The Kosterlitz-Thouless Phenomenon on a Fluid Random Surface

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ABSTRACT

The problem of a periodic scalar field on a two-dimensional dynamical random lattice is studied with the inclusion of vortices in the action. Using a random matrix formulation, in the continuum limit for genus zero surfaces the partition function is found exactly, as a function of the chemical potential for vortices of unit winding number, at a specific radius in the plasma phase. This solution is used to describe the Kosterlitz-Thouless phenomenon in the presence of 2D quantum gravity as one passes from the ultra-violet to the infra-red.

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1. Introduction

The Kosterlitz-Thouless (KT) phenomenon [1, 2, 3] has had far reaching implications for our understanding of effectively two-dimensional (2D) physical systems. Its myriad applications include those to 2D superfluids [1], crystal growth [4], lattice gauge theories [5], and superstring theories [6]. While this phenomenon is usually studied on a fixed regular lattice, it is known that many aspects of 2D statistical mechanics become more tractable when studied on lattices for which local connectivity is itself a dynamical variable, principally because they may be mapped to exactly solvable problems of random matrices [7]. In the language of two-dimensional quantum field theory this corresponds to the introduction of 2D quantum gravity. They seem to display the same critical phenomena as on fixed regular lattices but with mild modification of critical exponents; for example, the Ising model transition is third order rather than second [8]. The possibility of finding some non-perturbative results for the KT vortex plasma motivates a study of this phenomenon on such fluid random surfaces. In this paper, using the example of a circular scalar field at a particular radius, it is shown by an exact solution in the continuum limit how the gas of unbound vortices leads to topological order in two dimensions, making no assumptions about diluteness save the restriction to vortex winding number one (the most relevant).

The random matrix model employed is a particular case of those studied in [11] which may be reformulated as an O(2) model on a dynamical triangulation [18, 20]. On a single oriented link with periodic boundary conditions, representing the circular target space, one has an $N \times N$ complex matrix $M$ — the models are the non-critical string analogue of Weingarten’s matrix model [12] — the full partition function being

$$Z = \int DM \exp \left(-\text{Tr}[M^\dagger M] + \frac{\kappa}{N} \text{Tr}[M^\dagger M M^\dagger M + M M M^\dagger M^\dagger] + h \sqrt{N} \text{Tr}[M + M^\dagger]\right).$$

Expansion in $\kappa$, the bare cosmological constant, generates a dynamical quadrangulation (square simplices glued pairwise along edges) with the edges embedded on the periodic link; also included in the action is a coupling $h$ to vortices (and anti-vortices) of unit winding number, which cut small holes in the random surface that wind around the target circle. Gross and Klebanov have shown [15] that, in the double scaling limit [14], the vortex-free $c = 1$ matrix model partition function defined on a lattice is equivalent to that on the
continuous real line provided the lattice spacing \( a \) is less than some known critical value \( a_c \). At \( a = a_c \) a phase transition occurs (believed to be of KT type) due to the appearance of a marginal operator. The same holds true of the complex matrix models of ref.\[11\], from which one may deduce that (1) represents the case \( a = a_c \); this means that the periodic link has radius \( r = a_c/2\pi = \frac{1}{2}r_{sd} \), where \( r_{sd} \) is the self-dual radius. This places one deep in the purported vortex-plasma phase since unit charge vortices become relevant for \( r < \frac{1}{2}r_{sd} \).

It is precisely at \( r = \frac{1}{2}r_{sd} \) that one may reformulate the problem in terms of a solvable \( O(2) \) model by introducing an inducing field \[18, 20\]. Using an \( N \times N \) Hermitian matrix variable \( \phi \) one can rewrite (1) as

\[
Z \propto \int DM \, D\phi \, \exp \left( - \text{Tr}[M^\dagger M] - \text{Tr} \phi^2 + \sqrt{\frac{2\kappa}{N}} \text{Tr}[M^\dagger M\phi + MM^\dagger \phi] + h\sqrt{N} \text{Tr}[M + M^\dagger] \right),
\]

which after diagonalising \( \phi \) and performing the Gaussian \( M \)-integrals gives

\[
Z(h, b_0) = \int \prod_{i=1}^{N} d\lambda_i \exp \left( - N \sum_i 2\lambda_i^2 + \sum_{i \neq j} \log (\lambda_i - \lambda_j) - \sum_{i,j} \log (2b_0 - \lambda_i - \lambda_j) \right. \\
+ 4Nh^2 \sum_i (b_0 - \lambda_i)^{-1}.
\]

