INTEGRANT METRICS ON LIE GROUPS

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ABSTRACT. Index formulas for the curvature tensors of an invariant metric on a Lie group are obtained. The results are applied to the problem of characterizing invariant metrics of zero and non-zero constant curvature. Killing vector fields for such metrics are constructed and play an important role in the case of flat metrics.

1. Introduction. In an article that appeared recently F Hindeleh and the present author studied the integrability of Killing’s equations for invariant metrics on the three-dimensional Lie groups [8]. A case by case study was performed corresponding to each possible class of Lie algebras. Several cases studied led to spaces of constant curvature and one flat metric. The calculations were carried out in local coordinates because, although the metric concerned was invariant, some of the Killing vector fields constructed were not. An obvious question then is whether can one characterize those spaces for which an invariant metric on a Lie group engenders a space of constant curvature. On the other hand, a long time ago, Milnor [10] investigated curvature properties of left invariant metrics on Lie groups. In particular he obtained a characterization of flat invariant metrics.

The starting point for the present investigation is to note that the skew-adjointness condition introduced by Milnor is simply the Killing condition for invariant vector fields, an observation that seems to be novel. Several results in [10] pertaining to flat metrics are revisited in light of this remark. In Section 2 we derive a formula in indices for the curvature and Ricci tensors and scalar curvature of the invariant metric in terms of the structure constants of the Lie algebra. Undoubtedly similar formulas are already extant in the literature (see for example [3]); Milnor [10] himself provides a formula for Sectional curvature in Lemma 1.1. However, many such formulas are difficult to understand, let alone use, and depend on complicated coordinate-free notation. In addition, as far as we are aware, the formulas have not been adapted to the case of a solvable algebra as we do in Section 2. In Section 3 we reconsider flat metrics following [10] and explain how we can obtain and the significance of an adapted coordinate system. The construction involves a coordinate representation theorem for Lie algebras that are semi-direct products of abelian algebras. Finally in Section 4 we characterize invariant metrics of constant curvature.

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curvature and write down all such possible metrics: it is demonstrated that all the Lie algebras associated to a constant curvature metric are, in a sense, generalizations of the Lie algebra of the Euclidean group of the plane and in particular are solvable. A striking fact is that in the case of non-zero constant curvature, the nilradical is of codimension one, whereas in the flat case higher codimension is possible: a product of flat spaces is still flat whereas a product of non-zero constant curvature spaces is no longer of constant curvature after all.

The reader will observe that we have so far referred only to “invariant metrics” and “invariant vector fields”. In fact Milnor considered left-invariant metrics. Of course we can equally well take right-invariant metrics; in fact our preference is indeed for right-invariant metrics. Accordingly, in most of the text we have chosen to use simply the word “invariant” so that the results may be interpreted as pertaining to either left or right-invariant metrics. However, in Section 3 we shall have occasion to consider left and right invariant objects simultaneously so, out of necessity, the words “left” and “right” are introduced. Of course the roles of “left” and “right” could be interchanged mutatis mutandis. Generally, the summation convention on repeated indices, one a superscript the other a subscript, is in force unless it is impractical to do so: several summation signs appear in Section 2.

One of the driving forces in the recent development of differential geometry has been the study of Einstein spaces. However, most of the interest in Einstein spaces concerns compact spaces. In this regard invariant metrics on Lie groups are a bit disappointing for the following reasons. First of all, a Lie group admits an invariant metric with all Ricci curvatures positive if and only if it compact with finite fundamental group. However, in that case, the group actually admits a bi-invariant metric and the Lie algebra is semi-simple: all possible invariant metrics are obtained by translating the Killing form for each semi-simple factor. In other words one is not going to obtain new examples of Einstein spaces of positive curvature by examining invariant metrics on a Lie group. For a recent discussion of compact Einstein spaces in dimension four we refer to [1].

We note that the theory of invariant geometric objects is expanding in several directions. For example in [12] left invariant metrics are considered in the context of Finsler geometry. In [4], [5] and [7] invariant pseudo-Riemannian metrics have been studied. Finally [2, 6, 9] all are devoted to interesting directions within the context of invariant Riemannian metrics.

2. Formulas for curvature and Ricci tensors. In this Section we derive formulas for the curvature and Ricci tensors and scalar curvature of an invariant metric on a Lie group $G$.

We shall assume that we are working in an orthonormal basis $\{ e_i \}$ so that matrix determined by the metric is the identity and we write the brackets as

$$ [e_i, e_j] = C^k_{ij} e_k$$  \hspace{1cm} (1)

where $C^k_{ji} = -C^k_{ij}$.

