OPENNESS RESULTS FOR UNIFORM K-STABILITY

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Abstract. Assume that a projective variety together with a polarization is uniformly K-stable. If the polarization is canonical or anti-canonical, then the projective variety is uniformly K-stable with respects to any polarization sufficiently close to the original polarization.

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1. Introduction

In this article, we work over an arbitrary algebraically closed field \( k \) of characteristic zero. A variety is assumed to be a connected, reduced, separated and of finite type scheme over \( \text{Spec} \, k \). For the minimal model program, we refer the readers to [KM98] and [Kol13].

In this article, we show the following result:

Theorem 1.1 (see Corollaries 6.4 and 6.8). Let \((X, \Delta)\) be a projective slc pair and \( L \) be an ample \( \mathbb{Q} \)-line bundle on \( X \). Assume that \(((X, \Delta), L)\) is uniformly K-stable. If \( L = K_X + \Delta \) or \( L = -(K_X + \Delta) \), then there exists a Euclidean open neighborhood \( U \subset N^1(X)_R \) of \( L \) such that \(((X, \Delta), L')\) is uniformly K-stable for any \( \mathbb{Q} \)-line bundle \( L' \) with \( L' \in U \).

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Let \((X, \Delta)\) be a projective slc pair and \(L\) be an ample \(\mathbb{Q}\)-line bundle on \(X\). Motivated by the fundamental works [Tia97, Don02, Sze06, Sze15, Der16, BHJ15], we are interested in whether \(((X, \Delta), L)\) is uniformly K-stable or not. However, many basic properties of uniform K-stability remain unknown. For example, it is expected that the uniform K-stability of \(((X, \Delta), L)\) implies the uniform K-stability of \(((X, \Delta), L')\) for any ample \(\mathbb{Q}\)-line bundle \(L'\) such that \(L'\) is very close to \(L\) in \(N^1(X)_{\mathbb{R}}\). In fact, LeBrun and Simanca showed in [LS94, Corollary 2] that, if \((M, J, \omega)\) is a compact constant scalar curvature Kähler manifold with the automorphism group of \(M\) semisimple, then there exists a Euclidean open neighborhood \(U \subset H^{1,1}(M, \mathbb{R})\) of the class \([\omega]\) such that each element in \(U\) can be represented by the Kähler form of a constant scalar curvature Kähler metric. The problem is hard to prove in general. Theorem 1.1 gives a partial affirmative answer of the problem.

The idea of the proof of Theorem 1.1 is simple. Firstly we perturb both the boundary and the polarization. Secondly we perturb only the boundary. For the first step, when \(L = -(K_X + \Delta)\), we use Odaka’s theorem [Odk13b] and a valuative criterion for uniform K-stability of log Fano pairs established in [Fuj16a, Li16, Fuj16b, Fuj17] (see Theorem 5.7); when \(L = K_X + \Delta\), we use the result on the uniform bounds of Donaldson-Futaki invariants divided by certain norms [BHJ15, Corollary 9.3] (see Theorem 4.6 (1)). In particular, when \(L = -(K_X + \Delta)\), we will show the following theorem.

**Theorem 1.2** (see Theorem 5.7 (1)). Let \((X, \Delta)\) be an \(n\)-dimensional log Fano pair and set \(L := -(K_X + \Delta)\). Assume that \(((X, \Delta), L)\) is uniformly K-stable. Set

\[
\varepsilon := \frac{\delta(X, \Delta) - 1}{n \cdot \delta(X, \Delta) + n + 1}.
\]

Then, for any effective \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor \(B\) on \(X\) with \(\varepsilon L - B\) ample (resp., nef), \((X, \Delta + B)\) is a log Fano pair and \(((X, \Delta + B), L - B)\) is uniformly K-stable (resp., K-semistable). (For the definitions, see §2, §4 and §5.)

For the second step, we will show the following proposition.

**Proposition 1.3** (=Proposition 6.1). Let \((X, \Delta)\) be an \(n\)-dimensional projective demi-normal pair, \(L\) be an ample \(\mathbb{Q}\)-line bundle and \(N\) be an effective and nef \(\mathbb{Q}\)-divisor on \(X\). Then, for any semiample demi-normal test configuration \((X, \mathcal{L})/\mathbb{P}^1\) of \((X, L)\), we have

\[
n \mu_N(L) J^N \langle X, \mathcal{L} \rangle \geq DF_{\Delta+N}(X, \mathcal{L}) - DF_{\Delta}(X, \mathcal{L}).
\]
(For the definitions, see [3, 4])

As corollaries of the discussion in the first step and Proposition 1.3, we get Theorem 1.1. Moreover, in the proof, we show the following theorem:

**Theorem 1.4** (=Theorem 6.5. See also [Wei06, SW07]) Let \((X, \Delta)\) be an \(n\)-dimensional projective slc pair with \(n \geq 2\) and let \(L\) be an ample \(\mathbb{Q}\)-line bundle on \(X\). Assume that \(\mu_{KX+\Delta}(L) > 0\) and

\[
\frac{n^2}{n^2 - 1} \mu_{KX+\Delta}(L) L - (K_X + \Delta)
\]

is ample (resp., nef), where

\[
\mu_{KX+\Delta}(L) := \frac{(L^{n-1} \cdot (K_X + \Delta))}{(L^n)}.
\]

Then \(((X, \Delta), L)\) is uniformly K-stable (resp., K-semistable).

The article is organized as follows. In Section 2, we recall basic theories of demi-normal varieties and we see some properties of the cones of pseudo-effective or nef divisors in the \(\mathbb{R}\)-tensors of the Néron-Severi groups. In Section 3, we recall the definition of test configurations. Moreover, we establish a fundamental theory of demi-normal test configurations of demi-normal polarized pairs. Thanks to the theory, we do not need to consider almost trivial test configurations in the sense of [BHJ15, Definition 2.9]. In Section 4, we define the notions of uniform K-stability and K-semistability. Moreover, we recall fundamental results of Odaka [Odk12, Odk13a, Odk13b]. In Section 5, we recall the theory established in [Fuj16a, Li16, Fuj16b, Fuj17]. Moreover, we show in Theorem 5.7 that, if \(((X, \Delta), -(K_X + \Delta))\) is uniformly K-stable, then \(((X, \Delta + B), -(K_X + \Delta + B))\) is also uniformly K-stable for any effective and very small \(B\). In Section 6, we prove Theorem 1.1 by showing Proposition 1.3.

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## 2. Preliminaries

### 2.1. Demi-normal pairs

We recall the notion of demi-normal varieties. A standard reference is [KO13, §5].
Definition 2.1 ([Kol13 §5.1]). (1) An equi-dimensional variety $X$ is said to be a demi-normal variety if $X$ satisfies Serre’s $S_2$ condition and any codimension one point $\eta \in X$ satisfies that either $\mathcal{O}_{X, \eta}$ is regular or double normal crossing.

(2) Let $X$ be a demi-normal variety and let $\nu : \bar{X} \to X$ be the normalization. The conductor ideal of $X$ is defined to be

\[ \text{cond}_X := \text{Hom}_{\mathcal{O}_X} (\nu_* \mathcal{O}_{\bar{X}}, \mathcal{O}_X) \subset \mathcal{O}_X. \]

This ideal sheaf can be seen as an ideal sheaf $\text{cond}_{\bar{X}} \subset \mathcal{O}_{\bar{X}}$, named the conductor ideal of $\bar{X}/X$. We define

\[ D_X := \text{Spec}_X (\mathcal{O}_X / \text{cond}_X), \quad D_{\bar{X}} := \text{Spec}_{\bar{X}} (\mathcal{O}_{\bar{X}} / \text{cond}_{\bar{X}}), \]

and say them the conductor divisor of $X$, the conductor divisor of $\bar{X}/X$, respectively. Let $D_{\bar{X}}$ be the normalization of $D_X$. Then we can get the natural Galois involution $\tau_X : D_{\bar{X}} \to D_{\bar{X}}$.

(3) Let $X$ be a demi-normal variety. A divisor (resp., a $\mathbb{Q}$-divisor) on $X$ is a formal finite $\mathbb{Z}$-linear (resp., $\mathbb{Q}$-linear) sum $\sum a_i \Delta_i$ such that each $\Delta_i$ is an irreducible and reduced codimension one subvariety of $X$ with $\Delta_i \not\subset D_X$.

