Boundedness of some operators on weighted amalgam spaces

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Abstract Let $t \in (0, \infty)$, $p \in (1, \infty)$, $q \in [1, \infty]$, $w \in A_p$ and $v \in A_q$. We introduce the weighted amalgam space $(L^p, L^q)_t(\mathbb{R}^n)$ and show some properties of it. Some estimates on these spaces for the classical operators in harmonic analysis, such as the Hardy–Littlewood maximal operator, the Calderón–Zygmund operator, the Riesz potential, singular integral operators with the rough kernel, the Marcinkiewicz integral, the Bochner-Riesz operator, the Littlewood-Paley $g$ function and the intrinsic square function, are considered. Our main method is extrapolation. We obtain some new weak results for these operators on weighted amalgam spaces.

Keywords Muckenhoupt’s weight, operators, extrapolation, weak space, amalgam space.

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1 Introduction

In 1926, Wiener \cite{24} first introduced amalgam spaces to formulate his generalized harmonic analysis. For $p, q \in (0, \infty)$, the amalgam space $(L^p, L^q)(\mathbb{R})$ is defined by

$$\text{(L^p, L^q)(\mathbb{R}) := \left\{ f \in L^p_{\text{loc}} : \sum_{n \in \mathbb{Z}} \left( \int_n^{n+1} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\}.}$$

The first systematic study of these spaces was undertaken by Holland \cite{11} in 1975. A large number of authors \cite{3, 4, 8, 13, 20} research amalgam spaces or some applications of these spaces. A significant difference in considering amalgam spaces instead of $L^p$ spaces is that amalgam spaces give information about the local $L^p$, and global $L^q$, properties of the functions, while $L^p$ spaces do not make that distinction. To study the weak solutions of boundary value problems for a $t$-independent elliptic systems in the upper half plane, Auscher and Mourgoglou \cite{2} introduced the slice spaces in 2019. Moreover, Auscher and Prisenlos-Arribas \cite{1} studied the boundedness of Calderón–Zygmund operators, the Hardy-Littlewood maximal operator and the fractional integral operator on slice spaces. Recently, Ho \cite{14} obtained the boundedness of some classical operators, such as singular integral operators, the Fourier integral operator, the geometric maximal operator, the maximal Bochner–Riesz means, the parametric Marcinkiewicz integral and the multiplier operators on Hardy Orlicz-slice spaces introduced in \cite{27}.

The main purpose of this paper is to consider the boundedness of some operators on a new weighted amalgam space. To state our main results, we recall some necessary definitions.

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A weight $\omega$ is a positive and locally integrable function on $\mathbb{R}^n$. For $p \in (0, \infty)$, the weighted Lebesgue spaces $L^p_\omega(\mathbb{R}^n)$ is defined as the set of all measurable functions $f$ on $\mathbb{R}^n$ such that
\[
\|f\|_{L^p_\omega(\mathbb{R}^n)} := \left[ \int_{\mathbb{R}^n} |f(x)|^p \omega(x) \, dx \right]^{\frac{1}{p}} < \infty.
\]
The weak weighted Lebesgue space $L^{p,\infty}_\omega(\mathbb{R}^n)$ is defined as the set of all measurable functions $f$ on $\mathbb{R}^n$ such that
\[
\|f\|_{L^{p,\infty}_\omega(\mathbb{R}^n)} := \sup_{\alpha > 0} \alpha \omega(\{ x \in \mathbb{R}^n : |f(x)| > \alpha \})^{\frac{1}{p}} < \infty.
\]
For $p = \infty$,
\[
\|f\|_{L^\infty_\omega(\mathbb{R}^n)} := \operatorname{ess sup}_{x \in \mathbb{R}^n} |f(x)| < \infty.
\]

**Definition 1.1.** Let $w, v$ be weights, $0 < t < \infty$, $1 < p < \infty$, and $1 \leq q \leq \infty$. We define the weighted amalgam space $(L^p_w, L^q_v)_t := (L^p_w, L^q_v)_t(\mathbb{R}^n)$ as the space of all measurable functions $f$ on $\mathbb{R}^n$ satisfying $\|f\|_{(L^p_w, L^q_v)_t} < \infty$, where
\[
\|f\|_{(L^p_w, L^q_v)_t} := \left\| \left( \frac{1}{w(B(x,t))} \int_{B(x,t)} |f(y)|^p \omega(y) \, dy \right)^{\frac{1}{p}} \right\|_{L^q_v(\mathbb{R}^n)}.
\]

with the usual modification when $q = \infty$.

**Remark 1.1.** If $w = v = 1$, then $(L^p_w, L^q_v)_t(\mathbb{R}^n)$ is the slice space $(E^q_p)_t(\mathbb{R}^n)$ (see [1, 2]).

**Definition 1.2.** Let $w, v$ be weights, $0 < t < \infty$, $1 < p < \infty$, and $1 \leq q \leq \infty$. The weak weighted amalgam space $W(L^p_w, L^q_v)_t := W(L^p_w, L^q_v)_t(\mathbb{R}^n)$ is defined as the space of all measurable functions $f$ on $\mathbb{R}^n$ satisfying $\|f\|_{W(L^p_w, L^q_v)_t} < \infty$, where
\[
\|f\|_{W(L^p_w, L^q_v)_t} := \sup_{\lambda > 0} \lambda \left\| \chi_{\{x \in \mathbb{R}^n : |f(x)| > \lambda\}} \right\|_{(L^p_w, L^q_v)_t(\mathbb{R}^n)} < \infty.
\]

We still recall the definition of Muckenhoupt’s weights $A_p(1 \leq p \leq \infty)$. These weights introduced in [16] were used to characterize the boundedness of the Hardy-Littlewood maximal operator on weighted Lebesgue spaces. For a locally integrable function $f$, we define the centered Hardy–Littlewood maximal operator, for almost every $x \in \mathbb{R}^n$,
\[
Mf(x) := \sup_{\tau > 0} \frac{1}{|B(x, \tau)|} \int_{B(x, \tau)} |f(y)| \, dy.
\]

**Definition 1.3.** Let $1 < p < \infty$. A weight $w$ is said to be of class $A_p$ if
\[
\sup_{B \subset \mathbb{R}^n} \left( \frac{1}{|B|} \int_B w(x) \, dx \right)^{\frac{1}{p}} \left( \frac{1}{|B|} \int_B w(x)^{\frac{1}{p-1}} \, dx \right)^{p-1} < \infty.
\]

A weight $w$ is said to be of class $A_1$ if
\[
M(w)(x) \leq Cw(x) \quad \text{for almost all } x \in \mathbb{R}^n
\]
for some positive constant $C$. We define $A_\infty := \cup_{p \geq 1} A_p$.

Our main results are given as follows.

**Theorem 1.1.** Let $0 < t < \infty$, $1 < p < \infty$ and $w \in A_p$.

(a) If $1 < q \leq \infty$ and $v \in A_q$, then we have
\[
\|M(f)\|_{(L^p_w, L^q_v)_t(\mathbb{R}^n)} \leq C\|f\|_{(L^p_w, L^q_v)_t(\mathbb{R}^n)}.
\]
(b). If $q = 1$ and $v \in A_1$, then we have

$$
||M(f)||_{W(L^q_n, L^q_b)}(\mathbb{R}^n) \leq C ||f||_{(L^q_n, L^q_b)}(\mathbb{R}^n).
$$

The universal positive constant $C$ is independent of $f$ and $t$.

Let $\delta > 0$ and let $\Delta$ be the off-diagonal in $\mathbb{R}^n \times \mathbb{R}^n$, that is, $\Delta := \mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x): x \in \mathbb{R}^n\}$. The Calderón–Zygmund singular integral operator of non-convolution type is a bounded linear operator $T: L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ satisfying that, for all $f \in C_c^\infty(\mathbb{R}^n)$ and $x \notin \text{supp} (f)$,

$$
T(f)(x) := \int_{\mathbb{R}^n} K(x,y)f(y)dy,
$$

where the distributional kernel coincides with a locally integrable function $K$ defined away from the diagonal on $\mathbb{R}^n \times \mathbb{R}^n$. When $K$ also satisfies that, for $x, y \in \mathbb{R}^n$ with $x \neq y$,

$$
|K(x,y)| \leq \frac{C_0}{|x-y|^n},
$$

(1)

$$
|K(x,y) - K(x,y+h)| + |K(x,y) - K(x+h,y)| \leq \frac{C_1|h|^\delta}{|x-y|^{n+\delta}},
$$

(2)

whenever $|x-y| \geq 2|h|$, and we call $K$ the standard kernel.