The compatibility of the metric with its Levi-Civita connection $\Gamma^i_{jk}$ gives

$$ \Gamma^i_{ki} + \Gamma^i_{kj} = 0$$  \hspace{1cm} (2)

whereas the fact that the connection is torsion-free gives

$$ \Gamma^i_{ki} - \Gamma^i_{ik} = C^d_{ki}.$$  \hspace{1cm} (3)
From $\nabla e_k e_i - \nabla e_i e_k = [e_k, e_i]$, we have $\Gamma^k_{li} e_i = \Gamma^l_{ki} e_k = C^i_l e_i$. Now cycle the indices in eq(2) and use eq(3) to obtain

$$\Gamma^k_{ij} = \frac{1}{2} (C^i_{kj} + C^i_{kj} + C^i_{ij}). \quad (4)$$

Notice that in eq(4) the first two terms are symmetric in $i$ and $j$ and that the third term is skew-symmetric in $i$ and $j$. Now we find that the curvature is given by

$$R^k_{ij} = \Gamma^m_{jl} \Gamma^l_{km} - \Gamma^m_{kl} \Gamma^l_{jm} + \Gamma^i_{ml} C^m_{jk}. \quad (5)$$

Notice that if we write eqn.(5) invariantly we get the usual formula for curvature, that is,

$$R(X, Y)Z = \nabla_Y \nabla_X - \nabla_X \nabla_Y Z + \nabla_{[X, Y]}$$

where in the last term we have the Lie algebra bracket. After a very long calculation eq(6) gives for the curvature and Ricci tensors and scalar curvature:

$$4R^k_{ij} = \sum_m (\sum_j (C^m_{jk} C^l_{im} - C^m_{im})(C^m_{jl} C^l_{km} - C^m_{jm} C^m_{ij}) + C^m_{jl} C^m_{ik} C^m_{tm} + 2C^m_{jk} C^m_{jl} C^m_{il})$$

$$+(C^m_{ml} + C^m_{mj})(C^l_{im} + C^l_{km}) - (C^l_{ml} + C^l_{mk})(C^l_{im} + C^l_{jm}) \quad (7)$$

$$4R^l_{ij} = \sum_{k,m} (C^m_{jk} C^l_{im} + C^m_{im} C^l_{km} + 2C^m_{jk} C^m_{kl} - 2C^m_{jk} C^m_{kl} C^m_{jm} + C^m_{im} C^m_{jm}) \quad (8)$$

$$4R = -(4 \sum_m (\sum_j C^m_{jm}))^2 \quad (9)$$

Let us examine eq.(10) in the case that $g$ is solvable. We change notation and write $g$ as

$$[e_i, e_j] = C^k_{ij} e_k, [e_i, e_a] = C^k_{ia} e_k, [e_a, e_b] = C^k_{ab} e_k, \quad \text{ (10)}$$

where the $\{e_i\}, (1 \leq i \leq r)$ form a basis for the nilradical $\text{nil}(g)$ and $\{e_a\}, (r + 1 \leq a \leq n)$ form a basis for a vector space complement. In addition we have $C^k_{ji} = -C^k_{ij}, C^k_{ba} = -C^k_{ab}$ and given the coefficients $C^k_{ia}$ we shall define $C^k_{ai} = -C^k_{ia}$.

As such the formula for $-4R$ becomes, since the terms $4 \sum_m (\sum_j C^m_{jm})^2$ and

$$\sum_{j,k,m} C^m_{jk} C^m_{jm}$$

drop out because, for fixed $j$, the matrix $C^m_{jm}$ is nilpotent,

$$-4R = \sum_{j,a,b} (C^a_{j} e_b)^2 + \sum_{k,a} (C^k_{m} e_a)^2 + 2 \sum_{k,a} (C^k_{m} + C^k_{a})^2 + 2 \sum_{a} (C^a_{k})^2 \quad \text{ (11)}$$

Thus $R \leq 0$ and $R = 0$ if and only if $C^i_{jk} = 0, C^a_{ab} = 0$ and each for fixed $a$, the matrix $C^a_{k,m}$ is skew in $k$ and $m$.

