Correlation transfer in large-spin chains

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It is shown that transient spin-spin correlations in one-dimensional spin $S \gg 1$ chain can be enhanced for initially factorized and individually squeezed spin states. Such correlation transfer form "internal" to "external" degrees of freedom can be well described by using a semiclassical phase-space approach.

I. INTRODUCTION

Spin chains provide a natural mechanism for creation of quantum channels targeted on correlating of spatially separated particles. State $|1\rangle$ and entanglement $|2\rangle$ transfer can be achieved even in the simplest types of spin chains, which allows to employ them as a medium for a short distance quantum communication. Special attention attracts the idea (recently realized experimentally $|13\rangle$) of entanglement generation between distant spins $|14\rangle$. It was noted $|21\rangle$ that while the quality of entanglement transfer in spin chains increases with the dimension of interacting spins, the average fidelity still significantly diminishes with the distance between the spins. In order to enhance the pairwise entanglement we propose to use correlations stored in the initial factorized state of a chain consisting of large size spins. The main idea is to prepare each spin of the chain in an appropriate squeezed state, so that the correlation from the "internal degree of freedom" is transferred into the spin-spin entanglement during the evolution of the system. It worth noting, that the possibility of correlation transfer between subsystems of various physical systems has been widely discussed both from theoretical $|22\rangle$ and experimental $|23\rangle$ perspectives.

This article is organized as follows: first we show that for a particular type of spin-spin interaction it is possible to choose an optimal initial spin squeezed state, so that the maximum amount of entanglement between neighbors spins, as well as the correlation of a spin with the rest of the particles in the chain substantially increase; then, we argue that the dynamics of the correlation transfer for large spins can be fairly well described from the semiclassical point of view using the language of the phase-space distributions, which opens the possibility for large-scale simulations of correlation dynamics in spin chains.

II. THE MODEL

Let us consider an open chain of $N$ spins of size $S$ with homogeneous Ising-like interaction (the coupling constant is taken to be unity), governed by the Hamiltonian

$$H = \sum_{j=1}^{N} S_{zj}S_{zj+1}. \tag{1}$$

Initially the spins are prepared in a factorized state

$$|\psi_0\rangle = \prod_{j=1}^{N} |\psi_0\rangle_j,$$

where each spin is in the coherent state localized on the equator of the Bloch sphere subjected to a squeezing transformation (generated by $S^2$) and a rotation around $x$-axes in the maximum squeezing direction (in the corresponding tangent plane),

$$|\psi_0\rangle_j = e^{i\varphi_{zj}} e^{-i\mu_{zj}^2/2} |\pi/2,0\rangle_j, \tag{2}$$

$$|\pi/2,0\rangle_j = \frac{1}{2^S} \sum_{k=-S}^{S} \sqrt{\frac{2S!}{(S-k)!(S+k)!}} |k,S\rangle.$$

Since $|\psi_0\rangle$ is not an eigenstate of the Hamiltonian $|11\rangle$ some specific spin-spin correlations arise during the Hamiltonian evolution. The state of the system at time $t$ is

$$|\psi(t)\rangle = \prod_{j=1}^{N} \sum_{k=-S}^{S} \gamma_k E_k(t) P_{m,k}^{(j)} (\varphi) |m,S\rangle_j, \tag{3}$$

where

$$P_{m,k}^{(j)} (\varphi) = \langle j | m, S | e^{i\varphi_{zj}} | k, S \rangle_j, \quad \gamma_k = \frac{1}{2^S} \sqrt{\frac{(2S)!}{(S-k)!(S+k)!}},$$

$$E(t)_{kj} = \exp \left( -it \sum_{l} k_l m_{l+1} \right) e^{-i\mu_{zj}^2}.$$

The simplest characteristics of a correlation between a selected spin and the rest of the spin in the chain is the $I$-Concurrence $|31\rangle$,

$$\mathcal{C}_I = \frac{2S + 1}{2S} (1 - P_j (t)), \tag{4}$$
The degree of entanglement between a pair of spins can be described by the negativity \( N(\rho_{kl}) = \frac{2S + 1}{2S} \langle |\rho_{kl}^{t}\rangle |1 - 1\rangle \),

\( (5) \)

where \( |\rho| = Tr(\rho^{\dagger}\rho)^{1/2} \) is the trace norm of the partially transposed (on the \( k \)-th spin) reduced two-particle (\( k-l \)) density matrix \( \rho_{kl}^{t} \).

