Computation of sandwiched relative $\alpha$-entropy
of two $n$-mode gaussian states

K. R. Parthasarathy$^{1,*}$

$^1$Indian Statistical Institute, Theoretical Statistics and Mathematics Unit,
Delhi Centre, 7 S. J. S. Sansanwal Marg, New Delhi 110 016, India
(Dated: November 15, 2021)

Abstract

A formula for the sandwiched relative $\alpha$-entropy $\widetilde{D}_\alpha(\rho||\sigma) = \frac{1}{\alpha-1} \ln \text{Tr} \left( \sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right)^\alpha$ for
$0 < \alpha < 1$, of two $n$-mode gaussian states $\rho$, $\sigma$ in the boson Fock space $\Gamma(\mathbb{C}^n)$ is presented. This computation extensively employs the $\mathcal{E}_2$-parametrization of gaussian states in $\Gamma(\mathbb{C}^n)$ introduced in J. Math. Phys. 62 (2021), 022102.

To my revered Guru

Professor C R Rao

on his 101$^{st}$ birthday

$^*$ krp@isid.ac.in
1. INTRODUCTION

Sandwiched relative $\alpha$-entropy of two quantum states $\rho$, $\sigma$ was introduced concurrently by Wilde et. al. [1] and Müller Lennert et. al. [2] as

$$\widetilde{D}_\alpha(\rho||\sigma) = \begin{cases} 
\frac{1}{\alpha-1} \ln \text{Tr} \left( \sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right)^\alpha, & \text{if } \alpha \in (0, 1) \cup (1, \infty), \\
\text{Tr} \rho (\ln \rho - \ln \sigma), & \text{if } \alpha = 1, \\
\ln ||\sigma^{-1/2} \rho \sigma^{-1/2}||_\infty, & \text{if } \alpha = \infty.
\end{cases}$$ (1.1)

Note that

- $\text{Tr} \left( \sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right)^\alpha = \infty$ if $\text{supp}(\rho) \nsubseteq \text{supp}(\sigma)$.
- $\lim_{\alpha \to 1^-} \widetilde{D}_\alpha(\rho||\sigma)$ is equal to the quantum relative entropy $D(\rho||\sigma) = \text{Tr} \rho (\ln \rho - \ln \sigma)$.
- $\widetilde{D}_\alpha(\rho||\sigma)$ reduces to the Petz–Rényi relative entropy [3, 4] given by $D_\alpha(\rho||\sigma) = \frac{1}{\alpha-1} \ln \text{Tr} \{\rho^\alpha \sigma^{1-\alpha}\}$, $\alpha \in (0, 1) \cup (1, \infty)$ when $\rho$ and $\sigma$ commute. Thus sandwiched relative $\alpha$-entropy is viewed as a non-commutative generalization of the Petz–Rényi relative entropy $D_\alpha(\rho||\sigma)$.
- $\widetilde{D}_\alpha(\rho||\sigma)$ reduces to the relative max-entropy [5] $D_{\text{max}} = \ln ||\sigma^{-1/2} \rho \sigma^{-1/2}||_\infty$ in the limit $\alpha \to \infty$.
- $\widetilde{D}_\alpha(\rho||\sigma)$ is related to the quantum fidelity $F(\rho, \sigma) = \text{Tr} \left( \sigma^{1/2} \rho \sigma^{1/2} \right)^{1/2}$ when $\alpha = 1/2$.

Sandwiched relative $\alpha$-entropy $\widetilde{D}_\alpha(\rho||\sigma)$ finds several applications in quantum information tasks: It has been employed to prove strong converse theorems for quantum channels [1, 6]; for $\alpha > 1$ the sandwiched relative $\alpha$-entropy $\widetilde{D}_\alpha(\rho||\sigma)$ has a direct operational interpretation as strong converse error exponent in quantum hypothesis testing [7, 8].