3. Invariant flat metrics.

3.1. Skew-adjointess condition. Let $G$ be a Lie group and let $g$ denote its Lie algebra. It will be convenient now to introduce the invariant metric on $G$ and it will be denoted by $g$. Milnor in [10] prior to Lemma 1.2 introduces a “skew-adjointess” condition on invariant vector fields $U$, that is,

$$g([U, X], Y) + g(X, [U, Y]) = 0$$

for all invariant vector fields $X$ and $Y$. A key observation is that this condition is equivalent, for invariant $U$, to $U$ being a Killing vector field of $g$. Indeed if $X$ and
Y are invariant vector fields then \( g(X,Y) \) is a constant on \( G \). If \( U \) is any other vector field, then taking the Lie derivative we have that

\[
(\mathcal{L}_U g)(X,Y) + g([U,X],Y) + g(X,[U,Y]) = 0, \tag{12}
\]

so that eq(11) is equivalent to the usual Lie derivative condition.

Another way to write the Killing condition for \( U \) in terms of the covariant derivative is, for all \( X \) and \( Y \),

\[
g(\nabla_X U, Y) + g(X, \nabla_Y U) = 0. \tag{13}
\]

Yet another way to write the Killing condition for \( U \), if each of \( U, X, Y \) is invariant, is, for all such \( X \) and \( Y \),

\[
g(\nabla_X U, Y) = \frac{1}{2} g([Y,X], U). \tag{14}
\]

3.2. Flat invariant metrics. Milnor states the following theorem which is paraphrased slightly using the remark made in subsection 3.1.

**Theorem 3.1.** A Lie group admits a flat invariant metric if and only if its Lie algebra splits as a vector space orthogonal direct \( b \bigoplus u \) sum in which \( b \) is an abelian subalgebra, \( u \) is an abelian ideal and finally every (invariant) vector field in \( b \) is a Killing field.

Then certainly the algebra is solvable: indeed the derived series terminates at the third step. The idea of the proof is to show that there is a representation of \( \mathfrak{g} \) in \( \mathfrak{o}(\mathfrak{g}) \), the space of skew-adjoint endomorphisms of \( \mathfrak{g} \), whose kernel \( \mathfrak{u} \), consists of elements \( U \) of \( \mathfrak{g} \) such that \( \nabla_U V = 0 \) for all \( V \in \mathfrak{g} \). Milnor shows that \( \mathfrak{u} \) is an abelian ideal. Note that \([\mathfrak{g},\mathfrak{g}] < \mathfrak{u} < \text{nil}(\mathfrak{g})\) where \( \text{nil}(\mathfrak{g}) \) denotes the nilradical of \( \mathfrak{g} \).

**Corollary 1.** Every \( Y \in \mathfrak{b} \) is parallel.

*Proof.* This fact follows from eq(14) because given the hypotheses of the Theorem we have that \( \mathfrak{b} \) is orthogonal to the derived algebra \([\mathfrak{g},\mathfrak{g}]\). \( \square \)

The theorem asserts that the Lie algebra is a semi-direct product of abelian Lie algebras. We can write the non-zero brackets of the algebra in a basis \( \{e_1, e_2, ..., e_r, e_{r+1}, e_{r+2}, ..., e_n\} \) as \( [e_i,e_a] = C^j_{ia} e_k \) where \( 1 \leq i, k \leq r, r+1 \leq a \leq n \). Here \( \{e_1, e_2, ..., e_r\} \) and \( \{e_{r+1}, e_{r+2}, ..., e_n\} \) comprise bases for \( \mathfrak{u} \) and \( \mathfrak{b} \), respectively. It is easy to check that the Jacobi identity is equivalent to the condition \( C^k_{ia} C^l_{kb} - C^k_{ib} C^l_{ka} = 0 \) which says that \( \text{ad}(e_a) \) and \( \text{ad}(e_b) \) commute on the ideal \( \mathfrak{u} \).

We can change basis so that \( \{e_1, e_2, ..., e_r, e_{r+1}, e_{r+2}, ..., e_n\} \) is orthonormal. A change of basis in \( \mathfrak{u} \) has the effect of conjugating the matrix \( C^j_{ia} \) for each fixed \( a \). Since each \( \{e_a\} \), for \( r+1 \leq a \leq n \) is Killing, we have that for each \( a \) the matrix \( C^j_{ia} \) is skew-symmetric. In particular \( C^j_{ia} \) cannot be nilpotent. It follows from Engel’s theorem that \( e_a \) cannot belong to \( \text{nil}(\mathfrak{g}) \) and since \( \mathfrak{u} < \text{nil}(\mathfrak{g}) \) we must actually have \( \mathfrak{u} = \text{nil}(\mathfrak{g}) \).

The discussion can be summarized by means of the following theorem which forms a natural sequel to Theorem 3.1.

