Yang-Mills Instanton Sheaves

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(Dated: July 31, 2017)

Abstract

The $SL(2,\mathbb{C})$ Yang-Mills instanton solutions constructed recently by the biquaternion method were shown to satisfy the complex version of the ADHM equations and the monad construction. Moreover, we discover that, in addition to the holomorphic vector bundles on $CP^3$ similar to the case of $SU(2)$ ADHM construction, the $SL(2,\mathbb{C})$ instanton solutions can be used to explicitly construct instanton sheaves on $CP^3$. Presumably, the existence of these instanton sheaves is related to the singularities of the $SL(2,\mathbb{C})$ instantons on $S^4$ which do not exist for $SU(2)$ instantons.

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I. INTRODUCTION

Since the discovery of classical exact solutions of Euclidean $SU(2)$ (anti)self-dual Yang-Mills (SDYM) equation in 1970s, there has been many important and interesting research activities on YM instantons both in quantum field theory and algebraic geometry. On the physics side, applications of quantum instanton tunnelling in nonperturbative quantum field theory has resolved the QCD $U(1)_{A}$ problem [1] and, on the other hand, created the strong $CP$ problem with associated QCD $\theta$-vacua [2] structure. Mathematically, one important application of instantons in differential topology was the classification of four-manifolds [3].
The first BPST 1-instanton solution [4] was found in 1975. Soon later the CFTW $k$-instanton solutions [5] with $5k$ moduli parameters were constructed, and then the number of moduli parameters of the $k$-instanton solutions was extended to $5k + 4$ (5,13 for $k = 1,2$) based on the 4D conformal symmetry group. Finally, the complete solutions with $8k - 3$ moduli parameters for each $k$-th homotopy class were worked out in 1978 by mathematicians ADHM [7] using method in algebraic geometry. By using the monad construction combining with the Penrose-Ward transform, ADHM constructed the most general instanton solutions by establishing an one to one correspondence between anti-self-dual $SU(2)$-connections on $S^4$ and global holomorphic vector bundles of rank two on $CP^3$. The explicit closed forms of the complete $SU(2)$ instanton solutions for $k \leq 3$ had been worked out in [8].

In addition to $SU(2)$, the ADHM construction has been generalized to the cases of $SU(N)$ and many other SDYM theories with compact Lie groups [8, 9]. In a recent paper [10], the present authors generalized the quaternion calculation of $SU(2)$ ADHM construction to the biquaternion calculation with biconjugation operation, and constructed a class of non-compact $SL(2, C)$ YM instanton solutions with $16k - 6$ parameters for each $k$-th homotopy class. The number of parameters is consistent with the conjecture made by Frenkel and Jardim in [11] and was proved recently in [12] from the mathematical point of view. These new $SL(2, C)$ instanton solutions contain previous $SL(2, C) (M, N)$ instanton solutions as a subset constructed in 1984 [13].

One important motivation to study $SL(2, C)$ instanton solutions has been to understand, in addition to the holomorphic vector bundles on $CP^3$ in the ADHM construction which has been well studied in the $SU(2)$ instantons, the instanton sheaf structure on the projective space. Indeed, in constrast to the well known $SU(2)$ regular instanton solutions without singularities on $S^4$ spacetime, it was discovered that [10] there were singularities for $SL(2, C)$ instanton solutions on $S^4$ which can not be gauged away. Now recall that there is a fibration from $CP^3$ to $S^4$ with fibers being $CP^1$. A bundle $E$ on $CP^3$ can descend down to a bundle over $S^4$ if and only if no fiber of the twistor fibration is a jumping line for $E$. This is precisely the case for the $SU(2)$ ADHM construction.

For the case of $SL(2, C)$ instanton solutions, things get more interesting. Since some twistor lines are jumping lines, one may expect the existence of $SL(2, C)$ instanton sheaf structure on $CP^3$ after Penrose-Ward transform in the ADHM construction. In this paper, we will show that for the $SL(2, C)$ CFTW $k$-instanton solutions with $10k$ moduli parameters,
although the jumping lines exist on $S^4$, the vector bundle description on $CP^3$ remains valid as in the case of $SU(2)$ instantons. We then proceed to calculate the case of more general known $SL(2,C)$ 2-instanton solutions with 26 moduli parameters. We discover that, for some points on $CP^3$ and some subset of the $26D$ moduli space of 2-instanton solutions, the vector bundle description of $SL(2,C)$ 2-instanton on $CP^3$ breaks down, and one is led to use a description in terms of torsion free sheaves for these non-compact YM instantons or "instanton sheaves" on $CP^3$.[11].

This paper is organized as following. In section II, we review the biquaternion construction of $SL(2,C)$ YM instantons with $16k - 6$ moduli parameters[10]. In section III, we show that the $SL(2,C)$ YM instanton solutions constructed in section II are solutions of the complex version of the ADHM equations[14]. Along with the calculation, we identify the complex ADHM data $(B_{lm}, I_m, J_m)$ with $l, m = 1, 2$. We then identify the corresponding $\alpha$ and $\beta$ matrices in the monad construction of the holomorphic vector bundles on $CP^3$. In section IV, we show that the $SL(2,C)$ CFTW instanton solutions with $10k$ parameters on $S^4$ correspond to the locally free sheaves or holomorphic vector bundles on $CP^3$. We then examine the locally free conditions[11] of the complete $SL(2,C)$ 2-instanton solutions with 26 moduli parameters, and discover the existence of the $SL(2,C)$ 2-instanton sheaves on $CP^3$ for some subset of the $26D$ moduli space. Finally, a brief conclusion is given in section V. In the appendix, we show that for the $SL(2,C)$ 2-instanton solution presented in section II.B, the costability and stability conditions are equivalent. Thus in the calculation of section IV.B, we need only do the costable calculation.

II. BIQUATERNIONS AND SL(2,C) INSTANTONS

In this section, we review biquaternion construction of $SL(2,C)$ YM instanton solutions calculated in[10]. We begin with the discussion of $SL(2,C)$ YM equation. There are two linearly independent choices of $SL(2,C)$ group metric[15]

$$g^a = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad g^b = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$ (2.1)

where $I$ is the $3 \times 3$ unit matrix. In general, one can choose

$$g = \cos \theta g^a + \sin \theta g^b$$ (2.2)
where $\theta = \text{real constant}$. Note that this metric is not positive definite due to the non-compactness of $SL(2, C)$. On the other hand, it can be shown that, as a differential manifold, $SL(2, C)$ is isomorphic to $S^3 \times \mathbb{R}^3$, and one can easily calculate its third homotopy group

$$
\pi_3[SL(2, C)] = \pi_3[S^3 \times \mathbb{R}^3] = \pi_3(S^3) \cdot \pi_3(\mathbb{R}^3) = Z \cdot I = Z
$$

(2.3)

where $I$ is the identity group, and $Z$ is the integer group.

Wu and Yang [15] have shown that a complex $SU(2)$ gauge field is related to a real $SL(2, C)$ gauge field. Starting from $SU(2)$ complex gauge field formalism, we can write down all the $SL(2, C)$ field equations. Introduce the complex gauge field

$$
G^a_\mu = A^a_\mu + iB^a_\mu,
$$

(2.4)

the corresponding complex field strength is defined as ($g = 1$)

$$
F^a_{\mu\nu} \equiv H^a_{\mu\nu} + iM^a_{\mu\nu}, a, b, c = 1, 2, 3
$$

(2.5)

where

$$
H^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + \epsilon^{abc}(A^b_\mu A^c_\nu - B^b_\mu B^c_\nu),
$$

$$
M^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + \epsilon^{abc}(A^b_\mu B^c_\nu - A^b_\mu B^c_\nu).
$$

(2.6)

The $SL(2, C)$ YM equation can then be written as

$$
\partial_\mu H^a_{\mu\nu} + \epsilon^{abc}(A^b_\mu H^c_{\mu\nu} - B^b_\mu M^c_{\mu\nu}) = 0,
$$

$$
\partial_\mu M^a_{\mu\nu} + \epsilon^{abc}(A^b_\mu M^c_{\mu\nu} - B^b_\mu H^c_{\mu\nu}) = 0,
$$

(2.7)

and the $SL(2, C)$ SDYM equations are

$$
H^a_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} H^a_{\alpha\beta},
$$

$$
M^a_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} M^a_{\alpha\beta}.
$$

(2.8)

YM equation for the choice $\theta = 0$ can be derived from the following Lagrangian

$$
L = \frac{1}{4}(H^a_{\mu\nu} H^a_{\mu\nu} - M^a_{\mu\nu} M^a_{\mu\nu}).
$$

(2.9)
We now proceed to review the construction of $SL(2, C)$ YM instantons [10, 13]. We will use the convention $\mu = 0, 1, 2, 3$ and $\epsilon_{0123} = 1$ for 4D Euclidean space. In contrast to the quaternion in the $Sp(1)$ (=$SU(2)$) ADHM construction, the authors of [10] used biquaternion to construct $SL(2, C)$ YM instantons. A quaternion $x$ can be written as

$$x = x_\mu e_\mu, \quad x_\mu \in R, \quad e_0 = 1, e_1 = i, e_2 = j, e_3 = k$$

where $e_1, e_2$ and $e_3$ anticommute and obey

$$e_i \cdot e_j = -e_j \cdot e_i = \epsilon_{ijk} e_k; \quad i, j, k = 1, 2, 3,$$

$$e_1^2 = -1, e_2^2 = -1, e_3^2 = -1.$$  

(2.11)

(2.12)

The conjugate quarternion is defined to be

$$x^\dagger = x_0 e_0 - x_1 e_1 - x_2 e_2 - x_3 e_3$$

(2.13)

so that the norm square of a quarternion is

$$|x|^2 = x^\dagger x = x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

(2.14)

Occasionaly the unit quarternions can be expressed as Pauli matrices

$$e_0 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_i \rightarrow -i \sigma_i; \quad i = 1, 2, 3.$$  

(2.15)

A biquaternion (or complex-quaternion) $z$ can be written as

$$z = z_\mu e_\mu, \quad z_\mu \in C,$$

(2.16)

which occasionally can be written as

$$z = x + y i$$

(2.17)

where $x$ and $y$ are quaternions and $i = \sqrt{-1}$, not to be confused with $e_1$ in Eq.(2.10). The biconjugation [16] of $z$ is defined to be

$$z^\circ = z_\mu^\dagger e_\mu^\dagger = z_0^\dagger e_0^\dagger - z_1^\dagger e_1^\dagger - z_2^\dagger e_2^\dagger - z_3^\dagger e_3^\dagger = x^\dagger + y^\dagger i,$$

(2.18)

which was heavily used in the construction of $SL(2, C)$ instantons [10] in contrast to the complex conjugation

$$z^* = z_\mu^* e_\mu = z_0^* e_0 + z_1^* e_1 + z_2^* e_2 + z_3^* e_3 = x - y i.$$  

(2.19)
The norm square of a biquaternion is defined to be
\[ |z|^2 = z^\circ z = (z_0)^2 + (z_1)^2 + (z_2)^2 + (z_3)^2, \]  
(2.20)
which is a complex number in general as a subscript \( c \) is used in the norm.

