RAMANUJAN-NAGELL’S EQUATION AND SOME OF ITS VARIATIONS

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Abstract. The paper consists of two parts. The aim of the first, and main, part is to explain, in an elementary way, Hasse’s proof of Ramanujan-Nagell’s Theorem. In the second part, we formulate some natural extensions of Ramanujan-Nagell’s equation.

1. Ramanujan-Nagell’s Equation

Marin Mersenne, a French monk and mathematician, conjectured that the only integers \( n \leq 257 \) for which \( M_n = 2^n - 1 \) is a prime are \( n = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127, \) and \( 257. \) Since computers were not available at that time, it was hard to check his conjecture, actually we know now that Mersenne’s list is neither exhaustive, since \( M_{61}, M_{89}, M_{107} \) are primes, nor free from mistakes, since \( M_{67} \) and \( M_{257} \) are composites. So the question is to characterize the primes \( p \) for which \( M_p \) are primes. Lenstra, Pomerance, and Wagstaff, [6], conjectured that there are infinite primes \( p \) for which \( M_p \) is prime. The numbers \( M_n \) are called Mersenne numbers.

Perfect numbers is another interesting class of integers. An integer is called perfect if it is equal to the sum of its positive divisors other than the number itself. For example 6 is a perfect number, since \( 6 = 1 + 2 + 3. \) Another example is given by \( 28 = 1 + 2 + 4 + 7 + 14. \) All known perfect numbers are even and it is not known whether odd perfect numbers exist or not. One way to construct perfect numbers is via Mersenne primes, more precisely, we have the following

**Proposition 1.** Let \( p \) be a prime number. If the Mersenne number \( M_p = 2^p - 1 \) is prime then \( \frac{M_p (M_p + 1)}{2} \) is perfect.

Proof. Since \( 2^p - 1 \) is prime, the positive divisors of \( 2^{p-1} (2^p - 1) \) less than \( 2^{p-1} (2^p - 1) \) are

\[ 1, 2, ..., 2^{p-1}, (2^p - 1), 2 (2^p - 1), 2^2 (2^p - 1), ..., 2^{p-2} (2^p - 1). \]

Therefore the sum of the positive divisors of \( \frac{M_p (M_p + 1)}{2} = 2^{p-1} (2^p - 1) \) other than \( 2^{p-1} (2^p - 1) \) is given by

\[
\sum_{i=0}^{p-1} 2^i + (2^p - 1) \sum_{i=0}^{p-2} 2^i = (2^p - 1) + (2^p - 1) (2^{p-1} - 1)
\]

\[ = 2^{p-1} (2^p - 1). \]

Hence the proposition. □

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Thus to every Mersenne prime is associated a perfect even number. Actually, it can easily be seen that every even perfect number is of the form $\frac{M_p(M_p+1)}{2}$, where $M_p$ is some Mersenne prime, [3]. Since the largest known Mersenne prime is $2^{57885161} - 1$, by Proposition 1 the largest known perfect number is $2^{57885160}(2^{57885161} - 1)$.

Integers of the form $\frac{n(n+1)}{2}$ are called triangular numbers. From what was said above, every even perfect number is triangular. It was Ramanujan who first conjectured that only a finite number of Mersenne numbers can be triangular. Answering Ramanujan’s conjecture affirmatively is equivalent to proving that the equation

$$x^2 + 7 = 2^n,$$

has a finite number of integer solutions.

The first affirmative answer to Ramanujan’s conjecture, now called Ramanujan-Nagell’s Theorem, was given by T. Nagell, [5].

**Theorem** (Ramanujan-Nagell’s Theorem). The only integer solutions of the equation

$$x^2 + 7 = 2^n$$

are $(x, n) = (\pm 1, 3), (\pm 3, 4), (\pm 5, 5), (\pm 11, 7)$ and $(\pm 181, 15)$.

Hence, only a finite number of Mersenne numbers can be triangular.

