DECOMPOSITIONS OF RATIONAL GABOR REPRESENTATIONS

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Abstract. Let $\Gamma = \langle T_k, M_l : k \in \mathbb{Z}^d, l \in B\mathbb{Z}^d \rangle$ be a group of unitary operators where $T_k$ is a translation operator and $M_l$ is a modulation operator acting on $L^2(\mathbb{R}^d)$. Assuming that $B$ is a non-singular rational matrix of order $d$, with at least one non-integral entry, we obtain a direct integral irreducible decomposition of the Gabor representation which is defined by the isomorphism $\pi : (\mathbb{Z}_m \times B\mathbb{Z}^d) \times \mathbb{Z}^d \rightarrow \Gamma$ where $\pi(\theta, l, k) = e^{2\pi i \theta} M_l T_k$. We also show that the left regular representation of $(\mathbb{Z}_m \times B\mathbb{Z}^d) \times \mathbb{Z}^d$ which is identified with $\Gamma$ via $\pi$ is unitarily equivalent to a direct sum of $\text{card}(\Gamma, \Gamma)$ many disjoint subrepresentations of the type: $L_0, L_1, \cdots, L_{\text{card}(\Gamma, \Gamma)-1}$. It is shown that for any $k \neq 1$ the subrepresentation $L_k$ of the left regular representation is disjoint from the Gabor representation. Furthermore, we prove that there is a subrepresentation $L_1$ of the left regular representation of $\Gamma$ which has a subrepresentation equivalent to $\pi$ if and only if $|\det B| \leq 1$. Using a central decomposition of the representation $\pi$ and a direct integral decomposition of the left regular representation, we derive some important results of Gabor theory. More precisely, a new proof for the Density Condition for the rational case is obtained. We also derive characteristics of vectors $f$ in $L^2(\mathbb{R}^d)$ such that $\pi(\Gamma)f$ is a Parseval frame in $L^2(\mathbb{R}^d)$. Examples are also given in the paper.

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1. Introduction

The concept of applying tools of abstract harmonic analysis to time-frequency analysis, and wavelet theory is not a new idea [1, 2, 3, 7, 11]. For example in [1], Larry Baggett gives a direct integral decomposition of the Stone-von Neumann representation of the discrete Heisenberg group acting in $L^2(\mathbb{R})$. Using his decomposition, he was able to provide specific

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conditions under which this representation is cyclic. In Section 5.5, the author obtains a characterization of tight Weyl-Heisenberg frames in \( L^2(\mathbb{R}) \) using the Zak transform, and a precise computation of the Plancherel measure of a discrete type I group. In \([11]\), the authors present a thorough study of the left regular representations of various subgroups of the reduced Heisenberg groups. Using well-known results of admissibility of unitary representations of locally compact groups, they were able to offer new insights on Gabor theory.

Let \( B \) be a non-singular matrix of order \( d \) with real entries. For each \( k \in \mathbb{Z}^d \) and \( l \in B\mathbb{Z}^d \), we define the corresponding unitary operators \( T_k, M_l \) such that

\[
T_k f(t) = f(t - k) \quad \text{and} \quad M_l f(t) = e^{-2\pi i(l,t)} f(t)
\]

for \( f \in L^2(\mathbb{R}^d) \). The operator \( T_k \) is called a shift operator, and the operator \( M_l \) is called a modulation operator. Let \( \Gamma \) be a subgroup of the group of unitary operators acting on \( L^2(\mathbb{R}^d) \) which is generated by the set \( \{T_k, M_l : k \in \mathbb{Z}^d, l \in B\mathbb{Z}^d\} \). We write

\[
\Gamma = \langle T_k, M_l : k \in \mathbb{Z}^d, l \in B\mathbb{Z}^d \rangle.
\]

The commutator subgroup of \( \Gamma \) given by \( [\Gamma, \Gamma] = \{e^{2\pi i(l,Bk)} : l, k \in \mathbb{Z}^d\} \) is a subgroup of the one-dimensional torus \( \mathbb{T} \). Since \( [\Gamma, \Gamma] \) coincides with the center of the group, then \( \Gamma \) is a nilpotent group which is generated by \( 2d \) elements. Moreover, as a set, we shall regard \( \Gamma \) as a subset of \( \mathbb{T} \times \mathbb{R}^d \times \mathbb{R}^d \), and as such \( \Gamma \) is given the natural subspace topology. If \( B \in GL(d, \mathbb{Z}) \) then \( \Gamma \) is a locally compact abelian group. If \( B \) has some irrational entry then \( [\Gamma, \Gamma] \) is an infinite non-closed proper subgroup of the torus. Thus, if \( B \) has at least one irrational entry then \( \Gamma \) is a non-abelian group which is not locally compact. Finally, if the matrix \( B \) only has rational entries with at least one non-integral entry, then \( [\Gamma, \Gamma] \) is a finite non-trivial subgroup of the torus. As a result, \( \Gamma \) is a non-abelian discrete group which can also be regarded as a finite extension of an abelian group.

It is easily derived from the work in Section 4, \([11]\) that if \( B \) is an integral matrix, then the Gabor representation

\[
\pi : B\mathbb{Z}^d \times \mathbb{Z}^d \to \Gamma \subset \mathcal{U}(L^2(\mathbb{R}^d))
\]

defined by \( \pi(l,k) = M_l T_k \) is equivalent to a subrepresentation of the left regular representation of \( \Gamma \) if and only if \( B \) is a unimodular matrix. The techniques used by the authors of \([11]\) rely on the decompositions of the left regular representation and the Gabor representation into their irreducible components. The group generated by the operators \( M_l \) and \( T_k \) is actually a commutative group which is isomorphic to \( \mathbb{Z}_d \times B\mathbb{Z}_d \). The unitary dual and the Plancherel measure for such group are well-known and rather easy to write down. Thus, a precise direct integral decomposition of the left regular representation is easily obtained as well. Next, using the Zak transform, the authors decompose the representation \( \pi \) into a direct integral of its irreducible components as well. They are then able to compare both representations. As a result, one can derive from the work in the fourth section of \([11]\) that the representation \( \pi \) is equivalent to a subrepresentation of the left regular representation if and only if \( B \) is a unimodular matrix. The main objective of this paper is to generalize these ideas to the case where \( B \in GL(d, \mathbb{Q}) \).

Let us assume that \( B \) is an invertible rational matrix with at least one entry which is not an element of \( \mathbb{Z} \). Denoting the inverse transpose of a given matrix \( M \) by \( M^* \), it is not too hard to see that there exists a matrix \( A \in GL(d, \mathbb{Z}) \) such that \( \Lambda = B^* \mathbb{Z}^d \cap \mathbb{Z}^d = AZ^d \). A
precise algorithm for the construction of $A$ is described on Page 809 of [4]. Put
\begin{equation}
\Gamma_0 = \langle \tau, M_l, T_k : l \in B\mathbb{Z}^d, k \in \Lambda, \tau \in [\Gamma, \Gamma] \rangle
\end{equation}
and define $\Gamma_1 = \langle \tau, M_l : l \in B\mathbb{Z}^d, \tau \in [\Gamma, \Gamma] \rangle$. Then $\Gamma_0$ is a normal abelian subgroup of $\Gamma$. Moreover, we observe that $\Gamma_1$ is a subgroup of $\Gamma_0$ of infinite index. Let $m$ be the number of elements in $[\Gamma, \Gamma]$. We write $m = \text{card} ([\Gamma, \Gamma])$. Clearly, since $B$ has at least one rational non-integral entry, then $m > 1$. Furthermore, it is easy to see that there is an isomorphism
\[ \pi : (\mathbb{Z}_m \times B\mathbb{Z}^d) \rtimes \mathbb{Z}^d \to \Gamma \subset U(L^2(\mathbb{R}^d)) \]
defined by $\pi(j, Bl, k) = e^{2\pi i j} M_B T_k$. The multiplication law on the semi-direct product group $(\mathbb{Z}_m \times B\mathbb{Z}^d) \rtimes \mathbb{Z}^d$ is described as follows. Given arbitrary elements
\[ (j, Bl, k), (j_1, Bl_1, k_1) \in (\mathbb{Z}_m \times B\mathbb{Z}^d) \times \mathbb{Z}^d, \]
we define
\[ (j, Bl, k)(j_1, Bl_1, k_1) = ((j + j_1 + \omega(l_1, k)) \mod m, B(l + l_1), k + k_1) \]
where $\omega(l_1, k) \in \mathbb{Z}_m$, and $\langle Bl_1, k \rangle = \omega(l_1, k)$. We call $\pi$ a rational Gabor representation. It is also worth observing that $\pi^{-1}(\Gamma_0) = (\mathbb{Z}_m \times B\mathbb{Z}^d) \times A\mathbb{Z}^d$ and
\[ \pi^{-1}(\Gamma_1) = (\mathbb{Z}_m \times B\mathbb{Z}^d) \times \{0\} \simeq \mathbb{Z}_m \times B\mathbb{Z}^d. \]
Throughout this work, in order to avoid cluster of notations, we will make no distinction between $(\mathbb{Z}_m \times B\mathbb{Z}^d) \rtimes \mathbb{Z}^d$ and $\Gamma$.