(4) A pair $(X, \Delta)$ is said to be a demi-normal pair (resp., a normal pair) if $X$ is a demi-normal variety (resp., a normal variety) and $\Delta$ is a $\mathbb{Q}$-divisor on $X$ such that each coordinates belongs to the set $[0, 1]$, and $K_X + \Delta$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$, where $K_X$ is the canonical divisor on $X$.

(5) For any $n$-dimensional demi-normal projective variety $X$, for any ample $\mathbb{Q}$-line bundle $L$ on $X$, and for any $\mathbb{Q}$-divisor $\Delta$ on $X$, we set

\[ \mu_\Delta (L) := \frac{(L^{n-1} \cdot \Delta)}{(L^n)}. \]

We use the following proposition later.

Proposition 2.2 ([Kol13 Proposition 5.3]). Let $X$ be a demi-normal variety, let $\bar{X}, D_{\bar{X}}, \tau_X$ be as in Definition 2.1 (2). Then the triplet $(\bar{X}, D_{\bar{X}}, \tau_X)$ uniquely determines $X$.

Definition 2.3 (see [Kol13 §5.2] for example). Let $(X, \Delta)$ be a demi-normal pair and let $\nu : \bar{X} \to X$ be the normalization. Set $\bar{\Delta} := \nu_*^{-1} \Delta$ and let $D_{\bar{X}}$ be the conductor divisor of $\bar{X}/X$. Then $(\bar{X}, D_{\bar{X}} + \Delta)$ is a (possibly non-connected) normal pair.

Let $F$ be a prime divisor over $\bar{X}$, that is, there exists a projective birational morphism $\sigma : Y \to \bar{X}$ with $Y$ a (possibly non-connected) normal variety and $F$ a prime divisor on $Y$. 
(1) We set the log discrepancy $A_{(X, \Delta)}(F)$ of $(X, \Delta)$ along $F$ as
$$A_{(X, \Delta)}(F) := 1 + \operatorname{ord}_F(K_Y - \sigma^*(K_X + D + \Delta)).$$

(2) The pair $(X, \Delta)$ is said to be a semi log canonical pair (slc pair, for short) if $A_{(X, \Delta)}(F) \geq 0$ holds for any prime divisor $F$ over $\bar{X}$.

(3) The pair $(X, \Delta)$ is said to be a Kawamata log terminal pair (klt pair, for short) if $A_{(X, \Delta)}(F) > 0$ holds for any prime divisor $F$ over $\bar{X}$. If $(X, \Delta)$ is a klt pair, then $(X, \Delta)$ must be a normal pair.

(4) The pair $(X, \Delta)$ is said to be a log Fano pair if $(X, \Delta)$ is a projective klt pair and $-(K_X + \Delta)$ is an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor.

2.2. On cones of Cartier divisors. For a projective variety $X$, the $\mathbb{R}$-tensor of the Néron-Severi group $N^1(X)_\mathbb{R}$ of $X$ is a finite dimensional vector space over $\mathbb{R}$. Moreover, the nef cone $\operatorname{Nef}(X) \subset N^1(X)_\mathbb{R}$ is a closed and strongly convex cone. Moreover, if $X$ is normal, then the pseudo-effective cone $\overline{\operatorname{Eff}}(X) \subset N^1(X)_\mathbb{R}$ is also a closed and strongly convex cone. See [Laz04a] and [Nak04] for example.

**Lemma 2.4.** Let $(X, \Delta)$ be a log Fano pair. Then $\operatorname{Nef}(X) \subset N^1(X)_\mathbb{R}$ is spanned by the classes of finitely many semiample Cartier divisors. Moreover, $\overline{\operatorname{Eff}}(X) \subset N^1(X)_\mathbb{R}$ is spanned by the classes of finitely many effective Cartier divisors.

**Proof.** By [BCHM10, Corollary 1.4.3], there exists a small, projective and birational morphism $\sigma: \bar{X} \to X$ with $\bar{X}$ $\mathbb{Q}$-factorial. Since $(\bar{X}, \sigma^{-1}_*\Delta)$ is a klt pair and $-(K_{\bar{X}} + \sigma^{-1}_*\Delta)$ is nef and big, there exists a $\mathbb{Q}$-divisor $\bar{\Delta} \geq \sigma^{-1}_*\Delta$ such that $(\bar{X}, \bar{\Delta})$ is a log Fano pair (see also [Fuj16a, Lemma 2.1]). Thus $\bar{X}$ is a Mori dream space in the sense of [HK00] by [BCHM10, Corollary 1.3.2]. By [HK00, Definition 1.10 (2) and Proposition 1.11 (2)], $\operatorname{Nef}(\bar{X})$ is spanned by the classes of finitely many semiample Cartier divisors and $\overline{\operatorname{Eff}}(\bar{X})$ is spanned by the classes of finitely many effective Cartier divisors. It is easy to see that, under the natural linear inclusion
$$\sigma^*: N^1(X)_\mathbb{Q} \hookrightarrow N^1(\bar{X})_\mathbb{Q},$$
we have $\operatorname{Nef}(X) = \operatorname{Nef}(\bar{X}) \cap N^1(X)_\mathbb{R}$ and $\overline{\operatorname{Eff}}(X) = \overline{\operatorname{Eff}}(\bar{X}) \cap N^1(X)_\mathbb{R}$. Thus we get the assertion. □

The following proposition is intrinsically trivial.
Proposition 2.5. Let $X$ be an $n$-dimensional demi-normal projective variety, $L$ be an ample $\mathbb{Q}$-line bundle on $X$. Fix any norm $\| \cdot \|$ on $N^1(X)_{\mathbb{R}}$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that the set
\[
C_\delta := \{ (L + \xi) \in N^1(X)_{\mathbb{R}} \mid t \in \mathbb{R}_{\geq 0}, \xi \in N^1(X)_{\mathbb{R}}, \| \xi \| \leq \delta \}
\]
is a subset of $C^2_\varepsilon \cap C^3_\varepsilon$, where
\[
C^2_\varepsilon := \{ (L - a) \in N^1(X)_{\mathbb{R}} \mid t \in \mathbb{R}_{\geq 0}, a \in \text{Nef}(X)_{\mathbb{R}}, \| a \| \leq \varepsilon \},
\]
\[
C^3_\varepsilon := \{ (L + a) \in N^1(X)_{\mathbb{R}} \mid t \in \mathbb{R}_{\geq 0}, a \in \text{Nef}(X)_{\mathbb{R}}, \| a \| \leq \varepsilon \}.
\]

Proof. Set $\rho := \dim_{\mathbb{R}} N^1(X)_{\mathbb{R}}$. Since $L$ is ample, there exist linearly independent $a_1, \ldots, a_\rho \in N^1(X)_{\mathbb{R}}$ with $a_1, \ldots, a_\rho \in \text{Nef}(X)$ and there exist $t_1, \ldots, t_\rho \in \mathbb{R}_{>0}$ such that $L = \sum_{i=1}^\rho t_i a_i$ and $\sum_{i=1}^\rho t_i = 1$. Set $t_0 := \min_i t_i \in \mathbb{R}_{>0}$. We may assume that the norm $\| \cdot \|$ is given by
\[
\left\| \sum_{i=1}^\rho s_i a_i \right\| := \sum_{i=1}^\rho |s_i|.
\]

Take any $\delta \in (0, t_0)$ and any $\xi = \sum_{i=1}^\rho \xi_i a_i \in N^1(X)_{\mathbb{R}}$ with $\| \xi \| \leq \delta$ (i.e., $\sum_{i=1}^\rho |\xi_i| \leq \delta$). We note that $t_0^{-1} \delta t_i \geq |\xi_i|$ for any $i$. Moreover, we have
\[
L + \xi = (1 + t_0^{-1} \delta) \left( L - \frac{1}{1 + t_0^{-1} \delta} \sum_{i=1}^\rho (t_0^{-1} \delta t_i - \xi_i) a_i \right),
\]
\[
L + \xi = (1 - t_0^{-1} \delta) \left( L + \frac{1}{1 - t_0^{-1} \delta} \sum_{i=1}^\rho (t_0^{-1} \delta t_i + \xi_i) a_i \right).
\]
The classes
\[
\frac{1}{1 + t_0^{-1} \delta} \sum_{i=1}^\rho (t_0^{-1} \delta t_i - \xi_i) a_i,
\frac{1}{1 - t_0^{-1} \delta} \sum_{i=1}^\rho (t_0^{-1} \delta t_i + \xi_i) a_i
\]
are nef and those norms are bounded below by $(t_0^{-1} + 1) \delta/(1 - t_0^{-1} \delta)$. As a consequence, if we take $\delta \in (0, t_0)$ with $(t_0^{-1} + 1) \delta/(1 - t_0^{-1} \delta) \leq \varepsilon$, then we get $C_\delta \subset C^2_\varepsilon \cap C^3_\varepsilon$. \hfill \Box

3. Demi-normal test configurations

In this section, we see a fundamental theory for test configurations of demi-normal polarized varieties. In Section 3, we always assume that $X$ is a demi-normal projective variety and $L$ is an ample $\mathbb{Q}$-line bundle on $X$.

