Abstract In this article, we investigate inverse source problems for a wide range of PDEs of parabolic and hyperbolic types as well as time-fractional evolution equations by partial interior observation. Restricting the source terms to the form of separated variables, we establish uniqueness results for simultaneously determining both temporal and spatial components without non-vanishing assumptions at \( t = 0 \), which seems novel to the best of our knowledge. Remarkably, mostly we allow a rather flexible choice of the observation time not necessarily starting from \( t = 0 \), which fits into various situations in practice. Our main approach is based on the combination of the Titchmarsh convolution theorem with unique continuation properties and time-analyticity of the PDEs under consideration.

Keywords Inverse source problem, evolution equation, uniqueness, Titchmarsh convolution theorem, unique continuation

AMS Subject Classifications 35R11, 35R30, 35B60

1 Introduction

Let \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3, \ldots)\) be a bounded domain whose boundary \( \partial \Omega \) is of \( C^2 \) class. For \( f \in C^2(\bar{\Omega}) \), define the elliptic operators

\[
Af(x) := -\text{div}(a(x)\nabla f(x)) + c(x)f(x), \quad Lf(x) := (A + b(x) \cdot \nabla)f(x), \quad x \in \Omega,
\]

where \( \cdot \) and \( \nabla = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d} \right) \) refer to the inner product in \( \mathbb{R}^d \) and the gradient in \( x \), respectively. Here \( a = (a_{jk})_{1 \leq j, k \leq d} \in C^1(\bar{\Omega}; \mathbb{R}^{d \times d}) \), \( b = (b_1, \ldots, b_d) \in L^\infty(\Omega; \mathbb{R}^d) \) and \( c \in L^q(\Omega) \) with \( q \in (\frac{d}{2}, \infty] \) are \( x \)-dependent matrix-, vector- and scalar-valued functions, respectively. Further, there exists a constant \( \kappa > 0 \) such that

\[
a(x)\xi \cdot \xi \geq \kappa|\xi|^2, \quad \forall \ x \in \bar{\Omega}, \ \forall \ \xi \in \mathbb{R}^d, \quad c(x) \geq \kappa \text{ a.e. } x \in \Omega,
\]
where \(|\xi|^2 = \xi \cdot \xi\). Meanwhile, for \(\beta \in (0, 2]\) we denote the \(\beta\)-th order Caputo derivative in \(t\) by

\[
\partial_t^\beta f(t) := \begin{cases}
\frac{1}{\Gamma(\beta - \beta)} \int_0^t \frac{f(\tau)\,d\tau}{(t - \tau)^{\beta - \beta}}, & \beta \in (0, 1) \cup (1, 2), \\
f^{(\beta)}(t), & \beta = 1, 2,
\end{cases}
\]

where \(\Gamma(\cdot)\), \([\cdot]\) and \(\lfloor \cdot \rfloor\) stand for the Gamma function, the ceiling and the floor functions, respectively. In this paper, the order of the Caputo derivative is either a constant in \((0, 1]\) or an \(x\)-dependent piecewise constant function taking value in \((0, 1]\), which will be specified later in Section 2. We restrict the discussion of a variable order to \((0, 1]\) because there seems very few works on its generalization e.g. to \((0, 2]\).

Set \(R_+ := (0, \infty)\). In this paper, we are concerned with the following initial-boundary value problem for a (time-fractional) evolution equation

\[
\begin{cases}
\left(\rho(x) \partial_t^\alpha(x) + \mathcal{L}\right) u(t, x) = \mu(t) h(x), & (t, x) \in R_+ \times \Omega, \\
u(0, x) = \partial_t^{[\alpha]-1} u(0, x) = 0, & x \in \Omega, \\
\mathcal{R} u(t, x) = 0, & (t, x) \in R_+ \times \partial \Omega,
\end{cases}
\tag{1.1}
\]

where we make the following global assumptions:

\[
\mu \in L^1(R_+) : \text{compactly supported}, \quad \exists T_0 > 0 \text{ s.t. } \mu \not\equiv 0 \text{ in } (0, T_0),
\tag{1.2}
\]

\[
h \in L^2(\Omega), \quad \rho \in L^\infty(\Omega), \quad \exists \underline{\rho}, \overline{\rho} > 0 \text{ s.t. } \underline{\rho} \leq \rho \leq \overline{\rho} \text{ a.e. in } \Omega.
\tag{1.3}
\]

Here \(\mathcal{R}\) denotes either the Dirichlet boundary condition or the Neumann boundary condition associated with the principal part \(a\) of the elliptic operator \(A\), i.e.,

\[
\mathcal{R} f(x) = f(x) \quad \text{or} \quad \mathcal{R} f(x) = a(x) \nabla f(x) \cdot \nu(x), \quad x \in \partial \Omega,
\]

where \(\nu(x)\) is the outward unit normal vector to \(\partial \Omega\) at \(x \in \partial \Omega\). The solution to (1.1) is understood in the weak sense, whose precise definition will be given in Definition 2.1 (see Section 2).

This paper focuses on the uniqueness issue of the following inverse source problem concerning (1.1).

**Problem 1.1** Let \(u\) satisfy (1.1) with (1.2)-(1.3), \(I \subset R_+\) be a finite open interval and \(\omega \subset \Omega\) a nonempty open subset. Under certain assumptions, determine \(h\) with given \(\mu\) or determine \(h, \mu\) simultaneously by the partial interior observation of \(u\) in \(I \times \omega\).

The governing equation in (1.1) takes the form of a rather general (time-fractional) evolution equation which includes the traditional parabolic and hyperbolic ones. Correspondingly, Problem (1.1) also covers a wide range of inverse source problems arising in several scientific areas including medical imaging, optical tomography, seismology and environmental problems. For a constant \(\alpha \in (0, 2]\), Problem (1.1) corresponds to the recovery of a source in typical or anomalous diffusion processes appearing in geophysics, biology and environmental science (see e.g. [20],[10]). For \(\alpha = 2\), our inverse problem can be associated with the determination of an acoustic source with applications in area such as medical imaging and seismology. For instance, Problem (1.1) with \(\alpha = 2\) can be applied to some inverse diffraction and near-field holography problems (see e.g. [10] Chapter 2.2.5). We mention also that problem (1.1) with a variable
order $\alpha(x)$ is considered as a model for diffusion phenomenon in some complex media, where the variation of $\alpha$ is due to the presence of heterogeneous regions. In that context, Problem 1.1 can be seen as the determination of a source appearing in a diffusion process associated with problems in chemistry \cite{6}, biology \cite{11} and physics \cite{46,52}.

Among the various formulations of inverse problems, inverse source problems have gathered consistent popularity owing to their theoretical and practical significance. The interested reader can refer to \cite{17,35} for an overview of these problems. For $\alpha = 1, 2$, these problems have been studied extensively in the last decades. Without being exhaustive, we refer to \cite{7,15,16,19,26,40,51}, among which the approach of \cite{15,16,51} was based on the Bukhgeim-Klibanov method introduced in \cite{5}. For determining the temporal components in source terms of time-fractional diffusion equations and hyperbolic systems, we mention \cite{7,36,37,43} and \cite{4,12,13}, respectively. In the same spirit, \cite{14,32} were devoted to the determination of information about the support of general source terms in parabolic equations, and \cite{5,8} dealt with that of time-dependent point sources. It was proved in \cite{31} that some classes of time-dependent source terms appearing in similar problems to (1.1) with a constant $\alpha \in (0, 2)$ can be reconstructed from boundary measurements when $\Omega$ is a cylindrical domain. The approach of \cite{31} has been recently extended by \cite{21} to similar equations with time-dependent elliptic operators. Meanwhile, in \cite{33} the authors proved the recovery of general source terms from the full knowledge of the solution in a time interval $(t_0, T)$ with $t_0 \in (0, T)$.

Let us emphasize that almost all above mentioned results assumed the unknown functions to be independent of either the time variable or one space variable. For the first class of source terms, the known component in the source term may depend on time. Among all these results, the strategy allowing the recovery of such classes of source terms requires the non-vanishing assumption of the known component of the source term at $t = 0$. Recently, \cite{27} established one of the first results of recovering the source term in (1.1) without assuming $\mu(0) \neq 0$ or more generally, only requiring supp $\mu \subset [0, T)$. Moreover, \cite{27} also proved the simultaneous determination of $\mu, h$ in (1.1) provided that the restriction of $\mu$ to a subinterval of $\mathbb{R}^+$ is known and admits an analytic extension. On the same direction of \cite{27}, in this paper we will demonstrate the possibility of determining $\mu, h$ simultaneously in more general settings. By the way, we restrict the source term to the form $\mu(t)h(x)$ of complete separated variables in view of the obstruction for this problem described e.g. in \cite{27 §1.3.1}.

Meanwhile, another highlight of Problem 1.1 is the relaxation of the observation time. In most literature on inverse source problems, the observation was assumed to start from $t = 0$. However, in practice, usually the data is only available after the occurrence of some unpredictable accidents. Therefore, it is reasonable to generalize the observation time to a finite interval $I \subset \mathbb{R}^+$ not necessarily starting from $t = 0$. It turns out that such a relaxation gives affirmative answers to Problem 1.1 in most situations with unfortunate exceptions of $\alpha = 1, 2$.

The remainder of this article is organized as follows. In Section 2, we collect the preliminaries to deal with Problem 1.1 and state three main results on uniqueness according to the choices of $\alpha$ along with comments about these results. Then the next three sections are devoted to the proofs for different cases of $\alpha$ respectively. Finally, some technical details will be provided in Section 6.

2 Preliminaries and Main Results
For Banach spaces $X$ mild solutions (see [28, 29, 31]) with smooth coefficients. For $\bf{b}$ that $u \in C^{\alpha}_{\omega}(\Omega)$ for $\bf{b}$ the weight the governing system (1.1). We say $u \in L^1_{\loc}(\Omega; L^2(\Omega))$ is a weak solution to (1.1) if it satisfies the following conditions.

