Observation of topological edge modes in a quasi-periodic acoustic waveguide

David J. Apigo, Wenting Cheng, Kyle F. Dobiszewski, Emil Prodan, and Camelia Prodan

1Department of Physics, New Jersey Institute of Technology, Newark, NJ, USA
2Albert Dornsaim Honors College, New Jersey Institute of Technology, Newark, NJ, USA
3Department of Physics, Yeshiva University, New York, NY, USA

Topological edge modes are generated in an acoustic wave-guide by a simple quasi-periodic pattern and without any additional fine-tuning. The quasi-periodic wave-guides are characterized experimentally by standard acoustic measurements, a discrete effective theoretical model, and via a finite element approach utilizing COMSOL Multiphysics. The topological labels of the gaps are mapped using the integrated density of states with guidance from K-theory. The experiments and the COMSOL simulations confirm the existence of topological edge modes in these gaps.

The ideas based on topological concepts have revolutionized the field of condensed matter and led to the discovery of topological insulators and superconductors. The latter have been classified at the end of the previous decade and a table of strong topological phases has been conjectured. One of their common characteristics is the spontaneous emergence of disorder-immune boundary modes, whenever a sample is halved. Physics akin to that of topological condensed matter systems has been also predicted in classical wave-supporting materials and many examples of topological meta-materials have been reported in the literature.

In the same time, it has been pointed out that the periodic table of topological systems is highly enhanced if more complex systems are considered, such as the quasi-periodic or quasi-crystalline ones. K-theoretic arguments were used to show that every gap in the bulk spectrum of a quasi-periodic Hamiltonian is topological, in the sense that it will be completely filled by boundary spectrum under any boundary condition. This leads to genuine topological boundary states which cannot be removed by any tempering of the edge. The practical value of the findings is that the quasi-periodic Hamiltonians typically display a large number of gaps and, since they are all topological, we can generate localized wave-modes in both space and energy by simply halving the system.

Another finding in was that the statements just formulated above are independent of the exact form of the Hamiltonians. As such, the topological edge modes in quasi-periodic systems require minimal tuning and, for this reason, they can be easily scaled up or down to fit various practical applications.

In this work we put these general principles at work in a quasi-periodic acoustic wave-guide built out of acoustic chambers connected through small openings. Examining first a single chamber and dimer of two coupled chambers, we detect experimentally a frequency range where the chambers behave as single-mode resonators. Using this input, we build an effective discrete dynamical matrix which enable us to compute the topological gaps. Experimental measurements as well as COMSOL simulations confirm the existence of topological edge modes in one such spectral gap.

The quasi-periodic acoustic wave-guide is built out of connected acoustic chambers as illustrated in Fig. 1. The chambers are identical except for their lengths, which are modulated according to the algorithm:

\[ L_n = L_{\text{avg}} + \Delta L \left( \sin \left( (n+1) \theta + \phi \right) - \sin(n \theta + \phi) \right). \]  

To make the labels meaningful, we assume that the wave-guide is centered on the \( n = 0 \) chamber. In this, \( \theta \) is an angle incommensurate with 2\( \pi \), which will be kept fixed, and \( \phi \) is the phason, which should be let to vary. For example, a simple re-labeling \( n \to n + m \), which corresponds to re-centering of the wave-guide, will change \( \phi \) into \( (\phi + m \theta) \mod 2 \pi \). Since \( \theta \) is incommensurate, these re-labelings alone will sample the phason densely in \([0,2\pi]\) interval. \( L_{\text{avg}} \) in (1) is the average length of the acoustic chambers and \( \Delta L \) sets the magnitude of the fluctuations in \( L_n \).

The acoustic chambers were 3D printed out of polylactic acid (PLA) using an Ultimaker 3. Sinusoidal signals were produced by a Rigol DG1022 function generator, amplified by a Crown XLS 2502 power amplifier with the gain set to 6, and then applied on a CUI Inc. GF0501 speaker (diameter, \( d_s = 50 \) mm) placed at the edge of one acoustic chamber. A PCB Piezotronics Model-376C10 microphone and a PCB Piezotronics Model-485B12 power conditioner acquired the acoustic signals at the opposite edge of the system or at certain inserts. Their output is read by a National Instruments USB-6112 data acquisition box and then is further processed on a computer. The experimental setup is shown in Fig. 1 together with the relevant parameters of the system.

