The Complexity of Reasoning with FODD and GFODD

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Abstract

Recent work introduced Generalized First Order Decision Diagrams (GFODD) as a knowledge representation that is useful in mechanizing decision theoretic planning in relational domains. GFODDs generalize function-free first order logic and include numerical values and numerical generalizations of existential and universal quantification. Previous work presented heuristic inference algorithms for GFODDs. In this paper, we study the complexity of the evaluation problem, the satisfiability problem, and the equivalence problem for GFODDs under the assumption that the size of the intended model is given with the problem, a restriction that guarantees decidability. Our results provide a complete characterization placing these problems within the polynomial hierarchy. The same characterization applies to the corresponding restriction of problems in first order logic, giving an interesting new avenue for efficient inference when the number of objects is bounded. Our results show that for \( \Sigma_k \) formulas, and for corresponding GFODDs, evaluation and satisfiability are \( \Sigma_p^k \) complete, and equivalence is \( \Pi_p^k+1 \) complete. For \( \Pi_k \) formulas evaluation is \( \Pi_p^k \) complete, satisfiability is one level higher and is \( \Sigma_p^k+1 \) complete, and equivalence is \( \Pi_p^k+1 \) complete.

1 Introduction

The complexity of inference in first order logic has been investigated intensively. It is well known that the problem is undecidable, and that this holds even with strong restrictions on the types and number of predicates allowed in the logical language. For example, the problem is undecidable for quantifier prefix \( \forall^2 \exists^* \) with a signature having single binary predicate and equality [8]. Unfortunately, the problem is also undecidable if we restrict attention to satisfiability under finite structures [8, 22]. Thus, in either case, more refined notions of complexity are not appropriate. On the other hand, algorithmic progress in AI has made it possible to reason efficiently in some cases. In this paper we study such problems under the additional restriction that an upper bound on the intended model size is given explicitly. This restriction is natural for the intended applications in AI and it renders the problems decidable allowing for a more detailed complexity analysis.

This paper is motivated by recent work on decision diagrams, known as FODDs and GFODDs and the computational questions associated with them. Binary decision diagrams [3, 11] are a successful knowledge representation capturing functions over propositional variables, that allows for efficient manipulation and composition of functions, and diagrams have been used in various applications in program verification and AI [3, 1, 11]. Motivated by this success, several authors have
attempted generalizations to handle relational structure and first order quantification \[9, 31, 28, 15\]. In particular FODDs \[31\] and their generalization GFODDs \[15\] have been introduced and shown to be useful in the context of decision theoretic planning \[2, 18, 11, 12\] for problems with relational structure \[14, 16\].

GFODDs can be seen to generalize the function-free portion of first order logic (i.e., signatures with constants but without higher arity functions) to allow for non-binary numerical values generalizing truth values, and for numerical quantifiers generalizing existential and universal quantification in logic. Efficient heuristic inference algorithms for such diagrams have been developed focusing on the finite model case, and using the notion of “reasoning from examples” \[20, 21, 19\]. This paper formalizes the evaluation, satisfiability, and equivalence problems for such diagrams and analyses their complexity. To avoid the undecidability and get a more refined classification of complexity, we study a restricted form of the problem where the finite size of the intended model is given as part of the input to the problem. As we argue below this is natural and relevant in the applications of GFODDs for solving decision theoretic control problems. The same restrictions can be used for the corresponding (evaluation, satisfiability and equivalence) problems in first order logic, but to our knowledge this has not been studied before. We provide a complete characterization of the complexity showing an interesting structure. Our results are developed for the GFODD representation and require detailed arguments about the graphical representation of formulas in that language. The same lines of argument (with simpler proof details) yield similar results for first order logic. To translate our results to the language of logic, consider the quantifier prefix of a first order logic formula using the standard notation using \(\Sigma^k, \Pi^k\) to denote alternation depth of quantifiers in the formula. In particular, with this translation, our results show that:

1. Evaluation over finite structures spans the polynomial hierarchy, that is, evaluation of \(\Sigma^k\) formulas is \(\Sigma^p_k\) complete, and evaluation of \(\Pi^k\) formulas is \(\Pi^p_k\) complete.

2. Satisfiability, with a given bound on model size, is more complex: satisfiability of \(\Sigma^k\) formulas is \(\Sigma^{p^\epsilon}_k\) complete, and satisfiability of \(\Pi^k\) formulas is \(\Sigma^{p^\epsilon}_{k+1}\) complete.

3. Equivalence, under the set of models bounded by a given size, depends only on quantifier depth: both the equivalence of \(\Sigma^k\) formulas and equivalence of \(\Pi^k\) formulas are \(\Pi^{p^\epsilon}_{k+1}\) complete.

The positive results allow for constants in the signature but the hardness results, except for satisfiability for \(\Pi_1\) formulas, hold without constants. For signatures without constants, satisfiability of \(\Pi_1\) formulas is in \(\text{NP}\); when constants are allowed, it is \(\Sigma^{p^\epsilon}_2\) complete as in the general template.

These results are useful in that they clearly characterize the complexity of the problems solved heuristically by implementations of GFODD systems \[13, 16\] and can be used to partly motivate or justify the use of these heuristics. For example, the “model checking reductions” of \[15\] replace equivalence tests with model evaluation on a “representative” set of models, and choosing this set heuristically \[12\] leads to inference that is correct with respect to these models but otherwise incomplete. Our results show that this indeed leads to reduction of the complexity of the inference problem so that the reduction in accuracy is traded for improved worst case run time. Importantly, it shows that without compromising correctness the complexity of equivalence tests that are used to compress the representation will be higher. These issues and further questions for future work are discussed in the concluding section of the paper.

The rest of the paper is organized as follows. The next section recalls some definitions from complexity theory. Section 3 then presents FODDs and develops the results for this special case. We treat the FODD case separately for three reasons. First, this serves for an easy introduction into the results that avoids some of the complexity of GFODDs. Second, as will become clear, for
FODD we do not need the additional assumption on model size, so that the results are in a sense stronger. Finally, some of the proofs for GFODDs require alternation depth of at least two so that separate proofs are needed for FODDs in any case. Section 4 defines and develops the results for GFODDs. The final section concludes with a discussion and directions for future work.

2 Complexity Theory Preliminaries

We assume basic familiarity with complexity theory including the classes P, NP, and co-NP [13, 30, 24]. The polynomial hierarchy is defined from these inductively starting with $\Sigma^p_1 = \text{NP}$, $\Pi^p_1 = \text{co-NP}$. An algorithm is in the class $A^B$ if it uses computation in $A$ with a polynomial number of calls to an oracle in $B$. Then we have $\Sigma^p_{k+1} = \text{NP}^{\Sigma^p_k}$, and $\Pi^p_{k+1} = \text{co-NP}^{\Sigma^p_k}$. A problem is in $\Sigma^p_k$ iff its complement is in $\Pi^p_k$ and thus either of these can serve as the oracle in the definition.

3 Max FODD

We assume familiarity with basic concepts and notation in predicate logic (e.g., [23, 27, 4]). Decision diagrams are similar to expressions in first order logic. They are defined relative to a relational signature, with a finite set of predicates $p_1, p_2, \ldots, p_n$ each with an associated arity (number of arguments), a countable set of variables $x_1, x_2, \ldots$, and a set of constants $c_1, c_2, \ldots, c_m$. We do not allow function symbols other than constants (that is, a function with arity $\geq 1$). In addition, we assume that the arity of predicates is bounded by some numerical constant. A term is a variable or constant and an atom is either an equality between two terms or a predicate with an appropriate list of terms as arguments. Intuitively, a term is an object in the world of interest and an atom is a property which is either true or false.

First order decision diagrams (FODD) and their generalization (GFODD) were defined by [31, 15] inspired by previous work in [9]. A first order decision diagram is a rooted acyclic graph with directed edges. Each node in the graph is labeled. A non-leaf node is labeled with an atom from the signature and it has exactly two outgoing edges. The directed edges correspond to the truth values of the node’s atom. A leaf is labeled with a non-negative numerical value. We sometimes restrict diagrams to have only binary leaves with values 0 or 1. In this case we can consider the values to be the logical values false and true. An example FODD is shown in Figure 1. In this diagram and all other diagrams in the paper left going edges denote the true branch out of a node and right going edges represent the false branch.

Similar to the propositional case [3, 1], FODD syntax is restricted to comply with a predefined total order on atoms. In the propositional case the ordering constraint yields a normal form (a unique minimal representation for each function) which is in turn the main source of efficient reasoning. For FODDs, a normal form has not been established but the use of ordering makes for more efficient simplification of diagrams. In particular, following [31], we assume a fixed ordering on predicate names, e.g., $p_1 < p_2 < \ldots < p_n$, and a fixed ordering on variable names, e.g., $x_1 < x_2 < \ldots$ and constants $c_1 < c_2 < \ldots c_m$ and require that $c_i < x_j$ for all $i$ and $j$. The order is extended to atoms by considering them as lists. That is, $p_i(\ldots) < p_j(\ldots)$ if $i < j$ and $p_i(x_{k_1}, \ldots, x_{k_a}) < p_i(x'_{k_1}, \ldots, x'_{k_a})$ if $(x_{k_1}, \ldots, x_{k_a}) < (x'_{k_1}, \ldots, x'_{k_a})$. Node labels in the FODD must obey this order so that if node $a$ is above node $b$ in the diagram then the labels satisfy $a < b$. We assume that all diagrams are legally ordered in this way. The example of Figure 1 is ordered with predicate ordering $E < “=”$ and lexicographic variable ordering $v_1 < v_2 < v_3$. 

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The ordering assumption is helpful in that it simplifies the computations. We note, however, that for this paper the assumption makes for more complex analysis because we must show that hardness results hold even for this restricted case.

The semantics of FODDs assign a value \( \text{MAP}_B(I) \) for any diagram \( B \) and interpretation \( I \), where \( I \) is a “possible world” specifying a domain of objects and the truth values of predicates over these objects. The “multiple-paths” semantics defines this value by considering all possible valuations. A variable valuation \( \zeta \) is a mapping from the set of variables in \( B \) to domain elements in the interpretation \( I \). This mapping assigns each node label to a concrete (“ground”) atom in the interpretation and therefore to its truth value and in this way defines a single path from root to leaf. The value of this leaf is the value of the FODD \( B \) under the interpretation, \( I \), with variable valuation \( \zeta \) and is denoted \( \text{MAP}_B(I, \zeta) \).

Since different \( \zeta \) will generate different values \( \text{MAP}_B(I, \zeta) \) we need to aggregate these to yield a single value for \( I \). Let \( AG_\zeta \) be such a function that takes a set of real values and returns a real valued result. We define \( \text{MAP}_B(I) = AG_\zeta \text{MAP}_B(I, \zeta) \). An example of an aggregate function would be min or max. FODDs are defined by using max aggregation. GFODDs allow for more aggregation functions and allow each variable to be aggregated differently. We discuss GFODDs in the next section. For now, we define \( \text{MAP}_B(I) = \max_\zeta \text{MAP}_B(I, \zeta) \).

Consider evaluating the example of Figure 1 on interpretation \( I = ([1, 2, 3], \{ E(1, 3), E(3, 1), E(1, 2), E(2, 1) \}) \). Then for \( \zeta = \{v_1/1, v_2/2, v_3/3\} \) we have \( \text{MAP}_B(I, \zeta) = 0 \) but for \( \zeta = \{v_1/3, v_2/1, v_3/2\} \) we have \( \text{MAP}_B(I, \zeta) = 1 \) and therefore \( \text{MAP}_B(I) = \max_\zeta \text{MAP}_B(I, \zeta) = 1 \).

### 3.1 Computational Problems

Before defining the computational problems we must define the representation of inputs. We assume that FODDs and GFODDs are given using a list of aggregation operators (all of which are max aggregation).
operators for FODDs) and associated variables and a labelled graph representation of the diagram. This is clearly polynomially related to the number of variables and number of nodes in the GFODD. Some of our problems require interpretations as input. Here we assume a finite domain so as to avoid issues of representing the interpretation. Thus an interpretation is given as a list of objects serving as domain elements, and the extension of each predicate on these objects. We implicitly assume that each constant is identified with a domain element. Given that the signature is fixed and the arity of each predicate is constant, this implies that the size of $I$ is polynomially related to the number of objects in $I$. As illustrated in the example of Figure 1, a graph $G = (V, E)$ can be seen as an interpretation with domain $V$ and with one predicate formed by the edge relation. We can now define the computational problems of interest. The simplest problem requires us to evaluate a diagram on a given interpretation.

Definition 1 (FODD Evaluation) Given diagram $B$, interpretation $I$ with finite domain, and value $V \geq 0$: return Yes iff $MAP_B(I) \geq V$. In the special case when the leaves are restricted to $\{0, 1\}$ and $V = 1$ this can be seen as a returning Yes iff $MAP_B(I)$ is true.

To calculate $MAP_B(I)$ we can “run” FODD Evaluation multiple times, once for each leaf value as $V$, and return the highest achievable result. The second problem most naturally applies for diagrams with binary leaves. In this case we might want to check if a diagram is satisfiable.

Definition 2 (FODD Satisfiability) Given diagram $B$ with leaves in $\{0, 1\}$: return Yes iff there is some $I$ such that $MAP_B(I)$ is true.

When $B$ has more than two values in its leaves we can define a related problem:

Definition 3 (FODD Value) Given diagram $B$ and value $V \geq 0$: return Yes iff there is some $I$ such that $MAP_B(I) = V$.

Notice that FODD Value requires that $V$ is achievable but no value larger than $V$ is achievable on the same $I$ and, as the proofs below show, the extra requirement makes the problem harder. On the other hand, if we replace equality with $\geq V$ in FODD Value, the problem is equivalent to FODD Satisfiability because we can simply replace leaf values in the diagram with 0,1 according to whether they are $\geq V$.

Finally, diagrams allow for redundancies in the representation. It is therefore crucial for applications of FODDs and GFODDs that diagrams can be compressed into a form which is equivalent semantically but is smaller syntactically. Such transformations are known as reductions [31, 15]. One way to view reductions, which has been productive in practice, is to check whether an edge can be “removed” in the sense that instead of directing it to a sub-diagram we can direct it to a zero leaf and in this way potentially eliminate the sub-diagram rooted at the child. To perform this test we can produce the alternative diagram and test whether they are semantically equivalent. This is abstracted here as a comparison between arbitrary diagrams. Our hardness results, though, apply even to the special case of testing removal of a single edge.

Definition 4 (FODD Equivalence) Given diagrams $B_1$ and $B_2$: return Yes iff $MAP_{B_1}(I) = MAP_{B_2}(I)$ for all $I$.

We note that the corresponding satisfiability and equivalence problems for GFODDs are undecidable and we will have to constrain the set of legal $I$’s to have an explicitly given finite domain size to make them decidable and study more refined notions of complexity. This is not required for FODD.
3.2 Complexity Results

Evaluation of FODDs is essentially the same as evaluation of conjunctive queries in databases and can be analyzed similarly. We include the argument here for completeness.

**Theorem 5** FODD Evaluation is NP-complete.

**Proof.** Membership in NP is shown by the algorithm that guesses a valuation $\zeta$, calculates $\text{MAP}_B(I, \zeta)$ and returns Yes iff the leaf reached has value $\geq V$. Yes is returned iff some valuation yields a value $\geq V$ as needed.