Definition 3.1. (1) (see [Tia97, Don02]) A semiample test configuration (resp., an ample test configuration) $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ of $(X, L)$ consists of:
• a projective variety $\mathcal{X}$ together with a flat morphism $\mathcal{X} \to \mathbb{P}^1$,
• a $\pi$-semiample (resp., a $\pi$-ample) $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$,
• a holomorphic $\mathbb{G}_m$-action $\mathbb{G}_m \curvearrowright (\mathcal{X}, \mathcal{L})$ commuting with the multiplicative action $\mathbb{G}_m \curvearrowright \mathbb{P}^1$, and they satisfy that

$(\mathcal{X} \setminus \mathcal{X}_0, \mathcal{L}|_{\mathcal{X} \setminus \mathcal{X}_0})$ is $\mathbb{G}_m$-equivariantly isomorphic to $(\mathcal{X} \times (\mathbb{P}^1 \setminus \{0\}), p_1^* \mathcal{L})$ with the natural $\mathbb{G}_m$-action, where $\mathcal{X}_0$ is the scheme-theoretic fiber of $\mathcal{X} \to \mathbb{P}^1$ at $0 \in \mathbb{P}^1$ and $p_1: \mathcal{X} \times (\mathbb{P}^1 \setminus \{0\}) \to \mathcal{X}$ is the first projection.

(2) Let $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ be a semiample test configuration of $(X, L)$. $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ is said to be a demi-normal test configuration if

(i) $\mathcal{X}$ is a demi-normal variety, and
(ii) for any generic point $\eta \in \mathcal{X}_0$, the local ring $\mathcal{O}_{\mathcal{X}, \eta}$ is regular. If $X$ is normal, then a demi-normal test configuration is called a normal test configuration. (Note that, for the definition of a normal test configuration, the condition (2ii) follows immediately from the condition (2i).)

(3) For a semiample, demi-normal test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ and for a $\mathbb{Q}$-divisor $\Delta$ on $X$, let $\Delta_{\mathcal{X}}$ be the $\mathbb{Q}$-divisor on $\mathcal{X}$ defined by the closure of $\Delta \times (\mathbb{P}^1 \setminus \{0\})$ under the canonical isomorphism $\mathcal{X} \setminus \mathcal{X}_0 \simeq X \times (\mathbb{P}^1 \setminus \{0\})$.

We recall the notion of the partial normalization of test configurations which is important for our study.

**Definition 3.2** ([Odk13a, §3], [Odk13b, §5]). Let $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ be a semiample (resp., an ample) test configuration of $(X, L)$. Let $i: \mathcal{X} \setminus \mathcal{X}_0 \hookrightarrow \mathcal{X}$ be the inclusion and let $\nu: \bar{\mathcal{X}} \to \mathcal{X}$ be the normalization. Set

$$\mathcal{X}^{\nu} := \text{Spec}_X(i_*\mathcal{O}_{\mathcal{X} \setminus \mathcal{X}_0} \cap \nu_*\mathcal{O}_{\bar{\mathcal{X}}})^{\nu} \to \mathcal{X}.$$  

From the definition, $\nu$ factors through $\bar{\mathcal{X}} \to \mathcal{X}^{\nu} \to \mathcal{X}$. Of course, $(\mathcal{X}^{\nu}, \nu^*\mathcal{L})/\mathbb{P}^1$ is also a semiample (resp., an ample) test configuration of $(X, L)$. We call it the partial normalization of $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$.

**Proposition 3.3.** The above $(\mathcal{X}^{\nu}, \nu^*\mathcal{L})/\mathbb{P}^1$ is a demi-normal test configuration.

**Proof.** For any generic point $\eta \in \bar{\mathcal{X}}_0$, the morphism $\bar{\mathcal{X}} \to \mathcal{X}^{\nu}$ is an isomorphism at $\eta$ by [Odk13a, Lemma 3.9]. Thus it is enough to check that $\mathcal{X}^{\nu}$ satisfies Serre’s $S_2$ condition. Take any $x \in \mathcal{X}_0$. Take an affine open subvariety $x \in U \simeq \text{Spec} R$ around $x \in \mathcal{X}$. Then, around over $x \in \mathcal{X}$, $\mathcal{X}^{\nu}$ is written as the spectrum of the following $\mathbb{K}$-algebra

$$R^{\nu} := R[t^{-1}] \cap \bar{R},$$
where \( t \in R \) is the non-homogeneous coordinate of \( \mathbb{P}^1 \) and \( \bar{R} \) is the integral closure of \( R \) in the total quotient ring \( K \) of \( R \).

For any \( S \in \{ R^\nu, R[t^{-1}], \bar{R} \} \) and for any \( a \in K \), let us set
\[
D_S(a) := \{ s \in S | as \in S \}
\]
as in [HH94 (2.3)]. This is an ideal of \( S \). Moreover, let us consider the extension
\[
\tilde{S} := \{ a \in K | \text{ht} D_S(a) \geq 2 \}
\]
of \( S \). (We set \( \text{ht} S := +\infty \).) By [HH94 Proposition (2.4)] (see also [Ciu01 Remark 1.4]), the ring \( S \) satisfies Serre’s \( S_2 \) condition if and only if \( \tilde{S} = S \) holds. In particular, we have \( \bar{R}[t^{-1}] = R[t^{-1}] \) and \( \bar{\bar{R}} = \bar{R} \).

Take any \( a \in \tilde{R}^\nu \). Since \( \text{ht} D_{R^\nu}(a) \geq 2 \), we have \( \text{ht} D_{R^\nu}(a) \bar{R} \geq 2 \). Since \( D_{R^\nu}(a) \bar{R} \subset D_{\bar{R}}(a) \), we have \( a \in \tilde{\bar{R}} = \bar{R} \). On the other hand, we know that \( \bar{R}^\nu[t^{-1}] = R[t^{-1}] \). Since \( \text{ht} D_{\bar{R}^\nu}(a) \bar{R}[t^{-1}] \geq 2 \) and \( D_{R^\nu}(a) \bar{R}[t^{-1}] \subset D_{\bar{R}^\nu}[t^{-1}](a) \), we have \( a \in \bar{R}[t^{-1}] = R[t^{-1}] \). These imply that \( a \in \bar{R}^\nu \). Therefore, \( R^\nu \) satisfies Serre’s \( S_2 \) condition. \( \square \)

**Remark 3.4.** Let \((\mathcal{X}, \mathcal{L})/\mathbb{P}^1\) be a semialbe test configuration of \((X, L)\). Assume that the local ring \( \mathcal{O}_{\mathcal{X}, \eta} \) is regular for any generic point \( \eta \in X_0 \). Then the partial normalization \( \mathcal{X}^\nu \) is nothing but the \( S_2 \)-ification of \( \mathcal{X} \) (in the sense of [Gro65 Proposition (5.10.10) and (5.10.11)], [HH94 (2.3)], [Vas05 Definition 6.20] and [LN16 §4]) by Proposition 3.3. In the word of [Kol13 Definition 5.1], \( \mathcal{X}^\nu \) is the demi-normalization of \( \mathcal{X} \). In particular, any demi-normal test configuration is equal to its partial normalization.

**Lemma 3.5.** Let \((\mathcal{X}, \mathcal{L})/\mathbb{P}^1\), \((\mathcal{Y}, \mathcal{M})/\mathbb{P}^1\) be test configurations of \((X, L)\). Assume that there exists a \( \mathbb{G}_m \)-equivariant birational morphism \( \phi: \mathcal{X} \to \mathcal{Y} \) over \( \mathbb{P}^1 \). Then \( \phi \) lifts to the partial normalizations \( \phi^\nu: \mathcal{X}^\nu \to \mathcal{Y}^\nu \).

