Twisted $GL_n$ Loop Group Orbit and Solutions of the WDVV Equations

Johan van de Leur

Mathematical Institute,
University of Utrecht,
P.O. Box 80010, 3508 TA Utrecht,
The Netherlands
e-mail: vdleur@math.uu.nl

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Abstract

We show that all (n-component) KP tau-functions, which are related to the twisted loop group of $GL_n$, give solutions of the Darboux-Egoroff system of PDE’s. Using the Geometry of the Grassmannian we construct from the corresponding wave function the deformed flat coordinates of the Egoroff metric and from this the corresponding solution of the Witten–Dijkgraaf–E. Verlinde–H. Verlinde equations

1 Introduction

In the early 90’s B. Dubrovin [4] noticed that the local classification of massive topological field theories can be solved by classifying certain flat diagonal metrics

$$ds^2 = \sum_{i=1}^{n} h_i^2(u)(du_i)^2, \quad u = (u_1, \ldots, u_n),$$  \hspace{1cm} (1.1)

with

$$\partial_j h_i^2(u) = \partial_i h_j^2(u), \text{ and } \sum_{k=1}^{n} \partial_k h_i(u) = 0, \quad \partial_i = \frac{\partial}{\partial u_i}. \hspace{1cm} (1.2)$$

For the flat coordinates $t^i$, $1 \leq i \leq n$, of the metric (1.1), the functions

$$c_{kl}^m(t) = \sum_{i=1}^{n} \frac{\partial t^m}{\partial t^i} \frac{\partial u_i}{\partial t^k} \frac{\partial u_i}{\partial t^\ell},$$  \hspace{1cm} (1.3)

satisfy

$$\sum_{k=1}^{n} c_{ij}^k(t)c_{km}^\ell(t) = \sum_{k=1}^{n} c_{jm}^k(t)c_{ik}^\ell(t).$$  \hspace{1cm} (1.4)
If one writes down these equations for the function $F(t)$ for which
\[
\frac{\partial^3 F(t)}{\partial t^k \partial t^\ell \partial t^m} = c_{k\ell m}(t) = \sum_{i=1}^{n} \eta_{mi} c_{k\ell}^{i}(t), \quad \text{where} \quad \eta_{pq} = \sum_{i=1}^{n} h_{i}^{2}(u) \frac{\partial h_{i}}{\partial u} \frac{\partial u}{\partial t},
\] (1.5)
with the constraint
\[
\frac{\partial^3 F(t)}{\partial t^k \partial t^\ell \partial t^m} = \eta_{km},
\]
one obtains the well-known the Witten-Dijkgraaf-E. Verlinde-H. Verlinde (WDVV)-equations [19], [3].

Vanishing of the curvature of these metrics (1.1) can be written in the form of a system of partial differential equations in the canonical coordinates $u_i$ for the rotation coefficients
\[
\gamma_{ij} = \frac{\partial_j h_i(u)}{h_j(u)}, \quad i \neq j,
\] (1.6)
which is known under the name the Darboux-Egoroff system:
\[
\gamma_{ij}(u) = \gamma_{ji}(u), \\
\partial_k \gamma_{ij}(u) = \gamma_{ik}(u) \gamma_{kj}(u), \\
\sum_{k=1}^{n} \partial_k \gamma_{ij}(u) = 0,
\] (1.7)
From (1.2) and (1.6), we see that the Lamé coefficients $h_i$ satisfy:
\[
\partial_j h_i(u) = \gamma_{ij}(u) h_j(u), \quad i \neq j, \quad \partial_i h_i(u) = -\sum_{j \neq i} \gamma_{ij}(u) h_j(u).
\] (1.8)

The flat coordinates $t^1, \ldots, t^n$ of this metric can be found from the linear system
\[
\partial_i \partial_j t^k = \Gamma_{ij}^i \partial_i t^k + \Gamma_{ij}^j \partial_j t^k, \quad i \neq j; \quad \partial_i \partial_i t^k = \sum_{j=1}^{n} \Gamma_{ii}^j \partial_j t^k,
\] (1.9)
where $\Gamma_{ij}^k$ are Christoffel symbols:
\[
\Gamma_{ij}^i = \frac{\partial_i h_j}{h_i}, \quad \Gamma_{ii}^j = (2\delta_{ij} - 1) \frac{h_i \partial_j h_i}{h_j^2}.
\] (1.10)

R. Martini and the author constructed in [15] solutions of the Darboux-Egoroff system (1.7). These solutions were related to certain points in the $SL_n$ Loop group orbit of the highest weight vector of the homogeneous realization of the basic representation and were related to a reduction of the $n$-component KP hierarchy. They had no real representation theoretical explanation for these particular solution. In this paper we show that all elements in the twisted Loop group orbit of $GL_n$ lead to solutions of the Darboux-Egoroff system. The solutions of [15] are certain homogeneous solutions in this orbit. This makes it possible to construct non-homogeneous WDVV prepotentials $F$. Inspired by the papers [13] and [1], we construct in section 5 for all elements in the twisted Loop group orbit besides solutions of the Darboux-Egoroff system, also the related (deformed) flat coordinates and the WDVV prepotential $F$. 
2 Groups and Grassmannians

Consider the space $H_n$ of Laurent series in $t$ (this $t$ has nothing to do with the flat coordinates appearing in the previous section) with coefficients in $\mathbb{C}^n$

$$H_n = \{ \sum_j c_j t^j | c_j \in \mathbb{C}^n, c_j = 0 \text{ for } j << 0 \}.$$

Let $e_j$, $1 \leq j \leq n$, be a basis of $\mathbb{C}^n$. Define

$$v_{-n(k+1)+j} = v^{(j)}_{-k} := e_j t^k, \quad 1 \leq j \leq n, \quad k \in \mathbb{Z},$$

then an element of $H_n$ is a unique linear combination of, possibly infinitely many, $v_i$'s or equivalently $v^{(j)}_i$'s. The space $H_n$ has a natural filtration ($j \in \mathbb{Z}$)

$$\ldots H^{(j-1)} \subset H^{(j)} \subset H^{(j+1)} \subset \ldots,$$

where

$$H^{(j)}_n = \{ \sum_k b_k v_k | b_k \in \mathbb{C}, b_k = 0 \text{ for } k > j \}. \quad (2.2)$$

Let $H^*_n$ denote the space of linear functions $f$ on $H_n$ such that $f(H^{(j)}_n) = 0$ for $j << 0$. On the direct sum $\mathcal{H}_n = H_n \oplus H^*_n$ of these two spaces, one has a natural symmetric nondegenerate bilinear form $(\cdot, \cdot)$ for which the spaces $H_n$ and $H^*_n$ are isotropic, it is defined by $(f, v) = f(v)$ for $v \in H_n$ and $f \in H^*_n$. Define "dual basis" elements to the elements defined in (2.1) as follows

$$v^{(j)}_* = v^{(j)*} = e^{(*)}_j t^k, \quad (2.3)$$

then the bilinear form is given by the following formula's

$$(e^{(*)}_i t^j, e^{(*)}_j t^k) = \delta_{k+l-j, -1} \delta_{ij}, \quad (v^{(*)}_r, v^{(*)}_s) = \delta_{r-s} \delta_{ij}, \quad (v^*_r, v^*_s) = \delta_{r-s}. \quad (2.4)$$

Let $\mathbb{C}^{n*} = \oplus_{j=1}^n \mathbb{C} e^{(*)}_j$ then clearly

$$H^*_n = \{ \sum_j c^*_j t^j | c^*_j \in \mathbb{C}^{n*}, c^*_j = 0 \text{ for } j << 0 \}.$$

In analogy with (2.2), one also has the subspaces

$$H^{(j)*}_n = \{ \sum_k b_k v^*_k | b_k \in \mathbb{C}, b_k = 0 \text{ for } k > j \}. \quad (2.5)$$

Let $(\cdot, \cdot)$ be the symmetric bilinear form on $\mathbb{C}^n \oplus \mathbb{C}^{n*}$ for which $\mathbb{C}^n$ and $\mathbb{C}^{n*}$ are isotropic and $(e^*_i | e_j) = \delta_{ij}$. Let $v(t) \in H_n$ and $f(t) \in H^*_n$, then (2.4) is equivalent to

$$(f(t), v(t)) = \text{Res}_t (f(t) | v(t)), \quad (2.6)$$

where $\text{Res}_t \sum a_i t^i = a_{-1}$. Notice that the left hand side of (2.6) makes sense.

