1 Compressible Navier-Stokes equations with relaxation in divergence form

We consider a hyperbolic system of balance laws with relaxation given by

\begin{equation}
\begin{align*}
\rho_t + (\rho u)_x &= 0 \\
(\rho u)_t + (\rho u^2 + p(\rho))_x &= S_x \\
\tau((\rho S)_t + (\rho u S)_x) - u_x &= -S
\end{align*}
\end{equation}

where \( \tau > 0 \) and the pressure \( p \) satisfies the constitutive relation

\begin{equation}
\rho = \rho^\gamma
\end{equation}

for \( \gamma > 1 \). System (1.1) is to be solved on \([0, T) \times \mathbb{R}\) for some \( T \in (0, \infty) \) with \( (\rho, u, S) \) taking values in the state space \((0, \infty) \times \mathbb{R} \times \mathbb{R}\). It is equivalent to

\begin{align*}
\rho_t + (\rho u)_x &= 0 \\
\rho u_t + \rho uu_x + p(\rho)_x &= S_x \\
\tau(\rho S_t + \rho u S_x) - u_x &= -S.
\end{align*}

System (1.1) is a delayed version of the Navier-Stokes equations with a Maxwell-type law as a relaxation equation. A divergence-form variant of systems recently considered by Hu et al. [HW19, HRW22], it is a prototypical version of a model proposed by Ruggeri [Rug83] as discussed by Freistühler in [Fre22]. In the latter the question is raised whether, like the systems considered in [HW19, HRW22], also divergence-form relaxation systems such as (1.1) possess a dichotomy in the sense that for small perturbations of a homogeneous reference state the Cauchy problem for (1.1) has unique global
solutions which time-asymptotically decay to the reference state, while for (some) large data there is a blow-up of solutions, i.e. physically reasonable solutions can only exist on a finite time interval. This note serves to confirm the second part of the mentioned dichotomy.

In proving the claimed blow-up we closely follow the procedure given by Hu, Racke and Wang [HRW22]. In fact we show that system (1.1) has all the necessary features in order to apply the identical steps as in [HRW22] from a certain point on. Our system is somewhat similar to the one considered in [HW19], and in this note one may find many analogies to [HW19] including a dissipative-entropy equation and a finite speed of propagation result for the system (1.1).

The general strategy has its origins in two remarkable papers of T. Sideris. In [Sid84] he establishes a finite speed of propagation result for symmetric hyperbolic conservation laws which he uses in [Sid83] to show the “formation of singularities in three-dimensional compressible fluids”. The blow-up occurs for a certain averaged quantity involving $\rho u$. The strategy in [Sid83] is by now well-established and for instance used in [HW19] and [HRW22].

The system (1.1) belongs to a general class of quasi-linear equations considered by Godunov [God61] and Boillat [Boi74]. Equations in this Godunov-Boillat class have the property of possessing a dissipative entropy equation. In this note we will use, and shortly derive, the following mathematical entropy equation

\begin{equation}
0 = \left[ \frac{\rho - \rho(p)}{\gamma - 1} + \frac{\tau \rho S^2}{2} + \frac{\rho u^2}{2} \right]_t + \left[ \frac{\gamma}{\gamma - 1} p' + \frac{\gamma}{\gamma - 1} \rho u + \frac{\rho u^3}{2} + \frac{\tau \rho u S^2}{2} - uS \right]_x + S^2
\end{equation}

where $\rho > 0$ is some reference density. Assume we are given a solution $(\rho, u, S) \in C^1(\Omega)$ on some open subset $\Omega \subset (0, \infty) \times \mathbb{R})$. To obtain (1.6) multiply (1.3) by $p'(\rho)$ to get

\begin{equation}
0 = p'(\rho) \frac{p'(\rho)}{\gamma - 1} + \gamma u_x p(\rho) + up(\rho) \frac{\rho u^2}{2} = \rho u_x t + [\gamma up(\rho)]_x + (1 - \gamma) up(\rho)_x.
\end{equation}