**Theorem 1.2.** Let $T$ be a Calderón–Zygmund operator with kernel $K$ satisfying (1) and (2). Suppose that $0 < t < \infty$, $1 < p < \infty$ and $w \in A_p$.

(a). If $1 < q < \infty$ and $v \in A_q$, then we have

$$
||T(f)||_{(L^p_n, L^q_b)}(\mathbb{R}^n) \leq C ||f||_{(L^p_n, L^q_b)}(\mathbb{R}^n).
$$

(b). If $q = 1$ and $v \in A_1$, then we have

$$
||T(f)||_{W(L^1_n, L^1_b)}(\mathbb{R}^n) \leq C ||f||_{(L^1_n, L^1_b)}(\mathbb{R}^n).
$$

The universal positive constant $C$ is independent of $f$ and $t$.

We also recall the definition of $A_{p,q}$ weights which are closely related to the weighted boundedness of the fractional integral.

**Definition 1.4.** We say that a weight $w$ is said to be of class $A_{p,q}$, for $1 < p, q < \infty$, if there exists a constant $C > 0$ such that

$$
\sup_{B \subset \mathbb{R}^n} \left( \frac{1}{|B|} \int_B w(x)^q dx \right)^{\frac{1}{q}} \left( \frac{1}{|B|} \int_B w(x)^{-q'} dx \right)^{\frac{1}{p'}} \leq C < \infty,
$$

where $p'$ is the conjugate exponent of $p$, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

And say $w$ is in $A_{1,q}$ with $1 < q < \infty$, if there exist constant $C > 0$ such that

$$
\sup_{B \subset \mathbb{R}^n} \left( \frac{1}{|B|} \int_B w(x)^q dx \right)^{\frac{1}{q}} \left( \text{ess sup}_{B} \frac{1}{w(x)} \right) \leq C < \infty.
$$

**Proposition 1.1.** [18] Suppose that $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

(i). If $1 < p$, then $w \in A_{p,q}$ if and only if $w^q \in A_{p, \frac{n}{\alpha}}$;
(ii). If \(1 < p, \; w \in A_{p,q}, \) then \(w^q \in A_q\) and \(w^p \in A_p;\)

(iii). If \(p = 1\), then \(w \in A_{1,q}\) if and only if \(w^q \in A_1.\)

Proposition 1.2. [9] Let \(1 \leq p < \infty, \; w \in A_p,\) then

(i). A weight \(\omega\) is doubling, that is, for any ball \(B \in \mathbb{R}^n, \; \omega(2B) \leq C\omega(B),\) where the positive constant \(C\) is independent of \(B;\)

(ii). \(\omega \in A_\infty,\) for every ball \(B\) and every measurable set \(E \subset B,\) there exist \(C, \delta > 0\) such that

\[
\frac{\omega(E)}{\omega(B)} \leq C \left( \frac{|E|}{|B|} \right)^\delta.
\]

(3)

For \(\alpha \in (0, n),\) the Riesz potential \(I_\alpha\) is defined as follows.

Definition 1.5. Let \(0 < \alpha < n,\) the Riesz potential \(I_\alpha\) is defined by

\[
I_\alpha f(x) := \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(\xi)}{|x - \xi|^{n-\alpha}} d\xi,
\]

where \(\gamma(\alpha) = \pi^{\frac{n}{2}} 2^n \Gamma(\frac{\alpha}{2})/\Gamma(\frac{n-\alpha}{2}).\)

Theorem 1.3. Let \(0 < \alpha < n\) and \(0 < t < \infty.\) Suppose that \(1 < p, q < \frac{n}{\alpha}\) such that

\[
\frac{\alpha}{n} = \frac{1}{p_0} - \frac{1}{p} = \frac{1}{q_0} - \frac{1}{q}.
\]

Suppose that \(w \in A_{p_0,p}.\)

(a). If \(1 < q_0 < \infty, \) and \(v \in A_{q_0,q},\) then we have

\[
\|I_\alpha(f)\|_{(L^p_w, L^q_v)_{t}(\mathbb{R}^n)} \leq C\|f\|_{(L^{p_0}_{w_0}, L^{q_0}_{v_0})_{t}(\mathbb{R}^n)}.
\]

(b). If \(q_0 = 1, \) and \(v \in A_{1,q},\) then we have

\[
\|I_\alpha(f)\|_{W(L^p_w, L^q_v)_{t}(\mathbb{R}^n)} \leq C\|f\|_{(L^{p_0}_{w_0}, L^{q_0}_{v_0})_{t}(\mathbb{R}^n)}.
\]

The universal positive constant \(C\) is independent of \(f\) and \(t.\)

We end this section by explaining some notations. Given a weight \(w\) and a measurable set \(B,\) it can be denoted by \(w(B) := \int_B w(x)dx.\) For \(\alpha > 0\) and a ball \(B \subset \mathbb{R}^n,\) \(\alpha B\) is the ball with same center as \(B\) and radius \(\alpha\) times radius of \(B.\) We denote by \(B^c := \mathbb{R}^n \setminus B\) the complement of \(B.\) We write \(A \lesssim B\) to mean that there exists a positive constant \(C\) such that \(A \leq CB.\) \(A \approx B\) denotes that \(A \lesssim B\) and \(B \lesssim A.\) Throughout this paper, the letter \(C\) will be used for positive constants independent of relevant variables that may change from one occurrence to another.

2 Some Lemmas

We begin with some properties of weighted amalgam spaces in this section.

Lemma 2.1. [10] Give \(1 \leq p, q < \infty,\)

\[
[(L^p_{w_0}, L^q_v)_{t}(\mathbb{R}^n)]' = (L^{p'}_{w'}, L^{q'}_v)_{t}(\mathbb{R}^n),
\]

where \(\gamma(\alpha) = \pi^{\frac{n}{2}} 2^n \Gamma(\frac{\alpha}{2})/\Gamma(\frac{n-\alpha}{2}).\)
where \( \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1 \), \( v' = v^{1-p'} \), \( w' = w^{1-q'} \), and as for dual space of the weighted amalgam space, then we know

\[
[(L^p_w, L^q_v)_t(\mathbb{R}^n)]': = \left\{ f : \|f\|_{(L^p_w, L^q_v)_t(\mathbb{R}^n)}' := \sup_{\|g\|_{L^p_w L^q_v}_1(\mathbb{R}^n) \leq 1} \int_{\mathbb{R}^n} f(t)g(t)dt \right\}.
\]

**Lemma 2.2.** \([10]\) Given \( 1 \leq p, q \leq \infty \),

\[
\|fg\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{(L^p_w, L^q_v)_1(\mathbb{R}^n)}\|g\|_{(L^{p'}_w, L^{q'}_v)_1(\mathbb{R}^n)}.
\]

where \( \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1 \), \( v' = v^{1-p'} \), \( w' = w^{1-q'} \).