For hardness we reduce Hamiltonian path to this problem. As illustrated in Figure 1, given the number of nodes in a graph we can represent a generic Hamiltonian path verifier as a FODD $B$. To do this we simply produce a left going path $E(x_1, x_2), E(x_2, x_3), \ldots, E(x_{n-1}, x_n)$ which verifies existence of the edges, followed by equality tests to verify that all nodes are distinct. All “failure exits” on this path go to 0 and the success exit of the last test yields 1. Call this diagram $B$. Then, given any input $G$ for Hamiltonian path, we represent it as an interpretation $I$ and produce $(B, I, 1)$ as the input for FODD Evaluation. Clearly, $G$ has a Hamiltonian path iff $\text{MAP}_B(I) = 1$.

The other results for FODDs rely on the existence of small models:

**Lemma 6** For any FODDs $B$ with $k$ variables and constants, if $\text{MAP}_B(I) \geq V$ for some $I$ then there is an interpretation $I'$ with at most $k$ objects such that $\text{MAP}_B(I') \geq V$.

**Proof.** Let $I$ be as in the statement. Then there is a valuation $\zeta$ such that $\zeta$ reaches a leaf valued at least $V$ in $B$. Now let $I'$ be an interpretation including the objects that are used in the path traversed by $\zeta$ where the truth value of any predicate over arguments form these objects agrees with $I$. $I'$ has at most $k$ objects, $\zeta$ is a suitable valuation for $I'$ and $\text{MAP}_B(I', \zeta) \geq V$.

**Theorem 7** FODD Satisfiability is NP-Complete.

**Proof.** For membership we can guess an interpretation $I$, which by the previous lemma can be small, and guess a valuation $\zeta$ for that interpretation. We return Yes if and only if $\text{MAP}_B(I, \zeta) = 1$.

We show hardness with a reduction from 3SAT. Let $f$ be an arbitrary 3CNF formula. We create a new FODD variable for each literal in the CNF so that $v_{(i,j)}$ corresponds to the $j$th literal in the $i$th clause. Our FODD will have three portions. The first portion checks that the predicate $P_T()$ in the interpretation can be used to simulate Boolean assignments. The second portion ensures that if $v_{(i,j)}$ and $v_{(i',j')} correspond to the same Boolean variable then they map to the same object. The final portion tracks the structure of $f$ to guarantee the same value in the FODD.

The first part ensures that we have at least two different objects in our interpretation, $y_1$ and $y_2$. We then define a truth predicate $P_T$ and a corresponding block that ensures that the truth value of $P_T(y_1)$ is not equal to $P_T(y_2)$. As a result $P_T(y_1)$ and $P_T(y_2)$ correspond to true and false logical values. This is shown in the top part of Figure 2.
Next we create a series of consistency blocks corresponding to every variable \( x_i \) in the Boolean formula \( f \). For every variable \( x_i \) we create a shadow FODD variable \( w_i \) and equate it to all the \( v_{(i,j)} \) that correspond to \( x_i \). For example, consider the boolean formula \((x_1 \lor \neg x_2 \lor x_4) \land (\neg x_1 \lor x_2 \lor x_3) \land (x_1 \lor x_3 \lor \neg x_4)\).

The first block ensures that \( w_1, v_{(1,1)}, v_{(2,1)}, v_{(3,1)} \) are all assigned the same value. In addition to testing that the values are equivalent the block tests that each variable gets bound to the same object as \( y_1 \) or \( y_2 \). The only possible way to not get a 0 in these blocks is to ensure that each variable in the block has the same value and that it is equal to either \( y_1 \) or \( y_2 \). Figure 2 shows the first two portions chained together for our example.

To follow the structure of \( f \) we build a block for each clause and chain these blocks together. Each block has 3 nodes corresponding to the 3 literals in the clause. In particular, if the \( j \)th literal in the \( i \)th clause is positive the true edge (literal satisfied; call this success) continues to the next clause, and the false edge (literal failed) continues to the next literal. For a negative literal the true and false directions are swapped. The fail exit of the 3rd literal is attached to 0. Clause blocks have one entry and one exit and they are chained together. The success exit of the last clause is connected to the leaf 1. The only way to reach a value of 1 is if every clause block was satisfied by the valuations to \( v_{(i,j)} \). Figure 3 illustrates the clause blocks for our example.

Each of the portions, including the clause blocks, has one entry and one exit and we chain them together at the bottom of Figure 2. For a valuation to be mapped to 1 it must succeed in all three
portions. We claim that an arbitrary 3CNF formula $f$, is satisfied if and only if there is some interpretation $I$ such that $MAP_B(I) = 1$.

Consider first the case where $f$ is satisfiable. We introduce the interpretation $I$ that has two objects, $a$ and $b$, $P_T(a) = true$, and $P_T(b) = false$. Let $v$ be a satisfying assignment for $f$ and let $\zeta(v)$ be a valuation for $B$ on $I$ where $y_1 = a$, $y_2 = b$ and if $v$ maps $x_j$ to $1$ then $w_i$ and its block are mapped to $a$ and otherwise the block is mapped to $b$. Here, $\zeta(v)$ succeeds in all blocks, implying that $MAP_B(I, \zeta(v)) = 1$ and therefore $MAP_B(I) = 1$.

Consider next the case where $MAP_B(I) = 1$ for some $I$ and let $\zeta$ be such that $MAP_B(I, \zeta) = 1$. Then we claim that $\zeta$ identifies a satisfying assignment. First, since $\zeta$ succeeds in the first block we identify two objects that correspond via $P_T(\cdot)$ to truth values, without loss of generality assume that $P_T(y_1)$ is true. Then success in the second portion implies that we can identify an assignment to the Boolean variables, if the ith block is assigned to $y_1$ we let $x_i = 1$ and otherwise $x_i = 0$. Finally, success in the third portion implies that the clauses in $f$ are satisfied by the assignment to the $x_i$s. This completes the correctness proof.

Finally we address node ordering in the diagram. The only violation of ordering is the use of $P_T(\cdot)$ in the first block. Otherwise, we have all equalities above $P_T(\cdot)$, variable ordering $y_i < w_j < V_{a,b}$, and lexicographic ordering within a group. Now because our diagram forms one chain of blocks leading to a single sink leaf with value 1 we can move the three $P_T(\cdot)$ nodes to the bottom of the diagram in Figure 2. This does not change the mapped value for any valuation and thus does not affect correctness. We therefore conclude that $B$ is consistently sorted and $f$ is satisfiable iff $MAP_B(I) = 1$ for some $I$.

This proof illustrates the differences in arguments needed for FODDs and GFODDs vs. First Order Logic. For the latter the reduction can use the sentence $\exists v, (p_{x_1}(v) \lor p_{x_2}(v) \lor p_{x_4}(v)) \land (p_{x_3}(v)) \ldots$ to show the hardness result. However, this cannot be easily represented as a FODD because the literals appearing the clauses will violate predicate order and, if we try to reorder the nodes from a naive FODD encoding, the result might be exponentially larger. An alternative formulation can use $\exists x_1 \ldots (p(x_1) \lor \overline{p(x_2)} \lor p(x_4)) \land (\overline{p(x_1)}) \ldots$ to avoid the problem with predicate order. However, similar ordering issues now arise for the arguments. Our reduction introduces additional variables as well as the variable consistency gadget to get around these issues. The
same structure of reduction from 3SAT instances and their QBF generalizations will be used in the results for GFODD.

Equivalence is one level higher in the polynomial hierarchy:

**Theorem 8 FODD Equivalence** is $\Pi_2^p$-complete,

**Proof.** After showing membership we first present a reduction which ignores the need to order the nodes in the FODD and then show how to fix the construction.

**Membership in $\Pi_2^p$:** As above we use Lemma 6 to conclude that if the diagrams are not equivalent then there is a small interpretation that serves as a witness for the difference. We can then show that non-equivalence is in $\Sigma_2^p$. Given $B_1$, $B_2$ we guess an interpretation $I$ of the appropriate size, and then appeal to an oracle for FODD Evaluation to calculate $\text{MAP}_{B_1}(I)$ and $\text{MAP}_{B_2}(I)$. Using these values we return Yes or No accordingly. To calculate the map values, let $B$ be one of these diagrams, and let the leaf values of the diagram be $v_1, v_2, \ldots, v_k$. We make $k$ calls to FODD Evaluation with $(B, I, v_i)$ as input. $\text{MAP}_B(I)$ is the largest value on which the oracle returns Yes. If a witness $I$ for non-equivalence exists then this process can discover it and say No, and otherwise it will always say Yes. Therefore non-equivalence is in $\Sigma_2^p$, and equivalence is in $\Pi_2^p$.

**Reduction basics:** To show hardness, consider the problem of deciding arrowing from the Ramsey theory of graphs [29]. We say that $F \rightarrow (G, H)$ if for every 2-color edge-coloring of $F$ there is a red $G$ or a blue $H$. The arrowing problem is as follows: Given, 3 graphs $F, G, H$ as input, return Yes iff $F$ arrows the pair $(G, H)$. This problem was shown to be $\Pi_2^p$-complete by [29]. We reduce this problem to FODD equivalence. The signature includes equalities and two arity-2 predicates $E_F$ and $E_C$. Where $E_F$ captures the edge relation of the main graph $F$. $E_C$ is a coloring of all possible edges such that when $E_C(x_i, x_j)$ is true the edge is colored red and when it is false the edge is colored blue.

**The main construction:** To transform arrowing into an instance of FODD equivalence we build two FODDs with binary leaves. The first FODD is satisfied iff $I$ includes an embedding of $F$ in its edge relation $E_F$. The second FODD is satisfied iff the same condition holds and the coloring defined by $E_C$ has a red subgraph isomorphic to G or a blue subgraph isomorphic to H. We illustrate
Figure 5: A FODD verifying 1-1 mapping for the nodes of $F$.

the construction using an example input in Figure 4. Here the input graphs F, G, H are a positive instance of arrowing.
Figure 6: A fragment of a FODD verifying the graph F. Here every neighbor of $f_1$ is tested. Since $f_1$ is connected to $f_2$, $f_3$ we expect the corresponding atoms to be true and thus continue on the left branch; since it is not connected to vertex $f_4$, the gadget continues on the false side; $f_5$ is connected and we continue to the left again.

Figure 7: The complete FODD verifying the graph F.
To build a FODD which verifies that $I$ has an embedding of $F$, we map each node to a variable in the FODD and test that each node has its correct neighbors. We first build a “node mapping” gadget that makes sure that each variable in the FODD is mapped to a different object in the interpretation. This is done by following a path of $\binom{5}{2}$ inequalities, where off-path edges go to 0 and the final exit continues to the next portion. This gadget, for our example graph $F$ with 5 nodes, is shown in Figure 5. To test isomorphism to $F$ we test the neighbors of each node in sequence to verify an edge. The FODD fragment in Figure 6 shows how this can be tested for vertex $f_1$ in the example. If the edge is present in the graph we continue left (using the true branch) to the next neighbor and if the edge is not in $F$ we continue to the right child (the false branch). Edges off this path are directed to the zero leaf. The endpoint of the path will connect to the next portion of the FODD. In the example, $f_1$ has 3 of the 4 possible neighbors and therefore 3 of the steps go left and one goes right. This construction can be done for each node and the fragments can be connected together to yield the F-verifier. This is illustrated in Figure 7. Finally, the diagram $B_1$ is built by connecting the F-verifier at the bottom of node mapping gadget, and replacing the bottom node of the $F$ verifier with a leaf valued 1. We refer to this diagram as the “complete $F$ verifier” below. This construction can be done in polynomial time for any graph $F$. It should be clear from the construction that $MAP_{B_1}(I) = 1$ iff $I$ includes an isomorphic embedding of $F$ in its edge relation $E_F$. In addition, the verifier diagram is ordered where we have “$=$” $\prec E_F$, and where variables are ordered lexicographically.

The second digram $B_2$ includes the complete F-verifier and additional FODD fragments that are described next to capture the conditions on $G$ and $H$ respectively. In order to verify the embedding of colored subgraph $G$ we first define a node mapping capturing the mapping of $G$ nodes into $F$ nodes, and then verify that the required edges exist and that they have the correct color. The FODD fragment in Figure 8 shows how we can select a node mapping for vertex $g_1$. This fragment returns 0 unless $g_1$ is mapped to one of the nodes in $F$ that are identified in the $B_1$ portion. As depicted in Figure 9, this can be repeated for all the nodes in $G$, verifying that each node in $G$ is mapped to a node in $F$. Next we need to verify that the mapping is one to one. This can be done by using a path of inequalities between the variables referring to nodes of $G$. This FODD fragment is given in Figure 10. For correctness, we need to chain the two tests together, but this will violate node ordering. We therefore interleave the tests putting the uniqueness equality tests for a variable exactly after the equalities selecting its value. This change is possible because each such block has exactly one exit point. The resulting diagram, for our running example, is shown in Figure 11.
Figure 8: A FODD verifying the mapping of vertex $g_1$ to some vertex in $F$.

Figure 9: The complete FODD verifying the vertices of $G$ are mapped to vertices in $F$. 
Figure 10: A FODD verifying the mapping is one to one.

Figure 11: Node mapping construction for $G$ reordered to comply with sorted order.
To complete the embedding test, we need to check that the edges are preserved and that they have the correct color. We do this by first checking that the corresponding edges in $G$ are in $F$. We can do this using a left going path testing each edge in turn, where we test both $E_F(g_i, g_j)$ and $E_F(g_j, g_i)$ to account for the fact that the graph is undirected. This is illustrated on the left hand side of Figure 12. Note that here we test only for the edges in $G$ and do not need to verify nonexistence of the edges not in $G$ (it just happens here that $G$ is a clique so this is not visible in the example). The same FODD structure is repeated with predicate $E_C$ replacing $E_F$ to verify that the edges of $G$ are colored red, as shown on the right of Figure 12.

A similar construction with node mapping, edge verifier, and color verifier can be used for $H$. The node mapping construction is identical. Figure 13 shows the edge and color verifiers. The only difference in construction is that the color verifier tests that the edge is not in $E_C$ to capture the color blue and therefore has a mirror structure to the one verifying the $E_F$ edges. Note that in this case $H$ is not a complete graph and we are indeed only testing for the edges in $H$. This construction can be done in polynomial time for any $G$ and $H$.

Finally we connect the three portions together to obtain $B_2$ as follows. The final output of the complete $F$ verifier is connected to the root of the $G$ verifier. The final output of the $G$ verifier is connected to 1. The zero leaf of the $G$ verifier is removed and instead connected to the root of the $H$ verifier. The final output of the $H$ verifier is connected to 1. Therefore, there are exactly two edges leading to the 1 leaf in this diagram, corresponding to the positive outputs of the $G$ and $H$ verifiers. Figure 14 shows an overview of the two FODDs, $B_1$ and $B_2$, generated by the reduction from the original $F, G, H$.

The diagrams $B_1$ and $B_2$ are not consistent with any sorting order over node labels, and thus we need to modify them to get a consistent ordering. We show below how this can be done with only a linear growth in the size of the diagrams and without changing the semantics of $B_1$ and $B_2$. Before presenting this transformation we show that $F, G, H \in \text{Arrowing}$ iff $B_1$ and $B_2$ are equivalent.