The interest in the various normal subgroups of $\Gamma : \Gamma_0$ and $\Gamma_1$ stems from the theory of Mackey. Indeed, in Mackey theory [6, 9, 10], normal subgroups play an essential role in the parametrization of the unitary dual of group extensions. We shall present more details about how Mackey theory specializes to the class of groups considered in this paper in the third section.

The main results of this paper are summarized in the following propositions. Let
\[ \Gamma = \langle T_k, M_l : k \in \mathbb{Z}^d, l \in B\mathbb{Z}^d \rangle \]
and assume that $B$ is an invertible rational matrix with at least one entry which is not an integer. Let $L$ be the left regular representation of $\Gamma$.

**Proposition 1.** The left regular representation of $\Gamma$ is decomposed as follows:
\begin{equation}
L \simeq \bigoplus_{k=0}^{m-1} \int_{\mathbb{Z}^d \times A\mathbb{Z}^d} \text{Ind}_{\Gamma_0}^\Gamma \chi(k, \sigma) \ d\sigma.
\end{equation}
Moreover, the measure $d\sigma$ in (3) is a Lebesgue measure, and (3) is not an irreducible decomposition of $L$.

**Proposition 2.** The Gabor representation $\pi$ is decomposed as follows:
\begin{equation}
\pi \simeq \int_{\mathbb{R}^d \times A^* \mathbb{Z}^d} \text{Ind}_{\Gamma_0}^\Gamma \chi(1, \sigma) \ d\sigma.
\end{equation}
Moreover, $d\sigma$ is a Lebesgue measure defined on the torus $\mathbb{R}^d \times A^* \mathbb{Z}^d$ and (3) is an irreducible decomposition of $\pi$. 
It is worth pointing out here that the decomposition of \( \pi \) given in Proposition 2 is consistent with the decomposition obtained in Lemma 5.39, \cite{7} for the specific case where \( d = 1 \) and \( B \) is the inverse of a natural number larger than one.

**Proposition 3.** Let \( m \) be the number of elements in the commutator subgroup \([\Gamma, \Gamma]\). There exists a decomposition of the left regular representation of \( \Gamma \) such that

\[
L \simeq \bigoplus_{k=0}^{m-1} L_k,
\]

and for each \( k \in \{0, 1, \cdots, m - 1\} \), the representation \( L_k \) is disjoint from \( \pi \) whenever \( k \neq 1 \). Moreover, the Gabor representation \( \pi \) is equivalent to a subrepresentation of the subrepresentation \( L_1 \) of \( L \) if and only if \( |\text{det} B| \leq 1 \).

Although the problem of decomposing the representations \( \pi \) and \( L \) into their irreducible components is interesting in its own right, we shall also address how these decompositions can be exploited to derive interesting and relevant results in time-frequency analysis. The proof of Proposition 3 allows us to say that there exist a measurable set \( E \subset \mathbb{R}^d \) which is a subset of a fundamental domain for the lattice \( B^* \mathbb{Z}^d \times A^* \mathbb{Z} \), satisfying

\[
\mu(E) = \frac{1}{|\text{det}(B)\text{det}(A)| \dim (l^2(\Gamma/\Gamma_0))}
\]

where \( \mu \) is the Lebesgue measure on \( \mathbb{R}^d \times \mathbb{R}^d \), and a unitary map

\[
A : \int_E \bigoplus_{k=1}^{\ell(\sigma)} l^2 \left( \frac{\Gamma}{\Gamma_0} \right) d\sigma \to L^2(\mathbb{R}^d)
\]

which intertwines \( \int_E \bigoplus_{k=1}^{\ell(\sigma)} \text{Ind}_{\Gamma_0}^{\Gamma} (\chi_{(1,\sigma)}) \) with \( \pi \) such that, the multiplicity function \( \ell \) is bounded, the representations \( \chi_{(1,\sigma)} \) are characters of the abelian subgroup \( \Gamma_0 \) and

\[
\int_E \bigoplus_{k=1}^{\ell(\sigma)} \text{Ind}_{\Gamma_0}^{\Gamma} (\chi_{(1,\sigma)}) \ d\sigma
\]

is the central decomposition of \( \pi \) (Section 3.4.2, \cite{7}). Moreover, for the case where \( |\text{det}(B)| \leq 1 \), the multiplicity function \( \ell \) is bounded above by the number of cosets in \( \Gamma/\Gamma_0 \) while if \( |\text{det}(B)| > 1 \), then the multiplicity function \( \ell \) is greater than the number of cosets in \( \Gamma/\Gamma_0 \) on a set \( E' \subseteq E \) of positive Lebesgue measure. This observation that the upper-bound of the multiplicity function behaves differently in each situation may mistakenly appear to be of limited importance. However, at the heart of this observation, lies a new justification of the well-known Density Condition of Gabor systems for the rational case (Theorem 1.3 \cite{8}) which partly states that the representation \( \pi \) is cyclic if and only if \( |\text{det}(B)| \leq 1 \). In fact we shall offer a new proof of (a rational version of) the Density Condition for Gabor systems in Lemma 7.

It is also worth pointing out that the central decomposition of \( \pi \) as described above has several useful implications. Following the discussion on Pages 74-75, \cite{7}, the decomposition given in (5) may be used to:

1. Characterize the commuting algebra of the representation \( \pi \) and its center.
2. Characterize representations which are either quasi-equivalent or disjoint from \( \pi \) (see \cite{7} Theorem 3.17 and Corollary 3.18).
Additionally, using the central decomposition of $\pi$, in the case where the absolute value of the determinant of $B$ is less or equal to one, we are able to obtain a complete characterization of Parseval frames for $L^2(\mathbb{R}^d)$ which are of the type $\pi(\Gamma)f$ where $f$ is a vector in $L^2(\mathbb{R}^d)$.

**Proposition 4.** Let us suppose that $|\det B| \leq 1$. Then

$$\pi \simeq \int_{E} \left( \bigoplus_{k=1}^{\ell(\sigma)} \text{Ind}_{\Gamma_0}^{\Gamma} (\chi(1,\sigma)) \right) d\sigma$$

with $\ell(\sigma) \leq \text{card}(\Gamma/\Gamma_0)$. Moreover, $\pi(\Gamma)f$ is a Parseval frame in $L^2(\mathbb{R}^d)$ if and only if $f = \mathcal{A}(a(\sigma))_{\sigma \in E}$ such that for $d\sigma$-almost every $\sigma \in E$,

$$\|a(\sigma)(k)\|_{L_2(\mathbb{R}^d)}^2 = 1 \text{ for } 1 \leq k \leq \ell(\sigma)$$

and for distinct $k, j \in \{1, \cdots, \ell(\sigma)\}$, $\langle a(\sigma)(k), a(\sigma)(j) \rangle = 0$.

This paper is organized around the proofs of the propositions mentioned above. In Section 2, we fix notations and we revisit well-known concepts such as induced representations and direct integrals which are of central importance. The proof of Proposition 1 is obtained in the third section. The proofs of Propositions 2, 3 and some examples are given in the fourth section. Finally, the last section contains the proof of Proposition 4.

### 2. Preliminaries

Let us start by fixing our notations and conventions. All representations in this paper are assumed to be unitary representations. Given two equivalent representations $\pi$ and $\rho$, we write that $\pi \simeq \rho$. We use the same notation for isomorphic groups. That is, if $G$ and $H$ are isomorphic groups, we write that $G \simeq H$. All sets considered in this paper will be assumed to be measurable. Given two disjoint sets $A$ and $B$, the disjoint union of the sets is denoted $A \cup B$. Let $H$ be a Hilbert space. The identity operator acting on $H$ is denoted $1_H$. The unitary equivalence classes of irreducible unitary representations of $G$ is called the unitary dual of $G$ and is denoted $\hat{G}$.

Several of the proofs presented in this work rely on basic properties of induced representations and direct integrals. The following discussion is mainly taken from Chapter 6, [6]. Let $G$ be a locally compact group, and let $K$ be a closed subgroup of $G$. Let us define $q : G \to G/K$ to be the canonical quotient map and let $\varphi$ be a unitary representation of the group $K$ acting in some Hilbert space which we call $H$. Next, let $K_1$ be the set of continuous $H$-valued functions $f$ defined over $G$ satisfying the following properties:

1. $q(\text{support } (f))$ is compact,
2. $f(gk) = \varphi(k)^{-1}f(g)$ for $g \in G$ and $k \in K$.

