Dual Polyhedra, Mirror Symmetry
and
Landau-Ginzburg Orbifolds

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ABSTRACT

New geometrical features of the Landau-Ginzburg orbifolds are presented, for models with a typical type of superpotential. We show the one-to-one correspondence between some of the \((a, c)\) states with \(U(1)\) charges \((-1, 1)\) and the integral points on the dual polyhedra, which are useful tools for the construction of mirror manifolds. Relying on toric geometry, these states are shown to correspond to the \((1, 1)\) forms coming from blowing-up processes. In terms of the above identification, it can be checked that the monomial-divisor mirror map for Landau-Ginzburg orbifolds, proposed by the author, is equivalent to that mirror map for Calabi-Yau manifolds obtained by the mathematicians.
Mirror symmetry was first discovered in the context of string compactification [1, 2, 3]. Due to the relative sign of the two $U(1)$ charges, one $(2, 2)$ superconformal field theory allows two geometrical interpretations, i.e. topologically distinct Calabi-Yau manifolds $\mathcal{M}$ and $\mathcal{W}$. Assuming that mirror symmetry is true, some Yukawa couplings can be determined exactly [4, 5, 6]. However, recent analysis of mirror symmetry is purely geometrical.

Batyrev [7] proposed a powerful method for constructing the mirror manifolds of a certain class of Calabi-Yau manifolds. He showed that a pair of $(\Delta, \Delta^*)$ gives a Calabi-Yau manifold, where $\Delta$ is a (Newton) polyhedron corresponding to monomials and $\Delta^*$ is a dual (or polar) polyhedron describing the resolution of singularities, i.e. a point on a one- or two-dimensional face of $\Delta^*$ corresponds to a $(1, 1)$ form coming from resolution. Batyrev observed that the exchange of the roles of $(\Delta, \Delta^*)$ produces a mirror manifold.

Landau-Ginzburg models of $N = 2$ superconformal field theories are closely related to Calabi-Yau manifolds because of their (anti-)chiral ring structures. If we consider the theory with $c = 9$, the $(p, q)$ forms on a Calabi-Yau manifold can be identified with $(3 - p, q)$ states of the $(c, c)$ ring or $(-p, q)$ states of the $(a, c)$ ring, where $c$ ($a$) stands for (anti-)chiral and the states are labeled by the $U(1)$ charges. These $(c, c)$ and $(a, c)$ rings can be described in terms of the Landau-Ginzburg models.

In this paper, we will find a very simple relation between a $(-1, 1)$ state and a point on a face of $\Delta^*$, when a typical type of Landau-Ginzburg models are considered. Hence we can identify a $(-1, 1)$ state and a $(1, 1)$ form coming from blowing-up processes in a simple and exact way. Furthermore, we will show that the monomial-divisor mirror map for Landau-Ginzburg orbifolds proposed in ref. [8] is equivalent to that mirror map of Calabi-Yau manifolds [7, 5]. These are useful extensions of the results in the previous paper [8]. Our method gives us the possibility to study the new geometric content of a class of $N = 2$ superconformal field theories.

In this paper, we will restrict our attention to the superpotential of a form $W(X_i) = X_1a_1 + X_2a_2 + X_3a_3 + X_4a_4 + X_5a_5$, which corresponds to the Fermat type hypersurface in $WCP^4$. The Landau-Ginzburg orbifolds are obtained by quotienting with an Abelian symmetry group $G$ of $W(X_i)$, whose element $g$ acts as an $N \times N$ diagonal matrix, $g : X_i \rightarrow e^{2\pi i \theta_i g} X_i$, where $0 \leq \theta_i g < 1$. Of course the $U(1)$ twist $j : X_i \rightarrow e^{2\pi i q_i} X_i$ generates the symmetry group of $W(X_i)$, where $q_i = w_i$. $W(\lambda w_i X_i) = \lambda^4 W(X_i)$ and $\lambda \in \mathbb{C}^*$. In this paper, we further require that $w_5 = 1$ since the toric description of the corresponding Calabi-Yau mirror manifolds are well-known [9, 10].

