A new beam element for the analysis of laminated composites based on the asymptotic splitting method

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Abstract. A new beam multilayer finite element for solving the bending problems of laminated composites is proposed. The finite element model is based on the asymptotic splitting method, which allows one to obtain a solution to the linear spatial problem of the theory of elasticity without using simplifying hypotheses. The advantage of this approach is that all components of the stress tensor are taken into account and that the stress states of composite beams of a complex cross-section are examined in detail without using the plane strain or plane stress states.

1. Introduction

Composite constructions with a layered heterogeneous structure, such as layered beams or plates, are widely used today. Layered structures are characterized by low shear moduli depending on the orientation of the layers. In this case, the influence of shear deformations sharply increases and it must be taken into account.

For layered inhomogeneous beams, the classical beam theory (CBT), based on the Bernoulli-Euler hypotheses, gives low values of deflections and overestimates the total bending stiffness of the beam. Therefore, their analysis requires the use of more accurate beam theories. The main ideas and methods for solving the problems of mechanics of layered structures based on the introduction of simplifying hypotheses are given, in particular, in [1–5]. A detailed review of existing shear theories is given in [6,7].

It should be noted that most of the refined beam theories do not take into account the transverse distribution of the components of the stress tensor, that is, they consider the beam in the state of plane strain or plane stress. Therefore, today it is still relevant to build logically coherent and mathematically sound theories that allow working with the original spatial formulation of the problem, without introducing significant simplifying assumptions.

In this paper, a new finite element model is proposed for solving the problems of bending of composite laminated beams. The mathematical model is based on the method of asymptotic splitting [8,9], which allows one to obtain exact solutions of the linear spatial problem of the theory of elasticity without introducing simplifying hypotheses, and the GN theory of bending of layered beams [10,11]. The resulting finite element model gives accurate results for uniformly distributed, concentrated and linearly distributed loads and allows to work with various types of boundary conditions.
2. Mathematical formulation

Consider the problem of bending of an orthotropic laminated beam with an arbitrary symmetric cross-section. Let $L, b, h$ be the length, width and height of the beam, respectively; $i, h_i$ is the number of the layer (from top to bottom) and the relative height of the $i$th layer of the beam. According to the asymptotic splitting method [8,9] and GN theory [10,11], the bending equation of laminated beam is expressed as follows

$$
\frac{d^4 u_0}{dz^4} = \frac{P^*_x}{[EI]}, \quad p^*_x = p_x - \zeta \frac{d^2 p_x}{dz^2},
$$

where $u_0$ is the transverse deflection of the beam equal to the average value of the transverse displacements $u_x$ in the cross section; $p_x$ is the distributed load applied to the surface of the beam; $[EI]$ is the bending stiffness.

The material of the $i$th layer obeys Hooke’s law for an orthotropic material

$$
(\sigma_{\alpha\gamma})_i = (E_{\alpha\gamma})_i (\varepsilon_{\alpha\gamma})_i, \quad (\sigma_{\alpha\beta})_i = 2(\mu_{\alpha\beta})_i (\varepsilon_{\alpha\beta})_i, \quad \gamma, \alpha \neq \beta \in (x, y, z).
$$

The angle of rotation of the cross-section $\phi$, the bending moment $M_y$ and the shear force $Q_x$ are expressed in terms of $u_0$ according to the formulas

$$
\phi = \frac{du_0}{dz} - \frac{\tilde{K}_\phi}{[EI]} \left( Q_x + \zeta \frac{dp_x}{dz} \right), \quad M_y = -[EI] \left( \frac{d^2 u_0}{dz^2} + \frac{\tilde{\zeta}}{[EI]} p_x \right),
$$

$$
Q_x = -[EI] \left( \frac{d^3 u_0}{dz^3} + \frac{\tilde{\zeta}}{[EI]} \frac{dp_x}{dz} \right),
$$

where $\tilde{K}_\phi, \tilde{\zeta}$ are specific values that need to be found separately at the stage of the asymptotic splitting of the problem. If in (2.1), (2.2) we put $\tilde{K}_\phi, \tilde{\zeta}$ equal to zero, then we obtain the classical theory of bending based on the Bernoulli-Euler hypotheses.