**Lemma 2.3.** \([5]\) Let \( F \) be a given family of pairs \((f, g)\) of non-negative and not identically zero measurable functions on \( \mathbb{R}^n \). Suppose that for some fixed exponent \( p_0 \in [1, \infty) \), and every weight \( \omega \in A_{p_0} \),

\[
\int_{\mathbb{R}^n} f(x)^{p_0}\omega(x)dx \leq C_{\omega, p_0} \int_{\mathbb{R}^n} g(x)^{p_0}\omega(x)dx, \quad \forall (f, g) \in F.
\]

Then, for all \( p \in (1, \infty) \) and for all \( \omega \in A_p \),

\[
\int_{\mathbb{R}^n} f(x)^p \omega(x)dx \leq C_{\omega, p} \int_{\mathbb{R}^n} g(x)^p \omega(x)dx, \quad \forall (f, g) \in F.
\]

**Lemma 2.4.** Let \( t \in (0, \infty) \), \( r \in [1, \infty) \) and \( 1 < \alpha, p < \infty \). Given \( w \in A_p \), \( v \in A_q \), and \( q \in [1, rp] \), there exist a positive constant \( C \) such that

\[
C^{-1}\|f\|_{(L^p_w, L^q_v)_1(\mathbb{R}^n)} \leq \|f\|_{(L^p_w, L^q_v)_{\alpha, t}(\mathbb{R}^n)} \leq C_{\alpha, p}\|f\|_{(L^p_w, L^q_v)_1(\mathbb{R}^n)}.
\]

Where \( C \) is independent of \( f, t, \alpha \).

**Proof of Lemma 2.4.** Similar to the proof of \([19, \text{Theorem 3.6}]\), we only prove the right-hand of the above inequality. We split it to three steps. We first obtain the case \( q = p \) and \( 1 \leq r < \infty \). From this, we extrapolate concluding the desired estimate in the ranges \( 1 \leq q \leq rp \) and \( 1 < r < \infty \). Finally, we shall consider the case \( r = 1 \) and \( 1 \leq q < p \). From now on, we fix \( \alpha > 1 \). For the first step, let \( q = p \) and \( 1 \leq r < \infty \). By \( \omega \in A_p \), and (3), we conclude that

\[
\int_{\mathbb{R}^n} \frac{1}{w(B(x, \alpha t))} \int_{B(x, \alpha t)} |f(y)|^p w(y)dyv(x)dx = \frac{1}{w(B(x, \alpha ))} \int_{\mathbb{R}^n} |f(y)|^p w(y)v(B(y, \alpha t))dy \\
\leq \frac{\alpha \omega}{w(B(x, \alpha ))} \int_{\mathbb{R}^n} |f(y)|^p w(y)v(B(y, t))dy \\
\leq \alpha^{\omega} \int_{\mathbb{R}^n} \frac{1}{w(B(x, t))} \int_{B(x, t)} |f(y)|^p w(y)dyv(x)dx.
\]

To prove the second step, take an arbitrary \( p_0 \in [1, \infty) \) and consider \( F \) the family of pairs

\[
(f, g) := \left( \left[ \frac{1}{w(B(x, \alpha t))} \int_{B(x, \alpha t)} |f(y)|^p w(y)dy \right]^{\frac{1}{p_0}}, \left[ \frac{1}{w(B(x, t))} \int_{B(x, t)} |f(y)|^p w(y)dy \right]^{\frac{1}{p_0}} \right).
\]
Then, for any \( v \in A_{p_0} \), (4) gives
\[
\int_{\mathbb{R}^n} f(x)^{p_0} v(x) dx = \int_{\mathbb{R}^n} \frac{1}{w(B(x, \alpha t))} \int_{B(x, \alpha t)} |f(y)|^{p_0} w(y) dy v(x) dx \\
\leq \alpha^{p_0} \int_{\mathbb{R}^n} \frac{1}{w(B(x, t))} \int_{B(x, t)} |f(y)|^{p_0} w(y) dy v(x) dx \\
\leq \alpha^{p_0} \int_{\mathbb{R}^n} g(x)^{p_0} v(x) dx.
\]
Applying Lemma 2.3, then, for any given \( 1 < r < \infty \) and \( v \in A_r \),
\[
\int_{\mathbb{R}^n} \left[ \frac{1}{w(B(x, \alpha t))} \int_{B(x, \alpha t)} |f(y)|^{p_0} w(y) dy \right] \frac{r}{p} v(x) dx \leq C \alpha^{n r} \int_{\mathbb{R}^n} \left[ \frac{1}{w(B(x, t))} \int_{B(x, t)} |f(y)|^{p_0} w(y) dy \right] \frac{r}{p} v(x) dx.
\]
From this and letting \( p_0 = \frac{r}{p} \), we prove Lemma 2.4 under the restriction \( r \in (1, \infty) \), that is, whenever \( r \in (1, \infty) \), \( v \in A_r \) and \( q \in [1, rp] \), there exists a positive constant \( C \) such that the inequality
\[
\int_{\mathbb{R}^n} \left[ \frac{1}{w(B(x, \alpha t))} \int_{B(x, \alpha t)} |f(y)|^{p_0} w(y) dy \right] \frac{r}{p} v(x) dx \leq C \alpha^{n r} \int_{\mathbb{R}^n} \left[ \frac{1}{w(B(x, t))} \int_{B(x, t)} |f(y)|^{p_0} w(y) dy \right] \frac{r}{p} v(x) dx.
\]
holds true.

Then, we consider the case \( r = 1 \) and \( 1 \leq q < p \). Without loss of generality, we assume that
\[
\int_{\mathbb{R}^n} \left[ \frac{1}{w(B(x, t))} \int_{B(x, t)} |f(y)|^{p_0} w(y) dy \right] \frac{1}{p} v(x) dx < \infty.
\]
For a fixed \( \lambda > 0 \), set
\[
E_{\lambda} := \left\{ x \in \mathbb{R}^n : \left[ \frac{1}{w(B(x, \alpha t))} \int_{B(x, \alpha t)} |f(y)|^{p_0} w(y) dy \right] \frac{1}{p} \leq \lambda \right\}
\]
and
\[
O_{\lambda} := \mathbb{R}^n \setminus E_{\lambda} = \left\{ x \in \mathbb{R}^n : \left[ \frac{1}{w(B(x, t))} \int_{B(x, t)} |f(y)|^{p_0} w(y) dy \right] \frac{1}{p} > \lambda \right\}.
\]
Then, for each \( 0 < \gamma < 1 \), we also consider the set of global \( \gamma \)-density with respect to \( E_{\gamma} \) defined by
\[
E_{\gamma}^* := \left\{ x \in \mathbb{R}^n : \frac{|E_{\lambda} \cap B|}{|B(x, r)|} \geq \lambda, \forall B centered at x \right\}
\]
and denote its complement by
\[
O_{\lambda}^* := \left\{ x \in \mathbb{R}^n : \exists r > 0 \text{ s.t. } \frac{|O_{\lambda} \cap B(x, r)|}{|B(x, r)|} > 1 - \gamma \right\} = \{ x \in \mathbb{R}^n : M(\chi_{O_{\lambda}})(x) > 1 - \gamma \}.
\]
By [15], it can prove that \( E_{\gamma}^* \) is closed and \( \emptyset \not\subseteq E_{\gamma}^* \subset E_{\lambda} \).

Denote by \( \mathcal{R}(E_{\lambda}^*) := \bigcup_{x \in E_{\lambda}^*} \{ y \in \mathbb{R}^n : |y - x| < \alpha t \} \). For any \( y \in \mathcal{R}(E_{\lambda}^*) \), there exists a \( \bar{x} \in E_{\lambda}^* \) such that \( |ar{x} - y| < \alpha t \). Let \( z = y - \frac{y - \bar{x}}{2|x - y|} \). Then \( B(z, \frac{t}{2}) \subset B(\bar{x}, \alpha t) \cap B(y, t) \) and
\[
|B(\bar{x}, \alpha t) \setminus B(y, t)| \leq \left| B(\bar{x}, \alpha t) \setminus B \left( z, \frac{t}{2} \right) \right| = |B(\bar{x}, \alpha t)| - \left| B \left( z, \frac{t}{2} \right) \right| = \left( 1 - \frac{1}{2\alpha \lambda n} \right) |B(\bar{x}, \alpha t)|.
\]