Using the results of Intriligator and Vafa [19], we can construct the $(c, c)$ and $(a, c)$
rings. Also we could have the left and right $U(1)$ charges of the ground state $|h\rangle_{(a,c)}$ in the $h$-twisted sector of the $(a,c)$ ring. In terms of spectral flow, $|h\rangle_{(a,c)}$ is mapped to the $(c,c)$ state $|h\rangle_{(c,c)}$ with $h = hj^{-1}$. Then the charges of the $(a,c)$ ground state of $h$-twisted sector $|h\rangle_{(a,c)}$ are obtained to be

$$\begin{pmatrix} J_0 \\ \bar{J}_0 \end{pmatrix} |h\rangle_{(a,c)} = \left( -\sum_{\tilde{\theta}_i^{h'} > 0} (1 - q_i - \tilde{\theta}_i^{h'}) + \sum_{\tilde{\theta}_i^{h'} = 0} (2q_i - 1) \right) |h\rangle_{(a,c)}, \quad (1)$$

Using this result, we see that the $(-1,1)$ states written in the form $|j^l\rangle_{(a,c)}$ can always arise from the twisted sector with $I' = 0$, where $I'$ is the number of the invariant fields $X_i$ under the $h'$ action. Using the results of ref.[11], it was shown [8] that as long as we consider the Landau-Ginzburg models with no or one trivial field, the $(-1,1)$ states which can be represented by $|j^l\rangle_{(a,c)}$ may exist only in the twisted sector with $I' = 0$.

Let us turn our attention to geometry. Calabi-Yau manifolds are represented by hypersurfaces in $WCP$. In general, due to the $WCP$ identification $z_i \sim \lambda^{w_i}z_i$, $\lambda \in \mathbb{C}^*$, we have some fixed sets on a hypersurface. When we consider Calabi-Yau 3-folds, possible fixed sets are fixed points and fixed curves. To obtain a smooth Calabi-Yau manifold we have to blow up these singularities.

Those Calabi-Yau resolutions can be described in terms of toric geometry [7, 5]. Toric geometry describes the structure of a certain class of geometrical spaces in terms of simple combinatorial data. To investigate the mirror symmetry, Batyrev’s construction is useful. We will briefly summarize this method. Details are presented in [7, 8].

A (Newton) polyhedron $\Delta(w)$ is associated to monomials, where $w$ means the set of weights $w_i$. A dual polyhedron $\Delta^*(w)$ allows us to describe the resolution of singularities. Integral points on faces of dimension one or two of $\Delta^*(w)$ correspond to exceptional divisors. More precisely, points lying on a one-dimensional edge correspond to exceptional divisors over singular curves, whereas the points lying in the interior of two-dimensional faces correspond to the exceptional divisors over singular points. So, integral points on faces of $\Delta^*(w)$ correspond to the $(1,1)$ forms coming from blowing-up processes.

Since we consider the Fermat type quasihomogeneous polynomial, the corresponding Calabi-Yau hypersurface consists of monomials $z_i^{d/w_i}$ ($i = 1, \cdots, 5$). The associated 4-dimensional integral convex polyhedron $\Delta(w)$ is the convex hull of the integral vectors $m$ of the exponents of all quasi-homogeneous monomials of degree $d$ shifted by $(-1,\ldots,-1)$,
i.e. $\prod_{i=1}^{5} z_i^{m_i+1}$:

$$\Delta(w) := \{(m_1, \ldots, m_5) \in \mathbb{R}^5 | \sum_{i=1}^{5} w_i m_i = 0, m_i \geq -1\}. \quad (2)$$

This implies that only the origin is the point in the interior of $\Delta$. Its dual polyhedron is defined by

$$\Delta^* = \{ (x_1, \ldots, x_4) | \sum_{i=1}^{4} x_i y_i \geq -1 \text{ for all } (y_1, \ldots, y_4) \in \Delta \}. \quad (3)$$

In our case it is known that $(\Delta, \Delta^*)$ is a reflexive pair. An $l$-dimensional face $\Theta \subset \Delta$ can be represented by specifying its vertices $v_{i_1}, \ldots, v_{i_k}$. Then the dual face $\Theta^*$ is a $(4 - l - 1)$-dimensional face of $\Delta^*$ and defined by

$$\Theta^* = \{ x \in \Delta^* | (x, v_{i_1}) = \cdots = (x, v_{i_k}) = -1 \}, \quad (4)$$

where $(\ast, \ast)$ is the ordinary inner product.