Clearly, $G$ acts on the set $K_1$ by left translation. Now, to simplify the presentation, let us suppose that $G/K$ admits a $G$-invariant measure. We remark that in general, this is not always the case. However, the assumption that $G/K$ admits a $G$-invariant measure holds for the class of groups considered in this paper. We construct a unitary representation of $G$ by endowing $K_1$ with the following inner product:

$$\langle f, f' \rangle = \int_{G/K} \langle f(g), f'(g) \rangle_H d(gK) \text{ for } f, f' \in K_1.$$
Now, let $K$ be the Hilbert completion of the space $K_1$ with respect to this inner product. The translation operators extend to unitary operators on $K$ inducing a unitary representation $\text{Ind}_K^G(\varphi)$ which is defined as follows:

$$\text{Ind}_K^G(\varphi)(x)f(g) = f(x^{-1}g) \text{ for } f \in K.$$ 

We notice that if $\varphi$ is a character, then the Hilbert space $K$ can be naturally identified with $L^2(G/H)$. Induced representations are natural ways to construct unitary representations. For example, it is easy to prove that if $e$ is the identity element of $G$ and if $1$ is the trivial representation of $\{e\}$ then the representation $\text{Ind}_{\{e\}}^G(1)$ is equivalent to the left regular representation of $G$. Other properties of induction such as induction in stages will be very useful for us. The reader who is not familiar with these notions is invited to refer to Chapter 6 of the book of Folland [6] for a thorough presentation.

We will now present a short introduction to direct integrals; which are heavily used in this paper. For a complete presentation, the reader is referred to Section 7.4, [6]. Let \( \{H_\alpha\}_{\alpha \in A} \) be a family of separable Hilbert spaces indexed by a set $A$. Let $\mu$ be a measure defined on $A$. We define the direct integral of this family of Hilbert spaces with respect to $\mu$ as the space which consists of vectors $\varphi$ defined on the parameter space $A$ such that $\varphi(\alpha)$ is an element of $H_\alpha$ for each $\alpha \in A$ and

$$\int_A \|\varphi(\alpha)\|_{H_\alpha}^2 d\mu(\alpha) < \infty$$

with some additional measurability conditions which we will clarify. A family of separable Hilbert spaces $\{H_\alpha\}_{\alpha \in A}$ indexed by a Borel set $A$ is called a field of Hilbert spaces over $A$. A map

$$\varphi : A \rightarrow \bigcup_{\alpha \in A} H_\alpha \text{ such that } \varphi(\alpha) \in H_\alpha$$

is called a vector field on $A$. A measurable field of Hilbert spaces over the indexing set $A$ is a field of Hilbert spaces $\{H_\alpha\}_{\alpha \in A}$ together with a countable set $\{e_j\}_j$ of vector fields such that

1. the functions $\alpha \mapsto \langle e_j(\alpha), e_k(\alpha) \rangle_{H_\alpha}$ are measurable for all $j, k$,
2. the linear span of $\{e_k(\alpha)\}_k$ is dense in $H_\alpha$ for each $\alpha \in A$.

The direct integral of the spaces $H_\alpha$ with respect to the measure $\mu$ is denoted by $\int_A^\oplus H_\alpha d\mu(\alpha)$ and is the space of measurable vector fields $\varphi$ on $A$ such that

$$\int_A \|\varphi(\alpha)\|_{H_\alpha}^2 d\mu(\alpha) < \infty.$$

The inner product for this Hilbert space is:

$$\langle \varphi_1, \varphi_2 \rangle = \int_A \langle \varphi_1(\alpha), \varphi_2(\alpha) \rangle_{H_\alpha} d\mu(\alpha) \text{ for } \varphi_1, \varphi_2 \in \int_A^\oplus H_\alpha d\mu(\alpha).$$

3. The regular representation and its decompositions

In this section, we will discuss the Plancherel theory for $\Gamma$. For this purpose, we will need a complete description of the unitary dual of $\Gamma$. This will allow us to obtain a central decomposition of the left regular representation of $\Gamma$. Also, a proof of Proposition [1] will be given in this section.

Let $L$ be the left regular representation of $\Gamma$. Suppose that $\Gamma$ is not commutative and that $B$ is a rational matrix. We have shown that $\Gamma$ has an abelian normal subgroup of finite index
which we call $\Gamma_0$. Moreover, there is a canonical action $(74-79, [10])$ of $\Gamma$ on the group $\hat{\Gamma}_0$ (the unitary dual of $\Gamma_0$) such that for each $P \in \Gamma$ and $\chi \in \hat{\Gamma}_0$,

$$P \cdot \chi(Q) = \chi(P^{-1}QP).$$

Let us suppose that $\chi = \chi(\lambda_1, \lambda_2, \lambda_3)$ is a character in the unitary dual $\hat{\Gamma}_0$ where

$$(\lambda_1, \lambda_2, \lambda_3) \in \{0, 1, \cdots, m - 1\} \times \frac{\mathbb{R}^d}{B^* \mathbb{Z}^d} \times \frac{\mathbb{R}^d}{A^* \mathbb{Z}^d} \simeq \hat{\Gamma}_0,$$

and

$$\chi(\lambda_1, \lambda_2, \lambda_3) \left( e^{2\pi ik} M_{Bl}(T_{Aj}) \right) = e^{2\pi i (\lambda_2, Bl)} e^{2\pi i (\lambda_3, Aj)}.$$ 

Clearly, here $\mathbb{R}^d$ is identified with its dual $\hat{\mathbb{R}}^d$ and $\{0, 1, \cdots, m - 1\}$ parametrizes the unitary dual of the commutator subgroup $[\Gamma, \Gamma]$ which is isomorphic to the cyclic group $\mathbb{Z}_m$. For any $\tau \in [\Gamma, \Gamma]$, we have

$$\tau \cdot \chi(\lambda_1, \lambda_2, \lambda_3) = \chi(\lambda_1, \lambda_2, \lambda_3).$$

Moreover, given $M_l, T_k \in \Gamma$,

$$M_l \cdot \chi(\lambda_1, \lambda_2, \lambda_3) = \chi(\lambda_1, \lambda_2, \lambda_3), \text{ and } T_k \cdot \chi(\lambda_1, \lambda_2, \lambda_3) = \chi(\lambda_1, \lambda_2, \lambda_3 - k\lambda_1, \lambda_3).$$

Next, let $\Gamma_\chi = \{ P \in \Gamma : P \cdot \chi = \chi \}$. It is easy to see that the stability subgroup of the character $\chi(\lambda_1, \lambda_2, \lambda_3)$ is described as follows:

$$(8) \quad \Gamma_{\chi(\lambda_1, \lambda_2, \lambda_3)} = \{ \tau M_l T_k \in \Gamma : \tau \in [\Gamma, \Gamma], l \in B\mathbb{Z}^d, k\lambda_1 \in B^* \mathbb{Z}^d \}.$$ 

It follows from $(8)$ that the stability group $\Gamma_{\chi(\lambda_1, \lambda_2, \lambda_3)}$ contains the normal subgroup $\Gamma_0$. Indeed, if $\lambda_1 = 0$ then

$$\Gamma_{\chi(\lambda_1, \lambda_2, \lambda_3)} = \Gamma.$$ 

Otherwise,

$$(9) \quad \Gamma_{\chi(\lambda_1, \lambda_2, \lambda_3)} = \{ \tau M_l T_k \in \Gamma : \tau \in [\Gamma, \Gamma], l \in B\mathbb{Z}^d, k \in \left( \frac{1}{\lambda_1} B^* \mathbb{Z}^d \right) \cap \mathbb{Z}^d \}.$$ 