The elegant, and very elementary, proof given here is due to Hasse [2] (we follow Hasse’s proof given in Mordell [4]). To make the paper easy to read, and self contained, we will include the following

**Proposition 2.** Let $\mathcal{O}_Q(\sqrt{-7})$ be the ring of algebraic integers in the quadratic field $Q(\sqrt{-7})$ and $U_Q(\sqrt{-7})$ the group of units in $\mathcal{O}_Q(\sqrt{-7})$. Then

1. $\mathcal{O}_Q(\sqrt{-7}) = \left\{ \frac{a + b\sqrt{-7}}{2} \mid a, b \in \mathbb{Z} \text{ and } a \equiv b \pmod{2} \right\}$.
2. $\mathcal{O}_Q(\sqrt{-7})$ is a unique factorization domain.
3. $U_Q(\sqrt{-7}) = \{\pm 1\}$.

**Proof.**

1. Let $\alpha = u + v\sqrt{-7} \in \mathcal{O}_Q(\sqrt{-7})$. If $v = 0$, then $\alpha = u \in \mathbb{Q} \cap \mathcal{O}_Q(\sqrt{-7}) = \mathbb{Z}$.

Hence, we can assume that $v \neq 0$. The minimal polynomial $\min_{\alpha, Q}(t)$ of $\alpha$ over $\mathbb{Q}$ is given by,

$$\min_{\alpha, Q}(t) = (t - u - v\sqrt{-7})(t - u + v\sqrt{-7}) = t^2 - 2ut + u^2 + 7v^2.$$

Since $\alpha$ is an algebraic integer over $\mathbb{Q}$, then $\min_{\alpha, Q}(t) \in \mathbb{Z}[t]$.

Hence

$$2u \in \mathbb{Z} \text{ and } u^2 + 7v^2 \in \mathbb{Z}.$$
Therefore
\[(2u)^2 - 4(u^2 + 7v^2) = -28v^2 = -7(2v)^2 \in \mathbb{Z}.
\]
Consequently
\[2v \in \mathbb{Z}.
\]
Put
\[
\begin{cases}
  a = 2u \\
  b = 2v.
\end{cases}
\]
Since \(u^2 + 7v^2 \in \mathbb{Z}\), and since
\[a^2 + 7b^2 = 4(u^2 + 7v^2),
\]
we deduce that
\[a^2 \equiv b^2 \pmod{4}.
\]
Hence, either \(a\) and \(b\) are both odd, or both even. We conclude that
\[
\mathfrak{o}_{\mathbb{Q}(\sqrt{-7})} = \left\{ \frac{a + b\sqrt{-7}}{2} \mid a, b \in \mathbb{Z}, \ a \equiv b \pmod{2} \right\}.
\]
(2) Since every Euclidean domain is a principal ideal domain, hence a unique factorization domain, it is enough to prove that \(\mathfrak{o}_{\mathbb{Q}(\sqrt{-7})}\) is Euclidian.
For \(\frac{a + b\sqrt{-7}}{2} \in \mathfrak{o}_{\mathbb{Q}(\sqrt{-7})}\), consider the function \(N : \mathfrak{o}_{\mathbb{Q}(\sqrt{-7})} \setminus \{0\} \to \mathbb{N}\), given by
\[N\left(\frac{a + b\sqrt{-7}}{2}\right) = \frac{a^2 + 7b^2}{4}.
\]
Note that since \(\alpha\) is a nonzero algebraic integer, then \(N(\alpha)\) is a positive rational integer, i.e., \(N(\alpha) \in \mathbb{N}\). Hence if \(\alpha, \beta\) are elements of \(\mathfrak{o}_{\mathbb{Q}(\sqrt{-7})} \setminus \{0\}\), then
\[N(\alpha) \leq N(\alpha)N(\beta) = N(\alpha\beta).
\]
Let \(\alpha\) be an element of \(\mathfrak{o}_{\mathbb{Q}(\sqrt{-7})}\), \(\beta\) an element of \(\mathfrak{o}_{\mathbb{Q}(\sqrt{-7})} \setminus \{0\}\) and let \(\gamma = \frac{\alpha}{\beta} = c + d\sqrt{-7}\), where \(c\) and \(d\) are rational numbers. Let \(m\) and \(n\) be rational integers (elements in \(\mathbb{Z}\)), such that
\[(1.2) \quad |2d - n| < \frac{1}{2} \quad \text{and} \quad \left| c - m - \frac{n}{2} \right| < \frac{1}{2}.
\]
Then
\[
\alpha = \beta \left( c + d\sqrt{-7} \right)
= \beta \left( \left( c - m - \frac{n}{2} \right) + m + \frac{n}{2} \right) + \beta \left( \left( d - \frac{n}{2} \right) + \frac{n}{2} \right) \sqrt{-7}
= \beta \left( m + n \left( \frac{1 + \sqrt{-7}}{2} \right) \right) + \beta \left( \left( c - m - \frac{n}{2} \right) + \left( d - \frac{n}{2} \right) \sqrt{-7} \right).
\]
Put
\[q = m + n \left( \frac{1 + \sqrt{-7}}{2} \right), \quad r = \beta \left( \left( c - m - \frac{n}{2} \right) + \left( d - \frac{n}{2} \right) \sqrt{-7} \right).\]
Since \(\alpha, \beta\) and \(q\) are in \(\mathfrak{o}_\sqrt{-7}\), it follows that \(r = \alpha - \beta q \in \mathfrak{o}_\sqrt{-7}\).
Suppose that \(r \neq 0\), then, using (1.2), we get
\[
N(r) = N\left(\beta \left(\left(c - m - \frac{n}{2}\right) + \left(d - \frac{n}{2}\right)\sqrt{-7}\right)\right)
= N(\beta) N\left(\left(c - m - \frac{n}{2}\right) + \left(d - \frac{n}{2}\right)\sqrt{-7}\right)
= \left[\left(c - m - \frac{n}{2}\right)^2 + 7\left(d - \frac{n}{2}\right)^2\right] N(\beta)
< \left(\frac{1}{4} + \frac{7}{16}\right) N(\beta)
< N(\beta).
\]
Therefore \(\mathfrak{o}_\sqrt{-7}\) is Euclidean. Consequently \(\mathfrak{o}_\sqrt{-7}\) is a unique factorization domain.