Correctness of the construction: Consider the case when $F, G, H \in \text{Arrowing}$, that is, for every 2-color edge-coloring of $F$ there is a red $G$ or a blue $H$. We show that the two FODDs are equivalent by way of contradiction. Assume that $B_1$ and $B_2$ are not equivalent and let $I$ be any witness to this fact. $\text{MAP}_{B_2}(I) = 0$ implies $\text{MAP}_{B_1}(I) = 0$ as the only paths to 1 in $B_2$ go through
Consider the case when $F, G, H \notin \textit{Arrowing}$. Then there is a valid 2-color edge-coloring of $F$ which does not have a red $G$ and does not have a blue $H$. Construct the corresponding interpretation $I$ that represents $F$ and this edge-coloring. We claim that $\text{MAP}_{B_1}(I) = 1$ and $\text{MAP}_{B_2}(I) = 0$. The fact $\text{MAP}_{B_1}(I) = 1$ follows by construction mapping the nodes in $F$ to the variables that represent them. Now if $\text{MAP}_{B_2}(I) = 1$ then $\text{MAP}_{B_2}(I, \zeta) = 1$ for some $\zeta$ and we can trace the path that $\zeta$ traverses in $B_2$. This verifies either a red $G$ or a blue $H$ in $I$ and therefore in the corresponding coloring of $F$. This contradicts the assumption that the coloring is a witness for non-arrowing.
Fixing the construction to handle ordering: We next consider the node ordering in $B_1$ and $B_2$. The diagram $B_1$ is sorted, where predicate order puts equalities above $E_F$ and arguments are lexicographically ordered. For $B_2$ we consider the sub-block structure of the construction. Expanding each of the sub-blocks of $F$, $G$, $H$ in Figure 14 we observe that $B_2$ has the structure shown in Figure 15. We further observe that each block is internally sorted, but blocks of equalities, $E_F$ and $E_C$ are interleaved. By analyzing this structure we see that the blocks can be reordered at the cost of duplicating some of the blocks yielding the structure in Figure 16. It is easy to see that $B_2$ is satisfied in $I$ if and only if the reordered diagram is satisfied in $I$. The diagrams yield the same value for any valuation $\zeta$ which does not exit to 0 due to bad node mapping for $G$ or $H$. Thus the original version might yield 1 (e.g., through $G$ path) when the reordered diagram yields 0 on such a valuation (e.g., via the $H$ equalities). But in such a case there is another valuation that is identical to $\zeta$ except that it modified the bad node mapping (the $H$ equalities) and that yields 1 for the new diagram. The final diagram is consistent with predicate ordering “$=$” $E_F$ $E_C$ and variable ordering where $f_i$ $g_j$ $h_k$ for all $i,j,k$.

Finally, we further change $B_1$ by adding the equality blocks of $G$ and $H$ to the construction, so that the modified $B_1$ is as shown in Figure 17. Using the same argument as in $B_2$ one can see that this does not change the semantics of $B_1$. Moreover, with this change $B_1$ can be obtained from $B_2$ by “one edge removal” (of the edge below the $F$ verifier in $B_2$) so that the reduction holds for this more restricted case.

As mentioned above, FODD Value is defined similarly to FODD Satisfiability but appears to require more. The next result shows that this indeed pushes it one level higher in the hierarchy.

Theorem 9 FODD Value is $\Sigma^{p}_{2}$-complete.

Proof. The algorithm showing membership is as follows. We first observe that by Lemma 6 we can restrict our attention to small interpretations. Given input $B$ and $V$ we guess an interpretation $I$
of the appropriate size. We then make two calls to an NP-oracle for FODD Evaluation. Let $V'$ be either the least leaf value greater than $V$ or one greater than the max leaf if $V$ is the maximum. We query the oracle for FODD Evaluation on $(B, I, V)$ and $(B, I, V')$ return Yes iff the oracle returns Yes on the first and No on the second. The algorithm returns Yes iff there is an interpretation $I$ with value $V$.

For hardness we present a reduction from non-Equivalence of FODDs with binary leaves, which is shown to be $\Sigma_p^2$-hard in Theorem 8. We are given $B_1 = \max_{x_1} B_1(x_1)$ and $B_2 = \max_{x_2} B_2(x_2)$ as input for FODD non-Equivalence where $x_1, x_2$ stand for disjoint sets of variables. We construct the diagram $B = \max_{x_1} \max_{x_2} B(x_1, x_2)$ where $B(x_1, x_2) = B_1(x_1) + B_2(x_2)$ can be calculated directly on the graph representation of $B_1$ and $B_2$ using the apply procedure of [31] (see Figure 18 for a description and illustration of the apply procedure). Because $x_1$ and $x_2$ are disjoint, the diagram $B$ has the following behavior for any interpretation $I$: if $\text{MAP}_{B_1}(I) = 1$ and $\text{MAP}_{B_2}(I) = 1$ then
Figure 18: Overview of the apply procedure of \[31\] for binary operations over two FODDs. Recall that FODDs use an ordering over the atoms labeling nodes, so that atoms lower in the ordering are always higher in the diagram. Let \( p \) and \( q \) be the roots of \( B_1 \) and \( B_2 \) respectively. The procedure chooses a new root label (the smaller of \( p \) and \( q \)) and recursively combines the corresponding sub-diagrams, according to the relation between the two labels (\( <, =, \text{ or } > \)). In this example, we assume predicate ordering \( p_1 < p_2 \), and parameter ordering \( x_1 < x_2 \). Non-leaf nodes are annotated with numbers and numerical leaves are underlined for identification during the execution trace. For example, the top level call adds the functions corresponding to nodes 1 and 3. Since \( p_1(x_1) \) is the smaller label it is picked as the label for the root of the result. Then we must add both left and right child of node 1 to node 3. These calls are performed recursively and we avoid repetition using dynamic programming.

\[
\begin{align*}
\text{MAP}_{B}(I) = 2; \text{ otherwise if exactly one of them evaluates to 1 then MAP}_{B}(I) = 1; \text{ and otherwise MAP}_{B}(I) = 0. \text{ We produce } (B, V = 1) \text{ as input for FODD Value.}
\end{align*}
\]

Now, if \( B_1 \) and \( B_2 \) are not equivalent then there is an interpretation such that their maps are different, and without loss of generality we may assume \( \text{MAP}_{B_1}(I) = 1 \) and \( \text{MAP}_{B_2}(I) = 0 \). As argued above in this case \( \text{MAP}_{B}(I) = 1 \) as needed. For the other direction let \( I \) be such that \( \text{MAP}_{B}(I) = 1 \). Then, again using the argument above, we have \( \text{MAP}_{B_1}(I) = 1 \) and \( \text{MAP}_{B_2}(I) = 0 \) or vice versa and the diagrams are not equivalent. \( \square \)
4 Generalized FODD

The diagram portion of GFODDs is defined exactly as in FODDs. GFODDS however allow for other forms of aggregation of the values over MAP\(_B(I, \zeta)\). In particular, let the variables in \(B\) be given in some arbitrary order \(w_{i_1}, \ldots, w_{i_m}\). Then \(B\) is associated with another list of length \(m\) specifying aggregation over each \(w_{i_j}\) in that order. The definition by [15] allows for various aggregation operators (for example, \(\text{min}, \text{max}, \text{sum}, \text{average}\)) but here we use a more restricted set allowing each variable to be associated with either max or min aggregation. The semantics is defined as follows. First we calculate MAP\(_B(I, \zeta)\) for all \(\zeta\). Then we loop with \(j\) taking values from \(m\) to 1 aggregating values over \(w_{i_j}\) using its aggregation operator.

When leaves are in \(\{0, 1\}\) max and min aggregation correspond to existential and universal quantifiers in logic with the formula given in quantified normal form, that is, with all quantifiers at the front. For example, the expression \([\max_{w_1} \min_{w_2} \text{if } p(w_1, w_2) \text{ then } 1 \text{ else } 0]\), that can be captured with a diagram with one internal node, is the same as the logical formula \(\exists w_1, \forall w_2, p(w_1, w_2)\).

We say that a GFODD is a max-\(k\)-alternating GFODD if its set of aggregation operators has \(k\) blocks of aggregation operators, where the first includes max aggregation, the second includes min aggregation, and so on. We similarly define min-\(k\)-alternating GFODD where the first block has min aggregation operators. A GFODD has aggregation depth \(k\) if it is in one of these two classes.

The following result, mirroring the case in logic, will allow us to simplify some of the arguments by borrowing results from their “complements”. Let \(B\) be a GFODD associated with the ordered list of variables \(w_{i_1}, \ldots, w_{i_m}\), and aggregation list \(A_1, \ldots, A_m\) where each \(A_i\) is min or max. Let \(B' = \text{complement}(B)\) (with respect to maximum value \(M\)) be the diagram corresponding to \(B\) where we change leaf values and aggregation operators as follows: Let the max leaf value in \(B\) be of value \(\leq M\). Any leaf value \(v\) is replaced with \(M - v\). Each aggregation operator \(A_i\) is replaced with \(A_i'\) where \(M - v\). Notice that for diagrams with binary leaves this yields MAP\(_B(I) = 1 - \text{MAP}_{B'}(I)\), that is, negation. As an immediate application we get the following:

**Theorem 10** Let \(B\) be a GFODD with min and max aggregation and maximum leaf value \(\leq M\), and let \(B' = \text{complement}(B)\). For any interpretation \(I\), MAP\(_B(I) = M - \text{MAP}_{B'}(I)\).

**Proof.** By the construction of \(B'\), for any valuation \(\zeta\), we have that MAP\(_B(I, \zeta) = M - \text{MAP}_{B'}(I, \zeta)\). Considering the aggregation process, note that \(A_i\) MAP\(_B(\ldots w_i) = A_i' \left[ M - \text{MAP}_{B'}(\ldots w_i) \right] = M - A_i' \text{MAP}_{B'}(\ldots w_i)\). Now using this fact, we can argue by induction backward from the rightmost aggregation that for any prefix of variables \(P = w_{i_1}, \ldots, w_{i_p}\), valuation \(\zeta_p\) for these variables, and remaining variables \(R = w_{i_{p+1}}, \ldots, w_{i_m}\), we have \(A_R\) MAP\(_B(I, (P = \zeta_p; R)) = M - A_R' \text{MAP}_{B'}(I, (P = \zeta_p; R))\). When the prefix is empty we get the statement of the theorem. \(\square\)

Notice that for diagrams with binary leaves this yields MAP\(_B(I) = 1 - \text{MAP}_{B'}(I)\), that is, negation. As an immediate application we get the following:

**Theorem 11** The equivalence problem for min-GFODD is \(\Pi_2^0\)-complete.

**Proof.** By Theorem 10, two min diagrams \(B_1, B_2\) are equivalent if and only if their complements \(B_1', B_2'\) are equivalent where we can use the maximum among the leaf values of the two diagrams as \(M\). \(\square\)
The same argument holds for higher levels of quantifier alternation. The application to evaluation and satisfiability is not immediate. However, one might expect the complexity to vary with aggregation depth. The rest of this section formalizes this intuition and makes it precise.

Before presenting the details we recall one more useful fact about GFODDs. We say that a binary operation \( \text{op} \) is safe with respect to aggregation operator \( \text{agg} \) if it distributes with respect to it, that is \( b \text{ op} \text{agg}\{a_1, a_2, \ldots, a_n\} = \text{agg}\{(a_1 \text{ op} b), (a_2 \text{ op} b), \ldots, (a_n \text{ op} b)\} \). A list of safe pairs of binary operations and aggregation operators was provided by [15]. For the arguments below we recall that the binary operations + and \( \wedge \) are safe with respect to max and min aggregation. For example \( 5 + \max\{1, 2, 3, 4\} = \max\{6, 7, 8, 9\} \). With this definition we have:

**Theorem 12 (see Theorem 4 of [15])** Let \( B_1 = \langle V_1, D_1 \rangle \) and \( B_2 = \langle V_2, D_2 \rangle \) be GFODDs that do not share any variables and assume that \( \text{op}_c \) is safe with respect to all operators in \( V_1 \) and \( V_2 \). Let \( D = \text{apply}(B_1, B_2, \text{op}_c) \). Let \( V \) be any permutation of the list of variable in \( V_1 \) and \( V_2 \) so long as the relative order of operators in \( V_1 \) and \( V_2 \) remains unchanged, and let \( B = \langle V, D \rangle \). Then for any interpretation \( I \), \( \text{MAP}_B(I) = \text{MAP}_{B_1}(I) \text{ op}_c \text{MAP}_{B_2}(I) \).

Therefore, when adding (or taking the logical and of) functions represented by diagrams that are standardized apart we can use the apply procedure on the graphical representations of these functions, and at the same time we have some flexibility in putting together their list of aggregation functions. This will be useful in our reductions.

### 4.1 Computational Problems

Given the discussion above, GFODDs with binary leaves can be seen to capture the function free fragment of first order logic with equality. It is well known that satisfiability and therefore also equivalence of expressions in this fragment of first order logic is not decidable. In fact, the problem is undecidable even for very restricted forms of quantifier alternation (see survey and discussion in [8]). For example, the problem is undecidable for quantifier prefix \( \forall^2 \exists^* \) with a single binary predicate. The problem is also undecidable if we restrict attention to satisfiability under finite structures. Therefore, without further restrictions, we cannot expect much by way of classification of the complexity of the problems stated above for GFODDs.

We therefore restrict the problems so that the size of interpretations is given as part of the input. This makes the problems decidable and reveals the structure promised above. There are two motivations for using such a restriction. The first is that in some applications we might know in advance that the number of relevant objects is bounded by some large constant. For example, the main application of GFODDs to date has been for solving decision theoretic planning problems; in this context the number of objects in an instance (e.g., the number of trucks or packages in a logistics transportation problem) might be bounded by some known quantity. The second is that our results show that even under such strong conditions the computational problems are hard, providing some justification for the heuristic approaches used in FODD and GFODD implementations [14, 16, 17].

**Definition 13 (GFODD Model Evaluation)** Given diagram \( B \), interpretation \( I \) with finite domain, and value \( V \geq 0 \): return Yes iff \( \text{MAP}_B(I) \geq V \). Note that when the leaves are restricted to \( \{0, 1\} \) and \( V = 1 \) this can be seen as a returning Yes iff \( \text{MAP}_B(I) \) is true.

**Definition 14 (GFODD Satisfiability)** Given diagram \( B \) with leaves in \( \{0, 1\} \) and integer \( N \) in unary: return Yes iff there is some \( I \), with at most \( N \) objects, such that \( \text{MAP}_B(I) \) is true.

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Definition 15 (GFODD Value) Given diagram \( B \), integer \( N \) in unary and value \( V \geq 0 \): return Yes iff there is some \( I \), with at most \( N \) objects, such that \( \text{MAP}_{B}(I) = V \).

Definition 16 (GFODD Equivalence) Given diagrams \( B_1 \) and \( B_2 \) and integer \( N \) in unary: return Yes iff for all \( I \) with at most \( N \) objects, \( \text{MAP}_{B_1}(I) = \text{MAP}_{B_2}(I) \).

The assumption that \( N \) is in unary is convenient but not essential as our constructions will involve interpretations where the number of objects is linear in the size of \( B \) (bounded by the number of variables in \( B \)).

4.2 Complexity Results

Evaluation is similar to the FODD case but the proof is more complex due to the interaction between quantifier order and node ordering in the diagram.

Theorem 17 GFODD Evaluation for max-\( k \)-alternating GFODDs is \( \Sigma^p_k \)-complete. GFODD Evaluation for min-\( k \)-alternating GFODDs is \( \Pi^p_k \)-complete.