According to Mackey theory (see Page 76, [10]) and well-known results of Kleppner and Lipsman (Page 460, [10]), every irreducible representation of $\Gamma$ is of the type

$$\text{Ind}_{\Gamma_\chi}^\Gamma (\bar{\chi} \otimes \bar{\sigma})$$

where $\bar{\chi}$ is an extension of a character $\chi$ of $\Gamma_0$ to $\Gamma_\chi$, and $\bar{\sigma}$ is the lift of an irreducible representation $\sigma$ of $\Gamma_\chi/\Gamma_0$ to $\Gamma_\chi$. Also two irreducible representations

$$\text{Ind}_{\Gamma_\chi}^\Gamma (\bar{\chi} \otimes \bar{\sigma}) \text{ and } \text{Ind}_{\Gamma_\chi^{\prime}}^\Gamma (\bar{\chi}_{\chi} \otimes \bar{\sigma})$$

are equivalent if and only if the characters $\chi$ and $\chi_1$ of $\Gamma_0$ belong to the same $\Gamma$-orbit. Since $\Gamma$ is a finite extension of its subgroup $\Gamma_0$, then it is well-known that there is a measurable set which is a cross-section for the $\Gamma$-orbits in $\hat{\Gamma}_0$. Now, let $\Sigma$ be a measurable subset of $\hat{\Gamma}_0$ which is a cross-section for the $\Gamma$-orbits in $\hat{\Gamma}_0$. The unitary dual of $\Gamma$ is a fiber space which is described as follows:

$$\hat{\Gamma} = \bigcup_{\chi \in \Sigma} \left\{ \pi_{\chi, \sigma} = \text{Ind}_{\Gamma_\chi}^\Gamma (\bar{\chi} \otimes \bar{\sigma}) : \sigma \in \frac{\Gamma_\chi}{\Gamma_0} \right\}.$$ 

Finally, since $\Gamma$ is a type $I$ group, there exists a unique standard Borel measure on $\hat{\Gamma}$ such that the left regular representation of the group $\Gamma$ is equivalent to a direct integral of all
elements in the unitary dual of $\Gamma$, and the multiplicity of each irreducible representation occurring is equal to the dimension of the corresponding Hilbert space. So, we obtain a decomposition of the representation $L$ into a direct integral decomposition of its irreducible representations as follows (see Theorem 3.31, [7] and Theorem 5.12, [11])

\begin{equation}
L \simeq \int_{\Sigma} \int_{\hat{\Gamma}} \int_{\Gamma_0} \pi_{\chi, \sigma} \otimes 1_{l^2(\Gamma_\chi)}(\frac{\Gamma}{\Gamma_\chi}) d\omega_{\chi}(\sigma) d\chi
\end{equation}

and $\dim \left( l^2 \left( \frac{\Gamma}{\Gamma_\chi} \right) \right) \leq \text{card} \left( \Gamma / \Gamma_0 \right)$. The fact that $\dim \left( l^2 \left( \frac{\Gamma}{\Gamma_\chi} \right) \right) \leq \text{card} \left( \Gamma / \Gamma_0 \right)$ is justified because the number of representative elements of the quotient group $\frac{\Gamma}{\Gamma_\chi}$ is bounded above by the number of elements in $\Gamma_\chi$. Moreover the direct integral representation in (10) is realized as acting in the Hilbert space

\begin{equation}
\int_{\Sigma} \int_{\hat{\Gamma}} \int_{\Gamma_0} l^2 \left( \frac{\Gamma}{\Gamma_\chi} \right) \otimes l^2 \left( \frac{\Gamma}{\Gamma_\chi} \right) d\omega_{\chi}(\sigma) d\chi.
\end{equation}

Although the decomposition in (10) is canonical, the decomposition provided by Proposition [1] will be more convenient for us. Surprisingly, the proof of Proposition [1] is short and simple.

3.1. Proof of Proposition [1]. Let $e$ be the identity element in $\Gamma$, and let 1 be the trivial representation of the trivial group $\{e\}$. We observe that $L \simeq \text{Ind}^\Gamma_{\{e\}} (1)$. $\Gamma_0$ being an abelian group, then

\begin{equation}
L \simeq \text{Ind}^\Gamma_{\Gamma_0} \left( \text{Ind}^\Gamma_{\{e\}} (1) \right)
\end{equation}

\begin{equation}
\simeq \text{Ind}^\Gamma_{\Gamma_0} \left( \int_{\hat{\Gamma_0}} \chi_t dt \right)
\end{equation}

\begin{equation}
\simeq \int_{\hat{\Gamma_0}} \text{Ind}^\Gamma_{\Gamma_0} (\chi_t) dt.
\end{equation}

The second equivalence given above is coming from the fact that $\text{Ind}^\Gamma_{\{e\}} (1)$ is equivalent to the left regular representation of the group $\Gamma_0$. Since $\Gamma_0$ is abelian, its left regular representation admits a direct integral decomposition into elements in the unitary dual of $\Gamma_0$, each occurring once. Moreover, the measure $dt$ is a Lebesgue measure (also the Haar measure) supported on the unitary dual of the group, and the Plancherel transform is the unitary operator which is intertwining the representations

$\text{Ind}^\Gamma_{\{e\}} (1)$ and $\int_{\hat{\Gamma_0}} \chi_t dt$.

Based on the discussion above, it is worth mentioning that the representations occurring in (13) are generally reducible since it is not always the case that $\Gamma_0 = \Gamma_\chi$. We observe that $\hat{\Gamma_0}$ is parametrized by the group $\mathbb{Z}_m \times \frac{\mathbb{R}^d}{B^* Z^d} \times \frac{\mathbb{R}^d}{A^* Z^d}$. Thus, identifying $\hat{\Gamma_0}$ with $\mathbb{Z}_m \times \frac{\mathbb{R}^d}{B^* Z^d} \times \frac{\mathbb{R}^d}{A^* Z^d}$, we reach the desired result:

\begin{equation}
L \simeq \bigoplus_{k=0}^{m-1} \int_{\frac{\mathbb{R}^d}{B^* Z^d} \times \frac{\mathbb{R}^d}{A^* Z^d}} \text{Ind}^\Gamma_{\Gamma_0} (\chi(k,t)) dt.
\end{equation}
Remark 5. Referring to (2), we remark that $\Gamma_{\chi(1,t_2,t_3)} = \Gamma_0$, and in this case $\text{Ind}_{\Gamma_0}^{\Gamma} (\chi(1,t_2,t_3))$ is an irreducible representation of the group $\Gamma$.

4. Decomposition of $\pi$

In this section, we will provide a decomposition of the Gabor representation $\pi$. For this purpose, it is convenient to regard the set $\mathbb{R}^d$ as a fiber space, with base space the $d$-dimensional torus. Next, for any element $t$ in the torus, the corresponding fiber is the set $t + \mathbb{Z}^d$. With this concept in mind, let us define the periodization map

$$\mathcal{R} : L^2 (\mathbb{R}^d) \to \int_{\mathbb{R}^d} l^2 (\mathbb{Z}^d) \, dt$$

such that $\mathcal{R} f(t) = (f(t+k))_{k \in \mathbb{Z}^d}$. We remark here that we clearly abuse notation by making no distinction between $\mathbb{R}^d$ and a measurable subset of $\mathbb{R}^d$ which is a $\mathbb{Z}^d$-tiling set (or a fundamental domain for $\frac{\mathbb{Z}^d}{\mathbb{Z}^d}$). Next, the inner product which we endow the direct integral Hilbert space $\int_{\mathbb{R}^d} l^2 (\mathbb{Z}^d) \, dt$ with is defined as follows. For any vectors $f$ and $h \in \int_{\mathbb{R}^d} l^2 (\mathbb{Z}^d) \, dt$

the inner product of $f$ and $g$ is equal to

$$\langle f, h \rangle_{\mathcal{R}(l^2(\mathbb{R}^d))} = \int_{\mathbb{R}^d} \left( \sum_{k \in \mathbb{Z}^d} f(t+k) \overline{h(t+k)} \right) \, dt,$$

and it is easy to check that $\mathcal{R}$ is a unitary map.

4.1. Proof of Proposition [2]. For $t \in \mathbb{R}^d$, we consider the unitary character $\chi_{(1,-t)} : \Gamma_1 \to \mathbb{T}$ which is defined by $\chi_{(1,-t)} (e^{2\pi i z} M_t) = e^{2\pi i z} e^{-2\pi i (t,\cdot)}$. Next, we compute the action of the unitary representation $\text{Ind}_{\Gamma_1}^{\Gamma_1} \chi_{(1,-t)}$ of $\Gamma$ which is acting in the Hilbert space

$$\mathcal{H}_t = \left\{ f : \Gamma \to \mathbb{C} : f(PQ) = \chi_{(1,-t)} (Q)^{-1} f(P), Q \in \Gamma_1 \quad \text{and} \quad \sum_{P \Gamma_1 \in \mathcal{H}} |f(P)|^2 < \infty \right\}.$$

Let $\Theta$ be a cross-section for $\Gamma/\Gamma_1$ in $\Gamma$. The Hilbert space $\mathcal{H}_t$ is naturally identified with $l^2 (\Theta)$ since for any $Q \in \Gamma_1$, we have $|f(PQ)| = |f(P)|$. Via this identification, we may realize the representation $\text{Ind}_{\Gamma_1}^{\Gamma_1} \chi_{(1,-t)}$ as acting in $l^2 (\Theta)$. More precisely, for $a \in l^2 (\Theta)$,

$$\left( \text{Ind}_{\Gamma_1}^{\Gamma_1} \chi_{(1,-t)} (X) a \right) (T_j) = \begin{cases} a(T_{j-k}) & \text{if } X = T_k \\ e^{-2\pi i (\ell, l)} e^{-2\pi i (t, l)} a(T_j) & \text{if } X = M_t \\ e^{2\pi i \theta} a(T_j) & \text{if } X = e^{2\pi i \theta} \end{cases}.$$