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As for \( \bar{x} \in E^*_\lambda \). Then we obtain that
\[
\gamma |B(\bar{x}, \alpha t)| \leq |E_\lambda \cap B(\bar{x}, \alpha t)| = |E_\lambda \cap B(\bar{x}, \alpha t) \setminus B(y, t)| + |E_\lambda \cap B(\bar{x}, \alpha t) \cap B(y, t)| \\
\leq \left( 1 - \frac{1}{2^n \alpha^n} \right) |B(\bar{x}, \alpha t)| + |E_\lambda \cap B(y, t)|.
\]
Choosing \( \gamma = 1 - \frac{1}{2^n + \alpha^n} \) yields
\[
|E_\lambda \cap B(y, t)| \geq \frac{1}{2^n + \alpha^n} |B(\bar{x}, \alpha t)|,
\]
which, together with (3) by Proposition 1.2, further implies that, for any \( y \in \Re(E^*_\lambda) \),
\[
\frac{v(E_\lambda \cap B(y, t))}{|B(y, \alpha t)|} \geq C \frac{|E_\lambda \cap B(y, t)|}{|B(\bar{x}, \alpha t)|} \geq C \frac{1}{2^n + \alpha^n}.
\]
(6)

It follows from (6) that
\[
\int_{E^*_\lambda} \frac{1}{w(\alpha B)} \int_{B(x, t)} |f(y)|^p w(y) dy v(x) dx = \frac{1}{w(B(\alpha, \alpha t))} \int_{E^*_\lambda} \int_{\Re^n} |f(y)|^p \chi_{B(0, 1)}(\frac{y - x}{\alpha t}) v(x) w(y) dy dx \\
= \frac{1}{w(B(\alpha, \alpha t))} \int_{\Re(E^*_\lambda)} |f(y)|^p w(y) \int_{B(y, \alpha t)} v(x) dx dy \\
\leq C \frac{2^n + \alpha^n}{w(B(\alpha, \alpha t))} \int_{\Re(E^*_\lambda)} |f(y)|^p w(y) \int_{B(y, \alpha t) \cap E_\lambda} v(x) dx dy \\
\leq C \frac{2^n + \alpha^n}{w(B(\alpha, \alpha t))} \int_{B(x, t)} |f(y)|^p w(y) dy v(x) dx.
\]
(7)

Since \( v \in A_1 \), by Lemma 2.5, (5), (7) and the definition of \( O^*_\lambda \), then
\[
v \left( \left\{ x \in \Re^n : \left[ \frac{1}{w(B(\alpha, \alpha t))} \int_{B(x, t)} |f(y)|^p w(y) dy \right]^{\frac{1}{p}} > \lambda \right\} \right) \\
\leq v \left( \left\{ x \in O^*_\lambda : \left[ \frac{1}{w(B(\alpha, \alpha t))} \int_{B(x, t)} |f(y)|^p w(y) dy \right]^{\frac{1}{p}} > \lambda \right\} \right) \\
+ \left\{ x \in E^*_\lambda : \left[ \frac{1}{w(B(\alpha, \alpha t))} \int_{B(x, t)} |f(y)|^p w(y) dy \right]^{\frac{1}{p}} > \lambda \right\} \\
\leq v(\{ x \in \Re^n : M_\lambda(x) > 1 - \gamma \}) + \frac{1}{\lambda^p} \int_{E^*_\lambda} \frac{1}{w(B(\alpha, \alpha t))} \int_{B(x, t)} |f(y)|^p w(y) dy v(x) dx \\
\leq C_n v \left( \left\{ x \in \Re^n : \left[ \frac{1}{w(B(\alpha, \alpha t))} \int_{B(x, t)} |f(y)|^p w(y) dy \right]^{\frac{1}{p}} > \lambda \right\} \right) \\
+ C_n v \left( \frac{1}{\lambda^p} \int_{E^*_\lambda} \frac{1}{w(B(\alpha, \alpha t))} \int_{B(x, t)} |f(y)|^p w(y) dy v(x) dx \right).
Using this and the assumption that $1 \leq q < p$, then

$$
\int_{\mathbb{R}^n} \left[ \frac{1}{w(B(x, \alpha t))} \int_{B(x,t)} |f(y)|^p w(y) dy \right]^\frac{1}{p} \nu(x) dx 
\leq \int_{0}^{\infty} q^\lambda v \left( \left\{ x \in \mathbb{R}^n : \frac{1}{w(B(x, \alpha t))} \int_{B(x,t)} |f(y)|^p w(y) dy > \lambda \right\} \right) \frac{d\lambda}{\lambda} 
\leq C_{n,v} \int_{0}^{\infty} q^\lambda v \left( \left\{ x \in \mathbb{R}^n : \frac{1}{w(B(x, t))} \int_{B(x,t)} |f(y)|^p w(y) dy > \lambda \right\} \right) \frac{d\lambda}{\lambda} 
+ C_{n,v} \int_{0}^{\infty} q^\lambda \nu \left( \left\{ x \in \mathbb{R}^n : \frac{1}{w(B(x, t))} \int_{B(x,t)} |f(y)|^p w(y) dy > \lambda \right\} \right) \frac{d\lambda}{\lambda}
\leq C_{n,v} \int_{\mathbb{R}^n} \left[ \frac{1}{w(B(x, t))} \int_{B(x,t)} |f(y)|^p w(y) dy \right] v(x) dx 
+ C_{n,v} \int_{\mathbb{R}^n} \frac{1}{w(B(x, t))} \int_{B(x,t)} |f(y)|^p w(y) dy \int_{\mathbb{R}^n} \left[ \frac{1}{w(B(x, \alpha t))} \int_{B(x,\alpha t)} |f(y)|^p w(y) dy \right] \frac{d\lambda}{\lambda} q^\lambda v(x) dx 
\leq C_{n,v} \int_{\mathbb{R}^n} \left[ \frac{1}{w(B(x, t))} \int_{B(x,t)} |f(y)|^p w(y) dy \right] v(x) dx.
$$

Thus, we complete the proof of Lemma 2.4. \hfill \Box

Lemma 2.5. \cite{16} Let $1 < p \leq \infty$ and $w \in A_p$, then there exist a positive constant $C$ such that

$$
\left( \int_{\mathbb{R}^n} |Mf(x)|^p w(x) dx \right)^\frac{1}{p} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^\frac{1}{p}.
$$

For all $\lambda > 0$, if $p = 1$, and $w \in A_1$, then there exist a positive constant $C$ such that

$$
w(\{x \in \mathbb{R}^n : |Mf(x)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| w(x) dx.
$$

The universal positive constant $C$ is independent of $f$ and $\lambda$.

Lemma 2.6. \cite{7} Let $1 < p < \infty$ and $w \in A_p$, then there exist a positive constant $C$ such that

$$
\left( \int_{\mathbb{R}^n} |Tf(x)|^p w(x) dx \right)^\frac{1}{p} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^\frac{1}{p}.
$$

For all $\lambda > 0$, if $p = 1$, and $w \in A_1$, then there exist a positive constant $C$ such that

$$
w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| w(x) dx.
$$

The universal positive constant $C$ is independent of $f$ and $\lambda$.

Lemma 2.7. \cite{17} Let $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, and $w \in A_{p,q}$, then there exist a positive constant $C$ such that

$$
\left( \int_{\mathbb{R}^n} |I_{\alpha} f(x) w(x)|^q dx \right)^\frac{1}{q} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^\frac{1}{p}.
$$
If $p = 1$, and $w \in A_{1,q}$ with $q = \frac{n}{n-\alpha}$, then for all $\lambda > 0$, then there exist a positive constant $C$ such that

$$w \left( \{ x \in \mathbb{R}^n : |I_\alpha(f)(x)| > \lambda \} \right) \leq C \left( \frac{1}{\lambda} \int_{\mathbb{R}^n} |f(x)|w(x)^\frac{q}{r} dx \right)^q.$$  

The universal positive constant $C$ is independent of $f$ and $\lambda$.

3 The Proofs of Main Theorems

In this section, the proofs of these theorems are given. We first recall the definition of the uncentered Hardy–Littlewood maximal operator, for almost every $x \in \mathbb{R}^n$,

$$\overline{M}f(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|dy.$$  

Then, we have the following pointwise inequality.