We now review the biquaternion construction of \( SL(2, C) \) instantons which extends the quaternion construction of ADHM \( SU(2) \) instantons. The first step was to introduce the \((k + 1) \times k\) biquaternion matrix \( \Delta(x) = a + bx \)
\[
\Delta(x)_{ab} = a_{ab} + b_{ab}x, \quad a_{ab} = a^\mu_{ab}e_\mu, b_{ab} = b^\mu_{ab}e_\mu 
\]  
(2.21)
where \( a^\mu_{ab} \) and \( b^\mu_{ab} \) are complex numbers, and \( a_{ab} \) and \( b_{ab} \) are biquaternions. The biconjugation of the \( \Delta(x) \) matrix is defined to be
\[
\Delta(x)_{ab} = (\Delta(x)^\circ)_{ab} = \Delta(x)_{ba}^*e_\mu - \Delta(x)_{ba}^0 - \Delta(x)_{ba}^2 - \Delta(x)_{ba}^3. 
\]  
(2.22)
The quadratic condition of \( SL(2, C) \) instantons reads
\[
\Delta(x)^\circ \Delta(x) = f^{-1} = \text{symmetric, non-singular } k \times k \text{ matrix for } x \notin J, 
\]  
(2.23)
from which we can deduce that \( a^\circ a, b^\circ a, a^\circ b \) and \( b^\circ b \) are all symmetric matrices. The choice of \textit{biconjugation} operation was crucial for the construction of the \( SL(2, C) \) instantons. On the other hand, for \( x \in J \), \( \det \Delta(x)^\circ \Delta(x) = 0 \). The set \( J \) is called singular locus or "jumping lines". There are no jumping lines for the case of \( SU(2) \) instantons on \( S^4 \). In the \( Sp(1) \) quaternion case, the symmetric condition on \( f^{-1} \) implies \( f^{-1} \) is real; while for the \( SL(2, C) \) biquaternion case, it implies \( f^{-1} \) is complex which means \( [\Delta(x)^\circ \Delta(x)]_{ij}^\mu = 0 \) for \( \mu = 1, 2, 3 \).

To construct the self-dual gauge field, we introduce a \((k + 1) \times 1\) dimensional biquaternion vector \( v(x) \) satisfying the following two conditions
\[
v^\circ(x)\Delta(x) = 0, 
\]  
(2.24a)
\[
v^\circ(x)v(x) = 1 
\]  
(2.24b)
where \( v(x) \) is fixed up to a \( SL(2, C) \) gauge transformation
\[
v(x) \longrightarrow v(x)g(x), \quad g(x) \in 1 \times 1 \text{ Biquaternion.} 
\]  
(2.25)
Note that in general a \( SL(2, C) \) matrix can be written in terms of a \( 1 \times 1 \) biquaternion as
\[
g = \frac{q_\mu e_\mu}{\sqrt{q^\circ q}} = \frac{q_\mu e_\mu}{|q|_c}. 
\]  
(2.26)
The next step is to define the gauge field

\[ G_\mu(x) = v^\oplus(x) \partial_\mu v(x), \quad (2.27) \]

which is a 1 × 1 biquaternion. The \( SL(2, C) \) gauge transformation of the gauge field is

\[
G_\mu(x) \rightarrow G'(x) = (g^\oplus(x)v^\oplus(x))\partial_\mu(v(x)g(x))
= g^\oplus(x)G_\mu(x)g(x) + g^\oplus(x)\partial_\mu g(x) \quad (2.28)
\]

where in the calculation Eq. (2.24b) has been used. Note that, unlike the case for \( Sp(1) \), \( G_\mu(x) \) needs not to be anti-Hermitian.

One can then define the \( SL(2, C) \) field strength

\[ F_{\mu\nu} = \partial_\mu G_\nu(x) + G_\mu(x)G_\nu(x) - [\mu \leftrightarrow \nu], \quad (2.29) \]

and prove the self-duality of \( F_{\mu\nu} \). To count the number of moduli parameters for the \( SL(2, C) \) \( k \)-instantons, one can use transformations which preserve conditions Eq. (2.23), Eq. (2.24a) and Eq. (2.24b), and the definition of \( G_\mu \) in Eq. (2.27) to bring \( a \) and \( b \) in Eq. (2.21) into the following simple canonical form

\[
b = \begin{bmatrix} 0_{1 \times k} \\ I_{k \times k} \end{bmatrix}, \quad a = \begin{bmatrix} \lambda_{1 \times k} \\ -y_{k \times k} \end{bmatrix} \quad (2.30)
\]

where \( \lambda \) and \( y \) are biquaternion matrices with orders \( 1 \times k \) and \( k \times k \) respectively, and \( y \) is symmetric

\[ y = y^T. \quad (2.31) \]

Thus the constraints for the moduli parameters are

\[ a_{ci}^g a_{cj} = 0, \ i \neq j, \ \text{and} \ \ y_{ii} = y_{ji}. \quad (2.32) \]

The total number of moduli parameters for \( k \)-instanton can be calculated through Eq. (2.32) to be

\[ \# \text{ of moduli for } SL(2, C) \text{ } k\text{-instantons} = 16k - 6, \quad (2.33) \]
which is twice of that of the case of $Sp(1)$. Roughly speaking, there are $8k$ parameters for instanton "biquaternion positions" and $8k$ parameters for instanton "sizes". Finally one has to subtract an overall $SL(2, C)$ gauge group degree of freedom 6.

We provide two explicit examples of $SL(2, C)$ instantons here. These will be used in section IV for the discussion of instanton sheaves.

A. The $SL(2, C)$ CFTW $k$-instantons

We choose the biquaternion $\lambda_j$ in Eq.(2.30) to be $\lambda_j e_0$ with $\lambda_j$ a complex number, and choose $y_{ij} = y_j \delta_{ij}$ to be a diagonal matrix with $y_j = y_{j\mu} e_\mu$ a biquaternion. That is

$$\Delta(x) = \begin{bmatrix}
\lambda_1 & \lambda_2 & \ldots & \lambda_k \\
x - y_1 & 0 & \ldots & 0 \\
0 & x - y_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & x - y_k
\end{bmatrix}, \quad (2.34)$$

which satisfies the constraint in Eq.(2.32). Let

$$v = \frac{1}{\sqrt{\phi}} \begin{bmatrix} 1 \\ -q_1 \\ \vdots \\ -q_k \end{bmatrix}, \quad (2.35)$$

then

$$q_j = \frac{\lambda_j (x_\mu - y_{j\mu}) e_\mu}{|x - y_j|^2_c}, j = 1, 2, \ldots, k, \quad (2.36)$$

and

$$v = \frac{1}{\sqrt{\phi}} \begin{bmatrix}
1 \\
-\frac{\lambda_1 (x_\mu - y_{1\mu}) e_\mu}{|x - y_1|^2_c} \\
\vdots \\
-\frac{\lambda_k (x_\mu - y_{k\mu}) e_\mu}{|x - y_k|^2_c}
\end{bmatrix}, \quad (2.37)$$

with

$$\phi = 1 + \frac{\lambda_1 \lambda_1^@}{|x - y_1|^2_c} + \ldots + \frac{\lambda_k \lambda_k^@}{|x - y_k|^2_c} \quad (2.38)$$
where $\phi$ is a complex-valued function in general. One can calculate the gauge potential as

$$G_\mu = v^\rho \partial_\rho v = \frac{1}{4} [e^{\dagger}_\mu e_\nu - e^{\dagger}_\nu e_\mu] \partial_\nu \ln(1 + \frac{\lambda_1^2}{|x - y_1|^2} + \ldots + \frac{\lambda_k^2}{|x - y_k|^2})$$

$$= \frac{1}{4} [e^{\dagger}_\mu e_\nu - e^{\dagger}_\nu e_\mu] \partial_\nu \ln(\phi).$$

(2.39)

To get non-removable singularities, one needs to calculate zeros of

$$\phi = 1 + \frac{\lambda_1 \lambda_1^\rho}{|x - y_1|^2} + \ldots + \frac{\lambda_k \lambda_k^\rho}{|x - y_k|^2},$$

(2.40)

or

$$|x - y_1|^2 |x - y_2|^2 \ldots |x - y_k|^2 \phi = P_{2k}(x) + i P_{2k-1}(x) = 0.$$

(2.41)

