Cellularity of the $p$-Canonical Basis for Symmetric Groups

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Abstract

For symmetric groups we show that the $p$-canonical basis can be extended to a cell datum for the Iwahori-Hecke algebra $H$ and that the two-sided $p$-cell preorder coincides with the Kazhdan-Lusztig two-sided cell preorder. Moreover, we show that left (or right) $p$-cells inside the same two-sided $p$-cell for Hecke algebras of finite crystallographic Coxeter systems are incomparable (Property A).

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1 Introduction

The Hecke algebra of a crystallographic Coxeter system admits several geometric or algebraic categorifications (see [KL79; EW16]). The canonical bases arising from these categorifications coincide with the famous Kazhdan-Lusztig basis (see [KL80; EW14]) in the characteristic 0 setting and give rise to the $p$-canonical or $p$-Kazhdan-Lusztig basis in the characteristic $p > 0$ setting (see [JW17; RW18]).

The study of cells with respect to the $p$-canonical basis of the Hecke algebra was initiated in [Jen20] and continued in [JP19]. In this paper, we apply the Perron-Frobenius theorem to $p$-cells and to tackle several questions about the
various \(p\)-cell preorders. One of the most important features of the Kazhdan-Lusztig cell preorders is Property A which states that left (or right) Kazhdan-Lusztig cells in the same two-sided Kazhdan-Lusztig cell are incomparable with respect to the left (or right) cell preorder. We will prove Property A of \(p\)-cells for finite crystallographic Coxeter groups in this paper following ideas of [KM16].

Along the way, we introduce \(p\)-special modules and \(p\)-families which are generalizations of Lusztig’s special modules and families of irreducible representations of a Weyl group to the \(p\)-canonical basis. These concepts were originally introduced by Lusztig in [Lus79b; Lus79a; Lus82] and have played an important role in determining the complex irreducible characters of finite reductive groups (via character sheaves). The connections of \(p\)-special modules and \(p\)-families to the representation theory of finite reductive groups are completely unclear to the author and thus merit further study.

In the last part of the paper, we study consequences of Property A for \(p\)-cells of symmetric groups. One of the main results of [Jen20] is the characterization of \(p\)-cells for symmetric groups in terms of the Robinson-Schensted correspondence which gives a bijection \(w \mapsto (P(w), Q(w))\) between the symmetric group \(S_n\) and pairs of standard tableaux of the same shape with \(n\) boxes. As a consequence \(p\)-cells and Kazhdan-Lusztig cells coincide for all primes \(p\): The two-sided cell of \(w \in S_n\) is given by the set of elements in \(S_n\) whose \(Q\)-symbols have the same shape as \(Q(w)\). In other words, two-sided cells of \(S_n\) are in bijection with partitions of \(n\).

Even though the cells coincide, it was an open question to relate the two-sided \(p\)-cell preorder and the Kazhdan-Lusztig two-sided cell preorder on the level of two-sided cells. As a special case of the Lusztig-Vogan bijection (which was proven in [Bez09]) the Robinson-Schensted correspondence gives an order-preserving bijection between the set of two-sided cells equipped with the Kazhdan-Lusztig two-sided cell preorder and the set of partitions equipped with the dominance order. We extend this result to the two-sided \(p\)-cell preorder showing that the two-sided \(p\)-cell preorder and the Kazhdan-Lusztig two-sided cell preorder coincide for all primes \(p\).

When Graham and Lehrer introduced cellular algebras in [GL96], the Hecke algebra of a symmetric group together with the Kazhdan-Lusztig basis was one of the motivating examples. Finally, we show that the \(p\)-canonical basis can also be extended to a cell datum for the Hecke algebra of a symmetric group. Let’s recall the main result of [LM20, Theorem 1.1]:

**Theorem 1.1.** Let \(x, y \in S_n\) with \(x \leq y\). Then there exist \(N \geq n\) and \(v, w \in S_N\) with \(v \leq w\) such that

(i) \(v\) and \(w\) belong to the same right cell,

(ii) the singularity of \(X_w\) at \(v\) is smoothly equivalent to the singularity of \(X_y\) at \(x\).

As explained in [LM20, §4.3] this result allows to embed any example from [Wil17] in which the Kazhdan-Lusztig and the \(p\)-canonical basis differ into the same cell in some larger symmetric group and thus produces many examples where the \(p\)-canonical basis gives interesting basis of Specht modules. This adds to the interest in this newly defined cell datum for the Hecke algebra of a symmetric group.
1.1 Structure of the Paper

Section 2 We introduce notation for crystallographic Coxeter systems and their Hecke algebras. Then we recall important results about the diagrammatic category of Soergel bimodules and the $p$-canonical basis. Finally, we remind the reader of the Perron-Frobenius Theorem.

Section 3 We apply the Perron-Frobenius Theorem to $p$-cells to introduce $p$-special modules and $p$-families and to prove some of their elementary properties.

Section 4 We prove Property A for finite Weyl groups, showing that left $p$-cells within the same two-sided $p$-cell are incomparable with respect to the left $p$-cell preorder.

Section 5 This section deals with some consequences for $p$-cells of symmetric groups. First, we recall the characterization of $p$-cells in terms of the Robinson-Schensted correspondence. Then we show that the two-sided $p$-cell preorder corresponds to the dominance order on partitions. Finally, we prove that the $p$-canonical basis can be extended to a cellular datum for the Hecke algebra.

1.2 Acknowledgements

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2 Background

2.1 Crystallographic Coxeter Systems

Let $S$ be a finite set and $(m_{s,t})_{s,t\in S}$ be a matrix with entries in $\mathbb{N}\cup\{\infty\}$ such that $m_{s,s} = 1$ and $m_{s,t} = m_{t,s} \geq 2$ for all $s \neq t \in S$. Denote by $W$ the group generated by $S$ subject to the relations $(st)^{m_{s,t}} = 1$ for $s,t \in S$ with $m_{s,t} < \infty$. We say that $(W,S)$ is a Coxeter system and $W$ is a Coxeter group. The Coxeter group $W$ comes equipped with a length function $l : W \to \mathbb{N}$ and the Bruhat order $\leq$ (see [Hum90] for more details). A Coxeter system $(W,S)$ is called crystallographic if $m_{s,t} \in \{2,3,4,6,\infty\}$ for all $s \neq t \in S$. We denote the identity of $W$ by $e$. For $w \in W$ we define its left descent set via

$$\mathcal{L}(w) := \{s \in S | l(sw) < l(w)\}.$$  

The right descent set of $w$ is given by $\mathcal{R}(w) := \mathcal{L}(w^{-1})$. 

From now on, fix a generalized Cartan matrix $A = (a_{ij})_{i,j \in J}$ (see [Tit89, §1.1]). Let $(J, X, \{\alpha_i : i \in J\}, \{\alpha_i^\vee : i \in J\})$ be an associated Kac-Moody root datum (see [Tit89, §1.2] for the definition). Then $X$ is a finitely generated free abelian group, and for $i \in J$ we have elements $\alpha_i$ and $\alpha_i^\vee$ of $X$ and $X^\vee = \text{Hom}_\mathbb{Z}(X, \mathbb{Z})$ respectively that satisfy $a_{ij} = \alpha_j^\vee(\alpha_i)$ for all $i, j \in J$.

To $A$ we associate a crystallographic Coxeter system $(W, S)$ as follows: Choose a set of simple reflections $S$ of cardinality $|J|$ and fix a bijection $S \overset{\sim}{\to} J$, $s \mapsto i_s$. For $s \neq t \in S$ we define $m_{s,t}$ to be $2, 3, 4, 6, \text{ or } \infty$ if $a_{i_s,i_t}, a_{i_t,i_s}$ is $0, 1, 2, 3, \text{ or } \geq 4$ respectively. Most of the time, we will work with a Cartan matrix, so that $W$ is a finite Weyl group.

Fix a commutative ring $k$. $kV := X^\vee \otimes \mathbb{Z} k$ yields a balanced, potentially non-faithful realization of the Coxeter system over $k$. Set $kV^* := \text{Hom}_k(kV, k)$ and note that $kV^*$ is isomorphic to $X \otimes \mathbb{Z} k$. A realization obtained in this way is called a 

**Cartan realization** (see [AMRW17a, §10.1]). Throughout, we will assume our realization to satisfy:

**Assumption 2.1** (Demazure Surjectivity). The maps $\alpha_s : kV \to k$ and $\alpha_s^\vee : kV^* \to k$ are surjective for all $s \in S$.

This is automatically satisfied if $2$ is invertible in $k$ or if the Coxeter system $(W, S)$ is of simply-laced type and of rank $|S| > 2$.

We denote by $R = S(kV^*)$ the symmetric algebra of $kV^*$ over $k$ and view it as a graded ring with $kV^*$ in degree $2$. Given a graded $R$-bimodule $B = \bigoplus_{i \in \mathbb{Z}} B^i$, we denote by $B(1)$ the shifted bimodule with $B(1)^i = B^{i+1}$.

