Functorial quantization and
the Guillemin–Sternberg conjecture*

N.P. Landsman

Korteweg–de Vries Institute for Mathematics
University of Amsterdam
Plantage Muidergracht 24 NL-1018 TV AMSTERDAM
THE NETHERLANDS

email npl@science.uva.nl

January 12, 2022

Abstract

We propose that geometric quantization of symplectic manifolds is the arrow part of a
functor, whose object part is deformation quantization of Poisson manifolds. The ‘quantiza-
tion commutes with reduction’ conjecture of Guillemin and Sternberg then becomes a special
case of the functoriality of quantization. In fact, our formulation yields almost unlimited
generalizations of the Guillemin–Sternberg conjecture, extending it, for example, to arbitrary
Lie groups or even Lie groupoids. Technically, this involves symplectic reduction and Wein-
stein’s dual pairs on the classical side, and Kasparov’s bivariant K-theory for $C^*$-algebras
(KK-theory) on the quantum side.

*Dedicated to Alan Weinstein, at his 60th birthday. To appear in S.T. Ali et al (eds.), Proc. XXth Workshop on
Geometric Methods in Physics, Bialowieza, 2002.
1 Introduction

The theory of constraints and reduction in mechanics and field theory is important for physics, because the fundamental theories describing Nature (viz. electrodynamics, Yang–Mills theory, general relativity, and possibly also string theory) are a priori formulated as constrained systems (cf. [50]). The systematic investigation of classical constrained systems was initiated by Dirac, whose ideas were reformulated mathematically as the theory of symplectic reduction (see, e.g., [8, 31]). The procedure known as Marsden–Weinstein reduction [1, 38, 42] is a special case of this theory, which is easy to formulate, yet very rich in mathematical and physical applications. According to this procedure, a suitable action $G \circ M$ of a Lie group $G$ on a symplectic manifold $M$ produces another symplectic manifold, the reduced space $M^0$, which is a certain subspace of $M/G$. See [39] for a recent overview.

In general, the traditional idea of quantization has always been that a phase space, i.e., a symplectic space $M$, should be quantized by a Hilbert space $H(M)$, and that the classical observables, viz. the (real-valued) smooth functions on $M$ should be quantized by (self-adjoint) operators on $H$, which after all play the role of observables in quantum theory. What is the relationship between the quantization of $M$ and the quantization of the reduced space $M^0$?

The quantization of constrained systems was first analyzed in a general setting in [15], but there still exists no complete and satisfactory mathematical theory. Given some notion of quantization $Q$, the basic problem in such a theory would be to formulate a possible quantum analogue of the classical reduction procedure $R$, and compare the result of applying this procedure to the quantization of the unconstrained classical systems with the quantization of the classically reduced system. One would then hope that the order of quantization and reduction does not matter; this hope is symbolically expressed by \([Q,R] = 0\). This ‘quantization commutes with reduction’ principle can be turned into a mathematical conjecture once a precise meaning has been assigned to the operations $Q$ and $R$. See [20] for a survey of the literature on this problem in the context of geometric quantization, and cf. [31] for references (pre 1998) on other approaches.

In the context of (what we now call) Marsden–Weinstein reduction, Dirac proposed that the $G$ action on $M$ should be quantized by a unitary representation $U$ of $G$ on $H(M)$, while the so-called weak observables act on $H(M)$ by operators commuting with $U(G)$. The quantized reduction operation $RQ$ then consists in taking the $G$ invariant part $H(M)^G$ of $H(M)$, on which the weak observables then act by restriction. This idea makes rigorous mathematical sense in general only when $G$ is compact. If, in addition, $M$ is compact, one expects $H(M)$ to be finite-dimensional, and similarly for $H(M^0)$, so that the weakest possible form of the \([Q,R] = 0\) conjecture, in which the action of observables is ignored, would be

$$H(M)^G \cong H(M^0).$$

Here the $\cong$ sign stands for unitary isomorphism, and since the dimension is the only such invariant of a Hilbert space, one really is talking about a simple equality between numbers, i.e., $\dim(H(M)^G) = \dim(H(M^0))$.

Despite various refinements [20], some of which will be discussed below, the conjecture in which the conjecture has been studied in the mathematical literature. This literature started with the seminal paper [24], after which the conjecture in any form resembling [11] is usually named. Using geometric quantization, they proved under certain assumptions, among which the compactness of $M$ and $G$ are crucial. It is hard to think of a more favourable situation for quantization theory then the one assumed in [24]. Partly in order to generalize the Guillemin–Sternberg conjecture, in the mid-1990s a novel notion of quantization came up, which seems to incorporate all good features of geometric quantization whilst circumventing a number of its pitfalls; see [20, 33], and references therein. This definition of quantization is sometimes attributed to Raoul Bott.

In this approach, quantization is simply defined as the index of a suitable Dirac operator $D$ naturally associated to $M$; when $M$ carries a $G$ action $G \circ M$, this index is understood in the equivariant sense, so that the quantization of $M$, or rather of $G \circ M$, is an element of the representation ring $R(G)$ of $G$. The quantization of the reduced space $M^0$ remains an integer.
Taking the $G$ invariant part of a representation induces a map $R(G) \to \mathbb{Z}$, in terms of which the Guillemin–Sternberg conjecture can then be stated in a very elegant form. In that form, it was proved in [20, 11]; also see [29, 25] for other proofs and further references.

These ideas still only apply to the situation where $M$ and $G$ are compact (though cf. [37] for a special case where at least $M$ is noncompact), which is highly undesirable for applications to both physics and mathematics. Furthermore, it would be welcome to have some direct motivation for the notion of quantization as an (equivariant) index, and if possible also to incorporate some extra structure. For example, when no $G$ action is around, Bott’s definition of quantization merely produces a number, and the entire idea of quantizing functions on $M$ by operators is lost.

These problems can be addressed by combining geometric quantization with deformation quantization. In the latter, a Poisson manifold is quantized by an associative algebra, subject to a number of conditions. In the ‘formal’ setting, this should be an algebra over the commutative ring $\mathbb{C}[[\hbar]]$ of formal power series in one real variable $\hbar$, whereas in the ‘strict’ setting this should be a $C^\ast$-algebra over the commutative $C^\ast$-algebra $C(I)$ of continuous functions on the interval $I = [0, 1]$ [34]. As in the entire context of relating classical to quantum mechanics [61], the language of $C^\ast$-algebras is particularly attractive here. For our present purposes, it is sufficient to work with ordinary $C^\ast$-algebras (instead of $C^\ast$-algebras over $C(I)$); this amounts to quantizing at a fixed value of $\hbar$, as is usual also in geometric quantization. This simplification entails the need to impose prequantizability conditions on the symplectic manifolds in question.

In [31] we proposed that quantization should be seen as a functor between categories whose arrows are equivalence classes of bimodules. What this means is rather different in the classical and in the quantum case [32, 33]. In the former, the arrows between Poisson manifolds are isomorphism classes of symplectic dual pairs [26, 59]. In the latter, the arrows between (separable) $C^\ast$-algebras are homotopy classes of Kasparov bimodules [27]. Such bimodules are generalized Hilbert spaces equipped with a generalized Fredholm operator, such as (a bounded version of) a Dirac operator $\tilde{D}$.

In other words, quantization should map (isomorphism classes of) symplectic dual pairs into (homotopy classes of) Kasparov bimodules. More precisely, if Poisson manifolds $P_1$ and $P_2$ are quantized by (separable) $C^\ast$-algebras $Q(P_1)$ and $Q(P_2)$, respectively, then a symplectic dual pair $P_1 \leftrightarrow M \rightarrow P_2$ should be quantized by an element of the Kasparov group $KK(Q(P_1), Q(P_2))$. In the special case of a symplectic dual pair $pt \leftrightarrow M \rightarrow pt$, quantization should therefore produce an element of $KK(C, C) \cong \mathbb{Z}$, i.e., an integer. This is precisely what Bott’s index-theoretic definition of quantization does.