**Proof.** Let \( \nu_\mathcal{X}: \bar{\mathcal{X}} \to \mathcal{X}, \nu_\mathcal{Y}: \bar{\mathcal{Y}} \to \mathcal{Y} \) be the normalizations, and let \( i_\mathcal{X}: \mathcal{X} \setminus X_0 \to \mathcal{X}, i_\mathcal{Y}: \mathcal{Y} \setminus Y_0 \to \mathcal{Y} \) be the inclusions. We have a natural commutative diagram
\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\phi} & \mathcal{Y} \\
\nu_\mathcal{X} \downarrow & & \downarrow \nu_\mathcal{Y} \\
\bar{\mathcal{X}} & \xrightarrow{\phi^\nu} & \bar{\mathcal{Y}}.
\end{array}
\]
The morphism \( \bar{\mathcal{X}} \to \mathcal{X} \times_\mathcal{Y} \bar{\mathcal{Y}} \) over \( \mathcal{X} \) induces a homomorphism
\[
\phi^*(\nu_\mathcal{Y})_*\mathcal{O}_{\bar{\mathcal{Y}}} \to (\nu_\mathcal{X})_*\mathcal{O}_{\bar{\mathcal{X}}}
\]
of coherent $\mathcal{O}_X$-algebras. On the other hand, we have a natural homomorphism
\[ \phi^*(i_Y)_*\mathcal{O}_{Y\setminus Y_0} \rightarrow (i_X)_*\mathcal{O}_{X\setminus X_0} \]
of quasi-coherent $\mathcal{O}_X$-algebras. Thus we get a natural homomorphism
\[ \phi^* \left( (\nu_Y)_*\mathcal{O}_{\bar{Y} \cap (i_Y)_*\mathcal{O}_{Y\setminus Y_0}} \right) \rightarrow (\nu_X)_*\mathcal{O}_{\bar{X} \cap (i_X)_*\mathcal{O}_{X\setminus X_0}} \]
of coherent $\mathcal{O}_X$-algebras. The homomorphism induces a morphism $X^\nu \rightarrow X \times_Y Y^\nu$ over $X$.

\[ \Box \]

**Corollary 3.6.** Let $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ be a semiample, demi-normal test configuration of $(X, L)$. Let $\phi: (\mathcal{X}, \mathcal{L}) \rightarrow (\mathcal{Y}, \mathcal{M})/\mathbb{P}^1$ be the ample model of $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ in the sense of [BHJ15, Definition 2.16], i.e., $\phi$ is a projective birational morphism with $\phi_*\mathcal{O}_X = \mathcal{O}_Y$ and $(\mathcal{Y}, \mathcal{M})/\mathbb{P}^1$ is an ample test configuration of $(X, L)$ with $\phi^*\mathcal{M} \sim_{\mathbb{Q}} \mathcal{L}$. Then $(\mathcal{Y}, \mathcal{M})/\mathbb{P}^1$ is a demi-normal test configuration of $(X, L)$.

**Proof.** Take any generic point $\eta \in Y_0$. Since $\phi_*\mathcal{O}_X = \mathcal{O}_Y$ and the morphism $\mathcal{X} \rightarrow \mathbb{P}^1$ is flat, the morphism $\phi$ is an isomorphism over a neighborhood of $\eta$ by Zariski’s main theorem (see [Liu02, Proposition 4.4.2] for example). Thus it is enough to check that $Y$ satisfies Serre’s $S_2$ condition. Let $\nu: \bar{Y}^\nu \rightarrow Y$ be the partial normalization. By Remark 3.4, the partial normalization of $\mathcal{X}$ is $\mathcal{X}$ itself. Thus, together with Lemma 3.5, we get the following commutative diagram

\[ \begin{array}{ccc}
X & \xrightarrow{\phi} & Y^\nu \\
\downarrow & & \downarrow_{\nu} \\
\phi^\nu & \xrightarrow{\phi} & \bar{X} \\
\end{array} \]

Note that the composition of the inclusions
\[ \mathcal{O}_Y \hookrightarrow \nu_*\mathcal{O}_{Y^\nu} \hookrightarrow \nu_*\phi^\nu_*\mathcal{O}_X = \mathcal{O}_Y \]
is an identity. Thus $\mathcal{O}_Y = \nu_*\mathcal{O}_{Y^\nu}$. This implies that $Y^\nu = Y$. \[ \Box \]

**Definition 3.7.** Let $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ be a semiample, demi-normal test configuration of $(X, L)$ and let $\phi: (\mathcal{X}, \mathcal{L}) \rightarrow (\mathcal{Y}, \mathcal{M})/\mathbb{P}^1$ be the ample model of $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$. (By Corollary 3.6, $(\mathcal{Y}, \mathcal{M})/\mathbb{P}^1$ is an ample, demi-normal test configuration of $(X, L)$.) $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ is said to be a trivial test configuration of $(X, L)$ if $(\mathcal{Y}, \mathcal{M})/\mathbb{P}^1$ is $\mathbb{G}_m$-equivariantly isomorphic to $(X \times \mathbb{P}^1, p_1^*L)$ with the natural $\mathbb{G}_m$-action, where $p_1: X \times \mathbb{P}^1 \rightarrow X$ is the first projection.

**Definition 3.8.** Let $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ be a semiample (resp., an ample), demi-normal test configuration of $(X, L)$. Let $\nu: \bar{X} \rightarrow X$, $\nu: \bar{X} \rightarrow \mathcal{X}$ be the normalizations. Then $(\bar{X}, \nu^*\mathcal{L})/\mathbb{P}^1$ is a (possibly non-connected)
semiample (resp., ample), normal test configuration of \((\bar{X}, \nu^*L)\). We call this the associated normal test configuration of \((\bar{X}, \nu^*L)\).

The following proposition is useful in order to check whether a given demi-normal test configuration is trivial or not.

**Proposition 3.9.** Let \(((X, L))/\mathbb{P}^1\) be an ample, demi-normal test configuration of \((X, L)\) and let \(((\bar{X}, \nu^*L))/\mathbb{P}^1\) be the associated ample, normal test configuration of \((\bar{X}, \nu^*L)\). Then \(((\bar{X}, \nu^*L))/\mathbb{P}^1\) is the trivial test configuration if and only if (any connected component of) \(((\bar{X}, \nu^*L))/\mathbb{P}^1\) is so.

**Proof.** Let \(D_{\bar{X}} \subset \bar{X}\) be the conductor divisor of \(\bar{X}/X\), let \(\bar{D}_{\bar{X}}\) be its normalization and let \(\tau_{\bar{X}}: \bar{D}_{\bar{X}} \to \bar{D}_{\bar{X}}\) be the natural involution as in Definition 2.1 (1). Similarly, let \(D_{\bar{X}} \subset \bar{X}\) be the conductor divisor of \(\bar{X}/X\), let \(\bar{D}_{\bar{X}}\) be its normalization and let \(\tau_{\bar{X}}: \bar{D}_{\bar{X}} \to \bar{D}_{\bar{X}}\) be the natural involution. Assume that \(((\bar{X}, \nu^*L))/\mathbb{P}^1\) is the trivial test configuration. Then \(\bar{X}\) is \(\mathbb{G}_m\)-equivariantly isomorphic to \(\bar{X} \times \mathbb{P}^1\). Thus \(\bar{D}_{\bar{X}}\) is \(\mathbb{G}_m\)-equivariantly isomorphic to \(D_{\bar{X}} \times \mathbb{P}^1\). Moreover, the involutions \(\tau_{\bar{X}} \times \text{id}_{\mathbb{P}^1} = \tau_{X \times \mathbb{P}^1}: \bar{D}_{\bar{X}} \times \mathbb{P}^1 \to \bar{D}_{\bar{X}} \times \mathbb{P}^1\) must be equal to \(\tau_{\bar{X}}: \bar{D}_{\bar{X}} \to \bar{D}_{\bar{X}}\) under the above \(\mathbb{G}_m\)-equivariant isomorphism since \(\tau_{X \times \mathbb{P}^1}\) and \(\tau_{\bar{X}}\) are equal over \(\mathbb{P}^1 \setminus \{0\}\). Thus we get the assertion from Proposition 2.2. \(\square\)

4. Uniform K-stability

We recall the definition of Donaldson-Futaki invariants and various stability conditions.

