Stochastic Linear Bandits with Protected Subspace

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November 3, 2020

Abstract

We study a variant of the stochastic linear bandit problem wherein we optimize a linear objective function but rewards are accrued only orthogonal to an unknown subspace (which we interpret as a protected space) given only zero-order stochastic oracle access to both the objective itself and protected subspace. In particular, at each round, the learner must choose whether to query the objective or the protected subspace alongside choosing an action. Our algorithm, derived from the OFUL principle, uses some of the queries to get an estimate of the protected space, and (in almost all rounds) plays optimistically with respect to a confidence set for this space. We provide a $\tilde{O}(sd\sqrt{T})$ regret upper bound in the case where the action space is the complete unit ball in $\mathbb{R}^d$, $s < d$ is the dimension of the protected subspace, and $T$ is the time horizon. Moreover, we demonstrate that a discrete action space can lead to linear regret with an optimistic algorithm, reinforcing the sub-optimality of optimism in certain settings. We also show that protection constraints imply that for certain settings, no consistent algorithm can have a regret smaller than $\Omega(T^{3/4})$. We finally empirically validate our results with synthetic and real datasets.

1 INTRODUCTION

Consider the task of treating a disease characterized by some outlying biological marker. Often the medication necessary for treatment causes adverse side effects on other biological functionalities. During treatment, it is important to monitor such undesirable side effects by conducting various medical tests, while augmenting it with other medications to alleviate these adverse effects, and jointly calibrating the dosage of all these medications. Conducting tests may be expensive, thus it is desirable to find a treatment that has no side effects with efficient tests to optimally affect only the desired bio-marker. Such concerns are widespread in the treatment of disease - patients often receive multiple medications and the mitigation of drug related problems is a common concern, especially in the presence of comorbidities \cite{23}. Optimal blood pressure control, for instance, is described as a challenge in the treatment of type 2 diabetes \cite{23}, and antipsychotics prescribed for schizophrenia can result in side effects such as obesity, dyslipidemia and type 2 diabetes \cite{16}. Combination therapy (where a variety of medications are jointly prescribed) is often used to reduce the impact of adverse effects \cite{9}, and our work abstractly considers the problem of finding an optimal combination therapy guided by sequential medical testing during the course of a patient’s treatment to ensure recovery with the least cumulative side effects.

We approach this problem as an online decision making problem in which the results of various tests of bio-markers are regarded as bi-linear functions of treatments and patient characteristics. At each
round the physician may take an action (specify a therapy, dosages, schedules, etc.) \( a_t \) chosen from some given set \( \mathcal{A}_t \subset \mathbb{R}^d \). The physician has access to a test to monitor the result of the therapy, the result of which is given by \( X_t = \langle \theta_0, a_t \rangle + \eta_t \) with \( \eta_t \) representing some sub-gaussian noise, for some \( \theta_0 \in \mathbb{R}^d \). There may be other tests which should not be affected by the therapy (these test for side effects). Such tests are represented by \( \theta_i \in \mathbb{R}^d, i \in [L] \), and their outcomes are similarly sub-gaussian with mean \( \langle \theta_i, a_t \rangle \). In this setting, while the feedback from an action \( a_t \) is given by \( X_t \) above, the learners expected reward depends only on the components orthogonal to the protected space. That is, the expected reward given by \( \langle a_t, \theta_\bot \rangle \) where \( \theta_\bot \) is the component of \( \theta_0 \) orthogonal to \{\( \theta_i \)\}_{i \in [L]} \) is unseen. In some sense, this is the component of the therapy that does not contribute to side-effects. The objective is to minimize pseudo-regret, which is the total difference (that is, summed over all the rounds) between the expected reward obtained by a genie who knows the means of the outcomes of every test exactly for each patient, and the learner.

1.1 Contributions

1. We introduce the protected linear bandit as a model for online decision making with incomplete bandit feedback in which some subspace is considered to be protected, meaning that projections onto that space are subtracted from our reward. The optimal action is thus not the one that aligns most with the target vector, but rather the one which aligns most with the component of the target vector orthogonal to the unknown protected subspace. It is important to note that we do not have direct access to these projections, but only to inner products with individual vectors in the subspace.

2. We propose an algorithm for the above and derive an upper bound for its regret that grows as \( O(sd\sqrt{T}) \) in the number of rounds, similar to the best possible linear bandit regret for the case when the action space is the unit ball. The algorithm consists of two parts. First we remove redundancy in the set of protected vectors with a uniform exploration phase. We then restrict our attention to this independent set of constraints and play optimistically using an upper confidence bound based algorithm.

3. For general action spaces, we show that the partial feedback model can sometimes make it difficult for a learner. Example 5.1 shows an instance where naive optimism can lead to linear regret, and in Section 6 we show a \( \Omega(T^{2/3}) \) lower bound for any algorithm for a finite action (time-varying) space.

**Notation** We will denote by \( \text{Proj}_{\{\theta_i\}_{i \in [L]}} \) the projection operator onto the space spanned by \{\( \theta_i \)\}_{i \in [L]} and by \( \text{Proj}_{\{\theta_\bot\}_{i \in [L]}} \) the projection onto the orthogonal subspace. We use \( [L] = \{1, 2, \cdots, L\} \) to denote the set of the first \( L \) integers. We use \( ||x||_V \) to refer to the weighted norm \( \sqrt{x^T V x} \). Given a matrix \( P \in \mathbb{R}^{d \times L} \) (respectively, vector \( x \in \mathbb{R}^L \)) and a set \( S \subseteq [L] \), we denote by \( P_S \in \mathbb{R}^{d \times |S|} \) (respectively, \( x_S \)) the submatrix (vector) whose columns are the ones in \( P \) (\( x \)) indexed by \( S \). We denote the minimum eigenvalue by \( \lambda_{\text{min}}(\cdot) \). We denote by \( B^d_2 \) the unit 2-norm ball in \( \mathbb{R}^d \).

2 RELATED WORK

Multi armed bandits have been studied for decades at least since [18], and optimism in the face of uncertainty (OFUL) has proved to be an effective strategy in low regret algorithm design [4, 1]. We point an interested reader to [14] and references therein for works which are not directly related to ours.

Linear Bandits, where observable rewards are generated as the noisy inner products of actions and a hidden vector, were analyzed in [6, 1] and the regret of an optimistic algorithm was shown to grow as \( O(\sqrt{T \log T}) \) depending only on the dimension of the representation, independently of the number of arms. In our model, additional to the hidden vector (as in linear bandits) we have a
hidden protected space (spanned by multiple hidden constraint vectors). Our reward is the inner product of action and the component of the hidden vector orthogonal to the hidden protected space. Further, we do not observe the reward directly, instead we are allowed to make partial queries which, for diverse enough action space, can be used to infer the optimal action. Bandits with indirect access to rewards are studied under partial monitoring with finite [5,15], and infinite [5] action spaces. However, inferring optimal action in our model requires use of additional structure which absent in [3].

From a motivational standpoint, we share similarities with linear bandit with safety constraints where a learner is required to be safe. In [2], the authors study a linear bandit with known linear constraints where the actions should not violate these constraints. They propose an optimistic algorithm with initial safe exploration. This setting has been studied extensively, through the design of Thompson sampling based techniques [17], and extension to safe generalized linear bandits [3], safe contextual bandits [7], and safe reinforcement learning [10]. In a different model, [11] studied online learning where regret is constrained to be small compared to a known baseline. We differ from these works technically, as the constraints are unknown to us, unlike the above works. Further, we consider the protected space as reward shaping parameters, rather than hard constraints. Additionally, in the probabilistically approximately correct (PAC) learning framework, safety constrained optimization with unknown constraints and objectives with access to zeroth order oracles is studied in another line of work [20,21][8]. However, the convergence results in PAC-learning framework do not translate into regret minimization directly, as the former do not consider balancing exploration and exploitation.

We expand on the connections with Linear Bandits, Safety Constrained Linear Bandits, and Partial monitoring in Section 3.1.

3 MODEL

We consider a game between a player and a stochastic environment in which we have query access to $L + 1$ unknown vectors $\theta_0, \theta_1, \ldots, \theta_L \in \mathbb{R}^d$ with $\|\theta_i\|_2 \leq M$ for all $i$. The vectors $\theta_1, \theta_2, \ldots, \theta_L$, the protected vectors, are provided such that they span the protected subspace. In the context of our motivating problem, these represent low dimensional linear embeddings of the various tests for the biomarkers associated with side-effects. We are given a large number, $L$, of them, however they may represent a lower dimensional protected subspace of $\mathbb{R}^d$. We represent them also as columns of matrix $P(\theta_1, \theta_2, \ldots, \theta_L) \in \mathbb{R}^{d \times L}$. We refer to $\theta_0$ as the target vector. We would like to play arms that align as well as possible with $\theta_* = \text{Proj}_{\{\theta_i\}_{i \in [L]} \theta_0}$, that is, the orthogonal projection of $\theta_0$ onto the subspace orthogonal to the protected subspace. In the absence of the protected vectors, this would just be a linear bandit problem parameterized by $\theta_0$.

At any time $t$, the player can choose any action $A_t \in \mathcal{A}_t$, and an index $I_t \in \{0\} \cup [L]$, and receive a corresponding reward of $X_t = \langle A_t, \theta_{I_t} \rangle + \eta_t$ where $\eta_t$ is a conditionally $R$-subgaussian zero-mean noise.

