Particle production induced by vacuum decay in real time dynamics

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Abstract. We discuss particle production associated with vacuum decay, which changes the mass of a scalar field coupled to a background field which induces the decay. By utilizing the Stokes phenomenon, we can optimally track the time-evolution of mode function and hence calculate particle production analytically. In particular, we use real time formalisms for vacuum decay in Minkowski and de Sitter spacetime together with the Stokes phenomenon method. For each case, we take Wigner function method and stochastic inflation, respectively. With this formalism, the particle production can be viewed as that caused by nontrivial external fields.
1 Introduction

Quantum fields coupled to a time-dependent background appear in various contexts, such as cosmology or more general curved spacetime. On such backgrounds, the definition of the vacuum state is not unique, and change of vacuum states results in production of corresponding particles. Examples of such particle production are the Schwinger effect by the electromagnetic field [1, 2], the Hawking radiation in the black hole spacetime [3] and the gravitational particle creation by change of the expansion law of the universe [4, 5]. The efficiency of such particle production strongly depends on how abruptly the background changes [6]. In the case of the gravitational particle creation, the transition time scale of the background metric determines typical energy scale of the produced particle [7, 8]. This implies that the a sudden transition of background makes the effect of particle production efficient.

The vacuum decay such as a first-order phase transition is an example of such an abrupt transition as various parameters change discontinuously. If particle production associated with quantum tunneling is efficient, the tunneling dynamics could be affected by the back reaction of particle production. One of the most important possibilities is the issue of the Higgs field instability [9]. Since the Higgs field couples to almost all particles in the Standard Model, its tunneling might yield a considerable amount of the Standard Model particles. If this is the case, it would be necessary to reconsider the stability as well as the dynamics of the vacuum bubble after its nucleation.

Production of particles that are coupled to a tunneling scalar field has been studied in [10, 11] using a conventional instanton method [12, 13]. Despite our naive expectation of a sudden transition, it has been claimed that particle production is not so efficient: for a momentum $k$-mode, the number density of produced particles $n_k$ is exponentially suppressed for modes with energy $\omega_k \gg \Delta \tau^{-1}$ as $n_k \sim e^{-4\omega_k \Delta \tau}$. Here $\Delta \tau$ is an “imaginary transition time scale”, that is, the Euclidean time scale for the tunneling scalar moving from the false vacuum to the true one. One can find a similarity between this suppression factor and the one for the gravitational particle production case [7, 8], in which $n_k$ is exponentially suppressed by $e^{-4\omega_k \Delta t}$, where $\Delta t$ is a real transition time scale. However, the meaning of the imaginary transition time scale is not clear from the real time perspective.¹

¹In [14, 15], the vacuum transition rate is given without the notion of instanton and they give the meaning of tunneling rate within a real time formalism.
In this paper, in order to understand how the particle production associated with the vacuum decay from real time viewpoint, we analyze the vacuum decay with its real time formulations. There are several description of the vacuum decay alternative to the standard Euclidean methods: In [16], the Schwinger-Keldysh formalism with Wigner function method is used to describe the quantum tunneling in quantum field theory.\(^2\) In [19–23], a stochastic description of the quantum tunneling is discussed.\(^3\) In de Sitter spacetime, the stochastic inflation [26, 27] can give the time dependent probability distribution of a scalar field value, which can actually give the tunneling rate corresponding to that found in the standard instanton methods. Such “real time” formulations of the quantum tunneling are suitable for our purpose. However, we should emphasize that the initial state for the tunneling field seems different from that in the Euclidean description of the quantum tunneling. In particular, the “flyover” vacuum decay which we will use in the flat spacetime case requires certain initial distribution for momentum of the tunneling field. Nevertheless, the real time formalism can describe “vacuum decay” and give the decay rate similar to that in instanton methods. Therefore, we simply call such vacuum decay as quantum tunneling in this work.

On top of such real time formulations, we will use the Stokes phenomenon method [28, 29], which enables us to pick up a non-perturbative particle production, and also discuss how we can interpret the particle production caused by the vacuum decay in real time formulation.

This paper is organized as follows. In Sec. 2, we give a brief review of the relation between particle production and the Stokes phenomenon. Such a viewpoint enables us to find the optimal evaluation of particle production in non-trivial backgrounds. In Sec. 3, the production of a scalar particle coupling with a transiting scalar field in a flat spacetime background is investigated. First, we briefly review the real time formalism of the quantum tunneling and then evaluate a produced particle number density within the real time formalism. In Sec. 4, we consider the particle production in the de Sitter spacetime background. After revisiting the particle production in de Sitter spacetime, we extend our analyses to the case of a scalar field that couples to a transiting scalar field. In Sec. 5, we summarize our results and discuss the remaining issues.

We use the natural units \(c = \hbar = M_{\text{pl}} = 1\) throughout the paper.

2 Particle production as Stokes phenomenon

In this section, we briefly review how particle production is interpreted in terms of the Stokes phenomenon [28, 29]. Particle production caused by time-dependent background can be understood from the behavior of mode functions. For particles in nontrivial time-dependent background, WKB type (adiabatic) mode functions are useful in defining the vacuum state. Such adiabatic solutions show sudden change of their behavior at a certain point, which is the so-called Stokes phenomenon. The physical meaning of this sudden change is nothing but the production of particles.

\(^2\)In [17, 18], using the similar description, the effect of short wavelength modes on the long wavelength modes is discussed, which is called “activation” and gives rise to enhancement of the vacuum decay rate. In this sense, we have to be aware that what the real time formalism actually describes might be this “activation” rather than the very quantum tunneling in usual context.

\(^3\)We should notice that there are considerable differences among the methods in [19–23]. In [19, 20], they take a stochastic quantum noise into account throughout the tunneling process with the Madelung fluid description [24, 25], whilst a stochastic quantum noise is used only as an initial kick and a succeeding dynamics is described classically without any stochastic component in [21–23].
Let us consider a scalar field $\chi$ with a time-dependent mass $M_\chi^2(t)$ in the Minkowski background $ds^2 = -dt^2 + dx^2$. $\chi$ can be expanded as

$$\chi(t, x) = \int \frac{d^3k}{(2\pi)^{3/2}} \left( \hat{a}_k v_k(t) e^{i k \cdot x} + \hat{a}_k^\dagger \bar{v}_k(t) e^{-i k \cdot x} \right),$$  \hspace{1cm} (2.1)

where we have introduced annihilation and creation operators $\hat{a}_k$ and $\hat{a}_k^\dagger$, respectively. The mode equation of $\chi$ is given by

$$\ddot{v}_k + \omega_k^2 v_k = 0,$$  \hspace{1cm} (2.2)

where a dot denotes a time derivative and

$$\omega_k^2 = k^2 + M_\chi^2(t)$$  \hspace{1cm} (2.3)

is the effective frequency squared. In order to define a vacuum state, we take the WKB-type adiabatic solution for (2.2),

$$v_k = \frac{A_k}{\sqrt{2W_k(t)}} e^{-i \int_{t_c}^t dt' \omega_k(t')} + \frac{B_k}{\sqrt{2W_k(t)}} e^{i \int_{t_c}^t dt' \omega_k(t')},$$  \hspace{1cm} (2.4)