For our type of models, we then always obtain as vertices of $\Delta(w)$

$$
\begin{align*}
\nu_1 &= (d/w_1 - 1, -1, -1, -1), \quad \nu_2 = (-1, d/w_2 - 1, -1, -1), \quad \nu_3 = (-1, -1, d/w_3 - 1, -1), \\
\nu_4 &= (-1, -1, -1, d/w_4 - 1), \quad \nu_5 = (-1, -1, -1, -1),
\end{align*}
\quad (5)
$$

and for the vertices of the dual polyhedron $\Delta^*(w)$ one finds

$$
\begin{align*}
\nu_1^* &= (1, 0, 0, 0), \quad \nu_2^* = (0, 1, 0, 0), \quad \nu_3^* = (0, 0, 1, 0), \quad \nu_4^* = (0, 0, 0, 1), \\
\nu_5^* &= (-w_1, -w_2, -w_3, -w_4).
\end{align*}
\quad (6)
$$

For the Fermat type hypersurfaces of degree $d$, the explicit form of the monomial-divisor mirror map has been already studied. Through this map, integral points $\mu$ in $\Delta^*(w)$ are mapped to monomials of the homogeneous coordinates of $WCP^4$ by

$$
\mu = (\mu_1, \mu_2, \mu_3, \mu_4) \mapsto \frac{\prod_{i=1}^{4} z_i^{\mu_i d/w_i}}{(\prod_{i=1}^{5} z_i)^{(\sum_{i=1}^{4} \mu_i) - 1}}. \quad (7)
$$

In the following we will associate an integral point inside $\Delta^*(w)$, i.e. an exceptional divisor, with a $(-1, 1)$ state which can be written in the form $\langle j^{-1} \rangle_{(a,c)}$.

To explain our method, we define the phase symmetries $\rho_i$ which act on $X_i$ as

$$
\rho_i X_i = e^{2\pi i q_i} X_i, \quad (8)
$$
with trivial action for other fields. The operator $\rho_i$ can be represented by a diagonal matrix whose diagonal matrix elements are 1 except for $(\rho_i)_{i,i} = e^{2\pi i q_i}$. It is obvious that

$$j = \rho_1 \cdots \rho_5.$$  \hfill (9)

In ref. [8] the mirror map for the $(a,c)$ ground states in the $j^{-l}$-twisted sector $|j^{-l}\rangle_{(a,c)}$ are considered. In the $j^{-l}$-twisted sector, if a field $X_i$ is invariant then

$$\rho_i^{-l} = \rho_i^{-l} = \text{identity},$$  \hfill (10)

where $-l_i \equiv -l \mod a_i$ and one gets

$$j^{-l} = \prod_{-l_i \notin \mathbb{Z}} \rho_i^{-l_i}.  \hfill (11)$$

So, we may represent $|j^{-l}\rangle_{(a,c)} = |\prod_{-l_i \notin \mathbb{Z}} \rho_i^{-l_i}\rangle_{(a,c)}$. Furthermore, we can calculate the $U(1)$ charges of this state using eq.(11) and the result is

$$(- \sum_{-l_i \notin \mathbb{Z}} l_i q_i, \sum_{-l_i \notin \mathbb{Z}} l_i q_i).$$  \hfill (12)

The eq.(11) is the key equation for our purpose. This implies

$$-l_i q_i = u_i - l_i q_i \quad \text{for} \ i = 1 \sim 5,$$

where $-l_i$ are defined to be $-a_i + 1 \leq -l_i \leq 0$. Thus $u_i$ are uniquely determined. Clearly, $u_i$ are integers and $u_i = 0$ if $X_i$ is invariant under $j^{-l}$ action (see eq.(11)). $u_5$ always vanish since $w_5 = 1$.

Let $u \equiv (u_1, u_2, u_3, u_4)$ be an integral vector. In the following we will show that $u$ is on a face of $\Delta^*(w)$. So we should assert that $u$ is just an integral point inside $\Delta^*(w)$, which can be identified with the exceptional divisors. Through this identification, we obtain the one-to-one correspondence between the $(-1,1)$ state $|j^{-l}\rangle_{(a,c)}$ and the exceptional divisor. Moreover, once this identification is made, we can see that the monomial-divisor mirror map for Calabi-Yau manifolds, i.e. eq.(7), is equivalent to that mirror map for Landau-Ginzburg orbifolds conjectured in ref. [8].