To solve the bending equation, the boundary conditions are specified in terms of the functions (2.2). In the case of a hinged beam (H-H) or clamped from both ends beam (C-C) the boundary conditions are

$$
\text{H-H:} \quad u_0(z)|_{z=0,L} = M_y(z)|_{z=0,L} = 0, \quad \text{C-C:} \quad u_0(z)|_{z=0,L} = \phi(z)|_{z=0,L} = 0.
$$

The values $K_\phi, \zeta$ can be found by formulas using analytic solutions in a dimensionless form for the symmetric laminated beam from [10]

$$
K_\phi = -\frac{1}{I} \int_{-0.5}^{0.5} x(U_x)_i^{(1)} dx, \quad \zeta = \frac{D_2}{D_1}, \quad D_2 = \int_{-0.5}^{0.5} x(\tau_{zz})_i^{(2)} dx, \quad D_1 = \int_{-0.5}^{0.5} x(\tau_{zz})_i^{(1)} dx,
$$

$$
(\tau_{zz})_i^{(1)} = -x(E_x)_i, \quad (\tau_{xx})_i^{(1)} = -\int_{-0.5}^{0.5} (\tau_{zz})_i^{(1)} dx, \quad (U_x)_i^{(1)} = \frac{\nu}{2} (x^2 - 1),
$$

$$
(U_x)_i^{(1)} = \int_{-0.5}^{0.5} \left( \frac{(\tau_{zz})_i^{(1)}}{(\mu_{zz})_i} - (U_x)_i^{(1)} \right) dx + c_1, \quad (\tau_{xx})_i^{(2)} = -\frac{D_1}{b} - \int_{-0.5}^{0.5} (\tau_{zz})_i^{(1)} dx,
$$

$$
(\tau_{zz})_i^{(2)} = (E_x)_i (U_x)_i^{(1)} + \nu (\tau_{xx})_i^{(2)}, \quad \int_{-0.5}^{0.5} (\tau_{zz})_i^{(2)} dx = 0, \quad (\tau_{\alpha\beta})_i^{(1)} = 0, \quad \alpha, \beta \in (x, y),
$$

where the Poisson ratio is the same for all layers, the height of the cross-section varies from $-0.5$ to $0.5$. Formulas (2.3) are analytical solutions of auxiliary two-dimensional boundary value problems in the cross-section of the beam obtained during the asymptotic splitting procedure.
For some special cases, it is possible to obtain analytical solutions [10, 17, 18]. In the general case, boundary value problems are solved numerically, for example, by the finite element method or by the least-squares collocation method [12–16]. In the case of plane strain, boundary value problems can be reduced to systems of ordinary differential equations [9].

After solving the bending equation (2.1), all components of the stress tensor can be found by the formulas

\[
\sigma_{\alpha z} = -\langle \tau_{\alpha z} \rangle_h^2 \left( \frac{Q_x}{EI} \right)_h + \frac{1}{\langle EI \rangle_h} \left( \langle \tau_{\alpha z} \rangle_h^2 \right)_h^4 - \tilde{\zeta} \langle \tau_{\alpha z} \rangle_h^2 \frac{dp_x}{dz}, \quad \alpha, \beta \in x, y,
\]

\[
\sigma_{zz} = -\langle \tau_{zz} \rangle_h^2 \left( \frac{M_y}{EI} \right)_h + ((\tau_{zz})_h^2)h^3 - \tilde{\zeta} \langle \tau_{zz} \rangle_h^2 \left( \frac{p_x}{EI} \right)_h, \quad \tilde{K}_\phi = h^2 K_\phi,
\]

\[
\sigma_{\alpha\beta} = -\langle \tau_{\alpha\beta} \rangle_h^2 \left( \frac{M_y}{EI} \right)_h + ((\tau_{\alpha\beta})_h^2)h^3 - \tilde{\zeta} \langle \tau_{\alpha\beta} \rangle_h^2 \left( \frac{p_x}{EI} \right)_h, \quad \tilde{\zeta} = h^2 \zeta.
\]

The theory is valid as long as the following condition is satisfied

\[
\Delta = \frac{\tilde{\zeta}^2}{k} \frac{d^4 p_x}{dx^4} \ll 1.
\]

Thus, if the transverse load is a polynomial of degree 3 or lower, then the condition (2.5) is satisfied exactly.