In eq.(3) \( \lambda \) are the eigenvalues of \( \phi \), and there has been a redefinition \( \sqrt{2\kappa} = 1/4b_0 \) and rescaling \( \lambda \to 2\lambda\sqrt{N} \) and \( h^2 \to 8h^2/b_0 \). The case \( h = 0 \) has been studied extensively by I.Kostov and collaborators [19] as a loop gas problem, who found \( c = 1 \) critical exponents. Thus despite the presence of a marginal operator, at the radius chosen, over which one has little control, one can still study the KT phenomenon as \( h \) is turned on since the ultra-violet (UV) limit, corresponding to \( h = 0 \) as we shall see shortly, is that of a massless scalar theory.

At \( h \neq 0 \), the vortex–anti-vortex pairs appear as an extra singular term in the potential for this eigenvalue problem, which at large \( N \) may be treated by saddle-point methods \[21\]. Indeed, the operator coupling to \( h \) was identified already in the loop gas approach \[19\] and has a clear geometrical meaning. If one expands \( Z(2) \) in Feynman diagrams, a typical piece of planar diagram will appear as in Fig.1. A “bug” crawling on the surface goes \( 2\pi \) around the circle every time it crosses an arrowed propagator \( <M^\dagger M> \). Closed loops of these (the loop gas) delineate regions of constant scalar field, \( X \) say, which since the target lattice has
only one link, takes only one value. Vortex–anti-vortex pairs appear connected by cuts in the $X$ field. If the length of a cut is $l$, then in terms of the $\phi$ field it acts like a hole of length $2l$ on the surface with one marked point on the boundary. The other marked point, given by the other vortex, is not independent since the two are constrained to be length $l$ apart. This is the origin of the macroscopic loop operator coupling to $h^2$ in (3).

Thus the Boltzmann weights on a globally connected planar lattice take the form

$$\log Z(b_0, h) = \sum_{\text{graphs}} h \# \text{vortices} b_0^{-\text{total length of loops & cuts}}.$$  \hspace{1cm} (4)

One may therefore think of the arrowed lines as flux tubes which, because of the peculiar single valuedness of the target lattice, carry a unit charge’s worth of electric flux for a Coulomb field. $-\log h$ is the chemical potential for vortices, with small $h$ corresponding to the dilute regime. Flux tubes densely populate the surface and the continuum limit for area is achieved when they become infinite in length as $b_0$ is tuned to its critical value [19].

2. Saddle-Point Solution.

Let us now study the effect of turning on $h$ by finding the continuum limit of (3). The leading order of the large $N$ limit of $Z(h, b_0)$ describes surfaces of spherical topology. There exists a systematic approach to computing the $1/N$ corrections [18], corresponding to higher genus surfaces, but here we will concentrate on genus zero. Introducing an eigenvalue density $\rho(\lambda)$ for the problem (3), the saddle-point equation for $\rho$ is

$$\int_a^b d\mu \rho(\mu) \left( \frac{1}{\lambda - \mu} + \frac{1}{2b_0 - \lambda - \mu} \right) = 2\lambda - \frac{2h^2}{(b_0 - \lambda)^2},$$  \hspace{1cm} (5)

where the support of $\rho$ is on the interval $[a, b]$. This integral equation has been solved for $h = 0$ by Gaudin [22]. The same method will be used to solve $h \neq 0$ also. To transform (5) to an integral equation with Cauchy kernel make the following redefinitions

$$\lambda \rightarrow b_0 - \sqrt{A + B\lambda}, A + B = (b_0 - a)^2, A - B = (b_0 - b)^2.$$  \hspace{1cm} (6)
Then
\[ \int_{-1}^{1} d\mu \rho(\mu) \frac{1}{\lambda - \mu} = 2(\sqrt{A + B\lambda} - b_0) + \frac{2h^2}{A + b\lambda}. \tag{7} \]