Now, notice that the space $\mathcal{H}_n^{(j)} := H^{(j)}_n \oplus H^{(-j)*}_n$ is a maximal isotropic subspace of $\mathcal{H}_n$ with respect to our bilinear form $(\cdot, \cdot)$. Having these specific isotropic spaces in mind,
we want to define more general maximal isotropic subspaces. Let $\nabla \subset H_n$ be a maximal isotropic linear subspace, which satisfies the following conditions:

$$\nabla = V \oplus V^* \text{ with } V \subset H_n, \ V^* \subset H^*_n$$ (2.7)

and

$$H^{(j)}_n \subset V, \ H^{(j)*}_n \subset V^* \text{ for } j << 0.$$ (2.8)

All such subspaces form the points of an infinite (isotropic) Grassmannian $\overline{Gr}$. Clearly, since (2.7) holds, a $\nabla \in \overline{Gr}$ induces two unique spaces $V$ and $V^*$, which can be separately regarded as points of two Grassmannians

$$Gr = \{ V \subset H_n | H^{(j)}_n \subset V, \text{ for } j << 0 \} \text{ and } Gr^* = \{ V^* \subset H^*_n | H^{(j)*}_n \subset V^*, \text{ for } j << 0 \}.$$ 

It is obvious that the converse also holds, i.e., if $V \in Gr$ (or $V^* \in Gr^*$), then there exists a unique maximal space $V^* \in Gr^*$ (resp. $V \in Gr$) such that $(V^*, V) = 0$ and hence $\nabla = V \oplus V^*$ is a unique point of $\overline{Gr}$. The space $Gr$ is Sato's polynomial Grassmannian \[.\] Here it is coupled to $Gr^*$ and $\overline{Gr}$. The reason why we introduce the latter two spaces will become clear later on. It is well-known that the space $Gr = \bigcup_{m \in \mathbb{Z}} Gr_m$, disjoint union of the spaces

$$Gr_m = \{ V \in Gr | H^{(j)}_n \subset V \text{ and } \dim V/H^{(j)}_n = m - j \text{ for } j << 0 \}.$$ 

If $V \in Gr_m$, then $V^*$ belongs to

$$Gr^*_m = \{ V^* \in Gr | H^{(j)*}_n \subset V^* \text{ and } \dim V^*/H^{(j)*}_n = -m - j \text{ for } j << 0 \}.$$ 

In this situation, $\nabla = V \oplus V^*$ is an element of

$$Gr_m = \{ \nabla = V \oplus V^* \in \overline{Gr} | V \in Gr_m \text{ and } V^* \in Gr^*_m \}.$$ 

Suppose that the space $V$ is invariant under multiplication with $t$, i.e., $V$ is an element of the restricted Grassmannian

$$gr := \{ V \in Gr | tV \subset V \},$$ 

then it is clear from the above construction that its "dual space" $V^*$ is unique. Now let $f(t) \in V^*$, then for all $v(t) \in V$ we have $\text{Res}_t(f(t)|v(t)) = 0$. Since $tv(t) \in V$, also

$$\text{Res}_t(tf(t)|v(t)) = \text{Res}_t(f(t)|tv(t)) = 0 \text{ for all } v(t) \in V,$$

hence $tf(t) \in V^*$. This means that also $V^*$ and $\nabla = V \oplus V^*$ satisfy $tV^* \subset V^*$, $t\nabla \subset \nabla$ and that it makes sense to define

$$gr^* := \{ V^* \in Gr^* | tV^* \subset V^* \} \quad \text{and} \quad \overline{gr} := \{ \overline{V} \in \overline{Gr} | t\overline{V} \subset \overline{V} \}$$

and

$$gr_m := gr \cap Gr_m, \quad gr^*_m := gr^* \cap Gr^*_m \quad \text{and} \quad \overline{gr}_m := \overline{gr} \cap \overline{Gr}_m.$$ 

Consider $H^{(j)}_n$, $j \in \mathbb{Z}$, to be the a fundamental system of neighborhoods of zero, then $H_n$ becomes a topological vector space. Let $\mathfrak{g}_\infty$ be the algebra of all continuous endomorphisms
of $H_n$. If one considers an element of $H_n$ as an infinite vector with respect to the basis $v_i, i \in \frac{1}{2} + \mathbb{Z}$, then (see [11])

$$\Pi_\infty = \{(a_i)_{i,j} \in \frac{1}{2} + \mathbb{Z}| \text{for each } k \text{ the number of non-zero } a_{ij} \text{ with } i \leq k \text{ and } j \geq k \text{ is finite}\}.$$  

(2.9)

Denote by $\Pi_\infty$ the group of invertible elements of the associative algebra $\Pi_\infty$. Then $\Pi_\infty$ acts transitively on $Gr$. We can extend an element $g \in \Pi_\infty$ to an element, which by abuse of notation we denote by the same letter, $g$ of $O(\nabla)$, the orthogonal group of $\nabla$ (see [10]):

**Lemma 2.1** Let $g \in \Pi_\infty$ be such that

$$gv_j = \sum_{i \in \mathbb{Z} + \frac{1}{2}} A_{ij}v_i$$

and let $A = (A_{ij})_{i,j \in \mathbb{Z} + \frac{1}{2}}$ and $A^{-1} = (B_{ij})_{i,j \in \mathbb{Z} + \frac{1}{2}}$. Then

$$gv_j^* = \sum_{i \in \mathbb{Z} + \frac{1}{2}} B_{ji}v_i^*.$$  

**Proof.** Suppose that $gv_j^* = \sum_k B_{-\ell,k}v_{-\ell,k}$, then it follows from

$$\delta_{j,-\ell} = (v_j, v_{\ell}^*) = (\sum_i A_{ij}v_i, B_{-\ell k}v_{-\ell k}) = \sum_i A_{ij}B_{-\ell,i}$$

that $B = A^{-1}$. \qed

This lemma defines the (transitive) action of $\Pi_\infty$ on $Gr^*$ and hence on $\overline{Gr}$.

Let $L = \mathbb{C}[t, t^{-1}]$ be the algebra of Laurent polynomials in $t$. The identification (2.1) gives us an embedding

$$\phi : \text{Mat}_n(L) \to \Pi_\infty, \quad \phi(e_{ij} t^k) = \sum_{k \in \mathbb{Z}} E_{nk+i} \cdot n(k+t) + j - \frac{1}{2},$$  

(2.10)

where $e_{ij} = (\delta_{ir} \delta_{js})_{1 \leq r, s \leq n} \in \text{Mat}_n(\mathbb{C})$ and $E_{ij} = (\delta_{ir} \delta_{js})_{r, s \in \frac{1}{2} + \mathbb{Z}}$. This embedding gives rise to the embedding of the loop Lie algebra $gl_n(L)$ into $\Pi_\infty$ and the loop group $GL_n(L)$ in $\Pi_\infty$. Multiplication with $t$, i.e. with $\sum_{i=1}^{n} e_{ii}t$, commutes with the action of $GL_n(L)$ and thus $GL_n(L)$ acts on $gr$. This action is transitive, see [10]. Moreover, $\phi(t) = \phi(\sum_{i=1}^{n} e_{ii} t) = \sum_{k \in \frac{1}{2} + \mathbb{Z}} E_{kk} + n$. 