Defining the unitary operator $\mathfrak{J} : l^2 (\Theta) \to l^2 (\mathbb{Z}^d)$ such that $\mathfrak{J}(a)(j) = a(T_j)$, it is easily checked that:

$$\mathfrak{J}^{-1} [(\mathcal{R} f)(t)] = \begin{cases} \text{Ind}_{\Gamma_1}^{\Gamma_1} \chi_{(1,-t)} (T_k) [\mathfrak{J}^{-1} (\mathcal{R} f)(t)] & \text{if } X = T_k \\ \text{Ind}_{\Gamma_1}^{\Gamma_1} \chi_{(1,-t)} (M_t) [\mathfrak{J}^{-1} (\mathcal{R} f)(t)] & \text{if } X = M_t \\ \text{Ind}_{\Gamma_1}^{\Gamma_1} \chi_{(1,-t)} (e^{2\pi i \theta}) [\mathfrak{J}^{-1} (\mathcal{R} f)(t)] & \text{if } X = e^{2\pi i \theta} \end{cases}.$$
Thus, the representation $\pi$ is unitarily equivalent to

$$\int_{\mathbb{R}^d} \oplus_{\Delta} \text{Ind}_{\Gamma_1}^{\Gamma} \chi_{(1,-t)} \ dt.$$  

Now, we remark that $\text{Ind}_{\Gamma_1}^{\Gamma} \chi_{(1,-t)}$ is not an irreducible representation of the group $\Gamma$. Indeed, by inducing in stages (see Page 166, [9]), we obtain that the representation $\text{Ind}_{\Gamma_0}^{\Gamma} \left( \text{Ind}_{\Gamma_1}^{\Gamma} \left( \chi_{(1,-t)} \right) \right)$ and $\text{Ind}_{\Gamma_1}^{\Gamma} \chi_{(1,-t)}$ acts in the Hilbert space

$$K_t = \left\{ f: \Gamma_0 \to \mathbb{C} : f(PQ) = \chi_{(1,-t)}(Q^{-1})f(P), Q \in \Gamma_1 \text{ and } \sum_{\Gamma_1 \in \frac{\Gamma_0}{\Gamma_1}} |f(P)|^2 < \infty \right\}.$$  

Since $\chi_{(1,-t)}$ is a character, for any $f \in K_t$, we notice that $|f(PQ)| = |f(P)|$ for $Q \in \Gamma_1$. Thus, the Hilbert space $K_t$ is naturally identified with $l^2 \left( \frac{\Gamma_0}{\Gamma_1} \right) \simeq l^2 \left( AZ^d \right)$ where $AZ^d$ is a parametrizing set for the quotient group $\frac{\Gamma_0}{\Gamma_1}$. Via this identification, we may realize $\text{Ind}_{\Gamma_1}^{\Gamma} \chi_{(1,-t)}$ as acting in $l^2 \left( AZ^d \right)$. More precisely, for $T_j, j \in AZ^d$, we compute the action of $\text{Ind}_{\Gamma_1}^{\Gamma} \chi_{(1,-t)}$ as follows:

$$\left[ \text{Ind}_{\Gamma_1}^{\Gamma} \chi_{(1,-t)}(X)f \right](T_j) = \begin{cases} f(T_{j-k}) & \text{if } X = T_k \\ e^{-2\pi i(t,j)} f(T_j) & \text{if } X = M_l \\ e^{2\pi i t} f(T_j) & \text{if } X = e^{2\pi i t} \end{cases}.$$  

Now, let $F_{AZ^d}$ be the Fourier transform defined on $l^2 \left( AZ^d \right)$. Given a vector $f$ in $l^2 \left( AZ^d \right)$, it is not too hard to see that

$$\left[ F_{AZ^d} \left( \text{Ind}_{\Gamma_1}^{\Gamma} \chi_{(1,-t)}(X)f \right) \right](\xi) = \begin{cases} \chi_{\xi}(T_k) F_{AZ^d}f(\xi) & \text{if } X = T_k \\ e^{-2\pi i(t,j)} F_{AZ^d}f(\xi) & \text{if } X = M_l \\ e^{2\pi i t} F_{AZ^d}f(\xi) & \text{if } X = e^{2\pi i t} \end{cases}.$$  

As a result, given $X \in \Gamma$ we obtain:

$$F_{AZ^d} \circ \text{Ind}_{\Gamma_1}^{\Gamma} \chi_{(1,-t)}(X) \circ F_{AZ^d}^{-1} = \int_{\mathbb{R}^d} \chi_{(1,-t,\xi)}(X) \ d\xi,$$

where $\chi_{(1,-t,\xi)}$ is a character of $\Gamma_0$ defined as follows:

$$\chi_{(1,-t,\xi)}(X) = \begin{cases} \chi_{\xi}(T_k) & \text{if } X = T_k \\ e^{-2\pi i(t,j)} & \text{if } X = M_l \\ e^{2\pi i t} & \text{if } X = e^{2\pi i t} \end{cases}.$$  

Appealing to basic induction properties (see Page 41, [5]) we have

$$\text{Ind}_{\Gamma_1}^{\Gamma} \chi_{(1,-t)} \simeq \text{Ind}_{\Gamma_0}^{\Gamma} \left( \int_{\mathbb{R}^d} \chi_{(1,-t,\xi)} \ d\xi \right) \simeq \int_{\mathbb{R}^d} \left( \text{Ind}_{\Gamma_0}^{\Gamma} \chi_{(1,-t,\xi)} \right) \ d\xi.$$  

Putting (14) and (17) together, we arrive at: $\pi \simeq \int_{\mathbb{R}^d} \text{Ind}_{\Gamma_0}^{\Gamma} \chi_{(1,\sigma)} d\sigma$. Finally, the fact that $\text{Ind}_{\Gamma_0}^{\Gamma} \chi_{(1,\sigma)}$ is an irreducible representation is due to Remark [5]. This completes the proof.
Lemma 6. Let $\Gamma_1 = A_1 \mathbb{Z}^d$ and $\Gamma_2 = A_2 \mathbb{Z}^d$ be two lattices of $\mathbb{R}^d$ such that $A_1$ and $A_2$ are non-singular matrices and $|\det A_1| \leq |\det A_2|$. Then there exist measurable sets $\Sigma_1, \Sigma_2$ such that $\Sigma_1$ is a fundamental domain for $\frac{\mathbb{R}^d}{A_1 \mathbb{Z}^d}$ and $\Sigma_2$ is a fundamental domain for $\frac{\mathbb{R}^d}{A_2 \mathbb{Z}^d}$ and $\Sigma_1 \subseteq \Sigma_2 \subset \mathbb{R}^d$.

Proof. According to Theorem 1, [8], there exists a measurable set $\Sigma_1$ such that $\Sigma_1$ tiles $\mathbb{R}^d$ by the lattice $A_1 \mathbb{Z}^d$ and packs $\mathbb{R}^d$ by $A_2 \mathbb{Z}^d$. By packing, we mean that given any distinct $\gamma, \kappa \in A_2 \mathbb{Z}^d$, the set $(\Sigma_1 + \gamma) \cap (\Sigma_1 + \kappa)$ is an empty set and

$$\sum_{\lambda \in A_2 \mathbb{Z}^d} 1_{\Sigma_1} (x + \lambda) \leq 1 \text{ for } x \in \mathbb{R}^d.$$ 

Here $1_{\Sigma_1}$ denotes the characteristic function of the set $\Sigma_1$. We would like to construct a set $\Sigma_2$ which tiles $\mathbb{R}^d$ by $A_2 \mathbb{Z}^d$ such that $\Sigma_1 \subseteq \Sigma_2$. To construct such a set, let $\Omega$ be a fundamental domain for $\frac{\mathbb{R}^d}{A_2 \mathbb{Z}^d}$. It follows that, there exists a subset $I$ of $A_2 \mathbb{Z}^d$ such that $\Sigma_1 \subseteq \bigcup_{k \in I} (\Omega + k)$ and each $(\Omega + k) \cap \Sigma_1$ is a set of positive Lebesgue measure. Next, for each $k \in I$, we define $\Omega_k = (\Omega + k) \cap \Sigma_1$. We observe that

$$\Sigma_1 \subseteq \bigcup_{k \in I} (\Omega + k)$$

and each $(\Omega + k) \cap \Sigma_1$ is a set of positive Lebesgue measure. Next, for each $k \in I$, we define $\Omega_k = (\Omega + k) \cap \Sigma_1$. We observe that

$$\Sigma_1 = \bigcup_{k \in I} ((\Omega + k) \cap \Sigma_1) = \bigcup_{k \in I} \Omega_k.$$ 