where $W_k$ is recursively determined as

$$W_k^{(0)} = \omega_k,$$  \hspace{1cm} (2.5)

$$\left( W_k^{(n+1)} \right)^2 = \omega_k^2 - \frac{1}{2} \left[ \frac{\ddot{W}_k^{(n)}}{W_k^{(n)}} - \frac{3}{2} \left( \frac{\dot{W}_k^{(n)}}{W_k^{(n)}} \right)^2 \right],$$  \hspace{1cm} (2.6)

where $W_k^{(n)}$ denotes the quantity of $n$-th adiabatic order. We should stress that this $W_k^{(n)}$ is a divergent series. Therefore, we must truncate the expansion at the optimal order. The asymptotic behavior of such a WKB-type solution can be well described in terms of Stokes phenomenon: The approximated adiabatic solution is multivalued function having the cut near the so-called turning points $t_c$ satisfying $\omega_k(t_c) = 0$, and such turning point can be on a complex $t$-plane and usually associated with a complex conjugate point due to Schwarz’s reflection principle. Hereafter, we denote the turning point closest to the real time axis in the upper half plane as $t_c$. The behavior of the solution significantly changes beyond the Stokes lines connecting a pair of turning points. More specifically, the values of the $\alpha_k$ and $\beta_k$ suddenly change — in other words, particles are produced — around the Stokes line. The Stokes phenomenon determines which order $n$ is optimal to be truncated. After the truncation, (2.4) becomes

$$v_k = \frac{\alpha_k(t)}{\sqrt{2\omega_k(t)}} e^{-i \int_{t_c}^t dt' \omega_k(t')} + \frac{\beta_k(t)}{\sqrt{2\omega_k(t)}} e^{i \int_{t_c}^t dt' \omega_k(t')} ,$$  \hspace{1cm} (2.7)

where $\beta_k(t)$ which is responsible for particle production is given by

$$\beta_k(t) \sim -ie \frac{1}{2} F_k(t_c) S_k(t) ,$$  \hspace{1cm} (2.8)

where $F_k(t)$, called a singulant, is given by

$$F_k(t) = 2i \int_{t_c}^t dt' \omega_k(t') ,$$  \hspace{1cm} (2.9)
where we take the integration contour along the Stokes line connecting $t_c$ and $t_c^*$ until the contour crosses the real axes at $t = t_s$, and from $t_s$ the contour is along the real axes. $S_k(t)$, called the Stokes multiplier, is given by

$$S_k(t) = \frac{1}{2} \left[ 1 + \text{Erf} \left( -\frac{\text{Im} F_k(t)}{\sqrt{2 |\text{Re} F_k(t)|}} \right) \right],$$

and Erf$(x)$ denotes the error function. The Stokes line is the trajectory where the imaginary parts of the two exponents in (2.7) coincide: $\text{Im} F_k(t) = 0$. In terms of particle production, the singulant represents the amplitude of particle production, whereas the Stokes multiplier represents the time dependence of the particle number. We refer to appendix A of [30] for a review of the derivation of this formula. In case there are $N$ pairs of turning points, a resultant particle number $n_k$ is given by summing up the contributions from each Stokes lines with relative phases as [31]

$$n_k = |\beta_k(\infty)|^2 \sim \sum_{n=0}^{N-1} \exp \left( 2i \int_{t_{s,n}}^{t_{c,n}} \omega_k dt \right) e^{\frac{1}{2} F_{k,n}(t_{c,n}^*)} \right|^2,$$

where $t_{c,n}$, $t_{s,n}$ and $F_{k,n}$ denote the $n$-th turning point, the $n$-th intersection point between the $n$-th Stokes line and the real time axis, and the singulant with respect to $t_{c,n}$, respectively. The approximated formula (2.7) gives the optimal approximation for the WKB type solutions, and therefore for the particle production rate. We will use this description of particle production in the following discussions.

3 Flat spacetime case

In this section, we consider the production of a scalar field $\chi$ coupled to another scalar field $\phi$ that induces a quantum-tunneling from a false vacuum to the true one in flat spacetime background. Specifically, we consider the following system with two real scalar fields:

$$L = -\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) - \frac{1}{2} \partial^\mu \chi \partial_\mu \chi - \frac{1}{2} (M_0^2 + g\phi^2) \chi^2.$$

Here $\phi$’s potential $V(\phi)$ is a slightly tilted double-well shown in Fig. 1, and $\chi$ is a heavy scalar field that acquires an additional mass $g\phi^2$ through tunneling. In this case, the effective frequency of $\chi$ is given by

$$\omega_k^2 = k^2 + M_0^2 + g\phi^2.$$

We assume that $g$ is sufficiently small and the tunneling dynamics is not affected by the coupling between $\phi$ and $\chi$. The Schwinger-Keldysh formalism [2, 32] would be useful to describe the expectation value of physical quantity in time-dependent background. In this in-in formalism, the expectation value of the number density of $\chi$ is given by

$$\langle n_\chi(t) \rangle = \langle \phi_{in}, \chi_{in} | n_\chi(\phi(t); t) | \phi_{in}, \chi_{in} \rangle,$$

where $|\phi_{in}, \chi_{in}\rangle$ is some initial state of $\phi$ and $\chi$, and $n_\chi(\phi(t); t)$ is the number density operator which we will define later. Note that the number density operator should depend on $\phi$ since the $\chi$-production would be caused by the time-dependence of the effective mass $M_0^2 + g\phi^2$. 

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This means that we will take the in-state for $\chi$ to be an adiabatic vacuum. Using Schwinger-Keldysh formalism, we may formally express the number density as

$$\langle n(\chi(t)) \rangle = \int D\chi_f D\chi_b D\phi_f D\phi_b n(\phi; t) \exp \left[ -i \int d^4x (L_f - L_b) \right],$$

(3.4)

where the subscripts $f$ ($b$) denotes the forward (backward) variable and $L_{f,b} = L|_{\phi,\chi = \phi_{f,b},\chi_{f,b}}$.

Using the Wigner function, we can rewrite the above expression as

$$\langle n(\chi(t)) \rangle = \int D\chi_f D\chi_b \int d\phi_c d\pi_c \left[ \int D\phi_{c,0} D\pi_{c,0} W_0(\phi_{c,0}, \pi_{c,0}) n^W(\phi_c, \pi_c) F \right]$$

$$\times \exp \left[ -i \int d^4x \left( -\frac{1}{2} (\partial_\chi f)^2 - \frac{1}{2} M_0^2 \chi_f^2 \right) - (f \to b) \right] ,$$

(3.5)

where $\pi_{c,0}$ is the conjugate momentum of $\phi_{c,0}$,

$$F = \exp \left[ i \int d^4x \left( \frac{1}{2} g(\phi_c + \phi_q)^2 \chi_f^2 - \frac{1}{2} g(\phi_c - \phi_q)^2 \chi_b^2 \right) \right]$$

(3.6)

and we have used the Wigner representation of operators

$$O^W = \int d\phi_q d\pi_q O(\phi_c, \phi_q, \pi_c, \pi_q) e^{-2i\phi_q\pi_q}.$$  

(3.7)