In [34] we had no idea what the quantization functor should look like, and therefore missed the connection between the envisaged functoriality of quantization and the Guillemin–Sternberg conjecture. We now propose that a suitable generalization of Bott’s definition of quantization will do the job, and will check that the Guillemin–Sternberg conjecture is actually a special case of functoriality. Conversely, requiring the functoriality of quantization on suitable symplectic dual pairs leads to almost unlimited generalizations of the Guillemin–Sternberg conjecture. For example, one can now remove the restriction that $M$ and $G$ have to be compact, in which case the quantization functor constructs the quantization of a canonical $G$ action $G \odot M$ as a generalized equivariant index as defined in the K-theory of group $C^\ast$-algebras [13]. This relates the Guillemin–Sternberg conjecture to the Baum–Connes conjecture in noncommutative geometry [5, 13], in which it is postulated that the K-theory of a group $C^\ast$-algebra is exhausted by such indices. Moreover, techniques that have been developed in the context of the Baum–Connes conjecture [13, 30, 48] enable one to state a generalized Guillemin–Sternberg conjecture even for Lie groupoid actions. Finally, our approach incorporates and illuminates the use of shriek maps in K-theory [4, 12, 18, 14, 24], whose functoriality turns out to be a special case of the functoriality of quantization.

Since this paper relates two different areas of mathematics, we have tried to make it largely self-contained. Following a brief review of classical reduction, we recall the idea of looking at symplectic dual pairs as arrows between Poisson manifolds. We then review the Guillemin–Sternberg conjecture in its original form, and subsequently, following a recapitulation of Spin$^c$ structures and Dirac operators, in its modern form based on Bott’s definition of quantization. We then explain how the quantization context naturally leads to KK-theory, including the idea of interpret-
ing homotopy classes of Kasparov bimodules as arrows between $C^*$-algebras. We then show that the Guillemin–Sternberg conjecture is a special case of the functoriality of quantization. In the final two sections we consider generalizations of the Guillemin–Sternberg conjecture by relating quantization to K-homology and to foliation theory, respectively.

2 Classical reduction

A Poisson manifold $M$ is a manifold equipped with a Lie bracket $\{\, , \}$ on $C^\infty(M)$ with the property that for each $f \in C^\infty(M)$ the map $g \mapsto \{f, g\}$ defines a derivation of the commutative algebra structure of $C^\infty(M)$ given by pointwise multiplication. Hence this map is given by a vector field $\xi_f$, called the Hamiltonian vector field of $f$. Symplectic manifolds are special instances of Poisson manifolds, characterized by the property that the Hamiltonian vector fields exhaust the tangent bundle. In that case, the Poisson bracket comes from a symplectic form $\omega$ on $M$ in the usual way [1].

Suppose a Lie algebra $\mathfrak{g}$ acts on a Poisson manifold $M$ in strongly Hamiltonian fashion. This means that there exist Lie algebra homomorphisms $X \mapsto X^M$ from $\mathfrak{g}$ to the space $\Gamma(M, TM)$ of vector fields on $M$ and $X \mapsto J_X$ from $\mathfrak{g}$ to $C^\infty(M)$, with the property $X^M = \xi_{J_X}$. The functions $J_X$ may be assembled into a so-called momentum map $J : S \to \mathfrak{g}^*$, defined by $(J(\sigma), X) = J_X(\sigma)$. Here $\mathfrak{g}^*$ is the dual vector space of the Lie algebra $\mathfrak{g}$. This $\mathfrak{g}^*$ is canonically a Poisson manifold under the Lie–Poisson bracket, defined on linear functions (hence elements of $\mathfrak{g}^{**} = \mathfrak{g}$) by the Lie bracket. It follows that $J$ is a Poisson map. Note that a smooth map between two Poisson manifolds is called Poisson when its pullback is a Lie algebra homomorphism (and anti-Poisson when it is an anti-homomorphism). It may happen that the $\mathfrak{g}$ action comes from a $G$ action $G \subset M$, where $G$ is a Lie group with Lie algebra $\mathfrak{g}$: in that case, one has $X^M f(\sigma) = d\mathfrak{f}(\exp(-tX)\sigma)/dt|_{t=0}$. The $G$ action is called strongly Hamiltonian whenever the associated $\mathfrak{g}$ action is.

We now specialize to the case where $M$ is symplectic. The symplectic quotient or reduced space defined by the $G$ action, or physically by the constraint $J = 0$, is $M^0 = J^{-1}(0)/G$. In case that $0$ is a regular value of $J$ and the $G$ action is proper and free on $J^{-1}(0)$, $M^0$ is a manifold, which moreover carries a unique symplectic form $\omega^0$ with the property $i^*\omega = \pi^*\omega^0$. Here $i : J^{-1}(0) \hookrightarrow M$ is the inclusion and $\pi : J^{-1}(0) \to M^0$ is the projection map. Thus Marsden–Weinstein reduction produces a new symplectic manifold $(M^0, \omega^0)$ from a given symplectic manifold $(M, \omega)$ equipped with a strongly Hamiltonian $G$ action [1] [38] [39] [42]. If the stated assumptions are not met, singularities may arise in the reduced space (cf. [38] [40] [54]).

3 Symplectic dual pairs as arrows

On the classical side, a bimodule over a pair $P, Q$ of Poisson manifolds is by definition a so-called symplectic dual pair [25] [59] $Q \leftarrow M \rightarrow P$, simply called a dual pair in what follows. Here $M$ is a symplectic manifold, the map $Q \leftarrow M$ is Poisson, and $M \rightarrow P$ is anti-Poisson. Furthermore, the pullback of any function on $P$ should Poisson-commute on $M$ with the pullback of any function on $Q$. One of the motivating examples of a dual pair is $G\backslash M \leftarrow M \rightarrow g^*$, obtained from a strongly Hamiltonian $G$ action on $M$ with momentum map $J$. Similarly, $g^*_\subset \leftarrow M^\subset \rightarrow G\backslash M$ is a dual pair.

Two $Q$-$P$ dual pairs $Q \xleftarrow{\varphi_1} \check{M}_1 \xrightarrow{\varphi_2} P$, $i = 1, 2$, are said to be isomorphic when there is a symplectomorphism $\varphi : M_1 \to M_2$ for which $\varphi_2 \varphi = q_1$ and $p_2 \varphi = p_1$. We now interpret the equivalence class of a dual pair $Q \leftarrow M \rightarrow P$ as an arrow from $Q$ to $P$. Two compatible dual pairs $Q \leftarrow M_1 \to P$ and $P \leftarrow M_2 \to R$ can be composed when firstly $M_1 \times_P M_2$ is a coisotropic submanifold of $M_1 \times M_2$, and secondly the associated symplectic quotient of $M_1 \times_P M_2$ by its canonical foliation is a manifold. We then denote the product of the dual pairs in question by $P \leftarrow M_1 \circ_P M_2 \to R$. This product is well defined on equivalence classes, where it is associative, $\circ^{-}$.

$^1$In general, we write $P^{-}$ for a Poisson manifold $P$ equipped with minus its Poisson bracket, but we write $g^*$ for $(g^*)^{-}$.
since the operation ⊗ is associative up to isomorphism. For example, when $G$ is connected the product of the dual pairs $G\backslash M \leftarrow M \rightarrow g_\ast^+$ and $g_\ast^+ \leftarrow 0 \rightarrow pt$ is $G\backslash M \leftarrow M^0 \rightarrow pt$, where $M^0$ is the Marsden–Weinstein quotient $J^{-1}(0)/G$ as before.

As explained in [20], one can impose certain regularity conditions on both Poisson manifolds and dual pairs, which guarantee that all products exist and that one has identity arrows from $P$ to $P$. Thus one obtains a category $\text{Poisson}$ whose objects are (regular) Poisson manifolds and whose arrows are equivalence classes of (regular) dual pairs. The regularity condition on Poisson manifolds is very mild, and it is actually quite hard to construct an example that fails to satisfy it. On the other hand, many dual pairs one would like to use are not regular, such as $pt \leftarrow M \rightarrow pt$, where $pt$ is the space consisting of a point. Also, although $G\backslash M \leftarrow M \rightarrow g_\ast^+$ is regular, $pt \leftarrow M \rightarrow g_\ast^+$ is not. Nonetheless, the product of $pt \leftarrow M \rightarrow g_\ast$ and $g_\ast^+ \leftarrow 0 \rightarrow pt$ is well defined, and equal to

$$ (pt \leftarrow M \rightarrow g_\ast^+) \otimes_{g_\ast^+} (g_\ast^+ \leftarrow 0 \rightarrow pt) \cong pt \leftarrow M^0 \rightarrow pt. \quad (2) $$

Another example is the dual pair $X \leftarrow T^*X \xrightarrow{f\circ\pi} Y$ defined by a smooth map $X \xrightarrow{f} Y$. Here $X$ and $Y$ are manifolds with zero Poisson bracket, $f$ is smooth, and $T^*X$ has the canonical Poisson structure. The product of $X \leftarrow T^*X \rightarrow Y$ with the dual pair $Y \leftarrow T^*Y \rightarrow Z$ induced by $Y \xrightarrow{g} Z$ is

$$ (X \leftarrow T^*X \rightarrow Y) \otimes_Y (Y \leftarrow T^*Y \rightarrow Z) \cong X \leftarrow T^*X \xrightarrow{g\circ f\circ\pi} Z. \quad (3) $$

Note that the dual pairs defined by a $G$ action $G \circ M$ and by a map $X \xrightarrow{f} Y$ are both special cases of a very general functorial construction involving Lie groupoids $[32]$. Such examples indicate that products of dual pairs lying in a certain class often make sense when the regularity condition is not satisfied. Thus in the present paper we shall not impose the regularity conditions on dual pairs, refraining from a complete categorical structure. It will still be possible to map arrows of the above type into arrows in the category $\text{KK}$ defined below, and to check functoriality of this map, interpreted as quantization, with respect to the product $\otimes$. It is in this rather pragmatic sense that the notion of functoriality will be understood in what follows.