**Regret:** The sub-optimality of action $a$, $\Delta_a$, is given by

$$\Delta_a = \langle a_t^* - a, \text{Proj}_{\{\theta_i\}_{i \in [L]} \theta_0} \rangle$$

where

$$a_t^* = \arg \max_{a \in \mathcal{A}_t} \langle a, \text{Proj}_{\{\theta_i\}_{i \in [L]} \theta_0} \rangle$$

is the optimal action. The goal is to minimize pseudo-regret with respect to a genie who is aware of the true vectors $\{\theta_i\}_{i \in \{0\} \cup [L]}$ (and so would play $a_t^*$ at each round):

$$R_T = \sum_{t \in [T]} \Delta_{a_t} = \sum_{t \in [T]} \langle a_t^* - a_t, \text{Proj}_{\{\theta_i\}_{i \in [L]} \theta_0} \rangle.$$
**Assumption 3.1.** The action space $A_t$ at all times consists of all vectors with unit norm, i.e. $A_t = B_d^2$.

This assumption is helpful due to the nature of the reward function. Finite action spaces with optimistic algorithms can sometimes lead to problems such as the one in Example 3.1. In fact, we show in Section 6 that a particularly bad action space must result in $\Omega(T^3)$ regret for a consistent algorithm.

Because of Assumption 3.1, the optimal action $a^*_t$ is just $\frac{\text{Proj}_{\{\gamma_i \mid i \in [L]\}} \theta_0}{\|\text{Proj}_{\{\gamma_i \mid i \in [L]\}} \theta_0\|_2}$ at all time. This is the normalized projection of $\theta_0$ onto the space orthogonal to the protected subspace.

**Assumption 3.2.** There exists a subset $S \in [L]$ of size $|S| = s$ such that $\lambda_{\min}(\sum_{i \in S} \theta_i \theta_i^T) > 0$, while any larger set $S'$ has $\lambda_{\min}(\sum_{i \in S'} \theta_i \theta_i^T) = 0$. We assume knowledge of $s$.

This says that there is a $s$ dimensional subspace that contains all of the protected vectors. Our regret bounds will be in terms of $s$ rather than $L$.

We denote the greatest such $\lambda_{\min}(\sum_{i \in S} \theta_i \theta_i^T)$ (over all choices of $S$ with $|S| = s$) simply as $\lambda_{\min}$. This indicates the best spanning set of protected vectors.

Let $\mathcal{F}_t = \sigma(A_1, A_2, \cdots, A_t, \eta_1, \eta_2, \cdots, \eta_t)$ denote the $\sigma$-algebra generated by all actions and noises up to and including time $t$.

**Assumption 3.3.** The noise on the observed feedback, $\eta_t$, from taking an action is conditionally zero-mean $R$-subgaussian, meaning $E[\|\eta_t\|_{\mathcal{F}_{t-1}}] = 0$ and $E[e^{\lambda_{\max}} \|\eta_t\|_{\mathcal{F}_{t-1}}] \leq e^2 \lambda^2 R^2$.

This is standard, and used for deriving concentrations for the confidence sets for the unknown parameters.

### 3.1 Differences from Related Models

**Linear Bandits:** The standard linear bandit problem considers minimizing regret while learning a single unknown vector [6, 1] without other protected directions. In our setting, the regret depends on several unknown vectors; however, in each round, we only get a signal from one. As such, the noisy observations that are derived from the player’s action $\{X_s\}_{s \in [T]}$ do not immediately give us the sub-optimality of an action. When a player plays action $(A_t, I_t)$, it observes $X_t = \langle A_t, \theta_t \rangle + \eta_t$ and incurs regret $\langle a^*_t - A_t, \text{Proj}_{\{\gamma_i \mid i \in [L]\}} \theta_0 \rangle$. In particular, the player does not see a noisy version of $\langle A_t, \text{Proj}_{\{\gamma_i \mid i \in [L]\}} \theta_0 \rangle$. So aside from choosing the arm to pull, a player must also choose which vector to query with that arm. The analysis is further obscured by the fact that the rewards are a non-linear function of the unknown parameters. Finally, letting the set of protected vectors be empty ($L = 0$) recovers the standard linear bandit, so our setting is a generalization.

**Safety-constrained Linear Bandits:** Safety-constrained bandits, studied, for instance, in [11, 2], are typically supposed to guarantee a safety constraint with high probability at each round. For instance, [11] require that the cumulative regret of a learner not exceed the regret of a baseline learner by more than a small multiplicative factor. [2] have a safety constraint that is a geometric constraint on the arms that can be played at each round. Aside from maximizing the cumulative regret against $a^*_t \theta_0$, they have a known matrix $B$ and known constant $c$ such that the arm they pull at each round $a_t$ must satisfy $a^*_t B \theta_0 < c$ with high probability for some safety threshold $t$. Both of these are essentially constraints on the exploration of a learner. In contrast, we do not enforce any explicit exploration constraint. Rather, the difficulty of our problem is to learn the safety constraints simultaneously with the objective. Moreover, the aforementioned works (i) typically consider a single safety constraint as opposed to multiple, unknown directions $\{\theta_i\}_{i \in [L]}$, and (ii) they crucially assume ‘free’ access to an observation of the constraint violation at each action round, leading to very rapid learning of the linear constraint halfspace; in our setting, the exploration of the constraint/protection is partial (learn about one of the $\theta_i$) and has to be adaptively decided.
**Linear Partial Monitoring:** A reduction to the linear partial monitoring framework in [12], although possible, results in linear regret with existing guarantees. [12] provide a regret spectrum based on how informative the action space is, and derive a linear minimax bound for regret on games that are not globally observable. The following is a reduction to the linear partial monitoring setting.

Let \( \theta_\perp = \text{Proj}_{1}^{\perp} \theta_0 \). We may take \( \theta = e_{L+1} \otimes \theta_\perp + \sum_{i=0}^{L} e_i \otimes \theta_i \). An action \((i, a) \in [L] \times A\), is encoded as \( e_{L+1} \otimes a \), while \( A_{(i, a)} \) is taken to be \( e_i \otimes a \). The partial monitoring game described here is not globally observable, hence gives linear regret, since for all \( a_1, a_2 \in A \), we have \( e_{L+1} \otimes (a_1 - a_2) \notin \text{Span}_{i \in [L], a \in A} A_{(i, a)} \).

To overcome this difficulty, we leverage crucially the structure in \( \theta \) (specific to our problem), that the first \( d \) coordinates of \( \theta \) are actually a known function of the last \( (L + 1)d \) coordinates (a projection).

### 4 PROTECTED LIN-UCB

In this section, we present an algorithm for the regret minimization problem described in Section 3. Our algorithm, Algorithm 2, is developed following the Optimization in the Face of Uncertainty (OFU) principle [1], where we play optimistic actions that maximize the reward with high probability. For that purpose, we maintain and continually refine respective high probability confidence sets for the reward vector, and a subset of protected vectors that spans the protected space, namely the core set. As the dimension of the protected space is assumed to be known to be \( s \), it is possible to find a set of \( s \) protected vectors that span the space, and any additional vectors need not be considered. In the first phase of the algorithm, we use Algorithm 1 to reduce the number of relevant unknown vectors in an approximately optimal way.

**Coreset Estimation:** Because we need only concern ourselves with a spanning set of protected vectors, we first use the CORE-SET procedure to prune the set of protected vectors. We cannot simply pick \( s \) of the protected vectors arbitrarily, as these may not span the whole protected space, and even if they do, they may span the space inefficiently. We do this with a deterministic, isotropic phase in which we sample every unknown vector uniformly in every direction in a round robin manner until we are certain that some subset is within a multiplicative factor of being optimal.

**Algorithm 1:** CORE-SET for rank \( k \)

1. \( t \leftarrow 1; \)
2. while \( \forall S \subseteq [L], |S| = k, \lambda_{\min}(\sum_{i \in S} \hat{\theta}_i \hat{\theta}_i^T) \leq \frac{16L(R+R)(d \log 6 + \log \frac{1}{\delta})}{\sqrt{t}} \) do
   3. Query the standard basis for each of the protected vectors \( \{\theta_i\}_{i \in [L]} \) and update \( \{\Theta_i\}_{i \in [L]} \);
   4. \( t \leftarrow t + 1; \)
5. end
6. return \( \arg \max_{S \subseteq [L], |R| = k} \lambda_{\min}(\hat{P}_S) \), \( t \)

From this we get a set \( \tilde{S} \) for which with high probability, we have

\[
\lambda_{\min}(\sum_{i \in \tilde{S}} \theta_i \theta_i^T) \geq \frac{1}{3} \max_{S \subseteq [L]} \lambda_{\min}(\sum_{i \in S} \theta_i \theta_i^T).
\]

This is our notion of being optimal within a multiplicative factor. We restrict our attention to this set.