### 2.2 The Hecke Algebra

The Hecke algebra is the free $\mathbb{Z}[v, v^{-1}]$-algebra with $\{H_w \mid w \in W\}$ as basis, called the **standard basis**, and multiplication determined by:

$$H_s^2 = (v^{-1} - v)H_s + 1 \quad \text{for all } s \in S,$$

$$H_s H_y = H_{xy} \quad \text{if } l(x) + l(y) = l(xy).$$

There is a unique $\mathbb{Z}$-linear involution $(-)$ on $\mathcal{H}$ satisfying $\overline{v} = v^{-1}$ and $\overline{H_s} = H_{s^{-1}}$. The Kazhdan-Lusztig basis element $H_s$ is the unique element in $\mathcal{H} + \sum_{y < s} v\mathbb{Z}[v]H_y$ that is invariant under $(-)$. This is Soergel’s normalization from [Soe97] of a basis introduced originally in [KL79].

Let $\iota$ be the $\mathbb{Z}[v, v^{-1}]$-linear anti-involution on $\mathcal{H}$ satisfying $\iota(H_s) = H_s$ for $s \in S$ and thus $\iota(H_s) = H_{s^{-1}}$.

### 2.3 The Diagrammatic Category of Soergel Bimodules

In this section, we introduce the diagrammatic category of Soergel bimodules. The main reference for this is [EW16] (see also [Eli16] in the dihedral case and [EK10] in type A).

Let $BS$ be the diagrammatic category of Bott-Samelson bimodules as introduced in [JW17, §2.3]. It is a diagrammatic, strict monoidal category enriched over $\mathbb{Z}$-graded left $R$-modules.

Let $\mathbf{H}$ be the Karoubian envelope of the graded version of the additive closure of $BS$, in symbols $\mathbf{H} = \text{Karb}(BS)$. We call $\mathbf{H}$ the **diagrammatic category of Soergel bimodules**. In other words, in the passage from $BS$ to $\mathbf{H}$ we first allow
direct sums and grading shifts (restricting to degree preserving homomorphisms) and then the taking of direct summands. The following properties can be found in [EW16, Lemma 6.24, Theorem 6.25 and Corollary 6.26]:

**Theorem 2.2 (Properties of \( H \)).**

Let \( k \) be a complete, local, integral domain (e.g. a field or the \( p \)-adic integers \( \mathbb{Z}_p \)).

(i) \( H \) is a Krull-Schmidt category.

(ii) For all \( w \in W \) there exists a unique indecomposable object \( kB_w \in H \) which is a direct summand of \( w \) for any reduced expression \( w \) of \( w \) and which is not isomorphic to a grading shift of any direct summand of any expression \( v \) for \( v < w \). In particular, the object \( kB_w \) does not depend up to isomorphism on the reduced expression \( w \) of \( w \).

(iii) The set \( \{kB_w \mid w \in W \} \) gives a complete set of representatives of the isomorphism classes of indecomposable objects in \( H \) up to grading shift.

(iv) There exists a unique isomorphism of \( \mathbb{Z}[v, v^{-1}] \)-algebras

\[ \text{ch} : [H] \rightarrow H \]

sending \( [kB_s] \) to \( H_s \) for all \( s \in S \), where \([H]\) denotes the split Grothendieck group of \( H \). (We view \([H]\) as a \( \mathbb{Z}[v, v^{-1}] \)-algebra as follows: the monoidal structure on \( H \) induces a unital, associative multiplication and \( v \) acts via \( v[B] := [B(1)] \) for an object \( B \) of \( H \).)

It should be noted that we do not have a diagrammatic presentation of \( H \) as determining the idempotents in BS is usually extremely difficult.

### 2.4 The \( p \)-canonical Basis and \( p \)-Cells

In this section, we recall the definition of the \( p \)-canonical basis and its elementary properties (see [JW17; Jen20]). Let \( k \) be a field of characteristic \( p \geq 0 \). Note that the \( p \)-canonical basis depends on \( p \), but not on the explicit choice of \( k \).

**Definition 2.3.** Define \( p^H_w = \text{ch}(kB_w) \) for all \( w \in W \) where \( \text{ch} : [H] \rightarrow H \) is the isomorphism of \( \mathbb{Z}[v, v^{-1}] \)-algebras introduced earlier.

We will frequently use the following elementary properties of the \( p \)-canonical basis which can be found in [JW17, Proposition 4.2] unless stated otherwise:

**Proposition 2.4.** For all \( x, y \in W \) we have:

(i) \( p^H_x = p^H_x \), i.e. \( p^H_x \) is self-dual,

(ii) \( \iota(p^H_x) = p^H_{x-1} \), and thus in particular \( p_m^{y,x} = p_m^{y-1,x-1} \) as well as \( p_{h_y,x} = p_{h_y,x-1}, \)

(iii) \( p^H_x p^H_y = \sum_{z \in W} p^H_{z,y} p^H_{z,x} \) with self-dual \( p^H_{z,y}, p^H_{z,x} \in \mathbb{Z}_{\geq 0}[v, v^{-1}] \),

(iv) \( p^H_x = H_x \) for \( p = 0 \) (see [EW14]) and \( p \gg 0 \) (i.e. there are only finitely many primes for which \( p^H_x \neq H_x \)).
Let us recall the definition of $p$-cells (see [Jen20, §3.1] for more details) which is an obvious generalization of a notion introduced by Kazhdan-Lusztig in [KL79]:

**Definition 2.5.** For $h \in \mathcal{H}$ we say that $p_{H}w$ appears with non-zero coefficient in $h$ if the coefficient of $p_{H}w$ is non-zero when expressing $h$ in the $p$-canonical basis.

Define a preorder $\preceq_{R}$ (resp. $\preceq_{L}$) on $W$ as follows: $x \preceq_{R} y$ (resp. $x \preceq_{L} y$) if and only if $p_{H}x$ appears with non-zero coefficient in $p_{H}y$ (resp. $h p_{H}x$) for some $h \in \mathcal{H}$. Define $\preceq_{LR}$ to be the preorder generated by $\preceq_{R}$ and $\preceq_{L}$, in other words we have: $x \preceq_{LR} y$ if and only if $p_{H}x$ appears with non-zero coefficient in $h p_{H}y$ for some $h, h' \in \mathcal{H}$.

The left, right, or two-sided $p$-cells are the equivalence classes with respect to the corresponding preorders respectively.

Let $R$ be a fixed commutative ring with unit and let $A$ be an $R$-algebra with fixed $R$-basis indexed by $B$. For a subset $J \subseteq B$ we denote by $A(J)$ the $R$-span of the basis elements indexed by elements in $J$. Let $C$ be a left $p$-cell in $W$. To simplify notation, we write

$$
\begin{align*}
\preceq_{L} C &:= \{ x \in W \mid x \preceq_{L} y \text{ for some } y \in C \} \\
\preceq_{LR} C &:= \{ x \in W \mid x \preceq_{LR} y \text{ for some } y \in C \text{ and } y \notin C \}
\end{align*}
$$

Unless stated otherwise, we will consider throughout the $p$-canonical basis of $\mathcal{H}$ (and its extension of scalars to $\mathbb{C}$). Recall that $C$ gives rise to a cell module

$$
\mathcal{H}(C) = \frac{\mathcal{H}(\preceq_{L} C)}{\mathcal{H}(\preceq_{LR} C)}
$$

which is a left module for $\mathcal{H}$ (similarly for right or two-sided $p$-cells). The $p$-canonical basis elements indexed by elements in $C$ gives rise to a basis of $\mathcal{H}(C)$.

### 2.5 The Perron-Frobenius Theorem

Our main technical tool will be the Perron-Frobenius theorem (see [Per07; Fro08; Fro09]) which was published over a 100 years ago. A more modern exposition can be found in [Gan59, Vol. 2, Chapter XIII] or in [Hup90, Kapitel IV].

**Theorem 2.6** (Perron-Frobenius). Let $M \in \text{Mat}_{k \times k}(\mathbb{R}_{>0})$. Then there exists $\lambda \in \mathbb{R}_{>0}$, called the Perron-Frobenius eigenvalue of $M$, such that the following statements holds:

(i) $\lambda$ is an eigenvalue of $M$.

(ii) Any other eigenvalue $\mu \in \mathbb{C}$ of $M$ satisfies $|\mu| < \lambda$, so $\lambda$ gives the spectral radius of $M$.

(iii) The eigenvalue $\lambda$ has algebraic multiplicity 1.
(iv) There exists \( v \in \mathbb{R}^k_{>0} \) such that \( Mv = \lambda v \). There exists also \( \hat{v} \in \mathbb{R}^k_{>0} \) such that \( \hat{v}^T M = \lambda \hat{v}^T \).