4 The Guillemin–Sternberg conjecture

Guillemin and Sternberg $[21]$ considered the case in which the symplectic manifold $M$ is compact, prequantizable, and equipped with a positive-definite complex polarization $\mathcal{J}$. Recall that a symplectic manifold $(M,\omega)$ is called prequantizable when the cohomology class $[\omega]/2\pi$ in $H^2(M,\mathbb{R})$ is integral, i.e., lies in the image of $H^2(M,\mathbb{Z})$ under the natural homomorphism $H^2(M,\mathbb{Z}) \rightarrow H^2(M,\mathbb{R})$. In that case, there exists a line bundle $L_\omega$ over $M$ whose first Chern class $c_1(L_\omega)$ maps to $[\omega]/2\pi$ under this homomorphism; $L_\omega$ is called the prequantization line bundle over $M$. In general, this bundle is not unique.

Under these circumstances, the quantization operation $Q$ is well-defined through geometric quantization $[20]$: one picks a connection $\nabla$ on $L_\omega$ whose curvature is $\omega$, and defines the Hilbert space $H(M)$ as the space $H = H^0(M,\omega)$ of polarized sections of $L_\omega$ (i.e., of sections annihilated by all $\nabla_X, X \in \mathcal{J}$).

Now suppose that $M$ carries a strongly Hamiltonian action $G \circ M$ of a compact Lie group $G$ that leaves $\mathcal{J}$ invariant. The Hilbert space $H(M)$ then carries a natural unitary representation of $G$ determined by the classical data, as polarized sections of $L_\omega$ are mapped into each other by the pullback of the $G$ action. Moreover, it turns out that the reduced space $M^0$ inherits all relevant structures on $M$ (except, of course, the $G$ action), so that it is quantizable as well, in the same fashion. Thus $[11]$ becomes, in obvious notation, $H^0(M,L_\omega)^G \cong H^0(M^0,L_\omega^0)$, which Guillemin and Sternberg indeed managed to prove. The idea of the proof is to define a map from $H^0(M,L_\omega)^G$ to $H^0(M^0,L_\omega^0)$ by simply restricting a $G$ invariant polarized section of $L_\omega$ to $J^{-1}(0)$; this map is then shown to be an isomorphism $[21]$. 
The new approach to geometric quantization mentioned in the Introduction is based on the notion of a Spin\(^c\) structure on \(M\), which we briefly recall.\(^2\) A large number of approaches to Spin\(^c\) structures exist, of which the ones relating this concept to K-theory \[3, 37\], to K-homology \[8, 23\], to KK-theory \[14\], to E-theory \[13\] (all these approaches are, in turn, closely linked to index theory), and to Morita equivalence of \(C^*\)-algebras \[19, 50\] are particularly relevant to our theme. We will return to some of these in due course, but for the moment a purely differential-geometric approach is appropriate \[17, 20\].

Firstly, the compact Lie group Spin\(^c\)(\(n\)) is a nontrivial central extension of \(SO(n)\) by \(U(1)\), defined as Spin\(^c\)(\(n\)) = Spin\(^c\)(\(n\)) \(\times\) \(Z_2\) \(U(1)\), where Spin\(^c\)(\(n\)) is the usual twofold cover of \(SO(n)\), and \(Z_2\) is seen as the subgroup \([\{1, 1\}, \{−1, −1\}\]} \(\) of Spin\(^c\)(\(n\)) \(\times\) \(U(1)\). Thus one has the obvious homomorphisms \(π : \text{Spin}^c(\(n\)) \rightarrow SO(n) \cong \text{Spin}(\(n\))/\(Z_2\), given by projection on the first factor, and det : Spin\(^c\)(\(n\)) \rightarrow \(U(1)\), defined by \([x, z] \mapsto z^2\).

Let \(n = \dim(M)\). A Spin\(^c\) structure \((P, \Xi)\) on \(M\) is by definition a principal Spin\(^c\)(\(n\))-bundle \(P\) over \(M\) with an isomorphism \(P \times _x \mathbb{R}^n \cong TM\) of vector bundles. Here the bundle on the left-hand side is the bundle associated to \(P\) by the defining representation of \(SO(n)\). Various structures on \(M\) canonically induce a Spin\(^c\) structure on \(M\), such as a Spin structure or an almost complex structure. Note that a Spin\(^c\) structure on \(M\), when it exists, is not unique: up to homotopy, the class of possible Spin\(^c\) structures on \(M\) (with given orientation) is parametrized by the Picard group \(H^2(M, \mathbb{Z})\) \[20\].

A Spin\(^c\) structure defines a number of vector bundles over \(M\) associated to \(P\) by various representations of Spin\(^c\)(\(n\)). The first of these, which is isomorphic to the bundle \(TM\), has just been mentioned.\(^3\) The second is the canonical line bundle \(L = P \times _{\det} \mathbb{C}\) associated to \(P\) by the defining representation of \(U(1)\). Thirdly, Spin\(^c\)(\(n\)) has a canonical unitary representation \(\Delta_n\) on a finite-dimensional Hilbert space \(S\), the so-called (complex) spin representation, which for odd \(n\) is irreducible, and for even \(n\) decomposes into two irreducibles \(\Delta_n = \Delta_n^+ \oplus \Delta_n^-\) on \(S = S^+ \oplus S^-\). Thus one has an associated spinor bundle \(P = P \times _{\Delta_n} S\), which for even \(n\) decomposes into the direct sum \(S^\pm = P \times _{\Delta_n} S^\pm\). Thus the physical interpretation of Spin\(^c\) structures involves gravity, electromagnetism, and fermions.

A Spin\(^c\) structure on \(M\) defines a vector bundle action \(TM \rightarrow \text{End}(S)\) by Clifford multiplication, since both \(TM\) and \(S\) are subspaces of the Clifford bundle \(Cl(TM)\) over \(M\). This action may be seen as a map \(\psi : \Gamma(TM \otimes_M S) \rightarrow \Gamma(S)\). Furthermore, a connection on \(P\) induces on \(S\), which amounts to a covariant derivative \(\nabla : \Gamma(S) \rightarrow \Gamma(TM \otimes_M S)\). Identifying \(TM\) with \(TM\) through the Riemannian metric \(g\) determined by the Spin\(^c\) structure and composing these maps yields the Dirac operator \[\bar{\partial} : \Gamma(S) \rightarrow \Gamma(TM \otimes_M S) \rightarrow \Gamma(TM \otimes_M S) \rightarrow \Gamma(S)\].

This elliptic first-order linear differential operator is formally self-adjoint, and can be turned into a bounded self-adjoint operator \(\bar{\partial} : \mathcal{D}/\mathcal{D}^2(S) \rightarrow \mathcal{D}(S)\), where \(\mathcal{D}(S)\) stands for the Hilbert space of \(L^2\)-sections of the vector bundle \(S\). When \(M\) is even-dimensional, \(\bar{\partial}\) is odd with respect to the decomposition \(S = S^+ \oplus S^−\), so that one obtains the chiral Dirac operator \(\bar{\partial}^+ : \Gamma(S^+) \rightarrow \Gamma(S^-)\), with formal adjoint \(\bar{\partial}^- : \Gamma(S^-) \rightarrow \Gamma(S^+)\), by restriction. Similarly, one has \(\bar{\partial}^\pm : L^2(S^\pm) \rightarrow L^2(S^\mp)\).

6 Bott’s definition of quantization

The first step in Bott’s definition of quantization is to canonically associate a Spin\(^c\) structure to a given symplectic and prequantizable manifold \((M, \omega)\) \[20, 40\]. First, one picks an almost complex

\(^2\)Such a structure may more generally be defined on a real vector bundle \(E\) over \(M\); when \(E\) is the tangent bundle \(TM\) we obtain the special case discussed in the main text.

