THE LAPLACE TRANSFORM OF THE DIGAMMA FUNCTION: AN INTEGRAL DUE TO GLASSER, MANNA AND OLOA

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Abstract. The definite integral
\[ M(a) := \frac{4}{\pi} \int_{0}^{\pi/2} \frac{x^2 \, dx}{x^2 + \ln^2(2e^{-a} \cos x)} \]
is related to the Laplace transform of the digamma function
\[ L(a) := \int_{0}^{\infty} e^{-as} \psi(s + 1) \, ds, \]
by \( M(a) = L(a) + \gamma/a \) when \( a > \ln 2 \). We establish an analytic expression for \( M(a) \) in the complementary range \( 0 < a \leq \ln 2 \).

1. Introduction

The classical table of integrals by I. S. Gradshteyn and I. M. Ryzhik [7] contains a large collection organized in sections according to the form of the integrand. In each section one finds significant variation on the complexity of the integrals. For example, section 4.33 - 4.34, with the title Combinations of logarithms and exponentials, presents the elementary formula 4.331.1: for \( a > 0 \),

\[ \int_{0}^{\infty} e^{-ax} \ln x \, dx = -\frac{\gamma + \ln a}{a}, \]

where \( \gamma \) is the Euler constant

\[ \gamma = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k} - \ln n, \]
as well as the more elaborate 4.332.1 and 4.325.6:

\[ \int_{0}^{\infty} \ln x \, dx \frac{dx}{e^x + e^{-x} - 1} = \int_{0}^{1} \ln \left( \frac{1}{x} \right) \frac{dx}{x^2 - x + 1} = \frac{2\pi}{\sqrt{3}} \left( \frac{5}{6} \ln 2 - \ln \Gamma \left( \frac{1}{6} \right) \right). \]

The difficult involved in the evaluation of a definite integral is hard to measure from the complexity of the integrand. For instance, the evaluation of Vardi’s integral,

\[ \int_{\pi/4}^{\pi/2} \ln \tan x \, dx = \int_{0}^{1} \ln \left( \frac{1}{x} \right) \frac{dx}{1 + x^2} = \frac{\pi}{2} \ln \left( \frac{\Gamma \left( \frac{3}{4} \right) \sqrt{2\pi}}{\Gamma \left( \frac{1}{4} \right)} \right), \]

that appears as 4.229.7 in [7], requires a reasonable amount of Number Theory. The second form is 4.325.4, found in the section entitled Combinations of logarithmic functions of more complicated arguments and powers. The reader will find in [15] a discussion of this formula.

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It is a remarkable fact that combinations of elementary functions in the integrand often exhibits definite integrals whose evaluations are far from elementary. We have initiated a systematic study of the formulas in [7] in the series [1, 2, 9, 10, 11, 12]. The papers are organized according to the combinations appearing in the integrand. Even the elementary cases, such as the combination of logarithms and rational function discussed in [2] entail interesting results. The evaluations

\[ \int_0^b \frac{\ln t \, dt}{(1 + t)^{n+1}} = \frac{1}{n} \left[ 1 - (1 + b)^{-n} \right] \ln b - \frac{1}{n} \ln(1 + b) \]

\[ - \frac{1}{n(1 + b)^{n-1}} \sum_{j=1}^{n-1} \frac{1}{j!} \binom{n-1}{j} |s(j + 1, 2)| b^j, \]

for \( b > 0 \) and \( n \in \mathbb{N} \) produces an explicit formula for the case where the rational function has a single pole. Here \( s(n, k) \) are the Stirling numbers of the first kind counting the number of permutations of \( n \) letters having exactly \( k \) cycles. The case of a purely imaginary pole is expressed in terms of the rational function

\[ p_n(x) = \sum_{j=1}^{n} \frac{2^{2j}}{2j(2j)} \frac{x}{(1 + x^2)^j}, \]

as

\[ \int_0^x \frac{\ln t \, dt}{(1 + t^2)^{n+1}} = \frac{2^n}{2^{2n}} \left[ g_0(x) + p_n(x) \ln x - \sum_{k=0}^{n-1} \tan^{-1} x + p_k(x) \right], \]

with

\[ g_0(x) = \ln x \tan^{-1} x - \int_0^x \frac{\tan^{-1} t}{t} \, dt. \]