(v) Any \( w \in \mathbb{R}^k_{>0} \) which is an eigenvector for \( M \) is a scalar multiple of \( v \) and similarly for \( \hat{v} \).

(vi) If \( v \) and \( \hat{v} \) are normalized such that \( \hat{v}^T v = (1) \), then
\[
\lim_{n \to \infty} \frac{M^n}{\lambda^n} = v \hat{v}^T.
\]

3 \textit{p-Families and p-Special Modules}

In this section, we will apply some results of [KM16] to the \( p \)-canonical basis of the complex group ring of a finite Weyl group. Thus we assume that \((W,S)\) is a finite Weyl group throughout the section. For the sake of completeness, we give all the proofs.

Denote by \( \mathbb{C}W = \mathbb{C} \otimes_{\mathbb{Z}[v,w^{-1}]} \mathcal{H} \) the scalar extension of \( \mathcal{H} \) to \( \mathbb{C} \) where we specialize \( v \) to 1. For the rest of the paper fix a set of coefficients \( c = \{c_w\}_{w \in W} \) with \( c_w \in \mathbb{R}_{>0} \). c determines a Perron-Frobenius element
\[
p^{b c} = \sum_{w \in W} c_w \otimes p H_w \in \mathbb{C}W.
\]

Let \( C \) be a left \( p \)-cell in \( W \). We will denote by \( \mathbb{C}W(C) = \mathbb{C} \otimes_{\mathbb{Z}[v,w^{-1}]} \mathcal{H}(C) \) the extension of scalars of the corresponding cell module \( \mathcal{H}(C) \) to \( \mathbb{C} \). As an immediate consequence of Theorem 2.6 we get in this setting (see [KM16, Corollary 3]):

\textbf{Corollary 3.1.} The left (resp. right) action of \( p^{b c} \) on \( \mathbb{C}W(C) \) gives a Perron-Frobenius eigenvalue \( \rho_{b c}(C) \). There is up to isomorphism a unique irreducible \( \mathbb{C}W \)-module \( L_{C,c} \) occurring as composition factor of the cell module \( \mathbb{C}W(C) \) such that \( \rho_{b c}(C) \) is afforded by the action of \( p^{b c} \) on \( L_{C,c} \). Moreover, \( [\mathbb{C}W(C) : L_{C,c}] \) is equal to 1.

The following result shows that we can simplify our notation (see [KM16, Theorem 5]):

\textbf{Theorem 3.2.} \( L_{C,c} \) does up to isomorphism not depend on the choice of \( c \).

\textbf{Proof.} For simplicity, denote by \( n = |W| \) the cardinality of \( W \). Consider the map \( L_{C,c} : \mathbb{R}^n_{>0} \to \text{Irr} \mathbb{C}W \) which sends \( d \) to \( L_{C,d} \). Equipping \( \text{Irr} \mathbb{C}W \) with the discrete topology, we claim that this map is continuous. Since \( \mathbb{R}^n_{>0} \) is connected, any continuous function to a discrete set has to be constant. Therefore, the claim implies the theorem.

To prove the claim, we will show that the preimage \( X_L \subseteq \mathbb{R}^n_{>0} \) of any \( L \in \text{Irr} \mathbb{C}W \) under this map is closed. Assume \( X_L \) is non-empty and let \( d_1, d_2, \ldots \) be a sequence that converges to \( d \in \mathbb{R}^n_{>0} \). Let \( L_1, L_2, \ldots, L_k = L \) be the list of simple subquotients of \( \mathbb{C}W(C) \). Denote by \( M_{d_i} \) the linear operator on \( \mathbb{C}W(C) \) the element \( p^{b d_i} \in \mathbb{C}W \) induces. As \( d_i \in X_L \), we have for all \( i \in \mathbb{N} \) by Theorem 2.6:
\[
p^{b d_i}(C) = \max \{|\mu| \mid \mu \in \text{Spec}(M_{d_i},|L|)\}
\]
Therefore, we get in the limit as the spectrum of a matrix depends continuously on the matrix (see [Ser10, §5.2.3]):

$$\max\{|\mu| \mid \mu \in \text{Spec}(M_d |_{L_r})\} \geq \sup_{1 \leq j < k} \{|\mu| \mid \mu \in \text{Spec}(M_d |_{L_j})\}$$

Since the Perron-Frobenius eigenvalue has multiplicity one, it follows that $L_{C,d}$ is still isomorphic to $L$. It follows that $X_L$ is closed and thus finishes the proof of the claim (and the theorem).

Due to the last result, we can and will drop $c$ in the subscript and denote $L_{C,e}$ by $L_C$ from now on. The following result shows that $L_C$ is an invariant of two-sided $p$-cells (see [KM16, Theorem 6]):

**Theorem 3.3.** For any other left $p$-cell $C'$ which belongs to the same two-sided $p$-cell as $C$, we have $L_C \cong L_{C'}$ and $a_{C}(C) = a_{C}(C')$. 

**Proof.** Denote by $J$ the two-sided cell that contains $C$ and $C'$. We may assume that $C'$ is maximal with respect to $\preceq_L$ in $J$.

First, we will construct a non-zero homomorphism $\varphi : CW(C) \rightarrow CW(C')$ of $CW$-modules. For any $u \in C$ and $v \in C'$ there exist $h', h \in H$ such that $pH_u$ occurs with non-zero coefficient in $h'h_u$ by the definition of two-sided $p$-cells. It follows that $pH_u h$ intersects the set $\{ x \in W \mid x \geq_L v \}$ non-trivially. By our assumption of maximality the intersection $\{ x \in W \mid x \geq_L v \} \cap J$ coincides with $C'$. Thus right multiplication by $h$ and projection onto $CW(C')$ defines the non-zero homomorphism $\varphi$. Observe that $\varphi$ sends any linear combination of the $p$-canonical basis of $CW(C)$ with strictly positive coefficients to a non-zero linear combination of the $p$-canonical basis of $CW(C')$ with non-negative coefficients.

By Theorem 2.6 (iv) there exists an eigenvector $v$ of $p_h$ on $CW(C)$ with eigenvalue $a_{C}(C)$ that is a linear combination of the $p$-canonical basis with strictly positive coefficients. It follows that $\varphi(v)$ is non-zero and an eigenvector of $p_h$ in $CW(C')$ with eigenvalue $a_{C}(C)$. Moreover, $\varphi(v)$ is a linear combination of the $p$-canonical basis of $CW(C')$ with non-negative coefficients. Therefore, the corresponding eigenvalue is the Perron-Frobenius eigenvalue of $p_h$ on $CW(C')$ by Theorem 2.6 (v). This implies $a_{C}(C) = a_{C}(C')$. As $v \in L_C$, the simple subquotient $L_C$ is not annihilated by $\varphi$. Schur’s lemma implies that $\varphi$ induces an isomorphism $L_C \cong L_{C'}$. 

Kildetoft and Mazorchuk prove the following interesting results as well (see [KM16, Proposition 13]):

**Proposition 3.4.** (i) The number of left $p$-cells in the two-sided $p$-cell of $C$ is given by $\dim L_C$.

(ii) The two sided $p$-cells induce a partition of the irreducible representations of $W$ as follows:

$$\text{Irr}(W) = \bigcup_{J \subseteq W_{\text{two-sided } p\text{-cell}}} \{ L \in \text{Irr}(W) \mid [CW(J), L] \neq 0 \}$$
In particular, any simple subquotient of $\text{CW}(C)$ different from $L_C$ is not isomorphic to $L_{C'}$ for any other left $p$-cell $C'$.

Proof. Fix a total order $J_1, J_2, \ldots, J_k$ of the two-sided $p$-cells of $W$ such that $i < j$ implies $J_i \not\supseteq J_j$. For $0 \leq i \leq k$ denote by $I_i$ the linear span of $1 \otimes pH_x$ for $x \in J_s$ with $s \leq i$. Then

$$0 = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_k = \text{CW}$$

is a filtration of $\text{CW}$ by two-sided ideals.

Since $\text{CW}$ is a semisimple algebra, each simple $\text{CW}$-module $L$ occurs in the left regular representation with multiplicity $\dim(L)$. As (1) is a filtration by two-sided ideals, there exists an index $1 \leq i \leq k$ such that $L$ appears with multiplicity $\dim(I_i/I_{i-1})$ in $I_i/I_{i-1}$. On the other hand, $I_i/I_{i-1}$ is isomorphic to the direct sum of all the cell modules $\text{CW}(C)$ for $C$ a left $p$-cell in $J_i$. This implies part (ii) of the Proposition. Finally, observe that $L$ occurs exactly once in each of these left cell modules. This proves part (i) and finishes the proof of the Proposition.