Since the dynamics of $\phi$ is dominated by the “classical” part $\phi_c$ in the tunneling and the subsequent bubble dynamics [16], we approximate the Wigner representation $O^W$ as

$$O^W \approx O(\phi_c, 0, \pi_c, 0).$$  

(3.8)

Thus, we find the approximated expression for the number density as

$$\langle n(\chi(t)) \rangle \approx \int d\phi_{c,0} d\pi_{c,0} W_0(\phi_{c,0}, \pi_{c,0}) \left[ \int D\chi_f D\chi_b n(\phi_c, \pi_c; t) \exp \left( -i \int d^4x (\tilde{L}_f^\chi - \tilde{L}_b^\chi) \right) \right],$$

(3.9)
where
\[ \tilde{\mathcal{L}}_{\text{fb}} = -\frac{1}{2}(\partial \chi f b)^2 - \frac{1}{2}(M_0^2 + g \phi_c^2) \chi f b. \tag{3.10} \]
The quantity inside the square bracket can be understood as the number density of \( \chi \) that couples to the classical background scalar \( \phi_c \), namely,
\[ \langle n_\chi(t) \rangle \simeq \int d\phi_{c,0} d\pi_{c,0} W_0(\phi_{c,0}, \pi_{c,0}) \langle n_\chi(\phi_c; t) \rangle. \tag{3.11} \]

We can evaluate the particle number density \( \langle n_\chi(\phi_c; t) \rangle \) by using the Bogoliubov transformation with a classical background field \( \phi_c \). Then, the actual number density is given by \( \langle n_\chi(\phi_c; t) \rangle \) with the Wigner function weight. If the initial fluctuation at the false vacuum is very small, the initial Wigner function is well approximated by that for a massive free field
\[ W_0(\phi_{c,0}, \pi_{c,0}) \propto \exp \left[ -\int \frac{d^3k}{(2\pi)^3} (\Omega_k |\phi_{c,0}(k)|^2 + \Omega_k^{-1} |\pi_{c,0}(k)|^2) \right], \tag{3.12} \]

where \( \Omega_k = \sqrt{k^2 + m_F^2} = \sqrt{k^2 + (V''(0))^2} \) and \( m_F \) denotes the mass of \( \phi \) at the false vacuum. Since this Wigner function rapidly decreases for higher velocity \( \dot{\phi}_0 = \pi_{c,0} \), most of contribution comes from a mode that can barely pass over the potential barrier. Assuming that \( \dot{\phi}_0 \) is constant inside a spherical region \( S_r \) with a radius \( r \) in position space, the relation between \( \dot{\phi}_0 \) in momentum space and that in position space is
\[ \dot{\phi}_0(k) = \int_{S_r} d^3x e^{-ik \cdot x} \dot{\phi}_0 \sim (2\pi)^3 V \dot{\phi}_0, \tag{3.13} \]

where \( k \gg r^{-1} \) and \( V = \frac{4}{3} \pi r^3 \). Then, (3.12) is rewritten in terms of a position space as
\[ W_0(\dot{\phi}_0) \sim \sqrt{\frac{(2\pi)^3 V}{m_F}} \exp \left( -\frac{(2\pi)^3 V \dot{\phi}_0}{m_F} \right). \tag{3.14} \]

Since \( \phi \) must have an enough initial velocity in a volume larger than a critical value \( V_b \) in order for a nucleated bubble to expand, we take the critical value \( V_b \) as \( V \) hereafter. The critical bubble radius \( r_b \) is derived from balance between a bubble tension \( \sigma \) and a bubble pressure \( \varepsilon \) as
\[ 4\pi r_b^2 \sigma = \frac{4}{3} \pi r_b^3 \varepsilon. \tag{3.15} \]

Adopting the thin-wall approximation, \( \sigma \) is approximated as \( \varepsilon \sqrt{h} \) and then \( r_b = 3 \varepsilon \sqrt{h}/\varepsilon \), where \( h \) is the height of the potential barrier. Thus the property of the nucleated bubble is encoded in the Wigner function.

Now we move onto the evaluation of produced particle number density. As we have discussed, we can evaluate the \( \chi \)-production by considering the behaviour of the mode function of \( \chi \) with the background external field \( \phi \). Given that \( \phi \) acquires an initial velocity \( \dot{\phi}_0 \) at \( t = t_0 \) inside a spherical region \( S_B \) with the radius \( r_B \), \( \phi \) is approximately homogeneous in \( S_B \) and therefore the gradient energy of \( \dot{\phi} \) is negligible compared with the kinetic energy. Since \( V(\phi) \) is expanded around each vacuum and around the barrier as
\[ V(\phi) = \begin{cases} \varepsilon + \frac{1}{2} m_B^2 \phi^2 + O(\phi^3) & (\phi \approx 0) \\ h - \frac{1}{2} m_B^2 (\phi - \phi_B)^2 + O((\phi - \phi_B)^3) & (\phi \approx \phi_B) \\ \frac{1}{2} m_B^2 (\phi - v)^2 + O((\phi - v)^3) & (\phi \approx v) \end{cases}, \tag{3.16} \]
where $m_B^2 = V''(\phi_B)$, $m_T^2 = V''(v)$, the motion of $\phi$ inside $S_B$ is approximated, respectively, as

$$
\phi(t) \approx \begin{cases} 
\frac{\phi_0}{m_B} \sin [m_B(t - t_0)] & (\phi \approx 0) \\
\sqrt{\frac{m_B^2 - 2k \xi}{m_B}} \sinh [m_B(t - t_1)] + \phi_B & (\phi \approx \phi_B), \\
\sqrt{\frac{m_B^2 + 2k \xi}{m_T}} \sin [m_T(t - t_2)] + v & (\phi \approx v)
\end{cases}
$$

(3.17)

Here we assume that the particle production of $\chi$ is so inefficient that the energy of $\phi$ is almost conserved.

Let us find the turning points, the Stokes line, and the intersection point $t_s$ between the Stokes line and the real axes for the region near each extremum of $V(\phi)$. Around the false vacuum $\phi \approx 0$, the turning point is obtained by substituting the approximated expression (3.17) into the equation (3.16) into the contour vertical to and along the real time axis as

$$
\int_{t_c}^{t_s} dt \omega_k(t) + 2 \int_{Re t_c}^{t_s} dt \omega_k(t) = 0,
$$

(3.19)

and therefore $t_s$ is generally approximated as

$$
t_s \approx Re t_c - \frac{\Delta}{\omega_k(Re t_c)}.
$$

(3.20)

In the region around $\phi \approx 0$,

$$
\Delta_0 = \int_0^{Im t_c,0} \text{Im} [\omega_k(Re t_{c,0} + i \tau)] d\tau
\approx \int_0^{\sinh^{-1} x} \text{Im} \sqrt{\frac{g_0^2}{m_F} (x^2 - \sin^2 \xi) m_F^{-1} d\xi} = 0, \quad (\xi = m_F \tau)
$$

(3.21)

and hence, the Stokes line is the straight line connecting $t_{c,0}$ and $t_{c,0}^*$, namely $t_{s,0} = t_0$. The singulant along the Stokes line connecting the pair of these turning points is given by

$$
F_{k,0}(t_{c,0}^*) \approx 2i \int_{t_{c,0}}^{t_{c,0}^*} \sqrt{k^2 + M_0^2 + \frac{g_0^2}{m_F} \sin^2 [m_F(t - t_0)]} dt
= 4 \int_0^{\sinh^{-1} x} \sqrt{\frac{g_0^2}{m_F} (x^2 - \sin^2 \xi) m_F^{-1} d\xi} \quad (i \xi = m_F(t - t_0))
= 4 \sqrt{\frac{g_0^2}{m_F^4}} (-i x) \text{E} \left( i \sinh^{-1} x \left| - \frac{1}{x^2} \right\right),
$$

(3.22)

As long as there is no pole or branch cut between two turning points, we may deform the integration contour.
where \( E(\varphi|k^2) \) is the incomplete elliptic integral of the second kind in trigonometric form. Since it is practically difficult to use the function, we assume \( x \gg 1 \) and look for an approximated form of the function. We numerically find a fitting function of (3.22) to be

\[
F_{k,0}(t^*_c, 0) \approx 4\sqrt{\frac{g^2}{m^2_F}} x[\ln(4x) - 1],
\]

which allows us to proceed the following discussion with an analytic calculation.