For the $SL(2, C)$ CFTW $k$-instanton case, one encounters intersections of zeros of $P_{2k}(x)$ and $P_{2k-1}(x)$ polynomials with degrees $2k$ and $2k - 1$ respectively

$$P_{2k}(x) = 0, \quad P_{2k-1}(x) = 0.$$

(2.42)

**B. The General $SL(2, C)$ 2-instanton Solutions**

For this case we choose the following $\Delta(x)$ matrix with $y_{12} = y_{21}$

$$\Delta(x) = \begin{bmatrix} \lambda_1 & \lambda_2 \\ x - y_1 & -y_{12} \\ -y_{21} & x - y_2 \end{bmatrix},$$

(2.43)

$$\Delta^\rho(x) = \begin{bmatrix} \lambda_1^\rho & x^\rho - y_{12}^\rho \\ \lambda_2^\rho & -y_{12}^\rho \end{bmatrix}.$$  

(2.44)

The condition on $\Delta^\rho(x)\Delta(x)$

$$\Delta^\rho(x)\Delta(x) = \begin{bmatrix} \lambda_1^\rho \lambda_1 + (x^\rho - y_{12}^\rho)(x - y_1) + y_{12}^\rho y_{12} & \lambda_1^\rho \lambda_2 - (x^\rho - y_{12}^\rho)y_{12} - y_{12}^\rho(x - y_2) \\ \lambda_2^\rho \lambda_1 - y_{12}^\rho(x - y_1) - (x^\rho - y_{12}^\rho)y_{12} & \lambda_2^\rho \lambda_2 + y_{12}^\rho y_{12} + (x^\rho - y_{12}^\rho)(x - y_2) \end{bmatrix}$$

(2.45)

in Eq. (2.23) is

$$\lambda_2^\rho \lambda_1 - \lambda_1^\rho \lambda_2 = y_{12}^\rho(y_2 - y_1) + (y_1^\rho - y_2^\rho)y_{12}.$$  

(2.46)
which is linear in the biquaternion \( y_{12} \) instead of a quadratic equation, and \( y_{12} \) can be easily solved to be

\[
y_{12} = \frac{1}{2} \frac{(y_1 - y_2)}{|y_1 - y_2|^2_c} (\lambda_2^o \lambda_1 - \lambda_1^o \lambda_2). \tag{2.47}
\]

The number of moduli for \( SL(2, C) \) 2-instanton solutions is 26 as expected. This general 2-instanton solutions contain the previous CFTW 2-instanton solutions as a subset. We will see that although the vector bundle description on \( CP^3 \) remains valid for the case of \( SL(2, C) \) CFTW 2-instanton solutions, for some points on \( CP^3 \) and some subset of the 26D moduli space of the general \( SL(2, C) \) 2-instanton solutions, the vector bundle description breaks down, and one is led to use sheaf description for these non-compact YM instantons or “instanton sheaves” on \( CP^3 \).

For the singularities of general 2-instanton solutions, one needs to calculate zeros of the determinant

\[
det \Delta_{2-ins}(x)^\circ \Delta_{2-ins}(x) = |x - y_1|^2_c |x - y_2|^2_c + |\lambda_2|^2_c |x - y_1|^2_c + |\lambda_1|^2_c |x - y_2|^2_c
\]

\[
+ y_{12}^e (x - y_1)y_{12}^e (x - y_2) + (x - y_2)^o y_{12}(x - y_1)^o y_{12}
\]

\[
- y_{12}^e (x - y_1)\lambda_1^o \lambda_1 - \lambda_2^o \lambda_1 (x - y_1)^o y_{12}
\]

\[
- (x - y_2)^o y_{12} \lambda_2^o \lambda_2 - \lambda_2^o \lambda_1 y_{12}^o (x - y_2)
\]

\[
+ |y_{12}|^2_c (|\lambda_2|^2_c + |\lambda_1|^2_c) + |y_{12}|^4_c
\]

\[
= 0 \tag{2.48}
\]

where \( y_{12} \) is given by Eq. (2.47). In calculating the determinant, one notices that \( \Delta(x)^\circ \Delta(x) \) in Eq. (2.45) is a symmetric matrix with complex number entries. So there is no ambiguity in the determinant calculation.

III. SOLUTIONS OF COMPLEX ADHM EQUATIONS AND MONAD CONSTRUCTION

In this section, we will show that the \( SL(2, C) \) YM instanton solutions constructed in [10] are solutions of the complex version of the ADHM equations [14]. Along with the calculation, we will first identify the complex ADHM data \((B_{lm}, I_m, J_m)\) with \( l, m = 1, 2 \). We then identify the corresponding \( \alpha \) and \( \beta \) matrices in the monad construction on \( CP^3 \). These identifications will enable us to calculate in the next section the existence of points on \( CP^3 \).
with some instanton moduli where the vector bundle description of $SL(2, C)$ 2-instanton on $CP^3$ breaks down, and one is led to use sheaf description for $SL(2, C)$ non-compact YM instantons.

A. The Complex ADHM Equations

To do the calculation, we will need the explicit matrix representation (EMR) of the biquaternion

\[
e_0 \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 \to -i \sigma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad e_2 \to -i \sigma_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad e_3 \to -i \sigma_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.
\] (3.49)

So in the EMR, a biquaternion can be written as a $2 \times 2$ complex matrix

\[
z = z^0 e_0 + z^1 e_1 + z^2 e_2 + z^3 e_3
\]

\[
= \begin{bmatrix}
(a^0 + b^3) + i (b^0 - a^3) & (a^2 + b^1) + i (-b^2 - a^1) \\
(a^2 + b^1) + i (b^2 - a^1) & (a^0 - b^3) + i (b^0 + a^3)
\end{bmatrix}
\] (3.50)

where $a^\mu$ and $b^\mu$ are real and imaginary parts of $z^\mu$ respectively. The biconjugation is

\[
z^{\circ} = z_0 e_0 - z_1 e_1 - z_2 e_2 - z_3 e_3
\]

\[
= \begin{bmatrix}
(a^0 - b^3) + i (b^0 + a^3) & (a^2 - b^1) + i (b^2 + a^1) \\
(-a^2 - b^1) + i (-b^2 + a^1) & (a^0 + b^3) + i (b^0 - a^3)
\end{bmatrix}.
\] (3.51)

The norm square of a biquaternion used in this paper is defined to be

\[
z^{\circ} z = z z^{\circ}
\]

\[
= \begin{bmatrix}
z^0 + iz^3 & z^2 + iz^1 \\
-(z^2 - iz^1) & z^0 - iz^3
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(z^0)^2 + (z^1)^2 + (z^2)^2 + (z^3)^2 & 0 \\
0 & (z^0)^2 + (z^1)^2 + (z^2)^2 + (z^3)^2
\end{bmatrix}.
\] (3.52)

We are now ready to show that the $SL(2, C)$ YM instanton solutions constructed using biquaternion in the last section are indeed solutions of complex version of the ADHM equations. We will need to first identify the complex ADHM data $(B_{lm}, I_m, J_m)$ with $l, m = 1, 2$. 


For simplicity, we will do the calculation in the canonical form in Eq. (2.30) and Eq. (2.31) with constraints on the moduli parameters in Eq. (2.32). For the $k$-instanton case, the EMR of the $(k + 1) \times k$ biquaternion matrix $a$ in Eq. (2.30)

$$a = \begin{bmatrix}
\lambda_1 & \lambda_2 & \ldots & \lambda_k \\
y_{11} & y_{12} & \ldots & y_{1k} \\
y_{21} & y_{22} & \ldots & y_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
y_{k1} & y_{k2} & \ldots & y_{kk}
\end{bmatrix}$$ (3.53)

with $y_{ij} = y_{ji}$ can be written as a $2(k + 1) \times 2k$ complex matrix

$$a = \begin{bmatrix}
\lambda_1^0 - i\lambda_1^3 & - (\lambda_1^2 + i\lambda_1^1) & \lambda_2^0 - i\lambda_2^3 & - (\lambda_2^2 + i\lambda_2^1) & \ldots & \lambda_k^0 - i\lambda_k^3 & - (\lambda_k^2 + i\lambda_k^1) \\
y_{11}^0 - iy_{11}^3 & y_{11}^0 + iy_{11}^3 & y_{12}^0 - iy_{12}^3 & y_{12}^0 + iy_{12}^3 & \ldots & y_{1k}^0 - iy_{1k}^3 & y_{1k}^0 + iy_{1k}^3 \\
y_{21}^0 - iy_{21}^3 & y_{21}^0 + iy_{21}^3 & y_{22}^0 - iy_{22}^3 & y_{22}^0 + iy_{22}^3 & \ldots & y_{2k}^0 - iy_{2k}^3 & y_{2k}^0 + iy_{2k}^3 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
y_{k1}^0 - iy_{k1}^3 & y_{k1}^0 + iy_{k1}^3 & y_{k2}^0 - iy_{k2}^3 & y_{k2}^0 + iy_{k2}^3 & \ldots & y_{kk}^0 - iy_{kk}^3 & y_{kk}^0 + iy_{kk}^3
\end{bmatrix}$$ (3.54)

where $\lambda_j^i$ and $y_{jk}^i$ are all complex numbers. We then do the following rearrangement and
identification for the complex ADHM data