In this paper we derive a formula for the sandwiched relative $\alpha$-entropy $\widetilde{D}_\alpha(\rho||\sigma)$ for $0 < \alpha < 1$, of two $n$-mode gaussian states $\rho$, $\sigma$ in the boson Fock space $\Gamma(\mathbb{C}^n)$. We employ the $\mathcal{E}_2$-parametrization of gaussian states in $\Gamma(\mathbb{C}^n)$ proposed in Ref. [10] for this computation.

2. MATHEMATICAL PRELIMINARIES

We begin with the necessary mathematical preliminaries. All the theorems and proofs that are readily available in previous Refs. [10–14] are only stated.
Consider the Hilbert space $L^2(\mathbb{R}^n)$, or equivalently, the boson Fock space $\Gamma(\mathcal{H})$ over the complex Hilbert space $\mathcal{H} \equiv \mathbb{C}^n$ of finite dimension $n$. For any $u = (u_1, u_2, \ldots, u_n)^T$ in $\mathcal{H}$, define exponential vector $|e(u)\rangle$ in the boson Fock space $\Gamma(\mathcal{H})$ by

$$|e(u)\rangle = \sum_{k \in \mathbb{Z}_+^n} \frac{u^k}{\sqrt{k!}} |k\rangle$$

where $|k\rangle = |k_1, k_2, \ldots, k_n\rangle$, $k! = k_1!k_2!\ldots k_n!$ and $|k| = k_1 + k_2 + \ldots + k_n$. Then,

$$\langle e(u)|e(v)\rangle = e^{\langle u|v\rangle}.$$ 

The exponential vectors constitute a linearly independent and a total set in $\Gamma(\mathcal{H})$.

For any bounded operator $Z$ on $\Gamma(\mathcal{H})$ the generating function $G_Z(u, v)$, with $u, v$ in $\mathbb{C}^n$, is defined by [10]

$$G_Z(u, v) = \langle e(\bar{u})|Z|e(v)\rangle.$$ 

The operator $Z$ is said to belong to the class $\mathcal{E}_2(\mathcal{H}) \equiv \mathcal{E}_2$ if

$$\langle e(\bar{u})|Z|e(v)\rangle = c \exp \left( \langle \lambda^T u + \mu^T v + u^T A u + u^T \Lambda v + v^T B v \rangle \right), \quad \forall \ u, v \in \mathbb{C}^n, \quad (2.1)$$

where $c \neq 0$ is a scalar; $\lambda, \mu \in \mathbb{C}^n$; $A, B$ and $\Lambda$ are complex $n \times n$ matrices, with $A, B$ being symmetric. We list the properties [10] of $Z$ belonging to the operator semigroup $\mathcal{E}_2$:

1. If $Z \in \mathcal{E}_2$, then $Z^\dagger \in \mathcal{E}_2$.

2. If $Z_1, Z_2 \in \mathcal{E}_2$, then $Z_1 Z_2 \in \mathcal{E}_2$.

3. The ordered six-tuple $(c, \lambda, \mu, A, B, \Lambda)$ is the $\mathcal{E}_2$ parametrization of the operator $Z$.

   (a) The operator $Z \in \mathcal{E}_2$ is hermitian if and only if $c$ is real, $B = \bar{A}$ and $\Lambda$ is hermitian.

   (b) For any positive operator $Z$ in $\mathcal{E}_2$, its $\mathcal{E}_2$ parameters satisfy $c > 0$, $\bar{\lambda} = \mu$, $\bar{A} = B$ and $\Lambda \geq 0$.

4. If $K$ is a selfadjoint contraction in $\mathcal{H}$, then its second quantization $\Gamma(K)$ is a selfadjoint contraction in $\Gamma(\mathcal{H})$. Furthermore, $\Gamma(K) \in \mathcal{E}_2$ and $\Gamma(K) Z \Gamma(K)$ denoted by $Z'$ is an element in $\mathcal{E}_2$ with parameters $(c', \mu', A', \Lambda')$ given by $c' = c$, $\mu' = K \mu$, $A' = KAK^T$, $\Lambda' = K \Lambda K$. 