Around the barrier \( \phi \approx \phi_B \), the turning point is located at

\[
t_{c,1} = t_1 + i \frac{\pi}{2} m_B^{-1} \ln \left( y + \sqrt{y^2 - 1} \right) + \frac{i}{\sqrt{y^2 - 1}} \sqrt{\frac{g^2 \phi_B^2}{k^2 + M_0^2}} + O \left( y^{-2} \frac{g^2 \phi_B^2}{k^2 + M_0^2} \right),
\]

where \( y = \sqrt{\frac{m_B^2}{g^2 \phi_B^2 - 2(k - \varepsilon)}} \). Hereafter, we take \( y^{-1} \) and \( \sqrt{\frac{g^2 \phi_B^2}{k^2 + M_0^2}} \) as a perturbation for consistency with the later discussion since \( y \) is much larger than unity in the case that \( \phi \) barely passes over the potential barrier and we will regard \( \sqrt{\frac{g^2 \phi_B^2}{k^2 + M_0^2}} \) as much smaller than unity. Substituting the expression of \( \phi \) around \( \phi \approx \phi_B \) (3.17) into \( \Delta = \text{Re} \int_{t_c}^{t_{c,1}} dt \omega_k(t) \), we find

\[
\Delta_1 = \int_0^{\text{Im} t_{c,1}} \text{Im} \left[ \omega_k(\text{Re} t_{c,1} + i \tau) \right] d\tau
\]

\[
\approx \int_0^\frac{\pi}{2} \frac{\sqrt{k^2 + M_0^2}}{m_B} \text{Im} \left[ 1 + \left( \sqrt{1 - \frac{1}{y^2}} \cos \xi + i \sin \xi + \sqrt{\frac{g^2 \phi_B^2}{k^2 + M_0^2}} \right)^2 \right] d\xi
\]

\[
\approx \sqrt{\frac{k^2 + M_0^2}{m_B}} \left( \int_0^\frac{\pi}{2} \text{Im} \left[ 1 + e^{2i\xi} \right] d\xi \right)
\]

at the zeroth order in \( y^{-1} \) and \( \sqrt{\frac{g^2 \phi_B^2}{k^2 + M_0^2}} \). After performing numerical integration in the parentheses in the last equality of (3.25), we obtain

\[
\Delta_1 \approx 0.53 \times \frac{\sqrt{k^2 + M_0^2}}{m_B}
\]

\[
\Rightarrow t_{s,1} \approx t_1 + [\ln(2y) - 0.53]m_B^{-1}.
\]

However, this \( t_{s,1} \) is out of the range \( \phi \approx \phi_B \Leftrightarrow m_B(t - t_1) < 1 \) since \( y \gg 1 \). Therefore, this Stokes line should be regarded as unphysical one, and we neglect it.

Around the true vacuum \( \phi \approx v \), the turning point is located at

\[
t_{c,2} = t_2 + m_T^{-1} \sin^{-1} \left[ \sqrt{\frac{m_T^2}{g(\phi_B^2 + 2\varepsilon)}} \left( -\sqrt{g^2 v^2 + i \sqrt{k^2 + M_0^2}} \right) \right]
\]

\[
= t_2 + m_T^{-1} \ln \left( z + \sqrt{1 + z^2} \right) - \frac{m_T^{-1} \sqrt{g^2 v^2}}{\sqrt{1 + z^2}} \sqrt{\frac{g^2 v^2}{k^2 + M_0^2}} + O \left( z^{-2} \frac{g^2 v^2}{k^2 + M_0^2} \right),
\]

\text{at the zeroth order in } y^{-1} \text{ and } \sqrt{\frac{g^2 \phi_B^2}{k^2 + M_0^2}}.
where \( z = \sqrt{\frac{m_T^2}{g(v_0^2 + 2\epsilon)}} (k^2 + M_0^2) \). In order to proceed our discussion with analytic calculation, we assume \( \delta \equiv \sqrt{\frac{gv^2}{k^2+M_0^2}} \ll 1 \) and take it as a perturbation. Such an approximation is valid for the case where \( gv^2 \ll M_0^2 \) or for high frequency modes \( k^2 \gg gv^2 \). We substitute \( \phi \) around \( \phi \approx v \) in (3.17) into the expression of \( \Delta \), which yields

\[
\Delta^2 \approx \sqrt{k^2 + M_0^2} \int_0^{\sinh^{-1} z} \text{Im} \left[ \frac{1 - \sin^2 \tau - 2 \frac{i}{z} \left( 1 - \frac{\cosh \xi}{z \sqrt{1 + z^2}} \right) m_T^{-1} d\xi \right].
\] (3.30)

Again, we numerically look for an approximated expression of (3.30) for various \( z \) and \( \delta < 1 \) and find

\[
\Delta^2 \approx \frac{\sqrt{k^2 + M_0^2}}{m_T} C,
\] (3.31)

where

\[
C \begin{cases} 
\lesssim 0.1 & (z < 1) \\
\approx \delta & (z > 1).
\end{cases}
\] (3.32)

Hence, the Stokes line intersects the real time axis at

\[
t_{s,2} \approx t_2 - \left( \frac{C}{1 + \delta^2} + \frac{\delta}{\sqrt{1 + z^2}} \right) \sqrt{m_T},
\] (3.33)

which is actually within the range \( \phi \approx v \leftrightarrow m_T(t - t_2) < 1 \) since we have assumed \( \delta \ll 1 \). If \( \delta \) is much larger than unity, one finds that \( t_{s,2} \) is also out of the range of approximation and this Stokes line is irrelevant. This seems physically reasonable since, for \( \delta \gg 1 \), \( \chi \) becomes much heavier after the tunneling of \( \phi \) and the production rate of such a heavy particle should be suppressed. The singulant along the Stokes line connecting the pair of these turning points is given by

\[
F_{k,2}(t_{c,2}^*) = 2i \int_{t_{c,2}}^{t_{c,2}^*} \sqrt{k^2 + M_0^2 + g v^2 \left( \frac{1}{\sqrt{32}} \sin \left[ \sqrt{m_T} (t - t_2) \right] + v \right)^2} dt.
\] (3.34)