\(^3\)It induces both an orientation and a Riemannian metric on \(M\), by transferring the standard orientation and metric on \(\mathbb{R}^n\) to \(E\). Conversely, given an orientation and a Riemannian metric on \(M\), one should require a Spin\(^c\) structure on \(M\) to be compatible with these.
structure $\mathcal{J}$ on $M$ that is compatible with $\omega$ (in that $\omega(-,\mathcal{J}-)$ is positive definite and symmetric, i.e., a metric). This $J$ canonically induces a Spin$^c$ structure $P_{\mathcal{J}}$ on $TM$. But this is not the right one to use here. The Spin$^c$ structure $P$ needed to quantize $M$ is the one obtained by twisting $P_{\mathcal{J}}$ with the prequantization line bundle $L_\omega$. This means (cf. [20], App. D.2.7) that $P = P_{\mathcal{J}} \times_{\ker(\pi)} U(L_\omega)$, where $\pi : \text{Spin}^c(n) \to SO(n)$ was defined in the preceding section (note that $\ker(\pi) \cong U(1)$, and $U(L_\omega) \subset L_\omega$ is the unit circle bundle.\footnote{In fact, this construction needs to be corrected in some cases [20, 47], but this correction complicates the statement of the Guillemin–Sternberg conjecture, and will not be discussed here.}

When $M$ is compact, the operators $\tilde{\mathcal{D}}^\pm$ determined by the Spin$^c$ structure $(P, \cong)$ have finite-dimensional kernels, whose dimensions define the quantization of $(M, \omega)$ as

$$Q(M, \omega) = \text{index}(\mathcal{D}^+) = \dim \ker(\mathcal{D}^+) - \dim \ker(\mathcal{D}^-).$$

(4)

In fact, the corresponding Hilbert space operators $\tilde{\mathcal{D}}^\pm$ are Fredholm, and by elliptic regularity index$(\mathcal{D}^+)$ coincides with the Fredholm index $\dim \ker(\mathcal{D}^+) - \dim \ker(\mathcal{D}^-)$ of $\mathcal{D}^+$. This notion of quantization just associates an integer to $(M, \omega)$. This number turns out to be independent of the choice of the Spin$^c$ structure on $M$, as long as it satisfies the above requirement, and is entirely determined by the cohomology class $[\omega]$ (as remarked earlier, this is not true for the Spin$^c$ structure and the associated Dirac operator itself) [20].

This definition of quantization gains in substance when a compact Lie group $G$ acts on $M$ in strongly Hamiltonian fashion. In that case, the pertinent Spin$^c$ structure may be chosen to be $G$ invariant, and the spaces $\ker(\tilde{\mathcal{D}}^\pm)$ are finite-dimensional complex $G$ modules. Hence

$$G\text{-index}(\mathcal{D}^+) = [\ker(\mathcal{D}^+)] - [\ker(\mathcal{D}^-)]$$

(5)

defines an element of the representation ring $R(G)$ of $G$.\footnote{$R(G)$ is defined as the abelian group with one generator $[L]$ for each finite-dimensional complex representation $L$ of $G$, and relations $[L] = [M]$ when $L$ and $M$ are equivalent and $[L] + [M] = [L \oplus M]$. The tensor product of representations defines a ring structure on $R(G)$.} Thus, the quantization of $(M, \omega)$ with associated $G$ action may be defined as

$$Q(G \circ M, \omega) = G\text{-index}(\mathcal{D}^+) \in R(G).$$

(6)

As before, this element only depends on $[\omega]$ (and on the $G$ action). The same definition arises from the Hilbert space setting: the Hilbert spaces $L^2(S^\pm)$ carry unitary representations $U^\pm$ of $G$ in the obvious way, and the bounded Dirac operators $\tilde{\mathcal{D}}^\pm$ are equivariant under these, so that $\ker(\tilde{\mathcal{D}}^\pm)$ are unitary $G$ modules. Replacing $\mathcal{D}^\pm$ in (5) by $\tilde{\mathcal{D}}^\pm$ then yields an element of the ring of unitary finite-dimensional representations of $G$, which for a compact group is the same as $R(G)$. When $G$ is trivial, one may identify $R(G)$ with $\mathbb{Z}$ through the map $[V] - [W] \mapsto \dim(V) - \dim(W)$, so that (4) emerges as a special case of (6).

In this setting, the Guillemin–Sternberg conjecture makes sense as long as $M$ and $G$ are compact. The Hilbert space $H^0(M, L_\omega)^G$ in the original version of the conjecture is now replaced by the image $Q(G \circ M, \omega)_0$ of $Q(G \circ M, \omega)$ in $\mathbb{Z}$ under the map $[V] - [W] \mapsto \dim(V_0) - \dim(W_0)$, where $V_0$ is the $G$ invariant part of $V$, etc. The right-hand side of the conjecture is the quantization of the reduced space $M^0$ (which inherits a Spin$^c$ structure from $M$) according to (4). Denoting the pertinent Dirac operator on $M^0$ by $\mathcal{D}_0$, the Guillemin–Sternberg conjecture in the setting of Bott’s definition of quantization is therefore simply

$$G\text{-index}(\mathcal{D}^+)_0 = \text{index}(\mathcal{D}_0^+).$$

(7)

In this form, the conjecture was proved in [11]; it even holds when 0 fails to be a regular value of $J$. Also see [20, 46] for other proofs and further references.

Bott’s definition of quantization (4) or (6) isn’t actually all that far removed from the traditional idea of associating a group representation on a Hilbert space with a strongly Hamiltonian action on a symplectic manifold. In fact, when the symplectic form $\omega$ is sufficiently large, the space $\ker(\mathcal{D}^-)$ tends to vanish [10], so that $Q(G \circ M, \omega)$ is really a representation of $G$, up to isomorphism. This is relevant in the semiclassical regime, where one quantizes $(M, \omega/h)$ for small values of $h$.\footnote{Denoting the pertinent Dirac operator on $M^0$ by $\mathcal{D}_0$, the Guillemin–Sternberg conjecture in the setting of Bott’s definition of quantization is therefore simply

$$G\text{-index}(\mathcal{D}^+)_0 = \text{index}(\mathcal{D}_0^+).$$

(7)

In this form, the conjecture was proved in [11]; it even holds when 0 fails to be a regular value of $J$. Also see [20, 46] for other proofs and further references.

Bott’s definition of quantization (4) or (6) isn’t actually all that far removed from the traditional idea of associating a group representation on a Hilbert space with a strongly Hamiltonian action on a symplectic manifold. In fact, when the symplectic form $\omega$ is sufficiently large, the space $\ker(\mathcal{D}^-)$ tends to vanish [10], so that $Q(G \circ M, \omega)$ is really a representation of $G$, up to isomorphism. This is relevant in the semiclassical regime, where one quantizes $(M, \omega/h)$ for small values of $h$.\footnote{Denoting the pertinent Dirac operator on $M^0$ by $\mathcal{D}_0$, the Guillemin–Sternberg conjecture in the setting of Bott’s definition of quantization is therefore simply

$$G\text{-index}(\mathcal{D}^+)_0 = \text{index}(\mathcal{D}_0^+).$$

(7)}
7 From quantization to KK-theory

To motivate the use of Kasparov’s bivariant K-theory, or KK-theory, in the light of the Guillemin–Sternberg conjecture and Bott’s definition of quantization, let us recall a result from functional analysis (see, e.g., [18]). Recall that a bounded operator $F : H^+ \to H^-$ between two Hilbert spaces is called Fredholm when it is invertible up to compact operators, that is, when there exists a bounded operator $F' : H^- \to H^+$, called a parametrix of $F$, such that $FF' - 1$ and $F'F - 1$ are compact operators on $H^-$ and $H^+$, respectively. A key result is then that the space $\mathcal{F}(H^+, H^-)/\tilde{h}$ of homotopy equivalence classes $\left[F\right]$ of Fredholm operators $F$ (where the notion of homotopy is defined with respect to operator-norm continuous paths in the space of all Fredholm operators) is homeomorphic to $\mathbb{Z}$, where the pertinent homeomorphism is given by $\left[F\right] \mapsto \text{index}(F)$.