3
Consider \( Z > 0 \) with \( \mathcal{E}_2 \)-parameters \((c, \mu, A, \Lambda)\). Define a \( 2n \times 2n \) matrix
\[
M(A, \Lambda) = I_{2n} - \begin{pmatrix}
\text{Re}\Lambda & -\text{Im}\Lambda \\
\text{Im}\Lambda & \text{Re}\Lambda
\end{pmatrix} - 2 \begin{pmatrix}
\text{Re}A & \text{Im}A \\
\text{Im}A & -\text{Re}A
\end{pmatrix}
\]
where \( I_{2n} \) denotes \( 2n \times 2n \) identity matrix. If \( M(A, \Lambda) \geq 0 \) define
\[
c(A, \Lambda) = \sqrt{\det M(A, \Lambda)}.
\]

**Theorem 1.** Let \( Z \) be a positive operator in \( \mathcal{E}_2 \). Then \( Z \) is of trace class if and only if \( M(A, \Lambda) > 0 \). In such a case
\[
\text{Tr} Z = \frac{c}{c(A, \Lambda)} \exp \left[ (\mu_1^T, \mu_2^T) M(A, \Lambda)^{-1} \begin{pmatrix}
\mu_1 \\
\mu_2
\end{pmatrix} \right], \quad \mu = \mu_1 + i \mu_2, \quad \mu_1, \mu_2 \in \mathbb{R}^n.
\]

**Proof.** See proof of the Proposition VI.3 of Ref. [10]. \( \square \)

We parametrize any positive trace-class operator \( Z \in \mathcal{E}_2(\mathcal{H}) \) by a quadruple of \( \mathcal{E}_2 \)-parameters \((c, \mu, A, \Lambda)\) with \( c > 0, \mu \in \mathbb{C}^n, A, \Lambda \in \mathbb{M}_n(\mathbb{C}) \) with \( A \) being complex symmetric and \( \Lambda \) positive semi-definite.

**Theorem 2.** A state \( \rho \) in \( \Gamma(\mathcal{H}) \) is gaussian if and only if \( \rho \) belongs to \( \mathcal{E}_2(\mathcal{H}) \).

**Proof.** See proof of the Theorem V.7 of Ref. [10]. \( \square \)

**Corollary 1.** If \( Z \) is a positive trace class operator in \( \mathcal{E}_2(\mathcal{H}) \) then \( \frac{Z}{\text{Tr}Z} \) is a gaussian state.

**Proof.** Follows from the definition of \( \mathcal{E}_2(\mathcal{H}) \).

### A. Annihilation mean and covariance matrix of a gaussian state

At every element \( u \in \mathbb{C}^n \) one associates a pair of operators \( a(u), a^\dagger(u) \), called annihilation, creation operators [10–13], respectively in the boson Fock space \( \Gamma(\mathbb{C}^n) \). There exists a unique unitary operator
\[
W(u) = e^{a^\dagger(u) - a(u)}
\]
called the **Weyl operator** on \( \Gamma(\mathcal{H}) \). With every quantum state \( \rho \) in \( \Gamma(\mathcal{H}) \) we associate a complex-valued function
\[
\hat{\rho}(u) = \text{Tr} W(u) \rho, \quad u \in \mathbb{C}^n
\]
called the **quantum characteristic function** of \( \rho \) at \( u \).
A quantum state $\rho$ in $\Gamma(\mathcal{H})$ is called a $n$-mode gaussian state if there exists a vector $m \in \mathbb{C}^n$, called the annihilation mean vector, and a real symmetric $2n \times 2n$ matrix $S$ such that

$$\hat{\rho}(u) = \exp \left[ -2i \text{Im}(x - i y)^T m - \left( x^T, y^T \right) S \begin{pmatrix} x \\ y \end{pmatrix} \right]$$

$$= \exp \left[ -2i \left( x^T \text{Im} m - y^T \text{Re} m \right) - \left( x^T, y^T \right) S \begin{pmatrix} x \\ y \end{pmatrix} \right]$$

(2.7)

for all $u = x + i y$, $x, y \in \mathbb{R}^n$.