Since this integral is quite difficult to be analytically calculated in general cases, we evaluate it in the particular case, the double-well potential. Assuming \( V(\phi) \approx \frac{1}{2} \phi^2 (\phi - v)^2 \), (3.34) becomes

\[
F_{k,2}(t_{c,2}^*) = 2i \int_{t_{c,2}}^{t_{c,2}^*} \sqrt{k^2 + M_0^2 + g v^2 \left( \frac{1}{\sqrt{32}} \sin \left[ \sqrt{m_T} (t - t_2) \right] + v \right)^2} dt
\]

\[
= 2\delta^{-1} \int_{-\sqrt{32}/\delta}^{\sqrt{32}/\delta} \frac{1 - \delta^2 X^2 / 32}{1 + (X - \sqrt{32})^2} dX \sqrt{\frac{g}{\lambda}},
\] (3.35)

where \( X = i \left( \sin \left[ \sqrt{m_T} (t - t_2) \right] + \sqrt{32} \right) \). We numerically find a fitting function of (3.35) to be

\[
F_{k,2}(t_{c,2}^*) \approx D \delta^{-\epsilon} \sqrt{\frac{g}{\lambda}},
\] (3.36)
the particle production only at the point where $\phi$ is around each vacuum. In other words, when the change of the mass is small enough, as if the scalar particle $\chi$ experiences the same situation as that around $\phi = 0$, and the particle production takes place around the true vacuum as well. This is consistent with the expectation that the particle production efficiently occurs at the point where non-adiabaticity parameter $\omega_k/\omega_k^2$ takes the local maximum in real time. Therefore, we obtain the resultant number density of produced $\chi$ from (2.11) as

$$
\langle n_k(\dot{\phi}_0) \rangle \approx \sqrt{\pi} \frac{1}{2} \text{Erfc} \left( \sqrt{\frac{(2\pi)^3 V_b}{m_\phi}} \dot{\phi}_{\text{th}} \right) \langle n_k(\dot{\phi}_{\text{th}}) \rangle,
$$

(3.40)

Finally, we calculate the expectation value of produced number density of $\chi$ by integration weighted by the Wigner function. With (3.14), (3.11) becomes

$$
\langle n_k \rangle = \int_{\dot{\phi}_{\text{th}}}^{\infty} d\dot{\phi}_0 \sqrt{\frac{(2\pi)^3 V_b}{m_\phi}} \exp \left[ \frac{-(2\pi)^3 V_b}{m_\phi} \dot{\phi}^2_0 \right] \langle n_k(\dot{\phi}_0) \rangle,
$$

(3.39)

where $\dot{\phi}_{\text{th}}^2 \equiv 2(h - \varepsilon)$ is the initial energy threshold requisite for passing over the barrier. Here, the Gaussian factor of the Wigner function rapidly decreases for larger $\dot{\phi}_0$, and thus most of contribution of this integration comes from around $\dot{\phi}_0 \approx \dot{\phi}_{\text{th}}$. Therefore, we can approximate (3.39) as

$$
\langle n_k \rangle \approx \int_{\dot{\phi}_{\text{th}}}^{\infty} d\dot{\phi}_0 \sqrt{\frac{(2\pi)^3 V_b}{m_\phi}} \exp \left[ \frac{-(2\pi)^3 V_b}{m_\phi} \dot{\phi}^2_0 \right] \langle n_k(\dot{\phi}_{\text{th}}) \rangle
$$

$$
= \sqrt{\frac{\pi}{2}} \text{Erfc} \left( \sqrt{\frac{(2\pi)^3 V_b}{m_\phi}} \dot{\phi}_{\text{th}} \right) \langle n_k(\dot{\phi}_{\text{th}}) \rangle,
$$

(3.40)
where \( \text{Erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} dt \) is the complementary error function. Using the asymptotic form of \( \text{Erfc}(x) \) given by

\[
\text{Erfc}(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}} \tag{3.41}
\]

for \( x \gg 1 \) together with (3.23), (3.36) and (3.38), we can further proceed analytic calculation as

\[
\langle n_k \rangle \approx \sqrt{\frac{m_{\phi}}{(2\pi)^3 V_b \phi_{\text{th}b}^2}} \exp \left( \frac{-(2\pi)^3 V_b \phi_{\text{th}b}^2}{m_{\phi}} \right) \times \left( \exp \left[ -2\sqrt{\frac{k^2 + M^2_0}{\lambda v^2}} \ln \left( \frac{512 k^2}{e^2 g v^2} \right) \right] + \exp \left[ -D \delta^{-(1+\xi)} \sqrt{\frac{g}{\lambda}} \right] \right), \tag{3.42}
\]

Here, we have assumed the slightly tilted double-well potential \( V(\phi) \approx \frac{1}{2} \phi^2 (\phi - v)^2 \) in the second line for simplicity.\(^5\) If one would like to consider a much broader class of potentials, then they have to continue distinguishing \( m_F \) and \( m_T \). Since the first line corresponds to the transition rate, we can simply replace it by \( \Gamma \) and obtain the following expression:

\[
\langle n_k \rangle \approx \Gamma \left( \exp \left[ -2\Delta \ln \left( \frac{512}{e^2} \right) \sqrt{\frac{g}{\lambda}} \right] + \exp \left[ -D \Delta^{1+\xi} \sqrt{\frac{g}{\lambda}} \right] \right), \tag{3.43}
\]

where \( \Delta^2 \equiv (k^2 + M^2_0)/g v^2 \). Since the produced particle number density decays exponentially or even faster for high momentum modes, the total particle number density safely converges.

Let us discuss the physical meaning of the two contributions for the produced particle number given in (3.43). The first contribution coming from the Stokes line crossing \( t_0 \) is not the production due to the change of mass but that caused by the initial velocity of \( \phi \). Such a contribution exists no matter how small or large the mass difference is. The second contribution corresponds to the production by the transition of the mass of \( \chi \). This contribution seems more important than the first one, because the presence of the initial velocity might be significant only in the flyover vacuum decay, whereas the second contribution is really caused by the change of mass and therefore should always exist in any type of vacuum decay.

Finally, we should stress that we have only discussed the particle production from “one-way” process, \( \phi = 0 \) to \( \phi = v \). This is because we have approximated the dynamics of the tunneling scalar \( \phi \) to be homogeneous. Although this approximation makes analyses of particle production simpler, we find that the oscillation of the tunneling scalar will never stop. However, the bubble of true vacuum would form once \( \phi \) reaches true vacuum and extract the energy of \( \phi \), then \( \phi \) will never oscillate between true and false vacuum, which justifies the one-way process we have considered. Since the dominant effect for particle production comes from the homogeneous part inside the nucleating bubble, our estimation would be not so affected even if we take spatial dependence of \( \phi \) into account.

### 4 de Sitter spacetime case

In this section, we will consider the particle production of a massive scalar \( \chi \) coupled to a tunneling scalar \( \phi \).