In this way we find

\begin{equation}
up(\rho)_x = \left[ \frac{1}{\gamma - 1} p(\rho) \right]_t + \left[ \frac{\gamma}{\gamma - 1} up(\rho) \right]_x.
\end{equation}

From (1.3) and (1.4) we deduce

\begin{equation}
-\rho u_x + uS_x = \rho uu_t + pu^2 u_x + \frac{u^2}{2} (\rho_t + (pu)_x) = \left[ \frac{\rho u^2}{2} \right]_t + \left[ \frac{\rho u^3}{2} \right]_x
\end{equation}

\[2\]
and similarly from (1.5)

\[(1.9) \quad -S^2 + u_x S = \tau (\rho S_t + \rho u S_x) = \left[ \frac{\tau \rho S^2}{2} \right]_t + \left[ \frac{\tau \rho u S^2}{2} \right]_{x}. \]

Combining (1.7)-(1.9) gives the entropy dissipation equality (1.6) where we have added an affine linear expression in the time-derivative term and compensated for this appropriately in the space-derivative term.

## 2 Blow-up of solutions for some large data

We begin with two properties of the Cauchy problem for system (1.3)-(1.5). On the one hand we have existence of physically reasonable solutions of said Cauchy problem on a maximal time interval for suitable initial values since (1.3)-(1.5) may be written as a first-order symmetrizable hyperbolic system of quasi-linear differential equations. On the other hand there is a finite speed of propagation result in the spirit of [Sid84].

**Lemma 2.1.** Let \( \overline{\rho} > 0 \). Let \((\rho_0, u_0, S_0) : \mathbb{R} \to \mathbb{R}^3 \) satisfy \((\rho_0 - \overline{\rho}, u_0, S_0) \in H^2(\mathbb{R}) \) and \(\rho_0(x) > 0\) for all \(x \in \mathbb{R}\). Then there exists a maximal \(T \in (0, \infty]\) such that there is a unique solution \((\rho, u, S)\) of (1.3)-(1.5) satisfying

\[
(\rho - \overline{\rho}, u, S) \in C^0([0, T), H^2(\mathbb{R})) \cap C^1([0, T), H^1(\mathbb{R})),
\]

and

\[
(\rho(0), u(0), S(0)) = (\rho_0, u_0, S_0),
\]

and for all \((t, x) \in [0, T) \times \mathbb{R}\)

\[
\rho(t, x) > 0.
\]

**Proof.** See Chapter 2 in [Maj84]. \(\square\)

**Lemma 2.2.** Let \(T, R > 0\) and \(\overline{\rho} > 0\). Set

\[
(2.1) \quad \sigma := \sqrt{p'(\overline{\rho}) + \frac{1}{\tau \overline{\rho}}^2} > 0.
\]

Suppose \((\rho, u, S) \in C^1([0, T) \times \mathbb{R}) \) solves (1.3)-(1.5) and satisfies

- \((\rho, u, S) \in C^1_t([0, T] \times \mathbb{R}) \) for all \(t \in [0, T]\);
- \(\text{supp} (\rho(0, \cdot) - \overline{\rho}, u(0), S(0)) \subset (-R, R).\)
Then for all $t \in [0, T)$ and $x \in D_t := \{ x \in \mathbb{R} : |x| \geq R + \sigma t \}$ it holds

\begin{equation}
(\rho(t, x) - \bar{\rho}, u(t, x), S(t, x)) = (0, 0, 0).
\end{equation}

Proof. Write $U = (\rho, u, S)$ and define

\[ A_0(U) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \tau \rho \end{pmatrix}, \quad A_1(U) = \begin{pmatrix} u & \rho & 0 \\ p'(\rho) & u & -1 \\ 0 & -1 & \tau \rho u \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

