An Investigation of the Tomimatsu-Sato Spacetime

Wataru Hikida\textsuperscript{1} and Hideo Kodama\textsuperscript{2}

\textit{Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan}

\textbf{Abstract}

We investigate the structure of the $\delta = 2$ Tomimatsu-Sato spacetime. We show that this spacetime has degenerate horizons with two components, in contrast to the general belief that the Tomimatsu-Sato solutions with even $\delta$ do not have horizons.

\section{Introduction}

One of the most important problems in the relativistic astrophysics and gravitation theory is the final state of a massive star. In general relativity, such a collapse is usually assumed to end up with the formation of a regular black hole, but the possibility of the formation of a naked singularity is not ruled out. Concerning to this problem, Penrose \cite{1} proposed the conjecture that the latter possibility will not be realized under physically realistic conditions, which is called the \textit{cosmic censorship hypothesis}. This conjecture, however, still remains to be proved and there are many counterexamples in the spherically symmetrical cases.

The best known example of naked singularities is the shell-focusing singularities in the spherically symmetric dust collapse, so called the Tolman-Bondi model \cite{2, 3}. Another important example is the critical phenomena in gravitational collapse, which was first found for spherically symmetric collapse of a massless scalar field by Choptuik \cite{4}. In this example, although the naked singularity formation is not generic, a naked singularity solution and a self-similar structure appears as a critical point for a generic continuous initial data. Hence, there exist a generic set of regular solutions in any small neighborhood of the singular solutions in the initial data space. Due to this, the structure of the singular solutions play a crucial role in determining the behavior of solutions near the critical point, and the fate of the gravitational collapse.

Although these examples show that the formation of naked singularities is rather generic and solutions with naked singularities may play a key role for the spherically symmetric gravitational collapse, the assumption of the spherically symmetry is too restrictive to say anything on the cosmic censorship hypothesis in realistic systems. We must investigate systems with less symmetry. In such investigations, it is expected that the analysis of naked singular solutions provides useful information on the final fate of gravitational collapse, as the above example of the critical phenomena indicates. From this point of view, the Tomimatsu-Sato solutions are the best system to analyze as the starting point, because they are the simplest extension of the Kerr black hole solution.

Although it has been a long time since the Tomimatsu-Sato solutions were found, little is known about these solutions yet and sometimes wrong statements were made in the literature. For example, although it has been shown that the Tomimatsu-Sato solutions with odd $\delta$ have a Killing horizon at the segment $\rho = 0, |z| < \sigma$ in the Weyl coordinate, it has been believed that the solutions with even $\delta$ have no horizon because the corresponding segment for these solutions is not Killing horizon \cite{7}. In the present paper, we will show that this belief is false and the two points $\rho = 0, z = \pm \sigma$ are actually degenerate horizons for $\delta = 2$.

\section{Tomimatsu-Sato Spacetime}

In canonical form of the stationary axisymmetric metric \cite{5}

$$ds^2 = -f(dt - \omega d\phi)^2 + f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2],$$

(1)
the Kerr-TS family is expressed as [6]
\[ f = \frac{A}{B}, \quad \omega = \frac{2mq}{A} (1 - y^2) C, \quad e^{2\gamma} = \frac{A}{p^2(x^2 - y^2)^2}, \]  
(2)
where the functions \( A, B \) and \( C \) are polynomials with the degrees of \( 2\delta^2, 2\delta^3 \) and \( 2\delta^2 - 1 \), respectively, in the prolate spheroidal coordinates \( x \) and \( y \) defined by
\[ \rho = \sigma \sqrt{(x^2 - 1)(1 - y^2)}, \quad z = \sigma xy, \]  
(3)
and \( \sigma, p \) and \( q \) are related with mass \( m \) and angular momentum \( J \) as
\[ p^2 + q^2 = 1, \quad \sigma = \frac{mp}{\delta}, \quad J = m^2 q. \]  
(4)
In the Kerr (\( \delta = 1 \)) and the \( \delta = 2 \) Tomimatsu-Sato case, \( A, B \) and \( C \) are given by
\[ \begin{align*}
\delta = 1 & \quad : \quad A = p^2(x^2 - 1) - q^2(1 - y^2), \quad B = (px + 1)^2 + q^2 y^2, \quad C = -px - 1, \\
\delta = 2 & \quad : \quad A = p^4(x^2 - 1)^4 + q^4(1 - y^2)^4 \\
& \quad -2p^2 q^2(x^2 - 1)(1 - y^2)(2(x^2 - 1)^2 + 2(1 - y^2)^2 + 3(x^2 - 1)(1 - y^2)) \\
& \quad B = \{p^2(x^2 + 1)(x^2 - 1) - q^2(y^2 + 1)(1 - y^2) + 2px(x^2 - 1)\}^2 \\
& \quad +4q^2 y^2 \{px(x^2 - 1) + (px + 1)(1 - y^2)\}^2 \\
& \quad C = -p^2x(x^2 - 1)(2(x^2 + 1)(x^2 - 1) + (x^2 + 3)(1 - y^2)) \\
& \quad -p^2(x^2 - 1)\{4x^2(x^2 - 1) + (3x^2 + 1)(1 - y^2)\} + q^2(px + 1)(1 - y^2)^3.
\end{align*} \]  
(5)

