The renormalized $\phi^4_4$-trajectory by perturbation theory in a running coupling II: the continuous renormalization group

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Abstract

The renormalized trajectory of massless $\phi^4$–theory on four dimensional Euclidean space–time is investigated as a renormalization group invariant curve in the center manifold of the trivial fixed point, tangent to the $\phi^4$–interaction. We use an exact functional differential equation for its dependence on the running $\phi^4$–coupling. It is solved by means of perturbation theory. The expansion is proved to be finite to all orders. The proof includes a large momentum bound on amputated connected momentum space Green’s functions.
1 Introduction

In Wilson’s renormalization group \cite{W71, WK74}, renormalized theories come as renormalized trajectories of effective actions. The renormalization group leaves invariant a renormalized trajectory up to a flow of renormalization parameters. This invariance prescribes a renormalized theory without a detour to limit procedures.

Renormalization group invariance was used in \cite{Wi96} as a first principle to deduce a renormalized running coupling expansion for the renormalized trajectory of massless $\phi^4$–theory on four dimensional Euclidean space–time. The construction yielded a pair consisting of an effective potential $V(\phi|g)$ and a step $\beta$–function $\beta_L(g)$, both in perturbation theory in $g$, which obey the discrete flow equation

$$R_L V(\phi|g) = V(\phi|\beta_L(g)). \quad (1)$$

Here $R_L$ stands for a momentum space renormalization group transformation, based on the decomposition of $(-\Delta)^{-1}$ by means of an exponential regulator, which rescales by a factor of $L > 1$. Eq. \( (1) \) was shown to possess a unique solution consisting to first order in $g$ of a $\phi^4$–vertex together with a wave function term,

$$V(\phi|g) = g \left\{ \frac{1}{4!} \int d^D x \phi(x)^4 + \frac{\zeta^{(1)}}{2} \int d^D x \phi(x)(-\Delta)\phi(x) \right\} + O(g^2), \quad (2)$$

where $D = 4$, and consisting to all higher orders of polynomial vertices with all properties of a renormalized theory. The higher orders of \( (2) \) where determined inductively. The step $\beta$–function came out as

$$\beta_L(g) = g - \frac{3\log(L)}{(4\pi)^2}g^2 + O(g^3), \quad (3)$$

showing asymptotic freedom for the forward flow on the renormalized $\phi^4$–trajectory. The construction relied on the analysis of $R_L$ with a fixed scale $L > 1$.

The scale parameter $L$ is the ratio of an ultraviolet and an infrared cutoff. Remarkably, the effective potential $V(\phi|g)$ comes out independent of $L$. The only $L$–dependence remains in the step $\beta$–function \( (3) \). The differential $\beta$–function

$$\dot{\beta}(g) = L \frac{d}{dL} \beta_L(g) = -\frac{3}{(4\pi)^2}g^2 + O(g^3) \quad (4)$$

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on the other hand comes out independent of $L$. This suggests a differential approach for the pair $V(\phi|g)$ and $\dot{\beta}(g)$ which makes no reference to the scale $L$ at all. That is the present program. It yields an alternative construction for the renormalized $\phi^4$–trajectory.

The differential form of the discrete flow equation (1) is a functional differential equation. For the normal ordered potential $V(\phi|g) =: V(\phi|g)$: it reads

$$\left\{ \dot{\beta}(g) \frac{\partial}{\partial g} - \left( D\phi, \frac{\delta V}{\delta \phi} \right) \right\} V(\phi|g) = - \langle V(\phi|g), V(\phi|g) \rangle.$$  \(5\)

The operator $\left( D\phi, \frac{\delta V}{\delta \phi} \right)$ generates scale transformations. The right hand side of (5) is a bilinear renormalization group bracket $\langle A(\phi), B(\phi) \rangle$. It consists of contractions between $A(\phi)$ and $B(\phi)$, and is independent of $g$.

Restricted to the renormalized trajectory, the renormalization group becomes a one–dimensional dynamical system given by a flow of $g$ according to the ordinary differential equation

$$L \frac{d}{dL} g(L) = \dot{\beta}(g(L)).$$  \(6\)

The step $\beta$–function (3) is retained as a solution of (3).

The differential approach parallels completely its discrete brother [W96]. It yields an alternative construction of the same renormalization group invariant. The scheme will be to solve (3) inductively by perturbation theory in $g$. The $\phi^4$–trajectory will be selected as the unique solution which is given by (2) to first order in $g$, and higher orders of appropriate type. We remark that the first order wave function renormalization in (2) is peculiar to the four dimensional case.

Both the aim and the setup of this paper are identical to those in [W96]. However, the difference equations of [W96] are here replaced by differential equations. The formulation to be preferred is a matter of taste. The differential version has the advantage that it involves a quadratic non–linearity only. As a consequence, the inductive solution involves only pairwise convolution of lower order vertices.

2 Renormalization group

The setup will be as in [W96], with the difference that the scale $L$ is here variable.

Consider the following momentum space renormalization group transformation $\mathcal{R}_L$, depending on a scale parameter $L > 1$. Let $\mathcal{R}_L$ be composed
of a Gaussian fluctuation integral, with covariance $\Gamma_L$ and mean $\psi$, and a dilatation $S_L$ of $\psi$. Let the fluctuation covariance be defined by

$$\tilde{\Gamma}_L(p) = \frac{1}{p^2} \{ \tilde{\chi}(p) - \tilde{\chi}(Lp) \},$$

where $\tilde{\chi}(p) = \exp(-p^2)$. Let $d\mu_{\Gamma_L}(\zeta)$ be the associated Gaussian measure on field space with mean zero. Let $S_L$ be defined by $S_L\psi(x) = L^{1-D/2}\psi(x/L)$. We then define $\mathcal{R}_L$ as the renormalization group transformation

$$\mathcal{R}_L V(\psi) = -\log \int d\mu_{\Gamma_L}(\zeta) \exp \{-V(S_L\psi + \zeta)\}.$$

We will restrict our attention to $Z_2$-symmetric potentials, with $V(-\phi) = V(\phi)$. The field independent constant will be understood to be removed in (8).

The composition of two renormalization group transformations with scale $L$ is equal to one renormalization group transformation with scale $L^2$. Moreover, the transformation (8) satisfies

$$\mathcal{R}_{L_1} \mathcal{R}_{L_2} = \mathcal{R}_{L_1 L_2}, \quad L_1, L_2 > 1,$$

$$\lim_{L \to 1^+} \mathcal{R}_L = id.$$

It therefore defines a representation of the semi–group of dilatations with scale factors $L > 1$ on the space of effective potentials.

Due to the semi-group property the iteration of renormalization group transformations with fixed scale is identical with an increase of the scale in a single transformation. This interpolation is the motive of the present investigation, in conjunction with the previous construction in [Wi96]. The continuous point of view has the advantage to allow for infinitesimal renormalization group transformations, which can be expected to be close to the identity.

The renormalization group transformation (8) associates with a bare potential $V(\phi)$ an orbit $V(\phi|L) = \mathcal{R}_L V(\phi)$. This orbit satisfies the functional differential equation

$$\left\{ L \frac{\partial}{\partial L} - \left( \mathcal{D}\phi, \frac{\delta \phi}{\delta \phi} \right) - \frac{1}{2} \left( \frac{\delta}{\delta \phi}, C \frac{\delta}{\delta \phi} \right) \right\} V(\phi|L) =$$

$$-\frac{1}{2} \left( \frac{\delta}{\delta \phi} V(\phi|L), C \frac{\delta}{\delta \phi} V(\phi|L) \right).$$

(11)

Here $\mathcal{D}\phi(x) = (1 - D/2 - x\partial_x)\phi(x)$ generates dilatations of the field $\phi$. The operator $(\mathcal{D}\phi, \delta\phi)$ generates scale transformations of the potential,

$$\left( \mathcal{D}\phi, \frac{\delta \phi}{\delta \phi} \right) V(\phi) = \left. \frac{d}{dL} V(S_L\phi) \right|_{L=1}.$$
The operator \( C \) is given by
\[
C = S_{L^{-1}} \left( L \frac{\partial}{\partial L} \Gamma_L \right) S_{L^{-1}}^T,
\] (13)
and is \( L \)-independent. With an exponential regulator we have that \( C = 2\chi \). Eq. (11) follows from a functional heat equation for convolutions by a parameter dependent Gaussian measure, see [GJ87].