3. Finite element formulation

To solve the bending equation (2.1), we consider the interval \( \Omega = (0, L) \) directed along the longitudinal coordinate \( z \) with a length equal to the length of the beam. We divide \( \Omega \) into a certain number of finite elements \( \Omega_i \) forming the mesh \( T_h \) and introduce the Sobolev space \( H^1(\Omega) \)

\[
H^1(\Omega) = \{ v(z) \mid \|v\|_1^2 := \int_\Omega (|v|^2 + v^2)dz < \infty \}.
\]

For each element of the beam \( \Omega_i \), the equilibrium equations must be satisfied

\[
\frac{dQ_x}{dz} = -p_x, \quad \frac{dM_y}{dz} = Q_x.
\]

The system of equations (3.1) in terms of the variables \( u_0, \phi \) takes the form

\[
\begin{align*}
\frac{d}{dz} \left( \frac{EI}{K_\phi} \frac{du_0}{dz} - \phi \right) + A &= -p_x, \quad A = -\tilde{\zeta} \frac{dp_x}{dz}, \\
\frac{d}{dz} \frac{EI}{K_\phi} \frac{d\phi}{dz} + B &= \frac{EI}{K_\phi} \left( \frac{du_0}{dz} - \phi \right) + A, \quad B = -\tilde{\zeta} p_x + \tilde{K}_\phi p_x^*.
\end{align*}
\]

If the terms \( A, B \) are set to zero in (3.2), then we arrive at the system of equations of the refined Timoshenko theory with the shear correction factor \( k[GF] = [EI]K_\phi^{-1} \) [19].

Using the Bubnov-Galerkin method [20], we multiply equations (3.2) by the linear test functions \( \bar{u}_0, \phi \in H^1(\Omega) \) and integrate by parts along the length of the element \( L_{el} \)

\[
\begin{align*}
M_y \bar{u}_0 \bigg|^{L_{el}}_0 - \int_0^{L_{el}} \frac{EI}{K_\phi} \left( \frac{du_0}{dz} - \phi \right) \frac{du_0}{dz} dz &= \int_0^{L_{el}} (-p_x \bar{u}_0 + A \frac{du_0}{dz}) dz, \\
Q_x \bar{\phi} \bigg|^{L_{el}}_0 + \int_0^{L_{el}} \frac{EI}{K_\phi} \left( \frac{d\phi}{dz} - \phi \right) \phi dz &= \int_0^{L_{el}} (A \bar{\phi} + B \frac{d\phi}{dz}) dz.
\end{align*}
\]
The sought-for functions are expressed in terms of a linear combination of unknown values at the mesh nodes and piecewise linear basis functions \( N_i \in V_h \) on each element with the corresponding finite element space \( V_h \)

\[
V_h = \{ u^h \in H^1(\Omega), \quad u^h \text{ piecewise linear on } T_h \}, \quad N_i = \frac{z_j - z}{z_i - z_j}, \quad N_j = \frac{z - z_i}{z_i - z_j},
\]

where \( z_i \) are the coordinates of the nodes of the element \( \Omega_i \). We introduce the following finite element functions \( \bar{u}_0^h, \bar{\phi}^h, u_0^h, \phi^h \in V_h \) on each element \( \Omega_i \):

\[
\bar{u}_0^h = N_i + N_j, \quad \bar{\phi}^h = N_i + N_j, \quad u_0^h = N_i u_{0i} + N_j u_{0j}, \quad \phi^h = N_i \phi_{0i} + N_j \phi_{0j}.
\]

The test functions \( \bar{u}_0^h, \bar{\phi}^h \) belong to different subspaces of \( H^1 \) depending on the essential boundary conditions on the functions \( u_0, \phi \).