Lemma 3.1. For all $x \in \mathbb{R}^n$, $t > 0$, and $1 < r < \infty$, and all $f$ locally $r$ integrable, then

$$\left( \frac{1}{w(B(x,t))} \int_{B(x,t)} |M(f)(y)|^r w(y)dy \right)^\frac{1}{r} \leq \left( \frac{1}{w(B(x,2t))} \int_{B(x,2t)} |f(y)|^r w(y)dy \right)^\frac{1}{r}$$  

$$+ \overline{M} \left( \frac{1}{w(B(\cdot,t))} \int_{B(\cdot,t)} |f(z)|^r w(z)dz \right)^\frac{1}{r}(x).$$  

(8)

Proof of Lemma 3.1. Fix $x \in \mathbb{R}^n$ and $t > 0$, and split the supremum into $0 < \tau \leq t$ and $t < \tau$, and then

$$\left( \frac{1}{w(B(x,t))} \int_{B(x,t)} |M(f)(y)|^r w(y)dy \right)^\frac{1}{r}$$  

$$\leq \left( \frac{1}{w(B(x,t))} \int_{B(x,t)} \left( \sup_{0 < \tau \leq t} |B(y,\tau)| \int_{B(y,\tau)} |f(z)|dz \right)^r w(y)dy \right)^\frac{1}{r}$$  

$$+ \left( \frac{1}{w(B(x,t))} \int_{B(x,t)} \left( \sup_{t < \tau} |B(y,\tau)| \int_{B(y,\tau)} |f(z)|dz \right)^r w(y)dy \right)^\frac{1}{r} := I + II.$$  

For $I$, since $y \in B(x,t)$, $B(y,\tau) \subset B(x,2t)$. Then it follows from Lemma 2.5 that

$$I = \left( \frac{1}{w(B(x,t))} \int_{B(x,t)} \left( \sup_{0 < \tau \leq t} |B(y,\tau)| \int_{B(y,\tau)} |f(z)|dz \right)^r w(y)dy \right)^\frac{1}{r}$$  

$$\leq \left( \frac{1}{w(B(x,t))} \int_{B(x,t)} |M(f \cdot \chi_{B(x,2t)})(y)|^r w(y)dy \right)^\frac{1}{r}$$  

$$\lesssim \left( \frac{1}{w(B(x,2t))} \int_{B(x,2t)} |f(y)|^r w(y)dy \right)^\frac{1}{r}.$$  

For $II$, for any $z, \xi \in \mathbb{R}^n$, $z \in B(\xi, t)$ is equivalent to $z \in B(\xi, t)$. If $z \in B(y, \tau)$, $\xi \in B(z, t)$, then $\xi \in B(y, 2\tau)$. Besides, owing to $x \in B(y, t)$, then $x \in B(y, 2\tau)$. Applying the Fubini’s theorem and Hölder’s
inequality, then we get

\[
II = \left( \frac{1}{w(B(x,t))} \int_{B(x,t)} \left( \sup_{y \in B(y,\tau)} \frac{1}{|B(y,\tau)|} \int_{B(y,\tau)} |f(z)| dz \right)^r w(y) dy \right)^{\frac{1}{r}}
\]

\[
\leq \left( \frac{1}{w(B(x,t))} \int_{B(x,t)} \left( \sup_{y \in B(y,2\tau)} \frac{1}{|B(y,\tau)|} \int_{B(y,\tau)} |f(z)| dz d\xi \right)^r w(y) dy \right)^{\frac{1}{r}}
\]

\[
\leq \frac{1}{B^{(\cdot,t)}} \int_{B^{(\cdot,t)}} |f(z)| dz \right)^{\frac{1}{r}} (x)
\]

\[
\leq \frac{1}{w(B^{(\cdot,t)})} \int_{B^{(\cdot,t)}} |f(z)| w(z) dz \right)^{\frac{1}{r}} (x).
\]

Thus, we complete the proof of Lemma 3.1. □

**Proof of Theorem 1.1.** We first prove the case \( p \in (1, \infty) \) and \( q \in (1, \infty) \). By Lemma 2.4, Lemma 2.5, and Lemma 3.1, then

\[
||M(f)||_{(L_p,L_q)^r(\mathbb{R}^n)} = \left( \frac{1}{w(B^{(\cdot,t)})} \int_{B^{(\cdot,t)}} |M(f)(y)|^p w(y) dy \right)^{\frac{1}{p}} \left( \frac{1}{w(B^{(\cdot,t)})} \int_{B^{(\cdot,t)}} |f(y)|^p w(y) dy \right)^{\frac{1}{p}} \left( \frac{1}{w(B^{(\cdot,t)})} \int_{B^{(\cdot,t)}} |f(z)|^q w(z) dz \right)^{\frac{1}{q}} \left( \frac{1}{w(B^{(\cdot,t)})} \int_{B^{(\cdot,t)}} |f(z)|^q w(z) dz \right)^{\frac{1}{q}} \lesssim ||f||_{(L_p,L_q)^r(\mathbb{R}^n)}.
\]

And then for the case of \( p > 1 \) and \( q = 1 \).

By Lemma 2.1, there exists \( g \in (L_p^{f'}, L_{\infty}^{f'})(\mathbb{R}^n) \) such that

\[
\left\| \chi_{\{x \in \mathbb{R}^n : |Mf(x)| > \lambda \}} \right\|_{(L_p,L_q)^r(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \chi_{\{x \in \mathbb{R}^n : |Mf(x)| > \lambda \}} (x) g(x) dx.
\]

Since \( g(x) \leq |M(|g|^{\frac{1}{p}})|^\gamma (x) \) and \([M(|g|^{\frac{1}{p}})]^\gamma (x) \in A_1 \) for \( \gamma > 1 \), by Lemma 2.5, then it can obtain that

\[
\lambda \left\| \chi_{\{x \in \mathbb{R}^n : |Mf(x)| > \lambda \}} \right\|_{(L_p,L_q)^r(\mathbb{R}^n)} \leq \lambda \int_{\mathbb{R}^n} \chi_{\{x \in \mathbb{R}^n : |Mf(x)| > \lambda \}} (x) [M(|g|^{\frac{1}{p}})]^\gamma (x) dx \leq C \int_{\mathbb{R}^n} |f(x)| [M(|g|^{\frac{1}{p}})]^\gamma (x) dx.
\]

By taking the supremum over all \( \lambda > 0 \), then we get

\[
\|Mf\|_{W(L_p,L_q)^r(\mathbb{R}^n)} \leq C \int_{\mathbb{R}^n} |f(x)| [M(|g|^{\frac{1}{p}})]^\gamma (x) dx.
\]

By Lemma 2.2 and the fact \( M \) is bounded on \((L_w^{\infty}, L_{\infty}^{\infty})_1(\mathbb{R}^n)\),

\[
\|Mf\|_{W(L_p,L_q)^r(\mathbb{R}^n)} \leq C \|f\|_{(L_p,L_q)^r(\mathbb{R}^n)} \|M(|g|^{\frac{1}{p}})\|^\gamma_{(L_w^{\infty}, L_{\infty}^{\infty})_1(\mathbb{R}^n)} \leq \|f\|_{(L_p,L_q)^r(\mathbb{R}^n)} \|g\|_{(L_w^{\infty}, L_{\infty}^{\infty})_1(\mathbb{R}^n)}.
\]
Hence,
\[
\|Mf\|_{W(L^p_w, L^q_w)_t(\mathbb{R}^n)} \leq C\|f\|_{(L^p_w, L^q_w)_t(\mathbb{R}^n)}.
\]

This completes the proof of the Theorem 1.1. \qed

**Remark 3.1.** For any \( x \in \mathbb{R}^n \setminus \{0\} \),
\[
Hf(x) := \frac{1}{|x|^n} \int_{|y| \leq |x|} |f(y)|dy = \frac{C}{|B(0, |x|)|} \int_{B(0, |x|)} |f(y)|dy \leq CMf(x).
\]

By Theorem 1.1, we conclude that the Hardy operator \( H \) is bounded on \((L^p_w, L^q_w)_t(\mathbb{R}^n)\) for \( p, q > 1 \).

