0^{2\pi} |f(e^{is})|d\mu_z(s)\ d\mu_z(\theta)
\]

\[
+ 4 \left( \int_0^{2\pi} |f(e^{ih})|d\mu_z(h) - |f(z)| \right)
\]

for all $z \in \mathbb{D}$. Write $z = re^{i\theta}$. Raising both sides of (7.1) to power $p$, integrating from 0 to $2\pi$ with respect to $dt$, then raising both sides to power $q/p$, integrating from 0 to 1 with respect to $\nu(r)dr/(1 - r)^q$, and finally splitting the right-hand side into two parts, we obtain

\[
\|f'\|_{A^p_{q,\nu}}^q \lesssim F_1(f) + F_2(f),
\]

which is the first inequality in (1.4).

Set

\[
\Gamma = \Gamma(z, f) = \left\{ \theta \in [0, 2\pi) : \int_0^{2\pi} |f(e^{is})|d\mu_z(s) \leq |f(e^{i\theta})| \right\}, \quad z \in \mathbb{D}.
\]

Then elementary calculations together with the subharmonicity of $|f|$ yield

\[
\int_0^{2\pi} |f(e^{i\theta})| - \int_0^{2\pi} |f(e^{is})|d\mu_z(s)\ d\mu_z(\theta)
\]

\[
= 2 \int_{\Gamma} \left( |f(e^{i\theta})| - \int_0^{2\pi} |f(e^{is})|d\mu_z(s) \right) d\mu_z(\theta)
\]

\[
\leq 2 \int_{\Gamma} \left( |f(e^{i\theta})| - |f(z)| \right) d\mu_z(\theta), \quad z \in \mathbb{D}.
\]

It follows that

\[
\int_0^{2\pi} |f(e^{i\theta})| - \int_0^{2\pi} |f(e^{is})|d\mu_z(s)\ d\mu_z(\theta) + \left( \int_0^{2\pi} |f(e^{ih})|d\mu_z(h) - |f(z)| \right)
\]

\[
\leq 2 \left( \int_{\Gamma} + \int_0^{2\pi} \right) |f(e^{i\theta}) - f(z)| d\mu_z(\theta)
\]

\[
\leq 4 \int_0^{2\pi} |f(e^{i\theta}) - f(z)| d\mu_z(\theta), \quad z \in \mathbb{D}.
\]
Doing a corresponding integration procedure for this estimate as above and applying Theorem 2, we obtain
\[ F_1(f) + F_2(f) \lesssim \| f' \|^q_{A^p_{\nu,q}} + \| f \|^{q}_{H^p}, \]
which is the last inequality in (1.4). This completes the proof. \(\Box\)

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