The inversion formula for this problem is [23]
\[ \rho(\lambda) = -\frac{2}{\pi} \sqrt{1 - \lambda^2} \int_{-1}^{1} \frac{d\mu}{\sqrt{1 - \mu^2}} P \left( \frac{1}{\lambda - \mu} \right) \left[ \sqrt{A + B\mu} - b_0 + \frac{h^2}{A + B\mu} \right]. \tag{8} \]

A and B are determined by the normalisation and positivity conditions for \( \rho; \)
\[ \frac{B}{2} \int_{-1}^{1} d\mu \frac{\rho(\mu)}{\sqrt{A + B\mu}} = 1. \tag{9} \]
\[ \int_{-1}^{1} d\mu \frac{1}{\sqrt{1 - \mu^2}} \left( \sqrt{A + B\mu} - b_0 + \frac{h^2}{A + B\mu} \right) = 0. \]

To study the critical behaviour one need only know the form of \( \rho \) (8) for \( \lambda \to -1, \) though it may actually be found completely in terms of elliptic integrals. The continuum limit of surfaces is achieved as the bare cosmological constant is tuned to its critical value, given by the condition \( b = b_0, \) when the end of the support of \( \rho \) meets the singularities in (5). After some algebra, and performing the \( h \)-term integral, one can rewrite the solution (8) in the form
\[ \rho(\lambda) = \frac{4}{\pi^2} \sqrt{1 - \lambda^2} \left( BK(k) - (A + B\lambda)\Pi(\lambda) - \frac{\pi Bh^2}{2(A + B\lambda)\sqrt{A - B}} \right), \tag{10} \]
where
\[ \Pi(\lambda) = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} P \left( \frac{1}{\lambda - \cos 2\phi} \right), \tag{11} \]
an elliptic integral of the third kind, while \( k^2 = 2B/(A + B), \mu = \cos 2\phi, \) and \( K(E) \) denotes the complete elliptic integral of the 1st (2nd) kind. From the positivity condition one finds
\[ 2E(k)\sqrt{A + B} - \pi b_0 + \frac{\pi Bh^2}{\sqrt{A^2 - B^2}} = 0. \tag{12} \]
Using (10), the normalisation condition requires a little more manipulation before it can be
brought to the form

\[-\frac{2}{\pi^2}(A + B)((1 - k^2)K^2(k) - E^2(k)) - \frac{2h^2}{\pi\sqrt{A - B}}(K(k) + \text{n.s.}) = 1. \tag{13}\]

n.s. stands for terms non-singular in the limit \(b \to b_0\) \((k \to 1)\), which will be unimportant for the continuum limit. To approach it one introduces the cut-off \(\delta \to 0\) and renormalised parameters \(M\) and \(\xi\) as follows \[19\]:

\[\frac{A}{B} - 1 = M^2\delta^2\]

\[\sqrt{\frac{A}{B}} + \lambda = \xi\delta. \tag{14}\]

\(M^{-1}\) is the renormalised dielectric constant of the Coulomb gas, \(M\) the renormalised boundary cosmological constant coupling to the length of cuts in \(X\). Returning to (10) and evaluating it perturbatively in \(\delta\) one eventually finds \((\xi > M)\)

\[\rho(\xi, M) = -\frac{16\sqrt{A}}{\pi^2}\delta(\log M\delta)\sqrt{\xi^2 - M^2} - \frac{h^2}{2\pi\delta^2}\frac{\sqrt{\xi^2 - M^2}}{\xi^2 M} + \ldots \tag{15}\]

where \(\ldots\) stands for terms subleading as \(\delta \to 0\). The bare vortex parameter \(h\) couples to a relevant operator and therefore must be tuned to zero for finite renormalised parameter. The scaling law follows from (15) by requiring the 2nd term to be of the same order as the first

\[h^2 = -h_R^2\delta^3\log M\delta. \tag{16}\]