Hence in Bott’s definition of quantization [41] we may work with $\left[\tilde{D}^+\right]$ instead of with $\text{index}(\tilde{D}^+)$ ($= \text{index}(\tilde{D}^+)$). Thus we put

$$Q(pt \leftarrow M \to pt) = \left[\tilde{D}^+\right]. \quad (8)$$

As indicated by the notation, we regard the right-hand side of (8) as the quantization of (the isomorphism class of) the dual pair on the left-hand side. It will become clear shortly that this homotopy class is an element of the Kasparov group $KK(C, \mathbb{C})$, where we regard $\mathbb{C}$ as the $C^*$-algebra that quantizes the Poisson manifold $pt$. This group is isomorphic to $\mathbb{Z}$, and the image of $\left[F\right]^6$ under the isomorphism $KK(C, \mathbb{C}) \to \mathbb{Z}$ is precisely $\text{index}(F)$. Clearly, this isomorphism links (8) to (4).

To generalize this idea to more complicated dual pairs, we need Kasparov’s theory [27] (see also [21] for a full treatment and [13] [22] [55] for very useful introductions), which is a systematic machinery for dealing with homotopy classes of generalized Fredholm operators. The first step is to generalize the notion of a Hilbert space, which we here regard as a Hilbert $C$-$C$ bimodule, to the concept of a Hilbert $A$-$B$ bimodule, where $A$ and $B$ are separable $C^*$-algebras (which in our setting emerge as the quantizations of Poisson manifolds $P$ and $Q$). The correct generalization was introduced by Rieffel in a different context [41], and has already been used in the theory of constrained quantization in [31].

An $A$-$B$ Hilbert bimodule is an algebraic $A$-$B$ bimodule $E$ (where $A$ and $B$ are seen as complex algebras, so that $E$ is a complex linear space) with a compatible $B$-valued inner product. This is a sesquilinear map $(, ) : E \times E \to B$, linear in the second and antilinear in the first entry, satisfying $(x, y)^* = (y, x)$, $(x, x) \geq 0$, and $(x, x) = 0$ iff $x = 0$. The compatibility of the inner product with the remaining structures means that firstly $E$ has to be complete in the norm $\|x\|^2 = \|(x, x)\|$, secondly that $(x, y)b = (x, y)b$, and thirdly that $(a^*x, y) = (x, ay)$ for all $x, y \in E$, $b \in B$, and $a \in A$. The latter condition may be expressed by saying that $a$ is adjointable, with adjoint $a^*$; this is a nontrivial condition even when $a$ is bounded (note that an adjointable operator is automatically bounded). The best example of all this is the $A$-$A$ Hilbert bimodule $E = A$, with the obvious actions and the inner product $(a, b) = a^*b$.

An $A$-$C$ Hilbert bimodule is simply a Hilbert space equipped with a representation of $A$. A $C$-$B$ Hilbert bimodule is called a Hilbert $B$ module, or Hilbert $C^*$-module over $B$.

Adjointable operators on an $A$-$B$ Hilbert bimodule $E$ are the analogues of bounded operators on a Hilbert space; the collection of all adjointable operators indeed forms a $C^*$-algebra. The role of compact operators on $E$ is played by operators that can be approximated in norm by linear combinations of rank one operators of the form $z \mapsto x(y, z)$ for $x, y \in E$ (such operators are automatically adjointable). Again, as for Hilbert spaces, the space of all compact operators on $E$ is a $C^*$-algebra. In the example ending the preceding paragraph, the left $A$ action turns out to be by compact operators. A Fredholm operator, then, is an adjointable operator that is invertible up to compact operators.

Now an $A$-$B$ Kasparov bimodule is a pair of countably generated $A$-$B$ Hilbert bimodules $(E^+, E^-)$ with an ‘almost’ Fredholm operator $F : E^+ \to E^-$ that ‘almost’ intertwines the $A$ actions on $E^+$ and $E^-$. The first condition means that there is an adjointable operator $F^* : H^- \to H^+$

More precisely, of the homotopy class $\left[F, H^+, H^-\right]$, where $H^\pm$ are $C$-$C$ Hilbert bimodules under the action $z \mapsto z1$, $z \in C$. 

8
such that \( a(FF' - 1) \) and \( a(F'F - 1) \) are compact for all \( a \in A \), and the second states that \( aF - Fa \) is compact for all \( a \in A \). With the structure of \( E^\pm \) as \( A \)-\( B \) Hilbert bimodules understood, we denote such a Kasparov bimodule simply by \((F, E^+, E^-)\).

For \( B = \mathbb{C} \) this is sometimes called a Fredholm module \([23]\). A key example of a Fredholm module is given by \( E^\pm = L^2(S^\pm) \), and \( F = \overline{\partial}^+ \). When \( M \) is compact, this works for both \( A = \mathbb{C} \) and \( A = C(M) \), but when \( M \) isn’t one must take \( A = C_0(M) \). For general \( A \) and \( B \), it follows from the definitions that if \( A \) acts on \( E \) by compact operators, then the choice \( F = 0 \) yields a Kasparov bimodule. This applies, for instance, to the \( A \)-\( A \) Hilbert bimodule \((E^+ = A, E^- = 0)\).

A homotopy of \( A \)-\( B \) Kasparov bimodules is an \( A\text{-}C([0,1],B) \) Kasparov bimodule. The ensuing set \( KK(A,B) \) of homotopy classes of \( A \)-\( B \) Kasparov bimodules may more conveniently be described as the quotient of the set of all \( A \)-\( B \) Kasparov bimodules by the equivalence relation generated by unitary equivalence, translation of \( F \) along norm-continuous paths (of almost intertwining almost Fredholm operators), and the addition of degenerate Kasparov bimodules. The latter are those for which the operators \( a(F - Fa, a(FF' - 1) \) and \( a(F'F - 1) \) are not merely compact but zero for all \( a \in A \). Using the polar decomposition, one may always choose representatives for which all \( (F' - F^\ast)\alpha \) are compact (so that \( F \) is almost unitary), and this is often included in the definition of a Kasparov bimodule. In that case, the condition that \( (F' - F^\ast)\alpha = 0 \) is added to the definition of a degenerate Kasparov bimodule.

It is not difficult to see that \( KK(A,B) \) is an abelian group; the group operation is the direct sum of both bimodules and operators \( F \), and the inverse of the class of a Kasparov bimodule is found by swapping \( E^+ \) and \( E^- \) and replacing \( F : E^+ \to E^- \) by its parametrix \( F' : E^- \to E^+ \). Moreover, with respect to \( \ast \)-homomorphisms between \( C^\ast \)-algebras the association \( (A,B) \mapsto KK(A,B) \) is contravariant in the first entry, and covariant in the second.

Let us note that for any \( C^\ast \)-algebra \( A \) the group \( KK(C,A) \) is naturally isomorphic to the algebraic K-theory group \( K_0(A) \). \(^7\) Hence as far as \( K_0 \) is concerned, K-theory is a special case of KK-theory. Explicitly, the isomorphism \( KK(C,A) \to K_0(A) \) is the generalized index map\(^8\)

\[
[F, E^+, E^-] \mapsto [\ker(F)] - [\ker(F')].
\]

A remarkable aspect of Kasparov’s theory is the existence of a product

\[
KK(A,B) \times KK(B,C) \to KK(A,C),
\]

which is functorial in all conceivable ways. Disregarding \( F \), this would be easy to define, since one feature of algebraic bimodules that survives in the Hilbert case is the existence of a bimodule tensor product \([21]\): from an \( A \)-\( B \) Hilbert bimodule \( E \) and a \( B \)-\( C \) Hilbert bimodule \( \tilde{E} \) one can form an \( A \)-\( C \) Hilbert bimodule \( E \hat{\otimes}_B \tilde{E} \), called the interior tensor product of \( E \) and \( \tilde{E} \). However, the composition of the almost Fredholm operators in question is too complicated to be explained here (see \([9, 13, 14, 22, ? , 55]\). In any case, this product leads to the category \( KK \), whose objects are separable \( C^\ast \)-algebras, and whose arrows are Kasparov’s KK-groups.

To close this section, let us mention that we only use the ‘even’ part of KK-theory; in general, each KK group is \( \mathbb{Z}_2 \) graded, and what we have called \( KK(A,B) \) is really \( KK_0(A,B) \). This restriction is possible because symplectic manifolds happen to be even-dimensional.