Put

$$\Sigma_2 = \left( \Omega - \bigcup_{k \in I} (\Omega_k - k) \right) \cup \Sigma_1.$$ 

The disjoint union in the equality above is due to the fact that for distinct $k, j \in I$, the set $(\Omega_k - k) \cap (\Omega_j - j)$ is a null set. This holds because, $\Sigma_1$ packs $\mathbb{R}^d$ by $A_2 \mathbb{Z}^d$. Finally, we observe that

$$\Sigma_2 = \left( \Omega - \bigcup_{k \in I} (\Omega_k - k) \right) \cup \left( \bigcup_{k \in I} \Omega_k \right) ,$$

and

$$\Omega = \left( \Omega - \bigcup_{k \in I} (\Omega_k - k) \right) \cup \left( \bigcup_{k \in I} \Omega_k' \right) ,$$

where each $\Omega_k'$ is $A_2 \mathbb{Z}^d$-congruent with $\Omega_k$. Therefore $\Sigma_2$ is a fundamental domain for $\frac{\mathbb{R}^d}{A_2 \mathbb{Z}^d}$ which contains $\Sigma_1$. This completes the proof. \hfill \Box

Now, we are ready to prove Proposition 3. Part of the proof of the Proposition 3 relies on some technical facts related to central decompositions of unitary representations. A good presentation of this theory is found in Section 3.4.2, [7].
4.2. Proof of Proposition [3] From Proposition [2] we know that the representation \( \pi \) is unitarily equivalent to

\[
(18) \quad \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{Ind}_{\Gamma_0}^\Gamma \chi_{(1,\sigma)} \, d\sigma.
\]

We recall that \( \Gamma_0 \) is isomorphic to the discrete group \( \mathbb{Z}_m \times B\mathbb{Z}^d \times AZ^d \) and that \( \Gamma_1 \) is isomorphic to \( \mathbb{Z}_m \times B\mathbb{Z}^d \) where \( m \) is the number of elements in the commutator group of \( \Gamma \) which is a discrete subgroup of the torus. Next, since

\[
L \simeq \int_{\mathbb{R}^d} \text{Ind}_{\Gamma_0}^\Gamma (\chi_t) \, dt \simeq \bigoplus_{k=0}^{m-1} \int_{\mathbb{R}^d} \text{Ind}_{\Gamma_0}^\Gamma (\chi(k,\sigma)) \, d\sigma,
\]

letting

\[
(19) \quad L_k = \int_{\mathbb{R}^d} \text{Ind}_{\Gamma_0}^\Gamma (\chi(k,\sigma)) \, d\sigma,
\]

it is easily seen that \( L = L_0 \oplus \cdots \oplus L_{m-1} \). For distinct \( i \) and \( j \), the representations \( L_i \) and \( L_j \) described above are disjoint representations. This is due to the fact that if \( i \neq j \) then the \( \Gamma \)-orbits of \( \chi_{(i,\sigma)} \) and \( \chi_{(j,\sigma)} \) are disjoint sets, and therefore the induced representations \( \text{Ind}_{\Gamma_0}^\Gamma \chi_{(i,\sigma)} \) and \( \text{Ind}_{\Gamma_0}^\Gamma \chi_{(j,\sigma)} \) are disjoint representations. If we assume that \( |\det B| > 1 \) then \( |\det (B^*)| < 1 \). According to Lemma [3] there exist measurable cross-sections \( \Sigma_1, \Sigma_2 \) for \( \mathbb{R}^d \) and \( \mathbb{R}^d \) respectively such that \( \Sigma_1, \Sigma_2 \subset \mathbb{R}^2 \), \( \Sigma_1 \supset \Sigma_2 \) and \( \Sigma_1 - \Sigma_2 \) is a set of positive Lebesgue measure. Therefore,

\[
(20) \quad \pi \simeq \int_{\Sigma_1} \left( \text{Ind}_{\Gamma_0}^\Gamma (\chi_{(1,\sigma)}) \right) \, d\sigma \text{ and } L_1 \simeq \int_{\Sigma_2} \left( \text{Ind}_{\Gamma_0}^\Gamma (\chi_{(1,\sigma)}) \right) \, d\sigma
\]

and the representations above are realized as acting in the direct integrals of finite dimensional vector spaces:

\[
\int_{\Sigma_1} l^2 (\Gamma/\Gamma_0) \, d\sigma \text{ and } \int_{\Sigma_2} l^2 (\Gamma/\Gamma_0) \, d\sigma \text{ respectively.}
\]

We remark that the direct integrals described in (20) are irreducible decompositions of \( \pi \) and \( L_1 \). Now, referring to the central decomposition of the left regular representation which is described in (10) there exists a measurable subset \( E \) of \( \Sigma_2 \) such that the central decomposition of \( L_1 \) is given by (see Theorem 3.26, [7])

\[
\int_{E} \bigoplus_{k=1}^{\dim(l^2(\Gamma/\Gamma_0))} \text{Ind}_{\Gamma_0}^\Gamma (\chi_{(1,\sigma)}) \, d\sigma,
\]

and \( \dim(l^2(\Gamma/\Gamma_0)) \) is equal to the number of cosets in \( \Gamma/\Gamma_0 \). Furthermore, recalling that

\[
L_1 \simeq \int_{\Sigma_2} \left( \text{Ind}_{\Gamma_0}^\Gamma (\chi_{(1,\sigma)}) \right) \, d\sigma
\]

and letting \( \mu \) be the Lebesgue measure on \( \mathbb{R}^d \), it is necessarily the case that

\[
\mu (E) = \frac{\mu (\Sigma_2)}{\dim(l^2(\Gamma/\Gamma_0))} = \frac{1}{|\det (B)| \det (A) \dim(l^2(\Gamma/\Gamma_0))}.
\]
From the discussion provided at the beginning of the third section, the set \( E \) is obtained by taking a cross-section for the \( \Gamma \)-orbits in

\[
\frac{\mathbb{R}^d}{B \star \mathbb{Z}^d} \times \frac{\mathbb{R}^d}{A \star \mathbb{Z}^d}.
\]

Moreover, since \( \Sigma_1 \supset \Sigma_2 \) and since \( \Sigma_1 - \Sigma_2 \) is a set of positive Lebesgue measure then

\[
\pi \simeq \int_{\Sigma_1} \text{Ind}^\Gamma_{\Gamma_0} (\chi(1,\sigma)) \ d\sigma \simeq L_1 \oplus \int_{\Sigma_1 - \Sigma_2} \text{Ind}^\Gamma_{\Gamma_0} (\chi(1,\sigma)) \ d\sigma
\]

and \( \int_{\Sigma_1 - \Sigma_2} \text{Ind}^\Gamma_{\Gamma_0} (\chi(1,\sigma)) \ d\sigma \) is a subrepresentation of \( \pi \). Thus a central decomposition of \( \pi \) is given by

\[
\int_{E} \oplus_{k=1} u(\sigma) \text{dim}(l^2(\Gamma/\Gamma_0)) \text{Ind}^\Gamma_{\Gamma_0} (\chi(1,\sigma)) \ d\sigma
\]

and the function \( u : E \to \mathbb{N} \) is greater than one, on a subset of positive measure of \( E \). Therefore, according to Theorem 3.26, it is not possible for \( \pi \) to be equivalent to a subrepresentation of the left regular representation of \( \Gamma \) if \( |\det B| > 1 \). Now, let us suppose that \( |\det B| \leq 1 \). Then \( |\det B^*| \geq 1 \). Appealing to Lemma 6, there exist measurable sets \( \Sigma_1 \) and \( \Sigma_2 \) which are measurable fundamental domains for

\[
\frac{\mathbb{R}^d}{B^* \star \mathbb{Z}^d}, \text{ and } \frac{\mathbb{R}^d}{A^* \star \mathbb{Z}^d}
\]

respectively, such that \( \Sigma_1, \Sigma_2 \subset \mathbb{R}^d \) and \( \Sigma_1 \subseteq \Sigma_2 \). Next,

\[
L_1 \simeq \int_{\Sigma_2} \text{Ind}^\Gamma_{\Gamma_0} (\chi(1,\sigma)) \ d\sigma
\]

\[
\simeq \left( \int_{\Sigma_1} \text{Ind}^\Gamma_{\Gamma_0} (\chi(1,\sigma)) \ d\sigma \right) \oplus \left( \int_{\Sigma_2 - \Sigma_1} \text{Ind}^\Gamma_{\Gamma_0} (\chi(1,\sigma)) \ d\sigma \right)
\]

\[
\simeq \pi \oplus \left( \int_{\Sigma_2 - \Sigma_1} \text{Ind}^\Gamma_{\Gamma_0} (\chi(1,\sigma)) \ d\sigma \right).
\]

Finally, \( \pi \) is equivalent to a subrepresentation of \( L_1 \) and is equivalent to a subrepresentation of the left regular representation \( L \).