**Theorem 3.2.** Necessary and sufficient conditions for \( \{e_1, e_2, ..., e_r, e_{r+1}, e_{r+2}, ..., e_n\} \) to be an orthonormal basis for a flat right-invariant metric on a Lie group \( G \) are that the Lie algebra \( \mathfrak{g} \) should be of the form \([e_i,e_a] = C^j_{ia} e_j\), where \( 1 \leq i \leq r, r+1 \leq a \leq n \), the matrices \( C^j_{ia} \) should be skew-symmetric in \( i \) and \( j \) and that the \( C^j_{ia} \) for \( r+1 \leq a \leq n \) should pairwise commute.
In the Theorem the property of being skew-symmetric is not preserved under conjugation but the necessary and sufficient condition for a matrix to be conjugate to a skew-symmetric matrix is that it should be semi-simple and that any real eigenvalues should be zero.

3.3. A representation theorem for a class of Lie algebras. We now approach the construction of the metric in a different way that is of independent interest. The following theorem pertains generally to a solvable Lie algebra that is a semi-direct product of abelian Lie algebras.

**Theorem 3.3.** Suppose that the $n$-dimensional Lie algebra $\mathfrak{g}$ has a basis $\{e_1, e_2, \ldots, e_n\}$ and only the following non-zero brackets: $[e_i, e_a] = C_{ia}^j e_j$, where $(1 \leq i, j \leq r$, $r+1 \leq a, b, c \leq n)$; in other words $\mathfrak{g}$ is a semi-direct product of abelian Lie algebras. Then $\mathfrak{g}$ has an explicit faithful representation as a subalgebra of $gl(r+1, \mathbb{R})$.

**Proof.** The endomorphism $ad(e_a)$ for $r+1 \leq a \leq n$ correspond to $n \times n$ matrices in which there are non-zero entries only in the upper left $r \times r$ block: this $r \times r$ matrix is in fact $[C_{ja}^i]$. We denote by $E_a$ the $(r+1) \times (r+1)$ matrix whose upper left $r \times r$ block is $[C_{ja}^i]$ and whose remaining entries are zero. To obtain the representation, map $e_a$ to $E_a$ for $r+1 \leq a \leq n$; for each vector $e_i$ $(1 \leq i \leq r)$ map it to the $(r+1) \times (r+1)$ matrix $E_i$ whose only non-zero entry is a 1 in the $(i, r+1)^{th}$ position. Clearly the $E_i$’s commute. Note that the matrix product $E_i E_a$ is zero and so

$$[E_i, E_a] = \sum_{k=1}^{r} C_{ia}^k E_k.$$  \hspace{1cm} (15)

Finally, consider the Jacobi identity

$$[[e_a, e_b], e_c] + [[e_c, e_b], e_a] + [[e_b, e_c], e_a] = 0.$$  \hspace{1cm} (16)

On expanding we deduce that

$$C_{ia}^k C_{kb}^j - C_{ib}^k C_{ka}^j = 0.$$

It follows that $[E_a, E_b] = 0$ and we have the required representation. \hfill \Box

Continuing from the last Theorem for a specific algebra it is possible to obtain a matrix group representation of a Lie group $G$ whose Lie algebra is the given algebra, by exponentiating the matrices $\begin{bmatrix} 0 & x^i \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} g^a C_{ia}^j & 0 \\ 0 & 0 \end{bmatrix}$. We obtain the group matrix $\begin{bmatrix} e^{\gamma^a C_{ia}^j} & x^i \\ 0 & 1 \end{bmatrix}$. Here $(x^i, y^a)$ is a system of global coordinates on the group and the summation convention is in effect. The corresponding right-invariant one-forms are given by $(dx^i - x^j C_{ja}^i dy^a, dy^a)$ and the dual vector fields by $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a} + x^j C_{ja}^i \frac{\partial}{\partial x^i})$. On the other hand the left-invariant one-forms and vector fields are given by $(e^{-\gamma^a C_{ia}^j} dx^i, dy^a)$ and $(e^{\gamma^a C_{ia}^j} x^j, \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^a})$, respectively. In particular the one-forms $dy^a$ are bi-invariant.