---

\(^5\) Although the first term in the second line of (3.42), which comes from (3.23), seems to depend on \( v \), this is simply because we assume the double-well potential and then \( m_F^2 = \lambda v^2 \) and \( \phi_{\text{th}b}^2 = \lambda v^4/32 \) are satisfied.
4.1 Particle production without tunneling dynamics

For comparison with the later discussion, we start with the discussion on the particle production in the de Sitter spacetime without tunneling dynamics. We should stress that the particle production without tunneling in de Sitter spacetime is not physically expected. If we start with Bunch-Davies vacuum state, there would be no particle production since it is de Sitter invariant. Nevertheless, we demonstrate the “particle production” by introducing an adiabatic vacuum because such a discussion is useful to understand the case with tunneling dynamics. For simplicity, we will consider a conformally coupled massive scalar field $\chi$. The system is described by the following Lagrangian:

$$\sqrt{-g} L = -\frac{1}{2} \sqrt{-g} \left( \partial_{\mu} \chi \partial^{\mu} \chi + \frac{1}{6} R \chi^2 + M^2 \chi^2 \right).$$

Assuming the de Sitter background metric $ds^2 = \frac{1}{H^2} \eta (d\eta^2 + dx^2)$, we mode-expand $\chi$ as

$$\chi = \int \frac{d^3 k}{(2\pi)^{3/2} a(\eta)} \left( \hat{a}_k \psi_k(\eta) e^{ik \cdot x} + \hat{a}_k^\dagger \psi_k^*(\eta) e^{-ik \cdot x} \right),$$

where $a(\eta) = -\frac{1}{H \eta}$ is a scale factor, and we have introduced a creation (annihilation) operator $\hat{a}_k (\hat{a}_k^\dagger)$. The mode equation of the scalar field is given by

$$v_k'' + \omega_k^2 v_k = 0,$$

where a prime denotes a derivative with respect to the conformal time $\eta$ and

$$\omega_k^2(\eta) = k^2 + \frac{M^2}{(-H \eta)^2}.$$

Following the discussion in Sec. 2, let us discuss the particle production in our case. The turning points where $\omega_k(\eta_c) = 0$ in the complex $\eta$-plane are simply given by

$$\eta_c = 0 + i \frac{M}{H k}.$$

The value of the singulant along the line connecting these turning points is given by

$$\frac{1}{2} F_k(\eta_c^\ast) = i \int_{\eta_c}^{\eta_c^\ast} d\eta' \omega_k(\eta') = i \int_{\eta_c}^{\eta_c^\ast} d\eta' \sqrt{k^2 + \frac{M^2}{H^2 \eta'^2}}.$$

We integrate this along the path avoiding the simple pole $\eta' = 0$, and only the half-pole integration contributes because other parts cancel. Parametrizing $\eta = e^{i\theta} (\theta : \pi/2 \to 3\pi/2)$, we find

$$\frac{1}{2} F_k(\eta_c^\ast) = -\lim_{\epsilon \to 0} \int_{\pi/2}^{2\pi} e^{i\theta} d\theta \sqrt{e^{2k^2} + \frac{M^2 e^{-2i\theta}}{H^2}} = -\pi \frac{M}{H}.$$

Substituting this into (2.11), we obtain the asymptotic produced particle number as

$$n_k = |\beta_k|^2 = e^{-2\pi M/H}.$$

There are two problems in this analysis: One is that, since Stokes line is near the end point $\eta = 0$, the mode function beyond the Stokes line might not be available. Therefore, this
Bogoliubov coefficient might not have a clear physical meaning. The other is that $\beta_k$ does not depend on $k$, which seems to cause the infinite number of particle production, and the corresponding state is not normalizable and cannot be related to the original vacuum by unitary transformations.

The former issue might be circumvented by using the coordinate time $t$ instead of the conformal time $\eta$. Let us use the coordinate system $ds^2 = -dt^2 + e^{2Ht}d\mathbf{x}^2$, and the mode equation of the scalar is given by [30]

$$\ddot{f}_k + \omega_k^2 f_k = 0,$$

where we have parametrized $\chi$ as

$$\chi(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} a^{-3/2} \left( a_k f_k(t)e^{i\mathbf{k} \cdot \mathbf{x}} + a_k^\dagger f_k(t)e^{-i\mathbf{k} \cdot \mathbf{x}} \right)$$

and

$$\omega_k^2(t) = k^2 e^{-2Ht} + M^2.$$  

The WKB solution to the mode equation takes the form

$$f_k(t) = \frac{\alpha_k(t)}{\sqrt{\omega(t)}} e^{-i \int f^t \omega_k(t')} dt' + \frac{\beta_k(t)}{\sqrt{\omega(t)}} e^{i \int f^t \omega_k(t')}.$$  

The turning point is given by

$$t_c = -H^{-1} \left( \ln(M/k) + \frac{\pi}{2} i \right).$$

It is easy to derive the singulant along the Stokes line, and we find

$$\frac{1}{2} F_k(t^*_c) = i \int_{t_c}^{t^*_c} \omega_k(t') dt' = -\frac{M \pi}{H}.$$  

Thus, the asymptotic Bogoliubov coefficient is

$$\beta_k = i e^{-\frac{M \pi}{H}}$$

which is precisely the same as the one in (4.8). In this case, since the coordinate time $t$ varies from $t = -\infty$ to $t = \infty$, we expect that there exist the asymptotic mode function with the Bogoliubov coefficient (4.15) for sufficiently large $t$. However, the Bogoliubov coefficients and therefore the particle number is independent of the momentum as is the case with conformal time, which causes the divergence. Therefore, the particle production caused by these turning point should not be realized physically; otherwise the late time vacuum cannot be related to the initial one via unitary transformation.

However, we should notice that we have so far discussed the behavior with comoving momenta. More careful treatment is necessary to discuss the particle production with physical momenta. The singulant at time $t$ is given by

$$F_k(t) = \frac{2i e^{-Ht}\sqrt{M^2 e^{2Ht} + k^2} - 2i M \log \left( \frac{\sqrt{M^2 e^{2Ht} + k^2} + Me^{Ht}}{k} \right) + \pi M}{2H}. $$

---
Thus we find

\[
\text{Re } F_k(t) = \frac{\pi M}{2H}, \quad (4.17)
\]

\[
\text{Im } F_k(t) = H^{-1} \left( \omega_k(t) - M \log \left( \frac{\omega_k(t) + M}{ke^{-Ht}} \right) \right), \quad (4.18)
\]

with which the Stokes multiplier is given by

\[
S_k(t) = \frac{1}{2} \left( 1 + \text{Erf} \left[ -\left( \pi HM \right)^{-\frac{1}{2}} \left( \omega_k(t) - M \log \left( \frac{\omega_k(t) + M}{ke^{-Ht}} \right) \right) \right] \right). \quad (4.19)
\]