In this paper, we focus on the \( \delta = 2 \) Tomimatsu-Sato metric. The properties of the corresponding spacetime that have been found so far are summarized as follows [6] [7].

**Ring singularity:** This spacetime has a ring singularity at the root of \( B(x, y = 0) = 0 \). (The cross in Figure II)

**Ergosphere:** The timelike Killing vector becomes null at the roots of \( A \). There are two single roots for \( x \geq 1 \) (The dotted lines in Figure II), the smaller of which coincides with the ring singularity.

**Causality violation region:** Since \( \phi \) is the periodic angular coordinate, this spacetime has closed timelike loops in the region with negative \( g_{\phi\phi} \). (The shaded portion in Figure II)

**Directional singularity:** This metric has directional singularities at the points \( (\rho, z) = (0, \pm \sigma) \), where the value of a curvature invariant has different limits when the point is approached from different directions. (The two filled circles in Figure II)

**The nature of the surface** \( x = 1 \): In the Kerr case, \( x = 1(\rho = 0, |z| \leq \sigma) \) surface is an event horizon. However, in the \( \delta = 2 \) Tomimatsu-Sato case, this is not the case. Because the induced metric is Lorentzian and two Killing vectors \( \partial_t \) and \( \partial_{\phi} \) become parallel there, \( x = 1 \) surface cannot be a null surface.

### 3 Extension of Tomimatsu-Sato spacetime

Ernst [8] pointed out that the quasi-regular directional singularities at \( \rho = 0, z = \pm \sigma \) are hypersurfaces in reality, by introducing a polar coordinate system around each point. He further studied the behavior of geodesics crossing these surfaces, and concluded that they are timelike surfaces. Now, we show that his conclusion is false and these surfaces are horizons, by introducing an appropriate coordinate system.

First, we rewrite the metric in the former
\[ ds^2 = -e^{2\nu} dt^2 + e^{2\psi}(d\phi - \Omega dt)^2 + e^{2\mu} \left( \frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right), \]  
(7)
Figure 1: The structure of the spacetime. The crossed sign, dotted lines and the two filled circles denote the ring singularity, the infinite redshift surfaces and the directional singularities respectively. In the shaded portion, the closed timelike loops exist.

Figure 2: The relation between $(\rho, z)$ and $(X, W)$ coordinate system. The solid lines and the dotted lines are $X =$constant and $W =$constant lines respectively.