Set $\bar{U} := (\bar{\rho}, 0, 0)$ and consider for $V := U - \bar{U}$ the system

\[ V_t + A_0(\bar{U})^{-1} A_1(\bar{U}) V_x + A_0(\bar{U})^{-1} B V = G(\bar{U}, U, U_t, U_x) \]

with

\[ G(\bar{U}, U, U_t, U_x) = A_0(\bar{U})^{-1} [(A_0(\bar{U}) - A_0(U)) U_t + (A_1(\bar{U}) - A_1(U)) U_x]. \]

The eigenvalues of $A_0(\bar{U})^{-1} A_1(\bar{U})$ are

\[ \lambda_1(\bar{U}, \tau) = -\sqrt{p'(\bar{\rho}) + \frac{1}{\tau^2}}, \quad \lambda_2(\bar{U}, \tau) = 0, \quad \lambda_3(\bar{U}, \tau) = \sqrt{p'(\bar{\rho}) + \frac{1}{\tau^2}}. \]

Set $\sigma := \lambda_3$ and proceed like in [HW19] which follows the idea of [Sid84] to show the claim of the lemma.

The data for which solutions of (1.3)-(1.5) experience a blow-up is constructed from the function in the following lemma taken from [HR14] (see [HRW22], too):

**Lemma 2.3.** Let $L > 0$ and $M > 2$. Then the function

\[ u_{L,M}(x) := \begin{cases} 0, & x \in (-\infty, -M] \\ \frac{L}{2} \cos(\pi(x + M)) - \frac{L}{2}, & x \in (-M, -M + 1] \\ -L, & x \in (-M + 1, -1] \\ L \cos(\pi(x + M)), & x \in (-1, 1] \\ L, & x \in (1, M - 1] \\ \frac{L}{2} \cos(\pi(x + M)) + \frac{L}{2}, & x \in (M - 1, M] \\ 0, & x \in (M, \infty) \end{cases} \]

is an element of $H^2(\mathbb{R}) \cap C^1(\mathbb{R})$ with

\begin{equation}
||u_{L,M}||_{L^2}^2 \leq 2L^2M.
\end{equation}
Now we state and prove the theorem on blow-up of solutions to (1.3)-(1.5) for some large data.

**Theorem 2.4.** Let $R > 0$, $\overline{\rho} = 1$ and let $(\rho_0, S_0) \in C^1(\mathbb{R})$. Suppose $\rho_0(x) > 0$ for all $x \in \mathbb{R}$ and supp $(\rho_0 - 1, S_0) \subset (-R, R)$. Assume

\begin{equation}
\int_{\mathbb{R}} \rho_0(x) - 1 \, dx \geq 0.
\end{equation}

and set $\sigma := \sqrt{\gamma + \tau^{-1}}$. Choose $\tilde{\sigma}, L > 0$ such that

\begin{equation}
\tilde{\sigma}^2 = \max\{\sigma^2, (8 \max \rho_0)^{-1}\}
\end{equation}

and

\begin{equation}
L \min \rho_0 > \max\{\sqrt{8 \max \rho_0}, 16 \tilde{\sigma} \max \rho_0\}.
\end{equation}

Define

\begin{equation}
H_0 := \int_{\mathbb{R}} p(\rho_0(x)) - 1 - \gamma(\rho_0(x) - 1) \frac{\gamma - 1}{\gamma} + \frac{\tau \rho_0(x) S_0^2(x)}{2} \, dx
\end{equation}

and choose $M \geq \max\{4, R\}$ such that

\begin{equation}
H_0 + \max \rho_0 L^2 M \leq 2 \tilde{\sigma} M^2 \max \rho_0.
\end{equation}

Let $(\rho, u, S)$ be the solution of (1.3)-(1.5) with initial data $(\rho_0, u_{L,M}, S_0)$ and maximal time of existence $T > 0$ as given by Lemma 2.1. Then $T$ is finite and the critical averaged quantity initially satisfies

\begin{equation}
\int_{\mathbb{R}} x \cdot \rho_0(x) u_0(x) \, dx > \max\{16 \tilde{\sigma} M^2 \max \rho_0, M^2 \sqrt{8 \max \rho_0}\}.
\end{equation}

**Remark 2.5.** Before we enter the proof let us comment on where the above choices of constants enter in the proof. The relations (2.5) and (2.6) are needed to establish two a-priori estimates which appear in the deduction of the central Riccati-type differential inequality. Then again (2.6) together with (2.8) serves to find that the quantity on the right of (2.9) plays the role of a critical threshold for the Riccati-type inequality thereby allowing the conclusion of a finite maximal time of existence of the considered solution of (1.3)-(1.5).