The functional Laplacian on the left hand side of (11) can be removed by normal ordering at the expense of a more complicated bilinear term. We define a normal ordered potential \( V(\phi|L) \) by
\[
V(\phi|L) = \exp\left\{ -\frac{1}{2} \left( \frac{\delta}{\delta \phi}, v \frac{\delta}{\delta \phi} \right) \right\} V(\phi|L).
\] (14)
Strictly speaking \( V(\phi|L) \) is the pre–image of \( V(\phi|L) \) under the normal ordering operator and is therefore not decorated by normal ordering colons. To remove the functional Laplacian, the normal ordering covariance has to be chosen such that
\[
\frac{d}{dL} S_{L^{-1}} v(S_{L^{-1}}) T \bigg|_{L=1} = C.
\] (15)
Specifically, we choose
\[
\bar{v}(p) = \frac{\exp(-p^2)}{p^2},
\] (16)
a massless covariance with unit ultraviolet cutoff. The normal ordered potential then satisfies the functional differential equation
\[
\left\{ L \frac{\partial}{\partial L} - \left( D\phi, \frac{\delta}{\delta \phi} \right) \right\} V(\phi|L) = - \langle V(\phi|L), V(\phi|L) \rangle.
\] (17)
The non–linearity comes in form of a bilinear renormalization group bracket
\[
\langle A(\phi), B(\phi) \rangle = \frac{1}{2} \left( \frac{\delta}{\delta \phi_1}, C \frac{\delta}{\delta \phi_2} \right) \exp\left\{ \left( \frac{\delta}{\delta \phi_1}, v \frac{\delta}{\delta \phi_2} \right) \right\} A(\phi_1) B(\phi_2) \bigg|_{\phi_1=\phi_2=\phi},
\] (18)
where \( \phi_1 \) and \( \phi_2 \) denote two independent copies of \( \phi \). It consists of contractions between \( A(\phi) \) and \( B(\phi) \), each contraction being made of one hard propagator \( C \) and any number of soft propagators \( v \).
3 $\phi^4$–trajectory

Consider the following renormalization problem. We seek an effective potential $V(\phi|g)$ and a differential $\beta$–function $\dot{\beta}(g)$, both depending on a coupling parameter $g$, but not on $L$, with the following properties of a renormalized theory.

1. **Power series:** the effective potential $V(\phi|g)$ and the differential $\beta$–function $\dot{\beta}(g)$ are both formal power series

\[
V(\phi|g) = \sum_{r=1}^{\infty} \frac{g^r}{r!} V^{(r)}(\phi),
\]

\[
\dot{\beta}(g) = \sum_{r=1}^{\infty} \frac{g^r}{r!} \dot{\beta}^{(r)},
\]

in $g$. The question of the summability of (19) and (20) will be left aside.

2. **$\phi^4$–theory:** the effective potential $V(\phi|g)$ is to first order a $\phi^4$–vertex together with a wave function term,

\[
V^{(1)}(\phi) = \frac{1}{4!} \int d^D x \phi(x)^4 + \frac{\zeta^{(1)}}{2} \int d^D x \phi(x)(-\triangle)\phi(x),
\]

where $\zeta^{(1)}$ is a first order wave function parameter. The $r$’th order effective potential is a polynomial

\[
V^{(r)}(\phi) = \sum_{m=1}^{r+1} \frac{1}{(2m)!} \int d^D x_1 \cdots d^D x_{2m} \phi(x_1) \cdots \phi(x_{2m}) V^{(r)}_{2m}(x_1, \ldots, x_{2m})
\]

(22)

in the field $\phi$.

3. **Regularity:** the kernels in (22) are Euclidean invariant symmetric distributions. They are given by Fourier integrals

\[
V^{(r)}_{2m}(x_1, \ldots, x_{2m}) = \int \frac{d^D p_1}{(2\pi)^D} \cdots \frac{d^D p_{2m}}{(2\pi)^D} \exp \left( i \sum_{l=1}^{2m} p_l x_l \right)
\]

\[
(2\pi)^D \delta \left( \sum_{l=1}^{2m} p_l \right) \tilde{V}^{(r)}_{2m}(p_1, \ldots, p_{2m})
\]

(23)

With the $\delta$–function removed, their Fourier transforms are Euclidean invariant symmetric $C^\infty$–functions on momentum space $\mathbb{R}^D \times \cdots \times \mathbb{R}^D$. They
satisfy $L_{\infty, \epsilon}$–bounds

$$\| \partial^{|\alpha|} \tilde{V}^{(r)}_{2m} \|_{L_{\infty, \epsilon}} < \infty,$$  \hspace{1cm} (24)

for all $\epsilon > 0$. Here $\alpha = (\alpha_{l, \mu}) \in \mathbb{N}^D \times \cdots \times \mathbb{N}^D$ is a multi–index and $|\alpha| = \sum_{l, \mu} \alpha_{l, \mu}$. The $L_{\infty, \epsilon}$–norm is defined as

$$\| \tilde{V}^{(r)}_{2m} \|_{L_{\infty, \epsilon}} = \sup_{(p_1, \ldots, p_{2m}) \in M_{2m}} |\tilde{V}^{(r)}_{2m}(p_1, \ldots, p_{2m})| \exp \left(-\epsilon \sum_{l=1}^{2m} |p_l| \right),$$  \hspace{1cm} (25)

where $M_{2m}$ denotes the hyperplane of momenta with $\sum_{l=1}^{2m} p_l = 0$.

4. **Coupling parameter**: the four point kernel at zero momenta is

$$\tilde{V}^{(r)}_4(0, 0, 0, 0) = \delta_{r, 1},$$  \hspace{1cm} (26)

zero in higher orders than one.

5. **Invariance**: (19) and (20) satisfy the renormalization group equation

$$\left\{ \hat{\beta}(g) \frac{\partial}{\partial g} - \left( D\phi, \frac{\delta \phi}{\delta \phi} \right) \right\} \mathcal{V}(\phi|g) = -\langle \mathcal{V}(\phi|g), \mathcal{V}(\phi|g) \rangle.$$  \hspace{1cm} (27)

to all orders in $g$.

The differential $\beta$–function (20) defines a running (scale dependent) coupling $g(L)$ as solution to the ordinary differential equation

$$L \frac{d}{dL} g(L) = \hat{\beta}(g(L)).$$  \hspace{1cm} (28)

The effective potential $\mathcal{V}(\phi|g(L))$, as a function of $\phi$ and $L$, then satisfies the flow equation (17).

In perturbation theory, there exists a unique solution to this renormalization problem. We will construct it by induction on the order of $g$. The renormalization group equation (27) is independent of $L$. Its perturbative solution will also be independent of $L$. In other words, the solution will depend on cutoffs only through a running coupling. This property is called scaling. A potential $\mathcal{V}(\phi|g)$, together with a differential $\beta$–function $\hat{\beta}(g)$, which satisfies (27) is said to scale.
4 Scaling equations

Inserting the power series (19) and (20) into (27), we obtain a system of first order functional differential equations for their coefficients. It reads

\[
\left\{ r \dot{\beta}^{(1)} - \left( D \phi, \frac{\delta}{\delta \phi} \right) \right\} \mathcal{V}^{(r)}(\phi) = -\mathcal{K}^{(r)}(\phi). \tag{29}
\]

The right hand side of (29) depends only on lower orders \(\mathcal{V}^{(s)}(\phi)\), with \(1 \leq s \leq r - 1\), and is zero to first order. It is given by

\[
\mathcal{K}^{(r)}(\phi) = \sum_{s=2}^{r} \binom{r}{s} \dot{\beta}^{(s)} \mathcal{V}^{(r-s+1)}(\phi) + \sum_{s=1}^{r-1} \binom{r}{s} \left\langle \mathcal{V}^{(s)}(\phi), \mathcal{V}^{(r-s)}(\phi) \right\rangle. \tag{30}
\]

If \(\mathcal{V}^{(s)}(\phi)\) is given by (21) to first order and is a polynomial (22) for all orders \(2 \leq s \leq r - 1\), then also (30) is a polynomial

\[
\mathcal{K}^{(r)}(\phi) = \sum_{n=1}^{r+1} \frac{1}{(2n)!} \int d^Dx_1 \cdots d^Dx_{2n} \phi(x_1) \cdots \phi(x_{2n}) \mathcal{K}_{2n}^{(r)}(x_1, \ldots, x_{2n}). \tag{31}
\]