Let us come back now to Theorem 3.2. Notice that the $\{\frac{\partial}{\partial x^i}\}$ form a basis for the ideal $\mathfrak{u}$ and the $\{\frac{\partial}{\partial y^a}\}$ a basis for the subalgebra $\mathfrak{b}$ at the identity although they are left-invariant. Any right-invariant metric will be a quadratic form in the one-forms $(dx^i - x^j C_{ja}^i dy^a, dy^a)$. Since it is right-invariant every left-invariant vector field $\frac{\partial}{\partial y^a}$ will be Killing. On the other hand according to Theorem 3.2, every
right-invariant vector field \( \frac{\partial}{\partial x^i} + x^i C_{ja} \frac{\partial}{\partial x^j} \) is Killing and the flat metric is given by 
\[ \sum_i (dx^i - x^i C_{ja} dy^j)^2 + \sum_a (dy^a)^2. \]

It is instructive to verify that the metric just introduced is flat. We can do so in two ways. First of all the connection one-form is given by \( \omega = \begin{bmatrix} 0 & 0 \\ 0 & -C_{ja} dy^a \end{bmatrix} \). Hence \( d\omega = 0 \) and furthermore \( \omega \wedge \omega = 0 \) in view of eq(16). Alternatively, if we use eq(5) we find that the only non-zero connection components are given by \( \Gamma_{ja} = C_{ja} \); as such the only possibly non-zero curvature components are \( R_{j}}. However, once again these latter components are seen to be zero by virtue of eq(16).

**Proposition 1.** The vector fields \( e^b C_{ib} \frac{\partial}{\partial x^i} \) are parallel. Hence these vector fields together with the \( \frac{\partial}{\partial x^i} + x^i C_{ja} \frac{\partial}{\partial x^j} \) provides a complete parallelism for \( G \).

**Proof.** Since the vector fields \( e^b C_{ib} \frac{\partial}{\partial x^i} \) are Killing it will be sufficient to show that the one-forms dual via the metric \( g \) satisfy the Frobenius integrability conditions so that the distribution orthogonal to the \( e^b C_{ib} \frac{\partial}{\partial x^i} \) is integrable. Indeed the dual one-forms are \( e^b C_{ib}(dx^k - x^i C_{ja} dy^a) = e^{-b} C_{ib}(dx^k - x^j C_{ja} dy^a) = d(e^{-b} C_{ik} x^k) \) and so are exact.

In terms of the coordinates \((x^i, y^a)\) the geodesics are given by
\[ \ddot{x}^i = 2C_{ja} \dot{x}^j y^a - x^i C_{ja} C_{ib} y^b \dot{y}^a, \quad \ddot{y}^a = 0. \] (17)

### 3.4. Normalizing a flat invariant metric

It is clear that given a Lie algebra of the type that occurs in Theorem (3.3) we can take any number of copies of the abelian Lie algebra \( \mathbb{R} \) and add them to \( u \), without introducing any new non-zero brackets, so as to obtain a new Lie algebra of the same type. However, such an algebra is decomposable. Let us assume then from now on that \( g \) is indecomposable.

It is known that in a solvable Lie algebra \( g \) of dimension \( n \) the dimension \( r \) of the nilradical \( \mathfrak{nil}(g) \) is at least \( \frac{n+1}{2} \). Suppose first of all that \( r = n - 1 \). Then comparing with the construction of a Cartan subalgebra for the orthogonal algebra \( \mathfrak{o}(n) \), or the adjoint representation of \( \mathfrak{o}(n) \), we see that the single matrix \( C_{in}^k \) (here \( n \) is fixed) can be “diagonalized” by an orthogonal transformation (see for example [3]); that is, it can be reduced to the form \[ \begin{bmatrix} \lambda_1 J_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 J_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_p J_p \end{bmatrix} \] where each \( J_{\alpha} \) for \( 1 \leq \alpha \leq p \) is a matrix of the form \( \begin{bmatrix} J & 0 & \cdots & 0 \\ 0 & J & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & J \end{bmatrix} \), the matrix \( J \) is given by \( \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) and for some \( \lambda_\alpha \)'s in \( \mathbb{R} \), all of which are non-zero. The fact that all the \( \lambda_\alpha \)'s are non-zero is because we are assuming that the Lie algebra \( g \) is indecomposable. By scaling \( e_n \) we may reduce to the case \( \lambda_1 = 1 \). In the case \( n = 3 \) we have \( r = 2 \) in the notation of Theorem (3.3) and we obtain the Lie algebra of the Euclidean group of the plane.

If there is more than one such matrix \( C_{ia}^k \) (fixing \( a \) here), since they commute, they may all be simultaneously “diagonalized” by an orthogonal transformation so that they have the same block structure. We can now take linear combinations so that \[ [e_1, M^a e_a] = C^b_{ia} M^a e_b \] and define \( \overline{e_b} = M^a e_a \) and \( \overline{C}^k_{ib} = C^k_{ia} M^a \). As such the \( C_{ia} \)'s, for fixed \( a \), can now be assumed to be of the form
Lemma 4.1. Let $\langle \mathfrak{g}, \mathfrak{g} \rangle = \text{nil}(\mathfrak{g})$.