**Proof.** The proof is very close to section 3 of [HRW22], also with respect to notation. In fact after establishing the Riccati-type estimate the claim
follows like in [HRW22]. The general strategy follows [Si83]. First define for $t \in [0,T)$ the averaged quantities

$$(2.10) \quad m(t) := \int_{\mathbb{R}} \rho(t, x) - 1 \, dx$$

and

$$(2.11) \quad F(t) := \int_{\mathbb{R}} x \cdot \rho(t, x) u(t, x) \, dx.$$ 

$m$ and $F$ are well-defined and continuous functions because of the compact supports of $u(t, \cdot)$ and $\rho(t, \cdot) - 1$ by Lemma 2.2.

By conservation of density (1.3), the compact support of $\rho(t, \cdot) - 1$ and by (2.4) we have for $t \in [0,T)$

$$(2.12) \quad m(t) = m(0) \geq 0.$$ 

Setting

$$B_t := \{ x \in \mathbb{R} : |x| < M + \tilde{\sigma} t \} \subset D_t^c$$

and using Jensen’s inequality (see p.846 [WH18]) implies

$$(2.13) \quad \int_{B_t} p(\rho(t, x)) \, dx \geq \int_{B_t} p(\bar{\rho}) \, dx.$$ 

By Lemma 2.2 partial integration and (1.3) and (1.4) we have

$$F'(t) = \int_{\mathbb{R}} (\rho u^2)(t, x) \, dx + \int_{\mathbb{R}} p(\rho(t, x)) - p(\bar{\rho}) \, dx - \int_{\mathbb{R}} S(t, x).$$

Apply estimate (2.13) and Young’s inequality to conclude

$$(2.14) \quad F'(t) \geq \int_{\mathbb{R}} (\rho u^2)(t, x) \, dx - \frac{1}{2} \int_{\mathbb{R}} S^2(t, x) \, dx - (M + \tilde{\sigma} t).$$

Hölder’s inequality and (2.12) yield

$$(2.15) \quad F^2(t) \leq \int_{B_t} x^2 \rho(t, x) \, dx \int_{B_t} (\rho u^2)(t, x) \, dx$$

$$\leq 2(M + \tilde{\sigma} t)^3 \max \rho_0 \int_{B_t} (\rho u^2)(t, x) \, dx.$$
Let
\[ c_2 := \frac{\tilde{\sigma}}{M}, \]
\[ c_3 := \frac{1}{2 \max \rho_0 M^3}, \]
\[ c_1 := \frac{2c_2}{c_3}, \]
and assume for all \( t \in [0, T) \) the a-priori estimates
\begin{equation}
F(t) \geq c_1 > 0
\end{equation}
and
\begin{equation}
M + \tilde{\sigma} t = M(1 + c_2 t) \leq \frac{c_3}{2(1 + c_2 t)^3} F^2(t).
\end{equation}

Combining the estimate (2.14) for \( F' \) with the estimate (2.15) for \( F^2 \) and using the a-priori estimates (2.16) and (2.17) one finds the Riccati-type inequality
\begin{equation}
\frac{F'(t)}{F^2(t)} \geq \frac{c_3}{2(1 + c_2 t)^3} - \frac{1}{2c_1} \int \mathcal{S}^2(t, x) dx.
\end{equation}