The special case \( x = 1 \) becomes

\[ \int_0^1 \frac{\ln t \, dt}{(1 + t^2)^{n+1}} = -2^{-2n} \binom{2n}{n} \left( G + \sum_{k=0}^{n-1} \frac{\pi}{2k+1} \right), \]

where

\[ G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \]

is the Catalan’s constant. The values

\[ p_k(1) = \sum_{j=1}^{k} \frac{2^j}{2j(2j)} \]

do not admit a closed-form (in the sense of [13]), but they do satisfy the three term recurrence

\[ (2k + 1)p_{k+1}(1) - (3k + 1)p_k(1) + kp_{k-1}(1) = 0. \]

The study of definite integrals where the integrand is a combination of powers, logarithms and trigonometric functions was initiated by Euler [5], with the evaluation of

\[ \int_0^{\pi/2} x \ln(2 \cos x) \, dx = -\frac{7}{16} \zeta(3), \]
and
\begin{equation}
(1.12) \quad \int_{0}^{\pi/2} x^2 \ln(2 \cos x) \, dx = -\frac{\pi}{4} \zeta(3),
\end{equation}
that appear in his study of the Riemann zeta function at the odd integers. These type of integrals have been investigated in [8], [16]. The intriguing integral [3],
\begin{equation}
(1.13) \quad \int_{0}^{\pi/2} x^2 \ln^2(2 \cos x) \, dx = \frac{11\pi}{16} \zeta(4) = \frac{11\pi^5}{1440},
\end{equation}
was first conjectured on the basis of a numerical computation by Enrico Au-Yueng, while an undergraduate student at the University of Waterloo.

Recently O. Oloa [13] considered the integral
\begin{equation}
(1.14) \quad M(a) := \frac{4}{\pi} \int_{0}^{\pi/2} \frac{x^2 \, dx}{x^2 + \ln^2(2e^{-a \cos x})},
\end{equation}
and later established the value
\begin{equation}
(1.15) \quad M(0) = \frac{4}{\pi} \int_{0}^{\pi/2} \frac{x^2 \, dx}{x^2 + \ln^2(2 \cos x)} = \frac{1}{2} (1 + \ln(2\pi) - \gamma).
\end{equation}
Oloa’s method of proof relies on the expansion
\begin{equation}
(1.16) \quad \frac{x^2}{x^2 + \ln^2(2 \cos x)} = x \sin 2x + \sum_{n=1}^{\infty} (-1)^{n-1} \left( \frac{a_n}{n!} - \frac{a_{n+1}}{(n+1)!} \right) x \sin(2nx),
\end{equation}
where
\begin{equation}
(1.17) \quad a_n := \int_{0}^{1} (t)_n \, dt,
\end{equation}
with \((t)_n = t(t+1) \cdots (t+n-1)\) the Pochhammer symbol. The standard relation
\begin{equation}
(1.18) \quad (t)_n = \sum_{k=1}^{n} |s(n,k)| t^k,
\end{equation}
gives
\begin{equation}
(1.19) \quad a_n = \sum_{k=1}^{n} \frac{|s(n,k)|}{k+1}.
\end{equation}
M. L. Glasser and D. Manna [6] introduced the function
\begin{equation}
(1.20) \quad L(a) := \int_{0}^{\infty} e^{-as} \psi(s+1) \, ds,
\end{equation}
where \(\psi(x) = \frac{d}{dx} \ln \Gamma(x)\) is the digamma function. Integration by parts and using (1.1) gives
\begin{equation}
(1.21) \quad L(a) = -\gamma - a \ln a + a \int_{0}^{\infty} e^{-at} \ln \Gamma(t) \, dt.
\end{equation}
The main result in [6] gives a relation between \(M(a)\) and \(L(a)\).
Theorem 1.1. Assume that \( a > \ln 2 \). Then
\[
M(a) = L(a) + \frac{\gamma}{a}.
\]
That is, for \( a > \ln 2 \),
\[
M(a) = \frac{\gamma}{a} - \gamma - \ln a + a \int_0^\infty e^{-at} \ln \Gamma(t) \, dt.
\]