\[
\begin{bmatrix}
\lambda_0 - i\lambda_1^3 & \lambda_2 - i\lambda_2^3 & \cdots & \lambda_{k-2} - i\lambda_{k-2}^3 & (\lambda_1^2 + i\lambda_1^1) & (\lambda_2^2 + i\lambda_2^1) & \cdots & (\lambda_{k-1}^2 + i\lambda_{k-1}^1) \\
\lambda_1^2 - i\lambda_1^1 & \lambda_2^2 - i\lambda_2^1 & \cdots & \lambda_{k-2}^2 - i\lambda_{k-2}^1 & \lambda_1^0 + i\lambda_1^3 & \lambda_2^0 + i\lambda_2^3 & \cdots & \lambda_{k-1}^0 + i\lambda_{k-1}^3 \\
y_{11}^0 - iy_{11}^3 & y_{12}^0 - iy_{12}^3 & \cdots & y_{k-1,1}^0 - iy_{k-1,1}^3 & (y_{1,1}^0 + iy_{1,1}^1) & (y_{1,2}^0 + iy_{1,2}^1) & \cdots & (y_{k,1}^0 + iy_{k,1}^1) \\
y_{12}^0 - iy_{12}^3 & y_{22}^0 - iy_{22}^3 & \cdots & y_{k-1,2}^0 - iy_{k-1,2}^3 & (y_{1,2}^0 + iy_{1,2}^1) & (y_{2,2}^0 + iy_{2,2}^1) & \cdots & (y_{k,2}^0 + iy_{k,2}^1) \\
& \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
y_{1k}^0 - iy_{1k}^3 & y_{2k}^0 - iy_{2k}^3 & \cdots & y_{kk}^0 - iy_{kk}^3 & (y_{1k}^0 + iy_{1k}^1) & (y_{2k}^0 + iy_{2k}^1) & \cdots & (y_{kk}^0 + iy_{kk}^1) \\
\end{bmatrix}
\]  

\[a \rightarrow \begin{bmatrix}
J_1 & J_2 \\
B_{11} & B_{21} \\
B_{12} & B_{22} \\
\end{bmatrix}\]  

(3.56)

where we have done the rearrangement rule for an element \(z_{ij}\) in \(a\)

\[
z_{2n-1,2m-1} \rightarrow z_{n,m} ,
\]

\[
z_{2n-1,2m} \rightarrow z_{n,k+m} ,
\]

\[
z_{2n,2m-1} \rightarrow z_{k+n,m} ,
\]

\[
z_{2n,2m} \rightarrow z_{k+n,k+m} .
\]  

(3.57)

A simple example for the case of \(k = 2\) will be given in Eq.(4.70). In Eq.(3.56) \(B_{ij}\) are \(k \times k\)
complex matrices and $J_i$ are $2 \times k$ complex matrices. Similarly for $a^a$ we get

$$a^a = \begin{bmatrix}
\lambda_1^0 + i\lambda_2^0 & y_{11}^0 + iy_{11}^0 & y_{12}^0 + iy_{12}^0 & \ldots & y_{1k}^0 + iy_{1k}^0 \\
\lambda_2^0 + i\lambda_3^0 & y_{12}^0 + iy_{12}^0 & y_{22}^0 + iy_{22}^0 & \ldots & y_{2k}^0 + iy_{2k}^0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\lambda_k^0 + i\lambda_1^0 & y_{1k}^0 + iy_{1k}^0 & y_{2k}^0 + iy_{2k}^0 & \ldots & y_{kk}^0 + iy_{kk}^0
\end{bmatrix} \rightarrow \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0
\end{bmatrix}
$$

(3.58)

$$= \begin{bmatrix}
-I_2 & B_{22} & -B_{21} \\
I_1 & -B_{12} & B_{11}
\end{bmatrix}
$$

(3.60)

where $I_j$ are $k \times 2$ matrices. The next step is to impose the conditions in Eq.(2.32). The EMR of the biquaternion matrix $a^a a$

$$a^a a = \begin{bmatrix}
(a^a a)^{00} e_0 & 0 & \ldots & 0 \\
0 & (a^a a)^{00} e_0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & (a^a a)^{00} e_0
\end{bmatrix}
$$

(3.61)

can be written as

$$a^a a = \begin{bmatrix}
(a^a a)^{00} & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & (a^a a)^{00} & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & (a^a a)^{00} & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & (a^a a)^{00} & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & (a^a a)^{00} & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & (a^a a)^{00}
\end{bmatrix}
$$

(3.62)
After the rearrangement and identification, we get

\[
\begin{bmatrix}
(a^\ast a)^0_{11} & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & (a^\ast a)^0_{22} & \ldots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & (a^\ast a)^0_{kk} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & (a^\ast a)^0_{11} & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & (a^\ast a)^0_{22} & \ldots \\
0 & 0 & 0 & 0 & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & (a^\ast a)^0_{kk}
\end{bmatrix}
\]

\[a^\ast a \rightarrow (3.63)\]

which leads immediately to the complex ADHM equations

\[
\begin{align*}
[B_{11}, B_{12}] + I_1 J_1 &= 0, \\
[B_{21}, B_{22}] + I_2 J_2 &= 0, \\
[B_{11}, B_{22}] + [B_{21}, B_{12}] + I_1 J_2 + I_2 J_1 &= 0.
\end{align*}
\]

(3.65)

For the case of SU(2) ADHM instantons, we impose the conditions

\[
\begin{align*}
I_1 &= J^\dagger, J_2 = -I, J_1 = I^\dagger, J_2 = J, \\
B_{11} &= B_2^\dagger, B_{12} = B_1^\dagger, B_{21} = -B_1, B_{22} = B_2
\end{align*}
\]

(3.66)

to get the real ADHM equations

\[
\begin{align*}
[B_1, B_2] + IJ &= 0, \\
[B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J &= 0.
\end{align*}
\]

(3.67)

We thus complete the proof that the SL(2, C) YM instanton solutions we constructed using the biquaternion method in the last section are solutions of the complex version of the ADHM equations.

**B. The Monad Construction**

In the rest of this section, we construct the \( \alpha \) and \( \beta \) matrices in the monad construction
as functions of homogeneous coordinates $z, w, x, y$ of $CP^3$ which will be used in the next section. We define

$$\alpha = \begin{bmatrix} zB_{11} + wB_{21} + x \\ zB_{12} + wB_{22} + y \\ zJ_1 + wJ_2 \end{bmatrix}, \tag{3.68a}$$

$$\beta = [ -zB_{12} - wB_{22} - y \ zB_{11} + wB_{21} + x \ zI_1 + wI_2 ]. \tag{3.68b}$$

Similar to the real ADHM equations, it can be shown that the condition

$$\beta \alpha = 0 \tag{3.69}$$

is satisfied if and only if the complex ADHM equations in Eq.(3.65a) to Eq.(3.65c) holds.

In the monad construction of the holomorphic vector bundles, Eq.(3.69) implies $\text{Im} \, \alpha$ is a subspace of $\text{Ker} \, \beta$ which allows one to consider the quotient vector space $\text{Ker} \, \beta / \text{Im} \, \alpha$ at each point of $CP^3$. If the map $\beta$ is surjective and the map $\alpha$ is injective, then $\dim(\text{Ker} \, \beta / \text{Im} \, \alpha) = k + 2 - k = 2$ on every points of $CP^3$, thus one can use holomorphic vector bundles to describe instantons. This is the case of $SU(2)$ ADHM instantons. For the case of $SL(2, C)$ instantons, either $\beta$ may not be surjective or $\alpha$ may not be injective at some points of $CP^3$ for some ADHM data, the dimension of $(\text{Ker} \, \beta / \text{Im} \, \alpha)$ may vary from point to point on $CP^3$, and one is led to use sheaf description for these non-compact $SL(2, C)$ YM instantons or "instanton sheaves" on $CP^3[11]$. These instanton sheaves will be discussed in the next section.

IV. THE YANG-MILLS $SL(2, C)$ 2-INstantON SHEAVES

As were shown in section II, there were jumping lines of the $SL(2, C)$ CFTW $k$-Instanton solutions in Eq.(2.41) and the $SL(2, C)$ general 2-Instanton solutions in Eq.(2.48) on $S^4$ which can not be gauged away as in the $SU(2)$ case. As a result, the vector bundle descriptions on $CP^3$ for these cases may break down, and one is led to introduce the sheaf structure on $CP^3$. In this section, we will apply locally free conditions, or the costable and stable conditions introduced in [11] to explicitly show that the vector bundle description of the $SL(2, C)$ CFTW $k$-Instanton solutions on $CP^3$ remains valid, while that of general 2-instanton solutions breaks down.
A. \( SL(2, C) \) CFTW \( k \)-instanton Solutions are Locally Free

For illustration, we calculate the \( SL(2, C) \) CFTW 2-instanton case. The calculation of \( k \)-instanton case can be easily extended. For the 2-instanton case, \( \lambda = \lambda^0 e_0 \) and \( y \) is a digonal biquaternion matrix

\[
a = \begin{bmatrix} \lambda_1 & \lambda_2 \\ y_{11} & 0 \\ 0 & y_{22} \end{bmatrix} = \begin{bmatrix} p_1 + iq_1 & 0 & p_2 + iq_2 & 0 \\ 0 & p_1 + iq_1 & 0 & p_2 + iq_2 \\ y^0_{11} - iy^3_{11} - (y^2_{11} + iy^1_{11}) & 0 & 0 \\ y^2_{11} - iy^1_{11} & y^0_{11} + iy^3_{11} & 0 & 0 \\ 0 & 0 & y^0_{22} - iy^3_{22} - (y^2_{22} + iy^1_{22}) \\ 0 & 0 & y^2_{22} - iy^1_{22} & y^0_{22} + iy^3_{22} \end{bmatrix}
\]

\[
\rightarrow \begin{bmatrix} p_1 + iq_1 & p_2 + iq_2 & 0 & 0 \\ 0 & 0 & p_1 + iq_1 & p_2 + iq_2 \\ y^0_{11} - iy^3_{11} & 0 & - (y^2_{11} + iy^1_{11}) & 0 \\ 0 & y^0_{22} - iy^3_{22} & 0 & - (y^2_{22} + iy^1_{22}) \\ y^2_{11} - iy^1_{11} & 0 & y^0_{11} + iy^3_{11} & 0 \\ 0 & y^2_{22} - iy^1_{22} & 0 & y^0_{22} + iy^3_{22} \end{bmatrix} = \begin{bmatrix} J_1 & J_2 \\ B_{11} & B_{21} \\ B_{12} & B_{22} \end{bmatrix}
\] (4.70)
where we have made a rearrangement for $a$ in the second line of Eq.\([\text{4.70}]\). Similarly we have