Consider $H_n^{(-j)^*}$, it has as a "natural basis" the elements $v_i$ with $i < j$. Its dual space $H_n^{(-j)^*}$ has $v_i^*$ with $i < -j$ as "basis". To these spaces we associate special vectors in two semi-infinite wedge spaces (see [11], [12])

$$|H_n^{(-j)^*}| = v_{-\frac{1}{2}} \wedge v_{-\frac{3}{2}} \wedge v_{-\frac{5}{2}} \wedge v_{-\frac{7}{2}} \wedge \cdots,$$

$$\langle H_n^{(-j)^*} \rangle = \cdots \wedge v_{-j-\frac{3}{2}} \wedge v_{-j-\frac{5}{2}} \wedge v_{-j-\frac{7}{2}} \wedge v_{-j-\frac{9}{2}}.$$  

(2.11)

In fact we can associate to any element $W \in Gr_j$ and its dual space $W^* \in Gr_{-j}^*$ elements in these wedge spaces, viz., we know that there exists an $m \ll 0$ such that $H_n^{(m)} \subset W$.
and $H_n^{(m)}* \subset W*$. Then let $w_{j-\frac{1}{2}}, w_{j-\frac{3}{2}}, \ldots, w_{m+\frac{1}{2}}$ be a basis of $W\text{mod}H_n^{(m)}$ and $w_{j-\frac{1}{2}}, w_{j-\frac{3}{2}}, \ldots, w_{m+\frac{1}{2}}$ be a basis of $W*\text{mod}H_n^{(m)*}$ then we put

$$|W| = w_{j-\frac{1}{2}} \wedge w_{j-\frac{3}{2}} \wedge \cdots \wedge w_{m+\frac{1}{2}} \wedge v_{m-\frac{1}{2}} \wedge v_{m-\frac{3}{2}} \wedge \cdots,$$

$$\langle W* \rangle = \cdots \wedge v_{m-\frac{1}{2}} \wedge v_{m-\frac{3}{2}} \wedge v_{m+\frac{1}{2}} \wedge w_{m+\frac{3}{2}} \wedge \cdots \wedge w_{j-\frac{1}{2}}.$$ (2.12)

It is clear that up to a constant both $|W|$ and $\langle W* \rangle$ are independent of the choice of basis of $W\text{mod}H_n^{(m)}$ and $W*\text{mod}H_n^{(m)*}$. This gives us a map $\mu$ from $G\text{r}$ and $G\text{r}*$ into $\mathbb{P}F$ and $\mathbb{P}F*$, where $F = \oplus_{j \in \mathbb{Z}}F(j)$, $F* = \oplus_{j \in \mathbb{Z}}F(j)*$ and the spaces $F(j)$ and $F(-j)*$ are the vector spaces generated by the elements on the right-hand side of (2.12) (see [11], [9] or [12] for more details). Since $A_\infty$ and its subgroup $GL_n(L)$ act on $G\text{r}$ and $G\text{r}*$ we obtain a projective representation of these groups on $\mathbb{P}F$ and $\mathbb{P}F*$.

Let $\omega$ be the following orthogonal transformation of $V$ with respect to the bilinear form (2.4):

$$\omega(v_{m}^{(a)}) = (-)^{m+\frac{1}{2}}iv_{m}^{(a)*}, \quad \omega(v_{m}^{(a)*}) = (-)^{m+\frac{1}{2}}iv_{m}^{(a)}, \quad m \in \frac{1}{2} + \mathbb{Z}, \ 1 \leq a \leq n,$$ (2.13)

then $\omega(\overline{H}_n^{(0)}) = \overline{H}_n^{(0)}$ and $\omega(\overline{Gr}_m) = \overline{Gr}_{-m}$. If $g \in A_\infty$ is such that

$$gv_{j}^{(b)} = \sum_{a,k} A_{kj}^{(ab)} v_{k}^{(a)} \text{ then } gv_{j}^{(b)*} = \sum_{k,a} B_{kj}^{(b, a)} v_{k}^{(a)*} \text{ with } \sum_{j,b} A_{kj}^{(ab)} B_{kj}^{(b,c)} = \delta_k \delta_{ab},$$ (2.14)

then

$$\omega(gv_{j}^{(b)}) = \omega(g)\omega(v_{j}^{(b)}) = \omega(g)(-)^{j+\frac{1}{2}}iv_{j}^{(b)*}$$

$$= \sum_{a,k} A_{kj}^{(ab)} \omega(v_{k}^{(a)}) = \sum_{a,k} A_{kj}^{(ab)} (-)^{k+\frac{1}{2}}iv_{k}^{(a)*}.$$ This and a similar calculation for $gv_{j}^{(b)*}$ gives

$$\omega(g)v_{j}^{(b)} = \sum_{a,k} (-)^{k-j}B_{kj}^{(ba)}v_{k}^{(a)}, \quad \omega(g)v_{j}^{(b)*} = \sum_{a,k} (-)^{k-j}A_{kj}^{(ab)}v_{k}^{(a)*},$$ (2.15)

and hence induces on $GL_n(L)$ the automorphism:

$$\omega(A(t)) = ((A(-t))^T)^{-1}, \quad A(t) \in \text{Mat}_n(L),$$ (2.16)

where $A^T$ stand for the transposed of the matrix $A$. The fixed point set of this automorphism is the twisted loop group

$$GL_n(L)^{(2)} = \{A(t) \in GL_n(L) | A(-t) = ((A(t))^T)^{-1}\}.$$

All this suggests to define a new skew-symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $H_n$

$$\langle u, v \rangle = (u, \omega(v)) \quad u, v \in H_n,$$

or in other terms cf. (2.7)

$$\langle u(t), v(t) \rangle = i\text{Res}_t(u(t)|v(-t)) \quad u(t), v(t) \in H_n,$$ (2.17)
and new Grassmannians

\[ Gr^{(2)} = \{ W \in Gr_0 | \langle u, v \rangle = 0 \text{ for all } u, v \in W \}, \quad gr^{(2)} = gr \cap Gr^{(2)}. \]

We now want to show that \( gr^{(2)} \) the homogeneous space is of the twisted loop group \( GL_n(L)^{(2)} \). It is obvious, from the above discussion, that \( g \cdot H_n^{(0)} \in gr^{(2)} \) for any \( g \in GL_n(L)^{(2)} \). To prove the converse, we will use the following Theorem of [16]:

**Theorem 2.1** Any loop in \( g(t) \in GL_n(L) \) can be uniquely factorized as \( g(t) = g_u(t) \cdot g_+(t) \), with \( g_+(t) \in GL_n(C[t]) \) and \( g_u \in \Omega U_n = \{ h(t) \in U_n(L) | h(1) = I_n \} \). Moreover, \( gr = \Omega U_n \cdot H_n^{(0)} \) and isotropy group of \( H_n^{(0)} \) is the group \( U_n \) of constant loops.

Now let \( W \in gr^{(2)} \), then we can write \( W = g(t) \cdot H_n^{(0)} \) for certain \( g(t) \in \Omega U_n \). Clearly also \( \omega(g(t)) \cdot H_n^{(0)} = W \), hence \( (g(t))^{-1} \omega(g(t)) = u \in U_n \). Since \( \omega \) is an involution on \( GL_n(L) \) we find that \( u \omega(u) = 1 \), and since \( u \) is unitary we deduce that the following conditions hold for \( u \):

\[ u = u^T \quad \text{and} \quad u\pi^T = I_n. \]

So we can find a real orthogonal matrix \( v \) such that \( u = vd^T \), with \( d \) a diagonal matrix. Let \( s \) be a diagonal matrix that satisfies \( s^2 = d \), then \( w = w^T = vsv^T \) is a unitary matrix that satisfies \( u = w^2 \) and

\[ \omega(g(t)w) = g(t)u\omega(w) = g(t)w^2(w^T)^{-1} = g(t)w^2w^{-1} = g(t)w. \]

Since also \( W = g(t) \cdot H_n^{(0)} = g(t)w \cdot H_n^{(0)} \), we find that \( W \in GL_n(L)^{(2)} \cdot H_n^{(0)} \) and we have proven the following

**Theorem 2.2**

\[ gr^{(2)} = GL_n(L)^{(2)} \cdot H_n^{(0)}. \]

## 3 The Clifford Algebra and Tau-Functions

Using (2.11), we can associate to any point \( W \in Gr_j \) and the corresponding \( W^* \in Gr^*_j \) vectors \( |W| \in \mathbb{P}F^{(j)} \), \( |W^*| \in \mathbb{P}F^{(-j)*} \) and hence up to a constants unique vectors \( |W| \in F^{(j)} \) and \( |W^*| \in F^{(-j)*} \). Clearly \( \omega \) defined in (2.13) is not well defined on these vectors in the spaces \( F \) and \( F^* \). To solve this problem, we will define a Clifford algebra and its corresponding spin module and we will fix \( \omega \) on one of the vectors of the spin module. In this construction we will only consider the space \( F \), since the space \( W^* \) corresponding to \( W \in Gr \) is always unique.