First, we show that $u$ is a point on a dual face $\Theta^*$ of $\Delta^*(w)$. A dual face $\Theta^*$ is specified by some of the vertices $\{\nu_1, \nu_2, \nu_3, \nu_4, \nu_5\} \subset \Theta$ through the eq.(11). So, we have to show that there are some vectors $\nu_j$ satisfying

$$(u, \nu_j) = -1 \quad \text{for some} \ j.$$  \hfill (14)
To prove this, we consider the invariant field $X_j$ under $j^{-l}$ action. This implies $-lq_j = u_j$. It can be shown that the corresponding vector $\nu_j$ satisfies eq. (14). Denote by $(\nu_j)_i$ the $i$-th component of the vector $\nu_j$. Multiplying $u_i$ by $(\nu_j)_i$ and using eq. (13), one finds

$$(u, \nu_j) = -\sum_{i=1}^{5} l_i q_i,$$  \hfill (15)$$

where we have used $\sum_{i=1}^{5} q_i = 1$, $u_5 = 0$, and $-lq_j = u_j$.

It should be noticed that the right hand side of eq. (15) is nothing but the left $U(1)$ charge of the state $|j^{-l}\rangle_{(a,c)}$. Since we consider the $(-1,1)$ state, it has been proved that eq. (14) holds for the vector $\nu_j$ with $-lq_j = u_j$. Note that $\Theta^*$ is specified by the vectors $\nu_j \subset \Theta$ through eq. (4), whose corresponding fields $X_j$ with the same index are invariant under $j^{-l}$ action.

Now we turn our attention to the monomial-divisor mirror map. As mentioned above, the monomial-divisor mirror map for Calabi-Yau manifolds is summarized in eq. (7). In ref. [8], it was conjectured that the monomial-divisor mirror map for Landau-Ginzburg orbifolds is obtained to be

$$\left| \prod_{-lq_i \notin \mathbb{Z}} \rho_i^{-l_i} \right|_{(a,c)} \overset{\text{mirror pair}}{\leftrightarrow} \prod_{-lq_i \notin \mathbb{Z}} X_i^{l_i} |0\rangle,$$  \hfill (16)$$

where $X_i$ are the fields of the so-called transposed potential $W$ [12]. Evidently, $X_i = X_i^{-}$ for our Fermat type potentials.

By identifying the vector $\mu$ in eq. (7) with our vector $u$, we easily obtain

$$u = (u_1, u_2, u_3, u_4) \mapsto \frac{\prod_{i=1}^{4} z_i^{u_i d/w_i}}{(\prod_{i=1}^{5} z_i^{l_i})^{(\sum_{i=1}^{4} u_i) - 1}} = \prod_{-lq_i \notin \mathbb{Z}} z_i^{l_i}. \hfill (17)$$

Since the vector $u$ corresponds to the $(-1,1)$ state $|j^{-l}\rangle_{(a,c)}$ and $z_i$ to $X_i$, one finds that the two monomial-divisor mirror map, i.e. eq. (7) and (16), are equivalent.

Note that through the eqs. (13) and (17), the origin $\nu^*_0 = (0,0,0,0)$ in $\Delta^*(w)$ corresponds to the $(-1,1)$ state $|j^{-1}\rangle_{(a,c)}$ and its mirror partner corresponds to the monomial $z_1 z_2 z_3 z_4 z_5$. They always exist for our type of models.

Let us demonstrate our method by taking some examples. Since we have shown $u \in \Theta^*$, we change our notation of $u$ into $\nu^*_0$ ($\nu^*_i$, ··· if several $u$ exist).
First we consider the Landau-Ginzburg model with the superpotential

\[ W_1 = X_1^4 + X_2^4 + X_3^4 + X_4^8 + X_5^8, \]  

(18)

with \( U(1) \) charges of \( X_i \) being

\[ (\frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}), \]  

(19)

which were studied in ref.\[5, 8\]. The orbifold model \( W_1/j \) has a corresponding \( \mathbb{Z}_2 \) fixed curve which can be written

\[ z_1^4 + z_2^4 + z_3^4 = 0 \quad \text{in } WCP^4_{(1,1,2,2,2)}. \]  

(20)

In this model, we have one twisted ground state \( |j^{-4}\rangle_{(a,c)} \) which corresponds to the resulting one \( (1,1) \) form after blowing-up. Since \( j^{-4} = \rho_4^{-4} \rho_5^{-4} \), we can calculate \( \nu_6^* = (-1,-1,-1,0) \). Also it can be seen that \( \nu_6^* \) is on the dual face \( \Theta^* \) specified by the vectors \( \nu_1, \nu_2, \nu_3 \) (in the following we denote by \( \Theta^*(1,2,3) \) this face). From eq.\((16)\), \( \nu_6^* \) is mapped to the monomial \( X_4^4X_5^8 \). These are the same results as the ones obtained in \[5\].