1. $p_0 := \inf \{ e > 0 \mid e^{-e} u \in L^1(\Omega; L^2(\Omega)) \} < \infty$.
2. For all $p > p_0$, the Laplace transform $\hat{u}(p; \cdot)$ of $u(t, \cdot)$ with respect to $t$ solves the following boundary value problem

\[
\begin{aligned}
(L + p^\alpha \rho)\hat{u}(p; \cdot) &= \hat{\mu}(p) h \quad \text{in } \Omega, \\
\mathcal{R}\hat{u}(p; \cdot) &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

For a constant $\alpha \in (0, 2)$, the above definition of weak solutions is equivalent to that of mild solutions (see [28, 29, 31]) with smooth coefficients. For $\bf{b} \equiv 0$, we refer to [30, Theorem 2.3] (see also [29]) for the unique existence of a weak solution $u \in L^1_{\loc}(\Omega; L^2(\Omega))$ to (1.1). For $\alpha \in (0, 1]$ and $\bf{b} \not\equiv 0$, we refer to [28, pp. 13–15] for the unique existence of a weak solution $u \in L^1_{\loc}(\Omega; H^1(\Omega))$ to (1.1). For $\alpha \in L^\infty(\Omega)$ and $\bf{b} \equiv 0$, in Proposition 6.1, we prove the unique existence of a weak solution $u \in L^1_{\loc}(\Omega; L^2(\Omega))$ to (1.1). We point out that for $\alpha = 1, 2$, the weak solutions to (1.1) defined above coincide with the classical solution to the corresponding parabolic and hyperbolic equations, respectively.

Now we are well prepared to state our main results regarding Problem 1.1. Due to the essential difference of the problems and the corresponding results, we divide the statement into three contexts according to difference choices of $\alpha$, namely

1. $\alpha \equiv 2$,
2. $\alpha \in (0, 2)$ is a constant, and
3. $\alpha : \Omega \rightarrow (0, 1)$ is a piecewise constant.

In each case, we first establish a uniqueness principle for (1.1) and then, as a direct consequence, state a uniqueness result regarding Problem 1.1.

**Case 1** $\alpha \equiv 2$. We denote by $\dist$ the Riemannian distance of $\Omega$ equipped with the metric $g$, where $g = g(x)$ is the inverse of the matrix $(\rho^{-1}(x) a_{jk}(x))_{1 \leq j, k \leq d}$.

**Theorem 2.1** Let $u$ satisfy (1.1) with $\alpha \equiv 2$, where we assume (1.2)–(1.3) and additionally $\bf{b} \equiv 0$, $c \in L^\infty(\Omega)$, $\rho \in C^1(\Omega)$. For any nonempty open subset $\omega \subset \Omega$, if

$$T \geq T_0 + \sup_{x \in \Omega} \dist(x, \omega), \quad \text{where } \dist(x, \omega) := \inf_{y \in \omega} \dist(x, y),$$

(2.2)
then $u = 0$ in $(0, T) \times \omega$ implies $h \equiv 0$.

An immediate application of Theorem 2.1 to Problem 1.1 indicates the unique determination of the spatial component $h$ in the source term when the temporal component $\mu$ is known.

**Corollary 2.1** Let the conditions in Theorem 2.1 be fulfilled and $u_i$ satisfy (1.1) with $\alpha \equiv 2$ and $h_i \in L^2(\Omega)$ ($i = 1, 2$), where $\mu \in L^1(\mathbb{R}^+)$ is known. Then $u_1 = u_2$ in $(0, T) \times \omega$ implies $h_1 = h_2$ in $\Omega$.

To the best of our knowledge, the uniqueness principle of Theorem 2.1 and the associated uniqueness in Corollary 2.1 are the first results on determining the spatial component $h$ in a hyperbolic equation without assuming $\mu(0) \neq 0$ or $T = \infty$. Indeed, it seems that all other similar results required either some non-vanishing conditions at $t = 0$ (see e.g. [16, 19, 50, 51]) or infinite observation time (e.g. [12]). Especially, the non-vanishing condition in our context reads $\mu(0) \neq 0$, which restricts the problem under consideration to such a situation that the phenomenon of interest should start before observation. By removing this condition, we make the results of Theorem 2.1 and Corollary 2.1 more flexible to allow the measurement to start before the appearance of some unknown phenomenon.

**Case 2** $\alpha \in (0, 2)$ is a constant.

**Theorem 2.2** Let $u$ satisfy (1.1) with a constant $\alpha \in (0, 2)$, where we assume $\beta = \frac{1}{2} - \frac{1}{\alpha}$ and $h = 0$ in $\omega$. Moreover, we restrict $\rho \equiv 1$ if $\alpha \in (0, 1]$ and $b \equiv 0$ if $\alpha \in (1, 2)$. Then for any nonempty open subset $\omega \subset \Omega$,

1. If $\alpha \neq 1$, then for any $T \geq T_0$ and any $T_1 \in [0, T)$, the following implication holds true.

$$ (u = 0 \text{ in } (T_1, T) \times \omega) \implies (h \equiv 0 \text{ in } \Omega). \quad (2.3) $$

2. If $\alpha = 1$, then the implication (2.3) holds true provided that $T_1 = 0$.

Applying Theorem 2.2, we can prove the uniqueness for determining $\mu$ and $h$ simultaneously in the source term of (1.1), provided that $\mu$ satisfies (1.2) and is known in $(0, T_0)$.

**Corollary 2.2** Let the conditions in Theorem 2.2 be fulfilled and $u_i$ satisfy (1.1) with a constant $\alpha \in (0, 2)$, $\mu_i \in L^1(\mathbb{R}^+)$ being compactly supported and $h_i \in L^2(\Omega)$ ($i = 1, 2$), where we assume

$$ \mu_1 \text{ satisfies (1.2), } \mu_1 = \mu_2 \text{ in } (0, T_0), \quad (2.4) $$

$$ h_1 = h_2 \text{ in } \omega, \quad h_1 \neq 0 \text{ in } \Omega. \quad (2.5) $$

Then

1. If $\alpha \neq 1$, then for any $T \geq T_0$ and any $T_1 \in [0, T_0)$, the following implication holds true.

$$ (u_1 = u_2 \text{ in } (T_1, T) \times \omega) \implies (\mu_1 \equiv \mu_2 \text{ in } (0, T) \text{ and } h_1 \equiv h_2 \text{ in } \Omega). \quad (2.6) $$

2. If $\alpha = 1$, then the implication (2.6) holds true provided that $T_1 = 0$.

In [18, 27], similar results to Theorem 2.2 were established by requiring $T_1 = 0$. In this sense, Theorem 2.2 greatly improves the flexibility in the choice of the observation time. On the other hand, Theorem 2.2 extends the uniqueness principle in [27, Theorem 1.1] by removing the requirement $\text{supp } \mu \subset [0, T)$. Actually, the uniqueness principle of Theorem 2.2 even holds
true with measurement taken in \((0, T_0)\) where \(\mu\) is not uniformly vanishing (see (1.2)). As a direct consequence of Theorem 2.2, Corollary 2.2 claims the uniqueness of the simultaneous determination of both temporal and spatial components of the source term under the tolerable extra assumptions (2.4)–(2.5). To the best of our knowledge, Corollary 2.2 is the first result on completely determining a source term of separated variables stated in such a general context. The only comparable result seems to be [27, Theorem 1.3], which additionally requires that the restriction of \(\mu\) to \((0, T_0)\) admits a holomorphic extension to some neighborhood of \(\mathbb{R}_+\). In that sense, Corollary 2.2 generalizes [27, Theorem 1.3] considerably. In addition, for \(\alpha \in (0, 1]\) we also allow the presence of a convection term \(b\) in (1.1), which breaks the symmetry of the elliptic part and thus any solution representation of (1.1) using the eigensystem. Therefore, instead of the method in [27], we adopt the idea in [18] to combine the time-analyticity of the solution with the unique continuation property of parabolic equations.

**Case 3** \(\alpha : \Omega \rightarrow (0, 1)\) is a piecewise constant. More precisely, we assume that for a fixed \(N \in \mathbb{N}\), there exist constants \(\alpha_\ell \in (0, 1)\) and open subdomains \(\Omega_\ell \subset \Omega\) \((\ell = 1, 2, \ldots, N)\) with Lipschitz boundaries such that

\[
\Omega = \bigcup_{\ell=1}^{N} \Omega_\ell, \quad \Omega_\ell \cap \Omega_m = \emptyset \ (\ell \neq m),
\]

(2.7)

\[
\alpha(x) = \alpha_\ell \ (x \in \Omega_\ell, \ \ell = 1, \ldots, N), \quad 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_N < \min\{2\alpha_1, 1\}.
\]

(2.8)

**Theorem 2.3** Let \(u\) satisfy (1.1) with \(\alpha \in L^\infty(\Omega)\) satisfying (2.7)–(2.8), where we assume (1.2)–(1.3), \(b \equiv 0\) and there exists an open subdomain \(\mathcal{O} \subset \Omega\) such that

\[
\left(\bigcup_{\ell=1}^{N} \partial \Omega_\ell\right) \setminus \partial \Omega \subseteq \mathcal{O}, \quad h = 0 \text{ in } \mathcal{O}.
\]

(2.9)

Then for any open subset \(\omega \subset \Omega\) satisfying

\[
h = 0 \text{ in } \omega, \quad \omega \cap \mathcal{O} \neq \emptyset,
\]

(2.10)

any \(T \geq T_0\) and any \(T_1 \in [0, T)\), the implication (2.3) holds true.

For a better understanding of the readers, in Figure 1 we illustrate a typical situation of the geometrical assumptions in Theorem 2.3.

![Figure 1](image-url)

Figure 1. A typical situation of the geometrical assumptions in Theorem 2.3 with \(N = 4\). The subdomain \(\mathcal{O}\) is a connect open neighborhood of the interior boundaries of \(\Omega_\ell\), which \(\omega\) should intersect. The support of \(h\) and \(\mathcal{O} \cup \omega\) are disjoint.