Milnor [10] cites the Lie algebra of the Euclidean group of the plane as an example of a flat left invariant metric. In fact it is the only example in dimension three and there is no such indecomposable algebra in dimension four. In dimension five the only indecomposable example is given by $5.17(\mathfrak{p} = r = 0, s = 1)$ in [11] for which the non-zero brackets are $[\mathfrak{e}_1, \mathfrak{e}_5] = -\mathfrak{e}_2, [\mathfrak{e}_2, \mathfrak{e}_7] = \mathfrak{e}_1, [\mathfrak{e}_5, \mathfrak{e}_7] = -\alpha \mathfrak{e}_6, [\mathfrak{e}_6, \mathfrak{e}_7] = \alpha \mathfrak{e}_5, [\mathfrak{e}_3, \mathfrak{e}_8] = -\mathfrak{e}_4, [\mathfrak{e}_4, \mathfrak{e}_8] = \mathfrak{e}_3, [\mathfrak{e}_5, \mathfrak{e}_8] = -\beta \mathfrak{e}_6, [\mathfrak{e}_6, \mathfrak{e}_8] = \beta \mathfrak{e}_5.$

4. Invariant metrics of constant curvature. Having analyzed metrics of zero constant curvature it is natural to inquire about metrics of non-zero constant curvature $K$. We do not need to worry about the case $K > 0$ because according to Milnor [10] the only such simply connected example is the Lie group $SU(2)$, a result of Wallach. Accordingly we may assume that $K < 0$. In an orthonormal frame the only non-zero components of curvature are of the form $R^i_{\alpha\beta\gamma\delta}$ and each of them has the value $K$.

Again according to [10] (Theorem 1.6) a Lie group admits a left-invariant metric of strictly negative sectional curvature only if its Lie algebra is solvable. As such we may assume that the Lie algebra $\mathfrak{g}$ is of the form that appears in eq(10).

Now using eq(7) we find that

$$R^a_{bab} = 3 \sum_{m=1}^{r} (C^m_{ab})^2. \quad (18)$$

Thus the only way to avoid having some positive curvature is to have $r = n - 1$.

To continue we quote another Lemma from [10]. The Lemma pertains to a Lie group with invariant metric whose Lie algebra $\mathfrak{g}$ has an ideal $\mathfrak{u}$ of codimension one. A basis for a complement to $\mathfrak{u}$ is spanned by $\mathfrak{b}$ and the induced connection on $\mathfrak{u}$ is denoted by $\nabla$. The transformation $\text{ad}(\mathfrak{b})$ restricted to $\mathfrak{u}$ is denoted by $L$ and $L^*$ denotes its adjoint relative to the metric and $S = \frac{1}{2}(L + L^*)$ is the self-adjoint part of $L$.

Lemma 4.1. Let $u, v \in \mathfrak{u}$. Then we have the following covariant derivative relations:

$$\nabla_b b = 0, \nabla_b u = \frac{1}{2}(L - L^*)u, \nabla_u b = -Su$$

and

$$\nabla_u v = \nabla_u v + \langle Su, v \rangle \mathfrak{b}.$$
Now we apply the Lemma in our context where \( u \) is the nilradical of codimension one in the solvable algebra \( \mathfrak{g} \). We choose an orthonormal basis \( \{u^1, \ldots, u^{n-1}\} \) for \( \text{nil}(\mathfrak{g}) \) and \( b \) as unit vector that is orthogonal to \( \text{nil}(\mathfrak{g}) \). We recall that the sectional curvature \( \kappa(u, v) \) defined on a pair of orthonormal tangent vectors is given by \( \{u, R(v, u)v\} \). Then again quoting from [10] we find that \( \kappa(b, u^i) = -\lambda_i^2 \) where the \( \{\lambda_i\} \) comprise the set of eigenvalues of \( S \). Since, by assumption, our space is of constant curvature we must have, for \( 1 \leq i, j \leq n-1 \), that \( \lambda_i^2 = \lambda_j^2 \).