**Definition 4.1.** Let \((X, \Delta)\) be an \(n\)-dimensional projective demi-normal pair, let \(L\) be an ample \(\mathbb{Q}\)-line bundle on \(X\) and let \(((\mathcal{X}, \mathcal{L}))/\mathbb{P}^1\) be a semiample, demi-normal test configuration of \((X, L)\). Let

\[
\xymatrix{ \mathcal{V} \ar@{_{(}->}[d]_{\Theta} \ar@{_{(}->}[r]^\Pi & X \times \mathbb{P}^1 \ar@{_{(}->}[l]_{\mathcal{X}} \ar@{_{(}->}[l] \ar@{_{(}->}[l] \ar@{_{(}->}[l] \ar@{_{(}->}[l] \ar@{_{(}->}[l] }
\]
be the partial normalization of the graph of the natural birational map \(\mathcal{X} \to X \times \mathbb{P}^1\). Let \(p_1: X \times \mathbb{P}^1 \to X\) be the first projection.

(1) ([BHJ15 Definition 7.6]) We set

\[
J^{NA}(\mathcal{X}, \mathcal{L}) := \frac{1}{(L^n)} \left( (\Theta^* \mathcal{L} \cdot p_1^* L^n) - \frac{1}{n+1} (\mathcal{L}^{n+1}) \right).
\]
We set the log Donaldson-Futaki invariant as
\[
\text{DF}_\Delta(X, L) := \frac{1}{(L^n)} \left( (L^n \cdot (K_{X/P^1} + \Delta_X)) - \frac{n}{n+1} \mu_{K_{X/P^1}}(L)(L^{n+1}) \right),
\]
where \( K_{X/P^1} \) is \( K_X \) minus the pullback of \( K_{P^1} \).

**Proposition 4.2.** Let \((X, \Delta), L, (X, L)/P^1\) be as in Definition 4.1.

1. For any projective birational morphism \( \phi: (Z, \phi^*L) \to (X, L) \) between demi-normal test configurations of \((X, L)\), we have
\[
J^{\text{NA}}(X, L) = J^{\text{NA}}(Z, \phi^*L),
\]
\[
\text{DF}_\Delta(X, L) = \text{DF}_\Delta(Z, \phi^*L).
\]

2. Let \((\bar{X}, \nu^*L)/P^1\) be the associated normal test configuration of \((X, L)/P^1\). Then we have
\[
J^{\text{NA}}(X, L) = J^{\text{NA}}(\bar{X}, \nu^*L),
\]
\[
\text{DF}_\Delta(X, L) = \text{DF}_{\bar{X} + \Delta}(\bar{X}, \nu^*L),
\]
where \( \bar{\Delta} := \nu^{-1}\Delta \) on \( \bar{X} \) and \( D_X \) is the conductor divisor of \( \bar{X}/X \).

3. We have \( J^{\text{NA}}(X, L) \geq 0 \). Moreover, equality holds if and only if \((X, L)/P^1\) is a trivial test configuration of \((X, L)\).

**Proof.** \(1\) is trivial. See [Der16, BHJ15] for example.

\(2\) For the normalization \( \nu: \bar{X} \to X \), we know that \( \nu^*K_X = K_{\bar{X}} + D_X \). The assertion immediately follows from this fact.

\(3\) By \(1\), after replacing its ample model (see Corollary 3.6), we may assume that \((X, L)/P^1\) is an ample, demi-normal test configuration. If \( X \) is normal, then the assertion follows from [Der16, Theorem 1.3] and [BHJ15, Theorem 7.9]. For a general case, it follows from the normal case, \(2\) and Proposition 3.9.

**Definition 4.3** (see BHJ15, Der16 for example). Let \((X, \Delta)\) be a projective demi-normal pair and \( L \) be an ample \( \mathbb{Q} \)-line bundle on \( X \).

1. We say that \(((X, \Delta), L)\) is uniformly \( K \)-stable if there exists \( \delta \in (0, 1) \) such that for any semiample, demi-normal test configuration \((X, L)/P^1\) of \((X, L)\), the inequality \( \text{DF}_\Delta(X, L) \geq \delta \cdot J^{\text{NA}}(X, L) \) holds.

2. We say that \(((X, \Delta), L)\) is \( K \)-stable (resp., \( K \)-semistable) if, for any non-trivial, semiample, demi-normal test configuration \((X, L)/P^1\) of \((X, L)\), the inequality \( \text{DF}_\Delta(X, L) > 0 \) (resp., \( \geq 0 \)) holds.
We say that \((X, \Delta), L\) is K-polystable if \((X, \Delta), L\) is K-semistable and, the equality \(DF_\Delta(\mathcal{X}, \mathcal{L}) = 0\) for an ample, demi-normal test configuration \((\mathcal{X}, \mathcal{L})/\mathbb{P}^1\) of \((X, L)\) implies that \((\mathcal{X} \setminus \mathcal{X}_\infty, \Delta |_{\mathcal{X} \setminus \mathcal{X}_\infty}) \simeq (X \times \mathbb{A}^1, \Delta \times \mathbb{A}^1)\), where \(\mathcal{X}_\infty\) is the scheme-theoretic fiber of \(\mathcal{X} \to \mathbb{P}^1\) at \(\infty \in \mathbb{P}^1\).

Remark 4.4. (1) From Proposition 4.2, uniform K-stability implies K-stability. Moreover, it is obvious that K-stability implies K-polystability and K-polystability implies K-semistability.

(2) After Li and Xu found a pathological example [LX14 Example 3], the notion of test configurations trivial in codimension 2, almost trivial test configurations was given in [Sto11 Definition 1], [Odk15 Definition 3.3], respectively. See also [BHJ15 Definition 2.9]. Thanks to Propositions 3.3 and Corollary 3.6 the definitions of K-stability and K-semistability in our sense coincide with the ones in the senses of [Odk13a, Corollary 3.11] and [BHJ15, Definition 3.11]. In particular, we do not need to consider almost trivial test configurations in order to test K-stability of \((X, \Delta), L\) for demi-normal pairs \((X, \Delta)\).

The following theorems are important for the studies of K-stability.

Theorem 4.5 ([Odk13b Theorem 1.2]). Let \((X, \Delta)\) be a projective demi-normal pair and let \(L\) be an ample \(\mathbb{Q}\)-line bundle. If \((X, \Delta), L\) is K-semistable, then \((X, \Delta)\) is an slc pair.

Theorem 4.6. Let \((X, \Delta)\) be an \(n\)-dimensional projective slc pair.

1. ([BHJ15 Corollary 9.3], see also [Odk12 Theorem 1.1 (i)], [Der16 Theorem 1.2 (ii)]) Assume that \(L := K_X + \Delta\) is ample. Then, for any semiample, demi-normal test configuration \((\mathcal{X}, \mathcal{L})/\mathbb{P}^1\) of \((X, L)\), we have
   \[
   DF_\Delta(\mathcal{X}, \mathcal{L}) \geq \frac{1}{n} \cdot J^{NA}(\mathcal{X}, \mathcal{L}).
   \]
   In particular, \((X, \Delta), L\) is uniformly K-stable.

2. ([Odk12 Theorem 1.1 (ii)] and [BHJ15 Corollary 9.4], see also [Der16 Theorem (iii)]) Assume that \(K_X + \Delta \equiv 0\) and let \(L\) be an arbitrary ample \(\mathbb{Q}\)-line bundle on \(X\). Then \((X, \Delta), L\) is K-semistable. Moreover, \((X, \Delta), L\) is uniformly K-stable if and only if \((X, \Delta)\) is a klt pair.

3. ([Odk13b Theorem 1.3] and [BHJ15 Corollary 9.6]) Assume that \(L := -(K_X + \Delta)\) is ample. If \((X, \Delta), L\) is K-semistable, then \((X, \Delta)\) must be a klt pair (i.e., \((X, \Delta)\) is a log Fano pair).
Proof. Only (1) is unknown. However, the assertion immediately follows from Proposition 4.2 (2) and [BHJ15, §9] by considering the associated normal test configurations. □

We sometimes use the following negativity lemma.

**Lemma 4.7** (see [BHJ15, Lemma 6.14]). Let $X$ be an $n$-dimensional demi-normal projective variety, let $L$ be an ample $\mathbb{Q}$-line bundle on $X$ and let $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ be a semample, demi-normal test configuration of $(X, L)$. Assume that $D$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on $X$ supported on $X_0$ and $M_1, \ldots, M_{n-1}$ be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on $\mathcal{X}$ such that each $M_i$ is nef over $\mathbb{P}^1$. Then we have $(M_1 \cdots M_{n-1} \cdot D^2) \leq 0$.