Recall that on the infinite space \( \mathcal{P}_n = H_n^+ \oplus H_n^- \) we have a symmetric bilinear form \( (\cdot, \cdot) \) given by (2.4). Let \( CL(\mathcal{P}_n) \) be the Clifford algebra on this space, i.e. the associative algebra over \( C \) with unity 1 which has the following defining relations

\[ vw + uv = (u, v)1, \quad u, v \in \mathcal{P}_n. \quad (3.1) \]

We obtain a obtain a representation \( \psi \) of the clifford algebra on the space \( F \) by defining it on wedges as follows \( \{ w, w_j \in H_n, w^* \in H_n^* \text{ with } w_j = v_{-j-m-\frac{1}{2}} \text{ for certain } m \in \mathbb{Z} \text{ and } \}

\[ (w, w_j) = (v, v_{-j-m-\frac{1}{2}}) \]

and for \( w \in H_n, w^* \in H_n^* \) according to (2.13) we obtain the following: \( u \) and \( v \) are the real projections of \( w \) and \( w^* \), respectively, when \( (w, w_j) \) is defined.
all \( j \gg 0 \)

\[
\psi(w)(w_0 \wedge w_{-1} \wedge w_{-2} \wedge \cdots) = w \wedge w_0 \wedge w_{-1} \wedge w_{-2} \wedge \cdots,
\]

\[
\psi(w^*)(w_0 \wedge w_{-1} \wedge \cdots) = \sum_{i=0}^{\infty} (-i)! (w^*, w_i) w_0 \wedge w_{-1} \wedge \cdots w_{j-1} \wedge w_{j+1} \wedge \cdots. \tag{3.2}
\]

The space \( F \) is the spin module for this Clifford algebra. Let

\[
|0\rangle = v_{-\frac{1}{2}} \wedge v_{-\frac{3}{2}} \wedge v_{-\frac{5}{2}} \wedge v_{-\frac{7}{2}} \wedge \cdots,
\]

then \( F \) is the unique module generated by

\[
\psi(v_k)|0\rangle = \psi(v_k^*)|0\rangle = 0 \quad \text{for} \quad k < 0. \tag{3.3}
\]

It is straightforward to check that (cf. (2.12))

\[
w_{j-\frac{1}{2}} \wedge w_{j-\frac{3}{2}} \wedge \cdots \wedge w_{m+\frac{1}{2}} \wedge v_{m-\frac{1}{2}} \wedge v_{m-\frac{3}{2}} \wedge \cdots
= \psi(w_{j-\frac{1}{2}})\psi(w_{j-\frac{3}{2}})\cdots\psi(w_{m+\frac{1}{2}})\psi(v_{m-\frac{1}{2}})\psi(v_{m-\frac{3}{2}})\cdots\psi(v_{\frac{1}{2}})\psi(v_{\frac{3}{2}})|0\rangle. \tag{3.4}
\]

Define the fermionic fields \((z \in \mathbb{C}^\times)\)

\[
\psi^{+}(j)(z) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi^{+}(j)_k z^{-k - \frac{1}{2}} := \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi^{+}(j)_k z^{-k - \frac{1}{2}},
\]

\[
\psi^{-}(j)(z) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi^{-}(j)_k z^{-k - \frac{1}{2}} := \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi^{-}(j)_k z^{-k - \frac{1}{2}}, \tag{3.5}
\]

and bosonic fields \((1 \leq j \leq n)\) by

\[
\alpha^{(j)}(z) = \sum_{k \in \mathbb{Z}} \alpha^{(j)}_k z^{-k - 1} =: \psi^{+}(j)(z)\psi^{-}(j)(z),
\]

where \(: :\) stands for the normal ordered product defined in the usual way \((\lambda, \mu = + \text{ or } -)\):

\[
: \psi^{\lambda(i)}_k \psi^{\mu(j)}_\ell : = \begin{cases} 
\psi^{\lambda(i)}_k \psi^{\mu(j)}_\ell & \text{if } \ell \geq k, \\
-\psi^{\mu(j)}_\ell \psi^{\lambda(i)}_k & \text{if } \ell < k.
\end{cases} \tag{3.7}
\]

Notice that \(\psi^{+}(j)(z) = \psi(\delta(z-t)e_j)\) and \(\psi^{-}(j)(z) = \psi(\delta(z-t)e_j^*)\), where \(\delta(z-t) = z^{-1} \sum_{k \in \mathbb{Z}} (\frac{z}{t})^k\). One checks (using e.g. the Wick formula) that these bosonic operators satisfy the canonical commutation relation of the associative oscillator algebra:

\[
[\alpha^{(i)}_k, \alpha^{(j)}_\ell] = k \delta_{ij} \delta_{k, -\ell}, \tag{3.8}
\]

and one has

\[
\alpha^{(i)}_k|0\rangle = 0 \quad \text{for} \quad k \geq 0. \tag{3.9}
\]

In order to express the fermionic fields \(\psi^{\pm(i)}(z)\) in terms of the bosonic fields \(\alpha^{(i)}(z)\), we need some additional operators \(Q_i, i = 1, \ldots, n\), on \(F\). These operators are uniquely defined by the following conditions:

\[
Q_i|0\rangle = \psi^{+}(i)|0\rangle, \quad Q_i \psi^{\pm}(j)_k = (-1)^{\delta_{ij} + 1} \psi^{\pm}(j)_{k+bi_j} Q_i. \tag{3.10}
\]
They satisfy the following commutation relations:

\[ Q_j Q_k = -Q_k Q_j \quad \text{if} \quad i \neq j, \quad [\alpha_k^{(i)}, Q_j] = \delta_{ij} \delta_{k0} Q_j. \]  

(3.11)

**Theorem 3.1** (\[4\], \[3\])

\[ \psi^{\pm(i)}(z) = Q_i^{\pm1} z^{\pm \alpha_0^{(i)}} \exp(\mp \sum_{k<0} \frac{1}{k} \alpha_k^{(i)} z^{-k}) \exp(\mp \sum_{k>0} \frac{1}{k} \alpha_k^{(i)} z^{-k}). \]  

(3.12)

**Proof.** See \[14\].

The operators on the right-hand side of (3.12) are called vertex operators. They made their first appearance in string theory (cf. \[7\]).