Every gaussian state $\rho \equiv \rho(m, S)$ in $\Gamma(\mathbb{C}^n)$ is completely determined by the annihilation mean vector $m \in \mathbb{C}^n$ and the covariance matrix $S \in M_{2n}(\mathbb{R})$.

B. Relation between $(m, S)$ and the $\mathcal{E}_2$-parameters of a gaussian state

The following theorem establishes a connection between $(m, S)$ and the $\mathcal{E}_2$-parameters of a $n$-mode gaussian state.

**Theorem 3.** Consider a gaussian state $\rho(m, S)$ with mean vector $m \in \mathbb{C}^n$ and $2n \times 2n$ real symmetric covariance matrix $S$. Let the $\mathcal{E}_2$-parameters of $\rho(m, S)$ be $(c, \mu, A, \Lambda)$. Then

$$c = \left[ \det \left( \frac{1}{2} I_{2n} + S \right) \right]^{-1/2} \exp \left[ \left( \begin{array}{c} \text{Re} m \\ \text{Im} m \end{array} \right)^T J \left( \frac{1}{2} I_{2n} + S \right)^{-1} J \left( \begin{array}{c} \text{Re} m \\ \text{Im} m \end{array} \right) \right]$$

(2.8)

$$\mu = i \left( I_n, i I_n \right) \left( \frac{1}{2} I_{2n} + S \right)^{-1} J \left( \begin{array}{c} \text{Re} m \\ \text{Im} m \end{array} \right)$$

(2.9)

$$A = \frac{1}{4} \left( I_n, i I_n \right) \left( \frac{1}{2} I_{2n} + S \right)^{-1} \left( \begin{array}{c} I_n \\ i I_n \end{array} \right)$$

(2.10)

$$\Lambda = I_n - \frac{1}{2} \left( I_n, i I_n \right) \left( \frac{1}{2} I_{2n} + S \right)^{-1} \left( \begin{array}{c} I_n \\ -i I_n \end{array} \right)$$

(2.11)

where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$
In the opposite direction, we have

\[ S = M(-A, \Lambda)^{-1} - \frac{1}{2} I_{2n} \]  \hspace{2cm} (2.13)

\[
\begin{pmatrix}
\text{Re } m \\
\text{Im } m
\end{pmatrix} = M(-A, \Lambda)^{-1} \begin{pmatrix}
\text{Re } \mu \\
\text{Im } \mu
\end{pmatrix}.
\]  \hspace{2cm} (2.14)

**Proof.** See proofs of the Propositions VI.1 and VI.3 of Ref. [10]. \(\square\)

C. Gaussian symmetry transformation and structure theorem for \(n\)-mode gaussian state

Here we list some important features of gaussian states in \(\Gamma(\mathcal{H})\):

- Any unitary operator \(U \in E_2(\mathcal{H})\) is a gaussian symmetry i.e., \(U \rho U^\dagger\) is a gaussian state whenever \(\rho\) is a gaussian state (see Proposition V.10.1 of Ref. [10]). Every gaussian symmetry operation belongs to \(E_2(\mathcal{H})\).