(3) It is easy to see that an element \(u = \frac{a + b\sqrt{-7}}{2} \in \mathfrak{o}_\sqrt{-7}\) is a unit, i.e., invertible in \(\mathfrak{o}_\sqrt{-7}\), if and only \(N(u) = \frac{a^2 + 7b^2}{4} = 1\). But
\[
\frac{a^2 + 7b^2}{4} = 1 \text{ and } a, b \in \mathbb{Z} \iff a = \pm 2 \text{ and } b = 0.
\]
Hence
\[
U_{\sqrt{-7}} = \{\pm 1\}.
\]
\[\square\]

**Lemma 1.** If \(n = 7^l s\), where \(7 \nmid s\), then
\[
(1 + \sqrt{-7})^n \equiv 1 + n\sqrt{-7} \pmod{7^{l+1}}.
\]

**Proof.** Let us prove by induction that for all \(j \geq 1\),
\[
(1 + \sqrt{-7})^{7^j} \equiv 1 + 7^j\sqrt{-7} \pmod{7^{j+1}}
\]
From
\[
\binom{7}{k} = 7 \frac{1}{k} \left(\frac{6}{k - 1}\right),
\]
we deduce that for \(2 \leq k < 7\),
\[
\frac{6}{k - 1} \equiv 0 \pmod{k}.
\]
Hence for \(2 \leq k \leq 7\),
\[
\binom{7}{k} (\sqrt{-7})^k = 7 \frac{6}{k - 1} (\sqrt{-7})^k
\equiv 0 \pmod{7^2}.
\]
Consequently
\[
(1 + \sqrt{-7})^7 = \sum_{k=0}^{7} \binom{7}{k} (\sqrt{-7})^k
\equiv 1 + 7\sqrt{-7} \pmod{7^2}.
\]
Suppose

\[(1 + \sqrt{-7})^7 \equiv 1 + 7^k \sqrt{-7} \pmod{7^{k+1}}.\]

The congruence \((1.3)\) is equivalent to

\[(1 + \sqrt{-7})^{7^k} = 1 + 7^k \sqrt{-7} + 7^{k+1} \beta,\]

where \(\beta\) is an element of \(\mathcal{O}_Q(\sqrt{-7})\). Then

\[(1 + \sqrt{-7})^{7^{k+1}} = \left[ (1 + \sqrt{-7})^{7^k} \right]^7 = (1 + 7^k \sqrt{-7} + 7^{k+1} \beta)^7 \equiv 0 \pmod{7^{k+2}}\]

\[= (1 + 7^k \sqrt{-7})^7 + \sum_{j=1}^{7} \binom{7}{j} (7^{k+1} \beta)^j (1 + 7^k \sqrt{-7})^{7-j} \equiv (1 + 7^k \sqrt{-7})^7 \pmod{7^{k+2}}.\]