$$
\begin{align*}
a^v &= \begin{bmatrix}
\lambda_1^v & y_{11}^v & 0 \\
\lambda_2^v & 0 & y_{22}^v
\end{bmatrix} \\
&= \begin{bmatrix}
p_1 + iq_1 & 0 & y_{11}^0 + iy_{11}^3 & y_{11}^2 + iy_{11}^1 & 0 & 0 \\
0 & p_1 + iq_1 & -y_{11}^2 + iy_{11}^1 & y_{11}^0 - iy_{11}^3 & 0 & 0 \\
p_2 + iq_2 & 0 & 0 & 0 & y_{22}^0 + iy_{22}^3 & y_{22}^2 + iy_{22}^1 \\
0 & p_2 + iq_2 & 0 & 0 & -y_{22}^2 + iy_{22}^1 & y_{22}^0 - iy_{22}^3
\end{bmatrix} \\
\rightarrow \begin{bmatrix}
p_1 + iq_1 & 0 & y_{11}^0 + iy_{11}^3 & 0 & y_{11}^2 + iy_{11}^1 & 0 \\
p_2 + iq_2 & 0 & 0 & y_{22}^0 + iy_{22}^3 & 0 & y_{22}^2 + iy_{22}^1 \\
0 & p_1 + iq_1 & -y_{11}^2 + iy_{11}^1 & 0 & y_{11}^0 - iy_{11}^3 & 0 \\
0 & p_2 + iq_2 & 0 & -y_{22}^2 + iy_{22}^1 & 0 & y_{22}^0 - iy_{22}^3
\end{bmatrix} \\
&= \begin{bmatrix}
-I_2 & B_{22} & -B_{21} \\
I_1 & -B_{12} & B_{11}
\end{bmatrix},
\end{align*}
\tag{4.71}
$$

So we have the following identification

$$
I_1 = \begin{bmatrix} 0 & p_1 + iq_1 \\
0 & p_2 + iq_2 \end{bmatrix}, \quad I_2 = \begin{bmatrix} -(p_1 + iq_1) & 0 \\
-(p_2 + iq_2) & 0 \end{bmatrix}, \quad J_1 = \begin{bmatrix} p_1 + iq_1 & p_2 + iq_2 \\
0 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & 0 \\
p_1 + iq_1 & p_2 + iq_2 \end{bmatrix},
\tag{4.72a}
$$

$$
B_{11} = \begin{bmatrix} y_{11}^0 - iy_{11}^3 & 0 \\
0 & y_{22}^0 - iy_{22}^3 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} y_{11}^2 - iy_{11}^1 & 0 \\
0 & y_{22}^2 - iy_{22}^1 \end{bmatrix},
\tag{4.72b}
$$

$$
B_{21} = \begin{bmatrix} -(y_{11}^2 + iy_{11}^1) & 0 \\
0 & -(y_{22}^2 + iy_{22}^1) \end{bmatrix}, \quad B_{22} = \begin{bmatrix} y_{11}^0 + iy_{11}^3 & 0 \\
0 & y_{22}^0 + iy_{22}^3 \end{bmatrix}.
\tag{4.72c}
$$

It can be easily shown that for these parametrizations, the complex ADHM equations in Eq.\([\text{3.65a}]\) to Eq.\([\text{3.65c}]\) are satisfied.

The next step is to check whether there exists common eigenvector $v$ in the costable condition

$$
\begin{align*}
(zB_{11} + wB_{21})v &= -xv, \quad \tag{4.73a} \\
(zB_{12} + wB_{22})v &= -yv, \quad \tag{4.73b} \\
(zJ_1 + wJ_2)v &= 0. \quad \tag{4.73c}
\end{align*}
$$
If the common eigenvector \( v \) exists, then the dimension of \((\text{Ker } \beta / \text{Im } \alpha)\) discussed in the end of the last section will not be a constant. The holomorphic vector bundle description on \( CP^3 \) will break down. From Eq.(4.73a) and Eq.(4.73b), the possible solutions of \( v \) are either
\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]
or
\[
\begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]
but these can not be the solution of Eq.(4.73c). So \( SL(2, C) \) CFTW 2-instanton solutions are costable. Similar calculation can be done for the case of stable condition. Thus \( SL(2, C) \) CFTW 2-instanton solutions are locally free. Similar demonstration can be easily done for the \( SL(2, C) \) CFTW \( k \)-instanton solutions. We conclude that for the \( SL(2, C) \) CFTW \( k \)-instanton solutions with \( 10k \) moduli parameters, although the jumping lines exist on \( S^4 \), the vector bundle description on \( CP^3 \) remains valid as in the case of \( SU(2) \) instantons.

### B. Breakdown of Vector Bundle Description

In this subsection, we consider the case of complete known \( SL(2, C) \) 2-instanton solutions with 26 moduli parameters. We will see that, for some points on \( CP^3 \) and some subset of the \( 26D \) moduli space of 2-instanton solutions, the vector bundle description of \( SL(2, C) \) 2-instanton on \( CP^3 \) breaks down, and one is led to use sheaf description for these non-compact YM instantons or "instanton sheaves" on \( CP^3 \). In the appendix, we have shown that the costable and stable conditions for the complete \( SL(2, C) \) 2-instanton solutions given in section II.B are equivalent. So we need only examine the costable condition. To proceed, we first identify the ADHM data. Let

\[
\lambda_1 = \begin{bmatrix} \lambda_1^0 - i \lambda_1^3 - (\lambda_1^2 + i \lambda_1^1) \\ \lambda_1^2 - i \lambda_1^1 \end{bmatrix}, \quad \lambda_2 = \begin{bmatrix} \lambda_2^0 - i \lambda_2^3 - (\lambda_2^2 + i \lambda_2^1) \\ \lambda_2^2 - i \lambda_2^1 \end{bmatrix}
\]

and

\[
\lambda_1^\otimes = \begin{bmatrix} \lambda_1^0 + i \lambda_1^3 \\ - (\lambda_1^2 - i \lambda_1^1) \end{bmatrix}, \quad \lambda_2^\otimes = \begin{bmatrix} \lambda_2^0 + i \lambda_2^3 \\ - (\lambda_2^2 - i \lambda_2^1) \end{bmatrix}
\]
For further simplification, we define

\[
l = \begin{vmatrix} \lambda_0 & \lambda_1 \\ \lambda_0 & \lambda_2 \end{vmatrix} - \begin{vmatrix} \lambda_1 & \lambda_2 \\ \lambda_1 & \lambda_2 \end{vmatrix},
\]

\[
n = \begin{vmatrix} \lambda_0 & \lambda_1 \\ \lambda_0 & \lambda_2 \end{vmatrix} - \begin{vmatrix} \lambda_1 & \lambda_2 \\ \lambda_1 & \lambda_2 \end{vmatrix},
\]

\[
m = \begin{vmatrix} \lambda_0 & \lambda_1 \\ \lambda_0 & \lambda_2 \end{vmatrix} - \begin{vmatrix} \lambda_1 & \lambda_2 \\ \lambda_1 & \lambda_2 \end{vmatrix}.
\]

We choose the following biquaternions for \(y_{11}\) and \(y_{22}\)

\[
z = z^0 e_0 + z^1 e_1 + z^2 e_2 + z^3 e_3
\]

\[
= \begin{bmatrix} (a^0 + b^3) + i (b^0 - a^1) \end{bmatrix} (-a^2 + b^1) + i (-b^2 - a^1)
\]

\[
+ \begin{bmatrix} (a^2 + b^1) + i (b^2 - a^1) \end{bmatrix} (a^0 - b^3) + i (b^0 + a^3)
\]

\[
y_{11} = -de_0 = \begin{bmatrix} -d & 0 \\ 0 & -d \end{bmatrix}, \quad y_{22} = de_0 = \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}.
\]

One can then calculate \(y_{12}\) by using Eq. (2.47) to get

\[
y_{12} = \frac{-i}{2d} \begin{bmatrix} l & m + in \\ m + in & -l \end{bmatrix}.
\]

So we have the EMR of the biquaternion matrix

\[
\begin{bmatrix} \lambda_1 & \lambda_2 \\ y_{11} & y_{12} \\ y_{12} & y_{22} \end{bmatrix} = \begin{bmatrix} \lambda_0 - i\lambda_1 & -(\lambda_1^2 + i\lambda_1) & \lambda_0^2 - i\lambda_2 & -(\lambda_2^2 + i\lambda_2) \\ -d & 0 & (\frac{-i}{2d}) l & (\frac{-i}{2d}) (m - in) \\ (\frac{-i}{2d}) l & (\frac{-i}{2d}) (m - in) & d & 0 \\ (\frac{-i}{2d}) (m + in) & - (\frac{-i}{2d}) l & 0 & d \end{bmatrix}.
\]