Parallel to that of Corollary 2.2, we can apply Theorem 2.3 to prove the unique determination of \(\mu, h\) simultaneously in the source term of (1.1) under the same additional condition on \(\mu\).
Corollary 2.3 Let the conditions in Theorem 2.3 be fulfilled and $u_i$ satisfy (1.1) with $\alpha \in L^\infty(\Omega)$ satisfying (2.7)–(2.8), $\mu_i \in L^1(\mathbb{R}_+)$ being compactly supported and $h_i \in L^2(\Omega)$ ($i = 1, 2$), where we assume (2.4) and $h_1 = h_2$ in $\omega \cup \mathcal{O}$, $h_1 \not\equiv 0$ in $\Omega$. (2.11)

Then for any open subdomain $\omega \subset \Omega$ satisfying (2.10), any $T \geq T_0$ and any $T_1 \in [0, T_0)$, the implication (2.6) holds true.

To the best of our knowledge, Theorem 2.3 and Corollary 2.3 are the first results on the uniqueness of inverse source problems for time-fractional diffusion equations with variable orders. We restrict the order to a piecewise constant function due to its significance from the practical point of view. In that context, we obtain some uniqueness results comparable to the case of a constant order, provided that the support of spatial component in the source term does not pass through the regions where $\alpha$ takes different values (see condition (2.9)).

As was mentioned before, the highlight in Theorems 2.2–2.3 and Corollaries 2.2–2.3 is the relaxation of the choice of the observation time. Indeed, for $\alpha \neq 1$, all other related results required that the observation data should be taken from $t = 0$. In contrast, Theorems 2.2–2.3 and Corollaries 2.2–2.3 only assume that the measurement does not terminate earlier than $T_0$ in (1.2). Recalling that (1.2) means $\mu \not\equiv 0$ in $(0, T_0)$, such an assumption is the minimal necessary that one can expect. This remarkable relaxation not only allows the absence of data near the initial time, but also enables the delayed measurement carried out even after the disappearance of the source under consideration. Note that such restriction of the measurement requires the application of a result in [33], which cannot be applied to $\alpha = 1$. It is not clear whether the implication (2.3) still holds true for $\alpha = 1$ with $T_1 > 0$.

3 Proof of Theorem 2.1 and Corollary 2.1

Proof of Theorem 2.1 Let $u$ be the solution of (1.1) with $\alpha = 2$ and assume that the condition

$$u = 0 \quad \text{in} \quad (0, T) \times \omega$$

is fulfilled, where $T$ satisfies (2.2). We will show that (3.1) implies $h \equiv 0$.

According to the Duhamel’s principle, it is readily seen that $u$ takes the form of

$$u(t, \cdot) = \int_0^t \mu(t - \tau) v(\tau, \cdot) \, d\tau,$$

where $v$ solves

$$\begin{cases}
(\rho \partial^2_t + A)v = 0 & \text{in} \quad (0, T) \times \Omega, \\
v = 0, \partial_t v = h & \text{in} \quad \{0\} \times \Omega, \\
Rv = 0 & \text{on} \quad (0, T) \times \partial\Omega.
\end{cases}$$

By Lions and Magenes [39], we know $v \in C([0, T]; H^1(\Omega)) \cap C([0, T]; L^2(\Omega))$. Then for arbitrarily fixed $\psi \in C_0^\infty(\omega)$, it follows from (3.2) and (3.1) that

$$0 = \nu(\omega)(u(t, \cdot), \psi)_{C_0^\infty(\omega)} = \int_0^t \mu(t - \tau) v(\tau, \cdot) \, d\tau, \quad 0 < t < T,$$
where \( v_\psi(t) := \mathcal{D}(\omega)\langle v(t, \cdot), \psi \rangle \mathcal{C}_0^\infty(\omega) \). Since \( \mu \in L^1(0, T) \), we apply the Titchmarsh convolution theorem (see [48, Theorem VII]) to deduce that there exist \( \tau_1, \tau_2 \in [0, T] \) satisfying \( \tau_1 + \tau_2 \geq T \) such that
\[
\mu \equiv 0 \text{ in } (0, \tau_1), \quad v_\psi \equiv 0 \text{ in } (0, \tau_2).
\]
In view of the key assumption \([122]\), we have \( \tau_1 \leq T_0 \). Then it follows from \([22]\) that
\[
\tau_2 \geq T - \tau_1 \geq T - T_0 \geq \sup_{x \in \Omega} \text{dist}(x, \omega) =: T_1,
\]
which implies
\[
v_\psi(t) = \mathcal{D}(\omega)\langle v(t, \cdot), \psi \rangle \mathcal{C}_0^\infty(\omega) = 0, \quad t \in [0, T_1].
\]
Since in this identity, \( \psi \in C_0^\infty(\omega) \) was chosen arbitrary and \( T_1 \) is independent of \( \psi \), we deduce that
\[
v = 0 \quad \text{in } (0, T_1) \times \omega. \tag{3.3}
\]

On the other hand, recalling the metric \( g \) defined on \( \overline{\Omega} \), we introduce the Laplace-Beltrami operator \( \Delta_g \) associated with the Riemannian manifold \((\Omega, g)\). Then there exists \( K \in C(\overline{\Omega}; \mathbb{R}^d) \) such that
\[
(\partial_t^2 - \Delta_g + K \cdot \nabla + \rho^{-1}c)v = (\partial_t^2 + \rho^{-1}A)v = 0 \quad \text{in } (0, T) \times \Omega.
\]
Now we define the odd extension of \( v \) in \((-T, T) \times \Omega \), still denoted by \( v \), by
\[
v(t, \cdot) = -v(-t, \cdot) \quad \text{in } \Omega, \quad -T < t < 0.
\]
Using the fact that \( v(0, \cdot) = 0 \), we see that \( v \in H^3((-T, T) \times \Omega) \). Meanwhile, in view of \((3.3)\), we see that \( v \) satisfies
\[
\begin{aligned}
(\partial_t^2 - \Delta_g + K \cdot \nabla + \rho^{-1}c)v &= 0 \quad \text{in } (-T_1, T_1) \times \Omega, \\
v &= 0 \quad \text{in } (-T_1, T_1) \times \omega.
\end{aligned}
\]
Therefore, applying a global unique continuation result theorem similar to \([22\), Theorem 3.16] (see also \([26\), Theorem A.1]) derived from the local unique continuation result of \([12, 44]\), we obtain
\[
h = \partial_t v(0, \cdot) = 0 \quad \text{in } \{ x \in \Omega \mid \text{dist}(x, \omega) \leq T_1 \}.
\]
Then it follows from the definition of \( T_1 \) that \( \{ x \in \Omega \mid \text{dist}(x, \omega) \leq T_1 \} = \Omega \) or equivalently \( h \equiv 0 \) in \( \Omega \). This completes the proof of Theorem \([2.1]\) \( \square \)

**Proof of Corollary 2.1** Introducing \( u := u_1 - u_2 \), it is readily seen that \( u \) satisfies
\[
\begin{aligned}
(\rho(x)\partial_t^2 u + A)u(t, x) &= \mu(t)(h_1 - h_2)(x), \quad (t, x) \in \mathbb{R}_+ \times \Omega, \\
u(0, x) &= \partial_t u(0, x) = 0, \quad x \in \Omega, \\
R u(t, x) &= 0, \quad (t, x) \in \mathbb{R}_+ \times \partial \Omega.
\end{aligned}
\]
Meanwhile, the condition \( u_1 = u_2 \) in \((0, T) \times \omega\) implies \( u = 0 \) in \((0, T) \times \omega\). Since \( T \) satisfies \([122]\), Theorem \([2.1]\) immediately implies \( h_1 - h_2 \equiv 0 \) or equivalently \( h_1 \equiv h_2 \) in \( \Omega \). \( \square \)
4 Proof of Theorem 2.2 and Corollary 2.2

In this section, we investigate the case of a constant \( \alpha \in (0, 2) \) in (1.1). To this end, a similar solution representation to \( S_{1,2} \) is necessary, which requires some preparations.

In the case of \( \alpha \in (0, 1] \), we specify \( L \) as the unbounded elliptic operator acting on \( L^2(\Omega) \) with the domain \( D(L) = \{ f \in H^2(\Omega) \mid \Re f = 0 \text{ on } \partial \Omega \} \). From now on, for all \( s, \theta \in [0, \infty) \), we denote by \( D_{s,\theta} \) the set
\[
D_{s,\theta} := \{ s + re^{i\beta} \mid r > 0, \beta \in (-\theta, \theta) \}.
\]
According to [2] Theorem 2.1 (see also [38] Theorem 2.5.1), there exists \( \theta_0 \in (\frac{\pi}{2}, \pi) \) and \( s_0 \geq 0 \) such that \( D_{s_0,\theta_0} \) is in the resolvent set of \( L \). Moreover, there exists a constant \( C > 0 \) depending on \( L \) and \( \Omega \) such that
\[
\| (L + z)^{-1} \|_{B(L^2(\Omega; \rho \, dx))} + |z|^{-1}\| (L + z)^{-1} \|_{B(L^2(\Omega; \rho \, dx); H^2(\Omega))} \leq C|z|^{-1}, \quad z \in D_{s_0,\theta_0}.
\]
Here we employed the fact that \( L^2(\Omega) = L^2(\Omega; \rho \, dx) \) in the sense of the norm equivalence thanks to (1.3). We fix \( \theta_1 \in (\frac{\pi}{2}, \theta_0) \), \( \delta \in \mathbb{R}_+ \) and consider a contour \( \gamma(\delta, \theta_1) \) in \( \mathbb{C} \) defined by
\[
\gamma(\delta, \theta_1) := \gamma_- (\delta, \theta_1) \cup \gamma_0 (\delta, \theta_1) \cup \gamma_+ (\delta, \theta_1)
\]
on oriented in the counterclockwise direction, where
\[
\gamma_0 (\delta, \theta_1) := \{ s_0 + \delta e^{i\beta} \mid \beta \in [-\theta_1, \theta_1] \}, \quad \gamma_\pm (\delta, \theta_1) := \{ s_0 + r e^{\pm i\theta_1} \mid r \geq 0 \}
\]
with double signs in the same order.