where

$$e^{2\nu} = \frac{p^2(x^2 - 1)B}{D}, \quad \Omega = -\frac{4pqC}{D}, \quad e^{2\psi} = \frac{(1 - y^2)D}{p^2B}, \quad e^{2\mu} = \frac{B}{p^4(x^2 - y^2)^3}.$$  (8)

and $D$ is a polynomial determined by the equation

$$D = \frac{p^2(x^2 - 1)B^2 - 4\delta^2q^2(1 - y^2)C^2}{A}.\quad (9)$$

In the $\delta = 2$ Tomimatsu-Sato case, the explicit expression for $D$ is given by

$$D = p^6(x^2 - 1)(x^8 + 28x^6 + 70x^4 + 28x^2 + 1) - 16q^6(1 - y^2)^3$$
$$+ p^4q^2(32x^2(x^4 + 4x^2 + 1) - 4(1 - y^2)(x^2 - 1)^3 + (-6x^4 + 12x^2 + 10)(1 - y^2)^3)$$
$$- 4(1 - y^2)^3(x^4 + 6x^2 + 1)] + p^2q^4[(x^2 - 1)(64x^4 + (1 - y^2)^2(y^4 + 14y^2 + 1))$$
$$- 16(1 - y^2)^3(x^2 + 1)] + 8p^3q^2x(x^2 + 1)(x^4 + 6x^2 + 1) - 32pq^4x(1 - y^2)^3$$
$$+ 8p^3q^2x[(x^2 - 1)(8x^2(x^2 + 1) + (1 - y^2)^2(2y^2 - x^2 + 1)] - 4(1 - y^2)^3].\quad (10)$$

Here, we have renormalized $\sigma$ as $\sigma = 1$.

Next, we introduce the new coordinates $X$ and $W$ by

$$x^2 = \frac{1 + X}{1 + X - W}, \quad y^2 = \frac{(1 + X)(1 - W)}{1 + X - W}.\quad (11)$$

Figure 2 shows the relation between the $(\rho, z)$ and the $(X, W)$ coordinate system. In this $(X, W)$ coordinate system, the metric is written

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\phi - \Omega dt)^2 + e^{2\mu}\frac{(1 + X)W}{4(1 + X - W)} \left[ \frac{dX^2}{(1 + X)^2X} + \frac{dW^2}{(1 - W)W^2} \right].\quad (12)$$
In the region near $W = 0$, $A, B, C$ and $D$ behave as

$$
A = \left( \frac{W}{1 + X - W} \right)^4 (p^4 + X^4q^4 - 4p^2q^2X - 4p^2q^2X^3 - 6p^2q^2X^2),
$$

$$
B \sim 8(p + 1)(p^2 + q^2X^2) \left( \frac{W}{1 + X - W} \right)^2 + 8(p + 1)(p^2 - q^2X^3) \left( \frac{W}{1 + X - W} \right)^3,
$$

$$
C \sim -4p^2(1 + X)(p + 1) \left( \frac{W}{1 + X - W} \right)^2,
$$

$$
D \sim 64p^2(p + 1)^2 \left( \frac{W}{1 + X - W} \right) + 128p^2(p + 1)^2 \left( \frac{W}{1 + X - W} \right)^2. \quad (13)
$$

From this, it follows that at $W \sim 0$, $e^{2\nu}, e^\psi, \Omega$ and $e^\mu$ are expressed as

$$
e^{2\nu} \sim \frac{p^2 + q^2X^2}{8(p + 1)} \left( \frac{W}{1 + X} \right)^2 + \frac{-q^2X^3 + p^2}{8(p + 1)} \left( \frac{W}{1 + X} \right)^3,
$$

$$
\Omega \sim \frac{pq(1 + X)}{4(p + 1)} \left( \frac{W}{1 + X - W} \right),
$$

$$
e^{2\psi} \sim \frac{8(p + 1)X}{(p^2 + q^2X^2)},
$$

$$
e^{2\mu} \sim \frac{8(p + 1)(p^2 + q^2X^2)}{p^4W(1 + X)^2} + \frac{8(p + 1)(-q^2X^2 - q^2X^3)}{p^4(1 + X)^3}. \quad (14)
$$

This asymptotic behavior suggests that the $\delta = 2$ Tomimatsu-Sato spacetime can be extended regularly across the surface $W = 0$ in terms of the advanced coordinate $v$ and the $\phi_+$ coordinate defined by

$$
dv = dt + \frac{4(p + 1)dW}{p^2W^2\sqrt{1 - W}}, \quad d\phi_+ = d\phi + \frac{qdw}{pw\sqrt{1 - W}}. \quad (15)
$$