III. OPTIMIZATION OF THE CORRELATION TRANSFER

We have numerically optimized the I-Concurrence with respect to the squeezing parameter \( \mu \) and the rotation angle \( \vartheta \) for a) a \( j \)-th spin, \( j \neq 1, N \); b) the last (first) spin in the chain.

It results that the optimum squeezing parameter is the same in both cases and scales as \( \mu_{\text{opt}} \sim 0.126 S^{-0.858} \); the optimum rotation angle behaves as \( \vartheta_{\text{opt}} \sim \frac{1}{2} \arctan S^{-1/3} \). For \( j \)-th spin, \( j \neq 1, N \), max \( C_{I_j} \) scales with the number of spins in the chain as \( \sim N^{1/5} \) and depends on the size of spins as \( \max C_{I_j} \sim A^{g(S)} \), where \( A = 0.681 \) and \( g(S) = S^{1/5} \). For the last spins in the chain \( \max C_{I_{1,N}} \) scales with the number of spins as \( \sim N^{1/10} \) and depends on the spin size as \( \max C_{I_{1,N}} \sim B^{f(S)} \) where \( B = 0.523 \) and \( f(S) = S^{0.191} \). It is worth nothing that the instants, when the maximum value in the I-concurrence is reached does not depend nor on the spin size \( S \), neither on the longitude of the chain. The dynamics of the I-concurrence for the optimized initial spin squeezed (upper curve) and non-squeezed, \( \mu = 0 \), (middle curve) states in case an ”internal” spin is plotted in Fig. 1 and for the last spins of the chain in 2, here \( S = 5, N = 6 \).

The lowest curve represent the I-concurrence for spins-1/2. One may observe that a certain improvement of the entanglement between spins is achieved by inducing correlations in the initial factorized state of the system. This effect becomes even more pronounced for internal spins in the limit of large spins \( S \gg 1 \), when \( C_{I_j} \) tends to a constant value close to unity.

In the one-dimensional chain 1 the negativity 5 takes non-zero value only for consecutive spins. The negativity \( N_{j,j+1} \) is optimized for the squeezing parameter \( \mu_{\text{opt}} \approx 0.856 S^{-0.72} \), but scales in a different way for a1) internal pairs and for 2) external pairs, i.e. when one of the spins is located at the end of the chain: in the first case \( N_{j,j+1} \sim N^{-0.13} \) and \( N_{j,j+1} \sim D^{h(S)} \), where \( D = 0.721 \) and \( h(S) = S^{0.047} \); in the second case \( N_{1,2} \sim N^{-0.45} \) and \( N_{1,2} \sim G^{r(S)} \), where \( G = 0.341 \) and \( h(S) = S^{0.1} \). The evolution of \( N_{j,j+1} \) for the optimized initial spin squeezed (upper curve) and non-squeezed, \( \mu = 0 \), (middle curve) states in case an internal pairs is plotted in Fig. 3 and for the external pairs 4, here \( S = 7, N = 7 \).

IV. SEMICLASSICAL APPROACH

Unfortunately, the numerical optimization becomes quite involved for non-diagonal spin chain Hamiltonians and large spin size. In order to be able to carry out such optimization it is appropriate to use the phase-space methods. According to this approach we can reformulate the quantum mechanics on the language of distributions (symbols of operators) in the classical phase-space, where both states and observables are considered as smooth functions in such a way that average values are computed by convoluting symbols of the density matrix and the corresponding operators 34.
In addition, the quantum dynamics can be efficiently simulated in terms of the spin Wigner function defined as an invertible map of the density matrix \( \rho \) to a smooth distribution \( W_\rho(\Omega) \) on the two-dimensional sphere \( \Omega = (\theta, \phi) \in S_2 \)