The polynomial form (22) is therefore preserved by the renormalization group equation (27). Eq. (29) will be understood as a system of first order partial differential equations for the kernels in (22). It reads

\[
\left\{ \sigma_2^{(r)} + \sum_{m=1}^{2n} x_m \frac{\partial}{\partial x_m} \right\} \mathcal{V}_{2n}^{(r)}(x_1, \ldots, x_{2n}) = \mathcal{K}_{2n}^{(r)}(x_1, \ldots, x_{2n}) \tag{32}
\]

with real space scaling dimensions

\[
\sigma_2^{(r)} = n(2 + D) - r \dot{\beta}^{(1)}. \tag{33}
\]

We will look for solutions to (22) whose Fourier transforms (23) obey (24). We therefore switch from real space to momentum space. In momentum space (22) becomes a system of first order partial differential equations

\[
\left\{ \bar{\sigma}_{2n}^{(r)} - \sum_{m=1}^{2n} p_m \frac{\partial}{\partial p_m} \right\} \bar{V}_{2n}^{(r)}(p_1, \ldots, p_{2n}) = \bar{K}_{2n}^{(r)}(p_1, \ldots, p_{2n}), \tag{34}
\]

with momentum space scaling dimensions

\[
\bar{\sigma}_{2n}^{(r)} = D + n(D - 2) - r \dot{\beta}^{(1)}. \tag{35}
\]
The contribution $D$ in (35) comes from translation invariance. Anticipating \( \dot{\beta}(1) = 4 - D \), the scaling dimension of the $\phi^4$-vertex, the scaling dimensions (35) become order independent in four dimensions. They are given by

\[
\tilde{\sigma}_{2n} = 4 - 2n.
\]

(36)

The renormalized $\phi^4$-trajectory is a particular solution to the scaling equation (34). It is distinguished by the properties listed above.

5 Renormalization group PDEs

We will solve the scaling equations (29) by induction on the order of perturbation theory. The induction step consists of solving a system of first order renormalization group PDEs (34) of the general form

\[
\left\{ p \frac{\partial}{\partial p} - \sigma \right\} F(p) = G(p),
\]

(37)

where $\sigma \in \mathbb{Z}$, and where $G(p)$ is a given function of $p \in \mathbb{R}^N$. The integration of (37) calls for initial data. It is usually supplied in form of a bare action together with renormalization conditions at a distinguished scale. We will instead look for smooth solutions to (37). Recall that smoothness was one condition on the momentum space kernels (23) in our renormalization problem. Therefore, we will assume that $G \in C^\infty(\mathbb{R}^N)$ and look for solutions $F \in C^\infty(\mathbb{R}^N)$ of (37). The right hand side of (4.6) is a sum of multiple convolutions of lower order vertices with cutoff propagators. With an exponential cutoff, smoothness therefore iterates through the induction. But it is luxury in the sense that no higher momentum space derivatives are needed than those required by the Taylor expansions of the non–irrelevant momentum space kernels. The reader is invited to diminish the demands on regularity to this situation in the following statements.

5.1 Irrelevant case

Consider first the case when $\sigma < 0$. We can then immediately integrate (37). It is equivalent to

\[
L \frac{d}{dL} \left\{ L^{-\sigma} F(Lp) \right\} = L^{-\sigma} G(Lp).
\]

(38)

A special solution thereto is

\[
F(p) = \int_0^1 \frac{dL}{L} L^{-\sigma} G(Lp).
\]

(39)
The integral converges for $\sigma < 0$ and yields a smooth function $F(p)$. Furthermore, $F(p)$ inherits the symmetries of $G(p)$.

The integral (39) exhausts the irrelevant case. Suppose that $F_1(p)$ and $F_2(p)$ are two different smooth solutions to (38). Their difference $\triangle F(p) = F_2(p) - F_1(p)$ obeys the homogeneous equation
\[
\left\{ p \frac{\partial}{\partial p} - \sigma \right\} \triangle F(p) = 0,
\]
and is consequently a homogeneous function in $p$ of degree $\sigma$. But regularity at $p = 0$ demands $\triangle F(p) = 0$, when $\sigma < 0$.

Let $\sigma < 0$ and $G \in C^\infty(\mathbb{R}^N)$. Then we have that: (1) There exists a unique solution $F(p)$ of the PDE (37) in the space $C^\infty(\mathbb{R}^N)$. (2) It is given by the integral (39).

### 5.2 Non–irrelevant case

Consider then the case $\sigma \geq 0$. The integral (39) now fails to converge unless $G(p)$ provides a sufficiently high power of $p$. This can be achieved by a Taylor expansion of order $\sigma$. The remainder term behaves as $O(L^{\sigma+1})$ and compensates the power $L^{-1-\sigma}$ in (39). We thus expand
\[
F(p) = \sum_{|\alpha| \leq \sigma} \frac{p^\alpha}{\alpha!} \frac{\partial^{|\alpha|} F}{\partial p^\alpha}(0) + \sum_{|\alpha| = \sigma+1} \frac{p^\alpha}{\alpha!} \int_0^1 \int_0^1 dt \, (1-t)^{|\alpha|-1} \frac{\partial^{|\alpha|} F}{\partial p^\alpha}(tp),
\]
where $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}^N$ is an integer valued multi–index, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, $\alpha! = \alpha_1! \cdots \alpha_N!$, and $p^\alpha = p_1^{\alpha_1} \cdots p_N^{\alpha_N}$. Taking derivatives of (37), we find that
\[
\left\{ p \frac{\partial}{\partial p} - (\sigma - |\alpha|) \right\} \frac{\partial^{|\alpha|} F}{\partial p^\alpha}(p) = \frac{\partial^{|\alpha|} G}{\partial p^\alpha}(p),
\]
Each $p$–derivative thus reduces $\sigma$ by one unit. The derivatives of order $\sigma + 1$ in (41) therefore fall into the irrelevant case. From (39) it follows that
\[
\frac{\partial^{|\alpha|} F}{\partial p^\alpha}(p) = \int_0^1 \frac{dL}{L} L^{-(\sigma-|\alpha|)} \frac{\partial^{|\alpha|} G}{\partial p^\alpha}(Lp),
\]
for all derivatives of order $|\alpha| = \sigma + 1$ (in fact $|\alpha| \geq \sigma + 1$). The remainder term in (41) then involves a convergent integral. It yields a smooth function.
The Taylor coefficients in (41) on the other hand are determined by evaluation of (42) at \( p = 0 \). We have that

\[-(\sigma - |\alpha|) \frac{\partial^{|\alpha|} F}{\partial p^{\alpha}}(0) = \frac{\partial^{|\alpha|} G}{\partial p^{\alpha}}(0),\]

for all solutions to (42) which are regular at \( p = 0 \). Eq. (44) determines all relevant Taylor coefficients with \( |\alpha| < \sigma \) (in fact all non–marginal ones with \( |\alpha| \neq \sigma \)). The marginal Taylor coefficients are not only undetermined by (44) but also impose the condition

\[\frac{\partial^{|\alpha|} G}{\partial p^{\alpha}}(0) = 0\]

on the Taylor coefficients of \( G(p) \) of order \( \sigma \). Smooth solutions exist only if the PDE (42) satisfies (45). If this is the case, as will be supposed here, then the Taylor coefficients of \( F(p) \) of order \( \sigma \) are free parameters. In the perturbation theory for the \( \phi^4 \)-trajectory, we will use this freedom of the marginals at a given order of perturbation theory to satisfy the conditions on the existence of smooth solutions one order below. In this sense, the condition (45) on the marginal Taylor coefficients determines both the \( \beta \)-function and the wave function renormalization.

Let \( \sigma \geq 0 \) and \( G \in \mathcal{C}^\infty(\mathbb{R}^N) \). Then we have that: (1) The PDE (37) has smooth solutions \( F(p) \) if and only if all derivatives of \( G(p) \) of order \( \sigma \) satisfy (14). Let this be the case. (2) Then \( F(p) \) can be written in the form of a Taylor expansion (14) of order \( \sigma \). The Taylor remainder follows from (43). The Taylor coefficients of order \( \sigma \) are free parameters. The Taylor coefficients of order less than \( \sigma \) follow from (44).

