Entropy and holography constraints for inhomogeneous universes

Bin Wang\textsuperscript{a,b}, Elcio Abdalla\textsuperscript{a} and Takeshi Osada\textsuperscript{a}
\textsuperscript{a} Instituto de Física, Universidade de São Paulo, C.P.66.318, CEP 05315-970, São Paulo, Brazil
\textsuperscript{b} Department of Physics, Shanghai Teachers’ University, P.R. China

We calculated the entropy of a class of inhomogeneous dust universes. Allowing spherical symmetry, we proposed a holographic principle by reflecting all physical freedoms on the surface of the apparent horizon. In contrast to flat homogeneous counterparts, the principle may break down in some models, though these models are not quite realistic. We refined fractal parabolic solutions to have a reasonable entropy value for the present observable universe and found that the holographic principle always holds in the realistic cases.

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In view of the example of black hole entropy \cite{1}, an influential holographic principle relating the maximum number of degrees of freedom in a volume to its boundary surface area has been put forward recently \cite{2}. This principle is viewed as a real conceptual change in our thinking about gravity. The main aim of the holographic principle is to generalize its application to a broader class of situations, including cosmology. However, in a general cosmological setting, there is no unique appropriate notion which is analogous to the event horizon in black hole serving as a natural boundary. This makes the generalization particularly difficult. A remarkable progress has been made by Fischler and Susskind (FS) \cite{2}. They have shown that for flat and open Friedman-Lemaître-Robinson-Walker (FLRW) universes the holographic principle holds with the total entropy of the matter inside the particle horizon being smaller than the area of the horizon. Various different modifications of FS version of the holographic principle have been raised subsequently \cite{3,4}. Motivated by the fact that the first successful implication of AdS/CFT duality to solve the problem of the microscopic interpretation of black hole entropy appeared in (2+1)-dimensional models \cite{5}, we formulated the holography in such a case \cite{6,7}. Recently Bousso has provided a more elegant and a broader holographic principle and has applied to a number of examples including the collapsing FLRW cosmological models \cite{8}. Part of Bousso’s proposal has been proved in \cite{9}, however there is still some difficulties associated with it \cite{10}.

It is of great interest to take a closer look of holography in a generic realistic inhomogeneous cosmological setting. The first attempt was carried out by Tavakol and Ellis \cite{11} who considering Bousso’s proposal as well as a modified version of it. In both cases they found that operational difficulties exist in constructing the holographic principle in a realistic universe.

Compared to the homogeneous universe, there is a further difficulty in setting up the holographic principle in the real cosmos. In the homogeneous universes the comoving entropy density is assumed to be a constant and the total entropy is just the entropy density times the comoving volume. In addition to the vector defined to describe the gravitational entropy flux \cite{12}, and evolution of the density contrast studied to answer the possible existence of gravitational entropy \cite{13}, until now there is no exact calculation of the entropy of inhomogeneous universes.

In the present paper, we concentrate our attention on the parabolic Lemaitre-Tolman-Bondi (LTB) model which is the natural generalization of the flat dust FLRW model. Considering some characteristics of the realistic models, such as spherical symmetry, not referring to either initial or final moment, we find that it is appropriate to adopt the idea suggested in \cite{2} for homogeneous universes by defining the apparent horizon as a boundary hypersurface to construct the holographic principle here. We will show that choosing the apparent horizon in the formulation of the cosmic holography for the real cosmos is simple and valuable. From the first law of thermodynamics we will define the entropy density in the inhomogeneous universe and calculate the total entropy value within the apparent horizon. Different from the homogeneous flat universe, we will show that in the general parabolic LTB models, some fractal universe will violate the holographic principle. In order to describe the real cosmos, we will refine fractal parabolic solutions by comparing the calculated entropy in fractal models to that in our present observable universe. In the refined realistic models holographic principle can always be satisfied.

In normalized comoving coordinates the metric of the parabolic LTB model is
\begin{equation}
\text{d}s^2 = -\text{d}t^2 + R^2 (\text{d}r^2 + \sin^2 \theta \text{d}\phi^2) = h_{ab} \text{d}x^a \text{d}x^b + \tilde{r}^2 (x) (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2)
\end{equation}
where \( h_{ab} = \text{diag}[-1, R^2] \), and
\begin{equation}
R = \frac{1}{2} (9F)^{1/3} (t + \beta)^{2/3}
\end{equation}
is the area distance and \( R' \) plays the role of scale factor. \( \beta \) and \( F \) are two arbitrary functions of \( r \). The metric (1)
is spherically symmetric. We define the dynamical apparent horizon in terms of a condition $|\nabla \tilde{r}|^2 \equiv h^{ab} \partial_a \tilde{r} \partial_b \tilde{r} = 0$ to the areal radius, with the result

$$F(\tilde{r}_{AH}) = 3[t + \beta(\tilde{r}_{AH})]$$

(3)

and $\tilde{r}_{AH} = \frac{3}{2}[t + \beta(\tilde{r}_{AH})]$ is the physical apparent horizon, where $\tilde{r}_{AH}$ denotes the proper apparent horizon. Taking $\beta$ to be zero and the scale factor $R' \sim t^{2/3}$ in homogeneous flat dust universe, $\tilde{r}_{AH}$ agrees with the expression in [12] for the apparent horizon.