But

\[(1 + 7^k \sqrt{-7})^7 = \sum_{j=0}^{7} \binom{7}{j} (7^k \sqrt{-7})^j \equiv 0 \pmod{7^{k+2}}\]

\[= 1 + \binom{7}{1} (7^k \sqrt{-7}) + \sum_{j=2}^{7} \binom{7}{j} (7^k \sqrt{-7})^j \equiv 1 + 7^{k+1} \sqrt{-7} \pmod{7^{k+2}}.\]

Hence

\[(1 + \sqrt{-7})^{7^{k+1}} \equiv 1 + 7^{k+1} \sqrt{-7} \pmod{7^{k+2}}.\]

Therefore for all positive integers \(l\),

\[(1 + \sqrt{-7})^{7^l} \equiv 1 + 7^l \sqrt{-7} \pmod{7^{l+1}}.\]

Consequently

\[(1 + \sqrt{-7})^n = \left[ (1 + \sqrt{-7})^{7^l} \right]^s \equiv (1 + 7^l \sqrt{-7})^s \pmod{7^{l+1}}.\]
The Lemma follows from
\[
(1 + 7\sqrt{-7})^s = \sum_{j=0}^{7} \binom{s}{j} (7\sqrt{-7})^j
\]
\[
= 1 + \binom{s}{1} 7\sqrt{-7} + \sum_{j=2}^{7} \binom{s}{j} (7\sqrt{-7})^j
\]
\[
\equiv 1 + 7s\sqrt{-7} \pmod{7^{l+1}}
\]
\[
\equiv 1 + n\sqrt{-7} \pmod{7^{l+1}}.
\]

\[\Box\]

Proof. (of Ramanujan-Nagell’s Theorem) If \( n \) is even, then from the factorization
\[ 7 = 2^n - x^2 \]
\[ = (2^{\frac{n}{2}} - x)(2^{\frac{n}{2}} + x) \]
we deduce that \( 2^{\frac{n}{2}} + x = 7 \) and \( 2^{\frac{n}{2}} - x = 1 \). Hence \( 2^{\frac{n}{2}+1} = 2^3 \). Consequently, \( n = 4 \).

By substituting 4 for \( n \) in the equation, we get
\[ x^2 = 9. \]

So \((x, n) = (\pm 3, 4)\) is a solution.

From now on we assume that \( n \) is odd. If \( n = 3 \), then \( x = \pm 1 \), so \((x, n) = (\pm 1, 3)\) is another solution. Therefore we can assume that \( n \) odd and \( n \geq 5 \).

If \( x \) is a solution to the equation (1.1), then it is necessarily odd, hence \( x^2 + 7 \equiv 0 \pmod{4} \). If we put \( m = n - 2 \geq 3 \), then the equation (1.1) is equivalent to
\[ (1.4) \quad \frac{x^2 + 7}{4} = 2^m. \]

Using
\[ 2 = \left( \frac{1 + \sqrt{-7}}{2} \right) \left( \frac{1 - \sqrt{-7}}{2} \right) \]
we can write the equation (1.4) as
\[ (1.5) \quad \left( \frac{x + \sqrt{-7}}{2} \right) \left( \frac{x - \sqrt{-7}}{2} \right) = \left( \frac{1 + \sqrt{-7}}{2} \right)^m \left( \frac{1 - \sqrt{-7}}{2} \right)^m. \]

It is easy to see that \( \frac{1 + \sqrt{-7}}{2} \) and \( \frac{1 - \sqrt{-7}}{2} \) are irreducible, hence primes since by Proposition 2, \( \mathcal{O}_Q(\sqrt{-7}) \) is a unique factorization domain. Hence, from equation (1.5) either \( \frac{1 + \sqrt{-7}}{2} \) divides \( \frac{x + \sqrt{-7}}{2} \) or \( \frac{x - \sqrt{-7}}{2} \) but not both, for if it was the case then \( \frac{1 + \sqrt{-7}}{2} \) will divide \( \sqrt{-7} \). So there exist \( \frac{a + b\sqrt{-7}}{2} \in \mathcal{O}_Q(\sqrt{-7}) \) such that
\[ \sqrt{-7} = \frac{1 + \sqrt{-7}}{2} \left( \frac{a + b\sqrt{-7}}{2} \right). \]