There is a neat reformulation for the non–irrelevant case in terms of subtracted functions. Eq. (43) is equivalent to

\[ \left\{ L \frac{d}{dL} - \sigma \right\} F(Lp) = G(Lp).\]  

(46)

Smooth solutions to (46) also obey

\[ \left\{ L \frac{d}{dL} - (\sigma - n) \right\} \frac{d^n}{dL^n} F(Lp) = \frac{d^n}{dL^n} G(Lp),\]

(47)

for all \( n \geq 0 \). In particular, we have the identities

\[-(\sigma - n) \frac{d^n}{dL^n} F(Lp) \bigg|_{L=0} = \frac{d^n}{dL^n} G(Lp) \bigg|_{L=0}\]

(48)
as polynomials in $p$. We can then define subtracted functions

$$F^{(n)}(Lp) = F(Lp) - \sum_{m=0}^{n} \frac{L^m}{m!} \frac{d^m}{dL^m} F(Lp) \bigg|_{L=0}.$$  \hspace{1cm} (49)

From (46) and (48) it follows that they satisfy the subtracted version of (46), namely

$$\left\{ L \frac{d}{dL} - \sigma \right\} F^{(n)}(Lp) = G^{(n)}(Lp).$$  \hspace{1cm} (50)

But the subtracted function has the property $G^{(n)}(Lp) = O(L^{n+1})$ for all $p \in \mathbb{R}^N$. Therefore, the integral

$$F^{(n)}(p) = \int_0^1 \frac{dL}{L} L^{-\sigma} G^{(n)}(Lp)$$  \hspace{1cm} (51)

converges for $n \geq \sigma$. The Taylor remainder can thus be integrated as in the irrelevant case.

6 Perturbation theory

Renormalized perturbation theory is the inductive solution of the scaling equations (34). We have included a separate treatment of the first and second order scaling equations to illustrate the method.

6.1 First order

In the functional representation, the scaling equation (29) requires for $r = 1$ that

$$\left\{ \dot{\beta}^{(1)} - \left( \mathcal{D}\phi, \frac{\delta \phi}{\delta \phi} \right) \right\} \mathcal{V}^{(1)}(\phi) = 0.$$  \hspace{1cm} (52)

In other words, $\mathcal{V}^{(1)}(\phi)$ has to be an eigenvector of $(\mathcal{D}\phi, \delta\phi)$, and $\dot{\beta}^{(1)}$ has to be the eigenvalue. The $\phi^4$–vertex

$$\mathcal{O}_{4,0}(\phi) = \frac{1}{4!} \int d^D x \phi(x)^4$$  \hspace{1cm} (53)

is an eigenvector of $(\mathcal{D}\phi, \delta\phi)$. Its eigenvalue is $\dot{\beta}^{(1)} = 4 - D$. In four dimensions, the $\phi^4$–vertex becomes marginal. The wave function term

$$\mathcal{O}_{2,2}(\phi) = \frac{1}{2} \int d^D x \phi(x)(-\Delta)\phi(x)$$  \hspace{1cm} (54)
is marginal in any dimension. We therefore conclude that
\[ V^{(1)}(\phi) = O_{4,0}(\phi) + \zeta^{(1)} O_{2,2}(\phi) \]  
(55)
is in agreement with (52), whence \( \dot{\beta}^{(1)} = 0 \). The first order wave function parameter \( \zeta^{(1)} \) is not determined by (52).

6.2 Second order

The first order equation (52) is special as it is homogeneous. All higher order equations are inhomogeneous. To second order, (29) reads
\[ \left\{ 2\dot{\beta}^{(1)} - \left( D\phi, \frac{\delta\phi}{\delta\phi} \right) \right\} V^{(2)}(\phi) = -K^{(2)}(\phi), \]  
(56)
where
\[ K^{(2)}(\phi) = \dot{\beta}^{(2)} V^{(1)}(\phi) + 2 \left\langle V^{(1)}(\phi), V^{(1)}(\phi) \right\rangle. \]  
(57)
Eq. (57) contains two unknowns, \( \dot{\beta}^{(2)} \) and \( \zeta^{(1)} \). They belong to the two marginal couplings. In the momentum space kernel representation, (58) becomes
\[ \left\{ \delta^{(2)} - \sum_{m=1}^{2n} p_m \frac{\partial}{\partial p_m} \right\} \bar{V}^{(2)}_{2n}(p_1, \ldots, p_{2n}) = \bar{K}^{(2)}_{2n}(p_1, \ldots, p_{2n}), \]  
(58)
with
\begin{align*}
\bar{K}^{(2)}_2(p_1, p_2) & = \zeta^{(1)} A + \dot{\beta}^{(2)} \zeta^{(1)} p_1^2 + 2(\zeta^{(1)} p_1^2)^2 \bar{C}(p_1) + \bar{C} \star \bar{v} \star \bar{v}(p_1), \quad (59) \\
\bar{K}^{(2)}_4(p_1, p_4) & = \dot{\beta}^{(2)} + 8\zeta^{(1)} p_1^2 \bar{C}(p_1) + 6\bar{C} \star \bar{v}(p_1 + p_2), \quad (60) \\
\bar{K}^{(2)}_6(p_1, p_6) & = 20\bar{C}(p_1 + p_2 + p_3), \quad (61)
\end{align*}
where \( \star \) denotes convolution in momentum space times \((2\pi)^{-D}\). The kernels are understood to be symmetrized in their entries and to be restricted to the hyperplane of zero total momentum. Put \( p_{2n} = -\sum_{m=1}^{2n-1} p_m \) for instance. The constant \( A \) in (59) stands for the convergent one loop integral
\[ A = 4 \int \frac{D^D p}{(2\pi)^D} e^{-2p^2} = \frac{1}{(4\pi)^2}, \]  
(62)
with \( D = 4 \).
All higher kernels with \( n > 3 \) are zero in accordance with (22). The six point kernel has scaling dimension \( \tilde{\sigma}^{(r)}_6 = -2 \) and is irrelevant. From (39) we learn that it is given by\( ^1 \)

\[
\tilde{V}^{(2)}_6(p_1, \ldots, p_6) = - \int_0^1 \frac{dL}{L} L^2 \tilde{K}^{(2)}_6(Lp_1, \ldots, Lp_6). \tag{63}
\]

Notice that (63) is indeed a smooth function of the momenta. The four point kernel has scaling dimension \( \tilde{\sigma}^{(r)}_4 = 0 \) and is marginal. It requires a separate treatment of its zero momentum value, the four point coupling. We determine \( \dot{\beta}^{(2)} \) such that

\[
\tilde{K}^{(2)}_4(0, 0, 0, 0) = 0. \tag{64}
\]

From (64) it follows that the condition (64) reads

\[
\dot{\beta}^{(2)} = -6\tilde{C} \ast \tilde{\nu}(0) = \frac{-6}{(4\pi)^2}, \quad D = 4. \tag{65}
\]

Notice that it is independent of \( \zeta^{(1)} \). The value of the second order \( \phi^4 \)-coupling is not determined by renormalization invariance but rather by the choice of the expansion parameter. A natural choice is (24). The four point kernel is then equal to its first subtraction. It is therefore integrated to

\[
\tilde{V}^{(2)}_4(p_1, \ldots, p_4) = - \int_0^1 \frac{dL}{L} \tilde{K}^{(2)}_4(Lp_1, \ldots, Lp_4). \tag{66}
\]

The integral converges due to (64). The flow of the coupling parameter thus saves us from a logarithmic singularity of (64). We remark that, in three dimensions, the four point kernel is already irrelevant to second order and needs no subtraction at all. The quadratic kernel finally has scaling dimension \( \tilde{\sigma}^{(r)}_2 = 2 \) and is relevant. It calls for a Taylor expansion of order two with remainder term. Using Euclidean invariance, we write

\[
\tilde{V}^{(2)}_2(p, -p) = F(p^2), \tag{67}
\]

\[
\tilde{K}^{(2)}_2(p, -p) = 2G(p^2). \tag{68}
\]