After substituting the finite element functions (3.4) into (3.3) and assembling all the elements, we arrive at the system of algebraic equations

\[
Ku = F,
\]

where the unknowns are the values of the functions \( u_0, \phi \) at the nodes of the mesh. To avoid the shear locking, the terms responsible for the shear are integrated according to the Gauss formulas on a single-point pattern in the center of the element. Quadratic polynomial approximation was also used for function \( u_0 \) and \( \phi \) was assumed to be linear. The finite element model was implemented in the open-source package FENICS Project [21].

4. Convergence test

Here we show the convergence of the constructed finite element model. As a test we take the following material parameters for a three-layer beam with the same layer thickness from [22]:

\[
E_L = 25E_T, \quad \nu_{LT} = \nu_{TT} = 0.25, \quad G_{LT} = 0.5E_T, \quad G_{TT} = 0.2E_T.
\]

Table 1.

| Approximation       | Number of elements |
|---------------------|--------------------|
| Linear, Linear      | 0.648              |
| Quadratic, Linear   | 0.648              |
| Linear, Linear      | 1.184              |
| Quadratic, Linear   | 1.184              |

Table 1 shows the convergence of the transverse deflection values in the middle of the span of the beam to the classical beam theory solution for two different types of approximation of unknown \( u_0, \phi \). The load is assumed to be uniformly distributed. In this case, the values \( K_\phi, \zeta \) are assumed to be close to zero. Similarly, Table 1 shows the convergence of the transverse deflection values to the exact GN theory solution with values \( K_\phi = 8.8, \zeta = 8.6 \). When the number of elements is more than 10, the error of the solution for both approximations does not exceed 1%.
5. Four-point bending test of sandwich beams with soft core
Let us show the comparison with the experimental data from [23, 24] in the case of four-point bending of sandwich panels. Input parameters are given in Table 2 according to Figure 1.

![Figure 1. Four-point bending test setup.](image)

| Units | [MPa] | — | [mm] |
|-------|-------|---|-------|
| Reference | $E_{1,3}$ | $E_2$ | $G_{1,3}$ | $G_2$ | $\nu_{1,2,3}$ | $h_{1,3}$ | $h_2$ | $L$ | $a$ | $b$ |
| Sokolinsky et al. [23] | 72e3 | 58 | 28e3 | 22 | 0.33 | 0.05 | 19 | 203.2 | 76.2 | 60 |
| Piovar et al. [24] | 2.08e5 | 8.9 | 80.3e3 | 1.66 | 0.3 | 0.44 | 79.12 | 1000 | 345 | 100 |

Table 2. Parameters from experimental studies.

![Figure 2. Load-deflection diagrams of sandwich beams. Diagrams are taken from [23]. $K_\phi = 40.19$, $\zeta = 39.24$.](image)

![Figure 3. Load-deflection diagrams of sandwich beams. Diagrams are taken from [24]. $K_\phi = 343.56$, $\zeta = 341.62$.](image)

The first figure (Figure 2) shows the deflection curves in the middle of the span of the sandwich panel. It can be seen that the CBT gives unreliable values. The classical theory of sandwich panels (CST) [25] also underestimates deflections by about 20%. GN theory gives a difference of about 13% compared to the experiment. The difference in the results is most likely due to the fact that in the presence of a very soft core in the places of application of the load, as well as in the supports, local indentation effects [26] appear. The best match is shown by the high-order sandwich panels theory (HSAPT) [27], which takes into account the local indentation effects.

The second figure (Figure 3) shows a comparison with the experimental data from [24]. Local effects in the supports and in the places where the load was applied were neutralized as much as
possible with the help of additional steel plates smoothing the concentrated load. In this case, GN theory gives a good agreement on the deflections in comparison with the experimental data, as well as with spatial calculations in the ANSYS finite element package.

6. Conclusions

The proposed multilayer beam finite element can be effectively used to solve the problems of bending of laminated beams. The results obtained are in good agreement with the results of known theories, with special cases when the exact solution of the problem is known, as well as experimental data. It is advisable to expand the current finite element model to solve the problems of longitudinal-transverse bending and torsion of composite beams. The proposed approach can be effective for solving the problems of mechanics of composite plates and shells.

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