**Protected LinUCB:** For all \( i \in \tilde{S} \), we maintain one such ellipsoid \( \Theta_i \) centered at \( \hat{\theta}_i \) for each of the unknown vectors in the manner of the OFUL lin-UCB algorithm from [1]. We use these to infer a confidence interval for \( \langle a, \theta_\perp \rangle \). These sets are such that each of the unknown vectors is contained within their respective confidence sets at every round with high probability. We refer
to [1] for a detailed discussion on how such confidence sets are constructed, or Remark (4.1) for a summary. To keep track of the exploration for each we denote by \( T_{i,t} \) the number of times we have queried vector \( i \) until and including time \( t \), \( T_{i,t} = \sum_{s \leq t} \mathbf{1}_{I_s=i} \) and by \( V_{i,t} \) the distribution of the actions, \( V_{i,t} = \sum_{s \leq t} \mathbf{1}_{I_s=i} a_s^T \). We then play optimistically with respect to these confidence sets. Concretely, we maximize over all actions \( a_t \in A_t \) and all possible \( \bar{\theta}_i \in \Theta_i, i \in \tilde{S} \) and \( \bar{\theta}_0 \in \Theta_0 \) the value of \( \langle a_t, \text{Proj}_{\{\bar{\theta}_i \mid i \in \tilde{S}\}} \bar{\theta}_0 \rangle \). Note that the confidence set for \( \theta_\perp \) is not a geometric ellipsoid, and characterizing its shape exactly is quite difficult (see also Section 5.1). In each round a player must also chose an index determining the particular protected vector to be queried, and we make this decision based on which vector is least explored in the direction of the selected action.

**Remark 4.1.** The method described in [1] to construct confidence sets is as follows. After \( t \) rounds, suppose we have queried with arms \( \{a_s \}_{s \in [t]} \) and received rewards \( \{x_s = \bar{\theta}^T a_s + \eta_s \}_{s \in [t]} \). We use these to determine the maximum likelihood estimate

\[
\hat{\theta}_t = (\sum_{s \in [t]} a_s a_s^T + \rho I)^{-1}(\sum_{s \in [t]} a_s x_s) \tag{1}
\]

If the actions \( a \in A \) also satisfy \( ||a|| \leq M \), then Theorem 2 of [1] establishes that with probability \( 1 - \delta \),

\[
||\hat{\theta}_{i,t} - \theta_i||_{V_{i,t}} \leq \sqrt{\beta_{T_{i,t}}} \tag{2}
\]

where

\[
\sqrt{\beta_t} = R \sqrt{d \log \left( \frac{1 + t M^2 / \bar{\lambda}}{\delta} \right) + \lambda \sqrt{M}}.
\]

**5 REGRET UPPER BOUND FOR SPHERICAL ACTION SPACE**

In this section, we derive an upper bound on the regret of Algorithm 2. The algorithm begins by constructing a core-set \( \tilde{S} \) of the protected vectors that optimally span the protected subspace. This core-set has cardinality \( |\tilde{S}| = s \), the known dimension of the protected subspace and is constructed by paying a constant exploratory regret. Here we assume \( M, R, \rho = 1 \), but the results presented in the appendix are such that the dependence on these parameters is explicit.

We denote by \( \hat{\theta}_i \) the MLE estimate as in [1]. For a set \( S \subseteq [L] \) let \( P_S = \sum_{i \in S} \theta_i \hat{\theta}_i^T, \hat{P}_S = \sum_{i \in \tilde{S}} \hat{\theta}_i \hat{\theta}_i^T \), and by \( \lambda_{\text{min}}(\hat{P}_S) \) the minimum singular value of \( \hat{P}_S \). We have the following theorem that allows us to get a spanning set of protected vectors that span the protected space approximately optimally.

**Theorem 5.1.** **CORE-SET** terminates in at most \( t_0 = \frac{2304 L^2 (d \log 6 + \log \frac{L}{\delta})^2}{\lambda_{\text{min}}^2} \) iterations of the outer loop and returns a subset \( \tilde{S} \) such that, with probability at least \( 1 - \delta \), \( \lambda_{\text{min}}(\sum_{i \in \tilde{S}} \theta_i \hat{\theta}_i^T) \geq \frac{\lambda_{\text{min}}}{3} \).
Proof sketch. We establish error bounds on the protected vectors in Lemma A.2 and use these to bound the perturbation of the eigenvalues from a spanning set in Lemma A.3 (see Appendix A for details).

Once the core-set is found, we play optimistically with respect to confidence sets derived from estimates that only include the core-set vectors, reducing the number of parameters we need to learn. We have the following high probability regret bound for Algorithm (2):

**Theorem 5.2.** If we have $\mathcal{A}_t = B^d_2$, the regret of Algorithm (2) satisfies

$$\mathcal{R}_{[T]} \leq 12\sqrt{2^s + 1}\sqrt{Td \log(1 + \frac{TL}{d})}\sqrt{\beta_T\left(\frac{\delta}{2(L + 1)}\right)} + \frac{4608L^3d(d \log 6 + \log \frac{2L}{\delta})^2}{\lambda_{\min}^2}$$

with probability $1 - \delta$ where

$$\sqrt{\beta_T(\delta)} = R\sqrt{d \log(\frac{1}{\delta} + \frac{M^2}{\delta\rho})} + M\rho^\frac{1}{2}.$$

5.1 Key Ideas and Proof Sketch

We now describe the key ideas behind our algorithms and the main result, Theorem 5.2. Given only stochastic zero-order access to vectors $\{\theta_i\}_{i \in [L]}$, we must play the arm $a_t \in \mathcal{A}_t$ which maximizes $\langle a_t, \text{Proj}_{\{\theta_i\}_{i \in [L]}} \theta_0 \rangle$. Suppose for all $i \in L$, we know that the unknown vector $\theta_i$ was in some confidence set $\{\Theta_i\}$ with high probability (for example, by linear regression). Then, let the set of all possible $\theta_\perp$ be denoted $\Theta_\perp$ where each member is derived from a specific choice of $\{\theta_i\}_{i \in [L]}$ consistent with $\{\Theta_i\}_{i \in [L]}$. Clearly, this contains the true $\theta_\perp$ with high probability. Meanwhile, if we chose to play that action that gave us the maximum reward under any choice of $\theta_\perp \in \Theta_\perp$ then sub-optimality of an action is upper bounded by the uncertainty in the mean reward for that action, so a complete characterization of $\Theta_\perp$ would directly lead to a regret bound.

Concretely, suppose we keep track of the extent of exploration for each parameter using $V_{i,t} = \rho I + \sum_{s \leq t} 1_{a_s = a_t} a_s a^T_s$ (where $\rho$ is a regularization factor). It would be sufficient to upper bound $\Delta_{a_t}$ by some terms of the form $\sum_{i \in S} O(\log T)||a_t||V_{i,t}^{-1}$ that represent the exploration in the direction of $a_t$.

However, explicitly constructing $\Theta_\perp$ in the standard way as in [1] (or indeed, bounding $\Delta_{a_t}$ as above) presents new problems. To see why, in the standard linear bandit, for arm $a$ and the optimal parameter $\theta_0$, pulling arm $a$ repeatedly reduces uncertainty of $\theta_0$ in the direction of $a$. However, the object of our interest is $\text{Proj}_{\{\theta_i\}_{i \in [L]}} \theta_0$, i.e. the space orthogonal to the protected vectors. Thus, (i) the component of $a$ that lies in the protected space is not informative because any reduction in variance of a protected vectors in the span of the protected space does not change the variance of our estimate of the protected space, and (ii) the true reward depends on the protected vectors only through the space they span and not the vectors themselves. As such, it is not true that getting even infinite samples from an arm allows us to compute its mean reward with high confidence. Instances in which this fundamentally changes the regret bounds are presented in Sections 5.3 and 6. Thus, we do not attempt to construct a high probability $\Theta_\perp$ set. Instead, we find a confidence interval for the mean reward of only along the direction of the optimistic action.

Instead, for any $\tilde{\theta}_i \in \Theta_i$, and for the optimistic action $a$, we bound $\langle a, \text{Proj}_{\{\theta_i\}_{i \in [L]}} \text{Proj}_{\{\theta_i\}_{i \in [L]}} \theta_0 \rangle$ and show that this is very small using self-adjointness and idempotence of projection operators. We can now propagate the errors in the protected vectors linearly through our estimates of the subspace, thus crucially preserving sub-Gaussianity of noise. Please see Lemma B.3 in Appendix B for precise details.
5.2 Remarks

Here we discuss some of the key terms of the regret bound presented in Section 5.2.

**Remark 5.1** (Comparison with OFUL algorithm for Lin-UCB in [1]). The regret of the OFUL algorithm satisfies

\[ R_{T}^{L-UCB} \leq 4\sqrt{Td\log(1 + TL/d)}\sqrt{\beta_t(\delta)} \]

with probability \(1 - \delta\) where

\[ \sqrt{\beta_t(\delta)} = R\sqrt{d\log\left(\frac{1}{\delta} + \frac{TM^2}{\rho^2}\right) + M\rho^\frac{1}{2}}. \]

In comparison, our regret has a multiplicative \( \frac{s + 1}{\lambda_{\min}} \) factor. This comes from the fact that our rewards now depend on \(s + 1\) unrelated vectors. The dependence on \(\lambda_{\min}\) comes from the way perturbations of vectors affect perturbations of the space they span.

**Remark 5.2** (Dependence on \(A_t\)). An important way this model differs from that of the standard bandit problems is that we do not actually have bandit feedback, and never directly learn the rewards our arms earn. We resolve this by estimating the protected subspace in an online manner and constructing a “confidence set” for it that is related to the confidence ellipsoids of the vectors that span it. We study how feedback from the component vectors affects our confidence in the estimate of \(\theta_\perp\). A subspace estimate is constructed, for instance, in [13] for the setting in which the vectors spanning the subspace are queried i.i.d isotropically, i.e. with \(d\mathbb{E}[a_i a_i^T] = I\). This does not suffice for our purpose as we would like to be able to query the vectors adaptively. We elaborate on this restriction of the action space in Sections 6 and 5.3.