4.3. Examples. In this subsection, we shall present a few examples to illustrate the results obtained in Propositions 1, 2 and 3.

(1) Let us start with a trivial example. Let \( d = 1 \) and \( B = \frac{2}{3} \). Then

\[
B^* = \frac{3}{2}, \ A = 3, \text{ and } A^* = \frac{1}{3}.
\]

Next,

\[
L \simeq \bigoplus_{k=0}^2 \int_{[0,\frac{3}{2}] \times [0,\frac{1}{4}]} \text{Ind}^\Gamma_{\Gamma_0} \chi(k,\sigma) \ d\sigma
\]

and

\[
\pi \simeq \int_{[0,1] \times [0,\frac{1}{4}]} \text{Ind}^\Gamma_{\Gamma_0} \chi(1,\sigma) \ d\sigma.
\]
Now, the central decomposition of $L_1$ is given by
\[ \int_{[0, \frac{1}{2}) \times [0, \frac{1}{2})} \bigoplus_{j=1}^{3} \text{Ind}_{\Gamma_0}^\Gamma \chi(1, \sigma) \ d\sigma \]
and the central decomposition of the rational Gabor representation $\pi$ is
\[ \int_{[0, \frac{1}{2}) \times [0, \frac{1}{2})} \bigoplus_{j=1}^{2} \text{Ind}_{\Gamma_0}^\Gamma \chi(1, \sigma) \ d\sigma. \]
From these decompositions, it is obvious that the rational Gabor representation $\pi$ is equivalent to a subrepresentation of the left regular representation $L$.

(2) Let
\[ B = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}. \]
Then it is easy to see that
\[ B^* = \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{2}{3} \end{bmatrix}, \quad A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad A^* = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}. \]
Next, the left regular representation of $\Gamma$ can be decomposed into a direct integral of representations as follows:
\[ L \simeq \bigoplus_{k=0}^{5} \int_{S} \text{Ind}_{\Gamma_0}^\Gamma \chi(k, \sigma) \ d\sigma \]
where $S = S_1 \times A^* [0, 1)^2$ and
\[ S_1 = \left( [0, 1) \times \left[ 0, \frac{2}{3} \right) \right) \cup \left( \left[ 1, \frac{3}{2} \right) \times \left[ \frac{2}{3}, 1 \right) \right) \cup \left( \left[ -\frac{1}{2}, 0 \right) \times \left[ -\frac{1}{3}, 0 \right) \right) \]
is a common connected fundamental domain for the lattices $B^* \mathbb{Z}^2$ and $\mathbb{Z}^2$.

Illustration of set $S_1$
Moreover, we decompose the rational Gabor representation as follows: \( \pi \simeq \int_{\mathbb{S}} \text{Ind}_{\Gamma}^{\overline{\Gamma}} \chi_{(1, \sigma)} \, d\sigma \). One interesting fact to notice here is that: the rational Gabor representation \( \pi \) is actually equivalent to \( L_1 \) and \( L = L_0 \oplus L_1 \oplus L_2 \oplus L_3 \oplus L_4 \oplus L_5 \).

(3) Let \( \Gamma = \langle T_k, M_{Bl} : k, l \in \mathbb{Z} \rangle \) where

\[
B = \begin{bmatrix}
1 & 0 & 0 \\
-\frac{1}{5} & \frac{1}{5} & 0 \\
1 & -1 & 5
\end{bmatrix}.
\]

Then the inverse transpose of the matrix \( B \) is given by

\[
B^* = \begin{bmatrix}
1 & 1 & 0 \\
0 & 5 & 1 \\
0 & 0 & \frac{1}{5}
\end{bmatrix}.
\]

Next, we may choose

\[
A = \begin{bmatrix}
1 & 1 & 0 \\
0 & 5 & 5 \\
0 & 0 & 1
\end{bmatrix}
\]

so that \( A^* = \begin{bmatrix}
1 & 0 & 0 \\
-\frac{1}{5} & \frac{1}{5} & 0 \\
1 & -1 & 1
\end{bmatrix} \).

Finally, we observe that

\[
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 5 \\
1 & \frac{1}{5} & 0
\end{bmatrix} \begin{bmatrix}
0, 1 \end{bmatrix}^3
\]

is a common fundamental domain for both \( B^* \mathbb{Z}^3 \) and \( \mathbb{Z}^3 \). Put

\[
S = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 5 \\
1 & \frac{1}{5} & 0
\end{bmatrix} \begin{bmatrix}
0, 1 \end{bmatrix}^3 \times \begin{bmatrix}
1 & 0 & 0 \\
-\frac{1}{5} & \frac{1}{5} & 0 \\
1 & -1 & 1
\end{bmatrix} \begin{bmatrix}
0, 1 \end{bmatrix}^3.
\]

Then \( \mathbb{L} \simeq \bigoplus_{k=0}^{4} \int_{\mathbb{S}} \text{Ind}_{\Gamma_0}^{\overline{\Gamma}} \chi_{(k,t)} dt \) and \( \pi \simeq \int_{\mathbb{S}} \text{Ind}_{\Gamma_0}^{\overline{\Gamma}} \chi_{(1,t)} dt \).

5. Application to time-frequency analysis

Let \( \pi \) be a unitary representation of a locally compact group \( X \), acting in some Hilbert space \( \mathcal{H} \). We say that \( \pi \) is admissible, if and only if there exists some vector \( \phi \in \mathcal{H} \) such that the operator \( W_\phi^\pi : \mathcal{H} \rightarrow L^2 (X) \), \( W_\phi^\pi \psi (x) = \langle \psi, \pi (x) \phi \rangle \)

is an isometry of \( \mathcal{H} \) into \( L^2 (X) \).

**Lemma 7.** A representation of \( \Gamma \) is admissible if and only if the representation is equivalent to a subrepresentation of the left regular representation of \( \Gamma \).

**Proof.** Let \( \theta \) be a representation of \( \Gamma \) acting in a Hilbert space \( \mathcal{H} \). Let us suppose that \( \theta \) is admissible. Then there exists some function \( \phi \in \mathcal{H} \) such that the operator \( W_\phi^\theta : \mathcal{H} \rightarrow l^2 (\Gamma) \) defined by \( W_\phi^\theta \psi (x) = \langle \psi, \theta (x) \phi \rangle \) is an isometry of \( \mathcal{H} \) into \( l^2 (\Gamma) \). Moreover, \( W_\phi^\theta (\mathcal{H}) \) is a closed left-invariant subspace of \( l^2 (\Gamma) \). Next, it is easy to see that \( W_\phi^\theta \) intertwines the representation \( \theta \) with the left regular representation. As a result, \( \theta \) is equivalent to the left regular representation restricted to \( W_\phi^\theta (\mathcal{H}) \). For the converse, let us assume that \( \theta \) is equivalent to a subrepresentation of the left regular representation. It is easy to see that since \( \Gamma \) is discrete, then its left regular representation has an admissible vector. For example,
if we define $a \in l^2(\Gamma)$ such that $a(\gamma)$ is equal to one if $\gamma$ is the identity element and zero otherwise then $W^L_\gamma : l^2(\Gamma) \to l^2(\Gamma)$ is an isometry. Next, let us define a unitary operator $S : \mathcal{H} \to \mathcal{K} \subseteq l^2(\Gamma)$ such that $S$ intertwines the representation $\theta$ with the left regular representation restricted to $\mathcal{K}$. Let $P : l^2(\Gamma) \to \mathcal{K}$ be an orthogonal projection onto $\mathcal{K}$ which is a left-invariant Hilbert space. Then clearly, $P(a)$ is an admissible vector for the regular representation restricted to $\mathcal{K}$, and it is easy to see that $S^{-1}(P(a))$ is an admissible vector for the representation $\theta$. 

Given a countable sequence $\{f_i\}_{i \in I}$ of vectors in an Hilbert space $\mathbf{H}$, we say $\{f_i\}_{i \in I}$ forms a frame if and only if there exist strictly positive real numbers $A, B$ such that for any vector $f \in \mathbf{H}$,

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2.$$ 

In the case where $A = B$, the sequence of vectors $\{f_i\}_{i \in I}$ forms a tight frame, and if $A = B = 1$, $\{f_i\}_{i \in I}$ is called a Parseval frame. We remark that an admissible vector for the left regular representation of $\Gamma$ is a Parseval frame by definition.