- For any gaussian state \(\rho(m, S)\) in \(\Gamma(\mathbb{C}^n)\) there exists a sequence \(0 < t_1 \leq t_2 \leq \ldots \leq t_n \leq \infty\) and a symplectic matrix \(L \in \text{Sp}(2n, \mathbb{R})\) such that

\[
\rho(m, S) = U(m, L) \rho(t) U(m, L)^{-1}
\]  \hspace{2cm} (2.15)

where \(U(m, L) = W(m) \Gamma(L)\) is a unitary gaussian symmetry operator [11] consisting of a phase space translation \(W(m)\) and a disentangling unitary transformation \(\Gamma(L)\) on the boson Fock space \(\Gamma(\mathbb{C}^n)\). Here

\[
\rho(t) = \rho(0, D(t)) = \rho(t_1) \otimes \rho(t_2) \otimes \ldots \otimes \rho(t_n)
\]  \hspace{2cm} (2.16)

\[
\rho(t_j) = p(t_j) \sum_{k_j=0}^{\infty} e^{-k_j t_j} |k_j\rangle \langle k_j|, \quad p(t_j) = (1 - e^{-t_j}), \quad j = 1, 2, \ldots n
\]

corresponds to a \(n\)-mode gaussian thermal state characterized by zero mean and covariance matrix \(D(t)\) given by

\[
D(t) = L^T S L = \begin{pmatrix} D_0(t) & 0 \\ 0 & D_0(t) \end{pmatrix},
\]

\[
D_0(t) = \text{diag} \left[ \frac{1}{2} \coth \left( \frac{t_j}{2} \right), j = 1, 2, \ldots n \right].
\]  \hspace{2cm} (2.17)
Thus every gaussian state in $\Gamma(\mathbb{C}^n)$ is characterized by three equivalent fundamental parametrizations [10–13]:

1. $(m, S)$: mean annihilation vector $m \in \mathbb{C}^n$ and real symmetric covariance matrix $S \in M_{2n}(\mathbb{R})$.

2. $(t, L)$: Thermal parameters $t = (t_1, t_2, \ldots, t_n)$, $0 < t_1 \leq t_2 \leq \ldots \leq t_n \leq \infty$ and $L \in \text{Sp}(2n, \mathbb{R})$ such that $\rho(m, S) = U(m, L) \rho(t) U(m, L)^{-1}$ (see (2.15), (2.16), and (2.17)).

3. $(c, \mu, A, \Lambda)$: $E_2(\mathcal{H})$-parameters with $c > 0$, $\mu \in \mathbb{C}^n$, $A, \Lambda \in M_n(\mathbb{C})$ with a complex symmetric $A$ and positive semi-definite $\Lambda$.

3. COMPUTATION OF SANDWICHED RELATIVE $\alpha$-ENTROPY $\tilde{D}_\alpha(\rho||\sigma)$ OF TWO GAUSSIAN STATES $\rho, \sigma$

The $\alpha$-dependent sandwiched Rényi relative entropy [1, 2] between two states $\rho, \sigma$ is given by

$$\tilde{D}_\alpha(\rho||\sigma) = \frac{1}{\alpha - 1} \ln \text{Tr} \left[ (\sigma^{1-\alpha} \rho \sigma^\alpha)^\alpha \right].$$

Let $\rho, \sigma$ be two $n$-mode gaussian states with

$$\sigma' = U(\ell, L) \sigma (U(\ell, L))^{-1} = \rho(s) \quad (3.1)$$
$$\rho' = U(\ell, L) \rho (U(\ell, L))^{-1} \quad (3.2)$$

where $\rho(s)$ is an $n$-mode thermal state

$$\rho(s) = \rho(s_1) \otimes \rho(s_2) \otimes \ldots \otimes \rho(s_n), \quad (3.3)$$
$$\rho(s_j) = p(s_j) \sum_{k_j=0}^{\infty} e^{-k_j s_j} |k_j\rangle \langle k_j|, \quad p(s_j) = (1 - e^{-s_j}), \quad j = 1, 2, \ldots, n.$$  

characterized by the parameters $s = (s_1, s_2, \ldots, s_n)$, $0 < s_1 \leq s_2 \leq \ldots \leq s_n \leq \infty$. Note that $\rho(\infty) = |\Omega\rangle \langle \Omega|$ denotes the 1-mode Fock vacuum state and $p(\infty) = 1$.