By taking the norms we get
\[ (1.6) \quad 7 = 2 \left( \frac{a^2 + 7b^2}{4} \right). \]

A contradiction, since the equation (1.6) has no integer solutions. The same argument applies to \( \frac{1 - \sqrt{-7}}{2} \).
Actually we can prove more, namely \( \delta = \gcd \left( \frac{x + \sqrt{-7}}{2}, \frac{x - \sqrt{-7}}{2} \right) \) is a unit. The fact that \( \delta \mid \frac{x + \sqrt{-7}}{2} \) and \( \delta \mid \frac{x - \sqrt{-7}}{2} \), implies that \( \delta \mid x \) and \( \delta \mid \sqrt{-7} \). So, there exists \( \gamma \) and \( \mu \) in \( \mathfrak{o}_{Q(\sqrt{-7})} \) such that 
\[
x = \delta \gamma \quad \text{and} \quad \sqrt{-7} = \delta \mu.
\]
Hence
\[
(1.7) \quad x^2 = N(\delta) N(\gamma),
\]
and
\[
(1.8) \quad 7 = N(\delta) N(\mu).
\]
Since \( x \) is a solution of the equation \ref{1.1}, combining equations \ref{1.7} and \ref{1.8}, we deduce that
\[
(1.9) \quad x^2 + 7 = N(\delta) (N(\gamma) + N(\mu)) = 2^n.
\]
As a consequence of the equations \ref{1.8} and \ref{1.9}, we deduce that
\[
N(\delta) = 1,
\]
i.e., \( \delta \) is a unit. By Proposition \ref{2} (3), the group of units of \( \mathfrak{o}_{Q(\sqrt{-7})} \) is reduced to \( \{\pm 1\} \), hence \( \delta \in \{\pm 1\} \). Therefore either
\[
\frac{x + \sqrt{-7}}{2} = \pm \left( \frac{1 + \sqrt{-7}}{2} \right)^m \quad \text{and} \quad \frac{x - \sqrt{-7}}{2} = \mp \left( \frac{1 - \sqrt{-7}}{2} \right)^m,
\]
or
\[
\frac{x + \sqrt{-7}}{2} = \pm \left( \frac{1 - \sqrt{-7}}{2} \right)^m \quad \text{and} \quad \frac{x - \sqrt{-7}}{2} = \mp \left( \frac{1 + \sqrt{-7}}{2} \right)^m.
\]
i.e.,
\[
(1.10) \quad \left( \frac{1 + \sqrt{-7}}{2} \right)^m - \left( \frac{1 - \sqrt{-7}}{2} \right)^m = \pm \sqrt{-7}.
\]
Suppose that
\[
(1.11) \quad \left( \frac{1 + \sqrt{-7}}{2} \right)^m - \left( \frac{1 - \sqrt{-7}}{2} \right)^m = \sqrt{-7}.
\]
From now on \( \alpha = \frac{1 + \sqrt{-7}}{2} \) and \( \beta = \frac{1 - \sqrt{-7}}{2} \). From Proposition \ref{2} (1), we know that \( \alpha, \beta \in \mathfrak{o}_{Q(\sqrt{-7})} \).

The equation \ref{1.11} is equivalent to
\[
(1.12) \quad \alpha^m - \beta^m = \alpha - \beta.
\]
Then
\[
\alpha^2 = (1 - \beta)^2
= 1 + \beta^2 - 2\beta
= 1 + \beta^2 - \alpha \beta^2 \quad \text{(since } \alpha \beta = 2) 
\equiv 1 \pmod{\beta^2}.
\]
Consequently

\[ \alpha^m = \alpha \left( \alpha^2 \right)^{\frac{m-1}{2}} \equiv \alpha \pmod{\beta^2}. \]

Since \( m \geq 3 \), we deduce that

(1.13) \[ \alpha^m - \beta^m \equiv \alpha \pmod{\beta^2}. \]

Combining equations (1.12) and (1.13) we get

\[ \beta \equiv 0 \pmod{\beta^2}. \]