**Remark 5.3** (Knowledge of \(s\)). If \(s < L\), it is desirable to have regret that scales as \(s\) and not \(L\). This raises an additional difficulty, as demonstrated by the following example.

Suppose in the first instance, \(\theta_0 = [1, 1, 1], \theta_1 = [1, 0, 0], \theta_2 = [1, 0, 0]\), while in the second \(\theta_0 = [1, 1, 1], \theta_1 = [1, 0, 0], \theta_2 = [1, \Delta, 0]\). The true subspace dimension in the first is 1, while in the second, it is 2. The ideal action, \(\theta_\perp\) is \([0, 1, 1]\) in the first, while it is \([0, 0, 1]\) in the second.

For small \(\Delta\), it is difficult to decide between these, and deciding incorrectly leads to a sub-optimality that does not go to 0 as \(\Delta \to 0\). Note that this is very different from the analogous issue in the standard bandit problems, for example, in the multi-arm bandit (MAB), where a separation of \(\Delta\) leads only to a sub-optimality of \(\Delta\). To further complicate matters, such a suboptimality in a MAB is addressed as directly as possible by sampling the relevant arms of the bandit. In our case, the separation is in a direction orthogonal to \(\theta_\perp\), the direction we need to exploit. To avoid this, we assume knowledge of the dimension of the protected subspace dimension, and we are left with only the problem that a poorly described subspace, such as the one in the example, could amplify regret by a factor of \(\frac{1}{\lambda_{\min}}\).

If the protected subspace dimension \(s\) is unknown, but there is a lower bound on the separation parameter \(\lambda_{\min}\), we can modify Algorithm 2 which uses a stopping criterion based upon \(\lambda_{\min}\) rather than a subspace dimension to get a similar regret bound.

**Remark 5.4** (Solving the optimization problem in Line 5 of Algorithm 2). It is not obvious how to actually solve the optimisation problem in line 4 of the algorithm, because this is a maximization of a function that is not concave. In Appendix D we describe a simple way to solve this optimization explicitly for a fixed arm (that is, how to get the optimal \(\theta_0\) and \(\theta_1\) for a fixed \(a_t\)) if \(A_t = B_2^d\).

5.3 The Failure of Naive Optimism

A study of this algorithm reveals an interesting phenomenon. While Theorem 7 demonstrates a regret bound that scales in \(T\) as \(\tilde{O}(sd\sqrt{T})\) if we set the action space \(A_t\) to always be the unit ball \(B_2^d\), we also note in Section 6 that no consistent algorithm can do better than \(\Omega(T^{\frac{1}{2}})\) with no restriction on the action space. In fact, the naive optimism of Algorithm 2 can get stuck with linear regret, as demonstrated in the following example.
Example 5.1. For ease of notation, let \( u_\alpha \) denote the point \((\cos \alpha, \sin \alpha)\). Consider a problem with \( d = 2, L = 1 \), where \( \theta_0 = u_\frac{\pi}{4}, \theta_1 = u_0 \). For simplicity, suppose the player knows \( \theta_0 \) exactly. Suppose that at all times the player is given the choice of actions \( \mathcal{A}_t = \{a_1, a_2\} \) where \( a_1 = u_\frac{\pi}{4} \) and \( a_2 = u_\frac{\pi}{2} \).

Suppose at round \( t \), the vector \( \vec{\theta}_1 = u_0 + u_{-\frac{\pi}{4}} \) was in the confidence set for \( \theta_1 \), that is,

\[
||\vec{\theta}_1 - \theta_1||_{V_{i,t}} = ||u_{-\frac{\pi}{4}}||_{V_{i,t}} \leq \sqrt{\beta_{T,i}}.
\]

Then an optimistic evaluation of \( a_1 \) is at least as good as the evaluation that uses \( \vec{\theta} = u_0 + u_{-\frac{\pi}{4}} \). With this as the protected vector, the evaluation of \( a_1 \) is \( \cos^2 \frac{\pi}{8} \). Meanwhile, the evaluation of action \( a_2 \) can never exceed \( \cos^2 \frac{\pi}{8} \). An optimistic player will play \( a_1 \) at round \( t + 1 \). There is no hope of the player learning any better in the future, since \( \vec{\theta}_1 \) remains in the confidence ellipsoid

\[
||\vec{\theta}_1 - \theta_1||_{V_{i,t+1}} = ||u_{-\frac{\pi}{4}}||_{V_{i,t+1}} = \sqrt{||u_{-\frac{\pi}{4}}||_{V_{i,t}}^2 + \langle u_{-\frac{\pi}{4}}, u_{\frac{\pi}{4}} \rangle^2} \\
= ||u_{-\frac{\pi}{4}}||_{V_{i,t}} \leq \sqrt{\beta_{T,i}} \leq \sqrt{\beta_{T,i+1}}
\]

and so the learner will just play \( a_1 \) again. Such a learner suffers linear regret under a naively optimistic policy.

In this situation, because of the indirect nature of the true rewards, playing a sub-optimal action does not necessarily make us less optimistic about its mean reward.

6 REGRET LOWER BOUND FOR FINITE ACTION SPACE

In this section, we establish the difficulty of the protected linear bandit problem. Note that section 3 provides a \( O(\sqrt{T \log T}) \) upper bound on the regret of Algorithm 2 when the actions space is \( B_2^d \).

We suggested in section 4 that an adversarial action space could make the problem much harder. Here we provide a lower bound for the regret of any algorithm on a specially chosen instance.

Theorem 6.1. There is an instance of the Protected Linear Bandit problem such that any algorithm incurs a regret of \( \Omega(T^\frac{4}{3}) \).

Proof sketch. Consider a pair of instances, denoted with superscripts (1) and (2). For both, we set our ambient space to have dimension \( d = 2 \), and set \( s = L = 1 \). We denote by \( u_\alpha \in \mathbb{R}^2 \) the vector \((\cos \alpha, \sin \alpha)\). Take \( \alpha = T^{-\frac{1}{4}} \). We set \( \theta_0^{(1)} = \theta_0^{(2)} = u_\frac{\pi}{4} - \alpha \). In instance (1), we set \( \theta_1^{(1)} = u_0 \) while in instance (2), we set \( \theta_1^{(2)} = u_{-\alpha} \). In both instances, in each round, we allow the player an action space that consists of either the actions \( \{u_{-\alpha}, u_{2\alpha}\} \) or \( \{u_{-3\alpha}, u_{2\alpha}, u_{-3\alpha}\} \) with equal probability. These instances are chosen such that \( u_{2\alpha} \) is always optimal for the second instance, while whenever \( u_{-3\alpha} \) is available, it is optimal for the first instance. The event in which \( u_{-3\alpha} \) is picked more than half the times it is available must thus have a high probability under the interaction between the algorithm with the first instance and a low probability in the interaction with the second instance. The Bretagnolle-Huber inequality \([14]\) allows us to control the maximum difference in this probability by the KL divergence induced by the different interactions, which we prove to be bounded by a constant. The complete proof is given in Appendix \([\text{a}]\).
In this section, we validate our theoretical results with simulations on a synthetic instance, and an instance derived from the Warfarin dataset [22] that consists of clinical and pharmacogenetic data on Warfarin dosage in the presence of other medications. We compare our results with an \( \epsilon \)-Greedy baseline which we invent, as no other baseline is present for our problem. Our algorithm outperforms the \( \epsilon \)-Greedy algorithm, unless it is tuned with the knowledge of the instances and time horizons. For both experiments, we perform 10 parallel runs, and report the cumulative regrets (average, and average \( \pm 1 \times \) standard deviation).

Baseline Algorithm (\( \epsilon \)-Greedy): We compare our algorithm with a \( \epsilon \)-Greedy baseline that plays a pure exploration arm (that is, queries a uniformly random vector with a uniformly random action) with probability \( \frac{\epsilon}{\sqrt{t}} \) and uses these samples to estimate (using MLE) the protected and target vectors, and otherwise plays a pure exploitation based on these estimates.

Synthetic Data: A problem instance was generated randomly by drawing vectors randomly from \( \mathcal{N}(0, I_d) \) for \( d = 3 \) in each round which are then normalized. We set \( L = 2 \) and \( s = 1 \). We have set the regularization parameter \( \lambda = 0.1 \) and the failure probability \( \delta = 0.001 \). The regret due to the interaction of the player and the instance over \( T = 1000 \) rounds is plotted.

Warfarin Dataset: We consider the Warfarin dataset [22] and construct an instance to optimize Warfarin dosage in our setting. This dataset consists of dosages of Warfarin (an anticoagulant prescribed for Deep Vein Thrombosis, Stroke, Cardiomyopathy, etc) and other medications (‘Simvastatin’, ‘Atorvastatin’, ‘Fluvastatin’, etc.) as well as the resulting INR (International Normalized Ratio which indicates susceptibility to bleeding - this is provided as a number between roughly 1 and 4) and stability of Warfarin therapy (this is provided as a Boolean).

In this context, we consider the task of optimizing a therapy consisting of some combination of these medications to get optimal Stability while minimally affecting deviation from the normal range of INR (defined to be 2.5). We model the therapy (combination of medications) as a unit norm vector in \( \mathbb{R}^8 \) (interpreted as the dosages of each of 8 medications). We model the two tests (INR and Stability) as linear functions of the therapy vectors, and find their value (test vectors in \( \mathbb{R}^8 \)) offline using linear regression for INR, and logistic regression for Stability.