The proof in [6] begins with the representation
\[
\int_0^{\pi/2} \cos^\nu x \cos ax \, dx = \frac{\pi \Gamma(\nu + 2)}{2^{\nu+1}(\nu + 1) \Gamma(1 + \frac{\nu}{2}) \Gamma(1 + \frac{\nu}{2} - \frac{a}{2})},
\]
borrowed from 3.621.9 in [7]. Differentiating with respect to \( a \), evaluating at \( a = s \), and using \( \psi(1) = -\gamma \) yields
\[
\psi(s + 1) = \frac{2^{s+2}}{\pi} \int_0^{\pi/2} x \cos^s x \sin(sx) \, dx - \gamma.
\]
Replacing in (1.20) produces
\[
L(a) + \frac{\gamma}{a} = -\frac{4}{\pi} \Im \int_0^\infty \int_0^{\pi/2} x e^{s(\ln[2e^{-a} \cos x] - ix)} \, dx \, ds.
\]
The identity (1.22) now is an immediate consequence of the \( s \)-integral:
\[
\int_0^\infty e^{s(\ln[2e^{-a} \cos x] - ix)} \, ds = \frac{1}{ix - \ln[2e^{-a} \cos x]}.
\]

The authors also succeed in a series expansion of \( M(a) \) while they recognize as a hypergeometric function in two variables, and noted (quoting from [6]) strongly suggests that for general value of \( a \), no further progress is possible. This hypergeometric interpretation led the authors [6] to
\[
M(0) = 1 + \frac{1}{2} \int_0^1 t(1-t) \, _3F_2(1, 1, 2-t; 2, 3; 1) \, dt
\]
for which they invoke
\[
_3F_2(1, 1, 2-t; 2, 3; 1) = \frac{2(1 - \gamma - \psi(t + 1))}{1 - t}
\]
to enable them demonstrate a new proof of (1.15).

The graph of \( M(a) \) shown in Figure 1, obtained by the numerical integration of (1.14), has a well-defined cusp at \( a = \ln 2 \). In this paper, we provide analytic expressions for both branches of \( M(a) \). The region \( a > \ln 2 \), determined in [6], is reviewed in this section. The corresponding expressions for \( 0 < a < \ln 2 \) will be the content of the next section.

2. The case \( 0 < a < \ln 2 \)

Our starting point is the identity
\[
M(a) = -\frac{e^a}{2\pi} \Im \int_0^1 e^{-at} \int_0^\pi \frac{x(1 + e^{ix})t}{1 - e^a + e^{ix}} \, dx \, dt.
\]
The identity
\[(2.2) \quad \text{Im} \frac{x}{ix + \ln [2e^{-a} \cos x]} = \frac{x^2}{x^2 + \ln^2 [2e^{-a} \cos x]} \]
yields
\[(2.3) \quad M(a) = \frac{4}{\pi} \text{Im} \int_0^{\pi/2} \frac{x \, dx}{ix + \ln [2e^{-a} \cos x]}. \]

Under the assumption \( a > \ln 2 \), we have
\[(2.4) \quad \int_0^\infty e^s \ln [2e^{-a} \cos x] + ix \, ds = \frac{1}{ix + \ln [2e^{-a} \cos x]}, \]
which implies
\[(2.5) \quad M(a) = \frac{2}{\pi} \text{Im} \int_{-\pi/2}^{\pi/2} \int_0^\infty xe^s(\ln [2e^{-a} \cos x] + ix) \, dx \, ds, \]
where one uses the fact that the imaginary part of the integrand is an even function of \( x \). One more identity
\[(2.6) \quad e^{isx} \cdot e^s \ln [2e^{-a} \cos x] = e^s \ln [e^{-a}(1 + e^{ix})] \]
and the change of variables \( x \mapsto x/2 \), give reason to
\[(2.7) \quad M(a) = \frac{1}{2\pi} \text{Im} \int_{-\pi}^{\pi} \int_0^\infty xe^s(\ln [e^{-a}(1 + e^{ix})]) \, ds \, dx. \]

Then evaluate the \( s \)-integral to obtain
\[(2.8) \quad M(a) = -\frac{1}{2\pi} \text{Im} \int_{-\pi}^{\pi} \frac{x \, dx}{\ln [e^{-a}(1 + e^{ix})]}. \]

The formula
\[(2.9) \quad \frac{1}{\ln u} = \int_0^1 \frac{u' \, dt}{u - 1} \]
now gives \((2.1)\) from \((2.8)\).
Note 2.1. Even though the proof outlined here is valid for $a > \ln 2$, the identity (2.7) holds for $a > 0$.