A contradiction. Therefore

\[ -\sqrt{-7} = \alpha^m - \beta^m. \]

Hence

\[ -\sqrt{-7} = \left( \frac{1 + \sqrt{-7}}{2} \right)^m - \left( \frac{1 - \sqrt{-7}}{2} \right)^m \]

\[ = \frac{1}{2^m} \sum_{k=0}^{m} \binom{m}{k} (\sqrt{-7})^k \]

\[ = \frac{\sqrt{-7}}{2^{m-1}} \left[ \binom{m}{1} - \binom{m}{3} (\sqrt{-7})^2 + \cdots + (-1)^{\frac{m+1}{2}} \binom{m}{m} (\sqrt{-7})^{m-1} \right]. \]

Thus

\[ -2^{m-1} = \binom{m}{1} - \binom{m}{3} (\sqrt{-7})^2 + \cdots + (-1)^{\frac{m+1}{2}} \binom{m}{m} (\sqrt{-7})^{m-1} \]

\[ \equiv \binom{m}{1} \pmod{7} \]

\[ \equiv m \pmod{7}. \]

Since \( m - 1 \) is even, \( m - 1 \equiv 0, 2, \) or \( 4 \pmod{6} \).

- If \( m - 1 \equiv 0 \pmod{6} \), then

\[ m \equiv -2^{m-1} \pmod{7} \]

\[ \equiv - (2^6)^{\frac{m-1}{6}} \pmod{7} \]

\[ \equiv 6 \pmod{7}. \]

So we have the following system of congruences

\[ \begin{cases} 
  m \equiv 1 \pmod{6} \\
  m \equiv 6 \pmod{7}.
\end{cases} \]

By the Chinese remainder theorem

\[ m \equiv 13 \pmod{42}. \]

- If \( m - 1 \equiv 2 \pmod{6} \), then

\[ m \equiv -2^{m-1} \pmod{7} \]

\[ \equiv -2^2 (2^6)^{\frac{m-1}{6}} \pmod{7} \]

\[ \equiv 3 \pmod{7}. \]
So we have the following system of congruences
\[
\begin{align*}
m &\equiv 3 \pmod{6} \\
m &\equiv 3 \pmod{7}.
\end{align*}
\]
Hence
\[m \equiv 3 \pmod{42} .
\]