\[
W_\rho(\Omega) = \text{Tr}(\rho \hat{\omega}(\Omega)),
\]

\[
\rho = \frac{2S+1}{4\pi} \int_{S_2} d\Omega \hat{\omega}(\Omega) W(\Omega)
\]  

where \( \hat{\omega}(\Omega) \) is the kernel operator \([33]\),

\[
\hat{\omega}(\Omega) = \frac{2 \sqrt{\pi}}{\sqrt{2S+1}} \sum_{L=0}^{2S} \sum_{M=-L}^{L} Y_{LM}^*(\Omega) \hat{T}^{(S)}_{LM},
\]

being \( Y_{LM}(\Omega) \) the spherical harmonics and \( \hat{T}^{(S)}_{LM} \) the irreducible tensor operators \([36]\).

In the large spin (semiclassical) limit, \( S \gg 1 \), the Wigner function satisfies the Louville equation \([37]\),

\[
\partial_t W_\rho(\Omega) = 2\epsilon \{ W_\rho(\Omega), W_H(\Omega) \} + O(\epsilon^3),
\]

where \( \epsilon = (2S+1)^{-1} \), and

\[
\{ g, f \} = \frac{1}{\sin \theta} (\partial_\theta g \partial_\phi f - \partial_\phi g \partial_\theta f),
\]

is the Poisson brackets on the sphere, being \( W_H(\Omega) \) the symbol of the Hamiltonian. The solution of Eq. (9) is just the classical evolution of the initial distribution,

\[
W_\rho(\Omega|t) = W_\rho(\Omega(-t)),
\]

here \( \Omega(-t) = (\theta(-t), \phi(-t)) \) denotes classical trajectories on the sphere. The mean value of any observable \( \hat{f} \) is computed according to

\[
\langle \hat{f} \rangle = \int_{S_2} d\Omega W_f(\Omega) W_\rho(\Omega).
\]

where \( W_f(\Omega) \) is the Wigner symbol of \( \hat{f} \).

In the case of multi-partite systems the mapping kernel is a product of individual kernels Eqs. (8) so that,

\[
W_\rho(\Omega) = Tr(\rho \hat{\omega}_1(\Omega_1) \ldots \hat{\omega}_N(\Omega_N)),
\]

\[
\Omega = (\Omega_1, \ldots, \Omega_N),
\]

and the Wigner function of the reduced bi-partite system of \( j \) and \( j' \) spins has the form

\[
W(\Omega_j, \Omega_{j'}) = \left( \frac{2S+1}{4\pi} \right)^{N-2} \int d\Omega' W_\rho(\Omega'),
\]

where the integration is performed over all spins except for \( j \)-th and \( j' \)-th. Integrating \( W_\rho(\Omega_j, \Omega_{j'}) \) over one of the solid angles one obtains a single-particle Wigner function, e.g.

\[
W(\Omega_j) = \frac{2S+1}{4\pi} \int_{S_2} d\Omega' W(\Omega_j, \Omega_{j'}).
\]

Taking into account that the symbol of the Hamiltonian Eq. (10) is

\[
W_H(\Omega) = S(S+1) \sum_{j=1}^{N-1} \cos \theta_j \cos \theta_{j+1},
\]

and solving the equation of motion Eq. (9) one obtains the classical trajectories in the form:

\[
\theta_l = \theta_{l0} \quad \text{where} \quad l = 1, 2, \ldots, N
\]

\[
\phi_{10} = \phi_1 - \frac{t}{2\epsilon} \cos \theta_2, \quad \phi_{20} = \phi_2 - \frac{t}{2\epsilon} (\cos \theta_1 + \cos \theta_3), \quad \phi_{30} = \phi_3 - \frac{t}{2\epsilon} (\cos \theta_2 + \cos \theta_4), \ldots, \phi_{N0} = \phi_N - \frac{t}{2\epsilon} \cos \theta_{N-1},
\]
where $\theta_{0}$ and $\phi_{0}$ stand for the initial values of the corresponding angles. Using the above results we can represent the purity involved in Eq. (14) as follows