This agrees with KPZ scaling \[16\] at \(r = \frac{1}{2}r_{sd}\) but there is a logarithmic scaling violation; such violations are well-known at \(c = 1\), occurring also for the cosmological constant \[17\]. The latter is determined in terms of \(M\) by expanding (12) and (13) in \(\delta\);

\[\pi b_0 = 2\sqrt{2A}(1 - M^2\delta^2\log M\delta) + \frac{\pi h_R^2}{M\sqrt{2}}\delta^2\log M\delta + \ldots \tag{17}\]

\[1 = \frac{4A}{\pi^2}(1 - 2M^2\delta^2\log^2 M\delta) - \frac{\sqrt{2h_R^2}}{A\pi M}\delta^2\log^2 M\delta + \ldots \]

which, omitting \(\ldots\) now, determines \(A = \pi^2/4\) and

\[\Delta_R = (M^2 + \frac{\sqrt{2h_R^2}}{\pi^2 M})\log^2 M\delta, \tag{18}\]

where \(b_0 = \sqrt{2} + \delta^2\Delta_R\) defines the renormalised cosmological constant conjugate to renor-
malised area of the surface $A_R$. $\delta$ is thus the physical cutoff.

One can now eliminate $M$ in $\rho$ (15) in terms of the quantities $\Delta_R, h_R$. Apart from the logarithms, (18) is cubic in $M$ and it is convenient to use the parametric solution

$$M = 2\sqrt{\frac{\Lambda}{3} \cos \frac{\pi - \theta}{3}},$$

(19)

where

$$\cos \theta = \frac{h_R^2}{\sqrt{2\pi^2}} \left( \frac{3}{\Lambda} \right)^{3/2}, \quad \Lambda = \frac{\Delta_R}{\log^2 M \delta}.$$  

(20)

The equation (18) is sketched in Fig.2. (19) corresponds to the largest positive root, this being the physical one such that $M \to \infty$ as $\Delta_R \to \infty$. (15) and (19) constitute the solution for the scaling part of $\rho$, from which physical quantities may be calculated in the continuum limit.

The first thing to notice is that (18), which is the positivity condition for $\rho$, has no solution for $h_R^2 > \sqrt{2\pi^2}(\Lambda/3)^{3/2}$. In this case the singular term in the potential (5) has become too strong and there is no smooth large $N$ limit. Vortices contribute vacuum energy which can be negative $\sim e^{\Delta h A_R}$ and so the partition function at fixed cosmological constant does not converge in area $A_R$ if $\Delta_R < \Delta_R^c$ ($h_R$ too large).

As an intermediate step to calculating the partition function of spherical surfaces, it is convenient to first derive the macroscopic loop expectation at genus zero i.e. the partition function for surfaces with the disc topology.

$$\tilde{\omega}(z) = \int_0^\infty dl \, \omega(l) e^{-zl},$$

(21)

where $\omega(l)$ is the renormalised partition function for discs of renormalised perimeter length $l$ with one marked point on the boundary. Using (14) and (15)

$$\tilde{\omega}(z) = \int_M^\infty \frac{d\xi}{z + \xi} \rho(\xi, M)$$

$$= -\log M \delta \left( \frac{4}{\pi} \sqrt{z^2 - M^2} \left( \log \frac{z + \sqrt{z^2 - M^2}}{M} \right) \left[ \sqrt{2} - \frac{h_R^2}{\pi^2 M z^2} \right] + \frac{4h_R^2}{\pi^3} \left( \frac{\pi}{2z^2} - \frac{1}{z M} \right) \right).$$

(22)

In deriving (22) individual terms in the leading order which are analytic in $z$ have been
omitted since they correspond to surfaces of infinitesimal area bounded by finite perimeter [24]. (22) for \( h_R = 0 \) was given in [19]. As a useful check one may verify that \( \tilde{\omega} \) correctly solves the finite difference equation derived by Kostov directly from the continuum limit of (5) (see e.g. ref.[18]). The partition function for spherical surfaces may be derived from \( \tilde{\omega}(0) = d \log Z/dh^2 \),

\[
\tilde{\omega}(0) = 2\sqrt{2}(\log M\delta) \left( M + \frac{h_R^2}{2\pi^2 M^2} \right) = 2\sqrt{\frac{2\Delta_R}{3}} \left( \cos \frac{\pi - \theta}{3} + \frac{1}{2} \cos^2 \frac{\pi - \theta}{3} \right),
\]