After the rearrangement, we have the identification

\[
\begin{bmatrix} \lambda_0 & \lambda_0 & \lambda_0 & \lambda_0 \\ \lambda_0 & \lambda_0 & \lambda_0 & \lambda_0 \\ -d & 0 & (\frac{-i}{2d}) l & (\frac{-i}{2d}) (m - in) \\ (\frac{-i}{2d}) l & d & (\frac{-i}{2d}) (m - in) & 0 \\ 0 & (\frac{-i}{2d}) (m + in) & -d & (\frac{-i}{2d}) l \\ (\frac{-i}{2d}) (m + in) & 0 & - (\frac{-i}{2d}) l & d \end{bmatrix} = \begin{bmatrix} J_1 & J_2 \\ B_{11} & B_{21} \\ B_{12} & B_{22} \end{bmatrix}.
\]
where

\[
J_1 = \begin{bmatrix}
\lambda_0^0 - i\lambda_1^3 & \lambda_2^0 - i\lambda_2^3 \\
\lambda_1^2 - i\lambda_1^1 & \lambda_2^2 - i\lambda_2^1
\end{bmatrix},
J_2 = \begin{bmatrix}
-(\lambda_1^1 + i\lambda_1^1) & -(\lambda_2^1 + i\lambda_2^1) \\
\lambda_1^0 + i\lambda_1^1 & \lambda_2^0 + i\lambda_2^1
\end{bmatrix},
\]

\[B_{11} = \begin{bmatrix}
-d & (\frac{i}{2d})l \\
(\frac{i}{2d})l & d
\end{bmatrix},
B_{21} = \begin{bmatrix}
0 & (\frac{i}{2d}) (m - in) \\
(\frac{i}{2d}) (m - in) & 0
\end{bmatrix},
\]

\[B_{12} = \begin{bmatrix}
0 & (\frac{i}{2d}) (m + in) \\
(\frac{i}{2d}) (m + in) & 0
\end{bmatrix},
B_{22} = \begin{bmatrix}
-d & - (\frac{i}{2d})l \\
- (\frac{i}{2d})l & d
\end{bmatrix}.
\]

We are now ready to check the costable conditions in Eqs. (4.73a), (4.73b) and (4.73c). If the common eigenvector \(\nu\) exists for some ADHM data, then the dimension of \((\text{Ker} \beta / \text{Im} \alpha)\) will vary from point to point on \(CP^3\). For these cases, the holomorphic vector bundle description on \(CP^3\) will break down. For simplicity, let \(z = 1\) and Eqs. (4.73c) gives

\[
\det (J_1 + wJ_2) = 0
\]

or

\[
\left[ (\lambda_0^0 - i\lambda_1^3) (\lambda_2^0 - i\lambda_2^3) - (\lambda_0^0 - i\lambda_2^3) (\lambda_1^0 - i\lambda_1^3) \right] w
\]

\[+ \left[ (\lambda_1^0 - i\lambda_1^3) (\lambda_0^0 + i\lambda_2^3) - (\lambda_1^0 + i\lambda_1^3) (\lambda_2^0 - i\lambda_2^1) \right] w
\]

\[+ \left[ - (\lambda_1^0 + i\lambda_1^3) (\lambda_0^0 + i\lambda_2^3) + (\lambda_2^0 + i\lambda_2^1) (\lambda_1^0 + i\lambda_2^3) \right] w^2
\]

\[= 0,
\]

which can be written as

\[
(n + im) w^2 + (2il) w + (n - im) = 0
\]

whose solutions are

\[
w = \frac{1}{n + im} \left[ -il \pm \sqrt{- (l^2 + m^2 + n^2)} \right].
\]

We now examine Eq. (4.73a) which can be written as

\[
\begin{bmatrix}
-d + x & (\frac{i}{2d}) [l + w (m - in)] \\
(\frac{i}{2d}) [l + w (m - in)] & d + x
\end{bmatrix} v = 0.
\]
The existence of eigenvector implies

\[ x^2 - d^2 + \frac{1}{4d^2} [l + w (m - in)]^2 = 0. \]  \hfill (4.88)

The condition of Eq.(4.73b) can be written as

\[
\begin{pmatrix}
    y - wd & \left(\frac{\lambda}{2d}\right) (m + in - wl) \\
    \left(\frac{\lambda}{2d}\right) (m + in - wl) & y + wd
\end{pmatrix} v = 0.
\] \hfill (4.89)

The existence of eigenvector implies

\[ y^2 - w^2 d^2 + \frac{1}{4d^2} [m + in - wl]^2 = 0. \] \hfill (4.90)

The solutions for \(x\) and \(y\) are

\[ x = \pm \sqrt{d^2 - \frac{1}{4d^2} [l + w (m - in)]^2}, \] \hfill (4.91)

\[ y = \pm \sqrt{w^2 d^2 - \frac{1}{4d^2} (m + in - wl)^2}. \] \hfill (4.92)

Finally we have to identify the three different forms of the common eigenvector

\[
\begin{pmatrix}
    \frac{\lambda}{2d} [l + w (m - in)] \\
    d - x
\end{pmatrix} \sim \begin{pmatrix}
    \frac{\lambda}{2d} (m + in - wl) \\
    w - y
\end{pmatrix} \sim \begin{pmatrix}
    \lambda_0^2 - i \lambda_1^2 - w (\lambda_2^2 + i \lambda_2^1) \\
    - (\lambda_1^0 - i \lambda_1^2) + w (\lambda_2^1 + i \lambda_1^1)
\end{pmatrix}.
\] \hfill (4.93)

1. \textit{Example One}

For the first sample solution, we choose the moduli \(\lambda_1^0 = \lambda_2^0 = \lambda_1^2 = \lambda_2^2 = 0\), then we get \(l = 0, n = 0\) and \(m = \lambda_1^0 \lambda_2^0\). With these inputs, \(w = \frac{1}{im} \left[ 0 \pm \sqrt{-m^2} \right] = \pm 1\) and the constraints from common eigenvector become

\[
\begin{pmatrix}
    \pm \left(\frac{\lambda}{2d}\right) m \\
    d - \sqrt{d^2 - \frac{m^2}{4d^2}}
\end{pmatrix} \sim \begin{pmatrix}
    \left(\frac{\lambda}{2d}\right) m \\
    \pm d - \sqrt{d^2 - \frac{m^2}{4d^2}}
\end{pmatrix} \sim \begin{pmatrix}
    \mp i \lambda_2^1 \\
    - \lambda_1^0
\end{pmatrix}.
\] \hfill (4.94)

If we choose \(d^2 - \frac{m^2}{4d^2} = 0\), we have

\[ m = 2d^2 = \lambda_1^0 \lambda_2^1, \quad \lambda_1^0 = -\lambda_2^1, \quad x = y = 0. \] \hfill (4.95)
Let’s set $\lambda_1^0 = a$, $\lambda_2^1 = -a$ where $a$ is a complex number and $a \neq 0$, then the corresponding solutions of moduli parameters are

$$
\begin{pmatrix}
\lambda_1 & \lambda_2 \\
y_{11} & y_{12} \\
y_{12} & y_{22}
\end{pmatrix} =
\begin{pmatrix}
\lambda_1^0 - i\lambda_1^3 & - (\lambda_1^2 + i\lambda_1^1) & \lambda_2^0 - i\lambda_2^3 & - (\lambda_2^2 + i\lambda_2^1) \\
\lambda_1^2 - i\lambda_1^1 & \lambda_1^0 + i\lambda_1^3 & \lambda_2^2 - i\lambda_2^1 & \lambda_2^0 + i\lambda_2^3 \\
-d & 0 & \left(\frac{-i}{2d}\right) l & \left(\frac{-i}{2d}\right) (m - in) \\
0 & -d & \left(\frac{-i}{2d}\right) (m + in) & - \left(\frac{-i}{2d}\right) l \\
\left(\frac{-i}{2d}\right) l & \left(\frac{-i}{2d}\right) (m - in) & d & 0 \\
\left(\frac{-i}{2d}\right) (m + in) & - \left(\frac{-i}{2d}\right) l & 0 & d
\end{pmatrix}
$$

Note that since $\lambda_2^1 \neq 0$, this set of ADHM data is outside of CFTW case considered in section IV.A which was shown to be a locally free case. We thus have discovered that, for points $[x : y : z : w] = [0 : 0 : 1 : \pm 1]$ on $CP^3$ and the ADHM data given in Eq.(4.96), the vector bundle description of $SL(2, C)$ 2-instanton on $CP^3$ breaks down, and one is led to use sheaf description for these non-compact YM instantons or ”instanton sheaves” on $CP^3$.

Note in addition that for the case of $SU(2)$ 2-instanton, $\lambda_1^0$, $\lambda_2^1$ and $d$ are real numbers inconsistent with Eq.(4.95). So the complete $SU(2)$ 2-instanton solutions are locally free. This is consistent with the known vector bundle description of $SU(2)$ 2-instanton on $CP^3$.

In the language of monad construction discussed in section III.B, the map $\alpha$ fails to be injective for points $[x : y : z : w] = [0 : 0 : 1 : \pm 1]$ on $CP^3$ with ADHM data given in Eq.(4.96). Thus one is led to use sheaf description for these ”instanton sheaves” on $CP^3$.