**Proof.** Follows immediately from [BHJ15, Lemma 6.14] after taking the normalization of $\mathcal{X}$. □

5. **Uniform K-stability of log Fano pairs**

In this section, we always assume that $(X, \Delta)$ is an $n$-dimensional log Fano pair and $L := -(K_X + \Delta)$. We recall the theory established in [Fuj16a, Li16, Fuj16b, Fuj17]. More precisely, there is a simple criterion to test whether $((X, \Delta), L)$ is uniformly K-stable or not. We recall this in this section.

**Definition 5.1.** Let $F$ be a prime divisor over $X$. Fix a projective birational morphism $\sigma: Y \to X$ such that $Y$ is normal and $F$ is a prime divisor on $Y$.

1. For any Cartier divisor $M$ on $X$ and for any $x \in \mathbb{R}_{\geq 0}$, let $H^0(X, M - xF)$ be the sub $k$-vector space of $H^0(X, M)$ defined by
   \[ H^0(X, M - xF) := H^0(Y, \sigma^* M - xF) \subset H^0(Y, \sigma^* M) \]
   under the identification $H^0(X, M) = H^0(Y, \sigma^* M)$. Note that the definition does not depend on the choice of $\sigma$.

2. For any $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $M$ on $X$ and for any $x \in \mathbb{R}_{\geq 0}$, we set
   \[ \text{vol}_X(M - xF) = \limsup_{k \to \infty} \frac{\dim_k H^0(X, kM - kxF)}{k^n/n!} . \]
   The limsup is actually the limit (see [Laz04a, Laz04b]). Moreover, the function $\text{vol}_X(M - xF)$ is non-increasing and continuous over $x \in [0, \infty)$, and identically equal to zero for $x \gg 0$. 

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   The limsup is actually the limit (see [Laz04a, Laz04b]). Moreover, the function $\text{vol}_X(M - xF)$ is non-increasing and continuous over $x \in [0, \infty)$, and identically equal to zero for $x \gg 0$. 

Proof. Only (1) is unknown. However, the assertion immediately follows from Proposition 4.2 (2) and [BHJ15, §9] by considering the associated normal test configurations. □
Definition 5.2. (1) For any prime divisor $F$ over $X$, we set
\[ \hat{\beta}_{(X, \Delta)}(F) := 1 - \int_0^\infty \frac{\text{vol}_X(L - xF)dx}{A_{(X, \Delta)}(F)(L^n)}. \]
(2) (see [Tia87, Dem08]) We set
\[ \alpha(X, \Delta) := \sup \{ \alpha \in \mathbb{Q}_{\geq 0} \mid (X, \Delta + \alpha D) \colon \text{klt for any } D \geq 0 \text{ with } D \sim Q L \}. \]

The following theorem is important in this article.

Theorem 5.3 ([Fuj17, Theorem 1.5], see also [Li16, Theorem 3.7] and [Fuj16b, Theorem 6.6]). The followings are equivalent:

1. $((X, \Delta), L)$ is uniformly K-stable (resp., K-semistable),
2. there exists $\varepsilon \in (0, 1)$ (resp., $\varepsilon \in [0, 1)$) such that $\hat{\beta}_{(X, \Delta)}(F) \geq \varepsilon$, holds for any prime divisor $F$ over $X$.

Recently, the theory of delta-invariants introduced in [FO16] is much developed by [BJ17]. The following definition is not the original definition in [FO16, BJ17]. See [BJ17, Theorem C] in detail.

Definition 5.4 ([FO16, BJ17]). We set
\[ \delta(X, \Delta) := \inf_{F: \text{prime divisor over } X} \frac{1}{1 - \hat{\beta}_{(X, \Delta)}(F)}, \]
and we call it the delta-invariant of $(X, \Delta)$. From Theorem 5.3 the uniform K-stability (resp., the K-semistability) of $((X, \Delta), L)$ is equivalent to the condition $\delta(X, \Delta) > 1$ (resp., $\delta(X, \Delta) \geq 1$).

Lemma 5.5 (see [BJ17, Theorem A] and [FO16, Theorem 3.5]). We have the inequality
\[ \alpha(X, \Delta) \geq \frac{\delta(X, \Delta)}{n + 1}. \]

Proof. We give a proof for the readers’ convenience. Take any $D \geq 0$ with $D \sim Q L$. Set
\[ c := \max \{ c' > 0 \mid (X, \Delta + c'D) \text{ is log canonical} \}. \]
Then there exists a prime divisor $F$ over $X$ such that $A_{(X, \Delta + cD)}(F) = 0$ holds. We remark that $\hat{\beta}_{(X, \Delta)}(F) \geq 1 - \delta(X, \Delta)^{-1}$ holds. Take any resolution $\sigma: Y \to X$ with $F \subset Y$. Then $0 = A_{(X, \Delta + cD)}(F) = \ldots$
OPENNESS RESULTS FOR UNIFORM K-STABILITY

A_{(X,\Delta)}(F) - c \text{ord}_F \sigma^* D. Thus we get

\[ 1 - \frac{1}{\delta(X,\Delta)} \leq 1 - \frac{\int_0^\infty \text{vol}_Y(\sigma^* L - xF)dx}{A_{(X,\Delta)}(F)(L^n)} \leq 1 - \frac{\int_0^\infty \text{vol}_Y(\sigma^* L - \frac{c\cdot \epsilon}{A_{(X,\Delta)}(F)} \sigma^* D) dx}{A_{(X,\Delta)}(F)(L^n)} = 1 - \frac{1}{c(n+1)}. \]

Therefore we get \( c \geq \delta(X,\Delta)/(n+1). \)

We will use the following technical lemma later.

**Lemma 5.6** ([Fuj17, Claim 2.4] and [Fuj16b, Theorem 6.6]). Assume that there exists \( \epsilon \in (0, 1) \) such that, for any prime divisor \( F \) over \( X \), \( \hat{\beta}_{(X,\Delta)}(F) \geq \epsilon \) holds, that is, \( \delta(X,\Delta) \geq 1/(1 - \epsilon) \) holds. Then, for any semiample, normal test configuration \((X, L)/\mathbb{P}^1\) of \((X, L)\), we have the inequality

\[ DF_{\Delta}(X, L) \geq \frac{\epsilon}{n+1} \cdot J^{NA}(X, L). \]

The following is the main result in this section.

**Theorem 5.7.** (1) Take any \( \delta_0 \in \mathbb{R}_{>0} \) with \( \delta(X, \Delta) > \delta_0 \). Set

\[ \epsilon_0 := \frac{\delta(X, \Delta) - \delta_0}{n \cdot \delta(X, \Delta) + n + 1}. \]

Then, for any effective \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor \( B \) on \( X \) with \( \epsilon_0 L - B \) nef, \((X, \Delta + B)\) is a log Fano pair and \( \delta(X, \Delta + B) \geq \delta_0 \).

(2) Take any \( \delta_1 \in \mathbb{R}_{>0} \) with \( \delta(X, \Delta) < \delta_1 \). Set

\[ \epsilon_1 := \min \left\{ \frac{\delta(X, \Delta)}{n+1}, 1 - \frac{n+1}{n+1} \sqrt{\frac{\delta(X, \Delta)}{\delta_1}} \right\}. \]

Then, for any effective \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor \( B \) on \( X \) with \( \epsilon_1 L - B \) ample, \((X, \Delta + B)\) is a log Fano pair and \( \delta(X, \Delta + B) \leq \delta_1 \).