Actually, in the $(v, W, X, \phi_+)$ coordinate system, the metric is expressed as

$$
ds^2 = -e^{2\nu} \left( dv - \frac{4(p + 1)dW}{p^2W^2\sqrt{1 - W}} \right)^2 + e^{2\psi} \left[ \left( d\phi_+ - \frac{qdw}{pw\sqrt{1 - W}} \right) - \Omega \left( dv - \frac{4(p + 1)dW}{p^2W^2\sqrt{1 - W}} \right) \right]^2
$$

$$
+ e^{2\mu} \left( \frac{W}{1 + X - W} \right) \left[ \frac{dX^2}{(1 + X - W)} + \frac{dW^2}{(1 - W)W} \right]
$$

$$
= \frac{-p^2 + q^2X^2}{8(p + 1)} \left( \frac{W}{1 + X} \right)^2 [1 + O(W)]dv^2 + \frac{p^2 + q^2X^2}{p^2(p + 1)} \left( 1 + X \right)^2dvdW \left( 1 + O(W) \right)
$$

$$
+ \frac{8(p + 1)X}{(p^2 + q^2X^2)} \left( 1 + O(W) \right) \left( d\phi_+ - \frac{pqW}{4(p + 1)}dv + O(W)dvdW + O(1)dW \right)^2
$$

$$
+ O(1)dW^2 + \frac{2(p + 1)(p^2 + q^2X^2)}{p^4(1 + X)^2} [1 + O(W)] \frac{dX^2}{(1 + X)^2X}, \quad (16)
$$

which is regular at $W = 0$. This expression also shows that $W = 0$ surface is a null hypersurface, whose normal vector is a Killing vector. Hence, the $W = 0$ surface is a Killing horizon. Because $W = 0$ is the double root of $e^{2\nu}$, the horizon is degenerate, i.e., the surface gravity vanishes.

Finally, we examine the shape of horizon. The Tomimatsu-Sato spacetime has a Killing vector $\partial_\phi$. So the radius $R$ is given by

$$
g_{\phi\phi} |_{(X_0, W_0)} = R^2 |_{(X_0, W_0)}. \quad (17)
$$

Figure 4 shows the relation between $g_{\phi\phi}$ and $X$ at horizon ($W = 0$). There $p^2$ parameterizes the angular momentum as is seen from eq. (4): $p^2 = 1$ corresponds to the non-rotating case, and $p^2$ decreases as the angular momentum increases. Figure 4 expresses the shape of horizon for $p^2 = 0.5$. From this figure we see that the horizons of $\delta = 2$ Tomimatsu-Sato spacetime consist of two spheres.
4 Conclusion

We have investigated the structure of the $\delta = 2$ Tomimatsu-Sato spacetime. By introducing an appropriate coordinate system, we have shown that the two points in the Weyl coordinates, which have been recognized as the directional singularities, are really two-dimensional surfaces and that these surfaces are horizons. We have also shown that each of the two horizons has the topology of a sphere. This result is rather surprising because the $\delta = 2$ Tomimatsu-Sato solution is obtained from the Neugebauer-Kramer solution representing a superposition of two Kerr solutions, as the limit that the centers of two black holes coincide [9]. This may indicates a new possibility for the final states of gravitational collapse.

References

[1] R.Penrose Riv. Nuovo. Cimento. 1, 252 (1969)
[2] R.C.Tolman Proc. Nat. Acad. Sci. 20, 169 (1934)
[3] H.Bondi Mon. Not. R. Astron. Soc. 107, 410 (1947)
[4] M.W.Choptuik Phys. Rev. Lett. 70, 9 (1993)
[5] D. Kramer, H. Stephani, M. MacCallum and E. Herlt eds.: Exact Solutions of Einstein’s Field Equations(Cambridge Univ. Press, Cambridge, 1980)
[6] Akira Tomimatsu and Humitaka Sato Prog. Theor. Phys. 50, 95 (1973)
[7] G.W.Gibbons and R.A.Russell-Clark. Phys. Rev. Lett. 30, 398 (1973)
[8] F.J.Ernst. J. Math. Phys. 17, 1091 (1976)
[9] Ken-ichi Oohara and Humitaka Sato. Prog. Theor. Phys. 65, 1891 (1981)