Furthermore, we trade \( p^2 \) for a new variable \( u \). The scaling equation for the quadratic kernel then takes the form

\[
\left\{ 1 - u \frac{d}{du} \right\} F(u) = G(u). \tag{69}
\]

\(^1\)It is instructive to perform this integral. The result is \( \tilde{V}^{(2)}_6(p_1, \ldots, p_5) = 10(p_1 + p_2 + p_3)^{-2} \exp \left\{ - \frac{1}{2} \left( - (p_1 + p_2 + p_3)^2 \right) \right\} - 1 \}. This expression is regular at zero momentum and a bounded function of the momenta. It has the form of a cutoff propagator.
To solve it, we expand $F(u)$ in a Taylor formula

$$F(u) = F(0) + F'(0)u + \frac{u^2}{2} \int_0^1 dt \ (1-t) \ F''(tu).$$  \hspace{1cm} (70)$$

The Taylor coefficients follow from evaluating (69) and its $u$–derivatives at $u = 0$. We find

$$F(0) = G(0) = \frac{1}{2} \tilde{C} \ast \tilde{v} \ast \tilde{v}(0) + \zeta^{(1)} A, \hspace{1cm} \hspace{1cm} (71)$$

with a convergent two loop integral, which comes out as

$$\frac{1}{2} \tilde{C} \ast \tilde{v} \ast \tilde{v}(0) = \frac{1}{(4\pi)^2} \left(2 \log(2) - \log(3)\right). \hspace{1cm} (72)$$

The other Taylor coefficient is the second order wave function parameter

$$F'(0) = \zeta^{(2)}. \hspace{1cm} (73)$$

We leave it undetermined for the moment. Taking a $u$-derivative of (69), it follows that

$$-u \frac{d}{du} F'(u) = G'(u). \hspace{1cm} (74)$$

Regularity of $F'(u)$ at $u = 0$ requires that

$$0 = 2 \ G'(u) = \dot{\beta}^{(2)} \zeta^{(1)} + \frac{\partial}{\partial p^2} \tilde{C} \ast \tilde{v} \ast \tilde{v}(p) \bigg|_{p=0}. \hspace{1cm} (75)$$

This condition determines the first order wave function parameter. Its value in four dimensions is

$$\zeta^{(1)} = -\frac{1}{18(4\pi)^2}. \hspace{1cm} (76)$$

Eq. (73) explains the presence of a wave function term in (24). It is necessary to fulfill the condition under which smooth solutions of the second order equation (69) exist. The second order wave function parameter (73) is then fixed by the condition on the third order pendant to (75). The value of the second order mass parameter (71) is then

$$F(0) = \frac{1}{(4\pi)^2} \left(2 \log(2) - \log(3) - \frac{1}{36}\right). \hspace{1cm} (77)$$
A second $u$–derivative turns (74) into
\[
\left\{ -1 - u \frac{d}{du} \right\} F''(u) = G''(u).
\] (78)

The second derivative is thus irrelevant. Eq. (78) is equivalent to
\[
t \frac{d}{dt} \{ t F''(tu) \} = -t G''(tu).
\] (79)

Therefrom we conclude that
\[
F''(u) = - \int_0^1 \frac{dt}{t} t G''(tu).
\] (80)

The integrals (70) and (80) will not be evaluated. We content ourselves with the ascertainment that they converge and yield a smooth remainder term. The second order scheme is now complete.

### 6.3 Higher orders

The second order scheme generalizes to all higher orders. To order $r$, we first compute $\dot{\beta}^{(r)}$, then $\zeta^{(r-1)}$, and thereafter $\mathcal{V}^{(r)}(\phi)$, except for $\zeta^{(r)}$. Then we proceed to the order $r + 1$. Each step consists of solving a system of scaling equations (34) for the order $r$ momentum space kernels.

#### 6.3.1 Taylor coefficients

The momentum space kernels with non–negative scaling dimension will have to be Taylor expanded. In four dimensions, the quadratic needs to be expanded to second order, whereas the quartic kernel needs to be expanded to first order. A convenient way of writing the Taylor expansions is in terms of projectors
\[
\mathcal{P}_{2,0} \tilde{V}_{2n}^{(r)}(p_1, \ldots, p_{2n}) = \delta_{n,1} \tilde{V}_{2}^{(r)}(0, 0),
\] (81)
\[
\mathcal{P}_{2,2} \tilde{V}_{2n}^{(r)}(p_1, \ldots, p_{2n}) = \delta_{n,1} \frac{\partial}{\partial (p^2)} \tilde{V}_{2}^{(r)}(p, -p)|_{p=0},
\] (82)
\[
\mathcal{P}_{4,0} \tilde{V}_{2n}^{(r)}(p_1, \ldots, p_{2n}) = \delta_{n,2} \tilde{V}_{4}^{(r)}(0, 0, 0, 0).
\] (83)

They are defined to act linearly on polynomial functionals of the form (22). We have that
\[
\mathcal{P}_{2,0} \mathcal{V}^{(r)}(\phi) = \mu^{(r)} \mathcal{O}_{2,0}(\phi),
\] (84)
\[
\mathcal{P}_{2,2} \mathcal{V}^{(r)}(\phi) = \zeta^{(r)} \mathcal{O}_{2,2}(\phi),
\] (85)
\[
\mathcal{P}_{4,0} \mathcal{V}^{(r)}(\phi) = \lambda^{(r)} \mathcal{O}_{2,0}(\phi),
\] (86)
with coupling parameters
\[
\mu^{(r)} = \tilde{V}_2^{(r)}(0, 0), \\
\zeta^{(r)} = \frac{\partial}{\partial(p^2)} \tilde{V}_2^{(r)}(p, -p) \bigg|_{p=0}, \\
\lambda^{(r)} = \tilde{V}_4^{(r)}(0, 0, 0),
\]
and local vertices
\[
\mathcal{O}_{2,0}(\phi) = \frac{1}{2} \int d^D x \phi(x)^2, \\
\mathcal{O}_{2,2}(\phi) = \frac{1}{2} \int d^D x \phi(x)(-\triangle)\phi(x), \\
\mathcal{O}_{4,0}(\phi) = \frac{1}{4!} \int d^D x \phi(x)^4.
\]
They are all eigenvectors of \((\mathcal{D}\phi, \delta\phi)\) with eigenvalues
\[
\left(\mathcal{D}\phi, \frac{\delta\phi}{\delta\phi}\right) \mathcal{O}_{2,0}(\phi) = 2 \mathcal{O}_{2,0}(\phi), \\
\left(\mathcal{D}\phi, \frac{\delta\phi}{\delta\phi}\right) \mathcal{O}_{2,2}(\phi) = 0, \\
\left(\mathcal{D}\phi, \frac{\delta\phi}{\delta\phi}\right) \mathcal{O}_{4,0}(\phi) = (4 - D)\mathcal{O}_{4,0}(\phi).
\]
When applied to (29), the projectors yield the scaling equations
\[
\left( r\dot{\beta}^{(1)} - 2 \right) \mu^{(r)} \mathcal{O}_{2,0}(\phi) = -\mathcal{P}_{2,0} \mathcal{K}^{(r)}(\phi), \\
\left( r\dot{\beta}^{(1)} \right) \zeta^{(r)} \mathcal{O}_{2,2}(\phi) = -\mathcal{P}_{2,2} \mathcal{K}^{(r)}(\phi), \\
\left( r\dot{\beta}^{(1)} - (4 - D) \right) \lambda^{(r)} \mathcal{O}_{4,0}(\phi) = -\mathcal{P}_{4,0} \mathcal{K}^{(r)}(\phi).
\]
By means of (30), these are given by

\[ P_{2,0}(r)(\phi) = \sum_{s=2}^{r} \binom{r}{s} \hat{\beta}(s) \mu^{(r-s+1)} \mathcal{O}_{2,0}(\phi) \]

\[ + \sum_{s=1}^{r-1} \binom{r}{s} P_{2,0} \left\langle \mathcal{V}(s)(\phi), \mathcal{V}^{(r-s)}(\phi) \right\rangle, \tag{99} \]

\[ P_{2,2}(r)(\phi) = \sum_{s=2}^{r} \binom{r}{s} \hat{\beta}(s) \zeta^{(r-s+1)} \mathcal{O}_{2,2}(\phi) \]

\[ + \sum_{s=1}^{r-1} \binom{r}{s} P_{2,2} \left\langle \mathcal{V}(s)(\phi), \mathcal{V}^{(r-s)}(\phi) \right\rangle, \tag{100} \]

\[ P_{1,0}(r)(\phi) = \sum_{s=2}^{r} \binom{r}{s} \hat{\beta}(s) \lambda^{(r-s+1)} \mathcal{O}_{4,0}(\phi) \]

\[ + \sum_{s=1}^{r-1} \binom{r}{s} P_{4,0} \left\langle \mathcal{V}(s)(\phi), \mathcal{V}^{(r-s)}(\phi) \right\rangle. \tag{101} \]

Eqs. (96), (97), and (98) are linear equations for the coupling parameters (87), (88), and (89). They are algebraic equations rather than first order differential equations.