Proof. We prove membership by induction on \( k \) showing that the claim holds even for diagrams that do not satisfy the sorting order. Consider the input \((B,I,V)\). For the base case, \( k = 1 \), as in Theorems 5 we guess a valuation \( \zeta \), calculate \( v = \text{MAP}(B,\zeta) \), and return Yes iff \( v \geq V \). In the max case, if the true value is at least \( V \) then we say Yes for some \( \zeta \), and if the true value is less than \( V \) we never say Yes. Thus the problem is in NP. In the min case, if the true value is at least \( V \) then all \( \zeta \) yield Yes, and if the true value is less than \( V \) then some \( \zeta \) yields No. Thus the problem is in co-NP.

For the inductive step assume the claim holds for \( k - 1 \) and consider the input \((B,I,V)\) with an interpretation \( I \), value bound \( V \) and a max-\( k \)-alternating diagram \( B = \max_{w_1} \min_{w_2} \ldots Q^A_{w_k} B(w_1,\ldots,w_k) \) where in order to simplify notation each \( w_i \) may be a single variable or a set of variables (and we use the boldface notation to denote this fact).

Now for each tuple \( i \) of domain objects in \( I \) (which is appropriate for the number of variables in \( w_1 \)) let diagram \( B' \) be \( B' = \min_{w_2} \ldots Q^A_{w_k} B(w_1 = i,\ldots,w_k) \). Clearly \( B' \) is appropriate for evaluation on \( I \) and by the inductive hypothesis we can appeal to a \( \Pi^p_{k-1} \) oracle to solve GFODD Evaluation on \((B',I,V)\). Our algorithm guesses a value \( i \), calculates \( B' \), appeals to the oracle, and returns the same answer. Now, if the true value is \( < V \) then by definition any call to the oracle would yield No and we correctly answer No. If the true value is \( \geq V \) then for some \( i \) the oracle would return Yes. Therefore we nondeterministically return Yes and our algorithm is in NP\(^{\Sigma^p_{k-1}} \).

The argument for the other aggregation prefix is symmetric and argued in the same manner yielding an algorithm in co-NP\(^{\Sigma^p_{k-1}} \).

To show hardness we give a reduction from \( QBF_k \). Given a quantified 3CNF boolean formula we transform this into a GFODD \( B \) and interpretation \( I \) so that the following claim holds:

Claim 1: \( B \) evaluates to a value of 1 in \( I \) if and only if the quantified boolean formula is satisfied.
portion in that construction is not needed. On the other hand the construction and proof are more complex because of the alternation of quantifiers.

The interpretation $I$ has two objects, $a$ and $b$, and our truth assignment predicate assigns $P_T(a) = \text{true}$, and $P_T(b) = \text{false}$. Namely, $I = \{(a, b), P_T(a) = \text{true}, \text{ and } P_T(b) = \text{false}\}$.

Let the QBF formula be $Q_1 x_1 Q_2 x_2 \ldots Q_m x_m f$ where $Q$ is a quantifier $\forall$ or $\exists$ and the quantifiers come in $k$ alternating blocks. As above, we start the construction by creating a set of “shadow variables” corresponding to each QBF variable $x_i$. The corresponding GFODD variables include $w_i$ and the set of $v_{(a,b)}$ that refer to $x_i$ or $\neg x_i$ in the QBF. We define $w_i$ to be the set of variables in the block corresponding to $x_i$ and associate these variables with an aggregation operator $Q^A_i$ where if $Q_i$ is a $\exists$ then $Q^A_i$ is max and if $Q_i$ is a $\forall$ then $Q^A_i$ is min. Using these variables, we build FODD fragments we call variable consistency blocks. For each $x_i$, this gadget ensures that if two literals in the QBF refer to the same variable then the corresponding variables in the GFODD will have the same value. If this holds then a valuation goes through the block and continues to the next block. Otherwise, it exits to a default value, where for max blocks the default value is 0, and for min blocks the default value is 1.

Consider the expression

$$\forall x_1 \exists x_2 \forall x_3 \exists x_4 (x_1 \lor \neg x_2 \lor x_4) \land (\neg x_1 \lor x_2 \lor x_3) \land (x_1 \lor x_3 \lor \neg x_4),$$

which has the same clauses as in the previous proof but where we have changed the quantification. Figure 19 shows the variable consistency blocks for this example. Since, $v_{(1,1)}$, $v_{(2,1)}$, and $v_{(3,1)}$ refer to $x_1$ we need to ensure that when they are evaluated they are evaluated consistently and this is done by the first block. Because $x_1$ is a $\forall$ variable the default output value is 1. The consistency blocks are chained in the order of the quantification of the QBF. Once every consistency has been checked we continue to the clause blocks whose construction is exactly the same as in the previous proof (see Figure 3). This yields the diagram $B$ where we set the aggregation function to be $Q_1^A w_1, Q_2^A w_2, \ldots, Q_m^A w_m$ where $Q_i^A$ and $w_i$ are as above. Note that if the QBF has $k$ alternating blocks of quantifiers then $B$ has aggregation depth $k$. The output of the reduction is the pair $(B, I)$.