We then construct a Protected Linear Bandit instance, where all the available therapy records comprise the action space (i.e. \( d = 8 \) and 1832 arms), the INR test vector acts as the protected vector \( \theta_1 \) (i.e. \( L = s = 1 \)), and the Stability test vector acts as the reward vector \( \theta_0 \). We set \( \rho = 1 \), and \( \delta = 0.001 \) in Algorithm 2 and simulate the system for 3 parallel runs each with \( T = 1000 \) time steps.
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A Proof of CORE-SET Estimation

For this section, because we are choosing our matrix deterministically, we need not use the self-normalized bounds of [1]. We also need not use a regularization parameter because after a single round of querying a standard basis we will have an invertible $V_{i,t}$.

Let for some subset $S \subseteq [L]$, let $P_S = \sum_{i \in R} \theta_i \theta_i^T$, and let $\hat{P}_R = \sum_{i \in R} \hat{\theta}_i \hat{\theta}_i^T$. We denote by $\lambda_{\min}(P)$ the minimum eigenvalue of $P$.

**Theorem A.1.** $A$ terminates in at most $\frac{576L^2R^2(M+R)^2}{\lambda_{\min}^2}\left(\frac{d \log 6 + \log \frac{1}{\delta}}{T}\right)^2$ iterations of the outer loop and returns a subset $S$ such that, with probability at least $1 - \delta$, $\lambda_{\min}(\hat{P}_S) \geq \frac{1}{3} \max_{S \subseteq [L], |R| = k} \lambda_{\min}(P_S)$.

**Lemma A.2.** Suppose we sample each of the $L$ protected vectors using an orthonormal set of actions $T$ times for a total of $dLT$ isotropic samples. Then we have

$$||\hat{\theta}_i - \theta_i||_2 \leq 2R\sqrt{\frac{d \log 6 + \log \frac{1}{\delta}}{T}}$$

and thus

$$||(\hat{\theta}_i - \theta_i)(\hat{\theta}_i - \theta_i)^T||_2 \leq 8R^2\frac{d \log 6 + \log \frac{1}{\delta}}{T}$$

with probability at least $1 - L\delta$ for every $i \in [L]$.

**Proof.** From (20.3) of [14], each estimate $\hat{\theta}_i$ of $\theta_i$ satisfies with probability at least $1 - L\delta$

$$||\hat{\theta}_i - \theta_i||_{V_i, T_d} \leq 2R\sqrt{\frac{d \log 6 + \log \frac{1}{\delta}}{T}}$$

$$\Rightarrow \sqrt{T}||\hat{\theta}_i - \theta_i||_2 \leq 2R\sqrt{\frac{d \log 6 + \log \frac{1}{\delta}}{T}} \quad \text{by } V_i, T_d = TI \text{ by construction}$$

$$\Rightarrow ||\hat{\theta}_i - \theta_i||_2 \leq 2R\sqrt{\frac{d \log 6 + \log \frac{1}{\delta}}{T}}$$

$$\Rightarrow ||(\hat{\theta}_i - \theta_i)(\hat{\theta}_i - \theta_i)^T||_2 \leq 8R^2\frac{d \log 6 + \log \frac{1}{\delta}}{T} \quad \text{by } ||vv^T||_2 = v^Tv \text{ for any column vector } v \square$$

**Lemma A.3.** If we run CORE-SET for $T$ iterations of the outer loop, then for any $S \subseteq [L]$ we have

$$||\hat{P}_S - P_S||_2 \leq 8LR(M + R)\frac{d \log 6 + \log \frac{1}{\delta}}{\sqrt{T}}$$

with probability $1 - L\delta$.

**Proof.** This follows from explicit lower bounds we get for exploration from CORE-SET. With probability $1 - \delta$:

$$||\hat{P}_S - P_S||_2$$

$$= ||\sum_{i \in [L]} (\hat{\theta}_i \hat{\theta}_i^T - \theta_i \theta_i^T)||_2$$

$$= ||\sum_{i \in [L]} (\hat{\theta}_i (\hat{\theta}_i - \theta_i)^T + (\hat{\theta}_i - \theta_i) \theta_i^T)||_2$$
Definition of Rayleigh Quotient applied to the symmetric matrix and consider a symmetric noise matrix. We also have the following eigenvalue perturbation result.

Proof. Let this contradicts the termination condition, we know that the algorithm terminates in no more than rounds of the outer loop. On the other hand, because the algorithm hasnt terminated, we must have the termination condition for the algorithm is now 1

\[ \lambda_{\min}(P + R) \leq \lambda_{\min}(P) + \frac{\delta}{2} \]

by submultiplicativity of \| \cdot \|_2 norm

\[ \leq 2 \sum_{i \in [L]} \| \theta_i \|_2 \| \hat{\theta}_i - \theta_i \|_2 + \| \sum_{i \in [L]} (\hat{\theta}_i - \theta_i) (\hat{\theta}_i - \theta_i)^T \|_2 \]

by triangle inequality

\[ \leq 2 \sum_{i \in [L]} \| \theta_i \|_2 \| \hat{\theta}_i - \theta_i \|_2 + \sum_{i \in [L]} \| (\hat{\theta}_i - \theta_i) (\hat{\theta}_i - \theta_i)^T \|_2 \]

by Lemma A.2

\[ \leq 4 \sum_{i \in [L]} \| \theta_i \|_2 R \sqrt{\frac{d \log 6 + \log \frac{1}{\delta}}{T}} + \sum_{i \in [L]} 8 R^2 d \log 6 + \log \frac{1}{\delta} \]

by ||\theta_i||_2 \leq M

\[ \leq 4 LMR \sqrt{\frac{d \log 6 + \log \frac{1}{\delta}}{T}} + 8 LR^2 \frac{d \log 6 + \log \frac{1}{\delta}}{T} \]

by \( T, d, \frac{1}{\delta}, L \geq 1 \)

We also have the following eigenvalue perturbation result.

Lemma A.4. Let \( \lambda_{\min}(P) \) denote the minimum eigenvalue of symmetric matrix \( P \) with \( P \in \mathbb{R}^{d \times d} \), and consider a symmetric noise matrix \( E \in \mathbb{R}^{d \times d} \). Then

\[ \lambda_{\min}(P + E) \geq \lambda_{\min}(P) - ||E||_2. \]

Proof. Let \( \arg \min_{v: ||v||_2 = 1} v^T (P + E) v = \hat{v} \)

\[ \lambda_{\min}(P + E) = \min_{v: ||v||_2 = 1} v^T (P + E) v \]

\[ = \hat{v}^T P \hat{v} + \hat{v}^T E \hat{v} \]

\[ \geq \lambda_{\min}(P) + \hat{v}^T E \hat{v} \]

\[ \geq \lambda_{\min}(P) - ||E||_2 \]

(a): Definition of Rayleigh Quotient applied to the symmetric matrix \( P \). (b): for any \( v: ||v||_2 = 1 \), by Cauchy Schwartz, \( |v^T E v| \leq ||v||_2 ||E||_2 ||v||_2 \). For any scalar \( a, |a| < c \) for some \( c > 0 \), then \( a \geq -c \).

Proof of Theorem 5.1. We will use the shorthand \( \alpha \) to denote the constant \( 8 LMR(M + R)(d \log 6 + \log \frac{1}{\delta}) \) in Lemma A.3, so that we have \( ||\hat{P}_S - \hat{P}_S||_2 \leq \frac{\alpha}{\sqrt{T}} \) after \( T \) iterations of the outer loop. Also, the termination condition for the algorithm is now \( \lambda_{\min}(\hat{P}_{S'}) \geq \frac{2 \alpha}{\sqrt{T}} \) for some \( S' \subseteq [L] \).

Take \( S = \arg \max_{S \subseteq [L], |S| = \lambda_{\min}(P_S)} \lambda_{\min}(P_S) \). By Lemma A.4 after \( T = \frac{\alpha}{\lambda_{\min}(P_S)} \) rounds, we have with probability at least \( 1 - L \delta \) that

\[ \lambda_{\min}(P_S) - \lambda_{\min}(\hat{P}_S) \leq \frac{\alpha}{\sqrt{\lambda_{\min}(P_S)^3}} \]

On the other hand, because the algorithm hasnt terminated, we must have \( \lambda_{\min}(\hat{P}_S) \leq \frac{\alpha}{\sqrt{T}} \). Since this contradicts the termination condition, we know that the algorithm terminates in no more than \( \frac{9 \alpha^2}{\lambda_{\min}(P_S)} \) rounds of the outer loop.

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Suppose whenever we terminate we output \( \hat{S} \subset [L] \), \( \hat{S} = \arg \max_{S' \in [L]} \lambda_{\text{min}}(\hat{P}_{S'}) \). Then we have 
\[
\lambda_{\text{min}}(P_S') \geq \lambda_{\text{min}}(\hat{P}_{S'}) - \frac{\alpha}{\sqrt{T}} \geq \frac{1}{2} \lambda_{\text{min}}(\hat{P}_{S'}) \text{.}
\]
We have
\[
\lambda_{\text{min}}(P_S) \leq \lambda_{\text{min}}(\hat{P}_S) + \frac{\alpha}{\sqrt{T}} \leq \lambda_{\text{min}}(\hat{P}_S) + 2 \frac{\alpha}{\sqrt{T}} \leq 3 \frac{\alpha}{2} \lambda_{\text{min}}(\hat{P}_S) \leq 3(\lambda_{\text{min}}(\hat{P}_S) - \frac{\alpha}{\sqrt{T}}) \leq 3 \lambda_{\text{min}}(P_S) \text{.}
\]
by Lemma A.3 on \( P_S \), by \( \hat{S} = \arg \max_{S' \in [L]} \lambda_{\text{min}}(\hat{P}_{S'}) \), by termination condition, from the termination condition, by Lemma A.3 on \( P_S \).