Let \( \theta_2 \in (0, \theta_1 - \frac{\pi}{2}) \). Applying the above properties of \( L \), for \( \alpha \in (0, 1] \) and \( z \in D_{0,\theta_2} \), we can define an operator \( S_1(z) \in B(L^2(\Omega)) \) by
\[
S_1(z) u_0 = \frac{1}{2\pi i} \int_{\gamma(\delta, \theta_1)} e^{zp}(L + \rho^\alpha)^{-1} u_0 \, dp, \quad u_0 \in L^2(\Omega).
\]
We recall the following property of the map \( z \mapsto S_1(z) \).

**Lemma 4.1** (see [23] Lemma 2.4) For all \( \beta \in [0, 1] \), the map \( z \mapsto S_1(z) \) is analytic from \( D_{0,\theta_2} \) to \( B(L^2(\Omega); H^{2\beta}(\Omega)) \). Moreover, there exists a constant \( C > 0 \) depending only on \( L \) and \( \Omega \) such that
\[
\| S_1(z) \|_{B(L^2(\Omega; \rho \, dx); H^{2\beta}(\Omega))} \leq C|z|^\alpha (1 - \beta)^{-1} e^{s_0 \Re z}, \quad z \in D_{0,\theta_0}.
\]
In addition to this property, we combine (1.1) with the arguments for [28] Theorem 1.1 and [38] Remark 1 to deduce that for \( \mu \in L^\infty(\mathbb{R}_+) \), (1.1) admits a unique weak solution \( u \in C([0, \infty); L^2(\Omega)) \) taking the form
\[
u(t, \cdot) = \int_0^t \mu(t - \tau) S_1(\tau) h \, d\tau.
\]
(4.2)
Using some arguments similar to that for [27] Proposition 6.1, one can show that the identity (4.1) also holds true for \( \mu \in L^1(\mathbb{R}_+) \) with a compact support.

Likewise, in the case of \( \alpha \in (1, 2) \) with \( b \equiv 0 \), we turn to the eigensystem \( \{ (\lambda_n, \varphi_n) \}_{n \in \mathbb{N}} \) of the self-adjoint operator \( A = \rho^{-1} A \) acting on \( L^2(\Omega; \rho \, dx) \), with the boundary condition \( R u = 0 \) in \( \partial \Omega \), to define an operator \( S_2(t) \) for \( t > 0 \) by
\[
S_2(t) u_0 = t^{\alpha - 1} \sum_{n=1}^{\infty} E_{\alpha, \alpha} (-t^{\alpha} \lambda_n) (u_0, \varphi_n) \varphi_n, \quad u_0 \in L^2(\Omega),
\]
where $(\cdot, \cdot)_p$ denotes the inner product of $L^2(\Omega; \rho \, dx)$. Here $E_{\alpha,\alpha}(\cdot)$ is the Mittag-Leffler function defined by

$$E_{\alpha,\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \alpha)}, \quad z \in \mathbb{C}.$$  

Similarly as before, applying [41, Theorem 1.6] and following the arguments used in [29, Theorem 1.1], one can check that $S_2$ is well-defined as an element of $L_{1,0}^{1,\infty}(\mathbb{R}^+; \mathcal{B}(L^2(\Omega)))$. Moreover, we can show that (1.1) admits a unique weak solution $u \in C(\mathbb{R}^+; L^2(\Omega))$ taking the form

$$u(t, \cdot) = \int_0^t \mu(t - \tau) S_2(\tau) h \, d\tau. \quad (4.3)$$

For the proof of Theorem 2.2, we also invoke the Riemann-Liouville integral operator $I^\beta$ and the Riemann-Liouville derivative $D_t^\beta$ for $\beta \in (0, 2]$:

$$I^\beta f(t) := \begin{cases} f(t), & \beta = 0, \\ \frac{1}{\Gamma(\beta)} \int_0^t f(\tau) (t - \tau)^{1-\beta} \, d\tau, & \beta > 0 \quad (f \in C([0, \infty])), \end{cases} \quad D_t^\beta := \frac{d^{[\beta]}}{dt^{[\beta]}} \circ I_t^{[\beta]-\beta},$$

where $\circ$ denotes the composite. We need the following technical lemma.

**Lemma 4.2** Let $h \in L^1(0, T)$, $\beta \in (0, 1) \cup (1, 2)$, $\lambda > 0$ and $w \in L^1(0, T)$ be given by

$$w(t) := \int_0^t (t - \tau)^{\beta-1} E_{\beta,\beta}(-\lambda(t - \tau)^\beta) h(\tau) \, d\tau. \quad \text{Then}$$

$$(D_t^\beta + \lambda)w(t) = h(t), \quad t \in (0, T)$$

holds true in the sense of distributions.

The result of Lemma 4.2 is rather classical for smooth function $h$. However, we have not found a clear proof of such result for $h \in L^1(0, T)$. For this reason, we provide its full proof here.

**Proof of Lemma 4.2** Let $\beta \in (0, 1) \cup (1, 2)$ and fix a sequence $\{h_k\}_{k \in \mathbb{N}} \subset C_0^\infty(0, T)$ such that

$$\lim_{k \to \infty} \|h_k - h\|_{L^1(0, T)} = 0.$$  

One can check that, for $\{w_k\}_{k \in \mathbb{N}}$ defined by

$$w_k(t) := \int_0^t (t - \tau)^{\beta-1} E_{\beta,\beta}(-\lambda(t - \tau)^\beta) h_k(\tau) \, d\tau,$$

there holds

$$(D_t^\beta + \lambda)w_k(t) = h_k(t), \quad t \in (0, T). \quad (4.4)$$

Moreover, applying [41, Theorem 1.6] and Young’s convolution inequality, we get

$$\limsup_{k \to \infty} \|w_k - w\|_{L^1(0, T)} \leq C \|t^{\beta-1}\|_{L^1(0, T)} \limsup_{k \to \infty} \|h_k - h\|_{L^1(0, T)} = 0.$$  

In the same way, we obtain

$$\limsup_{k \to \infty} \|I_t^\beta w_k - I_t^\beta w\|_{L^1(0, T)} \leq C \|t^{[\beta]-\beta}\|_{L^1(0, T)} \limsup_{k \to \infty} \|w_k - w\|_{L^1(0, T)} = 0. \quad (4.5)$$
Therefore, fixing $\chi \in C_0^\infty(0,T)$ and applying (4.5), we find
\[
\lim_{k \to \infty} \mathcal{D}^p(0,T)(D_t^\beta w_k, \chi)_{C_0^\infty(0,T)} = \lim_{k \to \infty} \mathcal{D}^p(0,T)(\partial_t^\beta I^\beta [\cdot])_{C_0^\infty(0,T)}(w_k, \chi)_{C_0^\infty(0,T)} = \lim_{k \to \infty} \mathcal{D}^p(0,T)(\partial_t^\beta I^\beta [\cdot])_{C_0^\infty(0,T)}(w_k, \chi)_{C_0^\infty(0,T)} = \mathcal{D}^p(0,T)(D_t^\beta w, \chi)_{C_0^\infty(0,T)}.
\]
In the same way, we have
\[
\lim_{k \to \infty} \mathcal{D}^p(0,T)(w_k, \chi)_{C_0^\infty(0,T)} = \mathcal{D}^p(0,T)(w, \chi)_{C_0^\infty(0,T)},
\]
\[
\lim_{k \to \infty} \mathcal{D}^p(0,T)(h_k, \chi)_{C_0^\infty(0,T)} = \mathcal{D}^p(0,T)(h, \chi)_{C_0^\infty(0,T)}.
\]
Combining these identities with (4.6) and (4.4), we obtain
\[
\mathcal{D}^p(0,T)(h, \chi)_{C_0^\infty(0,T)} = \lim_{k \to \infty} \mathcal{D}^p(0,T)(h_k, \chi)_{C_0^\infty(0,T)} = \lim_{k \to \infty} \mathcal{D}^p(0,T)(D_t^\beta w_k + \lambda w_k, \chi)_{C_0^\infty(0,T)} = \mathcal{D}^p(0,T)(D_t^\beta w, \chi)_{C_0^\infty(0,T)}.
\]
This completes the proof of the lemma.

Now we are in a position to prove Theorem 2.2.

**Proof of Theorem 2.2** Let $u$ be the solution of (1.1) with a constant $\alpha \in (0,2)$. Without loss of generality, we only investigate the case of $\alpha \in (0,1)$ with $\rho \equiv 1$ because the case of $\alpha \in (1,2)$ with $b \equiv 0$ can be treated in the same manner. For clarity, we divide the proof into three steps.