Figure 19: Example of variable consistency blocks for reduction from QBF to GFODD Evaluation for formula $\forall x_1 \exists x_2 \forall x_3 \exists x_4 (x_1 \lor \neg x_2 \lor x_4) \land (\neg x_1 \lor x_2 \lor x_3) \land (x_1 \lor x_3 \lor \neg x_4)$. 

```
We claim that given an arbitrary instance \( f \) of \( QBF_k \), we have that \( f \) is satisfied if and only if \( MAP_B(I) = 1 \).

We start by showing a correspondence between assignments to the Boolean formula \( f \) and object assignments from \( I \) to \( B \). Let \( v \) be a Boolean assignment. If \( v \) assigns \( x_i \) to 1 then \( \zeta(v) \) maps the entire \( w_i \) block to \( a \). Otherwise \( \zeta(v) \) maps the block to \( b \). It is then easy to see that for all \( v \), \( \zeta(v) \) satisfies the consistency blocks and \( f(v) = 1 \) if and only if \( MAP_B(I, \zeta(v)) = 1 \). This, however, does not complete the proof because \( MAP_B(I) \) must also consider valuations \( \zeta \) that do not arise as maps of assignments \( v \).

We divide the set of valuations to the GFODD into two groups. The first group of legal valuations, called \( \text{Group 1} \) below, is the set of valuations that is consistent with some \( v \).

The second group, \( \text{Group 2} \), includes valuations that do not arise as \( \zeta(v) \) and therefore they violate at least one of the consistency blocks. Let \( \zeta \) be such a valuation and let \( Q^A_j \) be the first block from the left whose constraint is violated. By the construction of \( B \), in particular the order of equality blocks along paths in the GFODD, we have that the evaluation of the diagram on \( \zeta \) “exits” to a default value on the first violation. Therefore, if \( Q_j \) is a \( \forall \) then \( MAP_B(I, \zeta) = 1 \) and if \( Q_j \) is a \( \exists \) then \( MAP_B(I, \zeta) = 0 \).

We can now show the correspondence in truth values. Consider any partition of the blocks \( 1, \ldots, m \) into a prefix \( 1, \ldots, j \) and remainder \( (j + 1), \ldots, m \), and any Boolean assignment \( v \) to the prefix blocks. We claim that for all such partitions

\[
Q_{j+1}x_{j+1}, \ldots, Q_mx_m, \ f((x_1, \ldots, x_j) = v, (x_{j+1}, \ldots, x_m)) = Q_{j+1}^A w_{j+1}, \ldots, Q_m^A w_m, \ MAP_B(I, [(w_1, \ldots, w_{j-1}) = \zeta(v), (w_{j+1}, \ldots, w_m)]).
\]

Note that when \( j = 0 \), that is, the prefix is empty, the claim implies that \( MAP_B(I) \) is equal to \( f \), completing the proof. We prove the claim by induction, backwards from \( m \) to 0.

For the base case, \( j = m - 1 \), and the second part includes only one block. Consider any concrete substitution \( v \) and induced \( \zeta(v) \) for the first part, and any valuation \( \zeta_m \) for \( w_m \) so that the complete valuation is \( \zeta = \zeta(v), \zeta_m \). If \( \zeta \) (meaning via \( \zeta_m \)) is in group 2, then the map is 1 if \( Q_m \) is \( \forall \) and is 0 otherwise. Therefore, the value of \( Q_m^A w_m, MAP_B(I, [(w_1, \ldots, w_{m-1}) = \zeta(v), (w_m)]) \) is the same as that value when restricted to substitutions in group 1. But, as argued above, for group 1 this is the value returned by the QBF.

For the inductive step, the valuation \( v \) covers the first \( j - 1 \) blocks. Note that, by the inductive assumption, for any group 1 substitution \( v_j \) for \( x_j \) and corresponding, \( \zeta(v_j) \) for \( w_j \),

\[
Q_{j+1}x_{j+1}, \ldots, Q_mx_m, \ f((x_1, \ldots, x_j) = v, (x_j = v_j), (x_{j+1}, \ldots, x_m)) = Q_{j+1}^A w_{j+1}, \ldots, Q_m^A w_m, \ MAP_B(I, [(w_1, \ldots, w_{j-1}) = \zeta(v), (w_j = \zeta(v_j)), (w_{j+1}, \ldots, w_m)]).
\]

On the other hand, for any group 2 substitution \( \zeta_j \) for \( w_j \) and any values for \( (w_{j+1}, \ldots, w_m) \) we have that the leftmost block whose constraint is violated for the corresponding combined \( \zeta \) is block \( j \) and therefore \( MAP_B(I, [(w_2, \ldots, w_{j-1}) = \zeta(v), (w_j = \zeta_j), (w_{j+1}, \ldots, w_m)]) \) gets the default value for that block. Therefore, as in the base case, the aggregation over the \( j \)th block is determined by group 1 valuations, which are in turn identical to the QBF value and

\[
Q_jx_j, \ldots, Q_mx_m, \ f(x_1, \ldots, x_j) = v, (x_j, \ldots, x_m)) = Q_j^A w_j, \ldots, Q_m^A w_m, \ MAP_B(I, [(w_2, \ldots, w_{j-1}) = \zeta(v), (w_j, \ldots, w_m)]).
\]

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as required.

It turns out the the complexity of satisfiability is different for min and max diagrams, and their analysis requires different proofs. We therefore start with max-\(k\)-alternating GFODDs. The case of min GFODDs is analyzed after the analysis of equivalence because it is using similar techniques.

**Theorem 18**  
GFODD Satisfiability for max-\(k\)-alternating GFODDs (where \(k \geq 2\)) is \(\Sigma_p^{k-1}\)-complete.

**Proof.** We first show membership. Let \(B\) be a GFODD with aggregation max \(w_1, \min w_2, \ldots, Q_k A_k w_k\). Our algorithm nondeterministically chooses an interpretation \(I\) and a set of values for \(w_1, \ldots, w_k\) from the domain of objects for \(I\). Let \(i\) refer to this set of objects. We create a new GFODD, \(B' = \min w_2 \ldots Q_k A_k B(w_1 = i, \ldots, w_k)\) and appeal to a \(\Sigma_{k-1}^p\) oracle to solve GFODD evaluation on \((B', I)\). If the oracle query returns a 1 then we accept and otherwise we reject. The result is clearly correct using an algorithm in \(NP^{\Sigma_{k-1}^p}\).

The hardness argument is similar to the proof for GFODD evaluation. The main extension is that in the current proof we verify that any satisfying \(I\) embeds the interpretation from the previous proof. The reduction gets a QBF formula, \(Q_1 x_1 Q_2 x_2 \ldots Q_m x_m f\), with \(Q_i\) either a \(\forall\) or \(\exists\) quantifier. We first construct two diagrams \(B_1\) and \(B_2\) where \(B_2\), the QBF validation diagram, is exactly as in the proof of Theorem 17, that is, it includes consistency blocks followed by clause blocks. The diagram \(B_1\) has two portions. The first verifies that \(I\) has at least two objects and the truth values of \(P_T()\) on these objects are different. The second portion verifies that \(I\) has at most two objects. This is implemented using min variables such that if we identify three distinct objects we set the value to 0. The two portions are put together so as to respect predicate order, and the final diagram \(B_1\) is shown in Figure 20. The aggregation function for \(B_1\) is max\(y_1, \max y_2, \min z_1, \min z_2, \min z_3\).

Let \(I^* = \{[a, b], P_T(a) = \text{true}, \text{ and } P_T(b) = \text{false}\}\) be the intended interpretation as used in the previous proof. We have the following two claims:

(C1) for all \(I\), MAP\(B_1(I) = 1\) if and only if \(I\) is isomorphic to \(I^*\).

(C2) if \(I\) is isomorphic to \(I^*\) then MAP\(B_1(I) = 1\) if and only if \((Q_1 x_1 Q_2 x_2 \ldots Q_m x_m f) = 1\).

C2 is exactly the same as Claim 1 in the proof of Theorem 17. For C1, given \(I\) which is isomorphic to \(I^*\), the valuation of \(y_1, y_2\) to \(a, b\) and any valuation to the \(z\)'s yields a map value of...
1. Therefore, considering the aggregation order we see that for \((y_1, y_2) = (a, b)\) in \(B_1\) the minimum over \(z\) yields 1, and then the maximum over \(y\)’s is 1. For the other direction, we need to consider interpretations not isomorphic to \(I^*\). If \(I\) has only one object then its map is 0 for all valuations, and therefore the aggregated value is 0. If \(I\) has at least 3 objects then for any fixed valuation for \(y\) the minimum over \(z\) is 0, implying that the maximum over \(y\) also yields 0 and \(\text{MAP}_{B_1}(I) = 0\). Finally consider any \(I\) with two objects where \(P_T()\) has the same truth value on the two objects. In this case the map is 0 for any valuation and thus the final map value is 0. We have therefore shown that C1 holds.

For our reduction, we produce \(B = apply(B_1, B_2, \land)\) where for the aggregation we make use of Theorem \([12]\) and interleave the aggregation functions of \(B_1\) and \(B_2\) so that \(B\) has at most \(k\) alternations of quantifiers. This is always possible because the QBF starts with a \(\exists\) quantifier and \(k \geq 2\).

By the claims C1 and C2 and Theorem \([12]\) we get that \(\text{MAP}_{B}(I) = 1\) if and only if \(I\) is isomorphic to \(I^*\) and \((Q_1x_1Q_2x_2 \ldots Q_mx_mf) = 1\). Therefore, the QBF is true if and only if there exists an interpretation \(I\) (which must be isomorphic to \(I^*\)) that satisfies \(B\).

Equivalence is one level higher in the hierarchy; using a reduction from QBF we show how to “peel off” one level of quantifiers and push that into the “existential quantification” over interpretations that potentially witness non-equivalence.

**Theorem 19** GFODD Equivalence for diagrams with aggregation depth \(k\) (where \(k \geq 2\)) is \(\Pi^p_{k+1}\)-complete.

**Proof.** As noted above, by Theorem \([10]\) it suffices to show that this holds for max-\(k\)-alternating GFODDs. To show membership we show that the complement, nonequivalence, is in \(\Sigma^p_{k+1}\). Given two max-\(k\)-alternating GFODDs \(B_1\) and \(B_2\) as input, we guess an interpretation \(I\) of the appropriate size, and then appeal to an oracle for GFODD Evaluation to calculate \(\text{MAP}_{B_1}(I)\) and \(\text{MAP}_{B_2}(I)\). Using these values we return Yes or No accordingly. To calculate the map values, let \(B\) be one of these diagrams, and let the leaf values of the diagram be \(v_1, v_2, \ldots, v_\ell\). We make \(\ell\) calls to GFODD Evaluation with \((B, I, v_i)\) as input. Each call requires an oracle in \(\Sigma^p_k\) and \(\text{MAP}_{B}(I)\) is the largest value on which the oracle returns Yes. Clearly if a witness for nonequivalence exists then this process can discover it and say Yes (per non-equivalence), and otherwise it will always say No. Therefore non-equivalence is in \(\text{NP}^{\Sigma^p_k}\), that is \(\Sigma^p_{k+1}\) and equivalence is in \(\Pi^p_{k+1}\).

We reduce QBF satisfiability with \(k \geq 3\) alternations of quantifiers to equivalence of max-\((k-1)\)-alternating GFODDs. The reduction is conceptually similar to the one from the previous theorem but the details are more involved. In particular, here we assume a QBF whose first quantifier is \(\forall\), that is, \(\forall x_1, Q_2x_2 \ldots Q_mx_mf(x_1, x_2, \ldots, x_m)\) where this form has \(k\) blocks of quantifiers. To simplify the notation it is convenient to group adjacent variables having the same quantifiers into groups so that the QBF has the form \(\forall x_1 \ldots Q_kx_kf(x_1, x_2, \ldots, x_k)\) where \(x_i\) refers to a set of variables.

As above we define a notion of “legal interpretations” for our diagrams. A legal interpretation embeds the binary interpretation \(I^*\) from the previous proof and in addition includes a truth setting for all the variables in the first \(\forall\) block of the QBF. The reduction constructs diagrams \(B_1\), \(B_2\), and \(B = apply(B_1, B_2, \land)\) such that the following claims hold:

(C1) for all \(I\), \(\text{MAP}_{B_1}(I) = 1\) if and only if \(I\) is legal.
for all legal interpretations we have $\text{MAP}(C_2) = 0$ and by Theorem 12 we also have $\text{MAP}(B_2) = 0$. Therefore, by C2, for all legal interpretations of node labels, and then elaborating to fix the ordering. The set of predicates includes $P_T()$ which is as before and for every QBF variable $x_i$ in the first $\forall$ block we use a predicates $P_{x_i}()$. Notice that each $x_i$ is a member of $x_1$ (the first $\forall$ group) where the typeface distinguishes the individual variables in the first block, from blocks of variables. In the simplified construction, a legal interpretation has exactly two objects, say $a$ and $b$, where $P_T(a) \neq P_T(b)$ and where for each $P_{x_i}()$ we have $P_{x_i}(a) = P_{x_i}(b)$. That is, as above the assignment of an object to $v$ in $P_T(v)$ simulates an assignment to Boolean values, but the truth value of $P_{x_i}(v)$ is the same regardless of which object is assigned to $v$.

In our example QBF $\forall x_1 \exists x_2 \forall x_3 \exists x_4 (x_1 \lor -x_2 \lor x_4) \land (-x_1 \lor x_2 \lor x_3) \land (x_1 \lor x_3 \lor -x_4)$ the first block includes only the variable $x_1$ and the following interpretation is legal: $I = \{[a, b], P_T(a) = \text{true}, P_{x_1}(a) = P_{x_1}(b) = \text{false}\}$. The diagram $B_1$ has three portions where the first two are exactly as in the previous proof, thus verifying that $I$ has two objects and that $P_T()$ behaves as stated. The third portion verifies that each $P_{x_i}()$ behaves as stated, where we use a sequence of blocks, one for each $P_{x_i}()$. The combined diagram $B_1$ is shown in Figure 21 and the aggregation function is $\max_{y_1, y_2, \min_{z_1, z_2, z_3}}$. 

![Figure 21: The $B_1$ diagram for GFODD Equivalence reduction.](image)
To see that C1 holds consider all possible cases for non-legal interpretations. If \( I \) has at most one object the map is 0 for all valuations and thus the aggregation is 0. If \( I \) has at least 3 objects, then for any values for \( y_1, y_2 \) the min aggregation over \( z \) yields 0, and therefore the map is 0. If \( I \) has 2 objects but it violates the condition on \( P_T \) or \( P_x \) then again the map is 0 for any valuation and the aggregation is 0. On the other hand, if \( I \) is legal, then the \( z \) block never yields 0 and the correct mapping to \( y_1, y_2 \) yields 1. Therefore the aggregation is 1.

The diagram \( B_2 \) is similar to \( B_2 \) from the previous proof. The only difference is that we need to handle the first \( \forall \) block differently. As it turns out, all we need to do is replace the min aggregation for the \( w_1 \) block with maximum aggregation and accordingly replace the default value on that block to 0. To avoid confusion we recall that the current proof uses a slightly different notation from the previous one. In the current proof \( x_i \) is a set of variables from the QBF and therefore \( w_i \) is a set of blocks of variables all of which have the same aggregation function. The modified variable consistency diagram is shown in Figure 22. The clause blocks in this case have the same structure as in the previous construction but use \( P_{x_i}(V(i_1,i_2)) \) when \( x_i \) is a \( \forall \) variable from the first block and use \( P_T(V(i_1,i_2)) \) otherwise. This is shown in Figure 23. \( B_2 \) includes the variable consistency blocks followed by the clause blocks. Note that the new clause blocks are not sorted in any consistent order because the predicates \( P_{x_i}() \) and \( P_T() \) appear in an arbitrary ordering determined by the appearance of literals in the QBF. Other than this violation, all other portions of the diagrams described are sorted where the predicate order has \( =< P_T < P_{x_i} \) and where variables \( w_i \) are before \( v(i_1,i_2) \) and variables within group are sorted lexicographically. The combined aggregation function is \( \max_{w_1}, \max_{w_2}, \min_{w_3}, \ldots, Q^{A}_{w_k} \). We next show that claim C2 holds, which will complete the proof of the simplified construction.

Consider any legal \( I \), let the corresponding truth values for variables in \( x_1 \) be denoted \( \alpha \), and consider valuations for the QBF extending \( x_1 = \alpha \). Now consider any valuation \( v \) to the remaining variables in the QBF and the induced substitution to the GFODD variables \( \zeta(v) \) that is easily
Figure 23: The clause blocks for the GFOOD equivalence reduction. For all occurrences of the universal variables of the first block ($x_1$ in this example) we have been replaced $P_T$ with $Px_1$.

identified from the construction. Add any consistent group assignment to $w_1$ (that is, we assign $a$ or $b$ to each subgroup of variables in that group) to $\zeta(v)$ to get $\hat{\zeta}(v)$. By the construction of $B_2$ we have that $f([x_1 = \alpha, (x_2, \ldots, x_k) = v]) = \text{MAP}_{B_2}(I, \hat{\zeta}(v))$. To see this note that there are no quantifiers in this expression, there is a 1-1 correspondence between the valuations of $x_2, \ldots, x_k$ and $w_2, \ldots, w_k$, and that as long as the assignment to the $w_1$ block is group consistent it does not affect the value returned. We call this set of valuations, that arise as translations of substitutions for QBF variables, Group 1.

The second group, Group 2, includes valuations that do not arise as $\hat{\zeta}(v)$ and therefore they violate at least one of the consistency blocks. Let $\zeta$ be such a valuation and let $Q^A_j$ be the first block from the left whose constraint is violated. By the construction of $B_2$, in particular the order of equality blocks along paths in the GFOOD, we have that the evaluation of the diagram on $\zeta$ “exits” to a default value on the first violation. Therefore, if $j = 1$, that is the violation is in the block of $w_1$, $\text{MAP}_{B_2}(I, \zeta) = 0$ and for $j \geq 2$ if $Q_j$ is a $\forall$ then $\text{MAP}_{B_2}(I, \zeta) = 1$ and if $Q_j$ is a $\exists$ then $\text{MAP}_{B_2}(I, \zeta) = 0$.