(23)

by integrating w.r.t. \( h^2 \) at constant \( b_0 \). The constant of integration is \( c \sim \Delta_R^2 \) by dimensions, up to logarithms. Using (20) it is simpler to integrate w.r.t. \( \theta, dh^2 = -(\Lambda/3)^{3/2} \sqrt{2\pi^2} \sin \theta d\theta + \ldots \)

\[
\log Z \propto \frac{\Delta_R^2}{\log^2 \delta^2 \Delta_R} \left( \sin^2 \frac{\theta - \pi}{3} + \frac{1}{8} \log \cos \frac{\theta - \pi}{3} - \frac{1}{2} \sin^4 \frac{\theta - \pi}{3} \right) + c .
\]

(24)

This represents the continuum limit where all subleading terms + \ldots have been dropped. Equations (22) and (24) are the main results of this paper, representing the continuum limit on genus zero surfaces for the vortex gas with arbitrary coupling.

One can use these results to investigate the number of effective degrees of freedom in various regimes (see refs.[10, 9] for a similar non-perturbative study of the Sine-Gordon model coupled to 2D gravity). In particular one can use (24) to find the string susceptibility \( \gamma_{str} \), given by the fixed area partition function \( \log Z(A_R) \sim A_R^{3+\gamma_{str}} \exp(\Delta_{R}^c A_R) \). The UV limit \( A_R \to 0 \) occurs for \( \Delta_R \to \infty \), in which case (24) yields \( \gamma_{str} = 0 \), as appropriate for a massless scalar field. As the scale \( A_R \) is increased above that set by \( h_R^{-4/3} \) the screening effects of the vortex condensate come into play and the behaviour changes. To determine \( \gamma_{str} \) more generally one must identify the effective cosmological constant, given by the sum of contributions from gravity (\( \Delta_R \)) and matter (\( -\Delta_{R}^c \)), which is conjugate to area. This constant is zero at the critical point of (18) allowing identification of \( \Delta_{R}^c \). Since \( \theta_c = 0 \) one
finds
\[ M_c^2 \log^2 \delta M_c = \frac{\Delta^c_R}{3} \]
\[ \frac{h^2_R}{\sqrt{2} \pi^2} \left( \frac{3}{(\Delta^c_R)^2} \right)^{3/2} \log^3 \delta M_c = 1 \]
\[ \Rightarrow M_c = \left( \frac{h^2_R}{\sqrt{2} \pi^2} \right)^{1/3}. \]

(25)

As usual, the presence of logarithms complicates \( \Delta^c_R \) which moreover satisfies a transcendental equation. \( \Delta^c_R = 3M_c^2 \) neglecting logarithms.

To find \( \gamma_{str} \) in the infra-red (IR) limit \( A_R \to \infty \) one can proceed analytically. Defining \( \Delta_R = \Delta^c_R + \tilde{\Delta}_R \) and \( M = M_c + \tilde{M} \), for small \( \theta \) one may expand all formulae for \( \tilde{\Delta}_R \ll \Delta^c_R, \tilde{M} \ll M_c \) and find expressions for the small variables; even keeping track of logarithms is not too tedious. Expanding about (25) one finds

\[ \tilde{M} = \frac{1}{\log M_c \delta} \left( \frac{\Delta_R}{6} \sqrt{\frac{3}{\Delta^c_R}} + \frac{\theta}{3} \sqrt{\frac{\Delta^c_R}{3} + \ldots} \right) \]
\[ - \frac{1}{2} \theta^2 + \frac{\theta^4}{4!} + \ldots = \frac{3}{2} \left( \frac{3}{\Delta^c_R} \right)^2 M_c^2 \Delta_R \log^2 \delta M_c + \frac{2\tilde{M}}{M_c \log M_c \delta} + \ldots \]

(26)

and expanding \( \tilde{\Delta}_R \) in \( \theta \) gives

\[ \theta = 3 \sqrt{\frac{\Delta_R}{\Delta^c_R}} + O(\tilde{\Delta}_R^{3/2}) \]