We can describe now the \( n \)-component boson-fermion correspondence. Let \( C[x] \) be the space of polynomials in indeterminates \( x = \{x_k^{(i)}\}, \) \( k = 1, 2, \ldots, i = 1, 2, \ldots, n. \) Let \( N \) be a lattice with a basis \( \delta_1, \ldots, \delta_n \) over \( \mathbb{Z} \) and the symmetric bilinear form \( (\delta_i | \delta_j) = \delta_{ij}, \) where \( \delta_{ij} \) is the Kronecker symbol. Let

\[ \epsilon_{ij} = \begin{cases} -1 & \text{if } i > j \\ 1 & \text{if } i \leq j. \end{cases} \]  

(3.13)

Define a bimultiplicative function \( \epsilon : N \times N \to \{\pm1\} \) by letting

\[ \epsilon(\delta_i, \delta_j) = \epsilon_{ij}. \]  

(3.14)

Let \( \delta = \delta_1 + \ldots + \delta_n, \) \( M = \{\gamma \in N | (\delta | \gamma) = 0\}, \) \( \Delta = \{\alpha_{ij} := \delta_i - \delta_j | i, j = 1, \ldots, n, i \neq j\}. \) Of course \( M \) is the root lattice of \( s\ell_n(\mathbb{C}) \), the set \( \Delta \) being the root system.

Consider the vector space \( C[N] \) with basis \( e^\gamma, \gamma \in L, \) and the following twisted group algebra product:

\[ e^\alpha e^\beta = \epsilon(\alpha, \beta) e^{\alpha+\beta}. \]  

(3.15)

Let \( B = C[x] \otimes_C C[N] \) be the tensor product of algebras. Then the \( n \)-component boson-fermion correspondence is the vector space isomorphism

\[ \sigma : F \to B, \quad \text{with } \sigma : F^{(m)} \to B^{(m)} \]  

(3.16)

given by

\[ \sigma(\alpha_{-m_1}^{(i_1)} \ldots \alpha_{-m_s}^{(i_s)} Q_1^{k_1} \ldots Q_n^{k_n} | 0)) = m_1 \ldots m_s x_{m_1}^{(i_1)} \ldots x_{m_s}^{(i_s)} \otimes e^{k_1 \delta_1 + \ldots + k_n \delta_n}. \]  

(3.17)

The transported action of the operators \( \alpha_m^{(i)} \) and \( Q_j \) looks as follows:

\[
\begin{cases}
\sigma \alpha_m^{(j)} \sigma^{-1}(p(x) \otimes e^\gamma) = mx_m^{(j)} p(x) \otimes e^\gamma, \text{ if } m > 0, \\
\sigma \alpha_m^{(j)} \sigma^{-1}(p(x) \otimes e^\gamma) = \frac{\partial p(x)}{\partial x_m} \otimes e^\gamma, \text{ if } m > 0, \\
\sigma \alpha_0^{(j)} \sigma^{-1}(p(x) \otimes e^\gamma) = (\delta_j | \gamma) p(x) \otimes e^\gamma, \\
\sigma Q_j \sigma^{-1}(p(x) \otimes e^\gamma) = \epsilon(\delta_j, \gamma) p(x) \otimes e^{\gamma+\delta_j}.
\end{cases}
\]  

(3.18)
The transported action of the fermionic fields is as follows:

$$\sigma \psi^{\pm(j)}(z) \sigma^{-1} = e^{\pm \delta_j \pm \delta_j} \exp(\pm \sum_{k=1}^{\infty} x_k^{(j)}) \exp(\mp \sum_{k=1}^{\infty} \partial x_k^{(j)} - k)$$

(3.19)

We will now determine the second part of the boson–fermion correspondence, i.e., we want to determine $\sigma$ of the elements (3.4). Since for our purpose we are only interested in $Gr^0$ we will assume that this element is a wedge in $P^0$, i.e., let

$$\tau = A_{-\frac{1}{2}} \wedge A_{-\frac{3}{2}} \wedge A_{-\frac{5}{2}} \wedge \cdots \in F^0$$

with $A_{-p} = v_{-p}$ for all $p > P >> 0$. (3.20)

To such an element we can associate an element in $A = (A_{ij}) \in A_\infty$ such that $Av_k = A_{-k}$ for all $k > 0$. Notice that $A_{ij} = \delta_{ij}$ for $j < -P$. Then R. Martini and the author showed in [15] the following

**Proposition 3.1** Let $\sigma(\tau) = \sum_{\alpha \in M} \tau_\alpha(x)e^\alpha$. Assume that $\alpha = \sum_{j=1}^{n} k_j\delta_j$ and suppose that

$$Q^k_1 Q^k_2 \cdots Q^k_n |0\rangle = \lambda_\alpha v_{j_1-\frac{1}{2}} \wedge v_{j_2-\frac{3}{2}} \wedge v_{j_3-\frac{5}{2}} \wedge \cdots ,$$

with $j_{1} > j_{2} > j_{3} \cdots$ and $j_{-q} = -q$ for all $q > Q >> 0$ and $\lambda_\alpha = \pm 1$, then

$$\tau_\alpha(x) = \lambda_\alpha \det \left( \sum_{-R < \ell < 0}^{R} \sum_{j_{R+\frac{1}{2}}}^{j_{R+\frac{1}{2}}} \sum_{1 \leq j < n, q \in \mathbb{Z} + \frac{1}{2}} \left( \sum_{k=0}^{\infty} A_{r+nk, \ell} S_k(x^{(j)}) \right) E_{r, \ell} \right),$$

where $R = \max(P, Q)$ and $S_k(y)$ are the elementary Schur functions defined by $\sum_{k \in \mathbb{Z}} S_k(y) z^k = \exp(\sum_{k=1}^{\infty} y_k z^k)$. In particular if $1 \leq i < j \leq n$ and $\alpha = 0$, $\delta_i - \delta_j, \delta_j - \delta_i$, respectively, then $\lambda_0 = 1$, $\lambda_1 = (-1)^{n+1}$, $\lambda_2 = (-1)^{n+1}$, and $\lambda_3 = (-1)^{n+1}$, respectively.

We now want to know what happens if we apply $\omega$ to such tau-functions. Since $(\omega(u), \omega(v)) = (u, v)$, $\omega$ extends to an automorphism of order 4 on the Clifford algebra. Next notice that $|0\rangle = |H_n(0)^*\rangle \in \mathbb{P} F$. Since $\omega(H_n(0)^*) = H_n(0)^*$, it makes sense to extend $\omega$ to $F$ by fixing it on $|0\rangle$ as $\omega(|0\rangle) = |0\rangle$. It is then obvious that there is a one to one correspondence between elements $w \in F^0$ that satisfy $\omega(w) = \lambda w$ for certain $\lambda \in \mathbb{C}$ and points $W \in Gr^2$. Let $W \in Gr^2$ and

$$w = w_{-\frac{1}{2}} \wedge w_{-\frac{3}{2}} \wedge \cdots \wedge w_{-m+\frac{1}{2}} \wedge v_{-m+\frac{1}{2}} \wedge v_{-m+\frac{3}{2}} \wedge \cdots \in F^0$$

be the corresponding vector. Since $\langle w_i, w_j \rangle = \langle w_i, v_k \rangle = 0$, for all $0 > i, j > -m$ and all $k < -m$ we obtain that all $w_i$ are of the form $w_i = \sum_{-m < j < m} w_{ij} v_j$. Hence we can find vectors $w_{ij}, w_{ij}, \ldots, w_{im},$ of the same form such that $\langle w_i, w_j \rangle = \langle v_i, v_j \rangle$ for all $-m < i, j < m$. Thus $W = (w_{ij})_{-m < i, j < m}$ is a Symplectic matrix and must have determinant equal to 1. This makes it possible to write $w$ in two ways, viz,

$$w = \psi(w_{-\frac{1}{2}}) \psi(w_{-\frac{3}{2}}) \cdots \psi(w_{-m+\frac{1}{2}}) \psi(v_{-m+\frac{1}{2}}) \psi(v_{-m+\frac{3}{2}}) \cdots \psi(v_{m+\frac{3}{2}}) |0\rangle$$

(3.21)
and

\[ w = \psi(\omega((w_{\frac{1}{2}}, w_{\frac{1}{2}})^{-1}w_{\frac{1}{2}}))\psi(\omega((w_{\frac{1}{2}}, w_{\frac{1}{2}})^{-1}w_{\frac{1}{2}})) \cdots \]

\[ \cdots \psi(\omega((w_{\frac{1}{2}}, w_{\frac{1}{2}})^{-1}w_{\frac{1}{2}}))\psi(v_{m-\frac{1}{2}})\psi(v_{m-\frac{1}{2}}) \cdots \psi(v_{\frac{1}{2}})\psi(0). \]  