We can now show the correspondence in truth values. Consider any partition of the blocks $2, \ldots, k$ into a prefix $2, \ldots, j$ and remainder $(j+1), \ldots, k$, and any valuation $v$ to the prefix blocks. We claim that for all such partitions

$$Q_{j+1}x_{j+1}, \ldots, Q_kx_k, f(x_1 = \alpha, (x_2, \ldots, x_j) = v, x_{j+1}, \ldots, x_k) = Q_{j+1}w_{j+1}, \ldots, Q_kw_k, \text{MAP}_{B_2}(I, [w_1 = a, (w_2, \ldots, w_j) = \zeta(v), (w_{j+1}, \ldots, w_k)])$$

Note that when $j = 1$, that is, the prefix is empty, this yields that $Q_2x_2, \ldots, Q_kx_k, f(x_1 = \alpha, x_2, \ldots, x_k)$ is equal to $Q^A_2w_2, \ldots, Q^A_kw_k, \text{MAP}_{B_2}(I, [w_1 = a, w_2, \ldots, w_k])$. Now because the default value for violations of $w_1$ is 0 and because the aggregation for $w_1$ is max the latter expression is equal to $Q^A_1w_1, \ldots, Q^A_kw_k, \text{MAP}_{B_2}(I, [w_1, w_2, \ldots, w_k])$. This means that $\text{MAP}_{B_2}(I)$ is equal to $Q_2x_2, \ldots, Q_kx_k, f(x_1 = \alpha, x_2, \ldots, x_k)$ completing the proof of C2.

As above, we prove the claim by induction, backwards from $k$ to 1. For the base case, $j = k - 1$ and the second part includes only one block. Consider any concrete substitution for $w_k$ participating in the equation. If the substitution is in group 2, then the map is 1 if $Q_j$ is $\forall$ and is 0 otherwise.
Therefore, the value of $Q^A_k w_k, \text{MAP}_B(I, [w_1 = a, (w_2, \ldots, w_{k-1}) = \zeta(v), (w_k)])$ is the same as that value when restricted to substitutions in group 1. But, as argued above, for group 1 this is exactly the same value returned by the QBF.

For the inductive step, the valuation $v$ covers the first $j - 1$ blocks. Note that, by the inductive assumption, for any group 1 substitution $v_j$ for $x_j$ and corresponding, $\zeta(v_j)$ for $w_j$, $Q_{j+1}x_{j+1}, \ldots, Q_k x_k, f(x_1 = \alpha, (x_2, \ldots, x_{j-1}) = v, (x_j = v_j), x_{j+1}, \ldots, x_k) = Q^A_{j+1}w_{j+1}, \ldots, Q^A_k w_k, \text{MAP}_B(I, [w_1 = a, (w_2, \ldots, w_{j-1}) = \zeta(v), (w_j = \zeta(v_j)), (w_{j+1}, \ldots, w_k)])$. On the other hand, for any group 2 substitution $\zeta_j$ for $w_j$ and any values for $(w_{j+1}, \ldots, w_k)$ we have that the violating block for the corresponding combined $\zeta$ is block $j$ and therefore $\text{MAP}_B(I, [w_1 = a, (w_2, \ldots, w_{j-1}) = \zeta(v), (w_j = \zeta_j), (w_{j+1}, \ldots, w_k)])$ gets the default value for that block. Therefore, as in the base case, the map is determined by group 1 valuations, which are in turn identical to the QBF value and $Q_j x_j, \ldots, Q_k x_k, f(x_1 = \alpha, (x_2, \ldots, x_{j-1}) = v, (x_j, \ldots, x_k)) = Q^A_j w_j, \ldots, Q^A_k w_k, \text{MAP}_B(I, [w_1 = a, (w_2, \ldots, w_{j-1}) = \zeta(v), (w_j, \ldots, w_k)])$ as required. Therefore, the claim on the correspondence of values holds, and as argued above this completes the proof of C2.

**Extending the reduction to handle ordering:** We next extend the reduction to respect label order in diagram nodes. In addition, the modified construction will be such that $B_1$ and $B$ differ via the removal of one edge, and therefore the hardness result holds even for this restricted case.

The main idea in the extended construction is to replace the unary predicates $P_T$ and $P_{x_i}$ with one binary predicate $q(\cdot, \cdot)$ where the “second argument” in $q()$ serves to identify the corresponding predicate and hence its truth value. In addition we force $q()$ to be symmetric so that for any $A$ and $B$ the truth value of $q(A,B)$ is the same as the truth value of $q(B,A)$. In this way we have freedom to use either $q(A,B)$ or $q(B,A)$ as the node label which provides sufficient flexibility to handle the order issues. To implement this idea we need a few additional constructions.

Let the set of variables in the first $\forall$ block of the QBF be $x_1, x_2, \ldots, x_\ell$. The set of predicates in the extended reduction includes unary predicates $T(), y_1(), y_2(), x_1(), x_2(), \ldots, x_\ell()$, and one binary predicate $q(\cdot,\cdot)$. A legal interpretation includes exactly $\ell + 3$ objects which are uniquely identified by the unary predicates. We therefore slightly abuse notation and use the same symbols for the objects and predicates. In particular, the atoms $y_1(y_1), y_2(y_2), T(T)$, and $x_1(x_1), x_2(x_2), \ldots, x_\ell(x_\ell)$ are true in the interpretation and only these atoms are true for these unary predicates (e.g., $x_1(T)$ is false). The truth values of $q()$ reflect the constraints from above in addition to being symmetric. Thus the truth values of $q(y_1,T)$ and $q(T,y_1)$ are the same and they are the negation of the truth values of $q(y_2,T)$ and $q(T,y_2)$. For all $i$, the truth values of $q(y_1,x_i), q(x_i,y_1), q(y_2,x_i)$ and $q(x_i,y_2)$ are the same. The truth values of other instances of $q()$, for example, $q(x_2,T)$, can be set arbitrarily. For example, the following interpretation is legal when $\ell = 1$: $I = \{[a,b,c,d], y_1(a), y_2(b), T(c), x_1(d), q(c,a) = q(a,c) = \text{true}, q(c,b) = q(b,c) = \text{false}, q(d,a) = q(a,d) = q(d,b) = q(b,d) = \text{false}, q(\cdot,\cdot) = \text{false}\} \text{where } q(\cdot,\cdot) \text{refers to any instance not explicitly mentioned in the list.}$

We next define the diagram $B_1$ that is satisfied only in legal interpretations. We enforce exactly $\ell + 3$ objects using two complementary parts. The first includes $\binom{\ell + 4}{2}$ inequalities on a new set of $\ell + 4$ variables $z_1, \ldots, z_{\ell+4}$ with min aggregation. If we identify $\ell + 4$ distinct objects we set the value to 0. This is shown in Figure 24.

To enforce at least $\ell + 3$ objects and identify them we use the following gadget. For each of the unary predicates we have a diagram identifying its object and testing its uniqueness where we use both max and min variables. This is shown for the predicate $T()$ in Figure 25. The node $T(T)$
with max variable $T$ identifies the object $T$. The nodes $T(r_1), T(r_2)$ with min variables $r_1, r_2$ make sure that $T$ holds for at most one object. We chain the diagrams together as shown in Figure 26 where the variables $r_1, r_2$ are shared among all unary predicates. The corresponding aggregation function is $\max_{T, y_1, y_2, x_1, \ldots, x_\ell, \min r_1, r_2}$. This diagram associates each of the $\ell + 3$ objects with one of the unary predicates and in this way provides a reference to specific objects in the interpretation.

The symmetry gadget for $q()$ is shown in Figure 27 where the variables $m_1, m_2$ are min variables. If an input interpretation has two objects $A, B$ where $q(A, B)$ has a truth value different than $q(B, A)$ then minimum aggregation will map the interpretation to 0.

The truth value gadget for the simulation of $P_T$ is shown in Figure 28 where $y_1, y_2$ and $T$ are as above. The truth value gadget for the simulation of $P_{x_i}$ is shown in Figure 29 where $y_1, y_2$ and $x_i$ are as above. These diagram fragments refer to variables in other portions and they will be connected and aggregated together.
Figure 25: GFODD equivalence reduction. The diagram for the unary predicate $T()$ identifies one object for which $T()$ holds, and makes sure that $T()$ does not hold for two objects.

Figure 26: GFODD equivalence reduction. The diagram verifying that each unary predicate corresponds to exactly one object in the interpretation.
Figure 27: GFODD equivalence reduction. Diagram fragment verifying that \( q() \) is symmetric.

Figure 28: GFODD equivalence reduction. Diagram fragment verifying that \( q() \) simulates truth values of \( P_T() \) from the simple construction.

Figure 29: GFODD equivalence reduction. Diagram fragment verifying that \( q() \) simulates truth values of \( P_{x_i}() \) from the simple construction.
Figure 30: The diagram checking block consistency for the ordered version of the GFODD equivalence reduction.
Figure 31: The final version of the diagram $B_1$ for the ordered version of the GFODD equivalence reduction. The complete aggregation function is $\max_{T,y_1,y_2,x_1,...,x_\ell} \max_{w_1,w_2} \min_{m_1,m_2} \min_{r_1,r_2} \min_{z_1,...,z_{\ell+4}} \min_{w_3} ... Q^A_{w_k}$. 
Finally, we add a component that is not needed for verifying that \( I \) is legal but will be useful later when we include the clauses. In particular, we include a variable consistency block which is similar to the one in the simple construction (see Figure 22) but where we force each subgroup in \( w_i \) to bind to the same object as \( y_1 \) or \( y_2 \). This is shown in Figure 30.

We chain the diagrams together as shown in Figure 31 to get \( B_1 \) where we have moved the node labelled \( r_1 = r_2 \) to be above the block consistency gadget. Note that the diagram is sorted where for predicate order we have \( \preceq \) and for variables we have \( z_i, T, y_i, x_i \), and \( y_i < v_{(i, i_2)} \), and \( m_i < T < x_j \). The complete aggregation function is

\[
\max_{T,y_1,y_2,x_1,...,x_\ell} \min_{w_1,w_2,m_1,m_2} \min_{r_1,r_2} \min_{z_1,...,z_{\ell+4}} Q_{w_k}.^{1}
\]

We next show that the claim \( C_1 \) holds for the extended reduction.\(^1\) We first consider all possible cases for illegal interpretations.

- If the interpretation has \( \geq \ell + 4 \) objects then the top portion of the diagram yields 0 for some valuation of \( z \)'s. Consider then any valuation \( \zeta_p \) to the prefix of variables \( T, y_1, y_2, x_1, \ldots, x_\ell, w_1, w_2, m_1, m_2, r_1, r_2 \), and any \( \zeta \) which is an extension of this valuation, has the violating combination for \( z_1, \ldots, z_{\ell+4} \) and any valuation for the other variables. We have MAP\(_B\)(\( I, \zeta \)) = 0. Therefore, all aggregations from \( w_k \) to \( w_3 \) yield a value of 0 for this prefix. Now continuing backwards, the minimization over \( z \) yields a value of 0 for \( \zeta_p \), and this holds for any \( \zeta_p \). Therefore, all remaining aggregations yield 0 and the final value is 0.

- Next, consider the case where the interpretation has \( \leq \ell + 3 \) objects but one of the unary predicates is always false (i.e., it does not “pick” any object). The situation is similar to the previous case, but here we get a value of 1 for \( \zeta \) where \( r_1 = r_2 \) or where there is a block violation for some \( w_i \) block with min aggregation.

Consider any valuation \( \zeta_p \) to the prefix of variables \( T, y_1, y_2, x_1, \ldots, x_\ell, w_1, w_2, m_1, m_2 \), which is block consistent on \( w_1, w_2 \) and any \( \zeta \) which is an extension of this valuation, has \( r_1 \neq r_2 \), and any valuation for the other variables. We have two cases: if \( \zeta \) is in group 1 the map is 0 because the unary predicate test fails, and if \( \zeta \) is in group 2 the map is the default value of the first violated block \( w_1 \). Now, because there is at least one group 1 valuation, we can argue inductively backwards from \( k \) that all aggregations from \( k \) to 3 yield 0, and the same holds for the minimization over \( z \). (The induction is simpler than the one above because we always carry a value of 0 backwards, not an arbitrary value.) Next, because of this value of 0 for \( r_1 \neq r_2 \), the minimization over \( r \) yields 0 for \( \zeta_p \). Considering any variants of \( \zeta_p \) which are not block consistent on \( w_1, w_2 \) but otherwise identical we see that their value is also 0. As a result all remaining aggregations yield 0.

- Next consider the case where the previous two conditions are satisfied but where one of the unary predicates holds for two or more objects. The argument is identical to the previous case, except that \( \zeta \) has the violating pair for \( r_1, r_2 \) (instead of any \( r_1 \neq r_2 \)).

- Next consider any interpretation that has exactly \( \ell + 3 \) objects and where the unary predicates identify the objects corresponding to \( T, y_1, y_2, x_1, \ldots, x_\ell \) but where \( q() \) is not symmetric. In

\(^1\) Note that we can remove the variable correspondence block which complicates the argument (and does not test anything per legality of \( I \)) and still maintain correctness of \( C_1 \). But including it here simplifies the argument for diagram \( B \) below and thus simplifies the overall proof.
this case we consider a valuation $\zeta_p$ to the prefix of variables $T, y_1, y_2, x_1, \ldots, x_\ell, w_1, w_2$, and extensions $\zeta$ that have the violating pair for $m_1, m_2$. As above, starting with $\zeta$ that are block consistent on $w_1, w_2$ and have $r_1 \neq r_2$ we can argue that the aggregations down to $m$ yield 0, and as a result that the aggregation over $m$ yields 0. As $\zeta_p$ is arbitrary, it follows that the final value is 0.

- The only remaining cases are interpretations that are illegal only because the $q()$ simulation of $P_T()$ or $P_{x_i}()$ is not as required. In this case, the same argument as in the 2nd item in this list shows that the value is 0.
Figure 32: GFODD equivalence reduction. Diagrams capturing the clause blocks.
Figure 33: The final version of the diagram $B$ for the ordered version of the GFODD equivalence reduction. The complete aggregation function is $\max_{T, y_1, y_2, x_1, \ldots, x_\ell} \max_{w_1, w_2} \min_{m_1, m_2} \min_{r_1, r_2} \min_{z_1, \ldots, z_{\ell+4}} \min_{w_3} \ldots Q^A_{w_k}$.  

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Therefore, if \( I \) is illegal then \( MAP_B(I) = 0 \). Consider next any legal interpretation and the intended valuation \( \zeta_p \) to \( T, y_1, y_2, x_1, \ldots, x_\ell \). For any group 1 extension of this valuation and any valuation of the variables \( m, z, r \) the diagram yields 1. Therefore, we can argue inductively that all \( w_i \) aggregations down to \( w_3 \) yield 1, and the min aggregations over \( m, z, r \) yield 1 for \( \zeta_p \). Therefore, the max aggregation over \( T, y_1, y_2, x_1, \ldots, x_\ell \) yields 1, and \( MAP_B(I) = 1 \). This completes the proof of C1.

The diagram \( B \) is obtained by adding the clause blocks below the \( q() \) tests of \( B_1 \). The clause blocks have the same structure as above but they use \( q() \) instead of \( P_T() \) and \( P_x() \). This is shown in Figure 32. The final diagram for \( B_1 \) is shown in Figure 33 and it has the same aggregation function as \( B_1 \). Note that the diagram is also sorted using the same order as in \( B_1 \) with the addition that \( x_i < V_{(i_1,i_2)} \).

We next show that \( MAP_B(I) = 1 \) if and only if \( I \) is legal and \( Q_2x_2 \ldots Q_kx_kf((x_1 = \alpha), x_2, \ldots, x_k) = 1 \) where \( I \) embeds the substitution \( x_1 = \alpha \). This has the same consequences as having \( B = apply(B_1, B_2, \land) \) in the simple construction.

To prove the claim first note that by the construction \( B \) adds more tests on the path to a 1 leaf of \( B_1 \) and does not add any other paths to a value of 1. Therefore, for any \( I \) and any \( \zeta \), if \( MAP_B(I, \zeta) = 1 \) then \( MAP_{B_1}(I, \zeta) = 1 \) and as a result if \( MAP_B(I) = 1 \) then \( MAP_{B_1}(I) = 1 \). Therefore, by C1, if \( MAP_B(I, \zeta) = 1 \) then \( I \) is legal. Next, consider any legal \( I \) and any unintended valuation \( \zeta_p \) for the prefix \( T, y_1, y_2, x_1, \ldots, x_\ell \). As above, the aggregated value for this prefix is 0. Therefore, if \( MAP_B(I) = 1 \) and thus the max aggregation over these prefix variables yields 1, it must be through the intended valuation \( \zeta_p \) for the prefix \( T, y_1, y_2, x_1, \ldots, x_\ell \). However, when \( T, y_1, y_2, x_1, \ldots, x_\ell \) are fixed to their intended values, by the construction there is a 1-1 correspondence to valuations and values of \( B_2 \) in the simple construction (where we extend the notion of block consistent to enforce that \( w_i \) bind to \( y_1, y_2 \)). Therefore, the claim holds by C2 of the simple construction.

The result for GFODD Value is similar to the FODD case.

**Theorem 20** GFODD Value for diagrams with aggregation depth \( k \) (where \( k \geq 2 \)) is \( \Sigma_{k+1}^p \)-complete.

**Proof.** The proof of Theorem 9 goes through almost directly and requires only a slight wording variation. For membership we get the bound on interpretation size by the assumption on the input; then the algorithm is the same.

For the reduction, we use Theorem 12 to calculate \( B = apply(B_1, B_2, +) \). As stated in that theorem, we can mesh together the aggregation lists of \( B_1 \) and \( B_2 \) interleaving the max and min blocks from each diagram without increasing quantifier death and the diagram \( B \) has the same quantifier prefix and depth as those of \( B_1 \) and \( B_2 \).

Unlike max-\( k \)-alternating GFODDs, for min diagrams the search for a satisfying interpretation cannot be absorbed into the first aggregation operator. This fact pushes the problem one level higher in the hierarchy.

**Theorem 21** GFODD Satisfiability for min-\( k \)-alternating GFODDs (where \( k \geq 2 \)) is \( \Sigma_{k+1}^p \)-complete.

**Proof.** For membership, we guess an interpretation \( I \) of the appropriate size, and then appeal to a \( \Sigma_k^p \) oracle to solve GFODD evaluation for \( (B, I) \). This is clearly in \( NP^{\Sigma_k^p} \).
The hardness result uses a slight modification of the equivalence proof, which we sketch next. One can verify (see appendix) that all the details of the modification go through to establish the result.