In order to calculate the entropy within the apparent horizon we have to define the local entropy density first. From the standard big-bang cosmology [13] we learnt that when a relativistic particle becomes non-relativistic and disappears, its entropy is shared between the particle become thermal contact. Since photons and neutrinos never become non-relativistic, they share the entropy of the universe. It is reasonable to suppose that the entropy of the universe is mainly produced before the dust-filled era and this result should also hold in the inhomogeneous cosmology. Using the first law of thermodynamics, for the dust-filled universe ($p = 0$) we have the local entropy density $s = \rho / T$, where $T$ is the temperature of the universe and $\rho$ the radiational energy density given by $aT^4$.

Considering that in the expansion of the universe, the radiation always has the property of black body and supposing that the number density of the photon is conserved, we also have the relation $h \nu_0 / k T_0 = h \nu / k T$ in the inhomogeneous background where $\nu_0 = \nu(1 + z)$. From the geodesic equation the expression for the redshift is

$$1 + z = \frac{\text{d}t}{\text{d}\lambda} \Big|_{\lambda=0} = R' \frac{\text{d}R}{\text{d}\lambda} \Big|_{\lambda=0}^{-1} = \frac{R'}{R_0'}$$

(4)

where assumed that for $r \to 0$, $R' = 1$ [14]. We obtain the relation $TR' = T_0R'_0 = \text{Const.}$, which coincides with that the homogeneous case. Combining these considerations, the local entropy density in the inhomogeneous case can be expressed as

$$s(t, r) = a(T_0R'_0)^3 \frac{1}{R^3(t, r)} = C \frac{1}{R^3(t, r)}$$

(5)

where $C$ is a constant and $R'(t, r)$ has the form given in [3]. The total entropy measured in the comoving space inside the apparent horizon is

$$S = \int_{0}^{\tilde{r}_{AH}} s(t, r)4\pi R'^2 \text{d}r.$$  

(6)

For the homogeneous dust universe the local entropy density is only a function of $t$ proportional to $a^{-3}(t)$ from the first law. Eq(3) can thus reproduce the value $S = \frac{4\pi}{3} \sigma \tilde{r}_{AH}^3$, where $\sigma$ denotes the constant comoving entropy density.

With the method of calculating the total entropy of the inhomogeneous parabolic model at hand, we state our proposal of a holographic principle in inhomogeneous cosmology in the spirit of [13]: the entropy inside the apparent horizon can never exceed the area of apparent horizon in Planck units.

In order to get the entropy value and examine the holographic principle, we need detailed expressions of $F(r), \beta(r)$. Two particular forms of these arbitrary functions that lead to fractal behavior in parabolic models have been found in [19]. They are

Model 1 : $F = \alpha r^p$, $\beta = \beta_0 + \eta_0 r^q$

(7)

Model 2 : $F = \alpha r^p$, $\beta = \ln(e^{\beta_0} + \eta_1 r)$

(8)

where $\alpha \in [10^{-5}, 10^{-4}]$, $p$ and $\beta_0 \in [0.5, 4]$, $\eta_1 \in [1000, 1300]$, $q$ around 0.65 and $\eta_0$ around 50 are required to obtain fractal solutions.

The starting point of the dust-filled universe is at $t_0 = 10^{12}$ s and the present time of the universe $t = 15$ Gyr. Considering the very large numbers appearing in the numerical calculations, we adopt the units as those used in [11]. We express distances in Gpc, time unit in 3.26 Gyr, mass unit (MU) as $2.09 \times 10^{19} M_\odot$ and the temperature as $1 K = 3.7 \times 10^{-9} M_\odot$ to keep $c = G = k = 1$ ($k$ is the Boltzman constant). Using the present temperature $T = 2.7 K$, and the present size $10^{28} cm$ [14], in our units the constant $C$ in [5] amounts to $2.76 \times 10^{86}$.

Now, we start to investigate in detail these (fractal) parabolic models. Substituting (7) into (3), the proper apparent horizon can be gotten by solving the nonlinear equation

$$\alpha r_{AH}^p = 3[t + \beta_0 + \eta_0 r^q_{AH}]$$

(9)

where $\alpha < q < 0$. The solutions $r_{AH}$ can be found analytically [19] or numerically [28]. We have found through analytical analysis that (9) has no solutions when $p < q = 0.65$, which corresponds to having no apparent horizon in that range for $p$. This is not so that surprising if we recall the fact that not all homogeneous universe models have particle or event horizons. But compared to the flat dust FLRW model, which always has apparent horizon, this result indicates the difference between the general parabolic LTB model and its homogeneous counterpart. For $p > q$, the proper apparent horizon can be obtained by solving (7) and the physical apparent horizon is

$$\tilde{r}_{AH} = \frac{1}{2} \alpha r_{AH}^p.$$  

(10)

We found that changes of $p$ and $\alpha$ change a lot the behavior of the solution. Bigger $p$ or $\alpha$ leads to smaller results for $r_{AH}$. The difference caused by different values of $\alpha$
for big $p$ is smaller compared to that for small $p$. $\beta_0$ here does not affect much the result.