We extend $p_a_c$ to a function $W \to \mathbb{R}_{\geq 0}$ mapping $w$ contained in the left $p$-cell $C$ to $p_a_c(C)$. Using the characterization of the Perron-Frobenius eigenvalue as the largest real eigenvalue, we get immediately from Theorem 3.3:

**Corollary 3.5.** $p_a_c$ is constant on two-sided $p$-cells.

Moreover, we would like to prove:

**Conjecture 3.6.**

For $x, y \in W$ we have:

$$x \overset{p}{\leq} y \implies p_a_c(x) \geq p_a_c(y)$$

Our current techniques do not allow us to prove this as we do not understand the multiplication of the $p$-canonical basis well enough. Using Computer calculations we have verified the conjecture for finite Weyl groups of rank $\leq 4$. Proposition 3.4 allows us to define:

**Definition 3.7.** Let $J \subseteq W$ be a two-sided $p$-cell and $C \subseteq J$ a left (or right) $p$-cell. We call $L_C$ the $p$-special module of $J$ and the set

$$\{L \in \text{Irr}(W) \mid [\text{CW}(J), L] \neq 0\}$$

the $p$-family of $J$.

In the case of the Kazhdan–Lusztig basis and a left (or right) Kazhdan–Lusztig cell $C$, the module $L_C$ is a special representation in the sense of Lusztig (see [KM16, Proposition 9]). In [Lus84, Theorem 5.25] Lusztig shows that for the Kazhdan–Lusztig basis our notion of families coincides with the original definition. An alternative proof of this result based on the theory of primitive ideals can be found in [BV82; BV83].
Example 3.8. In type $B_2$ we have for the 2-Kazhdan–Lusztig basis (with the same conventions as in [JW17]):

$$^2H_{sts} = H_{sts} + H_{s}$$
$$^2H_{w} = H_{w} \text{ for all } w \in W \setminus \{sts\}$$

The 2-special modules in this case are:

- $\{id\} \rightsquigarrow$ the sign representation
- $\{s\} \rightsquigarrow \mathbb{C}$ where $s$ (resp. $t$) acts via $1$ (resp. $-1$)
- $W \setminus \{id, s, w_0\} \rightsquigarrow$ the geometric representation
- $\{w_0\} \rightsquigarrow$ the trivial representation

Thus the irreducible $CW$-modules fall into the following 2-families:

$$\{\text{sgn}\} \cup \{\text{sgn}_s\} \cup \{\text{sgn}_t, \text{geom}\} \cup \{\text{triv}\}$$

where $\text{sgn}_s$ denotes the special module associated to the two-sided 2-cell $\{s\}$. The difference to Lusztig’s families is that the family $\{\text{sgn}_s, \text{sgn}_t, \text{geom}\}$ splits up into two 2-families. Moreover, observe that tensoring with the sign representation does not give a permutation of the set of 2-families, whereas this is the case for Lusztig’s families.

4 Study of Left $p$-Cells Within the Same Two-Sided $p$-Cell

In this section, we will prove that left (or right) $p$-cells within the same two-sided $p$-cell are incomparable. Again we assume that $W$ is a finite Weyl group throughout the section. In order to do so, we will need the definition of an idempotent two-sided cell which reads in our setting as follows:

Definition 4.1. Let $J$ be a two-sided $p$-cell. $J$ is called idempotent if there exist elements $x, y, z \in J$ such that $^pH_x$ occurs with non-trivial coefficient in $^pH_y ^pH_z$.

Even though it might not be obvious at first, we get the following result (see [KM16, Proposition 13 (i)]):

Lemma 4.2. Each two-sided $p$-cell in $W$ is idempotent.

Proof. Let $J$ be a two-sided $p$-cell. Suppose $J$ is not idempotent. The set

$$\{^pH_x \mid x \leq J\}$$

induces a two-sided ideal in $H$ and thus in the semisimple algebra $CW$ which we will denote by $I_J$. We consider the ideal $I$ spanned by $1 \otimes ^pH_x$ for $x \in J$ in the semisimple quotient $CW/I_J$. As $J$ is not idempotent, $I$ is a non-zero nilpotent ideal, contradicting the semisimplicity of the quotient $CW/I_J$. (Recall that the Jacobson radical contains all nilpotent ideals.) \qed
Given a left $p$-cell $C$, Kildetoft and Mazorchuk study the set $X_C$ of all two-sided $p$-cells $J$ such that there exists $x \in J$ satisfying $pH_x \cdot \mathcal{H}(C) \neq 0$ (or equivalently $pH_x \cdot \mathcal{C}W(C) \neq 0$). Applying [KM16, Propositions 14 and 17] we get the following properties:

**Proposition 4.3.** (i) The set $X_C$ contains a minimum element, called the apex of $C$ and denoted $J(C)$.

(ii) For all $x \geq \frac{p}{2} J(C)$ we have $pH_x \cdot \mathcal{H}(C) \neq 0$.

(iii) For any two-sided $p$-cell $I$ and left $p$-cell $C' \subset I$, we have $J(C') = I$.

The following result is the main result of this section. In the setting of the Kazhdan–Lusztig basis, this result was originally deduced from properties of primitive ideals in enveloping algebras (see [Lus81, §4]). It is a weak form of [Lus03, P9].

**Theorem 4.4.** If $x \leq \frac{p}{L} y$ and $x \leq \frac{p}{2} y$, then $x \leq \frac{p}{L} y$.

We will break the proof of the theorem into several lemmata, following along the lines of [KM16, Proposition 18 and Corollary 19]. Let $J$ be a two-sided $p$-cell and $C \subseteq J$ a left $p$-cell.

**Lemma 4.5.** The following statements hold:

(i) Let $I_J$ be the $\mathbb{Z}[v,v^{-1}]$-span in $\mathcal{H}$ of all $pH_w$ such that $w \not\geq \frac{p}{2} J$. Then the cell module $\mathcal{H}(C)$ is naturally a $\mathcal{H}/I_J$-module. Similarly, $\mathcal{C}W(C)$ is naturally a $\mathcal{C}W/[v,v^{-1}] I_J$-module.

(ii) Define $a_c = \sum_{w \in J} c_w \otimes pH_w \in \mathcal{C}W$. Let $M_{c,C}$ be the matrix giving the action of $a_c$ on $\mathcal{C}W(C)$. Then $M_{c,C}$ has positive coefficients.

**Proof.** Observe that $I_J$ is a two-sided ideal in $\mathcal{H}$. Then the first part follows immediately from the definition of the apex of $C$ and the fact that the apex of $C$ is $J$ (see Proposition 4.3 (iii)). It remains to prove the second part.

First, we claim that all columns of $M_{c,C}$ are non-zero. Suppose that $M_{c,C}$ has a zero column indexed by $y \in C$. Since the structure coefficients of the $p$-canonical basis are Laurent polynomials with non-negative coefficients (see Proposition 2.4 (iii)), this implies

$$pH_x \cdot pH_y \in \mathcal{H}(\frac{p}{L} C)$$

for all $x \in J$. Denote by $I$ the $\mathbb{Z}[v,v^{-1}]$-span of all $pH_w$ in $\mathcal{H}/I_J$ for $x \in J$. It follows that

$$I : pH_y \subset \mathcal{H}(\frac{p}{L} C).$$

Observe that $I$ is a two-sided ideal in $\mathcal{H}/I_J$, so that we get:

$$I \cdot (\mathcal{H}/I_J) : pH_y \subset \mathcal{H}(\frac{p}{L} C)$$
From the transitivity of left cells, it follows that $p_H y$ generates the whole cell module $\mathcal{H}(C)$ under the action of $\mathcal{H}/I_J$. Therefore we must have

$$I \cdot \mathcal{H}(C) \subset \mathcal{H}(\frac{p}{L} C)$$

contradicting the fact that the apex of $C$ is $J$ (see Proposition 4.3 (iii)).

Next, we claim that all entries in every column of $M_{e,C}$ are non-zero. Consider the column corresponding to $y \in C$. Let $X \subseteq C$ be the set of all $x \in C$ such that the basis element $p_H x$ appears with a non-zero coefficient in $a_e p_H y$ in $\mathcal{H}(C)$. Above we have shown that $X$ is non-empty. Observe that $X$ contains all $x \in C$ such that the basis element $p_H x$ appears with a non-zero coefficient in $I \cdot p_H y$ in $\mathcal{H}(C)$. Since $I$ is a two-sided ideal in $\mathcal{H}/I_J$, it thus follows that the $\mathbb{Z}[v,v^{-1}]$-span of $p_H x$ for $x \in X$ in $\mathcal{H}(C)$ is invariant under the action of $\mathcal{H}/I_J$. By the transitivity of the cell module, we see that $X = C$. Therefore, all entries in $M_{e,C}$ are positive, which concludes the proof.