### 8 The Guillemin–Sternberg conjecture revisited

Let us return to a strongly Hamiltonian group action \( G \triangleleft M \), with associated dual pair \( pt \leftrightarrow M \to \mathfrak{g}^* \). To quantize this dual pair, we first note that the quantization of the Poisson manifold \( \mathfrak{g}^* \) is

\(^7\)When \( A \) has a unit, \( K_0(A) \) may be defined as the abelian group with one generator \([E]\) for each finitely generated projective (f.g.p.) right module over \( A \), and relations \([E] = [E']\) when \( E \) and \( E' \) are isomorphic, and \([E] + [E'] = [E \oplus E']\). For example, when \( X \) is a compact Hausdorff space one has \( K_0(C(X)) = K_0(X) \), the topological K-theory of Atiyah and Hirzebruch \([20]\). When \( A \) has no unit, \( K_0(A) \) is defined as the kernel of the canonical map \( K_0(\hat{A}) \to K_0(C) \), where \( \hat{A} = A \oplus \mathbb{C} \) is the unitization of \( A \).

\(^8\)The representatives \( F \) and \( F' \) of their respective homotopy classes have to be chosen such that their kernels in the \( A \) modules \( E^- \) and \( E^+ \) are indeed f.g.p.
the group $C^\ast$-algebra $C^\ast(G)$ \[22, 31\]; this is probably the best understood example in $C^\ast$-algebraic quantization theory.\footnote{Here $C^\ast(G)$ is a suitable completion of the convolution algebra on $G$ determined by a Haar measure \[16, 31\].} Although this holds for any $G$ with given Lie algebra, to obtain a unique functor we assume $G$ to be connected and simply connected. Hence the quantization of the dual pair $pt \leftrightarrow M \to g^\ast_+$ should be an element of the Kasparov group $KK(\mathbb{C}, C^\ast(G)) \cong K_0(C^\ast(G))$.

When $G$ is compact, which we assume throughout the remainder of this section, one may identify $K_0(C^\ast(G))$ with the representation ring $R(G)$; this is because finitely generated projective modules over $C^\ast(G)$ may be identified with finite-dimensional unitary representations of $G$. Now assume that $M$ is compact as well. Seen as an element of $R(G)$, the quantization of $pt \leftrightarrow M \to g^\ast_+$ is given by $G$-index$(\hat{D}^+)$, as in \[5\]; this is just a reinterpretation of Bott’s definition \[6\] of quantization. It is slightly more involved to explain the quantization of $pt \leftrightarrow M \to g^\ast_+$ when it is seen as an element of $KK(\mathbb{C}, C^\ast(G))$. Firstly, one turns the Hilbert spaces $L^2(S^\pm)$ into Hilbert $C^\ast(G)$ modules, as follows \[5, 17, 38\].

The canonical $G$ actions $U^\pm$ on $L^2(S^\pm)$ induce right actions $\tau^\pm$ of $C^\ast(G)$ by

$$
\tau^\pm(f) = \int_G dx f(x) U^\pm(x^{-1}),
$$

where $f \in C(G)$ (the action of a general element of $C^\ast(G)$ is then defined by continuity). Furthermore, one obtains a $C^\ast(G)$ valued inner product on $L^2(S^\pm)$ by the formula

$$
\langle \psi, \varphi \rangle : x \mapsto \langle \psi, U^\pm(x) \varphi \rangle,
$$

which defines an element of $C(G) \subset C^\ast(G)$. Completing $L^2(S^\pm)$ in the norm

$$
\| \psi \|^2 = \| \langle \psi, \psi \rangle \|_{C^\ast(G)}
$$

then yields Hilbert $C^\ast(G)$ modules $E^\pm(S)$. The operator $\hat{D}^+: L^2(S^+) \to L^2(S^-)$ extends to an adjointable operator $\hat{D}^+: E^+(S) \to E^-(S)$ by continuity, and the triple $(\hat{D}^+, E^+(S), E^-(S))$ defines a $\mathbb{C}$-$C^\ast(G)$ Kasparov bimodule, whose homotopy class is the desired element of $KK(\mathbb{C}, C^\ast(G))$, i.e.,

$$
Q(pt \leftrightarrow M \to g^\ast_+) = [\hat{D}^+, E^+(S), E^-(S)].
$$

The canonical isomorphism $KK(\mathbb{C}, C^\ast(G)) \to K_0(C^\ast(G)) = R(G)$ given by \[9\] indeed maps this element to $G$-index$(\hat{D}^+)$. 

Apart from the dual pair $pt \leftrightarrow M \to g^\ast_+$, the momentum map associated to the action $G \ltimes M$ equally well leads to a dual pair $g^\ast_- \leftrightarrow M^- \to pt$. This is to be quantized by an element of $KK(C^\ast(G), \mathbb{C}) \cong K^0(C^\ast(G))$, the so-called Kasparov representation ring of $G$ (cf. \[23\]). This time, we interpret the Hilbert spaces $L^2(S^\pm)$ as $C^\ast(G)$-$\mathbb{C}$ Hilbert bimodules, where the pertinent representations $\pi^\pm$ of $C^\ast(G)$ are given by a very slight adaptation of the procedure sketched in the preceding paragraph: to obtain left actions instead of right actions, we now put $\pi^\pm(f) = \int_G dx f(x) U^\pm(x)$. Since $\hat{D}^+ U^+(x) = U^-(x) \hat{D}^+$ for all $x \in G$, one now has $\hat{D}^+ \pi^+(f) = \pi^-(f) \hat{D}^+$ for all $f \in C^\ast(G)$. Since $\hat{D}^+$ is Fredholm one thus obtains an element $[\hat{D}^+, L^2(S^+), L^2(S^-)]$ of $KK(C^\ast(G), \mathbb{C})$, which we regard as the quantization of the dual pair $g^\ast_- \leftrightarrow S^- \to pt$.

The very simplest example is the dual pair $g^\ast_- \leftrightarrow 0 \to pt$, whose quantization is just

$$
Q(g^\ast_- \leftrightarrow 0 \to pt) = [0, \mathbb{C}, \mathbb{C}],
$$

where the $C^\ast(G)$-$\mathbb{C}$ Hilbert bimodules $\mathbb{C}$ carry the trivial representation of $G$. A simple computation of the Kasparov product

$$
KK(\mathbb{C}, C^\ast(G)) \times KK(C^\ast(G), \mathbb{C}) \to KK(\mathbb{C}, \mathbb{C}) \cong K_0(\mathbb{C}) \cong \mathbb{Z}
$$

yields

$$
[\hat{D}^+, E^+(S), E^-(S)] \times [0, \mathbb{C}, \mathbb{C}] = G$\text{-index}(\hat{D}^+)_0. \tag{14}
$$
induced in K-theory by the valued inner products (10) on the dense subspace $H$ of the Hilbert spaces $G$ in (7) is now reinterpreted as the image of $K$ plays a key role in the Baum–Connes conjecture [5]. Replacing this point of view, Bott’s quantization (4) of $(M, \omega)$ mest is just defined as the Kasparov group $KK(G) = KK(C(\mathbb{C}), \mathbb{C})$. The K-homology group of a manifold $M$ is independent of the connection picked to define $\mathcal{D}$, and is the analogue in K-homology of the fundamental class in ordinary homology defined by the orientation of $M$ [23]. From this point of view, Bott’s quantization [58] of $(M, \omega)$, which in our setting is the quantization of the dual pair $pt \leftarrow M \rightarrow pt$, is the image of the fundamental class of $M$ determined by the symplectic structure as explained, under the map $KK(C(\mathbb{C}), \mathbb{C}) \rightarrow KK(\mathbb{C}, \mathbb{C})$ obtained by forgetting the $C_0(M)$ actions on $L^2(S^\pm)$ (followed by the isomorphism $KK(\mathbb{C}, \mathbb{C}) \rightarrow \mathbb{Z}$).

In the presence of a proper $G$ action, one uses the equivariant K-homology group $K_0^G(M) = KK^G(C_0(M), \mathbb{C})$, which is defined like $KK(C_0(M), \mathbb{C})$, but with the additional stipulation that the Hilbert spaces $H^\pm$ in the Kasparov bimodule $(F; H^+, H^-)$ are unitary $G$ modules, in such a way that $F$ is equivariant and the representations of $C_0(M)$ on $E^\pm$ are covariant under $G$ [23] [57]. One now has a canonical map $K_0^G(M) \rightarrow K_0(C^*(G))$, called the analytic assembly map, which plays a key role in the Baum–Connes conjecture [5]. Replacing $K_0(C^*(G))$ with $KK(\mathbb{C}, C^*(G))$, this map is defined by a slight generalization of the construction of the element $[\mathcal{D}^+, E^+(S), E^-(S)]$ of $KK(\mathbb{C}, C^*(G))$ explained prior to [12]; cf. [57] for details. The basic idea is to define the $C_0(G)$-valued inner products (10) on the dense subspace $C_0(M)L^2(S^\pm)$, completing these subspaces in the norm (11) to obtain the Hilbert $C_0(G)$ modules $E^\pm(S)$.