The area of the apparent horizon in Planck units reads

$$A/\ell_p^2 = 4\pi b r_{AH}^2$$

(11)

where $b = 0.36 \times 10^{121}$ in our units. With (6) and the obtained value for $r_{AH}$, the entropy inside the apparent horizon is

$$S = 2.76 \times 10^{86} \pi \times$$

$$\int_0^{r_{AH}} \frac{\beta}{2} (t + \beta_0 + \eta_0 r^q)^2 \frac{dr}{(t + \beta_0 + \eta_0 q r^{q-1})^2}.$$ 

(12)

From the value of the constants in (11) and (12), one might naively expect that the holographic principle always holds. However this is not true.

Fig. 1 shows that at the beginning of the dust-filled era, $t_0 = 0.97 \times 10^{-5}$ in our units, when $\alpha = 10^{-4}$ or $10^{-5}$, the holographic principle will be violated if $p < 0.5$ or $p < 0.78$, respectively. This result does not change much for different values of $\beta_0$. The violation of the holographic principle here is really surprising because in homogeneous expanding universes, the holography has never been reported facing any challenge. This again shows the difference between fractal parabolic models and its special homogeneous counterpart.

We now face the question whether the holographic principle has to be challenged or it can be used to select a physically acceptable model. We prefer the second, more constructive, alternative.

It is well believed that the entropy of the present observable universe is of order $10^{90}$ ([6,8]). This reasonable entropy value can be used as a standard to select models describing the real cosmos.

Fig. 2: Inhomogeneous models which can accommodate reasonable entropy to meet the present observable value.

Fig.2(a) shows that for $\alpha = 10^{-4}, 3.50 \leq p \leq 4.0$, the entropy values of the universe described by the fractal model is around $10^{90}$, (b) shows that for $\alpha = 10^{-5}$, the range of $p$ changes to $3.91 \leq p \leq 4.0$ to meet the required reasonable entropy. The influence of different $\beta_0$ is small. These results show us how to characterize the constants of the model. It is worth noting that the range of $p$ violating the holographic principle has been excluded here, which corresponds to say that if these fractal models describe the real universe, they must satisfy the holographic principle.

The dependence of the entropy value on constants of the model shown in Fig.2 is similar to that of $r_{AH}$. Bigger values of $p$ leads to smaller values of entropy, and for the same $p$, bigger $\alpha$ brings smaller entropy.

Now we extend our discussion to Model 2. Using (8), the proper apparent horizon can be got from

$$\alpha r_{AH}^p = 3[t + \ln(e^{\beta_0} + \eta_1 r_{AH})].$$

(13)

In contrast to Model 1, apparent horizon can be found for all constants displaying fractal behavior. $p$ and $\alpha$ play the same crucial role to influence the result of $r_{AH}$. While the influence of $\beta_0, \eta_1$ is not important. The area of the apparent horizon in Planck units is expressed by (11) and the total entropy inside the apparent horizon is

$$S = 2.76 \times 10^{86} \pi \times$$

$$\int_0^{r_{AH}} \frac{\beta}{2} (t + \ln(e^{\beta_0} + \eta_1 r)^2 \frac{dr}{(t + \ln(e^{\beta_0} + \eta_1 r) r^{-1} + e^{\beta_0} + \eta_1 r)^2)}.$$ 

(14)

We found that the holographic principle always holds for this model. However for small values of $p$, the result for the entropy is again too big to meet the requirement of describing an observable realistic universe. In order to make the inhomogeneous model reasonable to describe the realistic universe, we need the observed entropy value as a criteria to choose reasonable constants in the model.
These regions of constants are required to delineate the inhomogeneous dust universe supports the behavior illustrated in homogenous cosmology [8], which shows that the entropy in the present observable universe. We found that the real cosmos by this fractal model. From Fig.3 we find that different $\eta_1$ does not change a lot of the final result. This behavior also holds for $\beta_0$.

In summary, considering properties of spherical symmetry and not relating to either initial or final moment for inhomogeneous dust universes, we introduced a simple holographic principle by asserting that all information about physical processes in the real cosmological setting can be stored on the surface of the apparent horizon. Investigating fractal parabolic models with the holographic principle, we found that the violation of the holographic principle appears in some fractal parabolic models, what has never been observed in any special flat homogeneous universe. In order to describe the real cosmos, we refined the fractal parabolic models by restricting constants choosing regions to get reasonable entropy values of the present observable universe. We found that the realistic models for an inhomogeneous universe satisfy the holographic principle. The slowly increasing of the entropy value with evolution of time in the inhomogeneous dust universe supports the behavior illustrated in homogeneus cosmology [8], which shows that the entropy in the universe is mainly created before the dust-filled era.

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\[\text{Fig. 3: Choosing parameters in order to meet the entropy value in the present observable universe.}\]