**Proof of theorem 1.2.** Let \( p = r\theta, \ q = s\theta \), where \( \theta > 1, \ r > 1 \) and \( s > 1 \). Then, by Lemma 2.1, for \( g \in (L^p_w, L^q_w)_t(\mathbb{R}^n) \), then
\[
\|Tf\|_{(L^p_w, L^q_w)_t(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |Tf(x)|^\theta |g(x)|dx \right)^{\frac{1}{\theta}}.
\]

Let \( w := |M(|g|^\frac{1}{\theta})|\gamma(x) \). The fact \( |M(|g|^\frac{1}{\theta})|\gamma(x) \in A_1, \ g(x) \leq |M(|g|^\frac{1}{\theta})|\gamma(x), \) and Lemma 2.6 yield
\[
\|Tf\|_{(L^p_w, L^q_w)_t(\mathbb{R}^n)} \leq \left( \int_{\mathbb{R}^n} |Tf(x)|^\theta |M(|g|^\frac{1}{\theta})|\gamma(x)dx \right)^{\frac{1}{\theta}} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^\theta |M(|g|^\frac{1}{\theta})|\gamma(x)dx \right)^{\frac{1}{\theta}}.
\]

By Lemma 2.2 and Theorem 1.1, thus,
\[
\|Tf\|_{(L^p_w, L^q_w)_t(\mathbb{R}^n)} \leq \|f\|_{(L^p_w, L^q_w)_t(\mathbb{R}^n)} \|M(|g|^\frac{1}{\theta})|\gamma\|_{(L^p_w, L^q_w)_t(\mathbb{R}^n)} \leq \|f\|_{(L^p_w, L^q_w)_t(\mathbb{R}^n)} \|g\|_{(L^p_w, L^q_w)_t(\mathbb{R}^n)}.
\]

And hence we complete the proof for the case of \( p, q > 1 \).

Now, we consider \( p > 1 \) and \( q = 1 \).

Taking \( g \in (L^p_w, L^\infty_w)_t(\mathbb{R}^n) \), by Lemma 2.1, we can write
\[
\|\chi\{x \in \mathbb{R}^n: |Tf(x)| > \lambda\}\|_{(L^p_w, L^1_w)_t(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \chi\{x \in \mathbb{R}^n: |Tf(x)| > \lambda\}g(x)dx.
\]

As for \( g(x) \leq |M(|g|^\frac{1}{\theta})|\gamma(x) \) with \( \gamma > 1 \), and by Lemma 2.6, then we know
\[
\lambda \|\chi\{x \in \mathbb{R}^n: |Tf(x)| > \lambda\}\|_{(L^p_w, L^1_w)_t(\mathbb{R}^n)} \leq \lambda \int_{\mathbb{R}^n} \chi\{x \in \mathbb{R}^n: |Tf(x)| > \lambda\}g(M(|g|^\frac{1}{\theta})\gamma(x)dx
\]
\[
\leq C \int_{\mathbb{R}^n} |f(x)|g(M(|g|^\frac{1}{\theta})\gamma(x)dx.
\]

Finally, take the supremum over \( \lambda > 0 \), hence, it can show that
\[
\|Tf\|_{W(L^p_w, L^1_w)_t(\mathbb{R}^n)} \leq C \int_{\mathbb{R}^n} |f(x)|g(M(|g|^\frac{1}{\theta})\gamma(x)dx.
\]

From Lemma 2.2 and Theorem 1.1, hence,
\[
\|Tf\|_{W(L^p_w, L^1_w)_t(\mathbb{R}^n)} \leq C\|f\|_{(L^p_w, L^1_w)_t(\mathbb{R}^n)} \|M(|g|^\frac{1}{\theta})|\gamma\|_{(L^p_w, L^\infty_w)_t(\mathbb{R}^n)} \leq \|f\|_{(L^p_w, L^1_w)_t(\mathbb{R}^n)} \|g\|_{(L^p_w, L^\infty_w)_t(\mathbb{R}^n)}.
\]

Hence, \( \|Tf\|_{W(L^p_w, L^1_w)_t(\mathbb{R}^n)} \leq C\|f\|_{(L^p_w, L^1_w)_t(\mathbb{R}^n)} \), the result holds. \qed
Proof of Theorem 1.3. For \( p_0, q_0 > 1 \), let
\[
\frac{1}{p} = \frac{1}{p_0} - \frac{\alpha}{n}, \quad \frac{1}{q} = \frac{1}{q_0} - \frac{\alpha}{n}.
\]
For \( \theta > 1 \) such that \( \frac{p}{\theta} > 1 \) and \( \frac{q}{\theta} > 1 \). Then, for \( g \in (L^{(p'/q')}_w, L^{(q'/p')}_{w'})_\lambda(\mathbb{R}^n) \), by Lemma 2.1, we can obtain that
\[
\|I_\alpha f\|_{(L^p_w, L^q_w)_{\lambda}(\mathbb{R}^n)} = \left\| |I_\alpha f(x)|^\theta g(x)dx \right\|^\frac{1}{\theta}.
\]
Noticing that for \( 0 < \eta < 1 \), \( M_\theta(|g|)(x) := |M_\theta(|g|)(x)|^\frac{1}{\theta} \in A_1 \), and letting \( w := |M_\theta(|g|)(x)|^\frac{1}{\theta} \), we have \( w^\alpha \in A_1 \) and hence \( w^\alpha \in A_{\theta, \alpha} \). Denote \( \frac{1}{\theta} = \frac{1}{p} + \frac{1}{q} \). Then \( w \in A_{\gamma, \theta} \). Moreover, by Lemmas 2.2 and 2.7, then
\[
\|I_\alpha f\|_{(L^p_L, L^q_L)_{\lambda}(\mathbb{R}^n)} \leq \left\| \int_{\mathbb{R}^n} |I_\alpha f(x)|^\gamma M_\eta(|g|)(x)dx \right\|^\frac{1}{\gamma} \leq C \left\| \int_{\mathbb{R}^n} |f(x)|^\gamma |w(x)|^\gamma dx \right\|^\frac{1}{\gamma} \leq C \left[ \|f\|^\gamma_{L^{(p_0'/q_0')}_w, L^{(q_0'/p_0')}_w}(\mathbb{R}^n) \|M_\eta(|g|)|^\gamma_{L^{(p_0'/q_0')}_w, L^{(q_0'/p_0')}_w}(\mathbb{R}^n) \right] \leq C \|f\|^\gamma_{L^{(p_0'/q_0')}_w, L^{(q_0'/p_0')}_w}(\mathbb{R}^n) \|M_\theta(|g|)|^\gamma_{L^{(p_0'/q_0')}_w, L^{(q_0'/p_0')}_w}(\mathbb{R}^n).
\]
Applying Theorem 1.1, then
\[
\|M_\theta(|g|)|^\gamma_{L^{(p_0'/q_0')}_w, L^{(q_0'/p_0')}_w}(\mathbb{R}^n) = \|M_\theta(|g|^\gamma)|^\gamma_{L^{(p_0'/q_0')}_w, L^{(q_0'/p_0')}_w}(\mathbb{R}^n) = \|M_\theta(|g|^\gamma)|^\gamma_{L^{(p_0'/q_0')}_w, L^{(q_0'/p_0')}_w}(\mathbb{R}^n) \leq C \|g\|^\frac{1}{\gamma}_{L^{(p_0'/q_0')}_w, L^{(q_0'/p_0')}_w}(\mathbb{R}^n) = C \|g\|^\frac{1}{\gamma}_{L^{(p_0'/q_0')}_w, L^{(q_0'/p_0')}_w}(\mathbb{R}^n).
\]
It is easy to check that
\[
\frac{1}{(p_0'/q_0')^\frac{\alpha}{\theta}} = (1 - \frac{\alpha}{n})^\frac{\gamma}{\theta} = (1 - \frac{\gamma}{p} - \frac{\alpha}{n})^\frac{\gamma}{\theta} = \theta(\frac{1}{\gamma} + \frac{\alpha}{n}) = 1 - \frac{1}{p/\theta} = \frac{1}{(p/\theta)^\gamma}
\]
and
\[
\frac{1}{(q_0'/p_0')^\frac{\alpha}{\theta}} = (1 - \frac{\alpha}{n})^\frac{\gamma}{\theta} = (1 - \frac{\gamma}{q} - \frac{\alpha}{n})^\frac{\gamma}{\theta} = \theta(\frac{1}{\gamma} + \frac{\alpha}{n}) = 1 - \frac{1}{q/\theta} = \frac{1}{(q/\theta)^\gamma}.
\]
So
\[
\|I_\alpha f\|_{(L^p_L, L^q_L)_{\lambda}(\mathbb{R}^n)} \leq C \|f\|_{L^{(p_0'/q_0')}_w, L^{(q_0'/p_0')}_w}(\mathbb{R}^n).
\]
Then, we consider the case of theorem 1.3 (b).
For \( p_0 > 1 \) and \( q_0 = 1 \), let
\[
\frac{1}{p} = \frac{1}{p_0} - \frac{\alpha}{n}, \quad \frac{1}{q} = \frac{1}{q_0} - \frac{\alpha}{n}.
\]
Take \( \theta = q = \frac{n}{n-\alpha} \). Then, for \( g \in (L^{(p'/q')}_w, L^{(q'/p')}_{w'})_\lambda(\mathbb{R}^n) \), by Lemma 2.1, we write
\[
\lambda \|\chi_{\{x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda\}}\|_{(L^p_w, L^q_w)_{\lambda}(\mathbb{R}^n)} = \lambda \|\chi_{\{x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda\}}\|_{(L^p_w, L^q_w)_{\lambda}(\mathbb{R}^n)} = \lambda \left( \int_{\mathbb{R}^n} \chi_{\{x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda\}}(x)g(x)dx \right)^\frac{1}{\theta}.
\]
And letting \( w := [M_\eta(|g|)(x)]^{\frac{1}{p}} \) with 0 < \( \eta < 1 \), we have \( w^\beta \in A_1 \) and hence \( w^\beta \in A_{\frac{p}{\eta}} \). Then \( w \in A_{1,\theta} \). By Lemma 2.2 and Lemma 2.7, it can obtain that