(3.22)

It is then straightforward to check that \( \omega(w) \) of the representation (3.21) exactly gives (3.22). This means that \( \omega(w) = w \) for all elements \( w \in F^{(0)} \) corresponding to \( W \in Gr^{(2)}. \)

Next notice that

\[ \omega(\psi^+(j)(z)) = \psi^+(j)(-z), \]

and hence

\[ \omega(\alpha^+(j)(z)) = \alpha^+(j)(-z), \]  

(3.23)

from which we deduce that

\[ \omega(\delta_j) = -\delta_j \quad \text{and} \quad \omega(x^+_k) = (-)^{k+1}x^+_k. \]

Here we write, as an abuse of notation, \( \omega \) for \( \sigma \omega \sigma^{-1}. \) Next we want to calculate what \( \omega \) does with \( Q_j. \) Notice first, using (3.12), that

\[ Q_j^\pm = \exp(\pm \sum_{k=0}^{n} \frac{1}{k} \alpha^+_k z^k) \psi^+(j)(-z) \exp(\pm \sum_{k=0}^{n} \frac{1}{k} \alpha^-_k z^{-k}) \psi^+(j)(-z) \]  

(3.24)

and that we may replace \( z \) in this formula (3.24) by \( -z, \) since the left-hand side is independent of \( z. \) So,

\[ \omega(Q_j^+) = i \exp(\mp \sum_{k=0}^{n} \frac{1}{k} \alpha^+_k (-z)^{-k}) \psi^+(j)(-z) \exp(\pm \sum_{k=0}^{n} \frac{1}{k} \alpha^-_k (-z)^{-k}) \psi^+(j)(-z) \]

\[ = i Q_j^+ (\tau^+_0(\delta_j)). \]

Thus we find for the operators \( e^{\pm \delta_j}: \)

\[ \omega(e^{\pm \delta_j}) = ie^{\mp \delta_j}(-)^{\delta_j}. \]

So we conclude that

\[ \omega \left( \tau_0(x^{(a)}_k) + \sum_{1 \leq i < j \leq n} \left( \tau_{\delta_i - \delta_j}(x^{(a)}_k) e^{\delta_i - \delta_j} + \tau_{\delta_j - \delta_i}(x^{(a)}_k) e^{\delta_j - \delta_i} \right) + \cdots \right) \]

\[ = \tau_0((-)^{k+1}x^{(a)}_k) - \sum_{1 \leq i < j \leq n} \left( \tau_{\delta_j - \delta_i}((-)^{k+1}x^{(a)}_k) e^{\delta_i - \delta_j} + \tau_{\delta_i - \delta_j}((-)^{k+1}x^{(a)}_k) e^{\delta_j - \delta_i} \right) + \cdots. \]  

(3.25)

Next assume that \( W \in gr^0. \) To this subspace corresponds an up to a multiple factor unique a vector \( w \in F^{(0)}. \) Since \( \sum_{i=1}^{n} te_i W \subset W, \) we can find special linearly independent vectors \( w_1, w_2, \ldots, w_n \in W \) such that \( t^\ell w_j = (\sum_{i=1}^{n} te_i) t^\ell w_j \in H_{-\ell n}^{(n)} \) for all \( 1 \leq j \leq n \) and such that

\[ w = w_1 \wedge w_2 \wedge \cdots \wedge w_n \wedge t w_1 \wedge t w_2 \wedge \cdots \wedge t^{\ell-1} w_n \wedge v_{-\ell n - \frac{\ell}{2}} \wedge v_{-\ell n - \frac{\ell}{2}} \wedge \cdots. \]
From this presentation of $w$ one easily sees that the action of
\[ \sum_{j=1}^{n} \alpha_k^{(j)} w = 0 \quad \text{for all } k > 0, \]
This leads to
\[ \sum_{j=1}^{n} \frac{\partial \tau_{\alpha}(x)}{\partial x^{(j)}_k} = 0 \quad \text{for all } k > 0 \quad (3.26) \]
and hence to the following

**Proposition 3.2** Tau-functions $\tau_W(x) = \sum_{\alpha \in M} \tau_{\alpha}(x^{(a)}) e^{\alpha}$ corresponding to $W \in gr^{(2)}$ satisfy the following conditions:

1. \( \sum_{j=1}^{n} \frac{\partial \tau_{\alpha}(x)}{\partial x^{(j)}_k} = 0 \quad \text{for all } k > 0, \)
2. \( \tau_{0}((-)^{k+1} x^{(a)}_k) = \tau_{0}(x^{(a)}_k), \)
3. \( \tau_{\delta_j - \delta_i}((-)^{k+1} x^{(a)}_k) = -\tau_{\delta_i - \delta_j}(x^{(a)}_k) \quad \text{for all } 1 \leq i, j \leq n, \ i \neq j. \)

4 **The KP hierarchy as a dynamical system**

It is well known, see e.g. [11], that $\tau_W$ corresponds to a $W \in Gr_0$ if and only if $\tau_W$ satisfies the KP hierarchy, i.e., the following equation:
\[ \text{Res}_{z=0} \sum_{j=1}^{n} \psi^{(j)}(z) \tau \otimes \psi^{-(j)}(z) \tau = 0, \ \tau \in F^{(0)}. \quad (4.1) \]

Using the boson-fermion correspondence we can write this equation as a family of equation on certain $n \times n$ wave functions ($\alpha \in \text{supp} \tau = \{\alpha \in M | \tau_{\alpha} \neq 0\}$)
\[ V^\pm(\alpha, x, z) = (V^\pm_{ij}(\alpha, x, z))_{i,j=1}^{n}, \quad (4.2) \]
(see [9] for more details) where
\[ V^\pm_{ij}(\alpha, x, z) := \varepsilon(\delta_j, \alpha + \delta_i) z^{(\delta_j | \pm \alpha + \alpha_{ij})} \]
\[ \times \exp(\pm \sum_{k=1}^{\infty} x^{(j)}_k z^k) \exp(\mp \sum_{k=1}^{\infty} \frac{\partial}{\partial x^{(j)}_k} z^{-k} \tau_{\alpha \pm \alpha_{ij}}(x) / \tau_{\alpha}(x)). \quad (4.3) \]
The equations are:
\[ \text{Res}_{z=0} V^+(\alpha, x, z) V^{-}(\beta, x', z)^T = 0 \quad \text{for all } \alpha, \beta \in \text{supp} \tau. \quad (4.4) \]
Define $n \times n$ matrices $W^\pm(m)(\alpha, x)$ by the following generating series (cf. (4.3)):
\[ \sum_{m=0}^{\infty} W^\pm(m)_{ij}(\alpha, x)(\pm z)^{-m} = \varepsilon_{ji} z^{\delta_{ij}-1} (\exp(\mp \sum_{k=1}^{\infty} \frac{\partial}{\partial x^{(j)}_k} z^{-k} \tau_{\alpha \pm \alpha_{ij}}(x)) / \tau_{\alpha}(x)). \quad (4.5) \]
Note that

\[ W^{\pm(0)}(\alpha, x) = I_n, \]

\[ W^{\pm(1)}_{ij}(\alpha, x) = \begin{cases} \varepsilon_{ji} \tau_{\alpha+\alpha_{ij}} / \tau_{\alpha} & \text{if } i \neq j \\ \tau_{\alpha}^{-1} \frac{\partial \tau_{\alpha}}{\partial x^{(i)}} & \text{if } i = j, \end{cases} \]