$$P_j(t) = \frac{2S+1}{4\pi} \int d\Omega_j W^2(\Omega_j, |t|).$$

In order to express the negativity in terms of the Wigner functions we first reconstruct the bi-partite density matrix from (15) and its partially transposed form,

$$\rho_{jj'} = \left(\frac{2S+1}{4\pi}\right)^2 \int d\Omega_j d\Omega_{j'} W_{\rho}(\Omega_j, \Omega_{j'}) \hat{\omega}(\Omega_j) \hat{\omega}(\Omega_{j'}),$$

$$\rho_{jj'}^\rho = \left(\frac{2S+1}{4\pi}\right)^2 \int d\Omega_j d\Omega_{j'} W_{\rho}(\Omega_j, \Omega_{j'}) \hat{\omega}^T(\Omega_j) \hat{\omega}(\Omega_{j'})$$

and afterwards compute the negativity in accordance with (5),

$$N_{jj'}(t) = \left(\frac{2S+1}{4\pi}\right)^2 \int W_{\rho_{jj'}^\rho}(\Omega_j, \Omega_{j'}|t)d\Omega_j d\Omega_{j'} - 1 \quad (17)$$

where $W_{\rho_{jj'}^\rho}(\Omega_j, \Omega_{j'}|t) = Tr(\sqrt{\rho_{jj'}^\rho(\Omega_j, \Omega_{j'})} \hat{\omega}(\Omega_j) \hat{\omega}(\Omega_{j'}))$.

Curiously, instead of the complicated expression for the negativity (17) we have found that the following approximation can be used,

$$N(t) = \int d\Omega_j d\Omega_{j'} |W^{tj}(\Omega_j, \Omega_{j'}|t)| - 1, \quad (18)$$

where $W^{tj}(\Omega_j, \Omega_{j'}|t) = Tr(\rho_{jj'}^\rho(\Omega_j, \Omega_{j'}) \hat{\omega}^t(\Omega_{j'}) \hat{\omega}(\Omega_j))$ is the Wigner symbol of the partially transposed density matrix. We have tested Eq. (18) for several randomly chosen density matrices.

One of the crucial ingredients of the phase-space approach is that the Wigner function of the initial state can be efficiently approximated in the limit $S \gg 1$ by using the asymptotic form of the mapping kernel (3). One can show that the symbol of $j$-th spin state in Eq. (12), $|\psi_j\rangle$, acquires the following approximated form

$$W_j(\Omega) \simeq \sum_{n,k=-S}^{S} \gamma_k \gamma_n e^{-i\mu(k^2-n^2)-i(n-k)\Phi+i\Phi (n+k+2S) d_{n,k}^S(\Theta)}, \quad (19)$$

where $d_{n,k}^S(\Theta)$ is the Wigner $d$-function, and

$$\cos\Theta = \cos\theta \cos\vartheta + \sin\theta \sin\vartheta \cos\phi \sin\vartheta,$$

$$\sin\Phi \sin\Theta = \sin\theta \sin\vartheta \cos\phi - \cos\theta \sin\vartheta,$$

which essentially simplifies the simulation of the quantum evolution for large spins.

In Fig. (5) we plot the initial Wigner function, which has a surprisingly non-trivial form (it was verified that the approximate expression Eq. (19) describes well the exact one, Eqs. (12, 18)).

In the Figs. 1 - 2 we compare the quantum (solid line) and the semi-classical (dotted line) dynamics. It is notable that a very good agreement between both types of calculations even for long times is observed. This allows to hope that the problem of optimization of the correlation transfer in more complicated Ising and Heisenberg type large spin chains can be appropriately analyzed by using the semi-classical approach.

Finally, we have shown that an optimization procedure is required in order to improve the dynamic correlation transfer between different degrees of freedom of large-spin chains. In order to perform such optimization the phase-space methods can be quite useful, especially in the limit of semiclassical-type systems.
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