As a preliminary analysis, we have performed measurements on a single acoustic chamber of length \( L_{\text{avg}} \) and on a dimer of such identical chambers, as shown in Fig. 2(a). The results are reported in Fig. 2(b,c). The measurements for a single chamber indicate a resonant peak around \( f_0 = 1.1 \) kHz followed by a large spectral gap. For the dimer, the results display two promi-
FIG. 1. Experimental setup. 3D printed PLA parts interlock to connect acoustic resonators, with the basic acoustic chamber shown in the insert. The wave-guide is mirrored relative to the mid-point where a domain wall is formed as indicated by the dashed line in the cross-section. In the shown configuration, a speaker is placed at the left-end and a piezoelectric microphone is inserted into a porthole. Various portholes exist along the wave-guide where the microphone can be inserted. The portholes that are not used are sealed with putty. A cross-section of the wave-guide is indicated below in gray. The larger/smaller diameters of the resonant chambers are $d_l = 48$ and $d_s = 30$ mm, respectively. The lengths $L_n$ were generated with (1) and their average was fixed at $L_{avg} = 40$ mm. For experimentation, the speaker was placed at Position 1 for bulk measurements and Position 4 for DW measurements. The microphone was placed at Position 2 and 3 for bulk and DW measurements, respectively.

Recent sharp peaks at $\nu_1 = 1.6$ kHz and $\nu_2 = 2.5$ kHz and broader peak centered at 5.5 kHz. The lowest $\nu_{1,2}$ peaks must originate from the coupling of the sharp $\nu_0$ mode seen in Fig. 2(b). This leads us to the important conclusion that the collective acoustic modes within the frequency range $[0.8, 4.5]$ kHz can be understood from the coupling of single-mode acoustic resonators, hence we can use the analysis developed in [27] for guidance. To determine the effective parameters, we write the most general coupling matrix between two identical modes of pulsation $\omega_0 = 2\pi v_0 a_0 \sqrt{1 + \alpha + \beta}$, leading to the splitting $\omega = \omega_0 \sqrt{1 + \alpha \pm \beta}$. Comparing with the data from Fig. 2(a,b), we obtain $\alpha = 2.6$ and $\beta = 1.5$, which indicate a strong coupling between the modes. We now consider the aperiodic wave-guide and set $\Delta L = 0.2l_{avg}$ and $\theta = \frac{2\pi}{32}$ in (1). This particular irrational fraction of $2\pi$ accepts a good rational approximation $\theta = \frac{6\pi}{32} + O(10^{-3})$, which will be used in the experiment. The fluctuations in the length of the acoustic chambers induce fluctuations in the pulsation $\omega_0$,

FIG. 2. a) The experimental setup for a coupled resonator test with function generator, power amplifier, signal conditioner, and computer not shown. c) Experimental signal acquired on an individual resonator (cross-section indicated) demonstrating peaks where resonance occurs. d) Experimental signal acquired on coupled resonators (cross-section indicated) demonstrating two prominent peaks where resonance occurs.

FIG. 3. Bulk resonant spectrum of the dynamical matrix (2) as a function of $\theta$. The red lines indicates the value $\theta = \frac{6\pi}{32}$ used in the experiments. The tuples seen inside the prominent gaps are the gap labels which indicate that both gaps are topological.
FIG. 4. COMSOL simulation of the air pressure in the acoustic wave-guide, referenced from the atmospheric pressure. Hard boundary conditions are considered along the walls and a sound source is placed at the left. The frequency of the sound source is indicated below each intensity plot. Note that green color indicates absence of sound.

$$\omega_0 \to \omega_n = \gamma_n \omega_0, \quad \gamma_n = \frac{L_{\text{avg}}}{L_n}.$$ 

The scaling property of the acoustic wave equation indicates that the coupling constants \( \alpha \) and \( \beta \) have to fluctuate in a similar manner. Hence, we write the dispersion equation of the collective modes as

$$\omega^2 \psi = D \psi,$$

where \( \psi = \sum_n A_n |n\rangle \) encodes the configuration of the coupled single-mode resonators and the dynamical matrix \( D \) takes form:

$$D = \omega_0^2 \sum_n \left( \gamma_n^2 (1+2\alpha)|n\rangle\langle n| + \beta_n |n\rangle\langle n+1| + |n+1\rangle\langle n| \right),$$

with \( \beta_n = \frac{L_{\text{avg}}^2}{L_n} \beta \). The bulk resonant spectrum of this dynamical matrix is reported in Fig. 3.

