Energy of Stable Half-Quantum Vortex in Equal-Spin-Pairing

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Abstract. In the triplet equal-spin-pairing states of both $^3\text{He} - A$ phase and $\text{Sr}_2\text{RuO}_4$ superconductor, existence of Half-Quantum Vortices ($HQV$s) are possible. The vortices carry half-integer multiples of magnetic quantum flux $\Phi_0 = \frac{hc}{2e}$. Our method is based on the explanation of the $HQV$ in terms of a $BCS$--like wave function with a spin-dependent boots. To obtain equilibrium condition for such systems, one has to take into account not only weak interaction energy but also effects of Landau Fermi liquid. We have considered $\ell = 2$ order effects of the Landau Fermi liquid. We have shown that the effects of Landau Fermi liquid interaction with $\ell = 2$ are negligible. In stable $HQV$, an effective Zeeman field exists. In the thermodynamic stability state, the effective Zeeman field produces a non-zero spin polarization in addition to the polarization of external magnetic field.

1. Introduction
One view to the liquid phase of $^3\text{He}$ contains two normal and superfluid parts which at $3 \times 10^{-3}$ Kelvin the superfluid part starts to be occurred mostly in triplet pairing [1, 2]. The spin triplet pairing in the superconductor compound $\text{Sr}_2\text{RuO}_4$ bellow 1.5 Kelvin is observed experimentally [3].

The triplet pairing contains particles with the same spin directions that leads to spin current. The spin current leads to interesting and important phenomena such as half-quantum vortices ($HQV$s) in equal-spin-pairing ($ESP$). Unlike common vortices, the half-quantum vortices contain half-integer multiplications of the flux quantum $\Phi_0 = \frac{hc}{2e}$. The origins of vortices in the type-II superconductors and $^3\text{He}/^4\text{He}$ superfluid are different. In the former case, the vortices are appeared in the presence of external magnetic field while appearance of the vortices in the latter cases is caused by the rotation of the vessel which $^3\text{He}/^4\text{He}$ is contained. Also external magnetic field influences the vortices in $^3\text{He}$ and can generate half-quantum vortices. Vakaryuk and Leggett[4] have shown that in equal-spin-pairing state, the stability condition of half-quantum vortices is obtained where not only weak interactions taken into account but also strong interactions. The general method for calculating the strong interactions are presented by the Landau Fermi liquid theory. They have considered only $\ell = 1$ term in this interaction. In this work, $\ell = 2$ term is accounted and it is found that term with $\ell = 2$ is very small compared with $\ell = 0, 1$ terms.
2. Theoretical approach

In the ESP state of a spin triplet condensate, the spin of particles in the Cooper pair is either aligned (up) or antialigned (down) with a common direction in the space which is called ESP axis\([1, 2]\). Therefore Cooper pairs may be described via a linear superposition of states \([\uparrow \uparrow]\) and \([\downarrow \downarrow]\). The pairs are condensed in the same orbital states that we show then using functions \(\varphi_\uparrow\) and \(\varphi_\downarrow\). The many-body wave function which describes a system of \(N/2\) pairs by \(\varphi_\uparrow\) and \(\varphi_\downarrow\) can be written as

\[
\Psi_{ESP} = A\{[\varphi_\uparrow(r_1, r_2) | \uparrow \uparrow]\} + \varphi_\downarrow(r_1, r_2) | \downarrow \downarrow]\}...[\varphi_\uparrow(r_{N-1}, r_N) | \uparrow \uparrow] + \varphi_\downarrow(r_{N-1}, r_N) | \downarrow \downarrow]\}
\]

in which \(A\) is the antisymmetrization operator with respect to particles coordinates \(r_i\) and spins. To prevent complications connected to the presence of the core of vortex annular geometry is considered. Also for simplicity it will be assumed that a neutral ESP system is studied. For annulus with the radius \(R\) and the wall thickness \(d\) it is assumed that \(d/R \ll 1\) so that one can ignore the effects of order \(d/R\) or higher. Therefore at the zero temperature case, the HQV state of the condensate can be described via \([4]\)

\[
\Psi_{HQV} = \exp\{i\ell_\parallel \sum_{i=\parallel} \theta_i + i\ell_\perp \sum_{i=\perp} \theta_i\}\Psi_{ESP},
\]

where \(\theta_i\) is the azimuthal coordinate of the \(i\)th particle on the annulus and the spin axis is along the symmetry axis of the annulus. The integer \(\ell_\sigma\) denotes the angular momentum of \(\sigma\), the component of pair wave function.