Therefore, the time-dependent Bogoliubov coefficient \( \beta_k(t) \) is

\[
\beta_k(t) = -\frac{ie^{-M\pi H}}{2} \text{Erf}[f(t)]. \quad (4.20)
\]

where

\[
f(t) \equiv (\pi HM)^{-\frac{1}{2}} \left( \omega_k(t) - M \log \left( \frac{\omega_k(t) + M}{ke^{-Ht}} \right) \right), \quad (4.21)
\]

and we have used the relation, \( \text{Erf}(-x) = -\text{Erf}(x) \) and \( \text{Erf}(x) = 1 - \text{Erfc}(x) \). In terms of the physical momentum \( k_{\text{phys}} = ke^{-Ht} \), the quantity \( f \) is written as

\[
f(t) = (\pi HM)^{-\frac{1}{2}} \left( \sqrt{k^2_{\text{phys}} + M^2} - M \log \left( \frac{\sqrt{k^2_{\text{phys}} + M^2}}{k_{\text{phys}}} + M \right) \right), \quad (4.22)
\]

which looks time independent. From this expression, we find (formally) time-independent particle spectrum produced in de Sitter background. Since the asymptotic form of complementary error function is given by (3.41) and \( f \sim k_{\text{phys}}/(\pi HM)^{\frac{1}{2}} \), the particle number density for high physical momentum decays as

\[
n_{k_{\text{phys}}} \sim \frac{HM e^{-\frac{2M \pi}{H}}}{4k^2_{\text{phys}}} e^{-2k^2_{\text{phys}}/(\pi HM)}. \quad (4.23)
\]

This would lead to a finite number of particle at any time \( t \). The total number of the particle is

\[
N_{\text{tot}} = \int \frac{d^3k}{(2\pi)^3} n_k = e^{3Ht} \int \frac{d^3k_{\text{phys}}}{(2\pi)^3} \frac{e^{-2M \pi}}{4} (\text{Erfc}(f))^2, \quad (4.24)
\]

where \( f \) in terms of physical momentum is given in (4.22). The integral would converge and give some finite value. For example, if we take \( H = 1, M = 10 \), numerical integration gives \( N_{\text{tot}} \sim 1.12 \times 10^{-14}e^{3t} \). Therefore the total number of particle inside the comoving volume \( N_{\text{tot}}/a^3 \) is finite.

We also note that the particle spectrum (4.8) is slightly different from the well-known results \([33, 34]\) given by

\[
n^d_S = \frac{1}{e^{2\pi M/H} - 1}. \quad (4.25)
\]

This spectrum is obtained by comparing the Bunch-Davies vacuum and its late time behavior.
\( T = \frac{H}{2\pi} \), known as the Gibbons-Hawking temperature [35]. Since we have discussed an adiabatic vacuum, the particle spectrum does not coincide with that of the Bunch-Davies vacuum. However, we find that the leading order is the same as (4.8) in the super-massive limit \( M/H \gg 1 \). In either way, this particle spectrum does not give the convergence of the momentum integral and should not be physical or more appropriate evaluation would be necessary.

4.2 Particle production with tunneling

In the following, we discuss the particle production induced by the tunneling dynamics of the background scalar field. In this case, we would expect the shift of the turning point by the tunneling dynamics. Let us consider the following system:

\[
\sqrt{-g}\mathcal{L} = -\frac{1}{2}\sqrt{-g}\left( \partial_{\mu}\phi\partial^{\mu}\phi + V(\phi) + \partial_{\mu}\chi\partial^{\mu}\chi + \frac{1}{6}R\chi^{2} + g\phi^{2}\chi^{2} \right),
\]

where \( \phi \) denotes a real scalar field. \( \phi \) is supposed to be the tunneling field, which initially sits at the false vacuum and eventually penetrates to the true vacuum. For our purpose, we apply the stochastic inflation formalism to the dynamics of \( \phi \) at the zeroth order in \( g \), which gives the coarse-grained dynamics of \( \phi \). In the stochastic inflation formalism [26, 27], one integrates large momentum modes out, which yields stochastic noise for lower frequency modes being regarded as a classical field. Even though we average over super-horizon modes, namely average over different Hubble patches, the expectation value in a single patch would asymptote to the super-horizon average, as long as we are interested in sufficiently long time interval.

Particularly, we focus on the one-point probability distribution function \( \rho[\phi(x)] \) obeying the following Fokker-Planck equation [27]

\[
\frac{\partial}{\partial t}\rho(\phi,x) = \frac{1}{3H} \frac{\partial}{\partial \phi} \left( V'(\phi)\rho(\phi,x) \right) + \frac{H}{8\pi^{2}} \frac{\partial^{2}}{\partial \phi^{2}} \rho(\phi,x),
\]

where the prime denotes the functional derivative with respect to \( \phi(x) \). The general solution of this Fokker-Planck equation is

\[
\rho(\phi,t) = \exp \left( -\frac{4\pi^{2}V(\phi)}{3H^{4}} \right) \sum_{n=0}^{\infty} a_{n}\Phi_{n}(\phi)e^{\frac{-\Lambda_{n}(t-\tau)}{H}},
\]

where \( a_{n} \) is a constant and \( \tau \) is the initial time, which we will take to be \( \tau = -\infty \). Here, \( \Phi_{n}(\phi) \) is the eigenfunction satisfying the following equation,

\[
\left[ -\frac{1}{2} \frac{\partial^{2}}{\partial \phi^{2}} + W(\phi) \right] \Phi_{n}(\phi) = \frac{4\pi^{2}\Lambda_{n}}{H^{3}} \Phi_{n}(\phi),
\]

where \( \Lambda_{n} \) is a non-negative eigenvalue and

\[
W(\phi) \equiv \frac{1}{2}[v'(\phi)^{2} - v''(\phi)],
\]

\[
v(\phi) \equiv \frac{4\pi^{2}}{3H^{4}}V(\phi).
\]
The lowest eigenvalue $\Lambda_0 = 0$ corresponds to the equilibrium mode, and we will take the mode up to the second lowest mode $n = 1$. For the case with double well potential $V = \frac{\lambda}{4}(\phi^2 - m^2/\lambda)^2$, we find

$$
\Lambda_1 = \frac{\sqrt{2}m^2}{3\pi H} \exp \left(-\frac{2\pi^2 m^4}{3\lambda H^4}\right).
$$

(4.32)

Notice that the eigenvalue $\Lambda_1$ is exponentially suppressed, and $\Lambda_1 \ll H$. Although the potential with false and true vacuum should not be exactly the same as the double well potential, the difference of the eigenvalues would not be so large. In either way, we will not use the explicit form of the scalar potential, and therefore the eigenvalue $\Lambda_1$, but assuming $\Lambda_1 = \Lambda = \text{const}(\ll H)$, the time-dependent expectation value is given by

$$
\langle \phi^2(t) \rangle = v^2 \left(1 - e^{-\Lambda(t-\tau)}\right),
$$

(4.33)

where $v$ denotes the vacuum expectation value of $\phi$ at the true vacuum, and we have assumed $\langle \phi \rangle = 0$ at the false vacuum. If the explicit form is assumed, one is able to find the eigenvalues $\Lambda_n$ explicitly e.g. by perturbative methods.

The time dependent expectation value yields the time dependent mass term for the scalar $\chi$, from which the particle production takes place besides that caused by de Sitter background. This expectation value becomes (infinitely) negative for $t \to -\infty$, which is not physically acceptable. Therefore, we may avoid such an issue by taking $\tau \to -\infty$. Instead of such a prescription, we will consider the follow modification

$$
\langle \phi^2(t) \rangle_{\text{reg}} = \frac{1}{2} v^2 \left(1 + \tanh \left(\frac{1}{2} \Lambda t\right)\right),
$$

(4.34)

which asymptotically reproduce the original expression for $t > 0$ while avoiding negative value of $\langle \phi^2 \rangle$ for $t < 0$.