**Notation:** we use $b = e^a - 1$ and assume $0 < a < \ln 2$, so that $0 < b < 1$.

Expanding the terms $(1 + e^{ix})^t$ and $1/(1 - be^{-ix})$ in power series produces

$$M(a) = -\frac{e^a}{2\pi} \int_0^1 \int_{-\pi}^\pi x e^{-at} \sum_{j=0}^\infty \sum_{k=0}^\infty b^j \frac{t^j}{k^j} \sin[x(k - j - 1)] dx \, dt.$$  

The term corresponding to $k = j + 1$ disappears and computing the $x$-integral we arrive at

$$M(a) = e^a \int_0^1 e^{-at} \sum_{j=0}^\infty \sum_{k=0}^\infty \frac{(-1)^j}{j+1-k} \frac{b^j}{t^j} \frac{t^j}{k^j} \sin[kx] \, dt.$$  

**Lemma 2.1.** Let $t \in \mathbb{R}$ and $j \in \mathbb{N} \cup \{0\}$. Then

$$\sum_{\nu=1}^\infty \frac{(-1)^\nu}{\nu} \left( \frac{t}{\nu+j} \right) = \binom{t}{j} [\psi(j+1) - \psi(t+1)].$$

**Proof.** The integral representation (3.268.2 in [7]):

$$\psi(p+1) - \psi(q+1) = -\int_0^1 \frac{x^p - x^q}{1-x} \, dx,$$

yields

$$\psi(p+1) - \psi(q+1) = \sum_{j=1}^\infty (-1)^{j-1} \left( \binom{p}{j} - \binom{q}{j} \right).$$

Therefore the result is a consequence of the identity

$$\binom{t}{k}^{-1} \sum_{m=1}^\infty \frac{(-1)^m}{m} \left( \frac{t}{m+k+1} \right) - \sum_{m=1}^\infty \frac{(-1)^m}{m} \left( \frac{t}{m} \right) = \sum_{m=1}^k \frac{(-1)^{m-1}}{m} \left( \frac{k}{m} \right).$$

Apply the difference operator $\Delta a(k) = a(k+1) - a(k)$ and use

$$\binom{t}{k+1}^{-1} \left( \frac{t}{m+k+1} \right) - \binom{t}{k}^{-1} \left( \frac{t}{m+k} \right) = \frac{m}{k+1} \binom{t}{k+1}^{-1} \left( \frac{t+1}{m+k+1} \right)$$

to write the derived equation as

$$-\binom{t+1}{k+1}^{-1} \sum_{m=1}^\infty (-1)^m \left( \frac{t+1}{m+k+1} \right) = \Delta \sum_{m=1}^k \frac{(-1)^m}{m} \left( \frac{k}{m} \right).$$

The left hand side of (2.14) reduces to $-1/(k+1)$ in view of the classical identity

$$\sum_{m=1}^\infty (-1)^{m-1} \left( \frac{t+1}{m+k+1} \right) = \binom{t}{k+1}.$$

A simple evaluation of the right hand side in (2.14) also produces $-1/(k+1)$. We conclude that, up to a constant term with respect to the index $k$, both sides of
are equal to the harmonic number $H_k$. The special case $k = 0$ shows that this constant vanishes.

Continuing from (2.10), we thus have

$$(2.17) \quad M(a) = e^a \int_0^1 \sum_{j=0}^\infty b^j \sum_{k=0}^j \frac{(-1)^j-k}{j+1-k} \left( t \right) \psi(j+1) dt$$

$$- \quad e^a b \int_0^1 e^{-at} \sum_{j=1}^\infty b^j \left( t \right) \psi(t+1) dt.$$

To simplify the first term in the previous expression observe

$$\sum_{j=0}^\infty b^j \sum_{k=0}^j \frac{(-1)^j-k}{j+1-k} = \sum_{k=0}^\infty b^k \left( t \right) \sum_{\nu=0}^k \frac{(-1)^\nu b^\nu}{\nu+1} = \frac{\ln(1+b)}{b} \sum_{k=0}^\infty \left( t \right) b^k = a e^a.$$

We deduce that in (2.17) the first term is $a/(1-e^{-a})$.