The thermal stability of system is obtained via minimization of energy of the system using the wave function Eq. (2) that one essentially needs full version of original Hamiltonian. In the simplest case, one may consider BCS Hamiltonian \(H_{BCS}\) along with spin triplet pairing term; however the mentioned Hamiltonian \(H\) is unable to provide the thermodynamic stability of the HQV. In such systems, the stability condition is obtained where not only weak interactions taken into account but also strong interactions. The general method for calculation the strong interactions are presented by the Landau Fermi liquid theory. Although the method was used for investigating the normal metals, the generalized version of the method is used for studying superconductor and superfluid. Thus Hamiltonian of the system involves two parts; 1) BCS Hamiltonian and 2) Landau Fermi liquid. Then \(H = H_{BCS} + H_{FL}\) in which \(H_{FL}\) is energy corrections of Fermi liquid effects and \(H_{BCS}\) denotes the usual BCS Hamiltonian with the spin triplet pairing term describing the weak coupling part of the theory. In the beginning, the expectation value of the weak coupling Hamiltonian on the state Eq.(2) is calculated. In this case, one can write it as a sum of three terms with different physical sources:

\[
E_{BCS} = E_0 + E_S + T.
\]

where \(E_0\) is the energy contribution originated from the freedom internal degrees of Cooper pairs. This contribution is independent on the center of mass motion of the Cooper pairs i.e., on quantum numbers \(\ell_\parallel\) and \(\ell_\perp\) and the magnetic field magnitude for the annulus radius \(R\) where is much larger than the BCS coherence length \(\xi_0\) \([5]\). In this work we assume a large enough annuls so that this term is ignorable.

The second term in the Eq. (3) is the energy of spin polarization of the system. We assume \(N_\sigma\) as the particles number with spin projection \(\sigma\). one can define \(S\) as a projection of the total number spin polarization on the symmetry axis as \(S \equiv (N_\uparrow - N_\downarrow)/2\). Therefor, spin polarization energy is obtained

\[
E_S = \frac{(gS\mu_B S)^2}{2\chi_{ESP}} - gS\mu_B B\cdot S,
\]

where \(g\) is the Landé g-factor of the Cooper pair, \(\mu_B\) is the Bohr magneton, \(B\) is the magnetic field, and \(\chi_{ESP}\) is the effective exchange constant.
in which $g_S$ is the gyromagnetic ratio for particles and $\chi_{ESP}$ is the ESP state spin susceptibility. It is important that the spin polarization $S$, is a variational parameter and the actual value of $S$ is obtained by minimization of energy.

Also $T$ in the Eq. (3) is the kinetic energy of the currents circulating in the system. We introduce the new parameters as [4]:

$$\ell_{s\Phi} \equiv \frac{\ell_{\uparrow} + \ell_{\downarrow}}{2} - \frac{\Phi}{\Phi_0}, \ell_{sp} \equiv \frac{\ell_{\uparrow} - \ell_{\downarrow}}{2},$$

(5)

where $\Phi$ is the total flux through the annulus and $\Phi_0 = \hbar c/2e$ is the flux quantum. By using the above introduced parameters $T$ takes the following form:

$$T = \frac{\hbar^2}{8 m^* R^2} \{ (\ell_{s\Phi}^2 + \ell_{sp}^2) N + 4 \ell_{sp} \ell_{s\Phi} \},$$

(6)

in which $m^*$ is the effective mass of particles containing the Fermi liquid corrections. It is related to the bare mass of particles $m$ by the usual relation of Fermi liquid theory as $m^* = m(1 + F_1/3)$ [1, 2]. In Eq. (6) the first term in the brackets is constant. Because it is proportional to the total number of particles $N \equiv N_\uparrow + N_\downarrow$ and given values of $\ell_{s\Phi}, \ell_{sp}$. The second term generates an effective Zeeman field in the HQV state due to proportional to the spin polarization $S$. The magnitude of this field and hence the value of the thermal equilibrium spin polarization should be obtain by energy minimization [4].

To make an HQV stable it should be considered the strong coupling effects [1, 6]. Also it needs to be accounted the change of Fermi liquid energy $E_{FL}$ due to the presence of spin and momentum currents in the HQV state. These currents are created by the spin dependent boost of Eq. (2). We use the standard formalism of Fermi liquid theory and investigate $\ell = 2$ order effects of Landau Fermi liquid. By considering the mentioned assumptions, we finally reach at the lengthy expression as follow:

$$E_{FL} = \frac{1}{2} \left( \frac{dn}{dc} \right)^{-1} Z_0 S^2 + N^{-1} \frac{\hbar^2}{8 m^* R^2} \frac{1}{3} \left[ (\ell_{s\Phi}^2 F_1 + \ell_{sp}^2 Z_1) N^2 + 4 (\ell_{s\Phi}^2 F_1 + \ell_{sp}^2 Z_1) S^2 + 4 \ell_{s\Phi} \ell_{sp} (F_1 + \frac{Z_1}{4}) S N \right]$$

$$+ \frac{N^{-1} \hbar^4}{64 m^* R^2 R_1} \left[ (F_2 + \frac{Z_2}{2}) \ell_{s\Phi}^2 + \ell_{sp}^2 \right] 2 \ell_{s\Phi} \ell_{sp} (F_1 + \frac{Z_1}{4}) S N \right]$$

$$+ \frac{N^{-1} \hbar^4}{12 m^*} \left( (8 F_2 + 2 Z_2) \ell_{s\Phi} \ell_{sp} (\ell_{s\Phi}^2 + \ell_{sp}^2) \right) S \right]$$

(7)

where $(dn/dc)$ is the density of states at the Fermi surface and $Z_0, Z_1, Z_2, F_0$ and $F_1$ are Landau parameters. The first term in the above equation, proportional to $Z_0$, is the energy cost generated by a spin polarization and the rest explains Fermi liquid corrections caused by the presence of currents.