The $\alpha$-dependent sandwiched relative entropy remains invariant when both the states $\rho, \sigma$ are changed by any unitary transformation $U$. Thus

$$\tilde{D}_\alpha(\rho||\sigma) = \tilde{D}_\alpha(\rho'||\sigma')$$
$$\tilde{D}_\alpha(\rho'||\rho(s))$$
$$\frac{1}{\alpha - 1} \ln \text{Tr} \left\{ (\rho(s)^{1-\alpha} \rho' \rho(s)^{1-\alpha})^\alpha \right\}. \quad (3.4)$$
Let us denote
\[ T_\alpha(\rho', \rho(s)) = \text{Tr} \left\{ \rho(s)^{1-\alpha} \rho' \rho(s)^{\frac{1-\alpha}{2\alpha}} \right\}^\alpha. \] (3.5)

Putting
\[ p(s) = \prod_{j=1}^n p(s_j), \]
we obtain from (3.3)
\[ \rho(s)^{1-\alpha} = p(s)^{1-\alpha} \sum_{k \in \mathbb{Z}_n^+} e^{-\sum_{j=1}^n k_j s_j \left( \frac{1-\alpha}{2\alpha} \right)} |k\rangle\langle k| \]
\[ = p(s)^{1-\alpha} \Gamma(K) \]
where
\[ K = \text{diag} \left( e^{-s_1 \left( \frac{1-\alpha}{2\alpha} \right)}, e^{-s_2 \left( \frac{1-\alpha}{2\alpha} \right)}, \ldots, e^{-s_n \left( \frac{1-\alpha}{2\alpha} \right)} \right) \] (3.7)
is the contraction diagonal matrix and \( \Gamma(K) \in \mathcal{E}_2(\mathcal{H}) \) is the corresponding positive contraction operator [10]. Thus
\[ \left\{ \rho(s)^{1-\alpha} \rho' \rho(s)^{1-\alpha} \right\}^\alpha = p(s)^{1-\alpha} \{ \Gamma(K) \rho' \Gamma(K) \}^\alpha. \] (3.8)

Consider the positive trace class operator \( Z \in \mathcal{E}_2((H)) \) defined by
\[ Z = \Gamma(K) \rho' \Gamma(K). \] (3.9)

Suppose the transformed gaussian state \( \rho' \) has its \( \mathcal{E}_2 \)-parameters \( (c, \mu, A, \Lambda) \). It follows that \( Z \) is an \( \mathcal{E}_2 \) operator with parameters \( (c', \mu', A', \Lambda') = (c, K \mu, K A K^T, K \Lambda K) \). Then (see (2.4))
\[ \text{Tr} Z = \frac{c}{c(A', \Lambda')} \exp \left[ \left( \mu_1' \mu_2'^T \right) M(A', \Lambda')^{-1} \left( \begin{array}{c} \mu_1' \\ \mu_2' \end{array} \right) \right], \] (3.10)
where \( c(A', \Lambda') = \sqrt{\det M(A', \Lambda')} \).

Now \( \rho_Z = \frac{Z}{\text{Tr} Z} \) is a gaussian state with \( (c', \mu', A', \Lambda') \) as its \( \mathcal{E}_2 \)-parameters. From the last part of Theorem 3 the covariance matrix \( S_Z \) of \( \rho_Z \) is given by
\[ S_Z = M(-A', \Lambda')^{-1} - \frac{1}{2} I_{2n}. \] (3.11)

Through Williamson resolution [11–13] of the covariance matrix viz.,
\[ D(t_Z) = L_Z^T S_Z L_Z = \begin{pmatrix} D_0(t_Z) & 0 \\ 0 & D_0(t_Z) \end{pmatrix}, \quad L_Z \in \text{Sp}(2n, \mathbb{R}), \] (3.12)
\[ D_0(t_Z) = \text{diag} \left[ \frac{1}{2} \coth \left( \frac{1}{2} t_Z \right), j = 1, 2, \ldots n \right]. \] (3.13)
we construct $\rho(t_Z)$, equivalent to $\rho_Z$ by a unitary gaussian symmetry, with thermal parameters $t_Z = ((t_Z)_1 \leq (t_Z)_2 \leq \ldots (t_Z)_n)$.