- If \( m - 1 \equiv 4 \pmod{6} \), then
  \[
  m \equiv -2^{m-1} \pmod{7} \\
  \equiv -2^4 (2^6)^{\frac{m-1}{7}} \pmod{7} \\
  \equiv 5 \pmod{7} .
  \]
So we have the following system of congruences
\[
\begin{align*}
m &\equiv 5 \pmod{6} \\
m &\equiv 5 \pmod{7}.
\end{align*}
\]
Consequently
\[m \equiv 5 \pmod{42} .
\]
Let us prove that \( m = 3, 5 \) and \( 13 \) are the only values of \( m \) coming from the equation \( 1.1 \) and satisfying \(-m \equiv 2^{m-1} \pmod{7} \). Suppose that there are two solutions \( m_1 \equiv m_2 \pmod{42} \) of the equation \(-m \equiv 2^{m-1} \pmod{7} \), i.e.,
\[
\begin{align*}
-m_1 &\equiv 2^{m_1-1} \pmod{7} \\\n-m_2 &\equiv 2^{m_2-1} \pmod{7} \\\nm_1 &\equiv m_2 \pmod{42} .
\end{align*}
\]
We can assume that \( m_2 > m_1 \) and \( m_2 - m_1 = 6 \times \mathcal{l} s \) where \( s \) is not divisible by \( 7 \). From
\[
\alpha^{m_2} = \alpha^{m_1} \alpha^{m_2 - m_1} \\
= \alpha^{m_1} \frac{1}{2^{m_2 - m_1}} (1 + \sqrt{-7})^{m_2 - m_1}
\]
and Lemma \[\text{[1]}\] we deduce that
\[
2^{m_2 - m_1} \alpha^{m_2} \equiv \alpha^{m_1} + (m_2 - m_1) \alpha^{m_1} \sqrt{-7} \pmod{7^{l+1}} .
\]
A similar argument gives
\[
2^{m_2 - m_1} \beta^{m_2} \equiv \beta^{m_1} - (m_2 - m_1) \beta^{m_1} \sqrt{-7} \pmod{7^{l+1}} .
\]
Hence
\[
2^{m_2 - m_1} (\alpha^{m_2} - \beta^{m_2}) \equiv \alpha^{m_1} - \beta^{m_1} + (m_2 - m_1) (\alpha^{m_1} + \beta^{m_1}) \sqrt{-7} \pmod{7^{l+1}} .
\]
Since
\[
2^{m_2 - m_1} = \left(2^6\right)^{\frac{m_2 - m_1}{7}} \\
= \left(1 + 6 \times 7\right)^{\mathcal{l} s} \\
= 1 + \sum_{j=1}^{\mathcal{l} s} \binom{\mathcal{l} s}{j} (6 \times 7)^j ,
\]
and for \( j \geq 1 \),
\[
\binom{7^i s}{j} (6 \times 7)^j = 6^j \frac{7^i (7^i s)!}{j! (7^i s - j)!} = 6^j \frac{7^{i+1}}{j} \left( \frac{7^i s - 1}{j - 1} \right)^s \equiv 0 \pmod{7^{i+1}} \text{ for } j \geq 1,
\]
we deduce that
\[
(1.15) \quad 2^{m_2 - m_1} \equiv 1 \pmod{7^{i+1}}.
\]
\[
(1.16) \quad \alpha^{m_2} - \beta^{m_2} = \alpha^{m_1} - \beta^{m_1} = -\sqrt{-7},
\]
Combining (1.16), (1.14), and (1.15) we get
\[
(m_2 - m_1) (\alpha^{m_1} + \beta^{m_1}) \sqrt{-7} \equiv 0 \pmod{7^{i+1}},
\]
Since
\[
(\alpha^{m_1} + \beta^{m_1}) \sqrt{-7} \not\equiv 0 \pmod{7},
\]
we deduce that
\[
m_2 - m_1 \equiv 0 \pmod{7^{i+1}}.
\]
A contradiction since the highest power of 7 dividing \( m_2 - m_1 \) is \( 7^i \). Hence \( m_1 = m_2 \).
Substitute the values \( n = m + 2 = 5, 7 \), and 15 in equation (1.1), we get
\( x = \pm 5, \pm 11, \) and \( \pm 181 \). Hence the main Theorem.

\[\square\]

2. SOME VARIATIONS OF RAMANUJAN-NAGELL’S EQUATION

If we want to know how many Mersenne numbers are multiples of triangular numbers then we have to solve
\[
k x^2 - k + 8 = 2^n.
\]
Finding the number of triangular numbers which are multiples of Mersenne numbers is reduced to solving
\[
x^2 + 8k = k 2^n + 1.
\]
The case \( k = 1 \) in the equations (2.1) and (2.2), called the "generalized-Ramanujan-Nagell equations", corresponds to Ramanujan-Nagell’s equation.
Knowing how many Mersenne numbers are squares of triangular numbers is reduced to solving
\[
2^n - 64 = (x^2 - 1)^2.
\]
Obviously, \( x = \pm 1, n = 6 \) is a solution.
If we want to know how many Mersenne numbers can be written as sums (resp. products) of let us say of two triangular numbers then we have to solve the Diophantine equation
\[
x^2 + y^2 + 6 = 2^n \ (\text{resp. } (x^2 - 1) (y^2 - 1) + 32 = 2^n).
\]
Obviously, \( x = \pm 1, y = \pm 1 \) and \( n = 3 \) (resp. \( x = \pm 1, y = \pm 1 \) and \( n = 5 \)) is a solution.
The following
\[(2.4) \quad ax^n + b = p^m,\]
where \(a, b, n\) and \(p\) are fixed positive integers, is another natural generalization of Ramanujan-Nagell’s equation.

Variations of Ramanujan-Nagell’s equation lead in general to very challenging Diophantine equations. If instead of considering relations between Mersenne and triangular numbers, we consider relations among other classes of special numbers such as Fermat numbers, Catalan numbers, Stirling numbers, Bell numbers, Pentagonal numbers,... we will get highly non trivial Diophantine equations. For a combinatorial description of the numbers mentioned above, and for some of their applications, see the beautiful book by Conway and Guy [1].

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