Again from the Lemma it follows that

\[
\kappa(u, v) = \pi(u, v) + \langle Su, v \rangle \langle u, Sv \rangle - \langle Su, u \rangle \langle Sv, v \rangle.
\] (19)

Since \( \lambda_i = \pm \lambda_j \) we can only have that \( \pi(u_i, u_j) \) is zero or \(-2\lambda_i^2\). However, we now have that \( \text{nil}(\mathfrak{g}) \) is a nilpotent ergo solvable unimodular Lie algebra of non-positive sectional curvature. Such a space must actually be flat according to Theorem 1.6 in [10]. Hence the only possibility is that all the \( \lambda_i \)'s are equal and \( S \) is a multiple of the identity. But now we invoke Theorem 2.4 in [10] which asserts that a Lie group that has a non-commutative nilpotent Lie algebra has directions both of strictly positive and strictly negative Ricci curvature. Hence \( \text{nil}(\mathfrak{g}) \) must actually be commutative.

To summarize:

**Theorem 4.2.** An invariant metric on a Lie group \( G \) is of constant non-zero curvature if and only if the Lie algebra \( \mathfrak{g} \) of \( G \) is solvable with a codimension one abelian nilradical and such that the linear transformation \( S \), the self-adjoint part of \( \text{ad}(b) \) restricted to \( \text{nil}(\mathfrak{g}) \), where \( b \) spans a complement to \( \text{nil}(\mathfrak{g}) \), is a multiple of the identity.

In concrete terms the Lie algebra occurring in the Theorem can be written as follows where \( a \) and each of the \( b_i \)'s is non-zero:

\[
[e_1, e_n] = a e_1 + b_1 e_2, \quad [e_2, e_n] = a e_2 - b_1 e_2, \quad [e_3, e_n] = a e_3 + b_2 e_4, \quad \ldots, \quad [e_{n-1}, e_n] = a e_{n-1} + b_{n-1} e_n, \quad [e_n, e_n] = a e_n + b_n e_{n+1},
\] (20)

In fact by scaling \( e_n \) we can even assume that \( a = 1 \).

In terms of the classification obtained in [11] the three dimensional cases correspond to algebras 3.3, 3.7, the four dimensional cases to 4.5(a = b = 1), 4.6(a = b) and the five dimensional cases to 5.7(a = b = c = 1), 5.13(a = p = 1) and 5.17(q = p).

The metric corresponding to (20) is given in adapted coordinates by

\[
\sum_{i=1}^{p}(dx_{2i-1} - (ax_{2i-1} + b_i x_{2i})dx_n)^2 + (dx_{2i} - (ax_{2i} - b_i x_{2i-1})dx_n)^2
\]
\[
+ \sum_{j=2p+1}^{n}(dx_j - ax_j dx_n)^2 + \epsilon^2(dx_n)^2.
\] (21)

In (21) \( \epsilon \) can assume any non-zero value: the curvature of the metric in (21) is \(-\frac{e^2}{2}\).

We shall assume for simplicity that \( \epsilon = 1 \).

In [8] Killing vector fields were constructed for all the three dimensional Lie groups in particular for cases 3.3, 3.6 and 3.7. It was pointed out that any left-invariant vector field is automatically a Killing vector for a right-invariant metric and conversely. In the case of algebra 3.7 some of the Killing vector fields were
rather complicated. In the present case we shall be content to reduce the metric to a standard form. Indeed consider the metric
\[ \sum_{k}^{n} \frac{d\tau_{k}^2 + d\tau_{2}^2 + \ldots + d\tau_{n}^2}{a^2 \tau_{k}^2}. \] (22)

It is the metric of \( n \)-dimensional hyperbolic space of constant curvature \( -a^2 \). The corresponding Killing vector fields are as follows:

\[ \Delta = \sum_{1}^{n} \tau_{k} D_{\tau_{k}}, \] (23)

\[ D_{\tau_{i}}, \quad (1 \leq i < n) \] (24)

\[ (\tau_{i})^2 = 2 \Delta, \quad (1 \leq i < n) \] (25)

\[ ((\tau_{1})^2 + (\tau_{2})^2 + \ldots + (\tau_{n})^2)D_{\tau_{i}} - 2\tau_{i}\Delta, \quad (1 \leq i < n). \]

Now the following transformation pulls back (22) to (21, \( \epsilon = 1 \)):

\[ x_{2i-1} = ae^{-ax_{n}}(x_{2i-1} \cos b_{i}x_{n} + x_{2i} \sin b_{i}x_{n}) \quad (1 \leq i \leq p), \] (26)

\[ x_{2i} = ae^{-ax_{n}}(-x_{2i-1} \sin b_{i}x_{n} + x_{2i} \cos b_{i}x_{n}) \quad (1 \leq i \leq p), \] (27)

\[ n = e^{-ax_{n}}, \quad n_{j} = ax_{j}e^{-ax_{n}} \quad (2p + 1 \leq j < n). \] (28)