One uses the entropy equality (1.6) to treat the \( \mathcal{S}^2 \)-term above:
\begin{equation}
\int \mathcal{S}^2(t, x) dx \leq \int \frac{p(\rho_0) - 1 - \gamma(\rho_0 - 1)}{\gamma - 1} + \frac{\rho_0 u_{L,M}^2}{2} + \tau \rho_0 S_{L,2}^2 dx
\end{equation}
\[ \leq H_0 + \frac{\max \rho_0}{2} \|u_{L,M}\|^2.\]

Remark 2.6. By Taylor’s Theorem one has for \( \rho > 0 \)
\[ p(\rho) = 1 + \gamma(\rho - 1) + \frac{\rho''(\xi)(\rho - 1)^2}{2} \]
for some \( \xi > 0 \). Note \( p''(\rho) > 0 \) for \( \rho > 0 \). These facts were used for the estimate in (2.19) (see also p.829 [HW19]).

Now we are in the exact same situation as in [HRW22] section 3, and all further steps repeat theirs. Collecting the constants once more in
\[ c_4 := \frac{H_0}{2c_1^2} \text{ and } c_5 := \frac{\max \rho_0}{4c_1^2}\]
and noting
\[ c_4 + c_5 \|u_{L,M}\|_{L^2}^2 \leq \frac{c_3^2}{8c_2} (H_0 + \max \rho_0 L^2 M) \leq \frac{c_3}{8c_2} \]
by (2.3), the definition of \( c_1 \) and (2.8) we finally find by integrating (2.18) that for all \( t \in [0, T) \)
\[ F(0)^{-1} \geq F(0)^{-1} - F(t)^{-1} \]
\[ \geq - \frac{c_3}{4c_2(1 + c_2 t)^2} + \frac{c_3}{4c_2} - c_4 - c_5 \|u_{L,M}\|_{L^2}^2 \]
\[ \geq - \frac{c_3}{4c_2(1 + c_2 t)^2} + \frac{c_3}{8c_2}. \]
But by definition of \( u_{L,M} \) (note its symmetry \( u_{L,M}(x) = -u_{L,M}(-x) \)), \( M \geq 4 \) and by (2.6) it holds
\[ F(0) > \frac{L}{2} \min \rho_0 M^2 \geq 16 \frac{\tilde{\sigma}}{M} \max \rho_0 M^3 = \frac{8c_2}{c_3} \]
which implies \( T < \infty \) since otherwise there exists a \( t > 0 \) such that (2.20) violates (2.21).

It remains to verify the a-priori estimates. We have
\[ F(0) \geq 2c_1 \]
by (2.21). If for some \( T^* \in (0, T] \) we have \( F \geq c_1 \) on \( [0, T^*) \) then we get from (2.20)
\[ F(t) \geq \frac{4c_2(1 + c_2 t)^2}{c_3} \geq 2c_1. \]
for all \( t \in [0, T^*) \). Since \( F \) is continuous these two facts imply \( F \geq c_1 \) on \( [0, T) \) (see p.10 [HR11]).

We show
\[ M + \tilde{\sigma} t = M(1 + c_2 t) \leq \frac{c_3}{4(1 + c_2 t)^3} F^2(t) \]
which implies (2.17). Proceeding in a similar fashion like before we have
\[ F^2(0) \geq \frac{L^2 M^4 \min \rho^2_0}{2} \geq 4M^4 \max \rho_0 = 4M^4 \frac{c_3}{c_3} = 2M \]
by (2.21) and (2.6). Hence the estimate (2.22) holds for \( t = 0 \). Again if for some \( T^* \in (0, T] \) the estimate (2.17) holds for all \( t \in [0, T^*) \) then by the Riccati-type inequality (2.20) and assumption (2.5) we find
\[ F^2(t) \geq \frac{16c_2^2}{c_3} (1 + c_2 t)^4 \geq \frac{4M}{c_3} (1 + c_2 t)^4 \]
closing the a-priori estimate (2.17) by a continuity argument. \( \square \)
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