In summary, with probability \( 1 - L \delta \), this procedure terminates in no more than \( \frac{576 L^2 M^2 R^2 (d \log 6 \log \frac{L}{\delta})^2}{\lambda_{\text{min}}^2} \) and outputs \( \hat{S} \) with \( \lambda_{\text{min}}(P_S) \geq \frac{1}{2} \lambda_{\text{min}} \).

\[\square\]

### B \hspace{1em} Regret upper bound for Theorem 5.2

**Theorem B.1.** If we have \( A_t = B_d^d \), the regret of Algorithm 3 satisfies
\[ R_{\{T\}} \leq 2 \sqrt{\left( \frac{3 \sqrt{S}}{\lambda_{\text{min}}} M + 1 \right)^2 (\delta \left( \frac{2}{2(L + 1)} + M^2 \right) + (s + 1) T d \log (1 + \frac{TL}{d \rho}) + \frac{1152L^3M^2R^2(M + R)^2 (d \log 6 + \log \frac{2L}{\sqrt{T}})^2}{\lambda_{\text{min}}^2} \) \]
with probability \( 1 - \delta \) where
\[ \sqrt{\beta_2(\delta)} = R \sqrt{d \log (1 + \frac{TM^2}{\rho} \delta)} + M \rho^{\frac{1}{2}}. \]

As a reminder, we use the notation \( f(a, \{ \tilde{\theta}_i \}) = \langle a, \text{Proj}_{\{\tilde{\theta}_i\}} a \rangle_0 \), \( (a_t, \{ \tilde{\theta}_i \}) = \arg \max_{a_t \in A, \{ \tilde{\theta}_i \} \in \hat{S}} f(a, \{ \tilde{\theta}_i \}) \) and \( i_t = \arg \max_{i \in \hat{S}} \|a_t\|_{V_t^{-1}} \sqrt{\beta_2(\delta)} \|V_t^{-1}\|_F \). We take \( \hat{S} \) to be the coreset returned by CORE-SET procedure, which satisfies \( \lambda_{\text{min}}(\hat{S}) \geq \frac{1}{3} \lambda_{\text{min}} \). We use \( \{ y_t \} \) as a shorthand for \( \{ \tilde{\theta}_i \} \in \hat{S} \).

**Property B.1.** If \( \text{Proj}^\perp a = a \) then for all \( b \), because \( \text{Proj}^\perp \) is self-adjoint and idempotent, we have \( \langle a, \text{Proj}^\perp b \rangle = \langle \text{Proj}^\perp a, b \rangle = \langle a, b \rangle \).

**Lemma B.2.** If \( A \) is \( B_d^d \), \( a_t \in B_d^d \) satisfies \( \text{Proj}_{\{\tilde{\theta}_i\}} a_t = a_t \).

**Proof.** Consider the action \( a' = \frac{\text{Proj}_{\{\tilde{\theta}_i\}} a_t}{\|\text{Proj}_{\{\tilde{\theta}_i\}} a_t\|_2} \). This satisfies
\[
\text{Proj}_{\{\tilde{\theta}_i\}} a_t = a_t \text{ satisfies} \text{Proj}_{\{\tilde{\theta}_i\}} a_t = a_t \text{.}
\]
Because \( a_t \) is optimal, we must have equality.
\[
\|\text{Proj}_{\{\tilde{\theta}_i\} \in [L]} a_t\|_2 = \|a_t\|_2 \implies \text{Proj}_{\{\tilde{\theta}_i\}} a_t = a_t \text{.}
\]
\[\square\]
Property B.2. For any choice of \( \{x_i\} \in \mathbb{R} \), we have \( \text{Proj}_{\{\bar{\theta}_i\}}^* \sum_{i \in S} \bar{\theta}_i x_i = 0 \)

This is true since \( \text{Proj}_{\{\bar{\theta}_i\}}^* \bar{\theta}_i = 0 \) for all \( i \in \tilde{S} \).

Property B.3. Because \( a_t \) is the optimistic action in Equation \( \theta \), we have \( \langle a_t, \text{Proj}_{\{\bar{\theta}_i\}}^* \bar{\theta}_0 \rangle \geq \langle a_s, \text{Proj}_{\{\bar{\theta}_i\}}^* \theta_0 \rangle \).

Lemma B.3. Suppose at time \( t \) a player plays \( (a_t, i_t) \). Then the suboptimality \( \Delta_{a_t} \) is upper bounded as

\[
\Delta_{a_t} \leq 2(3\sqrt{\lambda_\text{min}} M + 1)\|a_t\|_{V_i^{-1}} \sqrt{\beta_T(\delta)}
\]

Proof. We denote by \( V_{i,t} \) a matrix that represents the extent of exploration with the ith vector, \( V_{i,t} = \sum_{i \leq t} \mathbb{1}_{i = i} a_s a_s^T \), and by \( V_i \) the total exploration across all vectors, \( V_i = \sum_{s \leq t} a_s a_s^T \). We will denote by \( T_{i,t} \) the number of times \( i \) has been queried up to and including time \( t \), so \( T_{i,t} = \sum_{s \leq t} \mathbb{1}_{i = i} \).

Because the \( \theta_i, i \in \tilde{S} \) are a basis for the protected space, we can write \( \text{Proj}_{\{\theta_i\}} \theta_0 \) as

\[
\text{Proj}_{\{\theta_i\}} \theta_0 = \sum_{i \in S} \theta_i x_i
\]

for unique \( x_i \in \mathbb{R} \).

The suboptimality of an action is upper bounded as follows:

\[
\Delta_{a_t} = \langle a_s - a_t, \text{Proj}_{\{\bar{\theta}_i\}}^* \theta_0 \rangle
\]

\[
\leq \langle a_t, \text{Proj}_{\{\bar{\theta}_i\}}^* \bar{\theta}_0 \rangle - \langle a_t, \text{Proj}_{\{\theta_i\}}^* \theta_0 \rangle
\]

\[
= \langle a_t, \text{Proj}_{\{\bar{\theta}_i\}}^* (\bar{\theta} - \theta_0) \rangle + \langle a_t, (\text{Proj}_{\{\theta_i\}}^* - \text{Proj}_{\{\bar{\theta}_i\}}^*) \theta_0 \rangle
\]

\[
= \langle \text{Proj}_{\{\bar{\theta}_i\}}^* a_t, (\bar{\theta} - \theta_0) \rangle + \langle a_t, (\text{Proj}_{\{\theta_i\}}^* - \text{Proj}_{\{\bar{\theta}_i\}}^*) \theta_0 \rangle
\]

\[
= \langle \text{Proj}_{\{\bar{\theta}_i\}}^* a_t, (\bar{\theta} - \theta_0) \rangle + \langle a_t, \text{Proj}_{\{\bar{\theta}_i\}}^* (\text{Proj}_{\{\theta_i\}}^* - \text{Proj}_{\{\theta_i\}}^*) \theta_0 \rangle
\]

\[
= \langle a_t, (\bar{\theta} - \theta_0) \rangle + \langle a_t, \text{Proj}_{\{\theta_i\}}^* (\text{Proj}_{\{\theta_i\}}^* - \text{Proj}_{\{\theta_i\}}^*) \theta_0 \rangle
\]

\[
= \langle a_t, (\bar{\theta} - \theta_0) \rangle + \langle a_t, \text{Proj}_{\{\theta_i\}}^* \sum_{i \in S} (\theta_i - \bar{\theta}_i) x_i \rangle
\]

\[
= \langle a_t, (\bar{\theta} - \theta_0) \rangle + \langle a_t, \text{Proj}_{\{\theta_i\}}^* \sum_{i \in S} (\theta_i - \bar{\theta}_i) x_i \rangle
\]

\[
\leq \langle a_t, (\bar{\theta} - \theta_0) \rangle + \max_{i \in S} \langle a_t, \theta_i - \bar{\theta}_i \rangle \sum_{i \in S} \|x_i\|
\]

\[
\leq \|a_t\|_{V_i^{-1}} \|\theta_i - \bar{\theta}_i\|_{V_i^{-1}} + \|x_i\| \max_{i \in S} \|a_t\|_{V_i^{-1}} \|\theta_i - \bar{\theta}_i\|_{V_i^{-1}} \|x_i\|
\]

So

\[
\Delta_{a_t} \leq \|a_t\|_{V_i^{-1}} \|\bar{\theta} - \theta_0\|_{V_i^{-1}} + \max_{i \in S} \|a_t\|_{V_i^{-1}} \|\theta_i - \bar{\theta}_i\|_{V_i^{-1}} \|x_i\|
\]
From a union bound, we know that with probability \(1 - (L + 1)\delta\), equation (2) holds for all \(i \in [L] \cup \{0\}\). Because the coreset efficiently spans the protected space, we have

\[
\|x_S\|_1 \leq \sqrt{|S|}\|x_S\|_2
\]

by Cauchy-Schwartz

\[
\leq \sqrt{s} \frac{1}{\lambda_{\min}(S)} \left|\sum_{i \in S} \theta_i x_i\right|_2
\]

by \(\|\sum_{i \in S} \theta_i x_i\|_2 \geq \lambda_{\min}(S)\|x\|_2\)

\[
= \sqrt{s} \frac{1}{\lambda_{\min}(S)} \left|\text{Proj}(\theta)\theta_0\right|_2
\]

\[
\leq \sqrt{s} \frac{3}{\lambda_{\min}} \left|\theta_0\right|_2
\]

by \(\lambda_{\min}(S) \geq \frac{1}{3}\lambda_{\min}\) from Theorem A.1

So

\[
\|x_S\|_1 \leq \sqrt{s} \frac{3}{\lambda_{\min}} \left|\theta_0\right|_2
\]

(5)

The index of the query chosen alongside \(a_t\) is chosen to be the one such that for all \(t\), we have

\[
\|a_t\|_{V^{-1}_{i,t}} \sqrt{\beta_{T_{i,t}}(\delta)} \geq \|a_t\|_{V^{-1}_{i,t}} \sqrt{\beta_{T_{i,t}}(\delta)} \forall i \in \{0\} \cup \hat{S}
\]

(6)

Geometrically, this is the index corresponding to the vector that is least understood in the chosen direction, since an upper bound on the radius of a confidence ellipsoid for vector \(\theta_i\) in direction \(a_t\) is given by \(\|a_t\|_{V^{-1}_{i,t}} \sqrt{\beta_{T_{i,t}}(\delta)}\).