**Step 1** Let us fix $T \geq T_0$ and define
\[
v(t, \cdot) := I^{1-\alpha} u(t, \cdot), \quad t > 0.
\]
Using the fact that $u \in L^1_t(L^\infty_w(\Omega))$, we deduce that $v \in L^1(0,T; L^2(\Omega))$. In this step we will prove that actually $v \in W^{1,1}(0,T; H^2(\Omega))$. Following Definition 2.1 of weak solutions, we recall that for $p > p_0$, the Laplace transform $\hat{u}(p; \cdot)$ of $u(t, \cdot)$ with respect to $t$ solves the following boundary value problem
\[
\begin{cases}
\rho^n + \rho^{-1} \mathcal{A} \hat{u}(p; \cdot) = -\rho^{-1} b \cdot \nabla \hat{u}(p; \cdot) + \hat{\mu}(p) \rho^{-1} h & \text{in } \Omega, \\
R \hat{u}(p; \cdot) = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Therefore, for all $p > p_0$ and all $n \in \mathbb{N}$, we have
\[
((\rho^n + \rho^{-1} \mathcal{A}) \hat{u}(p; \cdot), \varphi_n)_{\rho} = -((\rho^{-1} b \cdot \nabla \hat{u}(p; \cdot), \varphi_n)_{\rho} + \hat{\mu}(p) (\rho^{-1} h, \varphi_n)_{\rho}.
\]
Integrating by parts, we obtain
\[
((\rho^n + \rho^{-1} \mathcal{A}) u(p; \cdot), \varphi_n)_{\rho} = ((\hat{u}(p; \cdot), (\rho^n + \rho^{-1} \mathcal{A}) \varphi_n)_{\rho} = (\lambda_n + p^n) (\hat{u}(p; \cdot), \varphi_n)_{\rho}
\]
and it follows
\[
(\hat{u}(p; \cdot), \varphi_n)_{\rho} = -((\rho^{-1} b \cdot \nabla \hat{u}(p; \cdot), \varphi_n)_{\rho} + \hat{\mu}(p) (\rho^{-1} h, \varphi_n)_{\rho}.
\]
In the same way, fixing
\[ w(t, \cdot) := \int_0^t S_2(t - \tau) \left[ -\rho^{-1} b \cdot \nabla u(\tau, \cdot) + \mu(\tau)\rho^{-1} h \right] d\tau, \quad t > 0, \]
we have
\[ (w(t, \cdot), \varphi_n)_\rho = \int_0^t (t - \tau)^{\alpha - 1} E_{\alpha, \alpha}(-\lambda_n(t - \tau)^\alpha + \rho^{-1} b \cdot \nabla u(\tau, \cdot) + \mu(\tau)\rho^{-1} h, \varphi_n)_\rho d\tau, \quad t > 0 \]
for all \( n \in \mathbb{N} \). Therefore, applying the Laplace transform in time for \( p > p_0 \) to the expression above, we get
\[ (\tilde{w}(p; \cdot), \varphi_n)_\rho = \frac{- (\rho^{-1} b \cdot \nabla \tilde{u}(p; \cdot), \varphi_n)_\rho}{\lambda_n + p^\alpha} + \frac{\tilde{\mu}(p) (\rho^{-1} h, \varphi_n)_\rho}{\lambda_n + p^\alpha} = (\tilde{u}(p; \cdot), \varphi_n)_\rho \]
for all \( n \in \mathbb{N} \). Then it follows that
\[ \tilde{w}(p; \cdot) = \tilde{u}(p; \cdot), \quad p > p_0, \]
and the uniqueness of the Laplace transform in time implies that
\[ u(t, \cdot) = w(t, \cdot) = \int_0^t S_2(t - \tau) \left[ -\rho^{-1} b \cdot \nabla u(\tau, \cdot) + \mu(\tau)\rho^{-1} h \right] d\tau, \quad t > 0. \quad (4.8) \]
In view of (4.8), fixing
\[ f_n(t) = (-\rho^{-1} b \cdot \nabla u(t, \cdot) + \mu(t)\rho^{-1} h, \varphi_n)_\rho, \quad n \in \mathbb{N}, \quad (4.9) \]
we deduce that
\[ u_n(t) := (u(t, \cdot), \varphi_n)_\rho = \int_0^t (t - \tau)^{\alpha - 1} E_{\alpha, \alpha}(-\lambda_n(t - \tau)^\alpha f_n(\tau) d\tau, \quad n \in \mathbb{N}, \ t > 0. \quad (4.10) \]

Since \( u \in L^1(0, T; H^1(\Omega)) \) by [23] pp.13–15, we deduce that \( -\rho^{-1} b \cdot \nabla u + \mu \rho^{-1} h \in L^1(0, T; L^2(\Omega)) \) and thus \( f_n \in L^1(0, T) \) for all \( n \in \mathbb{N} \) by (4.9). Then in view of Lemma 1.2, we know that
\[ D_t^\alpha u_n(t) = -\lambda_n u_n(t) + f_n(t), \quad n \in \mathbb{N}, \ t \in (0, T) \quad (4.11) \]
holds in the sense of distributions. Since \( -\rho^{-1} b \cdot \nabla u + \mu \rho^{-1} h \in L^1(0, T; L^2(\Omega)) \), we deduce that the sequence
\[ \sum_{n=1}^N f_n(t) \varphi_n \quad (N \in \mathbb{N}) \]
converges in the sense of \( L^1(0, T; L^2(\Omega)) \). In the same way, we can prove that the sequence
\[ \sum_{n=1}^N \lambda_n u_n(t) \varphi_n \quad (N \in \mathbb{N}) \]
converges in the sense of \( L^1(0, T; D(A^{-1})) \), where we recall that \( A = \rho^{-1} A \) acts on \( L^2(\Omega; \rho d\mathbf{x}) \) with the boundary condition \( \mathcal{R} u = 0 \) on \( \partial \Omega \). This proves that
\[ \sum_{n=1}^N D_t^\alpha u_n(t) \varphi_n = \sum_{n=1}^N [-\lambda_n u_n(t) + f_n(t)] \varphi_n \quad (N \in \mathbb{N}) \]
converges in the sense of $L^1(0, T; \mathcal{D}(A^{-1}))$. Thus, $D_t^su$ is well defined and

$$
D_t^s u = \sum_{n=1}^{\infty} D_t^s u_n(t) \varphi_n \in L^1(0, T; \mathcal{D}(A^{-1})).
$$

Let $H^2_0(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $H^2(\Omega)$. It is clear that $H^2_0(\Omega)$ is embedded continuously into $\mathcal{D}(A)$ and by duality, we deduce that $\mathcal{D}(A^{-1})$ embedded continuously into $H^{-2}(\Omega)$. It follows that $D_t^s u \in L^1(0, T; H^{-2}(\Omega))$. On the other hand, the function $v$ given by (4.14) satisfies $\partial_t v = D_t^s u \in L^1(0, T; H^{-2}(\Omega))$, and using the fact that $v \in L^1(0, T; H^2(\Omega))$, we get $v \in W^{1, 1}(0, T; H^{-2}(\Omega))$.

**Step 2** We fix $T_1 \in [0, T)$. In this step, we will show that the condition

$$
u = 0 \text{ in } (T_1, T) \times \omega \quad (4.12)
$$

implies

$$
u = 0 \text{ in } (0, T) \times \omega. \quad (4.13)
$$

Similarly to the proof of Theorem 4.1, we set $u_\psi(t) := D^{\ast}(\omega)\{u(t, \cdot), \psi\}_{C_0^\infty(\omega)}$ ($0 < t < T$) with arbitrarily fixed $\psi \in C_0^\infty(\omega)$. According to Step 1, the function $v$ defined by (4.7) lies in $W^{1, 1}(0, T; H^{-2}(\Omega))$ and thus

$$I^{1-\alpha}u_\psi = D^{\ast}(\omega)\{v(t, \cdot), \psi\}_{C_0^\infty(\omega)} \in W^{1, 1}(0, T). \quad (4.14)
$$

Moreover, in view of (4.11) and the fact that the sequence

$$
\sum_{n=1}^{N} D_t^s u_n(t) \varphi_n = \sum_{n=1}^{N} [-\lambda_n u_n(t) + f_n(t)] \varphi_n, \quad N \in \mathbb{N},
$$

converge in the sense of $L^1(0, T; \mathcal{D}(A^{-1}))$ to $D_t^s u$, we deduce that for a.e. $t \in (0, T)$, we have

$$
D_t^s u(t, \cdot) = \sum_{n=1}^{\infty} D_t^s u_n(t) \varphi_n = \sum_{n=1}^{\infty} [-\lambda_n u_n(t) + f_n(t)] \varphi_n
$$

$$
= \sum_{n=1}^{\infty} \left(-\rho^{-1} Au(t, \cdot) - \rho^{-1} b \cdot \nabla u(t, \cdot) + \mu(t)\rho^{-1} h, \varphi_n\right)_\rho \varphi_n
$$

$$
= -\rho^{-1} Au(t, \cdot) - \rho^{-1} b \cdot \nabla u(t, \cdot) + \mu(t)\rho^{-1} h.
$$

Here we recall that $f_n$ and $u_n$ were given by (4.9) and (4.10), respectively. On the other hand, from (4.12) we deduce that $-\rho^{-1} Au - \rho^{-1} b \cdot \nabla u = 0$ in $(T_1, T) \times \omega$. Combining this with the fact that $h = 0$ in $\omega$, we obtain

$$
D_t^s u = -\rho^{-1} Au - \rho^{-1} b \cdot \nabla u + \mu h = 0 \text{ in } (T_1, T) \times \omega. \quad (4.15)
$$

In the same way, using the fact that the function $v$ given by (4.7) lies in $W^{1, 1}(0, T; H^{-2}(\Omega))$, we obtain

$$
D_t^s u_\psi(t) = \frac{d}{dt} I^{1-\alpha} u_\psi(t) = \frac{d}{dt} (D^{\ast}(\omega)\{v(t, \cdot), \psi\}_{C_0^\infty(\omega)})
$$

$$
= D^{\ast}(\omega)\{\partial_t v(t, \cdot), \psi\}_{C_0^\infty(\omega)} = D^{\ast}(\omega)\{D_t^s u(t, \cdot), \psi\}_{C_0^\infty(\omega)}
$$

for a.e. $t \in (0, T)$, and (4.15) implies $D_t^s u_\psi = 0$ in $(T_1, T)$. Therefore, we have $u_\psi = D_t^s u_\psi = 0$ in $(T_1, T)$. Combining this with condition (4.13) and applying Theorem 1 yields

$$
u(t) = D^{\ast}(\omega)\{u(t, \cdot), \psi\}_{C_0^\infty(\omega)} = 0, \quad 0 < t < T;
$$
which implies \( \text{1.13} \) since \( \psi \in C_0^\infty(\omega) \) was chosen arbitrarily.

**Step 3** In this step, we show that \( \text{1.13} \) implies \( h \equiv 0 \) in \( \Omega \). We deal with the cases of \( \alpha \in (0, 1] \) and \( \alpha \in (1, 2) \) separately.