We will continue with the notation of the previous lemma. From Lemma 4.5 (ii) it follows that we can apply the Perron-Frobenius theorem to $M_{e,C}$. Let $\lambda$ denote the Perron-Frobenius eigenvalue of $M_{e,C}$. From Theorem 2.6 (vi) we deduce that the matrix

$$M_C := \lim_{m \to \infty} \frac{M_{e,C}^m}{\lambda^m}$$

is positive and satisfies $M_C^2 = M_C$. Observe that $M_C$ is the projector to the one-dimensional $\lambda$-eigenspace of $M_{e,C}$ and thus called the Perron-Frobenius projector.

**Theorem 4.6.** We can define an element $e_J \in CW/C \otimes_{\mathbb{Z}[v,v^{-1}]} I_J$ with the following properties:

(i) $e_J^2 = e_J$.

(ii) $e_J$ can be written as a linear combination of $1 \otimes p_H x$ for $x \in J$ with positive real coefficients.

(iii) $e_J$ acts on $CW(C)$ via $M_C$.

(iv) Let $C_1, C_2, \ldots, C_k$ be the left $p$-cells in $J$. In any total order of the $p$-canonical basis preserving elements of the same left $p$-cell as adjacent elements, we have that $e_J$ acts on $CW(J)$ via

$$M_C := \begin{pmatrix} M_{C_1} & 0 & \ldots & 0 \\ 0 & M_{C_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & M_{C_k} \end{pmatrix}$$

**Proof.** First, consider for $m \geq 1$ the element

$$a_{e,C}^m = \sum_{x \in J} d_{x,m} p_H x \in CW/C \otimes_{\mathbb{Z}[v,v^{-1}]} I_J$$
where \( d_{x,m} \in \mathbb{R}_{\geq 0} \). Since for \( x \in J \) the matrix giving the action of \( pH_x \) on \( CW(C) \) is non-zero by Proposition 4.3 (ii) and (iii) and has non-negative coefficients, the sequence \((d_{x,m})_{m \geq 1}\) is bounded for all \( x \in J \) and thus contains a convergent subsequence.

We have the following identity:

\[
\frac{d_{x,m+1}}{\lambda^{m+1}} = \sum_{x \in J} d_{x,m+1} pH_x = \frac{a_e}{\lambda} \frac{d_{m}}{\lambda^m} = \frac{a_e}{\lambda} \left( \sum_{y \in J} d_{y,m} pH_y \right) = \sum_{x,y \in J} N_{x,y} d_{y,m} pH_y
\]

where \( N \) is the matrix giving the action of \( a_e \) on the two-sided \( p \)-cell module \( CW(J) \). This implies for all \( x \in J \):

\[
d_{x,m+1} = \sum_{y \in J} N_{x,y} d_{y,m} = \cdots = \sum_{y \in J} \frac{(\lambda^m)^{x,y}}{\lambda^m} d_{y,1} = \sum_{y \in J} \frac{(\lambda^m)^{x,y}}{\lambda^m} d_{y,1} \tag{2}
\]

Let \( C_1, C_2, \ldots, C_k \) be an ordering of the left \( p \)-cells in \( J \) such that \( C_i \prec^L C_j \) implies \( i \leq j \). We choose a total order of the \( p \)-canonical basis in \( J \) refining the total order of the left \( p \)-cells in \( J \). Then observe that \( N \) has the following block upper-triangular form

\[
N = \begin{pmatrix}
M_{e,C_1} & * & \ldots & * \\
0 & M_{e,C_2} & \ddots & \vdots \\
\vdots & \ddots & \ddots & * \\
0 & \ldots & 0 & M_{e,C_k}
\end{pmatrix}
\]

where \( M_{e,C_l} \) is the matrix giving the action of \( a_e \) on the left \( p \)-cell module \( CW(C_l) \) for \( 1 \leq l \leq k \). Recall that the spectrum of \( N \) is the multiset union of the spectra of the diagonal block matrices \( M_{e,C_l} \) for \( 1 \leq l \leq k \). Therefore, Theorem 2.6 implies that \( \lambda \) is an eigenvalue with multiplicity \( k \) for \( N \) and that any other eigenvalue \( \mu \neq \lambda \) satisfies \( |\mu| < \lambda \).

Above we have shown that the sequence \((d_{x,m})_{m \geq 1}\) has a convergent subsequence for all \( x \in J \). This implies that \( \lim_{m \to \infty} \frac{X_m}{N_m} \) exists and consequently that for all \( x \in J \) the whole sequence \((d_{x,m})_{m \geq 1}\) converges to some \( d_x \in \mathbb{R}_{\geq 0} \). In addition, it follows that the geometric and algebraic multiplicity of \( \lambda \) for \( N \) coincide (see [Mey00, (7.10.33)]).

Next, we define

\[
e_J := \sum_{x \in J} d_x pH_x
\]

It follows immediately that \( e_J^2 = e_J \) as \( e_J \) is the projector to the \( \lambda \)-eigenspace of \( N \) and that \( e_J \) acts via \( M_C \) on \( CW(C) \). This concludes the proof of (i) and (iii).
We will prove part (iv) next. Observe that the action of $e_J$ on the two-sided $p$-cell module $CW(J)$ in the $p$-canonical basis is given by the following matrix:

\[
N_J := \lim_{m \to \infty} \frac{N^m}{\lambda^m} = \begin{pmatrix}
M_{C_1} & * & \ldots & * \\
0 & M_{C_2} & \ddots & \vdots \\
\vdots & \ddots & \ddots & * \\
0 & \ldots & 0 & M_{C_k}
\end{pmatrix}
\]

which is block upper-triangular with positive, idempotent matrices on the diagonal (by the considerations preceding the proposition). Since $N_J$ is a non-negative idempotent matrix, we may apply [Flo69, Theorem 2] to get that all off-diagonal blocks are 0. (Actually, the easy special case where neither a row nor a column of the non-negative idempotent matrix is zero suffices.) Therefore, $N_J$ is the direct sum of the positive, idempotent matrices $M_{C_i}$ for $1 \leq i \leq k$.

This completes the proof of part (iv).

It remains to prove part (ii). From (2), we deduce that:

\[
d_x = \sum_{y \in J} N_{x,y} c_{y,1} = \sum_{1 \leq i \leq k} \sum_{y \in C_i} (M_{C_j})_{x,y} c_{y,1} > 0
\]

This concludes the proof. Observe that $e_J$ projects in each cell module $CW(C_i)$ for $1 \leq i \leq k$ to the the unique (up to scalar) non-negative eigenvector of $M_{C_i}$ by Theorem 2.6 (iv) and (v). Moreover, the image of $e_J$ in $CW(J)$ gives a positive eigenvector for $N_J$ for the eigenvalue 1 which is not unique though as 1 is a semisimple eigenvalue of multiplicity $k$.

Recall that the coefficients $p \mu_{x,y}^z$ for $x, y, z \in W$ are the structure coefficients of the $p$-canonical basis (see Proposition 2.4 (iii)). The reader should compare the following corollary with [Lus03, P8] and keep in mind that $\gamma_{x,y,z}$ is the coefficient in front of the highest (or lowest) power of $v$ in $p \mu_{x,y}^{-1}$. One would obtain the third cell equivalence if cyclicity (see [Lus03, P7]) held.

**Corollary 4.7.** Let $x, y, z \in J$. If $p \mu_{x,y}^z$ is non-zero, then $y \overset{L}{\sim} z$ and $x \overset{R}{\sim} z$.

In other words, $z$ lies in the intersection of the right $p$-cell of $x$ with the left $p$-cell of $y$. Moreover, any $p \mu_{x,y}^z$ for $z \in J$ occurs with non-trivial coefficient in a product $p \mu_{x,z} \cdot p \mu_{z,y}$ with $x, y \in J$.

**Proof.** Theorem 4.6 (ii) shows that $e_J$ is a positive linear combination of all $p$-canonical basis elements indexed by elements in $J$. The block-diagonal form of the matrix $N_J$ giving the action of $e_J$ on the two-sided $p$-cell module $CW(J)$ (see Theorem 4.6 (iv)) demonstrates that $p \mu_{x,y}^z$ non-zero implies $y \overset{R}{\sim} z$.

To see that any $p \mu_{x,y}^z$ for $z \in J$ occurs with non-trivial coefficient in a product $p \mu_{x,z} \cdot p \mu_{z,y}$ with $x, y \in J$, recall that all the diagonal blocks in $N_J$ have positive coefficients.