### 6.3.2 The coefficient \( \dot{\beta}^{(r)} \)

In four dimensions, (98) reads

\[ P_{4,0}(r)(\phi) = \mathcal{K}_{4}^{(r)}(0, 0, 0, 0) \mathcal{O}_{4,0}(\phi) = 0, \tag{102} \]

since \( \dot{\beta}^{(1)} = 0 \). The choice (26) of the coupling parameter means \( \lambda^{(s)} = \delta_{s,1} \).

From (101) we then have

\[ \dot{\beta}^{(r)} \mathcal{O}_{4,0}(\phi) = - \sum_{s=1}^{r} \binom{r}{s} P_{4,0} \left\langle \mathcal{V}(s)(\phi), \mathcal{V}^{(r-s)}(\phi) \right\rangle. \tag{103} \]

Its right hand side is known from the induction hypothesis, and does not depend on \( \zeta^{(r-1)} \) because

\[ P_{4,0} \left\langle \mathcal{O}_{4,0}(\phi), \mathcal{O}_{2,2}(\phi) \right\rangle = 0, \tag{104} \]

\[ P_{4,0} \left\langle \mathcal{O}_{2,2}(\phi), \mathcal{O}_{2,2}(\phi) \right\rangle = 0, \tag{105} \]

due to the regularity of \( \tilde{C}(p) \) at \( p = 0 \). Eq. (103) determines \( \dot{\beta}^{(r)} \).
6.3.3 The coefficient $\zeta^{(r)}$

Consider then (97). Since $\dot{\beta}(1) = 0$, it becomes

$$\mathcal{P}_{2,2} \mathcal{K}^{(r)}(\phi) = \left. \frac{\partial}{\partial (p^2)} \mathcal{K}^{(r)}(p, -p) \right|_{p=0} \mathcal{O}_{2,2}(\phi) = 0. \tag{106}$$

From (100) it follows that

$$\left( \begin{array}{c} 1 \\ 2 \end{array} \right) \dot{\beta}^{(2)} \zeta^{(r-1)} \mathcal{O}_{2,2}(\phi) = - \sum_{s=3}^{r} \left( \begin{array}{c} r \\ s \end{array} \right) \dot{\beta}^{(s)} \zeta^{(r-s+1)} \mathcal{O}_{2,2}(\phi)$$

$$- \sum_{s=1}^{r-1} \left( \begin{array}{c} r \\ s \end{array} \right) \mathcal{P}_{2,2} \langle \mathcal{V}^{(s)}(\phi), \mathcal{V}^{(r-s)}(\phi) \rangle. \tag{107}$$

The right hand side is again independent of $\zeta^{(r-1)}$ because

$$\mathcal{P}_{2,2} \langle \mathcal{O}_{4,0}(\phi), \mathcal{O}_{2,2}(\phi) \rangle = 0, \tag{108}$$

$$\mathcal{P}_{2,2} \langle \mathcal{O}_{2,2}(\phi), \mathcal{O}_{2,2}(\phi) \rangle = 0. \tag{109}$$

But then (103) determines $\zeta^{(r-1)}$.

6.3.4 The coefficient $\mu^{(r)}$

The last coefficient is the effective mass parameter. Eq. (96) tells that

$$2 \mu^{(r)} \mathcal{O}_{2,0}(\phi) = \mathcal{P}_{2,0} \mathcal{K}^{(r)}(\phi), \tag{110}$$

and using (99) we then have

$$2 \mu^{(r)} \mathcal{O}_{2,0}(\phi) = \sum_{s=2}^{r} \left( \begin{array}{c} r \\ s \end{array} \right) \dot{\beta}^{(s)} \mu^{(r-s+1)} \mathcal{O}_{2,0}(\phi)$$

$$+ \sum_{s=1}^{r-1} \left( \begin{array}{c} r \\ s \end{array} \right) \mathcal{P}_{2,0} \langle \mathcal{V}^{(s)}(\phi), \mathcal{V}^{(r-s)}(\phi) \rangle. \tag{111}$$

The non–irrelevant part of $\mathcal{V}^{(r)}(\phi)$ is now complete, except for the wave function parameter $\zeta^{(r)}$ which we leave undetermined until the next order.

6.3.5 Irrelevant part

The irrelevant part is directly integrated. For $n \geq 3$, the momentum space scaling dimension (36) is negative. Those kernels are therefore all irrelevant. The scaling equation (34) is in this case integrated to

$$\tilde{\mathcal{V}}^{(r)}_{2n}(p_1, \ldots, p_{2n}) = - \int_{0}^{1} \frac{dL}{L} \frac{1}{L - \sigma^{(r)}_{2n}} \mathcal{K}^{(r)}_{2n}(Lp_1, \ldots, Lp_{2n}). \tag{112}$$
All kernels with $n > r+1$ are zero. The integrals converge due to the negative power counting.

The zero momentum value of the four point kernel has been transferred to the differential $\beta$-function through $\eqref{102}$. Its subtracted remainder is irrelevant and integrated to

$$\tilde{V}_4^{(r)}(p_1, \ldots, p_4) = -\int_0^1 \frac{dL}{L} \tilde{K}_4^{(r)}(Lp_1, \ldots, Lp_4). \quad (113)$$

The integral converges due to the subtraction at zero momentum, and

Finally, the two point kernel is reconstructed with the help of

$$\tilde{V}_2^{(r)}(p, -p) = F(p^2), \quad (114)$$
$$\tilde{K}_2^{(r)}(p, -p) = 2G(p^2), \quad (115)$$

and

$$F(u) = \mu^{(r)} + \zeta^{(r)} u + \frac{u^2}{2} \int_0^1 ds (1 - s) F''(su). \quad (116)$$

through

$$G''(u) = -\int_0^1 \frac{dt}{t} t G''(tu). \quad (117)$$

The iterative scheme is now complete, aside of the question of the convergence and smoothness of the right hand side of $\eqref{34}$.

\section{Bilinear renormalization group bracket}

The bilinear renormalization group bracket is responsible for inhomogeneous terms in our scaling equations. We write it out explicitly for even monomials $\mathcal{O}_{2n}(\phi)$ of the general form

$$\mathcal{O}_{2n}(\phi) = \frac{1}{(2n)!} \int d^D x_1 \cdots d^D x_{2n} \phi(x_1) \cdots \phi(x_{2n}) \mathcal{O}_{2n}(x_1, \ldots, x_{2n}). \quad (118)$$

Applied to two monomials of this form $\eqref{118}$, the bilinear bracket decomposes into

$$\langle \mathcal{O}_{2n}(\phi), \mathcal{O}_{2m}(\phi) \rangle = \sum_{l=|n-m|}^{n+m-1} N_{n,m,l} (\mathcal{O}_{2n} \ast \mathcal{O}_{2m})_{2l}(\phi). \quad (119)$$
and is itself a sum of monomials

\[ (O_{2n} \ast O_{2m})_{2l}(\phi) = \frac{1}{(2l)!} \int d^D x_1 \cdots d^D x_{2l} \phi(x_1) \cdots \phi(x_{2l}) (O_{2n} \ast O_{2m})_{2l}(x_1, \ldots, x_{2l}), \tag{120} \]

whose kernels are given by a multiple convolutions

\[ (O_{2n} \ast O_{2m})_{2l}(x_1, \ldots, x_{2l}) = \frac{1}{2(2l)!} \int dy_1 \cdots dy_{2(n+m-l)} C(y_1 - y_{n+m-l+1}) \prod_{k=2}^{n+m-l} v(y_k - y_{n+m-l+k}) \]

\[ \{ O_{2n}((x_1, \ldots, x_{n-l+m}, y_1, \ldots, y_{n+m-l})) \]

\[ O_{2m}((x_{n+m-l+1}, \ldots, x_{2l}, y_{n+m-l+1}, \ldots, y_{2(n+m-l)})) + ((2l)! - 1) \text{ permutations} \}, \tag{121} \]

The kernels are understood to be symmetric under permutations of their entries. The multiple convolution in (121) always involves one hard propagator \( C \) and \( n + m - l - 1 \) soft propagators \( v \), which is at the same time the number of loops. Furthermore, (115) involves a combinatorial factor

\[ N_{n,m,l} = \frac{(2l)!}{(n + m - l - 1)!(n - m + l)!(m - n + l)!}, \tag{122} \]

coming from the number of ways in which the contractions can be made. Eq. (121) can be interpreted as a fusion of two vertices. In the process of fusion links are created, consisting of propagators.