**Case 1** We first study the case of \( \alpha \in (0, 1] \) with \( \rho \equiv 1 \). In the same manner as before, for arbitrarily fixed \( \psi \in C_0^\infty(\omega) \), it follows from \( \text{4.13} \) and the solution representation \( \text{4.2} \) that
\[
0 = D'(\omega)(u(t, \cdot), \psi)_{C_0^\infty(\omega)} = \int_0^t \mu(t - \tau) v_\psi(\tau) \ dr, \quad 0 < t < T,
\]
where \( v_\psi(t) := D'(\omega)(S_1(t)h, \psi)_{C_0^\infty(\omega)} \in L^1(0, T) \) by Lemma \( \text{4.1} \). Therefore, parallel to the proof of Theorem \( \text{2.1} \), the Titchmarsh convolution theorem guarantees the existence of constants \( \tau_1, \tau_2 \in [0, T] \) satisfying \( \tau_1 + \tau_2 \geq T \) such that
\[
\mu \equiv 0 \text{ in } (0, \tau_1), \quad v_\psi \equiv 0 \text{ in } (0, \tau_2).
\]
Since \( \mu \not\equiv 0 \text{ in } (0, T_0) \) by the key assumption \( \text{1.2} \), we conclude \( \tau_1 < T_0 \) and thus
\[
\tau_2 \geq T - \tau_1 > T - T_0 \geq 0.
\]
In other words, we obtain
\[
v_\psi(t) = D'(\omega)(S_1(t)h, \psi)_{C_0^\infty(\omega)} = 0, \quad 0 < t < \tau_2,
\]
which implies
\[
v(t, \cdot) := S_1(t)h = 0 \text{ in } \omega, \quad 0 < t < \tau_2 \quad \text{(4.16)}
\]
again since \( \psi \in C_0^\infty(\omega) \) was chosen arbitrarily. On the other hand, similarly to the proofs of \( \text{23 Theorem 1.1} \) and \( \text{31 Theorem 1.4} \), we deduce that the map \( t \mapsto S_1(t)h \) admits an analytic extension from \( \mathbb{R}_+ \) to \( L^2(\Omega) \). Therefore, \( \text{4.16} \) implies
\[
v = 0 \text{ in } \mathbb{R}_+ \times \omega. \quad \text{(4.17)}
\]
In view of Lemma \( \text{4.1} \), the Laplace transform \( \hat{v}(p, \cdot) \) of \( v \) with respect to \( t \) is well-defined for all \( p > s_0 \). Further, following \( \text{23} \) (see also \( \text{23} \) Theorem 1.1), we see that \( \hat{v}(p, \cdot) \) solves the boundary value problem for an elliptic equation
\[
\begin{cases}
(p^2 + \mathcal{L})\hat{v}(p, \cdot) = h & \text{in } \Omega, \\
\mathcal{R}\hat{v}(p, \cdot) = 0 & \text{on } \partial\Omega,
\end{cases}
\quad \forall p > s_0.
\]
Meanwhile, \( \text{4.17} \) implies
\[
\hat{v}(p, \cdot) = 0 \text{ in } \omega, \quad p > s_0. \quad \text{(4.18)}
\]
On the other hand, let us introduce an initial-boundary value problem for a parabolic equation
\[
\begin{cases}
(\partial_t + \mathcal{L})w = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\
w = h & \text{in } \{0\} \times \Omega, \\
\mathcal{R}w = 0 & \text{in } \mathbb{R}_+ \times \partial\Omega,
\end{cases}
\]
as well as the Laplace transform \( \hat{w}(p, \cdot) \) of \( w \) with respect to \( t \) for \( p > p_0 \), where \( p_0 \geq s_0^2 \) is sufficiently large. Then straightforward calculation yields that for all \( p > p_0 \), there holds
\( \hat{v}(p^{1/\alpha}, \cdot) = \hat{w}(p, \cdot) \). Combining this with (4.18) and applying the uniqueness of the Laplace transform, we obtain
\[
\hat{w}(p, \cdot) = 0 \quad \text{in} \quad \mathbb{R}_+ \times \omega.
\]
Consequently, the unique continuation property of parabolic equations (e.g. [45, Theorem 1.1]) implies \( w \equiv 0 \) in \( \mathbb{R}_+ \times \Omega \) and thus \( h = w(0, \cdot) \equiv 0 \) in \( \Omega \). This completes the proof of Theorem 2.2 for \( \alpha \in (0, 1] \).

**Case 2**  
Now we turn to the case of \( \alpha \in (1, 2) \) with \( b \equiv 0 \). Combining the solution representation (4.3) with the above arguments, again we can conclude (4.16) with \( v(t, \cdot) := S_2(t)h \) in this case. On the other hand, applying again [41, Theorem 1.6] and utilizing arguments similar to those used in the proof of [24, Proposition 3.1], one can check that the map \( t \mapsto v(t, \cdot) \) admits an analytic extension as a function from \( \mathbb{R}_+ \) to \( L^2(\Omega) \). Therefore, again (4.16) implies (4.17).

Then for \( p > 0 \), it follows from [28, Theorem 1.1] that the Laplace transform \( \hat{v}(p; \cdot) \) of \( v \) with respect to \( t \) solves the boundary value problem for an elliptic equation
\[
\begin{aligned}
\left\{ \begin{array}{ll}
(p^\alpha + A)\hat{v}(p; \cdot) &= h \quad \text{in} \quad \Omega, \\
\mathcal{R}\hat{v}(p; \cdot) &= 0 \quad \text{on} \quad \partial\Omega.
\end{array} \right.
\end{aligned}
\]
Meanwhile, (4.17) implies (4.18) again. Therefore, repeating the above arguments (see also Step 2 in the proof of [27, Theorem 1.1]), again we arrive at \( h \equiv 0 \) in \( \Omega \). This completes the proof of Theorem 2.2 for \( \alpha \in (1, 2) \).

**Proof of Corollary 2.2**  
Since the proofs for the cases of \( \alpha \in (0, 1] \) and \( \alpha \in (1, 2) \) are again similar, we only deal with the former one without loss of generality.

Introducing the auxiliary function \( u := u_1 - u_2 \), we see that \( u \) satisfies
\[
\begin{aligned}
\left\{ \begin{array}{ll}
(\partial_t^\alpha + L)u &= F := \mu_1 h_1 - \mu_2 h_2 \quad \text{in} \quad \mathbb{R}_+ \times \Omega, \\
u &= 0 \quad \text{in} \quad \{0\} \times \Omega, \\
\mathcal{R}u &= 0 \quad \text{on} \quad \mathbb{R}_+ \times \partial\Omega.
\end{array} \right.
\end{aligned}
\tag{4.19}
\]
along with the additional condition \( u = 0 \) in \( (T_1, T_0) \times \omega \). Especially, thanks to the assumption \( T_1 < T_0 \) in (2.6), we have
\[
u = 0 \quad \text{in} \quad (T_1, T_0) \times \omega. \tag{4.20}
\]
On the other hand, it follows from (2.4)–(2.5) that
\[
F = \begin{cases} 
\mu_1 (h_1 - h_2) & \text{in} \quad (0, T_0) \times \Omega, \\
(\mu_1 - \mu_2)h_1 & \text{in} \quad \mathbb{R}_+ \times \omega.
\end{cases} \tag{4.21}
\]
and in particular,
\[
F = 0 \quad \text{in} \quad (0, T_0) \times \omega. \tag{4.22}
\]
Therefore, repeating the argument used in Step 1 of the proof of Theorem 2.2, we utilize (4.20) and (4.22) to conclude \( u \equiv 0 \) in \( (0, T_0) \times \omega \). Then a direct application of Theorem 2.2 implies \( h_1 - h_2 \equiv 0 \) or equivalently \( h_1 = h_2 \) in \( \Omega \).

From now on we can write \( F = \mu h_1 \) in (4.19) with \( \mu := \mu_1 - \mu_2 \) in \( \mathbb{R}_+ \times \Omega \) and it remains to show \( \mu \equiv 0 \) in \( (0, T) \). Now that \( u = 0 \) in \( (0, T) \times \omega \), we can take advantage of the solution
representation (4.12) to obtain
\[ 0 = \partial'(\omega)(u(t, \cdot), \psi)C_0^\infty(\omega) = \int_0^t \mu(t - \tau) v_\psi(\tau) \, d\tau, \quad 0 < t < T \]
with arbitrarily fixed \( \psi \in C_0^\infty(\omega) \), where \( v_\psi(t) := \partial'(\omega)(S_1(t)h_1, \psi)C_0^\infty(\omega) \). Again by the Titchmarsh convolution theorem, there exist constants \( \tau_1, \tau_2 \in [0, T] \) satisfying \( \tau_1 + \tau_2 \geq T \) such that
\[ \mu \equiv 0 \text{ in } (0, \tau_1), \quad v_\psi \equiv 0 \text{ in } (0, \tau_2). \]
Now it suffices to show \( \tau_2 = 0 \) by contradiction. If \( \tau_2 > 0 \), then we obtain \( S_1(t)h_1 = 0 \) in \( \omega \) for \( 0 < t < \tau_2 \). By the same argument used in Step 2 of the proof of Theorem 2.2, we can conclude \( h_1 = 0 \) in \( \Omega \), which contradicts with the assumption \( h_1 \not\equiv 0 \). Then there should hold \( \tau_2 = 0 \) and thus \( T \geq \tau_1 \geq T - \tau_2 = T \), that is, \( \mu \equiv 0 \) in \((0, T)\). This completes the proof of Corollary 2.2.

5 Proof of Theorem 2.3 and Corollary 2.3
In this section, we assume that \( b \equiv 0 \) and \( \alpha \in L^\infty(\Omega; (0,1)) \) fulfills (2.7)--(2.8). We fix \( \theta \in (\frac{\pi}{2}, \pi) \), \( \delta > 0 \) and we define the contour in \( \mathbb{C} \),
\[ \gamma(\delta, \theta) := \gamma_-(\delta, \theta) \cup \gamma_0(\delta, \theta) \cup \gamma_+(\delta, \theta), \]
oriented in the counterclockwise direction with
\[ \gamma_0(\delta, \theta) := \{ \delta e^{i\beta} | \beta \in [-\theta, \theta] \}, \quad \gamma_{\pm}(\delta, \theta) := \{ s e^{\pm i\theta} | s \in [\delta, \infty) \}, \]
with double signs in the same order. Then, following [28], we define the operator
\[ S(t)\psi := \frac{1}{2\pi i} \int_{\gamma(\delta,\theta)} e^{lp}(\rho\rho + A)^{-1} \psi \, dp, \quad t > 0 \]
(5.2)
Recall that here \( A = \rho^{-1}A \) acts on \( L^2(\Omega; \rho \, dx) \), with the boundary condition \( \mathcal{R}u = 0 \) in \( \partial\Omega \). Moreover, according to [28 Theorem 1.1], the operator \( S \) is independent of the choice of \( \theta \in (\frac{\pi}{2}, \pi) \), \( \delta > 0 \). In light of [28 Theorem 1.1] and [28 Remark 1], for \( \mu \in L^\infty(\mathbb{R}_+) \) compactly supported, problem (1.1) admits a unique weak solution (in the sense of Definition 2.1) given by
\[ u(t, \cdot) = \int_0^t \mu(t - \tau) S(\tau)(\rho^{-1}h) \, d\tau, \quad t > 0. \]
We prove in Proposition 6.1 that the result of [28] can be extended to problem (1.1) with \( \mu \in L^1(\mathbb{R}_+) \) compactly supported. Using the representation (5.2) of the weak solution of (1.1), we are now in position to complete the proof of Theorem 2.3 and Corollary 2.3.