**Proof.** Take any \( \epsilon \in (0, 1) \) with \( \epsilon < \delta(X, \Delta)/(n+1) \). Assume that an effective \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor \( B \) on \( X \) satisfies that \( \epsilon L - B \) is nef. By Lemma 2.4, there exists \( D \geq B \) with \( D \sim_{\mathbb{R}} \epsilon L \). Since \( \epsilon < 1 \), \( L - B \) is ample. Take any prime divisor \( F \) over \( X \) and fix any resolution \( \sigma: Y \to X \) with \( F \subset Y \). By Lemma 5.5 we have

\[ 0 \leq A_{(X, \Delta + \frac{\delta(X, \Delta)}{\epsilon(n+1)} D)}(F) = A_{(X, \Delta)}(F) - \frac{\delta(X, \Delta)}{\epsilon(n+1)} \text{ord}_F \sigma^* D. \]
Thus we get

\[
A_{(X,\Delta)}(F) \geq A_{(X,\Delta+B)}(F) \\
\geq A_{(X,\Delta+D)}(F) \geq \left(1 - \frac{\varepsilon(n+1)}{\delta(X,\Delta)}\right) A_{(X,\Delta)}(F) > 0.
\]

This implies that \((X, \Delta + B)\) is a log Fano pair. On the other hand, we have

\[
\operatorname{vol}_X(L - xF) \geq \operatorname{vol}_X((L - B) - xF) \\
\geq \operatorname{vol}_X((L - D) - xF) = (1 - \varepsilon)^n \operatorname{vol}_X\left(L - \frac{x}{1 - \varepsilon}F\right)
\]

for any \(x \in \mathbb{R}_{\geq 0}\). In particular, we have

\[
(L^n) \geq ((L - B)^n) \geq (1 - \varepsilon)^n (L^n).
\]

(1) For any prime divisor \(F\) over \(X\), we have

\[
\hat{\beta}_{(X,\Delta+B)}(F) \geq 1 - \frac{\int_0^\infty \operatorname{vol}_X(L - xF)dx}{\left(1 - \frac{\varepsilon_0(n+1)}{\delta(X,\Delta)}\right) (1 - \varepsilon_0)^n A_{(X,\Delta)}(F)(L^n)} \\
\geq 1 - \frac{1}{(\delta(X,\Delta) - \varepsilon_0(n+1))(1 - \varepsilon_0)^n} \\
\geq 1 - \frac{1}{\delta(X,\Delta) - \varepsilon_0(n \cdot \delta(X,\Delta) + n + 1)} = 1 - \frac{1}{\delta_0}.
\]

(2) For any \(B\) in the assumption of (2), we can find \(\varepsilon \in (0, \varepsilon_1)\) such that \(\varepsilon L - B\) is ample. Moreover, for any \(\delta_2 \in [\delta(X,\Delta), \delta_1)\), we can find a prime divisor \(F\) over \(X\) such that \(\hat{\beta}_{(X,\Delta)}(F) \leq 1 - 1/\delta_2\) holds. For such \(F\), we have

\[
\hat{\beta}_{(X,\Delta+B)}(F) \leq 1 - \frac{(1 - \varepsilon)^{n+1} \int_0^\infty \operatorname{vol}_X(L - xF)dx}{A_{(X,\Delta)}(F)(L^n)} \\
\leq 1 - \frac{(1 - \varepsilon)^{n+1}}{\delta_2} \leq 1 - \frac{1}{\delta_1 \delta_2 / \delta(X,\Delta)}.
\]

Thus \(\delta(X, \Delta+B) \leq \delta_1 \delta_2 / \delta(X,\Delta)\) for any \(\delta_2 \in [\delta(X,\Delta), \delta_1)\). Therefore, we get the inequality \(\delta(X,\Delta) \leq \delta_1\). □

Proof of Theorem 1.2 If \(\varepsilon L - B\) is nef, then \(((X, \Delta + B), L - B)\) is K-semistable by Theorem 5.7 (1). Assume that \(\varepsilon L - B\) is ample. We can take \(\delta_0 \in (1, \delta(X,\Delta))\) such that \(\varepsilon_0 L - B\) is ample, where \(\varepsilon_0 := (\delta(X,\Delta) - \delta_0)/(n \delta(X,\Delta) + n + 1)\). Now Theorem 1.2 is an immediate consequence of Theorem 5.7 (1). □
Corollary 5.8. Fix any norm $\| \cdot \|$ on $N^1(X)^\mathbb{R}$. Take any $\delta_0, \delta_1 \in \mathbb{R}_{>0}$ with $\delta(X, \Delta) \in (\delta_0, \delta_1)$. Then there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $(X, \Delta + B)$ is a log Fano pair with $\delta(X, \Delta + B) \in (\delta_0, \delta_1)$ for any effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $B$ on $X$ with $\|B\| \leq \varepsilon$.

Proof. Follows immediately from Theorem 5.7.

6. Perturbing boundaries

In this section, we prove Theorem 1.1. Technically, the following proposition is important in this article.

Proposition 6.1. Let $(X, \Delta)$ be an $n$-dimensional projective demi-normal pair, $L$ be an ample $\mathbb{Q}$-line bundle and $N$ be an effective and nef $\mathbb{Q}$-divisor on $X$. Then, for any semiample demi-normal test configuration $(\mathcal{X}, \mathcal{L})/\mathbb{P}^1$ of $(X, L)$, we have

$$n\mu_N(L)J^\text{NA}(\mathcal{X}, \mathcal{L}) \geq DF_{\Delta+N}(\mathcal{X}, \mathcal{L}) - DF_\Delta(\mathcal{X}, \mathcal{L}).$$

Proof. Let

$$\begin{CD}
\vartheta @> \psi \circ N \circ \Phi \circ \pi >> X \\
\mathcal{X} @> \Phi >> X \times \mathbb{P}^1
\end{CD}$$

be the partial normalization of the graph and let $p_1 : X \times \mathbb{P}^1 \to X$ be the first projection. We write $\phi := \Theta^* \mathcal{L}$, $\psi_L := \Pi^* p_1^* L$, $\psi_N := \Pi^* p_1^* N$ for simplicity. By Lemma 4.7, we have

$$(\phi^j \cdot \psi_{L}^{n-2-j} \cdot (\phi - \psi_L)^2 \cdot \psi_N) \leq 0$$

for any $0 \leq j \leq n - 2$. Thus we get $(\phi^n \cdot \psi_N) \leq n(\phi \cdot \psi_{L}^{n-1} \cdot \psi_N)$. Moreover, we note that

$$(L^n)(DF_{\Delta+N}(\mathcal{X}, \mathcal{L}) - DF_\Delta(\mathcal{X}, \mathcal{L})) = (\phi^n \cdot N_y) - \frac{n}{n+1} \mu_N(L)(\phi^{n+1}) \leq (\phi^n \cdot \psi_N) - \frac{n}{n+1} \mu_N(L)(\phi^{n+1})$$

since $N$ is effective. We also note that

$$(\phi^n \cdot \psi_N) - \frac{n}{n+1} \mu_N(L)(\phi^{n+1}) \leq n((\phi \cdot \psi_{L}^{n-1} \cdot \psi_N) - \mu_N(L)(\phi \cdot \psi_{L}^{n})) + n\mu_N(L)(L^n)J^\text{NA}(\mathcal{X}, \mathcal{L}) = n\mu_N(L)(L^n)J^\text{NA}(\mathcal{X}, \mathcal{L})$$

since $\psi_{L}^{n-1} \cdot (\psi_N - \mu_N(L)\psi_L) \equiv 0$ as a $\mathbb{Q}$-1-cycle.
As consequences of Proposition 6.1 we have many results. The following is a baby version of Theorem 6.5.

**Corollary 6.2.** Let \((X, \Delta)\) be a 1-dimensional projective slc pair such that \(K_X + \Delta\) is ample. Then \(((X, \Delta), L)\) is uniformly K-stable for any ample \(\mathbb{Q}\)-line bundle \(L\).

**Proof.** We may assume that \(L - (K_X + \Delta)\) is ample (by replacing \(L\) with high multiple). Take a general \(\mathbb{Q}\)-divisor \(A \geq 0\) with \(A \sim_{\mathbb{Q}} L - (K_X + \Delta)\). Then \((X, \Delta + A)\) is an slc pair. Thus, by Theorem 4.6, for any semiample, demi-normal test configuration \((\mathcal{X}, \mathcal{L})/\mathbb{P}^1\) of \((X, L)\), we have \(\operatorname{DF}_{\Delta+A}^\ast(\mathcal{X}, \mathcal{L}) \geq J^{\mathbb{N}}(\mathcal{X}, \mathcal{L})\). On the other hand, by Proposition 6.1, we have \(\operatorname{DF}_{\Delta}(\mathcal{X}, \mathcal{L}) \geq (1 - \mu_A(L))J^{\mathbb{N}}(\mathcal{X}, \mathcal{L})\). Thus we get the assertion since \(1 - \mu_A(L) = \mu_{K_X+\Delta}(L) > 0\). \(\Box\)

**Corollary 6.3.** Let \((X, \Delta)\) be an \(n\)-dimensional log Fano pair. Set \(L := -(K_X+\Delta)\). Assume that \(\delta(X, \Delta) > 1\). Take any \(\delta \in (1, \delta(X, \Delta))\). Set

\[
\varepsilon := \frac{\delta(X, \Delta) - \delta}{n \cdot \delta(X, \Delta) + n + 1}, \quad \delta_1 := \frac{\delta - 1}{(n + 1) \delta}.
\]

Take any nef \(\mathbb{Q}\)-divisor \(N\) on \(X\) with \(\varepsilon L - N\) nef. Then, for any semiample, normal test configuration \((\mathcal{X}, \mathcal{L}')/\mathbb{P}^1\) of \((X, L - N)\), we have

\[
\operatorname{DF}_{\Delta}(\mathcal{X}, \mathcal{L}') \geq (\delta_1 - n \mu_N(L - N))J^{\mathbb{N}}(\mathcal{X}, \mathcal{L}').
\]

(In particular, if \(\delta_1 > n \mu_N(L - N)\) (resp., if \(\delta_1 \geq \mu_N(L - N)\)), then \(((X, \Delta), L - N)\) is uniformly K-stable (resp., K-semistable).)