Now the equilibrium spin polarization in the HQV state is obtained. For this one must minimize the total energy $E_{BCS} + E_{FL}$ with respect to $S$. Then $S$ can be written as

$$S = (g_S \mu_B)^{-1} \chi B,$$

(8)

where $\chi$ is the spin susceptibility of the system. For $^3He - A$ the value of $\chi$ is approximately 0.37 times smaller than the normal state susceptibility at low temperatures [6]. We also find $\chi$ as:

$$\chi^{-1} = \frac{1}{\chi_{ESP}} + \left( \frac{dn}{dc} \right)^{-1} \left( Z_0 (g_S \mu_B)^{-2} + \frac{N^{-1} \hbar^2}{m^* R^2 (g_S \mu_B)^2} \left( \ell_{s\Phi}^2 F_1 + \ell_{sp}^2 Z_1 \right) \right)$$

$$+ \frac{N^{-1} \hbar^4}{16 m^* R^2 R_1 (g_S \mu_B)^2} \ell_{s\Phi}^2 \ell_{sp} (8 F_2 + Z_2) - \frac{N^{-1} \hbar^4 Z_2}{6 m^* (g_S \mu_B)^2},$$

(9)
The effective Zeeman field $B'$ is involved two parts: 1) the external Zeeman field $B$ and 2) the effective Zeeman field $B_{\text{eff}}$:

$$B' = B + B_{\text{eff}},$$

(10)

The effective Zeeman field $B_{\text{eff}}$ is due to the present of spin currents:

$$B_{\text{eff}} = -\frac{\hbar^2 (g_S \mu_B)}{2m^* R^2} \ell_{sp} \ell_s \Phi \{ 1 + \frac{F_1}{3} + \frac{Z_1}{12} + \frac{\hbar^2}{16 p_F R^2} (4F_2 + Z_2)(\ell_{sp}^2 + \ell_{sp}) \}.\quad (11)$$

The effective Zeeman field is a periodic function of the total external magnetic flux $\Phi$ with a period equal to $\Phi_0$. The sign of the effective field is altered when the total magnetic field is equal to half-integer values of the flux quantum[4].

Finally, the energy of the system $E = \langle H \rangle$ can be obtained by inserting account of $S$ to the relation of the total energy and ignoring the internal energy contribution. Then one can write:

$$E = -\frac{1}{2} \chi B' + \frac{\hbar^2 N}{8m R^2} \{ \ell_{sp}^2 + \ell_s^2 \} \{ 1 + \frac{Z_1}{12} + \frac{\hbar^2}{64 m p_F R^4} (4F_2 + Z_2)(\ell_{sp}^2 + \ell_{sp}) \}.$$

(12)

The contribution of energy of the spin polarization (first term of Eq. 12) is small for reasonable values of the external magnetic field. The third term of the equation is related to $\ell = 2$ and is of $\hbar^2/2m R^2 \ell_F$ order compared with the second term. Evidently, one can ignore them for analyzing the stability of HQVs. The stability region of the HQV's depends of the $(1 + Z_1/12)/(1 + F_1/3)$ that is, the ratio of superfluid spin density to superfluid density $\rho_{sp}/\rho_s$ [1]. The criteria of stability is directly obtained by minimizing of Eq. (12) and leads to $\rho_{sp}/\rho_s < 1$ [7]. In the $^3He - A$, the ratio is less than unity for all temperatures bellow critical temperature and then HQVs are possible for a large limit of the phases diagram[7].

3. Conclusions

In the equal-spin pairing condensation of HQV an effective Zeeman field $B_{\text{eff}}$ exists. In the thermodynamic stability state, the effective Zeeman field produces a non zero spin polarization in addition to the polarization of external magnetic field $B$. The thermal stability of system is obtained via minimization of energy of the system using the wave function that one essentially needs correct Hamiltonian. The best Hamiltonian of the system involves two parts: 1) BCS Hamiltonian and 2) Landau Fermi liquid. $\ell = 2$ order effects of Landau Fermi liquid have been considered in this paper. The third term in Eq. (12) contains Landau parameters $Z_2$ and $F_2$, which is of $\hbar^2/2m R^2 \ell_F$ order compared with the second term in this equation. This quantity takes approximately $10^{-7}$ where the radius of specimen ring is about 0.1 micron in annular geometry. Therefore, one can omit this term in Eq. (12) without concerns. Consequently for the stability condition of HQVs, it is suffice that the second term of Eq. (12) be considered and $\rho_{sp}/\rho_s < 1$ is obtained for stability condition.

References

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