The reduction of the second term in (2.17) employs the following result:

**Lemma 2.2.** Let $0 < a < \ln 2$ and $t \in \mathbb{R}$. Then

$$(2.18) \quad \int_0^1 e^{-at} \sum_{j=0}^\infty b^j \psi(j+1) dt = \ln(1-e^{-a}) + \int_1^\infty e^{-at} \frac{dt}{t}.$$

**Proof.** The Stirling numbers $s(j,k)$ satisfy

$$(2.19) \quad j! \left( t \right)_k = \sum_{k=0}^j |s(j,k)| t^k,$$

so that

$$(2.20) \quad \int_0^1 e^{-at} \sum_{j=0}^\infty b^j \psi(j+1) dt = \frac{e^{-a} b \gamma}{a} + e^{-a} \sum_{j=1}^\infty \left( b^{j+1} \alpha_j - b^j \alpha_{j-1} \right) \psi(j+1),$$

with

$$(2.21) \quad \alpha_j(a) = \frac{1}{j!} \sum_{k=0}^j \frac{|s(j,k)| k!}{a^{k+1}}.$$

The result now follows by summation by parts and the identity

$$(2.22) \quad \sum_{j=k}^\infty \frac{|s(j,k)| b^j}{j!} = \frac{\ln(1+b)}{k!}.$$

Therefore, the second term in (2.17) is

$$(2.23) \quad \text{second term} = \frac{\ln(1-e^{-a})}{1-e^{-a}} + \frac{1}{1-e^{-a}} \int_1^\infty e^{-at} \frac{dt}{t}.$$
Finally, the third term in (2.17) is
\[
\text{third term} = -\frac{e^a}{b} \int_0^1 e^{-at} \left( \sum_{j=1}^{\infty} \left( \frac{t}{j} \right)^b \right) \psi(t+1) \, dt = -\frac{e^a}{b} \int_0^1 (1-e^{-at})\psi(t+1) \, dt.
\]

A direct computation shows that \( \int_0^1 \psi(t+1) \, dt = 0 \), and integration by parts gives
\[
\text{third term} = \frac{a}{1-e^{-a}} \int_0^1 e^{-at} \ln(\Gamma(t+1)) \, dt.
\]
The identity \( \ln(\Gamma(t+1)) = \ln(\Gamma(t)) + \ln t \) now yields
\[
\text{third term} = \frac{a}{1-e^{-a}} \left( \int_0^1 e^{-at} \ln t \, dt + \int_0^1 e^{-at} \ln \Gamma(t) \, dt \right).
\]
Replacing (2.17), (2.19) and (2.21) into (2.17) provides the following expression for \( M(a) \):
\[
M(a) = \frac{\gamma}{a} + \frac{a + \ln(1-e^{-a}) - \gamma - \ln a}{1-e^{-a}} + \frac{a}{1-e^{-a}} \int_0^1 e^{-at} \ln \Gamma(t) \, dt.
\]
The term \( \gamma/a \) comes from the index \( j = 0 \) in the sum (2.18). Next, we make use of (1.1) to state our main result which, incidentally, is complementary to Theorem 1.1. This settles a conjecture of O. Oloa stated in [13].

**Theorem 2.1.** Assume \( 0 < a < \ln 2 \). Then
\[
M(a) = \frac{\gamma}{a} + \frac{a + \ln(1-e^{-a}) + \Gamma(0,a)}{1-e^{-a}} + \frac{1}{1-e^{-a}} \int_0^1 e^{-at} \psi(t+1) \, dt.
\]

This result enjoys a form similar to Theorem 1.1

**Corollary 2.1.** Assume \( 0 < a < \ln 2 \). Then
\[
M(a) = \frac{\gamma}{a} + \frac{a + \ln(1-e^{-a}) + \Gamma(0,a)}{1-e^{-a}} + \frac{1}{1-e^{-a}} \int_0^1 e^{-at} \psi(t+1) \, dt.
\]

where \( \Gamma(0,a) \) is the incomplete gamma function.

**Proof.** Split up the first integral in (2.25) and integrate by parts. \( \square \)

Differentiating (2.1) with respect to \( a \) at \( a = 0 \), and use the classical value
\[
\int_0^1 \ln \Gamma(t) \, dt = \frac{1}{2} \ln 2\pi
\]
and also
\[
\int_0^1 t \ln \Gamma(t) \, dt = \frac{\zeta'(2)}{2\pi^2} + \frac{1}{6} \ln 2\pi - \frac{\gamma}{12}
\]
obtained in [3], produces
\[
\int_0^{\pi/2} \frac{x^2 \ln(2 \cos x) \, dx}{(x^2 + \ln^2(2 \cos x))^2} = \frac{7\pi}{192} + \frac{\pi \ln 2\pi}{96} - \frac{\zeta'(2)}{16\pi}.
\]

Further differentiation of (2.1) produces the evaluation of a family of integrals similar to (2.28).
The integral in (2.1) can be expressed in an alternative form. Define

\[
\Lambda(z) := \lim_{n \to \infty} \left( \sum_{j=1}^{n} \frac{j}{j^2 + z^2} - \ln n \right).
\]

Observe that \(\Lambda(0) = \gamma\), so \(\Lambda(z)\) is a generalization of Euler’s constant.