As explained in [27], \( D \) belongs to the algebra generated by diagonal operators of the form \( \sum_n \Xi(n\theta)|n\rangle\langle n| \), with \( \Xi \) a continuous function over the unit circle, together with the shift operator \( S \). This algebra was shown in [27] to be isomorphic to the algebra of the non-commutative torus, which explains the similarity of the spectrum in Fig. 3 and the Hofstadter butterfly [28]. Furthermore, the \( K \)-theory of this algebra is known explicitly. In particular, if \( G \) is a gap in the spectrum and \( P_G \) is the projection onto the spectrum below \( G \), then the stable homotopies of \( P_G \) define a class in the \( k_{G'} \) group and such class is uniquely labeled by two integers \((n, m)\), related to the trace per volume of \( P_G \) as:

$$\text{Tr}_V[P_G] = n + m\theta,$$

which, at its turn, is equal to the integrated density of states (IDS) evaluated in the gap \( G \). The bulk-boundary principle for these quasi-periodic patterns states that the gap \( G \) is topological whenever \( m \neq 0 \) [27]. Since \( \text{IDS} \leq 1 \), we can see that \( m \) cannot be zero, unless the states are fully populated or empty. The gap labels, as computed from (3), are reported in Fig. 3 for the prominent spectral gaps.

We now turn our attention to the configuration of the wave-guide with a domain wall (DW), which is generated by mirror-reflection of the chambers with \( n \geq 0 \). This configuration was analyzed with COMSOL finite element simulations and the results are reported in Fig. 4 for different frequencies. Appropriate hard boundary conditions were considered along the walls for the sound wave equation and a source was placed at the left end. In these simulations, depending on the frequency of the source, one can see extended solutions, filling the entire wave-guide, as well as localized solutions at either the

FIG. 5. a) Experimental measurements for a DW configuration of the wave-guide. The plots represent normalized RMS voltage outputted by the microphone when the speaker/microphone were placed at Positions 1/2 (black line) and Positions 4/3 (red and blue line) (see Fig. 1). The red/blue measurements were performed on wave-guides generated with \( \phi = 0 \) and \( \phi = 3\theta \) in Eq. 1, respectively. The grayed regions suggest where the bulk spectrum is located. b) Dependence of the DW mode’s frequency on the phason \( \phi \), as derived from COMSOL simulations. The theoretical bulk spectrum is indicated by the gray bands. The red/blue points relate to the red/blue curves in panel (a).
left edge or at the domain wall. The results indicate a spectral bulk gap between 3.3 kHz and 4.4 kHz, consistent with the top gap mapped in Fig. 3. The most interesting feature is the DW-mode observed at $\nu = 4.13$ kHz, which is the predicted topological state.

The experimental measurements for the domain-wall configuration are reported in Figure 5(a). Scans with the source on the left-end and with the microphone at the interface were performed in the frequency range 500 to 5000 Hz in 2 Hz steps. In this configuration, the microphone will detect only the extended sound modes and, as such, we can generate a map of the bulk resonant spectrum. The result of this scan is shown as the black curve in Fig. 5(a). The data confirms the existence of a bulk spectral gap between 3.0 and 4.3 kHz, in good agreement with the COMSOL simulations, and a smaller gap below it. Scans with finer 0.1 Hz steps were performed in order to exclude existence of any spectrum inside these gaps. We believe they are exactly the spectral gaps labeled in Fig. 5.

To detect the DW-modes, the source was placed above a porthole located closer to the interface (Position 4 in Fig. 1) and the frequencies were scanned within the bulk spectral gap. The result is shown as the red curve in Fig. 5(a), which confirms the existence of the DW acoustic mode at a frequency consistent with the COMSOL simulation reported in Fig. 4. In order to demonstrate the topological nature of this DW-mode, we performed COMSOL simulations with varying phason values $\phi$.

This is precisely what we see in the simulations reported in Fig. 5(b). To confirm this predicted behavior, an additional experimental scan was performed with a waveguide generated with $\phi = 3\theta$ and the result is shown as the blue curve in Fig. 5(a). As one can see, the frequency of the DW mode has shifted as predicted by the COMSOL simulation.

The simulations in Fig. 4 indicate an extremely well localized DW mode. To probe experimentally the spatial profile of the mode, we have performed frequency scans with the microphone moved to the neighboring chambers from the interface. The results are reported in Fig. 6 and they indeed confirm the localization of the DW mode on the interface.

In conclusion, aperiodic patternings of acoustic waveguides were found to induce large bulk spectral gaps which, as theory predicted, are topological without performing any fine tuning. DW modes at the interface of two mirror-reflected wave-guides have been experimentally detected and characterized. Additionally, COMSOL simulations revealed the topological signature of these modes, namely, formation of chiral bands upon the variation of the phason parameter.

Our findings confirm once again that a simple quasi-periodic patterning of a wave-guide will produce topological edge modes. Such procedure can be easily applied to any scales to fit many possible applications. For example, using the well-known scaling rule for the sound wave equation, we find that we can produce topological edge modes in the MHz range by simply changing our length scale from millimeters to microns. Those type of wave-guides can be fabricated with conventional micro/nanofabrication techniques and preliminary COMSOL simulations indicated that they will perform as expected.

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