We present an elementary estimate on this fusion product. The estimate uses the \( L_{\infty,\epsilon} \)-norm in momentum space. The estimate works at this point for \( \epsilon \geq 0 \). Later it will be used for \( \epsilon > 0 \) only. Notice to begin with that

\[ \| \tilde{C} \|_{\infty, -2\epsilon} < \infty, \quad \| \tilde{v} \|_{1, -2\epsilon} < \infty, \tag{123} \]

for the propagators with exponential cutoff\(^2\). At \( \epsilon = 0 \) we have for instance \( \| \tilde{C} \|_{\infty} = 2 \) and \( \| \tilde{v} \|_{\infty} \leq 2\pi^{D/2}/(D - 2) \). If the Fourier transformed kernels now satisfy the bounds

\[ \| \tilde{O}_{2n} \|_{\infty, \epsilon} < \infty, \quad \| \tilde{O}_{2m} \|_{\infty, \epsilon} < \infty, \tag{124} \]

\(^2\)Here \( \| \tilde{C} \|_{\infty, -2\epsilon} = \sup_{p \in \mathbb{R}^D} \{ |\tilde{C}(p)| e^{2\epsilon|p|} \} \) and \( \| \tilde{v} \|_{1, -2\epsilon} = (2\pi)^{-D} \int d^D p |\tilde{v}(p)| e^{2\epsilon|p|} \) denote the \( L_{\infty, -2\epsilon} \) and \( L_{1, -2\epsilon} \)-norms in momentum space.
that is, are finite in the $L_{\infty,\epsilon}$-norm, then it follows that all summands in the decomposition of the bilinear operation have finite $L_{\infty,\epsilon}$-norms in momentum space. They obey

$$\| (O_{2n} \star O_{2m})_{2l} \|_{\infty,\epsilon} \leq \frac{1}{2} \| \tilde{C} \|_{\infty,-2\epsilon} \| \tilde{v} \|_{1,-2\epsilon}^{n+m-l-1} \| \tilde{O}_{2n} \|_{\infty,\epsilon} \| \tilde{O}_{2m} \|_{\infty,\epsilon}.$$  

(125)

Therefore, the renormalization group flow preserves the $L_{\infty,\epsilon}$-norm of momentum space kernels. The estimate immediately follows from the Fourier transform

$$(O_{2n} \star O_{2m})_{2l} (p_1, \ldots, p_{2l}) =$$

$$\frac{1}{2(2l)!} \int \frac{d^D q_1}{(2\pi)^D} \cdots \frac{d^D q_{n+m-l-1}}{(2\pi)^D} \tilde{C}(q_{n+m-l}) \prod_{k=1}^{n+m-l-1} \tilde{v}(q_k)$$

$$\left\{ \tilde{O}_{2n}(p_1, \ldots, p_{n-m+l}, q_1, \ldots, q_{n+m-l})$$

$$\tilde{O}_{2m}(p_{n-m+l+1}, \ldots, p_{2l}, -q_1, \ldots, -q_{n+m-l}) +$$

$$((2l)! - 1) \text{ permutations} \right\},$$

(126)

of (121). The $\delta$-functions from translation invariance have been removed. The sums of momenta in the kernels are zero through

$$p_{2l} = -\sum_{m=1}^{2l-1} p_m, \quad q_{n+m-l} = -\sum_{k=1}^{n-m+l} p_k - \sum_{k=1}^{n+m-l-1} q_k.$$  

(127)

The idea with the parameter $\epsilon$ is to use part of the exponential large momentum decay of the fluctuation and normal ordering propagators to compensate a possible large momentum growth of the kernels.

In an initial value problem for a scale dependent renormalization group flow with $L_{\infty}$-bounded initial data this might seem unnecessary. For instance, a pure $\phi^4$-vertex is given by a constant kernel and is thus $L_{\infty}$-bounded. The evolution preserves the $L_{\infty}$-bound for all finite scales $L$. However, we cannot expect the solution to be $L_{\infty}$-bounded uniformly in $L$. The limit $L \to \infty$ requires a separate treatment of zero momentum derivatives and Taylor remainders of the non-irrelevant kernels. The price to pay is a growth in momentum space. In the four dimensional case it is polynomial in powers and logarithms of momenta. With an exponential bound we are very far on the safe side.

3The $L_{\infty,\epsilon}$-norm for the momentum space kernels is defined as $\| \tilde{O}_{2n} \|_{\infty,\epsilon} = \sup_{(p_1, \ldots, p_{2n}) \in P_n} \{|\tilde{O}(p_1, \ldots, p_{2n})|e^{-\epsilon(|p_1| + \cdots + |p_{2n}|)}\}$ with $P_{2n} = \{(p_1, \ldots, p_{2n}) \in \mathbb{R}^D \times \cdots \times \mathbb{R}^D | p_1 + \cdots + p_{2n} = 0\}$ the hyperplane of total zero momentum.
8 Large momentum bound

We prove a large momentum bound for the solution to the renormalization group PDE (37) under the assumption of a large momentum bound on the inhomogeneous side.

The bound uses the $L_{\infty,\epsilon}$ norm for $\epsilon > 0$. Although being rather wasteful, it suffices to prove finiteness of the bilinear renormalization group bracket.

8.1 Irrelevant case

Let $\sigma < 0$. Suppose that the function $G(p)$ in (37) has a finite $L_{\infty,\epsilon}$-norm

$$\|G\|_{\infty,\epsilon} = \sup_{p \in \mathbb{R}^N} \left\{ |G(p)| e^{-\epsilon|p|} \right\} < \infty. \quad (128)$$

Its solution (39) then inherits an $L_{\infty,\epsilon}$-bound. From

$$|F(p)| e^{-\epsilon|p|} \leq \int_0^1 \frac{dL}{L} L^{-\sigma} |G(Lp)| e^{-\epsilon|p|} \leq \int_0^1 \frac{dL}{L} L^{-\sigma} e^{-(1-L)\epsilon|p|} \|G\|_{\infty,\epsilon} \quad (129)$$

it follows that

$$\|F\|_{\infty,\epsilon} \leq \frac{1}{-\sigma} \|G\|_{\infty,\epsilon}. \quad (130)$$

Eq. (130) shows that the irrelevant solution to the renormalization group PDE is not only finite but also decreases in the $L_{\infty,\epsilon}$-norm.

8.2 Marginal case

Let $\sigma = 0$. In this case we assemble $F(p)$ using a first order Taylor formula. Suppose then that we have $L_{\infty,\epsilon}$-bounds on the first derivatives

$$\|G_\mu\|_{\infty,\epsilon} = \sup_{p \in \mathbb{R}^N} \left\{ \left| \frac{\partial}{\partial p^\mu} G(p) \right| e^{-\epsilon|p|} \right\} < \infty. \quad (131)$$

If $F(p)$ is marginal, then its first derivatives are irrelevant with scaling dimension minus one. It follows that

$$\|F_\mu\|_{\infty,\epsilon} \leq \|G_\mu\|_{\infty,\epsilon}. \quad (132)$$

Therefrom it follows that

$$|F(p)| e^{-\epsilon|p|} \leq |F(0)| e^{-\epsilon|p|} + \sum_\mu |p_\mu| \int_0^1 dt |F_\mu(tp)| e^{-\epsilon|p|}$$

$$\leq |F(0)| + \sum_\mu |p_\mu| \int_0^1 dt e^{-(1-\epsilon)|p|} \|F_\mu\|_{\infty,\epsilon}. \quad (133)$$
The result is an $L_{\infty, \epsilon}$-bound

\[ \| F \|_{\infty, \epsilon} \leq |F(0)| + \frac{1}{\epsilon} \sum_{\mu} \| F_\mu \|_{\infty, \epsilon}. \]  (134)

This estimate is not uniform in $\epsilon$. It works for $\epsilon$ arbitrary small, but the bound grows with an inverse power of $\epsilon$. The large momentum growth is a consequence of the split in derivatives and Taylor remainder.