(27)

Expanding the partition function (24) in \( \theta \) one finds

\[ \log Z = \frac{(\Delta^c_R)^2}{\log^2 \delta^2 \Delta^c_R} \left( \alpha + \beta \theta^2 + \gamma \theta^4 + \ldots + (\epsilon + \nu \tilde{\Delta}_R + \eta \tilde{\Delta}^2_R + \ldots) \right), \]

where \( \alpha, \beta, \gamma, \ldots \) are known constants and the constant of integration \( c = \epsilon + \ldots \) is analytic as \( \tilde{\Delta}_R \to 0 \) since it does not depend upon \( h \), only \( \Delta_R \). Thus

\[ \log Z = \text{analytic} + \frac{\text{const.}}{\log^2 \delta^2 \Delta^c_R \sqrt{\Delta^c_R}} \tilde{\Delta}_R^{5/2} + O(\tilde{\Delta}_R^{7/2}), \]

(28)

confirming that \( \gamma_{str} = -1/2 \) in the IR limit, where the cuts in \( X \) disorder it to leave pure gravity. Note that the analytic terms as \( \tilde{\Delta}_R \to 0 \) come from the lower cutoff on the area integral, most importantly they do not contain logarithms of \( \tilde{\Delta}_R \).
The change in behaviour from the UV to the IR can be understood from the Coulomb gas in terms of screening effects [1]. At small areas $\Delta_R \to \infty$ and the renormalised $M^{-1} \to 0$ (18), all flux lines being of the order of the UV cutoff on the surface; this is the $c = 1$ regime where vortices are strongly bound to anti-vortices. At large areas $\Delta_R \to \Delta_R^c$ and $M \to M_c$ (25). When $h_R = 0$, since the surface is dense with flux loops, to achieve large area the dielectric constant is tuned to $\infty$ ($M_c = 0$). In the presence of the charge gas ($h_R \neq 0$) the inverse permittivity is $M - M_c$, the (non-universal) shift $M_c$ being due to dielectric polarization. It clearly required the non-perturbative treatment to see this since (25) is not analytic as $h_R \to 0$.

3. Conclusions.

Unfortunately it appears difficult to conceive of a general method for obtaining exact results for $r \neq \frac{1}{2} r_{sd}$ or higher vortex winding number. However one may add the winding number $\pm 2$ operator, $\tilde{m}_2 \text{Tr}[MM + M^\dagger M^\dagger]$, to (1) and the integrals are still Gaussian, yielding

\[
Z = \int \prod_i d\lambda_i \exp\left( \sum_{i \neq j} \log (\lambda_i - \lambda_j) - N \sum_i 2\lambda_i^2 - \frac{1}{2} \log ((2b_0 - \lambda_i - \lambda_j)^2 - (\tilde{m}_2)^2) + 8N\tilde{m}_1 \sum_i (b_0 - \frac{1}{2} \tilde{m}_2 - \lambda_i)^{-1} \right).
\]

instead of (3), with $h = \tilde{m}_1$. The geometrical meaning of the last two terms in the action (30) is illustrated in terms of flux lines in Fig.3. The saddle-point equation to be solved is

\[
\int_a^b d\mu \rho(\mu) \left( \frac{1}{\lambda - \mu} + \frac{1}{2b_0 - \tilde{m}_2 - \lambda - \mu} + \frac{1}{2 \tilde{m}_2 - \lambda - \mu} \right) = 2\lambda - \frac{2(\tilde{m}_1)^2}{(b_0 - \frac{1}{2} \tilde{m}_2 - \lambda)^2}
\]

but this is more difficult than (5); the author has not managed to solve it to the extent that useful information can be gleaned, for example on questions of multicriticality as $\tilde{m}_2$ is tuned as a function of $\tilde{m}_1$. One may however give a hand-wave argument that the IR limit is pure gravity. Consider $\tilde{m}_1 = 0$ and $\tilde{m}_2 \neq 0$ in (31) for simplicity. According to the
discussion of the previous section, the scaling regime is described by $(\lambda' + \frac{1}{2}\tilde{m}_2^R)\delta = b_0 - \lambda$ and