In particular, we reduce QBF satisfiability with \( k \geq 3 \) alternations of quantifiers to satisfiability of min-(\( k - 1 \))-alternating GFODDs. Here we assume a QBF whose first quantifier is \( \exists \), that is has the form \( \exists x_1 \ldots Q_k x_k f(x_1, x_2, \ldots, x_k) \) where \( x_i \) refers to a set of variables. We build \( B_1, B_2 \) and \( B \) exactly as above, with one exception: the leaf values on the diagram that checks for block consistency are flipped from the previous construction (because the corresponding aggregation operators are switched). The reduction still provides \( B_1, B_2, \) and \( B = apply(B_1, B_2, \wedge) \) such that the following claims hold:

\[
\begin{align*}
(C1) & \text{ for all } I, \text{ MAP}_{B_1}(I) = 1 \text{ if and only if } I \text{ is legal.} \\
(C2) & \text{ if } I \text{ is legal and it embeds the substitution } x_1 = \alpha \text{ then MAP}_{B_2}(I) = 1 \text{ if and only if } \forall x_2 \ldots Q_k x_k f((x_1 = \alpha), x_2, \ldots, x_k) = 1.
\end{align*}
\]

We then output the diagram \( B \) for GFODD satisfiability. Now, if the QBF is satisfied then there exists a value \( \alpha \) such that for \( x_1 = \alpha \) we have that \( Q_2 x_2 \ldots Q_k x_k f((x_1 = \alpha), x_2, \ldots, x_k) = 1 \). Therefore, by \( C2 \), for the legal \( I \) that embeds \( \alpha \), \( \text{MAP}_{B_2}(I) = 1 \). On the other hand, if the QBF is not satisfied then for all substitutions \( x_1 = \alpha \) we have \( Q_2 x_2 \ldots Q_k x_k f((x_1 = \alpha), x_2, \ldots, x_k) = 0 \). Therefore, by \( C2 \), all legal \( I \) (and any \( \alpha \) they embed) \( \text{MAP}_{B_2}(I) = 0 \) and by Theorem 12 we also have \( \text{MAP}_{B}(I) = 0 \). By \( C1 \), \( \text{MAP}_{B}(I) = 0 \) for non-legal interpretations. Therefore, \( B \) is not satisfiable. The analysis of the construction and the extension per ordering continues as in the previous proof and is almost identical. □

The proof of the previous theorem, as all other hardness proofs, uses a signature without any constants, i.e., we use equality and unary and binary predicates. For min-GFODDs (the case \( k = 1 \)) the use of constants affects the complexity of the problem. In particular, for a signature without constants, if a min-GFODD is satisfied by interpretation \( I \), then it is satisfied by the sub-interpretation of \( I \) with just one object (any object in \( I \) will do). Moreover, given diagram \( B \) and a specific \( I \) with one object, model evaluation is in \( P \) because there is only one valuation to consider. Therefore, in this case satisfiability is in \( NP \): we can guess the interpretation (i.e. truth values of predicates) and evaluate \( \text{MAP}_{B}(I) \) in polynomial time. On the other hand, if we allow constants in the signature the problem follows the same scheme as above and is \( \Sigma_2^p \)-complete.

**Theorem 22** GFODD Satisfiability for min-GFODDs is \( \Sigma_2^p \)-complete.

**Proof.** Membership is as in the general case. For hardness, we use the construction in the reduction of the previous proof which yields a GFODD with aggregation \( \text{min}^* \text{max}^* \) (i.e., the portion starting with \( w_3 \) does not exist) where the max variables are \( T, y_1, y_2, x_1, \ldots, x_\ell \). We then turn these variables into constants and remove the max aggregation to yield a min GFODD. Although turning these variables to constants is similar to pulling the max portion to the head of the formula one can verify that the arguments above regarding valuations and values hold yielding the same hardness result. □

5 Discussion

In this paper we explored the complexity of computations using FODD and GFODD, providing a classification placing them within the polynomial hierarchy, where, roughly speaking, equivalence is
one level higher in the hierarchy than evaluation and satisfiability. These results are useful in that they clearly characterize the complexity of the problems solved heuristically by implementations of GFODD systems [14, 16, 17] and can be used to partly motivate or justify the use of these heuristics. For example, the “model checking reductions” of [14] replace equivalence tests with model evaluation on a “representative” set of models, and [14] choose this set heuristically leading to inference that is correct with respect to these models but otherwise incomplete. Our results here show that this indeed leads to reduction of the complexity of the inference problem so that the reduction in accuracy is traded for improved worst case run time. As mentioned above, the proofs in the paper can be used (in simpler form) to show the same complexity results for the corresponding problems in first order logic. To our knowledge the complexity questions with an explicit bound on model size have not been previously studied in this context. Yet they can be useful in many contexts where such a bound can be given in a practical setting. For example, in such cases existing optimized QBF algorithms can be used for inference in this restricted form of first order logic.

There are several important directions for further investigation. The first involves using a richer set of aggregation operators. In particular the definition of GFODDs allows for any function to aggregate values, and functions such as sum, product, and average are both natural and useful for modeling and solving Markov Decision Processes, which have been the main application for FODDs. The work of [17] extends the model checking reductions to GFODDs with average aggregation. Clarifying the complexity of these problems and identifying the best algorithms for them is an important effort for the efficiency of such systems. In this context it is also interesting to clarify the relationship to query languages in databases that allow for similar aggregations and to formulations of “logic with counting” that has been developed in the context of database theory [22].

Considering this wider family of GFODDs also raises new computational questions beyond the ones explored in this paper. One such question arises from the connection to statistical relational models and specifically to lifted inference in such models (see e.g. [26, 25, 5]). In particular, consider MLNs [26] that can be seen to define a distribution over first order structures through a log linear probability model, where features of this model are defined by simple first order formula templates. It is easy to show how to encode such templates and their weights using a GFODD with product aggregation, and how these can be combined using a variant of the apply procedure. The main computational question in this context has been to calculate the probability of a query given the MLN model, and the number of objects \( n \) in the domain. Let \( \mathcal{I} \) be the set of models with \( n \) objects over the relevant signature. In our case this question translates to calculating

\[
\sum_{I \in \mathcal{I}} \text{MAP}_B(I)
\]

for an appropriate \( B \) that combines the query and the MLN model. This is closely related to the approaches that solve this problem via lifted weighted model counting [5]. A similarly interesting question would require us to calculate the best \( I \) for a particular \( B \)

\[
\arg\max_{I \in \mathcal{I}} \text{MAP}_B(I).
\]

In this case, if \( B \) captures say profit of some organization, then the computation optimizes the setting so as to maximize profit. Thirdly, we have defined a logic-inspired language but did not define or study any notion of implication. A natural notion of implication with numerical values, related to the one used by [7], is majorization:

\[
B_1 \models B_2 \iff \forall I \in \mathcal{I} \; \text{MAP}_{B_1}(I) \leq \text{MAP}_{B_2}(I).
\]
Efficient algorithms and complexity analysis for these new questions will expand the scope and applicability of GFODDs.

Finally, efficient algorithms for model evaluation play an important role in GFODD implementations. The work of [17] provides a generic algorithm inspired by the variable elimination algorithm from probabilistic inference. Several application areas, including databases, AI, and probabilistic inference have shown that more efficient algorithms are possible when the input formula or graph have certain structural properties such as low graph width, where for example the bottleneck in variable elimination arises due to the width parameter. We therefore conjecture that similar notions can be developed to provide more efficient evaluation for GFODDs with some structural properties. Coupled with model checking reductions, this can lead to realizations of GFODD systems that combine high expressive power going beyond first order logic with efficient algorithms.

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References

[1] R. Bahar, E. Frohm, C. Gaona, G. Hachtel, E. Macii, A. Pardo, and F. Somenzi. Algebraic decision diagrams and their applications. In IEEE /ACM ICCAD, pages 188–191, 1993.

[2] C. Boutilier, R. Reiter, and B. Price. Symbolic dynamic programming for first-order MDPs. In Proc. of IJCAI, pages 690–700, 2001.

[3] R. Bryant. Graph-based algorithms for boolean function manipulation. IEEE Transactions on Computers, C-35(8):677–691, 1986.

[4] C. Chang and J. Keisler. Model Theory. Elsevier, Amsterdam, Holland, 1990.

[5] G. Van den Broeck, N. Taghipour, W. Meert, J. Davis, and L. De Raedt. Lifted probabilistic inference by first-order knowledge compilation. In IJCAI, pages 2178–2185, 2011.

[6] R. Fagin. Finite-model theory - a personal perspective. Theoretical Computer Science, 116(1&2):3–31, 1993.

[7] G. J. Gordon, S. A. Hong, and M. Dudík. First-order mixed integer linear programming. In Proceedings of the Twenty-Fifth Conference on Uncertainty in Artificial Intelligence, pages 213–222, 2009.

[8] E. Grädel. Decidable fragments of first-order and fixed-point logic. From prefix-vocabulary classes to guarded logics. In Proceedings of Kalmar Workshop on Logic and Computer Science, Szeged, 2003.

[9] J. Groote and O. Tveretina. Binary decision diagrams for first order predicate logic. Journal of Logic and Algebraic Programming, 57:1–22, 2003.
[10] J. Hoey, R. St-Aubin, A. Hu, and C. Boutilier. Spudd: Stochastic planning using decision diagrams. In Proceedings of UAI, pages 279–288, 1999.

[11] S. Hölldobler, E. Karabaev, and O. Skvortsova. FluCaP: a heuristic search planner for first-order MDPs. JAIR, 27:419–439, 2006.

[12] S. Hölldobler and O. Skvortsova. A logic-based approach to dynamic programming. In AAAI-04 workshop on learning and planning in Markov Processes – advances and challenges, 2004.

[13] S. Homer and A. L. Selman. Computability and Complexity Theory. Springer-Verlag, New York, 2001.

[14] S. Joshi, K. Kersting, and R. Khardon. Self-taught decision theoretic planning with first order decision diagrams. In Proc. of ICAPS, pages 89–96, 2010.

[15] S. Joshi, K. Kersting, and R. Khardon. Decision theoretic planning with generalized first order decision diagrams. AIJ, 175:2198–2222, 2011.

[16] S. Joshi and R. Khardon. Probabilistic relational planning with first order decision diagrams. JAIR, pages 231–266, 2011.

[17] S. Joshi, R. Khardon, A. Raghavan, P. Tadepalli, and A. Fern. Solving relational mdps with exogenous events and additive rewards. In ECML, 2013.

[18] K. Kersting, M. Van Otterlo, and L. De Raedt. Bellman goes relational. In Proc. of ICML, 2004.

[19] R. Khardon, H. Mannila, and D. Roth. Reasoning with examples: Propositional formulae and database dependencies. Acta Informatica, pages 267–286, 1999.

[20] R. Khardon and D. Roth. Reasoning with models. Artificial Intelligence, 87:187–213, 1996.

[21] R. Khardon and D. Roth. Learning to reason. Journal of the ACM, 44(5):697–725, 1997.

[22] Leonid Libkin. Elements of Finite Model Theory. Springer, 2004.

[23] J.W. Lloyd. Foundations of Logic Programming. Springer Verlag, 1987. Second Edition.

[24] C. H. Papadimitriou. Computational complexity. Addison-Wesley, 1994.

[25] L. De Raedt, A. Kimmig, and H. Toivonen. Problog: A probabilistic prolog and its application in link discovery. In IJCAI, pages 2462–2467, 2007.

[26] Matthew Richardson and Pedro Domingos. Markov logic networks. Machine Learning, 62(1-2):107–136, 2006.

[27] S. Russell and P. Norvig. Artificial Intelligence: a modern approach. Prentice Hall, 1995.

[28] S. Sanner and C. Boutilier. Practical solution techniques for first order MDPs. AIJ, 173:748–788, 2009.

[29] M. Schaefer. Graph ramsey theory and the polynomial hierarchy. J. Comput. Syst. Sci., 62(2):290–322, 2001.

44
[30] M. Sipser. *Introduction to the Theory of Computation*. Thomson South-Western, 3rd edition, 2012.

[31] C. Wang, S. Joshi, and R. Khardon. First order decision diagrams for relational MDPs. *JAIR*, 31:431–472, 2008.
A Detailed Proofs

Theorem 23  GFOOD Satisfiability for min-k-alternating GFOODs (where k ≥ 2) is \( \Sigma^p_{k+1} \)-hard.

Proof. We reduce QBF satisfiability with \( k \geq 3 \) alternations of quantifiers to satisfiability of min-(\( k - 1 \))-alternating GFOODs. The reduction borrows most of the construction from the previous theorem, swapping between min and max aggregation to adjust it to the new context.

In particular, here we assume a QBF whose first quantifier is \( \exists \), that is, \( \exists x_1, Q_2 x_2 \ldots Q_m x_m f(x_1, x_2, \ldots, x_m) \) where this form has \( k \) blocks of quantifiers. To simplify the notation it is convenient to group adjacent variables having the same quantifiers into groups so that the QBF has the form \( Q_1 x_1 \ldots Q_k x_k f(x_1, x_2, \ldots, x_k) \) where \( x_i \) refers to a set of variables.

As above we define a notion of "legal interpretations" for our diagrams. A legal interpretation embeds the binary interpretation \( * \) from previous proofs and in addition includes a truth setting for all the variables in the first \( \exists \) block of the QBF. The reduction constructs diagrams \( B_1, B_2, \) and \( B = \text{apply}(B_1, B_2, \land) \) such that the following claims hold:

(\( C1 \)) for all \( I, \text{MAP}_{B_1}(I) = 1 \) if and only if \( I \) is legal.
(\( C2 \)) if \( I \) is legal and it embeds the substitution \( x_1 = \alpha \) then \( \text{MAP}_{B_2}(I) = 1 \) if and only if \( Q_2 x_2 \ldots Q_k x_k f((x_1 = \alpha, x_2, \ldots, x_k)) = 1 \).

We then output the diagram \( B \) for GFOOD satisfiability. Now, if the QBF is satisfied then there exists a value \( \alpha \) such that for \( x_1 = \alpha \) we have that \( Q_2 x_2 \ldots Q_k x_k f((x_1 = \alpha, x_2, \ldots, x_k)) = 1 \). Therefore, by \( C2 \), for the legal \( I \) that embeds \( \alpha \), \( \text{MAP}_{B_2}(I) = 1 \).

On the other hand, if the QBF is not satisfied then for all substitutions \( x_1 = \alpha \) we have \( Q_2 x_2 \ldots Q_k x_k f((x_1 = \alpha, x_2, \ldots, x_k)) = 0 \). Therefore, by \( C2 \), all legal \( I \) (and any \( \alpha \) they embed) \( \text{MAP}_{B_2}(I) = 0 \) and by Theorem[12] we also have \( \text{MAP}_{B}(I) = 0 \). By \( C1 \), \( \text{MAP}_{B}(I) = 0 \) for non-legal interpretations. Therefore, \( B \) is not satisfiable.

We now proceed with the details, starting first with a simplified construction ignoring ordering of node labels, and then elaborating to fix the ordering. The set of predicates includes \( P_T(*) \) which is as before and for every QBF variable \( x_i \) in the first \( \exists \) block we use a predicates \( P_{x_i}(*) \). Notice that each \( x_i \) is a member of \( x_1 \) (the first \( \exists \) group) where the typeface distinguishes the individual variables in the first block, from blocks of variables. In the simplified construction, a legal interpretation has exactly two objects, say \( a \) and \( b \), where \( P_T(a) \neq P_T(b) \) and where for each \( P_{x_i}(*) \) we have \( P_{x_i}(a) = P_{x_i}(b) \). That is, as above the assignment of an object to \( v \) in \( P_T(v) \) simulates an assignment to Boolean values, but the truth value of \( P_{x_i}(v) \) is the same regardless of which object is assigned to \( v \).

Consider our example QBF modified to start with \( \exists \) quantifier \( \exists x_1 \forall x_2 \exists x_3 \forall x_4 (x_1 \lor \bar{x}_2 \lor x_3) \land (x_1 \lor x_3 \lor \bar{x}_4) \). The first block includes only the variable \( x_1 \) and the following interpretation is legal: \( I = \{ [a, b], P_T(a) = \text{true}, P_T(b) = \text{false}, P_{x_1}(a) = P_{x_1}(b) = \text{false} \} \).

The diagram \( B_1 \) has three portions where the first two are exactly as in the previous proof, thus verifying that \( I \) has two objects and that \( P_T(*) \) behaves as stated. The third portion verifies that each \( P_{x_i}(*) \) behaves as stated, where we use a sequence of blocks, one for each \( P_{x_i}(*) \). The combined diagram \( B_1 \) is shown in Figure[21] and the aggregation function is \( \min_{z_1, z_2, z_3}, \max_{y_1, y_2} \).

To see that \( C1 \) holds consider all possible cases for non-legal interpretations. If \( I \) has at most one object the map is 0 for all valuations and thus the aggregation is 0. If \( I \) has at least 3 objects, then the min aggregation over \( z \) yields 0. If \( I \) has 2 objects but it violates the condition on \( P_T \) or \( P_{x_i} \) then again the map is 0 for any valuation and the aggregation is 0. On the other hand,