This allows us to upper bound \(\Delta_{a_t}\) in terms of the history as

\[
\Delta_{a_t} \leq \|a_t\|_{V^{-1}_{t-1}} \left|\theta_0\right|_2 \|a_t\|_{V^{-1}_{t-1}} \left|\theta_i\right|_2 \|x_S\|_1
\]

by (4)

\[
\leq 2\|a_t\|_{V^{-1}_{t-1}} \sqrt{\beta_{T_{t-1}}(\delta)} + 2 \max_{i \in [L]} \|a_t\|_{V^{-1}_{t-1}} \sqrt{s\beta_{T_{t}}(\delta)} \frac{3}{\lambda_{\min}} \left|\theta_0\right|_2
\]

by (2) and (5)

\[
\leq 2(3\sqrt{s} M + 1)\|a_t\|_{V^{-1}_{t-1}} \sqrt{\beta_{T_{t}}(\delta)} \leq 2(3\sqrt{s} M + 1)\|a_t\|_{V^{-1}_{t-1}} \sqrt{\beta_{T}(\delta)}
\]

by (6), and \(T_{i,t} \leq T\)

The following is a standard identity for the covariance matrices.

**Lemma B.4.** The log det of the covariance matrix satisfies

\[
\log \det V_{i,T} = d \log \lambda + \sum_{t=1}^{T} 4_{t=1} \log(1 + \|a_t\|_{V^{-1}_{i,t-1}}^2)
\]

**Proof.** This follows from Sylvester’s identity [19].

\[
\log \det V_{i,t} = \log \det(V_{i,t-1} + a_t a_t^T)
\]

\[
= \log \det V_{i,t-1} + \log \det(I + a_t a_t^T V_{i,t-1}^{-1})
\]

\[
= \log \det V_{i,t-1} + \log \det(I + a_t^T V_{i,t-1}^{-1} a_t)
\]

\[
= \log \det V_{i,t-1} + \log \det(I + \|a_t\|_{V^{-1}_{i,t-1}}^2)
\]

\[
\square
\]

There is also a simple upper bound on the log det that comes from the arithmetic-geometric means inequality.

**Lemma B.5.**

\[
\log \det V_{i,T} \leq d \log \left(\rho + \frac{T_{i,T} L}{d}\right)
\]

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Proof. Let $\lambda_j(V_{i,t})$ denote the eigenvalues of $V_{i,t}$.

$$
\log \det V_{i,T} = \log \prod_{j \leq d} \lambda_j(V_{i,t}) \leq d \log \left( \frac{\sum_{j \leq d} \lambda_j(V_{i,t})}{d} \right) = d \log \left( \frac{\operatorname{Tr} V_{i,t}}{d} \right) \leq d \log(\rho + \frac{|T_i|L}{d}).
$$

\[ \square \]

Proof of Theorem B. We first get an upper bound for the regrets separately for times in which each of the protected vectors are queried.

$$
\log \det V_{i,T} = d \log \frac{\sum_{t \leq T} \mathbb{1}_{i_{t-1} \neq i}}{\sum_{t \leq T} \mathbb{1}_{i_{t-1} = i}} \log(1 + ||a_t||_i^2) \leq \log(1 + x) \geq \frac{x}{1 + x}, \ also \ \Delta_{i,t} \leq 2M
$$

Writing this as an upper bound on the sub-optimality, we have:

$$
\sum_{t \leq T} \mathbb{1}_{i_{t-1} \neq i} \Delta_{i,t} \leq 4T_i \beta_T((3 \frac{\sqrt{s}}{\lambda_{\min}} M + 1)^2 \beta_T(\delta) + M^2)(\log \det V_{i,T} - d \log \rho)
$$

We can now combine the regrets from each of the protected vectors:

$$
\sum_{t \leq T} \Delta_{i,t} = \sum_{i \in [0] \cup \tilde{S}} \sum_{t \leq T} \mathbb{1}_{i_{t-1} = i} \Delta_{a_t}
\leq \sum_{i \in [0] \cup \tilde{S}} \sqrt{4T_i \beta_T((3 \frac{\sqrt{s}}{\lambda_{\min}} M + 1)^2 \beta_T(\delta) + M^2)(\log \det V_{i,T} - d \log \rho)}
\leq \sum_{i \in [0] \cup \tilde{S}} \left( \frac{\beta_T((3 \frac{\sqrt{s}}{\lambda_{\min}} M + 1)^2 \beta_T(\delta) + M^2)}{d \log(1 + \frac{T_i |L|}{d \rho})} \right)
$$

In each round, we concentrate our querying only on the vectors in the core set $\tilde{S}$ with $|\tilde{S}| = s$, so we have the first equality below. Next applying Cauchy-Schwartz inequality we obtain the result.

$$
\mathcal{R}_{[T]} = \sum_{i \in [0] \cup [L]|_{t \in [T]}} \mathbb{1}_{i_{t-1} = i} \Delta_{a_t}
\leq \sum_{i \in [0] \cup \tilde{S}} \left( \frac{\beta_T((3 \frac{\sqrt{s}}{\lambda_{\min}} M + 1)^2 \beta_T(\delta) + M^2)(s + 1)Td \log(1 + \frac{T_i |L|}{d \rho})} {d \log(1 + \frac{T_i |L|}{d \rho})} \right)
$$

From [1] we know that with probability $1 - \delta$, the confidence ellipsoids constructed contain the true parameters for each individual vector for $\beta_T(\delta) = \left( R \sqrt{\frac{d \log((1 + 2M^2)/\delta) + \rho M^2}{d}} \right)^2$. By a union bound, with probability $1 - L\delta$, these inequalities hold for all $L$ simultaneously.
Finally, we must add the regret accrued during the initial core-set estimation phase. By Theorem A.1 with probability \(1 - \frac{\delta}{2}\), this phase lasts at most \(\frac{576L^2 R^2 (M + R)^2 d (d \log 6 + \log \frac{12}{\delta})^2}{\lambda_{\text{min}}^2}\) rounds, and adds at most \(\frac{1152L^3 M R^2 (M + R)^2 d (d \log 6 + \log \frac{2L}{\delta})^2}{\lambda_{\text{min}}^2}\) to the regret. The rest of the rounds accrue a regret of at most \(2\sqrt{((\frac{3\sqrt{2}}{\lambda_{\text{min}}}M + 1)^2 \beta_T(\frac{\delta}{2(L + 1)}) + M^2)(s + 1)Td \log(1 + \frac{TL}{d\rho})}\) with probability \(1 - \frac{\delta}{2}\). So the total regret is upper bounded by

\[
2\sqrt{\left((\frac{3\sqrt{2}}{\lambda_{\text{min}}}M + 1)^2 \beta_T(\frac{\delta}{2(L + 1)}) + M^2)(s + 1)Td \log(1 + \frac{TL}{d\rho}) + \frac{1152L^3 M R^2 (M + R)^2 d (d \log 6 + \log \frac{2L}{\delta})^2}{\lambda_{\text{min}}^2}}
\]

with probability at least \(1 - \delta\).

\[\square\]

C Proof of Regret Lower Bound (Theorem 6.1)

**Theorem C.1.** There is an instance of the Protected Linear Bandit problem such that any algorithm incurs a regret of \(\Omega(T^{\frac{3}{4}})\).

**Proof.** Consider a pair of instances, denoted with superscripts (1) and (2). For both, we set our ambient space to have dimension \(d = 2\), and set \(s = L = 1\). We denote by \(u_\alpha \in \mathbb{R}^2\) the vector \((\cos \alpha, \sin \alpha)\). Take \(\alpha = T^{-\frac{1}{4}}\). We set \(\theta(1) = \theta(2) = u_{\frac{\pi}{4} - \alpha}\). In instance (1), we set \(\theta(1) = u_0\) while in instance (2), we set \(\theta(2) = u_{-\alpha}\). In both instances, in each round, we allow the player an action space that consists of either the actions \(\{u_{\pi - \alpha}, u_{2\alpha}\}\) or \(\{u_{\pi - \alpha}, u_{2\alpha}, u_{3\alpha}\}\) with equal probability. We will denote by \(\Pi_1, \Pi_2\) respectively the orthogonal projection for \(\theta_1\) in instance (1) and (2). We have

\[
\Pi_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Pi_2 = \begin{pmatrix} \sin^2 \alpha & -\frac{1}{2} \sin(2\alpha) \\ -\frac{1}{2} \sin(2\alpha) & \cos^2 \alpha \end{pmatrix}.
\]

Note that for \(x < 1\), we have \(x \geq \sin x \geq \frac{\sqrt{2}}{2} x\) and \(1 \geq 1 - \cos x \geq 1 - x\), and for all \(x, \cos x \geq 1 - \frac{x^2}{2}\). We will assume \(T \geq 256 \implies 0 \leq \alpha \leq \frac{1}{4}\).