2. Example Two

For the second sample solution, we choose the moduli $\lambda_1^1 = \lambda_2^3 = \lambda_2^3 = \lambda_0^0 = \lambda_2^1 = \lambda_2^3 = 0$, then we get $l = 0$, $n = \lambda_1^0\lambda_2^2$ and $m = 0$. With these inputs, $w = \pm i$ and the constraints
from common eigenvector become

\[
\begin{bmatrix}
\pm \left( \frac{i}{2d} \right) \frac{n}{\sqrt{-d^2 + \frac{n^2}{4d^2}}} \\
\frac{i}{2d} n - \frac{n^2}{4d^2}
\end{bmatrix}
\sim
\begin{bmatrix}
\pm \left( \frac{i}{2d} \right) n \\
\frac{i}{2d} n - \frac{n^2}{4d^2}
\end{bmatrix}
\sim
\begin{bmatrix}
\pm i \lambda_2^2 \\
-\lambda_1^0
\end{bmatrix}.
\]

If we choose \( d^2 - \frac{n^2}{4d^2} = 0 \), we have

\[ n = 2d^2 = \lambda_1^0 \lambda_2^2, \quad \lambda_1^0 = -\lambda_2^2, \quad x = y = 0. \] (4.97)

Let’s set \( \lambda_1^0 = a, \lambda_2^2 = -a \) where \( a \) is a complex number and \( a \neq 0 \), then the corresponding solutions of moduli parameters are

\[
\begin{bmatrix}
\lambda_1 & \lambda_2 \\
y_{11} & y_{12} \\
y_{12} & y_{22}
\end{bmatrix} =
\begin{bmatrix}
\lambda_1^0 - i\lambda_1^2 & -\left( \lambda_1^2 + i\lambda_1^1 \right) & \lambda_2^0 - i\lambda_2^2 & -\left( \lambda_2^2 + i\lambda_2^1 \right) \\
\lambda_2^0 - i\lambda_2^2 & \lambda_1^0 + i\lambda_1^2 & \lambda_2^0 - i\lambda_2^2 & \lambda_2^0 + i\lambda_2^2 \\
-d & 0 & (\frac{i}{2d}) l & (\frac{i}{2d}) (m - in) \\
0 & -d & (\frac{i}{2d}) (m + in) & -\left( \frac{i}{2d} \right) l
\end{bmatrix}
\]

\[
= \begin{bmatrix}
a & 0 & 0 & a \\
0 & a & -a & 0 \\
\frac{i}{\sqrt{2}} a & 0 & 0 & \frac{ia}{\sqrt{2}} \\
0 & \frac{i}{\sqrt{2}} a & \frac{ia}{\sqrt{2}} & 0 \\
\frac{ia}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}} a
\end{bmatrix}, \quad a \neq 0. \] (4.98)

Note that since \( \lambda_2^2 \neq 0 \), this set of ADHM data is again outside of CFTW case considered in section IV.A. Eq.(4.98) gives the second example of sheaf description of the\( SL(2, C) \) 2-instanton solution on \( CP^3 \). Note again that for the case of \( SU(2) \) 2-instanton, \( \lambda_1^0, \lambda_2^2 \) and \( d \) are real numbers inconsistent with Eq.(4.95). We thus have discovered that, for points \([x : y : z : w] = [0 : 0 : 1 : \pm i] \) on \( CP^3 \) and the ADHM data given in Eq.(4.98), the vector bundle description of \( SL(2, C) \) 2-instanton on \( CP^3 \) breaks down. Note that for this case, the map \( \alpha \) in the monad construction fails to be injective for points \([x : y : z : w] = [0 : 0 : 1 : \pm i] \) on \( CP^3 \) with ADHM data given in Eq.(4.98).

To further catch the geometric picture we remark that outside the above points \([x : y : z : w] = [0 : 0 : 1 : \pm 1] \) or \([0 : 0 : 1 : \pm i] \) (or setting \( w = 1 \) instead of \( z = 1 \) with
similar formulas) the vector bundle description is valid, and more generally for any other biquaternion ADHM data a certain set of finitely many points can be found similarly as above, so that the corresponding vector bundle description is necessarily valid outside these finitely many points (this does not mean, however, that the vector bundle description has to be broken at these finitely many points). On the other hand, as we have shown that a CFTW $k$-instanton solution gives a global vector bundle on $CP^3$, its small perturbation as a general $SL(2, C)$ 2-instanton solution shall remain a global vector bundle solution (which is due to the fact that if a system of linear equations have no common solution with a certain parameter (i.e. a vector bundle case), then the same can be said with all nearby parameters corresponding to the perturbations; the parameters here can be specified by our ADHM data).

We conclude that for a certain proper subset of our biquaternion ADHM data (including the examples above), the vector bundle description breaks down only at those finitely many points (whose positions depend on the details of the ADHM data). However for ADHM data outside this proper subset the vector bundle description remains valid on the whole $CP^3$. Mathematically [11] the breakdown of the vector bundle description is related to the third Chern number $c_3$ of the obtained sheaf, which could be nonzero in the sheaf case in contrast to the vector bundle case in which $c_3$ is necessarily zero because the bundle is of rank two (two dimensional).

Secondly, as remarked in Introduction ADHM construction highly depends on one to one correspondence between ASD connections on the one side and certain holomorphic objects on the other side-twistor space, which is mainly accomplished by Penrose-Ward transform. By using the knowledge and information on the twistor side one may therefore reach an understanding for ASD connections. This idea works most effectively for the vector bundle case to which the $SU(2)$ instantons perfectly belong. However in the present $SL(2, C)$ case the holomorphic objects are no longer merely vector bundles on the twistor space as we have discussed in this paper, which renders the corresponding transformation in between less clarified. For instance, a vector bundle on $CP^3$ can descend down to $S^4$ if and only if its set of jumping lines does not include any fiber of the fibration map $CP^3 \to S^4$. It is a worthy work to examine the singularities of the Penrose-Ward transformed object on $S^4$. Presumably the singularities may appear when the preceding jumping-line condition is not met, or when the holomorphic object on $CP^3$ is a sheaf instead of a vector bundle. Since
the cases vary and the nature of the problem appears quite different from case to case, we shall leave the study of singularities on $S^4$ to a future work.

Finally it is a natural question to ask whether our biquaternion ADHM solutions give all solutions to complex ADHM equations. As remarked in the Introduction the number $16k - 6$ of parameters obtained by our biquaternion ADHM method is basically the expected number in mathematics. Yet an interested reader will soon find the solution for $k = 1$ with $I_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $I_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $J_1 = J_2 = 0$ and all $B_{lm} = 0$ seem to lie outside the biquaternion ADHM data. Now if one takes $I_1 = \begin{bmatrix} t & 0 \end{bmatrix}$, $I_2 = \begin{bmatrix} 0 & t \end{bmatrix}$, $J_1 = -I_1^\dagger$, $J_2 = I_2^\dagger$ and all $B_{lm} = 0$, which is seen to be a biquaternion ADHM solution and is equivalent to $(B_{lm}, gI_m, J_m g^{-1})$ in general for any nonzero complex number $g$, then one sees, by setting $g = t^{-1}$ and letting $t \to 0$ the above biquaternion solution under equivalence indeed reproduces the above (non-biquaternion) solution. It is to be expected that by a certain limiting procedure the biquaternion ADHM solutions (up to equivalence) reproduce all solutions to complex ADHM equations.

V. CONCLUSION

In the ADHM construction of $SU(2)$ YM instantons, one establishes an one to one correspondence between anti-self-dual $SU(2)$ connections on $S^4$ and global holomorphic vector bundles of rank two on $CP^3$ satisfying certain reality conditions. In this paper, we try to extend this correspondence to the case of non-compact $SL(2, C)$ YM instanton. As the first step of this program, the $SL(2, C)$ YM instanton solutions constructed recently by the biquaternion method were shown to satisfy the complex version of the ADHM equations and the monad construction. We can then identify the complex ADHM data for these $SL(2, C)$ YM instanton along the calculation.

The next step was to calculate the costable and stable conditions of these ADHM data. For the case of $SL(2, C)$ CFTW $k$-instanton solutions with $10k$ moduli parameters, although there exist twistor lines which are jumping lines on $S^4$, the corresponding ADHM data are locally free and the vector bundle description of $SL(2, C)$ CFTW $k$-instanton on $CP^3$ remains valid as in the case of $SU(2)$ instantons. We then proceed to calculate the second case of complete known $SL(2, C)$ 2-instanton solutions with 26 moduli parameters. We discover that, for some points on $CP^3$ and some subset of the complex ADHM data of $SL(2, C)$ 2-instanton solutions, the vector bundle description of $SL(2, C)$ 2-instanton on $CP^3$
breaks down, and one is led to use sheaf description for these non-compact YM instantons or “instanton sheaves” on $CP^3$.

Although the existence of instanton sheaf has been discussed previously, their explicit constructions have not been worked out yet. We expect that the explicit forms of the $SL(2, C)$ YM instanton sheaf solutions constructed in this paper will be helpful to the further developments on this subject both physically and mathematically.

Acknowledgments

The work of J.C. Lee is supported in part by the Ministry of Science and Technology and S.T. Yau center of NCTU, Taiwan. The work of I-H. Tsai has been possible due to an opportunity for him to visit S.T. Yau center of NCTU to which he owes his thanks.

Appendix A: Equivalence of Costable and Stable Conditions for 2-instanton

In this appendix, we show that for the $SL(2, C)$ 2-instanton solution presented in section II.B of this paper, the costability and stability conditions are equivalent. Thus in the calculation of section IV.B, we need only do the costable calculation. Note that, as was pointed out in [11], there is an explicit example of a stable solution of the complex ADHM equation which is not costable.