Recall that $J^{-1}$ is a two-sided $p$-cell as well. Applying the $\mathbb{Z}[v, v^{-1}]$-linear anti-involution $\iota$, we obtain $p \mu_{x,y}^z = p \mu_{y^{-1}, x^{-1}}^{x^{-1}, y^{-1}}$. Combined with the first part of the corollary applied to $J^{-1}$ we see that $p \mu_{x,y}^z$ non-zero gives $x^{-1} \overset{R}{\sim} z$ which is equivalent to $x \overset{R}{\sim} z$. \(\square\)
Corollary 4.8. The left $p$-cells within $J$ are incomparable with respect to the left $p$-cell preorder. In other words, we have for $x, y \in J$:

$$x \overset{p}{\preceq}_L y \Rightarrow x \overset{p}{\sim}_L y$$

Proof. Let $I$ be the $\mathbb{Z}^{|v, v^{-1}|}$-span of all $p_H^x$ in $\mathcal{H}/I_J$ for $z \in J$. If $x \overset{p}{\preceq}_L y$, then there exists $h \in \mathcal{H}$ such that $p_H^x$ occurs with non-trivial coefficient in $h \cdot p_H^y$. From the previous corollary it follows that there are $v, w \in J$ such that $p_H^v, p_H^w$ is non-zero. Thus, $p_H^v$ occurs with non-trivial coefficient in $h \cdot p_H^w$. Since $I$ is a two-sided ideal in $\mathcal{H}/I_J$, it follows that the image of $h \cdot p_H^v, p_H^w$ in $\mathcal{H}/I_J$ lies in $I$. Rewriting $(h \cdot p_H^v, p_H^w)$ in $\mathcal{H}/I_J$ in terms of the $p$-canonical basis and applying the previous corollary shows that $x$ lies in the same left $p$-cell as $w$ which in turn lies in the same left cell as $y$. \hfill $\square$

Corollary 4.9. Two-sided $p$-cells are the smallest subsets that are at the same time a union of left $p$-cells and one of right $p$-cells.

Example 4.10. Let us illustrate the results of this section in type $B_2$ continuing Example 3.8. Let us consider the most interesting two-sided 2-cell $J = \{2, 12, 212, 21, 121\}$ which decomposes into two left 2-cells

$$J = \{2, 12, 212\} \cup \{21, 121\}$$

and the element $a_c = \sum_{x \in J} p_H^x$ (i.e. all constants are chosen to be 1). The matrix giving the action of $a_c$ on $\mathcal{C}W(J)$ looks as follows

$$N := \begin{pmatrix} 3 & 4 & 3 \\ 4 & 6 & 4 \\ 3 & 4 & 3 \\ 6 & 4 \\ 8 & 6 \end{pmatrix}$$

where we omitted the 0 entries. $N$ has as Perron-Frobenius eigenvalue $\lambda = 6 + 4\sqrt{2}$. The eigenvectors to this eigenvalue for multiplication with $N$ on the right are given by

$$\hat{v}_1^T := \begin{pmatrix} 1 & \sqrt{2} & 1 & 0 & 0 \end{pmatrix} \text{ and } \hat{v}_2^T := \begin{pmatrix} 0 & 0 & 0 & \sqrt{2} & 1 \end{pmatrix}.$$

Similarly, the eigenvectors to this eigenvalue for multiplication with $N$ on the left are given by $v_1 := \hat{v}_1$ and

$$v_2 := \begin{pmatrix} 0 \\ 0 \\ \sqrt{2} \\ 2 \end{pmatrix}.$$
After normalization, these eigenvectors allow us to easily calculate $N_J$:

$$N_J := \lim_{m \to \infty} \frac{N^m}{\lambda^m} = \left( \begin{array}{cccc} \frac{1}{\sqrt{4}} & \frac{\sqrt{2}}{\sqrt{4}} & \frac{1}{\sqrt{4}} & \frac{\sqrt{2}}{\sqrt{4}} \\ \frac{\sqrt{2}}{\sqrt{4}} & \frac{1}{\sqrt{4}} & \frac{\sqrt{2}}{\sqrt{4}} & \frac{1}{\sqrt{4}} \\ \frac{1}{\sqrt{4}} & \frac{\sqrt{2}}{\sqrt{4}} & \frac{1}{\sqrt{4}} & \frac{\sqrt{2}}{\sqrt{4}} \\ \frac{\sqrt{2}}{\sqrt{4}} & \frac{1}{\sqrt{4}} & \frac{\sqrt{2}}{\sqrt{4}} & \frac{1}{\sqrt{4}} \end{array} \right) = \frac{1}{4} (v_1 \hat{v}^T + v_2 \hat{v}^T)$$

For the idempotent $e_J$ we get:

$$e_J := \frac{2 - \sqrt{2}}{8} \otimes (pH_2 + pH_{212} + pH_{121})$$

$$+ \frac{\sqrt{2} - 1}{4} \otimes (pH_{21} + pH_{12}) \in CW/C \otimes \mathbb{Z}[\nu, \nu^{-1}] H(\frac{p}{2} \leq 1212)$$

5 $p$-Cells for Symmetric Groups

5.1 Two-Sided $p$-Cell Preorder and the Dominance Order

The Robinson-Schensted correspondence (see [BB05, §A.3.3] or [Ful97, §4.1]) gives a bijection between the symmetric group $S_n$ and pairs of standard tableaux of the same shape with $n$ boxes. The row-bumping algorithm gives a way to explicitly calculate the image $(P(w), Q(w))$ of $w \in S_n$ under the Robinson-Schensted correspondence.

Throughout this section we assume that we used the Cartan matrix in finite type $A_{n-1}$ as input. In this case the Coxeter system $(W, S)$ can be identified with the pair $(S_n, \{s_1, \ldots, s_{n-1}\})$ consisting of the symmetric group together with the set of simple transpositions.

$p$-Cells for symmetric groups admit a beautiful description in terms of the Robinson-Schensted correspondence (see [Jen20, Theorem 4.33]):

Theorem 5.1. For $x, y \in S_n$ we have:

$$x \stackrel{\mathcal{L}}{\preceq} y \iff Q(x) = Q(y)$$

$$x \stackrel{\mathcal{R}}{\preceq} y \iff P(x) = P(y)$$

$$x \mathcal{L}_R y \iff Q(x) and Q(y) have the same shape$$

In particular, Kazhdan–Lusztig cells and $p$-cells of $S_n$ coincide.

The last result shows that partitions of $n$ correspond to two-sided cells of $S_n$. Recall the definition of the dominance order on the set of partitions of $n$:

Definition 5.2. Let $\lambda = (\lambda_1, \lambda_2, \ldots), \mu = (\mu_1, \mu_2, \ldots)$ be two partitions of $n$. We have $\lambda \preceq \mu$ in the dominance order if and only if

$$\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i$$

for all $k \geq 1$. 
The following fundamental property of the dominance order (see [Bry73, Proposition 2.8]) will be useful for us:

**Proposition 5.3.** For two partitions $\lambda$, $\mu$ of $n$ the following holds:

$$\lambda \leq \mu \iff \lambda^T \geq \mu^T$$

where $\lambda^T$ denotes the conjugate or transpose partition of $\lambda$.

We will not explicitly need the following definition, but we give it for the sake of completeness (see [BB05, §A3.7 and §A3.8] for details):

**Definition 5.4.** Given a standard tableau $T$, the combinatorial algorithm called evacuation proceeds as follows:

First, delete the entry 1 and perform a backward slide on the cell that contained it. Then repeat this for the entry 2, etc. Finally, the tableau that records in reverse the order in which the cells of $T$ have been vacated is called the evacuation of $T$ and denoted by $e(T)$.

For our next result, we will use the following compatibility of the Robinson-Schensted correspondence with the multiplication of the longest element $w_0 \in S_n$ (see [BB05, Fact A3.9.1] or [Knu98, Theorem D] for a proof):

**Theorem 5.5.** If $x \in S_n$ corresponds to $(P, Q)$ under the Robinson-Schensted correspondence, then we have under the Robinson-Schensted correspondence that

(i) $xw_0$ corresponds to $(P^T, e(Q)^T)$,

(ii) $w_0x$ corresponds to $(e(P)^T, Q^T)$ and

(iii) $w_0xw_0$ corresponds to $(e(P), e(Q))$

where $e(P)$ is the evacuation of $P$.

To simplify notation, we will denote by $J_{\lambda} \subseteq S_n$ the two-sided cell corresponding to a partition $\lambda$ of $n$. In order to prove the main result, we will rely on the following observation which is motivated by [Wil03, Proposition A.2.1]:

**Proposition 5.6.** The dominance order on partitions of $n$ is generated by the weak Bruhat order under the Robinson-Schensted correspondence.

**Proof.** Let $\lambda, \mu$ be two partitions of $n$ such that $\lambda \geq \mu$. We will prove the statement by induction on the length of a maximal chain between $\lambda$ and $\mu$. For $\lambda = \mu$ there is nothing to show.

For the induction step, suppose $\lambda > \mu$. It is enough to show that there exists $s \in S$, $x \in J_\mu$ and a partition $\nu$ of $n$ such that

- $xs < x$,
- $xs \in J_\nu$ and
- $\lambda \geq \nu > \mu$. 

The idea is to obtain $\nu$ from $\mu$ by applying a single raising operation. It is a well-known fact that the raising operations generate the dominance order (see for example [Mac95, (1.16)]).