\[
\lambda \| \chi_{\{ x \in \mathbb{R}^n : |I_n f(x)| > \lambda \}} \|_{(L^p_\omega, L^q_\omega)_*} = \lambda \left[ \int_{\mathbb{R}^n} \chi_{\{ x \in \mathbb{R}^n : |I_n f(x)| > \lambda \}} M_\eta g(x) dx \right] \frac{1}{\lambda} \\
\leq C \int_{\mathbb{R}^n} |f(x)| w(x) dx \\
\leq C \| f \|_{(L^p_\omega, L^q_\omega)_*} \| [M_\eta(g)]^{\frac{1}{p}} \|_{(L^p_\omega, L^q_\omega)_*}.
\]

From Theorem 1.1, it follows that

\[
\| [M_\eta(g)]^{\frac{1}{p}} \|_{(L^p_\omega, L^q_\omega)_*} = \| [M(|g|^p)]^{\frac{1}{p}} \|_{(L^p_\omega, L^q_\omega)_*} = \| M(|g|^p) \|_{(L^p_\omega, L^q_\omega)_*} = \frac{1}{\eta^{\frac{1}{p}} \theta^{\frac{1}{q}}} C \| g \|_{(L^p_\omega, L^q_\omega)_*},
\]

since \( \frac{\theta}{\eta^{\frac{1}{p}}} = (1 - \frac{1}{p}) \theta = (1 - \frac{1}{p} - \frac{1}{p^n}) = 1 - \frac{1}{p^n} = \frac{1}{p^{n-1}} \), then we have

\[
\| I_n f \|_{W(L^p_\omega, L^q_\omega)_*} \leq C \| f \|_{(L^p_\omega, L^q_\omega)_*}.
\]

Thus, the result holds.

\[
\square
\]

4 Further Remarks

Notice that the proof of Theorem 1.2 only needs to use the boundedness of \( M \) on \( (L^p_\omega, L^q_\omega)_* \) and the boundedness of any operator on \( L^p_\omega(\mathbb{R}^n) \). So we introduce some operators, their simple proofs use Theorem 1.1 and the boundedness of the operator on \( L^p_\omega(\mathbb{R}^n) \) and is omitted.

Let \( S^{n-1}(n \geq 2) \) be the unit sphere in \( \mathbb{R}^n \) equipped with the normalized Lebesgue measure \( d\sigma \), \( \Omega(x) \) is homogeneous of degree zero on \( \mathbb{R}^n \) and \( \Omega \in L^q(S^{n-1}) \) with \( 1 < \theta \leq \infty \) and such that

\[
\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0
\]

(9)

where \( x' = \frac{x}{|x|} \) for any \( x \neq 0 \), the homogeneous singular integral operator \( T_\Omega \) can be defined by

\[
T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x')}{|y|^n} f(x - y) dy,
\]

and the Marcinkiewicz integral of higher dimension \( \mu_\Omega \) by

\[
\mu_\Omega f(x) = \left( \int_0^\infty \left( \int_{|x-y| \leq t} \frac{\Omega(x-y)}{|x-y|^{n-1}} f(y) dy \right)^2 dt \right)^{\frac{1}{2}}.
\]

Lemma 4.1. [6] For \( \Omega \in L^q(S^{n-1}) \) and \( 1 < \theta < \infty \), if \( \theta' \leq p < \infty \) and \( w \in A_{\frac{p}{\theta'}} \), then there exist \( C > 0 \) such that

\[
\| T_\Omega f \|_{L^p_\omega(\mathbb{R}^n)} \leq C \| f \|_{L^q_\omega(\mathbb{R}^n)}.
\]

If \( p = 1 \), \( w \in A_1 \), then there exist a constant \( C > 0 \) such that

\[
\| T_\Omega f \|_{L^1_\omega(\mathbb{R}^n)} \leq C \| f \|_{L^1_\omega(\mathbb{R}^n)}.
\]
Theorem 4.1. Let $0 < t < \infty$, $\Omega \in L^\theta(\mathbb{S}^{n-1})$, $1 < \theta, \gamma < \infty$, $\theta' \leq p < \infty$, $w \in A_{p'/\theta'}$.
(a) If $\gamma' \leq q < \infty$, and $v \in A_{q/\gamma'}$, then

$$||T_{\Omega}(f)||_{(L^p_v,L^s_w)(\mathbb{R}^n)} \leq C||f||_{(L^p_v,L^s_w)(\mathbb{R}^n)}.$$ 

(b) If $q = 1$ and $v \in A_1$, then

$$||T_{\Omega}(f)||_{W(L^p_v,L^s_w)(\mathbb{R}^n)} \leq C||f||_{(L^p_v,L^s_w)(\mathbb{R}^n)}.$$ 

The positive constant $C$ is independent of $f$ and $t$.

Lemma 4.2. [26] For $\Omega \in L^\theta(\mathbb{S}^{n-1})$ and $1 < \theta \leq \infty$, if $p$, $\theta$, $w$ satisfy one of the following conditions:

(i). $\theta' < p < \infty$, and $w \in A_{p'/\theta'}$,

(ii). $1 < p < \theta$, and $w^{1-\theta'} \in A_{p'/\theta'}$,

(iii). $1 < p < \infty$, and $w \in A_p$.

Then there exist constant $C > 0$ such that

$$||\mu_{\Omega}(f)||_{L^p_w(\mathbb{R}^n)} \leq C||f||_{L^p_w(\mathbb{R}^n)}.$$ 

If $p = 1$, $w \in A_1$, then there exist a positive constant $C$ such that

$$||\mu_{\Omega}(f)||_{L^1_w(\mathbb{R}^n)} \leq C||f||_{L^1_w(\mathbb{R}^n)}.$$ 

Theorem 4.2. Let $0 < t < \infty$, $\Omega \in L^\theta(\mathbb{S}^{n-1})$, $1 < p < \infty$ and $w \in A_p$.
(a) If $1 < q < \infty$ and $v \in A_q$, then there exist constant $C > 0$ such that

$$||\mu_{\Omega}(f)||_{(L^p_v,L^s_w)(\mathbb{R}^n)} \leq C||f||_{(L^p_v,L^s_w)(\mathbb{R}^n)}.$$ 

(b) If $q = 1$ and $v \in A_1$, then there exist constant $C > 0$ such that

$$||\mu_{\Omega}(f)||_{W(L^p_v,L^s_w)(\mathbb{R}^n)} \leq C||f||_{(L^p_v,L^s_w)(\mathbb{R}^n)}.$$ 

The universal positive constant $C$ is independent of $f$ and $t$.