It follows that the element of $KK(\mathbb{C}, C^*(G))$ that quantizes the dual pair $pt \leftarrow M \rightarrow g^*_+$, a la Bott, is just the image of the pertinent fundamental class of $M$ under the analytic assembly map. The functoriality condition [10] remains well defined, but the computation [14] is invalid for noncompact groups, so that for noncompact $G$ the left-hand side of the Guillemin–Sternberg conjecture is simply given by the left-hand side instead of the right-hand side of [12]. This yields a generalization of the Guillemin–Sternberg conjecture to noncompact groups, where $G$-index$(\mathcal{D}^+)_0$ in (7) is now reinterpreted as the image of $G$-index$(\mathcal{D}^+)$ in $K_0(C^*(G))$ under the map $K_0(C^*(G)) \rightarrow \mathbb{Z}$ induced in K-theory by the $^*$-homomorphism $f \mapsto \int_M dx f(x)$ from $C^*(G)$ to $\mathbb{C}$.

9 Guillemin–Sternberg for noncompact groups

The above reformulation of the Guillemin–Sternberg conjecture as a special case of the functoriality of Bott’s definition of quantization paves the way for far-reaching generalizations of this conjecture. Firstly, one can now consider noncompact $G$ and $M$, as long as the $G$ action on $M$ is proper. It is convenient to use the language of K-homology (cf. [23]). The K-homology group of a manifold $M$ is just defined as the Kasparov group $K_0(M) = KK(C_0(M), \mathbb{C})$. A Spin$^c$ structure on $M$ defines an element $[\mathcal{D}^+]$ of $K_0(M)$ through its associated Dirac operator. This so-called fundamental class never vanishes. It is independent of the connection picked to define $\mathcal{D}$, and is the analogue in K-homology of the fundamental class in ordinary homology defined by the orientation of $M$ [23]. From this point of view, Bott’s quantization [58] of $(M, \omega)$, which in our setting is the quantization of the dual pair $pt \leftarrow M \rightarrow pt$, is the image of the fundamental class of $M$ determined by the symplectic structure as explained, under the map $KK(K_0(C_0(M), \mathbb{C}) \rightarrow KK(\mathbb{C}, \mathbb{C})$ obtained by forgetting the $C_0(M)$ actions on $L^2(S^\pm)$ (followed by the isomorphism $KK(\mathbb{C}, \mathbb{C}) \rightarrow \mathbb{Z}$).

In the presence of a proper $G$ action, one uses the equivariant K-homology group $K_0^G(M) = KK^G(C_0(M), \mathbb{C})$, which is defined like $KK(C_0(M), \mathbb{C})$, but with the additional stipulation that the Hilbert spaces $H^\pm$ in the Kasparov bimodule $(F; H^+, H^-)$ are unitary $G$ modules, in such a way that $F$ is equivariant and the representations of $C_0(M)$ on $E^\pm$ are covariant under $G$ [23] [57]. One now has a canonical map $K_0^G(M) \rightarrow K_0(C^*(G))$, called the analytic assembly map, which plays a key role in the Baum–Connes conjecture [5]. Replacing $K_0(C^*(G))$ with $KK(\mathbb{C}, C^*(G))$, this map is defined by a slight generalization of the construction of the element $[\mathcal{D}^+, E^+(S), E^-(S)]$ of $KK(\mathbb{C}, C^*(G))$ explained prior to [12]; cf. [57] for details. The basic idea is to define the $C_0(G)$-valued inner products (10) on the dense subspace $C_0(M)L^2(S^\pm)$, completing these subspaces in the norm (11) to obtain the Hilbert $C_0(G)$ modules $E^\pm(S)$.

It follows that the element of $KK(\mathbb{C}, C^*(G))$ that quantizes the dual pair $pt \leftarrow M \rightarrow g^*_+$, a la Bott, is just the image of the pertinent fundamental class of $M$ under the analytic assembly map. The functoriality condition [10] remains well defined, but the computation [14] is invalid for noncompact groups, so that for noncompact $G$ the left-hand side of the Guillemin–Sternberg conjecture is simply given by the left-hand side instead of the right-hand side of [12]. This yields a generalization of the Guillemin–Sternberg conjecture to noncompact groups, where $G$-index$(\mathcal{D}^+)_0$ in (7) is now reinterpreted as the image of $G$-index$(\mathcal{D}^+)$ in $K_0(C^*(G))$ under the map $K_0(C^*(G)) \rightarrow \mathbb{Z}$ induced in K-theory by the $^*$-homomorphism $f \mapsto \int_M dx f(x)$ from $C^*(G)$ to $\mathbb{C}$.

10We here assume that $G$ is unimodular, which guarantees that (10) is positive. This was shown for discrete $G$ in Lemma 3 in [23], but the proof apparently works for unimodular groups in general. In general, the construction in the preceding section produces a Hilbert module over the reduced group $C^*$-algebra $C^*_r(G)$ [9]. This is sufficient for the Baum–Connes conjecture, but not for our generalized Guillemin–Sternberg conjecture.

11Cf. [25] for an exposition of the link between the analytic assembly map and $C^*$-algebraic deformation quantization, following Connes’s discussion of this map in E-theory [13].

12A complication arises when $M$ does not admit a $G$ invariant Spin$^c$ structure. For techniques to overcome this cf. [23] [57].
As a first example, consider the case where $G = \Gamma$ is discrete and infinite. One then simply has $M^0 = M/\Gamma$, and $D_0^+$ is just the operator on $M/\Gamma$ whose lift is $D^+$. Using Atiyah’s $L^2$-index theorem [2], our generalized Guillemin–Sternberg conjecture is equivalent to

$$G\text{-}index(D^+_0) = \text{tr} \circ \pi_* \circ G\text{-}index(D^+).$$

Here $\pi_* : K_0(C^*(\Gamma)) \to K_0(C^r(\Gamma))$ is the K-theory map functorially induced by the canonical projection $\pi : C^*(\Gamma) \to C^r(\Gamma)$, and $\text{tr} : K_0(C^*(\Gamma)) \to \mathbb{C}$ is defined by the pairing of the trace $f \mapsto f(e)$ on $C^r(\Gamma)$ (seen as a cyclic cocycle) with K-theory [13].

10 Foliation theory and quantization

A second generalization of the Guillemin–Sternberg conjecture arises when one considers strongly Hamiltonian actions of Lie groupoids on symplectic manifolds; the pertinent symplectic reduction procedure was first studied in [43], and is actually a special case of the product $\otimes$ [11, 31]. Furthermore, the appropriate construction of elements of $K_0(C^*(G))$ has been given in [13, 48]. A very interesting special case comes from foliation theory, as follows (cf. [12, 13, 24, 44, 45]). Let $(V_i, F_i)$, $i = 1, 2$, be foliations with associated holonomy groupoids $G(V_i, F_i)$ (assumed to be Hausdorff for simplicity). A smooth generalized map $f$ between the leaf spaces $V_1/F_1$ and $V_2/F_2$ is defined as a smooth right principal bibundle $M_f$ between the Lie groupoids $G(V_1, F_1)$ and $G(V_2, F_2)$. Classically, such a bibundle defines a dual pair $T^*F_1 \leftarrow T^*M_f \to T^*F_2$ [32]. Here $TF_i \subset TV_i$ is the tangent bundle to the foliation $(V_i, F_i)$, whose dual bundle $T^*F_i$ has a canonical Poisson structure. Quantum mechanically, $f$ defines an element $[12, 24]

$$f_i \in KK(C^*(G(V_1, F_1)), C^*(G(V_2, F_2))).$$

In our functorial approach to quantization, $f_i$ is interpreted as the quantization of the dual pair $T^*F_1 \leftarrow T^*M_f \to T^*F_2$. The functoriality of quantization among dual pairs of the same type then follows from the computations in [24, 32]. The construction and functoriality of shriek maps in [4, 12] is a special case of this, in which the $V_i$ are both trivially foliated.

References

[1] Abraham, R. and Marsden, J.E. (1985). Foundations of Mechanics, 2nd ed. Addison Wesley, Redwood City.

[2] Atiyah, M.F. (1976). Elliptic operators, discrete groups and von Neumann algebras. Astérisque 32-33, 43–72.

[3] Atiyah, M.F., Bott, R. and Shapiro, A. (1964). Clifford modules. Topology 3 suppl. 1, 3–38.