We see from (4.3) that \( V^{\pm}(\alpha, x, z) \) can be written in the following form:

\[ V^{\pm}(\alpha, x, z) = \sum_{m=0}^{\infty} W^{\pm(m)}(\alpha, x)(\pm z)^{-m} R^{\pm}(\alpha, \pm z) S^{\pm}(x, z), \]

where

\[ R^{\pm}(\alpha, z) = \sum_{i=1}^{n} \varepsilon(\delta_i, \alpha) E_{ii}(\pm z)^{\pm(\delta_i|\alpha)}, \]

\[ S^{\pm}(x, z) = \sum_{i=1}^{n} e^{\pm \sum_{j=1}^{\infty} x^{(i)} z^{(j)}} E_{ii}. \]

Here \( E_{ij} \) stands for the \( n \times n \) matrix whose \((i, j)\) entry is 1 and all other entries are zero. Now let \( \partial = \sum_{j=1}^{n} \frac{\partial}{\partial x^{(j)}} \), then \( V^{\pm}(\alpha, x, z) \) can be written in terms of formal pseudo-differential operators (see [9] for more details). Let

\[ P^{\pm}(\alpha) \equiv P^{\pm}(\alpha, x, \partial) = I_n + \sum_{m=1}^{\infty} W^{\pm(m)}(\alpha, x) \partial^{-m}, \quad R^{\pm}(\alpha) = R^{\pm}(\alpha, \partial), \]

then

\[ V^{\pm}(\alpha, x, z) = P^{\pm}(\alpha) R^{\pm}(\alpha) S^{\pm}(x, z) \]

and one can prove that \( P^{-}(\alpha) = P^{+}(\alpha)^{-1} \) and the following Lemma:

**Proposition 4.1** Let \( \alpha, \beta \in \text{supp } \tau \), then \( P^{+}(\alpha) \) satisfies the Sato equations:

\[ \frac{\partial P^{+}(\alpha)}{\partial x^{(j)}_{k}} = -(P^{+}(\alpha) E_{jj} \partial^{k} P^{+}(\alpha)^{-1}) P^{+}(\alpha) \]

and \( P^{+}(\alpha), \ P^{+}(\beta) \) satisfy

\[ (P^{+}(\alpha) R^{+}(\alpha - \beta) P^{+}(\beta)^{-1})_{-} = 0 \text{ for all } \alpha, \beta \in \text{supp } \tau. \]

This is another formulation of the \( n \)-component KP hierarchy (see [9]). Introduce the following formal pseudo-differential operators \( L(\alpha), \ C^{(j)}(\alpha) \):

\[ L(\alpha) \equiv L(\alpha, x, \partial) = P^{+}(\alpha) \partial P^{+}(\alpha)^{-1}, \]

\[ C^{(j)}(\alpha) \equiv C^{(j)}(\alpha, x, \partial) = P^{+}(\alpha) E_{jj} P^{+}(\alpha)^{-1}. \]
then related to the Sato equation is the following linear system
\begin{align}
L(\alpha)V^+(\alpha,x,z) &= zV^+(\alpha,x,z), \\
C^{(i)}(\alpha)V^+(\alpha,x,z) &= V^+(\alpha,x,z)E_{ii}, \\
\frac{\partial V^+(\alpha,x,z)}{\partial x_k^{(i)}} &= (L(\alpha)^k C^{(i)}(\alpha))_+ V^+(\alpha,x,z).
\end{align}

To end this section we write down explicitly some of the Sato equations (4.12) on the matrix elements \( W^{(s)}_{ij} \) of the coefficients \( W^{(s)}(x) \) of the pseudo-differential operator
\[ P = P^+(\alpha) = I_n + \sum_{m=1}^{\infty} W^{(m)}(x)\partial^{-m}. \]

We shall write \( W = W^{(1)} \) and \( W_{ij} \) for \( W^{(1)}_{ij} \) to simplify notation, then the simplest Sato equation is
\[ \frac{\partial P}{\partial x_1^{(k)}} = [\partial E_{kk}, P] + [W, E_{kk}]P. \]

In particular we have for \( i \neq k \):
\[ \frac{\partial W_{ij}}{\partial x_1^{(k)}} = W_{ik}W_{kj} - \delta_{jk}W_{ij}^{(2)}. \]

The equation (4.16) is equivalent to the following equation for \( V = V^+(\alpha) \):
\[ \frac{\partial V}{\partial x_1^{(k)}} = (E_{kk}\partial + [W, E_{kk}])V. \]

5 The Darboux-Egoroff system

Define
\[ w_{ij}(x) = W^{(1)}_{ij}(0,x), \]
then from the previous section we know that \( w_{ij}(x) \) satisfies
\[ \frac{\partial w_{ij}(x)}{\partial x_1^{(k)}} = w_{ik}(x)w_{kj}(x) \quad i \neq k \neq j. \]

If we moreover assume that the wave function corresponds to a point \( W \in gr^{(2)} \), we also have
\[ \sum_{k=1}^{n} \frac{\partial w_{ij}(x)}{\partial x_1^{(k)}} = 0, \]
and
\[ w_{ij}((-)^{k+1}x_k^{(a)}) = w_{ji}(x_k^{(a)}). \]
This makes it possible to obtain solutions of the Darboux-Egoroff system, viz define
\[ \gamma_{ij}(x) = w_{ij}(x)|_{x_k^{(a)}=0} \text{ for all } k \geq 1, \ 1 \leq \ell \leq n, \]
then these \( \gamma_{ij} \) satisfy the equations (1.7). Thus we have obtained the main theorem of this paper.
Theorem 5.1 Let $W \in gr^{(2)} = GL_n(L)^{(2)} \cdot H_N^{(0)}$ and let $\tau(x) = \sum_{\alpha \in M} \tau_{\alpha}(x)e^\alpha$ be the corresponding tau-function. Then the
\[
\gamma_{ij}(x) = \epsilon_{ji} \left( \frac{\tau_{\delta_i - \delta_j}(x)}{\tau_0(x)} \right) x_{2k}^{(i)} = 0 \text{ for all } k \geq 1, 1 \leq \ell \leq n
\]
are solutions of the Darboux-Egoroff system (5.7) for $u_i = x_1^{(i)}$.

It is obvious that one can construct even more tau-functions that correspond to $W \in gr_0$ and which lead to solutions of the Darboux-Egoroff system. Namely, if we take a tau-function which comes from a $W \in gr^{(2)}$, then we can always multiply it with an element $e^\beta$ for $\beta \in M$. The $\tau_{\beta}(x)$ and the $\tau_{\beta + \delta_i - \delta_j}(x)$ of this new tau-function also lead to solutions of the Darboux-Egoroff system.

It is easy to see from theorem 3.2 that the wave functions satisfy
\[
\sum_{i=1}^{n} \frac{\partial V^{\pm}(\alpha, x, z)}{\partial x_1^{(i)}} = zV^{\pm}(\alpha, x, z) \tag{5.6}
\]
This means that we do not really have formal pseudo-differential operators, but rather formal matrix-valued Laurent series in $z^{-1}$. The Sato equation takes the following simple form. Let $P(z) = P^+(\alpha, x, z) = I + Wz^{-1} = \cdots$ then
\[
\frac{\partial P(z)}{\partial x_1^{(i)}} = -(P(z)E_{jj}P(z)^{-1}z^k)P(z).
\]
and equation (4.18) turns into
\[
\frac{\partial V^{+}(\alpha, x, z)}{\partial x_1^{(k)}} = (zE_{kk} + [W, E_{kk}])V^{+}(\alpha, x, z). \tag{5.7}
\]