Thus

$$\text{Tr} \rho_Z^\alpha = \text{Tr} \rho(t_Z)^\alpha = \frac{p(t_Z)^\alpha}{p(\alpha t_Z)}.$$  \hspace{1cm} (3.14)

Therefore

$$\text{Tr} Z^\alpha = \frac{[p(t_Z)]^\alpha}{p(\alpha t_Z)} (\text{Tr} Z)^\alpha.$$  \hspace{1cm} (3.15)

The following theorem summarizes the above computations:

**Theorem 4.** Let $\rho, \sigma \equiv \rho(\ell, S)$ be two $n$-mode gaussian states in $\Gamma(\mathbb{C}^n)$, with $\ell, S$ denoting the annihilation mean and covariance matrix of $\sigma$. Let $U(\ell, L)$ be the gaussian symmetry leading to the standard form of $\sigma$ i.e.,

$$\rho(s) = U(\ell, L) \sigma (U(\ell, L))^{-1} = \rho(s_1) \otimes \rho(s_2) \otimes \ldots \otimes \rho(s_n),$$

$$\rho(s_j) = p(s_j) \sum_{k_j=0}^{\infty} e^{-k_j s_j} |k_j\rangle \langle k_j|; \quad p(s_j) = (1 - e^{-s_j}), \quad j = 1, 2, \ldots n.$$  \hspace{1cm} (3.16)

Let $(c, \mu, A, \Lambda)$ be the $E_2$-parameters of the transformed gaussian state $\rho' = U(\ell, L) \rho (U(\ell, L))^{-1}$. Consider the positive trace-class operator $Z \in E_2(H)$, characterized by its $E_2$-parameters $(c', \mu', A', \Lambda') = (c, K \mu, K A K^T, K \Lambda K)$, where $K = \text{diag} \left( e^{-s_1 \frac{1}{2 \alpha}}, e^{-s_2 \frac{1}{2 \alpha}}, \ldots, e^{-s_n \frac{1}{2 \alpha}} \right)$ is a selfadjoint contraction in $H$. Suppose the gaussian state $\rho_Z = \frac{Z}{\text{Tr} Z}$, constructed from $Z$, has its thermal parameters $t_Z = ((t_Z)_1 \leq (t_Z)_2 \leq \ldots (t_Z)_n)$. Then, the $\alpha$-dependent sandwiched relative entropy $\tilde{D}_\alpha(\rho||\sigma) = \frac{1}{\alpha - 1} \ln \text{Tr} \left( \sigma \frac{1 - \alpha}{2 \alpha} \rho \sigma \frac{1 - \alpha}{2 \alpha} \right)^\alpha$, for $0 < \alpha < 1$, of $\rho$ and $\sigma$ is given by

$$\tilde{D}_\alpha(\rho||\sigma) = \tilde{D}_\alpha(\rho'||\rho(s))$$

$$= \frac{1}{\alpha - 1} \ln T_a(\rho', \rho(s)).$$  \hspace{1cm} (3.17)

where

$$T_a(\rho', \rho(s)) = \frac{p(s)^{1-\alpha} p(t_Z)^\alpha}{p(\alpha t_Z)} (\text{Tr} Z)^\alpha,$$ \hspace{1cm} (3.18)

$$p(s) = \prod_{j=1}^{n} p(s_j), \quad p(t_Z) = \prod_{j=1}^{n} p((t_Z)_j)$$
\[
Tr Z = \frac{c}{c(A', \Lambda')} \exp \left[ \left( \begin{pmatrix} \mu_1' \\ \mu_2' \end{pmatrix}, \lambda \right) M(A', \Lambda')^{-1} \begin{pmatrix} \mu_1' \\ \mu_2' \end{pmatrix} \right], \quad \mu = \mu_1 + i \mu_2, \quad \mu_1, \mu_2 \in \mathbb{R}^n.
\]

**Proof.** Follows from the detailed computations given above.

For an alternate approach on the computation of sandwiched relative $\alpha$-entropy between two gaussian states see Ref. [15].