Proof of Theorem 2.3 Let \( u \) be the solution of (1.1) with a variable \( \alpha \) satisfying (2.7) and \( \omega \subset \Omega \) satisfy (2.4). Similarly to the proof of Theorem 2.2, we divide the proof into two steps.

Step 1 First we prove that, for any \( T \geq T_0 \) and \( T_1 \in [0, T) \), the condition (1.12) implies (1.13). For this purpose, we investigate each subdomain \( \Omega_\ell \) \( (\ell = 1, \ldots, N) \), on which \( \alpha(x) = \alpha_\ell \) reduces to a constant. If \( \omega \cap \Omega_\ell \neq \emptyset \), then obviously \( \mathcal{A}u = 0 \) in \( (T_1, T) \times (\omega \cap \Omega_\ell) \). Meanwhile,
the assumption (2.10) implies \( h = 0 \) in \( \omega \cap \Omega_\ell \). Then in view of the governing equation in (1.1), we obtain
\[
\partial_t^\alpha u = -\rho^{-1}(Au + \mu h) = 0 \quad \text{in} \quad (T_1, T) \times (\omega \cap \Omega_\ell).
\]
Repeating the argument used in Step 1 of the proof of Theorem 2.2, we can deduce \( u = 0 \) in \((0, T) \times (\omega \cap \Omega_\ell)\), which indicates (4.13) by collecting all nonempty \( \omega \cap \Omega_\ell \) (\( \ell = 1, \ldots, N \)).

**Step 2** We show that (4.13) implies \( h \equiv 0 \) in \( \Omega \) in this step. According to Proposition 6.1, \( u \) takes the form
\[
u(t, \cdot) = \int_0^t \mu(t - \tau) v(\tau, \cdot) \, d\tau
\]
with \( v(t, \cdot) := S(t)\rho^{-1}h \). Applying condition (2.8) and Lemma 6.1, we deduce that \( v \in L^1_{\text{loc}}(\mathbb{R}_+; L^2(\Omega)) \) with
\[
\|v(t, \cdot)\|_{L^2(\Omega)} \leq C\rho^{-1}h \max\left(\tau^{2\alpha_N-\alpha_1-1}, \tau^{2\alpha_1-\alpha_N-1}, 1\right), \quad t > 0.
\]
Combining this result with the arguments of Step 2 of the proof of Theorem 2.2, we can show that (4.13) implies the existence of a constant \( \tau_2 > 0 \) such that
\[
v = 0 \quad \text{in} \quad (0, \tau_2) \times \omega.
\]
Following [28, Theorem 1.1], we deduce that the map \( t \mapsto v(t, \cdot) \) admits an analytic extension from \( \mathbb{R}_+ \) to \( L^2(\Omega) \). Therefore, the condition (5.4) implies
\[
v = 0 \quad \text{in} \quad \mathbb{R}_+ \times \omega.
\]
On the other hand, the estimate (5.3) implies
\[
\int_0^\infty e^{-pf}\|v(t, \cdot)\|_{L^2(\Omega)} \, dt < \infty, \quad p > 0
\]
and applying the properties of the operator \( S \) borrowed form [28, Theorem 1.1], we obtain
\[
\hat{v}(p; \cdot) = (p^\alpha + A)^{-1}(\rho^{-1}h), \quad p > 0.
\]
Thus, \( \hat{v}(p; \cdot) \) solves the boundary value problem for an elliptic equation
\[
\begin{aligned}
\left\{ \begin{array}{ll}
(p^\alpha \rho + A)\hat{v}(p; \cdot) = h & \text{in} \ \Omega, \\
\mathcal{R}\hat{v}(p; \cdot) = 0 & \text{on} \ \partial\Omega.
\end{array} \right.
\end{aligned}
\]
Simultaneously, (5.5) implies that
\[
\hat{v}(p; \cdot) = 0 \quad \text{in} \ \omega, \quad p > 0.
\]
Combining this with (2.10) and (2.11), we deduce that for all \( p > 0 \), the restriction of \( \hat{v}(p; \cdot) \) to \( \mathcal{O} \) satisfy
\[
\begin{aligned}
\left\{ \begin{array}{ll}
(p^\alpha \rho + A)\hat{v}(p; \cdot) = 0 & \text{in} \ \mathcal{O}, \\
\hat{v}(p; \cdot) = 0 & \text{in} \ \omega \cap \mathcal{O}.
\end{array} \right.
\end{aligned}
\]
Therefore, applying the unique continuation property of elliptic equations (see e.g. [44, Theorem 1]) yields
\[
\hat{v}(p; \cdot) = 0 \quad \text{in} \ \mathcal{O}, \quad p > 0.
\]
This, together with the fact that \( R\hat{v}(p; \cdot) = 0 \) on \( \partial\Omega \) and \( \partial\Omega_{\ell} \subset \partial\Omega \cup \mathcal{O} \) for all \( \ell = 1, \ldots, N \) by (2.9), leads to

\[
R\hat{v}(p; \cdot) = 0 \quad \text{on} \quad \partial\Omega_{\ell}, \quad p > 0, \quad \ell = 1, \ldots, N.
\]

Therefore, combining this boundary condition with (5.6) and (5.7), we find that the restriction of \( \hat{v}(p; \cdot) \) to \( \Omega_{\ell} (\ell = 1, \ldots, N) \) satisfies

\[
\begin{align*}
(p^{\alpha_{1}}\rho + A)\hat{v}(p; \cdot) &= h \quad \text{in} \quad \Omega_{\ell}, \\
R\hat{v}(p; \cdot) &= 0 \quad \text{on} \quad \partial\Omega_{\ell}, \quad p > 0, \\
\hat{v}(p; \cdot) &= 0 \quad \text{in} \quad \Omega_{\ell} \cap \mathcal{O},
\end{align*}
\]

Here we notice that \( \Omega_{\ell} \cap \mathcal{O} \neq \emptyset \) by (2.9). Thus, utilizing the arguments used in Step 3 of the proof of [27, Theorem 1.1], we conclude \( h = 0 \) in \( \Omega_{\ell} \) for all \( \ell = 1, \ldots, N \) and consequently \( h \equiv 0 \). This completes the proof of Theorem 2.3.

**Proof of Corollary 2.3** Similarly to the proof of Corollary 2.2, the auxiliary function \( u := u_1 - u_2 \) satisfies

\[
\begin{align*}
(p^{\alpha_{1}}\rho + A)u &= F := \mu_{1}h_{1} - \mu_{2}h_{2} \quad \text{in} \quad \mathbb{R}^{+} \times \Omega, \\
u &= 0 \quad \text{in} \quad \{0\} \times \Omega, \\
R u &= 0 \quad \text{on} \quad \mathbb{R}^{+} \times \partial\Omega
\end{align*}
\]

with \( u = 0 \) in \((T_1, T) \times \omega\), where \( F \) satisfies (4.21). Especially, we have \( F \equiv \mu_{1}(h_{1} - h_{2}) \) in \((0, T_0) \times \omega\), where \( h_{1} - h_{2} = 0 \) in \( \omega \cup \mathcal{O} \) by (2.11). Repeating the argument used in Step 1 of the proof of Theorem 2.3, we can conclude \( u = 0 \) in \((0, T_0) \times \omega\), which further implies \( h_{1} - h_{2} \equiv 0 \) or equivalently \( h_{1} = h_{2} \equiv 0 \) in \( \Omega \) by Theorem 2.3.

Finally, in a similar way to the end of the proof of Corollary 2.2, we arrive at \( \mu_{1} \equiv \mu_{2} \) in \((0, T)\), which completes the proof of Corollary 2.3.

**6 Appendix**

Let \( \alpha \in L^{\infty}(\Omega) \) be such that there exist two constants \( \alpha_{0}, \alpha_{M} \in (0, 1) \) such that

\[
0 < \alpha_{0} \leq \alpha \leq \alpha_{M} < 1, \quad \alpha_{M} < 2\alpha_{0}.
\]  

In this section, we prove the unique existence of a weak solution to the problem (1.1) when \( b \equiv 0, \alpha \in L^{\infty}(\Omega) \) satisfies \( 6.1 \) and \( \rho \in L^{1}(\mathbb{R}^{+}) \) is compactly supported. We start with the following intermediate result.

**Lemma 6.1** Let \( \theta \in \left(\frac{\pi}{2}, \pi\right) \). The map \( t \rightarrow S(t) \) defined by (5.2) lies in \( L^{1}_{\text{loc}}(\mathbb{R}^{+}; \mathcal{B}(L^{2}(\Omega))) \) and there exists a constant \( C > 0 \) depending only on \( A, \rho, \theta, \Omega \) such that

\[
\|S(t)\|_{\mathcal{B}(L^{2}(\Omega))} \leq C \max\{t^{2\alpha_{M} - \alpha_{0} - 1}, t^{2\alpha_{0} - \alpha_{M} - 1}\}, \quad t > 0.
\]

**Proof.** Throughout this proof, by \( C > 0 \) we denote generic constants depending only on \( A, \rho, \theta, \Omega \), which may change from line to line. In light of [28, Proposition 2.1], for all \( \beta \in (0, \pi) \), we have

\[
\| (A + (re^{i\beta})^\alpha)^{-1} \|_{\mathcal{B}(L^{2}(\Omega))} \leq C \max\{r^{\alpha_{0} - 2\alpha_{M}}, r^{\alpha_{M} - 2\alpha_{0}}\}, \quad r > 0, \quad \beta_{1} \in (-\beta, \beta).
\]
where $C > 0$ depends only on $\mathcal{A}, \rho, \Omega$ and $\beta$. Using the fact that the operator $S$ is independent of the choice of $\delta > 0$ (see the discussion at the beginning of Section 5), we can decompose

$$S(t) = S_-(t) + S_0(t) + S_+(t), \quad t > 0,$$

where

$$S_m(t) = \frac{1}{2\pi i} \int_{\gamma_m(t^{-1}, \theta)} e^{tp^\alpha} (p^\alpha + A)^{-1} dp, \quad m = 0, \mp, \ t > 0.$$

Here we recall that the contours $\gamma_m(\delta, \theta)$ for $m = 0, \mp$ and $\delta > 0$ were given by (5.1).