Next let
\[
\Phi^{+}(x, z) = V^{\pm}(0, x, z)|_{x_2^{(i)} = 0 \text{ for all } k \geq 1, 1 \leq \ell \leq n}, \tag{5.8}
\]
then it is straightforward to check that
\[
\Phi^{-}(x, z) = \Phi^{+}(x, -z).
\]
Thus
\[
\text{Res}_z \Phi^{+}(x, z)\Phi^{+}(x', -z)^T = 0
\]
from which one deduces, when one takes $x = x'$, that
\[
\Phi^{+}(x, z)\Phi^{+}(x, -z)^T = I_n.
\]
Let $\Gamma(x) = (\gamma_{ij}(x))_{1 \leq i, j \leq n}$, then $\Phi(x, z) := \Phi^{+}(x, z)$ satisfies:
\[
\Phi(x, z)\Phi(x, -z)^T = I_n,
\]
\[
\sum_{j=1}^{n} \frac{\partial \Phi(x, z)}{\partial x_1^{(j)}} = z\Phi(x, z), \tag{5.9}
\]
\[
\frac{\partial \Phi(x, z)}{\partial x_1^{(k)}} = (zE_{kk} + [\Gamma(x), E_{kk}])\Phi(x, z),
\]
Following [4], we want solutions of this system for $z = 0$. However, just putting $z = 0$ in (5.9) does not make sense. There is a way to construct such solutions, viz. let $\tau = g(t)0$, with $g(t) = \sum_i A(i)t^i \in GL(L)^{(2)}$, so in particular $g(-t)^T = g(t)^{-1}$, then

$$\psi(g(t)t^{-1}e_j)\tau \neq 0 \quad \text{and} \quad \psi(g(t)t^ke_j)\tau = 0, \quad \text{for all } 1 \leq j \leq n, \ k \geq 0.$$ 

Using the fermionic fields we can rewrite this to

$$\text{Res}_z \sum_i \sum_{k=1}^n A(i)_{kj}z^{i-1}\psi^{+(k)}(z)\tau \neq 0 \quad \text{and} \quad \text{Res}_z \sum_i \sum_{k=1}^n A(i)_{kj}z^{i+\ell}\psi^{+(k)}(z)\tau = 0$$

for all $1 \leq j \leq n, \ell \geq 0$ and thus

$$\text{Res}_z z^{-1}V^+(0, x, z)g(z) \neq 0 \quad \text{and} \quad \text{Res}_z z^{\ell}V^+(0, x, z)g(z) = 0 \quad \text{for all } \ell \geq 0.$$ 

Now define

$$\Psi(x, z) := z^{-1}\Phi(x, z)g(z),$$

then this satisfies

$$\Psi(x, z)\Psi(x, -z)^T = -z^{-2}I_n,$$

$$\sum_{j=1}^n \frac{\partial\Psi(x, z)}{\partial x^{(j)}}(j)^k = z\Psi(x, z),$$

$$\frac{\partial\Psi(x, z)}{\partial x^{(k)}} = (zE_{kk} + [\Gamma(x), E_{kk}])\Psi(x, z),$$

$$\text{Res}_z z^{\ell}\Psi(x, z) = 0 \quad \text{for all } \ell > 0.$$ 

We thus get (c.f. [4],[5]):

**Proposition 5.1** Let $\Psi(x, z)$ be constructed as above. Define

$$\psi(x) = (\psi_{ij}(x))_{1 \leq i, j \leq n} := \text{Res}_z \Psi(x, z),$$

Then these $\psi_{ij}$’s satisfy the equations

$$\frac{\partial\psi_{ij}}{\partial x^{(k)}} = \gamma_{ik}\psi_{kj}, \quad k \neq i, \quad \sum_{k=1}^n \frac{\partial\psi_{ij}}{\partial x^{(k)}} = 0. \quad (5.11)$$

with $\gamma_{ij}$ given by (5.5) and the formula’s

$$h_i = \psi_{i1},$$

$$\eta_{\alpha\beta} = \sum_{i=1}^n \psi_{i\alpha}\psi_{i\beta} = \delta_{\alpha\beta},$$

$$\frac{\partial h_{\alpha}}{\partial x^{(i)}} = \psi_{i1}\psi_{i\alpha}, \quad \text{Res}_z z^{\ell}\psi_{i1} = 0 \quad \text{for all } \ell \geq 0.$$ 

$$c_{\alpha\beta}^{\gamma} = c_{\alpha\beta\gamma} = \sum_{i=1}^n \psi_{i\alpha}\psi_{i\beta}\psi_{i\gamma},$$

determine (locally) a semisimple Frobenius manifold on the domain $x^{(i)}_1 \neq x^{(j)}_1$ and $\psi_{11}\psi_{21} \cdots \psi_{n1} \neq 0$. 

Define
\[ \Theta(x, z)' := z^2 \sum_{i=1}^{n} \psi_{1i} E_{ii} \Psi(x, z), \]
\[ \Theta(x, z) = (\theta_1(x, z) \theta_2(x, z) \theta_3(x, z) \cdots \theta_n(x, z)) := z (\psi_{21} \psi_{31} \cdots \psi_{n1}) \Psi(x, z), \]
then it is straightforward to check that
\[ \text{Res}_z z^k \Theta(x, z) = 0, \quad \text{Res}_z z^{k-1} \Theta(x, z)' = 0 \quad \text{for all} \quad k \geq 0, \]
\[ \Theta(x, z)' \Theta(x, -z)' = -z^2 \sum_{i=1}^{n} h_i^2(x) E_{ii}, \]
\[ z^{-1} \Theta(x, z)' \Theta(x, -z)' = (h_1^2(x) h_2^2(x) h_3^2(x) \cdots h_n^2(x))^T. \]
From which we deduce that the flat coordinates \( t^i \) are given by
\[ \theta_j(x, z) = \delta_{j,1} + t_j(x) z + \sum_{k=2}^{\infty} \theta_j^{(k)}(x) z^k. \]
These \( \theta_j(x, z) \) are the deformed flat coordinates, see e.g. [6]. So we are in the situation of the paper [1] and we can construct the prepotential \( F(t(x)) \). Using the formula's (5.10) and (5.11) one calculates that
\[ \frac{\partial^2 \Theta(x, z)}{\partial x_1^i \partial x_1^j} = \Gamma_{ij}^k(x) \frac{\partial \Theta(x, z)}{\partial x_1^k}, \quad i \neq j \]
\[ \frac{\partial^2 \Theta(x, z)}{\partial x_1^i \partial x_1^j} = \sum_{j=1}^{n} \Gamma_{ji}^j(x) \frac{\partial \Theta(x, z)}{\partial x_1^j} + z \frac{\partial^2 \Theta(x, z)}{\partial x_1^i \partial x_1^j}, \]
where the Christoffel symbols are given by (1.10) and hence that
\[ \frac{\partial^2 \Theta(x, z)}{\partial t^k \partial t^l} = \sum_{m=1}^{n} c_{k\ell m}(x) z \frac{\partial \Theta(x, z)}{\partial t^m}. \]
Since \( \frac{\partial \Theta(x, z)}{\partial t^m} \) is a linear combination of \( \frac{\partial \Theta(x, z)}{\partial x_1^{(k)}} \)'s,
\[ z^{-1} \frac{\partial \Theta(x, z)}{\partial t^k} \Theta(x, -z)' \]
is independent of \( z \), which means that all coefficients, except the constant coefficient, are zero. In particularly the coefficient of \( z^2 \) gives:
\[ \theta_m^{(2)}(x) = -\frac{\partial \theta_1^{(3)}(x)}{\partial t^m} + \sum_{i=1}^{n} t_i \frac{\partial \theta_i^{(2)}(x)}{\partial t^m}. \]
The coefficient of \( z^2 \) of (5.16) leads to
\[ \frac{\partial^2 \theta_m^{(2)}(x)}{\partial t^k \partial t^l} = c_{k\ell m}(x), \]
\[ \frac{\partial F(x)}{\partial t^m} = \theta_m^{(2)}(x) \] and we obtain the Theorem of [1] for the \( GL(L)^{(2)} \)-group orbit.
Theorem 5.2 The function $F(x) = F(t(x), x)$ defined by

$$F(x) = -\frac{1}{2}\theta_1^{(3)}(x) + \frac{1}{2} \sum_{i=1}^{n} t^i(x)\theta_1^{(2)}(x)$$

satisfies equation (1.5).

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