The following proposition is well-known for the more general case where $B$ is any invertible matrix (not necessarily a rational matrix.) Although this result is not new, the proof of Proposition 8 is new, and worth presenting in our opinion. We also remark that Proposition 8 is a generalization of Lemma 5.30 which is given in the Monograph [7].

**Proposition 8.** Let $B$ be a rational matrix. There exists a vector $g \in L^2(\mathbb{R}^d)$ such that the system $\{M_i T_k g : l \in \mathbb{BZ}^d, k \in \mathbb{Z}^d\}$ is a Parseval frame in $L^2(\mathbb{R}^d)$ if and only if $|\det B| \leq 1$.

**Proof.** The case where $B$ is an element of $GL(d, \mathbb{Z})$ is easily derived from [11], Section 4. We shall thus skip this case. So let us assume that $B$ is a rational matrix with at least one entry not in $\mathbb{Z}$. We have shown that the representation $\pi$ is equivalent to a subrepresentation of the left regular representation of $L$ if and only if $|\det B| \leq 1$. Since $\Gamma$ is a discrete group, then its left regular representation is admissible if and only if $|\det B| \leq 1$. Thus, the representation $\pi$ of $\Gamma$ is admissible if and only if $|\det B| \leq 1$. Suppose that $|\det B| \leq 1$. Then $\pi$ is admissible and there exists a vector $f \in L^2(\mathbb{R}^d)$ such that the map $W_f^\pi$ defined by

$$W_f^\pi h(e^{2\pi i \theta} M_i T_k f) = \langle h, e^{2\pi i \theta} M_i T_k f \rangle$$

is an isometry. As a result, for any vector $h \in L^2(\mathbb{R}^d)$, we have

$$\left( \sum_{\theta \in [\Gamma, \Gamma]} \sum_{l \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} |\langle h, e^{2\pi i \theta} M_i T_k f \rangle|^2 \right)^{1/2} = \|h\|_{L^2(\mathbb{R}^d)}.$$ 

Next, for $m = \text{card } ([\Gamma, \Gamma])$,

$$\sum_{\theta \in [\Gamma, \Gamma]} \sum_{l \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} |\langle h, e^{2\pi i \theta} M_i T_k f \rangle|^2 = \sum_{l \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} \sum_{\theta \in [\Gamma, \Gamma]} \left| \langle h, M_i T_k (m^{1/2} f) \rangle \right|^2.$$ 

Therefore, if $g = m^{1/2} f$ then

$$\left( \sum_{l \in \mathbb{Z}^d} \sum_{k \in \mathbb{Z}^d} |\langle h, M_i T_k g \rangle|^2 \right)^{1/2} = \|h\|_{L^2(\mathbb{R}^d)}.$$
For the converse, if we assume that there exists a vector \( g \in L^2(\mathbb{R}^d) \) such that the system \( \{ M_l T_k g : l \in B \mathbb{Z}^d, k \in \mathbb{Z}^d \} \) is a Parseval frame in \( L^2(\mathbb{R}^d) \) then it is easy to see that \( \pi \) must be admissible. As a result, it must be the case that \(|\det B| \leq 1\). \( \square \)

5.1. **Proof of Proposition 4.** Let us suppose that \(|\det B| \leq 1\). From the proof of Proposition 3 we recall that there exists a unitary map

\[ \mathfrak{A} : \int_{\sigma}^{\mathbb{E}} \left( \bigoplus_{k=1}^{\ell} \left( \frac{\Gamma}{\Gamma_0} \right)^2 \right) d\sigma \rightarrow L^2(\mathbb{R}^d) \]

which intertwines the representations \( \int_{\mathbb{E}}^{\mathbb{E}} \left( \bigoplus_{k=1}^{\ell} \text{Ind}_{\Gamma_0}^\Gamma (\chi(1,\sigma)) \right) d\sigma \) with \( \pi \) such that

\[ \int_{\mathbb{E}}^{\mathbb{E}} \left( \bigoplus_{k=1}^{\ell} \text{Ind}_{\Gamma_0}^\Gamma (\chi(1,\sigma)) \right) d\sigma \]

is the central decomposition of \( \pi \), and \( \mathbb{E} \subset \mathbb{R}^d \) is a measurable subset of a fundamental domain for the lattice \( B^* \mathbb{Z}^d \times A^* \mathbb{Z}^d \) and the multiplicity function \( \ell \) satisfies the condition: \( \ell(\sigma) \leq \dim \mathbb{E} \left( \frac{\Gamma}{\Gamma_0} \right) \). Next, according to the discussion on Page 126, the vector \( a \) is an admissible vector for the representation \( \tau = \int_{\mathbb{E}}^{\mathbb{E}} \left( \bigoplus_{k=1}^{\ell} \text{Ind}_{\Gamma_0}^\Gamma (\chi(1,\sigma)) \right) d\sigma \) if and only if \( a \in \int_{\mathbb{E}}^{\mathbb{E}} \left( \bigoplus_{k=1}^{\ell} \mathbb{E} \left( \frac{\Gamma}{\Gamma_0} \right)^2 \right) d\sigma \) such that for \( d\sigma \)-almost every \( \sigma \in \mathbb{E} \),

\[ \|a(\sigma)(k)\|^2_{L^2(\frac{\Gamma}{\Gamma_0})} = 1 \text{ for } 1 \leq k \leq \ell(\sigma) \]

and for distinct \( k, j \in \{1, \ldots, \ell(\sigma)\} \) we have \( \langle a(\sigma)(k), a(\sigma)(j) \rangle = 0 \). Finally, the desired result is obtained by using the fact that \( \mathfrak{A} \) intertwines the representations \( \tau \) with \( \pi \).

5.2. **Concluding remarks.** The case where \( B \) is invertible and has at least one irrational entry is not treated in this paper because it is rather special. In fact, if \( B \) is invertible and not rational, then the group generated by the operators \( T_k \) and \( M_l T_l \), \( k, l \in \mathbb{Z}^d \) is no longer a locally compact group if given the subspace topology of \( \mathbb{T} \times \mathbb{R}^d \times \mathbb{R}^d \). However, this does not represent a serious obstruction. One way around this issue, would be to give \( \Gamma \) the discrete topology. As a discrete group, the regular representation of \( \Gamma \) is admissible. However, the center of \( \Gamma \) not being compact, then the Gabor representation is not admissible. Thus, in this situation, it is not possible for the Gabor representation to be equivalent to a subrepresentation of the left regular representation of the group. Moreover, the fact that \( \Gamma \) is a non-type \( I \) group with non-type \( I \) regular representation complicates things a great deal from a representation theoretic viewpoint.

**References**

[1] L. Baggett, Processing a radar signal and representations of the discrete Heisenberg group, Colloq. Math. 60/61 (1990), no. 1, 195–203.

[2] L. Baggett, K. Merrill, Abstract harmonic analysis and wavelets in \( \mathbb{R}^n \), The functional and harmonic analysis of wavelets and frames, 17–27, Contemp. Math., 247, Amer. Math. Soc., Providence, RI, 1999.

[3] L. Baggett, An abstract interpretation of the wavelet dimension function using group representations, J. Funct. Anal. 173 (2000), no. 1, 1–20.

[4] G. Cariolaro, Unified Signal Theory, 1st Edition, 2011, Springer.

[5] L. Corwin, F. Greenleaf, Representations of nilpotent Lie groups and their applications. Part I. Basic theory and examples, Cambridge Studies in Advanced Mathematics, 18. Cambridge University Press, Cambridge, 1990.
[6] G. Folland, A course in abstract harmonic analysis, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[7] H. Führ, Abstract harmonic analysis of continuous wavelet transforms. Lecture Notes in Mathematics, 1863. Springer-Verlag, Berlin, 2005.

[8] D. Han and Y. Wang, Lattice tiling and the Weyl-Heisenberg frames, GAFA 11 (2001), no. 4, 742–758.

[9] A. Kleppner, R. Lipsman, The Plancherel formula for group extensions. I, II. Ann. Sci. École Norm. Sup. (4) 5 (1972), 459–516.

[10] R. Lipsman, Group representations. A survey of some current topics. Lecture Notes in Mathematics, Vol. 388. Springer-Verlag, Berlin-New York, 1974.

[11] A. Mayeli, V. Oussa, Regular Representations of Time Frequency Groups, Math. Nachr. 287, No. 11-12, 1320-1340 (2014).

[12] G. Pfander, P. Rashkov, Y. Wang, A geometric construction of tight multivariate Gabor frames with compactly supported smooth windows. J. Fourier Anal. Appl. 18 (2012), no. 2, 223–239.

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