We first identify the ADHM data of the solutions we found. The corresponding biquaternion matrices after imposing the rearrangement rule are

\[
a \rightarrow \begin{bmatrix}
\lambda_0^0 - i\lambda_1^1 & \lambda_2^0 - i\lambda_3^3 & - (\lambda_1^2 + i\lambda_2^1) & - (\lambda_2^2 + i\lambda_3^1) \\
\lambda_1^0 - i\lambda_2^1 & \lambda_3^0 - i\lambda_4^4 & \lambda_4^0 + i\lambda_1^1 & \lambda_1^0 + i\lambda_2^2 \\
y_1^0 - iy_1^1 & y_2^0 - iy_2^1 & - (y_1^2 + iy_1^1) & - (y_2^2 + iy_2^1) \\
y_1^0 - iy_2^1 & y_2^0 - iy_1^1 & - (y_1^2 + iy_2^1) & - (y_2^2 + iy_1^1) \\
y_1^2 - iy_1^1 & y_2^2 - iy_2^1 & y_1^0 + iy_1^1 & y_2^0 + iy_2^1 \\
y_1^2 - iy_2^1 & y_2^2 - iy_1^1 & y_1^0 + iy_2^1 & y_2^0 + iy_2^1 \\
\end{bmatrix} = \begin{bmatrix}
J_1 & J_2 \\
B_{11} & B_{21} \\
B_{12} & B_{22} \\
\end{bmatrix}
\]

(A.1)
and

\[
a® = \begin{bmatrix}
\lambda_1^0 + i\lambda_1^3 & \lambda_1^2 + i\lambda_1^1 & y_1^0 + iy_1^1 & y_1^0 + iy_1^2 & y_1^2 + iy_1^1 & y_1^2 + iy_1^2 \\
\lambda_2^0 + i\lambda_2^3 & \lambda_2^2 + i\lambda_2^1 & y_2^0 + iy_2^1 & y_2^0 + iy_2^2 & y_2^2 + iy_2^1 & y_2^2 + iy_2^2 \\
-\lambda_3^0 + i\lambda_3^1 & \lambda_3^2 - i\lambda_3^1 & y_3^0 - iy_3^1 & y_3^0 - iy_3^2 & y_3^2 - iy_3^1 & y_3^2 - iy_3^2 \\
-\lambda_4^0 + i\lambda_4^1 & \lambda_4^2 - i\lambda_4^1 & y_4^0 - iy_4^1 & y_4^0 - iy_4^2 & y_4^2 - iy_4^1 & y_4^2 - iy_4^2 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-I_2 & B_{22} & -B_{21} \\
-I_1 & -B_{12} & B_{11}
\end{bmatrix}.
\] (A.2)

We have the identification for the ADHM data

\[
I_1 = \begin{bmatrix}
-\lambda_1^0 + i\lambda_1^1 & \lambda_1^2 - i\lambda_1^3 \\
-\lambda_2^0 + i\lambda_2^1 & \lambda_2^2 - i\lambda_2^3 \\
\end{bmatrix}, \quad I_2 = \begin{bmatrix}
-(\lambda_1^0 + i\lambda_1^3) & -(\lambda_1^2 + i\lambda_1^1) \\
-(\lambda_2^0 + i\lambda_2^3) & -(\lambda_2^2 + i\lambda_2^1) \\
\end{bmatrix},
\] (A.3)

\[
J_1 = \begin{bmatrix}
\lambda_1^0 - i\lambda_1^3 & \lambda_1^2 - i\lambda_1^3 \\
\lambda_2^0 - i\lambda_2^3 & \lambda_2^2 - i\lambda_2^3 \\
\end{bmatrix}, \quad J_2 = \begin{bmatrix}
-(\lambda_1^2 + i\lambda_1^1) & -(\lambda_2^2 + i\lambda_2^1) \\
\lambda_1^0 + i\lambda_1^3 & \lambda_2^0 + i\lambda_2^3 \\
\end{bmatrix},
\] (A.4)

\[
B_{11} = \begin{bmatrix}
y_{11}^0 - iy_{11}^1 & y_{12}^0 - iy_{12}^1 \\
y_{12}^0 - iy_{12}^1 & y_{22}^0 - iy_{22}^1 \\
\end{bmatrix}, \quad B_{12} = \begin{bmatrix}
y_{11}^0 - iy_{11}^1 & y_{12}^0 - iy_{12}^1 \\
y_{12}^0 - iy_{12}^1 & y_{22}^0 - iy_{22}^1 \\
\end{bmatrix},
\] (A.5)

\[
B_{21} = \begin{bmatrix}
y_{11}^0 + iy_{11}^1 & y_{12}^0 + iy_{12}^1 \\
y_{12}^0 + iy_{12}^1 & y_{22}^0 + iy_{22}^1 \\
\end{bmatrix}, \quad B_{22} = \begin{bmatrix}
y_{11}^0 + iy_{11}^1 & y_{12}^0 + iy_{12}^1 \\
y_{12}^0 + iy_{12}^1 & y_{22}^0 + iy_{22}^1 \\
\end{bmatrix}.
\] (A.6)

Now the costable condition and stable condition are

\[
(zB_{11} + wB_{21}) v = -xv,
\] (A.7a)

\[
(zB_{12} + wB_{22}) v = -yv,
\] (A.7b)

\[
(zJ_1 + wJ_2) v = 0;
\] (A.7c)

\[
(zB_{11}^\dagger + wB_{21}^\dagger) v = -\bar{x}v,
\] (A.8a)

\[
(zB_{12}^\dagger + wB_{22}^\dagger) v = -\bar{y}v,
\] (A.8b)

\[
(zI_1^\dagger + wI_2^\dagger) v = 0.
\] (A.8c)

Let's consider the $B_{ij}$ part first. Eq. (A.7a) can be written as

\[
\left( z \begin{bmatrix}
y_{11}^0 - iy_{11}^1 & y_{12}^0 - iy_{12}^1 \\
y_{12}^0 - iy_{12}^1 & y_{22}^0 - iy_{22}^1 \\
\end{bmatrix} + w \begin{bmatrix}
y_{11}^0 - iy_{11}^1 & y_{12}^0 - iy_{12}^1 \\
y_{12}^0 - iy_{12}^1 & y_{22}^0 - iy_{22}^1 \\
\end{bmatrix} \right) v = -xv.
\] (A.9)
The Hermitian conjugate reads
\[
\left( \bar{z} \begin{bmatrix} y_{11}^0 + i y_{12}^3 & y_{12}^0 + i y_{12}^3 \\ y_{12}^0 + i y_{12}^3 & y_{22}^0 + i y_{22}^3 \end{bmatrix} + \bar{w} \begin{bmatrix} y_{11}^2 + i y_{11}^1 & y_{12}^2 + i y_{12}^1 \\ y_{12}^2 + i y_{12}^1 & y_{22}^2 + i y_{22}^1 \end{bmatrix} \right) v = -\bar{x}v \quad (A.10)
\]
or
\[
\left( \bar{z}B_{11}^\dagger + \bar{w}B_{21}^\dagger \right) v = -\bar{x}v,
\]
which is the same with Eq. (A.8a). Similarly, one can show that Eq. (A.8b) is the Hermitian conjugate of Eq. (A.7b).

We now consider the I, J part. Eq. (A.7c) can be written as
\[
\left( z \begin{bmatrix} \lambda_1^0 - i\lambda_1^3 & \lambda_2^0 - i\lambda_2^3 \\ \lambda_1^0 - i\lambda_1^3 & \lambda_2^0 - i\lambda_2^3 \end{bmatrix} + w \begin{bmatrix} - (\lambda_1^1 + i\lambda_1^1) - (\lambda_1^2 + i\lambda_1^2) \\ \lambda_1^0 + i\lambda_1^1 & \lambda_2^0 + i\lambda_2^3 \end{bmatrix} \right) v = 0. \quad (A.12)
\]
There is an eigenvector \( v \) if and only if \( \det(z J_1 + w J_2) = 0 \) or
\[
\begin{align*}
&\left[ (\lambda_1^0 - i\lambda_1^3) (\lambda_2^0 - i\lambda_2^3) - (\lambda_1^0 - i\lambda_1^3) (\lambda_1^3 - i\lambda_1^3) \right] z^2 \\
&\quad + \left[ (\lambda_1^0 - i\lambda_1^3) (\lambda_2^0 + i\lambda_2^3) - (\lambda_1^2 + i\lambda_1^2) (\lambda_2^2 - i\lambda_2^2) \right] z w \\
&\quad + \left[ - (\lambda_1^2 + i\lambda_1^2) (\lambda_2^0 + i\lambda_2^3) + (\lambda_2^2 + i\lambda_2^2) (\lambda_1^0 - i\lambda_1^3) \right] w^2 \\
&= 0 \quad (A.13)
\end{align*}
\]
On the other hand, Eq. (A.7c) can be written as
\[
\left( \bar{z} \begin{bmatrix} -\lambda_1^2 + i\lambda_1^1 & \lambda_1^0 - i\lambda_1^3 \\ -\lambda_2^2 + i\lambda_2^1 & \lambda_2^0 - i\lambda_2^3 \end{bmatrix} \right)^\dagger + \bar{w} \left( - (\lambda_1^1 + i\lambda_1^1) - (\lambda_1^2 + i\lambda_1^2) \right)^\dagger v = 0 \quad (A.14)
\]
or
\[
\left( \bar{z} \begin{bmatrix} -\lambda_1^2 - i\lambda_1^1 & -\lambda_2^2 - i\lambda_2^3 \\ \lambda_1^0 + i\lambda_1^3 & \lambda_2^0 + i\lambda_2^3 \end{bmatrix} \right) + \bar{w} \left( - (\lambda_1^1 - i\lambda_1^1) - (\lambda_2^2 - i\lambda_2^2) \right) v = 0. \quad (A.15)
\]
An eigenvector \( v \) exists if and only if \( \det(\bar{z}I_1 + \bar{w}I_2) = 0 \) or
\[
\begin{align*}
&\left[ (\lambda_1^0 + i\lambda_1^3) (\lambda_2^0 + i\lambda_2^3) - (\lambda_1^0 + i\lambda_1^3) (\lambda_1^2 + i\lambda_1^2) \right] \bar{z}^2 \\
&\quad + \left[ (\lambda_1^1 + i\lambda_1^3) (\lambda_2^0 - i\lambda_2^3) - (\lambda_1^2 - i\lambda_1^2) (\lambda_2^2 + i\lambda_2^2) \right] \bar{z} \bar{w} \\
&\quad + \left[ - (\lambda_1^2 - i\lambda_1^1) (\lambda_2^0 - i\lambda_2^3) + (\lambda_2^2 - i\lambda_2^1) (\lambda_1^0 - i\lambda_1^3) \right] \bar{w}^2 \\
&= 0 \quad (A.16)
\end{align*}
\]
It is easy to see that Eq. (A.13) and Eq. (A.16) are complex conjugate to each other and are equivalent.

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