**Proof.** By Lemma 2.4, we may assume that \(N\) is effective. By Theorem 6.7, \((X, \Delta + N)\) is a log Fano pair with \(\delta(X, \Delta + N) \geq \delta\). Thus we get

\[
\delta_1 \cdot J^{\mathbb{N}}(\mathcal{X}, \mathcal{L}') \leq \operatorname{DF}_{\Delta+N}^\ast(\mathcal{X}, \mathcal{L}')
\]

\[
\leq \operatorname{DF}_{\Delta}(\mathcal{X}, \mathcal{L}') + n \mu_N(L - N) \cdot J^{\mathbb{N}}(\mathcal{X}, \mathcal{L}').
\]

from Lemma 5.6 and Proposition 6.1. \(\Box\)

**Corollary 6.4.** Let \((X, \Delta)\) be an \(n\)-dimensional log Fano pair and set \(L := -(K_X + \Delta)\). Assume that \(\delta(X, \Delta) > 1\). Set

\[
\varepsilon := \frac{\delta(X, \Delta) - 1}{(n^2 + n + 1) \delta(X, \Delta) + n^2 + n - 1}.
\]

Take any nef \(\mathbb{Q}\)-divisor \(N\) on \(X\) with \(\varepsilon L - N\) ample. Then \(((X, \Delta), L - N)\) is uniformly K-stable. In particular, by Proposition 2.4, for any norm \(\| \cdot \|\) on \(N^1(X)_{\mathbb{R}}\), there exists \(\varepsilon' \in (0, 1)\) such that \(((X, \Delta), L')\) is uniformly K-stable for any \(\mathbb{Q}\)-line bundle \(L'\) on \(X\) with \(\|L' - L\| \leq \varepsilon'\).
Proof. We may assume that \( n \geq 2 \). We just apply Corollary 6.3 for \( \delta := \left( \delta(X, \Delta) + 1 \right) / 2 \). Note that

\[
\left( n^2 + n + 1 \right) \delta(X, \Delta) + n^2 + n - 1 - 2 \left( n \cdot \delta(X, \Delta) + n + 1 \right) = (n^2 - n + 1) \delta(X, \Delta) + n^2 - n - 3 \geq n^2 - n \geq 0
\]

since \( n \geq 2 \). Thus we get

\[
\varepsilon \leq \frac{\delta(X, \Delta) - 1}{2 (n \cdot \delta(X, \Delta) + n + 1)} = \frac{\delta(X, \Delta) - \delta}{n \cdot \delta(X, \Delta) + n + 1}.
\]

Moreover, the condition \( \mu_N(L - N) < \left( \delta - 1 \right) / \left( n(n + 1) \delta \right) \) is equivalent to the condition

\[
\left( (L - N)^{n-1} \cdot \left( \frac{\delta - 1}{(n^2 + n + 1) \delta - 1} \right) L - N \right) > 0.
\]

Note that \( (\delta - 1) / ((n^2 + n + 1) \delta - 1) \) is equal to \( \varepsilon \).\( \Box \)

**Theorem 6.5** (cf. [Wei06, SW07]). Let \((X, \Delta)\) be an \( n \)-dimensional projective slc pair with \( n \geq 2 \) and let \( L \) be an ample \( \mathbb{Q} \)-line bundle on \( X \). Assume that \( \mu_{K_X+\Delta}(L) > 0 \) and

\[
\frac{n^2}{n^2 - 1} \mu_{K_X+\Delta}(L) L - (K_X + \Delta)
\]

is ample (resp., nef). Then \((X, \Delta, L)\) is uniformly K-stable (resp., K-semistable).

**Proof.** Take any \( \varepsilon \in \mathbb{Q} \) with \( 0 < \varepsilon \ll 1 \) (resp., \( -1 \ll \varepsilon < 0 \)) such that

\[
\left( \frac{n^2}{n^2 - 1} \mu_{K_X+\Delta}(L) - \varepsilon \right) L - (K_X + \Delta)
\]

is ample. Take a general effective \( \mathbb{Q} \)-divisor \( A \) with small coefficients \( \mathbb{Q} \)-linearly equivalent to \( (n^2 / (n^2 - 1)) \mu_{K_X+\Delta}(L) - \varepsilon) L - (K_X + \Delta) \). Then \((X, \Delta + A)\) is an slc pair and

\[
K_X + \Delta + A \sim_{\mathbb{Q}} \left( \frac{n^2}{n^2 - 1} \mu_{K_X+\Delta}(L) - \varepsilon \right) L.
\]

Thus, for any semiample, semi-normal test configuration \((\mathcal{X}, \mathcal{L})/\mathbb{P}^1\) of \((X, L)\), we have

\[
\text{DF}_{\Delta+A}(\mathcal{X}, \mathcal{L}) \geq \frac{1}{n} \left( \frac{n^2}{n^2 - 1} \mu_{K_X+\Delta}(L) - \varepsilon \right) J^{\text{NA}}(\mathcal{X}, \mathcal{L})
\]

by Theorem 4.6 (I). On the other hand, by Proposition 6.1, we have

\[
\text{DF}_{\Delta+A}(\mathcal{X}, \mathcal{L}) - \text{DF}_{\Delta}(\mathcal{X}, \mathcal{L}) \leq n \mu_A(L) \cdot J^{\text{NA}}(\mathcal{X}, \mathcal{L}).
\]
Therefore we get
\[
DF_{\Delta}(X, L) \geq \left( n - \frac{1}{n} \right) \varepsilon \cdot J^{NA}(X, L)
\]
by combining those inequalities. \(\square\)

**Remark 6.6.** (1) Assume that \(\mu_{K_X+\Delta}(L) = 0\) and \(-(K_X + \Delta)\) is nef. Then, Dervan pointed out to the author that, we can easily show that \(K_X + \Delta\) is numerically trivial. When \(K_X + \Delta\) is numerically trivial, the uniform K-stability and the K-semistability of \(((X, \Delta), L)\) is well-understood by Theorem 4.6 (2).

(2) The author found Theorem 6.5 under the additional hypothesis “\(K_X + \Delta\) is ample” in order to prove Corollary 6.8. Codogni and Dervan pointed out to the author that we do not need the assumption.

**Corollary 6.7.** Let \((X, \Delta)\) be an \(n\)-dimensional projective slc pair such that \(L := K_X + \Delta\) is ample. Then \(((X, \Delta), L + N)\) is uniformly K-stable for any nef \(\mathbb{Q}\)-divisor \(N\) with \(L - (n^2 - 1)N\) big.

**Proof.** We may assume that \(n \geq 2\) by Corollary 6.2. Set \(M := L + N\). Since \(M - n^2N\) is big, we have \(\mu_N(M) < 1/n^2\). Note that
\[
\frac{n^2}{n^2 - 1} \mu_L(M)M - L = \frac{1}{n^2 - 1} (1 - n^2 \mu_N(M))M + N
\]
is ample. Thus \(((X, \Delta), L + N)\) is uniformly K-stable by Theorem 6.5. \(\square\)

**Corollary 6.8.** Let \((X, \Delta)\) be a projective slc pair such that \(L := K_X + \Delta\) is ample. Fix any norm \(\| \cdot \|\) on \(N^1(X)_{\mathbb{R}}\). Then there exists \(\delta > 0\) such that \(((X, \Delta), L')\) is uniformly K-stable for any \(\mathbb{Q}\)-line bundle \(L'\) on \(X\) with \(\|L' - L\| \leq \delta\).

**Proof.** Follows immediately from Corollaries 6.2, 6.7 and Proposition 2.5. \(\square\)

**Proof of Theorem 1.1.** Follows immediately from Corollaries 6.4, 6.8 and Theorem 4.6 (3). \(\square\)

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