In order to complete the proof of the lemma, it suffices to prove

$$\|S_m(t)\|_{B(L^2(\Omega))} \leq C \max \left( t^{2\alpha_M - \alpha_0 - 1}, t^{2\alpha_M - \alpha_0 - 1}, 1 \right), \quad t > 0, \ m = 0, \mp.$$  \hspace{1cm} (6.4)

Indeed, it follows from (6.1) that $2\alpha_0 - \alpha_M - 1 > -1$ and $2\alpha_M - \alpha_0 - 1 > \alpha_M - 1 > -1$. Thus, applying the estimates (6.4), we can easily deduce that $S \in L^1_{\text{loc}}(\mathbb{R}^+; B(L^2(\Omega)))$ satisfies the estimate (6.2).

For $m = 0$, using (6.3), we find

$$\|S_0(t)\|_{B(L^2(\Omega))} \leq C \int_{\theta}^{\theta} \left\| \left( A + (t^{-1} e^{i\beta})^\alpha \right)^{-1} \right\|_{B(L^2(\Omega))} d\beta \leq C \max \left( t^{2\alpha_M - \alpha_0 - 1}, t^{2\alpha_M - \alpha_0 - 1} \right),$$

which implies (6.1) for $m = 0$. For $m = \mp$, again we employ (6.3) to estimate

$$\|S_\mp(t)\|_{B(L^2(\Omega))} \leq C \int_{t^{-1}}^{\infty} e^{t \cos \theta} \left\| \left( A + (r e^{i\theta})^\alpha \right)^{-1} \right\|_{B(L^2(\Omega))} dr \leq C \int_{t^{-1}}^{\infty} e^{t \cos \theta} \max \left( t^{\alpha_0 - 2\alpha_M}, t^{\alpha_M - 2\alpha_0} \right) dr.$$

For $t > 1$, we obtain

$$\|S_\mp(t)\|_{B(L^2(\Omega))} \leq C \int_{t^{-1}}^{\infty} e^{t \cos \theta} \max \left( t^{\alpha_0 - 2\alpha_M}, t^{\alpha_M - 2\alpha_0} \right) dr \leq C t^{-1} \int_{0}^{\infty} e^{t \cos \theta} \left( t^{\alpha_0 - 2\alpha_M} / t^{\alpha_M - 2\alpha_0} \right) dr + C \left( t^{2\alpha_M - \alpha_0 - 1} + 1 \right).$$

In the same way, for $t \in (0, 1]$, we get

$$\|S_\mp(t)\|_{B(L^2(\Omega))} \leq C \int_{1}^{\infty} e^{t \cos \theta} \max \left( t^{\alpha_0 - 2\alpha_M}, t^{\alpha_M - 2\alpha_0} \right) dr \leq C \max \left( t^{2\alpha_0 - \alpha_M - 1}, t^{2\alpha_M - \alpha_0 - 1} \right).$$

Combining these two estimates, we obtain

$$\|S_\mp(t)\|_{B(L^2(\Omega))} \leq C \max \left( t^{2\alpha_0 - \alpha_M - 1}, t^{2\alpha_M - \alpha_0 - 1}, 1 \right), \quad t > 0.$$  \hspace{1cm} (6.3)

This proves that (6.4) also holds true for $m = \mp$. This completes the proof of the lemma.

Using Lemma 4.1, we will show that Theorem 1.1 can be extended to $\mu \in L^1(\mathbb{R}^+)$ with a compact support.
Proposition 6.1 Let $h \in L^2(\Omega)$, $\alpha \in L^\infty(\Omega)$ satisfy \[(1.1)\] and $\mu \in L^1(\mathbb{R}_+)$ be compactly supported. Then there exists a unique weak solution $u \in L^1_{\text{loc}}(0, T; L^2(\Omega))$ to \[(1.1)\].

Proof. Suppose that $\text{supp} \mu \subset [0, T]$ for some $T > 0$ and choose a sequence $\{\mu_n\}_{n \in \mathbb{N}} \subset C_0^\infty(0, T + 1)$ satisfying
\[
\lim_{n \to \infty} \|\mu_n - \mu\|_{L^1(\mathbb{R}_+)} = 0. \tag{6.5}
\]
In light of Lemma 6.1, for $t \geq 0$ we can introduce
\[
u_n(t, \cdot) := \int_0^t \mu_n(t - \tau) S(\tau)(\rho^{-1} h) d\tau, \quad n \in \mathbb{N},
\]
and we have
\[
u(t, \cdot) := \int_0^t \mu(t - \tau) S(\tau)(\rho^{-1} h) d\tau
\]
as elements of $L^1_{\text{loc}}(\mathbb{R}_+; L^2(\Omega))$. We prove that for all $p > 0$, the Laplace transform $\hat{\nu}(p)$ of $u$ is well-defined in $L^2(\Omega)$ and we have
\[
\lim_{n \to \infty} \|\hat{\nu}_n(p) - \hat{\nu}(p)\|_{L^2(\Omega)} = 0. \tag{6.6}
\]
Applying estimate \[(6.2)\], we obtain
\[
\|e^{-pt}u_n(t, \cdot)\|_{L^2(\Omega)} \leq \int_0^t e^{-p\tau} \|S(\tau)\|_{L^2(\Omega)} \|\rho^{-1} h\|_{L^2(\Omega)} e^{-p(t-\tau)} \|\mu(t-\tau)\| d\tau \leq C \left(e^{-pt} \max\left(t^{2\alpha_0 - \alpha_M - 1}, t^{2\alpha_M - \alpha_0 - 1}, 1\right) \ast (e^{-pt} \|\mu(t)\|)\right)
\]
for all $t > 0$ and $p > 0$, where $\ast$ denotes the convolution in $\mathbb{R}_+$. Therefore, applying Young’s convolution inequality and condition \[(1.1)\], we deduce
\[
\|\hat{\nu}(p)\|_{L^2(\Omega)} \leq \int_0^\infty \|e^{-pt}u(t, \cdot)\|_{L^2(\Omega)} dt \leq C \left(\int_0^\infty e^{-pt} \max\left(t^{2\alpha_0 - \alpha_M - 1}, t^{2\alpha_M - \alpha_0 - 1}, 1\right) dt\right) \left(\int_0^\infty e^{-pt}\|\mu(t)\| dt\right) < \infty.
\]
for all $p > 0$. This proves the well-posedness of $\hat{\nu}(p)$ for all $p > 0$ in the sense of $L^2(\Omega)$. In the same way, for all $t > 0$, $p > 0$ and $n \in \mathbb{N}$, we get
\[
\|e^{-pt}(u_n - u)(t, \cdot)\|_{L^2(\Omega)} \leq \int_0^t e^{-p\tau} \|S(\tau)\|_{L^2(\Omega)} \|\rho^{-1} h\|_{L^2(\Omega)} e^{-p(t-\tau)} \|\mu_n - \mu\| (t - \tau) d\tau \leq C \left(e^{-pt} \max\left(t^{2\alpha_0 - \alpha_M - 1}, t^{2\alpha_M - \alpha_0 - 1}, 1\right) \ast (e^{-pt}\|\mu_n - \mu\|(t))\right).
\]
Thus, applying Young’s convolution inequality again, we have
\[
\|\hat{\nu}_n(p) - \hat{\nu}(p)\|_{L^2(\Omega)} \leq \int_0^\infty \|e^{-pt}(u_n - u)(t, \cdot)\|_{L^2(\Omega)} dt \leq C \left(\int_0^\infty e^{-pt} \max\left(t^{2\alpha_0 - \alpha_M - 1}, t^{2\alpha_M - \alpha_0 - 1}, 1\right) dt\right) \left(\int_0^\infty e^{-pt}\|\mu_n(t) - \mu(t)\| dt\right) \leq C\|\mu_n - \mu\|_{L^1(\mathbb{R}_+)}
\]
for all $p > 0$ and $n \in \mathbb{N}$, and \[(6.5)\] implies \[(6.6)\]. On the other hand, in view of \[(2.8)\, \text{Theorem 1.1 and Remark 1}\], since $\mu_n \in L^\infty(0, T)$, we have
\[
(p^\alpha + A)\hat{\nu}_n(p) = \left(\int_0^\infty e^{-pt}\mu_n(t, \cdot) dt\right) \rho^{-1} h, \quad p > 0.
\]
for all \( n \in \mathbb{N} \). In addition, (6.5) implies that

\[
\lim_{n \to \infty} \hat{\mu}_n(p) = \hat{\mu}(p), \quad p > 0.
\]

Therefore, we obtain

\[
\lim_{n \to \infty} \| \hat{u}_n(p) - \hat{\mu}(p)(A + p^\alpha - 1)(\rho^{-1}h) \|_{L^2(\Omega)} = 0
\]

and (6.6) implies that \( \hat{u}(p) = (A + p^\alpha - 1)\hat{\mu}(p)\rho^{-1}h, \quad p > 0 \). From the definition of the operator \( A \), we deduce that \( \hat{u}(p) \) solves the boundary value problem (2.1) for all \( p > 0 \). Therefore, we conclude that \( u \) is the unique weak solution of (1.1) and the proof is completed. 

\[\blacksquare\]

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