$$
\int_{b'}^\infty d\mu' \rho(\mu') \left( \frac{1}{\lambda' - \mu'} - \frac{1}{2\lambda' + \mu'} - \frac{1}{2\tilde{m}_2^R + \lambda' + \mu'} \right) = 2(\delta\lambda' + \frac{1}{2}\tilde{m}_2^R - b_0), \quad (32)
$$

where $\tilde{m}_2^R = \tilde{m}_2/\delta$ is consistent with KPZ scaling. The IR limit is equivalent to $\tilde{m}_2^R \to \infty$ in which case the 3rd term in the kernel of (32) may be dropped, leaving the saddle point problem of the O(1) model in Kostov’s classification of the dense phase of $O(n)$ models [19] i.e., pure gravity.

For the more general models of ref.[11] one can ask about the extent to which similar methods of solution may be applied, in particular the introduction of induction fields to obtain plaquettes from an action quadratic in $M$. A particularly interesting problem concerns the effects of an extrinsic curvature dependence, which can smooth badly behaved surfaces [13]. This may be included in the Weingarten-type [12] models by introducing new flavours of link matrix. For example at $c = 2$, two flavours $M, N$ suffice since the worldsheet normals are $\pm 1$. The extrinsic curvature, given by $\frac{1}{2}(1 - \cos \theta) = 0, 1$ for parallel ($\theta = \pi$) or anti-parallel ($\theta = 0$) neighboring plaquettes where $\theta$ is the angle subtended, can be included by using an action

$$
\sum_l \text{Tr}[M^\dagger(l)M(l) + N^\dagger(l)N(l)] + K\text{Tr}[N^\dagger(l)M(l) + M^\dagger(l)N(l)] - \kappa \sum_{\text{plaq}} \text{Tr}[P_M + P_N], \quad (33)
$$

where $\sum_l$ is the sum over all links of a square lattice; $P_M$ is the product of $M$’s around plaquettes of orientation $+$ and $P_N$ the product of $N$’s around plaquettes of orientation $-$; $K$ is the bare extrinsic curvature coupling. The $c$-dimensional generalisation has $Z_c$ symmetry and $c(c - 1)$ flavours, and one assumes that the continuous symmetry would be restored at any critical point. These models are difficult to solve in full generality, but simplified versions which still incorporate extrinsic curvature can be reduced to eigenvalue problems. This will be dealt with in a future publication [25]

To summarize, in this paper the matrix model of random surfaces introduced in ref.[11] has been further investigated and applied to the two-dimensional vortex plasma in the presence of gravity. The exact solution on genus zero was found at half the self-dual radius as
a function of renormalised cosmological constant and vortex chemical potential (22) (24) .
In general these matrix models seem to give new possibilities for exact solution where the
traditional Hermitian models are too difficult, the principle improvements being:

1. Vortex operators are manifest and easily manipulated.
2. Exact solutions are possible even in the presence of non-tree-like embeddings.
3. Extrinsic curvature can be trivially introduced and, in certain cases [25], leads also to
   solvable models.
4. The matrix models have gauge symmetry [11] and are in fact closely related to conven-
   tional non-linear lattice gauge theories, for which there is an abundance of expertise
   available.

The accompanying geometrical picture confirms the KT hypothesis of unbound vortices,
which in the renormalised scheme are joined by Coulomb flux tubes, cuts in the periodic
scalar field that disorder it at large length scales. Hopefully this approach can be further
developed to extend the remarkable non-perturbative answers which matrix models furnish.

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Note Added: The reader’s attention is also drawn to recent numerical studies of the XY
model on random surfaces [26, 27].
FIGURE CAPTIONS

Fig.1. A typical piece of planar Feynman graph for action (1). A single vortex–anti-vortex pair and three “vacuum” flux loops are shown arrowed. Dotted line is $< \phi \phi >$ propagator.

Fig.2. A sketch of the cubic equation (18).

Fig.3. Vortex flux configurations corresponding to the expansion of action (30) in powers of $\tilde{m}_1$ and $\tilde{m}_2$. 
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