In this case, the frequency for a comoving momentum $k$ mode is given by

$$
\omega_k(t) = \sqrt{k^2 e^{-2Ht} + \frac{1}{2} M^2 \left(1 + \tanh \left(\frac{1}{2} \Lambda t\right)\right)}
$$

(4.35)

where $M^2 \equiv g v^2$. Because of the complication of the frequency, it is impossible to find the analytic expression for the singulant $F_k(t) = 2i \int_{t_{\tau}}^{t} \omega_k(t')dt'$, although it would be possible to calculate it numerically. Therefore, we discuss the high and the low momentum modes separately with approximation. For the former case, according to (4.13), the turning point is located at the point with Re $t \gg 1$, which would mean, the creation of the particle takes place at late time. For sufficiently large $k$, the production time on the real axis is large enough to regard $\langle \phi^2 \rangle \sim v^2$, and the frequency is effectively given by

$$
\omega_k(t) \sim \sqrt{k^2 e^{-2Ht} + M^2},
$$

(4.36)

and therefore, the Bogoliubov coefficient would become that in (4.20).

Let us consider the production of low frequency mode. We approximate the frequency $\omega_k$ as follows:

$$
\omega_k(t) = \sqrt{k^2 e^{-2Ht} + \frac{1}{2} M^2 \left(1 + \tanh \left(\frac{1}{2} \Lambda t\right)\right)}
$$

$$
= e^{\frac{\Lambda t}{2}} \sqrt{k^2 e^{-2Ht-\Lambda t} + \frac{M^2}{1 + e^{\Lambda t}}}
$$

$$
\sim e^{\frac{\Lambda t}{2}} \sqrt{k^2 e^{-2Ht} + M^2}.
$$

(4.37)
Here, we have used the approximations $e^{-2Ht-\Lambda t} \sim e^{-2Ht}$ and $\frac{1}{1+e^{\Lambda t}} \sim 1$. The former is justified since $\Lambda \ll H$, and the latter is consistent as long as $-\Lambda \text{Re} t \gg 1$. With this approximation, the turning point is the same as that in (4.13).

With the approximated frequency, we find the singulant to be

$$F_k(t) = i \int_{t_c}^{t} \omega_k(t') dt'$$

$$= \left[ \frac{2i \frac{\Lambda}{2H} \sqrt{k^2 e^{-2Ht} + M^2 (M^2 e^{2Ht} + k^2)}}{k^2 (\Lambda - 2H)} \right] _{t_c}^t 2F_1 \left( 1, \frac{\Lambda}{2H} + 1; \frac{2H + \Lambda}{4H}; -\frac{e^{2Ht} M^2}{k^2} \right).$$

The most relevant quantity is $F_k(t^*_c)$ given by

$$F_k(t^*_c) \sim -\frac{\pi M}{H} \left( 1 + \frac{\Lambda (\log \left( \frac{k}{M} \right) + 1)}{2H} \right),$$

where we have taken the leading order term in $\Lambda$. Therefore, the Bogoliubov coefficient is asymptotically given by

$$\beta_k \sim i e^{F_k(t^*_c)} = i e^{-\frac{\pi M}{H} \left( 1 + \frac{\Lambda (\log \left( \frac{k}{M} \right) + 1)}{2H} \right)} \left( \frac{M}{k} \right)^{\frac{\pi M \Lambda}{H^2}},$$

and the particle number density is given by

$$n_k \sim e^{-\frac{2\pi M}{H} \left( 1 + \frac{\Lambda (\log \left( \frac{k}{M} \right) + 1)}{2H} \right)} \left( \frac{M}{k} \right)^{\frac{\pi M \Lambda}{H^2}}.$$}

Thus we have found the produced particle spectrum corrected by the tunneling dynamics. There is an extra factor depending on momentum, which becomes larger for smaller momentum $k$. This seems reasonable since the low frequency modes are more sensitive to the change of the effective mass, particularly because $\chi$ is originally a massless particle. However, we should also note that we have used the coarse-grained expression for $\langle \phi(t) \rangle$, and for very small momentum modes, the wavelength can be larger than the coarse-grained scale roughly given by $H^{-1}$. In such a case, the stochastic inflation might not be an appropriate formalism to describe the production of $\chi$. In this sense, the divergent behavior of $k \to 0$ would be corrected by more appropriate formalism. We also note that if we are interested in total number density of the produced particle, the IR divergence would not spoil because the negative power of $k$ given by the tunneling dynamics correction is so small that the phase space volume $\int d^3k$ would cancel the negative power of $k$, and there would not be any IR divergence.

Finally, let us comment on the particle production in de Sitter spacetime. In de Sitter space, the notion of particle or its production would not be useful because the frequency never becomes stationary. In a realistic scenario, inflationary de Sitter phase will end at some point and the Universe eventually asymptotes to the present Universe, where the particles would have (almost) time-independent frequency. In such a case, the production of particles would be physically meaningful. Even in such a case, our discussion would not be altered much, and there would be corresponding turning points at which particle production takes place. There, our estimation of the particle production due to tunneling would be useful.

\[\text{Here, we have assumed } e^{\Lambda t} \sim 0, \text{ and this cannot be justified if we are interested in } \text{Re} t > 0. \text{ However, as long as we are interested in the time scale shorter than } \Lambda^{-1}(\gg H^{-1}), \text{ we can approximate } e^{\Lambda t} \sim C = \text{const.} \text{ and in such a case, we can use the approximated formula by replacing } M^2 \text{ with } M^2(1 + C)^{-1}.\]
5 Summary and discussion

We have analyzed particle production with a tunneling background field by the Stokes phenomenon method and the real time formalism in the flat and in the de Sitter spacetime.

In the case of the flat spacetime, analyzing the Stokes line, we have found that particle production is efficient only when the tunneling background field is around each vacuum. We have obtained the number density of produced particle (3.43). This result is valid not only for a double-well potential but also for a broad class of potential with at least two metastable vacua. We should note that we have used homogeneous approximation for the tunneling scalar, which causes the continuous oscillation of the tunneling scalar between true and false vacuum. We expect that taking the spatial dependence would change our result would change our result slightly, but not so significantly because we have considered a heavy scalar whose Compton wavelength is much shorter than the nucleated bubble radius.

In the case of the de Sitter spacetime, since it is known that particle production occurs without any background dynamics, first we have revisited such particle production by using the Stokes phenomenon method. Although the total produced particle number density after comoving momentum integration suffers from divergence similarly to the previous studies, we have shown that a physical momentum distribution is convergent. Comparing with this result, we have also analyzed particle production with a tunneling background field using the stochastic inflation formalism, which gives a time-dependent distribution of a scalar field value in de Sitter spacetime. We have found that the efficiency of production for low momentum modes is enhanced by power-law due to the background tunneling dynamics. The resultant particle number density also seems to suffer from the IR divergence; however, it results from the pure (eternal) de Sitter phase and thus it is not problematic in realistic inflationary models where inflationary phase will end at some point.

Finally, we have to add one more comment. Application to Higgs instability would require to include gravity because the spacetime would become anti-de Sitter with large negative curvature after vacuum decay. We expect that our discussion with Stokes phenomenon analyses in real time formalism would be applicable to such cases. We intend to consider such extensions in future work.

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