if $I$ is legal, then for any assignment to $z$, the correct mapping to $y_1, y_2$ yields 1. Therefore the aggregation over $z$ yields 1.

The diagram $B_2$ is similar to $B_2$ from the previous proof. The only difference is that we need to handle the first $\exists$ block differently. As it turns out, all we need to do is replace the max aggregation for the $w_1$ block with min aggregation and accordingly replace the default value on that block to 1. The modified variable consistency diagram is shown in [we flip leaf values in diagram] Figure 22

The clause blocks in this case have the same structure as in the previous construction but use $P_{x_i}(V_{(i_1,i_2)})$ when $x_i$ is a $\exists$ variable from the first block and use $P_T(V_{(i_1,i_2)})$ otherwise. This is shown in Figure 23. $B_2$ includes the variable consistency blocks followed by the clause blocks. Note that the new clause blocks are not sorted in any consistent order because the predicates $P_{x_i}$ and $P_T()$ appear in an arbitrary ordering determined by the appearance of literals in the QBF. Other than this violation, all other portions of the diagrams described are sorted where the predicate order has $= \prec P_T \prec P_{x_i}$ and variables $w_i$ are before $V_{(i_1,i_2)}$ and variables within group are sorted lexicographically. The combined aggregation function is $\min_{w_1}, \min_{w_2}, \max_{w_3}, \ldots , Q^A_{w_k}$. We next show that claim C2 holds, which will complete the proof of the simplified construction.

Consider any legal $I$, let the corresponding truth values for variables in $x_1$ be denoted $\alpha$, and consider valuations for the QBF extending $x_1 = \alpha$. Now consider any valuation $v$ to the remaining variables in the QBF and the induced substitution to the GFODD variables $\zeta(v)$ that is easily identified from the construction. Add any consistent group assignment to $w_1$ (that is, we assign $a$ or $b$ to all variables in that group) to $\zeta(v)$ to get $\hat{\zeta}(v)$. By the construction of $B_2$ we have that $f([x_1 = \alpha, (x_2, \ldots, x_k) = v]) = MAP_{B_2}(I, \hat{\zeta}(v))$. To see this note that there are no quantifiers in this expression, there is a 1-1 correspondence between the valuations of $x_2, \ldots, x_k$ and $w_2, \ldots, w_k$, and that as long as the assignment to the $w_1$ block is group consistent it does not affect the value returned. We call this set of valuations, that arise as translations of substitutions for QBF variables, Group 1.

The second group, Group 2, includes valuations that do not arise as $\hat{\zeta}(v)$ and therefore they violate at least one of the consistency blocks. Let $\zeta$ be such a valuation and let $Q^A_j$ be the first block from the left whose constraint is violated. By the construction of $B_2$, in particular the order of equality along paths in the GFODD, we have that the evaluation of the diagram on $\zeta$ “exits” to a default value on the first violation. Therefore, if $j = 1$, that is the violation is in the block of $w_1$, $MAP_B(I, \zeta) = 1$ and for $j \geq 2$ if $Q_j$ is a $\forall$ then $MAP_B(I, \zeta) = 1$ and if $Q_j$ is a $\exists$ then $MAP_B(I, \zeta) = 0$.

We can now show the correspondence in truth values. Consider any partition of the blocks $2, \ldots, k$ into a prefix $2, \ldots, j$ and remainder $(j+1), \ldots, k$, and any valuation $v$ to the prefix blocks. We claim that for all such partitions

$$Q_{j+1}x_{j+1}, \ldots, Q_kx_k, f(x_1 = \alpha, (x_2, \ldots, x_j) = v, x_{j+1}, \ldots, x_k)$$

$$= Q_{j+1}w_{j+1}, \ldots, Q_kw_k, MAP_{B_2}(I, [w_1 = a, (w_2, \ldots, w_j) = \zeta(v), (w_{j+1}, \ldots, w_k)])$$

Note that when $j = 1$, that is, the prefix is empty, this yields that $Q_2x_2, \ldots, Q_kx_k, f(x_1 = \alpha, x_2, \ldots, x_k)$ is equal to $Q_2^A w_2, \ldots, Q_k^A w_k, MAP_{B_2}(I, [w_1 = a, w_2, \ldots, w_k])$. Now because the default value for violations of $w_1$ is 1 and because the aggregation for $w_1$ is min the latter expression is equal to $Q_2^A w_1, \ldots, Q_k^A w_k, MAP_{B_2}(I, [w_1, w_2, \ldots, w_k])$. This means that $MAP_{B_2}(I)$ is equal to $Q_2x_2, \ldots, Q_kx_k, f(x_1 = \alpha, x_2, \ldots, x_k)$ completing the proof of C2.

As above, we prove the claim by induction, backwards from $k$ to 1. For the base case, $j = k - 1$ and the second part includes only one block. Consider any concrete substitution for $w_k$ participating
in the equation. If the substitution is in group 2, then the map is 1 if \(Q_j\) is \(\forall\) and is 0 otherwise. Therefore, the value of \(Q_j^4w_k, MAP_B_2(I, [w_1 = a, (w_2, \ldots, w_{k-1}) = \zeta(v), (w_k)])\) is the same as that value when restricted to substitutions in group 1. But, as argued above, for group 1 this is exactly the same value returned by the QBF.

For the inductive step, the valuation \(v\) covers the first \(j - 1\) blocks. Note that, by the inductive assumption, for any group 1 substitution \(v_j\) for \(x_j\) and corresponding, \(\zeta(v_j)\) for \(w_j, Q_{j+1}x_{j+1}, \ldots, Q_kx_k, f(x_1 = \alpha, (x_2, \ldots, x_{j-1}) = v, (x_j = v_j), x_{j+1}, \ldots, x_k) = Q_j^4w_{j+1}, \ldots, Q_k^4w_k, MAP_B_2(I, [w_1 = a, (w_2, \ldots, w_{j-1}) = \zeta(v), (w_j = \zeta(v_j)), (w_{j+1}, \ldots, w_k)])\). On the other hand, for any group 2 substitution \(\zeta_j\) for \(w_j\) and any values for \((w_{j+1}, \ldots, w_k)\) we have that the violating block for the corresponding combined \(\zeta\) is block \(j\) and therefore \(MAP_B_2(I, [w_1 = a, (w_2, \ldots, w_{j-1}) = \zeta(v), (w_j = \zeta_j), (w_{j+1}, \ldots, w_k)])\) gets the default value for that block. Therefore, as in the base case, the map is determined by group 1 valuations, which are in turn identical to the QBF value and \(Q_jx_j, \ldots, Q_kx_k, f(x_1 = \alpha, (x_2, \ldots, x_{j-1}) = v, (x_j, \ldots, x_k)) = Q_j^4w_j, \ldots, Q_k^4w_k, MAP_B_2(I, [w_1 = a, (w_2, \ldots, w_{j-1}) = \zeta(v), (w_j, \ldots, w_k)])\) as required. Therefore, the claim on the correspondence of values holds, and as argued above this completes the proof of C2.

**Extending the reduction to handle ordering:** We next extend the reduction so as to respect label order in diagram nodes.

The main idea in the extended construction is to replace the unary predicates \(P_T\) and \(P_x\) with one binary predicate \(q(\cdot, \cdot)\) where the “second argument” in \(q()\) serves to identify the corresponding predicate and hence its truth value. In addition we force \(q()\) to be symmetric so that for any \(A\) and \(B\) the truth value of \(q(A, B)\) is the same as the truth value of \(q(B, A)\). In this way we have freedom to use either \(q(A, B)\) or \(q(B, A)\) as the node label which provides sufficient flexibility to handle the order issues. To implement this idea we need a few additional constructions.

Let the set of variables in the first \(\exists\) block of the QBF be \(x_1, x_2, \ldots, x_\ell\). The set of predicates in the extended reduction includes unary predicates \(T(), y_1(), y_2(), x_1(), x_2(), \ldots, x_\ell()\), and one binary predicate \(q(\cdot, \cdot)\). A legal interpretation includes exactly \(\ell + 3\) objects which are uniquely identified by the unary predicates. We therefore slightly abuse notation and use the same symbols for the objects and predicates. In particular, the atoms \(y_1(y_1), y_2(y_2), T(T), \) and \(x_1(x_1), x_2(x_2), \ldots, x_\ell(x_\ell)\) are true in the interpretation and only these atoms are true for these unary predicates (e.g., \(x_1(T)\) is false). The truth values of \(q()\) reflect the constraints from above in addition to being symmetric. Thus the truth values of \(q(y_1, T)\) and \(q(T, y_1)\) are the same and they are the negation of the truth values of \(q(y_2, T)\) and \(q(T, y_2)\). For all \(i\), the truth values of \(q(y_1, x_i), q(x_i, y_1), q(y_2, x_i)\) and \(q(x_i, y_2)\) are the same. The truth values of other instances of \(q()\), for example, \(q(x_2, T)\), can be set arbitrarily. For example, the following interpretation is legal when \(\ell = 1: I = \{[a, b, c, d], y_1(a), y_2(b), T(c), x_1(d), q(c, a) = q(a, c) = \text{true}, q(c, b) = q(b, c) = \text{false}, q(d, a) = q(a, d) = q(d, b) = q(b, d) = \text{false}, q(\cdot, \cdot) = \text{false}\}\) where \(q(\cdot, \cdot)\) refers to any instance not explicitly mentioned in the list.

We next define the diagram \(B_1\) that is satisfied only in legal interpretations. We enforce exactly \(\ell + 3\) objects using two complementary parts. The first includes \(\ell + 4\) inequalities on a new set of \(\ell + 4\) variables \(z_1, \ldots, z_{\ell+4}\) with min aggregation. If we identify \(\ell + 4\) distinct objects we set the value to 0. This is shown in Figure 24.

To enforce at least \(\ell + 3\) objects and identify them we use the following gadget. For each of the unary predicates we have a diagram identifying its object and testing its uniqueness where we use both max and min variables. This is shown for the predicate \(T()\) in Figure 25. The node \(T(T)\)
with max variable $T$ identifies the object $T$. The nodes $T(r_1), T(r_2)$ with min variables $r_1, r_2$ make sure that $T$ holds for at most one object. We chain the diagrams together as shown in Figure 26 where the variables $r_1, r_2$ are shared among all unary predicates. The corresponding aggregation function is $\min r_1, r_2, \max T, y_1, y_2, x_1, \ldots, x_{\ell}$. This diagram associates each of the $\ell + 3$ objects with one of the unary predicates and in this way provides a reference to specific objects in the interpretation.

The symmetry gadget for $q()$ is shown in Figure 27 where the variables $m_1, m_2$ are min variables. If an input interpretation has two objects $A, B$ where $q(A, B)$ has a truth value different than $q(B, A)$ then minimum aggregation will map the interpretation to 0.

The truth value gadget for the simulation of $P_I$ is shown in Figure 28 where $y_1, y_2$ and $T$ are as above. The truth value gadget for the simulation of $P_{x_i}$ is shown in Figure 29 where $y_1, y_2$ and $x_i$ are as above. These diagram fragments refer to variables in other portions and they will be connected and aggregated together.

Finally, we add a component that is not needed for verifying that $I$ is legal but will be useful later when we include the clauses. In particular, we include a variable consistency block which is similar to the one in the simple construction but where we force $w_i$ to bind to the same object as $y_1$ or $y_2$. This is shown in [we flip leaf values in diagram] Figure 30.

We chain the diagrams together as shown in Figure 31 to get $B_1$ where we have moved the node labelled $r_1 = r_2$ to be above the block consistency gadget. [here too leaf values in consistency block need to be flipped] Note that the diagram is sorted where for predicate order we have $= < T < y_1 < y_2 < x_i < q$ and for variables we have $z_i, T, y_i, x_i < r_j < w_k$, $y_i < y_{(l_1,l_2)}$, and $m_i < T < x_j$. The complete aggregation function is

$$\min \min \min \min \max \max \ldots Q^A \min L_{w_k}$$

We next show that the claim C1 holds for the extended reduction.

We first consider all possible cases for illegal interpretations.

- If the interpretation has $\geq \ell + 4$ objects then the top portion of the diagram yields 0 for some valuation of $z$’s regardless of the values of other variables. Therefore aggregation over $z$ yields 0, and then aggregations over $r$ and $m$ yield 0.

- Next, consider the case where the interpretation has $\leq \ell + 3$ objects but one of the unary predicates is always false (i.e., it does not “pick” any object). The situation is similar to the previous case, but here we get a value of 1 for $\zeta$ where $r_1 = r_2$ or where there is a block violation for some $w_i$ block with min aggregation.

Consider any valuation $\zeta_p$ to the prefix of variables up to $x_{\ell}$ in aggregation order which is block consistent on $w_1, w_2$, has $r_1 \neq r_2$, and any valuation for the other variables. We have two cases: if $\zeta$ is in group 1 the map is 0 because the unary predicate test fails, and if $\zeta$ is in group 2 the map is the default value of the first violated block $w_i$. Now, because there is at least one group 1 valuation, we can argue inductively backwards from $k$ that all aggregations from $k$ to 3 yield 0. (The induction is simpler than the one above because we always carry a value of 0 backwards, not an arbitrary value.) Therefore, the maximization over $T, y_1, y_2, x_1, \ldots, x_{\ell}$ yields 0, and the minimization over $z, w_1, w_2$ yields 0. In the minimization over $r$, we have 0.

\[\text{Note that we can remove the variable correspondence block which complicates the argument (and does not test anything per legality of } I \text{) and still maintain correctness of C1. But including it here simplifies the argument for diagram } B \text{ below and thus simplifies the overall proof.}\]
for the cases where \( r_1 \neq r_2 \), and 1 otherwise. Therefore the minimization over \( r \) yields 0 as well, and the minimization over \( m \) yields 0.

- Next consider the case where the previous two conditions are satisfied but where one of the unary predicates holds for two or more objects. The argument is identical to the previous case, except that \( \zeta \) has the violating pair for \( r_1, r_2 \) (instead of any \( r_1 \neq r_2 \)).

- Next consider any interpretation that has exactly \( \ell + 3 \) objects and where the unary predicates identify the objects corresponding to \( T, y_1, y_2, x_1, \ldots, x_\ell \) but where \( q() \) is not symmetric. In this case we consider (as in the second item on this list) any valuation \( \zeta_p \) to the prefix of variables up to \( x_\ell \) in aggregation order which is block consistent on \( w_1, w_2 \), has \( r_1 \neq r_2 \), has the violating pair for \( m_1, m_2 \) and any valuation for the other variables. As above, we can argue that the aggregations down to \( w_3 \) and then down to \( r \) yield 0 when \( r_1 \neq r_2 \) and \( m_1, m_2 \) has the violating pair. As a result, for this setting of \( m \), minimization over \( r \) yields 0, and therefore the minimization over \( m \) also yields 0.

- The only remaining cases are interpretations that are illegal only because the \( q() \) simulation of \( P_T() \) or \( P_{x_i}() \) is not as required. In this case, the same argument as in the 2nd item in this list shows that the value is 0.

Therefore, if \( I \) is illegal then \( \text{MAP}_{B_1}(I) = 0 \). Consider next any legal interpretation and the intended valuation \( \zeta_p \) to \( T, y_1, y_2, x_1, \ldots, x_\ell \). For any group 1 extension of this valuation and any valuation of the variables \( m, z, r \) the diagram yields 1. Therefore, we can argue inductively that all \( w_i \) aggregations down to \( w_3 \) yield 1. This implies that the maximization over \( T, y_1, y_2, x_1, \ldots, x_\ell \) yields 1, and the remaining minimizations yield 1. This completes the proof of C1.

The diagram \( B \) is obtained by adding the clause blocks below the \( q() \) tests of \( B_1 \). The clause blocks have the same structure as above but they use \( q() \) instead of \( P_T() \) and \( P_{x_i}() \). This is shown in Figure 32. The final diagram for \( B \) is shown in Figure 33 as above the only difference is that we flip the leaf values in the consistency block and it has the same aggregation function as \( B_1 \). Note that the diagram is also sorted using the same order as in \( B_1 \) where we have in addition that \( x_i < V_{(i_1, i_2)} \).

We next show that \( \text{MAP}_{B}(I) = 1 \) if and only if \( I \) is legal and \( Q_2 x_2 \ldots Q_k x_k f((x_1 = \alpha), x_2, \ldots, x_k) = 1 \) where \( I \) embeds the substitution \( x_1 = \alpha \). This has the same consequences as having \( B = \text{apply}(B_1, B_2, \wedge) \) in the simple construction.

To prove the claim first note that by the construction \( B \) adds more tests on the path to a 1 leaf of \( B_1 \) and does not add any other paths to a value of 1. Therefore, for any \( I \) and any \( \zeta \), if \( \text{MAP}_{B}(I, \zeta) = 1 \) then \( \text{MAP}_{B_1}(I, \zeta) = 1 \) and as a result if \( \text{MAP}_{B}(I) = 1 \) then \( \text{MAP}_{B_1}(I) = 1 \). Therefore, by C1, if \( \text{MAP}_{B}(I) = 1 \) then \( I \) is legal. Next, consider any legal \( I \) and any unintended valuation \( \zeta_p \) for the block of variables \( T, y_1, y_2, x_1, \ldots, x_\ell \). As above, the aggregated value down to \( w_3 \) for this prefix is 0. Therefore, if \( \text{MAP}_{B}(I) = 1 \) and thus the max aggregation over these variables yields 1 (for the prefix valuation for \( M, R, Z, w_1, w_2 \)), it must be through the intended valuation for \( T, y_1, y_2, x_1, \ldots, x_\ell \). However, when \( T, y_1, y_2, x_1, \ldots, x_\ell \) are fixed to their intended values, by the construction there is a 1-1 correspondence to valuations and values of \( B_2 \) in the simple construction (where we extend the notion of block consistent to enforce that \( w_i \) bind to \( y_1, y_2 \)). Therefore, the claim holds by C2 of the simple construction. \( \square \)