We also define the Bochner-Riesz operator of order $\delta > 0$ in terms of Fourier transform by

$$ (T^\delta_R f)(\xi) = \frac{1}{(1 - |\xi|^2/R^2)^\delta} \hat{f}(\xi), $$

where $\hat{f}$ denote the Fourier transform of $f$. These operators can be defined by

$$ T^\delta_R f(x) = (f * \phi_{1/R})(x), $$

where $\phi(x) = [(1 - |x|^2)^{\delta/4}]^\vee(x)$, and $f^\vee$ is the inverse Fourier transform of $f$.

The associate maximal operator is defined by

$$ T^\delta_* f(x) = \sup_{R > 0} |T^\delta_R f(x)|. $$
Lemma 4.3. \textit{[12, 21, 25]} Let $n \geq 2$. If $1 < p < \infty$ and $w \in A_p$, then there exist constant $C$ such that
\[
\left( \int_{\mathbb{R}^n} |T_{n-1}^{(n-1)/2} f(x)|^q w(x) dx \right)^\frac{1}{q} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^q w(x) dx \right)^\frac{1}{q}.
\]
For any $\alpha, R > 0$, and $\phi_1$, if $C > 0$, then there exist constant $C$ such that
\[
w(\{x \in \mathbb{R}^n : T_{R}^{(n-1)/2}(f)(x) > \alpha\}) \leq \frac{C}{\alpha} \int_{\mathbb{R}^n} |f(x)|w(x) dx.
\]
The positive constant $C$ is independent of $f$ and $\alpha$.

Theorem 4.3. Suppose that $0 < t < \infty$, $1 < p < \infty$, $w \in A_p$.
\(\text{(a). If } 1 < q < \infty, \text{ and } v \in A_q, \text{ then we have}
\|T_{n-1}^{(n-1)/2}(f)\|_{(L^p_w, L^q_w)(\mathbb{R}^n)} \leq C \|f\|_{(L^p_w, L^q_w)(\mathbb{R}^n)}.
\)
\(\text{(b). For any } R > 0, \text{ if } q = 1 \text{ and } v \in A_1, \text{ then we have}
\|T_{R}^{(n-1)/2}(f)\|_{W(L^p_w, L^q_w)(\mathbb{R}^n)} \leq C \|f\|_{(L^p_w, L^q_w)(\mathbb{R}^n)}.
\)
The universal constant $C > 0$ is independent of $f$ and $t$.

Suppose that $\varphi(x) \in L^1(\mathbb{R}^n)$ satisfies
\[
\int_{\mathbb{R}^n} \varphi(x) dx = 0.
\]
The generalized Littlewood-Paley $g$ function $g_{\varphi}$ is defined by
\[
g_{\varphi}(f) = \left( \int_0^\infty |\varphi_t * f(x)|^2 \frac{dt}{t} \right)^\frac{1}{2},
\]
where $\varphi_t(x) = \frac{1}{t^n} \varphi(\frac{x}{t})$.

Lemma 4.4. \textit{[18]} Suppose that $\varphi \in L^1(\mathbb{R}^n)$ satisfies (10) and the following condition
\[
|\varphi(x)| \leq \frac{C}{(1 + |x|)^{n+1}}
\]
and
\[
|\nabla \varphi(x)| \leq \frac{C}{(1 + |x|)^{n+2}}.
\]
If $1 < p < \infty$, $w \in A_p$, then there exist constant $C > 0$ such that
\[
\|g_{\varphi}(f)\|_{L^p_w(\mathbb{R}^n)} \leq C \|f\|_{L^p_w(\mathbb{R}^n)}.
\]
If $p = 1$ and $w \in A_1$, then there exist constant $C > 0$ such that
\[
\|g_{\varphi}(f)\|_{L^1_{w, \infty}(\mathbb{R}^n)} \leq C \|f\|_{L^1_{w, \infty}(\mathbb{R}^n)}.
\]

Theorem 4.4. Suppose that $\varphi(x)$ satisfy (10), (11), and (12), for $1 < p < \infty$ and $w \in A_p$.
\(\text{(a). If } 1 < q < \infty \text{ and } v \in A_q, \text{ then there exist constant } C > 0 \text{ such that}
\|g_{\varphi}(f)\|_{(L^p_w, L^q_w)(\mathbb{R}^n)} \leq C \|f\|_{(L^p_w, L^q_w)(\mathbb{R}^n)}.
\)
(b). If \( q = 1 \) and \( v \in A_1 \), then there exist constant \( C > 0 \), such that

\[
\| g_\varphi(f) \|_{W(L_v^q, L_v^1)_t(R^n)} \leq C \| f \|_{(L_v^q, L_v^1)_t(R^n)}.
\]

The positive constant \( C \) is independent of \( f \) and \( t \).

The intrinsic square functions were first introduced by Wilson [22, 23] which are defined as follows. For \( 0 < \alpha \leq 1 \), let \( \mathcal{C}_\alpha \) be the family of functions \( \varphi \) defined on \( \mathbb{R}^n \) such that \( \varphi \) has support containing in \( \{ x \in \mathbb{R}^n : |x| \leq 1 \} \) and satisfy (10), for all \( x, y \in \mathbb{R}^n \),

\[
|\varphi(x) - \varphi(y)| \leq |x - y|^{\alpha}.
\]

And for \( (y, t) \in \mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0, \infty) \), and \( f \in L_{loc}^1(\mathbb{R}^n) \). Let

\[
A_\alpha(f)(y, t) = \sup_{\varphi \in \mathcal{C}_\alpha} |f * \varphi_t(y)| = \sup_{\varphi \in \mathcal{C}_\alpha} \left| \int_{\mathbb{R}^n} \varphi_t(y - z)f(z)dz \right|.
\]

Then we define the intrinsic square function of \( f \) (of order \( \alpha \)) by

\[
S_\alpha(f)(x) = \left( \int_{\mathbb{R}^n} (A_\alpha(f)(y, t))^2 dy dt \right)^{\frac{1}{2}},
\]

where \( \Gamma(x) = \{ (y, t) \in \mathbb{R}^{n+1}_+ : |x - y| < t \} \) and \( \varphi_t(x) = \frac{1}{t^n} \varphi(x/t) \).

Lemma 4.5. [22] Let \( 0 < \alpha \leq 1 \), if \( 1 < p < \infty \), and \( w \in A_p \), then there exist constant \( C > 0 \) such that

\[
\| S_\alpha(f) \|_{L_w^p(\mathbb{R}^n)} \leq C \| f \|_{L_w^p(\mathbb{R}^n)}.
\]

If \( p = 1 \), \( w \in A_1 \), then there exist \( C > 0 \) such that

\[
\| S_\alpha(f) \|_{L_w^\infty(\mathbb{R}^n)} \leq C \| f \|_{L_w^1(\mathbb{R}^n)}.
\]

Theorem 4.5. Let \( 0 < \alpha \leq 1 \). For \( 1 < p < \infty \) and \( w \in A_p \).

(a). If \( 1 < q < \infty \) and \( v \in A_q \), then there exist constant \( C > 0 \), such that

\[
\| S_\alpha(f) \|_{(L_v^q, L_v^1)_t(\mathbb{R}^n)} \leq C \| f \|_{(L_v^q, L_v^1)_t(\mathbb{R}^n)}.
\]

(b). If \( q = 1 \) and \( v \in A_1 \), then there exist constant \( C > 0 \), such that

\[
\| S_\alpha(f) \|_{W(L_v^q, L_v^1)_t(\mathbb{R}^n)} \leq C \| f \|_{(L_v^q, L_v^1)_t(\mathbb{R}^n)}.
\]

The positive constant \( C \) is independent of \( f \) and \( t \).

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