[4] Atiyah, M.F. and Singer, I.M. (1968). The index of elliptic operators I. Ann. Math. 87, 485–530.

[5] Baum, P., Connes, A. and Higson, N. (1994). Classifying space for proper actions and K-theory of group $C^*$-algebras. Contemp. Math. 167, 241–291.

[6] Baum, P. and Douglas, R.G. (1982). $K$ homology and index theory. Proc. Sympos. Pure Math. 38, 117–173.

[7] Bayen, F., Flato, M., Fronsdal, C., Lichnerowicz, A. and Sternheimer, D. (1978). Deformation theory and quantization. I, II. Ann. Phys. (N.Y.) 110, 61–110, 111–151.

[8] Binz, E., J., Śniatycki, J. and Fischer, H. (1988). The Geometry of Classical Fields. North–Holland, Amsterdam

---

13 The best way to see this is to interpret $TF_i$ as the Lie algebroid of $G(V_i, F_i)$, and to pass to the canonical Poisson structure on the dual bundle $A^*(G)$ to the Lie algebroid $A(G)$ of any Lie groupoid $G$. 

12
[9] Blackadar, B. (1999). K-theory for Operator Algebras, 2nd ed. Cambridge University Press, Cambridge.

[10] Braverman, M. (1998). Vanishing theorems for the kernel of a Dirac operator. arXiv:math.DG/9805127

[11] Bursztyn, H., and Weinstein, A. (2003). Picard groups in Poisson geometry. Moscow Math. J., to appear. arXiv:math.SG/0304048

[12] Connes, A. (1982). A survey of foliations and operator algebras. Proc. Sympos. Pure Math. 38, 521–628.

[13] Connes, A. (1994). Noncommutative Geometry, Academic Press, San Diego.

[14] Connes, A. and Skandalis, G. (1984). The longitudinal index theorem for foliations. Publ. Res. Inst. Math. Sci. 20, 1139–1183.

[15] Dirac, P.A.M. (1964). Lectures on Quantum Mechanics. Belfer School of Science, Yeshiva University, New York.

[16] Dixmier, J. (1977). $C^*$-Algebras, North–Holland, Amsterdam.

[17] Duistermaat, J. J. (1996). The Heat Kernel Lefschetz Fixed Point Formula for the Spin-c Dirac Operator. Birkhäuser, Boston.

[18] Douglas, R.G. (1998). Banach Algebra Techniques in Operator Theory. Second edition. Springer-Verlag, New York.

[19] Gracia-Bondía, J.M., Várilly, J.C. and Figueroa, H. (2001). Elements of Noncommutative Geometry. Birkhäuser, Boston.

[20] Guillemin, V., Ginzburg, V. and Karshon, Y. (2002). Moment Maps, Cobordisms, and Hamiltonian Group Actions. American Mathematical Society, Providence, RI.

[21] Guillemin, V. and Sternberg, S. (1982). Geometric quantization and multiplicities of group representations. Inv. Math. 67, 515–538.

[22] Higson, N. (1990). A primer on $KK$-theory. Proc. Sympos. Pure Math. 51, Part 1, 239–283.

[23] Higson, N. and Roe, J. (2000). Analytic K-homology. Oxford University Press, Oxford.

[24] Hilsum, M. and Skandalis, G. (1987). Morphismes $K$-orientés d’espaces de feuilles et fonctorialité en théorie de Kasparov (d’après une conjecture d’A. Connes). Ann. Sci. 'Ecole Norm. Sup. (4) 20, 325–390.

[25] Karasev, M.V. (1989). The Maslov quantization conditions in higher cohomology and analogs of notions developed in Lie theory for canonical fibre bundles of symplectic manifolds. I, II. Selecta Math. Soviet. 8, 213–234; 235–258.

[26] Karoubi, M. (1978). K-theory: An Introduction. Springer-Verlag, Heidelberg.

[27] Kasparov, G.G. (1981). The operator $K$-functor and extensions of $C^*$-algebras. Math. USSR Izvestija SSSR 16, 513–572.

[28] Kasparov, G.G. (1988). Equivariant $KK$-theory and the Novikov conjecture. Invent. Math. 91, 147–201.

[29] Kucerovsky, D. (1997). The $KK$-product of unbounded modules. K-theory 11, 17–34.

[30] Le Gall, P. (1999). Théorie de Kasparov équivariante et groupoïdes. I. K-Theory 16, 361–390.
[31] Landsman, N.P. (1998). Mathematical Topics Between Classical and Quantum Mechanics. Springer, New York.

[32] Landsman, N.P. (2000). The Muhly-Renault-Williams theorem for Lie groupoids and its classical counterpart. Lett. Math. Phys. 54, 43–59. arXiv:math-ph/0008005

[33] Landsman, N.P. (2001). Quantized reduction as a tensor product. Ref. [36], pp. 137–180. arXiv:math-ph/0008004

[34] Landsman, N.P. (2002). Quantization as a functor. Contemp. Math. 315, 9–24. arXiv:math-ph/0107023

[35] Landsman, N.P. (2003). Deformation quantization and the Baum-Connes conjecture. Comm. Math. Phys. 237, 87–103. arXiv:math-ph/0210015

[36] Landsman, N.P., Pflaum, M. and Schlichenmaier, M., eds. (2001). Quantization of Singular Symplectic Quotients. Birkhäuser, Basel.

[37] Lawson, H. B. and Michelsohn, M.-L. (1989). Spin Geometry. Princeton University Press, Princeton, NJ.

[38] Marsden, J.E. and Weinstein, A. (1974). Reduction of symplectic manifolds with symmetry. Rep. Math. Physics 5, 121-130.

[39] Marsden, J.E. and Weinstein, A. (2001). Comments on the history, theory, and applications of symplectic reduction. Ref. [36], pp. 1–19.

[40] Meinrenken, E. (1998). Symplectic surgery and the Spin\(^c\)-Dirac operator. Adv. Math. 134, 240–277.

[41] Meinrenken, E. and Sjamaar, R. (1999). Singular reduction and quantization. Topology 38, 699–762.

[42] Meyer, K. (1973). Symmetries and integrals in mechanics. Peixoto, M.M., ed., Dynamical systems, pp. 259–272. Academic Press, New York.

[43] Mikami, K. and Weinstein, A. (1988). Moments and reduction for symplectic groupoids. Publ. RIMS Kyoto Univ. 24, 121–140.

[44] Moerdijk, I. (1993). Foliations, groupoids and Grothendieck etendues. Rev. Academia Cien-

[45] Mrčun, J. (1996). Stability and Invariants of Hilsum–Skandalis Maps, Ph.D. thesis, University of Utrecht.

[46] Paradan, P.-E. (2001). Localization of the Riemann-Roch character. J. Funct. Anal. 187, 442–509.

[47] Paradan, P.-E. (2001). Spin\(^c\) quantization and the K-multiplicities of the discrete series. arXiv:math.DG/0103222

[48] Paterson, A. (2002). The equivariant analytic index for proper group actions. Preprint.

[49] Pflaum, M. (2001). Analytic and Geometric Study of Stratified Spaces. Lecture Notes in Mathematics 1768. Springer, Berlin.

[50] Plymen, R. J. (1986). Strong Morita equivalence, spinors and symplectic spinors. J. Operator Theory 16, 305–324.

[51] Rieffel, M.A. (1974). Induced representations of C\(^\ast\)-algebras. Adv. Math. 13, 176–257.
[52] Rieffel, M.A. (1990). Lie group convolution algebras as deformation quantization of linear Poisson structures. Amer. J. Math. 112, 657–686.

[53] Sjamaar, R. (1996). Symplectic reduction and Riemann-Roch formulas for multiplicities. Bull. Amer. Math. Soc. (N.S.) 33, 327–338.

[54] Sjamaar, R. and Lerman, E. (1991). Stratified symplectic spaces and reduction. Ann. Math. 134, 375–422.

[55] Skandalis, G. (1991). Kasparov’s bivariant $K$-theory and applications. Exposition. Math. 9, 193–250.

[56] Sundermeyer, K. (1982). Constrained Dynamics. Lecture Notes in Physics 169. Springer, Berlin.

[57] Valette, A. (2002). Introduction to the Baum-Connes Conjecture. Birkhäuser, Basel.

[58] Valette, A. (2003). On the Baum-Connes assembly map for discrete groups. With an appendix by D. Kucerovsky. Preprint.

[59] Weinstein, A. (1983). The local structure of Poisson manifolds. J. Diff. Geom. 18, 523–557.