Let $i \in \mathbb{N}$ be minimal such that $\lambda_i > \mu_i$ and observe that it is the first index for which the parts of $\lambda$ and $\mu$ differ. It follows $\mu_{i-1} = \lambda_{i-1} \geq \lambda_i > \mu_i$ and $\mu_{i+1} \neq 0$ as $\mu$ and $\lambda$ are both partitions of $n$.

We will distinguish two cases:

1. **Case:** $\mu_{i+1} > \mu_{i+2}$. Define the partition $\nu$ as follows:

   $$v_j := \begin{cases} 
   \mu_i + 1 & \text{if } j = i, \\
   \mu_{i+1} - 1 & \text{if } j = i + 1, \\
   \mu_j & \text{otherwise.}
   \end{cases}$$

   It follows immediately that $\nu$ is a partition of $n$ satisfying $\lambda \trianglerighteq \nu > \mu$.

   Our convention of drawing partitions is that $\nu_i$ gives the number of boxes in row $i$. With this convention in mind, divide $\nu$ into three parts:
   
   - the first $i - 1$ rows,
   - the $i$-th and $(i + 1)$-st row and
   - the remaining rows.

   Let $T$ be the standard tableau of shape $\nu$ that is column superstandard in each of the three pieces (of course up to shift so that the numbers do not repeat). Suppose that there are $k$ boxes in the first $i - 1$ rows. Then rows $i$ and $i + 1$ of $T$ look as follows:

   $$
   \begin{array}{ccccccc}
   k + 1 & k + 3 & \cdots & k + 2\nu_{i+1} - 1 & k + 2\nu_{i+1} + 1 & k + 2\nu_{i+1} + 2 & \cdots & k + \nu_{i+1} + \nu_i \\
   k + 2 & k + 4 & \cdots & k + 2\nu_{i+1} \\
   \end{array}
   $$

   Define a permutation $y \in S_n$ as follows: Let $(y(1), y(2), \ldots, y(n))$ be the reading word of $T$ (i.e. the sequence obtained by reading the entries of $T$ from bottom to top, left to right). Then the sequence contains the following piece for part 2:

   $$(\ldots, k + 2, k + 4, \ldots, k + 2\nu_{i+1}, k + 1, k + 3, \ldots, k + 2\nu_{i+1} - 1, k + 2\nu_{i+1} + 1, k + 2\nu_{i+1} + 2, \ldots, k + \nu_{i+1} + \nu_i, \ldots)$$

   Observe that $k + \nu_{i+1} + \nu_i$ and $k + \nu_{i+1} + \nu_i - 1$ are in neighbouring positions and let $s \in S$ be the simple transposition swapping these two positions.

   It follows $ys > y$ as $s$ introduces a new inversion. We have that $ys$ lies in $J_\mu$ because when applying the row-bumping algorithm the entry $k + \nu_i + \nu_{i+1}$ is bumped one row down by $k + \nu_i + \nu_{i+1} - 1$ and thus ends up in row $i + 1$. This finishes the proof in the first case by setting $x := ys$. 

2. Case: \( \mu_{i+1} = \mu_{i+2} \). Let \( m \in \mathbb{N} \) be maximal such that \( \mu_m = \mu_{i+1} \). Define the partition \( \nu \) as follows:

\[
\nu_j := \begin{cases} 
\mu_i + 1 & \text{if } j = i \text{ and } \mu_i = \mu_i + 1, \\
\mu_{i+1} + 1 & \text{if } j = i + 1 \text{ and } \mu_i > \mu_{i+1}, \\
\mu_m - 1 & \text{if } j = m, \\
\mu_j & \text{otherwise.}
\end{cases}
\]

It follows that \( \nu \) is a partition of \( n \) satisfying \( \lambda \geq \nu > \mu \). Indeed, observe that the smallest index \( k \in \mathbb{N} \) such that \( \sum_{l=1}^{k} \lambda_l = \sum_{l=1}^{k} \mu_l \) satisfies \( k \geq m \) in order to see \( \lambda \geq \nu \).

Next, observe that \( \nu \) and \( \mu \) differ in exactly two adjacent columns \( l \) and \( l+1 \). By Proposition 5.3, we have \( \nu^T < \mu^T \) and \( \mu^T \) is obtained from \( \nu^T \) by raising a box from row \( l+1 \) to row \( l \). We will define the permutation \( y \in S_n \) and the simple transposition \( s \in S_n \) as before, with the only difference that \( T \) will be of shape \( \mu^T \) instead of \( \nu^T \). It follows that \( ys \) lies in \( J_{\nu^T} \) and \( \mu_{T} = \mu_{sT} \) is obtained from \( \nu_{T} \) by raising a box from row \( l+1 \) to row \( l \). We will define the permutation \( y \in S_n \) and the simple transposition \( s \in S_n \) as before, with the only difference that \( T \) will be of shape \( \mu^T \) instead of \( \nu^T \). It follows that \( ys \) lies in \( J_{\nu^T} \) and \( \mu_{T} = \mu_{sT} \) is obtained from \( \nu_{T} \) by raising a box from row \( l+1 \) to row \( l \). We will define the permutation \( y \in S_n \) and the simple transposition \( s \in S_n \) as before, with the only difference that \( T \) will be of shape \( \mu^T \) instead of \( \nu^T \). It follows that \( ys \) lies in \( J_{\nu^T} \) and \( \mu_{T} = \mu_{sT} \) is obtained from \( \nu_{T} \) by raising a box from row \( l+1 \) to row \( l \).

The goal of this section is to prove the following result:

**Theorem 5.7.** Let \( J, J' \subseteq S_n \) be two-sided cells that correspond to the partitions \( \lambda, \lambda' \) of \( n \) respectively. Then we have the following:

\[
J \preceq_J J' \iff \lambda \preceq \lambda'.
\]

In particular, the two-sided \( p \)-cell preorder coincides with the Kazhdan–Lusztig two-sided cell preorder for all primes \( p \).

*Proof.* (\( \Rightarrow \)) In this case, we have \( h, h' \in H \) such that for some \( w \in J \) the \( p \)-canonical basis element \( p\mathcal{H}w \) occurs with non-trivial coefficient in \( h^p\mathcal{H}w'h' \) where \( w' \) is the longest element in a standard parabolic subgroup contained in \( J' \). Since the corresponding Schubert variety is smooth, we have \( p\mathcal{H}w' = \mathcal{H}w' \). This implies \( w \leq 2^{\frac{w}{2}} \) and thus \( J \preceq J' \).

Applying the characterization of the two-sided Kazhdan–Lusztig preorder in terms of the dominance order (see [Gec06, Theorem 5.1]), we get \( \lambda \preceq \lambda' \).

It is important to note that Geck uses the opposite dominance order because his connection to two-sided Kazhdan–Lusztig cells is not based on the Robinson–Schensted correspondence, but on leading matrix coefficients of irreducible representations of \( \mathcal{H} \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{R}(\nu) \). It depends on the parametrization of irreducible representations of \( S_n \) in a way that is compatible with induced sign characters from Young subgroups (see [Gec06, Example 3.8]). In his correspondence, he introduces the transpose of a partition which by Proposition 5.3 inverses the dominance order (compare [Gec06, Corollary 5.6] with Theorem 5.1).

(\( \Leftarrow \)) By induction on the maximal chain between \( \lambda \) and \( \mu \) it is enough to prove \( J \preceq J' \) in the case where \( \lambda < \lambda' \) are adjacent in the dominance order (so that
\( \lambda \leq \mu \leq \lambda' \) implies \( \mu = \lambda \) or \( \mu = \lambda' \). In the proof of Proposition 5.6 we have shown that in this case there exists \( x \in J, s \in S \) such that \( xs < x \) and \( xs \in J' \). Due to \( (xs)s = x > xs \) it follows that \( pH_x \) occurs with non-trivial coefficient in \( pH_{xs}H_s \). This implies \( x \not\leq_r xs \) and thus \( J < p \leq J' \) finishing the proof. \( \square \)

The last result in the setting of the Kazhdan–Lusztig basis is a special case of a conjecture made independently by Lusztig in [Lus89, §10.8] and Vogan in [Vog00]. Proofs can be found in [DPS98, p. 2.13.1] and [Gec06, Theorem 5.1] for symmetric groups, in [Shi96] for affine Weyl groups of type \( \tilde{A}_n \) or of rank \( \leq 4 \) and in [Bez09, Theorem 4 b)] in general.

It remains an open question to relate the left (or right) \( p \)-cell preorder and the Kazhdan–Lusztig left (or right) cell preorder. It should also be noted that a characterization of either of these preorders on standard tableau under the Robinson-Schensted correspondence is not known.

### 5.2 Cellularity of the \( p \)-Canonical Basis

First, we will recall the definition of a cellular algebra given in [GL96, Definition 1.1]. Let \( R \) be a fixed commutative ring with unit and let \( A \) be an \( R \)-algebra.