**ACKNOWLEDGEMENT**

This paper would have been impossible to prepare without the help of Professor A R Usha Devi. In spite of her heavy duties as Chairperson of the Physics Department, Bangalore University, Bengaluru, she has prepared the manuscript based on my handwritten computations. I also thank Mrs. Shyamala Parthasarathy for her ready support in communicating my handwritten papers between Delhi and Bengaluru as and when needed.

**REFERENCES**

[1] M. M. Wilde, A. Winter, and D. Yang, *Strong converse for the classical capacity of entanglement-breaking and Hadamard channels via a sandwiched Rényi relative entropy*, Commun. Math. Phys. 331 (2014), 593–622.

[2] M. Müller-Lennert, F. Dupuis, O. Szehr, S. Fehr, and M. Tomamichel, *On quantum Rényi entropies: A new definition and some properties*, J. Math. Phys. 54 (2013), 122203.

[3] A. Rényi, *On measures of entropy and information*, Proceedings of the 4th Berkeley Symposium on Mathematics, Statistics and Probability; edited by J. Neyman, 1, University of California Press, Berkeley, 1961, pp. 547–561.

[4] D. Petz, *Quasi-entropies for finite quantum systems*, Reports in Mathematical Physics, 23 (1986), pp. 57–65.

[5] N. Datta *Min- and max-relative entropies and a new entanglement monotone*, IEEE Trans. Inform. Theory, 55 (2009), pp. 2816–2826.
[6] M. Tomamichel, M. M. Wilde, A. Winter, *Strong converse rates for quantum communication*, IEEE International Symposium on Information Theory (ISIT), (2015), pp. 2386–2390.

[7] T. Cooney, M. Mosonyi, M. M. Wilde, *Strong converse exponents for a quantum channel discrimination problem and quantum-feedback-assisted communication*, Comm. Math. Phys., 344 (2014), pp. 797–829.

[8] M. Mosonyi, T. Ogawa, *Quantum hypothesis testing and the operational interpretation of the quantum Rényi relative entropies*, Commun. Math. Phys., 334 (2015), pp. 1617–1648.

[9] M. Hayashi, M. Tomamichel, *Correlation detection and an operational interpretation of the Rényi mutual information*, J. Math. Phys. 57 (2016), 102201.

[10] T. C. John, and K. R. Parthasarathy, *A common parametrization for finite mode gaussian states, their symmetries, and associated contractions with some applications*, J. Math. Phys. 62 (2021), 022102.

[11] K. R. Parthasarathy, *A pedagogical note on the computation of relative entropy of two n-mode gaussian states*, arXiv:2102.06708v2 [quant-ph].

[12] K. R. Parthasarathy, *What is a gaussian state?*, Commun. Stoch. Anal., 4 (2010), pp. 143–160.

[13] K. R. Parthasarathy, *The symmetry group of Gaussian states in L^2(\mathbb{R}^n)*, Prokhorov and contemporary probability theory, Springer Proc. Math. Stat., 33, Springer, Heidelberg, 2013, pp. 349–369.

[14] K. R. Parthasarathy and Ritabrata Sengupta, *From particle counting to gaussian tomography*, Infinite Dimensional Analysis, Quantum Probability and Related Topics, 18 (2015), 1550023.

[15] K. P. Seshadreesan, L. Lami, and M. M. Wilde, *Rényi relative entropies of quantum Gaussian states*, J. Math. Phys. 59 (2018), 072204.