In the first instance, action \(u_{\pi - \alpha}\) has reward \(\langle u_{\frac{\pi}{4} - \alpha}, \Pi_1 u_{\pi - \alpha}\rangle = \sin \alpha \cos \alpha\), action \(u_{2\alpha}\) has reward \(\langle u_{\frac{\pi}{4} - \alpha}, \Pi_1 u_{2\alpha}\rangle = \sin(2\alpha) \cos \alpha\), and action \(u_{3\alpha}\) has reward \(\langle u_{\frac{\pi}{4} - \alpha}, \Pi_1 u_{3\alpha}\rangle = \sin(3\alpha) \cos \alpha\). When available, \(u_{3\alpha}\) is the optimal action. When it is not available, \(u_{2\alpha}\) is optimal. In either cases, the sub-optimality gap is at least \(\cos \alpha(\sin(3\alpha) - \sin(2\alpha)) \geq 1 - \alpha\). So \(\cos \alpha(\sin(3\alpha) - \sin(2\alpha)) \geq (1 - \frac{2}{\pi}\alpha)(\frac{1}{2}\alpha) \geq \frac{1}{4}\alpha\) for small enough \(\alpha \leq \frac{1}{4}\).

In the second instance, action \(u_{\pi - \alpha}\) has reward \(\langle u_{\frac{\pi}{4} - \alpha}, \Pi_2 u_{\pi - \alpha}\rangle = 0\), action \(u_{2\alpha}\) has reward \(\langle u_{\frac{\pi}{4} - \alpha}, \Pi_2 u_{2\alpha}\rangle = \sin(3\alpha)\), and action \(u_{3\alpha}\) has reward \(\langle u_{\frac{\pi}{4} - \alpha}, \Pi_2 u_{3\alpha}\rangle = \sin(2\alpha)\). Here, \(u_{2\alpha}\) is always optimal. The sub-optimality gap is at least \(\sin(3\alpha) - \sin(2\alpha) \geq \frac{3}{2}\alpha\) for \(\alpha \leq \frac{1}{4}\).

Denote by \(A\) the event in which the action \(u_{3\alpha}\) is played fewer than half the times it is observed.

Let \(\mathbb{P}_1(A)\) denote the probability of event \(A\) under the distribution induced by interaction of the algorithm with instance 1, and let \(\mathbb{P}_2(A^c)\) denote the probability of playing action \(u_{3\alpha}\) at least half the times it is observed under the distribution induced by interaction with instance 2. The regret of the algorithm on instance (1) is then at least \(R_1 \geq T^{-\frac{3}{4}} \frac{T}{4} \mathbb{P}_1(A)\), while the regret on instance (2) is at least \(R_2 \geq T^{-\frac{3}{4}} \frac{T}{4} \mathbb{P}_2(A^c)\). By the Bretagnolle-Huber inequality [14], we have

\[
\mathbb{P}_1(A) + \mathbb{P}_2(A^c) \geq \frac{1}{2} e^{-D(\mathbb{P}_1 || \mathbb{P}_2)}.
\]

So

\[
R_1 + R_2 \geq T^{-\frac{3}{4}} \frac{T}{4} (\mathbb{P}_1(A) + \mathbb{P}_2(A^c)) \geq \frac{1}{8} T^{\frac{3}{4}} e^{-D(\mathbb{P}_1 || \mathbb{P}_2)}.
\]

Finally, we can bound \(D(\mathbb{P}_1 || \mathbb{P}_2)\) in terms of the KL-divergences of the reward distributions in each round. We have \(D(\mathbb{P}_1 || \mathbb{P}_2) \leq 21\), as proved below

\[
D(\mathbb{P}_1 || \mathbb{P}_2) = \mathbb{E}_1 \sum_{t \in [T]} D(P_{A_1^{s_{(1,i,t)},1}}, P_{A_1^{s_{(1,i,t)},2}})
\]

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\[
= \sum_{t \in [T]} P_1(I_t^{(1)} = 0) E_1[(A_t^{(1)}, \theta_t^{(1)}) - \theta_t^{(2)}] | I_t^{(1)} = 0] \\
+ \sum_{t \in [T]} P_1(I_t^{(1)} = 1) E_1[(A_t^{(1)}, \theta_t^{(1)}) - \theta_t^{(2)}] | I_t^{(1)} = 1] \\
= \sum_{t \in [T]} P_1(I_t^{(1)} = 1) E_1[(A_t^{(1)}, \theta_t^{(1)}) - \theta_t^{(2)}] | I_t^{(1)} = 1] \\
\leq \sum_{t \in [T]} E_1[(A_t^{(1)}, u_0 - u_0)] \\
\leq T(u_\pi - u_0)^2 \\
= (\cos(4\alpha) - \cos(3\alpha))^2 T \\
\leq \left( \frac{9\alpha^2}{2} \right)^2 T \\
\leq 21\alpha^4 T = 21
\]
Thus we have
\[
R_1 + R_2 \geq \frac{\frac{1}{8} T^3}{6} e^{-21}
\]
for \( T \geq 256 \). This means that any algorithm performs poorly on at least one of the two instances
\[
\max \{R_1, R_2\} \geq \frac{e^{-21}}{16} T^\frac{3}{4}.
\]

D Computing the optimistic parameters

Let \( f(a, P, \theta) := \langle a, \text{Proj}_{\{\theta, i \in [L] \}} \theta \rangle \). In line 4 of Algorithm 2 we must compute
\[
(a_t, P_t, \theta_t) = \arg \max_{a \in A_t, P \in \mathcal{P}, \theta_0 \in \Theta_0} f(a, P, \theta_0) \tag{8}
\]
Note that this is not a concave function. To avoid having to do a grid search, we use a trick that only works exactly if the maximizer is orthogonal to the protected space. This happens, for instance, when the action space is \( B^2 \) as shown in B.2.
We can use this observation to construct a lower bound to this function that is tight only for the optimal parameter values. This surrogate can now be used in line 4 of Algorithm 2.

Lemma D.1. For any action \( a \), we have \( \hat{P}(a) \in \mathcal{P}, \hat{\theta}_0(a) \in \Theta_0 \), and \( \max_{P \in \mathcal{P}, \theta \in \Theta_0} f(a, P, \theta_0) \geq f(a, \hat{P}(a), \hat{\theta}_0(a)) \) for
\[
\hat{\theta}_0(a) := \hat{\theta}_0 + \frac{\sqrt{\beta_{T_{\theta_0} - 1} a}}{||a||_{V_0,t}}; \quad \hat{\theta}_i(a) := \hat{\theta}_i + (2\alpha_i - 1) \frac{\sqrt{\beta_{T_{\theta_0} - 1} a}}{||a||_{V_i,t}},
\]
where
\[
\alpha_i = \text{clip}_{[0, 1]} \left[ \frac{\langle a, \hat{\theta}_i \rangle + \frac{\sqrt{\beta_{T_{\theta_0} - 1} ||a||}}{||a||_{V_{i,t}}}}{2 \sqrt{\beta_{T_{\theta_0} - 1} ||a||_{V_{i,t}}}} \right].
\]
Moreover, for action \( a^* \) that maximizes (8), we have \( \max_{P \in \mathcal{P}, \theta_0 \in \Theta_0} f(a^*, P, \theta_0) = f(a^*, \hat{P}(a^*), \hat{\theta}_0(a^*)) \).
Proof. For any action $a$ the first part hold. For this to be useful, we must first verify that these parameter values are feasible. Indeed,

$$||\theta^0 - \hat{\theta}_0||_{V_{0,t-1}} = ||\hat{\theta}_0 + \sqrt{\beta T_{0,t-1}} a ||_{V_{0,t}} - \hat{\theta}_0||_{V_{0,t-1}} = ||\sqrt{\beta T_{0,t-1}} a ||_{V_{0,t}} = \sqrt{\beta T_{0,t-1}}$$

and similarly

$$||\theta^i - \hat{\theta}_i||_{V_{i,t-1}} = ||\hat{\theta}_i + (2\alpha - 1) \sqrt{\beta T_{i,t-1}} a ||_{V_{i,t}} - \hat{\theta}_0||_{V_{0,t-1}} = ||(2\alpha - 1) ||\sqrt{\beta T_{0,t-1}} a ||_{V_{0,t}} \leq \sqrt{\beta T_{0,t-1}}$$

for $\alpha \in [0, 1]$. Because this is a feasible point, for a maximization problem, for any action, this is a lower bound to the solution.

At the maximizer $a^*$, we already know that there is a choice of $\hat{\theta}_i$ for which $\langle a^*, \hat{\theta}_i \rangle = 0$ for all $i \in [L]$. Any other choice of $\hat{\theta}_i$ that is orthogonal to $a^*$ will result in the same value of $f(a^*, P, \theta_0)$, since $\hat{\theta}_i$ only acts through a projection collective column space which is orthogonal to $a^*$ anyway. For this choice of $a^*$, and a protected space that is orthogonal to it, the optimal $\hat{\theta}_0$ is standard. \qed