**Definition 5.8.** A cell datum for \( A \) is a quadruple \((\Lambda, *, M, C)\) consisting of:

- A finite partially ordered set \( \Lambda \),
- An \( R \)-linear anti-involution \( * \) of \( A \),
- For every \( \lambda \in \Lambda \) a finite, non-empty set \( M(\lambda) \) of indices, and
- An injective map \( C : \bigcup_{\lambda \in \Lambda} M(\lambda) \times M(\lambda) \to A \). If \( \lambda \in \Lambda \) and \( S, T \in M(\lambda) \) write \( C^\lambda_{S,T} = C(S, T) \in A \).

and satisfying the following conditions:

- (i) The image of \( C \) gives an \( R \)-basis of \( A \).
- (ii) \((C^\lambda_{S,T})^* = C^\lambda_{T,S}\).
- (iii) If \( \lambda \in \Lambda \) and \( S, T \in M(\lambda) \) then we have for any element \( a \in A \)

\[
ac^\lambda_{S,T} = \sum_{S' \in M(\lambda)} r_a(S', S)c^\lambda_{S', T} \pmod{A(\lambda)}
\]

where the coefficients \( r_a(S', S) \in R \) are independent of \( T \) and where \( A(\lambda) \) denotes the \( R \)-submodule of \( A \) generated by \( \{C^\mu_{V,W} \mid \mu < \lambda, V, W \in M(\mu)\} \).

Examples of algebras that can be equipped with a cell datum include matrix rings \( \text{Mat}_{d \times d}(R) \), \( R[x]/(x^n) \), the Temperley Lieb algebra and many more (see [GL96, §1, §4-6] for the details).

For the rest of this section we assume again that we used the Cartan matrix in finite type \( A_{n-1} \) as input for the Hecke category.

The goal of this section is to extend the \( p \)-canonical basis to a cell datum of \( \mathcal{H} \). The cell datum will be chosen as follows:
• Let $\Lambda$ be the set of partitions of $n$, equipped with the dominance order.
• Let $\ast$ be the $\mathbb{Z}[v, v^{-1}]$-linear anti-involution $\iota$ of $\mathcal{H}$.
• For $\lambda \in \Lambda$, let $M(\lambda)$ be the set of standard tableaux of shape $\lambda$.
• If $w \in S_n$ corresponds to $(P, Q)$ of shape $\lambda$ under the Robinson-Schensted correspondence, then define $C$ to map $(P, Q)$ to $\rho H_w$. Since the Robinson-Schensted correspondence gives a bijection between $S_n$ and the set of pairs of standard tableaux of the same shape with $n$ boxes, the map $C$ is obviously injective.

Before we can prove the main theorem, we will need the following classical result about the Robinson-Schensted correspondence (see [Ful97, §4.1, Corollary to Symmetry Theorem]):

**Theorem 5.9** (Symmetry Theorem for $S_n$).
If $w \in S_n$ corresponds to $(P(w), Q(w))$, then $w^{-1}$ corresponds to $(Q(w), P(w))$ under the Robinson-Schensted correspondence.

Moreover, we need the following classical result by Knuth (see [Knu70, Theorem 6]):

**Theorem 5.10.** Let $x, y \in S_n$. Then $x$ and $y$ are Knuth equivalent if and only if $P(x) = P(y)$.

In addition, we will require the following description of descent sets under the Robinson-Schensted correspondence (see [BB05, Fact A3.4.1] and [Wil03, Proposition 2.7.1] for a proof):

**Definition 5.11.** Let $P$ be a standard tableau. The **tableau descent set** of $P$, denoted by $D(P)$, is the set of integers $i > 0$ for which $i + 1$ lies strictly below and weakly to the left of $i$ in $P$.

**Lemma 5.12.** For $w \in S_n$ we have:
(i) $s_i \in \mathcal{L}(w) \iff i \in D(P(w))$
(ii) $s_i \in \mathcal{R}(w) \iff i \in D(Q(w))$

As last ingredient, we need to show that when acting on a left cell module the resulting structure coefficients are independent of the $Q$-symbols involved.

**Lemma 5.13.** Let $x, x', y, y' \in S_n$ be such that $x \mathcal{L}_L y$, $x' \mathcal{L}_L y'$, $P(x) = P(x')$, and $P(y) = P(y')$. Then we have for all $s \in S$ the following:

$$\rho P_{s,x} = \rho P_{s,x'}$$

In particular, we may introduce the notation $\rho r_s(P(y), P(x)) := \rho P_{s,x}$ as this coefficient does not depend on the $Q$-symbols of $x$ and $y$.

**Proof.** First, observe that $\mathcal{L}(x) = \mathcal{L}(x')$ by Lemma 5.12 as the $P$-symbols of $x$ and $x'$ coincide. We will consider the only interesting case where $s \notin \mathcal{L}(x)$.

By Theorem 5.10, there exists a sequence of elementary Knuth operations relating $x$ and $x'$ as their $P$-symbols coincide. Each elementary Knuth operation
corresponds to a right star operation with respect to a rank 2 standard parabolic subgroup (see [Jen20, Lemma 4.30]) and thus we will use the corresponding terms interchangeably in the following.

Next, we claim that we may apply the whole sequence of right star operations to $y$ as well. Since $x$ and $y$ lie in the same left cell, we have $\mathcal{R}(x) = \mathcal{R}(y)$ by combining Theorem 5.1 and Lemma 5.12. Therefore, the first right star operation can be applied to $y$ as well. By [Jen20, Theorem 4.13], applying the first right star operation to $x$ and to $y$ gives two elements that still lie in the same left cell. Therefore, we can repeat the argument to see that we can apply the whole sequence of elementary Knuth transformations to $y$ to obtain an element $y''$.

Since Knuth equivalent permutations have the same $P$-symbol (see Theorem 5.10), we have $P(y') = P(y) = P(y'')$. Our argument also shows that $x'$ and $y''$ lie in the same left cell. Therefore, we have $Q(y') = Q(x') = Q(y'')$ by Theorem 5.1. It follows that $y' = y''$ as both their $P$ and $Q$-symbols coincide.

The result now follows from repeated application of [Jen20, Corollary 4.10].

Finally, we can prove our main result of this section:

**Theorem 5.14.** The quadruple $(\Lambda, \ast, M, C)$ gives a cell datum for $\mathcal{H}$.

*Proof.* (i) follows from the fact that the $p$-canonical basis is a basis for the $\mathbb{Z}[v, v^{-1}]$-algebra $\mathcal{H}$ and that the Robinson-Schensted correspondence gives a bijection between the indexing sets involved.

Combining $\iota^p(\mathcal{H}_x) = \mathcal{H}_{x^{-1}}$ for all $x \in W$ (see [Jen20, Proposition 2.5 (iv)]) with Theorem 5.9 gives (ii).

It remains to prove condition (iii). First, let us recall that we have for $s_i \in S$ and $x \in W$:

$$\mathcal{H}_{s_i}^p \mathcal{H}_x = \begin{cases} (v + v^{-1})^p \mathcal{H}_x & \text{if } s_i \in \mathcal{L}(x), \\
\sum_{y \leq s_i, x}^p \mathcal{H}_y & \text{otherwise.} \end{cases} \quad (3)$$

Let $J$ be the two-sided cell of $x$. We want to reduce the multiplication formula (3) modulo $\mathcal{H}(\frac{p}{2} \leq J)$. If $\mathcal{H}_y^p$ for some $y \in W$ occurs with non-zero coefficient in (3), then we have $y \leq^p x$. Corollary 4.8 implies that if $\mathcal{H}_y^p$ does not lie in $\mathcal{H}(\frac{p}{2} \leq J)$, then we have $y \leq^L x$. In particular, $x$ and $y$ have the same $Q$-symbol under the Robinson-Schensted correspondence by Theorem 5.1.

Let $\lambda$ be the partition of $n$ that corresponds to the two-sided cell $J$. By Theorem 5.7, the $\mathbb{Z}[v, v^{-1}]$-submodule $\mathcal{H}(\frac{p}{2} \leq J)$ coincides with $\mathcal{H}(\langle \lambda \rangle)$.

Lemma 5.12 shows that for the left descent set of a permutation coincides with the left descent set of its $P$-symbol. Suppose that $x$ corresponds to $(P, Q)$ under the Robinson-Schensted correspondence to simplify notation.

Combining the arguments above with Lemma 5.13 we may rewrite and reduce (3) modulo $\mathcal{H}(\langle \lambda \rangle)$ as follows:

$$\mathcal{H}_{s_i}^p C_{P, Q} = \begin{cases} (v + v^{-1})^p C_{P, Q} & \text{if } i \in \mathcal{D}(P), \\
\sum_{\mathcal{P}' \in M(\lambda)} \mathcal{P}' \mathcal{R}_s(P, P) C_{P', Q} & \text{otherwise.} \end{cases} \quad (mod H(\langle \lambda \rangle)) \quad (4)$$
This finishes the proof of (iii).

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