### 8.3 Relevant case

Let $\sigma > 0$. This case requires a generalization of the bound in the marginal case. The Taylor expansion is pushed to order $\sigma + 1$. Then the derivatives become irrelevant. We assume $L_{\infty, \epsilon}$-estimates on all derivatives

\[ \| G_\alpha \|_{\infty, \epsilon} = \sup_{p \in \mathbb{R}^N} \left\{ \left| \partial^{|\alpha|} G(p) \right| e^{-|p|} \right\} < \infty \]  (135)

of order $|\alpha| = \sigma + 1$. Since they are irrelevant with scaling dimension minus one it follows that the corresponding derivatives of $F(p)$ obey

\[ \| F_\alpha \|_{\infty, \epsilon} \leq \| G_\alpha \|_{\infty, \epsilon}, \]  (136)

and are also $L_{\infty, \epsilon}$-bounded. From the Taylor formula it then follows that

\[ |F(p)| e^{-|p|} \leq \sum_{|\alpha| \leq \sigma} \frac{|p|^{|\alpha|}}{\alpha!} e^{-|p|} |F_\alpha(0)| + \sum_{|\alpha| = \sigma + 1} \frac{|p|^{|\alpha|}}{\alpha!} \int_0^1 dt (1 - t)^\sigma |F_\alpha(tp)| e^{-|p|} \]

\[ \leq \sum_{|\alpha| \leq \sigma} \frac{|p|^{|\alpha|}}{\alpha!} e^{-|p|} |F_\alpha(0)| + \]

\[ \sum_{|\alpha| = \sigma + 1} \frac{|p|^{\sigma+1}}{\alpha!} \int_0^1 dt (1 - t)^\sigma e^{-(1-t)|p|} \| F_\alpha \|_{\infty, \epsilon}, \]  (137)

and thus

\[ \| F \|_{\infty, \epsilon} \leq \sum_{|\alpha| \leq \sigma} \frac{1}{\alpha!} A_{\epsilon,|\alpha|} |F_\alpha(0)| + \sum_{|\alpha| = \sigma + 1} \frac{1}{\alpha!} B_{\epsilon,\sigma+1} \| F_\alpha \|_{\infty, \epsilon}, \]  (138)

with constants

\[ A_{\epsilon,|\alpha|} = \sup_{p \in \mathbb{R}^N} \left\{ |p|^{|\alpha|} e^{-|p|} \right\}, \quad B_{\epsilon,\sigma+1} = \frac{\Gamma(\sigma + 1)}{\epsilon^{\sigma+1}}. \]  (139)
Thus we again have an $L_{\infty,\epsilon}$-bound on the function $F(p)$. This completes the large momentum bound on $F(p)$. Exactly the same strategy applies to the derivatives of $F(p)$ as well. The irrelevant derivatives inherit immediately large momentum bounds. The relevant derivatives require Taylor expansions. We omit to spell out explicitly the necessary bounds on the derivatives of $G(p)$.

8.4 Iteration and regularity

The iterative scheme determines order by order $\beta^{(s)}$, $\zeta^{(s-1)}$, $\mu^{(s)}$, and the irrelevant remainders $\tilde{V}_{irr,2n}^{(s)}(p_1, \ldots, p_{2n-1})$. It is finite to all orders of perturbation theory because of the following iteration of regularity. Suppose that we have shown the following to all orders $s \leq r - 1$:

I) $\beta^{(s)}$, $\zeta^{(s-1)}$, and $\mu^{(s)}$ are finite numbers. II) $\tilde{V}_{irr,2n}^{(s)}(p_1, \ldots, p_{2n-1})$ is a smooth function on $\mathbb{R} \times \cdots \times \mathbb{R}$ for all $1 \leq n \leq s + 1$, symmetric in the momenta, and $O(D)$-invariant. III) $\|\tilde{V}_{irr,2n,\alpha}^{(s)}\|_{\infty,\epsilon}$ is finite for all $\epsilon > 0$, $1 \leq n \leq s + 1$, and $|\alpha| \geq 0$. Here $\alpha$ is a multi-index which labels momentum derivatives.

Then the same statements hold at order $s = r$. Since they are trivially fulfilled to order one they iterate to all orders of perturbation theory.

To prove the iteration of regularity we once more inspect each step of the iterative scheme. First, the irrelevant remainders $\tilde{K}_{irr,2n}^{(r)}(p_1, \ldots, p_{2n-1})$ are smooth functions on $\mathbb{R} \times \cdots \times \mathbb{R}$, symmetric under permutations and $O(D)$-invariant. They and all their momentum derivatives satisfy $L_{\infty,\epsilon}$-bounds. They are composed of two contributions. The first immediately inherits a bound from the induction hypothesis. The second is a sum of renormalization group brackets of lower orders. Therefore, they consist of multiple convolutions with propagators. The integrals converge, are smooth functions of the external momenta, and satisfy $L_{\infty,\epsilon}$-bounds. Second, we have linear equations for the coefficients $\beta^{(r)}$, $\zeta^{(r-1)}$, and $\mu^{(r)}$ with finite coefficients. Third, the integration of the inhomogeneous renormalization group PDEs, yields solutions with the desired properties.

9 Conclusions

The aim of perturbative renormalization theory is to derive power series expansions for Green’s functions which are free of divergencies. The BPHZ theorem states that this can be accomplished by writing the Green’s functions in terms of renormalized parameters. An elegant proof of the BPHZ theorem was given by Callan [C76]. A polished version of which is due to Lesniewski.
Their method is similar to ours in that it is based on renormalization group equations for the renormalized Green’s functions, the Callan-Symanzik equations. The method proposed here is different in that it does not resort to any kind of graphical analysis, not to analysis of sub-graphs, and not to skeleton expansions.

A new generation of proofs of the BPHZ theorem was initiated with the work of Polchinski [P84]. His proof has been simplified further by Keller, Kopper, and Salmhofer [KKS90]. Our method is similar in that it uses an exact renormalization differential equation. The details are however quite different. The main difference is that Polchinski begins with a cutoff theory. He then shows how the cutoff can be removed in a way such that the effective interaction remains finite. Our method directly addresses the limit theory without cutoffs, expressed in terms of a renormalization group transformation with cutoffs. Roughly speaking, we are here simultaneously changing Polchinski’s renormalization conditions and integrating an amount of fluctuations. Unlike Polchinski and followers we use a renormalization group differential equation with dilatation term. A way to think of (5) is as an equation for a renormalization group fixed point of a system which has been enhanced by one degree of freedom, the running coupling. This fixed point problem can only be formulated with rescaling and with dilatation term.

Another renormalization group approach to renormalized perturbation theory comes from Gallavotti [G85, GN85] and collaborators. Pedagogical accounts of tree expansions can be found in [BG93, PHRWS88]. There the result of renormalization is expressed in terms of a renormalized tree expansion. The program of [W96] with an iterated transformation with fixed $L$ is related to the tree expansion. Both are built upon a cumulant expansion for the effective interaction. The renormalization procedure is however quite different. Like Polchinski, Gallavotti starts from a cutoff theory. It is organized in terms of trees, which describe the sub-structure of divergencies in Feynman diagrams. The divergencies are transformed into a flow of the non-irrelevant couplings. This part is similar to ours. The basic difference with Gallavotti is that we do not organize our expansion in terms of trees. A hybrid approach between Polchinski and Gallavotti is due to Hurd [H89].

An important question is whether this construction of renormalized trajectories extends beyond perturbation theory. Another important question is whether it extends to renormalized trajectories at non-trivial fixed points. We hope to return with answers to these questions in the future.

\[4\] It certainly works in the cases where perturbation theory converges.
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