As a more complicated example, we take the following Landau-Ginzburg superpotential

\[ W_2 = X_1^3 + X_2^3 + X_3^6 + X_4^9 + X_5^{18}, \]  

(21)

with \( U(1) \) charges

\[ (\frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{9}, \frac{1}{18}). \]  

(22)

This model is considered in \[5, 8\]. The orbifold model \( W_2/j \) has one corresponding \( \mathbb{Z}_2 \) fixed curve, one corresponding \( \mathbb{Z}_3 \) fixed curve and corresponding \( \mathbb{Z}_6 \) fixed points on the intersections of these curves. They can be written as

\[ \mathbb{Z}_2 \text{ fixed curve } z_1^3 + z_2^3 + z_4^9 = 0 \]  

(23)

\[ \mathbb{Z}_3 \text{ fixed curve } z_1^3 + z_2^3 + z_3^6 = 0 \]  

(24)

\[ \mathbb{Z}_6 \text{ fixed points } z_1^3 + z_2^3 = 0 \quad \text{in } WCP^4_{(6,6,3,2,1)}. \]  

(25)

After blowing-up, one obtains four \( (1,1) \) forms whose mirror partners can be associated to monomial deformation. It is easy to find the corresponding \( (-1,1) \) states written in the form \( |j^{-1}\rangle_{(a,c)} \), the vectors \( \nu_j^* \), dual faces \( \Theta^* \) on which \( \nu_j^* \) are lying, and mirror partners. The results are displayed in Table \[6\] where we have omitted the bar over \( X_i \).

This result agrees with the one obtained in \[\[6\]\]. Note that we do not need any geometrical informations such as the number of fixed sets or the relations among them.


| $(a,c)$ state | $\nu_j^*$ vector | dual face | mirror partner |
|----------------|------------------|-----------|----------------|
| $|j^{-3}\rangle_{(a,c)}$ | $\nu_0^* = (-1, -1, 0, 0)$ | $\Theta^*(1, 2)$ | $X_3^3 X_4^3 X_5^3 |0\rangle$ |
| $|j^{-6}\rangle_{(a,c)}$ | $\nu_7^* = (-2, -2, -1, 0)$ | $\Theta^*(1, 2, 3)$ | $X_4^6 X_5^6 |0\rangle$ |
| $|j^{-9}\rangle_{(a,c)}$ | $\nu_8^* = (-3, -3, -1, -1)$ | $\Theta^*(1, 2, 4)$ | $X_3^3 X_5^9 |0\rangle$ |
| $|j^{-12}\rangle_{(a,c)}$ | $\nu_9^* = (-4, -4, -2, -1)$ | $\Theta^*(1, 2, 3)$ | $X_4^3 X_5^{12} |0\rangle$ |

Table 1: The monomial-divisor mirror map for Landau-Ginzburg orbifolds of $W_2$

In this model, there are two $(-1, 1)$ states represented by $X_1|j^{-2}\rangle_{(a,c)}$ and $X_2|j^{-2}\rangle_{(a,c)}$. The mirror partners for the corresponding $(1, 1)$ forms cannot be described by monomials. Unfortunately, we have not succeeded in associating these states with toric data yet.

In conclusion, we have discovered a simple and direct connection between the Landau-Ginzburg and the toric descriptions of our class of Calabi-Yau manifolds. In other words, we find that a Landau-Ginzburg orbifold and a corresponding Calabi-Yau manifold have just the same toric data, as far as models with a typical type of superpotentials are concerned. Our method enables us to calculate toric data easily without referring to the geometrical nature of $\Delta^*(w)$. We can uniquely identify a $(-1, 1)$ state $|j^{-l}\rangle_{(a,c)}$ with a $(1, 1)$ form resulting from resolution. This would be a useful technique for analyzing the Yukawa couplings, especially when Calabi-Yau manifolds with several $(1, 1)$ forms are considered.

The Fermat type superpotential considered in this paper corresponds to the Gepner model of A-type [13]. However, there are non-Fermat-type potentials with only the singularities which can be treated through toric geometry. For example, there is the hypersurface embedded in $WCP^4_{(1,2,2,2,3)}$, which gets two toric divisors after blowing up [14]. For this model, our method for the calculation of the vector $u$ could not be applied as it is. Some extensions are needed. We will report it elsewhere [15].

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