We conclude that (21) is indeed of constant curvature \( -a^2 \). Moreover, we may obtain the Killing vector fields for (21) by transforming the Killing vector fields for (22). Indeed \( \Delta \) in (23) transforms to \( -\frac{1}{a}D_{\tau_{n}} \) and \( D_{\tau_{i}} \) in (24) to \( \frac{1}{a}e^{-ax_{n}}D_{\tau_{i}} \): thus (23) and (24) account for all the left-invariant vector fields which are automatically Killing vector fields of a right invariant metric. As regards (25) the vector fields \( \tau_{j}D_{\tau_{i}} - \tau_{i}D_{\tau_{j}} \) are transformed to \( x_{j}D_{x_{i}} - x_{i}D_{x_{j}} \) where \( 1 \leq i < j < n \). Finally for (26) there are three types of Killing vector fields: if \( 2p + 1 \leq j < n \)

\[ e^{-ax_{n}}((1 + a^2 \sum_{m=1}^{n-1} x_{2m}^2)D_{x_{j}} + 2ax_{j}(D_{x_{n}} + \sum_{i=1}^{p} b_{i}(x_{2i-1}D_{x_{2i}} - x_{2i}D_{x_{2i-1}}))), \] (29)

whereas if \( 1 \leq i \leq p, \)

\[ e^{-ax_{n}}((1 + a^2 \sum_{m=1}^{n-1} x_{2m}^2)(\cos b_{i}x_{n}D_{x_{2i-1}} + \sin b_{i}x_{n}D_{x_{2i}}) + 2a(x_{2i-1} \cos b_{i}x_{n}) \]

\[ + x_{2i} \sin b_{i}x_{n})(D_{x_{n}} + \sum_{q=1}^{p} b_{q}(x_{2q-1}D_{x_{2q}} - x_{2q}D_{x_{2q-1}}))), \] (30)

and

\[ e^{-ax_{n}}((1 + a^2 \sum_{m=1}^{n-1} x_{2m}^2)(-\sin b_{i}x_{n}D_{x_{2i-1}} + \cos b_{i}x_{n}D_{x_{2i}}) + 2a(-x_{2i-1} \sin b_{i}x_{n}) \]

\[ + x_{2i} \cos b_{i}x_{n})(D_{x_{n}} + \sum_{q=1}^{p} b_{q}(x_{2q-1}D_{x_{2q}} - x_{2q}D_{x_{2q-1}}))), \] (31)

We remark finally that these formulas for Killing vector fields remain valid in the flat case where \( a = 0. \)
REFERENCES

[1] M. Anderson, A survey of Einstein Metrics on 4-Dimensional Manifolds, Handbook of Geometric Analysis, 3, International Press, Boston, 2010.

[2] T. Arias-Marco and O. Kowalski, Classification of 4-dimensional homogeneous weakly Einstein manifolds, Czechoslovak Math. J., 65 (2015), 21–59.

[3] A. Besse, Einstein Manifolds, 1st ed., Springer, Berlin, Heidelberg, New York, 1987.

[4] S. Chen and K. Liang, Left-invariant pseudo-Einstein metrics on Lie groups, J. Nonlinear Math. Phys., 19 (2012), 1250015, 11 pp.

[5] Z. Chen, D. Hou and C. Bai, A left-symmetric algebraic approach to left invariant flat pseudo-metrics on Lie groups, J. Geom. Phys., 62 (2012), 1600–1610.

[6] P. Gadea, J. González-Dávila and J. Oubina, Cyclic metric Lie groups, Monatsh. Math., 176 (2015), 219–239.

[7] M. Guediri, Novikov algebras carrying an invariant Lorentzian symmetric bilinear form, J. Geom. Phys., 82 (2014), 132–144.

[8] F. Hindeleh and G. Thompson, Killing’s equations for invariant metrics on Lie groups, Journal of Geometry and Mechanics, 3 (2011), 323–335.

[9] H. Kodama, A. Takahara and H. Tamaru, The space of left-invariant metrics on a Lie group up to isometry and scaling, Manuscripta Math., 135 (2011), 229–243.

[10] J. Milnor, Curvatures of left invariant metrics on Lie groups, Advances in Math., 21 (1976), 293–329.

[11] J. Patera, R. T. Sharp, P. Winternitz and H. Zassenhaus, Invariants of real low dimension Lie algebras, J. Math. Phys., 17 (1976), 986–994.

[12] H. Wang and S. Deng, Left invariant Einstein-Randers metrics on compact Lie groups, Canad. Math. Bull., 55 (2012), 870–881.

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