**Lemma 2.3.** Let \(a > 0\), \(c = 1 - e^{-a}\) and define \(A = \ln 2 + \gamma\). Then

\[
\int_{0}^{1} e^{-at} \ln \Gamma(t) \, dt = \frac{A(a-c)}{a^2} - \frac{c}{2a} \Lambda \left( \frac{a}{2\pi} \right) + 2c \sum_{j=1}^{\infty} \frac{\ln j}{a^2 + 4\pi^2 j^2}.
\]

**Proof.** Expand the exponential into a MacLaurin series and use the values

\[
\int_{0}^{1} t^n \ln \Gamma(t) \, dt = \frac{1}{n+1} \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \binom{n+1}{2k-1} \frac{(2k)!}{k(2\pi)^{2k}} [A\zeta(2k) - \zeta'(2k)]
\]

\[
- \frac{1}{n+1} \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \binom{n+1}{2k} \frac{(2k)!}{2(2\pi)^{2k}} \zeta(2k) + \frac{\ln \sqrt{2\pi}}{n+1}
\]

given as (6.14) in [4]. Then calculate by interchanging the resulting double sums. □

The next corollary follows from the identity \(M(a) = L(a) + \frac{\gamma}{a}\).

**Corollary 2.2.** Assume \(0 < a < \ln 2\) and let \(c = 1 - e^{-a}\). Then

\[
\int_{0}^{\infty} e^{-at} \ln \Gamma(t) \, dt = -\frac{\gamma + \ln a}{ace^a} + \frac{A(a-c)}{a^2c} - \frac{1}{2a} \Lambda \left( \frac{a}{2\pi} \right) + 2 \sum_{j=1}^{\infty} \frac{\ln j}{a^2 + 4\pi^2 j^2}.
\]

We now state two identities involving the function \(f(t) = 2^{-t} \ln \Gamma(t)\). The proof of these identities was supplied to the authors by O. Espinosa.

**Lemma 2.4.** The identities

\[
\int_{0}^{\infty} f(t) \, dt = 2 \int_{0}^{1} f(t) \, dt - \frac{\gamma + \ln \ln 2}{\ln 2}
\]

\[
\int_{0}^{\infty} tf(t) \, dt = 2 \int_{0}^{1} (t+1) f(t) \, dt - \frac{(\gamma + \ln \ln 2)(1 + 2 \ln 2) - 1}{\ln^2 2}
\]

hold.

**Proof.** The function \(f(t)\) satisfies \(f(t+1) = \frac{1}{2} f(t) + \frac{1}{2} 2^{-t} \ln t\). Splitting the integral

\[
\int_{0}^{\infty} f(t) \, dt = \int_{0}^{1} f(t) \, dt + \int_{0}^{\infty} f(t+1) \, dt
\]

and using (1.1) gives the first result. The proof of (2.33) is similar, it only requires differentiating (1.1) with respect to the parameter \(a\). □

The reader will check that (2.32) is equivalent to the continuity of \(M(a)\) at \(a = \ln 2\). The identity (2.33) provides a proof of the last result is worthy of singular (pun-intended) interest.

**Theorem 2.2.** The jump of \(M'(a)\) at \(a = \ln 2\) is 4.
3. Conclusions

The integral

\[ M(a) := \frac{4}{\pi} \int_0^{\pi/2} \frac{x^2 \, dx}{x^2 + \ln^2(2e^{-a}\cos x)}, \]

satisfies

\[ M(a) = \frac{\gamma}{a} + \int_0^\infty e^{-at} \psi(t + 1) \, dt, \quad (3.1) \]

for \( a > \ln 2 \) and

\[ M(a) = \frac{\gamma}{a} + \frac{(a + \ln(1 - e^{-a}) + \Gamma(0, a))}{1 - e^{-a}} + \frac{1}{1 - e^{-a}} \int_0^1 e^{-at} \psi(t + 1) \, dt \]

for \( 0 < a \leq \ln 2 \).

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